Entropy of embedded surfaces in quasi-fuchsian manifolds

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Abstract

We compare critical exponent for quasi-Fuchsian groups acting on the hyperbolic 3-space and entropy of invariant disks embedded in $\mathbb{H}^3$. We give a rigidity theorem for all embedded surfaces when the action is Fuchsian and a rigidity theorem for negatively curved surfaces when the action is quasi-Fuchsian.

1 Introduction

The aim of this paper is to compare two geometric invariants of Riemannian manifolds: critical exponent and volume entropy. The first one is defined through the action of the fundamental group on the universal cover, the second one is defined for compact manifolds as the exponential growth rate of the volume of balls in the universal cover. These two invariants have been studied in many cases, we pursue this study for quasi-Fuchsian manifolds.

Let $\Gamma$ be a group acting on a simply connected Riemannian manifold $(X, g)$. If the action on $X$ is discrete we define the critical exponent by

$$\delta(\Gamma) := \limsup_{R \to \infty} \frac{1}{R} \text{Card}\{ \gamma \in \Gamma | d(\gamma \cdot o, o) \leq R \},$$

where $o$ is any point in $X$. It does not depends on this particular base point thanks to triangle inequality. If we want to insist on the space on which $\Gamma$ acts we will write $\delta(\Gamma, X)$.

The volume entropy $h(g)$ of a Riemannian compact manifold $(\Sigma, g)$ is defined by

$$h(g) := \lim_{R \to \infty} \frac{\log \text{Vol}_g(B_g(o, R))}{R},$$

where $B_g(o, R)$ is the ball of radius $R$ and center $o$ in the universal cover of $\Sigma$. We will also use the notation $h(X)$ for simply connected manifolds $X$ as the exponential growth rate of its balls.

It is a classical fact, using a simple volume argument that the volume entropy coincides with the critical exponent of $\pi_1(\Sigma)$ acting on $\tilde{\Sigma}$. Moreover, a famous theorem of G. Besson, G. Courtois and S. Gallot [BCG95] said that the entropy allows to distinguish hyperbolic metric in the set of all metrics, $\text{Met}(\Sigma)$. Remark that entropy is sensitive to homothetic transformations: for any $\lambda > 0$ we have $h(\lambda^2 g) = \frac{1}{\lambda} h(g)$. Assume that $\Sigma$ admits an hyperbolic metric $g_0$ and let $\text{Met}_0(\Sigma)$ be the set of metrics on $\Sigma$ whose volume is equal to $\text{Vol}(\Sigma, g_0)$, then Besson, Courtois, Gallot’s Theorem says for all $g \in \text{Met}_0(\Sigma)$:

$$h(g) \geq h(g_0).$$

with equality if and only if $g = g_0$. 

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Our aim is to study the behavior of the volume entropy for a subset of all the metrics on a surface. This subset is the metrics induced by an incompressible embedding into a quasi-Fuchsian manifolds. It has not the cone structure of $\text{Met}(\Sigma)$: it is not invariant by all homothetic transformations. Hence we will look at the behavior of $h(g)$ without normalization by the volume.

Let $S$ be a compact surface of genus $g \geq 2$ and $\Gamma = \pi_1(S)$ its fundamental group. A Fuchsian representation of $\Gamma$ is a faithful and discrete representation in $\text{PSL}_2(\mathbb{R})$. A quasi-Fuchsian representation is a perturbation of Fuchsian representation in $\text{PSL}_2(\mathbb{C})$. More precisely it is a discrete and faithfull representation of $\Gamma$ into $\text{Isom}(\mathbb{H}^3)$, such that the limit set on $\partial \mathbb{H}^3$ is a Jordan curve. A celebrated theorem of R. Bowen [Bow79], asserts that for quasi-Fuchsian representations, critical exponent is minimal and equal to 1 if and only if the representation is Fuchsian.

We choose an isometric, totally geodesic embedding of $\mathbb{H}^2$ in $\mathbb{H}^3$ (The equatorial plane in the ball model for example). This embedding gives a inclusion $i : \text{Isom}(\mathbb{H}^2) \to \text{Isom}(\mathbb{H}^3)$.

Let $\rho$ be a Fuchsian representation of $\Gamma$. The group $\Gamma$ acts naturally on $\mathbb{H}^2$, respectively $\mathbb{H}^3$, by $\rho$, respectively $i \circ \rho$. For every points $o \in \mathbb{H}^2$ we have

$$d_{\mathbb{H}^3}(i \circ \rho(\gamma)o, o) = d_{\mathbb{H}^2}(\rho(\gamma)o, o),$$

since $\mathbb{H}^2$ is totally geodesic in $\mathbb{H}^3$. The critical exponent for these two actions of $\Gamma$ are then equal

$$\delta(\Gamma, \mathbb{H}^3) = \delta(\Gamma, \mathbb{H}^2) = 1.$$

In light of this trivial example, two questions rise up. What is the entropy of a $\Gamma$ invariant disk which is not totally geodesic? What happens when we modify the Fuchsian representation in $\text{PSL}_2(\mathbb{C})$?

We will answer to the first question. Since $\rho$ is a Fuchsian representation, the critical exponent of $\Gamma$ acting on $\mathbb{H}^3$ through $i \circ \rho$ is 1, and we have the following

**Theorem 1.1.** Suppose $\Gamma$ is Fuchsian. Let $\Sigma$ be a $\Gamma$ invariant disk embedded in $\mathbb{H}^3$. We have

$$h(\Sigma) \leq \delta(\Gamma, \mathbb{H}^3),$$

(4)

equality occurs if and only if $\Sigma$ is the totally geodesic hyperbolic plane preserved by $\Gamma$.

Remarks that $\delta(\Gamma, \mathbb{H}^3) = h(\Sigma, g_0)$, hence the last theorem can be rewritten as follow :

**Theorem 1.2.** For all metrics $g$ obtained as induced metrics by an incompressible embedding in a Fuchsian manifold we have

$$h(g) \leq h(g_0)$$

(5)

with equality if and only if $g = g_0$.

We did not renormalize by the volume, this explains the dichotomy between (4) and (5).

We will prove this theorem in the next section. The inequality is trivial since the induce distance between two points is always greater than the distance in $\mathbb{H}^3$: $d_{\Sigma} \geq d_{\mathbb{H}^2}$, but the rigidity is not. We have no geometrical (curvature) hypothesis on $\Sigma$, therefore it is not obvious at all to show that the inequality is strict as soon as $\Sigma$ is not totally geodesic. Indeed we cannot use the "usual" techniques of negative curvature like Bowen-Margulis measure, or even the uniqueness of geodesic between two points.

We obtain an answer to the second question under a geometrical hypothesis on the curvature:
Theorem 1.3. Let $\Gamma$ be a quasi-Fuchsian group and $\Sigma \subset \mathbb{H}^3$ a $\Gamma$–invariant embedded disk. We suppose that $\Sigma$ endowed with the induced metric has negative curvature. We then have
\[ h(\Sigma) \leq I(\Sigma, \mathbb{H}^3)\delta(\Gamma, \mathbb{H}^3), \]
where $I(\Sigma, \mathbb{H}^3)$ is the geodesic intersection between $\Sigma$ and $\mathbb{H}^3$. Moreover, equality occurs if and only if the length spectrum of $\Sigma/\Gamma$ is proportional to the one of $\mathbb{H}^3/\Gamma$.

The geodesic intersection will be defined in section 3.1. Roughly, it is the average ratio of the length between two points of $\Sigma$ for the extrinsic and intrinsic distance. We need the curvature assumption to define and use this invariant.

As a corollary of Theorem 1.3, we obtain

Corollary 1.4. Under the assumptions of Theorem 1.3 we have
\[ h(\Sigma) \leq \delta(\Gamma, \mathbb{H}^3), \]
with equality if and only if $\Gamma$ is fuchsian and $\Sigma$ is the totally geodesic hyperbolic plane, preserved by $\Gamma$.

Theorem 1.3 has to be compared to results obtained by G. Knieper who compared entropy for two different metrics on the same manifolds and our proof of Theorem 1.3 follows his paper [Kni95]. As in his paper, we obtain that the intersection is larger than 1 as soon as $\Gamma$ is not Fuchsian.

It is also related to the work of M. Bridgeman and E. Taylor [BT00], indeed we answer by the negative to Question 2 of their paper. And finally, we can see our work as an extension of U. Hamenstaedt’s paper [Ham02], where she compared the geodesic intersection between the boundary of convex hull and $\mathbb{H}^3$ for quasi-Fuchsian manifolds.

As we said, the two proofs are very different one from each others. For the Fuchsian case, we give precise estimates for the length of some paths of the hyperbolic plane. We show that in some sense the length between two points on $\Sigma$ is much greater than the extrinsic distance between those two points. For quasi-Fuchsian manifolds, we use well known techniques of negative curvature geometry: we compare the Patterson Sullivan measures for $\mathbb{H}^3$ and for $\Sigma$.

2 Fuchsian case

In this section we are going to prove Theorem 1.1. This theorem has a strong condition on $\Gamma$, i.e. it is conjugate to a subgroup of $\text{PSL}_2(\mathbb{R})$ but we make no geometrical assumptions on $\Sigma$. As we said, there could be more than one geodesic between two points on $\Sigma$.

We already remarked that the inequality is trivial, as is the equality when $\Sigma$ is totally geodesic. Therefore, the only thing left to prove is the strict inequality when $\Sigma$ is not totally geodesic or in other words if $\Sigma \neq \mathbb{H}^2$ then $h(\Sigma) < 1$.

The proof of the theorem is based on the comparison between the distances on equidistant surfaces of the totally geodesic $\Gamma$–invariant hyperbolic plane. We are going to prove several lemmas which together gives Theorem 1.1. The strict inequality follows directly from Lemmas 2.2 and 2.8. We denote by $D$ the totally geodesic, $\Gamma$–invariant plane. The induced metric on $D$ is the usual hyperbolic metric, and we will denote it by $\mathbb{H}^2$. We are first going to see that between all the equidistant surfaces, $\mathbb{H}^2$ has the biggest entropy. Then we will make this argument work when only one part of the surface is "above" $D$. The idea to prove it, is to consider another distance $d_m$ on $D$, which will be used as an intermediary between $\Sigma$ and $\mathbb{H}^2$. We will explain, after the definition of $d_m$ how the two comparisons will be proved.
Let us begin to parametrize $\mathbb{H}^3$ by $\mathbb{H}^2 \times \mathbb{R}$ as follows: take an orientation for the unit normal tangent space of $\mathbb{H}^2$, then to a point $x \in \mathbb{H}^3$ we associate $s(x)$ the orthogonal projection from $\mathbb{H}^3$ to $\mathbb{H}^2$. It is the first parameter of the parametrization. The oriented distance along this geodesic gives the second one. Hence the parametrisation, called Fermi coordinates, is defined by

$$
\mathbb{H}^3 \rightarrow \mathbb{H}^2 \times \mathbb{R} \\
(z \rightarrow (s(z), \tilde{d}(z, s(z)))
$$

where $\tilde{d}$ is the oriented distance defined by the choice of the orientation on the unit normal tangent of $\mathbb{H}^2$. With this parametrization, the metric on $\mathbb{H}^3$ is

$$
g_{\mathbb{H}^3} = \cosh^2(r)g_0 + dr^2.
$$

Look at $S(r)$ the equidistant disk at distance $r$ of $\mathbb{H}^2$, its metric, induces by the one on $\mathbb{H}^3$, is $g_r = \cosh^2(r)g_0$. It is isometric to a hyperbolic plane of curvature $\frac{1}{\cosh(r)}$, and its volume entropy is $h(S(r)) = \frac{h(0)}{\cosh(r)} = \frac{1}{\cosh(r)}$, hence the entropy is maximal if and only if $r = 0$. For the general case, we are going to refine this argument showing that it is sufficient that a small part of $\Sigma$ is over $\mathbb{H}^2$ for the entropy to be strictly less than 1.

Let $\Sigma$ be an embedded $\Gamma$-invariant disk in $\mathbb{H}^3$. We assume that $\Sigma \neq \mathbb{D}$, and we endowed $\Sigma$ with its induced metric. Let $x, y$ be two points on $\Sigma$. Let $c_{x,y}^\Sigma$ be a geodesic on $\Sigma$ linking $x$ to $y$. We parametrize $c_{x,y}^\Sigma$ by its Fermi coordinates, $(c, r)$. We then have

$$
d_{\Sigma}(x, y) = \int_0^L \|c_{x,y}^\Sigma(t)\|_{g_0} dt \\
= \int_0^L \sqrt{r'(t)^2 + \cosh^2(r(t))|c'(t)|^2_{g_0}} dt. \\
\geq \int_0^L \cosh(r(t))|c'(t)|_{g_0} dt. \tag{6}
$$

We now endowed $\mathbb{D}$ with another distance than the one coming from hyperbolic metric. It will play the role of intermediary to compare $d_{\Sigma}(x, y)$ on $\Sigma$ with $d_{g_0}(s(x), s(y))$ on $\mathbb{H}^2$.

We call $\sigma$ the restriction of $s$ on $\Sigma$. Since $\Sigma \neq \mathbb{D}$, there exists $x_0 \in \mathbb{D} \setminus \Sigma$, $\varepsilon > 0$ and $\eta > 0$ such that

$$
d_{\mathbb{D}}(\sigma^{-1}B(x_0, 2\varepsilon), \mathbb{D}) > \eta.
$$

This means that all the points in the pre-image of $B(x_0, 2\varepsilon)$ by $\sigma$ are at distance greater than $\eta$ from $\mathbb{D}$. We will assume that $2\varepsilon$ is smaller than the injectivity radius of $\mathbb{H}^2/\Gamma$ in order that the translations of $B(x_0, 2\varepsilon)$ by $\Gamma$ are disjoint. We have taken $2\varepsilon$ in order to simplify the proof of Lemma 2.4.

We now consider on $\mathbb{D}$ the metric $g_m$ defined by putting weight on the translations of $B(x_0, 2\varepsilon)$ by $\Gamma$.

**Definition 2.1.** We define $g_m$ by

$$
g_m := \cosh(\eta)^2 g_0,
$$

on $\Gamma \cdot B(x_0, 2\varepsilon)$, and

$$
g_m := g_0,
$$

elsewhere.
Lemma 2.2. We have

\[ \ell_m(c) = \int_0^1 \|\dot{c}(t)\|_{g_m} dt. \]

This gives a distance \( d_m \) on \( \mathbb{D} \) by choosing:

\[ d_m(x, y) := \inf_{c} \{\ell_m(c) \mid c(0) = x, c(1) = y\}. \]

In order to prove Theorem 1.1 we will compare the entropy of \((\mathbb{D}, d_m)\) with the one of \( \Sigma \) and the one of \( \mathbb{H}^2 \). The comparison with the entropy of \( \Sigma \) is quite easy and follows quickly from the definition of \( d_m \) and the inequality \( \mathfrak{H} \). The comparison with the entropy of \( \mathbb{H}^2 \) is more subtle. Indeed, there exist geodesics of \( \mathbb{H}^2 \) which are geodesics for \((\mathbb{D}, d_m)\) (any lift of a closed geodesic which does not cross the ball \( B(x_0, 2\varepsilon) / \Gamma \) on \( \mathbb{H}_c^2 / \Gamma \)). We will first prove that two points of \( \mathbb{D} \) which are joined by a geodesic of \( \mathbb{H}^2 \) which crosses often \( \Gamma \cdot B(x_0, 2\varepsilon) \) are much farther away from each other for \( d_m \) distance, cf Lemme 2.4. Then, we will use a large deviation theorem for the geodesic flow (Theorem 2.6), to show that there are few geodesics which do not cross the ball \( B(x, 2\varepsilon) \) which does not cross the ball \( \mathbb{D} \).

The comparison between \( h(\Sigma) \) and the critical exponent of \((\mathbb{D}, d_m)\) follows from the inequality \( \mathfrak{H} \) and the definition of \( d_m \).

**Lemma 2.2.** We have

\[ h(\Sigma) \leq \delta((\mathbb{D}, d_m)). \]

**Proof.** Let \( x \in \Sigma \) and \( o = \sigma(x) \in \mathbb{D} \). Since \( \Sigma / \Gamma \) is compact, we have

\[ h(\Sigma) = \lim_{R \to \infty} \frac{1}{R} \log \text{Card}\{\gamma \in \Gamma \mid d_{\Sigma}(\gamma x, x) \leq R\}. \]

And by definition

\[ \delta((\mathbb{D}, d_m)) = \lim_{R \to \infty} \frac{1}{R} \log \text{Card}\{\gamma \in \Gamma \mid d_m(\gamma o, o) \leq R\}. \]

It is sufficient to prove that \( d_{\Sigma}(x, y) \geq d_m(s(x), s(y)) \), for all \( x, y \in \Sigma \). Let \( c_\Sigma = (c, r) \) be a geodesic on \( \Sigma \) joining \( x \) to \( y \). Recall that we have

\[ d_{\Sigma}(x, y) \geq \int_0^L \cosh(r(t))\|c'(t)\|_{g_m} dt. \]

If \( c(t) \notin \Gamma \cdot B(x_0, 2\varepsilon) \), then \( \|c'(t)\|_{g_m} = \|c'(t)\|_{g_0} \). In particular

\[ \|c'(t)\|_{g_m} \leq \cosh(r(t))\|c'(t)\|_{g_0}. \]

If \( c(t) \in \Gamma \cdot B(x_0, 2\varepsilon) \), then by definition of \( g_m \), \( \|c'(t)\|_{g_m} = \cosh(\eta)\|c'(t)\|_{g_m} \) and since \( \Sigma \) is "far" from \( \mathbb{D} \), \( r(t) > \eta \). In particular,

\[ \|c'(t)\|_{g_m} \leq \cosh(r(t))\|c'(t)\|_{g_0}. \]
Finally

\[ d_{S}(x, y) \geq \int_{0}^{L} \|c'(t)\|_{2m} \, dt \]
\[ \geq l_{m}(c) \]
\[ \geq d_{m}(s(x), s(y)). \]

Our next aim is to compare the distance \( d_{m} \) and \( d_{H^{2}} \). Let us fix some notations before stating the first lemma. For all \( v \in T^{1}H^{2} \), let \( \zeta_{R}^{v} \) be the probability measure on \( T^{1}H^{2} \), defined for all borelian \( E \subset T^{1}H^{2} \) by:

\[ \zeta_{R}^{v}(E) = \frac{1}{R} \int_{0}^{R} \chi_{E}(\phi_{t}^{H^{2}}(v)) \, dt \]

where \( \chi_{E} \) is the indicator function of \( E \). For a borelian \( E \) which is a unitary tangent bundle of a subset of \( D \), \( E := T^{1}A \), we have:

\[ \zeta_{R}^{v}(E) = \frac{1}{R} \text{Leb}\{t \in [0, R]|c_{v}(t) \in A\} \]

since \( \phi_{t}^{H^{2}}(v) \in E \) is equivalent to \( c_{v}(t) = \pi\phi_{t}^{H^{2}}(v) \in A \).

Let \( L \) be the Liouville measure on the unitary tangent bundle of the quotient surface \( T^{1}H^{2}/\Gamma \). Recall that the metric \( g_{m} \) is given by \( g_{m} = \cosh^{2}(\eta)g_{0} \) on \( T^{1}B(x_{0}, 2\varepsilon) \). We fix the following \( K := T^{1}(\Gamma \cdot B(x_{0}, \varepsilon)) \).

**Definition 2.3.** Let \( \kappa > 0 \) be such \( L(K/\Gamma) - 2\kappa > 0 \). We define the following sets,

\[ \mathcal{E}(R) := \{v \in T^{1}H^{2} | \zeta_{R}^{v}(K) > L(K/\Gamma) - \kappa\}, \]

and for all points \( o \in H^{2} \), we note

\[ \mathcal{E}_{o}(R) := \{v \in T^{1}oH^{2} | \zeta_{R}^{v}(K) > L(K/\Gamma) - \kappa\}. \]

A geodesic of length \( R \) whose direction is given by a vector \( v \in \mathcal{E}(R) \) crosses \( \pi K \) "often", that is at least a number proportional to \( R \), cf. Figure 1. Indeed, if \( v \in \mathcal{E}(R) \) we have

\[ \frac{1}{R} \text{Leb}\{t \in [0, R]|c_{0}(t) \cap \pi K \neq \emptyset\} > L(K/\Gamma) - \kappa > \kappa > 0, \]

since \( \dot{c}_{0}(t) \in K \) is equivalent to \( c_{0}(t) = \pi c_{0}(t) \in \pi K \) by definition of \( K \).

The next argument is the key in the proof of Theorem [14]. It shows that we can compare the length of a geodesic in \( H^{2} \) which crosses often \( \pi K \) with its \( d_{m} \) length.

**Lemma 2.4.** There exists \( C > 1 \), such that for all \( R > 0 \), for all \( v \in \mathcal{E}_{o}(R) \) and for all \( x \in \{\exp(tv) | t \in [R, 2R]\} \), we have:

\[ d_{m}(o, x) \geq C d_{S}(o, x). \]
Proof. Let $c_0$ be the geodesic for $g_0$ and $c_m$ be a minimizing geodesic for $g_m$ between $o$ and $x$. Let $d$ be the hyperbolic distance between $o$ and $x$, $d = d_{\mathbb{H}^2}(o, x)$, and we parametrize $c_0$ by unit speed we thus have $c_0(d) = x$. Let $N(R)$ be the number of intersections between $\pi K$ and $c_0([0, R])$, that is $N$ is the number of connected components of $c_0([0, R]) \cap \pi K$. On one hand, all components of $c_0([0, R]) \cap \pi K$ are inside balls of radius $\epsilon$, hence $c_0$ "stays" at most $2\epsilon$ in each components. On the other hand, the hypothesis $v \in E_o(R)$, implies that 

$$\frac{1}{R} \text{Leb}\{t \in [0, R] | c_0(t) \cap \pi K \neq \emptyset \} > L(K/\Gamma) - \kappa = \kappa > 0.$$ 

These two facts imply that $2\epsilon N(R) \geq \kappa R$, that is to say

$$N(R) \geq \frac{\kappa}{2\epsilon} R. \quad (8)$$

For $i \leq N(R)$, let $t_i \in [0, d]$ such that $c_0(t_i) \in \pi K$ and $c_0[t_{i-1}, t_i] \setminus \pi K$ is connected: we just have chosen a point $x_i = c_0(t_i)$ in each balls of $\pi K$ crossing $c_0$. There exists $\gamma_i \in \Gamma$ such that $x_i \in B(\gamma_i x_0, \epsilon)$ hence $B(x_i, \epsilon) \subset B(\gamma_i x_0, 2\epsilon)$ on which the metric $g_m$ is $g_m = \cosh^2(\eta)g_0$. See Figure 2 Therefore the geodesic $c_0$ is divided into $N(R)$ segments: $[x_i, x_{i+1}]$, such that for every $i$ we know that on the ball $B(x_i, \epsilon)$ the metric $g_m$ is given by $g_m = \cosh^2(\eta)g_0$. We want a lower bound on $d_m(o, x)$, therefore we can estimate the length of $c_m$ with the metric given by $\cosh^2(\eta)g_0$ on the smaller balls $B(x_i, \epsilon) \subset B(\gamma_i x_0, 2\epsilon)$ and, $g_0$ on the rest of the plane.
We call $y_i$ the middle of $[x_i, x_{i+1}]$. We now restrain our attention on one segment $[y_i, y_{i+1}]$. Let $0 < a < 1$ whose dependence on $\eta$ will be made clear in the rest of the proof. We are going to analyse two different cases.

**Assume $c_m$ crosses $B(x_i, a\epsilon)$**

Let $\Delta_i$ be the lines (geodesics in $\mathbb{H}^2$) orthogonal to $c_0$ and passing through $y_i$. Let $z_i^1$ and $z_i^2$ be the end points of the diameter of $B(x_i, \epsilon)$ defined by $z_i^1 = c_0(t_i - \epsilon)$ and $z_i^2 = c_0(t_i + \epsilon)$, and call $D_i^1$ and $D_i^2$ the lines orthogonal to $c_0$ and passing through $z_i^1$ and $z_i^2$. See figure 9. We want to consider the intersections between $c_m$ and the lines $\Delta_i$, $D_i^1$ and $D_i^2$. There might have many intersections. We will call first (resp. last) intersection of $c_m$ with a line $D$ the point $c_m(t_f)$ (resp $c_m(t_l)$) where $t_f := \inf\{t \mid c_m(t) \in D\}$ (resp $t_l := \sup\{t \mid c_m(t) \in D\}$).

Let $A_i'$, $B_i'$ and $C_i'$ be the last intersections of $c_m$ with, respectively, $\Delta_i$, $D_i^1$ and $D_i^2$. Let $B_i, C_i$ and $A_{i+1}$ be the first intersections of $c_m$ with, respectively, $D_i^1$, $D_i^2$ and $\Delta_{i+1}$. This divides $c_m$ in five connected components: $[A_i', B_i']$, $[B_i, B_i']$, $[B_i', C_i']$, $[C_i, C_i']$, $[C_i', A_{i+1}]$.

Our work will be to give a lower bound for the length of each components cf. Figure 9. Since it might happen that $B_i = B_i'$ and $C_i = C_i'$ the bound on the length of those two components will be trivial: $d_m(B_i, B_i') \geq 0$ and $d_m(C_i, C_i') \geq 0$.

The $g_m$-length of $c_m$ from $A_i'$ to $B_i$ is equal (or larger) to its $g_0$-length since the metric $g_m$ is equal to the metric $g_0$ outside $K$. Moreover the $g_0$-length of $c_m$ from $A_i'$ to $B_i$ is greater than $d_{g_0}(y_i, z_i^1)$ since the orthogonal projection decreases lengths. We then have

$$d_m(A_i', B_i) \geq d_{g_0}(y_i, z_i^1).$$

For the same reasons we have

$$d_m(C_i', A_{i+1}) \geq d_{g_0}(z_i^2, y_{i+1}).$$

We want to give a lower bound for the $g_m$-length of $c_m$ between $B_i'$ and $C_i$. We made the assumption that $c_m$ crosses the ball $B(x_i, a\epsilon)$ hence $c_m$ stays at least $2\epsilon - 2a\epsilon$ in the ball $B(x_i, \epsilon)$. In other words if $c_m$ is unitary for $g_0$ we have $\text{Leb}\{t \mid c_m(t) \cap B(x_i, \epsilon) \neq \emptyset\} \geq 2\epsilon - 2a\epsilon$. In the
ball $B(x, \varepsilon)$, the metric $g_m$ is equal to $\cosh(\eta)^2 g_0$ hence the $g_m$-length satisfies

$$d_m(B'_i, C_i) \geq \int_{\{t \mid c_m(t) \cap B(x, \varepsilon) \neq \emptyset\}} \| \dot{c}_m(t) \|_{m} dt$$

$$= \int_{\{t \mid c_m(t) \cap B(x, \varepsilon) \neq \emptyset\}} \cosh(\eta) dt$$

$$\geq \varepsilon \cosh(\eta)(2 - 2\alpha).$$

Choose $\alpha > 0$ such that $\cosh(\eta)(2\varepsilon - 2\alpha\varepsilon) > 2\varepsilon$, that is to say $\alpha \leq 1 - \frac{1}{\cosh(\eta)}$. In order to fix the idea we set $\alpha := \frac{1}{2}(1 - \frac{1}{\cosh(\eta)})$. This implies

$$d_m(B'_i, C_i) \geq \varepsilon \cosh(\eta)(2 - 2\alpha)$$

$$= \varepsilon \cosh(\eta) \left( 2 - \left( 1 - \frac{1}{\cosh(\eta)} \right) \right)$$

$$= \cosh(\eta) + 1 \varepsilon$$

$$= 2\varepsilon + \varepsilon[\cosh(\eta) - 1]$$

$$= d_{g_0}(z_i^1, z_i^2) + \varepsilon[\cosh(\eta) - 1].$$

Finally we proved

$$d_m(A_i, A_{i+1}) \geq d_m(A'_i, A_{i+1}) \geq d_{g_0}(y_i, y_{i+1}) + \varepsilon[\cosh(\eta) - 1].$$  \hspace{1cm} (9)
Assume $c_m$ does not cross $B(x_i, a\epsilon)$

Let $\Delta_i$ be the line orthogonal to $c_0$ and passing through $y_i$ and $\Omega_i$ the one through $x_i$. Call $A'_i$ the last intersection of $c_m$ and $\Delta_i$ and $E_i$ the first intersecton of $c_m$ with $\Omega_i$. Since $c_m$ does not cross $B(x_i, a\epsilon)$, $E_i$ is in one of the connected component of $\Omega_i \setminus B(x_i, a\epsilon)$. Named $e_i$ the intersection of $S(x_i, a\epsilon)$ (the sphere of center $x_i$ and diameter $a\epsilon$) and $\Omega_i$ in the same connected component as $E_i$, this is also the orthogonal projection of $E_i$ on $B(x_i, a\epsilon)$. See figure 4.

We parametrise the geodesic $\Omega_i$ by $R$, we give $\omega: R \to H^2$ such that $\omega(0) = x_i$ and the orientation is chosen in order to have $\omega(a\epsilon) = e_i$. The function $t \to d_{gh}(\omega(t), \Delta_i)$ is convex, which has a minimum at 0, it is hence increasing on $\mathbb{R}^+$. Therefore, $d_{gh}(\Delta_i, E_i) \geq d_{ghz}(\Delta_i, e_i)$.

It follows that

$$d_m(A'_i, E_i) \geq d_{ghz}(A'_i, E_i) \geq d_{gh}(\Delta_i, E_i) \geq d_{gh}(\Delta_i, e_i).$$

Let us compute $d_{gh}(\Delta_i, e_i)$. We fix the following notations:

$$L = d_{gh}(\Delta_i, e_i)$$
$$l = d_{gh}(y_i, x_i)$$
$$H = d_{gh}(y_i, e_i)$$

Now Pythagore's theorem in hyperbolic geometry for the triangle $(y_i, x_i, e_i)$ gives

$$\cosh(l) \cosh(a\epsilon) = \cosh(H).$$
Let $\theta$ be the angle $\widehat{x_iy_i}$. We have

$$\cos(\theta) = \frac{\tanh(l)}{\tanh(H)}.$$ 

and

$$\sin(\pi/2-\theta) = \frac{\sinh(L)}{\sinh(H)}.$$ 

Hence

$$\sinh(L) = \sinh(H) \frac{\tanh(l)}{\tanh(H)} = \cosh(H) \tanh(l) = \cosh(a\varepsilon) \sinh(l).$$

From this equation, we cannot conclude that $L > l + u$ for some $u > 0$. Indeed if $L$ goes to 0 so does $l$. To avoid this problem we are going to assume that $l$ is greater than the injectivity radius of $S$.

Remark the following property of sinh which is a consequence of easy calculus. For all $x_0 > 0$ and $\varpi > 1$, there exists $u > 0$, such that for all $x > x_0$, we have $\varpi \sinh(x) \geq \sinh(x + u)$. Now we choose $y_i$ on $c_0$ in order to have $d_{g_0}(x_i, y_i) \geq s/2$ where $s$ is the injectivity radius of $\mathbb{H}^2/\Gamma$. Consequently, applying the previous property with $\varpi = \cosh(a\varepsilon)$ and $x_0 = s/2$, there exists $u > 0$ such that

$$\cosh(a\varepsilon) \sinh(l) \geq \sinh(l + u).$$

Since sinh is increasing we deduce that

$$L \geq l + u.$$ 

Altogether, we show that there exists $u > 0$ such that

$$d_m(A_i', E_i) \geq d_{g_0}(y_i, x_i) + u.$$ 

By the same arguments we can show that

$$d_m(E_i', A_{i+1}) \geq d_{g_0}(x_i, y_{i+1}) + u.$$ 

($E_i'$ is the last intersection of $c_m$ with $\Omega_i$). Hence, if $c_m$ does not meet $B(x_i, a\varepsilon)$, the $g_m$-length of $c_m$ between $A_i$ and $A_{i+1}$ satisfies, (taking trivial bounds for first and last intersections)

$$d_m(A_i, A_{i+1}) \geq d_{g_0}(y_i, y_{i+1}) + 2u. \tag{10}$$

**Conclusion**  Let $\alpha := \min\{\varepsilon[\cosh(\eta) - 1]; 2u\}$. From (9) and (10) we have:

$$d_m(A_i, A_{i+1}) \geq d_{g_0}(y_i, y_{i+1}) + \alpha.$$ 

Summing on $i$ we get

$$d_m(o, x) \geq d_{g_0}(o, x) + N(R)\alpha.$$ 

Equation (8) and the fact that $d_{g_0}(o, x) \leq 2R$ imply that

$$N(R) \geq \frac{\kappa}{2\varepsilon} R \geq \frac{\kappa}{4\varepsilon} d_{g_0}(o, x).$$ 

\footnote{this is where we use the upper bound on $d_{g_0}(o, x)$.}
Figure 4: $c_m$ does not cross $B(x, a\varepsilon)$.

Subsequently,

$$d_m(o, x) \geq \left(1 + \frac{\alpha \kappa}{4\varepsilon}\right) d_{g_0}(o, x).$$

This proves the Lemma with $C = (1 + \frac{\alpha \kappa}{4\varepsilon})$.

We are now going to compare the entropy of $(D, d_m)$ with the one of $\mathbb{H}^2$. Let us define

$$\mathcal{F}_o(R) = \{\exp(tv) | t \in \mathbb{R}^+, v \in \mathcal{E}_o(R)\}.$$

We note by $B_m(o, 2R)$ the ball of radius $2R$ for the $d_m$ distance.

**Lemma 2.5.** Let $C' := \min(2, C)$ where $C$ satisfies the Lemma 2.4. We have for all $o \in D$, and all $R > 0$:

$$B_m(o, 2R) \subset B_{\mathbb{H}^2}(o, 2R/C') \cup \left( B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o(R) \right).$$

**Proof.** Indeed we have $B_m(o, 2R) = \left( B_m(o, 2R) \cap \mathcal{F}_o(R) \right) \cup \left( B_m(o, 2R) \cap \mathcal{F}_o(R) \right)$. Let $x \in B_m(o, 2R) \cap \mathcal{F}_o(R)$. Since $d_{\mathbb{H}^2}(o, x) \leq d_m(o, x)$, it follows that $d_{\mathbb{H}^2}(o, x) \leq 2R$. There are only two possibilities. If $d_{\mathbb{H}^2}(o, x) \leq R$, we have in particular $d_{\mathbb{H}^2}(o, x) \leq \frac{2R}{C'}$. However, if $d_{\mathbb{H}^2}(o, x) \geq R$, we apply Lemma 2.4 and we get $d_{\mathbb{H}^2}(o, x) \leq \frac{2R}{C'} \leq \frac{2R}{C'}$. Therefore,

$$B_m(o, 2R) \cap \mathcal{F}_o(R) \subset B_{\mathbb{H}^2}(o, \frac{2R}{C'}) \cap \mathcal{F}_o(R) \subset B_{\mathbb{H}^2}(o, \frac{2R}{C'}).$$

Since we also have for $R > 0$, $B_m(o, 2R) \subset B_{\mathbb{H}^2}(o, 2R)$, this gives

$$B_m(o, 2R) \cap \mathcal{F}_o(R) \subset B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o(R),$$

and prove the lemma.

The Liouville measure on $T^1\mathbb{H}^2$ is the product of the riemannian measure of $\mathbb{H}^2$ with the angular measure on every fiber. We denote this product by $L = d\mu(x) \times d\theta(x)$. Our aim is to show that the set $\mathcal{E}_o(R)$ is small and the volume of $B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o(R)$ is small compared to
the one of $B_{2R}(a, 2R)$. For this we are going to use a large deviation theorem of Y. Kifer [Kif90] which gives an upper bound on the mass of the vector which do not behave as the Liouville measure.

Let $\mathcal{P}$ be the set of probability measures on $T^1\mathbb{H}^2/\Gamma$ endowed with the weak topology. Let $\mathcal{P}^t$ be the subset of $\mathcal{P}$ of probability measures invariant by the geodesic flow. We also denote by $L$ the Liouville measure on the quotient $T^1\mathbb{H}^2/\Gamma$. Recall that for a vector $v \in T^1\mathbb{H}^2/\Gamma$ we denote by $\zeta^R_0$ the probability measure given for all borelians subset $E \subset T^1\mathbb{H}^2/\Gamma$ by

$$\zeta^R_0(E) = \frac{1}{R} \int_0^R \chi_E \left( \phi^R_{t}/\Gamma(v) \right) dt.$$

**Theorem 2.6.** [Kif90] Let $\overline{\mathcal{A}}$ be a compact subset of $\mathcal{P}$,

$$\limsup_{T \to \infty} \frac{1}{T} \log L \{ v \in T^1\mathbb{H}^2/\Gamma \mid \zeta^T_0 \in \overline{\mathcal{A}} \} \leq - \inf_{\mu \in \overline{\mathcal{A}} \cap \mathcal{P}^t} f(\mu)$$

where $f(\mu) = 1 - h_\mu(\phi^R_{t}/\Gamma)$ and $h_\mu(\phi^R_{t}/\Gamma)$ is the entropy of the geodesic flow $\phi^R_{t}/\Gamma$ with respect to $\mu$.

The fact that the theorem can be applied on this setting is explained after the Theorem 3.4 in [Kif90]. In this reference the function $f$ is given by a formula which seems different. One can look at [PPS89, Chapter 7], where the authors explain in details why the geodesic flow of negatively curved surfaces satisfies the hypothesis of Kifer’s Theorem, and that one can take $f(\mu) = 1 - h_\mu(\phi^R_{t}/\Gamma)$.

**Lemma 2.7.** There exists $o \in \mathbb{H}^2$, $\alpha > 0$ and $R_0 > 0$ such that for all $R > R_0$

$$\theta_o(\mathcal{E}^c_\alpha(R)) \leq e^{-\alpha R}.$$

**Proof.** Let us keep the notations of Lemma 2.4, $K = T^1\Gamma \cdot B(x, \epsilon)$ and we consider the following subset of $\mathcal{P}$

$$A := \{ \mu \in \mathcal{P} \mid \mu(K/\Gamma) \leq L(K/\Gamma) - \kappa \}.$$

This set is not closed for the weak topology. Its closure satisfies

$$\overline{A} \subset \{ \mu \in \mathcal{P} \mid \mu(T^1\Gamma \cdot B^o(x, \epsilon)/\Gamma) \leq L(K/\Gamma) - \kappa \},$$

where $B^o(x, \epsilon)$ is the open ball. There might be equality between the two sets, but we won’t use it.

However, since the unitary tangent of the sphere $S(x, \epsilon)$ is transverse to the flow, we have:

$$\{ v \in T^1\mathbb{H}^2/\Gamma \mid \zeta^R_0 \in A \} = \{ v \in T^1\mathbb{H}^2/\Gamma \mid \zeta^R_0 \in \overline{A} \}.$$

In other words, the measures $\zeta^R_0$ do not charge $T^1S(x, \epsilon)$.

Since $L \notin \overline{A}$ and $L$ is the unique measure of maximal entropy satisfying $h(L) = 1$, we have

$$- \inf_{\mu \in \overline{A}} f(\mu) = -\alpha < 0.$$

Besides, it is clear that the set $\mathcal{E}_c(R) = \{ v \in T^1\mathbb{H}^2 \mid \zeta^R_0(K) \leq L(K/\Gamma) - \kappa \}$ is $\Gamma$-invariant from the $\Gamma$ invariance of $K$. By definition and the previous remark we get

$$\mathcal{E}_c(R)/\Gamma = \{ v \in T^1\mathbb{H}^2/\Gamma \mid \zeta^R_0 \in A \} = \{ v \in T^1\mathbb{H}^2/\Gamma \mid \zeta^R_0 \in \overline{A} \}.$$
The Theorem 2.6 says that there exists $R_0 > 0$ such that for all $R > R_0$ we have

$$L(E^c(R)/\Gamma) \leq e^{-\alpha R}.$$ 

The product structure of $L$ implies the existence of a point $o \in \mathbb{H}^2/\Gamma$ such that

$$\theta_o (E^c_o(R)/\Gamma)) \leq e^{-\alpha R}.$$ 

The Lemma follows, choosing any lift of $o$ in $\mathbb{H}^2$. \hfill $\square$

We finish the proof of Theorem 1.1 with Lemma 2.8, which compare the critical exponent between $d_m$ and hyperbolic distance. Lemmas 2.2 and 2.8, conclude the proof.

**Lemma 2.8.** There exists $u > 0$ such that

$$\delta((D, d_m)) \leq 1 - u.$$ 

**Proof.** We are going to show that the volume entropy of $(D, d_m)$ satisfies the inequality, that would imply the similar result on critical exponent.

Let $o \in D$ be a point satisfying Lemma 2.7. From Lemma 2.5, we have

$$B_m(o, 2R) \subset B_{\mathbb{H}^2}(o, 2R) \cup \left( B_{\mathbb{H}^2}(o, 2R) \cap F^c_o(R) \right).$$ 

On one hand we have the classical upper bound $\text{Vol}(B_{\mathbb{H}^2}(o, 2R)) = O(e^{2R/C'}).$ On the other hand the volume form on $\mathbb{H}^2$ can be written in polar coordinates as $\sinh(r) dr d\theta$, hence for all $R > R_0$ we get

$$\text{Vol}(B_{\mathbb{H}^2}(o, 2R) \cap F^c_o(R)) = \int_0^{2R} \int_{E^c_o(R)} \sinh(r) d\theta dr \leq \int_0^{2R} e^{-\alpha R} e^r dr \leq e^{(2-\alpha)R}.$$ 

Let $u > 0$, defined by $1 - u = \max(\frac{1}{2}, (1 - \alpha/2)) < 1$. The last two upper bounds give

$$\text{Vol}(B_m(o, 2R)) = O(e^{2R/C'}) + O(e^{(2-\alpha)R}) = O(e^{2(1-u)R}).$$ 

We finish by taking the log and the limit. \hfill $\square$

### 3 Quasi-Fuchsian case

#### 3.1 Geodesic intersection

Let $\Sigma$ be an incompressible surface in $M$. We designed by $\phi^T_{\mathbb{H}^3}, \phi^T_{\Sigma}$ the geodesic flows on the unitary tangent spaces $T^1\mathbb{H}^3, T^1\Sigma$ respectively. We named $\pi$ the projection from $T^1\mathbb{H}^3$ to $\mathbb{H}^3$. The restriction of $\pi$ to $T^1\Sigma$ will still be denoted by $\pi$. There is two distances we can consider on $\Sigma$. The intrinsic one, defined as the infimum of the length of curves staying on $\Sigma$ and the extrinsic one, where we take the distance in $\mathbb{H}^3$. We will denote $d_\Sigma$ and $d$ this two distances.

First of all let us remark that there is no riemannian metric on $\Sigma$ which induces $d$. If such a metric existed, our Theorem 1.3 would be a particular case of [Kni95].

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Proposition 3.1. If $\Sigma$ is not totally geodesic, there is no riemannian metric on $\Sigma$ which induces $d$.

Proof. Assume there is such a riemannian metric, named $g'$. Let $\epsilon > 0$ be such that the exponential map for $g'$ is an embedding at every point. Let $c_{g'} : [0, \epsilon] \to \Sigma$ be a minimizing geodesic for $g'$ on $\Sigma$, then for all $t \in [0, \epsilon]$,

$$d_{g'}(c_{g'}(0), c_{g'}(t)) + d_{g'}(c_{g'}(t), c_{g'}(\epsilon)) = d_{g'}(c_{g'}(0), c_{g'}(\epsilon))$$

But since we suppose that $g'$ induces $d$ we have the same equality for $d$

$$d(c_{g'}(0), c_{g'}(t)) + d(c_{g'}(t), c_{g'}(\epsilon)) = d(c_{g'}(0), c_{g'}(\epsilon))$$

and this implies that $c_{g'}$ is a geodesic for $H^3$. Hence every points of $\Sigma$ is included in a totally geodesic disc, therefore $\Sigma$ is totally geodesic.

Consider the following function

$$a : T^1\Sigma \times \mathbb{R} \to \mathbb{R}$$

Let $t_1, t_2 \in \mathbb{R}$ and $v \in T^1\Sigma$, we have by the triangle inequality,

$$a(v, t_1 + t_2) = d(\pi \phi_t^\Sigma v, \pi(v))$$

hence $a$ is a subadditive cocycle for the geodesic flow $\phi_t^\Sigma$. Since $a$ is $\Gamma$ invariant it defines a subadditive cocycle on $T^1\Sigma$, still denoted by $a$.

The following is a consequence of Kingman’s subadditive ergodic theorem [Kin73].

Theorem 3.2. Let $\mu$ be a $\phi_t^\Sigma$ invariant probability measure on $T^1\Sigma$. Then

$$I_\mu(\Sigma, M, v) := \lim_{t \to \infty} \frac{a(v, t)}{t}$$

exists for $\mu$ almost $v \in T^1\Sigma$ and defines a $\mu$-integrable function on $T^1\Sigma$, invariant under the geodesic flow and we have :

$$\int_{T^1\Sigma} I_\mu(\Sigma, M, v) d\mu = \lim_{t \to \infty} \int_{T^1\Sigma} \frac{a(v, t)}{t} d\mu.$$ 

Moreover if $\mu$ is ergodic $I_\mu(\Sigma, M, v)$ is constant $\mu$-almost everywhere. In this case, we write $I_\mu(\Sigma, M)$.

3.2 Patterson Sullivan measures

We called $\Lambda$ the limit set of $\Gamma$ acting on $H^3$. Since $\Gamma$ acts cocompactly on $\Sigma$, and on the convex core $C(\Lambda)$, the three geometric spaces $\Gamma$ (seen as its Cayley graph), $\Sigma$ and $C(\Lambda)$ are quasi-isometric. We assume from now on that $(\Sigma, g)$ has negative curvature, hence there is a unique geodesic in each homotopy class of curves, and for every pair of points in $\Sigma$ there is a unique
Let \( \Sigma \) be a geodesic on \( \Sigma \), and denoted by \( c_\Sigma(\pm \infty) \) its limit points on \( \Lambda \). There is a unique \( \mathbb{H}^3 \)-geodesic \( c_{3\Sigma} \) whose endpoints are \( c_\Sigma(\pm \infty) \). Since \( \Sigma \) is quasi-isometric to \( C(\Lambda) \), the two geodesics \( c_{3\Sigma} \) and \( c_\Sigma \) are at bounded distance.

Let \( p \in \Sigma \) and call \( pr_p^{\Sigma} \) the projection from \( \Sigma \) to \( \Lambda \) defined as follows. For any point \( x \in \Sigma \) call \( c_{p,x}^{\Sigma} \) the geodesic on \( \Sigma \) which joint \( p \) to \( x \), then

\[
pr_p^{\Sigma}(x) = c_{p,x}^{\Sigma}(+\infty).
\]

We will denote the equivalent projection in \( \mathbb{H}^3 \) by \( pr_p^{3\Sigma} \). There is two small distinctions to notice between \( pr_p^{3\Sigma} \) and \( pr_p^{\Sigma} \). First \( pr_p^{3\Sigma} \) is defined for every points in \( \mathbb{H}^3 \), whereas \( pr_p^{\Sigma} \) is only defined for points in \( \Sigma \). Second is that the codomain of \( pr_p^{\Sigma} \) is exactly \( \Lambda \) whereas the codomain of \( pr_p^{3\Sigma} \) is all \( S^2 \).

As we just said, for all \( \xi \in \Lambda \) the geodesics, \( c_{p,\xi}^{\Sigma} \) and \( c_{p,\xi}^{3\Sigma} \) are at bounded distance, and this bound depends only on the quasi-isometry between \( \Sigma \) and \( C(\Lambda) \). There exists \( C_1 \) such that for all \( \xi \in \Lambda \) the Hausdorff distance between geodesics \( c_{p,\xi}^{\Sigma} \) and \( c_{p,\xi}^{3\Sigma} \) is less than \( C_1 \).

Let \( x \in \Sigma \), \( R > 0 \) and consider the quasi-isometry between \( \Sigma \) and \( C(\Lambda) \). There exists \( C_1 \) such that for all \( \xi \in \Lambda \) the Hausdorff distance between geodesics \( c_{p,\xi}^{\Sigma} \) and \( c_{p,\xi}^{3\Sigma} \) is less than \( C_1 \).

Let \( x \in \Sigma \), \( R > 0 \) and consider the quasi-isometry between \( \Sigma \) and \( C(\Lambda) \). There exists \( C_1 \) such that for all \( \xi \in \Lambda \) the Hausdorff distance between geodesics \( c_{p,\xi}^{\Sigma} \) and \( c_{p,\xi}^{3\Sigma} \) is less than \( C_1 \).

The same argument shows that

\[
pr_p^{\Sigma}(B_{3\Sigma}(x,R) \cap \Sigma) \subset pr_p^{3\Sigma}(B_{3\Sigma}(x,R+C_1)) \cap \Lambda \subset pr_p^{3\Sigma}(B_{3\Sigma}(x,R+C_1)).
\]

The distance on \( \Sigma \) and on \( \mathbb{H}^3 \) are locally equivalent: for every \( R > 0 \) there exists \( C_2 \) such that all balls satisfy the following

\[
B_{\Sigma}(x,R-C_2) \subset B_{3\Sigma}(x,R) \cap \Sigma \subset B_{\Sigma}(x,R+C_2)
\]

Set \( C = \max(C_1, C_2) \) we then have

**Theorem 3.3.**

\[
pr_p^{\Sigma}(B_{\Sigma}(x,R-C)) \cap \Lambda \quad \subset \quad pr_p^{3\Sigma}(B_{3\Sigma}(x,R-C)) \cap \Lambda \quad \subset \quad pr_p^{3\Sigma}(B_{3\Sigma}(x,R+C)) \cap \Lambda
\]

Before proving Theorem 1.3 we will recall some basic facts about Patterson-Sullivan measure. Some classical references for this are the papers of Patterson and Sullivan themselves, \([\text{Pat76}]\) and \([\text{Sul79}]\), the lecture of J-F. Quint \([\text{Qui06}]\) and the monograph of T. Roblin \([\text{Rob03}]\). Let \( (X,g) \) be a simply connected manifolds with negative curvature and \( X(\infty) \) its geometric boundary. If \( \Gamma \) is a discrete group acting on \( (X,g) \) we can associated a family of measures \( \{\mu^g_\gamma\}_{\gamma \in \Gamma} \) on \( X(\infty) \) constructed as follows. Let \( x,y \) two points of \( X \) and consider the Poincaré series:

\[
P(s) := \sum_{\gamma \in \Gamma} e^{-sd(\gamma x,y)}.
\]

The convergence of \( P(s) \) is independent of \( x \) and \( y \) by the triangle inequality. It converges for \( s > \delta(\Gamma) \) and diverges for \( s < \delta(\Gamma) \). If the action is cocompact, \( \delta(\Gamma) = h(g) \) and the series
diverges at $h(g)$. Then we define the probability measure
\[
\mu_{p,x}^g(s) := \frac{\sum_{\gamma \in \Gamma} e^{-sd(\gamma x, p)} \delta_{\gamma x}}{\sum_{\gamma \in \Gamma} e^{-sd(p, p)}}.
\]

By compactness of the set of probability measures on $X(\infty)$, we obtain a measure on $X(\infty)$ by taking a weak limit of a sequence $\mu_{p,x}^g(s_n)$\(^3\):
\[
\mu_p^g := \lim_{s_n \to h(g)} \mu_p^g(s_n)
\]

It is supported on the accumulation points of $G$, that is to say the limit set.

These measures called Patterson-Sullivan measures have the following properties. They are quasi-conformal,
\[
\frac{d\mu_p^g}{d\mu_q^g}(\xi) = e^{-h(g)\beta_\xi(p,q)},
\]
where $\beta_\xi(p,q) = \lim_{z \to \xi} d_g(p,z) - d_g(q,z)$.

And $\Gamma$-equivariant,
\[
\mu_p^g \circ \gamma = \mu_p^{g\gamma^{-1}}
\]
for all $\gamma \in \Gamma$.

Moreover we know these measures behave locally like $h(g)$–Hausdorff measures. See [Qui06, Lemma 4.10] for example.

**Lemma 3.4** (Shadow’s lemma). For $R > 0$ sufficiently large, there exists $c > 1$ such that for all $x \in X$
\[
\frac{1}{c} e^{-h(g)d_\delta(x,p)} \leq \mu_p^g(pr_p^g(B_g(x,R))) \leq ce^{-h(g)d_\delta(x,p)}.
\]

Suppose that $X/\Gamma$ is compact, from Patterson-Sullivan measure, we can construct an invariant measure on $T^1X/\Gamma$. Let $\Lambda^{(2)}$ be $\Lambda \times \Lambda \setminus \text{diagonal}$, there is a natural identification of $\Lambda^{(2)} \times \mathbb{R}$ and $T^1X$, a vector $v \in T^1X$ is identified with $(c_v(+\infty), c_v(-\infty), \beta_{c_v}(+\infty)(p, \pi v))$. The Bowen-Margulis measure is defined by
\[
d\mu_{BM}(\xi, \eta, t) = e^{2h(g)\langle \xi | \eta \rangle_p} d\mu_p^g(\xi) d\mu_p^g(\eta) dt
\]
where $\langle \xi | \eta \rangle_p$ is the Gromov product, given by
\[
\langle \xi | \eta \rangle_p = \lim_{t \to \infty} \left( t - \frac{1}{2} d(c_p, \xi(t), c_{p,\eta}(t)) \right).
\]

From the quasi-conformal property of $\mu_p^g$, it follows that $\mu_{BM}$ is invariant by $\Gamma$ and defined a measure on $T^1X/\Gamma$. The invariance by the geodesic flow is clear by definition and it is shown in [Nic89] that $\mu_{BM}$ is ergodic.

Finally we will need the following theorem, which is classical for compact manifolds endowed with two different negatively curved metrics. Since we treat a case slightly different we give the proof.

**Theorem 3.5.** If $\mu_p^\Sigma$ and $\mu_p^M$ are equivalent, then the marked length spectrum of $\Sigma$ is proportional to the marked length spectrum of $M$.

\(^3\)It is a classical result of Sullivan that there is in fact a unique limit, up to normalization.
Remark that in the Fuchsian case, any surface equidistant to the totally geodesic one has a metric proportional to $H^2$ and therefore satisfies the hypothesis of the Theorem. It seems likely that it is the only case where the length spectrum is proportional to the one of the ambient manifold, however it is still unknown.

Before proving this theorem let us introduce Gromov distance on the boundary.

**Definition 3.6.** [GdlH90] Let $\xi, \eta$ be two points on $\partial X$. The Gromov distance is defined by

$$D_X(\xi, \eta) = \exp(-a \langle \xi | \eta \rangle),$$

where $a > 0$ is sufficiently small for $D_X$ to be distance.

Remark that for the same space, there is not a unique Gromov distance, but two of them are Hölder equivalent.

The proof of Theorem 3.5 is in two steps. The first one we prove that if the Patterson Sullivan measures are equivalent then the Gromov distances $D_\Sigma$ and $D_{\mathbb{H}^3}$ are Hölder equivalent. The second one we prove that this last condition implies the proportionality of the length spectrum.

**Lemma 3.7.** If $\mu_\Sigma^p$ and $\mu_{\mathbb{H}^3}^p$ are equivalent, then the two Gromov distances are Hölder equivalent.

**Proof.** Let us consider on $\Lambda^{(2)}$ the Bowen-Margulis currents defined by

$$\nu_\Sigma(\xi, \eta) = \frac{d\mu_\Sigma^p(\xi)d\mu_\Sigma^p(\eta)}{D_\Sigma(\xi, \eta)^{2\delta(\Sigma)}},$$

$$\nu_{\mathbb{H}^3}(\xi, \eta) = \frac{d\mu_{\mathbb{H}^3}^p(\xi)d\mu_{\mathbb{H}^3}^p(\eta)}{D_{\mathbb{H}^3}(\xi, \eta)^{2\delta(\mathbb{H}^3)}}.$$

From the definitions of Busemann functions ($\beta_\xi(x) = \lim_{u \to \xi} d(x, u) - d(o, u)$) and Gromov product ($< \xi, \eta > = \lim_{u \to \xi, v \to \eta} \frac{1}{2}(d(u, o) + d(v, o) - d(u, v))$) we have for $\Sigma$ as well as for $\mathbb{H}^3$ that

$$\beta_\xi(\gamma p) + \beta_\eta(\gamma p) = 2 < \gamma^{-1} \xi, \gamma^{-1} \eta > - 2 < \xi, \eta > .$$

Using $D(\xi, \eta) = e^{-<\xi, \eta>}$ and conformal property of Patterson Sullivan measures, the currents $\nu_\Sigma$ and $\nu_{\mathbb{H}^3}$ are $\Gamma$-invariant.

By assumption $\mu_\Sigma^p$ and $\mu_{\mathbb{H}^3}^p$ are equivalent, therefore $\nu_\Sigma$ and $\nu_{\mathbb{H}^3}$ are also equivalent. The ergodicity and the $\Gamma$-invariance implies the existence of $c > 0$ such that

$$\nu_\Sigma = c \nu_{\mathbb{H}^3}.$$

Since $\mu_\Sigma^p$ and $\mu_{\mathbb{H}^3}^p$ are equivalent there exists a function $f : \Lambda \to \mathbb{R}^+$ such that $\mu_\Sigma^p(\xi) = f(\xi) \mu_{\mathbb{H}^3}^p$. We have

$$f(\xi)f(\eta)d_{\mathbb{H}^3}^p(\xi, \eta) = c d_\Sigma^p(\xi, \eta).$$

We see that $f$ is equal almost everywhere to a continuous function. We can therefore suppose that $f$ is continuous on $\Lambda$ hence strictly positive. By compactness, there exists $C > 1$ such that $\frac{1}{C} \leq f(\xi) \leq C$. Finally we get what we stated

$$\frac{c}{C^2} d_\Sigma^p(\xi, \eta) \leq d_{\mathbb{H}^3}^p(\xi, \eta) \leq C^2 c d_\Sigma^p(\xi, \eta).$$

We now show the second part
**Lemma 3.8.** If $D_{\Sigma}$ and $D_{\mathbb{H}^3}$ are Hölder equivalent the marked length spectrum are equivalent

**Proof.** In [PPS89] Section 3.5] the authors show that in a very general setting we have:

$$\lim_{n \to \infty} \frac{1}{n} \log |g^-, g^+; A_n(\xi), \xi| = \ell(g),$$

where $\ell(g)$ is the displacement of $g$ and $|g^-, g^+; A_n(\xi), \xi| = \frac{D(g^-, g^+; A_n(\xi)) D(g^+, g^-; \xi)}{D(g^-, \xi) D(g^+, g^-; \xi)}$.

In particular, we can apply this result to $\Sigma$ and $\mathbb{H}^3$ we get

$$\lim_{n \to \infty} \frac{1}{n} \log |g^-, g^+; A_n(\xi), \xi|_{\Sigma} = \ell_\Sigma(g),$$

and

$$\lim_{n \to \infty} \frac{1}{n} \log |g^-, g^+; A_n(\xi), \xi|_{\mathbb{H}^3} = \ell_{\mathbb{H}^3}(g).$$

By assumption on the distances $D_{\Sigma}, D_{\mathbb{H}^3}$, there exists $C > 1$ such that we have

$$\frac{1}{C} |g^-, g^+; A_n(\xi), \xi|_{\mathbb{H}^3} \leq |g^-, g^+; A_n(\xi), \xi|_{\Sigma} \leq C |g^-, g^+; A_n(\xi), \xi|_{\mathbb{H}^3}.$$

Hence

$$\ell_\Sigma(g) = r \ell_{\mathbb{H}^3}(g).$$

**Theorem 3.5** follows directly from Lemmas 3.7 and 3.8.

### 3.3 Entropy comparison

We finally get to the proof of Theorem 1.3. First we prove the inequality using the behaviour of Patterson-Sullivan measures and a volume comparison of a subset of $\Sigma$, the proof follows the same lines as [Kni95] Theorem 3.4]. Then we prove the equality case using Theorem 3.5.

**Theorem 3.9.** Let $(\Sigma, g)$ has negative curvature,

$$h(g) \leq I_{BM}(\Sigma, M) \delta(\Gamma).$$

With equality if and only if the marked length spectrum of $\Sigma$ is proportional to the marked length spectrum of $M$. In this case, the proportionality factor is given by $\ell_\Sigma(g) I(\Sigma, M) = \ell_M(g)$.

**Proof.** The geodesic flow is ergodic with respect to the Bowen-Margulis measure $\mu_{BM}$, hence for $\mu_{BM}$-almost all $v \in T^1 \Sigma$ we have:

$$\lim_{t \to \infty} \frac{a(v, t)}{t} = I_{BM}(\Sigma, M).$$

Let $v$ and $v'$ be two unit vectors on the same weak stable manifold. Then $d(c_v(t), c_{v'}(0)) \leq d(c_v(t), (c_v(t)) + d(c_v(t), c_v(0)) + d(c_v(0), c_{v'}(0)))$, and the same inequality holds interchanging the role of $v$ and $v'$. Moreover $d(c_v(t), (c_v(t))$ decreases exponentially since $v$ and $v'$ are on the same weak stable manifold. Hence $\lim_{t \to \infty} \frac{a(v, t)}{t}$ exists if and only if $\lim_{t \to \infty} \frac{a(v', t)}{t}$ does.

Let $v_{\xi}(\xi)$ denotes the unitary vector in $T^1_{\mathbb{S}}$ such that $c_{v_{\xi}(\xi)}(\xi) = \xi$. The previous fact and the product structure of $d\mu_{BM}$ assures that for $\mu_{BM}$-almost all $\xi \in \partial\Sigma$ we have

$$\lim_{t \to \infty} \frac{a(v_{\xi}(\xi), t)}{t} = I_{BM}(\Sigma, M).$$
For all \( \epsilon > 0 \) and \( T > 0 \) we define the set
\[
A_p^{T, \epsilon} = \left\{ \xi \in \partial \Sigma \left| \frac{a(v_p(\xi), t)}{t} - I_{\mu}(\Sigma, M) \leq \epsilon, \quad t \geq T \right. \right\}.
\]

For all \( d \in [0, 1] \) and all \( \epsilon > 0 \), there exists \( T > 0 \) such that \( \mu(\Sigma)^{A_p^{T, \epsilon}} \geq d \). For \( t > T \) consider the subset \( \{ v_p(t)(\xi) \in A_p^{T, \epsilon} \} \subset S_t(p, t) \) of the geodesic sphere of radius \( t \) and center \( p \) on \( \Sigma \).

Choose \( \{ B_5(x_i, R_i) | i \in I \} \) a covering of this subset of fixed radius \( R > 0 \) such that \( x_i \in S_5(p, t) \) and \( B_5(x_i, R/4) \) are pairwise disjoint. Then, by the local behaviour of \( \mu(\Sigma) \), there exists a constant \( c > 1 \), independent of \( t \), such that
\[
\frac{1}{t} e^{-h(\xi)t} \leq \mu(\Sigma)^{\text{pr}_p(B_5(x_i, R))} \leq c e^{-h(\xi)t}.
\]
It is clear that \( A_p^{T, \epsilon} \subset \bigcup_{i \in I} \text{pr}_p(B_5(x_i, R)) \) and therefore,
\[
d \leq \mu(\Sigma) \left( \bigcup_{i \in I} \text{pr}_p(B_5(x_i, R)) \right) \leq \sum_{i \in I} \mu(\Sigma)^{\text{pr}_p(B_5(x_i, R))} \leq c \text{Card}(I) e^{-h(\xi)t}.
\]

Since \( \mathbb{H}^3 / \Gamma \) is convex cocompact, \( C_Q(\Lambda) / \Gamma \) is compact, where \( C_Q(\Lambda) \) is the \( Q \) neighbourhood of the convex core of \( \Lambda \). Hence for any \( Q > 0 \),
\[
\delta(\Gamma) = \lim_{R \to \infty} \text{Vol}(B_{\mathbb{H}^3}(a, R) \cap C_Q(\Lambda)).
\]

Now take \( Q \) sufficiently large such that \( \Sigma \) is inside \( C_Q(\Lambda) \). There exists \( K \) such that \( B_5(x_i, R/4) \subset B_{\mathbb{H}^3}(x_i, R + K) \cap C_Q(\Lambda) \).

From the definition of the set \( A_p^{T, \epsilon} \), we then have that the disjoint union \( \bigcup_{i \in I} B_5(x_i, R/4) \subset B_{\mathbb{H}^3}(p, t(I_{\mu, \mathbb{H}^3}(\Sigma, \mathbb{H}^3) + \epsilon) + R + K) \cap C_Q(\Lambda) \). It follows that,
\[
e^{-h(\xi)t} \leq \frac{c}{d} \text{Card}(I)
\leq \frac{c}{dV} \sum_{i \in I} \text{vol}_{\mathbb{H}^3}(B_{\mathbb{H}^3}(x_i, R/4) \cap C_Q(\Lambda))
\leq \frac{c}{dV} \text{vol}_{\mathbb{H}^3}(B_{\mathbb{H}^3}(p, t(I_{\mu, \mathbb{H}^3}(\Sigma, \mathbb{H}^3) + \epsilon) + R + K) \cap C_Q(\Lambda))
\]

Hence
\[
h(\xi) \leq \frac{1}{t} \left( \log \frac{c}{dV} + \log \text{vol}_{\mathbb{H}^3}(B_{\mathbb{H}^3}(p, t(I_{\mu, \mathbb{H}^3}(\Sigma, \mathbb{H}^3) + \epsilon) + R + K) \cap C_Q(\Lambda)) \right)
\]
Taking the limit \( t \to \infty \) we get
\[
h(\xi) \leq (I_{\mu, \mathbb{H}^3}(\Sigma, \mathbb{H}^3) + \epsilon) \delta(\Gamma)
\]
and we conclude since \( \epsilon \) is arbitrary.

For the proof of the equality case in Theorem [1.3] we will use the result equivalent to [Kn95] Corollary 3.6 in our context, that is:

**Lemma 3.10.** [Kn95] Let \( p \in \Sigma \) and \( \mu(p) \) the Patterson-Sullivan measure with respect to \( p \) and \( g \), there exists a constant \( L \) such that for \( \mu(p) \) almost all \( \xi \in \partial \Sigma \) there is a sequence \( t_n \to \infty \) such that
\[
|d(p, \pi_\mu^{\Sigma}(v_p(\xi))) - I_{\mu, \mathbb{H}^3}(\Sigma, \mathbb{H}^3)t_n| \leq L.
\]

Proof. It follows from Lemma 3.5 of [Kn95], that our lemma is true provided there exists a constant \( C > 0 \) such that, for all \( t_1, t_2 > 0 \) and all \( v \in T^1 \Sigma \),
\[
a(v, t_1) + a(\phi_v^t v, t_2) \leq C + a(v, t_1 + t_2).
\]

Let \( v \in T^1 \Sigma \) and \( c_v^\Sigma \) be the geodesic on \( \Sigma \) directed by \( v \). Recall that there exists \( C_1 \) such that the \( \mathbb{H}^3 \)-geodesic from \( \pi(v) \) to \( c_v^\Sigma(t_1 + t_2) \) is at bounded distance \( C_1 \) of \( c_v^\Sigma(t_1 + t_2) \), independent of \( t_1 \) and \( t_2 \). The \( \mathbb{H}^3 \)-geodesic from \( p \) to \( c_v^\Sigma(t_1) \) and the one from \( c_v^\Sigma(t_1) \) to \( c_v^\Sigma(t_1 + t_2) \) are also at bounded distance \( C_1 \) of \( c_v^\Sigma \). This implies the desired property with \( C = 2C_1 \).

Equality case in \( \mathbb{H}^3 \) Suppose that \( h(g) = \mu_{\muBM}(\Sigma, \mathbb{H}^3)\delta(\Gamma) \). Choose a point \( p \in \Sigma \) and \( \xi \in \Lambda \), set \( y_n := \pi_\Sigma^\Sigma v_p(\xi) \). From the above lemma, for \( \mu_p^\Sigma \) almost all \( \xi \) we have
\[
|d(p, y_n) - \muBM(\Sigma, \mathbb{H}^3)\xi| \leq L.
\]

Set \( R > 0 \) a fixed constant, by local property of the Patterson-Sullivan measure on \( \mathbb{H}^3 \), there is \( c_1 \) such that
\[
\frac{1}{c_1} e^{-\delta(\Gamma)d(p, y_n)} \leq \muBM^\Sigma(pBM\Sigma(y_n, R)) \leq c_1 e^{-\delta(\Gamma)d(p, y_n)},
\]
by Theorem \[3.3\]
\[
prBM\Sigma(BBM\Sigma(x, R - C)) \cap \Lambda \subset prBM\Sigma(BBM\Sigma(x, R) \cap \Sigma) \subset prBM\Sigma(BBM\Sigma(x, R + C)).
\]

Hence there is a constant \( c_2 \) such that
\[
\frac{1}{c_2} e^{-\delta(\Gamma)d(p, y_n)} \leq \muBM^\Sigma(prBM\Sigma(y_n, R) \cap \Sigma) \leq c_2 e^{-\delta(\Gamma)d(p, y_n)},
\]
by the local property of the Patterson-Sullivan measure on \( \Sigma \), there is \( c_3 \) such that
\[
\frac{1}{c_3} e^{-h(\Sigma)dz(p, y_n)} \leq \muBM^\Sigma(prBM\Sigma(y_n, R)) \leq c_3 e^{-h(\Sigma)dz(p, y_n)},
\]
and by Theorem \[3.3\]
\[
prBM\Sigma(BBM\Sigma(x, R - C)) \subset prBM\Sigma(BBM\Sigma(x, R) \cap \Sigma) \subset prBM\Sigma(BBM\Sigma(x, R + C)).
\]

Hence there is \( c_4 \) such that
\[
\frac{1}{c_4} e^{-h(\Sigma)dz(p, y_n)} \leq \muBM^\Sigma(prBM\Sigma(y_n, R) \cap \Sigma) \leq c_4 e^{-h(\Sigma)dz(p, y_n)},
\]
by the choice of \( y_n \) and since \( h(\Sigma) = \muBMBM(\Sigma, \mathbb{H}^3)\delta(\Gamma) \)
\[
eq L e^{-\delta(\Gamma)d(p, y_n)} \leq e^{-h(\Sigma)dz(p, y_n)} \leq L e^{-\delta(\Gamma)d(p, y_n)}.
\]

Hence there is \( c_5 > 0 \) such that
\[
\frac{1}{c_5} e^{-\delta(\Gamma)d(p, y_n)} \leq \muBM^\Sigma(prBM\Sigma(y_n, R) \cap \Sigma) \leq c_5 e^{-\delta(\Gamma)d(p, y_n)},
\]
Finally we have a constant \( c_6 \) such that
\[
c_6 \leq \frac{\muBM^\Sigma(prBM\Sigma(y_n, R) \cap \Sigma)}{\muBM^\Sigma(prBM\Sigma(y_n, R) \cap \Sigma)} \leq c_6.
\]

Since \( prBM\Sigma(BBM\Sigma(y_n, R) \cap \Sigma) \to \xi \) the measures \( \muBM^\Sigma \) and \( \muBM^\Sigma \) are equivalent. We conclude by Theorem \[3.5\]
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