The geometry of the solution space of first order Hamiltonian field theories II: non-Abelian gauge theories

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August 31, 2022

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Abstract

We go on with the program started in the companion paper [CDI+] of defining a Poisson bracket structure on the space of solutions of the equations of motion of first order Hamiltonian field theories. The case of non-Abelian gauge theories is addressed by using a suitable version of the coisotropic embedding theorem.

Introduction

In the first part of this series of papers [CDI+] and in [CDI+20b, CDI+20a] the authors started the analysis of the construction of a covariant Poisson brackets for first order Hamiltonian classical field theories. The philosophy of the paper was that of equipping the space of solutions of the equations of motion, which is the space capturing the covariance (w.r.t. the relativity group of the theory) properties of the theory, with a Poisson structure. In particular the authors showed that, within the multisymplectic formulation of first order Hamiltonian field theories, such a space, always referred to as solution space, can be canonically equipped with a pre-symplectic structure. In particular such a structure revealed to be symplectic within those theory non exhibiting any gauge invariance since its kernel is made exactly by the generators of gauge transformations. For theories without gauge symmetries the Poisson bracket was defined in the usual way by constructing the Poisson tensor as the inverse of the symplectic structure. Moreover, in [CDI+] the first easiest example of gauge theory was analysed, i.e. Electrodynamics. In that case the authors, following an idea in [DGMS93], constructed the Poisson bracket by the aid of a flat connection that could be fixed on a particular bundle associated with the theory. As already anticipated in [CDI+] such idea does not work for all gauge theories since for some theories, such as Yang-Mills theories, for topological reasons a flat connection of the type used in Electrodynamics can not be fixed.

However, in the present paper we will show that, within the general formulation of first order Hamiltonian field theories presented in [CDI+], the so called coisotropic embedding theorem can be used as a tool to define a Poisson bracket even in cases where a flat connection can not be fixed. The price one should pay is that the Poisson bracket will be defined on an enlargement of the solution space where additional degrees of freedom appear. In the case of Yang-Mills theories, which is the main example we will deal with in this paper, the number of additional degrees of freedom emerging
via the procedure described turns to be equal to the dimension of $T^* g \cong g \times g^*$ (g denoting the Lie algebra of the Lie group associated with the gauge theory) and, therefore, can be interpreted as the ghost and antighost (or ghost momenta following the terminology of [HT94]) appearing in the BRST approach to the quantization of gauge theories.

The paper is organized as follows. In Sect. 1 we recall the content of the coisotropic embedding theorem. In particular we recall its original formulation in Sect. 1.1.1 and then we give a proof based on the use of a connection in 1.1.2. Then we argue how to use such a theorem to construct a Poisson bracket on a pre-symplectic manifold, distinguishing the three different cases where the connection used is closed (Sect. 1.2.1), horizontally closed, i.e. flat (Sect. 1.2.2) and non-closed (Sect. 1.2.3). We will see that in the closed and horizontally closed cases, the Poisson bracket defined on the enlarged space emerging from the coisotropic embedding theorem can be reduced to a Poisson bracket on the original manifold, the solution space. In particular, it will turn out that the Poisson bivector field on the enlarged space in terms of which the Poisson bracket is defined, can be projected to a Poisson bivector field on the solution space.

Then, since the pre-symplectic manifold we want to analyse is the solution space of Yang-Mills theories, we devote Sect. 2 to recall, very briefly and avoiding all the proofs, the multisymplectic formulation of first order Hamiltonian field theories extensively discussed in [CDI+] and reference therein.

In Sect. 3 the main aim of the paper is addressed, that is, the construction of the Poisson bracket on the solution space of Yang-Mills theories. In particular, we first address in Sect. 3.1 a preliminar (finite-dimensional) example, the magnetic monopole, in order to introduce the construction in a more manageable setting. Then, in section 3.2 we deal with the example of free Electrodynamics already addressed in [CDI+] within the approach in terms of the coisotropic embedding theorem presented in this paper. In particular, we will show that free Electrodynamics lies in what we called above the "closed case" and, thus, the Poisson bracket defined on the enlarged space obtained via the coisotropic embedding theorem can be projected to the solution space giving rise exactly to the Poisson bracket written in [CDI+]. Finally, we proceed with the main example of the paper, that is Yang-Mills theories, in Sect. 3.3 and 3.4. In particular, in Sect. 3.3 we recall the multisymplectic formulation of Yang-Mills theories, whereas in Sect. 3.4 we proceed with the construction of the Poisson bracket via the coisotropic embedding theorem.

1 The coisotropic embedding theorem and Poisson brackets

In this section we show how the coisotropic embedding theorem can be seen as a tool to define a Poisson bracket on a suitable enlargement of a pre-symplectic manifold and under which geometrical conditions such a Poisson bracket can be 'projected' to a Poisson bracket on the original pre-symplectic manifold.
1.1 The coisotropic embedding theorem

We devote this section to recall the content of the coisotropic embedding theorem. Consider a presymplectic manifold \((M, \omega)\). The coisotropic embedding theorem gives a canonical way of embedding \((M, \omega)\) into a symplectic manifold, say \((\tilde{M}, \Omega)\) such that \((M, \omega)\) is a coisotropic submanifold, i.e., \(i^*\Omega = \omega\), \(i\) denoting the embedding map.

In Sect. 1.1.1 we sketch the proof of the theorem in its standard formulation, which is due to M. Gotay [Got82]. We also refer to [GS90] for the proof of the theorem and for its equivariant version.

The original formulation is to some extent too general for our purposes and not well suited for calculations. For this reason, in Sect. 1.1.2 we use a proof of the theorem based on the use of an additional structure, i.e., a connection which makes the theorem less general but better suited to our applications.

1.1.1 Classical coisotropic embedding theorem

Assume \(\omega\) to be of constant rank and denote by \(K_m\) the kernel of \(\omega\) at \(m \in M\). Denote by \(K\) the characteristic bundle over \(M\), i.e., the subbundle of \(TM\) with typical fibre \(K_m\). Its dual bundle, say \(K^\ast\), is again a vector bundle over \(M\). The base manifold \(M\) can be immersed into \(K^\ast\) as the zero section, \(\sigma_0\), of the fibre bundle. Along the image of \(\sigma_0\) the tangent space to \(K^\ast\) canonically splits as \(T_{\sigma_0}(m)K^\ast = T_m M \oplus K^\ast_m\).

Remark 1.1. This is true only along the image of \(\sigma_0\), since as far as one moves away from it, only the vertical vectors of \(K^\ast\) are canonically defined whereas their complement at each point should be specified via an additional choice.

Consider a complement of \(K_m\) into \(T_m M\), say \(W_m\). Then \(T_{\sigma_0(m)}K^\ast = W_m \oplus K_m \oplus K^\ast_m\). Therefore, using the fact that \(K_m \oplus K^\ast_m\) is a symplectic vector space\(^1\) and that the original pre-symplectic structure \(\omega\) is non-degenerate when contracted along elements of \(W_m\), a symplectic structure can be constructed on \(\sigma_0(M)\) in the following way:

\[
\Omega_0 = \tau^* \omega + \omega_{K \oplus K^\ast} \circ \text{pr},
\]

where \(\tau\) is the projection of the fibre bundle \(K^\ast \to M\) and \(\text{pr}\) is the projection \(T_{\sigma_0(m)}K^\ast \to K_m \oplus K^\ast_m\). The extension of \(\Omega_0\) to the whole \(K^\ast\) gives a differential form \(\Omega_0^\text{ext}\) such that:

\[
\sigma_0^* \Omega_0^\text{ext} = \omega.\]

Nevertheless \((K^\ast, \Omega_0^\text{ext})\) is not yet the symplectic manifold we were searching for because \(\Omega_0^\text{ext}\), even if non-degenerate, is, in general, not closed outside \(\sigma_0(M)\). However, as it is proven in [GS90, lemma 4]

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\(^1\)The symplectic structure is given by the natural pairing between \(K_m\) and its dual, \(\omega_{K_m \oplus K^\ast_m}(V + \rho, V' + \rho') = \langle \rho, V' \rangle - \langle \rho', V \rangle\) where \(V, V' \in K_m\) and \(\rho, \rho' \in K^\ast_m\).
39.1 at page 318], a differential form $\alpha$ defined in a tubular neighborhood of $\sigma_0(\mathcal{M})$, say $\tilde{M}$, such that:

$$d\alpha = -d\Omega^0_{\text{ext}} \quad \text{and} \quad \alpha\big|_{\sigma_0(\mathcal{M})} = 0,$$

(3)
can always be added to $\Omega^0_{\text{ext}}$. Because of the properties of $\alpha$, the manifold $(\tilde{M}, \Omega)$, where $\Omega = \Omega^0_{\text{ext}}|_{\tilde{M}} + \alpha$, is the symplectic manifold we were searching for.

The construction of the differential form $\alpha$, as presented in [GS90], requires the choice of a retraction of $K^*$ into $\mathcal{M}$ and to construct its flow. This may be not easy from the computational point of view. For this reason, in the next section we propose a way of constructing $\Omega$ related with the choice of a connection on the bundle $\mathcal{M} \to \mathcal{M}/K$ ($K$ denoting the distribution associated with $K_m$) which, even if is less general, is more practical from the computational point of view and that also allows for studying the possibility of defining a Poisson structure on the original (pre-symplectic) manifold $\mathcal{M}$.

1.1.2 The coisotropic embedding theorem via connections

The splitting $T_m\mathcal{M} = W_m \oplus K_m$ considered in the previous section can be given in terms of a connection on the bundle $\mathcal{M} \to \mathcal{M}/K$. Indeed, such a connection is specified by selecting a $1-1$ tensor field on $\mathcal{M}$:

$$P : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}^v(\mathcal{M}),$$

(4)
for some 1-forms $P^j$ on $\mathcal{M}$. Denoting by $\{ \mu_j \}_{j=1, \ldots, \dim K^*}$ a system of coordinates on $K^*_m$, the term $\omega_{K^* \oplus K^* \circ \text{pr}}$ reads:

$$\omega_{K^* \oplus K^* \circ \text{pr}} = d\mu_j \wedge P^j.$$

(6)
With this in mind, the structure $\Omega^0_{\text{ext}}$ on $K^*$ reads:

$$\Omega^0_{\text{ext}} = \tau^*\omega + d\mu_j \wedge P^j.$$

(7)
This structure is not closed because:

$$d\mu_j \wedge P^j = d(\mu_j P^j) - \mu_j dP^j.$$

(8)
Therefore, the term which is missing to make $\Omega^0_{\text{ext}}$ closed is exactly the term:

$$\alpha = \mu_j dP^j,$$

(9)

$^2$Note that $\dim \mathcal{M}$ may be infinite and, thus, $j$ may should be considered to be an infinite-dimensional label.
which has the properties of the differential form $\alpha$ of the previous section since it vanishes on $\sigma_0(\mathcal{M})$ where $\mu_j = 0$.

Therefore, the symplectic manifold obtained via the coisotropic embedding theorem is any tubular neighborhood of $\sigma_0(\mathcal{M})$ into $K^*$ equipped with the symplectic structure:

$$\Omega = \tau^*\omega + d\mu_j \wedge P_j + \mu_j dP_j.$$  \hfill (10)

### 1.2 Poisson brackets on pre-symplectic manifolds

Now, we are going to see how, depending on the properties of the connection chosen, the coisotropic embedding theorem can be used to construct Poisson brackets. In particular we will see that if the connection is closed or has zero curvature, a Poisson bracket can be directly defined on the original pre-symplectic manifold. On the other hand, if the connection has non-zero curvature, a Poisson bracket can be defined only on the enlarged manifold $\tilde{\mathcal{M}}$.

#### 1.2.1 The closed case

Let us focus for a moment on the case where $dP_j = 0$ and let us see how a Poisson structure can be induced on the original pre-symplectic manifold $\mathcal{M}$ starting from $\Omega$.

The symplectic structure $\Omega$, at each point $\tilde{m}$ of $\tilde{\mathcal{M}}$, is the sum of a non-degenerate structure having components only on $W^m$ and a non-degenerate structure having components only on $K_m \oplus K^*_m$:

$$\Omega_{\tilde{m}}(\tilde{X}_W + \tilde{X}_K + \tilde{X}_{K^*}, \cdot) = \tau^*\omega_{\tilde{m}}(\tilde{X}_W, \cdot) + d\mu_j \wedge P_j(\tilde{X}_K + \tilde{X}_{K^*}, \cdot) =: \Omega_{W^m}(\tilde{X}_W, \cdot) + \Omega_{K_m \oplus K^*_m}(\tilde{X}_K + \tilde{X}_{K^*}, \cdot)$$ \hfill (11)

where $\tilde{X}_W \in W^m$, $\tilde{X}_K \in K_m$, $\forall \tilde{m} \in \tilde{\mathcal{M}}$ and $\tilde{X}_{K^*} \in K^*_m$, $\forall \tilde{m} \in \tilde{\mathcal{M}}$. This means that $\Omega$ results as the direct sum of two closed forms:

$$\Omega = \Omega_W \oplus \Omega_{K \oplus K^*}$$ \hfill (12)

which, restricted respectively to the distributions $W$ and $K \oplus K^*$, are non-degenerate. Consequently, the Poisson bi-vector field on $\tilde{\mathcal{M}}$, say $\Lambda$, associated with the symplectic structure $\Omega$, reads:

$$\Lambda = \Lambda_W \oplus \Lambda_{K \oplus K^*}$$ \hfill (13)

where $\Lambda_W$ is a Poisson bi-vector field belonging, at each $\tilde{m} \in \tilde{\mathcal{M}}$, to $W_m \wedge W_m$ whereas $\Lambda_{K \oplus K^*}$ is a Poisson bi-vector field belonging, at each $\tilde{m} \in \tilde{\mathcal{M}}$ to $K_m \oplus K^*_m \wedge K_m \oplus K^*_m$. Now, since $\Lambda$ is the Poisson bi-vector field associated with a symplectic structure, it satisfies the following:

$$[\Lambda, \Lambda]_S = 0$$ \hfill (14)
where $[,]_S$ denote the Schouten-Nijenhuis brackets, which is equivalent to the fact that the bracket associated to $\Lambda$, $\{f, g\} = \Lambda(df, dg)$, satisfies the Jacobi identity. Now, since $\Lambda_W$ is the inverse (restricted to $W$) of the closed (and non-degenerate when restricted to $W$) form $\Omega_W$, it satisfies:

$$[\Lambda_W, \Lambda_W]_S = 0$$

(15)

itself. Another way of proving this latter equality is the following. The Schouten bracket of $\Lambda$ with itself reads:

$$[\Lambda, \Lambda]_S = [\Lambda_W, \Lambda_W]_S + 2[\Lambda_W, \Lambda_{K\oplus K^*}]_S + [\Lambda_{K\oplus K^*}, \Lambda_{K\oplus K^*}]$$

(16)

where the second term on the right hand side vanishes because, being $P$ defined on $\mathcal{M}$, $W$ is invariant with respect to elements in $K^*$ and, being $P$ a connection, $W$ is invariant with respect to $K$, i.e., $W$ commutes both with $K^*$ and with $K$. What is more, the last term on the right hand side vanishes because:

$$\Lambda_{K\oplus K^*} = \frac{\partial}{\partial \mu_j} \wedge V_j$$

(17)

and:

$$\left[ \frac{\partial}{\partial \mu_j}, V_k \right] = 0.$$  

(18)

With this in mind (15) is a straightforward consequence.

Now, the bi-vector field $\Lambda_W$ satisfying (15) can be used to define a Poisson bracket on the pre-symplectic manifold $\mathcal{M}$. Indeed, since $W$ is a distribution on $\mathcal{M}$, seen as a distribution on $\tilde{\mathcal{M}}$ it clearly commutes with the distribution $K^*$ generated by $\frac{\partial}{\partial \mu_j}$. This means that:

$$[\Lambda_W, \frac{\partial}{\partial \mu_j}]_S = 0 \quad \forall \ j = 1, ..., \dim K^*.$$  

(19)

Consequently, the bi-vector field $\Lambda_W$ is projectable onto $\mathcal{M}$ via $\tau : \tilde{\mathcal{M}} \rightarrow \mathcal{M}$ to the following bi-vector field:

$$\lambda_W = \tau_\ast \Lambda_W \in \bigwedge^2(\mathcal{M}).$$

(20)

The latter also has a vanishing Schouten bracket with itself because of the following equalities:

$$[\lambda_W, \lambda_W]_S = [\tau_\ast \Lambda_W, \tau_\ast \Lambda_W]_S = \tau_\ast [\Lambda_W, \Lambda_W] = 0$$

(21)

and, thus, it defines a Poisson bracket on $\mathcal{M}$ in the following way:

$$\{f, g\} = \lambda_W(df, dg)$$

(22)

for $f, g \in \mathcal{F}(\mathcal{M})$. 

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1.2.2 Zero-curvature case

The previous construction can be extended to the case where \( P_j \) is not closed but only horizontally-closed, i.e., to the case where:

\[
d_H P^j = dP^j ((\mathbb{1} - P)(\cdot), (\mathbb{1} - P)(\cdot)) = 0. \tag{23}
\]

Indeed, when \( P_j \) is not closed, the structure \( \Omega \) reads:

\[
\Omega = \Omega_W \oplus \Omega_{K \oplus K^*} \oplus \alpha \tag{24}
\]

where \( \Omega_W \) and \( \Omega_{K \oplus K^*} \) are closed and, in general, \( \alpha = \mu_j dP^j \) has components both on \( W \) and on \( K \oplus K^* \), i.e.:

\[
\alpha_{\tilde{m}}(\tilde{X}_W + \tilde{X}_K + \tilde{X}_{K^*}, \cdot) = \alpha_{\tilde{m}}(\tilde{X}_W, \cdot) + \alpha_{\tilde{m}}(\tilde{X}_K + \tilde{X}_{K^*}, \cdot) =: \\
\alpha_{W_{\tilde{m}}}(\tilde{X}_W, \cdot) + \alpha_{K \oplus K^*_{\tilde{m}}}(\tilde{X}_K + \tilde{X}_{K^*}, \cdot) = \\
\mu_j d_H P^j_{\tilde{m}}(\tilde{X}_W, \cdot) + \mu_j d_V P^j(\tilde{X}_K + \tilde{X}_{K^*}, \cdot), \tag{25}
\]

where \( d_V P^j(X, Y) = dP^j(P(X), P(Y)) \) and \( d_H P^j(X, Y) = dP^j((\mathbb{1} - P)(X), (\mathbb{1} - P)(Y)) \).

Now, if \( d_H P^j = 0 \), i.e., if \( P \) has zero curvature, \( \Omega \) reads:

\[
\Omega = \Omega_W \oplus \tilde{\Omega}_{K \oplus K^*}. \tag{26}
\]

where \( \tilde{\Omega}_{K \oplus K^*} = \Omega_{K \oplus K^*} + \alpha_{K \oplus K^*} \). Therefore, again \( \Lambda \) reads:

\[
\Lambda = \Lambda_W \oplus \tilde{\Lambda}_{K \oplus K^*}. \tag{27}
\]

with \( \Lambda_W \) satisfying:

\[
[\Lambda_W, \Lambda_W] = 0 \tag{28}
\]

since it comes from a closed 2-form \( \Omega_W \). Therefore, also in this case the previous construction can be performed, giving rise to the Poisson bracket (22) on \( \mathcal{M} \).

1.2.3 The non-closed case

The general case where neither \( d_H P^j = 0 \) nor \( d_V P^j = 0 \) does not allow to directly define a Poisson bracket on \( \mathcal{M} \) using the Poisson bracket defined on \( \mathcal{M} \).

Indeed, in that case the structure \( \Omega \) reads:

\[
\Omega = \tilde{\Omega}_W \oplus \tilde{\Omega}_{K \oplus K^*}, \tag{29}
\]

\(^3\text{Recall that } d_V P^j = dP^j(P(\cdot), P(\cdot)).\)
where:
\[ \tilde{\Omega}_W = \Omega_W \oplus \alpha_W, \quad \tilde{\Omega}_{K \oplus K^*} = \Omega_{K \oplus K^*} \oplus \alpha_{K \oplus K^*}, \]  
with:
\[ \alpha_W = \mu_j d_H P^j, \quad \alpha_{K \oplus K^*} = \mu_j d_V P^j. \]

However, in this case even if \( \tilde{\Omega}_W \) and \( \tilde{\Omega}_{K \oplus K^*} \) are non-degenerate when restricted to \( W \) and \( K \oplus K^* \) respectively, they are not closed. Therefore, the corresponding bi-vector fields \( \tilde{\Lambda}_W \) and \( \tilde{\Lambda}_{K \oplus K^*} \) such that:
\[ \Lambda = \tilde{\Lambda}_W \oplus \tilde{\Lambda}_{K \oplus K^*}, \]  
do not satisfy:
\[ [\tilde{\Lambda}_W, \tilde{\Lambda}_W]_S = 0, \quad [\tilde{\Lambda}_{K \oplus K^*}, \tilde{\Lambda}_{K \oplus K^*}]_S = 0, \]  
and, thus, \( \tilde{\Lambda}_W \) can not define a Poisson bracket on \( \tilde{\mathcal{M}} \).

The only natural construction in this case seems to be to consider the Poisson bracket associated with \( \Omega \) on the whole \( \tilde{\mathcal{M}} \) restricted to the subalgebra of functions on \( \mathcal{M} \) being pull-back (via \( \tau \)) of functions on \( \mathcal{M} \). That is, one can consider two functions \( \tilde{f}, \tilde{g} \in \mathcal{F}(\tilde{\mathcal{M}}) \) such that \( \tilde{f} = \tau^* f \) and \( \tilde{g} = \tau^* g \) with \( f, g \in \mathcal{F}(\mathcal{M}) \) and to consider their bracket computed with respect to \( \Lambda \)
\[ \{ \tilde{f}, \tilde{g} \} = \Lambda(d\tilde{f}, d\tilde{g}) = \tilde{\Lambda}_W(d\tilde{f}, d\tilde{g}). \]

Even if, due to the fact that \( \Lambda \) is a bi-vector field coming from a symplectic structure, this bracket satisfies the Jacobi identity, it can not be used to induce a bracket on \( \mathcal{M} \) because, in general, since the term \( \alpha_W \) added to \( \Omega_W \) contains a dependence on the variables \( \mu_j \), the function \( \{ \tilde{f}, \tilde{g} \} \) is not the pull-back of a function on \( \mathcal{M} \) as well. Indeed, in general it will be of the form
\[ \{ \tilde{f}, \tilde{g} \} = \tilde{h} + H \]
where \( \tilde{h} = \tau^* h \) with \( h \in \mathcal{F}(\mathcal{M}) \) and \( H \in \mathcal{F}(\tilde{\mathcal{M}}) \).

In the next sections we will see how all these constructions can be used to define a Poisson bracket on the solution space of non-Abelian gauge theories. For this reason, even if it is already extensively described in the companion paper [CDI+], we first devote Sect. 2 to recall the multisymplectic formulation of first order Hamiltonian field theories and how in this formalism the solution space of the theory is canonically a pre-symplectic manifold (in particular, it is genuinely pre-symplectic within gauge theories). Then, in Sect. 3 we will apply the construction of the present section to the solution space of Yang-Mills theories.

## 2 Multisymplectic formulation of Classical Field Theories

In this section we recall the multisymplectic formulation of first order Hamiltonian field theories for which we refer to [IS17, CDI+22, CDI+] and references therein. Since it is already extensively discussed...
in the companion paper [CDI+], here, for the sake of completeness, we recall the main features of the formalism but we will avoid all the proofs.

Sect. 2.1 is devoted to recall the construction of the carrier space where one settles the multisymplectic description of first order Hamiltonian field theories. In Sect. 2.2 we recall how to formulate the variational principle à la Schwinger-Weiss. Finally, in Sect. 2.3 we recall how a canonical 2-form on the solution space of the theory emerges.

### 2.1 Carrier space: the Covariant Phase Space

A multisymplectic formulation of a classical field theory starts with:

- A reference space-time, i.e., an $n$-dimensional $(n = d + 1)$ smooth differential manifold $\mathcal{M}$, eventually with boundary $\partial \mathcal{M}$, where we use a system of local coordinates $(x^\mu)$, $\mu = 0, \ldots, d$ defined on an open set $U_\mathcal{M} \subset \mathcal{M}$. Actually, it is not required for $\mathcal{M}$ to be a space-time in the sense that it carries a Lorentzian metric, but rather it is only required to be orientable, i.e., to carry a volume form, say $\text{vol}_\mathcal{M} \in \Omega^n(\mathcal{M})$.

- A fibre bundle over $\mathcal{M}$, say $\pi : E \to \mathcal{M}$, whose typical fibre is denoted by $E$ and where we use the system of local fibred coordinates $(x^\mu, u^a)$, $\mu = 0, \ldots, d$, $a = 1, \ldots, r$ defined on an open set $U_E \subset E$. Sections of $\pi$ are local functions on $\mathcal{M}$ with values in $E$, they are denoted by $\phi$ and represent the configuration fields of the theory. In the system of local fibred coordinates chosen they read:
  $$\phi : U_\mathcal{M} \to U_E : x^\mu \mapsto \phi^a(x).$$
  (36)

The carrier space where we settle the description of the field theory is the reduced dual of the first order jet bundle $J^1\pi$ (see [CDI+]), which we call Covariant Phase Space and we denote by $\mathcal{P}(E)$. As in [CDI+], we denote by $(x^\mu, u^a, \rho^\mu_a)$, $\mu = 0, \ldots, d$, $a = 1, \ldots, r$, a system of adapted fibred coordinates on $\mathcal{P}(E)$ defined on an open set $U_{\mathcal{P}(E)} \subset \mathcal{P}(E)$.

- A Hamiltonian, i.e., a local section of the projection $\kappa$ appearing in Eq.5 of [CDI+]:
  $$H : U_{\mathcal{P}(E)} \to U_J\pi : (x^\mu, u^a, \rho^\mu_a) \mapsto (x^\mu, u^a, \rho^\mu_a, \rho_0 = H(x, u, \rho)),$$
  (37)
  which defines the local function $H^5$. Now, the generic 1-semibasic $n$-form on $E$ reads (see [CDI+]):
  $$w = \rho_0^\mu du^a \wedge i_{\partial^\mu}\text{vol}_\mathcal{M} + \rho_0\text{vol}_\mathcal{M}.$$ 
  (38)

---

4Note that within the literature sometimes people refer to Covariant Phase Space as the solution space of the equations of motion, whereas for us it will be called simply solution space or Euler-Lagrange space.

5In the rest of the paper we will refer to Hamiltonian equivalently to indicate the section $H$ and the function $H$. 

The choice of a Hamiltonian defines a unique differential \( n \)-form on \( \mathcal{P}(\mathbb{E}) \) which is the pull-back via \( H \) of \( w \):

\[
\Theta_H = (-H)^* w = \rho^a \partial_a \omega \wedge i_{\partial_a} \omega = H(x, u, \rho) = H(x, u, \rho) \text{vol}_\mathbb{M} - H(x, u, \rho) \text{vol}_\mathbb{M}.
\]

(39)

Its differential is denoted by \(-d\Theta_H\) and is a multisymplectic form\(^7\), from which the name of the formalism we are using.

### 2.2 The space of dynamical fields and the Schwinger-Weiss variational principle

The dynamical content of the theory will be implemented by means of a variational principle à la Schwinger-Weiss. The first ingredient to formulate it, is an action functional to extremize. With the ingredients introduced in the previous section in hand, an action functional can be defined in the following way:

\[
\mathcal{F}_\chi = \int_\mathbb{M} \chi^* \Theta_H = \int_\mathbb{M} \left( P^\mu \partial_\mu \phi^a - H(\chi) \right) \text{vol}_\mathbb{M},
\]

(40)

where \( \chi \) is a section of \( \delta_1 \). In particular we chose \( \chi \) to be a section of the type introduced in Def.2.1 of [CDI\textsuperscript{+}], i.e., a section of \( \delta_1 \) which is the composition of a section \( \phi \) of \( \pi \) and a section \( P \) of \( \delta_0 \), \( \chi = P \circ \phi =: (\phi, P) \).

Note that \( \mathcal{F} \) is a real-valued function on \( \Gamma_{\text{split}}(\delta_1) \). \( \Gamma_{\text{split}}(\delta_1) \), being (a subset of) a space of smooth sections, is a Frechet space. However, in order to avoid the amount of technicalities needed to define a Cartan-like differential calculus on Frechet spaces\(^8\), it would be desirable for \( \mathcal{F} \) to be defined actually on a Banach manifold. The way we will address this problem is to extend (by continuity) \( \mathcal{F} \) to the closure of \( \Gamma_{\text{split}} \) with respect to a Banach norm in which \( \mathcal{F} \) is continuous. The existence of such a norm is clearly not guaranteed in general and it depends on the particular \( \mathcal{F} \) one has in hand. However, from now on we assume it to exist in general and we prove that this is true case by case in the examples considered. We denote the closure of \( \Gamma_{\text{split}}(\delta_1) \) by \( \mathcal{F}_{\mathcal{P}(\mathbb{E})} \) and we still denote by \( \chi = (\phi, P) \) elements of \( \mathcal{F}_{\mathcal{P}(\mathbb{E})} \). The induced space of \( \phi \)'s will be denoted by \( \mathcal{F}_\mathbb{E} \). We will refer to \( \mathcal{F}_\mathbb{E} \) as the space of configuration fields and to \( \mathcal{F}_{\mathcal{P}(\mathbb{E})} \) as the space of dynamical fields of the theory.

Now, given a tangent vector to \( \mathcal{F}_{\mathcal{P}(\mathbb{E})} \) at \( \chi \), say \( \mathcal{X}_\chi \), the variation of \( \mathcal{F} \) in the direction of \( \mathcal{X}_\chi \) reads [IS17, CDI\textsuperscript{+}):

\[
\delta_{\mathcal{X}_\chi} \mathcal{F}_\chi = \int_\mathbb{M} \chi^* [i_{\mathcal{X}} d\Theta_H] + \int_{\partial_\mathbb{M}} \chi^* [i_{\partial_\mathbb{M}} d\Theta_H],
\]

(41)

where \( \mathcal{X} \) is a vector field over \( \mathcal{P}(\mathbb{E}) \) defined on an open neighborhood of the image of \( \chi \) which is an

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\(^6\)The minus sign is a matter of convention.

\(^7\)We recall that a multisymplectic form is a closed non-degenerate \( n \)-form, where by non-degenerate we mean that \( i_X d\Theta_H = 0 \implies X = 0 \).

\(^8\)For which we refer to [MK97] and references therein.
extension of $\mathcal{X}_\chi^9$ and $\chi_{\partial\mathcal{M}} = \chi\big|_{\partial\mathcal{M}}$ is the restriction of $\chi$ to $\partial\mathcal{M}$. The terms appearing in (41), being all linear in $\mathcal{X}_\chi$, can be interpreted in terms of differential forms on $\mathcal{F}_{\mathcal{P}(\mathcal{E})}$ as explained in [IS17,CDI+], and (41) can be written as:

$$d\mathcal{S} = \mathcal{E}L + \Pi_\partial^* \alpha^{\partial\mathcal{M}}$$

(42)

where $\mathcal{E}L \in \Omega^1(\mathcal{F}_{\mathcal{P}(\mathcal{E})})$ is defined by:

$$\mathcal{E}L|\mathcal{X}_\chi) = \int_{\partial\mathcal{M}} \chi^* [i_Xd\Theta_H] ,$$

(43)

the map $\Pi_\partial$ is the restriction map to the boundary:

$$\Pi_\partial : \mathcal{F}_{\mathcal{P}(\mathcal{E})} \to \mathcal{F}_{\partial\mathcal{M}} : \chi \mapsto \chi\big|_{\partial\mathcal{M}} ,$$

(44)

($\mathcal{F}_{\partial\mathcal{M}}$ being the space of restrictions of $\chi$’s to $\partial\mathcal{M}$) and $\Pi_\partial^* \alpha^{\partial\mathcal{M}} \in \Omega^1(\mathcal{F}_{\mathcal{P}(\mathcal{E})})$ is defined by:

$$\Pi_\partial^* \alpha^{\partial\mathcal{M}}(\mathcal{X}_\chi) = \int_{\partial\mathcal{M}} \chi^* [i_X\Theta_H] ,$$

(45)

where $\alpha^{\partial\mathcal{M}} \in \Omega^1(\mathcal{F}_{\partial\mathcal{M}})$ is defined by:

$$\alpha^{\partial\mathcal{M}}(\mathcal{X}_{\chi_{\partial\mathcal{M}}}) = \int_{\partial\mathcal{M}} \chi^* [i_X\Theta_H] ,$$

(46)

($\mathcal{X}_{\chi_{\partial\mathcal{M}}} \in \mathcal{T}_{\chi_{\partial\mathcal{M}}} \mathcal{F}_{\partial\mathcal{M}}$).

Now, the Schwinger-Weiss variational principle states that the dynamical fields appearing in nature are those for which the variation (along any direction) of the action functional only depends on boundary terms, that is, on terms only depending on the restrictions of the $\chi$’s to $\partial\mathcal{M}$. Evidently, the term $\Pi_\partial^* \alpha^{\partial\mathcal{M}}$ is a boundary term and, thus, the dynamical fields appearing in nature are those $\chi \in \mathcal{X}_{\mathcal{F}_{\mathcal{P}(\mathcal{E})}}$ for which:

$$\mathcal{E}L|\mathcal{X}_\chi) = 0 \ \forall \ \mathcal{X}_\chi \in \mathcal{T}_{\chi_{\mathcal{F}_{\mathcal{P}(\mathcal{E})}}} .$$

(47)

As it is proven in [IS17,CDI+], such $\chi$’s satisfy:

$$\chi^* [i_Xd\Theta_H] = 0 \ \forall \ X \in \mathcal{X}(\mathcal{U}(\chi)) ,$$

(48)

where $\mathcal{U}(\chi)$ is an open neighborhood of the image of $\chi$. The latter equation, in the system of local coordinates chosen on $\mathcal{P}(\mathcal{E})$ reads:

$$\begin{cases}
\frac{\partial \phi^\mu}{\partial x^\nu} = \left(\frac{\partial H}{\partial \phi^\mu}\right)_{\chi}, \\
\frac{\partial p^\mu}{\partial x^\nu} = -\left(\frac{\partial H}{\partial \phi^\mu}\right)_{\chi}
\end{cases}$$

(49)

It means that $X$ agree with $\mathcal{X}_\chi$ when evaluated on the image of $\chi$. Note that since the argument of the integral is evaluated along the image of $\chi$, this term does not depend on the particular extension chosen.
which are the so-called De Donder-Weyl equations also called sometimes covariant Hamilton equations.

The space of zeroes of $\mathbb{E}_L$, i.e., the space of solutions of the De Donder-Weyl equations is what we will refer to as the solution space throughout the whole paper and we will denote by:

$$\mathcal{E}_L \,:=\, \left\{ \chi \in \mathcal{F}_{\mathcal{P}(E)} : \mathbb{E}_L \chi = 0 \right\}.$$  \hspace{1cm} (50)

We will assume it to be a smooth immersed submanifold of $\mathcal{F}_{\mathcal{P}(E)}$.

### 2.3 The canonical pre-symplectic structure and the canonical formalism near the boundary

As it is extensively discussed in [CDI$^+$], the space $\mathcal{E}_L$ carries a canonical 2-form that can be used to define, in some cases, a Poisson bracket on $\mathcal{E}_L$. It is related with the differential 1-form $\alpha$ of Sect. 2.2 in the following way.

The differential of $\Pi^* \Omega_{\partial \mathcal{M}}(\mathcal{E}_L, \mathcal{Y}_\chi)$ is a 2-form on $\mathcal{F}_{\mathcal{P}(E)}$ given by:

$$-d \Pi^* \alpha_{\partial \mathcal{M}}(\mathcal{E}_L, \mathcal{Y}_\chi) =: \Pi^* \Omega_{\partial \mathcal{M}}(\mathcal{E}_L, \mathcal{Y}_\chi) = \int_{\partial \mathcal{M}} [i_X i_Y d\Theta_H],$$  \hspace{1cm} (51)

where $X$ and $Y$ are any two extensions of $\mathcal{X}_\chi$ and $\mathcal{Y}_\chi$ to vector fields defined in an open neighborhood of the image of $\chi$. It can be equivalently defined for any slice $\Sigma$ of $\mathcal{M}$ by formally substituting $\partial \mathcal{M}$ with $\Sigma$:

$$\Pi^* \Omega_{\Sigma}(\mathcal{E}_L, \mathcal{Y}_\chi) := \int_{\Sigma} [i_X i_Y d\Theta_H].$$  \hspace{1cm} (52)

Now, in [CDI$^+$] it is proven that $\Pi^* \Omega_{\Sigma}$ does not depend on the particular $\Sigma$ chosen if evaluated on $\chi \in \mathcal{E}_L$.

In [IS17, CDI$^+$] it is also proved that locally $\mathcal{E}_L$ is isomorphic with the space of solutions of the pre-symplectic Hamiltonian system $(\mathcal{F}_{\mathcal{P}(E)}^{\Sigma}, \Omega^{\Sigma}, \mathcal{H})$, where $\mathcal{F}_{\mathcal{P}(E)}^{\Sigma}$ is the space of fields restricted to $\Sigma$, $\Omega^{\Sigma}$ is the 2-form on $\mathcal{F}_{\mathcal{P}(E)}^{\Sigma}$ appearing in (51) and $\mathcal{H}$ is a function on $\mathcal{F}_{\mathcal{P}(E)}^{\Sigma}$ which, by fixing a coordinate system on $\mathcal{M}$ such that $\Sigma$ is the hypersurface with $x^0 = x^0_{\Sigma}$, reads:

$$\mathcal{H}(\chi_{\Sigma}) = \int_{\Sigma} \left[ -\beta^k_a \partial_k \varphi^a + H(\chi_{\Sigma}) \right] vol_\Sigma,$$  \hspace{1cm} (53)

where $\chi_{\Sigma} \in \mathcal{F}_{\mathcal{P}(E)}^{\Sigma}$ has the following coordinate expression:

$$\chi_{\Sigma}(\underline{z}) = \left( \phi^a|_{\Sigma}(\underline{z}), P^0_a|_{\Sigma}(\underline{z}), P^k_a|_{\Sigma}(\underline{z}) \right) =: \left( \varphi^a(\underline{z}), p_a(\underline{z}), \beta^k_a(\underline{z}) \right).$$  \hspace{1cm} (54)

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10The minus sign is a matter of convention.

11By slice we mean a co-dimension one hypersurface of $\mathcal{M}$ which split the space-time $\mathcal{M}$ in two regions, say $\mathcal{M}_+$ and $\mathcal{M}_-$.

12This can be done without loss of generality by virtue of the embedding theorem being $\Sigma$ a smooth submanifold of $\mathcal{M}$. 

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where $\mathcal{X} \in \Sigma$ and where we have distinguished the component $P^0_a$ which, in the system of coordinate chosen, is transversal to $\Sigma$ from the tangent components $P^k_a$. By means of "solutions of the presymplectic system" we mean the integral curves of the vector field $\Gamma$ on $\mathcal{F}_P^\Sigma(E)$ which is solution of the canonical equation:

$$i_\Gamma \Omega^\Sigma = d\mathcal{H}.$$  \hfill (55)

It is well known [GNH78] that, being $\Omega^\Sigma$ pre-symplectic, the latter equation can not be considered as it is as a well-posed equation for $\Gamma$ but rather one should apply the so-called pre-symplectic constraint algorithm (PCA). Such algorithm (that we recalled in Appendix A1 of [CDI+]) allows to find, eventually, a smooth immersed submanifold of $\mathcal{F}_P^\Sigma(E)$ that we will denote by $\mathcal{M}_\infty \hookrightarrow \mathcal{F}_P^\Sigma(E)$, on which the canonical equation:

$$i_{\Gamma_\infty} \Omega^\Sigma_\infty = d\mathcal{H}_\infty$$  \hfill (56)

is well posed, where $\Omega^\Sigma_\infty = i_* \Omega^\Sigma$, $\mathcal{H}_\infty = i_* \mathcal{H}$ and $\Gamma_\infty \in \mathcal{X}(\mathcal{M}_\infty)$. The integral curves of such a $\Gamma_\infty$ immersed into $\mathcal{F}_P^\Sigma(E)$ via $i_\infty$ are the solutions of the original pre-symplectic system considered.

In [IS17,CDI+] it is proved that such solutions are at least locally in one-to-one correspondence with extrema of $\mathcal{S}$. In [CDI+] it is also proved that when $\Omega^\Sigma_\infty$ is symplectic it gives a Poisson bracket which is equivalent to the one defined by the canonical structure $\Pi^*_\Sigma \Omega^\Sigma$ in the sense that they are related via a diffeomorphism.

Therefore, the idea here is to use again the structure $\Omega^\Sigma_\infty$ emerging from the PCA applied to Yang-Mills theories to define a Poisson bracket on $\mathcal{E}(\mathcal{L}_\#).$

The main problem here, is that in this case $\Omega^\Sigma_\infty$ turns out to be pre-symplectic and, therefore, it does not give rise directly to a Poisson bracket. Regarding this problem, the authors already started to analyse the case in which $\Omega^\Sigma_\infty$ is still pre-symplectic in [CDI+] where the first example of gauge theory, that is, free Electrodynamics is considered. There, the difficulty of defining a Poisson bracket in the pre-symplectic case was overcome by the aid of a flat connection by using an idea developed in [DGMS93]. However, as we will see, in this example a flat connection can not be chosen and we will use the coisotropic embedding theorem as a tool to define a Poisson bracket on a suitable enlargement of $\mathcal{E}(\mathcal{L}_\#).$

3 Covariant Poisson bracket on the solution space of Yang-Mills theories

3.1 A preliminar example: the magnetic monopole

Having in mind the Lagrangian description of the magnetic monopole given in [BMSS83], we propose the following multisymplectic Hamiltonian description.

We consider the one-dimensional space-time $\mathcal{M} = [a]$ being an interval of the real line with (global)
coordinate \{t\}. The bundle \(E\) underlying the theory is:

\[
\pi : I \times \mathbb{R}_+ \times SU(2) \to \mathbb{R},
\]

on which we will use the following set of local coordinates \(\{t, r, s\}\) where \(r\) is a (global) coordinate on \(\mathbb{R}_+\) and \(\{s\}\) is a system of local coordinates on \(SU(2)\). Having in mind the description given in [BMSS83], we will not consider the whole Covariant Phase space \(\mathcal{P}(E) = I \times T^*\mathbb{R}_+ \times T^*SU(2)\) with local coordinates \(\{t, r, p_r, s, \alpha_j\}_{j=1,2,3}\), but rather a submanifold of it, say \(\mathcal{M} = I \times T^*\mathbb{R}_+ \times \mathcal{N}\), where \(\mathcal{N}\) is the submanifold of \(T^*SU(2)\) given by the condition \(\alpha_3 = n\), for some real constant \(n\).

The space of dynamical fields in this case is the space of smooth sections of the bundle \(\pi : \mathcal{M} \to I\), which, equipped with the sup-norm, is a Banach manifold.

We consider the following Hamiltonian on \(\mathcal{M}\):

\[
H = \frac{1}{2} \left( p_r^2 + \alpha_1^2 + \alpha_2^2 \right),
\]

(58)

giving rise to the one-form:

\[
\Theta_H = p_r dr + \alpha_j \theta^j - H dt,
\]

(59)

where \(\theta_j\) are the left-invariant one-forms on \(SU(2)\).

In this case a one-codimension hypersurface of \(\mathcal{M}\) is just a point, say \(\{\bar{t}\}\) in \(\mathbb{R}\). Consequently, the 2-form \(\Omega^2\) is the following differential form on the finite-dimensional manifold \(T^*\mathbb{R}_+ \times \mathcal{N}\):

\[
\Omega^2 = dp_r \wedge dr + d\alpha_j \wedge \theta^j + \alpha_j d\theta^j,
\]

(60)

which, by using the fact that \(d\theta^i = \frac{1}{2} \epsilon^j_{\ k\ l} \theta^k \wedge \theta^l\), reads:

\[
\Omega^2 = dp \wedge dr + (d\alpha_1 - \alpha_2 \theta_3) \wedge \theta_1 + (d\alpha_2 + \alpha_1 \theta_3) \wedge \theta_2 - n \theta_1 \wedge \theta_2.
\]

(61)

What is more, being \(\mathcal{M}\) one-dimensional, the Hamiltonian functional \(H\) coincide with the Hamiltonian function \(H\)

\[
\mathcal{H} = \frac{1}{2} \left( p^2 + \alpha_1^2 + \alpha_2^2 \right).
\]

(62)

Since the manifold \(T^*\mathbb{R}_+\) does not play any relevant role in the construction of the coisotropic embedding and of the related Poisson bracket, we will neglect it and we will refer to \(\mathcal{M}\) as \(\mathcal{M} = \mathcal{N}\) with pre-symplectic structure

\[
\omega = (d\alpha_1 - \alpha_2 \theta_3) \wedge \theta_1 + (d\alpha_2 + \alpha_1 \theta_3) \wedge \theta_2 - n \theta_1 \wedge \theta_2,
\]

(63)

\(\alpha_j\) denote the coordinates on the fibres of \(T^*SU(2)\) associated with the choice of the left-invariant one-forms as a basis for the one-forms on \(SU(2)\).
with $\theta_j$ being the pull-back of the left-invariant one-forms on $SU(2)$ to $\mathcal{M}$. The kernel of the latter pre-symplectic structure reads

$$K_m = \text{span}\left\{ \bar{X}_3^\dagger \right\}$$

(64)

where $X_3$ denotes the third left-invariant vector field on $SU(2)$, $X_3^\dagger$ denotes its canonical lift to $T^*SU(2)$, and $\bar{X}_3^\dagger$ denotes the projection of the latter vector field to $\mathcal{M}$.

In this case the complement $W_m$ of $K_m$ inside $T_m\mathcal{M}$ is taken in the following way. Consider the local coordinates on $SU(2)$ given by $\{s, \beta_j\}_{j=1,2,3}$ where $\beta_j$ are the coordinates over the fibres associated with the choice of the right-invariant forms as a basis of differential one-forms. Consider the basis of vector fields over $T^*SU(2)$ given by the canonical lift of the right-invariant vector fields, say $Y_j^\dagger$, and the vertical vector fields $\partial/\partial \beta_j$. Project them to $\mathcal{M}$ to the basis of vector fields $\{\bar{Y}_j^\dagger, \partial/\partial \beta_j - d\alpha_3 \left( \partial/\partial \beta_j \right) \partial/\partial \alpha_3 \}$. Then, subtract their components along $\bar{X}_3^\dagger$, obtaining

$$W = \text{span}\left\{ Z_j = \bar{Y}_j^\dagger - \theta_3(Y_j^\dagger)\bar{X}_3^\dagger, \quad B_j = \frac{\partial}{\partial \beta_j} - d\alpha_3 \left( \frac{\partial}{\partial \beta_j} \right) \frac{\partial}{\partial \alpha_3} \right\}_{j=1,2,3}.$$  

(65)

This choice of $W$ is the one given by the connection whose projector onto the module of horizontal vector fields reads

$$P = \theta_3 \otimes \bar{X}_3^\dagger.$$  

(66)

To construct the symplectic manifold of the coisotropic embedding theorem we consider the dual of the characteristic bundle of $(\mathcal{M}, \omega)$, i.e. $K^* \simeq T^*SU(2)$ where we chose the system of local coordinates $\{s, \alpha_1, \alpha_2, \mu\}$. It project onto $\mathcal{M}$ via the projection

$$\tau : T^*SU(2) \to \mathcal{M} : (s, \alpha_1, \alpha_2, \mu) \mapsto (s, \alpha_1, \alpha_2).$$  

(67)

The structure $\Omega_0$ along the zero section of $\tau$ reads

$$\Omega_0 = \omega + d\mu \wedge \theta_3.$$  

(68)

Its extension over a tubular neighborhood of the zero section reads

$$\Omega_0^{ext} = \tau^*\omega + d\mu \wedge \theta_3$$  

(69)

but it is not closed since $d\omega = 0$ but $d(d\mu \wedge \theta_3) = 0$. Indeed

$$d\mu \wedge \theta_3 = d(\mu \theta_3) - \mu d\theta_3,$$  

(70)

thus the term $\alpha$ needed to made $\Omega_0^{ext}$ closed is exactly

$$\alpha = \mu d\theta_3,$$  

(71)
which, indeed, vanish along the zero section of $\tau$. Therefore, the symplectic manifold of the coisotropic embedding theorem in this case is $\tilde{\mathcal{M}} = T^*SU(2)$ with the symplectic form

$$\Omega = \tau^*\omega + d\mu \wedge \theta_3 + \mu d\theta_3.$$  \hfill (72)

Recalling that $d\theta_3 = -\theta_1 \wedge \theta_2$ we are in the case where $\alpha$ has only "components" along $W$, since it does not contain terms in $\theta_3$. The term $-\theta_1 \wedge \theta_2$ is indeed the curvature of the connection associated with the one form $\theta_3$ which is non-vanishing. Therefore

$$\Omega = \tilde{\Omega}_W + \Omega_{K \oplus K}.$$  \hfill (73)

where $\tilde{\Omega}_W = \tau^*\omega - \mu \theta_1 \wedge \theta_2$.

Note that nor $\tilde{\Omega}_W$ nor $\Omega_{K \oplus K}$ are closed and, thus, $\tilde{\Omega}_W$ can not be used to induce a Poisson bi-vector field directly on $\tilde{\mathcal{M}}$ and evaluate the associate bracket (which will obey the Leibnitz rule and the Jacobi identity) on the subalgebra of functions on $\tilde{\mathcal{M}}$ being pull-back via $\tau$ of functions on $\mathcal{M}$. As discussed in section 1.2.3 this bracket does not restrict to such a subalgebra and, in general, will present anomalies terms depending on the additional variables $\mu$.

First of all, the basis of differential one-forms on $\mathcal{M}$ dual to the basis of vector fields $\{Z_j, B_j\}_{j=1,2,3}$ is readily computed to be made by $\{\eta_j, \rho_j\}_{j=1,2,3}$ where $\eta_j$ are the right-invariant differential forms and $\rho_j = d\beta_j - \frac{1}{2}\epsilon_{jkl}\eta_k \wedge \eta_l$. Using the basis $\{\eta_j, \rho_j, \theta_3, d\mu\}_{j=1,2,3}$ of differential one-forms on $\tilde{\mathcal{M}}$, the symplectic structure $\Omega$ reads

$$\Omega = \delta^{jk}\rho_j \wedge \eta_k - \frac{1}{2}\epsilon^{kl}_j\left(\beta_l + (n - \mu)\theta_3^l\right)\eta_j \wedge \eta_k + d\mu \wedge \theta_3.$$  \hfill (74)

where $\theta_3 = \theta_3(\tilde{Y}^\dagger_i)$. The inverse of such a differential two-form is computed to be the following Poisson bi-vector field

$$\Lambda = \delta^{jk}Z_j \wedge B_k - \frac{1}{2}\epsilon_{jkl}\left(\beta_l + (n - \mu)\theta_3^l\right)B_j \wedge B_k + \frac{\partial}{\partial \mu} \wedge \tilde{X}^\dagger_i.$$  \hfill (75)

That this Poisson bracket do not restrict to the subalgebra of functions on $\tilde{\mathcal{M}}$ being the pull-back of functions on $\mathcal{M}$ is readily seen by noting that the term $B_j \wedge B_k$ explicitly depends on $\mu$ and, thus, $\Lambda$ is not projectable to $\mathcal{M}$.

As an example, we explicitly write the Poisson algebra closed by the pull-back via $\tau$ of following functions

$$J_k = \beta_k - n\theta_3^k;$$  \hfill (76)

which reads

$$\{\tilde{J}_j, \tilde{J}_k\} = \epsilon_{jkl}\tilde{J}_l + \mu \epsilon_{jkl}\theta_3^l.$$  \hfill (77)

where $\tilde{J}_k = \tau^*J_k$.

**Remark 3.1.** The functions $J_k$ above are the restriction to the zero section of $\tau$ of the conserved currents associated with the canonical lift of the left-action of $SU(2)$ on $\mathcal{M}$.

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3.2 A second example: free Electrodynamics

As extensively described in Ex. 2.7 and Sect. 5.3 of [CDI⁺], a multisymplectic formulation of free Electrodynamics on the Minkowski space-time leads to a final stable manifold being:

\[ \mathcal{M}_\infty = \prod_{k=1,2,3} \mathcal{H}^2(\Sigma, \text{vol}_\Sigma) \times \prod_{k=1,2,3} \mathcal{H}^1(\text{div}; \Sigma, \text{vol}_\Sigma)^k, \]  

(78)

whose points are denoted, in a system of local coordinates, by \( m_\infty = (a_k, p^k) \) and where \( \Sigma \) is any 1-codimension hypersurface of the Minkowski space-time.

\( \mathcal{M}_\infty \) is a pre-symplectic manifold equipped with the pre-symplectic structure:

\[ \Omega^\Sigma_{\infty}(\mathcal{X}, \mathcal{Y}) = \int_\Sigma \left( \mathcal{X}_{a_k} p^k - \mathcal{Y}_{p^k} \mathcal{X}_{a_k} \right) \text{vol}_\Sigma, \]  

(79)

whose kernel, at each point \( m_\infty \) reads:

\[ K_{m_\infty} = \left\{ \mathcal{X} : \mathcal{X}_{a_k} = \partial_k \phi \right\}, \]  

(80)

for some \( \psi \in \mathcal{H}^\frac{5}{2}(\Sigma, \text{vol}_\Sigma) \).

Denoting by \( K \) the characteristic distribution of \( \Omega^\Sigma_{\infty} \), on the bundle \( \mathcal{M}_\infty \to \mathcal{M}_\infty/K \) the connection represented by the following \( 1-1 \) tensor field can be fixed:

\[ P_k = P_k \otimes \frac{\delta}{\delta a_k}, \]  

(81)

where by \( \left\{ \frac{\delta}{\delta a_k}, \frac{\delta}{\delta p^j} \right\}_{k=1,2,3} \) we denote a basis of vector field over \( \mathcal{M}_\infty \) and where:

\[ P_k = \partial_k \int_\Sigma G_\Delta(\bar{x}, y) \delta^{jl} \partial_j \delta a_l(y) d^3y, \]  

(82)

with \( G_\Delta \) being the Green’s function of the Laplacian operator, \( \bar{x}, y \) points in \( \Sigma \) and \( \left\{ \delta a_j \right\}_{j=1,2,3} \) a dual basis of \( \left\{ \frac{\delta}{\delta a_j} \right\}_{j=1,2,3} \). Such a connection, gives rise to the following splitting of \( T_{m_\infty} \mathcal{M}_\infty \approx \mathcal{M}_\infty \):

\[ T_{m_\infty} \mathcal{M}_\infty \approx \mathcal{M}_\infty = \prod_{k=1,2,3} \mathcal{H}^\frac{5}{2}(\text{div}; \Sigma, \text{vol}_\Sigma) \oplus \text{grad} \mathcal{H}^\frac{5}{2} \times \prod_{k=1,2,3} \mathcal{H}^\frac{1}{2}(\text{div}; \Sigma, \text{vol}_\Sigma)^k, \]  

(83)

where the complement of \( K_{m_\infty} = \text{grad} \mathcal{H}^\frac{5}{2} \) represents the horizontal distribution associated with the connection.

The enlarged manifold obtained extending \( \mathcal{M}_\infty \) via the dual of \( K_{m_\infty} \) in the spirit of the coisotropic embedding theorem is:

\[ \tilde{\mathcal{M}} \approx T_m \tilde{\mathcal{M}} = \prod_{k=1,2,3} \mathcal{H}^\frac{5}{2}(\text{div}; \Sigma, \text{vol}_\Sigma) \oplus \text{grad} \mathcal{H}^\frac{5}{2} \times \prod_{k=1,2,3} \mathcal{H}^\frac{1}{2}(\text{div}; \Sigma, \text{vol}_\Sigma)^k \times \text{grad} \mathcal{H}^\frac{5}{2}, \]  

(84)
where a point will be denoted, in a system of local coordinates, by \( \tilde{m} = (\tilde{a}_k, \partial_k \psi, p^k, \mu^k) \).

In this case, a direct computation shows that:

\[ dP_k = 0 \]  

and, thus, we are in the case of Sect. 1.2.1. Consequently, denoting an element of \( T_{\tilde{m}}\tilde{M} \) by \( \mathcal{X} = (\tilde{X}_a, \partial_k \psi, X_p, \mu^k) \) the symplectic structure on \( \tilde{M}_\infty \) is:

\[ \tilde{\Omega}(\mathcal{X}, \mathcal{Y}) = \int_\Sigma (\tilde{X}_a \partial_k \psi \delta_{ap} - \tilde{X}_a \delta_{kp} \partial_k \psi) \text{vol}_\Sigma + \tilde{X}_a \partial_k \psi \delta_{kp} \partial_k \psi \delta_{ap} \text{vol}_\Sigma = \tilde{\Omega}_{\kappa \oplus \kappa^*}(\mathcal{X}, \mathcal{Y}). \]  

The inverse of \( \tilde{\Omega} \) reads:

\[ \tilde{\Lambda} = \Lambda_W \oplus \Lambda_{\kappa \oplus \kappa^*}, \]  

where:

\[ \Lambda_W = \frac{\delta}{\delta \tilde{a}_k} \wedge \frac{\delta}{\delta p^k}, \]  

with an integration over \( \Sigma \) implied in the wedge product. Such a bivector field is a Poisson bivector field being the inverse of a closed and non-degenerate form on \( W \) and it is projectable to the Poisson bivector field over \( M_\infty \):

\[ \lambda = \frac{\delta}{\delta \tilde{a}_k} \wedge \frac{\delta}{\delta p^k}, \]  

giving rise to the following Poisson bracket between any two functions on \( M_\infty \):

\[ \{ f, g \} = \int_\Sigma \left( \frac{\delta f}{\delta \tilde{a}_k} \frac{\delta g}{\delta p^k} - \frac{\delta f}{\delta p^k} \frac{\delta g}{\delta \tilde{a}_k} \right) \text{vol}_\Sigma, \]  

which coincides with the bracket in Eq. 145 of [CDI+].

### 3.3 Multisymplectic formulation of Yang-Mills theories

Let us now pass to the main aim of the paper, that is the construction of the Poisson structure in the case of Yang-Mills theories.

Let us start by describing the multisymplectic formulation of Yang-Mills theories on the Minkowski space-time. Here, the space-time reads \( \mathcal{M} = (\mathbb{R}^4, \eta) \), \( \eta \) being the Minkowski metric. Yang-Mills fields are connection one-forms on a principal fibre bundle with structure group \( G \) whose Lie algebra will be denoted by \( \mathfrak{g} \). Usually \( G \) is a compact and connected group, indeed, the most relevant examples of Yang-Mills type theories are Electrodynamics, the theory of weak interactions and Quantum Chromodynamics of strong interactions where \( G \) is respectively \( U(1) \), \( SU(2) \) and \( SU(3) \). Their unification is, indeed, described as a Yang-Mills theory with \( G = U(1) \times SU(2) \times SU(3) \). Connection
where space-time indices are raised/lowered by means of the Minkowski metric and Lie algebra indices whose sections are couples of connection one-forms and bivector fields with values in \( \mathfrak{g} \) are the coefficients of the curvature of the connection \( \nabla \). With this Hamiltonian, the action functional is seen to be:

\[
\chi = (A, P) : U_{\mathcal{M}} \rightarrow U_{\mathcal{P}(\mathcal{E})} : x^\mu \mapsto \left( A_\mu^a(x) dx^\mu \otimes \xi_a, P_{\mu\nu}^a(x, A(x)) \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} \otimes \xi_a \right),
\]

where \( U_{\mathcal{M}} \) is an open set in \( \mathcal{M} \) and \( \{ \xi_a \}_{a=1,...,\text{dim}\mathfrak{g}} \) is a basis of \( \mathfrak{g} \).

The Yang-Mills Hamiltonian \( H \) is:

\[
H(x, \alpha, \rho) = \frac{1}{4} \rho_{ab}^\alpha \rho_{b}^\alpha + \frac{1}{2} \epsilon_{abc}^{a} \rho_{ab}^\alpha \alpha_{c}^\alpha,
\]

where space-time indices are raised/lowered by means of the Minkowski metric and Lie algebra indices by means of the Killing-Cartan metric on \( \mathfrak{g} \) and where \( \epsilon_{abc}^{a} \) are the structure constants of the Lie algebra \( \mathfrak{g} \). With this Hamiltonian, the action functional is seen to be:

\[
\mathcal{S}_x = \int_{\mathcal{M}} \chi^* \Theta_H = - \int_{\mathcal{M}} \left[ P_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{4} P_{\mu\nu}^a P_{\mu\nu}^a \right] \text{vol}_{\mathcal{M}},
\]

where:

\[
F_{\mu\nu}^a = \frac{1}{2} \left( \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + \epsilon_{bc}^a A_{\mu}^b A_{\nu}^c \right) = (\nabla A)^{a}_{\mu\nu} =: \nabla_{\mu} A_{\nu}^a,
\]

are the coefficients of the curvature of the connection \( A \) and where \( \nabla \) represents the covariant derivative with respect to the connection \( A \).

For technical reasons that will be clear in the sequel, we will chose the space of dynamical fields made by \( \mathcal{H}^3 \) potentials and \( \mathcal{H}^1 \) momenta fields. The construction of the space of dynamical fields goes as follows. We consider the space \( \Gamma_0^{\text{split}}(\delta_1) \) of smooth, splitting sections of \( \delta_1 \) with compact support, on which the action functional \( 94 \) is well defined. On \( \Gamma_0^{\text{split}}(\delta_1) \) we consider the norm

---

\(^{14}\text{This restriction is allowed because the manifold we obtain, even if it is not strictly speaking the reduced dual of } \pi, \text{ is still a multisymplectic manifold.}\)
\[ \| \chi \|_{\varepsilon} = \sum_{\mu,a} \| A^a_\mu \|_{\mathcal{H}^0} + \sum_{\mu,\nu,a} \| P^{\mu\nu}_a \|_{\mathcal{H}^1}. \]

As the following proposition proves, \( \mathcal{S} \) is continuous in the norm \( \| \cdot \|_{\varepsilon} \) and, therefore, it can be extended by continuity to the closure of \( \Gamma^0_{\text{split}}(\delta_1) \) with respect to \( \| \cdot \|_{\varepsilon} \), i.e., to:

\[ \Gamma^0_{\text{split}}(\delta_1) \ni \| \varepsilon \|_{\varepsilon} =: \mathcal{F}_{P(\varepsilon)} = \prod_{\mu,a} \mathcal{H}^3(\mathcal{M}, \text{vol}_{\#})^a_\mu \times \prod_{\mu,\nu,a} \mathcal{H}^2(\mathcal{M}, \text{vol}_{\#})^{\mu\nu}_a. \tag{96} \]

Now, let us prove the continuity of \( \mathcal{S} \).

**Proposition 3.2 (Continuity of \( \mathcal{S} \)).** The action functional:

\[ \mathcal{S}_\chi = - \int_\mathcal{M} \left[ P^{\mu\nu}_a F^a_{\mu\nu} + \frac{1}{4} P^{\mu\nu}_a P^{\rho\sigma}_a \right] \text{vol}_{\#}, \tag{97} \]

is well defined on \( \Gamma^0_{\text{split}}(\delta_1) \) and is continuous in the norm \( \| \cdot \|_{\varepsilon} \).

**Proof.** That \( \mathcal{S} \) is well defined on \( \Gamma^0_{\text{split}}(\delta_1) \) is obvious.

Regarding the continuity, the following estimate holds:

\[ | \mathcal{S}_\chi - \mathcal{S}_{\tilde{\chi}} | = \left| \int_\mathcal{M} \left( P^{\mu\nu}_a F^a_{\mu\nu} + \frac{1}{4} P^{\mu\nu}_a P^{\rho\sigma}_a - \tilde{P}^{\mu\nu}_a F^a_{\mu\nu} - \frac{1}{4} \tilde{P}^{\mu\nu}_a P^{\rho\sigma}_a \right) \text{vol}_{\#} \right| \leq \]

\[ \leq \int_\mathcal{M} \left( \left| P^{\mu\nu}_a - \tilde{P}^{\mu\nu}_a \right| F^a_{\mu\nu} \right) \text{vol}_{\#} + \int_\mathcal{M} \left| \left| \tilde{P}^{\mu\nu}_a (F^a_{\mu\nu} - F^a_{\mu\nu}) \right| \right| \text{vol}_{\#} + \]

\[ + \frac{1}{4} \int_\mathcal{M} \left| \tilde{P}^{\mu\nu}_a (P^{\mu\nu}_a - \tilde{P}^{\mu\nu}_a) \right| \text{vol}_{\#} + \frac{1}{4} \int_\mathcal{M} \left| \tilde{P}^{\mu\nu}_a (P^{\mu\nu}_a - \tilde{P}^{\mu\nu}_a) \right| \text{vol}_{\#}, \tag{98} \]

where \( \tilde{\chi} = (\tilde{A}, \tilde{P}) \) and \( \tilde{F}^{\mu\nu}_a = \frac{1}{2} \left( \partial_\nu \tilde{A}^a_\mu - \partial_\mu \tilde{A}^a_\nu + \epsilon_{\mu\nu\sigma}^a \tilde{A}^b_\sigma \tilde{A}^c_\nu \right) \). Now:

\[ \mathcal{S}_1 \leq \sum_{\mu,\nu,a} \| P^{\mu\nu}_a - \tilde{P}^{\mu\nu}_a \|_{L^2} \| F^a_{\mu\nu} \|_{L^2} \leq \sum_{\mu,\nu,a} \| P^{\mu\nu}_a - \tilde{P}^{\mu\nu}_a \|_{\mathcal{H}^2} \| F^a_{\mu\nu} \|_{L^2}, \tag{99} \]

\[ \mathcal{S}_2 \leq \sum_{\mu,\nu,a} \| \tilde{P}^{\mu\nu}_a \|_{L^2} \left( \frac{1}{2} \left( \| \partial_\nu (\tilde{A}^a_\mu - \tilde{A}^a_\nu) \|_{L^2} + \| \partial_\mu (\tilde{A}^a_\mu - \tilde{A}^a_\nu) \|_{L^2} \right) \right) + \]

\[ + \sum_{\mu,\nu,a} \left( \| \tilde{P}^{\mu\nu}_a \|_{L^2} + | \epsilon_{\mu\nu\sigma}^a | \| \tilde{P}^{\mu\nu}_a \tilde{A}^b_\mu \|_{L^2} \right) \| \tilde{A}^c_\nu - \tilde{A}^c_\nu \|_{\mathcal{H}^3} \leq \sum_{\mu,\nu,a} \left( \| \tilde{P}^{\mu\nu}_a \|_{L^2} + | \epsilon_{\mu\nu\sigma}^a | \| \tilde{P}^{\mu\nu}_a \tilde{A}^b_\mu \|_{L^2} \right) \| \tilde{A}^c_\nu - \tilde{A}^c_\nu \|_{\mathcal{H}^3} \leq \] \[ \leq \frac{1}{4} \sum_{\mu,\nu,a} \| P^{\mu\nu}_a \|_{L^2} \| P^{\mu\nu}_a - \tilde{P}^{\mu\nu}_a \|_{L^2} \leq \frac{1}{4} \sum_{\mu,\nu,a} \| P^{\mu\nu}_a \|_{L^2} \| P^{\mu\nu}_a - \tilde{P}^{\mu\nu}_a \|_{\mathcal{H}^2} \tag{100} \]

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\[
\mathcal{J}_4 \leq \frac{1}{4} \sum_{\mu, \nu, a} \| \hat{P}_a^{\mu \nu} \|_{L^2} \| P_{a \mu} - \hat{P}_a^{\mu \nu} \|_{L^2} \leq \frac{1}{4} \sum_{\mu, \nu, a} \| \hat{P}_a^{\mu \nu} \|_{L^2} \| P_{a \mu} - \hat{P}_a^{\mu \nu} \|_{H^2}. \tag{102}
\]

Therefore, it is evident that if \( \| \chi - \hat{\chi} \|_E \to 0 \), then, since both \( \| A_{a \mu}^\alpha - \hat{A}_{a \mu}^\alpha \|_{H^3} \) and \( \| P_{a \mu \nu} - \hat{P}_{a \mu \nu} \|_{H^2} \) goes to zero for all \( \mu, \nu, a \), \( | S_{\chi} - S_{\hat{\chi}} | \to 0 \).

**Remark 3.3.** Actually, the regularity chosen is not the minimal requirement for the \( A_{a \mu} \)’s, since we really only need their covariant derivatives to be square integrable whereas the functions themselves may be not integrable still leaving the action functional well defined. We restrict to this class of functions, since it is very useful from the mathematical point of view. However, physically speaking, it must be clear that this exclude some (actually, never observed in nature) situations, such as the magnetic monopole, where the potential \( A_{a \mu}^\alpha \) is non-zero at infinity.

Thus, our space of dynamical fields reads:

\[
F_{P(E)} = \prod_{a, \mu} H^3(\mathcal{M}, \text{vol}_\mathcal{M})_{a \mu}^a \times \prod_{a, \mu, \nu} H^2(\mathcal{M}, \text{vol}_\mathcal{M})_{a \mu \nu}^{\mu \nu},
\tag{103}
\]

which is a Hilbert manifold (actually an Hilbert space) whose tangent space at each point reads:

\[
T_{\chi} F_{P(E)} = \prod_{a, \mu} H^3(\mathcal{M}, \text{vol}_\mathcal{M})_{a \mu}^a \times \prod_{a, \mu, \nu} H^2(\mathcal{M}, \text{vol}_\mathcal{M})_{a \mu \nu}^{\mu \nu}.
\tag{104}
\]

Now, let us consider the slice \( \Sigma = \{ m \in \mathcal{M} : x^0 = x_0 \} \). By virtue of the trace theorem [DL90] the space \( F_{P(E)}^\Sigma \) turns out to be:

\[
F_{P(E)}^\Sigma = \prod_{a, \mu} H^3(\Sigma, \text{vol}_\Sigma)_{a \mu}^a \times \prod_{a, \mu, \nu} H^2(\Sigma, \text{vol}_\Sigma)_{a \mu \nu}^{\mu \nu}.
\tag{105}
\]

Elements is \( F_{P(E)}^\Sigma \), will be denoted by \( \chi_\Sigma \) and we will use the following coordinate expression for them:

\[
\chi_\Sigma = \left( A_{a \mu}^\alpha \big|_{\Sigma}(\bar{x}), P^0_{a \mu} \big|_{\Sigma}(\bar{x}), P_{a \mu}^{jk} \big|_{\Sigma}(\bar{x}) \right) = \left( a_{a \mu}^\alpha(\bar{x}), \gamma_{a \mu}^0(\bar{x}), \beta_{a \mu}^{jk}(\bar{x}) \right),
\tag{106}
\]

where \( \bar{x} \) is the coordinatization of a point in \( \Sigma \). Again, being \( F_{P(E)}^\Sigma \) a Hilbert manifold and, actually a Hilbert space, its tangent space at each point is isomorphic to the Hilbert space itself:

\[
T_{\chi_\Sigma} F_{P(E)}^\Sigma = \prod_{a, \mu} H^3(\Sigma, \text{vol}_\Sigma)_{a \mu}^a \times \prod_{a, \mu, \nu} H^2(\Sigma, \text{vol}_\Sigma)_{a \mu \nu}^{\mu \nu}.
\tag{107}
\]

Now, the 2-form \( \Omega^\Sigma \) reads:

\[
\Omega^\Sigma(\chi_\Sigma, \chi_\Sigma) = \int_{\Sigma} \left[ \frac{\gamma_{a \mu}^0 \gamma_{a \mu}^{jk} - \gamma_{a \mu}^0 \gamma_{a \mu}^{jk}}{\gamma_{a \mu}^0 \gamma_{a \mu}^{jk}} \right] \text{vol}_\Sigma,
\tag{108}
\]

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where \( \chi_a \) and \( \chi_p \) are the \( a_p \) and the \( p_a \) components of a tangent vector to \( \mathcal{F}_{\mathcal{P}}(E) \) at \( \chi \). On the other hand, the Hamiltonian functional is:

\[
\mathcal{H}(\chi) = \int_{\Sigma} \left[ \frac{1}{2} p_a p_a + \frac{1}{4} \beta_a \beta_a + p_a \nabla_a a_a + \frac{1}{2} \beta_a \partial_a \right] vol_{\Sigma},
\]

where:

\[
\nabla_a a_a = \partial_a a_a + \epsilon_a \epsilon_b a_b c_c,
\]

\[
\nabla_a a_a = \frac{1}{2} (\partial_a a_a - \partial_a a_a + \epsilon_a \epsilon_b a_b c_c).
\]

Now, we use the PCA to find solutions of the pre-symplectic system \( (\mathcal{F}_{\mathcal{P}}(E), \Omega, \mathcal{H}) \). Following what we recalled in Appendix A1 of [CDI⁺], first we should determine \( T_{\chi\Sigma} : \mathcal{F}_{\mathcal{P}}(E) \to \ker \Omega \). By looking at Eq. (108), \( \ker \Omega \) is made by tangent vectors having only components along \( a_p \) and by tangent vectors having only components along \( \beta_a \). Let us denote a basis of \( \ker \Omega \) by:

\[
\left\{ \frac{\delta}{\delta a_a}, \frac{\delta}{\delta \beta_a} \right\}_{a=1, \ldots, dim, j=k=1, \ldots, 3}.
\]

Then, the first manifold of the PCA is obtained by imposing that such tangent vectors lie also in the kernel of \( d\mathcal{H} \) at each \( \chi \). The following conditions emerge:

\[
\begin{align*}
\frac{d}{d \chi} \mathcal{H} & = 0 \implies \nabla^*_a p_a = 0, \\
\frac{d}{d \chi} \mathcal{H} & = 0 \implies \beta_a = -\gamma^{b} p_a \partial_a \nabla a_a ,
\end{align*}
\]

where \( \nabla^*_a p_a = \partial_a p_a + \epsilon_a \epsilon_b p_a a_b \) is the adjoint of the covariant differential. The second constraint can be eliminated by replacing \( \beta_a \) with the expression on the right hand side in terms of the \( a_a \) while the first constraint cannot be eliminated. Therefore, the first manifold obtained via the PCA is:

\[
\mathcal{M}_1 = \left\{ (a_a, p_a) : \nabla^*_a p_a = 0 \right\} =: \prod_{k,a} \mathcal{H}^2(\Sigma, vol_\Sigma)_a \times \prod_{k,a} \mathcal{H}^2(\Sigma, \nabla a_a, 0, vol_\Sigma)_a ,
\]

which can be immersed into \( \mathcal{F}_{\mathcal{P}}(E) \) via the second of Eq. (110) and by fixing any arbitrary \( a_a \). We denote such immersion by \( i_1 \). We will prove that \( \mathcal{M}_1 \) is actually a Hilbert space itself by proving that it is a closed subspace of \( \mathcal{F}_{\mathcal{P}}(E) \). But, for the moment let us proceed with the PCA. In the second step of the PCA we should determine \( T_{\chi\Sigma} : \mathcal{F}_{\mathcal{P}}(E) \to \ker \Omega \). It is made by tangent vectors \( \chi \) to \( \mathcal{F}_{\mathcal{P}}(E) \) at \( \chi \in i_1(\mathcal{M}_1) \) such that \( i_1(\chi_\Sigma, \Omega_\Sigma) = 0 \). It is readily seen that by virtue of the constraint \( \nabla a_a = 0 \), \( T_{\chi\Sigma} : i_1(\mathcal{M}_1) \) is made by tangent vectors having \( a_a \) component equal to the covariant differential of a
Lie algebra-valued function, say $\psi^a\!\!\!\!\!\!\!$ and $p_a^k$ component equal to $[p^k, \psi]^a$, $[\cdot, \cdot]$ being the Lie bracket in $g$:

$$T_{\chi \Sigma} i_1(M_1) = \left\{ \chi_{\chi \Sigma} \in T_{\chi \Sigma} F_{P(S)} | i_1(M_1) : \chi_{\chi \Sigma}^a = \nabla k \psi^a, \chi_{\chi \Sigma}^k = [p^k, \psi]_a \right\} . \quad (116)$$

Note that, consistently, $\chi_{\chi \Sigma}$ is a tangent vector to $i_1(M_1) \subset F_{P(S)}$ since $\nabla_k \psi^a$ is an $H^2(\Sigma, \text{vol}_\Sigma)$ function and $[p^k, \psi]_a$ is an $H^2(\Sigma, \text{vol}_\Sigma)$ function because $p_a^k \in H^2(\Sigma, \text{vol}_\Sigma)_a \supset H^{-2}(\Sigma, \text{vol}_\Sigma)_a = H^2(\Sigma, \text{vol}_\Sigma)_a$, thus:

$$\int [p^k, \psi]_\Sigma \text{vol}_\Sigma = \int \epsilon_a^b \epsilon_c^b p_a^k \psi^c \text{vol}_\Sigma = \epsilon_a^b \epsilon_c^b \langle p_a^k, \psi^c \rangle_{H^2} , \quad (117)$$

and, consequently:

$$\| [p^k, \psi]_\Sigma \|_{H^2} \leq \| \epsilon_a^b \epsilon_c^b \|_{H^{-2}} \| \psi^c \|_{H^2} \leq \infty . \quad (118)$$

Let us denote elements of $T_{\chi \Sigma} i_1(M_1)$ by $\chi_{\chi \Sigma}^\text{GAUGE}$. A direct computation shows that $i_{\chi_{\chi \Sigma}^\text{GAUGE}} \text{vol} H = 0 \forall \psi^a$, that is, the PCA stops after finding the first manifold $M_1 =: M_\infty$.

Therefore, the final manifold obtained out of the PCA is the Hilbert manifold:

$$M_\infty = \left\{ (a^a_k, p^k_a) : \nabla_k p^k_a = 0 \right\} =: \prod_{k,a} H^2(\Sigma, \text{vol}_\Sigma)_a \times \prod_{k,a} H^2(\Sigma, \nabla p^k_a = 0, \text{vol}_\Sigma)_a . \quad (119)$$

That this is a Hilbert manifold and, actually, a Hilbert space, is due to the fact that $H^2(\Sigma, \text{vol}_\Sigma)_a$ is a Hilbert space and that $H^2(\Sigma, \nabla p^k_a = 0, \text{vol}_\Sigma)_a$ is a closed (and, thus, Hilbert) subspace of $H^1(\Sigma, \text{vol}_\Sigma)_a$. Let us prove this last claim. Being $\nabla$ the covariant derivative associated with a $H^2$ connection, it acts as a linear operator between the Hilbert spaces $H^2(\Sigma, \text{vol}_\Sigma)_a$ and $H^2(\Sigma, \text{vol}_\Sigma)_a$:

$$\nabla : H^2(\Sigma, \text{vol}_\Sigma)_a \rightarrow H^2(\Sigma, \text{vol}_\Sigma)_a : f^a \mapsto \nabla_k f^a . \quad (120)$$

\footnote{We will chose $\psi^a$ to lie in $H^2(\Sigma, \text{vol}_\Sigma)$ so that we are ensured that its covariant gradient is again a $H^2(\Sigma, \text{vol}_\Sigma)$ function. Indeed, this is due to the fact that $H^2(\Sigma, \text{vol}_\Sigma)$ is a Banach algebra (see [Ada75]) which ensures the following inequality

$$\sum_{k,a} \| \nabla_k \psi^a \|_{H^2} \leq \sum_{k,a} \| \partial_k \psi^a \|_{H^2} + \sum_{k,a} \| \epsilon^b \epsilon^c \| \| a^b_k \psi^c \|_{H^2} \leq \sum_{k,a} \| \psi^a \|_{H^2} + \sum_{k,a} \| \epsilon^b \epsilon^c |B_{b,c,k}| a^b_k \psi^c \|_{H^2} \| \psi^c \|_{H^2} , \quad (114)$$

for some constants $B_{b,c,k}$, where the inequality $\sum_{k,a} \| \partial_k \psi^a \|_{H^2} \leq 3 \sum_{k,a} \| \psi^a \|_{H^2}$ is due to

$$\sum_{k,a} \| \partial_k \psi^a \|_{H^2} = \sum_{k,a} \| k \tilde{\psi} \|_{L^2} \leq 3 \sum_{a} \| k \tilde{\psi} \|_{L^2} = 3 \sum_{a} \| \psi^a \|_{H^2} , \quad (115)$$

where $\tilde{\psi}$ is the Fourier transform of $\psi^a$.}

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Its adjoint is an operator from $\mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^k_a$ that can be identified with $\mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^a_k$ itself, to $\mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^a_k$ that can be identified with $\mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^a_k$:

$$\nabla^* : \mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^k_a \rightarrow \mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^a_k : p^k_a \mapsto \nabla^* k^k_a.$$  

(121)

The following holds.

**Proposition 3.4** (Closedness of $\nabla$ in $\mathcal{H}^{1/2}$). The operator $\nabla$ is a closed operator from $\mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^a_k$ to $\mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^a_k$.

**Proof.** Consider a sequence of functions in $\mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^a_k$, say $\{ f_n^a \}_{n \in \mathbb{N}}$ converging to some $f^a \in \mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^a_k$ in the $\mathcal{H}^{1/2}$-norm. Then $\nabla f_n^a$ converges to $\nabla f^a$ in the $\mathcal{H}^{1/2}$-norm. Indeed:

$$\sum_{k,a} \| \nabla f_n^a - \nabla f^a \|_{\mathcal{H}^{1/2}} = \sum_{k,a} \| \nabla_k (f_n^a - f^a) \|_{\mathcal{H}^{1/2}} = \sum_{k,a} \| \partial_k (f_n^a - f^a) + e_{b,c} a^b_k (f_n^c - f^c) \|_{\mathcal{H}^{1/2}} =$$

$$\leq \sum_{k,a} \| \partial_k (f_n^a - f^a) \|_{\mathcal{H}^{1/2}} + \sum_{k,a} \| e_{b,c} \|_{\mathcal{H}^{1/2}} \| a^b_k (f_n^c - f^c) \|_{\mathcal{H}^{1/2}} \leq$$

$$\leq 3 \sum_{a} \| f_n^a - f^a \|_{\mathcal{H}^{1/2}} + \sum_{k,a} \| e_{b,c} \|_{\mathcal{H}^{1/2}} \| a^b_k \|_{\mathcal{H}^{1/2}} \| f_n^c - f^c \|_{\mathcal{H}^{1/2}} ,$$

(122)

where the last inequality is due to the content of footnote 13 and to the fact that $\mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)$ is a Banach algebra. Because of the latter inequality, $\sum_{k,a} \| \nabla f_n^a - \nabla f^a \|_{\mathcal{H}^{1/2}}$ approaches zero when $\| f_n^a - f^a \|_{\mathcal{H}^{1/2}}$ approaches zero. Thus, by definition of closed operator, $\nabla$ is closed.

Therefore, by means of the closed range theorem, the kernel of the adjoint of $\nabla$, i.e., $\mathcal{H}^{1/2} (\Sigma, \nabla^* p^k_a = 0, \text{vol}_\Sigma)^k_a$, is a closed split subspace of $\mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^k_a$ whose orthogonal complement coincide with the image of $\nabla$, say $\nabla \mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^a_k$. That is, the following splitting into closed (and, thus, Hilbert) subspaces exists:

$$\mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^k_a = \mathcal{H}^{1/2} (\Sigma, \nabla^* p^k_a = 0, \text{vol}_\Sigma)^k_a \oplus \nabla \mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^a_k ,$$

(123)

and the $p^k_a$'s of $\mathcal{M}_\infty$ lie exactly in the first component of such splitting. Therefore, we can conclude that $\mathcal{M}_\infty$ is the following Hilbert space:

$$\mathcal{M}_\infty = \prod_{k,a} \mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^k_a \times \prod_{k,a} \mathcal{H}^{1/2} (\Sigma, \nabla^* p^k_a = 0, \text{vol}_\Sigma) ,$$

(124)

whose tangent space at each point reads:

$$T_{(\alpha, p)} \mathcal{M}_\infty = \prod_{k,a} \mathcal{H}^{1/2} (\Sigma, \text{vol}_\Sigma)^k_a \times \prod_{k,a} \mathcal{H}^{1/2} (\Sigma, \nabla^* p^k_a = 0, \text{vol}_\Sigma) .$$

(125)
Such tangent space also coincide with the space of solutions of the linearization of the constraint $\nabla_k p^k_a = 0$, i.e., with the space of functions $\mathbb{X}_a^k$ and $\mathbb{X}_p^k$ (representing the components of the tangent vector) satisfying:

$$\nabla_k \mathbb{X}_p^k = [p^k, \mathbb{X}_a^k]_a,$$

as the following proposition proves.

**Proposition 3.5.** The space of solutions of:

$$\nabla_k \mathbb{X}_p^k = [p^k, \mathbb{X}_a^k]_a,$$  

is an affine space modelled over the vector space $\mathcal{H}^2(\Sigma, \nabla_k p^k_a = 0, \text{vol}_\Sigma)$.

**Proof.** As we proved above, $\mathcal{H}^2(\Sigma, \text{vol}_\Sigma)$ splits as $\mathcal{H}^2(\Sigma, \nabla_k p^k_a = 0, \text{vol}_\Sigma) \oplus \nabla \mathcal{H}^2(\Sigma, \text{vol}_\Sigma)$. Let us denote by $\tilde{\mathbb{X}}_p^k$ and $\nabla_k \mathbb{X}_\phi^a$ the components of the $p^k$-component of $\mathbb{X}$ in such a splitting. Then, equation (127) reads:

$$\Delta \mathbb{X}_\phi^a = [p^k, \mathbb{X}_a^k]_a,$$  

where $\Delta$ is the covariant Laplacian. The last equation has a unique solution for any fixed $\mathbb{X}_a^k$ given by the action of the Green function of $\Delta$ on the right hand side. This means that solutions of (127) are parametrized by all the $\tilde{\mathbb{X}}_p^k$ (belonging to $\mathcal{H}^2(\Sigma, \nabla_k p^k_a = 0, \text{vol}_\Sigma)$) and by a particular solution of (128), i.e., it is an affine space modelled over the vector space $\mathcal{H}^2(\Sigma, \nabla_k p^k_a = 0, \text{vol}_\Sigma)$.

Now, on the final manifold of the PCA, the equation:

$$i_{\Gamma_\infty} \Omega^\Sigma_\infty = d \mathcal{H}_\infty,$$  

where $\Omega^\Sigma_\infty = i_{\infty}^* \Omega^\Sigma := i_{\infty}^* \Omega^\Sigma, \mathcal{H}_\infty = i_{\infty}^* \mathcal{H} := i_{\infty}^* \mathcal{H}$ is well posed for a $\Gamma_\infty \in \mathfrak{X}(\mathcal{M}_\infty)$.

### 3.4 Poisson bracket on the solution space via coisotropic embedding

Following what we said in Sect. 2.3 the idea is to use the structure $\Omega^\Sigma_\infty$ to define a Poisson bracket on the solution space of the theory. However, as we saw above, $\Omega^\Sigma_\infty$ is pre-symplectic. Indeed, $\mathbb{X}_\text{Gauge}^{\Gamma_\infty} \in T\chi_\Sigma i_1(\mathcal{M}_\infty)^\perp$ and, furthermore, it is actually tangent to $i_\infty(\mathcal{M}_\infty)$, that is, it is $i_\infty$-related with a tangent vector to $\mathcal{M}_\infty$ which has, again, $a^a_k$ component equal to $\nabla_k \psi^a$ and $p^a_k$ component equal to $[p^k, \psi]_a$, and which we will still denote by $\mathbb{X}_\text{Gauge}^{\Gamma_\infty}$. Therefore, in order to define a Poisson bracket on $\mathcal{M}_\infty$, we may use the coisotropic embedding theorem as explained in Sect. 1.2. In particular, the idea is to use a well known connection on the bundle $\mathcal{M}_\infty \rightarrow \mathcal{M}_\infty/\ker \Omega^\Sigma_\infty$; i.e., the so-called *Coulomb connection* introduced in [NR79, Sin78] to perform the construction outlined in Sect. 1. Such a connection is represented by the following $1-1$ tensor field over $\mathcal{M}_\infty$:

$$\mathcal{P} = P^a_k \otimes \frac{\delta}{\delta a^a_k} + [p^k, \mathcal{G}]_a \otimes \frac{\delta}{\delta p^a_k} \in \mathcal{T}^1(\mathcal{M}_\infty),$$  

(130)
where in the tensor product an integration over $\Sigma$ is implied and where:

$$P^a_k = \nabla_k \int_\Sigma G_\Delta \eta^j \nabla \delta a^j \text{vol}_\Sigma, \quad G^a = \int_\Sigma G_\Delta \eta^j \nabla \delta a^j \text{vol}_\Sigma,$$

(131)

with $G_\Delta$ being the Green function of the covariant Laplacian operator $\Delta = \eta^{jk} \nabla_j \nabla_k$ and $\{ \delta a^j, \delta p^j \}$ being a basis of differential one forms dual to the basis of vector fields

$$\left\{ \frac{\delta}{\delta a^j}, \frac{\delta}{\delta p^j} \right\}_{a=1, \ldots, \text{dim} g, k=1, 2, 3}.$$

Note that, (130) is actually a connection on the bundle $\mathcal{M}_\infty \to \mathcal{M}_\infty / \ker \Omega^\Sigma$ because it is the identity on vertical tangent vectors, i.e.:

$$\mathcal{P}(X^{\text{GAUGE}}) = X^{\text{GAUGE}},$$

(132)

and, as it is proven in [NR79, Sin78], it is equivariant with respect to the vertical automorphisms of the bundle. We will denote by $\mathcal{R} := 1 - \mathcal{P}$ the projector over horizontal vector fields. The latters are indeed defined to be the image of $\mathcal{R}$, say $\text{Im} \mathcal{R}$.

Now, in order to apply the procedure outlined in Sect. 1 we should identify the dual of the vector space spanned at each point of $\mathcal{M}_\infty$ by $X^{\text{GAUGE}}$. By looking at the expression of $X^{\text{GAUGE}}$, the subspace of $T^{a,p}(\mathcal{M}_\infty)$ spanned by $X^{\text{GAUGE}}$ at $(a,p)$, is parametrized by the $\psi^a \in \mathcal{H}^{\mathcal{Z}}(\Sigma, \text{vol}_\Sigma)^a$ and is actually the subspace of $T^{a,p}(\mathcal{M}_\infty)$ given by the image of the operator $\nabla \oplus [p^k, \cdot]$ acting on $\mathcal{H}^{\mathcal{Z}}(\Sigma, \text{vol}_\Sigma)^a$:

$$\mathcal{V} := \prod_{k,a} \nabla \mathcal{H}^{\mathcal{Z}}(\Sigma, \text{vol}_\Sigma)^a_k \times [p^k, \mathcal{H}^{\mathcal{Z}}(\Sigma, \text{vol}_\Sigma)]^a_a,$$

(133)

where $\nabla$ is the covariant derivative associated with the fixed connection $a$ of the point $(a,p) \in \mathcal{M}_\infty$. That $\mathcal{V}$ is a Hilbert space itself, is a consequence of the fact that $\nabla \oplus [p^k, \cdot]$ is a closed operator acting on $\mathcal{H}^{\mathcal{Z}}(\Sigma, \nabla, \text{vol}_\Sigma)^a_a$. Indeed, the following two propositions hold.

**Proposition 3.6 (Closedness of $\nabla$ in $\mathcal{H}^{\mathcal{Z}}$).** $\nabla$ is a closed operator from $\mathcal{H}^{\mathcal{Z}}(\Sigma, \text{vol}_\Sigma)^a_a$ into $\mathcal{H}^{\mathcal{Z}}(\Sigma, \nabla, \text{vol}_\Sigma)^a_a$.

**Proof.** The proof is analogous to the of Prop. 3.4.

**Proposition 3.7 (Closedness of $[p^k, \cdot]$).** $[p^k, \cdot]$ is a closed operator from $\mathcal{H}^{\mathcal{Z}}(\Sigma, \text{vol}_\Sigma)^a_a$ to $\mathcal{H}^{\mathcal{Z}}(\Sigma, \nabla, \text{vol}_\Sigma)^a_a$.

**Proof.** This is a consequence of the discussion between Eq. (117) and (118)

Being $\mathcal{V}$ a Hilbert space, it has a well defined dual space (isomorphic with $\mathcal{V}$ itself). Let us denote it by $\mathcal{V}^*$ let us denote by $\mathcal{X}^{\mu, k}_{\text{GAUGE}}$ its elements. Then, the construction of Sect. 1.2 can be applied. In particular, since $dP^a_k$ is different from zero for the connection chosen here, we are in the case described in Sect. 1.2.3. The symplectic manifold one constructs here is a tubular neighborhood of the zero section of the bundle $\mathcal{K}^*$ over $\mathcal{M}_\infty$ with typical fibre $\mathcal{V}^*$. Denote by $\tilde{\mathcal{M}}$ such a manifold and by
(a^k_a, p^k_a, \mu^k_a) a system of local coordinates defined over an open set \( U_M \subset \tilde{M} \). Then the symplectic structure on \( M \) reads:

\[
\tilde{\Omega} = \tau^* \Omega^\Sigma_{\infty} \big|_{\text{Im} \tau} + d\mu_a^k \wedge P^a_k + d\mu_a^k \wedge [p^k, \mathcal{G}]_a + \mu_a^k dP^a_k,
\]

(134)

where \( \tau \) is the projection from \( \tilde{M} \) to \( M_{\infty} \). Then, given two functions on \( M_{\infty} \), say \( f \) and \( g \), the structure \( \tilde{\Omega} \) above, allows to write the Poisson bracket between \( \tilde{f} := \tau^* f \) and \( \tilde{g} := \tau^* g \), which reads:

\[
\{ \tilde{f}, \tilde{g} \} = \tilde{\Omega}(\tilde{x}_f, \tilde{x}_g),
\]

(135)

As a specific example, consider the following functions on \( M_{\infty} \):

\[
f = p_a^{k_1}(x_1) \quad g = p_a^{k_2}(x_2).
\]

(136)

Their pull-back to \( \tilde{M} \) reads:

\[
\tilde{f} = p_a^{k_1}(x_1) \quad \tilde{g} = p_a^{k_2}(x_2).
\]

(137)

Then, a direct computation shows that:

\[
\{ \tilde{g}, \tilde{f} \} = \tilde{x}_f(\tilde{g}) = -2 \int_{\Sigma} \left[ \mu_a^k(y) \nabla^a_k \mathcal{G}(y, x) \delta^{k_1 k_2} \epsilon^a_{a_1 a_2} \delta(x_a, x_{a_1}) \delta(x_a, x_{a_2}) + \right. \\
+ \mu_a^k(y) \mathcal{G}(y, x) p_b^k(y) \delta^{k_1 k_2} \epsilon^b \epsilon_{a a_1 a_2} \delta(x_a, x_{a_1}) \delta(x_a, x_{a_2}) \left. \right] d^3 x d^3 y.
\]

(138)

**Conclusions**

In the present manuscript we concluded the program, started in the companion paper [CDI⁺], of constructing a Poisson bracket on the space of the solutions of the equations of motion (always referred to as solution space) of a large class of classical field theories, namely, first order Hamiltonian ones.

To resume, in this series of papers we showed how, within the multisymplectic formulation of first order Hamiltonian field theories, the solution space is canonically equipped with a pre-symplectic 2-form.

In particular we saw that for some theories (analysed in [CDI⁺]) it is actually (strongly) symplectic and, thus, it gives rise automatically to a Poisson bracket given by the bivector field being the inverse of the symplectic structure.

Moreover, we saw that for those theories exhibiting a gauge symmetry, the canonical 2-form has a non-trivial kernel. We started the analysis of gauge theories in [CDI⁺] and concluded it in the present paper. In particular we saw that, being the 2-form pre-symplectic, via the coisotropic embedding
theorem a symplectic, and, thus, a Poisson, structure on a suitable enlargement of the solution space can be always defined. Moreover, we saw that for those theories for which a global gauge fixing in the space of fields can be performed, such Poisson structure projects to a Poisson structure on the solution space, while for those theories exhibiting Gribov’s ambiguities the use of additional degrees of freedom, interpreted as ghost fields, is necessary in order to define a Poisson structure. What is more, we classified these two cases via a geometrical structure, that is a connection on the characteristic bundle associated with the solution space equipped with the pre-symplectic structure. In particular the two cases are selected by the connection being flat or not.

The definition of a Poisson bracket on the solution space of Einstein’s equations within the Palatini’s formulation of General Relativity via the approach adopted in the present paper will be addressed in a subsequent paper.

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