Posterior variability of inclusion shape based on
tomographic measurement data

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Abstract. We treat the problem of recovering the unknown shape of a single inclusion with unknown constant permittivity in an otherwise uniform background material, from uncertain measurements of trans-capacitance at electrodes outside the material. The ubiquitous presence of measurement noise implies that the practical measurement process is probabilistic, and the inverse problem is naturally stated as statistical inference. Formulating the inverse problem in a Bayesian inferential framework requires accurately modelling the forward map, measurement noise, and specifying a prior distribution for the cross-sectional material distribution. Numerical implementation of the forward map is via the boundary element method (BEM) taking advantage of a piecewise constant representation. Summary statistics are calculated using MCMC sampling to characterize posterior variability for synthetic and measured data sets.

1. Introduction

Electrical capacitance tomography (ECT) is an imaging modality in which one attempts to recover the spatially-varying permittivity of an insulating medium from measurement of capacitance outside the boundary of the medium. Applications for ECT can be found e.g. in imaging of multi-phase flow in oil and chemical industries [1, 2]. Other fields of application include characterizing different phases in fluidized beds, mixing processes, and combustion chambers. ECT systems can be implemented with low-cost instrumentation and due to its robustness and small failure probability is suitable for operating under harsh environmental conditions as well as in the presence of strong external electromagnetic fields.

By measuring the capacitances between electrodes around the periphery, the permittivity distribution inside closed bodies can be spatially resolved [3]. The sensor system used in this work measures electric (displacement) charges on sensing electrodes. A voltage is applied at a single excitation electrode while all other (15) electrodes are held at zero potential, and the resulting charge on each electrode is measured. Electrodes are excited in turn leading to 16 measurements cycles. Differing electrical measurement configurations are also possible, though all are based on the principal of applying fixed voltages at excitation electrodes and measuring the field induced at the remaining measurement electrodes [4]. Linearity of the relationship between applied voltage and resulting charge means that all measurement schemes are linear functions of the set we consider.

We use the boundary element method (BEM) to numerically simulate the forward problem, i.e. to calculate electric charges given the material distribution $\varepsilon(x, y)$ and applied
voltages. BEM is used rather than the well-established finite element method (FEM) since we are interested in recovering the shape of a few inclusions when there are distinct material phases [4, 5]. Then the boundary identification problem in ECT allows representations in which the permittivity is piecewise constant within the problem domain $\Omega$.

The majority of ECT reconstructions reported in the literature apply deterministic approaches such as regularized least squares to solve the inverse problem. Deterministic inversion consists of applying a regular approximation to the inverse of the forward map to give a single estimate of the unknown parameters. In contrast we treat the inverse problem as one of statistical inference. [6, 7] by recasting the ECT inverse problem in a Bayesian inferential framework [8]. The solution of the inverse problem then corresponds to summarizing the posterior probability distribution conditioned on measured data and prior knowledge. Bayesian methods have many advantages over deterministic methods, such as robust predictive densities, posterior error estimates, and direct support for optimal decisions. Estimated quantities are calculated as expectations over all solutions consistent with the data. This is especially important when one is mainly interested in properties (functions) of the unknown parameters such as inclusion area, as in this work. Conditioning that calculation on a single estimate typically leads to errors that can be substantial. That is because the single ‘most likely’ solution, found by a regularized inversion, is typically unrepresentative of the bulk of feasible solutions in high dimensional nonlinear problems such as ECT [9].

We model the boundary of the unknown inclusion as a simple polygon, with a prior distribution specified to not bias estimates of inclusion area (see section 4.2). Smoothness of mean boundaries is inherent in the data and it is not necessary to further impose smoothness, as is common [5]. Exploration of the posterior distribution is performed using Markov chain Monte Carlo (MCMC) sampling with Metropolis-Hastings-Green (MHG) dynamics [10, 11]. Proposal of a candidate state is based on multiple moves chosen to ensure convergence of the resulting MCMC sampler in reasonable time. We give examples of estimates of inclusion area along with uncertainties in these estimates. Performance of the MCMC sampler is evaluated by calculating the integrated autocorrelation time (IACT) since that can be interpreted as the number of correlated samples with the same variance-reducing power as one independent sample.

2. Mathematical model of ECT measurements

Reconstruction of the cross-sectional material distribution within the pipe from boundary measurements requires a mathematical model of the measurement process. This forward model establishes the functional mapping from the permittivity distribution $\varepsilon(x)$ to the measured displacement currents. Under the assumption of non-conducting materials and negligible magnetic fields and wave propagation effects, measurements can be modelled using an electrostatic field problem in the interior of the screen. This leads to a Dirichlet boundary value problem (BVP) for the electric potential [3]:

$$\nabla \cdot (\varepsilon \nabla u) = 0$$

$$u|_{\Gamma_0} = u_0$$

$$u|_{\Gamma_e} = 0,$$

where $\Gamma_0$ is the boundary of the excitation electrode with applied voltage $u_0$, while $\Gamma_e$ denotes the union of all other boundaries held at virtual earth including the surrounding shield. The total boundary is then $\partial \Omega = \Gamma_0 \cup \Gamma_e$. Given the permittivity $\varepsilon$ and Dirichlet conditions on $\partial \Omega$ this BVP has a unique solution $u$. Displacement charges on the $i^{th}$ electrode with boundary $\Gamma_i$ may then be calculated as

$$Q_i = \int_{\Gamma_i} \varepsilon \frac{\partial u}{\partial n} ds$$
Figure 1. Cross-sectional view. Inclusion with permittivity $\varepsilon_2$ in a background material $\varepsilon_1$

where $n$ denotes the inward normal vector. Simulation of the 16 measurement cycles requires 16 solutions of the BVP (1) in turn setting each electrode to the driving electrode with surface $\Gamma_0$ and then evaluating displacement charges as in equation (2) at the remaining electrodes.

The BVP equation (1) is numerically solved using the BEM by approximating relevant boundaries between material phases by boundary elements. Figure 1 illustrates an elliptic inclusion ($\varepsilon_2$) in an otherwise constant background material with permittivity $\varepsilon_1$. At the interface $\partial\Omega_2$ between the sub-domains $\Omega_1$ and $\Omega_2$ the natural transition conditions are continuity of electric potential and displacement current normal to the interface, i.e.

$$u^{(1)} = u^{(2)},$$

$$-\varepsilon_1 \frac{\partial u^{(1)}}{\partial n} = \varepsilon_2 \frac{\partial u^{(2)}}{\partial n},$$

where $n$ represents the outward normal vector, $u^{(1)}$ and $u^{(2)}$ denote the electric potential in sub-domains $\Omega_1$ and $\Omega_2$, respectively. In order to solve the forward problem, domain boundaries $\partial\Omega$ and $\partial\Omega_2$ are approximated by $N$ piecewise linear segments. For simplicity, constant boundary elements are used in this work, i.e. the electric potential and its normal derivative are assumed to be constant on each of the $N$ elements.

3. ECT sensor details

The sensor used is based on the measurement of displacement currents [4]. The cross-section of a PVC pipe with a diameter of 100 mm is used as measurement plane. The 16 measurement electrodes are equally spaced around the surface of the pipe. The electrodes are 16 mm wide with a spacing of 21 mm and have a length of 40 mm. Each electrode can be alternately used as a transmitter or receiver. An excitation frequency of 40 MHz is used for the displacement current measurements.

A single measurement frame consists of 16 projections, according to the 16 available transmitting electrodes. For one projection a specific electrode acts as transmitter while all the others sense the displacement current. A measurement frame consequently consists of $16 \times 15 = 240$ entries. The coupling capacitances between the electrodes are in the range between 1 fF to 5 pF. This means that at the same time very small capacitances need to be measured and a high dynamic range is necessary, posing high demands on the sensor hardware. The existing setup features frame rates of 100 frames/s.

4. Bayesian inferential formulation in ECT

Our representation of the true parameters is the set of points, or vertices, defining a polygonal boundary. This set denoted $\theta \in \mathbb{R}^N$ defines a ‘state’ in our reconstruction algorithm. The
forward map in section 2 relates the state $\theta$ to ideal or noise-free measurements $d_{\text{ideal}}$.

The omnipresence of measurement noise implies that in practice a range of data may be measured for a given state $\theta$. Let $\pi(d|\theta)$ denote the probability density function (pdf) over allowable measurements $d$ for a given true state $\theta$. In many cases the measurement noise may be treated as additive with a multivariate Gaussian distribution, due to the linear response of measurement instrumentation [8]. Making a set of measurements corresponds to drawing a sample $d$ from $\pi(d|\theta)$. The inverse problem in the Bayesian framework corresponds to working out what one can say about the parameter $\theta$ given measurements $d$. As a function of $\theta$, $\pi(d|\theta)$ is called the likelihood function and is generally not a probability density. Inference about $\theta$ is based on the posterior density $\pi(\theta|d)$

$$\pi(\theta|d) = \frac{\pi(d|\theta)\pi(\theta)}{\pi(d)}$$

where $\pi(\theta)$ denotes the prior density, expressing the information of $\theta$ independent of the measurement of $d$. The denominator $\pi(d)$ is a normalizing constant that we do not explicitly need to calculate for our present purposes. The posterior density $\pi(\theta|d)$ gives the probability density over allowable states $\theta$ conditioned on measurements and prior information. Summarizing the posterior distribution corresponds to solving the inverse problem since that gives knowledge of the allowable values of parameters with uncertainties, etc.

4.1. Likelihood function in ECT

Analysis of repeated measurements of a fixed object showed that measurement errors are well modelled as additive zero mean (multivariate) Gaussian, i.e. $(d_{\text{ideal}} - d) \sim \mathcal{N}(0, \Sigma)$ [12]. Figure 2 shows the measured matrix of correlation coefficients. It can be seen that the matrix of correlation coefficients only has significant diagonal elements, while off-diagonal elements are small. Based on this result, the measurement error covariance matrix $\Sigma$ is modelled by a diagonal matrix with variances $\sigma^2_i$. The likelihood function $\pi(d|\theta)$ then has the form

$$\pi(d|\theta) \propto \exp \left\{ -\frac{1}{2} (q_m - d)^T \Sigma^{-1} (q_m - d) \right\}$$

where $q_m$ denotes the vector of measured electric charges at the sensing electrodes and $\Sigma$ is the measurement error covariance matrix (model-data mismatch covariance matrix).
4.2. Prior modeling
In industrial ECT the estimation of process parameters like material fraction is of primary interest. As mentioned earlier, we represent the boundary of the inclusion as a polygon with some fixed number of vertices. The simple choice of specifying a uniform prior distribution over vertex position or, equivalently, omitting the prior density has the effect of significantly biasing posterior estimates of area. Using the change of variables relation for probability distributions we see that a uniform density in vertex position gives a density over area that scales as \((\text{area})^{-1/2}\). Since this puts greater weight on small areas, estimated areas will always be smaller than the true area. To avoid this bias we specify the prior density in terms of area directly, given in equation (7). The circumference of the inclusion \(l(\theta)\) with respect to the circumference of a circle with an area equal to the area of the polygon \(\text{area}(\theta)\) is calculated. The variance \(\sigma^2\) is chosen to be small to penalize small and large areas.

\[
\pi(\theta) \propto \exp\left\{\frac{-1}{2\sigma^2} \left(\frac{l(\theta)}{2\sqrt{\text{area}(\theta)\pi}} - 1\right)\right\}, \quad (7)
\]

Redundancy in the polygonal representation can also lead to numerical inefficiency without contributing to the quality of reconstructions. These difficulties can be circumvented by including further terms in the prior distribution. We avoid corner clustering of vertices by requiring that the variance of the edge lengths is also small. Furthermore, polygons with angles between adjacent edges \((\geq 30^\circ)\) are rejected to avoid ‘spikes’ in the boundary.

4.3. Posterior sampling
The resultant posterior distribution is explored using MCMC sampling with MHG dynamics by generating a Markov chain with equilibrium distribution \(\pi\) [14]. These algorithms use a proposal density \(q(\theta, \theta')\) to suggest a new state, i.e. a possible move \(\theta \rightarrow \theta'\), which is accepted or rejected according to a rule that ensures the desired ergodic behavior. Pseudocode for the MCMC sampling algorithm can be found e.g. in [11]. Choice of the proposal density is largely arbitrary, with convergence guaranteed when the resulting MCMC is irreducible and aperiodic. However, the choice of proposal distribution critically affects efficiency of the resulting sampler, with design of a good proposal being something of an art.

For ECT, the parameter vector \(\theta = (x_1, ..., x_n, y_1, ..., y_n)^T\) gives vertices \(x_i, y_i\) of a polygon representing the boundary of a material inclusion. We use several types of update that propose a new state \(\theta'\), usually referred to as moves. Multiple moves can be built into the MCMC sampler simply by defining separate reversible transition probabilities for each move [13, 14]. At least one of the moves must be irreducible on the state-space to ensure that the equilibrium distribution of the Markov chain is independent of the initial choice \(\theta^{(0)}\) of the parameter vector. We find that a combination of \(N = 4\) moves gives a suitably efficient MCMC. These are translation, rotation, and scaling of the polygon, and moving the position of one vertex of the polygon. The last of these move ensures irreducibility but, by itself, would lead to a very slow algorithm. The remaining moves are designed to give an efficient algorithm. A new candidate \(\theta'\) is proposed from \(\theta\) by randomly choosing one of these four moves and using a random step size \(\lambda^t\) tuned for each move.

5. Shape recovery and variability
Two experiments were performed to validate the proposed MCMC algorithm and sample-based solution to the inverse ECT problem. Both use a circular inclusion with a diameter 20 mm \((\varepsilon_r = 3.5)\) in an air-filled pipe \((\varepsilon_r = 1.0)\). For the first experiment \(n_{\text{samp}} = 2000000\) samples are drawn from the posterior distribution, with a simulated data set corrupted by noise. The data was created using 10000 boundary elements in the forward map. For the the second experiment
Figure 3. Scatter plots. (a) Entire domain $\Omega$ with circular inclusion. (b) Detail plot. The gray contour corresponds to the true shape.

Figure 4. Summary statistics. (a) Histogram of reconstructed sample areas. (b) Histogram of reconstructed sample circumferences.

$n_{\text{samp}}=1000000$ samples are drawn from a posterior distribution for a measured data set. Noise standard deviation in both cases was $\sigma_{\text{meas}}=\sigma_{\text{simul}}=6.7\times10^{-4}$. A burn-in period of $n_{\text{burn}}=50000$ samples was found suitable after testing the algorithm with different initial states and different ratios of moves. For the results presented, the four proposal moves were chosen with equal probability, giving an acceptance rate of about 2%.

5.1. Experiment 1 – circular inclusion (simulated data)

Figure 3 illustrates the posterior variability in inclusion shape. As it can be seen, the samples from the posterior distribution are scattered around the true shape shown by the gray contour. Histograms of reconstructed inclusion area and circumference are shown in Figure 4. Sampled area as well as sampled circumference are scattered around their true values $A_{\text{true}}=3.14\times10^{-4}$ m$^2$ and $l_{\text{true}}=6.28\times10^{-2}$ m. The estimated parameters and their posterior variability are summarized in Table 1. Figure 5 depicts the MCMC output trace and the autocorrelation in updates of the log-likelihood. The faster the autocorrelation function decays the less correlation is between consecutive states of the chain and consequently more reliable estimates are obtainable. In particular, the autocorrelation function should be – after falling off smoothly to zero – distributed with some noise about the x-axis. The IACT is quantified in the outermost right column of Table 1. The last row gives the mean of the log-likelihood. Furthermore, the posterior variability represented by the standard deviation of the parameters is very small implying high reliability in estimated parameters.
**Figure 5.** MCMC output trace (top chart) and autocorrelation (bottom chart) of log-likelihood if sample states in the first experiment.

**Table 1.** Posterior variability for the first experiment.

| Quantities             | true values | mean   | standard deviation | IACT  |
|------------------------|-------------|--------|--------------------|-------|
| x-coordinate of center [m] | 0.00        | -2.09×10^{-4} | 2.46×10^{-4}    | 5.75×10^{-1} |
| y-coordinate of center [m] | 2.50×10^{-2} | 2.52×10^{-2} | 6.76×10^{-5}    | 2.24×10^{-3} |
| Area [m²]              | 3.14×10^{-4} | 3.17×10^{-4} | 5.03×10^{-6}    | 1.98×10^{-2} |
| Circumference [m]      | 6.28×10^{-2} | 6.24×10^{-2} | 6.95×10^{-5}    | 3.09×10^{-4} |
| Log-likelihood         | –           | -35.76  | 0.26               | 6.48×10^{-2} |

5.2. Experiment 2 – circular contour (measured data)

Figure 6(a) shows the posterior variability in inclusion shape and position for measured data. Samples from the posterior distribution are scattered corresponding to the circular shape of the PVC rod. Due to the lack of a reference measurement system, the true shape is not depicted. However, the knowledge of the geometry of the rod allows validation of the reconstruction results in terms of area and circumference. The centered gray circle-shaped contour represents the initial state of the Markov chain. MAP and the conditional mean (CM) estimates are presented in Figure 6(b). For this experiment these two estimates almost coincide indicating that the posterior distribution has a well defined single mode. Table 2 summarizes the inversion results

**Figure 6.** Reconstruction results. (a) Scatter plot. Randomly chosen points of the posterior distribution are plotted. (b) Point estimates. MAP estimate (gray) and the CM estimate (dashed black) calculated from the posterior distribution.
Table 2. Posterior variability for the second experiment.

| Quantities                  | true values | mean    | standard deviation | IACT  |
|-----------------------------|-------------|---------|--------------------|-------|
| x-coordinate of center [m]  | – 3.71\times 10^{-2} | 2.32\times 10^{-5} | 5.89\times 10^{2} |
| y-coordinate of center [m]  | – -1.14\times 10^{-2} | 3.02\times 10^{-5} | 4.65\times 10^{2} |
| Area [m^2]                  | 3.14\times 10^{-4} | 3.13\times 10^{-4} | 6.88\times 10^{-6} |
| Circumference [m]           | 6.28\times 10^{-2} | 6.24\times 10^{-2} | 1.57\times 10^{-4} |
| Log-likelihood              | – -46.10    | 1.72\times 10^{-1} | 3.99\times 10^{6} |

of the second experiment. Mean, standard deviation and the IACT are evaluated for area, circumference and center coordinates of the circular inclusion.

6. Conclusions
We have demonstrated sample-based recovery of a single inclusion with unknown shape and unknown constant permittivity from measured and simulated electrical capacitance data. Formulating the inverse problem in a Bayesian inferential framework requires accurate simulation of the measurement process using the BEM, characterizing the statistics of measurement noise, and specifying an a priori distribution over inclusions that does not bias the quantity of interest. MCMC sampling is effective in sampling the posterior distribution and allows accurate estimation of process metrics such as area and circumference of an inclusion, as well as certainties of those estimates.

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