UNIQUENESS OF FORM EXTENSIONS AND DOMINATION OF SEMIGROUPS

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ABSTRACT. In this article, we present a new method to study uniqueness of form extensions in a rather general setting. The method is based on the theory of ordered Hilbert spaces and the concept of domination of semigroups. Our main abstract result transfers uniqueness of form extension of a dominating form to that of a dominated form. This result can be applied to a multitude of examples including various magnetic Schrödinger forms on graphs and on manifolds.

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Introduction

In many situations in mathematics and physics one is given a Laplace-type operator on a domain of smooth functions and looks for self-adjoint extensions. Existence of such self-adjoint extensions follows from general theory of forms. Indeed, there always exists an extension with Dirichlet boundary condition and an extension with Neumann boundary condition. Accordingly, these are the most common extensions. Now, it may well be that these two extensions agree and the question whether this happens is of quite some interest. In this context we also mention the even stronger property of essential self-adjointness, i. e., uniqueness of a self-adjoint extension studied extensively on manifolds, see e.g. [3,24,25,36], and on graphs, see e. g. [4,5,12,13,19,21,45,46].

On the structural level, the question whether the Laplacian with Dirichlet and the Laplacian with Neumann boundary conditions agree is connected to an additional feature of the Laplacian viz that it may generate a Markov semigroup. More specifically, in typical situation all self-adjoint extensions of the Laplacian which generate a Markov semigroup can be shown to lie between the Laplacian with Dirichlet boundary conditions and the Laplacian with Neumann boundary conditions (see [8] for open subsets of Euclidean space, [11] for locally finite graphs and [34] for recent results dealing with general Dirichlet forms). Thus, equality of these two boundary conditions then amounts to uniqueness of a self-adjoint extension generating a Markov process. This phenomenon is known as Markov uniqueness. Clearly, it is of substantial...
interest in any operator theoretic treatment of Markov processes and has therefore received ample attention, see e.g. [6,9,17,20,22,33,34,37,40].

The corresponding questions can be asked for Laplacians on functions as well as for the more involved Laplacians on bundles or with magnetic or electric potential. In fact, recent years have seen quite a few articles (e.g. [3,5,27,28]) dealing with uniqueness questions for extensions of Laplacians with magnetic potential or Laplacians on bundles over graphs and manifolds.

In this paper we present a new approach to studying equality of Dirichlet and Neumann Laplacian on bundles or with magnetic potential. Our approach even deals with uniqueness questions in a substantially more general context. Its key element is to consider the question of uniqueness within the framework of dominating semigroups. The study of domination of semigroups has a long history. A treatment of domination in terms of a Kato inequality for the generators of the semigroups was given in the influential works of Simon [39] and Hess, Schrader, Uhlenbrock. Here, [39] deals with comparison of two semigroups acting on the same $L^2$-space and [14] deals with comparison of two semigroups acting on rather general and even possibly different Hilbert spaces. For semigroups acting on the same $L^2$-space a characterization of domination via forms was later developed by Ouhabaz in [30,31] and then extended to forms acting on vector-valued functions by Manavi, Vogt, Voigt [29]. For our considerations we rely on the framework developed by us in the companion work [23]. This framework can be seen as an abstraction of [29] to the setting of rather general Hilbert spaces originally studied in [14].

Our main abstract theorem (Theorem 2.3) gives stability of uniqueness under a domination condition. To the best of our knowledge it is the very first result of its kind. It is highly relevant as often the question of uniqueness of form extensions is easier or already proven for the dominating form.

As a corollary we obtain that, roughly speaking, Dirichlet and Neumann Laplacian on a bundle agree whenever the corresponding Laplacians on the underlying space agree (Corollaries 2.4 and 2.6). This type of result can then be applied to various examples including magnetic Schrödinger operators on manifolds and on graphs (see Sections 3 and 4). In particular, we recover in a new and direct way all known examples for graphs and domains in Euclidean space and provide new examples which were not treated earlier. Another application of our main theorem to Kac regularity of domains has been given in a recent preprint [44] by the third-named author.

In retrospect it is not completely surprising that domination of semigroups plays a role in investigation of uniqueness issues. Indeed, as already discussed above, domination of semigroups is equivalent to the validity of a variant of Kato’s inequality for the generators and this inequality is a key element in all previous proofs of essential self-adjointness of Laplacians with magnetic potential. However, even in those cases where this was shown earlier, the actual line of reasoning is quite different from ours: It proceeds by treating the magnetic situation by mimicking the proof given for the Laplacian and invoking Kato’ inequality. In this sense, our paper provides the first conceptual connection between uniqueness of extensions and the theory of dominating semigroups. Note, however, that our result does not deal with essential self-adjointness but rather with a somewhat weaker statement that can be thought of as a form of Markov uniqueness (see discussion above). In the final analysis the reason that our methods do not give stability of essential self-adjointness may come from the fact that domination of semigroups concerns the order structure and can therefore only be expected to be
of use in connection with extensions respecting some form of order such as extensions given Markov semigroups.

The framework of this article are symmetric forms and selfadjoint operators as this is the situation in the applications we have in mind. However, it is to be expected that sectorial forms and their generators can be treated by similar means as well as the domination theory developed in \cite{29} works for such forms.

The article is organized as follows: In Section 1 we review the basics on quadratic forms on Hilbert spaces and domination of operators. In Section 2 we prove the main theorem of this article (Theorem 2.3), a criterion for form uniqueness in terms of domination. In the subsequent two sections we discuss the above mentioned applications, namely Schrödinger forms on vector bundles on manifolds in Section 3 and Schrödinger forms on vector bundles over graphs in Section 4.

The article has its origin in the master’s thesis of one of the authors (M. W.).

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1. Quadratic Forms and Domination of Semigroups

Our considerations are cast in the framework of semigroups associated to lower bounded forms. The main abstract result of this article (presented in the next section) gives stability of form uniqueness under domination. Here domination is meant in the sense of \cite{23} and form uniqueness will be phrased via suitable cores of the form. Specifically, these cores will have an ideal property. A crucial role is played by forms on \( L^2 \)-spaces satisfying the first Beurling Deny criterion. The necessary background and notation is discussed in this section. If not noted otherwise the material can be found in standard textbooks such as \cite{32,42}.

Let \( H \) be a Hilbert space. A quadratic form \( q: D(q) \to \mathbb{R} \) defined on a subspace \( D(q) \) of \( H \) induces a sesquilinear form

\[
D(q) \times D(q) \to \mathbb{C}, (f,g) \mapsto \frac{1}{4} \sum_{k=1}^{4} q(f + i^k g)
\]

We will not distinguished between a quadratic form and the induced sesquilinear form and write \( q \) for both of them. Moreover, all quadratic forms in this article are assumed to be densely defined.

The form \( q \) is said to be lower bounded if there exists \( \lambda \in \mathbb{R} \) such that \( q(f) \geq -\lambda \|f\|^2_H \) for all \( f \in D \). In this case the form \( q_\mu := q + \mu \langle \cdot, \cdot \rangle \) is an inner product on \( D(q) \) and the norms associated to \( q_\mu, q_{\mu'} \) are equivalent for \( \mu, \mu' > \lambda \). If the choice of \( \mu \) does not matter, we will write \( \langle \cdot, \cdot \rangle_q \) for \( q_\mu \) and \( \|\cdot\|_q \) for the associated norm and call it the form norm of \( q \). A lower bounded form \( q \) is closed if \( (D(q), \langle \cdot, \cdot \rangle_q) \) is complete (and, hence, a Hilbert space). A dense subspace of \( (D(q), \|\cdot\|_q) \) is also called a form core.

For every closed form \( q \) there exists a unique self-adjoint operator \( T \) with \( D(T) \subset D(q) \) and \( \langle Tf, g \rangle = q(f,g) \) for all \( f \in D(T), \ g \in D(T) \). The domain of \( T \) is then dense in \( D(q) \).
with respect to $\|\cdot\|_q$. The operator $T$ is called the generator of $q$ (note that this convention makes the generator a lower semibounded operator).

Forms on $L^2$-spaces and, in particular, those well compatible with its lattice structure will be of main relevance for our study. These are discussed next. Let $(X,B,\mu)$ be a measure space. A form $q$ on $L^2(X,\mu)$ is real if $f \in D(q)$ implies $\bar{f} \in D(q)$ and $q(f) = q(\bar{f})$. Of course this condition is void when working over $\mathbb{R}$. For real functions $f,g$ we define

$$f \wedge g := \min\{f,g\}.$$  

A real form $q$ satisfies the first Beurling-Deny criterion if $f \in D(q)$ implies $|f| \in D(q)$ and $q(|f|) \leq q(f)$. In this case also $q(f^+) \leq q(f)$ for any real valued $f$, where $f^+$ denotes the positive part of $f$ i.e. $f^+ := \max\{f,0\}$. The importance of the first Beurling Deny criterion comes from the fact that a form $q$ with generator $T$ satisfies the first Beurling Deny criterion if and only if the associated semigroup $(e^{-tT})_{t \geq 0}$ is positivity preserving, that is, $e^{-tT}f \geq 0$ for all $t \geq 0$ and $f \geq 0$.

Domination will provide the main condition for our stability result. The relevant aspects of domination will be discussed next (see [14,29,31,39] as well for further discussion). The basic idea is to 'estimate' a form or semigroup on a Hilbert space $\mathcal{H}$ by a form or semigroup on an $L^2$-space. In order to achieve this comparison one needs a map from $\mathcal{H}$ to the $L^2$-space mimicking the modulus of a complex number and allowing for a version of the polar decomposition of a complex number as well. Following [14] such maps will be discussed next.

Let $(X,B,\mu)$ be a measure space and $\mathcal{H}$ a Hilbert space. We denote by $L^2_+(X,\mu)$ the space all nonnegative functions in $L^2(X,\mu)$ and, more generally, whenever $V \subset L^2(X,\mu)$ is a subspace, we write $V_+$ and sometimes also $V^+$ for $V \cap L^2_+(X,\mu)$.

A map $|\cdot| : \mathcal{H} \to L^2_+(X,\mu)$ is called absolute map if

$$|\langle \xi, \eta \rangle| \leq \langle |\xi|, |\eta| \rangle$$

for all $\xi, \eta \in \mathcal{H}$ with equality if $\xi = \eta$. So, in particular, $|\xi| = 0$ implies $\xi = 0$. The elements $\xi$ and $\eta$ of $\mathcal{H}$ are called paired if

$$\langle \xi, \eta \rangle = \langle |\xi|, |\eta| \rangle.$$  

If for all $\xi \in \mathcal{H}$ and $f \in L^2_+(X,\mu)$ there exists $\eta \in \mathcal{H}$ such that $|\eta| = f$ and $\xi$ and $\eta$ are paired the absolute map is called absolute pairing.

Clearly, absolute maps are generalizations of the modulus function. Similarly, an absolute pairing can be thought of as giving not only a modulus function but also a polar decomposition. More specifically, whenever $\xi$ and $\eta$ are paired one may think about $\eta$ as being of the form $|\eta| \text{sgn} \xi$. In fact, the reader may just bear the following examples in mind, where such an interpretation is rather immediate.

**Example 1.1** (Direct integral): Let $((H_x)_{x \in X}, \mathfrak{M})$ be a measurable field of Hilbert spaces over $(X,B,\mu)$ in the sense of [31], Definition IV.8.9, and $\mathcal{H} = \int_X^\oplus H_x d\mu(x)$. The norm on $H_x$ is denoted by $|\cdot|_x$, $x \in X$. Then, the map $|\cdot| : \mathcal{H} \to L^2_+(X,\mu)$ given by $|\xi|(x) = |\xi(x)|_x$ is an absolute pairing. Indeed, for $\xi \in \mathcal{H}$ and $f \in L^2_+(X,\mu)$ let

$$\eta(x) = \begin{cases} \frac{f(x)}{|\xi(x)|_x} \xi(x) & : \xi(x) \neq 0, \\ f(x) \xi(x) & : \xi(x) = 0, \end{cases}$$
where \( \eta \in \mathfrak{M} \) with \(|\zeta| = 1 \) (the existence of such an element is proven in [41], Lemma IV.8.12). Then \( \eta \) and \( \xi \) are paired with \(|\eta| = f \). The other properties of an absolute pairing are easy to check.

A special case of this construction is given by a constant field of Hilbert spaces, where all \( H_x, x \in X \), are equal to one fixed separable Hilbert space \( H \). In this case, \( H \) is also denoted by \( L^2(X, \mu; H) \) and given by the vector space of all measurable maps (with respect to the corresponding Borel-\( \sigma \)-algebras) \( \xi : X \longrightarrow H \) with \( \int |\xi(x)|^2 d\mu(x) < \infty \), where two such maps are identified if they agree \( \mu \)-almost everywhere.

For the applications discussed in the later part of this article, we will restrict attention to the special instance of the previous example given by Hermitian vector bundles. There the Hilbert spaces in question are finite dimensional and form a continuous field. For definiteness reasons we provide an explicit discussion next.

**Example 1.2** (Hermitian bundle): A Hermitian vector bundle is given by topological spaces \( E \) and \( X \) and a continuous surjective map \( \pi : E \longrightarrow X \) with the property that each fiber \( \pi^{-1}(x) \), \( x \in X \), carries the structure of a finite dimensional vector space with an inner product \( \langle \cdot, \cdot \rangle_x \) and that there is a finite dimensional Hilbert space \( (H, \langle \cdot, \cdot \rangle) \) such that to each point \( p \in X \) there exists a neighborhood \( U \) and a homeomorphism \( \varphi : U \times H \longrightarrow \pi^{-1}(U) \), called local trivialization, such that \( \varphi_x := \varphi(x, \cdot) \) is an isometric isomorphism between the inner product space \( H \) and \( \pi^{-1}(x) \) for each \( x \in U \). We will denote such a bundle by \( \pi : E \longrightarrow X \) and say that \( E \) is a bundle over \( X \) (with projection \( \pi \)). We subsequently also often skip the \( \pi \) and the \( X \) in the notation and just call \( E \) the bundle. (For further details we refer to [26], where the term Euclidean vector bundle is used.)

Let now \( \pi : E \longrightarrow X \) be a Hermitian vector bundle, \( \mu \) a Borel measure on \( X \) and assume that \( X \) is a Lindelöf space, that is a topological space such that every open cover has a countable subcover. Examples of Lindelöf spaces include \( \sigma \)-compact spaces and separable metric spaces. A function \( \eta : X \longrightarrow E \) with \( \pi \circ \xi = \text{id}_X \) is called a section. By \( L^2(X, \mu; E) \) we denote the set of measurable (with respect to the corresponding Borel-\( \sigma \)-algebras) sections \( \xi \) with \( \int_X |\xi(x)|^2 dm(x) < \infty \), where \(|\cdot|_x \) is the norm induced from \( \langle \cdot, \cdot \rangle_x \) and sections are identified which agree \( \mu \)-almost everywhere. This space is a Hilbert space and

\[
|\cdot| : L^2(X, \mu; E) \longrightarrow L^2_+(X, \mu), \quad |\xi|(x) = |\xi(x)|_x
\]

is an absolute pairing.

Indeed, this is just a special case of the previous example with constant field of Hilbert spaces. More precisely, by the Lindelöf property there is a countable family of local trivializations \( \varphi_k : U_k \times H \longrightarrow \pi^{-1}(U_k) \) such that \( X = \bigcup_k U_k \). Setting \( V_k = U_k \setminus \bigcup_{j=1}^{k-1} U_j \), one obtains

\[
\psi : X \times H \longrightarrow E, \quad \psi|_{V_k} = \varphi_k|_{V_k}
\]

which is a measurable bijection with measurable inverse. By definition, the induced map

\[
\Psi : L^2(X, \mu; H) \longrightarrow L^2(X, \mu; E), \quad \Psi(\xi)(x) = \psi(x, \xi(x))
\]

is an isometric isomorphism and \(|\Psi(\xi)| = |\xi| \) for all \( \xi \in L^2(X, \mu; H) \).

Let now an absolute pairing \(|\cdot| : \mathcal{H} \longrightarrow L^2_+(X, \mu) \) be given. Then, the subspace \( U \) of \( \mathcal{H} \) is a generalized ideal of the subspace \( V \) of \( L^2(X, \mu) \) if for all \( \xi \in U \) we have \(|\xi| \in V \) and for any \( f \in V \) with \( 0 \leq f \leq |\xi| \) there exists an \( \eta \in U \) such that \(|\eta| = f \) and \( \xi \) and \( \eta \) are paired.
Let $a$ be a closed form on $H$ and $b$ a closed form on $L^2(X,\mu)$. We say that $a$ is dominated by $b$ if $D(a)$ is a generalized ideal of $D(b)$ and

$$\Re a(\xi,\eta) \geq b(|\xi|,|\eta|)$$

whenever $\xi,\eta \in D(a)$ are paired. In the situations we have in mind the dominating form $b$ will additionally satisfy the first Beurling Deny criterion. The relevance of domination comes from the following fundamental result.

**Theorem:** Let $A$, $B$ be the generators of the forms $a$ and $b$ respectively and assume that $b$ satisfies the first Beurling Deny criterion. Then, the form $a$ is dominated by $b$ if and only if the semigroup $(e^{-tA})_{t\geq 0}$ is dominated by $(e^{-tB})_{t\geq 0}$ in the sense that

$$|e^{-tA}\xi| \leq e^{-tB}|\xi|$$

for all $\xi \in H$ and all $t \geq 0$.

This result has quite some history: For $H = L^2(X,\mu)$ is was shown in [30, 31]. This was then extended to $H = L^2(X,\mu;H)$ for a Hilbert space $H$ in [29] (where even sectorial forms are treated). The general case stated here (and actually an even more general case) is given in [23]. For the applications in the later part of the article the result of [29] suffices.

As discussed already, our aim is to study uniqueness of form extensions. So, we will be interested in form cores. These cores will need to have a special structure which we introduce next. Let $|\cdot|: H \to L^2(X,\mu)$ be an absolute pairing and $U, V$ subspaces of $H$. The space $U$ is called an ideal of $V$ if $f \in U$, $g \in V$ and $|g| \leq |f|$ implies $g \in U$. If $U, V \subset L^2(X,\mu)$, this is meant with respect to the absolute pairing given by the pointwise modulus. So, in this case our notion of ideals coincides with the usual definition in the theory of Banach lattices.

**Remark:** The concept of ideal and of generalized ideal certainly have the same flavor. So, it is worth noting that there is no general relation between these two concepts. Indeed, they arise in rather different situations. Ideals come about as subspaces of the same Hilbert space, whereas the notion of generalized ideals applies to subspaces of two different Hilbert spaces, which are linked by an absolute pairing. So, in this respect the terminology is somewhat unsatisfactory. Still, we stick with it, as it seems to be the standard notation used in the field. Connections between ideals and generalized ideals in the case $H = L^2(X,\mu)$ are studied in [29], Proposition 3.6 and Corollary 3.7.

## 2. A criterion for form uniqueness

In this section we present the main theorem of this article, which allows one to transfer form uniqueness of a dominating form to that of the dominated form.

We begin with two technical lemmas that may be of interest in other situations as well. The first lemma shows that the form norm is compatible with taking minima.

**Lemma 2.1:** Let $(X,B,\mu)$ be a measure space and $q$ be a lower bounded quadratic form on $L^2(X,\mu)$ satisfying the first Beurling-Deny criterion. If $f, g \in D(q)$ are real-valued, then $f \wedge g \in D(q)$ and

$$\|f \wedge g\|_q^2 \leq \|f\|_q^2 + \|g\|_q^2.$$
Theorem 2.3: Let \( H \) be a Hilbert space, \((X, B, \mu)\) a measure space, and \(|\cdot|: H \rightarrow L^2_{+}(X, \mu)\) an absolute pairing. Let \( a \) be a closed form in \( H \), \( b \) a closed form in \( L^2(X, \mu) \) satisfying the first Beurling-Deny criterion, and \( D_a \subset D(b) \) ideals such that the following conditions hold:

- \( a \) is dominated by \( b \)
- \( D_b^+ \cap |D(a)| \subset |D_a| \)

If \( D_b \) is a form core for \( b \), then \( D_a \) is a form core for \( a \).
Proof. Let \(-\lambda < 0\) be a common lower bound for \(a\) and \(b\). As \(a\) is closed, \(D(a)\) is a Hilbert space with the inner product \(\langle \cdot, \cdot \rangle_a = (1 + \lambda)\langle \cdot, \cdot \rangle_H + a(\cdot, \cdot)\) and analogously for \(b\).

We show that \(D_a \subset D(a)\) is dense with respect to \(\| \cdot \|_a\) by proving that \(D_a^+ = \{0\}\) in \((D(a), \langle \cdot, \cdot \rangle_a)\). For this purpose, let \(h \in D(a)\) such that
\[
0 = \langle h, u \rangle_a = (1 + \lambda)\langle h, u \rangle + a(h, u)
\]
for all \(u \in D_a\).

Consider \(v \in D_b^+\) such that \(v \leq |h|\). Since \(a\) is dominated by \(b\), \(D(a)\) is a generalized ideal of \(D(b)\). Hence, as \(h\) belongs to \(D(a)\), there exists \(\tilde{h} \in D(a)\) such that \(|\tilde{h}| = v\) and \(h, \tilde{h}\) are paired. This implies in particular,
\[
v = |\tilde{h}| \in D_b^+ \cap |D(a)| \subset |D_a|,
\]
where we used the assumption on \(D_b\) for the last inclusion. Since \(D_a\) is an ideal in \(D(a)\) we then obtain \(\tilde{h} \in D_a\). By the orthogonality assumption on \(h\) above this implies
\[
0 = \langle h, \tilde{h} \rangle_a.
\]

Now, as \(a\) is dominated by \(b\), we have \(|h| \in D(b)\) and from the preceding equality and domination we infer
\[
(*) \quad 0 = (1 + \lambda)\langle h, \tilde{h} \rangle + \text{Re} a(h, \tilde{h}) \geq (1 + \lambda)\langle |h|, v \rangle + b(|h|, v).
\]

After these considerations the proof can be finishes as follows: As \(D_b\) is a core for \(D(b)\) Lemma 2.2 can be applied with \(v = |h|\) and there exists a sequence \((v_n)\) in \(D_b\) such that \(0 \leq v_n \leq |h|\) and \(|v_n - |h||_b \to 0\). Applying inequality \((*)\) to \(v = v_N\) we obtain
\[
0 \geq (1 + \lambda)\langle |h|, v_N \rangle + b(|h|, v_N) = \langle |h|, v_N \rangle_b \to |||h||_b^2.
\]
Hence \(|h| = 0\) and therefore also \(h = 0\). Thus, \(D_a^+ = \{0\}\). \(\square\)

Remark: 
- In applications, the situation will often be as follows: We are given forms \(a_0\) on \(D_a\), \(b_0\) on \(D_b\) (usually not closed) and minimal extensions \(a_{\text{min}}\), \(b_{\text{min}}\) (the closures of \(a_0\), \(b_0\)) and maximal extensions \(a_{\text{max}}\), \(b_{\text{max}}\). If \(b_{\text{min}} = b_{\text{max}}\), then the theorem yields \(a_{\text{min}} = a_{\text{max}}\). This situation is discussed in detail in the subsequent two sections.
- If \(H = L^2(X, \mu)\), \(|\cdot|\) is the pointwise modulus, and \(D_a = D_b\), then the condition \(D_a^+ \subset |D_a|\) is automatically satisfied.

We now turn to a corollary that contains the concrete situation of our applications in the next sections. There, we consider a Lindelöf space \(X\) and \(\mu\) a Borel measure on \(X\) and a Hermitian vector bundle \(E\) over \(X\) (compare Example 1.2 above for details). In this situation, we denote by \(L^2_c(X, \mu)\) the space of square integrable functions that vanish outside a compact set and by \(L^2_c(X, \mu; E)\) the space of square integrable sections in \(E\) that vanish outside a compact set.

**Corollary 2.4:** Let \(X\) be a Lindelöf space, \(\mu\) a Borel measure on \(X\) and \(E\) a Hermitian vector bundle over \(X\). Assume that \(b\) is a closed form in \(L^2_c(X, \mu)\) satisfying the first Beurling-Deny criterion and \(a\) a closed form in \(L^2_c(X, \mu; E)\) that is dominated by \(b\). If \(D(b) \cap L^2_c(X, \mu)\) is a form core for \(b\), then \(D(a) \cap L^2_c(X, \mu; E)\) is a form core for \(a\).

**Proof.** We will apply Theorem 2.3 to \(D_b = D(a) \cap L^2_c(X, \mu; E)\) and \(D_b = D(b) \cap L^2_c(X, \mu)\). It is obvious that these are ideals in \(D(a)\) and \(D(b)\) respectively.

Now let \(g \in D_b^+ \cap |D(a)|\). Then there is an \(f \in D(a)\) such that \(|f| = g \in L^2_c(X, \mu)\). Thus, \(f \in D(a) \cap L^2_c(X, \mu; E)\) and \(g = |f| \in |D_a|\). \(\square\)
Remark: • If \(b\) is a regular Dirichlet form, \(D(b) \cap L^2_c(X, \mu)\) is a form core for \(b\). Indeed, \(D(b) \cap C_c(X) \subset D(b) \cap L^2_c(X, \mu)\) is dense in \(D(b)\) by definition. Note, however, that we do not use the second Beurling-Deny criterion in our reasoning.

• The preceding corollary concerns bundles \(E\) over an underlying topological space \(X\). As discussed in Example [1.2] the space of the \(L^2\)-section in the bundle can also be considered as a direct integral of Hilbert spaces over \(X\). In fact, it is easily possible to generalize the corollary to the setting of direct integrals discussed in Example [1.1] but we do not need this for the purposes of this article.

In applications to manifolds one is in an even more regular situation. More specifically, in the smooth case, one is usually interested in the closure of the form defined on smooth functions (sections) as minimal form. We make the following definition adapted to this situation.

Definition 2.5: Let \(M\) be a Riemannian manifold and \(E \rightarrow M\) a smooth Hermitian vector bundle and denote by \(\Gamma_c(M; E)\) the space of compactly supported smooth sections in \(M\). A form \(a\) on \(L^2(M; E)\) is called smoothly inner regular if \(D(a) \cap \Gamma_c(M; E)\) is dense in \(D(a) \cap L^2_c(M; E)\) with respect to \(\|\cdot\|_a\).

From the definition of smooth inner regularity and the above corollary, the following corollary can easily be deduced.

Corollary 2.6: Let \(M\) be a Riemannian manifold and \(E \rightarrow M\) a smooth Hermitian vector bundle. Let \(b\) be a closed form on \(L^2(M)\) satisfying the first Beurling-Deny criterion and \(a\) a closed, smoothly inner regular form on \(L^2(M; E)\) that is dominated by \(b\). If \(D(b) \cap C^\infty_c(M)\) is a form core for \(b\), then \(D(a) \cap \Gamma_c(M; E)\) is a form core for \(a\).

3. Schrödinger forms on weighted Riemannian manifolds

In this section we study quadratic forms associated to Schrödinger operator on vector bundles over Riemannian manifolds. This kind of operators and the associated forms have been extensively studied in the last decades, let us just mention [2, 3, 10, 15] as references covering all necessary basics for this section.

After introducing the quadratic forms in question, we prove that the Schrödinger forms on vector bundles are dominated by the corresponding forms acting on functions (Proposition [3.7]), which implies the uniqueness result in this setting (Theorem [3.8]). Finally we discuss how this result enables us to apply capacity estimates to uniqueness question for Schrödinger forms on vector bundles (Corollary [3.10]).

Throughout this section let \((M, g, \mu)\) be a weighted Riemannian manifold, that is, \((M, g)\) is a Riemannian manifold and \(\mu = e^{-\psi} \text{vol}_g\) for some \(\psi \in C^\infty(M)\). All (local) Lebesgue and Sobolev spaces are taken with respect to the measure \(\mu\).

Definition 3.1 (Regular Schrödinger bundle [2]): A regular Schrödinger bundle is a triple \((E, \nabla, V)\) consisting of

- a complex Hermitian vector bundle \(E\) over \(M\),
- a metric covariant derivative \(\nabla\) on \(E\),
- a potential \(V \in L^1_\text{loc}(M; \text{End}(E))_+\).

For a vector bundle \(E \rightarrow M\) we denote by \(\Gamma(M; E)\) the space of smooth sections and by \(\Gamma_c(M; E)\) the subspace of compactly supported smooth sections. If \(E \rightarrow M\) is Hermitian, we write \(\langle \cdot | \cdot \rangle\) and \(|\cdot|\) for the inner product and induced norm on the fibers, respectively.
Example 3.2: If \( \eta \in \Gamma(M; T^* M) \), then \( \nabla = d + i\eta \) is a metric covariant derivative on the trivial complex line bundle \( M \times \mathbb{C} \to M \). Thus, magnetic Schrödinger operators with electric potential are naturally included in this setting.

Definition 3.3 (Schrödinger form with Neumann boundary conditions): Let

\[
W^{1,2}(M; E) = \{ \Phi \in L^2(M; E) \ | \ \nabla \Phi \in L^2(M; E \otimes T^*_c M) \},
\]

where \( \nabla \) is to be understood in the distributional sense. The space \( W^{1,2}_{loc}(M; E) \) is defined accordingly.

The Schrödinger form with Neumann boundary conditions \( \mathcal{E}^{(N)}_{\nabla, V} \) is defined by

\[
D(\mathcal{E}^{(N)}_{\nabla, V}) = \{ \Phi \in W^{1,2}(M; E) \ | \ (V \Phi, \Phi) \in L^2(M) \},
\]

\[
\mathcal{E}^{(N)}_{\nabla, V}(\Phi) = \int_M |\nabla \Phi|^2 d\mu + \int_M (V \Phi, \Phi) d\mu.
\]

Just as in the scalar case one shows that \( \mathcal{E}^{(N)}_{\nabla, V} \) is closed.

Definition 3.4 (Schrödinger form with Dirichlet boundary conditions): The Schrödinger form with Dirichlet boundary conditions \( \mathcal{E}^{(N)}_{\nabla, V} \) is the closure of the restriction of \( \mathcal{E}^{(N)}_{\nabla, V} \) to \( \Gamma_c(M; E) \).

In the scalar case when \( \nabla \) is simply the exterior derivative on functions, we will write \( E^{(N)}_V \) and \( E^{(D)}_V \) for \( \mathcal{E}^{(N)}_{\nabla, V} \) and \( \mathcal{E}^{(D)}_{\nabla, V} \) respectively, and simply \( E^{(N)} \) and \( E^{(D)} \) if \( V = 0 \). It is well-known that the forms \( E^{(N)}_{\nabla} \) and \( E^{(D)}_{\nabla} \) are Dirichlet forms (see e.g. [8], Section 1.2). In particular, they satisfy the first Beurling-Deny criterion.

We will now establish that \( \mathcal{E}^{(N)}_{\nabla, V} \) is smoothly inner regular in the sense of Definition 2.5. This result should be well-known, but since we could not find a reference, we outline its proof here. We write \( W^{1,2}_c(M; E) \) for \( W^{1,2}(M; E) \cap L^2_c(M; E) \), the space of all sections in \( W^{1,2} \) with compact support.

Lemma 3.5: The space \( \Gamma_c(M; E) \) is dense in \( W^{1,2}_c(M; E) \) and \( D(\mathcal{E}^{(N)}_{\nabla, V}) \cap L^2_c(M; E) \).

Proof. Let \( \Omega \subset M \) be open and relatively compact. Relative compactness ensures that \( \Omega \) can be covered by finitely many charts \( (U_1, \varphi_1), \ldots, (U_n, \varphi_n) \) such that \( \bigcup_{k=1}^n U_k \) is relatively compact and such that there exist trivializations \( \psi_k \) for \( E|_{U_k} \).

Let \( (\lambda_k) \) be a partition of unity subordinate to \( (U_k) \). For \( \Phi \in L^2(M; E) \) define \( \Phi_k \) by

\[
\Phi_k : \varphi_k(U_k) \xrightarrow{\varphi_k^{-1}} U_k \xrightarrow{\lambda_k \Phi} E|_{U_k} \xrightarrow{\varphi_k^{-1}} U_k \times \mathbb{R}^n \xrightarrow{pr} \mathbb{R}^n.
\]

In particular, if \( \Phi \in W^{1,2}(M; E) \), then \( \Phi_k \in W^{1,2}_c(\varphi_k(U_k); \mathbb{C}^n) \). Moreover, since \( \bigcup_k U_k \) is relatively compact, there are constants \( c, C > 0 \) such that

\[
e\|\Phi_k\|_{W^{1,2}(\varphi_k(U_k); \mathbb{C}^n)}^2 \leq \int_{U_k} (|\lambda_k \Phi|^2 + |\nabla(\lambda_k \Phi)|^2) d\mu \leq C\|\Phi_k\|_{W^{1,2}(\varphi_k(U_k); \mathbb{C}^n)}^2
\]

for all \( \Phi \in W^{1,2}_c(M; E) \), \( k \in \{1, \ldots, N\} \), as can be easily seen from the local coordinate expression for \( \nabla \). It is well-known that \( C^\infty_c(\varphi_k(U_k); \mathbb{C}^n) \) is dense in \( W^{1,2}_c(\varphi_k(U_k); \mathbb{C}^n) \). Using the norm estimate above and the partition of unity, this gives the density of \( \Gamma_c(M; E) \) in \( W^{1,2}_c(M; E) \).

This argument can easily be extended to the case of non-vanishing potentials, we just wanted to avoid the additional notation. \( \square \)
In the following lemma we collect some useful calculus rules for the weak covariant derivative. They can be derived from the corresponding rules for smooth sections by first noticing that they are local, hence it suffices to verify them for compactly supported sections, and then using the density of $C_c(M; E)$ in $W^{1,2}_c(M; E)$.

**Lemma 3.6:** If $v \in W^{1,2}_c(M)$, $\Phi, \Psi \in W^{1,2}_c(M; E)$ and $f : C \to \mathbb{C}$ is smooth and Lipschitz, then $\nabla(v\Phi) \in L^1_c(M; E \otimes T^*_cM)$, $d(\Phi\Psi) \in L^1_c(M; T^*_cM)$, $f \circ v \in W^{1,2}_c(M)$ and

$$\nabla(v\Phi) = v\nabla\Phi + \Phi \otimes dv,$$
$$d(\Phi\Psi) = \langle \nabla\Phi|\Psi \rangle + \langle \Phi|\nabla\Psi \rangle,$$
$$d(f \circ v) = (f' \circ v)dv.$$ 

**Proposition 3.7:** If $(E, \nabla, V)$ is a regular Schrödinger bundle and $W \in L^1_c(M_\pm)$ such that $V_x \geq W_x$ in the sense of quadratic forms for a.e. $x \in M$, then $\mathcal{E}^{(N)}_{\nabla, V}$ is dominated by $\mathcal{E}^{(N)}_W$.

**Proof.** Step 1. If $\Phi \in D(\mathcal{E}^{(N)}_{\nabla, V})$, then $|\Phi| \in D(\mathcal{E}^{(N)}_W)$:

Let $|\Phi| = (|\Phi|^2 + \varepsilon^2)^{1/2}$. For $\Phi \in \Gamma_c(M; E)$ it was shown in [13], Section 2, that $d|\Phi| \leq |\nabla\Phi|$. Since $\Gamma_c(\Omega; E)$ is dense in $W^{1,2}_c(\Omega; E)$ for $\Omega \subset M$ open, relatively compact and the inequality is local, we conclude that $d|\Phi| \leq |\nabla\Phi|$ for all $\Phi \in W^{1,2}_c(M; E)$.

Since $|\Phi| \to |\Phi|$ in $L^2_c(M)$, we can use the lower semicontinuity of the energy integral to get

$$\int_\Omega |d|\Phi|^2 \, d\mu \leq \liminf_{\varepsilon \to 0} \int_\Omega |d|\Phi| \varepsilon |^2 \, du \leq \int_\Omega |\nabla\Phi|^2 \, d\mu$$

for all open, relatively compact $\Omega \subset M$. This inequality implies $|d|\Phi|^2 \leq |\nabla\Phi| \in L^2(M)$.

Finally, $W|\Phi|^2 \leq (V\Phi|\Phi) \in L^2(M)$ by assumption. Thus $d|\Phi| \in D(\mathcal{E}^{(N)}_W)$.

**Step 2.** If $v \in D(\mathcal{E}^{(N)}_W)$ and $\Phi \in D(\mathcal{E}^{(N)}_{\nabla, V})$ with $0 \leq v \leq |\Phi|$, then $v\text{sgn }\Phi \in D(\mathcal{E}^{(N)}_{\nabla, V})$:

Let $\Psi_\varepsilon = v\Phi/|\Phi|$. Using $|\Phi| \geq \varepsilon$ and the chain rule, we get $d(\Psi_\varepsilon) = -\frac{d|\Phi|}{|\Phi|} \Phi/|\Phi|^2$. The product rule implies

$$\nabla \frac{\Phi}{|\Phi|} = \frac{1}{|\Phi|} \nabla \Phi - \Phi \otimes \frac{d|\Phi|}{|\Phi|^2}.$$ 

As $|\Phi| \geq \varepsilon$, the right-hand side is in $L^2_c(M; E \otimes T^*_cM)$. Hence we can apply the product rule a second time to get

$$\nabla \Psi_\varepsilon = \frac{\Phi}{|\Phi|} \otimes dv + \frac{v}{|\Phi|} \nabla \Phi - \Phi \otimes \frac{d|\Phi|}{|\Phi|^2}.$$ 

Thus

$$|\nabla \Psi_\varepsilon| \leq |dv| \frac{|\Phi|}{|\Phi|} + \frac{v}{|\Phi|} (|\nabla \Phi| + |d|\Phi|) \leq |dv| + 2|\nabla \Phi|.$$ 

Moreover,

$$(V\Psi_\varepsilon|\Psi_\varepsilon) = \frac{v^2}{|\Phi|^2} (V\Phi|\Phi) \leq (V\Phi|\Phi).$$

As in Step 1, a limiting argument gives $v\text{sgn }\Phi \in D(\mathcal{E}^{(N)}_{\nabla, V})$.

**Step 3.** If $v, \Phi$ as in Step 2, then $\Re \mathcal{E}^{(N)}_{\nabla, V}(\Phi, v\text{sgn }\Phi) \geq \mathcal{E}^{(N)}_W(|\Phi|, v)$:
Let \( \Psi_\varepsilon = v\Phi/|\Phi|_\varepsilon \) as in Step 2. By what we have already established in Steps 1 and 2,

\[
\text{Re}\langle \nabla \Phi | \nabla \Psi_\varepsilon \rangle = \text{Re}\left( \frac{\text{Re}(\nabla \Phi)}{|\Phi|_\varepsilon} \otimes dv + \frac{v}{|\Phi|_\varepsilon} \nabla \Phi - v \otimes \frac{d|\Phi|_\varepsilon}{|\Phi|_\varepsilon^2} \right)
\]

\[
= \frac{1}{|\Phi|_\varepsilon} \left( \text{Re}(\nabla \Phi)|\Phi|_\varepsilon |dv| + \frac{v}{|\Phi|_\varepsilon} (|\nabla \Phi|^2 - \frac{1}{|\Phi|_\varepsilon} \text{Re}(\nabla \Phi)|\Phi|_\varepsilon d|\Phi|_\varepsilon) \right)
\]

\[
= \langle d|\Phi|_\varepsilon|dv\rangle + \frac{v}{|\Phi|_\varepsilon} (|\nabla \Phi|^2 - |d|\Phi|_\varepsilon|^2)
\]

\[
\geq \langle d|\Phi|_\varepsilon|dv\rangle.
\]

Furthermore,

\[
\langle V\Phi | \Psi_\varepsilon \rangle = \frac{v}{|\Phi|_\varepsilon} \langle V\Phi | \Phi \rangle \geq |\Phi|_\varepsilon W|\Phi| v.
\]

Letting \( \varepsilon \to 0 \) gives the desired inequality. \( \square \)

**Remark:** a) A distributional version of Kato’s inequality in this setting was first proven by Hess, Schrader, Uhlenbrock [15] (for compact manifolds and vanishing potentials) based on arguments originally due to Kato [16] for magnetic Schrödinger operators. Their considerations do not include discussion of domains of the operators or forms and, therefore, do not allow one to conclude domination. Our reasoning, which relies on the same method, can be seen as a completion of their result.

b) For open manifolds, the same domination has been proven by Güneysu [10], proof of Proposition 2.2 for Dirichlet boundary conditions (and vanishing potentials) using methods from stochastic analysis and the semigroup characterization of domination. Of course, both our result and the result of [10] apply to those situations where Dirichlet- and Neumann boundary conditions agree.

**Lemma 3.5** and **Proposition 3.7** ensure that the conditions of Corollary 2.6 are met, so that we obtain the following main result of this section.

**Theorem 3.8:** Let \( (E, \nabla, V) \) be a regular Schrödinger bundle and \( W \in L^1_{\text{loc}}(M) \) such that \( V_x \geq W_x \) in the sense of quadratic forms for a.e. \( x \in M \). If \( \mathcal{E}_W^{(N)} = \mathcal{E}_W^{(D)} \), then \( \mathcal{E}_{\nabla, V}^{(N)} = \mathcal{E}_{\nabla, V}^{(D)} \).

To wrap it up, we discuss how Theorem 3.8 enables us to apply capacity estimates to form uniqueness problems for Schrödinger operators on bundles. The scalar case has been treated in [9]; we follow their terminology.

**Definition 3.9** (Cauchy boundary, capacity): Denote by \( \tilde{M} \) the metric completion of \( M \). The Cauchy boundary \( \partial_C M \) of \( M \) is defined as \( \partial_C M = \tilde{M} \setminus M \). The capacity of an open subset \( \Omega \) of \( \tilde{M} \) is defined as

\[
\text{cap}(\Omega) = \inf\{||u||_{W^{1,2}(M)}^2 \mid u \in W^{1,2}(M), 0 \leq u \leq 1, u|_{\Omega \cap M} = 1\}.
\]

As usual, the infimum of the empty set is taken to be \( \infty \). The capacity is extended to arbitrary subsets \( \Sigma \) of \( \Omega \) by setting

\[
\text{cap}(\Sigma) = \inf_{\Omega \supset \Sigma \text{ open}} \text{cap}(\Omega).
\]

A subset \( \Sigma \) of \( \tilde{M} \) is called polar if \( \text{cap}(\Sigma) = 0 \).

**Corollary 3.10:** Consider the following assertions:
The Cauchy boundary $\partial_C M$ of $M$ is polar.

(ii) $\mathcal{E}^{(N)} = \mathcal{E}^{(D)}$.

(iii) $\mathcal{E}^{(N),V} = \mathcal{E}^{(D),V}$ for all regular Schrödinger bundles $(E, \nabla, V)$.

Then (i) $\implies$ (ii) $\iff$ (iii). Moreover, if there exists an exhaustion $(B_k)$ of $\tilde{M}$ such that $\text{cap}(B_k \setminus M) < \infty$ for all $k \in \mathbb{N}$, then all three assertions are equivalent.

Proof. The implication (ii) $\implies$ (iii) is content of the last theorem, while (iii) $\implies$ (ii) is trivial. The remaining statements follow from [9], Theorem 1.7 and Lemma 2.2. □

Remark: It is one achievement of our abstract main theorem to make concepts from potential theory such as the capacity in this corollary directly applicable to Schrödinger forms, which are not Dirichlet forms, without having to go through the uniqueness proof for the scalar case again.

4. Applications to magnetic Schrödinger forms on graphs

In this section we will study discrete analogs of the Laplacian respectively magnetic Schrödinger operators in Euclidean space. Analysis on graphs has been an active field of research in recent years and uniqueness of extensions of operators respectively forms on graphs have been intensively studied. We just point to [11, 12, 19, 45, 46] for non-magnetic forms and the recent series of articles [27, 28] for magnetic forms as a few examples.

Compared to the Euclidean case, the discrete setting allows more clarity in the presentation as some mere technical complications do not appear. In particular, Corollary 2.4 can be applied directly since $L^\infty_c(X)$ and $C_c(X)$ coincide for discrete spaces.

We will start with some basic definitions, including those of magnetic Schrödinger forms on graphs (Definitions 4.3 and 4.4), essentially following [18], [19] for graphs and Dirichlet forms over discrete spaces and [28] for vector bundles over graphs and magnetic Schrödinger operators. Then we show that the form with magnetic field is dominated by the form without magnetic field (Proposition 4.6) before we finally give the uniqueness result (Theorem 4.8) and discuss some examples.

Definition 4.1 (Weighted graph): A weighted graph $(X, b, c, m)$ consists of an (at most) countable set $X$, an edge weight $b: X \times X \rightarrow [0, \infty)$, a killing term $c: X \rightarrow [0, \infty)$ and a measure $m: X \rightarrow (0, \infty)$ subject to the following conditions for all $x, y \in X$:

(b1) $b(x, x) = 0$,

(b2) $b(x, y) = b(y, x)$,

(b3) $\sum_{z \in X} b(x, z) < \infty$.

Observe that we do not assume our graphs to be locally finite, that is, $\{y \in X : b(x, y) > 0\}$ may be infinite as long the edge weights are still summable (at each vertex). We shall regard $X$ as a discrete topological space. Consequently, $C_c(X)$ is the space of functions on $X$ with finite support. We regard $m$ as a measure on the power set $\mathcal{P}(X)$ of $X$ via

$$m(A) := \sum_{x \in A} m(x), \ A \subset X,$$

and denote the corresponding $L^2$-space by $\ell^2(X, m)$.

Any graph comes with a formal Laplacian $\tilde{L}$ acting on

$$\{ f : X \rightarrow \mathbb{R} : \sum_y b(x, y) |f(y)| < \infty \text{ for all } x \in X \}$$
by
\[ \hat{L} f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(y) - f(y)) + \frac{c(x)}{m(x)} f(x) \]
for \( x \in X \).

For the next definition recall that a Hermitian vector bundle over a discrete base space \( X \) is just a family of isometrically isomorphic finite-dimensional Hilbert spaces indexed by \( X \).

**Definition 4.2** (Unitary connection): Let \( X \) be a discrete space and \( E \rightarrow X \) a Hermitian vector bundle. A connection on \( E \) is a family \( \Phi = (\Phi_{x,y})_{x,y \in X} \) of unitary maps \( \Phi_{x,y} : E_y \rightarrow E_x \) such that \( \Phi_{x,y} = \Phi_{y,x}^{-1} \).

For a Hermitian vector bundle \( E \) over a discrete space \( X \) equipped with a measure \( m \) we denote by
\[ \Gamma(X; E) = \{ u : X \rightarrow \prod_{x \in X} E_x \mid u(x) \in E_x \}, \]
\[ \Gamma_c(X; E) = \{ u \in \Gamma(X; E) \mid \text{supp } u \text{ finite} \}, \]
\[ \ell^2(X, m; E) = \{ u \in \Gamma(X; E) \mid \sum_{x \in X} \langle u(x), u(x) \rangle_x m(x) < \infty \} \]
the space of all sections, the space of all sections with compact support and the space of all \( L^2 \)-sections. The latter becomes a Hilbert space equipped with the inner product
\[ \langle \cdot, \cdot \rangle_{\ell^2(X, m; E)} : \ell^2(X, m; E) \times \ell^2(X, m; E) \rightarrow \mathbb{C}, (u, v) \mapsto \sum_{x \in X} \langle u(x), v(x) \rangle_x m(x). \]

A bundle endomorphism \( W \) of a Hermitian vector bundle \( E \) over a discrete base space \( X \) is a family \( (W(x))_{x \in X} \) of linear maps \( W(x) : E_x \rightarrow E_x \).

For the remainder of the section, \((X, b, c, m)\) is a weighted graph, \( E \) a Hermitian vector bundle over \( X \) with unitary connection \( \Phi \) and \( W \) a bundle endomorphism of \( E \) that is pointwise positive, that is, \( \langle W(x) v, v \rangle_x \geq 0 \) for all \( x \in X, v \in E_x \).

Now we can define the basic object of our interest, the magnetic Schrödinger form (with Dirichlet and Neumann boundary conditions).

**Definition 4.3** (Magnetic form with Neumann boundary conditions): For \( u \in \Gamma(X; E) \) let
\[ \hat{Q}_{\Phi, b, W}(u) = \frac{1}{2} \sum_{x,y} b(x, y) \| u(x) - \Phi_{x,y} u(y) \|_x^2 + \sum_x \langle W(x) u(x), u(x) \rangle_x \in [0, \infty]. \]

The magnetic Schrödinger form with Neumann boundary conditions is defined via
\[ D(\hat{Q}_{\Phi, b, W}^{(N)}) = \{ u \in \ell^2(X, m) \mid \hat{Q}_{\Phi, b, W}(u) < \infty \}, \]
\[ \hat{Q}_{\Phi, b, W}^{(N)}(u) = \hat{Q}_{\Phi, b, W}(u). \]

To shorten notation, we write \( \| \cdot \|_{\Phi, b, W} \) for the form norm of \( \hat{Q}_{\Phi, b, W}^{(N)} \). By the same arguments as in the Dirichlet form case, the form \( \hat{Q}_{\Phi, b, W}^{(N)} \) is closed (see [19], Lemma 2.3).

**Definition 4.4** (Magnetic form with Dirichlet boundary conditions): The magnetic Schrödinger form with Dirichlet boundary conditions \( \hat{Q}_{\Phi, b, W}^{(D)} \) is the closure of the restriction of \( \hat{Q}_{\Phi, b, W}^{(N)} \) to \( C_c(X) \).
If $E_x = \mathbb{C}$ endowed with the standard inner product and $\Phi_{x,y} = 1$ for all $x, y \in X$, we will suppress $\Phi$ in the index and simply write $Q_{b,W}^{(D)}$ (resp. $Q_{b,W}^{(N)}$). We may also drop other indices if they are clear from the context. The interest in these forms is particularly a result of the fact that $Q_{b,c}^{(D)}$ and $Q_{b,c}^{(N)}$ are Dirichlet forms. Indeed, all regular Dirichlet forms over a discrete measure space are of the form $Q_{b,c}^{(D)}$ for some graph $(X, b, c)$ (cf. [19], Lemma 2.2). This is one motivation to study also graphs that are not locally finite.

For our subsequent considerations we also note that the generators of both $Q_{b,c}^{(D)}$ and $Q_{b,c}^{(N)}$ are restrictions of $\tilde{L}$ (see [11]). So, if the restriction $L_0 := \tilde{L}|_{C_c(X)}$ of $\tilde{L}$ to $C_c(X)$ maps into $\ell^2(X, m)$ and is essentially self-adjoint then $Q_{b,c}^{(D)} = Q_{b,c}^{(N)}$ follows.

As a next step to establish criteria for $Q_{b,c}^{(N)} = Q_{b,c}^{(D)}$ we will show that the form with magnetic field is dominated by the non-magnetic form. First we prove an easy technical lemma.

**Lemma 4.5:** Let $V$ be a Hilbert space, $a, b \in V$, and $\alpha, \beta \geq 0$ with $\alpha \leq \|a\|$, $\beta \leq \|b\|$. Define

$$\tilde{a} = \begin{cases} \alpha \|a\| & : a \neq 0 \\ 0 & : a = 0 \end{cases}$$

and likewise $\tilde{b}$. Then

$$\|\tilde{a} - \tilde{b}\|^2 \leq |\alpha - \beta|^2 + \|a - b\|^2.$$

**Proof.** If $a = 0$ or $b = 0$, the inequality is obvious. Hence assume that $a, b \neq 0$.

In the following computation we use the inequality $2\lambda \mu \leq \lambda^2 + \mu^2$ for $\lambda, \mu \in \mathbb{R}$.

$$\|\tilde{a} - \tilde{b}\|^2 = \|\tilde{a}\|^2 + \|\tilde{b}\|^2 - 2 \text{Re} \langle \tilde{a}, \tilde{b} \rangle$$

$$= \alpha^2 + \beta^2 - 2 \text{Re} \langle \tilde{a}, \tilde{b} \rangle$$

$$= |\alpha - \beta|^2 + 2\alpha\beta - \text{Re} \langle \tilde{a}, \tilde{b} \rangle$$

$$= |\alpha - \beta|^2 + 2|a||b|(|a||b| - \text{Re} \langle a, b \rangle)$$

$$\leq |\alpha - \beta|^2 + 2\|a\||b\| - 2 \text{Re} \langle a, b \rangle$$

$$\leq |\alpha - \beta|^2 + \|a\|^2 + \|b\|^2 - 2 \text{Re} \langle a, b \rangle$$

$$= |\alpha - \beta|^2 + \|a - b\|^2. \quad \square$$

We will now prove that the magnetic form is dominated by the form without magnetic field. We note that a pointwise Kato’s inequality was already given in [28], Lemma 3.3. This work does not involve a discussion of validity of this inequality on the domains of the operators. Hence, it can not be used to conclude domination. In this sense our result can be seen as a completion of their result (compare Remark 3 for a discussion of a similar issue as well).

**Proposition 4.6:** Assume that $\langle W(x)u(x), u(x) \rangle_x \geq c(x)|u(x)|^2$ for all $x \in X$, $u(x) \in E_x$. Then $Q_{\Phi,b,W}^{(N)}$ is dominated by $Q_{b,c}^{(N)}$. 

Proof. We have to show that $D(Q_{\Phi,b,W}^{(N)})$ is a generalized ideal in $D(Q_{b,c}^{(N)})$ and that

$$\text{Re} Q_{\Phi,b,W}^{(N)}(u, \tilde{u}) \geq Q_{b,c}^{(N)}(|u|, |	ilde{u}|)$$

holds for all $u, \tilde{u} \in D(Q_{\Phi,b,W}^{(N)})$ satisfying $\langle u(x), \tilde{u}(x) \rangle_x = |u(x)||\tilde{u}(x)|$ for all $x \in X$.

First, let $u \in D(Q_{\Phi,b,W}^{(N)})$. Then $|u| \in \ell^2(X,m)$ and

$$\tilde{Q}_{\Phi,b,W}(u) = \frac{1}{2} \sum_{x,y} b(x,y)|u(x) - \Phi_{x,y}u(y)|^2 + \sum_x |W(x)u(x), u(x)|$$

$$\geq \frac{1}{2} \sum_{x,y} b(x,y)||u(x)| - |u(y)||^2 + \sum_x c(x)|u(x)|^2$$

$$= \tilde{Q}_{b,c}(|u|),$$

hence $|u| \in D(Q_{b,c}^{(N)})$.

Next let $v \in D(Q_{b,c}^{(N)})$ with $0 \leq v \leq |u|$. Obviously, $\|v \text{sgn } u\|_{\ell^2} \leq \|v\|_{\ell^2}$, thus $v \text{sgn } u \in \ell^2(X,m;F)$.

Applying Lemma 4.5 to $V = E_x, a = u(x), b = \Phi_{x,y}u(y), \alpha = v(x), \beta = v(y)$, we obtain

$$|v(x) \text{sgn } u(x) - \Phi_{x,y}v(y) \text{sgn } u(y)|^2 \leq |v(x) - v(y)|^2 + |u(x) - \Phi_{x,y}u(y)|^2.$$ Summation over $x, y$ implies

$$\tilde{Q}_{\Phi,b,0}(v \text{sgn } u) \leq Q_{b,0}^{(N)}(v) + Q_{\Phi,b,0}^{(N)}(u).$$

Furthermore,

$$\sum_x \langle W(x)v(x) \text{sgn } u(x), v(x) \text{sgn } u(x) \rangle \leq \sum_x |u(x)|^2 \langle W(x) \text{sgn } u(x), \text{sgn } u(x) \rangle$$

$$= \sum_x \langle W(x)u(x), u(x) \rangle,$$

hence

$$\tilde{Q}_{\Phi,b,W}(v \text{sgn } u) \leq Q_{b,0}^{(N)}(v) + \tilde{Q}_{\Phi,b,0}^{(N)}(u) + \sum_x c(x)|u(x)|^2 \leq Q_{b,c}^{(N)}(v) + Q_{\Phi,b,W}^{(N)}(u),$$

that is, $v \text{sgn } u \in D(Q_{\Phi,b,W}^{(N)})$.

Let $u, \tilde{u} \in D(Q_{\Phi,b,W}^{(N)})$ such that $\langle u(x), \tilde{u}(x) \rangle_x = |u(x)||\tilde{u}(x)|$ for all $x \in X$. Then we have

$$\text{Re}(u(x) - \Phi_{x,y}u(y), \tilde{u}(x) - \Phi_{x,y}\tilde{u}(x))$$

$$= \text{Re}(\langle u(x), \tilde{u}(x) \rangle - \langle u(x), \Phi_{x,y}\tilde{u}(y) \rangle - \langle \Phi_{x,y}u(y), \tilde{u}(x) \rangle + \langle u(y), \tilde{u}(y) \rangle)$$

$$= |u(x)||\tilde{u}(x)| + |u(y)||\tilde{u}(y)| - \text{Re}(\langle u(x), \Phi_{x,y}\tilde{u}(y) \rangle - \text{Re}(\Phi_{x,y}u(y), \tilde{u}(x))$$

$$\geq |u(x)||\tilde{u}(x)| + |u(y)||\tilde{u}(y)| - |u(x)||\tilde{u}(y)| + |u(y)||\tilde{u}(x)|$$

$$= (|u(x)| - |u(y)|)(|\tilde{u}(x)| - |\tilde{u}(y)|).$$

After multiplication with $b(x,y)$ and summation over $x, y \in X$ we get

$$\text{Re} Q_{\Phi,b,W}^{(N)}(u, \tilde{u}) \geq Q_{b,c}^{(N)}(|u|, |	ilde{u}|). \quad \Box$$

Corollary 4.7: The form $Q_{b,c}^{(N)}$ is dominated by $Q_{b,0}^{(N)}$. 
Proof. This follows from Proposition 4.6 by taking \( E_x = \mathbb{C} \), \( W(x) = c(x) \) and \( \Phi_{x,y} = 1 \) for all \( x, y \in X \).

Having proven the domination property, the announced main result of this section is now an easy consequence of Corollary 2.4. In a very informal way it says that adding a magnetic and electric field does not disturb the form uniqueness.

**Theorem 4.8:** Assume that \( \langle W(x)u(x),u(x) \rangle \geq c(x)|u(x)|^2 \) for all \( x \in X \), \( u(x) \in E_x \). If \( Q^{(D)}_{b,c} = Q^{(N)}_{b,c} \), then \( Q^{(D)}_{\Phi,b,W} = Q^{(N)}_{\Phi,b,W} \).

**Proof.** We have proven in Proposition 4.6 that \( Q^{(D)}_{\Phi,b,W} \) is dominated by \( Q^{(N)}_{\Phi,b,c} \). An application of Corollary 2.4 for \( a = Q^{(N)}_{\Phi,b,W} \) and \( b = Q^{(N)}_{\Phi,b,c} \) yields the claim. \( \square \)

To apply the theorem, we need \( Q^{(D)}_{b,c} = Q^{(N)}_{b,c} \). There are quite a few conditions under which this equality holds.

**Example 4.9** (See [19] for details and proofs): If \( \tilde{L}C_c(X) \subset \ell^2(X,m) \) and \( \sum_{n=1}^{\infty} m(x_n) = \infty \) for any sequence \( (x_n) \) in \( X \) such that \( b(x_n,x_{n+1}) > 0 \) for all \( n \in \mathbb{N} \), then \( L_0 \) is essentially self-adjoint and all form extensions of \( Q^{(D)} \) coincide with \( Q^{(D)} \). The given conditions are in particular satisfied if \( \inf_{x \in X} m(x) > 0 \).

It turns out that the concept of intrinsic pseudo metrics (introduced for general not necessarily local Dirichlet forms in [7]) provides a suitable framework for many conditions for uniqueness of form extensions. Here, a pseudo metric \( d: X \times X \to [0, \infty) \) is called intrinsic if

\[
\frac{1}{m(x)} \sum_{y} b(x,y) d(x,y)^2 \leq 1
\]

for all \( x \in X \). A pseudo metric \( d \) is said to have finite jump size if there is an \( s \in \mathbb{R} \) such that \( b(x,y) = 0 \) for all \( x, y \in X \) with \( d(x,y) > s \). A pseudo metric \( d \) is called a path pseudo metric if there is a function \( \sigma: X \times X \to [0, \infty) \), satisfying \( \sigma(x,y) = \sigma(y,x) \) and \( \sigma(x,y) > 0 \) iff \( b(x,y) > 0 \) for all \( x, y \in X \), such that

\[
d(x,y) = d_\sigma(x,y) := \inf_{\gamma} \sum_{k=1}^{n} \sigma(x_{k-1},x_k)
\]

where the infimum is taken over all paths \( (x_0, \ldots, x_n) \) connecting \( x \) and \( y \). An intrinsic path pseudo metric \( d_\sigma \) is called strongly intrinsic if

\[
\frac{1}{m(x)} \sum_{y} b(x,y) \sigma(x,y)^2 \leq 1
\]

for all \( x \in X \).

The following conditions are taken from [12], Theorem 1 and 2. Further examples can be found there.

**Example 4.10:** Let \( d \) be an intrinsic pseudo metric on \((X,b,c,m)\). If the weighted degree function

\[
\text{Deg}: X \to [0, \infty), \quad \text{Deg}(x) = \frac{1}{m(x)} \left( \sum_{y \in X} b(x,y) + c(x) \right)
\]
is bounded on the combinatorial neighborhood of each distance ball, then $Q^{(D)} = Q^{(N)}$.

**Example 4.11:** If $(X, b, 0, m)$ is locally finite and there is an intrinsic path metric $d$ such that $(X, d)$ is metrically complete, then $L_0$ is essentially self-adjoint and consequently $Q^{(D)} = Q^{(N)}$.

The following condition is given in [13, Lemma 2.6, and 1], Theorem 1. Its connection to intrinsic metrics is discussed in [13, Theorem 2.8].

**Example 4.12:** If the graph is complete, then $Q^{(D)} = Q^{(N)}$. Here, completeness means that there is a non-decreasing sequence $(\eta_k)$ in $C_c(X)$ such that $\eta_k \to 1$ pointwise and

$$\frac{1}{m(x)} \sum_y b(x, y)|\eta_k(x) - \eta_k(y)|^2 \leq \frac{1}{k}$$

for all $x \in X$, $k \in \mathbb{N}$.

**Remark:** The previous example shows the strength of our method as it has not been treated in earlier works. Notice that we have only $Q^{(D)} = Q^{(N)}$ and not the stronger assertion that $L_0$ is essentially self-adjoint as in the examples before.

**References**

[1] C. Anné and N. Torki-Hamza. The Gauss-Bonnet operator of an infinite graph. *Anal. Math. Phys.*, 5(2):137–159, 2015.

[2] F. Bei and B. Güneysu. Kac regular sets and Sobolev spaces in geometry, probability and quantum physics. *ArXiv e-prints*, August 2017.

[3] M. Braverman, O. Milatovic, and M. Shubin. Essential self-adjointness of Schrödinger-type operators on manifolds. *Russian Math. Surveys*, 57(4):641, 2002.

[4] Y. Colin de Verdière, N. Torki-Hamza, and F. Truc. Essential self-adjointness for combinatorial Schrödinger operators II—metrically non complete graphs. *Math. Phys. Anal. Geom.*, 14(1):21–38, 2011.

[5] Y. Colin de Verdière, N. Torki-Hamza, and F. Truc. Essential self-adjointness for combinatorial Schrödinger operators III—Magnetic fields. *Ann. Fac. Sci. Toulouse Math. (6)*, 20(3):599–611, 2011.

[6] A. Eberle. *Uniqueness and non-uniqueness of semigroups generated by singular diffusion operators*. Lecture Notes in Mathematics, 1718. Springer-Verlag, 1999.

[7] R. L. Frank, D. Lenz, and D. Wingert. Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory. *J. Funct. Anal.*, 266(8):4765–4808, 2014.

[8] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes*. De Gruyter Studies in Mathematics Series. De Gruyter, 1994.

[9] A. Grigor’yan and J. Masamune. Parabolicity and stochastic completeness of manifolds in terms of the Green formula. *J. Math. Pures Appl. (9)*, 100(5):607–632, 2013.

[10] B. Güneyes. Kato’s inequality and form boundedness of Kato potentials on arbitrary Riemannian manifolds. *Proc. Amer. Math. Soc.*, 142(4):1289–1300, 2014.

[11] S. Haeseler, M. Keller, D. Lenz, and R. Wojciechowski. Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions. *J. Spectr. Theory*, 2:397–432, 2012.

[12] X. Huang, M. Keller, J. Masamune, and R. Wojciechowski. A note on self-adjoint extensions of the Laplacian on weighted graphs. *J. Funct. Anal.*, 265(8):1556–1578, 2013.

[13] B. Hua and Y. Lin. Stochastic completeness for graphs with curvature dimension conditions. *Adv. Math.*, 306:279–302, 2017.

[14] H. Hess, R. Schrader, and D. A. Uhlenbrock. Domination of semigroups and generalization of Kato’s inequality. *Duke Math. J.*, 44(4):893–904, 1977.

[15] H. Hess, R. Schrader, and D. A. Uhlenbrock. Kato’s inequality and the spectral distribution of Laplacians on compact Riemannian manifolds. *J. Differential Geom.*, 15(1):27–37 (1981), 1980.

[16] T. Kato. Schrödinger operators with singular potentials. *Israel J. Math.*, 13(1-2):135–148, 1972.

[17] T. Kawabata and M. Takeda. On uniqueness problem for local Dirichlet forms. *Osaka J. Math.*, 33(4):881–893, 1996.
[18] M. Keller and D. Lenz. Unbounded Laplacians on graphs: basic spectral properties and the heat equation. *Math. Model. Nat. Phenom.*, 5(04):198–224, 2010.

[19] M. Keller and D. Lenz. Dirichlet forms and stochastic completeness of graphs and subgraphs. *J. Reine Angew. Math.*, 666:189–223, 2012.

[20] K. Kuwae. Reflected Dirichlet forms and the uniqueness of Silverstein’s extension. *Potential Anal.*, 16(3):221–247, 2002.

[21] K. Kuwae and T. Shioya. Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry. *Comm. Anal. Geom.*, 11(4):599–673, 2003.

[22] K. Kuwae and Y. Shiozawa. A remark on the uniqueness of Silverstein extensions of symmetric Dirichlet forms. *Math. Nachr.*, 288(4):389–401, 2015.

[23] D. Lenz, M. Schmidt and M. Wirth. Domination of quadratic forms. *In preparation*.

[24] P. Li and G. Tian. On the heat kernel of the Bergmann metric on algebraic varieties. *J. Amer. Math. Soc.*, 8(4):857–877, 1995.

[25] J. Masamune. Essential self-adjointness of Laplacians on Riemannian manifolds with fractal boundary. *Comm. Partial Differential Equations*, 24(3-4):749–757, 1999.

[26] J. W. Milnor and J. D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76.

[27] O. Milatovic and F. Truc. Self-adjoint extensions of discrete magnetic Schrödinger operators. *Ann. Henri Poincaré*, 15(5):917–936, 2014.

[28] O. Milatovic and F. Truc. Maximal accretive extensions of Schrödinger operators on vector bundles over infinite graphs. *Integral Equations Operator Theory*, 81(1):35–52, 2015.

[29] A. Manavi, H. Vogt, and J. Voigt. Domination of semigroups associated with sectorial forms. *J. Operator Theory*, 54(1):9–26, 2005.

[30] E. Ouhabaz. Invariance of closed convex sets and domination criteria for semigroups. *Potential Anal.*, 5(6):611–625, 1996.

[31] E. Ouhabaz. $L^p$ contraction semigroups for vector valued functions. *Positivity*, 3(1):83–93, 1999.

[32] M. Reed and B. Simon. *Methods of modern mathematical physics*. 2. Fourier analysis, self-adjointness, volume 2. Elsevier, 1975.

[33] D. W. Robinson and A. Sikora. Markov uniqueness of degenerate elliptic operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 10(3):683–710, 2011.

[34] D. W. Robinson. Uniqueness of diffusion operators and capacity estimates. *J. Evol. Equ.*, 13(1):229–250, 2013.

[35] M. Schmidt. *Energy forms*. PhD thesis, Friedrich-Schiller-Universität Jena, 2017.

[36] M. Shubin. Essential self-adjointness for semi-bounded magnetic Schrödinger operators on non-compact manifolds. *J. Funct. Anal.*, 186(1):92–116, 2001.

[37] M. L. Silverstein. Symmetric Markov processes. *Lecture Notes in Mathematics*, Vol. 426. Springer-Verlag, 1974.

[38] B. Simon. An Abstract Kato’s Inequality for Generators of Positivity Preserving Semigroups. *Indiana Univ. Math. J.*, 26(6), 1977.

[39] B. Simon. Kato’s inequality and the comparison of semigroups. *J. Funct. Anal.*, 32(1):97–101, 1979.

[40] B. Simon. Maximal and minimal Schrödinger forms. *J. Operator Theory*, 1:37–47, 1979.

[41] M. Takesaki. *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.

[42] J. Weidmann. *Linear Operatoren in Hilberträumen. Teil I Grundlagen*. B. G. Teubner, 2 edition, 2000.

[43] D. Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, extended edition, 2007.

[44] M. Wirth Stability of Kac regularity under domination of quadratic forms. *ArXiv e-prints*, September 2017.

[45] R. K. Wojciechowski. *Stochastic completeness of graphs*. PhD thesis, City University of New York, 2008.

[46] R. K. Wojciechowski. *Heat kernel and essential spectrum of infinite graphs*. *Indiana Univ. Math. J.*, 58(3):1419–1441, 2009.
