Correlations in the chaotic spectrum of pressure modes in rapidly rotating stars

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The oscillation spectrum of pressure waves in stars can be determined by monitoring their luminosity. For rapidly rotating stars, the corresponding ray dynamics is mixed, with chaotic and regular zones in phase space. Our numerical simulations show that the chaotic spectra of these systems exhibit strong peaks in the autocorrelation which are at odd with Random Matrix Theory predictions. We explain these peaks through a semiclassical theory based on the peculiar distribution of the actions of classical periodic orbits. Indeed this distribution is strongly bunched around the average action between two consecutive rebounds and its multiples. In stars this phenomenon is a direct consequence of the strong decrease of the sound speed towards the star surface, but it would arise in any other physical system with a similar bunching of orbit actions. The peaks discussed could be observed by space missions and give insight on the star interiors.

Introduction
Most of the information that we can obtain from stars stems from the light that they emit. Variations of this light can be monitored for many stars, enabling to detect periodic patterns produced by oscillation modes of stars. In the case of the Sun, it has been possible to theoretically construct these modes and compare them with observations, giving crucial pieces of information on the internal structure [1]. Ultra-precise photometric data from the recent space missions COROT [2] and Kepler [3] include many rapidly rotating stars, for which a theory of oscillation modes is needed to infer physical properties of their internal structure [4].

Recently, ray dynamics and semiclassical techniques have been used to describe the oscillation modes in such rapidly rotating stars [5]. Indeed, acoustic waves have a short-wavelength limit in the same way as quantum or electromagnetic waves do, and the ray dynamics is also governed by Hamiltonian equations of motion [6]. Numerical simulations of this dynamics for a stellar model showed that when the rotation rate increases, stable and chaotic regions coexist in phase space. As a consequence, the stationary acoustic modes can be divided in modes localized in the regular zones or in the chaotic zones, with markedly different properties [5]. For regular island modes an asymptotic formula was built in [7], showing that they are characterized by regular spacings. Chaotic modes, which have been studied in the context of quantum mechanics by the field of quantum chaos [8], are expected to be distributed in accordance with the predictions of Random Matrix Theory (RMT) [9, 10], and should not display regularities.

In this Letter we show that chaotic modes in models of rapidly rotating stars display pseudo-regularities which can be clearly seen from peaks in the autocorrelation of the spectra. They could correspond to peaks extracted from the observed frequency spectra of δ Scuti stars [11], a class of rapidly rotating pulsating stars. Understanding their origin would then be key to derive physical constraints on star interiors. These peaks are also of interest from a theoretical standpoint, since they are unseen in the autocorrelation of RMT spectra [10]. Here, we explain the pseudo-regularities in the chaotic mode spectrum from a theory based on the general properties of acoustic rays in stars, coupled with the semiclassical theory of correlations in chaotic spectra [12].

Pseudo-regularities in the chaotic spectrum of stars
The acoustic ray dynamics is derived making standard approximations: we neglect the Coriolis force and the perturbations of the gravitational potential, and use an adiabatic approximation. In the linear limit, one obtains a Helmholtz-type equation. As the system is cylindrically symmetric, it can be reduced to a two-dimensional problem. The pressure perturbations Ψ(x, t) satisfy:

\[
\left( \frac{\partial^2}{\partial t^2} + \omega_c^2 \right) \Psi(x, t) - c_s^2 \nabla^2 \Psi(x, t) = 0
\]

where \(c_s\) is the sound speed, that decreases from the center to the surface as the square root of the temperature and \(\omega_c\) is a cutoff frequency which increases sharply close to the boundary, confining the wave inside the star.

We seek solutions of the form \(\Psi = e^{i\Lambda \phi(x,t)}\) where \(\Lambda^{-1}\) is a small dimensionless parameter and get an equation for the phase \(\phi(x,t)\) [6]:

\[
-\Lambda \phi(x,t) = (\omega_c^2 + c_s^2 k^2)^{1/2} = H
\]

where \(k = \nabla \phi\) is the wavector and \(H\) is the ray dynamics hamiltonian. In the following we express acoustic frequencies in terms of \(\omega_p = (GM/R_p^3)^{1/2}\), where \(R_p\) is the polar radius of the star. Rotation rates are given in units of \(\Omega_k = (GM/R_{eq}^3)^{1/2}\), where \(G\) is Newton’s constant, \(M\) the mass and \(R_{eq}\) the equatorial radius. \(\Omega_k\) corresponds to the critical rotation where the centrifugal acceleration equates gravity at the equator.
The study of the dynamics through the Poincaré Surface of Section (PSS) reveals three main types of structures for a wide range of rotation rates (see Fig. 1): stable islands are built around periodic orbits, whispering gallery rays remain close to the surface and chaotic ergodic trajectories fill all available space. Stationary modes are associated to these different phase space regions. As presented in [13], modes associated with the stable islands have simple spectra of the form \(\omega_{n\ell} = n\ell\alpha + \ell\delta \ell + \alpha\) and in the case of the most important 2-period island modes this formula was explained by an asymptotic theory [13].

By contrast, chaotic modes associated with the ergodic phase space region are not predicted in general to follow any simple asymptotic formula. According to the Bohigas-Giannoni-Snuit conjecture [8] chaotic mode spectra should have correlations given by RMT, that have the property of repelling each other at short distance. Conversely, generic regular modes should follow a Poisson distribution with no level repulsion [14]. Using a two-dimensional code that computes the stationary oscillations of a polytropic model of rotating star [15] we produced spectra at five different rotations. All modes are axisymmetric and either symmetric (even) or antisymmetric (odd) with respect to the equator. We then selected the chaotic modes by removing from the full spectra the island modes and whispering gallery modes. In Fig. 2 we display the distribution of the ratio of consecutive level spacings \(r_n = (\omega_{n+1} - \omega_n)/(\omega_n - \omega_{n-1})\) of chaotic modes and in Table I we indicate the average value of \(\bar{r}_n = \min(r_n, 1)\) [16, 17]. For most rotations the spectra follow closely the RMT statistics, with the exception of the \(\Omega/\Omega_k = 0.706\) spectrum, intermediate between Poisson and RMT. We will explain this peculiarity below by the presence, at this rotation, of partial barriers in the chaotic zone.

The ratio of consecutive level spacings is a short-range quantity; to investigate the correlations at longer range we computed the two-point autocorrelation function \(R_2(\xi) = \langle d(\omega - \frac{\xi}{\Delta}) d(\omega + \frac{\xi}{\Delta}) \rangle\), where \(d(\omega)\) is the spectral density and \(\langle \cdot \rangle\) is a frequency average. As shown in Fig. 3 a deviation from RMT statistics [10] appears in the form of a peak centered at value \(\Delta \approx 0.15 - 0.15\omega_p\) (panel a). This peak is robust in the sense that it appears at every rotation rate, though its position shifts slightly. Other peaks emerge from the noise, especially as rotation increases (fig. 3 panels c and d) around \(\frac{\Delta}{2}\) and \(\frac{3}{2}\Delta\). In the following, we will explain the origin of the most robust peaks and their evolution with respect to rotation, using semiclassical methods. We will also link the presence of the additional peaks to the existence of partial barriers.

**Semiclassical analysis** Semiclassics (here large \(\omega\) limit) is built on the Gutzwiller trace formula, which relates the spectral density to a sum over periodic orbits of the
shows the distribution of travel times for four rotations at odd chaotic modes. Datasets: a) 206 levels from 28.35 \( \omega_p \) to 46.89 \( \omega_p \), b) 223 levels from 28.15 \( \omega_p \) to 44.09 \( \omega_p \), c) 217 levels from 26.02 \( \omega_p \) to 40.29 \( \omega_p \) and d) 283 levels from 23.57 \( \omega_p \) to 36.22 \( \omega_p \). Dashed lines mark the position of the most robust peak, at much longer range than the mean level spacing (\( \approx 0.09 \omega_p \)).

The idea is to consider the form factor as:

\[
K(T) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dT \exp(i \xi T) C(\xi)
\]  

which is the Fourier transform of the autocorrelation:

\[
C(\xi) = \langle [\bar{d}(\omega - \frac{1}{2} \xi) - \bar{d}(\omega + \frac{1}{2} \xi)][d(\omega + \frac{1}{2} \xi) - d(\omega - \frac{1}{2} \xi)] \rangle = \langle \left( \text{Re} \sum_{i} A_i e^{iS_i(\omega - \frac{1}{2} \xi)} \right) \left( \text{Re} \sum_{j} A_j e^{iS_j(\omega + \frac{1}{2} \xi)} \right) \rangle.
\]

For short times, the off-diagonal terms \( i \neq j \) are washed-out by the frequency average, allowing to approximate the form factor as \( K(T) \approx \sum_j A_j^2 \delta(T - T_j) \), i.e. the density of periodic orbits weighted with intensities \( A_j^2 \).

The next step of Berry’s method is to use the general result of Hannay and Ozorio de Almeida [18] which implies that \( \sum_j A_j^2 \delta(T - T_j) \propto T \) for sufficiently large

Figure 3. Frequency autocorrelations computed at four rotations at odd chaotic modes. Datasets: a) 206 levels from 28.35 \( \omega_p \) to 46.89 \( \omega_p \), b) 223 levels from 28.15 \( \omega_p \) to 44.09 \( \omega_p \), c) 217 levels from 26.02 \( \omega_p \) to 40.29 \( \omega_p \) and d) 283 levels from 23.57 \( \omega_p \) to 36.22 \( \omega_p \). Dashed lines mark the position of the most robust peak, at much longer range than the mean level spacing (\( \approx 0.09 \omega_p \)).

Figure 4. Normalized number of n-chord trajectories, with n = 1, ..., 15, vs their travel time \( T \) at rotation \( \Omega/\Omega_k = 0.589 \). The inset is a close-up of the first packet, the dashed line marks its mean value, very close to \( T_0 \).

To make this result explicit we express the amplitude and density as functions of the metric and topological entropies \( h_1, h_2 : A(T) \approx Te^{-\frac{T}{2h_1}}, T \) and \( \sum_j \delta(T - T_j) \rightarrow \rho(T) \approx \frac{1}{\pi} e^{h_2 T} \) and suppose these two entropies to be equal [19, 20]. These formulas give: \( K(T) \rightarrow A^2(T) \rho(T) \propto T \), which is the RMT prediction (higher order terms can also be computed [21, 24]). This result is valid for \( T \gg T_{\text{min}} \), where \( T_{\text{min}} \) is the period of the shortest orbit, but \( T \) small enough for the diagonal approximation to hold [12]. In our system, however, the peculiar distribution of periodic orbits \( \rho(T) \) modifies the form factor behavior for short times.

Like classical trajectories in billiards, acoustic rays bounce on the reflective caustic close to the surface. Thus any periodic orbit can be divided into chords, each chord connecting two successive surface bouncing points. Periodic orbits thus belong to a class of trajectories that we call n-chords, whose endpoints have to lie on the surface. While it is extremely difficult to find all periodic orbits of our system, we can nevertheless infer some of their properties from the study of large samples of n-chord trajectories.

Fig. 4 shows the distribution of travel times for n-chords in the chaotic region of phase space at rotation \( \Omega/\Omega_k = 0.589 \). In stars, the travel time is not proportional to the geometric length, as e.g. in billiards. Indeed, since the sound velocity is much smaller near the surface of the star, all trajectories spend much more time near the surface than in the core. This results in a travel time distribution that concentrates around specific values evenly spaced out (see Fig 4). As the number of chords increases the individual packets grow wider (similarly to the law of large numbers), until the
width becomes much larger than the interpeak distance, and the distribution turns into a smooth curve. This is quantified by the ratio \( \sigma_0/T_0 \), \( \sigma_0 \) being the standard deviation of the first packet and \( T_0 \) the average distance between the mean values of consecutive packets. In the case of a billiard whose boundary is shaped like our star surface, \( \sigma_0/T_0 = 0.32 \) meaning that the packet structure disappears after a few rebounds. In the stellar case \( \sigma_0/T_0 \) grows with rotation but remains below 0.08. The packets are thus discernible for much longer times.

We model the distribution of Fig. 4 as \( P(T) = \sum_n P_n(T) \), with \( P_n(T) \) the probability density function of n-chords travel times:

\[
P_n(T) = \frac{1}{\sqrt{2\pi n \sigma_0}} \exp\left(-\frac{(T - n T_0)^2}{2(\sqrt{n \sigma_0})^2}\right)
\]

where the parameters \( \sigma_0 \) and \( T_0 \) are directly related to the sound speed profile in the star. As mentioned before we cannot find directly the periodic orbits, but we claim that the constraints imposed by \( P(T) \) are strong enough to explain the correlations seen in the spectra.

To take into account the exponential growth rate of the number of periodic orbits with \( T \), the density of periodic orbits in our system is thus modeled as \( \rho(T) \approx \frac{1}{T} e^{h_1 T} \times P(T) \). Assuming \( A(T) \approx T e^{\frac{1}{2} h_1 T} \) gives:

\[
K(T) \to A^2(T) \, \rho(T) \propto T \, P(T)
\]

For long times \( P(T) \) becomes flat and we recover the sum rule \( 18 \sum_j A_j^2 \delta(T - T_j) \propto T \), which is consistent with the existence of RMT statistics at short spectral distance (Fig. 2). For shorter times \( P(T) \) creates a specific regime departing from RMT.

The Fourier transform of \( TP(T) \) is shown in the left inset of Fig. 5 for different rotations, it shows a peak, as in the spectral data (Fig. 3), that moves towards low frequencies for increasing rotation. The peaks extracted from spectral data and obtained from semiclassical theory are in good agreement (Fig. 5 main panel). The discrepancy of about 5% is similar to what was obtained for island modes in [3], and can be attributed to the relatively low values of numerically computed frequencies. Specific conditions are needed for such a peak to occur. Indeed the peak disappears if \( \sigma_0/T_0 \) takes values typical of chaotic billiards (Fig. 5 right inset).

Other peaks The semiclassical theory outlined above explains the origin of the main peak in the autocorrelation. Other peaks are sometimes visible in the autocorrelations of Fig. 5 especially for the rotation \( \Omega/\Omega_k = 0.706 \) where \( P(r) \) deviates strongly from RMT (see Fig. 2). We attribute these peaks to the presence of partial barriers in the chaotic zone around the main stable island. Partial barriers isolate some zones of phase space, from which trajectories escape more slowly than in the rest of the chaotic zone [27, 28]. To find them we compute trajectories with initial condition near the main island and monitor their evolution on the PSS. Without barrier all points would spread ergodically, however Fig. 5...
shows that a subset of points remains near the island for a long time. Close to $\Omega/\Omega_k = 0.706$, the region enclosed by partial barriers grows in size and traps trajectories more efficiently. For the relatively low frequency waves here considered, they will act as barriers and quantize independently a subset of modes. This will weaken the level repulsion at short spectral distance as seen in Fig. 2. Besides, as trapped trajectories revolve around a 6-periodic orbit, we expect the modes to quantize like island modes $\Omega_k$, leading to a $\Delta/3$ regularity.

**Conclusion** In this Letter, we have shown that specificities in the distribution of the actions of periodic orbits can create a peak in the autocorrelation of chaotic spectra. Such a peak can potentially be detected in the frequency spectrum of chaotic pressure modes in rapidly rotating stars. Recent data have confirmed the existence of peaks in the autocorrelation spectrum of the rapidly rotating δ Scuti stars. They have been attributed so far to regular island modes but our results indicate that chaotic modes should also produce such peaks. These two kinds of peaks should be distinct yet close. In addition to astrophysical observations, the phenomenon described here could be tested with an experimental setup such as e.g. electromagnetic waves in a cavity with a strong gradient of the refractive index along the radial direction.

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jectory at the center of the island. The peak of chaotic modes corresponds to the inverse of a mean travel time between two rebounds for chaotic trajectories. In view of the 1-chord distribution shown in the inset of Fig. 4, these two quantities should be close. This is confirmed by frequency computations.