On the Maximum Number of $k$-Hooks of Partitions of $n$

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Abstract. Let $\alpha_k(\lambda)$ denote the number of $k$-hooks in a partition $\lambda$ and let $b(n, k)$ be the maximum value of $\alpha_k(\lambda)$ among partitions of $n$. Amdeberhan posed a conjecture on the generating function of $b(n, 1)$. We give a proof of this conjecture. In general, we obtain a formula that can be used to determine $b(n, k)$. This leads to a generating function formula for $b(n, k)$. We introduce the notion of nearly $k$-triangular partitions. We show that for any $n$, there is a nearly $k$-triangular partition which can be transformed into a partition of $n$ that attains the maximum number of $k$-hooks. The operations for the transformation enable us to compute the number $b(n, k)$.

Keywords: partition, hook length, generating function, nearly $k$-triangular partition

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1 Introduction

The objective of this paper is to derive a generating function formula for the maximum number of $k$-hooks in the Young diagrams of partitions of $n$. For $k = 1$, the problem was posed by Amdeberhan [1]. Let $\alpha_1(\lambda)$ be the number of 1-hooks in the partition $\lambda$, or equivalently, the number of distinct parts in $\lambda$. Let

$$b_n = \max\{\alpha_1(\lambda) : \lambda \in P(n)\},$$

where $P(n)$ denotes the set of partitions of $n$.

Amdeberhan [1] posed the following conjecture.

Conjecture 1.1 We have

$$\sum_{n \geq 0} b_n q^n = \frac{1}{1 - q} \left( \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}} - 1 \right),$$  \hspace{1cm} (1.1)

where $(q, q)_{\infty} = (1 - q)(1 - q^2)(1 - q^3) \ldots$. 

Following the notation $\alpha_1(\lambda)$ of Amdeberhan, we use $\alpha_k(\lambda)$ to denote the number of $k$-hooks in $\lambda$, and let

$$b(n, k) = \max\{\alpha_k(\lambda) : \lambda \in P(n)\}.$$ 

The main result of this paper is the following generating function formula for $b(n, k)$.

**Theorem 1.2** For $n \geq 0$ and $k \geq 1$, we have

$$\sum_{n \geq 0} b(n, k)q^n = \frac{1}{1-q} \left( \sum_{t \geq 1} q^{t^2} \frac{1-q^{tk}}{1-q^k} - 1 \right). \quad (1.2)$$

Clearly, Theorem 1.2 reduces to Theorem 1.1 when $k = 1$. Pak [6] gave the following generating function of the statistic $\alpha_1(\lambda)$, where he used $\gamma(\lambda)$ to denote $\alpha_1(\lambda)$:

$$\sum_{n \geq 0} \sum_{\lambda \in P(n)} \gamma(\lambda)q^n = \frac{q}{(1-q)(q; q)_\infty}. \quad (1.3)$$

In general, the statistic $\alpha_k(\lambda)$ has been studied by Han [5, Eq. 1.5]. More precisely, he derived the following generating function formula:

$$\sum_{n \geq 0} \sum_{\lambda \in P(n)} x^{\alpha_k(\lambda)} q^{|\lambda|} = \prod_{j \geq 1} \frac{(1 + (x-1)q^{kj})^k}{1-q^j}. \quad (1.4)$$

Taking logarithms of both sides of (1.4) and differentiating with respect to $x$, we obtain the following relation by setting $x = 1$:

$$\sum_{n \geq 0} \sum_{\lambda \in P(n)} \alpha_k(\lambda)q^n = \frac{kq^k}{(1-q^k)(q; q)_\infty}. \quad (1.5)$$

It can be seen that identity (1.5) becomes (1.3) when $k = 1$.

To prove Theorem 1.2 we introduce a class of partitions, called nearly $k$-triangular partitions.

**Definition 1.3** For fixed $m \geq 0$ and $k \geq 1$, let $m = lk + r$, where $0 \leq r \leq k - 1$. Let $T_m^{(k)}$ denote the nearly $k$-triangular partition with $m$ parts as given by

$$T_m^{(k)} = ((l+1)k, \ldots, (l+1)k, lk, \ldots, lk, \ldots, 2k, \ldots, 2k, k, \ldots, k).$$

For example, Figure 1.1 illustrates a nearly 3-triangular partition with eight parts. Throughout the paper, we use the symbol * to mark these cells with hook length $k$.

It can be seen that there is exactly one cell in each row of $T_m^{(k)}$ with hook length $k$. 

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Figure 1.1: A nearly 3-triangular partition $T^{(3)}_s$.

Let us recall some basic notation and terminology on partitions as used in [2]. A partition $\lambda$ of a positive integer $n$ is a finite nonincreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = n$. The entries $\lambda_i$ are called the parts of $\lambda$. The number of parts of $\lambda$ is called the length of $\lambda$, denoted by $l(\lambda)$. The weight of $\lambda$ is the sum of parts, denoted by $|\lambda|$.

A partition can be represented by a Young diagram. For each cell $u$ in the Young diagram of $\lambda$, we define the hook length $h_u(\lambda)$ of $u$ by the number of cells $v$ in the Young diagram of $\lambda$ such that $v = u$, or $v$ appears below $u$ in the same column, or $v$ lies to the right of $u$ in the same row. A hook with length $k$ is called a $k$-hook, see Figure 1.2.

Figure 1.2: A 3-hook and the three cells with hook length 3.

This paper is organized as follows. In Section 2, we give a formula that can be used to determine $b_n$ in Theorem 2.1. We then derive the generating function for $b_n$. In Section 3, we define two operations $D_i$ and $P_i$ on Young diagrams. Using these operations one can transform a partition $\lambda$ with $m$ $k$-hooks into a nearly $k$-triangular partition $T^{(k)}_s$, where $s \geq m$. This leads to a proof of Theorem 3.1. In Section 4, we define an operation $Q_j$ on Young diagrams. We obtain a formula for $b(n, k)$ as well as a formula for the generating function.
2 Proof of Conjecture 1.1

In this section, we give a proof of Conjecture 1.1.

**Theorem 2.1** Assume that $m$ is a nonnegative integer and $n$ is an integer such that $(\frac{m+1}{2}) \leq n \leq (\frac{m+2}{2}) - 1$. Then we have $b_n = m$.

**Proof.** Recall that $\alpha_1(\lambda)$ is the number of distinct parts of $\lambda$ and $b_n$ is the maximum value of $\alpha_1(\lambda)$ when $\lambda$ ranges over partitions of $n$. We claim that we have $b_n < m + 1$ for any $n < (\frac{m+2}{2})$. Assume that $\lambda$ is a partition with $m + 1$ distinct parts. It is clear that $|\lambda| \geq 1 + 2 + \cdots + (m + 1) = \binom{m+2}{2}$. In other words, if $n < (\frac{m+2}{2})$ then we have $b_n < m + 1$. So the claim is verified.

Next we proceed to show that $b_n \geq m$ for any $n \geq (\frac{m+1}{2})$. Let $\lambda = (m, m - 1, \ldots, 2, 1^{n-(\frac{m+1}{2})+1})$. Clearly, $\lambda$ has $m$ distinct parts and $|\lambda| = n$. Thus $b_n \geq m$ if $n \geq (\frac{m+1}{2})$. So we reach the conclusion that $b_n = m$ for $(\frac{m+1}{2}) \leq n \leq (\frac{m+2}{2}) - 1$. This completes the proof. 

We are ready to prove Conjecture 1.1 with the aid of Theorem 2.1.

**Proof of Conjecture 1.1.** First, we may express the generating function of $b_n$ in terms of the generating function of $b_{n+1} - b_n$. More precisely, we have

\[
(1 - q) \sum_{n \geq 0} b_n q^n = \sum_{n \geq 0} (b_{n+1} - b_n) q^{n+1}.
\]

To compute $b_{n+1} - b_n$, we denote the interval $[(\frac{m+1}{2}), (\frac{m+2}{2}) - 1]$ by $I_m$. By Theorem 2.1, we see that $b_n$ is determined by the interval which $n$ lies in. There are two cases:

Case 1: $n$ and $n + 1$ lie in the same interval $I_m$. Then we have $b_{n+1} = b_n = m$, that is, $b_{n+1} - b_n = 0$.

Case 2: $n$ and $n + 1$ lie in two consecutive intervals $I_m$ and $I_{m+1}$, that is, $n = (\frac{m+2}{2}) - 1$. In this case, we have $b_n = m$ and $b_{n+1} = m + 1$. So we have $b_{n+1} - b_n = 1$.

Combining the above two cases, we find that

\[
\sum_{n \geq 0} (b_{n+1} - b_n) q^{n+1} = \sum_{m \geq 0} q^{(\frac{m+2}{2})} = \sum_{m \geq 0} q^{(\frac{m+1}{2})} - 1.
\]

By Gauss’ identity \[3, \text{ Eq. 1.4.10}\]

\[
\sum_{n=0}^{\infty} q^{\frac{(n+1)}{2}} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},
\]

(2.2)
we obtain that

\[(1 - q) \sum_{n \geq 0} b_n q^n = \sum_{m \geq 0} q^{(m+1)2} - 1 = \frac{(q^2; q^2)_\infty}{(q; q)_\infty} - 1. \tag{2.3}\]

Thus, identity (1.1) can be deduced from (2.3) by dividing both sides by \((1 - q)\). This completes the proof.

3 Nearly \(k\)-triangular partitions

In this section, we introduce the structure of nearly \(k\)-triangular partitions, and we show that such a partition has minimum weight given the number of \(k\)-hooks. This property will be used in the next section to determine \(b(n, k)\).

For \(m \geq 0\) and \(k \geq 1\), the weight of the nearly \(k\)-triangular partition \(T_m^{(k)}\) is given by

\[t(m, k) = m \left( \left\lfloor \frac{m}{k} \right\rfloor + 1 \right) k - \left( \left\lfloor \frac{m}{k} \right\rfloor + 1 \right) \frac{k^2}{2}. \tag{3.1}\]

It can be seen that there is exactly one cell in each row of \(T_m^{(k)}\) that has hook length \(k\). Hence \(T_m^{(k)}\) is a partition with \(m\) \(k\)-hooks. The following theorem states that \(T_m^{(k)}\) has minimum weight among partitions with \(m\) \(k\)-hooks.

**Theorem 3.1** For \(m \geq 0\) and \(k \geq 1\), if \(\lambda\) is a partition with \(m\) \(k\)-hooks, then we have \(|\lambda| \geq t(m, k)\).

To prove Theorem 3.1, we introduce two operations \(D_i\) and \(P_i\) defined on Young diagrams. They can also be considered as operations on partitions. We shall show that one can transform a partition \(\lambda\) with \(m\) \(k\)-hooks into a nearly \(k\)-triangular partition \(T_m^{(k)}\) by applying the operations \(D_i\) and \(P_i\).

The operations \(D_i\) and \(P_i\) are defined as follows. Let \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)\) be a partition. The operation \(D_i\) means to remove the \(i\)-th row of the Young diagram of \(\lambda\). The operation \(P_i\) applies to partitions \(\lambda\) for which \(\lambda_i > \lambda_{i+1}\). More precisely, \(P_i(\lambda)\) is obtained from \(\lambda\) via the following steps. Assume that the cells with hook length \(k\) are marked by \(*\).

**Step 1.** Remove the last cell \(u\) from the \(i\)-th row of the Young diagram of \(\lambda\), and denote the resulting partition by \(\mu\). If the Young diagram of \(\lambda\) contains no marked cell in the column occupied by \(u\), then we set \(P_i(\lambda) = \mu\).

**Step 2.** At this step, there is a cell of hook length \(k\) in the column of \(\lambda\) that contains \(u\). Denote this marked cell by \(w_1\) and assume that it is in the \(j\)-th row of \(\lambda\). Evidently, the marked cell \(w_1\) in \(\lambda\) is of hook length \(k - 1\) in \(\mu\). There are two cases:

- Case 1:
- Case 2:
Case 1: $\mu_j = \mu_{j-1}$. Notice that the cell $w'_1$ above $w_1$ is of hook length $k$ in $\mu$. That is to say that $w'_1$ is a marked cell in $\mu$. We set $P_i(\lambda) = \mu$;

Case 2: $\mu_j < \mu_{j-1}$. We add one cell $v$ at the end of the $j$-th row in $\mu$ and denote the resulting partition by $\nu$. Clearly, $w_1$ is also a marked cell in $\nu$. If the Young diagram of $\mu$ contains no marked cell in the $(\mu_j + 1)$-th column, then we set $P_i(\lambda) = \nu$. Otherwise, go to the next step.

Step 3. There is a marked cell in the $(\mu_j + 1)$-th column of $\mu$. Let $w_2$ denote this marked cell and suppose that it is in the $h$-th row. Evidently, the cell $w_2$ has hook length $k + 1$ in $\nu$. There are two cases:

Case 1: $\nu_k = \nu_{k+1}$. Notice that the cell $w'_2$ below $w_2$ is of hook length $k$ in $\nu$. In this case, we set $P_i(\lambda) = \nu$;

Case 2: $\nu_k > \nu_{k+1}$. We remove the last cell $u'$ in the $h$-th row of $\nu$ and denote the resulting partition by $\mu'$. Now, $w_2$ is also a marked cell in $\mu'$. If the Young diagram of $\nu$ contains no marked cell in the column occupied by $u'$, we set $P_i(\lambda) = \mu'$. Otherwise, there is a marked cell in the column occupied by $u'$ in $\nu$. We set $u = u'$, $\lambda = \nu$, $\mu = \mu'$ and go back to Step 2.

Figure 3.3 gives an illustration of the operation $P_i$. The cells with the symbol $-$ are the removed cells and the cells with the symbol $+$ are the added cells.

Figure 3.3: The operation $P_i$.

The following property of the operation $P_i$ is easy to verify, and hence the proof is omitted.

**Lemma 3.2** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition such that $\lambda_i > \lambda_{i+1}$ and let $\alpha_k(\lambda, i)$ denote the number of marked cells in the $i$-th row of $\lambda$. Then we have $|\lambda| \geq |P_i(\lambda)|$ and

$$\alpha_k(\lambda) - \alpha_k(\lambda, i) \leq \alpha_k(P_i(\lambda)) - \alpha_k(P_i(\lambda), i).$$
We are now in a position to present a proof of Theorem 3.1 by using the operations \( P_i \) and \( D_i \).

**Proof of Theorem 3.1.** Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) be a partition with \( m \) \( k \)-hooks. We shall give a procedure to transform \( \lambda \) into \( T_s^{(k)} \) where \( s \geq m \) by using the operations \( D_i \) and \( P_i \). Notice that \( D_i \) decreases the weight of a partition and \( P_i \) either keeps the weight or decreases the weight by one. Moreover, it can be seen that during the process of the transformation \( D_i \) preserves the number of \( k \)-hooks and \( P_i \) keeps or increases the number of \( k \)-hooks. Hence we arrive at the conclusion that \(|\lambda| \geq t(s, k) \geq t(m, k)\).

Clearly, we have \( r \geq m \). We aim to construct a sequence \( \beta^{(r)}, \beta^{(r-1)}, \ldots, \beta^{(1)}, \beta^{(0)} \) of nearly \( k \)-triangular partitions starting with \( T_r^{(k)} \) and ending with \( T_s^{(k)} \), where \( s \geq m \). In the construction of \( T_s^{(k)} \) from \( \lambda \), we denote the intermediate partitions by \( \lambda^{(r)}, \lambda^{(r-1)}, \ldots, \lambda^{(1)}, \lambda^{(0)} \) with \( \lambda^{(r)} = \lambda \) and \( \lambda^{(0)} = T_s^{(k)} \). We compare the partitions \( \beta^{(i)} \) and \( \lambda^{(i)} \) to construct \( \beta^{(i-1)} \) and \( \lambda^{(i-1)} \) by the following process, where, \( \beta^{(i)} \) and \( \lambda^{(i)} \) have the same number of parts and \( \lambda^{(i)} \) and \( \beta^{(i)} \) differ only in the first \( i \) rows.

First, we compare the last entry of \( \lambda^{(r)} \) with the last entry of \( \beta^{(r)} \). Recall that \( \lambda^{(r)} = \lambda \) and \( \beta^{(r)} = T_r^{(k)} \). There are two cases:

**Case 1:** The \( r \)-th entry of \( \lambda^{(r)} \) is equal to the \( r \)-th entry of \( \beta^{(r)} \). Then we set \( \beta^{(r-1)} = T_r^{(k)} \) and \( \lambda^{(r-1)} = \lambda^{(r)} \).

**Case 2:** The \( r \)-th entry of \( \lambda^{(r)} \) is not equal to the \( r \)-th entry of \( \beta^{(r)} \). There are two subcases:

**Case 2.1:** \( \lambda^{(r)}_r < \beta^{(r)}_r = k \). Clearly, there are no cells of hook length \( k \) in the \( r \)-th row of the Young diagram of \( \lambda^{(r)} \). Applying the operation \( P_r \) to \( \lambda^{(r)} \), the \( r \)-th entry of \( \lambda^{(r)} \) decreases by one. In view of Lemma 3.2, we have \( \alpha_k(P_r(\lambda^{(r)})) \geq \alpha_k(\lambda) \) and \( |P_r(\lambda^{(r)})| \leq |\lambda| \). Hence, applying the operation \( P_r \) to \( \lambda^{(r)} \) iteratively, more precisely, \( \lambda^{(r)}_r \) times, we obtain a new partition \( \mu \) with \( r-1 \) parts. It is clear that \( \alpha_k(\mu) \geq \alpha_k(\lambda^{(r)}) \) and \( |\mu| \leq |\lambda^{(r)}| \). We set \( \beta^{(r-1)} = T_{r-1}^{(k)} \) and \( \lambda^{(r-1)} = \mu \).

**Case 2.2:** \( \lambda^{(r)}_r > \beta^{(r)}_r = k \). Evidently, there is a cell with hook length \( k \) in the \( r \)-th row of \( \lambda^{(r)} \). Applying the operation \( P_r \) to \( \lambda^{(r)} \) \( (\lambda^{(r)}_r - k) \) times, we obtain a new partition \( \mu \). It is clear that the \( r \)-th entry of \( \mu \) equals the \( r \)-th entry of \( \beta^{(r)} \). By Lemma 3.2, we find that \( \alpha_k(\mu) \geq \alpha_k(\lambda^{(r)}) \) and \( |\mu| \leq |\lambda^{(r)}| \). We set \( \beta^{(r-1)} = T_r^{(k)} \) and \( \lambda^{(r-1)} = \mu \).

Then we proceed to compare the \( i \)-th entry of \( \lambda^{(i)} \) with the \( i \)-th entry of \( \beta^{(i)} \) for \( r-1 \geq i \geq 1 \), where we assume that \( \lambda^{(i)} \) and \( \beta^{(i)} \) have been constructed by the above procedure. Assume that \( \lambda^{(i)} \) has \( t \) parts and \( \beta^{(i)} = T_t^{(k)} \). There are two cases:

**Case 1:** The \( i \)-th entry of \( \lambda^{(i)} \) is equal to the \( i \)-th entry of \( \beta^{(i)} \). Then we set \( \beta^{(i-1)} = T_t^{(k)} \) and \( \lambda^{(i-1)} = \lambda^{(i)} \).

**Case 2:** The \( i \)-th entry of \( \lambda^{(i)} \) is not equal to the \( i \)-th entry of \( \beta^{(i)} \). There are three subcases:

**Case 2.1:** \( i \equiv t \pmod{k} \) and \( \lambda^{(i)}_i - \lambda^{(i)}_{i+1} < k \). In this case, let \( d = \lambda^{(i)}_i - \lambda^{(i)}_{i+1} \). Clearly,
there are no cells of hook length $k$ in the $i$-th row of the Young diagram of $\lambda^{(i)}$, see Figure 3.4. Applying the operation $P_i$ to $\lambda^{(i)}$ $d$ times, we obtain a new partition $\mu$ with $\mu_i = \mu_{i+1}$. It is clear that there are no marked cells in the $i$-th row of $\mu$. In view of Lemma 3.2 we see that $\alpha_k(\mu) \geq \alpha_k(\lambda^{(i)})$ and $|\mu| \leq |\lambda^{(i)}|$. Now we apply the operation $D_i$ to $\mu$ to obtain a new partition $\nu$ with $\nu_i = \mu_i + 1$. It is clear that there are no marked cells in the $i$-th row of $\nu$. In view of Lemma 3.2, we see that $\alpha_k(\nu) \geq \alpha_k(\lambda^{(i)})$ and $|\nu| < |\lambda^{(i)}|$. Notice that the partitions $\nu$ and $T^{(k)}_{i-1}$ differ only in the first $i-1$ rows. We set $\beta^{(i-1)} = T^{(k)}_{i-1}$ and $\lambda^{(i)}_{i-1} = \nu$.

Case 2.2: $i \equiv t \pmod{k}$ and $\lambda^{(i)}_i - \lambda^{(i)}_{i+1} < k$. Denote $d = \lambda^{(i)}_i - \lambda^{(i)}_{i+1} - k$. Evidently, there is a cell with hook length $k$ in the $i$-th row of $\lambda^{(i)}$, see Figure 3.5. Applying the operation $P_i$ to $\lambda^{(i)}$ $d$ times, we obtain a new partition $\mu$. It is clear that there is also a marked cell in the $i$-th row. In view of Lemma 3.2 we see that $\alpha_k(\mu) \geq \alpha_k(\lambda^{(i)})$ and $|\mu| \leq |\lambda^{(i)}|$. Now the partitions $\mu$ and $T^{(k)}_{i-1}$ differ only in the first $i-1$ rows. We set $\beta^{(i-1)} = T^{(k)}_{i-1}$ and $\lambda^{(i)}_{i-1} = \mu$.

Case 2.3: $i \not\equiv t \pmod{k}$. In this case, we have $\lambda^{(i)}_i - \lambda^{(i)}_{i+1} > 0$, see Figure 3.6. Let $d = \lambda^{(i)}_i - \lambda^{(i)}_{i+1}$. Applying the operation $P_i$ to $\lambda^{(i)}$ $d$ times, we obtain a new partition $\mu$. Clearly, there is a marked cell in the $i$-th row of $\mu$. By Lemma 3.2 we deduce that...
\[ \alpha_k(\mu) \geq \alpha_k(\lambda^{(i)}) \quad \text{and} \quad |\mu| \leq |\lambda^{(i)}|. \]

Now, the partitions \( \mu \) and \( T^{(k)}_t \) differ only in the first \( i - 1 \) rows. We set \( \beta^{(i-1)} = T^{(k)}_t \) and \( \lambda^{(i-1)} = \mu \).

Figure 3.6: The case for \( i \not\equiv t \pmod{k} \).

Repeat the above process, we eventually obtain a nearly \( k \)-triangular partition \( \lambda^{(0)} = \beta^{(0)} = T^{(k)}_s \). In the construction of \( \lambda^{(0)} \) from \( \lambda^{(r)} \), the operations \( P_i \) and \( D_i \) imply the following relations:

\[
m = \alpha_k(\lambda^{(r)}) \leq \cdots \leq \alpha_k(\lambda^{(i)}) \leq \alpha_k(\lambda^{(i-1)}) \leq \cdots \leq \alpha_k(\lambda^{(0)}) = s
\]

and

\[
|\lambda| = |\lambda^{(r)}| \geq \cdots \geq |\lambda^{(i)}| \geq |\lambda^{(i-1)}| \geq \cdots \geq |\lambda^{(0)}| = t(s,k).
\]

Since \( s \geq m \), we have \( |\lambda| \geq |\lambda^{(0)}| = t(s,k) \geq t(m,k) \). This completes the proof. \( \blacksquare \)

As a consequence of Theorem 3.1, we obtain the following upper bound on \( b(n,k) \).

**Corollary 3.3** Assume \( m \geq 0 \) and \( k > 0 \) and \( n \) is a nonnegative integer such that \( n < t(m+1,k) \), then \( b(n,k) \leq m \).

Figure 3.7 illustrates the transformation from \( \lambda = (10,7,4,3,3,3,3) \) to a nearly \( 3 \)-triangular partition \( T^{(3)}_5 = (6,6,3,3,3) \). It can be checked that both \( \lambda \) and \( T^{(3)}_5 \) have five \( 3 \)-hooks and \( |\lambda| > t(5,3) = 21 \).

4 **Proof of Theorem 1.2**

In this section, we show that the number \( b(n,k) \), that is, the maximum number of \( k \)-hooks among partitions of \( n \), can be determined by the number \( t(m,k) \), namely, the weight of the nearly \( k \)-triangular partition \( T^{(k)}_m \). In the previous section, we have obtained an upper bound on \( b(n,k) \). To determine \( b(n,k) \), we shall give a lower bound on \( b(n,k) \), as stated in Theorem 4.2.

**Theorem 4.1** Assume \( m \geq 0 \), \( k \geq 1 \) and \( n \) is a nonnegative integer such that \( t(m,k) \leq n \leq t(m+1,k) - 1 \), then we have \( b(n,k) = m \).
Figure 3.7: \( \lambda = (10, 7, 4, 3, 3, 3, 3) \) and \( T_5^{(3)} = (6, 6, 3, 3, 3) \).

Note that Theorem 4.1 reduces to Theorem 2.1 when \( k = 1 \). The following theorem gives a lower bound on \( b(n, k) \).

**Theorem 4.2** Assume \( m \geq 0 \) and \( k > 0 \) and \( n \) is a nonnegative integer such that \( n \geq t(m, k) \), then we have \( b(n, k) \geq m \).

To prove Theorem 4.2, we introduce an operation \( Q_j \) defined on Young diagrams. In fact, \( Q_j \) is same as the operation \( P_i \) except for the first step. Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) be a partition. The operation \( Q_j \) applies to partitions \( \lambda \) for which \( \lambda_j - 1 > \lambda_j \). More precisely, \( Q_j(\lambda) \) is constructed via the following steps.

Step 1. Add a cell \( v \) at the end of the \( j \)-th row of the Young diagram of \( \lambda \), and denote the resulting partition by \( \mu \). If the Young diagram of \( \lambda \) contains no marked cells in the \( (\lambda_j + 1) \)-th column of \( \lambda \), then we set \( Q_j(\lambda) = \mu \);

Step 2. At this step, it is clear that there is only one cell of hook length \( k \) in the \( (\lambda_j + 1) \)-th column of \( \lambda \). Denote this marked cell by \( w \) and assume that it is in the \( h \)-th row of \( \lambda \). Note that the marked cell \( w \) in \( \lambda \) is of hook length \( k + 1 \) in \( \mu \). There are two cases:

Case 1: \( \mu_h = \mu_{h+1} \). Clearly, the cell \( w' \) below \( w \) is of hook length \( h \) in \( \mu \). In this case, we set \( Q_j(\lambda) = \mu \)

Case 2: \( \mu_h > \mu_{h+1} \). We apply the operation \( P_h \) to \( \mu \) and denote the resulting partition by \( \nu \). It is easily seen that the \( h \)-th entry of \( \mu \) decreases by one. Consequently, \( w \) has
hook length $k$ in $\nu$. We set $Q_j(\lambda) = \nu$.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, we regard the $(r + 1)$-th entry as 0 when we apply $Q_{r+1}$ to $\lambda$. Under this convention, $Q_{r+1}$ increases the number of parts of $\lambda$ by one.

The following property of the operation $Q_j$ is similar to Lemma 3.2. The proof is omitted.

**Lemma 4.3** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition such that $\lambda_j < \lambda_{j-1}$. Then we have

$$\alpha_k(\lambda) - \alpha_k(\lambda, j) \leq \alpha_k(Q_j(\lambda)) - \alpha_k(Q_j(\lambda), j)$$

(4.1)

and

$$|\lambda| \leq |Q_j(\lambda)| \leq |\lambda| + 1.$$  

(4.2)

We are now ready to prove Theorem 4.2 by using the operation $Q_j$.

**Proof of Theorem 4.2.** For each $n \geq t(m, k)$, it suffices to show that there exists a partition $\lambda$ of $n$ with at least $m$ $k$-hooks. We proceed by induction on $n$.

First, when $n = t(m, k)$, the nearly $k$-triangular partition $T_m(k)$ is a partition of $t(m, k)$ with $m$ $k$-hooks, that is the theorem holds for $n = t(m, k)$.

We now assume that there is a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of $N$ with $s$ $k$-hooks, where $N \geq t(m, k)$ and $s \geq m$. The following procedure gives the construction of a partition of $N + 1$ with at least $s$ $k$-hooks.

Apply $Q_{r+1}$ to $\lambda$ and denote the resulting partition by $\mu^{(1)}$. From Lemma 4.3 we see that $\alpha_k(\mu^{(1)}) \geq s$ and $|\lambda| \leq |\mu^{(1)}| \leq |\lambda| + 1$. There are two cases:

**Case 1:** $|\mu^{(1)}| = |\lambda| + 1 = N + 1$. Then $\mu^{(1)}$ is a partition of $N + 1$ with at least $s$ $k$-hooks.

**Case 2:** $|\mu^{(1)}| = |\lambda| = N$. We continue to construct a sequence of partitions,

$$\mu^{(2)} = Q_{r+2}(\mu^{(1)}), \ldots, \mu^{(i+1)} = Q_{r+i+1}(\mu^{(i)}), \ldots, \mu^{(k-1)} = Q_{r+k-1}(\mu^{(k-2)}).$$

It is clear that $\mu^{(i)}$ has $r + i$ parts and contains at least $i$ parts equal to one. It follows from (4.1) that $\alpha_k(\mu^{(i)}) \geq s$ for $2 \leq i \leq k - 1$. There are two subcases:

**Case 2.1:** There exists a partition $\mu^{(i)}$ such that $|\mu^{(i)}| = N + 1$, where $2 \leq i \leq k - 1$. Then $\mu^{(i)}$ is a partition of $N + 1$ with at least $s$ $k$-hooks.

**Case 2.2:** $|\mu^{(i)}| = N$ for $2 \leq i \leq k - 1$. Now, we construct $\mu^{(k)}$ from $\mu^{(k-1)}$. Recall that $\mu^{(k-1)}$ contains at least $k - 1$ parts equal to one with at least $s$ $k$-hooks. Set $\mu^{(k)}$ to be the partition obtained from $\mu^{(k-1)}$ by adding one as a new part. Now, we have $|\mu^{(k)}| = N + 1$. Moreover, there are at least $k$ parts equal one and there is a cell of hook length $k$ in the first column of $\mu^{(k)}$. It can be seen that the positions of the marked cells in the other column of $\mu^{(k-1)}$ stay unchanged in $\mu^{(k)}$. This yields
\[ \alpha_k(\mu^{(k)}) \geq \alpha_k(\mu^{(k-1)}) \geq s. \] Thus we have obtained a partition \( \mu^{(k)} \) of \( N + 1 \) with at least \( s \) \( k \)-hooks.

Figure 4.8 gives an illustration of the construction of a partition of 12 with three \( 3 \)-hooks from the nearly \( 3 \)-triangular partition with three parts.

Combining Corollary 3.3 and Theorem 4.2, we obtain Theorem 4.1. We conclude this paper with a proof of Theorem 1.2.

Proof of Theorem 1.2. Multiplying both sides by \( (1 - q) \), (1.2) can be rewritten as

\[ (1 - q) \sum_{n \geq 0} b(n, k)q^n = \sum_{t \geq 1} q^{(1)k^2} \frac{1 - q^{tk^2}}{1 - q^{tk}} - 1. \] (4.3)

It is easily checked that

\[ (1 - q) \sum_{n \geq 0} b(n, k)q^n = \sum_{n \geq 0} (b(n + 1, k) - b(n, k))q^{n+1}. \] (4.4)

To compute \( b(n + 1, k) - b(n, k) \), we denote the interval \([t(m, k), t(m + 1, k) - 1]\) by \( I_{m}^{(k)} \). By Theorem 4.1 we see that \( b(n, k) \) is determined by the interval containing \( n \). There are two cases:

Case 1: \( n \) and \( n + 1 \) belong to the same interval \( I_{m}^{(k)} \). By Theorem 4.1 we have \( b(n + 1, k) = b(n, k) = m \), that is, \( b(n + 1, k) - b(n, k) = 0 \).

Case 2: \( n \) and \( n + 1 \) lie in two consecutive intervals \( I_{m}^{(k)} \) and \( I_{m+1}^{(k)} \), that is, \( n = t(m + 1, k) - 1 \). By Theorem 4.1 we obtain that \( b(n, k) = m \) and \( b(n + 1, k) = m + 1 \). It follows that \( b(n + 1, k) - b(n, k) = 1 \).

Combining the above two cases, we deduce that

\[ \sum_{n \geq 0} (b(n + 1, k) - b(n, k))q^{n+1} = \sum_{m \geq 0} q^{t(m+1,k)} = \sum_{m \geq 0} q^{t(m,k)} - 1. \]

It follows that

\[ (1 - q) \sum_{n \geq 0} b(n, k)q^n = \sum_{m \geq 0} q^{t(m,k)} - 1. \] (4.5)
Recall that
\[ t(m, k) = m \left( \left\lfloor \frac{m}{k} \right\rfloor + 1 \right) k - \left( \frac{\left\lfloor \frac{m}{k} \right\rfloor + 1}{2} \right) k^2. \]

Write \( m = jk + r \), where \( 0 \leq r \leq k - 1 \). Then we get
\[ t(jk + r, k) = (jk + r) (j + 1) k - \left( \frac{j + 1}{2} \right) k^2 \]
\[ = \left( \frac{j + 1}{2} \right) k^2 + r(j + 1)k. \tag{4.6} \]

Substituting (4.6) into (4.5), we find that
\[ \sum_{m \geq 0} q^{t(m,k)} - 1 = \sum_{j \geq 0} \sum_{r=0}^{k-1} q^{(\frac{j+1}{2})k^2 + r(j+1)k} - 1 \]
\[ = \sum_{j \geq 0} q^{(\frac{j+1}{2})k^2} \frac{1 - q^{(j+1)k^2}}{1 - q^{(j+1)k}} - 1. \]

In view of (4.5), we obtain that
\[ (1 - q) \sum_{n \geq 0} b(n, k) q^n = \sum_{t \geq 1} q^{(\frac{1}{2})k^2} \frac{1 - q^{tk^2}}{1 - q^{tk}} - 1. \]

This completes our proof. \( \blacksquare \)

Using the Jacobi triple product identity \cite[Eq. 1.6.1]{Andrews1998}, we may express the generating function \( \sum_{n \geq 0} b(n, k) q^n \) in the following form:
\[ \frac{1}{1 - q} \left( \frac{q^{2k^2}; q^{2k^2}}{(q^{k^2}; q^{2k^2})_\infty} + \frac{1}{2} \sum_{r=1}^{k-1} (-q^{rk}; q^{k^2})_\infty (-q^{k^2-rk}; q^{k^2})_\infty (q^{k^2}; q^{k^2})_\infty - \frac{k + 1}{2} \right). \]

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References

[1] T. Amdeberhan, Theorems, problems and conjectures, \texttt{arXiv:math.CO/1207.4045}.

[2] G.E. Andrews, The Theory of Partitions, Cambridge University Press, 1998.

[3] G.E. Andrews and B.C. Berndt, Ramanujan’s Lost Notebook, Part II, Springer, 2005.
[4] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, 1990.

[5] G.N. Han, The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications, Ann. Inst. Fourier (Grenoble) 60 (2010) 1–29.

[6] I. Pak, Partition bijections: A survey, Ramanujan J. 12 (2006) 5–75.