Dimerization induced mobility edges and multiple reentrant localization transitions in non-Hermitian quasicrystals

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Non-Hermitian effects could create rich dynamical and topological phase structures. In this work, we show that the collaboration between lattice dimerization and non-Hermiticity could generally bring about mobility edges and multiple localization transitions in one-dimensional quasicrystals. Non-Hermitian extensions of the Aubry-André-Harper (AAH) model with staggered onsite potential and dimerized hopping amplitudes are introduced to demonstrate our results. Reentrant localization transitions due to the interplay between quasiperiodic gain/loss and lattice dimerization are found. Quantized winding numbers are further adopted as topological invariants to characterize transitions among phases with distinct spectrum and transport nature. Our study thus enriches the family of non-Hermitian quasicrystals by incorporating effects of lattice dimerization, and offering a convenient way to modulate localization transitions and mobility edges in non-Hermitian systems.

I. INTRODUCTION

Non-Hermitian systems have attracted increasing attention over the past decade due to their abundant dynamical, topological and transport properties (see Refs. \textsuperscript{[14–17]} for reviews). Theoretical studies have uncovered physics and phenomena that are unique to non-Hermitian settings, such as the exceptional points \textsuperscript{[7–9]}, non-Hermitian skin effects \textsuperscript{[10,13]}, anomalous localization transitions \textsuperscript{[14–17]} and enlarged symmetry classifications of topological matter \textsuperscript{[18–22]}. Experimental progress has also made it possible to observe novel nonunitary dynamics and non-Hermitian topological phases in a variety of platforms, ranging from cold atoms \textsuperscript{[23–25]}, photonics \textsuperscript{[26–29]}, acoustics \textsuperscript{[30–32]}, electrical circuits \textsuperscript{[33–35]} to NV center in diamonds \textsuperscript{[36]}. These developments further suggest potential applications of non-Hermitian effects in topological lasers \textsuperscript{[37–39]} and high-performance sensors \textsuperscript{[40–43]}.

Recently, the interplay between non-Hermitian effects and spatial quasiperiodicity has been found to generate a rich class of matter called non-Hermitian quasicrystal (NHQC) \textsuperscript{[44–72]}. In an NHQC, onsite gain and loss or nonreciprocal hoppings could induce \textit{PT}-breaking transitions of the spectrum from real to complex and localization transitions of eigenstates between extended and localized phases. Moreover, in certain situations, a critical phase could appear in between the extended and localized phases, which holds energy-dependent mobility edges separating delocalized and localized states \textsuperscript{[50,51]}. In a Hermitian quasicrystal, critical mobility edge phases could generally appear when the system possesses long-range hoppings \textsuperscript{[73–75]} or dimerized lattice structures \textsuperscript{[76–78]}. However, much less is known about generic ways of inducing and controlling mobility edges and localization transitions in non-Hermitian systems \textsuperscript{[64,65]}. In this work, we introduce a simple scheme to generate critical phases with mobility edges and create multiple reentrant localization transitions in NHQCs by adding experimentally implementable spatial dimerization to the system. In Sec. \textsuperscript{II} we introduce general theoretical tools that can be used to unveil the spectral, localization, topological transitions and mobility edges in NHQCs. In Sec. \textsuperscript{III} we present models that will be considered in this work and discuss their common features. In Sec. \textsuperscript{IV} we reveal how staggered onsite potentials or dimerized hopping amplitudes could produce critical regions separating extended and localized phases in NHQCs, and how the change of dimerization effects could induce reentrant topological transitions among NHQC phases with distinct spectral and transport nature. In Sec. \textsuperscript{V} we summarize our results and discuss potential future studies.

II. THEORY

We first outline the theoretical framework that will be adopted to investigate NHQCs in this work. We focus on one-dimensional (1D) systems with the following form of lattice Hamiltonian

\begin{equation}
H = \sum_{n \in \mathbb{Z}} (J_R^a |n⟩⟨n+1| + J_L^a |n⟩⟨n+1| + V_n |n⟩⟨n|). \tag{1}
\end{equation}

Here $J_R^a$ ($J_L^a$) describes the hopping amplitude from the $(n+1)$’s ($n$’s) to the $n$’s ($|n⟩\langle n+1|)$’s site of the lattice. $V_n$ is the amplitude of onsite potential. For a system with periodic boundary condition (PBC), we take the lattice site index $n = 1, ..., L$ and identify the basis $|n⟩ = |n+L⟩$. The Hamiltonian $H$ can be made non-Hermitian by either setting $J_R^a \neq (J_L^a)^*$ (asymmetric hopping) or $V_n \neq V_n^*$ (onsite gain and loss). $H$ further describes a quasicrystal if we set $J_R^a$ or $V_n$ to a quasiperiodic function of $n$. In the lattice representation, the spectrum and states of $H$ can be obtained by solving the eigenvalue
equation $H|\psi\rangle = E|\psi\rangle$, yielding

$$J_n^R \psi_{n+1} + J_n^L \psi_{n-1} + V_n \psi_n = E \psi_n. \quad (2)$$

Here $\psi_n$ represents the amplitude of wave function $|\psi\rangle = \sum_n \psi_n |n\rangle$ on the nth lattice site. For a lattice of length \( L \), there are \( L \) eigenenergies \( E = \{E_j|j = 1, \ldots, L\} \), whose values can in general be complex if \( H \neq H^\dagger \).

In a system described by \( H \), the presence of non-Hermiticity may induce three types of transitions \[43\]. On the spectral side, if \( H \) possesses the PT or other types of symmetry that could make it pseudo-Hermitian, its spectrum can be real and the growth of its non-Hermitian parameters may induce a real-to-complex (e.g., the PT-breaking) transition in the spectrum. Such a transition can be identified by looking at the maximum of the imaginary parts of \( E \) over all eigenstates, i.e.,

$$\text{max}(\text{Im}E) = \max_{j\in\{1,\ldots,L\}} (|\text{Im}E_j|). \quad (3)$$

It is clear that the value of \( \text{max}(\text{Im}E) \) grows from zero to finite when the spectrum of \( H \) changes from real to complex, and vice versa. Besides, not all eigenstates may take complex energies after the spectrum transition. To reveal this fact, we introduce the density of states (DOSs) with complex eigenenergies

$$\rho = N(\text{Im}E \neq 0)/L, \quad (4)$$

where \( N(\text{Im}E \neq 0) \) denotes the number of states whose energies have finite imaginary parts. In the deep non-Hermitian region we would expect \( \rho \rightarrow 1 \).

An NHQC could generally possess a localized phase, an extended phase, and a critical phase in which localized and extended states coexist and are separated by a mobility edge \[50, 51\]. Different characteristics of these phases can be extracted from the level-spacing statistics and inverse/normalized participation ratios of the states. Assuming the eigenenergies \( \{E_j|j = 1, \ldots, L\} \) are sorted by their real parts and denoting the real gap between the \( j \)th and \( (j-1) \)th energy levels as \( \varepsilon_j = \text{Re}E_j - \text{Re}E_{j-1} \), we define adjacent gap ratios (AGRs) as

$$g_j = \min(\varepsilon_j, \varepsilon_{j+1})/\max(\varepsilon_j, \varepsilon_{j+1}) \quad \text{for} \quad j = 2, \ldots, L - 1.$$ 

Here \( \min(a, b) \) and \( \max(a, b) \) yield the minimum and maximum of \( a \) and \( b \), respectively. The statistical feature of level-spacing is encoded in the average of AGRs over all states, which is defined as

$$\overline{g} = \frac{1}{L} \sum_j g_j. \quad (5)$$

In the limit \( L \rightarrow \infty \), we have \( \overline{g} \rightarrow 0 \) if all bulk states are extended, and \( \overline{g} \) approaches a constant in the localized phase \[50,83\]. In the critical phase, the value of \( \overline{g} \) is non-universal and changes between zero and its upper bound obtained for the localized phase. Meanwhile, if \( |\psi_j\rangle = \sum_{n=1}^L \psi_{nj} |n\rangle \) is a normalized eigenstate of \( H \) with energy \( E_j \), we define its inverse participation ratio (IPR) and normalized participation ratio (NPR) as

$$\text{IPR}_j = \sum_{n=1}^L |\psi_{nj}|^4 \quad \text{and} \quad \text{NPR}_j = (L \sum_{n=1}^L |\psi_{nj}|^4)^{-1}. \quad (6)$$

In the thermodynamic limit \( L \rightarrow \infty \), we have \( \text{IPR}_j \rightarrow 0 \) (\( \text{NPR}_j \rightarrow 1 \)) if \( |\psi_j\rangle \) is an extended (a localized) state, where the localization length \( \xi_j \) could be energy-dependent \[84\]. With the help of IPRs and NPRs, we can distinguish the three phases and capture the transitions among them from the following quantities:

$$\text{max}(\text{IPR}) = \max_{j\in\{1,\ldots,L\}} (\text{IPR}_j), \quad (6)$$

$$\text{min}(\text{IPR}) = \min_{j\in\{1,\ldots,L\}} (\text{IPR}_j), \quad (7)$$

$$\eta = \log_{10}(\langle \text{IPR}\rangle/\langle \text{NPR}\rangle). \quad (8)$$

Here \( \langle \text{IPR}\rangle = \frac{1}{L} \sum_{j=1}^L \text{IPR}_j \) and \( \langle \text{NPR}\rangle = \frac{1}{L} \sum_{j=1}^L \text{NPR}_j \) denote the average of IPRs and NPRs over all states. When \( L \rightarrow \infty \), we would have \( \text{max}(\text{IPR}) \rightarrow 0 \) if the system resides in an extended phase, \( \text{min}(\text{IPR}) > 0 \) if the system stays in a localized phase, and \( \eta \) to be finite if the system is in a critical phase with mobility edges \[17\]. The AGRs, IPRs and NPRs can thus be used to distinguish extended, localized and critical phases of an NHQC from complementary perspectives. Note in passing that in this work, our calculations are performed under the PBC. In this case, the non-Hermitian skin effect is absent and will not affect the spatial distribution of bulk states. Therefore, we compute the IPR and NPR for all bulk states using right eigenvectors, whereas using left eigenvectors will generate equivalent results concerning Eqs. \(6\) - \(8\).

When the energies of an NHQC possess imaginary parts, they may develop loop structures on the complex plane. Interestingly, the emergence of these complex energy loops was found to be related to the localization transitions in NHQCs. One can then introduce spectral winding numbers as topological order parameters to characterize phases with different transport nature and signify the transitions among them in an NHQC \[17,84\]. The generic definition of such a winding number is

$$w_\ell = \lim_{L \rightarrow \infty} \int_0^{2\pi} \frac{d\theta}{2\pi i} \partial_\theta \ln(\text{det}[H(\theta) - \mathcal{E}_\ell])). \quad (9)$$

Here \( \theta \) can be viewed as a flux penetrating through the ring formed by the system under PBC. The model Hamiltonian \( H \) in Eq. \[1\] can be changed to its \( \theta \)-dependent form \( H(\theta) \) by either setting \( J_n^{R(L)} \) to \( J_n^{R(L)} e^{(-i)\theta}/N \) or adding a phase shift \( \theta/N \) to \( V_n \), where \( N \) is the number of cells of the lattice. For the models considered in this work, the ways of introducing phase factor \( \theta \) to \( H \) are summarized in Table \[1\]. \( \mathcal{E}_\ell \) is a model-dependent base energy on the complex plane, and \( w_\ell \) counts the number of times the spectrum of \( H(\theta) \) winds around \( \mathcal{E}_\ell \) when \( \theta \) is swept over a period. If the system has no critical phases,
there is only one $\mathcal{E}_t = \mathcal{E}_1$, which can be chosen as the real part of energy of the first eigenstate of $H$ whose IPR deviates from zero. The transition of the system from the extended to localized phases would then accompany the quantized jump of $w_1$. If the extended and localized phases of the system are further separated by a critical phase, another base energy $\mathcal{E}_t = \mathcal{E}_2$ is needed, which can be chosen as the real part of energy of the last eigenstate of $H$ whose IPR departs from zero. In this case, we expect $w_1$ ($w_2$) to take a quantized jump when the system goes from the extended (critical) phase to the critical (localized) phase [64]. Within a given phase, the winding number will be pinned to a quantized value and can thus be interpreted as a topological order parameter of the corresponding NHQC phase.

We are now ready to introduce explicit models with staggered onsite potentials or dimerized hopping amplitudes, and show how the interplay between lattice dimerization and non-Hermitian effects could create rich phase and transition patterns in NHQCs.

III. MODEL

To uncover the impact of lattice dimerization on NHQCs, we start with the model introduced in Ref. [49], whose Hamiltonian takes the form $H_0 = \sum_{n \in \mathbb{Z}} (J|n\rangle\langle n + 1| + \text{H.c.} + V e^{i2\pi\alpha}|n\rangle\langle n|)$. Here $J, V \in \mathbb{R}$ and $\alpha = \frac{\sqrt{5}-1}{2}$. As a non-Hermitian extension of the AAH model [85–87], $H_0$ forms a minimal construction of an NHQC. It was proved that when $|V| < |J|$, the spectrum of $H_0$ is real and all its eigenstates are extended. When $|V| > |J|$, the spectrum becomes complex and all eigenstates are localized. Therefore, the system described by $H_0$ undergoes a P'T-breaking transition together with a localization transition at $|V| = |J|$, which can be topologically characterized by the quantized jump of a spectral winding number [69]. Note that no critical phases and mobility edges are found in $H_0$, and all states undergo the same localization transition when $|V|$ switches from below to above $|J|$, with the common Lyapunov exponent $\lambda = \ln|V/J|$ [49].

Another NHQC model that will be employed has the Hamiltonian $H_1 = \sum_{n \in \mathbb{Z}} (J|n\rangle\langle n + 1| + \text{H.c.} + V \cos(2\pi\alpha + i\gamma)|n\rangle\langle n|)$, where $i\gamma$ introduces an imaginary phase shift in the superlattice potential. It was found that when $\gamma = \gamma_c = \ln|2J/V|$, the states of $H_1$ could undergo a transition from an extended phase with real spectrum ($|\gamma| < |\gamma_c|$) to a localized phase with complex spectrum ($|\gamma| > |\gamma_c|$) [48]. This transition is further accompanied by the quantized jump of a spectral winding number $w$ for zero to $-1$ [48]. However, there are also no signatures of critical mobility edge phases in the system described by $H_1$, and all eigenstates experience the same spectral and localization transitions at $\gamma = \gamma_c$.

In this work, we use dimerized versions of $H_0$ and $H_1$ to showcase our main results. We consider three different extensions, which are denoted by M1–M3. Their Hamiltonians share the form of Eq. (1) with components listed in Table I. In M1–M3, the strength of hopping dimerization and staggered onsite potential are separately controlled by the parameters $\Lambda$ and $\Delta$. For the calculations presented below, we choose $J = 1$ as the unit of energy, let $\alpha = (\sqrt{5}-1)/2$ be the inverse golden ratio and assuming PBC throughout. It will be shown that the dimerization effects could generically induce critical phases with mobility edges, and furthermore trigger alternated and reentrant localization transitions in NHQCs.

IV. RESULTS

We now investigate the spectral, localization and topological transitions in M1–M3 with the tools introduced in Sec. III. We first consider the effect of lattice dimerization in onsite potential (M1) and hopping amplitudes (M2) in Subsec. IV A and IV B. Both of these two types of dimerization are found to induce critical phases with mobility edges and multiple localization transitions in the presence of non-Hermitian quasiperiodic potential. In Subsec. IV C, we unveil intriguing patterns of reentrant localization transitions in M3, which are due to the interplay between onsite dimerization and dissipation.

A. M1: Effect of staggered onsite potential

Referring to Table I, the M1 corresponds to a non-Hermitian variant of the AAH model plus a staggered onsite potential. Following Eq. (2), the Hamiltonian of M1 reads $H = \sum_n (|n\rangle\langle n + 1| + \text{H.c.} + V_n|n\rangle\langle n|)$ and its eigenvalue equation takes the form $\psi_{n+1} + \psi_{n-1} + V_n \psi_n = E \psi_n$, where $V_n = V e^{i2\pi\alpha} + (-1)^n \Delta$ and we have set $J = 1$ as the unit of energy. Since $V_0 = V^* \alpha$, M1 possesses the P'T-symmetry, implying that its spectrum could be real in certain parameter regions. Under the PBC, the spectrum of M1 for a typical set of onsite dimerization $\Delta$ are shown in Figs. (1a)–(c). We observe that with the increase of $\Delta$, the spectrum could change from purely real [Fig. (1a)] to a mixture of real and complex energies [Fig. (1b)], and finally be purely complex [Fig. (1c)]. Therefore, the presence of staggered onsite potential in M1 could not only induce a P'T-breaking transition of the spectrum, but also create an intermediate region with coexisting real and complex eigenenergies, which is absent when $\Delta = 0$. Compared with the single transition of $H_0$ as mentioned in Sec. III, the introduction of a staggered onsite potential could clearly enrich the spectral patterns of NHQCs. In Fig. (1d), we further show the maximum of the imaginary part of spectrum versus the quasiperiodic non-Hermitian potential $V$ and onsite dimerization $\Delta$. For $V \in (0, 1)$, we find a P'T-breaking transition of the spectrum from real (max $\text{Im}E = 0$) to complex (max $\text{Im}E > 0$) with the increase of either $V$ or $\Delta$. For $V > 1$, the system enters a phase with only complex energies, and the change...
TABLE I. NHQCs with staggered onsite potential or dimerized hopping amplitudes. The system parameters $J, V, \Delta, \Lambda, \gamma \in \mathbb{R}$, and the lattice site index $n \in \mathbb{Z}$. $\alpha$ is irrational and set as $\frac{\sqrt{5} - 1}{2}$ throughout this work. $N = L/2$ is the number of dimerized cells in the lattice of length $L$. In the calculation of winding numbers, the Hamiltonians $H$ of M1–M3 are replaced by $H(\theta)$ via changing their $V_n$ in the third column to the corresponding $V_n(\theta)$ in the last column.

| Model index | Hopping amplitudes $J_{n,R}^{i,R}$ | Onsite potential $V_n$ | Onsite potential with phase shift $V_n(\theta)$ |
|-------------|-----------------------------------|------------------------|-----------------------------------------------|
| M1          | $J$                               | $Ve^{i2\pi \alpha n} + (-1)^n \Delta$ | $Ve^{i(2\pi \alpha n + \theta/N)} + (-1)^n \Delta$ |
| M2          | $J + (-1)^n \Lambda$              | $Ve^{i2\pi \alpha n}$    | $Ve^{i(2\pi \alpha n + \theta/N)}$            |
| M3          | $J$                               | $V \cos(2\pi \alpha n + i\gamma) + (-1)^n \Delta$ | $V \cos(2\pi \alpha n + i\gamma + \theta/N) + (-1)^n \Delta$ |

FIG. 1. Spectral properties of M1 under the PBC. The length of lattice is $L = 2584$ for all panels. (a)–(c) show the spectrum of M1 at $V = 0.5$ with the increase of staggered onsite potential $\Delta$. (d) exhibits the maximum of imaginary parts of energies versus $V$ and $\Delta$. (e) denotes the DOSs with nonvanishing imaginary parts of energies. (d) and (e) share the same color bar.

FIG. 2. State properties of M1, computed with the length of lattice $L = 2584$ under the PBC. (a) shows the averaged AGRs [Eq. (5)], (b) and (c) show the maximum [Eq. (6)] and minimum [Eq. (7)] of IPRs over all states at different system parameters ($V, \Delta$). (d) presents the function $\eta$ [Eq. (8)], which is finite only in the critical phase with mobility edge.

of $\Delta$ would not create further spectral transitions. In Fig. 1(c), we present the DOSs [Eq. (1)] with complex eigenenergies $\rho$ versus $V$ and $\Delta$, which show three qualitatively distinct regions. At the bottom left corner (in blue), we have $\rho = 0$ and all eigenstates of $H$ have real energies. At the top right corner (in yellow), we have $\rho = 1$ and the eigenenergies of all states are complex. In between the former two regions, we find $0 < \rho < 1$, implying the eigenstates therein form a mixture of real and complex energies. Notably, this intermediate region is absent without the dimerization in onsite potential. Furthermore, as will be discussed shortly, the states in these three regions possess distinct localization and topological features.

To decode the localization transitions in M1, we study its averaged AGRs [Eq. (5)], IPRs [Eqs. (6)–(7)] and the measure $\eta$ [Eq. (8)] characterizing the presence of a critical phase with mobility edge. In Fig. 2(a), we find three different regions in $\overline{\eta}$ versus $V$ and $\Delta$, which have one-to-one correspondences with the three regions of spectrum in Fig. 1(c). Therefore, we expect M1 to show an extended phase ($\overline{\eta} = 0$) with real spectrum, a localized phase ($\overline{\eta} \approx 0.3$) with purely complex spectrum, and a critical phase $0 < \overline{\eta} \lesssim 0.3$ in which eigenstates with real and complex energies coexist and are separated by a mobility edge. These inferences are further confirmed by the numerical results of max(IPR) ($= 0$ only if all eigenstates are extended), min(IPR) ($> 0$ only if all eigenstates are localized) and $\eta$ (finite only if localized and extended states coexist), as presented in Figs. 2(b)–2(d). Therefore, the presence of onsite dimerization and its interplay with the non-Hermitian quasiperiodicity could indeed yield a critical phase with mobility edge, and induce multiple transitions in NHQCs.

The transitions among extended, critical, and localized phases in M1 can be further attached with quantized jumps of topological invariants. According to Eq. (9), we construct a pair of winding numbers $w_1$ and $w_2$ to characterize the transitions from extended to critical and from critical to localized phases in M1, respectively. Note that for M1–M2, the phase shift in Eq. (8) is introduced by setting $Ve^{i2\pi \alpha n} \rightarrow Ve^{i(2\pi \alpha n + 2\theta/L)}$ in $V_n$, where $L/2$ corresponds to the number of dimerized cells. By calculating
We now investigate a non-Hermitian AAH model with dimerized hopping amplitudes, whose Hamiltonian takes the form

\[ H = \sum_n (J_n |n+1\rangle \langle n| + H.c. + V_n |n\rangle \langle n|), \]

with \( J_n = 1 + (-1)^n \Lambda \) and \( V_n = V e^{i2\pi\alpha n} \) as given by M2 in Table I. The resulting eigenvalue equation is

\[ J_n \psi_{n+1} + J_{n-1} \psi_{n-1} + V_n \psi_n = E \psi_n, \]

and the uniform part of hopping is set to \( J = 1 \). In experiments, the dimerized hopping amplitudes can be realized in various physical platforms. For example, in cold atom systems, it can be engineered by superimposing two standing optical waves of wavelengths \( \lambda \) and \( 2\lambda \) to generate a lattice potential of the form \( U \sin^2(2\pi x/\lambda + \phi/2) + U' \sin^2(4\pi x/\lambda + \pi/2) \), where \( x \) denotes the position and \( \phi \) represents a controllable phase between the two standing wave fields of strengths \( U \) and \( U' \). Since \( J_n = J_{-n}^* \) and \( V_n = V_{-n}^* \), M2 also obeys the \( \mathcal{PT} \)-symmetry, which means that its spectrum can be real in some parameter domains. Note that a different non-Hermitian extension of M2 was considered in Ref. [94], and dimerization-induced intermediate phases with mobility edges were also observed.

When \( V \in (0,1) \), M2 undergoes four spectral transitions with the increase of hopping dimerization \( \Lambda \). Typical spectra of the system before and after these transitions are shown in Figs. 4(a)–4(d). With the increase of \( \Lambda \), M2 first goes through a \( \mathcal{PT} \)-breaking transition from a real spectrum [Fig. 4(a)] to a mixed spectrum with both real and complex eigenvalues [Fig. 4(b)]. At a larger \( \Lambda \), states with real energies vanish and the spectral becomes purely complex [Fig. 4(c)]. However, with the further increase of \( \Lambda \), real eigenvalues in the spectrum re-emerge and the system goes back to a phase with coexisting real and complex energies, as shown in Fig. 4(d). When \( \Lambda \) becomes even larger, M2 could again enter the phase with real spectrum and the \( \mathcal{PT} \)-symmetry is recovered, which corresponds to the region with \( \max |\text{Im}E| = 0 \) at the top left corner of Fig. 4(e). The DOSs with complex energies shown in Fig. 4(f) further confirms the observed alternating transitions among real, mixed and complex spectrum regions. The physical mechanism behind these reentrant spectral transitions may be understood as follows. With the increase of \( \Lambda \), the averaged hopping amplitudes

\[ (w_1, w_2), \]

we obtain the topological phase diagram of the system, as presented in Fig. 3. We find a region (in blue) with winding numbers \((w_1, w_2) = (0,0)\), which is in coincidence with the extended phase with real spectrum, a second region (in green) with \((w_1, w_2) = (1,0)\), which is consistent with the critical phase with partially real spectrum, and a third region (in yellow) with \((w_1, w_2) = (1,1)\) coinciding with the localized phase with purely complex spectrum in Figs. 4 and 2. These results clearly demonstrate that the spectral winding numbers \((w_1, w_2)\) could be employed as topological order parameters to distinguish the three NHQC phases of M1 and characterize the transitions among them. Put together, the existence of staggered onsite potential could indeed induce \( \mathcal{PT} \)-breaking, multiple localization and topological transitions, and also yield a critical phase with mobility edge. It can therefore be used as a flexible knob to tune and create intriguing phases and transitions in NHQCs.

In Sec. [VIB] we show that multiple localization transitions and mobility edges could also be induced in NHQCs by adding dimerization to the hopping amplitudes.
among lattice sites, which may be expressed as a function $f(J - \Lambda, J + \Lambda)$ does not change monotonically with $\Lambda$. Since the spectrum transitions in M2 are originated from the competition between hopping and onsite energy scales, the non-monotonic behavior of $f(J - \Lambda, J + \Lambda)$ in $\Lambda$ implies the non-monotonic process of spectral transitions.

Similar to the cases encountered in M1, the spectral transitions in M2 are also accompanied by state transitions among extended, critical, and localized phases. This is demonstrated by the results presented in Fig. 5 which describe the averaged AGRs [Fig. 5(a)], IPRs [Fig. 5(b)–(c)] and the measure of critical phase with mobility edges $\eta$ [Fig. 5(d)]. From the minimum and maximum of IPRs, we can clearly see two distinct phase boundaries at $\max(\text{IPR}) = 0 \rightarrow 0$ and $\min(\text{IPR}) = 0 \rightarrow 0$, respectively, across which the system changes from the extended to critical and from the critical to localized phases. These phase boundaries coincide with the parameters at which the DOSs with complex energies change from $0 \rightarrow \rho \in (0, 1)$ and from $\rho \in (0, 1) \rightarrow 1$ in Fig. 4(f). The critical phases in which extended and localized states coexist are highlighted by the regions with finite values of $\eta$ in Fig. 5(d). These two sets of states are separated in energy by mobility edges. Notably, there are two such critical regions separated by a localized phase for $V \in (0, 1)$. These phases are originated from the interplay between dimerized hoppings and non-Hermitian quasiperiodic potential. Therefore, with the increase of $\Lambda$, the system could undergo multiple and reentrant transitions among extended, critical, and localized NHQC phases. Each transition is accompanied by a drastic change in the structure of spectrum.

With the help of Eq. (10), we obtain the spectral winding numbers $(w_1, w_2)$ of M2, leading to the topological phase diagram presented in Fig. 6. Together with the results shown in Figs. 4 and 5, we see that there are indeed three distinct topological NHQC phases in M2. The extended phase (with real spectrum) and localized phase (with complex spectrum) possess winding numbers $(w_1, w_2) = (0, 0)$ and $(w_1, w_2) = (1, 1)$, whereas the critical phase (with mobility edge) has $(w_1, w_2) = (1, 0)$. When $V \in (0, 1)$, we could encounter four topological transitions with the increase of hopping dimerization $\Lambda$. At a fixed $\Lambda (\neq 0, 1)$, M2 will first change from the extended to critical phase, and finally enter the localized phase with the increase of non-Hermitian potential $V$. The critical phases with $(w_1, w_2) = (1, 0)$ separating the extended and localized ones are present only if $\Delta \neq 0$ in M2. Put together, we conclude that both dimerized onsite and hopping modulations could induce critical regions with mobility edges in NHQCs and control transitions among phases with distinct spectral, localization and topological nature.

C. M3: Multiple reentrant localization transitions

In the last example, we consider the case in which the non-Hermiticity is introduced by an imaginary phase shift in the AAH model with a staggered onsite potential, leading to the M3 in Table 1. The Hamiltonian of M3 takes the form $H = \sum_n (|n\rangle\langle n+1| + H.c. + V_n |n\rangle\langle n|)$ with $V_n = V \cos(2\pi \alpha n + i\gamma) + (-1)^n \Delta$. The eigenvalue equation reads $\psi_{n+1} + \psi_{n-1} + V_n \psi_n = E \psi_n$. Since $V_n = V_{-n}$, $H$ here also possesses the $\mathcal{PT}$-symmetry and its spec-
trum could be real. With the increase of the staggering strength $\Delta$, spectral and localization transitions would also occur in M3 as demonstrated below. Most notably, triggered by onsite dimerization, the system is found to be able to roam alternately between localized and critical mobility edge phases in a non-monotonic way.

In Fig. 8 we present selected spectra of M3 under different strengths of lattice dimerization. We observe that with a finite amount of imaginary phase shift $i\gamma$, the number of real eigenvalues in the spectrum could change with $\Delta$ in a highly non-monotonic manner. Specifically, the real eigenvalues in the spectrum vanish when $\Delta$ changes from 0.2 to 0.7, but re-emerge when $\Delta$ goes from 0.7 to 0.9. This process could repeat until $\Delta$ is large enough and all states end in taking complex energies. These rich and non-monotonic behaviors of spectrum are closely related to the reentrant localization and topological transitions of M3.

In Fig. 9 we present the DOSs with nonzero imaginary parts of energy $\rho$ and AGRs $\bar{\gamma}$ versus the phase shift $\gamma$ and onsite dimerization $\Delta$. Close to the Hermitian limit ($\gamma \simeq 0$), we find a transition of the system from real ($\rho = 0$) to complex ($\rho > 0$) spectrum with the increase of $\Delta$, which goes together with the change of $\bar{\gamma}$ from zero to a finite value $0 < \bar{\gamma} \lesssim 0.3$. The further increase of $\Delta$ causes a second transition, after which $\rho$ and $\bar{\gamma}$ approach 1 and 0.3. Surprisingly, when $\gamma$ is away from zero but still smaller than the critical value $\gamma_c = \ln[2|J/V|]$, we find $\rho$ and $\bar{\gamma}$ to show reentrant behaviors between the phases with $\rho \in (0, 1)$, $\bar{\gamma} \in (0, 0.3)$, and $\rho = 1$, $\bar{\gamma} \simeq 0.3$, which are clearly demonstrated by the dashed and dotted lines shown in Figs. 9(a) and 9(b). These reentrant transitions, which are absent if $\Delta = 0$ in M3, are expected to be consistent with the alternation of the system between localized and critical phases through a series of topological localization transitions, as confirmed by the results presented in Figs. 9(a) and 9(b). For example, near $\gamma = 0.5$, the system is found to follow a sequence of transitions from extended $\rightarrow$ critical $\rightarrow$ localized $\rightarrow$
critical → localized → critical, and finally ends in a localized phase with the increase of onsite dimerization $\Delta$. This transition sequence is further accompanied by the changes of winding numbers following $(w_1, w_2) = (0, 0) \rightarrow (-1, 0) \rightarrow (-1, -1) \rightarrow (-1, 0) \rightarrow (-1, -1) \rightarrow (-1, 0)$, and finally arrives at $(w_1, w_2) = (-1, -1)$. Note that for M3, the phase shift in Eq. (6) is introduced by setting $i \gamma \rightarrow i \gamma + 2 \theta/L$ in $V_n$. Therefore, we conclude that the lattice dimerization could not only produce critical phases with mobility edges, but also induce multiple and reentrant topological localization transitions in NHQCs. Notably, the reentrant transitions found here are different from those observed in Hermitian quasicrystals [7], as in our case these transitions are only found at finite amounts of imaginary phase shift $i \gamma$, implying that they are unique phenomena originated from the interplay between non-Hermiticity and spatial dimerization. Moreover, the alternating jumps of topological order parameters endow richer structures to the reentrant transitions in NHQCs compared with their Hermitian counterparts [76-78], providing us with more insights to non-Hermitian disordered systems from a topological perspective.

V. SUMMARY

In this work, we found lattice dimerization induced critical phases with mobility edges and multiple reentrant localization transitions in NHQCs. These findings were explicitly demonstrated in non-Hermitian extensions of the AAH model with staggered onsite potentials or dimerized hoppings amplitudes. Especially, the interplay between dimerization and non-Hermitian effects works as a flexible knob to control the phase transitions and critical properties of NHQCs. Moreover, topological order parameters were employed to precisely distinguish extended, critical, localized phases of NHQCs and capture the transitions among them. Put together, our results unveil the richness of NHQC phases in the presence of lattice dimerization, uncover the chance of generating reentrant localization transitions in NHQCs, and suggest a way to manipulate the transport nature of NHQCs by tuning dimerization effects. In Refs. [76-78], multiple and/or reentrant localization transitions are found in quasicrystals with lattice dimerization or during the process in which one type of quasicrystal changes to another. In all the cases, the systems under consideration are described by Hermitian Hamiltonians. Compared with these former studies, our work uncovers the role played by the collaboration of non-Hermitian effects and lattice dimerization in generating reentrant localization and topological phase transitions that are unique to non-Hermitian settings. In future work, it would be interesting to consider lattice dimerization in different types of NHQC models, such as those with nonreciprocal hoppings, and explore more exotic phases of dimerized NHQCs in the presence of many-body interactions and time-periodic drives. The possible interplay between lattice dimerization and the recently found pseudo mobility edges [91] due to non-Hermitian skin effects also deserve more thorough explorations.

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