A reinterpretation of the Taub singularity

Bjørn Jensen *

Institute of Physics, University of Oslo, P.O. Box 1048, N-0316 Blindern, Oslo 3, Norway

Jaromír Kučera †

Dep. of Theor. Physics and Astrophysics, Masaryk University
Kotlářská 2, 611 37 Brno, Czech Republic

(June 20, 1994)

Abstract

We reinterpret the well known Taub-singularity in terms of a cylinder symmetric geometry. It is shown that a cylindrical analog to the Einstein-Rosen bridge as well as a cosmic string will be present in the geometry.

PACS numbers: 02.40.tm, 04.20.Jb
It is a somewhat curious fact that the static spherical symmetric vacuum solution of Einstein’s field equations is unique while no unique solution exist for the corresponding cylinder symmetric problem. Even though the most general form of the cylinder symmetric vacuum solution, the Levi-Civita metric \[1\], has been known for a long time physical interpretations has only been achieved for a relatively restricted class of solutions \[2\]. The aim of these notes is to provide a reinterpretation of the well known Taub singularity \[3\] in terms of a cylinder symmetric geometry. Along the way we give by this approach an interpretation of two previously ill-understood solutions of the Levi-Civita metric.

The structure of these notes is as follows. We will first use a well known connection between plane and cylinder symmetry in order to construct the space-time manifold of a cylinder symmetric shell of finite thickness which is filled with an incompressible perfect fluid. This connection has not, as far as we know, been utilized before in the study of cylinder symmetric systems in the literature. A cylindrical analog to the Taub-singularity resides in the vacuum on one side of the shell while the other vacuum is flat. These vacuum solutions have appeared in previous studies but have so far defied interpretation \[2,4\]. We match this construction using the Israel-formalism to the conic flat space which appear outside cosmic gauge strings. The resulting structure can be interpreted either as a cosmic gauge string with a naked singularity in the interior region or as a naked cylinder-symmetric singularity with a cosmic string in the interior. It is possible to travel from the conic region and into the singular region after having been traveling through a flat part which we interpret as a cylindrical analog to the Einstein-Rosen bridge.

In \[5,6\] a static solution of Einstein’s equations was obtained which was interpreted as an infinite plane wall of finite thickness composed of an incompressible ideal perfect fluid with a constant energy density. The pressure inside the wall is everywhere greater than zero and the boundaries of the wall are derived from the physical condition of vanishing pressure
p = 0. This particular solution has a strange and unexpected property. Despite the fact that the energy density is assumed to be constant inside the wall the interior solution can in no way be made mirror symmetric about the central plane in the wall if the solution is to be free from curvature singularities. It follows that the exterior solution on one side of the wall is flat while the exterior solution on the other side must display a non-vanishing intrinsic curvature. This property is in fact not a curious property of the specific solution only. It can be shown in general that no singularity free plane symmetric source which obeys the dominant energy condition will give rise to the same geometry on both sides of the wall.

It was a wish to get a deeper understanding of this curious property that led to the present notes. We will first use this specific solution and a connection between plane and cylinder symmetry in order to derive the space-time manifold of a thick cylindrical shell.

The connection between plane and cylinder symmetry which we want to utilize is a simple one. Every static cylinder symmetric interval can be written in the form

\[ ds^2 = e^{2\nu(r)} dt^2 - dr^2 - e^{2\chi(r)} d\varphi^2 - e^{2\lambda(r)} dz^2. \]  

In these coordinates we will assume \( 0 \leq \varphi \leq 2\pi \) and we will leave the range of \( r \) and \( z \) unrestricted. Similarly, every static plane symmetric interval can be brought on the form

\[ ds^2 = e^{2\nu(x)} dt^2 - dx^2 - e^{2\lambda(x)} (dy^2 + dz^2). \]

Hence, by imposing the equality \( g_{\varphi\varphi} = g_{zz} \) in eq. (1) we clearly see that the cylinder symmetric case is locally a special case of the plane symmetric one. We will first discuss which axisymmetric vacuum solutions are possible with this sort of restriction.

The most general axisymmetric vacuum solutions, known as the Kasner solutions, can in general be written in the convenient form

\[ ds^2 = - \left( dr^2 + r^4 \left( \frac{\Delta + 1}{A^2 + z^2} \right) d\varphi^2 + r^2 \left( \frac{\Delta^2 + 1}{A^2 z^2} \right) dz^2 \right) + r^4 \left( \frac{\Delta + 1}{A^2 + z^2} \right) dt^2. \]
Here $A$ is an arbitrary constant with the limits $A \rightarrow \pm \infty$ included. Imposing the “plane symmetry” condition on this metric restricts the constant $A$ to the two possibilities $A = 3$ and $A = -1$. For $A = 3$ we obtain the line-element

$$ds^2 = -\left(dt^2 + r^{4/3}d\varphi^2 + r^{4/3}dz^2\right) + r^{-2/3}dt^2$$ \hspace{1cm} (4)

which is regular everywhere except at $r = 0$ where a curvature singularity resides. It is clear that this geometry can be considered as a cylindrical analog to the Taub solution with the Taub singularity residing at $r = 0$ \[3\]. The metric structure does not reduce to the Minkowski metric in cylinder coordinates in any region. This is a well known problem which arises in the interpretation of cylinder symmetric solutions in general. However, all the curvature invariants vanish in the limit $r \rightarrow \infty$. For $A = -1$ the interval eq.(3) reduces to

$$ds^2 = -\left(dt^2 + d\varphi^2 + dz^2\right) + r^2dt^2,$$ \hspace{1cm} (5)

which is regular and flat everywhere. However, a coordinate singularity is present at $r = 0$.

At this point it is interesting to make a connection to the form of the Levi-Civita metric used in \[2\]. In that work the Levi-Civita metric was written as

$$ds^2 = R^{8\sigma^2 - 4\sigma}(K^2dR^2 + L^2dz^2) + M^2R^{2-4\sigma}d\phi^2 - N^2R^{4\sigma}dt^2 \quad (R \geq 0).$$ \hspace{1cm} (6)

$K, L, M, N$ and $\sigma$ was considered as arbitrary constants. Imposing the plane symmetry condition implies that

$$\sigma = \pm \frac{1}{2}.$$ \hspace{1cm} (7)

In previous studies these solutions have proven particularly difficult to interperet \[2\]. $\sigma = 1/2$ leads directly to eq.(5) in our work while $\sigma = -1/2$ leads to eq.(4) via a coordinate transformation. We therefore believe that the present considerations are of particular importance for the understanding of these solutions.
We next turn our attention to the metric field inside the cylindrical wall. We assume a cylindrical wall of finite thickness and composed of a fluid with a non-zero constant and positive energy density $\rho$. The pressure is everywhere positive $p > 0$ except at the boundaries of the wall where $p = 0$ holds. We will denote the radial coordinate in this region by $R$. On the assumption that the metric structure takes the form eq.(1) we find that Einsteins equations for the case of an incompressible ideal perfect fluid, which is at rest in the coordinate system, are given by

\begin{align*}
G_{tt} &= \chi'' + \lambda'' + \chi'^2 + \chi'\chi' = -\rho \\
G_{rr} &= \nu'\lambda' + \nu'\chi' + \lambda'\chi' = p \\
G_{\varphi\varphi} &= \nu'' + \lambda'' + \nu'^2 + \lambda'^2 + \nu'\lambda' = p \\
G_{zz} &= \nu'' + \chi'' + \nu'^2 + \chi'^2 + \nu'\chi' = p.
\end{align*}

$'$ denotes differentiation with respect to the radial coordinate $R$. We now impose the “plane symmetry” restriction $\lambda(R) = \chi(R)$. Then the form of the metric, and hence Einsteins equations, for the cylinder and plane symmetrical situations coincide. In [6] the plane symmetric situation was described using Cartesian coordinates $x, y, z$. The coordinates are chosen in such a way that the vector $\partial_z$ is always perpendicular to the plane wall while $\partial_x$ and $\partial_y$ are parallel to it. Identifying the $z$-coordinate in [6] with $R$, the $x$-coordinate with $\varphi$ and the $y$-coordinate with $z$ in eq.(3), we can easily transform the solution found in [6] to cylindrical coordinates. We then get

\begin{align*}
e^{2\lambda(R)} &= \cos^{(4/3)} \beta R \\
e^{2\nu(R)} &= \left[ -Q \sin \beta R + F(-1/2, -1/6; 1/2; \sin^2 \beta R) \right]^2 \\
p(R) &= \rho (e^{-\nu(R)} - 1).
\end{align*}

Here $\beta = \frac{1}{2} \sqrt{3\rho}$, $F(a, b; c; x)$ denotes the hypergeometric function, and $Q$ is a constant of integration. A detailed discussion of this solution allow us to restrict our attention to situations with $0 < Q < F(-1/2, -1/6; 1/2; 1)$ and $0 \leq R < \pi/(2\beta)$ (due to periodicity of the trigonometric functions in eqs.(12,13)) [6]. However, in order not to violate the dominant
energy condition $Q$ should be further restricted to the interval $0 < Q < Q_m$ where $Q_m \sim 0.86$ (the upper limit can only be determined numerically). Then two kinds of boundary conditions, both characterized by $p = 0$, follow from eq. (9) which now takes the form

$$\lambda'(2\nu' + \lambda') = 0.$$  \hfill (15)

It follows that one kind of boundary is characterized by

$$\lambda' = 0,$$  \hfill (16)

while a second one is given by

$$\lambda' = -2\nu'.$$  \hfill (17)

When eq.(16) is combined with the condition $0 \leq R < \pi/(2\beta)$ it follows that $R = 0$. Equation (17) is much harder to solve but it is readily shown numerically that it possess one solution such that $R < \pi/(2\beta)$ \cite{6}.

We now want to match this interior solution smoothly to the two axisymmetric vacuum solutions considered earlier. It is seen from the conditions of continuity of the metric tensor and its first derivatives that only metric eq.(5) is possible to join to the boundary of the first kind eq.(16) while only metric eq.(4) is possible to join to the boundary of the second kind eq.(17). Since the metric and its derivatives are continuous across the matching surfaces it should be possible to set up a single coordinate system for the entire manifold. It is rather straightforward to extend the $R$-coordinate into the flat interior part via the coordinate transformation $r = Q\beta(R - 1/(Q\beta))$, $t \to Q\beta t$. The metric eq.(5) then takes the form

$$ds^2 = -\left( dR^2 + d\varphi^2 + dz^2 \right) + Q^2\beta^2[R - 1/(Q\beta)]^2 dt^2,$$  \hfill (18)

where now $R \leq 0$. The metric component $g_{\varphi\varphi}$ in eq.(18) is independent of $R$. Hence a space-like hyper-surface $t = \text{const.}, z = \text{const.}$ has a cylindrical geometry with a $S^1 \times \mathbb{R}$ product topology. It is also possible to extend the coordinatization into the Taub-part of the
manifold. However, contrary to the claim in [6] it is not possible to match the wall smoothly to the metric eq.(4) in such a way that the singularity disappears, i.e. it is not possible to replace the Taub-singularity with the plane wall solution in [6]. This follows immediately from the condition for the continuity of the gravitational potential $g_{tt}$ and the corresponding derivative relation across the matching surface. However the solutions can be matched smoothly when the presence of the singularity structure is tolerated (this must partially be done numerically due to the complex form of the metric coefficients in the interior of the wall). The singularity will then reside at a finite distance from the surface of the wall. It follows that the plane solution found in [6] is the first non-singular source found for the Taub-singularity. Other previous known sources for the Taub-singularity display infinitely thin walls with and without a scalar field as part of the vacuum structure [11–13].

We will now match the above structure consisting of the plane and the two vacuum solutions to the exterior region of a cosmic gauge string. The geometry in this region is assumed to take the form

$$ds^2 = F dt^2 - B^2(R + R_0)^2 d\phi^2 - dz^2 - dR^2 \quad (R < 0).$$

(19)

Here $F$, $B$ and $R_0$ are constants to be determined. When $B \neq 1$ this geometry displays a conic deficit angle which is a well known feature of the cosmic gauge string exterior space-time [14]. The matching will require that a thin cylindrical shell $S$ in general will be induced. Let $\vec{N}$ be a space-like unit vector orthogonal to this surface and we will let it point from the cosmic string geometry and into the region described by eq.(18). We will let $S$ reside at a fixed radial coordinate $R = R_s < 0$. The properties of $S$ can be computed from the extrinsic curvatures $K_{ij}$ of this surface relative to the two flat geometries by the use of the Israel-formalism [15]. When $S_{ij}$ denotes the energy-momentum tensor of the matter in $S$ we have

$$- S_{ij} = \gamma_{ij} - g_{ij} \text{Tr} \gamma_{ij}$$

(20)
where $\gamma_{ij} = K^+_{ij} - K^-_{ij}$. $K^+_{ij}$ is the curvature relative to the flat metric eq.(18) while $K^-_{ij}$ is the extrinsic curvature of $S$ relative to the gauge string metric eq.(19). From the condition for the continuity of the gravitational potentials at $R = R_S$ we get $F = Q^2 \beta^2 (R_S - 1/(Q\beta))^2$ and $B = (R_S + R_0)^{-1}$. From the definition $K_{ij} = -N_{ij}$ we then have

$$S_{tt} = \frac{Q^2 \beta^2 (R_S - 1/(Q\beta))^2}{R_S + R_0} \tag{21}$$

$$S_{\phi\phi} = \frac{1}{R_S - 1/(Q\beta)} \tag{22}$$

$$S_{zz} = \frac{1}{R_S - 1/(Q\beta)} - \frac{1}{R_S + R_0}. \tag{23}$$

The metric eq.(19) is trivially regular provided $R + R_0 < 0$. However this breaks both the weak and the dominant energy-conditions since $S_{tt} \leq 0$ (hatted indices refer to the tetrade observer at rest in the geometry). The strong energy condition is not satisfied either since the Tolman-mass density $S^i_{\ i} - S^z_{\ z} - S^\phi_{\ \phi} = 2 Q \beta (Q \beta R_S - 1)^{-1}$ is negative. This is unexpected since the intrinsic curvatures in the cosmic gauge string metric vanish. The geometry eq.(19) is boost-invariant in the longitudinal direction. It is worth noting that this symmetry is not reflected in the energy-momentum tensor of the matter in $S$. The surface $S$ is a cylindrical surface and in what way it “bends” can be found quite easily from the sign of $K_{\phi\phi}$. Relative to the metric eq.(19) we have $K_{\phi\phi} = (R_S + R_0)^{-1}$. Since $K_{\phi\phi} < 0$ when $R_S + R_0 < 0$ we have that $\vec{N}$ can be looked upon as pointing “into” a cylinder defined by $S$. In this situation we can interpret the resulting structure as representing a cosmic string with a naked singularity in the interior region. The weak energy condition is satisfied when $R_0$ takes on sufficiently positive values. In this situation eq.(19) is no longer regular but will display a conic singularity at $R = -R_0$ which we take as an indication for the presence of a cosmic string. Such singularities are very weak in the sense that the singular axis can be replaced with a smooth solution of Einsteins equations $^{[16]}$. We now have that $K_{\phi\phi} > 0$ and we can look upon $\vec{N}$ as pointing “out of” a cylinder defined by $S$. In this situation $S$ actually “encapsulates” a cosmic string such that the string defines a symmetry axis in the manifold.
It is interesting to investigate the motion of freely falling particles moving in the radial direction in the flat region eq.(18). In this geometry we get from the geodesic equation that the radial acceleration of the particle relative to a coordinate-basis observer is

$$\ddot{R} = \frac{-E^2}{Q\beta(|R| + 1/(Q\beta))}.$$  \hspace{1cm} (24)

\(\ddot{R}\) denotes twice differentiation with respect to an affine parameter and the conserved energy of the particle \(E\) obeys \(E = Q^2\beta^2(R - 1/(Q\beta))^2\dot{t}\). It is clear that the particles experiences a positive acceleration and will consequently experience the shell with the perfect fluid as source for attractive gravitation. This surface displays a positive gravitational mass and is therefore a natural source for an attractive gravitational pull. This fact explains the negative Tolman-mass density of the shell \(S\) which screens the field of the wall. Note that the acceleration is everywhere finite. Hence nothing seems to prevent a freely falling particle to enter the wall and subsequently to get in “direct” contact with the singularity. In the “Taub-part” of the manifold the sign of \(\ddot{R}\) changes. Hence the particle will experience the singularity as source for a repulsive gravitational “force”. The repulsive nature of the Taub-singularity is in accord with the repulsive nature of other well known singularities such as the singularities in the Reisner-Nordström and Kerr-geometries. However the Taub-singularity differs from these since it is not hidden behind any event-horizon.

The Taub singularity challenges our understanding of the singularity structure in the general theory of relativity. When interpreted as a “sheet” singularity it can obviously be generated by a source with a finite gravitating energy density (which obeys the energy conditions) and without the occurrence of topology changing or causality violating surfaces or regions. It is usually believed that singularities in Einsteins theory is connected with such “pathologies” or (near) infinite energy densities. Due to the fact that no physically acceptable sources (i.e. sources that obey the dominant energy condition) will generate the same geometry on both sides of a wall and to conform with the above belief we are therefore tempted to interpret the Taub solution as describing a cylindrical “sheet” singularity and
not a “planar” sheet singularity as is the usual interpretation when it is generated by physically acceptable sources. The existence of the Taub-singularity is in this interpretation directly coupled to the existence of a topology changing region described by eq.(18).

**Acknowledgments**

One of us (J.K.) thanks the University of Oslo for hospitality during the time which parts of this work was performed and acknowledges the Norwegian Research Council NAVF/SEP for financial support. We also thanks Svend Hjelmeland, a master student at UiO, for interesting conversations concerning the properties of the Levi-Civita metric.

---

1When these notes was at the end of completion we received a preprint gr-qc/9405074 where it also is shown that Einstein-Rosen like bridges may exist “inside” cosmic gauge strings.
REFERENCES

[1] T. Levi-Civita, *Atti Accad.Lincei Rendi* **28** (1919) 101.

[2] W.B. Bonnor, and W. Davidson, *Class. Quantum Grav.* **9** (1992) 2065.

[3] A.H. Taub, *Ann.Math.* **53** (1951) 472.

[4] W.B. Bonnor, and M.A.P. Martins, *Class.Quant.Grav.* **8** (1991) 727.

[5] R.M. Avakyan, and J. Horský, *Sov.Astrophys.J.* **11** (1975) 454.

[6] J. Novotný, J. Kučera, and J. Horský, *Gen.Rel.Grav.* **19** (1987) 1195.

[7] A.D. Dolgov, and I.B. Khriplovich, *Gen.Rel.Grav.* **21** (1989) 13.

[8] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt E, *Exact Solutions of Einstein’s Field Equations* (VEB Deutscher Verlag der Wissenschaften, Berlin, (1980)) p.220.

[9] E. Frehland, *Commun.math.Phys.* **23** (1971) 127.

[10] E. Frehland, *Commun.math.Phys.* **26** (1972) 307.

[11] J. Horský, and J. Novotný, *J.Phys.A* **2** (1969) 251.

[12] J. Ipser, and P. Sikivie, *Phys.Rev.D* **30** (1984) 712.

[13] Ø. Grøn, and H.H. Soleng, *Phys.Lett.A* **165** (1992) 191.

[14] J.R. Gott III, *Astrophys.J.* **288** (1985) 422.

[15] W. Israel, *Nuovo Cimento* **44B** (1966) 1.

[16] W.A. Hiscock, *Phys.Rev.D* **31** (1985) 3288.

[17] Se f.ex. P.S. Joshi, *Global Aspects in Gravitation and Cosmology* (Clarendon Press, Oxford, (1993)).