Feedback control on geometric phase in dissipative two-level systems

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The effect of feedback on a two-level dissipative system is studied in this paper. The results show that it is possible to control the phase in the open system even if its state can not be manipulated from an arbitrary initial one to an arbitrary final one. The dependence of the geometric phase on the control parameters is calculated and discussed.

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Control on a quantum system is a major task required for quantum information processing. Several approaches to reach this goal have been proposed in the past decade, which can be divided into open loop and closed loop problems, according to how the controls enter the dynamics. In the open loop problems, the controls enter the dynamics through the system Hamiltonian. It affects the time evolution of the system state, but not its spectrum, i.e., the eigenvalues of the target density matrix $\rho_f$ remain unchanged in the dynamics due to the unitarity of the evolution. In the closed loop problems, a feedback is required from the controller to enter the system based on measurement performed for the open system. This feedback control strategy can be traced back to 1980’s[1,2,3] when it is used to explain the observation[4] of subshot-noise fluctuations in an in-loop photon current. In these works, the authors did this using quantum Langevin equations and semiclassical techniques, the latter approach was made fully quantum-mechanical by Plimak[5]. For linear systems, all of these approaches and the trajectory approach[6,7,8] are equally easy to use to find analytical solutions. Nevertheless, the trajectory approach has advantages that (1) it is applicable for quantum system with non-linear dynamics, and (2) it is very easy to consider the limit of Markovian (instantaneous) feedback, i.e., a master equation describing the unconditional system dynamics with feedback may be possible to derive[6,7,8]. The Markovian feedback can be used to modify the stationary state in a two-level dissipative quantum system, and this feedback manifests interesting effects on the time-optimal control in the quantum system governed by the Lindbald master equation[6].

Geometric phases in quantum theory attracted great interest since Berry[10] showed that the state of a quantum system acquires a purely geometric feature in addition to the usual dynamical phase when it is varied slowly and eventually brought back to its initial form. The Berry phase has been extensively studied[11,12] and generalized in various directions[13,14,15,16,17], such as geometric phases for mixed states[14], for open systems[16], and with a quantized field driving[17]. In a recent paper[18], Sjöqvist calculated the geometric phase for a pair of entangled spins in a time-independent uniform magnetic field. This is an interesting development in holonomic quantum computation and it shows how the prior entanglements modify the Berry phase. This study was generalized[19] to the case of spin pairs in a rotating magnetic field, which showed that the geometric phase of the whole entangled bipartite system can be decomposed into a sum of geometric phases of the two subsystems, provided the evolution is cyclic. A renewed interest in geometric phenomena in quantum physics has been recently motivated by the proposal of using geometric phases for quantum computing. Geometric phases depend only on the geometry of the path executed, and are therefore resilient to certain type of errors. The idea is to explore this inherent robustness provided by the topological properties of some quantum systems as a means of constructing built-in fault tolerant quantum logic gates. Various strategies have been proposed to reach this goal, some of them making use of purely geometric evolution[20,21,22]. Others make use of hybrid strategies that combine together geometric and dynamical evolution[23,24]. Several proposals for geometric quantum computations have been suggested and realized in different context, including NMR experiments[23], ion traps[25,26,27,28,29], cavity QED[30], atomic ensembles[31,32], Josephson junction[33], anyonic system[34] and quantum dot[35,36].

For open systems governed by the Lindblad master equation, it was shown that the controls can not fully compensate the effect of decoherence, indicating that the state of open systems can not be manipulated from an arbitrary initial state to an arbitrary final state. This gives rise to a question that can the geometric phase of such an open system be controlled?

In this paper, we shall study the effect of feedback on the geometric phase in a dissipative two-level system governed by the Lindblad master equation. Consider an atom with two relevant levels $\{|g\rangle, |e\rangle\}$ and lower operator $\sigma_- = |g\rangle\langle e|$. Let the atomic decay rate be $\gamma$ and let it be driven by a classical magnetic field $B(t)$. Within the Markovian approximation for the system-environment couplings, the time evolution of the two-level system is described by the Lindblad master equation,

$$i\frac{\partial}{\partial t}\rho = [H_0, \rho] + \mathcal{L}(\rho),$$

$$H_0 = \mu \vec{B} \cdot \vec{\sigma},$$
Here a closed-loop control $F$ is introduced, which is trig-
gerated immediately only after a detection click, namely a quantum jump occurs. This scheme was used to generate and protect entangled steady state in cavity QED system and the jump feedback $F\sigma^-\rho_+F^\dagger$ can be understood as follows. The unitary operator $F$ is applied only immediately after a detection event, which is described by term $\sigma^-\rho_+$. Intuitively the stationary states depend on the feedback operator $F$. So, once the measurement prescription has been chosen, the freedom to design a feedback to produce a stationary state lies in the different choices for the feedback operator $F$. Although an enormous range of possibilities for $F$ is allowed, even considering the limitations imposed by experimental constraints, we here choose (with the constraint $FF^\dagger = 1$) 

$$F = e^{i\vec{\sigma} \cdot \vec{A}} = \cos A + \frac{i\vec{\sigma} \cdot \vec{A}}{A} \sin A,$$ \hspace{1cm} (2)

where we denote $A = |\vec{A}|$. In fact the feedback $F$ written in this form covers all allowed possibilities. Writing the reduced density matrix

$$\rho(t) = \frac{1}{2} + \frac{1}{2} \bar{\rho}(t) \cdot \bar{\sigma}$$ \hspace{1cm} (3)

we show after a simple algebra that

$$[H_0, \rho] = \frac{iA\bar{\sigma}}{2} \cdot \left[\bar{B} \times \bar{p} - \bar{\rho} \times \bar{B}\right],$$ \hspace{1cm} (4)

and

$$\mathcal{L}(\rho) = \bar{\sigma} \cdot \left(-\frac{i}{2} \bar{p} \bar{\sigma} - \frac{i}{4} \gamma \bar{\sigma} - \frac{i\gamma}{2} \left(\frac{1}{2} + \bar{\rho} \cdot \bar{\sigma}\right) \cdot \left[\cos 2A \bar{\sigma} + \sin 2A \bar{\sigma} + 2 \frac{A_x \sin A^2}{A^2} \bar{\sigma}\right]\right),$$ \hspace{1cm} (5)

Here $\bar{\sigma} = (0, 0, 1)$, and the master equation can be rewritten as,

$$\frac{\partial}{\partial t} \left( \begin{array}{c} p_x \\ p_y \\ p_z \end{array} \right) = \left( \begin{array}{ccc} -\frac{1}{2} & -2\mu B_x & 2\mu B_y - \frac{1}{2} \left[ -\sin \frac{A_x}{A} A_y + 2 \frac{A_x A_y}{A^2} \sin^2 A \right] \\ 2\mu B_x & -\frac{1}{2} & -2\mu B_y - \frac{1}{2} \left[ -\sin \frac{A_x}{A} A_x + 2 \frac{A_x A_x}{A^2} \sin^2 A \right] \\ -2\mu B_y & 2\mu B_x & -\frac{1}{2} \left[ \cos 2A + 2 \frac{A_x A_x}{A^2} \cos^2 A \right] \end{array} \right) \left( \begin{array}{c} p_x \\ p_y \\ p_z \end{array} \right) + \frac{1}{2} \left( \begin{array}{c} -\frac{\gamma}{2} [-\frac{A_y}{A} \sin 2A + 2 \frac{A_x A_y}{A^2} \sin^2 A] \\ -\frac{\gamma}{2} [\frac{A_x}{A} \sin 2A + 2 \frac{A_x A_x}{A^2} \sin^2 A] \\ -\frac{\gamma}{2} [\cos 2A + 2 \frac{A_x A_x}{A^2} \cos^2 A] \end{array} \right).$$ \hspace{1cm} (6)

For an open system, its state in general is not pure and the evolution of the system is not unitary. For non-unitary evolution, the geometric phase can be calculated as follows. First, solve the eigenvalue problem for the reduced density matrix $\rho(t)$ and obtain its eigenvalues $E_k(t)$ as well as the corresponding eigenvectors $|E_k(t)\rangle$; Second, substitute $E_k(t)$ and $|E_k(t)\rangle$ into\[37],

$$\gamma_g(\tau) = \arg \sum_k \left| \langle E_k(t=0)|E_k(\tau)\rangle e^{-\int_0^\tau (E_k(t)|E_k(t)\rangle)dt} \right|,$$

where $\gamma_g$ is the geometric phase for the system undergoing nonunitary evolution\[37]. In the following we shall choose

$$|\Psi(0)\rangle = \cos \frac{\theta}{2} (e) + \sin \frac{\theta}{2} (g)$$ \hspace{1cm} (8)

as the initial state for the dissipative two-level system, in terms of $\bar{\rho} = (p_x, p_y, p_z)$ the initial state can be expressed as

$$p_x = \sin \theta, p_y = 0, p_z = \cos \theta.$$ \hspace{1cm} (9)

The final state of the dissipative two-level system is then

$$\rho = \left( \begin{array}{ccc} \frac{1}{2} + \frac{1}{2} p_z & \frac{1}{2} p_x + \frac{i}{2} p_y \\ \frac{1}{2} p_x - \frac{i}{2} p_y & \frac{1}{2} - \frac{1}{2} p_z \end{array} \right),$$

with $\alpha$ and $\phi$ defined by,

$$\cos \alpha \equiv \frac{p_z}{\sqrt{p_x^2 + p_y^2 + p_z^2}},$$

$$\tan \phi \equiv \frac{p_y}{p_x}. \hspace{1cm} (11)$$

The eigenvalues and corresponding eigenstates of the reduced density matrix follows

$$E_\pm = \frac{1}{2} \pm \frac{1}{2} \sqrt{p_x^2 + p_y^2 + p_z^2},$$ \hspace{1cm} (12)

and

$$|E_\pm\rangle = \cos \frac{\alpha}{2} e^{i\phi} |e\rangle + \sin \frac{\alpha}{2} |g\rangle,$$

$$|E_\mp\rangle = \sin \frac{\alpha}{2} e^{-i\phi} |e\rangle - \cos \frac{\alpha}{2} |g\rangle.$$ \hspace{1cm} (13)

respectively. It is easy to check that,

$$\frac{1}{2} - \frac{1}{2} \sqrt{p_x^2(0) + p_y^2(0) + p_z^2(0)} = 0.$$ \hspace{1cm} (14)
straightforward calculations show that
\[ \gamma(f) = 0 \]
was chosen to be \( \omega = 0.005 \mu B_0 \) and [0, \( \pi \)] represents time interval of the system evolution, \( \tau = 2 \pi / \omega \). In these plots, we set \( \mu B_0 = 1 \), and both \( A \) and \( \beta \) are in units of \( \pi \). (a) \( \gamma = 0.001 \), (b) \( \gamma = 0.005 \), (c) \( \gamma = 0.01 \), (d) \( \gamma = 0.05 \), (e) \( \gamma = 0.1 \), (f) \( \gamma = 0.5 \), (g) \( \gamma = 1 \) and (h) \( \gamma = 3 \).

so the geometric phase \( \gamma_g \) reduces to
\[ \gamma_g = \arg \left( \langle E_+(t = 0)|E_+(t)\rangle e^{-\int_0^t \langle E_+(t)|\dot{E}_+(t)\rangle \, dt} \right), \] (15)
straightforward calculation shows that
\[ (E_+(t)) = i \cos^2 \left( \frac{\alpha}{2} \right) \frac{\partial}{\partial t}, \] (16)
and
\[ \frac{\partial \phi}{\partial t} = -\frac{\partial \phi}{\partial \cos \phi} \frac{\partial \cos \phi}{\partial t} = -\sqrt{p_x^2 + p_y^2}, \] (17)
\[ \frac{\partial \cos \phi}{\partial t} = \frac{\dot{p}_x \sqrt{p_x^2 + p_y^2} - p_x \dot{p}_x p_y + p_y \dot{p}_y}{p_x^2 + p_y^2}. \] (18)

We can use these equations to perform numerical simulations of the geometric phase for the open system. In the numerical simulation, we choose \( \tilde{B}(t) = B_0 (\cos \Theta \cos \Phi, \cos \Theta \sin \Phi, \sin \Theta) \) as the varying magnetic field with \( \Phi = \omega t \). Without atomic decay, i.e., \( \gamma = 0 \), the two-level system evolves freely, and its dynamics is governed by \( i \partial \rho / \partial t = \{ H_0, \rho \} \), where \( H_0 \) is given in Eq.(14). The instantaneous eigenstates of \( H_0 \) are \( |+\rangle = \cos \frac{\Theta}{2} e^{i \Phi} |e\rangle + \sin \frac{\Theta}{2} e^{i \Phi} |g\rangle \), \(-\rangle = \sin \frac{\Theta}{2} e^{-i \Phi} |e\rangle - \cos \frac{\Theta}{2} |g\rangle \), with the corresponding eigenvalues \( e_+ = \pm \mu B_0 \), respectively. For this system to evolve adiabatically, the adiabatic condition requires \( \omega \ll \mu B_0 \). In the numerical simulation, \( \omega \) was chosen to be \( \omega = 0.005 \mu B_0 \) and \( \tau = 2 \pi / \omega \).

To be specific, we set \( A_z = A \sin \beta, A_y = A \cos \beta \), and \( A_z = 0 \), where \( A \) is a constant. So the feedback is characterized by \( \beta \) and \( A \). The plots presented in Fig.1 are for the geometric phases acquired by the dissipative two-level system as a function of \( A \) and \( \beta \). For very small atomic decay rate \( \gamma \to 0 \) (Fig.1(a)), the geometric phase approaches a constant \( \gamma_g \sim (1 - \cos \theta) \) (in units of \( \pi \)). As \( \gamma \) increases, the range of the geometric phase acquired by the dissipative system increases, implying that the geometric phase can be controlled even with large atomic decay rate \( \gamma \). This is different from the control on quantum states, where the control can not fully compensate the decoherence. Figure 1 also shows that the geometric phase is a periodic function of \( A \), this can be understood by examining Eq.(2), where the feedback control is given. In addition to the above observation, we can find from figure 1 that the geographic phase is regular for small and large \( \gamma \), while it is irregular for intermediate values of \( \gamma \). The physics behind this feature is the following. For very small \( \gamma \), \( \mathcal{L}(\rho) \) is negligible, hence \( H_0 \) dominates over \( \mathcal{L}(\rho) \) in the dynamics and the geometric phase is mainly determined by \( H_0 \). When \( \gamma \) is large enough such that \( \mathcal{L}(\rho) \) dominates the dynamics, the geometric phase then comes from the dynamics governed by \( \mathcal{L}(\rho) \). With a specific \( A = \pi / 4 \), the geometric phase as a function of \( \beta \) and \( \gamma \) is plotted in figure 2. In all these plots, we set \( \Theta = \theta \), thus the initial state is an eigenstate of the free Hamiltonian \( H_0 \). For \( A = \pi / 2 \), the feedback is \( F = i (\sin \beta \sigma_x + \cos \beta \sigma_y) \) whereas for \( A = \pi \), \( F \) becomes 1, i.e., there is no feedback operating on the system. This can be found in Fig.2(d), where the geometric phase acquired is independent of \( \beta \). From figure 2(a) and 2(e)
we find that figure 2(c) is exactly the same as figure 2(a) by replacing $\beta$ by $\beta + \pi$, indicating that the geometric phase remains unchanged with $A \to \pi - A$ and $\beta \to \beta + \pi$. This feature can be understood as follows. Recall that $\tilde{A} = (A \sin \beta, A \cos \beta, 0)$, $F$ can be written as $F = \cos A + i(\sigma_x \sin \beta + \sigma_y \cos \beta) \sin A$, it is clear that $F$ remains unchanged by replacing $A$ and $\beta$ with $\pi - A$ and $\pi + \beta$, respectively.

To sum up, in this paper, we have studied the effect of feedback on the geometric phase of a dissipative two-level system. The dependence of the phase on the feedback parameters are calculated and discussed. The results suggested that we can manipulated the phase by a properly designed feedback control. For small and large atomic dissipative rates with respect to the amplitude of the driving magnetic field $\mu B_0$, the geometric phase is a periodic function of the feedback parameters, the physics behind these features is also presented.

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