REFINED UPPER BOUNDS FOR THE LINEAR DIOPHANTINE PROBLEM OF FROBENIUS

MATTHIAS BECK AND SHELEMYAHU ZACKS

Abstract. We study the Frobenius problem: given relatively prime positive integers $a_1, \ldots, a_d$, find the largest value of $t$ (the Frobenius number $g(a_1, \ldots, a_d)$) such that $\sum_{k=1}^{d} m_k a_k = t$ has no solution in nonnegative integers $m_1, \ldots, m_d$. We introduce a method to compute upper bounds for $g(a_1, a_2, a_3)$, which seem to grow considerably slower than previously known bounds. Our computations are based on a formula for the restricted partition function, which involves Dedekind-Rademacher sums, and the reciprocity law for these sums.

1. Introduction

Given positive integers $a_1 < a_2 < \cdots < a_d$ with $\gcd(a_1, \ldots, a_d) = 1$, the linear Diophantine problem of Frobenius asks for the largest integer $t$ for which we cannot find nonnegative integers $m_1, \ldots, m_d$ such that

$$t = m_1 a_1 + \cdots + m_d a_d .$$

We call this largest integer the Frobenius number $g(a_1, \ldots, a_d)$; its study was initiated in the 19th century. One fact which makes this problem attractive is that it can be easily described, for example, in terms of coins of denominations $a_1, \ldots, a_d$; the Frobenius number is the largest amount of money which cannot be formed using these coins. For $d = 2$, it is well known (most probably at least since Sylvester [?]) that

$$(1) \quad g(a_1, a_2) = a_1 a_2 - a_1 - a_2 .$$

For $d > 2$, all attempts to find explicit formulas have proved elusive. Two excellent survey papers on the Frobenius problem are [?] and [?].

Our goal is to establish upper bounds for $g(a_1, \ldots, a_d)$. The literature on such bounds is vast; it includes results by Erdős and Graham [?]

$$(2) \quad g(a_1, \ldots, a_d) \leq 2a_d \left\lfloor \frac{a_1}{d} \right\rfloor - a_1 ,$$

Selmer [?]

$$(3) \quad g(a_1, \ldots, a_d) \leq 2a_{d-1} \left\lfloor \frac{a_d}{d} \right\rfloor - a_d ,$$

and Vitek [?]

$$(4) \quad g(a_1, \ldots, a_d) \leq \left\lfloor \frac{1}{2}(a_2 - 1)(a_d - 2) \right\rfloor - 1 .$$

2000 Mathematics Subject Classification. 11D04, 05A15, 11Y16.

Key words and phrases. The linear Diophantine problem of Frobenius, upper bounds, Dedekind-Rademacher sums, reciprocity laws.
Here \(a_1 < a_2 < \cdots < a_d\), and \(|x|\) denotes the greatest integer not exceeding \(x\). Davison [?] established the lower bound

\[
g(a_1, a_2, a_3) \geq \sqrt{3a_1a_2a_3} - a_1 - a_2 - a_3.
\]

Experimental data [?] shows that Davison’s bound is sharp in the sense that it is very often very close to \(g(a_1, a_2, a_3)\). On the other hand, the upper bounds given by (2), (3), and (4) seem to be quite large compared to the actual Frobenius numbers. In this paper, we derive a method of achieving sharper upper bounds for the Frobenius number. Our results are based on a formula for the restricted partition function (Section 2), which involves Dedekind-Rademacher sums, and the reciprocity law for these sums (Section 3). The main result is derived in Section 4; computations which illustrate our new bounds can be found in Section 5.

We focus on the first non-trivial case \(d = 3\); any bound for this case yields a general bound, as one can easily see that \(g(a_1, \ldots, a_d) \leq g(a_1, a_2, a_3)\) if \(a_1, a_2,\) and \(a_3\) are relatively prime. If not then we can reduce by one variable at a time: Again by the definition of the Frobenius number, \(g(a_1, \ldots, a_d) \leq g(a_1, \ldots, a_{d-1})\) if \(a_1, \ldots, a_{d-1}\) are relatively prime. If not, we can use a formula of Brauer and Shockely [?] if \(n = \gcd(a_1, \ldots, a_{d-1})\) then

\[
g(a_1, \ldots, a_d) = n \ g\left(\frac{a_1}{n}, \ldots, \frac{a_{d-1}}{n}, a_d\right) + (n-1) a_d.
\]

Hence

\[
g(a_1, \ldots, a_d) \leq n \ g\left(\frac{a_1}{n}, \ldots, \frac{a_{d-1}}{n}\right) + (n-1) a_d.
\]

2. The restricted partition function

We approach the Frobenius problem through the study of the restricted partition function

\[
p_{\{a_1, \ldots, a_d\}}(n) = \# \left\{ (m_1, \ldots, m_d) \in \mathbb{Z}_{\geq 0}^d : m_1 a_1 + \cdots + m_d a_d = n \right\},
\]

the number of partitions of \(n\) using only \(a_1, \ldots, a_d\) as parts. In view of this function, the Frobenius number \(g(a_1, \ldots, a_d)\) is the largest integer \(n\) such that \(p_{\{a_1, \ldots, a_d\}}(n) = 0\).

In the \(d = 3\) case, we can additionally assume that \(a = a_1, b = a_2,\) and \(c = a_3\) are pairwise relatively prime, a simplification due to Johnson’s formula [?]: if \(n = \gcd(a, b)\) then

\[
g(a, b, c) = n \ g\left(\frac{a}{n}, \frac{b}{n}, c\right) + (n-1) c.
\]

(This identity is a special case of [10].)

In the case that \(a, b, c\) are pairwise relatively prime, Beck, Diaz, and Robins derived the following result for the partition function \(p_{\{a,b,c\}}\) [?, Theorem 3]:

\[
\begin{align*}
 p_{\{a,b,c\}}(n) = & \frac{n^2}{2abc} + \frac{n}{2} \left( \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} \right) + \frac{1}{12} \left( \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \right) \\
 & + S_n(b,c; a) + S_n(c,a; b) + S_n(a,b; c).
\end{align*}
\]
Here [?, Equation (14)]

\[(8)\]

\[S_t(a,b;c) = c - 1 + \sum_{m=0}^{c-1} \left( -a^{-1} \left( \frac{bm + t}{c} \right) \right) \left( \frac{m}{c} \right),\]

where \(aa^{-1} \equiv 1 \mod c\) and \((x) = x - \lfloor x \rfloor - 1/2\), is a special case of a Dedekind-Rademacher sum; we will discuss these sums in the next section.

To bound the Frobenius number (from above), we need to bound \(P_{t(a,b,c)}\) (from below), whose only nontrivial ingredients are the Dedekind-Rademacher sums. A classical bound for the Dedekind-Rademacher sum yielded in [?] the inequality

\[g(a,b,c) \leq \frac{1}{2} \left( \sqrt{abc(a+b+c)} - a - b - c \right),\]

which is of comparable size to the other upper bounds given by (2), (3), and (4). However, we will show that one can obtain bounds of smaller magnitude.

### 3. Dedekind-Rademacher Sums

The Dedekind-Rademacher sum [?] is defined for \(a, b \in \mathbb{Z}, x, y \in \mathbb{R}\) as

\[R(a,b;x,y) = \sum_{k=0}^{b-1} \left( \left( \frac{a(k+y)}{b} + x \right) \right)^* \left( \left( \frac{k+y}{b} \right) \right)^*,\]

where

\[((x))^* = \begin{cases} \lfloor x \rfloor & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}\]

Rademacher’s sum generalizes the classical Dedekind sum \(R(a,b;0,0)\) [?]. An easy bound for the Dedekind-Rademacher sum \(R(a,b;x,0)\) can be obtained through the Cauchy-Schwartz inequality: if \(a\) and \(b\) are relatively prime then

\[|R(a,b;x,0)| = \left| \sum_{k=0}^{b-1} \left( \left( \frac{ak}{b} + x \right) \right)^* \left( \left( \frac{k}{b} \right) \right)^* \right| \leq \sqrt{ \left( \sum_{k=0}^{b-1} \left( \left( \frac{ak}{b} + x \right) \right)^2 \right) \left( \sum_{k=0}^{b-1} \left( \left( \frac{k}{b} \right) \right)^2 \right) } \]

\[= \sqrt{ \left( \sum_{k=0}^{b-1} \left( \left( \frac{k}{b} + x \right) \right)^2 \right) \left( \sum_{k=1}^{b-1} \left( \frac{k}{b} - \frac{1}{2} \right)^2 \right) } \]

\[= \sqrt{ \left( \sum_{k=0}^{b-1} \left( \frac{k}{b} + \frac{1}{b} - \frac{1}{2} \right)^2 \right) \left( \frac{b}{12} - \frac{1}{4} + \frac{1}{6b} \right) } \]

\[= \sqrt{ \left( \frac{b}{12} + \frac{1}{6b} \right) \left( \frac{b}{12} - \frac{1}{4} + \frac{1}{6b} \right) } \]
(In the third and fourth step we use the periodicity of \((x)^*\). An important property of \(R(a, b; x, y)\) is Rademacher’s reciprocity law \([2]\): if \(a\) and \(b\) are relatively prime then

\[
R(a, b; x, y) + R(b, a; y, x) = Q(a, b; x, y).
\]

Here

\[
Q(a, b; x, y) = \begin{cases} 
-\frac{1}{4} + \frac{1}{12} \left( \frac{b}{a} + 1 \right) & \text{if both } x, y \in \mathbb{Z}, \\
\left( \frac{x}{a} \right)^* \left( \frac{y}{a} \right)^* \frac{1}{2} \left( \frac{\psi_2(y)}{a^2} \right) + \frac{b}{a} \psi_2(ax + by) + \frac{b}{a} \psi_2(x) & \text{otherwise},
\end{cases}
\]

where

\[
\psi_2(x) = (x - \lfloor x \rfloor)^2 - (x - \lfloor x \rfloor) + 1/6
\]
denotes the periodic second Bernoulli function. Among other things, this reciprocity law allows us to compute \(R(a, b; x, y)\) in polynomial time, by means of a Euclidean-type algorithm using the first two variables: simply note that we can replace \(a\) in \(R(a, b; x, y)\) by the least residue of \(a\) modulo \(b\).

To express \(S\) in terms of \(R\), we rewrite \((8)\) as

\[
S_t(a, b; c) = \sum_{m=0}^{c-1} \left( \frac{-a^{-1}(bm + t)}{c} \right)^* \left( \frac{m}{c} \right)^* + \left\{ \begin{array}{l}
\frac{1}{4} \left( \frac{-a^{-1}t}{c} \right)^* - \frac{1}{2} \left( \frac{-a^{-1}t}{c} \right)^* \quad \text{if } c|t, \\
-\frac{1}{2} \left( \frac{-a^{-1}t}{c} \right)^* - \frac{1}{2} \left( \frac{-a^{-1}t}{c} \right)^* \quad \text{otherwise}.
\end{array} \right.
\]

Accordingly,

\[
S_t(a, b; c) = R \left( -a^{-1}b, c; -\frac{a^{-1}t}{c} \right) + \left\{ \begin{array}{l}
\frac{1}{4} \left( \frac{-a^{-1}t}{c} \right)^* - \frac{1}{2} \left( \frac{-a^{-1}t}{c} \right)^* \quad \text{if } c|t, \\
-\frac{1}{2} \left( \frac{-a^{-1}t}{c} \right)^* - \frac{1}{2} \left( \frac{-a^{-1}t}{c} \right)^* \quad \text{otherwise}.
\end{array} \right.
\]

To ease our computations, we bound this as

\[
S_t(a, b; c) \geq R \left( -a^{-1}b, c; -\frac{a^{-1}t}{c} \right) - \frac{1}{2}.
\]

4. **Upper bounds for \(g(a, b, c)\)**

To bound \(S_t(a, b; c)\) from below (which yields an upper bound for \(g(a, b, c)\)), we use an interplay of \((10)\) and \((9)\) to obtain a bound for the Dedekind-Rademacher sum corresponding to \(S_t\), according to \((11)\). The idea is to reduce the arguments of the Dedekind-Rademacher sum after the application of \((10)\), which means that the bound given by \((9)\) will be more accurate. To illustrate this, let \(c_1\) be the least nonnegative residue of \(-a^{-1}b\) modulo \(c\). Then

\[
R \left( -a^{-1}b, c; -\frac{a^{-1}t}{c} \right) = R \left( c_1, c; -\frac{a^{-1}t}{c}, 0 \right) = Q \left( c_1, c; -\frac{a^{-1}t}{c}, 0 \right) - R \left( c, c_1; 0, -\frac{a^{-1}t}{c} \right).
\]

If \(c_1 = 1\) then the right-hand side can be simplified, as \(R \left( c, 1; 0, -\frac{a^{-1}t}{c} \right) = 0\). If \(c_1 \neq 1\) then the Dedekind-Rademacher sum on the right-hand side of \((12)\) can be bounded (via \((9)\)) sharper then the Dedekind-Rademacher sum on the left-hand side. In fact, by a repeated application of \((10)\), we can achieve bounds which are even better. To keep the computations somewhat simple, we apply \((10)\) once more and illustrate what this process yields in terms of lower bounds for \(S_t\). Let \(c_2\) be
the least nonnegative residue of $c$ modulo $c_1$. If $c_2 = 1$ then

\[
R \left( -a^{-1}b, c; -\frac{a^{-1}t}{c}, 0 \right) = Q \left( c_1, c; -\frac{a^{-1}t}{c}, 0 \right) - R \left( c, c_1; 0, -\frac{a^{-1}t}{c} \right)
\]

\begin{align*}
&= Q \left( c_1, c; -\frac{a^{-1}t}{c}, 0 \right) - R \left( c_2, c_1; 0, -\frac{a^{-1}t}{c} \right) \\
&= Q \left( c_1, c; -\frac{a^{-1}t}{c}, 0 \right) - Q \left( c_2, c_1; 0, -\frac{a^{-1}t}{c} \right) + R \left( c_1, c_2; -\frac{a^{-1}t}{c}, 0 \right).
\end{align*}

(13)

as $R \left( c_1, 1; -\frac{a^{-1}t}{c}, 0 \right) = 0$. If $c_2 \neq 1$ then (12) can be refined as

\[
R \left( -a^{-1}b, c; -\frac{a^{-1}t}{c}, 0 \right) = Q \left( c_1, c; -\frac{a^{-1}t}{c}, 0 \right) - R \left( c_2, c_1; 0, -\frac{a^{-1}t}{c} \right)
\]

\begin{align*}
&= Q \left( c_1, c; -\frac{a^{-1}t}{c}, 0 \right) - Q \left( c_2, c_1; 0, -\frac{a^{-1}t}{c} \right) + R \left( c_1, c_2; -\frac{a^{-1}t}{c}, 0 \right).
\end{align*}

(14)

The Dedekind-Rademacher sum on the right-hand side can be bounded according to (9) as

\[
R \left( c_1, c_2; -\frac{a^{-1}t}{c}, 0 \right) \geq -\sqrt{\left( \frac{c_2}{12} + \frac{1}{6c_2} \right) \left( \frac{c_2}{12} - \frac{1}{4} + \frac{1}{6c_2} \right)}.
\]

(15)

We still need to bound $Q$. $\psi_2$ has a minimum of $-1/12$ (at $x = 1/2$) and a maximum of $1/6$ (at $x = 0$). These extreme values yield for

\[
Q \left( c_1, c; -\frac{a^{-1}t}{c}, 0 \right) = \begin{cases} 
-\frac{1}{4} - \frac{1}{12} \left( \frac{c}{c_1} + \frac{1}{c_1} + \frac{c}{c_1} \right) & \text{if } c|t, \\
\frac{1}{2} \left( \frac{c}{6c_1} + \frac{1}{6c_1} + \frac{c}{c_1} \psi_2 \left( -\frac{a^{-1}t}{c} \right) \right) & \text{otherwise},
\end{cases}
\]

the lower bound

\[
Q \left( c_1, c; -\frac{a^{-1}t}{c}, 0 \right) \geq -\frac{1}{4} + \frac{c_1}{12c_1} + \frac{1}{12c_1} - \frac{c}{24c_1} = Q_{\text{low}}(c_1, c),
\]

(16)

as well as the upper bound

\[
Q \left( c_2, c_1; 0, -\frac{a^{-1}t}{c} \right) \leq \frac{c_2}{12c_1} + \frac{1}{12c_2} - \frac{c}{24c_1} = Q_{\text{up}}(c_2, c_1).
\]

(17)

These inequalities yield the following.

**Proposition.** Suppose $a$ and $b$ are relatively prime to $c$. Let $c_1$ be the least nonnegative residue of $-a^{-1}b$ modulo $c$, and let $c_2$ be the least nonnegative residue of $c$ modulo $c_1$.

(i) If $c_1 = 1$ then $S_t(a, b; c) \geq -\frac{c}{24} + \frac{1}{6c} - \frac{3}{4}.$

(ii) If $c_1 \neq 1$ and $c_2 = 1$ then $S_t(a, b; c) \geq \frac{c_1}{12c} + \frac{1}{12c_1} - \frac{c}{24c_1} - \frac{1}{6c_1} - \frac{c_1}{12} - \frac{3}{4}.$

(iii) If $c_1 \neq 1$ and $c_2 \neq 1$ then

\[
S_t(a, b; c) \geq \frac{c_1}{12c} + \frac{1}{12c_1} - \frac{c}{24c_1} - \frac{c_2}{12c_1} - \frac{1}{12c_1c_2} - \frac{c_1}{12c_2} - \frac{3}{4} - \sqrt{\left( \frac{c_2}{12} + \frac{1}{6c_2} \right) \left( \frac{c_2}{12} - \frac{1}{4} + \frac{1}{6c_2} \right)}.
\]
Proof. (i) Use (12) with $c_1 = 1$ in (11):
\[
S_t(a, b; c) \geq R \left( -a^{-1}b, c; -a^{-1} \frac{t}{c}, 0 \right) - \frac{1}{2} = Q \left( 1, c; -a^{-1} \frac{t}{c}, 0 \right) - \frac{1}{2} \geq -\frac{1}{4} + \frac{1}{6c} - \frac{c}{24} - \frac{1}{2}.
\]
Here the last inequality follows from (16).

(ii) Use (13) in (11) together with the bounds (16) and (17):
\[
S_t(a, b; c) \geq R \left( -a^{-1}b, c; -a^{-1} \frac{t}{c}, 0 \right) - \frac{1}{2} = Q \left( c_1, c; -a^{-1} \frac{t}{c}, 0 \right) - Q \left( 1, c_1; 0, -a^{-1} \frac{t}{c} \right) - \frac{1}{2} \geq -\frac{1}{4} + \frac{c_1}{12c} + \frac{1}{2} \frac{c}{24c_1} - \frac{c}{24} - \frac{1}{2}.
\]

(iii) Use (14) with the bounds given in (15), (16), and (17):
\[
S_t(a, b; c) \geq R \left( -a^{-1}b, c; -a^{-1} \frac{t}{c}, 0 \right) - \frac{1}{2} = Q \left( c_1, c; -a^{-1} \frac{t}{c}, 0 \right) - Q \left( c_2, c; 0, -a^{-1} \frac{t}{c} \right) + R \left( c_1, c_2; -a^{-1} \frac{t}{c}, 0 \right) - \frac{1}{2} \geq -\frac{1}{4} + \frac{c_1}{12c} + \frac{1}{12c_1c} - \frac{c}{24c_1} - \frac{1}{2} \frac{c}{24} - \frac{1}{2}.
\]

These lower bounds can be combined with (7) and the quadratic formula to give an upper bound on the Frobenius number.

Theorem. Suppose $a, b,$ and $c$ are pairwise relatively prime. Denote the lower bounds for $S_t(b, c; a)$, $S_t(c, a; b)$, and $S_t(a, b; c)$ according to the previous proposition by $\alpha$, $\beta$, and $\gamma$, respectively. Then
\[
g(a, b, c) \leq \sqrt{\frac{1}{4} (a + b + c)^2 - \frac{1}{4} (a^2 + b^2 + c^2) - 2abc(\alpha + \beta + \gamma) - \frac{1}{2} (a + b + c)}.
\]

One should note that $\alpha + \beta + \gamma$ is negative. We can see that the growth behavior of this upper bound is dominated by $-2abc(\alpha + \beta + \gamma)$ under the square root. This means that if we can make $-(\alpha + \beta + \gamma)$ somewhat smaller than $\min(a, b, c)$ then we get a bound which grows considerably less than the bounds given by (2), (3), and (4). In fact, we can see this difference in example computations already when we use the bounds $\alpha, \beta, \gamma$ as given by our proposition. What is more important, however, is the fact that we can easily obtain even better bounds by improving our proposition through additional applications of Rademacher’s reciprocity law (10). We illustrate this with the following algorithm, whose result is a bound on $S_t(a, b; c)$, which can be used in the above theorem (instead of the bounds coming from the proposition).
Algorithm. Input: a, b, c (pairwise relatively prime) and N (number of iterations). Output: lower bound $S$ for $S_t(a,b,c)$.

c_1 := -a^{-1}b \mod c \ (\text{least nonnegative residue})
S := 0
n := 1

REPEAT {
    c_2 := c \mod c_1 \ (\text{least nonnegative residue})
    S_1 := S + \text{Q}_{\text{low}}(c_1,c)
    S_2 := S_1 - \text{Q}_{\text{up}}(c_2,c_1)

    IF $c_1 = 1$ THEN $S := S_1$
    Else $S := S_2$

    IF $c_1 = 1$ OR $c_2 = 1$ OR $n = N$ THEN BREAK

    c := c_2
    c_1 := c_1 \mod c_2 \ (\text{least nonnegative residue})
    n := n + 1
}

IF $c_1 > 1$ AND $c_2 > 2$ THEN $S := S - \sqrt{(c_2/12 + 1/(6 \ c_2) - 1/4) \ (c_2/12 + 1/(6 \ c_2))}$
S := S - 1/2

The algorithm repeats the steps described in the proposition $N$ times, at each step bounding $Q$ coming from Rademacher reciprocity according to (16) and (17). It stops prematurely if one of the variables is 1, in which case the remaining Dedekind-Rademacher sum is zero.

5. Computations

In the present section we illustrate the newly proposed upper bound for $g(a,b,c)$ numerically. In order to compare the results also with the lower bound given by Davison we present here the values

$$f(a,b,c) = g(a,b,c) + a + b + c.$$  

For these Frobenius numbers, Davison’s lower bound is

$$f(a,b,c) \geq \sqrt{3} \ z,$$

where $z = \sqrt{abc}$. In [?] we presented together with David Einstein an algorithm for the exact computation of $f(a,b,c)$. Einstein computed 20000 “admissible” (see [?]) values of $f(a,b,c)$ for relatively prime arguments chosen at random from the set $\{3, \ldots, 750\}$. In [?] we arrived at the empirical conjecture that $f(a,b,c) \leq \sqrt{abc^{5/4}}$. The objectives of our current presentations are:
In Figure we plot the new upper bound as a function of \( z \), and compare the points with the conjectured upper bound. The difference between the known upper bound for \( f(a, q) \) and the new upper bound is then \( z \). In Figure we plot the known upper bound as a function of the new upper bound. This is the known upper bound as a function of \( z \) and the new upper bound \( z \). In Figure we plot the new upper bound \( z \) and the theorems to obtain an upper bound for \( f(a, q) \). In Figure we plot the new upper bound \( z \).

Theorem 2. For the proposition and the algorithm for \( N = z \) in the minimum of the known bounds given by Proposition 1, the algorithm for \( N = z \) in the minimum values of \( a, q \), and randomly chosen from data. In all computations we used elements for two of these objectives. We computed the new upper bound and the known upper bound, and for these objectives we computed the new upper bound and the known upper bound for two.

\[
\text{In (iii)} \text{ to compare the new upper bound to the conjectured upper bound.}
\]

\[
\text{In (iv)} \text{ to compare an upper bound with the conjectured upper bound.}
\]

\[
\text{In (v)} \text{ to compute the new upper bound to the true value of } f(a, q).
\]

\[
\text{In (vi)} \text{ to compare an upper bound with the conjectured upper bound.}
\]

\[
\text{In (vii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (viii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (ix)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (x)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xi)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xiii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xiv)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xv)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xvi)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xvii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xviii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xix)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xx)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxi)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxiii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxiv)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxv)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxvi)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxvii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxviii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxix)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxx)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxxi)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxiii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxiv)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxv)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxvi)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxvii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxviii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxix)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxx)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxxi)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxiii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxiv)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxv)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxvi)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxvii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxviii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxix)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxx)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxxi)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxiii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxiv)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxv)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxvi)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxvii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxviii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxix)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxx)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxxi)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxiii)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxiv)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[
\text{in (xxv)} \text{ to compare an upper bound with the known upper bound, which is, according to (v),}
\]

\[...\]
Gain from increasing the number $N$ of iterations, the algorithm becomes exponentially faster one or two iterations; accordingly, there is not much

II is reasonable to expect even better results when one uses more than two iterations in the algo-

6. FINAL REMARKS

values. In 98% of the cases the ratio of the new upper bound to the true $f(a',b,c)$ is greater than

values, that even for large values of $f(a',b,c)$ there are cases where the new upper bound yields close

Finally, in Figure 2 we plot the points of the new upper bound versus the true $f(a',b,c)$ values. We

upper bound are smaller than $\frac{1}{2^2}$. This gives additional credence to the empirical conjecture in

FIGURE 2. The new and old upper bounds compared

\begin{center}
\begin{tikzpicture}
\begin{axis}[
width=\textwidth,height=\textwidth,
grid=major,
axis lines=left,
xtick={0,50000,100000,150000,200000,250000},
xticklabels={0,50000,100000,150000,200000,250000},
ytick={20000,40000,60000,80000,100000},
yticklabels={20000,40000,60000,80000,100000},
node near coords,]
\addplot+[only marks] table [x=Known Upper Bound, y=New Upper Bound] {data.csv};
\end{axis}
\end{tikzpicture}
\end{center}
efficiently. According to $Q$, it is not clear how to compute their $\gamma$-dimensional analogues of Dekking-Rauzy substitutions. However, it is not clear how to compute them for $\gamma < 2$ if $\gamma$ involves higher-order arguments of $\gamma$. Theorem 4.111. There are forbidden substitutions to $\gamma$-dimensional analogues of Dekking-Rauzy substitutions. However, it is not clear how to compute them for $\gamma < 2$ if $\gamma$ involves higher-order arguments of $\gamma$. Theorem 4.111.

Consistencies.

We close with a remark on the General Problem of Problems. This is with an arbitrary number of

The new upper bounds and the conjecture.

**Figure 3.** The new upper bounds and the conjecture.
This general setting.

Figure 4. The new upper bounds compared to the Frobenius numbers.
Figure 5. The new upper bounds compared (1 iteration vs. 2 iterations)