Background field calculations and nonrenormalization theorems
in 4d supersymmetric gauge theories
and their low–dimensional descendants

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Abstract

We analyze the structure of multiloop supergraphs contributing to the effective
Lagrangians in 4d supersymmetric gauge theories and in the models obtained from
them by dimensional reduction. When $d = 4$, this gives the renormalization of
the effective charge. For $d < 4$, the low-energy effective Lagrangian describes the
metric on the moduli space of classical vacua. These two problems turn out to be
closely related. In particular, we establish the relationship between the 4d non-
renormalization theorems (in minimal and extended supersymmetric theories) and
their low–dimensional counterparts.

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1 Introduction

Soon after the discovery of supersymmetry, it was understood that it imposes stringent constraints on renormalization pattern of 4d field theories. In particular, there is no renormalization of a superpotential $W$ which is a function of the chiral fields $\Phi_i(x, \theta)$. It is the so called $F$ term in the Lagrangian given by the the integral over the chiral subspace of superspace, $\int d^2 \theta W$.)

The gauge coupling term is also given by the integral over the chiral subspace of gauge invariant quantity, $\int d^2 \theta \text{Tr} W^\alpha W_\alpha$, what suggests its nonrenormalization. It is well known that the situation is more complicated in this case. There is no, indeed, charge renormalization in $\mathcal{N}=4$ super–Yang–Mills (SYM) theory. In $\mathcal{N}=2$ theories, only 1–loop contribution in the $\beta$ function survives. In $\mathcal{N}=1$ theories, multiloop contributions to the $\beta$ function are related to renormalization of $Z$–factors, which allows one to evaluate higher loops exactly in pure SYM theory and express them via anomalous dimensions of the matter fields in the theories involving chiral matter multiplets [1].

The simplest way to prove all the listed nonrenormalization theorems is to analyze the structure of the relevant supergraphs. We refer the reader to the textbooks [2–4] for the proof of nonrenormalization theorems for superpotential, but recall in some more details (following Refs. [5]) how it is done for gauge couplings, the subject of our interest here.

Consider for simplicity Abelian theory — supersymmetric electrodynamics. It involves the massless photon and photino described the vector superfield $V$ and massive charged particles of spin 0 and 1/2 described by two chiral superfields $\Phi^i = \{S, T\}$ with opposite electric charges. To relate the physical charge $e_{\text{phys}}$ measured in infrared, i.e. below the matter masses, to the bare charge $e_0$ defined at ultraviolet scale $\Lambda_{\text{UV}}$, we have to evaluate the supergraphs describing vacuum loops in the presence of a soft background gauge field $V$. The relevant 1–loop and 2–loop graphs are depicted in Fig. 1.

![Figure 1: Contributions to the effective action, a) one loop; b) two loops. Solid lines are chiral field superpropagators $\langle \Phi_1 \Phi_2 \rangle$ and the dashed line stands for vector superfields. The bar on the solid line marks the $\Phi$ end.](image)

The one–loop graph gives a nonvanishing correction to effective Lagrangian of the
vector field $V$ at momenta much smaller than the mass of the matter fields,

$$L_{\text{eff}}^{1-\text{loop}}(V) = \text{Re} \left\{ \left[ \frac{1}{e_0^2} + \frac{1}{4\pi^2} \ln \frac{\Lambda_{\text{UV}}}{m_0} \right] \frac{1}{2} \int d^2 \theta W^2 \right\}, \quad (1)$$

where $e_0$ and $m_0$ are the bare charge and mass and the ultraviolet cut off $\Lambda_{\text{UV}}$ is introduced as a mass of Pauli-Villars regulators.

The statement is that the two–loop graph and also higher loop contributions to $L_{\text{eff}}$ vanish identically. Indeed, according to the supergraph Feynman rules [3, 4], each vertex involves the integral $\int d^8 z = \int d^4 x d^4 \theta$ and the whole contribution of the graph in Fig. 1b is $\Delta L_{\text{eff}}(V) = \frac{1}{e} \int d^4 \theta \mathcal{K}^{(2)}(z_1)$, where

$$\mathcal{K}^{(2)}(z_1) = \frac{ie^2}{2} \sum_{i,j=1,2} \int d^8 z_2 \langle \Phi^i \Phi^j \rangle \langle \Phi^j \Phi^i \rangle \langle v_1 v_2 \rangle .$$

Here $\Phi^i$ stands for the charged chiral superfield $S$ or $T$ and the vector superfield $v$ is a quantum deviation from the classical background $V$. Furthermore, $\langle \Phi^i \Phi^j \rangle$, $\langle v_1 v_2 \rangle$ are quantum superpropagators evaluated in external background $V$ and subscripts 1,2 refer to the superspace coordinates $z_{1,2}$. Now, $\langle v_1 v_2 \rangle$ does not depend on external field and on its gauge. The charged field propagators are gauge–dependent:

$$\langle \Phi^i \Phi^j \rangle \rightarrow e^{-iq_i \Lambda_1} \langle \Phi^i \Phi^j \rangle e^{iq_j \Lambda_2}, \quad (2)$$

where $q_i = \{1, -1\}$ are electric charges of the fields $\Phi^i$.

The point is, however, that the integrand $\mathcal{K}^{(2)}$ is gauge–independent and should thereby be locally$^2$ expressed via the gauge–invariant superfield $W_\alpha$. But $W_\alpha$ is a chiral superfield and the integral over $d^4 \theta$ of any function of $W_\alpha$ vanishes. Therefore $\int d^4 \theta \mathcal{K}^{(2)} = 0$ Q.E.D. The same reasoning apply also to an arbitrary multiloop graph.

Another way to prove the same statement is based on the fact that the effective charge is a holomorphic function of $m_0$ [6]. Indeed, the higher powers of log $m$ produce dependence of the effective charge on the phase of $m_0$, i.e., on the vacuum angle – the effect which does not occur in perturbation theory where terms presenting total derivatives do not contribute.

We hasten to comment that this does not mean that multiloop contributions to $\beta$ function in $N = 1$ supersymmetric QED vanish. Higher loops appear when expressing the bare mass $m_0$ entering Eq. (1) via the physical mass $m_{\text{phys}}$. The physical mass is renormalized in spite of the fact that the mass term in the Lagrangian is not. Indeed, the physical mass can be defined as the pole of the fermion propagator $\propto 1/(Z \hat{p} - m_0)$, where $Z$ describes the renormalization of the kinetic term

$$Z \int d^4 \theta \left( \bar{S} e^V S + \bar{T} e^{-V} T \right) . \quad (3)$$

\textit{Locality follows from the presence of an infrared cutoff (nonzero mass) in the theory. We postpone the discussion of what happens in massless theories till Sect. 2.5.}
We have $m_0 = Zm_{\text{phys}}$ what leads to an exact relation expressing the charge renormalization via the matter $Z$ factor,

$$\frac{1}{e^2_{\text{phys}}} = \frac{1}{e^2_0} + \frac{1}{4\pi^2} \ln \frac{\Lambda_{\text{UV}}}{m_{\text{phys}}} - \frac{1}{4\pi^2} \ln Z. \quad (4)$$

In particular, using knowledge of $Z$ at one–loop level,

$$Z = \frac{m_0}{m_{\text{phys}}} = 1 - \frac{e^2_0}{4\pi^2} \ln \frac{\Lambda_{\text{UV}}}{m_{\text{phys}}} + \ldots, \quad (5)$$

we obtain the two–loop renormalization of the charge

$$\frac{1}{e^2_{\text{phys}}} = \frac{1}{e^2_0} + \frac{1}{4\pi^2} \ln \frac{\Lambda_{\text{UV}}}{m_{\text{phys}}} - \frac{1}{4\pi^2} \ln \left[ 1 - \frac{e^2_0}{4\pi^2} \ln \frac{\Lambda_{\text{UV}}}{m_{\text{phys}}} \right] + \frac{e^2_0}{16\pi^4} \ln \frac{\Lambda_{\text{UV}}}{m_{\text{phys}}} + \ldots. \quad (6)$$

In case of $\mathcal{N}=2$ supersymmetric electrodynamics, the above consideration shows an absence of higher loops because $Z = 1$ in this case. Indeed, the $\mathcal{N}=2$ SQED involves an extra neutral chiral superfield $\Upsilon$. Besides the $\Upsilon$ kinetic term, the Lagrangian contains the superpotential term $\propto \int d^2\theta \, \Upsilon ST$. The latter is not renormalized: this is the standard $F$ term nonrenormalization theorem. The point is that this superpotential term is related by extended supersymmetry to the charged field kinetic term. Hence, nonrenormalization of the superpotential implies in $\mathcal{N}=2$ theory nonrenormalization of the kinetic term. In other words, in $\mathcal{N}=2$ theory, $m_{\text{phys}} = m_0$ and hence only the 1-loop term in the $\beta$ function survives. It is valid for non-Abelian case as well. Moreover, if the matter content of the particular $\mathcal{N}=2$ theory is such that the one-loop $\beta$ function vanishes, the theory is finite. The $\mathcal{N}=4$ SYM theory belongs to this class.

The $\mathcal{N}=2$ SQED represents a new phenomenon — an infinite degeneracy of vacuum states. Indeed, different vacua of the theory are characterized by a value of $\Upsilon$ which serves as a modulus parameter in the set (moduli space) of vacua. In nonsupersymmetric theories, the vacuum degeneracy is always associated with spontaneous breaking of a continuous global symmetry what implies physical equivalence of different vacua — chiral description for pions in QCD is an example. It is not the case for moduli spaces in supersymmetric theories. In the $\mathcal{N}=2$ SQED different values of $\Upsilon$ imply different masses of the charged field, i.e. different physics. The effective low-energy Lagrangian of this theory has the following form:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \text{Re} \left\{ \int d^2\theta \left[ \frac{1}{e^2_0} + \frac{1}{4\pi^2} \ln \frac{\Lambda_{\text{UV}}}{\Upsilon} \right] W^2 + \int d^4\theta \left[ \frac{1}{e^2_0} + \frac{1}{4\pi^2} \left( \ln \frac{\Lambda_{\text{UV}}}{\Upsilon} + 1 \right) \right] \bar{\Upsilon} \Upsilon, \right\}, \quad (7)$$

where we normalized $\Upsilon$ in such a way that its background value is equal to the mass of the charged field.\(^3\) For the lowest component $\nu$ of the modulus field $\Upsilon$ it gives

$$\mathcal{L}_{\text{eff}} = \frac{1}{2e^2(\nu)} \partial_\mu \bar{\nu} \partial^\mu \nu = \frac{1}{2} \left[ \frac{1}{e^2_0} + \frac{1}{8\pi^2} \ln \frac{|\Lambda_{\text{UV}}|^2}{|\nu|^2} \right] \partial_\mu \bar{\nu} \partial^\mu \nu, \quad (8)$$

\(^3\)The bare mass $m_0$ provides just a shift in $\Upsilon$. 

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what can be viewed as a metric in moduli space. We see a remarkable relation between
the moduli metric and the effective charge renormalization in perturbation theory.

This relationship allows one to construct an alternative proof of the nonrenormalization
theorem. Actually, extended supersymmetry *dictates* the coefficient of \( W^2 \) in the first term
in (7) to be a holomorphic function of \( T \). This generalizes the comment above referring to
holomorphic dependence on \( m_0 \) in \( \mathcal{N} = 1 \) theories. Nonvanishing higher–order corrections
would spoil this holomorphy and are not allowed. We will discuss this in more details in
Sect. 3.

For non-Abelian theory such as \( \mathcal{N} = 2 \) pure SYM theory with the SU(2) gauge group
the low energy effective Lagrangian involves only Abelian degrees of freedom and has
a similar form. In perturbative calculations, the massive matter fields in the loops are
substituted in this case by the charged massive vector fields \( W^\pm_\mu \) and their superpartners.
The coefficient of the logarithm is of the opposite sign, of course, reflecting asymptotic
freedom at short distances,

\[
\frac{1}{g^2(\nu)} = \frac{1}{g_0^2} - \frac{1}{4\pi^2} \ln \frac{|\Lambda_{UV}|^2}{|\nu|^2} = \frac{1}{4\pi^2} \ln \frac{|\nu|^2}{|\Lambda_{IR}|^2}. \tag{9}
\]

Moreover, in this case all nonperturbative terms which are powers of \( |\Lambda_{IR}|^4/|\nu|^4 \) are also
known thanks to the Seiberg-Witten exact solution \([7]\). Nonperturbative effects in moduli
dynamics are even more crucial in case of \( \mathcal{N} = 1 \) gauge theories where they could lead to
appearance of superpotential for moduli \([8]\).

Similar questions can be posed and solved for low-dimensional descendants, i.e. theories
obtained by dimensional reduction of the corresponding 4d theories (see \([9]\) for a recent review).
Typically, the Coulomb branch moduli space is enhanced in the descendants compared to the 4d case, involving now the components of 4d vector potential in the
reduced spatial directions.\(^4\) E.g., in 4d \( \mathcal{N} = 1 \) theories, there is no Coulomb branch whatsoever, but it appears in lower dimensions. In particular, reduction of the \( \mathcal{N} = 1 \) SYM to
one dimension leads to the effective Lagrangian representing a nonstandard “symplectic”
\( \mathcal{N} = 2 \) \( \sigma \) model defined on a 3r–dimensional target space \((r\) is the rank of the group) with
certain conditions for the metric \([10,11]\). The 2d effective Lagrangian is a Kähler \( \sigma \) model
living on target space of \( r \) complex dimensions.

If we start from \( \mathcal{N} = 2 \) SYM in 4 dimensions, the effective Lagrangians become fancier
and prettier. The 1d effective Lagrangian represents then \([12]\) a generalization of the \( \sigma \) model
suggested in \([13]\) with \((3 + 2)r = 5r\) dimensional target space.\(^5\) The 2d effective
Lagrangian \([12,14]\) belongs to the class of twisted \( \mathcal{N} = 4 \) \( \sigma \) models \([15]\). Finally, 3d
effective Lagrangians are hyper–Kähler \( \sigma \) models. For each unit of \( r \), the moduli space
involves 2 variables coming from adjoint scalar, one variable from the component of the

\[^4\]The term *Coulomb branch* was coined for \( d = 4 \) where it involves besides scalar fields also massless
photon mediating Coulomb interaction. Being reluctant to invent new words, we will use this term for
flat directions associated with the gauge field and/or its superpartners also for \( d = 2 \) and \( d = 1 \) where the
only reminiscent of Coulomb phase is the absence of mass.

\[^5\]The fermions belong to the spinor representation of \( \text{SO}(5) \equiv \text{Sp}(2) \) and one can call it the symplectic
model of the second kind. For the symplectic models of the first kind with 2 complex supercharges, the
fermions are doublets in \( \text{SO}(3) \equiv \text{Sp}(1) \).
vector potential in the reduced dimension and one variable representing a dual 3d photon. The moduli spaces represent Atiyah–Hitchin (AH) manifolds for unitary groups \cite{16} and hyper–Kähler manifolds obtained from AH manifolds after certain factorizations for other simple Lie groups \cite{17}.

The metric on the target spaces of these σ models can be determined by evaluating perturbative loop corrections to the effective Lagrangian. The problem is conceptually very similar to that of effective charge renormalization in \( d = 4 \). Indeed, as was noted in \cite{18}, the 1–loop corrections to the metric in lower dimensions are rigidly related to the 1–loop coefficient in the 4–dimensional \( \beta \) function.

Nonrenormalization theorems can also be formulated in lower dimensions. In particular, for the descendants of \( \mathcal{N} = 2 \) theories, all higher loop corrections to the metric beyond one loop vanish. This is very similar to what happens in 4 dimensions. The proof of the low–dimensional nonrenormalization theorems is based on the constraints on the form of the effective Lagrangian following from supersymmetry. For \( d = 4 \), such constraints lead to holomorphic dependence on moduli. In lower dimensions, a generalization of this is harmonic dependence on extended moduli. For example, for the groups of rank 1, the presence of 4 complex supercharges in the twisted σ model (\( d = 2 \)) as well as in the Diaconescu–Entin model (\( d = 1 \)) requires that the metric represents a harmonic O(4) [O(5)] invariant function. With

\[
\mathcal{L}_{\text{eff}} = \frac{1}{2e^2} h(A) \partial_\mu A \partial^\mu A
\]  

the only possibility is

\[
d = 2 : \quad A \in \mathbb{R}^4, \quad h(A) = 1 + \frac{C}{A^2};
\]

\[
d = 1 : \quad A \in \mathbb{R}^5, \quad h(A) = 1 + \frac{C}{|A|^3}. \quad (11)
\]

But this means that the would be higher loop corrections to the metric \( \Delta h_{d=2}^{l\text{-loops}} \propto 1/|A|^{2l} \) and \( \Delta h_{d=1}^{l\text{-loops}} \propto 1/|A|^{3l} \) vanish for \( l > 1 \). The metric \( h(A) \) is harmonic for \( d = 3 \) too though the moduli \( A \) do not exhaust all light degrees of freedom, the latter involving also the massless gauge field that is present in the effective Lagrangian. (see Sect. 3 for more comments about the connection between the 4d nonrenormalization theorems and their lower dimensional counterparts.)

For the descendants of \( \mathcal{N} = 1 \) theories, multiloop corrections do not vanish. Two–loop corrections were calculated in the Abelian case in \cite{19}. For \( d=1 \), the metric has the form

\[
h(A) = 1 + \frac{e^2}{2|A|^3} - \frac{3e^4}{4|A|^6} + \ldots, \quad A \in \mathbb{R}^3.
\]

This corresponds to massless SQED with one flavor.

The question arises whether these multiloop corrections are related to multiloop corrections in 4d \( \beta \) function as they do for the first loop. In Ref. \cite{19}, the calculations were
performed in component formalism and this relationship was not clearly seen. In this paper, we reproduce the calculations in the $N=1$ superfield formalism and establish such a relationship at the two–loop level. We noted above that in $d=4$, multiloop corrections to the $\beta$ function are related to mass renormalization. One can ask whether it is also true in some sense in lower dimensions. The answer to this question is negative.

In the main body of the paper we present a systematic study of one and two loop corrections to the low energy effective Lagrangian of the Abelian and non-Abelian gauge theories for all dimensions $d \leq 4$. At the perturbative level, it fixes the metric in the moduli space as well as dynamics of light degrees of freedom associated with the $d$ dimensional gauge fields (present for $d = 3, 4$).

The plan of the paper is the following. Sect. 2 is the central part of the paper. There we consider SQED and perform the superfield calculation of the 2–loop correction in different dimensions. In particular, we demonstrate by diagrammatic methods how the exact relation (4) works order by order in perturbation theory. In Sect. 3, we discuss the $N=2$ extension of SQED and emphasize the universal reason by which the higher–order corrections to the metric vanish in all dimensions: extended supersymmetry requires it to be harmonic. We also illustrate how the two–loop corrections to the metric cancel out: supergraph techniques allow one to reproduce in a simple way the result of [19] and extend it to all dimensions. In Sect. 4, we extend the calculations to the non–Abelian case emphasizing their relationship to Abelian ones. It also demonstrates the validity of the the known exact expression for the $4d \beta$ function [1] at the two-loop level. The last section is reserved as usual to final conclusive remarks and acknowledgments.

## 2 Multiloop corrections to the metric: Abelian case

### 2.1 Notation and definitions

Let us start with fixing the notation. The density of Lagrangian of massive supersymmetric QED reads\(^\text{6}\)

$$
\mathcal{L} = \text{Re} \left\{ \frac{1}{2e^2} \int d^2 \theta W^2 + \int d^4 \theta \left[ \bar{S} e^V S + T e^{-V} T \right] + 2m \int d^2 \theta ST \right\},
$$

(13)

where

$$
V = C + i \theta \chi - i \bar{\theta} \bar{\chi} - \frac{i}{\sqrt{2}} N \theta^2 - \frac{i}{\sqrt{2}} \bar{N} \bar{\theta}^2 - 2 \theta \sigma^\mu \bar{\theta} A_\mu
$$

$$
+ \left[ 2i \theta^2 \bar{\theta} \left( \bar{\lambda} - i \frac{4}{\sqrt{2}} \bar{\sigma}^\mu \partial_\mu \chi \right) + \text{h.c.} \right] + \theta^2 \bar{\theta}^2 \left( D - \frac{1}{4} \partial^2 C \right)
$$

(14)

\(^\text{6}\) Our conventions are close to that of Ref. [2], $\theta^2 = \theta^\alpha \theta_\alpha$, $\int d^2 \theta \theta^2 = \int d^2 \bar{\theta} \bar{\theta}^2 = 1$; $(\sigma^\mu)_{\alpha \beta} = \{1, \sigma\}_{\alpha \beta}$, $(\bar{\sigma}^\mu)_{\beta \alpha} = \bar{\epsilon}^{\beta \gamma} \epsilon^{\alpha \delta} (\sigma^\mu)_{\delta \gamma} = \{1, -\sigma\}_{\beta \alpha}$, but we use the metric $\eta_{\mu \nu} = \text{diag}(1, -1, -1, -1)$ and include the extra factor 2 in the definition of $V$. Note that, for any superfield $X$, $\int d^2 \theta X = - \frac{D^2}{4} X$ and $\int d^2 \bar{\theta} X = - \frac{\bar{D}^2}{4} X$.\]

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and

\[ W_\alpha = \frac{1}{8} \tilde{D}^2 D_\alpha V = i (\lambda_\alpha + i \theta_\alpha D - \theta^\beta F_{\alpha \beta} - i \theta^2 \sigma^\mu \partial_\mu \bar{\lambda}) \]  \tag{15}

with

\[ D_\alpha = \frac{\partial}{\partial \theta^\alpha} - (\sigma^\mu)_{\alpha \dot{\beta}} \bar{\theta}^{\dot{\beta}} i \partial_{\mu}, \quad \tilde{D}_\dot{\alpha} = - \frac{\partial}{\partial \theta^{\dot{\alpha}}} + \theta^\alpha (\sigma^\mu)_{\alpha \dot{\beta}} i \partial_{\mu}, \]  \tag{16}

\[ \{D_\alpha, \tilde{D}_{\dot{\beta}}\} = 2i (\sigma^\mu)_{\alpha \dot{\beta}} \partial_{\mu}, \quad D^2 \tilde{D}^2 D^2 = -16 \partial^2 D^2, \quad \tilde{D}^2 D^2 \tilde{D}^2 = -16 \partial^2 \tilde{D}^2. \]  \tag{17}

The theory is defined by two parameters \( e^2 \) and \( m \), which generically are complex numbers. Their imaginary parts lead to terms which are total derivatives in the Lagrangian density. We can include such terms into consideration viewing \( 1/e^2 \) and \( m \) as spurion chiral fields.

In the 4d case one more scale parameter associated with the UV cut off should be added but for lower dimensions it is not needed.

The theory is invariant under gauge transformations,

\[ S \rightarrow e^{-i \Lambda} S, \quad T \rightarrow e^{i \Lambda} T, \quad V \rightarrow V + i (\overline{\Lambda} - \Lambda) \]  \tag{18}

with the chiral function \( \Lambda \). The superfield strength \( W_\alpha \) is gauge invariant. We will also extensively use the superconnection \( \Gamma_\mu \),

\[ \Gamma_\mu = \frac{1}{4} (\bar{\sigma}_\mu)^{\dot{\beta} \alpha} \tilde{D}_\dot{\beta} D_\alpha V = A_\mu + \ldots, \]  \tag{19}

which transforms as

\[ \Gamma_\mu \rightarrow \Gamma_\mu - \partial_\mu \Lambda. \]  \tag{20}

The superconnection \( \Gamma_\mu \) is defined such that the superfields \( \nabla_\mu S = (\partial_\mu - i \Gamma_\mu) S \) and \( \nabla_\mu T = (\partial_\mu + i \Gamma_\mu) T \) are transformed under the gauge transformations (18) in the same way as \( S, T \). Covariant derivatives acting on the right chiral superfields \( \bar{S}, \bar{T} \) have the form \( \partial_{\dot{\mu}} \pm i \Gamma_{\dot{\mu}} \), where \( \Gamma_{\dot{\mu}} \) is complex conjugate of (19).

Note, that the superconnections \( \Gamma_\mu \) are, of course, constrained superfields, it is the components of \( V \) which are unconstrained. The superderivatives of \( \Gamma_\mu \) are related to the superfield strengths \( W_\alpha, \tilde{W}_{\dot{\alpha}} \),

\[ \tilde{D}_{\dot{\alpha}} \Gamma_\mu = (\sigma_\mu)_{\dot{\alpha} \alpha} W^\alpha, \quad D_\alpha \Gamma_\mu = (\sigma_\mu)_{\alpha \dot{\alpha}} \tilde{W}^\dot{\alpha}. \]  \tag{21}

Our conventions for the supergraphs are close to those in Refs. [3, 4]. For convenience, we have included the factors \( D^2/4 \) and \( \tilde{D}^2/4 \) usually attributed to the vertices in the definition of the propagators. In the absence of external field, the free 4-dimensional propagators are

\[ \langle S_1 S_2 \rangle = \langle T_1 T_2 \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{ip(x_2 - x_1)} \frac{i}{p^2 - |m|^2} \frac{\tilde{D}^2 D_2^2 \delta^4(\theta_1 - \theta_2)}{16}, \]  

\[ \langle V_1 V_2 \rangle = -2e^2 \int \frac{d^4 k}{(2\pi)^4} e^{ik(x_2 - x_1)} \frac{i}{k^2} \left[ 1 + (\alpha - 1) \frac{D_1^2 \tilde{D}^2_2 + \tilde{D}_1^2 D_2^2}{16k^2} \right] \delta^4(\theta_1 - \theta_2), \]  \tag{22}
where we should substitute $\pm \mu$ for $i\partial/\partial x_{1,2}^\mu$ in the expressions (16) for $D_\alpha$ and $\bar{D}_\alpha$. Note the useful relation

$$\frac{\bar{D}_1^2 D_2^2 \delta^4(\theta_1 - \theta_2)}{16} = \exp\{p_\mu[\theta_2 \sigma_\mu \bar{\theta}_2 + \theta_1 \sigma_\mu \bar{\theta}_1 - 2\theta_1 \sigma_\mu \bar{\theta}_2]\}.$$  \hspace{1cm} (23)

The expression for the propagator of gauge superfield $V$ depends on the parameter $\alpha$ in the gauge fixing term $L_{g.f.}$. We will use the Feynman gauge, $\alpha = 1$. This gauge appears to be special providing for a benign behavior of the propagator in the infrared. We will return to the discussion of this point later.

The vertices $\langle \bar{S}V^n S \rangle$ and $\langle \bar{T}V^n T \rangle$ are read out directly from the Lagrangian (13). Although the propagators $\langle \Phi_1 \Phi_2 \rangle$ and $\langle \bar{\Phi}_1 \bar{\Phi}_2 \rangle$ are also present in the theory, they involve the extra factors $\theta^2$ or $\bar{\theta}^2$, and one can check that the contribution to $\mathcal{K}^{(2)}(\Gamma)$ of the graph like in Fig.1b, but with the bars on the same line, vanishes.

### 2.2 Dimensional reduction

The above definitions refer to SQED in 3+1 dimensions. To pass to the dimensionally reduced descendants of the theory, one can follow the standard logics (cf. e.g. [20]) and put the system in a small spatial box $|x_k| \leq L_k/2$, $(k = 1, 2, 3)$, imposing periodic boundary conditions. The gauge invariant integrals

$$I_k = \frac{1}{L_k} \int_{-L_k/2}^{L_k/2} A_k dx_k$$  \hspace{1cm} (25)

represent the moduli of the theory, they count the set of degenerate classical vacua. The moduli space is also a 3-dimensional torus, $I_k$ are periodic coordinates living on the interval $|I_k| \leq \pi/L_k$. A supersymmetric extension of $I_k$

$$\tilde{I}_k = \frac{1}{L_k} \int_{-L_k/2}^{L_k/2} \Gamma_k dx_k$$  \hspace{1cm} (26)

invariant under supergauge transformations (20) adds fermionic moduli.

Reduction of the $k$-th spatial coordinate is introduced as the limit $L_k \to 0$. In this limit, $I_k$ coincides with $A_k$ (and $\tilde{I}_k$ with $\Gamma_k$) while the interval where $A_k$ is defined becomes infinite. For unreduced coordinates we take an opposite limit $L_k \to \infty$ such that the corresponding moduli interval becomes a point $A_k = 0$. Thus, we consider the limit when the moduli torus shrinks along unreduced directions and becomes unbounded along the reduced ones.

The $d$-dimensional descendant of SQED represents a theory with the Lagrangian density given by the same expression (13) but with the fields depending only on $d$ coordinates. The number of moduli fields is given by co-dimension $\hat{d}$ defined as

$$\hat{d} = 4 - d.$$  \hspace{1cm} (27)
Thus, the moduli space of the theory has dimension $\hat{d}$ and is parametrized by the vector $A = \{A_{\hat{k}}\} \in \mathbb{R}^{\hat{d}}$, where $\hat{k}$ marks reduced coordinates. We call this the Coulomb branch following a clear analogy with the $\mathcal{N}=2$ case [7], which can be viewed as a dimensional reduction from the $\mathcal{N}=1$ gauge theory in six dimensions. The coordinate dependent excitations $A_{\hat{k}}(x^0, ..., x^{\hat{d}})$ of moduli are called moduli fields. The supersymmetric extension of these moduli fields is given by superfields $\Gamma_{\hat{k}}$ defined in Eq. (19). It is important for us, however that the co-dimensional components $\Gamma_{\hat{k}}$ (the supermoduli, $\hat{k} = d, \ldots, 4$) are gauge invariant fields as it seen from Eq. (20). Moreover, $\Gamma_{\hat{k}}$ are real in contrast to the space–time $\bar{\mu} = 0, \ldots, d-1$ components, for which $\Gamma_{\bar{\mu}} = i \partial_{\bar{\mu}} V$.

The set of light bosonic fields on the Coulomb branch is given by $A_{\mu}$. This includes, besides gauge independent moduli fields $A_{\hat{k}}$, the remaining gauge dependent $A_{\bar{\mu}}$ living in the $d$-dimensional space, $\bar{\mu} = 0, \ldots, d-1$. We will see that the effective Lagrangian depends not only on $W_{\alpha}$, but also on the supermoduli $\Gamma_{\hat{k}}$ and, if limiting ourselves by the terms having not more than two derivatives of bosonic fields, can be written in the form

$$L_{\text{eff}} = \text{Re} \left\{ \frac{1}{2e^2} \int d^2 \theta h(\Gamma) W^2 \right\}, \quad \Gamma = \{\Gamma_1, ..., \Gamma_d\}.$$  \hspace{1cm} (28)

It differs from the original gauge term in Eq. (13) by the function $h(\Gamma)$ which introduces dependence on gauge–invariant moduli superfields. Note that the superfields $\Gamma$ are not chiral. However, it is only the chiral part of $h(\Gamma)$ which contributes. Indeed, acting by $\bar{D}_\alpha$ on the Lagrangian (28) we get zero as it follows from Eq. (21) and $W_{\alpha} W_{\beta} W_{\gamma} = 0$.

Trading the $\theta$ integration in Eq. (28) for differentiation and using Eq. (21) we obtain the component form of the effective Lagrangian,

$$L_{\text{eff}} = \text{Re} \left\{ \frac{1}{2e^2} \left[ h(A) \left( -\frac{1}{2} F_{\mu\nu}^2 + D^2 + 2 \lambda \sigma^\mu i \partial_\mu \lambda \right) - \frac{i}{2} \frac{\partial h}{\partial A_{\hat{k}}} F_{\mu\nu} \lambda \sigma^\mu \sigma^\nu \sigma_k \bar{\lambda} - \frac{\partial^2 h}{\partial A_{\hat{k}}^2} \frac{\partial A_{\hat{k}}}{\partial A_{\hat{l}}} \lambda^2 \bar{\lambda} \right] \right\}. \hspace{1cm} (29)$$

We have omitted here the terms which represent the total derivatives, like $\theta$-term. The part of the Lagrangian (29) which contains bosonic moduli fields $A_{\hat{k}}$,

$$L_{\text{eff}}^{\text{moduli}} = \frac{h(A)}{2e^2} \partial_\mu A_{\hat{k}} \partial^\mu A_{\hat{k}},$$  \hspace{1cm} (30)

shows that the metric in the moduli space has a simple conformally flat form,

$$ds^2 = h^{\hat{i}\hat{j}}(A) dA_{\hat{k}} dA_{\hat{i}}, \quad h^{\hat{i}\hat{j}}(A) = h(A) \delta^{\hat{i}\hat{j}}.$$  \hspace{1cm} (31)

We also can present $L_{\text{eff}}$ as an integral over full superspace,

$$L_{\text{eff}} = \text{Re} \left\{ \frac{1}{e^2} \int d^4 \theta K(\Gamma) \right\},$$  \hspace{1cm} (32)
where the function \( K(A) \) is related to the metric \( h(A) \) via derivatives,

\[
h(A) = \frac{1}{2} \frac{\partial^2 K(A)}{\partial A^\mu \partial A_\mu}.
\] (33)

The equivalence of (28) and (32) is revealed when substituting \( \int d^2 \theta \) by \(-\bar{D}^2/4\) and using the relations (21). Note that \( K \) depends on all four components of \( A_\mu \) while the metric \( h \) is the function of \( \hat{d} \)-dimensional moduli only. Equation (33) implies that we can consider \( K \) as a function of the moduli \( A \) up to harmonic terms such as

\[
\frac{1}{\hat{d}} A_\mu A^\mu + \frac{1}{\hat{d}} A^2.
\] (34)

This harmonic ambiguity in \( K \) is particularly important in the limit of four dimensions when \( \hat{d} \to 0 \).

We can illustrate this in the leading classical approximation. In this approximation the metric is

\[
h^{(0)}(A) = 1,
\] (35)

and the function \( K \) satisfying Eq. (33) and depending only on \( A \) is

\[
K^{(0)}(A) = -\frac{1}{\hat{d}} A^2,
\] (36)

which is singular at \( \hat{d} \to 0 \). Adding the harmonic term (34), we come to

\[
K^{(0)}(A) = -\frac{1}{\hat{d}} A_\mu A^\mu,
\] (37)

which allows for a smooth \( \hat{d} \to 0 \) limit. Although, in contrast to Eq. (36), this form of \( K \) is not gauge invariant both forms lead to the same gauge invariant action.

### 2.3 One-loop corrections

The one-loop corrections due to the charged matter fields, Fig. 1a, were calculated in Ref. [10, 18]. For any dimension \( \hat{d} \) in the interval \( 1 \leq \hat{d} \leq 4 \), the one-loop expression for \( h^{(1)}(A) \) can be presented in the form of the Feynman integral (after the Wick rotation of \( p_0 \))

\[
h^{(1)}(A) = 2e^2 \int \frac{d^d p}{(2\pi)^d} \left[ \frac{1}{(p^2 + A^2 + |m|^2)^2} - \frac{1}{(p^2 + A^2 + \Lambda_{UV}^2)^2} \right],
\] (38)

where \( A_k \) is substituted for co-dimensional components of the momentum \( p_k \) and the second term in the square brackets introduces Pauli-Villars regularization needed only for \( \hat{d} = 4 \). Generically, \( \Lambda_{UV} \) might be complex, but we assume it to be real and positive. Probably, the simplest way to find \( h \) is to calculate the loop in the background of constant potential \( A_\mu \) and constant auxiliary field \( D \). Then \( A_\mu \) enters as a shift \( p_\mu \to p_\mu + A_\mu \).

while $D$ splits fermion and boson masses. Expanding over $D$ up to the second order and comparing the result with the corresponding term in Eq. (29), one arrives at (38). The result of integration is

$$h^{(1)}(A) = \frac{2\Gamma(d/2)}{(4\pi)^{d/2}} \frac{e^2}{(A^2 + |m|^2)^{d/2}} - \left( m \to \Lambda_{\text{UV}} \right).$$

Note that the integral (38) is convergent in the infrared and the result (39) implies the choice of low normalization point, $\mu \ll (|m|^2 + A^2)^{1/2}$, for the effective Lagrangian. Let us add also that, once $h(A)$ is known, the function $K(A)$ is defined by integration. We will give later explicit expressions for $K(A)$ for different integer $d$.

### 2.4 Two-loop calculations

At the two-loop level, only the graph depicted in Fig. 1b (or rather two such graphs with chiral superfields $S, T$ in the loop) contribute. Note that the graphs involving 4-point vertices $\Phi \Phi vv$, do not contribute because in the Feynman gauge we are using the quantum superfield propagator $\langle v_1 v_2 \rangle \propto \delta^4(\theta_1 - \theta_2)$, which vanishes at the coinciding superspace points. Comparing the general form (32) of the effective Lagrangian with the expression corresponding to the graph in Fig. 1b, we get the expression for the two-loop part of $K$,

$$K^{(2)}(\Gamma(z_1)) = ie^2 \int d^d x_1 d^d x_2 d^4\theta_2 \langle S_1 \bar{S}_2 \rangle \langle S_2 \bar{S}_1 \rangle \langle v_1 v_2 \rangle.$$

Generically, the expression for the superpropagator $\langle S_1 \bar{S}_2 \rangle$ in external gauge field is rather complicated. It has the form

$$\langle S_1 \bar{S}_2 \rangle = -\frac{i}{16} \left[ \nabla_\mu \nabla_\mu + i W^\alpha \nabla_\alpha + \frac{i}{2}(\nabla^\alpha W_\alpha) + |m|^2 \right]^{-1} \nabla_1^2 \nabla_2^2 \delta^d(x_1 - x_2) \delta^4(\theta_1 - \theta_2),$$

where all derivatives are covariant, see Ref. [3] for details.

However, we can use any background for determination of $K^{(2)}(\Gamma)$. A convenient choice is to keep only the lowest component $\Gamma_k|_{\theta = \bar{\theta} = 0} = A_k$ and assume that all higher components vanish. For such a choice, the superfield $W_\alpha$ vanishes and the propagator of the charged superfields has a particular simple form. In momentum space it can be obtained from the free 4d propagator (see the footnote at the previous page) by substituting moduli $A_k$ for the co-dimensional components of momentum $p_k$.

$$\langle S_1 \bar{S}_2 \rangle = \int \frac{d^d p}{(2\pi)^d} e^{ip(x_1 - x_2)} \frac{i}{p^2 - A^2 - |m|^2} \frac{\bar{D}_1^2 D_2^2 \delta^4(\theta_1 - \theta_2)}{16}.$$
Bearing all this in mind and using the relation (23) and its corollary\textsuperscript{9}

$$
\delta^4(\theta_1 - \theta_2) \frac{D_1^2 D_2^2}{16} \delta^4(\theta_1 - \theta_2) = \delta^4(\theta_1 - \theta_2),
$$

we can write after Wick rotation

$$
\mathcal{K}^{(2)}(A) = -2e^4 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + A^2 + |m|^2} \frac{1}{(p + k)^2 + A^2 + |m|^2}.
$$

(44)

In this expression \( p \) and \( k \) refer only to the \( d \)-dimensional part of momenta, the co-dimensional part of \( p \) is written explicitly via \( A \).

Let us start the calculation with representing the integral over \( p \) in the dispersion form,

$$
\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + A^2 + |m|^2} = \int_4^{2(m^2)} ds \frac{\rho(s, m, A)}{s + k^2},
$$

(45)

$$
\rho(s, m, A) = \frac{1}{\sqrt{s - 4A^2 - 4|m|^2}} \frac{1}{2^{d-3} \pi^{(d-1)/2} \Gamma(\frac{3-d}{2})}
$$

To make the integral over \( s \) convergent at \( d = 4 \), we regularize it in the UV by the Pauli-Villars regulators, i.e., by subtracting from the integrand the expression where \( m \) is substituted by \( \Lambda_{UV} \),

$$
\rho_{\text{reg}}(s, m, A) = \rho(s, m, A) - \rho(s, \Lambda_{UV}, A).
$$

(46)

The integral over \( k \) in (44) involves for \( d \leq 2 \) an infrared divergence coming from the small \( k \) region. It also needs the UV regularization for \( d = 4 \). By substituting the photon propagator

$$
\frac{1}{k^2} \rightarrow \frac{1}{k^2 + \mu^2} - \frac{1}{k^2 + \Lambda_{UV}^2}.
$$

(47)

we take care of both IR and UV divergences. (We have chosen the ultraviolet regulator for the photon propagator to be the same as the Pauli–Villars regulator, though in principle they could be different.) Integrating then over \( k \), we get

$$
\mathcal{K}^{(2)}(A) = e^4 \int_4^{\infty} ds \frac{\rho_{\text{reg}}(s, m, A)}{(2d-1)\pi^{(d-2)/2} \Gamma(\frac{4-d}{2}) \sin \frac{d-2}{2}} \frac{s^{(2-d)/2} - (\mu^2)^{(2-d)/2}}{s - \mu^2}
$$

$$
- (\mu^2 \rightarrow \Lambda_{UV}^2).
$$

(48)

We need the piece where \( \mu^2 \) is substituted by \( \Lambda_{UV}^2 \) only for \( d = 4 \). In this case the \( \Lambda_{UV}^2 \) part cancels completely the part from the first line in Eq. (48) and both \( \mathcal{K}^{(2)} \) and \( h^{(2)} \) vanish at \( d = 4 \). We will discuss this vanishing in more detail in the next Section.

\textsuperscript{9}Note that one needs at least 4 spinor derivatives to deal with the second \( \delta \) function and to obtain a nonvanishing result. This was exactly the reason by which we were allowed to substitute the covariant derivative \( \nabla_\alpha \) by \( D_\alpha \) in the propagator \( \langle S_1 \bar{S}_2 \rangle \).
For $d < 4$ we obtain

$$K^{(2)}(A) = \frac{\sqrt{\pi} \Gamma(\frac{d}{2} - 1) \Gamma(\frac{d}{2} - 1)}{4(2\pi)^d \Gamma(\frac{d}{2} + 1)} \frac{e^4}{(A^2 + |m|^2)^{d-1}} - \frac{2\Gamma(\frac{d}{2} - 1) \Gamma(\frac{d}{2})}{(4\pi)^d} \frac{e^4}{\mu^{d-2}(A^2 + |m|^2)^{d/2}} - (m \to \Lambda_{UV}),$$

and the corresponding metric $h^{(2)}$ is

$$h^{(2)}(A) = -\frac{\sqrt{\pi} \Gamma(\frac{d}{2} - 1) \Gamma(\frac{d}{2} + 1)}{4(2\pi)^d \Gamma(\frac{d}{2} + 1)} \frac{e^4(A^2 - |m|^2)}{(A^2 + |m|^2)^{d+1}} + \frac{4\Gamma(\frac{d}{2} - 1) \Gamma(\frac{d}{2} + 1)}{(4\pi)^d} \frac{e^4(A^2 - |m|^2)}{\mu^{d-2}(A^2 + |m|^2)^{2d/2}} - (m \to \Lambda_{UV}).$$

The apparent singularities at $d = 1, 2$ cancel out as they should. Indeed, introducing the infrared and ultraviolet regularizations as in Eq. (47) makes the integral (44) finite for any $d$.

The singular in the infrared parameter $\mu$ part of this result coincides with the one produced by the one-loop correction to matrix element of $K^{(1)}$, see [19] for detailed discussion. Indeed, denoting $a_k$ deviations in the moduli $A_k$, we get for this matrix element

$$\langle K^{(1)} \rangle_{1\text{-loop}} = \frac{1}{2} \frac{\partial^2 K^{(1)}}{\partial A_k \partial A_l} (a_k a_l) = \frac{e^2}{2} \frac{\partial^2 K^{(1)}}{\partial A_k \partial A_l} \int \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{k^2 + \mu^2} - \frac{1}{k^2 + \Lambda_{UV}^2} \right]$$

$$= -\frac{e^2 h^{(1)}(A) \Gamma(\frac{d}{2} - 1)}{(4\pi)^{d/2} \mu^{d-2}} - (\mu \to \Lambda_{UV}),$$

The matching of infrared-singular parts in expressions (49) and (51) means that, when we pass to the Wilsonian effective action, the IR parameter $\mu$ in Eqs. (49) and (50) becomes the normalization point for the Wilsonian action. We implied in the above expressions that the normalization point $\mu$ is much less than $(A^2 + |m|^2)^{1/2}$.

### 2.4.1 Dimension 1

In SQED reduced to one dimension, the moduli space is three-dimensional with the following metric $h$:

$$h_{1d}(A) = 1 + \frac{e^2}{2(A^2 + |m|^2)^{3/2}} - \frac{3e^4(A^2 - |m|^2)}{4(A^2 + |m|^2)^4} + \frac{3e^4(2A^2 - 3|m|^2)}{16\mu(A^2 + |m|^2)^{7/2}}.$$  

Here the tree, Eq. (35), one-loop, Eq. (39) and two-loop, Eq. (50), terms are combined. The effective Lagrangian is given by Eq. (29) where the only nonvanishing components of $F_{\mu\nu}$ are $F_{0k} = \dot{A}_k$.

The function $K(\Gamma)$ is

$$K_{1d}(\Gamma) = -\frac{1}{3} \Gamma^2 + \frac{e^2 \text{arcsinh}(\Gamma^2/|m|^2)^{1/2}}{(\Gamma^2)^{1/2}} + \frac{e^4}{8(\Gamma^2 + |m|^2)^2} - \frac{e^4}{4\mu(\Gamma^2 + |m|^2)^{3/2}}.$$  

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2.4.2 Dimension 2

The moduli space in the SQED reduced to two dimensions $x^0, x^1$ is parametrized by $\mathbf{A} = \{A_2, A_3\}$ with the metric

$$h_{2d}(\mathbf{A}) = 1 + \frac{e^2}{2\pi} \frac{1}{\mathbf{A}^2 + |m|^2} + \frac{e^4}{4\pi^2} \left[ \frac{\mathbf{A}^2 - |m|^2}{(\mathbf{A}^2 + |m|^2)^{3/2}} \ln \frac{\mathbf{A}^2 + |m|^2}{\mu^2} - \frac{|m|^2}{(\mathbf{A}^2 + |m|^2)^{3/2}} \right], \tag{54}$$

as it follows again from Eqs. (35), (39) and the $d \to 2$ limit of Eq. (50).\(^{10}\) The effective Lagrangian (29) contains besides moduli fields $\mathbf{A} = \{A_2, A_3\}$ the vector field $A_\mu(x^0, x^1)$ ($\mu = 0, 1$). Although $A_\mu$ contains no propagating degrees of freedom, it induces a contact interaction for fermionic fields, like the auxiliary field $D$ does. The function $K(\Gamma)$ in this case has the form

$$K_{2d}(\Gamma) = -\frac{1}{2} \Gamma^2 - \frac{e^2}{4\pi} \int_0^{\frac{|m|}{2\pi}} \ln(t + 1) \, dt \, \frac{1}{\Gamma^2 + |m|^2} \left[ \ln \frac{\Gamma^2 + |m|^2}{\mu^2} + 2 \right]. \tag{55}$$

Now, the bosonic part of the effective Lagrangian has the sigma model form (30). The full effective Lagrangian represents its supersymmetric extension with two complex supercharges. A theorem due to Alvaréz-Gaume and Freedman [22] says that there is only one such Lagrangian representing a Kähler supersymmetric $\sigma$ model (incidentally, any 2-dimensional manifold is Kählerian). In other words, the effective Lagrangian can be presented in the form

$$\mathcal{L}_{\text{eff}} = \frac{1}{e^2} \int d^4\theta K(\bar{\Phi}, \Phi), \tag{56}$$

where $\Phi = \phi + \sqrt{2} \theta \lambda + \theta^2 F$ is a chiral superfield with the lowest component $\phi = (A_2+iA_3)/\sqrt{2}$. This was emphasized in Ref. [18], where the Kähler potential $K(\bar{\Phi}, \Phi)$ was calculated at the one-loop level. One can explicitly check that the component form of the Lagrangian (56) coincides, indeed, with (29) if the Kähler potential $K$ is chosen as

$$K(\bar{\Phi}, \Phi) = -K_{2d}(\Gamma^2 = 2\bar{\Phi} \Phi). \tag{57}$$

2.4.3 Dimension 3

Let us denote $A$ the only modulus that survives at $d=3$. The metric becomes

$$h_{3d}(A) = 1 + \frac{e^2}{4\pi} \frac{1}{(A^2 + |m|^2)^{1/2}} + \frac{e^4}{16\pi^2} \frac{A^2 - |m|^2}{(A^2 + |m|^2)^2}, \tag{58}$$

and the function $K(\Gamma)$ is

$$K_{3d}(\Gamma) = -\Gamma^2 - \frac{e^2}{2\pi} \left[ \Gamma \arcsinh \frac{\Gamma}{|m|} - \sqrt{\Gamma^2 + |m|^2} \right] + \frac{e^4}{16\pi^2} \ln \frac{\Gamma^2 + |m|^2}{A_{\text{UV}}^2}. \tag{59}$$

\(^{10}\) Actually, the parameter $\mu$ in Eq. (54) differs from that in Eq. (50) by a certain constant factor chosen such that the two–loop term in Eq. (54) is a pure logarithm in the limit $m = 0$. 15
If the field $A$ were the only bosonic field in the effective theory, the metric could be made trivial by field redefinition. However, at $d=3$, unlike what happens at $d=1,2$, the vector field $A_\mu$ describes one propagating degree of freedom.

The effective Lagrangian (29) takes the following 3d form:

$$L_{\text{eff}} = \frac{1}{2e^2} \left\{ h \left[ -\tilde{F}_\mu^2 + (\partial_\mu A)^2 + (\lambda \sigma^\mu i \partial_\mu \bar{\lambda} + \text{h.c.}) \right] - h' \tilde{F}_\mu \lambda \sigma^\mu \bar{\lambda} - \frac{1}{4} \left( h'' - \frac{(h')^2}{2h} \right) \lambda^2 \bar{\lambda}^2 \right\}. \quad (60)$$

Here $\tilde{F}_\mu = \epsilon_{\mu\nu\gamma} F^{\nu\gamma}/2$ and we excluded the auxiliary field $D$. Let us now introduce the dual photon field $\pi$ performing the duality transformation, i.e. adding the term $\partial_\mu \pi \tilde{F}_\mu/e^2$ to the Lagrangian and integrating over $\tilde{F}_\mu$. We obtain

$$L_{\text{eff}} = \frac{1}{2e^2} \left\{ \frac{1}{h} [(h \partial_\mu A)^2 + (\partial_\mu \pi)^2] + h(\lambda \sigma^\mu i \partial_\mu \bar{\lambda} + \text{h.c.}) - \frac{1}{4} \left( h'' + \frac{(h')^2}{h} \right) \lambda^2 \bar{\lambda}^2 \right\}. \quad (61)$$

To compare this expression with the generic form (56) of Kähler model, let us relate our fields with components of the chiral superfield $\Phi$ in the following way:

$$\Phi = \phi + \sqrt{2} \theta \psi + \theta^2 F, \quad \phi = \sigma + i \frac{\pi}{\sqrt{2}}, \quad \sigma = -\frac{1}{2} K'(A) \quad \psi = h(A) \bar{\lambda}. \quad (62)$$

We verify then that the effective Lagrangian (61) can be presented in the Kähler form (56) with the Kähler function $K(\frac{\Phi + \bar{\Phi}}{\sqrt{2}})$ depending only on the sum $\Phi + \bar{\Phi}$ and satisfying the relation

$$\frac{1}{2} K''(\sigma) = \frac{1}{h[A(\sigma)]} = 1 - \frac{e^2}{4\pi (\sigma^2 + |m|^2)^{1/2}} - \frac{e^4}{16\pi^2} \left[ \frac{\sigma \arcsinh(\sigma/|m|)}{(\sigma^2 + |m|^2)^{3/2}} - \frac{2|m|^2}{(\sigma^2 + |m|^2)^2} \right], \quad (63)$$

where $\sigma = (\phi + \bar{\phi})/\sqrt{2} = -K'(A)/2$.

Thus, we came to the Kähler model where metric does not depend on the extra modulus $\pi$ associated with the dual photon. It means that $\pi$ is a coordinate along an isometry of the metric. The phase nature of $\pi$ becomes more visible if one would use the superfield $\tilde{\Phi} = \mu \exp(\Phi/\mu)$ instead of $\Phi$ (with $\mu$ being an arbitrary scale parameter). The metric depends only on $\tilde{\Phi} \tilde{\Phi}$.

### 2.5 $d = 4$: Anatomy of zero

As was discussed in the Introduction, in massive 4d SQED, the 2–loop contribution to $L_{\text{eff}}$ vanishes. We have seen before how this comes about when explicitly evaluating the expression (44) for $\mathcal{K}^{(2)}$ carefully regularized in the ultraviolet and infrared [see the remark after Eq. (48)]. The zero was obtained after cancellation of infrared and ultraviolet contributions to the integral. Indeed, consider the expression (50) for the metric at $d < 4$. It was obtained from the first term in Eq. (48) for $\mu \ll m$. The second term corresponding to the UV regularization of the photon propagator does not contribute at $d < 4$, but for $d = 4$ it is important. In the limit $4 - d = d = \epsilon \to 0$, Eq. (50) gives

$$h^{(2)} = -\frac{e^4}{32\pi^4} \log \frac{\Lambda^2_{\text{UV}}}{|m|^2}. \quad (64)$$
This is canceled by the UV contribution coming from the second term in Eq.(47).

It is instructive to explore in more details the mechanism for this cancellation in four dimensions redoing the calculations in a different and more transparent way. Note first of all that there are no moduli in \( d = 4 \) and the integral (44) gives just a number – not a function of connections. On the other hand, in 4 dimensions the correction (if any) is proportional to the original Lagrangian of the gauge field

\[
\frac{C}{2e_0^2} \int d^2\theta W^2 = \frac{C}{4e_0^2} \int d^4\theta \Gamma_\mu^2. 
\]  
(65)

Hence, \( C = 0 \).

Let us forget about this vanishing for the moment and try to calculate the contribution in the same way as we did it for lower dimensions. As earlier, we consider the background involving only the component \( A_\mu \) that is constant. The expression for \( \mathcal{K}^{(2)} \) can be written in the form (before Euclidean rotation),

\[
\mathcal{K}^{(2)} = -2e_0^2 \int \frac{d^4k}{(2\pi)^4k^2} \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p_\mu + \Gamma_\mu)^2 - m^2} \frac{1}{(p_\mu + k_\mu + \Gamma_\mu)^2 - m^2}. 
\]  
(66)

With \( \Gamma_\mu \) representing just a shift of the variable of integration, the integral obviously does not depend on \( \Gamma_\mu \). This notwithstanding, expand the right side of Eq. (66) in \( \Gamma_\mu \) and keep the quadratic terms. Using Lorentz symmetry and performing Wick rotations in the momentum integrals, we obtain

\[
\mathcal{K}^{(2)} = e_0^2 \int \frac{d^4k}{(2\pi)^4k^2} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} \frac{1}{(p + k)^2 + m^2} \frac{\partial^2}{\partial p_\mu^2} \left[ \frac{1}{p^2 + m^2} \frac{1}{(p + k)^2 + m^2} \right]. 
\]  
(67)

The integral is still zero. The integrand can be represented as

\[
\frac{\partial^2}{\partial p_\mu^2} \left[ \frac{1}{p^2 + m^2} \frac{1}{(p + k)^2 + m^2} \right] = -\frac{8m^2}{(p^2 + m^2)^3} \frac{1}{(p + k)^2 + m^2} - \frac{8m^2}{p^2 + m^2} \frac{1}{[(p + k)^2 + m^2]^3} + \frac{8(p^2 + pk)}{(p^2 + m^2)^2[(p + k)^2 + m^2]^2}. 
\]  
(68)

Now, the first two terms in the right side are infrared in nature. Indeed, the integral over the first term is saturated by \( p^2 \sim m^2 \). In the limit \( m \to 0 \), it can be assimilated to the integral of \( \delta \) function,

\[
-\frac{8m^2}{(p^2 + m^2)^3} \to -4\pi^2 \delta^4(p). 
\]

Similarly, the second term goes over to \( \delta^4(p + k) \) in the massless limit. On the other hand, the third term is not infrared. The corresponding integral is saturated in the region \( p^2 \sim k^2 \) and integration over \( k \) is logarithmic, i.e., \( m^2 \ll k^2 \ll \Lambda_{UV}^2 \).

Everything is prepared now to determine the effective Lagrangian at the normalization point \( \mu \gg m \) or, in other terms, in massless theory. Indeed, the effective Lagrangian is defined after integrating over all modes with characteristic energy exceeding the separation scale \( \mu \). Up to now, we assumed that \( \mu \ll m \). But in the opposite limit, the infrared
contributions in the integral coming from the virtualities $p^2 \sim m^2$ should simply be discarded because this range of momenta should not be counted. These momenta will reappear in the matrix elements in the effective theory. Only the third term in R.H.S. of Eq. (68) is left. It gives a nonvanishing contribution to the integral.

The latter is conveniently evaluated as the infrared contribution taken with opposite sign. We obtain

$$\mathcal{K}^{(2)}(\mu) = 2e_0^4\pi^2 \Gamma^2_\mu \int \frac{d^4k}{(2\pi)^4k^2} \int \frac{d^4p}{(2\pi)^4} \frac{\delta^4(p)}{(p + k)^2} = \frac{e_0^4\pi^2}{2(2\pi)^6} \Gamma^2_\mu \int \frac{\Lambda_{\text{UV}} d^4k}{k^4} = \frac{e_0^4}{64\pi^4} \ln \frac{\Lambda_{\text{UV}}}{\mu}. \quad (69)$$

Bearing in mind the relation (65) and the definitions (13), (32), we obtain the contribution $(e_0^2/16\pi^4) \ln(\Lambda_{\text{UV}}/\mu)$ in $1/e^2(\mu)$ in accordance with Eq. (6).

By construction, the 2–loop contribution to $1/e^2(\mu)$ that we have just evaluated depends on the graph in Fig. 1b with very small virtuality of one of the matter field lines, i.e. this line is effectively cut off. The diagram thus obtained describe the one–loop polarization operator of the matter field $\Sigma^{(1)}(p)$. To make things absolutely clear, we illustrate the procedure just described in Fig. 2.\footnote{A factorization of similar nature was observed in $\mathcal{N} = 4$ SYM for more complicated case of scattering amplitudes \cite{23} and even for non–supersymmetric QED in a self–dual gauge background \cite{24}.}

Note that there are two ways of cutting the two–loop graph and that cancels the original combinatorial factor 1/2.

![Figure 2: Two–loop effective action via one–loop polarization operator.](image)

This explicit analysis expresses the two-loop effective charge as

$$\frac{1}{e^2_{\text{two-loop}}(\mu)} = \frac{1}{e_0^2} + \frac{1}{4\pi^2} \ln \frac{\Lambda_{\text{UV}}}{\mu} - \frac{1}{4\pi^2} \ln Z_{\text{one-loop}}. \quad (70)$$

with $Z_{\text{one-loop}} = 1 - (e_0^2/4\pi^2) \ln(\Lambda_{\text{UV}}/\mu)$. Thereby, it relates the 2–loop $\beta$ function to the 1–loop anomalous dimension of the matter field in accordance with the general result \cite{1}. Moreover, in terms of the Wilsonian Lagrangian which for $\mu \gg m$ includes the matter part, the whole two-loop result is attributed to the matrix element of the one-loop corrections to the matter part \cite{5}. In this sense, there is no higher-loop corrections to the gauge coupling in the Wilsonian Lagrangian, they all dwell in the matter $Z$ factor.

The same, of course, follows from the analysis of the regime $\mu \ll m$, as we explained in the Introduction: we saw that the 2–loop contribution to $\mathcal{L}_{\text{eff}}$ is zero in this case, but the nonzero 2–loop contribution in $1/e^2_{\text{phys}}$ depended on the physical mass renormalization, which was determined by the renormalization of the kinetic term related to anomalous dimensions. As a result, we obtained the result (6) which coincides with (70) up to the interchange $m_{\text{phys}} \leftrightarrow \mu$.\footnote{A factorization of similar nature was observed in $\mathcal{N} = 4$ SYM for more complicated case of scattering amplitudes \cite{23} and even for non–supersymmetric QED in a self–dual gauge background \cite{24}.}
2.5.1 Remark on the gauge dependence

In the derivation above we used the Feynman gauge choice, \( \alpha = 1 \). In a generic gauge, the propagator of the vector superfield (22) involves an extra term \( \propto (\alpha - 1)(D_1^2 \bar{D}_2^2 + \bar{D}_1^2 D_2^2)\delta^4(\theta_1 - \theta_2)/k^4 \), which is more singular in infrared than the main \( 1/k^2 \) term. When \( \alpha \neq 1 \), the tadpole diagram, involving the vector field loop and the \( \Phi \Phi v^2 \) vertex, does not vanish and should be added to the two-loop graph (b) in Fig. 1.

At the two-loop level it is not difficult to verify that the effective Lagrangian does not depend on the gauge choice. Extra contributions due to the change of the propagator are canceled out in the sum of two diagrams. The situation is a little bit more subtle for \( Z \)-factors of the matter fields. The gauge-dependent term brings about logarithmic infrared divergences there \([25, 26]\). At the one–loop level, they were explicitly evaluated in \([27]\).

One may ask now how is it possible to interpret Eq. (70) relating the \( Z \)-factor to the running charge if this \( Z \)-factor is gauge–dependent? However, one can see from explicit calculations that the gauge-dependent part in the \( Z \)-factor is due to infrared range of integration over virtual momenta, where they are of the same order as external ones. The ultraviolet dependence of the \( Z \)-factor does not depend on the gauge. In other terms, we can formulate this as a statement of gauge independence of the Wilsonian \( Z \)-factor, in which the infrared part should be omitted by construction. It is this Wilsonian \( Z \)-factor which enters the relation (70).

A particular way to calculate the gauge-independent Wilsonian \( Z \)-factors is to introduce a nonvanishing mass for the quantum vector field \( v \) by adding \( \int d^4 \theta \mu^2 v^2/4e^2 \) to the Lagrangian. The propagator then becomes

\[
\langle v_1 v_2 \rangle = -2e^2 \int \frac{d^4k}{(2\pi)^4} e^{ik(x_2-x_1)} \frac{i}{k^2 - \mu^2} \left[ 1 + \frac{\alpha - 1}{k^2 - \alpha \mu^2} \frac{D_1^2 \bar{D}_2^2 + \bar{D}_1^2 D_2^2}{16} \right] \delta^4(\theta_1 - \theta_2)
\]

It is sufficient to choose both \( \mu^2 \) and \( \alpha \mu^2 \) to be much larger than the virtuality of external matter line \( p^2 - m^2 \) to get rid of the gauge-dependent infrared part in \( Z \).

2.5.2 Three and higher loops

Let us see now what happens at the 3–loop level and higher. The relevant 3–loop supergraphs (with the proper combinatorial factors) are drawn in Fig. 3. Expanding over \( \Gamma_\mu \) produces the d’Alembert operator acting on the matter field momenta. The infrared contributions shown in Fig. 4 are expressed via the two–loop contribution to the matter polarization operator \( \Sigma^{(2)} \) (it is straightforward to see that the combinatorial factors come out right). Besides, there is an extra infrared contribution coming from the graph in Fig. 3a which is depicted in Fig. 5. Note that the combinatorial factor \( 1/2 \) present in Fig. 3a is not changed here! To understand why, let us look at the analytic expression corresponding to this graph. The singular in momentum \( p \) part has the form

\[
K^{(3)} \propto \int d^4 p \int d^4 \theta_2 \langle S_1 \bar{S}_2 \rangle^2.
\]

(71)
Figure 3: Three-loop effective action.

Figure 4: Infrared contributions associated with $\Sigma^{(2)}$.

Substituting here the superpropagators from Eq. (22), bearing in mind the relation (23), and expanding the exponential there up to the terms $\propto p^2$, we see that the integrand behaves as $1/p^2$ rather than $1/p^4$ at small $p$. In component language, this means that only the contribution of the fermion components of the corresponding superpropagator is relevant such that $\text{Tr}(1/p)^2 \propto 1/p^2$. Substituting $p_\mu \to p_\mu + \Gamma_\mu$ and expanding over $\Gamma_\mu$, we obtain the structure $\Box(1/p^2)$, which is equivalent to inserting a cross in one of the matter lines and does not bring about an extra numerical factor. Combining all contributions,

Figure 5: The infrared contribution associated with $\frac{1}{2}[\Sigma^{(1)}]^2$.

we obtain

$$K^{(2)} + K^{(3)} + \ldots \propto \delta Z^{(1)} + \frac{1}{2} (\delta Z^{(1)})^2 + \delta Z^{(2)} + \ldots = \ln(1 + \delta Z^{(1)} + \delta Z^{(2)})$$

(72)

in accordance with Eq. (4). Similarly, one can separate the infrared contributions in the higher loops. They give the higher terms of the expansion of the logarithm.

Again, all this was done under assumption that the gauge is chosen such that the vector propagator is proportional to $\delta^4(\theta_1 - \theta_2)/k^2$ and the coefficient of $(D_1^2 D_2^2 + \bar{D}_1^2 D_2^2)\delta^4(\theta_1 - \theta_2)/k^4$ vanishes (otherwise, the $Z$-factor would be singular in the infrared). Note that it is not enough now to stay in the Feynman gauge $\alpha = 1$. In the latter, the structure
\( \propto (D_1^2 \bar{D}_2^2 + D_2^2 \bar{D}_1^2) \delta^4(\theta_1 - \theta_2)/k^4 \) is absent at the tree level, but is generated after calculating loop corrections. To cope with this, one should pose \( \alpha = 1 + Ce^2 \ln(\Lambda_{UV}^2/k^2) + O(e^4) \). The dependence of \( \alpha \) on \( k \) corresponds actually to a nonlocal gauge choice [26]. It is not obvious to prove that everything comes out correctly with this procedure in any order of perturbation theory. We hope to return to this question in some future work.

3 Extended \( \mathcal{N} = 2 \) SQED: Harmonicity and cancellations

3.1 Harmonicity

As was mentioned in the introduction, the \( \mathcal{N} = 2 \) extension of SQED involves an extra massless chiral superfield \( \Upsilon \), the following terms are added to the Lagrangian (13):

\[
\Delta \mathcal{L} = \text{Re} \left\{ \frac{1}{2e^2} \int d^4 \theta \bar{\Upsilon} \Upsilon + 2 \int d^2 \theta \bar{\Upsilon} \Upsilon^{ST} \right\}.
\]

The mass of the charged matter field is given by a background value of the lowest \( \Upsilon \) component, \( m = \nu \). This explicitly realizes the moduli nature of the mass.

Before going over to the \( \mathcal{N} = 2 \) case, let us discuss an interesting feature of the one- and two-loop results for the \( \mathcal{N} = 1 \) theory referring to their mass dependence. At \( d = 4 \) there is no moduli \( A \) and the function \( h \) in Eq. (29) depends only on the mass parameter \( m \). In the one-loop order \( h^{(1)} \propto \log |\Lambda_{UV}/m| \) and this dependence can be viewed as a real part of \( \log(\Lambda_{UV}/m) \) which is an analytic function of \( m \). In other words, \( h^{(1)} \) is a harmonic function in the plane \( m_1, m_2 \) which are real and imaginary parts of \( m = m_1 + im_2 \). Both the function and the argument refer to the coefficients of the chiral \( F \) terms in four dimensions.

However, this holomorphy is broken by higher loops. Indeed, \( Z \) factor entering Eqs. (4), (5) is real, and the function

\[
\ln \left[ 1 - \frac{e_0^2}{4\pi^2} \left| \ln \frac{\Lambda_{UV}}{m_{\text{phys}}} \right| \right]
\]

is not holomorphic. Only in case of extended \( \mathcal{N} = 2 \) supersymmetry \( Z \) stays equal to 1, higher loop contributions in the effective charge vanish, and holomorphy is maintained.

Actually, one can derive that higher loops vanish from the requirement of holomorphy, which is a corollary of extended supersymmetry [7]. Indeed, in \( U(1) \) theory the supersymmetric effective action can be expressed in terms of the gauge-invariant \( \mathcal{N} = 2 \) superfield

\[
\mathcal{W} = \Upsilon + i\sqrt{2} \tilde{e}^\alpha W_\alpha - \tilde{\theta}^2 \bar{D}^2 \Upsilon
\]

as

\[
\mathcal{L}_{\text{eff}} = \int d^2 \theta d^2 \bar{\theta} F(\mathcal{W}) + \text{ h.c.}
\]
The function $F(W)$ is called prepotential. Doing the integral over $d^2\tilde{\theta}$ in Eq. (76), we can express the effective Lagrangian as

$$L_{\text{eff}} = \text{Re} \left\{ \int d^2\theta F''(\Upsilon)W^2 + 2 \int d^4\theta F'(\Upsilon)\bar{\Upsilon} \right\}. \quad (77)$$

Comparing the first term with the corresponding term in Eq. (13), we see that $2F''(\Upsilon)$ can be interpreted as the inverse effective charge $1/\epsilon_{\text{eff}}^2(\Upsilon)$. By construction, it is a holomorphic function of the moduli.

In the region where $\epsilon_{\text{eff}}^2(\Upsilon)$ is small, the prepotential can be evaluated perturbatively. At the one–loop level, $F''(\Upsilon) \propto \ln \Upsilon$, which is holomorphic. The point is that nonvanishing second or higher loops would imply nonholomorphic dependence like in Eq. (74) (with $\Upsilon$ substituted for $m$), which is not allowed.

In $\mathcal{N} = 2$ SQED, the one–loop calculation gives an exact result for the prepotential. In non-Abelian theories, there are also nonperturbative contributions associated with instantons. These contributions (they are important in the strong coupling region, $\nu \sim \Lambda_{\text{IR}}$) were determined exactly for $SU(2)$ theory by Seiberg and Witten [7]. In this paper, we limit our discussion to perturbative effects.

Passing to $d < 4$ we see from Eq. (38) that the harmonicity in $m_1, m_2$ is lost already at the one-loop level.\textsuperscript{12} What we have instead, however, is the harmonicity in $2 + \hat{d}$ space where the moduli $A$ are added to $m_1, m_2$, i.e. harmonicity exists in the extended moduli/parameter space.

Again, looking at the two-loop results we see that this harmonicity is not supported by higher loops. However, similar to 4d theories, the extended $\mathcal{N} = 2$ supersymmetry makes harmonicity exact. [See the discussion around Eq. (11). We remind that the vector $A$ in Eq. (11) involves besides the components of vector potential in reduced dimensions also the lowest component of $\Upsilon$, which can be viewed as a linear combination $\nu = A_4 + iA_5$.] This, of course, implies vanishing of higher loops in $\mathcal{N} = 2$ theories.

Let us make few more comments on the $d = 3$ case, where the picture is slightly more complicated. In three (and obviously also in four) dimensions, light degrees of freedom associated with the Abelian gauge field come into play. As was explained above, for $d = 3$, gauge field is dual to the scalar one and one obtains an extra moduli, the dual photon $\pi$. We have explained before how this duality transformation works in the $\mathcal{N} = 1$ case, when restoring the Kähler form of the effective Lagrangian. In the $\mathcal{N} = 2$ case, a similar procedure leads to hyper–Kähler supersymmetric $\sigma$ model living on the 4–dimensional Taub-NUT manifold with the metric

$$ds^2 = h(A) dA^2 + h^{-1}(A) (d\pi + A(A) dA)^2, \quad h(A) = 1 + \frac{e^2}{4\pi |A|}, \quad (78)$$

where $A(A)$ represents the vector potential of an Abelian Dirac monopole satisfying $\partial \times A = \partial h$.\textsuperscript{13}

\textsuperscript{12} A similar one-loop phenomenon taking place in four dimensions was discussed in Refs. [28]. Moduli originated from a string construction led there to hierarchical structure of masses.

\textsuperscript{13} In the Abelian case, the result (78) is exact. In the non-Abelian $SU(2)$ case, a similar expression with $e^2$ substituted by $-2g^2$ describes the asymptotics of the metric at large $|A|$, whereas the full Atiyah–Hitchin metric involves also nontrivial nonperturbative contributions [16].
The function $h(A)$ is harmonic, which is not accidental. A well-known mathematical fact is that the Kähler potential of a hyper-Kähler manifold involving the $U(1)$ isometry is obtained by a Legendre transformation (physically, this is a duality transformation) out of a 3-dimensional harmonic function [29]. This harmonic function is nothing but the prepotential $K$ in the expression

$$L = \frac{1}{e^2} \int d^4 \theta K(\Gamma). \quad (79)$$

Now, $\Gamma$ has 3 components, with $\Gamma_3$ representing the superconnection in the reduced dimension and $(\Gamma_4 + i \Gamma_5)/\sqrt{2} = \Upsilon$. The metric $h(A)$ obtained from $K$ by $h = -(1/2) \partial^2 K/(\partial \Gamma_3)^2$ is also harmonic.

### 3.2 Cancellations

The vanishing of two and higher loop contributions to the metric can be confirmed by direct perturbative calculations. In [19], this was done at the two-loop level for the $d = 1$ theory by evaluating explicitly the relevant graphs: individual contributions to the effective Lagrangian canceled out in the sum. The calculation was done in components and the mechanism for this calculation was not obvious, however. We would like to note here that the cancellation becomes transparent if doing the calculations in the supergraph technique.

In the extended case, in addition to the two-loop graph in Fig. 1b, there is an extra contribution to the effective Lagrangian due to the $\Upsilon$ exchange depicted by the graph in Fig. 6.

![Figure 6: Two-loop supergraph with $\Upsilon$ exchange.](image)

The 4d $\Upsilon$ propagator has the form

$$\langle \Upsilon_1 \bar{\Upsilon}_2 \rangle = 2e^2 \int \frac{d^4k}{(2\pi)^d} e^{ik(x_2-x_1)} \left( \frac{i}{k^2} \frac{\bar{D}_1^2 D_2^2}{16} \right) \delta^4(\theta_1 - \theta_2). \quad (80)$$

The differential operators $\bar{D}_1^2/4$ and $D_2^2/4$ can be absorbed into the vertices completing $\int d^2 \theta_1, \int d^2 \bar{\theta}_2$ up to $\int d^4 \theta_{1,2}$. What is left coincides up to the opposite sign with the vector field propagator $\langle v_1 v_2 \rangle$ in Eq. (22). It is straightforward to see then that the contribution of the graph in Fig. 6 to $K^{(2)}$ exactly cancels the contribution of Fig. 1b.
At the three–loop level, we have now four extra supergraphs depicted in Fig. 7. The full contribution to the effective Lagrangian cancels out, but the mechanism of this cancellation is not evident as was the case for the two–loop graphs. The situation here is similar to what we had at the two–loop level in the component formalism. One should not be surprised here: the cancellation is the corollary of extended $\mathcal{N} = 2$ supersymmetry and should not be manifest neither in the component nor in the $\mathcal{N} = 1$ supergraph formalism. We believe that the cancellation would become manifest for any loop if working in the formalism of $\mathcal{N} = 2$ harmonic supergraphs [30].

![Figure 7: Three–loop supergraphs with Υ exchange.](image)

4 Non–Abelian theory

As we discussed in the Introduction, $\mathcal{N} = 1$ SYM theory in low dimensions, $d < 4$, involves the moduli associated with Abelian components of gauge potentials in reduced dimensions. The effective low–energy Lagrangian depends on the corresponding moduli fields (and also massless gauge fields at $d = 3$). For $SU(2)$ theory they are $A_\mu^3$ and their superpartners $\lambda_\alpha^3$. The effective Lagrangian is obtained after integrating out the heavy charged fields $A_\mu^\pm$, $\lambda_\alpha^\pm$. The mass of the latter depends on the moduli $A_\mu^3$. We will see below that the calculations of effective Lagrangian at the one and two–loop level in supergraph technique are effectively reduced to Abelian ones.

The Lagrangian of the theory has the form

$$\mathcal{L} = \text{Re} \frac{1}{g^2} \int d^2 \theta \text{Tr}\{W^\alpha W_\alpha\},$$

(81)

where

$$W_\alpha = \frac{D^2}{8} e^{-V} D_\alpha e^V.$$  

We will restrict ourselves by the case $G = SU(2)$. A earlier, we perform our calculations in the background field method [3], i.e. substituting $V \rightarrow V + v$, where $V$ is now a classical background field, which we assume to be Abelian.

We choose the Feynman gauge adding to the Lagrangian the term $-\text{Tr} \int d^4 \theta \nabla^2 v \nabla^2 v/16g^2$, where $\nabla_\alpha = D_\alpha - i \Gamma_\alpha$ is the covariant spinor derivative.\(^{14}\) The terms quadratic in $v$ acquire

\(^{14}\)For non-Abelian theories, the study of gauge dependence is more involved than for SQED, and we will not try here to explore this issue.
the form
\[ \mathcal{L}^{(2)} = \frac{1}{2g^2} \text{Tr} \int d^4 \theta \, v \left[ \nabla_\mu \nabla^\mu + i W^\alpha \nabla_\alpha + i \overline{W}^\dot{\alpha} \nabla_\dot{\alpha} \right] v . \]  

(82)

The covariant d’Alembert operator can be split up as
\[ \nabla_\mu \nabla^\mu = \nabla_\mu \nabla^\mu + \Gamma^2_\kappa \]  

(\kappa = 0, \ldots, d - 1; \kappa = d, \ldots, 4).

The moduli \( A_\kappa \) are the lowest components of \( \Gamma^3_\kappa \). They give mass to the charged fields \( v^\pm \) and will not be treated perturbatively.

Fixing the gauge leads to appearance of ghosts. The ghosts have the same algebraic nature as the parameters of gauge transformation, i.e. they are adjoint chiral superfields. Note that the number of ghost degrees of freedom is two times more than the number of gauge parameters (for example, for usual Yang–Mills theory, there are \( N^2 - 1 \) gauge parameters and \( N^2 - 1 \) complex ghosts \( c \)). In supersymmetric case [3, 4], this means that we have two different ghost chiral superfields \( c, \bar{c} \). On top of this, there is also the Kallosh–Nielsen ghost \( b \), which appears due to the fact that the gauge fixing term is by itself field–dependent. But the Kallosh–Nielsen ghost contributes only at the 1–loop level and does not appear in the multiloop graphs.

The calculation of the effective Lagrangian at the one–loop level is straightforward now. Consider first the loop of the \( v \) field. The vertices of its interaction with the background field \( V \) can be read out from the Lagrangian (82). They involve at most one spinor derivative. Considering the graph with two such vertices and the propagators
\[ \langle v_1^+ v_2^- \rangle = -\frac{2i g^2}{k^2 - A^2} \delta^4(\theta_1 - \theta_2) , \]

one can observe that the loop vanishes. Indeed, each propagator involve the factor \( \delta^4(\theta_1 - \theta_2) \), their product is zero and two covariant derivatives coming from the vertices are not able to cope with this.

We are left with the ghost loops. To reduce their calculation to that in SQED, it is convenient to introduce \( G_f, f = 1, 2, 3 \) such that
\[ G_1 \equiv c, \quad G_2 \equiv c', \quad \bar{G}_1 \equiv c', \quad \bar{G}_2 \equiv \bar{c}, \quad G_3 \equiv b, \quad \bar{G}_3 \equiv \bar{b} . \]

Only charged ghosts \( G^+_i, \bar{G}^+_i, i = 1, 2, 3 \), interact with the external field. Their propagators and interaction vertices are the same as for the chiral matter fields \( S, T \) in massless SQED. Each ghost loop gives the same contribution as the SQED matter loops up to a sign. Thus, the non–Abelian one–loop result is obtained from the Abelian one [see Eq.(39)], if multiplying it by \(-3\) and substituting \( e^2 \rightarrow g^2 \). This conforms with the previous component calculations for the metric [9, 18],

\[ h_{1d}^{SU(2)}(A) = 1 - \frac{3g^2}{2|A|^3} + \ldots , \]

\[ h_{2d}^{SU(2)}(A) = 1 - \frac{3g^2}{2\pi |A|^2} + \ldots , \]

\[ h_{3d}^{SU(2)}(A) = 1 - \frac{3g^2}{4\pi A} + \ldots . \]  

(83)
We are going to show now that the same factor $-3$ relates the Abelian and non-Abelian contributions to the metric at the two-loop level. The interaction of the ghosts with the quantum superfield $v^a$ can be derived by standard methods. The cubic term has the form (see Eqs. (6.2.20), (6.2.22) of Ref. [3])

$$
\mathcal{L}_{\text{ghost}} = \frac{ig}{4} \int d^4 \theta \, \epsilon^{abc} v^a (\bar{G}_1 + G_2)^b (G_1 + \bar{G}_2)^c.
$$

The relevant two-loop diagrams are drawn in Fig. 8. The corresponding analytic expressions have the same structure (44), (66) as in the Abelian case, but the values of color and combinatorial factors are such that the net contribution of the three graphs in Fig. 8 is zero [31].

Figure 8: Two-loop graphs with ghosts, $-2$ is a relative combinatorial coefficient.

Thus, we are left only with the graph with three gauge field lines depicted in Fig. 9. Expanding (81) in $v$ and “converting” one of the factors $\bar{D}^2$ to $-4 \int d^2 \bar{\theta}$, we express the cubic interaction term as

$$
\mathcal{L}^{(3)} = -\frac{1}{8g^2} \int d^4 \theta \, \text{Tr} \left\{ (\bar{D}^2 D_\alpha v)(D^\alpha v)v \right\}.
$$

(84)

[Strictly speaking, covariant derivatives $\nabla_\alpha$ enter, but we can substitute them by $D_\alpha$ by the same token as in the Abelian case; see the footnote after Eq.(42)].

In color vector notations, the vertex (84) involves the factor $\epsilon^{abc}$ so that two of the lines are charged with respect to the external field and acquire the mass $|A|$, and the third line is neutral and remains massless.

Note that the vertex has several terms distinguished by the way the factors $\bar{D}^2 D_\alpha$ and $D^\alpha$ are attributed to different lines. Therefore, we actually have not one but several diagrams. In principle, one could draw $3! = 6$ such diagrams, but those of them that involve more that 4 covariant derivative factors on a given line vanish [3, 32] and we are left with only four terms written in a symbolic way as

$$
\begin{align*}
&\left( \frac{\bar{D}^2 D_\alpha \langle v_1 v_2 \rangle D^\beta}{\langle v_1 v_2 \rangle} \right) - \left( \frac{\bar{D}^2 D_\alpha \langle v_1 v_2 \rangle D^\beta}{\langle v_1 v_2 \rangle} \right) \left( \frac{D^\alpha \langle v_1 v_2 \rangle D^\beta}{\langle v_1 v_2 \rangle} \right) + \left( \frac{\bar{D}^2 D_\alpha \langle v_1 v_2 \rangle D^\beta}{\langle v_1 v_2 \rangle} \right) + \left( \frac{D^\alpha \langle v_1 v_2 \rangle \bar{D}^2 D_\beta}{\langle v_1 v_2 \rangle} \right)
\end{align*}
$$

(85)
Each row in the individual term in Eq. (85) corresponds to a propagator of vector superfield with covariant derivatives acting from the left and from the right.

Next, we are using the $D$–algebra rules $[3,31,32]$, which allow one to flip the covariant derivative factors from left to right and back on a given line and from one line to another at a given vertex. After a simple massaging, all the terms in Eq. (85) are reduced to a standard form

$$
\begin{pmatrix}
\bar{D}^2 D^2 \langle v_1 v_2 \rangle \\
\bar{D}^2 D^2 \langle v_1 v_2 \rangle \\
\langle v_1 v_2 \rangle 
\end{pmatrix}
$$

(86)

The analytic expression corresponding to Eq. (86) is the same as for the Abelian graph in Fig. 1b. Collecting accurately all the coefficients, we obtain the net coefficient $-3$ compared to the Abelian case, as was announced.$^{15}$

It is curious to observe that in 1$d$ theory the metric has the form

$$
h(A) = 1 - \frac{3g^2}{2|A|^3} + \frac{9g^4}{4|A|^6} + \ldots ,
$$

(87)

which coincides with the expansion of

$$
\frac{1}{1 + \frac{3g^2}{2|A|^3}}.
$$

(88)

We do not find reasons to believe, however, that this reproduces correctly also the higher order corrections in the metric.

In $\mathcal{N} = 2$ non-Abelian theory, we should add to the graph in Fig. 9 also the graphs involving the loop of the adjoint chiral multiplet $\Upsilon^a$. Again, the structure of the integral is exactly the same as for the graph Fig. 1b, but it involves an extra color factor 3. This exactly cancels the contribution of the graph in Fig. 9, as expected.$^{15}$

---

$^{15}$The result is not new, of course. It was obtained in [31] by direct calculation of the of the effective action in external gauge background. Our calculation is much simpler, however. We do not calculate directly the effective action, but the function $K$ such that $\mathcal{L}_{\text{eff}} = \int d^4 \theta K$. This allowed us to keep the background very simple - a constant gauge potential. The effective action in such a background vanishes, but the function $K$ does not.
Going back to $\mathcal{N} = 1$ theory, note that the coefficient $-3$ is universal and appears in any dimension including dimension 4. For $d = 4$, we do not have moduli and, to determine the one–particle irreducible effective action in the external background field at scale $\mu$, we should evaluate the graphs integrating over momenta $p > \mu$. To determine the effective charge (the coefficient of $\text{Tr}\{F_{\mu\nu}^2\}$), it suffices to restrict oneself to an Abelian background. After this, we can repeat the above reasoning and reduce the task of calculating the non–Abelian graph in Fig. 9 to that for the graph in Fig. 1b. The factor $-3$ is thereby reproduced. The effective charge is

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} - \frac{3}{4\pi^2} \ln \frac{\Lambda_{\text{UV}}}{\mu} - \frac{3g_0^2}{16\pi^4} \ln \frac{\Lambda_{\text{UV}}}{\mu} + \ldots .$$

This implies

$$\beta(\alpha_s) = -\frac{d}{d\ln \mu} \alpha_s(\mu) = -\frac{3\alpha_s^2}{\pi} \left(1 + \frac{\alpha_s}{\pi} + \ldots \right),$$

what coincides, of course, with the expansion of the exact $\beta$ function [1]

$$\beta(\alpha_s) = -\frac{3c_V}{2\pi} \frac{\alpha_s^2}{1 - \frac{c_V}{2\pi} \alpha_s}.$$  

We can give now an interpretation of this result repeating our discussion of Abelian theory in the previous section. An actual calculation of the effective action for $d = 4$ in the region $\mu \gg m$ (obviously, non–Abelian theory is massless) requires expansion of the integrand over $\Gamma_\mu$ and subtracting from $\int (\text{total derivative}) = 0$ the infrared contribution. This amounts to cutting certain lines and expresses the result via polarization operators of the corresponding superfields in lower orders. At the two–loop level and working in the Feynman gauge, we observed that the graphs with ghosts cancel, which implies that the 2–loop $\beta$ function is expressed via the 1–loop $Z$-factor of the vector superfield,

$$\frac{1}{g_{\text{two-loops}}^2(\mu)} = \frac{1}{g_0^2} - \frac{3}{4\pi^2} \ln \frac{\Lambda_{\text{UV}}}{\mu} + \frac{1}{4\pi^2} \ln Z_{\text{one-loop}}^v .$$

Comparing this with (89–91), we see that $Z_{\text{one-loop}}^v$ just coincides with the one–loop charge renormalization. Indeed, this was the result of the explicit calculation of Refs. [32]. For sure, it is specific for SYM in the gauge chosen. In an ordinary gauge theory, renormalization factor of the gluon propagator has, generically, nothing to do with the renormalization of the effective charge.

Our guess is that the higher loop diagrams have a similar behavior under condition that one works in the nonlocal gauge of Ref. [26] (a refinement of the Feynman gauge – see the discussion at the end of Sect. 2.5.2), which kills infrared singular contributions in the propagators. Namely, in any order (i) ghost loops cancel out and (ii) renormalization factor of the vector propagator coincides with the effective charge renormalization. This expresses the $n$-th contribution to the $\beta$ function via the $n-1$-th one in the way prescribed by Eq. (91). In terms of operator Wilsonian action, the second and higher loops appear as its matrix elements.
5 Conclusions

In this paper, we have observed that the corrections to the effective Lagrangian of supersymmetric gauge theories in different dimensions obtained from the 4d theory by dimensional reduction procedure can be expressed in a closed universal form in the framework of the supergraph background field formalism, see Eqs.(39,50). For $\mathcal{N}=2$ theories, all corrections beyond first loop vanish. As was discussed in details in Sect.3, the universal reason for that is the requirement of harmonicity (a generalization of holomorphy requirement for 4d theories) following from extended supersymmetry. One–loop corrections are always harmonic, even in $\mathcal{N}=1$ theories when mass parameters are considered as extra moduli, but higher loop corrections are not and so they vanish for $\mathcal{N}=2$.

Another methodic point where we tried to shed some more light refers to four dimensions and is the origin of the exact relation (4) expressing higher loop corrections to the effective charge in supersymmetric SQED via $Z$–factor of charged matter fields. A general proof of it was discussed in the Introduction and is well known. It is instructive, however, to reproduce this result by direct calculation of Feynman graphs. We did it in Sect.2.5. It turns out that the contribution of an arbitrary multiloop graph depends on a kinematical region where one of the charged field lines goes on shell, and the result depends on a subgraph describing a contribution to the charged polarization operator.

In Sect.4, we perform a similar calculation for non-Abelian supersymmetric pure gauge theory. In that case, the contribution of a diagram describing an $n$-loop correction to the effective charge is expressed via subgraphs describing $Z$–factor of the gauge field (the only charged physical field in the theory). In other words, a recurrent formula exists expressing $n$–loop contribution to the $\beta$ function via a $n−1$-th one. This leads to the known result (91). We showed it explicitly at the two–loop level. It would be interesting to extend this analysis to higher loops.

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