Gauge Independence in a Higher-Order Lagrangian Formalism via Change of Variables in the Path Integral

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Abstract

In this paper we work out the explicit form of the change of variables that reproduces an arbitrary change of gauge in a higher-order Lagrangian formalism.

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1 Introduction

It is a standard lore in the path integral formalism, that any result (such as, e.g., the Schwinger-Dyson equations, the Ward identities, etc.), that can be (formally) proven via change of integration variables, can equivalently be (formally) obtained via an integration by parts argument. And vice-versa. The latter method is typically the simplest. In 1996 it was shown in Ref. [1], by using integration by parts, how to formulate a higher-order field-antifield formalism that is independent of gauge choice. In this paper we work out the explicit form of the change of variables that reproduces a given change of gauge in an higher-order formalism. Perhaps not surprisingly, the construction relies on identifying appropriate homotopy operators.

2 The $\Delta$ Operator

From a modern perspective [2] the primary object in the Lagrangian field-antifield formalism [3, 4, 5] is the $\Delta$ operator, which is a nilpotent Grassmann-odd differential operator

$$\Delta^2 = 0, \quad \varepsilon(\Delta) = 1,$$

and which depends on antisymplectic variables $z^A$ and their corresponding partial derivatives $\partial_B$. Their commutator* reads

$$[\partial_B, z^A] = \delta^A_B.$$

3 $Sp(2)$-Symmetric Formulation

We mention for completeness that there also exists an $Sp(2)$-symmetric Lagrangian field-antifield formulation [6]. This formulation is endowed with two Grassmann-odd nilpotent, anticommuting $\Delta^a$ operators

$$\Delta^{\{a} \Delta^{b\}} = 0, \quad \varepsilon(\Delta^a) = 1, \quad a, b \in \{1, 2\}.$$

Often (but not always!) the resulting $Sp(2)$-symmetric formulas look like the standard formulas with $Sp(2)$-indices added and symmetrized in a straightforward manner. In this paper, we will focus on the standard formulation and usually only mention the corresponding $Sp(2)$-symmetric formulation when it deviates in a non-trivial manner.

4 Planck Number Grading

Planck’s constant $\hbar$ is here treated as a formal parameter (as opposed to an actual number) in the spirit of deformation quantization (as opposed to geometric quantization). The Planck number grading $Pl$ is defined via the rules

$$Pl(\hbar) = 1, \quad Pl(z^A) = 0, \quad Pl(\partial_A) = -1,$$

and extended to normal-ordered† differential operators in the natural way. More precisely, a derivative $\partial_A$ inside an operator $F$ gets assigned Planck number $-1$ (0) for the parts that act outside (inside)

*The word super is often implicitly implied. For instance, the word commutator means the supercommutator $[F, G] \equiv FG - (-1)^{FG}GF$.
†Normal-ordering means that all the $z$’s appear to the left of all the $\partial$’s. Antinormal-ordering means the opposite.
the operator, respectively. We mention for later convenience the superadditivity of Planck number grading
\[ \text{Pl}(FG) \geq \text{Pl}(F) + \text{Pl}(G) \],
\[ \text{Pl}([F,G]) \geq \text{Pl}(F) + \text{Pl}(G) + 1 \], \tag{4.2} \]
where the uppercase letters \( F \) and \( G \) denote operators.

5 Higher-Order \( \Delta \) Operator

In the standard field-antifield formalism [3, 4, 5], the \( \Delta \) operator is a second-order operator. (See also Section 18.) In the higher-order generalization [1], which is the main topic of this paper, the \( \Delta \) operator is assumed to have Planck number grading \[ \text{Pl}(\Delta) \geq -2 \]. \tag{5.1} \]

Evidently, the Planck number inequality (5.1) means that the normal-ordered \( \Delta \) operator is of the following triangular form\(^\dagger\)
\[ \Delta = \sum_{n=-2}^{\infty} \sum_{m=0}^{n+2} \left( \frac{\lambda}{\pi} \right)^n \Delta_{n,m} , \quad \Delta_{n,m} = \Delta_{n,m}^A \ldots \Delta_{m}^A (z) \rightarrow \partial_{A_m} \ldots \partial_{A_1} . \tag{5.2} \]
The higher-order terms in the \( \Delta \) operator can e.g., be physically motivated as quantum corrections, which arise in the correspondence between the path integral and the operator formalism.

6 Path Integral

The (formal) path integral
\[ Z_X = \int d\mu \ w , \quad w \equiv e^{iW} , \quad x \equiv e^{iX} , \tag{6.1} \]
in the \( W\)-\( X\)-formalism [8, 9, 10, 11, 12, 13, 14] consists of three parts:

1. A path integral measure \( d\mu = \rho [dz] [d\lambda] \), where \( \lambda^a \) are Lagrange multipliers implementing the gauge fixing conditions, and \( z^A = \{ \phi^a; \phi^{*a} \} \) are the antisymplectic variables, i.e., fields \( \phi^a \) and antifields \( \phi^{*a} \). Here \( \rho = \rho(z) \) is a density with \( \varepsilon(\rho) = 0 \) and \( \text{Pl}(\ln \rho) \geq -1 \).

2. A gauge-generating quantum master action \( W \), which satisfies the quantum master equation (QME)\(^\S\)
\[ (\Delta w) = 0 , \quad w \equiv e^{iW} , \quad \text{Pl}(W) \geq 0 . \tag{6.2} \]
The path integral (6.1) will in general depend on \( W \), since \( W \) contains all the physical information about the theory, such as, e.g., the original action, the gauge generators, etc. [15, 16]. The

\(^\dagger\)In contrast to the original proposal [1], we also allow the three terms \( \Delta_{-2,0} \), \( \Delta_{-1,0} \) and \( \Delta_{-1,1} \) with negative \( n \) in eq. (5.2). The two last terms arise naturally in the \( Sp(2) \)-symmetric formulation [6, 12]. The two first terms affect the classical master eq. See also Sections 18-19 for the second-order case.

\(^\S\)The parenthesis in eq. (6.2) is here meant to emphasize that the QME is an identity of functions (as opposed to differential operators), i.e., the derivatives in \( \Delta \) do not act outside the parenthesis. Note however that similar parenthesis will not always be written explicitly in order not to clog formulas. In other words, it must in general be inferred from the context whether an equality means an identity of functions or an identity of differential operators.
triangular form (5.2) of the $\Delta$ operator implies that the QME (6.2) is perturbative in Planck’s constant $\hbar$, i.e.,
\[
\mathcal{P} \left( w^{-1} \Delta(h, z, \partial) w \right) = \mathcal{P} \left( \Delta \left( h, z, \partial + \frac{i}{\hbar} (\partial W) \right) \right) \geq -2 .
\]

Besides the triangular form (5.1), which is imposed to ensure perturbativity, there are additional “boundary” and rank conditions to guarantee the pertinent classical master equation and proper classical master action $S$ [15, 16].

3. A gauge-fixing quantum master action $X$, which satisfies the transposed quantum master equation
\[
(\Delta^T x) = 0 , \quad x \equiv e^{\frac{i}{\hbar} X} , \quad \mathcal{P}(X) \geq 0 .
\]
The path integral (6.1) will in general not depend on $X$, cf. Section 13 and Section 16.

The transposed operator $F^T$ has the property that
\[
\int d\mu \ (F^T f) g = (-1)^{\varepsilon_f \varepsilon_g} \int d\mu \ f (Fg) .
\]
Here the lowercase letters $f, g, \ldots$ denote functions, while the upper case letters $F, G, \ldots$ denote operators. One can construct any transposed operator by successively apply the following rules
\[
(F + G)^T = F^T + G^T , \quad (F G)^T = (-1)^{\varepsilon_f \varepsilon_g} G^T F^T , \quad (z^A)^T = z^A , \quad \partial_A^T = -\rho^{-1} \partial_A \rho .
\]
In particular the transposed operator $\Delta^T$ is also nilpotent
\[
(\Delta^T)^2 = 0 .
\]
The transposed derivative $\partial_A^T$ satisfies a modified Leibniz rule:
\[
\partial_A^T (f g) = (\partial_A^T f) g - (-1)^{\varepsilon_A \varepsilon_f} f (\partial_A g) .
\]

Let us mention for completeness that the $\Delta$ operator (which takes functions to functions) and the $W$-$X$-formalism can be recast in terms of Khudaverdian’s operator $\Delta_E$ (which takes semidensities to semidensities) [17, 18, 19, 20, 21, 22, 23, 24, 25].

7 Higher-Order Quantum BRST Operators

The quantum BRST operators $\sigma_W$ and $\sigma_X$ take operators into functions (i.e., left multiplication operators). They are defined as
\[
\sigma_W F := \frac{\hbar}{i} w^{-1} ([\Delta, F] w) \overset{(6.2)}{=} \frac{\hbar}{i} w^{-1} (\Delta F w) ,
\]
\[
\sigma_X F := \frac{\hbar}{i} x^{-1} ([\Delta^T, F] x) \overset{(6.4)}{=} \frac{\hbar}{i} x^{-1} (\Delta^T F x) ,
\]
respectively, where $F$ is an operator. They are nilpotent, Grassmann-odd,
\[
\sigma_W^2 = 0 = \sigma_X^2 , \quad \varepsilon(\sigma_W) = 1 = \varepsilon(\sigma_X) ,
\]
and perturbative in the sense that
\[
\mathcal{P}(\sigma_W F) \geq \mathcal{P}(F) \leq \mathcal{P}(\sigma_X F) .
\]
In the $Sp(2)$-symmetric formulation the quantum BRST operators $\sigma_W^a$ and $\sigma_X^a$ carry an $Sp(2)$-index since the $\Delta^a$ operator does.

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*The word classical means here independent of Planck’s constant $\hbar$.}
8 Higher quantum antibrackets

The 1-quantum antibracket is

\[ \hat{\Phi}_\Delta^1 \equiv D \equiv [\Delta, \cdot], \quad D^2 \equiv 0, \quad \varepsilon(D) = 1, \quad \Pi(DF) \geq \Pi(F) - 1. \tag{8.1} \]

The \( n \)-quantum antibracket consists of nested commutators of \( n \) operators with the \( \Delta \)-operator \([26, 27, 28, 29, 30, 31, 32]\). We will not need the full definition here, but it can in principle be deduced uniquely via polarization of the following recursive formula \([32]\)

\[ \hat{\Phi}_\Delta^n (B, \ldots, B) \equiv [\ldots [[\Delta, B], B], \ldots, B] = [\hat{\Phi}_\Delta^{n-1} (B, \ldots, B), B], \quad \varepsilon(B) = 0, \quad \hat{\Phi}_\Delta^0 = \Delta. \tag{8.2} \]

Philosophically speaking, the \( n \)-quantum antibrackets (8.2) are secondary/derived objects, which can be obtained from the underlying concept of a fundamental \( \Delta \)-operator. The pertinent Lie bracket is the 2-quantum antibracket \( \text{derived bracket} \ [26, 27, 28] \)

\[ [F, G] := \frac{1}{2} [DF, DG] - (-1)^{(\varepsilon_F + 1)(\varepsilon_G + 1)} (F \leftrightarrow G) = -(-1)^{\varepsilon_F} \hat{\Phi}_\Delta^2 (F, G), \tag{8.3} \]

where

\[ \hat{\Phi}_\Delta^2 (F, G) := \frac{1}{2} [DF, G] + (-1)^{\varepsilon_F \varepsilon_G} (F \leftrightarrow G). \tag{8.4} \]

The 2-quantum antibracket is Grassmann-odd

\[ \varepsilon ([F, G]) = \varepsilon(F) + \varepsilon(G) + 1, \tag{8.5} \]

and perturbative

\[ \Pi([F, G]) \geq \Pi(F) + \Pi(G). \tag{8.6} \]

The 1-quantum antibracket \( D \) generates the 2-quantum antibracket \([28]\)

\[ [DF, DG] = D[F, G] = [DF, G] - (-1)^{\varepsilon_F} [F, DG]. \tag{8.7} \]

The 3-quantum antibracket is defined as

\[ \hat{\Phi}_\Delta^3 (\Psi_1, \Psi_2, \Psi_3) := \frac{1}{6} \sum_{\text{cycl. } 1,2,3} [[D\Psi_1, \Psi_2], \Psi_3] - (1 \leftrightarrow 2), \quad \varepsilon(\Psi_i) = 1. \tag{8.8} \]

The Jacobi identity for the 2-quantum antibracket is satisfied up to \( D \)-exact terms

\[ \sum_{\text{cycl. } 1,2,3} [[[\Psi_1, \Psi_2], \Psi_3] = \frac{1}{2} D\hat{\Phi}_\Delta^3 (\Psi_1, \Psi_2, \Psi_3), \quad \varepsilon(\Psi_i) = 1, \tag{8.9} \]

or equivalently, in the polarized language \([32]\)

\[ 6\hat{\Phi}_\Delta^3 (\hat{\Phi}_\Delta^2 (B, B), B) = D\hat{\Phi}_\Delta^3 (B, B, B), \quad \varepsilon(B) = 0. \tag{8.10} \]

**Proof of eq. (8.10):**

\[
6\hat{\Phi}_\Delta^3 (\hat{\Phi}_\Delta^2 (B, B), B) - D\hat{\Phi}_\Delta^3 (B, B, B) = 3[DB, [DB, B]] + 3[D[DB, B], B] - D[[DB, B], B] \\
= 4[[DB, B], DB] + 2[D[DB, B], B] = 4[DB, [DB, B]] - 2[[DB, DB], B] = 0. \tag{8.11}
\]
9 Grassmann-even $Sp(2)$ quantum brackets

In the $Sp(2)$-symmetric case, besides the $Sp(2)$-symmetric higher quantum antibrackets (which we will not discuss here), there is a tower of Grassmann-even quantum brackets. The pertinent 1-quantum bracket is

$$ D \equiv \frac{1}{2} \hbar \epsilon_{ab} [\Delta^a, [\Delta^b, \cdot ]], \quad D^2 \equiv 0, \quad \varepsilon(D) = 0, \quad \text{Pl}(D F) \geq \text{Pl}(F) - 1. \quad (9.1) $$

The 2-quantum bracket is defined as

$$ [F,G] := \frac{1}{2} [DF,G] + \frac{1}{2} [F,DG] = -(-1)^{\varepsilon_F \varepsilon_G} [G,F]. \quad (9.2) $$

(Hopefully it does not lead to confusion that we use the same notation for the Grassmann-even quantum brackets $D$ and $[,]$ in this Section 9 as we use for the Grassmann-odd quantum antibrackets $D$ and $[,]$ in the previous Section 8.) Up to $D$-exact terms, the 2-quantum bracket is

$$ [F,G] - \frac{1}{2} D[F,G] = \frac{1}{4} \hbar \epsilon_{ab} [F, [\Delta^a, [\Delta^b, G]]] - (-1)^{\varepsilon_F \varepsilon_G} (F \leftrightarrow G), \quad (9.3) $$

The 2-quantum bracket is Grassmann-even

$$ \varepsilon([F,G]) = \varepsilon(F) + \varepsilon(G), \quad (9.4) $$

and perturbative

$$ \text{Pl}([F,G]) \geq \text{Pl}(F) + \text{Pl}(G). \quad (9.5) $$

The 1-quantum bracket $D$ generates the 2-quantum bracket

$$ [DF,DG] = D[F,G] = [DF,G] + [F,DG]. \quad (9.6) $$

We note for later the identity

$$ [F,DG] - [DF,G] = D[F,G] - \frac{\hbar}{i} \epsilon_{ab} [\Delta^a, [[\Delta^b, F],G]]. \quad (9.7) $$

The Jacobi identity for the 2-quantum antibracket is satisfied up to $D$-closed terms

$$ \sum_{\text{cycl. 1,2,3}} [[B_1,B_2],B_3] \sim 0, \quad \varepsilon(B_i) = 0. \quad (9.8) $$

In detail, in the polarized language [32]

$$ 6[[\Psi,\Psi],\Psi] = D[[\Psi,\Psi],\Psi] + \frac{\hbar}{i} \epsilon_{ab} [\Delta^a, [[\Delta^b, \Psi],\Psi]], \quad \varepsilon(\Psi) = 1. \quad (9.9) $$

**Proof of eq. (9.9):**

$$ 6[[\Psi,\Psi],\Psi] - D[[\Psi,\Psi],\Psi] = \frac{\hbar}{i} \epsilon_{ab} [\Delta^a, [[\Delta^b, \Psi],\Psi]] $$

$$ = 3[[\Psi,\Psi],D\Psi] + 3[D[[\Psi,\Psi],\Psi] - D[[\Psi,\Psi],\Psi] = -4[D\Psi,[[D\Psi,\Psi]] + 2[[D\Psi,D\Psi],\Psi] = 0. \quad (9.10) $$
10 Space of Solutions to QME

We can generate a new solution to the QME (6.2) via a finite transformation

\[ w \rightarrow w' = (e^{D\Psi} w) , \quad \varepsilon(\Psi) = 1 , \quad \text{P}(\Psi) \geq 0 , \quad (10.1) \]

where \( D \) is the Grassmann-odd 1-quantum antibracket (8.1). The composition of two finite transformations is again a finite transformation

\[ e^{D\Psi_1}e^{D\Psi_2} = e^{DBCH(D\Psi_1,D\Psi_2)} = e^{DBCH(\Psi_1,\Psi_2)} , \quad \varepsilon(\Psi_i) = 1 . \quad (10.2) \]

The second and third expression in eq. (10.2) contain the Baker–Campbell–Hausdorff (BCH) series expansion (with the Lie bracket replaced with the commutator \([\cdot,\cdot]\) and the 2-quantum antibrackets \([\[\cdot,\cdot]\],\cdot\], respectively). In detail, the latter reads

\[ \text{BCH}(\Psi_1,\Psi_2) = \Psi_1 + \int_0^1 dt \sum_{n=0}^{\infty} \left( \frac{-1}{n+1} \right)^n \left( e^{-t[\Psi_2,\cdot]} e^{-[\Psi_1,\cdot]I} - 1 \right)^n \Psi_2 \]

\[ = \Psi_1 + \Psi_2 + \frac{1}{2} \left[ \Psi_1, \Psi_2 \right] + \frac{1}{12} \left[ \left[ \Psi_1, \left[ \Psi_1, \Psi_2 \right] \right] + \frac{1}{12} \left[ \left[ \Psi_1, \Psi_2 \right], \Psi_2 \right] \right] + O(\Psi_i^4) \quad (10.3) \]

Here we have used the Jacobi identity (8.9).

11 \( Sp(2) \) case

There is \( Sp(2) \)-symmetric analogue of Section 10. We can generate a new solution via the finite transformation

\[ w \rightarrow w' = (e^{DB} w) , \quad \varepsilon(B) = 0 , \quad \text{P}(B) \geq 0 . \quad (11.1) \]

where \( D \) is the Grassmann-even 1-quantum bracket (9.1). The composition of two finite transformations is again a finite transformation

\[ e^{DB_1}e^{DB_2} = e^{DBCH(B_1,B_2)} , \quad \varepsilon(B_i) = 0 , \quad (11.2) \]

with a BCH formula in eq. (11.2) for the bosons \( B_i \) similar to the formula (10.3) for the fermions \( \Psi_i \).

12 Maximal Deformation

One may formally argue [7] that any two solutions to the QME (6.2) are connected via a finite transformation (10.1), i.e., the group of finite transformations (10.1) acts transitively on the space of solutions to the QME (6.2).

The infinitesimal generator of an infinitesimal transformation (10.1)

\[ \delta w = ([\Delta,\Psi]w) \quad (6.2) = (\Delta \Psi w) , \quad (12.1) \]

is evidently the 1-antibracket \( D\Psi \equiv [\Delta,\Psi] \) for an infinitesimal operator \( \Psi \) with \( \varepsilon(\Psi) = 1 \) and \( \text{P}(\Psi) \geq 0 \). Phrased equivalently, eq. (12.1) means that the change in the master action is given by the quantum BRST transformation

\[ \delta W = \frac{\hbar}{i} \delta \ln w = \sigma_W \Psi . \quad (12.2) \]
The same story holds for $X$ instead of $W$ if we replace the operator $\Delta$ with the transposed operator $\Delta^T$, e.g.,

$$
\delta x = ([\Delta^T, \Psi] x) = (\Delta^T \Psi x), \quad \delta X = \frac{\hbar}{2} \delta \ln w = \sigma_X \Psi .
$$

(12.3)

When discussing $X$ (as opposed to $W$) we will implicitly assume that the pertinent quantum (anti)brackets are generated by the transposed operator $\Delta^T$.

Moreover, to obtain the $Sp(2)$-symmetric formulation, formally replace the operator $\Delta \rightarrow \Delta^a$ and $\Psi \rightarrow \Psi^a \equiv \frac{1}{2} \hbar i \delta \ln w = \sigma_X \Psi$. Note that $\mathcal{P}(\Psi^a) \geq 0$ holds.

13 Gauge-Independence via Integration by Parts

The gauge-independence of the path integral can be formally proved via integration by parts [1]

$$
\delta Z \equiv Z_{X+\delta X} - Z_X = \int d\mu w \delta x \stackrel{(12.3)}{=} \int d\mu w (\Delta^T \Psi x) \stackrel{\text{int. by parts}}{=} \int d\mu (\Delta w) (\Psi x) \stackrel{(6.2)}{=} 0 .
$$

(13.1)

The main purpose of this paper is to re-prove gauge-independence via change of variables in the path integral, cf. Section 16. To this end, we introduce two types of homotopy operators, cf. Sections 14–Sections 15.

14 Homotopy Operator $h^A$ (\Delta)

The pertinent homotopy operator $h^A (\Delta)$ is best explained for operators $\Delta$ on antinormal-ordered form

$$
\Delta(\partial, z) = \sum_{m=0}^{\infty} \Delta_m(\partial, z) , \quad \Delta_m(\partial, z) = \partial_{A_m} \cdots \partial_{A_1} \Delta^{A_1 \cdots A_m}_m(z) .
$$

(14.1)

We stress that the derivatives $\partial_{A_m} \cdots \partial_{A_1}$ in eq. (14.1) also act beyond (i.e., to the right of) $\Delta^{A_1 \cdots A_m}_m(z)$. Then the homotopy operator is defined on a homogeneous component $\Delta_m(\partial, z)$ as

$$
h^A_m (\Delta_m) := \begin{cases} 
-\frac{1}{m} [z^A, \Delta_m] = (-1)^{\varepsilon_A} \partial_{A_{m-1}} \cdots \partial_{A_1} \Delta^{A_1 \cdots A_{m-1}A}_m(z) & \text{for } m \geq 1 , \\
0 & \text{for } m = 0 .
\end{cases}
$$

(14.2)

The definition (14.2) is extended to an arbitrary operator $\Delta$ by linearity. The homotopy operator (14.2) satisfies the following homotopy property

$$
(-1)^{\varepsilon_A} \partial_{A} h^A (\Delta(\partial, z)) = \Delta(\partial, z) - \Delta(0, z)
$$

(14.3)

for antinormal-ordered operators (14.1). Two homotopy operators (14.2) commute:

$$
h^A h^B \Delta = (-1)^{\varepsilon_A \varepsilon_B} h^B h^A \Delta .
$$

(14.4)
15 Bilinear Homotopy Operator $B^A(f, \Delta)$

Given a function $f$ and an operator $\Delta$, the bilinear homotopy operator $B^A(f, \Delta)$ is defined via

$$(-1)^{c_A} B^A(f, \Delta) \equiv f : \frac{1}{1-\partial_B h} \rightarrow^A h(\Delta)1 \equiv f : \sum_{n=0}^{\infty} \left( \partial_B h \right)^n : \rightarrow^A h(\Delta)1$$

$$\equiv f \rightarrow^A h(\Delta)1 + \left( f \partial_B h \right) \rightarrow^B \rightarrow^A \rightarrow^C \rightarrow^A \rightarrow^A \rightarrow^A \rightarrow^A + \ldots (15.1)$$

where the ordering symbol “$:\rightarrow$” here means that all derivatives $\partial_B$ should be to the left of all the homotopy operators $\rightarrow h$. One may prove that the bilinear homotopy operator $B^A(f, \Delta)$ has the following important homotopy property

$$(-1)^{c_A} \left( \partial_A B^A(f, \Delta) \right) = (-1)^{c_A} (\Delta^T f) - f(\Delta1).$$

16 Gauge-Independence via Change of Variables

The infinitesimal change $\delta z^A$ of (passive) coordinates $z^A$ can be viewed as an infinitesimal vector field\footnote{We are here and below guilty of infusing some active picture language into a passive picture, i.e., properly speaking, the active vector field has the opposite sign.}

$$\delta z^A = \frac{1}{wx} B^A(\Psi x, \rightarrow^w) , \quad \mathcal{P}(\Psi) \geq 0.$$  

(16.1)

One may show that the Planck number $\mathcal{P}(\delta z^A) \geq -1$ of the vector field is greater than or equal to $-1$, as it should be. The Boltzmann density (= integrand) of the path integral (6.1) is $\rhowx$. The divergence of the vector field (16.1) with respect to the Boltzmann density is

$$\text{div}_{\rhowx} \delta z \equiv \frac{(-1)^{c_A}}{\rhowx} \left( \partial_A \rho x w \delta z^A \right) \overset{(16.1)}{=} \frac{(-1)^{c_A}}{wx} \left( \partial_A B^A(\Psi x, \rightarrow^w) \right)$$

$$\overset{(15.2)}{=} \frac{1}{wx} \left\{ (\Psi x)(\Delta w) + w(\Delta^T \Psi x) \right\}$$

$$\overset{(6.2)+(6.4)}{=} \frac{1}{x} \left[ \left( \Delta^T \Psi x \right) x \right] \overset{(7.2)}{=} \frac{i}{h} \sigma X \Psi \overset{(12.3)}{=} \frac{\delta x}{x}. (16.2)$$

On one hand, an infinitesimal change of integration variables in path integral cannot change the value of path integral. On the other hand, it induces an infinitesimal Jacobian factor. Hence

$$0 = \int [d\lambda][dz] (-1)^{c_A} \left( \partial_A \rho x w \delta z^A \right) = \int d\mu \ w x \text{div}_{\rho x w} \delta z \overset{(16.2)}{=} \int d\mu \ w \delta x \overset{(6.1)}{=} Z_{X+\delta X} - Z_{X} \equiv \delta Z,$$  

which, in turn, can mimic an arbitrary infinitesimal change of gauge-fixing. Thus we have formally proven via change of variables that the path integral $Z_X$ does not depend on gauge-fixing $X$. Eq. (16.3) is the main result.

17 Higher antibrackets

The $n$-antibracket [33, 1, 32] is the restriction of the quantum $n$-antibracket (8.2) from operators to functions

$$\Phi^n_\Delta(f_1, \ldots, f_n) := \hat{\Phi}^n_\Delta(f_1, \ldots, f_n)1. (17.1)$$
In particular, the 2-antibracket \((f,g)\) of two function \(f\) and \(g\) is defined as
\[
(f,g) := (−1)^{ε_f} \left[ \hat{Δ}, f \right], g \right] 1 = −[f,g]1 = −(−1)^{(ε_f+1)(ε_g+1)}(g,f)
\]  

(17.2)

18 Second-Order \(Δ\) operator

It is natural to ponder how to build a nilpotent \(Δ\)-operator, that takes scalar functions in scalar functions, from the following given geometric data:

1. An anti-Poisson structure
\[
(f, g) = (f \xrightarrow{A} E^{AB} \xrightarrow{B} g) = −(−1)^{(ε_f+1)(ε_g+1)}(g,f)\, \quad ε(E^{AB}) = ε_A + ε_B + 1, \quad \text{Pl}(E^{AB}) ≥ 0,
\]

which satisfies the Jacobi identity
\[
\sum_{f,g,h \text{ cycl.}} (−1)^{(ε_f+1)(ε_h+1)}(f, (g,h)) = 0.
\]

(18.1)

2. A density \(ρ\) with \(ε(ρ) = 0\) and \(\text{Pl}(\ln ρ) ≥ −1\).

3. A Grassmann-odd vector field \(V = V^A \xrightarrow{A} \partial\), with \(ε(V) = 1\) and \(\text{Pl}(V) ≥ −2\), that is compatible with the anti-Poisson structure:
\[
(V(f,g)) = (Vf,g) − (−1)^{ε_f}(f,Vg).
\]

(18.3)

Often we assume that the antibracket (18.1) is non-degenerate/invertible. Then the vector field is locally a Hamiltonian vector field \(V = (H, \cdot)\). This Hamiltonian \(H\) can be absorbed into the density by redefining the density \(\tilde{ρ} = ρ e^{2H}\).

To guarantee nilpotency \(Δ^2 = 0\), the minimal solution (to the above posed problem in Section 18) is the following second-order \(Δ\) operator
\[
Δ = Δ_ρ + V + ν, \quad ε(Δ) = 1, \quad \text{Pl}(Δ) ≥ −2,
\]

(18.4)

where \(Δ_ρ\) is the odd Laplacian
\[
Δ_ρ = \frac{(−1)^{ε_A}}{2ρ} \partial_A ρ E^{AB} \partial_B = −\frac{(−1)^{ε_A}}{2} \partial_A E^{AB} \partial_B,
\]

(18.5)

where \(ν\) is an odd scalar function
\[
ν = ν_ρ + \frac{1}{2} \text{div}_ρ V − \frac{1}{2} V^A E^{AB} V^B, \quad ε(ν) = 1, \quad \text{Pl}(ν) ≥ −2,
\]

(18.6)

and where the odd scalar \(ν_ρ\) is constructed from \(ρ\) and \(E^{AB}\), cf. Refs. [21, 22, 23, 24, 25]. The transposed vector field is
\[
V^T = −V − \text{div}_ρ V.
\]

(18.7)

The transposed operator \(Δ^T\) corresponds to letting the vector field \(V \to −V\) change sign:
\[
Δ^T = Δ|_{V \to −V}.
\]

(18.8)

To obtain the \(Sp(2)\)-symmetric formulation, formally replace \(Δ_ρ \to Δ^a_ρ; (\cdot, \cdot) \to (\cdot, \cdot)^a; V \to V^a; ν \to ν^a; \) etc. Note that some equations, such as, e.g., (18.2) and (18.3) should be symmetrized in the \(Sp(2)\) indices. We will not here discuss an \(Sp(2)\)-analogue of eq. (18.6).
Now let us check how the higher-order formalism of the previous Sections 2-16 applies to the second-order $\Delta$ operator (18.4). The QME (6.2) becomes
\[
\frac{1}{2}(W, W) + \frac{i}{\hbar} \left( (\Delta_\rho + V) W \right) + (\frac{1}{4})^2 \nu = 0 , \tag{19.1}
\]
and the BRST operator (7.1) becomes
\[
\sigma_W f = \frac{i}{\hbar} \left( (\Delta_\rho + V) f \right) + (W, f) . \tag{19.2}
\]
The homotopy operator (14.2) becomes
\[
\rightarrow h \rightarrow ^{A} (\Delta w) = \frac{1}{2} E^{AB} \partial_B w + (-1)\varepsilon_A V^A w + (z^A, \ln \sqrt{\rho}) w , \tag{19.3}
\]
\[
\rightarrow h \rightarrow ^{B} (\Delta w) = \frac{(-1)^\varepsilon_A}{2} E^{AB} w . \tag{19.4}
\]
The infinitesimal change of variables becomes
\[
2 x \delta z^A \overset{(16.1)}{=} \frac{2}{w} B^A (\Psi, \Delta w) = (\Psi x) (\ln w, z^A) + 2 V^A = (\Psi, z^A) , \tag{19.5}
\]
where $\Psi$ is an infinitesimal operator. For an infinitesimal function $\psi$, eq. (19.5) reduces further to
\[
2 \delta z^A \overset{(19.5)}{=} \frac{i}{\hbar} \psi (W - X, z^A) + 2 \psi V^A - (\psi, z^A) \overset{\psi}{=} \frac{i}{\hbar} \psi (\sigma_W z^A - \sigma_X z^A) - (\psi, z^A) . \tag{19.6}
\]
Finally, consider a finite change of solution to the QME (6.2)
\[
w' \equiv e^{\frac{i}{\hbar} W'} = (e^{-D\psi} w) , \quad D\psi \overset{(8.1)}{=} [\Delta, \psi] = (\Delta\psi) - \text{ad}\psi , \quad \text{ad}\psi \equiv (\psi, \cdot) , \tag{19.7}
\]
where $\psi$ is a finite function, with $\varepsilon(\psi) = 1$ and $\text{Pl}(\psi) \geq 0$. An application of the BCH formula shows that the corresponding change in the action reads [14, 35, 36]
\[
W' = e^{\text{ad}\psi} W + i\hbar E(\text{ad}\psi)(\Delta\psi) = e^{\text{ad}\psi} W + i\hbar \frac{e^{\text{ad}\psi} - 1}{\text{ad}\psi} (\Delta\psi) , \quad w' = (e^{\text{ad}\psi} w) e^{-E(\text{ad}\psi)(\Delta\psi)} , \tag{19.8}
\]
where
\[
E(x) := \int_0^1 dt e^{xt} = \frac{e^x - 1}{x} . \tag{19.9}
\]
**Proof of eq. (19.8):** For a vector field $\xi$ and a function $f$, the BCH formula simplifies to
\[
e^\xi e^f = e^{\xi + B(\cdot;\xi)} f , \tag{19.10}
\]
where
\[
B(x) := \frac{x}{e^x - 1} = \frac{1}{E(x)} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + O(x^6) \tag{19.11}
\]
is the generating function for Bernoulli numbers. Therefore eq. (19.10) can be inverted into
\[
e^{\xi + f} = e^{\xi E(-[\cdot;\xi]) f} , \tag{19.12}
\]
which, in turn, leads to eq. (19.8) with $\xi = \text{ad}\psi$ and $f = -(\Delta\psi)$.

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