Decrease of the portfolio computational workload following the Capital Asset Pricing theory

T Stoilov, K Stoilova, M Vladimirov

1 Institute of Information and Communication Technologies – Bulgarian Academy of Sciences, Sofia, Bulgaria
2 Nikola Vaptsarov Naval Academy, Varna
3 Varna University of Economics, Varna, Bulgaria

e-mail: toodor@hsi.iccs.bas.bg

Abstract. The paper derives an evaluation algorithm for decreasing the computational workload in definition and solution of portfolio optimization problem. The algorithm applies numerical relations, which lead to transformation of the classical portfolio problem to optimization one, which contains parameters from Capital Market Theory. The decrease of the computational workload results from the inclusion of beta coefficients in the portfolio problem instead of evaluation of the covariance portfolio matrix. The algorithm is illustrated by solution of portfolio problem with mutual funds on Bulgarian Stock Exchange.

Keywords: Active investments, mathematical models in portfolio theory, portfolio optimization

1. Introduction

The foundation of Modern Portfolio Theory (MPT) starts with the formal definition of the portfolio problem by Markowitz in 1952. The portfolio problem is defined as optimization one for maximization of the portfolio return, subject to the constraint value of the portfolio variance [1]. The solution of this problem gives the relative values of the investment, which has to be used for buying assets which contribute to the portfolio performance. For the direction of evolution of the MPT one can refer to the overview presented in [2].

The formal presentation of the portfolio return is given by linear relation of expected assets’ returns multiplied by the corresponding weights. The portfolio risk is defined as quadratic relation between the assets’ weights and the covariance matrix of assets’ returns. From practical considerations the evaluation of the covariance matrix requires a big amount of calculations. Nevertheless of the symmetric structure of the covariance matrix, a portfolio with $N=100$ different assets/stocks would require the evaluation of 5050 covariance components, $\binom{N(N-1)}{2} + N$. Thus, the evaluation of a covariance matrix will insist time consuming resources for finding a set of data which are obligatory for the formal definition of the portfolio problem.
This paper derives an algorithm which constraint the need to find all components of the covariance matrix. The formal ground of the algorithm is the usage of the Capital Asset Pricing Model, launched by Sharpe [4]. Because the MPT has its own formal models and applies portfolio formal relations, below it is given short description of the portfolio problem.

2. Mean variance portfolio problem

The portfolio optimization targets the construction of such allocation of an investment between a set of assets by means to make a trade-off between the portfolio expected return and portfolio risk. The mean variance definition of the portfolio optimization problem targets maximization of the portfolio expected return and minimizing the volatility as measure of portfolio risk [3]. The classical portfolio problem, defined by Markowitz, has two forms:

- maximization of the portfolio return for a given level of risk

\[
\max \left[ E^T w \right] \quad \text{subject to} \quad w^T \Sigma w \leq \sigma_{\max}^2 \tag{1}
\]

or

- minimization of the portfolio risk for a required level of return

\[
\min \left[ w^T \Sigma w \right] \quad \text{subject to} \quad E^T w \geq E_{\min} \tag{2}
\]

where

- \( E_i \) - the mean return of asset \( i = 1, \ldots, N \), \( E^T = (E_1, \ldots, E_N) \),
- \( \Sigma \) - the covariance matrix between the returns of the assets,
- \( w^T = (w_1, \ldots, w_N) \), \( w_i \) - the relative part of the investment, allocated to asset \( i = 1, \ldots, N \), as a solution (weight) of the portfolio problem,
- \( \sigma_{\max}^2 \) - given level of portfolio risk,
- \( E_{\min} \) - required level of portfolio return.

The MPT assumes that the portfolio risk can be decreased by holding in appropriate combination the risky assets. The input parameters which have to be evaluated for the classical problem (1) concern the vector of mean returns of the assets, \( E^T = (E_1, \ldots, E_N) \), where \( N \) is the set of returns, participating in the portfolio and the covariation matrix

\[
\text{COV}(.) = \sum_{N \times N} = \begin{bmatrix}
\text{cov}_{11} & \ldots & \text{cov}_{1N} \\
\ldots & \ldots & \ldots \\
\text{cov}_{N1} & \ldots & \text{cov}_{NN}
\end{bmatrix}.
\]

Using the notations from [3,4] these input parameters are evaluated from a historical set of data about the assets’ returns \( R_i^{(n)} \), \( i = 1, \ldots, N \) (set of returns), \( n \) is the discrete in time in which the assets returns \( R_i^{(n)} \) are estimated. Thus, \( E_i = \sum_{k=1}^{n} p_i^{(k)} R_i^{(k)} \), where \( p_i^{(k)} \) is probability that at time \( k \) the asset \( i \) has current level of return \( R_i^{(k)} \). The classical mean variance (MV) theory assumes normal distribution of the assets returns and each discrete \( k \) has equal probability, the mean asset returns is evaluated as an average value

\[
E_i = \frac{1}{n} \sum_{k=1}^{n} R_i^{(k)} \tag{3}
\]

Respectively, the covariance components \( \text{cov}_{ij} \) between couples of assets \( i, j \in 1, \ldots, N \) is evaluated as
Following [4] it holds

\[ \text{cov}_{ij} = \frac{1}{n} \sum_{k=1}^{n} \left( R_i^{(k)} - E_i \right) \left( R_j^{(k)} - E_j \right), \quad \forall i, j \in 1, ..., N \]  

(4)

which means that the diagonal components of the covariance matrix \( \Sigma \) contain the volatilities (risk) of the assets. The covariance matrix is a symmetrical one, which reduces the computational load for evaluating its components. Unfortunately, with the increase of \( N \), the set of assets, participating in the portfolio it is not practical to implement relation (4) for all \( \frac{N(N-1)}{2} + N = \frac{N(N+1)}{2} \) components of the covariation matrix \( \Sigma \). The decrease of the computational workflow is a prerequisite for implementation of active portfolio management. The research targets the derivation of sequence of computations which result in algorithm for faster solution of the classical portfolio problem (1). It is used additional relations originated from the Capital Market Theory [4]. Then, the solution of the portfolio problem is solved in Excel environment using real data from the Bulgarian stock exchange with actual data of 2019.

3. Relations originated by the Capital Asset Pricing Model (CAPM)

The Capital Market Theory originated a model, titled CAPM [4, 6]. This model introduces two additional formal components in the MPT: the risk free asset and the market point \( (E_M, \sigma_M) \). The CAPM introduces relationship between the Market return \( E_M \), the portfolio mean return \( E_p \) or asset mean return \( E_i \) and the risk free value \( r_f \). This relation is linear, named Security Market Line (SML). This line is derived according to assumption that the assets’ returns \( E_i \) depend in linear regression to the market returns \( R_M \), respectively \( E_M \). For analytical derivation of SML geometrical considerations are used, given in fig.1.

![Figure 1. Presentation of asset i and Market point](image)

The security \( i \) and the market has return and risk values respectively \( (E_i, \sigma_i) \) and \( (E_M, \sigma_M) \). The notation \( \sigma_i \) is the standard deviation of the returns \( R_i \) around the mean value \( E_i \) and it is a measure for the risk of asset \( i \). Respectively, \( \sigma_M \) is the risk of the market point \( M \). With the two assets \( (E_i, \sigma_i) \) and \( (E_M, \sigma_M) \) it is possible to make a combination in a portfolio with assets weights, respecting the portfolio constraint

\[ w_i + w_M = 1 \, . \]  

(6)
The resulting portfolio will have mean return and risk is given

\[ E_{p(w)} = w_iE_i + w_M E_M, \]
\[ \sigma^2_{p(w)} = w_i^2 \sigma_i^2 + w_M^2 \sigma_M^2 + 2w_i w_M \text{cov}_{iM}, \]

(7)

where \( \text{cov}_{iM} \) is the covariance value between the returns of asset \( i \) and the market. If the weights \( w_i \) and \( w_M \) change according to relation (6), the portfolio return and risk \( (E_p, \sigma_p) \) will move to the curve \( \overline{iM} \).

Following the CAPM, the point \( M \) is a tangent one to the curve \( \overline{iM} \) and passes through the risk free point \( (r_f, 0) \). The tangent line in arbitrary point \( S_i \) of the curve \( \overline{iM} \) will have slope \( s_i = \frac{dE_p}{d\sigma_p} \).

To find analytical description of the derivative \( \frac{dE_p}{d\sigma_p} \) here it is used the sequential of the differentiation

\[ \frac{dE_p}{d\sigma_p} = \frac{\partial E_p/\partial w_i}{\partial \sigma_p/\partial w_i}. \]

(8)

Using (6), relation (7) can be described only with the weight \( w_i \) or

\[ w_i = 1 - w_M \]
\[ E_{p(w_i)} = w_i E_i + (1 - w_M) E_M \]
\[ \sigma_p(w_i) = \sqrt{w_i^2 \sigma_i^2 + (1 - w_i)^2 \sigma_M^2 + 2w_i (1 - w_i) \text{cov}_{iM}}. \]

(9)

Using (9) the partial derivatives give

\[ \frac{\partial E_p}{\partial w_i} = E_i - E_M \]
\[ \frac{\partial \sigma_p}{\partial w_i} = \frac{(\sigma_i^2 + \sigma_M^2 - 2 \text{cov}_{iM})w_i + \text{cov}_{iM} - \sigma_M^2}{\sigma_p}. \]

Assuming that asset \( i \) move to point \( M \), then \( w_i = 0, w_M = 1 \). The ratio of the partial derivatives gives the slope of the tangent line \( \overline{pM} \)

\[ \frac{dE_{(w_i)}}{d\sigma_{(w_i)}|_{w_i=0}} = \frac{E_i - E_M}{(\text{cov}_{iM}/\sigma_M^2)}; \quad \sqrt{\sigma_{p_{w_i=0}}} = \sqrt{\sigma_M^2 - \sigma_M}. \]

But this tangent line at point \( M \) will match with the line \( \overline{pM} \) from fig.1, which has a slope equal to \( \frac{E_M - P}{\sigma_M} \). Hence, the equality holds

\[ E_i = r_f + (E_M - r) \frac{\text{cov}_{iM}}{\sigma_M}. \]

(10)

This relation is the analytical form of the SML. The relations

\[ \frac{\text{cov}_{iM}}{\sigma_M^2} = \beta_i \]

(11)
is named coefficient “beta”, which is well known for the portfolio theory. Relation (10) holds not only for asset $i$, but for arbitrary portfolio

$$E_p = r_f + (E_M - r_f)\beta_p.$$  \hfill (12)

This research uses relations (4), (7), (10), (11) to simplify the evaluations of the components of the covariation matrix $\Sigma = [\text{cov}_{ij}], \ i,j \in 1,...,N$.

4. Evaluation of the portfolio “beta”, $\beta_p$

From (11) it follows

$$\text{cov}_{PM} = \beta_p \sigma_M^2$$  \hfill (13)

$$\text{cov}_{1M} = \beta_1 \sigma_M^2$$

$$\text{cov}_{2M} = \beta_2 \sigma_M^2.$$  

The definition of the coefficient $\text{cov}_{PM}$ is

$$\text{cov}_{PM} = \sum_{k=1}^{n} p^{(k)} (R_p^{(k)} - E_p)(R_M^{(k)} - E_M).$$

By substituting

$$R_p^{(k)} = w_1 R_1^{(k)} + w_2 R_2^{(k)} \quad \text{and} \quad E_p = w_1 E_1 + w_2 E_2$$

it follows

$$\text{cov}_{PM} = \sum_{k=1}^{n} p^{(k)} \left( w_1 R_1^{(k)} + w_2 R_2^{(k)} - w_1 E_1 - w_2 E_2 \right) (R_M^{(k)} - E_M)$$

or

$$\text{cov}_{PM} = w_1 \text{cov}_{1M} + w_2 \text{cov}_{2M}.$$  \hfill (14)

Relation (13) shows that the covariance of the portfolio is a weighting sum of the covariance of the assets towards the market portfolio. By substituting (13) in (14) it follows analogical relations for the portfolio “beta”

$$\beta_p = w_1 \beta_1 + w_2 \beta_2.$$  \hfill (15)

5. Substitution of $\beta$ in the portfolio goal function

The goal of this research is to decrease the computational workload for the definition and solution of the portfolio problem. This can be achieved by substitution the components of the covariance matrix $\Sigma = [\text{cov}_{ij}], \ i,j \in 1,...,N$ with the beta coefficients. Thus, it is expected to decrease the computations for the definition of the input data for the portfolio problem (1). Problem (1) can be worked out in a new formal view applying relation (11) and (12) of the SML. Thus, in the initial portfolio problem it can be introduced the new coefficients $\beta_l$

$$\max_w \left( E_p(w) \right)$$

$$\sigma_p^2(w) \leq \sigma_{\text{max}}^2$$

$$\sum_{i=1}^{N} w_i = 1.$$  

To simplify the notations it is assumed $N=2$. Hence, the goal function of (16) is worked out in the form

$$\max_w E_p = \max_w \left[ r_f + (E_M - r_f)\beta_p \right].$$
where $\beta_p$ is evaluated according (15). The goal function does not contain the covariance matrix. For the evaluation of $\beta_p$ is needed only $\beta_1$ and $\beta_2$ which are only two components for the portfolio. For the general case of $N$ assets in the portfolio the optimization problem (16) needs only $N$ evaluations of coefficients $\beta_i$, $i \in 1, \ldots, N$ instead of, $\left(\frac{N(N-1)}{2} + N\right)$.

6. Substitution of $\beta$ in the constraint $\sigma_p(w)$ of the portfolio problem

The portfolio risk analytically is expressed by relation (7). This relation does not contain in explicit way the coefficients $\beta_i$, $i \in 1, \ldots, N$. For that case the next analytical expressions are implemented. The multiplication between the two covariations towards the market gives

$$cov_{1M}cov_{2M} = \sum_{k=1}^{n} \left( p^{(k)}_1 (R^{(k)}_1 - E_1) \right) \left( R^{(k)}_M - E_M \right) \sum_{k=1}^{n} p^{(k)}_2 (R^{(k)}_2 - E_2) \left( R^{(k)}_M - E_M \right) = \sum_{k=1}^{n} p^{(k)}_1 \left( R^{(k)}_1 - E_1 \right) \left( R^{(k)}_M - E_M \right) \sum_{k=1}^{n} p^{(k)}_2 \left( R^{(k)}_2 - E_2 \right) \left( R^{(k)}_M - E_M \right)^2 = cov_{12} \sigma^2_M.$$

Respectively, the square is

$$cov_{1M}cov_{1M} = \sum_{k=1}^{n} p^{(k)}_1 \left( R^{(k)}_1 - E_1 \right) \left( R^{(k)}_M - E_M \right) \sum_{k=1}^{n} p^{(k)}_2 \left( R^{(k)}_2 - E_2 \right) \left( R^{(k)}_M - E_M \right)^2 = \sigma^2 \sigma^2_M.$$

From these relations using (13) it holds

$$cov_{1M}cov_{2M} = \beta_1 \sigma^2_M \beta_2 \sigma^2_M = cov_{12} \sigma^2_M$$

or

$$\beta_1 \beta_2 \sigma^2_M = cov_{12} \quad (17)$$

Respectively,

$$cov_{1M}cov_{1M} = \beta_1 \sigma^2_M \beta_1 \sigma^2_M = \tau^2 \tau^2_M$$

or

$$\beta_1^2 \sigma^2_M = \sigma^2_i \quad (18)$$

After substitution of $\sigma_i$ and $cov_{12}$ in (7) the expression of the portfolio risk is expressed with the coefficients $\beta_i$ or

$$\sigma^2_p = \sigma^2_M (w_1^2 \beta_1^2 + w_2^2 \beta_2^2 + 2 \beta_1 \beta_2 w_1 w_2) = \sigma^2_M (w_1 \beta_1 + w_2 \beta_2)^2 \quad (19)$$

The portfolio optimization problem (1) takes analytical form

$$\max_{w_1, w_2} \left[ r_f + (E_M - r_f) (\beta_1 w_1 + \beta_2 w_2) \right]$$

$$\sigma^2_p = \sigma^2_M (w_1 \beta_1 + w_2 \beta_2)^2 \leq \sigma^2_{\max} \quad (20)$$

This formal description has advantages in comparison with (1) because it is needed to evaluate only the $\beta_i$, $i \in 1, \ldots, N$ coefficients, which is a set with $N$ number. The classical formulation (1) insists the evaluation of the covariance matrix which has, $\left(\frac{N(N-1)}{2} + N\right)$ components, which are much more in comparison with the case with $N$ beta coefficients.

The form of the portfolio problem (20) is implemented in Excel environment with real data from the Bulgarian Stock Exchange.
7. Numerical experiments from Bulgarian Stock Exchange

The numerical experiments were provided with 5 Bulgarian funds [5]: Advance Invest, PFBK Invest, Raiffeisen Bank, Astra Energy, Sky energy. These funds have highest capitalization in comparison with other funds in Bulgaria. Their corresponding parameters \( E_i, \sigma_i^2, \delta_i, \beta_i, \text{cov}_{im}, \ i = 1, \ldots, N \) are given on lines 9 till 14 of the Excel figure 2. The covariances \( \text{cov}_{im} \ i = 1, 5 \) are evaluated towards the Bulgarian Stock market index Sofix (M). All parameters are evaluated on annual base for 2019. The solutions of the portfolio problem are given in cells C17 till C22, fig.2. The weights are the solutions of portfolio problem (1) where \( r_f=0.02\% \) (G15), \( \sigma_M=0.5963 \) (G12), \( \sigma_M^2 = 0.3556 \) (G13). Only one working column is prepared with the multiplication \( \beta_iw_i \) and sell D23 contains the sum \( \sum_{i=1}^{5} \beta_iw_i \).

The portfolio risk is evaluated in cell F23:

\[
\sqrt{\sigma_\beta^2 (w_1 \beta_1 + w_2 \beta_2)^2} \leq \sigma_{\text{max}} = 0.01 .
\]

The goal function is the portfolio return, evaluated in cell F24: \( r_f + (E_M - r_f)(\beta_1w_1 + \beta_2w_2) \). The portfolio problem insists nonnegative weights, \( w_i \geq 0 \) and \( \sum_{i=1}^{5} w_i = 1 \). The portfolio solution is

\[
w^T = (0.325 \ 0.087 \ 0.148 \ 0.186 \ 0.251)^T, \ Risk=0.01, \ Return=0.02054 .
\]

The portfolio Return exceeds the risk free value, which is a prerequisite for the implementation of this investment.

---

**Figure 2. Portfolio problem’s solution with Bulgarian mutual funds**

This form of the portfolio problem performs less computation in comparison with the classical evaluations of portfolio risk and return according to (7). The benefit comes from fewer evaluations of the components of the covariance matrix as follows

- case with \( \beta \): evaluation of \( N \) values of covariances towards the market points;
case with relation (4) : \( \frac{N(N+1)}{2} + N = \frac{N^2 + N}{2} + N \) values are needed for the estimation of the covariance matrix.

The fewer amount of calculations for beta is a prerequisite for the implementation of active portfolio management by means to cope the investment decisions with the fast changing market behaviour.

8. Conclusions

This research makes appropriate formal derivations by means to describe the portfolio optimization problem in new analytical form. The portfolio problem is derived into new form by substituting portfolio parameters from MPT with beta parameters from the CAPM. This form allows to be used less input parameters and respectively less computational workload for the evaluations of the portfolio parameters. The less computational workload allows the portfolio problem to be defined and solved faster in case of dynamical changes of the market. Additionally, the portfolio problem can be solved with less power of the computing system.

Acknowledgments

This work has been supported by project ДН12/10, 20.12.2017 of the Bulgarian National Science fund: Integrated bi-level optimization in information service for portfolio optimization

References

[1] Markowitz H 1952 Portfolio selection J. of Finance 7 77-91.
[2] Kolm PN, Tutuncu R and Fabozzi FJ 2014 60 Years of Portfolio optimization: Practical challenges and current trends European Journal of Operational Research 234 (2) 356-371
[3] Gruber M, Brown S, Goetzmann W and Elton E 2010 Modern portfolio theory and investment analysis, 8th ed, Wiley 752 p.
[4] Sharpe W 2000 Portfolio theory and capital markets (McGraw Hill 316 p ISBN 0-07-155320-8)
[5] Stoilov T, Stoilova K and Vladimirov M 2019 Financial investments by Portfolio Optimization. J. Materials Sciences and Engineering 618 doi:10.1088/1757-899X/618/1/012030
[6] Rossi M 2016 The capital asset pricing model: a critical literature review J. Global Business and Economics review 10 (5) 604-617 On-line ISSN 1745-1329