Comments on the temperature dependence of the gauge topology

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Recent efforts in lattice evaluation of the topological susceptibility had shown that at high temperatures it is given by well-separated instantons (even in QCD with light fermions, where those are highly suppressed). Recent development of the semiclassical theory suggest that below $T_{\text{max}} \sim 2.5T_c$, where Polyakov line has values between one and zero, the topology ensemble can be represented by a plasma of instanton constituents (called instanton-dyons or instanton-monopoles). It has been shown that such ensemble undergoes deconfinement and chiral transitions, semi-qualitatively reproducing the lattice results. There are ongoing efforts to locate them on the lattice, or use (flavor-dependent) periodicity phases of the deformed versions of QCD on the lattice and semiclassically, in order to test this theory. We here propose another possibly useful tool: the topological susceptibility of a sub-lattice.

I. INTRODUCTION

Recently the field of lattice gauge topology has been re-activated, due to two independent developments.

One is several recent extensive lattice studies of the topological susceptibility $\chi(T)$ in a wide range of temperatures $T$, from zero to about 2 GeV. (Its motivation is partly a relation to axion models of the dark matter.) In Fig.1 (upper) from [3] one can see the continuum extrapolated value of $\chi^{1/4}$ versus $T/T_c$ (the lower red shaded region) compared for $T/T_c > 2.5$ with the dilute instanton gas approximation (DIGA). The upper gray region corresponds to the results of ref. [1]. Not shown in this plot are results from the work [2], which impressively followed $\chi(T)$ to $T$ as large as 2 GeV, also with the conclusion that DIGA is correct at high enough $T$. Ref. [4] also had measured higher moment of the topological charge fluctuations, $b_2$. The data from this work shown in the lower part of Fig.1 also show that for $T/T_c > 2.5$ the DIGA value – following from $E_{\text{vacuum}} \sim \cos(\theta)$ – seem to be reached. Accepting these conclusions, we discuss why the behavior changes below this temperature, and what is the correct description of the topology below it.

Another recent development are works devoted to of the ensemble of the instanton constituents, called instanton-monopoles or instanton-dyons. As shown in the pioneering papers by Kraan and van Baal [3] and Lee and Lu Lee:1998bh, an instanton consists of $N_c$ (number of colors) of those. They share unit topological charge of the instanton according to certain fractions $\nu_i, i = 1...N_c$ such that $\sum_i \nu_i = 1$.

The main rational for the instanton to get disassembled into those constituents is the fact that the mean Polyakov line below certain $T$ deviates from 1, forcing all objects to interpolate to a nonzero ‘holonomy values’ of $A_0$. In QCD with physical quarks this happens at $T_{\text{Polyakov}} \approx 2.5T_c$ [9].

We suggest that deviation of the DIGA from susceptibilities below this temperature is not a mere coincidence, and that at $T < 2.5T_c$ the instantons are disassembled into instanton-dyons. In these comments we discuss how different versions of the topological susceptibility can help us to tell if this is indeed the case.

![Graph](image.png)

FIG. 1: (Upper) The topological susceptibility $\chi^{1/4}$ versus the temperature $T/T_c$, from [3] (Lower) The higher order fluctuation parameter $b_2$, from [1].

II. VARIOUS DEFINITIONS OF THE TOPOLOGICAL SUSCEPTIBILITY

In order to see how those theoretical ideas can match lattice measurements, one needs to clarify and distinguish various existing definition of $\chi$. As we will see,
few existing definitions are not at all identical, and should lead to very different results.

We start with the canonical definition of the topological susceptibility $\chi_{\text{canonical}}$, following standard route of the statistical mechanics. The vacuum or thermal ensemble with nonzero theta-angle $\theta$ is defined by an additional term in action, which adds to the partition function an extra factor $e^{iQ}$, where $Q$ is the total topological charge of the volume $V_4$ under consideration. Since $i\theta$ play the same role as a chemical potential, the topological susceptibility can thus be given the same standard definition as any other susceptibility, namely

$$
\chi_{\text{canonical}} = \left( \frac{1}{V_4 Z} \right) \left( \frac{\partial^2}{\partial (i\theta)^2} Z(\theta) \right)
$$

(1)

The volume $V_4$ in this definition should be the one of a subsystem, of even much larger heat bath. Grand canonical ensemble with the chemical potential implies that there is free exchange of particles through the boundaries of the subsystem. Large heat bath ensures that the values of variables like $T$ and $\mu = i\theta$ are fixed, without any fluctuations.

The standard lattice definition

$$
\chi_{\text{lat}} = \frac{\langle Q^2 \rangle}{V_4}
$$

(2)

may look identical to the canonical one given above, but is in fact quite different, due to the fact that in this case the system with $V_4$ is the whole lattice. It is topologically a torus with no boundaries, since periodic boundary conditions of the fields are imposed. Both electric $Q_E$ and magnetic $Q_M$ charge of this volume must be zero, and the topological charge $Q$ must be integer-valued.

Another definition $\chi_{\text{sublat}}$ has been proposed by Verbaaschot and myself [8]. Since it was done many years ago, let us remind it. We proposed to cut the lattice into two subsystems, $a$ and $b$, with subvolumes $V_4^a + V_4^b = V_4$ and define the corresponding susceptibilities by the same expression above. The simplest arrangement is to cut by two planes normal to one of the coordinates, producing two “slices” with

$$
V_4^a = L^3 x, V_4^b = L^3 \bar{x}, \bar{x} = L - x
$$

(3)

This definition needs some extra work, but it has two important advantages over the $\chi_{\text{lat}}$. One is that now the sub-volumes do have a boundary, and they do not have a requirement that $Q_E = Q_M = 0$. As we will discuss below, quarks and Dirac strings can “leak” through it.

Note also, that in this setting one obtains not a number but the function $\chi^a(x)$, which can be used to define the “screening length of the topological charge”, known also as the $\eta'$ mass. In this case one gets an idea what is a “large enough box”, since for $m(\eta')x \gg 1$ the dependence on $x$ disappears.

![Graph](image)

**FIG. 2:** An example of the susceptibility in sub-box $\chi_{\text{sublat}}$ as a function of the fraction of the total box, $x/L$. The thin parabolic line corresponds to randomly placed instantons and antiinstantons, the dots are for the interacting instanton liquid, from [8]. Strong screening of the topological charge in this model is evident. Thin lines show different fits, from which the value of $m_{\eta'}$ was extracted.

### III. INSTANTONS AND THE HIGH-T REGION

We start discussing the differences between various definitions of $\chi$ using the context of the instanton ensemble, in which $\chi_{\text{sublat}}$ was originally introduced.

Let us start with QCD in the chiral limit, with (massless) quarks. In this case any configuration with nonzero topological charge $Q$ has $Q$ quark zero modes. Therefore, the fermionic determinant is zero if $Q \neq 0$, so the gauge ensemble include only configurations with $Q = 0$ and thus $\chi_{\text{lat}} = 0$.

Let us first, for simplicity, focus on $T > T_c$, where there is no quark condensate and the chiral symmetry remains unbroken. In this case the topological objects can exist only as some clusters with the total topological charge $Q = 0$. The simplest of those are the instanton-antiinstanton molecules. The ensemble made of those has been discussed by Ilgenfritz and myself [11]. We do not discuss them here as they are not relevant for topological susceptibility.

While $\chi_{\text{lat}} = 0$, the sublattice definition would lead to a non-zero value $\chi_{\text{sublat}} \neq 0$, because the instantons and the antiinstantons may happen to be located in different subvolumes, see Fig.2. The quarks, created by $I$ and absorbed by $\bar{I}$ may “leak” through the boundary!

Note, that for the particular geometry of sub-box proposed, by moving a plane and changing $x$ one changes the sub-volumes $V_4^a, V_4^b$ but not the area of the surface $A_3$ separating them. Since the leakage is expected to be proportional to this area, $\chi_{\text{sublat}} \sim A_3$, not volume, it should become $x$-independent at large $x$.

Suppose now we allow small but non-zero quark masses $m$ (for simplicity, the same for $N_f$ quark flavors). Quark “veto” on $Q \neq 0$ configurations such as individual in-
stantons is now lifted. Since in a dilute ensemble the instantons can be considered to be non-interacting, one should use the Poisson distribution, and therefore

\[ \frac{\chi(T)}{T^4} \sim \frac{n(T)}{T^4} \sim \left( \frac{\Lambda}{T} \right)^{b} \prod_{i=1}^{N_f} \left( \frac{m_f}{T} \right) \]

(4)

where \( b = 11N_c/3 + 2N_f/3 \). So, the high-T limit corresponds to very small \( \chi_{\text{lat}} \), decreasing as relatively large power of \( T \), times a rather small product of quark masses. (We will discuss \( SU(2) \) gauge theory, \( SU(3) \) gauge theory and QCD with 3 dynamical quarks: the (inverse) powers of the temperature in those cases are \( 22/3 = 7.66 \), 11 and 6, respectively.)

![Figure 3: Some configurations which produce no contributions to the \( \chi_{\text{lat}} \), but contribute to \( \chi_{\text{sublat}} \)](image)

**IV. THE INSTANTON-DYONS**

Semiclassical theory of instantons, incorporating a nonzero Polyakov line VEV and thus a nonzero mean value of the gauge field \( < A_4 > \neq 0 \), lead in 1998 to the discovery of the instanton-dyons \([4, 5]\). It is nearly two decades since these papers, but only recently a heavy work on building a semiclassical theory of their ensemble was intensified. Last year alone has produced about a dozen papers on that. Those will not be discussed below, for a brief review of some of them see \([6]\).

When the mean Polyakov line \( < P > \) is between 1 to 0, gauge topology is expected to be described by an ensemble of the instanton-dyons, with different temperature-dependent actions and non-integer \([15]\) topological charges, driven by \( < P(T) > \).

In the simplest case of the \( SU(2) \) gauge theory there are two types of dyons, selfdual \( M \) with \( S_M / S_0 = \nu \) and \( L \) with \( S_L / S_0 = \bar{\nu} \), plus anti-selfdual anti-dyons. The parameter \( \nu \) is related to the Polyakov line by \( < P > = \cos(\pi \nu) \). In the \( SU(2) \) gauge theory there are two \( M \)-type dyons, related to complex-conjugated eigenvalues of \( < P > \). In this case \( \bar{\nu} = 1 - 2\nu \) and \( < P > = (1/3) + (2/3)\cos(2\pi \nu) \).

Plugging in the lattice input \( < P(T) > \) one can plot the dyon action: see example in Fig.4 (for QCD). This plot can be used to identify the region in which the instanton-dyon theory is semiclassical. The semiclassical density depends on the action by \( n \sim S^2 \exp(-S) \): the power of the action represent half of bosonic zero modes. This formula has a maximum at \( S = 2 \), and we take it as the lowest possible action at which it makes sense. The left side of Fig.4 indicate the lowest temperature at which this condition is fulfilled \( T_{\text{min}} \sim 80 \text{ MeV} \). The right side – high \( T \) – shows that while the action of the \( L \) dyon grows, that of \( M \) decreases, so \( T_{\text{max}} \sim 370 \text{ MeV} \). The phase transition is indicated as a transition from a symmetric to asymmetric phase. These considerations of course refer to simple non-interacting dyons. We use them simply to convey the range of \( T_{\text{min}} < T < T_{\text{max}} \) in which this approach is expected to work.

The semiclassical formulae for the density of instanton-dyons are higher than instantons, because they have smaller actions. This is the generic reason why the \( \chi(T) \) at \( T < 2.5 T_c \) gets larger than the DIGA prediction. Another generic reason is that \( M \)-type dyons have no quark zero modes and thus are not suppressed by fermion masses.

Proper studies of the dyon ensemble – such as \([10]\) from which we borrowed Fig.6 – include their mutual interaction as well as back reaction to the holonomy potential, determining its value \( \nu \) from the global minimum of the

![Figure 5: Some configurations which produce very different contributions to the \( \chi_{\text{lat}} \) and \( \chi_{\text{sublat}} \)](image)
free energy. As one can see from this figure, there is no symmetric phase, and there are always more \( M \) dyons than \( L \). Also the deconfinement and chiral transitions become in QCD-like theories just a smooth cross-overs, happening at roughly the same temperature.

![Diagram showing the densities of \( L \) and \( M \) type dyons, versus the action \( S = 8\pi^2/g^2(T) \), in \( SU(2) \) QCD with two light quark flavors. (Lower plot) The mean Polyakov line \( < P > \) and quark condensate \( \Sigma \), versus the same variable \( S \).](image)

Now let us go back to the topological susceptibilities. Suppose first there are no fermions in the theory. Can sub-lattices have non-integer topological charges? Yes: the Dirac string can penetrate through the boundary and nothing prevents a configuration shown in Fig. 5 (right). So, \( \chi_{\text{sublatt}} \) would obtain contributions with non-integer values of \( Q_{a} \), \( Q_{b} \).

This is in sharp contrast to \( \chi_{\text{lat}} \), since lattice configurations can only have integer values of \( Q \). At this point, one always ask why in the dyon theory any lattice configurations have integer \( Q \). It is because the lattice, unlike the sublattice, must have zero total magnetic charge \( Q_{\text{magnetic}} = 0 \).

This simple discussion nicely illustrates the drastic difference between the topology on the lattice and the sublattices: the former are not canonical but in some way share properties of the microcanonical ensembles, with fixed charges.

Let us no return to QCD, switching on the light quarks. The crucial observation is that only twisted \( L \)-type dyon has physical – anti-periodic – quark zero modes. Therefore, if quarks are massless, those can only exist inside “neutral clusters”, such as \( LL \) molecules. We however will not discuss the molecular component here, as any topologically neutral objects are irrelevant for the topological susceptibility.

V. THE TOPOLOGICAL SCREENING AND THE \( \eta' \) MASS

At \( T < T_c \) the chiral symmetry is broken. How exactly it happens from the point of view of topological object has been worked out in the instanton liquid model, see [7] for a review. The nature of the topological objects involved –an instanton or only its \( L \)-type constituent – is unimportant: any one with a fermionic zero mode is generating the corresponding \( \lambda \) Hooft vertex. As \( T \to T_c \) quarks travel through longer and longer chains of alternating topological objects. The length of a chain scales as \( V_4 \), not as its dimension \( (V_4)^{1/4} \), which in the thermodynamical limit become infinite. That is why Dirac eigenvalues reach zero and quark condensate is formed. As a result, pions get massless and one can use chiral perturbation theory to describe \( \chi(T) \) at \( T < T_c \). This is all well known and we do not need to describe it.

There are however some issues related to \( \chi_{\text{sublatt}} \) and the topological screening length \( m_{\eta'} \), we would like to comment on. Let us start with the following (well known) puzzle: its numerical value \( 1/m_{\eta'} = 1.958 \text{GeV} \sim 0.2 \text{fm} \) is several times smaller compared to the typical distance between the topological objects, e.g. \( L \)-dyons at \( T_{\text{min}} \), which is about \( 1 \text{fm} \). One may wonder if indeed the quark-induced interaction can generate so strong correlations inside such chains. The calculation for dyons are in progress, and so we can only mention that it indeed worked out in the instanton liquid, even in its simplest form, see \( \chi_{\text{sublatt}} \) already shown in Fig. 2.

Another comment refers to the limit of large number of colors \( N_c \to \infty \). As famously noted by Witten [12], in this limit the \( \eta' \) is expected to be light,

\[
m_{\eta'}^2 \sim 1/N_c \to 0
\]  

(5)

One should also recall that the so called compressibility of the instanton ensembles, the fluctuations of \( N(V) = (g^2/32\pi^2) \int d^4 x G_{\mu\nu}^2(x) \), satisfies the following low energy theorem

\[
<N(V)^2> - <N(V)>^2 = \frac{4}{b} <N(V)> 
\]  

(6)

where \( b = 11N_c/3 + 2N_f/3 \). For \( N_c \to \infty \) the r.h.s. vanishes, which means the quantity \( N(V) \) has in this limit no fluctuations. This in turn implies, that the isoscalar scalar meson \( \sigma \) must become heavy.

In the real world QCD with \( N_c = 3 \) these two masses have the opposite relation,

\[
m_{\eta'} \approx 958 \text{MeV} > m_{\sigma} \approx 500 \text{MeV}
\]  

(7)
but at some $N_c$ they should become equal, and then continue to move, up and down. According to instanton liquid study by Schaefer, the $m(\eta')$ does indeed decreases with $N_c$ as in (3). What happens with $m_{\omega}(N_c)$ remains unknown. Lattice studies of these issues would be of significant interest.

Witten [12] and Veneziano [13] famously related the topological susceptibility to the $\eta'$ mass. However, their argument is for $\chi$ in the limit of infinite number of colors, not in physical QCD [16]. A similar expression for derived in [8] is based on $\chi_{\text{sublat}}(x)$ in physical QCD and, ironically, corresponds to the limit of small (rather then large) volume limit.

VI. SUMMARY

These comments can finally be summarized in a sketch shown in Fig. 7 below $T \sim T_{\text{max}} \sim 2.5T_c$ a dilute instanton gas changes to an ensemble of instanton-dyons. In this lower region $\chi_{\text{lat}} \neq \chi_{\text{sublat}}$, they have different $T$-dependences. If it can be evaluated on the lattice, it will perhaps reveal the dis-assembly of instantons into the constituents, with non-integer topological charges, directly.

FIG. 7: A sketch indicating different forms of the topological ensembles as a function of the temperature.