Derivations of transient thermal Green’s functions in three-dimensional general anisotropic media

Jiakuan Zhou*, Xueli Han

1School of Aerospace Engineering, Beijing Institute of Technology, Beijing, 100081, China

*Corresponding author’s e-mail: jiakuan_zhou@qq.com

Abstract. In this paper, three-dimensional transient thermal Green’s functions in general anisotropic media are derived in relatively concise forms via the Radon transform. Both situations in full-space and half-space are provided. For the case in full-space, the governing equation of heat conduction problem in three-dimension is reduced to a similar one in one-dimension whose solution is existent. For the case in half-space, both Dirichlet and flux-free boundary conditions are considered, and the solutions are derived by an image method. Applying the inverse Radon transform to solutions in transform domain, Green’s functions in physical domain are subsequently expressed as an integral over a unit sphere. If written in terms of usual spherical coordinate, moreover, these solutions are regular integrals over finite intervals and can be evaluated easily and effectively. Numerical examples are presented to verify the accuracy and applicability of the present derivations, and to demonstrate the effects of distinguishing boundary conditions.

1. Introduction

Nowadays more and more non-conventional material are extensively utilized in aerospace and high-end manufacturing industries, while these new materials usually exhibit anisotropic properties. Moreover, most heat conduction problems refer to unsteady state from the view point of energy exchange. Thus the analysis of dynamic heat conduction problem in general anisotropic media is essential. Unfortunately, besides the time variable in dynamic problems, the cross-derivatives of spatial variables due to the anisotropy really make it harder to find solutions that satisfy partial differential equations of anisotropic heat conduction equations and boundary conditions [1]. However, the boundary element method (BEM) as well as the Green’s function method contributes a lot [2]. Thus the construction of Green’s functions becomes vital.

There are many classic techniques to construct Green’s functions nowadays. For example, Wang et al. [3] straightly extended Sommerfeld’s [4] image method to derive the temperature field in static heat conduction problem, and Ochmann[5] even extended traditional image method to complex image method, whereas difficulty of conceiving image sources remains when solving problems with complicated boundary conditions. Integral transform method is extensively used to change problems to transform domain that may be handled easier. Among which Fourier transform and Laplace transform are extremely widespread to deal with partial differential equations [6-9]. It is worth mentioning that the complex-variable method can be useful when performing inverse transforms, exclusively for two-dimensional (2D) problems [10]. Moreover, the technique of coordinate transformation which can map the original issue into an equivalent but simple one is also practical, particularly for materials with symmetries [11-12]. However, the involved interfaces of multi-layer problems seriously increase the
difficulty, and further algebraic constrains are required [13]. Alternative technique such as potential function is also meritorious for its particular advantages [14]. As mentioned above, however, the investigations of transient heat conduction problem in three-dimensional (3D) general anisotropic media are rarely available due to its complexity.

The objective of the present paper is to derive transient thermal Green’s functions in 3D general anisotropic media by Radon transform. The Radon transform, though not utilized as widely as the Fourier transform and the Laplace transform, is also useful in dealing with partial differential equations [15]. The most interesting characteristic of the Radon transform is its capability of reducing the 3D governing equation to a similar one in 1D [16-17]. By virtue of this characteristic, the governing equation of transient heat conduction problem in 3D general anisotropic media can be reduced to a similar one in 1D whose solution in full space is existent. In half-space, both Dirichlet and flux-free boundary conditions are considered, and corresponding solutions are obtained by image method. Green’s functions in physical domain are subsequently expressed as an integral over a unit sphere after inverse transform. These solutions in spherical coordinate are regular integrals over finite intervals and can be evaluated easily and effectively.

Finally, numerical examples are given to verify the accuracy and applicability of the present derivations, and to demonstrate the effects of distinguishing boundary conditions. Moreover, the high efficiency and concise forms of the present Green’s functions make it appropriate to incorporate in BEM.

2. Green’s functions in full-space
Along this paper, it is assumed that sub-indices \( i, j \) range from 1 to 3, and repeated indices imply summation. The symbols \( \partial_i \), \( \partial_t \) and \( \partial s \) indicate \( \frac{\partial}{\partial x_i} \), \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial s} \) respectively. Moreover, vectors are in bold italic type.

Consider a fixed Cartesian coordinate system \((x_1,x_2,x_3)\), the transient heat conduction equation in 3D general anisotropic media subjected to a unit point heat source is

\[ -\partial_i q_i - \partial_t T = -\delta(t)\delta(x - X) \]  

(1)

Where \( \delta(t) \) and \( \delta(x - X) \) are the Dirac delta functions, \( x \) and \( X \) are the field position and the source position respectively.

The heat flux \( q_i \) is related to the temperature field \( T \) through the Fourier’s law of

\[ q_i = -k_{ij} \partial_j T \]  

(2)

Where \( k_{ij} \) is the thermal diffusivity coefficient. \( k_{ij} \) is symmetric and positive definite according to the thermodynamic considerations. Substituting equation (2) to equation (1), the governing equation in terms of \( T \) becomes

\[ k_{ij} \partial_j \partial_i T - \partial T = -\delta(t)\delta(x - X) \]  

(3)

\[ T |_{t=0} = 0 \text{ and } \lim_{|x| \to \infty} T \to 0 \]  

(4)

Applying the Radon transform in equation (A.1) to equations (3) and (4), it yields

\[ k_{ij} \eta_j \eta_i \partial^2 T - \partial T = -\delta(t)\delta(s - n \cdot X) \]  

(5)

\[ \hat{T} |_{t=0} = 0 \text{ and } \lim_{s \to \infty} \hat{T} \to 0 \]  

(6)

with properties of equation (A.6) to (A.8) are used. Equation (5) is the governing equation of 1D transient heat conduction problem, whose solution can be found in Duffy’s [18] summarization as

\[ g(x,t|\xi,\tau) = \frac{H(t-\tau)}{\sqrt{4\pi a^2(t-\tau)}} \exp \left[ -\frac{(x-\xi)^2}{4a^2(t-\tau)} \right] \]  

(7)
Thus the corresponding solution to equation (5) has the same form of equation (7) as

$$\hat{T} = H(t) \frac{1}{\sqrt{4\pi t}} \exp \left[ -\frac{(s-n \cdot X)^2}{4\Gamma t} \right] \text{ where } \Gamma = k_n n_j$$

(8)

Then applying the inverse Radon transform defined by equation (A.4) to $\hat{T}$, the Green’s function of temperature field in physical domain are obtained as

$$T = -\frac{1}{8\pi^2} \int_0^{2\pi} d\beta \int_0^{\pi} H(t) \left\{ \frac{(s-n \cdot X)^2}{8\sqrt{\pi (\Gamma t)^{3/2}}} - \frac{1}{4\sqrt{\pi (\Gamma t)^{3/2}}} \right\} \exp \left[ -\frac{(s-n \cdot X)^2}{4\Gamma t} \right] \sin \alpha d\alpha$$

(9)

An application of similar procedure to the Fourier’s law of equation (2) yields

$$\hat{q}_i = -Y_i \partial_i \hat{T} \text{ where } Y_i = k_n n_j$$

(10)

Therefore the heat flux $q_i$ in the transform domain becomes

$$\hat{q}_i = H(t) Y_i \left( \frac{s-n \cdot X}{4\Gamma t} \right) \exp \left[ -\frac{(s-n \cdot X)^2}{4\Gamma t} \right]$$

(11)

and in physical domain we have

$$q_i = \frac{-1}{8\pi^2} \int_0^{2\pi} d\beta \int_0^{\pi} H(t) Y_i \left( \frac{(s-n \cdot X)^3}{16\sqrt{\pi (\Gamma t)^{3/2}}} - \frac{3(s-n \cdot X)}{8\sqrt{\pi (\Gamma t)^{3/2}}} \right) \exp \left[ -\frac{(s-n \cdot X)^2}{4\Gamma t} \right] \sin \alpha d\alpha$$

(12)

Equation (9) and equation (12) are the thermal Green’s functions which are called fundamental solutions also. Now that they are expressed as regular integrals over finite intervals in usual spherical coordinate, they can be evaluated easily and effectively by the Gauss quadrature formula. Therefore the Green’s functions may contribute significantly to BEM.

3. Green’s functions in half-space

Now, we consider a 3D general anisotropic media in upper half-space with a point source inside. The interface is coincident with the plan $x_3 = 0$, where Dirichlet boundary condition and flux-free boundary condition are considered respectively. To solve these problems, the image method is employed. For the Dirichlet boundary condition, the temperature in the interface satisfies

$$T|_{x_3=0} = 0$$

(13)

We can find this Green’s function by introducing an image source of $\delta(x - X')$, that owns opposite property of the original source, located at the mirror point $X' = (X_1, X_2, -X_3)$. Thus the half-space problem can be turned into an equivalent problem in full-space, and the corresponding governing equation becomes

$$k_n \partial_i \partial_i T - \partial_i T = \delta(t) \delta(x - X') - \delta(t) \delta(x - X)$$

(14)

Following the similar procedures of solving Green’s functions in full-space, and noticing the linearity of equation (A.5), we can obtain the Green’s functions in half-space under Dirichlet boundary condition as

$$T = \frac{-1}{8\pi^2} \int_0^{2\pi} d\beta \int_0^{\pi} H(t) \left( \frac{(s-n \cdot X)^2}{8\sqrt{\pi (\Gamma t)^{3/2}}} - \frac{1}{4\sqrt{\pi (\Gamma t)^{3/2}}} \right) \exp \left[ -\frac{(s-n \cdot X)^2}{4\Gamma t} \right] \sin \alpha d\alpha$$

$$-\frac{-1}{8\pi^2} \int_0^{2\pi} d\beta \int_0^{\pi} H(t) \left( \frac{(s-n \cdot X)^2}{8\sqrt{\pi (\Gamma t)^{3/2}}} - \frac{1}{4\sqrt{\pi (\Gamma t)^{3/2}}} \right) \exp \left[ -\frac{(s-n \cdot X')^2}{4\Gamma t} \right] \sin \alpha d\alpha$$

(15)
\[ q_i = -\frac{1}{8\pi^2} \int_0^{2\pi} d\beta \int_0^\pi H(t) \left[ \frac{(s-n \cdot X)^3}{16\sqrt{\pi}(\Gamma t)^{3/2}} - \frac{3(s-n \cdot X)}{8\sqrt{\pi}(\Gamma t)^{5/2}} \right] \exp \left[ -\frac{(s-n \cdot X)^2}{4\Gamma t} \right] \sin \alpha \, d\alpha \]

\[ -\frac{1}{8\pi^2} \int_0^{2\pi} d\beta \int_0^\pi H(t) \left[ \frac{(s-n \cdot X')^3}{16\sqrt{\pi}(\Gamma t)^{3/2}} - \frac{3(s-n \cdot X')}{8\sqrt{\pi}(\Gamma t)^{5/2}} \right] \exp \left[ -\frac{(s-n \cdot X')^2}{4\Gamma t} \right] \sin \alpha \, d\alpha \]

For the flux-free boundary condition, the temperature in the interface satisfies

\[ \partial_t T \bigg|_{x=x_0} = 0 \]

Therefore the image source should own same property of the original source and can be expressed as \(-\delta(x - x')\), and the equivalent governing equation in full-space becomes

\[ k_i \partial_t \partial_i T - \partial_i T = -\delta(t)\delta(x - x') - \delta(t)\delta(x - X) \]  

Similarly, the Green’s functions in half-space under flux-free boundary condition are

\[ T = -\frac{1}{8\pi^2} \int_0^{2\pi} d\beta \int_0^\pi H(t) \left[ \frac{(s-n \cdot X)^2}{8\sqrt{\pi}(\Gamma t)^{3/2}} - \frac{1}{4\sqrt{\pi}(\Gamma t)^{5/2}} \right] \exp \left[ -\frac{(s-n \cdot X)^2}{4\Gamma t} \right] \sin \alpha \, d\alpha \]

\[ + -\frac{1}{8\pi^2} \int_0^{2\pi} d\beta \int_0^\pi H(t) \left[ \frac{(s-n \cdot X')^2}{8\sqrt{\pi}(\Gamma t)^{3/2}} - \frac{1}{4\sqrt{\pi}(\Gamma t)^{5/2}} \right] \exp \left[ -\frac{(s-n \cdot X')^2}{4\Gamma t} \right] \sin \alpha \, d\alpha \]

Under both boundary conditions, the Green’s functions in half-space are the linear superposition of two Green’s functions with one pair of mirror point sources in full-space. The feature of linear superposition could also be utilized in deriving Green’s functions in bimaterials and multi-materials. The property of image source depends on the boundary condition, totally for the purpose that its field can be offset in the interface so as to satisfy the boundary condition.

4. Numerical examples

In numerical examples, we choose Graphite as transversely isotropic (and orthotropic also) material, and Graphite as anisotropic one, with thermal diffusivity coefficients being given in reference [17] as

\[
K_G = \begin{bmatrix}
355 \\
355 \\
89
\end{bmatrix} \text{W m}^{-1} \text{C}^{-1} \text{C} \text{m}^{-1}
\]

\[
K_G' = \begin{bmatrix}
321.75 \\
6.30 \\
87.74 \\
353.81 \\
-16.62 \\
123.44
\end{bmatrix} \text{W m}^{-1} \text{C}^{-1} \text{C} \text{m}^{-1}
\]

All the numerical results from the present solutions are implemented by Gauss quadrature formula, which is precise and efficient since the inverse transform integrals have been expressed as regular integrals over finite intervals.

First, we examine the accuracy of equation (9), the fundamental solution. When the material is orthotropic one, the classic solution is available as

\[ T = \frac{1}{8\sqrt{k_{11}k_{22}k_{33}(\pi t)^{3/2}}} \exp \left[ -\frac{(x_1 - X_1)^2}{k_{11}} + (x_2 - X_2)^2/k_{22} + (x_3 - X_3)^2/k_{33} \right] \frac{4t}{k_{11}} \]

The material constants of Graphite are used. The field point and source point are located at \(x = (1,2,3)\) and \(X = (0,0,0)\) respectively. The results are showed and compared in table 1. It’s clear that they are highly consistent. This comparison demonstrates the accuracy and applicability of the present work.
Table 1. The comparison of fundamental solutions for orthotropic case.

| Time (s) | Present results (℃) | Classical results (℃) |
|----------|----------------------|-----------------------|
| 0.005    | 5.971654606984E-05   | 5.971654606984E-05   |
| 0.01     | 3.761887185283E-04   | 3.761887185283E-04   |
| 0.05     | 3.37004418268E-04    | 3.37004418268E-04    |
| 0.1      | 1.589191522187E-04   | 1.589191522187E-04   |
| 0.5      | 1.789739072619E-05   | 1.789739072619E-05   |
| 1        | 6.512583233228E-06   | 6.512583233228E-06   |
| 5        | 5.960807940548E-07   | 5.960807940548E-07   |
| 10       | 2.113542530565E-07   | 2.113542530565E-07   |

Next, results of 3D Green’s functions for general anisotropic media are validated for Graphite*. Here we suppose that the field point and source point are located at $x = (0,1,0.5)$ and $X = (0,0,2)$ respectively. Temperature $T$ and heat flux $q_3$ are evaluated from $t = 0s$ to $t = 0.07s$, and are showed in figure 1. The half-space I and the half-space II denote the results for the Dirichlet boundary condition and flux-free boundary condition respectively. It can be seen that both temperature and heat flux rise sharply and almost the same at first. However, divergences appear once rising to the peak. Both the absence of media in lower half-space and the differences of boundary condition contribute to divergences. Generally, the absence of lower half-space and the Dirichlet boundary condition, especially for $T = 0$, accelerate the thermal equilibrium process, while another condition postpones this process.

![Figure 1](image1.png)

Figure 1. Variation of temperature $T$ (a) and heat flux $q_3$ (b) with time in full- and half-space

Distributions of temperature field and heat flux along a line $(0,1,x)$ axis are also investigated and showed in figure 2, with $t = 0.01s$ and source point at $X = (0,0,2)$. It is obvious that boundary conditions are fully satisfied, such as $T = 0$ of Dirichlet boundary condition and $q_3 = 0$ of flux-free boundary condition. All $q_1$ in full-space and half-space are nearly 0 in anywhere of the path, due to the coincidental assumption of field position and source position. Specifically, the plane where field point and source point exist happen to be perpendicular to the $x_1$-direction. Similar results are also mentioned by Buroni [17].
5. Conclusion
In this paper, 3D transient thermal Green’s functions in general anisotropic media are derived in relatively concise forms via the Radon transform. The final expressions are inverse transform integrals over a unit sphere, and can be evaluated efficiently in terms of usual spherical coordinate. Both solutions in full-space and half-space are obtained, and two different boundary conditions are investigated respectively. Moreover, numerical examples are provided to verify the accuracy of the Green’s functions, and some interesting features are discussed. The new derivation method can be further employed in solving problems in multi-materials, and the present solutions can be significant supplements to BEM.

6. Appendix Radon transform
The Radon transform and some properties are listed below. Further details can be found in references [15]. The Radon transform of an arbitrary function \( f(x) \) in \( \mathbb{R}^3 \) is defined as

\[
\hat{f}(s, \mathbf{n}) = \mathcal{R}[f(x)] = \int f(x)\delta(s - \mathbf{n} \cdot \mathbf{x})dx
\]

(A.1)

Where \( \mathbf{n} \) is a unit vector and can be written in terms of spherical coordinate polar angle \( \alpha \) and azimuthal angle \( \beta \) as

\[
\mathbf{n} = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha)
\]

(A.2)

\( s \) is the perpendicular distance from the origin to the hyperplane as

\[
s = \mathbf{n} \cdot \mathbf{x} = \sin \alpha \cos \beta x_1 + \sin \alpha \sin \beta x_2 + \cos \alpha x_3
\]

(A.3)

The inverse transform of the Radon transform in spherical coordinate is

\[
f(x) = -\frac{1}{8\pi^2} \int_0^{2\pi} d\beta \int_0^\pi \hat{f}(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) \sin \alpha d\alpha \quad \text{where} \quad \hat{f}(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) = \frac{\partial^2}{\partial s^2} f(s, \mathbf{n}) \bigg|_{s=n \cdot x}
\]

(A.4)

And some basic properties of

(a). Linearity

\[
\mathcal{R}[c_1 f + c_2 g] = c_1 \mathcal{R}[f] + c_2 \mathcal{R}[g]
\]

(A.5)

(b). Shifting property

\[
\mathcal{R}(f(x - X)) = \int f(x - X)\delta(s - \mathbf{n} \cdot \mathbf{x})dx = \hat{f}(s - \mathbf{n} \cdot \mathbf{X}, \mathbf{n})
\]

(A.6)

(c). Transform of derivatives

\[
\mathcal{R}(\partial_\alpha f) = n_\alpha \partial_\alpha \hat{f}(s, \mathbf{n}) \quad \text{and} \quad \mathcal{R}(\partial_\beta f) = n_\beta \partial_\beta \hat{f}(s, \mathbf{n})
\]

(A.7)

(d). Transform of Dirac delta function

Figure 2. Distribution of temperature \( T \) (a) and heat flux \( q_i \) (b) along \((0,1,x_i)\) in full- and half-space.
\[ \Re \delta(x) = \delta(s) \quad (A.8) \]

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