PROPAGATION DYNAMICS OF NONLOCAL DISPERAL EQUATIONS WITH INHOMOGENEOUS BISTABLE NONLINEARITY

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Abstract. This paper is concerned with the nonlocal dispersal equations with inhomogeneous bistable nonlinearity in one dimension. 

1. Introduction. In the present paper, we focus on the nonlocal dispersal equation with spatially dependent bistable nonlinearity

\[ u_t(x,t) = J * u(x,t) - u(x,t) + f(x,u), \quad x, t \in \mathbb{R}, \]  

where the nonlinearity \( f(x,u) \) and kernel function \( J(x) \) satisfy the following assumptions.

(F) The nonlinearity \( f \) satisfies that

\[
\begin{cases}
    f(x,u) = f_1(u) & \text{for } x \geq 0, \quad u \in [0,1], \\
    f(x,u) = f_2(u) & \text{for } x \leq -L, \quad u \in [0,1], \\
    f_2(u) \leq f(x,u) \leq f_1(u), \quad f_u(x,u) < 1 & \text{for } x \in [-L,0], \quad u \in [0,1],
\end{cases}
\]

where \( L > 0, f_1 \) and \( f_2 \) are two given bistable nonlinearities satisfying that

\[
f_i \in C^{1,1}([0,1]), \quad f_i(0) = f_i(1) = 0, f'_i(0), f''_i(1) < 0, \]

\[
f_i < 0 \text{ in } (0,\theta_i), f_i > 0 \text{ in } (\theta_i,1), 0 < \theta_1 < \theta_2 < 1, \text{ and } \int_0^1 f_i(s)ds > 0, \quad i = 1,2.
\]

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The kernel function $J \in C^1(\mathbb{R})$ satisfies that
\[
\begin{cases}
J(x) = J(-x), & J(x) \geq 0, \quad \int_{\mathbb{R}} J(y) dy = 1, \\
\int_{\mathbb{R}} |J'(y)| dy < \infty, & \int_{\mathbb{R}} J(y) e^{-\lambda y} dy < \infty \quad \text{for all } \lambda > 0.
\end{cases}
\]

A typical example is $f(x, u) = u(1 - u)(a(x) - u)$.

The nonlocal dispersal equation has attracted so much attention because of its extensive applications to account for diffusion phenomena involving jumps in biology, physics and chemistry [1, 12]. Traveling waves as one kind of special solutions with invariant profile and fixed speed as well as entire solutions have been adequately investigated. It is well-known that the importance of the study of entire solutions of reaction-diffusion equations (nonlocal dispersal equations) is frequently recalled in the literature. Since the pioneering works of Hamel and Nadirashvili [13, 14], there have been tremendous advances in studying the existence of entire solutions for various models. In particular, when the nonlinearity is homogeneous (i.e. $L = 0$ and $f_1 = f_2$), the theories of traveling waves and entire solutions for nonlocal dispersal equation 1 with various types of nonlinearities have been well established, the related results refer to [2, 6, 7, 8, 9, 15, 18, 23, 24, 26, 27] and references therein. Specifically, when the kernel $J(x)$ is compactly supported, Sun et al. [23] constructed a two-dimensional manifold of entire solutions which behave as two traveling wave solutions coming from both directions for bistable nonlocal dispersal equation.

However, for the inhomogeneous nonlinearity, several works are devoted to transition fronts (see [3, 19]) of nonlocal dispersal equations [17, 20] and forced waves in shifting habitats [25]. Traveling waves and spreading speed of monostable nonlocal dispersal equations with space periodic nonlinearity were studied in [21]. Li et al. [16] further obtained the existence of entire solutions for space periodic nonlinearity. Particularly, Eberle [10, 11] constructed a heteroclinic orbit connecting two traveling waves for bistable local dispersal equation 1 in cylinders. Meanwhile, Berestycki and Rodríguez [5] considered a bistable nonlocal dispersal equation with a gap in one dimension. As far as we know, there is no result on entire solutions for the nonlocal dispersal equation 1 with inhomogeneous nonlinearity $f(x, u)$ satisfying (F).

In this paper, we aim to construct an entire solution connecting traveling waves with the two different nonlinearities, motivated by [10, 11]. By constructing suitable sub- and super-solutions, we establish the existence and uniqueness of the entire solution behaving as the traveling wave coming from one side and eventually going to the other side, which is different with the one constructed in [23] for the case $L = 0$ and $f_2 = f_1$. It should be pointed out that there are many differences to [4], though the similar method is applied. More precisely, since the nonlocal operator is not a compact operator as the Laplacian operator in [4], we have to establish the Lipschitz continuity of entire solutions in order to prove its uniqueness as in [23]. Compared to [23], we drop the assumption the kernel $J(x)$ is compactly supported.

The crucial part of this paper is to figure out the long time asymptotic behavior of the entire solution. Since the lack of compactness of the nonlocal operator, we can not use the Lyapunov function argument as [10, 11] to show that the entire solution converges to a translation of the other traveling wave as time goes to positive infinity. However, inspired by the idea in [7], we use a “squeezing” technique to address this
Theorem 1.1. Let assumptions (F) and (J) hold. Then there exists a unique entire solution $u(x,t)$ of (1) with $0 < u(x,t) < 1$, $u_t(x,t) > 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$ such that

$$u(x,t) - \phi_1(x + ct) \to 0 \text{ as } t \to -\infty \text{ uniformly in } x \in \mathbb{R}$$

and

$$u(x,t) - \phi_2(x + ct + \beta) \to 0 \text{ as } t \to +\infty \text{ uniformly in } x \in \mathbb{R}$$

for some $\beta \in \mathbb{R}$, where $(\phi_i, c_i) (i = 1, 2)$ are the traveling wave solutions solving

$$\begin{cases}
J \ast \phi_i - \phi_i + f_i(\phi_i) - c_i \phi_i' = 0, \\
\phi_i(-\infty) = 0, \phi_i(+\infty) = 1, \\
0 < \phi_i < 1.
\end{cases} \quad (4)$$

Theorem 1.2. The entire solution $u(x,t)$ constructed in Theorem 1.1 is Lyapunov stable in the following sense: For any given $\epsilon > 0$, there exists $\delta > 0$ such that for any uniformly continuous $v_0(x) \in [0,1]$ with $\sup_{x \in \mathbb{R}} |v_0(x) - u(x+a,t_0)| < \delta$, the solution $v(x,t,v_0(x))$ of (1) with initial value $v_0(x)$ satisfies

$$|v(x,t,v_0) - u(x+a,t+t_0)| < \epsilon$$

for $x \in \mathbb{R}$ and $t \geq 0$, where $a$, $t_0 \in \mathbb{R}$ are two constants.

The rest of this paper is organized as follows. In Section 2, we recall some results of the bistable traveling waves for homogeneous nonlinearities and prove the comparison principle for (1). Section 3 is devoted to constructing the unique entire solution. In Section 4, we study the asymptotic profile as time goes to positive infinity. Finally, we establish Lyapunov stability of the entire solution in Section 5.

2. Preliminaries. In this section, some known results on the traveling waves of (1) are outlined and the comparison principle is established.

2.1. Traveling waves. It follows from Theorem 2.7 in [2] and Theorem 2.7 in [23] that (4) admits a solution $\phi_1(z)$ satisfying

$$\begin{cases}
\beta_0 e^{\alpha_0 z} \leq \phi_1(z) \leq \alpha_0 e^{\lambda_0 z}, \ z \leq 0, \\
\beta_1 e^{-\lambda_1 z} \leq 1 - \phi_1(z) \leq \alpha_1 e^{-\lambda_1 z}, \ z > 0,
\end{cases} \quad (5)$$

where $\alpha_0$, $\alpha_1$, $\beta_0$ and $\beta_1$ are positive constants, $\lambda_0$ and $\lambda_1$ are the positive roots of

$$c\lambda_0 = \int_{\mathbb{R}} J_1(y)e^{-\lambda_0 y}dy - 1 + f_1'(0) \quad \text{and} \quad c\lambda_1 = \int_{\mathbb{R}} J_1(y)e^{-\lambda_1 y}dy - 1 + f_1'(1),$$

respectively. Moreover, we have

$$\begin{cases}
\tilde{\beta}_0 e^{\alpha_0 z} \leq \phi_1'(z) \leq \tilde{\alpha}_0 e^{\lambda_0 z}, \ z \leq 0, \\
\tilde{\beta}_1 e^{-\lambda_1 z} \leq \phi_1'(z) \leq \tilde{\alpha}_1 e^{-\lambda_1 z}, \ z > 0,
\end{cases} \quad (6)$$

for some constants $\tilde{\alpha}_0$, $\tilde{\alpha}_1$, $\tilde{\beta}_0$ and $\tilde{\beta}_1 > 0$. At last, note that $f_1 \in C^{1,1}([0,1])$, there exists some $L_f > 0$ such that

$$|f_1(u + v) - f_1(u) - f_1(v)| \leq L_f uv \quad \text{for } 0 \leq u, v \leq 1.$$
2.2. Comparison principle. We show that the following comparison principle holds by a contradiction argument.

**Proposition 1.** Suppose that assumptions (F) and (J) hold. Furthermore, let \(u(x,t), v(x,t)\) be continuous and bounded functions on \(\mathbb{R} \times [0,T]\) for some \(T > 0\) satisfy

\[
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} - \left( \int_{\mathbb{R}} J(x-y) [u(y,t) - u(x,t)] dy \right) - f(x,u(x,t)) &\geq 0, \\
u(x,0) &\geq 0, \quad x \in \mathbb{R},
\end{aligned}
\]

\[
\begin{aligned}
\frac{\partial v(x,t)}{\partial t} - \left( \int_{\mathbb{R}} J(x-y) [v(y,t) - v(x,t)] dy \right) - f(x,v(x,t)) &\leq 0, \\
v(x,0) &\leq 0, \quad x \in \mathbb{R},
\end{aligned}
\]

respectively. Then, \(u(x,t) \geq v(x,t)\) in \(\mathbb{R} \times [0,T]\).

Furthermore, if \(u(x,0) \neq v(x,0)\) for \(x \in \mathbb{R}\), then \(u(x,t) > v(x,t)\) for \(x \in \mathbb{R}, \; t \in [0,T]\).

**Proof of Proposition 1.** Let \(\tilde{w}(x,t) = u(x,t) - v(x,t)\). In fact, it is sufficient to show \(\tilde{w}(x,t) \geq 0\) for \((x,t) \in \mathbb{R} \times [0,\epsilon_0 T]\) with \(\epsilon_0 \in (0,1)\). Otherwise, suppose that \(\inf_{\mathbb{R} \times [0,\epsilon_0 T]} \tilde{w}(x,t) < 0\). Denote \(\tilde{w}(x,t) = e^{Zt} \tilde{w}(x,t)\), where \(Z = \|f_u(x,u)\|_\infty + 1\). It follows that \(\inf_{\mathbb{R} \times [0,\epsilon_0 T]} \tilde{w}(x,t) < 0\). Then there exists a sequence \((x_n, t_n) \in \mathbb{R} \times (0,\epsilon_0 T)\) such that

\[
\lim_{n \to +\infty} \tilde{w}(x_n, t_n) = \inf_{\mathbb{R} \times [0,\epsilon_0 T]} \tilde{w}(x,t) < 0.
\]

Observe that

\[
\begin{aligned}
\tilde{w}_t(x,t) &= Z \tilde{w}(x,t) + e^{Zt} \tilde{w}_t(x,t) \\
&\geq Z \tilde{w}(x,t) + e^{Zt} \int_{\mathbb{R}} J(x-y) \left[ \tilde{w}(y,t) - \tilde{w}(x,t) \right] dy + e^{Zt} \left[ f(x,u(x,t)) - f(x,v(x,t)) \right] \\
&= Z \tilde{w}(x,t) + \int_{\mathbb{R}} J(x-y) \left[ \tilde{w}(y,t) - \tilde{w}(x,t) \right] dy + f_u(x, u_\theta(x,t)) \tilde{w}(x,t),
\end{aligned}
\]

where \(u_\theta(x,t)\) is between \(v(x,t)\) and \(u(x,t)\). This implies that

\[
\begin{aligned}
\tilde{w}(x_n, t_n) - \tilde{w}(x_n, 0) &\geq \int_0^{t_n} \left[ J * \tilde{w}(x_n, s) - \tilde{w}(x_n, s) + Z \tilde{w}(x_n, s) + f_u(x, u_\theta(x,s)) \tilde{w}(x_n, s) \right] ds \\
&\geq \int_0^{t_n} \left[ J * \tilde{w}(x_n, s) + (\|f_u(x,u)\|_\infty - f_u(x, u_\theta(x,s))) \right] \inf_{\mathbb{R} \times [0,\epsilon_0 T]} \tilde{w}(x,t) ds.
\end{aligned}
\]

Letting \(n\) converge to infinity, we have

\[
\inf_{\mathbb{R} \times [0,\epsilon_0 T]} \tilde{w}(x,t) \geq (1 + \|f_u(x,u)\|_\infty - \min_{\mathbb{R} \times [0,1]} |f_u(x, u_\theta(x,s))|) \epsilon_0 T \inf_{\mathbb{R} \times [0,\epsilon_0 T]} \tilde{w}(x,t).
\]
Since \( \inf_{\mathbb{R} \times [0, \epsilon_0 T]} \tilde{w}(x, t) < 0 \), we can choose \( \epsilon_0 \) sufficiently small such that
\[
\inf_{\mathbb{R} \times [0, \epsilon_0 T]} \tilde{w}(x, t) \geq (1 + \|f_u(x, u)\|_\infty - \min_{\mathbb{R} \times [0, 1]} |f_u(x, u_0(x, s))|)\epsilon_0 T \inf_{\mathbb{R} \times [0, \epsilon_0 T]} \tilde{w}(x, t).
\]
Thus, we get a contradiction. Therefore, we obtain \( u(x, t) \geq v(x, t) \) for \( x \in \mathbb{R} \) and \( t \in [0, T] \). The remaining part of this proposition can be proved similarly by replacing the auxiliary function \( \tilde{w}(x, t) = e^{2t} \tilde{w}(x, t) \) with \( \tilde{w}(x, t) = e^{2t} \tilde{w}(x, t) - ct \) for small \( \epsilon > 0 \). In fact, we can similarly obtain
\[
\inf_{\mathbb{R} \times [0, \epsilon_0 T]} \tilde{w}(x, t) \geq \left(1 + \|f_u(x, u)\|_\infty - \min_{\mathbb{R} \times [0, 1]} |f_u(x, u_0(x, s))| - \frac{\epsilon}{\inf_{\mathbb{R} \times [0, \epsilon_0 T]} \tilde{w}(x, t)}\right)\epsilon_0 T < 1.
\]
This finishes the proof.

3. Construction of the entire solution. In this section, we focus on the construction of the entire solution which behaves like a traveling wave approaching from infinity. The main idea is to establish suitable sub- and super-solutions, which are defined as follows.
\[
W^-(x, t) = \begin{cases} 
\phi_1(x + c_1 t - \xi(t)) - \phi_1(-x + c_1 t - \xi(t)), & x \geq 0, \\
0, & x < 0,
\end{cases}
\]
and
\[
W^+(x, t) = \begin{cases} 
\phi_1(x + c_1 t + \xi(t)) + \phi_1(-x + c_1 t + \xi(t)), & x \geq 0, \\
2\phi_1(c_1 t + \xi(t)), & x < 0,
\end{cases}
\]
here \( \xi(t) \) is the solution of the following equation
\[
\dot{\xi}(t) = Me^{\lambda(c_1 t + \xi)}, \quad t < -T, \quad \xi(-\infty) = 0,
\]
where \( M, \lambda \) and \( T \) are positive constants to be specified later. A direct calculation yields that
\[
\xi(t) = \frac{1}{\lambda} \ln \frac{1}{1 - c_1^{-1} Me^{\lambda c_1 t}}.
\]
For the function \( \xi(t) \) to be defined, one must have \( 1 - c_1^{-1} Me^{\lambda c_1 t} > 0 \). Besides, there is
\[
c_1 t + \xi(t) \leq 0 \text{ for } -\infty < t \leq -T,
\]
where \( T := \frac{1}{\lambda c_1} \ln \frac{c_1 + M}{c_1} \).

Now we verify that \( W^\pm \) are sub- and super-solutions of \( 1 \) for \( t \leq -T \). Define
\[
\mathcal{L} u = u_t - (J * u - u) - f(x, u).
\]
3.1. Sub-solution. We first deal with the sub-solution $W^-(x,t)$. Since it is obvious for $x < 0$, we only consider the case $x \geq 0$. A straightforward calculation yields that

$$
\mathcal{L}W^-(x,t) = (c_1 - \xi(t))[\phi'_1(x + c_1t - \xi(t)) - \phi'_1(-x + c_1t - \xi(t))] \\
- \int_0^{+\infty} J(x - y)[\phi_1(y + c_1t - \xi(t)) - \phi_1(-y + c_1t - \xi(t))]\,dy \\
+ [\phi_1(x + c_1t - \xi(t)) - \phi_1(-x + c_1t - \xi(t))] - f(x, W^-) \\
= -\xi(t)[\phi'_1(x + c_1t - \xi(t)) - \phi'_1(-x + c_1t - \xi(t))] \\
+ \int_0^{-\infty} J(x - y)[\phi_1(y + c_1t - \xi(t)) - \phi_1(-y + c_1t - \xi(t))]\,dy \\
+ f_1(\phi_1(x + c_1t - \xi(t))) - f_1(\phi_1(-x + c_1t - \xi(t))) - f(x, W^-).
$$

Recall that $f(x, u) = f_1(u)$ for $x \geq 0$ and $\phi'_1 > 0$, it follows that

$$
\mathcal{L}W^-(x,t) \leq -\dot{\xi}(t)[\phi'_1(x + c_1t - \xi(t)) - \phi'_1(-x + c_1t - \xi(t))] \\
+ f_1(\phi_1(x + c_1t - \xi(t))) - f_1(\phi_1(-x + c_1t - \xi(t))).
$$

Then we continue to show $\mathcal{L}W^-(x,t) \leq 0$ in two cases $0 < x < -c_1t + \xi(t)$ and $x \geq -c_1t + \xi(t)$.

Case 1. For $0 < x < -c_1t + \xi(t)$, similar to [4], we have

$$
\phi'_1(x + c_1t - \xi(t)) - \phi'_1(-x + c_1t - \xi(t)) > m[\phi_1(x + c_1t - \xi(t)) - \phi_1(-x + c_1t - \xi(t))]
$$

for some $m > 0$. Consequently,

$$
\mathcal{L}W^-(x,t) \leq -\dot{\xi}(t)[\phi'_1(x + c_1t - \xi(t)) - \phi'_1(-x + c_1t - \xi(t))] \\
+ f_1(\phi_1(x + c_1t - \xi(t))) - f_1(\phi_1(-x + c_1t - \xi(t)) \geq -mMe^\lambda(c_1t + \xi(t)) \\
+ Lf\alpha_0e^{\lambda_0(-x + c_1t - \xi(t))} \\
= [\phi_1(x + c_1t - \xi(t)) - \phi_1(-x + c_1t - \xi(t))]e^{\lambda(c_1t + \xi(t))} \geq -mM \\
+ Lf\alpha_0e^{\lambda_0(-x + c_1t - \xi(t))}.
$$

Therefore, if we choose $M \geq \frac{L_f\alpha_0}{m}$ then $\mathcal{L}W^-(x,t) \leq 0$.

Case 2. For $x \geq -c_1t + \xi(t)$, if $\lambda_0 \geq \lambda_1$, then we obtain

$$
\mathcal{L}W^-(x,t) \leq -Me^{\lambda(c_1t + \xi(t))} \left[ \tilde{\beta}_1e^{-\lambda_1(-x + c_1t - \xi(t))} - \tilde{\alpha}e^{\lambda_0(-x + c_1t - \xi(t))} \right] \\
+ Lf\alpha_0e^{\lambda_0(-x + c_1t - \xi(t))} \left[ \alpha_1e^{-\lambda_1(-x + c_1t - \xi(t))} - \beta_0e^{\lambda_0(-x + c_1t - \xi(t))} \right] \\
\leq -e^{\lambda_0(-x + c_1t - \xi(t))} \left[ M\tilde{\beta}_1e^{(\lambda_0 - \lambda_1)x}e^{-(\lambda_0 + \lambda_1)(c_1t + \xi(t))} \right] \\
- M\tilde{\alpha}e^{\lambda(c_1t + \xi(t))} - Lf\alpha_0.
$$

Since $c_1t + \xi(t) < 0$ for $t < -T$ and $\lambda < \min\{\lambda_0, \lambda_1\}$, $\mathcal{L}W^-(x,t) \leq 0$ provided

$M \geq \frac{2L_f\alpha_0}{\tilde{\beta}_1}$. 

For $\lambda_0 < \lambda_1$, which means $f_1'(0) > f'_2(1)$, we have
\[
\mathcal{L}W^-(x, t) 
\leq M e^{\lambda(c_1 t + \xi(t))} \phi'_1(-x + c_1 t - \xi(t)) + f_1(\phi_1(x + c_1 t - \xi(t))) 
- f_1(\phi_1(-x + c_1 t - \xi(t))) - f_1(\phi_1(x + c_1 t - \xi(t)) - \phi_1(-x + c_1 t - \xi(t))).
\]
Moreover, if $x + c_1 t - \xi(t) > L$ for some $L \gg 1$ such that $\phi_1(x + c_1 t - \xi(t)) \in [1 - \sigma, 1]$ and $\phi_1(-x + c_1 t - \xi(t)) \in [0, \sigma]$. It follows that
\[
\mathcal{L}W^-(x, t) \leq M e^{\lambda(c_1 t + \xi(t))} \phi'_1(-x + c_1 t - \xi(t)) + [f'_1(1) - f'(0)]\phi_1(-x + c_1 t - \xi(t)) 
+ o(\phi_1(-x + c_1 t - \xi(t))) 
\leq e^{\lambda_0(-x + c_1 t - \xi(t))}(M e^{\lambda(c_1 t + \xi(t))} + (f'_1(1) - f'(0)) - o(1)).
\]
Since $t < -T$ such that $c_1 t + \xi(t) < -1$, we have $\mathcal{L}W^-(x, t) \leq 0$. Finally, if $0 < x + c_1 t - \xi(t) \leq L$, then, from the above case $\lambda_0 \geq \lambda_1$, there holds
\[
\mathcal{L}W^-(x, t) \leq -e^{\lambda_0(-x + c_1 t - \xi(t))} \left[M \tilde{\beta}_1 e^{(\lambda_0 - \lambda_1)x} e^{-\lambda_0 + \lambda_1 - \lambda} c_1 t + (\lambda + \lambda_0 + \lambda_1)\xi(t))
- M \tilde{\alpha} e^{\lambda(c_1 t + \xi(t))} - L f_0 \right]
\leq -e^{\lambda_0(-x + c_1 t - \xi(t))} \left[M \tilde{\beta}_1 e^{(\lambda_0 - \lambda_1)L} e^{(\lambda_0 + \lambda_1) c_1 t + (\lambda + \lambda_0 + \lambda_1) \xi(t))}
- M \tilde{\alpha} e^{\lambda(c_1 t + \xi(t))} - L f_0 \right].
\]
Thanks to $t < -T$, $\lambda < \min\{\lambda_0, \lambda_1\}$ and $M > \frac{2L f_0 e^{(\lambda_1 - \lambda_0)L}}{\tilde{\beta}_1}$, we obtain
\[
\mathcal{L}W^-(x, t) \leq 0.
\]

3.2. **Super-solution.** We intend to testify the super-solution $W^+(x, t)$ in two steps.

**Step 1.** $W^+(x, t) = 2 \phi_1(c_1 t + \xi(t))$ for $x < 0$. Following a direct calculation, we have
\[
\mathcal{L}W^+(x, t) = 2(c_1 + \xi(t)) \phi'_1(c_1 t + \xi(t)) - \int_{\mathbb{R}} J(x - y)W^+(y, t) dy 
+ 2 \phi_1(c_1 t + \xi(t)) - f(x, 2 \phi_1(c_1 t + \xi(t))) 
\geq 2(c_1 + \xi(t)) \phi'_1(c_1 t + \xi(t)) - f_1(2 \phi_1(c_1 t + \xi(t))) 
- \int_{0}^{+\infty} J(x - y)[\phi_1(y + c_1 t + \xi(t)) + \phi_1(-y + c_1 t + \xi(t))] dy.
\]
Denote
\[
I = \int_{0}^{+\infty} J(x - y)[\phi_1(y + c_1 t + \xi(t)) + \phi_1(-y + c_1 t + \xi(t))] dy.
\]
Then, it follows that
\[
I = \int_{0}^{-c_1 t - \xi(t)} J(x - y)[\phi_1(y + c_1 t + \xi(t)) + \phi_1(-y + c_1 t + \xi(t))] dy 
+ \int_{-c_1 t - \xi(t)}^{+\infty} J(x - y)[\phi_1(y + c_1 t + \xi(t)) + \phi_1(-y + c_1 t + \xi(t))] dy 
\leq \alpha_0 e^{\lambda_0(c_1 t + \xi(t))} \int_{0}^{+\infty} J(x - y) [e^{\lambda_0 y} + e^{-\lambda_0 y}] dy + 2 \int_{-c_1 t - \xi(t)}^{+\infty} J(x - y) dy.
\]
Let
\[ J_{\lambda_0} = \int_{0}^{+\infty} J(y) \left[ e^{\lambda_0 y} + e^{-\lambda_0 y} \right] dy. \]

It follows from assumption \((J)\) that there exists \(K_J > 0\) such that
\[ J(x) \leq K_J e^{2\lambda_0 x} \text{ for } x \geq 0. \]

Furthermore, there hold
\[ I \leq J_{\lambda_0} e^{\lambda_0 (c_1 t + \xi(t))} + 2K_J \int_{-c_1 t + \xi(t)}^{+\infty} e^{2\lambda_0 (x-y)} dy \]
\[ \leq J_{\lambda_0} e^{\lambda_0 (c_1 t + \xi(t))} + \frac{2K_J}{\lambda_0} e^{2\lambda_0 (c_1 t + \xi(t))}. \]

Therefore, according to \(t < -T\) such that \(\phi_1(c_1 t + \xi(t)) \ll 1\), we get
\[ LW^+(x,t) \geq 2(c_1 + \tilde{\xi}(t)) \phi_1(c_1 t + \xi(t)) - J_{\lambda_0} e^{\lambda_0 (c_1 t + \xi(t))} \]
\[ \quad - \frac{2K_J}{\lambda_0} e^{2\lambda_0 (c_1 t + \xi(t))} - f_1(2\phi_1(c_1 t + \xi(t))) \]
\[ \geq \left( 2c_1 \tilde{\beta}_0 - \alpha_0 J_{\lambda_0} - 2f'_1(0) \tilde{\beta}_0 - \phi_1(0) \right) e^{\lambda_0 (c_1 t + \xi(t))} \]
\[ + \left( 2M \tilde{\beta}_0 - \frac{2K_J}{\lambda_0} \right) e^{2\lambda_0 (c_1 t + \xi(t))}. \]

Since
\[ M > \frac{K_J}{\tilde{\beta}_0 \lambda_0} \quad \text{and} \quad J_{\lambda_0} < \frac{2c_1 \tilde{\beta}_0 + 2|f'_1(0)| \tilde{\beta}_0}{\alpha_0}, \]

consequently, we have \(LW^+(x,t) \geq 0\) for \(t < -T\).

**Step 2.** For \(x \geq 0\), \(W^+(x,t) = \phi_1(x + c_1 t + \xi(t)) + \phi_1(-x + c_1 t + \xi(t))\). Then we can obtain
\[ LW^+(x,t) \]
\[ = (c_1 + \tilde{\xi}(t)) [\phi'_1(x + c_1 t + \xi(t)) + \phi'_1(-x + c_1 t + \xi(t))] - \int_{R} J(x-y)W^+(y,t)dy \]
\[ + [\phi_1(x + c_1 t + \xi(t)) + \phi_1(-x + c_1 t + \xi(t))] - f_1(W^+) \]
\[ = \tilde{\xi}(t) [\phi'_1(x + c_1 t + \xi(t)) + \phi'_1(-x + c_1 t + \xi(t))] \]
\[ + \int_{-\infty}^{0} J(x-y) [\phi_1(y + c_1 t + \xi(t)) + \phi_1(-y + c_1 t + \xi(t)) - 2\phi_1(c_1 t + \xi(t))]dy \]
\[ + f_1(\phi_1(x + c_1 t + \xi(t))) + f_1(\phi_1(-x + c_1 t + \xi(t))) \]
\[ - f_1(\phi_1(x + c_1 t + \xi(t)) + \phi_1(-x + c_1 t + \xi(t))). \]

Denote
\[ II = \int_{-\infty}^{0} J(x-y) [\phi_1(y + c_1 t + \xi(t)) + \phi_1(-y + c_1 t + \xi(t)) - 2\phi_1(c_1 t + \xi(t))]dy. \]
We consider two cases.

**Case 1.** \(0 \leq x \leq -c_1 t - \xi(t).\) Without loss of generality, let \(\phi_1(0) = \theta_1.\) Since \(\phi_1' > 0\) and \(t < -T\) such that \(\phi_1(c_1 t + \xi(t)) < \frac{1}{2} \theta_1,\) we have

\[
II = \int_{c_1 t + \xi(t)}^{0} J(x - y) [\phi_1(y + c_1 t + \xi(t)) + \phi_1(-y + c_1 t + \xi(t))
- 2\phi_1(c_1 t + \xi(t))] dy + \int_{-\infty}^{c_1 t + \xi(t)} J(x - y) \phi_1(y + c_1 t + \xi(t))
+ \phi_1(-y + c_1 t + \xi(t)) - 2\phi_1(c_1 t + \xi(t))] dy
\geq \int_{c_1 t + \xi(t)}^{0} J(x - y) C_{\lambda_0} e^{\lambda_0 (c_1 t + \xi(t))} \left[ e^{\lambda_0 y} + e^{-\lambda_0 y} - 2 \right.
- \left(2 + e^{(\lambda_0 + \eta)y} + e^{-(\lambda_0 + \eta)y}\right) C_{\eta} e^{\eta (c_1 t + \xi(t))}] dy
\geq - (2 + J_\eta) C_{\lambda_0} C_{\eta} e^{(\lambda_0 + \eta)(c_1 t + \xi(t))}.
\]

The second inequality follows from

\[
|\phi_1(x) - C_{\lambda_0} e^{\lambda_0 x}| \leq C_{\eta} e^{(\lambda_0 + \eta)x} \text{ for } x \leq 0 \text{ and some } 0 < \eta < \lambda_0,
\]

which can be easily obtained by 5 and 6. As a consequence,

\[
\mathcal{L}W^+(x, t) \geq M e^{(\lambda + \lambda_0)(c_1 t + \xi(t))} \tilde{\beta}_0 \left( e^{\lambda_0 x} + e^{-\lambda_0 x} \right) - (2 + J_\eta) C_{\lambda_0} C_{\eta} e^{(\lambda_0 + \eta)(c_1 t + \xi(t))}
- L_f \alpha_0^2 e^{2\lambda_0 (c_1 t + \xi(t))}
\geq e^{(\lambda + \lambda_0)(c_1 t + \xi(t))} \left[ 2M \beta_0 - (2 + J_\eta) C_{\lambda_0} C_{\eta} e^{(\lambda - \lambda_0)(c_1 t + \xi(t))}
- L_f \alpha_0^2 e^{(\lambda - \lambda_0)(c_1 t + \xi(t))} \right]
\geq e^{(\lambda + \lambda_0)(c_1 t + \xi(t))} \left[ 2M \beta_0 - (2 + J_\eta) C_{\lambda_0} C_{\eta} - L_f \alpha_0^2 \right].
\]

The last inequality holds since \(\lambda < \min\{\lambda_0, \lambda_1, \eta\}.\) Thus \(\mathcal{L}W^+(x, t) \geq 0,\) provided that

\[
M \geq \frac{(2 + J_\eta) C_{\lambda_0} C_{\eta} + L_f \alpha_0^2}{2 \beta_0}.
\]

**Case 2.** Here \(x > -c_1 t - \xi(t).\) From Case 1, we know

\[
II \geq - \int_{c_1 t + \xi(t)}^{0} J(x - y) C_{\lambda_0} e^{\lambda_0 (c_1 t + \xi(t))}
\left(2 + e^{(\lambda_0 + \eta)y} + e^{-(\lambda_0 + \eta)y}\right) C_{\eta} e^{\eta (c_1 t + \xi(t))} dy
\geq - C_{\lambda_0} C_{\eta} K_f \left( e^{(\lambda_0 + \eta)(c_1 t + \xi(t))} \right)
\int_{c_1 t + \xi(t)}^{0} e^{-2\lambda_0 (x - y)} \left(3 + e^{-(\lambda_0 + \eta)y}\right) dy
\geq - C_{\lambda_0} C_{\eta} K_f e^{\lambda_0 (-x + c_1 t + \xi(t)) - \lambda_0 x} \left( \frac{3}{\lambda_0} e^{\eta (c_1 t + \xi(t))} + \frac{1}{\eta} \right).
\]
Moreover, if \( \lambda_1 \leq \lambda_0 \), then
\[
\mathcal{L} W^+(x, t) \\
\geq M e^{\lambda(x_1 t + \xi(t))} \left( \tilde{\beta}_1 e^{-\lambda_1(x_1 t + \xi(t))} + \tilde{\beta}_0 e^{\lambda_0(-x_0 + c_1 t + \xi(t))} \right) \\
- L f x_0 e^{\lambda_0(-x + c_1 t + \xi(t))} - C_{\lambda_0} C_{\eta} K_j e^{\lambda_0(-x + c_1 t + \xi(t))} \left( \frac{3}{\lambda_0} e^{\eta(c_1 t + \xi(t))} + \frac{1}{\eta} \right) \\
\geq e^{\lambda_0(\varphi_1(-x_1 t + \xi(t))) + \lambda_0(-x + c_1 t + \xi(t))} \left[ M \tilde{\beta}_1 e^{-(\lambda_0 - \lambda_1)x_0 - \lambda_0 + \lambda_1)(c_1 t + \xi(t)))} \\
- \frac{3C_{\lambda_0} C_{\eta} K_j}{\lambda_0} e^{(\eta - \lambda_0)(c_1 t + \xi(t))} - \left( L f x_0 + \frac{C_{\lambda_0} C_{\eta} K_j}{\eta} \right) e^{-\lambda_0(c_1 t + \xi(t))} \right].
\]
Remember that \( \lambda < \min\{\lambda_0, \lambda_1, \eta\} \) and \( t < -T \). As a matter of fact, take \( M \) sufficiently large such that
\[
M \tilde{\beta}_1 - \frac{3C_{\lambda_0} C_{\eta} K_j}{\lambda_0} - \left( L f x_0 + \frac{C_{\lambda_0} C_{\eta} K_j}{\eta} \right) \geq 0. \tag{7}
\]
This yields that \( \mathcal{L} W^+(x, t) \geq 0 \).

If \( \lambda_0 < \lambda_1 \), we have \( f_1'(0) > f_1(1) \). In addition, for \( x > -c_1 t - \xi(t) + \mathcal{M} \) with \( \mathcal{M} \gg 1 \) such that \( \phi_1(x + c_1 t + \xi(t)) \in [1 - \sigma, 1] \), we have
\[
f_1(\phi_1(x + c_1 t + \xi(t))) + f_1(\phi_1(-x + c_1 t + \xi(t))) - f_1(W^+) \\
= [f_1'(0) - f_1'(1)] \phi_1(-x + c_1 t + \xi(t)) = o(\phi_1(-x + c_1 t + \xi(t))).
\]
Meanwhile,
\[
\mathcal{L} W^+(x, t) \geq e^{\lambda_0(x_1 t + \xi(t)) + \lambda_0(-x + c_1 t + \xi(t))} \left( M \tilde{\beta}_1 e^{(\lambda_0 - \lambda_1)x_0 - \lambda_0 + \lambda_1)(c_1 t + \xi(t))} \\
- \frac{3C_{\lambda_0} C_{\eta} K_j}{\lambda_0} e^{(\eta - \lambda_0)(c_1 t + \xi(t))} + \left( [f_1'(0) - f_1'(1)] \tilde{\beta}_0 \right) \\
- \frac{C_{\lambda_0} C_{\eta} K_j}{\eta} e^{\lambda_0(-x + c_1 t + \xi(t))}. \right.
\]
Then \( \mathcal{L} W^+(x, t) \geq 0 \) by 7 and \( -x < c_1 t + \xi(t) - \mathcal{M} \ll -1 \) such that
\[
[f_1'(0) - f_1'(1)] \tilde{\beta}_0 - \frac{C_{\lambda_0} C_{\eta} K_j}{\eta} e^{-\lambda_0 x} \geq 0.
\]
When \( -c_1 t - \xi(t) < x < -c_1 t - \xi(t) + \mathcal{M} \), it follows from the case \( \lambda_0 \geq \lambda_1 \) that
\[
\mathcal{L} W^+(x, t) \geq e^{\lambda_0(x_1 t + \xi(t)) + \lambda_0(-x + c_1 t + \xi(t))} \left[ M \tilde{\beta}_1 e^{(\lambda_0 - \lambda_1)x_0 - \lambda_0 + \lambda_1)(c_1 t + \xi(t))} \\
- \frac{3C_{\lambda_0} C_{\eta} K_j}{\lambda_0} e^{(\eta - \lambda_0)(c_1 t + \xi(t))} - \left( L f x_0 + \frac{C_{\lambda_0} C_{\eta} K_j}{\eta} \right) e^{-\lambda_0(c_1 t + \xi(t))} \right] \\
\geq e^{\lambda_0(x_1 t + \xi(t)) + \lambda_0(-x + c_1 t + \xi(t))} \left[ M \tilde{\beta}_1 e^{(\lambda_0 - \lambda_1)M - \lambda_0(c_1 t + \xi(t))} \\
- \frac{3C_{\lambda_0} C_{\eta} K_j}{\lambda_0} e^{(\eta - \lambda_0)(c_1 t + \xi(t))} - \left( L f x_0 + \frac{C_{\lambda_0} C_{\eta} K_j}{\eta} \right) e^{-\lambda_0(c_1 t + \xi(t))} \right].
\]
Similar to the case \( \lambda_1 \leq \lambda_0 \), we can obtain \( \mathcal{L} W^+(x, t) \geq 0 \) for \( t < -T \) by letting \( M \) be sufficiently large.
Proof of Theorem 1.1. Let \( u_n(x,t) \) be the unique solution of the Cauchy problem

\[
\begin{aligned}
(u_n)_t(x,t) &= J \ast u_n(x,t) - u_n(x,t) + f(x,u_n), \quad x \in \mathbb{R}, \ t > -n, \\
\phi_n(x,-n) &= W^-(x,-n), \quad x \in \mathbb{R}.
\end{aligned}
\]

Since \( W^-(x,-n) = u_n(x,-n) \leq W^+(x,-n) \), the comparison principle yields for \( n > T \)

\[
W^-(x,t) \leq u_n(x,t) \leq W^+(x,t) \text{ for } x \in \mathbb{R}, \ t \in [-n,-T].
\]

Moreover, since \( m := \min_{\mathbb{R} \times [0,1]} (1 - f_n(x,u)) > 0 \) by the assumption \( f_n(x,u) < 1 \), it is not difficult to see that \( \delta u_n(x,t) = u_n(x+\eta,t) - u_n(x,t) \) with \( \eta \in \mathbb{R} \) is a sub-solution of

\[
\begin{aligned}
\{ v'(t) &= L\eta - mv(t), \ t > -n, \\
\}
\]

where \( L \geq \int_{\mathbb{R}} |J'(y)|\,dy \) and \( M \) is some positive constant. It is then similar to the proof of Proposition 2.4 in [23] that \( u_n(x,t) \) and \( (u_n)_t(x,t) \) are Lipschitz continuous with respect to \( x \). Besides, a direct calculation gives that \( |u_n(x,t)| < C \) and \( |(u_n)_t(x,t)| < C \) for some positive constant \( C \). Then applying Arzela-Ascoli Theorem, there exists a subsequence of \( \{u_n\}_{n=1}^{\infty} \) and \( \{(u_n)_t\}_{n=1}^{\infty} \), still denoted by \( \{u_n\}_{n=1}^{\infty} \) and \( \{(u_n)_t\}_{n=1}^{\infty} \), such that \( \{u_n\}_{n=1}^{\infty} \) and \( \{(u_n)_t\}_{n=1}^{\infty} \) converge to a function \( u(x,t) \) and \( u_t(x,t) \). Now we have obtained the entire solution. In fact, the regularity of this entire solution is not so well. But we can show the entire solution is Lipschitz continuous in \( x \in \mathbb{R} \), which is important to show the uniqueness of the entire solution.

**Proposition 2.** Let \( u(x,t) \) be the entire solution in Theorem 1.1. Then \( u(x,t) \) satisfies

\[
|u(x+\eta,t) - u(x,t)| \leq M' \eta,
\]

and

\[
\left| \frac{\partial u(x+\eta,t)}{\partial t} - \frac{\partial u(x,t)}{\partial t} \right| \leq M'' \eta
\]

for \( M', M'' > 0 \).

The proof is similar to that of Proposition 3.1 in [23] by virtue of the fact \( f_n(x,u) < 1 \) for \( x \in \mathbb{R} \) and \( u \in [0,1] \), so we omit it here.

As for the uniqueness, since it is easy to see \( u(x,t) \to \phi_1(x + c_1 t) \) as \( t \to -\infty \) by the sub- and super-solutions established above, we can similarly show that the conclusion of Lemma 3.1 in [4] also holds true here. Then referring to the process of Section 3 in [4], we can obtain the uniqueness of the entire solution by constructing similar sub- and super-solutions. Thus we have shown the first part of Theorem 1.1.

**4. Long time asymptotic profile.** In this section, we are going to show the entire solution established in previous section converges to a shift of \( \phi_2(x + c_2 t) \) as \( t \to +\infty \). The main idea here is first to construct suitable sub-and super-solutions taking advantage of the traveling wave \( \phi_2(x + c_2 t) \) to get proper lower and upper bounds, which is a classical and regular method in study of traveling waves. Then we apply a “squeezing” technique which is used to prove the asymptotic stability of
the traveling waves for nonlocal dispersal equation in [7] to obtain the convergence of the entire solution. Here, we assume that for some $X > 0$, $\omega$, $\sigma > 0$ and

$$\omega < \min \left\{ \left| \frac{f'_i(t)}{4} \right|, \left| \frac{f''_i(t)}{4} \right|, \left| \frac{f'_j(1)}{4} \right|, \left| \frac{f''_j(1)}{4} \right| \right\},$$

there hold

$$\phi_1(x), \phi_2(x) \leq \frac{\sigma}{2} \text{ for } x \leq -X \text{ and } \phi_1(x), \phi_2(x) \geq 1 - \frac{\sigma}{2} \text{ for } x \geq X,$$

besides,

$$f'_i(s), f''_i(s) \leq -\omega \text{ for } s \in [0, \sigma] \cup [1 - \sigma, 1].$$

**Theorem 4.1.** Let assumptions \((F)\) and \((J)\) hold, and \(u(x, t)\) be the unique entire solution of \(1\) with \(0 < u(x, t) < 1, u_t(x, t) > 0\) for all \(t \in \mathbb{R}\) and \(x \in \mathbb{R}\) such that

$$u(x, t) - \phi_1(x + c_1 t) \to 0 \text{ as } t \to -\infty \text{ uniformly in } x \in \mathbb{R}. \quad (8)$$

Then

$$u(x, t) - \phi_2(x + c_2 t + \beta) \to 0 \text{ as } t \to +\infty \text{ uniformly in } x \in \mathbb{R}$$

for some \(\beta \in \mathbb{R}\).

**Lemma 4.2.** Suppose assumptions \((F)\) and \((J)\) hold. Then there exist \(T_-, \beta_-\), \(\delta_-\), \(\omega > 0\) such that

$$u(x, t) \geq \max \left\{ \phi_2(x + c_2 t - \beta_-) - \delta_- e^{-\omega t}, 0 \right\}$$

for \(x \in \mathbb{R}\) and \(t \geq T_-\).

**Proof of Lemma 4.2.** In fact, it follows from \(8\) that there exists \(t_- \in \mathbb{R}\) such that

$$|u(x, t_-) - \phi_1(x + c_1 t_-)| \leq \frac{\epsilon_-}{2}$$

for any \(\epsilon_- > 0\) and \(x \in \mathbb{R}\). Denote

$$\tilde{u}(x, t) = u(x, t + t_-) \text{ and } \tilde{u}(x, t) = \max \left\{ \phi_2(\xi_-(x, t)) - C^v e^{-\omega t}, 0 \right\},$$

where

$$\xi_-(x, t) = x + c_2(t + t_-) - \beta^0 + C^v e^{-\omega t} - C^v \text{ for } t \geq 0, \ x \in \mathbb{R}.$$  

We first show \(\tilde{u}(x, 0) \geq \tilde{u}(x, 0) \text{ for } x \in \mathbb{R}\). Then there exists \(x_1 > 0\) such that

$$\phi_1(x + c_1 t_-) \geq 1 - \frac{\epsilon_-}{2} \text{ and } \phi_2(x + c_2 t_-) \geq 1 - \frac{\epsilon_-}{2} \text{ for } x \geq x_1.$$  

As \(u_t > 0\) and \(\phi'_i > 0 \ (i = 1, 2)\), we have

$$1 - \frac{\epsilon_-}{2} \leq u(x, t) \leq 1 \text{ and } 1 - \frac{\epsilon_-}{2} \leq \phi_i \leq 1, \ i = 1, 2$$

for \(t \geq t_-, \ x \geq x_1\). Then,

$$|\phi_i - u(x, t)| \leq \epsilon_-, \ i = 1, 2 \text{ for } t \geq t_-, \ x \geq x_1.$$  

Meanwhile, there exists \(x_2 \in \mathbb{R}\) such that

$$0 < \phi_i(x + c_i t_-) \leq \frac{\epsilon_-}{2}, \ i = 1, 2 \text{ for } x \leq x_2.$$  

Particularly, there holds

$$|u(x, t_-) - \phi_2(x + c_2 t_-)| \leq \epsilon_- \text{ for } x \in \mathbb{R} \setminus (x_1, x_2).$$

In addition, \(\min_{x \in [x_1, x_2]} u(x, t) > 0\) since \(0 < u(x, t) < 1\). Then we can choose \(\beta^0 > 0\) such that

$$\phi_2(x + c_2 t_- - \beta^0) \leq u(x, t_-) \text{ for } x \in [x_1, x_2].$$
From all the discussion above we obtain \( \tilde{u}(x, 0) \geq u(x, 0) \). Next, we prove

\[
\mathcal{L} u(x, t) = u(x, t) - (J * u(x, t) - u(x, t)) - f(x, u) \leq 0 \quad \text{for } t \geq 0, \quad x \in \mathbb{R}.
\]

In fact, we only need to consider \( u(x, t) = \phi_2(\xi_-(x, t)) - C_v e^{-\omega t} \) because 0 is obviously a sub-solution of 1. Hence,

\[
\mathcal{L} u(x, t) = (c_2 - C_v \omega e^{-\omega t}) \phi_2'(\xi_-(x, t)) + C_v \omega e^{-\omega t} - \int_{\mathbb{R}} J(x-y) \phi_2(\xi_-(y, t)) dy
\]

\[
+ \phi_2(\xi_-(x, t)) - f(x, u(x, t))
\]

\[
\leq - C_v \omega e^{-\omega t} \phi_2'(\xi_-(x, t)) + f_2(\phi_2(\xi_-(x, t)))
\]

\[
- f_2(\phi_2(\xi_-(x, t)) - C_v e^{-\omega t} + C_v \omega e^{-\omega t}.
\]

In the following, we continue the proof in three cases.

**Case** \( \xi_-(x, t) \leq -X \). In this case,

\[
\phi_2(\xi_-) - C_v e^{-\omega t} \leq \phi_2(\xi_-) \leq \sigma.
\]

Then, by \( \omega < \frac{|f_2'(0)|}{4} \),

\[
\mathcal{L} u(x, t) \leq (f_2'(0) + 2\omega) C_v e^{-\omega t} \leq 0.
\]

**Case** \( \xi_-(x, t) \geq X \). For this case, by \( 2C_v < \sigma \),

\[
1 - \sigma < 1 - \frac{\sigma}{2} - C_v \leq \phi_2(\xi_-) - C_v e^{-\omega t} \leq \phi_2(\xi_-) < 1.
\]

Then, since \( \omega < \frac{|f_2'(1)|}{4} \),

\[
\mathcal{L} u(x, t) \leq (f_2'(1) + 2\omega) C_v e^{-\omega t} \leq 0.
\]

**Case** \( \xi_- \in [-X, X] \). Let \( \tau_0 = \min_{\xi \in [-X, X]} \phi_2'(\xi) \). Then,

\[
\mathcal{L} u(x, t) \leq - \tau_0 C_v \omega e^{-\omega t} + \|f_2'\|_{\infty} C_v e^{-\omega t} + C_v \omega e^{-\omega t}
\]

\[
\leq (\|f_2'\|_{\infty} C_v + +C_v \omega - \tau_0 C_v \omega) e^{-\omega t}.
\]

Choose \( C_v = \tau_0 \omega^{-1} (\|f_2'\|_{\infty} C_v + +C_v \omega) \). This gives \( \mathcal{L} u(x, t) \leq 0 \). We finish the proof by letting \( \beta_- = \beta^0 + C_v \) and \( \delta_- = C_v \).

Similarly, we can prove the following lemma.

**Lemma 4.3.** Under the assumptions of Lemma 4.2, there exist \( T_+ > T_- \), \( \beta_+ \), \( C_+ \), \( \omega > 0 \) such that

\[
\min \{ \phi_1(x + c_1 t + \beta_+) + C_+ e^{-\omega t}, 1 \} \geq u(x, t)
\]

for \( x \in \mathbb{R} \), \( t \geq T_+ \), where \( \omega \), \( T_- \) are defined in Lemma 4.2.

Now we are in a position to establish an important lemma.

**Lemma 4.4.** Under the assumptions of Lemma 4.2, there exist \( T > \max \{ T_+, T_- \} \), \( \beta \), \( C \), \( \omega > 0 \) such that

\[
\min \{ \phi_2(x + c_2 t + \beta) + C e^{-\omega t}, 1 \} \geq u(x, t)
\]

for \( x \in \mathbb{R} \), \( t \geq T \), where \( \omega \), \( T_- \), \( T_+ \) are defined in Lemmas 4.2 and 4.3.
**Proof of Lemma 4.4.** Define \( \hat{u}(x,t) = u(x,t+T) \) and \( \overline{u}(x,t) = \min\{\phi_2(\xi_+(x,t)) + v(t), 1\} \), where

\[
\xi_+(x,t) = x + c_2(t + T) + \beta^+ + V(t), \quad V(t) = K_V \int_0^t v(s)ds,
\]

and

\[
v(t) = (\epsilon_1 + C_3)e^{-\omega t} - C_3e^{-\lambda^1 c_2 t}
\]

for \( 0 < \epsilon_1, \quad C_3 < \frac{\sigma}{4} \) and \( t \geq 0 \). Here, \( \lambda^1 \) is the positive root of

\[
c_2 \lambda^1 = \int_{\mathbb{R}} J(x-y)e^{-\lambda^1 y}dy - 1 + f'_2(1).
\]

Then

\[
v(0) = \epsilon_1, \quad 0 < v(t) < \frac{\sigma}{2} \quad \text{for} \quad t \geq 0
\]

and

\[
\hat{v}(t) = -\omega v(t) - (w - \lambda^1 c_2)C_3e^{-\lambda^1 c_2 t} \quad \text{for} \quad t \geq 0.
\]

Now we intend to show \( \hat{u}(x,0) \leq \overline{u}(x,0) \). Recall the fact that \( |1 - \phi_2(x)| \leq C^2 e^{-\lambda^1 x} \) for \( x \in \mathbb{R}, \quad C^2 > 0 \). Meanwhile, we assume that \( 0 < \omega < \lambda^1 c_2 \). Additionally, by the assumptions of \( f(x,u) \), there exists \( C_0 > 0 \) such that

\[
|f_1(s) - f_2(s)| \leq C_0|1 - s| \quad \text{for} \quad s \in [0,1].
\]

Since \( \phi_1(-\infty) = 0 \) and \( \phi_1(+\infty) = 1 \), there exists \( x_3 < 0 \) such that

\[
\phi_1(x + c_1 T + \beta^+) < \frac{\epsilon_1}{2} \quad \text{and} \quad \phi_2(x + c_2 T) < \epsilon_1 \quad \text{for} \quad x < x_3
\]

as well as \( x_4 > 0 \) such that

\[
\phi_2(x + c_2 T - \beta_-) \geq 1 - \epsilon_1 \quad \text{and} \quad \phi_2(x + c_2 T) \geq 1 - \epsilon_1
\]

for \( x > x_4 \). Note that we can take \( T > 0 \) sufficiently large such that

\[
\max\{C_+ , \delta_- \} e^{-\omega T} < \frac{\epsilon_1}{2} \quad \text{and} \quad X - c_2 T \leq -x_0.
\]

Therefore, it is obvious that

\[
u(x,T) \leq \min\{\phi_2(x + c_2 T) + \epsilon_1, 1\} \leq \overline{u}(x,0) \quad \text{for} \quad x \in \mathbb{R} \setminus (x_3, x_4).
\]

Furthermore, since \( \max_{x \in [x_3, x_4]} u(x,T) < 1 \), one can choose \( \beta^+ > 0 \) such that

\[
\phi_2(x + c_2 T + \beta^+) \geq u(x,T) \quad \text{for} \quad x \in [x_3, x_4].
\]

This yields that \( \hat{u}(x,0) \leq \overline{u}(x,0) \).

In the following, we are going to prove

\[
\mathcal{L} \overline{u}_t(x,t) = \mathcal{L} \overline{u}_x(x,t) - (J * \overline{u}(x,t) - \overline{u}(x,t)) - f(x,\overline{u}) \geq 0 \quad \text{for} \quad t \geq 0, \quad x \in \mathbb{R}.
\]

Here we only need to consider \( \overline{u}(x,t) = \phi_2(\xi_+(x,t)) + v(t) \) since 1 is obviously a super-solution of 1. A direct calculation gives that

\[
\mathcal{L} \overline{u}_t(x,t) = \xi_+(x,t)\phi_2'(\xi_+(x,t)) + \hat{v}(t) - \int_{\mathbb{R}} J(x-y)\phi_2(\xi_+(y,t))dy
\]

\[
+ \phi_2(\xi_+(x,t)) - f(x,\phi_2(\xi_+(x,t)) + v(t))
\]
\[
\dot{V}(t)\phi_2'(\xi_+(x,t)) + \dot{v}(t) + f_2(\phi_2(\xi_+(x,t))) - f(x, \phi_2(\xi_+(x,t)) + v(t)) \\
\geq \begin{cases}
\dot{V}(t)\phi_2'(\xi_+(x,t)) + \dot{v}(t) + f_2(\phi_2(\xi_+(x,t))) - f_2(\phi_2(\xi_+(x,t)) + v(t)) \quad \text{for } x < -L, \\
\dot{V}(t)\phi_2'(\xi_+(x,t)) + \dot{v}(t) + f_2(\phi_2(\xi_+(x,t))) - f_1(\phi_2(\xi_+(x,t)) + v(t)) \quad \text{for } x \geq -L.
\end{cases}
\]

We continue to prove \( L\Pi(x,t) \geq 0 \) in three cases.

**Case 1.** If \( \xi_+ > X \) and \( x \geq -L \), then

\[ L\Pi(x,t) \geq \dot{V}(t)\phi_2'(\xi_+(x,t)) + \dot{v}(t) + f_1(\phi_2(\xi_+(x,t))) - f_1(\phi_2(\xi_+(x,t)) + v(t)) - C_0|1 - \phi_2(\xi_+(x,t))| \geq \dot{V}(t)\phi_2'(\xi_+(x,t)) + \dot{v}(t) - (f_1'(1) + \omega)\dot{v}(t) - C_2C_0e^{-\lambda\omega}x^{e_2}x^{\lambda_2}(L-e_2T) \]

Particularly, choose \( C_3 \) such that

\[ \dot{v}(t) = -\omega v(t) + C_2C_0e^{-\lambda\omega}x^{e_2}x^{\lambda_2}(L-e_2T), \]

which means

\[ C_3 = \frac{C_2C_0e^{\lambda_2e_2}(L-e_2T)}{\lambda_1e_2 - \omega}. \]

As a consequence,

\[ L\Pi(x,t) \geq -(f_1'(1) + 2\omega)v(t) \geq 0. \]

In addition, if \( \xi_+ > X \) and \( x < -L \), then

\[ L\Pi(x,t) \geq -\omega v(t) - \omega v(t) - (w - \lambda_1e_2)C_3e^{e_2}x^{\lambda_2}x^{e_2} - (f_2'(1) + \omega)v(t) \geq 0. \]

**Case 2.** If \( \xi_+ < -X \), then \( x < -L \). Similar to Case 1 with \( x < -L \), we can obtain \( L\Pi(x,t) \geq 0 \).

**Case 3.** For \( \xi_+ \in [-X,X] \), we know \( x < -L \). Therefore,

\[ L\Pi(x,t) \geq \dot{V}(t)\phi_2'(\xi_+(x,t)) + \dot{v}(t) - \|f_2'\|_\infty v(t) \geq \dot{V}(t)\tau_0 - \omega v(t) - \|f_2'\|_\infty v(t) \geq (K\Pi \tau_0 - \omega - \|f_2'\|_\infty)v(t), \]

where \( \tau_0 \) is defined as in Lemma 4.2. It follows from

\[ K\Pi \geq \frac{\omega + \|f_2'\|_\infty}{\tau_0} \]

that \( L\Pi(x,t) \geq 0 \). This finishes the proof by letting \( \beta = \beta^+ + \epsilon_1 + C_3 \) and \( C = C_3 + \epsilon_1 \).

Now we are ready to show that the entire solution converges to a shift of \( \phi_2(x + c_2t) \) using a “squeezing” technique [7]. We should declare that assumption (A) (see (B2) in [7]) holds.

(A) There exists a positive nonincreasing function \( \zeta(n) \) defined on \([1, +\infty)\) such that for any \( u_1(x,t), u_2(x,t) \) satisfying \(-1 \leq u_1(x,t), u_2(x,t) \leq 2, L u_1(x,t) \geq 0, Lu_2(x,t) \leq 0 \) and \( u_1(x,0) \geq u_2(x,0) \), there holds

\[ \min_{x \in [\pm n]} \{u_1(x,1) - u_2(x,1)\} \geq \zeta(n) \int_0^1 [u_1(y,0) - u_2(y,0)]dy \text{ for } m \geq 1. \]
In fact, it follows from the comparison principle that $u(x, t) \geq v(x, t)$ for $x \in \mathbb{R}$ and $t \geq 0$. Furthermore, denote $\tilde{w}(x, t) = e^{KT}[u(x, t) - v(x, t)] \geq 0$ with $K \geq 1 + \max_{x \in [-1, 2]} |f_s(x, s)|$. Then

$$\tilde{w}_t(x, t) \geq K\tilde{w}(x, t) + f(x, u(x, t)) - f(x, v(x, t))$$

$$\geq J \ast \tilde{w}(x, t) + [K - (1 + \max_{s \in [-1, 2]} |f_s(x, s)|)]\tilde{w}(x, t)$$

$$\geq J \ast \tilde{w}(x, t),$$

which implies that $\tilde{w}(x, t) \geq \tilde{w}(x, 0)$ for $t \geq 0$. Therefore, $\tilde{w}_t(x, t) \geq J \ast \tilde{w}(x, 0)$ and $\tilde{w}(x, t) \geq J \ast \tilde{w}(x, 0)$. Repeating the same progress on $\{t_0, 2t_0, ..., \bigl[(N-1), Nt_0\bigr]\}$, we have $\tilde{w}(x, Nt_0) \geq t_N J \ast \tilde{w}(x, 0)$ for $N \geq 1$. In addition, since $J(x) \geq 0$ and $\int_{\mathbb{R}} J(y)dy = 1$, there exists $N_0 = N_0(M) \geq 1$ such that by letting $t_*= \frac{T}{N_0}$ we have

$$\tilde{w}(x, T) \geq \left(\frac{T}{N_0}\right)^N J \ast \tilde{w}(x, 0) \geq \left(\frac{T}{N_0}\right)^N c(M) \int_0^1 \tilde{w}(y, 0)dy,$$

where

$$c(M) = \min_{x \in [-M-1, M+1]} J(x) > 0.$$

Thus, there exists a positive function $\eta(x, t) \in C([0, \infty), [0, \infty))$ such that

$$u(x, t) - v(x, t) \geq \eta(x, t) \int_0^1 [u(y, 0) - v(y, 0)]dy$$

for $x \in \mathbb{R}, t > 0,$

which implies the assumption (A) holds.

Now, we start to prove the following lemma, which plays an important role in the proof of Theorem 4.1.

**Lemma 4.5.** Suppose that assumptions (F) and (J) hold. Then there exists a small $\epsilon_0$ such that if for some $\tau \geq 0$, $\xi \in \mathbb{R}$, $\delta \in (0, \frac{\epsilon_0}{2}]$, and $h > 0$, there holds

$$\phi_2(x + c_2\tau + \xi) - \delta \leq u(x, t) \leq \phi_2(x + c_2\tau + \xi + h) + \delta \quad \text{for } x \in \mathbb{R}, \quad (9)$$

then for every $t > \tau + 1$, there exist $\tilde{\xi}(t)$, $\tilde{\delta}(t)$ and $\tilde{h}(t)$ satisfying

$$\tilde{\xi}(t) \in \left[\xi - \delta \frac{\gamma}{C_m} , \xi + h + \delta \frac{\gamma}{C_m}\right],$$

$$\tilde{\delta}(t) \leq e^{-\omega(t-\tau-1)} [\delta + \epsilon_0 \min\{h, 1\}],$$

$$\tilde{h}(t) \leq \tau h - \frac{\gamma \epsilon_0}{C_m} \min\{h, 1\} + 2\delta \frac{\gamma}{C_m},$$

such that (9) holds with $(\tau, \xi, \delta, h)$ replaced by $(t, \tilde{\xi}(t), \tilde{\delta}(t), \tilde{h}(t))$. Here, $C_m := \max\{\delta_-, C\}$ and $\gamma := \max\{Cw, \frac{Kv}{\omega}\}$, where the parameters are defined in Lemmas 4.2 and 4.4.

**Proof of Lemma 4.5.** In view of Lemmas 4.2 and 4.4, it is easy to see that

$$\phi_2(x + c_2t - \beta_-) - \delta_+ e^{-\omega t} \leq u(x, t) \leq \phi_2(x + c_2t + \beta) + Ce^{-\omega t}.$$

Furthermore, as in the proofs of Lemmas 4.2 and 4.4,

$$\phi_2 \left(x + c_2t - \beta^0 + C^ve^{-\omega t} - C^e\right) - \delta_- e^{-\omega t} \leq u(x, t)$$

and

$$\phi_2 \left(x + c_2t + \beta^+ + \frac{K_V}{\omega} - \frac{K_V}{\omega} e^{-\omega t}\right) + Ce^{-\omega t} \geq u(x, t).$$
Denote
\[ \kappa = \max\{-\beta^0 - \gamma\}, \quad h = \beta^+ + \frac{K\nu}{\omega} - \kappa. \]
In addition, by the definition of \( C_m\), \( \gamma \) and letting \( \tilde{u}(x, t) = u(x - \kappa, t)\), we have
\[ \phi_2 \left( x + c_2 t + \gamma e^{-\omega t} \right) - C_m e^{-\omega t} \leq \tilde{u}(x, t) \leq \phi_2 \left( x + c_2 t + h + \gamma e^{-\omega t} \right) + C_m e^{-\omega t}. \]

Let \( \overline{h} = \min\{h, 1\} \) and \( \vartheta_0 = \frac{1}{2} \min \phi'_2(x) \). Then
\[ \int_0^1 [\phi_2(y + \overline{h}) - \phi_2(y)] \, dy \geq 2 \vartheta_0 \overline{h}. \]

Therefore, at least one of the following two inequalities is true
\[ (i) \int_0^1 [\tilde{u}(y, 0) - \phi_2(y)] \, dy \geq \vartheta_0 \overline{h}, \quad (ii) \int_0^1 [\phi_2(y + \overline{h}) - \tilde{u}(y, 0)] \, dy \geq \vartheta_0 \overline{h}. \]

Next, we consider the case (i) since the case (ii) is similar. According to the assumption (A), for \( \zeta = \zeta(M + c_2 + 2) \) with \( M \gg 1 \) such that \( \frac{2}{C_m} \phi'(x) < 1 \) for \( |x| \geq M \), and for every \( x \in [M - c_2 - \gamma, M + c_2 + \gamma] \), there holds
\[ \tilde{u}(x, 1) - \left[ \phi_2 \left( x + \gamma e^{-\omega} \right) - C_m e^{-\omega} \right] \geq \zeta \int_0^1 [\tilde{u}(y, 0) - (\phi_2 - C_m)] \, dy \geq \zeta \vartheta_0 \overline{h}. \]

Now define
\[ \epsilon_0 = \min \left\{ \frac{\sigma}{2} \min \frac{C_m}{\gamma}, \frac{\zeta \vartheta_0 \gamma}{2C_m \phi'_2(x)} \right\}. \]

Accordingly, there exists \( \tilde{\nu} \in (0, 1) \) such that
\[ \phi_2 \left( x + \gamma e^{-\omega} + 2\epsilon_0 \overline{h} \frac{\gamma}{C_m} \right) - \phi_2 \left( x + \gamma e^{-\omega} \right) = \phi'_2 \left( x + \gamma e^{-\omega} + 2\epsilon_0 \overline{h} \frac{\gamma}{C_m} \right) 2\epsilon_0 \overline{h} \frac{\gamma}{C_m} \leq \zeta \vartheta_0 \overline{h} \]
for all \( x \in [-M - c_2 - (\gamma - 1), M + c_2 + (\gamma - 1)] \). Hence,
\[ \tilde{u}(x, 1) \geq \phi_2 \left( x + \gamma e^{-\omega} + 2\epsilon_0 \overline{h} \frac{\gamma}{C_m} \right) - C_m e^{-\omega} \]
for \( x \in [-M - c_2 - (\gamma - 1), M + c_2 + (\gamma - 1)] \).

For \( |x| \geq M + c_2 + (\gamma - 1) \), by the choice of \( M \), we know that
\[ \phi_2 \left( x + \gamma e^{-\omega} \right) \geq \phi_2 \left( x + \gamma e^{-\omega} + 2\epsilon_0 \overline{h} \frac{\gamma}{C_m} \right) - \epsilon_0 \overline{h}. \]

Then, it follows that
\[ u(x, 1) \geq \phi_2 \left( x + \kappa + \gamma e^{-\omega} + 2\epsilon_0 \overline{h} \frac{\gamma}{C_m} \right) - \epsilon_0 \overline{h} - C_m e^{-\omega} \] for \( x \in \mathbb{R} \).

Note that \( \varrho := \delta e^{-\omega} + \epsilon_0 \overline{h} \leq \sigma \), repeat the operation above with 1 replaced by \( 1 + \varrho \), which implies that
\[ u(x, 1 + \varrho) \geq \phi_2 \left( x + c_2 t + \kappa + \gamma e^{-\omega} + 2\epsilon_0 \overline{h} \frac{\gamma}{C_m} + \gamma e^{-\omega t'} \right) - pe^{-\omega t'} \]
\[ \geq \phi_2 \left( x + c_2 (\varrho t + \kappa + \epsilon_0 \gamma \frac{\gamma}{C_m} - \frac{\delta \gamma}{C_m}) - (\delta + \epsilon_0 \overline{h}) e^{-\omega t'} \right). \]
Thus we finish the proof by setting
\[ t = 1 + t', \quad \tilde{\xi}(t) = \frac{\gamma \epsilon_0 \tilde{h}}{C_m}, \quad \tilde{\delta} = (\delta + \epsilon_0 \tilde{h}) e^{-\omega(t-1)} \]
\[ \tilde{h} = \left[ h + \delta \frac{\gamma}{C_m} e^{-\omega t} \right] - \tilde{\xi}(t) = h - \frac{\epsilon_0}{C_m} + \delta \frac{\gamma}{C_m} \left[ 2 - e^{-\omega t} \right]. \]

Now we shall prove Theorem 4.1.

**Proof of Theorem 4.1.** We shall divide the proof in three steps.

**Step 1.** Following from Lemmas 4.2 and 4.4, there exist \( T^*, M^* > 0 \) such that
\[ \phi_2(x + c_2 T^* - M^*) - C_m \leq u(x, T^*) \leq \phi_2(x + c_2 T^* + M^*) + C_m \quad \text{for } x \in \mathbb{R}. \quad (10) \]
Here, \( C_m \) is defined as in the proof of Lemma 4.4. Define
\[ \epsilon^* = \min \left\{ \frac{\sigma}{2}, \frac{\epsilon_0}{4} \right\} \quad \text{and} \quad k_0 = \epsilon_0 \frac{\gamma}{C_m} - 2 \epsilon^* \frac{\gamma}{C_m} \geq \frac{\gamma \epsilon_0}{2C_m} > 0. \]
Meanwhile, fix \( t^* \geq 2 \) such that
\[ e^{-\omega(t^*-1)} \left[ 1 + \frac{\epsilon_0}{\epsilon^*} \right] \leq 1 - k^*. \]
Then, replace \( C_m \) with \( \epsilon^* \) and denote \( M^* \), \( T^* \) by \( \frac{h_0}{2}, T_0 \). Assume that \( h_0 \geq 1 \), otherwise, we directly go to Step 2.

By (10), applying Lemma 4.5 with \( \tau = T_0, \xi = -\frac{h_0}{2}, h = h_0, \delta = \epsilon^*, \tilde{\sigma} \) holds with \( \tau = T_0 + t^* \), some \( \xi \in \left[ -\frac{h_0}{2} - \frac{\gamma}{C_m}, \frac{h_0}{2} + \frac{2 \gamma}{C_m} \right], \delta = \epsilon^* \) and \( h = h_0 - k^* \), by the definition of \( k^*, t^* \), it follows that
\[ \tilde{\delta}(T_0 + t^*) \leq [\epsilon^* + \epsilon_0] e^{-\omega t^*} \leq \epsilon^* \] and \( \tilde{h}(T_0 + t^*) \leq h_0 - \frac{\gamma}{C_m} \epsilon_0 + 2 \epsilon^* \frac{\gamma}{C_m} \leq h_0 - k^*. \]

Repeat the same process, it yields that (9) holds for \( \tau = T_0 + Nt^*, \delta = \epsilon^* \) and \( h = h_0 - Nk^* \) with \( N \) such that \( h_0 - (N - 1)t^* \geq 1 \). Thus, there exists \( T_1 > T_0 \) such that (9) holds for \( \tau = T_1, \delta = \epsilon^* \) and \( h = 1 \) and some \( \xi \in \mathbb{R} \).

**Step 2.** In this step, we use a mathematical induction to show that for every nonnegative integer \( k \), (9) holds for \( \xi = \xi^k \in \mathbb{R} \) and
\[ \tau = T^k := T_1 + kt^*, \delta = \delta^k := (1 - k^*)^k \epsilon^*, h = h_k := (1 - k^*)^k. \]

It is obvious that the assertion holds for \( k = 0 \) by Step 2. Then suppose that the assertion is true for \( k = l \geq 0 \). We show that it is true for \( k = l + 1 \). In fact, as can be seen in Lemma 4.5 with \( \tau = T^l \) and \( t = T^{l+1} \), one can obtain that (9) holds with \((\tau, \xi, \delta, h)\) replaced by \((\tilde{\tau}, \tilde{\xi}, \tilde{\delta}, \tilde{h})\) satisfying
\[ \tilde{\xi} \in \left[ \xi^l - \delta^l \frac{\gamma}{C_m}, \xi^l + \delta^l \frac{\gamma}{C_m} \right], \]
\[ \tilde{\delta} \leq (\delta^l + \epsilon_0 h^l) = [1 - k^l]^l \epsilon^* \left( 1 + \frac{\epsilon_0}{\epsilon^*} \right) e^{-\omega(t^*-1)} \leq (1 - k^l)^{l+1} \epsilon^*, \]
\[ \tilde{h} \leq h^l + \epsilon_0 \frac{\gamma}{C_m} + 2 \delta^l \frac{\gamma}{C_m} = [1 - k^l]^l \left[ 1 - \epsilon_0 \frac{\gamma}{C_m} + 2 \epsilon^* \frac{\gamma}{C_m} \right] = [1 - k^l]^{l+1} \epsilon^* \]
by the definition of $\epsilon^*$, $k^*$ and $t^*$. This means that 9 holds for $\tau = T^{k+1}$, some $\xi = \xi^{k+1} \in \left[ \xi^† - \delta^† \frac{2}{C_m}, \xi^† + \delta^† \frac{2}{C_m} \right]$, $\delta = (1 - k^*)^{1 + \epsilon^*}$ and $h = (1 - k^*)^{1 + \epsilon^*}$. Now we finish the mathematical induction.

**Step 3.** So far, we have known that 9 holds for $(\tau, \xi, \delta, h) = (T^k, \xi^k, \delta^k, h^k)$ for all $k = 0, 1, \ldots$. Furthermore, 9 holds with $\tau \in [T^k, \infty)$, $\delta = \delta^k$, $h = h^k + 2k^k \frac{\gamma}{C_m}$ and $\xi = \xi^k - \delta^k \frac{2}{C_m}$, $k = 0, 1, \ldots$.

Now we define
\[
\delta(t) = \delta^k, \quad \xi(t) = \xi^k - \delta^k \frac{\gamma}{C_m}, \quad h(t) = h^k + 2k^k \frac{\gamma}{C_m}
\]
for $t \in [T^k, T^{k+1}]$, $k = 0, 1, \ldots$. Then,
\[
\phi_2(x + c_2 t + \xi(t)) - \delta(t) \leq u(x, t) \leq \phi_2(x + c_2 t + \xi(t) + h(t)) + \delta(t) \quad \text{for } t \geq T_1, \quad x \in \mathbb{R}.
\]

It follows from the definition of $\delta(t)$ and $h(t)$ that
\[
\begin{align*}
\delta(t) &= \delta^k = (1 - k^*)^{1 + \epsilon^*} \leq \epsilon^* \exp \left[ \left( \frac{t - T_1}{t^*} - 1 \right) \ln(1 - k^*) \right] \quad \text{for } t \geq T_1, \\
h(t) &= h^k + 2k^k \frac{\gamma}{C_m} \leq \left[ 1 + 2\epsilon^* \frac{\gamma}{C_m} \right] \exp \left[ \left( \frac{t - T_1}{t^*} - 1 \right) \ln(1 - k^*) \right] \quad \text{for } t \geq T_1.
\end{align*}
\]

Moreover, since for any $t \geq \tau \geq T_1$,
\[
\xi(t) \in \left[ \xi(\tau) - \delta(\tau) \frac{\gamma}{C_m}, \xi(\tau) + h(\tau) + \delta(\tau) \frac{\gamma}{C_m} \right],
\]
there holds
\[
|\xi(t) - \xi(\tau)| \leq h(\tau) + 2\delta(\tau) \frac{\gamma}{C_m},
\]
which implies that $\xi(\infty) := \lim_{t \to +\infty} \xi(t)$ exists and
\[
|\xi(\infty) - \xi(\tau)| \leq h(\tau) + 2\delta(\tau) \frac{\gamma}{C_m} \leq \left[ 1 + 4\epsilon^* \frac{\gamma}{C_m} \right] e^{\left( \frac{t - T_1}{t^*} - 1 \right) \ln(1 - k^*)} \quad \text{for } t \geq T_1.
\]

Therefore, we have that
\[
|u(x, t) - \phi_2(x + c_2 t + \xi(\infty))| \to 0 \quad \text{as } t \to +\infty.
\]

Furthermore, the convergence is exponential. Then we complete the proof.

5. **Lyapunov stability of the entire solution.** We investigate the Lyapunov stability of the entire solution obtained in Theorem 1.1 in this section. That is, the aim here is to prove Theorem 1.2. The following lemma plays an important role in proving Theorem 1.2.

**Lemma 5.1.** Let $u(x, t)$ be the unique entire solution in Theorem 1.1. Then for any $\varphi \in (0, \frac{1}{2}]$, there exist constants $T_\varphi = T_\varphi(\varphi) > 1$ and $K_\varphi = K_\varphi(\varphi) > 0$ such that
\[
u_\varphi(x, t) \geq K_\varphi \quad \text{for any } t \geq T_\varphi \text{ and } x \in \Omega_\varphi(t),
\]
where
\[
\Omega_\varphi(t) = \{ x \in \mathbb{R} : \varphi \leq u(x, t) \leq 1 - \varphi \}.
\]
In view of that for all \( \tau \), we obtain \( u_k(t_k, x_k) \to 0 \) as \( k \to +\infty \).

Now suppose there exist sequences \( t_k \in [T_\varphi, +\infty) \) and \( x_k \in \Omega_\varphi(t) \) such that
\[
\frac{\partial u_k(t_k, x_k)}{\partial t} = 0
\]
for some function \( u \) satisfying 1 with \( \frac{\partial u_k(t_k, x_k)}{\partial t} = 0 \) and \( (u_k(x, t))_t \geq 0 \). By the comparison principle, we further have
\[
\frac{\partial u_k(x, t)}{\partial t} = 0 \quad \text{for } t \geq 0.
\]
However, this is impossible because by Theorem 4.1
\[
u_k(x, t) = \phi_2(x + c_t \beta + a) \quad \text{for some } a \in [-M, M].
\]

For the second case, \( x_k \) remains bounded by the definition of \( \Omega_\varphi(t) \). Therefore, we assume that \( x_k \to x^* \) as \( k \to +\infty \) and let
\[
u_k(x, t) := u(x + x_k, t).
\]
Then, each \( v_k(x, t) \) is defined for all \( (x, t) \in \{x \in \mathbb{R} \mid x \leq -1\} \times [T_\varphi, +\infty) \) by the definition of \( \Omega_\varphi(t) \). Similarly, there exists a subsequence, again denoted by \( \{v_k\}_k \), such that
\[
u_k \to \nu^* \quad \text{as } k \to +\infty
\]
for some function \( \nu^* \) satisfying 1 with \( \frac{\partial \nu^*(t_\varphi)}{\partial t} = 0 \), \( (\nu^*(x, t))_t \geq 0 \). Then by the comparison principle, we obtain \( \frac{\partial \nu^*(x, t)}{\partial t} = 0 \) for \( t \geq t_\varphi \), but this is impossible since by Theorem 4.1,
\[
u^*(x, t) - \phi_2(x + \beta + x^* + c_t) \to 0 \quad \text{as } t \to +\infty.
\]
This ends the proof. \( \square \)

**Proof of Theorem 1.2.** We first define a pair of sub- and super-solutions as follows.
\[
U^\pm(x, t) = u(x + a, t + t_0 \pm \delta \omega(1 - e^{-\omega t})) \pm \omega e^{-\omega t},
\]
where \( \omega \) is defined as in Section 4 and \( t_0 \geq T_\varphi \) and \( \delta, \omega > 0 \) are constants. Besides, we claim that \( U^\pm \) are super- and sub-solutions of 1. Then, by the comparison principle, we have
\[
U^-(x, t) \leq v(x, t) \leq U^+(x, t) \quad \text{for } x \in \mathbb{R}, \quad t \geq 0.
\]
In view of that for all \( |\tau| \leq \omega \) with \( \omega =: \omega(\epsilon) = \frac{\omega}{2M} \) and \( M = 2 + \max_{u \in [0, 1]} (f_L(x, u))_u \), we have
\[
\sup_{x \in \mathbb{R}, \quad t \in \mathbb{R}} |u(x, t) - u(x, t + \tau)| \leq \sup_{x \in \mathbb{R}, \quad t \in \mathbb{R}} |u(t, t)\omega| \leq \frac{\epsilon}{2}.
\]
It then follows from 11 and 12 that
\[
|v(x, t, v_0) - u(x + a, t + t_0)| < \epsilon
\]
Now we prove the claim. We show \( U^+(x,t) \) is a super-solution of \( 1 \) for \( t \geq 0 \). Then it can be similarly shown that \( U^-(x,t) \) is a sub-solution of \( 1 \) for \( t \geq 0 \). Since it is easy to see that \( U^+(x,0) \geq v_0(x) \) for \( x \in \mathbb{R} \) and \( \delta < \frac{\omega}{2} \), we only need to show that
\[
LU^+(x,t) = U^+_t(x,t) - (J * U^+(x,t) - U^+(x,t)) - f(x,U^+)
\geq 0.
\]
We go further to show \( LU^+(x,t) \geq 0 \) in two cases.

**Case 1.** For any \( u = u(x+a,t+\gamma + \tilde{\delta} \omega (1 - e^{-\omega t})) \in [0,\sigma] \cup [1-\sigma,1] \) with \( \sigma \) defined as in Section 4,
\[
LU^+(x,t) \geq - \min \left\{ \frac{|f_u(x,0)|}{2}, \frac{|(f_u(x,1))|}{2} \right\} \omega e^{-\omega t} - \omega \omega e^{-\omega t}
\geq \omega e^{-\omega t} (2\omega - \omega)
\geq 0.
\]

Let \( \tilde{\delta} \) be sufficiently large such that
\[
\tilde{\delta} \geq \frac{\omega K_\sigma}{\frac{\omega}{2} + ||f_u(x,u)||_\infty}.
\]
Thus, we have \( LU^+(x,t) \geq 0 \). Now, we obtain the claim. The proof of Theorem 1.2 is completed.

**Remark 1.** In this paper, we have considered the existence, uniqueness, asymptotic behavior and Lyapunov stability of entire solutions of the nonlocal dispersal equation \( 1 \) under the assumption \( J(x) = J(-x), x \in \mathbb{R} \). It is well-known from [8, 22, 28, 29] that the asymmetry of \( J \) has a great influence on the profile of the traveling waves and the sign of the wave speeds, which further makes the properties of the entire solution more diverse. Naturally, an interesting problem is to consider entire solutions of \( 1 \) under asymmetric conditions.

**Remark 2.** The method used here can be also applied to consider the bistable lattice differential equations with \( f_i \) satisfying \( (F) \)
\[
\dot{u}_i(t) = u_{i+1}(t) + u_{i-1}(t) - 2u_i(t) + f_i(u_i(t)), \quad i \in \mathbb{Z}, \quad t \in \mathbb{R}.
\]
(13) The existence, uniqueness, asymptotic behavior and the Lyapunov stability of entire solutions to (13) can be similarly obtained.

While for bistable random diffusion equations with \( f(x,u) \) satisfying \( (F) \)
\[
u_t = \nu_{xx} + f(x,u), \quad x \in \mathbb{R}, \quad t \in \mathbb{R},
\]
the construction of the entire solution behaving as the traveling wave pertaining to \( f_1 \) coming from infinity is similar. In fact, since regularity of the entire solution is
known, the Lyapunov function method can be used to show the approach of the entire solution to the traveling wave pertaining to $f_2$, one can refer to [10, 11].

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