Lie-admissible algebras and Operads

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Abstract
A Lie-admissible algebra gives a Lie algebra by anticommutativity. In this work we describe remarkable types of Lie-admissible algebras such as Vinberg algebras, pre-Lie algebras or Lie algebras. We compute the corresponding binary quadratic operads and study their duality. Considering Lie algebras as Lie-admissible algebras, we can define for each Lie algebra a cohomology with values in an Lie-admissible module. This permits to study some deformations of Lie algebras, in particular classes of Lie-admissible algebras such as Vinberg algebras or pre-Lie algebras.

1 Lie-admissible algebras

1.1 Definition
Let \( A = (A, \mu) \) be a finite dimensional algebra on a commutative field \( K \) of characteristic zero. In this notation, \( \mu \) is the law of \( A \), that is a linear mapping \( \mu : A \otimes A \rightarrow A \)
on the vector space \( A \).

We denote by \( a_\mu : A^{\otimes 3} \rightarrow A \) the associator of the law \( \mu : \)
\[
a_\mu (X_1, X_2, X_3) = \mu (X_1, \mu (X_2, X_3)) - \mu (\mu (X_1, X_2), X_3).
\]
Let \( \sum_n \) be the symmetric group of degree \( n \). For every \( \sigma \in \sum_3 \), we put
\[
\sigma (X_1, X_2, X_3) = (X_{\sigma^{-1}(1)}, X_{\sigma^{-1}(2)}, X_{\sigma^{-1}(3)}).
\]

**Definition 1.1** The algebra \( A = (A, \mu) \) is called Lie-admissible if the law \( \mu \) satisfies
\[
\sum_{\sigma \in \sum_3} (-1)^{\varepsilon(\sigma)} a_\mu \circ \sigma = 0. \tag{*}
\]

This definition ([A]) is equivalent to say that the mapping \([.,.]_\mu : A \otimes A \rightarrow A\) defined by \([X, Y] = \mu(X, Y) - \mu(Y, X)\) is a Lie bracket. We will denote by \( A_L \) the corresponding Lie algebra.

1
1.2 Examples

1. Every associative algebra (not necessarily unitary) is a Lie-admissible algebra.
2. An algebra \( \mathcal{A} = (A, \mu) \) is a Vinberg algebra (also called left symmetric algebra) if its law satisfies
   \[
   \mu(X, \mu(Y, Z)) - \mu(Y, \mu(X, Z)) = \mu(\mu(X, Y), Z) - \mu(\mu(Y, X), Z).
   \]
   It is also a Lie-admissible algebra.
3. Of course a Lie algebra law is a law of Lie-admissible algebra, the Jacobi conditions implying (\(*\)).
4. A pre-Lie algebra is defined by a law \( \mu \) such that
   \[
   \mu(\mu(X, Y), Z) - \mu(X, \mu(Y, Z)) = \mu(\mu(X, Z), Y) - \mu(X, \mu(Z, Y)).
   \]
   The bracket \([.,.]_{\mu}\) being a Lie bracket, a pre-Lie algebra is a Lie-admissible algebra.

**Remarks.** Every associative algebra is a Vinberg algebra and a pre-Lie algebra. A Vinberg algebra is a pre-Lie algebra if and only if all the associators are equal
   \[
   a_{\mu} \circ \sigma = a_{\mu} \circ v
   \]
   for every \( v \) and \( \sigma \) in \( \Sigma_3 \).
   A Lie algebra is associative (respectively Vinberg, respectively pre-Lie algebra) if and only if it is 2-step nilpotent.

1.3 Geometrical interpretation

Let \( \mathfrak{g} \) be a real Lie algebra provided with an affine structure. The associated covariant derivative \( \nabla \) satisfies
   \[
   \left\{ \begin{array}{l}
   \nabla_X Y - \nabla_Y X - [X, Y] = 0 \\
   \nabla X \nabla Y - \nabla Y \nabla X = \nabla [X,Y]
   \end{array} \right.
   \]
Then the product \( \mu(X,Y) = \nabla_X Y \) endows the vector space \( \mathfrak{g} \) with a structure of Vinberg algebra. As we have \([X,Y] = \mu(X,Y) - \mu(Y,X)\), every Lie algebra provided with an affine structure is subordinated to a Lie-admissible algebra which is in this case a Vinberg algebra. More generally, every Lie algebra is subordinated to a Lie-admissible algebra. In fact it is sufficient to consider Levi Civita connections (which always exists). As the torsion vanishes, the covariant derivative satisfies the first of the previous equations and \( \mu(X,Y) = \nabla_X Y \) is a law of Lie-admissible algebra such that \([X,Y] = [X,Y]_{\mu}\).

This permits to consider the set of Lie-admissible algebras as the set of invariant linear connections torsionless on Lie algebras. In fact, if \( \mu \) is a law of admissible algebra, putting \( \nabla_X Y = \mu(X,Y) \) then \( \nabla \) defines a linear connection if the Bianchi identities are satisfied. Let us note \( R(X,Y) \) the curvature tensor corresponding to \( \nabla \). As \( \mu \) is an Lie-admissible law, we have
\( R(X,Y).Z + R(Y,Z).X + R(Z,X).Y = 0 \) then the first identity is realized. For the second we can prove that

\[
(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0.
\]

Using the relations \((\nabla_X R)(Y, Z) = \nabla_X (R(Y, Z)) - R(\nabla_X Y, Z) - R(Y, \nabla_X Z)\),
and \(\nabla_{[X,Y]} = \nabla_{\nabla_X Y} - \nabla_{\nabla_Y X}\), we deduce the second identity.

### 1.4 Actions of the symmetric group \( \Sigma_3 \)

**Definition 1.2** Let \( G \) be a subgroup of \( \Sigma_3 \). We say that the algebra law is \( G \)-associative if

\[
\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} a_{\mu} \circ \sigma = 0.
\]

The subgroups of \( \Sigma_3 \) are well known. We have \( G_1 = \{Id\}, G_2 = \{Id, \tau_{12}\}, G_3 = \{Id, \tau_{23}\}, G_4 = \{Id, \tau_{13}\}, G_5 = \{Id, (231), (312)\} = A_3 \) (the alternating group), \( G_6 = \Sigma_3 \) where \( \tau_{ij} \) is the transposition between \( i \) and \( j \) and \( (231) \) a 3-cycle.

We deduce the following type of Lie-admissible algebras:

1. If \( \mu \) is \( G_1 \)-associative then \( \mu \) is an associative law.
2. If \( \mu \) is \( G_2 \)-associative then \( \mu \) is a law of Vinberg algebra (also called left symmetric algebras [N]). If \( A \) is finite-dimensional, the associated Lie algebra is provided with an affine structure.
3. If \( \mu \) is \( G_3 \)-associative then \( \mu \) is a law of pre-Lie algebra [G], also called right symmetric algebras. Let us note that if \( x.y \) is pre-Lie law then \( x \circ y = y.x \) is a Vinberg law.
4. If \( \mu \) is \( G_4 \)-associative then \( \mu \) satisfies

\[
(X.Y).Z - X.(Y.Z) = (Z.Y).X - Z.(Y.X)
\]

5. If \( \mu \) is \( G_5 \)-associative then \( \mu \) satisfies the generalized Jacobi condition :

\[
(X.Y).Z + (Y.Z).X + (Z.X).Y = X.(Y.Z) + Y.(Z.X) + Z.(X.Y)
\]

Moreover if the law is antisymmetric, then it is a law of Lie algebra.

6. If \( \mu \) is \( G_6 \)-associative then \( \mu \) is a Lie-admissible law.

**Example.** Let us consider the vector space \( C^\infty(\mathbb{R}, \mathbb{R}) \) of real infinitely derivable functions with values in \( \mathbb{R} \). We can endow this space with the following algebra structures:

1. \( \mu_1(f, g) = f.g \) and \( (C^\infty(\mathbb{R}, \mathbb{R}), \mu_1) \) is associative (type 1)
2. \( \mu_2(f, g) = f.g' \) and \( (C^\infty(\mathbb{R}, \mathbb{R}), \mu_2) \) is a Vinberg algebra (type 2)
3. \( \mu_3(f, g) = f'.g \) and \( (C^\infty(\mathbb{R}, \mathbb{R}), \mu_3) \) is a pre-Lie algebra (type 3)
4. \( \mu_4(f, g) = f'.g + f.g' \) and \( (C^\infty(\mathbb{R}, \mathbb{R}), \mu_4) \) is invariant by \( G_4 \) (type 4).

Suppose that \( \mu \) is invariant by \( G_5 \). If this law is also antisymmetric then it is a Lie algebra law. This shows that the definition of \( G_5 \)-invariance gives another generalization of "non-commutative" Lie algebra than the notion of Leibniz algebra.
1.5 $G$-cogebras

In this section we introduce the notion of cogebra dualizing the $G_i$-associative algebras. This leads to present the definition in the category $\text{Vect}_K$ of $K$-vector spaces.

**Definition 1.3** A $G_i$-associative algebra is a pair $(A, \mu)$ where $A$ is a vector space and $\mu : A \otimes A \longrightarrow A$ a linear mapping satisfying the following axiom ($G_i$-ass):

The square

$$
\begin{array}{ccc}
  A \otimes A \otimes A & \xrightarrow{(\mu \otimes \text{Id})_{G_i}} & A \otimes A \\
  (\text{Id} \otimes \mu)_{G_i} & \downarrow & \mu \\
  A \otimes A & \xrightarrow{\mu} & A
\end{array}
$$

commutes, where $(\text{Id} \otimes \mu)_{G_i}$ is the linear mapping defined by:

$$(\text{Id} \otimes \mu)_{G_i} = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)}(\text{Id} \otimes \mu) \circ \sigma.$$ 

Let $\Delta$ be a comultiplication on a vector space $C$:

$$\Delta : C \longrightarrow C \otimes C.$$ 

We define the bilinear mapping $G_i \circ (\Delta \otimes \text{Id})$ by

$$G_i \circ (\Delta \otimes \text{Id}) = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)}(\Delta \otimes \text{Id}) \circ (\Delta \otimes \text{Id}).$$

**Definition 1.4** A $G_i$-cogebra is a vector space $C$ provided with a comultiplication $\Delta : C \longrightarrow C \otimes C$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
  C & \xrightarrow{\Delta} & C \otimes C \\
  \Delta & \downarrow & \downarrow_{G_i \circ (\Delta \otimes \Delta)} \\
  C \otimes C & \xrightarrow{G_i \circ (\Delta \otimes \text{Id})} & C \otimes C \otimes C.
\end{array}
$$

The next results relate $G_i$-algebras and $G_i$-cogebras.

**Proposition 1.1** The dual space of a $G_i$-cogebra is a $G_j$-associative algebra.

**Proof.** Let $(C, \Delta)$ a $G_i$-cogebra. Let us consider the maps $\lambda : C^* \otimes C^* \rightarrow (C \otimes C)^*$ given by $\lambda(f \otimes g)(v \otimes u) = f(u) \otimes g(v)$ and $\tau : C^* \otimes C^* \rightarrow C^* \otimes C^*$ defined by $\tau(u \otimes v) = v \otimes u$. Let us consider the law $\mu = \Delta^* \circ \lambda \circ \tau$. It provides the dual space $C^*$ with a $G_j$-associative algebra. In fact we have

$$\mu(f_1 \otimes f_2)(x) = \Delta^* \circ \lambda(f_2 \otimes f_1)(x) = (f_2 \otimes f_1)(\Delta(x)).$$
for all $f_1, f_2 \in C^*$, $x \in C$. We denote $X \otimes Y = \Delta(x), X_1 \otimes X_2 = \Delta(X)$ and $Y_1 \otimes Y_2 = \Delta(Y)$ (we use the sigma notation and we forget the sigma). This gives

$$\mu(f_1 \otimes f_2)(x) = f_1(X)f_2(Y).$$

The associator of $\mu$ is written:

$$\mu(\mu(f_1 \otimes f_2 \otimes f_3))(x) - \mu(f_1 \otimes \mu(f_2 \otimes f_3))(x) = (f_1 \otimes f_2 \otimes f_3)(X_1 \otimes X_2 \otimes Y - X \otimes Y_1 \otimes Y_2)$$

and the property of the comultiplication $\Delta$ is equivalent to

$$\sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma \circ (\Delta \otimes Id)(\Delta(x)) = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma \circ (Id \otimes \Delta)(\Delta(x))$$

that is

$$\sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma(\Delta(X) \otimes Y) = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma(X \otimes \Delta(Y))$$

or

$$\sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma(X_1 \otimes X_2 \otimes Y - X \otimes Y_1 \otimes Y_2) = 0.$$

This relation proves that the associator of $\mu$ satisfies the same relation.

**Proposition 1.2** The dual vector space of a finite dimensional $G_i$-associative algebra has a $G_i$-cogebra structure.

**Proof.** Let $A$ a finite dimensional $G_i$-associative algebra and let $\{e_i, i = 1, ..., n\}$ be a basis of $A$. If $\{f_i\}$ is the dual basis then $\{f_i \otimes f_j\}$ is a basis of $A^* \otimes A^*$. Let us put

$$\Delta(f) = \sum_{i,j} f(\mu(e_i \otimes e_j)) f_i \otimes f_j.$$

In particular

$$\Delta(f_k) = \sum_{i,j} C_{ij}^{k} f_i \otimes f_j$$

where $C_{ij}$ are the structure constants of $\mu$ related to the basis $\{e_i\}$. Then $\Delta$ is the comultiplication of a $G_i$-cogebra.

## 2 Lie-Admissible operads

### 2.1 Binary quadratic operads

Let $\mathbb{K}[\sum_n]$ be the $\mathbb{K}$-algebra of the symmetric group $\sum_n$. An operad $\mathcal{P}$ is defined by a sequence of $\mathbb{K}$-vector spaces $\mathcal{P}(n), n \geq 1$ such that $\mathcal{P}(n)$ is a module over $\mathbb{K}[\sum_n]$ and with composition maps

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n + m - 1) \quad i = 1, ..., n$$

satisfying some "associative" properties, the May Axioms [M].
Any \( \mathbb{K}[\sum_{n}] \)-module \( E \) generates a free operad noted \( \mathcal{F}(E) \) ([G]) satisfying \( \mathcal{F}(E)(1) = \mathbb{K}, \mathcal{F}(E)(2) = E \). In particular if \( E = \mathbb{K}[\sum_{2}] \), the free module \( \mathcal{F}(E)(n) \) admits as a basis the "parenthesized products" of \( n \) variables indexed by \( \{1, 2, \ldots, n\} \). For instance a basis of \( \mathcal{F}(E)(2) \) is given by \((x_1.x_2)\) and \((x_2.x_1)\), and a basis of \( \mathcal{F}(E)(3) \) is given by

\[
\{((x_i.x_j).x_k), (x_i.(x_j.x_k)), i \neq j \neq k \neq i, i, j, k \in \{1, 2, 3\}\}.
\]

Let \( E \) be a \( \mathbb{K}[\sum_{2}] \)-module and \( R \) a \( \mathbb{K}[\sum_{3}] \)-submodule of \( \mathcal{F}(E)(3) \). We denote \( \mathcal{R} \) the ideal generated by \( R \), that is the intersection of all the ideals \( \mathcal{I} \) of \( \mathcal{F}(E) \) such that \( \mathcal{I}(1) = 0, \mathcal{I}(2) = 0 \) and \( \mathcal{I}(3) = R \).

We call binary quadratic operad generated by \( E \) and \( R \) the operad \( \mathcal{P}(\mathbb{K}, E, R) \), also denoted \( \mathcal{F}(E)/\mathcal{R} \) and defined by

\[
\mathcal{P}(\mathbb{K}, E, R)(n) = (\mathcal{F}(E)/\mathcal{R})(n) = \frac{\mathcal{F}(E)(n)}{\mathcal{R}(n)}
\]

Thus an operad \( \mathcal{P} \) is binary quadratic operad if and only if there exists a \( \mathbb{K}[\sum_{2}] \)-submodule \( E \) and \( \mathbb{K}[\sum_{3}] \)-submodule of \( \mathcal{F}(E)(3) \) such that \( \mathcal{P} \simeq \mathcal{F}(E)/\mathcal{R} \).

**Example.** The associative operad \( \text{Ass} \), the Lie operad \( \text{Lie} \), the Leibniz operad \( \text{Leib} \) are binary quadratic operads ([G.K],[L]).

### 2.2 Lie-Admissible operads

Let \( \mathcal{F}(E) \) be the free operad generated by \( E = \mathbb{K}[\sum_{2}] \). Consider \( R \) the \( \mathbb{K}[\sum_{3}] \)-submodule generated by the vector

\[
u = x_1.(x_2.x_3) + x_2.(x_3.x_1) + x_3.(x_1.x_2) - x_2.(x_1.x_3) - x_3.(x_2.x_1)
-x_1.(x_3.x_2) - (x_1.x_2).x_3 - (x_2.x_3).x_1 - (x_3.x_1).x_2 + (x_2.x_1).x_3
+(x_3.x_2).x_1 + (x_1.x_3).x_2\]

The Lie-Admissible operad, noted \( \text{LieAdm} \) is the binary quadratic operad defined by

\[
\text{LieAdm} = \mathcal{F}(E)/\mathcal{R}
\]

As we have distinguished 6 types of Lie-admissible algebras, we can determine each corresponding operad. We obtain the following binary quadratic operads:

1. \( \text{Ass} \) corresponding to the associative algebras.
2. \( \text{Vinb} \) corresponding to the Vinberg algebras. Here the \( \mathbb{K}[\sum_{3}] \)-submodule is generated by the vectors

\[
x_1.(x_2.x_3) - (x_1.x_2).x_3 - x_2.(x_1.x_3) + (x_2.x_1).x_3.
\]

3. \( \text{PreLie} \) corresponding to the pre-Lie algebras (\( G_3 \)-associative). The vectors which generate the ideal \( R \) are:

\[
x_1.(x_2.x_3) - (x_1.x_2).x_3 - x_3.(x_2.x_1) + (x_3.x_2).x_1.
\]
4. \( G_4 - \text{Ass} \) corresponding to the \( G_4 \)-associative algebras. \( R \) is generated by
\[
x_1.(x_2.x_3) - (x_1.x_2).x_3 - x_3.(x_2.x_1) + (x_3.x_2).x_1.
\]

5. \( G_5 - \text{Ass} \) corresponding to the \( G_5 \)-associative algebras. \( R \) is generated by
\[
x_1.(x_2.x_3) + x_2.(x_3.x_1) - (x_2.x_3).x_1 + x_3.(x_1.x_2) - (x_3.x_1).x_2.
\]

6. \text{LieAdm}.

### 2.3 The dual operad \text{LieAdm}^1

Let us consider the binary quadratic operad \( \mathcal{P}(K, E, R) \), the dual binary quadratic operad is
\[
\mathcal{P}^! = \mathcal{P}(K^{op}, E^\vee, R^\perp)
\]
where \( E^\vee \) is the dual of \( E \) tensorised by the signature of \( \sum_n \) and \( R^\perp \) the orthogonal complement to \( R \) in \( \mathcal{F}(E^\vee)(3) = \mathcal{F}(E)(3)^\vee \).

Before studying the dual operads of each Lie-Admissible operad defined above, let us introduce some classes of associative algebras.

**Definition 2.1** An associative algebra \( A \) is called 3-order abelian if we have
\[
X_1X_2X_3 = X_{\sigma(1)}X_{\sigma(2)}X_{\sigma(3)}
\]
for all \( \sigma \in \sum_3 \) and for all \( X_i \in A \).

The unitary 3-order abelian algebras are the commutative algebras. But there exists non commutative 3-order abelian algebras. Let us consider, for example, the five dimensional associative algebra defined by
\[
e_1^2 = e_2, \quad e_1^3 = e_3, \quad e_1e_4 = e_5, \quad e_4e_1 = e_3 + e_5, \quad e_4^2 = e_3.
\]

If \( A \) is a 3-order abelian algebra, then the subalgebra \( D(A) \) generated by the product \( xy \) is abelian. Then \( A \) is an extension
\[
0 \to V \to A \to A_1 \to 0
\]
where \( A_1 \) is abelian and \( V \) satisfying \( vx = xv \) for all \( x \in A_1 \) and \( v \in V \). In this case, the corresponding Lie algebra is 2-step nilpotent. In fact, as we have \( abc = bac \), then \([a, b]c = 0. \) We have also \( c[a, b] = 0 \) thus \([a, b], c \) = 0. Return to the geometrical interpretation. Such a Lie algebra is provided with an affine connection which have neither curvature nor torsion. Moreover the operator of connection satisfies
\[
\nabla_X = 0
\]
for all \( X \) in the center and
\[
\nabla_X \nabla_Y = \nabla_Y \nabla_X
\]
for any generators $X$ and $Y$.
Let us consider now the scalar product on $\mathcal{F}(E)(3)$ defined by
\[
< i(jk), i(jk) > = \text{sgn}( \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix})
\]
\[
< (ij)k, (ij)k > = -\text{sgn}( \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix})
\]

Let $R$ be the $\mathbb{K}[\sum_3]$-submodule which determines the LieAdm operad. The annihilator $R^\perp$ respect to this scalar product is of dimension 11. Let $R'$ be the $\mathbb{K}[\sum_3]$-submodule of $\mathcal{F}(E)(3)$ generated by the relations
\[
(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)} - x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}),
\]
\[
(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)} - (x_{\sigma(1)}x_{\sigma(3)})x_{\sigma(2)},
\]
\[
(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)} - (x_{\sigma(2)}x_{\sigma(1)})x_{\sigma(3)}).
\]

Then $\dim R' = 11$ and $< u, v > = 0$ for all $v \in R'$ where $u$ is the vector which generates $R$. This implies $R' \simeq R^\perp$ and $(\mathcal{F}(E)/R)^\perp$ is by definition to binary the quadratic operad $\mathcal{F}(E)/R^\perp$.

**Proposition 2.1** The dual operad of LieAdm is the binary quadratic operad whose corresponding algebras are associative and satisfying
\[
abc = acb = bac
\]
that is there are 3-order abelian.

Return to the different classes of Lie-admissible algebras. One knows that $\mathcal{A}ss^1 = \mathcal{A}ss\ ((\text{G.K}))$ and $\mathcal{P}re\mathcal{L}ie^1 = \mathcal{P}erm\ ((\text{Ch}))$. Recall that this last corresponds to the associative product with the identities :
\[
abc = acb
\]

**Proposition 2.2** The dual operads of Vinb, $G_4 - \mathcal{A}ss$, $G_5 - \mathcal{A}ss$ are the quadratic operads with corresponding algebras the associative algebras satisfying respectively :
- for Vinb$^1$ : $abc = bac$
- for $G_4 - \mathcal{A}ss^1$ : $abc = cba$
- for $G_5 - \mathcal{A}ss^1$ : $abc = bca = cab$.

**Sketch of proof.** $R_2^\perp$ is the $\mathbb{K}[\sum_3]$-sub-module of $\mathcal{F}(E)(3)$ generated by the vectors
\[
(x_1, (x_2.x_3) - (x_1.x_2).x_3), (x_1, (x_2.x_3) - x_2.(x_1.x_3)) ,
\]
\[
(x_1, (x_2.x_3) - (x_2.x_1).x_3)
\]

for all $x_1, x_2, x_3 \in E.$
Likewise

\[ R_+^4 = \langle (x_1.x_2.x_3) - (x_1.x_2).x_3, (x_1.x_2.x_3) - x_3.(x_2.x_1) \rangle, \]
\[ R_5^+ = \langle (x_1.x_2.x_3) - (x_1.x_2).x_3, (x_1.x_2.x_3) - x_3.(x_1.x_2) \rangle, (x_1.x_2.x_3) - (x_3.x_1).x_2 \rangle \]

and \( \dim R_+^4 = 9, \dim R_5^+ = 10 \). Then it is sufficient to note that \((F(E)/R)^\perp\) is by definition the binary quadratic operad \(F(E)/R^\perp\).

### 2.4 Koszul duality

Recall that a quadratic operad \( P \) is called Koszul operad if for every \( P \)-free algebra \( F_P(V) \) one has \( H^i_P(F_P(V)) = 0, \quad i > 0 \).

**Proposition 2.3** The operads \( \text{Ass}, \text{Vinb}, \text{PreLie} \) are Koszul operads. The operads \( G_4 - \text{Ass} \) and \( G_5 - \text{Ass} \) are not Koszul operads.

**Proof.** In fact, from [G.K] and [G] the operads \( \text{Ass}, \text{PreLie} \) are Koszul operads. Considering the relations between \( \text{PreLie} \) and \( \text{Vinb} \), this last satisfies the same property. For the two others, we will show that there are not Koszul operads using their Poincare series and the Ginzburg-Kapranov criterium. This series is defined for a general operad \( P \) by

\[ g_P(x) := \sum_{i=1}^{\infty} (-1)^{i} \dim P(n) \frac{x^n}{n!}, \]

The Poincare series for a Koszul operad and for its dual operad are connected by the functional equation

\[ g_P(g_P(x)) = x. \]

But we have

\[ g_{G_4 - \text{Ass}}(x) = -x + x^2 - \frac{3}{2} x^3 + \frac{59}{4!} x^4 + \ldots, \]
\[ g_{G_4 - \text{Ass}}'(x) = -x + x^2 - \frac{1}{2} x^3 - \frac{1}{4} x^4 + \ldots \]
\[ g_{G_5 - \text{Ass}}(x) = -x + x^2 - \frac{10}{3!} x^3 + \frac{39}{4!} x^4 + \ldots, \]
\[ g_{G_5 - \text{Ass}}'(x) = -x + x^2 - \frac{1}{3} x^3 + \frac{1}{12} x^4 + \ldots \]

These series do not satisfy the functional equation. This proves the proposition.

### 3 The categories \( G_i\text{-ASS} \)

#### 3.1 Functorial correspondance

Let \( \text{LIE} \) and \( \text{LIE} - AD \) be the categories of Lie algebras and Lie-admissible algebras on \( \mathbb{K} \). The correspondance

\[ T : \text{LIE} - AD \rightarrow \text{LIE} \]
is a functor of which dual functor corresponds to the application

\[ T^* : LIE \rightarrow LIE - AD \]

with \( T^*(\mu) = \frac{1}{\dim B} \mu \) (we identify a Lie algebra with its law).

### 3.2 The functor \( A \otimes - \)

It is well known that the category \( ASS \) of associative algebras on \( \mathbb{K} \) is a tensorial category. This property is not true for \( LIE - AD \) and for all the subcategories \( G_i - ASS \) for \( i \neq 1 \). In this section we will prove that the functor \( A \otimes - \) determines a duality between the algebras over the operad \( G_i - Ass \) and algebras on the dual operad \( G_i - Ass' \).

**Theorem 3.1** Let \( A \) be a \( G_i \)-associative algebra. Then \( A \otimes - \) is a covariant functor

\[ A \otimes - : (G_i - ASS)^l \rightarrow G_i - ASS \]

where \( (G_i - ASS)^l \) is the category of algebras corresponding to the algebras on the dual operad \( G_i - Ass' \).

**Proof.** Let \( B \) be an algebra. Then \( A \otimes B \) is an algebra for the classical product \((a_1 \otimes b_1).(a_2 \otimes b_2) = a_1.a_2 \otimes b_1.b_2\). This product is in \( G_i - Ass \) if and only if \( B \) is an algebra on \( G_i - Ass' \). In fact

\[
((a_1 \otimes b_1).(a_2 \otimes b_2)).(a_3 \otimes b_3) = (a_1.a_2).a_3 \otimes (b_1.b_2).b_3 = (a_1.a_2).a_3 \otimes b_1.b_2.b_3
\]

because \( B \) is associative. Let \( \mu \) is the law of the algebra \( A \otimes B \) and put \( X_i = a_i \otimes b_i \).

First, let us prove the theorem for the category \( G_2 - ASS \). In this case \( A \) is a Vinberg algebra. Then

\[
\mu(X_1, \mu(X_2, X_3)) - \mu(X_2, \mu(X_1, X_3)) = \mu(\mu(X_1, X_2), X_3) + \mu(\mu(X_2, X_1), X_3)
\]

\[
= a_1.(a_2.a_3) \otimes b_1.b_2.b_3 - a_2.(a_1.a_3) \otimes b_2.b_1.b_3 - (a_1.a_2).a_3 \otimes b_1.b_2.b_3
\]

\[
+ (a_2.a_1).a_3 \otimes b_2.b_1.b_3
\]

\[
= (a_1.a_2.a_3) - (a_1.a_2).a_3 + (a_2.a_1).a_3 \otimes b_1.b_2.b_3
\]

because \( B \) is in the category \( G_2 - ASS' \),

\[ = 0 \otimes b_1.b_2.b_3 \]

because \( A \) is a Vinberg algebra

\[ = 0. \] Then \( A \otimes B \) also is a Vinberg algebra.

The demonstration is the same for the other \( (G_i - Ass) \)-algebras, taking the adapted relation (i.e for \( i = 3 \), we consider \( \mu(\mu(X, Y), Z) - \mu(X, \mu(Y, Z)) = \mu(\mu(X, Z), Y) - \mu(X, \mu(Z, Y)) \), etc...).

**Applications.** This theorem permits to construct interesting classes of Vinberg algebras and to give new examples of Lie algebras provided with affine structure. For example suppose \( dim A = dim B = 2 \). From [R], \( A \) is isomorphic to one of the following algebras:

1. A is commutative and isomorphic to
\[A_1 : \begin{cases} X_1.X_1 = X_1 \\ X_1.X_2 = X_2.X_1 = X_2 \\ X_2.X_2 = X_2 \end{cases} \]
\[A_2 : \begin{cases} X_1.X_1 = X_1 \\ X_1.X_2 = X_2.X_1 = X_2 \\ X_2.X_2 = 0 \end{cases} \]
\[A_3 : \begin{cases} X_1.X_1 = X_1 \\ X_1.X_2 = X_2.X_1 = X_2 \\ X_2.X_2 = -X_1 \end{cases} \]
\[A_4 : \{X_1.X_1 = X_2\} \]
\[A_5 : \{X_1.X_1 = X_1\} \]
\[A_6 : \{X_i.X_j = 0\} \]

2. \(A\) is non commutative and isomorphic to

\[A_7 : \begin{cases} X_1.X_1 = \frac{b^2 + 2c}{e}X_1 - \frac{b^2 + c}{e^2}X_2 \\ X_1.X_2 = bX_1 - \frac{b^2 + c}{e}X_2 \\ X_2.X_1 = bX_1 - \frac{b^2 + c}{e}X_2 \\ X_2.X_2 = cX_1 - bX_2 \end{cases} \]
\[A_8 : \begin{cases} X_1.X_1 = aX_1 + cX_2 \\ X_1.X_2 = X_2 \\ X_2.X_1 = X_2.X_2 = 0 \end{cases} \]
\[A_9 : \begin{cases} X_1.X_1 = aX_1 \\ X_1.X_2 = (a + 1).X_2 \\ X_2.X_1 = aX_2 \\ X_2.X_2 = 0 \end{cases} \]

In this case the Lie algebra associated to \(A\) is the two-dimensional solvable abelian Lie algebra.

Let us classify the Vinberg algebra of dimension 2.

1. \(B\) is commutative and isomorphic to \(A_i, i = 1, \ldots, 7\).

2. \(B\) is non commutative and isomorphic to

\[B_7 : \begin{cases} e_1.e_1 = e_1 \\ e_1.e_2 = e_2 \\ e_2.e_1 = e_2.e_2 = 0 \end{cases} \]

If \(A\) and \(B\) are commutative, the corresponding Lie algebra is the 4-dimensional abelian Lie algebra.

Suppose \(A\) is commutative and \(B\) is not commutative. In this case the bracket of the Lie algebra associated to \(A \otimes B\) satisfy

\[ [X_i \otimes e_j, X_k \otimes e_l] = X_i.X_k \otimes e_je_l - X_k.X_i \otimes e_le_j = X_i.X_k \otimes [e_j, e_l] \]

with \([e_1, e_2] = e_2\).

When we put \(f_{ij} = X_i \otimes e_j\), we obtain the following list of Lie algebras, \(A_i \otimes B_7\):
Let $P$ be a binary quadratic operad. A $P$-algebra is given by a $\mathbb{K}$-vector space $V$ and an operad morphism

$$\varphi : P \longrightarrow End(V)$$

where $End(V)$ is the operad of endomorphisms of $V$ defined by

$$End(V)(n) = Hom_{\mathbb{K}}(V^\otimes n, V).$$

4 Cohomology of Lie-admissible algebras

4.1 Cohomology of LieAdm-algebras

Let $P$ be a binary quadratic operad. A $P$-algebra is given by a $\mathbb{K}$-vector space $V$ and an operad morphism

$$\varphi : P \longrightarrow End(V)$$

where $End(V)$ is the operad of endomorphisms of $V$ defined by

$$End(V)(n) = Hom_{\mathbb{K}}(V^\otimes n, V).$$
Let $A$ be a $LieAdm$-algebra. The cochain complex of the cohomology $H^n_{LieAdm}(A, A)$ is given by

$$C^n_{LieAdm}(A) = \text{Hom}_L((LieAdm^1)^\vee (n) \otimes \Sigma_n A^{\otimes n}, A)$$

where $V^\vee = V^* \otimes \Sigma_n (sgn)$ for $V$ a $K$-vector space provided with an action of the group $\sum_n V^* = \text{Hom}(V, K)$ and $sgn$ the signature representation.

As every $LieAdm^1$-algebra is an associative 3-order commutative algebra, then the complex $C^n_{LieAdm}(A)$ is

$$A \rightarrow \text{Hom}(A \otimes A, A) \rightarrow \text{Hom}(\wedge^3(A), A) \rightarrow \text{Hom}(\wedge^4(A), A) \rightarrow ...$$

The differential operator is defined by the composition mapping of the operad $LieAdm$. To determine its expression, let us recall the definition of Nijenhuis-Gerstenhaber products.

Let $f$ and $g$ be $n$-, respectively $m$-, linear mappings on a vector space $V$. We define the $(n + m - 1)$-linear mapping $f \circ_6 g$ by

$$f \circ_6 g(X_1, ..., X_{n+m-1}) = \sum_{i=1,...,n} \sum_{\sigma \in \Sigma_{n+m-1}} (-1)^{\varepsilon(\sigma)} (-1)^{(i-1)(m-1)} f(X_{\sigma(1)}, ..., X_{\sigma(i-1)}, g(X_{\sigma(i)}, ..., X_{\sigma(i+m-1)}), X_{\sigma(i+m)}, ..., X_{\sigma(n+m-1)})$$

where $\Sigma_n$ refers to the $p$-symmetric group. This product corresponds to the composition product in the operad $LieAdm$.

For example if $f = g = \mu$ and $n = m = 2$ then

$$\mu \circ_6 \mu(X_1, X_2, X_3) = \sum_{\sigma \in \Sigma_3} (-1)^{\varepsilon(\sigma)} \{ \mu(\mu(X_{\sigma(1)}, X_{\sigma(2)}), X_{\sigma(3)}) - \mu(X_{\sigma(1)}, \mu(X_{\sigma(2)}, X_{\sigma(3)}) \}$$

and $\mu$ is a Lie-admissible law if and only if $\mu \circ_6 \mu = 0$. Likewise if $\mu$ is a bilinear mapping and $f$ and endomorphism of $V$, then

$$\mu \circ_6 f(X_1, X_2) = \mu(f(X_1), X_2) - \mu(f(X_2), X_1) + \mu(X_1, f(X_2)) - \mu(X_2, f(X_1))$$

and

$$f \circ_6 \mu(X_1, X_2) = f(\mu(X_1, X_2)) - f(\mu(X_2, X_1)) = f([X_1, X_2]_\mu)$$

as soon as $\mu \circ_6 \mu = 0$.

In [N], Nijenhuis defines the following product denoted by $f g$ :

$$f \circ_1 g(X_1, ..., X_{n+m-1}) = \sum_{i=1,...,n} (-1)^{(i-1)(m-1)} f(X_1, ..., X_{i-1}, g(X_i, ..., X_{i+m-1}),$$

$$X_{i+1}, ..., X_{n+m-1}).$$

We have

$$f \circ_6 g = \sum_{\sigma \in \Sigma_{n+m-1}} (-1)^{\varepsilon(\sigma)} f \circ_1 g \circ \sigma = \sum_{\sigma \in \Sigma_{n+m-1}} (-1)^{\varepsilon(\sigma)} f g \circ \sigma.$$
Let $P$ be the antisymmetric operator. It is defined by

$$\begin{align*}
P(f)(X_1, \ldots, X_n) &= \sum_{\sigma \in \Sigma_n} (-1)^{\varepsilon(\sigma)} f(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}).
\end{align*}$$

It is clear that $f \circ_6 g = P(f \circ_1 g)$

**Lemma 4.1** We have the following identities :

$$P(P(f) \circ_1 g) = (n + m - 1)! P(f \circ_1 g) = P(f \circ_1 P(g)).$$

This can be prove directly.

We deduce that the following bracket

$$[f, g]^{\circ_6} = f \circ_6 g - (-1)^{(n-1)(m-1)} g \circ_6 f,$$

satisfies :

1. $[g, f]^{\circ_6} = (-1)^{(n-1)(m-1)+1}[f, g]^{\circ_6}$
2. $(-1)^{(n-1)(m-1)}[[f, g]^{\circ_6}, h]^{\circ_6} + (-1)^{(m-1)(n-1)}[[g, h]^{\circ_6}, f]^{\circ_6} + (-1)^{(p-1)(m-1)}[[h, f]^{\circ_6}, g]^{\circ_6} = 0$

where $h$ is a $p$-linear mapping on $V$.

**Consequence.** $[\cdot, \cdot]^{\circ_6}$ is a bracket of a graded Lie algebra (on the space of multilinear mappings).

**Definition 4.1** Let $\phi$ be a in $C_{\text{Lie,Adm}}^n(A)$, where $A$ is a LieAdm-algebra of law $\mu$. The differential operator

$$\delta_{\mu} : C_{\text{Lie,Adm}}^{n+1}(A) \longrightarrow C_{\text{Lie,Adm}}^n(A)$$

is defined by

$$\delta_{\mu} \phi = -[\mu, \phi]^{\circ_6}$$

### 4.2 Particular cases

i) $n = 2$

Let $\varphi$ be a bilinear mapping. Then

$$\begin{align*}
-\delta_{\mu} \varphi (X_1, X_2, X_3) &= \\
&\mu (\varphi (X_1, X_2), X_3) - \mu (X_1, \varphi (X_2, X_3)) + \mu (\varphi (X_2, X_3), X_1) \\
&-\mu (X_2, \varphi (X_3, X_1)) - \mu (X_3, \varphi (X_1, X_2)) + \mu (\varphi (X_3, X_1), X_2) \\
&-\mu (\varphi (X_2, X_1), X_3) + \mu (X_2, \varphi (X_1, X_3)) - \mu (\varphi (X_3, X_2), X_1) \\
&+\mu (X_3, \varphi (X_2, X_1)) + \mu (X_1, \varphi (X_3, X_2)) - \mu (\varphi (X_1, X_3), X_2) \\
&+\varphi (\mu (X_1, X_2), X_3) - \varphi (X_1, \mu (X_2, X_3)) + \varphi (\mu (X_2, X_3), X_1) \\
&-\varphi (X_2, \mu (X_3, X_1)) - \varphi (X_3, \mu (X_1, X_2)) + \varphi (\mu (X_3, X_1), X_2) \\
&-\varphi (\mu (X_2, X_1), X_3) + \varphi (X_2, \mu (X_1, X_3)) - \varphi (\mu (X_3, X_2), X_1) \\
&+\varphi (X_3, \mu (X_2, X_1)) + \varphi (X_1, \mu (X_3, X_2)) - \varphi (\mu (X_1, X_3), X_2)
\end{align*}$$
ii) \( n = 1 \)
Let \( f \) be in \( C^1 \). Then
\[
\delta_\mu f(X_1, X_2) = -\mu(f(X_1), X_2) - \mu(X_1, f(X_2)) + f(\mu(X_1, X_2))
+ \mu(f(X_2), X_1) + \mu(X_2, f(X_1)) - f(\mu(X_2, X_1))
\]

iii) \( n = 0 \)
Here we can define directly the definition of \( \delta_\mu(X) \). Let \( X \) be in \( A \). Consider the map
\[
h_X : Y \mapsto \mu(X, Y) - \mu(Y, X)
\]
\[
-\delta_\mu h_X(X_1, X_2) = \mu(h_X(X_1), X_2) + \mu(X_1, h_X(X_2)) - h_X(\mu(X_1, X_2))
- \mu(h_X(X_2), X_1) - \mu(X_2, h_X(X_1)) + h_X(\mu(X_2, X_1))
\]
\[
= \mu(\mu(X, X_1), X_2) - \mu(\mu(X, X_2), X_1) + \mu(X_2, \mu(X_1, X))
- \mu(\mu(X, X_2), X_1) + \mu(\mu(X, X_2), X_1) - \mu(X_2, \mu(X, X_1))
+ \mu(X_2, \mu(X_1, X))
\]
Then
\[
\delta_\mu h_X(X_2, X_1) - \delta_\mu h_X(X_1, X_2) = \mu \circ_0 \mu(X, X_1, X_2) = 0
\]
\[
\delta_\mu h_X(X_2, X_1) = \delta_\mu h_X(X_2, X_1)
\]
Let us consider
\[
C^0 = \{ X \in A \mid P(\delta_\mu h_X) = \delta_\mu h_X \}
\]
For \( X \in C^0 \), \( \delta_\mu h_X = 0 \). Then we can define \( B^1(A, A) \) putting
\[
\delta(X) = h_X
\]
and \( Z^1(A, A) = \{ f \in C^1/\delta f = 0 \} \).
Then \( H^0(A, A) \) is well defined.

**Remark.** In the way, we can define the cohomology \( H^*_G_{\text{Ass}}(A, A) \) for a \( G_{\text{Ass}} \)-algebra. We denote by \( f \odot_i g \) the corresponding Nijenhuis product.
We have yet given the expression of this product for \( i = 1 \) and \( i = 6 \). In other cases we put :
\[
f \odot_2 g(X_1, ..., X_{n+m-1}) = \sum_{\sigma \in \Sigma_{m+1}} (-1)^{\varepsilon(\sigma)} \{ \sum_{i=1}^{n+m-2}(-1)^{(i-1)(m-1)} f(X_{\sigma(1)}, \ldots, X_{\sigma(i-1)}, g(X_{\sigma(i)}), \ldots, X_{\sigma(m+i-1)}, X_{\sigma(m+i+1)}, \ldots, X_{\sigma(n+m-2)}, X_{n+m-1})
+ (-1)^{(n-1)} f(X_{\sigma(1)}, \ldots, X_{\sigma(n-1)}, g(X_{\sigma(n)}), \ldots, X_{\sigma(m+n-2)}, X_{n+m-1}) \}.
\]
where

\[ \delta \] then satisfies

A 2-cochain corresponding to the Chevalley’s cohomology is alternated and

**Remark.**

with value in \((\mathfrak{g})\) in these two cases, the cochain spaces are the same, excepted for \(n = 2\). Recall also that the second cohomology in nothing other than the Chevalley cohomology.

**Definition 4.2** Let \(\mathfrak{g}\) be a Lie algebra. We call Lie-admissible cohomology of \(\mathfrak{g}\) with value in \(\mathfrak{g}\) the cohomology \(H^*_\text{Lie,Adm}(\mathfrak{g}, \mathfrak{g})\).

**Remark.**

A 2-cochain corresponding to the Chevalley’s cohomology is alternated and satisfies

\[
\delta_c \varphi (X_1, X_2, X_3) = \varphi (X_1, X_2) . X_3 + \varphi (X_2, X_3) . X_1 + \varphi (X_3, X_1) . X_2 + \varphi (X_1, X_2, X_3) + \varphi (X_2, X_3, X_1) + \varphi (X_3, X_1, X_2). 
\]

where \(\delta_c\) is the Chevalley operator. If we compute \(\delta \varphi \in B^2_{\text{Lie,Adm}}(\mathfrak{g}, \mathfrak{g})\) we obtain

\[
\delta \varphi (X_1, X_2) = 2(\varphi (X_1, X_2) . X_3 - X_3 . \varphi (X_1, X_2) + \varphi (X_2, X_3) . X_1 - X_1 . \varphi (X_2, X_3) + \varphi (X_3, X_1) . X_2 - X_2 . \varphi (X_3, X_1) + \varphi (X_1, X_2, X_3) - \varphi (X_3, X_1, X_2) + \varphi (X_2, X_3, X_1) - \varphi (X_1, X_2, X_3) + \varphi (X_3, X_1, X_2) - \varphi (X_2, X_3, X_1))
\]

then

\[ \delta \varphi = 4 \delta_c \varphi. \]
5 Lie-admissible modules on a Lie algebra

5.1 Module on a Lie-admissible algebra

Let \( A = (A, \mu) \) be a Lie-admissible algebra and \( M \) a vector space on \( \mathbb{K} \).

**Definition 5.1** \( M \) is an \( A \)-module if there is bilinear mapping

\[
\lambda : A \otimes M \to M \\
\rho : M \otimes A \to M
\]

satisfying :

\[
\lambda(X, \lambda(Y, v) - \lambda(Y, \lambda(X, v)) - \lambda([X, Y]_\mu, v) - \lambda(X, \rho(v, Y) + \rho(\lambda(X, v), Y) + \\
\rho(v, [X, Y]_\mu) - \rho(\rho(v, X), Y) - \rho(\lambda(Y, v), X) + \lambda(Y, \rho(v, X)) + \rho(\rho(v, Y), X) = 0
\]

for all \( X, Y \in A \) and \( v \in M \).

For example, the vector space \( A \) is an \( A \)-module.

**Proposition 5.1** Let \( A = (A, \mu) \) be a Lie-admissible algebra and \( M \) an \( A \)-module defined by the mapping \( \lambda \) and \( \rho \). Then the bilinear mapping

\[
\hat{\lambda} : A \otimes M \to M
\]

defined by

\[
\hat{\lambda}(X, v) = \lambda(X, v) - \rho(v, X)
\]

provides the vector space \( M \) with an \( A_L \)-module where \( A_L \) is the Lie algebra \((A, [\cdot, \cdot]_\mu)\).

We find again the same result established by Nijenhuis in [N] for the Vinberg algebra.

5.2 Modules on \( G_i \)-associative algebras

If \( A_V = (A, \mu_V) \) is a Vinberg algebra an \( A_V \)-module \( M \) is given by the mappings \( \lambda \) and \( \rho \) satisfying the two conditions

\[
\begin{aligned}
\lambda(X, \lambda(Y, v) - \lambda(Y, \lambda(X, v)) - \lambda([X, Y]_\mu, v) = 0 \\
\lambda(X, \rho(v, Y) - \rho(\lambda(X, v), Y) - \rho(v, \mu(X, Y)) + \rho(\rho(v, Y), X) = 0.
\end{aligned}
\]

Considering \( A_V \) as a Lie-admissible algebra, then \( M \) is also a module on this Lie-admissible algebra.

The notion of \( A \)-module is well known in the cases of Lie-admissible algebras of type 1 (associative). It is easy to write the definitions of modules on algebras of type 4 and 5. We find :

- type 4 :

\[
\begin{aligned}
\lambda(\mu(X, Y), v) - \lambda(X, \lambda(Y, v) - \rho(\rho(v, Y), X) + \rho(v, \mu(X, Y)) = 0 \\
\rho(\lambda(X, v), Y) - \lambda(X, \rho(v, Y) - \rho(\lambda(Y, v), X) + \lambda(Y, \rho(v, X)) = 0
\end{aligned}
\]
\[ \lambda(\mu(X, Y), v) - \lambda(X, \lambda(Y, v)) + \rho(\lambda(X, v), Y) - \lambda(X, \rho(v, Y) - \rho(v, \mu(X, Y))) + \rho(\rho(v, X), Y) = 0. \]

We can see that if \( \mu \) is antisymmetric (i.e. a law of Lie algebra) then we find again the definition of module on Lie algebra considering \( \rho(v, X) = -\lambda(X, v) \).

### 5.3 Lie-admissible modules on Lie algebras

Let \( A = (A, \mu) \) be a Lie algebra. Consider this algebra as a Lie-admissible algebra that we will note \( A_{\text{ad}} = (A, \mu) \) to distinguish the two structures. It is clear that every module \( M \) on the Lie algebra \( A \) is also a module on the Lie-admissible algebra \( A_{\text{ad}} \). But the converse is false.

**Definition 5.2** We call Lie-admissible module on the Lie algebra \( A = (A, \mu) \) every module on the Lie-admissible algebra \( A_{\text{ad}} = (A, \mu) \).

Let \( g \) be the solvable non abelian 2-dimensional Lie algebra. There exists a basis \( \{X_1, X_2\} \) such that \([X_1, X_2] = X_2\). Every one dimensional module on the Lie algebra \( g \) is given by the mapping \( \lambda \) (here \( \rho = -\lambda \)) defined by

\[
\begin{align*}
\lambda(X_1, v) &= \alpha v \\
\lambda(X_2, v) &= 0
\end{align*}
\]

On the other hand, a Lie-admissible module on \( g \) is determined by the mapping \( \lambda \) and \( \rho \) given by

\[
\begin{align*}
\lambda(X_1, v) &= \alpha v \\
\lambda(X_2, v) &= \beta v \\
\rho(v, X_1) &= \gamma v \\
\rho(v, X_2) &= \beta v
\end{align*}
\]

Suppose now that \( M \) is a \( n \)-dimensional Lie-admissible module on \( g \). Then if \( A, B, C, D \) are the matrices on the linear operators \( \lambda(X_1, .), \lambda(X_2, .), \rho(. , X_1), \rho(. , X_2) \) in a given basis of \( M \), then these matrices satisfy

\[ [(B - D), (C - A)] = B - D. \]

We describe in this way all the structures of Lie-admissible modules on \( g \).

Now consider the Lie algebra \( g = sl(2, \mathbb{C}) \). By a similar computation we can see that every \( n \)-dimensional Lie-admissible module on \( sl(2, \mathbb{C}) \) is described by the following matrix representations:

\[
\begin{align*}
[A_1 - B_1, A_2 - B_2] &= 4(A_2 - B_2) \\
[A_1 - B_1, A_3 - B_3] &= -4(A_3 - B_3) \\
[A_2 - B_2, A_3 - B_3] &= 2(A_1 - B_1)
\end{align*}
\]

Such representation also is completely reducible.
6 Deformations

Let us denote $\mathcal{L}A_n$ the algebraic variety of $n$-dimensional Lie-admissible algebras on an algebraic closed field $K$ of characteristic 0. We have a natural fibration

$$
\pi : \mathcal{L}A_n \to \mathcal{L}_n,
$$

where $\mathcal{L}_n$ indicates the algebraic variety of $n$-dimensional Lie algebras:

$$
\pi(\mu) = [\cdot, \cdot]_\mu
$$

This fiber owns a global section

$$
s : \mathcal{L}_n \to \mathcal{L}A_n
$$

defined by

$$
s(\mu) = \frac{1}{2}\mu.
$$

Let us denote $T_{s(\mu)}\mathcal{L}A_n$ and $T_{s(\mu)}\pi^{-1}(\langle \mu \rangle)$ the tangent spaces to the variety and to the fiber at the point $s(\mu)$. We know that $T_{s(\mu)}\mathcal{L}A_n$ is identified to the space of the 2-cocycles $Z^2_{LieAdm}(A, A)$.

**Lemma 6.1** $T_{s(\mu)}\pi^{-1}(\mu) \simeq \{\phi : A^\otimes 2 \to A / \phi(x, y) = \phi(y, x)\} \simeq S^2(A)$.

**Proof.** In fact $\frac{1}{2}\mu + t\phi$ is a linear deformation of $\frac{1}{2}\mu$ in the fiber $\pi^{-1}(\mu)$ then $\phi(x, y) - \phi(y, x) = 0$. More every symmetric bilinear mapping is a cocycle that is in $Z^2_{LieAdm}(A, A)$.

Recall the geometrical problem concerning the existence of affine structure on solvable Lie algebras. A Lie algebra is provided with an affine structure if and only if there exists a Vinberg algebra whose associated Lie algebra corresponds to the given. We know that there exists nilpotent Lie algebras without affine structure. In this case the fiber $\pi^{-1}(\mu)$ does not cut the subvariety of Vinberg laws. For $\phi \in S^2(A) \subset Z^2_{LieAdm}(A, A)$ fixed, the straightline $\frac{1}{2}\mu + t\phi$ is in the fiber $\pi^{-1}(\mu)$. This line cut the subvariety of Vinberg if and only if there is $t_0$ such that $\frac{1}{2}\mu + t_0\phi$ is a Vinberg law. Considering $\phi$ for $t_0\phi$ (we can always suppose that $t_0 = 1$ we obtain :

**Proposition 6.2** The deformation $1/2\mu + \phi$ is in $Vinb_n$ if and only if the symmetric mapping satisfies

$$
4\phi(\mu(X_2, X_1), X_3) + 2\mu(X_1, \phi(X_2, X_3)) + 2\phi(X_1, \mu(X_2, X_3)) + 4\phi(X_1, \phi(X_2, X_3)) - 2\mu(X_2, \phi(X_1, X_3)) - 2\phi(X_2, \mu(X_1, X_3)) - 4\phi(X_2, \phi(X_1, X_3)) + \mu(\mu(X_2, X_1), X_3) = 0.
$$

for all $X_1, X_2, X_3 \in \mathfrak{g}$. 


This proposition can be considered as a criterium of existence of an affine structure on a given Lie algebra.

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