THE S-PROCEDURE VIA DUAL CONE CALCULUS

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Abstract. Given a quadratic function $h$ that satisfies a Slater condition, Yakubovich’s S-Procedure (or S-Lemma) gives a characterization of all other quadratic functions that are copositive with $h$ in a form that is amenable to numerical computations. In this paper we present a deep-rooted connection between the S-Procedure and the dual cone calculus formula $(K_1 \cap K_2)^* = K_1^* + K_2^*$, which holds for closed convex cones in $\mathbb{R}^2$. To establish the link with the S-Procedure, we generalize the dual cone calculus formula to a situation where $K_1$ is nonclosed, nonconvex and nonconic but exhibits sufficient mathematical resemblance to a closed convex cone. As a result, we obtain a new proof of the S-Lemma and an extension to Hilbert space kernels.

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1. Introduction. Yakubovich’s S-Lemma [9], also called S-Procedure, is a well-known result from robust control theory that characterizes all quadratic functions that are copositive with a given other quadratic function. A function $g$ is called copositive with $h$ if $h(x) \geq 0$ implies $g(x) \geq 0$.

Theorem 1.1 (S-Lemma, [9]). Let $g,h : \mathbb{R}^n \to \mathbb{R}$ be quadratic functions such that $h(x_0) > 0$ at some point $x_0 \in \mathbb{R}^n$. Then $g$ is copositive with $h$ if and only if there exists $\xi \geq 0$ such that $g(x) - \xi h(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Note that $g$ and $h$ are neither assumed to be convex nor homogeneous, and that the condition $g(x) - \xi h(x) \geq 0$ for all $x \in \mathbb{R}^n$ is easy to check, for a quadratic function $x \mapsto x^T Q x + 2 \ell^T x + c$ can always be formulated so that the matrix $Q$ is symmetric, and then the function is nonnegative everywhere on $\mathbb{R}^n$ if and only if the matrix $[Q \; \ell]$ is positive semidefinite. The importance of this characterization is that it can be checked numerically.

Theorem 1.1 arose as a generalization of earlier results by Finsler [4], Hestenes & McShane [5] and Dines [3]. Megretsky & Treil [6] later extended the result further. The S-Lemma has surprisingly powerful consequences in robust optimization and control theory, as this result allows to replace certain nonconvex optimization problems by convex polynomial time solvable ones, and semi-infinite programming problems by optimization models with finitely many constraints. Indeed, Theorem 1.1 says that in an optimization problem in which the coefficients $Q, \ell, c$ of the polynomial $g$ play the role of decision variables, the infinitely many constraints

$$g(x) \geq 0, \quad \forall x \in \mathbb{R}^n \text{ s.t. } h(x) \geq 0$$

can be replaced by a single matrix inequality

$$
\begin{bmatrix}
Q & \ell \\
\ell^T & c
\end{bmatrix} - \xi 
\begin{bmatrix}
A & b \\
b^T & d
\end{bmatrix} \succeq 0,
$$

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where $A, b, d$ are chosen such that $h(x) = x^T Ax + 2b^T x + d$, and where $\xi$ is an auxiliary decision variable introduced by this lifting.

For an overview of the history of the S-Lemma and its applications, see [7] and [2]. Three existing known approaches to proving Theorem 1.1 described in [7] are due to Yakubovich [9], Ben-Tal & Nemirovski [1] and Sturm & Zhang [8], and Yuan [10].

In this paper we give a new proof of the S-Lemma that is based on a generalization of the dual cone calculus formula $(K_1 \cap K_2)^* = K_1^* + K_2^*$, which is known to hold true for closed convex cones $K_1, K_2 \subseteq \mathbb{R}^2$, to a situation where $K_1$ is nonclosed, nonconvex and nonconic but exhibits sufficient mathematical resemblance to a closed convex cone. For this purpose we introduce a weak notion of convexity, homogenization-convexity, the theory of which will be developed in Section 2. Our proof extends quite straightforwardly to an S-Lemma for Hilbert space kernels. The techniques we employ are elementary. The main ideas of the proof merely require linear algebra in two dimensions. The S-Lemma and its extension to Hilbert space kernels are then obtained by a lifting.

Among the existing proofs of the S-Lemma, Yakubovich’s original proof is closest in spirit to the proof presented in this paper. Yakubovich employed a result of Dines [3], which shows that the joint range $\{(f(x), g(x)) : x \in \mathbb{R}^n\}$ of two homogeneous quadratic functions $f, g$ on $\mathbb{R}^n$ is convex. Our own approach is based on showing that the projection of the set $\{(x_1 x_1^T : x \in \mathbb{R}^n)\}$ into a 2-dimensional subspace satisfies the weaker notion of homogenization-convexity. Once this is established, the S-Lemma follows from our generalized dual cone calculus formula.

1.1. Notation. The inner product on any Hilbert space $V$ is denoted by $\langle \cdot, \cdot \rangle$. This inner product defines the canonical self-duality isomorphism on $V$ and the canonical norm $\|\cdot\|$. The topological closure and boundary of a set $C \subseteq V$ under the induced topology are denoted by $\text{clo}[C]$ and $\partial C$. The convex, conic and homogeneous hulls of $C$ are denoted by

$\text{conv}(C) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N}, \lambda_i \geq 0, x_i \in C, \forall i, \sum_{i=1}^{n} \lambda_i = 1 \right\},$

$\text{cone}(C) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N}, \lambda_i \geq 0, x_i \in C, \forall i \right\},$

$\text{hom}(C) := \{\tau x : \tau \geq 0, x \in C\}.$

The relation between these three concepts is that $\text{cone}(C) = \text{hom}(\text{conv}(C))$.

**Definition 1.2.** For any $C \subseteq V$ we refer to the set $\text{clo}[\text{hom}(C)]$ as the homogenization of $C$.

We denote the unit sphere in $(V, \langle \cdot, \cdot \rangle)$ by $S(V)$ and the spherical projection by $q : V \setminus \{0\} \to S(V)$

$$q(x) = \frac{x}{\|x\|}.$$
Note that the spherical projection is not defined at the origin of $V$. Nonetheless, by abuse of language, if $C \subseteq V$ we write $q(C)$ for $q(C \setminus \{0\})$. The set of recession directions of a set $C \subseteq V$ is given by

$$\text{rec}(C) := \{ s \in S(V) : \forall \tau, \epsilon > 0, \exists x \in C \text{ s.t. } \| s - q(x) \| < \epsilon, \| x \| > \tau \}. $$

For any $x_1, x_2 \in V \setminus \{0\}$ we write

$$[x_1, x_2] = \{ \xi x_2 + (1 - \xi) x_1 : \xi \in [0, 1] \}$$

for the straight-line segment between $x_1$ and $x_2$. For $y_1, y_2 \in S(V)$, we write $[y_1, y_2] := q([x_1, x_2])$, where $x_1 \in q^{-1}(y_1)$ and $x_2 \in q^{-1}(y_2)$. It is easy to check that the definition of $[y_1, y_2]$ does not depend on the specific choice of $x_1$ and $x_2$. A subset $S \subseteq S(V)$ is spherically convex if $[y_1, y_2] \subseteq S$ for all $y_1, y_2 \in S$.

2. **Homogenization-Convexity.** Our approach to the S-Lemma hinges on a weak notion of convexity that we shall now define.

**Definition 2.1.** A set $C \subseteq \mathbb{R}^2$ is homogenization-convex if the homogenization $\text{clo}[\text{hom}(C)]$ of $C$ is a convex subset of $\mathbb{R}^2$.

A few alternative characterizations provide further insight:

**Lemma 2.2.** The following conditions on a set $C \subseteq \mathbb{R}^2$ are equivalent:

i) $C$ is homogenization-convex.

ii) $\text{clo}[q(C)]$ is spherically convex in $S(V)$

iii) $\text{clo}[\text{hom}(C)] = \text{clo}[\text{cone}(C)]$.

iv) $q([x_1, x_2]) \subseteq \text{clo}[q(C)]$ for all $x_1, x_2 \in C$.

**Proof.** i)$\Leftrightarrow$ ii)$\Leftrightarrow$ iii) follow immediately from $q^{-1}(\text{clo}[q(C)]) = \text{clo}[\text{hom}(C)] \setminus \{0\}$ and from the characterization of cone($C$) as the smallest convex set $K$ such that $C \subseteq K$ and hom($K$) = $K$. ii)$\Rightarrow$ iv) follows from the definition of spherical convexity. iv)$\Rightarrow$ ii): Let $y_1, y_2 \in \text{clo}[q(C)]$ and $x_i \in q^{-1}(y_i)$ ($i = 1, 2$). If $x_1 \sim \pm x_2$, then $[y_1, y_2] = \{ y_1, y_2 \} \subseteq \text{clo}[q(C)]$. Otherwise, $x_1$ and $x_2$ are linearly independent, and for all $\lambda \in [0, 1]$, $q(\lambda y_1 + (1 - \lambda) y_2) = \lim_{n \to \infty} q(\lambda x_1^n + (1 - \lambda) x_2^n) \in \text{clo}[q(C)]$ for some sequences $(x_i^n)_n \subseteq C$ for which $q(x_i^n) \to y_i$, ($i = 1, 2$).

It follows from Lemma 2.2 iii) that if $C$ is convex then $C$ is homogenization-convex. The examples of Figure 2.1 illustrate that the reverse relationship is not true. See also Figure 2.2 for examples of sets that are not homogenization-convex. The following example is relevant to our proof of the S-Lemma:
Example 2.3. Let \( x(t) = a_0 + ta_1 + t^2a_2 \) and \( y(t) = b_0 + tb_1 + t^2b_2 \) for some real coefficients \( a_1, b_1 \) \((i = 0, 1, 2)\). Then \( C := \{(x(t), y(t)) : t \in \mathbb{R}\} \) is homogenization-convex.

Proof. When \((a_2, a_1)\) and \((b_2, b_1)\) are linearly dependent, then there exist \( \eta_1, \eta_2 \in \mathbb{R} \), not both zero, such that \( \eta_1a_1 + \eta_2b_1 = 0 \) \((i = 1, 2)\), and then \( C \) is a subset of the line \( \{z \in \mathbb{R}^2 : \eta_1z_1 + \eta_2z_2 = \eta_1a_0 + \eta_2b_0\}\). Since \( C \) is connected by arcs, it must be an interval, hence convex. This implies that \( C \) is homogenization-convex. In the case where \((a_2, a_1)\) and \((b_2, b_1)\) are linearly independent, there exist \( \xi_1, \xi_2 \in \mathbb{R} \) such that \( \xi_1(a_2, a_1) + \xi_2(b_2, b_1) = (0, 1) \), so that \( \xi_1x(t) + \xi_2y(t) = t + c \) for some \( c \in \mathbb{R} \). Furthermore, we may assume without loss of generality that \( b_2 \neq 0 \). The set of loci \( \{(x, y) : x = x(t), y = y(t), t \in \mathbb{R}\} \) is then characterised by the equation

\[
y = b_2(\xi_1x + \xi_2y - c)^2 + b_1(\xi_1x + \xi_2y - c) + b_0.
\]

This is the general equation of a parabola. Hence, \( C = \partial K \), where \( K \) is the set of points enclosed by the parabola. \( K \) being a convex set with unique recession direction \((a_2, b_2)\), the homogenization-convexity of \( C \) is a special case of Example 2.5 below.

Example 2.4. Let \( C = \partial K \) where \( K \) is a closed convex subset of \( \mathbb{R}^2 \) with complement \( K^c = \mathbb{R}^2 \setminus K \) and such that \( 0 \in \text{int}[K^c] \). Then \( C \) is homogenization-convex.

Proof. Consider the map

\[
\sigma : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\},
\]

\[
v \mapsto \inf \{\tau \geq 0 : \tau v \in K\},
\]

defined for all \( v \in K \), where \( \inf \emptyset := +\infty \) as usual. Choose arbitrary points \( x_1, x_2 \in C \). If \( x_1, x_2 \) are linearly dependent, then \( q([x_1, x_2]) \subseteq \{q(x_1), q(x_2)\} \subseteq \text{clo}[q(C)] \). Else \( x_1, x_2 \) are linearly independent, and for any point \( x \in [x_1, x_2] \), we have \( x \neq 0 \), so that \( q(x) \) is well defined. Since \( x \in K \), we have \( \sigma(x) \leq 1 \), and since \( 0 \in \text{int}[K^c], \sigma(x) > 0 \). Furthermore, \( \sigma(x)x \in \partial K = C \), so that \( x = \sigma(x)^{-1}(\sigma(x)x) \in \text{hom}(C) \). Since \( x \) was chosen arbitrarily, this shows that \( q([x_1, x_2]) \subseteq \text{clo}[q(C)] \), and the claim follows from Lemma 2.2 iv).

Example 2.5. Let \( C = \partial K \) where \( K \) is a closed convex subset of \( \mathbb{R}^2 \) with at most one recession direction. Then \( C \) is homogenization-convex.
Proof. We may assume without loss of generality that $0 \in K$, for otherwise our claim is true by virtue of Example 2.4. Consider the map

$$\varsigma(v) := \sup\{\tau \geq 0 : \tau v \in K\},$$

defined for all $v \in K$. Then $\varsigma(\cdot)$ takes finite values on $K \setminus (q^{-1}(\text{rec}(K)) \cup \{0\})$. Since $\varsigma(v)v \in \partial K = C$ when $\varsigma(v)$ is finite, it follows that

$$\text{hom}(K) \setminus q^{-1}(\text{rec}(K)) \subseteq \text{hom}(C) \subseteq \text{hom}(K). \quad (2.1)$$

By assumption, $\text{rec}(K)$ is either empty or a singleton. If $\dim(K) = 1$, then $C = K$. Otherwise, taking closures in (2.1) reveals that $\overline{\text{hom}(K)} = \overline{\text{hom}(C)}$, and by convexity of $K$, $\text{cone}(K) \subseteq \text{hom}(K)$. Therefore,

$$\overline{\text{hom}(C)} = \overline{\text{cone}(C)} \supseteq \overline{\text{cone}(K)} \supseteq \overline{\text{hom}(C)},$$

and the claim follows from Lemma 2.2 iii). $\Box$

2.1. Dual Cone Calculus. Any subset $C \subseteq \mathbb{R}^n$ is associated with a dual cone $C^* = \{y \in \mathbb{R}^n : \langle x, y \rangle \geq 0, \forall x \in C\}$. When $K_1, K_2$ are closed polyhedral cones, then the dual cone formula

$$(K_1 \cap K_2)^* = K_1^* + K_2^* \quad (2.2)$$

applies. In particular, this formula holds true for all closed cones $K_1, K_2 \subseteq \mathbb{R}^2$, since all cones in $\mathbb{R}^2$ are polyhedral. The following property of dual cones is also well known,

$$C^* = (\overline{\text{cone}(C)})^*, \quad (2.3)$$

$$C = \overline{\text{cone}(C \cap K)} \supseteq \overline{\text{cone}(C)} \supseteq \overline{\text{cone}(K)} \supseteq \overline{\text{cone}(C \cap K)}. \quad (2.5)$$

In this section we set out to generalizing the relation (2.2) to the case where $K_1$ is merely a homogenization-convex set and $K_2$ is a closed convex cone with nonempty interior.

**Lemma 2.6.** Let $C \subseteq \mathbb{R}^2$ be homogenization-convex and $K \subseteq \mathbb{R}^2$ a closed convex cone with nonempty interior. Then

$$\overline{\text{cone}(C \cap K)} = \overline{\text{cone}(C)} \cap K. \quad (2.5)$$

**Proof.** We only need to prove the inclusion $\supseteq$, since the reverse relation is trivial. Let $x \in \text{cone}(C \cap K) \setminus \{0\}$. Then there exist $x_1, x_2 \in C$ and $\lambda_1, \lambda_2 \geq 0$ such that $x = \lambda_1 x_1 + \lambda_2 x_2$. If either $\lambda_1$ or $\lambda_2$ is zero or if $x_1, x_2 \in K$, then it is trivially true that $x \in \text{cone}(C \cap K)$. Furthermore, if $x_1, x_2$ are linearly dependent, then $x = \tau x_i$ for some $\tau > 0$ and $i \in \{1, 2\}$, and by homogeneity of $K$, $x_i \in C \cap K$ and $x \in \text{cone}(C \cap K)$. We may therefore assume that $x_1, x_2$ are linearly independent, $\lambda_1, \lambda_2 > 0$, and that $x_1 \notin K$.

Like all closed convex cones in $\mathbb{R}^2$, $K$ is of the form $K = \{x : \phi_i(x) \geq 0, \phi_2(x) \geq 0\}$ for some linear forms $\phi_i : \mathbb{R}^2 \to \mathbb{R}$, ($i = 1, 2$). We may furthermore assume that both are nonzero, as the case $\phi_1 = 0 = \phi_2$ is trivial, and the case $\phi_1 \neq 0 = \phi_2$ follows from a simplification of the argument we are about to give. Without loss of generality,
we may assume that $\phi_1(x_1) < 0$. Since $0 \leq \phi_1(x) = \lambda_1 \phi_1(x_1) + \lambda_2 \phi_1(x_2)$, we then have $\phi_1(x_2) > 0$.

We first treat the case $\phi_2(x_1) \geq 0$. The linear independence of $x_1$ and $x_2$ implies that $y_1 := \xi x_1 + (1 - \xi)x_2 \neq 0$, where $\xi = \phi_1(x_2)/(\phi_1(x_2) - \phi_1(x_1)) \in (0, 1)$. By construction, $\phi_1(y_1) = 0$. The homogenization-convexity of $C$ further implies $q(y_1) \in [q(x_1), q(x_2)] \subseteq \text{clo}[q(C)]$. This shows the existence of a sequence $(y_n^n)_n \in \mathbb{N} \subseteq C$ such that $\phi_1(y_n^n) > 0$ and $q(y_n^n) \to q(y_1)$. Defining $\rho := \lambda_1/(\lambda_1 + \lambda_2)$ and $z := \rho x_1 + (1 - \rho)x_2$, we have $x = (\lambda_1 + \lambda_2)z$ and $\phi_1(z) = (\lambda_1 + \lambda_2)^{-1}\phi_1(x) \geq 0$. Since $\phi_1(y_1) = 0$, it must be the case that $\rho \leq \xi$, so that $\eta := \rho/\xi \in (0, 1]$, and furthermore, $z = \eta y_1 + (1 - \eta)x_2$. Since $\phi_2(x_1), \phi_2(z) \geq 0$ and $y_1 \in [x_1, z]$, we also have $\phi_2(y_1) \geq 0$, so that $y_1 \in K$. Since $K$ has nonempty interior and $y_1 \neq 0$, we have $y_1^n \in C \cap K$ for all $n \gg 1$, and without loss of generality, we may assume that this holds for all $n \in \mathbb{N}$. Next, if $\phi_2(x_2) \geq 0$, set $y_2 = x_2$ and $y_2^n = x_2$ for all $n \in \mathbb{N}$. Otherwise, interchanging the roles of $x_1$ and $x_2$ and of $\phi_1$ and $\phi_2$, a repeat of the above construction yields the existence of a point $y_2 \in K \setminus \{0\}$ and of a sequence $(y_2^n)_n \subseteq C \cap K$ such that $q(y_2^n) \to q(y_2)$ and $z \in [y_1, y_2]$. This shows

$$
\begin{align*}
x &= (\lambda_1 + \lambda_2)z \in \text{cone}(\{y_1, y_2\}) \\
&\subseteq \text{clo}[\text{cone}(\{y_i^n : n \in \mathbb{N}, i = 1, 2\})] \\
&\subseteq \text{clo}[\text{cone}(C \cap K)].
\end{align*}
$$

It remains to treat the case $\phi_2(x_1) < 0$. In this case, $x \in K$ implies $x_2 \in K$. The above construction can then be repeated using the point $x_1$ for both $\phi_1$ and $\phi_2$, revealing the existence of points $y_i \neq 0$ such that $\phi_1(y_i) = 0$ and $z \in [y_i, x_2]$, ($i = 1, 2$). Without loss of generality, we may assume that $y_2 \in [y_1, x_2]$, whence $y_2 \in K$ and there exists a sequence $(y_2^n)_n \subseteq C \cap K$ such that $q(y_2^n) \to q(y_2)$. We therefore have

$$
\begin{align*}
x &= (\lambda_1 + \lambda_2)z \in \text{cone}(\{y_2, x_2\}) \\
&\subseteq \text{clo}[\text{cone}(\{y_i^n : n \in \mathbb{N} \cup \{x_2\}\})] \\
&\subseteq \text{clo}[\text{cone}(C \cap K)].
\end{align*}
$$

In summary, we have established that $\text{clo}[\text{cone}(C \cap K)] \supseteq \text{cone}(C \cap K) \setminus \{0\}$. Our claim now follows by taking closures on both sides of this inclusion. \( \square \)

We are now ready to state and prove the main result of this paper, for the purpose of which we are going to make the following regularity assumption,

$$
\text{clo}[\text{cone}(C \cap K)] = \text{clo}[\text{cone}(C)] \cap K. \tag{2.6}
$$

**Theorem 2.7.** Let $C \subseteq \mathbb{R}^2$ be homogenization-convex and $K \subseteq \mathbb{R}^2$ a closed convex cone such that the regularity assumption (2.6) holds. Then

$$(C \cap K)^* = C^* + K^*.$$
Proof. Using Lemma 2.6 and the classical dual cone calculus formulas, we find
\[(C \cap K)^* \overset{(2.3)}{=} (\text{clo} \text{cone} (C \cap K))^* \]
\[\overset{(2.5)}{=} (\text{clo} \text{cone}(C) \cap K)^* \]
\[\overset{(2.6)}{=} (\text{clo} \text{cone}(C)) \cap K)^* \]
\[\overset{(2.7)}{=} (\text{clo} \text{cone}(C))^* + K^* \]
\[\overset{(2.3)}{=} C^* + K^*. \]

Next, let us give a sufficient criterion that is easier to check than Condition (2.6).

**Lemma 2.8.** Let \( C \subseteq \mathbb{R}^2 \) and \( K \subseteq \mathbb{R}^2 \) a convex cone. If \( C \cap \text{int}[K] \neq \emptyset \), then Condition (2.6) holds.

**Proof.** The proof works in arbitrary normed vector spaces \( V \). We only need to prove that the inclusion \( \supseteq \) holds in (2.6), the reverse relation being trivial. Let \( x_0 \in C \cap \text{int}[K] \), and let \( (x_n)_{n \in \mathbb{N}} \subset \text{cone}(C) \) be a sequence such that \( x_n \to x \in K \). Then for every \( \varepsilon > 0 \) we have \( x_n + \varepsilon x_0 \in \text{cone}(C) \cap K \) for all \( n \) large enough. Therefore, \( x + \varepsilon x_0 \in \text{clo}[\text{cone}(C) \cap K] \). This being true for all \( \varepsilon > 0 \), we have \( x \in \text{clo}[\text{cone}(C) \cap K] \), as claimed.

**Corollary 2.9.** \( C \subseteq \mathbb{R}^2 \) be homogenization-convex, and let \( \psi, \phi : \mathbb{R}^2 \to \mathbb{R} \) be linear forms, with \( \phi \) chosen such that there exists \( x_0 \in C \) where \( \phi(x_0) > 0 \). Then the following conditions are equivalent,

i) \( \psi(x) \geq 0 \) for all \( x \in C \) such that \( \phi(x) \geq 0 \),

ii) there exists \( \xi \geq 0 \) such that \( \psi(x) - \xi \phi(x) \geq 0 \) for all \( x \in C \).

**Proof.** This is a special case of Theorem 2.7 with \( K = \{ x : \phi(x) \geq 0 \} \) and where the sufficient criterion of Lemma 2.8 applies.

Next, we lift Corollary 2.9 into arbitrary real Hilbert spaces, resulting in the following result.

**Theorem 2.10.** Let \( (V, \langle \cdot, \cdot \rangle) \) be a real Hilbert space, \( \psi, \phi : V \to \mathbb{R} \) continuous linear forms, \( W := (\ker(\phi) \cap \ker(\psi))^\perp \) and \( \pi_W \) the orthogonal projection of \( V \) onto \( W \) along \( \ker(\phi) \cap \ker(\psi) \). Let \( C \) be a subset of \( V \) such that \( \phi(x_0) > 0 \) for some \( x_0 \in C \) and such that \( \pi_W C \) is homogenization-convex in \( W \). Then the following conditions are equivalent:

i) \( \psi(x) \geq 0 \) for all \( x \in C \) such that \( \phi(x) \geq 0 \),

ii) there exists \( \xi \geq 0 \) such that \( \psi(x) - \xi \phi(x) \geq 0 \) for all \( x \in C \).

**Proof.** Applying Corollary 2.9 to \( \phi |_W, \psi |_W \) and \( \pi_W C \) on the two-dimensional subspace \( W \), we find that i) \( \Leftrightarrow \psi |_W (x) \geq 0 \) for all \( x \in \pi_W C \) such that \( \phi |_W (x) \geq 0 \Leftrightarrow \exists \xi \geq 0 \) such that \( \psi |_W (x) - \xi \phi |_W (x) \geq 0 \) for all \( x \in \pi_W C \Leftrightarrow \) ii).}

It is important to understand that Theorem 2.10 is more than just a generalization of Corollary 2.9 to arbitrary real Hilbert spaces, for rather than assuming that \( C \) be homogenization-convex in \( V \) (if the definition is appropriately extended to arbitrary Hilbert spaces), the theorem merely gets away with the weaker assumption that the projected set \( \pi_W C \) be homogenization-convex. This distinction is crucial, as in our proof of the S-Lemma, \( C \) is not homogenization-convex, while \( \pi_W C \) is
homogenization-convex due to the two dimensional nature of $W$. In fact, $\pi_W C$ is in general not homogenization-convex when $\dim(W) \geq 3$, and this is the main reason why the S-Lemma does not hold for quadratic functions copositive with more than one quadratic form.

Note further that if the set $C$ is actually convex (rather than just homogenization-convex), Theorem 2.10 becomes a special case of Farkas’ Theorem, see [11].

2.2. Proof of the S-Lemma. Next, we shall see that, despite its Farkas flavour, Theorem 2.10 in fact a generalisation of the S-Lemma, and (2.6) is a weakening of the standard regularity assumption: denoting the set of real symmetric $n \times n$ matrices by $\mathcal{S}_n$, and combining the tools developed above, we obtain a proof of Theorem 1.1:

Proof. Let $g$ be given by $g(x) = x^TQx + 2 \ell^Tx + c$, where $Q \in \mathcal{S}_n$, $\ell \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then $g(x) = \langle A, [x \ 1]^T[x \ 1] \rangle$, where $\langle A, X \rangle = \text{tr}(A^TX)$ is the trace inner product defined on the space $\mathcal{S}_{n+1}$ of symmetric $(n+1) \times (n+1)$ matrices, and where

$$A = \begin{bmatrix} Q & \ell \\ \ell^T & c \end{bmatrix}.$$ 

Likewise, there exists $B \in \mathcal{S}_{n+1}$ such that $h(x) = \langle B, [x \ 1]^T[x \ 1] \rangle$. Let $C \subset \mathcal{S}_{n+1}$ be defined by $C = \{zz^T : z = [x \ 1]^T, x \in \mathbb{R}^n\}$. Using the notation just introduced, the claim of the theorem is that the following two conditions are equivalent,

(i) $\langle A, X \rangle \geq 0$ for all $X \in C$ such that $\langle B, X \rangle \geq 0$,

(ii) there exists $\xi \geq 0$ such that $\langle A - \xi B, X \rangle \geq 0$ for all $X \in C$.

We note that $\psi : X \mapsto \langle A, X \rangle$ and $\phi : X \mapsto \langle B, X \rangle$ are linear forms on $\mathcal{S}_{n+1}$. Furthermore, if $X_0 = [x_0 \ 1]^T[x_0 \ 1]$, then $X_0 \in C$ and $\phi(X_0) = h(x_0) > 0$. Thus, the equivalence of i) and ii) follows from Theorem 2.10 if it can be established that $\pi_W C$ is homogenization-convex, where $\pi_W$ is the orthogonal projection of $(\mathcal{S}_{n+1}, \langle \cdot, \cdot \rangle)$ onto $W := (\ker(\phi) \cap \ker(\psi))^\perp = \text{span}\{A, B\}$. Let $X_1, X_2 \in C$. Then $X_i = [x_i \ 1]^T[x_i \ 1]$ for some $x_i \in \mathbb{R}^n$, $(i = 1, 2)$. For $t \in \mathbb{R}$, define $x(t) := x_2 - t(x_2 - x_1)$ and $X(t) := [x(t) \ 1]^T[x(t) \ 1] = G_0 + tG_1 + t^2G_2$, where

$$G_0 = \begin{bmatrix} x_1^T \\ x_0^T \end{bmatrix},$$

$$G_1 = -\begin{bmatrix} x_2^T \\ x_0^T \end{bmatrix} \begin{bmatrix} x_2 - x_1 \end{bmatrix}^T - \begin{bmatrix} x_2 - x_1 \end{bmatrix}^T \begin{bmatrix} x_2^T \\ x_0^T \end{bmatrix},$$

$$G_2 = \begin{bmatrix} x_2^T \\ x_0^T \end{bmatrix} \begin{bmatrix} x_2 - x_1 \end{bmatrix}^T.$$

Let $E_1, E_2 \in \mathcal{S}_{n+1}$ be an orthonormal basis of $W$. Then $\pi_W X(t) = a(t)E_1 + b(t)E_2$, where $a(t) := \langle G_0, E_1 \rangle + t\langle G_1, E_1 \rangle + t^2\langle G_2, E_1 \rangle$ and $b(t) := \langle G_0, E_2 \rangle + t\langle G_1, E_2 \rangle + t^2\langle G_2, E_2 \rangle$. Defining $T := \{[a(t) b(t)]^T : t \in \mathbb{R}\}$, Lemma 2.3 shows that $T$ is homogenization-convex in $\mathbb{R}^2$. By virtue of Lemma 2.2 iv), this implies that $\pi_W C$ is homogenization-convex, as claimed. ∎

2.3. Generalization to Hilbert Space Kernels. The proof given above generalizes to infinite-dimensional spaces:

**Theorem 2.11.** Let $(V, \langle \cdot, \cdot \rangle)$ be a real Hilbert space, and let $g, h : V \to \mathbb{R}$ be continuous quadratic functions defined on $V$ by

$$g : x \mapsto c_g + 2 \langle v_g, x \rangle + \langle x, M_g x \rangle,$$

$$h : x \mapsto c_h + 2 \langle v_h, x \rangle + \langle x, M_h x \rangle,$$
where \( M_g, M_h : V \to V \) are self-adjoint operators, \( v_g, v_h \in V \) and \( c_g, c_h \in \mathbb{R} \). Let \( h \) us further assume that there exists \( x_0 \in V \) where \( h(x_0) > 0 \). Then \( g \) is copositive with \( h \) if and only if there exists \( \xi \geq 0 \) such that \( g(x) - \xi h(x) \geq 0 \) for all \( x \in V \).

Proof. The proof is identical to that of Theorem 1.1 bar the following construction: let \( H := V \oplus \mathbb{R} \), where \( \oplus \) denotes the direct sum of Hilbert spaces, and let us write \( \langle \cdot, \cdot \rangle_H \) for the inner product on \( H \). Let \( \mathcal{S} \) be the space of self-adjoint operators on \( H \). By the Hellinger-Toeplitz Theorem, such operators are automatically continuous, and it is easy to see that \( A, B \in \mathcal{S} \), where

\[
A: (x, t) \mapsto (M_g x + \tau v_g, \langle v_g, x \rangle + \tau c_g)
\]

\[
B: (x, t) \mapsto (M_h x + \tau v_h, \langle v_h, x \rangle + \tau c_h)
\]

Let \( \{e_i : i \in \mathbb{N} \} \) be an orthonormal basis of \( H \). The following operators are in \( \mathcal{S} \),

\[
E_{ij} : y \mapsto \frac{1}{1 + \delta_{ij}} (\langle e_i, y \rangle_H e_j + \langle e_j, y \rangle_H e_i),
\]

where \( \delta_{ij} \) is the Kronecker delta. Defining

\[
\langle E_{ij}, E_{kl} \rangle_S := \begin{cases} 1 & \text{if } \{i, j\} = \{k, l\}, \\ 0 & \text{otherwise,} \end{cases}
\]

the \( E_{ij} \) generate a Hilbert space \((S, \langle \cdot, \cdot \rangle_S)\) for which \( \{E_{ij} : i, j \in \mathbb{N} \} \) is an orthonormal basis. In fact, \( S \) is the set of compact operators in \( \mathcal{S} \), and the topology defined by the trace inner product \( \langle \cdot, \cdot \rangle_S \) is the uniform topology, since \( \langle E_{ij}, X \rangle_S = \langle e_i, X e_j \rangle_H \) for all \( X \in S \). Every \( x \in V \) defines an operator \( R(x) \in \mathcal{S} \),

\[
R(x) : z \mapsto \langle (x, 1), z \rangle_H (x, 1),
\]

and if \( (x, 1) = \sum_{i \in \mathbb{N}} \xi_i e_i \) then \( R(x) = \sum_{ij} \xi_i \xi_j E_{ij} \) and \( \sum_{ij} \xi_i^2 \xi_j^2 = (\sum_i \xi_i^2)(\sum_j \xi_j^2) < \infty \). This shows that \( C := \{R(x) : x \in V\} \subset S \). Extending the map

\[
\psi : C \to \mathbb{R},
\]

\[
R(x) \mapsto \langle (x, 1), A(x, 1) \rangle_H
\]

by linearity and continuity, we obtain a bounded linear operator on the Hilbert space \( \text{clo}[\text{span}(C)], \langle \cdot, \cdot \rangle_S \). Likewise, \( B \) defines a bounded linear operator \( \phi \) on the same space. Replacing \( \mathcal{S}_{m+1} \) by \( \text{clo}[\text{span}(C)] \) in the proof of Section 2.2, a repetition of the arguments presented there proves the claim of Theorem 2.11. \( \square \)

We remark that the condition

\[
g(x) - \xi h(x) \geq 0, \quad \forall x \in V
\]

is equivalent to requiring that

\[
K : V \times V \to \mathbb{R},
\]

\[
(x, y) \mapsto \langle x, (M_g - \xi M_h) y \rangle + \langle v_g - \xi v_h, x + y \rangle + c_g - \xi c_h
\]

be a positive definite kernel.
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