Abstract

We give a description of the boundary of a complex of free factors that is analogous to E. Klarreich’s description of the boundary of a curve complex. The argument uses the geometry of folding paths developed in [BF11] and the structure theory of trees on the boundary of Outer space developed recently by Coulbois, Hilion, Lustig and Reynolds.

1 Introduction

The complex of free factors, denoted $F = F_N$, for the free group $F_N$ is an analogue for the complex of curves for a surface. The simplicial complex $F$ arises as the nerve of the intersection pattern for thin regions in Outer space, and hence codes the geometry of Outer spaces relative to these thin regions. Vertices of $F$ are conjugacy classes of non-trivial proper free factors of the rank-$N$ free group $F_N$, and higher dimensional simplices correspond to chains of inclusions of free factors.

Equip $F$ with the simplicial metric. It was shown in [BF11] that $F$ is Gromov hyperbolic; the goal of the present note is to give a concrete description of the boundary $\partial F$ of $F$. Kapovich-Rafi [KR] have shown that hyperbolicity of $F$ can be deduced from the hyperbolicity of the free splitting complex, which was shown by Handel-Mosher [HM12], and an alternative proof of this was given by Hilion-Horbez [HH].

Let $\partial CV_N$ denote the boundary of the Culler-Vogtmann Outer space $CV_N$ [CV86]; the points of $\partial CV_N$ are represented by very small actions of $F_N$ on $\mathbb{R}$-trees. Associated to $T \in \partial CV_N$ is a (algebraic) lamination $L(T)$,

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which intuitively records information about which elements of \( F_N \) act with short translation length in \( T \). A lamination is an \( F_N \)-invariant, flip invariant, closed subset \( X \subseteq \partial^2 F_N = \partial F_N \times \partial F_N \setminus \text{diag} \). A finitely generated subgroup \( H \leq F_N \) is a virtual retract of \( F_N \) by M. Hall’s Theorem, hence \( H \) is quasi-convex in \( F_N \), and \( \partial H \) embeds in \( \partial F_N \); say that \( H \) carries a leaf of \( X \) if \( X \cap \partial^2 H \neq \emptyset \).

A lamination \( X \) is called arational if no leaf of \( X \) is carried by a proper free factor of \( F_N \); a tree \( T \in \partial CV_N \) is called arational if \( L(T) \) is arational. Let \( AT \subseteq \partial CV_N \) denote the set of arational trees, equipped with the subspace topology. Define a relation \( \sim \) on \( AT \) by \( S \sim T \) if and only if \( L(S) = L(T) \), and give \( AT/\sim \) the quotient topology. Our main result is:

**Theorem 1.1.** The space \( \partial F \) is homeomorphic to \( AT/\sim \).

This theorem is a very strong analogue of E. Klarreich’s description of the boundary \( \partial C(S) \) of the complex of curves \( C(S) \) associated to a non-exceptional surface \( S \); Klarreich showed that \( \partial C(S) \) is homeomorphic to \( AF/\sim \), where \( AF \subseteq PMC(S) \) is the subspace consisting of arational measured foliations, and where for \( E, F \in AF \) one has \( E \sim F \) if and only if the underlying topological foliations are equivalent [Kla99].

The argument for our main result follows the outline of Klarreich’s paper, but the details are quite different; the difficulty comes from pushing an analogy between Outer space and Teichmüller space.

The paper is organized as follows. Relevant background about Outer space, very small \( F_N \)-trees, laminations, and \( F \) is found in Section 2. The proof of the main result can be roughly divided into four steps. The first step is to show that arational trees are indeed very close analogues of arational measured foliations on surfaces; this is accomplished in Sections 3 and 4. The main result is Theorem 4.4, a duality result, which says that if \( T \) is arational and shares a length 0 current with \( S \), then \( S \) is also arational and \( S \sim T \). The second step is to obtain control over the way that trees can fail to be arational; this is accomplished in Sections 5 and 6. Here, we bring a study of standard geodesics in Outer space, which serve as surrogates for Teichmüller geodesics, and the main result there is Lemma 6.10 which shows that if \( G_t \) is a folding line that converges to a tree \( T \notin AT \), then the image of \( G_t \) for large \( t \) in \( F \) is a uniformly bounded set. The last two steps involve running Klarreich’s argument and collecting some basic facts about the point set topology of the spaces \( AT, AT/\sim \), and \( \partial F \). This is the content of Section 7, where the main result is proved.

Technically, our arguments use the geometry of Outer space and folding paths as developed in [BF11], the structure theory of trees in \( \partial CV_N \) devo-
oped recently by Coulbois, Hilion, Lustig and Reynolds [CHL09, CHL08a, CHL08b, CHL08c, CHL07, Rey11, Rey10, Rey12, CHR11].

Note: Very recently, the main result of this paper was also announced by Hamenstädt [Ham12].

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2 Background

Let $F_N$ denote the free group of rank $N$. Throughout, we consider isometric actions of $F_N$ on $\mathbb{R}$-trees; all actions are assumed minimal. Let $T$ be a tree; a subset $I \subseteq T$ is called an arc if $I$ is isometric to a segment in $\mathbb{R}$. An arc is non-degenerate if it contains more than one point. For a subset $Y$ of an $F_N$-tree $T$, the stabilizer of $Y$, denoted $\text{Stab}(Y)$, is the set-wise stabilizer of $Y$. In this section, we collect some definitions and basic results.

2.1 Outer space and very small $F_N$-trees

A subset $Y \subseteq T$ that is the convex hull of three points is called a tripod if $Y$ is not a segment. An action $F_N \curvearrowright T$ on a tree $T$ is very small if for any non-degenerate arc $I \subseteq T$, either $\text{Stab}(I) = \{1\}$ or $\text{Stab}(I)$ is a maximal cyclic subgroup of $F_N$, and if for any tripod $Y \subseteq T$, $\text{Stab}(Y) = \{1\}$. An action $F_N \curvearrowright T$ is discrete (or simplicial) if the $F_N$-orbit of any point of $T$ is a discrete subset of $T$.

The unprojectivised Outer Space of rank $N$, denoted $\text{cv}_N$, is the topological space whose underlying set consists of free, minimal, discrete, isometric actions of $F_N$ on $\mathbb{R}$-trees. For $T \in \text{cv}_N$ we frequently consider the quotient graph $T/F_N$; this is a marked metric graph, i.e. there is an identification $\pi_1(T/F_N) \cong F_N$ defined up to conjugation and edges of $T/F_N$ have positive lengths.

A minimal $F_N$-tree is completely determined by its translation length function [CMS7]; this gives an inclusion $\text{cv}_N \subseteq \mathbb{R}^{F_N}$ and a topology on $\text{cv}_N$. The non-trivial points in the closure $\overline{\text{cv}_N}$ in $\mathbb{R}^{F_N}$ are very small isometric actions of $F_N$ on $\mathbb{R}$-trees [CL95, BF94]. The Culler-Vogtmann Outer space, denoted $\text{CV}_N$, is the image of $\text{cv}_N$ in the projective space $\mathbb{P}\mathbb{R}^{F_N}$; the points of $\text{CV}_N$ are thought of as free, simplicial $F_N$-trees of co-volume one. $\text{CV}_N$ is canonically a complex of simplices-with-missing-faces (which we simply
call *simplices*), with an (open) simplex corresponding to varying the lengths of edges (and keeping them positive) on a fixed marked graph. The closure $\overline{\text{CV}}_N$ of $\text{CV}_N$ in $\mathbb{P}R_F^N$ is compact and $\partial \text{CV}_N = \overline{\text{CV}}_N - \text{CV}_N$ is the projectivization of $\partial \text{CV}_N = \overline{\text{CV}}_N - \text{CV}_N$ and consists of very small trees that are either non-free or non-simplicial.

The group $\text{Out}(F_N)$ acts on $\text{cv}_N$, $\overline{\text{cv}}_N$, $\text{CV}_N$ and $\overline{\text{CV}}_N$: given a tree $T$ with length function $l_T$ and an element $\Phi \in \text{Out}(F_N)$, for $g \in F_N$, set $l_T(\Phi(g)) := l_T(\varphi(g))$, where $\varphi$ is any lift of $\Phi$ to $\text{Aut}(F_N)$.

Let $T \in \partial \text{cv}_N$, and let $H \leq F_N$ be finitely generated. If $H$ does not fix a point in $T$, then we let $T_H$ stand for the minimal $H$-invariant subtree of $T$; $T_H$ is the union of axes of hyperbolic elements of $H$. If $T$ has trivial arc stabilizers, which is always the case when $T$ has dense orbits, then for any finitely generated $H \leq F_N$, there is a unique minimal tree for $H$: either $T_H$ in the case of the previous sentence, or the unique fixed point of $H$, if $H$ contains no hyperbolic element.

### 2.2 Algebraic Laminations and Currents

We review algebraic laminations associated to $F_N$-trees; see \cite{CHL08a} and \cite{CHL08b} for details. Let $\partial F_N$ denote the Gromov boundary of $F_N$ — i.e. the boundary of any Cayley graph of $F_N$; boundaries of hyperbolic spaces are reviewed below (equivalently, $\partial F_N$ is the space of ends of $F_N$). Let $\partial^2(F_N) := \partial F_N \times \partial F_N \setminus \Delta$, where $\Delta$ is the diagonal. The left action of $F_N$ on a Cayley graph induces actions by homeomorphisms of $F_N$ on $\partial F_N$ and $\partial^2 F_N$. Let $i : \partial^2 F_N \to \partial^2 F_N$ denote the involution that exchanges the factors. A lamination is a non-empty, closed, $F_N$-invariant, $i$-invariant subset $L \subseteq \partial^2 F_N$.

Associated to $T \in \partial \text{cv}_N$ is a lamination $L(T)$, which is constructed as follows. Let

$$L_\epsilon(T) := \{(g^{-\infty}, g^{\infty}) | l_T(g) < \epsilon\}$$

and define $L(T) := \cap_{\epsilon > 0} L_\epsilon$. 

Let $T \in \partial \text{cv}_N$, and let $H \leq F_N$ be finitely generated. Then $H$ is virtually a retract of $F_N$ and, hence, is quasi-convex in $F_N$; so $\partial^2 H$ embeds in $\partial^2 F_N$. We say that $H$ carries a leaf of $L(T)$ if there is a leaf $l \in L(T)$ such that $l \in \partial^2 H$. We note that $H$ carries a leaf of $L(T)$ if and only if either some element of $H$ fixes a point in $T$, or the action $H \curvearrowright T_H$ is not discrete.

A *(measured geodesic)* current is an $F_N$-invariant Radon measure $\nu$ on $\partial^2 F_N$, i.e $\nu$ is a Borel measure that is finite on compact subsets of $\partial^2 F_N$. Let $\text{Curr}(F_N)$ denote the set of currents; equip $\text{Curr}(F_N)$ with the weak* topology. The group $\text{Out}(F_N)$ acts on $\text{Curr}(F_N)$ on the left as follows: let
$C \subseteq \partial^2 F_N$ be compact, let $\Phi \in \text{Out}(F_N)$, and let $\nu \in \text{Curr}(F_N)$, then $\Phi(\nu)(C) := \nu(\varphi^{-1}(C))$, where $\varphi \in \text{Aut}(F_N)$ is any lift of $\Phi$.

If $g \in F_N$ is such that the conjugacy class of $g$ does not contain an element of the form $h^k$ for $h \in F_N$ and $k > 1$, then there is a counting current, denoted $\eta_g$, associated to the conjugacy class of $g$. We also set $\eta_{g^k} = k\eta_g$ and frequently write $g$ instead of $\eta_g$. In [KL09], Kapovich and Lustig establish the following:

**Proposition 2.1.** [KL09, Theorem A] There is a unique $\text{Out}(F_N)$-invariant, continuous length pairing that is $\mathbb{R}_{\geq 0}$ homogeneous in the first coordinate and $\mathbb{R}_{\geq 0}$-linear in the second coordinate

$$\langle \cdot, \cdot \rangle : \text{cv}_N \times \text{Curr}(F_N) \to \mathbb{R}_{\geq 0}$$

Further, $\langle T, \eta_g \rangle = l_T(g)$ for all $T \in \text{cv}_N$ and all counting currents $\eta_g$.

The support $\text{Supp}(\nu)$ of a current $\nu$ is a lamination on $F_N$; $\text{Supp}(\nu)$ has an isolated point if and only if $\nu$ has an atom. Kapovich and Lustig give the following characterization of zero length:

**Proposition 2.2.** [KL10, Theorem 1.1] Let $T \in \text{cv}_N$, and let $\nu \in \text{Curr}(F_N)$. Then $\langle T, \nu \rangle = 0$ if and only if $\text{Supp}(\nu) \subseteq L(T)$.

We let $\mathbb{P}\text{Curr}(F_N)$ denote the space of projective classes (i.e. homothety classes) of currents. The action of $\text{Out}(F_N)$ on $\mathbb{P}\text{Curr}(F_N)$ is not minimal, but there is a unique minset $\mathbb{P}M_N \subseteq \mathbb{P}\text{Curr}(F_N)$ that is the closure of projective currents corresponding to primitive conjugacy classes of $F_N$ [Mar97, Kap06]; let $M_N$ denote the preimage of $\mathbb{P}M_N$ in $\mathbb{P}\text{Curr}(F_N)$.

### 2.3 Gromov Hyperbolic Spaces

We give a very brief review of Gromov hyperbolic spaces and their boundaries. Let $(X, d)$ be a metric space, and let $p \in X$ be a basepoint. For $x, y \in X$, the Gromov product of $x$ and $y$ (relative to $p$) is defined as

$$(x, y) = (x, y)_p := \frac{1}{2}(d(x, p) + d(y, p) - d(x, y))$$

The metric space $(X, d)$ is called Gromov hyperbolic if there is some $\delta \geq 0$ such that for any $x, y, z \in X$, one has

$$(x, z) \geq \min\{(x, y), (y, z)\} - \delta$$
If \( (X, d) \) is a geodesic metric space, then hyperbolicity of \( (X, d) \) also can be characterized by geodesic triangles being \textit{thin}.

If \( (X, d) \) is Gromov hyperbolic, then one says that a sequence of points \( \{x_n\} \) converges if \( (x_n, x_m) \to \infty \) as \( m, n \to \infty \). Two convergent sequences \( \{x_n\}, \{y_n\} \) are equivalent if \( (x_n, y_n) \to \infty \). The boundary \( \partial X \) of \( X \) is defined to be the collection of equivalence classes of convergent sequences in \( X \); two equivalence classes of sequences are close in \( \partial X \) if any pair of representatives have large Gromov product for all large \( n \). That all this is well-defined follows from hyperbolicity.

Given metric spaces \( (X, d) \) and \( (X', d') \) and a number \( C \), a function \( f : X \to X' \) is called a \textit{C-quasi-isometric embedding} if for all \( x, y \in X \)

\[
\frac{1}{C} d(x, y) - C \leq d'(f(x), f(y)) \leq Cd(x, y) + C
\]

The map \( f \) is a \textit{quasi-isometry} if in addition, for any \( z' \in X' \), there is \( z \in X \) such that

\[
d'(f(z), z') \leq C
\]

If the spaces \( X \) and \( X' \) are equipped with an action of a group \( G \), one arrives at the obvious notion of \textit{G-equivariant quasi-isometry}. Any quasi-isometry \( X \to X' \) between Gromov hyperbolic spaces induces a homeomorphism \( \partial X \to \partial X' \).

A \textit{quasi-geodesic} in \( X \) is a quasi-isometrically embedded copy of an interval of \( \mathbb{R} \). Two quasi-geodesic rays \( r, r' : [0, \infty) \to X \) with \( r(0) = r'(0) = 0 \) are \textit{equivalent} if their images have finite Hausdorff distance in \( X \). The boundary \( \partial X \) coincides with the collection of equivalence classes of quasi-geodesic rays (based at \( p \)), where two classes of rays are close if a pair of representatives stay close for a large initial segment of \( [0, \infty) \).

2.4 The Complex of Free Factors

The \textit{complex of free factors}, denoted \( \mathcal{F} \), has as vertices conjugacy classes of non-trivial proper free factors of \( F_N \), where conjugacy classes \([A^1], \ldots, [A^{k+1}]\) span a simplex in \( \mathcal{F} \) if and only if there are representatives \( A^1, \ldots, A^{k+1} \) such that after possibly reordering \( A^1 < \ldots < A^{k+1} \). Regard \( \mathcal{F} \) as a metric space by identifying each simplex with a standard simplex, and endow the resulting space with the path metric. Being its 1-skeleton, the \textit{graph of free factors} is quasi-isometric to the complex of free factors. When the rank \( N = 2 \) the complex \( \mathcal{F} \) is a discrete set, but after a natural modification of the definition it becomes homeomorphic to the Farey graph. In this paper we will always assume \( N \geq 3 \). We have:
Proposition 2.3. [BF11] Main Theorem] The metric space $\mathcal{F}$ is hyperbolic.

Throughout the sequel, we shall use the term factor to mean a conjugacy class of non-trivial, proper free factors of $F_N$; oftentimes, we will blur the distinction between conjugacy classes and the subgroups representing them, since we expect little confusion to arise from this. A conjugacy class of an element or a finitely generated subgroup of $F_N$ is simple if it is contained in a factor.

There is a coarsely well-defined projection $\pi : cv_N \to F_N$; associate to $T \in cv_N$ the collection of factors represented by subgraphs of $T/F_N$. It is noted in [BF11, Section 3] that $\pi(T)$ has diameter at most 4 and that if the volume of an immersion representing a factor $F$ in $T/F_N$ is uniformly bounded, then $d_F(\pi(T), F)$ is uniformly bounded as well. The projection $\pi$ descends to a projection $CV_N \to F_N$, also denoted $\pi$.

Given a number $K$, say that a function $\iota : [0, \infty) \to X$ is a reparameterized quasi-geodesic if there are $0 = t_0 < t_1 < \ldots < t_m < \ldots \in [0, \infty)$ such that $\text{diam}(\iota([t_i, t_{i+1}])) \leq K$ and $|i - j| \leq d(\iota(t_i), \iota(t_j)) + 2$.

Proposition 2.4. [BF11] Corollary 5.5 and Proposition 9.2] Let $G_t$ be a geodesic in $cv_N$. Then $\pi(G_t)$ is a reparameterized quasi-geodesic with uniform constant.

Here and throughout, the phrase uniform constant is taken to mean a constant that depend only on $N = \text{Rank}(F_N)$.

2.5 Geometry of Outer space

We now review the Lipschitz distance in $CV_N$, optimal maps, train track structures and folding paths. For more details the reader is referred to [FM11, Bes11, BF11].

A point of $CV_N$ can be thought of as a graph $G$ equipped with a marking $\pi_1(G) \cong F_N$ and a metric of volume 1. If $G, G' \in CV_N$ there is a canonical homotopy equivalence $G \to G'$ which commutes with markings. A map $f : G \to G'$ is a difference of markings if it belongs to this homotopy class and has constant slope on every edge. We denote by $\sigma(f)$ the largest slope, i.e. the Lipschitz constant for $f$, and we put

$$d(G, G') = \inf \log \sigma(f)$$

where $f$ ranges over all difference of markings. This is the Lipschitz distance in $CV_N$; it is not symmetric.
Any difference of markings map \( f : G \to G' \) with \( d(G, G') = \log \sigma(f) \) is called an optimal map – these always exist. If \( f : G \to G' \) is an optimal map, the union of all edges on which the slope of \( f \) is \( \sigma(f) \) is the tension graph \( \Delta_f \). Two directions (i.e. half-edges) in \( \Delta_f \) based at a vertex \( v \) are equivalent if \( f \) takes both to the same direction in \( G' \). Equivalence classes are gates.

A train track structure on a finite graph is a collection of equivalence relations, one on the set of directions at each vertex, such that at every vertex there are at least two gates. The tension graph may have vertices with only one gate, but there is always a subgraph \( \Delta \subset \Delta_f \) with an induced train track structure, and in fact \( \Delta = \Delta_g \) for a perturbation \( g \) of \( f \).

Let \( \Delta \) be a graph with a train track structure. A turn (i.e. a pair of directions at a vertex) is illegal if the two directions are equivalent, otherwise it is legal. A path in \( \Delta \) is legal if all turns it crosses are legal.

Let \( f : G \to G' \) be an optimal map. A loop \( \alpha \) in \( G \) is a witness (or it is maximally stretched) if \( l_{G'}(f_\ast(\alpha)) = \sigma(f)l_G(\alpha) \), where \( f_\ast(\alpha) \) is the immersed loop homotopic to \( f(\alpha) \). Equivalently, \( \alpha \) is contained in \( \Delta_f \) and it is legal. There is always a witness that crosses every edge at most twice and crosses at least one edge exactly once. In particular, such a loop has length \( < 2 \) and it represents the conjugacy class of a basis element of \( F_N \).

Now suppose that \( f : G \to G' \) is an optimal map with \( \Delta_f = G \) and with \( \geq 2 \) gates at every vertex. For this discussion it is convenient to rescale \( G \) so that the slope of \( f \) is 1 on every edge. Thus we are now viewing \( G, G' \) as elements of \( \text{cv}_N \).

Set \( G_0 = G \) and for small \( t \geq 0 \) let \( G_t \) be obtained from \( G \) by identifying initial segments of length \( t \) within each gate. We have natural factorizations \( G_0 \to G_t \to G' \). A path \( G_t, t \in [0, L] \) in \( \text{cv}_N \) from \( G \) to \( G' \) induced by \( f \) is a (greedy) folding path (induced by \( f \)) if \( G_0 = G, G_L = G' \) and for \( t \leq t' \) there are maps \( f_{t,t'} : G_t \to G_{t'} \) such that \( f_{t,t} = \text{id}, f_{0,L} = f \) and \( f_{t_2,t_3}f_{t_1,t_2} = f_{t_1,t_3} \), and so that for any \( t_0 < L \) the path \( G_t, t \in [t_0, t_0 + \epsilon] \) is obtained as above by identifying small segments within each gate with the induced maps \( G_{t_0} \to G_t \). We refer to this particular parametrization as the natural parametrization. Given \( f : G \to G' \) as above, there is a unique folding path induced by \( f \).

The image of a folding path in \( \text{cv}_N \) is a folding path in \( \text{cv}_N \), usually parametrized by arc-length. Every folding path is a geodesic, i.e. for \( t_1 < t_2 < t_3 \) we have \( d(G_{t_1}, G_{t_0}) = d(G_{t_1}, G_{t_2}) + d(G_{t_2}, G_{t_0}) \), but there are many geodesics that are not folding paths. In fact, not every pair of points in \( \text{cv}_N \) can be connected by a folding path. However, there is always a standard geodesic joining a given pair of points: it is a geodesic which is the
concatenation of a path inside a simplex and a folding path.

3 Laminations and Dendrites

An $F_N$-tree $T \in \partial cv_N$ is called indecomposable if for any non-degenerate arcs $I, J \subseteq T$, there are $g_1, \ldots, g_r \in F_N$ such that $I \subseteq g_1 J \cup \ldots \cup g_r J$ and such that $g_i J \cap g_{i+1} J$ is non-degenerate. The goal of this section is to prove the following maximality condition about laminations associated to indecomposable trees.

**Proposition 3.1.** Let $T \in \partial cv_N$ be indecomposable. If $U \in \partial cv_N$ satisfies $L(T) \subseteq L(U)$, then $L(T) = L(U)$.

To prove this fact, we will need to consider actions by homeomorphisms of $F_N$ on dendrites, which are compact, locally connected, uniquely arcwise connected metrizable spaces, see e.g. [Why63]. The connection to actions in $\partial cv_N$ comes from [CHL07].

The weak topology, also called the observers’ topology in [CHL07], on $T$ has as subbasis the collection of directions (i.e. complementary components) at points of $T$; let $T_w$ denote $T$ with the weak topology. Let $\overline{T}$ be the metric completion of $T$. Then there are two topologies on $\overline{T} = \overline{T} \cup \partial T$: the Gromov (metric) topology and the weak topology $\overline{T}_w$ defined in the same way as on $T$. The weak topology is weaker than the metric topology, and $\overline{T}_w$ is a dendrite. It is shown in [CHL07] that if $T$ has dense orbits, then the quotient space $\partial F_N / L(T)$ is homeomorphic to $\overline{T}_w$. There is a natural embedding of $T_w$ into $\overline{T}_w$; note that $T_w$ is uniquely arcwise connected but is not compact. The action of $F_N$ on $T$ induces an action by homeomorphisms on $\overline{T}_w$ for which $T_w$ is invariant.

Note that $T_w$ is the subspace consisting of points of $\overline{T}_w$ that are contained in the interior of an embedded path in $\overline{T}_w$, that is, the set of points $x$ of $\overline{T}_w$ that are separating. Call the points of $\overline{T}_w \setminus T_w$ endpoints. Connected subsets of $\overline{T}_w$ are path connected. Since the metric topology agrees with the weak topology on finite subtrees of $T$, we have that segments in $T$ are segments in $T_w$, and tripods in $T$ are tripods in $T_w$. Hence the action of $F_N$ on the space $T_w$ is very small. Any segment in $\overline{T}_w$ with endpoints in $\overline{T}_w \setminus T_w$ meets $T_w$ in an open dense sub-segment. If $T$ is indecomposable, then so is $T_w$.

**Proposition 3.2.** Let $p : X \to Y$ be a surjective map between two dendrites. Assume that:
(i) $X = \hat{T}_w$ for $T \in \partial cv_N$ indecomposable, and

(ii) $F_N$ acts on $Y$, and $p$ is $F_N$-equivariant.

Then one of the following holds:

(a) $p$ is a homeomorphism,

(b) $Y$ is a point, or

(c) there is an open interval $Z \subset Y$ such that for every $z \in Z$ we have $|p^{-1}(z)| > 2$.

Before we begin the proof we will make an observation. Assume that the conclusion of the above proposition fails. Suppose $[a, b]$ and $[c, d]$ are two segments in $X$ with $[a, b] \cap [c, d] = [u, v]$ a nondegenerate segment.

Claim 3.3. Assume that the conclusion of the above proposition fails. Suppose $[a, b]$ and $[c, d]$ are two segments in $X$ with $[a, b] \cap [c, d] = [u, v]$ a nondegenerate segment. If $p(a) = p(b)$ and if $p(c) = p(d)$, then $p(u) = p(v)$.

The proof of the claim uses only that dendrites are uniquely arcwise connected.

Proof. First note that up to symmetry, there are 3 possible configurations, which are shown in Figure 1.

![Figure 1](image-url)

We will contradict the assumption that Proposition 3.2 fails.

For the first configuration, if $p(a) = p(b) = r$ and $p(c) = p(d) = s$ but $r \neq s$, then take for $Z$ the open interval $(r, s)$. Every $z \in Z$ has a preimage point in each interval $(a, c), (c, b), (b, d)$, a contradiction.

In the second configuration, if $p(u) \neq p(v)$, take $Z = (p(u), p(v))$, and notice that the preimage of each $z \in Z$ intersects $[a, b]$ in at least two points, and $[c, u] \cup [d, v]$ in at least one point, a contradiction.

In the last configuration, if $p(u) \neq p(d) = p(c)$ take $Z = (p(u), p(d))$. Each $z \in Z$ has at least two preimages in $(a, b)$ and at least one in $(c, u)$, a contradiction.
Proof of Proposition 3.2. First note that if \( p \) collapses a nondegenerate segment, then indecomposability of \( X \) and equivariance forces \( p \) to be constant, implying that \( Y \) is a point. So assume that \( p \) does not collapse any nondegenerate interval and that \( p \) is not a homeomorphism. This gives that \( p \) is not injective, so there are distinct \( a, b \in X \) with \( p(a) = p(b) \). By the pidgeon hole principle, after replacing \([a, b]\) by a smaller interval, we may assume that \( a, b \in T \) and that \( p(a) = p(b) \) has valence 2; indeed, since \( Y \) has a countable basis, \( Y \) contains at most countably many points with valence \( > 2 \) (this is a theorem of Whyburn).

Again by the pidgeon hole principle, there are distinct \( c, d \in [a, b] \) with \( p(c) = p(d) \) and with the \( T \)-distance between \( c, d \) arbitrarily small. Apply indecomposability to \( I = [a, b] \) and \( J = [c, d] \) to deduce that \( I \subset \cup g_i(J) \) for \( i = 1, 2, \ldots, k \) and \( g_i(J) \cap g_{i+1}(J) \) is nondegenerate. We may also assume that \( k \) is minimal, so in particular \( g_1(J) \) and \( g_k(J) \) will contain the endpoints of \([a, b]\). Apply the Claim to the segments \( I \) and \( g_i(J) \) (by equivariance, the endpoints of \( g_i(J) \) are mapped to the same point). Thus the endpoints of \( g_i(J) \cap I \) map to the same point \( y_i \) in \( Y \). We now claim that \( p(a) = p(b) = y_1 = \cdots = y_k \). It is clear that \( p(a) = y_1 \) since \( a \) is an endpoint of \( g_1(J) \cap I \) (up to switching \( a \) and \( b \)) and similarly \( p(b) = y_k \). To see that \( y_1 = y_2 \) apply the Claim to \( g_1(J) \) and \( g_2(J) \) etc.

We now have points \( a = t_0 < t_1 < \cdots < t_m = b \) in \([a, b]\) with \( p(t_i) = y \) for every \( i \). We may take \( m \) as large as we want by making \( J \) small. The images of the intervals \([t_i, t_{i+1}]\) are dendrites \( D_i \) containing \( y \), and since the valence of \( y \) is 2, as soon as we have \( m \geq 3 \) two of the dendrites, say \( D_i \) and \( D_j \), will have nondegenerate overlap. Take \( Z \) to be an open interval in the overlap. Then any point in \( Z \) will have at least two preimages in \([t_i, t_{i+1}]\) and at least two in \([t_j, t_{j+1}]\). \( \square \)

We are now in position to prove Proposition 3.1. For the proof we will need the main result of \([\text{CH}]\); we note that if \( T \in \partialcv_N \) has dense orbits, then the map \( Q \) used to define \( Q\)-index in \([\text{CH}] \) is the quotient map \( Q = Q_T : \partial F_N \to \partial F_N / L(T) = T_w \). Here is a simplified version of the result of Coulbois-Hilion:

**Proposition 3.4.** \([\text{CH}]\). Theorem 5.3] Let \( T \) be a very small \( F_N \)-tree with dense orbits, and let \( Q = Q_T : \partial F_N \to \partial F_N / L(T) = T_w \). Then there are at most countably many points \( z \in T_w \) for which \(|Q^{-1}(z)| > 2 \).

**Remark 3.5.** If \( T \) in the statement of Proposition 3.1 is of pseudo-surface type (defined in \([\text{CH}]\)), then the statement follows immediately from \([\text{CH}]\).
Theorem 5.10], and for geometric trees the statement follows from the quasi-isometric classification of leaves. Our proof of Proposition 3.1 is new in the indecomposable fake-Levitt case.

Proof of Proposition 3.1. Suppose that $U \in \partial \text{cv}_N$ satisfies $L(T) \subseteq L(U)$. It follows from [Lev94] that $U$ can be assumed to have dense orbits; indeed, if $U$ does not have dense orbits, then we can collapse the simplicial part of $U$ to get a tree with dense orbits, and one easily sees from the definition of $L(\cdot)$ that the associated lamination can only be enlarged; see [Lev94] or [Rey12] for details. One has that the quotient map $\partial F_N \to \partial F_N / L(U) = \hat{U}_w$ factors through $\partial F_N \to \partial F_N / L(T) = \hat{T}_w$, so we get a surjective map $p : \hat{T}_w \to \hat{U}_w$, which is $F_N$-equivariant.

Now apply Proposition 3.2. Since $U$ contains more than one point, conclusion (b) is not possible. If conclusion (c) holds, then there are uncountably many points of $\hat{U}$ whose pre-image in $\partial F_N$ contains strictly more than two points; but this is impossible by Proposition 3.4. Hence, $p$ is a homeomorphism, so $L(T) = L(U)$.

4 Arational Trees

We recall a notion of reduction for very small trees, introduced in [Rey12]. For $T \in \partial \text{cv}_N$ and $F$ a factor, say that $F$ reduces $T$ if $F$ acts with dense orbits on some subtree $Y \subseteq T$. It should be emphasized that $Y$ can consist of a single point. If $Y$ contains two points, then $Y$ necessarily has infinite diameter, and in this case the minimal subtree $T_F$ for $F$ is dense in $Y$.

Use $\mathcal{R}(T)$ to denote the set of all factors reducing $T$. It is noted in [Rey12] that if $F'$ is a factor carrying a leaf of $L(T)$, then there is $F \in \mathcal{R}(T)$ with $F \leq F'$; so, regarded as subsets of $\mathcal{F}$, $\mathcal{R}(T)$ is 1-dense in the set of all factors carrying a leaf of $L(T)$.

When $\mathcal{R}(T) = \emptyset$, the tree $T$ is arational; this is equivalent to the statement that no leaf of $L(T)$ is carried by a factor by [Rey12]. Toward establishing an intuitive analogy with surfaces, we note that the analogous laminations are precisely the arational laminations—i.e. the minimal and filling laminations.

We have the following classification of arational trees:

**Proposition 4.1.** [Rey12, Theorem 1.1] Let $T \in \partial \text{cv}_N$. The following are equivalent:

(i) $T$ is arational,
(ii) \( T \) is indecomposable, and if \( T \) is not free, then \( T \) is dual to an arational measured lamination on a surface with one boundary component.

When \( T \) is geometric one can prove the equivalence of (i) and (ii) using the methods of Section 6, see particularly the proof of Lemma \[6.12\]. When \( T \) is non-geometric, one forms a geometric resolution and a folding path converging to \( T \). In general, under folding parts of the approximating graphs become geometric. When \( T \) is also arational, the folding path gives a sequence of strong approximations without forward invariant subgraphs whose unrescaled volumes go to 0, and \( T \) is free and indecomposable (see the proof of Lemma \[6.16\] where a similar argument is used).

If \( X \subseteq \partial^2 F_N \), say that a leaf \( l = (x, y) \in \partial^2 F_N \) is diagonal over \( X \) if there are leaves \((x_1, x_2), (x_2, x_3), \ldots, (x_{r-1}, x_r) \in X\), such that \( x = x_1 \) and \( y = x_r \); and say that \( X \) is diagonally closed if every leaf that is diagonal over \( X \) belongs to \( X \). Laminations associated to trees are always diagonally closed [CHL08b].

We collect the following information about laminations associated to arational trees; for the statement, \( L'(T) \) denotes the Cantor-Bendixson derivative of \( L(T) \), i.e. \( L'(T) = (L(T))' \) is the set of non-isolated points of \( L(T) \), and we set \( L''(T) = (L'(T))' \) and \( L'''(T) = (L''(T))' \).

**Proposition 4.2.** Let \( T \in \partial \text{cVN} \).

(i) If \( T \) is free and indecomposable, then \( L'(T) \) is minimal, no leaf of \( L'(T) \) is carried by a factor, and \( L(T) \) is obtained from \( L'(T) \) by adding finitely many \( F_N \)-orbits of isolated leaves, each of which is diagonal and not periodic.

(ii) If \( T \) is dual to an arational measured lamination on a surface with one boundary component, then \( L'''(T) \) is minimal, no leaf of \( L(T) \) is carried by a factor, and \( L(T) \) is the smallest diagonally closed lamination containing \( L'''(T) \).

**Proof.** Statement (i) follows from the main results of [Rey11] and [CHR11]. Statement (ii) follows from straightforward considerations about surface laminations and foliations, and the reader is assumed to have some familiarity with this; see [CB88] and [CHL08b] for background and the next paragraph for details.

Let \( S \) be a hyperbolic surface with one totally geodesic boundary component, equipped with a filling, minimal geodesic measured lamination \( \Lambda \) to which \( T \) is dual. The universal cover \( \tilde{S} \) can be identified with a closed convex subset of \( \mathbb{H}^2 \) whose boundary consists of lines covering \( \partial S \) and the
set of ends of $\tilde{S}$ is $\partial F_N$. By construction, the lift $\tilde{\Lambda}$ of $\Lambda$ gives a subset of the dual lamination $L(T)$. Now $\tilde{\Lambda}$ is not diagonally closed, and we will enlarge it by adding diagonal leaves. A complementary component of $\tilde{\Lambda}$ containing a boundary component of $\tilde{S}$ is the universal cover of the *crown set* (see [CBSS]), i.e. it is obtained from a hyperbolic half-plane invariant under a hyperbolic isometry $g$ (the deck transformation preserving the boundary) by deleting a pairwise disjoint collection of hyperbolic half-planes invariant under $g$, with adjacent half-spaces cobounding cusps. Thus the set of ends of this region can be identified with $\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$ where $\pm \infty$ correspond to the ends of the boundary component and elements of $\mathbb{Z}$ to the cusps.

Since $L(T)$ is closed and diagonally closed, we see that it contains $F_N$-orbits of leaves connecting any pair of distinct points in $\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$.

To see that $L(T)$ contains no additional leaves, observe that any biinfinite geodesic that does not belong to the collection just described must intersect $\Lambda$ transversally, so it gets nonzero measure and does not belong to $L(T)$. Finally, we have that $L'(T)$ is obtained from $L(T)$ by deleting isolated leaves, and these are in the orbit of leaves connecting $m$ and $n$ for $|m-n| > 1$, then $L''(T)$ is obtained from $L'(T)$ by further deleting orbits of leaves connecting $n$ to $\pm \infty$, and lastly, $L'''(T)$ is obtained by deleting the orbit of the boundary.

To see that no leaf is carried by a factor $A$ consider the lift of $\Lambda$ and the added leaves to the $A$-cover of $S$ and restrict to the convex core; here, the lifts of leaves of $\Lambda$ must be compact, c.f. [Rey11].

Hence, we get the following:

**Corollary 4.3.** Let $T, U \in \partial \text{cv}_N$. If $T$ is arational and if $L'''(T) \subseteq L(U)$, then $L(T) = L(U)$.

**Proof.** Using Proposition 4.2, we get that $L'''(T) \subseteq L(U)$ implies that $L(T) \subseteq L(U)$. Apply Proposition 3.1 to conclude. $\square$

For any $T \in \overline{\text{CV}}_N$ we define

$$T^* = \{\mu \in \mathcal{P}M_N \mid \langle T, \mu \rangle = 0\}$$

Thus if $T \in \text{CV}_N$ then $T^* = \emptyset$. An elementary limiting argument shows that if $T \in \partial \text{CV}_N$ then $T^* \neq \emptyset$, but for efficiency of the exposition this is postponed to Remark 4.7 below.

**Theorem 4.4.** Let $T \in \partial \text{cv}_N$. If $T$ is arational, and if $\mu \in M_N$ satisfies $\langle T, \mu \rangle = 0$, then $\text{Supp}(\mu) = L'''(T)$. In particular, if $U$ is another very small tree satisfying $\langle U, \mu \rangle = 0$, then $U$ is arational, $L(T) = L(U)$ and $T^* = U^*$.  

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Proof. If \( \mu \in M_N \) satisfies \( \langle T, \mu \rangle = 0 \), then Proposition 2.2 gives that \( \text{Supp}(\mu) \subseteq L(T) \). The support of a current cannot contain non-periodic isolated leaves, since translates of such leaves have accumulation points (recall that currents are Radon measures).

We now show that the support of a current in \( M_N \) cannot contain a periodic leaf corresponding to a conjugacy class not carried by a factor. Here we will use the fact that a conjugacy class \( g \) is not simple if and only if there is a basis with respect to which the Whitehead graph of \( g \) is connected and has no cut points.

Choose a basis for \( F_N \), and let \( \mu_n \) be a sequence of currents corresponding to primitive conjugacy classes \( g_n \) with \( \mu_n \) converging to \( \mu \). Each \( g_n \) has Whitehead graph that either is disconnected or has a cut point. After passing to a subsequence, \( g_n \) all have the same Whitehead graph \( W \). If \( l \) is a periodic leaf of \( \text{Supp}(\mu) \) corresponding to \( g \in F_N \), then \( g_n \) contains \( g^2 \) as a subword; it follows that the Whitehead graph for \( g \) is contained in \( W \), so \( g \) is carried by a factor (since the statement about Whitehead graphs is true for every basis). We conclude by Proposition 4.2 that \( \text{Supp}(\mu) \subseteq L''(T) \), and hence \( \text{Supp}(\mu) = L''(T) \).

Now let \( U \in \partial cv_N \) be some other tree such that \( \langle U, \mu \rangle = 0 \). Again by Proposition 2.2, we have that \( L''(T) \subseteq L(U) \). Apply Corollary 4.3 to conclude.

Corollary 4.5. Let \( S, T \in \partial cv_N \). If \( S^* \subseteq T^* \) and \( \mathcal{R}(S) \neq \emptyset \), then \( \mathcal{R}(S) \cap \mathcal{R}(T) \neq \emptyset \).

Proof. First suppose that there is a factor \( F \) fixing a point in \( S \). Choose a basis for \( F \), and note that \( \{ \eta_g | g \text{ has length } \leq 2 \} \subseteq S^* \subseteq T^* \), so by Serre’s Lemma \( F \) fixes a point in \( T \) as well.

If no factor fixes a point in \( S \), then one finds \( F \in \mathcal{R}(S) \) such that \( F \sim S_F \) is arational (e.g. choose \( F \) of minimal rank). Let \( \mu \) be a current that is supported on \( L''(S_F) \); we have that \( \text{Supp}(\mu) \) fills \( F \) and that \( \mu \in S^* \subseteq T^* \). Hence, by Proposition 2.2 \( L''(S_F) \subseteq L(T) \), so by Theorem 4.4 either \( F \) acts arationally on \( T_F \) or else \( F \) fixes a point in \( T \); in either case, \( F \in \mathcal{R}(T) \).

We obtain the following result, which suggests that arational trees lie “at infinity” with respect to \( \mathcal{F} \).

Corollary 4.6. Let \( T_n \in CV_N \) be a sequence of trees converging to an arational tree \( T \), and let \( Y_n = \pi(T_n) \) denote a projection to \( \mathcal{F} \). For any basepoint \( 0 \in F \), we have \( d(0, Y_n) \rightarrow \infty \).

We follow Feng Luo’s argument, an adaptation of [Kob88].
Proof. We may assume that $Y_n$ is a factor generated by a uniformly bounded loop in $T_n/F_N$ representing a conjugacy class $g_n$. This guarantees that $Y_n$ is at uniformly bounded distance from $\pi(T_n)$. Choose a basepoint $0 \in \mathcal{F}$. Toward contradiction, suppose that $d(0, Y_n)$ does not go to infinity; by passing to a subsequence, we can assume that $d(0, Y_n) = r$ for every $n$. Then there are paths $0 = A^0_n, A^1_n, \ldots, A^{r-1}_n, A^r_n = Y_n \in \mathcal{F}$.

Choose simplicial trees $T^i_n$ in which $A^i_n$ is elliptic and let $g^i_n$ be conjugate into both $A^i_n$ and $A^{i-1}_n$ for $i = 1, \ldots, r$; also arrange that $g^r_n = g_n$ and that $T^0_n$ does not depend on $n$. After possibly passing to a further subsequence and rescaling, we can assume that $\lim T^i_n = T^i \in \partial CV_N$ and $\lim g^i_n = \eta^i \in PM_N$ for every $i$. By the Kapovich-Lustig continuity we have $\langle T^i, \eta^i \rangle = 0 = \langle T^{i-1}, \eta^i \rangle$ so by induction on $i$ and using Theorem 4.4 we see that none of $T^i$'s are arational and in particular $\eta^r$ does not have length 0 in any arational tree.

To get a contradiction, we argue $\langle T, \eta^r \rangle = 0$. By construction, $\langle T_n, g_n \rangle \leq C$ so $\langle T_n/\mu_n, g_n/|g_n| \rangle \leq C/\mu_n|g_n|$ where $|g_n|$ is the length of $g_n$ in a fixed rose, and $\mu_n$ are rescaling constants so that $T_n/\mu_n \to T$. Again by the continuity of the length pairing and the fact that both $|g_n|$ and $\mu_n$ are bounded below, it suffices to argue that $|g_n| \to \infty$, i.e. that it is possible to choose $g_n$'s to be all distinct after a subsequence. If not, then after a subsequence all $T_n$ belong to the same simplex and the limit $T$ is a simplicial tree with trivial edge stabilizers, so certainly not arational. \square

Remark 4.7. A similar argument shows that if $T \in \partial CV_N$ then $T^* \neq \emptyset$. If $T$ is in the closure of a simplex there are simple elliptic elements, and otherwise construct $\mu \in T^*$ as the limit of $g_n/|g_n|$ as above.

5 Primitive Elements and Vertex Groups

By a vertex group we mean a vertex stabilizer in a very small simplicial $F_N$-tree. We associate to $g \in F_N$, respectively $A \leq F_N$, the smallest free factor containing it, which we denote by $\text{Fill}(g)$, respectively $\text{Fill}(A)$; $g$ ($A$) is simple if $\text{Fill}(g)$ ($\text{Fill}(A)$) is a proper subgroup of $F_N$, hence we get a map $\text{Fill} : \{\text{non-trivial simple elements (subgroups)}\} \to \mathcal{F}$.

Lemma 5.1. There is a constant $C$ such that for every very small simplicial $F_N$-tree $T$, the set of simple elements fixing a point of $T$ map under $\text{Fill}$ to a set of diameter at most $C$ in $\mathcal{F}$.

We assume that the reader is familiar with Whitehead’s algorithm [LS01]. The argument is an adaptation of the proof of [BF11, Lemma 3.2].
Proof. Let $T \in \partial \text{cv}_N$ be simplicial. If $T$ has an edge $e$ with trivial stabilizer, then collapsing every edge not in the orbit of $e$ gives a tree $T'$ corresponding to a free splitting of $F_N$, and every simple elliptic element of $T$ is elliptic in $T'$. The image under $\text{Fill}$ of the simple elliptic elements of $T'$ has diameter at most 2 in $\mathcal{F}$.

So, assume that $T$ has no edge with trivial stabilizer; collapse edges outside of a fixed orbit of edges and replace $T$ with the resulting 1-edge splitting. This increases the diameter of the image of $\text{Fill}$ by at most 2. We have two cases to consider, corresponding to whether $T/F_N$ is a segment or a loop.

First suppose that $T/F_N$ is a segment, so $T$ corresponds to a splitting $A \ast_w B$. By Swarup’s theorem [Swa86] (see also [BF94, Lemma 4.1]), we have, possibly after interchanging $A$ and $B$, that $A = \langle a_1, \ldots, a_k, w \rangle$ and $B = \langle b_1, \ldots, b_l \rangle$, where $\{a_1, \ldots, a_k, b_1, \ldots, b_l\}$ is a basis for $F_N$ and $w \in B$.

If $w$ is contained in a factor $B'$ of $B$, then $A$ is contained in the factor $\langle a_1, \ldots, a_k \rangle \ast B' \leq F_N$ and the Lemma follows. So assume that $w$ fills $B$, and after possibly changing the basis of $B$, that the Whitehead graph of $w$ with respect to the basis $\{b_i\}$ is connected and has no cut points.

Now let $g$ be a simple conjugacy class in $A$, i.e. a cyclic word in the $a_i$’s and $w$. If $w$ does not appear, the image of $\text{Fill}(g)$ is at distance $\leq 1$ from $\langle a_1, \ldots, a_k \rangle$ and we are done. Likewise, if some $a_i$ does not appear, then $\text{Fill}(g)$ is at distance $\leq 1$ from $\langle a_1, \ldots, \hat{a_i}, \ldots, a_k \rangle \ast B$, hence at distance $\leq 4$ from $\langle a_1, \ldots, a_k \rangle$, and, again, we are done.

We now apply the Whitehead algorithm that transforms $g$ to a cyclic word that does not involve all the basis elements. At each step a Whitehead automorphism is applied whose effect on $g$ is that it gets shorter. The Whitehead automorphism can be read off from the Whitehead graph of $g$ in the basis $\{a_i\} \cup \{b_j\}$, which has a cut vertex called the special letter. Note that the Whitehead graph of $g$ contains as a subgraph the Whitehead graph of $w$ with one edge removed (since $w$ doesn’t get “closed up” in $g$) and this subgraph is connected by our assumption on $w$.

If the special letter is $a_i^\pm 1$, then all $b_j^\pm 1$’s are on one side of the cut vertex, and therefore $B$ is either fixed or gets conjugated by $a_i^\pm 1$. The word $w$ inside $g$ stays unaffected.

Now assume the special letter is $b_j^\pm 1$. Say $w = wx_1y$ as a word in $\{b_j\}$. Thus $x \neq y^{-1}$, and in the Whitehead graph $W$ of $g$ with respect to $\{a_i\} \cup \{b_j\}$, $b_j^\pm$ is a cut vertex. The Whitehead graph of $g$ with respect to $\{a_1, \ldots, a_k, w\}$ is obtained from $W$ by removing all vertices $b_j^\pm$ except for $x$ and $y^{-1}$ and all edges incident to them, and renaming $x$ to $w$ and $y^{-1}$ to
Therefore this Whitehead graph is disconnected and \( g \) is contained in a factor \( C \) complementary to \( B \), so the Lemma again follows. (The factor \( C \) is obtained from \( \langle a_1, \ldots, a_k \rangle \) by applying the Whitehead automorphism with special letter \( w \). Topologically, one can blow up the rose on \( a_1, \ldots, a_k, w \) by inserting an ideal edge which is not crossed by \( g \); then in the graph of spaces corresponding to \( A +_w B \) collapse the 2-cell from this ideal edge to get a graph representing \( F_N \), containing the rose on \( b_1, \ldots, b_l \) and a representative of \( g \) disjoint from this rose.)

To summarize, the Whitehead algorithm runs as long as some \( a_i^{\pm 1} \) is a special letter and during this time \( B \) is fixed up to conjugacy. When some \( a_i \) is completely erased from \( g \) or some \( b_j^{\pm 1} \) becomes the special letter, we are done by the above discussion.

When \( T/F_N \) is a loop the argument is similar: we can write the vertex group as \( \langle a_1, \ldots, a_{N-1}, w^c \rangle \) where \( F_N = \langle a_1, \ldots, a_{N-1}, c \rangle \) and \( w \in \langle a_1, \ldots, a_{N-1} \rangle \). Again we may assume that the Whitehead graph of \( w \) in \( \langle a_1, \ldots, a_{N-1} \rangle \) is connected and has no cut points. Let \( g \) be simple and written as a cyclic word in \( a_1, \ldots, a_{N-1}, w^c \). We may assume it involves all of these generators. If \( w = xw_1y \) then the Whitehead graph of \( w \) in the basis \( \{ a_1, \ldots, a_{N-1}, c \} \) is obtained from the Whitehead graph of \( w \) in \( \{ a_1, \ldots, a_{N-1} \} \) by removing an edge joining \( y^{-1} \) and \( x \), adding edges from \( c^{-1} \) to \( x \) and from \( c^{-1} \) to \( y^{-1} \) and perhaps adding more edges. In particular, the subgraph spanned by the \( a_i^{\pm 1} \) and by \( c^{-1} \) is connected and has no cut points. Since \( c \) and \( c^{-1} \) are not connected to each other, neither \( c \) nor \( c^{-1} \) can be cut points. If say \( a_i \) is a cut point, \( c \) is the only vertex on one side and the associated automorphism is of the form \( c \mapsto ca_i \) and all \( a_j \) fixed. It follows that \( a_i \) is either \( x^{-1} \) or \( y \); either way the automorphism preserves \( \langle a_1, \ldots, a_{N-1} \rangle \) and conjugates \( w \). The proof concludes as before.

To extend Lemma [5.1] to all trees in \( \partial cv_N \), we use the following result. Recall that for \( T \in \partial cv_N \), \( \mathcal{R}(T) \) denotes the collection of all factors reducing \( T \).

**Proposition 5.2.** [Rey12, Theorem 1.3] Let \( T \in \partial cv_N \), and assume that \( T \) is not arational. There is a simplicial tree \( T_0 \) such that for any \( F \in \mathcal{R}(T) \), some element of \( F \) fixes a point in \( T_0 \).

It follows that the diameter of \( \mathcal{R}(T) \) in \( \mathcal{F} \) is at most two more than the diameter of the \( \text{Fill} \)-image of the set of simple elliptic elements in \( T_0 \), hence we get:

**Corollary 5.3.** Let \( T \in \partial cv_N \), and assume that \( T \) is not arational. The set \( \mathcal{R}(T) \) has uniformly bounded diameter in \( \mathcal{F} \).
6 Sequences of Geodesics

In this section we examine possible accumulation sets of sequences of geodesics in CV
\(N\). The main result is Theorem 6.6.

6.1 Limits of Sequences of Geodesics

Fix a basis \(B\) for \(F_N\); for \(g \in F_N\), let \(|g|\) denote the word length of \(g\) in \(B\). We work in Outer space with graphs normalized to have volume 1, and we sometimes consider universal covers. The following is essential for the remainder of the paper:

**Remark 6.1.** All (projectivized) currents come from \((\mathbb{P}M_N) M_N\).

For a tree \(T \in \text{cv}_N\) and for \(g \in F_N\), we use \(\langle T, g \rangle\) to mean \(\langle T, \eta_g \rangle\), which is the translation length of \(g\) in \(T\) by Proposition 2.1. We will use Proposition 2.1 below without reference.

Suppose that we have a sequence of geodesics \([S_n, T_n]\) in \(\text{cv}_N\). We assume that \(d(S_n, T_n) = \log \lambda_n\), that \(T_n/\mu_n\) converges to \(T\) and that \(S_n/\kappa_n\) converges to \(S\) for some \(\lambda_n, \mu_n, \kappa_n \geq 1\).

**Lemma 6.2.** In this situation, \(\inf \frac{\kappa_n \lambda_n}{\mu_n} > 0\).

**Proof.** Fix some \(g \in F_N\). Then \(\lambda_n \langle S_n, g \rangle \geq \langle T_n, g \rangle\) i.e.

\[
\frac{\kappa_n \lambda_n}{\mu_n} \langle S_n/\kappa_n, g \rangle \geq \langle T_n/\mu_n, g \rangle.
\]

Passing to the limit and assuming \(\kappa_n \lambda_n/\mu_n \to 0\) gives \(\langle T, g \rangle = 0\), which is impossible, since \(g\) was arbitrary. \(\square\)

**Lemma 6.3.** If \(\frac{\kappa_n \lambda_n}{\mu_n}\) is bounded above then \(T^* \supseteq S^*\).

**Proof.** We may assume \(\kappa_n \lambda_n/\mu_n = 1\). As above, for any \(g \in F_N\) we have \(\langle S, g \rangle \geq \langle T, g \rangle\). Let \(\nu\) be some current, and let \(g_n \in F_N\) be such that \(g_n/|g_n| \to \nu\). Passing to the limit gives that \(\langle S, \nu \rangle = 0\) implies that \(\langle T, \nu \rangle = 0\).

The same line of reasoning shows:

**Lemma 6.4.** Let \(S,T \in \text{cv}_N\). If there is a Lipschitz map \(S \to T\), then \(S^* \subseteq T^*\).
To the sequence of geodesics $[S_n, T_n]$ we associate a closed subset of projectivized measured currents $C([S_n, T_n])$ defined as the set of (projective classes of) those $\nu$ that can be represented as $\lim \gamma_n/|\gamma_n|$ where $\gamma_n$ is a maximally stretched simple loop in $S_n$ (i.e. a legal simple loop in the tension graph for some optimal map $S_n \to T_n$, see Section 2.5). Without further comment, we always allow passing to a subsequence of $[S_n, T_n]$.

**Lemma 6.5.** Let $[S_n, T_n]$ be geodesics such that $d(S_n, T_n) = \log \lambda_n, T_n/\mu_n \to T$ and such that $S_n/\kappa_n \to S$. If $\kappa_n \lambda_n/\mu_n \to \infty$, then $C([S_n, T_n]) \subseteq S^*$.

**Proof.** Since $\gamma_n$ is legal we have $\lambda_n \langle S_n, \gamma_n \rangle = \langle T_n, \gamma_n \rangle$ i.e.

$$\langle S_n/\kappa_n, \gamma_n/|\gamma_n| \rangle = \frac{\mu_n}{\lambda_n \kappa_n} \langle T_n/\mu_n, \gamma_n/|\gamma_n| \rangle$$

On the other hand, continuity of $\langle \cdot, \cdot \rangle$ gives $\langle T_n/\mu_n, \gamma_n/|\gamma_n| \rangle \to \langle T, \nu \rangle < \infty$, so $\langle S, \nu \rangle = 0$.

The following result summarizes the previous lemmas.

**Theorem 6.6.** Let $[S_n, T_n]$ be a sequence of geodesics, $U_n \in [S_n, T_n]$, and assume $S_n/\kappa_n \to S, U_n/\rho_n \to U, T_n/\mu_n \to T$. Then

(i) If $\rho_n e^{d(U_n, T_n)}/\mu_n$ is bounded, $T^* \supseteq U^*$; in particular, $T$ is not free simplicial if $U$ is not.

(ii) If $\rho_n e^{d(U_n, T_n)}/\mu_n$ is not bounded, then $S^* \cap U^* \neq \emptyset$.

**Proof.** The conclusion in (i) follows from Lemma [6.3]. In case (ii) we may take $\rho_n e^{d(U_n, T_n)}/\mu_n \to \infty$ and therefore $\kappa_n e^{d(S_n, T_n)}/\mu_n \to \infty$ since the ratio $\kappa_n e^{d(S_n, U_n)}/\rho_n$ is bounded below by Lemma [6.2]. Then observe that $C([S_n, T_n]) \subseteq C([U_n, T_n])$ and by Lemma [6.5] $C([S_n, T_n]) \subseteq S^*$ and $C([U_n, T_n]) \subseteq U^*$. Since $C([S_n, T_n]) \neq \emptyset$, we have $S^* \cap U^* \neq \emptyset$.

**Corollary 6.7.** Suppose that $S_n/\kappa_n$ converges to $S$, that $T_n/\mu_n$ converges to $T$, and let $[S_n, T_n]$ be a geodesic. If $S^* = T^*$, then any tree $U$ representing a point in the accumulation set of $[S_n, T_n]$ in $\overline{CV}_N$ satisfies $U^* \cap T^* \neq \emptyset$. In particular, if $S$ and $T$ are arational, then so is $U$, and $U^* = S^* = T^*$.

**Corollary 6.8.** Suppose $S_n/\lambda_n \to S, T_n \to T$, and let $[S_n, T_n]$ be a geodesic. If $S$ and $T$ are arational with $S^* \neq T^*$, then the accumulation set of $[S_n, T_n]$ in $\overline{CV}_N$ includes points of $CV_N$. 

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Proof. The accumulation set is connected, it includes $S, T$ and every tree $U$ in it satisfies either $U^* \cap S^* \neq \emptyset$ or $U^* \subseteq T^*$. By Theorem 4.4 the first alternative is equivalent to $U^* = S^*$ and the second to $U^* = \emptyset$ or $U^* = T^*$. Since the set of trees $U$ with $U^* = S^*$ is closed and disjoint from the set where $U^* = T^*$, the accumulation set must include some trees with $U^* = \emptyset$.

6.2 Reducing Factors are Visible

Recall the construction of folding lines from Section 2.5.

A train track structure on a graph is recurrent if there is a legal loop crossing every edge, and it is birecurrent if there is a legal loop that crosses every edge with either orientation.

**Lemma 6.9.** Let $T \in CV_N$, and let $\Sigma$ be a fixed simplex in $CV_N$. There exists a point $T_0$ in $\Sigma$ such that any optimal map $f : T_0 \to T$ induces a recurrent train track structure on $T_0/F_N$.

**Proof.** Let $T_0 \in \Sigma$ be a point that minimizes the function $\Sigma \to \mathbb{R}, \; Y \mapsto d(Y, T)$, by which we mean the log of the smallest Lipschitz constant of a difference of markings map $Y \to T$. Such a point exists since the map is proper.

Let $G = T_0/F_N$ and consider an optimal map $f : T_0 \to T$. First note that the tension graph $\Delta$ must be all of $G$ for otherwise we could increase the metric on $\Delta$ and decrease it in the complement thus reducing the distance to $T$. Likewise, all vertices of $\Delta$ must have $\geq 2$ gates, for otherwise we may perturb $f$ to another optimal map whose tension graph is a proper subgraph, again reaching contradiction.

Now consider the directed graph $D$ whose vertices are oriented edges of $G$, where there is a directed edge from $e$ to $e'$ if $G$ has a legal path of the form $e \cdots e'$. Two vertices in a directed graph are equivalent if there are directed paths joining each with the other. A finite directed graph always has an equivalence class $S$ of vertices so that there are no directed edges from a vertex in $S$ to a vertex outside $S$. Note that if $S$ contains some edge $e$ with both orientations, then all edges in $S$ come with both orientations. Now we have the following possibilities.

Case 1. $S = D$. Then the train track structure on $G$ is birecurrent.

Case 2. $S$ contains every edge of $G$ with a single orientation. Then $G$ is recurrent (and has a coherent orientation).

Case 3. $S$ consists of edges in a subgraph $G'$ with both orientations. This is impossible since the 2 gate condition guarantees that there is a legal
path of length 2 with one edge in $G'$ and one outside.

**Case 4.** $S$ consists of edges in a subgraph $G'$ with a single orientation. Orient the edges in $G'$ according to $S$ and note that for every vertex of $G'$ all incoming edges must belong to the same gate. Also orient all half-edges outside $G'$ but incident to a vertex $v$ of $G'$ towards $v$. All such half-edges must belong to the same gate as all incoming edges within $G'$.

We now perform a sort of backward flow relative to this orientation of $G'$. Fix a small $\epsilon > 0$. For any vertex $x \in G'$, flow $x$ backwards along an incoming edge for time $\epsilon$. This gives a multi-valued function $\varphi_\epsilon$ on the set of vertices. The ambiguity comes from the fact that there may be more than one incoming edge at $x$. In a similar way, we have a multivalued function $\tilde{\varphi}_\epsilon$ defined on the set of vertices of $\tilde{G}' \subset T_0$ with values in $T_0$. However, all points in $\tilde{\varphi}_\epsilon(x)$ are identified under folding, so the composition $f_\epsilon = f \tilde{\varphi}_\epsilon$ is a well-defined function from the vertex set of $\tilde{G}'$ to $T$. On the vertices outside $G'$ we define $f_\epsilon$ as $f$ and we then extend to the edges linearly. The slope on the edges in $G'$ or disjoint from $\tilde{G}'$ remains the same as before, and on the remaining edges the slope of $f_\epsilon$ is strictly smaller than before. Thus the tension graph is a proper subgraph of $G$ and we can change the metric as before to reduce the distance to $T$, a contradiction.

**Remark 6.10.** The result continues to hold if $T$ is in $\partial CV_N$, provided it does not belong to the closure of $\Sigma$. We do not need this generalization.

**Lemma 6.11.** Let $\Delta_0, \Gamma_i \in CV_N$ and assume $\Gamma_i$ converges to $T \in \partial CV_N$. Let $\gamma_i$ be a folding path from $\Delta_i$ to $\Gamma_i$, where $\Delta_i$ is in the same simplex as $\Delta_0$ and is given by Lemma 6.9, i.e. every optimal map $\Delta_i \to \Gamma_i$ has a recurrent train track structure. Then one of the following holds, after a subsequence.

(i) $\Delta_i$ converges to $\Delta \in CV_N$ and certain initial segments of $\gamma_i$ converge uniformly on compact sets to a folding path (ray) $\gamma$ from $\Delta$ that converges to $S \in \partial CV_N$ with $S^* \subseteq T^*$, or

(ii) $\Delta_i$ converges to a tree $S \in \partial cv_N$, and every element elliptic in $S$ is also elliptic in $T$.

**Proof.** Let $f_i : \Delta_i \to \Gamma_i$ be optimal maps. After a subsequence, all $\Delta_i$ belong to the same open simplex and underlying graphs can be identified; only the metric depends on $i$. If a subsequence of $\Delta_i$ projects to a sequence contained in a compact subset of $CV_N$, then we are in case (i). Otherwise, the injectivity radii of $\Delta_i$ go to zero, so that after a subsequence $\Delta_i \to S$. As $\Delta_i$ degenerate to $S$, there is a core subgraph $G \subseteq \Delta_i$ which is the union of loops whose lengths go to 0; its volume goes to 0.
Pass to a subsequence so that all train track structures on $\Delta_i$ agree. Since they are recurrent, there is an element $g \in F_N$ whose representatives in $\Delta_i$ are legal and cross every edge of $\Delta_i$. Let $s$ be a loop contained in $G$, then $\text{length}_{\Gamma_i}(s)/\text{length}_{\Gamma_i}(g) \leq \text{length}_{\Delta_i}(s)/\text{length}_{\Delta_i}(g) \to 0$; hence elliptic elements in $S$ are also elliptic in $T$.

Now suppose that we are in case (i), i.e. after a subsequence, $\Delta_i$ converge to $\Delta \in CV_N$. If we show the convergence statement, then the claim $S^* \subseteq T^*$ will follow from Theorem 6.6. Parametrize all folding paths via arc length. Consider the set

$$T = \{ t_0 \in [0, \infty) \mid \gamma_i|\{0, t_0\} \text{ converges uniformly after a subsequence} \}$$

It follows from the Arzéla-Ascoli Theorem, using the fact that small metric closed balls are compact, that small $t_0 > 0$ belong to $T$, and more generally, $T = [0, t_0)$ for some $t_0 > 0$ (possibly $t_0 = \infty$). By a diagonal argument there is a subsequence so that $\gamma_i|\{0, t_0\}$ converges uniformly on compact sets to a ray $r_t$ in $CV_N$. We show that $r_t$ is a folding path. The point here is that being a folding path is a local condition.

For $t \in [0, t_0)$, we have that $\gamma_i(t)$ converge to $r_t$ so we have maps $r_t \to \gamma_i(t)$ and $\gamma_i(t) \to r_t$ that are $(1 + e_i(t))$-Lipschitz with $e_i(t) \to 0$. Composing with these maps, we obtain for $0 \leq t_1 \leq t_2 < t_0$ maps $f_{t_1, t_2} : r_{t_1} \to r_{t_2}$ as limits of folding maps $\gamma_i(t_1) \to \gamma_i(t_2)$ (this may require a further subsequence; e.g. do it for rational $t_1, t_2$ and arrange that for $t_1 < t_2 < t_3$ the map $f_{t_1, t_3}$ is the composition $f_{t_2, t_3} f_{t_1, t_2}$ and then define $f_{t_1, t_2}$ in general by taking limits). The limiting maps are optimal and there are at least two gates at every vertex, by a straightforward limiting argument.

It remains to argue that $r_t$ is a (greedy) folding path when restricted to $[0, t_2]$ for $t_2 < t_0$, induced by the optimal map $f_{0, t_2}$. It is convenient to rescale the graphs and reparametrize $[0, t_2]$ so that all maps $f_{t_1, t_2}$ are isometric on small segments. If edges $e_1, e_2$ in $r_{t_1}$ form an illegal turn, their images in $r_{t_2}$ overlap on an initial segment of length $> \epsilon$, say. For large $i$ the images $e_1', e_2'$ of $e_1, e_2$ in $\gamma_i(t_1)$ are nearly isometric to $e_1, e_2$ and have possible overlap much less than $\epsilon$, but their images in $\gamma_i(t_2)$ have overlap $> \epsilon$. This means that for $0 < \delta < \epsilon$ the images of $e_1', e_2'$ in $\gamma_i(t_1 + \delta)$ have overlap of about $\delta$ and by taking the limit we see that the turn $e_1, e_2$ is folding with speed 1.

From Lemma 6.11 one has that if $T$ is arational, and if $\Delta$ and $\Gamma_i$ are as in the statement, then we always are in case (i).

The proof of the following three lemmas uses the Rips Theory; see [BF95] for background. It will be convenient to use the following terminology. A
simple subgroup is reducing for $T \in \partial \text{cv}_N$ if it acts with dense orbits on a subtree of $T$. A quasi-surface is a 2-complex $K$ obtained from a graph $\Gamma$ by attaching a collection of compact surfaces with negative Euler characteristic along the boundary.

Let $K$ be a quasi-surface. Note that when $\pi_1(K)$ is free, then $K$ has a “collapsible boundary component”. More precisely, fix an isomorphism $\pi_1(K) \cong F_N$, represent $F_N$ by a rose $R$, and represent each component of the underlying graph of $K$ by an immersion to $R$. Thus a map to $R$ is also defined on the boundary of the attaching surfaces and we may extend to each surface. After homotopy, the preimage of a regular value $y$ will consist of trees. An endpoint indicates that $y$ is crossed by only one boundary component, and only once. This is our “collapsible” boundary component – for more details see [BF94, Lemma 4.1]. Now there are two possibilities. One is that this boundary component is attached to a circle component of $\Gamma$ by a degree 1 map, which we may take to be a homeomorphism. In this case the boundary component is “free”. Otherwise, the boundary component is attached along arcs to other parts of $K$ and cutting along these arcs produces a free splitting of $F_N$ showing that all attached surfaces represent simple subgroups.

Using quasi-surfaces the reader can easily construct examples of trees that satisfy any one of the three alternatives below, but not the other two.

**Lemma 6.12.** Suppose $T$ is very small and has trivial arc stabilizers. Then either

(i) every point stabilizer is simple, or

(ii) there is a cyclic point stabilizer which is not simple and all other point stabilizers not conjugate to it are simple, or

(iii) $T$ has a reducing subgroup $A \in \text{R}(T)$ such that $A|T$ is dual to a filling measured lamination on a compact surface with negative Euler characteristic.

**Proof.** First assume that $T$ is geometric, i.e. dual to a measured lamination on a finite 2-complex $K$. Then $K$ can be transformed using the Rips machine into a standard form. If the lamination contains compact leaves (which, in the standard form, are just points in an edge of $K$), then (i) holds. Likewise, if the lamination contains a component of thin (Levitt) type, one can find a morphism $T' \to T$ where $T'$ has the same set of elliptic elements and is dual to a lamination on a finite complex that has compact leaves (by “cutting a slit in a naked band”), so again (i) holds. The remaining possibility is that
all components of the lamination are of surface type, i.e. $K$ is obtained from a complex $K_0$ not carrying any leaves by attaching surfaces carrying minimal laminations. Since each component of $K_0$ represents a point stabilizer, which is free, we may replace it with a graph. Thus $K$ is a quasi-surface. By the above discussion, $K$ contains a collapsible boundary component. If this component is not free, then (iii) holds. If it is free note that this free boundary component and its powers are the only possible nonsimple elliptic elements, since we may collapse from a free boundary component to a complex of the form $K'$ wedge a homotopically nontrivial graph with $K'$ carrying all other elliptic elements (since the surfaces have negative Euler characteristic).

When $T$ is not geometric, one can find a geometric resolution $T' \to T$ so that $T'$ has the same set of elliptic elements. The tree $T'$ is also very small and has trivial arc stabilizers. If (i) or (ii) holds for $T'$ then it also holds for $T$. If (iii) holds for $T'$ then $A|T' \to A|T$ is an isomorphism, so (iii) holds for $T$ as well. The isomorphism statement is Skora’s theorem \cite{Skoro96} that predates the Rips machine. For a relatively simple proof using the Rips machine see \cite{Bes02}. The idea is that any further folding of $A|T'$ would be resolved by a complex obtained from $S$ by identifying distinct leaves, and therefore would contain many leaves that contain at least 4 disjoint rays. By the classification of measured laminations this forces toral (axial) components, which are impossible for very small trees.

Note that the Skora theorem also follows from \cite{CH} as in Proposition 3.1. 

**Lemma 6.13.** If $G_t$ is a folding line that converges to $T \in \partial cv_N$ and if there is a non-degenerate arc $I = [x,y] \subseteq T$ with non-trivial stabilizer, then either $\text{Stab}(I) = \text{Stab}(x)$ or $\text{Stab}(I) = \text{Stab}(y)$.

**Proof.** Choose $a \in \text{Stab}(x)$, $b \in \text{Stab}(y)$, and let $c$ generate $\text{Stab}(I)$; so $ab$ and $bc$ are also elliptic in $T$. We will show that $ac$ is also elliptic, which proves the claim.

Note that if $g \in F_N$ is elliptic in $T$, then the length of $g|G_t$ is necessarily bounded; indeed, the number of illegal turns in $g|G_0$ is an upper bound for the number of illegal turns in $g|G_t$, so if $g|G_t$ is unbounded, then $g|G_t$ must contain a long legal segment. Choose a basepoint $b \in G_0$; the images $b_t$ of $b$ in $G_t$ give basepoints in $G_t$. We think of all elements of $F_N$ as loops based at $b_t$. Choose graphs $H^t_a, H^t_b, H^t_c, H^t_{ab}, H^t_{bc}$ with immersions into $G_t$ representing $a, b, c, ab, bc$, respectively; each $H^t_i$ looks like a balloon, i.e. a circle, possibly with a (long) segment, called a string, attached to it. After contracting the strings to a point, we get graphs of bounded size for all $t$. 

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If all strings are short, then \( bc \) is clearly represented by immersions of bounded size for all \( t \), implying that \( bc \) is elliptic in \( T \). If the strings for \( H^t_a \) are not short, then since \( ab|G_t \) is bounded, the string for \( H^t_a \) contains all but a bounded amount of the string for \( H^t_b \) for all \( t \); similarly, the string for \( H^t_b \) contains all but a bounded amount of the string for \( H^t_c \). Hence the string for \( H^t_a \) contains all but a bounded amount of the string for \( H^t_c \), and it follows that \( ac \) is bounded in \( G_t \) as well and is elliptic in \( T \).

**Lemma 6.14.** The alternative in Lemma 6.12 holds also for trees \( T \in \partial \text{CV}_N \) that are limits of folding paths \( G_t \).

**Proof.** In the geometric case the proof is the same except that now we allow annuli with circle leaves as surfaces when building the quasi-surface. If we find a free boundary component, then its surface cannot be an annulus since otherwise the dual tree would not be minimal, so (i) or (ii) holds as before. If we find a collapsible boundary component, we can cut as before and (iii) holds. If all surfaces are annuli \( T \) is simplicial and Lemma 6.13 implies that \( T \) has edges with trivial stabilizer, so (i) holds. The nongeometric case follows as before.

In the next lemma it is convenient to work with naturally parametrized folding paths in \( \text{cv}_N \).

**Lemma 6.15.** Let \( G_t, t \in [0, \infty) \), be a folding path in \( \text{cv}_N \) converging to a tree \( T \in \partial \text{cv}_N \). If \( \lim_{t \to \infty} \text{vol}(G_t) = 0 \), then \( T \) has dense orbits.

**Proof.** We will use the fact that \( T \) is the equivariant Gromov-Hausdorff limit of \( G_t \); see [Pau89]. The claim would be essentially obvious if \( G_t \) had a vertex with at least three gates. In general we argue as follows. For every \( \epsilon > 0 \) there is \( t_0 \) so that \( \text{vol}(G_{t_0}) < \epsilon \) and for every point \( \hat{x} \in \tilde{G}_{t_0} \) there is a tripod \( \{\tilde{a}, \tilde{b}, \tilde{c}\} \subset \tilde{G}_{t_0} \) whose center is within \( \epsilon \) of \( \hat{x} \), \([\tilde{a}, \tilde{b}]\) and \([\tilde{a}, \tilde{c}]\) are legal, and all three segments between the center and the endpoints have length > 1. For \( t > t_0 \) the images of \([\tilde{a}, \tilde{b}]\) and \([\tilde{a}, \tilde{c}]\) may get folded past the image of the center, but not by more than \( \text{vol}(G_{t_0}) \), see [BFH05]. Thus branch points are dense in \( T \).

**Lemma 6.16.** Suppose \( G_t, t \in [0, \infty), \) is a folding path converging to \( T \in \partial \text{cv}_N \), and assume that \( T \) is not arational. Then there is a factor \( B < F_N \) such that \( B \in \mathcal{R}(T) \), and such that \( B \) has uniformly bounded volume along \( G_t \) for \( t \) large.

In particular, the projection of \( G_t \) to \( F \) is bounded and for large \( t_0 \) the projection of \( G_t, t \in [t_0, \infty) \) is uniformly bounded.
Proof. First we argue that it suffices to find a simple reducing subgroup \( A < F_N \) such that \( \text{vol}(A|G_t) \) is uniformly bounded for large \( t \). Indeed, let \( B \) be the smallest factor that contains \( A \). Then \( A|G_t \rightarrow G_t \) factors through \( B|G_t \) and \( A|G_t \rightarrow B|G_t \) is surjective (otherwise we find a smaller factor that contains \( A \)) so \( \text{vol}(B|G_t) \leq \text{vol}(A|G_t) \) is also bounded for large \( t \).

Second, recall \([BF11, \text{Corollary 3.5}]\) that if \( B \) is a factor then the distance in \( F \) between \( B \) and \( \pi(G_t) \) is bounded by a function of \( \text{vol}(B|G_t) \), so the last sentence follows from the first paragraph.

Third, if the length of the folding path is finite, or equivalently if after rescaling \( G_t \) so that folding maps are isometries on small segments the volume does not go to 0, we may take \( A \) to be the fundamental group of a component of the subgraph whose volume goes to 0. So we will now assume that the folding path has infinite length.

For now let \( C \) be any simple subgroup reducing \( T \). It follows from \([BF11, \text{Lemma 4.1}]\) that \( C|G_t \) cannot contain a legal segment of length \( > 2 \) inside a topological edge; otherwise the volume of \( C|G_t \) would grow exponentially, and \( C \) would not reduce \( T \).

Fix a large number \( M \), much larger than the possible number of illegal turns in any train track structure. If the number of illegal turns in each topological edge in \( C|G_t \) is \( \leq M \) for large \( t \), then \( C|G_t \) has uniformly bounded volume and we are done. Choose \( t_0 \) such that the number of illegal turns in topological edges of \( C|G_t \) has stabilized for \( t \geq t_0 \) (by the Unfolding Principle of \([BF11]\) the number of such turns cannot increase), and suppose that for some edges this number is \( > M \) and focus on \( M \) consecutive such illegal turns. By our choice of \( M \) there are many turns in this collection that project to the same illegal turn in \( G_t \). This gives many loops \( g|G_t \) of uniformly bounded length for \( t \) large.

For \( t_n \rightarrow \infty \), we have scaling constants \( \lambda_n \) such that \( \hat{G}_{t_n}/\lambda_n \) converges to \( T \in \partial \text{cv}_N \). From our assumption that the folding path has infinite length we see that \( \lambda_n \rightarrow \infty \) and so any \( g \) constructed above is elliptic in \( T \).

If \( C|G_t \) contains legal loops, consider the subgraph \( D_t \subset C|G_t \) which is the union of all legal loops. This subgraph is clearly forward invariant and eventually the number of components and their ranks stabilize. Take the simple subgroup \( A \) to be represented by one of these stable components. Then by Lemma \([6.15]\) \( A \) is reducing, and \( A|G_t \) has uniformly bounded volume for all large \( t \).

So we are done unless \( C|G_t \) doesn’t contain any legal loops for large \( t \), or equivalently the complement in \( C|G_t \) of the set of 1-gate vertices is a forest. In this case \( C \) is elliptic in \( T \). Indeed, loops in \( C|G_t \) have uniformly bounded legal segments (sufficiently long legal segments would close up to form legal
loops) and so grow slower than legal loops (see the Derivative Formula in [BF11]).

We now consider three cases according to the three alternatives in Lemma 6.12 which also applies to $T$ by Lemma 6.14. If (i) holds then the elements $g$ constructed above are simple and we are done. If (iii) holds, we will start with $C = A$, a non-elliptic reducing simple subgroup, and by the above discussion we are done. Finally, assume (ii) holds. We constructed paths $g_i$ in $C|G_t$ connecting consecutive equivalent illegal turns. Under the assumption (ii), if $g_i$ are not simple, they are all conjugate to powers of a fixed element $g$. Note that if consecutive paths $g_i, g_{i+1}$ do not have common powers, then $g_ig_{i+1}$ is not conjugate to a power of $g$, since $C$ is simple. We conclude that the concatenation of the $g_i$’s is a large power of an element conjugate to $g$, which we rename $g$. Now $C|G_t$ is not just a loop representing a power of $g$, so inside $C|G_t$ we can find a loop representing an element of the form $x_1g^{n_1}x_2g^{n_2} \cdots x_kg^{n_k}$ with $k$ and lengths of $x_i$ bounded, and this element is elliptic in $T$, not conjugate to a power of $g$, and stays of bounded length along $G_t$. Here $n_i$ can be large, but we may replace them with uniformly bounded numbers and the new element is still elliptic in $T$, it is uniformly bounded in $G_t$ for large $t$, and it is simple since it is not conjugate to a power of $g$.

7 The Boundary of the Complex of Free Factors

Let $\partial \mathcal{F}$ denote the boundary of the complex of free factors, and let $\mathcal{AT} \subseteq \partial \mathcal{CV}_N$ denote the set of (projective classes of) arational trees. Define an equivalence relation $\sim$ on $\mathcal{AT}$, where $T \sim S$ if and only if $L(T) = L(S)$ (equivalently, $T^* = S^*$, see Proposition 2.2 and Theorem 4.4). We note that $\sim$ is precisely the relation of “forgetting the measure” for elements of $\mathcal{AT}$; see [CHL07]. Give $\mathcal{AT}$ the subspace topology, and consider the quotient map $p : \mathcal{AT} \to \mathcal{AT}/\sim$.

Lemma 7.1. The quotient map $p : \mathcal{AT} \to \mathcal{AT}/\sim$ is closed, and point pre-images are compact.

Proof. Let $K \subseteq \mathcal{AT}$ be closed; we show that $C = p^{-1}(p(K))$ is closed. Let $\{T_n\}$ be a convergent sequence in $C$, say $T_n$ converges to $T \in \mathcal{AT}$; let $Y_n \in K$ such that $p(Y_n) = p(T_n)$. This means that $Y_n^* = T_n^*$. Now, let $\eta_n \in Y_n^* = T_n^*$. After passing to a further subsequence we may assume that $Y_n \to Y \in \partial \mathcal{CV}_N$ and that $\eta_n \to \eta \in M_N$. By Proposition 2.1 we have $\langle Y, \eta \rangle = 0 = \langle T, \eta \rangle$; so Theorem 4.4 gives that $Y$ is arational and that $L(T) = L(Y)$. It follows that $Y \in K$ and $p(T) = p(Y)$, so $T \in C$. 

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The statement that equivalence classes are compact can be proved similarly using the compactness of \( \partial \mathcal{C}V_N \). If \( T_i \) converge to \( T \) in \( \partial \mathcal{C}V_N \) and if \( T_i \in \mathcal{A}T \) are all equivalent, then choose some \( \nu \in T_i^* \). By Proposition 2.1 we have \( \nu \in T^* \) and then \( T \in \mathcal{A}T \) is equivalent to all \( T_i \) by Theorem 4.4.

The following result justifies our use of sequential arguments.

**Corollary 7.2.** The quotient space \( \mathcal{A}T / \sim \) is metrizable and second countable.

*Proof.* Closed surjective maps with compact point preimages preserve the properties of being metrizable and second countable [Eng89, Theorems 3.7.19 and 4.4.15].

We can now give a description of \( \partial \mathcal{F} \).

**Proposition 7.3.** There is a continuous map \( \partial \pi : \mathcal{A}T \to \partial \mathcal{F} \), such that if \( T_i \in \mathcal{C}V_N \) converge to \( T \in \mathcal{A}T \), then \( \pi(T_i) \) converge to \( \partial \pi(T) \).

*Proof.* Let \( T_i \in \mathcal{C}V_N \) converge to \( T \in \mathcal{A}T \); we need to see that \( \pi(T_i) \) converges to a point of \( \partial \mathcal{F} \) that depends only on \( T \). Toward contradiction, suppose this is not the case. Then we get subsequences \( X_n \) and \( Y_n \) such that the Gromov product \( (\pi(X_n), \pi(Y_n)) \) is uniformly bounded. Consider (say a standard) geodesics \( [X_n, Y_n] \); Proposition 2.4 gives that these geodesics are mapped by \( \pi \) to uniform quasi-geodesics in \( \mathcal{F} \). Hence we find \( Z_n \) on \( X_n \to Y_n \) with \( \pi(Z_n) \) of uniformly bounded distance from any basepoint in \( \mathcal{F} \). On the other hand, Lemma 6.7 and Theorem 4.4 give that any limit \( Z \) of \( \{Z_n\} \) must be arational. Finally, Corollary 4.6 gives a contradiction. Hence, we have a function \( \partial \pi : \mathcal{A}T \to \partial \mathcal{F} \).

The continuity statement follows similarly. Let \( T_i \in \mathcal{A}T \) converge to \( T \in \mathcal{A}T \), but assume that \( \partial \pi(T_i) \) does not converge to \( \partial \pi(T) \). After a subsequence we may assume that \( (\partial \pi(T_i), \partial \pi(T)) \) is bounded above. Choose trees \( X_i, Y_i \in \mathcal{C}V_N \) so close to \( T_i \) and \( T \) respectively that \( (\pi(X_i), \pi(Y_i)) \) is also bounded above and so that \( X_i \to T, Y_i \to T \). As above, there is \( U_i \) on a geodesic from \( X_i \) to \( Y_i \) with \( \pi(U_i) \) at bounded distance from a basepoint, which is impossible.

**Proposition 7.4.** For arational trees \( S \) and \( T \), we have \( \partial \pi(S) = \partial \pi(T) \) if and only if \( L(S) = L(T) \).

*Proof.* By the same argument as in the proof of Proposition 7.3, we get that \( L(S) = L(T) \) implies that \( \partial \pi(S) = \partial \pi(T) \). So assume that \( L(S) \neq L(T) \), let \( S_n \) converge to \( S \) and \( T_n \) converge to \( T \); consider standard geodesics \( [S_n, T_n] \).
By Lemma 6.8 we have that \([S_n, T_n]\) accumulates on some portion of \(\text{CV}_N\), hence after passing to a subsequence, we find points on \([S_n, T_n]\) projecting to points of \(\mathcal{F}\) of uniformly bounded distance from any base point. Hence \((\pi(S_n), \pi(T_n))\) is uniformly bounded, so \(\partial \pi(S) \neq \partial \pi(T)\).

Proposition 7.5. The map \(\partial \pi\) is surjective. Further, if \(\{T_n\}\) converge to a tree \(T\) that is not arational, then no subsequence of \(\{\pi(T_n)\}\) converges to a point of \(\partial \mathcal{F}\).

Proof. Let \(X \in \partial \mathcal{F}\), and let \(X_n \in \mathcal{F}\) converge to \(X\). Choose \(T_n \in \pi^{-1}(X_n)\), and pass to a subsequence so that \(\{T_n\}\) converges to \(T\) in \(\text{CV}_N\). We will show that \(T \in \mathcal{A}T\), which implies \(\partial \pi(T) = X\).

Toward contradiction, suppose that \(T\) is not arational. Recall that \(T\) has its reducing set \(\mathcal{R}(T) \subseteq \mathcal{F}\), which is nonempty and uniformly bounded, see Corollary 5.3. Fix \(n\) large so that for \(m > n\) \(X_m\) belongs to a small neighborhood of the end \(X\) and is far from \(\mathcal{R}(T)\). In particular, we may assume that geodesics connecting \(X_m\) and \(X_{m'}\) for \(m, m' > n\) are also far from \(\mathcal{R}(T)\).

Now consider for \(m \gg n\) a folding path \([T'_m, T_m]\) where \(T'_m\) is in the same simplex as \(T_n\) and the train track structure on \(T'_m/F_N\) is recurrent (see Lemma 6.9), and let \(m \to \infty\). Apply Lemma 6.11 and first assume that case (ii) applies, so \(T'_m \to S\) and elliptic elements in \(S\) are elliptic in \(T\). Then there is a factor \(A\) which is elliptic in \(S\), and thus it is reducing for \(T\), and it is also coarsely equal to \(\pi(T_n)\), contradicting the fact that \(\pi(T_n)\) is far from \(\mathcal{R}(T)\).

Now, suppose that case (i) of Lemma 6.11 applies, so after a subsequence initial segments of \([T'_m, T_m]\) converge to a ray \(r_n\) that converges to \(S^* \subseteq T^*\). By Corollary 4.5 \(S\) is not arational and \(\mathcal{R}(S)\) coarsely equals \(\mathcal{R}(T)\). Using Lemma 6.16 we see that the projection of \(r_n\) to \(\mathcal{F}\) is eventually contained in a uniformly bounded neighborhood of \(\mathcal{R}(S)\), and therefore of \(\mathcal{R}(T)\).

To obtain a contradiction just note that the projections of \([T'_m, T_m]\) are uniform quasi-geodesics by Proposition 2.4, so they don’t come close to \(\mathcal{R}(T)\) for large \(n, m\) and the geodesics \(T'_m \to T_m\) cannot accumulate to \(r_n\).

Lemma 7.6. The map \(\partial \pi : \mathcal{A}T \to \partial \mathcal{F}\) is closed.

Proof. Let \(C \subseteq \mathcal{A}T\) be closed, and let \(K = \partial \pi(C)\). Let \(X_n \in K\) converge to \(X \in \partial \mathcal{F}\); we want to find \(Y \in C\) with \(\partial \pi(Y) = X\). Choose \(Y_n \in (\partial \pi)^{-1}(X_n) \cap C\), and pass to a subsequence to ensure that \(Y_n\) converge to \(Y \in \partial \text{CV}_N\).
We claim that $Y \in \mathcal{A}T$. This follows from Proposition 7.5 applied to a sequence $\{T_n\}$ in $\text{CV}_N$ approximating $\{Y_n\}$ so that $T_n \to Y$ and $\pi(T_n) \to X$. Now the fact that $\partial \pi(Y) = X$ follows from Proposition 7.3.

Summarizing, we have:

**Theorem 7.7.** The space $\partial F$ is homeomorphic to the quotient space $\mathcal{A}T/\sim$.

**Proof.** The map $\partial \pi : \mathcal{A}T \to \partial F$ factors through $p : \mathcal{A}T \to \mathcal{A}T/\sim$ to give a continuous, bijective, closed map $\mathcal{A}T/\sim \to \partial F$.

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