Cancellation of divergences in unitary gauge calculation on 

$H \to \gamma\gamma$ process via one W loop, and application

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Abstract

Following the thread of R. Gastmans, S. L. Wu and T. T. Wu, the calculation in the unitary gauge for the $H \to \gamma\gamma$ process via one W loop is repeated, without the specific choice of the independent integrated loop momentum at the beginning. We start from the ‘original’ definition of each Feynman diagram, and show that the 4-momentum conservation and the Ward identity of the W-W-photon vertex can guarantee the cancellation of all terms among the diagrams which are to be integrated to give divergences higher than logarithmic. The remaining terms are to the most logarithmically divergent, hence is independent from the set of integrated loop momentum. This way of doing calculation is applied to the $H \to \gamma Z$ process via one W loop in the unitary gauge, the divergences proportional to $M_{Z}^{2}/M_{W}^{3}$ including quadratic ones are all cancelled, and terms proportional to $M_{Z}^{2}/M_{W}^{3}$ are shown to be zero. The way of dealing with this quadratic divergence has subtle implication on the employment of the Feynman rules.
1. INTRODUCTION

The Glashow-Weinberg-Salam electroweak (EW) theory is a $SU(2) \times U(1)$ Yang-Mills gauge theory, with the gauge symmetry 'broken' by a scalar field via the Englert-Brout-Higgs-Guralnik-Hagen-Kibble Mechanism. This has been confirmed from experiment, i.e., the massive $W^\pm, Z$ particles v.s. the massless photon, and a 'remaining' neutral scalar particle which is generally referred to as the Higgs particle. This is remarkably different from the other set of the standard model, quantum chromodynamics (QCD) with its SU(3) gauge symmetry not broken. Though the confinement mechanism is yet not fully understood, itself is a manifestation of the non-broken—simply that all physical states are colour singlet, i.e., invariant w.r.t. the SU(3) group.

In general, a realistic calculation of the S-matrix or scattering amplitude employing the quantized field theory of the standard model need to fix a specific gauge and it is adopted that the result should be independent from the choice of the gauge. So for the feasibility of loop calculations, the 'renormalizable gauge', e.g., $R_\xi$ gauge, is favoured. Both EW and QCD can take this gauge. On the other hand, for the EW theory, one can also take a 'physical gauge', unitary gauge, where only the physical degrees of freedom present. According to Weinberg [1], this gauge can be defined by imposing the condition of the scalar field relevant to its vacuum expectation value and the 'symmetry breaking'. Such a gauge may not be introduced for QCD. This does not matter as long as that all physical observables like decay width, scattering amplitude, etc., are the same for any gauges. However, recently, strong implication for the contrary case is recognized, based on the careful revisit on the $H \rightarrow \gamma\gamma$ decay width in the unitary gauge and $R_\xi$ gauge [2–6]. The first indication of this fact is that calculations in the unitary gauge for loop diagrams should be extensively studied.

The main purpose of this paper is to repeat the calculation by Gastmans, Wu and Wu in [5, 6] (in this paper we refer to these two papers and the works in them as GWW) to get some insights as following. We start from the 'original expression' of each of the Feynman diagrams that the momenta of the propagators are kept as the original, not to be expressed by the independent integrated loop momentum. They are related by the 4-momentum conservation of each vertex, expressed by the $\delta$ function attached with each vertex. By investigating the details of the cancellation one can demonstrate the origin of the inner relation of the loop momenta of these diagrams pointed by GWW. This can shed
light on cases of more complex diagrams and/or with more loops to find the proper inner relation of the loop momenta of those diagrams (in this paper we will discuss the application for $H \rightarrow \gamma Z$ process via one W loop in the unitary gauge as the most simple example). Such a study from this most general/original footing has yet not available in literature (e.g., see citations for GWW work) until now.

This study is important since, in the unitary gauge, high level divergent terms of loop diagrams appear and should properly cancel. Or else one can not get the correct result, either not possible to make the comparison with results from other gauges. For any diagram whose divergence higher than logarithmic, to shift the integral momentum can lead to extra terms with lower divergence (or finite). In such case, the proper set of diagrams with correct inner relations of the loop momenta must be treated together to get the correct result, as pointed by GWW. Only a part of diagrams of the set shifting the momenta will change the result. On the other hand, starting from the original expression of the standard perturbative expansion of the S-matrix, one can keep the momentum of each propagator as the origin, and the integration on each momentum, as well as the 4-momentum conservation $\delta$ function of each vertex. This means that there are no ambiguity of this starting point. In the following of this paper we will see that the inner relations of the momenta is the natural result of the cancellation which is led from the concrete expression of the diagrams (hence the symmetry relation of the diagrams).

In the calculation on Higgs particle decay into two photons via the W loop in the unitary gauge, there are two kinds of divergent terms of each diagram:

(1) Those higher than logarithmic, which need the exact cancellation and only logarithmic divergent (and/or finite) terms can remain. In our procedure, their cancellation is the result that different diagrams give exactly the same integrand with opposite signs. All these need not the setting of the specific integrated momentum. The special relation, e.g., that of GWW, is the natural result of the general relation and cancellation without the need of being set à priori.

(2) The other divergence is the logarithmic terms, whose cancellation is also for sure and the choice of the loop momenta can be independent for each diagram since shift of the momenta will not change the result. Indeed, the most easy calculation way is to use the Feynman-Schwinger parametrization to deal with the proper part of the terms and to get the result.
The debate in literature on GWW result is related with the dimensional regularization. We will not discuss the solution of this problem, i.e., which is correct, dimensional regularization or Dyson subtraction, or some relations among them. However, in the following we will show that the higher level (higher than logarithmic) divergent terms of 4 dimension can NOT be dealt with a shift of momenta in the D (sufficiently smaller than 4) dimension where it is convergent and 'seems' the shift correct. So this shows that the cancellation of the higher level (higher than logarithmic) divergent terms of 4 dimension is even à priori w.r.t. the dimensional regularization if one chooses dimensional regularization as the way out.

The above content is studied mainly in Sec. 2. In Sec. 3, we will discuss the symmetrical integration. In Sec. 4, we show that the 'original' form of the amplitude, can help to clarify, e.g., why in the standard Ward identity of QED, the 'external' momentum—the momentum of the self energy—seems only flow via the fermion line. In Sec. 5, this method is applied to $H \rightarrow \gamma Z$ process via one W loop in the unitary gauge. The divergences proportional to $M_W^2/M^3$ (In all this paper, as GWW, M is used to represent the W mass) including quadratic ones are all cancelled, and terms proportional to $M_W^2/M^3$ are shown to be zero. This is one of the crucial steps for the investigation on the possible difference \[2\] between the unitary gauge and $R_\xi$ gauge for the $H \rightarrow \gamma Z$ process. Furthermore, the dealing with the quadratic divergences proportional to $M_W^2/M^3$ in $H \rightarrow \gamma Z$ has subtle implication on the employment of the Feynman rules: Better to start with all the integrations on the propagator momenta and all the $\delta$ functions of all the vertices.

2. CALCULATION

1. To say that a high level divergent term (higher than logarithmic) is changed when shifting the integrated momentum, first of all one should ask, what is the 'original' to be changed? There may not exist the 'original' one for a single diagram, once taking into account that different Feynman diagrams are related and hence the loop momenta (à la GWW); however, one can have some definiteness starting from the original form derived from the expansion of the S-matrix according to the standard Dyson-Wick procedure. Since the S-matrix is the exponential of a spacetime integral; once taking these spacetime integrals at each order, one gets $\delta$ functions, one for each vertex, relating all the momenta of the
If we start from such a form for each diagram, without integrating the \( \delta \) functions, there will be no indefiniteness.

As convention, The S-matrix and T-matrix have the relation \( S = I + iT \), and the matrix element between initial and final states \( iT_{fi} = i(2\pi)^4 \delta(P_f - P_i)M_{fi} \), for the case without the presence of an external classical field which breaks the space-time displacement invariance. Here we keep all the momenta respectively corresponding to each propagator and hence all \( \delta \) functions respectively corresponding to each vertex. The one corresponding to the initial-final state energy momentum conservation is contained in these \( \delta \) functions. After integrating over them, one will get the above form of T-matrix element with the \( M_{fi} \) is the integration of the independent loop momenta only, without the \( \delta \) functions attached to the vertices. This is the standard procedure in developing the Dyson-Wick perturbation theory in the interaction picture [7]. The four-momentum conservation \( \delta \) function attached to each of the vertices is the result of integration of space-time contained in the perturbative expansion of the S-matrix, and is the manifestation of space-time displacement invariance (the shifting of spacetime not changing the result). In the following, we do the calculation and do not ‘integrate out’ the \( \delta \) functions of each vertex until have to. So here we deal with the matrix elements \( T_{fi} \) rather than \( M_{fi} \):

\[
T_1 = \frac{-ie^2gM}{(2\pi)^4} \int d^4q_1d^4q_2d^4q_3(2\pi)^4\delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \\
\times (g^\alpha_\beta - \frac{q_1^\alpha q_1^\beta}{M^2})(g^{\alpha\gamma} - \frac{q_3^\alpha q_3^\gamma}{M^2})(g^{\beta\gamma} - \frac{q_2^\beta q_2^\gamma}{M^2}) \frac{V_{\beta\mu\nu}(q_1, -k_1, -q_3) V_{\alpha\nu\gamma}(q_3, -k_2, -q_2)}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)}
\]

Here we do not explicitly write the matrix element subscript \( f_i \), and all the \( \delta \) functions are understood as four-dimensional one, i.e., \( \delta(P_1 - P_2) := \delta^4(P_1 - P_2) \). As GWW, we also omit the polarization vector, so \( T_1 \) should be understood as \( T_{1\mu\nu} \). Here the integration is easy to be considered in 4 as well as in D dimension. In D dimension, it is \( d^4q_1d^4q_2d^4q_3(\delta^4)^3 \rightarrow d^Dq_1d^Dq_2d^Dq_3(\delta^D)^3 \) (Though we will not discuss the (in)correctness of dimensional regularization, we just clarify the procedure of cancellation in the whole section also valid in D dimension if this procedure is needed).

\[
T_2 = \frac{ie^2gM}{(2\pi)^4} \int d^4q_1d^4q_2d^4q_3(2\pi)^4\delta(P - q_1 + q_2)\delta(q_1 - q_2 - k_1 - k_2) \\
\times (g^\alpha_\beta - \frac{q_1^\alpha q_1^\beta}{M^2})(g^{\alpha\gamma} - \frac{q_2^\alpha q_2^\gamma}{M^2}) \frac{2g_{\mu\nu}g_{\beta\gamma} - g_{\mu\beta}g_{\nu\gamma} - g_{\mu\gamma}g_{\nu\beta}}{(q_1^2 - M^2)(q_2^2 - M^2)}
\]

\begin{equation}
\end{equation}
\[ T_3 = \frac{-ie^2 gM}{(2\pi)^4} \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_2 - q_3) \delta(q_2 - k_1 - q_2) \] (3)
\[ \times (g_\alpha^\beta - \frac{q_{1\alpha} q_1^\beta}{M^2})(g^{\rho\sigma} - \frac{q_3^\rho q_3^\sigma}{M^2})(g^{\alpha\gamma} - \frac{q_2^\alpha q_2^\gamma}{M^2}) \frac{V_{\beta\mu\nu}(q_1, -k_2, -q_3) V_{\sigma\mu\nu'}(q_3, -k_1, -q_2)}{(q_1^2 - M^2)(q_2^2 - M^2)(q_3^2 - M^2)} \]

These just correspond to \( M_{1,2,3} \) of GWW times the \( \delta \) function of the whole energy-momentum conservation. The momentum of each propagator recover back to the original one of the definition of the propagator. If integrating the \( \delta \) functions, by choosing the loop momenta as GWW do, one can get the similar equations as Eqs. (2.2-2.4) of GWW. If we set the independent integrated momentum \( k \) in other ways, we can get other forms. Here the relation between \( T_1 \) and \( T_3 \), i.e., \( \mu \leftrightarrow \nu \), and \( k_1 \leftrightarrow k_2 \) is clear to be read out. The Feynman diagrams are shown in Figure. This corresponds to that the momentum space Feynman rules (see GWW) are slightly modified (in fact 'recovered', see the classical paper of Dyson, especially its Eq. (20) and discussions before and after it) as: Any propagator with momentum \( q \) has an extra \( \left[ \int \frac{dq}{(2\pi)^4} \right] \) 'operator', i.e., should this integration on \( q \) to be done in the calculation of the Feynman diagram; any vertex has an extra factor \( (2\pi)^4 \delta(\sum_i q_i) \), with \( q_i \), each momentum of all the propagators meeting in the vertex, incoming.

In this way, and to this step, one has no problem of 'choosing' the loop momenta, as long as before one has to integrate on one or more of the \( \delta \) functions. This is EXACTLY the definition of the amplitude without any ambiguity, provided that one adopts each term in the Dyson-Wick perturbation theory is well defined and the propagators and vertices are well defined. And this is just the Feynman rules itself. Not to keep the integration on each propagator momentum and the \( \delta \) function of each vertex is the 'simplified version' of Feynman rules—if still right in all cases (In our opinion, seems not, see Sec. 5). Some old quantum field theory text books in fact gave the original/not-simplified version, as is used here. The simplified one is only popular in modern time. However, to our knowledge, no textbook (or Dyson’s paper) persuaded to keep the \( \delta \) function until have to be integrated as we will do in the following, this is one of the reasons why we study this problem in details in this section.

In the Dyson-Wick perturbative theory framework and in the interaction picture, the propagator is definite (provided properly defined in relation with the standard scattering theory, and we will discuss in Sec. 3), the \( \delta \) functions from each vertex has set the relations of all the propagator momenta definitely. This means once one keeps all these relations and
FIG. 1: The one W loop Feynman diagrams. Taking the waving line with momentum $k_2$ and polarization vectors $\epsilon^\nu_\lambda$ represents the photon or Z particle respectively, they correspond to $T_1$, $T_2$ and $T_3$ of $H \to \gamma\gamma$ process or $T_{1Z}$, $T_{2Z}$ and $T_{3Z}$ of $H \to \gamma Z$ process. In general, the inner integrated momenta should be considered as not correlated between different diagrams, so here we mark those of $T_2$ and $T_3$ with prime or double primes. In the manuscript, this is also implied though not explicitly written. But for the purpose of cancellation which is determined by the integrand, they are taken to be the same in the concrete step of derivation. This is easily to be tracked.
calculate according to mathematics without mistakes, one has no uncertainty (or one is free from) in choosing independent 'loop' momenta for each diagram or some of the terms.

This also means that if at some step to write down the amplitude with further 'freedom/restriction' which is not included in the above Feynman rules, especially the propagator integrations and vertex δ functions, or is contradicting to this definite result, this step may introduce more things deviating from the exact definition of the amplitude in the above standard framework. So that step is a wrong step rather than some 'ambiguity', since it is not included in the theory at beginning. Even our following discussion on H → γZ process (Sec. 5) indicates that to set the independent integrated loop momentum at the beginning seems some of the necessary freedom/possibility contained in the Feynman rules lost.

2. To better illustrate the calculation procedure, here we mention the formulae we use repeatedly in the calculation. I.e., Eqs. (2.5-2.12) of GWW [5]. For these equations, we will refer to as GWW2.5, GWW2.6,..., GWW2.12 in this paper. Complementary to GWW2.9, we add an equation as \( V^{\alpha\mu\gamma}(p_1, -k_1, p_3) p_3^\gamma = -(p_1^2 g_{\alpha\mu} - p_{1\alpha} p_{2\mu}) \), i.e., the case contracted with the other inner momentum \( p_3 \). In GWW, they only list the formula for \( p_1^\gamma V^{\alpha\mu\gamma}(p_1, -k_1, p_3) \). However, for simplicity, we still refer this complemented formulae as GWW2.9. And here we would like to address that all these formulations still valid in D dimension. The property of 3 vertex, which is named as Ward Identity (W.I.) in GWW, is the key role in the evaluation. For the feasibility to investigate the H → γZ process in the later part of this paper, we list the equations for this process corresponding to GWW (2.5-2.12) in Appendix A. They straightforwardly go to GWW (2.5-2.12) by simply taking \( M_Z = 0 \).

We also would like to mention that, the explicitly employment of \( k_{1\mu} e^\mu = k_{2\nu} e^\nu = 0 \) (GWW2.5) everywhere means that it is valid without any account of any divergence factors timing with it.

Here the integral is easy to be considered as in 4 as well as in D dimension. In both cases, one can arrange each of the diagrams in terms associate with the minus power of the W mass M (with the extra overall M factors from the coupling taken out, as GWW). We will check once they can cancel in 4 dimension, whether they can cancel in D dimension (the terms can be taken as convergent)—this will be explicitly mentioned in the following.

First is the \( M^{-6} \) term, which only appears in \( T_1 \) and \( T_3 \), are explicitly read out as zero, respectively, because of the property of the W.I., GWW2.11, 2.12. So, in four dimension, this means that the divergence of power 6th does not exist, and even does not leave any
terms with lower divergence which may draw any uncertainty to the following analysis. This
W.I., frankly the basic property of the three-boson vertex applied to the specific process
we investigate, can not be broken or made to any deviation. This also is the case for D
dimension, i.e., in D dimension (for this\(M^{-6}\) case should be D < 0), terms proportional to
\(M^{-6}\) also equals to zero. All the following analysis respect this fact. Any direct result from
the W.I. without referring to any other relations is solid without any ambiguity.

3. \(M^{-4}\) terms

In all the following, the \(-\frac{ie^2g^M}{(2\pi)^2}\) factor will not explicitly written, and all terms should
multiply with this factor to get the proper terms in the corresponding T amplitude of Eqs.
(113). So the \(M^{-4}\) terms from Eq. (11) is (analogy to Eq. (3.1) of GWW, only with the extra
\(\delta\) functions, integrations, and without the overall constant factors mentioned above)

\[
T_{11} = \frac{1}{M^4} \int d^4q_1d^4q_2d^4q_3(2\pi)^4\delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \tag{4}
\]

\[
\times q_1\beta^\gamma_1 g^\rho\sigma q_2^\alpha q_2^\gamma V_{\beta\mu\nu}(q_1, -k_1, -q_3) V_{\sigma\nu\gamma}(q_3, -k_2, -q_2)
\]

\[
\frac{(q_1^2 - M^2)(q_2^2 - M^2)(q_3^2 - M^2)}{(q_1^2 - M^2)(q_2^2 - M^2)}.
\]

Now since \(q_1^\beta V_{\beta\mu\nu}(q_1, -k_1, -q_3) = (q_3^2 - M^2)g_{\mu\nu} - q_{3\mu}q_{3\nu} + M^2g_{\mu\nu}\), according to GWW2.10,

\(T_{11} = T_{111} + T_{112} + T_{113}\).

In this form, \(T_{112}\) is explicitly read to be zero:

\[
T_{112} = \frac{1}{M^4} \int d^4q_1d^4q_2d^4q_3(2\pi)^4\delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \tag{5}
\]

\[
\times q_1\alpha^\gamma_1 q_2^\alpha q_2^\gamma V_{\sigma\nu\gamma}(q_3, -k_2, -q_2)
\]

\[
\frac{(q_1^2 - M^2)(q_2^2 - M^2)(q_3^2 - M^2)}{(q_1^2 - M^2)(q_2^2 - M^2)}.
\]

since the factor \(-q_2^\gamma q_3^\rho g^\rho\sigma V_{\sigma\nu\gamma}(q_3, -k_2, -q_2) = 0\) according to GWW2.12.

\[
T_{111} = \frac{1}{M^4} \int d^4q_1d^4q_2d^4q_3(2\pi)^4\delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \tag{6}
\]

\[
\times q_1\alpha q_2^\alpha q_2^\gamma g_{\mu\nu} g^\rho\sigma V_{\sigma\nu\gamma}(q_3, -k_2, -q_2)
\]

\[
\frac{(q_1^2 - M^2)(q_2^2 - M^2)}{(q_1^2 - M^2)(q_2^2 - M^2)}.
\]

Again to employ GWW2.9 (our compensatory) for \(V_{\sigma\nu\gamma}(q_3, -k_2, -q_2)q_2^\gamma\), it is

\[
T_{111} = \frac{1}{M^4} \int d^4q_1d^4q_2d^4q_3(2\pi)^4\delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \tag{7}
\]

\[
\times q_1 \cdot q_2(q_3^2 g_{\mu\nu} - q_{3\mu}q_{3\nu})
\]

\[
\frac{(q_1^2 - M^2)(q_2^2 - M^2)}{(q_1^2 - M^2)(q_2^2 - M^2)}.
\]

In this step, one can take into account that, \(\int dx f(x)\delta(x - a) = \int dx f(a)\delta(x - a)\), to use
the relation \(q_3 = q_1 - k_1 = q_2 + k_2\), to get

\[
(q_3^2 g_{\mu\nu} - q_{3\mu}q_{3\nu}) = q_1 \cdot q_2 g_{\mu\nu} - q_{1\mu}q_{2\nu} + (q_1 \cdot k_2 - q_2 \cdot k_1 - k_1 \cdot k_2)g_{\mu\nu}.
\]
In fact, all the Ward identities for the 3-boson vertex we use here also have employed the energy momentum conservation at the vertex. So

\[ T_{111} = \frac{1}{M^4} \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2) \]  
\[ \times q_1 \cdot q_2 \frac{q_1 \cdot q_2 g_{\mu\nu} - q_1 \mu q_2 \nu + (q_1 \cdot k_2 - q_2 \cdot k_1 - k_1 \cdot k_2) g_{\mu\nu}}{(q_1^2 - M^2)(q_2^2 - M^2)}. \]  

Now we check the \( M^{-4} \) term in \( T_3 \), which is called \( T_{31} \), and find the similar derivation can be done, to get \( T_{31} = T_{311} + T_{312} + T_{313} \). \( T_{312} = 0 \), according to GWW2.12, and

\[ T_{311} = \frac{1}{M^4} \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_2 - q_3) \delta(q_3 - k_1 - q_2) \]  
\[ \times q_1 \cdot q_2 \frac{q_1 \cdot q_2 g_{\mu\nu} - q_2 \mu q_1 \nu + (q_1 \cdot k_1 - q_2 \cdot k_2 - k_1 \cdot k_2) g_{\mu\nu}}{(q_1^2 - M^2)(q_2^2 - M^2)}. \]  

It is easy to see that in \( T_{111} \) and \( T_{311} \), \( q_3 \) only appears in the \( \delta \) functions and can be integrated to get 1. So to take the momenta of the corresponding propagators in \( T_1 \) and \( T_2 \) as the same is the result of the expression, e.g., that of the vertex and its contraction with the special ‘momentum over mass’ term of unitary gauge, hence the symmetric relation between these 2 diagrams. This is explicitly expressed here and has been employed before calculation as criteria for setting the independent integrated momentum in GWW. Here we see that two contraction respectively with the two vertices both lead to terms of \( q_3 \) and can be expressed symmetrically by \( q_1 \) and \( q_2 \).

By definition, for any finite \( q_1, q_2 \), and \( q_3 \), the corresponding \( \delta \) function can be always integrated to get 1. And since the trivial contribution of infinite points of the propagator momenta (see discussions in the following on symmetrical integration), there is no singularity to introduce extra term by this definition. Consequently the integrands in \( T_{111} \) and \( T_{311} \) are not dependent on \( q_3 \). So these two integrals are well defined and can be summed together, and then with \( M^{-4} \) terms from \( T_2 \), with the corresponding integrated propagator momenta taken as the same. The final result is definitely zero without uncertainty:

\[ T_{111} + T_{311} = \frac{1}{M^4} \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2) \]  
\[ \times q_1 \cdot q_2 \frac{2q_1 \cdot q_2 g_{\mu\nu} - q_1 \mu q_2 \nu - q_2 \mu q_1 \nu}{(q_1^2 - M^2)(q_2^2 - M^2)}, \]  

since \((q_1 \cdot k_2 - q_2 \cdot k_1 - k_1 \cdot k_2) g_{\mu\nu} + (q_1 \cdot k_1 - q_2 \cdot k_2 - k_1 \cdot k_2) g_{\mu\nu} = 0\), taking now \( q_1 - q_2 = k_1 + k_2 \).
from the $\delta$ function. The $M^{-4}$ term in $T_2$ is

$$T_{21} = \frac{-1}{M^4} \int d^4q_1 d^4q_2 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2) \times q_1 \cdot q_2 \frac{2q_1 \cdot q_2 g_{\mu\nu} - q_1 \mu q_2 \nu - q_2 \mu q_1 \nu}{(q_1^2 - M^2)(q_2^2 - M^2)},$$

so

$$T_{111} + T_{311} + T_{21} = 0.$$

The calculation is more simple than the explicit one with the independent loop momentum as in GWW.

As mentioned above, the expression of integrand of $T_{111}$ and $T_{311}$ are exactly expressed by $q_1, q_2$, and the $\delta$ functions are exactly the same form when integrating over $q_3$. Then the summation is just negative to $T_{21}$. So the cancellation is definite, just read from the expressions themselves. This general result—result from the most general expression without setting the momenta beforehand—'enforces' the set of propagator momenta by GWW when using the independent integrated momentum. If some one deliberately set one of the above three terms from the three diagrams with a different momentum ($q_1$ or $q_2$) from the other two, and since these are high divergent terms, this may lead $T_{111} + T_{311} + T_{21}$ to be low divergent or finite non-zero value. In $D(<4, \text{small enough})$ dimension, these are considered as NOT divergent terms, and the shift allowed. Then each of the terms itself can be calculated individually. If one of them makes a shift but others not, the limit $D$ to $4$ may give a non zero result. So the above relation and cancellation must be taken first even for $D$ dimension.

Here we would like to draw the attention of the reader that since all the metric tensor appears in this part with an external index (i.e., $\mu$ or $\nu$) which to be contracted with the photon polarization vectors, one will not get any terms dependent on the dimension, $D$, from the self-contraction of the metric tensor. So the conclusion is similar for $D$ as well as 4 dimensions. This argument is also valid for $M^{-2}$ terms but need to be checked for $M^0$ terms for there would appear self contraction of the metric tensor, which equals to $D$ rather than four in $D$ dimension.

Now it is seen that the 4th power divergence is exactly zero, and no finite (nonzero) term proportional to $M^{-4}$. The remaining terms from $T_{11}$ and $T_{31}$ is $T_{113}$ and $T_{313}$ which are
proportional to $M^{-2}$:

$$T_{113} = \frac{1}{M^2} \int d^4q_1d^4q_2d^4q_3(2\pi)^4\delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \quad (12)$$

$$\times \frac{q_1 \cdot q_2 V_{\mu\nu\gamma}(q_3, -k_2, -q_2)q_2^\gamma}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)}.$$  

$$T_{313} = \frac{1}{M^2} \int d^4q_1d^4q_2d^4q_3(2\pi)^4\delta(P - q_1 + q_2)\delta(q_1 - k_2 - q_3)\delta(q_3 - k_1 - q_2) \quad (13)$$

$$\times \frac{q_1 \cdot q_2 V_{\nu\mu\gamma}(q_3, -k_1, -q_2)q_2^\gamma}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)}.$$  

They are to be considered together with other $M^{-2}$ terms.

4. $M^{-2}$ terms

Besides the above $M^{-2}$ terms, we need to investigate the following: From $T_1$

$$T_{12} = -\frac{1}{M^2} \int d^4q_1d^4q_2d^4q_3(2\pi)^4\delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \quad (14)$$

$$\times g_\alpha^\beta g_\rho^\sigma q_2^\rho q_2^\sigma V_{\beta\mu\nu}(q_1, -k_1, -q_3) V_{\sigma\nu\gamma}(q_3, -k_2, -q_2)$$

$$\frac{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)}{4}.$$  

$$T_{13} = -\frac{1}{M^2} \int d^4q_1d^4q_2d^4q_3(2\pi)^4\delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \quad (15)$$

$$\times g_\alpha^\beta q_3^\rho q_3^\sigma g_\alpha^\gamma \frac{V_{\beta\mu\nu}(q_1, -k_1, -q_3) V_{\sigma\nu\gamma}(q_3, -k_2, -q_2)}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)}.$$  

$$T_{14} = -\frac{1}{M^2} \int d^4q_1d^4q_2d^4q_3(2\pi)^4\delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \quad (16)$$

$$\times q_1 q_1^\beta g_\rho^\sigma g_\alpha^\gamma \frac{V_{\beta\mu\nu}(q_1, -k_1, -q_3) V_{\sigma\nu\gamma}(q_3, -k_2, -q_2)}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)}.$$  

The term from $T_2$ is

$$T_{22+23} = \frac{1}{M^2} \int d^4q_1d^4q_2d^4q_3(2\pi)^4\delta(P - q_1 + q_2)\delta(q_1 - q_2 - k_1 - k_2) \quad (17)$$

$$\times \frac{2q_1^2 g_{\mu\nu} + 2q_2^2 g_{\mu\nu} - 2q_1 \mu q_1 \nu - 2q_2 \nu q_2}{(q_1^2 - M^2)(q_2^2 - M^2)}.$$  

This corresponds to $(M22+M23$ of GWW).  

We see $T_{12}$ and $T_{14}$ both give $(q_3^2 - M^2)$ factor in numerator when the Ward identity directly applied, which can reduce the corresponding factor in denominator and combine with $T_{22+23}$. The following shows the fact.
As above, by applying WI GWW2.10 to \( V_{\sigma\gamma}(q_3, -k_2, -q_2)q_2^\gamma \) and \( q_1^\beta V_{\beta\rho}(q_1, -k_1, -q_3) \) respectively to \( T_{12} \) and \( T_{14} \), both are written with 3 terms, respectively: \( T_{12} = T_{121} + T_{122} + T_{123} \) and \( T_{14} = T_{141} + T_{142} + T_{143} \).

This discussion also applies to \( T_3 \), so the following is to investigate

\[
T_{121} + T_{141} + T_{321} + T_{341} + T_{22+23}.
\]

For this part, we use \( q_2 = q_1 - P \), so can use W.I. again, while the extra term \( -P^\beta V_{\beta\mu}(q_1, -k_1, -q_3) \) will combine with the corresponding extra term from \( T_{141} \), since

\[
T_{141} = -\frac{1}{M^2} \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2)
\]

\[
\times \frac{q_2^\beta}{(q_1^2 - M^2)(q_2^2 - M^2)} V_{\mu\nu\gamma}(q_3, -k_2, -q_2) q_1^\gamma.
\]

We use \( q_1 = q_2 + P \), and the extra term is \( V_{\mu\nu\gamma}(q_3, -k_2, -q_2) P^\gamma \).

\[
T_{121} + T_{141} = -\frac{1}{M^2} \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2)
\]

\[
\times \frac{2q_3^\mu g_{\mu\nu} - 2q_3\mu g_{3\nu} + 2k_1 \cdot k_2 g_{\mu\nu} + 3q_3\mu k_{3\nu} - 3k_2\mu g_{3\nu} - 4k_2\mu k_{1\nu}}{(q_1^2 - M^2)(q_2^2 - M^2)}
\]

\[
= -\frac{1}{M^2} \int d^4q_1 d^4q_2 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2)
\]

\[
\times \frac{(q_1^2 + q_2^2) g_{\mu\nu} - 2k_1 \cdot q_1 g_{\mu\nu} + 2k_2 \cdot q_2 g_{\mu\nu} + 2k_1 \cdot k_2 g_{\mu\nu} - 2q_1\mu q_{2\nu} + 3q_1\mu k_{1\nu} - 3k_2\mu q_{2\nu} - 4k_2\mu k_{1\nu}}{(q_1^2 - M^2)(q_2^2 - M^2)}
\]

Here \( q_3\mu = q_1\mu \), and \( q_3\nu = q_2\nu \). Two \( q_3^\mu \)'s are expressed as functions of \( q_1^2 \) and \( q_2^2 \) respectively to get a more symmetric form. Then we integrate over \( q_3 \) to get 1 for the second step.

\[
T_{321} + T_{341} = -\frac{1}{M^2} \int d^4q_1 d^4q_2 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2)
\]

\[
\times \frac{(q_1^2 + q_2^2) g_{\mu\nu} - 2k_2 \cdot q_1 g_{\mu\nu} + 2k_1 \cdot q_2 g_{\mu\nu} + 2k_2 \cdot k_2 g_{\mu\nu} - 2q_1\mu q_{2\nu} + 3q_1\mu k_{2\nu} + 3k_1\nu q_{2\mu} - 3k_1\nu k_{2\mu} - 4k_2\mu k_{1\nu}}{(q_1^2 - M^2)(q_2^2 - M^2)}
\]

From the above two equations and Eq.(14), one can easily found

\[
T_{121} + T_{141} + T_{321} + T_{341} + T_{22+23} = 0,
\]

by observing \( 2q_1\mu q_{1\nu} + 2q_2\mu q_{2\nu} - 2q_1\mu q_{2\nu} - 2q_1\nu q_{2\mu} = 2k_2\mu k_{1\nu} \). This is also valid for D dimension.
Now the remaining $M^{-2}$ terms are all from $T_1$ and $T_2$. They will be investigated separately as GWW. Those from $T_1$
are
\[ T_{113} + T_{13} + T_{122} + T_{142}. \]

$T_{113}$ and $T_{13}$ are shown in Eq. (12) and Eq. (15), respectively. They can be directly calculated with the W.I. of GWW2.9, say, applying to $V_{\mu\nu\gamma}(q_3, -k_2, -q_2)q_2^\gamma$, $V_{\mu\nu\rho}(q_1, -k_1, -q_3)q_3^\rho$, and $q_3^\rho V_{\sigma\nu\gamma}(q_3, -k_2, -q_2)$. For Eqs. (14, 16), we have discussed their first term $T_{121}$ and $T_{141}$, while the second terms of them are

\[
T_{122} = \frac{1}{M^2} \int d^4q_1d^4q_2d^4q_3(2\pi)^4 \delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \quad (22)
\times q_2^\beta V_{\beta\mu\nu}(q_1, -k_1, -q_3)q_3^\nu g_3
\]

\[
T_{142} = \frac{1}{M^2} \int d^4q_1d^4q_2d^4q_3(2\pi)^4 \delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \quad (23)
\times \frac{q_3\mu q_3^\nu V_{\nu\sigma\gamma}(q_3, -k_2, -q_2)}{(q_1^2 - M^2)(q_2^2 - M^2)(q_3^2 - M^2)} q_1^\gamma.
\]

We here again apply the W.I. and combine them together to obtain a simple form

\[
T_{113} + T_{13} + T_{122} + T_{142} = \frac{1}{M^2} \int d^4q_1d^4q_2d^4q_3(2\pi)^4 \delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \quad (24)
\times \left(2q_1^2q_2^\mu q_2^\nu + 2q_2^2q_1^\mu q_1^\nu - 4q_1 \cdot q_2 q_1^\mu q_2^\nu + q_1 \cdot q_2 q_3^\mu g_\mu + q_1 \cdot q_2 q_3^\nu g_\nu - 2q_1^2g_\mu - 2q_2^2g_\nu \right).
\]

It looks as quadratic, but easy to see in fact to the most linear, since

\[
2q_1 \cdot q_2 = -(q_1 - q_2)^2 + q_1^2 + q_2^2, \text{ then } (2q_1^2q_2^\mu q_2^\nu + 2q_2^2q_1^\mu q_1^\nu - 4q_1 \cdot q_2 q_1^\mu q_2^\nu) \text{ equals to}
\]

\[
2q_1^2(q_2^\mu - q_1^\mu)q_2^\nu + 2q_2^2q_1^\mu(q_1^\nu - q_2^\nu) + 2(q_1 - q_2)^2g_\mu = 2q_1^2(-k_2^\mu)q_2^\nu + 2q_2^2q_1^\mu k_1^\nu + 2(k_1 + k_2)^2q_1^\mu q_2^\nu;
\]

and $q_1 \cdot q_2 q_3^\mu g_\mu - q_1^2 q_2^\nu g_\mu = -(k_1 + k_2)^2 q_3^\mu g_\mu + \frac{q_1^2 + q_2^2}{2} q_3^\mu g_\mu - q_1^2 q_2^\nu g_\mu$.

However, $\frac{q_1^2 + q_2^2}{2} q_3^\mu g_\mu - q_1^2 q_2^\nu g_\mu = (q_1^2 q_2 \cdot k_2 - q_2^2 q_1 \cdot k_1)g_\mu$ hence is also linear.

Now we write $T_{113} + T_{13} + T_{122} + T_{142}$ as the summation of two parts:

\[
\frac{1}{M^2} \int d^4q_1d^4q_2d^4q_3(2\pi)^4 \delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \quad (25)
\times 2q_1^2(-k_2^\mu)q_2^\nu + 2q_2^2q_1^\mu k_1^\nu + (q_1^2 q_2 \cdot k_2 - q_2^2 q_1 \cdot k_1)g_\mu.
\]
\[
\frac{1}{M^2} \int d^4 q_1 d^4 q_2 d^4 q_3 \frac{(2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2)}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)} \times 2(k_1 + k_2)^2 q_{1\mu} q_{2\nu} - \frac{(k_1 + k_2)^2}{2} q_3^2 g_{\mu\nu},
\]

(26)
i.e., the linear and logarithmic divergent terms respectively. Then the above linear term, after further taking out logarithmic and finite terms from it, should combine with the corresponding term from \(T_3\), then is deduced to get as two terms, one is logarithmic divergent, the other is finite.

Some details are:

\[
2q_1^2(-k_2\mu)q_{2\nu} = -2(q_3^2 - M^2)k_{2\mu}q_{2\nu} - 4q_3 \cdot k_1 k_2 q_{2\nu} - 2M^2 k_{2\mu}q_{2\nu},
\]

\[
2q_2^2 q_{1\mu} k_{1\nu} = 2(q_3^2 - M^2)q_{1\mu} k_{1\nu} - 4q_3 \cdot k_2 q_{1\mu} k_{1\nu} + 2M^2 q_{1\mu} k_{1\nu},
\]

\[
(q_1^2 q_2 \cdot k_2 - q_2^2 q_1 \cdot k_1) g_{\mu\nu} = (q_3^2 - M^2)(q_2 \cdot k_2 - q_1 \cdot k_1) g_{\mu\nu} + 4q_2 \cdot k_2 q_1 \cdot k_1 g_{\mu\nu} + M^2 (q_2 \cdot k_2 - q_1 \cdot k_1) g_{\mu\nu}.
\]

So it is the following linear term (with \((q_3^2 - M^2)\) factor reduced with the common one in the denominator, and \(q_3\) integrated)

\[
\frac{1}{M^2} \int d^4 q_1 d^4 q_2 \frac{(2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2)}{(q_1^2 - M^2)(q_2^2 - M^2)} (-2k_{2\mu}q_{2\nu}+2q_{1\mu}k_{1\nu}+(q_2 \cdot k_2 - q_1 \cdot k_1) g_{\mu\nu})
\]

(27)
to be combined with that from \(T_3\):

\[
\frac{1}{M^2} \int d^4 q_1 d^4 q_2 \frac{(2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2)}{(q_1^2 - M^2)(q_2^2 - M^2)} (-2k_{1\nu}q_{2\mu}+2q_{1\nu}k_{2\mu}+(q_2 \cdot k_1 - q_1 \cdot k_2) g_{\mu\nu}),
\]

(28)
and deduces to logarithmic term. Half of their summation is then:

\[
\frac{1}{M^2} \int d^4 q_1 d^4 q_2 \frac{(2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2)}{(q_1^2 - M^2)(q_2^2 - M^2)} (2k_{2\mu}k_{1\nu} - k_1 \cdot k_2 g_{\mu\nu}).
\]

(29)
This term can again be separated into a logarithmic term and a finite term,

\[
\frac{1}{M^2} \int d^4 q_1 d^4 q_2 d^4 q_3 \frac{(2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2)}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)} q_3^2 (2k_{2\mu}k_{1\nu} - k_1 \cdot k_2 g_{\mu\nu})
\]

\[
+ \frac{1}{M^2} \int d^4 q_1 d^4 q_2 d^4 q_3 \frac{(2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2)}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)} (-M^2)(2k_{2\mu}k_{1\nu} - k_1 \cdot k_2 g_{\mu\nu}).
\]

Hence effectively \(T_{113} + T_{13} + T_{122} + T_{142} = T1LG + T1F\), with

\[
T1LG = \frac{1}{M^2} \int d^4 q_1 d^4 q_2 d^4 q_3 \frac{(2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2)}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)} \times \left[(-2q_3^2 k_1 \cdot k_2 + 4k_1 \cdot q_3 k_2 \cdot q_3) g_{\mu\nu} - 4k_1 \cdot q_3 k_{2\mu} q_{3\nu} - 4k_2 \cdot q_3 q_{3\mu} k_{1\nu} + 2q_3^2 k_{2\mu} k_{1\nu} + 4k_1 \cdot k_2 q_{3\mu} q_{3\nu}\right];
\]

(30)
and

\[ T1F = \int d^4q_1d^4q_2d^4q_3 \frac{(2\pi)^4 \delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2)}{(q_1^2 - M^2)(q_2^2 - M^2)(q_3^2 - M^2)} \times \left[ 2q_1\mu k_1\nu - 2k_2\mu q_2\nu - 2k_2\mu k_1\nu + (q_2 \cdot k_2 - q_1 \cdot k_1 + k_1 \cdot k_2)g_{\mu\nu} \right]. \]  

(31)

The summation in fact is deduced into a quite simple form.

It can be checked, by the product with the the overall factor, the finite integral T1F is exactly GWW3.42 \((M_{1132})\) when taking the choice of integrated loop momentum as GWW. The logarithmic one T1LG is also easy to recover GWW3.41, when taking the choice of integrated loop momentum as GWW. It is in fact that employing whichever loop momentum, by the help of Feynman-Schwinger parametrization, one can always prove it as zero (or to be discussed in dimensional regularization which we ignore). A direct calculation of T1LG by the symmetrical integration without the help of Feynman-Schwinger parametrization will be done in Sec. 3.

This is the same for D dimension. We would like to remind that the 4 appear in the \(M^{-2}\) is the result of combination of 4 terms, not from the dimension. In the logarithmic \(M^0\) terms, things could be different.

Here we also see that the quadratic divergent terms are deduced directly to logarithmic, by summing \(T_1\) and \(T_3\). The linear divergent terms are all safely combined and summed to get logarithmic ones, according to the momentum relations, without the requirement of the average on \(k \leftrightarrow -k\) as in GWW.

From the derivation of the cancellation of the linear divergence, we see it is quite non trivial as to separate and recombine the terms. As the above, it again shows the subtle loop momentum relation exist between \(T_1\) and \(T_3\), which is correctly set before calculation by GWW, and then after the average on \(k\) and \(-k\), the linear divergence cancelled while the remaining logarithmic term are correct.

If we do not care about this, directly deal with the \(T_{113} + T_{13} + T_{122} + T_{142}\) as the summation of linear and logarithmic terms \((25) + (26)\) in D dimension, where D smaller than 4 and the integral is finite. Since it is finite, we directly employ Feynman-Schwinger parametrization and do the shift. We find that the linear divergence is of no problem cancelled. Terms giving logarithmic divergence for \(D \to 4\) and proportional to \(1/M^2\) appear. But it is not T1LG which can give FINITE result when \(D \to 4\); it is just divergence not able to be cancelled (by those from \(T_3\)). This again means that even one considers the extrapolation to D dimension,
the relation of loop momenta still need to be respected hence divergent terms higher than logarithmic cancelled. The dimension regularization can not take the place of that.

5. $M^0$ terms

The third part of $T_{12}$, $T_{14}$, i.e., $T_{123}$, $T_{143}$, as well as the corresponding ones of $T_3$, are $M^0$ terms and to be investigated here together with the corresponding terms from $T_2$ (Pay attention that $T_2$ lack of a overall minus sign):

\[ T_{123} = - \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2) \]
\[ \times q_2^\beta V_{\beta\mu\nu}(q_1, -k_1, -q_3) g_\nu^\gamma \]
\[ \times \frac{q_2^\gamma}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)} \]

\[ T_{143} = - \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2) \]
\[ \times \frac{g_\mu^\sigma V_{\sigma\nu\gamma}(q_3, -k_2, -q_2)}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)} q_1^\gamma \]

\[ T_{15} = \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2) \]
\[ \times g_\alpha^\beta g_\rho^\sigma V_{\beta\mu\nu}(q_1, -k_1, -q_3) V_{\sigma\nu\gamma}(q_3, -k_2, -q_2) \]
\[ \frac{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)} \]

\[ T_{24} = - \int d^4q_1 d^4q_2 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2) \]
\[ \times g_\alpha^\beta g_\rho^\sigma 2g_{\mu\nu}g_{\beta\gamma} - g_{\mu\beta}g_{\nu\gamma} - g_{\nu\gamma}g_{\mu\beta} \]
\[ \frac{(q_1^2 - M^2)(q_2^2 - M^2)}{(q_1^2 - M^2)(q_2^2 - M^2)} \]

These four terms (half of $T_{24}$) summed with T1F, the final result is

\[ \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2) \]
\[ \times \frac{-6k_1 \cdot k_2 g_{\mu\nu} + 6k_2m\cdot k_{1\nu} + 3M^2 g_{\mu\nu} + (-3q_3^2 g_{\mu\nu} + 12q_3m q_{3\nu})}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)} \]

In D dimension, only difference is that the terms in the last bracket of the numerator of second line is replaced by $-(D - 1)q_3^2 g_{\mu\nu} + 4(D - 1)q_3m q_{3\nu}$. The dealing with those of $T_3$ is just the same.

Similar way of calculation employing Feynman-Schwinger parametrization and Dyson subtraction as GWW will give the exact same result as GWW. We will not write in details here.
3. DYSON SUBTRACTION, DIMENSIONAL REGULARIZATION AND SYMMETRICAL INTEGRATION

Eq. (36) still has logarithmic divergence, simply summed with that from \( T_3 \) can not symmetrize each components of \( q_3 \). So one need to use Feynman-Schwinger parametrization to do the calculation, as done by GWW. Since this is logarithmic divergence, the shift is allowed. But one also need do the Dyson subtraction to get the U(1) gauge invariant result. In the GWW calculation, the symmetrical integration in 4 dimension

\[
\int d^4l \frac{l^2 g_{\mu\nu} - 4l_{\mu}l_{\nu}}{(l^2 - M^2 + i\epsilon)^3} = 0
\]

is employed.

On the other hand, for this logarithmic divergence, one can first do the calculation in \( D \) dimension, then discontinuously extrapolate to 4 dimension, an extra term other than doing symmetrical integration in 4 dimension as above

\[
\lim_{D \to 4} \int d^Dl \frac{l^2 g_{\mu\nu} - 4l_{\mu}l_{\nu}}{(l^2 - M^2 + i\epsilon)^3} = -\frac{i\pi^2}{2} g_{\mu\mu}
\]

can just play the role of Dyson subtraction term to let Eq. (36) invariant under U(1) gauge and get the similar result. However, this may suggest that TILG also must be calculated in \( D \) dimension and discontinuously extrapolate to 4 dimension and gives a finite terms, not zero as the symmetrical integration, just the non zero term at \( M \to 0 \), as discussed in literature [11].

Whatever considered as uncertainty if only considering this process and in this unitary gauge (In Appendix B, we see that in \( R_1 \) gauge, however, these two approaches give the similar result), we will not investigate this topic, but just discuss the symmetrical integration and its relations with the definition of the free particle propagator in the formal theory of scattering. We will employ the symmetrical integration to deal with the divergences proportional to \( M_Z^2/M^4 \) in the \( H \to \gamma Z \) process.

In the framework of perturbative theory in the interaction picture of the canonical quantization, The *Feynman* propagator, is the vacuum expectation value of the time ordered product of field operators. The time-boundary condition for these operators are as of the free field since the interaction is adiabatically removed at time equals to plus/minus infinity, and for interactions decrease fast enough with distance, the wave function in infinitely distant place should be the superposition of the plane wave and the spherical wave. Because
of this, first of all, in the 4-dimension expression, two poles of the denominator guarantee the Heaviside function for the time order, hence for the proper causality.

The coordinate space Feynman propagator (taking a scalar field as example) is

\[ D_F(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik \cdot (x-y)}. \]  

(37)

In doing the integration of \( k^0 \) on the complex number plane, the behaviour of the integrand when \( k^0 \) goes to infinity must be suppressed, by the imaginary part of \( k^0 \), via the (negative real) exponential factor. This is convinced by the Jordan Lemma. Indeed, the exponential factor in the integral of Eq. (37) is the necessary factor for the application of the residue theorem to do the integration of Eq. (37). Hence for a specific Feynman diagram, though the behaviour of the integrand in the phase space could be complex, but for each \( q \), the momentum of the propagator, the large \( q^0 \) behaviour must be suppressed, or else the definition of the propagator is deviated from what is in the original form, such as Eq. (37). At the same time, for the pole to be in the closed contour so as to get non zero result, we must require the modulo:

\[ \lim_{q^0 \to \infty, q \to \infty} \frac{q}{q^0} < 1. \]  

(38)

So the behaviour of the \( q^0 \) control all the four components in infinity.

And this means, if without the exponential functions, or once it is integrated to get the \( \delta \) functions, one has to keep in mind the interaction picture perturbative theory in fact based on the on shell real physical (free) particle Hilbert space, the 'virtual off-shell states' mainly come from the way to write the propagator from a three dimensional integral to the four dimensional covariant one. The extra integration is introduced as the contour integral on the complex plane. With the condition in the above equation to keep the pole in the contour, the infinite behaviour is 'enforced' to be suppressed by the exponential, so that the Jordan lemma valid and the 3-dimensional and 4-dimensional forms equal to each other. This indicates that, without the exponential, the behaviour at infinity of the propagator momenta, is still in need to be suppressed, to fulfill the requirement of the Jordan Lemma. For the high level divergent terms that exactly the same and cancel, we need not to take account this fact. But this fact is needed as the basis for doing symmetrical integration, convergent or divergent.

In the realistic calculation, the Wick rotation is performed. Before the rotation, since the exponential factor, as convinced by the Jordan lemma, integration of \( k^0 \) on the infinite
points of the imaginary axis is suppressed and this is also the necessary condition for the rotation, i.e., no poles in 1,3 quadrants of the complex $k^0$ plane. After the rotation, these four components are in more symmetric footing. The ‘periodic (or vanishing) boundary conditions’ for space components is similar, as to considering the Lippmann-Schwinger Equation for the static scattering state. The three dimensional Green function of the three dimensional Lippmann-Schwinger Equation is also defined by a contour integral on the complex momentum plane with an exponential, to guarantee the wave function in infinite distance is the superposition of the plane wave and the spherical wave, when the interaction potential decrease fast enough. The exponential suppresses the infinite contribution of the momentum.

Based on the above discussions, adopting the symmetrical integration, we can directly calculate the T1LG to show it is zero without the help of Feynman-Schwinger parametrization \[12\]. Integrating the $\delta$ functions, and keeping $q_3$ as the independent integrated variable, we get

$$T_{1LG} = \frac{1}{M^2} \int d^4q_3 \frac{(2\pi)^4 \delta(P-k_1-k_2)}{((q_3+k_1)^2-M^2)(q_3^2-M^2)((q_3-k_2)^2-M^2)} N_{\mu\nu},$$ \hfill (39)$$

with

$$N_{\mu\nu} = \left[-2q_3^2 k_1 \cdot k_2 + 4k_1 \cdot q_3 k_2 \cdot q_3 \right] g_{\mu\nu} - 4k_1 \cdot q_3 k_2 \cdot q_3 g_{\mu\nu} - 4k_2 \cdot q_3 q_3 g_{\mu\nu} + 2q_3^2 k_2 \cdot q_3 g_{\mu\nu} + 4k_1 \cdot k_2 q_3 g_{\mu\nu}.$$ \hfill (40)$$

Obviously,

$$\frac{1}{M^2} \int d^4q_3 \frac{(2\pi)^4 \delta(P-k_1-k_2)}{((q_3+k_1)^2-M^2)(q_3^2-M^2)(q_3^2-M^2)} N_{\mu\nu} = 0,$$ \hfill (41)$$

adopting the symmetrical integration (and hereafter we discuss straightforwardly in 4 dimension). Useful in the following, we point out

$$\int d^4q_3 \frac{k_1 \cdot q_3}{(q_3^2-M^2)(q_3^2-M^2)(q_3^2-M^2)} = 0,$$ \hfill (42)$$

and

$$\int d^4q_3 \frac{(k_1 \cdot q_3)^2}{(q_3^2-M^2)(q_3^2-M^2)(q_3^2-M^2)} = 0,$$ \hfill (43)$$

and also valid for any denominator which is respectively symmetrical on all components of $q_3$. To get the above equation we notice, under symmetrical integration, $q_{3\alpha} q_{3\beta} = 1/4 q_3^2 \delta_{\alpha\beta}$. Then we employ the fact $k_1^2 = 0$. This can be developed as to show the nilpotency under symmetrical integration for $k \cdot q$, where $k^2 = 0$ and $q$ is the integrated variable. This discussion can also be valid for cases a ‘symmetric’ factor like $N_{\mu\nu}$ introduced.
In the denominator of T1LG, two factors break the symmetry of all the components of $q_3$, so we first write is as

$$
\frac{1}{((q_3 + k_1)^2 - M^2)(q_3^2 - M^2)((q_3 - k_2)^2 - M^2)}
$$

$$
= \frac{1}{2} \left( \frac{(q_3^2 - M^2)(q_3^2 - M^2)((q_3 - k_2)^2 - M^2)}{((q_3 + k_1)^2 - M^2)(q_3^2 - M^2)((q_3 - k_2)^2 - M^2)} + \frac{-2k_1 \cdot q_3}{((q_3 + k_1)^2 - M^2)(q_3^2 - M^2)((q_3 - k_2)^2 - M^2)} \right)
$$

$$
+ \frac{1}{2} \left( \frac{1}{((q_3 + k_1)^2 - M^2)(q_3^2 - M^2)((q_3 - k_2)^2 - M^2)} + \frac{2k_2 \cdot q_3}{((q_3 + k_1)^2 - M^2)(q_3^2 - M^2)((q_3 - k_2)^2 - M^2)} \right).
$$

The first terms in 2nd and 3rd lines of the above equation, times $N_{\mu\nu}$ can be proved to be zero under the symmetrical integration of $q_3$. Taking $\frac{1}{((q_3 + k_1)^2 - M^2)(q_3^2 - M^2)((q_3 - k_2)^2 - M^2)}$ as example:

$$
\int d^4q_3 \left( \frac{N_{\mu\nu}}{((q_3 + k_1)^2 - M^2)(q_3^2 - M^2)((q_3 - k_2)^2 - M^2)} \right)
$$

$$
= \int d^4q_3 \left( \frac{-2k_1 \cdot q_3 N_{\mu\nu}}{(q_3^2 - M^2)((q_3 + k_1)^2 - M^2)} \right)
$$

$$
= \int d^4q_3 \left( \frac{(-2k_1 \cdot q_3)^2 N_{\mu\nu}}{(q_3^2 - M^2)((q_3 + k_1)^2 - M^2)} \right)
$$

$$
= \cdots
$$

$$
= \lim_{m \to \infty} \int d^4q_3 \left( \frac{(-2k_1 \cdot q_3)^m N_{\mu\nu}}{(q_3^2 - M^2)((q_3 + k_1)^2 - M^2)} \right).
$$

The last line is easy to be proved to be zero by an integration by parts. This result is irrelevant with the number of $(q_3^2 - M^2)$ factors at beginning. Likely irritation of 'symmetrization' as the above equation, one can find

$$
\int d^4q_3 \left( \frac{1}{((q_3 + k_1)^2 - M^2)(q_3^2 - M^2)((q_3 - k_2)^2 - M^2)} N_{\mu\nu} \right)
$$

$$
= \lim_{m \to \infty} \int d^4q_3 \left( \frac{(-2k_1 \cdot q_3)^m}{((q_3 + k_1)^2 - M^2)(q_3^2 - M^2)((q_3 - k_2)^2 - M^2)} N_{\mu\nu} \right)
$$

$$
+ \frac{(2k_2 \cdot q_3)^m}{((q_3 + k_1)^2 - M^2)(q_3^2 - M^2)((q_3 - k_2)^2 - M^2)} N_{\mu\nu}/2,
$$

also easy to be proved to be zero by the integration by parts. As a matter of fact, this investigation is valid as long as the number of the 'asymmetric' factors such as $(q_3 + k_1)^2 - M^2$ is finite, i.e., a loop with finite number of vertices (so in the unitary gauge this result can not be straightforwardly valid to infinite orders).
4. STANDARD WARD IDENTITY IN QED

From the above we learn that for any diagram, there is a definitely 'original' definition without any ambiguity of the loop integral momenta. This experience of calculation demonstrated here are expected to be applied to higher loop level calculations to eliminate any ambiguity for the choice of the loop integrated momenta.

This also helps for the understanding of the momentum flow in discussing the Ward identity in QED.

Things can be easily made clear at one loop for QED. The electron self energy can be written as

$$\Sigma(p_1) = \Sigma(p_1, p_2) \sim \int d^4q d^4k \delta(p_1 - q - k)\delta(q + k - p_2)\gamma^\mu \frac{1}{q - m} \gamma^\mu \frac{1}{k^2}$$  (48)

It represent a factor like

$$\int dx \int dy \bar{\Psi}(x) A^\mu(x) \gamma^\mu \Psi(y)$$  (49)

as a part of the Wick expansion of some order of the perturbation expansion of the S-matrix (the · and ¶ signalize the 'contraction', i.e., vacuum expectation value of the time-ordered product of the field operators). We can move one δ function $$\delta(q + k - p_2) = \delta(p_1 - p_2)$$ out, only deal with

$$\Sigma(p_1) \sim \int d^4q d^4k \delta(p_1 - q - k)\gamma^\mu \frac{1}{q - m} \gamma^\mu \frac{1}{k^2},$$  (50)

and in deed track back to the original form

$$\Sigma(p_1) \sim \int d^4x_1 \int d^4q d^4k e^{i(p_1 - q - k)\cdot x_1}\gamma^\mu \frac{1}{q - m} \gamma^\mu \frac{1}{k^2}$$  (51)

($$2\pi^4$$ factor taken away).

So we have

$$\frac{\partial \Sigma(p_1)}{\partial p_1^\nu} \sim \int d^4x_1 \int d^4q d^4k (ia^\nu_1)e^{i(p_1 - q - k)\cdot x_1}\gamma^\mu \frac{1}{q - m} \gamma^\mu \frac{1}{k^2}$$

$$= \int d^4x_1 \int d^4q d^4k (-\frac{\partial}{\partial q_\nu} e^{i(p_1 - q - k)\cdot x_1})\gamma^\mu \frac{1}{q - m} \gamma^\mu \frac{1}{k^2}$$

$$= \int d^4x_1 \int d^4q d^4k e^{i(p_1 - q - k)\cdot x_1}\gamma^\mu (\frac{\partial}{\partial q_\nu} \frac{1}{q - m})\gamma^\mu \frac{1}{k^2}$$  (52)

with a surface term eliminated for getting the last line (This is consistent with the static single particle wave function space boundary condition referring to Lippman-Schwinger Equation). This explains why seems $$p_1$$ only flow via the fermion line—of course it can only flow
via the photon line, to get a ‘useless’ formulation not relevant to the W.I.. But the key point is that \( p_1 \) can not be taken as to flow via both photon and fermion lines, which leads to a double counting.

5. **DIVERGENT TERMS PROPORTIONAL TO \( \frac{M^2}{M^4} \) FOR \( H \to \gamma Z \) PROCESS VIA ONE W LOOP**

Though less experimentally significant \([8,9]\), calculation of the \( H \to \gamma Z \) process via one W loop in the unitary gauge encounters new quadratic divergence proportional to \( \frac{M^2}{M^4} \) (taking into account the W mass from the coupling, \( \frac{M^2}{M^4} \)). Even, if this divergence is not possible to be got rid of so that unitary gauge can not give finite result, the validity of unitary gauge may be questioned. In this section we demonstrate the cancellation of all the divergences proportional to \( \frac{M^2}{M^4} \), which is crucial for the complete calculation on this process in the unitary gauge and for comparing with results from other gauges.

There are totally 4 Feynman diagrams, one of them is zero \([2]\), the other three are similar as the \( H \to \gamma \gamma \) process via one W loop (Figure 1), except that \( k_2 \) represent the 4-momentum of the Z particle, with the corresponding polarization vectors \( e^{\nu}_{Z\lambda}, \lambda = 1, 2, 3 \), since Z is massive. We still have \( k_2 e^{\nu}_{Z\lambda} = 0, \forall \lambda \) (see Eq. (A1)). There is also an extra \( \cot \theta_W \) factor for the WWZ vertex, where \( \theta_W \) is the Weinberg angle.

Correspondingly, the amplitudes \( T_{1Z}, T_{2Z}, T_{3Z} \) are formally similar as Eq. (1-3), except the extra \( \cot \theta_W \) factor and the \( k_2 \) representing the 4-momentum of Z particle. We again omit the polarization vectors for the photon and Z particle as above.

Since there is still a WW\( \gamma \) vertex, the \( M^{-6} \) terms in \( T_{1Z} \) and \( T_{3Z} \) are again zero. For the \( M^{-4} \) terms, in \( T_{1Z} \) and \( T_{3Z} \), respectively, there are 3 ways of the combination of two of three \( q_i^{\alpha_i} q_i^{\beta_i}/M^2 \) \((i = 1, 2, 3)\) from the W propagators. One combination is zero because of the W.I. of the WW\( \gamma \) vertex. Another gives a \( M^2_Z/M^4 \) terms because of the W.I. of the WWZ vertex (Eq. (A10)). The third corresponds to those of the \( H \to \gamma \gamma \) process, gives likely terms besides extra \( M^2_Z/M^4 \) terms.

Those \( M^{-4} \) terms of \( T_{2Z} \) is formally similar as those of \( T_2 \).

Collecting all the \( M^{-4} \) terms (the \( M^{-2} \) terms because of the application of Eq. (A7) are to the most logarithmically divergent, not discussed here), taking the integrated variables the same for these three diagrams, one finds all the uncancelled terms of \( M^{-4} \) order are
proportional to \( M_Z^2 \), so are zero when Z mass goes to zero. They are (times \( -ie^2 qM \cot \theta W \)):

\[
T_c = \frac{1}{M^4} \int d^4q_1 d^4q_2 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2) \frac{-M_Z^2}{(q_1^2 - M^2)(q_2^2 - M^2)} \times q_1 \cdot q_2 g_{\mu\nu},
\]

\[
T_{11B} = \frac{1}{M^4} \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2) \times \frac{M_Z^2}{(q_1^2 - M^2)(q_2^2 - M^2)(q_3^2 - M^2)} (-q_2^2 q_{2\mu} q_{2\nu} + 2q_1 \cdot q_2 q_{3\mu} q_{3\nu})
\]

\[
T_{31B} = \frac{1}{M^4} \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_2 - q_3) \delta(q_3 - k_1 - q_2) \times \frac{M_Z^2}{(q_1^2 - M^2)(q_2^2 - M^2)(q_3^2 - M^2)} (-q_2^2 q_{1\mu} q_{1\nu} + 2q_1 \cdot q_2 q_{3\mu} q_{3\nu})
\]

In the above equations, \((q_1 - q_2) = (k_1 + k_2)\) is used. In the last line of each, we have decomposed them into various terms with various levels of divergences.

Before discussing the \( M_Z^2/M^4 \) terms, we first collect the uncancelled \( M_Z^2/M^2 \) terms appearing in above equations for further investigation. The ones directly read from the Eqs. (54) and (55) respectively are the fourth term in last line

\[
T_{11B1} = \frac{1}{M^4} \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2) \times \frac{M_Z^2}{(q_1^2 - M^2)(q_2^2 - M^2)(q_3^2 - M^2)} M^2 k_{2\mu} q_{2\nu},
\]

24
\[ T_{31B1} = \frac{1}{M^4} \int d^4 q_1 d^4 q_2 d^4 q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_2 - q_3) \delta(q_3 - k_1 - q_2) \] (57)
\[ \times \frac{M^2_Z}{(q_1^2 - M^2)(q_2^2 - M^2)(q_3^2 - M^2)} (-M^2 k_{2\mu} q_{1\nu}). \]

The denominators are not the same, since different kinematic configuration. Similar attention should be paid in the following.

The linear divergent terms in Eqs. (54) and (55) respectively are the third term in last line. Similar as in Sec.2, the \((q_3^2 - M^2)\) reduces the corresponding factor in the denominator, and then the terms independent on \(q_3\). Integrating on \(q_3\) in both and they can combine and gives

\[ \frac{1}{M^4} \int d^4 q_1 d^4 q_2 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2) \frac{-M^2_Z}{(q_1^2 - M^2)(q_2^2 - M^2)} k_{2\mu} k_{1\nu}. \] (58)

Again \((q_1 - q_2) = (k_1 + k_2)\) is used.

Further, dividing Eq. (58) by 2, each respectively recover a \((q_3^2 - M^2)\) factor in the numerator and denominator, and recover a third \(\delta\) function and integration on \(q_3\) corresponding to \(T_{1Z}\) and \(T_{3Z}\). The \(q_3^2\) term of numerator respectively cancels the fifth (last) term in last line respectively of Eqs. (57) and (55). This cancellation can be straightforwardly shown by the symmetrical integration discussed in Sec.3, while the remaining \(M^2_Z/M^2\) terms are

\[ T_{11B2} = \frac{1}{2M^4} \int d^4 q_1 d^4 q_2 d^4 q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2) \] (59)
\[ \times \frac{M^2_Z}{(q_1^2 - M^2)(q_2^2 - M^2)(q_3^2 - M^2)} (M^2 k_{2\mu} k_{1\nu}), \]

\[ T_{31B2} = \frac{1}{2M^4} \int d^4 q_1 d^4 q_2 d^4 q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_2 - q_3) \delta(q_3 - k_1 - q_2) \] (60)
\[ \times \frac{M^2_Z}{(q_1^2 - M^2)(q_2^2 - M^2)(q_3^2 - M^2)} (M^2 k_{2\mu} k_{1\nu}). \]

Similar cancellation employing symmetrical integration as above is also done for first term of last line respectively of Eqs. (54) and (55) with first term of last line of the equation of \(T_c\), and the remaining terms are

\[ T_{11B3} = \frac{1}{4M^4} \int d^4 q_1 d^4 q_2 d^4 q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2) \] (61)
\[ \times \frac{M^2_Z}{(q_1^2 - M^2)(q_2^2 - M^2)(q_3^2 - M^2)} (M^2 g_{\mu\nu}), \]
\[ T_{31B3} = \frac{1}{4M^4} \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2) \delta(q_3 - k_1 - k_2) \delta(q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2) \delta(q_3 - k_2 - q_3) \delta(q_3 - k_1 - q_2) \delta(q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2) \delta(q_3 - k_2 - q_3) \delta(q_3 - k_1 - q_2) \]

(62)

\[
\times \frac{M_2^2}{(q_1^2 - M^2)(q_2^2 - M^2)q_3^2 - M^2)}(M^2 g_{\mu\nu}).
\]

All the above remaining non zero terms are finite. We will not discuss the calculations and results in this paper. They are possibly non zero since that both W and Z masses go to 0 is a limit.

Now the quadratically divergent terms:

\[ T_{eq} = \frac{1}{M^4} \int d^4q_1 d^4q_2 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2) \delta(q_1 + q_2) \delta(q_1 - q_2 - k_1 - k_2) \delta(q_3 - k_2 - q_3) \delta(q_3 - k_1 - q_2) \delta(q_3 - k_2 - q_3) \delta(q_3 - k_1 - q_2) \]

(63)

\[
\times (-\frac{q_1^2 + q_2^2}{2}) g_{\mu\nu},
\]

\[ T_{11Bq} = \frac{1}{M^4} \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2) \]

(64)

\[
\times \frac{M_2^2}{(q_1^2 - M^2)(q_2^2 - M^2)q_3^2 - M^2)}q_1 q_3 \mu q_3 \nu,
\]

\[ T_{31Bq} = \frac{1}{M^4} \int d^4q_1 d^4q_2 d^4q_3 (2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_2 - q_3) \delta(q_3 - k_1 - q_2) \]

(65)

\[
\times \frac{M_2^2}{(q_1^2 - M^2)(q_2^2 - M^2)q_3^2 - M^2)}q_1 q_3 \mu q_3 \nu.
\]

From Eqs. (64) and (65), we suspect that one can respectively reduce a factor \((q_2^2 - M^2)\) or \((q_1^2 - M^2)\) from numerator and denominator, and to combine with the corresponding term in Eq. (63). For Eq. (63), one adds the \((q_3^2 - M^2)\) factor in numerator and denominator and the corresponding integration and \(\delta\) function for a re-scaled \(q_3\) (see following Eqs. (67, 68)), for the sake of the cancellation of the quadratic divergence by the symmetrical integration; then reduces the denominator factor \((q_2^2 - M^2)\) or \((q_1^2 - M^2)\) as in Eqs. (64) and (65) for the proper combination. We also employ the fact only \(k_1 \cdot q_3\) appears and the symmetrical integration discussed in Sec. 3 can be straightforwardly applied here, so terms proportional to

\[
\frac{2\sqrt{2}k_1 \cdot q_3}{(q_3 - \sqrt{2}k_1)^2 - 2M^2}
\]

(66)

is zero under symmetrical integration. Hence no linear divergences.

The concrete procedure and results are:

We combine the \(q_2^2\) term of \(T_{eq}\) with \(T_{11Bq}\) and \(q_1^2\) term of \(T_{eq}\) with \(T_{31Bq}\), and employ the former for the illustration in details. The combination of \(q_2^2\) term of \(T_{eq}\) with \(T_{11Bq}\) are
divided into 5 terms, the first three are from $T_{eq}$, and the other 2 terms from $T_{11Bq}$. In the following equations, the number of $\delta$ functions implies the integration and the number of integration variables, and common factors as $M^2_Z/M^4$ are taken away.

\[
\frac{(-1/2)q_3^2g_{\mu\nu}}{(q_1^2 - M^2)(q_3^2 - M^2)} \delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3/a)\delta(q_3 - (k_2 + q_2)a) \tag{67}
\]

\[
\frac{(1/2)M^2g_{\mu\nu}}{(q_1^2 - M^2)(q_3^2 - M^2)} \delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3/a)\delta(q_3 - (k_2 + q_2)a) \tag{68}
\]

\[
\frac{(-1/2)M^2g_{\mu\nu}}{(q_1^2 - M^2)(q_3^2 - M^2)} \delta(P - q_1 + q_2)\delta(k_1 + k_2 - q_1 + q_2) \tag{69}
\]

\[
\frac{q_3q_{3\mu}g_{3\nu}}{(q_1^2 - M^2)(q_3^2 - M^2)} \delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2) \tag{70}
\]

\[
\frac{q_{3\mu}q_{3\nu}}{(q_1^2 - M^2)(q_3^2 - M^2)(q_3^2 - M^2)} \delta(P - q_1 + q_2)\delta(q_1 - k_1 - q_3)\delta(q_3 - k_2 - q_2), \tag{71}
\]

with $a = \sqrt{2}$.

Since $q_2$ only appear in $\delta$ function for the first line (67), we first integrate it with the 3rd $\delta$ function, and then integrate the $q_1$ with the second $\delta$ function, we get

\[
\frac{(-1/4)q_3^2g_{\mu\nu}}{((q_3 + \sqrt{2}k_1)^2 - 2M^2)(q_3^2 - M^2)}. \tag{72}
\]

It combines with the fourth line (70) (also use $q_3$ as independent integrated momentum). The quadratic divergence is cancelled by means of symmetrical integration; then the linear divergence also zero under symmetrical integration; then the remained logarithmic term canceling second line (68) to leave finite term

\[
\frac{(-1/4)M^4g_{\mu\nu}}{((q_3 + \sqrt{2}k_1)^2 - 2M^2)(q_3^2 - M^2)^2}. \tag{73}
\]

For the remaining 3rd line (69) and 5th line (71), they are logarithmic, and the 3rd line (69) again need rescaled variable $q_3$ introduced. But now we need take $q_1$ as independent variable: In one word, this re-scaling converts the multiple to the shift, and cancels the divergence employing symmetrical integration. Rather, we can consider the re-introducing $q_3$ and the corresponding $\delta$ functions in above sections and those in this section before the dealing with the quadratic divergence cancellation as the the case of re-scaling factor $a = 1$.

Now we see that the total result on the terms proportional to $M^2_Z/M^4$ not only finite but simply zero. And the $M^2_Z/M^2$ terms they left after all the above cancellation are finite.

From this section, we learn that the calculation in the way introduced in this paper, especially not to integrate the $\delta$ functions before have to, provides the exact definition of the
Feynman diagrams and provides the necessary way of the conversion of integrated variables to let the unphysical ultraviolet divergences cancelled. The above study is on a little more complex process, especially that $T_c$ is not coming from a single diagram but the summation of parts from all the three diagrams, and that $T_{eq}$ need a re-scaled variable to be re-introduced. One may conclude that for the most general case of the calculation of the Feynman diagrams, the correct way of setting the independent integrated variables at the beginning as done by GWW, may not be available. So calculation without integration on all the $\delta$ functions until have to is a more proper or maybe necessary way of the employment of the Feynman rules.

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Appendix A: Mathematics for $H \rightarrow \gamma Z$ corresponding to Eqs. (2.5-2.12) of GWW

(These formulation will recover to the complemented GWW formulations which we employed in the above for calculating the $H \rightarrow \gamma \gamma$ process once taking $M_Z = 0$.)

\[
k_1^2 = 0, \quad k_2^2 = M_Z^2, \quad k_{1\mu} = k_{2\nu} = 0. \quad (A1)
\]

\[
(k_1 + k_2)^2 = 2k_1 \cdot k_2 + M_Z^2 = M_H^2. \quad (A2)
\]

\[
V_{\alpha\beta\gamma}(p_1, p_2, p_3) = (p_2 - p_3)_{\alpha}g_{\beta\gamma} + (p_3 - p_1)_{\beta}g_{\gamma\alpha} + (p_1 - p_2)_{\gamma}g_{\alpha\beta}; \quad (A3)
\]

\[
p_1 + p_2 + p_3 = 0 \quad (incoming).
\]

\[
p_1^2V_{\alpha\beta\gamma}(p_1, p_2, p_3) = (p_2^2g_{\beta\gamma} - p_3^2p_{3\gamma}) - (p_2^2g_{\beta\gamma} - p_2p_{2\gamma}) \quad (A4)
\]

\[
V_{\alpha\beta\gamma}(p_1, p_2, p_3)p_3^\gamma = -(p_1^2g_{\alpha\beta} - p_{1\alpha}p_{1\beta}) + (p_2^2g_{\alpha\beta} - p_{2\alpha}p_{2\beta})
\]
\[ p_1^2 V_{\alpha\mu\gamma}(p_1, -k_1, p_3) = p_3^2 g_{\mu\gamma} - p_{3\mu} p_{3\gamma} \quad (A5) \]

\[ V_{\alpha\mu\gamma}(p_1, -k_1, p_3) p_3^2 = -(p_1^2 g_{\alpha\mu} - p_{1\alpha} p_{1\mu}) \]

\[ p_1^2 V_{\alpha\nu\gamma}(p_1, -k_2, p_3) = p_3^2 g_{\nu\gamma} - p_{3\nu} p_{3\gamma} - M_Z^2 g_{\nu\gamma} \quad (A6) \]

\[ V_{\alpha\nu\gamma}(p_1, -k_2, p_3) p_3^2 = -(p_1^2 g_{\alpha\nu} - p_{1\alpha} p_{1\nu}) + M_Z^2 g_{\alpha\nu} \]

\[ p_1^2 V_{\alpha\mu\gamma}(p_1, -k_1, p_3) = (p_3^2 - M^2) g_{\mu\gamma} - p_{3\mu} p_{3\gamma} + M^2 g_{\mu\gamma} \quad (A7) \]

\[ V_{\alpha\mu\gamma}(p_1, -k_1, p_3) p_3^2 = -[(p_1^2 - M^2) g_{\alpha\mu} - p_{1\alpha} p_{1\mu}] - M^2 g_{\alpha\mu} \]

\[ p_1^2 V_{\alpha\nu\gamma}(p_1, -k_2, p_3) = (p_3^2 - M^2) g_{\nu\gamma} - p_{3\nu} p_{3\gamma} + (M^2 - M_Z^2) g_{\nu\gamma} \quad (A8) \]

\[ V_{\alpha\nu\gamma}(p_1, -k_2, p_3) p_3^2 = -[(p_1^2 - M^2) g_{\alpha\nu} - p_{1\alpha} p_{1\nu}] - (M^2 - M_Z^2) g_{\alpha\nu} \]

\[ p_1^2 V_{\alpha\mu\gamma}(p_1, -k_1, p_3) p_3^2 = 0 \quad (A9) \]

\[ p_1^2 V_{\alpha\nu\gamma}(p_1, -k_2, p_3) p_3^2 = -M_Z^2 p_{3\nu} = M_Z^2 p_{1\nu} \quad (A10) \]

**Appendix B: \( R_1 \) gauge**

The goldstone contribution is the only one which can contribute to the term not zero for \( M = 0 \), they are (with the \( i e^2 g M^2_H / M \) factor):

\[ A = \int d^4q_1 d^4q_2 d^4q_3 \frac{(2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2)}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)M^2} (4 q_{3\mu} q_{3\nu}) \quad (B1) \]

\[ B = \int d^4q_1 d^4q_2 d^4q_3 \frac{(2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2)}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)M^2} (-q_3^2 g_{\mu\nu}) \quad (B2) \]

\[ C = \int d^4q_1 d^4q_2 d^4q_3 \frac{(2\pi)^4 \delta(P - q_1 + q_2) \delta(q_1 - k_1 - q_3) \delta(q_3 - k_2 - q_2)}{(q_1^2 - M^2)(q_3^2 - M^2)(q_2^2 - M^2)M^2} (M^2 g_{\mu\nu}) \quad (B3) \]

After the Feynman-Schwinger parametrization, \( A \) contributes \( a_1 = 4l_\mu l_\nu \) and \( a_2 = -4\alpha_1\alpha_2 k_2 \mu k_1 \nu \). \( B \) contributes \( b_1 = -l^2 g_{\mu\nu} \) and \( b_2 = 2\alpha_1\alpha_2 k_1 \cdot k_2 g_{\mu\nu} \), both with an \( M^{-2} \) factor. \( \alpha_1 \alpha_2 \) are the Feynman-Schwinger parameters. \( C \) contributes \( g_{\mu\nu} \) without the \( M^{-2} \) factor.

Only the convergent term of \( A \) and \( B \), \( a_2 b_2 \) can not give the gauge invariant amplitude. However, the dimensional regularization calculation with the discontinuous extrapolation to 4 dimension on \( (a_1 + b_1) + c \) can compensate to give the gauge invariant result. On the other hand, to take the symmetrical integration in 4 dimension, where the corresponding
divergent terms cancel, then the Dyson subtraction term (take the $M_H$ as coupling constant not zero) give the same result. I.e., both give

$$g_{\mu \nu} \left( \frac{M^2}{M^2 - 2\alpha_1 \alpha_2 k_1 \cdot k_2} - 1 \right) = g_{\mu \nu} \frac{2\alpha_1 \alpha_2 k_1 \cdot k_2}{(M^2 - 2\alpha_1 \alpha_2 k_1 \cdot k_2)}$$  \hspace{1cm} (B4)$$

and can combine with the convergent terms to recover the gauge invariance and give the non zero term for $M=0$.

So, in the unitary gauge, employing dimensional regularization and Dyson subtraction will give different result. In $R_1$ gauge, these two approaches give the similar result. In this consideration, the difference for different gauges is really a fatal problem which need more investigation (As a matter of fact, the set of Feynman Rules can be seen as the same when $\xi \to \infty$, however, even in this limit, the Lagrangians are not the same, formally seems two kinds of realization for the standard model from the naïve view point of canonical quantization). For any ‘renormalizable gauge’, the method introduced in this paper are not necessary for this special process, since there is no divergence higher than logarithmic.

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[10] e.g., in $T_1$, $q_1 = k + (k_1 + k_2)/2$, $q_2 = k - (k_1 + k_2)/2$, and $q_3 = k - (k_1 + k_2)/2$, respectively.
[11] The equivalence theorem or the decoupling theorem may not be the smoking gun to justify the result. We can even make the argument in the the simplest case, $M$ simply taking the value 0, not to consider $M_H \gg M$. In this way, the Higgs particle should decouple from the
W and does not contribute to this \( H \rightarrow \gamma \gamma \) process, since now no Higgs mechanism. However, \( M_Z \rightarrow 0 / M \rightarrow 0 \) limit for \( H \rightarrow \gamma Z \) process may still gives non-zero term since this is a limit.

[12] This part of discussion not only serves as demonstration of the symmetrical integration which will be used for \( H \rightarrow \gamma Z \), but serves part of the answer to the question raised by Professor Tai Tsun Wu to one of the authors: Can one calculate T1LG, which corresponding to \( M_{131} \) in [5], and the following, without the help of the Feynman-Schwinger Parametrization?