Fermions with integer spins

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Fermion fields with the internal degrees of freedom described in Clifford space carry in any dimension $d$ a half integer spin $-\frac{1}{2}$. There are two kinds of spins in Clifford space. The \textit{spin-charge-family} theory (\cite{1, 2, 3}, and the references therein), assuming even $d = (13 + 1)$, uses one kind of spins, $S^{ab}$, to describe in $d = (3+1)$ spins and charges of fermions, while the other kind, $\tilde{S}^{ab}$, takes care of families. The creation and annihilation operators, written each as a product of nilpotents and projectors of an odd Clifford character, fulfill the anticommutation relations as required in the second quantization procedure for fermions. It is proven in this paper that also in Grassmann space there exist the creation and annihilation operators of an odd Grassmann character, which fulfill as well the anticommutation relations for fermion fields. However, while the internal spins determined by the generators of the Lorentz group of the Clifford objects $S^{ab}$ and $\tilde{S}^{ab}$ are half integer, the internal spins determined by the Grassmann objects $S^{ab}$ are integer. Grassmann space offers no families. We discuss here the quantization procedure — first and second — of the fields in both spaces, presenting for the Grassmann case also the action, the basic states and the solution of the "Weyl" equation for free massless fermion fields in order to try to understand why the Clifford algebra "wins in the competition" for the physical (observable) degrees of freedom for fermions.

\section{I. INTRODUCTION}

It is proven in this paper that besides Clifford space also Grassmann space offers the description of the internal degrees of freedom of fermions in the second quantized procedure. In both cases there exist the creation and annihilation operators, which fulfill the anticommutation relations required for fermions Eqs.\textsuperscript{[42, 61]}. But while the internal spins determined by the generators of the Lorentz group of the Clifford objects $S^{ab}$ and $\tilde{S}^{ab}$ — in the \textit{spin-charge-family} theory $S^{ab}$ determine the spin degrees of freedom and $\tilde{S}^{ab}$ the family degrees of freedom — are half integer, the internal spin determined by $S^{ab}$ (expressible with $S^{ab} + \tilde{S}^{ab}$, which appear in the \textit{spin-charge-family} theory to describe spins an families) is integer.

Correspondingly Clifford space offers according to the \textit{spin-charge family} theory the description of spins, charges and families, all in the fundamental representations of the subgroups of the Lorentz group $SO(d - 1, 1)$, while Grassmann space offers spins and charges in the adjoint representations.
of the subgroups of the Lorentz group $SO(d - 1, 1)$ and no family degrees of freedom. Fermions with integer spins would lead to completely different nucleons, nuclei, atoms, molecules, matter than observed so far.

Assuming that "nature knows" all the mathematics, the question arises: Why our universe "uses" Clifford rather than Grassmann coordinates, although both lead in the second quantization procedure to the anticommutation relations required for fermion degrees of freedom? Is the answer that Clifford degrees of freedom offer the appearance of families, the half integer spin and charges in the fundamental representations of the groups? Can the choice of the Clifford degrees of freedom explain why the simple starting action of the spin-charge-family theory of one of us (N.S.M.B.) [3–9] is doing so far extremely well in manifesting the so far observed properties of fermion and boson fields in the low energy regime?

These questions, too demanding for this paper, offer only a trial to make first steps towards understanding them.

In the working hypothesis (of N.S.M.B.) that "nature knows all the mathematics" both "coordinates", Grassmann and Clifford, could be used to describe the internal degrees of freedom of fermions. In a trial to understand why Grassmann space "is not the choice of nature", at least not in the low energy regime, it must be noticed that the existence of two kinds of $\gamma$'s — $\gamma^a$'s and $\tilde{\gamma}^a$'s of the spin-charge-family theory — enables to describe not only the spin and charges of fermions, but also the families of fermions, in the first and second quantized theory of fields, and also to understand the origin of scalar fields, which are responsible for masses of family members (quarks and leptons) and of the weak gauge fields [30].

This work is a part of the project of both authors, which includes the fermionization procedure of boson fields (or the bosonization procedure of fermion fields), discussed in Refs. [11, 12, 14] for any dimension $d$ (by the authors of this contribution, while one of them, H.B.F.N. [13], has succeeded with another author to do the fermionization for $d = (1+1)$), and which would hopefully also help to understand a little better the content and dynamics of our universe.

In the spin-charge-family theory [3–9, 20] — which offers explanations for all the assumptions of the standard model, with the appearance of families, the scalar higgs and the Yukawa couplings included, offering also the explanation for the matter-antimatter asymmetry in our universe and for the appearance of the dark matter — a very simple starting action for massless fermions and bosons in $d = (13 + 1)$ is assumed, in which massless fermions interact with only gravity, the vielbeins $f^a\alpha$ (the gauge fields of moments $p_a$) and the two kinds of the spin connections ($\omega_{\alpha\beta\gamma}$ and
\( \tilde{\omega}_{ab\alpha} \), the gauge fields of the two kinds of the Clifford algebra objects \( \gamma^a \) and \( \tilde{\gamma}^a \), respectively).

\[
\mathcal{A} = \int d^4 x \ E \left( \frac{1}{2} \langle \bar{\psi} \gamma^a p_{0a} \psi \rangle + h.c. \right) + \int d^4 x \ E (\alpha R + \tilde{\alpha} \tilde{R}),
\]

(1)

with \( p_{0a} = f^\alpha_a p_0 + \frac{1}{2E} \{ p_0, EF^\alpha_a \} \). \( p_{0a} = p_a - \frac{1}{2} S^{ab} \omega_{aba} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{aba} \) and \( R = \frac{1}{2} \{ f^{[a} f^{b]} \} (\omega_{aba, \beta} - \omega_{ca\alpha} \omega_{\beta b\gamma}) + h.c., \tilde{R} = \frac{1}{2} \{ f^{[a} f^{b]} \} (\tilde{\omega}_{aba, \beta} - \tilde{\omega}_{ca\alpha} \tilde{\omega}_{\beta b\gamma}) + h.c. \). The two kinds of the Clifford algebra objects, \( \gamma^a \) and \( \tilde{\gamma}^a \), connected with the left and the right multiplication of the Clifford objects, respectively, Eq. (49), (there is no third kind of the Clifford operators), anticommute and determine the infinitesimal generators of the Lorentz transformations in the internal space of fermions — \( S^{ab} \) for \( SO(13, 1) \) and \( \tilde{S}^{ab} \) for \( \tilde{SO}(13, 1) \) — arranging states into representations.

\[
\{ \gamma^a, \gamma^b \}^+_+ = 2\eta^{ab} = \{ \tilde{\gamma}^a, \tilde{\gamma}^b \}^+_+ = 0,
\]

\[
S^{ab} = \frac{i}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a),
\]

\[
\tilde{S}^{ab} = \frac{i}{4} (\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a).\]

(2)

One kind of the objects, the generators \( S^{ab} \), are used in the spin-charge-family theory to determine spins and charges of spinors of any family, another kind, \( \tilde{S}^{ab} \), determines the family quantum numbers. Here \( f^{[a} f^{b]} = f^{[a} f^{b]} - f^{ab} f^{\beta a} \).

The scalar curvatures \( R \) and \( \tilde{R} \) determine dynamics of the gauge fields — the spin connections and the vielbeins — manifesting in \( d = (3 + 1) \) as all the known vector gauge fields as well as the scalar fields \( 5 \), which offer the explanation for the appearance of the higgs and the Yukawa couplings, of the ordinary matter/antimatter asymmetry \( 4 \) and the dark matter \( 10 \), provided that the symmetry breaks from the starting \( SO(13, 1) \) to \( SO(3, 1) \times SU(3) \times U(1) \).

In Grassmann space the infinitesimal generators of the Lorentz transformations \( S^{ab} \) are expressible with anticommuting coordinates \( \theta^a \) and their conjugate momenta \( p^{\theta a} \)

\[
\{ \theta^a, \theta^b \}^+_+ = 0, \quad \{ p^{\theta a}, p^{\theta b} \}^+_+ = 0, \quad \{ p^{\theta a}, \theta^b \}^+_+ = i\eta^{ab},
\]

\[
S^{ab} = \theta^a p^{\theta b} - \theta^b p^{\theta a}.
\]

(3)

Taking into account that \( \gamma^a \) and \( \tilde{\gamma}^a \) are expressible in terms of \( \theta^a \) and their conjugate momenta \( p^{\theta a} \)

\[
\gamma^a = (\theta^a - i p^{\theta a}), \quad \tilde{\gamma}^a = i(\theta^a + i p^{\theta a}),
\]

(4)
one recognizes

$$S^{ab} = S^{ab} + \tilde{S}^{ab},$$

(5)

from where one concludes, if taking into account Eq. (1), that in the Grassmann case the covariant momenta $p_{0\alpha}$ are

$$p_{0\alpha} = p_{\alpha} - \frac{1}{2} S^{ab} \Omega_{aba},$$

(6)

with $\Omega_{aba}$ as the only kind of the connection fields (instead of the two kinds in the Clifford case — $\omega_{aba}$, which is the gauge fields of $S^{ab}$ and $\tilde{\omega}_{aba}$, which is the gauge fields of $\tilde{S}^{ab}$).

It follows for $S^{ab}$

$$\{S^{ab}, S^{cd}\}_- = i \{S^{ad} \eta^{bc} + S^{bc} \eta^{ad} - S^{ac} \eta^{bd} - S^{bd} \eta^{ac}\},$$

$$S^{ab\dagger} = \eta^{aa} \eta^{bb} S^{ab}.$$

(7)

The same relations are true also if $S^{ab}$ is replaced with either $S^{ab}$ or $\tilde{S}^{ab}$. These infinitesimal generators of the Lorentz group — the two kinds of the Clifford operators and the Grassmann operators — operate as follows

$$\{S^{ab}, \gamma^e\}_- = -i (\eta^{ae} \gamma^b - \eta^{be} \gamma^a),$$

$$\{\tilde{S}^{ab}, \tilde{\gamma}^e\}_- = -i (\eta^{ae} \tilde{\gamma}^b - \eta^{be} \tilde{\gamma}^a),$$

$$\{S^{ab}, S^{cd}\}_- = 0,$$

$$\{S^{ab}, \theta^e\}_- = -i (\eta^{ae} \theta^b - \eta^{be} \theta^a),$$

$$\{S^{ab}, p^{be}\}_- = -i (\eta^{ae} p^{b} - \eta^{be} p^{a}),$$

$$\{M^{ab}, A^{d...e...g}\}_- = -i (\eta^{ae} A^{d...h...g} - \eta^{be} A^{d...a...g}),$$

(8)

where $M^{ab}$ are defined in the Clifford case by the sum of $L^{ab}$ plus either $S^{ab}$ (if $\gamma^a$‘s are chosen to describe the basis, otherwise $\tilde{S}^{ab}$ replace $S^{ab}$), while in the Grassmann case $M^{ab}$ is $L^{ab} + S^{ab}$ (which is, Eq. (5), $M^{ab} = L^{ab} + S^{ab} + \tilde{S}^{ab}$).

We present in what follows the first and the second quantization of the fields, the internal degrees of freedom of which are described by Grassmann coordinates $\theta^a$, and, for comparison, as well as of the fields, the internal degrees of freedom of which are described by the Clifford coordinates $\gamma^a$ (or $\tilde{\gamma}^a$). In both cases fermions interact with gravity only: with vielbeins and — in the Grassmann case with the spin connection fields $\Omega_{aba}$ as the gauge fields of $S^{ab}$, and in the Clifford case with $\omega_{aba}$ as the gauge fields of $S^{ab}$ and $\tilde{\omega}_{aba}$ as the gauge fields of $\tilde{S}^{ab}$. 
The action is in the Clifford case taken from the \textit{spin-charge-family} theory, Eq. (1), while in the Grassmann case it is defined in Eq. (33).

In all these cases we mostly comment free massless fermion fields; masses of the fields in \( d = (3 + 1) \) are in the \textit{spin-charge-family} theory due to their interactions with the gravitational scalar fields with the space index \( s > 4 \), determined by the scalar vielbein or spin connection fields (1–9, and the references therein). Also in the Grassmann case the mass terms manifesting in \( d = (3 + 1) \) are assumed to follow due to the interactions of massless fermion and boson fields with the scalar gravitational fields, vielbeins or spin connection fields, with the space index \( s > 4 \).

II. SECOND QUANTIZATION IN GRASSMANN AND IN CLIFFORD SPACE

We present in this section properties of fermion fields with the integer spin in \( d \)-dimensional space, expressed in terms of the Grassmann algebra objects, and the fermion fields with the half integer spin, expressed in terms of the Clifford algebra objects. Since the Clifford algebra objects are expressible with the Grassmann algebra objects, Eq. (4), the norms of both are determined by the integral in Grassmann space, Eqs. (25, 27) [32].

a. Fields with the integer spin in Grassmann space

A point in \( d \)-dimensional Grassmann space of anticommuting coordinates \( \theta^a \), \( (a = 0, 1, 2, 3, 5, \ldots, d) \), is determined by a vector \( \{ \theta^a \} = (\theta^0, \theta^1, \theta^2, \theta^3, \theta^5, \ldots, \theta^d) \). A linear vector space over the coordinate Grassmann space has correspondingly the dimension \( 2^d \), due to the fact that \((\theta^a_i)^2 = 0\) for any \( a_i \in (0, 1, 2, 3, 5, \ldots, d) \).

Correspondingly are fields in Grassmann space expressible in terms of the Grassmann algebra objects

\[
B = \sum_{k=0}^{d} a_{a_1a_2\ldots a_k} \theta^{a_1} \theta^{a_2} \ldots \theta^{a_k} |\phi_{\text{og}} \rangle, \quad a_i \leq a_{i+1},
\]

where \( |\phi_{\text{og}} \rangle \) is the vacuum state, here assumed to be \( |\phi_{\text{og}} \rangle = |1 \rangle \), so that \( \frac{\partial}{\partial \theta^a} |\phi_{\text{og}} \rangle = 0 \) for any \( \theta^a \). The Kalb-Ramond boson fields \( a_{a_1a_2\ldots a_k} \) are antisymmetric with respect to the permutation of indexes, since the Grassmann coordinates anticommute \( \{ \theta^a, \theta^b \}_+ = 0 \), Eq. (3).

The left derivative \( \frac{\partial}{\partial \theta^a} \) on vectors of the space of monomials \( B(\theta) \) is defined as follows

\[
\frac{\partial}{\partial \theta^a} B(\theta) = \frac{\partial B(\theta)}{\partial \theta^a},
\]

\[
\left\{ \frac{\partial}{\partial \theta^a}, \frac{\partial}{\partial \theta^b} \right\}_+ B = 0, \text{ for all } B.
\]

(10)
The commutation relations are for $p^{\theta a} = i\frac{\partial}{\partial \theta a}$ defined in Eq. (3), where the metric tensor $\eta^{ab}$ ($= \text{diag}(1, -1, -1, \ldots, -1)$) lowers the indexes of a vector $\{\theta^a\}$: $\theta_a = \eta_{ab} \theta^b$ (the same metric tensor lowers the indexes of the ordinary vector $x^a$ of commuting coordinates).

Defining

$$\theta^a = \eta^{ab} \theta^b,$$

it follows

$$\left(\frac{\partial}{\partial \theta a}\right)^\dagger = \eta^{aa} \theta^a, \quad (p^{\theta a})^\dagger = -i\eta^{aa} \theta^a. \quad (12)$$

Making a choice for the complex properties of $\theta^a$, and correspondingly of $\frac{\partial}{\partial \theta a}$, as follows

$$\{\theta^a\}^* = (\theta^0, \theta^1, -\theta^2, \theta^3, -\theta^5, \theta^6, \ldots, -\theta^{d-1}, \theta^d),$$

$$\{
\frac{\partial}{\partial \theta a}\}^* = \left(\frac{\partial}{\partial \theta_0}, \frac{\partial}{\partial \theta_1}, -\frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial \theta_3}, -\frac{\partial}{\partial \theta_5}, \frac{\partial}{\partial \theta_6}, \ldots, -\frac{\partial}{\partial \theta_{d-1}}, \frac{\partial}{\partial \theta_d}\right), \quad (13)$$

it follows for the two Clifford algebra objects $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta a})$, and $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta a})$, Eq. (4), that $\gamma^a$ is real if $\theta^a$ is real, and $\gamma^a$ is imaginary if $\theta^a$ is imaginary, while $\tilde{\gamma}^a$ is imaginary when $\theta^a$ is real and $\gamma^a$ is real if $\theta^a$ is imaginary, just as it is required in Eq. (19).

Applying the operator $S^{ab}$ of Eq. (3) on the "state" $\frac{1}{\sqrt{2}}(\theta^a \pm \epsilon \theta^b)$, it follows

$$S^{ab}(\theta^a \pm \epsilon \theta^b) = \pm(-i\epsilon(\theta^a \pm \epsilon \theta^b), \quad (14)$$

$\epsilon = i$, if $\eta^{aa} = \eta^{bb}$ and $\epsilon = -1$, if $\eta^{aa} \neq \eta^{bb}$.

We define here the commuting objects $\gamma^a_G$, which will be helpful when looking for the appropriate action for Grassmann fermions, Eq. (33). These operators will be needed also when looking for the definition of appropriate discrete symmetry operators in the Grassmann case. Following the definition of the discrete symmetry operators in the Clifford algebra case [21] in $((d-1)+1)$ space-time and in $(3+1)$ space-time, the discrete symmetry operators ($C_G, T_G, P_G$) in $((d-1)+1)$ and ($C_{NG}, T_{NG}, P_{NG}$) in $(3+1)$ will be defined, respectively.

$$\gamma^a_G = (1 - 2\theta^a \eta^{aa} \frac{\partial}{\partial \theta a}) = -i\eta^{aa} \gamma^a \tilde{\gamma}^a, \quad \{\gamma^a_G, \gamma^b_G\}_- = 0. \quad (15)$$

Index $a$ is not the Lorentz index in the usual sense. $\gamma^a_G$ are commuting operators for all $(a, b)$. They are real and Hermitian.

$$\gamma^a_G = \gamma^a_G, \quad (\gamma^a_G)^* = \gamma^a_G. \quad (16)$$

Correspondingly it follows: $\gamma^a_G \gamma^a_G = I$, $\gamma^a_G \gamma^a_G = I$. $I$ represents the unit operator.
By introducing the generators of the infinitesimal Lorentz transformations in Grassmann space, as presented in Eq. (3), and making use of the Cartan subalgebra of commuting operators, Eq. (13), the basic states in Grassmann space can be arranged into representations of the Cartan subalgebra operators, Eq. (14), Ref. [2, 15]. All these states have integer spin. The starting state in $d$-dimensional space, for example, with the eigenvalues of the Cartan subalgebra equal to either $i$ or $1$ is: $(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\ldots(\theta^{d-1} + i\theta^d)|\phi_{og}\rangle$, with $|\phi_{og}\rangle = |1\rangle$, Eq. (14). All the states of the representation, which starts with this state, follow by the application of those $S^{ab}$, which do not belong to the Cartan subalgebra of the Lorentz algebra. $S^{01}$, for example, transforms this starting state into $(\theta^0\theta^3 + i\theta^1\theta^2)(\theta^5 + i\theta^6)\ldots(\theta^{d-1} + i\theta^d)|\phi_{og}\rangle$, while $S^{01} - iS^{02}$ transforms this state into $(\theta^0 + \theta^3)(\theta^1 - i\theta^2)(\theta^5 + i\theta^6)\ldots(\theta^{d-1} + i\theta^d)|\phi_{og}\rangle$.

b. Fields with the half integer spin in Clifford space

Let us present as well the properties of the fermion fields with the half integer spin, expressed by the Clifford algebra objects (1–5, 9, 16) and the references therein

$$ F = \sum_{k=0}^{d} a_{a_1a_2\ldots a_k} \gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_k}|\psi_{oc}\rangle, \quad a_i \leq a_{i+1}, \quad (17) $$

where $|\psi_{oc}\rangle$ is the vacuum state. The Kalb-Ramond fields $a_{a_1a_2\ldots a_k}$ are again in general boson fields, which are antisymmetric with respect to the permutation of indexes, since the Clifford objects have the anticommutation relations, Eq. (2), $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}$. The linear vector space over the Clifford coordinate space has, as in the Grassmann case, the dimension $2^d$, due to the fact that $(\gamma^a)^2 = \eta^{aa}$ for any $a_i \in (0, 1, 2, 3, 5, \ldots, d)$.

As written in Eq. (4), $\gamma^a$ are expressible in terms of the Grassmann coordinates and their conjugate momenta, as $\gamma^a = (\theta^a - ip^{\theta a})$, and $\tilde{\gamma}^a = i(\theta^a + ip^{\theta a})$, with the anticommutation relation of Eq. (2), $\{\gamma, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}, \tilde{\gamma}^b\}_+$, $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$. Taking into account Eqs. (11, 12, 14) one finds

$$ (\gamma^a)^\dagger = \gamma^a \eta^{aa}, \quad (\tilde{\gamma}^a)^\dagger = \tilde{\gamma}^a \eta^{aa}, $$

$$ \gamma^a \gamma^a = \eta^{aa}, \quad \gamma^a(\gamma^a)^\dagger = I, \quad \tilde{\gamma}^a \tilde{\gamma}^a = \eta^{aa}, \quad \tilde{\gamma}^a(\tilde{\gamma}^a)^\dagger = I, \quad (18) $$

where $I$ represents the unit operator. Making a choice for the $\theta^a$ properties as presented in Eq. (13), it follows for the Clifford objects

$$ \{\gamma^a\}^* = (\gamma^0, \gamma^1, -\gamma^2, \gamma^3, \gamma^5, \ldots, -\gamma^{d-1}, \gamma^d), $$

$$ \{\tilde{\gamma}^a\}^* = (-\gamma^0, -\gamma^1, \gamma^2, -\gamma^3, \gamma^5, -\gamma^6, \ldots, \gamma^{d-1}, -\gamma^d), \quad (19) $$
Applying the operators $S^{ab}$ and $\tilde{S}^{ab}$, Eq. (2), on $\frac{1}{2}(\gamma^a + \frac{i\eta^a}{ik}\gamma^b)$ and on $\frac{1}{2}(1 + \frac{i}{k} \gamma^a \gamma^b)$ one obtains

\[
\begin{align*}
S^{ab}\frac{1}{2}(\gamma^a + \frac{i\eta^a}{ik}\gamma^b) &= \frac{k}{2} \frac{1}{2}(\gamma^a + \frac{i\eta^a}{ik}\gamma^b), \\
\tilde{S}^{ab}\frac{1}{2}(\gamma^a + \frac{i\eta^a}{ik}\gamma^b) &= \frac{k}{2} \frac{1}{2}(\gamma^a + \frac{i\eta^a}{ik}\gamma^b), \\
S^{ab}\frac{1}{2}(1 + \frac{i}{k} \gamma^a \gamma^b) &= \frac{k}{2} \frac{1}{2}(1 + \frac{i}{k} \gamma^a \gamma^b) \\
\tilde{S}^{ab}\frac{1}{2}(1 + \frac{i}{k} \gamma^a \gamma^b) &= -\frac{k}{2} \frac{1}{2}(1 + \frac{i}{k} \gamma^a \gamma^b).
\end{align*}
\]  

(20)

One could make a choice of $\tilde{\gamma}^a$ instead of $\gamma^a$ and change correspondingly relations in Eq. (20).

All three choices for the linear vector space — spanned over either the coordinate Grassmann space, or over the vector space of $\gamma^a$, as well as over the vector space of $\tilde{\gamma}^a$ — have the dimension $2^d$.

We can express Grassmann coordinates $\theta^a$ and momenta $p^{ba} = i \frac{\partial}{\partial \theta^a}$ in terms of $\gamma^a$ and $\tilde{\gamma}^a$ as well [34]

\[
\begin{align*}
\theta^a &= \frac{1}{2}(\gamma^a - i\tilde{\gamma}^a), \\
\frac{\partial}{\partial \theta^a} &= \frac{1}{2}(\gamma^a + i\tilde{\gamma}^a).
\end{align*}
\]  

(21)

It then follows, taking into account Eqs. (22, 49), $\frac{\partial}{\partial \theta^a} |\psi\rangle_1 = \eta^{ab} |\psi\rangle_1$.

Correspondingly we can use either $\gamma^a$ or $\tilde{\gamma}^a$ instead of $\theta^a$ to span the vector space. In this case we change the vacuum from the one with the property $\frac{\partial}{\partial \theta^a} |\phi\rangle = 0$ to $|\psi\rangle_0 >$ with the property [2, 7, 9]

\[
<\psi_0|\gamma^a|\psi_0> = 0, \quad \tilde{\gamma}^a|\psi_0> = i\gamma^a|\psi_0>, \quad \tilde{\gamma}^a\gamma^b|\psi_0> = -i\gamma^b\gamma^a|\psi_0>,
\]

\[
\tilde{\gamma}^a\tilde{\gamma}^b|\psi_0> |_{a \neq b} = -\gamma^a\gamma^b|\psi_0>, \quad \tilde{\gamma}^a\tilde{\gamma}^b|\psi_0> |_{a = b} = \eta^{ab}|\psi_0>.
\]  

(22)

This is in agreement with the requirement

\[
\begin{align*}
\gamma^a \mathbf{F}(\gamma)|\psi_0> &= (a_0 \gamma^a + a_{a_1} \gamma^a \gamma^a + a_{a_1a_2} \gamma^a \gamma^a \gamma^a + \cdots + a_{a_1 \cdots a_d} \gamma^a \gamma^a \cdots \gamma^a) |\psi_0>, \\
\tilde{\gamma}^a \mathbf{F}(\tilde{\gamma})|\psi_0> &= (i a_0 \gamma^a - i a_{a_1} \gamma^a \gamma^a + i a_{a_1a_2} \gamma^a \gamma^a \gamma^a + \cdots + \\
i (-1)^d a_{a_1 \cdots a_d} \gamma^a \gamma^a \cdots \gamma^a) |\psi_0>.
\end{align*}
\]  

(23)

The basic states in Clifford space can be arranged in representations, in which any state is the eigenstate of the Cartan subalgebra operators of Eq. (13). The state, for example, in $d$-dimensional space with the eigenvalues of either $S^{03}, S^{12}, S^{56}, \ldots, S^{d-1}$ or $\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \ldots, \tilde{S}^{d-1}$ equal to $\frac{1}{2}(i, 1, 1, \ldots, 1)$ is $(\gamma^0 - \gamma^3)(\gamma^1 + i\gamma^2)(\gamma^5 + i\gamma^6) \cdots (\gamma^{d-1} + i\gamma^d)$, where the states are expressed in
terms of $\gamma^a$. The states of one representation follow from the starting state by the application of $S^{ab}$, which do not belong to the Cartan subalgebra operators, while $\tilde{S}^{ab}$, which operate on family quantum numbers, cause jumps from the starting family to the new one.

A. Norms of vectors in Grassmann and Clifford space

Let us look for the norm of vectors in Grassmann space, $\mathbf{B} = \sum_{k=0}^{d} a_{a_1a_2...a_k} \theta^{a_1} \theta^{a_2} \ldots \theta^{a_k} |\phi_{og} >$, and in Clifford space, $\mathbf{F} = \sum_{k=0}^{d} a_{a_1a_2...a_k} \gamma^{a_1} \gamma^{a_2} \ldots \gamma^{a_k} |\psi_{oc} >$, where $|\phi_{og} >$ and $|\phi_{oc} >$ are the vacuum states in the Grassmann and Clifford case, respectively. In what follows we refer to Ref. [2].

a. Norms of Grassmann vectors

Let us define the integral over the Grassmann space [2] of two functions of the Grassmann coordinates $< \mathbf{B}|\theta > < \mathbf{C}|\theta >$, by requiring

$$\{d\theta^a, \theta^k\}_+ = 0, \quad \int d\theta^a = 0, \quad \int d\theta^a \theta^a = 1, \quad \int d^d \theta \theta^1 \ldots \theta^d = 1,$$

$$d^d \theta = d\theta^d \ldots d\theta^0, \quad \omega = \prod_{k=0}^{d} (\frac{\partial}{\partial \theta_k} + \theta^k), \quad (24)$$

with $\frac{\partial}{\partial \theta_k} \theta^e = \eta^{ae}$. We shall use the weight function $\omega = \prod_{k=0}^{d} (\frac{\partial}{\partial \theta_k} + \theta^k)$ to define the scalar product $< \mathbf{B}|\mathbf{C} >$

$$< \mathbf{B}|\mathbf{C} > = \int d^{d-1} x d^d \theta^a \omega < \mathbf{B}|\theta > < \theta|\mathbf{C} > = \sum_{k=0}^{d} \int d^{d-1} x b_{b_1...b_k}^* c_{a_1...a_k}, \quad (25)$$

where, according to Eq. (11), it follows: $< \mathbf{B}|\theta > = < \phi_{og} | \sum_{p=0}^{d} (-i)^p a_{a_1...a_p} \theta^{a_1} \theta^{a_2} \ldots \theta^{a_p} \eta^{a_{p+1}a_{p+2}} \ldots \eta^{a_{d-1}a_d} |\phi_{og} > = 1$ for $|\phi_{og} > = |1 >$, as assumed in Eq. (9).

The norm $< \mathbf{B}|\mathbf{B} >$ is correspondingly always nonnegative. Let us notice that the choice that the Hermitian conjugated value of $\theta^a$ is $\frac{\partial}{\partial \theta^a} (\theta^a)^\dagger = \eta^{a\alpha} \frac{\partial}{\partial \theta_{\alpha}}$, Eq. (11)), makes that we easily evaluate in any $d$ the scalar product $< \phi_{og} (\frac{\partial}{\partial \theta^a} \theta^{a_1} \theta^{a_2} \ldots \theta^{a_{d-2}} \theta^{d-1}\theta^d) |\phi_{og} > = 1$ for $|\phi_{og} > = |1 >$ (without integration over coordinate space of $\theta^a$'s).

b. Norms of Clifford vectors

To evaluate norms in the Clifford space for vectors of Eq. (17) we can use as well Eqs. (24, 25), if expressing $\gamma^a$ (or equivalently $\tilde{\gamma}^a$) in terms of $\theta^a$ and $p^{\beta a}$: $< (\theta^a - ip^{\beta a})|\mathbf{F} >$ (or equivalently $< (\theta^a + ip^{\beta a})|\tilde{\mathbf{F}} >$, Eq. (4). In this case $|\psi_{oc} > = |\phi_{og} > = |1 >$. It follows

$$< \mathbf{F}|\mathbf{G} > = \int d^{d-1} x d^d \theta^a \omega < \mathbf{F}|\gamma > < \gamma|\mathbf{G} > = \sum_{k=0}^{d} \int d^{d-1} x a_{a_1...a_k}^* b_{b_1...b_k}. \quad (26)$$
Similarly one obtains, if expressing \( \tilde{F} = \sum_{k=0}^{d} a_{a_1 a_2 \ldots a_k} \tilde{\gamma}^{a_1} \tilde{\gamma}^{a_2} \ldots \tilde{\gamma}^{a_k} |\phi_{oc} > \) and \( \tilde{G} = \sum_{k=0}^{d} b_{b_1 b_2 \ldots b_k} \tilde{\gamma}^{b_1} \tilde{\gamma}^{b_2} \ldots \tilde{\gamma}^{b_k} |\phi_{oc} > \) and taking \( |\psi_{oc} > = |\phi_{og} > = |1 >, \) the scalar product

\[
< \tilde{F} | \tilde{G} > = \int d^{d-1} x d^{d} \theta \omega < \tilde{\gamma} | \tilde{\gamma} > = \sum_{k=0}^{d} \int d^{d-1} x a^{*}_{a_1 \ldots a_k} a_{b_1 \ldots b_k} \cdot \]

(27)

To simplify the evaluation we use instead \([3, 16]\) in the Clifford case the vacuum state \( |\phi_{oc} >, \) Eq. (59), which is the product of projectors, Eq. (50). It takes care of the orthogonality of states as we would evaluate the integration in Grassmann space.

Correspondingly we can write

\[
\int d^{d} \theta \omega (a_{a_1 a_2 \ldots a_k} \tilde{\gamma}^{a_1} \gamma^{a_2} \ldots \gamma^{a_k}) (a_{a_1 a_2 \ldots a_k} \gamma^{a_1} \gamma^{a_2} \ldots \gamma^{a_k}) = a^{*}_{a_1 a_2 \ldots a_k} a_{a_1 a_2 \ldots a_k} \cdot
\]

(28)

The norm of each scalar term in the sum of \( F \) is nonnegative.

c. We have learned:

In both spaces — Grassmann and Clifford — norms of basic states can be defined so that the states, which are eigenvectors of the Cartan subalgebra, are orthogonal and normalized using the same integral.

Studying the second quantization procedure in Subsect. II C we learn that not all \( 2^d \) states can be represented as creation and annihilation operators, either in the Grassmann or in the Clifford case, since they must — in both cases — fulfill the requirements for the second quantized operators, either for states with integer spins in Grassmann space or for states with a half integer spin in Clifford space.

B. Actions in Grassmann and Clifford space

We construct an action for free massless fermion in which the internal degrees of freedom is described: i. in Grassmann space, ii. in Clifford space. In the first case the internal degrees of freedom manifest integer spins, in the second case the half integer spin.

While the action in Clifford space is well known since long [22], the action in Grassmann space will be defined here. In both cases we present an action for free massless fermions in \(( (d - 1) + 1)\) space [35]. States in Grassmann space as well as states in Clifford space will be arranged to be the eigenstates of the Cartan subalgebra — with respect to \( S^{ab} \) in Grassmann space and with respect to \( S^{ab} \) and \( \tilde{S}^{ab} \) in Clifford space, Eq. (B4), and orthogonal and normalized with respect to Eq. (24).

In both spaces the requirement that states are obtained by the application of creation operators on the vacuum state — \( \hat{b}_i^\theta \) obeying the commutation relations of Eq. (42) on the vacuum state
|φ\_og \rangle = |1\rangle \in \text{Grassmann space, and } \hat{b}^a_i \text{ obeying the commutation relation of Eq. (61) on the vacuum states } |\psi\_oc \rangle, \text{ Eq. (59), in Clifford space — reduces the number of states, in Clifford space more than in Grassmann space. But while in Clifford space all physically applicable states are reachable by either } S^{ab} \text{ (defining family members quantum numbers) or by } \tilde{S}^{ab} \text{ (defining family quantum numbers), the states in Grassmann space, belonging to different representations with respect to the Lorentz generators, seem not to be connected.}

\textbf{a. Action in Clifford space}

In Clifford space the action for a free massless fermion must be Lorentz invariant

\[ \mathcal{A} = \int d^4x \frac{1}{2} \left( \psi^\dagger \gamma^0 \gamma^a p_a \psi \right) + \text{h.c.}, \]

\[ p_a = i \frac{\partial}{\partial x^a}, \]

leading to the equations of motion

\[ \gamma^a p_a |\psi^{\alpha} > = 0, \]

which fulfill also the Klein-Gordon equation

\[ \gamma^a p_a \gamma^b p_b |\psi^{\alpha}_i > = p^a p_a |\psi^{\alpha}_i > = 0, \]

for each of the basic states \(|\psi^{\alpha}_i >\). Correspondingly \(\gamma^0\) appears in the action since we pay attention that

\[ S^{ab\dagger} \gamma^0 = \gamma^0 S^{ab}, \quad S^{\dagger} \gamma^0 = \gamma^0 S^{-1}, \]

\[ S = e^{-\frac{i}{2} \omega_{ab}(S^{ab}+L^{ab})}. \]

All the states, belonging to different values of the Cartan subalgebra — they differ at least in one value of either the set of \(S^{ab}\) or the set of \(\tilde{S}^{ab}\), Eq. (14) — are orthogonal according to the scalar product, defined as the integral over the Grassmann coordinates, Eq. (21), for a chosen vacuum state \(|1\rangle\). Correspondingly the states generated by the creation operators, Eq. (56), on the vacuum state, Eq. (59), are orthogonal as well (both last equations will appear later).

\textbf{b. Action in Grassmann space}

We define here the action in Grassmann space, for which we require — similarly as in the Clifford case — that the action for a free massless fermion is Lorentz invariant.

\[ \mathcal{A} = \frac{1}{2} \left\{ \int d^d x \ d^d \theta \ \omega \left( \phi^\dagger (1 - 2 \theta^0 \frac{\partial}{\partial \theta^0}) \frac{1}{2} (\theta^a p_a + \eta^{\alpha a} \theta^a p_\alpha) \phi \right) \right\}, \]

(33)
We use the integral over $\theta^a$ coordinates with the weight function $\omega$ from Eq. (24). Requiring the Lorentz invariance we add after $\phi^\dagger$ the operator $\gamma_0^a (\gamma_0^a = (1 - 2 \theta^0_0 \frac{\partial}{\partial \theta^0_0}))$, which takes care of the Lorentz invariance. Namely

$$S^{ab\dagger} (1 - 2 \theta^0_0 \frac{\partial}{\partial \theta^0_0}) = (1 - 2 \theta^0_0 \frac{\partial}{\partial \theta^0_0}) S^{ab},$$

$$S^\dagger (1 - 2 \theta^0_0 \frac{\partial}{\partial \theta^0_0}) = (1 - 2 \theta^0_0 \frac{\partial}{\partial \theta^0_0}) S^{-1},$$

$$S = e^{-\frac{i}{2} \omega_{ab} (L^{ab} + S^{ab})},$$

(34)

while $\theta^a, \frac{\partial}{\partial \theta^a}$ and $p^a$ transform as Lorentz vectors. The equations of motion follow from the action, Eq. (33),

$$\frac{1}{2} [(1 - 2 \theta^0_0 \frac{\partial}{\partial \theta^0_0}) \theta^a + ((1 - 2 \theta^0_0 \frac{\partial}{\partial \theta^0_0}) \theta^a)^\dagger] p_a |\phi^k_i> = 0,$$

(35)

as well as the Klein-Gordon equation

$$\{\theta^a p_a, \frac{\partial}{\partial \theta^b} p_b\} + |\phi^k_i> = p^a p_a |\phi^k_i> = 0,$$

(36)

for superposition (of each of the two groups) of the basic states $|\phi^k_i>$, $k = (1, 2)$, Eqs. (31, 45).

We shall see that, if one identifies the creation operators in both spaces with the products of odd numbers of either $\theta^a$ — in the Grassmann case — or $\gamma^a$ — in the Clifford case — and the annihilation operators as the Hermitian conjugate operators of the creation operators, the creation and annihilation operators fulfill the anticommutation relations, required for fermions. The internal parts of states are then defined by the application of the creation operators on the vacuum state.

But while the Clifford algebra defines states with the half integer "eigenvalues" of the Cartan subalgebra operators of the Lorentz algebra, the Grassmann algebra defines states with the integer "eigenvalues" of the Cartan subalgebra operators.

**c. We learn:**

In both spaces — in Clifford and in Grassmann space — there exists the action, which leads to the equations of motion and to the corresponding Klein-Gordon equation for free massless fields.

**C. Second quantization of Grassmann and Clifford vectors**

States in Grassmann space as well as states in Clifford space are organized to be — within each of the two spaces — orthogonal and normalized with respect to Eq. (24, 25, 26). All the states in each of spaces are chosen to be eigenstates of the Cartan subalgebra — with respect to $S^{ab}$ in Grassmann space, and with respect to $S^{ab}$ and $\tilde{S}^{ab}$ in Clifford space, Eq. (B4).
In both spaces the requirement that states are obtained by the application of creation operators on the vacuum state — $\hat{b}^\theta_i$ obeying the commutation relations of Eq. (42) on the vacuum state $|\phi_\text{og}\rangle = |1\rangle$ for Grassmann space, and $\hat{b}^\alpha_i$ obeying the commutation relation of Eq. (61) on the vacuum states $|\psi_\text{oc}\rangle$, Eq. (59), for Clifford space — reduces the number of states arranged into the representations of the Lorentz group. The reduction of degrees of freedom depends on whether $d = 2(2n + 1)$ or $d = 4n$, $n$ is a positive integer. The second quantization procedure with creation operators expressed by the product of Grassmann or Clifford objects requires that the product has an odd number of objects, either in Grassmann or in Clifford space.

We shall pay attention in this paper almost only to spaces with $d = 2(2n + 1)$. We define in Grassmann space creation operators by an odd number of factors of superposition of $\theta^a$’s and annihilation operators by Hermitian conjugation of the corresponding creation operators. In Clifford space we define creation operators by an odd number of factors of superposition of $\gamma^a$’s and the annihilation operators by Hermitian conjugate creation operators. Each basic state in any of two spaces is a product of factors chosen to be eigenstates of the Cartan subalgebra of the Lorentz algebra, Eqs. (134, 39, 53, 52).

But while in Clifford space all physically applicable states are reachable either by $S^{ab}$ or by $\tilde{S}^{ab}$, the states belonging to different groups with respect to the Lorentz generators in Grassmann space are not connected by the Lorentz operators.

Let us construct creation and annihilation operators for fermions while using a. Grassmann vector space, b. Clifford vector space. We shall see that from $2^d$ states in any of these two spaces there are reduced number of states generated by the creation operators, which fulfill the requirements for the creation and their Hermitian conjugate annihilation operators.

**a. Second quantization in Grassmann space**

There are $2^d$ states in Grassmann space, orthogonal to each other with respect to Eqs. (24, 25). To any coordinate there exists the conjugate momentum. We pay attention in what follows mostly to spaces with $d = 2(2n + 1)$. In $d = 2(2n + 1)$ spaces there are $\frac{d!}{2^{\frac{d}{2}}}$ states, divided into two separated groups of states, Eq. (46). All states of one group are reachable from a starting state by the application of $S^{ab}$. The states, which contribute in the second quantization procedure, are Grassmann odd products of eigenstates of the Cartan subalgebra. Any Grassmann odd state can be written as a creation operator, operating on the vacuum state, while the Hermitian conjugated creation operator is the corresponding annihilation operator. Creation and annihilation operators of an odd Grassmann character fulfill the commutation relations of Eq. (48, 12). Let us see how it
works.

If $\hat{b}_i^{\theta}$ is a creation operator, which creates a state in the Grassmann space when operating on a vacuum state $|\psi_{og}\rangle$ and $\hat{b}_i^{\theta} = (\hat{b}_i^{\theta \dagger})^\dagger$ is the corresponding annihilation operator, then for a set of creation operators $\hat{b}_i^{\theta \dagger}$ and the corresponding annihilation operators $\hat{b}_i^{\theta}$ it must be

$$\hat{b}_i^{\theta} |\phi_{og}\rangle = 0, $$
$$\hat{b}_i^{\theta \dagger} |\phi_{og}\rangle \neq 0. \quad (37)$$

We first pay attention on only the internal degrees of freedom of the Grassmann fermions: the spin.

Choosing $\hat{b}_a^{\theta} = \theta^a$, then it follows that $(\hat{b}_a^{\theta \dagger})^\dagger = \frac{\partial}{\partial \theta^a}$. One correspondingly finds

$$\hat{b}_a^{\theta \dagger} = \theta^a, \quad \hat{b}_a^{\theta} = \frac{\partial}{\partial \theta^a}, $$
$$\{\hat{b}_a^{\theta}, \hat{b}_b^{\theta \dagger}\} + |\phi_{og}\rangle > = \delta_{ab} |\phi_{og}\rangle >, $$
$$\{\hat{b}_a^{\theta \dagger}, \hat{b}_b^{\theta}\} + |\phi_{og}\rangle > = 0, $$
$$\{\hat{b}_a^{\theta}, \hat{b}_b^{\theta \dagger}\} + |\phi_{og}\rangle > = 0, $$
$$\hat{b}_a^{\theta} |\phi_{og}\rangle > = \theta^a |\phi_{og}\rangle >, $$
$$\hat{b}_a^{\theta \dagger} |\phi_{og}\rangle > = 0. \quad (38)$$

The vacuum state $|\phi_{og}\rangle >$ is in this case chosen to be $|1\rangle >$.

The identity $I (I^\dagger = I)$ cannot be taken as a creation operator, since its annihilation partner does not fulfill Eq. (37).

We can use the products of superposition of $\theta^a$’s as creation operators and correspondingly products of superposition of $\frac{\partial}{\partial \theta^a}$’s as annihilation operators, provided that they fulfill the requirements for the creation and annihilation operators, Eq. (42), with the vacuum state $|\phi_{og}\rangle > = |1\rangle >$. In general they would not. Only an odd number of $\theta^a$ in any product would have the required anticommutation properties.

To construct creation operators it is convenient to take products of such superposition of vectors $\theta^a$ and $\theta^b$ that each factor is the “eigenstate” of one of the Cartan subalgebra members of the Lorentz algebra (34). Let us start with the creation operator, which is a products of $\frac{d}{2}$ “eigenstates” of the Cartan subalgebra $S^{ab}$: $\hat{b}_a^{\theta \dagger} = \frac{1}{\sqrt{2}} (\theta^{ai} \pm \epsilon \theta^{bi})$. Then the corresponding annihilation operator has as well $\frac{d}{2}$ factors: $\hat{b}_a^{\theta} = \frac{1}{\sqrt{2}} (\frac{\partial}{\partial \theta^a} \pm \epsilon^* \frac{\partial}{\partial \theta^b})$, $\epsilon = i$, if $\eta^{ai} a_i = \eta^{bi} b_i$ and $\epsilon = -1$, if $\eta^{ai} a_i \neq \eta^{bi} b_i$. \]
Let us in \( d = 2(2n + 1) \), \( n \) is a positive integer, start with the state

\[
|\phi_1\rangle = \hat{b}_1^{\theta_1} |1\rangle,
\]

\[
\hat{b}_1^{\theta_1} = \left( \frac{1}{\sqrt{2}} \right)^{\frac{d}{2}} (\theta_0 - i\theta_3)(\theta_1 + i\theta^2)(\theta_5 + i\theta_6) \cdots (\theta_{d-1} + i\theta_d),
\]

\[
S^{ab}(\theta^a \mp \theta^b) = \pm (\theta^a \mp \theta^b), \quad (a = 0, b \neq 0),
\]

\[
S^{ab}(\theta^a \pm i\theta^b) = \pm (\theta^a \pm i\theta^b), \quad (a, b) \neq 0,
\]

\[
S^{ab}(\theta^a\theta^b) = 0.
\]  

(39)

The rest of states, belonging to the same Lorentz representation, follow from the starting state by the application of the operators \( S^{cf} \), which do not belong to the Cartan subalgebra operators.

One can find creation and annihilation operators for \( d = 4n \) in App. A.

1. We proposed in Eq. (39) the starting creation operator \( \hat{b}_1^{\theta_1} \), the upper index indicates one of the two groups, the lower index indicates the starting state. By taking into account Eqs. (11, 12) the starting creation operator and its annihilation partner are for \( d = 2(2n + 1) \) equal to

\[
\hat{b}_1^{\theta_1\dagger} = \left( \frac{1}{\sqrt{2}} \right)^{\frac{d}{2}} (\theta_0 - i\theta_3)(\theta_1 + i\theta^2)(\theta_5 + i\theta_6) \cdots (\theta_{d-1} + i\theta_d),
\]

\[
\hat{b}_1^{\theta_1} = \left( \frac{1}{\sqrt{2}} \right)^{\frac{d}{2}} \left( \frac{\partial}{\partial \theta^{d-1}} - i \frac{\partial}{\partial \theta^d} \right) \cdots \left( \frac{\partial}{\partial \theta^0} - \frac{\partial}{\partial \theta^3} \right).
\]

for \( d = 2(2n + 1) \).

(40)

The rest of creation operators belonging to this group (group 1) in \( d = 2(2n + 1) \) follow by the application of all the operators \( S^{cf} \), which do not belong to the Cartan subalgebra operators. The corresponding annihilation operators are the Hermitian conjugated values of a particular creation operator. For \( d = 2(2n + 1) \) one finds by the application of \( S^{01} \) another creation operator and the corresponding annihilation operator as follows

\[
\hat{b}_2^{\theta_1\dagger} = \left( \frac{1}{\sqrt{2}} \right)^{\frac{d}{2} - 1} (\theta^0\theta^3 + i\theta^1\theta^2)(\theta^5 + i\theta^6) \cdots (\theta_{d-1} + i\theta_d),
\]

\[
\hat{b}_2^{\theta_1} = \left( \frac{1}{\sqrt{2}} \right)^{\frac{d}{2} - 1} \left( \frac{\partial}{\partial \theta^{d-1}} - i \frac{\partial}{\partial \theta^d} \right) \cdots \left( \frac{\partial}{\partial \theta^3} - \frac{\partial}{\partial \theta^0} \right),
\]

in general:

\[
\hat{b}_i^{\theta_1\dagger} \propto S^{ab}\cdots S^{cf}\hat{b}_1^{\theta_1\dagger},
\]

\[
\hat{b}_i^{\theta_1} = (\hat{b}_i^{\theta_1\dagger})^\dagger.
\]  

(41)

It was taken into account in the above equation that any \( S^{ac} \) \( (a \neq c) \), which does not belong to Cartan subalgebra, Eq. (B4), transforms \( (\frac{1}{\sqrt{2}})^2(\theta^a + i\theta^b)(\theta^c + i\theta^d) \) \( (a \neq c \text{ and } a \neq d, b \neq c \text{ and } c \neq d) \).
b \neq d, \eta^{aa} = \eta^{bb}) into \frac{1}{\sqrt{2}}(\theta^a\theta^b + \theta^c\theta^d)$. The states are normalized and the simplest phases are assumed. One evaluates that $S^{ab}(\theta^a \pm \epsilon \theta^b) = \mp i \eta^{aa}(\theta^a \pm \epsilon \theta^b)$, $\epsilon = 1$ for $\eta^{aa} = 1$ and $\epsilon = i$ for $\eta^{aa} = -1$, while either $S^{ab}$ or $S^{cd}$, applied on $(\theta^a \theta^b \pm \epsilon \theta^c \theta^d)$, gives zero. The vacuum state is in all these cases $|1\rangle$.

Although all the states, generated by creation operators, which include one $(I \pm \epsilon \theta^a \theta^b)$ or several $(I \pm \epsilon \theta^{a_1} \theta^{b_1}) \cdots (I \pm \epsilon \theta^{a_k} \theta^{b_k})$, are eigenstates of the Cartan subalgebra operators and are orthogonal with respect to the scalar product, Eq. (25), to all the other states, their Hermitian conjugate values include $I^\dagger$, which, when applying on the vacuum state $|\phi_{og}\rangle = |1\rangle$, do not give zero. Correspondingly such creation operators do not have appropriate annihilation partners, which would fulfill Eqs. (37, 38).

However, creation operators which are products of several $\theta$’s, $\theta^{a_1} \cdots \theta^{a_n}$, let say $n$ with $n = 2, 4 \ldots \frac{d}{2} - 1$ (always of an even number of $\theta$’s, since $S^{ab}$ is a Grassmann even operator, Eq. (3)), and are ”eigenstates” of the Cartan subalgebra operators (Eq. (B4), namely $S^{ab} \theta^a \theta^b |1\rangle = 0$), fulfill the requirements of Eq. (42), provided that the rest of expression has an odd number of factors — $(\frac{d}{2} - n)$.

\[
\begin{align*}
\{\hat{b}_{i_1}^{\theta_1}, \hat{b}_{j_1}^{\theta_1}\} + |\phi_{og}\rangle &= \delta_{ij} |\phi_{og}\rangle, \\
\{\hat{b}_{i_1}^{\theta_1}, \hat{b}_{j_1}^{\theta_1}\} + |\phi_{og}\rangle &= 0 |\phi_{og}\rangle, \\
\{\hat{b}_{i_1}^{\theta_1}, \hat{b}_{j_1}^{\theta_1}\} + |\phi_{og}\rangle &= 0 |\phi_{og}\rangle, \\
\hat{b}_{j_1}^{\theta_1} |\phi_{og}\rangle &= |\phi^1_{og}\rangle \\
\hat{b}_{j_1}^{\theta_1} |\phi_{og}\rangle &= 0 |\phi_{og}\rangle \\
(k, l) &= 1, 2.
\end{align*}
\]

It is not difficult to see that states included into one representation, which started with $\hat{b}_{i_1}^{\theta_1}|1\rangle$ as presented in Eq. (40) for $d = (2n + 1)2$ have the properties, required by Eq. (42):

**i.a.** In any $d$-dimensional space the product \( \frac{\partial}{\partial \theta^{a_1}} \cdots \frac{\partial}{\partial \theta^{a_k}} \), with all different $a_i$ (also if all or some of them are equal, since $(\frac{\partial}{\partial \theta^a})^2 = 0$), if applied on the vacuum $|1\rangle$, is equal to zero. Correspondingly the second equation and the fifth equation of Eq. (42) are fulfilled.

**i.b.** In any $d$-dimensional space the product of different $\theta$’s — $\theta^{a_1} \theta^{a_2} \cdots \theta^{a_k}$ with all different $\theta^{a_i}$ ($a_i \neq a_j$ for all $a_i$ and $a_j$) — applied on the vacuum $|1\rangle$, is different from zero. Since all the $\theta$’s, appearing in Eqs. (40, 41), are different, forming orthogonal and normalized states, the fourth equation of Eq. (42) is fulfilled.

**i.c.** The third equation of Eq. (42) is fulfilled provided that there is an odd number of $\theta^a$ in
the expression for a creation operator. Then, when in the anticommutation relation different $\theta^a$'s appear (like in the case of $d = 6 \{\theta^0\theta^3\theta^5, \theta^1\theta^2\theta^6\}_+$), such a contribution gives zero. When two or several equal $\theta$'s appear in the anticommutation relation, the contribution is zero (since $(\theta^a)^2 = 0$).

\textbf{i.d.} Also for the first equation in Eq. (42) it is not difficult to show that it is fulfilled only for a particular creation operator and its Hermitian conjugate: Let us show this for $d = 1 + 3$ and the creation operator $\frac{1}{\sqrt{2}}(\theta^0 - \theta^3)\theta^1\theta^2$ and its Hermitian conjugate (annihilation) operator: $\frac{1}{\sqrt{2}}\{\frac{\partial}{\partial \theta^2} \frac{\partial}{\partial \theta^5} (\frac{\partial}{\partial \theta^0} - \frac{\partial}{\partial \theta^3}), \frac{1}{\sqrt{2}}(\theta^0 - \theta^3)\theta^1\theta^2\}_+$. Applying $(\frac{\partial}{\partial \theta^0} - \frac{\partial}{\partial \theta^3})$ on $(\theta^0 - \theta^3)$ gives two, while $\frac{\partial}{\partial \theta^2}$ applied on $\theta^1\theta^2$ gives one.

\textbf{ii.} There is one additional group of creation and annihilation operators in $d = 2(2n + 1)$, which follow from the starting state

$$|\phi^2_1\rangle = \hat{b}^{\theta^2}_0|1\rangle,$$

$$\hat{b}^{\theta^2}_0|1\rangle = \left(\frac{1}{\sqrt{2}}\right)^2(\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d),$$

for $d = 2(2n + 1)$. (43)

This state can not be obtained from the previous group of states, presented in Eqs. (40, 41) by the application of $S^{ef}$, since each $S^{ef}$ changes an even number of factors, never an odd one. Correspondingly the above starting state forms a new group of states in $d = 2(2n + 1)$. All the other states of this new group of states follow from the starting one by the application of $S^{ef}$. The corresponding creation and annihilation operators are

$$\hat{b}^{\theta^2}_1 = \left(\frac{1}{\sqrt{2}}\right)^2(\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d),$$

$$\hat{b}^{\theta^2}_1 = \left(\frac{1}{\sqrt{2}}\right)^2(\frac{\partial}{\partial \theta^d} - \frac{i}{\partial \theta^3})\cdots(\frac{\partial}{\partial \theta^0} + \frac{\partial}{\partial \theta^3}),$$

for $d = 2(2n + 1)$. (44)

As in the case of the first group all the rest of creation operators are obtainable from the starting one by the application of $S^{ac}$, and the annihilation operators by the Hermitian conjugation of the creation operators.

$$\hat{b}^{\theta^2}_i \propto S^{ab} \cdots S^{ef} \hat{b}^{\theta^2}_1,$$

$$\hat{b}^{\theta^2}_i = (\hat{b}^{\theta^2}_1)^\dagger.$$

Also all these creation and annihilation operators fulfill the requirements for the creation and annihilation operators, presented in Eq. (42) because of the same reasons as in the first case.
One can choose as the starting creation operator of the second group of operators by changing
sign instead of in the factor \((\theta^0 - \theta^3)\) in the starting creation operator of the first group in any of
the rest of factors in the product. In each case the same group will follow.

Let us now count the number of states of the odd Grassmann character in \(d = 2(2n + 1)\).

There are in \((d = 2)\) two creation \(((\theta^0 \mp \theta^1, \text{ for } \eta^{\alpha \beta} = \text{diag}(1, -1))\) and correspondingly two
annihilation operators \((\partial_{\theta^0} \mp \partial_{\theta^1})\), each belonging to its own group with respect to the Lorentz
transformation operators, both fulfilling Eq. (42).

It is not difficult to see that the number of all creation operators of an odd Grassmann character
in \(d = 2(2n + 1)-\)dimensional space is equal to \(\frac{d!}{d^2 d^2!}\).

We namely ask: In how many ways can one put on \(\frac{d^2}{2}\) places \(d\) different \(\theta^a\)’s. And the answer is
— the central binomial coefficient for \(x^{\frac{d}{2}} 1^{\frac{d}{2}}\) — with all \(x\) different. This is just \(\frac{d!}{\frac{d^2}{2}}\). But we have
counted all the states with an odd Grassmann character, while we know that these states belong
to two different groups of representations with respect to the Lorentz group.

Correspondingly one concludes: There are two groups of states in \(d = 2(2n + 1)\) with an odd
Grassmann character, each of these two groups has

\[
\frac{1}{2} \cdot \frac{d!}{\frac{d^2}{2}}
\]  

members.

In \(d = 2\) we have two groups with one state, which have an odd Grassmann character, in \(d = 6\)
we have two groups of 10 states, in \(d = 10\) we have two groups of 126 states with an odd Grassmann
characters. And so on.

Correspondingly we have in \(d = 2(2n + 1)\)-dimensional spaces two groups of creation operators
with \(\frac{1}{2} \cdot \frac{d!}{\frac{d^2}{2}}\) members each, creating states with an odd Grassmann character and the same number
of annihilation operators. Creation and annihilation operators fulfill anticommutation relations
presented in Eq. (42), .

The rest of creation operators [and the corresponding annihilation operators] with the opposite
Grassmann character than the ones studied so far — like \(\theta^0 \theta^1 \left[ \partial_{\theta^0} \partial_{\theta^1} \right] \) in \(d = (1 + 1) (\theta^0 \mp \theta^3)(\theta^1 \pm i\theta^2) \left[ (\partial_{\theta^1} \mp i \partial_{\theta^2}) (\partial_{\theta^0} \mp \partial_{\theta^3}) \right], \theta^0 \theta^3 \theta^1 \theta^2 \left[ \partial_{\theta^0} \partial_{\theta^1} \partial_{\theta^2} \partial_{\theta^3} \right] \) in \(d = (3 + 1), \) do not fulfill the
anticommutation relations required for fermions in Eq. (42), with \(\hat{b}_i^{\theta 1} \) and \(\hat{b}_i^{\theta 1\dagger} \) replaced by \(\hat{b}_i^{\theta k} \) and
\(\hat{b}_i^{\theta k\dagger}, k = (1, 2) \) and correspondingly with \(\{\hat{b}_i^{\theta k}, \hat{b}_j^{\theta l\dagger}\} \phi_{\text{og}} = \delta_{k,l} \delta_{i,j} \phi_{\text{og}}, (k,l) = (1, 2), (i,j) \) running from \((1, \ldots, \frac{d!}{\frac{d^2}{2}})\).

All the states \(\phi_i^k >, k = (1, 2)\), generated by the creation operators, Eq. (42), on the vacuum
state \(\phi_{\text{og}} > (= \mid 1 >)\) are the eigenstates of the Cartan subalgebra operators and are orthogonal.
and normalized with respect to the norm of Eq. (24)

\[ < \phi_k^i | \phi_j^{k'} > = \delta_{ij} \delta^{kk'}, \]

\[ (k, k') = (1, 2), \ (i, j) = (1, 2, \ldots, \frac{d!}{2^d \sqrt{d!}}). \]  

(47)

If we now extend the creation and annihilation operators to the ordinary coordinate space, \((x^0, \vec{x})\), the relations among creation and annihilation operators at a chosen time read

\[ \{ \hat{b}^{\theta_i}(\vec{x}), \hat{b}^{\theta_j}(\vec{x}') \} + |\phi_{og} > = \delta_{ij} \delta^{kk'} \delta(\vec{x} - \vec{x}') |\phi_{og} >, \]

\[ \{ \hat{b}^{\theta_i}(\vec{x}), \hat{b}^{\theta_j}(\vec{x}') \} + |\phi_{og} > = 0 |\phi_{og} >, \]

\[ \{ \hat{b}^{\theta_i}(\vec{x}), \hat{b}^{\theta_j}(\vec{x}') \} + |\phi_{og} > = 0 |\phi_{og} >, \]

\[ \hat{b}^{\theta_i}(\vec{x}) |\phi_{og} > = 0 |\phi_{og} >, \]

\[ \hat{b}^{\theta_j}(\vec{x}) |\phi_{og} > = |\phi_j^k(\vec{x}) > \]

\[ |\phi_{og} > = |1 >. \]  

(48)

Again indexes \(k = (1, 2)\) in \((\hat{b}^{\theta_1}, \hat{b}^{\theta_1})\), \((\hat{b}^{\theta_2}, \hat{b}^{\theta_2})\) represent creation and annihilation operators of one of the two groups of states, reachable by \(S^{ab}\).

**b. Second quantization in Clifford space**

In Grassmann space the requirement that products of "eigensates" of the Cartan subalgebra operators form the creation and annihilation operators, obeying the relations of Eq. (42), reduces the number of states from \(2^d\) (allowed in the first quantization procedure) to two isolated groups of \(\frac{d!}{2^d \sqrt{d!}}\) vectors each. (There are no generators of the Lorentz transformations \(S^{ab}\) that would connect both groups of states and correspondingly there are no families.)

Let us study what happens, when, let say, \(\gamma^a\)’s are used to create the basis and correspondingly also to create the creation and annihilation operators. Here we briefly follow Ref. [19].

Let us point out that \(\gamma^a\) is expressible with \(\theta^a\) and its derivative \((\gamma^a = (\theta^a + \frac{\partial}{\partial \theta^a}))\), Eq. (14), and that we again require that creation (annihilation) operators create (annihilate) states, which are "eigensates" (Eq. C2) of the Cartan subalgebra operators, Eq. (13). Then the application of \(\gamma^a\) on any Clifford algebra object determined by \(\gamma^a\)’s can be evaluated as follows, Eq. (22, 23),

\[ (\tilde{\gamma}^a A = i(-)^{(A)} A \gamma^a)|\psi_{oc} >, \]  

(49)

where \((-)^{(A)} = -1\), if \(A\) is an odd Clifford algebra object and \((-)^{(A)} = 1\), if \(A\) is an even Clifford algebra object, while \(|\psi_{oc} >\) is the vacuum state, replacing the vacuum state \(|\psi_o > = |1 >\), used
in Grassmann case, with the one of Eq. (59), in accordance with the relation of Eqs. (4, 25, 24), Refs. [19, 20]. We could as well make a choice of \( \tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \rho^a}) \) instead of \( \gamma^a \)'s to create the basic states [37].

Making a choice of the Cartan subalgebra "eigenstates" of \( S_{ab} \), Eq. (20), one defines nilpotents \( ab \left( k \right) \) and projectors \( ab \left[ k \right] \):

\[
\begin{align*}
ab \left( k \right) & = \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b), \quad (k)^2 = 0, \\
ab \left( k \right) & = \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b), \quad [k]^2 = [k],
\end{align*}
\]

where \( k^2 = \eta^{aa}\eta^{bb} \). Recognizing that the Hermitian conjugate values of \( ab \left( k \right) \) and \( ab \left[ k \right] \) are

\[
\begin{align*}
ab \left( k \right) ^\dagger & = \eta^{aa}(-k), \quad [k]^\dagger = [k],
\end{align*}
\]

while the corresponding "eigenvalues" of \( S_{ab} \) and \( \tilde{S}_{ab} \) on nilpotents and projectors, Eq. (Eq. (20), are

\[
\begin{align*}
S_{ab} \left( k \right) & = \frac{k}{2} \ab \left( k \right), \quad S_{ab} \left[ k \right] = \frac{k}{2} \ab \left[ k \right], \\
\tilde{S}_{ab} \left( k \right) & = \frac{k}{2} \ab \left( k \right), \quad \tilde{S}_{ab} \left[ k \right] = -\frac{k}{2} \ab \left[ k \right],
\end{align*}
\]

we find for \( d = 2(2n + 1) \) that from the starting state made as a product of an odd number of only nilpotents

\[
\begin{align*}
|\psi_1^1\rangle & = \hat{b}_1^{1\dagger}|\psi_{oc}\rangle, \\
\hat{b}_1^{1\dagger} & = \begin{pmatrix} 03 & 12 & 35 & d-3 & d-2 & d-1 & d \\ (+) & (+) & (+) & (+) & (+) & (+) & (+) \end{pmatrix}, \\
\hat{b}_1^1 & = \begin{pmatrix} d-1 & d & d-3 & d-2 & 35 & 12 & 01 \\ (-) & (-) & (-) & (-) & (-) & (-) & (-) \end{pmatrix},
\end{align*}
\]

having correspondingly an odd Clifford character, all other states of the same Lorentz representation, there are \( 2^d-1 \) members, follow by the application of \( S^{cd} \) (which do not belong to the Cartan subalgebra) on the starting state [38], Eq. (14), \( (S^{cd} |\psi_1^1\rangle = |\psi_1^1\rangle) \).

\[
\begin{align*}
\hat{b}_1^{1\dagger} \propto S_{ab}..S_{ef} \hat{b}_1^{1\dagger}, \quad |\psi_1^1\rangle & = S_{ab}..S_{ef} |\psi_1^1\rangle, \\
\hat{b}_1^1 \propto \hat{b}_1^{1\dagger} S_{ef}..S_{ab},
\end{align*}
\]

with \( S^{ab\dagger} = \eta^{aa}\eta^{bb} S_{ab} \). We make a choice of the proportionality factors so that the corresponding states \( |\psi_1^1\rangle = \hat{b}_1^{1\dagger}|\psi_{oc}\rangle \) are normalized [19, 20].

The operators \( \tilde{S}_{cd} \), which do not belong to the Cartan subalgebra of \( \tilde{S}_{ab} \), Eq. (14), generate "eigenstates" of the Cartan subalgebra operators \( (\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1d}) \), with the eigenvalues
which determine the "family" quantum numbers. There are \(2^d - 1\) families. From the starting new member with a different "family" quantum number the whole Lorentz representation of family members with this "family" quantum number follows by the application of \(S^{ef}\): \(S^{ab} \cdots \tilde{S}^{ef} \sum |\psi_1^\alpha\rangle = |\psi_1^\alpha\rangle\). All states of one Lorentz representation of any particular "family" quantum number have an odd Clifford character, since neither \(S^{cd}\) nor \(\tilde{S}^{cd}\), both of an even Clifford character, can change this character.

Any vector \(|\psi_1^\alpha\rangle\) follows from the starting vector, Eqs. (53), by the application of either \(\tilde{S}^{ef}\), which change the family quantum number, or \(S^{gh}\), which change the member of a particular family or with the corresponding product of \(S^{ef}\) and \(\tilde{S}^{ef}\)

\[
|\psi_1^\alpha\rangle \propto \tilde{S}^{ab} \cdots \tilde{S}^{ef} S^{mn} \cdots S^{pr} |\psi_1^1\rangle. \tag{55}
\]

Correspondingly we define \(\hat{b}^\alpha_i\) (up to a constant) to be

\[
\hat{b}^\alpha_i \propto \tilde{S}^{ab} \cdots \tilde{S}^{ef} S^{mn} \cdots S^{pr} \hat{b}^\alpha_1 \propto S^{mn} \cdots S^{pr} \hat{b}^\alpha_1 S^{ab} \cdots S^{ef}. \tag{56}
\]

This last expression follows due to the property of the Clifford object \(\tilde{\gamma}^a\) and correspondingly of \(\tilde{S}^{ab}\), presented in Eqs. (49, C8).

For \(\hat{b}^\alpha_i (= (\hat{b}^{\alpha\dagger}_i)^\dagger)\) we accordingly have

\[
\hat{b}^\alpha_i = (\hat{b}^{\alpha\dagger}_i)^\dagger \propto S^{ef} \cdots S^{ab} \hat{b}^\alpha_1 S^{pr} \cdots S^{mn}. \tag{57}
\]

The proportionality factor ought to be chosen so that the corresponding states \(|\psi_1^\alpha\rangle = \hat{b}^\alpha_i |\psi_\text{oc}\rangle\) are normalized when the vacuum state \(|\psi_\text{oc}\rangle\) is normalized, \(<\psi_\text{oc}| |\psi_\text{oc}\rangle = 1\), while all the states belonging to the physically acceptable states, like \([+i]|[-][+][-][+]\) \(\cdots\) \((+)(+)|\psi_\text{oc}\rangle\), must not give zero for either \(d = 2(2n + 1)\) or for \(d = 4n\). We also want that states, obtained by the application of either \(S^{cd}\) or \(\tilde{S}^{cd}\) or both, are orthogonal. To make a choice of the vacuum it is needed to know the relations of Eq. (C4). It must be

\[
<\psi_\text{oc}| \cdots \left(k\right)^\dagger \cdots \left(k'\right)^\dagger |\psi_\text{oc}\rangle = \delta_{kk'}, \]
\[
<\psi_\text{oc}| \cdots \left[k\right]^{\dagger} \cdots \left[k'\right]^{\dagger} |\psi_\text{oc}\rangle = \delta_{kk'}, \]
\[
<\psi_\text{oc}| \cdots \left[k\right]^{\dagger} \cdots \left(k'\right)^\dagger |\psi_\text{oc}\rangle = 0. \tag{58}
\]

We must choose the vacuum state that fulfills the above requirements as well as the requirements \(\hat{b}^{\alpha\dagger}_i |\psi_\text{oc}\rangle \neq 0\) and \(\hat{b}^\alpha_i |\psi_\text{oc}\rangle = 0\) for all members \(i\) of any family \(\alpha\). Since any \(\tilde{S}^{eg}\) changes \((+)\) \((+)\)
into $[+][+]$ and $[+][+]$, while $(+)^{\dagger}=(+)=[-]$, the vacuum state $|\psi_{oc}\rangle$ must be

$$|\psi_{oc}\rangle = a_{03}^{[+]}a_{12}^{[-]}a_{56}^{[-]}d^{-1}d + a_{03}^{[+]}a_{12}^{[-]}a_{56}^{[-]}d^{-1}d + a_{03}^{[+]}a_{12}^{[-]}a_{56}^{[-]}d^{-1}d$$

for $d = 2(2n + 1)$.

$n$ is a positive integer. There are $2d^{d-1}$ summands, since we can start with the vacuum state $a_{03}^{[-]}a_{12}^{[-]}a_{56}^{[-]}d^{-1}d|0\rangle$, which fulfills the requirement for $\hat{b}_{1}^{\dagger}|\psi_{oc}\rangle \neq 0$ and $\hat{b}_{1}^{\dagger}|\psi_{oc}\rangle = 0$, and then we must step by step replace all possible pairs of $[[-][-] \cdots [-]]$ in the starting part $[[-][-] \cdots [-]]$ into $[+][+] \cdots [+]$ and include new terms into the vacuum state so that the last $(2n + 1)$ summands have for $d = 2(2n + 1)$ case, $n$ is a positive integer, only one factor $[-]$ and all the rest $[+]$, each $[-]$ at different position.

The vacuum state has then the normalization factor $1/\sqrt{2^{d/2-1}}$,

while there is

$$2^{d-1}2^{d-1}$$

number of creation operators, defining the orthonormalized states when applying on the vacuum state of Eqs. (59) and the same number of annihilation operators, which are Hermitian conjugated to creation operators. Operators $\tilde{S}^{ab}$ connect members of different families, operators $S^{ab}$ generate all the members of one family.

Paying attention on only internal degrees of freedom, that is on spin, the creation and annihilation operators must fulfill the relations

$$\{\hat{b}_{i}^{\alpha}, \hat{b}_{j}^{\beta}\}^{\dagger}_{+}|\psi_{oc}\rangle = \delta_{\beta}^{\alpha} \delta_{j}^{i}|\psi_{oc}\rangle,$$

$$\{\hat{b}_{i}^{\alpha}, \hat{b}_{j}^{\beta}\}^{\dagger}_{+}|\psi_{oc}\rangle = 0|\psi_{oc}\rangle,$$

$$\{\hat{b}_{i}^{\alpha}, \hat{b}_{j}^{\beta}\}|\psi_{oc}\rangle = 0|\psi_{oc}\rangle,$$

$$\hat{b}_{i}^{\alpha}|\psi_{oc}\rangle = 0|\psi_{oc}\rangle,$$

$$\hat{b}_{i}^{\alpha}|\psi_{oc}\rangle = |\psi_{i}^{\alpha}\rangle,$$  \hspace{1cm} (61)

with $(i,j)$ determining family members quantum numbers and $(\alpha,\beta)$ denoting family quantum numbers.

Only Clifford odd objects fulfill the commutation relations of Eq. (61), since the Clifford even objects, like $\{(\gamma^{0} - \gamma^{3})(\gamma^{1} + i\gamma^{2}), (\gamma^{0} - \gamma^{3})(\gamma^{1} + i\gamma^{2})\}^{\dagger}_{+}$, do not.

The reader can find the detailed proofs for the above statements in Refs. [19, 20].
Let us extend the creation and annihilation operators to the ordinary coordinate space $(x^0, \vec{x})$

\[
\{\hat{b}^i_+(\vec{x}), \hat{b}^{\beta\dagger}_j(\vec{x}')\}_+ |\phi_{oc}\rangle = \delta^i_\beta \delta^j_\gamma \delta(\vec{x} - \vec{x}') |\phi_{oc}\rangle,
\]
\[
\{\hat{b}^\dagger_+(\vec{x}), \hat{b}^{\gamma}_i(\vec{x}')\}_+ |\phi_{oc}\rangle = 0 |\phi_{oc}\rangle,
\]
\[
\{\hat{b}^\alpha_+(\vec{x}), \hat{b}^{\beta\dagger}_j(\vec{x}')\}_+ |\phi_{oc}\rangle = 0 |\phi_{oc}\rangle,
\]
\[
\hat{b}^\alpha_+(\vec{x}) |\phi_{oc}\rangle = 0 |\phi_{oc}\rangle,
\]
\[
\hat{b}^{\alpha\dagger}_j(\vec{x}) |\phi_{oc}\rangle = |\psi^\alpha_i(\vec{x})\rangle,
\]

(62)

with the vacuum state $|\phi_{oc}\rangle$ defined in Eq. (59).

c. Discrete symmetries in Grassmann space and in Clifford space in $d$ and in $d = (3 + 1)$ space

We treated so far free massless fermions in Grassmann and in Clifford space. The fermion "nature" of states are in both spaces demonstrated by the fact that the corresponding creation and annihilation operators fulfill the anticommutation relations of Eq. (48) in Grassmann case and of Eq. (62) in Clifford space. Fermions — in both spaces — are in general in superposition of eigenstates of the Cartan subalgebra operators. Let us, to simplify this section discussing discrete symmetries of fermions in $((d-1)+1)$-dimensional space and in $(3+1)$-dimensional space, introduce a common name for the creation operator for a fermion. Let $\Psi^\dagger_p[\psi_p]$ be the creation operator creating a fermion in the state $\Psi_p$ (which is a function of $\vec{x}$) and let $\Psi_p(\vec{x})$ be the second quantized field creating a fermion at position $\vec{x}$ either in the Grassmann (expressible with the superposition of $\hat{b}^{\beta\dagger}_i(\vec{x})$) or in the Clifford case (expressible with the superposition of $\hat{b}^{\beta\dagger}_i(\vec{x})$). Then

\[
\Psi^\dagger_p[\psi_p] = \int \Psi^\dagger_p(\vec{x}) \Psi_p(\vec{x}) d^{(d-1)}x,
\]

(63)

describes on a vacuum state a single particle in the state $\Psi_p$ with a positive energy

\[
\{\Psi^\dagger_p[\psi_p] = \int \Psi^\dagger_p(\vec{x}) \Psi_p(\vec{x}) d^{(d-1)}x \} |vac\rangle
\]

so that the anti-particle state becomes

\[
\{C\Psi^\dagger_p[\psi_p] = \int \Psi_p(\vec{x}) (C \Psi_p(\vec{x})) d^{(d-1)}x \} |vac\rangle .
\]

We distinguish in $d$-dimensional space two kinds of discrete operators $C, P$ and $T$ operators with respect to the internal space which we use.
In the Clifford case we have

\[ C_H = \prod_{\gamma^a \in \mathbb{C}} \gamma^a K, \]
\[ T_H = \gamma^0 \prod_{\gamma^a \in \mathbb{R}} \gamma^a K I x^0, \]
\[ P^{(d-1)}_H = \gamma^0 I \vec{x}, \]
\[ I_x x^a = -x^a, \quad I_{x^0} x^a = (-x^0, \vec{x}), \quad I_{\vec{x}} \vec{x} = -\vec{x}, \]
\[ I_{\vec{x}} x^a = (x^0, -x^1, -x^2, -x^3, x^5, x^6, \ldots, x^d). \] (64)

The product \( \prod \gamma^a \) is meant in the ascending order in \( \gamma^a \).

In the Grassmann case we correspondingly define

\[ C_G = \prod_{\gamma^a_G \in \mathbb{C}} \gamma^a_G K, \]
\[ T_G = \gamma^0_G \prod_{\gamma^a_G \in \mathbb{R}} \gamma^a_G K I x^0, \]
\[ P^{(d-1)}_G = \gamma^0_G I \vec{x}, \] (65)

\( \gamma^a_G \) is defined in Eq. (15) as \( \gamma^a_G = (1 - 2\theta^a \eta^{aa} \partial / \partial \theta^a) \), while \( I_x x^a = -x^a, I_{x^0} x^a = (-x^0, \vec{x}), I_{\vec{x}} \vec{x} = -\vec{x}, I_{\vec{x}} x^a = (x^0, -x^1, -x^2, -x^3, x^5, x^6, \ldots, x^d) \), like in the Clifford case. Let be noticed, that since \( \gamma^a_G = (1 - 2\theta^a \eta^{aa} \partial / \partial \theta^a) \) is always real as there is \( \gamma^a i \bar{\gamma}^a \), while \( \gamma^a \) is either real or imaginary, we use in Eq. (65) \( \gamma^a \) to make a choice of appropriate \( \gamma^a_G \). In what follows we shall use the notation as in Eq. (65).

Let us define in the Clifford case and in the Grassmann case the operator "emptying" (which empties the Dirac sea) \[ \prod_{\mathbb{R}^d} \gamma^a K \] (arXiv:1312.1541), so that operation of "emptying\(_{NH}\)" after the charge conjugation \( C_H \) in the Clifford case and "emptying\(_{NG}\)" after the charge conjugation \( C_G \) in the Grassmann case (each of them, \( C_H \) and \( C_G \), transforms the state put on the top of either the Clifford or the Grassmann Dirac sea into the corresponding negative energy state) creates the anti-particle state to the starting particle state, each anti-particle state put on the top of the Dirac sea and any of the two solving the Weyl equation, either in the Clifford case, Eq. (30), or in the Grassmann case, Eq. (35), for free massless fermions

\[ "\text{emptying}_{NH}\" = \prod_{\mathbb{R}^d} \gamma^a K \quad \text{in Clifford space}, \]
\[ "\text{emptying}_{NG}\" = \prod_{\mathbb{R}^d} \gamma^a_G K \quad \text{in Grassmann space}, \] (66)
although we must keep in mind that indeed the anti-particle state is a hole in the Dirac sea from the Fock space point of view. The operator “emptying” is bringing the single particle operator $\mathcal{C}_H$ in the Clifford case and $\mathcal{C}_G$ in the Grassmann case into the operator on the Fock space in each of the two cases. Then the anti-particle state creation operator — $\Psi^\dagger_a [\Psi_p]$ — to the corresponding particle state creation operator — can be obtained also as follows

$$\Psi^\dagger_a [\Psi_p] |\text{vac}\rangle = \mathcal{C}_H \mathcal{G}_G \Psi^\dagger_p (\vec{x}) \Psi_p (\vec{x}) d^{(d-1)}x |\text{vac}\rangle ,$$

with indexes $\mathcal{H}$ and $N_H$ denoting the Clifford case and with $G$ and $N_G$ denoting the Grassman case.

Each of the operators $\mathcal{C}_H$ and $\mathcal{C}_G$

$$\mathcal{C}_H = \text{"emptying}_{NH}$ \cdot \mathcal{C}_H ,$$

$$\mathcal{C}_G = \text{"emptying}_{NG}$ \cdot \mathcal{C}_G ,$$

operating on $\Psi_p (\vec{x})$ transforms the positive energy fermion state (which solves the corresponding Weyl equation for a massless free fermion in each of cases) put on the top of the Dirac sea into the positive energy anti-fermion state, which again solves the corresponding Weyl equation for a massless free anti-fermion put on the top of the Dirac sea. Let us point out that either the operator “emptying$_{NH}$” or the operator “emptying$_{NG}$” transforms the single particle operator either $\mathcal{C}_H$ or $\mathcal{C}_G$ into the operator operating in the Fock space.

We use the Grassmann even, Hermitian and real operators $\gamma^a_G$, Eq. (15), to determine discrete symmetries in Grassmann space for $(d + 1) - 1$ space (as presented in Eq. (65)) and for $(3 + 1)$ space. In the Clifford case and $(3 + 1)$ space we follow Ref. [21]. In the Grassmann case we determine the discrete symmetries in $(3 + 1)$ space as follows

$$\mathcal{C}_G = \prod_{\gamma^m \in \mathbb{R} \gamma^m} \gamma^m_G K I_x^{x^6 x^8 ... x^d} ,$$

$$\mathcal{T}_G = \gamma^0_G \prod_{\gamma^m \in \mathbb{R} \gamma^m} K I_{x^0} I_x^{x^6 x^8 ... x^{d-1}} ,$$

$$\mathcal{P}^{(d-1)}_{NG} = \gamma^0_G \prod_{s=5}^{d} \gamma^s_G I_{\vec{x}} ,$$

$$\mathcal{C}_G = \prod_{\gamma^s_G \in \mathbb{R} \gamma^s} \gamma^s_G I_x^{x^6 x^8 ... x^d} ,$$

$$\mathcal{C}_G \mathcal{P}^{(d-1)}_{NG} = \gamma^0_G \prod_{\gamma^s_G \in \mathbb{R} \gamma^s, s=5}^{d} \gamma^s_G I_{\vec{x}} I_x^{x^6 x^8 ... x^d} ,$$

$$\mathcal{C}_G \mathcal{T}_G \mathcal{P}^{(d-1)}_{NG} = \prod_{\gamma^0_G \in \mathbb{R} \gamma^0} \gamma^0_G I_x K .$$

(69)
Let us illustrate "Grassmann fermions" in the case of $d = 5 + 1$, before the break, as well as after the break of $d = 5 + 1$ into $d = 3 + 1$, when the fifth and the sixth dimension determine the charge in $d = 3 + 1$. There are two decuplets in this case [15], both of an odd Grassmann character, which can be second quantized. There are two triplets in the first decuplet— $(\psi^I_1, \psi^I_2, \psi^I_3)$ and $(\psi^I_4, \psi^I_5, \psi^I_6)$ — both solving Eq. (35) for massless free fermions in Grassmann space with the space function $e^{-ip^a x^a}$. The Grassmann even operator $C_{NG} P_{NG}^{(d-1)}$, Eq. (69), transforms $\psi^I_1$ with $p^a = (|p^0|, 0, 0, |p^3|, 0, 0)$ into the anti-particle state $\psi^I_6$, with the positive energy $|p^0|$ and with $-|p^3|$, for example. Correspondingly $C_{NG} P_{NG}^{(d-1)}$, Eq. (69), transforms the particle state $\psi^I_3$ with the positive energy into the anti-particle state $\psi^I_4$ with the positive energy. All these states belong to the same representation, the same decuplet.

Applying the Grassmann even operators on one of the states of one of the decuplets — $C_G = \gamma_0^2 \gamma_5^G$, Eq. (65), $C_{NG} P_{NG}^{(d-1)} = \gamma_0^1 \gamma_3^G \gamma_5^G I_x s I_3 K$, Eq. (69) — one remains within the same decuplet. (To get the positive energy antiparticle states the operator emptying$_{NH}$ and emptying$_{NG}$ in the Clifford and the Grassmann case are needed, Eqs. (66, 68)). The reader can find more discussions in Refs. [15, 21].

**d. What do we learn from the second quantization procedure in Grassmann and in Clifford space?**

We proved that basic states in both spaces can be written by creation operators operating on an appropriate vacuum state. The creation and their Hermitian conjugated annihilation operators fulfill in both spaces anticommutation relations as required for fermions, Eqs. (48, 62).

In both spaces the creation operators are chosen to create states that are eigenstates of the corresponding Cartan subalgebra of the Lorentz algebra, the generators of which are $S^{ab}$, Eq. (5), in the Grassmann case and $\hat{S}^{ab}$ (generating spins) and $\tilde{S}^{ab}$ (generating families) in the Clifford case, Eqs. (2, 4). In both spaces creation and annihilation operators have an odd character: In the Grassmann case the creation operators are sums of products of $\theta^a$’s and the corresponding Hermitian conjugated annihilation operators are sums of products of $\eta^{aa} \frac{\partial}{\partial \theta^a}$’s, Eqs. (44, 45). In the Clifford case the creation operators are sums of products of $\gamma^a$’s arranged into nilpotents and projectors and annihilation operators are the corresponding Hermitian conjugated operators, Eq. (53, 54).

While in the Grassmann case the vacuum state is simple, $|\phi_{og} > = |1 >$, in the Clifford case the vacuum state is a sum of products of $2^\frac{d}{2} - 1$ projectors, Eq. (59).

In $2(2n + 1)$-dimensional spaces there are in the Clifford case $2^\frac{d}{2} - 1$ states in one representation...
TABLE I: The creation operators of the two decuplets, I and II, of the orthogonal group SO(5,1) in Grassmann space are presented. Applied on the vacuum state $|\psi_{og}\rangle = |1> >$ the creation operators form "eigenstates" of the Cartan subalgebra, Eq. (11), $S^{12}, S^{56}$ for SO(5,1). The creation operators within each decuplet are reachable from any member by a product of $S^{ab}$'s (which do not belong to the Cartan subalgebra). Creation operators and their Hermitian conjugated annihilation operators fulfill the anticommutation relations for fermions, Eq. (10). The product of the discrete operators $C_{NG}$ and $P_{NG}^{(d-1)}$, Eq. (9), $(C_{NG}P_{NG}^{(d-1)}) = \gamma_G^{0}\gamma_G^{5} I_x I_6$ in $d = (5+1)$ transforms, for example, $\psi^a_1$ into $\psi^a_6$, $\psi^a_2$ into $\psi^a_4$ and $\psi^a_3$ into $\psi^a_1$. Solutions of the Weyl equation, Eq. (8), with the negative energies belong to the "Grassmann sea", solutions with the positive energy belong to the particles and antiparticles.

Reachable from (any) starting state by products of $S^{ab}$, while products of $\tilde{S}^{ab}$ transform each of these states by changing their family quantum numbers. There are correspondingly $2^{d-1} \times 2^{d-1}$ states reachable with products of $S^{ab}$'s or $\tilde{S}^{ab}$'s or of both, $S^{ab}$'s and $\tilde{S}^{ab}$'s. Each state is obtained by the corresponding creation operator on the vacuum state and it is annihilated by its Hermitian conjugated operator.

In $2(2n+1)$-dimensional spaces there are in the Grassmann case two decoupled representations, each with $\frac{1}{2} \cdot \frac{d_l}{2} \frac{d_r}{2}$ states. Each of states can be obtained by the corresponding creation operator and any state is annihilated by the Hermitian conjugated creation operator. While all of $2^{\frac{d_l}{2}-1} \times 2^{\frac{d_r}{2}-1}$ states in Clifford space are reachable by even Clifford objects, either by products of $S^{ab}$'s or by products of $\tilde{S}^{ab}$'s or by products of both, in Grassmann space the two groups of representations are
III. CONCLUSIONS

We have learned in the present study that one can use either Grassmann or Clifford space to express the internal degrees of freedom of fermions in any even dimensional space, either for \( d = 2(2n+1) \) or \( d = 4n \). In both spaces the creation operators and their Hermitian conjugated annihilation operators fulfill the anticommutation relation requirements, needed for fermions, provided that they are expressed as superposition of odd products of \( \theta^a \)'s (\( \theta^a = \frac{\partial}{\partial \theta^a} \eta^{aa} \), Eq. (11)) in the Grassmann case, or in the Clifford space as odd products of Clifford objects (either \( \gamma^a = (\theta^a + \frac{\partial}{\partial \theta^a}) \), Eq. (4) and correspondingly \( \gamma^a = (\theta^a + \frac{\partial}{\partial \theta^a}) \), or \( \tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta^a}) \), and correspondingly \( \tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta^a}) \)), Eq. (4). But while in the Clifford case states appear in the fundamental representations of the Lorentz group, carrying half integer spins, states in the Grassmann case are in adjoint representations of the Lorentz group. The Clifford case, offering two kinds of the Clifford objects (\( \gamma^a \) and \( \tilde{\gamma}^a \)), enables to describe besides the spin degrees of freedom of fermion fields also their family degrees of freedom. The Grassmann case offers no families. Assuming that “nature has both choices” for describing the internal degrees of freedom of fermion fields, the question arises why Grassmann space is not chosen, or better, why Clifford space wins.

In the case that spin degrees in \( d \geq 5 \) manifest as charges in \( d = (3 + 1) \), fermions in the Grassmann case manifest charges in the adjoint representations. On the other hand in the Clifford case — this is used in the spin-charge-family theory, assuming the Lorentz group \( SO(13,1) \) which offers the explanation for the origin of all the properties of the observed quarks and leptons and all the gauge fields with which fermions interact — the spin and charges appear in the fundamental representations of the corresponding groups, offering also the family degrees of freedom.

We present in this paper the action (Eq. (33)) describing free massless fermions with the internal degrees of freedom describable in Grassmann space, Eqs. (33, 34). The action leads to the equation of motion (Eq. (35)), analogous to the Weyl equation in Clifford space (Eq. (30)), fulfilling as well the Klein-Gordon equation (Eq. (36)).

Since the Clifford objects \( \gamma^a \) and \( \tilde{\gamma}^a \) are expressible with the Grassmann coordinates \( \theta^a \) and their conjugate moments \( \frac{\partial}{\partial \theta^a} \), either basic states in Grassmann space, Eq. (9), or basic states in Clifford space, Eq. (53), can be normalized with the same integral, Eq. (24, 25, 26).

To understand better the difference in the description of the fermion internal degrees of freedom with either Clifford or Grassmann space, let us replace in the starting action of the spin-charge-
family theory, Eq. [1], using the Clifford algebra to describe fermion degrees of freedom, the covariant momentum \( p_{0a} = f^a \rho_0 \), \( p_{0a} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha} \), with \( p_{0a} = p_\alpha - \frac{1}{2} S^{ab} \Omega_{ab\alpha} \), where \( S^{ab} = S^{ab} + \tilde{S}^{ab} \), Eq. [5], and \( \Omega_{ab\alpha} \) are the spin connection gauge fields of \( S^{ab} \) (which are the generators of the Lorentz transformations in Grassmann space!), while \( f^a \rho_0 \) replaces the ordinary momentum when massless objects start to interact with the gravitational field through the vielbeins and the spin connections. Let us add that it follows, if varying the action with respect to either \( \omega_{ab\alpha} \) or \( \tilde{\omega}_{ab\alpha} \) when no fermions are present, that both spin connections are uniquely determined by the vielbeins ([3, 5, 9] and references therein) and correspondingly in this particular case \( \Omega_{ab\alpha} = \omega_{ab\alpha} = \tilde{\omega}_{ab\alpha} \).

Let us use instead of \( p_a \) in the action for free massless fermion fields in Grassmann space (describing the internal degrees of freedom), Eq. [33], the above covariant momentum \( p_{0a} = f^a \rho_0 - \frac{1}{2} S^{ab} \Omega_{ab\alpha} \). It follows in this case that the representations of the Lorentz group in \( d = 2(2n + 1) = 13 + 1 \) and of their subgroups \( SO(7,1), SU(3) \) and \( U(1) \) are in the adjoint representations of the groups.

The spin-charge-family theory (using Clifford objects) offers the explanation for all the assumptions of the standard model of elementary fields, fermions and bosons, vector and scalar gauge fields, with the appearance of families included, explaining also the phenomena like the existence of the dark matter [10], of the matter-antimatter asymmetry [4], offering correspondingly the next step beyond both standard models — cosmological one and the one of the elementary fields.

We do notice, however, that the Grassmann degrees of freedom do not offer the appearance of families at all.

We also notice that the second quantization procedure allows in \( d = 2(2n + 1) \)-dimensional space for each member of a Weyl representation in Clifford space (for each of \( 2^{d-1} \) “family members”) \( 2^{\frac{d}{2} - 1} \) “families”, all together therefore \( 2^{\frac{d}{2} - 1} \times 2^{\frac{d}{2} - 1} \) basic states which can be second quantized. From \( 2^d \) Clifford objects, only those of an odd Clifford character contribute to the second quantization — half of them as creation and half of them as annihilation operators, \( 2^{\frac{d}{2} - 1} \) projectors from the rest of objects form the vacuum state.

We notice that in the case that Grassmann space describes the internal degrees of freedom of fermions and \( d = 2(2n + 1) \) only twice two isolated groups of \( \frac{1}{2} \frac{d}{d+1} \) states of an odd Grassmann character can be second quantized.

To come to the low energy regime the symmetry must break, first from \( SO(13,1) \) to \( SO(7,1) \times SU(3) \times U(1) \) and then further to \( SO(3,1) \times SU(3) \times U(1) \), in both spaces, in Grassmann and in Clifford. In Clifford case there are two kinds of generators and correspondingly two kinds of
symmetries. We learned in Refs. [23–25] that when breaking symmetries only some of families remain massless and correspondingly observable in \( d = (3 + 1) \).

This study is to learn more about possibilities that "nature experiences". One of the authors (N.S.M.B.) wants to learn: a. Why is the simple starting action of the spin-charge-family theory doing so well in manifesting the observed properties of the fermion and boson fields? b. Under which condition can more general action lead to the starting action of Eq. (1)? c. What would more general action, if leading to the same low energy physics, mean for the history of our Universe? d. Could the fermionization procedure of boson fields or the bosonization procedure of fermion fields, discussed in Ref. [12] for any even dimension \( d \) (by the authors of this contribution, while one of them (H.B.F.N. [13]) has succeeded with another author to do the fermionization for \( d = (1 + 1) \)) tell more about the "decisions" of the universe in the history?

Although we have not yet learned enough to be able to answer these questions, yet we have learned at least that the description of the fermion internal degrees of freedom in Grassmann space is possible. However, such fermions would manifest integer spins and charges in the adjoint representations of the corresponding groups, would not offer families, and would not be in agreement with the spin and charges and other observations so far. We also have learned that if there are no fermion present only one kind of dynamical fields manifests, since either \( \omega_{aba} \) or \( \tilde{\omega}_{aba} \) are uniquely expressed by vielbeins ([9] Eq. (C9) and references therein), which could mean that the appearance of the two kinds of the spin connection fields might be due to the break of symmetries.

Acknowledgement:

N.S. Mankoč Borštnik would like to thank Faculty of Mathematics and Physics, University of Ljubljana, for offering the computer and other facilities.

Appendix A: Creation and annihilation operators in Grassmann and Clifford space for \( d = 4n \)

We discuss in Subsect. II C mainly cases with \( d = 2(2n + 1) \), since if assuming no conserved charges in the fundamental theory with fermions, which carry only spins and interact with only the gravity — as the spin-charge-family theory assumes — the dimensions \( 4n \), \( n \) is positive integer, as well as all odd dimensions, are excluded under the requirement of mass protection [20].

Let us nevertheless add in this appendix comments on the second quantization procedure in \( d = 4n \) spaces.

i. Grassmann space
In Eq. (39) we define in Grassmann space a possible starting creation operator for \( d = 2(2n+1) \) spaces. In \( d = 4n \) we correspondingly start with the state
\[
|\phi_1^1\rangle = \hat{b}_1^{\phi_1\dagger} |1\rangle ,
\]
\[
\hat{b}_1^{\phi_1\dagger} = \left( \frac{1}{\sqrt{2}} \right)^{d-1} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \cdots (\theta^{d-3} + i\theta^{d-2})\theta^{d-1}\theta^d , \tag{A1}
\]
generated by the creation operator \( \hat{b}_1^{\phi_1\dagger} \), which is, as it aught to be — like in the \( d = 2(2n+1) \) case — of an odd Grassmann character to fulfill the anticommutation relations for fermions, Eq. (48). Again the rest of states, belonging to the same Lorentz representation, follow from the starting state
\[
|\phi_1\rangle \quad \text{— of an odd Grassmann character to fulfill the anticommutation relations for fermions, Eq. (48).}
\]
Their annihilation partners follow by Hermitian conjugation.

One finds therefore for the (chosen) starting creation and the corresponding annihilation operator
\[
\hat{b}_2^{\phi_1\dagger} = \left( \frac{1}{\sqrt{2}} \right)^{d-2} (\theta^0\theta^3 + i\theta^1\theta^2)(\theta^5 + i\theta^6) \cdots (\theta^{d-3} + i\theta^{d-2})\theta^{d-1}\theta^d ,
\]
\[
\hat{b}_2^{\phi_1} = \left( \frac{1}{\sqrt{2}} \right)^{d-2} \frac{\partial}{\partial\theta^0} \frac{\partial}{\partial\theta^1} (\frac{\partial}{\partial\theta^3} - i\frac{\partial}{\partial\theta^2}) \cdots (\frac{\partial}{\partial\theta^5} - i\frac{\partial}{\partial\theta^6} \cdots \frac{\partial}{\partial\theta^d}) , \tag{A2}
\]
The application of \( S^{01} \), for example, generates
\[
\hat{b}_2^{\phi_2\dagger} = \left( \frac{1}{\sqrt{2}} \right)^{d-2} (\theta^0\theta^3 + i\theta^1\theta^2)(\theta^5 + i\theta^6) \cdots (\theta^{d-3} + i\theta^{d-2})\theta^{d-1}\theta^d ,
\]
\[
\hat{b}_2^{\phi_2} = \left( \frac{1}{\sqrt{2}} \right)^{d-2} \frac{\partial}{\partial\theta^0} \frac{\partial}{\partial\theta^1} (\frac{\partial}{\partial\theta^3} - i\frac{\partial}{\partial\theta^2}) \cdots (\frac{\partial}{\partial\theta^5} - i\frac{\partial}{\partial\theta^6} \cdots \frac{\partial}{\partial\theta^d}) , \tag{A3}
\]
There is the additional group of creation and annihilation operators in \( d = 4n \), which follows from the starting creation operator \( \hat{b}_1^{\phi_2\dagger} \)
\[
\hat{b}_1^{\phi_2\dagger} = \left( \frac{1}{\sqrt{2}} \right)^{d-1} (\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \cdots (\theta^{d-3} + i\theta^{d-2})\theta^{d-1}\theta^d ,
\]
\[
\hat{b}_1^{\phi_2} = (\hat{b}_1^{\phi_2\dagger})^\dagger = \left( \frac{1}{\sqrt{2}} \right)^{d-1} \frac{\partial}{\partial\theta^0} \frac{\partial}{\partial\theta^1} (\frac{\partial}{\partial\theta^3} - i\frac{\partial}{\partial\theta^2}) \cdots (\frac{\partial}{\partial\theta^5} + \frac{\partial}{\partial\theta^6} \cdots \frac{\partial}{\partial\theta^d}) , \tag{A4}
\]
for \( d = 4n \).

All the rest of creation operators follow from the starting creation operator of each of the two groups by the (left) application of products of \( S^{ab} \)
\[
\hat{b}_i^{\phi_k\dagger} \propto S^{ab} \cdots S^{ef} \hat{b}_1^{\phi_k\dagger} ,
\]
\[
\hat{b}_i^{\phi_k} = (\hat{b}_i^{\phi_k\dagger})^\dagger , \quad k = 1, 2 . \tag{A5}
\]
Only creation and annihilation operators with an odd Grassmann character, fulfill, applied on the vacuum state \(|1>\), the anticommutation relations required for fermions, Eq. (42).

i. Clifford space

In Eq. (53) we define in Clifford space a possible starting creation operator for \(d = 2(2n + 1)\) spaces. In \(d = 4n\) we correspondingly start with the state with an odd number of nilpotents and with one projector

\[
|\psi_i > = \hat{b}_{1\dagger} |\psi_{oc} > ,
\]

\[
\hat{b}_1 = \begin{pmatrix}
03 & 12 & 35 & d-3 & d-2 & d-1 & d \\
\end{pmatrix}
\]

\[
\hat{b}_1 \dagger = \begin{pmatrix}
+ - i + & + & \cdots & + & + \\
\end{pmatrix}
\]

\[
(\hat{b}_1 \dagger)^\dagger = \begin{pmatrix}
+ & - & \cdots & - & -( -i ) \\
\end{pmatrix}
\]

(A6)

All the other creation operators, creating all the members of the representation of this particular family, are obtainable by the application of products of \(S_{ab}\) on this creation operator from the left hand side. There are \(2^{d-1}\) members of each family. All the other families follows from the starting one by the application of products of \(\tilde{S}_{ab}\). There are \(2^{\frac{d}{2}} - 1\) families with \(2^{\frac{d}{2}} - 1\) members each.

A general creation operator in \(d = 4n\) follows by the application of \(S_{ab}\) and \(\tilde{S}_{ab}\) on the starting creation operator of Eq. (A6) and the corresponding annihilation operator is its Hermitian conjugated value.

Correspondingly we define \(\hat{b}_i^{\alpha\dagger}\) (up to a constant) to be

\[
\hat{b}_i^{\alpha\dagger} \propto \tilde{S}_{ab} \cdots \tilde{S}_{ef} S_{mn} \cdots S_{pr} \hat{b}_{1\dagger}
\]

\[
\begin{pmatrix}
03 & 12 & 35 & d-3 & d-2 & d-1 & d \\
\end{pmatrix}
\]

\[
\hat{b}_i^{\dagger} = (\hat{b}_i^{\alpha\dagger})^\dagger \propto S_{ef} \cdots S_{ab} \hat{b}_{1\dagger} S_{pr} \cdots S_{mn}
\]

\[
d = 4n.
\]

(A7)

These creation and annihilation operators — again of an odd Clifford character in \(4n\) — fulfill the anticommutation relations of Eq. (62), if applied on the vacuum state of Eq. (59),

\[
|\psi_{oc} > = \begin{pmatrix}
03 & 12 & 35 & d-3 & d-2 & d-1 & d \\
\end{pmatrix}
\]

\[
|0 >
\]

\[
d = 4n,
\]

(A8)

\(n\) is a positive integer. There are \(2^{\frac{d}{2} - 1}\) summands, since we step by step replace all possible pairs of \([-\cdots -]\) in the starting part \([-i][-]\cdots [-\cdots -] [+]\) into \([+] \cdots [+]\) and include new terms into the vacuum state so that the last \(2n + 1\) summand has for \(d = 4n\) also the factor \([+]\) in the starting term \([-i][-][-\cdots [-\cdots -] [+]\) changed into \([-\cdots -]\). The vacuum state has then the
normalization factor $1/\sqrt{2^{d/2-1}}$.

Appendix B: Lorentz algebra and representations in Grassmann and Clifford space

The Lorentz transformations of vector components $\theta^a, \gamma^a$, or $\tilde{\gamma}^a$ — usable for the description of the internal degrees of freedom of fermion fields obeying in the second quantization the anticommutation relations for fermions — and of vector components $x^a$, which are real (ordinary) commuting coordinates, $\theta^a = \Lambda^a_b \theta^b$, $\gamma^a = \Lambda^a_b \gamma^b$, $\tilde{\gamma}^a = \Lambda^a_b \tilde{\gamma}^b$ and $x^a = \Lambda^a_b x^b$, leave forms $a_{a_1a_2...a_i} \theta^{a_1} \theta^{a_2} ... \theta^{a_i}$, $a_{a_1a_2...a_i} \gamma^{a_1} \gamma^{a_2} ... \gamma^{a_i}$, $a_{a_1a_2...a_i} \tilde{\gamma}^{a_1} \tilde{\gamma}^{a_2} ... \tilde{\gamma}^{a_i}$ and $b_{a_1a_2...a_i} x^{a_1} x^{a_2} ... x^{a_i}$, $i = (1, \ldots, d)$, invariant.

While $b_{a_1a_2...a_i} (= \eta_{a_1b_1} \eta_{a_2b_2} \ldots \eta_{a_ib_i} b_{b_1b_2...b_i})$ is a symmetric tensor field, $a_{a_1a_2...a_i} (= \eta_{a_1b_1} \eta_{a_2b_2} \ldots \eta_{a_ib_i} a_{b_1b_2...b_i})$ are antisymmetric Kalb-Ramond fields.

The requirements: $x^a x^b \eta_{ab} = x^c x^d \eta_{cd}, \theta^a \theta^b \varepsilon_{ab} = \theta^c \theta^d \varepsilon_{cd}, \gamma^a \gamma^b \varepsilon_{ab} = \gamma^c \gamma^d \varepsilon_{cd}$ and $\tilde{\gamma}^a \tilde{\gamma}^b \varepsilon_{ab} = \tilde{\gamma}^c \tilde{\gamma}^d \varepsilon_{cd}$ lead to $\Lambda^a_b \Lambda^c_d \eta_{ac} = \eta_{bd}$. Here $\eta^{ab}$ (in our case $\eta^{ab} = \text{diag}(1,-1,-1,\ldots,-1)$) is the metric tensor lowering the indexes of vectors ($\{x^a\} = \eta^{ab} x_b$, $\{\theta^a\} = \eta^{ab} \theta_b$, $\{\gamma^a\} = \eta^{ab} \gamma_b$ and $\{\tilde{\gamma}^a\} = \eta^{ab} \tilde{\gamma}_b$) and $\varepsilon_{ab}$ is the antisymmetric tensor. An infinitesimal Lorentz transformation for the case with $\text{det} \Lambda = 1, \Lambda^0_0 \geq 0$ can be written as $\Lambda^a_b = \delta^a_b + \omega^a_b$, where $\omega^0_b + \omega_b^0 = 0$.

In Eqs. (48) the commutation relations among the above objects are presented.

1. Lorentz properties of basic vectors

What follows is taken from Ref. [2] and Ref. [9], Appendix B.

Let us first repeat some properties of the anticommuting Grassmann and Clifford coordinates, taking into account Eqs. (34). An infinitesimal Lorentz transformation of the proper ortochronous Lorentz group is then

\[
\begin{align*}
\delta \theta^c &= -\frac{i}{2} \omega_{ab} S^{ab} \theta^c = \omega^c_a \theta^a, \\
\delta \gamma^c &= -\frac{i}{2} \omega_{ab} S^{ab} \gamma^c = \omega^c_a \gamma^a, \\
\delta \tilde{\gamma}^c &= -\frac{i}{2} \omega_{ab} \tilde{S}^{ab} \tilde{\gamma}^c = \omega^c_a \tilde{\gamma}^a, \\
\delta x^c &= -\frac{i}{2} \omega_{ab} L^{ab} x^c = \omega^c_a x^a, \\
\end{align*}
\]  

(B1)

where $\omega_{ab}$ are parameters of a transformation and $\gamma^a$ and $\tilde{\gamma}^a$ are expressible by $\theta^a$ and $\frac{\partial}{\partial \theta^a}$ in Eqs. (34).
Let us write the operator of finite Lorentz transformations as follows

\[ S = e^{-\frac{i}{2} \omega_{ab}(S_{ab} + L_{ab})}, \]  

(B2)

\( S_{ab} \) have to be replaced by \( S_{ab} \) and \( \tilde{S}_{ab} \) in the Clifford case. We see that the Grassmann \( \theta^a \) and the ordinary \( x^a \) coordinates and the Clifford objects \( \gamma^a \) and \( \tilde{\gamma}^a \) transform as vectors

\[ \theta^c = e^{-\frac{i}{2} \omega_{ab}(S_{ab} + L_{ab})} \theta^c e^{\frac{i}{2} \omega_{ab}(S_{ab} + L_{ab})} \]

\[ = \theta^c - \frac{i}{2} \omega_{ab} \{ S_{ab}, \theta^c \} - \cdots = \theta^c + \omega_{ca} \theta^a + \cdots = \Lambda^c_a \theta^a, \]

\[ x^c = \Lambda^c_a x^a, \quad \gamma^c = \Lambda^c_a \gamma^a, \quad \tilde{\gamma}^c = \Lambda^c_a \tilde{\gamma}^a. \]  

(B3)

Correspondingly one finds that compositions like \( \gamma^a p_a \) and \( \tilde{\gamma}^a p_a \), here \( p_a = \frac{\partial}{\partial x^a} \), transform as scalars (remaining invariants), while \( S_{ab} \omega_{abc} \) and \( \tilde{S}_{ab} \tilde{\omega}_{abc} \) transform as vectors. Objects like

\[ R = \frac{1}{2} f^{[a}f^{b]}(\omega_{aba,c} - \omega_{ca} \omega_{c[b}) \quad \text{and} \quad \tilde{R} = \frac{1}{2} \tilde{f}^{[a}\tilde{f}^{b]}(\tilde{\omega}_{aba,c} - \tilde{\omega}_{ca} \tilde{\omega}_{c[b}) \]  

from Eq. (1) transform with respect to the Lorentz transformations as scalars.

Making a choice of the Cartan subalgebra set of the algebra \( S_{ab}, S_{ab} \) and \( \tilde{S}_{ab} \), Eqs. (2, 5, 7),

\[ S_{03}, S_{12}, S_{56}, \ldots, S_{d-1}d, \]

\[ S_{03}, S_{12}, S_{56}, \ldots, S_{d-1}d, \]

\[ \tilde{S}_{03}, \tilde{S}_{12}, \tilde{S}_{56}, \ldots, \tilde{S}_{d-1}d, \]  

(B4)

one can arrange the basic vectors so that they are eigenstates of the Cartan subalgebra, belonging to representations of \( S_{ab} \), or of \( S_{ab} \) and \( \tilde{S}_{ab} \), with \( ab \) from Eq. (B4).

Appendix C: Technique to generate spinor representations in terms of Clifford algebra objects

Here we briefly repeat the main points of the technique for generating spinor representations from Clifford algebra objects, following Ref. [2, 16]. We advise the reader to look for details and proofs in these references. No requirements for the second quantization is taken into account.

We assume the objects \( \gamma^a \), Eq. (4), which fulfill the Clifford algebra relations of Eq. (2),

\[ \{ \gamma^a, \gamma^b \}_+ = I.2\eta^{ab} \text{, for } a, b \in \{ 0, 1, 2, 3, 5, \ldots, d \} \text{, for any } d, \text{ even or odd. } I \text{ is the unit element in the Clifford algebra, while } \{ \gamma^a, \gamma^b \}_\pm = \gamma^a \gamma^b \pm \gamma^b \gamma^a. \]

The “Hermiticity” property for \( \gamma^a \)'s and \( \tilde{\gamma}^a \)'s, Eq. (18), follows from Eq. (11),

\[ \gamma^a = \eta^{a\alpha} \gamma^\alpha \]

\[ \tilde{\gamma}^a = \eta^{a\varepsilon} \tilde{\gamma}^\varepsilon \]

leading to \( \gamma^a \gamma^a = I \), \( \gamma^a \tilde{\gamma}^a = I \).
The Clifford algebra objects \( S^{ab} \) close the Lie algebra of the Lorentz group \( \{ S^{ab}, S^{cd} \}_- = i(\eta^{ad} S^{bc} + \eta^{bc} S^{ad} - \eta^{ac} S^{bd} - \eta^{bd} S^{ac}) \), Eq. (7). One finds from Eq. (18) that \( (S^{ab})^\dagger = \eta^{aa} \eta^{bb} S^{ab} \) and that \( \{ S^{ab}, S^{ac} \}_+ = \frac{1}{2} \eta^{aa} \eta^{bc} \).

Recognizing that two Clifford algebra objects \( (S^{ab}, S^{cd}) \) with all indexes different commute, we select (out of many possibilities) the Cartan subalgebra set of the algebra of the Lorentz group of Eq. (B4).

To make the technique simple, we introduce the graphic representation \( \{ 16 \} \), Eq. (50),

\[
\begin{align*}
\langle k \rangle^{ab} &= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik} \gamma^b), \\
[k]^{ab} &= \frac{1}{2}(1 + i \frac{\eta^{aa}}{k} \gamma^b),
\end{align*}
\]

where \( k^2 = \eta^{aa} \eta^{bb} \). One can easily check by taking into account the Clifford algebra relation (Eqs. (4) (11)) and the definition of \( S^{ab} \) (Eq. (24)) that if one multiplies from the left hand side by \( S^{ab} \) the Clifford algebra objects \( \langle k \rangle^{ab} \) and \( [k]^{ab} \), it follows that, Eq. (52),

\[
\begin{align*}
S^{ab} \langle k \rangle^{ab} &= \frac{1}{2} k \langle k \rangle^{ab}, \\
S^{ab} [k]^{ab} &= \frac{1}{2} k [k]^{ab}.
\end{align*}
\]

This means that \( \langle k \rangle^{ab} \) and \( [k]^{ab} \) acting from the left hand side on the vacuum state \( |\psi_{oc}\rangle \), Eqs. (59) (63) for \( d = 2(2n + 1) \) and \( d = 4n \) respectively, are eigenvectors of \( S^{ab} \).

We further find

\[
\begin{align*}
\gamma^{a} \langle k \rangle^{ab} &= \eta^{aa} [\langle k \rangle^{ab}], \\
\gamma^{b} \langle k \rangle^{ab} &= -ik [\langle k \rangle^{ab}], \\
\gamma^{a} [k]^{ab} &= \langle k \rangle^{ab}, \\
\gamma^{b} [k]^{ab} &= -ik \eta^{aa} \langle k \rangle^{ab}.
\end{align*}
\]

It follows that \( S^{ac} \langle k \rangle^{ab} \frac{cd}{cd} = -\frac{i}{2} \eta^{aa} \eta^{cc} \langle k \rangle^{ab} \frac{cd}{cd} \), \( S^{ac} \langle k \rangle^{ab} \frac{cd}{cd} = \frac{i}{2} \langle k \rangle^{ab} \frac{cd}{cd} \), \( S^{ac} [k]^{ab} \frac{cd}{cd} = \frac{i}{2} \langle k \rangle^{ab} \frac{cd}{cd} \), \( S^{ac} [k]^{ab} \frac{cd}{cd} = -\frac{i}{2} \eta^{aa} \eta^{cc} \langle k \rangle^{ab} \frac{cd}{cd} \).

It is useful to deduce the following relations

\[
\begin{align*}
\langle k \rangle\langle k \rangle &= 0, \quad \langle k \rangle\langle -k \rangle = \eta^{aa} \langle k \rangle^{ab} \frac{ab}{ab}, \\
\langle k \rangle\langle k \rangle^{ab} \frac{ab}{ab} &= \langle k \rangle, \\
\langle k \rangle\langle -k \rangle &= \eta^{aa} \langle k \rangle^{ab} \frac{ab}{ab}, \\
\langle -k \rangle\langle -k \rangle &= 0, \\
\langle -k \rangle\langle -k \rangle^{ab} \frac{ab}{ab} &= \langle -k \rangle, \\
\langle -k \rangle\langle -k \rangle^{ab} \frac{ab}{ab} &= \langle -k \rangle, \\
\langle k \rangle\langle -k \rangle &= 0, \\
\langle k \rangle\langle -k \rangle^{ab} \frac{ab}{ab} &= \langle k \rangle, \\
\langle k \rangle\langle -k \rangle^{ab} \frac{ab}{ab} &= \langle -k \rangle, \\
\langle k \rangle\langle -k \rangle^{ab} \frac{ab}{ab} &= \langle -k \rangle.
\end{align*}
\]
We recognize in the first equation of the first row and the first equation of the second row the demonstration of the nilpotent and the projector character of the Clifford algebra objects \( ab(k) \) and \( ab[k] \), respectively.

Whenever the Clifford algebra objects apply from the left hand side, they always transform \( ab(k) \) to \([-k]\), never to \([k]\), and similarly \( ab[k] \) to \((-k)\), never to \((k)\).

We define in Eq. (59, A8) the vacuum state \( |\psi_{oc} > \) so that one finds

\[
\begin{align*}
\langle ab(k) (k) > &= 1, \\
\langle ab[k] [k] > &= 1.
\end{align*}
\] (C5)

Taking the above equations into account it is easy to find a Weyl spinor irreducible representation for \( d \)-dimensional space, with \( d \) even or odd. (We advise the reader to see Ref. [2, 16] in particular for \( d \) odd.)

For \( d \) even, we simply set the starting state as a product of \( d/2 \), let us say, only nilpotents \( ab(k) \) for \( d = 2(2n + 1) \), Eq. (53), or nilpotents and one projector, Eq. (A6), for \( d = 4n \), one for each \( S^{ab} \) of the Cartan subalgebra elements (Eq. (B4)), applying it on the vacuum state, Eqs. (59, A8). Then the generators \( S^{ab} \), which do not belong to the Cartan subalgebra, applied to the starting state from the left hand side, generate all the members of one Weyl spinor.

\[
|\psi_{oc} > = \bigotimes_{0d}^{35} (k_{0d})(k_{12})(k_{35}) \cdots (k_{d-1 d-2}) |\psi_{oc} >,
\]

\[
|\psi_{oc} > = \bigotimes_{0d}^{35} [-k_{0d}][-k_{12}](k_{35}) \cdots (k_{d-1 d-2}) |\psi_{oc} >,
\]

\[
|\psi_{oc} > = \bigotimes_{0d}^{35} [-k_{0d}](k_{12})[-k_{35}] \cdots (k_{d-1 d-2}) |\psi_{oc} >,
\]

\[
|\psi_{oc} > = \bigotimes_{0d}^{35} (k_{0d})[-k_{12}][-k_{35}] \cdots [-k_{d-1 d-2}] |\psi_{oc} >,
\]

for \( d = 2(2n + 1) \), \( n \) = positive integer. \hspace{1cm} (C6)

\[
|\psi_{oc} > = \bigotimes_{0d}^{35} (k_{0d})(k_{12})(k_{35}) \cdots (k_{d-1 d-2}) |\psi_{oc} >,
\]

\[
|\psi_{oc} > = \bigotimes_{0d}^{35} [-k_{0d}][k_{12}](k_{35}) \cdots (k_{d-1 d-2}) |\psi_{oc} >,
\]

\[
|\psi_{oc} > = \bigotimes_{0d}^{35} [-k_{0d}][k_{12}][-k_{35}] \cdots (k_{d-1 d-2}) |\psi_{oc} >,
\]

\[
|\psi_{oc} > = \bigotimes_{0d}^{35} (k_{0d})[-k_{12}][-k_{35}] \cdots [-k_{d-1 d-2}] |\psi_{oc} >,
\]

for \( d = 4n \), \( n \) = positive integer. \hspace{1cm} (C7)
1. Technique to generate "families" of spinor representations in terms of Clifford algebra objects

When all $2^d$ states are considered as a Hilbert space, we found in this paper that for $d$ even there are $2^{d/2-1}$ "family members" and $2^{d/2-1}$ "families" of spinors, which can be second quantized. (The reader is advised to see also Refs. [2, 9, 16, 17, 26, 27].) We shall here pay attention on only even $d$.

One Weyl representation forms a left ideal with respect to the multiplication with the Clifford algebra objects. We proved in Refs. [9, 17], and the references therein that there is the application of the Clifford algebra object from the right hand side, which generates "families" of spinors.

Right multiplication with the Clifford algebra objects namely transforms the state with the quantum numbers of one "family member" belonging to one "family" into the state of the same "family member" (into the same state with respect to the generators $S^{ab}$ when the multiplication from the left hand side is performed) of another "family".

We defined in Refs. [2, 17] the Clifford algebra objects $\tilde{\gamma}^a$’s as operations which operate formally from the left hand side (as $\gamma^a$’s do) on any Clifford algebra object $A$ as follows, Eq. (49),

$$\tilde{\gamma}^a A = i(\gamma^a A - A \gamma^a), \quad (C8)$$

with $(\gamma^a A = -1$, if $A$ is an odd Clifford algebra object and $(\gamma^a A = 1$, if $A$ is an even Clifford algebra object.

Then it follows, in accordance with Eq. (11), that $\tilde{\gamma}^a$ obey the same Clifford algebra relation as $\gamma^a$.

$$(\tilde{\gamma}^a \gamma^b + \gamma^b \tilde{\gamma}^a)A = -i i((\gamma^a \gamma^b + \gamma^b \gamma^a) = I \cdot 2 \eta^{ab} A \quad (C9)$$

and that $\tilde{\gamma}^a$ and $\gamma^a$ anticommutate

$$(\tilde{\gamma}^a \gamma^b + \gamma^b \tilde{\gamma}^a)A = i(\gamma^a A - A \gamma^a) = 0 \quad (C10)$$

We may write

$$\{\gamma^a, \tilde{\gamma}^b\}_+ = 0, \quad \text{while} \quad \{\tilde{\gamma}^a, \gamma^b\}_+ = I \cdot 2 \eta^{ab}. \quad (C11)$$
One accordingly finds
\[
\tilde{\gamma}^a_{\ ab}(k): = -i \gamma^a(k) \gamma^b = -i \eta^{aa}_{\ ab}[k],
\]
\[
\tilde{\gamma}^b_{\ ab}(k): = -i \gamma^b(k) = -k [k],
\]
\[
\tilde{\gamma}^a_{\ ab}[k]: = i [k] \gamma^a = i (k),
\]
\[
\tilde{\gamma}^b_{\ ab}[k]: = i [k] \gamma^b = -k \eta^{aa}_{\ ab}(k).
\]
(C12)

If we define, Eq. (2),
\[
\tilde{\gamma}^a_{\ ab}(k): = -i \gamma^a(k) \gamma^b = -i \eta^{aa}_{\ ab}[k],
\]
\[
\tilde{\gamma}^b_{\ ab}[k]: = i [k] \gamma^b = -k \eta^{aa}_{\ ab}(k),
\]
(C13)

it follows
\[
\tilde{S}^{ab}A = A \frac{1}{4} (\gamma^b \gamma^a - \gamma^a \gamma^b),
\]
(C14)

manifesting accordingly that \( \tilde{S}^{ab} \) fulfill the Lorentz algebra relation as \( S^{ab} \) do. Taking into account Eq. (49), we further find
\[
\{ \tilde{S}^{ab}, S^{ab} \}_- = 0, \quad \{ \tilde{S}^{ab}, \gamma^c \}_- = 0, \quad \{ S^{ab}, \gamma^c \}_- = 0.
\]
(C15)

One also finds
\[
\{ \tilde{S}^{ab}, \Gamma \}_- = 0, \quad \{ \tilde{\gamma}^a, \Gamma \}_- = 0, \quad \{ \tilde{S}^{ab}, \tilde{\Gamma} \}_- = 0, \quad \text{for } d \text{ even,}
\]
\[
\Gamma^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}} \gamma^a), \quad \text{if } d = 2n,
\]
\[
\tilde{\Gamma}^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}} \tilde{\gamma}^a), \quad \text{if } d = 2n,
\]
(C16)

where handedness \( \Gamma \) is a Casimir of the Lorentz group, which means that in \( d \) even transformation of one "family" into another with either \( \tilde{S}^{ab} \) or \( \tilde{\gamma}^a \) leaves handedness \( \Gamma \) unchanged.

We advise the reader to read [2] where the two kinds of Clifford algebra objects follow as two different superpositions of a Grassmann coordinate and its conjugate momentum.

We present for \( \tilde{S}^{ab} \) some useful relations
\[
\tilde{S}^{ab}(k) = \frac{k}{2} \gamma^a(k),
\]
\[
\tilde{S}^{ab}[k] = -\frac{k}{2}[k],
\]
\[
\tilde{S}^{ac}_{\ ab cd}(k)(k) = \frac{i}{2} \eta^{aa}_{\ cd}[k][k][k][k],
\]
\[
\tilde{S}^{ac}_{\ ab [k]}[k] = -\frac{i}{2} \gamma^a(k),
\]
\[
\tilde{S}^{ac}_{\ ab cd}(k)[k] = -\frac{i}{2} \eta^{aa}_{\ cd}[k][k],
\]
\[
\tilde{S}^{ac}_{\ ab cd}[k](k) = \frac{i}{2} \eta^{cc}_{\ ab cd}(k)[k].
\]
(C17)
We transform the state of one "family" to the state of another "family" by the application of $\tilde{S}^{ac}$ (formally from the left hand side) on a state of the first "family" for a chosen $a,c$. To transform all the states of one "family" into states of another "family", we apply $\tilde{S}^{ac}$ to each state of the starting "family". It is, of course, sufficient to apply $\tilde{S}^{ac}$ to only one state of a "family" and then use generators of the Lorentz group ($S^{ab}$) to generate all the states of one Dirac spinor $d$-dimensional space.

One must notice that nilpotents $^{ab}(k)$ and projectors $^{ab}[k]$ are "eigenvectors" not only of the Cartan subalgebra $S^{ab}$ but also of $\tilde{S}^{ab}$. Accordingly only $\tilde{S}^{ac}$, which do not carry the Cartan subalgebra indices, cause the transition from one "family" to another "family".

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[30] However, even in the Grassmann case with gravity only the scalar fields would appear in the \( d = (3+1) \), takinf care of masses of fermions and some boson fields.

[31] \( f^\alpha_a \) are inverted vielbeins to \( e^a_\alpha \) with the properties \( e^{a}_\alpha f^\alpha_b = \delta^\alpha_b, \ e^{a}_\alpha f^\beta_a = \delta^\beta_\alpha, \ E = \det(e^a_\alpha) \). Latin indices \( a, b, ..., m, n, ..., s, t, .. \) denote a tangent space (a flat index), while Greek indices \( \alpha, \beta, ..., \mu, \nu, ..., \sigma, \tau, ... \) denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index \((a, b, c, ... \) and \( \alpha, \beta, \gamma, ... \)\), from the middle of both the alphabets the observed dimensions \( 0, 1, 2, 3 \) \((m, n, ... \) and \( \mu, \nu, ..., \sigma, \tau, ... \)\), indexes from the bottom of the alphabets indicate the compactified dimensions \((s, t, .. \) and \( \sigma, \tau, ... \)\). We assume the signature \( \eta^{ab} = diag\{1, -1, -1, \cdots, -1\} \).

[32] Observations in this paper might help also when fermionizing boson fields or bosonizing fermion fields [11].

[33] In Ref. 2, the definition of \( \theta^a \) was differently chosen. Correspondingly also the scalar product needed a (slightly) different weight function in Eq. 25.

[34] In Ref. 28, the author suggested in Eq. (47) a choice of superposition of \( \gamma^a \) and \( \tilde{\gamma}^a \), which resembles the choice of one of the authors (N.S.M.B.) in Ref. 2 and both authors in Ref. 16, 17 and in present article.

[35] In \( d = (3+1) \) space masses of fermions are in both cases, Grassmann and Clifford, caused by the interaction of fermions with vielbeins and spin connections (of one kind in Grassmann case and of two kinds in Clifford case) with the space index \( s \geq 5 \) and with nonzero vacuum expectation values.

[36] The main reason that we treat here mostly \( d = 2(2n + 1) \) spaces is that one Weyl representation, expressed by the product of the Clifford algebra objects, manifests in \( d = (1 + 3) \) all the observed properties of quarks and leptons, if \( d \geq 2(2n + 1), n = 3 \), and that the breaks of the starting symmetry down to \( d = (3 + 1) \) can lead to massless fermions [23, 24].

[37] In the case that we would choose \( \tilde{\gamma}^a \)'s instead of \( \gamma^a \)'s, Eq.41, the role of \( \tilde{\gamma}^a \) and \( \gamma^a \) should be then correspondingly exchanged in Eq. 49.

[38] The smallest number of all the generators \( S^{ac} \), which do not belong to the Cartan subalgebra, needed to
create from the starting state all the other members, is $2^{2^{d-1}} - 1$. This is true for both even dimensional spaces – $2(2n + 1)$ and $4n$. 