Estimates of Dirichlet heat kernel for symmetric Markov processes

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Abstract

We consider a large class of symmetric pure jump Markov processes dominated by isotropic unimodal Lévy processes with weak scaling conditions. We first establish sharp two-sided heat kernel estimates for these processes in $C^{1,\rho}$ open sets, $\rho \in (\alpha/2, 1]$ where $\alpha$ is the upper scaling parameter in the weak scaling conditions. As a corollary of our main result, we obtain a sharp two-sided Green function estimates and a scale invariant boundary Harnack inequality with explicit decay rates in $C^{1,\rho}$ open sets.

1 Introduction

The study of the heat kernel of a semigroup is an area of interactions among probability, analysis and geometry. Transition density function provides direct access to path properties of a Markov process. While, it is also the fundamental solution (or heat kernel) of the heat equation with the infinitesimal generator of the corresponding process. Dirichlet heat kernel describes operator with zero exterior conditions. For instance the Green function and the solutions to Cauchy and Poisson problems with Dirichlet conditions are expressed by the heat kernel, cf. (4.1) below. In this paper, we consider a large class of symmetric pure jump Markov processes dominated by isotropic unimodal Lévy processes with weak scaling conditions and we shall estimate the transition density $p_D(t, x, y)$ of such Markov processes killed upon leaving an open set $D \subset \mathbb{R}^d$ with $C^{1,\rho}$ smoothness of the boundary. Put differently, we shall establish a sharp two sided estimates of the Dirichlet heat kernel of the integro-differential operators with maximum principle. Such operators are commonly used to model nonlocal phenomena

The precise estimates for the Dirichlet heat kernel of the Laplacian (and the Brownian motion) were given in 2002 by Zhang [45] for bounded $C^{1,1}$ domains (see [44] for bounds of the Dirichlet heat kernel of the Laplacian on bounded Lipschitz domain).

For the fractional Laplacian, in 2010 Chen, Kim and Song [16] gave sharp (two-sided) explicit estimates for the Dirichlet heat kernel $p_D(t, x, y)$ of the fractional Laplacian in any $C^{1,1}$
open set $D$ and over any finite time interval (see [3] for an extension to non-smooth open sets). When $D$ is bounded, one can easily deduce large time Dirichlet heat kernel estimates from short time estimates by a spectral analysis.

The approach developed in [16] provides a road map for establishing sharp two-sided Dirichlet heat kernel estimates of other discontinuous processes and the result of [16] has been generalized to more general stochastic processes: purely discontinuous symmetric Lévy processes ([21, 15, 22, 6]), symmetric Lévy processes with Gaussian component ([18, 14]), symmetric non-Lévy processes ([17, 33]) and non-symmetric stable processes with gradient perturbation ([19, 34]).

Let $P_y(\tau_D > t)$ be the survival probability of the corresponding process and $p(t, x, y) = p_{\mathbb{R}^d}(t, x, y)$ be the (free) heat kernel for $D = \mathbb{R}^d$. Another form of two-sided heat kernel estimates is the following factorization;

\begin{equation}
(1.1)
\end{equation}

In fact, (1.1) holds for more general sets like Lipschitz open set. See [3, 4, 22]. See [7] for a direct approach to get the sharp estimates on the survival probabilities of unimodal Lévy processes.

Even though extensions of the result in [16] were obtained for a quite large class of symmetric Lévy processes including general unimodal Lévy processes whose Lévy densities satisfying weak scaling conditions in [22, 6], the extension to symmetric Markov processes whose jumping kernels satisfying similar weak scaling conditions is unknown. In this paper we extend the results of [6] and [33] to more general sets (we assume $C^1$-regularity) and more general processes which are non-isotropic and non-Lévy. Our results cover not only a large class of symmetric Markov processes whose jumping kernels satisfying weak scaling conditions but also cover a large class of symmetric Markov processes whose jumping kernels decaying exponentially with damping exponent $\beta \in (0, \infty)$ and symmetric finite range Markov processes.

For two nonnegative functions $f$ and $g$, the notation $f \asymp g$ means that there are positive constants $c_1$ and $c_2$ such that $c_1g(x) \leq f(x) \leq c_2g(x)$ in the common domain of definition for $f$ and $g$. We will use the symbol “:=,” which is read as “is defined to be.” For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. We will use $dx$ to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure.

For $0 < \alpha \leq \bar{\alpha} < 2$, let $\phi$ be an increasing function on $[0, \infty)$ satisfying that there exist positive constants $\underline{c} \leq 1$ and $1 \leq C$ such that

\begin{equation}
(\text{WS})
\end{equation}

\[\underline{c} \left( \frac{R}{r} \right)^{\underline{\alpha}} \leq \phi(R) \leq \frac{C}{\phi(r)^{\bar{\alpha}}}, \quad \text{for } 0 < r \leq R.\]

We will always assume that $\phi$ satisfies (WS) and, using this $\phi$ we define

\[\nu(r) := \frac{1}{\phi(r)^d} \quad \text{for } r > 0.\]

Note that by (WS) and (1.2), there exists $c = c(\bar{\alpha}, C, d)$ such that

\[\nu(r) \leq c \nu(2r) \quad \text{for any } r > 0.\]

Since (WS) implies

\[\int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu(|x|)dx \leq c \left( \int_0^1 s^{-\bar{\alpha}+1}ds + \int_1^\infty s^{-\bar{\alpha}+1}ds \right) < \infty,\]

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there exists a pure jump isotropic unimodal Lévy process \( Z \) whose Lévy measure is \( \nu(\{|x|\})dx \).

In this paper we will always assume that \( \kappa : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty) \) is a symmetric measurable function and there exists \( L_0 > 1 \) such that
\[
L_0^{-1} \leq \kappa(x, y) \leq L_0, \quad x, y \in \mathbb{R}^d.
\]

Let \( J : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty) \) be a symmetric measurable function, which will be the jumping kernel of our process. We consider two sets of conditions on \( J \).

The constant 1 in the condition (J1.1) plays no special role. One can change 1 to any small positive real number.

In this paper, we will always assume that \( \chi \) is a nondecreasing function on \( (0, \infty) \) with \( \chi(r) \equiv \chi(0), r \in (0, 1] \), and there exist \( \gamma_1, \gamma_2, L_1, L_2 > 0 \) and \( \beta \in [0, \infty] \) such that
\[
L_1 e^{\gamma_1 r^\beta} \leq \chi(r) \leq L_2 e^{\gamma_2 r^\beta}, \quad r > 1.
\]

The second set of the conditions on \( J \) is following:

\[ J(x, y) = \kappa(x, y)\nu(|x - y|)\chi(|x - y|)^{-1}, \]

which is equal to
\[
\begin{cases}
\kappa(x, y) (\phi(|x - y|)|x - y|d \cdot \chi(|x - y|))^{-1} & \text{if } \beta \in [0, \infty), \\
\kappa(x, y) (\phi(|x - y|)|x - y|d)\cdot 1_{\{|x - y| \leq 1\}} & \text{if } \beta = \infty.
\end{cases}
\]

Clearly (J2) implies (J1.1) and (J1.2). Moreover, if (J2) holds and \( \beta \neq \infty \), then (J1) holds.

We consider the Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) associated with the jumping kernel \( J \):
\[
\mathcal{E}(u, v) := \frac{1}{2} \int \int (u(x) - u(y))(v(x) - v(y))J(x, y)dxdy,
\]

and \( \mathcal{F} := \{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u, u) < \infty\} \). Under the conditions (J1.1) and (J1.2), by [1] Theorem 2.1 and [2] Theorem 2.4 \( (\mathcal{E}, \mathcal{F}) \) is a regular (symmetric) Dirichlet form on \( L^2(\mathbb{R}^d, dx) \). Moreover, the corresponding Hunt process \( Y \) is conservative and \( Y \) has Hölder continuous transition density \( p(t, x, y) \) on \( (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \) (see [2]).

We now state the estimates for the transition density \( p(t, x, y) \) of \( Y \) with the jumping intensity kernel \( J \) satisfies either the conditions (J1.2) and (J1.3), or the condition (J2) separately. The proof of the upper bound of Theorem 1.1 below is almost the same as that of [2] (2.6)] using the condition (J1.3) instead of [2] (1.5). So we will skip the proof of the upper bound. The proof of the lower bound is given in Section 6.

**Theorem 1.1.** Suppose that \( Y \) is a symmetric pure jump Hunt process whose jumping intensity kernel \( J \) satisfies the conditions (J1.2) and (J1.3). Then, for each \( M > 0 \) and \( T > 0 \), there is a positive constant \( C \geq 1 \) which depends on \( \phi, L_0, M \) and \( T \) such that for every \( (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \) with \( |x - y| < M \), the function \( p(t, x, y) \) has the following estimates:
\[
C^{-1}([\phi^{-1}(t)]^{-d} \wedge \nu(|x - y|)) \leq p(t, x, y) \leq C([\phi^{-1}(t)]^{-d} \wedge \nu(|x - y|))
\]

where \( \phi^{-1}(t) \) is the inverse function of \( \phi(t) \).
For each $a, \gamma, T > 0$, we define a function $F_{a, \gamma, T}(t, r)$ on $(0, T] \times [0, \infty)$ as

$$F_{a, \gamma, T}(t, r) := \begin{cases} 
[\phi^{-1}(t)]^{-d} \wedge t \nu(r)e^{-\gamma r} & \text{if } \beta \in [0, 1], \\
[\phi^{-1}(t)]^{-d} \wedge t \nu(r) & \text{if } \beta \in (1, \infty) \text{ with } r < 1, \\
t \exp \left\{ -a \left( \frac{r}{(Tr)^{\frac{a}{r}}} \right) \right\} & \text{if } \beta \in (1, \infty) \text{ with } r \geq 1, \\
(t/(Tr)^{\frac{a}{r}}) = \exp \left\{ -ar \left( \frac{r}{(Tr)^{\frac{a}{r}}} \right) \right\} & \text{if } \beta = \infty \text{ with } r \geq 1.
\end{cases}$$ (1.6)

Theorem 1.2. Suppose that $Y$ is a symmetric pure jump Hunt process whose jumping intensity kernel $J$ satisfies the condition (J2). Then, the process $Y$ has a continuous transition density function $p(t, x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. For each $T > 0$, there are positive constants $C_1, c_1$ and $c_2 \geq 1$ which depend on $\phi, L_0, \beta, \chi$ and $T$ such that for every $t \in (0, T]$ the function $p(t, x, y)$ has the following estimates:

$$c_2^{-1}F_{c_1, \gamma, T}(t, |x - y|) \leq p(t, x, y) \leq c_2 F_{c_1, \gamma, T}(t, |x - y|).$$

Theorem 1.2 for the case $\beta \in (1, \infty]$ basically comes from [13] Theorems 1.2 and 1.4. Even though they use 1 instead of $T$, the proof is the same. When $\beta \in [0, 1)$, the upper bound in Theorem 1.2 comes from [31] Theorem 2, Proposition 1. The lower bound in Theorem 1.2 will be proved as a special case of the preliminary lower bound on the heat kernel of the killed process in Section 6.

For any open set $D \subset \mathbb{R}^d$, the first exit time of $D$ by the process $Y$ is defined by the formula $\tau_D := \inf\{t > 0 : Y_t \notin D\}$ and we use $Y_D$ to denote the process obtained by killing the process $Y$ upon exiting $D$. By the strong Markov property, it can easily be verified that $p_D(t, x, y) := p(t, x, y) - \mathbb{E}_x[p(t - \tau_D, Y_{\tau_D}, y); t > \tau_D]$ is the transition density of $Y_D$. Using the continuity and estimate of $p$, it is routine to show that $p_D(t, x, y)$ is symmetric and continuous (e.g., see the proof of Theorem 2.4 in [24]).

Let $\rho \in (0, 1]$ and $D \subset \mathbb{R}^d$ (when $d \geq 2$) be a $C^{1, \rho}$ open set with $C^{1, \rho}$ characteristics $(R_0, \Lambda)$, that is, there exist a localization radius $R_0 > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$ there exist a $C^{1, \rho}$-function $\varphi = \varphi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(0) = 0$, $\nabla \varphi(0) = (0, \ldots, 0)$, $\|\nabla \varphi\|_{\infty} \leq \Lambda$, $|\nabla \varphi(x) - \nabla \varphi(w)| \leq \Lambda|x - w|^{\rho}$ and an orthonormal coordinate system $CS_z$ of $z = (z_1, \ldots, z_{d-1}, z_d) := (\tilde{z}, z_d)$ with origin at $z$ such that $D \cap B(z, R_0) = \{y = (\tilde{y}, y_d) \in B(0, R_0) \mid y_d > \varphi(\tilde{y})\}$. The pair $(R_0, \Lambda)$ will be called the $C^{1, \rho}$ characteristics of the open set $D$. Note that a $C^{1, \rho}$ open set $D$ with characteristics $(R_0, \Lambda)$ can be unbounded and disconnected, and the distance between two distinct components of $D$ is at least $R_0$. By a $C^{1, \rho}$ open set in $\mathbb{R}$ with a characteristic $\rho > 0$, we mean an open set that can be written as the union of disjoint intervals so that the infimum of the lengths of all these intervals is at least $R_0$ and the infimum of the distances between these intervals is at least $R_0$.

We introduce a condition on the regularity of $\kappa(x, y)$:

(K$_{\eta}$) There are $L_3 > 0$ and $\eta > \alpha/2$ such that $|\kappa(x, x + h) - \kappa(x, x)| \leq L_3|h|^{\eta}$ for every $x, h \in \mathbb{R}^d$, $|h| \leq 1$.

Note that the condition (K$_{\eta}$) implies that

$$|\kappa(x + h_1, x + h_2) - \kappa(x, x)| \leq 2L_3(|h_1|^{\eta} + |h_2|^{\eta}), \quad \text{for } |h_1|, |h_2| < 1. \quad (1.7)$$

We also introduce a condition on the regularity of $\phi$:

(SD) $\phi \in C^1(0, \infty)$ and $r \to -\nu'(r)/r$ is decreasing.
(See Remark 1.4 below.)

We are now ready to state the following theorem, which is one of the main results of this paper. Let $\delta_D(x)$ be a distance between $x$ and $D^c$, and let

$$\Psi(t,x) := \left(1 + \sqrt{\frac{\phi(\delta_D(x))}{t}}\right).$$

(1.8)

**Theorem 1.3.** Suppose that $Y$ is a symmetric pure jump Hunt process whose jumping intensity kernel $J$ satisfies the conditions (J1), (SD) and (K_p). Suppose that $\rho \in (\pi/2,1]$ and $D$ is a bounded $C^{1,\rho}$ open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda)$. Then, for each $M > 0$ and $T > 0$, there exist $c_1 = c_1(M, \phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T, d)$ and $c_2 = c_2(M, \phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T, d, \text{diam}(D)) > 0$ such that the transition density $p_D(t,x,y)$ of $Y^D$ has the following estimates.

1. For any $(t,x,y) \in (0,T] \times D \times D$, we have
   $$c_1^{-1}\Psi(t,x)\Psi(t,y)p(t,x,y) \leq p_D(t,x,y) \leq c_1\Psi(t,x)\Psi(t,y)p(t,x,y).$$

2. For any $(t,x,y) \in [T,\infty) \times D \times D$, we have
   $$c_2^{-1}e^{-t\lambda_D}\sqrt{\phi(\delta_D(x))} \leq p_D(t,x,y) \leq c_2e^{-t\lambda_D}\sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))},$$
   where $-\lambda_D < 0$ is the largest eigenvalue of the generator of $Y^D$.

**Remark 1.4.** The conditions (SD) and (WS) hold for a large class of pure jump isotropic unimodal Lévy processes including all subordinated Brownian motions with weak scaling conditions (see (1.9) below): let $W = (W_t, \mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$ and $S = (S_t)$ an independent driftless subordinator with Laplace exponent $\varphi_1$. The Laplace exponent $\varphi_1$ is a Bernstein function with $\varphi_1(0+) = 0$. Since $\varphi_1$ has no drift part, $\varphi_1$ can be written in the form

$$\varphi_1(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \mu(dt).$$

Here $\mu$ is a $\sigma$-finite measure on $(0,\infty)$ satisfying $\int_0^\infty (t \wedge 1) \mu(dt) < \infty$. $\mu$ is called the Lévy measure of the subordinator $S$.

The subordinate Brownian motion $Z = (Z_t, \mathbb{P}_x)$ is defined by $Z_t = W_{S_t}$. The Lévy measure of $Y$ has a density with respect to the Lebesgue measure given by $x \to \nu_d(|x|)$ with

$$\nu_d(r) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{r^2}{4t}\right) \mu(t)dt, \quad r \neq 0.$$  

Thus $r \to \nu_d(r)$ is smooth for $r > 0$, and

$$-\frac{\nu_d'(r)}{r} = 2\pi \nu_{d+2}(r), \quad r > 0$$

which is decreasing. Suppose that

$$2\lambda \left(\frac{R}{r}\right)^{\alpha/2} \leq \frac{\varphi_1(R)}{\varphi_1(r)} \leq C \left(\frac{R}{r}\right)^{\beta/2}$$

for $0 < r \leq R$. (1.9)

Then, by [5, Theorem 26]

$$\nu_d(r) \asymp r^{-d}\varphi_1(r^{-2}).$$

Let $\hat{\varphi}(r) := r^{-d}\nu_d(r)^{-1}$ (so that $\nu_d(r) = \hat{\varphi}(r)^{-1}r^{-d}$). Then $\hat{\varphi}$ is smooth, and since $\hat{\varphi}(r) \asymp \varphi_1(r^{-2})^{-1}$, it satisfies (WS).
When either $D$ is unbounded or $\beta = \infty$, we need precise information on $J$ for large $|x - y|$ which is encoded in (J2). Moreover, when $\beta \in (1, \infty]$, we need to impose an addition assumption for $D$ in order to obtain the sharp lower bound of $p_D(t, x, y)$: We say that the path distance in an open set $U$ is comparable to the Euclidean distance with characteristic $\lambda_1$ if for every $x$ and $y$ in $U$ there is a rectifiable curve $l$ in $U$ which connects $x$ to $y$ such that the length of $l$ is less than or equal to $\lambda_1 |x - y|$. Clearly, such a property holds for all bounded $C^{1,\eta}$ connected open sets, $C^{1,\eta}$ connected open sets with compact complements, and connected open sets above graphs of $C^{1,\eta}$ functions.

**Theorem 1.5.** Suppose that $Y$ is a symmetric pure jump Hunt process whose jumping intensity kernel $J$ satisfies the conditions (J2), (SD) and (K_a). Suppose that $\rho \in (\pi/2, 1]$ and $D$ is a $C^{1,\rho}$ open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda)$. Then, for each $T > 0$, the transition density $p_D(t, x, y)$ of $Y^D$ has the following estimates.

1. There is a positive constant $c_1 = c_1(\beta, \chi, \phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T, d)$ such that for all $(t, x, y) \in (0, T] \times D \times D$ we have

$$p_D(t, x, y) \leq c_1 \Psi(t, x) \Psi(t, y) \begin{cases} F_{1,2}^{\gamma_1,\gamma_1,\gamma_1,\gamma_1,\gamma_1,\gamma_1,\gamma_1,\gamma_1,\gamma_1}((t, |x - y|/6) & \text{if } \beta \in [0, \infty), \\
F_{1,2}^{\gamma_1,\gamma_1,\gamma_1,\gamma_1,\gamma_1,\gamma_1,\gamma_1,\gamma_1,\gamma_1}((t, |x - y|/4) & \text{if } \beta = \infty,
\end{cases}$$

where $C_{1,2}$ is the constant in Theorem 1.2.

2. There is a positive constant $c_2 = c_2(\beta, \chi, \phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T, d)$ such that for all $t \in (0, T]$ we have

$$p_D(t, x, y) \geq c_2 \Psi(t, x) \Psi(t, y) \begin{cases} [\phi^{-1}(t)]^{-d} \wedge t e^{-\gamma_2|x-y|^{\beta}} \nu(|x-y|) & \text{if } \beta \in [0, 1], \\
[\phi^{-1}(t)]^{-d} \wedge t \nu(|x-y|) & \text{if } \beta \in (1, \infty) \text{ and } |x-y| < 1, \\
\text{ or } \beta = \infty \text{ and } |x-y| \leq 4/5.
\end{cases}$$

3. Suppose in addition that the path distance in $D$ is comparable to the Euclidean distance with characteristic $\lambda_1$. Then, there are positive constants $c_3 = c_i(\beta, \chi, \phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T, d, \lambda_i)$, $i = 3, 4$, such that if $x, y \in D$ and $t \in (0, T]$, we have

$$p_D(t, x, y) \geq c_3 \Psi(t, x) \Psi(t, y) \begin{cases} F_{4,2,2,2,2,2,2,2,2,2,2}((t, |x-y|)) & \text{if } \beta \in (1, \infty) \text{ and } |x-y| \geq 1, \\
F_{4,2,2,2,2,2,2,2,2,2,2}((t, 5|x-y|/4)) & \text{if } \beta = \infty \text{ and } |x-y| \geq 4/5.
\end{cases}$$

4. If $\beta \in (1, \infty)$, there is a positive constant $c_5 = c_5(\beta, \chi, \phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T, d)$ such that for every $x, y$ in the different components of $D$ with $|x-y| \geq 1$ and $t \in (0, T]$ we have

$$p_D(t, x, y) \geq c_5 \Psi(t, x) \Psi(t, y) t e^{-\gamma_2(5|x-y|/4)^{\beta}} \nu(|x-y|).$$

5. Suppose in addition that $\beta = \infty$ and $D$ is bounded and connected. Then the claim of Theorem 1.3 (2) holds.

Recall that the Green function $G_D(x, y)$ of $Y$ on $D$ is defined as $G_D(x, y) = \int_0^\infty p_D(t, x, y)dt$. As an application of Theorem 1.3 and 1.4, we derive the sharp two sided estimate on the Green function $G_D(x, y)$ of $Y$ on bounded $C^{1,\rho}$ open sets. For notational convenience, let

$$a(x, y) := \sqrt{\phi(\delta_D(x))} \sqrt{\phi(\delta_D(y))}$$

(1.10)
and
\[ g(x, y) := \begin{cases} 
\frac{\phi(|x - y|)}{|x - y|^d} \left( 1 \wedge \frac{\phi(\delta_D(x))}{\phi(|x - y|)} \right)^{1/2} \left( 1 \wedge \frac{\phi(\delta_D(y))}{\phi(|x - y|)} \right)^{1/2} & \text{when } d \geq 2 \\
\frac{a(x, y)}{|x - y|} \wedge \left( \frac{a(x, y)}{\phi^{-1}(a(x, y))} + \left( \int_{|x-y|}^{\infty} \frac{\phi(s)}{s^2} ds \right)^{1/2} \right) & \text{when } d = 1 
\end{cases} \]

where \( x^+ := x \vee 0 \).

**Theorem 1.6.** Suppose that \( \rho \in (\pi/2, 1] \) and \( D \) is a bounded \( C^{1,\rho} \)-open set in \( \mathbb{R}^d \) with characteristics \((R_0, \Lambda)\). Let \( Y \) be a symmetric pure jump Hunt process whose jumping intensity kernel \( J \) satisfies \((K_\eta)\) and \((SD)\). Suppose either (1) the jumping intensity kernel \( J \) satisfies the condition \((J1)\) or (2) \( D \) is connected and the jumping intensity kernel \( J \) satisfies the condition \((J2)\) with \( \beta = \infty \). Then for every \((x, y) \in D \times D\), we have \( G_D(x, y) \asymp g(x, y) \).

**Remark 1.7.** When \( d = 1 \), if either \( \alpha < 1 \) or \( \overline{\alpha} > 1 \), one can write Green function estimates in simpler forms. (see, [22, Corollary 7.4 and Remark 7.5])

We also obtain the uniform and scale-invariant boundary Harnack inequality with explicit decay rates in \( C^{1,\rho} \)-open sets as an application of Theorem 1.5 and 1.6.1. A function \( f : \mathbb{R}^d \to \mathbb{R} \) is said to be harmonic in the open set \( D \) with respect to \( Y \) if for every open set \( U \subset D \) whose closure is a compact subset of \( D \), \( \mathbb{E}_x[|f(Y_{t\eta})]| < \infty \) for every \( x \in U \) and
\[ f(x) = \mathbb{E}_x[f(Y_{t\eta})] \quad \text{for every } x \in U. \quad (1.11) \]
It is said that \( f \) is regular harmonic in \( D \) with respect to \( Y \) if \( f \) is harmonic in \( D \) with respect to \( Y \) and \((1.11)\) holds for \( U = D \).

The next condition guarantees that \( C^2_c(\mathbb{R}^d) \) is in the domain of Feller generator.

(L) \( Y \) is Feller and there exists a function \( q(r) \) such that \( J(x, y) \leq q(|x - y|) \) and
\[ \lim_{R \to \infty} \int_{|h| > R} q(|h|) dh = 0. \quad (1.12) \]
Note that, if \((J2)\) holds, then clearly \((1.12)\) holds, and by Theorem 1.2 \( Y \) is Feller. Thus the condition \((L)\) is weaker than the condition \((J2)\).

The next condition on \( J \) is necessary for the boundary Harnack inequality to hold (see, [8, Assumption C and Example 5.14]).

(C) For any \( 0 < r < R \leq 2 \) there exists \( C^* = C^*(\phi, d, r/R) \) such that for any \( x_0 \in \mathbb{R}^d \),
\[ x \in B(x_0, r) \text{ and } y \in B(x_0, R)^c, \quad (C^*)^{-1}J(x_0, y) \leq J(x, y) \leq C^*J(x_0, y). \]
Note that when \( \beta \in [0, 1] \), the condition \((J2)\) implies \((C)\). On the other hand, when \( \beta \in (1, \infty] \), under the condition \((J2)\) the boundary Harnack inequality does not hold.

**Theorem 1.8.** Suppose that \( D \) is \( C^{1,\rho} \)-open set in \( \mathbb{R}^d \) with characteristics \((R_0, \Lambda)\) with \( \rho \in (\pi/2, 1] \). Let \( Y \) be a symmetric pure jump Hunt process whose jumping intensity kernel \( J \) satisfies the conditions \((J1)\), \((L)\), \((C)\), \((K_\eta)\) and \((SD)\). Then, there exists \( c = c(\phi, \eta, L_0, L_3, d, \Lambda) \) such that for any \( 0 < r < R_0 \wedge 1 \), \( z \in \partial D \) and any nonnegative function \( f \) in \( \mathbb{R}^d \) which is regular harmonic in \( D \cap B(z, r) \) with respect to \( Y \), and vanishes in \( D^c \cap B(z, r) \), we have
\[ \frac{f(x)}{f(y)} \leq c \sqrt[\phi(\delta_D(x))]{\phi(\delta_D(y))} \quad \text{for any } x, y \in D \cap B(z, r/2). \]
The rest of this paper is organized as follows. In Section 2, we solve a martingale-type problem for $Y$ which yields a Dynkin-type formula. Section 3 deals with the isotropic Lévy process $Z$ with Lévy measure $\nu(|x|)dx$. We compute some key upper bound of the generator of $Z$ on our testing function for $C^{1,\rho}$ open sets. In Section 4, we give the key estimates on exit distributions (Theorem 4.2). Section 5 contains the proof of the upper bound of $p_D(t,x,y)$. When $|x-y| < c$, we use Meyer’s construction. Then, by using Lemma 5.1 twice, we prove the upper bound of $p_D(t,x,y)$ without using the lower bound of $p(t,x,y)$. In Sections 6 and 7, we prove the lower bound estimates for $p_D(t,x,y)$. We first consider the case $\delta_D(x) \land \delta_D(y) \geq t^{1/\alpha}$; that is, $x$ and $y$ are kept away from the boundary of $D$. Such results in Section 6 and the key estimates on the exit distributions obtained in Section 4 are used in Section 7 to prove the lower bound for all $x,y \in D$. Finally in Section 8, as an application of the Theorem 4.2, we derived Green function estimates and the uniform scale-invariant Boundary Harnack inequality with explicit decay rates in $C^{1,\rho}$ open sets.

Throughout the rest of this paper, the positive constants $L_0, L_1, L_2, L_3, \gamma_1, \gamma_2$ can be regarded as fixed. In the statements and the proofs of results, the constants $c_i = c_i(a,b,c,\ldots)$, $i = 1,2,3,\ldots$, denote generic constants depending on $a,b,c,\ldots$, whose exact values are unimportant. These are given anew in each statement and each proof. The dependence of the constants on the dimension $d \geq 1$ will not be mentioned explicitly.

For a function space $\mathbb{H}(U)$ on an open set $U$ in $\mathbb{R}^d$, we let $\mathbb{H}_c(U) := \{f \in \mathbb{H}(U) : f$ has compact support}$, $\mathbb{H}_0(U) := \{f \in \mathbb{H}(U) : f$ vanishes at infinity}$ and $\mathbb{H}_b(U) := \{f \in \mathbb{H}(U) : f$ is bounded}$.

## 2 Generator of $Y$

In this section, we assume that $Y$ is the symmetric pure jump Hunt process with the jumping intensity kernel $J$ satisfying the conditions (J1.1), (J1.2) and (K$_\eta$). Recall that these conditions imply that $Y$ is strong Feller (see [12, Theorem 3.1]).

We define an operator $\mathcal{L}$, by

$$\mathcal{L}g(x) := P.V. \int (g(x+h) - g(x))J(x,x+h)dh := \lim_{\varepsilon \downarrow 0} \mathcal{L}^\varepsilon g(x), \quad (2.1)$$

where

$$\mathcal{L}^\varepsilon g(x) := \int_{|h| > \varepsilon} (g(x+h) - g(x))J(x,x+h)dh,$$

whenever these exist pointwise. Let $g \in C_0^2(\mathbb{R}^d)$ and $\varepsilon < r < 1$, then by (J1.1), we have

$$\mathcal{L}^\varepsilon g(x) = \kappa(x,x) \int_{|h| < \varepsilon} (g(x+h) - g(x) - h \cdot \nabla g(x))\nu(|h|)dh$$

$$+ \int_{\varepsilon < |h| < r} (g(x+h) - g(x))(\kappa(x,x+h) - \kappa(x,x))\nu(|h|)dh$$

$$+ \int_{r < |h| < 1} (g(x+h) - g(x))\kappa(x,x+h)\nu(|h|)dh + \int_{1 < |h|} (g(x+h) - g(x))J(x,x+h)dh.$$

Since $(K_\eta)$ holds, we have that

$$|(g(x+h) - g(x))(\kappa(x,x+h) - \kappa(x,x))| \leq ||\nabla g||_{\infty}(L_3 + 2L_0)|h|^{\eta+1}, \quad x,h \in \mathbb{R}^d.$$
Since (WS) and the inequality \( \eta > \frac{\alpha}{2} > \frac{\alpha}{1} \) imply \( \int_{|h|<r} |h|^2 \nu(|h|)dh \leq \int_{|h|<r} |h|^\eta \nu(|h|)dh < \infty \), \( \mathcal{L}g \) is well defined and \( \mathcal{L}^c g \) converges to \( \mathcal{L}g \) locally uniformly on \( \mathbb{R}^d \). Furthermore, for every \( 0 < r < 1 \),

\[
\mathcal{L}g(x) = \kappa(x,x) \int_{|h|<r} (g(x+h) - g(x) - h \cdot \nabla g(x))\nu(|h|)dh \\
+ \int_{|h|<r} (g(x+h) - g(x))(\kappa(x,x+h) - \kappa(x,x))\nu(|h|)dh \\
+ \int_{r \leq |h| \leq 1} (g(x+h) - g(x))\kappa(x,x+h)\nu(|h|)dh + \int_{|h| \leq 1} (g(x+h) - g(x))J(x,x+h)dh. \tag{2.2}
\]

Now we are ready to prove the following lemma.

**Lemma 2.1.** There is \( C_{2.1} = C_{2.1}(\phi, \eta, L_0, L_3) > 0 \) such that for any function \( g \in C^2_c(\mathbb{R}^d) \) and \( 0 < r < 1 \),

\[
||\mathcal{L}g||_{\infty} \leq C_{2.1}(r^{2}||\partial^2 g||_{\infty} + r^{\eta+1}||\nabla g||_{\infty} + ||g||_{\infty}). \tag{2.3}
\]

**Proof.** By (2.2), (1.4), (K_\eta) and (J1.2), we obtain that

\[
|\mathcal{L}g(x)| \leq c_0 \left( L_0 ||\partial^2 g||_{\infty} \int_0^r \frac{s}{\phi(s)}ds + L_3 ||\nabla g||_{\infty} \int_0^r \frac{s^{\eta}}{\phi(s)}ds \right. \\
+ 2L_0 ||g||_{\infty} \left. \int_r^1 \frac{ds}{s\phi(s)} \right) + c_1 ||g||_{\infty}. \tag{2.4}
\]

For \( s \leq r \), since \( \phi(r)/\phi(s) \leq C(r/s)^{\alpha} \) by (WS) and \( \eta > \frac{\alpha}{2} > \frac{\alpha}{1} \), we have

\[
\int_0^r \frac{s^{\eta}}{\phi(s)}ds \leq \frac{C}{\phi(r)} \frac{1}{\eta + 1 - \alpha} r^{\eta+1}. \tag{2.5}
\]

For \( r < s \), since \( \phi(s/r)^{\alpha} \leq \phi(s)/\phi(r) \) by (WS), we have

\[
\int_r^\infty \frac{ds}{s\phi(s)} \leq \frac{C^{-1}}{\phi(r)} r^{\alpha} \int_r^\infty s^{-1-\alpha}ds < (C \alpha)^{-1} \frac{1}{\phi(r)}. \tag{2.6}
\]

Applying (2.5) and (2.6) to (2.4), we conclude that (2.3) hold. \( \square \)

**Lemma 2.2.** For any \( u \in C^2_c(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \), there exists a \( \mathbb{P}_x \)-martingale \( M_t^u \) with respect to the filtration of \( Y \) such that

\[
M_t^u = u(Y_t) - u(Y_0) - \int_0^t \mathcal{L}u(Y_s)ds
\]

\( \mathbb{P}_x \)-a.s. In particular, for any stopping time \( S \) with \( \mathbb{E}_x S < \infty \) we have

\[
\mathbb{E}_x u(Y_S) - u(x) = \mathbb{E}_x \int_0^S \mathcal{L}u(Y_s)ds. \tag{2.7}
\]
Proof. Let \((A, D(A))\) be the \(L^2\)-generator of the semigroup \(T_t\) with respect to \(Y\). Due to [41, Proposition 2.5], we have \(C^2_0(\mathbb{R}^d) \subset D(A)\) and \(A|_{C^2_0(\mathbb{R}^d)} = \mathcal{L}|_{C^2_0(\mathbb{R}^d)}\). Since \(T_t\) is strongly continuous (see [28, Theorem 1.3.1 and Lemma 1.3.2]) we have that for any \(f \in D(A)\) and \(t \geq 0\),

\[
\left\| (T_t f - f) - \int_0^t T_s Af ds \right\|_{L^2} = 0
\]

(see e.g. [28, Proposition 1.5]). Hence for \(u \in C^2_0(\mathbb{R}^d)\),

\[
T_t u(x) - u(x) = \int_0^t T_s \mathcal{L} u(x) ds, \quad \text{a.e. } x \in \mathbb{R}^d,
\]

(2.8)

and \(\mathcal{L} u\) is bounded by Lemma 2.1.

Let us denote \(g_t(x) = \int_0^t T_s \mathcal{L} u(x) ds\). First, we show that \(g_t \in C_b(\mathbb{R}^d), \ t > 0\). Note that \(g_t(x) \leq t||\mathcal{L} u||_{\infty}\). Hence, since \(T_\varepsilon\) is strong Feller for any \(\varepsilon > 0\), we have \(T_\varepsilon g_{t-\varepsilon} \in C_b(\mathbb{R}^d)\) for all \(\varepsilon \in (0, t)\). Moreover,

\[
|g_t(x) - T_\varepsilon g_{t-\varepsilon}(x)| = |g_\varepsilon(x)| \leq \varepsilon ||\mathcal{L} u||_{\infty}.
\]

Hence, \(g_t\) is continuous and (2.8) holds for any \(x \in \mathbb{R}^d\). This and Markov property imply that

\[
M_t^u = u(Y_t) - u(Y_0) - \int_0^t \mathcal{L} u(Y_s) ds
\]

is \(\mathbb{P}_x\)-martingale for any \(x \in \mathbb{R}^d\). Since \(|M_t^u| \leq 2||u||_{\infty} + t||\mathcal{L} u||_{\infty}\), by the optional stopping theorem (2.7) follows. 

Lemma 2.3. There exists a constant \(C_{2.3} = C_{2.3}(\phi, \eta, L_0, L_3) > 0\) such that, for any \(r \in (0, 1]\), \(x_0 \in \mathbb{R}^d\), and any stopping time \(S\) (with respect to the filtration of \(Y\)), we have

\[
\mathbb{P}_x (|Y_S - x_0| \geq r) \leq C_{2.3} \frac{\mathbb{E}_x[S]}{\phi(r)}, \quad x \in B(x_0, r/2).
\]

Proof. Fix \(x_0 \in \mathbb{R}^d\). Since this lemma is clear for \(\mathbb{E}_x[S] = \infty\), we consider the case that \(\mathbb{E}_x[S] < \infty\) for \(x \in B(x_0, r/2)\). Define a radial function \(g \in C^\infty_c(\mathbb{R}^d)\) such that \(-1 \leq g \leq 0\), with

\[
g(y) := \begin{cases} 
-1, & \text{if } |y| < 1/2 \\
0, & \text{if } |y| \geq 1.
\end{cases}
\]

Then,

\[
\sum_{i=1}^d \left\| \frac{\partial}{\partial y_i} g \right\|_{\infty} + \sum_{i,j=1}^d \left\| \frac{\partial^2}{\partial y_i \partial y_j} g \right\|_{\infty} = c_1 < \infty.
\]

For any \(r \in (0, 1]\), define \(g_r(y) = g(\frac{x_0 - y}{r})\) so that \(-1 \leq g_r \leq 0\),

\[
g_r(y) = \begin{cases} 
-1, & \text{if } |x_0 - y| < r/2 \\
0, & \text{if } |x_0 - y| \geq r,
\end{cases}
\]

(2.9)

and

\[
\sum_{i=1}^d \left\| \frac{\partial}{\partial y_i} g_r \right\|_{\infty} < c_1 r^{-1} \quad \text{and} \quad \sum_{i,j=1}^d \left\| \frac{\partial^2}{\partial y_i \partial y_j} g_r \right\|_{\infty} < c_1 r^{-2}.
\]

(2.10)
By Lemma 2.1 there exists $c_2 = c_2(\alpha, \xi, \overline{\alpha}, \overline{C}, \eta, L_0, L_3) > 0$ such that for $0 < r < 1$,
\[
\|L_g_r\|_\infty \leq \frac{c_2}{\phi(r)}. \tag{2.11}
\]

Combining Lemma 2.2, (2.9) and (2.11), we find that for any $x \in B(x_0, r/2)$ with $\mathbb{E}_x S < \infty$, we have
\[
P_x(|Y_S - x_0| \geq r) = \mathbb{E}_x \left[1 + g_r(Y_S) \mid |Y_S - x_0| \geq r\right]
\leq \mathbb{E}_x [1 + g_r(Y_S)] = -g_r(x) + \mathbb{E}_x [g_r(Y_S)] = \mathbb{E}_x \left[\int_0^S L_g_r(Y_t) dt\right] \leq \|L_g_r\|_\infty \mathbb{E}_x[S]
\leq C_{2.2} \frac{\mathbb{E}_x[S]}{\phi(r)}.
\]
\[\blacksquare\]

Recall that for any open set $D \subset \mathbb{R}^d$, $\tau_D = \inf\{t > 0 : Y_t \notin D\}$ denote the first exit time of $D$ by the process $Y_t$.

**Corollary 2.4.** There exists a constant $C_{2.4} = C_{2.4}(\phi, \eta, L_0, L_3) > 0$ such that, for any $r \in (0, 1]$, $x_0 \in \mathbb{R}^d$, and any open sets $U$ and $D$ with $D \cap B(x_0, r) \subset U \subset D$, we have
\[
P_x(Y_{\tau_U} \in D) \leq C_{2.4} \frac{\mathbb{E}_x[\tau_U]}{\phi(r)}, \quad x \in D \cap B(x_0, r/2). \tag{2.12}
\]

**Proof.** Since $D \setminus U \subset B(x_0, r)^c$, by Lemma 2.3 we have that for $x \in D \cap B(x_0, r/2)$
\[
P_x(Y_{\tau_U} \in D) \leq P_x (|Y_{\tau_U} - x_0| \geq r) \leq C_{2.3} \frac{\mathbb{E}_x[\tau_U]}{\phi(r)}.
\]
\[\blacksquare\]

### 3 Analysis on $Z$

Recall that $Z$ is a pure jump isotropic unimodal Lévy process with Lévy measure $\nu(|x|)dx$. Moreover, we assume (SD) holds in this section. The Lévy-Khintchine (characteristic) exponent of $Z$ has the form
\[
\psi(|\xi|) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot x)) \nu(|x|)dx, \quad \xi \in \mathbb{R}^d. \tag{3.1}
\]

Let $Z^d$ be the last coordinate of $Z$ and $M_t = \sup_{s \leq t} Z_s^d$ and let $L_t$ be the local time at 0 for $M_t - Z_t^d$, the last coordinate of $Z$ reflected at the supremum. We consider its right-continuous inverse, $L_t^{-1}$ which is called the ascending ladder time process for $X_t^1$. Define the ascending ladder-height process as $H_s = Z_{L_s^{-1}}^d = M_{L_s^{-1}}$. The Laplace exponent of $H_s$ is
\[
\kappa(|\xi|) = \exp \left\{ \frac{1}{\pi} \int_0^{\infty} \frac{\log \psi(\theta|\xi|)}{1 + \theta^2} d\theta \right\}, \quad \xi \geq 0.
\]
(See [27 Corollary 9.7].) The renewal function $V$ of the ascending ladder-height process $H$ is defined as
\[
V(x) = \int_0^\infty \mathbb{P}(H_s \leq x) ds, \quad x \in \mathbb{R}.
\]
then $V(x) = 0$ if $x < 0$ and $V$ is non-decreasing. Also $V$ is subadditive (see [2] p.74), that is,

$$V(x + y) \leq V(x) + V(y), \quad x, y \in \mathbb{R},$$

(3.2)

and $V(\infty) = \infty$. Since resolvent measures of $Z^d_t$ are absolutely continuous, it follows from [43, Theorem 2] that $V(x)$ is absolutely continuous and harmonic on $(0, \infty)$ for the process $Z^d_t$. Also, $V'$ is a positive harmonic function for $Z^d_t$ on $(0, \infty)$, hence $V$ is actually (strictly) increasing.

For $r > 0$, define Pruitt’s function $h(r) = \int_{\mathbb{R}^d} (1 \wedge |z|^2 r^{-2}) \nu(dz)$ (e.g., see [39]). By [5, Corollary 3] and [7, Proposition 2.4] (see also [2] p.74), we first note that

$$h(r) \asymp [V(r)]^{-2} \asymp \kappa(r^{-1}) \asymp \psi(r^{-1}) \quad \text{for any } r > 0.$$

Clearly, (WS) implies that $s \to \phi(s^{-1})$ also satisfies (WS), that is, using the notation in [5], $s \to \phi(s^{-1}) \in \text{WLSC}(\mathcal{L}, 0) \cap \text{WUSC}(\mathcal{L}, 0)$. So by (1.2) and [5, Proposition 28], we have that

$$\psi(r) \asymp \phi(r^{-1}) \quad \text{for any } r > 0.$$

Combining these observations, we conclude that

$$V(r) \asymp [\phi(r)]^{1/2} \quad \text{and} \quad \nu(r) \asymp [V(r)]^{-2} r^{-d} \quad \text{for any } r > 0.$$  

(3.3)

So by (WS), there exists $C_V := (\underline{c}, \overline{C}, d) > 1$ such that

$$C_V^{-1} \left(\frac{R}{r}\right)^{\alpha/2} \leq \frac{V(R)}{V(r)} \leq C_V \left(\frac{R}{r}\right)^{\alpha/2} \quad \text{for any } 0 < r \leq R.$$  

(3.4)

Define $w(x) := V((x_d)^+) \text{ and } \mathbb{H} := \{x = (\bar{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$. Since the renewal function $V$ is harmonic on $(0, \infty)$ for $Z^d$, by the strong Markov Property $w$ is harmonic in $\mathbb{H}$ with respect to $Z$.

**Proposition 3.1.** $x \to V(x)$ is twice-differentiable for any $x > 0$, and there exists $C_{3.1} > 0$ such that

$$|V''(x)| \leq C_{3.1} \frac{V'(x)}{x \wedge 1} \quad \text{and} \quad V'(x) \leq C_{3.1} \frac{V(x)}{x \wedge 1}, \quad x > 0.$$

**Proof.** Let $f((\bar{y}, x_d)) = V'((x_d)^+)$ for $\bar{y} \in \mathbb{R}^{d-1}$. Then $f$ is harmonic in $\mathbb{H}$. The assumption (A) satisfies from (SD), by Theorem 1.1 therein, we get for any $x > 0$

$$\left|\frac{\partial}{\partial x_d} f((\bar{0}, x))\right| \leq C_{3.1} \frac{f((\bar{0}, x))}{x \wedge 1} \quad \text{and} \quad \frac{\partial}{\partial x_d} w(\bar{0}, x) \leq C_{3.1} \frac{w(\bar{0}, x)}{x \wedge 1}.$$  

This implies the claim of proposition, because $V(x) = w(\bar{0}, x)$ and $V'(x) = f(\bar{0}, x)$, $x > 0$.  

**Proposition 3.2.** For $\lambda > 0$, there exists $C_{3.2} = C_{3.2}(d, \lambda) > 0$ such that for any $r > 0$, we have

$$\sup_{\{x \in \mathbb{R}^d : 0 < x_d \leq \lambda r\}} \int_{B(x, r)^c} w(y) \nu(|x - y|) dy < C_{3.2} \frac{V(r)}{r}.$$  

(3.5)
Proof. Since \( w(x + z) = V(x_d + z_d) \leq V(x_d) + V(|z|) \) for \( x_d > 0 \), it follows that
\[
\int_{B(x,r)^c} w(y)\nu(|x - y|)dy = \int_{B(0,r)^c} w(x + z)\nu(|z|)dz \\
\leq V(x_d)\int_{B(0,r)^c} \nu(|z|)dz + \int_{B(0,r)^c} V(|z|)\nu(|z|)dz.
\]
By \([7] (2.23)\) and Lemma 3.5, we have that
\[
\sup_{\{x \in \mathbb{R}^d : 0 < x \leq \lambda r \}} \int_{B(x,r)^c} w(y)\nu(|x - y|)dy \leq c_1 \left( \frac{V(\lambda r)}{|V(r)|^2} + \frac{1}{V(r)} \right).
\]
Since \( V \) is subadditive, \( V(\lambda r) \leq (\lambda + 1)V(r) \) for any \( \lambda > 0 \), and therefore we conclude the result. \( \square \)

For any function \( f : \mathbb{R}^d \to \mathbb{R} \) and \( x \in \mathbb{R}^d \), we define an operator as follows:
\[
\mathcal{L}_Z f(x) := \text{P.V.} \int_{\mathbb{R}^d} (f(y) - f(x))\nu(|x - y|)dy := \lim_{\varepsilon \downarrow 0} \mathcal{L}_Z^\varepsilon f(x),
\]
\[
\mathcal{D}(\mathcal{L}_Z) := \left\{ f \in C^2(\mathbb{R}^d) : \text{P.V.} \int_{\mathbb{R}^d} (f(y) - f(x))\nu(|x - y|)dy \text{ exists and is finite.} \right\},
\]
where
\[
\mathcal{L}_Z^\varepsilon f(x) := \int_{B(x,\varepsilon)^c} (f(y) - f(x))\nu(|x - y|)dy. \tag{3.6}
\]

Recall that \( C^2_0(\mathbb{R}^d) \) be the collection of \( C^2 \) functions in \( \mathbb{R}^d \) vanishing at infinity. It is well known that \( C^2_0(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{L}_Z) \) and that, by the rotational symmetry of \( Z \), \( A_Z|_{C^2_0(\mathbb{R}^d)} = \mathcal{L}_Z|_{C^2_0(\mathbb{R}^d)} \) where \( A_Z \) is the infinitesimal generator of \( Z \) (e.g. [10] Theorem 31.5). Hence, we see that Dynkin formula holds for \( \mathcal{L}_Z \): for each \( g \in C^2_0(\mathbb{R}^d) \) and any bounded open subset \( U \) of \( \mathbb{R}^d \) we have
\[
\mathbb{E}_x \int_0^{\tau_U} \mathcal{L}_Z g(Z_t)dt = \mathbb{E}_x[g(Z_{\tau_U})] - g(x). \tag{3.7}
\]

**Theorem 3.3.** For any \( x \in \mathbb{R}^d \), \( \mathcal{L}_Z w(x) \) is well-defined and \( \mathcal{L}_Z w(x) = 0 \).

Proof. By subadditivity of \( V \), \( |w(y) - w(x)| \leq V(|y - x_d|) \leq V(|x - y|) \) for \( x \in \mathbb{R}^d \). By \([7] \text{Lemma 3.5}\), it follows that for any \( \varepsilon \in (0, 1/2) \)
\[
\left| \int_{B(x,\varepsilon)^c} (w(y) - w(x))\nu(|x - y|)dy \right| \leq \int_{B(0,\varepsilon)^c} V(|z|)\nu(|z|)dz < \frac{c_1}{V(\varepsilon)} < \infty. \tag{3.8}
\]
Thus \( \mathcal{L}_Z^\varepsilon w(x) \) is well defined in \( \mathbb{R}^d \) and
\[
\mathcal{L}_Z w(x) = \int_{B(x,\varepsilon)^c} (w(y) - w(x) - 1_{\{|x - y| < 1\}}(x - y) \cdot \nabla w(x))\nu(|x - y|)dy.
\]
Since Proposition 3.1 implies \( V''(s) \) exists and so \( w \) is twice differentiable in \( \mathbb{R}^d \), we have that
\[
x \mapsto \int_{B(x,\varepsilon)} (w(y) - w(x) - (x - y) \cdot \nabla w(x))\nu(|x - y|)dy
\]
converges to 0 locally uniformly in \( \mathbb{H} \) as \( \varepsilon \downarrow 0 \). From (3.8), we see that \( \mathcal{L}_Z w(x) \) converges to

\[
\mathcal{L}_Z w(x) = \int_{\mathbb{R}^d} (w(y) - w(x) - 1_{\{|x-y|<1\}}(x-y) \cdot \nabla w(x)) \nu(|x-y|)dy
\]

locally uniformly in \( \mathbb{H} \) as \( \varepsilon \downarrow 0 \).

For every \( x \in \mathbb{H} \), \( z \in B(x, (\varepsilon \wedge t_d)/2) \) and \( y \in B(z, \varepsilon)^c \), it holds that \( |y-z|/2 \leq |x-y| \leq 3|y-z|/2 \). Since \( r \to \nu(r) \) is decreasing, using Proposition 3.1

\[
1_{\{|y-z|>\varepsilon\}}|w(y) - w(z)| \leq c_2 \sup_{\varepsilon/2<s<x} V''(s) |y-z|^2 1_{\{|y-z|<\varepsilon/2\}} \nu(|y-z|/2)
\]

\[
+ (w(y) + V(xd + 1)) 1_{\{|y-z|>\varepsilon/2\}} \nu(|y-z|/2).
\]

So applying the dominated convergence theorem with Proposition 3.2 and the fact that \( \nu \) is a Lévy density, we obtain that \( x \to \mathcal{L}_Z^\varepsilon w(x) \) is continuous for each \( \varepsilon \). Therefore, the function \( \mathcal{L}_Z w(x) \) is continuous in \( \mathbb{H} \).

Let \( U_1 \) and \( U_2 \) be relatively compact open subsets of \( \mathbb{H} \) satisfying \( \overline{U_1} \subset U_2 \subset \overline{U_2} \subset \mathbb{H} \) and \( 0 < r_0 := \text{dist}(U_1, U_2^c) < 1 \). By Proposition 3.2

\[
\int_{U_1} \int_{U_2^c} w(y) \nu(|x-y|)dydx \leq |U_1| \sup_{x \in U_1} \int_{U_2^c} w(y) \nu(|x-y|)dy \leq |U_1| \sup_{x \in U_1} \int_{B(x, r_0)^c} w(y) \nu(|x-y|)dy < \infty. \tag{3.9}
\]

Since \( w \) is harmonic, \( w(Z_{\tau_{U_1}}) \in L^1(\mathbb{P}_x) \) and

\[
\sup_{x \in U_1} \mathbb{E}_x [1_{U_2^c}(Z_{\tau_{U_1}})w(Z_{\tau_{U_1}})] \leq \sup_{x \in U_1} \mathbb{E}_x [w(Z_{\tau_{U_1}})] = \sup_{x \in U_1} w(x) < \infty. \tag{3.10}
\]

From (3.9) and (3.10), the conditions \( \mathbb{H} \) (2.4), (2.6) hold and by \( \mathbb{H} \) Lemma 2.3, Theorem 2.11(ii), we have that for any \( f \in C_0^2(\mathbb{H}) \),

\[
0 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (w(y) - w(x))(f(y) - f(x)) \nu(|x-y|)dxdy. \tag{3.11}
\]

For \( f \in C_0^2(\mathbb{H}) \) with \( \text{supp}(f) \subset \overline{U_1} \subset U_2 \subset \overline{U_2} \subset \mathbb{H} \),

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |w(y) - w(x)||f(y) - f(x)| \nu(|x-y|)dxdy
\]

\[
= \int_{U_1} \int_{U_2^c} |w(y) - w(x)||f(y) - f(x)| \nu(|x-y|)dxdy + 2 \int_{U_1} \int_{U_2^c} |w(y) - w(x)||f(y)| \nu(|x-y|)dxdy
\]

\[
\leq c_2 \int_{U_1 \times U_2} |y-x|^2 \nu(|x-y|)dxdy + 2 \|f\|_\infty \int_{U_1} \int_{U_2^c} \nu(|x-y|)dxdy,
\]

(3.12)

and it is finite from (3.9) and the fact that \( \nu \) is a Lévy density. Applying the dominated convergence theorem with (3.11) and (3.12), for any \( f \in C_0^2(\mathbb{H}) \), we have

\[
0 = \lim_{\varepsilon \downarrow 0} \int_{\{|x,y\} \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| > \varepsilon \}} (w(y) - w(x))(f(y) - f(x)) \nu(|x-y|)dxdy
\]

\[
= -2 \lim_{\varepsilon \downarrow 0} \int_{\mathbb{H}} f(x) \left( \int_{\{y \in \mathbb{R}^d : |x-y| > \varepsilon \}} (w(y) - w(x)) \nu(|x-y|)dy \right) dx = -2 \int_{\mathbb{H}} f(x) \mathcal{L}_Z w(x) dx.
\]
We have used Fubini’s theorem and the fact that $\mathcal{L}_z^w \to \mathcal{L}_Z w$ converges uniformly on the support $f$. Hence, by the continuity of $\mathcal{L}_z w$, we have $\mathcal{L}_z w(x) = 0$ in $\mathbb{H}$. □

**Proposition 3.4.** Suppose that $\rho \in (\pi/2, 1]$ and $D$ is a $C^{1,\rho}$ open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda)$. For any $z \in \partial D$ and $r \leq 1 \wedge R_0$, we define

$$h_r(y) = h_{r,z}(y) := V(\delta_D(y))1_{D \cap B(z,r)}(y).$$

There exists $C_{3.4} = C_{3.4}(\phi, \Lambda, \rho, d) > 0$ independent of $z$ such that $\mathcal{L}_Z h$ is well-defined in $D \cap B(z, r/4)$ and

$$|\mathcal{L}_Z h_r(x)| \leq \frac{C_{3.4}}{V(r)} \quad \text{for all } x \in D \cap B(z, r/4). \quad (3.13)$$

**Proof.** Since the case of $d = 1$ is easier, we give the proof only for $d \geq 2$. For $x \in D \cap B(z, r/4)$, let $z_x \in \partial D$ be the point satisfying $\delta_D(x) = |x - z_x|$. Let $\varphi$ be a $C^{1,\rho}$ function and $CS = CS_{z_x}$ be an orthonormal coordinate system with $z_x$ chosen so that $\varphi(0) = 0$, $\nabla \varphi(0) = (0, \ldots, 0)$, $||\nabla \varphi||_\infty \leq \Lambda$, $|\nabla \varphi(y) - \nabla \varphi(\tilde{z})| \leq \Lambda|y - \tilde{z}|^\rho$, and $x = (0, x_d)$, $D \cap B(z, R_0) = \{y = (\tilde{y}, y_d) \in B(0, R_0) \in CS : y_d > \varphi(\tilde{y})\}$. We fix the function $\varphi$ and the coordinate system $CS$, and we define a function $g_x(y) = V(\delta_{\mathbb{H}}(y)) = V(y_d)$, where $\mathbb{H} = \{y = (\tilde{y}, y_d) \in CS : y_d > 0\}$ is the half space in $CS$.

Note that $h_r(x) = g_x(x)$, and that $\mathcal{L}_Z (h_r - g_x) = \mathcal{L}_Z h_r$ by Theorem 3.3. So, it suffices to show that $\mathcal{L}_Z (h_r - g_x)$ is well defined and that there exists a constant $c_1 = c_1(\phi, \infty, \overline{\Omega}, \Lambda, \rho, d) > 0$ independent of $x \in D \cap B(z, r/4)$ and $z \in \partial D$ such that

$$\int_{D \cap \mathbb{H}} |h_r(y) - g_x(y)| \nu(|x - y|)dy \leq c_1 V(r)^{-1}. \quad (3.14)$$

We define $\tilde{\varphi} : B(\overline{0}, r) \to \mathbb{R}$ by $\tilde{\varphi}(\overline{y}) := 2A|\overline{y}|^{1+\rho}$. Since $\nabla \varphi(0) = 0$, by the mean value theorem we have $-\tilde{\varphi}(\overline{y}) \leq \varphi(\overline{y}) \leq \tilde{\varphi}(\overline{y})$ for any $y \in D \cap B(x, r/2)$ and so that

$$\{z = (\tilde{z}, z_d) \in B(x, r/2) : z_d \geq \tilde{\varphi}(\tilde{z})\} \subset D \cap B(x, r/2)$$

and so that

$$\{z = (\tilde{z}, z_d) \in B(x, r/2) : z_d \geq -\tilde{\varphi}(\tilde{z})\}.$$ 

Let $A := \{y \in (D \cup \mathbb{H}) \cap B(x, r/4) : -\tilde{\varphi}(\overline{y}) \leq y_d \leq \tilde{\varphi}(\overline{y})\}$ and $E := \{y \in B(x, r/4) : y_d > \tilde{\varphi}(\overline{y})\} \subset D$. We will prove (3.14) by showing that $I + II + III \leq c_1 V(r)^{-1}$, where

$$I := \int_{B(x,r/4)^c} (h_r(y) + g_x(y))\nu(|x - y|)dy,$$

$$II := \int_A (h_r(y) + g_x(y))\nu(|x - y|)dy, \quad \text{and} \quad III := \int_E |h_r(y) - g_x(y)|\nu(|x - y|)dy.$$

First, since $h_r \leq V(r)$, by (2.23) and Proposition 3.2, we have

$$I \leq V(r) \int_{B(x,r/4)^c} \nu(|x - y|)dy + \sup_{\{z \in \mathbb{R}^d : 0 < z_d < r\}} \int_{B(z,r/4) \cap \mathbb{H}} g_x(y)\nu(|z - y|)dy$$

$$\leq c_2 V(r)^{-1} + \sup_{\{z \in \mathbb{R}^d : 0 < z_d < r\}} \int_{B(z,r/4) \cap \mathbb{H}} w(y)\nu(|z - y|)dy \leq (c_2 + C_{3.2}) V(r)^{-1}.$$

For $y \in A$, since $\tilde{\varphi}(\overline{y}) \leq 2A|\overline{y}|$ and $V$ is increasing and subadditive, we observe that $h_r(y) + g_x(y) \leq 2V(2\tilde{\varphi}(\overline{y})) \leq 8(\Lambda + 1)V(|\overline{y}|)$. For $s \leq r/4$, note that $m_{d-1}\{y : |\overline{y}| =
Since \( s, \tilde{\varphi}(y) \leq y_d \leq \tilde{\varphi}(y) \rangle \leq c_3 s^{d+p-1} \) where \( m_{d-1}(dy) \) is the surface measure on \( \mathbb{R}^{d-1} \). Thus
\[
\int_{|y|=s} 1_A(y)V(|y|)\nu(|y|)m_{d-1}(dy) \leq c_3 V(s)\nu(s)s^{d+p-1} \text{ for } 0 < s < r/4.
\]
From (3.4), we note that \( V(s)^{-1} \leq C_V V(r)^{-1}(r/s)^{\overline{\pi}/2} \) for \( s \leq r \). Hence, by (3.3), (12) and the fact that \( \overline{\pi}/2 < \rho \),
\[
\begin{align*}
\text{II} & \leq 8(\Lambda + 1) \cdot c_3 \int_0^r V(s)\nu(s)s^{d+p-1}ds \\
& \leq c_4 \int_0^r V(s)^{-1}s^{\rho-1}ds \leq c_4 \cdot C_V V(r)^{-1}r^{\overline{\pi}/2} \int_0^r s^{\rho-1-\overline{\pi}/2}ds \\
& = c_4 \cdot C_V V(r)^{-1} \frac{1}{\rho - \overline{\pi}/2} \rho^\rho \leq c_5 V(r)^{-1}
\end{align*}
\]
for some positive constant \( c_5 = c_3(\overline{\pi}, \varepsilon, \overline{\pi}, \Lambda, \rho, d) \).

When \( y \in E \), we have that \( |y_d - \delta_D(y)| \leq (1 + 2\Lambda)\tilde{\varphi}(y) \rangle \). Indeed, if \( 0 < y_d = \delta_E(y) \leq \delta_D(y) \rangle \) and \( y \in E \), \( \delta_D(y) \rangle \leq y_d + |\varphi(y)| \leq y_d + \tilde{\varphi}(y) \rangle \). If \( y_d = \delta_E(y) \rangle \geq \delta_D(y) \rangle \) and \( y \in E \), \( 0 < y_d - \tilde{\varphi}(y) \rangle \leq (1 + \Lambda)\delta_D(y) \rangle \) so that \( y_d - \delta_D(y) \rangle \leq \tilde{\varphi}(y) \rangle + \Lambda\delta_D(y) \rangle \). Hence, by the mean value theorem and the scale invariant Harnack inequality for \( Z^d \) \( \langle 12 \rangle \) Theorem 1.4 \rangle \) applying to \( V \) we have that
\[
|h_r(y) - g_d(y)| = |V(\delta_D(y)) - V(y)|d \leq \sup_{u \in (y_d \delta_D(y), y_d \delta_D(y))} V'(u)|\delta_D(y) - y_d|
\]
\[
\leq c_6 \inf_{u \in (y_d \delta_D(y), y_d \delta_D(y))} V'(u)|\tilde{\varphi}(y)\rangle |\tilde{\varphi}(y)\rangle^{1+\rho}.
\]
Since \( E \subset \{ \overline{y}, y_d \rangle : |\overline{y} \rangle < r/4, \tilde{\varphi}(y) \rangle < y_d < \tilde{\varphi}(y) \rangle + r/2 \rangle \), using with (3.15) and the polar coordinates for \( |\overline{y} \rangle = v \), we first see that
\[
\text{III} \leq 2\Lambda c_6 \int_E V'(y_d - \tilde{\varphi}(y)\rangle |\overline{y} \rangle^{1+\rho} \nu(|x - y|)dy
\]
\[
\leq c_7 \int_0^{r/4} \int_{\overline{\varphi}(v)}^{\overline{\varphi}(v)+r/2} V'(y_d - \tilde{\varphi}(v)\rangle )\nu\langle (v^2 + |y_d - x_d|^2)^{1/2}v^{d+p-1}dy_ddv
\]
Let \( s := y_d - \tilde{\varphi}(v) \). Since \( (v^2 + |y_d - x_d|^2)^{1/2} \geq (v + |y_d - x_d|)/2 \) and \( \nu \rangle \) is decreasing, by (12) and (3.3), we have that
\[
\nu((v^2 + |y_d - x_d|^2)^{1/2})v^{d+p-1} \leq \nu((v + |s + \tilde{\varphi}(r) - x_d|)/2)(v + |s + \tilde{\varphi}(v) - x_d|^d_{d+p-1})
\]
\[
\leq c_8 V(v + s + \tilde{\varphi}(v) - x_d)\rangle ^{-2}(v + |s + \tilde{\varphi}(v) - x_d|)^{-\rho-1}
\]
For \( g(s) := \sup_{u \geq s} V(u)/u \), \( V'(s) \leq C_3(4) V(s)/s \leq C_3(4) g(s) \) by Proposition 3.1 \rangle \). Therefore we have that
\[
\text{III} \leq c_7 \cdot c_8 \int_0^{r/2} \int_0^{r/2} V'(s) V(v + |s + \tilde{\varphi}(v) - x_d)\rangle ^{-2}(v + |s + \tilde{\varphi}(v) - x_d|)^{\rho-1}dsdv
\]
\[
\leq c_9 \int_0^{r/2} \int_0^{r/2} g(s) V(v + |s + \tilde{\varphi}(v) - x_d)\rangle ^{-2}(v + |s + \tilde{\varphi}(v) - x_d|)^{\rho-1}dsdv.
\]
Applying (3.6) Lemma 4.4 \rangle \) with non-increasing functions \( g(s) \) and \( f(s) := V(s)^{-2}s^{\rho-1} \rangle \) and \( x(r) = s + \tilde{\varphi}(r) \rangle \), we have that
\[
\text{III} \leq c_{10} \int_0^{3r/4} G(u)V(u)^{-2}u^{\rho-1}du.
\]
where \( G(u) = \int_0^u g(s)ds \). By subadditivity of \( V \) and (3.3), \( G(u) \leq 2 \int_0^u (V(s)/s) ds \leq c_{11} V(u) \).

Using (3.4) again and the fact that \( \pi/2 < \rho \), we conclude that

\[
\text{III} \leq c_{10} \cdot c_{11} \int_0^R V(u)^{-1} u^{\rho-1} du \leq c_{12} V(r)^{-1} r^{\pi/2} \int_0^R u^{-\pi/2+\rho-1} du \leq c_{13} V(r)^{-1}
\]

for some positive constant \( c_{13} := c_{13}(\alpha, \rho, L, \rho, d) \).

\[ \square \]

### 4 Estimates on exit distributions for \( Y \)

In this section we give some key estimates on exit distributions for \( Y \). Throughout this section, we assume that \( Y \) is the symmetric pure jump Hunt process with the jumping intensity kernel \( J \) satisfying the conditions (J1.1), (J1.2), (SD) and (K0).

For any \( x \in \mathbb{R}^d \), stopping time \( S \) (with respect to the filtration of \( Y \)), and nonnegative measurable function \( f \) on \( \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \) with \( f(s, y, y) = 0 \) for all \( y \in \mathbb{R}^d \) and \( s \geq 0 \), we have the following Lévy system:

\[
\mathbb{E}_x \left[ \sum_{s \leq S} f(s, Y_s, Y_s) \right] = \mathbb{E}_x \left[ \int_0^S \left( \int_{\mathbb{R}^d} f(s, y, y) J(y, y) dy \right) ds \right]
\]

(e.g., see [23, Appendix A]).

Throughout this section, we assume that \( D \) is a \( C^{1,\rho} \) open set for some \( \rho \in (\pi/2, 1] \) with \( C^{1,\rho} \) characteristics \((R_0, \Lambda)\), and without loss of generality, we will assume that \( R_0 < 1 \) and \( \Lambda > 1 \). Recall that the function \( h_r(y) = h_{r,z}(y) \) is defined in Proposition 3.4.

**Lemma 4.1.** Let \( r \leq R_0/2 \). For any \( z \in \partial D \) and \( k \in \mathbb{N} \), let \( B_k := \{ y \in D \cap B(z, r/4) : \delta_D \cap B(z,r/4)(y) \geq 2^{-k} \} \). Then, for every \( u \in \mathbb{R}^d \) and \( k \in \mathbb{N} \) with \( |u| < 2^{-k} < 2^{-8}r \),

\[
\mathcal{L}^u h_{r,z}(w) := \lim_{\varepsilon \downarrow 0} \int_{|w-u-y| < \varepsilon} (h_{r,z}(y) - h_{r,z}(w-u)) J(w, u+y) dy
\]

is well defined in \( B_k \) and there exists \( C_{4.1} \geq C_{4.1}(\phi, L_0, L_3, \Lambda, \rho, \eta, d) > 0 \) independent of \( z \in \partial D \), \( k \in \mathbb{N} \) with \( 2^{-k+8} < r \leq R_0 \) such that

\[
|\mathcal{L}^u h_{r,z}(w)| \leq \frac{C_{4.1}}{V(r)} \quad \text{for all } w \in B_k, \ |u| < 2^{-k}.
\]

**Proof.** We fix \( z \in \partial D \) and use the short notation \( h_r(y) = h_{r,z}(y) \). For any \( w \in B_k \) and \( |u| < 2^{-k} < 2^{-8}r \), let \( x := w - u \in B(z, r/4) \). Define \( \kappa_u(x, y) := \kappa(u + x, u + y) \), and for \( \varepsilon < 2^{-k-1} \) denote \( A_{\varepsilon}(x) \) and \( L(x) \) by

\[
A_{\varepsilon}(x) := \int_{\varepsilon/|x-y| < 1} (h_r(y) - h_r(x))(\kappa_u(x, y) - \kappa_u(x, x)) \nu(|x-y|) dy,
\]

\[
L(x) := \int_{1/|x-y| < 1} (h_r(y) - h_r(x))(J(x, y) - \kappa_u(x, x)) \nu(|x-y|) dy,
\]

so that

\[
\int_{\varepsilon/|x-y| < 1} (h_r(y) - h_r(x)) \kappa_u(x, y) \nu(|x-y|) dy + \int_{1/|x-y| < 1} (h_r(y) - h_r(x)) J(x, y) dy
\]

\[
= A_{\varepsilon}(x) + \kappa_u(x, x) \cdot \mathcal{L}^u h_r(x) + L(x)
\]

\[ \Box \]
where $\mathcal{L}_Z^2$ is defined in (3.6). By the definition of $h_r$, (1.2), (1.4) and (1.2), for $r \leq R_0 < 1$, we first obtain that
\[
|L(x)| \leq c_0 \left| \int_{1 < |x-y|} J(x,y)dy \right| \leq c_1 V(r)^{-1}. \tag{4.2}
\]

On the other hand,
\[
A_\varepsilon(x) \leq \left( \int_{|x-y| < r/2} + \int_{r/2 < |x-y|} \right) |h_r(y) - h_r(x)| |\kappa_u(x,y) - \kappa_u(x,x)| \nu(|x-y|)dy =: I(x) + II(x).
\]
For $|x - y| < r/2$, $|h_r(y) - h_r(x)| \leq V(|x - y|)$ by subadditivity of $V$, and $V(r)/V(|x - y|) \leq C_V(r/|x - y|)^{\bar{\alpha}/2}$ by (3.4). Also $|\kappa_u(x,y) - \kappa_u(x,x)| \leq L_3|x - y|^\eta$ by the assumption $\text{(K_\eta)}$. Hence (1.2) and (3.4) imply that
\[
I(x) \leq c_2 \int_{|x-y| < r} V(|x-y|)^{-1} |x-y|^{-\eta}dy \leq c_2 C_V V(r)^{-1} r^{\bar{\alpha}/2} \int_{|x-y| < r} |x-y|^{-\bar{\alpha}/2 + \eta}dy \leq c_3 V(r)^{-1} \tag{4.3}
\]
for some positive constant $c_3 := c_3(\bar{\alpha}, C, L_3, \eta, d)$. The last inequality holds since $\eta > \bar{\alpha}/2$.

To obtain the upper bound of $II(x)$, note that $|h_r(y) - h_r(x)| \leq 2V(|x - y|)$ for $r/2 \leq |x - y|$, using the subadditivity of $V$. Indeed, if $y \in (D \cap B(z,r))^c$, then $h_r(y) = 0$ and by subadditivity of $V$, $|h_r(y) - h_r(x)| \leq V(r) \leq 2V(|x - y|)$. If $y \in D \cap B(z,r)$, by subadditivity of $V$, $|h_r(y) - h_r(x)| \leq V(\delta_D(y) - \delta_D(x)) \leq V(|x - y|)$. Hence by [7] Lemma 3.5 and (1.4), we obtain that
\[
II(x) \leq 2L_0 \int_{r/2 < |x-y|} |h_r(y) - h_r(x)| \nu(|x-y|)dy \leq c_3 V(r)^{-1}. \tag{4.4}
\]

Thus, from (4.3) and (4.4), we have that
\[
\lim_{\varepsilon \downarrow 0} A_\varepsilon(x) \text{ exists and } \lim_{\varepsilon \downarrow 0} A_\varepsilon(x) \leq (c_3 + c_5) V(r)^{-1}. \tag{4.5}
\]

Finally from Proposition 3.4 we have that $\lim_{\varepsilon \downarrow 0} \mathcal{L}_Z^2 h_r(x)$ exists and
\[
\left| \lim_{\varepsilon \downarrow 0} \mathcal{L}_Z^2 h_r(x) \right| \leq C_{3.4} V(r)^{-1}. \tag{4.6}
\]

Hence, combining (4.2), (4.5) and (4.6) we have the conclusion.

\[\square\]

**Theorem 4.2.** For any $x \in D$, let $z_x \in \partial D$ be a point satisfying $\delta_D(x) = |x - z_x|$.

(1) There are constants $A_{1.2} = A_{1.2}(\phi, L_0, L_3, \Lambda, \rho, \eta) \in (0, 1)$ and $C_{1.2} = C_{1.2}(\phi, L_0, L_3, \Lambda, \rho, \eta) > 0$, such that for any $s \leq A_{1.2} R_0/2$ and $x \in D$ with $\delta_D(x) < s$,
\[
\mathbb{E}_x \left[ \tau_{D \cap B(z_x,s)} \right] \leq C_{1.2} V(s) V(\delta_D(x)). \tag{4.7}
\]

(2) There is a constant $C_{4.2} = C_{4.2}(\phi, L_0, L_3, \Lambda, \rho, \eta) > 0$, such that for any $s \leq R_0/2$, $\lambda \geq 4$ and $x \in D$ with $\delta_D(x) < \lambda^{-1} s/2$,
\[
\mathbb{P}_x \left( Y_{\tau_{D \cap B(z_x,\lambda^{-1}s)}} \in \{ 2\Lambda |\bar{y}| < y_d, \lambda^{-1} s < |y| < s \ in \ CS_{z_x} \} \right) \geq C_{4.2} \frac{V(\delta_D(x))}{V(s)}. \tag{4.8}
\]

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Proof. Without loss of generality, we assume that \( z_x = 0 \). For \( R \leq R_0/2 \), let \( h_R(y) = V(\delta_D(y))1_{D \cap B(0,R)}(y) \). Let \( f \geq 0 \) be a smooth radial function such that \( f(y) = 0 \) for \( |y| > 1 \) and \( \int_{\mathbb{R}^d} f(y) dy = 1 \). For \( k \geq 1 \), define \( f_k(y) := 2^{kd} f(2^k y) \) and \( h_R^{(k)} := f_k * h_R \in C^2(\mathbb{R}^d) \), and let \( B^\lambda_k := \{ y \in D \cap B(0,\lambda^{-1}R/4) : \delta_D(0,\lambda^{-1}R)(y) \geq 2^{-k} \} \) for \( \lambda \geq 4 \).

Since \( h_R^{(k)} \) is a \( C^2 \) function, \( \mathcal{L}^{(k)}h_R \) is well defined everywhere. By Lemma 4.1, for \( w \in B^\lambda_k \) and \( u \in B(0,2^{-k}) \) the following limit exists and

\[
\lim_{\varepsilon \downarrow 0} \int_{|w-y| > \varepsilon} (h_R(y) - h_R(w) - h_R(w - u)) J(w,y) dy
= \lim_{\varepsilon \downarrow 0} \int_{|w-u-y| > \varepsilon} (h_R(y') - h_R(w - u)) J(w, u + y') dy' = \mathcal{L}^u h_R(w)
\]

exits and \(-C_4 V(R)^{-1} \leq \mathcal{L}^u h_R(w) \leq C_4 V(R)^{-1}\). We note that

\[
\int_{|w-y| > \varepsilon} (h_R^{(k)}(y) - h_R^{(k)}(w)) J(w,y) dy
= \int_{|w-y| > \varepsilon} \int_{\mathbb{R}^d} f_k(u) (h_R(y) - h_R(w) - h_R(w - u)) du J(w,y) dy
= \int_{|u| < 2^{-k}} f_k(u) \int_{|w-y| > \varepsilon} (h_R(y) - h_R(w - u)) J(w,y) dy du.
\]

By letting \( \varepsilon \downarrow 0 \) and using the dominated convergence theorem, it follows that for \( w \in B^\lambda_k \) and \( 2^{-k+8} < \lambda^{-1} R \),

\[
|\mathcal{L}^{(k)}h_R(w)| = \left| \int_{|u| < 2^{-k}} f_k(u) \mathcal{L}^u h_R(w) du \right| \leq C_4 V(R)^{-1} \int_{|u| < 2^{-k}} f_k(u) du = C_4 V(R)^{-1}. \quad (4.9)
\]

Applying Lemma 2.2 to \( B_k^\lambda \) and \( h_R^{(k)} \), and using (4.9), for any \( x \in B^\lambda_k \), we have

\[
\mathbb{E}_x \left[ h_R^{(k)} \left( Y_{\tau_{B_k^\lambda}} \right) \right] - C_4 V(R)^{-1} \mathbb{E}_x \left[ \tau_{B_k^\lambda} \right] \leq h_R^{(k)}(x) - \mathbb{E}_x \left[ h_R^{(k)} \left( Y_{\tau_{B_k^\lambda}} \right) \right] + C_4 V(R)^{-1} \mathbb{E}_x \left[ \tau_{B_k^\lambda} \right].
\]

By letting \( k \to \infty \), for any \( x \in D \cap B(0,\lambda^{-1} R) \), we obtain

\[
V(\delta_D(x)) = h_R(x) \geq \mathbb{E}_x \left[ h_R \left( Y_{\tau_{D \cap B(0,\lambda^{-1}R)}} \right) \right] \geq C_4 V(R)^{-1} \mathbb{E}_x \left[ \tau_{D \cap B(0,\lambda^{-1}R)} \right] \quad (4.10)
\]

and

\[
V(\delta_D(x)) = h_R(x) \leq \mathbb{E}_x \left[ h_R \left( Y_{\tau_{\mathbb{R}^d \cap B(0,\lambda^{-1}R)}} \right) \right] + C_4 V(R)^{-1} \mathbb{E}_x \left[ \tau_{\mathbb{R}^d \cap B(0,\lambda^{-1}R)} \right]. \quad (4.11)
\]

For any \( z \in D \cap B(0,\lambda^{-1} R) \) and \( y \in D \cap (B(0,R) \setminus B(0,\lambda^{-1} R)) \), by the fact that \( \nu \) is decreasing and \( (1.3) \), \( \nu(|y - z|) \geq \nu(2|y|) \geq C_1 \nu(|y|) \). So by (1.4), (J1.1) and (4.1), we obtain

\[
\mathbb{E}_x \left[ h_R \left( Y_{\tau_{D \cap B(0,\lambda^{-1}R)}} \right) \right] \geq L_0^{-1} \mathbb{E}_x \int_{D \cap (B(0,R) \setminus B(0,\lambda^{-1}R))} \int_{\tau_{D \cap B(0,\lambda^{-1}R)}} \nu(|y_t - y|) d h_R(y) dy
\geq L_0^{-1} c_1 \mathbb{E}_x \left[ \tau_{D \cap B(0,\lambda^{-1}R)} \right] \int_{D \cap (B(0,R) \setminus B(0,\lambda^{-1}R))} \nu(|y|) d h_R(y) dy. \quad (4.12)
\]

Let \( A := \{ \langle \tilde{y}, y \rangle : 2\Lambda |\tilde{y}| < |y| \} \). For any \( y \in A \cap B(0,R) \), since \( y_d > 2\Lambda |\tilde{y}| > 2\Lambda |\tilde{y}|^{\nu+1} > \varphi(\tilde{y}) \), we have \( A \cap B(0,R) \subset D \cap B(0,R) \) and

\[
\delta_D(y) \geq (1 + \Lambda)^{-1} (y_d - \varphi(\tilde{y})) \geq (2\Lambda)^{-1} (y_d - \Lambda |\tilde{y}|) > (4\Lambda)^{-1} y_d \geq (4\Lambda((2\Lambda)^{-2} + 1)^{1/2})^{-1}|y|.
\]

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Combining this and (3.4), \( V(\delta_D(y)) \geq c_2 V(|y|) \). By changing to polar coordinates with \(|y| = t\), Proposition 3.3 and Proposition 3.4 we obtain that

\[
\int\!\!\int_{D \cap (B(0, \lambda^{-1} R) \setminus B(0, \lambda^{-1} R))} \nu(|y|) h(y) dy \geq c_2 \int\!\!\int_{A \cap (B(0, \lambda^{-1} R) \setminus B(0, \lambda^{-1} R))} \nu(|y|) V(|y|) dy \geq c_3 \int_{\lambda^{-1} R}^{R} \nu(t) V(t) t^{d-1} dt \geq c_4 \int_{\lambda^{-1} R}^{R} V(t)^{-1} t^{-1} dt \geq c_4 \cdot C \int_{\lambda^{-1} R}^{R} \frac{V(t)}{V(t)^2} dt = c_4 \cdot C (V(\lambda^{-1} R)^{-1} - V(R)^{-1}). \tag{4.13}
\]

Combining (4.12) and (4.13), there exists \( C_5 := C_5(\overline{\alpha}, \overline{C}, L_0, \Lambda, d) > 0 \) such that

\[
\mathbb{E}_x \left[ h_x \left( Y_{\tau_{D \cap B(0, \lambda^{-1} R)}}, \lambda \right) \right] \geq C_5 \mathbb{E}_x \left[ \tau_{D \cap B(0, \lambda^{-1} R)} \right] (V(\lambda^{-1} R)^{-1} - V(R)^{-1}). \tag{4.14}
\]

Using (3.4) again, \( V(\lambda^{-1} R) \leq V(\lambda_0^{-1} R) \leq C_4 \sum_\alpha 2 V(R) \) for any \( \lambda \geq \lambda_0 \geq 4 \). Let \( \lambda_0 := (2C_4)(C_5 + C_4) / C_5 \geq 1 \). Then combining (4.11) and (4.14), we have that for \( \lambda \geq \lambda_0 \)

\[
V(\delta_D(x)) \geq (C_5 V(\lambda^{-1} R)^{-1} - (C_5 + C_4) V(R)^{-1}) \mathbb{E}_x[\tau_{D \cap B(0, \lambda^{-1} R)}] \geq (C_5 + C_4 V(\lambda^{-1} R)^{-1}) \mathbb{E}_x[\tau_{D \cap B(0, \lambda^{-1} R)}]. \tag{4.15}
\]

Thus, we have proved (4.17) with \( A \lambda^{-1} = \lambda_0^{-1} \) and \( s = \lambda^{-1} R \) where \( \Lambda \geq \lambda_0 \).

From (4.11) and Corollary 2.4 for \( \delta_D(x) < \lambda^{-1} R / 2 \) and \( \lambda \geq 4 \), we first note that

\[
V(\delta_D(x)) \leq V(R) \mathbb{P}_x \left( Y_{\tau_{D \cap B(0, \lambda^{-1} R)}} \in D \right) + C_4 V(\lambda^{-1} R)^{-1} \mathbb{E}_x \left[ \tau_{D \cap B(0, \lambda^{-1} R)} \right] \leq (C_4 V(R) V(\lambda^{-1} R)^{-2} + C_4 V(R)^{-1}) \mathbb{E}_x \left[ \tau_{D \cap B(0, \lambda^{-1} R)} \right] = c_6 V(R) (V(\lambda^{-1} R)^{-2} + V(R)^{-2}) \mathbb{E}_x \left[ \tau_{D \cap B(0, \lambda^{-1} R)} \right]. \tag{4.17}
\]

By (3.4) and the subadditivity of \( V \), \( V(s)^{-1} \geq 3^{-1} (3\lambda^{-1} R / s)^{\alpha / 2} C_4 V(\lambda^{-1} R)^{-1} \) for \( s \leq 3\lambda^{-1} R \). Combining this with (3.3) and using the polar coordinate with \(|y| = t\), we have that

\[
\int\!\!\int_{A \cap (B(0, 3\lambda^{-1} R) \setminus B(0, 2\lambda^{-1} R))} \nu(|y|) dy \geq c_7 \int_{3\lambda^{-1} R}^{3\lambda^{-1} R} \nu(t) t^{d-1} dt \geq c_8 \int_{3\lambda^{-1} R}^{3\lambda^{-1} R} V(t)^{-2} t^{-1} dt \geq c_9 V(\lambda^{-1} R)^{-2}. \tag{4.18}
\]

For any \( z \in B(0, \lambda^{-1} R) \) and \( y \in B(0, 3\lambda^{-1} R) \setminus B(0, 2\lambda^{-1} R) \), by (1.3) and the fact that \( \nu \) is decreasing, \( \nu(|y - z|) \geq \nu(|y| + |z|) \geq \nu(3|y| / 2) \geq c_10 \nu(|y|) \). So by (1.1), (J1.1) (4.11) and (4.18), we obtain

\[
\mathbb{P}_x \left( Y_{\tau_{D \cap B(0, \lambda^{-1} R)}} \in A \cap \left( B(0, 3\lambda^{-1} R) \setminus B(0, 2\lambda^{-1} R) \right) \right) \geq L_0^{-1} \mathbb{E}_x \left[ \int_{0}^{\tau_{D \cap B(0, \lambda^{-1} R)}} \int_{A \cap (B(0, 3\lambda^{-1} R) \setminus B(0, 2\lambda^{-1} R))} \nu(|Y_s - y|) dy ds \right] \geq L_0^{-1} c_{10} \mathbb{E}_x \left[ \int_{0}^{\tau_{D \cap B(0, \lambda^{-1} R)}} \int_{A \cap (B(0, 3\lambda^{-1} R) \setminus B(0, 2\lambda^{-1} R))} \nu(|y|) dy ds \right] \geq c_9 c_{10} L_0^{-1} V(\lambda^{-1} R)^{-2} \mathbb{E} \left[ \tau_{D \cap B(0, \lambda^{-1} R)} \right]. \tag{4.19}
\]
Hence combining (4.17), (4.19), and the fact that $V$ is increasing, we conclude that for $\lambda \geq 4$,
\[
V(\delta_d(x)) \leq c_{11} V(R) (V(\lambda^{-1}r)^{-2} + V(r)^{-2}) V(\lambda^{-1}R)^2 \cdot \mathbb{P}_x \left( Y_{\delta_d(B(0,\lambda^{-1}R))} \in A \cap (B(0,3\lambda^{-1}R) \setminus B(0,2\lambda^{-1}R)) \right)
\]
\[
\leq 2c_{11} V(R) \mathbb{P}_x \left( Y_{\delta_d(B(0,\lambda^{-1}R))} \in A \cap (B(0,R) \setminus B(0,\lambda^{-1}R)) \right) .
\]
Thus, we have proved (4.8) with $s = R$.

5 Upper bound estimates

In this section, we derive the upper bound estimate on $p_D(t,x,y)$ for $t \leq T$ in $C^{1,\rho}$ open set $D$ with $C^{1,\rho}$ characteristics $(R_0,\Lambda)$ for $\rho \in (\pi/2,1]$. As before, we will assume that $R_0 < 1$ and $\Lambda > 1$ and fix such $C^{1,\rho}$ open set $D$ with $\rho \in (\pi/2,1]$ throughout this section. We first introduce the next lemma which give a guideline to obtain the upper bound estimate on $p_D(t,x,y)$ (for its proof, see [14, Lemma 3.1] and [6, Lemma 1.10]). Applying (4.7) and Theorem 1.2 to
\[
\text{Proof.}
\]
\[
\text{Hence combining (4.17), (4.19), and the fact that } V \text{ satisfies the conditions (J1.1) and (J1.2), suppose that } E \subset \mathbb{R}^d \text{ be disjoint open subsets and } U := E \setminus (U_1 \cup U_3) . \text{ If } x \in U_1, y \in U_3 \text{ and } t > 0, \text{ we have}
\]
\[
p_E(t,x,y) \leq \mathbb{P}_x \left( Y_{\tau_{U_1}} \in U_2 \right) \cdot \sup_{s,t,z \in U_2} p_E(s,z,y)
\]
\[
+ \int_0^t \mathbb{P}(\tau_{U_1} > s) \mathbb{E}_y(\tau_{E} > t - s) ds \cdot \sup_{u \in U_1, z \in U_3} J(u,z) . \quad (5.1)
\]
\[
\leq \mathbb{P}_x \left( Y_{\tau_{U_1}} \in U_2 \right) \cdot \sup_{s,t,z \in U_2} p(s,z,y) + (t \wedge \mathbb{E}_x[\tau_{U_1}]) \cdot \sup_{u \in U_1, z \in U_3} J(u,z) . \quad (5.2)
\]

For the remainder of the section, we assume that $Y$ is the symmetric pure jump Hunt process with the jumping intensity kernel $J$ satisfying the conditions (J1.1), (J1.2), (K$\eta$) and (SD). Let
\[
a_{T,R_0} := |V(A_{4,1/4}R_0/4)|^2 T^{-1}
\]
where $A_{4,1/4}$ is the constant in Theorem 4.2.1. Denote $V^{-1}$ be the inverse function of $V$, then $V^{-1}(\sqrt{a_{T,R_0} \cdot t}) \leq A_{4,1/4}R_0/4$ for any $t \leq T$.

Lemma 5.2. There exists $C_{5.2} = C_{5.2}(\phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T)$ such that for any $t \leq T$ and $x \in D$, we have that
\[
\mathbb{P}_x(\tau_D > t) \leq C_{5.2} \left( 1 \wedge \frac{V(\delta_d(x))}{\sqrt{t}} \right) .
\]

Proof. Let $r_1 := V^{-1}(\sqrt{a_{T,R_0} \cdot t})$. We only consider the case $V(\delta_d(x)) < \sqrt{a_{T,R_0} \cdot t}$ which implies $\delta_d(x) < A_{4,1/4}R_0/4 < R_0/4$. Let $U_1 := D \cap B(z_x,2r_1) \subset D$ where $z_x \in \partial D$ with $\delta_d(x) = |x - z_x|$. Then, using Chebyshev’s inequality and Corollary 2.4, we first obtain that
\[
\mathbb{P}_x(\tau_D > t) = \mathbb{P}_x(\tau_U > t, \tau_D = \tau_U) + \mathbb{P}_x(\tau_D > \tau_U > t)
\]
\[
\leq \mathbb{P}_x(\tau_U > t) + \mathbb{P}_x(\tau_U \in D) \leq (t^{-1} + C_{2.4}(\phi(2r_1)) \mathbb{E}_x[\tau_U] .
\]
From (3.3) and the fact that \( V \) is increasing and subadditive,
\[
\phi(2r_t) \asymp [V(2r_t)]^2 \asymp [V(V^{-1}(\sqrt{a_{T,R_0}}t))]^2 = a_{T,R_0}t.
\]
Therefore, using (4.7) in Theorem 4.2 we conclude that
\[
\mathbb{P}_x(\tau_D > t) \leq c_1 \frac{1}{t} \mathbb{E}_x \tau_{U_1} \leq c_1 C_{5.3} \frac{1}{t} V(r_t) V(\delta_D(x)) \leq c_2 \frac{V(\delta_D(x))}{\sqrt{t}}.
\]

We will use the following inequality several times, which follows from (WS): there exist \( C_t := (\mathcal{C}_{-1}^{-1} \lor \mathcal{L}^{-2})^{1/2} > 1 \) such that
\[
C_t^{-1} \left( \frac{r}{R} \right)^{1/2} \leq \frac{\phi^{-1}(r)}{\phi^{-1}(R)} \leq C_t \left( \frac{r}{R} \right)^{1/2} \quad \text{for } 0 < r \leq R.
\] (5.3)

Recall the functions \( F_{a,T} (t,r) \) and \( \Psi(t,x) \) are defined in (1.6) and (1.8), respectively.

**Proposition 5.3.** Let \( a \leq a_{T,R_0} \).

1. Suppose that \( D \) is bounded and the jumping intensity kernel \( J \) satisfies the condition (J1). Then there exists a positive constant \( C_{5.3} = C_{5.3}(\beta, \phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T, \text{diam}(D), a) \) such that for any \((t,x,y) \in (0,T] \times D \times D \) with \( V^{-1}(\sqrt{a \cdot t}) \leq |x-y| \), we have
\[
p_D(t,x,y) \leq C_{5.3} \Psi(t,x) \cdot \left( \left[ \phi^{-1}(t) \right]^{-d} \land t \nu(|x-y|) \right).
\]

2. Suppose that the jumping intensity kernel \( J \) satisfies the condition (J2). Then there exists a positive constant \( C_{5.3} = C_{5.3}(\beta, \phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T, a) \) such that for any \((t,x,y) \in (0,T] \times D \times D \) with \( |x-y| \geq V^{-1}(\sqrt{a \cdot t}) \cdot 1_{\beta \in [0,1]} + 2 \cdot 1_{\beta \in (1,\infty)} + (2 + V^{-1}(\sqrt{a \cdot t})) \cdot 1_{\beta = \infty} \), we have
\[
p_D(t,x,y) \leq C_{5.3} \Psi(t,x) \cdot \begin{cases} F_{1.2}^{1/2} \wedge_{\gamma_1,T} (t,|x-y|/3) & \text{if } \beta \in [0,1], \\
(2t/T|x-y|)^{1/2} & \text{if } \beta = \infty.
\end{cases}
\]

**Proof.** Since we assume that \( D \) is bounded in (1), by applying Theorem 1.1 instead of Theorem 1.2, the proof of (1) is similar to the that of (2), so we only give the proof of (2). Let \( r_t := V^{-1}(\sqrt{a \cdot t})/9 \). If \( \delta_D(x) \geq r_t/2 \), using subadditivity of \( V \), we see that \( \Psi(t,x) \asymp 1 \). Thus, by Theorem 1.2 and the fact that \( r \rightarrow F_{c,\gamma,T}(t,r) \) is decreasing, we obtain the conclusion.

Let \( 0 < \delta_D(x) \leq r_t/2 \). Since \( 9r_t \leq V^{-1}(\sqrt{a_{T,R_0}}t) = A_{1.2} R_0/4 < 1 \), \( |x-y| \geq 9r_t \) for all \( \beta \in [0,1] \). Let \( U_1 := B(z_x, r_t) \cap D, U_3 := \{ z \in D : |z-x| \geq |x-y|/2 \} \) and \( U_2 := D \setminus (U_1 \cup U_3) \). Then \( x \in U_1, y \in U_3 \) and \( U_1 \cap U_3 = \emptyset \). Note that \( |x-y|/2 \leq |x-y| - |z-x| \leq |y-z| \) for any \( z \in U_2 \). Therefore, by virtue of Theorem 1.2 we have we obtain
\[
\sup_{s<t,z \in U_2} p(s,z,y) \leq C_{1.2} \sup_{s<t,|z-y| > |x-y|/2} F_{1.2}^{1/2} \wedge_{\gamma_1,T} (s,|z-y|) \leq c_1 F_{1.2}^{1/2} \wedge_{\gamma_1,T} (t,|x-y|/2). \quad (5.4)
\]
In fact, if \( \beta \in (1,\infty) \), we have \( |z-y| \geq |x-y|/2 > 1 \) and so \( F_{1.2}^{1/2} \wedge_{\gamma_1,T} (s,|z-y|) \) is increasing in \( s \). If \( \beta \in [0,1] \), we have \( |z-y| \geq |x-y|/2 \geq V^{-1}(\sqrt{a \cdot t})/2 \) and so
\[
\phi^{-1}(s) - d \land \nu(|z-y|) e^{-\gamma_1 t |z-y|^\beta} \leq \nu(|z-y|) e^{-\gamma_1 t |z-y|^\beta} \quad \text{using (3.3), (5.3) and (5.3)}.
\] Also, \( s \nu(r) e^{-\gamma_1 r} \) is increasing
Proposition 5.4. (1) Suppose that the jumping intensity kernel \( J \) satisfying \((J1)\) and \( D \) is bounded. There exists a positive constant \( C_{5.4} = C_{5.4}(\beta, \phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T, \text{diam}(D)) \) such that for any \((t, x, y) \in (0, T] \times D \times D\), we have
\[
p_D(t, x, y) \leq C_{5.4} \Psi(t, x)\Psi(t, y) \left( [\phi^{-1}(t)]^{-d} \wedge t\nu(|x - y|) \right).
\]
(2) There exists a positive constant \( C_{5.4} = C_{5.4}(\beta, \phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T) \) such that for any \((t, x, y) \in (0, T] \times D \times D\), we have
\[
p_D^X(t, x, y) \leq C_{5.4} \Psi(t, x)\Psi(t, y) \left( [\phi^{-1}(t)]^{-d} \wedge t\nu(|x - y|) \right).
\]

Proof. Using Theorem 1.1 and Proposition 5.3 (1) instead of Theorem 1.2 and Proposition 5.3 (2) respectively, the proof of (1) is almost identical to the one of (2). So we only give the proof of (2).

The semigroup property, Theorem 1.2 (for \( \beta = 0 \)), 5.3 and Lemma 5.2 yield
\[
p_D^X(t/2, x, y) \leq \left( \sup_{z, w \in D} p_D^X(t/4, z, w) \right) \int_D p_D^X(t/4, x, z)dz
\leq c_1[\phi^{-1}(t/4)]^{-d} \mathbb{P}_x(\tau_D > t/4) \leq c_2[\phi^{-1}(t)]^{-d}\Psi(t, x).
\]
Thus, by Proposition 5.3 (5.3) and Theorem 1.2 (for \( \beta = 0 \)), we obtain
\[
p_D^X(t/2, x, y) \leq c_3 \Psi(t/2, x) \left( [\phi^{-1}(t/2)]^{-d} \land (t/2) \nu(|x - y|) \right) \leq c_4 \Psi(t, x)p^X(t/2, x, y).
\]
Combining these with Theorem 1.2 (for \( \beta = 0 \)), the symmetry \( p_D^X \) and the semigroup property of \( p^X \), we conclude that
\[
p_D^X(t, x, y) = \int_D p_D^X(t/2, x, z) \cdot p_D^X(t/2, z, y)dz \leq c_2^2 \Psi(t, x)\Psi(t, y)p^X(t, x, y)
\leq c_5 \Psi(t, x)\Psi(t, y) \left( [\phi^{-1}(t)]^{-d} \land t\nu(|x - y|) \right).
\]

\( \square \)

Suppose that the jumping intensity kernel \( J \) satisfying (J2). By Meyer’s construction (e.g., see [23, §4.1]), when \( \beta \in (0, \infty) \) the process \( Y \) can be constructed from \( X \) by removing jumps of size greater than 1 with suitable rate. Let \( p_D^X(t, x, y) \) be the transition density function of \( X \) on \( D \). For \( \beta \in (0, \infty) \), we define
\[
J(x) := \int_{\mathbb{R}^d} \kappa(x, y)\nu(|x - y|) \left( 1 - \chi(|x - y|)^{-1} \right) dy
\]
where \( \chi(|x - y|) \) is defined in (1.3). Then \( \|J\|_{\infty} \leq c_1 \int_{|z| > 1} \nu(|z|)dz < \infty \). By (11) Lemma 3.6] we have
\[
p_D(t, x, y) \leq e^{\|J\|_{\infty}}p_D^X(t, x, y) \quad \text{for any } (t, x, y) \in (0, T] \times D \times D.
\] (5.8)
Thus, the sharp upper bound of of \( p_D(t, x, y) \) for \( |x - y| < M \) for some \( M > 0 \) follows from the one of \( p_D^X(t, x, y) \) and (5.8). Therefore Combining (5.8), Propositions 5.3(2) and 5.4(2), we have the following result.

**Proposition 5.5.** Suppose that the jumping intensity kernel \( J \) satisfying (J2). There exists a positive constant \( C_{5,5} = C_{5,5}(\beta, \phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T) \) such that for every \((t, x, y) \in (0, T] \times D \times D \) we have
\[
p_D(t, x, y) \leq C_{5,5} \Psi(t, x) \cdot \begin{cases}
F_{\delta, \gamma_1, \gamma_1, T}(t, |x - y|/3) & \text{if } \beta \in [0, \infty), \\
F_{\delta, \gamma_1, T}(t, |x - y|/2) & \text{if } \beta = \infty,
\end{cases}
\] (5.9)
where \( C_{5,5} \) is the constant in Theorem 1.3 and \( \gamma_1 \) is the constant in (1.3).

Now we are ready to prove the upper bound of Theorem 1.3(1) and Theorem 1.5(1).

**Proofs of the upper bounds of \( p_D(t, x, y) \) in Theorems 1.3(1) and 1.5(1).** In Proposition 5.5(1), we have proved the upper bound of \( p_D(t, x, y) \) in Theorem 1.3(1). So we only give the proof of the upper bound of \( p_D(t, x, y) \) in Theorem 1.5(1).

Let \( r_t := V^{-1}(\sqrt{\delta T, R_0} - t) \) so that \( r_t \leq A_{1,2}R_0/4 < 1/4 \). By Proposition 5.5 and the symmetry of \( p_D(t, x, y) \), we only need to prove the upper bound of \( p_D(t, x, y) \) for the case \( \delta_D(x) \vee \delta_D(y) < r_t \), which we will assume throughout the proof.

If \( \beta = \infty \) and \( 6 < |x - y| \leq 6(1 \lor C_{1,2}^{-1}) \), by (5.8) and Proposition 5.4, we have
\[
p_D(t, x, y) \leq c_1 \Psi(t, x)\Psi(t, y)(t/T) \leq c_4 \Psi(t, x)\Psi(t, y)(t/T)^{(1.2)^{1/6}}|x - y|/6.
\]
If either the case \( \beta \in [0, \infty) \) and \( |x - y| \leq 6(1 \lor C_{1,2}^{-1}) \) holds or the case \( \beta = \infty \) and \( |x - y| \leq 6 \) holds, by (5.8) and Proposition 5.4(2), we have
\[
p_D(t, x, y) \leq e^{\|J\|_{\infty}}p_D^X(t, x, y) \leq c_2 \Psi(t, x)\Psi(t, y) \left( [\phi^{-1}(t)]^{-d} \land t\nu(|x - y|) \right).
\]
Thus, the upper bound of \( p_D(t, x, y) \) in Theorem 1.5(1) holds for \( |x - y| \leq 6(1 \lor C^{-1}_{1.2}) \).

For the remainder of the proof, we assume that \( \delta_D(x) \lor \delta_D(y) < r_1 \) and \( |x - y| > 6(1 \lor C^{-1}_{1.2}) \).

For any \( x \) with \( \delta_D(x) < r_1 \), let \( z_x \in \partial D \) such that \( \delta_D(x) = \|z_x - x\| \). Let \( U_1 := B(z_x, r_1) \cap D \), \( U_3 := \{z \in D : \|z - x\| \geq |x - y|/2\} \), and \( U_2 := D \setminus (U_1 \cup U_3) \). Note that \( x \in U_1 \) and \( y \in U_3 \) and \( |x - y|/2 \leq |z - y| \) for \( z \in U_2 \). Thus, by Proposition \( 5.3 \) we have

\[
\sup_{s < t, x \in U_2} p_D(s, z, y) \\
\leq C_{5.3} \left( \frac{V(\delta_D(y))}{\sqrt{s}} \cdot \left( F_{1.2} \gamma_{1,1,t}(s, |z - y|/3) \cdot 1_{\beta \in [0,\infty)} + F_{1.2} \gamma_{1,1,t}(s, |z - y|/2) \cdot 1_{\beta = \infty} \right) \right) \\
\leq C_{5.3} \left( \frac{V(\delta_D(y))}{\sqrt{s}} \cdot \left( F_{1.2} \gamma_{1,1,t}(s, |z - y|/3) \cdot 1_{\beta \in [0,\infty)} + F_{1.2} \gamma_{1,1,t}(s, |z - y|/2) \cdot 1_{\beta = \infty} \right) \right) \\
\leq C_{5.3} \left( \frac{V(\delta_D(y))}{\sqrt{s}} \cdot \left( F_{1.2} \gamma_{1,1,t}(t, |x - y|/6) \cdot 1_{\beta \in [0,\infty)} + F_{1.2} \gamma_{1,1,t}(t, |x - y|/4) \cdot 1_{\beta = \infty} \right) \right). \tag{5.10}
\]

The last inequality is clear for \( \beta \in [0, \infty) \) by the definition of \( F_{1.2} \gamma_{1,1,t}(t, r) \) and for \( \beta = \infty \) we used the fact that \( s \to s^{-1/2}(s/Tr)^{ar} \) is increasing if \( ar \geq 1 \). Hence from (5.9) and (5.10), we obtain

\[
\mathbb{P}_x \left( \tau_{u_1} \in U_2 \right) \left( \sup_{s < t, x \in U_2} p_D(s, z, y) \right) \\
\leq c_4 \frac{V(\delta_D(x)) V(\delta_D(y))}{\sqrt{t}} \cdot \left\{ \begin{array}{ll}
F_{1.2} \gamma_{1,1,t}(t, |x - y|/6) & \text{if } \beta \in [0, \infty), \\
F_{1.2} \gamma_{1,1,t}(t, |x - y|/4) & \text{if } \beta = \infty.
\end{array} \right. \tag{5.11}
\]

Also from Lemma 5.2 we have

\[
\int_0^t \mathbb{P}_x(\tau_{u_1} > s) \mathbb{P}_y(\tau_D > t - s) ds \leq \int_0^t \mathbb{P}_x(\tau_D > s) \mathbb{P}_y(\tau_D > t - s) ds \\
\leq C_{5.2}^2 \frac{V(\delta_D(x)) V(\delta_D(y))}{\sqrt{t}} \int_0^t s^{-1/2}(t - s)^{-1/2} ds \leq c_5 t \frac{V(\delta_D(x)) V(\delta_D(y))}{\sqrt{t}}. \tag{5.12}
\]

For \( (u, z) \in U_1 \times U_3 \) and \( |x - y| > 6 > 6r_1 \), note that \( |u - z| \geq |x - y| - |x - u| - |z - y| \geq |x - y|/3 \). Thus, if \( \beta \in [0, \infty) \), by (1.5) and (1.2),

\[
\left( \sup_{u \in U_1, z \in U_3} J(u, z) \right) \leq c_6 e^{-\gamma_1(x-y)/3} \nu(|x - y|/3) \leq c_7 t^{-1} F_{1.1,1}(t, |x - y|/3).
\]

Combining this with (5.12), we obtain

\[
\int_0^t \mathbb{P}_x(\tau_{u_1} > s) \mathbb{P}_y(\tau_D > t - s) ds \cdot \left( \sup_{u \in U_1, z \in U_3} J(u, z) \right) \\
\leq c_8 \frac{V(\delta_D(x)) V(\delta_D(y))}{\sqrt{t}} F_{1.1,1}(t, |x - y|/3). \tag{5.13}
\]

If \( \beta = \infty \), since \( |u - z| > 1 \), \( J(u, z) = 0 \) on \( U_1 \times U_3 \).

Therefore by applying (5.11) and (5.13) for \( \beta \in [0, \infty) \) and by applying (5.11) for \( \beta = \infty \) in (5.1) of Lemma 5.1 we prove the upper bound of \( p_D(t, x, y) \) in Theorem 1.5(1) for \( \delta_D(x) \lor \delta_D(y) < r_1 \) and \( |x - y| > 6(1 \lor C^{-1}_{1.2}) \). \( \square \)
6 Preliminary lower bound estimates

In this section, we discuss a preliminary lower bound for \( p_D(t,x,y) \). In this section we will always assume that \( Y \) is the symmetric pure jump Hunt process with the jumping intensity kernel \( J \) satisfying either the conditions (J1.2) and (J1.3) or the condition (J2). Since \( Y \) satisfies conditions imposed in [12], using [12] Theorem 5.2 and Lemma 2.5, the proof of the next lemma is the same as that of [22] Lemma 3.2. Thus, we omit the proof.

**Lemma 6.1.** Let \( a,b \) and \( T \) be positive constants. Then there exists a constant \( C_{6.1} = C_{6.1}^{(a,b,L_0,\phi,T)} > 0 \) such that for all \( \lambda \in (0,T] \) we have

\[
\inf_{y \in \mathbb{R}^d} \mathbb{P}_y (\tau_{B(z,2b\phi^{-1}(\lambda))} > a\lambda) \geq C_{6.1}
\]

Let \( D \) be an arbitrary non-empty open set, and \( a \) and \( T \) be positive constants. We use the convention that \( \delta_D(\cdot) \equiv \infty \) when \( D = \mathbb{R}^d \) to derive the lower bound of \( p(t,x,y) \) in Theorem 1.1 and 1.2 simultaneously.

Using [12] Theorem 5.2 and Lemma 6.1, the proof of the next lemma is similar to that of [22] Proposition 3.3] for \( |x - y| \leq a\phi(t)/2 \leq a\phi(T)/2 \). Thus, we omit the proof.

**Proposition 6.2.** Let \( D \) be an arbitrary open set and let \( a \) and \( T \) be positive constants. Suppose that \( (t,x,y) \in (0,T] \times D \times D \), with \( \delta_D(x) \geq a\phi^{-1}(t) \geq 2|x - y| \). Then there exists a positive constant \( C_{6.2} = C_{6.2}^{(a,L_0,\phi,T)} \) such that \( p_D(t,x,y) \geq C_{6.2} |\phi^{-1}(t)|^{-d} \).

**Proposition 6.3.** Let \( D \) be an arbitrary open set and let \( a \) and \( T \) be positive constants.

1. Suppose that the jumping intensity kernel \( J \) satisfies the conditions (J1.2) and (J1.3). Then for every \( M > 0 \), there exists a constant \( C_{6.3} = C_{6.3}^{(a,M,L_0,\phi,T)} > 0 \) such that for all \( (t,x,y) \in (0,T] \times D \times D \), with \( \delta_D(x) \wedge \delta_D(y) \geq a\phi^{-1}(t) \) and \( a\phi^{-1}(t) \leq 2|x - y| \leq 2M \), we have \( p_D(t,x,y) \geq C_{6.3} t\nu(|x - y|) \).

2. Suppose that the jumping intensity kernel \( J \) satisfies the condition (J2). Then there exists a constant \( C_{6.3} = C_{6.3}^{(a,L_0,\phi,T)} > 0 \) such that for every \( (t,x,y) \in (0,T] \times D \times D \), with \( \delta_D(x) \wedge \delta_D(y) \geq a\phi^{-1}(t) \) and \( a\phi^{-1}(t) \leq 2|x - y| \), we have \( p_D(t,x,y) \geq C_{6.3} t\nu(|x - y|) \).

**Proof.** We first give the proof of (2). By Lemma 6.1, there exists \( c_1 = c_1(a,L_0,\phi,T) > 0 \) such that

\[
\inf \{ |z - y| \leq 4^{-1}a\phi^{-1}(t) \} \mathbb{P}_z (\tau_{B(z,6^{-1}a\phi^{-1}(t))} > t) \geq c_1.
\]

Thus by the strong Markov property

\[
\mathbb{P}_x (Y^D_t \in B(y, 2^{-1}a\phi^{-1}(t))) \geq c_1 \mathbb{P}_x (Y^D_t \text{ hits the ball } B(y, 4^{-1}a\phi^{-1}(t)) \text{ by time } t).
\]

Using this and the Lévy system in (4.3), we obtain

\[
\mathbb{P}_x (Y^D_t \in B(y, 2^{-1}a\phi^{-1}(t))) \\
\geq c_1 \mathbb{P}_x (Y^D_t \wedge B(x,6^{-1}a\phi^{-1}(t))) \in B(y, 4^{-1}a\phi^{-1}(t)) \text{ and } t \wedge \tau_{B(x,6^{-1}a\phi^{-1}(t))} \text{ is a jumping time } \\
= c_1 \mathbb{E}_x \left[ \int_0^{t \wedge \tau_{B(x,6^{-1}a\phi^{-1}(t))}} \int_{B(y,4^{-1}a\phi^{-1}(t))} J(Y_s,u)du ds \right].
\]

(6.2)
Lemma \[6.1\] also implies that

\[
E_x \left[ t \wedge \tau_{B(x,6^{-1}a\phi^{-1}(t))} \right] \geq t \mathbb{P}_x \left( \tau_{B(x,6^{-1}a\phi^{-1}(t))} \geq t \right) \geq c_2 t \quad \text{for all } t \in (0, T]. \tag{6.3}
\]

Let \( w \) be the point on the line connecting \( x \) and \( y \) (i.e., \(|x - y| = |x - w| + |w - y|\)) such that \(|w - y| = 7 \cdot 2^{-5}a\phi^{-1}(t)\), then \( B(w, 2^{-5}a\phi^{-1}(t)) \subset B(y, 4^{-1}a\phi^{-1}(t))\). Moreover, for every \((z,u) \in B(x, 6^{-1}a\phi^{-1}(t)) \times B(w, 2^{-5}a\phi^{-1}(t))\), we have

\[
|z - u| < 6^{-1}a\phi^{-1}(t) + 2^{-5}a\phi^{-1}(t) + |x - w| = |x - y| + (6^{-1} + 2^{-5} - 7 \cdot 2^{-5})a\phi^{-1}(t) < |x - y|
\]

and thus \( B(w, 2^{-5}a\phi^{-1}(t)) \subset \{u : |z - u| < |x - y|\} \). Combining this result with (\(J2\)), (1.4) and (6.3), we obtain

\[
\mathbb{E}_x \left[ \int_0^{t \wedge \tau_{B(x,6^{-1}a\phi^{-1}(t))}} \int_{B(y,4^{-1}a\phi^{-1}(t))} J(Y_s, u) duds \right] \\
\geq \mathbb{E}_x \left[ \int_0^{t \wedge \tau_{B(x,6^{-1}a\phi^{-1}(t))}} \int_{B(w,2^{-5}a\phi^{-1}(t))} J(Y_s, u) 1_{|y - u| < |x - y|} duds \right] \\
\geq L_0^{-1} \mathbb{E}_x \left[ t \wedge \tau_{B(x,6^{-1}a\phi^{-1}(t))} \right] |B(w, 2^{-5}a\phi^{-1}(t))| \nu(|x - y|)/\chi(|x - y|) \\
> c_3 t |\phi^{-1}(t)|^d \nu(|x - y|)/\chi(|x - y|). \tag{6.4}
\]

Then, using the semigroup property along with Proposition \[5.2\], \[6.3\] and \[6.4\], the proposition follows from the proof of [22 Proposition 3.5].

The proof of (1) is identical to the that of (2) except that we apply (\(J1.3\)) in \[6.3\] instead of (\(J2\)) and \[1.4\]. \(\Box\)

Combining Propositions \[5.2\] and \[6.3\] we obtain the following preliminary lower bound of \(p_D(t, x, y)\). Note that the lower bound in Proposition \[6.4\] (1) is the sharp interior lower bound of \(p_D(t, x, y)\) under the conditions (\(J1.2\)) and (\(J1.3\)). Moreover, under the condition (\(J2\)), the lower bound in Proposition \[6.4\] (2) that yields the sharp interior lower bound of \(p_D(t, x, y)\) for the case \(\beta \in [0, 1]\) and the case \(\beta \in (1, \infty]\) with \(|x - y| < 1\).

\textbf{Proposition 6.4.} Let \(D\) be an arbitrary open set and let \(a\) and \(T\) be positive constants.

\(1\) Suppose that the jumping intensity kernel \(J\) satisfies the conditions (\(J1.2\)) and (\(J1.3\)). Then, for every \((t, x, y) \in (0, T] \times D \times D\) and \(M > 0\), with \(\delta_D(x) \wedge \delta_D(y) \geq a\phi^{-1}(t)\) and \(|x - y| < M\), there exists a constant \(C_{6.4} = C_{6.4}(a, M, L_0, \phi, T) > 0\) such that

\[
p_D(t, x, y) \geq C_{6.4} \left( |\phi^{-1}(t)|^{-d} \wedge t \nu(|x - y|) \right).
\]

\(2\) Suppose that the jumping intensity kernel \(J\) satisfies the condition (\(J2\)). Then, for every \((t, x, y) \in (0, T] \times D \times D\), with \(\delta_D(x) \wedge \delta_D(y) \geq a\phi^{-1}(t)\), there exists a constant \(C_{6.4} = C_{6.4}(a, L_0, \phi, T) > 0\) such that

\[
p_D(t, x, y) \geq C_{6.4} \left( |\phi^{-1}(t)|^{-d} \wedge t \nu(|x - y|)/\chi(|x - y|) \right).
\]

For the remainder of this section, assume that the jumping intensity kernel \(J\) satisfies the condition (\(J2\)) for \(\beta \in (1, \infty]\) with \(|x - y| \geq 1\). Also, we assume that \(D\) is an connected open set with the following property: there exist \(\lambda_1 \in [1, \infty)\) and \(\lambda_2 \in (0, 1]\) such that for every
$r \leq 1$ and $x, y$ in the same component of $D$ with $\delta_D(x) \wedge \delta_D(y) \geq r$ there exists in $D$ a length parameterized rectifiable curve $l$ connecting $x$ to $y$ with the length $|l|$ of $l$ less than or equal to $\lambda_1|x - y|$ and $\delta_D(l(u)) \geq \lambda_2 r$ for $u \in [0, |l|]$.

Now we prove the preliminary lower bound of $p_D(t, x, y)$ separately for the case $\beta = \infty$ and the case $\beta \in (1, \infty)$. We will closely follow the proofs of [11, Theorem 3.6] and [13, Theorem 5.5].

**Proposition 6.5.** Let $\beta = \infty$. Suppose that $T > 0$ and $a \in (0, (4\phi^{-1}(T))^{-1}]$. Then there exist constants $C_{(6.5)} = C_{(6.5)}(a, L_0, \phi, T, \lambda_1, \lambda_2) > 0$, $i = 1, 2$, such that for any $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geq a\phi^{-1}(t)$, $|x - y| \geq 1$, and $t \leq T$ we have

$$p_D(t, x, y) \geq C_{(6.5)} \left(\frac{t}{T|x - y|}\right)^{\frac{1}{(6.5)}}.$$

Proof. Let $R_1 := |x - y| \geq 1$, and by the assumption on $D$, there is a length parameterized curve $l \subset D$ connecting $x$ and $y$ such that the total length $|l| \leq \lambda_1 R_1$ and $\delta_D(l(u)) \geq \lambda_2 a\phi^{-1}(t)$ for every $u \in [0, |l|]$. Define $k$ be the integer satisfying $(4 \leq 4\lambda_1 R_1 \leq k < 4\lambda_1 R_1 + 1 \leq 5\lambda_1 R_1$ and $r_i := 2^{-1}\lambda_2 a\phi^{-1}(t) \leq 8^{-1}$. For each $i = 0, 1, 2, \ldots, k$, let $x_i := l(i|l|/k)$ and $B_i := B(x_i, r_i)$, then $\delta_D(x_i) \geq 2r_i$ and $B_i \subset B(x_i, 2r_i) \subset D$. Since $4\lambda_1 R_1 \leq k$ for each $y_i \in B_i$, we have

$$|y_i - y_{i+1}| \leq |y_i - x_i| + |x_i - x_{i+1}| + |x_{i+1} - y_{i+1}| \leq \frac{1}{8} + \frac{|l|}{k} + \frac{1}{8} \leq \frac{\lambda_1 R_1}{4\lambda_1 R_1} + \frac{1}{4} = \frac{1}{2} \quad (6.5)$$

Moreover $\delta_D(y_i) \geq \delta_D(x_i) - |y_i - x_i| \geq r_i \geq r_i/k$. Thus by Proposition $(6.4)(2)$, there are constants $c_i = c_i(a, L_0, \phi, T) > 0$, $i = 1, 2$, such that for $(y_i, y_{i+1}) \in B_i \times B_{i+1}$ we have

$$p_D(t/k, y_i, y_{i+1}) \geq c_i \left(\frac{1}{\phi^{-1}((t/k))d} \wedge \frac{t/k}{\phi(|y_i - y_{i+1}|/y_i - y_{i+1})d}\right) \geq c_2 t/(Tkk). \quad (6.6)$$

The last inequality comes from $t/k \leq T/4$ for the first part and $(6.5)$ for the second part. Note that $r_i \geq c_3(t/kT)^{1/2}$ for some $c_3 = c_3(a, \phi, T, \lambda_2)$ by $(5.3)$. Hence, combining these observations and the fact that $k \approx R_1$, we conclude that

$$p_D(t, x, y) \geq \int_{B_1} \cdots \int_{B_{k-1}} p_D(t/k, x, y_i) \cdots p_D(t/k, y_{k-1}, y) dy_{k-1} \cdots dy_1 \geq (c_2 t/(Tk))^{k} \prod_{i=1}^{k-1} |B_i| \geq (c_4 t/(Tk))^{c} k \geq c_5 t/(T}\geq c_6 t/(T)^{1-\epsilon} R_1 \geq c_8 t/(T)^{1-\epsilon} R_1.$$

\[ \square \]

**Proposition 6.6.** Let $\beta \in (1, \infty)$. Suppose that $T > 0$ and $a \in (0, (4\phi^{-1}(T))^{-1}]$. Then there exist constants $C_{(6.6)} = C_{(6.6)}(a, \beta, \chi, L_0, \phi, T, \lambda_1, \lambda_2) > 0$, $i = 1, 2$ such that for any $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geq a\phi^{-1}(t)$, $|x - y| \geq 1$, and $t \leq T$ we have

$$p_D(t, x, y) \geq C_{(6.6)} \exp \left\{-C_{(6.6)} \left(|x - y| \left(\frac{T}{t}\right)^{\frac{\beta}{3}} \wedge (|x - y|)^{\beta}\right)\right\}.$$

Proof. Let $R_1 := |x - y|$. If either $1 \leq R_1 \leq 2$ or $R_1 (\log(TR_1/t))^{(3-1)/\beta} \geq (R_1)^{\beta}$, the proposition holds by virtue of Proposition $(6.4)(2)$. Thus for the remainder of this proof we assume that $R_1 > 2$ and $R_1 (\log(TR_1/t))^{(3-1)/\beta} < (R_1)^{\beta}$, which is equivalent to $1 \leq R_1 (\log TR_1/t)^{-1/\beta}$ and $R_1 \exp(-R_1^{\beta}) < t/T$.  

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Let $k \geq 2$ be a positive integer such that

$$R_1\left(\log \frac{TR_1}{t}\right)^{-1/\beta} \leq k < R_1\left(\log \frac{TR_1}{t}\right)^{-1/\beta} + 1 < 2R_1\left(\log \frac{TR_1}{t}\right)^{-1/\beta} \quad (6.7)$$

then $R_1/k \geq 2^{-1}(\log(TR_1/t))^{1/\beta} \geq 2^{-1}(\log 2)^{1/\beta} =: c_0$.

By Proposition 6.4(2) and (6.8), and using the facts that $t/k$ from Proposition 6.5 and 6.6 with $c$ for some constants $(a,\alpha,\phi,\chi,\beta,T,\lambda_1)$ such that for any $(R)\subset\mathbb{R}$ then $\delta_\beta$ for any $(\lambda, i)\in B_i \times B_{i+1}$, we conclude that

$$|y_i - y_{i+1}| \leq |x_i - x_{i+1}| + 2r_t \leq \frac{|l|}{k} + c_0 \leq (\lambda_1 + 1)\frac{R_1}{k}. \quad (6.8)$$

By Proposition 6.4(2) and (6.8), and using the facts that $t/k \leq T/2$ and $R_1/k \geq c_0$, we have that for any $(y_i, y_{i+1}) \in B_i \times B_{i+1}$,

$$p_D(t/k, y_i, y_{i+1}) \geq c_1\left(\frac{1}{c_0^{-1}(t/k)^{\beta}} \wedge \frac{t}{k}\right) \cdot \nu(|y_i - y_{i+1}|)/\chi(|y_i - y_{i+1}|) \geq c_2\frac{t}{k} \cdot e^{-c_3(R_1/k)^{\beta}}$$

for some constants $c_i = c_i(a, L_0, \phi, \chi, \beta, T, \lambda_1)$, $i = 2, 3$. Since $\phi(R_1/k) \leq c_4(R_1/k)^\gamma$ by (WS) with $R_1/k \geq c_0$, using (6.4), we have that

$$p_D(t/k, y_i, y_{i+1}) \geq c_2 \cdot c_4 \frac{t}{TR_1} \left(\frac{k}{R_1}\right)^{\gamma \alpha + \beta - 1} \cdot e^{-c_3(R_1/k)^\beta}$$

$$\geq c_2 \cdot c_4 \frac{t}{TR_1} \left(\log \frac{TR_1}{t}\right)^{-\gamma \alpha + \beta - 1} \cdot \left(\frac{t}{TR_1}\right)^{c_3} \geq c_5 \left(\frac{t}{TR_1}\right)^{c_6} \quad (6.9)$$

for some $c_i = c_i(a, L_0, \phi, \chi, \beta, T, \lambda_1)$, $i = 5, 6$. Note that $r_t \geq c_7(t/TR_1)^{1/2}$ for some $c_7 = c_7(a, \beta, \phi, \lambda_2)$ by (5.3) and the fact that $t/TR_1 \leq 1/2$. Combining this with (6.9), (6.7) and by the semigroup property, we conclude that

$$p_D(t, x, y) \geq \int_{B_1} \cdots \int_{B_{k-1}} p_D(t/k, x, y_1) \cdots p_D(t/k, y_{k-1}, y)dy_1 \cdots dy_{k-1}$$

$$\geq c_8 \exp\left\{-c_{9} k \log(TR_1/t)\right\}$$

$$\geq c_8 \exp\left\{-c_{9} \left(2R_1 \log\left(\frac{TR_1}{t}\right)^{-1/\beta}\right) \log \frac{TR_1}{t}\right\}$$

$$\geq c_8 \exp\left\{-2c_{9} \cdot R_1 \log\left(\frac{TR_1}{t}\right)^{-1/\beta}\right\}. \quad \square$$

**Proofs of the lower bounds in Theorems 1.1 and 1.2.** The lower bound of $p(t, x, y)$ in Theorem 1.1 follows from Proposition 6.4(1) with $D = \mathbb{R}^d$. The lower bound of $p(t, x, y)$ in Theorem 1.2 for the case $\beta \in [0, 1]$ and the case $\beta \in (1, \infty)$ with $|x - y| < 1$ follows from Proposition 6.4(2) with $D = \mathbb{R}^d$ and the remaining cases of Theorem 1.2 follows from Propositions 6.3 and 6.6 with $D = \mathbb{R}^d$. \quad \square
7 Lower bound estimates

In this section, we first obtain the boundary decay in Lemma 7.4 using (4.8), Lemma 6.1 and Lemmas 7.1 and 7.2 below. Using the semigroup property, and then applying Lemma 7.4 and Lemma 7.2.

Suppose that $E \subset \mathbb{R}^d$ be an open set and $U_1, U_2 \subset E$ be disjoint open subsets. If $x \in U_1$, $y \in U_2$ and $t > 0$, we have

$$p_E(t, x, y) \geq t \ P_x(\tau_{U_1} > t) \ P_y(\tau_{U_2} > t) \inf_{(u, w) \in U_1 \times U_2} J(u, w).$$

For the remainder of the section, we assume that $Y$ is the symmetric pure jump Hunt process with the jumping intensity kernel $J$ satisfying the conditions (J1.1), (J1.2) and (K1). For any $T > 0$, let

$$\hat{a}_T := \hat{a}_{T, R_0} := \frac{R_0}{80 \phi^{-1}(T)},$$

and for $x \in D$ we use $z_x$ to denote a point on $\partial D$ such that $|z_x - x| = \delta_D(x)$.

We first give the survival probability where $x$ is near the boundary of $D$ in the following lemma.

**Lemma 7.2.** Let $a \leq \hat{a}_T$. Then, there exists a constant $C_{\Gamma(7.2)} = C_{\Gamma(7.2)}(a, \phi, L_0, L_3, \rho, \eta, \Lambda, T) > 0$ such that for every $t \leq T$ and $x \in D$ with $\delta_D(x) < \alpha^{-1}(t)$ we have

$$\mathbb{P}_x(\tau_{B(z_x, 10\alpha^{-1}(t))} \cap D > t/3) \geq C_{\Gamma(7.2)} \frac{V(\delta_D(x))}{\sqrt{t}}. \quad (7.1)$$

**Proof.** Without loss of generality, we assume that $z_x = 0$. Consider a coordinate system $CS := CS_0$ such that $B(0, R_0) \cap D = \{y = (\tilde{y}, y_d) : \tilde{y} \in B(0, R_0) \text{ in } CS : y_d > \psi(\tilde{y})\}$, where $\psi$ is a $C^{1,\rho}$ function such that $\psi(0) = 0$, $\nabla \psi(0) = (0, \ldots, 0)$, $\|\nabla \psi\|_\infty \leq \Lambda$, and $|\nabla \psi(\tilde{y}) - \nabla \psi(w)| \leq \Lambda|\tilde{y} - \hat{w}|^{\rho}$, define $\varphi_1(\tilde{y}) := 2\Lambda|\tilde{y}|^{\rho+1}$ and $V := \{y = (\tilde{y}, y_d) : \tilde{y} \in B(0, R_0) \text{ in } CS : y_d > \varphi_1(\tilde{y})\}$. Since $\varphi_1(\tilde{y}) \geq 2\Lambda|\tilde{y}|^{\rho+1}$ for $y \in B(0, R_0)$, the mean value theorem yields $V \subset B(0, R_0) \cap D$.

Let $U_1 := B(0, 2a\alpha^{-1}(t)) \cap D, U_2 := B(0, 10a\alpha^{-1}(t)) \cap D$, and

$$W := \{y = (\tilde{y}, y_d) \in B(0, 8a\alpha^{-1}(t)) \setminus B(0, 2a\alpha^{-1}(t)) \text{ in } CS : y_d > \varphi_1(\tilde{y})\} \subset V. \quad (7.2)$$

Since $\Lambda|\hat{w}| = \varphi_1(\hat{w})/2 < w_d/2$ for $w \in W$, we have

$$\delta_D(w) > \frac{(w_d - \varphi(\hat{w}))}{(1 + \Lambda)} > \frac{(w_d - \Lambda|\hat{w}|)}{(1 + \Lambda)} > \frac{w_d}{2(1 + \Lambda)} \quad \text{for } w \in W. \quad (7.3)$$

Moreover, since $|\hat{w}| \leq (2\Lambda)^{-1}|w| \leq \Lambda^{-1}4a\alpha^{-1}(t) \leq a\alpha^{-1}(t)$ for $w \in W$, we have

$$w_d^2 = |w|^2 > |\hat{w}|^2 \geq (2a\alpha^{-1}(t))^2 - (a\alpha^{-1}(t))^2 \geq (a\alpha^{-1}(t))^2 \quad \text{for } w \in W. \quad (7.4)$$

Combining (7.3) and (7.4), we obtain $\delta_D(w) > 2^{-1}(1 + \Lambda)^{-1}a\alpha^{-1}(t)$ and $B(w, r_1a\alpha^{-1}(t)) \subset U_2$ for $w \in W$, where $r_1 := (2(1 + \Lambda))^{-1}$. By virtue of the strong Markov property, Lemma 6.1.
and (4.8), we have
\[
P_x(\tau U_2 > t/3) \geq P_x(\tau U_2 > t/3, Y_{\tau U_1} \in W) = E_x[P_{Y_{\tau U_1}}(\tau U_2 > t/3) : Y_{\tau U_1} \in W]
\]
\[
> E_x[P_{Y_{\tau U_1}}(\tau B_{Y_{\tau U_1}, r_1 \alpha^{-1}(t)}) > t/3 : Y_{\tau U_1} \in W] > \left( \inf_{s \in \mathbb{R}^d} P_x(\tau B_{s, r_1 \alpha^{-1}(t)}) > t/3 \right) P_x(Y_{\tau U_1} \in W)
\]
\[
> C_{6.1} P_x(Y_{\tau U_1} \in W) \geq C_{6.1} \cdot 1.2 \cdot V(\delta_D(x)) / V(8a\phi^{-1}(t)) \geq c_1 V(\delta_D(x)) / \sqrt{t}.
\]
By the subadditivity of \(V\) and (3.13), \(V(8a\phi^{-1}(t)) \leq (8a + 1)V(\phi^{-1}(t)) \approx \sqrt{t}\), and therefore we obtain the last inequality.

We introduce the following definition for the subsequent lemma.

**Definition 7.3.** Let \(0 < \kappa_1 \leq 1/2\). We say that an open set \(D\) is \(\kappa_1\)-fat if there is \(R_1 > 0\) such that for all \(x \in D\) and all \(r \in (0, R_1]\) there is a ball \(B(A_r(x), \kappa_1 r) \subset D \cap B(x, r)\). The pair \((R_1, \kappa_1)\) are called the characteristics of the \(\kappa_1\)-fat open set \(D\).

Note that a \(C^{1,\rho}\) open set \(D\) with characteristics \((R_0, \Lambda)\) is a \(\kappa_1\)-fat set with characteristics \((R_1, \kappa_1)\) depending only on \(R_0, \Lambda, \rho, \eta, R_0, \Lambda, T\), and without loss of generality, we assume that \(R_0 \leq R_1\) (by choosing \(R_0\) smaller if necessary). Let \(A_r(x)\) is always the point \(A_r(x) \in D\) in Definition 7.3 for \(D\).

Recall that the function \(\Psi\) is defined in (1.3).

**Lemma 7.4.** There exists a constant \(C_{7.4} = C_{7.4}(\phi, L_0, L_3, \rho, \eta, R_0, \Lambda, T) > 0\) such that, for every \(t \leq T\) and \(x \in D\), we can find \(x_1\) with \(\delta_D(x_1) \geq 2^{-1} \kappa_1 \hat{a}_T \phi^{-1}(t)\) and \(|x_1 - x| \leq 6 \hat{a}_T \phi^{-1}(t)\) such that

\[
\int_{B(x_1, 4^{-1} \kappa_1 \hat{a}_T \phi^{-1}(t))} p_D(t/3, x, z)dz \geq C_{7.3} \Psi(t, x).
\]

**Proof.** Let \(r_t := \hat{a}_T \phi^{-1}(t) \leq R_0 / 80 \leq 1/80\) and we consider the case \(\delta_D(x) < 2^{-1} \kappa_1 r_t\) first. In this case we let \(x_1 := A_{6r_t}(z_x)\) and denote \(B_{x_1} := B(x_1, 4^{-1} \kappa_1 r_t)\) and \(B_{x_2} := B(z_x, 5 \kappa_1 r_t) \cap D\) so that \(B_{x_1} \cap B_{x_2} = \emptyset\). For any \(u \in B_{x_2}\) and \(w \in B_{x_1}\),

\[
|u - w| \leq |u - z_x| + |z_x - x_1| + |x_1 - w| \leq 12 \kappa_1 r_t \leq 1.
\]

Since \(\phi(12 \kappa_1 r_t) = \phi(\phi^{-1}(t)) = t\) by (WS), using (J1.1), (1.2) and (1.4), we have that

\[
\inf_{(u, w) \in B_{x_2} \times B_{x_2}} J(u, w) \geq L_0^{-1} \phi(12 \kappa_1 r_t)^{-1} |12 \kappa_1 r_t|^{-d} \geq c_1 t^{-1} |\phi^{-1}(t)|^{-d}
\]

for some constant \(c_1 := c_1(\phi, L_0, R_0, \Lambda, T) > 0\). Therefore, Lemmas 7.1, 7.2 and 6.1 implies that

\[
\int_{B_{x_1}} p_D(t/3, x, z)dz \geq \frac{t}{3} \int_{B_{x_1}} P_x(\tau_{B_{x_1}} > t/3) P_x(\tau_{B_{x_1}} > t/3) \cdot \inf_{(u, w) \in B_{x_2} \times B_{x_2}} J(u, w)dz \\
\geq \frac{1}{3} \int_{B_{x_1}} P_x(\tau_{B_{x_1}} > t/3) \cdot C_{6.1} \int_{B_{x_1}} dz \cdot c_1 |\phi^{-1}(t)|^{-d} \\
= c_2 P_x(\tau_{B_{x_1}} > t/3) \geq c_2 \cdot C_{7.2} V(\delta_D(x)) / \sqrt{t}.
\]
For \( \delta_D(x) \geq 2^{-1} \kappa_1 r_t \), let \( x_t = x \) and \( B_{x_1} := B(x_t, 4^{-1} \kappa_1 r_t) \). By Lemma 6.1,
\[
\int_{B_{x_1}} p_{D}(t/3, x, z) dz \geq \int_{B_{x_1}} p_{B_{x_1}}(t/3, x, z) dz = \mathbb{P}_{x}(\tau_{B_{x_1}} > t/3) > C_{6.1}
\]
and this proves the lemma.

We are now ready to give the proof of the lower bound estimates for \( p_{D}(t, x, y) \). Recall our assumption that \( \rho \in (\overline{\mathbb{R}}/2, 1) \) and \( D \) is a \( C^{1,\rho} \) open set. When the jumping intensity \( J \) of \( Y \) satisfies (J2), for the cases \( \beta \in (1, \infty) \) with \( |x - y| \geq 1 \) and \( \beta = \infty \) with \( |x - y| \geq 4/5 \), we assume in addition that the path distance in each connected component of \( D \) is comparable to the Euclidean distance with characteristic \( \lambda_1 \). Note that combining this assumption with \( C^{1,\rho} \) assumption entails that \( D \) satisfies the assumption made before Proposition 6.5.

**Proofs of the lower bound of** \( p_{D}(t, x, y) \) **in Theorems 1.3(1), 1.5(2) and 1.5(3).** Let \( r_t := \tilde{a}_T \phi^{-1}(t) \leq R_0/80 \leq 1/80 \). By Lemma 6.3, for any \( x, y \in D \), there exists \( x_1, y_1 \in D \) such that \( \delta_D(x_1) \wedge \delta_D(y_1) > 2^{-1} \kappa_1 r_t \) and \( |x_1 - x| \vee |y_1 - y| \leq 6r_t \), and
\[
\int_{B_{x_1}} p_{D}(t/3, x, z) dz \int_{B_{y_1}} p_{D}(t/3, y, z) dz \geq C_{7.4}^2 \Psi(t, x, y),
\]
where \( B_{x_1} := B(x_1, 4^{-1} \kappa_1 r_t) \) and \( B_{y_1} := B(y_1, 4^{-1} \kappa_1 r_t) \). Thus by the semigroup property,
\[
p_{D}(t, x, y) = \int_{D} \int_{D} p_{D}(t/3, x, u)p_{D}(t/3, u, w)p_{D}(t/3, w, y) du dw \geq \int_{B_{x_1}} p_{D}(t/3, x, u) du \int_{B_{y_1}} p_{D}(t/3, y, w) dw \left( \inf_{(u, w) \in B_{x_1} \times B_{y_1}} p_{D}(t/3, u, w) \right) \geq C_{7.4}^2 \Psi(t, x, y) \inf_{(u, w) \in B_{x_1} \times B_{y_1}} p_{D}(t/3, u, w).
\]

We now carefully calculate the lower bounds of \( p_{D}(t/3, u, w) \) on \( B_{x_1} \times B_{y_1} \). Since \( |x - x_1| \vee |y - y_1| \leq 6r_t \), for \( u \in B_{x_1} \) and \( w \in B_{y_1} \) we have
\[
|x - y| - 6^{-1} \leq |x - y| - (12 + (\kappa_1/2))r_t \leq |u - w| \leq |x - y| + (12 + (\kappa_1/2))r_t \leq |x - y| + 6^{-1}
\]
and \( \delta_D(u) \wedge \delta_D(w) \geq 4^{-1} \kappa_1 r_t \).

We first assume that the jumping intensity kernel \( J \) satisfies the condition (J2). Let \( \beta \in [0, 1] \). If \( |x - y| \leq 15r_t \), then \( |u - w| \leq 28r_t < 1 \) and \( \phi(|u - w|)|u - w|^d \leq c_t[\phi^{-1}(t)]^d \) since \( \phi(|u - w|) \leq \phi(28 \kappa_1 r_t) \approx \phi(\phi^{-1}(t)) = t \) by (WS). If \( |x - y| > 15r_t \), then \( |u - w| \leq |x - y| + 6^{-1} \) and \( \phi(|u - w|)|u - w|^d \leq c_2 \phi(|x - y|)|x - y|^d \) since \( r \rightarrow \phi(r) \) is increasing and using (WS). Combining these observations with Proposition 6.4(2), (1.2) and (1.5),
\[
p_{D}(t/3, u, w) \geq c_3 \left[ \phi^{-1}(t) \right]^{-d} \land te^{-72|x - y|^d \nu(|x - y|)} \geq c_4 \left[ \phi^{-1}(t) \right]^{-d} \land te^{-72|x - y|^d \nu(|x - y|)}.
\]
If \( \beta \in (1, \infty) \) and \( |x - y| \leq 4/5 \), then (1.3) yields \( |u - w| \leq |x - y| + 6^{-1} < 1 \). Similar to the above case, considering the cases \( |x - y| \leq 15r_t \) and \( |x - y| > 15r_t \) separately, we have
\[
p_{D}(t/3, u, w) \geq c_5 \left[ \phi^{-1}(t) \right]^{-d} \land t \cdot \nu(|x - y|)).
\]
Moreover,
\[
(1) \text{ if } \beta \in (1, \infty) \text{ and } 4/5 \leq |x - y| < 2, \text{ then } |u - w| \times 1. \text{ Thus by Proposition 6.4(2), we have } p_{D}(t/3, u, w) \geq c_6.t.
\]
Hence combining (7.7) with these observations, we have proved the lower bound of $p_D(t, x, y)$ in Theorem 1.5(2).

Suppose the jumping intensity kernel $J$ satisfies the condition (J1) and $M > 0$. Let $|x - y| < M$. Similar to the $\beta \in [0, 1]$ case, applying Proposition 6.4(1) instead of Proposition 6.4(2) and considering $|x - y| \leq 15r_1 \land M$ and $15r_1 \land M < |x - y| \leq M$ separately, we have $p_D(t/3, u, w) \geq c_7 \left(|\phi^{-1}(t)|^{-d} \land t \cdot \nu(|x - y|)\right)$. Hence combining (7.7) with this, we have proved the lower bound of $p_D(t, x, y)$ in Theorem 1.5(1).

We now return to the assumption that the jumping intensity kernel $J$ satisfies the condition (J2), further assume that the path distance in $D$ is comparable to the Euclidean distance. If $4/5 \leq |x - y|$, then (7.8) yields $|u - w| \approx |x - y|$. Recall that we have already discussed the case $\beta \in (1, \infty)$ and $4/5 \leq |x - y| < 2$ in (1). We now consider $p_D(t/3, u, w)$ in each of the remaining cases.

(2) If $\beta = \infty$ and $4/5 \leq |x - y| < 2$, then by Propositions 6.3 and 6.5 we have

$$p_D(t/3, u, w) \geq c_9 \frac{4t}{5T|x - y|} \geq c_8 \left(\frac{4t}{5T|x - y|}\right)^{5|x - y|/4}.$$

(3) If $\beta \in (1, \infty)$ and $2 \leq |x - y|$, then $1 < |u - w|$ and from Proposition 6.6 and (7.8) we obtain

$$p_D(t/3, u, w) \geq c_9 t \exp \left\{-c_{10} \left(|u - w| \left(\log \frac{T|u - w|}{t}\right)^{\frac{\beta}{\beta - 1}} \land |u - w|^{\beta}\right)\right\}$$

$$\geq c_9 t \exp \left\{-c_{10} \left((5|x - y|/4) \left(\log \frac{T(|x - y| + 6^{-1})}{t}\right)^{\frac{\beta - 1}{\beta}} \land (5|x - y|/4)^{\beta}\right)\right\}$$

$$\geq c_9 t \exp \left\{-c_{11} \left(|x - y| \left(\log \frac{T|x - y|}{t}\right)^{\frac{\beta}{\beta - 1}} \land |x - y|^{\beta}\right)\right\}.$$

The last inequality comes from the inequality $\log r \leq \log(r + b) \leq 2\log r$ for $r \geq 2 \vee b > 0$.

(4) If $\beta = \infty$ and $2 \leq |x - y|$, then $1 < |u - w|$ and from Proposition 6.5 and (7.8) we have

$$p_D(t/3, u, w) \geq c_{12} \left(\frac{t}{T|u - w|}\right)^{c_{13}|u - w|} \geq c_{12} \left(\frac{t}{T(|x - y| + 6^{-1})}\right)^{c_{13}5|x - y|/4}$$

$$\geq c_{12} \left(\frac{t}{T|x - y|}\right)^{c_{13}5|x - y|/2} \geq c_{12} \left(\frac{4t}{5T|x - y|}\right)^{c_{13}5|x - y|/2}.$$

The second last inequality holds by virtue of the inequality $r^2 \geq r + b$ for $r \geq 2 \vee b > 0$.

Hence combining (7.7) with the above observations (1) – (4) on the lower bound of $p_D(t/3, u, w)$, we have proved the lower bound of $p_D(t, x, y)$ in Theorem 1.5(3). □

**Proof of Theorem 1.5(4).** Let $D(x)$ and $D(y)$ be connected components containing $x$ and $y$, respectively. By definition of a $C^{1, \alpha}$ open set, the distance between $x$ and $y$ is at least $R_0$. Using Lemma 7.4, we find that $x_1 \in D(x)$ and $y_1 \in D(y)$. Define $B_{x_1}$ and $B_{y_1}$ in the same way as when beginning the proof of Theorem 1.5(2) and 1.5(3) so that (7.6) holds.

For any $u \in B_{x_1}$ and $w \in B_{y_1}$, since $3R_0/4 \leq 3|x - y|/4 \leq |u - w| \leq 5|x - y|/4$, by Proposition 6.4(2) and (5.3),

$$p_D(t/3, u, w) \geq c_1 \nu(|u - w|) e^{-\gamma_2 u - w|^{\beta}} \geq c_2 \nu(|x - y|) e^{-\gamma_2 (5|x - y|/4)^{\beta}}.$$

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By the semigroup property, combining (7.6) and this observation, we conclude that

\[ p_D(t, x, y) \geq \int_{B_{2t}} \int_{B_{t/3}} p_D(t/3, x, w)p_D(t/3, u, w)p_D(t/3, w, y)dwdw \]

\[ \geq \int_{B_{2t}} p_D(t/3, x, u)du \int_{B_{t/3}} p_D(t/3, y, w)dw \cdot \inf_{(u, w) \in B_{2t} \times B_{t/3}} p_D(t/3, u, w) \]

\[ \geq c_3\Psi(t, x)\Psi(t, y) \cdot tv(|x - y|)e^{-\gamma_2(5|x - y|/4)^\beta}. \]

\[ \square \]

**Proofs of Theorems 1.3(2) and 1.5(5).** Using Lemmas 5.2 and 7.2 instead of [22] (5.1) and (5.10)], and by the fact that (3.3) and \( D \) is bounded and connected when \( J \) satisfies the condition \((J2)\) and \( \beta = \infty \), we can obtain the large time heat kernel estimates for \( p_D(t, x, y) \) following the proofs of [22 Theorem 1.5(iii)] and [20 Theorem 1.5(iii)], so we omit the proofs.

**8 Green function and boundary Harnack inequality**

In this section we give the Green function estimates and establish the boundary Harnack inequality as applications of the Dirichlet heat kernel estimates.

**Proof of Theorem 1.6.** When \( d \geq 2 \), the proof of Green function estimates is almost identical to the one of [22 Section 7]. Thus we skip the proof.

Suppose \( d = 1 \). Note that by the inequality in Proposition 8.1, we have

\[ V'(r) \leq c \frac{V(r)}{r} \quad \text{for } 0 < r \leq M, \quad (8.1) \]

Using (8.1) instead of [22 (7.3)], one can obtain the Green function estimates by following the proofs in [22 Section 7] line by line. Indeed, for any \( T > 0 \), let

\[ K_T(a, r) := a + \phi(r) \int_1^1 \left( 1 - \frac{ua}{\phi(r)} \right) \frac{1}{u^2\phi(1/u)}du + \phi(r) \left( 1 - \frac{a}{\phi(r)} \right) \]

which is defined in [22 (7.4)]. By the same proof of [22 Theorem 7.3(iii)], we have that

\[ G_D(x, y) \asymp K_T(a(x, y), |x - y|) \]

where \( a(x, y) = \sqrt{\phi(\delta_D(x))}/\sqrt{\phi(\delta_D(y))} \). Recall that \( C_I \) is the constant in [5.3]. Let \( T_1 := (2V(2C_I)^{4\beta}) \phi(\delta_D(\delta_D)) \). Since \( 0 < a(x, y) \leq \phi(\delta_D(\delta_D)) = (2^{-1} \wedge (2C_I)^{-\beta})T_1 \) and \( \phi(|x - y|) \leq \phi(\delta_D(\delta_D)) \leq T_1/2 \), it is enough to show that for any \( T > 0 \) and for any \( 0 < a \leq (2^{-1} \wedge (2C_I)^{-\beta})T \) and \( 0 < \phi(r) \leq T/2 \),

\[ K_T(a, r) \asymp \frac{a}{r} \wedge \left( \frac{a}{\phi^{-1}(a)} + \left( \int_r^{\phi^{-1}(a)} \phi(s) \frac{1}{s^2} \right)^+ \right) \quad (8.2) \]

where \( x^+ := x \vee 0 \).

When \( 0 < a < \phi(r) \leq T/2 \), the proof of (8.2) is the same as that of [22 Lemma 7.2]. Now we assume that \( \phi(r) \leq a \leq (2^{-1} \wedge (2C_I)^{-\beta})T \) using [5.3], we have \( c_1^{-1}V(r)^2 \leq \phi(r) \leq c_1V(r)^2 \) for some constant \( c_1 > 1 \). Thus by the change of variable \( u = V(r)^2/V(s)^2 \), we have that

\[ \int_{\phi(r)/T}^{\phi(r)/a} \frac{du}{u\phi^{-1}(u\phi(r))} \leq c_1V(r)^2/a \int_{V(r)^2/c_1T}^{1/V(c_1T)^2} \frac{du}{u\phi^{-1}(u\phi(r))} \]

\[ \leq \int_1^{1/V(c_1T)^2} \phi^{-1}(c_1^{-1}V(s)^2) \frac{V'(s)}{V(s)}ds. \]
Since \( \phi^{-1}(c_1^{-1}V(s)^2) \geq \phi^{-1}(c_1^{-2}\phi(s)) \geq c_2 s \) by (3.3) and (5.3), combining this with (8.1), we have that
\[
\int_{\phi(r)/a}^{\phi(r)/r} \frac{a \cdot du}{uh^n(u-h\phi(r))} \leq c_3 a \int_{1/V(c_1)}^{1/V(c_1)/2} \frac{1}{s^2} ds \leq c_4 \frac{a}{\phi^{-1}(a)}.
\] (8.3)

For the last inequality, we again used (3.3) and (5.3). Applying (8.3) to the proof of the upper bound for \( K_T(a, r) \) in [22] (7.6), and following the rest of the proof of [22] Theorem 7.3(iii) for the \( \phi(r) \leq a \leq (2^{-1} \land (2C')^{-1})T \) case, we obtain (8.2) and hence we prove Theorem 1.6 for all dimension.

To prove Theorem 1.8 we use the above estimates of Green function and the following the scale and translate invariant boundary Harnack inequality.

**Proposition 8.1.** Suppose that \( D \) is an open set in \( \mathbb{R}^d \). Let \( Y \) be a symmetric pure jump Hunt process whose jumping intensity kernel \( J \) satisfies the conditions (J1), (L), (C) and (K_\eta). Then, there exists \( c = c(\phi, \eta, L, L_0, \lambda) \) such that for any \( 0 < r < 1 \), \( z \in \partial D \) and any nonnegative functions \( f, g \) in \( \mathbb{R}^d \) which are regular harmonic in \( D \cap B(z, r) \) with respect to \( Y \), and vanish in \( D^c \cap B(z, r) \), we have
\[
\frac{f(x)}{f(y)} \leq c \frac{g(x)}{g(y)} \quad \text{for any } x, y \in D \cap B(z, 2r/3).
\]

**Proof.** To prove the claim we use [8]. We only have to check assumptions stated in [8]. Note that (C) is an uniform version of [8, Assumption C]. Thus there is a constant \( c_{0,7} \) in [8] satisfies \( c_{2,7}(x_0, R_1, R_2) = C^*(\phi, d, R_1/R_2) \) for any \( x_0 \in \mathbb{R}^d \) and \( 0 < R_1 < R_2 \leq 2 \). Let \( 0 < r < 1 \) and \( 2/3 < a < 2 \).

We first check the bounds on the constants \( c_{2,8} \) and \( c_{2,9} \) in [8]. In our case, the constants \( c_{2,8} \) and \( c_{2,9} \) in [8] can be taken as
\[
c_{2,8}(x_0, ar, 2r) := \inf_{\{|r| \leq |x_0-y| \leq 2r\}} J(x_0, y) \quad \text{and} \quad c_{2,9}(x_0, r) \leq C^* \left( \int_{\mathbb{R}^d \setminus B(x_0, 2r)} J(x_0, y) dy \right)^{-1}
\]
where \( C^* = C^*(\phi, d, 1/2) \) is the constant in (C). (see [8] (2.8) and (2.9)) and the last display of [8, Proposition 2.9]). Then by (J1.3) and (WS) for \( c_{2,8} \), and by (J1.1), (J1.2) (1.4) and (WS) for \( c_{2,9} \), we have that
\[
c_{2,8}(x_0, ar, 2r) \geq c_1 \phi(r)^{-1} r^{-d} \quad \text{and} \quad c_{2,9}(x_0, r) \leq c_2 \phi(r)
\] (8.4)
where the constant \( c_1 > 0 \) depends on \( \phi, a \) and \( d \), and the constant \( c_2 > 0 \) depends on \( \phi, L_0 \) and \( d \).

We now check [8, Assumption A–D] (and its scale and translate invariant version) holds. First of all, since \( p(t, x, y) \) is continuous, clearly the transition operators \( T_t \) of \( Y \) is strong Feller. Recall that we assume that \( T_t \) is Feller, that is, \( T_t \) maps \( C_0(\mathbb{R}^d) \) into \( C_0(\mathbb{R}^d) \). Since \( Y \) is symmetric, [8, Assumption A] holds.

Let \( \hat{A} \) be the corresponding generator on \( C_0(\mathbb{R}^d) \) defined as
\[
\hat{A}u := \lim_{t \to 0} \frac{T_t u - u}{t} \quad \text{(strong limit)} \quad \text{and} \quad D(\hat{A}) := \{ u \in C_0(\mathbb{R}^d) : \hat{A}u < \infty \}. 
\]
Recall the operator $L_g(x) = P.V. \int (g(y) - g(x))J(x, y)dy$ defined in (2.1). Then

$$C_c^2(\mathbb{R}^d) \subset D(\hat{A}) \quad \text{and} \quad \hat{A}u = Lu \quad \text{for any } u \in C_c^2(\mathbb{R}^d). \hspace{1cm} (8.5)$$

Indeed, we first obtain that for any $u \in C_c^2(\mathbb{R}^d)$, $Lu \in C_0(\mathbb{R}^d)$ by (L) and so,

$$\|Tu(Lu) - Lu\|_\infty \to 0 \quad \text{as } t \to 0. \hspace{1cm} (8.6)$$

Since, from Lemma 2.2, $M_t = u(Y_t) - u(Y_0) - \int_0^t Lu(Y_s)ds$ is $\mathbb{P}_x$-martingale with respect to the filtration of $Y$, we have that

$$\frac{T_tu(x) - u(x)}{t} \to \mathbb{E}_x\left[\int_0^t L(Y_s)ds\right].$$

Thus we obtain that for any $u \in C_c^2(\mathbb{R}^d)$,

$$\sup_x \left| \frac{T_tu(x) - u(x)}{t} - Lu(x) \right| = \sup_x \left| \frac{1}{t} \int_0^t T_sLu(x) - Lu(x)ds \right| \leq \frac{1}{t} \int_0^t \|T_s(Lu) - Lu\|_\infty ds,$$

and combining this with (8.6), we conclude (8.5). Therefore, [8] Assumption B] holds with $D = C_c^2(\mathbb{R}^d)$.

For $0 < R_1 < R_2$, let $A(x, R_1, R_2) = \{y \in \mathbb{R}^d : R_1 < |x - y| < R_2\}$ be the open annulus around $x$, and $\overline{A}(x, R_1, R_2)$ the closure of $A(x, R_1, R_2)$. For every compact set $K$ and open set $U$ satisfying $K \subset U \subset \mathbb{R}^d$, let

$$\mathcal{F}_{K, U} := \{f \in C_c^2(\mathbb{R}^d) : f \equiv 1 \text{ in } K, f \equiv 0 \text{ in } U^c, \text{ and } 0 \leq f(x) \leq 1\},$$

and $\varrho(K, U) := \inf_{f \in \mathcal{F}_{K, U}} \sup_x Lf(x)$. Then by Lemma 2.1 and (WS), for $2/3 < a < b \leq 1$ there exist $c_3 = c_3(\phi, \eta, L_0, L_3, a, b)$ such that for any $x_0 \in \mathbb{R}^d$ and $0 < r < 1$

$$\varrho(x_0, ar, br) := \varrho(\overline{A}(x_0, ar, br), A(x_0, 2r/3, 3r)) + \varrho(\overline{B}(x_0, ar), B(x_0, br)) \leq c_3(\varphi(r))^{-1}. \hspace{1cm} (8.7)$$

Let $B_u := B(x_0, u)$ be a ball centered at $x_0$ with radius $u > 0$. Let $d \geq 1$, $0 < r < 1$ and $x, y \in B_r$. By Theorem 111 (with $M = 2$ and $T = \phi(2)$) and the semigroup property we have, for $t_0 = \phi(|x - y|)$,

$$G_{B_r}(x, y) \leq \int_0^{t_0} p(s, x, y)ds + \int_{t_0}^{\infty} p_{B_r}(s + t_0, x, y)ds$$

$$\leq C \int_0^{t_0} s\nu(|x - y|)ds + \int_{B_r} p(t_0, z, y)G_{B_r}(x, z)dz$$

$$\leq C \int_0^{t_0} (t_0^{a/2}\nu(|x - y|)) + (\varphi^{-1}(t_0))^{-d}\mathbb{E}_x\tau_{B_r})$$

$$= C \frac{|x - y|^a}{\varphi(|x - y|)} + \mathbb{E}_x\tau_{B_r}). \hspace{1cm} (8.8)$$

Let $5/6 < a < 1$. For $x \in B_{5r/6}$ and $y \in B_r \setminus B_{ar}$, $(a - 5/6)r \leq |x - y| \leq 2r$. Hence, by (8.4) and (WS) we obtain

$$c_{(2.10)}(x_0, 5r/6, ar, r) := \sup_{x \in B_{5r/6}, \ y \in B_r \setminus B_{ar}} G_{B_r}(x, y) \leq c_4 \frac{\varphi(r)}{r^d}. \hspace{1cm} (8.9)$$
where the constant $c_i$ depends on $\phi, a$ and $d$. Hence [\textit{8 Assumption D}] holds.

We have observed that [\textit{8 Assumption A–Assumption D}] hold. In addition, by (8.4), (8.7) and (8.9), the upper bound of the constants $c_{(3.9)}, c_{(3.11)}$ and $c_{(1.1)}$ in [\textit{8}] from the expressions of the constants $c_{(3.9)}, c_{(3.11)}$ and $c_{(1.1)}$ in [\textit{8} (3.9)–(3.11)] so that for any $x_0 \in \mathbb{R}^d$ and $0 < r < 1,$

\[
\begin{align*}
  c_{(3.9)}(x_0, 5r/6, 11r/12, r) &\leq c_6 \frac{\phi(r)}{r^d}, \\
  c_{(3.11)}(x_0, 5r/6, 11r/12, r) &\leq 2c_{(3.9)}(x_0, 5r/6, 11r/12, r) \\
  \cdot \max \left( \frac{\hat{\rho}(x_0, 11r/12, 2r)}{c_{(2.8)}(x_0, 11r/12, 2r)}, |B(0, 1)| C^*(\phi, d, 1/2) r^d \right) &\leq c_7 \phi(r), \quad \text{and} \\
  c_{(1.1)}(x_0, 2r/3, r) &\leq \left( \frac{\hat{\rho}(x_0, 3r/4, 5r/6)}{c_{(3.11)}(x_0, 5r/6, r)} + C^*(\phi, d, 9/10)^4 \right) \leq c_8
\end{align*}
\]

where the constants $c_i, i = 6, 7, 8$ are depending only on $\phi, \eta, L_0, L_3$ and $d$. Therefore, we obtain the scaling and translation invariant version of [\textit{8} (BHI)] for $r < 1$, with the constant $c_{(1.1)} = c_{(1.1)}(x_0, 2r/3, r)$ which is independent of $r < 1$ and $x_0 \in \mathbb{R}^d$. $\square$

Alternatively, one can check the conditions in [\textit{35] Section 4}, which also provides [\textit{35 Corollary 4.2}, the scaling and translation invariant version of (BHI).

We now use the above proposition to prove Theorem 1.8

**Proof of Theorem 1.8.** Suppose that $\rho \in (\pi/2, 1]$ and $D$ is a $C^{1,\rho}$ open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda)$. Since $D$ is a $C^{1,\rho}$ open set, it is easy to see that for any $z \in \partial D$ there exits a bounded $C^{1,\rho}$ open set $U$ in $\mathbb{R}^d$ whose characteristics depend only on $R_0$ and $\Lambda$ (independent of $z \in \partial D$) such that $B(z, 7R_0/8) \cap D \subset U \subset B(z, R_0) \cap D$ (if $d = 1$ we can take $U = (z, z + R_0)$ or $U = (z - R_0, z)$). Choose a point $z_0 \in U \setminus B(z, 3R_0/4)$ and let $g_1(x) = G_U(x, z_0)$. Since $g_1$ is regular harmonic in $D \cap B(z, 3R_0/4)$, appleying Proposition 8.1 we obtain

\[
\frac{f(x)}{f(y)} \leq c_1 \frac{g_1(x)}{g_1(y)}, \quad x, y \in D \cap B(z, r/2).
\]

Theorem 1.6 implies the claim of the theorem. $\square$

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