ROUGH CAT(0) SPACES

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ABSTRACT. We investigate various notions of rough CAT(0). These conditions define classes of spaces that strictly include the union of all Gromov hyperbolic length spaces and all CAT(0) spaces.

AMS classification: Primary 51M05, 51M10. Secondary: 54B10, 51F99.

KEYWORDS: Gromov hyperbolic spaces, CAT(0) spaces, rough CAT(0) spaces.

1. Introduction

Gromov hyperbolic spaces and CAT(0) spaces have been intensively studied, in particular with regard to their boundary theories, which display many common features as for instance the presence of canonical boundary topologies. It is thus natural to ask whether there is a ‘unified theory’ including Gromov hyperbolic spaces, CAT(0) spaces, and more, together with as much common boundary theory as possible. In this paper we discuss various possible variants of such a ‘unified theory’ of so-called rough CAT(0) spaces, also taking into consideration some existing weak notions of nonpositive curvature. We first investigate properties of the interior of such spaces, such as the property of having (roughly) unique geodesics, and then produce non-trivial examples of rough CAT(0) spaces. In a sequel of this paper [8], we investigate the boundary theory within the ‘unified theory’ of Gromov hyperbolic and CAT(0) spaces introduced here.

There are already a wide variety of conditions related to rough CAT(\(\kappa\)). In geometric group theory alone, there are notions such as semi-hyperbolic groups in the sense of Alonso and Bridson [1], and (bi)automatic [14] and (bi)combable groups (for all of which, see also [4] and the references therein), but these notions are much weaker than CAT(0), as are metric geometry notions such as the Ptolemaic condition ([13], [9]), Busemann convexity [10], \(k\)-convexity [12], and \(L\)-convexity [19]. Rough CAT(0) is closer to CAT(0) than are any of these conditions, allowing us to prove for rough CAT(0) spaces (here and in [8]) analogues of more parts of the CAT(0) theory than can be proved for any of these other notions. Another related notion is the CAT(-1,\(\varepsilon\)) notion of Gromov ([16], [12]) which implies Gromov hyperbolicity: this is very closely related to rough CAT(\(\kappa\)) for \(\kappa < 0\), a notion we briefly consider and show to be equivalent to Gromov hyperbolicity.

Recall that in the context of geodesic metric spaces, \(\delta\)-hyperbolic spaces, \(\delta \geq 0\), are spaces with the property that for every geodesic triangle, each side of the triangle is contained in a \(\delta\)-neighborhood of the union of the other two sides. On the other hand, CAT(0) spaces are geodesic spaces with metric \(d\) having the property that for any two points \(u\) and \(v\) on a geodesic triangle the comparison points \(\bar{u}\) and \(\bar{v}\) in some Euclidean comparison triangle satisfy \(d(u, v) \leq |\bar{u} - \bar{v}|\). It is thus natural to introduce some amount of ‘additive fudge’ to this comparison property in order to obtain the notion of a rough CAT(0) space.
We work in length spaces and thus replace geodesic triangles and segments by $h$-short triangles made of $h$-short segments, which were introduced by Väisälä [21] in the context of Gromov hyperbolicity: a $h$-short segment, $h \geq 0$, is a path whose length is larger by at most $h$ than the distance between its endpoints. Comparison triangles can be defined using the distances between vertices, and one could then attempt to define a rough CAT(0) condition by introducing a uniform additive fudge to the CAT(0) condition. There is, however, a problem with choosing a fixed $h$, since then even the Euclidean plane would not be rough CAT(0): $h$-short segments are not forced to remain a uniformly bounded distance apart when the distance between their common endpoints increases; see Example 3.3. Thus, $h$ must depend on how far apart are the vertices of a $h$-short triangle.

Since the definition can be formulated in this generality, we introduce rough CAT($\kappa$) spaces with $-\infty \leq \kappa \leq 0$; the case $\kappa > 0$ is trivial and we discard it. We write rCAT($\kappa$) as an abbreviation of “rough CAT($\kappa$)” . We define a notion of rCAT($\kappa$) spaces with an explicit upper bound on $h$ which, although useful for many purposes, seems a little contrived. We therefore also define a variant condition rCAT($\kappa ; \ast$), where the positive upper bound on $h$ is an arbitrary positive function of the vertices of the triangle. This variant is aesthetically more pleasing, but turns out to be equivalent to the original notion of rCAT($\kappa$), a fact that will prove to be quite useful in Section 5.

We also define weak and very weak rCAT($\kappa$) conditions. The weak rCAT($\kappa$) condition, which is equivalent to the full strength CAT($\kappa$) condition at least when $k < 0$, is equivalent to a certain 4-point subembedding condition which makes it clear that it is stable under many limiting processes. The very weak rCAT(0) condition will be seen to be equivalent to the bolicity condition of Kasparov and Skandalis [17], [18] that was introduced in the context of their work on the Baum-Connes and Novikov Conjectures.

Some of the results mentioned above are established in Section 3 and the remaining ones are proven in Section 4 where, motivated by the fact that CAT(0) spaces are uniquely geodesic, we explore a rough unique geodesic property for (weak) rough CAT(0) spaces; see Theorem 4.2. We also prove in Section 4 that every CAT(0) space is $(2 + \sqrt{3})$-rCAT(0).

Knowing that the class of rCAT(0) spaces includes both Gromov hyperbolic length spaces and CAT(0) spaces, it is natural to ask whether there are rCAT(0) spaces that are neither CAT(0) nor Gromov hyperbolic. In Section 5 we give two constructions (products and gluing) for getting new rCAT(0) spaces from old ones, which easily produce such examples.

In Theorem 5.1 we show that the $l^2$-product of rough CAT(0) spaces is also rough CAT(0). A rough CAT(0) space that is neither CAT(0) nor Gromov hyperbolic is thus obtained by taking the $l^2$-product of a Gromov hyperbolic space that is not CAT(0) and a CAT(0) space that is not Gromov hyperbolic (e.g. the $l^2$-product of the unit circle and the Euclidean plane).

Theorem 5.5 shows that gluing rough CAT(0) spaces along bounded isometric subspaces also gives rough CAT(0) spaces, but Example 5.10 shows that this mechanism breaks down as soon as we ask for unbounded gluing sets, even if they are convex. Finally, Proposition 5.11 shows that normed vector spaces do not produce interesting examples, since they must be CAT(0) if they are rough CAT(0).
We wish to thank the referee for carefully reading the paper and for spotting an error in an earlier version of Example 5.10.

2. Preliminaries

Let \((X, d)\) be a metric space. We shall not distinguish notationally between paths \(\gamma : I \to X, I \subset \mathbb{R}\), and their images \(\gamma(I)\). Suppose \((X, d)\) is rectifiably connected. We define the *intrinsic metric associated with* \(X, d\) by

\[
l(x, y) := \inf\{\text{len}(\gamma) : \gamma \text{ is a path in } X \text{ containing } x, y\}.
\]

\((X, d)\) is a *length space* if \(l = d\). A path \(\gamma\) of length \(d(x, y)\) joining \(x, y \in X\) is called a *geodesic segment*, and is often denoted \([x, y]\). \((X, d)\) is a *geodesic space* if all pairs of points can be joined by geodesic segments, that is, the above infimum is always attained.

**Definition 2.1.** A \(h\)-short segment, \(h \geq 0\), in the length space \((X, d)\) is a path \(\gamma : [0, L] \to X, L \geq 0\), satisfying

\[
\text{len}(\gamma) \geq d(\gamma(0), \gamma(L)) \geq \text{len}(\gamma) - h.
\]

We denote \(h\)-short segments connecting points \(x, y \in X\) by \([x, y]_h\). It is convenient to use \([x, y]_h\) also for the image of this path, so instead of writing \(z = \gamma(t)\) for some \(0 \leq t \leq L\), we often write \(z \in [x, y]_h\). Given such a path \(\gamma\) and point \(z = \gamma(t)\), we denote by \([x, z]_h\) and \([z, y]_h\) respectively the subpaths \(\gamma|_{[0, t]}\) and \(\gamma|_{[t, L]}\), respectively; note that both of these are \(h\)-short segments. We sometimes write \(\gamma[x, z]_h\) and \(\gamma[z, y]_h\) in place of \([x, z]_h\) and \([z, y]_h\) if we need to specify the short path (or geodesic) of which we are taking a subpath.

The above notation requires further explanation because of its ambiguity: given points \(x, y\) in a length space \(X\), there are always many short segments \([x, y]_h\) for each \(h > 0\), so the notation \([x, y]_h\) involves a choice. When we use this notation in any part of this paper (by a *part*, we mean a definition or a statement or proof of a result), the choice of such a path does not affect the truth of the underlying statements. However, all subsequent uses of \([x, y]_h\) in the same part of the paper refer to the same choice of short segment, and subsequent uses of \([x, z]_h\) and \([z, y]_h\) for \(z \in [x, y]_h\) refer to subpaths of this choice of \([x, y]_h\). Even once we fix \(\gamma = [x, y]_h : [0, L] \to X\), the definitions of such subpaths \([x, z]_h\) and \([z, y]_h\) may require a choice of \(t \in [0, L]\) for which \(z = \gamma(t)\) (since \([x, y]_h\) might not be an arc). The first use of \([x, z]_h\) or \([z, y]_h\) in any part of the paper involves such a choice, and all subsequent uses of either \([x, z]_h\) or \([z, y]_h\) in the same part is consistent with this choice of \(t\).

Note that a 0-short segment is a geodesic segment; in this case, we simply write \([x, y]_h\) instead of \([x, y]_h\), and we also write \((x, y)\) for the subpath of \([x, y]\) with endpoints removed. Geodesic segments are used in this paper only in the context of the model spaces \(M^2_{\kappa}\).

**Remark 2.2.** The fact that \((X, d)\) is assumed to be a length space ensures that for any \(x, y \in X\) and \(h > 0\), there exists an \(h\)-short segment \([x, y]_h\).

Given a number \(\kappa \in \mathbb{R}\), the metric model space \(M^2_{\kappa}\) is defined as follows. \(M^2_0\) is the Euclidean plane, \(M^2_{\kappa}\), \(\kappa > 0\), is obtained from the sphere by multiplying the metric with \(1/\sqrt{\kappa}\), and \(M^2_{\kappa}\), \(\kappa < 0\), is obtained from the hyperbolic plane by multiplying the metric with \(1/\sqrt{-\kappa}\). For more details we refer for instance to [3] Chapter 1.2.
When $\kappa = -\infty$, $M_\kappa^2$ is the union of the real and imaginary axes of $\mathbb{R}^2$ with the length metric attached. This is a much smaller space than what $M_\kappa^2$ would be if it were defined as a cone at infinity of the space $M_\kappa^2$ for $\kappa \in (-\infty, 0)$. We are, however, only interested in embeddings and subembeddings of three or four points in our model space, and for these our simple definition of $M_\kappa^2$ suffices.

Since only the case $\kappa = 0$ will be considered for the bulk of this paper, the distance between $a, b \in M_\kappa^2$ is denoted by $|a - b|$, no matter what value $\kappa$ has. For $\kappa > 0$, let $D_\kappa$ denote the diameter of $M_\kappa^2$, for $-\infty \leq \kappa \leq 0$, set $D_\kappa$ to be infinity.

The following result is referred to as Alexandrov’s lemma and will be instrumental for the considerations in Section 3.

**Lemma 2.3** (Alexandrov’s lemma). Let $\kappa \in \mathbb{R}$ and consider distinct points $A, B, B', C \in M_\kappa^2$: if $\kappa > 0$, we assume that $|B - C| + |C - B'| + |B - A| + |A - B'| < 2D_\kappa$. Suppose that $B$ and $B'$ lie on opposite sides of the line $AC$. (Note that the triangle inequality and the assumption above imply that $|B - B'| < D_\kappa$.) Consider geodesic triangles $T := (A, B, C)$ and $T' := (A, B', C)$. Let $\alpha, \beta, \gamma$ (resp. $\alpha', \beta', \gamma'$) be the vertex angles of $T$ (resp. $T'$) at $A, B, C$ (resp. $A, B', C$).

Suppose that $\gamma + \gamma' \geq \pi$. Then

$$|B - C| + |C - B'| \leq |B - A| + |A - B'|.$$ 

Let $\overline{T} \subset M_\kappa^2$ be a geodesic triangle with vertices $\overline{A}, \overline{B}, \overline{B}'$ such that $|\overline{A} - \overline{B}| = |A - B|$, $|\overline{A} - \overline{B}'| = |A - B'|$, and $|\overline{B} - \overline{B}'| = |B - C| + |C - B'| < D_\kappa$. Let $\overline{C}$ be the point in $[\overline{B}, \overline{B}']$ with $|\overline{B} - \overline{C}| = |B - C|$. Let $\overline{\alpha}, \overline{\beta}, \overline{\beta}'$ be the vertex angles of $\overline{T}$ at vertices $\overline{A}, \overline{B}, \overline{B}'$. Then

$$\overline{\alpha} \geq \alpha + \alpha', \quad \overline{\beta} \geq \beta, \quad \overline{\beta}' \geq \beta', \quad |\overline{A} - \overline{C}| \geq |A - C|.$$ 

Moreover, an equality in any of these implies the equality in the others, and occurs if and only if $\gamma + \gamma' = \pi$.

A geodesic triangle $T(x, y, z)$ in a geodesic space $X$ is a collection of three points $x, y, z \in X$ together with a choice of geodesic segments $[x, y]$, $[x, z]$ and $[y, z]$. Given such a geodesic triangle $T(x, y, z)$, a comparison triangle is a geodesic triangle in $M_\kappa^2$, $T(\overline{x}, \overline{y}, \overline{z})$, such that corresponding distances coincide: $d(x, y) = |\overline{x} - \overline{y}|$, $d(y, z) = |\overline{y} - \overline{z}|$, $d(z, x) = |\overline{z} - \overline{x}|$. A point $\overline{u} \in [\overline{x}, \overline{y}]$ is a comparison point for $u \in [x, y]$ if $d(x, u) = |\overline{x} - \overline{u}|$.

For details on the definition and characterizations of CAT($\kappa$) we refer the reader for instance to [5, Chapter II.1]. Let $X$ be geodesic and $\kappa \in \mathbb{R}$. Let $T(x, y, z)$ be a geodesic triangle in $X$ with perimeter less than $2D_\kappa$, and consider a comparison triangle $T(\overline{x}, \overline{y}, \overline{z})$ for $T(x, y, z)$ in $M_\kappa^2$. We say that $T(x, y, z)$ satisfies the CAT($\kappa$) condition if for any $u, v \in T(x, y, z)$,

$$d(u, v) \leq |u - v|.$$ 

In the case that $\kappa \leq 0$ we call $X$ a CAT($\kappa$) space if all geodesic triangles in $X$ satisfy the CAT($\kappa$) condition. For $\kappa > 0$ we say $X$ is a CAT($\kappa$) space if all geodesic triangles of perimeter less than $2D_\kappa$ satisfy the CAT($\kappa$) condition. Equivalently, $u$ can be assumed to be one of the vertices of the triangle $T(x, y, z)$ and $v$ can be assumed to be on the opposite side. Even more, $v$ can be assumed to be a midpoint of the opposing side.

Another way of characterizing geodesic CAT($\kappa$) spaces, $\kappa \in \mathbb{R}$, is by using the so-called 4-point condition. Suppose $x_i \in X$ and $\overline{x}_i \in M_\kappa^2$ for $0 \leq i \leq 4$, with $x_0 = x_4$. 

and \( \bar{x}_0 = \bar{x}_4 \). We say that \((\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)\) is a subembedding of \((x_1, x_2, x_3, x_4)\) in \( M^2_\kappa \) if
\[
\begin{align*}
d(x_i, x_{i-1}) &= |\bar{x}_i - \bar{x}_{i-1}|, & 1 \leq i \leq 4, \\
d(x_1, x_3) &\leq |\bar{x}_1 - \bar{x}_3| & \text{and } d(x_2, x_4) &\leq |\bar{x}_2 - \bar{x}_4|.
\end{align*}
\]

The metric space \((X, d)\) satisfies the 4-point condition, if every 4-tuple in \( X \) has a subembedding in \( M^2_\kappa \). When \( X \) is geodesic, this turns out to be equivalent to \( X \) being CAT(\( \kappa \)). Also, \( X \) is CAT(0) if and only if the CN inequality of Bruhat and Tits is satisfied, that is, for all \( x, y, z \in X \) and all \( m \in X \) with \( d(y, m) = d(m, z) = d(y, z)/2 \),
\[
d(x, y)^2 + d(x, z)^2 \geq 2d(x, m)^2 + \frac{1}{2}d(y, z)^2.
\]

We refer the reader to [15, 11, 21], or [5 Part III.H] for the theory of Gromov hyperbolic spaces. A metric space \((X, d)\) is \( \delta \)-hyperbolic, \( \delta \geq 0 \), if
\[
\langle x, z; w \rangle \geq \langle x, y; w \rangle \wedge \langle y, z; w \rangle - \delta, \quad x, y, z, w \in X,
\]
where \( \langle x, z; w \rangle \) is the Gromov product defined by
\[
2 \langle x, z; w \rangle = d(x, w) + d(y, w) - d(x, y).
\]

The following is a version of the well-known Tripod Lemma, almost as stated in [21 2.15], the only minor difference being that it is stated for short arcs rather than short paths.

**Lemma 2.4.** Suppose that \( \gamma_1 \) and \( \gamma_2 \) are unit speed \( h \)-short paths from \( o \) to \( x_1 \) and \( x_2 \), respectively, in a \( \delta \)-hyperbolic space. Let \( y_1 = \gamma_1(t) \) and \( y_2 = \gamma_2(t) \) for some \( t \geq 0 \), where \( d(o, y_1) \leq \langle x_1, x_2; o \rangle \). Then \( d(y_1, y_2) \leq 4\delta + 2h \).

A map \( f : (X, d_X) \to (Y, d_Y) \) is an \((A, B)\)-quasi-isometry if there are constants \( A > 0, B \geq 0 \) such that
\[
\frac{1}{A} d_X(x_1, x_2) - B \leq d_Y(f(x_1), f(x_2)) \leq A d_X(x_1, x_2) + B
\]
for any \( x_1, x_2 \in X \), and such that \( \text{dist}(y, f(X)) \leq B, y \in Y \). Here, \( \text{dist}(x, A) := \inf \{d(x, y) : y \in A\} \) is the distance of a point \( x \) from a set \( A \). A \( B \)-rough isometry is a \((1, B)\)-quasi-isometry; \( B \) is called the roughness constant of \( f \).

We write \( A \wedge B \) and \( A \vee B \) for the minimum and maximum, respectively, of two numbers \( A, B \).

### 3. Rough CAT(\( \kappa \)) spaces: basic results

In this section, we define rough CAT(0) spaces and some weaker variants of them, and prove some basic results involving these conditions.

**Definition 3.1.** A \( h \)-short triangle \( T := T_h(x_1, x_2, x_3) \) with vertices \( x_1, x_2, x_3 \in X \) is a collection of \( h \)-short segments \([x_1, x_2]_h\), \([x_2, x_3]_h \) and \([x_3, x_1]_h \). Given such a \( h \)-short triangle \( T \), a comparison triangle will mean a geodesic triangle \( \bar{T} := T(\bar{x}_1, \bar{x}_2, \bar{x}_3) \) in the comparison space \( M^2_\kappa \), \( -\infty \leq \kappa < \infty \) such that \( |\bar{x}_i - \bar{x}_j| = d(x_i, x_j), i, j \in \{1, 2, 3\} \). Furthermore, we say that \( \bar{u} \in \bar{T} \) is a comparison point for \( u \in T \), say \( u \in [x_1, x_2]_h \), if
\[
|\bar{x} - \bar{u}| \leq \text{len}([x, u]_h) \quad \text{and} \quad \bar{u} - \bar{y} \leq \text{len}([u, y]_h).
\]

If \( \kappa = -\infty \), the comparison triangle is called a comparison tripod.
Note that $\bar{u}$ is not uniquely determined by $u$ as in the case of comparison points for triangles in CAT($\kappa$) spaces. Also, it immediately follows from the definition that
\[
|\bar{x} - \bar{u}| \geq \text{len}([x, u]_h) - h \quad \text{and} \quad |\bar{u} - \bar{y}| \geq \text{len}([u, y]_h) - h.
\]

In order to avoid cluttered notation, we do not specify the comparison space in the notation $T(\cdot, \cdot, \cdot)$; the space will always be clear from the context.

**Remark 3.2.** Clearly, we can always find comparison triangles in $M^2_\kappa$ for any $h$-short triangle $T_h(x, y, z)$ in any length space $X$, as long as $d(x, y) + d(y, z) + d(z, x) \leq 2D_\kappa$. In fact this amounts to the well-known fact that triangles in $\text{CAT}(\kappa, \ast)$ spaces rather than geodesic spaces), but for $\kappa = 0$ (the main case that interests us!), the value of $h$ must depend on how far apart are the vertices of the $h$-short triangle. The following example shows that a fixed $h > 0$ “would not work” when $\kappa = 0$ in the sense that even the Euclidean plane would fail to satisfy such a condition.

**Example 3.3.** Let $h > 0$ be fixed, and take $x, y = z$ to be the points given in coordinate form as $(-R, 0)$ and $(R, 0)$, respectively, for some $R > 0$. Let $T := T_h(x, y, z)$ be the short triangle consisting of the pair of line segments from $x$ to $y$ and $y$ to $z$ (the latter being degenerate), plus a path from $z$ to $x$ consisting of the two line segments from $z$ to $u := (0, t)$ and $u$ to $x$, where $t = \sqrt{hR + h^2/4}$; it is clear that $T$ is an $h$-short triangle. The comparison triangle $\overline{T}$ is the (geodesic) planar triangle with the same vertices. If we take $v$ to be the origin, then $t$, and so $d(u, v)$, tends to infinity as $R$ tends to infinity, while the distance between any comparison points in $\overline{T}$ remains bounded.

**Definition 3.4.** Let $-\infty \leq \kappa \leq 0$, $C > 0$, and $h \geq 0$. Suppose $(X, d)$ is a length space and that $T_h(x, y, z)$ is a $h$-short triangle in $X$. We say that $T_h(x, y, z)$ satisfies the $C$-**rough CAT($\kappa$)** condition if given a comparison triangle $T(\bar{x}, \bar{y}, \bar{z})$ in $M^2_\kappa$ associated with $T_h(x, y, z)$, we have
\[
d(u, v) \leq |\bar{u} - \bar{v}| + C,
\]
whenever $u, v$ lie on different sides of $T_h(x, y, z)$ and $\bar{u}, \bar{v} \in T(\bar{x}, \bar{y}, \bar{z})$ are corresponding comparison points.

We define a short function (for a metric space $X$) to be any function $H : X \times X \times X \to (0, \infty)$.

**Definition 3.5.** Let $-\infty \leq \kappa \leq 0$ and $C > 0$. We say that a length space $(X, d)$ is $C$-**rough rCAT($\kappa; \ast$), or simply $C$-**rCAT($\kappa; \ast$)** if there exists a short function $H$ such that the following condition holds: if $T_h(x, y, z)$ is a $h$-short triangle in $X$ for $h := H(x, y, z)$, then $\overline{T}_h(x, y, z)$ satisfies the $C$-**rough CAT($\kappa$)** condition.
It is often useful to use a specific short function $H$, so we say that $X$ is $C$-$r\text{CAT}(\kappa)$ if it is $C$-r\text{CAT}(\kappa; *) with standard short function $H$ defined by

$$H(x, y, z) = \frac{1}{1 \lor d(x, y) \lor d(x, z) \lor d(y, z)}.$$ 

For both the $r\text{CAT}(\kappa; *) or $r\text{CAT}(\kappa)$ conditions, we call the associated parameter $C$ the roughness constant; we omit this parameter if its value is unimportant.

In the Euclidean plane, it follows from Example 3.3 that $h$ can be no larger than some multiple of $1/(d(x, y) \lor d(x, z) \lor d(y, z))$ in order for a $T_h(x, y, z)$ to satisfy a given rough CAT(0) condition. It is also easy to see that any given rough CAT(0) condition requires that $h$ to be bounded, regardless of how close together $x, y, z$ are. These considerations show that our definition of a standard short function gives a short function that is in general as large as it can possibly be, modulo multiplication by a fixed constant, if we want all CAT(0) spaces to be $r\text{CAT}(0)$.

In spite of this justification, our choice of standard short function still seems a little contrived, and the definition of $r\text{CAT}(\kappa; *) spaces seems more natural than that of $r\text{CAT}(\kappa)$ spaces. However, we will show in Corollary 4.4 that these two classes are equivalent, with quantitative dependence of roughness constants.

Trivially if $X$ is $C$-r\text{CAT}(\kappa; *) with a given short function $H$, it is $C$-r\text{CAT}(\kappa; *) with any other pointwise smaller short function $G$, so we may always assume that the short function $H$ is pointwise no larger than the standard short function.

In the above definitions, we could allow $\kappa$ to be positive, as long as we restrict $x, y, z$ so that $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ (as in the definition of CAT($\kappa$) for $\kappa > 0$). However, it is trivial that every length space is $C$-rough CAT($\kappa$) for $C > D_\kappa$, so the class of all rough CAT($\kappa$) spaces is of no interest. For this reason, we insist that $-\infty \leq \kappa \leq 0$ from now on.

It is well known (and easily shown) that the CAT($\kappa$) condition is equivalent to a weaker version of the same definition where the comparison inequality is assumed only when one point is a vertex, and one can even restrict the other point to being a midpoint of a side. This leads us to the following definitions.

**Definition 3.6.** Let $-\infty \leq \kappa \leq 0$ and $C > 0$. A weak $C$-rough CAT($\kappa$) condition is similar to the $C$-rough CAT($\kappa$) condition defined in Definition 3.3 except that it is required to hold only when $v = x$ and $u \in [y, z]_h$. A very weak $C$-rough CAT($\kappa$) condition is also similar to the $C$-rough CAT($\kappa$) condition, except that it is required to hold only when $v = x$ and $u \in [y, z]_h$ is a $h$-midpoint of $[y, z]_h$, that is, if it has the property that the Euclidean midpoint $\bar{u}$ of $[y, z]$ is a comparison point for $u$. Weak and very weak $C$-r\text{CAT}(\kappa; *) spaces, are then defined by making the associated changes to the above definitions of $C$-r\text{CAT}(\kappa; *) spaces, and we can analogously define weak and very weak r\text{CAT}(\kappa) spaces.

By elementary geometry, we see that if $x, y, z$ are points in the Euclidean plane and $u$ lies on the line segment from $y$ to $z$ with $|u - y| = t|z - y|$, then

$$ (d(x, u))^2 \leq (1 - t)(d(x, y))^2 + t(d(x, z))^2 - t(1 - t)(d(y, z))^2. \tag{3.7} $$

It follows that the weak $C$-r\text{CAT}(0) condition can be written in the following more explicit form: if $u = \lambda(s)$, where $\lambda : [0, L] \to X$ is a $h$-short path from $y$ to $z$ parametrized by arclength, $h$ satisfies the usual bound, and we have both $td(y, z) \leq s$ and $(1 - t)d(y, z) \leq L - s$ for some $0 \leq t \leq 1$, then

$$ (d(x, u) - C)^2 \leq (1 - t)(d(x, y))^2 + t(d(x, z))^2 - t(1 - t)(d(y, z))^2. \tag{3.8} $$
The very weak $C$-rCAT(0) condition can be written in a similar form, but with the restriction $t = 1/2$.

The following result summarizes what we can say about the relationships between all these variants of rCAT(\(\kappa\)) spaces.

**Theorem 3.9.**

(a) For $-\infty \leq \kappa < 0$, the classes of rCAT(\(\kappa\)), rCAT(\(\kappa; \ast\)), weak rCAT(\(\kappa\)), and weak rCAT(\(\kappa; \ast\)) spaces all coincide with the class of Gromov hyperbolic spaces, and all containment implications hold with quantitative dependence of parameters.

(b) The classes of rCAT(0) spaces and rCAT(0; \(\ast\)) spaces coincide, again with quantitative control of parameters, and the same is true of weak rCAT(0) and weak rCAT(0; \(\ast\)) spaces.

(c) The class of rCAT(0) spaces is strictly larger than the union of the classes of Gromov hyperbolic and CAT(0) spaces.

Part (a) of this theorem follows from Theorem 3.18 below, while part (b) follows from Corollary 4.4, and an example to prove part (c) was given in the Introduction (see also Section 5).

There are a few other possible relationships between these variant rCAT(\(\kappa\)) spaces whose truth we cannot determine. Specifically we do not know if very weak rCAT(\(\kappa\)) spaces are necessarily weak rCAT(\(\kappa\)) (either for \(k < 0\) or \(k = 0\)), and we do not know if weak rCAT(0) spaces are necessarily rCAT(0). While the class of rCAT(0) spaces is the main focus of our interest in this paper, the (weak) rCAT(\(\kappa\)) characterization of Gromov hyperbolicity in the above theorem may also be of some interest.

We now wish to discuss another connection to existing notions of non-positive curvature. In their work on the Baum-Connes and Novikov Conjectures, Kasparov and Skandalis [17], [18] introduced the class of bolic spaces which, as our class of rCAT(0) spaces, includes both Gromov hyperbolic spaces and CAT(0) spaces. It turns out that in the case of length spaces, bolicity is equivalent to very weak rCAT(0). To see this, we first note that by work of Bucher and Karlsson [7], bolicity is reduced to a condition reminiscent of the CN inequality of Bruhat and Tits ([5, p. 163] and [6]).

**Definition 3.10.** A metric space \(X\) is called \(\delta\)-bolic, for some \(\delta > 0\), if there is a map \(m : X \times X \to X\) with the property that for all \(x, y, z \in X\)

\[
2d(m(x, y), z) \leq \sqrt{2d(x, z)^2 + 2d(y, z)^2 - d(x, y)^2 + 4\delta}.
\]

**Proposition 3.11.** Let \(X\) be a length space. If \(X\) is very weak $C$-rCAT(0), \(C > 0\), then it is \(\delta\)-bolic with \(\delta = C/2\). If \(X\) is \(\delta\)-bolic, \(\delta > 0\), then it is very weak $C$-rCAT(0) with \(C = 4\delta + \sqrt{2}\).

**Proof.** Let \(X\) be a very weak $C$-rCAT(0) space. Let \(x, y, z \in X\) and let \(T_h(x, y, z)\) be some \(h\)-short triangle with comparison triangle \(T(\bar{x}, \bar{y}, \bar{z})\). Let \(m(y, z)\) be some \(h\)-midpoint of \([y, z]_h\). This defines a map \(m : X \times X \to X\). By definition, the Euclidean midpoint \(\bar{m}\) of \([\bar{y}, \bar{z}]\) is a comparison point for \(m(y, z)\). Using the comparison triangle property, the Euclidean parallelogram law and the very weak $C$-rCAT(0) condition,
we obtain
\[ d(x, y)^2 + d(x, z)^2 = |\bar{x} - \bar{y}|^2 + |\bar{x} - \bar{z}|^2 \]
\[ = 2|\bar{x} - \bar{m}|^2 + \frac{1}{2}|\bar{y} - \bar{z}|^2 \]
\[ \geq 2(d(x, m) - C)^2 + \frac{1}{2}d(y, z)^2. \]

Thus \( X \) is \( C/2 \)-bolic.

Let now \( X \) be a \( \delta \)-bolic length space with some \( \delta > 0 \). Let \( T_h(x, y, z) \) be some \( h \)-short triangle and \( T(\bar{x}, \bar{y}, \bar{z}) \) a corresponding comparison triangle in the Euclidean plane. Furthermore, let \( m \) be some \( h \)-midpoint for \([y, z]_h\), that is, \( m \) admits the Euclidean midpoint \( \bar{m} \) of \([\bar{y}, \bar{z}] \) as a comparison point. By definition we thus obtain that \( d(y, m) \leq d(y, z)/2 + h \) and \( d(m, z) \leq d(y, z)/2 + h \). By applying the bolic inequality for \( y, z, m \in X \) and \( m(y, z) \in X \), and the fact that \( h = 1/(1 \vee d(x, y) \vee d(x, z) \vee d(y, z)) \), it follows that
\[
2d(m(y, z), m) \leq \sqrt{2d(y, m)^2 + 2d(m, z)^2 - d(y, z)^2 + 4\delta} \\
\leq \sqrt{4(d(y, z)/2 + h)^2 - d(y, z)^2 + 4\delta} \\
\leq 2\sqrt{2} + 4\delta.
\]

Applying bolicity for \( x, y, z \in X \) and \( m(y, z) \in X \) now yields
\[
2d(x, m) \leq 2d(x, m(y, z)) + 2d(m(y, z), m) \\
\leq \sqrt{2d(x, y)^2 + 2d(x, z)^2 - d(y, z)^2 + 8\delta + 2\sqrt{2}}.
\]

By using the comparison triangle property and the Euclidean parallelogram equality we finally deduce
\[
2(d(x, m) - 4\delta - \sqrt{2})^2 \leq d(x, y)^2 + d(x, z)^2 - \frac{1}{2}d(y, z)^2 \\
= |\bar{x} - \bar{y}|^2 + |\bar{x} - \bar{z}|^2 - \frac{1}{2}|\bar{y} - \bar{z}|^2 \\
= 2|\bar{x} - \bar{m}|^2,
\]
which implies the very weak \( C \)-rCAT(0) inequality with \( C = 4\delta + \sqrt{2} \). \( \square \)

The CAT(\( \kappa \)) condition (for geodesic spaces \( X \)) is normally stated as the \( C = h = 0 \) variant of our rCAT(\( \kappa \)) definition, but it can also be written as a so-called 4-point condition. We prove an rCAT(\( \kappa \)) analogue of this, but first we need a simple lemma.

**Lemma 3.12.** Let \( x, y \) be a pair of points in the Euclidean plane \( \mathbb{R}^2 \), with \( l := |x - y| > 0 \). Fixing \( h > 0 \), and writing \( L := l + h \), let \( \gamma : [0, L] \rightarrow \mathbb{R}^2 \) be a \( h \)-short segment from \( x \) to \( y \), parametrized by arclength. Then there exists a map \( \lambda : [0, L] \rightarrow [x, y] \) such that \( \lambda(0) = x \), \( \lambda(L) = y \), and
\[
|\lambda(t) - x| \leq |\gamma(t) - x|, \quad 0 \leq t \leq L, \\
|\lambda(t) - y| \leq |\gamma(t) - y|, \quad 0 \leq t \leq L, \\
\delta(t) := \text{dist}(\gamma(t), \lambda(t)) \leq M := \frac{1}{2}\sqrt{2lh + h^2}, \quad 0 \leq t \leq L.
\]

In particular if \( h \leq 1/(1 \vee l) \), then \( \delta(t) \leq \sqrt{3}/2 \) for all \( 0 \leq t \leq L \).
Proof. The desired result is invariant under isometries of the plane, so we choose the points \( x = (x_1, 0) \) and \( y = (y_1, 0) \) to be located on the first coordinate axis with \( x_1 < y_1 \). We also write \( \gamma = (\gamma_1, \gamma_2) \) in Euclidean coordinates. Now define \( \lambda(t) = (\lambda_1(t), 0) \), where \( \lambda_1(t) = (\gamma_1(t) \lor x_1) \lor y_1 \). It is clear that \( \lambda(0) = x, \lambda(L) = y \), and that \( \lambda \) satisfies (3.13) and (3.14).

It is clear that to maximize \( \delta(t) := \text{dist}((\gamma(t), \lambda(t))) \), we should pick \( \gamma \) to be the concatenation of two straight line paths, one from \( x \) to \( \gamma(t) \) of length \( t \) and one from \( \gamma(t) \) to \( y \) of length \( L - t \). But then \( \gamma(t) \) traces out an ellipse and it is routine to verify that \( \delta(t) \leq \delta(L/2) = M \).

\[ \square \]

**Definition 3.16.** Let \( (X, d) \) be a metric space, \(-\infty \leq \kappa \leq 0\), and \( C \geq 0 \). Suppose \( x_i \in X \) and \( \bar{x}_i \in M^2_\kappa \) for \( 0 \leq i \leq 4 \), with \( x_0 = x_4 \) and \( \bar{x}_0 = \bar{x}_4 \). We say that \((\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)\) is a C-rough subembedding of \((x_1, x_2, x_3, x_4)\) in \( M^2_\kappa \) if

\[
d(x_i, x_i-1) = |\bar{x}_i - \bar{x}_i-1|, \quad 1 \leq i \leq 4,
\]

\[
d(x_1, x_3) \leq |\bar{x}_1 - \bar{x}_3|, \quad \text{and}
\]

\[
d(x_2, x_4) \leq |\bar{x}_2 - \bar{x}_4| + C.
\]

**Definition 3.17.** A metric space \( (X, d) \) satisfies the \((C, \kappa)\)-rough 4-point condition, where \( C \geq 0 \) and \(-\infty \leq \kappa \leq 0\), if every 4-tuple in \( X \) has a \( C \)-rough subembedding in \( M^2_\kappa \). When \( \kappa = 0 \), we omit it from the notation.

**Theorem 3.18.** For a length space \( (X, d) \) and \(-\infty \leq \kappa \leq 0\), the following conditions are equivalent, with quantitative dependence of parameters:

(a) \( X \) is weak \( C \)-rCAT(\( \kappa \)) for some \( C > 0 \);

(b) \( X \) is weak \( C \)-rCAT(\( \kappa; \ast \)) for some \( C > 0 \);

(c) \( X \) satisfies the \((C', \kappa)\)-rough 4-point condition for some \( C' > 0 \).

Moreover if \(-\infty \leq \kappa < 0\), then these conditions are quantitatively equivalent to \( \delta \)-hyperbolicity and to rCAT(\( \kappa \)).

**Proof.** Trivially (a) implies (b), with the same \( C \). We next prove that (b) implies (c) with \( C' = 2C \). Suppose \( X \) is a weak \( C \)-rCAT(\( \kappa; \ast \)) space with short function \( H \). Let \((x_1, x_2, x_3, x_4)\) be a 4-tuple in \( X \), and let \( h := H(x_1, x_3, x_2) \land H(x_1, x_3, x_4) \). Choose \( h \)-short triangles \( T_2 := T_h(x_1, x_3, x_2) \) and \( T_4 := T_h(x_1, x_3, x_4) \), and comparison triangles \( \mathcal{T}_1 := T(\bar{x}_1, \bar{x}_3, \bar{x}_2) \) and \( \mathcal{T}_2 := T(\bar{x}_1, \bar{x}_3, \bar{x}_4) \), such that \( \mathcal{T}_2 \) and \( \mathcal{T}_4 \) have a common side \([\bar{x}_1, \bar{x}_3]\), and that \( \bar{x}_2 \) and \( \bar{x}_4 \) lie on opposite sides of the line through \( \bar{x}_1 \) and \( \bar{x}_3 \). Let \( z \) be the point of intersection of \([\bar{x}_2, \bar{x}_4]\) and the line through \( \bar{x}_1 \) and \( \bar{x}_3 \).

Suppose first that \( z \in [\bar{x}_1, \bar{x}_3] \); this is always the case if \( \kappa = -\infty \) but it may fail for finite \( \kappa \). Picking a \( h \)-short segment \([x_1, x_3]_h\), let \( z \in [x_1, x_3]_h \) be such that \( d(x_1, z) = |\overline{x}_1 - \overline{z}| \); note that for \( i = 2, 4 \), the points \( \overline{z}, \overline{y}_i \) are comparison points in the triangle \( \mathcal{T}_i \) for \( z, y_i \), respectively. By the triangle and weak rough CAT(\( \kappa \)) inequalities,

\[
d(x_2, x_4) \leq d(x_2, z) + d(z, x_4) \leq |\overline{x}_2 - \overline{z}| + |\overline{z} - \overline{x}_4| + 2C = |\overline{x}_2 - \overline{x}_4| + 2C.
\]

Note that \( d(x_1, x_3) = |\overline{x}_1 - \overline{x}_3| \). Thus \((\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4)\) is a \( C' \)-rough subembedding in \( M^2_{\kappa} \) of \((x_1, x_2, x_3, x_4)\), with \( C' = 2C \).

Alternatively suppose that the segments \([\overline{x}_1, \overline{x}_3]\) and \([\overline{x}_2, \overline{x}_4]\) do not intersect (and so \( \kappa \in \mathbb{R} \)). Let \( Q \) be the quadrilateral consisting of the union of the four geodesic segments \([\overline{x}_1, \overline{x}_2]\), \([\overline{x}_2, \overline{x}_3]\), \([\overline{x}_3, \overline{x}_4]\), and \([\overline{x}_4, \overline{x}_1]\). Then \( M^2_{\kappa} \setminus Q \) has two components: we call the one containing \((\overline{x}_1, \overline{x}_3)\) the inner component, and we define the inner and outer angles at the vertices in \( Q \) in the natural way; note that the inner angle at \( \overline{x}_i, i = 1, 3 \), is the sum of the angles at the same point in the triangles \( \mathcal{T}_2 \) and \( \mathcal{T}_4 \).
If both inner angles were less than \( \pi \), then it would follow by continuity that there exists a point \( \tilde{u} \in [\tilde{x}_1, \tilde{x}_3] \) such that the inner angle at \( \tilde{u} \) for the quadrilateral with vertices \( \tilde{x}_1, \tilde{x}_2, \tilde{u}, \tilde{x}_4 \) is \( \pi \). But then it follows from Alexandrov’s Lemma (Lemma 2.3) that \( \tilde{z} = \tilde{u} \), contradicting the fact that \( \tilde{z} \notin [\tilde{x}_1, \tilde{x}_3] \).

Thus we may assume without loss of generality that the inner angle at \( \tilde{x}_1 \) in \( Q \) is at least \( \pi \). It follows that there exists a geodesic triangle in \( M^2_\kappa \) with vertices \( \tilde{x}_2 = \tilde{x}_3, \tilde{x}_3 = \tilde{x}_3, \) and \( \tilde{x}_3 = \tilde{x}_1, \) together with a point \( \tilde{x}_1 \in [\tilde{x}_2, \tilde{x}_4], \) such that
\[
|\tilde{x}_i - \tilde{x}_{i-1}| = d(x_i, x_{i-1}), 1 \leq i \leq 4,
\]
and
\[
|\tilde{x}_2 - \tilde{x}_1| = |\tilde{x}_2 - \tilde{x}_1| + |\tilde{x}_1 - \tilde{x}_4| = d(x_2, x_1) + d(x_1, x_4) \geq d(x_2, x_4).
\]

By Alexandrov’s lemma, \( |\tilde{x}_1 - \tilde{x}_3| \geq |\tilde{x}_1 - \tilde{x}_3| = d(x_1, x_3) \). Thus \( (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) \) is a 0-rough subembedding in \( M^2_\kappa \) of \( (x_1, x_2, x_3, x_4) \). Putting this case together with the first case, we see that weak \( C_\kappa \)-rCAT\((\kappa; *) \) implies the \((2C, \kappa)\)-rough 4-point condition.

We prove that (c) implies (a) first for \( \kappa = 0 \). Suppose \( x, y, z \) lie in a length space \( X \) that satisfies the \( C' \)-rough 4-point condition, and write \( h := H(x, y, z) \), where \( H \) is the standard short function. Suppose also that \( T = T_h(x, y, z) \) is a \( h \)-short geodesic triangle, \( u \in [y, z]_h \), and \( v := x \).

Let \( (\tilde{y}, \tilde{u}, \tilde{z}, \tilde{x}) \) be a \( C' \)-rough subembedding for \( (x_1, x_2, x_3, x_4) = (y, u, z, x) \). Apply Lemma 3.12, with \( \gamma \) being the piecewise linear path from \( \tilde{y} \) to \( \gamma(t) := \tilde{u} \) to \( \tilde{z} \), to get an associated point \( \lambda(t) := \tilde{u} \) on the line segment \([\tilde{y}, \tilde{z}] \) such that \( |\tilde{u} - \tilde{u}'| \leq \sqrt{3}/2 \). Thus \( d(u, x) \leq |\tilde{u} - \tilde{x}| + C_1 \), where \( C_1 := C' + \sqrt{3}/2 \).

The Euclidean triangle \( T(\tilde{x}, \tilde{y}, \tilde{z}) \) satisfies \( |\tilde{x} - \tilde{y}| = d(x, y) \) and \( |\tilde{x} - \tilde{z}| = d(y, z) \), but it is not necessarily a comparison triangle for \( T \) because we know only that \( |\tilde{z} - \tilde{y}| \geq d(z, y) \). However if we take \( T' := T(x', y', z') \) to be a comparison triangle in \( M^2_0 \) for \( T \), and define \( u' \in [y', z'] \) via the equation
\[
|u' - y'| = |\tilde{u}' - \tilde{y}'| / |\tilde{z}' - \tilde{y}'| \, ,
\]
then it follows from (3.7) that \( |\tilde{u}' - \tilde{x}| \leq |u' - x'| \). Moreover by combining the subembedding properties, (3.13), (3.14), (3.19), and the fact that \( |y' - z'| \leq |\tilde{y} - \tilde{z}| \), we see that
\[
|u' - y'| \leq |\tilde{u}' - \tilde{y}'| \leq |\tilde{u} - \tilde{y}| = |u - y| \leq t \, ,
\]
\[
|u' - z'| \leq |\tilde{u}' - \tilde{z}'| \leq |\tilde{u} - \tilde{z}| = |u - z| \leq L - t \, ,
\]
and so \( u' \) is a comparison point for \( u \). Since \( d(u, x) \leq |\tilde{u} - \tilde{x}| + C_1 \), we have shown the \( C_1 \)-rCAT\((0) \) condition for this choice of data, comparison triangle, and this particular choice of comparison point \( u' \).

A general comparison point \( u'' \) for \( u \) on the side \([y', z'] \) of \( T' \) must satisfy \( |u'' - u'| \leq h \leq 1 \), and so it follows that the \( C' \)-rough 4-point condition implies a weak \( C_\kappa \)-rCAT\((0) \) condition for \( C = C' + 1 + \sqrt{3}/2 \).

It remains to prove that when \( -\infty \leq \kappa < 0 \), (c) implies (a), and both are equivalent to Gromov hyperbolicity. The \( \delta \)-hyperbolicity condition can be written in the form
\[
d(x, z) + d(y, w) \leq (d(x, y) + d(z, w)) \vee (d(x, w) + d(y, z)) + 2\delta \, ,
\]
and this condition holds (with \( \delta = \log 3/\sqrt{-\kappa} \)) for all points \( x, y, z, w \in M^2_\kappa \); see [11 Theorem 1.5.1]. If instead \( x, y, z, w \) lie in a space \( X \) that satisfies the \((C', \kappa)\)-rough 4-point condition, then this condition and the \( \delta \)-hyperbolicity of \( M^2_\kappa \) immediately imply the \((\delta + C')\)-hyperbolicity of \( X \).
Taking \( h \leq 1 \) in Lemma 2.1, it is readily deduced that every \( \delta \)-hyperbolic space is \( C\text{-rCAT}(-\infty) \) for \( C = 4h + 2 \), and so a fortiori \( C\text{-rCAT}(\kappa) \) for every \( \kappa \). We note in particular that the \((C', \kappa)\)-rough 4-point condition implies the \( C\text{-rCAT}(\kappa) \) condition for \( C = 4C' + 2 + 4\log 3/\sqrt{-\kappa} \). \( \square \)

It follows rather easily from Theorem 3.18 that \( \text{CAT}(\kappa) \) spaces are \( \text{rCAT}(\kappa) \) with roughness constant \( C = C(\kappa) \) when \( k < 0 \); alternatively, this follows from the well-known geodesic stability of Gromov hyperbolic spaces (see for instance [5, Part III.H] or [21]). The fact that every \( \text{CAT}(\kappa) \) space is an \( \text{rCAT}(\kappa) \) space is also true when \( \kappa = 0 \); see Corollary 4.6.

**Remark 3.20.** The \( \text{CAT}(0) \) analogue of Theorem 3.18 in [5, II.3.9] assumes that \( X \) is a complete space with approximate midpoints. Such an assumption readily implies that \( X \) is a length space, so we use this latter assumption in our theorem and in Corollary 3.22 below, since we do not wish to restrict the theory of \( \text{rCAT}(0) \) spaces to complete spaces.

It is shown in Bridson and Haefliger [5, II.3.10] that \( \text{CAT}(0) \) is preserved by various limit operations, including pointed Gromov-Hausdorff limits and ultralimits; in particular both generalized tangent space and asymptotic cones of \( \text{CAT}(0) \) spaces are \( \text{CAT}(0) \) spaces (see [5] for the definition of all of these concepts). The trick is to use the 4-point condition and the rather weak limit concept of a 4-point limit. Essentially the same arguments, with the 4-point condition replaced by our rough 4-point condition, give us similar results for \( \text{rCAT}(0) \) spaces which we now state. We omit the proofs since they are obtained by routine adjustments to the proofs of II.3.9 and II.3.10 in [5]. For completeness, we begin with a definition of 4-point limits.

**Definition 3.21.** A metric space \( (X, d) \) is a 4-point limit of a sequence of metric spaces \( (X_n, d_n) \) if for every \( x_1, x_2, x_3, x_4 \in X \), and \( \varepsilon > 0 \), there exist infinitely many integers \( n \) and points \( x_{i,n} \in X_n \), \( 1 \leq i \leq 4 \), such that \( |d(x_i, x_j) - d_n(x_{i,n}, x_{j,n})| < \varepsilon \) for \( 1 \leq i, j \leq 4 \).

**Corollary 3.22.** Suppose the length space \( (X, d) \) is a 4-point limit of the weak \( C\text{-rCAT}(0; *) \) spaces \( (X_n, d_n) \). If \( C_n \leq C \) for all \( n \), then \((X, d)\) is a weak \( C'\text{-rCAT}(0) \) space for some \( C' \) dependent only on \( C \). If \( C_n \to 0 \) and \( X \) is complete, then \((X, d)\) is a \( \text{CAT}(0) \) space.

**Corollary 3.23.** Suppose \((X, d)\) is a length space and \((X_n, d_n)\) form a sequence of \( C\text{-rCAT}(0) \) spaces. Then there exists a constant \( C' \) dependent only on \( C \) such that:

(a) If \((X, d)\) is a (pointed or unpointed) Gromov-Hausdorff limit of \((X_n, d_n)\) then \((X, d)\) is a \( C'\text{-rCAT}(0) \) space.

(b) If \((X, d)\) is an ultralimit of \((X_n, d_n)\), then \((X, d)\) is a \( C'\text{-rCAT}(0) \) space.

(c) If \( X \) is \( \text{rCAT}(0) \), then any asymptotic cone \( \text{Cone}_\omega X := \lim_\omega (X, d/n) \) is a \( \text{CAT}(0) \) space for every non-principal ultrafilter \( \omega \).

In each of the cases above, the existence of an approximate midpoint for arbitrary \( x, y \in X \) (meaning a point \( m \) such that \( d(x, m) \vee d(y, m) \leq \varepsilon + d(x, y)/2 \) for fixed but arbitrary \( \varepsilon > 0 \)) follows easily from the hypotheses, and so \((X, d)\) is a length space if it is complete.

### 4. Rough \( \text{CAT}(0) \) and Roughly Unique Geodesics

In this section, we explore the rough unique geodesic property of (weak) rough \( \text{CAT}(0; *, *) \) spaces. Recall that \( \text{CAT}(0) \) spaces are uniquely geodesic. The \( \text{rCAT}(0) \)
condition for a \( h \)-short triangle \( T(x, y, y) \) readily gives the following rough version of this.

**Observation 4.1.** Let \( x, y \) be a pair of points in a \( C \)-rCAT(0) space \((X, d)\), and let \( h := 1/(1 \lor d(x, y)) \). Let \( \gamma_i : [0, L_i] \to X \), \( i = 1, 2 \) be a pair of \( h \)-short segments from \( x \) to \( y \), parametrized by arclength, with \( L_1 \leq L_2 \). Then \( d(\gamma_1(t), \gamma_2(t)) \leq C \), \( 0 \leq t \leq L_1 \).

The following theorem improves the above observation.

**Theorem 4.2.** Let \( x, y \) be a pair of points in a weak \( C \)-rCAT(0; * ) space \((X, d)\), with \( L := d(x, y) \). For \( i = 1, 2 \), let \( h_i > 0 \) and let \( \gamma_i : [0, L + h_i] \to X \) be a \( h_i \)-short segment from \( x \) to \( y \), parametrized by arclength; we assume that \( h_1 \leq h_2 \). Then

\[
d(\gamma_1(t), \gamma_2(t)) \leq 2C + h_2 + \sqrt{2Lh_1 + h_1^2} + \sqrt{2Lh_2 + h_2^2}, \quad 0 \leq t \leq L + h_1,
\]

In particular, if \( h_2 \leq 1/(1 \lor L) \), then

\[
d(\gamma_1(t), \gamma_2(t)) \leq 2C + 1 + \sqrt{3}, \quad 0 \leq t \leq L + h_1.
\]

**Proof.** The result follows from the triangle inequality if \( t \leq h_2 \). Let us therefore assume that \( h_2 \leq L + h_1 \) and fix \( t \in (0, L + h_1] \). Let \( t' := t \lor L \).

Throughout this proof \( i \) can equal either 1 or 2. We write \( z_i := \gamma_i(t) \) and choose a path \( \gamma_3 : [0, L + h] \to X \) from \( x \) to \( y \), parametrized by arclength, for some \( 0 < h \leq H(x, y, z_1) \land H(x, y, z_2) \). Let \( T^i = T_h(x, y, z_i) \) be a \( h \)-short triangle which includes \( \gamma_3 \) as a side, let \( \overline{T}^i = T(\bar{x}, \bar{y}, \bar{z}_i) \) be the concatenation of the two sides of \( T^i \) other than \([\bar{x}, \bar{y}]\) forms a \( h_i \)-short path, so by weak \( C \)-rCAT(0; * ) and Lemma 3.12

\[
d(z_1, z_2) \leq d(z_1, u_1) + d(u_1, u_2) + d(u_2, z_2) \\
\leq |\bar{z}_1 - \bar{u}_1| + h_2 + |\bar{u}_2 - \bar{z}_2| + 2C \\
\leq 2C + h_2 + \frac{\sqrt{2Lh_1 + h_1^2}}{2} + \frac{\sqrt{2Lh_2 + h_2^2}}{2},
\]

as required. \( \Box \)

**Remark 4.3.** It is clear from the above proof that the upper bound can be improved if \( h_1 \leq h \) (where \( h \) is as in the proof). In this case, we get

\[
d(\gamma_1(t), \gamma_2(t)) \leq C + h_2 + \frac{\sqrt{2Lh_2 + h_2^2}}{2}, \quad 0 \leq t \leq L + h_1,
\]

and if \( h_2 \leq 1/(1 \lor L) \), then

\[
d(\gamma_1(t), \gamma_2(t)) \leq C + 1 + \frac{\sqrt{3}}{2}, \quad 0 \leq t \leq L + h_1.
\]

Using the above theorem and remark, we readily get the first statement of the following corollary. The weak and very weak variants follow by an examination of the proof of Theorem 4.2. Alternatively, the weak rCAT(0) part of this corollary follows from Theorem 3.18 (with the same constant \( C'' \)).
Corollary 4.4. A C-rCAT(0; ∗) space is C′-rCAT(0), for C′ = 3C + 2 + \sqrt{3}. A weak (or very weak) C-rCAT(0; ∗) space is weak (or very weak) C″-rCAT(0), for C″ = 2C + 1 + \sqrt{3}/2.

We now state a variant of Theorem 4.2 for CAT(0) spaces; we omit the very similar (but less technical) proof.

Theorem 4.5. Let x, y be a pair of points in a CAT(0) space (X, d), with L := d(x, y). For i = 1, 2, let h_i ≥ 0 and let γ_i : [0, L + h_i] → X be a h_i-short segment from x to y, parametrized by arclength; we assume that h_1 ≤ h_2. Then
\[
d(γ_1(t), γ_2(t)) ≤ h_2 + \frac{\sqrt{2Lh_1 + h_1^2}}{2} + \frac{\sqrt{2Lh_2 + h_2^2}}{2}, \quad 0 ≤ t ≤ L + h_1,
\]
In particular, if h_1 = 0 and h_2 ≤ 1/(1 ∨ L), then
\[
d(γ_1(t), γ_2(t)) ≤ 1 + \frac{\sqrt{3}}{2}, \quad 0 ≤ t ≤ L.
\]

The above theorem has the following easy corollary.

Corollary 4.6. A CAT(0) space is C-rCAT(0) for C = 2 + \sqrt{3}.

We record here a Rough Convexity lemma for rCAT(0) spaces. This is a rough analogue of [5, Proposition II.2.2], and can be proved in a similar way, so we leave its proof as an exercise.

Lemma 4.7. Suppose a_1, a_2, b_1, b_2 are points in a C-rCAT(0) space. Let γ_i : [0, 1] → X be constant speed h_i-short paths parametrized by arclength from a_i to b_i, i = 1, 2, where h_i = 1/(1 ∨ d(a_i, b_i)). Then
\[
d(γ_1(t), γ_2(t)) ≤ (1 − t)d(a_1, a_2) + td(b_1, b_2) + 2C.
\]
If either a_1 = a_2 or b_1 = b_2, then we can replace 2C by C in the above estimate.

Remark 4.8. Note that \(\mathbb{R}^2\) with the Euclidean metric is CAT(0), while \(\mathbb{Z}^2\) with the \(l^1\)-metric is not even very weak rCAT(0). Thus rCAT(0) is not invariant under quasi-isometry. By comparison, we note the well-known facts that Gromov hyperbolicity is invariant under quasi-isometry in the context of geodesic spaces, while the CAT(0) property is only invariant under isometry.

5. Examples

We already know that the class of rCAT(0) spaces include both Gromov hyperbolic (by Theorem 4.13 and the fact that rCAT(κ) implies rCAT(0) for κ < 0) and CAT(0) spaces (by Corollary 4.6). Here we give two constructions (products and gluing) for getting new rCAT(0) spaces from old ones, making it easy to construct rCAT(0) spaces that are neither CAT(0) nor Gromov hyperbolic.

For metric spaces \((X_1, d)\) and \((X_2, d)\), the \(l^2\)-product \((Z, |·|)\) is given by \(X := X_1 \times X_2\) and
\[
d((x_1, x_2), (y_1, y_2)) = \sqrt{(d(x_1, y_1))^2 + (d(x_2, y_2))^2}.
\]
It is well known that \((X, d)\) is a metric space. Note that we are using \(d\) to indicate three different metrics: in all cases, the reader should infer from the context which one is meant. We also use \(\text{len}(γ)\) to indicate length of a path \(γ\) in any one of these spaces.
The product of CAT(0) spaces is CAT(0). A proof follows immediately from the equivalence of CAT(0) with the CN inequality of Bruhat and Tits. Since it is not clear if rCAT(0) is equivalent to a rough version of the CN inequality (that is, bolicity or very weak rCAT(0), as shown in Proposition 3.11), no such easy proof of the rough analogue of this result is available. Nevertheless it is true according to the following theorem. Quantitative dependence of the roughness constant is most neatly stated by Corollary 4.4.

**Theorem 5.1.** If \((X_1, d)\) and \((X_2, d)\) are both \(C\)-rCAT(0; \(\ast\)) spaces, then their \(l^2\)-product \((X, d)\) is a \((\sqrt{2} C)\)-rCAT(0; \(\ast\)) space.

To prove the above theorem, we first need a lemma.

**Lemma 5.2.** Suppose \((X, d)\) is the \(l^2\)-product of two length spaces \((X_1, d)\) and \((X_2, d)\). If \(\gamma = (\gamma_1, \gamma_2): [0, T] \to X_1 \times X_2\) is a path in \(X\), then

\[
\text{len}(\gamma) \geq \sqrt{\text{len}(\gamma_1)^2 + \text{len}(\gamma_2)^2},
\]

with equality if \(\gamma_1\) and \(\gamma_2\) are traversed at the same relative rate, i.e. if

\[
\text{len}(\gamma_1) \text{len}(\gamma_2|_{[0,t]}) = \text{len}(\gamma_2) \text{len}(\gamma_1|_{[0,t]}), \quad 0 < t < T.
\]

**Proof.** The triangle inequality for the Euclidean plane immediately gives the following inequality for non-negative numbers \(a_i, b_i, 1 \leq i \leq n\):

\[
\sum_{i=1}^{n} \sqrt{a_i^2 + b_i^2} \geq \sqrt{\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2}.
\]

By taking \(a_i := d(\gamma_1(t_i), \gamma_1(t_{i-1}))\) and \(b_i := d(\gamma_2(t_i), \gamma_2(t_{i-1}))\) in the above inequality, where the numbers \(0 = t_0 < t_1 \leq \cdots \leq t_n = T\) form a partition of \([0, T]\), we deduce (5.3).

If \(\gamma_1\) and \(\gamma_2\) are traversed at the same relative rate, then the vectors \((a_i, b_i)\) defined in the last paragraph are positive scalar multiples of each other, so we get equality in the planar triangle inequality, which upon taking a supremum over all such partitions gives equality in (5.3).

If the paths are not traversed at the same relative rate then we split \(\gamma\) into two subpaths \(\gamma^i = (\gamma^i_1, \gamma^i_2), i = 1, 2,\) where \(\gamma^1 = \gamma|_{[0,T_1]}, \gamma^2 = \gamma|_{[T_1,T]},\) and \(0 < T_1 < T\) is such that the equation in (5.4) fails for \(t = T_1\). Letting \(a_i = \text{len}(\gamma^i_1)\) and \(b_i = \text{len}(\gamma^i_2)\), it follows that \((a_1, b_1)\) and \((a_2, b_2)\) are not scalar multiples of a single vector, and so

\[
\text{len}(\gamma) = \sum_{i=1}^{2} \text{len}(\gamma^i) \geq \sum_{i=1}^{2} \sqrt{a_i^2 + b_i^2} > \sqrt{\left(\sum_{i=1}^{2} a_i\right)^2 + \left(\sum_{i=1}^{2} b_i\right)^2} = \sqrt{\left(\text{len}(\gamma_1)\right)^2 + \left(\text{len}(\gamma_2)\right)^2}.
\]

We are now ready to prove Theorem 5.1. Note that it follows implicitly from the following proof that the “if” clause for equality in Lemma 5.2 is actually an “if and only if”.
Proof of Theorem 5.1} Suppose $a = (a_1, a_2), b = (b_1, b_2)$ are a pair of points in $X$. Suppose $\gamma_i$ is a rectifiable path from $a_i$ to $b_i$ of length $L_i$, $i = 1, 2$. By reparametrization if necessary, we assume that $\gamma_i$ is of constant speed, and then define $\gamma = (\gamma_1, \gamma_2)$. It follows from Lemma 5.2 that $\text{len}(\gamma) = \sqrt{L_1^2 + L_2^2}$. Since $X$ is a length space, we can choose $\gamma_i$ so that $L_i$ is arbitrarily close to $d(a_i, b_i)$, $i = 1, 2$, and it then follows that $\text{len}(\gamma)$ is arbitrarily close to $d(a, b)$. Thus $X$ is a length space.

Letting $a, b \in X$ be as above, it follows from Lemma 5.2 that if $\gamma = (\gamma_1, \gamma_2)$ is a $h$-short path from $a$ to $b$, then $\gamma_i$ is a $h'$-short path from $a_i$ to $b_i$, $i = 1, 2$, where $h' > 0$ depends only on $d(a_1, b_1), d(a_2, b_2)$, and $h$, with $h' \to 0$ as $h \to 0$ (for fixed $a, b$).

Suppose now that we are given points $x, y, z \in X$, a $h$-short triangle $T := T_h(x, y, z)$, and points $u, v \in X$ on different sides of $T$. By projecting this data onto $X$, $i = 1, 2$, it follows that we get an associated $h'$-short triangle $T_i := T_h(x_i, y_i, z_i)$, and points $u_i, v_i \in X_i$ on different sides of $T_i$; here $h' > 0$ depends only on the distances between pairs of vertices of $T_i$, $i = 1, 2$, and on $h$, with $h' \to 0$ as $h \to 0$ (for fixed $x, y, z$). We assume that the positive number $h$, and hence $h'$, is sufficiently small to guarantee that the $C$-rough CAT(0) condition holds for $T_1$ and $T_2$.

Suppose now that for each side $\gamma = (\gamma_1, \gamma_2)$ of $T$, the projected paths $\gamma_1$ and $\gamma_2$ are traversed at the same relative rate. Let $u$ be on the side $[x,y]$, and let $v$ be on the side $[y,z]$. The $C$-rough CAT(0) condition applied to the projected pairs of points gives $d(u_i, v_i) \leq |\bar{u}_i - \bar{v}_i| + C$, where $\bar{u}_i, \bar{v}_i$ are comparison points for $u_i, v_i$ on the comparison triangle $\bar{T}_i = T(\bar{x}_i, \bar{y}_i, \bar{z}_i)$, $i = 1, 2$. It follows readily that if we define $\bar{T} = T(\bar{x}, \bar{y}, \bar{z})$, where $\bar{x} = (\bar{x}_1, \bar{x}_2)$, etc., if we define $\bar{u}, \bar{v}$ analogously, and if we identify the plane in $\mathbb{R}^4$ containing $\bar{T}$ with $M^2$, then $\bar{T}$ is a comparison triangle for $T$; $\bar{u}, \bar{v}$ are comparison points for $u, v$; and the triangle inequality implies that $d(u, v) \leq |u - v| + \sqrt{2}C$, as required.

In view of the above, the theorem follows readily once we prove the following claim: if we fix a pair of points $x = (x_1, x_2), y = (y_1, y_2) \in X$ with $d(x_i, y_i) > 0$ for $i = 1, 2$, and we pick a $h$-short path $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to X$ from $x$ to $y$, then $\gamma_1$ and $\gamma_2$ are traversed at almost the same relative rate. More precisely, if we define $L(t; \gamma_i) := \text{len}(\gamma_i|[0,t]) / \text{len}(\gamma_i)$, then for all numbers $0 \leq t \leq 1$ and for our fixed pair of points $x, y$, we claim that there exists $\varepsilon$ dependent only on $h$ such that $|L(t; \gamma_1) - L(t; \gamma_2)| < \varepsilon$, and such that $\varepsilon \to 0$ as $h \to 0$.

Let $F$ be the set of all rectifiable paths from $x$ to $y$, let $D := d(x, y)$, let $D_i := d(x_i, y_i)$, and let $D(t, \gamma_i) = d(\gamma_i(t), x_i)$, for $i = 1, 2$. Since $\gamma$ is $h$-short, and so $\gamma_i$ are $h'$-short, with $h' \to 0$ as $h \to 0$, the claim follows if we prove that $(D(t; \gamma_1), D(t; \gamma_2))$ stays uniformly close to the main diagonal of the rectangle $[0, D_1] \times [0, D_2]$.

Given $\gamma = (\gamma_1, \gamma_2) \in F$, we define a path $\lambda_\gamma : [0, 1] \to [0, 1]^2$ by the equation $\lambda_\gamma(t) = (D(t; \gamma_1), D(t; \gamma_2))$. Note that $\lambda_\gamma$ is a path from $(0, 0)$ to $(D_1, D_2)$ and we need to show that this path remains close to the diagonal (with a tolerance tending to 0 as $h \to 0$).

Given $p := (p_1, p_2) \in [0, 1]^2$, let $F_p$ be the set of all paths $\nu \in F$ such that $\lambda_\nu(t) = p$, for some point $t \in [0, 1]$. We denote the associated value of $t$ as $t(p, \nu)$; note that $t(p, \nu)$ may not be unique, but any non-uniqueness corresponds only to a harmless choice of a point in a subinterval of $[0, 1]$ on which $\nu$ remains stationary so, for the sake of having a fixed definition, we choose $t(p, \nu)$ to be the smallest number with the above defining property. Cutting $\nu \in F_p$ into two subpaths $\nu^1, \nu^2$ at the point
its rectangular domain. By compactness it readily follows that the minimum value

Note that the function \( f : [0, D_1] \times [0, D_2] \to \mathbb{R} \) defined above is continuous and it takes on its minimum value \(|(p_1, p_2)| + (D_1 - p_1, D_2 - p_2)| =: f(p)\).

We now consider spaces obtained by gluing a pair of length spaces \((X_i, d_i), i = 1, 2\), along isometric closed subspaces \(S_i \subset X_i, i = 1, 2\) where \(f_i : S \to S_i\) are isometries from some fixed metric space \((S, d_S)\) to \((S_i, d_i|_{S_i})\). This means that we are creating a new space \(X = X_1 \sqcup S X_2\) as the quotient of the disjoint union of \(X_1\) and \(X_2\) under the identification of \(f_1(s)\) with \(f_2(s)\) for each \(s \in S\). The glued metric \(d\) on \(X\) is defined by the equations \(d|_{X_i \times X_i} = d_i, i = 1, 2,\) and

\[
d(x_1, x_2) = \inf_{s \in S} (d_1(x_1, f_1(s)) + d_2(f_2(s), x_2)), \quad x_1 \in X_1, \ x_2 \in X_2.
\]

Then \(d\) is also a length metric [5, I.5.24]. For simplicity of notation, we identify \(X_1, X_2,\) and \(S\) with the naturally associated subspaces of \(X\), so that \(S = X_1 \cap X_2\).

**Theorem 5.5.** If \(X = X_1 \sqcup S X_2\) where \((S, d_S)\) is of diameter \(D < \infty\) and \((X_i, d_i)\) is a \(C\)-rCAT(0) space for \(i = 1, 2\), then \(X\) is a \(C'\)-rCAT(0) space for some \(C' = C'(C, D)\).

A comparable gluing result for CAT(0) spaces \(X_1\) and \(X_2\) requires that \(S_i\) be convex in \(X_i\) (meaning that it contains all geodesics in \(X_i\) between every pair of points in \(S_i\)) and complete for \(i = 1, 2\), but the boundedness of \(S\) is dropped. The conclusion is then that \(X\) is CAT(0); see [3, II.11.1].

Before proving Theorem 5.5, we need some elementary lemmas concerning planar geometry. The first is a “small perturbation” result.

**Lemma 5.6.** Suppose \(T(x, y, z)\) and \(T(x', y', z')\) are triangles in the Euclidean plane, and that two of \(|x - z| - |x' - z'|, |y - z| - |y' - z'|, |x - y| - |x' - y'|\) equals zero, with the third being at most \(h := 1/(1 + (|x - y| \vee |x - z| \vee |y - z|)^2)\). Suppose also that \(u \in [x, z], u' \in [x', z'], v \in [x, y],\) and \(v' \in [x', y']\), with \(|x - u| = |x' - u'|\) and \(|x - v| = |x' - v'|\). Then \(|u' - v| \leq |u - v| + 2\).

**Proof.** We write \(a = |x - z|, a' = |x' - z'|, b = |y - z|, b' = |y' - z'|, c = |x - y|, c' = |x' - y'|, d = |z - v|, d' = |z' - v'|, e = |u - v|,\) and \(e' = |u' - v'|, l = |x - u|\). Since two of the three sidelengths are preserved, we may assume by symmetry between \(y\) and \(z\) that \(c = c'\). Define the numbers \(s, t, t' \in [0, 1]\) by \(t = l/a, t' = l/a'\), and \(s = |x - v|/c\). We assume that \(a \lor a' \geq 1\), since otherwise the result follows trivially from the triangle inequality. Thus

\[
|t - t'| \leq hl/aa' \leq h/(a \lor a') \leq h.
\]
Using (3.7), we get the following four equations, which we use implicitly in the rest of the proof:

\[
\begin{align*}
d^2 &= (1 - s)a^2 + sb^2 - s(1 - s)c^2, \\
(d')^2 &= (1 - s)(a')^2 + s(b')^2 - s(1 - s)c^2, \\
e^2 &= (1 - t)(sc)^2 + td^2 - t(1 - t)a^2, \\
(e')^2 &= (1 - t')(sc)^2 + t'(d')^2 - t'(1 - t')(a')^2.
\end{align*}
\]

Note that \(t(1 - t)a^2 = l(a - l)\) and similarly \(t'(1 - t')(a')^2 = l(a' - l)\). It is readily verified that \(h \leq 1/(1 \lor a' \lor b' \lor c)^2\), and trivially \(h \leq 1/(1 \lor a \lor b \lor c)^2\).

If \(a = a'\) and \(b' \leq b\), then it follows from the above equations that \(d' \leq d\) and \(e' \leq e\), so we are done. If \(a = a'\) and \(b < b' \leq b + h\), we see that \((d')^2 = d^2 + 2shb + sh^2 \leq d^2 + 3\), and hence that \((e') \leq e^2 + 3\). Thus \(e' \leq e + \sqrt{3}\) in this case.

Suppose instead that \(b = b'\) and \(a - h \leq a' \leq a\). Then \((d')^2 \leq d^2\), and so

\[(e')^2 \leq e^2 + (t' - t)d^2 + l(a - a') \leq e^2 + 2.\]

In the last inequality, we used the estimate \((t' - t)d^2 \leq 1\), which in turn follows from the earlier estimate \(|t' - t| \leq h\) and the fact that \(d \leq a \lor b\). We deduce that \(e' \leq e + \sqrt{2}\) in this case.

Lastly, suppose that \(b = b'\) and \(a < a' \leq a + h\). Then \((d')^2 \leq d^2 + (1 - s)(ah + h^2) \leq d^2 + 3\), and as in the previous case

\[(e')^2 \leq e^2 + 3t' + (t - t')(sc)^2 \leq e^2 + 4.\]

Thus \(e' \leq e + 2\) in this case. \(\square\)

We now state a lemma that we call the Zipper Lemma because in the important case \(\delta_x = \delta_y > 0\), we get one triangle from another by “zipping up” two sides (shortening them by the same amount).

**Lemma 5.7.** Suppose \(x, y, z, z', u, u'\) are points in the Euclidean plane and write \(\delta_x := |x - z| - |x - z'|\) and \(\delta_y := |y - z| - |y - z'|\). Suppose also that \(u \in [x, z]\), \(u' \in [x, z']\), and \(|x - u| = |x - u'|\). Then

(a) If \(v \in [x, y]\) and \(|\delta_x| \leq \delta_y\) then \(|u' - v| \leq |u - v|\).

(b) If \(v \in [y, z]\) and \(v' \in [y, z']\) with \(|y - v| = |y - v'|\) and \(\delta_x = \delta_y \geq 0\), then \(|u' - v'| \leq |u - v|\).

**Proof.** By the Cosine Rule applied to the triangles \(T(x, u', v)\) and \(T(x, z', y)\), it is clear that the distance from \(u'\) to \(v \in [x, y]\) decreases as we move \(z'\) directly towards \(y\) while keeping \(x, y\) and \(z\) fixed, since both are associated with the (common) angle at \(x\) in both triangles decreasing. Thus it suffices to prove that the angle at \(x\) in the triangle \(T(x, z', y)\) is smaller than the angle at \(x\) in the triangle \(T(x, z, y)\) in the special cases \(\delta_y = \delta_x > 0\) and \(\delta_y = -\delta_x > 0\).

We first prove (a) for \(\delta_x = \delta_y > 0\). Without loss of generality, we assume that \(x, y\) are given in Cartesian coordinates by \((c, 0)\) and \((-c, 0)\), respectively. Let

\[2a := |z - x| - |z - y| = |z' - x| - |z' - y| ,\]

so that \(a \leq c\). The lemma is clear if either \(a = c\) or \(a = 0\), so we assume that \(0 < a < c\) and write \(b = \sqrt{c^2 - a^2}\) and \(e = c/a\). Thus \(z\) and \(z'\) both lie on one branch
of the hyperbola
\[ \frac{w_1^2}{a^2} - \frac{w_2^2}{b^2} = 1, \]
where \((w_1, w_2)\) are the Cartesian coordinates of a point \(w\) on this hyperbola.

We assume for now that \(z, z'\) lie on the right branch of this hyperbola, i.e. that \(|z - x| < |z - y|\). Let \(r = |z - x|\) and let \(\theta\) be the angle at \(x\) in the triangle \(T(x, y, z)\). Then \(z = (z_1, z_2)\) satisfies the equation
\[ r^2 = (z_1 - ae)^2 + z_2^2 = (z_1 - ae)^2 + (e^2 - 1)(z_1^2 - a^2) = (ez_1 - a)^2, \]
and so \(r = ez_1 - a\), since we are on the right branch of the hyperbola. Also \(z_1 = r \cos(\pi - \theta) + ae = ae - r \cos \theta\), and so \(r = e(ae - r \cos \theta - a)\). Rearranging this equation we get
\[ r = \frac{a(e^2 - 1)}{1 + e \cos \theta}. \]
It is clear from this equation that the angle \(\theta\) decreases as \(r\) decreases, so we are done.

If instead \(|z - x| > |z - y|\), the analysis is similar except that now \(r = a - ez_1\), and so we instead get
\[ r = \frac{a(e^2 - 1)}{-1 + e \cos \theta}, \]
and again it is clear that \(\theta\) decreases as \(r\) decreases.

We now prove (a) for \(\delta_y = -\delta_x > 0\). We could do this in a similar manner to the proof for \(\delta_y = \delta_x\) above, but using an ellipse rather than a hyperbola. However we will instead give a slightly shorter calculus proof. Let \(a := |x - z|\), \(b := |y - z|\), and \(c := |x - y|\), and let \(\theta\) be the angle at \(x\) in the triangle \(T(x, y, z)\). The desired conclusion is obvious in the degenerate cases \(b = a + c\) and \(c = a + b\), and the degenerate case \(a = b + c\) cannot arise by the triangle inequality since \(|x - z'| > a\) and \(|y - z'| < b\). We may therefore assume that we are in the non-degenerate case with \(\sin \theta > 0\). For the rest of this paragraph prime superscripts indicate derivatives with respect to a parameter \(t\). Specifically, holding \(c\) fixed, and considering \(a = a(t)\), \(b = b(t)\), and \(\theta = \theta(t)\) to be functions of \(t\) with \(a'(t) = 1\) and \(b'(t) = -1\), it suffices to show that \(\theta'(t) < 0\) for all \(0 \leq t < \delta_y\). The fact that the triangle is non-degenerate at \(t = 0\) implies that it is non-degenerate for all \(0 \leq t < \delta_y\), and so \(\sin \theta(t) > 0\) on \([0, \delta_y]\). Differentiating the equation \(b^2 = a^2 + c^2 - 2ac \cos \theta\), we get
\[ -b(t) = a(t) - c(t) \cos \theta(t) + a(t)c(t) \sin \theta(t) \theta'(t), \]
and so
\[ \theta'(t) = \frac{-a(t) - b(t) + c(t) \cos \theta(t)}{a(t)c(t) \sin \theta(t)}. \]
The desired inequality \(\theta'(t) < 0\) follows easily for all \(0 \leq t < \delta_y\).

Finally we prove (b). Let \(a := |x - z|\), \(b := |y - z|\), and \(c := |x - y|\) as before, and also let \(p := |z - u|\), \(q := |z - v|\), and \(e := |u - v|\). Without loss of generality, we assume that \(a, b, p, q > 0\), \(p < a\), and \(q < b\). Again we use calculus and reserve prime superscripts for \(t\)-derivatives below. Holding \(c\) fixed and taking \(a'(t) = b'(t) = p'(t) = q'(t) = -1\), with \(\theta(t)\) being the angle at \(z\) for \(T(x, y, z)\), it suffices to show that \(e'(t) < 0\). Differentiating the Cosine Rule for the triangles \(T(x, y, z)\) and \(T(u, v, z)\) with respect to \(t\), we get
\[ 0 = (a(t) + b(t))(\cos \theta(t) - 1) + (a(t)b(t) \sin \theta(t)) \theta'(t), \]
\[ e(t)e'(t) = (p(t) + q(t))(\cos \theta(t) - 1) + (p(t)q(t) \sin \theta(t)) \theta'(t). \]
Combining these equations, we get
\[ \frac{e(t)e'(t)}{p(t)q(t)} = \left( \frac{1}{a(t)} + \frac{1}{b(t)} - \frac{1}{p(t)} - \frac{1}{q(t)} \right) (1 - \cos \theta(t)), \]
and so it is clear that \( e'(t) \leq 0 \), as required.

We now state a useful perturbation of the previous lemma. The proof is easy: for (a), first apply Lemma 5.6 to lengthen \(|z - x|\) by \( h \), and then apply Lemma 5.7 and for (b), apply Lemma 5.6 twice and then Lemma 5.7.

**Lemma 5.8.** Suppose \( x, y, z, z', u, u' \) are points in the Euclidean plane and write \( \delta_x := |x-z| - |x-z'| \) and \( \delta_y := |y-z| - |y-z'| \). Suppose also that \( u \in [x, z], u' \in [x, z'] \), and \(|x - u| = |x - u'| \) and we write and \( h := 1/(1 + (|x - y| \lor |x - z| \lor |y - z|)^2) \). Then

(a) If \( v \in [x, y] \) and \( |\delta_x| \leq \delta_y + h \) then \(|u' - v| \leq |u - v| + 2\).
(b) If \( v \in [y, z] \) and \( v' \in [y, z'] \) with \(|y - v| = |y - v'| \) and \( \delta_x + h_1 = \delta_y + h_2 \geq 0 \) for some \( 0 \leq h_1, h_2 \leq h \), then \(|u' - v'| \leq |u - v| + 4\).

**Proof of Theorem 5.5.** Let \( d \) be the glued metric on \( X \). We first claim that any \( h \)-short path \( \gamma : [0, L] \to X \) for a pair of points \( x, y \in X_1 \) lies within a distance \( D/2 + 2h \) of a \( h \)-short path for this pair in \( X_1 \).

Suppose without loss of generality that \( \gamma \) is parametrized by arclength and not contained in \( X_1 \). The only parts of \( \gamma \) that do not fully lie in \( X_1 \) consist of disjoint subpaths \( \gamma_i, i \in I \), where \( I \) is a countable index set and the endpoints of every \( \gamma_i \) lie in \( S \).

The distance between these endpoints is the same in either \( X_1 \) or \( X_2 \), and \( X_1 \) is a length space, so we can replace these subpaths by subpaths in \( X_1 \) whose combined length is at most the same as the combined length of the \( \gamma_i \) subpaths, as long as least one \( \gamma_i \) is non-geodesic, an assumption that we add for the moment. We therefore get a new \( h \)-short path \( \gamma' : [0, L'] \to X_1 \) from \( x \) to \( y \), parametrized by arclength, with \( L' \leq L \). By the triangle inequality, \( d(\gamma(t), \gamma'(t)) \leq (D + h)/2 + h \) for all \( 0 \leq t \leq L' \). This establishes the claim under the assumption that at least one of the subpaths \( \gamma_i \) is non-geodesic.

The argument when every \( \gamma_i \) is geodesic is similar except that we may not be able to replace them by geodesic subpaths in \( X_1 \). As long as \( L < d(x, y) + h \), we can choose the replacement subpaths so short as to guarantee that the resulting path \( \gamma' : [0, L'] \to X_1 \) from \( x \) to \( y \) is \( h \)-short, is parametrized by arclength, and again satisfies \( d(\gamma(t), \gamma'(t)) \leq (D + h)/2 + h \) for all \( 0 \leq t \leq L \).

The only remaining problem is when \( L = d(x, y) + h \). It follows that the parts of \( \gamma \) other than the \( \gamma_i \) subpaths cannot all be geodesic, so we can take a non-geodesic subpath of \( \gamma \) that is disjoint from every \( \gamma_i \) and has length at most \( h/2 \). We replace this non-geodesic subpath by a shorter subpath that remains within \( X_1 \). We now have a path of length less than \( d(x, y) + h \) and we can proceed as in the last paragraph to construct a \( h \)-short path \( \gamma' : [0, L'] \to X_1 \) from \( x \) to \( y \), parametrized by arclength, satisfying \( d(\gamma(t), \gamma'(t)) \leq (D + h)/2 + h/2 \) for all \( 0 \leq t \leq L' \). This finishes the proof of the claim.

In view of the above claim and Corollary 4.4, it suffices to prove the rCAT(0) condition for all \( h \)-short triangles with given vertices \( x, y, z \), where \( h \leq H \) for some \( H = H(x, y, z) > 0 \), and considering only \( h \)-short sides within \( X_i \) for any pair of vertices that both lie in \( X_i, i = 1, 2 \).
Thus it suffices to prove an $r\text{CAT}(0)$ condition for a $h$-short triangle with vertices $x, y, z$, where $x, y \in X_1$ and $z \in X_2$, and the path in the triangle from $x$ to $y$ is $\gamma_{xy} : [0, L_{xy}] \to X_1$, with similar notation for the other two sides. We assume that $h \leq H$, where $H := 1/3(1 + (d(x, y) \vee d(x, z) \vee d(y, z))^2)$.

We may further assume that both $\gamma^1_{xz} := \gamma_{xz}|_{[0, M_{xz}]}$ and $\gamma^1_{yz} := \gamma_{yz}|_{[0, M_{yz}]}$ lie in $X_1$, and both $\gamma^2_{xz} := \gamma_{xz}|_{[M_{xz}, L_{xz}]}$ and $\gamma^2_{yz} := \gamma_{yz}|_{[M_{yz}, L_{yz}]}$ lie in $X_2$, for some choice of numbers $M_{xz}$ and $M_{yz}$. Of these four subpaths, we call the two with superscript “1” the initial segments of the associated side of the triangle, and the other two the final segments of the associated side. We write $s_x := \gamma^1_{xz}(M_{xz})$ and $s_y := \gamma^1_{yz}(M_{yz})$.

Symmetry reduces the task of verifying the $r\text{CAT}(0)$ condition for points $u, v$ to the following five cases:

(a) $u$ lies on the initial segment of $\gamma_{xz}$, and $v$ lies on $\gamma_{xy}$.
(b) $u$ lies on the final segment of $\gamma_{xz}$, and $v$ lies on $\gamma_{xy}$.
(c) $u, v$ lie on final segments of $\gamma_{xz}$ and $\gamma_{yz}$, respectively.
(d) $u, v$ lie on initial segments of $\gamma_{xz}$ and $\gamma_{yz}$, respectively.
(e) $u$ lies on the final segment of $\gamma_{xz}$, and $v$ lies on the initial segment of $\gamma_{yz}$.

In Case (a), we first apply the Zipper Lemma Lemma 5.8(a), with all data as in that lemma except for the Lemma’s $z'$ and $u'$: we take $z' = s_y$ and the $u'$ is taken to be a point on a $h$-short path $\lambda$ from $x$ to $s_y$ whose distance to $x$ is $d(x, u)$, if such a point exists (which we assume for now). Now $d(u, v) \leq d(u, u') + d(u', v)$, and by the $r\text{CAT}(0)$ condition for the triangle with vertices $x, s_x, s_y$, we see that $d(u, u') \leq D + C$. Combining the $r\text{CAT}(0)$ conditions for the triangles with vertices $x, y, s_y$ with the Zipper Lemma and this estimate for $d(u, u')$, we deduce the desired $r\text{CAT}(0)$ inequality for the pair $u, v$ in the triangle with vertices $x, y, z$.

If there is no point $u'$ on $\lambda$ with $d(u', x) = d(u, x)$, then take $u' = s_y$, and so $d(u', x) < d(u, x)$. As in the last paragraph, we get an $r\text{CAT}(0)$ inequality for the pair $u'', v$, where $u''$ is a point on $\gamma_{xy}$ such that $d(u'', x) = d(u', x)$. But since $d(u', x) \geq d(s_x, x) - D \geq d(u, x) - D - h$,

and so $d(u', u) \leq D + 2h$. Since this quantity is bounded, the $r\text{CAT}(0)$ condition for $u'', v$ implies an $r\text{CAT}(0)$ condition for $u, v$ (with a parameter $C$ that is larger by $2D + 4h$).

We next consider Case (b). First construct a “comparison quadrilateral” $\tilde{Q}$ with vertices $\tilde{x}, \tilde{y}, \tilde{s}_y, \tilde{s}_x$ for the quadrilateral $Q$ with vertices $x, y, s_y, s_x$. Theorem 3.18 ensures that we can do this in a certain sense, but we need less than guaranteed by that: in fact we need only that distances between each of the four pairs of adjacent vertices is preserved (such a “comparison quadrilateral” exists for any quadrilateral in any metric space). We form a new metric space $(G, d_G)$ by gluing a filled Euclidean triangle with sides of length $|\tilde{s}_x - \tilde{s}_y| = d(s_x, s_y), d(y, z) - d(y, s_y), d(x, z) - d(x, s_x)$, to the Euclidean plane along the line segment from $\tilde{s}_x$ to $\tilde{s}_y$. If we can prove a variant $r\text{CAT}(0)$ condition for the triangle with vertices $x, y, z$, and $u, v$ as in Case (b) where we have all the usual inequalities and equations of Definition 3.14 but with the (geodesic) comparison triangle $\tilde{T}$ in $G$ rather than the Euclidean plane, then the usual $\text{CAT}(0)$ condition follows by combining this variant $r\text{CAT}(0)$ condition with the usual $\text{CAT}(0)$ condition for the comparison triangle in $G$; the fact that $G$ is $\text{CAT}(0)$ follows from the $\text{CAT}(0)$ gluing theorem referred to after the statement of Theorem 5.5. Since $u$ is on the final segment of $\gamma_{xz}$ and $v \in X_1$, we see that $d(u, v) = d(u, s) + d(s, v)$ for some $s \in S$, and so $d(u, v)$ is within a distance
2D of \(d(u, s_x) + d(s_x, v)\). Similarly if \(\bar{u}, \bar{v}\) are the comparison points for \(u, v\) in \(\overline{T}\), then 
\[d(\bar{u}, \bar{v})\] is within a distance 2D of \(d(\bar{u}, s_x) + d(bs_x, \bar{v})\). Since \(d(u, s_x)\) and \(d(\bar{u}, s_x)\) differ by at most \(h\), it follows that the desired variant rCAT(0) condition for \(u, v\) follows from the usual rCAT(0) condition for the pair of points \(s_x, v\), as proven in Case (a) (once we increase the parameter \(C\) by 8\(D + 2h\)).

Case (c) follows easily from the fact that \(u, v\) lie on a \(h\)-short triangle in \(X_2\) with vertices \(z, u_0, v_0\), and \(d(u_0, v_0) < D\); we leave the details to the reader.

We next handle Case (d). Suppose first that we can find a point \(w \in X_1\) such that 
\[d(w, s_x) \leq D + h, \quad d(w, s_y) \leq D + h, \quad \text{and} \quad d(z, x) - d(w, x) + h_1 = d(z, y) - d(w, y) + h_2 \geq 0,\]
for some \(0 \leq h_1, h_2 \leq 3h\). We pick \(h\)-short paths \(\gamma_{zw}\) and \(\gamma_{yw}\) from \(x\) to \(w\), and from \(y\) to \(w\), respectively, and associated points \(u'\) on \(\gamma_{zw}\) and \(v'\) on \(\gamma_{yw}\) such that 
\[d(u', x) = d(u, x) \quad \text{and} \quad d(v', y) = d(v, y)\] (as for \(u''\) in Case (a), we let \(u'\) and/or \(v'\) equal \(w\) if one or other of these last equations cannot be satisfied). Applying Lemma 5.8(b) with 3\(h\) playing the role of \(h\) in the lemma, it is clear that the distance apart of the comparison points for \(u, v\) in the comparison triangle 
\[T_1 = T(\bar{x}, \bar{y}, \bar{z})\]
for the triangle with vertices \(x, y, z\) is either larger, or smaller by at most 4, than the distance apart of the comparison points for \(u, v\) in the comparison triangle 
\[T_2 = T(\bar{x}, \bar{y}, w)\]
for the triangle with vertices \(x, y, w\), assuming that we choose the comparison points so that 
\[d(u, x) = |\bar{u} - \bar{x}|, \quad \text{and} \quad d(v, y) = |\bar{v} - \bar{y}|, \quad \text{and} \quad d(v', y) = |\bar{v'} - \bar{y}|\]. Putting the rCAT(0) condition for the pair \(u, u'\) together with the estimates 
\[d(u, u') \leq D + h + C\]
and 
\[d(v, v') \leq D + h + C\],
we get an rCAT(0) condition for \(u, v\), as required.

It remains to find a point \(w\) with the desired properties. Let \(\lambda : [0, L] \to X_1\) be a \(h\)-short path from \(s_x\) to \(s_y\), parametrized by arclength. Let \(w = \lambda(t)\) for some \(t \in [0, L]\). Then 
\[d(w, s_x) \leq D + h \quad \text{and} \quad d(w, s_y) \leq D + h\]
everywhere \(w = \lambda\). Also let 
\[\delta_x := d(x, z) - d(x, w) \quad \text{and} \quad \delta_y := d(y, z) - d(y, w)\].

When \(t = 0\), the \(h\)-shortness of \(\gamma_{xz}\) implies that 
\[L_{xz} - M_{xz} + h \leq \delta_x + 2h \leq L_{xz} - M_{xz} + 2h,\]
and the \(h\)-shortness of \(\gamma_{yz}\) implies that 
\[-D \leq -d(s_y, w) \leq h + d(z, y) - d(z, s_y) - d(s_y, w) \leq \delta_y + h \leq L_{yz} - M_{yz} + h.\]

In particular, \(\delta_x + 2h \geq \delta_y\). Similarly when \(t = L\) we get that \(\delta_y + 2h \geq \delta_x\). It follows that for some \(t \in [0, L]\) we have 
\[\delta_x + h_1 = \delta_y + h_2, \quad \text{for some non-negative numbers} \quad h_1, h_2 \quad \text{with} \quad h_1 + h_2 = 3h.\]
(In fact we get \(\delta_x + h_1 = \delta_y + h_2\) for some non-negative numbers \(h_1, h_2\) satisfying \(h_1 + h_2 \leq 2h\) and \(h_1 h_2 = 0\), but it suits us to increase both numbers so that \(h_1 + h_2 = 3h.\))

Note that 
\[L \leq d(s_x, s_y) + h \leq d(s_x, z) + d(z, s_y) + h \leq L_{xz} - M_{xz} + L_{yz} - M_{yz} + h,\]
and so 
\[d(w, x) + d(w, y) \leq (M_{xz} + t) + (L - t + M_{yz}) \leq L_{xz} + L_{yz} + h \leq d(z, x) + d(z, y) + 3h.\]

It follows that \(\delta_x + \delta_y + 3h \geq 0\), and so \(\delta_x + h_1 = \delta_y + h_2 \geq 0\), as required.

Case (e) follows from Case (d) in the same way as Case (b) follows from Case (a). □

It is often useful to glue an infinite number of spaces together, sometimes along a single point or set, or sometimes at different places along some base space. The
following general gluing theorem says that for either of these types of gluing of C-rCAT(0) spaces along uniformly bounded gluing sets, we get another rCAT(0) space.

**Theorem 5.9.** Suppose we have a collection of C-rCAT(0) spaces $X_i$, $i \in I$, where $I$ is some index set containing 0 as an element. We write $I^* = I \setminus \{0\}$. Suppose further that in each $X_i$, $i \in I^*$, we have a closed subspace $S_i$ that is glued isometrically to a closed subspace $T_i$ of $X_0$. Suppose further that $S_i$ (and $T_i$) is of diameter at most $D < \infty$, $i \in I^*$. Then the resulting space $X$ is a C'-rCAT(0) space for some $C' = C'(C, D)$.

**Sketch of proof.** Using a similar argument to the proof of the claim at the beginning of the proof of Theorem 5.5, we see that for sufficiently small $h$, a $h$-short path between $x \in X_i$ and $j \in X_j$, $i, j \in I$, is within a bounded Hausdorff distance of a $h$-short path path that only passes through $X_i$, $X_j$, and $X_0$. Thus we may restrict ourselves to examining $h$-short triangles whose sides are of this type, and an rCAT(0) condition for any pair of points on such a triangle with vertices in $X_i$, $X_j$, and $X_k$ follows from at most three appeals to Theorem 5.5 (to glue $X_i$, $X_j$, and $X_k$ to $X_0$).

As mentioned earlier, if we glue a pair of CAT(0) spaces along a pair of isometric convex subspaces, we get a CAT(0) space. It is tempting therefore to suspect that if we glue a pair of rCAT(0) spaces along a pair of isometric convex subspaces (or even isometric “roughly convex” subspaces, whatever this should mean), we get an rCAT(0) space. However the following example shows that this is false.

**Example 5.10.** First let $X_1, X_2$ be two disjoint isometric copies of the closed Euclidean upper half-plane $X := \{(x, y) : y \geq 0\}$. We write $I_i : X \to X_i$, $i = 1, 2$, for the natural (isometric) identification maps. We will isometrically glue the $y = 0$ edges of $X_1$ and $X_2$ to the edges given by opposite sides of a “warped ladder” $Y$.

To construct $Y$, we begin with its two “sides” consisting of two disjoint copies of $\mathbb{R}$: for $i = 1, 2$, let $\nu_i : \mathbb{R} \to Y$ be isometric maps to these sides of $Y$. Next we define the **rung** $R_n$, $n \in \mathbb{Z}$, of the ladder $Y$ to be a line segment of length $\exp(-|n|)$, with one endpoint glued to $\nu_1(n)$ and the other to $\nu_2(n)$. These line segments $R_n$ are pairwise disjoint, and disjoint from the lines $\nu_i(\mathbb{R})$ except where glued at their endpoints. We give $Y$ the glued metric $d_Y$. Thus adjacent rungs are always a distance 1 apart but the two sides are warped in the sense that the distance from $\nu_1(n)$ to $\nu_2(n)$ decays exponentially in $|n|$.

We now isometrically glue together $X_1$, $Y$, and $X_2$ along their edges to get the glued space $Z = (X_1 \sqcup \mathbb{R} \sqcup \mathbb{R}) \sqcup \mathbb{R} X_2$. More precisely, for each $x \in \mathbb{R}$ and $i = 1, 2$, we identify $I_i(x, o)$ with $\nu_i(x)$. We denote the glued metric by $d$. As usual, it is convenient to consider $X_1, X_2, Y$ to be subsets of $Z$.

Rather trivially $X_1, X_2$ are CAT(0), and so rCAT(0). Also $Y$ is rCAT(0): probably the easiest way to see this is to note that $Y$ is roughly isometric to $\mathbb{R}$ and so Gromov hyperbolic. Thus $Z$ is obtained by isometrically gluing three rCAT(0) spaces along convex subsets and, if the isometric gluing of two rCAT(0) spaces along a closed convex set were always rCAT(0), then $Z$ would be rCAT(0) (just apply such a result twice).

However we claim that $(Z, d)$ is not rCAT(0). To see this note first that

$$d_n(y) := d(I_i((0, y)), I_i((n, 0))) = \sqrt{y^2 + n^2}$$

is independent of $i$ and is an even function of $n \in \mathbb{Z}$, and the unique geodesic of length $d_n$ in $Z$ is a line segment in $X_i$ between these points.
Clearly $Z$ is proper, and $d_n(\cdot)$ is increasing and unbounded as a function of $|n|$, so we readily deduce that there exists a geodesic segment from $I_1((0, y))$ to $I_2((0, y))$ for all $y > 0$. Moreover
\[ d_n(y) - d_0(y) = \sqrt{y^2 + n^2} - y \to 0 \quad (y \to \infty). \]

Fixing $n \in \mathbb{N}$ and choosing $y$ so large that $d_n(y) - d_0(y) < e^{-n+1} - e^{-n}$, we ensure that one geodesic segment between $I_1((0, y))$ and $I_2((0, y))$ in $Z$ must cross $Y$ along a rung $R_N$ for some $N \geq n$. By symmetry, another geodesic segment between these two points goes via $R_{-N}$. Letting $n \to \infty$, we therefore have a pair of geodesic segments with the same endpoints such that the distance between their midpoints is greater than $2n$, and so can be arbitrarily large. Such a configuration is incompatible with the rCAT(0) condition.

Finally, we show that there are no interesting examples among the class of normed real vector spaces. As is well known, such spaces are CAT(0) if and only if they are inner product spaces [3, II.1.14]. It is straightforward to use the dilation structure of such spaces to show that they must be CAT(0) if they are rCAT(0); we give the details for completeness.

**Proposition 5.11.** Suppose $(V, \| \cdot \|)$ is a normed real vector space with distance $d(x, y) = \|x - y\|$. Then $V$ is rCAT(0) if and only if it is CAT(0).

**Proof.** Suppose $(V, d)$ is $C$-rCAT(0). Being a normed vector space, $V$ is certainly a geodesic space. We wish to prove the CAT(0) condition for a fixed but arbitrary geodesic triangle $T$ with vertices $x, y, z \in V$. The translation invariance of $d$ allows us to assume without loss of generality that $x = 0$. Let $\mathcal{T}$ be a comparison triangle in $M_0^1$ with vertex at 0 corresponding to $x = 0$, let $u, v$ be points on different sides of $T$ and let $\bar{u}, \bar{v}$ be the respective comparison points on $\mathcal{T}$.

We now exploit the dilation invariance of $V$. Given a geodesic $\gamma : [0, L] \to V$ from $a \in V$ to $b \in V$, we get a dilated geodesic $R\gamma : [0, L] \to V$ from $Ra$ to $Rb$ for any given $R > 0$ by defining $(R\gamma)(t) = R\gamma(t)$. If we dilate our geodesic triangle $T$ in this manner, we get a geodesic triangle which we call $RT$, and it is clear that the similarly dilated Euclidean triangle $R\mathcal{T}$ is a comparison triangle for $RT$, and that $R\bar{u}, R\bar{v}$ are respective comparison points for $Ru, Rv \in RT$. Furthermore if $d(u, v) = |\bar{u} - \bar{v}| + \varepsilon$ for some $\varepsilon > 0$, then $d(Ru, Rv) = |R\bar{u} - R\bar{v}| + R\varepsilon$, so by taking $R > C/\varepsilon$ we contradict the rCAT(0) inequality. Thus the rCAT(0) condition can only hold if the CAT(0) condition holds. \[ \square \]

**Remark 5.12.** It follows from the above theorem that we cannot change the $l^2$-product in Theorem 5.1 to an $l^p$-product for any $p \neq 2$, since certainly the $l^p$-product of two Euclidean lines is rCAT(0) only when $p = 2$.

**References**

[1] J.M. Alonso and M.R. Bridson, ‘Semihyperbolic groups’, Proc. London Math. Soc. (3), 70 (1995), 56–114.

[2] D. Burago, Y. Burago and S. Ivanov, ‘A Course in Metric Geometry’, Graduate Studies in Mathematics, 33, AMS, 2001.

[3] A.F. Beardon, The geometry of discrete groups, Springer Verlag, New York, 1983.

[4] M.R. Bridson, ‘Non-positive curvature and complexity for finitely presented groups’ in International Congress of Mathematicians. Vol. II, 961–987, Eur. Math. Soc., Zürich 2006.

[5] M.R. Bridson and A. Haefliger, ‘Metric spaces of non-positive curvature’, Springer Verlag, Berlin, 1999.
[6] F. Bruhat and J. Tits, ‘Groupes réductifs sur un corps local. I. Doneés radicielles valuées’, Inst. Études Sci. Publ. Math. 41 (1972), 5–251.
[7] M. Bucher and A. Karlsson, ‘On the definition of bolic spaces’, Expo. Math. 20 (2002), 269–277.
[8] S. Buckley and K. Falk, ‘The boundary at infinity of a Rough CAT(0) space’, preprint.
[9] S. M. Buckley, K. Falk, D. J. Wraith, ‘Ptolemaic spaces and CAT(0)’, Glasgow J. Math. 51 (2009), 301–314.
[10] H. Busemann, ‘Spaces with non-positive curvature’, Acta Math. 80 (1948), 259–310.
[11] M. Coornaert, T. Delzant, and A. Papadopoulos, Géométrie et théorie des groupes, Lecture Notes in Mathematics 1441, Springer, Berlin, 1990.
[12] T. Delzant and M. Gromov, ‘Courbure mésoscopique et théorie de la toute petite simplification’, J. Topology 1 (2008), 804–836.
[13] T. Foertsch, A. Lytchak, V. Schroeder, ‘Nonpositive curvature and the Ptolemy inequality’, Int. Math. Res. Not. IMRN 2007, no. 22, Art. ID rnm100, 15 pp.
[14] S. M. Gersten and H. Short, ‘Rational subgroups of biautomatic groups’, Ann. of Math. 134 (1991), 125–158.
[15] E. Ghys and P. de la Harpe (Eds.), ‘Sur les groupes hyperboliques d’après Mikhael Gromov’, Progress in Math. 38, Birkhäuser, Boston, 1990.
[16] M. Gromov, ‘Mesoscopic curvature and hyperbolicity’ in Global differential geometry: the mathematical legacy of Alfred Gray, 58–69, Contemp. Math. 288, Amer. Math. Soc., Providence, RI, 2001.
[17] G. Kasparov and G. Skandalis, ‘Groupes “boliques” et conjecture de Novikov’, Comptes Rendus 158 (1994), 815–820.
[18] G. Kasparov and G. Skandalis, ‘Groups acting properly on “bolic” spaces and the Novikov conjecture’, Ann. Math. 158 (2003), 165–206.
[19] S.-I. Ohta, ‘Convexities of metric spaces’, Geom. Dedicata 125 (2007), 225–250.
[20] Y.G. Reshetnyak, ‘On the theory of spaces of curvature not greater than $K$’, Mat. Sb. 52 (1960), 789–798.
[21] J. Väisälä, ‘Gromov hyperbolic spaces’, Expo. Math. 23 (2005), no. 3, 187–231.

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