We consider higher dimensional topological Taub-NUT/Bolt-AdS solutions where a cosmological constant is treated as a pressure. The thermodynamic quantities of these solutions are explicitly calculated. Furthermore, we find these thermodynamic quantities satisfy the Clapeyron equation. In particular, a new thermodynamically stable region for the NUT solution is found by studying the Gibbs free energy. Intriguingly, we also find that like the AdS black hole case, the G – T diagram of the Bolt solution has two branches which are joined at a minimum temperature. The Bolt solution with the large radius, at the lower branch, becomes stable beyond a certain temperature while the Bolt solution with the small radius, at the upper branch, is always unstable.

I. INTRODUCTION

It has been found that the area of the event horizon of a black hole is proportional to its physical entropy in search for similarities between black hole physics and thermodynamics [1]. It has been successively suggested that by using the thermodynamic relationship between the thermal energy, temperature, and entropy, the first law of black hole thermodynamics can be expressed in the similar forms to the first law of standard thermodynamics [2]. It has been also found that there is a phase transition in the Schwarzschild-AdS black hole through investigations of complete analogy between black hole system and standard thermodynamic system [3]. Since then, it has been studied for the phase transitions and critical phenomena in a variety of black hole solutions [4]-[7] and extensively investigated in various thermodynamic issues of black hole in higher dimensional AdS space [8]-[18].

It has recently been suggested that by considering \((d+1)\)-dimensional AdS black holes, the thermodynamic pressure \(p\) is given by

\[
p = -\frac{1}{8\pi \Lambda} \frac{u(2u + 1)}{8\pi l^2},
\]

in units where \(G = c = \hbar = k_B = 1\), and \(u\) is of the form \(d + 1 = 2u + 2\) with a positive integer \(u\). Several series of relevant investigations have been performed [19]-[42].

Recently, it has been shown that the thermodynamic volume in the Taub-NUT-AdS case can be negative. This negative thermodynamic volume may be interpreted in that the environment (universe) applies work to the system (Taub-NUT-AdS black hole) in the process of the Taub-NUT-AdS black hole formation, while the positive thermodynamic volume may be interpreted as applying the work on the environment (universe) by the system (the whole black hole) considering the process of forming the black hole [30]. They also have found that there is the first order phase transition from Taub-NUT-AdS to Taub-Bolt-AdS with considering the phase structure of these black holes [30]. This issue has been studied for the Kerr-Bolt-AdS case [35] and investigated extensively in higher dimensional NUT/Bolt case. In these higher dimensional cases, it has been particularly found that the Taub-NUT-AdS solution has a thermodynamically stable range as a function of the temperature for any odd \(u\) and there is the transition from Taub-NUT-AdS to Taub-Bolt-AdS for all odd \(u\) only [43].

Furthermore, it has been shown that this higher dimensional NUT/Bolt case with a discrete parameter \(k\) can be generalized [44]. In the context of the extended thermodynamics, the thermodynamic properties of the case \(k = 1\) has been studied [44] only. Thus, it would be interesting to be a similar discussion of the generalizations in the \(k = 0, -1\) topological solutions. More intriguingly, their thermodynamic phase structure would be investigated through exploring the behaviour of the Gibbs free energy since understanding its behaviour is essential for uncovering possible thermodynamic phase transitions. In this paper, we address these questions.

The paper is organized as follows: in the next section we investigate thermodynamic properties in topological Taub-NUT/Bolt-AdS spaces for any \(u\). We explicitly obtain the general forms of thermodynamic quantities such as the entropy, the enthalpy, the specific heat, the temperature, the thermodynamic volume, the Gibbs free energy, and the latent heat. In particular, by introducing the Gibbs free energy we discuss their phase structure and their instability. In the last section we give our conclusion.

II. TOPOLOGICAL TAUB-NUT/BOLT-ADS SPACES

We consider topological Taub-NUT/Bolt-AdS metric in higher dimensional spacetime and the general solution in the Euclidean section is given by [45]-[51] (for the generalized versions of the issue, see e.g., [44])

\[
ds^2 = f(r) \left\{ dt^2 + 4N \sum_{i=1}^{u} f_k^2 \left( \frac{\theta_i}{2} \right) d\phi_i \right\}^2 + \frac{dr^2}{f(r)} + (r^2 - N^2) \sum_{i=1}^{u} \left( d\theta_i^2 + f_k^2(\theta_i) d\phi_i^2 \right),
\]

\[ (2.1) \]
where $N$ represents a NUT charge for the Euclidean section, and the metric function $f(r)$ is found to be

$$f(r) = \frac{k}{(r-N)^2} \int \frac{(a^2-N^2)^k}{a^2} \left(1 + \frac{(a^2-N^2)^{u+1}}{r^2 a^2} \right) da - \frac{2kr}{(r-N)^2}.$$  

(2.2)

with a cosmological parameter $l$ and a geometric mass $m$. The discrete parameter $k$ takes the values 1, 0, -1 and defines the form of the function $f_k(\theta_i)$

$$f_k(\theta_i) = \begin{cases} \sin \theta_i, & \text{for } k = 1 \\ \theta_i, & \text{for } k = 0 \\ \sinh \theta_i, & \text{for } k = -1, \end{cases}$$  

(2.3)

and the space $M^2$ corresponds to a two dimensional sphere for $k = 1$, plane for $k = 0$, and pseudohyperboloid for $k = -1$, respectively. For $k = 1$, the NUT solution occurs when solving $f(r)|_{r=N}=0$. The inverse of the temperature $\beta$ is obtained by requiring regularity in the Euclidean time $t_E$ and radial coordinate $r$ [51].

$$\beta = \frac{\sqrt{r}}{f(r)} \bigg|_{r=N} = \frac{4(u+1)\pi}{\sigma} N,$$  

(2.4)

where $\sigma$ is a positive integer and $\beta$ is the period of $t_E$. The $\sigma$ appears since the period cannot be bigger than $4(u+1)\pi N$, so that the Misner-string singularities vanish. However, when $\sigma$ has an integer value, the period can smaller than $4(u+1)\pi N$. This property is not guaranteed for $k=0$ and $k=-1$ since there is no Misner-string and no periodicity of time $t$. However, it was shown that this holds for the $k=0,-1$ cases through checking the self-consistency of the thermodynamic relations [51].

Let us first consider the NUT solution ($r = N$). Then after taking $\sigma = 1$ for convenience without loss of generality, we obtain the following formula for the temperature:

$$\frac{1}{\beta} = T = \frac{k}{4(u+1)\pi N},$$  

(2.5)

where $k$ has 1 or 0 since there are no hyperbolic NUT solutions. Thus, for the NUT solution we consider the cases $k = 1, 0$ only.

Using counter term subtraction method, we get the regularized action [45-51]

$$I_{\text{NUT}} = \frac{(4\pi)^n N^{2u-1}(2uN^2-k^2)}{16\pi^2} \Gamma(\frac{1}{2} - u) \Gamma(u+1)\beta.$$  

(2.6)

where the gamma function $\Gamma(t)$ is defined as $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$.

Employing the Gibbs-Duhem relation $S = \beta M - I$ and substituting in (1.1), the entropy is found to be

$$S_{\text{NUT}} = \frac{(4\pi)^n N^{2u-1}}{16\pi^2} \left\{ 16\pi^2 N^2 p - (2u-1)k \right\} \times \Gamma(\frac{1}{2} - u) \Gamma(u+1)\beta.$$  

(2.7)

Here $M$ is the conserved mass $M = u(4\pi)^{u-1} m$, and this will be identified with enthalpy $H$ ($M \equiv H = U + pV)$ [19], which leads to

$$H_{\text{NUT}} = u(4\pi)^{u-1} \Gamma(\frac{3}{2} - u) \Gamma(u+1) \left\{ \frac{N^{2u-1}}{\sqrt{\pi(2u-1)}} k \right.$$  

$$- \frac{16\sqrt{\pi(2u+1)} N^{2u+1}}{u(2u-1)(2u+1)^p} \right\}.$$  

(2.8)

One the other hand, by thermal relation $C = -\beta \partial_p S$, the specific heat is given as

$$C_{\text{NUT}} = \frac{2\pi(4u+1)(u+1)^2 S_{\text{NUT}} T^2}{\pi(2u-1)(u+1)^2 T^2 - kp},$$  

(2.9)

which is negative for $p > \pi(2u-1)(u+1)^2 T^2/k$ while is positive $p < \pi(2u-1)(u+1)^2 T^2/k$ and diverges at $p = \pi(2u-1)(u+1)^2 T^2/k$.

The thermodynamic volume is also obtained as

$$V_{\text{NUT}} = -\frac{u(4\pi)^n N^{2u-1}}{2\sqrt{\pi}} \Gamma(-\frac{1}{2} - u) \Gamma(u+1).$$  

(2.10)

Then from the above thermodynamic quantities, the generalized Smarr formula due to dimensional scaling arguments for any value of $k$ is given as

$$\frac{1}{2} H - \frac{u}{2u-1} TS + \frac{1}{2u-1} pV = 0,$$  

(2.11)

which is precisely matched with that of static $d$-dimensional black holes with negative cosmological constant [10], [19], [20].

Using an thermal relation $U = H - pV$, the internal energy of Taub-NUT is obtained as

$$U_{\text{NUT}} = \frac{u(4u+1)}{8\pi} \left\{ (2\sqrt{\pi} N)^{2u-1} k - (2u+1)(2\sqrt{\pi} N)^{2u+1} N^2 \right\} \times \Gamma(-\frac{1}{2} - u) \Gamma(u+1).$$  

(2.12)

![FIG. 1. Plot of the entropy $S_4$ (yellow solid curve), specific heat $C_4$ (green solid curve) and pressure $p$ (red solid curve for $p = 4\pi T^2$, and blue solid curve for $p = 12\pi T^2$, respectively) as a function of the temperature $T$ in four dimensions for $k = 1$.](image)
since both the entropy and the specific heat are positive when \( \sqrt{\frac{2}{\pi}} < T < \sqrt{\frac{2}{\pi}} \). However, when the Gibbs free energy is introduced, the NUT solution in some areas of this region is still unstable since the Gibbs free energy \( (2.13) \) is positive for \( p < 4\pi T^2 \) or \( p > 12\pi T^2 \). For any \( u \), the NUT solution is thermally unstable since the Gibbs free energy is negative for \( 4\pi T^2 < p < 12\pi T^2 \). For \( k \), the NUT solution in some areas of the pure AdS spacetime is also more thermally unstable than the pure AdS spacetime since the Gibbs free energy is negative for \( 4\pi T^2 < p < 12\pi T^2 \). For any \( u \), the NUT solution is thermally unstable since the Gibbs free energy is negative for \( \pi(u+1)^2T^2 \). The NUT solution is thermally stable for any \( u \) since the Gibbs free energy is positive for \( \pi(u+1)^2T^2 \). For any \( u \), the NUT solution is thermally unstable since the Gibbs free energy is negative for \( 4\pi T^2 < p < 12\pi T^2 \). For any \( u \), the NUT solution is thermally unstable since the Gibbs free energy is negative for \( \pi(u+1)^2T^2 \). The NUT solution is thermally stable for any \( u \) since the Gibbs free energy is positive for \( \pi(u+1)^2T^2 \).

For \( k = 1 \), as shown in Fig 2., the Gibbs free energy is positive for odd \( u \) while the Gibbs free energy is negative for even \( u \). In this range of the temperature \( T \) the phase transition from the Taub-NUT-AdS spaces to the pure AdS spacetime occurs for odd \( u \) whereas that of opposite direction occurs for even \( u \). It is quite natural that this thermodynamic behavior of the Taub-NUT-AdS spaces changes due to odd \( u \) or even \( u \) since the sign of the gamma function in the Gibbs free energy \( (2.13) \) alternates due to odd \( u \) or even \( u \).

Remarkably, this topological Taub-NUT-AdS solution has the Hawking-Page transition between the Taub-NUT-AdS spaces and the pure AdS spacetime, which occurs on the line \( \tilde{p}_{\text{coex}} = \pi T \sqrt{\frac{2}{\pi}} \frac{(2u+1)(u+1)^2 T^2}{k} \), or

\[
S_{\text{NUT}} = \frac{\Gamma(1/2-u)\Gamma(u+2)}{2\sqrt{\pi}} \left[ \frac{-k}{\Gamma(2\sqrt{\pi}(u+1))} \right]^{2u} \left( \frac{1}{T} \right)^{2u}.
\]

As shown in Fig 3., there is a NUT solution phase for \( p < \tilde{p}_{\text{coex}} \) at any fixed temperature while there is a NUT phase for \( p > \tilde{p}_{\text{coex}} \) and the two states exist together for \( p = \tilde{p}_{\text{coex}} \). The coexistence lines of the topological Taub-NUT-AdS solution phases and the divergent lines of the topological Taub-NUT-AdS solution specific heat move more to the left as the dimension of spacetime increases.

The shift of entropy occurs on the coexistence lines of the topological Taub-NUT-AdS solution phases, and so the displacement of entropy is given by

\[
\Delta S = \frac{\Gamma(\frac{1}{2}-u)\Gamma(u+2)}{2\sqrt{\pi}} \left[ \frac{k}{\Gamma(2\sqrt{\pi}(u+1))} \right]^{2u} \left( \frac{1}{T} \right)^{2u},
\]

which leads to the latent heat

\[
L_{\text{NUT}} = \frac{\Gamma(\frac{1}{2}-u)\Gamma(u+2)}{2\sqrt{\pi}} \left[ \frac{k}{\Gamma(2\sqrt{\pi}(u+1))} \right]^{2u} \left( \frac{1}{T} \right)^{2u-1}
\]

by an thermal relation \( L = T\Delta S \), and the latent heat in terms of \( p \) is obtained as

\[
L_{\text{NUT}} = \frac{\Gamma(\frac{1}{2}-u)\Gamma(u+2)}{2\sqrt{\pi}} \left( \frac{1}{T} \right)^{u-1}.
\]

Then at any given temperature \( T \) for any \( u \), the displacement of entropy and the latent heat are zero in the case \( k = 0 \) and for odd \( u \) are negative in the case \( k = 1 \). The former is trivial since the temperature \( T \) is zero as well as the Gibbs free energy \( G_{\text{NUT}} \). However, the latter is related with energy supplied by the process of forming the Taub-NUT-AdS system from the pure AdS spacetime. It indicates a net release of latent energy back into the environment because of evaporating of Taub-NUT-AdS system. The latent heat \( L_{\text{NUT}} \), like the AdS black hole case [35], vanishes as the temperature goes to infinity instead of a certain finite value (critical temperature) and a second order phase transition cannot take place at a certain finite temperature. The pressure \( p \) vanishes for asymptotically flat spacetime and the latent heat \( L_{\text{NUT}} \) becomes infinity. This means that it is not possible to spontaneously form the black hole from the Minkowski spacetime.

For AdS spacetime the Gibbs free energy is \( G_{\text{AdS}} = 0 \) and for the Taub-NUT-AdS solution

\[
G_{\text{NUT}} = U_{\text{NUT}} - T S_{\text{NUT}} + p V_{\text{NUT}}.
\]
Their differentials on the coexistence line are $dG_{\text{ADs}} = 0$ and $dG_{\text{NUT}} = -S_{\text{NUT}} dT + V_{\text{NUT}} dp$, which lead to

$$0 = dG_{\text{ADs}} - dG_{\text{NUT}} = S_{\text{NUT}} dT - V_{\text{NUT}} dp \quad (2.19)$$

where the coexistence line with two states may be defined by $G_{\text{ADs}} = G_{\text{NUT}}$. Hence, the insertion of the (2.27) and (2.10) into (2.19) yields

$$\frac{dp}{dT} = \frac{S_{\text{NUT}}}{V_{\text{NUT}}} = \frac{2\pi^2(2u+1)(u+1)^2}{k}. \quad (2.20)$$

Furthermore, since the thermodynamic volume and the entropy become zero for AdS spacetime we can write

$$\Delta V = V_{\text{NUT}} \quad \text{and} \quad \Delta S = S_{\text{NUT}}, \quad (2.21)$$

and have the Clapeyron equation $dp/dT = \Delta S/\Delta V$.

As checking the self-consistency of the thermodynamic relations, we can reproduce the above result (2.20) through $\frac{dp}{dT} = \frac{2\pi^2(2u+1)(u+1)^2}{k}$ which shows that the Clapeyron equation still holds for the NUT solution.

Let us consider the Boltzmann solution ($r = r_B > N$). Requiring $f(r)|r=r_B > N$ and $f'(r)|r=r_B = N(u+1)$, the Boltzmann solution occurs. In Taub-Boltzmann metric, the inverse of the temperature, the action, and the mass are respectively

$$\beta = \frac{4\pi}{f'(r)} \bigg|_{r=r_B} = \frac{4\pi^2 r_B}{kT^2 + (2u+1)(u+1)^2(N^2 - k^2)}, \quad (2.22)$$

$$I_{\text{Bolt}} = \frac{(4\pi)^{n-1}}{2} \left[ \frac{(2u+1)(-1)^{n}N^{2n+2}r_B}{2u+1} \right] + \sum_{i=0}^{u} \left[ \left( \frac{u}{i} \right) \frac{(-1)^{i}N^{2i+1}2u-2i}{(2u-2i+1)r_B} \right] \beta. \quad (2.23)$$

Here the Bolt radius $r_B$ is

$$r_{B,\pm} = \frac{[2u+2][2u+1](2u+2)^2N^2(2u+1)N^2 - k^2]}{2u+1}(2u+1)N, \quad (2.24)$$

where $r_{B,\pm}$ denotes the large radius of the Bolt solution and $r_{B,-}$ is the small radius of the Bolt solution. The discriminant of the square root in the Bolt radius is always positive for the $k = 0, -1$ cases but sometimes negative for $k = 1$, and so the former has no upper limit on $N$ while the latter has the maximum magnitude of the NUT charge $N_{\text{max}}$

$$N \leq \sqrt{2u(2u+1)/(2u+1)^2} = N_{\text{max}}. \quad (2.25)$$

Furthermore, from the Bolt radius (2.24) one can find in the cases $k = 0, -1$ the large radius $r_{B,\pm}$ exists only for $N > 0$ since the Bolt solution occurs for $r_B > N$.

Using parallel way as in the case of the NUT solution, the enthalpy $H_{\text{Bolt}}$, the entropy $S_{\text{Bolt}}$, and thermodynamics volume $V_{\text{Bolt}}$ for the Bolt solution yields respectively

$$H_{\text{Bolt}} = \frac{u(4\pi)^{n-1}}{2} \left[ \sum_{i=0}^{u} \left( \frac{u}{i} \right) (-1)^i N^{2i+1} \frac{2u-2i-1}{(2u-2i+1)} k \right] + 8\pi \sum_{i=0}^{u+1} \left( \frac{u+1}{i} \right) (-1)^i N^{2i} \frac{2u-2i+1}{(2u+2i+1)} p \quad (2.26)$$

$$S_{\text{Bolt}} = \frac{(4\pi)^{n-1}}{4} \left[ \sum_{i=0}^{u} \left( \frac{u}{i} \right) \frac{(2u-1)(-1)^i N^{2i+1} \frac{2u-2i-1}{2u-2i-1} k}{2u-2i+1} \right] + \left\{ \sum_{i=0}^{u} \left( \frac{u}{i} \right) \frac{8\pi(2u+2i+1)(-1)^i N^{2i+1} \frac{2u-2i+1}{(2u+2i+1)} p}{u(u-1)(2u+2i+1)} \right\} \beta, \quad (2.27)$$

$$V_{\text{Bolt}} = \frac{(4\pi)^{n-1}}{2u+1} \left\{ \frac{2\pi^2(2u+1)(u+1)^2}{k} \beta \right\} + \pi \sum_{i=0}^{u} \left( \frac{u}{i} \right) \frac{8\pi^2(2u+2i+1)(-1)^i N^{2i+1} \frac{2u-2i+1}{(2u+2i+1)} p}{u(u-1)(2u+2i+1)} \beta, \quad (2.28)$$

where the incomplete beta function $B(x, a, b)$ is defined as $B(x, a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$. Like the previous NUT case, these thermodynamic quantities satisfy the generalized Smarr formula (2.11).

The specific heat is given as

$$C_{\text{Bolt}, \pm} = \frac{u(4\pi)^{n-1}}{2u+1} \left( \frac{4\pi^2 r_B}{kT^2 + (2u+1)(u+1)^2(N^2 - k^2)} \right)^{2u} + \left\{ \sqrt{\pi} \left( \frac{k^2 p^2 - 2\pi^2 u(1+u)^2 p^2 T^2 + \frac{8\pi^4 u^2(1+u)^2 p^2 T^4}{u(u-1)(2u+1)}(2u+2i+1) N^2 - k^2} \right) \right\} \times \left( \frac{4\pi^2 r_B}{kT^2 + (2u+1)(u+1)^2(N^2 - k^2)} \right)^{2u} + \frac{4\pi^2 r_B}{kT^2 + (2u+1)(u+1)^2(N^2 - k^2)} \times \left( \frac{k^2 p^2 - 2\pi^2 u(1+u)^2 p^2 T^2 + \frac{8\pi^4 u^2(1+u)^2 p^2 T^4}{u(u-1)(2u+1)}(2u+2i+1) N^2 - k^2} \right) \times \left( \frac{4\pi^2 r_B}{kT^2 + (2u+1)(u+1)^2(N^2 - k^2)} \right)^{2u} + \frac{4\pi^2 r_B}{kT^2 + (2u+1)(u+1)^2(N^2 - k^2)}$$

where $C_{\text{Bolt}, +}$ denotes the specific heat with the radius $r_{B,+}$, and $C_{\text{Bolt}, -}$ is the specific heat with the radius $r_{B,-}$. Here, $A$ and $B_+\pm$ are

$$A = (2u+1) \left[ k^2 p^2 - 2\pi^2 u(1+u)^2 p^2 T^2 + \frac{8\pi^4 u^2(1+u)^2 p^2 T^4}{u(u-1)(2u+1)}(2u+2i+1) N^2 - k^2 \right], \quad B_\pm = \sqrt{A} \pm \pi u(1+u)(2u+1) T^2,$$

and the hypergeometric function $1F_2(a; b; c; z)$ is defined for $|z| < 1$ by the power series

$$1F_2(a; b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}. \quad (2.30)$$

For example, the four-dimensional specific heat is obtained as

$$C_{\text{Bolt}, \pm} = -\frac{4\pi^2 p^2}{4u+1} \left( \frac{k^2 p^2 - 2\pi^2 u(1+u)^2 p^2 T^2 + \frac{8\pi^4 u^2(1+u)^2 p^2 T^4}{u(u-1)(2u+1)}(2u+2i+1) N^2 - k^2} \right) \times \left( \frac{4\pi^2 r_B}{kT^2 + (2u+1)(u+1)^2(N^2 - k^2)} \right)^{2u} + \frac{4\pi^2 r_B}{kT^2 + (2u+1)(u+1)^2(N^2 - k^2)}$$

which is well matched with the result in [51]. Here, as far as we know, even if including the Bolt solution without out a cosmological constant treated as a pressure, the specific heat (2.29) is firstly shown in the analytical expressions. In fact, since the specific heat of the Bolt solution for arbitrary dimensions has a highly complicated
high-order polynomial terms, the analytical expressions have not been obtained even for the case $k = 1$. Furthermore for $k = 1$ the four-dimensional specific heat $C_{\text{Bolt},+}$ (2.31) diverges at $T = T_2$ $T_2 = \sqrt{\frac{g}{\pi}} \sqrt{\frac{3\pi}{2\pi}}$, \hfill (2.32) and for $k = 1$ the higher dimensional specific heat also diverges at $T$ $T = \frac{\sqrt{\pi}}{\sqrt{2\pi}} \sqrt{1 + \frac{u}{u,u + u^2}} \frac{u + u^2}{u + u^2 + 1}$. \hfill (2.33) The internal energy of AdS-Taub-Bolt $U$ is given as $U_{\text{Bolt}} = \frac{(4\pi)^{u-1}}{4\pi} u N^2 u - 1 \left\{ (2u + 1) N^2 - k P^2 \right\} \times B\left(\frac{N^2}{r_B^2} \frac{1}{2} - u, u + 1\right)$, \hfill (2.34) and the Gibbs free energy $G_{\text{Bolt}} = u \left(\frac{N}{r_B} \right)^{2u} B\left(\frac{N^2}{r_B^2} \frac{1}{2} - u, u + 1\right) + 16\pi N r_B \left\{ \left(1 - \frac{N^2}{r_B^2} \right)^{u+1} - \left(\frac{N}{r_B} \right)^{2u+1} B\left(\frac{N^2}{r_B^2} \frac{1}{2} - u, u + 1\right) \right\} p$. \hfill (2.35)

As shown in Fig 4., two branches (the red solid curve and the red dashed curve) are joined at the temperature $T_2$. At this temperature two phases occur, that is to say that the upper branch (red dashed curve) is the phase of the Taub-Bolt-AdS system with the small radius $r_B$ and the lower branch (red solid curve) is the phase of the Taub-Bolt-AdS system with the large radius $r_B$. Like the AdS black hole case, at $T_2$ the four-dimensional specific heat of the Bolt solution also diverges and the higher dimensional cases diverge at $T$ (2.33). Since the Gibbs free energy is positive, the Bolt solution for the upper branch is unstable whereas since the lower branch includes the negative value of the Gibbs free energy, this solution in such region becomes stable. Thus, when $G_{\text{Bolt}} = 0$, like the Taub-NUT-AdS black hole case, there are the Hawking-Page transition between the Taub-Bolt-AdS black hole and the pure AdS spacetime, which occurs on the lines for the upper branch $r_{B,+}$ $\bar{p}_{\text{coex},+} = \left[ \frac{\pi^2 u (2u + 1)(u + 1)^2 T^4}{k^5} \times \frac{D^{u+1} p (D, \frac{1}{2} - u, u + 1)}{(1-D)^{u+1} \sqrt{D+u} D^{u} B(D, \frac{1}{2} - u, u + 1)} \right]^{1/3} \hfill (2.36)$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{Plot of the (2u + 2)-dimensional Gibbs free energy $G_{\text{Bolt}}$ as a function of temperature $T$ for any $u$ (red solid/dashed curve for $u = 1$, yellow solid/dashed curve for $u = 2$, brown solid/dashed curve for $u = 3$, and green solid/dashed curve for $u = 4$, respectively) for pressure $p = 3$ and $k = 1$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5}
\caption{For the four-dimensional NUT solution and Bolt solution, plot of $p$ as a function of $T$. Here $T_2$ and $T_3$ are the same values used in Fig 4.}
\end{figure}

Furthermore as shown in Fig 5., the thermodynamic instability of the Taub-NUT/Bolt-AdS system is classified by the value of the temperature $T$. When $T > T_0$, the Taub-NUT-AdS system evaporation to a stable cold remnant. When $T_0 < T < T_1$, the Taub-NUT-AdS system is a thermally stable configuration. When $T_1 < T < T_2$, and then the Taub-NUT-AdS system evaporation again since the four-dimensional entropy $S_4$ is negative. When $T_2 < T < T_3$, the Taub-Bolt-AdS system with two phases occurs at $T_2$ but still evaporates since the Gibbs free energy of the Taub-Bolt-AdS system is positive. Finally, when $T_3 < T < T_3$, the Taub-Bolt-AdS system with the large radius $r_{B,+}$ becomes stable since the Gibbs free energy of the Taub-Bolt-AdS system is negative. This thermodynamic nature of the Taub-NUT/Bolt-AdS system may hold for higher dimensional cases since $p - T$ phase diagrams in higher dimension are similar shapes [43].

From now on, considering the action difference of Taub-NUT-AdS and Taub-Bolt-AdS, we investigate the instability of the Taub-NUT/Bolt-AdS system. Their action difference, $I_D$ is defined as $I_{\text{Bolt}} - I_{\text{NUT}}$ $I_D = \frac{(4\pi)^u}{8\pi r_B^2} \left\{ \frac{2N (N^2 - r_B^2)(u + 1)}{N r_B^2} \left(1 - \frac{N^2}{r_B^2} \right)^u \right\} + (2uN^2 - k P^2) \left\{ (u + 1) N^2 u \left(\frac{N^2}{r_B^2} \frac{1}{2} - u, u + 1\right) \right\} - \frac{2N x_k}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} - u\right) \Gamma(2 + u) \right\}$. \hfill (2.37)
Considering the four-dimensional case \((u = 1)\), for \(k = 1\) taking the cosmological parameter \(l\) and the pressure \(p\) as fixed parameters, the action difference \(\mathcal{I}_D\) becomes negative as \(N\) increases. This means that the Taub-Bolt-AdS system with \(r_{B,+}\) is a more thermally stable configuration than the Taub-NUT-AdS system for \(\mathcal{I}_D < 0\), and so there is the first order phase transition from Taub-NUT-AdS system to Taub-Bolt-AdS system with \(r_{B,+}\) at a critical NUT charge\[36, 43\]. As shown in Fig 6., for the case \(k = 0\) Taub-Bolt-AdS system with \(r_{B,+}\) is stable only since the lower branch is always negative whereas the upper branch is always positive. It is shown that the curves of \(\mathcal{I}_4\) move more to the right as the pressure \(p\) decreases (the cosmological parameter \(l\) grows up) since \(p\) is inversely proportional to \(l\). For \(u = 2\) a similar result is obtained as with the \(u = 1\) case. Furthermore, \(I - N\) diagrams in higher dimension are similar shapes \[13\]. Thus, this thermodynamic instability of the Taub-NUT/Bolt-AdS system may hold for higher dimensional cases, and is well matched with the result in \[51\].

\[9\] M. Cvetic and S. S. Gubser, JHEP 9904, 026 (1999) [hep-th/9902195].
\[13\] T. K., Dey, Phys. Lett. B 595, 484 (2004) [hep-th/0406169].
\[14\] T. K. Dey, S. Mukherji, S. Mukhopadhyay and S. Sarkar, JHEP 0704, 014 (2007) [hep-th/0609038].
\[15\] N. Banerjee and S. Dutta, JHEP 0707, 047 (2007) [arXiv:0705.2682 [hep-th]].
\[16\] T. K. Dey, S. Mukherji, S. Mukhopadhyay and S. Sarkar, JHEP 0709, 026 (2007) [arXiv:0706.3996 [hep-th]].
\[17\] R. A. Konoplya and A. Zhidenko, Phys. Rev. D 78, 104017 (2008) [arXiv:0809.2048 [hep-th]].
\[18\] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, Phys. Rev. D 60, 064024 (2001) [arXiv:1005.4832 [hep-th]].
\[19\] Y. S. Myung, Eur. Phys. J. C 72, 2116 (2012) [arXiv:1203.1367 [hep-th]].
\[20\] C. S. Peca and J.P.S. Lemos, Phys. Rev. D 59, 124007 (1999) [gr-qc/9805004].
\[21\] M. Cvetiè and S. S. Gubser, JHEP 9904, 024 (1999) [hep-th/9902195].
\[22\] S. Gunasekaran, R. B. Mann and D. Kubiznak, JHEP 1111, 004 (2011) [arXiv:1109.2433 [gr-qc]].
\[23\] S. W. Hawking and D. N. Page, Commun. Math. Phys. 87, 577 (1983).
\[24\] T. K. Dey, Phys. Lett. B 595, 484 (2004) [hep-th/0406169].
\[25\] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).
\[26\] D. Kubiznak and R. B. Mann, JHEP 1211, 110 (2012) [arXiv:1208.6251 [hep-th]].
\[27\] M. Cvetic, G. W. Gibbons, D. Kubiznak and C. N. Pope, Phys. Rev. D 84, 024037 (2011) [arXiv:1012.2888 [hep-th]].
\[28\] R. -G. Cai, Phys. Rev. D 65, 084014 (2002) [hep-th/0109133].
\[29\] T. K. Dey, Phys. Lett. B 595, 484 (2004) [hep-th/0406169].
\[30\] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. 31, 161 (1973).
