CONTINUITY OF SET OF BILIPSCHITZ CLASSES
IN EUCLIDEAN SPACE

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Introduction

This paper is devoted to studying biLipschitz equivalence of Delone sets.

Let $M$ be a metric space with distance $d_M(x, y)$. Denote by $B_{\rho}(x)$ and $B^\circ_{\rho}(x)$ respectively the closed and the open balls with radius $\rho$ centered at $x$. A set $A \subset M$ is a Delone set, if for some $0 < r < R$ the following conditions hold.

- $B^\circ_r(x) \bigcap B^\circ_r(y) = \emptyset$ for every $x, y \in A$.
- $\bigcup_{x \in A} B_R(x) = M$.

Two Delone sets $A \subset M_1$ and $B \subset M_2$ are biLipschitz equivalent, if there exist a real $\lambda \geq 1$ and a bijection $F : A \to B$ such that the inequality

$$\frac{1}{\lambda} d_{M_1}(x, y) \leq d_{M_2}(F(x), F(y)) \leq \lambda d_{M_1}(x, y)$$

holds for every $x, y \in A$.

Map $F$ for which such an inequality holds is called $\lambda$-biLipschitz.

The question about biLipschitz equivalence was raised by M. Gromov in [1]. In particular, the following problem was stated:

Given a metric space $M$ determine if every two Delone sets $A, B \subset M$ are biLipschitz equivalent.

If $M = \mathbb{E}^1$ — a Euclidean line, then the answer is obviously positive. Also positive answers were obtained by P. Papasoglu (see [2]) for homogeneous trees, O. Bogopolsky (see [3]) for hyperbolic spaces $\mathbb{H}^d$ and K. Whyte (see [4]) for non-amenable spaces.

In case of Euclidean space $M = \mathbb{E}^d$ of dimension $d \geq 2$ D. Burago and B. Kleiner (see [5]) and independently C. McMullen (see [6]) proved the following result:

**Theorem 1.** For every integer $d \geq 2$ there exists a Delone set $A \subset \mathbb{E}^d$ which is not biLipschitz equivalent to the integer net $\mathbb{Z}^d$.

In [5] theorem 1 is proved for $d = 2$, but the proof is easily generalized for every dimension $d \geq 2$. Therefore in $\mathbb{E}^d$ for every $d \geq 2$ there exist at least 2 biLipschitz classes.

The main result of this paper is

**Theorem 2.** For every integer $d \geq 2$ the set of biLipschitz classes in $\mathbb{E}^d$ has cardinality continuum.

**Proof of theorem 2**

Obtain the upper estimate for cardinality of the set of biLipschitz classes.

Use the following result of A. Garber (see [7, lemma 2])

**Lemma 3.** Let $A \subset \mathbb{E}^d$ be a Delone set. Then there exists a Delone set $D \subset \mathbb{Z}^d$ such that $A$ and $D$ are biLipschitz equivalent.
From lemma 3 follows that every biLipschitz class has at least one member among subsets of \( \mathbb{Z}^d \). Therefore cardinality of the set of biLipschitz classes is at most cardinality of family of all subsets containing in \( \mathbb{Z}^d \), i.e. continuum. The upper estimate proved.

To prove the lower estimate obtain a continuum family of pairwise non-equivalent Delone sets. These sets will be members of some special class.

From this point we consider only rectangular coordinates in \( \mathbb{E}^d \). Parallelepipeds (cubes) with edges parallel to coordinate lines are called coordinate.

Let \( Q \) be a coordinate cube. Denote by \( m(Q) \) its vertex with the least sum of coordinates.

Consider a tiling \( T \) of \( \mathbb{E}^d \) into coordinate cubes whose edge lengths belong to \([1, L]\). The set \( \mathcal{A} = \{ m(Q) : Q \in T \} \) is obviously a Delone set. Delone sets obtained in such a way are called \( L \)-special.

Introduce some notation. Let \( P_{MN} = \mathbb{Z}^d \cap (\{0, MN\} \times \{0, N\}^{d-1}) \), \( P_{MN}^i = \mathbb{Z}^d \cap (\{iN, (i+1)N\} \times \{0, N\}^{d-1}) \) for \( i = 0, 1, \ldots, M-1 \),

\[
\begin{align*}
  u &= (0, 0, \ldots, 0), \\
  v &= (MN, 0, 0, \ldots, 0).
\end{align*}
\]

Call points \( x, y \in P_{MN} \) corresponding if \( y - x = (N, 0, 0, \ldots, 0) \).

**Lemma 4.** Let \( \lambda \geq 1 \), \( \varepsilon \in (0, \frac{1}{4}) \) \( a \in (0, 1) \). Then there exist \( k > 0 \) and \( M_0 \in \mathbb{N} \) such that for every \( M, N \in \mathbb{N} \), \( M > M_0 \) and for arbitrary \( \lambda \)-biLipschitz map \( F : P_{MN} \to \mathbb{E}^d \) at least one of the following statements hold:

1. There exist corresponding points \( x, y \) such that

\[
\frac{|F(y) - F(x)|}{|y - x|} > (1 + k) \frac{|F(v) - F(u)|}{|v - u|},
\]

2. There exists \( i \) such that number of pairs of corresponding points \( x \in P_{MN}^i, y \in P_{MN}^{i+1} \) for which holds

\[
\frac{|F(y) - F(x) - \frac{1}{M}(F(v) - F(u))|}{\frac{1}{M}|F(v) - F(u)|} < \varepsilon,
\]

is at least \( aN^d \).

Proof for \( d = 2 \) is in [5, Lemma 3.2]. Proof for an arbitrary \( d \) is obtained by a straightforward repeating the arguments of [5].

**Lemma 5.** Let \( I = [0, 1] \), \( \alpha \in (0, \frac{1}{2}) \), and let \( P, Q \subset I^d \) be closed sets with a boundary being a finite polyhedron. If \( P \cup Q = I^d \), \( \text{int} P \cap \text{int} Q = \ldots \)
∅ and also \( \text{Vol}_d(P) \geq \alpha \) and \( \text{Vol}_d(Q) \geq \alpha \) then \((d-1)\)-dimensional volume \( \text{Vol}_{d-1}(\partial P \cap \partial Q) \geq \frac{\alpha}{2^{d-1}} \).

**Proof.** Denote by \( \pi \) the projection onto hyperplane \( x_1 = 0 \).

Conduct the proof by induction over \( d \).

**Induction base:** \( d = 2 \). If \( \text{Vol}_1(\pi(P) \cap \pi(Q)) \geq \frac{\alpha}{2} \) then statement of lemma is obviously true. Otherwise the following inequalities hold:

\[
1 - \alpha \geq 1 - \text{Vol}_2(Q) = \text{Vol}_2(P) \geq \text{Vol}_1(\pi(P)) - \text{Vol}_1(\pi(P) \cap \pi(Q)).
\]

Hence \( \text{Vol}_{d-1}(\pi(P)) < 1 - \frac{\alpha}{2} \).

Therefore there exists \( t_P \in (0, 1) \) such that \( P \cap \{x_2 = t_P\} = \emptyset \). Similarly, there exists \( t_Q \in (0, 1) \) such that \( Q \cap \{x_2 = t_Q\} = \emptyset \). It follows that projection of \( P, Q \) onto line \( x_2 = 0 \) is a segment \([0, 1]\). Hence \( \text{Vol}_d(P) < 1 - \frac{\alpha}{2} \).

**Induction step.** Similarly to previous if \( \text{Vol}_{d-1}(\pi(P) \cap \pi(Q)) \geq \frac{\alpha}{2} \) statement of lemma is obvious. Otherwise \( \text{Vol}_{d-1}(\pi(P)) < 1 - \frac{\alpha}{2} \). Then every section of \( Q \) by a hyperplane \( x_1 = t \) has a \((d-1)\)-volume \( \geq \frac{\alpha}{2} \). Similarly, every section of \( P \) by a hyperplane \( x_1 = t \) has a \((d-1)\)-volume \( \geq \frac{\alpha}{2} \). By induction assumption, every section of \( \partial P \cap \partial Q \) has a \((d-2)\)-volume \( \geq \frac{\alpha}{2^{d-2}} \), and the statement of lemma is now obvious.

**Lemma 6.** Given \( \lambda \geq 1 \), \( L \geq 1 \) and rational \( c > 1 \) there exists a finite point set \( B_0 \) and a parallelepiped \( \Pi = \prod_{i=1}^{d} [0, b_i) \) \( b_i \in \mathbb{N} \) such that:

1. \( B_0 \subset \Pi \).

2. There exists a tiling \( T_0 \) of \( \Pi \) into coordinate cubes with edges \( 1 \) and \( c \) such that \( \{m(Q) : Q \in T_0\} = B_0 \).

3. For every Delone set \( B \) such that \( B \cap \Pi = B_0 \) and \( (0,0,\ldots,0,b_d) \in B \) and for every \( \lambda \)-biLipschitz bijection \( F : B \to A \) where \( A \) is \( L \)-special, the set \( F(B_0) \) has at least one exceptional point.

**Proof.** Conduct the construction of \( B_0 \) in 3 steps:

1. Choose \( \varepsilon, a \) which have the same meaning as in lemma 4; choose a parameter \( H_0 \).

2. Choose \( N \) and \( M \).

3. Choose \( H \) fulfilling \( H \geq H_0 \) and construction of \( B_0 \) itself.

Describe the construction beginning from the last step. Let \( \varepsilon, a, M, N, H_0 \) be already chosen on previous steps.

Take a parallelepiped

\[
\Phi_0,(0,0,\ldots,0) = [0, 1)^{d-1} \times [0, M).
\]
Consider its tiling into unit cubes. Colour these cubes checkerboardwise into black an white, starting with black.

Take in parallelepiped $\Phi_{0,(0,0,...,0)}$ parallelepipeds

$$\Phi_{1,\frac{1}{M} (j_1,j_2,...,j_{d-1},0)} = [0, \frac{1}{M}]^{d-1} \times [0,M) + \frac{1}{N} \cdot (j_1,j_2,...,j_{d-1},0)$$

where $j_i = 0, 1, \ldots, N-1$. From this point colouring of $\Phi_{0,(0,0,...,0)}$ will change only inside parallelepipeds of type $\Phi_{1,z}$. Divide each of these parallelepipeds into cubes with edge equal to $\frac{1}{M}$ and colour them checkerboardwise starting from black.

Continue the process. On $\nu$-th step in each parallelepiped of type $\Phi_{\nu-1,z}$ take the parallelepipeds

$$\Phi_{\nu,z} = [0, \frac{1}{M^{\nu-1}}]^{d-1} \times [0,M) + z$$

take the parallelepipeds

$$\Phi_{\nu,z + \frac{1}{M^{\nu-1}} (j_1,j_2,...,j_{d-1},0)} = [0, \frac{1}{M^{\nu-1}}]^{d-1} \times [0,M) + z + \frac{1}{N} \cdot \frac{M^{\nu-1}}{M} \cdot (j_1,j_2,...,j_{d-1},0).$$

From this point colouring will change only inside these parallelepipeds. Divide each of these parallelepipeds into cubes with edge equal to $\frac{1}{M}$ and colour them checkerboardwise starting from black.

Repeat while $\nu \leq \nu_0 = \lceil \log_2 \frac{1}{\varepsilon} \rceil + 2$.

Note that if $\Phi_{0,(0,0,...,0)}$ is divided into cubes with edge $\frac{1}{M}$ then each of them is coloured in one colour — black or white. Call them *coloured cubes*.

Make a homothety of parallelepiped $\Phi_{0,(0,0,...,0)}$ together with colouring of coefficient $H$ and center at origin. Choose $H$ such that coloured cubes were taken into cubes that have integer edges and also could be divided into cubes with edge $c$. Inequality $H \geq H_0$ also must hold.

Images of black coloured cubes divide into unit cubes and images of white cubes — into cubes with edge $c$. The obtained tiling of $\Pi = H \cdot \Phi_{0,(0,0,...,0)}$ denote by $T_0$. Let $B_0 = \{ m(Q) : Q \in T_0 \}$.

Describe the second step. Let $\varepsilon, a H_0$ be chosen before, choose $N$ and $M$.

Let $y - x = (0,0,\ldots,0,1)$. Choose $N$ such that if $P = [0, \frac{1}{N}]^d$ then for every vector $e$ fulfilling

$$|F(y) - F(x) - e| < \varepsilon \cdot |e|,$$

holds the inequality

$$|F(y') - F(x') - e| < 2\varepsilon \cdot |e|$$

if only $x' \in x + P$, $y' \in y + P$ and $F$ is $\lambda$-biLipschitz.

Let $P_1 P_2$ be cubes with edge $\frac{1}{M}$ coloured on $l$-th step black and white respectively. Let each be divided into $N^d$ equal cubes and let $Q_1 \subset P_1$, $Q_2 \subset P_2$ be such cubes. Choose $M$ such that independently from choice of $H$ holds true

$$\frac{|(P_1 \cap B_0) - (Q_1 \cap B_0)|}{|(P_2 \cap B_0) - (Q_2 \cap B_0)|} \geq \frac{1 + c}{2}.$$
This inequality is obviously true if only

\[ \text{Vol}_d (Q_1 \cap (\cup_z \Phi_{l+1,z})) \leq \frac{1}{c-1} \text{Vol}_d (Q_1) \text{ and} \]

\[ \text{Vol}_d (Q_2 \cap (\cup_z \Phi_{l+1,z})) \leq \frac{1}{c-1} \text{Vol}_d (Q_2) \]

which is true for big enough \( M \). Also take \( M > M_0 \) where \( M_0 \) comes from lemma 4 and \( N \mid M \).

Describe the first step.

Let \( a = \frac{3+\varepsilon}{2} \). Show that there exists a choice of \( \varepsilon \) and \( H_0(\varepsilon) \) such that \( B_0 \) constructed as before fulfilled the conditions of lemma 6.

Suppose that for every \( \varepsilon \) and \( H_0 \) there is a Delone set \( B \supset B_0 \) fulfilling the conditions of lemma 6 and \( \lambda \)-biLipschitz bijection \( F : B \to A \) such that \( F(B_0) \)

consists only of standard points.

Let \( u = (0,0,\ldots,0), v = (0,0,\ldots,0,HM) \). If the first case of statement of lemma 4 holds there exist corresponding points \( x, y \) such that

\[ |F(y) - F(x)| > (1 + k) |F(v) - F(u)| \]

In this case instead of \( u, v \) consider a pair \( x, y \) and restriction of \( F \) to a subset of \( B_0 \) contained in parallelepiped \( x + [0,M^{-1}] \times [0,H] \).

Apply to this set all the arguments similarly as to \( B_0 \). If such a substitution can be made \( \log_{1+k} \lambda^2 \) times, then from \( u, v \) we come to \( u', v' \) such that

\[ \frac{|F(u') - F(v')|}{|u' - v'|} > \lambda^2 \frac{|F(v) - F(u)|}{|v - u|} \]

which makes a contradiction to \( \lambda \)-biLipschitz property of \( F \).

Therefore on some step we have the second case of lemma 4. Let the adjoint cubes for which this case holds have numbers \( i \cdot i + 1 \). Let also \( i \)-th cube be originally white and, respectively, \( (i + 1) \)-th black. Let \( \tilde{F} = G_{\lambda} \circ F \).

Let \( C \) be a set of points of \( i \)-th cube such that have distance at least \( 10\lambda L \) from its boundary, \( C' \) are all points of \( i + 1 \)-th cube. Using our assumptions obtain two inequalities involving \( \text{Vol}_{d-1}(\partial \tilde{F}(C)) \).

\[ \text{Vol}_{d-1}(\partial \tilde{F}(C)) \leq \beta_0 \cdot |C|^{\frac{d-1}{d+1}} \]

\[ \text{Vol}_{d-1}(\partial \tilde{F}(C)) \geq \beta_2 \varepsilon^{-1} \cdot |C|^{\frac{d-1}{d+1}} \]

For small enough \( \varepsilon \) they contradict each other and that completes the proof of lemma 6.

Proving lemmas 7 and 8 \( H \) is assumed big enough depending on \( \varepsilon \), i.e. \( H \geq H_0(\varepsilon) \).

**Lemma 7.** Inequality

\[ \text{Vol}_{d-1}(\partial \tilde{F}(C)) \leq \beta_0 \cdot |C|^{\frac{d-1}{d+1}} \]

holds true, where \( \beta_0 \) is a constant depending on \( d, \lambda, L \) and \( c \) (but not \( \varepsilon \)).
Proof. This inequality follows immediately from the fact that if \( \tilde{F}(x) \) has common boundary with \( F(C) \) then \( x \) is a point of \( \tilde{i} \)-th cube, that does not depend on \( C \). The number of such points does not exceed \( \beta_1 \cdot |C|^{\frac{d-1}{d}} \), hence \( (d-1) \)-volume of boundary of corresponding cubes does not exceed \( \beta_0 \cdot |C|^{\frac{d-1}{d}} \).

Let \( s = 4 \varepsilon \frac{\lambda}{4(1+\varepsilon)} |F(v) - F(u)| \).

Let \( K \) be a real independent from \( \varepsilon \) and such that

\[
\left(1 + \frac{(2K+2)^d - (2K)^d}{K^d}\right) \frac{4}{3+c} < \frac{8}{7+c}.
\]

Take a full (in respect to inclusion relationship) packing of coordinate cubes with centers in \( F(C) \) and edges equal to \( Ks \). Let it consist of \( W \) cubes.

Denote by \( U \) a union of coordinate cubes with the same centers and edges equal to \( 2Ks \). Since the chosen packing is full all points of \( F(C) \) are contained in \( U \).

Let \( \tau \) be a translation by vector \( \frac{1}{M}(F(v) - F(u)) \). Consider \( \frac{\varepsilon}{\lambda} \)-neighbourhood of \( \tau(F(C)) \). Denote by \( C'' \) the set of points of \( \mathcal{A} \) that belong to this neighbourhood. Since the second case of lemma 4 assumed true and due to choice of \( a \) and \( M \) obtain:

\[
|C' \cap C''| \geq a \frac{1+c}{2} \cdot |C| = \frac{3+c}{4} \cdot |C|.
\]

But \( \tilde{F}(C' \cap C'') \) is contained in the union of cubes with the same centers as \( \tau(U) \) and edge equal to \( (2K + 2)s \), because for big \( H \) holds \( s > 1 \). Denote this union by \( U_1 \).

Note that \( \text{Vol}_d(U) \geq WK^d s^d \), \( \text{Vol}_d(U_1) \leq \text{Vol}_d(U) + ((2K+2)^d - (2K)^d) W s^d \).

According to choice of \( K \) obtain

\[
\text{Vol}_d(\tilde{F}(C)) \leq \frac{4}{3+c} \text{Vol}_d(U_1) \leq \frac{8}{7+c} \text{Vol}_d(U).
\]

Rewrite the last inequality as \( \text{Vol}_d(U \setminus \tilde{F}(C)) \geq \frac{c-1}{7+c} \text{Vol}_d(U) \). Due to an estimate for \( \text{Vol}_d(U) \) already obtained,

\[
\text{Vol}_d(U \setminus \tilde{F}(C)) \geq \frac{c-1}{7+c} WK^d s^d.
\]

Let \( \mu \in (0,1) \) be such that \( \mu + (1-\mu) \cdot \frac{\varepsilon}{2(1+\varepsilon)} < \frac{\varepsilon}{2(1+\varepsilon)} \). Note that \( \mu \) does not depend on \( \varepsilon \). Then in at least \( \mu W \) cubes of \( U \) set \( \tilde{F}(C) \) occupies volume at most \( \left(1 - \frac{c-1}{2(1+\varepsilon)}\right) \cdot (2Ks)^d \). Call the cubes marked.

Lemma 8. Suppose \( \varepsilon \) small enough, then in our assumptions on \( F \)

\[
\text{Vol}_{d-1}(\partial \tilde{F}(C)) \geq \beta_1 \varepsilon^{-1} \cdot |C|^{\frac{d-1}{d}},
\]

where \( \beta_2 \) depends on \( d, \lambda, \) and \( c \).

Proof. If \( \varepsilon < \frac{\lambda}{10K^{d-1}} \) and \( H \) is big enough then in every marked cube \( \tilde{F}(C) \) occupies volume at least \( \beta_3 \cdot (2Ks)^d \). Indeed, if \( F(x) \) is a center of marked cube
then due to $\lambda$-biLipschitz property of $F$ all points of $C \cap B_{Ks}(x)$ are taken inside this cube. For $H$ big enough $s$ is also big, then

$$\operatorname{Vol}_d(\tilde{F}(C \cap B_{Ks}(x))) = |C \cap B_{Ks}(x)| \geq 2\beta_3 \cdot (2Ks)^d,$$

and on the other hand, a part of volume of $\tilde{F}(C \cap B_{Ks}(x))$ not exceeding $(2Ks + 2)^d - (2Ks)^d$ can be excluded from the marked cube. But for big enough $s$ it does not exceed $\beta_3 \cdot (2Ks)^d$. Hence the inequality.

Accurate to lemma 5 inside marked cubes $\partial \tilde{F}(C)$ has $(d - 1)$-volume at least $\beta_4 s^{d-1}$ where $\beta_4$ depends on $d$, $\lambda$, and $c$.

Since cubes of packing do not intersect, no $8^d + 1$ cubes of $U$ have a common point. Then for some $\beta_5$, depending on $d$, $\lambda$, and $c$ holds

$$\operatorname{Vol}_{d-1}(\partial \tilde{F}(C)) \geq \beta_5 W s^{d-1}. $$

Since $F(C) \subset U$ obtain $|C| \leq W(2Ks + 2)^d \leq \beta_6 W s^d$. Again $H$ and $s$ are assumed big enough. Therefore

$$\operatorname{Vol}_{d-1}(\partial \tilde{F}(C)) \geq \beta_7 |C| s^{-1}. $$

Due to $\lambda$-biLipschitz property of $F$ holds

$$\frac{1}{M} |F(v) - F(u)| \leq \beta_8 |C|^{\frac{1}{d}}. $$

Using the definition of $s$ obtain

$$s \leq \beta_9 |C|^{\frac{1}{d}}, $$

which together with the last inequality for $\operatorname{Vol}_{d-1}(\partial \tilde{F}(C))$ implies the statement of lemma.

**Lemma 9.** Given real $\lambda \geq 1$, $L \geq 1$, rational $c > 1$ and positive integer $j \in \mathbb{N}$ there exists a finite point set $\tilde{D}$ and a parallelepiped $\Pi = \prod_{i=1}^d [0, b_i)$ where $b_i$ are positive integer such that:

1. $\tilde{D} \subset \Pi$.

2. There exists a tiling $T_0$ of $\Pi$ into coordinate cubes with edges 1 and $c$ such that $\{m(Q) : Q \in T_0\} = \tilde{D}$.

3. For every Delone set $D$ fulfilling $D \cap \Pi = \tilde{D}$ and $(0, 0, \ldots, 0, b_d) \in D$ for every $\lambda$-biLipschitz bijection $F : D \to A$ with $L$-special Delone set $A$, the set $F(\tilde{D})$ contains at least $j$ exceptional points.

**Proof.** If $j = 1$ then the desired statement is exactly lemma 6. If $j > 1$ take a parallelepiped $\Pi(\lambda, L, c, j)$ with first $d - 1$ edges equal to corresponding edges of $\Pi(\lambda, L, c, 1)$ and the last edge $j$ times greater than the corresponding edge of $\Pi(\lambda, L, c, 1)$. Divide $\Pi(\lambda, L, c, j)$ into $j$ parallelepipeds congruent to $\Pi(\lambda, L, c, 1)$.
Take in each of them a set congruent to $\tilde{D}(\lambda, L, c, 1)$. Denote the obtained set by $\tilde{D}(\lambda, L, c, j)$. Obviously, it fulfills the statement of lemma.

Return to the proof of theorem 2. Let $\{c_i\}_{i=1}^{\infty}$ be a sequence of rationals from $(1, 2]$, e.g. $c_i = 1 + \frac{1}{i}$.

In notation of lemma 9 let $D_1 = \tilde{D}(1, 2, c_1, 1)$. By induction define

$$D_j = \tilde{D}(j, 2, c_j, \sum_{i=1}^{j-1} \#D_i + 1).$$

Let $r_j = 100j \cdot \text{diam}(D_{j+1})$. Without loss of generality, let $r_j$ be strictly increasing.

Let $G$ be an additive group of rationals with denominator equal to some positive integer exponent of 2. From each class of $\mathbb{R}/G$ choose one number and for each chosen number take the sequence of digits after the point in its binary representation. Obtain a continuum set of non-confinal $(0, 1)$-sequences, i.e. every two sequences have an infinite set of indices for which the corresponding members are different.

For every taken sequence $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ construct a 2-special Delone set $D_\alpha$ as follows. Take $D_1$ so that the corresponding parallelepiped was coordinate with integer vertices. Further, if $\alpha$ has zero as $j$-th digit take a copy of $D_{j+1}$ at $\lceil r_j \rceil$ to the right from $D_j$; if $\alpha$ has unit as $j$-th digit then take a copy of $D_{j+1}$ at $\lceil 100j r_j \rceil$ to the right from $D_j$. Also corresponding to $D_{j+1}$ parallelepiped should be coordinate with integer vertices. In addition, include into $D_\alpha$ all points of $\mathbb{Z}^d$ which are outside all the parallelepipeds corresponding to $D_j$. These points will be standard for $D_\alpha$.

Prove that any two constructed sets are not biLipschitz equivalent.

Let there exist $\lambda$-biLipschitz bijection $F : D_\alpha \to D_\beta$. Lemma 10 states some property of this bijection.

**Lemma 10.** For $j > 2 \lambda$ there exists a point of $D_\alpha$ in a copy of $D_j$ (see construction of $D_\alpha$) which is sent by $F$ into some point of copy of $D_j$ in $D_\beta$.

**Proof.** Indeed, since image of $D_j$ contains many enough exceptional points, there is a point $x$ of $D_\alpha$ in a copy of $D_j$ that is sent into an exceptional point, and, moreover, $F(x)$ does not belong to copies of $D_i$ for $i < j$. If $F(x)$ belongs to copy of $D_j$ then the proof is complete. Let $y = F(x)$ belong to copy of $D_{j+k}$, $k > 0$. Consider the image of $D_{j+k}$ under $F^{-1}$ that is also $\lambda$-biLipschitz. Let $z \in D_{j+k}$. Then

$$|F^{-1}(y) - F^{-1}(z)| \leq \lambda |y - z| \leq \lambda \text{diam}(D_{j+k}) < r_{j+k-1}.$$

Therefore all exceptional points in image of $D_{j+k}$ under $F^{-1}$ belong to copies of $D_i$ for $i < j + k$, and it immediately implies a contradiction since there are many enough exceptional points in the image of $D_{j+k}$.

Continue the proof of theorem 2.

Take $j > 2\lambda$ such that $\alpha$ and $\beta$ differ in $j$-th digit. Without loss of generality, $\alpha_j = 0$ and $\beta_j = 1$. By lemma 10 there are $x_j$ and $x_{j+1}$ in copies of $D_j$ and...
$D_{j+1}$ respectively in $D_\alpha$ which are taken to points $y_j$ and $y_{j+1}$ of corresponding copies in $D_\beta$. By construction of $D_\alpha$ and $D_\beta$,

$$100j \cdot \text{diam}(D_{j+1}) < |x_j - x_{j+1}| < (100j + 2) \cdot \text{diam}(D_{j+1}),$$

and

$$10000j^2 \cdot \text{diam}(D_{j+1}) < |y_j - y_{j+1}| < (10000j^2 + 2) \cdot \text{diam}(D_{j+1}).$$

Therefore

$$\frac{|y_j - y_{j+1}|}{|x_j - x_{j+1}|} > 99j > \lambda.$$

A contradiction with $\lambda$-biLipschitz property of $F$ makes proof of theorem 2 complete.

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