INTEGRAL POINTS AND RELATIVE SIZES OF COORDINATES OF ORBITS IN $\mathbb{P}^N$

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ABSTRACT. We give a generalization to higher dimensions of Silverman’s result on finiteness of integer points in orbits. Assuming Vojta’s conjecture, we prove a sufficient condition for morphisms on $\mathbb{P}^N$ so that $(S, D)$-integral points in each orbit are Zariski-non-dense. This condition is geometric, and for dimension 1 it corresponds precisely to Silverman’s hypothesis that the second iterate of the map is not a polynomial. In fact, we will prove a more precise formulation comparing local heights outside $S$ to the global height. For hyperplanes, this amounts to comparing logarithmic sizes of the coordinates, generalizing Silverman’s precise version in dimension 1. We also discuss a variant where we can conclude that integral points in orbits are finite, rather than just Zariski-non-dense. Further, we show unconditional results and examples, using Schmidt’s subspace theorem and known cases of Lang–Vojta conjecture. We end with some extensions to the case of rational maps and to the case when the arithmetic of the orbit under one map is controlled by the geometric properties of another. We include many explicit examples to illustrate different behaviors of integral points in orbits in higher dimensions.

1. Introduction

Dynamics is a field studying asymptotic behavior of iterations of a self-map. We will denote the $m$-fold iterations $\phi \circ \cdots \circ \phi$ by $\phi^{(m)}$; we will not use higher derivatives so this should not cause any confusion. Dynamics was classically studied over $\mathbb{C}$, but more recently arithmetic dynamics, studying number-theoretic behaviors of self-maps defined over number fields, has now also become a very active field. For example, one would like to know how frequently “integral points” occur on the orbit $O_\phi(P) = \{\phi^{(m)}(P) : m = 0, 1, 2, \ldots\}$ of a rational point $P$. Silverman [13] answered this question for rational functions in one variable, as follows:

Theorem 1 (Silverman). Let $\phi \in \mathbb{Q}(z)$ be a rational function of degree $\geq 2$.
(i) If $\phi^{(2)}$ is not a polynomial (i.e. not in $\mathbb{Q}[z]$), then $O_\phi(P) \cap \mathbb{Z}$ is a finite set for any $P \in \mathbb{Q}$.
(ii) Assume that neither $\phi^{(2)}$ nor $\frac{1}{\phi^{(2)}(1/z)}$ is a polynomial. If we write $\phi^{(m)}(P) = a_m/b_m$ in a reduced form, then for any $P \in \mathbb{Q}$ with $|O_\phi(P)| = \infty$,
$$\lim_{m \to \infty} \frac{\log |a_m|}{\log |b_m|} = 1.$$
Viewing a rational function as a morphism on $\mathbb{P}^1$, the hypothesis of being a polynomial is equivalent to having a totally ramified fixed point at $\infty$. So the upshot is that unless $\phi$ satisfies a special ramification property, only finitely many points in any orbit are integral and in fact orbit points asymptotically become further and further away from being integral. The proof of this theorem uses the powerful Diophantine approximation theorem of Roth, together with a combinatorial description of ramification on iterations based on Riemann–Hurwitz formula.

In this article, we will prove several results that generalize Silverman’s theorem to higher-dimensions. Dynamics in higher-dimension is in general very difficult. Some results are known for powers of $P$ to higher-dimensions. Dynamics in higher-dimension is in general very difficult. Theorem 2 of degree $d \geq 2$, and let $D$ be a divisor on $\mathbb{P}^N$. If $\deg D_{nc}^{(m)} > N + 1$ for some $n$, then Vojta’s conjecture on the divisor $D_{nc}^{(m)}$ implies that for any $P \in \mathbb{P}^N(\mathbb{Q})$, $\mathcal{O}_\phi(P) \cap (\mathbb{P}^N \setminus D)(\mathbb{Z})$ is Zariski-non-dense.

Theorem 2 (cf. Theorem 4). Let $\phi : \mathbb{P}^N \rightarrow \mathbb{P}^N$ be a morphism defined over $\mathbb{Q}$ of degree $d \geq 2$, and let $D$ be a divisor on $\mathbb{P}^N$. If $\deg D_{nc}^{(m)} > N + 1$ for some $n$, then Vojta’s conjecture on the divisor $D_{nc}^{(m)}$ implies that for any $P \in \mathbb{P}^N(\mathbb{Q})$, $\mathcal{O}_\phi(P) \cap (\mathbb{P}^N \setminus D)(\mathbb{Z})$ is Zariski-non-dense.

If $D$ is the hyperplane $X_N = 0$, then by writing $\phi^{(m)}(P) = [a_0^{(m)} : \cdots : a_N^{(m)}]$ with $a_i^{(m)} \in \mathbb{Z}$ with common divisor 1, $\phi^{(m)}(P)$ is in $(\mathbb{P}^N \setminus D)(\mathbb{Z})$ if and only if $a_i^{(m)} = \pm 1$. So the theorem says that the Zariski-closure of integer-coordinate points in orbits is not all of $\mathbb{P}^N$. Note that Theorem 2 for $N = 1$ is exactly Theorem 1 (i). Indeed, using Riemann–Hurwitz, Silverman proves that $\phi^{(2)}$ is not a polynomial if and only if some iterate has at least three distinct poles. Since Zariski-non-denseness is equivalent to finiteness on $\mathbb{P}^1$, Theorem 2 for $N = 1$ completely agrees with...
Silverman’s result. This is an upgrade from a prior work [18], as the hypothesis in
the main theorem there was strictly stronger for \( N = 1 \) than Silverman’s theorem.

Just as in dimension 1, we can also obtain a more precise version involving the
number of digits of the coordinates, albeit assuming Vojta’s conjecture for more
divisors:

**Theorem 3** (cf. Corollary 3). Let \( D \) be the hyperplane \( X_N = 0 \), and let \( c = \sup_n \frac{\deg D^{(n)} - (N+1)}{d^n} \). Assuming Vojta’s conjecture on \( \mathbb{P}^N \), for any \( P \in \mathbb{P}^N(\mathbb{Q}) \) and

\[
\epsilon > 0, \quad \left\{ \phi^{(m)}(P) : \frac{\log \left| \lambda^{(m)}_N \right|}{\log \max_i \left| \lambda^{(m)}_i \right|} \leq c - \epsilon \right\}
\]
is Zariski-non-dense.

To make up for having to use a deep conjecture to treat general maps, we in-
clude many explicit examples throughout the paper which do not require Vojta’s
conjecture. In fact, the examples are important components of this paper. In par-
ticular, Examples 1, 3, and 4 provide different circumstances where assuming this
conjecture can be circumvented, and Examples 6, 7, and 8 analyze interesting ra-
tional maps without assuming any conjecture. They should shed some light to the
spectrum of behaviors for integral points in orbits in higher-dimensions.

We now describe the results and examples in this paper in slightly more de-
tail. In Section 2, we recall definitions of global and local heights and of normal-
crossings divisors, and then discuss Vojta’s conjecture. In Section 3, we will prove
the number-field versions of Theorems 2 and 3 involving several places. These
theorems and their several variants are all consequences of Theorem 4. One of
the variants (Corollary 2) allows us to conclude “finiteness” rather than “Zariski-
non-denseness” under the assumption that the orbit of \( P \) is generic, that is, any
infinite subset of \( O_\phi(P) \) is Zariski-dense. It is one of the far-reaching and difficult
conjectures of Zhang [20] that the orbit of most points is generic, but we will show
some examples where finiteness can be concluded unconditionally.

In Section 4, we discuss results and examples for which Vojta’s conjecture is not
necessary. For example, if \( D^{(n)}_{nc} \) is a union of hyperplanes, then Schmidt’s subspace
theorem gives us the same results unconditionally (Proposition 1). This situation
has a bonus that if \( D^{(n)}_{nc} \) is defined over \( \mathbb{Q} \), then the exceptions to the subspace
theorem are also hyperplanes defined over \( \mathbb{Q} \). This observation is exploited in
Example 1 and we discuss its connection with dynamical (rank-one) Mordell–Lang
conjecture. Another situation where the results become unconditional comes from a
weaker form of Vojta’s conjecture called Lang–Vojta conjecture for integral points.
We take advantage of known cases of this conjecture in Proposition 2 and Example
3. We also mention an example (Example 5) which demonstrates that our criterion
given in Theorem 2 for Zariski-non-denseness of integral points in orbits is not yet
satisfactory.

In Section 5, we discuss some extensions. The first extension is removing the
assumption that \( \phi \) is a *morphism*. For this, we will use the notion of \( D \)-ratio,
introduced by Lee [11]. Using his height inequality for rational maps, we prove
an extension to rational maps (Theorem 5). We also give an explicit example of
this theorem for which Vojta’s conjecture is unnecessary (Example 8). The second
extension is a case when the arithmetic of the orbit under one map is controlled by
the geometry of another (Theorem 6). We mention several open questions at the
very end.
2. Background on Vojta’s Conjecture

In this section, we briefly introduce Vojta’s conjecture \cite{Vojta90}. It is an extremely deep inequality of heights, and its consequences are vast, including Mordell’s conjecture (Faltings \cite{Faltings83}), Schmidt’s subspace theorem, Bombieri–Lang conjecture and the \textit{abc} conjecture (Vojta \cite{Vojta95}). There are multiple versions of the conjecture, including the so-called Lang–Vojta conjecture \cite{Lang76} Proposition 4.1.2 which specializes to integral points and a uniform one over \(\mathbb{Q}\) \cite{Vojta90} Conjecture 5.2.6, but the version we will use in this article is an inequality for rational points over a fixed number field \cite{Vojta90} Conjecture 3.4.3.

We first recall important basic properties of heights, and set some notations. Let \(k\) be a number field, and let \(M_k\) be the set of absolute values up to equivalence. For each \(v \in M_k\), let \(|\cdot|_v\) be the absolute value in the class of \(v\) which is the \(\frac{|k : \mathbb{Q}|}{|k|}\)-th power of the extension of the normalized absolute value on \(\mathbb{Q}\), so that the product formula is simply \(\prod_v |x|_v = 1\) for \(x \in k^*\). We sometimes use the additive notation \(v(x) = -\log |x|_v\). When \(S\) is a finite subset of \(M_k\) containing all of the archimedean ones, the ring \(R_S\) of \(S\)-integers is the set of all \(x \in k\) such that \(|x|_v \leq 1\) for all \(v \notin S\). We define \textit{Weil height} on \(\mathbb{P}^N(\mathbb{Q})\) by

\[
h([x_0 : \cdots : x_N]) = \sum_{v \in M_k} \log \max(|x_0|_v, \cdots, |x_N|_v),
\]

which is well-defined. When \(\phi : \mathbb{P}^N \rightarrow \mathbb{P}^M\) is a rational (algebraic) map, we define the \textit{degree} of \(\phi\) to be the common degree of the homogeneous polynomials defining coordinates of \(\phi\). This can also be viewed as the degree of the map on the Picard group. We have the obvious height inequality \(h(\phi(P)) \leq dh(P) + O(1)\), and whenever \(\phi\) is a morphism (i.e. it is defined everywhere without indeterminacy), we also have an important inequality in the opposite direction:

\[
(1) \quad h(\phi(P)) > dh(P) + O(1).
\]

Let \(X\) be a projective variety over \(k\), assumed to be irreducible unless otherwise stated. For any Cartier divisor \(D\) on \(X\), we can define a Weil height \(h_D\) by writing \(D\) as a difference of ample divisors and using the heights on the projective spaces. We can also define \textit{local height} \(\lambda_v(D, -) : X(k)\langle |D| \rightarrow \mathbb{R}\) for \(v \in M_k\) and a divisor \(D\), using an \(M_k\)-bounded metric on the line bundle \(\mathcal{L}(D)\). In essence, \(\lambda_v(D, P)\) is \(-\log |f(P)|_v\), where \(f\) is a local equation for \(D\), but one needs to glue this together in a coherent way using \(M_k\)-bounded functions. As \(\lambda_v(D, P)\) is big when \(P\) is \(v\)-adically close to \(D\), this is the number-theoretic analog of the proximity function in Nevanlinna theory. For details, see \cite{Faltings83}, \cite{Lang76}. With our normalization, we have

\[
(2) \quad \sum_{v \in M_k} \lambda_v(D, P) = h_D(P) + O(1), \quad \forall P \in X(k)\langle |D|\rangle.
\]

Local height functions also satisfy functoriality with respect to pullbacks: given \(\phi : Y \rightarrow X\),

\[
(3) \quad \lambda_v(D, \phi(Q)) = \lambda_v(\phi^*(D), Q) + O(1), \quad \forall Q \in Y(k)\langle \phi^{-1}(|D|)\rangle,
\]

and in fact the inequality with \(\leq\) holds even for rational maps \(\phi : Y \rightarrow X\).

On \(\mathbb{P}^N\), if a divisor \(D\) is defined (globally) by the homogeneous polynomial \(F\) of degree \(d\), then a local height is simply

\[
(4) \quad \lambda_v(D, [x_0 : \cdots : x_N]) = v(F(x_0, \ldots, x_N)) - d \min(v(x_0), \ldots, v(x_N)).
\]
In particular, choosing $F$ to have coefficients in the ring of integers, we see that

$$λ_v(D, P) \geq 0 \text{ for any non-archimedean } v.$$ 

We also note that $h_D(P) = dh(P) + O(1)$, but a similar relation does not hold with a single $λ_v(D, -)$.

For $k = \mathbb{Q}$, let us write $P \in \mathbb{P}^N(\mathbb{Q})$ as $[a_0 : \cdots : a_N]$, where $a_i \in \mathbb{Z}$ with common divisor 1. Then letting $H = (X_N = 0)$,

$$|λ_{v_p}(H, P)| = \log \max_i |a_i|_{v_p} - \log |a_N|_{v_p} = \log(\text{p-part of } |a_N|)$$

for each place $v_p$ corresponding to the prime $p$. If $S \subset M_\mathbb{Q}$ a finite subset including the archimedean place $v_\infty$, we then denote the prime-to-$S$ part of an integer $x$ by $|x|_S$. By above,

$$\log |a_N|'_S = \sum_{v \not\in S} λ_v(H, P).$$

Thus, $\{P \in (\mathbb{P}^N \setminus H)(\mathbb{Q}) : \sum_{v \not\in S} λ_v(H, P) = 0\}$ is precisely the set $(\mathbb{P}^N \setminus H)(R_S)$ of points with $S$-integer coordinates, i.e. $[a_0 : \cdots : a_{N-1} : 1]$ with $a_i \in R_S$. In general, we say a set is $(S, D)$-integral if it is of the form

$$\{P \in (X \setminus D)(k) : λ_v(D, P) \leq c_v \text{ for } v \not\in S\},$$

where almost all $c_v$ are 0. We will abuse the notation and write $(X \setminus D)(R_S)$ for an $(S, D)$-integral set.

We say that a divisor on a smooth variety is normal-crossings if near each point the divisor is defined by $x_1 \cdots x_k = 0$, where $x_1, \ldots, x_k$ is a part of a local (analytic) coordinate system. Note that by definition, the multiplicity of each irreducible component in a normal-crossings divisor is 1. We are now ready to state Vojta’s conjecture [15 Conjecture 3.4.3].

**Conjecture 1** (Vojta’s Conjecture). Let $X$ be a smooth projective variety over $k$, $K$ a canonical divisor of $X$, $A$ an ample divisor and $D$ a normal-crossings divisor. Fix height functions $λ_v(D, -)$, $h_K$, and $h_A$. Let $S \subset M_k$ be a finite set of places. Then given $ε > 0$, there exists a Zariski-closed $Z_ε \subset X$ such that

$$\sum_{v \in S} λ_v(D, P) + h_K(P) < εh_A(P) + O(1)$$

for all $P \in X(k)$ not on $Z_ε$.

Since local heights have logarithmic poles along $D$, this conjecture states that a point cannot get too close $v$-adically to $D$ for $v \in S$, and how close a rational point can approximate $D$ is controlled by the geometry of the variety, namely how negative the canonical divisor is. We note that the normal-crossings assumption on the divisor is absolutely essential, and this condition will be important in the rest of the paper. Since we mostly work with projective spaces, we also state the following version.

**Conjecture 2** (Vojta’s Conjecture for $\mathbb{P}^N$). Let $D$ be a normal-crossings divisor on $\mathbb{P}^N$ defined over $k$, and $S$ be a finite set of places. Then given $ε > 0$, there exists a finite union $Z_ε$ of hypersurfaces and a constant $C$ such that for any $P \in \mathbb{P}^N(k) \setminus Z_ε$,

$$\sum_{v \in S} λ_v(D, P) < (N + 1 + ε)h(P) + C.$$
This is precisely Roth’s theorem when $N = 1$ and $S$ consists of a single archimedean absolute value. In fact, if $D$ is a union of hyperplanes in $\mathbb{P}^N$ in general position, this conjecture can be shown to be equivalent to Schmidt’s subspace theorem. Thus, one can view Vojta’s conjecture as a higher-degree extension of the subspace theorem.

3. PROOFS OF THE THEOREMS

We will first prove the following technical theorem, from which Theorems 2 and 3 and other variants can be easily derived.

**Theorem 4.** Let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism defined over $\overline{\mathbb{Q}}$ of degree $d \geq 2$. Let $D$ be a divisor on $\mathbb{P}^N$ defined over $\overline{\mathbb{Q}}$, and let $D^{(n)}_{nc}$ be the normal-crossings part of $(\phi^{(n)})^*(D)$. Let $c_n = \frac{\deg(D^{(n)} - (N+1))}{\deg(D) \cdot d^n}$, and let $k$ be a number field that contains the fields of definition of $\phi$, $D$, and $D^{(n)}_{nc}$. Assume Vojta’s conjecture for the divisor $D^{(n)}_{nc}$. Then for any $P \in \mathbb{P}^N(k)$, for any finite set $S \subset M_k$, and for any $\epsilon > 0$, the set

$\{ \phi^{(n)}(P) : \frac{\sum_{v \notin S} \lambda_v(D, \phi^{(n)}(P))}{\deg(D) \cdot h(\phi^{(n)}(P))} \leq c_n - \epsilon \}$

is Zariski-non-dense. In particular, $\mathcal{O}_Z(P) \cap (\mathbb{P}^N \setminus D)(R_S)$ is Zariski-non-dense if $c_n > 0$ for some $n$.

**Proof.** By applying Vojta’s conjecture to the divisor $D^{(n)}_{nc}$ on $\mathbb{P}^N$, there exists a constant $C$ such that

$\sum_{v \in S} \lambda_v \left( D^{(n)}_{nc}, Q \right) < (N + 1 + \epsilon) h(Q) + C \tag{8}$

holds for all $Q \in \mathbb{P}^N(k)$ except for points on a finite union $Z_\epsilon$ of hypersurfaces. Since $\phi$ is a morphism, the degree of $D^{(n)} = (\phi^{(n)})^* D$ is $d^n \deg(D)$, and thus we also have

$\sum_{v \in S} \lambda_v \left( D^{(n)} - D^{(n)}_{nc}, Q \right) \leq h(D^{(n)} - D^{(n)}_{nc})(Q) + C'_1 \leq \left( d^n \deg(D) - \deg(D^{(n)}_{nc}) \right) h(Q) + C_1. \tag{9}$

Then for $Q \notin Z_\epsilon$,

$\sum_{v \in S} \lambda_v(D, \phi^{(n)}(Q)) = \sum_{v \in S} \lambda_v((\phi^{(n)})^*(D), Q) + C_2$ \hspace{1cm} \text{functoriality}\hspace{1cm} \text{[4]}

$\leq \left( d^n \deg(D) - \deg(D^{(n)}_{nc}) + (N + 1) + \epsilon \right) h(Q) + C_3 \hspace{1cm} \text{[5] and [6]}

$\leq \left( 1 - c_n + \frac{\epsilon}{d^n \deg(D)} \right) \deg(D) h(\phi^{(n)}(Q)) + C_4 \hspace{1cm} \phi$ a morphism and \hspace{1cm} \text{[7]}

$= \left( 1 - c_n + \frac{\epsilon}{d^n \deg(D)} \right) h_D(\phi^{(n)}(Q)) + C_5.$
Hence, for any $m \geq n$, we let $Q = \phi^{(m-n)}(P)$ and conclude that
\[
\sum_{v \notin S} \lambda_v(D, \phi^{(m)}(P)) > \left( c_n - \frac{\epsilon}{d^n \deg(D)} \right) h(D(\phi^{(m)}(P))) - C_6
\]
\[
> \left( c_n - \frac{\epsilon}{d^n \deg(D)} \right) \deg(D) h(\phi^{(m)}(P)) - C_7
\]
as long as $\phi^{(m-n)}(P) \notin Z_\epsilon$. Note that if $0_\phi(P)$ is finite, then this theorem is trivial. Otherwise, $h(\phi^{(m)}(P)) \to \infty$ as $m \to \infty$ by Northcott’s theorem. By dividing both sides of the inequality by $\deg(D)h(\phi^{(m)}(P))$, $C_7$ can be incorporated into a change in $\epsilon$ for large enough $m$'s. Moreover, if $\phi^{(m-n)}(P) \in Z_\epsilon$, then $\phi^{(m)}(P) \in \phi^{(n)}(Z_\epsilon)$, which is a Zariski-closed set not equal to the whole of $\mathbb{P}^N$. Therefore, the result follows, as the given set is contained in the union of $\phi^{(n)}(Z_\epsilon)$ with a finitely many of the orbit points. The last sentence of the theorem is immediate from the discussion of $(S, D)$-integral sets in the previous section. □

**Remark 1.** What we actually prove is the following: there exists a constant $C$ such that $\{ \phi^{(m)}(P) : \sum_{v \notin S} \lambda_v(D, \phi^{(m)}(P)) \leq (c_n - \epsilon) \deg(D) \cdot h(\phi^{(m)}(P)) - C \}$ is Zariski-non-dense. The constant $C$ comes partially from (7), so it is not effective. In the following, we will prove similar results in various settings, all of which can be stated as differences of heights, although we state them with ratios for simplicity.

**Corollary 1.** Let $c_n = \frac{\deg D_{nc}^{(n)} - (N+1)}{d^n \deg(D)}$ and $c = \sup_n c_n$. Then assuming Vojta’s conjecture for $\mathbb{P}^N$, for all $\epsilon > 0$,
\[
\left\{ \phi^{(m)}(P) : \frac{\sum_{v \notin S} \lambda_v(D, \phi^{(m)}(P))}{\deg(D) h(\phi^{(m)}(P))} \leq c - \epsilon \right\}
\]
is Zariski-non-dense.

**Proof.** This is vacuous if $c \leq 0$ (from (14), each $\lambda_v(D, -)$ is nonnegative), and if not, there exists $n$ such that $c - \epsilon < c_n \leq c$ with $c_n > 0$. Then using $\epsilon' = \frac{c_n - (c - \epsilon)}{2}$ in Theorem 4 shows the result, as $c_n - \epsilon' > c - \epsilon$. □

**Remark 2.** In truth, if $\{c_n : n \in I\}$ is a monotone increasing sequence whose limit is $c$, then we only need to assume Vojta’s conjecture for divisors $D_{nc}^{(n)}$ with $n \in I$.

Note that $c$ is completely determined geometrically, and it does not depend on the choice of $S$. On the other hand, the (possibly reduced) proper subvariety that contains $I$ will depend on $\epsilon$ and $S$ (conjecturally, the hypersurface part does not depend on $S$, but the additional higher-codimensional part will certainly depend on $S$).

We next discuss another variation of the main results. An infinite set of rational points is called *generic* if any infinite subset is Zariski-dense. Zhang [20] has conjectured that any polarized dynamical system has a point whose orbit is generic. If we assume genericity of the orbit, then we can conclude finiteness, rather than just Zariski-non-denseness, as follows.
Corollary 2. Let $c = \sup_n \deg\frac{D^{(n)} - (N+1)}{d = \deg D}$. Let us assume that the orbit of $P$ is generic (in particular, $\vert \mathcal{O}_V(P) \vert$ is infinite). Assuming Vojta’s conjecture for $\mathbb{P}^N$, for all $\epsilon > 0$, then
$$
\sum_{v \notin S} \lambda_v(D, \phi^{(m)}(P)) \frac{\deg(D)h(\phi^{(m)}(P))}{\deg(D)h(\phi^{(m)}(P))} > c - \epsilon
$$
holds for sufficiently large $m$. In particular,
(i) If there exists $n$ such that $\deg D^{(n)} > N + 1$, then $\mathcal{O}_V(P) \cap (\mathbb{P}^N \setminus D)(R_S)$ is a finite set.
(ii) If $c = 1$, then
$$
\lim_{m \to \infty} \sum_{v \notin S} \lambda_v(D, \phi^{(m)}(P)) \frac{\deg(D)h(\phi^{(m)}(P))}{\deg(D)h(\phi^{(m)}(P))}
$$
exists and equal to 1.

Proof. By the definition of genericity, the only Zariski-non-dense subset of the orbits is a finite set, so the first statement follows immediately from Corollary 1. (i) then follows as before from the fact that $h(\phi^{(m)}(P)) \to \infty$, and (ii) follows from the fact that the numerator inside the limit is $\leq \deg D(\phi^{(m)}(P)) = \deg(D)h(\phi^{(m)}(P)) + O(1)$, together with the squeeze theorem.

We now specialize Theorem 4 and its corollaries to $k = \mathbb{Q}$ and $D = H = (X_N = 0)$, and derive Theorem 5. For $P \in \mathbb{P}^N(\mathbb{Q})$, let us write $\phi^{(m)}(P) = [a_0^{(m)} : \cdots : a_N^{(m)}]$, where $a_i^{(m)} \in \mathbb{Z}$ with common divisor 1.

Corollary 3. Let $c = \sup_n \frac{\deg H^{(n)} - (N+1)}{d = \deg D}$, and let $S \subset M_\mathbb{Q}$ be a finite subset containing $v_\infty$.
(i) If there exists $n$ such that $\deg H^{(n)}_V > N + 1$, then assuming Vojta’s conjecture for $H^{(n)}_V$, for any $P \in \mathbb{P}^N(\mathbb{Q})$, the $(S, H)$-integral points $\{\phi^{(m)}(P) : |a_N^{(m)}|_S = \pm 1\}$ in the orbit of $P$ is Zariski-non-dense.
(ii) Assuming Vojta’s conjecture for $\mathbb{P}^N$, for all $\epsilon > 0$,
$$
\left\{ \phi^{(m)}(P) : \frac{\log |a_N^{(m)}|_S}{\log \max_i |a_i^{(m)}|} \leq c - \epsilon \right\}
$$
is Zariski-non-dense.
(iii) If the orbit of $P$ is generic, then assuming Vojta’s conjecture for $\mathbb{P}^N$, for all $\epsilon > 0$,
$$
\frac{\log |a_N^{(m)}|_S}{\log \max_i |a_i^{(m)}|} > c - \epsilon
$$
holds for all sufficiently large $m$. In particular, if there exists $n$ such that $\deg H^{(n)}_V > N + 1$, $\{\phi^{(m)}(P) : |a_N^{(m)}|_S = \pm 1\}$ is a finite set, and if $c = 1$,
$$
\lim_{m \to \infty} \frac{\log |a_N^{(m)}|_S}{\log \max_i |a_i^{(m)}|} = 1.
$$
Remark 3. If $S = \{ v_\infty \}$, $|x|_S = |x|$ for any integer, so we obtain Theorem \ref{thm:main}.
Moreover, in the case of $\mathbb{P}^1$, if the orbit is infinite (i.e. $P$ is not preperiodic), then it is automatically generic. Therefore, the last part of (iii) generalizes Silverman’s coordinate-size result (Theorem \ref{thm:Silverman} (ii)).

Proof. These all follow directly from Theorem \ref{thm:main} and Corollaries \ref{cor:cor1} and \ref{cor:cor2} using (\ref{eq:main}) and (\ref{eq:Silverman}). \hfill \Box

\section{Unconditional Results and Examples}

In this section, we discuss some cases for which we obtain results such as Theorem \ref{thm:main} unconditionally without assuming any conjecture. One major case comes from Schmidt’s subspace theorem (Proposition \ref{prop:Schmidt} and Example \ref{ex:Schmidt}). There are other special cases for which Vojta’s conjecture can be proved, and we discuss these examples as well (Proposition \ref{prop:Vojta} and Examples \ref{ex:Vojta} and \ref{ex:Vojta2}).

First, we discuss cases for which Schmidt’s subspace theorem applies. Since this theorem is equivalent to Vojta’s conjecture for $X = \mathbb{P}^N$ and $D$ a union of hyperplanes in general position, whenever normal-crossings divisors are linear, we get results in the previous section unconditionally. We now make this precise. Similar to the definition of $D^{(n)}_{nc}$, we define a linear normal-crossings part $D^{(n)}_{lin}$ of $D^{(n)}$ to be a highest-degree normal-crossings subdivisor over $\overline{\mathbb{Q}}$ of $D^{(n)}$ whose support is a union of hyperplanes. In other words, among all the different general-position unions of $\overline{\mathbb{Q}}$-hyperplanes contained in $|D^{(n)}|$, $D^{(n)}_{lin}$ has the most number of components.

Proposition 1. Let $\phi: \mathbb{P}^N \to \mathbb{P}^N$ be a morphism defined over $\overline{\mathbb{Q}}$, let $D$ be an effective divisor on $\mathbb{P}^N$ defined over $\overline{\mathbb{Q}}$, and $P \in \mathbb{P}^N(\overline{\mathbb{Q}})$. Suppose there exists $n$ such that $c'_n = \frac{\deg D^{(n)}_{lin} - (N+1)}{\deg D}$ is positive. Let $k$ contain the fields of definition of $\phi$, $D$, $D^{(n)}_{lin}$, and $P$, and let $S$ be a finite subset of $M_k$. Then for all $\epsilon > 0$,

\begin{equation}
\left\{ \phi^{(m)}(P) : \frac{\sum_{v \not\in S} \lambda_v(D, \phi^{(m)}(P))}{\deg(D) h(\phi^{(m)}(P))} \leq c'_n - \epsilon \right\}
\end{equation}

is Zariski-non-dense. Further, if no hyperplane defined over $k$ contains infinitely many points of $O_{\phi}(P)$, then (\ref{eq:Schmidt}) is a finite set.

As before, we also have a version for $(S,D)$-integral points, a version with $c' = \sup_n c'_n$, and a version with $| \cdot |_S$.

Remark 4. Much progress has been made on Schmidt’s subspace theorem and its exceptional hyperplanes. For example, we have some upper bounds for the number of exceptional hyperplanes. However, in general we still do not have a bound for the heights of the coefficients of the exceptional linear subspaces, and so we do not have an effective bound of $m$ which lies in (\ref{eq:Schmidt}).

Proof. All of these follow directly from the corresponding statements involving $D^{(n)}_{nc}$ instead of $D^{(n)}_{lin}$. One notable observation is the fact that Schmidt’s subspace theorem actually lets us conclude that the exceptions to the inequality of Vojta’s conjecture are contained in a finite union of hyperplanes defined over $k$. For general
normal-crossings divisors, we only know that the exception is a union of hypersurfaces, so this is much stronger. To conclude finiteness of (11), we only need to show that each hyperplane over \( k \) contains only finitely many points of \( \mathcal{O}_\phi(P) \).

**Example 1.** Let \( \phi \) be the morphism \( [Y^4 + Z^4 : X^3(Y + Z) : YZ^3] \) on \( \mathbb{P}^2 \). Letting \( D = (Z = 0) \), \( D^{(2)} = (X^3(X + Y + Z)Y^3Z^9 = 0) \), and \( D^{(2)}_{\text{lin}} = D^{(2)}_{\text{nc}} = (XYZ(X + Y + Z) = 0) \), satisfying the hypothesis of the proposition. Hence, \((S, D)\)-integral points in the orbit are Zariski-non-dense, and more precisely,

\[
\phi^{(m)}(P) = \left[ a^{(m)} : b^{(m)} : c^{(m)} \right] : \frac{\log |c^{(m)}|_{\mathbb{S}}}{\log \max(|a^{(m)}|, |b^{(m)}|, |c^{(m)}|)} \leq \frac{1}{16} - \epsilon
\]

is Zariski-non-dense.

Now, let \( P = [100 : 2000 : 1] \), and we will show that no line defined over \( \mathbb{Q} \) contains infinitely many points of \( \mathcal{O}_\phi(P) \). Unlike the ratio of the number of digits of coordinates, the ratio of the coordinates is unaffected even when there is a common factor. Since the \( Z \)-coordinate of \( P \) is much smaller than the first two coordinates and the first two coordinates of \( \phi \) are quartic in \( X \) and \( Y \) while the last coordinate is only linear, the ratio of the last coordinate of \( \phi^{(m)}(P) \) to either of the first two coordinates becomes smaller and smaller in absolute value as \( m \to \infty \). Therefore, a hyperplane with a nonzero coefficient for \( Z \) cannot contain infinitely many points of \( \mathcal{O}_\phi(P) \). The orbit points of \( P \) clearly will not lie on \( X = 0 \) or \( Y = 0 \), so we are left to check \( X - \alpha Y = 0 \) for \( \alpha \in \mathbb{Q}^* \). Any orbit point lying on this line has its previous iterate a rational point on \( Y^4 + Z^4 - \alpha(X^4 + X^3Y + X^3Z) = 0 \). When this is a smooth curve, it has genus 3, so this immediately gives finiteness. Using the Jacobian criterion, the derivatives with respect to \( Y \) and \( Z \) give \( 4Y^3 - \alpha X^3 = 4Z^3 - \alpha X^3 = 0 \). So \( X \neq 0 \) and letting \( \eta_1, \eta_2 \) be some cube root of the rational number \( \frac{4}{7} \), we have \( Y = \eta_1 X \) and \( Z = \eta_2 X \). Then the derivative with respect to \( X \) yields \( 4X^3 + 3X^2Y + 3X^2Z = X^3(4 + 3\eta_1 + 3\eta_2) = 0 \). There are two possibilities. When \( \eta_1 = \eta_2 \), then \( \eta_i < 0 \), so \( \alpha < 0 \). This line does not contain any orbit points of \( P \), as their coordinates are all positive. When \( \eta_1 = -\eta_2 \), then \( \Re(\eta_i) = -\frac{\alpha}{2} \) so \( \alpha = 4\eta_2^3 = \frac{256}{27} \). For \( m \) even, the \( Y \)-coordinate of \( \phi^{(m)}(P) \) is bigger than the \( X \)-coordinate, so it will not be on this line. The \( X \)-coordinate of \( \phi(P) \) is about 7600 times as big as the \( Y \)-coordinate, and for points with a much larger \( X \)-coordinate than the \( Y \)-coordinate, the first coordinate of \( \phi^{(2)} \) is dominated by \( X^{16} \) while \( X^4Y^{12} \) dominates the second coordinate. Hence, the ratio of the \( X \)-coordinate to the \( Y \)-coordinate becomes bigger and bigger upon every \( \phi^{(2)} \). Again, we observe that a common factor will not affect the ratio of the coordinates. Therefore, the first two coordinates of points in \( \mathcal{O}_\phi(P) \) will never have a ratio of \( \frac{256}{27} \).

Therefore, we conclude that \( (12) \) is a finite set for \( P = [100 : 2000 : 1] \), unconditionally without assuming any conjectures. Note that to conclude finiteness, it was very useful to know that the exceptions are lines (so that we can make arguments involving ratios of coordinates) and that the coefficients are in \( \mathbb{Q} \).

The argument above only utilizes standard methods, but it is somewhat adhoc and it is difficult to generalize to arbitrary \( \phi \) of similar shape and arbitrary \( P \). On the other hand, given a specific situation, one can usually come up with a similar argument to show that a line defined over \( \mathbb{Q} \) does not contain infinitely many of its orbit points.
Example 2. Example 1 can be generalized to higher-dimensions. For example, let \( \phi = \prod L_i : X_0^d : \cdots : X_{N-2}^d : X_{N-1}X_N^{d-1} \) on \( \mathbb{P}^N \), where \( L_1, \ldots, L_d \) are linear forms in general position such that \( \phi \) is a morphism. Then \((\phi^{(N+1)})^{-1}(X_N = 0)\) is the vanishing locus of \((\prod L_i)X_0 \cdots X_N\), so Proposition 1 applies. On the other hand, it is more difficult in general to conclude that \( (\phi^{(N+1)})^{-1}(X_N = 0) \) is a finite set, as exceptions are no longer 1-dimensional.

Remark 5. As is evident in Example 1 even in cases for which Schmidt’s subspace theorem applies, it would be beneficial to have an affirmative answer to dynamical Mordell–Lang question in order to conclude finiteness rather than Zariski-non-denseness. This question asks whether an infinite intersection of the orbit with a subvariety forces the subvariety to be preperiodic, and it has been proved affirmatively in various settings using Skolem–Mahler–Lech method (see for example [1]).

The higher-rank case, involving several maps which commute with each other, was introduced in [8]. They prove an affirmative answer in low-dimensional cases and also demonstrated a couple of counterexamples. The higher-rank case has now been proved to completely fail in general [12], though no counterexample has yet been found for orbits of a single map. The case relevant to Example 1 namely the case of self-maps on \( \mathbb{P}^2 \) with respect to a line, is not known.

As a next situation when the results become unconditional, we take advantage of the known cases of the “integral point” version of Vojta’s conjecture. This version is sometimes called Lang–Vojta conjecture, and it is a special case [13, Proposition 4.1.2] of Conjecture 1 in Section 2. This conjectures that when \( X \setminus D \) is of log general type, the \((S, D)\)-integral points are Zariski-non-dense. We will now use a known case of this to obtain an unconditional result:

Proposition 2. Let \( \phi : \mathbb{P}^N \rightarrow \mathbb{P}^N \) be a morphism defined over \( \overline{\mathbb{Q}} \) of degree \( d \geq 2 \). Let \( D \) be a divisor on \( \mathbb{P}^N \) defined over \( \overline{\mathbb{Q}} \), and let \( P \in \mathbb{P}^N(\overline{\mathbb{Q}}) \). Let \( k \) be a number field that contains the fields of definition of \( \phi \), of irreducible components over \( \overline{\mathbb{Q}} \) of \( D \), and of \( P \), and let \( S \subset M_k \) be a finite subset. Suppose there exists \( n \) with \( (\phi^{(n)})^* D = D_1 + D_2 \) such that

- \( D_1 \) contains \( N + 2 \) distinct geometrically-irreducible components,
- \( \exists \alpha < \deg(D_2) \) with \( \sum_{v \in S} \lambda_v(D_2, \phi^{(m)}(P)) \leq \alpha h(\phi^{(m)}(P)) + O(1) \forall m \gg 0 \).

Then

\[
\phi^{(m)}(P) : \frac{\sum_{v \in S} \lambda_v(D, \phi^{(m)}(P))}{(\deg D) h(\phi^{(m)}(P))} \leq \frac{\deg(D_2) - \alpha}{(\deg D)d^n}
\]

is Zariski-non-dense.

Proof. The proof is similar in spirit to the proof of Theorem 1. Let \( m \gg 0 \) be such that \( \phi^{(m)}(P) \) is in the set in question and let \( Q = \phi^{(m-n)}(P) \). Note that

\[
\sum_{v \not\in S} \lambda_v(D_2, Q) = (\deg D_2) h(Q) - \sum_{v \in S} \lambda_v(D_2, Q) - C_1 \geq ((\deg D_2) - \alpha) h(Q) - C_1.
\]

On the other hand,

\[
\sum_{v \not\in S} \lambda_v(D_1, Q) + \sum_{v \not\in S} \lambda_v(D_2, Q) = \sum_{v \not\in S} \lambda_v((\phi^{(n)})^* D, Q) + \sum_{v \not\in S} \lambda_v(D, \phi^{(m)} P) + C_2
\]

\[
\leq \frac{\deg(D_2) - \alpha}{(\deg D)d^n} \cdot (\deg D) h(\phi^{(m)}(P)) + C_2 \leq (\deg(D_2) - \alpha) h(Q) + C_3.
\]
Putting these two together, we see that $\sum_{v \in S} \lambda_v(D_1, Q)$ is bounded by $C'_1 + C_3$. For a fixed number field, \cite{4} shows that there are only finitely many non-archimedean places for which the minimum positive value of $\lambda_v(D_1, -)$ is below $C'_1 + C_3$. Therefore, so $Q$ must belong to a set of $(S, D_1)$-integral points. But Lang-Vojta conjecture is known for $\mathbb{P}^N$ and a divisor with $N + 2$ distinct geometrically-irreducible components (originally \cite{15} Corollary 2.4.3 and a special case of \cite{16} Corollary 0.3), so the conclusion follows by taking image by $\phi^{(n)}$.

**Remark 6.** Of course, once $(\phi^{(n)})^*(D)$ contains $N + 2$ distinct geometrically-irreducible components, a set $Z$ of $(S, (\phi^{(n)})^*(D))$-integral points in all of $\mathbb{P}^N$ is Zariski-non-dense. Hence, the set of orbit points which are $(S, D)$-integral is contained in $(\phi^{(n)})(Z)$, which is also Zariski-non-dense. Thus, integral-point statements such as Theorem \cite{2} are trivial, while the number-of-digits comparison statements such as Proposition \cite{4} are less immediate.

**Example 3.** Let $\phi = [X^3: L \cdot Q : YZ^2]$ on $\mathbb{P}^2$, where $L$ is a $\mathbb{Q}$-linear form and $Q$ is a geometrically irreducible $\mathbb{Q}$-quadratic form such that neither goes through $[0 : 0 : 1]$ or $[0 : 1 : 0]$. This is a morphism. Let $P \in \mathbb{P}^2(\mathbb{Q})$ be such that the $Y$-coordinate is much larger than the other two coordinates. $(\phi^{(2)})^*(Z = 0) = (L \cdot Q \cdot YZ^2, L)$, so $D_1 = (L \cdot Q \cdot YZ)$ has four distinct components. Then $D_2 = (Y) + 3(Z)$, and as $L \cdot Q$ contains a nonzero $Y^3$-term while the other two coordinates of $\phi$ do not, the $Y$-coordinate is always the largest in $\mathcal{O}_\phi(P)$. Therefore, $\lambda_{v_{\infty}}((Y), (\phi^{(m)}(P)) = 0$, and hence we have

$$\lambda_{v_{\infty}}(D_2, (\phi^{(m)}(P)) \leq 3h(\phi^{(m)}(P)).$$

Therefore, the proposition unconditionally tells us that

$$\left\{ \phi^{(m)}(P) : \frac{\log |a_2^{(m)}|}{\log \max(|a_0^{(m)}|, |a_1^{(m)}|, |a_2^{(m)}|)} \leq \frac{1}{9} - \epsilon \right\}$$

is contained in a finite union of algebraic curves. Note that since $(\phi^{(2)})^*(Z = 0)$ does not contain four distinct lines, Proposition \cite{4} does not apply.

We next discuss one other situation where our results become unconditional. Sometimes, Vojta’s inequality can be verified even for non-normal-crossings divisors, and our next example takes advantage of this.

**Example 4.** Let $k = \mathbb{Q}$, $S = \{ \infty \}$, and $D_1 = (XYZ(Y + Z) = 0)$ on $\mathbb{P}^2$. Since three lines of $D$ go through $[1 : 0 : 0]$, $D$ is not normal-crossings. On the other hand, using \cite{4} and writing $P = [a : b : c]$ with integers with gcd 1, the LHS of (2) becomes

$$\lambda_{v_{\infty}}(D_1, [a : b : c]) - 3h(P) = \log \max( |a|, |b|, |c| )^4 \quad \text{and} \quad 3 \log \max( |a|, |b|, |c| )$$

$$= \log \max( |a|, |b|, |c| ) |a||b||c| \leq 0,$$

because whichever coordinate has the maximum absolute value, it is canceled by the denominator. Hence, Vojta’s inequality is trivially satisfied for $D_1$, and so if $(\phi^{(n)})^* D$ contains $D_1$, one can replace $D_{nc}^{(n)}$ with $D_1$ and obtain Theorem \cite{2} and its consequences unconditionally. For example, let $\phi = [X^3 + 2Y^3 + 3Z^3 : X^2Y + YZ^2 + Z^3 : X^2Y - YZ^2 - Z^3]$ and let $D = ((Y + Z)(Y - Z) = 0)$. Since the common intersections of the last two coordinates of $\phi$ are $[0 : 1 : 0], [0 : 1 : -1], [1 : 0 : 0]$. 

and the first coordinate of \( \phi \) is nonzero at these points, \( \phi \) is a morphism. Now, \( \phi^* D = (4X^2YZ(Y + Z) = 0) \), which contains \( D_1 \) as a subdivisor. Thus, we conclude unconditionally from Theorem 4 that
\[
\left\{ \phi^{(m)}(P) : \frac{\sum_{v \notin S} \lambda_v(D, \phi^{(m)}(P))}{2h(\phi^{(m)}(P))} \leq \frac{1}{6} - \epsilon \right\}
\]
is contained in a finite union of algebraic curves for any \( \epsilon > 0 \) and \( P \in \mathbb{P}^2(\mathbb{Q}) \).

We close this section with an example which demonstrates that our theorems are not yet satisfactory for determining Zariski-non-density of integral points in orbits.

**Example 5.** Let \( \phi = [X^3 : Y^3 : Z^2(Y - Z)] \). This is a morphism, and since the last two coordinates of \( \phi \) only involve \( Y \) and \( Z \), this property continues to hold for all \( \phi^{(n)} \). On the other hand, any homogeneous polynomial in \( Y \) and \( Z \) factors into linear terms, all of which go through the point \([1 : 0 : 0]\). Thus, for \( D = (Z = 0) \), \( D_n^{(n)} \) is at most two lines for every \( n \), so this map does not satisfy the hypothesis of Proposition 4.

However, we can actually show that an orbit of \( P = [a : b : 1] \) with \( a, b \in \mathbb{Z} \) and \( a > b > 1 \) contains only finitely many \((S,D)\)-integral points. To see this, let us write \( \phi^{(m)}(P) = [a_m : b_m : c_m] \) with \( a_m, b_m, c_m \in \mathbb{Z} \) with gcd 1. Adding primes dividing \( b \) to \( S \), we see that \( b_m \) is always an S-unit. Hence, if \( \phi^{(m+1)}(P) \) is integral, then \( c_m \) and \( b_m - c_m \) are integers which divide \( b_m^3 \), so they are also S-units. Looking at \( \frac{b_m - c_m}{b_m} + \frac{c_m}{b_m} = 1 \), we see that having infinitely many \((S,D)\)-integral orbit points contradicts the S-unit equation.

Note that in this particular example,
\[
\lim_{m \to \infty} \frac{\log c_m}{\log \max(a_m, b_m, c_m)}
\]
does not go to 1 in general. Since the last two coordinates of \( \phi \) only involve \( Y \) and \( Z \) and the last coordinate of \( \phi^{(2)} \) is not a pure power of \( Z \), it follows from Silverman’s result on \( \mathbb{P}^1 \) that \( \lim \frac{\log \max(b_m, c_m)}{\log c_m} = \lim \frac{\log b_m}{\log c_m} = 1 \). If we further assume that \( b \) is a prime, then one can show by induction that \( c_m \) is not divisible by \( b \), so there is no cancelation when we compute the next iterate. Therefore,
\[
\lim_{m \to \infty} \frac{\log c_m}{\log \max(a_m, b_m, c_m)} = \lim_{m \to \infty} \frac{\log c_m}{\log a_m} = \lim_{m \to \infty} \frac{\log b_m}{\log a_m} = \frac{\log b}{\log a}.
\]
Thus, the asymptotic ratio of the number of digits of the coordinates of iterates actually depends on the initial point, rather than controlled by the geometry of the divisor and \( \phi \). In this example, \( \phi \) has a totally ramified fixed point at \([1 : 0 : 0]\), so in some sense this is an “exceptional case,” just as the polynomials are exceptions in dimension 1. On the other hand, finiteness of integral points still holds as above. We need further research to understand this example in a more theoretical framework.

### 5. Generalizations and Further Examples

In this section, we give two directions to which Theorem 4 can be extended, and give several illuminating examples. Theorem 5 deals with a certain class of rational maps, and Theorem 6 deals with a case when the arithmetic of orbits under one map can be controlled by the geometry of another map.
When there exists \( n \) such that \( \deg D^{(n)}_{nc} > N + 1 \), we know (at least under assuming Vojta’s conjecture) from Theorem \(^4\) that the set of orbit points which are \((S, D)\)-integral is Zariski-non-dense. On the other hand, when an iterate of \( \phi \) is a polynomial, i.e. there is a hyperplane \( H \) such that \((\phi(k))^*(H) = a^k H\), one can easily get Zariski-dense \( H \)-integral points in an orbit. In this case, the set \( \bigcup_{i=0}^{\infty}(\phi(i))^{-1}(H) \) is totally invariant under \( \phi \), and so it is conjectured to be a union of hyperplanes (the proof for \( \mathbb{P}^2 \) in \([3]\) is valid). Moreover, since \((\phi(k-i))(H)\) is irreducible, \((\phi(i))^{-1}(H)\) can only have one irreducible component. Therefore, \( \deg H^{(n)}_{nc} = 1 \) for all \( n \) when an iterate of \( \phi \) is a polynomial.

It is now natural to ask about Zariski-density of integral points in orbits when \( 2 \leq \sup_n \deg D^{(n)}_{nc} \leq N + 1 \). Characterizing such maps is an important first step. For example, in dimension 1, if iterates are never polynomials, Riemann-Hurwitz tells us that there is an iterate that has three distinct poles. Therefore, sup, \( n \), \( \deg D^{(n)}_{nc} \) is totally invariant under \( \phi \), and so it is conjectured to be a union of hyperplanes (the proof for \( \mathbb{P}^2 \) in \([3]\) is valid). Moreover, since \((\phi(k-i))(H)\) is irreducible, \((\phi(i))^{-1}(H)\) can only have one irreducible component. Therefore, \( \deg H^{(n)}_{nc} = 1 \) for all \( n \) when an iterate of \( \phi \) is a polynomial.

Example 6. Let \( \phi \) be \([X_0^2 : X_1^2 : X_1X_2^2 : X_2X_3^2 : \cdots : X_{N-1}X_N^2] \) on \( \mathbb{P}^N \). This is a rational map, undefined for example at \([0 : 0 : \cdots : 0 : 1]\) but the indeterminacy locus is contained in the hyperplane \( X_0 = 0 \). Let \( H = (X_N = 0) \). Then the support of \( \phi^i(H) \) is defined by \( X_{N-1}X_N = 0 \), the support of \( (\phi(2))^i(H) \) is defined by \( X_{N-2}X_{N-1}X_N = 0 \), and continuing in a similar way, one sees that the support of \( (\phi(k))^i(H) \) for \( k \geq N - 1 \) is defined by \( X_1 \cdots X_N = 0 \). Thus, sup, \( n \), \( \deg D^{(n)}_{nc} = N \) for this map.

If we let \( P = [2^N : 2^{N-1} : \cdots : 2 : 1] \), then we immediately see that \( \phi^{(m)}(P) \) is always integral with respect to \( H \). If we let \( a_i^{(m)} \) denote the power of 2 of the \( i \)-th coordinate of \( \phi^{(m)}(P) \) in the reduced form, all the coordinates of \( \phi^{(m+1)}(P) \) are divisible by \( 2^{N-i} \) and it is easy to prove by induction that \( a_0^{(m)} > a_1^{(m)} > \cdots > a_{N-1}^{(m)} \). Hence, \( a_0^{(m+1)} = 3a_0^{(m)} - a_{N-1}^{(m)} > 2a_0^{(m)} \), so all orbit points are distinct. Thus, the orbit contains infinitely many \( H \)-integral points.

With a bit more work, we can also show that the orbit is Zariski-dense. Since each coordinate of \( \phi \) is a monomial, one can view it as a map on \( \mathbb{G}^m_{\mathbb{Z}} \). Since the orbit points are integral, if they are not Zariski-dense, the Mordell–Lang conjecture on semiabelian varieties \([10]\) tells us that they are contained in a finite union of translates of subtori. Thus, for infinitely many \( m \)-s, \( (a_0^{(m)}, \ldots, a_{N-1}^{(m)}) \) satisfy a (possibly non-homogeneous) linear equation \( c_0 Y_0 + \cdots + c_{N-1} Y_{N-1} = c \). Now, from...
above discussion, we have the following relation:

\[
\begin{pmatrix}
\alpha_0^{(m+1)} \\
\vdots \\
\alpha_N^{(m+1)}
\end{pmatrix}
= \begin{pmatrix}
3 & 0 & \cdots & \cdots & \cdots & 0 & -1 \\
0 & 3 & 0 & \cdots & \cdots & 0 & -1 \\
0 & 1 & 2 & 0 & \cdots & 0 & -1 \\
0 & 0 & 1 & 2 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 1 & 2 & -1 \\
0 & \cdots & \cdots & 0 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\alpha_0^{(m)} \\
\vdots \\
\alpha_N^{(m)}
\end{pmatrix}.
\]

Let \( A \) denote the \( N \times N \) matrix above. The Jordan decomposition of \( A \) is one \( 1 \times 1 \) block of eigenvalue 3 and one \( (N - 1) \times (N - 1) \) block of eigenvalue 2, and the transformation matrix is

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 1 & 1 & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & 1 & \cdots & 0 & : \\
0 & 1 & 0 & \cdots & 0 & 0
\end{pmatrix},
\]

where the first column is an eigenvector of eigenvalue 3 and the second column of 2. From this, it is easy to compute that the first row of \( A^m \) has entries in \( 3^m \cdot \mathbb{Q}[m] \) and the other rows have entries in \( 2^m \cdot \mathbb{Q}[m] \). So the coefficient of \( Y_0 \) is zero. Now, because of the form of the transformation matrix, for any \( i \geq 3 \), \((i,j)\)-entry of \( A^m \) agrees with \((2,j)\)-entry for \( 2^m m^{i-2} \), \( \ldots \), \( 2^m m^{N-2} \), terms, so \( a_i^{(m)} \) agrees with \( a_1^{(m)} \) for \( 2^m m^{i-2} \) and higher-degree terms. Hence, \( a_1^{(m)}, \ldots, a_N^{(m)} \) are clearly linearly independent. On the other hand, \( 2^m (a_0 + a_1 m + \cdots + a_{N-2} m^{N-2}) = \alpha \) can only have finitely many solutions in \( m \), since by clearing the denominators of \( \alpha_i \)'s and \( \alpha \), the LHS is at least \( 2^m \) in absolute value if the polynomial part is nonzero. This concludes the proof that all the orbit points of \( P \) are distinct and \( H \)-integral, and that they form a Zariski-dense set.

Note that this example is not covered by Theorem 4 as it is only a rational map. We will quickly discuss a rare rational map whose arithmetic of orbits can be analyzed by Theorem 4 and its consequences; we will then present a theory which treats more general rational maps, including Example 6.

**Example 7.** Let \( \phi = [Y^5 : (\prod_{i=1}^4 L_i) Z : Z^5] \) on \( \mathbb{P}^2 \) defined over \( \mathbb{Q} \), where \( L_i(X,Y,Z) = \alpha_i X + \beta_i Y + \gamma_i Z \) are lines in general position with \( \alpha_i \neq 0 \). This is undefined at \([1 : 0 : 0]\), and

\[
\phi^{(2)} = \left[ Z^5 \prod_{i=1}^4 L_i(X,Y,Z)^5 : Z^5 \prod_{i=1}^4 \left( \alpha_i Y^5 + \beta_i Z \right) \prod_{j=1}^4 L_j(X,Y,Z) + \gamma_i Z^5 \right] : Z^{25}.
\]

Upon canceling the common \( Z^5 \), we see that \( \phi^{(2)} \) is actually a morphism of degree 20, since plugging in \( Z = 0 \) into the second coordinate results in \( Y = 0 \), and plugging these into the first coordinate leads to \( X = 0 \). For \( D = (X = 0) \), \( (\phi^{(2)})^* D \) contains \( L_i \)'s, so applying Theorem 4 (actually Proposition 1 and Corollary 3) to the map \( \phi^{(2)} \) twice, first to the orbit of \( P \in \mathbb{P}^2(\mathbb{Q}) \) and second to the orbit of \( \phi(P) \),
we unconditionally conclude that

\[
\left\{ \phi^{(m)}(P) : \frac{\log |a_0^{(m)}|}{\log \max(\{ |a_0^{(m)}|, |a_1^{(m)}|, |a_2^{(m)}| \})} \leq \frac{1}{20} - \epsilon \right\}
\]

is contained in a finite union of algebraic curves.

Of course, a rational map usually stays a rational map upon composition, and in general (arithmetic) dynamics for rational maps is much more difficult than for morphisms. To obtain Theorem 4-like results for rational maps, we note that the reason for the morphism assumption in Theorem 4 is that the height inequality (1) does not hold for rational maps. On the other hand, once something similar to (1) is obtained, the proof should go through. We remark that Lee [11] has recently proved a height inequality for rational maps whose indeterminancy is contained in a hyperplane. We recall his definition of "D-ratio," comparing the coefficients of irreducible divisors in two divisors: one is the pullback of the hyperplane via the blowup map, and the other is the pullback of the hyperplane via the morphism that resolves the indeterminancy.

**Definition 1.** Let \( \phi : \mathbb{P}^N \rightarrow \mathbb{P}^N \) be a rational map whose indeterminancy locus is contained in a hyperplane \( H \). Let \( \pi : V \rightarrow \mathbb{P}^N \) be a sequence of monoidal transformations (with smooth centers) which resolves the indeterminancy, with \( \tilde{\phi} : V \rightarrow \mathbb{P}^N \) the resolved morphism. Then the D-ratio \( r(\phi) \) of \( \phi \) is defined to be the minimum \( r \) such that the \( \mathbb{Q} \)-divisor \( \frac{r}{\deg \phi}(\tilde{\phi})^*H - \pi^*H \) is linearly equivalent to a divisor which is a nonnegative linear combination of the strict transform of \( H \) and the exceptional divisors from \( \pi \). The D-ratio depends on the choices of \( H \) and on the blowups that resolve \( \phi \).

Lee [11, Proposition 4.5] shows that \( r(\phi) \geq 1 \) and equal to 1 if \( \phi \) is a morphism. Further, he shows the following height inequality [11, Theorem A]:

\[
\frac{r}{\deg \phi} h(\phi(P)) > h(P) - C \quad \text{for } P \notin H.
\]

We will now use this to obtain an analogous of Theorem 4 for rational maps.

**Theorem 5.** Let \( \phi : \mathbb{P}^N \rightarrow \mathbb{P}^N \) be a dominant rational map defined over \( \overline{\mathbb{Q}} \) whose indeterminancy is contained in a hyperplane \( H \) and whose degree is \( d \geq 2 \). Let \( r \) be its D-ratio. Let \( D \) be a divisor on \( \mathbb{P}^N \) defined over \( \overline{\mathbb{Q}} \), and let \( D_{nc}^{(n)} \) the normal-crossings part of \((\phi^{(n)})^*(D)\). Let \( e_n = \deg(\phi^{(n)}) \). Let \( k \) be a number field that contains the fields of definition of \( \phi \), \( D \), and \( D_{nc}^{(n)} \). Let us assume Vojta’s conjecture for the divisor \( D_{nc}^{(n)} \). If

\[
e_n - \frac{\deg D_{nc}^{(n)} - (N + 1)}{\deg(D)} < \left( \frac{d}{r} \right)^n,
\]

then for any \( P \in \mathbb{P}^N(k) \) with \( \mathcal{O}_\phi(P) \cap H = \emptyset \) and for any finite set \( S \subset M_k \), the set \( \mathcal{O}_\phi(P) \cap (\mathbb{P}^N \setminus D)(R_S) \) is Zariski-non-dense. More precisely, if we let \( e_n = 1 - \left( e_n - \frac{\deg D_{nc}^{(n)} - (N + 1)}{\deg(D)} \right) \left( \frac{k}{\mathbb{Q}} \right)^n \), then

\[
\left\{ \phi^{(m)}(P) : \frac{\sum_{v \in S} \lambda_v(D, \phi^{(m)}(P))}{\deg(D) h(\phi^{(m)}(P))} \leq e_n - \epsilon \right\}
\]
is Zariski-non-dense for any $\epsilon > 0$.

**Remark 7.** We make several observations.

(i) For morphisms, $r = 1$ and $e_n = d^n$, so the condition for integral points in orbits to be Zariski-non-dense agrees with Theorem 2 and the $c_n$ agrees with that in Theorem 3. As $r$ gets bigger, one generally needs more degrees of the normal-crossings part to conclude Zariski-non-denseness of integral points in orbits.

(ii) Although [11] assumes that maps are polynomials for all of the dynamical results, his height inequality [13] holds away from the hyperplane, so for our purposes it is enough to assume that the forward orbit does not intersect the hyperplane.

(iii) By applying this theorem with $\psi = \phi^{(n)}$ and observing that the orbit of $P$ under $\phi$ is a finite union of orbits under $\psi$ (cf. the end of Example 7), we see that (14) can be replaced by

$$e_n - \frac{d \deg D^{(n)}_{nc} - (N + 1)}{\deg(D)} < \max\left(\frac{d}{r}, \frac{e_n}{r(\phi^{(n)})}\right).$$

(iv) By [11] Proposition 4.5, it follows that $e_n \geq (d/r)^n$. If $\phi$ is a polynomial on $A^N = \mathbb{P}^N \setminus H$, then $H^{(n)}_{nc} = H$ for all $n$, so we see that (14) will always fail. This is consistent with the fact that polynomial mappings can have infinitely many integral points in orbits.

**Proof.** The proof is very similar to that of Theorem 4 but we need to accommodate for the fact that $\phi$ is only a rational map. Specifically, the degree of $\phi^{(n)}$ might not be $d^n$, and the height inequality [11] needs to be replaced by the one with the D-ratio [13]. Note that $\deg((\phi^{(n)})^* D) = e_n \deg(D)$. Vojta’s conjecture for $D^{(n)}_{nc}$ tells us that there exists a constant $C$ and a finite union $Z_\epsilon$ of hypersurfaces such that

$$\sum_{v \in S} \lambda_v \left( D^{(n)}_{nc}, Q \right) < \left( N + 1 + \frac{\epsilon d^n \deg(D)}{r^n} \right) h(Q) + C$$

holds for all $Q \in \mathbb{P}^N(k) \setminus Z_\epsilon$. We also have

$$\sum_{v \in S} \lambda_v \left( D^{(n)} - D^{(n)}_{nc}, Q \right) \leq h(D^{(n)} - D^{(n)}_{nc})(Q) + C_1 \leq \left( e_n \deg(D) - \deg D^{(n)}_{nc} \right) h(Q) + C_1'.$$

Thus, for $Q \notin Z_\epsilon$ such that $O_{\phi}(Q) \cap H = \emptyset$,

$$\sum_{v \in S} \lambda_v(D, \phi^{(n)}(Q)) \leq \sum_{v \in S} \lambda_v((\phi^{(n)})^*(D), Q) + C_2$$

$$< \left( e_n \deg(D) - \deg D^{(n)}_{nc} + N + 1 + \frac{\epsilon d^n \deg(D)}{r^n} \right) h(Q) + C_3$$

$$\leq \left( e_n - \frac{\deg D^{(n)}_{nc} - (N + 1)}{\deg(D)} \right) \left( \frac{r}{d} \right)^n \deg(D) h(\phi^{(n)}(Q))$$

$$+ \epsilon \deg(D) h(\phi^{(n)}(Q)) + C_4 \quad \text{by (13)}$$

$$\leq \left( e_n - \frac{\deg D^{(n)}_{nc} - (N + 1)}{\deg(D)} \right) \left( \frac{r}{d} \right)^n h_D(\phi^{(n)}(Q)) + c h_D(\phi^{(n)}(Q)) + C_5.$$
For any \( m \geq n \), letting \( Q = \phi^{(m-n)}(P) \), we have
\[
\sum_{\nu \not \in S} \lambda_{\nu}(D, \phi^{(m)}(P)) > (c_n - \epsilon) h_D(\phi^{(m)}(P)) - C_6,
\]
as long as \( \phi^{(m-n)}(P) \notin \mathbb{Z} \) and \( \mathcal{O}_\phi(P) \cap H = \emptyset \). The rest of the argument is the same as Theorem 4.

**Example 8.** This is an example of Theorem 5 for which Vojta’s conjecture is not necessary. Let \( M_i \) be the linear form \( X_0 + X_1 + X_2 + X_4 - X_i \) and let \( \phi \) be the self-map
\[
\begin{bmatrix}
M_0 \prod_{j=1}^{d-1} L_{0,j} : M_1 \prod_{j=1}^{d-1} L_{1,j} : M_2 \prod_{j=1}^{d-1} L_{2,j} : X_0^{d-2}X_1X_2 : M_3 \prod_{j=1}^{d-1} L_{3,j}
\end{bmatrix}
\]
on \( \mathbb{P}^4 \) (note that the last coordinate is indexed with 3’s for convenience), where \( L_{i,j} \)'s are linear forms satisfying the following properties:

1. \( L_{i,j} \)'s have nonzero coefficients for \( X_4 \), and \( \{L_{0,1}, \ldots, L_{2,d-1}, M_0, M_1, M_2\} \) is in general position (i.e. any five of them are linearly independent).
2. \( \{P : L_{0,j}(P) = 0\} \cap \cdots \cap \{P : L_{3,j}(P) = 0\} \) is empty on the hyperplanes \( X_i = 0 \) in \( \mathbb{P}^4 \) for \( i = 0, \ldots, 2 \).
3. For any \( i \in \{0, 1, 2\} \) and \( k \in \{0, \ldots, 3\} \), the \( \mathbb{P}^2 \) defined by \( X_i = M_k = 0 \) does not contain any point of \( \bigcap_{\ell \neq k} \{P : L_{\ell,j}(P) = 0\} \).
4. For any \( i \in \{0, 1, 2\} \) and \( k, m \in \{0, \ldots, 3\} \) with \( k \neq m \), the \( \mathbb{P}^1 \) defined by \( X_i = M_k = M_m = 0 \) does not contain any point of \( \bigcap_{\ell \neq k, m} \{P : L_{\ell,j}(P) = 0\} \).

We will have an explicit example of the \( L_{i,j} \)'s at the end of this example, but it is intuitively easy to see that linear forms in general enough position will satisfy the above. Indeed, for (2), restricting linear forms to \( X_i = 0 \) makes them into hyperplanes in \( \mathbb{P}^3 \), and four hyperplanes in \( \mathbb{P}^3 \) in general position do not meet (over \( \mathbb{Q} \)). For (3), each linear form becomes a line on \( \mathbb{P}^2 \) upon restriction, three of which in general position do not meet. Finally, for (4), a linear form becomes a point on \( \mathbb{P}^1 \) upon restriction, and two general points do not coincide. Thus, linear forms in general position should work.

Now, let us see that the conditions above *unconditionally* ensure Zariski- non-denseness of integral points in orbits. First, we will show that the only indeterminancy of \( \phi \) is \([0 : 0 : 0 : 0 : 1]\). If \( P \) is an indeterminancy point, from the second to last coordinate, \( P \) is on the hyperplane \( X_i = 0 \) for some \( i = 0, 1, 2 \). Let us first suppose that none of the \( M_k \)'s vanishes at \( P \). Then one of the linear forms from each coordinate of \( \phi \) must vanish at \( P \), contradicting (2). Next, suppose that exactly one \( M_k \) vanishes at \( P \). Then for each \( \ell \in \{0, \ldots, 3\} \) not equal to \( k \), one of the linear forms \( L_{\ell,j} \) must vanish at \( P \), contradicting (3). Thirdly, suppose that exactly two of the \( M_k \)'s vanish at \( P \), say \( M_k = M_m \). Then for each \( \ell \neq k, m \), one of the linear forms \( L_{\ell,j} \) must vanish at \( P \), contradicting (4). Finally, any three of the \( M_k \)'s and \( X_i \) are linearly independent, so their simultaneous vanishing at \( P \) would imply that \( X_0 = X_1 = X_2 = X_3 = 0 \), namely \( P = [0 : 0 : 0 : 0 : 1] \).

Now we blow up \( P \). The key computation turns out to be completely symmetric with \( X_0, \ldots, X_3 \), so let us work on a patch where \( X_0 = x_0, X_1 = x_0x'_1, X_2 = x_0x'_2, X_3 = x_0x'_3, X_4 = 1 \). Since the blowup does not affect \( \phi \) outside the center,
we only need to look at the exceptional divisor, defined by $x_0 = 0$ on this patch, to see if this blowup resolves the indeterminacy. Here, $\phi$ is defined by

\[
\left[ x_0(x'_1 + x'_2 + x'_3) \prod L_{0,j}(0, 0, 0, 1) : x_0(1 + x'_2 + x'_3) \prod L_{1,j}(0, 0, 0, 1) : x_0(1 + x'_1 + x'_3) \prod L_{2,j}(0, 0, 0, 1) : x_0x_0^{-1}x'_1x'_2 : x_0(1 + x'_1 + x'_2) \prod L_{3,j}(0, 0, 0, 1) \right],
\]

so one factor of $x_0$ comes out. By condition (1), all the linear forms evaluated at $(0, 0, 0, 1)$ are nonzero. Thus, if this map were undefined, from the second, third, and the fifth coordinates, we must have $x'_2 + x'_3 = x'_1 + x'_3 = x'_1 + x'_2 = -1$. But then $x'_1 + x'_2 + x'_3 = -3/2$, and the first coordinate will be nonzero. Therefore, this map is well-defined everywhere on the exceptional divisor, and hence the indeterminacy of $\phi$ is resolved by this one blowup. Defining $H$ to be the hyperplane $X_3 = 0$, we see from the calculation above that $\phi^*(H)$ has $(d-1)E$. Thus, $\pi^*H = H + E$ and $\bar{\phi}^*H = dH + (d - 1)E$, so the D-ratio is

\[
d \cdot \max \left( \frac{1}{d}, \frac{1}{d-1} \right) = \frac{d}{d-1}.
\]

On the other hand, $(\phi^{(2)})^*H$ is defined by $M_0^{d-2}M_1M_2 \prod L_{0,j}^{d-2} \prod L_{1,j} \prod L_{2,j}$, so by (1) this divisor contains $3d$ linear forms in general position. Thus, \([14]\) for the second iteration becomes

\[
d^2 - 3d + 5 < (d-1)^2.
\]

This is satisfied for $d \geq 5$, so Theorem \([3]\) and Proposition \([1]\) apply. We unconditionally conclude that if $\phi^{(m)}(P) = [a^{(m)}_0 : \cdots : a^{(m)}_4]$ with $a^{(m)}_3 \neq 0$ for all $m$, then

\[
\left\{ \phi^{(m)}(P) : \frac{\log |a^{(m)}_3|}{\log \max |a^{(m)}_i|} \leq \frac{d - 4}{(d-1)^2} - \epsilon \right\}
\]

is Zariski-non-dense for any $\epsilon > 0$.

As for a specific example of the $L_{i,j}$’s, let $d = 5$ and let

\[
L_{i,j} = X_0 + (4i + 2 + j)X_1 + (4i + 2 + j)^2X_2 + (4i + 2 + j)^3X_3 + (4i + 2 + j)^4X_4.
\]

For (1), the independence of five of the $L_{i,j}$’s comes from the Vandermonde determinant. By a tiresome computation, we can also confirm that one or more of $M_0, M_1, M_2$ together with $L_{i,j}$’s are in general position as well, confirming (1) (if we want to avoid computation, we can just use the $L_{i,j}$’s, resulting in $\deg D^{(2)}_{\text{lin}} = 3d-3$ and needing to assume $d \geq 8$ instead). For (2), we check that a Vandermonde determinant with one of the columns removed is nonzero. For the last two conditions, let us assume that $i = 0$, i.e. $X_0 = 0$; other cases are similar. For (3) with $k = 0$, we have $X_0 = 0$ and $X_1 = -X_2 - X_3$, so we need to check that the determinant

\[
\begin{vmatrix}
a^2 - a & a^3 - a & a^4 \\
b^2 - b & b^3 - b & b^4 \\
c^2 - c & c^3 - c & c^4
\end{vmatrix}
\]

is nonzero. This determinant is $-abc(a-b)(a-c)(b-c)((a-1)(b-1)(c-1)+1)$, so this is clearly nonzero for distinct integers $\geq 3$. Similarly, for $k = 1$ (resp. $k = 2, 3$), we check that the $3 \times 3$ determinant where each row has the shape $a, a^3 - a, a^4$ (resp. $a^2, a^3 - a, a^4$, and $a^2 - a, a^3, a^4$) is nonzero for distinct integers $\geq 3$. 

Theorem 6. Let each over a fixed number field, it is important that we do not need a field extension for two coordinates where we can choose linear forms. Choosing $D$ and $P$ come up with other constructions to have unconditional examples of Theorem 5 on example seems to be efficient, but of course unnecessary. It will be interesting to is tion: the fact that the points are in an orbit was only minimally used in all of the $\phi$ the map $D$ and assuming Vojta’s Conjecture (Conjecture 1) is a height inequality $\phi$ and assuming Vojta’s conjecture for the divisor $D_{Nc,\psi}$, the normal-crossings part of $(\psi^{(n)})^*(D)$. Assume that $\deg D_{Nc,\psi}^{(n)} > N + 1$ for some $n$, and further assume the “inverse image property”: there exists a number field $k$ such that for each $m$, there exists $Q_m \in \mathbb{P}^N(k)$ such that $\phi^{(m)}(P) = \psi^{(n)}(Q_m)$. Then letting $c_n = \frac{\deg D_{nc,\psi}^{(n)} - (N+1)}{d^n \deg(D)}$ and assuming Vojta’s conjecture for the divisor $D_{nc,\psi}^{(n)}$, for all finite subset $S$ of $M_k$ and $\epsilon > 0$, the set
\[
\left\{ \phi^{(m)}(P) : \sum_{v \notin S} \frac{\lambda_v(D, \phi^{(m)}(P))}{\deg(D) h(\phi^{(m)}(P))} \leq c_n - \epsilon \right\}
\]
is Zariski-non-dense.

Remark 9. Note that the choice of $\psi$ can depend on $P$. This theorem is the analog of Theorem 4 and analogs of Corollaries 1 and 2 also hold for this situation.

Proof. Let $D_{nc,\psi}^{(n)} = (\psi^{(n)})^*(D)$. The argument is essentially the same as the proof of Theorem 5, replacing $\phi$ by $\psi$ when appropriate. Enlarging $k$ if necessary, we
may assume that \( D^{(n)}_{nc, \psi} \) and \( \psi \) and \( \phi \) are defined over \( k \). By Vojta’s conjecture for the divisor \( D^{(n)}_{nc, \psi} \) over \( k \), there exist constants \( C \) and \( C_1 \) and a union \( Z_\epsilon \) of hypersurfaces such that

\[
\sum_{v \in S} \lambda_v \left( D^{(n)}_{nc, \psi}, Q \right) < (N + 1 + \epsilon \deg(D)) h(Q) + C \quad Q \in \mathbb{P}^N(k) \setminus Z_\epsilon
\]

\[
\sum_{v \in S} \lambda_v \left( D^{(n)}_{nc, \psi} - D^{(n)}_{nc, \psi}, Q \right) \leq h_{D^{(n)}_{nc, \psi} - D^{(n)}_{nc, \psi}}(Q) + C_1
\]

\[
\leq (d^n \deg(D) - \deg D^{(n)}_{nc, \psi}) h(Q) + C'_1.
\]

As before, as long as \( Q_m \notin Z_\epsilon \), the addition of the two inequalities gives us

\[
\sum_{v \in S} \lambda_v (D, \phi^{(m)}(P)) = \sum_{v \in S} \lambda_v (D, \psi^{(n)}(Q_m)) = \sum_{v \in S} \lambda_v (D^{(n)}_{nc, \psi}, Q_m) + C_2
\]

\[
< \left( d^n \deg(D) - \deg D^{(n)}_{nc, \psi} + N + 1 + \epsilon \deg(D) \right) h(Q_m) + C_3
\]

\[
\leq (1 - c_n + \epsilon) \deg(D) h(\psi^{(n)}(Q_m)) + C_4
\]

\[
\leq (1 - c_n + \epsilon) h_D(\phi^{(m)}(P)) + C_5.
\]

The rest of the argument is the same as Theorem 4. \( \square \)

**Example 9.** When \( \psi \) commutes with \( \phi \), the hypothesis of the theorem is satisfied. Indeed, given a fixed preimage \( Q \) of \( P \) via \( \psi^{(n)} \), we have \( \phi^{(m)}(P) = \phi^{(m)}(\psi^{(n)}(Q)) = \psi^{(n)}(\phi^{(m)}(Q)) \). Thus, we can set \( Q_m = \phi^{(m)}(Q) \) and \( k \) to be a number field containing the fields of definition of \( Q \) and \( \phi \). On the other hand, a general morphism does not have another nontrivial morphism that commutes with it just for the dimension reasons, and in fact commuting pairs of endomorphisms on \( \mathbb{P}^N \) have been classified by Dinh–Sibony [6].

We make a final remark that all the results we obtained can be combined, so for example Theorem 4 can work with rational maps using \( D \)-ratio and Theorem 5 becomes unconditional if the relevant divisor satisfies the hypothesis of Proposition 2.

We end the paper with some open questions. A first obvious one is computing \( c = \sup_n \frac{\deg D^{(n)}_{nc} - (N + 1)}{\deg(D)} \). The normal-crossings condition is not an easy definition to work with, so in general it is very difficult to compute \( c \). It would be great to have an example of \( c = 1 \) for which all the relevant \( D^{(n)}_{nc} \) are hyperplanes in general position. This can potentially give us an unconditional example of Corollary 2(ii).

Secondly, as remarked before Example 6, it would be beneficial to geometrically characterize morphisms on \( \mathbb{P}^N \) for which \( 2 \leq \sup_n \deg D^{(n)}_{nc} \leq N + 1 \), as this falls between the polynomial cases and the cases that can be treated by Theorem 4. Some examples were discussed in [19], but a more complete characterization should enable us answer whether integral points in orbits under these maps are Zariski-dense. Thirdly, Example 5 shows that the hypothesis in Theorem 2 is not yet a necessary condition. Is there a better geometric criterion, for instance by analyzing whether there are totally ramified fixed subvarieties? Fourthly, it would be great to generalize arguments in Example 4 to work for more general maps which have enough degrees in \( D^{(n)}_{lin} \). As remarked there, the methods to go from Zariski-non-dense to finite are standard but adhoc. This problem is related to the (rank-one)
dynamical Mordell–Lang question, so it has an importance on its own. Finally, it would be illuminating to find \( \phi \) and \( \psi \) which do not commute but still satisfy the “inverse image property” of Theorem 6 for some point \( P \). If the space can be decomposed into two directions in some sense and if the two maps commute in one direction and the point is preperiodic in another direction, it might be possible. These are all interesting questions, meriting further study.

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