On the structure of the S-matrix in general boundary quantum field theory in curved space

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We use the general boundary formulation (GBF) of quantum field theory to compute the S-matrix for a general interacting scalar field in curved spacetime. This S-matrix is obtained as the asymptotic limit of the GBF-amplitude associated with a finite spacetime region. Two types of regions are of interest here. The first type is bounded by the disjoint union of two (non-compact) hypersurfaces. For the special choice of Cauchy hypersurfaces, the asymptotic GBF-amplitude reduces to the standard S-matrix. The second type of region is enclosed by one connected and timelike boundary, called the hypercylinder region. As a by-product we obtain the general expression of the Feynman propagator for the field defined in both regions. Our work generalizes previous results obtained in Minkowski and de Sitter spacetimes to (a wide class of) curved spacetimes.

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I. INTRODUCTION

The purpose of this paper is the derivation of the general structure of the scattering matrix for a quantum scalar field defined on a broad class of flat and curved spacetimes. In the usual treatment, the S-matrix refers to scattering processes for states defined on (asymptotic) spacelike Cauchy surfaces. The results we present generalize this situation to the case of states defined on certain classes of timelike hypersurfaces. This new type of S-matrices can be explicitly computed, the corresponding Feynman rules of perturbation theory can derived, and an appropriate notion of probability can be extracted from them, by adopting the general boundary formulation (GBF) of quantum field theory [1–4]. The novelty of the GBF resides firstly in associating to arbitrary hypersurfaces in spacetime Hilbert spaces of states. Cauchy surfaces are then only a special choice, not obligatory within the GBF. Secondly, amplitudes are associated with spacetime regions and determined by a linear map from the Hilbert spaces defined on the boundaries of these regions to the complex numbers. Such structures are required to satisfy a set of axioms that guarantees their coherence. Finally, the GBF provides a consistent probabilistic interpretation for these amplitudes.

The new types of scattering matrices can offer a novel perspective on standard QFT, by shedding light on geometrical aspects of the theory. Moreover, this extension provided by the GBF is a necessity in situations where the usual S-matrix fails for some reason, as will be the case for example in Anti-de Sitter spacetime where no asymptotic temporal regions exist. Another situation where the GBF is expected to offer the appropriate tools is in describing the dynamics of fields in the presence of an eternal black hole: no free temporal asymptotic states can be defined since the interaction with the black hole will never vanish in time. However, far away from the black hole, a notion of free spatial asymptotic states is available, and one can compute amplitudes for these states within the GBF.

Our main aim here is to contribute to the development of the GBF. Indeed this work can be seen as a generalization of previous results obtained in Minkowski [5, 6], Euclidean [7] and de Sitter spacetimes [8, 9]. Inspired by these papers, we will consider two classes of spacetime regions. The first one is characterized by a boundary given by the disjoint union of hypersurfaces, that we do not require to be Cauchy or even spacelike. Apart from the metric nature of the hypersurfaces involved, this is close to the usual situation in the sense that the dynamics taking place in these regions is understood as an evolution from an initial
hypersurface to a final one. The second class of regions are radically different: Their boundary is completely connected and timelike, two aspects not treatable within the standard formulation of QFT. "Evolution" takes place inside the regions enclosed by the boundary.

The outline of the paper is as follows. In the next section we introduce the two types of spacetime regions we will be interested. We describe there the classical theory of a real scalar field, expressing the solution of the Klein-Gordon equation in terms of boundary field configurations. In Sec. [III] the GBF quantization prescription is described and the main structures corresponding to the different regions considered are defined. In Sec. [IV] the quantum amplitudes for states of the free theory are computed. This result is obtained following different steps. First, we evaluate the path integral of the field propagator for the regions of interest in Sec. [IV]A. Then, vacuum and coherent states are introduced in Sec. [IV]B and Sec. [IV]C respectively. Finally, in Sec. [IV]D the expressions of the free amplitudes are given. The interacting theory is treated in Sec. [V] starting with the interaction of the scalar field with a source field (Sec. [V]A and Sec. [V]B), and subsequently using functional derivative techniques to obtain the amplitudes for the general interacting theory in Sec. [V]C. We summarize and discuss our results in Sec. [VII].

II. CLASSICAL THEORY

We consider a real massive minimally coupled Klein-Gordon field \( \phi \) in a 4-dimensional curved spacetime manifold with Lorentzian signature and metric tensor \( g_{\mu\nu} \). The action in a spacetime region \( M \) is

\[
S_{M,0}(\phi) = \frac{1}{2} \int_M d^4x \sqrt{|g|} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right),
\]

where the integration is extended over the region \( M \) and we use the notation \( \partial_\mu = \partial / \partial x^\mu \), \( g \) is the determinant of the metric tensor, \( g \equiv \det g_{\mu\nu} \), and \( m \) indicates the mass of the field. The subscript 0 refers to the free theory. The variation of the action yields the Klein-Gordon equation,

\[
\left( \frac{1}{\sqrt{|g|}} \partial_\mu \left( \sqrt{|g|} g^{\mu\nu} \partial_\nu \phi \right) + m^2 \right) \phi(x) = 0.
\]

The action for a solution of the Klein-Gordon equation can be computed performing an integration by parts in (1) and using equation (2), yielding

\[
S_{M,0}(\phi) = \frac{1}{2} \int_{\partial M} d^3s \sqrt{|g^{(3)}|} n_\mu \phi (g^{\mu\nu} \partial_\nu \phi),
\]

where \( s \) denotes some coordinates on the boundary \( \partial M \) of \( M \), \( g^{(3)} \) is the determinant of the induced metric on the boundary and \( n_\mu \) is the outward normal to \( \partial M \).

We will be interested in studying the dynamics of the field in two different types of spacetime regions called slice regions and hypercylinder regions respectively. The next two subsections are devoted to the definition of these regions and the expression of the action for the field defined there in terms of the boundary field configurations \( \varphi \),

\[
\phi(x)|_{x \in \partial M} = \varphi(s),
\]

where \( s \) are generic coordinates on \( \partial M \).

A. Slice regions

In this section we will follow the treatment of [13]. For the first type of regions, we suppose a smooth coordinate system \((\tau, \mathbf{x})\) which defines a foliation of the spacetime. The coordinates on the leaves of the foliation are denoted by \( x = (x^1, x^2, x^3) \in \mathbb{R}^3 \), while \( \tau \in \mathbb{R} \) indicates the foliation parameter indexing the leaves. We also require the metric to be block diagonal with respect to the foliation, i.e. \( g^{\alpha\beta} = 0 = g^{x^i\tau} \) for all \( i \in \{1, 2, 3\} \). It is important to notice that \( \tau \) does not necessarily need to be the time of the physical theory, and consequently \( x \) are not required to be spatial coordinates.
The first spacetime region we consider is bounded by the disjoint union of two constant-τ hypersurfaces, \( \Sigma_1 = \{ (\tau, x) : \tau = \tau_1 \} \) and \( \Sigma_2 = \{ (\tau, x) : \tau = \tau_2 \} \), namely the region is \( M = [\tau_1, \tau_2] \times \mathbb{R}^3 \). We refer to this region as the slice region, and we use the subscript \( [\tau_1, \tau_2] \) in the relevant quantities. The free action (3) in the slice region takes the form

\[
S_{[\tau_1, \tau_2],0}(\phi) = \frac{1}{2} \int d^3x \left( \sqrt{|g^{(3)}_{\tau_1} g^{(3)}_{\tau_2}|} \phi(\tau_2, x) (\partial_\tau \phi)(\tau_2, x) - \sqrt{|g^{(3)}_{\tau_1} g^{(3)}_{\tau_2}|} \phi(\tau_1, x) (\partial_\tau \phi)(\tau_1, x) \right),
\]

where \( g^{(3)}_{\tau_1} \) and \( g^{(3)}_{\tau_2} \) denote the metric restricted to the hypersurfaces \( \Sigma_1 \) and \( \Sigma_2 \) respectively. We assume that a solution of the Klein-Gordon equation can be written as

\[
\phi(\tau, x) = (X_\tau(x) Y_\tau(x) (\phi)) + (X_\tau(\tau) Y_\tau(\tau) (\phi)),
\]

where each \( X_\tau(\tau) \) is to be understood as a linear operator from the space of "initial" data \( Y_\tau \) to solutions on hypersurfaces at fixed values of \( \tau \). In particular each \( X_\tau(\tau) \) acts as an operator on a mode decomposition of \( Y_\tau \) and we use the same notation \( X_\tau(\tau) \) for the eigenvalues of the corresponding operator. In the following we shall also assume that all these operators commute among each other. In the case of non-commuting operators, the computations become more involved, as can be seen from Appendix A of [13]. Also we note that since the Klein-Gordon equation is real all the operators, the computations become more involved, as can be seen from Appendix A of [13].

The symplectic form on the space of smooth solutions of (2) on \( \Sigma_{[\tau_1, \tau_2]} \) is

\[
\omega(\phi_1, \phi_2) = \frac{1}{2} \int g^{(3)} g^{(3)} \sqrt{|g^{(3)} g^{(3)}|} (\phi_1 (\partial_\tau \phi_2 - \phi_2 (\partial_\tau \phi_1)),
\]

where

\[
\Delta_1(\tau_1, \tau_2) := \partial_\tau \Delta(\tau_2) \big|_{\tau = \tau_1}, \quad \Delta_2(\tau_1, \tau_2) := \partial_\tau \Delta(\tau) \big|_{\tau = \tau_2}.
\]

The symplectic form on the space of smooth solutions of (2) on \( \Sigma_{[\tau_1, \tau_2]} \) is

\[
\omega(\phi_1, \phi_2) = \frac{1}{2} \int \Sigma d^3x \sqrt{|g^{(3)} g^{(3)}|} (\phi_1 (\partial_\tau \phi_2 - \phi_2 (\partial_\tau \phi_1)),
\]

which is independent of the choice of leaf \( \Sigma \) of the foliation. (See [10]. In case of the leaves being spacelike, this is just the standard symplectic form.) This implies that the operator

\[
\mathcal{W} := \sqrt{|g^{(3)} g^{(3)}|} \Delta_2(\tau, \tau) = -\sqrt{|g^{(3)} g^{(3)}|} \Delta_1(\tau, \tau)
\]

is independent of \( \tau \). Note that this immediately implies \( \mathcal{W}^{(1,2)}_{[\tau_1, \tau_2]} = \mathcal{W}^{(2,1)}_{[\tau_1, \tau_2]} \). For later convenience, it is useful to introduce a decomposition of the boundary field configurations in terms of modes \( \varphi_k(x) \), where \( k \) denotes the set of three parameters labeling the modes:

\[
\varphi(x) = \int d^3k c(k) \varphi_k(x).
\]
For discrete values of the parameters $\mathbf{a}$, the integral will be replaced by a sum. We assume that this set of modes forms an orthonormal basis in the space of boundary field configurations,

$$
\int d^3 \varphi_k(\mathbf{x}) \overline{\varphi_k(\mathbf{x}')} = \delta(\mathbf{x} - \mathbf{x}').
$$

(14)

In Minkowski spacetime, for regions bounded by two equal time hypersurfaces, this set corresponds to the set of plane wave modes.

B. Hypercylinder regions

In order to define the second region of interest, we introduce a second foliation of the spacetime, defined by a smooth coordinate system $(r, t, \vartheta, \varphi)$, where $t \in \mathbb{R}$ is now the time variable and $r \in \mathbb{R}^+, \vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ are spatial spherical coordinates. In the following we will use the collective notation $\Omega$ for the variables $\vartheta$ and $\varphi$. The leaves of this new foliation are the hypersurfaces of constant $r$. We will refer to these leaves as the hypercylinders. Again we require the metric to be block diagonal with respect to the foliation.

In terms of this new foliation we define two kinds of regions: The region bounded by the disjoint union of two hypercylinders of different radii, which is analogous to the slice regions defined above. This region $M = [R_1, R_2] \times \mathbb{R} \times [0, \pi] \times [0, 2\pi]$, where $R_1$ and $R_2$ are the respective radii of the two hypercylinders, will be denoted by the subscript $[R_1, R_2]$. The second kind of region is bounded by one hypercylinder and we will use the subscript $R$ to refer to the hypercylinder of radius $R$. Notice in this case the full connectedness of the boundary of the spacetime region in which the dynamics of the scalar field will be considered.

The free action (3) in these regions takes the form

$$
S_{[R_1, R_2], 0}(\phi) = -\frac{1}{2} \int dt \, d\Omega \left( \sqrt{|g_{R_2}^{(3)} g_{R_2}^{(3)TT}|} \phi(R_2, t, \Omega)(\partial_r \phi)(R_2, t, \Omega) - \sqrt{|g_{R_1}^{(3)} g_{R_1}^{(3)TT}|} \phi(R_1, t, \Omega)(\partial_r \phi)(R_1, t, \Omega) \right),
$$

(15)

$$
S_{R, 0}(\phi) = -\frac{1}{2} \int dt \, d\Omega \sqrt{|g_{R_1}^{(3)} g_{R_1}^{(3)TT}|} \phi(R, t, \Omega)(\partial_r \phi)(R, t, \Omega),
$$

(16)

where $g_{R}^{(3)}$ denotes the metric restricted to the hypercylinder of radius $R$. Assuming a decomposition of the solution of the Klein-Gordon equation analogous to (4), in the two hypercylinder regions the field expressed in terms of the boundary configurations reads

$$
\phi(r, t, \Omega) = \left( \frac{\Delta(r, R_2)}{\Delta(R_1, R_2)} \varphi_1 \right)(t, \Omega) + \left( \frac{\Delta(R_1, r)}{\Delta(R_1, R_2)} \varphi_2 \right)(t, \Omega),
$$

(17)

$$
\phi(r, t, \Omega) = \left( \frac{X_a(r)}{X_a(R)} \varphi \right)(t, \Omega),
$$

(18)

where we have adopted the same notation as in the previous subsection. Expression (15) needs a comment. In the cases studied so far, i.e. a scalar theory in Minkowski space $\mathbb{R}^4$, in de Sitter space $\mathbb{R}^4_\Lambda$, and in 2d Euclidean space $\mathbb{R}^2$, the Klein-Gordon equation expressed in spherical (or polar) coordinates reduces to a certain Bessel equation, with two independent solutions provided by the spherical Bessel functions of the first and second kind respectively. These Bessel functions have different behavior at the origin: the first one is regular while the second one diverges at the origin. Consequently, since the hypercylinder region (the disk region in the 2d Euclidean theory) contains the origin, only the spherical Bessel function of the first kind is admissible to obtain a smooth solution of the Klein-Gordon equation. We are assuming a similar situation here, where $X_a$ represents the regular solution to the radial part of the Klein-Gordon equation$^1$. We can now write the action,

$$
S_{[R_1, R_2], 0}(\phi) = \frac{1}{2} \int dt \, d\Omega \left( \varphi_1 \varphi_2 \right) W_{[R_1, R_2]}(\varphi_1, \varphi_2),
$$

(19)

---

$^1$ This will be the case for spaces conformal to (a portion of) Minkowski spacetime.
where the $W_{[R_1, R_2]}^{(i,j)}$ is a $2 \times 2$ matrix with elements $W_{[R_1, R_2]}^{(i,j)}(i, j = 1, 2)$, given by

\[
\begin{align*}
W_{[R_1, R_2]}^{(1,1)} &= \sqrt{|g_{R_1}^{(3)} g_{R_2}^{(3)}|} \frac{\Delta_1(R_1, R_2)}{\Delta(R_1, R_2)}, & W_{[R_1, R_2]}^{(1,2)} &= \sqrt{|g_{R_1}^{(3)} g_{R_2}^{(3)}|} \frac{\Delta_2(R_1, R_1)}{\Delta(R_1, R_2)}, \\
W_{[R_1, R_2]}^{(2,1)} &= -\sqrt{|g_{R_1}^{(3)} g_{R_2}^{(3)}|} \frac{\Delta_1(R_1, R_2)}{\Delta(R_1, R_2)}, & W_{[R_1, R_2]}^{(2,2)} &= -\sqrt{|g_{R_1}^{(3)} g_{R_2}^{(3)}|} \frac{\Delta_2(R_1, R_2)}{\Delta(R_1, R_2)},
\end{align*}
\]

(20)

\[
\Delta_1(R_1, R_2) := \partial_r \Delta(r, R_2)|_{r=R_1}, & \quad \Delta_2(R_1, R_2) := \partial_r \Delta(R_1, r)|_{r=R_2}.
\]

(21)

And the free action of the field in the hypercylinder of radius $R$ is

\[
S_{R,0}(\phi) = -\frac{1}{2} \int dt d\Omega \sqrt{|g_{R}^{(3)} g_{R}^{(3)}|} \varphi(t, \Omega) \left( \frac{X' \varphi}{X \varphi} \right)(t, \Omega),
\]

(22)

where the prime denotes a derivate with respect to the radial coordinate. The space of smooth solutions of the Klein-Gordon equation in the hypercylinder regions is equipped with the symplectic form

\[
\omega(\phi_1, \phi_2) = \frac{1}{2} \int dt d\Omega \sqrt{|g_{R}^{(3)} g_{R}^{(3)}|} (\phi_1 \partial_r \phi_2 - \phi_2 \partial_r \phi_1),
\]

(23)

which is independent of the choice of the hypercylinder, namely \ref*{23} is independent of the radius of the hypercylinder. We then have that the operator

\[
\mathcal{W} := \sqrt{|g_{R}^{(3)} g_{R}^{(3)}|} \Delta_2(r, R) = -\sqrt{|g_{R}^{(3)} g_{R}^{(3)}|} \Delta_1(r, R)
\]

(24)

is independent of $r$, which in turn implies $W_{[R_1, R_2]}^{(1,2)} = W_{[R_1, R_2]}^{(2,1)}$. As in the previous section we introduce a decomposition of the boundary field configuration in terms of modes $\varphi_{\omega}(t, \Omega),^2$

\[
\varphi(t, \Omega) = \int d^3 p \varphi_{\omega}(t, \Omega).
\]

(25)

An orthonormal relation analogous to \ref*{14} will be assumed.

\section{III. Quantum Theory}

We adopt a Schrödinger-Feynman quantization, namely the quantum states of the field are described by wave functionals on the space of field configurations, and the path integral quantization is implemented. According to the axioms of the GBF, to each oriented hypersurface $\Sigma$ we associate the Hilbert space $\mathcal{H}_\Sigma$ of wave functionals of field configurations on $\Sigma$ with the inner product

\[
\langle \psi_\Sigma | \psi'_\Sigma \rangle := \int \mathcal{D}\varphi \, \psi_\Sigma(\varphi) \, \psi'_\Sigma(\varphi),
\]

(26)

where the integral is over all field configurations $\varphi$ on the hypersurface $\Sigma$. Next, standard transition amplitudes are generalized in the GBF to amplitudes given by the linear map $\rho_M : \mathcal{H}_{\partial M} \rightarrow \mathbb{C}$ associated to spacetime regions $M$. Boundary state spaces and amplitudes are required to satisfy a number of consistency axioms $^2$. The amplitude $\rho_M$ for a state $\psi$ is defined as

\[
\rho_M(\psi) = \int \mathcal{D}\varphi \, \psi(\varphi) \, Z_M(\varphi),
\]

(27)

^2 In Minkowski and de Sitter spacetimes, such basis was expressed in terms of spherical harmonics. Consequently part of the numbers $p$ were the angular momentum quantum numbers $(l, m)$ taking only discrete values.
where $Z_M$ is the field propagator encoding the information on the dynamics of the field in the spacetime region $M$,

$$Z_M(\varphi) = \int_{\phi|_{\partial M} = \varphi} D\phi \, e^{iS_M(\phi)}, \quad (28)$$

$S_M(\phi)$ is the action of the field in the region $M$ and the integration is extended over all field configurations $\phi$ matching the boundary configuration $\varphi$ on the boundary $\partial M$. If we consider the hypercylinder region of radius $R$, the field propagator of the theory reads

$$Z_R(\varphi) = \int_{\phi|_{R} = \varphi} D\phi \, e^{iS_R(\phi)}, \quad (29)$$

where $S_R(\phi)$ is the action in the hypercylinder region. In the case of the slice regions, since the boundary hypersurface is the disjoint union of two disconnected hypersurfaces the boundary state space is the tensor product of two Hilbert spaces $H_1 \otimes H_2$, where the dualization of the Hilbert space associated with the hypersurfaces at $\tau_2$ comes from the different orientation of this hypersurface. A state in this Hilbert space is then $\psi_{\tau_1} \otimes \psi_{\tau_2}$ and the amplitude for this state takes the form

$$\rho_{[\tau_1, \tau_2]}(\psi_{\tau_1} \otimes \psi_{\tau_2}) = \int D\varphi_1 D\varphi_2 \psi_{\tau_1}(\varphi_1) \psi_{\tau_2}(\varphi_2) Z_{[\tau_1, \tau_2]}(\varphi_1, \varphi_2), \quad (30)$$

where the field propagator is

$$Z_{[\tau_1, \tau_2]}(\varphi_1, \varphi_2) = \int_{\phi|_{[\tau_1]} = \varphi_1, \phi|_{[\tau_2]} = \varphi_2} D\phi \, e^{iS_{[\tau_1, \tau_2]}(\phi)}. \quad (31)$$

The most important condition amplitudes are required to satisfy is the composition property, that relates the amplitudes associated with two spacetime regions with the amplitude associated with the union of these regions. In particular consider two slice regions sharing a boundary, namely $M_{[\tau_1, \tau_2]}$ and $M_{[\tau_2, \tau_3]}$. The composition of the amplitudes $\rho_{[\tau_1, \tau_2]}$ and $\rho_{[\tau_2, \tau_3]}$ must equal the amplitude $\rho_{[\tau_1, \tau_3]}$. This translates into the following identity for the propagator:

$$Z_{[\tau_1, \tau_3]}(\varphi_1, \varphi_3) = \int D\varphi_2 \, Z_{[\tau_1, \tau_2]}(\varphi_1, \varphi_2) Z_{[\tau_2, \tau_3]}(\varphi_2, \varphi_3). \quad (32)$$

The analogous composition relation for the hypercylinder region is

$$Z_{R_2}(\varphi_2) = \int D\varphi_1 \, Z_{[R_1, R_2]}(\varphi_1, \varphi_2) Z_{R_2}(\varphi_2). \quad (33)$$

**IV. FREE THEORY**

Firstly we consider the quantum free scalar theory in slice and hypercylinder regions. We start with the expression of the free field propagators associated to the different regions. Then we define the vacuum and coherent states and finally compute the free amplitude for the coherent states in the two regions of interest.

**A. Field propagators**

The free field propagator $Z_{[\tau_1, \tau_2]}$ for the slice region $M_{[\tau_1, \tau_2]}$ can be evaluated by shifting the integration variable by a classical solution $\phi_{cl}$ which matches the boundary configurations $\varphi_1$ and $\varphi_2$ at $\tau = \tau_1, \tau = \tau_2$ respectively,

$$Z_{[\tau_1, \tau_2], 0}(\varphi_1, \varphi_2) = \int_{\phi|_{[\tau_1]} = \varphi_1, \phi|_{[\tau_2]} = \varphi_2} D\phi \, e^{iS_{[\tau_1, \tau_2], 0}(\phi)} = N_{[\tau_1, \tau_2], 0} e^{iS_{[\tau_1, \tau_2], 0}(\varphi_1, \varphi_2)}, \quad (34)$$
where the free action $S_{[\tau_1,\tau_2],0}(\varphi_1,\varphi_2)$ is given by (35) and the normalization factor is formally given by

$$N_{[\tau_1,\tau_2],0} = \int_{\phi|_{\Sigma_1}=\phi|_{\Sigma_2}=0} \mathcal{D}\phi e^{iS_{[\tau_1,\tau_2],0}(\phi)}.$$  (35)

In [13] it was shown that the explicit form of this normalization factor may be fixed by requiring the fulfillment of the composition property (32), resulting in the following expression

$$N_{[\tau_1,\tau_2],0} = \left( \frac{\det iW^{(1,2)}_{[\tau_1,\tau_2]}}{2\pi} \right)^{-1/2}.$$  (36)

Applying the same technique the free field propagator in the hypercylinder region $M_R$ results to be

$$Z_{R,0}(\varphi) = N_{R,0} e^{iS_{R,0}(\varphi)},$$  (37)

with $S_{R,0}(\varphi)$ given by (16). The composition rule (33) leads to the following relation for the normalization factors of two hypercylinders of radii $R_1$ and $R_2$

$$N_{R_2,0} = N_{R_1,0} \left( \frac{X_a(R_2)}{X_a(R_1)} \right)^{-1/2}.$$  (38)

An explicit expression for $N_{R,0}$, satisfying this condition, can be obtained using the vacuum state, that will be introduced in the next subsection.

**B. Vacuum states**

According to the axioms of the GBF, a vacuum state is associated with each leaf of the foliations considered. We assume that the wave functional describing this state on a generic hypersurface $\Sigma$ has the form of a Gaussian,

$$\psi_{\Sigma,0}(\varphi) = C_\Sigma \exp \left( -\frac{1}{2} \int d^3s \varphi(s)(A_\Sigma \varphi)(s) \right),$$  (39)

where $s$ denotes a generic coordinate system on $\Sigma$, $A_\Sigma$ is some operator and $C_\Sigma$ is a normalization factor satisfying

$$|C_\Sigma|^2 = \det \left( \frac{A_\Sigma + \bar{A}_\Sigma}{2\pi} \right)^{1/2}.$$  (40)

The general form of the operator $A_\Sigma$ has been derived in [11],

$$A_\Sigma = -i \sqrt{|g^3_\Sigma|} \frac{\partial_n (\Upsilon(s^0))}{\Upsilon(s^0)},$$  (41)

where $g^3_\Sigma$ is the three-metric induced on $\Sigma$, $\partial_n$ is the normal derivative to $\Sigma$. In particular we have that $\partial_n = \sqrt{|g^0_\Sigma|} \partial/\partial s^0$. Also, we have introduced the notation

$$\Upsilon(s^0) := c_a X_a(s^0) + c_b X_b(s^0),$$  (42)

where $c_a$ and $c_b$ are complex numbers satisfying the condition $\overline{c_a} c_b - \overline{c_b} c_a \neq 0$, and $s^0$ is the parameter indexing the foliation. The normalization factor of the vacuum state can then be expressed as

$$C_\Sigma = \det \left( i \sqrt{|g^3_\Sigma|} \frac{\partial_n (\Upsilon(s^0))}{\Upsilon(s^0)} \right)^{1/4}.$$  (43)
One easily checks that this expression satisfies (41) as well as the relation between two normalization factors referring to two leaves of the foliation, obtained evolving the vacuum state with the free field propagator associated with this slice region.

The axioms of the GBF fix the vacuum state to have the unit amplitude in the case of the free theory, \( \rho_{M,0}(\psi_0) = 1 \). This implies a relation between the normalization factor of the free field propagator and the one of the vacuum state.

For the slice regions we have

\[
1 = \rho_{[\tau_1, \tau_2],0}(\psi_{\tau_1,0} \otimes \overline{\psi_{\tau_2,0}}) = \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \psi_{\tau_2,0}(\varphi_2) \psi_{\tau_1,0}(\varphi_1) Z_{[\tau_1, \tau_2],0}(\varphi_1, \varphi_2),
\]

which implies the following equality for the normalization factors

\[
1 = C_{\tau_2} C_{\tau_1} N_{[\tau_1, \tau_2],0} \det \left( \frac{A_{\tau_1} - iW_{\tau_1,0}}{2\pi} \right)^{-1/2} \det \left( \frac{A_{\tau_2}}{2\pi} \right)^{-1/2}.
\]

The normalization factor (43) specialized to the slice region considered here solves this equality, where (36) has to be used. For the hypercylinder region, the amplitude of the vacuum state allows us to fix the normalization factor \( N_{R,0} \) of the free field propagator,

\[
1 = \rho_{R,0}(\psi_{R,0}) = \int \mathcal{D}\varphi \psi_{R,0}(\varphi) Z_{R,0}(\varphi),
\]

from which we obtain the relation

\[
1 = N_{R,0} C_R \det \left( \frac{A_R - iW_R}{2\pi} \right)^{-1/2},
\]

where \( W_R = -\sqrt{g_R^{(3)} g_R^{(1)}} X_\psi(R)/X_\eta(R) \), with the prime denoting a derivative with respect to the radial coordinate. A solution to (47) is provided by

\[
N_{R,0} = \det \left( \frac{2\pi (c_\psi c_\eta - \overline{c_\psi} c_\eta) X_\psi^2(R)}{\sqrt{g_R^{(3)} g_R^{(1)}} \Delta_2(R, R)} \right)^{-1/4}.
\]

We can easily check that this expression satisfies relation (48).

### C. Coherent states

In this section we introduce coherent states since they have been a useful tool in \([6, 9]\) to compute amplitudes. We define the coherent states for the Klein-Gordon field in Schrödinger representation in terms of a complex function \( \eta \),

\[
\psi_{\Sigma, \eta}(\varphi) = K_{\Sigma, \eta} \exp \left( \int d^3 s \eta(s) \varphi(s) \right) \psi_{\Sigma,0}(\varphi),
\]

where the normalization factor \( K_{\Sigma, \eta} \) is given by

\[
K_{\Sigma, \eta} = \exp \left( -\frac{1}{2} \int d^3 s \eta(s) A_{\Sigma}^{-1} (\eta(s) + \overline{\eta(s)}) \right)
\]

wherein \( A_{\Sigma} := (A_{\Sigma} + \overline{A_{\Sigma}}) \). The inner product of two coherent states defined by the complex functions \( \eta_1 \) and \( \eta_2 \) results to be

\[
\langle \psi_{\Sigma, \eta_1} | \psi_{\Sigma, \eta_2} \rangle = \exp \int d^3 s \left( \overline{\eta_2(s)} A_{\Sigma}^{-1} \eta_1(s) - \frac{1}{2} \eta_1(s) A_{\Sigma}^{-1} \eta_1(s) - \frac{1}{2} \eta_2(s) A_{\Sigma}^{-1} \eta_2(s) \right).
\]
Coherent states remain coherent under free evolution. This means that in the slice regions we have the relation

\[
\psi_{\tau_2,\eta_2}(\phi_2) = \int D\phi_1 \psi_{\tau_1,\eta_1}(\phi_1) Z_{[\tau_1,\tau_2],0}(\phi_1,\phi_2),
\]

(52)

\[
= K_{\tau_1,\eta_1} C_{\tau_1} N_{[\tau_1,\tau_2],0} \int D\phi_1 \exp \left( \frac{1}{2} \int d^3\phi \left[ \eta_1 + iW_{[\tau_1,\tau_2]}^{(1,2)} \phi_1 - \frac{1}{2} \phi_1 \right] \left( A_{\tau_1} - iW_{[\tau_1,\tau_2]}^{(1,1)} \phi_1 \right) \right) \times \exp \left( \frac{1}{2} \int d^3\phi \left[ \phi_2 W_{[\tau_1,\tau_2]}^{(2,2)} \phi_2 \right] \right).
\]

(53)

We shift the integration variable by the quantity \((\eta_1 + iW_{[\tau_1,\tau_2]}^{(1,2)} \phi_2)/(A_{\tau_1} - iW_{[\tau_1,\tau_2]}^{(1,1)})\) and obtain,

\[
\psi_{\tau_2,\eta_2}(\phi_2) = K_{\tau_1,\eta_1} \exp \left( \frac{1}{2} \int d^3\phi \left[ \frac{\eta_1}{A_{\tau_1} - iW_{[\tau_1,\tau_2]}^{(1,1)}} \left( \frac{1}{2} \phi_1 \right) + 2\eta_1 \frac{iW_{[\tau_1,\tau_2]}^{(1,2)}}{A_{\tau_1} - iW_{[\tau_1,\tau_2]}^{(1,1)}} \phi_2 \right] \right) \psi_{\tau_1,0}(\phi_2),
\]

(54)

where the following equality, compatible with both (40) and (45), has been used,

\[
C_{\tau_1} N_{[\tau_1,\tau_2],0} \det \left( \frac{A_{\tau_1} - iW_{[\tau_1,\tau_2]}^{(1,1)}}{2\pi} \right)^{-1/2} = C_{\tau_2}.
\]

(55)

Equation (54) implies the following relations,

\[
\eta_1 \frac{iW_{[\tau_1,\tau_2]}^{(1,2)}}{A_{\tau_1} - iW_{[\tau_1,\tau_2]}^{(1,1)}} = \eta_2,
\]

(56)

\[
K_{\tau_1,\eta_1} \exp \left( \frac{1}{2} \int d^3\phi \left[ \frac{\eta_1}{A_{\tau_1} - iW_{[\tau_1,\tau_2]}^{(1,1)}} \right] \right) = K_{\tau_2,\eta_2}.
\]

(57)

Substituting (56) in (57) and using expressions (41) and (9), we can show with simple algebra that relation (57) is indeed satisfied. From (56) we derive the equality

\[
(\Upsilon(\tau_1)\eta_1)(\bar{x}) = (\Upsilon(\tau_2)\eta_2)(\bar{x}).
\]

(58)

We define the quantity \(\xi(x) = (\Upsilon(\tau))\eta(\bar{x})\) which is independent of \(\tau\). This allows to define coherent states in the interaction picture as

\[
\psi_{\tau,\xi}(\phi) = K_{\tau,\xi} \exp \left( \int d^3x \frac{\xi(x)}{\Upsilon(\tau)} \phi(x) \right) \psi_{\tau,0}(\phi),
\]

(59)

and the normalization factor results to be

\[
K_{\tau,\xi} = \exp \left( \frac{1}{2} \int d^3x \xi(x) B \left( \frac{\Upsilon(\tau)}{\Upsilon(\tau)} \xi(x) + \xi(x) \right) \right)
\]

(60)

wherein we use the operator \(B := (iW_{[\tau_1,\tau_2]}^{(1,2)}[c_{\alpha} - c_{\beta}c_{\alpha}] \Delta(\tau_1,\tau_2)]^{-1}\) which is independent of any \(\tau\) as can be seen using (9) and (12). Analogous formulas hold for coherent states defined on the hypercylinder.

D. Free amplitudes

We are now in the position to compute explicitly the free amplitude for a coherent state in the interaction picture. In the slice region we consider a coherent state defined by the complex function \(\xi_1\) at \(\tau_1\) and a
second coherent state at \( \tau_2 \) defined by \( \xi_2 \). The free amplitude for the coherent state \( \psi_{\tau_1, \xi_1} \otimes \psi_{\tau_2, \xi_2} \) defined on the boundary of the region \( M_{[\tau_1, \tau_2]} \) is
\[
\rho_{[\tau_1, \tau_2], 0}(\psi_{\tau_1, \xi_1} \otimes \psi_{\tau_2, \xi_2}) = \int D\varphi_1 D\varphi_2 \psi_{\varphi_2}(\varphi_2) \psi_{\tau_1, \xi_1}(\varphi_1) Z_{[\tau_1, \tau_2], 0}(\varphi_1, \varphi_2),
\]
\[
= \exp -\frac{1}{2} \int d^3x \left( \xi_1(\vec{x}) B\xi_1(\vec{x}) + \xi_2(\vec{x}) B\xi_2(\vec{x}) - 2\xi_2(\vec{x}) B\xi_2(\vec{x}) \right).
\]
This amplitude does not depend on \( \tau_1 \) and \( \tau_2 \), since the product \( W^{(1,2)}_{[\tau_1, \tau_2]} \Delta(\tau_1, \tau_2) \) does not depend on \( \tau_1 \) and \( \tau_2 \) as can be easily seen from the considerations following the definition of the symplectic structure at the end of Section II A. This was expected since we are considering the free evolution of states in the interaction picture.

The free amplitude for a coherent state defined by the complex function \( \xi \) on the hypercylinder reads
\[
\rho_{R, 0}(\psi_{R, \xi}) = \int D\varphi \psi_{R, \xi}(\varphi) Z_{R, 0}(\varphi),
\]
\[
= \exp \frac{1}{2} \int dt d\Omega \xi(t, \Omega) B \left( \frac{\xi(t, \Omega) + \xi(t, \Omega)}{c_0} \right).
\]
This amplitude is independent of the radius \( R \) of the hypercylinder.

V. INTERACTING THEORY

Following [6, 9], as an intermediate step toward the study of the general interacting theory, we first consider in the next section the interaction of the scalar field with a real source field \( \mu \). The action describing this interaction is indicated with the subscript \( \mu \),
\[
S_{M, \mu}(\phi) = S_{M, 0}(\phi) + \int_M d^4x \sqrt{-g(x)} \mu(x) \phi(x).
\]
We assume that the field \( \mu \) is confined in the interior of the region \( M \), i.e. \( \mu(x) = 0 \) for \( x \in \partial M \) and \( x \notin M \). The corresponding propagator is evaluated with the same technique applied in Section IV A namely by shifting the integration variable of the path integral by a classical solution of the free theory matching the boundary field configurations on the boundary \( \partial M \). The computation of the amplitude in the presence of the source field \( \mu \) in the two different regions will be presented in the next two subsections. Finally, the case of a general interacting theory will be treated in the last subsection using functional derivative techniques.

A. Slice region

Specializing to the slice region, the field propagator can be expressed in terms of the free one as
\[
Z_{[\tau_1, \tau_2], \mu}(\varphi) = \frac{N_{[\tau_1, \tau_2], \mu}}{N_{[\tau_1, \tau_2], 0}} Z_{[\tau_1, \tau_2], 0}(\varphi_1, \varphi_2) \exp \left( i \int d^3x \left( \mu_1(x) \varphi_1(x) + \mu_2(x) \varphi_2(x) \right) \right),
\]
where we have introduced the quantities
\[
\mu_1(x) := \int_{\tau_1}^{\tau_2} d\tau \sqrt{-g(\tau, x)} \Delta(\tau, x) \mu(\tau, x), \quad \mu_2(x) := \int_{\tau_1}^{\tau_2} d\tau \sqrt{-g(\tau, x)} \Delta(\tau, x) \mu(\tau, x).
\]
The normalization factor \( N_{[\tau_1, \tau_2], \mu} \) is formally equal to
\[
N_{[\tau_1, \tau_2], \mu} = \int_{\phi|_{\tau_1} = \phi|_{\tau_2} = 0} D\phi e^{iS_{[\tau_1, \tau_2], \mu}(\phi)},
\]
The amplitude for the boundary state using (64) and introducing the complex functions with vanishing boundary conditions, and using (70), after a tedious calculation we arrive at the following expression.

\[
\alpha(x) = \mu(x),
\]

with vanishing boundary conditions,

\[
\alpha(\tau_1, x) = \alpha(\tau_2, x) = 0.
\]

\(\alpha\) can be written as

\[
\alpha(x) = \int_{\tau_1}^{\tau_2} \frac{1}{\sqrt{|g|}} \partial_{\mu} \left( \sqrt{|g|} g^{\mu \nu} \partial_{\nu} \right) \alpha(x) = \mu(x),
\]

(67)

Then, the quotient \(N[\tau_1, \tau_2], \mu/N[\tau_1, \tau_2], 0\) takes the form

\[
\frac{N[\tau_1, \tau_2], \mu}{N[\tau_1, \tau_2], 0} = \exp \left( \frac{i}{2} \int \frac{d^4 x}{\sqrt{-g(x)}} \alpha(x) \mu(x) \right).
\]

(70)

The amplitude for the boundary state \(\psi_{\tau_1, \xi_1} \otimes \psi_{\tau_2, \xi_2}\) in the presence of the source field \(\mu\) is

\[
\rho[\tau_1, \tau_2], \mu(\psi_{\tau_1, \xi_1} \otimes \psi_{\tau_2, \xi_2}) = \int \mathcal{D} \psi_1 \mathcal{D} \psi_2 \psi_{\tau_1, \xi_1}(\varphi_1) \psi_{\tau_2, \xi_2}(\varphi_2) \rho[\tau_1, \tau_2], \mu(\varphi_1, \varphi_2).
\]

(71)

Using (64) and introducing the complex functions \(\tilde{\xi}_1\) and \(\tilde{\xi}_2\) defined as

\[
\tilde{\xi}_1(\varphi) := \xi_1(\varphi) + i (\Upsilon(\tau_1) \mu_1)(\varphi), \quad \tilde{\xi}_2(\varphi) := \xi_2(\varphi) - i (\Upsilon(\tau_2) \mu_2)(\varphi),
\]

(72)

the amplitude (71) can be expressed in terms of the free amplitude for the new boundary state \(\psi_{\tau_1, \tilde{\xi}_1} \otimes \psi_{\tau_2, \tilde{\xi}_2}\),

\[
\rho[\tau_1, \tau_2], \mu(\psi_{\tau_1, \xi_1} \otimes \psi_{\tau_2, \xi_2}) = \rho[\tau_1, \tau_2], 0(\psi_{\tau_1, \xi_1} \otimes \psi_{\tau_2, \xi_2}) \frac{K_{\tau_1, \xi_1} K_{\tau_2, \xi_2}}{K_{\tau_1, \tilde{\xi}_1} K_{\tau_2, \tilde{\xi}_2}} \frac{N[\tau_1, \tau_2], \mu}{N[\tau_1, \tau_2], 0}.
\]

(73)

Substituting the expressions of the free amplitude (61), the normalization factor of the coherent states (60) and using (70), after a tedious calculation we arrive at the following expression

\[
\rho[\tau_1, \tau_2], \mu(\psi_{\tau_1, \xi_1} \otimes \psi_{\tau_2, \xi_2}) = \rho[\tau_1, \tau_2], 0(\psi_{\tau_1, \xi_1} \otimes \psi_{\tau_2, \xi_2}) \exp \left( \int \frac{d^4 x}{\sqrt{-g(x)}} \tilde{\xi}(x) \mu(x) \right)
\]

\[
\times \exp \left( \frac{i}{2} \int \frac{d^4 x}{\sqrt{-g(x)}} [\alpha(x) + \beta(x)] \mu(x) \right),
\]

(74)

where the function \(\tilde{\xi}\) is

\[
\tilde{\xi}(x) = \frac{1}{\sqrt{W[\tau_1, \tau_2]}} \left( \frac{1}{c_\alpha c_\beta - c_\beta c_\alpha} \Delta(\tau_1, \tau_2) \Upsilon(\tau) \xi_1(\varphi) + \Upsilon(\tau) \xi_2(\varphi) \right),
\]

(75)

and the function \(\beta\) in the last exponential of (74) results to be

\[
\beta(\tau, \varphi) = \int_{\tau_1}^{\tau_2} d\tau' \sqrt{-g(\tau', \varphi)} \frac{1}{\sqrt{W[\tau_1, \tau_2]}} \left( \frac{\Upsilon(\tau') \Upsilon(\tau) c_\beta - c_\alpha c_\alpha}{c_\alpha c_\beta - c_\beta c_\alpha} \Delta(\tau_1, \tau_2) - \theta(\tau - \tau') \Delta(\tau, \tau') \right) \mu(\tau', \varphi).
\]

(76)

Then the sum of \(\alpha\) and \(\beta\) gives

\[
\alpha(\tau, \varphi) + \beta(\tau, \varphi) = \int_{\tau_1}^{\tau_2} d\tau' \sqrt{-g(\tau', \varphi)} \frac{1}{\sqrt{W[\tau_1, \tau_2]}} \left( \frac{\Upsilon(\tau') \Upsilon(\tau) c_\beta - c_\alpha c_\alpha}{c_\alpha c_\beta - c_\beta c_\alpha} + \theta(\tau - \tau') \Delta(\tau, \tau') \right) \mu(\tau', \varphi),
\]

(77)
and with simple algebra we can rewrite the terms in parenthesis as
\[
\frac{1}{c_a c_b - c_b c_a} \left( \theta(\tau' - \tau) \Upsilon(\tau) \Upsilon(\tau') + \theta(\tau - \tau') \Upsilon(\tau) Y(\tau') \right),
\]
(78)
Then, a convenient way to express the sum of \(\alpha\) and \(\beta\) above is
\[
\alpha(x) + \beta(x) = \int d^4x' \sqrt{-g(x')} G_F(x, x') \mu(x'),
\]
(79)
with \(G_F\) given by
\[
G_F(x, x') = \int d^3k \frac{1}{W_{[\tau_1, \tau_2]}(\tau_1, \tau_2)} \left( \theta(\tau' - \tau) \left( \Upsilon(\tau) \varphi_k (x) \Upsilon(\tau') \varphi_k (x') \right) + \theta(\tau - \tau') \left( \Upsilon(\tau) \varphi_k (x') \Upsilon(\tau') \varphi_k (x) \right) \right),
\]
(80)
where the mode expansion \([13]\) together with orthonormal relation \([14]\) have been used. In Minkowski spacetime for a scalar field in a slice region where \(\tau\) represents the Minkowski time, \(G_F\) coincides with the standard Feynman propagator, \([5, 6]\). The same happens in de Sitter space with \(\tau\) equal to the de Sitter conformal time, \([8, 9]\). This justifies our suggestive notation \(G_F\): \([80]\) plays the rôle of the Feynman propagator for the scalar field theory defined in the slice region. In particular, \(G_F\) satisfies the inhomogeneous Klein-Gordon equation in both variables \(x\) and \(x'\)
\[
\left( \frac{1}{\sqrt{|g|}} \partial_{\mu} \left( \sqrt{|g|} g^{\mu\nu} \partial_{\nu} \right) + m^2 \right) G_F(x, x') = (-g(x))^{-1/2} \delta(x - x'),
\]
(81)
the delta function is a four dimensional delta. A representation of the Feynman propagator analogous to \([80]\) in Minkowski spacetime can be found in Chap. 13 of \([12]\). Finally, the amplitude \([74]\) reads
\[
\rho_{[\tau_1, \tau_2], \mu} (\psi_{\tau_1, \xi_1} \otimes \overline{\psi}_{\tau_2, \xi_2}) = \rho_{[\tau_1, \tau_2], 0} (\psi_{\tau_1, \xi_1} \otimes \overline{\psi}_{\tau_2, \xi_2}) \exp \left( \int d^4x \sqrt{-g(x)} \hat{\xi}(x) \mu(x) \right) \times \exp \left( \frac{1}{2} \int d^4x d^4x' \sqrt{g(x)g(x')} \mu(x) G_F(x, x') \mu(x') \right).
\]
(82)
This expression is independent of \(\tau_1\) and \(\tau_2\). The limit of \([82]\) for asymptotic values of \(\tau_1\) and \(\tau_2\) is trivial, and we can interpret it as the S-matrix for the scalar theory in the presence of a source field.

**B. Hypercylinder region**

We consider now the interacting theory in the hypercylinder region. The field propagator takes the form
\[
Z_{R, \mu}(\varphi) = \frac{N_{R, \mu}}{N_{R, 0}} Z_{R, 0}(\varphi) \exp \left( i \int dt \, M(t, \Omega) \varphi(t, \Omega) \right),
\]
(83)
where
\[
M(t, \Omega) = \int_0^R dr \sqrt{-g(r, t, \Omega)} \mu(r, t, \Omega) \frac{X_a(r)}{X_a(R)}.
\]
(84)
The same expression \([70]\) is valid for the quotient \(N_{R, \mu}/N_{R, 0}\) but now \(\alpha\) is given by
\[
\alpha(r, t, \Omega) = \int_0^R dr' \sqrt{-g(r', t, \Omega)} \frac{1}{W_{[R_1, R_2]}(R_1, R_2)} g(r, r') \mu(r', t, \Omega),
\]
(85)
where \(g(r,r')\) is

\[
g(r,r') = \theta(r-r') (X_a(r)X_b(r') - X_b(r)X_a(r')) - X_a(r)X_b(r') + X_a(r) \frac{X_b(R)}{X_a(R)} X_a(r').
\]

The amplitude for a boundary coherent state \(\psi_{R,\xi}\) in the presence of a source field results to be

\[
\rho_{R,\mu}(\psi_{R,\xi}) = \int \mathcal{D}\varphi \psi_{R,\xi}(\varphi) Z_{R,\mu}(\varphi).
\]

Following the same treatment implemented for the slice region, we introduce the quantity \(\tilde{\xi}\)

\[
\tilde{\xi}(t,\Omega) = \xi(t,\Omega) + i (\Upsilon(R) M)(t,\Omega),
\]

where we have adopted the notation (82) adapted to the hypercylinder and the amplitude (87) can then be expressed in terms of the free amplitude for the new coherent state defined by \(\tilde{\xi}\),

\[
\rho_{R,\mu}(\psi_{R,\tilde{\xi}}) = \rho_{R,0}(\psi_{R,\tilde{\xi}}) \frac{K_{R,\xi}}{K_{R,\tilde{\xi}}} \frac{N_{R,\mu}}{N_{R,0}}.
\]

With (62) and the expression of the normalization factor of coherent states on the hypercylinder, we obtain

\[
\rho_{R,\mu}(\psi_{R,\xi}) = \rho_{R,0}(\psi_{R,\xi}) \exp \left(\int \mathrm{d}^4x \sqrt{-g(x)} \hat{\xi}(x) \mu(x)\right) \exp \left(i \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g(x)} |\alpha(x) + \beta(x)| \mu(x)\right),
\]

where the complex function \(\hat{\xi}\) is

\[
\hat{\xi}(r,t,\Omega) = -\frac{1}{W_{[1,2]}(R_1, R_2)c_b} (X_a(r)\xi)(t,\Omega),
\]

and the \(\beta\) results to be equal to

\[
\beta(r,t,\Omega) = -\int_0^R \mathrm{d}r' \sqrt{-g(r',t,\Omega)} \frac{1}{W_{[1,2]}(R_1, R_2)c_b} X_a(r') \frac{\Upsilon(R)}{X_a(R)} X_a(r') \mu(r',t,\Omega).
\]

Now we can perform the sum of \(\alpha\) and \(\beta\) appearing in the last exponential in the r.h.s. of (90),

\[
\alpha(r,t,\Omega) + \beta(r,t,\Omega) = -\int_0^R \mathrm{d}r' \sqrt{-g(r',t,\Omega)} \frac{1}{W_{[1,2]}(R_1, R_2)c_b} \left(\theta(r-r') X_a(r') \Upsilon(r) + \theta(r'-r) X_a(r) \Upsilon(r')\right) \mu(r',t,\Omega).
\]

As in the previous subsection, this sum can be reexpressed in the form (79) with \(G_F\) now given by

\[
G_F(x,x') = -\int \frac{\mathrm{d}^3p}{W_{[1,2]}(R_1, R_2)c_b} \left(\theta(r-r') \left(X_a(r') \varphi_{E}(t',\Omega)\right) \left(\Upsilon(r) \varphi_{E}(t,\Omega)\right)
+ \theta(r'-r) \left(X_a(r) \varphi_{E}(t,\Omega)\right) \left(\Upsilon(r') \varphi_{E}(t',\Omega)\right)\right),
\]

where the expansion of the boundary field configuration in terms of the set of modes \(\varphi_{E}\) has been used. One can check that \(G_F\) satisfies the inhomogeneous Klein-Gordon equation (81). The amplitude in the presence of the source field \(\mu\) for the coherent state \(\psi_{R,\mu}\) defined on the hypercylinder results to have the same structure as (82), i.e.

\[
\rho_{R,\mu}(\psi_{R,\tilde{\xi}}) = \rho_{R,0}(\psi_{R,\tilde{\xi}}) \exp \left(\int \mathrm{d}^4x \sqrt{-g(x)} \tilde{\xi}(x) \mu(x)\right) \times \exp \left(i \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g(x)} g(x') \mu(x) G_F(x,x') \mu(x')\right).
\]

Nothing in this expression depends on the radius \(R\) of the hypercylinder, and therefore (95) coincides with its asymptotic limit for \(R \to \infty\). Hence, we interpret it as the S-matrix for the scalar field theory in the presence of a source.
C. General interaction

The asymptotic amplitude for a general interacting theory can be worked out perturbatively applying functional derivatives techniques. In particular, the action of the scalar field with an arbitrary potential \( V \) in the spacetime region \( M \) is given by

\[
S_{M,V}(\phi) = S_{M,0}(\phi) + \int_M d^4x \, V(x, \phi(x)).
\]

We may write the exponential of \( i \) times this action as an infinite series of variational operators acting on the corresponding term in the presence of a source field

\[
\exp(iS_{M,V}(\phi)) = \exp \left( i \int_M d^4x \, \sqrt{-g(x)} \, V \left( x, -i \frac{\delta}{\delta \mu(x)} \right) \right) \exp \left( iS_{M,\mu}(\phi) \right) \bigg|_{\mu=0},
\]

where \( S_{M,\mu} \) is the action in the presence of a source interaction, defined in \((63)\). We assume that the potential \( V \) vanishes outside the region \( M \). Inserting the above expression in the field propagator \((28)\) leads to

\[
Z_{M,V}(\varphi) = \exp \left( i \int_M d^4x \, \sqrt{-g(x)} \, V \left( x, -i \frac{\delta}{\delta \mu(x)} \right) \right) Z_{M,\mu}(\varphi) \bigg|_{\mu=0},
\]

and the amplitude for the general interacting theory is then

\[
\rho_{M,V}(\psi) = \exp \left( i \int_M d^4x \, \sqrt{-g(x)} \, V \left( x, -i \frac{\delta}{\delta \mu(x)} \right) \right) \rho_{M,\mu}(\psi) \bigg|_{\mu=0}.
\]

This result applies to both types of regions considered here.

VI. EXAMPLES

We provide in this section some examples showing the consistency of the general expressions derived in previous sections with some results so far obtained in Minkowski and de Sitter spacetimes. In particular we indicate the main operators involved and mode decompositions of the boundary field configurations from which the vacuum state, the various amplitudes and Feynman propagators can be recovered.

A. Slice region in Minkowski and de Sitter spacetimes

As a first example with consider a special type of slice region in Minkowski spacetime defined as follows. The foliation parameter \( \tau \) coincides with the global time variable, and then the three coordinates \( \mathbf{x} \) are the usual spatial coordinates. This corresponds to the standard situation in which the spacetime is foliated by equal Minkowski time hyperplanes. Plane waves represent a useful orthonormal basis to expand the usual spatial coordinates. This corresponds to the standard situation in which the spacetime is foliated

\[
X_a(\tau) = \cos(\omega\tau), \quad X_b(\tau) = \sin(\omega\tau) \quad \text{with} \quad \omega := \sqrt{-\Delta_{\mathbf{x}} + m^2},
\]

\( \Delta_{\mathbf{x}} \) being the Laplacian in the coordinates \( \mathbf{x} \). For the slice region \([\tau_1, \tau_2]\) the matrix \( W_{[\tau_1, \tau_2]} \) results to be

\[
W_{[\tau_1, \tau_2]} = \frac{\omega}{\sin(\omega(\tau_2 - \tau_1))} \begin{pmatrix} \cos \omega(\tau_2 - \tau_1) & -1 \\ -1 & \cos \omega(\tau_2 - \tau_1) \end{pmatrix}.
\]

The election \( c_a = 1 \) and \( c_b = i \) provides a suitable vacuum state, which coincides with formula \((15)\) of \([6]\).

In the case of a massive scalar field in de Sitter space, the slice region considered in \([9]\) is bounded two equal conformal de Sitter time hypersurfaces. As in Minkowski spacetime, the modes \( \varphi_{\mathbf{k}}(\mathbf{x}) \) are plane waves, \( e^{ik\mathbf{x}} \). Now, the operators \( X_i \) arise from a Bessel equation,

\[
X_a(\tau) = \tau^{3/2} J_\nu(k\tau), \quad X_b(\tau) = \tau^{3/2} Y_\nu(k\tau) \quad \text{with} \quad k = |\mathbf{k}|.
\]
$J_\nu$ and $Y_\nu$ are the Bessel functions of the first and second kind respectively, with index $\nu = \sqrt{9/4 - (mR)^2}$ and $R$ denotes the inverse of the Hubble constant. For the slice region $[\tau_1, \tau_2]$ the elements of the matrix $W_{[\tau_1, \tau_2]}$ are

$$W_{[\tau_1, \tau_2]}^{(1,1)} = -\frac{R^2}{\tau_1^2} \left( \frac{3}{2\tau_1} + k J'_\nu(k\tau_1) Y_\nu(k\tau_2) - Y'_\nu(k\tau_1) J_\nu(k\tau_2) \right),$$

(103)

$$W_{[\tau_1, \tau_2]}^{(1,2)} = W_{[\tau_1, \tau_2]}^{(2,1)} = -\frac{2R^2}{\pi \Delta(\tau_1, \tau_2)},$$

(104)

$$W_{[\tau_1, \tau_2]}^{(2,2)} = \frac{R^2}{\tau_2^2} \left( \frac{3}{2\tau_2} + k J'_\nu(k\tau_1) Y_\nu(k\tau_2) - Y'_\nu(k\tau_1) J_\nu(k\tau_2) \right),$$

(105)

where $\Delta(\tau_1, \tau_2) = (\tau_1 \tau_2)^{3/2} (J_\nu(k\tau_1) Y_\nu(k\tau_2) - Y_\nu(k\tau_1) J_\nu(k\tau_2))$ and a prime indicates the derivative with respect to the argument. The vacuum state is obtained by fixing $c_a = 1$ and $c_b = i$.

Inserting all the above formulas in the free amplitude $|F\rangle$, the complex function $\xi$ (75) and the Feynman propagator (80) provides the correct expressions for the corresponding quantities in Minkowski [6] and de Sitter spacetimes [9].

### B. Hypercylinder in Minkowski and de Sitter spacetimes

We now turn to the hypercylinder in Minkowski space. The modes $\varphi_\nu(t, \Omega)$ are given by the product of spherical harmonics $Y_\nu^m(\Omega)$ and the exponential factor $(2\pi)^{-1/2}e^{itE}$, with $E \in \mathbb{R}$. Hence, we use $p$ as a collective notation for the set of (discrete and continuum) quantum numbers $(l, m, E)$. We can define

$$X_a(r) = a_l(E, r) \quad \text{and} \quad X_b(r) = b_l(E, r),$$

(106)

where $a_l$ and $b_l$ are defined as

$$a_l(E, r) = \begin{cases} j_l(r\sqrt{E^2 - m^2}), & \text{if } E^2 > m^2, \\ i_l(r\sqrt{m^2 - E^2}), & \text{if } E^2 < m^2, \end{cases} \quad \text{and} \quad b_l(E, r) = \begin{cases} n_l(r\sqrt{E^2 - m^2}), & \text{if } E^2 > m^2, \\ i_l(r\sqrt{m^2 - E^2}), & \text{if } E^2 < m^2. \end{cases}$$

(107)

$j_l, n_l, i_l$ and $i_l$ are the spherical Bessel functions of the first and second kind and the modified spherical Bessel functions of the first and second kind respectively. In the region between two hypercylinders of radii $R_1$ and $R_2$, the matrix $W_{[R_1, R_2]}$ has the form

$$W_{[R_1, R_2]} = \frac{1}{\Delta(R_1, R_2)} \begin{pmatrix} R_1^2 \Delta_1(R_1, R_2) & 1/p \\ 1/p & -R_2^2 \Delta_2(R_1, R_2) \end{pmatrix},$$

(108)

where

$$p := \begin{cases} \sqrt{E^2 - m^2}, & \text{if } E^2 > m^2, \\ i\sqrt{m^2 - E^2}, & \text{if } E^2 < m^2, \end{cases}$$

(109)

and the functions $\Delta(R_1, R_2) = a_l(E, R_1)b_l(E, R_2) - b_l(E, R_1)a_l(E, R_2)$ and $\Delta_i(R_1, R_2) (i = 1, 2)$ are defined as in (21). The vacuum state is fixed by the choice $c_a = 1$ and $c_b = i$.

In de Sitter space, the field configuration on the hypercylinder are expanded in the basis of the modes $\varphi_\nu(t, \Omega) = e^{t^{3/2}}H_\nu(kt)Y_\nu^m(\Omega)$, where $H_\nu$ is the MacDonald function of index $\nu$ and $p$ denotes the quantum numbers $(l, m, k)$. The operators $X_i$ are given by the spherical Bessel functions of the first and second kind only,

$$X_a(r) = j_l(kr) \quad \text{and} \quad X_b(r) = n_l(kr).$$

(110)

The matrix $W_{[R_1, R_2]}$ reads

$$W_{[R_1, R_2]} = \frac{R^2}{t^2} \frac{1}{\Delta(R_1, R_2)} \begin{pmatrix} -R_1^2 \Delta_1(R_1, R_2) & 1/k \\ 1/k & R_2^2 \Delta_2(R_1, R_2) \end{pmatrix}.$$

(111)

Our choice for the vacuum state corresponds to $c_a = 1$ and $c_b = i$.

With all these expressions at our disposal we can immediately obtain the free amplitude (02), the complex function $\xi$ (91) and the Feynman propagator (94), that coincide with those evaluated in [6] and [9].
VII. SUMMARY AND OUTLOOK

In this paper we have implemented the quantization of a real massive scalar field in a certain class of 4-dimensional curved spaces according to the prescriptions of the general boundary formulation of quantum field theory. The class of spacetimes we considered are specified by the requirement we asked for the form of the metric tensor: We demand it to be block diagonal in the two different foliations we chose. Although this assumption seems to be very restrictive, it is nevertheless satisfied for many (if not all) the spacetimes where quantum field theories have been studied so far. It is important to emphasize that these foliations refer to the same spacetime; we introduce them because they are useful to define two types of spacetime regions characterized by different boundaries: While two hypersurfaces (not required to be Cauchy surfaces or spacelike hypersurfaces) bound the first type of regions, called the slice regions, the second kind has only one connected boundary which is referred to as the hypercylinder. Moreover, we assume that the solutions of the Klein-Gordon equation in the different regions can be written in a special form, i.e., in terms of operators satisfying certain conditions specified in Section II A and Section II B.

Our main objective was the derivation of a general expression for the S-matrix in the case of a general interacting theory. This has been achieved by constructing the relevant quantities, i.e., the state spaces associated to the boundaries of the slice and hypercylinder regions, the field propagators encoding the dynamics of the field in these regions, the vacuum and coherent states. Then, applying the same procedure of [5, 6] we computed the amplitude for boundary coherent states in the regions considered in three cases: First we evaluated the amplitude for the free theory in the interaction picture, then the interaction with a source field was considered and finally we used functional derivative techniques to treat the general interacting theory. The asymptotic limit of these amplitudes has been interpreted as representing the S-matrices for the scalar field defined in the two types of regions.

The structures of these S-matrices in slice and hypercylinder regions are similar. In the case of the source interaction, the asymptotic amplitudes for coherent states factor in three terms: The amplitude of the free theory, an exponential containing a function that establishes a one to one correspondence between complex solutions of the equation of motion and coherent states, and a term bilinear in the source function in which the Feynman propagator appears. In the case of an interacting scalar field in Minkowski spacetime studied in [2, 3], the Feynman propagator obtained in the slice region, given as in the standard treatment by a time interval region, and the one in the hypercylinder region were shown to be equivalent. Then equating the complex function $\xi$ appearing in (the corresponding formulas of ours) (82) and (89), an isomorphism was constructed between the state spaces associated to the boundaries of the two regions and the equivalence of the free amplitudes under the action of this isomorphism was shown. In this way the interacting theory in the hypercylinder region turned out to provide the same asymptotic amplitudes as the standard treatment based on the time interval region. An analogous result was recovered in de Sitter space [8, 9]. A natural question would then be if a similar situation is to be expected here. To address this question, that we expect should be answered in the positive at least for field theories defined on spacetimes conformal to Minkowski spacetime, many strategies (not all independent) can be envisaged: defining a map from the space of classical solutions of the equation of motion in the slice region to the space of solutions in the whole spacetime and a second map from this space to the space of solutions in the hypercylinder region and then composing these two maps to relate solutions in the two regions of interest; implementing a Bogolubov transformation between the two sets of modes in which the field $\phi$ has been expanded; finding a relation between the vacuum states defined on the boundary of the two regions; showing the equivalence of the different boundary conditions the Feynman propagator satisfies in the slice and hypercylinder regions. So far no general result has been obtained along these lines.

Other important aspects that deserve to be investigated concern analytic properties and unitarity of the S-matrix. A first step in the study of unitary implementability of quantum dynamics of a scalar field in curved spacetime within the GBF was taken in [13], where results were obtained for the free theory as well as in the presence of a source interaction. The technique used in that paper was based on a certain composition property of the field propagator. With the general structure of the S-matrix at our disposal, this question can now be handled from a new perspective.

There are other directions of generalizing the work presented here, and we mention a few of them. The first natural extension will be to compute the S-matrix for fields defined in more general regions. In particular, compact spacetime regions will play a major rôle, e.g., causal diamonds and 4-balls. Second, more interesting field theories need to be considered: Fields of higher spin have to be taken into account in order to investigate
QED or Yang-Mills Theory\(^3\) from a GBF perspective.

Besides its contribution to the development of the GBF, our result should also be useful in the study of quantum field theories in spacetimes with boundaries, e.g., mirrors or horizons, that appear in many contexts \([13]\), like the Casimir effect \([16]\). Moreover, considering the hypercylinder region enables us to define S-matrices for spacetimes where this has not been possible in the standard formalism due to the lack of free temporal asymptotic states, such as Anti-de Sitter (AdS) spacetime or in the case of a field in the presence of an eternal black hole. A boundary S-matrix in AdS for states defined at timelike infinity (which plays an important rôle in the conjectured AdS/CFT correspondence \([17]\)) was proposed in \([18, 19]\). The GBF is likely to provide the appropriate tool for a rigorous derivation of this type of S-matrix. In the same spirit the hypercylinder geometry appears to be the necessary ingredient in the proposal by ’t Hooft, see \([20]\), to describe the scattering of particles against a black hole. Therefore a next step will be the application of our work to these systems.

So far, two methods of quantization have been studied within the GBF framework: the Feynman-Schrödinger quantization used e.g. in this paper and the holomorphic quantization developed in \([21]\). Now the expressions found here for the Feynman propagator in (certain) curved spacetimes enable us to also write down the other members of the Green functions family, in particular the Wightman, Schwinger and Hadamard functions. Thus we have put us in a position where we can (at least formally) define canonical commutation relations (CCR) for field operators on general hypersurfaces. If and which physical information is contained in these CCR has to be explored. Based on the CCR the development of a formulation of canonical quantization within the GBF seems to be possible. It will then be interesting to investigate the relations between the GBF and the algebraic approach to QFT.

The holomorphic quantization method introduced in \([21]\) at present applies to linear field theories only. If a dictionary can be established for holomorphic and Schrödinger states, our results can shed some light on how to include interactions into the holomorphic quantization scheme.

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