The existence of $p$-convex tensor products of $L_p(X)$–spaces for the case of an arbitrary measure

A. Ya. Helemskii

Abstract

We obtain a far-reaching generalization (in several directions) of the theorem of A. Lambert on the existence of the projective tensor product of operator sequence spaces. This result is obtained in the context of spaces, generalizing $p$-multinormed spaces of Dales et al. which are based on an arbitrary, perhaps non-discrete measure.

Keywords: $L$–space, $L$–boundedness, $p$–convex tensor product, convenient measure space, inflation.

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1. Introduction

The main result of the present paper is the following theorem. All notions that participate in it, will be gradually explained. Below $X$ denotes an arbitrary measure space which is not atomic with finite set of atoms and which is supposed, for simplicity, to be separable.

The main Theorem. An arbitrary pair of near-$L_p(X)$-spaces, where $1 \leq p < \infty$, has a $p$-convex tensor product.

A far-away predecessor of this result is a theorem of A. Lambert [15, §3.1.1] on projective tensor products of his “operator sequence spaces”; the latter are situated, in a sense, between normed spaces and abstract operator spaces. Afterwards, a group of mathematicians (Dales, Polyakov, Daws, Pham, Ramsden, Laustsen, Oikhberg, Troitsky; see [6] and also [3, 4, 5]) introduced more general structures than Lambert spaces, called $p$-multinormed spaces; $p \in [1, \infty]$. After their papers, in the frame-work of the so-called non-coordinate approach, the

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$L$-spaces were introduced in \cite{13}. The latter, provided $L := L_p(X)$, can be considered as “multinormed spaces based on arbitrary measures”. Indeed, in the case of $X := \mathbb{N}$ with the counting measure they transform to $p$-multinormed spaces of \cite{6} (and if, in addition, $p := 2$, to Lambert spaces).

“The main theorem” was proved in \cite{13} under the additional condition that the set of atoms in $X$ is either empty or infinite. For a time there was a suspicion that for an arbitrary $X$ the theorem is false. (It was based on the known bad properties of $L_p(X)$ as a tensor factor, see \cite{8, §12.1}). However, recently T. Oikhberg kindly sent to the author a preprint \cite{17}, where he has constructed an isomorphism between the categories of $\ell_p$-spaces and $L_p(X)$-spaces for arbitrary $X$. Apart from the independent value of this result, its proof was based on a construction that, as it happened, allowed us to dispense of the afore-mentioned additional condition on $X$. As a matter of fact, one can even consider tensor products of the so-called “stratified spaces”, more general than $L_p(X)$-spaces. But this is outside of the scope of this paper.

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2. $L$–spaces and $L$–boundedness.

As usual, we denote by $\mathcal{B}(E)$ the space of all bounded operators on a normed space $E$, endowed with the operator norm. Two projections $P$ and $Q$ on $E$ are called transversal, if $PQ = QP = 0$. The symbol $\otimes$ is used for the algebraic tensor product of linear spaces and linear operators, and also for elementary tensors.

Choose and fix (so far arbitrary) normed space $L$, which we shall call the base space.

Our principal example of a base space is $L_p(X)$, where $1 \leq p < \infty$, and $X$ is a measure space, which is not reduced to a finite set of atoms, or, equivalently, $L_p(X)$ is infinite-dimensional. To make our text shorter, we shall always assume that all our measures have a countable basis.

The amplification of a given linear space $E$ is the tensor product $L \otimes E$. Usually we shortly denote it by $LE$, and an elementary tensor, say $\xi \otimes x; \xi \in L, x \in E$, by $\xi x$. Note that $LE$ is a left module over the algebra $\mathcal{B}$ with the outer multiplication “$\cdot$”, well defined by $a \cdot (\xi x) := a(\xi)x$.

**Definition 2.1.** A norm on $LE$ is called $L$–norm on $E$, if the left $\mathcal{B}(L)$-
module $LE$ is contractive, that is if we always have the estimate $\|a \cdot u\| \leq \|a\|\|u\|$. This estimate, as well as an equivalent estimate $\|a \otimes 1_E\| \leq \|a\|$, will be called contractibility property. The space $E$, endowed by an $L$-norm, is called $L$-space. If we only know that we have the indicated estimate for operators of rank 1, we speak about near-$L$-norms and, accordingly, about near-$L$-spaces.

**Remark 2.2.** The class of near-$L$-spaces that we shall need in the proof of the main theorem, is bigger than the class of $L$-spaces. Let $L := L_p(X)$, $E$ be a normed space, and the norm on $LE$ is given by the identification of that space with the corresponding dense subspace in $L_p(X,E)$. Then we obtain a near-$L$-space but, generally speaking, not an $L$-space: the indicated estimate fails already in the case $p = 2, E = \ell_1$ (see [8, p.147]).

As to the papers, cited above, they consider, after the translation into the "index-free" language, the case $X := \mathbb{N}$ with the counting measure. In particular, the notion of an $L_p(X)$-space for such an $X$ is equivalent to that of a $p$-multinormed space in [6].

**Proposition 2.3.** Suppose that $E$ is a normed space, and a cross-norm $\|\cdot\|$ is given on $LE$. Then $\|\cdot\|$ is a near-$L$-norm on $E$ iff for all $f \in L^*$ we have $\|f \otimes 1_E\| \leq \|f\|$.

$\triangleright$ Every $a \in B(L)$ of rank 1 acts as $\eta \mapsto f(\zeta)\xi$ for some $\xi$ and $f$, so $\|a\| = \|f\|\|\xi\|$. It is easy to verify that $a \cdot u = \xi\left((f \otimes 1_E)u\right)$, for all $u \in LE$. Therefore the estimates $\|a \cdot u\| \leq \|a\|\|u\|$ and $\|(f \otimes 1_E)u\| \leq \|f\|\|u\|$ are equivalent. $\triangleright$

A near-$L$-space $E$ becomes a normed space in the "classical" sense, if for $x \in E$, we set $\|x\| := \|\xi x\|$, where $\xi \in L$ is an arbitrary vector with $\|\xi\| = 1$. Clearly, the result does not depend on the choice of $\xi$. The obtained normed space is called the underlying space of a given $L$-space, and the latter is called an $L$-quantization of a former. We use such a term by analogy with quantizations in operator space theory; see, e.g., [9], [10] or [12].

It is easy to verify that the complex plane $\mathbb{C}$ has the only $L$-quantization, given by the identification of $LC$ with $L$. However, as a rule, general normed spaces have a lot of $L$-quantizations. In particular, by endowing $LE$ with the norm of (non-completed) projective, respectively injective tensor product of normed spaces, we obtain two, generally speaking, different $L$-quantization, called maximal, respectively minimal. (See [13] for details.)

Suppose we are given an operator $\varphi : E \rightarrow F$ between linear spaces. Denote,
for brevity, the operator $1_L \otimes \varphi : LE \rightarrow LF$ (taking $\xi x$ to $\xi \varphi(x)$) by $\varphi_\infty$ and call it amplification of $\varphi$. Obviously, $\varphi_\infty$ is a morphism of left $B(L)$-modules.

**Definition 2.4.** An operator $\varphi : E \rightarrow F$ between $L$-spaces is called $L$-bounded or $L$-contractive, if the operator $\varphi_\infty$ is bounded or contractive, respectively.

As to numerous examples and counterexamples see, e.g., [13], and also [6].

To define amplifications of bilinear operators, we need a certain additional structure, called in what follows $\Diamond$-operation or “diamond operation” on $L$. This is a bilinear operator $\Diamond : L \times L \rightarrow L$ of norm one. We shall write $\xi \Diamond \eta$ instead of $\Diamond(\xi, \eta)$.

For “most” $X$, $L_p(X)$ has a natural, in a sense, diamond operation (see [13, §3], and also, in the case of a discrete measure, [15, §1.2.2]). But we emphasize that our main theorem is valid for arbitrary $\Diamond$.

Now let $\mathcal{R} : E \times F \rightarrow G$ be a bilinear operator between linear spaces. Its amplification is the bilinear operator $\mathcal{R}_\infty : LE \times LF \rightarrow LG$, well-defined (because of the bilinearity) on elementary tensors by $\mathcal{R}_\infty(\xi x, \eta y) = (\xi \Diamond \eta) \mathcal{R}(x, y)$.

**Definition 2.5.** A bilinear operator $\mathcal{R}$ between $L$-spaces is called $L$-bounded or $L$-contractive, if its amplification is (just) bounded, or contractive, respectively.

In the case $L = \ell_2$ and a particular $\Diamond$, taking sequences $\{\xi_n\}$ and $\{\eta_m\}$ into (arbitrarily renumerated) double sequence $\{\xi_n \eta_m\}; m, n \in \mathbb{N}$, we obtain, in equivalent terms, the definition of an $L$-bounded bilinear operator, given by Lambert. Again, see [13] for numerous examples.

3. $p$-convex tensor product and preliminaries of its existence

From now on, and up to the end of the paper, we assume that $L := L_p(X); p \in [1, \infty)$ (i.e. we are within the context of our main example of $L$), and that we fix an arbitrary $\Diamond$-operation on our base space.

Let $Y$ be a measurable subset in $X$. Consider the projection $P_Y \in B(L)$, acting as $f \mapsto f\chi$, where $\chi$ is a characteristic function of $Y$. A projection of that kind will be called proper. Clearly, two proper projections are transversal iff the intersection of the respective measurable subsets has measure 0.
Let $E$ be a linear space. We call a projection $P \in \mathcal{B}(L)$ a support of an element $u \in LE$, if $P \cdot u = u$.

**Definition 3.1.** A near-$L$–space $E$ is called $p$–convex, if for any $u, v \in LE$, with transversal proper supports, we have $\|u + v\| \leq (\|u\|^p + \|v\|^p)^{\frac{1}{p}}$.

The introduced class of $L$-spaces, being a generalization for arbitrary $p$ of column operator spaces, is, in our opinion, the most interesting. For the special case $L := \ell_p$, the given definition is equivalent to the definition of a $p$-convex $p$–multinormed space, given in [6]. Also it worth mentioning, in this connection, the theory of $p$–operator spaces of Daws [7]; see also earlier papers of Pisier [19] and Le Merdy [16].

As an example, one can easily show that every $L$-space with the minimal quantization is $p$-convex. Another example is provided by the near-$L$-space from Remark 2.2.

Now let $E$ and $F$ be two arbitrary chosen near-$L$–spaces.

**Definition 3.2.** A pair $(\Theta, \theta)$, consisting of a $p$-convex $L$-space $\Theta$ and an $L$–contractive bilinear operator $\theta : E \times F \to \Theta$, is called (non-completed) $p$-convex tensor product of $E$ and $F$ if, for every $p$-convex $L$-space $G$ and every $L$–bounded bilinear operator $\mathcal{R} : E \times F \to G$, there exists a unique $L$–bounded operator $R : \Theta \to G$ such that the diagram

$$
\begin{array}{c}
E \times F \\
\downarrow \theta \\
\Theta \\
\downarrow R \\
G
\end{array}
\quad
\begin{array}{c}
\quad \mathcal{R} \\
\quad R
\end{array}
$$

is commutative, and moreover we have $\|R_\infty\| = \|\mathcal{R}_\infty\|$.

In what follows, the property of the pair in question will be called the universal property.

We emphasize that $\Theta$ and $G$ are supposed (in comparison to $E$ and $F$) to be $L$-spaces, and not just near-$L$-spaces.

**Remark 3.3.** We see that the $L$–spaces $\Theta$ and $G$ are assumed to be $p$–convex. Other assumptions lead to other types of tensor products. For instance, if we shall take the class of all $L$-spaces, we shall come to an essentially different concept, the so-called general tensor product of our $E$ and $F$. This variety has its own existence theorem; this is Theorem 4.6 in [13]. Nevertheless $p$-convex tensor products, being in the case $p = 2$ intimately connected with the projective tensor products.
of operator spaces, discovered by Blecher/Paulsen [1] and Effros/Ruan [11], seem to be most interesting.

Thus, all notions that participate in the formulation of our main theorem, are explained, and we can proceed to its proof.

As it was mentioned in Introduction, this theorem earlier was proved under the additional assumption that $X$ either has no atoms or has an infinite set of atoms. Such a measure space we shall call convenient.

We recall the construction of our desired tensor product in the case of a convenient $X$. Take, as underlying linear space of $\Theta$, just $E \otimes F$, and as $\theta$ the canonical bilinear operator $\vartheta : (x, y) \mapsto x \otimes y$. So, our task is to introduce a suitable norm on $L(E \otimes F)$.

We first need an “extended” version of our fixed diamond operation, this time between elements of amplifications of linear spaces. Namely, for $u \in LE, v \in LF$ we consider the element $u \diamondsuit v := \vartheta_\infty(u, v) \in L(E \otimes F)$. In other words, this “diamond operation” is well defined by $\xi x \diamondsuit \eta y := (\xi \diamondsuit \eta)(x \otimes y)$, with $\xi, \eta \in L, x \in E, y \in F$.

An isometry on $L$ will be called proper, if its image is the image of a proper projection. Two isometries will be called disjoint, if the intersection of their images is $\{0\}$.

As is well known (in equivalent terms), if $X$ is convenient, then $L_p(X)$ possesses an infinite family of mutually disjoint proper isometries. See, e.g., [2, Cor. 9.12.18] and also [14, §14] or [21, III.A].

The following preparatory statement is crucial in our construction.

Proposition 3.4 ([13 Prop. 5.6]). Let $X$ be convenient. Then every $U \in L(E \otimes F)$ can be represented as

$$a \cdot \sum_{k=1}^n I_k \cdot (u_k \diamondsuit v_k),$$

where $a \in B(L), u_k \in LE, v_k \in LF$ and $I_k$ are pairwise disjoint proper isometries on $L$.

Now we have the right to take $U \in L(E \otimes F)$ and assign to it the number

$$\|U\|_{pL} := \inf \left\{ \|a\| \left( \sum_{k=1}^n \|u_k\|_p \|v_k\|_p \right)^{\frac{1}{p}} \right\},$$
where the infimum is taken over all possible representations of $U$ in the indicated form. It turns out that it is just what we need:

**Theorem 3.5.** (Theorem 5.18). The function $U \mapsto \|U\|_{pL}$ is a $L$-norm on $E \otimes F$, and the pair $(E \otimes_{pL} F, \vartheta)$, where $E \otimes_{pL} F$ denotes $E \otimes F$, endowed with the indicated $L$-norm, is a $p$-convex tensor product of $E$ and $F$.

**Remark 3.6.** It was assumed in the cited theorem that $E$ and $F$ are $L$-spaces, and the $\diamond$-operation has the property $\|\xi \diamond \eta\| = \|\xi\| \|\eta\|$. However one can easily notice that its proof uses the estimate $\|a \otimes 1_E\| \leq \|a\|; a \in B(L)$ from Definition 2.1 only for operators of rank 1, and only the property of $\diamond$ to have norm 1.

We proceed to the main contents of the present paper. How can one behave, if $X$ is not convenient, that is the set of its atoms is not empty and finite? It turns out that it is possible to reduce the “unconvenient” case to the “convenient” one.

For an arbitrary linear space, say $H$, let us consider the algebraic direct sum of a countable family of copies. So, it consists of eventually zero sequences $\bar{\xi} = (\xi_1, \xi_2, \ldots); \xi_k \in H$. If $H$ has a norm, we set $\|\bar{\xi}\| := (\sum_k \|\xi_k\|^p)^{\frac{1}{p}}$ and call the resulting normed space standard extension of $H$.

Now, for our fixed $X$, we denote by $NX$ the measure space which is the disjoint union of a countable family of copies of $X$: $NX := X_1 \sqcup X_2 \sqcup \cdots$. Clearly, $NX$ is convenient. Therefore the space $L_p(NX)$ satisfies, with $NX$ in the role of $X$, the conditions of Theorem 3.5.

Let $\mathbb{L}$ be the algebraic direct sum of a countable family of copies of $L$. Then we have the right to consider on the spaces $\mathbb{L}E$ and $\mathbb{L}F$ the norm of the standard extension of $LE$ and $LF$, respectively.

We do not know, whether an arbitrary $L$-space is also a $\mathbb{L}$-space with respect to the norm of the standard extension of the given $L$-norm; may be not. Nevertheless, the following fact is valid.

**Proposition 3.7.** If $E$ is a near-$L$-space, then it is also a near-$\mathbb{L}$-space.

$\triangleright$ It is easy to verify that the norm on $\mathbb{L}E$, as well as the norm on $LE$, is a cross-norm with respect to the norm of the underlying space of the given near-$L$-space. Therefore, by virtue of Proposition 2.3, it suffices to show that for every $f \in \mathbb{L}^*$ we have $\|f \otimes 1_E\| \leq \|f\|$. In what follows, we omit the easy case $p = 1$.

For $\xi \in L$ and $n \in \mathbb{N}$ we denote by $\bar{\xi}^n \in \mathbb{L}$ the sequence with the $n$-th term $\xi$ and all others zeroes. Introduce the functionals $f_n : L \rightarrow \mathbb{C} : \xi \mapsto f(\bar{\xi}^n)$. Fix, for a moment, $n$ and consider an element $\bar{u} = (u_1, u_2, \ldots) \in \mathbb{L}E$ with $u_n := \xi x$ for some
Clearly, it is a diamond operation on $L$, $x \in E$ and $u_m = 0$ for $m \neq n$. We see that $f \otimes 1_E(\bar{u}) = \sum_m (f_m \otimes 1_E)(u_m)$.

Since sums of such elements give the whole $LE$, the same equality is valid for all $\bar{u} \in LE$.

But since we know what is $E$, the same Proposition 2.3 gives $\|f_m \otimes 1_E\| \leq \|f_m\|$, for all $m$. Further, it is known (and easy to verify) that $\|f\| = (\sum_m \|f_m\|^q)^{1/q}$, $q$ is the number, conjugate to $p$. Therefore for every $\bar{v} = (v_1, \ldots) \in LE$ we obtain

$$\|(f \otimes 1_E)(\bar{v})\| \leq \sum_m \|f_m\| \|v_m\| \leq (\sum_m \|f_m\|^q)^{1/q} (\sum_m \|v_m\|^p)^{1/p} = \|f\| \|\bar{v}\|.$$  

The spaces $L$ and $\mathbb{L}$ are connected by the isometry $J : L \to \mathbb{L} : \xi \mapsto (\xi, 0, 0, \ldots)$ and the coisometry $Q : \mathbb{L} \to L : (\xi, \xi_2, \ldots, \xi_n, \ldots) \mapsto \xi$; of course, $QJ = 1_L$. For every linear space $G$ we shall write $J_G$ instead of $J \otimes 1_G : LG \to \mathbb{L}G$ and $Q_G$ instead of $Q \otimes 1_G : \mathbb{L}G \to LG$.

Our task is to construct a pair $(\Theta, \theta : E \times F \to \Theta)$, satisfying the conditions of Definition 3.1. We shall show that, similarly to Theorem 3.5, we can take $E \otimes F$ as the underlying linear space of $\Theta$, and the canonical bilinear operator as $\theta$.

Where to look for the desired norm on $L(E \otimes F)$?

Using the recipe of Proposition 3.7, we transform $E$ and $F$ into near-$\mathbb{L}$-spaces. Then we introduce the bilinear operator $\hat{\diamond} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ by

$$\hat{\xi} \hat{\diamond} \hat{\eta} := J(Q_E \hat{\xi} \otimes Q_F \hat{\eta}).$$

Clearly, it is a diamond operation on $\mathbb{L}$.

But since $NX$ (though, perhaps, not $X$) is convenient, there exists a $p$-convex tensor product of $E$ and $F$ as that of near-$\mathbb{L}$-spaces with respect to any diamond operation on $\mathbb{L}$; in particular, we choose $\hat{\diamond}$. Moreover, as the $\mathbb{L}$-space $\Theta$ we can take $E \otimes F$ with the respective norm on $\mathbb{L}(E \otimes F)$ that we shall denote by $\|\cdot\|_{pL}$.

Finally, we introduce a norm on $L(E \otimes F)$, induced by the injection $J_{E \otimes F}$. In other words, for $U \in L(E \otimes F)$ we set $\|U\|_{pL} := \|J_{E \otimes F}(U)\|_{pL}$.

This norm will turn out to be our desired $L$-norm on the desired tensor product. If there is no danger of confusion, we shall omit indices in the notation of the respective norms.

We must verify the needed requirements.

Take $a \in B(L), U \in L(E \otimes F)$ and set $\bar{a} := JaQ \in B(\mathbb{L}), \bar{U} := J_{E \otimes F}(U) \in \mathbb{L}(E \otimes F)$. Then

$$\|a \cdot U\| = \|(Ja \otimes 1_{E \otimes F})(U)\| = \|(JaG) \cdot (J_{E \otimes F}(U))\| = \|\bar{a} \cdot \bar{U}\| \leq \|\bar{a}\| \|\bar{U}\| = \|a\| \|U\|,$$
so that we have the contractibility property. If \( U_k \in L(E \otimes F); k = 1, 2 \) have transversal proper projections \( P_k \) in \( L \), then
\[
P_k := JP_kQ \text{ are transversal proper projections in } L,
\]
that are supports of \( J_{E \otimes F}U_k \). Therefore the \( p \)-convexity of \( E \otimes F \) as an \( L \)-space implies the \( p \)-convexity of \( E \otimes F \) as an \( L \)-space.

Finally, for \( u \in LE, v \in LF \) the equality \( J_{E \otimes F}(u \diamond v) = J_Eu \otimes J_Fv \) and \( \mathbb{L} \)-contractibility of \( \vartheta \) imply that \( \|u \diamond v\| \leq \|u\|\|v\| \), that is the desired \( \mathbb{L} \)-contractibility of \( \vartheta \).

Now the main thing remains: the universal property. In this connection, the following notion will be useful.

**Definition 3.8.** Let \( G \) be a \( p \)-convex \( \mathbb{L} \)-space and simultaneously a \( p \)-convex \( \mathbb{L} \)-space. Then the latter space is called an **inflation** of the former space, if \( J_G \) is an isometry, and \( Q_G \) is a coisometry.

For example, if \( G \) is the minimal \( \mathbb{L} \)-space, then it is easy to show that \( G \) as the minimal \( \mathbb{L} \)-space is an inflation of the former. As another example, suppose that \( G \) belongs to the class \( SQ_p \), i.e. it is a subspace of a quotient space of some \( L_p(Y) \). We make it an \( L \)-space and an \( \mathbb{L} \)-space by the identification of \( LG \) and \( \mathbb{L}G \) with the corresponding subspaces in \( L_p(X, G) \) and \( L_p(NX, G) \). Then the second space is an inflation of the first one. The required properties follow from Theorems 1.35 and 1.41 in [6].

In the following three propositions we suppose that \( G \) is a given \( L \)-space that has an inflation, and we fix the latter.

Let \( \mathcal{R} : E \times F \rightarrow G \) be a bilinear operator which is \( L \)-bounded as an operator between near-\( L \)-spaces with respect to our initial \( \diamond \)-operation. We denote by \( \mathcal{\hat{R}} \) the same bilinear operator as an operator between near-\( \mathbb{L} \)-spaces. Speaking about its \( \mathbb{L} \)-boundedness, we mean the diamond operation \( \hat{\circ} \), defined above.

**Proposition 3.9.** Our \( \mathcal{\hat{R}} \) is also \( \mathbb{L} \)-bounded, and we have \( \|\mathcal{\hat{R}}_\infty\| = \|\mathcal{R}_\infty\| \).

\( \triangleright \) The estimate \( \|\mathcal{\hat{R}}_\infty\| \leq \|\mathcal{R}_\infty\| \) follows from the formula
\[
\mathcal{\hat{R}}_\infty(\bar{u}, \bar{v}) = J_G\left(\mathcal{R}_\infty(Q_E\bar{u}, Q_F\bar{v})\right); \bar{u} \in \mathbb{L}E, \bar{v} \in \mathbb{L}F,
\]
which is an easy corollary of the definition of \( \hat{\circ} \). The inverse estimate follows from the formula
\[
\mathcal{R}_\infty(u, v) = Q_G\left(\mathcal{R}_\infty(J_Eu, J_Fv)\right),
\]
an easy corollary of the obvious equality \( \xi \hat{\circ} \eta = Q(J_\xi \hat{\circ} J_\eta) \). \( \triangleright \)
Consider the operators $R_\infty : L(E \otimes F) \to LG$ and $\bar{R}_\infty : L(E \otimes F) \to LG$. These are the amplifications of the operator $R : E \otimes F \to G$ that is associated with $\mathcal{R}$ and $\bar{\mathcal{R}}$, respectively.

**Proposition 3.10.** We have $\|R_\infty\| \leq \|\bar{R}_\infty\|$.

This estimate follows from the formula

$$R_\infty(U) = Q_G \left( \bar{R}_\infty(J_{E \otimes F}(U)) \right).$$

Obviously, one should only verify the latter equality on $U$ of the form $\xi(x \otimes y); \xi \in L, x \in E, y \in F$. Then

$$R_\infty(U) = (QJ\xi)\bar{R}(x \otimes y) = Q_G \left( \bar{R}_\infty(J\xi(x \otimes y)) \right) = Q_G \left( \bar{R}_\infty(J_{E \otimes F}(U)) \right).$$

**Proposition 3.11.** If $G$ is as above, then for an arbitrary $L$-bounded bilinear operator $\mathcal{R} : E \times F \to G$ and the associated linear operator $R : E \otimes F \to G$ we have $\|R_\infty\| = \|\mathcal{R}_\infty\|$.

Consider $G$ with the $L$-norm of the given inflation. Because of the universal property of the tensor product of our $E$ and $F$ as near-$L$-spaces, we have $\|\mathcal{R}_\infty\| = \|\bar{R}_\infty\|$. Combining this with two previous propositions, we obtain the estimate $\|R_\infty\| \leq \|\mathcal{R}_\infty\|$.

Further, it follows from the formula $u \diamond v = Q_{E \otimes F}(J_Eu \diamondsuit J_Fv)$, which can be easily verified on elementary tensors, that for all $u \in LE, v \in LF$ we have

$$\|\mathcal{R}_\infty(u, v)\| = \|R_\infty(u \diamond v)\| \leq \|R_\infty\| \|Q_{E \otimes F}(J_Eu \diamondsuit J_Fv)\| \leq \|R_\infty\| \|J_Eu \diamondsuit J_Fv\|.$$

But $\vartheta : (x, y) \mapsto x \otimes y$ is $L$-contractive with respect to the corresponding near-$L$-norms and the operation $\diamondsuit$. Therefore $\|J_Eu \diamondsuit J_Fv\| \leq \|J_Eu\| \|J_Fv\|$. Consequently we have $\|\mathcal{R}_\infty(u, v)\| \leq \|R_\infty\| \|u\| \|v\|$, that is $\|\mathcal{R}_\infty\| \leq \|R_\infty\|$.

4. **Existence of inflations and completion of the proof of the main theorem**

Thus, we see that for concluding the proof of the main theorem it suffices to know that every $L$-space has at least one inflation. This for some time we did not know. It is natural to begin with the testing of the standard extension of the given $L$-norm. However the existence of near-$L$-spaces that are not $L$-spaces (see Remark 2.2) makes one to have doubts; it seems to us that it does not fit.
Nevertheless, by virtue of a recent result of T. Oikhberg, mentioned in Introduction, one can show that inflations do always exist. Indeed it is easily seen that, for arbitrary measure spaces \(X\) and \(Y\) with infinite-dimensional separable \(L_p(X)\) and \(L_p(Y)\), his argument actually allows us to construct a certain \(L_p(Y)\)-norm on some \(G\), embarking from a given \(L_p(X)\)-norm on the same \(G\). We use his method for a proof of the following fact.

**Theorem 4.1.** Let \(X\) be as in the formulation of the main theorem. Then for \(L := L_p(X); p \in [1, \infty)\) every \(L\)-space has an inflation.

Before the proof, we note that we shall construct an inflation, which essentially differs from the standard extension of the given \(L\)-space; see the discussion above.

In what follows, if \(Z\) is a measurable subset of some measure space, say \(Y\), we shall denote its normalized in \(L_p(Y)\) characteristic function by \(\hat{\chi}(Z)\).

\[\ldots\]

\[\triangleleft\] We need the following preparatory statement.

**Lemma.** (Here we strictly follow the argument of Oikhberg). Let \(Y\) be an arbitrary measure space, \(\tilde{L}\) a subspace in \(L_p(Y)\), which is the linear span of several characteristic functions of measurable sets. Then there exists a projection on \(\tilde{L}\) in \(\mathcal{B}((L_p(Y)))\) of norm 1.

\[\triangleleft\] There exist disjoint subsets of non-zero measure \(Z_k\) in \(Y\), such that \(\tilde{L} = \text{span}\{\hat{\chi}(Z_k)\}\). Take in \(L_q(Y) = L_p(Y)^*\) (here \(q\) is the conjugate number to \(p\)) norm 1 functions \(\tilde{\xi}_k\), such that \(\tilde{\xi}_k = 0\) outside \(Z_k\), and \(\langle \tilde{\xi}_k, \hat{\chi}(Z_k) \rangle = 1\). Consider the operator \(P : L_p(Y) \to L_p(Y) : \eta \mapsto \sum_{k=1}^n \langle \tilde{\xi}_k, \eta \rangle \hat{\chi}(Z_k)\). Clearly, \(P\) is identical on \(\tilde{L}\). Further, since for \(\eta \in L_p(Y)\) we have \(P(\eta) = \sum_{k=1}^n \langle \tilde{\xi}(Z_k), \eta_k \rangle \hat{\chi}(Z_k)\), where \(\eta_k = \eta\) on \(Z_k\) and \(\eta_k = 0\) outside \(Z_k\), we easily see that \(\|P(\eta)\| \leq \|\eta\|\).

So, we are given, for \(L := L_p(X)\), an \(L\)-space \(G\). At first we want to introduce a certain norm on \(L^0G\), where \(L^0\) is a dense subspace in \(L\), consisting of simple functions.

Every \(u \in L^0G\) can be represented, for some family \(Y_k; r = 1, \ldots, n\) of pairwise disjoint subsets of non-zero measure in \(NX\), as \(\sum_{k=1}^n \hat{\chi}(Y_k)x_k, x_k \in G\). Take in \(X\) an arbitrary family \(Z_k\) of pairwise disjoint subsets of non-zero measure and set \(v := \sum_{k=1}^n \hat{\chi}(Z_k)x_k \in LG\). We put \(\|u\| := \|v\|\).

The subsequent argument consists of several natural stages.

1. The number \(\|u\|\) does not depend on the choice of the subsets \(Z_k\).

\[\triangleleft\] Let \(Z'_k\) be another family, and \(v' := \sum_{k=1}^n \hat{\chi}(Z'_k)x_k \in LG\). Consider the operator \(J : \text{span}\{\hat{\chi}(Z_k)\} \to \text{span}\{\hat{\chi}(Z'_k)\} : \hat{\chi}(Z_k) \mapsto \hat{\chi}(Z'_k);\) clearly, it is an
isometric isomorphism. By the preceding lemma, there exists a projection $P : L \to \text{span}\{\hat{\chi}(Z_k)\}$ of norm 1. Therefore $\|JP\| = 1$, so that the contractibility property for $L$-spaces implies $\|v\| = \|JPv\| \leq \|v\|$. A similar argument provides the inverse estimate. $\triangleright$

2. The number $\|u\|$ does not depend on the representation of $u$ as a sum of elementary tensors of the indicated form.

If we have another representation of our $u$, then, breaking the subsets, corresponding to both families, into the same disjoint unions and using the linear independence of the respective characteristic functions, we see that both representations lead to the same representation. To show that the resulting representation gives the same number as the initial one, it suffices, in its turn, to show that the number does not change after breaking one of the initial subsets into two disjoint subsets of non-zero measure, say, after breaking $Y_1$ into $Y'$ and $Y''$. Thus, the new representation has the form $\hat{\chi}(Y')z_1 + \hat{\chi}(Y'')z_2 + \sum_{k=2}^n \hat{\chi}(Y_k)x_k$ for some $z_1, z_2 \in G$. Since the tensor factors $\hat{\chi}(Y')$ and $\hat{\chi}(Y'')$ are linearly independent, it follows that $z_l = \lambda_l x_1; l = 1, 2$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$.

Recall the subsets $Z_k \subset X$. By stage 1, we can assume that we can break $Z_1$ into two disjoint sets of non-zero measure, say, $Z'$ and $Z''$. Therefore, if we consider the indicated new representation of our $u$, then the mentioned recipe gives the number $\|v''\|$, where $v'' := \lambda_1 \hat{\chi}(Z')x_1 + \lambda_2 \hat{\chi}(Z'')x_1 + \sum_{k=2}^n \hat{\chi}(Z_k)x_k$.

Obviously we have $\|\lambda_1 \hat{\chi}(Z') + \lambda_2 \hat{\chi}(Z'')\| = |\lambda_1|^p + |\lambda_2|^p = 1$. This easily implies that there exist operators $J_1, J_2$ of norm 1, acting on the space $\text{span}\{\hat{\chi}(Z'), \hat{\chi}(Z''), \hat{\chi}(Z_k); k = 2, \ldots, n\}$ and such that

$$J_1(\hat{\chi}(Z_1) = \lambda_1 \hat{\chi}(Z') + \lambda_2 \hat{\chi}(Z'') \quad \text{and} \quad J_2(\lambda_1 \hat{\chi}(Z') + \lambda_2 \hat{\chi}(Z'')) = \hat{\chi}(Z_1).$$

Further, the lemma provides a projection $P : L \to \text{span}\{\hat{\chi}(Z'), \hat{\chi}(Z''), \hat{\chi}(Z_k); k = 2, \ldots, n\}$ of norm 1. Therefore, the contractibility property for $G$ as for an $L$-space, being applied to the operators $J_1P$ and $J_2P$, gives $\|v\| = \|v''\|$. $\triangleright$

So, stages 1 and 2 together show that the number $\|u\|; u \in L^0 G$ is well defined.

3. The function $u \mapsto \|u\|$ is an $L^0$-norm on $G$.

$\triangleleft$ Obviously, this function is a norm on $L^0 G$. So, it remains to show that for $a \in B(L^0)$ and $u \in L^0 G$ we have $\|a \cdot u\| \leq \|a\| \|u\|$.

Let $u$ has its initial representation, and $a \cdot u$ is represented as $\sum_{i=1}^m \hat{\chi}(Y_i^1)y_i$, where $Y_i^1$ are some pairwise disjoint subsets in $NX$ of non-zero measure. Take
Consequently, \( \| \) that clearly coincides with \( \sum \) several characteristic functions of measurable sets, such that \( \leq \) seminorm. Let us show that for all \( \leq \) By the lemma, there is a projection of norm 1, say \( \leq \). Our lemma provides projections \( \leq \) \( \leq \leq \) \( \leq \). Further, the construction of the norm on \( \leq \) isomorphisms. Further, the construction of the norm on \( \leq \) disjoint subsets of non-zero measure in \( \leq \):

Thus, we have a well defined function \( \leq \leq \leq \leq \leq \leq \). But, since \( \leq \) are linearly independent, \( \leq \leq \leq \leq \leq \leq \). Consequently, \( \leq \) coincides with \( \leq \leq \leq \leq \). Therefore, \( \leq \leq \leq \leq \leq \). Thus, we have a well defined function \( \leq \leq \leq \leq \) on \( \leq \leq \leq \leq \leq \leq \leq \leq \), which is obviously a seminorm. Let us show that for all \( \leq \) \( \leq \leq \leq \leq \) \( \leq \leq \). At first suppose that \( \leq \leq \leq \leq \leq \). By the lemma, there is a projection of norm 1, say \( \leq \), of \( \leq \leq \leq \leq \leq \leq \) on a linear span of several characteristic functions of measurable sets, such that \( \leq \leq \leq \leq \leq \leq \leq \leq \leq \leq \leq \). Take \( \leq \leq \leq \). Since all \( \leq \) can be approximated by simple functions, the same lemma gives a projection of norm 1, say \( \leq \), on a linear span of several
characteristic functions of measurable sets, such that \( \|a \cdot \xi_k - Qa \cdot \xi_k\| < \epsilon \). Consider an operator on \( L \), acting as \( QaP \). Then stage 3 gives \( \|(QaP)u\| \leq \|QaP\|\|u\| \leq \|a\|\|u\| \). Hence \( \|a-u\| \leq \|a-u-(QaP)u\| + \|(QaP)\cdot u\| < \epsilon + \|a\|\|u\| \). This, of course, implies \( \|a\cdot u\| \leq \|a\|\|u\| \).

Finally, let \( u \) be an arbitrary element in \( LG \), and \( u = \sum_{k=1}^{N} \xi_k x_k \), where \( \xi_k \in L \), \( x_k \in G \). Since \( a \) is bounded, we see that for some sequence \( u_n \in L^0G \) we simultaneously have \( u_n \to u \) and \( a \cdot u_n \to a \cdot u; n \to \infty \). Therefore taking limit in the already obtained estimate \( \|a \cdot u_n\| \leq \|a\|\|u_n\| \) we have our desired estimate.

Now let us show that our seminorm is actually a norm.

Take \( u \neq 0 \) and represent it as \( \sum_{k=1}^{N} \xi_k x_k \), with linearly independent \( \xi_k \in L \) and \( x_1 \neq 0 \). There exist \( a \in B(L) \) such that \( a(\xi_1) = \xi_1 \) and \( a(\xi_k) = 0; k > 1 \), and also, for every \( k = 1, ..., N \) a sequence \( \xi_{k,n} \in L^0; n \in \mathbb{N} \), converging to \( \xi_k \). Set \( u_n := \sum_{k=1}^{N} \xi_{k,n} x_k \); we have \( \|a \cdot u_n\| \geq \|a(\xi_1) x_1\| - \sum_{k=2}^{N} \|a(\xi_{k,n}) x_k\| \). But clearly \( a(\xi_{1,n}) \) converges to \( \xi_1 \), and \( a(\xi_{k,n}) \) converges to 0 for other \( k \). Hence for sufficiently big \( n \) we have \( \|a \cdot u_n\| \geq \epsilon \) for some \( \epsilon > 0 \). Combining this with the estimation above, we obtain that \( \|u_n\| \geq \epsilon/\|a\| \). Therefore \( \|u\| > 0 \). \( \triangleright \)

5. The \( p \)-convexity is preserved by passing from \( L \)-spaces to \( L^0 \)-spaces.

\( \triangleright \) Suppose, at first, that \( u_1, u_2 \) with transversal supports lie in \( L^0G \). By definition of the norm in \( L^0G \), there exist a family \( Z_1^1, ..., Z_N^1, Z_1^2, ..., Z_M^2 \) of pairwise disjoint subsets of non-zero measure in \( X \) such that, for some \( v_1, v_2 \in LG \) of the form \( v_1 := \sum_{k=1}^{N} \hat{\chi}(Z_k^1) x_k \), \( v_2 := \sum_{k=1}^{N} \hat{\chi}(Z_k^2) y_k \), respectively, we have \( \|u_1\| = \|v_1\| \), \( \|u_2\| = \|v_2\| \) and \( \|u_1 + u_2\| = \|v_1 + v_2\| \). Since \( v_1, v_2 \) obviously have transversal supports, and our \( L \)-space is \( p \)-convex, we have \( \|u_1 + u_2\|^p \leq \|u_1\|^p + \|u_2\|^p \).

Now take arbitrary \( u_1, u_2 \in LG \) with transversal supports. Clearly for \( k = 1, 2 \) there exist a sequence \( u_k^n \in L^0G \) with the same support as \( u_k \), converging to \( u_k \). Then, passing to limits, we obtain the desired estimate. \( \triangleright \)

It is clear that all statements and arguments in stages 1-5 are valid, if we replace \( L := L_p(\mathbb{N}X) \) by \( L_p(Y) \) for an arbitrary measure space \( Y \). Now we concentrate on our concrete situation.

**End of the proof.** It remains to show that \( J_G \) is an isometry, and \( Q_G \) is a coisometry.

\( \triangleright \) Take at first \( u \in L^0G \) and represent it as \( \sum_{k=1}^{N} \hat{\chi}(Y_k) x_k \) with pairwise disjoint \( Y_k \subset X \). Then \( J_G(u) \), as an element of the subspace \( L^0G \) of \( LG \) has the same representation, only now we must consider \( Y_k \) as subsets in the first summand \( X_1 \)},
in \(\mathbb{N}X = X_1 \sqcup X_2 \sqcup \ldots\). Therefore, calculating \(\|J_G(u)\|\) by the prescribed recipe, we can take as \(Z_k\) the initial \(Y_k\), and the same \(x_k\). But then the respective \(v\) is just \(u\), therefore \(\|J_G(u)\| = \|u\|\). Thus, the restriction of \(J_G : LG \to \mathbb{L}G\) on a dense subspace in \(LG\) is an isometry, so the same is true for \(J_G\).

Turn to \(Q_G\). Since we have \(Q_GJ_Gu = u\) for all \(u \in LG\), and \(J_G\) is an isometry, we only have to show that \(Q_G\) is contractive.

Take \(\bar{u} \in \mathbb{L}\) and observe that \(J_GQ_G\bar{u} = (JQ)\bar{u}\). Therefore \(\|Q_G\bar{u}\| = \|J_GQ_G\bar{u}\| \leq \|JQ\|\|\bar{u}\| = \|\bar{u}\|\).

The proof of Theorem 4.1 is concluded. ⊳

Combining this theorem with Proposition 3.11, we obtain our main theorem.

5. \(p\)-convex tensor product as a functor

Now let us do some final observation. Recall an important notion in geometry of normed spaces. Suppose that we assign to every pair, say \(E,F\), of normed spaces the space \(E \otimes F\) endowed with some norm. The most interesting are assignments, satisfying the so-called metric mapping property [[8, §12]] (see also “uniform cross-norms” in [20, §6.1]): for every bounded operators \(\varphi : E_1 \to F_1, \psi : E_2 \to F_2\) we have \(\|\varphi \otimes \psi\| \leq \|\varphi\|\|\psi\|\). (In other terms, such an assignment, extended to bounded operators, is a bifunctor on the category of normed spaces and contractive operators.) We shall show that the \(p\)-convex tensor product has a natural analogue of that “functorial property” for near-\(L\)-spaces. As usual, \(L := L_p(X)\), and the only condition on \(X\) is that \(L\) is infinite-dimensional and separable.

**Proposition 5.1.** Let \(\varphi : E_1 \to E_2, \psi : F_1 \to F_2\) be \(L\)-bounded operators between near-\(L\)-spaces. Then the operator \(\varphi \otimes \psi : E_1 \otimes_p L E_2 \to F_1 \otimes_p L F_2\) is \(L\)-bounded, and we have \(\|(\varphi \otimes \psi)_{\infty}\| \leq \|\varphi_{\infty}\|\|\psi_{\infty}\|\).

⊳ We first assume that \(X\) is convenient. Then, taking \(U \in L(E_1 \otimes E_2)\), we have the right to present it as \(a \cdot \sum_k I_k \cdot u_k \otimes v_k; a \in B(L), u_k \in LE_1, v_k \in LF_1\) with mutually disjoint proper isometries \(I_k\) (see Proposition 3.4). Then, since amplifications of our operators are morphisms of left \(B(L)\)-modules, we have \((\varphi \otimes \psi)_{\infty}(U) = a \cdot \sum_k I_k \cdot (\varphi \otimes \psi)_{\infty}(u_k \otimes v_k)\). By the formula \((\varphi \otimes \psi)_{\infty}(u \otimes v) = \varphi_{\infty}(u) \otimes \psi_{\infty}(v)\), easily verified on elementary tensors, the latter expression is \(a \cdot \sum_k I_k \cdot (\varphi_{\infty}(u_k) \otimes \psi_{\infty}(v_k))\). Therefore, by definition of the norm on \(F_1 \otimes_p L F_2\) for
convenient $X$, we have
\[
\|(\varphi \otimes \psi)_\infty(U)\| \leq \|a\| \left( \sum_k \|\varphi_\infty(u_k)\|^p \|\psi_\infty(v_k)\|^p \right)^{\frac{1}{p}} \leq \|a\| \left( \sum_k \|u_k\|^p \|v_k\|^p \right)^{\frac{1}{p}}.
\]

It remains to take the respective infimum over all representations of $U$ in the prescribed form.

Turn to an arbitrary $X$. Using the standard extension of given $L$-norms, consider our four spaces as near-$L$-spaces (see Proposition 3.7). Denote our given operators as acting between near-$L$-spaces as $\tilde{\varphi}$ and $\tilde{\psi}$, respectively. Thus, since $L = L_p(NX)$, and $NX$ is convenient, we have $\|(\tilde{\varphi} \otimes \tilde{\psi})_\infty\| \leq \|\tilde{\varphi}_\infty\| \|\tilde{\psi}_\infty\|$. But, using the definition of near-$L$-norms on our four spaces, we easily obtain that $\|\tilde{\varphi}_\infty\| = \|\varphi_\infty\|$ and $\|\tilde{\psi}_\infty\| = \|\psi_\infty\|$. At the same time for $U \in L(E_1 \otimes_{pl} E_2)$ we have
\[
\|(\varphi \otimes \psi)_\infty(U)\| = \|J_{F_1 \otimes F_2}((\varphi \otimes \psi)_\infty(U))\| = \|((\varphi \otimes \psi)_\infty(U), 0, 0, \ldots)\| = \|(\tilde{\varphi} \otimes \tilde{\psi})_\infty(U, 0, 0, \ldots)\| = \|((\tilde{\varphi} \otimes \tilde{\psi})_\infty(U, 0, 0, \ldots)\| = \|((\tilde{\varphi} \otimes \tilde{\psi})_\infty(U, 0, 0, \ldots)\| = \|((\tilde{\varphi} \otimes \tilde{\psi})_\infty(U, 0, 0, \ldots)\|,
\]
and consequently $\|(\varphi \otimes \psi)_\infty\| \leq \|(\tilde{\varphi} \otimes \tilde{\psi})_\infty\|$. The desired estimate immediately follows. ⊲

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A. Ya. Helemskii
Faculty of Mechanics and Mathematics
Moscow State (Lomonosov) University
Moscow 119991 Leninskie Gory
E-mail: helemskii@rambler.ru