RIGHT-ANGLED ARTIN GROUPS WITH NON-PATH-CONNECTED BOUNDARY

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ABSTRACT. We place conditions on the presentation graph $\Gamma$ of a right-angled Artin group $A_\Gamma$ that guarantee the standard CAT(0) cube complex on which $A_\Gamma$ acts geometrically has non-path-connected boundary.

1. Introduction

In [7], Gromov showed that if $G$ is a hyperbolic group acting geometrically on two metric spaces $X$ and $Y$, then the boundaries of $X$ and $Y$ are homeomorphic. The same is not true for CAT(0) spaces; in [6] Croke and Kleiner demonstrate a group that acts geometrically on two CAT(0) spaces with non-homeomorphic boundaries, and it was later shown ([14]) that the same group has uncountably many distinct CAT(0) boundaries. The group is the right-angled Artin group whose presentation graph is the path on four vertices $P_4$, and so has presentation

$$\langle a, b, c, d \mid [a, b] = [b, c] = [c, d] = 1 \rangle.$$ 

In [5], it is shown that the boundary of the standard CAT(0) cube complex on which this group acts is non-path-connected. The boundary of such a cube complex is connected if and only if the the presentation graph of the group is connected (and so the group is one-ended). In this paper, the method in [5] is generalized to a class of right-angled Artin groups whose presentation graphs admit a certain type of splitting. The main theorem here is as follows:

**Theorem 1.1.** Let $\Gamma$ be a connected graph. Suppose $\Gamma$ contains an induced subgraph $\langle \{a, b, c, d\}, \{\{a, b\}, \{b, c\}, \{c, d\}\} \rangle$ (isomorphic to $P_4$), and there are subsets $B \subset \text{lk}(c)$ and $C \subset \text{lk}(b)$ with the following properties:

1. $B$ separates $c$ from $a$ in $\Gamma$, with $d \notin B$;
2. $C$ separates $b$ from $d$ in $\Gamma$, with $a \notin C$;
3. $B \cap C = \emptyset$.

Then $\partial S_\Gamma$ is not path connected.

Here, $S_\Gamma$ is the standard CAT(0) cube complex on which the right-angled Artin group $A_\Gamma$ with presentation graph $\Gamma$ acts geometrically, and $\text{lk}(v)$ is the set of vertices of $\Gamma$ sharing an edge with $v$. We in fact show a slightly stronger result, with the hypothesis $B \cap C = \emptyset$ replaced with the statement of Claim 3.7. The hypotheses here essentially require a copy of $P_4$ in $\Gamma$ that is either not contained in a cycle, or has every cycle containing it separated by chords based at $b$ and $c$. It is a known fact of graph theory that any graph that does not split as a join contains an induced subgraph isomorphic to $P_4$, and any graph $\Gamma$ that splits as a non-trivial join has $\partial S_\Gamma$ path connected, so the hypothesis that $\Gamma$ contain a copy of $P_4$ is satisfied in any interesting case.
If a connected boundary of a CAT(0) space is locally connected, then it is a Peano space (a continuous image of $[0,1]$) and therefore path connected. The boundaries of some right-angled Coxeter groups are therefore known to be path connected ([11] and [4]), because they are locally connected. However, a consequence of a theorem in [10] is that for right-angled Artin groups, $\partial S_G$ is locally connected iff $\Gamma$ is a complete graph; i.e. $A_\Gamma \cong \mathbb{Z}^n$ and $\partial S_G \cong S^{n-1}$. Thus no approach involving local connectivity works for right-angled Artin groups.

In [12], the construction of [6] is generalized to demonstrate a class of groups with non-unique boundary. These groups are of the form

$$G = (G_1 \times \mathbb{Z}^n) \ast_{\mathbb{Z}^n} (\mathbb{Z}^n \times \mathbb{Z}^m) \ast_{\mathbb{Z}^m} (\mathbb{Z}^m \times G_2),$$

where $G_1$ and $G_2$ are infinite CAT(0) groups. It is easily verified that if $G_1$ and $G_2$ are right-angled Artin groups, then $G$ is a right-angled Artin group whose presentation graph satisfies the conditions of the main theorem of this paper; in fact, the method of this paper should work even if $G_1$ and $G_2$ are arbitrary infinite CAT(0) groups.

It seems this boundary path connectivity problem may be related to the question of when two right-angled Artin groups are quasi-isometric. In [1], Behrstock and Neumann show that all right-angled Artin groups whose presentation graphs are trees of diameter greater than 2 are quasi-isometric; in [2], Bestvina, Kleiner, and Sageev show that right-angled Artin groups with atomic presentation graphs (no valence 1 vertices, no separating vertex stars, and no cycles of length $\leq 4$) have $A_\Gamma$ quasi-isometric to $A_{\Gamma'}$ iff $\Gamma \cong \Gamma'$. The connection between these results and the result of this paper is that if $\Gamma$ is a tree of diameter greater than 2, then $\Gamma$ satisfies the hypotheses of the main theorem here, and therefore $\partial S_G$ has non-path-connected boundary; if $\Gamma$ is atomic, then $\Gamma$ cannot satisfy the hypotheses of the main theorem here.

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2. Preliminaries

**Definition 2.1.** Given a (undirected) graph $\Gamma$ with vertex set $S = \{a_1, \ldots, a_n\}$, the corresponding **right-angled Artin group** $A_\Gamma$ is the group with presentation

$$\langle a_1, \ldots, a_n \mid [a_i, a_j] \text{ if } i < j \text{ and } \{a_i, a_j\} \text{ is an edge of } \Gamma \rangle.$$ 

We call $\Gamma$ the **presentation graph** for $A_\Gamma$.

**Definition 2.2.** If $A_\Gamma$ is a right-angled Artin group with Cayley graph $\Lambda_\Gamma$, let $\tau \in S$ be the label of the edge $e$ of $\Lambda_\Gamma$. An **edge path** $\alpha \equiv (e_1, e_2, \ldots, e_n)$ in $\Lambda_\Gamma$ is a map $\alpha : [0, n] \rightarrow \Lambda_\Gamma$ such that $\alpha$ maps $[i, i+1]$ isometrically to the edge $e_i$. For $\alpha$ an edge path in $\Lambda_\Gamma$, let $\text{let}(\alpha) \equiv \{\tau_1, \ldots, \tau_n\}$, and let $\tau = \tau_1 \cdots \tau_n$. If $\beta$ is another geodesic with the same initial and terminal points as $\alpha$, then call $\beta$ a **rearrangement** of $\alpha$.

**Lemma 2.3.** If $w = g_1 \cdots g_k$ is a word in $A_\Gamma$ (with each $g_i \in S^\pm$) that is not of minimal length, then two letters of $g_1 \cdots g_k$ **delete**: that is, for some $i < j$, $g_i = g_j^{-1}$, the sets $\{g_i, g_j\}$ and $\{g_i+1, \ldots, g_j-1\}$ commute, and $w = g_1 \cdots g_i-1 g_{i+1} \cdots g_{j-1} g_{j+1} \cdots g_k$.

**Proof.** Let $w = h_1 \cdots h_m$ be a minimal length word representing $w$, and draw a van Kampen diagram $D$ for the loop $g_1 \cdots g_k h_m^{-1} \cdots h_1^{-1}$. For each boundary edge
$e_i$ corresponding to a $g_i$, trace a band across the diagram by picking the opposite edge of $e_i$ in the relation square containing $e_i$, and continuing to pick opposite edges (without going backwards). Note that such a band cannot cross itself, and so this band must end on another boundary edge of $D$. Since $k > m$, there is some boundary edge $e_i$ corresponding to some $g_i$ that has its band $B_i$ end on a boundary edge $e_j$ corresponding to $g_j$, with $i < j$. Note this implies $g_i = g_j^{-1}$.

Now, either all the bands corresponding to $g_{i+1}, \ldots, g_{j-1}$ cross $B$ (implying each of $g_{i+1}, \ldots, g_{j-1}$ commutes with $g_i$ and $g_j$), or some band corresponding to one of $g_{i+1}, \ldots, g_{j-1}$ ends on a boundary edge corresponding to another of $g_{i+1}, \ldots, g_{j-1}$. Picking an “innermost” such band and repeating the above argument gives the desired result.

**Remark 2.4.** Note that the bands in the van Kampen diagram $D$ share the same labels along their ‘sides’. This means that deleting the band $B$ from the diagram and matching up the separate parts of what remains (along paths with the same labels) gives a van Kampen diagram $D'$ for the loop $w = g_1 \cdots g_{i-1} g_{i+1} \cdots g_j^{-1} g_j+1 \cdots g_k h_{m-1} \cdots h_1^{-1}$.

**Remark 2.5.** Given a non-geodesic edge path $(e_1, \ldots, e_k)$ in the Cayley graph $\Lambda \Gamma$ for $A \Gamma$, we say edges $e_i$ and $e_j$ delete if their corresponding labels delete in the word $e_1 \cdots e_k$.

**Lemma 2.6.** Suppose $A \Gamma$ is a right-angled Artin group, and $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$ are geodesics between the same two points in the Cayley graph $\Lambda \Gamma$ for $A \Gamma$. There exist geodesics $(\gamma_1, \tau_1), (\gamma_1, \delta_1), (\delta_2, \gamma_2)$, and $(\tau_2, \gamma_2)$ with the same end points as $\alpha_1, \beta_1, \alpha_2, \beta_2$ respectively, such that:

1. $\tau_1$ and $\tau_2$ have the same labels,
2. $\delta_1$ and $\delta_2$ have the same labels, and
3. $\text{lett}(\tau_1)$ and $\text{lett}(\delta_1)$ are disjoint and commute.

Furthermore, the paths $(\tau_1^{-1}, \delta_1)$ and $(\delta_2, \tau_2^{-1})$ are geodesic.
Proof. Let $D$ be a van Kampen diagram for the loop $(\alpha_1, \alpha_2, \beta_2^{-1}, \beta_1^{-1})$, and let $\alpha_1 = (a_1, \ldots, a_k)$, $\beta_1 = (b_1, \ldots, b_m)$. Let $a_{i_1}, \ldots, a_{i_j}$ be (in order) the edges of $\alpha_1$ whose bands in $D$ end on $\beta_1$. Note that by Lemma 2.3, $\beta_1$ can be rearranged to begin with an edge labeled $a_{i_1}$, since $a_{i_1}$ and $b_{\ell_1}$ delete in $(\alpha_1^{-1}, \beta_1)$ for some $\ell_1$ and all the bands based at $b_1, \ldots, b_{\ell_1}, a_1, \ldots, a_{i_1-1}$ cross the band based at $a_{i_1}$ and ending at $b_{\ell_1}$. Similarly, $\beta_1$ can be rearranged to begin with an edge labeled $a_{i_1}$ followed by an edge labeled $a_{i_2}$, and continuing in this manner, we obtain a rearrangement of $\beta_1$ that begins with $\gamma_1 = (a_{i_1}, \ldots, a_{i_j})$, and we let $\delta_1$ be the remainder of this rearrangement. This argument also implies $\alpha_1$ can be rearranged to begin with $\gamma_1$, and we let $\tau_1$ be the remainder of this rearrangement. Note that if $e$ is an edge of $\tau_1$, no edge of $\delta_1$ is labeled $e$ or $e^{-1}$, since bands with those labels must have crossed in $D$. We obtain $\gamma_2$, $\tau_2$ and $\delta_2$ in the analogous way from $\alpha_2$ and $\beta_2$, and note that in a van Kampen diagram $B'$ for $(\tau_1, \delta_2, \tau_2^{-1}, \delta_1^{-1})$, no band based on $\tau_1$ can end on $\delta_2$, since $(\tau_1, \delta_2)$ is geodesic, and no band based on $\tau_1$ ends on $\delta_1$, since $\tau_1$ and $\delta_1$ share no labels or inverse labels. Therefore all bands on $\tau_1$ end on $\tau_2$, so $\tau_1$ and $\tau_2$ have the same labels, as do $\delta_1$ and $\delta_2$. \hfill \qed

Definition 2.7. Under the hypotheses of the previous lemma, we call $\tau_1$ the down edge path at $x$, and we call $\delta_2$ the up edge path at $x$. If $\alpha_1$ and $\beta_1$ have the same length, we call the above figure the diamond at $x$ for $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$.

Definition 2.8. $P_4$ is the (undirected) graph on four vertices $a, b, c, d$, with edge set $\{\{a, b\}, \{b, c\}, \{c, d\}\}$.

Definition 2.9. The union of two graphs $(V_1, E_1)$ and $(V_2, E_2)$ is the graph $(V_1 \cup V_2, E_1 \cup E_2)$.

Definition 2.10. The join of two graphs $(V_1, E_1)$ and $(V_2, E_2)$ is the graph $(V_1 \cup V_2, E_1 \cup E_2 \cup (V_1 \times V_2))$. 

Figure 1
Definition 2.11. A graph is **decomposable** if it can be expressed as joins and unions of isolated vertices.

The following is Theorem 9.2 in [9].

**Theorem 2.12.** A finite graph $G$ is decomposable iff it does not contain $P_4$ as an induced subgraph.

In particular, if a connected graph $G$ does not contain $P_4$ as an induced subgraph, then it must split as the join $G_1 \vee G_2$, for some subgraphs $G_1, G_2$ of $G$.

Definition 2.13. For a graph $\Gamma$ and a vertex $a$ of $\Gamma$, $\text{lk}(a) = \{ b \in \Gamma \mid \{a, b\} \text{ is an edge of } \Gamma \}$.

Let $\Lambda_\Gamma$ be the Cayley graph for the group $\Gamma$.

**Definition 2.14.** The standard complex $S_\Gamma$ for the group $\Gamma$ is the CAT(0) cube complex whose one-skeleton is $\Lambda_\Gamma$, with each cube given the geometry of $[0, 1]^n$ for the appropriate $n$.

For more on cube complexes and the definitions below, see [13].

**Definition 2.15.** A midcube in a cube complex $C$ is the codimension 1 subspace of an $n$-cube $[0, 1]^n$ obtained by restricting exactly one coordinate to $\frac{1}{2}$. A hyperplane is a connected nonempty subspace of $C$ whose intersection with each cube is either empty or consists of one of its midcubes.

**Lemma 2.16.** If $D$ is a hyperplane of the cube complex $C$, then $C - D$ has exactly two components.

Given a graph $\Gamma$, a vertex $v$ of $\Gamma$, and the corresponding standard complex $S_\Gamma$, note that if a hyperplane of $S_\Gamma$ intersects an edge of $S_\Gamma$ with label $v$, then every edge intersected by this hyperplane is also labeled $v$. Thus we can refer to hyperplanes in $S_\Gamma$ as $v$-hyperplanes, for $v$ a vertex of $\Gamma$. If $x$ is a vertex of $S_\Gamma$, then $x$ and $xv$ are separated by a $v$-hyperplane $D$. Let $xS_{\text{lk}(v)}$ denote the cube complex generated by the coset $x\langle \text{lk}(v) \rangle$; then $D$ and $xS_{\text{lk}(v)}$ are isometric and parallel, of distance $\frac{1}{2}$ apart.

**Definition 2.17.** A metric space $(X, d)$ is **proper** if each closed ball is compact.

**Definition 2.18.** Let $(X, d)$ be a proper CAT(0) space. Two geodesic rays $c, c' : [0, \infty) \to X$ are called asymptotic if for some constant $K$, $d(c(t), c'(t)) \leq K$ for all $t \in [0, \infty)$. Clearly this is an equivalence relation on all geodesic rays in $X$. We define the boundary of $X$ (denoted $\partial X$) to be the set of equivalence classes of geodesic rays in $X$. We denote the union $X \cup \partial X$ by $\overline{X}$.

The next proposition guarantees that the topology we wish to put on the boundary is independent of our choice of basepoint in $X$.

**Proposition 2.19.** Let $(X, d)$ be a proper CAT(0) space, and let $c : [0, \infty) \to X$ be a geodesic ray. For a given point $x \in X$, there is a unique geodesic ray based at $x$ which is asymptotic to $c$.

For a proof of this (and more details on what follows), see [3].

We wish to define a topology on $\overline{X}$ that induces the metric topology on $X$. Given a point in $\partial X$, we define a neighborhood basis for the point as follows: Pick a basepoint $x_0 \in X$. Let $c$ be a geodesic ray starting at $x_0$, and let $\epsilon > 0$, $r > 0$. Let $S(x_0, r)$ denote the sphere of radius $r$ centered at $x_0$, let $B(x_0, r)$ denote
the open ball of radius $r$ centered at $x_0$ and let $p_r : X - B(x_0, r) \to S(x_0, r)$ denote the projection to $S(x_0, r)$. Define

$$U(c, r, \epsilon) = \{ x \in X : d(x, x_0) > r, d(p_r(x), c(r)) < \epsilon \}.$$ 

This consists of all points in $X$ whose projection to $S(x_0, r)$ is within $\epsilon$ of the point of the sphere through which $c$ passes. These sets together with the metric balls in $X$ form a basis for the cone topology. The set $\partial X$ with this topology is sometimes called the visual boundary. In this article, we will call it the boundary of $X$.

**Proposition 2.20.** If $X$ and $Y$ are proper CAT(0) spaces, then $\partial (X \times Y) \cong \partial X \ast \partial Y$, where $\ast$ denotes the spherical join.

If the graph $\Gamma$ splits as a non-trivial join $\Gamma_1 \vee \Gamma_2$, then the group $A_\Gamma$ splits as the direct product $A_{\Gamma_1} \times A_{\Gamma_2}$, and so we have $S_\Gamma \cong S_{\Gamma_1} \times S_{\Gamma_2}$. The previous proposition then gives that $\partial S_\Gamma \cong \partial S_{\Gamma_1} \ast \partial S_{\Gamma_2}$. Any non-trivial spherical join is path connected, and so $\partial S_\Gamma$ is path connected.

**Lemma 2.21.** There is a bound $\delta > 0$ such that if $\alpha$ is a CAT(0) geodesic path in $S_\Gamma$, then there is a Cayley graph geodesic path $\beta$ in $\Lambda_\Gamma$ (contained naturally in $S_\Gamma$) such that each vertex of $\beta$ is within distance $\delta$ of $\alpha$, and each point of $\alpha$ is within $\delta$ of a vertex of $\beta$.

A proof of this can be found in Section 3 of [8].

### 3. Result

The goal of this section is to prove the following theorem:

**Theorem 3.1.** Let $\Gamma$ be a connected graph. Suppose $\Gamma$ contains an induced subgraph $((\{a, b, c, d\}, \{\{a, b\}, \{b, c\}, \{c, d\}\})$ (isomorphic to $P_4$), and there are subsets $B \subset \text{lk}(c)$ and $C \subset \text{lk}(b)$ with the following properties:

1. $B$ separates $c$ from $a$ in $\Gamma$, with $d \notin B$;
2. $C$ separates $b$ from $d$ in $\Gamma$, with $a \notin C$;
3. $B \cap C = \emptyset$.

Then $\partial S_\Gamma$ is not path connected.

In fact, we prove a stronger result, with the hypothesis $B \cap C = \emptyset$ replaced by the statement of Claim 3.7. For the remainder of this section, suppose $a, b, c, d \in \Gamma$, $B \subset \text{lk}(c)$, and $C \subset \text{lk}(b)$ are as in Theorem 3.1. Note that $b \in B$, $c \in C$. We wish to consider the following rays in $\Lambda_\Gamma$ (equivalently in $S_\Gamma$), based at the identity vertex $*$:

$$r = cdab(cb)^2cdab(cb)^6 \cdots = \prod_{i=1}^{\infty} (cb)^{k_i} cdab$$

and

$$s = dbcb^2adbc(b^2c)^2b^2adbc(b^2c)^6b^2a \cdots = \prod_{i=1}^{\infty} dbc(b^2c)^{k_i} b^2a$$

where the $k_i$ are defined recursively with $k_0 = -1$, $k_{i+1} = 2k_i + 2$.

Define the following vertices of $r$, for $n \geq 0$:
v_n = \left( \prod_{i=1}^{n} (cb)^{k_i} cdab \right) (cb)^{k_{n+1}} cd

v'_n = v_n a

Define the following vertices of s, for n ≥ 0:

w_n = \left( \prod_{i=1}^{n} dbc(b^2 c)^{k_i} b^2 a \right)

w'_n = w_n d

We have v_0 = cd, v'_0 = cda, v_1 = cdab(cb)^2 cd, w_0 = *, w'_0 = d, w_1 = dbcb^2 a. It will be helpful to refer to Figure 2 for many of the claims that follow.

The following is proved in [5].

**Claim 3.2.** For n ≥ 0, v_n = w'_n c^{k_{n+1}+1} and v'_n b^{k_{n+1}+1} = w_{n+1}.

Since b ∈ B and c ∈ C, we then have v_n(C) = w'_n(C) and w_n(B) = v'_n−1(B).

If Q_c denotes the component of c in Γ − B, and Q_b denotes the component of b in Γ − C, then AΓ can be represented as \langle Q_c \cup B \rangle *B (Γ − Q_c) or \langle Q_b \cup C \rangle *C (Γ − Q_b), and so at each vertex x of AΓ, the cosets x\langle B \rangle and x\langle C \rangle separate AΓ. Therefore, if xs_B and xs_C denote the cube complexes generated by \langle B \rangle and \langle C \rangle respectively at a vertex x of SΓ, then xs_B and xs_C separate SΓ. Note that SΓ − xs_B has at least
two components: one containing \( xc^{-1} \), and one containing \( xa \). Similarly, \( S_T - xS_C \) has at least two components: one containing \( xb^{-1} \), and one containing \( xd \).

For each \( i \), define the following components of \( S_T \):

1. \( V_i^+ \) is the component of \( S_T - v_iS_B \) containing \( v_i a \);
2. \( V_i^- \) is the component of \( S_T - v_iS_B \) containing \( v_i c^{-1} \);
3. \( W_i^+ \) is the component of \( S_T - w_iS_C \) containing \( w_i d \);
4. \( W_i^- \) is the component of \( S_T - w_iS_C \) containing \( w_i b^{-1} \).

Note \( V_i^+ \) contains the vertices of \( r \) after \( v_i \), and \( W_i^+ \) contains the vertices of \( s \) after \( w_i \). For each \( V_i^\pm \), (respectively \( W_i^\pm \)), let \( V_i^{\pm} \) denote the closure of \( V_i^\pm \) in \( S_T \), so \( V_i^\pm = V_i^{\pm} \cup v_iS_B \) (\( W_i^\pm = W_i^{\pm} \cup w_iS_C \)). For a subset \( S \) of \( S_T \), let \( L(S) \) denote the limit set of \( S \) in \( \partial S_T \).

**Claim 3.3.**

1. The sets \( V_i^\pm, W_i^\pm \) are convex.
2. \( L(V_i^+) \cap L(V_i^-) = L(v_iS_B) \) and \( L(W_i^+) \cap L(W_i^-) = L(w_iS_C) \).
3. The set \( L(v_iS_B) \) (respectively \( L(w_iS_C) \)) separates \( L(V_i^+) \) and \( L(V_i^-) \) (respectively \( L(W_i^+) \) and \( L(W_i^-) \)) in \( \partial X \).

**Proof.** For (1), the only way out of the set \( V_i^- \) is through the convex subcomplex \( v_iS_B \).

For (2), if \( q \) is a ray in \( L(V_i^+) \cap L(V_i^-) \), then there are geodesic rays \( q_1 \in V_i^+ \), \( q_2 \in V_i^- \) that are a bounded distance from \( q \), and therefore from one another. Thus both \( q_1 \) and \( q_2 \) remain a bounded distance from \( v_iS_B \), as required.

For (3), suppose \( \alpha : [0, 1] \to \partial S_T \) is a path connecting \( x \in L(V_i^+) \) and \( y \in L(V_i^-) \). Choose \( w \in v_iS_B \), and for each \( t \in [0, 1] \), let \( \beta_t : [0, \infty) \to S_T \) be the geodesic ray from \( w \) to \( \alpha(t) \in \partial S_T \). This gives a continuous map \( H : [0, 1] \times [0, \infty) \to S_T \) where \( H(t, s) = \beta_t(s) \). Note \( H(0, s) \subset V_i^+ \), \( H(1, s) \subset V_i^- \). For each \( n \geq 0 \), let \( z_n \) be a point of \( H([0, 1] \setminus \{n\}) \) in \( v_iS_B \); then \( L(\cup_{n=1}^{\infty} \{z_n\}) \subset Im(\alpha) \cap L(v_iS_B) \) as required. \( \square \)

In [6], it is shown that \( r \) and \( s \) track distinct \( \text{CAT}(0) \) geodesics in \( S_T \), so \( L(r) \) and \( L(s) \) are distinct one-element sets.

**Claim 3.4.** For \( n \geq 1 \), the sets \( L(w_{2n-1}S_C) \) and \( L(r) \) are separated in \( \partial S_T \) by \( L(w_{2n+1}S_B) \).

**Proof.** First note that \( L(r) \in L(V_i^+) \) for each \( i \geq 1 \). Let \( D_{2n} \) be the \( d \)-hyperplane that separates \( w_{2n} \) from \( w_{2n}' \) (and also separates \( v_{2n} \) from the previous vertex of \( r \)), and let \( A_{2n} \) be the \( a \)-hyperplane that separates \( v_{2n} \) from \( v_{2n}' \) (and also separates \( w_{2n-1} + 1 \) from the previous vertex of \( s \)). Note that \( w_{2n-1}S_C \) is contained in the same component of \( S_T - D_{2n} \) as \( \ast \) since \( d \notin C \) and therefore no path in \( \langle C \rangle \) based at \( w_{2n-1} \) crosses \( D_{2n} \). Also note \( A_{2n} \subset V_{2n-1}^- \). Since \( D_{2n} \) and \( A_{2n} \) cannot cross (since \( d \) does not commute with \( a \)), and \( D_{2n} \) is not in the same component as \( v_{2n+1}S_B \) in \( S_T - A_{2n} \), we have that \( w_{2n-1}S_C \subset V_{2n+1}^- \). The previous claim gives the result. \( \square \)

**Claim 3.5.** For \( n \geq 1 \), the sets \( L(v_{2n-1}S_B) \) and \( L(r) \) are separated in \( \partial S_T \) by \( L(w_{2n+1}S_C) \).

**Proof.** The proof is analogous to the proof of the previous claim, replacing the hyperplanes \( D_{2n} \) and \( A_{2n} \) with the hyperplanes \( A_{2n-1} \) and \( D_{2n} \) respectively. \( \square \)
Remark 3.6. The previous two claims imply that if there is a path in $\partial S_T$ between a point of $L(w_1S_C)$ and $L(r)$, the path must pass through (in order) $L(v_3S_B)$, $L(w_5S_C)$, $L(v_7S_B)$, $L(w_9S_C)$, and so on.

We will now show that the sets $L(v_iS_B)$ (resp. $L(w_iS_C)$) are eventually ‘close’ to $L(s)$ (resp. $L(r)$), implying the path described in Remark 3.6 cannot exist.

Claim 3.7. $C \cap lk(a) \cap lk(d) = C \cap lk(a) \cap lk(c) = \emptyset$, and $B \cap lk(a) \cap lk(d) = B \cap lk(d) \cap lk(b) = \emptyset$.

Proof. If $e \in C \cap lk(a) \cap lk(d)$, then $(a,e,d,c)$ is a path from $a$ to $c$ in $\Gamma$. Since $B$ separates $a$ from $c$ and $d \notin B$, we must have $e \in B$, but $B \cap C = \emptyset$. Similarly, if $e \in C \cap lk(a) \cap lk(c)$, then $(a,e,c)$ is a path from $a$ to $c$ in $\Gamma$, and so $e \in B$, contradiction. The remaining statements are proved identically. □

For $i \geq 1$, let $r_i$ (respectively $s_i$) be the segment of $r$ (respectively $s$) between * and $v_i'$ (respectively * and $w_i'$). Let $\beta_i$ be a Cayley graph geodesic ray based at $w_i'$ with labels in $B$, and let $\gamma_i$ be a Cayley graph geodesic ray based at $v_i'$ with labels in $C$.

Claim 3.8. Any Cayley graph geodesic from * to a point of $\gamma_i$ must pass within 4 units of $v_i'$. Any Cayley graph geodesic from * to a point of $\beta_i$ must pass within 4 units of $w_i'$.

Proof. First observe that if $(r_i, \gamma_i)$ is not $\Lambda_T$-geodesic, then an edge of $\gamma_i$ must delete with an edge of $r_i$. Since $a, b, d \notin C$, the labels of these deleting edges must be $c$ and $c^{-1}$. However, the labels of these edges must also be in $lk(a) \cap lk(c)$, by Lemma 2.3 (see Figure 2). Therefore $(r_i, \gamma_i)$ is a Cayley geodesic.

Now, suppose there is a $\Lambda_T$-geodesic $\rho$ between * and a point of $\gamma_i$ with $d(\rho, v_i') > 4$. Let $\alpha$ denote the segment of $(r_i, \gamma_i)$ between * and the endpoint of $\rho$. Consider a diamond based at $v_i'$ for $\rho$ and $\alpha$ as in Lemma 2.6. Let $\tau$ and $\delta$ be the down edge path and up edge path respectively at $v_i'$, and note $\tau$ and $\delta$ have length at least 3. Every $\Lambda_T$-geodesic from * to $v_i'$ must end with an edge labeled $a$, so every label of $\delta$ is in $lk(a)$. If an edge of $\tau$ has label $d$, then every label of $\delta$ is in $C \cap lk(a) \cap lk(d)$, but this set is empty by Claim 3.7. By Lemma 2.3 every other edge of $\tau$ has its label in $lk(d) \cap \{a, b, c, d\}$, so the remaining edges of $\tau$ must be labeled $c$, but $C \cap lk(a) \cap lk(c)$ is also empty. Thus $d(\rho, v_i') \leq 4$. The proof of the second statement is identical. □

Claim 3.9. $\partial S_T$ is not path connected.

Proof. Observe that since $v_{n-1}' b^k w_{n+1} = w_n$ by Claim 3.2 and $C \subset lk(b)$, any ray $\alpha$ based at $w_n$ with labels in $C$ stays a bounded distance from the ray based at $v_{n-1}'$ with the same labels. Combining Claim 3.8 and Lemma 2.21 we have that a CAT(0) geodesic from * to a point of $L(\alpha)$ must pass within $\delta+4$ of $v_{n-1}'$, where $\delta$ is the tracking constant given by Lemma 2.21. We therefore have that any sequence of points $\{p_i\}_{i=1}^{\infty}$ with each $p_i \in L(w_iS_C) \subset \partial S_T$ must converge to $L(r) \in \partial S_T$. Similarly, any sequence of points $\{q_i\}_{i=1}^{\infty}$ with each $q_i \in L(v_iS_B) \subset \partial S_T$ must converge to $L(s) \in \partial S_T$. Therefore, by Remark 3.6, given any $\epsilon$, any path from a point of $L(w_iS_C)$ to $L(r)$ eventually bounces back and forth infinitely between the $\epsilon$-neighborhood of $L(s)$ and the $\epsilon$-neighborhood of $L(r)$, which is impossible; therefore, no such path exists. □
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