Thermodynamic resources in continuous-variable quantum systems

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A system’s deviation from its ambient temperature has long been known to be a resource—a consequence of the second law of thermodynamics, which constrains all systems to drift towards thermal equilibrium. Here we consider how such constraints generalize to continuous-variable quantum systems comprising interacting identical bosonic modes. Introducing a class of operationally-motivated bosonic linear thermal operations to model energetically-free processes, we apply this framework to identify uniquely quantum properties of bosonic states that refine classical notions of thermodynamic resourcefulness. Among these are (1) a spectrum of temperature-like quantities; (2) well-established non-classicality measures with operational significance. Taken together, these provide a unifying resource-theoretic framework for understanding thermodynamic constraints within diverse continuous-variable applications.

The simple harmonic oscillator is an iconic system in quantum science, used to describe a diverse spectrum of bosonic quantum systems—from the optical modes of light to phononic excitations within trapped ions. These continuous-variable (CV) systems enable one to encode and process continuous quantum degrees of freedom, allowing CV variants of many quantum algorithms, as well as cryptographic and metrological protocols [1–3]. Such variants can exhibit significant practical advantages over discrete-variable counterparts—from the relative ease of creating ultra-large entangled clusters [4, 5] to hybrid factoring algorithms that require only one pure CV mode [6].

In the context of thermal physics, quantum harmonic oscillators present a compelling mechanistic model for temperature: as we lower the temperature $T$ of a harmonic oscillator, we also monotonically lower the variance $\eta$ of its momentum and position quadrature fluctuations. Indeed, this one-to-one correspondence was a key ingredient in early attempts to understand the specific heat of solids [7]. While these initial studies considered only semi-classical settings, the subsequent flourishing of quantum technologies has made it imperative to consider more general CV states. An instructive case is that of squeezed states, which are thermal states whose statistical fluctuations in certain quadratures are suppressed below the zero-temperature level [8]. Such states also have definitive thermodynamic value: heat engines using squeezed thermal reservoirs, for example, appear to perform beyond Carnot efficiency [9, 10]. This suggests that squeezing itself can be leveraged to do work, much like the temperature gradients that power conventional engines.

What other quantum effects are thermodynamically useful, and is it meaningful to speak of temperatures in general, non-equilibrium settings? Such questions motivate the need for a systematic characterization of...
the thermodynamic resources contained within bosonic CV systems. A resource-theoretic treatment, which has catalyzed profound advances in understanding the thermodynamics of discrete-variable systems \cite{11,14}, could stimulate further developments in bosonic heat engines \cite{9,15–19}, by singling out uniquely quantum resources that can be harnessed for work.

Our approach draws inspiration from the second law of thermodynamics, which may be paraphrased as follows: “When constrained to operations that cannot access additional sources of free energy, temperature gradient is a non-increasing monotone.” Here we ask, what other properties of a quantum system are monotones embodying different forms of free energy, or generalized notions of temperature?

In particular, we construct a framework of quantum thermodynamics for identical, linearly-interacting bosonic CV systems. We start by defining bosonic linear thermal operations (BLTO): the processes that can be enacted in such systems without requiring additional sources of free energy. An operational restriction to BLTO leads to several families of second law–like statements. Firstly, we identify a spectrum of generalized temperatures for general bosonic states, all of which (1) align with standard notions of temperature for classical states, and furthermore, (2) equilibrate towards the ambient temperature under BLTO operations. Secondly, we illustrate that many existing indicators of operational performance and quantifiers of non-classicality—including phase-space signal-to-noise ratios, squeezing of formation \cite{20}, phase-space sensing resolution \cite{21}—are all non-increasing under BLTO. This thus establishes that many well-known quantifiers of the state’s resourcefulness for information-processing and sensing tasks are in fact types of thermodynamic currency.

I. FRAMEWORK

Notation and preliminaries. Continuous-variable quantum systems occur in many different physical mediums, but it useful to adopt the terminology of one medium for clarity. Here we will adopt the terminology of quantum optics, with the understanding that the results presented can be readily adapted to other physical settings.

In such contexts, a single continuous-variable (CV) system is known a bosonic mode, or quantum mode (qumode). Each qumode is associated with a conjugate pair of quadrature operators $(\hat{q}, \hat{p})$, analogous to the classical position and momentum and satisfying the canonical commutation relation $[\hat{q}, \hat{p}] = i\hbar$. In the case of an $m$-mode system, we denote the quadrature operators by $\hat{x} \equiv (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{2m}) \equiv (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \ldots, \hat{q}_m, \hat{p}_m)$. For a state whose density operator is $\rho$, we denote the associated first quadrature moments $\langle \hat{x}_j \rangle_{\rho}, \langle \hat{x}_k \rangle_{\rho}$.

The second phase-space moments are represented by the covariance matrix $V_{\rho}$ of $\rho$, defined by

$$\langle V_{\rho} \rangle_{j,k} := \frac{1}{2} \{ \{ \hat{x}_j - \langle \hat{x}_j \rangle_{\rho}, \hat{x}_k - \langle \hat{x}_k \rangle_{\rho} \} \}_{\rho}, \tag{1}$$

where $\{.,.\}$ denotes the anti-commutator. We make a choice of units with $\hbar = 2$, whereby the covariance matrix of the vacuum state is the identity matrix. The uncertainty constraint on a state’s covariance matrix reads $V + i\Omega_{\rho} \geq 0$, where

$$\Omega_{\rho} = \Omega_{2m} = \bigoplus_{k=1}^{m} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2}$$

is called the symplectic form on $m$ modes.

We denote by $\gamma$ the density operator of a qumode in the standard thermal state at ambient temperature. Assuming the standard Hamiltonian $H = \frac{1}{2} \hbar \omega (\hat{q}^2 + \hat{p}^2)$ and ambient temperature $T$, the resulting thermal states are Gaussian with zero first moments and quadrature fluctuations

$$\langle q^2 \rangle = \langle p^2 \rangle = \eta := \coth \left( \frac{\hbar \omega}{k_B T} \right). \tag{3}$$

When $T = 0$, the thermal state coincides with the vacuum state $|0\rangle$, whose uniform quadrature variance $\eta = 1$ is called the vacuum shot noise. The parameter $\eta$ increases monotonously with increasing temperature, growing linearly with $T$ in the limit where $T \gg \hbar \omega / k_B$. This is in line with the semi-classical picture, where quadrature fluctuations are taken to be a proxy for temperature. Thus, we will treat $\eta$ as our measure of temperature, with the understanding that it has a one-to-one correspondence with $T$.

Bosonic linear thermal operations. A key idea of the second law of thermodynamics is that, without additional energetic input, a system gravitates towards statistical equilibrium; in particular, equilibrium entails an equality between the system’s temperature and that of its environment. Thus, when limited to energetically free operations (operations that have no access to external free energy), it is impossible to increase a system’s temperature differential relative to its surrounding environment. In this vein, this temperature differential can be regarded as a thermodynamic resource—a quantity that cannot be created by free operations.

To generalise these ideas to systems of identical bosonic modes, we thus need to first define the class of operations that we consider to be energetically free. From a practical perspective, the following operations appear natural: (1) the introduction of an additional bosonic mode initialized in state $\gamma$, the thermal state at the ambient temperature; (2) the coupling of any two modes through linear energy-conserving interactions, and (3) the discarding of any number of modes. The combination of these operations clearly preserves the set of thermal states at the
ambient temperature, and thus cannot create free energy when there is none to begin with. This then leads us to a formal definition of bosonic linear thermal operations:

**Definition 1** (Bosonic linear thermal operation [BLTO]). Denote the initial system by \( S \), and the number of its constituent modes by \( m \equiv m_S \). A bosonic linear thermal operation (BLTO) is a process realizable through the following steps:

1. Adding an ancillary system \( A \) consisting of an arbitrary number \( m_A \) of elementary modes in uncorrelated thermal states: \( \gamma \otimes m_A \).
2. Application of any passive linear unitary on the composite \( SA \).
3. Partial trace over a subsystem \( A' \) comprising an arbitrary number \( m_{A'} \) of modes, leaving an output system \( S' \) of \( m' \equiv m_S \equiv m + m_A - m_{A'} \) modes.

As the definition suggests, our framework treats thermal states at the designated ambient temperature as “free of cost” in a resource-theoretic sense: the set \( \{ \gamma \equiv \gamma \otimes k \}_{k \in \mathbb{N}} \) is closed under BLTO. Note that while the operations map Gaussian states to Gaussian states, our results under an operational restriction to BLTO apply just as well to non-Gaussian initial and final states.

**II. BOSONIC “SECOND LAWS”**

We will now derive several laws governing the state transitions of modes subject to BLTO evolution. These laws in effect establish BLTO resource monotones: state functions that vary monotonically under BLTO, thus supplementing the classical thermodynamic free energies. In this sense, these laws generalize the second law of thermodynamics for pertinent physical systems, much like the “second laws” of Refs. [22] and related works. We present the laws in three categories: laws associated with temperature-like quantities, laws concerning the thermal degradation of phase-space displacement considered as a signal carrier, and laws of non-classicality degradation.

**A. Thermalization of generalized temperatures**

In equilibrium thermodynamics, a system’s temperature determines how it exchanges heat with other systems. In particular, interaction with a heat bath causes the system’s temperature to approach that of the bath. We define, and prove similar thermalization results for, several families of monotones that generalize the notion of temperature to non-equilibrium bosonic states. Recall that the thermal state has covariance matrix \( \eta \mathbb{I} \), the fixed parameter \( \eta \) corresponding to the bath’s temperature. In the context of generalized temperatures, we will refer to the value \( \eta \) as the thermal level. We will consider a value \( \eta' \) > \( \eta \) to be super-thermal, and a value \( \eta' < \eta \) to be sub-thermal.

The generalized temperatures will be based on the directional variances of a state: for a state \( \rho \) with covariance matrix \( V_\rho \), the directional variance along some unit vector \( v \) in the phase space \( V \) is given by \( V_\rho^T V_\rho \). This quantifies the variance in the measurement of a quadrature parallel to \( v \). Note that all directional variances of a thermal state are identically thermal (i.e., equal to \( \eta \)).

**Definition 2** (Principal directional temperatures). For an \( m \)-mode state \( \rho \), we define its \( k \)-th principal directional temperature (principal temperature for short) \( \tau_k(\rho) \), for \( k \in \{1,2,\ldots,2m\} \), as follows: \( \tau_1(\rho) \) is defined as the largest directional variance in the entire phase space; \( \tau_2(\rho) \) is the largest directional variance in the subspace orthogonal to a direction associated with \( \tau_1(\rho) \), and so on, with each subsequent value defined by maximizing over the subspace remaining after the preceding ones.

The principal temperatures are in fact just the \( 2m \) eigenvalues of the covariance matrix \( V_\rho \) of \( \rho \), and therefore efficiently computable from \( V_\rho \). Experimentally, they can be inferred from the statistics of quadrature measurements. Our first result (proof in Supplemental Material [3] then states:

**Theorem 1.** Under bosonic linear thermal operations (BLTO), each of the principal temperatures shifts closer to the thermal level \( \eta \), never passing the latter. Specifically, if a BLTO maps \( \rho \mapsto \sigma \), then:

1. \( \rho \) has no fewer super-thermal principal temperatures than does \( \sigma \);
2. \( \rho \) has no fewer sub-thermal principal temperatures than does \( \sigma \);
3. When arranged in decreasing order, each of \( \sigma \)'s super-thermal principal temperatures is no higher than the corresponding one of \( \rho \);
4. When arranged in increasing order, each of \( \sigma \)'s sub-thermal principal temperatures is no lower than the corresponding one of \( \rho \).

While the principal temperatures can be inferred from measurement statistics, their directions do not necessarily correspond to a set of phase-space quadratures. For example, if two thermal modes at different temperatures...
are coupled through an even beamsplitter, and one of the
outgoing modes is then squeezed, the resulting state’s
principal temperatures correspond to directions in phase
space whose simultaneous interpretation as mode quadra-
tures is forbidden by the uncertainty principle. This
motivates us to define another family of temperature-like
measures, with a more direct physical meaning:

**Definition 3** (Principal mode temperatures). For an
$m$-mode state $\rho$, we define its $k^{th}$ principal mode temper-

ature $\mu_k(\rho)$, for $k \in \{1, 2, \ldots, m\}$, as follows: $\mu_1(\rho)$ is
defined as the largest (arithmetic) mean principal tem-

perature of a single mode that can be obtained from $\rho$
by energy-conserving operations; $\mu_2(\rho)$ is the largest sin-
gle-mode mean principal temperature obtainable from the
remaining modes, and so on.

Fig. 2 schematically illustrates the definition of the
mode temperatures. As with the principal directional

temperatures, we have a thermalization law on the prin-
cipal mode temperatures (proof in Supplemental Mate-
rial [A3]):

**Theorem 2.** Under bosonic linear thermal operations
(BLTO), each of the principal mode temperatures shifts
closer to the thermal level $\eta$, never passing the latter.

The detailed implications of this law mirror the ex-

panded explanation provided in Theorem 1. Figure 3
provides a visual summary of these two theorems. Note
that the principal mode temperatures are not the same as
the symplectic eigenvalues: the latter correspond to the
temperatures of thermal modes required in preparing the
state, rather than ones that can be extracted from the
state. The symplectic eigenvalues are subject to a some-
what weaker law under BLTO (proof in Supplemental
Material [A4]):

**Theorem 3.** Under bosonic linear thermal operations
(BLTO), the sub-thermal symplectic eigenvalues cannot
shift further away from the thermal level. Specifically, if
a BLTO maps $\rho \mapsto \sigma$, then

1. $\rho$ has no fewer sub-thermal symplectic eigenvalues
   than does $\sigma$;

2. When arranged in increasing order, each of $\sigma$’s sub-
   thermal symplectic eigenvalues is no lower than the
   corresponding one of $\rho$.

It is well-known (see, e.g., [8]) that the symplectic
eigenvalues quantify the temperatures of thermal states
required in preparing a Gaussian state by Gaussian oper-
ations (cf. Fig. 4). The last theorem then tells us that the
sub-thermal symplectic eigenvalues directly quantify the
amount of sub-thermal temperature differential required
in preparing the state under BLTO. The super-thermal
symplectic eigenvalues, on the other hand,are not mono-
tones in that they may sometimes increase under BLTO,
albeit not without the initial presence of squeezedness in
the state.

**B. Signal deterioration laws**

Our next result is a straightforward observation about
the phase-space quadrature moments:

**Observation 4.** If a bosonic linear thermal opera-
tion (BLTO) achieves the transformation $\rho \mapsto \sigma$, then

$$
\sum_{k=1}^{2m} |\langle \hat{x}_k \rangle_\sigma|^2 \leq \sum_{j=1}^{2m} |\langle \hat{x}_j \rangle_\rho|^2 .
$$

Thus, if the phase-space displacement in the state is
used as a medium to carry information, then the max-
imum strength of the signal deteriorates under BLTO.
However, recall Theorem 1: the super-thermal variances
undergo a diminution under BLTO—possibly counter-
acting the signal attenuation. Thus, we ask: can the
noise reduction possibly compensate for the signal at-
tenuation, resulting in an improvement of the signal-to-
noise ratio (SNR)? In order to answer this question, we
must formally define the SNR. For an $m$-mode state $\rho$
with first moments $\langle \hat{x} \rangle_\rho$, the first moment’s component
along the direction of an arbitrary unit vector $v \in \mathbb{R}^{2m}$
in phase space is given by $v^T \langle \hat{x} \rangle_\rho$. The corresponding direc-
tional variance, in terms of the covariance matrix $V_\rho$, is

$$
\sqrt{v^T V_\rho v}.
$$

Thus, we can define the directional SNR as the
ratio between these two quantities:

$$
\frac{v^T \langle \hat{x} \rangle_\rho}{\sqrt{v^T V_\rho v}}.
$$
The optimal SNR of $\rho$ then is the maximum directional SNR over the entire phase space. In fact, as with the generalized temperatures, we define an entire family of SNR’s:

**Definition 4** (Principal directional SNR’s). For an $m$-mode state $\rho$, we define its $k^{th}$ principal directional signal-to-noise ratio $\text{SNR}_k(\rho)$, for $k \in \{1, 2, \ldots, 2m\}$, as follows: $\text{SNR}_1(\rho)$ is the optimal directional SNR over the entire phase space; $\text{SNR}_2(\rho)$ is the optimum over the subspace orthogonal to a direction achieving $\text{SNR}_1(\rho)$, and so on.

In the same spirit that the principal mode temperatures were defined, we define the following operationally-motivated variants of the principal directional SNR’s, restricting the directions to be simultaneously obtainable as quadratures in the phase space:

**Definition 5** (Principal mode SNR’s). For an $m$-mode state $\rho$, we define its $k^{th}$ principal mode SNR $\text{MSNR}_k(\rho)$, for $k \in \{1, 2, \ldots, m\}$, as follows: $\text{MSNR}_1(\rho)$ is defined as the largest directional SNR in a single mode that can be obtained from $\rho$ by energy-conserving operations; $\text{MSNR}_2(\rho)$ is the largest directional SNR in a single mode obtainable from the remaining modes, and so on.

Note that all the principal directional and mode SNR’s of a thermal state are zero, by virtue of the first moments being zero. In general, we have:

**Theorem 5.** Under bosonic linear thermal operations (BLTO), the principal directional and mode SNR’s can never increase. Specifically, if a BLTO maps $\rho \rightarrow \sigma$ with an $m'$-mode output, then

1. $\text{SNR}_k(\sigma) \leq \text{SNR}_k(\rho) \quad \forall k \in \{1, 2, \ldots, 2m'\}$;
2. $\text{MSNR}_k(\sigma) \leq \text{MSNR}_k(\rho) \quad \forall k \in \{1, 2, \ldots, m'\}$.

Thus, the SNR in every principal component of the phase-space displacement can only deteriorate under BLTO, showing that the signal attenuation effect always trumps any reduction in noise. It is important to note that this result is not of the “data-processing principle” type: that any *specific* information contained in the initial state could only possibly be lost, would be true not only under BLTO but any processing. Rather, Theorem 5 is about the usefulness of the displacement degrees of freedom as a *potential* information encoding medium—if these degrees of freedom were used to carry information, then their usefulness for this purpose would deteriorate under BLTO. In particular, if we relaxed BLTO by allowing displacement unitaries, Theorem 5 would no longer hold, while of course the data-processing principle would still hold.

**III. NON-CLASSICALITY DEGRADATION AND OTHER INHERITED LAWS**

Some notable measures already defined in the literature, and known to have operational significance in other contexts, turn out to be BLTO monotones:

1. The recently-developed resource theory for CV non-classicality [21] identified passive linear circuits with classical ancillary systems and measurement–feed-forward as the class of operations that cannot increase non-classicality as manifested by the negativity of the Glauber–Sudarshan $P$ function. Since BLTO fall within these operations, all non-classicality measures found in [21] are also BLTO monotones. These include convex roof extensions of phase-subspace variances, as well as Fisher information–based measures that quantify the usefulness of a state in the task of detecting phase-space displacement operations. The stronger constraints in BLTO imply that similar Fisher information-based results would hold in connection with the task of detecting a bosonic phase shift.

2. In any resource theory, the distance of a given state from the free states (under any contractive metric) is a monotone. Under BLTO, the thermal states are the only free states. Thus, we can construct numerous monotones of the form $D(\rho, \gamma)$, where $D$ is contractive. In particular, the “relative entropy of athermality”, $S(\rho\|\gamma)$, has been identified as a direct analog of the classical Helmholtz free energy for discrete-variable systems [22]. This and all similar metric-based measures naturally function as BLTO monotones, provided they have well-defined values.

3. The *squeezing of formation* [20] is defined as the aggregate of the single-mode squeezing required for preparing a given state from unsqueezed resources. This is a BLTO monotone, since BLTO do not allow any squeezing operations or squeezed ancillary modes. Interestingly, it is known [20] that the squeezing of formation can in general be strictly (indeed, unboundedly) smaller than the squeezing resource determined by the canonical Euler (or Bloch–Messiah) preparation of a Gaussian state (Fig. 4), which we may call the *squeezing of unitary formation*. Since BLTO severely restrict the ancillary systems that can be used, it is plausible that the squeezing of unitary formation is also a BLTO monotone; this question remains open.

**IV. ILLUSTRATIVE EXAMPLES**

We now present some illustrations of our results. First, Fig. 5 depicts the application of our results to the problem of determining which states are reachable under
BLTO from a given initial state. To simplify the illustration, we consider only the second moments of all states and ignore their other features. The initial states in these examples were chosen arbitrarily to represent a diverse range of cases. However, we shall now consider a practically relevant special case, wherein the initial state is a squeezed thermal state of the same temperature as the bath. In order to motivate this example, consider the semiclassical regime. Here the system’s state can deviate from equilibrium with the bath in only one way, namely as a thermal state at temperatures different from the bath’s. On the other hand, modes in their full quantum description can contain a fundamentally quantum-mechanical form of athermality: squeezing. Indeed, squeezed thermal states have been investigated as resources to power nano-scale engines at efficiencies surpassing classical bounds [9, 10].

By considering squeezed thermal states at the bath temperature, we can study this quantum thermodynamic resource in isolation.

Fig. 6 depicts some examples of this category. Evidently, the presence of squeezing in the initial state enables reaching states outside of the solid black set; this can be interpreted as the conversion of the quantum form of athermality, manifested by squeezing, to the classical form of a temperature differential relative to the bath. This interpretation is all the more vivid in the case of the two-mode initial state, where the accessible region contains thermal states at a range of temperatures higher than the bath’s—a purely classical thermodynamic resource. In light of such examples, it is not surprising that squeezed thermal states can be used to overcome classical performance limitations in engines and other applications.
FIG. 6: The region enclosed by the solid blue line marks all single-mode states accessible by BLTO starting from a single-mode (left) and a two-mode (right) squeezed thermal state at the same temperature as the bath’s. The solid black line shows all squeezed thermal states at the bath temperature. The examples illustrate that genuinely quantum resource in the form of squeezing can be converted to a classical form of resource—temperature differential relative to the bath.

V. DISCUSSION

In this article, we have built a quantitative framework for isolating those features of a bosonic CV quantum system that could constitute thermodynamic resources. Our approach takes inspiration from the second law, identifying quantifiers of thermodynamic resourcefulness by determining if they can ever increase under a practical class of bosonic linear thermal operations (BLTO). Our framework naturally retrieves temperature gradients as non-increasing monotones in the classical limit, while revealing a far richer spectrum of generalised temperature-like quantities when squeezing and entanglement are involved. Many of these quantities acquire immediate operational meaning in terms of phase-space fluctuations, while others are directly related to existing measures of non-classicality or figures of merit for operational tasks in metrology and communication. In applying our framework to two-mode squeezed states, we illustrated that quantum notions of non-classicality (squeezing, entanglement, etc.) can be directly converted to classical notions of free energy (temperature gradients), demonstrating that CV non-classicality has definitive thermodynamic value.

There are many interesting avenues to extend our work. In particular, there can be many alternatives to what operations we consider to be thermodynamically free. Here our choice of bosonic linear operations was heavily motivated by practical consideration of the bosonic setting, where non-Gaussian operations, or those that involve interacting modes with different free Hamiltonians, would almost certainly involve expensive non-linearity. However, in other contexts, these restrictions could be bypassed. It would certainly be interesting to see how our results change if we allowed bosonic non-linear operations such as parametric down-conversion, or hybrid models such as the Jaynes–Cummings interaction. Meanwhile, what states one considers free provides another freedom of choice. Indeed, the recently-proposed resource theory of local activity posits that thermal states at all temperatures are free [23].

An exciting future direction would be to further understand the operational consequences of our generalised temperatures. One particularly promising avenue is in sensing and metrology. Indeed, closely related notions of non-classicality have already been found to capture the usefulness of a state for sensing phase-space displacement [21, 24], while BLTO operations naturally emerge when considering sensing under energetic constraints.

Note

During the preparation of this article, we became aware of closely related work on Gaussian thermal operations [25], where arbitrary quadratic local and interaction Hamiltonians are considered free.
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SUPPLEMENTAL MATERIAL A: Proofs

1. The form of a generic BLTO

Towards proving our results, it will help to strip the definition (Def. 1) of a BLTO down into its bare mathematical form using the symplectic geometry of the phase space. Considering the generic BLTO depicted in Fig. 1, denote as before the m-mode phase space of the input system $S$ by $V \cong \text{Sp}(2m, \mathbb{R})$; let $V' \equiv V_S$, denote the phase space of the output system $S'$, and $V_A$, $V_M$ those of the ancillary systems. Being a passive linear unitary, $U$ induces on the composite phase space $V \oplus V_A$ a symplectic transformation $M$ that is, besides, orthogonal by virtue of the passivity of $U$. Denoting the phase space quadrature operators of $S$ as $(\hat{x})_{j \in \{1, 2, \ldots, 2m\}} \equiv (q_1, p_1, q_2, p_2, \ldots, q_m, p_m)$, those of $A$ as $(\hat{x})_{j \in \{2(m+1), 2m+2, \ldots, (m+m_A)\}}$, those of $S'$ as $(\hat{x})_{k \in \{1, 2, \ldots, 2m'\}}$, and those of $A'$ as $(\hat{x})_{k \in \{2m'+1, 2m'+2, \ldots, (m'+m_A')\}}$ we have

$$\hat{x}' = \sum_{j=1}^{2(m+m_A)} M_{kj} \hat{x}_j. \quad (A1)$$

Noting that the phase-space first moments of thermal modes are identically zero, the resulting transformation of the system first moments looks as follows:

$$\langle \hat{x}'_k \rangle_\sigma = \sum_{j=1}^{2m} M_{kj} \langle \hat{x}_j \rangle_\rho. \quad (A2)$$

Meanwhile, the second moments are encapsulated in the covariance matrix. In order to understand how the latter transforms, we note from the properties of the thermal state that

$$V_\sigma = \Pi_{V'} M (V_\rho \oplus \eta I_A) M^T \Pi_{V'}, \quad (A3)$$

where $\Pi_{V'}$ is the projector onto the phase space $V'$ of $S'$. It will be useful for the upcoming proofs to note that the combined operator $\Pi_{V'} M$ effects a symplectic projection.

2. Proof of Observation 4

The orthogonality of $M$ implies the conservation of the euclidean norm in phase space:

$$\sum_k |\langle \hat{x}_k \rangle_\sigma|^2 = \sum_j |\langle \hat{x}_j \rangle_\rho|^2. \quad (A4)$$

Restricting the index $k$ to the output system $S'$ immediately yields Observation 4.

3. Proof of theorems 1 and 2

We first translate our definitions and theorems to mathematical language; to this end, we start by introducing some notation.

Definition A.1 (Eigenvalues). For a symmetric matrix $V$ acting on a (finite $m$)-dimensional real vector space $V$, the $k$th largest eigenvalue of $V$, for $k \in \{1, 2, \ldots, m\}$, is given by

$$\lambda_k [V] := \max_{V_k \subseteq V} \min_{\nu_k \in V_k} \nu_k^T V \nu_k, \quad (A5)$$

where $V_k$ varies over all $k$-dimensional subspaces of $V$.

Definition A.2. For a symmetric $V$ acting on a real, (finite $2m$)-dimensional symplectic vector space $(V, \Omega_V)$, define for $k \in \{1, 2, \ldots, m\}$

$$\nu_k [V] := \frac{1}{2} \max_{V_{2k} \subseteq V} \min_{V_k \subseteq V_{2k}} \text{Tr}[\Pi_{V_k} V], \quad (A6)$$

where $V_{2k}$ varies over all $2k$-dimensional symplectic subspaces of $V$, and $V_k$ over all $2$-dimensional symplectic subspaces of each $V_{2k}$.

Note that the $\nu_k$ are not the symplectic eigenvalues of $V$. However, they can be expressed as the eigenvalues of an operator, following the line of argument used in Ref. [21]. Appendix D:

Observation A.1. For any given $V$, define $W := \frac{1}{2} (V + \Omega V^T)$. Then,

$$\nu_k [V] = \lambda_{2k} [W]. \quad (A7)$$

Proof. First, note that

$$\text{Tr}[\Pi_{V_k} V] = q^T V q + \Omega V^T p, \quad (A8)$$

where $q$ is an arbitrary unit vector in $V_2$ and $p = \Omega V^T q$ is the quadrature conjugate to $q$. Thus,

$$\text{Tr}[\Pi_{V_k} V] = q^T (V + \Omega V^T) q = 2q^T W q. \quad (A9)$$

$W$ has a special structure in terms of $2 \times 2$ blocks:

$$W = \begin{pmatrix} W^{1,1} & W^{1,2} & \cdots & W^{1,j} \\ W^{2,1} & W^{2,2} & \cdots & W^{2,j} \\ \vdots & \vdots & \ddots & \vdots \\ W^{j,1} & W^{j,2} & \cdots & W^{j,j} \end{pmatrix}, \quad W^{i,j} = \begin{pmatrix} W^R_{i,j} & -W^I_{i,j} \\ W^I_{i,j} & W^R_{i,j} \end{pmatrix}. \quad (A10)$$
with the diagonal blocks satisfying $W_{j,j} = 0$. This makes the expression for $v_k[V]$ amenable to an isomorphism onto a complex vector space of half the dimension: We form $\hat{W} \in \mathbb{C}^{m \times m}$ with elements $\hat{W}_{ij} := W_{j,j}^{i,i} + iW_{j,j}^{i,j}$, and similarly a vector $\mathbf{r} = (r_{1,z},r_{1,p},r_{2,x},r_{2,p},\ldots,)$ $\in \mathbb{V}$ is mapped to $\mathbf{\hat{r}} = (r_{1,z} + ir_{1,p},r_{2,x} + ir_{2,p},\ldots) \in \mathbb{V} \cong \mathbb{C}^m$. Then $\mathbf{\hat{r}}^T \hat{W} \mathbf{\hat{r}}$; in addition, an orthogonal basis in $\mathbb{C}^m$ corresponds to a symplectic basis in $\mathbb{V}$. Therefore,

$$v_k[V] = \max_{\nu_k \subseteq \mathbb{V}} \min_{q \in \nu_k} \mathbf{q}^T \hat{W} \mathbf{q} = \max_{\nu_k \subseteq \mathbb{V}} \min_{q \in \nu_k} \mathbf{\hat{q}}^T \hat{W} \mathbf{\hat{q}} = \lambda_k[\hat{W}].$$

(A11)

That these are the doubly degenerate eigenvalues of $W$ is seen by inverting the isomorphism to map from the diagonalized form of $\hat{W}$ back to the real $2m$-dimensional matrix $\text{diag}(\lambda_1[\hat{W}],\lambda_1[\hat{W}],\lambda_2[\hat{W}],\lambda_2[\hat{W}],\ldots)$.

Observation A.2. $\lambda_j[W] \geq \lambda_k[W]$ and $\nu_j[W] \geq \nu_k[W]$ whenever $j < k$.

Observation A.3. If $\dim(\mathbb{V}) = 2m$, then $\lambda_k[-W] = -\lambda_{2m+1-k}[W]$ and $v_k[-W] = -v_{m+1-k}[W]$ for all applicable $k$.

It is straightforward to see why this holds for the $\lambda$’s, considering that they are the eigenvalues of a Hermitian operator in a finite-dimensional vector space. It also holds for the $\nu$’s, since by virtue of Observation A.1 they, too, are the eigenvalues of a Hermitian operator.

Note. In the remainder, any expression with $\pm$ and/or $\mp$ signs is to be interpreted as a conjunction of exactly two sub-expressions: the one obtained by consistently applying the top sign throughout, and the other by consistently applying the bottom one. The scope of every such consistent application will be clear from the context.

Definition A.3 (Principal directional temperatures). For an $m$-mode state $\rho$ with covariance matrix $V_\rho$, we define its $k$th largest principal directional temperature (principal temperature for short) $\tau_k(\rho)$, for $k \in \{1,2,\ldots,2m\}$, as

$$\tau_k(\rho) \equiv \tau_k^\dagger := \lambda_k[V_\rho].$$

(A12)

Definition A.4 (Principal mode temperatures). For an $m$-mode state $\rho$ with covariance matrix $V_\rho$, we define its $k$th principal mode temperature (mode temperature for short) $\mu_k(\rho)$, for $k \in \{1,2,\ldots,m\}$, as

$$\mu_k(\rho) \equiv \mu_k^\dagger := v_k[V_\rho].$$

(A13)

Observation A.4. The principal directional and mode temperatures as defined above are arranged in non-increasing order. It follows from Observation A.2 that the same collections of values, arranged in non-decreasing order, are given respectively by

$$\tau_k^\dagger := -\lambda_k[-V_\rho],$$

(A14)

$$\mu_k^\dagger := -v_k[-V_\rho].$$

(A15)

Based on the above observations, we now reproduce theorems 1 and 2 of the main text formally in terms of the $\lambda$’s and $\nu$’s:

Theorem A.5 (Theorems 1 and 2 of main text). For a given $m$-mode state $\rho$ and $m'$-mode state $\sigma$ (Fig. 7), denote the corresponding covariance matrices as $(V_\rho,V_\sigma)$, and define

$$k^\pm_\rho := \{ k : \lambda_k[\pm V_\rho] > \pm \eta \};$$

$$k^\pm_\sigma := \{ k : \lambda_k[\pm V_\sigma] > \pm \eta \};$$

$$k^{\text{Sp} \pm}_\rho := \{ k : \nu_k[\pm V_\rho] > \pm \eta \};$$

$$k^{\text{Sp} \pm}_\sigma := \{ k : \nu_k[\pm V_\sigma] > \pm \eta \}. $$

(A16)

Then, $\rho \overset{\text{CVTO}}{\rightarrow} \sigma$ only if

1. $k^\pm_\rho \geq k^\pm_\sigma$ and $k^{\text{Sp} \pm}_\rho \geq k^{\text{Sp} \pm}_\sigma$; and, furthermore,

2. $\lambda_k[\pm V_\rho] \geq \lambda_k[\pm V_\sigma]$ for all $k \leq k^\pm_\rho$, and $\nu_k[\pm V_\rho] \geq \nu_k[\pm V_\sigma]$ for all $k \leq k^{\text{Sp} \pm}_\rho$.

Proof. We will go through the proof for the $\nu$’s, which require relatively more careful treatment; we omit the proof for the $\lambda$’s, which proceeds on similar lines but more straightforwardly. Recall that $V_\rho$ and $V_\sigma$ are symmetric positive-semidefinite matrices acting on the respective phase spaces of $\mathbb{S}$ and $\mathbb{S}'$, viz. $\mathbb{V} \cong (\mathbb{V}_\rho,\Omega_\rho) \cong (\mathbb{R}^{2m},\Omega_{2m})$ and $(\mathbb{V}',\Omega') \cong (\mathbb{V}_\sigma,\Omega_{m'}) \cong (\mathbb{R}^{2m'},\Omega_{2m'})$ respectively. Eq. (A13) tells us that $\rho \overset{\text{CVTO}}{\rightarrow} \sigma$ only if there is an orthogonal, symplectic $M$ (acting globally on the symplectic space $\mathbb{V} \oplus \mathbb{V}_A$, where $A$ is an ancilla consisting of an arbitrary number $m_A \in \mathbb{N}$ of modes) such that

$$V_\sigma = \Pi_{\mathbb{V}'} M (V_\rho \oplus nI_A) M^T \Pi_{\mathbb{V}'} ,$$

(A17)

where $\Pi_{\mathbb{V}'}$ effects an orthogonal projection onto the phase space $\mathbb{V}'$ of $\mathbb{S}'$, a symplectic subspace of $\mathbb{V} \oplus \mathbb{V}_A$. Now, for $1 \leq k \leq m'$,
\[ \nu_k [\pm V_\sigma] := \frac{1}{2} \max_{V_{2k} \subseteq V^2} \min_{V_{2k} \subseteq V^2} \text{Tr} [\pm \Pi V_2 V_\sigma] \]
\[ = \frac{1}{2} \max_{V_{2k} \subseteq V^2} \min_{V_{2k} \subseteq V^2} \text{Tr} [\pm \Pi V_2 \Pi V^M (V_\rho \oplus \eta I_A) M^T \Pi V^\nu] \]
\[ \leq \frac{1}{2} \max_{V_{2k} \subseteq V^2} \min_{V_{2k} \subseteq V^2} \text{Tr} [\pm \Pi V_2 (V_\rho \oplus \eta I_A)] = \nu_k [\pm V_\rho \oplus \eta I_A]. \quad (A18) \]

The second line follows from \((A17)\), and the last line from the fact that the maximization therein subsumes the cases covered by that in the line before. We will now prove that the inequalities \((A18)\) for \(1 \leq k \leq m'\) are collectively equivalent to the conjunction of (the symplectic parts of) conditions 1 and 2 in the statement of Theorem A.5.

We shall first prove that the former implies the latter. Firstly, it follows from the definition of \(k_{\rho, \pm}^{Sp}\), that for \(1 \leq k \leq k_{\rho, \pm}^{Sp}\),
\[ \nu_k [\pm V_\sigma] > \pm \eta. \quad (A19) \]
Meanwhile, for \(k > k_{\rho, \pm}^{Sp}\),
\[ \nu_k [\pm V_\rho \oplus \eta I_A] \leq \pm \eta. \quad (A20) \]
This necessitates \(k_{\rho, \pm}^{Sp} \geq k_{\rho, \pm}^{Sp}\), i.e. condition 1. Provided this holds, we have for \(k \leq k_{\rho, \pm}^{Sp}\) that
\[ \nu_k [\pm V_\rho \oplus \eta I_A] = \nu_k [\pm V_\rho]. \quad (A21) \]
This establishes that inequality \((A18)\) for \(1 \leq k \leq m'\) implies conditions 1 and 2. For the converse, suppose 1 and 2 hold. For \(k \leq k_{\rho, \pm}^{Sp}\),
\[ \nu_k [\pm V_\rho \oplus \eta I_A] = \nu_k [\pm V_\rho] > \pm \eta. \quad (A22) \]
by the definition of \(k_{\rho, \pm}^{Sp}\), thus securing \((A18)\) by virtue of condition 2. On the other hand, for \(k > k_{\rho, \pm}^{Sp}\), condition 1 implies that \(k > k_{\rho, \pm}^{Sp}\), so that
\[ \nu_k [\pm V_\sigma] \leq \pm \eta. \quad (A23) \]
For \((A18)\) to hold, we require this quantity to be bounded above by \(\nu_k [\pm V_\rho \oplus \eta I_A]\) for some \(A\) consisting of an arbitrary number of modes. We can achieve this by making the dimensionality of the phase space of \(A\) larger than \(2 (m' - k_{\rho, \pm}^{Sp})\), so that for \(m' \geq k > k_{\rho, \pm}^{Sp}\),
\[ \nu_k [\pm V_\rho \oplus \eta I_A] = \pm \eta. \]
\[ \square \]

4. Proof of Theorem 3

Recall Eq. \((A3)\) relating the input and output covariance matrix under a BLTO:
\[ V_\sigma = \Pi V^M (V_\rho \oplus \eta I_A) M^T \Pi V^\nu, \quad (A24) \]
where \(M\) is some orthogonal symplectic matrix. Let \(\tilde{V} := M (V_\rho \oplus \eta I_A) M^T\). Since \(M\) is symplectic, the symplectic spectrum of \(\tilde{V}\) is identical to that of \(V_\rho \oplus \eta I_A\).

Let \(\eta_1 [V_\rho], \eta_2 [V_\rho], \ldots, \eta_m [V_\rho]\) denote the symplectic eigenvalues of \(V_\rho\) in non-decreasing order. Define
\[ k_\rho := \{| j : \eta_j [V_\rho] < \eta |\}, \quad (A25) \]
i.e., the number of sub-thermal symplectic eigenvalues of \(V_\rho\). The symplectic spectrum of \(V_\rho \oplus \eta I_A\)—and, therefore, that of \(V\)—is then given by
\[ \left( \eta_1 [\tilde{V}], \eta_2 [\tilde{V}], \ldots, \eta_{m+m_A} [\tilde{V}] \right) = (\eta_1 [V_\rho], \eta_2 [V_\rho], \ldots, \eta_{k_\rho} [V_\rho], \eta, \eta, \ldots, \eta_{k_\rho+1} [V_\rho], \ldots, \eta_m [V_\rho]), \quad (A26) \]

5. Proof of Theorem 5

Once again, a mathematical translation of definitions \(4\) and \(5\) will help us prove this theorem.

Definition A.5 (definitions \(4\) and \(5\) of main text). For an \(n\)-mode state \(\rho\) with covariance matrix \(V_\rho\), we define its \(k\)th largest principal directional SNR for \(k \in \]

with \(\eta\) appearing \(m_A\) times on the RHS. Since \(V_\sigma\) is obtained from \(\tilde{V}\) by simply removing all rows and columns other than those associated with \(S'\), the symplectic eigenvalues of \(V_\sigma\) and those of \(\tilde{V}\) are related by the interlacing condition \(27\)
\[ \eta_j [V_\sigma] \geq \eta_j [\tilde{V}]. \quad (A27) \]
But for \(j \leq k_\rho\), \(\eta_j [\tilde{V}] = \eta_j [V_\rho]\). Theorem 5 follows. \[ \square \]
\[ \{1, 2 \ldots, 2m\} \text{ as} \]
\[ \text{SNR}_k(\rho) := \sqrt{\min_{\nu \in \mathcal{V}_\ell} \max_{\nu \in \mathcal{V}_\ell \setminus 0} \frac{v^T \langle \hat{x} \rangle_{\rho} \langle \hat{x} \rangle^T_{\rho} v}{v^T v}}, \quad (A28) \]

and its \( k \)th largest principal mode SNR for \( k \in \{1, 2 \ldots, m\} \) as
\[ \text{MSNR}_k(\rho) = \sqrt{\min_{\nu \in \mathcal{V}_{2\ell} \setminus 0} \max_{\nu \in \mathcal{V}_{2\ell} \setminus 0} \frac{v^T \langle \hat{x} \rangle_{\rho} \langle \hat{x} \rangle^T_{\rho} v}{v^T v}}. \quad (A29) \]

Note that \( \text{SNR}(\rho \otimes \gamma_A) = \text{SNR}(\rho) \oplus 0_A \) and likewise for the MSNR’s. The proof of theorem 5 then follows in a straightforward manner along the same lines as the previous proof.

Here we find it opportune to note a simple way to compute the principal directional SNR’s:

**Observation A.6.** For an \( m \)-mode state \( \rho \) with first moments and covariance matrix given by \( \langle \hat{x} \rangle_{\rho}, V_{\rho} \), define
\[ R_{\rho} := V_{\rho}^{-1/2} \langle \hat{x} \rangle_{\rho} \langle \hat{x} \rangle^T_{\rho} V_{\rho}^{-1/2}. \quad (A30) \]

Then, for \( k \in \{1, 2 \ldots, 2m\} \),
\[ \text{SNR}_k(\rho) = \sqrt{\lambda_k [R_{\rho}]]. \quad (A31) \]

**Proof.** Let us consider the LHS, with the shorthand \( \ell := 2m - k + 1 \):
\[ |\text{SNR}_k(\rho)|^2 = \min_{\nu \in \mathcal{V}_\ell} \max_{\nu \in \mathcal{V}_\ell \setminus 0} \frac{v^T \langle \hat{x} \rangle_{\rho} \langle \hat{x} \rangle^T_{\rho} v}{v^T v} \]
\[ = \min_{\Pi : \Pi^2 = 1} \max_{\nu \in \mathcal{V}_\ell \setminus 0} \frac{v^T \Pi^2 \langle \hat{x} \rangle_{\rho} \langle \hat{x} \rangle^T_{\rho} \Pi v}{v^T v}. \quad (A32) \]

The strict positive-definiteness of \( V_{\rho} \) (by the uncertainty principle) ensures that \( u \equiv V_{\rho}^{-1/2} \Pi v \) is nonzero whenever \( \Pi v \) is, and vice-versa; it also ensures that for every \( \ell \)-dimensional subspace \( \mathcal{V}_\ell \) of \( \mathcal{V} \), there exists a \( \Pi \) such that \( \text{span} \left( V_{\rho}^{-1/2} \Pi \mathcal{V} \right) = \mathcal{V}_\ell \). Thus,
\[ |\text{SNR}_k(\rho)|^2 = \min_{\nu \in \mathcal{V}_\ell \setminus 0} \max_{\nu \in \mathcal{V}_\ell \setminus 0} \frac{u^T R_{\rho} u}{u^T u} = \lambda_k [R_{\rho}]. \quad (A33) \]

\[ \square \]

Note that this interpretation as the eigenvalues of some operator fails for the mode SNR’s since the latter’s definition lacks the symplectic symmetry enjoyed by the definition of the mode temperatures.