Dimension free $L^p$-bounds of maximal functions associated to products of Euclidean balls

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Abstract. A few years ago, Bourgain proved that the centered Hardy-Littlewood maximal function for the cube has dimension free $L^p$-bounds for $p > 1$. We extend his result to products of Euclidean balls of different dimensions. In addition, we provide dimension free $L^p$-bounds for the maximal function associated to products of Euclidean spheres for $p > \frac{N}{N-1}$ and $N \geq 3$, where $N - 1$ is the lowest occurring dimension of a single sphere. The aforementioned result is obtained from the latter one by applying the method of rotations from Stein’s pioneering work on the spherical maximal function.

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1 Introduction

For any convex body $B$ and any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, denote the centered Hardy-Littlewood maximal function of $f$ associated to $B$ by

$$M_B f(x) := \frac{1}{|B|} \sup_{t > 0} \int_B |f(x + ty)| \, dy.$$ 

The history of dimension free bounds starts with the celebrated result by Stein [10], who discovered that for $1 < p \leq \infty$, the centered Hardy-Littlewood maximal operator

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associated to the Euclidean ball in \( \mathbb{R}^n \) has an \( L^p \)-bound that does not depend on the dimension \( n \). This result has been obtained by averaging over spheres, and using the \( L^p \)-boundedness of the spherical maximal operator for \( n > 2 \) and \( p > \frac{n}{n-1} \) (see also [11]). One can ask whether this holds for any convex body \( B \) associated to the Euclidean ball in \( \mathbb{R}^n \) been obtained since then. Bourgain [2] proved that the centered maximal operator is \( L^2 \)-bounded independently of \( n \) for any convex, centrally symmetric body in \( \mathbb{R}^n \). To do this, he showed that there are a unique linear map \( A: \mathbb{R}^n \to \mathbb{R}^n \) with \( \det A = 1 \) and a constant \( L = L(B) \) such that for every \( \xi \in S^{n-1} \), we have

\[
\int_{A(B)} |\langle x, \xi \rangle|^2 \, dx = L(B)^2. \tag{1.1}
\]

If \( B = A(B) \), we say that \( B \) is in isotropic position, which can always be assumed since \( \|M_B\|_{p \to p} = \|M_{A(B)}\|_{p \to p} \). The \( L^2 \) result then follows from exploiting the decay of the Fourier transform of \( \chi_B \), also proven in [2]. Precisely, there is a universal constant \( C \) such that for every \( B \) in isotropic position, we have the following estimates.

\[
|\hat{\chi}_B(\xi)| \leq C|L\xi|^{-1}, \quad |1 - \hat{\chi}_B(\xi)| \leq C|L\xi|, \quad |\langle \xi, \nabla \chi_B(\xi) \rangle| \leq C \tag{1.2}
\]

The \( L^2 \) result has been improved to the range \( p > \frac{3}{2} \) in [3], and independently by Carbery [5]. Going further, Müller [8] showed that for \( p > 1 \), we have \( \|M_B\|_{p \to p} \leq C(p, \sigma, Q) \), with some geometric invariants \( \sigma = \sigma(B) \) and \( Q = Q(B) \). For isotropic \( B \), these are defined as

\[
\begin{align*}
\sigma(B)^{-1} &= \max\{\text{Vol}_{n-1}(\{x \in B : \langle x, \xi \rangle = 0\}) : \xi \in S^{n-1}\}, \\
Q(B) &= \max\{\text{Vol}_{n-1}(\pi_\xi(B)) : \xi \in S^{n-1}\},
\end{align*}
\]

where \( \pi_\xi \) is the projection onto the subspace orthogonal to \( \xi \). Müller also showed that if \( B \) is an \( \ell^q \)-ball for some \( 1 \leq q < \infty \), then \( \sigma(B) \) and \( Q(B) \) do not depend on \( n \), bounding the corresponding maximal function independently of the dimension. However, in the case \( q = \infty \), i.e. \( B = [-1/2, 1/2]^n \), we have \( Q(B) = \sqrt{n} \), and the problem remained open until recently. However, in [3], Bourgain succeeded to show that also in the case of the cube, there exists a dimension free bound. A survey of all these results, with attention to further details, has recently been published by Deleaval, Guédon, and Maurey [6]. Following the latest result by Bourgain, the purpose of this work is to prove the following theorem.

**Theorem 1.** Let \( B := B_1 \times \cdots \times B_\ell \) be a direct product of \( \ell \geq 1 \) Euclidean balls \( B_k \) in \( \mathbb{R}^{n_k} \), \( n_k \geq 1 \), and put \( n := n_1 + \cdots + n_\ell \). Assume further that \( 1 < p \leq \infty \). Then

\[
\|M_Bf\|_p \leq C_p\|f\|_p \tag{1.3}
\]

for every \( f \in L^p(\mathbb{R}^n) \), where \( C_p \) depends only on \( p \), but not on \( \ell \) or the dimensions \( n_k \) of the factors.
We will see that, similarly to the case of the cube, the invariant $Q(B)$ from [8] grows like $\sqrt{t}$, so that we cannot make use of Müller’s bounds.

An outline of our approach goes as follows. A naïve way would be to estimate

$$M_B f(x) \leq \frac{1}{|B|} \sup_{t_1, \ldots, t_\ell > 0} \int_{B_t} \cdots \int_{B_1} |f(x + (t_1 y^{(1)}, \ldots, t_\ell y^{(\ell)}))| \, dy^{(1)} \cdots dy^{(\ell)}$$

getting iterated maximal functions, where we write $x = (x^{(1)}, \ldots, x^{(\ell)})$ with $x^{(k)} \in \mathbb{R}^{n_k}$.

Due to Stein’s dimension free bound for the Euclidean ball, we can estimate each iterated maximal function to get

$$\|M_B f\|_p \leq C_p^\ell \|f\|_p$$

(1.4)

for $1 < p < \infty$. Let $B^{(N)}$ be the Euclidean ball in $\mathbb{R}^N$ with Lebesgue measure 1. Since the Fourier transform satisfies $\chi_{B^{(N)}} = O(|\xi|^{-N+1})$ as $|\xi| \to \infty$, we see that the Fourier transform of $\chi_B$ has a decay of at least this rate (with $N = \min n_k$) on certain subspaces of $\mathbb{R}^n$, while the decay is even better elsewhere, behaving in a way very similar to the cube.

Hence, in order to show Theorem 1, we shall make use of some of the central arguments in Bourgain’s approach for the cube to attain a bound that depends on $p$ and max $n_k$, but not on $\ell$. From here on, we aim to combine this with (1.4) to achieve a bound as in Theorem 1. For this, we will provide a similar theorem for spheres. Let $S := S_1 \times \cdots \times S_\ell$ be a product of Euclidean spheres $S_k$ in $\mathbb{R}^{n_k}$. Let $\sigma_S$ be the product of the spherical Lebesgue measures of the $S_k$ and define the maximal operator $M_S$ on $S(\mathbb{R}^n)$ by

$$M_S f(x) := \frac{1}{|S|} \sup_{t > 0} \int_S |f(x + t\omega)| \, d\sigma_S(\omega).$$

Here, $|S|$ denotes the $(n - \ell)$-dimensional volume of $S$.

**Theorem 2.** Let $S$ be as above and $n_k \geq 3$ for each $k$. Put $N = \min_{1 \leq k \leq \ell} n_k$ and assume that $p > \frac{N}{N - 1}$. Then we have

$$\|M_S f\|_p \leq C_p \|f\|_p$$

(1.5)

for every $f \in S(\mathbb{R}^n)$, where $C_p$ only depends on $p$.

As in the well-known case of $\ell = 1$, the lower bound for $p$ in Theorem 2 is optimal.

We will prove Theorem 2 by using Stein’s approach to see that we can increase the dimension of each factor of $S$ without increasing $C_p$. With Stein’s argument, we also get $\|M_B f\|_p \leq \|M_S f\|_p$ if $B$ is the convex hull of $S$, hence Theorem 2 is sufficient for $N$ large enough, depending on a fixed value of $p$. To show Theorem 2, we proceed with applying an interpolation similar as in Carbery’s proof for $p > 3/2$, in which he makes use of the fact that $\langle \xi, \nabla \tilde{\chi}_B(\xi) \rangle$ has bounded $L^2$-multiplier norm for a general convex body $B$. In his final remark, he states that if this derivative has bounded $L^2$-multiplier
norm for a bigger range of \( q \), we would obtain a better bound on \( p \) than \( \frac{3}{2} \), which is the case if we can bound the higher fractional derivatives

\[
\left( \frac{d}{dr} \right)^z \hat{\sigma}_{S^{N-1}} (r \xi)
\]

of the Fourier transforms of spherical measures, where \( \text{Re}(z) = \frac{N+1}{2} \). Bounding these uses several ideas from Müller’s proof, where he proceeded similarly for arbitrarily higher derivatives to bound them in terms of certain geometric invariants. Also, a lot of the calculations will rely on the explicit forms and the decay of \( \hat{z}^* \sigma_{S^{N-1}} \) and \( \hat{z}^* \sigma_{S} \).

Hence, if we fix \( p > 1 \), we have attained a dimension free bound, for \( N \) large enough, by generalizing Stein’s and Carbery’s ideas. The remaining cases are covered by our generalization of Bourgain’s arguments for the cube, achieving a bound only depending on the finitely many remaining \( N \) and thus only on \( p \).

Throughout this paper, we write \( B_N^r \) for the \( N \)-dimensional Euclidean ball of radius \( r \), and \( B^N \) if \( r = 1 \). We put \( r_N := \pi^{-1/2} \Gamma(\frac{N}{2} + 1)^{1/N} \). Similarly, \( S^{N-1}_r \) denotes the \((N-1)\)-dimensional sphere of radius \( r \) in \( \mathbb{R}^N \), with \( S^{N-1} \) being the sphere of radius 1. The Fourier transform of \( f \in L^1(\mathbb{R}^n) \) is written as

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i (x, \xi)} f(x) \, dx,
\]

and it will also be used in the distributional sense.

## 2 Independence of the number of factors

This part is mainly a walkthrough of [4], where we will omit any proof that does not need any further modification.

Let \( B, \ell, \) and \( n \) be as in Theorem 1 and let \( m(\xi) = \hat{\chi}_B(\xi) \). We group the variables by setting

\[
V_k := \left\{ \sum_{j=1}^{k-1} n_j + 1, \ldots, \sum_{j=1}^k n_j \right\}, \quad k \in \{1, \ldots, \ell\}.
\]

The goal of this section is to show the following weaker result.

**Proposition 2.1.** Let \( N := \max_{1 \leq k \leq \ell} n_k \). Then

\[
\| M_B f \|_p \leq C_{p,N} \| f \|_p \tag{2.1}
\]

for every \( f \in L^p(\mathbb{R}^n) \), \( 1 < p \leq \infty \).

It is enough to consider the case \( B = (B^N_{r_N})^\ell \): Since \( \| M_B \|_{p \to p} \) is invariant under linear transformations of \( B \) (as already mentioned in [2]), we can assume that \( |B_k| = 1 \) for every \( k \in \{1, \ldots, \ell\} \) (hence \( B_k = B^N_{r_N} \)). By change of coordinates, we can assume
\[ B = \prod_{j=1}^{N} (B_{r_j}^j)^{\ell_j} \] for certain \( \ell_j \in \mathbb{N} \), where, without loss of generality, we allow that \( \ell_j = 0 \).

Suppose that we already found constants \( C_{p,j} \) independent of \( \ell_j \) such that
\[ \| M_{(B_j^j)^{\ell_j}} f \|_p \leq C_{p,j} \| f \|_p. \]

Then we can argue similarly as in \([1,4]\) to get
\[ \| M_B f \|_p \leq \prod_{j=1}^{\ell} C_{p,j} \| f \|_p. \]

Let \( B = (B_{r_N}^N)^{\ell} \), i.e. \( n = N\ell \). Then \( B \) is in isotropic position, with
\[ L(B)^2 = \int_{B} |\langle x, \xi \rangle|^2 \, dx = \sum_{k=1}^{\ell} \int_{B} |\langle x^{(k)}, \xi^{(k)} \rangle|^2 \, dx \]
\[ + \sum_{k,k'=1}^{\ell} \int_{B} \langle x^{(k)}, \xi^{(k)} \rangle \langle x^{(k')}, \xi^{(k')} \rangle \, dx = 0 \]
\[ = \sum_{k=1}^{\ell} \int_{B_{r_N}^N} |\langle y, \xi^{(k)} \rangle|^2 \, dy = L(B_{r_N}^N)^2 \]
for every \( \xi \in S^{n-1} \), since \( |B_{r_N}^N| = 1 \). Of course, any Euclidean ball itself is in isotropic position due to rotation invariance. We will first show that Müller’s bounds won’t apply to even this special case of our situation.

**Lemma 2.2.** We have
\[ Q(B) = \sqrt{\ell} \cdot \pi^{-(n-1)/2} \frac{\Gamma(N/2 + 1)}{\Gamma(N/2 + 1/2)}. \]

**Proof.** Let \( \xi \in S^{n-1} \). We will estimate \( |\pi_{\xi}(B)| \), the \((n-1)\)-dimensional volume of the orthogonal projection of \( B \) onto \( \xi^\perp \). The geometric arguments from \([1]\) and a limiting argument show that
\[ |\pi_{\xi}(B)| = \int_{\partial B} (\langle n(x), \xi \rangle)_+ \, d\sigma(x), \]
where \( \sigma \) is the Lebesgue surface measure of \( \partial B \) and \( n(x) \) is the corresponding normal vector (well-defined \( \sigma \)-almost everywhere on \( \partial B \)). Write
\[ \partial B = \bigcup_{k=1}^{\ell} H_k, \quad H_k = (B_{r_N}^N)^{k-1} \times S_{r_N}^{N-1} \times (B_{r_N}^N)^{\ell-k-1}. \]
Then the $H_k$ are the pendants to the faces of a cube, and by Fubini and rotation invariance of the spherical measure, we get

$$|\pi_\xi(B)| = \sum_{k=1}^{\ell} \int_{H_k} \langle n(x), \xi \rangle_+ d\sigma(x)$$

$$= \sum_{k=1}^{\ell} \int_{S_{S_N}^{N-1}} \langle \langle \omega, \xi^{(k)} \rangle_+ \rangle d\sigma_{S_{S_N}^{N-1}}(\omega)$$

$$= \sum_{k=1}^{\ell} |\xi^{(k)}| \int_{\omega_1 > 0} \omega_1 d\sigma_{S_{S_N}^{N-1}}(\omega)$$

$$= r_N |B_{r_N}^{N-1}| \sum_{k=1}^{\ell} |\xi^{(k)}|.$$  

We can maximize $|\pi_\xi(B)|$ by choosing $\xi^{(k)} = \ell^{-1/2} e_1 \in \mathbb{R}^N$. Since

$$r_N |B_{r_N}^{N-1}| = r_N |B_{N-1}^{N} - B_{r_N}^{N-1}| = \pi^{-\ell/2} \Gamma\left(\frac{N+1}{2}\right),$$

the lemma follows.

Consider the same decomposition as in Bourgain’s proof. Let $H : \mathbb{R}^n \to \mathbb{R}$ such that $\hat{H}(\xi) = e^{-|\xi|^2}$ and put $\Omega(s) = \chi_B * (H_{2s} - H_{2s+1})$ for $s \geq 1$. Then

$$\chi_B = (\chi_B * H) + \sum_{s=1}^{\infty} \Omega(s).$$

Using only the well-known estimates $|\xi||m(\xi)| < C L(B)^{-1} < C'$ and $|\langle \xi, \nabla m(\xi) \rangle| < C$ for general convex bodies, Lemma 3 from [2] and the exponential decay of $H$ give us (see (1.16) in [3])

$$\|\sup_{t>0} |f * (\Omega(s))_t|\|_2 < C 2^{-s/2} \|f\|_2$$

for $s \geq 1$ and

$$\|\sup_{t>0} |f * (H * \chi_B)_t|\|_2 < C \|f\|_2.$$

For $1 < p < 2$, it suffices to find a bound

$$\|\sup_{t>0} |f * (\chi_B * H_{2s})_t|\|_p \leq C_{p,s} \|f\|_p,$$

$s \in \mathbb{N}$, so that $C_{p,s}$ is suitable for interpolation with the $L^2$-estimates. For this, Bourgain takes the ideas from [5] to conclude that it suffices to find an $L^p$-bound for the operator $T$, defined by

$$\hat{Tf}(\xi) = |\xi|m(\xi)e^{-|\xi|^2} \hat{f}(\xi),$$  

(2.2)
and to estimate
\[
\sup_{|\xi|=1} \int_{\mathbb{R}^n} |\langle x, \xi \rangle|^{\ell} (\chi_B * H_{2-\ell})(x) \, dx < C_k, \quad k \geq 1.
\]  
(2.3)

For (2.3), we can make use of the fact that \( B \) is symmetric in each coordinate, applying Khinchin’s inequality as in [4, p. 279].

To estimate the operator \( T \) in (2.2), Bourgain uses a duality argument as in [8, p. 306] and Stein’s dimension free bound on the Riesz transforms (see [10]), which leaves him with proving Lemma 3 from [4]. In our situation, Proposition 2.1 follows from the following, similar Lemma.

**Lemma 2.3.** Let
\[
N := \max_{1 \leq k \leq \ell} n_k.
\]
For \( R \geq 2 \) and \( j \in \{1, \ldots, n\} \), let \( \mu_j = \partial_j (\chi_B * H_{1/R}) \). Then for every \( 2 \leq p < \infty \), \( 0 < \varepsilon < 1 \), and \( f \in L^p \) we have
\[
\left\| \left( \sum_{j=1}^n |f * \mu_j|^2 \right)^{1/2} \right\|_p \leq C_{p,\varepsilon,N} R^{12N-\varepsilon} \|f\|_p, 
\]
(2.4)
with \( C_{p,\varepsilon,N} \) independent of \( R \) and \( \ell \).

Fix \( 2 \leq p < \infty \), \( R \geq 2 \), and \( 0 < \varepsilon < 1 \). The direct interpolation from [4, p. 280] shows that
\[
\left\| \left( \sum_{j=1}^n |f * \mu_j|^2 \right)^{1/2} \right\|_p \leq C_p R^{1-2/p} \|f\|_p.
\]
(2.5)
We proceed with Bourgain’s Fourier localization. By means of Pisier’s result on contractive semigroups [9, p. 390], we get the following.

**Lemma 2.4.** Let \( \eta = (1 - |x|)_+ \), \( t > 0 \) and let \( T_j : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \) be the convolution by \( \eta_t \) in the \( j \)-th variable. For every \( k \in \{1, \ldots, \ell\} \) let
\[
S_k := \bigcap_{j \in V_k} T_j.
\]
Furthermore, put for every \( k \in \{0, \ldots, \ell\} \)
\[
A_k := \sum_{\substack{\{1, \ldots, \ell\} \ni j \in A \atop |A| = k}} \prod_{j \notin A} S_j \prod_{j \in A} (\text{Id} - S_j).
\]
(2.6)
Then for \( 1 < q < \infty \), \( \|A_k\|_{q \to q} \leq C_q^k \) with \( C_q \) independent of \( k \).

Fix \( t := R^{-\varepsilon} \) and let \( A_k \) be as in (2.6). Then \( A_k \geq 0 \) and \( \sum_{k=0}^\ell A_k = \text{Id} \). For some \( K \geq 1 \) to be chosen later, we decompose \( f \in L^p \) as
\[
f = \sum_{k=0}^K A_k f + g.
\]
To achieve a good $L^2$-estimate on $\left( \sum_{j=1}^{n} |g \ast \mu_j|^2 \right)^{1/2}$, we prove a variant of Lemma 6 in [4], where we write $\xi = (\zeta_1, \ldots, \zeta_\ell)$ with each $\zeta_k \in \mathbb{R}^N$.

**Lemma 2.5.** For every $\delta > 0$ and $k \geq 1$, we have

$$|m(\xi)| \leq C_{k,N} \left( 1 + \sum_{|\zeta_j| \leq R} |\zeta_j|^2 \right)^{-k/2} R^k. \quad (2.7)$$

**Proof.** To adapt the original proof to our setting, we need to show

$$|\hat{\chi}_{B_{2R}^N}(\zeta)| \leq e^{-c|\zeta|^2}$$

for $|\zeta| \leq 1$ and

$$|\hat{\chi}_{B_{2R}^N}(\zeta)| \leq C = e^{-c'}$$

for $|\zeta| > 1$, with $c' > 0$, $C < 1$, $\zeta \in \mathbb{R}^N$. Since

$$\hat{\chi}_{B_{2R}^N}(\zeta) = r_N^{N/2} |\zeta|^{-N/2} J_{N/2}(2\pi r_N |\zeta|)$$

with $J_{\nu}$ being the Bessel function of order $\nu$, we use the well-known series expansion

$$J_{\nu}(x) = \pi^{-1/2} x^{\nu} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k + 1/2)}{\Gamma(k + \nu + 1)} \frac{x^{2k}}{(2k)!} \quad (2.10)$$

to get

$$\hat{\chi}_{B_{2R}^N}(\zeta) = \pi^{N/2} r_N^{N/2} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k + 1/2)}{\Gamma(k + N/2 + 1)} \frac{(2\pi r_N |\zeta|)^{2k}}{(2k)!}$$

$$= \pi^{-1/2} \Gamma(N/2 + 1) \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k + 1/2)}{\Gamma(k + N/2 + 1)} \frac{(2\pi r_N |\zeta|)^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(2\pi r_N |\zeta|)^{2k}}{(2k)!} \prod_{j=0}^{k-1} j + 1/2 \prod_{j=N/2}^{k} j + 1/2 \quad (2.11)$$

Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(t) = \sum_{k=0}^{\infty} (-1)^k \frac{(2\pi r_N t)^{2k}}{(2k)!} \prod_{j=0}^{k-1} j + 1/2 \prod_{j=N/2}^{k} j + 1/2.$$ 

Then we have

$$f(0) = 1, \quad \frac{d}{dt} \left( f(t) - e^{-ct^2} \right) \bigg|_{t=0} = 0,$$

and

$$\frac{d^2}{dt^2} \left( f(t) - e^{-ct^2} \right) \bigg|_{t=0} = -2\pi r_N \frac{1}{N} + 2c \quad (2.12)$$
for each $c > 0$. Hence $f(t) - e^{-ct^2}$ has a local minimum at $t = 0$ for $c > \frac{\pi r_{N \xi}}{N}$, implying that (2.8) holds near 0 with such a choice of $c$. Thus we are left with showing that $|\hat{\chi}_{B_N^N}(\zeta)| < 1$ if $\zeta \neq 0$, which implies $|\hat{\chi}_{B_N^N}(\zeta)| \leq C_\alpha < 1$ on $\mathbb{R}_{>0}$ for each $\alpha > 0$. For that, we use that

$$
\hat{\chi}_{B_N^N}(\zeta) = \frac{\Gamma(\frac{N}{2} + 1)}{\Gamma(\frac{N+1}{2})^{1/2}} \int_{-1}^{1} e^{it \cdot 2\pi r_{N\xi}|\zeta|} (1 - t^2)^{\frac{N-1}{2}} \, dt
$$

$$
= \frac{\Gamma(\frac{N}{2} + 1)}{\Gamma(\frac{N+1}{2})^{1/2}} \cdot 2 \int_{0}^{1} \cos(t \cdot 2\pi r_{N\xi}|\zeta|) (1 - t^2)^{\frac{N-1}{2}} \, dt. \quad (2.13)
$$

Since the integrand in (2.13) is continuous and $|\cos(t \cdot 2\pi r_{N\xi}|\zeta|)| < 1$ for all but finitely many $t \in [0, 1]$ (if $|\zeta| \neq 0$), we get

$$
|\hat{\chi}_{B_N^N}(\zeta)| < \frac{\Gamma(\frac{N}{2} + 1)}{\Gamma(\frac{N+1}{2})^{1/2}} \cdot 2 \int_{0}^{1} (1 - t^2)^{\frac{N-1}{2}} \, dt = 1.
$$

Thus (2.8) and (2.9) hold, allowing us to conclude the Lemma as in [3]. We will recall the argument.

Let $I_0 = \{j \in \{1, \ldots, \ell\} : |\zeta_j| > 1\}$. Since for $j \notin I_0$, we have $|\zeta_j| \leq 1$ and thus $|m(\zeta_j)| < e^{-c|\zeta_j|^2}$ by (2.8), we get

$$
\prod_{j \notin I_0} |m(\zeta_j)| \leq \exp \left( -C \sum_{j \notin I_0} |\zeta_j|^2 \right),
$$

and also, by (2.9)

$$
\prod_{j \in I_0} |m(\zeta_j)| \leq e^{-c|I_0|}.
$$

Together with the obvious estimate

$$
\sum_{|\zeta_j| \in R^d} |\zeta_j|^2 \leq R^{2\delta} |I_0| + \sum_{j \notin I_0} |\zeta_j|^2 \leq R^{2\delta} \left( |I_0| + \sum_{j \notin I_0} |\zeta_j|^2 \right),
$$

this leads to

$$
|m(\xi)| \leq \exp \left( -c R^{-2\delta} \sum_{|\zeta_j| \in R^d} |\zeta_j|^2 \right). \quad (2.14)
$$

But $e^{-C|x|} = O((1 + |x|)^{-\delta/2})$ for every $k \in \mathbb{N}_{>0}$, and hence, (2.14) implies

$$
|m(\xi)| \leq C_k \left( 1 + R^{-2\delta} \sum_{|\zeta_j| \in R^d} |\zeta_j|^2 \right)^{-\delta/2} \leq C_k \left( R^{-2\delta} + R^{-2\delta} \sum_{|\zeta_j| \in R^d} |\zeta_j|^2 \right)^{-\delta/2},
$$

immediately concluding the Lemma. \textbf{q.e.d.}
With Lemma 2.5 we can establish a bound
\[
\left\| \left( \sum_{j=1}^{n} |g \ast \mu_j|^2 \right)^{1/2} \right\|_2 \leq C_K R^{1-\frac{4K}{p}} \|f\|_2;
\]
(2.15)
with \(C_K\) only depending on \(K\). To achieve (2.15), one simply has to replace \(\xi_j\) by \(\zeta_j\) and \(\hat{\eta}(t\xi_j)\) by \(\hat{\eta}(t\zeta_j, 1) \cdots \hat{\eta}(t\zeta_j, N)\) in the corresponding proofs from [4]. By interpolation with (2.5), the choice
\[
K = \left\lceil \frac{10(p-1)}{\varepsilon} \right\rceil
\]
gives us
\[
\left\| \left( \sum_{j=1}^{n} |g \ast \mu_j|^2 \right)^{1/2} \right\|_p \leq C_{p,\varepsilon} \|f\|_p.
\]
(2.16)
Hence it only remains to estimate
\[
\left\| \left( \sum_{j=1}^{n} |A_k f \ast \mu_j|^2 \right)^{1/2} \right\|_p \leq C_{p,\varepsilon} R^{12N-\varepsilon} \|f\|_p
\]
for \(0 \leq k \leq K\). For \(S \subset \{1, \ldots, n\}\), put
\[
\Gamma_S := \prod_{j \not\in S} S_j \prod_{j \in S} (\text{Id} - S_j).
\]
Then, by the triangle inequality, we have
\[
\left( \sum_{j=1}^{n} |A_k f \ast \mu_j|^2 \right)^{1/2} \leq \left( \sum_{j=1}^{n} \left| \sum_{|S|=k} \Gamma_S f \ast \mu_j \right|^{2} \right)^{1/2}
\]
(2.17)
\[+ \left( \sum_{j=1}^{n} \left| \sum_{|S|=k} \Gamma_S f \ast \mu_j \right|^{2} \right)^{1/2} \]  (2.18)
Bourgain applies a stochastic method to decouple the variables, which reduces (2.17) to the case \(k = 0\) and (2.18) to the case \(k = 1\). We can acquire the same by replacing \(T_j\) by \(S_j\) in that procedure. With that, we only have to find suitable constants \(b_0 = b_0(R)\), \(b_1 = b_1(R)\) such that
\[
\left\| \left( \sum_{j=1}^{n} |A_0 f \ast \mu_j|^2 \right)^{1/2} \right\|_p \leq b_0 \|f\|_p
\]
(2.19)
and
\[
\left\| \left( \sum_{k=1}^{\ell} |\Gamma_k G_k f|^2 \right)^{1/2} \right\|_p \leq b_1 \|f\|_p,
\]
(2.20)
where \(\Gamma_k = (\text{Id} - S_k) \prod_{j \not= k} S_j\) and
\[
G_k f = \left( \sum_{j \in V_k} |f \ast \mu_j|^2 \right)^{1/2}.
\]
Let $B_p = B_{p,R}$ be minimal such that

$$\left\| \left( \sum_{j=1}^{n} |f * \mu_j|^2 \right)^{1/2} \right\|_p \leq B_p \| f \|_p. \quad (2.21)$$

To estimate $b_1$, we need to rely on Lemmas 7-9 from [1]. The proofs of these will become more complicated in our setting, with estimates that will depend on $N$. For the proofs, we will provide slightly more details than in [1]. Instead of using the properties of the convolution operators $T_j$, we need to convolve with a function that is roughly stable under small translations. Bourgain considers the function

$$\varphi(x) := \frac{c}{1 + x^4}$$

with $c$ so that $\int_{\mathbb{R}} \varphi(x) \, dx = 1$. Then $C^{-1} \varphi(x - y) \leq \varphi(x) \leq C \varphi(x - y)$ for every $x \in \mathbb{R}$ and $|y| \leq 1$, $\varphi(x) = O(e^{-C|x|})$ as $|x| \to \infty$, and $|1 - \varphi(x)| < C x^2$ for every $x \in \mathbb{R}$.

Fix $t_0 := R^{-3\varepsilon}$ and let $\tilde{L}_j : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ be the convolution by $\varphi_{t_0}$ in the $j$-th variable, $j \in \{1, \ldots, n\}$. For $k \in \{1, \ldots, \ell\}$, let

$$L_k := \prod_{j \in V_k} \tilde{L}_j \quad \text{and} \quad L^{(k)} := \prod_{j \notin V_k} \tilde{L}_j = \prod_{k' \neq k} L_{k'}. $$

With these notions, we show the following version of Lemma 7 from [1].

**Lemma 2.6.** Let $M \in \mathbb{N}$, $q = 2^M$, and let $f_1, \ldots, f_\ell \in L^q(\mathbb{R}^n)$ be positive functions. Then

$$\left\| \sum_{j=1}^{\ell} L_j^{(j)} f_j \right\|_q \leq C_{q,N} \sum_{k=0}^{M-1} \left\| \prod_{j=1}^{\ell} L_j \left( \sum_{j=1}^{\ell} f_{j_2}^{2k} \right) \right\|_{2M-k}^{2-k} + C_{q,N} \left( \sum_{j=1}^{\ell} \| f_j \|_q^q \right)^{1/q} \quad (2.22)$$

$$\leq C_{q,N} \left\| \sum_{j=1}^{\ell} f_j \right\|_q. \quad (2.23)$$

**Proof.** The proof of (2.23) is easy. We will show (2.22) by induction on $M$, with the case $M = 0$ ($q = 1$) being obvious. Fix $M > 0$ and assume that

$$\left\| \sum_{j=1}^{n} L_j^{(j)} f_j \right\|_{q/2} \leq C_q \sum_{k=0}^{M-2} \left\| \prod_{j=1}^{n} L_j \left( \sum_{j=1}^{n} f_2^{2k} \right) \right\|_{2M-1-k}^{2-k} + C_q \left( \sum_{j=1}^{n} \| f_j \|_{q/2}^{q/2} \right)^{2/q}.$$

Then we have

$$\left\| \sum_{j=1}^{n} L_j^{(j)} f_j \right\|_q \leq q! \sum_{1 \leq j_1 \leq \ldots \leq j_q \leq \ell} \int \prod_{k=1}^{q} (L^{(j_k)} f_{j_k}(x)) \, dx.$$
If we split
\[
\sum_{1 \leq j_1 \leq \ldots \leq j_q \leq \ell} \int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx = \sum_{j_1 \leq \ldots \leq j_q} \int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx \\
+ \sum_{j_1 \leq \ldots \leq j_q} \int_{j_1 < j_2} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx,
\]
we can estimate the first sum as follows, using Hölder’s inequality with \( q_2 \) and \( q-q_2 \)
\[
\sum_{j_1 \leq \ldots \leq j_q} \int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx \leq \int_{\mathbb{R}^n} \left( \sum_{j=1}^n (L^{(j)} f_j(x))^2 \right)^{\frac{q}{2}} \cdot \left( \sum_{j=1}^n L^{(j)} f_j(x) \right)^{q-2} \, dx \\
\leq \left\| \sum_{j=1}^n (L^{(j)} f_j)^2 \right\|_{q/2} \left\| \sum_{j=1}^n L^{(j)} f_j \right\|_q^{q-2}.
\]
(2.24)
In the case \( j_1 < j_2 \), we estimate
\[
\int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx
\]
directly. Without loss of generality, assume \( j_1 = 1 \) and put
\[
g_j = \left( \prod_{1 \leq k \leq \ell \setminus \{1, j\}} L_k \right) f_j
\]
for \( j \in \{1, \ldots, \ell\} \). Denote \( x = (x^{(1)}, x') \) with \( x^{(1)} \in \mathbb{R}^N \) and \( x' \in \mathbb{R}^{n-N} \), and let \( \Phi: \mathbb{R}^N \to \mathbb{R}, \Phi(y) = \prod_{k=1}^N \varphi(y_k) \). Then
\[
\Phi_{t_0}(y) = \prod_{k=1}^N \varphi_{t_0}(y_k),
\]
and we get
\[
\int_{\mathbb{R}^n} \prod_{k=1}^q (L^{(j_k)} f_{j_k}(x)) \, dx
\]
\[
= \int_{\mathbb{R}^{n-N}} \int_{\mathbb{R}^N} g_1(x^{(1)}, x') \cdot \prod_{k=2}^q L_{1g_{j_k}}(x^{(1)}, x) \, dx^{(1)} \, dx' \\
= \int_{\mathbb{R}^{n-N}} \int_{(\mathbb{R}^N)^{q-1} \mathbb{R}^N} g_1(x^{(1)}, x') \cdot \prod_{k=2}^q (g_{j_k}(x^{(1)} - y^{(k)}, x') \Phi_{t_0}(y^{(k)})) \, dx^{(1)} \, dy^{(2)}, \ldots, y^{(q)} \, dx'.
\]
Now fix \( x' \). Using that \( \phi_{t_0}(\tau) \geq \phi_{t_0}(t_0) = \frac{c}{2t_0} \), averaging over \( |\tau| \leq t_0 \) implies

\[
\int_{(\mathbb{R}^N)^q} \int_{\mathbb{R}^N} g_1(x^{(1)}, x') \cdot \prod_{k=2}^q \left( g_{j_k}(x^{(1)} - y^{(k)}, x') \Phi_{t_0}(y^{(k)}) \right) \, dx^{(1)} \, d(y^{(2)}, \ldots, y^{(q)}) \\
\leq C^{-N} \int_{[-t_0,t_0]^N} \Phi_{t_0}(\tau) \int_{(\mathbb{R}^N)^q} \int_{\mathbb{R}^N} g_1(x^{(1)}, x') \\
\cdot \prod_{k=2}^q \left( g_{j_k}(x^{(1)} - y^{(k)}, x') \Phi_{t_0}(y^{(k)}) \right) \, dx^{(1)} \, d(y^{(2)}, \ldots, y^{(q)}) \, d\tau \\
= C^{-N} \int_{[-t_0,t_0]^N} \Phi_{t_0}(\tau) \int_{(\mathbb{R}^N)^q} \int_{\mathbb{R}^N} g_1(x^{(1)} - \tau, x') \\
\cdot \prod_{k=2}^q \left( g_{j_k}(x^{(1)} - y^{(k)}, x') \Phi_{t_0}(y^{(k)} + \tau) \right) \, dx^{(1)} \, d(y^{(2)}, \ldots, y^{(q)}) \, d\tau \\
\leq C^{N(q-1)} C^{-N} \int_{[-t_0,t_0]^N} \Phi_{t_0}(\tau) \int_{(\mathbb{R}^N)^q} \int_{\mathbb{R}^N} g_1(x^{(1)} - \tau, x') \\
\cdot \prod_{k=2}^q \left( g_{j_k}(x^{(1)} - y^{(k)}, x') \Phi_{t_0}(y^{(k)}) \right) \, dx^{(1)} \, d(y^{(2)}, \ldots, y^{(q)}) \, d\tau \\
\leq C_{q,N} \int_{\mathbb{R}^N} \prod_{k=1}^q \left( L_1 g_{j_k}(x^{(1)}, x') \right) \, dx^{(1)}.
\]

Altogether, we have

\[
\int_{\mathbb{R}^n} \prod_{k=1}^q \left( L^{(j_k)} f_{j_k}(x) \right) \, dx \leq C_{q,N} \int_{\mathbb{R}^n} \prod_{k=1}^q \left( L_1 g_{j_k}(x) \right) \, dx \\
= C_{q,N} \int_{\mathbb{R}^n} \left( \prod_{j=1}^n L_j \right) f_1(x) \cdot \prod_{k=2}^q L^{(j_k)} f_{j_k}(x) \, dx.
\]

The same argument holds if \( j_1 \neq 1 \), and hence by Hölder’s inequality with \( q \) and \( \frac{q-1}{q} \),

\[
\sum_{j_1 \leq \ldots \leq j_q} \int_{\mathbb{R}^n} \prod_{k=1}^q \left( L^{(j_k)} f_{j_k}(x) \right) \, dx \leq C_{q,N} \sum_{j_1 = 1}^n \sum_{j_2 = 1}^n \ldots \sum_{j_q = 1}^n \left( \prod_{j=1}^n L_j \right) f_j(x) \cdot \prod_{k=2}^q L^{(j_k)} f_{j_k}(x) \, dx \\
\leq C_{q,N} \left\| \left( \prod_{j=1}^n L_j \right) \left( \sum_{j=1}^n f_j \right) \right\|_q \left\| \sum_{j=1}^n L^{(j)} f_j \right\|_{q-1}^{q-1}. \quad (2.25)
\]

With (2.24) and (2.25), we get

\[
\left\| \sum_{j=1}^n L^{(j)} f_j \right\|_q^q \leq C_q \left\| \sum_{j=1}^n \left( L^{(j)} f_j \right)^2 \right\|_{q/2} \left\| \sum_{j=1}^n L^{(j)} f_j \right\|_{q}^{-2}.
\]
Since
\[ + C_q \left( \prod_{j=1}^{n} L_j \right) \left( \sum_{j=1}^{n} f_j \right) \left\| \sum_{j=1}^{n} L^{(j)} f_j \right\| \frac{q-1}{q}, \]
\[ \text{Since} \]
\[ \left\| \sum_{j=1}^{n} (L^{(j)} f_j)^2 \right\|^{1/2} \leq \left\| \sum_{j=1}^{n} L^{(j)} f_j \right\|_q \]
and \((L^{(j)} f_j)^2 \leq L^{(j)} f_j^2\), this leads to
\[ \left\| \sum_{j=1}^{n} L^{(j)} f_j \right\|_q \leq C_q \left\| \sum_{j=1}^{n} L^{(j)} f_j^2 \right\|^{1/2} + C_q \left( \prod_{j=1}^{n} L_j \right) \left( \sum_{j=1}^{n} f_j \right) \left\| \right\|, \]
and our induction hypothesis concludes the lemma. \textbf{q.e.d.}

We directly show a version of Lemma 9 from [4], which is a corollary of Lemma 8.

\textbf{Lemma 2.7.} Let \( M \geq 1, q = 2^M \), and \( f_1, \ldots, f_\ell \in L^q(\mathbb{R}^n) \). Then
\[ \left\| \left( \sum_{k=1}^{\ell} |L^{(k)} G_k f_k|^2 \right)^{1/2} \right\|_q \leq C_{q,N} R^{12N^2} \left\| \left( \sum_{k=1}^{\ell} |f_k|^2 \right)^{1/2} \right\|_q. \]  \hspace{1cm} (2.26)

\textit{Proof.} Since \( \mu_j = \partial_j (\chi_B * H_{1/R}) = \partial_j (\chi_B) * H_{1/R} \) by taking distributional derivatives and convolutions, and \( H_{1/R} \) is the density function of a probability measure, we can use Jensen’s inequality to estimate
\[ \left\| \left( \sum_{k=1}^{\ell} |L^{(k)} G_k f_k|^2 \right)^{1/2} \right\|_q \leq \left\| H_{1/R} * \left( \sum_{k=1}^{\ell} |L^{(k)} G_k f_k|^2 \right)^{1/2} \right\|_q \]
\[ \leq \left\| \left( \sum_{k=1}^{\ell} L^{(k)} |G_k f_k|^2 \right)^{1/2} \right\|_q, \]
where
\[ G_k f_k = \left( \sum_{j \in V_k} |(\partial_j \chi_B) * f_k|^2 \right)^{1/2}. \]

Hence it suffices to show
\[ \left\| \left( \sum_{k=1}^{\ell} L^{(k)} |G_k f_k|^2 \right)^{1/2} \right\|_q \leq C_q R^{24N^2} \left\| \left( \sum_{k=1}^{\ell} |f_k|^2 \right)^{1/2} \right\|_q. \]  \hspace{1cm} (2.27)

Take \( \psi \in \mathcal{S}(\mathbb{R}^n) \). We want to establish \( \langle \partial_j \chi_B, \psi \rangle \) for every \( j \in \{1, \ldots, n\} \). First, assume
\[ j = \ell = 1. \text{ Then we have} \]

\[-\langle \partial_j \chi_B, \psi \rangle = \int_{B_N^N} \partial_1 \psi(x) \, dx = \int_{B_N^{N-1}} \int_{\mathbb{R}^{N-1}} \partial_1 \psi(x_1, x') \, dx_1 \, dx' \]

\[= \int_{B_N^{N-1}} \psi(\sqrt{r_N^2 - |x'|^2}, x') - \psi(-\sqrt{r_N^2 - |x'|^2}, x') \, dx' \]

\[= \int_{B_N^{N-1}} \psi(\omega) \frac{\omega_1}{r_N} \, d\sigma(\omega). \]

For general \( j \) and \( \ell \), choose the unique \( k = k(j) \) with \( j \in V_k \). Then

\[-\langle \partial_j \chi_B, \psi \rangle = \int_{B^k_{S_{N-1}}} \int_{B^\ell_{S_{N-1}}} \psi(x^{(1)}, \ldots, \omega^{(k)}; \ldots, x^{(\ell)}) \frac{\omega^{j-(k-1)N}}{r_N} \, d\sigma(\omega) \, dx', \quad (2.28)\]

where \( B^k = \prod_{k' \neq k} B_{k'} \) and \( x' = (x_1, \ldots, x_{(k-1)N}, x_{kN+1}, \ldots, x_n) \). Now let

\[ \tau_j f(x) = \int_{S_{r_N}^{N-1}} f(x^{(1)}, \ldots, x^{(k)} + \omega, \ldots, x^{(\ell)}) \frac{\omega^{j-(k-1)N}}{r_N} \, d\sigma(\omega) \]

Then

\[ \| \tau_j \|_{1\rightarrow 1} = 2|B_{r_N}^{N-1}|, \]

and hence

\[ \sum_{j \in V_k} |(\partial_j \chi_B) * 2| \leq \sum_{j \in V_k} |\chi_{B^k} * \tau_j | \leq 2|B_{r_N}^{N-1}| \sum_{j \in V_k} \tau_j |f_k|^2 * \chi_{B^k}, \]

taking the last convolution only in the variables of \( B^k \). Application of Lemma 2.6 gives us

\[ \left\| \sum_{k=1}^\ell L^{(k)} |G_k f_k|^2 \right\|_q^{1/2} \leq 2|B_{r_N}^{N-1}| \left\| \sum_{k=1}^\ell L^{(k)} \left( \sum_{j \in V_k} \tau_j \left( |f_j|^2 * \chi_{B^k} \right) \right) \right\|_{q/2}^{1/2} \]

\[\leq C_{q,N} \sum_{i=1}^{M-1} \left( \prod_{k=1}^\ell L_k \right) \left( \sum_{k=1}^\ell \left( \sum_{j \in V_k} \tau_j \left( |f_k|^2 * \chi_{B^k} \right) \right)^{2^{i-1}} \right) \left\| \sum_{k=1}^\ell \tau_j \left( |f_k|^2 * \chi_{B^k} \right) \right\|^q_{q/2} \]. \quad (2.29)\]

We can easily estimate

\[ \left( \sum_{k=1}^\ell \left\| \sum_{j \in V_k} \tau_j \left( |f_k|^2 * \chi_{B^k} \right) \right\|_{q/2}^q \right)^{1/2} \leq (2N|B_{r_N}^{N-1}|)^{1/2} \cdot \left( \left( \sum_{k=1}^\ell |f_k(x)|^2 \right)^{1/2} \right). \quad (2.30)\]
Note that from the properties of $\varphi$, we can deduce

\[
\varphi_{t_0}(x + \rho) = \frac{1}{t_0} \frac{e}{1 + \frac{(x+\rho)^4}{t_0^4}} \leq \frac{1}{t_0} \varphi(x + \rho) \leq \frac{C^{(r_N)}}{t_0^4} \varphi(x) = \frac{C_N}{t_0^4} \leq \frac{C_N}{t_0^4} \varphi_{t_0}(x)
\]

for $x \in \mathbb{R}$ and $|\rho| \in [-r_N, r_N]$. Hence

\[\Phi_{t_0}(x - y) \leq C_N t_0^{-4N} \Phi_{t_0}(x)\]

for every $x, y \in \mathbb{R}^N$ with $|y| \leq r_N$. Thus for any positive $g \in L^q(\mathbb{R}^N)$, we have

\[
\left( \Phi_{t_0} * \int_{S^N_{-1}} (\tau_{\omega} g) \cdot \frac{|\omega|}{\tau_{N}} \, d\sigma(\omega) \right)(x) \leq 2 |B_{r_N}^{-1} |C_N t_0^{-4N} \Phi_{t_0} * g(x) \right)
\]

\[
= C_N t_0^{-4N} \int_{\mathbb{R}^N} \int_{B_{r_N}^N} \Phi_{t_0}(y) \, dz \, g(x - y) \, dy
\]

\[
\leq C_N t_0^{-8N} \int_{\mathbb{R}^N} \int_{B_{r_N}^N} \Phi_{t_0}(y - z) \, dz \, g(x - y) \, dy
\]

\[
= C_N R^{24N \varepsilon} \cdot (\chi_{B_{r_N}^N} * \Phi_{t_0}) * g(x).
\]

Taking $1 \leq i \leq N - 1$, this implies

\[
\left\| \left( \prod_{k=1}^{\ell} L_k \right) \left( \sum_{k=1}^{\ell} \left( \sum_{j \in \mathcal{V}_k} \tau_j(|f_k|^2 \chi_{B^k}) \right)^{2^{i-1}} \right) \right\|_{2M-i}^{2^{-i}}
\]

\[
\leq \left\| \left( \sum_{k=1}^{\ell} C_N N R^{24N \varepsilon} \cdot |f_k|^2 \chi_{B} \right)^{2^{i-1}} \right\|_{2^{M-i}}^{2^{-i}}
\]

\[
\leq C_N R^{12N \varepsilon} \left\| \left( \sum_{k=1}^{\ell} |f_k|^2 \right)^{1/2} \right\|_q.
\]

The estimates (2.29), (2.30), and (2.31) conclude the proof. q.e.d.

Now, we can estimate (2.19) and (2.20) by arguing as in [4, Section 4]. Since

\[A_0 = \prod_{k=1}^{\ell} S_k = \prod_{j=1}^{n} T_j,
\]

we can establish

\[b_0 \leq C_p(R^{3\varepsilon} + B_p R^{-\frac{2\varepsilon}{p}})
\]

with $B_p$ as in (2.21). For (2.20), we can use Lemmas 2.6 and 2.7 to obtain

\[b_1 \leq C_{p,N}(R^{12N \varepsilon} + B_p R^{-\frac{2\varepsilon}{p}}).
\]
This leads to
\[ B_p \leq C_{p,\varepsilon}(1 + b_0 + b_1) < C_{p,\varepsilon,N}(R^{12N-\varepsilon} + B_p R^{-\frac{2\varepsilon}{p'}}), \]
giving us
\[ B_p \leq C_{p,\varepsilon,N} R^{12N-\varepsilon} \]
and thus proving Lemma 2.3 and Proposition 2.1.

3 Stein’s approach revisited

Let \( B = B_1 \times \cdots \times B_\ell, n = n_1 + \cdots + n_\ell \) be as in Theorem 1. Let \( N := \min_{1 \leq k \leq \ell} n_k \). Fix \( 1 < p \leq \infty \). We show that if \( N > \frac{n}{p-1} \), i.e. \( p > \frac{N}{n-1} \), we can deduce (1.3) in Theorem 1 from Theorem 2 and that (1.5) from Theorem 2 holds if we show the following theorem.

**Theorem 2'.** Let \( S := (S_R^{-N-1})^{\ell} \), with \( R \) so that \(|S| = 1\). Then
\[ \|M_Sf\|_p \leq C_{p,N}\|f\|_{p'} \quad (3.1) \]

We can freely change the radii of each sphere because for every \( r, s > 0 \) and \( n > 1 \), we have
\[ \int_{S_r^{n-1}} f(s\omega) \, d\sigma_{S_r^{n-1}}(\omega) = \frac{1}{n-1} \int_{S_r^{n-1}} f(\omega) \, d\sigma_{S_r^{n-1}}(\omega). \]
Thus, if \( S' = S_1' \times \cdots \times S_\ell' \) is a product of spheres with \( \dim S_k' = \dim S_k \) for each \( k \), we have \( \|M_{S'} f\|_p \leq C\|f\|_p \) for all \( f \in S \) if \( \|M_S f\|_p \leq C\|f\|_p \) for all \( f \in S \).

We can not get a pointwise estimate \( M_B f(x) \leq M_S f(x) \) as in the case \( \ell = 1 \), but we can indeed get an \( L^p \)-estimate by a similar argument. Assume that each \( B_k \) and each \( S_k \) has radius 1 (thus \( S_k = S^{n_k-1} \)), and let \( \sigma_k \) be the respective surface measure for each \( S_k \).

**Lemma 3.1.** We have
\[ \|M_B f\|_p \leq \|M_S f\|_p \quad (3.2) \]
for each \( f \in S(\mathbb{R}^n) \).

**Proof.** Using polar coordinates and the fact that \(|S_k| = n_k|B_k|\), we can estimate
\[
M_B f(x) = \frac{1}{|B|} \sup_{t > 0} \left( \prod_{k=1}^{\ell} \int_{S_t^{n_k-1}} \cdots \int_{S_t^{n_\ell-1}} |f(x + t(s_1\omega_1, \ldots, s_\ell\omega_\ell))| \, d\sigma_t(\omega_\ell) \cdots d\sigma_1(\omega_1) \, ds \right) \\
\leq \left( \prod_{k=1}^{\ell} \int_{S_t^{n_k-1}} \cdots \int_{S_t^{n_\ell-1}} |f(x + t(\omega_1, \ldots, \omega_\ell))| \, d\sigma_t(\omega_\ell) \cdots d\sigma_1(\omega_1) \, ds \right) \\
= \int_{[0,1^\ell]} \prod_{k=1}^{\ell} n_k s_k^{n_k-1} M_{S_t^{n_k-1}} f(x) \, ds.
\]
A simple application of Minkowski’s integral inequality yields
\[
\|M_Bf\|_p \leq \int_{[0,1]^\ell} \prod_{k=1}^\ell n_k s_k^{n_k-1} \cdot \|M_Sf\|_p \, ds = \|M_Sf\|_p,
\]
which is \(3.2\).

We now generalize Stein’s method of rotations from \[10\] for our situation with the following Lemma.

**Lemma 3.2.** Let \(S = S^{n_1-1} \times \cdots \times S^{n_{\ell-1}}, k \in \{1, \ldots, \ell\}\), and set
\[
S^+ = \prod_{j=1}^{k-1} S^{n_j-1} \times S^{n_k} \times \prod_{j=k+1}^\ell S^{n_j-1}.
\]

Let \(1 < p \leq \infty\) and assume that there is a constant \(C > 0\) such that \(\|M_Sf\|_p \leq C\|f\|_p\) for every \(f \in S(\mathbb{R}^n)\). Then also
\[
\|M_{S^+}f\|_p \leq C\|f\|_p \tag{3.3}
\]
for every \(f \in S(\mathbb{R}^n)\).

**Proof.** We can assume \(k = 1\). For any \(u \in S^{n_1}\), denote by
\[
S^{n_1-1}_u := \{x \in S^{n_1} : x \perp u\}
\]
the rotated \((n_1 - 1)\)-dimensional spheres in \(\mathbb{R}^{n_1+1}\), and let \(\sigma^u\) be the surface measure of \(S^{n_1-1}_u\) so that \(\sigma^u(S^{n_1-1}_u) = |S^{n_1-1}_u|\). Furthermore, let \(\sigma^+\) be the surface measure of \(S^{n_1}\). Define a new measure \(\mu\) on \(S^{n_1}\) by putting for every Lebesgue-measurable set \(A \subset \mathbb{R}^{n_1+1}\)
\[
\mu(A) := \int_{S^{n_1}} \int_{S^{n_1-1}_u} \chi_A(\omega) \, d\sigma^u(\omega) \, d\sigma^+_1(u).
\]

By \[10\], we have \(\mu = |S^{n_1-1}||\sigma^+_1\), and
\[
\|M_{S^{n_1-1}_u}f\|_p = \|M_{S^{n_1-1}}f\|_p \leq C\|f\|_p,
\]
where \(S_u = S^{n_1-1}_u \times S_2 \times \cdots \times S_\ell\). Hence we can calculate
\[
M_{S^+}f(x) = \frac{1}{|S^+|} \sup_{t > 0} \frac{1}{|S^{n_1-1}|} \int_{S^{n_1}} \int_{S^{n_1-1}_u} \cdots \int_{S_\ell} |f(x + t(\omega_1, \ldots, \omega_\ell))| \, d\sigma^+_1(u)
\]
\[
\leq \frac{1}{|S^{n_1}|} \int_{S^{n_1}} \sup_{t > 0} \frac{1}{|S^{n_1-1}_u|} \int_{S_2} \cdots \int_{S_\ell} |f(x + t(\omega_1, \ldots, \omega_\ell))| \, d\sigma^+_1(u)
\]
\[
= \frac{1}{|S^{n_1}|} \int_{S^{n_1}} M_{S_u}f(x) \, d\sigma^+_1(u),
\]
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By Minkowski’s integral inequality, we then get
\[
\|M_{S} f\|_p \leq \frac{1}{|S|^{1}} \int_{S} \|M_{S_u} f\|_p \, d\sigma^+_u(u) \leq C\|f\|_p.
\]
This concludes the lemma. \[\text{q.e.d.}\]

Now assume that we’ve already shown Theorem 2’ and take $B$ as in Theorem 1. Fix $1 < p < \infty$ and let $N_0 = \lfloor \frac{p}{p-1} \rfloor$. By change of coordinates, we can assume $B = B' \times B''$, where
\[
B' = \prod_{n_k \geq N_0} B_k \quad \text{and} \quad B'' = \prod_{n_k < N_0} B_k.
\]
Then
\[
\|M_B\|_{p \to p} \leq \|M_{B'}\|_{p \to p} \cdot \|M_{B''}\|_{p \to p}.
\]
Furthermore, assume that for $n_k \geq N_0$, each $B_k$ has radius 1, while for $n_k < N_0$, each $B_k$ has volume 1. Since $N_0$ only depends on $p$, we get
\[
\|M_{B''}\|_{p \to p} \leq C_p
\]
from Proposition 2.1. Let $S' = \prod_{n_k \geq N_0} S_{n_k-1}$. Then by Lemma 3.1 and successive application of Lemma 3.2, we obtain
\[
\|M_{B'}\|_{p \to p} \leq \|M_{S'}\|_{p \to p} \leq \|M_{(S_{N_0-1})^{\ell'}}\|_{p \to p},
\]
where $\ell' = \{|k \in \{1, \ldots, \ell\} : n_k \geq N_0\}$. But since we can freely vary the radius of each sphere, Theorem 2’ implies
\[
\|M_{(S_{N_0-1})^{\ell'}}\|_{p \to p} \leq C_{N_0} = C_p.
\]
This concludes Theorem 1.

4 Higher Fourier derivatives of spherical measures

We are left with proving Theorem 2’. Let $N > 2$, $p > \frac{N}{N-1}$, and $S = (S_{R}^{N-1})^{\ell}$ with $R$ so that $|S| = 1$, i.e. $R^{N-1} = \frac{\Gamma(N/2)}{2\pi^{N/2}}$. First, we use the approach from [5] to show that we only have to bound
\[
\| \sup_{t \leq 2} \int_{S} |f(x + t\omega)| \, d\sigma_S(\omega) \|_p \leq C_{p,N}\|f\|_p \quad (4.1)
\]
for $p < 2$. For our setting, we use a different proof, which can be found in Lemma 6.15 and the argument in subsection 6.5.1 from [6]. This result makes use of Lemma 3 from [2]. Since the proofs only rely on the properties of the respective Fourier transforms, a short inspection of them shows that these lemmas still hold when we take finite (signed) Borel measures on $\mathbb{R}^n$ instead of $L^1$-kernels, in the following sense.
Lemma 4.1 (Lemma 3 from [2]). Let \( \nu \) be a finite Borel measure on \( \mathbb{R}^n \) so that \( \hat{\nu} \) is differentiable, and put

\[
\alpha_j := \sup_{2^j \leq |\xi| \leq 2^{j+2}} |\hat{\nu}(\xi)|, \quad \beta_j := \sup_{2^j \leq |\xi| \leq 2^{j+2}} |\langle \nabla \hat{\nu}(\xi), \xi \rangle|
\]

for every \( j \in \mathbb{Z} \). Then for every \( f \in L^2 \), we have

\[
\left\| \sup_{t>0} |f * \nu_t| \right\|_2 \leq C \Gamma(\nu) \|f\|_2
\]

with

\[
\Gamma(\nu) := \sum_{j \in \mathbb{Z}} \alpha_j^{1/2} (\alpha_j^{1/2} + \beta_j^{1/2}). \tag{4.2}
\]

Lemma 4.2 (Lemma 6.15 from [6]). Let \( \nu \) be a finite Borel measure on \( \mathbb{R}^n \) and \( K \in L^1(\mathbb{R}^n) \) such that \( \hat{\nu} \) and \( \hat{K} \) are both differentiable. Assume there is a constant \( C \) so that for every \( \theta \in S^{n-1} \) and every \( u \in \mathbb{R}^* \)

\[
|\hat{\mu}(u\theta)| \leq C \cdot \min\{|u|, |u|^{-1}\}, \tag{4.3}
\]

\[
|\langle \theta, \nabla \hat{\mu}(u\theta) \rangle| \leq C \cdot \min\{1, |u|^{-1}\} \tag{4.4}
\]

with \( \mu = \nu, K \). Then we have

\[
\Gamma(\nu * K_{2^k}) \leq C' 2^{-k/2} \tag{4.5}
\]

for every \( k \in \mathbb{Z} \), with \( C' \) only depending on \( C \) and \( \Gamma(\nu * K_{2^k}) \) as in (4.2).

Let \( P \) be the Poisson kernel, i.e. \( \hat{P}(\xi) = e^{-|\xi|} \). We will show later that the Borel measure \( \sigma_S = P \, dx \) satisfies (4.3) and (4.4). By Stein’s maximal theorem for semigroups, \( \| \sup_{t>0} P_t * f \|_q < C \|f\|_q \) for \( 1 < q < \infty \), and one can take \( \hat{K}(\xi) = e^{-|\xi|} - e^{-2|\xi|} \) in Lemma 4.2 to proceed as in [6], getting the strong \( L^2 \)-boundedness property from Carbery’s proof as required, and being left with showing (4.1). Here, we use part (ii) of the proposition in [5], which also holds for any finite Borel measure with bounded Fourier transform. For this, we need to consider fractional derivatives. For a finite Borel measure \( \nu \) on \( \mathbb{R}^n \) and \( z \in \mathbb{C} \), denote the fractional derivative of \( \hat{\nu} \) of order \( z \) by

\[
(\langle \xi, \nabla \rangle)^z \hat{\nu}(\xi) = \left( \frac{d}{dr} \right)^z \hat{\nu}(r\xi)|_{r=1} = \int (2\pi i(x, \xi))^z e^{2\pi i(x, \xi)} \, d\nu(x) \tag{4.6}
\]

whenever the right hand side is well-defined. Let

\[
m(\xi) = \hat{\sigma_S}(\xi).
\]

According to the proposition in [5], we need to show that there is \( 1 > \alpha > \frac{1}{p} \) so that the fractional derivative \( \langle \xi, \nabla \rangle^\alpha m \) has bounded \( L^p \)-multiplier norm independent of \( \ell \). Our basic idea will be to estimate the fractional derivatives of \( m \) of order \( z \) with \( \text{Re} \, z = 0 \) as \( L^1 \)-multipliers, and with \( \text{Re} \, z = \frac{N-1}{2} \) as \( L^2 \)-multipliers, followed by applying Stein’s
interpolation theorem. It turns out that \( \frac{N-1}{2} \) is the best possible upper bound on \( \Re z \) for that estimate, and that we can also establish (4.3) and (4.4) while bounding these fractional derivatives. However, we encounter some technical difficulties like in [8]. To deal with these, we introduce the Riesz fractional derivative of a function \( f : [0, 2] \to \mathbb{C} \), defined as
\[
I^{-z}f(t) = \frac{-1}{\Gamma(-z)} \int_0^t (u-t)^{-z-1} f(u) \, du,
\]
for \( \Re z < 0 \) and \( 0 < t \leq 2 \). This operator can be extended analytically to the complex plane. For any \( k \in \mathbb{N} > 0 \), assume that \( f \) as above is \( k \) times differentiable and let \( \Re z < k \). Then
\[
I^{-z}f(t) = E_{k,f}(z,t) + (-1)^k \frac{1}{\Gamma(k - z)} \int_0^t (u-t)^{-z+k-1} f^{(k)}(u) \, du,
\]
where
\[
E_{k,f}(z,t) = \sum_{j=0}^{k-1} (-1)^j \frac{2^{-z+j} f^{(j)}(2)}{\Gamma(j + 1 - z)}.
\]
It follows that
\[
I^{-k}f(t) = (-1)^k f^{(k)}(t)
\]
if \( f \) is \( k \) times differentiable. Now consider the holomorphic family of multipliers \( (m_x)_{x \in \mathbb{C}} \) defined by
\[
m_x(\xi) := \left. I^{-z}m(t\xi) \right|_{t=1}.
\]
From Müller’s work (see also Lemma 7.3 in [6]), it follows that for every \( 0 < \alpha < 1 \),
\[
m_\alpha(\xi) - (\langle \xi, \nabla \rangle)^\alpha m(\xi)
\]
is an \( L^q \)-multiplier for \( 1 \leq q \leq \infty \), bounded by \( \frac{1}{\Gamma(1-\alpha)} \). Thus we only need to bound the \( m_x \), and from (4.7) and (4.8), it follows that we need to bound the usual derivatives
\[
\frac{d^r}{dr^r} m(r\xi) \text{ for } 1 \leq r \leq 2.
\]
Let us give a quick idea of our approach to bound these derivatives. We have
\[
m(\xi) = \prod_{j=1}^{\ell} \hat{m}(\zeta_j),
\]
writing again \( \xi = (\zeta_1, \ldots, \zeta_\ell) \) with each \( \zeta_j \in \mathbb{R}^N \), and
\[
\hat{m}(\zeta) := \hat{\sigma}_{S^N_r}^{-1}(\zeta).
\]
By the general Leibniz formula, we have
\[
\left( \frac{d}{dr} \right)^k m(r\xi) = \sum_{\sum_{|\alpha|=k} \alpha_j = \ell} \binom{k}{\alpha} \prod_{j=1}^{\ell} \left( \frac{d}{dr} \right)^{\alpha_j} \hat{m}(r\zeta_j).
\]
It turns out that the well known bound \( \tilde{m}(\zeta) = O(|\zeta|^{-\frac{N+2}{2}}) \) as \( |\zeta| \to \infty \) extends to \( \frac{d^{\alpha}}{dr^\alpha} \tilde{m}(r\zeta) = O(|\zeta|^{\alpha - \frac{N+2}{2}}) \), \( |\zeta| \to \infty \). Hence we can get a good estimate on the factors where \( |\zeta| \) is sufficiently large. One can also use the integral representation of Bessel functions to see that for \( k > 0 \), \( \frac{d^{k}}{dr^k} \tilde{m}(r\zeta) \) is close to 0 if \( \zeta \) is close to 0. However, we also need some estimate for the derivatives of \( \tilde{m} \) if \( \zeta \) is neither sufficiently close to 0, nor sufficiently large. All the necessary estimates are established in the following lemma.

**Lemma 4.3.** For each \( \alpha \in \mathbb{N} \) and \( \zeta \in \mathbb{R}^N \), the following estimates hold.

(i) There are constants \( C_{\alpha,N}, \tilde{C}_{\alpha,N} \) such that for every \( |\zeta| \geq C_{\alpha,N} \) and every \( r \in [1, 2] \)

\[
\left| \left( \frac{d}{dr} \right)^\alpha \tilde{m}(r\zeta) \right| \leq \tilde{C}_{\alpha,N}(2\pi r|\zeta|)^{-\frac{N}{2}+\alpha}.
\]

(ii) If \( 0 \leq 2\pi r|\zeta| \leq C \) for some constant \( C > 1 \), then there is \( c > 0 \), depending only on \( C \), such that for every \( r \in [1, 2] \), we have

\[
|\tilde{m}(r\zeta)| \leq e^{-c(2\pi r|\zeta|)^2}.
\]

(iii) If \( \alpha > 0 \) and \( 1 \leq 2\pi r|\zeta| \leq C \) for some constant \( C > 1 \), then for every \( r \in [1, 2] \), we have

\[
\left| \left( \frac{d}{dr} \right)^\alpha \tilde{m}(r\zeta) \right| \leq C'(2\pi r|\zeta|)^{2\alpha} \leq (2\pi r|\zeta|)^{2\alpha} \cdot e^{-c'(2\pi r|\zeta|)^2},
\]

with \( 0 < C' < 1 \), \( C' \) only depending on \( C \), and \( c' \) chosen so that \( C' \leq e^{-cC^2} \).

(iv) If \( \alpha > 0 \) and \( 2\pi r|\zeta| < 1 \), then there is \( C' < 1 \) such that for \( 1 \leq r \leq 2 \),

\[
\left| \left( \frac{d}{dr} \right)^\alpha \tilde{m}(r\zeta) \right| \leq C'(2\pi r|\zeta|)^{2\alpha}.
\]

**Proof.** (i) We will show that

\[
\left( \frac{d}{dr} \right)^\alpha \tilde{m}(r\zeta) = 2\pi^{\alpha+1}r^{-\frac{N}{2}-1}R^\frac{N}{2}+\alpha|\zeta|^{-\frac{N}{2}+\alpha} \sum_{k=0}^{\alpha} \binom{\alpha}{k} \pi r R|\zeta|^{-k}B_{\alpha,k}(2\pi r|\zeta|), \tag{4.11}
\]

with

\[
B_{\alpha,k}(t) = \frac{\Gamma\left(\frac{N}{2} + k\right)}{\Gamma\left(\frac{N}{2}\right)} \sum_{j=0}^{a-k} (-1)^{j+k} \binom{\alpha - k}{j} J_{\frac{N}{2}-a-k+2j}(t). \tag{4.12}
\]

Since

\[
\tilde{m}(\zeta) = 2\pi R^\frac{N}{2}|\zeta|^{-\frac{N}{2}} J_{\frac{N}{2}}(2\pi R|\zeta|),
\]

\[
\tilde{m}(r\zeta) = \tilde{m}(\zeta) r^{-\frac{N}{2}+\alpha} J_{\frac{N}{2}+\alpha}(r|\zeta|).
\]

Hence

\[
\left( \frac{d}{dr} \right)^\alpha \tilde{m}(r\zeta) = 2\pi^{\alpha+1}r^{-\frac{N}{2}+\alpha} (2\pi R|\zeta|)^{\alpha} \sum_{k=0}^{\alpha} \binom{\alpha}{k} \pi r R|\zeta|^{-k} B_{\alpha,k}(2\pi r|\zeta|),
\]

as claimed in (4.11).
we obtain
\[
\left( \frac{d}{dr} \right)^\alpha \tilde{m}(r\zeta) = (2\pi)^N r^{N-1-\alpha} |\zeta|^\alpha \left( \frac{d}{dt} \right)^\alpha \left[ (2\pi t)^{-\frac{N-2}{2}} J_{\frac{N}{2}-2}(2\pi t) \right] \bigg|_{t=r|\zeta|}. \tag{4.13}
\]

It is well-known that for every \( \nu \in \mathbb{R} \), we have
\[
\frac{d}{dt} J_\nu(t) = \frac{1}{2} (J_{\nu-1}(t) - J_{\nu+1}(t)),
\]
which extends to
\[
\left( \frac{d}{dt} \right)^\alpha J_\nu(t) = \frac{1}{2^\alpha} \sum_{j=0}^\alpha (-1)^j \binom{\alpha}{j} J_{\nu+2j}(t).
\]

By the Leibniz formula, we thus get
\[
\left( \frac{d}{dt} \right)^\alpha \left[ t^{-\nu} J_\nu(t) \right] = \sum_{k=0}^\alpha (-1)^k \binom{\alpha}{k} \frac{\Gamma(\nu+k)}{\Gamma(\nu)} t^{-\nu-k} J_{\nu-k}^{(\alpha-k)}(t)
\]
\[
= \sum_{k=0}^\alpha \sum_{j=0}^{\alpha-k} 2^{-\alpha+k} (-1)^{j+k} \binom{\alpha}{k} \frac{\Gamma(\nu+k)}{\Gamma(\nu)} t^{-\nu-k} \binom{\alpha-k}{j} J_{\nu+2j+\alpha-k}(t),
\]
which we can insert into (4.13). For \( \nu = \frac{N-2}{2} \), combining this equation with (4.12) yields
\[
\left( \frac{d}{dt} \right)^\alpha \left[ (2\pi t)^{-\frac{N-2}{2}} J_{\frac{N}{2}-2}(2\pi t) \right] = (2\pi)^\alpha \sum_{k=0}^\alpha 2^{-\alpha+k} \binom{\alpha}{k} (2\pi t)^{-\frac{N-2}{2}+k} B_{\alpha,k}(2\pi t).
\]

Thus (4.13) gives us
\[
\left( \frac{d}{dr} \right)^\alpha \tilde{m}(r\zeta) = (2\pi)^N r^{N-1+\alpha} |\zeta|^\alpha \sum_{k=0}^\alpha 2^{-\alpha+k} \binom{\alpha}{k} (2\pi r|\zeta|)^{-\frac{N-2}{2}+k} B_{\alpha,k}(2\pi r|\zeta|)
\]
\[
= 2\pi^{\alpha+1} r^{-\frac{N-2}{2}} \Gamma(\frac{N}{2}+\alpha) |\zeta|^{-\frac{N-2}{2}+\alpha} \sum_{k=0}^\alpha \binom{\alpha}{k} |r|^{-\alpha-k} B_{\alpha,k}(2\pi r|\zeta|),
\]
which is (4.11). Since \( R \) depends only on \( N \), all the parameters in (4.11) depend only on \( N \) and \( \alpha \). Since for every half-integer \( \nu \), \( J_\nu(t) = \mathcal{O}(t^{-1/2}) \) as \( t \to \infty \), (i) follows for each fixed \( r \in [1, 2] \), hence by continuity uniformly in \( r \).

(ii) From the series expansion for Bessel functions (2.10), we get
\[
\tilde{m}(\zeta) = 2\pi R^{\frac{N}{2}} |\zeta|^{-\frac{N-2}{2}} \Gamma(\frac{1}{2}) \left( \frac{2\pi |\zeta|}{2\pi R} \right)^{N/2} \sum_{k=0}^\infty (-1)^k \frac{\Gamma(k+1/2)}{\Gamma(k+N/2)} \frac{(2\pi R|\zeta|)^{2k}}{(2k)!} \prod_{j=0}^{k-1} \frac{j + 1/2}{j + N/2}.
\]

Hence we can apply an argument similar to (2.12) to see that
\[
|\tilde{m}(\zeta)| \leq e^{-c(2\pi R|\zeta|)^2}
\]

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holds for some \( c' > 0 \) if \( |\zeta| \) is close to 0, say \( 2\pi R|\zeta| < \varepsilon \) for some \( \varepsilon > 0 \). Otherwise, the integral representation of Bessel functions gives us

\[
\tilde{m}(\zeta) = \frac{\Gamma(N/2)}{\Gamma(N/2 - 1/2)} \sqrt{\pi} \int_{-1}^{1} e^{2\pi i R|\zeta| t} (1 - t^2)^{N/2 - 3/2} dt. \tag{4.15}
\]

But we have

\[
\left| \int_{-1}^{1} e^{2\pi i R|\zeta| t} (1 - t^2)^{N/2 - 3/2} dt \right| \leq 2 \int_{0}^{1} \left| e^{2\pi i R|\zeta| t} (1 - t^2)^{N/2 - 3/2} \right| dt \\
\leq 2 \int_{0}^{1} (1 - t^2)^{N/2 - 3/2} dt \\
= \frac{\Gamma(1/2)\Gamma(N-2)}{\Gamma(N/2)},
\]

with equality holding if and only if \( \zeta = 0 \). Combining this with (4.15) leads to

\[
|\tilde{m}(\zeta)| < 1
\]

for \( \zeta \neq 0 \), implying \(|\tilde{m}(\zeta)| \leq C' < 1 \) for \( \varepsilon \leq 2\pi R|\zeta| \leq 2C \). Choosing a positive \( c < c' \) such that \( C' \leq e^{-c} \) implies

\[
|\tilde{m}(r\zeta)| \leq e^{-c(2\pi R|\zeta|)^2}
\]

for \( 2\pi R|\zeta| \leq C \) and \( 1 \leq r \leq 2 \).

(iii) Similarly as in (ii), we get

\[
\left( \frac{d}{dr} \right)^{\alpha} \tilde{m}(r\zeta) = \frac{\Gamma(N/2)}{\Gamma(N/2 - 1/2)} (2\pi i R|\zeta|)^{\alpha} \int_{-1}^{1} e^{2\pi i Rr|\zeta| t^{\alpha}} (1 - t^2)^{N/2 - 3/2} dt. \tag{4.16}
\]

But we have

\[
\left| \int_{-1}^{1} e^{2\pi i Rr|\zeta| t^{\alpha}} (1 - t^2)^{N/2 - 3/2} dt \right| \leq 2 \int_{0}^{1} t^{\alpha} (1 - t^2)^{N/2 - 3/2} dt \\
= \frac{\Gamma(\alpha + 1)\Gamma(N-1)}{\Gamma(\alpha + N/2)}.
\]

Together with (4.16), this gives us

\[
\left| \left( \frac{d}{dr} \right)^{\alpha} \tilde{m}(r\zeta) \right| \leq (2\pi R|\zeta|)^{\alpha} \frac{\Gamma(\alpha + 1)\Gamma(N/2)}{\Gamma(1/2)\Gamma(\alpha + N/2)}.
\]
Hence (iii) follows if we can estimate

$$\frac{\Gamma(\frac{\alpha+1}{2})\Gamma(N/2)}{\Gamma(1/2)\Gamma(N+\alpha/2)} < 1,$$

But this can be established using Stirling’s formula: For each $x > 0$, we have

$$\left(\frac{2\pi}{x}\right)^{\frac{1}{2}} \left(\frac{x}{e}\right)^x \leq \Gamma(x) \leq \left(\frac{2\pi}{x}\right)^{\frac{1}{2}} \left(\frac{x}{e}\right)^x e^{\frac{\alpha - 1}{2}}.

Thus

$$\frac{\Gamma(\frac{\alpha+1}{2})\Gamma(N/2)}{\Gamma(1/2)\Gamma(N+\alpha/2)} \leq \frac{1}{\sqrt{\pi}} \left(\frac{\pi(\alpha + N)}{N(\alpha + 1)}\right)^{\frac{1}{2}} \left(\frac{\alpha + 1}{2e}\right)^{\frac{\alpha+1}{2}} \left(\frac{N}{2e}\right)^{\frac{N}{2}} e^{\frac{\alpha - 1}{2}}

\leq \left(\frac{\alpha + N}{N(\alpha + 1)}\right)^{\frac{1}{2}} \left(\frac{\alpha + 1}{2e}\right)^{\frac{\alpha+1}{2}} \left(\frac{N}{2e}\right)^{\frac{N}{2}} e^{\frac{\alpha - 1}{2}}

= \frac{1}{\sqrt{2}} \left(\frac{\alpha + 1}{2}\right)^{\frac{N}{2}} \left(\frac{N}{2}\right)^{\frac{N}{2}} e^{\frac{\alpha - 1}{2}}

< \frac{e^{-1/3}}{\sqrt{2}}.$$

(iv) By deriving the series expansion (4.14) in the proof of (ii), we get

$$\left(\frac{d}{dr}\right)^\alpha \hat{m}(r|\zeta) = (2\pi R|\zeta|)^{2(\frac{\alpha+1}{2})} \sum_{2k\geq\alpha} (-1)^k r^{2k-\alpha}\frac{(2\pi R|\zeta|)^{2(k-\frac{\alpha+1}{2})}}{(2k-\alpha)!} \frac{k-1}{j + N/2}.$$

Since $\alpha > 0$, we use that

$$(2\pi R|\zeta|)^{2(\frac{\alpha+1}{2})} \leq (2\pi R|\zeta|)^2,$$

since $2\pi R|\zeta| < 1$. To conclude the proof, we have to estimate

$$\sum_{k=\lceil\frac{\alpha+1}{2}\rceil}^{\infty} (-1)^k r^{2k-\alpha}\frac{t^{2(k-\frac{\alpha+1}{2})}}{(2k-\alpha)!} \frac{k-1}{j + N/2} \leq C' < 1 \quad (4.17)$$

for $t = 2\pi R|\zeta| < 1$. If $\alpha$ is odd, then $\lfloor\frac{\alpha+1}{2}\rfloor = \frac{\alpha+1}{2}$, and we get

$$\sum_{k=\lceil\frac{\alpha+1}{2}\rceil}^{\infty} (-1)^k r^{2k-\alpha}\frac{t^{2(k-\frac{\alpha+1}{2})}}{(2k-\alpha)!} \frac{k-1}{j + N/2} = r \sum_{k=0}^{\infty} (-1)^k r^{2k+\alpha}\frac{(rt)^{2k}}{(2k+1)!} \frac{k+\alpha+1}{j + N/2} \cdot \rho_{\alpha+1} \quad (4.18)$$

But since $rt < 2$, one can use $(rt)^{2k+2} \leq 4(rt)^{2k}$ and $(2k+3)! \geq 6 \cdot (2k+1)!$ for each $k \in \mathbb{N}$ to see that the summands of the alternating right hand side series in (4.18) are strictly decreasing in $k$. Hence

$$\left| r \sum_{k=0}^{\infty} (-1)^k r^{2k+\alpha}\frac{(rt)^{2k}}{(2k+1)!} \frac{k+\alpha+1}{j + N/2} \right| \leq r \sum_{k=0}^{\frac{\alpha+1}{2}} \frac{j + 1/2}{j + N/2} = \frac{2}{N} < 1,$$

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If $\alpha$ is even, then $\lfloor \frac{\alpha + 1}{2} \rfloor = \frac{\alpha}{2}$, and similarly, we get

$$
\sum_{k=\alpha/2}^{\infty} (-1)^k r^{2k-\alpha} \frac{t^{2k-\alpha}}{(2k-\alpha)!} \prod_{j=0}^{k-1} \frac{j + 1/2}{j + N/2} = \sum_{k=0}^{\infty} (-1)^{k+\alpha/2} \frac{(rt)^{2k+\alpha/2-\alpha}}{(2k)!} \prod_{j=0}^{k-1} \frac{j + 1/2}{j + N/2}. \quad (4.19)
$$

Here, a similar argument provides that since $rt < 2$, the summands of the alternating right hand side series of (4.19) are decreasing from $k \geq 1$. Thus

$$
\left| \sum_{k=0}^{\infty} (-1)^{k+\alpha/2} \frac{(rt)^{2k+\alpha/2-\alpha}}{(2k)!} \prod_{j=0}^{k-1} \frac{j + 1/2}{j + N/2} \right| \leq \max \left\{ \frac{1}{N} + \frac{(rt)^{4}}{4!}, \frac{1}{N} + \frac{3}{N + 2}, \frac{5}{2!} \frac{(rt)^{2}}{N + 2} \right\} \leq \max \left\{ \frac{5}{3N}, \frac{1}{N + 2} \right\} = \frac{5}{3N} < 1.
$$

q.e.d.

**Remark.** The proofs of (iii) and (iv) even work for $0 < r \leq 2$, which will be necessary to show the conditions in Lemma 4.3.

Other than the Leibniz formula (4.10), we need another form of $\frac{d^i}{dr^i} m(r \xi)$ to apply the estimates from Lemma 4.3.

**Lemma 4.4.** For each $j \in \mathbb{N}$, and $a = (a_1, \ldots, a_j) \in \{1, \ldots, \ell\}^j$, define the sequence $\alpha(a) \in \mathbb{N}^\ell$ by $\alpha(a)_k = |\{j' : a_{j'} = k\}|$.

Then

$$
\left( \frac{d}{dr} \right)^j m(r \xi) = \sum_{a \in \{1, \ldots, \ell\}^j} \prod_{k=1}^{\ell} \frac{(d)}{dr} \alpha(a)_k \tilde{m}(r \zeta_k). \quad (4.20)
$$

**Proof.** We use induction on $j$, where the case $j = 0$ is obvious. So assume that (4.20) is true for some $j \in \mathbb{N}$. Then

$$
\left( \frac{d}{dr} \right)^{j+1} m(r \xi) = \sum_{a \in \{1, \ldots, \ell\}^j} \frac{d}{dr} \left[ \prod_{k=1}^{\ell} \frac{(d)}{dr} \alpha(a)_k \tilde{m}(r \zeta_k) \right] = \sum_{a \in \{1, \ldots, \ell\}^j} \prod_{k=1}^{\ell} \frac{(d)}{dr} \alpha(a)_k \tilde{m}(r \zeta_k) \prod_{k' \neq k} \frac{(d)}{dr} \alpha(a)_{k'} \tilde{m}(r \zeta_k) = \sum_{a \in \{1, \ldots, \ell\}^j} \prod_{k=1}^{\ell} \frac{(d)}{dr} \alpha(a)_{k'} + \delta_{k,k'} \tilde{m}(r \zeta_k).
$$

Since for each $a \in \{1, \ldots, \ell\}^j$, $k \in \{1, \ldots, \ell\}$, the sequence $(a, k) \in \{1, \ldots, \ell\}^{j+1}$ has the property

\[ \alpha((a, k))_{k'} = \alpha(a)_{k'} + \delta_{k,k'}, \]
it follows that
\[
\left(\frac{d}{dr}\right)^{j+1} m(r\xi) = \sum_{a \in \{1, \ldots, \ell\}} \prod_{k=1}^{\ell} \left(\frac{d}{dr}\right)^{\alpha(a)_k} \tilde{m}(r\zeta_k).
\]

q.e.d.

With this, we can estimate the derivatives of \(m\) straightforward.

**Lemma 4.5.** Let \(K = \lceil \frac{N-1}{2} \rceil\). For every \(\xi \in \mathbb{R}^n\) and every \(r \in [1, 2]\), we have
\[
(1 + |\xi|)^{-\frac{N-1}{2} - K} \left| \left(\frac{d}{dr}\right)^{K} m(r\xi) \right| \leq C_N. \tag{4.21}
\]

**Proof.** Fix \(0 \neq \xi = (\zeta_1, \ldots, \zeta_\ell)\) and put \(t = (t_1, \ldots, t_\ell), t_k := 2\pi R|\zeta_k|, 1 \leq k \leq \ell\). By Lemma 4.3 (i), we can find a constant \(A_N\) such that for \(2\pi R|\zeta_k| > A_N\) and \(0 \leq j < K\),
\[
\left| \left(\frac{d}{dr}\right)^j \tilde{m}(r\zeta_k) \right| \leq \frac{1}{2}. \tag{4.22}
\]

By the Leibniz formula, we get
\[
\left| \left(\frac{d}{dr}\right)^{K} m(r\xi) \right| = \sum_{j=0}^{K} {K \choose j} \left(\frac{d}{dr}\right)^{j} \left[ \prod_{t_k \leq A_N} \tilde{m}(r\zeta_k) \right] \cdot \left(\frac{d}{dr}\right)^{K-j} \left[ \prod_{t_k > A_N} \tilde{m}(r\zeta_k) \right]. \tag{4.23}
\]

By rearranging the variables, we can assume that \(\{k : t_k \leq A_N\} = \{1, \ldots, \ell'\}\) for some \(\ell' \leq \ell\).
Assume further that \(K = N/2\). In that case, we thus have to find bounds
\[
(1 + |\xi|)^{-1/2} \left| \left(\frac{d}{dr}\right)^{j} \left[ \prod_{k=\ell'+1}^{\ell} \tilde{m}(r\zeta_k) \right] \right| \leq C_N \tag{4.24}
\]
and
\[
\left| \left(\frac{d}{dr}\right)^{j} \left[ \prod_{k=1}^{\ell} \tilde{m}(r\zeta_k) \right] \right| \leq C_N \tag{4.25}
\]
for each \(j \leq K\), with \(C_N\) being independent of \(\ell\) and \(\ell'\). As of (4.25), we shall assume that also \(\{k : t_k < 1\} = \{1, \ldots, \ell''\}\) for some \(\ell'' \leq \ell'\). Then by using the Leibniz formula as in (4.23), instead of (4.25) we have to find bounds
\[
\left| \left(\frac{d}{dr}\right)^{j} \left[ \prod_{k=\ell''+1}^{\ell} \tilde{m}(r\zeta_k) \right] \right| \leq C_N \tag{4.25'}
\]
and
\[
\left| \left(\frac{d}{dr}\right)^{j} \left[ \prod_{k=1}^{\ell''} \tilde{m}(r\zeta_k) \right] \right| \leq C_N, \tag{4.26}
\]

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\[ j \leq K, C_N \text{ independent of } \ell, \ell', \text{ and } \ell'' \]. One further assumption will be \( \ell'' < \ell' < \ell \), since otherwise, at least one of our required estimates would be trivial. We will now show (4.24),

\[
\left( \frac{d}{dr} \right)^j \left[ \prod_{k=\ell'+1}^{\ell} \tilde{m}(r\zeta_k) \right] = \sum_{\alpha \in \mathbb{N}^{\ell'-\ell}} \sum_{k=1}^{\ell} \left( \frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_{k+\ell'}) = \sum_{|\alpha| = j} \sum_{k=1}^{\ell} \left( \frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_{k+\ell'}) + \sum_{|\alpha| = j} \sum_{\forall k \alpha_k < K} \left( \frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_{k+\ell'}) \quad (4.27)
\]

Due to the choice of \( A_N \), (4.22) yields

\[
(1 + |\xi|)^{-\frac{1}{2}} \sum_{|\alpha| = j} \sum_{\forall k \alpha_k < K} \left( \frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_{k+\ell'}) \leq \sum_{|\alpha| = j} \sum_{\forall k \alpha_k < K} \left( \frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_{k+\ell'}) < C. \quad (4.28)
\]

The second sum in (4.27) is equal to 0 if \( j < K \). If \( j = K \), we use part (i) of Lemma 4.3 to get

\[
(1 + |\xi|)^{-\frac{1}{2}} \sum_{|\alpha| = j} \sum_{\exists k \alpha_k = K} \left( \frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_{k+\ell'}) \leq \sum_{|\alpha| = j} \sum_{\exists k \alpha_k = K} \left( \frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_{k+\ell'}) \leq \sum_{k=\ell'+1}^{\ell} (1 + |\xi|)^{-\frac{1}{2}} \tilde{C}_N |\zeta_k|^{1/2} \frac{2^{\ell - \ell'} - 1}{2^{\ell' - 1}} < \frac{\tilde{C}_N (\ell - \ell')}{2^{\ell' - 1}} < C_N \quad (4.29)
\]

Combining (4.27), (4.28), and (4.29) gives us (4.24).

Coming to (4.25), let \( \ell' = (t_{\ell' + 1}, \ldots, t_{\ell}) \). By Lemma 4.3 (ii) and (iii), we get

\[
\left| \left( \frac{d}{dr} \right)^j \left[ \prod_{k=\ell'+1}^{\ell'} \tilde{m}(r\zeta_k) \right] \right| = \sum_{|\alpha| = j} \sum_{\alpha \in \mathbb{N}^{\ell' - \ell''}} \left( \frac{d}{dr} \right)^{\alpha_k} \tilde{m}(r\zeta_{k+\ell''})
\]

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To conclude the estimate, we will partition the set $\{\ell'\}$. Hence by (4.30), we get

\[
\left( \frac{d}{dr} \right)^j \left[ \prod_{k=1}^{\ell''} \tilde{m}(r \xi_k) \right] = \sum_{a \in \{1, \ldots, \ell''\}} \prod_{k=1}^{\ell''} \left( \frac{d}{dr} \right)^{\alpha(a)_k} \tilde{m}(r \xi_k).
\]

Let $t'' = (t_1, \ldots, t_{\ell''})$. Lemma 4.3 (ii) and (iv) now lead to

\[
\left| \left( \frac{d}{dr} \right)^j m(r \xi) \right| \leq e^{-c_N |t''|^2} \prod_{\alpha(a)_k \neq 0} t_k^2. \tag{4.30}
\]

To conclude the estimate, we will partition the set $\{1, \ldots, \ell''\}^j$ as follows. For each $a \in \{1, \ldots, \ell''\}$, we have $|\{k : \alpha(a)_k \neq 0\}| \leq j$. For $i \in \{1, \ldots, j\}$, put

\[ A_i := \{ a \in \{1, \ldots, \ell''\} : |\{k : \alpha(a)_k \neq 0\}| = i \}. \]

Then clearly

\[
\sum_{a \in A_i} \prod_{\alpha(a)_k \neq 0} t_k^2 \leq \sum_{k_1, \ldots, k_i = 1} t_{k_1}^2 \cdots t_{k_i}^2.
\]

Hence by (4.30), we get

\[
\left( \frac{d}{dr} \right)^j \left[ \prod_{k=1}^{\ell''} \tilde{m}(r \xi_k) \right] \leq e^{-c_N |t''|^2} \sum_{i=1}^{j} \sum_{a \in A_i} \prod_{\alpha(a)_k \neq 0} t_k^2 \\
\leq e^{-c_N |t''|^2} \left( \sum_{k_1 = 1}^{\ell''} t_{k_1}^2 + \sum_{k_1, k_2 = 1}^{\ell''} t_{k_1}^2 t_{k_2}^2 + \cdots + \sum_{k_1, \ldots, k_j = 1}^{\ell''} t_{k_1}^2 \cdots t_{k_j}^2 \right) \\
= e^{-c_N |t''|^2} \sum_{k=1}^{\ell''} \frac{|t''|^{2k}}{2^k - 1} < C_N.
\]

This concludes the proof for $K = N/2$. The arguments for the case $K = N-1$ are basically the same. We also have to estimate (4.25) and (4.26), but we have to drop the factor $(1 + |\xi|)^{-1/2}$ in (4.24). Then the only difference is that the sum

\[
\sum_{k=\ell' + 1}^{\ell} (1 + |\xi|)^{-1/2} \frac{\tilde{C}_N |\xi_k|^{1/2}}{2^{\ell' - \ell' - 1}}
\]

is bounded by
in (4.20) becomes
\[ \sum_{k=e+1}^{\ell} \frac{\tilde{C}_N}{2^{\ell-e-1}}, \]
which makes the estimate even slightly easier. \textbf{q.e.d.}

Now, we are able to show that \( \sigma_S - P dx \) fulfills (4.3) and (4.4), which allows us to apply Carbery’s interpolation argument.

\begin{lemma}
There is a constant \( C_N \) such that for every \( \theta \in S^{n-1} \) and every \( u \in \mathbb{R}^* \)
\[ |m(u\theta) - \hat{P}(u\theta)| \leq C_N \cdot \min\{|u|, |u|^{-1}\}, \]
\[ |\langle \theta, \nabla (m - \hat{P})(u\theta) \rangle| \leq C_N \cdot \min\{|1, |u|^{-1}\}. \tag{4.31} \]
\[ |\langle \theta, \nabla (m - \hat{P})(u\theta) \rangle| \leq C \cdot \min\{1, |u|^{-1}\}. \tag{4.32} \]
\end{lemma}

\begin{proof}
We clearly have
\[ |u \cdot \hat{P}(u\theta)| \leq C, \quad |1 - \hat{P}(u\theta)| \leq C|u|, \quad \text{and} \quad |\langle \theta, \nabla \hat{P}(u\theta) \rangle| \leq C \cdot \min\{1, |u|^{-1}\}. \]
From (1.2), we also know that
\[ |u \cdot m(u\theta)| \leq C_N. \]
We know that \( m \) and \( \hat{P} \) are both bounded. Also, from (4.14), a similar argument as in (2.12) yields that
\[ |m(u\theta)| \geq e^{-c_N u^2} \]
for \( |u| \leq 2 \) and some \( c_N > 0 \) chosen sufficiently small. Hence
\[ |1 - m(u\theta)| \leq 1 - e^{-c_N u^2} \leq C_N u^2 \leq C_N |u| \]
if \( |u| \leq 2 \), while \( |1 - m(u\theta)| < |u| \) is obvious for \( |u| > 2 \). Since
\[ |m(u\theta) - \hat{P}(u\theta)| \leq |m(u\theta)| + |\hat{P}(u\theta)| \quad \text{and} \quad |m(u\theta) - \hat{P}(u\theta)| \leq |1 - m(u\theta)| + |1 - \hat{P}(u\theta)|, \]
(4.31) follows. For (4.32), it remains to show that
\[ |\langle \theta, \nabla m(u\theta) \rangle| \leq C_N \cdot \min\{1, |u|^{-1}\}, \]
If \( |u| \geq 1 \), this follows from (1.2), since
\[ |u \langle \theta, \nabla m(u\theta) \rangle| = |\langle u\theta, \nabla m(u\theta) \rangle| \leq C. \]
Otherwise, assume that \( 0 < u < 1 \). Write \( \theta = (\zeta_1, \ldots, \zeta_L) \) with \( \zeta_k \in \mathbb{R}^N \) as before. Since \( |\theta| = 1 \), we can find \( C_N \) such that \( 2\pi R|\zeta_k| \leq C_N \) for each \( k \). By Lemma 4.3 (iii) and (iv) and the remark after its proof, we have
\[ \left| \frac{d}{du} \tilde{m}(u\zeta_k) \right| < (2\pi R|\zeta_k|)^2. \]
Hence
\[ |\langle \theta, \nabla m(u\theta) \rangle| = \left| \frac{d}{du} m(u\theta) \right| = \sum_{k=1}^{\ell} \left| \frac{d}{du} \tilde{m}(u\zeta_k) \right| \prod_{k' \neq k} |\tilde{m}(u\zeta_{k'})| \leq \sum_{k=1}^{\ell} (2\pi R\zeta_k)^2 = (2\pi R)^2. \]
This concludes the estimates. \textbf{q.e.d.}
This leaves us with finding some \( \alpha \) with \( \frac{1}{p} < \alpha < 1 \) such that \( m_\alpha \) is an \( L^p \)-multiplier. We can’t interpolate the corresponding multiplier operators of the family \( (m_z)_{0 \leq Re \, z < \frac{N-1}{2}} \) directly, but interpolation is still possible. For this, assume that \( \frac{N}{N-1} < p < \frac{N}{N-2} \), and let \( \frac{1}{p} = 1 - \theta + \frac{\theta}{2} \), i.e. \( \theta = 2 - 2/p \). Then \( \frac{N-1}{2} \theta < 1 < \frac{N}{2} \theta \). Fix \( \varepsilon \) such that \( 0 < \varepsilon < \frac{N}{2} \theta - 1 \), and set
\[
\alpha := \frac{N - 1}{2} \theta - \varepsilon.
\]
Then \( \alpha > 1 - \frac{\theta}{2} = \frac{1}{p} \). If \( N \) is odd, \( \frac{N-1}{2} \) is an integer, and one can use the formulas \((4.7)\) and \((4.8)\) to argue as in Section 7.3 and Lemma 7.5 of [3], leaving us with bounding
\[
\left( \frac{d}{dr} \right)^j m(r \xi)
\]
for \( j \in \{0, \ldots, \frac{N-1}{2} - 1\} \), which we technically did not prove yet. But this is covered by the estimates \((1.23)\), \((1.25)\), and \((1.26)\) in the proof of Lemma \ref{Lemma1}. We get that \( m_z \) is an \( L^1 \)-multiplier for \( Re \, z = -\varepsilon \) and an \( L^2 \)-multiplier for \( Re \, z = \frac{N-1}{2} - \varepsilon \) so that the family
\[
(m_{\frac{N-1}{2}z-\varepsilon})_{0 \leq Re \, z \leq 1}
\]
is analytic and of admissible growth in the sense of Stein’s interpolation theorem. For \( p > \frac{N}{N-2} \), we can interpolate with the endpoint \( \infty \). This already suffices to prove Theorem \ref{Theorem1}, since one could simply go from \( N \) to \( N + 1 \) if \( N \) is even, but for even \( N \), Theorem \ref{Theorem2} still holds. In that case, we need to interpolate twice, paying further attention to the upcoming bounds of the multiplier operators. Suppose we have an analytic family of operators \( (T_z) \) with \( 0 \leq Re \, z \leq 1 \) so that
\[
\|T_{it}\|_{p_0 \rightarrow p_0} \leq M_0(t) \quad \text{and} \quad \|T_{1+it}\|_{p_1 \rightarrow p_1} \leq M_1(t),
\]
and that we have \( b < \pi \) such that
\[
\sup_{t \in \mathbb{R}} e^{-b|t|} \log M_j(t) < \infty,
\]
\( j = 0, 1 \). Then \( \|T_\theta\|_{p \rightarrow p} \leq M(\theta) \) for \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \) and for instance from [7, p. 38], it follows that
\[
M(t) = \exp \left[ \frac{\sin(\pi t)}{2} \int_{\mathbb{R}} \left( \frac{\log M_0(y)}{\cosh(\pi y) - \cos \pi t} + \frac{\log M_1(y)}{\cosh(\pi y) + \cos \pi t} \right) dy \right]. \tag{4.33}
\]
For \( 0 < \varepsilon < 1 \), we consider the multipliers \( m_\varepsilon^z(\xi) = (1 + |\xi|)^{-\frac{N-1}{2} - \varepsilon - z} \) \( m_z(\xi) \), similarly as in [8]. For \( -\varepsilon \leq Re \, z \leq \frac{N}{2} \), Lemma 7.4 in [6] gives us the precise bounds
\[
|m_{r+it}^z(\xi)| \leq C_N \varepsilon^{-1} (1 + t^2)^{\frac{N-1}{4} + \frac{\varepsilon}{2}},
\]
where \( C_N \) depends on the bounds of
\[
(1 + |\xi|)^{-1/2} \left( \frac{d}{dr} \right)^j m(r \xi)
\]
for $j \in \{0, \ldots, \frac{N-1}{2} - 1\}$, which again can be achieved using the arguments in the proof of Lemma 4.5. Hence, for every $t \in \mathbb{R}$, the family $(m_{\frac{N-1}{2} - \varepsilon + it})_z$ of $L^2$-multipliers has admissible growth for $0 \leq \text{Re} z \leq 1$, and by interpolation, (4.33) implies

$$|m_{\frac{N-1}{2} - \varepsilon + it}(\xi)| \leq e^{C_N|t|}.$$ 

Thus we also have admissible growth for $\text{Re} z = \frac{N-1}{2} - \varepsilon$, $N$ even, and taking $\varepsilon$ and $\alpha$ as above, this proves Theorem 2.

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