A problem of restriction $g_n \downarrow g_{n-1}$ for Lie algebras of series $A, B, C, D$.

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Using the Zhelobenko’s approach we investigate a branching of an irreducible representation of $g_n$ under the restriction of algebras $g_n \downarrow g_{n-1}$, where $g_n$ is a Lie algebra of type $B_n, C_n, D_n$ or a Lie algebra of type $A$, where in this case we put $g_n = gl_{n+1}$, $g_{n-1} = gl_{n-1}$. We give a new explicit description of the space of the $g_{n-1}$-highest vectors, then we construct a base in this space. The case $n = 2$ is considered separately for different algebras, but a passage from $n = 2$ to an arbitrary $n$ is the same for all series $A, B, C, D$. This new procedure has the following advantage: it establishes a relation between spaces of $g_{n-1}$-highest vectors for different series of algebras. This procedure describes an extension of Gelfand-Tsetlin tableaux to the left.

1 Introduction

In the book [1] Zhelobenko used realizations of a representation of a simple Lie algebra $g$ in the space of all functions on a corresponding Lie group and in the space of functions on a subgroup of unipotent upper-triangular matrices. In [1] conditions that define a representation of a given highest weight in these realization are presented. These realization are very convenient for an investigation of restriction problems. A problem of a restriction $g \downarrow \mathfrak{t}$, where $g$ is a Lie algebra and $\mathfrak{t}$ is its subalgebra is a problem of a description of $\mathfrak{t}$-highest vectors in an irreducible representation of $g$. A solution of such a problem is a key step in a construction of a Gelfand-Tsetlin type base in a representation of a Lie algebra.

In [1] the cases of restriction problems $gl_n \downarrow gl_{n-1}$ and $sp_{2n} \downarrow sp_{2n-2}$ are considered. In the first case a construction of a base in the space of $gl_{n-1}$-highest vectors encoded by Gelfand-Tsetlin tableaux is given. Then using a realization in the space of functions on the subgroup of unipotent upper-triangular matrices in [1] it is shown that these restriction problems are equivalent. Using this equivalence the problem $sp_{2n} \downarrow sp_{2n-2}$ is solved.
Later the problems \( g_n \downarrow g_{n-1} \), where \( g_n = \mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \text{ or } \mathfrak{o}_{2n} \) in the realization in the space of functions on unipotent upper-triangular matrices was investigated by V.V. Shtepin in [2], [3], [4]. He obtained solutions of these problems but a relation with the problem \( \mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1} \) was not discovered.

A.I. Molev in [5], [6], [7] (see also [8]) obtained a solution of a problem of construction of a Gelfand-Tsetlin type base for a finite dimensional representation of a Lie algebra \( g_n = \mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \text{ or } \mathfrak{o}_{2n} \). Molev gave a construction of base vectors and obtained formulas for the action of generators of the algebra in this base. But he used another technique. To obtain a solution of the problem \( g_n \downarrow g_{n-1} \) an action of a Yangian on the space of \( g_{n-1} \)-highest vectors with a fixed highest weight was constructed. For all series of algebras Gelfand-Tsetlin type tableaux constructed by Molev have the following property. There right part has a structure that depends on a series of the algebra. But as \( n \) increases a tableau grows to the left and the structure of the extension of the tableau does not depend on the type of the algebra.

This fact is a starting point of the present paper. The main result is a new construction of a solution of the problem \( g_n \downarrow g_{n-1} \) (Theorem 4) establishing a relation between solutions for different series (Corollary 2). We investigate the problem \( g_n \downarrow g_{n-1} \) in the realization in the space of functions on the whole group.

First of all for series \( B, C, D \) we give explicit conditions that define an irreducible representation of a given highest weight in the realization in function on the whole group (see Theorem 1 where the conditions are formulated, and Section 2.5.2 where these conditions are rewritten explicitly), in [1] it is done only for the series \( A \). Then we obtain a description of functions on the group that correspond to \( g_{n-1} \)-highest vectors (Theorem 2 and Theorem 3).

Finally in the main Sections 3, 5 we give a procedure of a construction of a Gelfand-Tsetlin type base in this space. Firstly separately for the series \( A, B, C, D \) it is done for \( n = 2 \) (see Section 3), and then for all these series simultaneously we describe a passage from the problem \( g_2 \downarrow g_1 \) to \( g_n \downarrow g_{n-1} \) (Section 4 Theorem 4). This procedure is interpreted as an extension of a Gelfand-Tsetlin tableau to the left. We call this procedure an extension of a problem of restriction \( g_n \downarrow g_{n-1} \supset g_2 \downarrow g_1 \).

2 Introduction

2.1 Algebras \( \mathfrak{o}_{2n+1}, \mathfrak{o}_{2n}, \mathfrak{sp}_{2n} \)

The Lie algebras \( \mathfrak{o}_{2n}, \mathfrak{sp}_{2n} \) are considered as subalgebras in the algebra of all matrices \( 2n \times 2n \), whose rows are indexed \( i, j = -n, \ldots, -1, 1, \ldots, n \), the algebra...
\( \mathfrak{o}_{2n+1} \) is a subalgebra in the space of all \((2n+1) \times (2n+1)\)-matrices, whose rows and columns are indexed by \(i, j = -n, \ldots, 0, 1, \ldots, n\).

The algebras \( \mathfrak{o}_{2n+1} \) and \( \mathfrak{o}_{2n} \) are generated by

\[
F_{i,j} = E_{i,j} - E_{-j,-i},
\]

where \(i, j = -n, \ldots, -1, 0, 1, \ldots, n\) in the case \( \mathfrak{o}_{2n+1} \) and \(i, j = -n, \ldots, -1, 1, \ldots, n\) in the case \( \mathfrak{o}_{2n} \).

The algebra \( \mathfrak{sp}_{2n} \) is generated by

\[
F_{i,j} = E_{i,j} - \text{sign}(i) \text{sign}(j) E_{-j,-i},
\]

where \(i, j = -n, \ldots, -1, 1, \ldots, n\).

Subalgebras \( \mathfrak{o}_{2n-1}, \mathfrak{o}_{2n-2}, \mathfrak{sp}_{2n-2} \) are generated by \(F_{i,j}\) for \(i, j = -n, \ldots, -2, 2, \ldots, n\) in the case \(C, D\) and \(i, j = -n, \ldots, -2, 0, 2, \ldots, n\) in the case \(B\).

Also we use the algebra \( \mathfrak{gl}_{n+1} \) of all matrices \((n+1) \times (n+1)\), whose rows and columns are indexed by \(i, j = -n, \ldots, -1, 1, \ldots, n\). This algebra is generated by \(E_{i,j}\), \(i, j = -n, \ldots, -1, 1\), choose a subalgebra \( \mathfrak{gl}_{n-1} \), spanned by \(E_{i,j}\), \(i, j = -n, \ldots, -2\).

2.2 Functions on the group

We use a realization of a representation of a Lie algebra in the space of functions on the Lie group \(G = O_{2n+1}, Sp_{2n}, O_{2n}, GL_{n+1}\) (see [1]). Onto a functions \(f(g), g \in G\) an elements \(X \in G\) acts by right shifts according to the formula

\[
(Xf)(g) = f(gX).
\]

Fix a highest weight \([m] = [m_{-n}, \ldots, m_{-1}]\) in the cases \(O_{2n+1}, Sp_{2n}, O_{2n}\) and \([m] = [m_{-n}, \ldots, m_{-1}, m_1 = 0]\) in the case \(GL_{n+1}\). In the cases \(G = Sp_{2n}, GL_{n+1}\) the highest weight is integer and in the case \(O_{2n+1}, O_{2n}\) it can be half-integer. In the cases \(GL_{n+1}, O_{2n+1}, Sp_{2n}\) the weight is non-negative and in the case \(O_{2n}\) we can have \(m_{-1} < 0\).

2.3 Determinants and a formula for the highest vector

Let \(a^j_i, i, j = -n, \ldots, 0, \ldots, n\) be a function of a matrix element on the group \(GL_{2n+1}\). Here \(j\) is a row index and \(i\) is a column index. Later we consider restrictions of these functions onto \(G \subset GL_{2n+1}\). Thus relations between \(a^j_i\) appear.

Put

\[
a_{i_1, \ldots, i_k} := \det(a^j_i)_{i = i_1, \ldots, i_k}^{j = -n, \ldots, -n+k-1}.
\]
That is we take a determinant of a submatrix of a matrix \((a^j_i)\), formed by rows \(-n, ..., -n + k - 1\) and columns \(i_1, ..., i_k\). By formulas (3), (4) we conclude that an operator \(E_{i,j}\) acts onto a determinant by changing column indices according to the ruler

\[
E_{i,j}a_{i_1, ..., i_k} = a_{\{i_1, ..., i_k\}]_{j \rightarrow i},
\]

where \(j \rightarrow i\) is an operation of a substitution of \(i\) instead of \(j\), if \(j \notin \{i_1, ..., i_k\}\) then we obtain 0. An operator \(F_{i,j}\) for the series \(B, C, D\) acts by formulas (1), (2).

In the case of the series \(D\) put

\[
\bar{a}_{i_1, ..., i_n} := \det(a_{j=\{i_1, ..., i_n\}}^{j=-n, ..., -2, 1}).
\]

Then

\[
(a_{-n, ..., -2, -1})^{-1} = \bar{a}_{-n, ..., -2, 1}.
\]

This equality in the case \(n = 2\) is checked by direct computation, the case of an arbitrary \(n\) is considered as in Lemma 7.

Now let us give a formula for the highest vector (see [1]). The vector

\[
v_0 = \prod_{k=-n}^{-2} (a_{-n, ..., -k})^{m-k-m_{-k}+1}a_{-n, ..., -2, -1}^{-n} a_{-n, ..., -2, -1}^{-1},
\]

for the series \(A, B, C, D\) and \(m_{-1} \geq 0\),

\[
v_0 = \prod_{k=-n}^{-2} (a_{-n, ..., -k})^{m-k-m_{-k}+1}a_{-n, ..., -2, -1}^{-n} a_{-n, ..., -2, -1}^{-1},
\]

for the case \(D\) and \(m_{-1} < 0\),

is a highest vector for \(g_n\) with the weight \([m_{-n}, ..., m_{-1}]\) for the series \(B, C, D\) and \([m_{-n}, ..., m_{-1}, 0]\) for the series \(A\). Note that in the case of an integer highest weight this is a polynomial function. And in the case of \(B, D\) and a half-integer highest weight this is not a polynomial function since it contains a factor \(a_{-n, ..., -2, -1}^{-m_{-1}}\) or \(\bar{a}_{-n, ..., -2, -1}^{-m_{-1}}\).

2.4 Functions on the subgroup \(Z\)

2.4.1 The action on the functions on \(Z\)

For a typical matrix \(X \in G\) one has a Gauss decomposition

\[
X = \zeta \delta z,
\]

into a product of a lower-triangular unipotent matrix, a diagonal matrix and an upper-triangular unipotent matrix. We denote a subgroup of upper-triangular
unipotent matrices as $Z$. For an element $g \in G$ and a matrix $X$ one has the Gauss decomposition

$$Xg = \tilde{\zeta}\tilde{\delta}z.$$

Put $\tilde{\delta} = \text{diag}(\tilde{\delta}_{-n}, \tilde{\delta}_{-n+1}, \ldots)$. The action of $G$ on the functions on $Z$ is given by the formula

$$(gf)(z) = \tilde{\delta}^{m-n} \tilde{\delta}^{m-n+1} \ldots \tilde{\delta}^{-1} f(\tilde{z}).$$

### 2.4.2 Conditions that define an irreducible representation in the functions on $Z$

Functions on $Z$ that form an irreducible representation with a given highest weight are selected by the following condition (see [1]).

1. A function satisfies the indicator system.

The indicator system is a system of PDE of type

$$L_{-n,-n+1}^{r_{-n}} f = 0, \ldots, L_{-2,-1}^{r_{-2}} f = 0, \quad L_{-1,0}^{r_{-1}} f = 0 \quad \text{in the cases} \quad A, C,$$

$$L_{-n,-n+1}^{r_{-n}} f = 0, \ldots, L_{-2,-1}^{r_{-2}} f = 0, \quad L_{-1,0}^{r_{-1}} f = 0 \quad \text{in the case} \quad B,$$

$$L_{-n,-n+1}^{r_{-n}} f = 0, \ldots, L_{-2,-1}^{r_{-2}} f = 0, \quad L_{-1,1}^{r_{-1}} f = 0 \quad \text{in the case} \quad D$$

Here $L_{-i,-j}$ is an operator acting on a function $f(z)$ and doing a left infinitesimal shift of a function $f(z)$ onto $F_{-i,-j}$ for the series $B, C, D$ and a shift by $E_{-i,-j}$ for the series $A$.

An exponent $r_{-i}$ is written as follows

$r_{-n} = m_{-n} - m_{-n+1}, \ldots, r_{-2} = m_{-2} - m_{-1}, \quad r_{-1} = m_{-1}$ in the cases $A, C,$

$r_{-n} = m_{-n} - m_{-n+1}, \ldots, r_{-2} = m_{-2} - m_{-1}, \quad r_{-1} = 2m_{-1}$ in the case $B,$

$r_{-n} = m_{-n} - m_{-n+1}, \ldots, r_{-2} = m_{-2} - |m_{-1}|, \quad r_{-1} = m_{-2} + |m_{-1}|$ in the case $D.$

(10)

It turns out that such a function is a polynomial in matrix elements.\footnote{Even in the case of half-integer highest weight}

In [1] and also in [4] in the case of the series $D$ the exponents $r_{-2} = m_{-2} - m_{-1}$ and $r_{-1} = m_{-2} + m_{-1}$ are used. The reason is that in [1], [4] and in the present paper a different choice of the highest vector in the realization in functions on $G$ is done (see [3]). And the highest vector must satisfy the indicator system.
2.5 Conditions that define an irreducible representations in the functions on $G$

2.5.1 The general Theorem

In [1] in the case $GL_{n+1}$ the following statement is proved. In the realization in the space of functions on the group $G$ an irreducible representation with the highest weight $[m_{-n}, ..., m_{-1}, 0]$ and a highest vector are selected by conditions

1. $L_- f = 0$, where $L_-$ is a left infinitesimal shift by an arbitrary element of $GL_{n+1}$, corresponding to a negative root.
2. $L_{-i,-j} f = m_{-i} f$, where $L_{-i,-j}$, $i = 1, ..., n$ is a left infinitesimal shift by an element of $GL_{n+1}$, corresponding to a Cartan element $E_{-i,-j}$.
3. $f$ satisfies the indicator system.

Let us prove an analogous statement for the series $B$, $C$, $D$.

**Theorem 1.** In the realization in functions on the whole group for the series $B$, $C$, $D$ an irreducible representation with the highest vector given in Section is selected by the conditions 1, 3, by the condition 2 where we change $E_{-i,-j}$ to $F_{-i,-j}$ in the definition of $L_{-i,-j}$.

**Proof.** The scheme of the proof is the following. Firstly we derive formulas for the action of a left infinitesimal shift. Using them we prove the the highest vector satisfies the conditions 1-3. The main difficulty is to prove that the highest vector satisfies the indicator system. Then we easily prove that an arbitrary vector of the representation satisfies the conditions 1-3. Secondly we prove that among functions that satisfy conditions 1-3 there is nothing but functions that form a representation with the highest vector.

Take a determinant

$$a_{i_1, \ldots, i_k} = \det(a^{i_j}_{i_1, \ldots, i_k})_{i_1=1, \ldots, i_k}$$

introduce a notation

$$a_{i_1, \ldots, i_k}^{-n, \ldots, -n + k - 1},$$

where upper indices are row indices. Then the operator $L_{-i,-j}$ of the left infinitesimal shift acts on the upper indices $-n, ..., -n + k - 1$ by the following. For the series $A$ the left infinitesimal shift by $E_{-i,-j}$ act as follows

$$L_{-i,-j} a_{i_1, \ldots, i_k}^{-n, \ldots, -n + k - 1} = a_{i_1, \ldots, i_k}^{(-n, \ldots, -n + k - 1)_{-i,-j}}.$$ (11)
and the left infinitesimal shift by $F_{-i, -j}$ for the series $B$, $C$, $D$ is expressed through these operators.

Analogously

$$L_{-i,-j} a_{1, \ldots, i_k}^{\{-n, \ldots, -n+k-1\}} = a_{1, \ldots, i_k}^{\{-n, \ldots, -n+k-1\}} \text{ if } -i \in \{-n, \ldots, -n+k-1\}, \; 0 \text{ otherwise} . \tag{12}$$

On a product of determinants $L_{-i,-j}, L_{-i,-j}$ act according to the Leibnitz ruler.

From these formulas we see that the conditions 1-3 for an irreducible representations with the highest vector defined in Section 2.3 do hold.

First we prove that conditions 1-3 hold for the highest vector. The fact that conditions 1 and 2 hold is proved by very easy direct computations. We need to prove that the condition 3 holds.

From the formula (11) it follows that operators $L_{-i,-i+1}$ for $i = -n, \ldots, -3$ for all series act onto (5) by acting only onto $a_{-n, \ldots, -i}^{m-i}$. Also $L_{-2,-1}$ for the series $A, B, C$, and $D$ in the case $m_1 \geq 0$, $L_{-2,1}$ for the series $D$ in the case $m_1 < 0$ act onto (5) by acting only onto $a_{-n, \ldots, -2}^{m-2-m_1}$. The operator $L_{-1,1}$ for the series $A, C$, $L_{-1,0}$ for the series $B$ acts on to a determinant of order $n$.

But there are also special operators. For the series $D$ and $m_1 \geq 0$ the operator $L_{-2,-1}$ acts onto determinants of orders $n-1$ and $n$. And in the case $m_1 < 0$ so does the operator $L_{-2,-1}$.

Since the power $r_{-i} = m_{-i} - m_{-i+1}$ of the determinant $a_{-n, \ldots, -n+i-1}$ in (5) is an integer then due to (11) the equations $L_{-i,-i+1} v_0 = 0, i = -n, \ldots, -2$ hold. For the same reason the equation $L_{-1,1}^{m_{-1}+1} v_0 = 0$ for the series $A, C$ hold.

Let us check that $L_{-1,0}^{m_{-1}+1} v_0 = 0$ for the series $B$. For the series $B$ one has

$$L_{-1,0} a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1} = a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1}, \; L_{-1,0}^2 a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1} = -a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1},$$

$$L_{-1,0}^3 a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1} = 0.$$

From these formulas we obtain that $L_{-1,0}^{2m_{-1}+1} v_0 = 0$ for the series $B$ and an integer highest weight.

One also has

$$L_{-1,0} (a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1})^{1/2} = \frac{1}{2} a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1} (a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1})^{-1/2},$$

$$L_{-1,0}^2 a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1} = -\frac{1}{2} a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1} (a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1})^{-1/2} - \frac{1}{2} (a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1})^2 (a_{-n, \ldots, -2, -1}^{n, \ldots, -2, -1})^{-3/2} = 0.$$
To prove these formulas one must use the relation
\[ a_{-n, \ldots, -2} - a_{-n, \ldots, -1}a_{-n, \ldots, -2} = \frac{1}{2}(a_{-n, \ldots, -2}a_{-n, \ldots, -1})^2. \]
This relation can be derived by analogy with Lemma 7 below. From these formulas we conclude that \( L_{-1,0}^{2m-1+1}v_0 = 0 \) for the series \( B \) and a half-integer highest weight.

Now consider the case of the series \( D \) and \( m-1 \geq 0 \). Let us check that \( L_{-2,1}^{m-2+m-1+1}v_0 = 0 \). One has

\[
\begin{align*}
L_{-2,1}a_{-n, \ldots, -2} &= a_{-n, \ldots, -1}, \\
L_{-2,1}a_{-n, \ldots, -1} &= 0,
\end{align*}
\]

Thus \( v_0 \) vanishes under the action of \( L_{-2,1} \) to the power which equals one plus the power of \( a_{-n, \ldots, -2} \) and the double power of \( a_{-n, \ldots, -1} \), that is \( v_0 \) vanishes under the action of \( L_{-2,1} \) to the power \( 1 + (m-2-m-1) + 2(m-1) = 1 + m-2 + m-1 \).

For the series \( D \) and \( m-1 < 0 \). Let us check that \( L_{-2,-1}^{m-2+m-1+1}v_0 = 0 \). One has

\[
\begin{align*}
L_{-2,-1}a_{-n, \ldots, -2} &= a_{-n, \ldots, -1}, \\
L_{-2,-1}a_{-n, \ldots, -1} &= 0,
\end{align*}
\]

Thus \( v_0 \) vanishes under the action of \( L_{-2,-1} \) to the power \( 1 + (m-2-m-1) + 2m-1 = 1 + m-2 + m-1 \).

Thus the highest vector satisfies the indicator system.

The fact that the conditions 1-3 hold for an arbitrary vector of the representation follows from the fact that left and right shifts commute. And an arbitrary vector of the representation is a linear combination of right shifts of the highest vector.

Thus we need to check that among functions that satisfy the conditions 1-3 there are no other functions.

Let us use the following statement form [1].

**Proposition 1.** If one restricts form \( G \) to \( Z \) functions that form an irreducible representation with the highest vector \( [2] \) one obtains a bijection with the space of functions on \( Z \) that from an irreducible representation with the same highest weight. These function on \( Z \) are selected by conditions from Section 2.4.2.

So let us be given a function on \( G \), that satisfy the conditions 1-3. It’s restriction on \( Z \) satisfies the condition 1 from the Section 2.4.2. Thus this restriction belongs to the irreducible representation in the realization in functions.
on $Z$. Using the statement above we conclude that the initial function on $G$ belongs to an irreducible representation with a highest vector $(8)$. \hfill \Box

2.5.2 Solutions of the indicator system and the equations $L_{-i,-1}f = m_{-1}f$.

In the proof of Theorem 1 the formulas for the action of $L_{-i,-1}$ were derived. They give us the following statement.

**Lemma 1.** Solutions of the system of equations $L_{-i,-1}f = m_{-1}f$ that are functions of determinants are described as follows. If one expands the function $f$ into a sum (maybe infinite) of products of determinants then in each summand the sum of powers of determinants of order $n - i + 1$ equals $r_{-1}$ for $i = n, ..., 2$. Also a sum of powers of determinants of order $n$ equals $m_{-1}$ in the case $m_{-1} ≥ 0$ and $-m_{-1}$ in the case $m_{-1} < 0$.

Let us find conditions that select functions that satisfy the indicator system.

**Lemma 2.** If the case of an integer non-negative highest weight the solution of the indicator system are polynomials that satisfy the condition of Lemma 1.

*Proof.* Since the highest vector $(8)$ is a polynomial in determinants and the space of such function is invariant under the right action of algebra, then an arbitrary vector can be represented as a polynomial in determinants. Then the statement of Lemma follows immediately from (11). \hfill \Box

**Lemma 3.** If the highest weight is half-integer then among functions that satisfy solutions of the indicator system are functions of type

\[
f = (a_{-n, ..., -2, -1})^{\frac{1}{2}} f_1 + a_{-n, ..., -2, 0} (a_{-n, ..., -2, -1})^{-\frac{1}{2}} f_2 \text{ for the series } B \\
f = (a_{-n, ..., -2, -1})^{\frac{1}{2}} f_1 + a_{-n, ..., -1, 1} (a_{-n, ..., -2, -1})^{-\frac{1}{2}} f_2 \text{ for the series } D \text{ and } m_{-1} ≥ 0,
\]

\[
f = (a_{-n, ..., -2, 1})^{\frac{1}{2}} f_1 + a_{-n, ..., -1, 1} (a_{-n, ..., -2, 1})^{-\frac{1}{2}} f_2 \text{ for the series } D \text{ and } m_{-1} < 0.
\]

(13)

where $f_1$ and $f_2$ are polynomials in determinants.

*Proof.* Consider the case of the series $B$.

Let us first prove that a solution of the indicator system looks as (13). A function of determinants satisfies automatically the condition 1 of Theorem 1, so our function is a vector of a representation with the highest vector $(8)$.

Consider first the case of the highest vector $(a_{-n, ..., -2, -1})^{\frac{1}{2}}$. An arbitrary element of the representation is represented as a linear combination of
\[ F_{p-1}^{p-1} F_{p-2}^{p-2} \cdots F_{n+1}^{p-n} (a_{n}, \ldots, -2, -1)^{\frac{p}{2}}, \]

this vector is non-zero only in the case \( p-n = \cdots = p-2 = 0 \), and \( p-1 = 0 \) or \( 1 \). Indeed for \( p-1 = 1 \) we get the vector \( \frac{1}{2}a_{n}, \ldots, -2, 0(a_{n}, \ldots, -2, -1)^{-\frac{3}{2}} \). And for \( p-1 = 2 \) we obtain

\[
- \frac{1}{2}a_{n}, \ldots, -2, 1(a_{n}, \ldots, -2, -1)^{-\frac{3}{2}} - \frac{1}{4}(a_{n}, \ldots, -2, 0)^2(a_{n}, \ldots, -2, -1)^{\frac{1}{2}} = \]

\[
= \frac{1}{2}(a_{n}, \ldots, -2, -1)^{-\frac{3}{2}}(a_{n}, \ldots, -2, 1(a_{n}, \ldots, -2, -1) + \frac{1}{2}(a_{n}, \ldots, -2, 0)^2) = 0,
\]

where we used relation from Lemma 8 below.

Now consider the case of an arbitrary highest vector. It can be represented as follows

\[ v_0 = v'_0(a_{n}, \ldots, -2, -1)^{\frac{p}{2}}, \]

where \( v'_0 \) is a polynomial in determinants. An arbitrary vector \( f \) is a linear combination of \( F_{p-1}^{p-1} F_{p-2}^{p-2} \cdots F_{n+1}^{p-n} (a_{n}, \ldots, -2, -1)^{\frac{p}{2}} \). This vector is of type (13).

Now we must prove that every vector of type (13), which satisfies the condition of Lemma 1 is a solution of the indicator system.

Operators \( L_{-i, -i+1} \), \( i = n, \ldots, 2 \) act onto determinants of orders \( n-i+1 \) that is onto determinants of orders \( 1, \ldots, n-1 \). Such determinants occur in \( f_1, f_2 \), in particular they occur in non-negative integer powers, the sum of powers of determinants of order \( i \) equals to \( r_{n+i-1} \) since Lemma 1 holds. Thus conditions \( L_{-i, -i+1}^r f = 0 \) for \( i = n, \ldots, 2 \) hold.

Now consider the equation \( L_{-1, 0}^{2m-1} f = L_{-1, 0}^{2m-1} f + 1 f = 0 \), where \( [m-1] \) is an integer part. The operator \( L_{-1, 0}^{2m-1} f + 1 \) acts according to the Leibnitz ruler onto each summand in (13) as follows.

Either \( L_{-1, 0}^{2m-1} f + 1 \) acts onto the second factor \( f_1 \) or \( f_2 \). Then we obtain 0, since the sum of powers of determinants of order \( n \) in \( f_1 \) and \( f_2 \) equals \( [m-1] \), such functions are annihilated by \( L_{-1, 0}^{2m-1} f + 1 \).

Either \( L_{-1, 0}^{2m-1} f + 1 \) acts onto the second factor \( f_1 \) or \( f_2 \), and \( L_{-1, 0}^{2m-1} f + 1 \) acts onto the first factor. We obtain 0 by the same reason.

Either \( L_{-1, 0}^{2m-1} f + 1 \) acts onto the second factor, and \( L_{-1, 0}^{k} f \) acts onto the first factor, where \( k \geq 2 \). But the first factor is a vector of a representation with the highest weight \( \left[ \frac{3}{2}, \frac{3}{2} \right] \), thus it is annihilated by \( L_{-1, 0}^{2} f \).

Thus (13) vanishes under the action of \( L_{-1, 0}^{2m-1} f + 1 \). In the case of the series \( B \) the Lemma is proved.

Now consider the case of the series \( D \). Let \( m-1 \geq 0 \).

Suppose that the highest vector is \( (a_{n}, \ldots, -2, -1)^{\frac{p}{2}} \). The an arbitrary vector of the representation is a linear combination of vectors of type \( F_{p-1}^{p-1} F_{p-2}^{p-2} \cdots F_{n+1}^{p-n} (a_{n}, \ldots, -2, -1)^{\frac{p}{2}} \).
This vector is non-zero only if \( p_{-2} = \ldots = p_{-n} = 0 \) and \( p_{-1} = 0 \) or 1. Indeed when \( p_{-1} = 1 \) we obtain the vector \( a_{-n, \ldots, -1, 1} (a_{-n, \ldots, -2, -1})^\frac{1}{2} \). And when \( p_{-1} = 2 \) we obtain

\[
a_{-n, \ldots, 1, 2}(a_{-n, \ldots, -2, -1})^\frac{1}{2} - (a_{-n, \ldots, -1, 1})^2(a_{-n, \ldots, -2, -1})^\frac{1}{2} =
\]

\[
(a_{-n, \ldots, -2, -1})^\frac{1}{2}(a_{-n, \ldots, 1, 2}a_{-n, \ldots, -2, -1} - (a_{-n, \ldots, -1, 1})^2) = 0.
\]

In the derivation of this formulas relations from Lemma 8 were used. The further considerations in the case \( D \) and \( m_{-1} \geq 0 \) are analogous to considerations in the case \( B \). The case \( D \) with \( m_{-1} < 0 \) is considered analogously.

\[\square\]

**Lemma 4.** In the case of the series \( D \) for \( m_{-1} < 0 \) the analogues of Lemmas 2, 3 take place. But we must change the determinants of order \( n \) by the ruler \( a_{i_1, \ldots, i_n} \rightarrow \bar{a}_{\{i_1, \ldots, i_n\}} \), where \( \{1\} \leftrightarrow \{1\} \) is an interchange of indices \( -1 \) and 1.

**Definition 1.** Functions that satisfy conditions of Lemmas 2, 3 we call the admissible functions of the determinants.

Since determinant and functions of them satisfy automatically the condition 1 of Theorem 1 we obtain the following corollary

**Lemma 5.** The admissible functions of the determinants are exactly the functions that form an irreducible representation with the highest vector \( (8) \).

### 2.6 \( g_{n-1} \)-highest vectors

**Lemma 6.** A vector that is highest with respect to \( g_{n-1} \) can be represented as an admissible function of determinants such that it depends on determinants \( a_{i_1, \ldots, i_k} \) that vanish under the action of elements corresponding to positive roots of \( g_{n-1} \).

**Proof.** Let us show that one can represent a vector as a function of determinants that are highest with respect to \( g_{n-1} \). Let us use a realization in function on \( Z \) and the Proposition 1.

In [1] it is shown that in this realization the vectors that are highest with respect to \( g_{n-1} \) are polynomials in matrix elements \( z_{-k, -1}, z_{-k, 1} \). Admissible function of determinants that are highest with respect to \( g_{n-1} \) (see Section 2.6.1 where such determinants are listed), under restriction to \( Z \) give all possible such functions. By Proposition 1 the restriction to \( Z \) is a bijection. Hence in the

\[\footnote{In [1] the cases \( GL_{n+1}, Sp_{2n} \) are considered, the cases \( O_{2n}, O_{2n+1} \) are considered analogously.}\]
realization in the functions on $G_n$ every highest with respect to $g_{n-1}$ vector is an admissible function of determinants that are highest with respect to $g_{n-1}$.

Let us give an explicit description of the functions selected by Lemma 6.

### 2.6.1 Determinants that are highest with respect to $g_{n-1}$

Using formulas (5), (1), (2) we obtain that determinants that are highest with respect to $g_{n-1}$ are

\[ a_{-n}, a_{\pm 1}, a_{-n,-n+1}, a_{-n,\pm 1}, a_{-1,1}, \ldots, a_{-n,...,-3,-2}, a_{-n,...,-3,\pm 1}, a_{-n,...,-4,1,1} \]

and

1. $a_{-n,...,-2,-1}, a_{-n,...,-2,1}, a_{-n,...,-3,1,1}$ in the case of the series $A$,

2. $a_{-n,...,-2,-1}, a_{-n,...,-2,1}, a_{-n,...,-3,1,1}, a_{-n,...,-3,2,2}$ in the case of the series $C$,

3. $a_{-n,...,-2,-1}, a_{-n,...,-2,1}, a_{-n,...,-3,1,1}, a_{-n,...,-2,0}$ in the case of the series $B$,

4. $a_{-n,...,-3,2}, a_{-n,...,-2,-1}, a_{-n,...,-2,1}, a_{-n,...,-3,1,1}, a_{-n,...,-3,1,2}, a_{-n,...,-3,2,2}$,
   $a_{-n,...,-3,1,2}$ in the case of the series $D$ and $m_{-1} < 0$.

5. $a_{-n,...,-3,2}, a_{-n,...,-2,-1}, a_{-n,...,-2,1}, a_{-n,...,-3,1,1}, a_{-n,...,-3,1,2}, a_{-n,...,-3,2,2}$,
   $a_{-n,...,-3,1,2}$ in the case of the series $D$ and $m_{-1} \geq 0$.

But no all of these functions are independent.

**Lemma 7.** For functions on $Sp_{2n}$ one has a relation $a_{-n,...,-3,-2,2} = -a_{-n,...,-3,-1,1}$

**Proof.** For a typical matrix $X \in Sp_4$ one has a Gauss decomposition \([3]\). The matrices $\zeta, \delta$ and $z$ can be represented as exponents of Lie algebra elements $\zeta = e^A, \delta = e^B, z = e^C$. The matrix $A$ is a linear combination of $F_{i,j}, i > j$, $B$ is a linear combination of $F_{i,i}, C$ is a linear combination of $F_{i,j}, i < j$. Using this fact we obtain a parametrization of matrices $A, B, C$, then after taking an exponent a parametrization of matrices $\zeta, \delta \in z$, and finally a presentation of an arbitrary matrix $X \in Sp_4$. Then we can check the equality $a_{-n,...,-3,-2,2} = -a_{-n,...,-3,-1,1}$ by direct computations \([4]\). Since $Sp_4^0$ is dense in $Sp_4$ the equality holds everywhere on $Sp_4$.

Consider the case of an arbitrary group $Sp_{2n}$. Let $\mathcal{F}_{i,j}(\alpha)$ be a matrix with units on the diagonal, with $\alpha$ on the place $(i,j)$ and with $-\alpha$ on the

---

\(^{3}\text{Of course it is better to do it using a computer.}\)
place \((-j, -i)\). The matrices \(F_{i,j}(\alpha)\) belong to \(Sp_{2n}\). A multiplication by this matrix on the right is equivalent to doing an elementary transformation of the matrix \(X\): we add to the \(j\)-th row the \(i\)-th row with a coefficient \(\alpha\), and we add simultaneously to the \(-i\)-th row the \(-j\)-the row with a coefficient \(-\alpha\). A multiplication by \(F_{i,j}(\alpha)\) on the left is equivalent to an analogous transformation of rows.

Multiplying by \(F_{i,j}(\alpha)\) on the left and on the right we can transform \(X\) preserving \(a_{-n, \ldots, -3, -2, 2}\) and \(a_{-n, \ldots, -3, -1, 1}\) to the following form

\[
\begin{pmatrix}
  x_{-n,-n} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & x_{-n,n} \\
  \vdots & & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
  0 & \cdots & x_{-3,-3} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
  0 & \cdots & 0 & x_{-2,-2} & x_{-2,-1} & x_{-2,1} & x_{-2,2} & 0 & \cdots & 0 \\
  0 & \cdots & 0 & x_{-1,-2} & x_{-1,-1} & x_{-1,1} & x_{-1,2} & 0 & \cdots & 0 \\
  0 & \cdots & 0 & x_{1,-2} & x_{1,-1} & x_{1,1} & x_{1,2} & 0 & \cdots & 0 \\
  0 & \cdots & 0 & x_{2,-2} & x_{2,-1} & x_{2,1} & x_{2,2} & 0 & \cdots & 0 \\
  x_{n,-n} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & x_{n,n}
\end{pmatrix}
\tag{14}
\]

Since matrices \(F_{i,j}(\alpha)\) belong to \(Sp_{2n}\), this matrix belongs to \(Sp_{2n}\). We have

\[
a_{-n, \ldots, -3, -2, 2} = x_{-n,-n} \cdots x_{-3,-3} \cdot \det \begin{pmatrix} x_{-2,-2} & x_{-2,2} \\ x_{-1,-2} & x_{-1,2} \end{pmatrix},
\]

\[
a_{-n, \ldots, -3, -1, 1} = x_{-n,-n} \cdots x_{-3,-3} \cdot \det \begin{pmatrix} x_{-2,-1} & x_{-2,1} \\ x_{-1,-1} & x_{-1,1} \end{pmatrix}.
\]

The submatrix

\[
\begin{pmatrix}
  x_{-2,-2} & x_{-2,-1} & x_{-2,1} & x_{-2,2} \\
  x_{-1,-2} & x_{-1,-1} & x_{-1,1} & x_{-1,2} \\
  x_{1,-2} & x_{1,-1} & x_{1,1} & x_{1,2} \\
  x_{2,-2} & x_{2,-1} & x_{2,1} & x_{2,2}
\end{pmatrix}
\tag{15}
\]

belongs to \(Sp_{4}\). Using the equality \(a_{-2,2} = -a_{-1,1}\) for \(Sp_{4}\) we obtain that \(a_{-n, \ldots, -3, -2, 2} \in Sp_{2n}\) and \(a_{-n, \ldots, -3, -1, 1} \in Sp_{2n}\).

Analogously one prove the following statement.

**Lemma 8.** In the case of the series \(B\) one has an equality

\[
a_{-n, \ldots, -2, 1} a_{-n, \ldots, -2, -1} = \frac{1}{2} a_{-n, \ldots, -2, 0}^2.
\]
in the case of the series \( D \) and \( m_{-1} < 0 \) one has equalities the

\[ a_{-n,...,-3,2} \cdot a_{-n,...,-3,-2} = -a_{-n,...,-3,1} \cdot a_{-n,...,-3,-1}, \quad \bar{a}_{-n,...,-2,-1} = 0, \]
\[ a_{-n,...,-3,1} \cdot a_{-n,...,-3,-2} = -a_{-n,...,-3,1} \cdot \bar{a}_{-n,...,-3,1}, \]
\[ a_{-n,...,-3,2} \cdot \bar{a}_{-n,...,-3,-2} = -a_{-n,...,-3,1} \cdot \bar{a}_{-n,...,-3,-2}, \]
\[ a_{-n,...,-3,-2,2} = -a_{-n,...,-3,-1,1}, \quad \bar{a}_{-n,...,-3,1,2} = 0, \]

in the case of the series \( D \) and \( m_{-1} \geq 0 \) one has the equalities

\[ a_{-n,...,-3,2} \cdot a_{-n,...,-3,-2} = -a_{-n,...,-3,1} \cdot a_{-n,...,-3,-1}, \quad a_{-n,...,-2,1} = 0, \]
\[ a_{-n,...,-3,1} \cdot a_{-n,...,-3,-2} = -a_{-n,...,-3,1} \cdot a_{-n,...,-3,-2}, \]
\[ a_{-n,...,-3,1,2} = 0, \quad a_{-n,...,-3,-2,2} = a_{-n,...,-3,-1,1}, \]
\[ a_{-n,...,-3,1,2} \cdot a_{-n,...,-3,-2} = -a_{-n,...,-3,1} \cdot a_{-n,...,-3,-2}, \]

So we have proved the following.

**Theorem 2.** The vectors that are highest with respect to \( g_{n-1} \) are admissible functions of variables

\[ a_{-n}, a_{1 \pm 1}, a_{-n,-n+1}, a_{n \pm 1}, a_{-1,1}, \ldots, a_{-n,...,-3,2}, a_{-n,...,-3,1}, a_{-n,...,-4,1,1}, \]

and also

1. \( a_{-n,...,-2,1}, a_{-n,...,-2,1}, a_{-n,...,-3,-1,1} \) in the cases \( A \) and \( C \),
2. \( a_{-n,...,-2,1}, a_{-n,...,-2,0} \) in the case \( B \),
3. \( a_{-n,...,-3,2}, \bar{a}_{-n,...,-2,1}, a_{-n,...,-3,-1,1}, \bar{a}_{-n,...,-3,-1,2} \) in the case \( D \) and negative \( m_{-1} < 0 \),
4. \( a_{-n,...,-3,2}, a_{-n,...,-2,1}, a_{-n,...,-3,-1,1}, a_{-n,...,-3,1,2} \) in the case \( D \) and positive \( m_{-1} \geq 0 \).

In the case \( D \) we use determinants that are dependent, since if we express the dependent determinants through others we obtain non-admissible functions.

### 2.6.2 Relations between determinants.

Since we consider determinants of submatrices of a big matrix, we have the Plucker relations

**Lemma 9.** \( \sum a_{\sigma(i_1, \ldots, i_k) a_{j_1, \ldots, j_l}} = 0 \), we take a summation over all permutations of the set of indices \( i_1, \ldots, i_k, j_1, \ldots, j_l \).
If one passes to the realization in the space of the functions on the group \( Z \), then one can express all determinants through the independent matrix elements \( z_{-k,-1}, z_{-k,1}, k = -n, \ldots, -2 \), and \( z_{-1,1} \) in the cases \( A, C \); \( z_{-1,0} \) in the case \( B \).

One has

\[
a_{-n,\ldots,-k} = 1, \quad a_{-n,\ldots,-k-1,1} = z_{-k,1}, \quad a_{-n,\ldots,-k-1,-1} = z_{-k,-1},
\]
\[
a_{-n,\ldots,-k-2,-1,1} = z_{-k,1}z_{-k-1,1}, \quad a_{-n,\ldots,-k-1,1} = z_{-k,1}z_{-k-1,-1},
\]
\[
a_{-n,\ldots,-2,0} = z_{-1,0} \text{ for the series } B,
\]
\[
a_{-n,\ldots,-3,2} = -z_{-2,1}z_{-2,-1} \text{ for the series } D,
\]
\[
\bar{a}_{-n,\ldots,-2,1} = 1, \quad \bar{a}_{-n,\ldots,-3,-1,1} = z_{-2,-1}, \quad \bar{a}_{-n,\ldots,-3,-1,2} = -z_{-2,-1}^2 \text{ for the series } D \text{ and } m_{-1} < 0,
\]
\[
a_{-n,\ldots,-2,-1} = 1, \quad a_{-n,\ldots,-3,-1,1} = -z_{-2,1}, \quad a_{-n,\ldots,-3,-1,2} = -z_{-2,1}^2 \text{ for the series } D \text{ and } m_{-1} \geq 0.
\]

Analyzing these expressions we come to the conclusion

**Lemma 10.** There are no other relations other then those, presented in Lemmas 8, 9.

Using results of the Section 2.6.1 and Theorem 2 we come to the conclusion.

**Theorem 3.** A function represents a \( g_{n-1} \)-highest vector in a representation with the highest vector \( 0 \) if and only if it is a polynomial in determinants listed in Theorem 2 or a function of these determinants of type (13) such that the following property holds. If one expands this function as into a sum of products of determinants the sum of powers of determinants \( a_{-n,\ldots,-k-1,-k}, a_{-n,\ldots,-k-1,-1}, \]
\( a_{-n,\ldots,-k-1,1}, a_{-n,\ldots,-k-2,-1,1}, \) and \( a_{-n,\ldots,-3,-2} \) in the case \( k = 2 \) and the series is \( D \) equals \( v_{-k} \) for \( k = n, \ldots, 2 \) and also

1. For the series \( A, C \) the sum of powers \( a_{-n,\ldots,-2,-1}, a_{-n,\ldots,-2,1}, a_{-n,\ldots,-3,-1,1} \) equals \( m_{-1} \).
2. For the series \( B \) the sum of powers \( a_{-n,\ldots,-2,-1}, a_{-n,\ldots,-2,0}, a_{-n,\ldots,-3,-1,1} \) equals \( m_{-1} \).
3. For the series \( D \) and \( m_{-1} < 0 \) the sum of powers \( \bar{a}_{-n,\ldots,-2,1}, \bar{a}_{-n,\ldots,-3,-1,1}, \bar{a}_{-n,\ldots,-3,-1,2} \) equals \( -m_{-1} \).
4. For the series \( D \) the sum of powers \( a_{-n,\ldots,-2,-1}, a_{-n,\ldots,-3,-1,1}, a_{-n,\ldots,-3,1,2} \) equals \( m_{-1} \).
3 The problem of restriction \( g_2 \downarrow g_1 \)

3.1 The case \( sp_4 \)

Let us be given a representation of \( sp_4 \) with the highest weight \([m_{-2}, m_{-1}]\). Consider the problem of restriction \( sp_4 \downarrow sp_2 \). In [1] it is shown that the problems \( gl_3 \downarrow gl_1 \) and \( sp_4 \downarrow sp_2 \) are equivalent. Thus \( sp_2 \)-highest vectors are encoded by integer tableaux that satisfy the betweenness conditions

\[
\begin{array}{ccc}
  m_{-2} & m_{-1} & 0 \\
  k_{-2} & k_{-1} & \ \\
  s_{-2}
\end{array}
\]  

(16)

Let us give a formula for a function corresponding to a tableau. In [9] it is shown that to a \( gl_2 \)-highest vector encoded by a tableau \((16)\) with \( k_{-2} = s_{-2} \) there corresponds a polynomial

\[
a_{1}^{m_{-2} - k_{-2}} a_{-2,1}^{k_{-2} - m_{-1}} a_{-2,1}^{m_{-1} - k_{-1}} a_{-2,1}^{k_{-2} - 1}.
\]  

(17)

To obtain a vector corresponding to the tableau \((16)\) one must apply to this monomial \((s_{-2} - k_{-1})! / (k_{-2} - s_{-2})! \), one obtains modulo multiplication by a constant the polynomial

\[
a_{1}^{m_{-2} - k_{-2}} a_{-2,1}^{k_{-2} - m_{-1}} \sum_{p_{-1}, p_{-1,1}, p_{1, p_{-2,1}}} \frac{1}{p_{-1}! p_{1, p_{-2,1}}!} a_{-1,1}^{p_{-1}} a_{-2,1}^{p_{1, p_{-2,1}}} a_{-2,1}^{p_{-2,1}},
\]  

(18)

where a summation is taken over all non-negative \( p_{-1}, p_{-1,1}, p_{1}, p_{-2,1} \), such that

\[
p_{-1} + p_{-2} = k_{-2} - m_{-1}, \quad p_{-1,1} + p_{-2,1} = m_{-1} - k_{-1}, \quad p_{-1} + p_{-1,1} = k_{-2} - s_{-2}.
\]  

(19)

3.2 The case \( o_5 \)

Let us be given a representation of \( o_5 \) with the highest weight \([m_{-2}, m_{-1}]\). Consider a problem restriction \( o_5 \downarrow o_3 \). In the Zhelobenko’s realization the problem \( o_{2n+1} \downarrow o_{2n-1} \) is considered in [10], where a relation with the restriction problem \( gl_{n+1} \downarrow gl_{n-1} \) is established. The \( o_3 \)-highest vectors in a \( o_5 \)-representation are encoded by a number \( \sigma \) and a tableaux that satisfy the betweenness conditions whose elements are simultaneously integers of half-integers

\[
\begin{array}{ccc}
  m_{-2} & m_{-1} & 0 \\
  \sigma, & k_{-2} & k_{-1} \ \\
  s_{-2}
\end{array}
\]  

(20)
where \( \sigma = 0, 1 \). If \( k_{-1} = 0 \) then \( \sigma = 0 \).

In [10] the realization in functions on the subgroup \( Z \) is used. If one passes to the realization in functions on the whole group one obtains

\[
a_{-2,0}^m k_{-2} \sum_{p_{-1}, p_{-1,1}, p_{-2,1}} \frac{1}{p_{-1}! p_{-1,1}! p_{-2,1}!} a_{-1,1}^{p_{-1,1}} a_{-2}^{p_{-2,1}} (a_{-2,1})^{p_{-2,1}}, \quad (21)
\]

where a summation is taken over all non-negative integers \( p_{-1}, p_{-1,1}, p_{-2,1} \), such that

\[
p_{-1} + p_{-2} = k_{-2} - m_{-1}, \quad p_{-1,1} + p_{-2,1} = m_{-1} - k_{-1}, \quad p_{-1} + p_{-1,1} = k_{-2} - s_{-2}.
\]

(22)

To obtain a function described in Theorem 3 one must make a change

\[
a_{-2,1} \mapsto \frac{a_{-2,0}}{2a_{-2,1}}.
\]

Let us prove this fact without a reference to [10].

Consider the case of an integer highest weight. Then admissible functions are just polynomials in determinants. Consider first the case of an integer highest weight. Since \( f \) is an element of an irreducible representation with the highest vector \( v_0 \), it can be represented as a linear combination of vectors of type \( F^p_{-1, -2} F^q_{0, -1} v_0 \), where \( v_0 = a_{-2}^{m_{-2} - m_{-1}} a_{-2,1}^{m_{-1}} \). One has

\[
F^2_{0, -1} a_{-2, -1} = 2 F_{0, -1} a_{-2, -2, 0} = 2a_{-2, 0}^2 - 2a_{-2, 1}^2 = -4a_{-2, -1} a_{-2, 1}, \quad (23)
\]

thus in the case \( q = 2q' \) one has

\[
F^p_{-1, -2} F^{2q'}_{0, -1} v_0 = \text{const} E^p_{-1, -2} E^{q'}_{1, -1} v_0.
\]

The polynomial on the right satisfies the conditions of Theorem 3 for the algebra \( \mathfrak{gl}_3 \) and the highest weight \([m_{-2}, m_{-1}, 0]\). Thus there exists an isomorphism between the span of \( F^p_{-1, -2} F^{2q'}_{0, -1} v_0 \) and the space of representation of \( \mathfrak{gl}_3 \) with the highest weight \([m_{-2}, m_{-1}, 0]\). Using the previous Section one concludes that in the span of \( F^p_{-1, -2} F^{2q'}_{0, -1} v_0 \) there exists a base (21), given by tableaux (20) where \( \sigma = 0 \).

In the considered case the eigenvalues of \( F_{-2, -2} \) and \( F_{-1, -1} \) correspond to eigenvalues of \( E_{-2, -2} \) and \( E_{-1, -1} = E_{1, 1} \). The later are equal to \( s_{-2} \) and \( -2(k_{-2} + k_{-1}) + (m_{-2} + m_{-1}) + s_{-2} \).

In the case \( q = 2q' + 1 \) one has

\[
F^p_{-1, -2} F^{2q' + 1}_{0, -1}(a_{-2}^{m_{-2} - m_{-1}} a_{-2,1}^{m_{-1}}) = \text{const}(E^p_{-1, -2} E^{q'}_{1, -1}(a_{-2}^{m_{-2} - m_{-1}} a_{-2,1}^{m_{-1}})) a_{-2,1}.
\]

If one removes \( a_{-2,0} \) then one obtains on the right a polynomial which satisfies the conditions of Theorem 3 for the algebra \( \mathfrak{gl}_3 \) and the highest weight.

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[m_2 - 1, m_1 - 1, 0]. In the space of such polynomials there exists a base \( [18] \) encoded by an integer tableaux of type

\[
\begin{align*}
m_{-2} - 1 & \quad m_{-1} - 1 \quad 0 \\
k_{-2} - 1 & \quad k_{-1} - 1 \\
s_{-2} - 1
\end{align*}
\] (24)

Note that here \( k_{-1} - 1 \geq 0 \)

Thus in the span of \( F_{1,-2}^q v_0 \) there exists a base \( [21] \) encoded by \( [20] \) where \( \sigma = 1 \) and \( k_{-1} \geq 1 \).

The eigenvalues of \( F_{-2,-2} \) and \( F_{-1,-1} \) correspond to eigenvalues of \( E_{-2,-2} - 1 \) and \( E_{-1,-1} - E_{1,1} \). The later are equal to \( s_{-2} - 1 \) and \(-2(k_{-2} - 1 + k_{-1} - 1) + (m_{-2} - 1 + m_{-1} - 1) + s_{-2} - 1 = -2(k_{-2} + k_{-1}) + (m_{-2} + m_{-1}) + s_{-2} + 1 \).

Thus in the case of integer highest weight there exists a base \( [21] \) encoded by \( [20] \) where \( \sigma = 0, 1 \).

Now consider the case of half-integer highest weight. One has

\[
F_{0, -1}^{-1/2} = \frac{1}{2} a_{-2,0}^{-1/2}, \quad F_{0, -1}^{1/2} = 0. \quad (25)
\]

The highest vector can be written as \( v_0 = a_{-2}^{m_{-2} - m_{-1}} a_{-2, -1}^{[m_{-1}] + 1/2} \), where \([m_{-1}]\) is the integer part. A vector of an irreducible representation is a linear combination of vectors \( F_{1,-2}^{q} F_{0,-1}^{q} (a_{-2}^{m_{-2} - m_{-1}} a_{-2, -1}^{[m_{-1}] + 1/2}) \).

If \( q = 2q' \) then using \( [23] \), \( [25] \) we obtain that that this vector equals to

\[
F_{1,-2}^{p} F_{0,-1}^{q'} (a_{-2}^{m_{-2} - m_{-1}} a_{-2, -1}^{[m_{-1}] + 1/2}) = const E_{1,-2}^{p} E_{1,-1}^{q'} (a_{-2}^{m_{-2} - m_{-1}} a_{-2, -1}^{[m_{-1}] + 1/2}) a_{-2, -1}^{1/2},
\]

thus we have a natural isomorphism between the span of \( F_{1,-2}^{p} F_{0,-1}^{q'} v_0 \) and the space of an irreducible representation of \( gl_3 \) with the highest weight \([m_{-2} - \frac{1}{2}, m_{-1} - \frac{1}{2}, 0]\). In this space there exists a base encoded by an integer tableau of type

\[
\begin{align*}
m_{-2} - \frac{1}{2} & \quad m_{-1} - \frac{1}{2} \quad 0 \\
k_{-2} - \frac{1}{2} & \quad k_{-1} - \frac{1}{2} \\
s_{-2} - \frac{1}{2}
\end{align*}
\] (26)

Using the formula for the vector corresponding to a tableau in the case \( gl_3 \) we obtain that in the span of \( F_{1,-2}^{p} F_{0,-1}^{q'} v_0 \) the exist a base \( [18] \) encoded by half-integer tableau \( [20] \) with \( \sigma = 0 \).

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The eigenvalues of \( F_{-2,-2} \) and \( F_{-1,-1} \) correspond to eigenvalues of \( E_{-2,-2} - \frac{1}{2} \) and \( E_{-1,-1} - E_{1,1} - \frac{1}{2} \). One can easily check that \( s_{-2} \) is the \((-2)\)-th component of the weight and \((-1)\)-th component of the weight is calculated as \(-2(k_{-2} + k_{-1}) + (m_{-2} + m_{-1}) + s_{-2} \).

Now let \( q = 2q' + 1 \), one has

\[
F_{-1,-2}^p F_{0,-1}^{2q'+1} (a_{-2}^{m_{-2}-m_{-1}} a_{-2,-1}^{[m_{-1}]+1/2}) = \text{const}(E_{-1,-2}^p E_{1,-1}^{q'} a_{-2}^{m_{-2}-m_{-1}} a_{-2,-1}^{[m_{-1}]+1/2}) a_{-2,0} a_{-2,-1},
\]

thus there exists an isomorphism between the span of \( F_{-1,-2}^p F_{0,-1}^{2q'+1} v_0 \) and the space of an irreducible representation of \( \mathfrak{gl}_3 \) with the highest weight \([m_{-2} - \frac{1}{2}, m_{-1} - \frac{1}{2}, 0] \). Thus in the span the exists a base of type \([13] \) encoded by half-integer tableau \([20] \) with \( \sigma = 1 \). The eigenvalues of \( F_{-2,-2} \) and \( F_{-1,-1} \) correspond to eigenvalues of \( E_{-2,-2} - \frac{1}{2} \) and \( E_{-1,-1} - E_{1,1} + \frac{1}{2} \). One can easily check that \( s_{-2} \) is the \((-2)\)-th component of the weight and \((-1)\)-th component of the weight is calculated as \(-2(k_{-2} + k_{-1}) + (m_{-2} + m_{-1}) + s_{-2} + 1 \).

Thus in the case of half-integer highest weight there exists a base \([21] \) encoded by \([20] \) where \( \sigma \) = 0, 1.

The weight is calculated by the following ruler: \( s_{-2} \) is a \((-2)\)-component of the weight of the \([21] \) and it’s \((-1)\)-component is equals to \(-2(k_{-2} + k_{-1}) + (m_{-2} + m_{-1}) + s_{-2} + \sigma \).

3.3 The case \( \mathfrak{o}_4 \)

Let us be given an irreducible representation of \( \mathfrak{o}_4 \) with the highest weight \([m_{-2}, m_{-1}] \), where \( m_{-2} \geq [-m_{-1}] \).

Let us construct a base using a restriction of algebras \( \mathfrak{o}_4 \downarrow \mathfrak{o}_2 \). One has \( \mathfrak{o}_4 = \mathfrak{o}_2 \oplus \mathfrak{o}_2 \), where these copies of \( \mathfrak{o}_2 \) are

\[
\begin{align*}
\text{span} & < F_{-1,-2}, F_{-2,-2} - F_{-1,-1}, F_{-2,-1} >, \\
\text{span} & < F_{1,-2}, F_{-2,-2} + F_{-1,-1}, F_{-2,1} >.
\end{align*}
\]

There exists the following base in the representation

\[
\frac{(m_{-2} - m_{-1} - k)! (m_{-2} + m_{-1} - l)!}{(m_{-2} - m_{-1})! (m_{-2} + m_{-1})!} F_{-1,-2}^k v_0, \quad 0 \leq k \leq m_{-2} - m_{-1}, \quad 0 \leq l \leq m_{-2} + m_{-1}, \quad k, l \in \mathbb{Z},
\]

and \( v_0 \) is a highest vector. The weight of this vector equals to

\[
(m_{-2} - k - l, m_{-1} + k - l)
\]

Let us give another indexation of vectors \([27] \).
3.3.1 The case $m \geq 0$

The highest vector is written as follows

$$v_0 = a_{m-2}^{m-1} a_{m-1}^{m-1}, \quad (29)$$

The operator $F_{-1, -2}$ acts onto determinants that are highest with respect to $a_2$ as follows

$$a_{-2} \mapsto a_{-1}, \quad a_1 \mapsto -a_2, \quad \text{other determinant } \mapsto 0.$$

The operator $F_{1, -2}$ acts onto determinants that are highest with respect to $a_2$ as follows

$$a_{-2} \mapsto a_{1}, \quad a_{-1} \mapsto -a_2, \quad a_{-2, -1} \mapsto -2a_{-1, 1}, \quad a_{-1, 1} \mapsto a_{1, 2}, \quad \text{other determinant } \mapsto 0.$$

Thus (27) modulo multiplication on a constant equals to

$$\sum_{p_{-1}, p_1, p_2, p_2'} \left( \frac{(-1)^{p_2 + p_2' + p_{-1, 1} + p_{1, 2}}}{p_{-1}! p_1! p_2! p_{2}'! (p_{-1, 1} + p_{1, 2})} \right) a_{-2}^{p_{-2}} a_{-1}^{p_{-1, 1}} a_1^{p_1} a_2^{p_2} a_{-2, -1}^{p_{-2, -1}} a_{-1, 1}^{p_{-1, 1}} a_{1, 2}^{p_{1, 2}}, \quad (30)$$

where the powers satisfy the equalities

$$p_{-2} + p_{-1} + p_1 + p_2 + p_2' = m_{-2} - m_{-1}, \quad p_{-2, -1} + p_{-1, 1} + 2p_{1, 2} = m_{-1}.$$  

$$p_{-1} + p_2' = k, \quad p_1 + p_2 + p_{-1, 1} + p_{1, 2} = l.$$

The powers $p_1, p_{-1}, p_2, p_2', p_{-2, -1}, p_{-1, 1}, p_{1, 2}$ are integer and non-negative.

By Lemma 8 one can express all determinants through $a_{-2}, a_1, a_{-1}, a_{-2, -1},$ one obtains that (27) modulo multiplication by a constant can be rewritten as follows

$$a_{-2}^{m_{-2} - m_{-1} - k - l} a_1^{k} a_{-1}^{m_{-1}} a_{2, -1}^{m_{-2}}, \quad (30)$$

but here the power of $a_{-2}$ can be negative.

Define numbers $k_{-2}, s_{-2}$ by formulas

$$k_{-2} = m_{-2} - k, \quad s_{-2} = m_{-2} - k - l. \quad (31)$$

Let us compose an integer or half-integer tableau

$$m_{-2} \quad m_{-1}$$

$$k_{-2}$$

$$s_{-2}$$

(32)
For its elements the following restrictions hold\footnote{Thus for $s_2$ in \ref{eq:32} the betweness condition do not hold}:

\begin{align}
  m_{-2} & \geq k_{-2} \geq m_{-1} \\
  m_{-2} & \geq |s_{-2}|
\end{align}

Using \ref{eq:28}, we obtain the following statement

**Proposition 2.** $(-2)$-component of the weight of the vector encoded by \ref{eq:32}, equals $s_{-2}$, $(-1)$-component of the weight equals $-2k_{-2} + (m_{-2} + m_{-1}) + s_{-2}$

3.3.2 The case $m_{-1} < 0$

The highest vector is written as

\begin{equation}
  v_0 = a_{-2}^{m_{-2}-m_{-1}} a_{-2,1}^{-m_{-1}},
\end{equation}

The vector \ref{eq:27} modulo multiplication by a constant equals

\[
\sum \left( -1 \right)^{p_2' + p_2'' + p_{-1,1} + p_{-1,2}} a_{-2}^{p_2 - 1} a_{-1}^{p_1} a_{1}^{p_1'} a_{2}^{p_2'} a_{2}^{p_{-2,1}} a_{-1,1}^{p_{-1,1}} a_{-1,2}^{p_{-1,2}},
\]

where summation is taken over non-negative integers such that

\[
  p_{-2} + p_{-1} + p_1 + p_2' + p_2'' = m_{-2} - m_{-1}, \quad p_{-2,1} + p_{-1,1} + 2p_{1,2} = -m_{-1},
\]

\[
  p_1 + p_2' = k, \quad p_2'' + p_{-1} + p_{-1,1} + p_{-1,2} = l
\]

Due to relations from Lemma\ref{lem:8} all determinants can be expressed through $a_{-2}$, $a_1$, $a_{-1}$, $a_{-2,1}$, as a result we obtain an expression

\begin{equation}
  a_{-2}^{m_{-2}-m_{-1}-k-1} a_{-1}^{k} a_{1}^{-m_{-1}} a_{2,1}^{m_{-1}},
\end{equation}

Put

\begin{equation}
  k_{-2} = m_{-2} - l, \quad s_{-2} = m_{-2} - k - l.
\end{equation}

Compose an integer or half-integer tableau

\begin{equation}
\begin{array}{ccc}
  m_{-2} & - & m_{-1} \\
  k_{-2} & & \\
  s_{-2} & & \\
\end{array}
\end{equation}
This is the needed indexation.

Using (28) we obtain the following statement. Note that we give a formula not \((-1)\), but for the \((+1)\)-th component of the weight.

**Proposition 3.** \((-2)\)-th component of a weight of a vector encoded by \((37)\) equals to \(s - 2\), \((+1)\)-th component of the weight equals to \(2k - 2 - (m - 2 - m) - s - 2\).

One can say that in the considered case 1 and \(-1\) change their roles.

### 4 The extension of restriction problems \(\mathfrak{gl}_3 \downarrow \mathfrak{gl}_1 \supset \mathfrak{gl}_2 \downarrow \mathfrak{gl}_0\)

In this Section we investigate a relation of restriction problems \(\mathfrak{gl}_2 \downarrow \mathfrak{gl}_0\) and \(\mathfrak{gl}_3 \downarrow \mathfrak{gl}_1\). This relation describes the fact that the Gelfand-Tsetlin tableaux that appears in the problem \(\mathfrak{gl}_3 \downarrow \mathfrak{gl}_1\) are obtained from tableaux for the problem \(\mathfrak{gl}_2 \downarrow \mathfrak{gl}_0\) by extension to the left.

Thus consider the problem of restriction \(\mathfrak{gl}_3 \downarrow \mathfrak{gl}_1\) and consider a tableau

\[
\begin{array}{ccc}
m - 2 & m - 1 & 0 \\
k - 2 & k - 1 \\
s - 2
\end{array}
\]  

(39)

According to the Section 3.1, to this tableau there corresponds a polynomial

\[
a_1^{m - 2 - k - 2} a_2^{k - 1} \sum_{\substack{p - 1 \geq 0 \geq m - 2 - m - 1 \geq \sum_{\text{all non-negative integers } p, p - 1, p - 2, 1}}}
\frac{1}{p - 1! p_1! p - 2, 1!} a_{p - 1} a_{p - 1, 1} a_{p - 2, 1} a_{p - 2, 1},
\]

where a summation is taken over all non-negative integers \(p, p - 1, p_2, 1\), such that

\[
p - 1 + p - 2 = k - 2 - m - 1, \quad p - 1, 1 + p - 2, 1 = m - 1 - k - 1, \quad p - 1 + p - 1, 1 = k - 2 - s - 2.
\]  

(40)

**Lemma 11.** Take a span of products of type

\[
a_{p - 2} a_{p - 1} a_{p - 1} a_{p - 2, 1} a_{p - 2, 1} a_{p - 1, 1},
\]

\[
p - 2 + p - 1 + p_1 = m - 2 - m - 1, \quad p - 2, 1 + p - 2, 1 + p - 1, 1 = m - 1,
\]

(41)
with a fixed sum $p_{-2,1} + p_{-1,1}$, introduce a number $k_{-1}$ by a formula $p_{-2,1} + p_{-1,1} = m_{-1} - k_{-1}$. Then in this span there exists a base, indexed by tableaux

\[
m_{-2} \quad m_{-1} \\
\quad k_{-2} \quad k_{-1} \\
\quad s_{-2}
\]

The space described in Lemma is actually a span of $\mathfrak{gl}_3$-tableaux with a fixed $\mathfrak{gl}_2$-tableau in the right upper corner. Thus we call the elements of this space solutions of the problem of extension $\mathfrak{gl}_3 \downarrow \mathfrak{gl}_1 \supset \mathfrak{gl}_2 \downarrow \mathfrak{gl}_0$ with given $m_{-2}, m_{-1}, k_{-1}$.

Since for the considered monomials the power $p_{-2,1} = k_{-1}$ is fixed we can divide all the polynomials by $a_{-2,-1}^p - a_{-2,-1}^{p-1}$. We obtain the following statement

**Lemma 12.** Take a span of products of type

\[
a_{-2}^{p-2} a_{-1}^{p-1} a_{1}^p a_{-2,1}^{p-2,1} a_{-1,1}^{p-1,1},
\]

\[
p_{-2} + p_{-1} + p_1 = m_{-2} - m_{-1}, \quad p_{-2,1} + p_{-1,1} = m_{-1} - k_{-1},
\]

Then in this span there exists a base, indexed by tableaux

\[
m_{-2} \quad m_{-1} \\
\quad k_{-2} \quad k_{-1} \\
\quad s_{-2}
\]

The space in which a base in Lemma 12 is constructed is not defined by numbers $m_{-2}, m_{-1}, k_{-1}$, but by differences $m_{-2} - m_{-1}, m_{-1} - k_{-1}$, which must be integer and non-negative. That is why we allow below the numbers $m_{-2}, m_{-1}, k_{-1}$ to be simultaneously half-integer.

**5 The extension problems $g_n \downarrow g_{n-1} \supset g_2 \downarrow g_1$**

For simplicity put $n = 3$. The case of an arbitrary $n$ is discussed at the end of the Section. The space of $g_2$-highest vectors in the problem $g_3 \downarrow g_2$ is spanned by products of type

\[
f = a_{-3}^{p-3} a_{-1}^{p-1} a_1^p a_{-3,2}^{p-3,-2} a_{-3,1}^{p-3,1} a_{-1,1}^{p-1,1} \cdot f_2,
\]

where
f_2 = a_{-3,-2,-1}^{p,-3,-2,1} a_{-3,-1,1}^{p,-3,-1,1} a_{-3,2,1}^{p,-3,2,1} для серии A,C ,

f_2 = a_{-3,-2,-1}^{p,-3,-2,1} a_{-3,-1,1}^{p,-3,-1,1} a_{-3,2,0}^{p,-3,2,0} для серии B ,

f_2 = a_{-3,1,a}^{p,-3,-2,1} a_{-3,-1,1}^{p,-3,-1,1} a_{-3,1,2}^{p,-3,1,2} для серии D и m_{-1} < 0

f_2 = a_{-3,1,a}^{p,-3,-2,1} a_{-3,-1,1}^{p,-3,-1,1} a_{-3,1,2}^{p,-3,1,2} для серии D и m_{-1} \geq 0

The conditions for exponents are written in the Theorem 3. Below we write them explicitly.

5.1 The case of series A, C, B

In the case of series A, C the powers are non-negative and satisfy

\[ p_{-3} + p_{-1} + 1 = m_{-3} - m_{-2}, \quad p_{-3,-2} + p_{-3,-1} + p_{-3,1} + p_{-1,1} = m_{-2} - m_{-1}, \]
\[ p_{-3,-2,-1} + p_{-3,-1,1} + p_{-3,2,1} = m_{-1}, \]

and in the case B the powers are non-negative and satisfy

\[ p_{-3} + p_{-1} + 1 = m_{-3} - m_{-2}, \quad p_{-3,-2} + p_{-3,-1} + p_{-3,1} + p_{-1,1} = m_{-2} - m_{-1}, \]
\[ p_{-3,-2,-1} + p_{-3,-1,1} + p_{-3,-2,0} = m_{-1}, \]

Consider solutions such that

\[ p_{-3,1} + p_{-1,1} = m_{-2} - k_{-2}; \]
\[ p_{-3,1} + p_{-3,-2} + p_{-3,-2,-1} + p_{-3,-1,1} + p_{-3,2,1} \text{ where } p_{-3,-2,0} \text{ is fixed} \]

For the series A, B, C the mapping of (41) to a polynomial that is a solution of the extension \( gl_{13} \downarrow gl_{1} \supset gl_{2} \downarrow gl_{0} \) with numbers \( m_{-3}, m_{-2}, k_{-2} \) by the ruler

\[ f \mapsto a_{-3,a}^{p,-3,1} a_{-2,1}^{p,-3,2,1} a_{-1,1}^{p,-3,1,1} \] (43)

gives a well-defined mapping from the space of polynomials (41), that satisfy (42), into the space of solutions of the problem \( gl_{13} \downarrow gl_{1} \supset gl_{2} \downarrow gl_{0} \). To prove this we need to check that (43) respects relations between determinants. The problem is that in \( f \) on the left side of the (45) determinant for the problem \( g_{3} \downarrow g_{2} \) occur and on the right side of (43) determinants for the problem \( gl_{3} \downarrow gl_{1} \) occur. The relations between determinants are written in Lemma 10. It says that the the determinants that are not mapped into unit under (43) satisfy only Plucker relations, which take place for determinant on both sides.
One sees that (43) maps isomorphically into the space of admissible linear combinations of (41), that satisfy (42), to the space of solutions of the extension $\mathfrak{gl}_3 \downarrow \mathfrak{gl}_1 \supset \mathfrak{gl}_2 \downarrow \mathfrak{gl}_0$ with $m_{-3}, m_{-2}, k_{-2}$. Thus we obtain.

**Lemma 13.** In the space of solutions (41), that satisfy (42) there exists a base indexed by an integer or half-integer tableau

\[
\begin{array}{cccc}
m_{-3} & m_{-2} \\
k_{-3} & k_{-2} \\
s_{-3}
\end{array}
\]  

(44)

If one does not put conditions (42) but allows these indices to take all possible values one obtains

**Corollary 1.** In the space of solutions of (41), there exists a base indexed by an integer or half-integer tableau of type

\[
\begin{array}{cccc}
m_{-3} \\
k_{-3} & D \\
s_{-3}
\end{array}
\]  

(45)

where $D$ is a $g_2$-tableau with the highest weight $[m_{-2}, m_{-1}, 0]$ in the case of the series $C$, $B$ and $[m_{-2}, m_{-1}, 0]$ in the case of the series $A$.

This tableau is a span of (41) with fixed $m_{-3}, k_{-3}, s_{-3}$, but different indices $p_{-3,-1}, p_{-3,-2}...$ which were fixed before. More precise put $q_1 = p_{-3,1} + p_{-1,1}$, $q_{-1} = p_{-3,-1}$, $q_{-2} = p_{-3,-2}$, $q_{-2,-1} = p_{-3,-2,-1}$, $q_{-1,1} = p_{-3,-1,1}$, $q_{-2,1} = p_{-3,-2,1}$ in the case of the series $A$, $C$ or $q_{-2,0} = p_{-3,-2,0}$ in the case of the series $B$. To the vector (44) associate a monomial

\[
b_1^{q_{-1}} b_2^{q_{-2}} b_3^{q_{-2,-1}} b_{-1,1}^{q_{-1,1}} b_{-2,1}^{q_{-2,1}}
\]  

in the cases $A$, $C$,

\[
b_1^{q_{-1}} b_2^{q_{-2}} b_3^{q_{-2,-1}} b_{-1,1}^{q_{-1,1}} b_{-2,0}^{q_{-2,0}}
\]  

in the case $B$, (46)

In the span of such monomial there exist a base given by $g_2$-tableaux $D$. Then the vector (45) is a linear combination of (44) with the same coefficients as we take for monomials (46) to obtain a polynomial corresponding to a tableau.

### 5.2 The case $D$.

#### 5.2.1 The case $m_{-1} \geq 0$

In the considered case the following inequalities hold
\[ p_{-3} + p_{-1} + p_1 = m_{-3} - m_{-2}, \quad p_{-3,-2} + p_{-3,-1} + p_{-3,1} + p_{-1,1} + p_{-3,2} = m_{-2} - m_{-1}, \]
\[ p_{-3,-2,-1} + p_{-3,-1,1} + 2p_{-3,1,2} = m_{-1}. \]

(47)

All power are non-negative.

There exist relations between determinants. Let us remove dependent determinants. By Lemma 3 we can remove \( a_{-3,2}, a_{-3,-1,1}, a_{-3,1,2} \), then the powers of \( a_{-3,-2}, a_{-3,-1}, a_{-3,1}, a_{-3,-2,-1} \) are changed by the following rulers

1. The power of \( a_{-3,-2} \) is changed to \( q_{-3,-2} = p_{-3,-2} - p_{-3,2} - p_{-3,-1,1} - 2p_{-3,-1,2} \).
2. The power of \( a_{-3,-1} \) is changed to \( q_{-3,-1} = p_{-3,-1} + p_{-3,2} \).
3. The power of \( a_{-3,1} \) is changed to \( q_{-3,1} = p_{-3,1} + p_{-3,2} + p_{-3,-1,1} + 2p_{-3,-1,2} \).
4. The power of \( a_{-3,-2,-1} \) is changed to \( q_{-3,-2,-1} = m_{-1} \).

Note that

\[ q_{-3,-2} + q_{-3,-1} + q_{-3,1} = m_{-2} - m_{-1}, \]

the power \( q_{-3,-2} \) can become negative, but \( q_{-3,-1}, q_{-3,1} \) are positive.

Consider solutions such that

\[ q_{-3,-1} + p_{-1,1} = m_{-2} - k_{-2}, \]
\[ q_{-3,1}, q_{-3,-2}, q_{-3,-2,-1}, \text{ are fixed} \tag{48} \]

From one hand one has

\[ m_{-2} - k_{-2} = q_{-3,-1} + p_{-1,1} \geq 0, \quad \Rightarrow m_{-2} \leq k_{-2}, \]

from the other hand one has

\[ k_{-2} = m_{-2} - p_{-3,-1} - p_{-3,2} - p_{-1,1} = m_{-1} + p_{-3,-2} + p_{-3,1} \geq m_{-1}. \]

Thus \( k_{-2} \) satisfies the inequalities

\[ m_{-2} \geq k_{-2} \geq m_{-1}. \]

The same arguments as in the previous Section show that the mapping

\[ f \mapsto a_1^{p_1} a_{-1}^{p_{-1}} a_2^{p_{-2}} a_{-1,1}^{p_{-1,1}} a_{-2,1}^{q_{-3,1}} \tag{49} \]
sends isomorphically the space of functions (47) that satisfy (48), into the space of solutions of the extension $\mathfrak{gl}_3 \downarrow \mathfrak{gl}_1 \supset \mathfrak{gl}_2 \downarrow \mathfrak{gl}_0$ with $m_{-3}, m_{-2}, k_{-2}$. Thus one obtains the following statement.

**Lemma 14.** In the case $m_{-1} \geq 0$ in the space of functions (47), that satisfy (48) there exists a base indexed by integer or half-integer tableaux

\[
\begin{array}{cc}
m_{-3} & m_{-2} \\
k_{-3} & k_{-2} \\
s_{-3} & &
\end{array}
\]  

(50)

Analogously to Corollary 1 for series $A, B, C$, for the series $D$ and $m_{-1} \geq 0$ we obtain

**Lemma 15.** In the case $m_{-1} \geq 0$ in the space of functions (47), that satisfy (48) there exists a base given by integer or half-integer tableaux of type

\[
\begin{array}{cc}
m_{-3} & \mathcal{D} \\
k_{-3} & \\
s_{-3} & &
\end{array}
\]

where $\mathcal{D}$ is a tableau for $\mathfrak{a}_4$ with the highest weight $[m_{-2}, m_{-1}]$.

**5.2.2 The case $m_{-1} < 0$**

In the case $m_{-1} < 0$ we operate as follows.

In the case under consideration the following inequalities do hold

\[
\begin{align*}
p_{-3} + p_{-1} + p_1 &= m_{-3} - m_{-2}, \\
p_{-3,-2} + p_{-3,-1} + p_{-3,1} + p_{-1,1} + p'_{-3,2} + p''_{-3,2} &= m_{-2} - m_{-1}, \\
p_{-3,-2,-1} + p_{-3,-1,1} + 2p_{-3,-1,2} &= -m_{-1}.
\end{align*}
\]  

(51)

All powers are non-negative.

Using relations from Lemma 8 we can express determinants $a_{-3,2}, a_{-3,-1,1}, a_{-3,-1,2}$ through other determinants, the powers of determinants $a_{-3,-2}, a_{-3,-1}, a_{-3,1}, a_{-3,-2,1}$ are changed by the following ruler.

1. The power of $a_{-3,-2}$ is changed to $q_{-3,-2} = p_{-3,-2} - p_{-3,2} - p_{-3,-1,1} - 2p_{-3,-1,2}$.

2. The power of $a_{-3,1}$ is changed to $q_{-3,1} = p_{-3,1} + p_{-3,2}$.

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3. The power of $a_{-3,-1}$ is changed to $q_{-3,-1} = p_{-3,-1} + p_{-3,2} + p_{-3,-1,1} + 2p_{-3,-1,2}$.

4. The power of $a_{-3,-2,1}$ is changed to $q_{-3,-2,1} = -m_{-1}$.

Note that

$$q_{-3,-2} + q_{-3,-1} + q_{-3,1} = m_{-2} - m_{-1}.$$  

The power $q_{-3,-2}$ can be negative but $q_{-3,-1}$, $q_{-3,1}$ is non-negative.

Consider functions such that

$$q_{-3,1} + p_{-1,1} = m_{-2} - k_{-2},$$

$$q_{-3,-1}, q_{-3,-2}, q_{-3,-2,1},$$ are fixed

Then

$$m_{-2} \geq k_{-2} \geq -m_{-1}.$$  

The same arguments as in the previous Section show that the mapping

$$f \mapsto a_{1,1}^{p_{1}} a_{-1}^{p_{-1}} a_{-2}^{p_{-2}} a_{-1,1}^{p_{-1,1}} a_{-2,1}^{q_{-3,-1}}$$

is an isomorphism between the space of polynomials \((47)\) that satisfy \((48)\) into the space of solution of the extension problem $\mathfrak{gl}_3 \downarrow \mathfrak{gl}_1 \supset \mathfrak{gl}_2 \downarrow \mathfrak{gl}_0$ with $m_{-3}, m_{-2}, k_{-2}$. We obtain the following Statements.

**Lemma 16.** When $m_{-1} < 0$ in the space of functions \((41)\), that satisfy \((48)\) there exists a base indexed by integer or half-integer tableaux

$$m_{-3} \quad m_{-2} \\ k_{-3} \quad k_{-2} \\ s_{-3}$$  

(54)

**Lemma 17.** In the case $m_{-1} < 0$ in the space of functions \((47)\), that satisfy \((48)\) there exists a base indexed by integer or half-integer tableaux of type

$$m_{-3} \\ k_{-3} \mathcal{D} \\ s_{-3}$$

where $\mathcal{D}$ is a tableau for $\mathfrak{o}_4$ with the highest weight $[m_{-2}, m_{-1}]$.  

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5.3 Considerations in the case of arbitrary $n$

To consider the case of an arbitrary $n$ we consider the sequence of extensions

$$g_n \downarrow g_{n-1} \supset g_n \downarrow g_{n-1} \supset \ldots \supset g_3 \downarrow g_2.$$  

As it is done above, one can show that all these extensions are isomorphic to the extension $\mathfrak{gl}_1 \downarrow \mathfrak{gl}_1 \supset \mathfrak{gl}_2 \downarrow \mathfrak{gl}_0$. The isomorphism for further extensions is analogous to those constructed in Section 5.1 for all series.

5.4 Summary

We come to the Theorem

**Theorem 4.** Let us be given a representation of $g_n$ with the highest weight $[m-n, \ldots, m-1]$ in the cases $B$, $C$, $D$ and $[m-n, \ldots, m-1, 0]$ in the case $A$. Consider the problem $g_n \downarrow g_{n-1}$. Then in the space of $g_{n-1}$-vectors there exists a base given by tableau such that in the whole tableau (including $\mathcal{D}$, except $s_{-2}$ in the of the series $D$) the betweenness conditions hold

$$
\begin{array}{cccc}
m_{-n} & \ldots & m_{-3} \\
k_{-n} & \ldots & k_{-3} & \mathcal{D} \\
s_{-n} & \ldots & s_{-3}
\end{array}
$$

where $\mathcal{D}$ is a tableau for $g_2$ with the highest weight $[m_{-2}, m_{-1}]$ for $B$, $C$, $D$ and $[m_{-2}, m_{-1}, 0]$ for $A$. All elements are simultaneously integer or half-integer for series $B$, $D$.

**Corollary 2.** The spaces of highest vectors with a fixed tableau $\mathcal{D}$ are isomorphic for series $A$, $B$, $C$, $D$.

Let us calculate a weight of a vector corresponding to a tableau. The $(-k)$-th component of the weight is a sum of powers of determinants that contain $(-k)$ minus the sum of powers of determinants that contain $k$.

**Proposition 4.** The lower row is a $g_{n-1}$-weight of the corresponding $g_{n-1}$-highest vector.

**Proof.** Consider the case $n = 3$, the case of an arbitrary $n$ is considered analogously. The index $(-2)$ is contained only in those determinants that participate in construction of $\mathcal{D}$. Using the statement for $g_2$-tableau we conclude that $s_{-2}$ is a $(-2)$-component of the weight.

Now consider the $(-3)$-th component of the weight. The sum of powers, that contain $(-3)$ equals to $k_{-2}$. The sum of powers of determinants that
contain \((-3)\) and that participate in the correspondence \((43), (49), (53)\) equals to \(s_{-3} - k_{-2}\). Thus the sum of all powers equals to \(k_{-2}\).

Let us give a formula for the eigenvalue of \(F_{-1,-1}\) or \(F_{1,1}\).

**Proposition 5.** The eigenvalue of \(F_{-1,-1}\) on the tableau \((55)\)

\[
-2 \sum_{i=1}^{n} k_{-i} + \sum_{i=1}^{n} m_{-i} + \sum_{i=1}^{n} s_{-i} \quad \text{in the case } p_{2n}, o_{2n},
\]

\[
-2 \sum_{i=1}^{n} k_{-i} + \sum_{i=1}^{n} m_{-i} + \sum_{i=1}^{n} s_{-i} + \sigma \quad \text{in the case } o_{2n+1},
\]

\[
-2 \sum_{i=1}^{n} k_{-i} + \sum_{i=1}^{n} m_{-i} + \sum_{i=1}^{n} s_{-i} \quad \text{in the case } o_{2n},
\]

In the case \(o_{2n}\) and \(m_{-1} < 0\) the eigenvalue of \(F_{1,1}\) on the tableau \((55)\)

\[
-2 \sum_{i=1}^{n} k_{-i} + \sum_{i=1}^{n} m_{-i} - m_{-1} + \sum_{i=1}^{n} s_{-i}
\]

**Proof.** Consider first the cases \(A, B, C\) and \(D\) with \(m_{-1} \geq 0\). Note that \(\sum_{k=-n}^{3}(m_{-i} - k_{-i})\) is a sum of powers of determinants of order less \(n-2\) that contain 1, and \(\sum_{k=-n}^{3}(k_{-i} - s_{-i})\) is a sum of powers of determinants of order less \(n-1\) that contain \(-1\). This follows from the fact that correspondences \((43), (49), (53)\) the indices \(-1\) and 1 are preserved. And after this correspondence the sum of powers of determinants that contain 1 or \(-1\) can be calculated using the standard rulers of calculation of the weight of a \(g_{3}\)-tableau.

Now we have to calculate analogous sums for determinant of orders \(n-2\) and \(n-1\). For series \(A, C\) these sums equal \((m_{-2} - k_{-2}) + (m_{-1} - k_{-1})\) and \((k_{-2} - s_{-2})\). For series \(B\) these sums equal \((m_{-2} - k_{-2}) + (m_{-1} - k_{-1}) + \sigma\) and \((k_{-2} - s_{-2}) + \sigma\). This follows from the ruler of calculation of the weight of a \(g_{2}\)-tableau.

Now take the difference of these sums, then one obtains the statement of the Proposition for series \(A, C\). Thus in these cases we know the sum of powers of determinants that contain 1 and the sum of powers of determinants that contain \(-1\). Taking the difference of them we obtain the needed expression.

For series \(D\) and \(m_{-1} \geq 0\) the considerations are slightly different: the formula for the difference of these sums of powers for the determinant of orders \(n-2\) and \(n-1\) is given in Proposition 2.

The case \(D\) with \(m_{-1} < 0\) is considered analogously, but one must interchange 1 and \(-1\).
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