The Hopf algebra structure of $GL(1, H_q)$ and the isomorphism between $SP_q(1)$ and $SU_q(2)$

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Abstract

In this Letter, we introduce the Hopf algebra structure of the quantum quaternionic group $GL(1, H_q)$ and discuss the isomorphism between the quantum symplectic group $SP_q(1)$ and the quantum unitary group $SU_q(2)$.

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1. Introduction

During the past few years, quantum groups [1,2] and $q$-deformed enveloping algebras [3] have been intensively studied both by mathematicians and mathematical physics. From a mathematical point of view, these algebraic structures are just special classes of Hopf algebras [4].

In modern mathematical research, classical Lie groups and Lie algebras are naturally related not only to the problems of algebras but also to that of geometry and then to that of analysis. These materials are quite basic and play quite important roles in many respects. However, the situation is completely different when we turn our attention to the quantum groups. Quantum groups are not groups in the ordinary sense, they may be thought of as matrix groups in which the matrix elements are themselves noncommutative, obeying sets of bilinear product relations [2].

Quaternionic quantum algebra and its coalgebra structure is important in physics. The relevance of quaternionic vector spaces was recently demonstrated in quantum mechanics by Horwitz et al. [5] and Adler [6]. One of the specific properties of quaternionic mechanics is the dependence of its physical content on the definition of the tensor product [5,7].

Quaternionic Lie groups can be defined in terms of matrices with quaternionic elements. To make contact with the usual formulation of Lie groups in terms matrices with complex matrix elements it is useful to represent each quaternion in terms of 2x2 complex matrices. Therefore, the quantum deformation of the quaternion algebra $H$ may be introduce using the idea of the quantum matrix theory [1,2].

The notion of quantum quaternions $GL(1, H_q)$ of one parameter $q$ was introduced independently by Marchiafava and Rembielinski [8] and the author [9]. In this Letter, we obtain the Hopf algebra structure of the quantum quaternionic group $GL(1, H_q)$ and show that $SP_q(1) \cong SU_q(2)$. In Section 2, we introduce the one parameter quantum deformation of the quaternionic group $GL(1, H)$ along the lines of work of [8] and [9]. In Section 3, we show that there is an isomorphism between $SP_q(1)$ and $SU_q(2)$ just as the classical case. The Hopf algebra structure of $GL(1, H_q)$ is introduced in Section 4.

2. Review of $GL(1, H_q)$

We shall define the $q$-deformation of the quaternionic group $GL(1, H)$, denoted
by GL(1,H_q), as an algebra A_q equipped with a ∗-operation, where A_q has the following properties:

(1) The unital associative algebra A_q generated by a_0, a_1, a_2, a_3 and q-commutation relations \cite{8,9}

\[
x_{\pm}a_{k} = q^{\pm 1}a_{k}x_{\pm}, \quad k = 2, 3, \quad a_{2}a_{3} = a_{3}a_{2}
\]

\[
a_{0}a_{1} = a_{1}a_{0} - \frac{i}{2}\lambda_{-}(a_{2}^{2} + a_{3}^{2}),
\]

where i = -1, q is a nonzero real number, \(\lambda_{-} = q - q^{-1}\) and \(x_{\pm} = a_{0} \pm ia_{1}\).

(2) The operation of conjugation in A_q is a map ∗ : A_q → A_q and acts on the generators \(a_{k} (k = 0, 1, 2, 3)\) of A_q as follows \cite{8,9}

\[
a_{k}^* = a_{k}, \quad k = 0, 1, \quad a_{2}^* = \frac{1}{2}(\lambda_{+}a_{2} - i\lambda_{-}a_{3}),
\]

\[
a_{3}^* = \frac{i}{2}(\lambda_{-}a_{2} - i\lambda_{+}a_{3}),
\]

where \(\lambda_{+} = q + q^{-1}\). It is easy to show that the map ∗ is involutive, that is, it satisfies

\[(a_{k}^*)^* = a_{k}, \quad k = 0, 1, 2, 3.\]

The map ∗ will be called the anti-involution on A_q.

(3) Assume that any quaternion h has a representation

\[h = a_{0}e_{0} + a_{1}e_{1} + a_{2}e_{2} + a_{3}e_{3}\]

with the generators of A_q and it will be called the q-quaternion, and in this case we shall say that q-quaternion h belongs to GL(1,H_q).

Here the quaternion units \(e_{k}\) have multiplicative properties defined by

\[e_{i}e_{j} = -\delta_{ij}e_{0} + \epsilon_{ijk}e_{k}\]

and

\[e_{0}^{2} = 1, \quad e_{0}e_{k} = e_{k}e_{0}, \quad k = 1, 2, 3\]

where \(\delta_{ij}\) denotes the Kronecker delta and

\[e_{ijk} = \frac{1}{2}(i - j)(j - k)(k - i)\]

and also

\[a_{k}e_{l} = e_{l}a_{k}, \quad k, l = 0, 1, 2, 3.\]
The quaternionic conjugation is defined by

\[ e_k^* = 2\delta_{0,k}e_0 - e_k, \quad k = 0, 1, 2, 3. \]  

(8)

It is easy to verify that the relations (1) do obey the quantum Yang-Baxter equations [1].

Relations between starred and unstarred generators are determined by

\[ x_\pm a_0^* = q^{\mp 1}a_0^* x_\pm, \quad a_k^* a_l = a_l a_k^*, \quad k = 2, 3, \]

(9)

for \( l = 2, 3 \). Note that \( (x_\pm)^* = x_\mp \).

The conjugation of a \( q \)-quaternion \( h \in GL(1, H_q) \) given by (4) is introduced as

\[ \bar{h} = e_0 a_0^* - e_1 a_1^* - e_2 a_2^* - e_3 a_3^*. \]

(10)

Hence, we can introduce the \( q \)-norm of the \( q \)-quaternion \( h \) [8,9]

\[ \mathcal{N}_q(h) = h\bar{h} = \bar{h}h = a_0^2 + a_1^2 + \frac{1}{2} \lambda_+(a_2^2 + a_3^2) \]

(11)

using the relations (1) and (2) with (5)-(7). It is easy to verify that \( \mathcal{N}_q(h) \) commutes with all the generators \( a_k \) of \( A_q \), that is \( calN_q(h) \) belongs to the center of \( GL(1, H_q) \). Note that if for \( h_1 \) and \( h_2 \) are two \( q \)-quaternions with the generators \( a_k \) and \( b_k \) and the generators \( a_k \) commute with \( b_k \), then the (quaternionic) product \( h_1 h_2 \) is a \( q \)-quaternion, i.e., \( h_1 h_2 \in GL(1, H_q) \). It is also verified that

\[ \mathcal{N}_q(h_1 h_2) = \mathcal{N}_q(h_1)\mathcal{N}_q(h_2). \]

(12)

Since the \( q \)-norm, \( \mathcal{N}_q(h) \), is central, the components (the generators \( a_k \) of \( A \)) of \( q \)-quaternion \( h \) may be normalized so that \( \mathcal{N}_q(h) = 1 \), that is,

\[ a_0 a_1 - a_1 a_0 = \frac{1 - q^2}{1 + q^2} (1 - a_0^2 - a_1^2). \]

Thus, we obtain the quantum subgroup \( SL(1, H_q) = SP_q(1) \) of quantum quaternionic group \( GL(1, H_q) \) with \( \mathcal{N}_q(h) = 1 \). In the limit \( q \to 1 \) of the deformation parameter, we obtain a classical quaternionic group.

The addition of two \( q \)-quaternions must satisfy the following properties in order to be a \( q \)-quaternion. Let \( h \) and \( h' \) be two \( q \)-quaternion with the generators \( a_k \) and \( b_k \) for \( k = 0, 1, 2, 3 \) as follows:

\[ x_\pm^1 x_\pm^2 = q^{-2} x_\pm^2 x_\pm^1, \quad x_\pm^1 x_\pm^2 = x_\pm^2 x_\pm^1, \]

\[ x_+^1 b_k = q^{-1} b_k x_+^1, \quad x_-^1 b_k = q^{-1} b_k x_-^1 - \lambda_- a_k x_-^2, \quad k = 2, 3, \]

4
\[ x_k^2 a_k = q a_k x_k^2, \quad x_k^2 a_k = q a_k x_k^2 + \lambda_1 b_k x_k^1, \quad k = 2, 3, \]
\[ x_k^1 x_k^2 = x_k^2 x_k^1 + \lambda_1 (y_k^1 y_k^2 + y_k^2 y_k^1), \]
\[ y_k^1 y_k^2 = q^{-2} y_k^2 y_k^1, \quad y_k^1 y_k^2 = y_k^2 y_k^1 + \lambda_1 x_k^1 x_k^2, \]
\[ y_k^1 y_k^2 = q^{-2} y_k^2 y_k^1, \quad y_k^1 y_k^2 = y_k^2 y_k^1 + \lambda_1 x_k^1 x_k^2. \]

(It is easy to see that these relations are consistent with the braid statics in [10]). Then the generators \( c_k = a_k + b_k \) \((k = 0, 1, 2, 3)\) for which
\[
h'' = h + h' = (x^1_+ + x^2_+) + (y^1_+ + y^2_+) j \\
= (a_0 + b_0)e_0 + (a_1 + b_1)e_1 + (a_2 + b_2)e_2 + (a_3 + b_3)e_3
\]
also obey the relations (1) so that \( h'' \) is in GL(1,H\(q\)). Here \( j = e_2 \) \((ij = e_3)\).

3. The Isomorphism Between SP\(_q\)(1) and SU\(_q\)(2)

We begin with some information. It is well known that if \( \varphi : G_1 \longrightarrow G_2 \) is a one-to-one homomorphism of groups, then the map \( \varphi \) is an isomorphism of \( G_1 \) onto the subgroup \( \varphi(G_2) \) of \( G_2 \). So we can consider \( G_1 \) as a subgroup of \( G_2 \). First we will construct a one-to-one homomorphism
\[
\varphi : \text{GL}(1,H\(q\)) \longrightarrow \text{GL}_q(2)
\]
and then for \( h \in \text{GL}(1,H\(q\)) \) we are going to use as the \( q \)-norm of \( h \) the \( q \)-determinant of \( \varphi(h) \). Here the group \( \text{GL}_q(2) \) is the quantum group of 2x2 nonsingular matrices whose matrix elements obey certain \( q \)-dependent commutation relations (similar to (1)) [2].

For each \( q \)-quaternion \( h \) given by (4), define \( \varphi(h) \) as a 2x2 matrix by
\[
\varphi(h) = \begin{pmatrix}
a_0 + i a_1 & a_2 + i a_3 \\
-a_2 + i a_3 & a_0 - i a_1
\end{pmatrix},
\]
where
\[
\varphi(e_1) = i \sigma_3, \quad \varphi(e_2) = i \sigma_2, \quad \varphi(e_3) = i \sigma_1
\]
with the Pauli matrices satisfying
\[
\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k, \quad \sigma_i \sigma_j + \sigma_j \sigma_i = 0, \quad i \neq j.
\]
It is easy to verify that the map \( h \longmapsto \varphi(h) \) is one-to-one and
\[
\varphi(h_1 h_2) = \varphi(h_1) \varphi(h_2)
\]
for $h_1, h_2 \in \text{GL}(1, H_q)$ provided that the components (the generators of $A_q$) of $h_1$ commute with those of $h_2$ (recall that, in this case $h_1 h_2 \in \text{GL}(1, H_q)$). So this map is a homomorphism.

Next, if $h \in \text{GL}(1, H_q)$ we have
\begin{equation}
\text{Det}_q(\varphi(h)) = (a_0 + ia_1)(a_0 - ia_1) + q(a_2 + ia_3)(a_2 - ia_3) = N_q(h).
\end{equation}

Now we discuss the isomorphism between $\text{SP}_q(1)$ and $\text{SU}_q(2)$. $\text{SP}_q(1)$ is the set of all $q$-quaternions of unit length ($q$-norm) and $\text{SU}_q(2)$ is the set of all complex 2x2 matrices $A$ such that (see, for example, [11])
\begin{equation}
(A^*)^T A = I \quad \text{and} \quad \text{Det}_q(A) = 1,
\end{equation}
where $T$ denotes the matrix transposition and $A^*$ stands for the Hermitian conjugate of $A$. That is, $A \in \text{SU}_q(2)$ if and only if $A \in \text{GL}_q(2)$ with (18). The operation in $\text{SP}_q(1)$ is multiplication of $q$-quaternions, in $\text{SU}_q(2)$ it is matrix multiplication. Consider the map
\begin{equation}
\varphi : \text{SP}_q(1) \longrightarrow \text{SU}_q(2).
\end{equation}

We have seen that the map $\varphi$ induces a one-to-one homomorphism of $\text{GL}(1, H_q)$ into $\text{GL}_q(2)$, thus restriction of $\varphi$ to $\text{SP}_q(1)$ is still a one-to-one homomorphism. Therefore, we just need to show that
\begin{enumerate}
  \item $h \in \text{SP}_q(1) \implies \varphi(h) \in \text{SU}_q(2)$ and
  \item for every $A \in \text{SU}_q(2)$ there exists some $\varphi(h)$ with $h \in \text{SP}_q(1)$.
\end{enumerate}

Note that the matrix $\varphi(h)$ in (15) is a $\text{GL}_q(2)$ matrix. This is easy to show that using the relations (1).

Let
\[ A = \begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix} \]
be in $\text{SU}_q(2)$. Then we find that
\[ d = a^∗ \quad \text{and} \quad c = -q^{-1}b^∗ \]
with $\text{Det}_q(A) = 1$.

On the other hand, using the relations (2), we can write $([\varphi(h)]^*)^T \varphi(h) = I$ since $N_q(h) = 1$. Here $I$ denotes the 2x2 unit matrix. Also $\text{Det}_q(\varphi(h)) = 1$. Now
\[ a_2 - ia_3 = q^{-1}(a_2 + ia_3)^∗ \]
with (2), so that
\[ \varphi(h) = \begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -q^{-1}(a_2 + ia_3)^* & (a_0 + ia_1)^* \end{pmatrix}. \] (22)

It is easy to verify that the matrix elements of \( \varphi(h) \) satisfy the relations of \( \text{SU}_q(2) \) provided \( (ab)^* = b^*a^* \), that is, \( \varphi(h) \in \text{SU}_q(2) \). So, if \( a = a_0 + ia_1 \) and \( b = a_2 + ia_3 \) we may take \( h = a_0 + ia_1 + (a_2 + ia_3)i \) and have \( \varphi(h) = A \) (and \( \mathcal{N}_q(h) = 1 \)). This proves that the map \( \varphi \) in (19) is an isomorphism.

4. The Hopf Algebra Structure of \( GL(1, H_q) \)

Now we introduce three operators \( \Delta, \epsilon \) and \( S \) on \( GL(1,H_q) \), which are called the comultiplication, the counit, and the antipode (co-inverse), respectively.

(1) The comultiplication \( \Delta \) is defined by
\[ \Delta(h) = h \otimes h. \] (23)

The action of comultiplication \( \Delta \) on the generators \( a_k \) of \( A_\Delta \) can be introduced as follows:
\begin{align*}
\Delta(a_0) &= a_0 \otimes a_0 - (a_1 \otimes a_1 + a_2 \otimes a_2 + a_3 \otimes a_3), \\
\Delta(a_1) &= a_0 \otimes a_1 + a_1 \otimes a_0 + a_2 \otimes a_3 - a_3 \otimes a_2, \quad (4.2) \\
\Delta(a_2) &= a_0 \otimes a_2 + a_2 \otimes a_0 + a_3 \otimes a_1 - a_1 \otimes a_3, \\
\Delta(a_3) &= a_0 \otimes a_3 + a_3 \otimes a_0 + a_1 \otimes a_2 - a_2 \otimes a_1,
\end{align*}
and
\[ \Delta(e_0) = e_0 \otimes e_0, \] (4.3)

where \( \otimes \) denotes the tensor product. Note that the relations (24) are invariant under (1). For simplicity, we prove the invariance of one of the relations in (1) here. Proofs of the remaining formulas are similar. A direct calculation shows that
\begin{align*}
\Delta(x_+)\Delta(a_2) &= q\Delta(a_2)(x_+ \otimes x_+) - \frac{\lambda}{2} [\Delta(a_2) + i\Delta_2](y_+ \otimes y_-) \\
&- \frac{1}{2} [\lambda_+ \Delta(a_2) - i\lambda_- \Delta_2](y_+ \otimes y_-) \\
&= q\Delta(a_2)\Delta(x_+)
\end{align*}

where \( y_\pm = a_2 \pm ia_3 \) and
\[ \Delta_2 = a_0 \otimes a_3 - a_3 \otimes a_0 + a_1 \otimes a_2 + a_2 \otimes a_1. \]
Using (23), it is also easy to show that
\[
\Delta(N_q(h)) = N_q(h) \otimes N_q(h).
\] (26)

The comultiplication \( \Delta \) is an algebra homomorphism which is co-associative, that is,
\[
(\mathcal{I} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathcal{I}) \circ \Delta,
\] (4.5)
where \( \circ \) stands for the composition of maps and \( \mathcal{I} \) is the identity map.

(2) The counit \( \varepsilon \) is introduced by
\[
\varepsilon(h) = e_0
\] (28)
whose action on the generators \( a_k \) of \( \mathcal{A}_q \) can be defined by
\[
\varepsilon(a_k) = \delta_{0,k} e_0, \quad k = 0, 1, 2, 3
\] (29)
and also
\[
\varepsilon(e_0) = e_0.
\] (30)
The counit \( \varepsilon \) is an algebra homomorphism such that
\[
(\varepsilon \otimes \mathcal{I}) \circ \Delta = \mathcal{I} = (\mathcal{I} \otimes \varepsilon) \circ \Delta.
\]
Thus we have verified that \( GL(1, H_q) \) is a bialgebra with multiplication \( m \) satisfying the associativity axiom:
\[
m \circ (m \otimes \mathcal{I}) = m \circ (\mathcal{I} \otimes m)
\]
where \( m(h_1 \otimes h_2) = h_1 h_2 \).

A bialgebra with the extra structure of the antipode is called a Hopf algebra [4].

(3) We can introduce the antipode \( S \) as follows:
\[
S(h) = h^{-1}.
\] (31)
The action of antipode \( S \) on the generators \( a_k \) of \( \mathcal{A}_q \) can be defined by
\[
S(a_k) = N_q^{-1}(h)(2 \delta_{0,k} a_0 - a_k^*)
\] (32)
for \( k = 0, 1, 2, 3 \). The antipode \( S \) is an algebra anti-homomorphism which satisfies
\[
m \circ [(S \otimes \mathcal{I}) \circ \Delta] = \varepsilon = m \circ [(\mathcal{I} \otimes S) \circ \Delta].
\] (33)
The comultiplication, counit and antipode which are specified above supply GL(1,H_q) with a Hopf algebra structure.

5. Discussion

We have introduced the quantum deformation of one parameter of 1x1 quaternionic group, GL(1,H_q) along the lines of the work of [9]. We have constructed the Hopf algebra structure of GL(1,H_q) and have discussed an isomorphism between SP_q(1) and SU_q(2). We hope that the methods used in this paper will be helpful for an explicit construction of GL(n,H_q) and its Hopf algebra structure.

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