The spherical image of singular varieties of bounded mean curvature

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Abstract

In this paper we study singular varieties of arbitrary codimension and bounded mean curvature in a viscosity sense. These varieties, introduced by White, naturally arise in the study of the area-blow up of sequences of smooth submanifolds with uniformly bounded mean curvature and provide a non-variational analogous for the class of varifolds of bounded mean curvature. We introduce notions of mean curvature and second fundamental form and we establish a Coarea-type formula which allows to compute the area of the spherical image of the singular variety in terms of its curvature. Such a formula is new even for integral varifolds of bounded (or zero) mean curvature. Then we extend the celebrated Almgren sphere theorem to these class of singular varieties. Our proof, which is based on Almgren’s ideas, combines the aforementioned Coarea-type formula, the barrier principle of White and the Alexandrov’s technique of moving planes to derive crucial quantitative informations without assuming a-priori regularity for the variety (no analogous of Allard’s regularity theorem is available in our non-variational setting).

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1 Introduction

The general setting. In this paper we study some geometric-analytic aspects of certain singular varieties introduced by Brian White in [Whi16] to analyse the area-blow up of sequences of smooth submanifolds of arbitrary codimension and mean curvature uniformly bounded by a non negative number $h$.

1.1 Definition. (see [Whi16] 2.1) Suppose $1 \leq m < n$ are integers, $\Omega$ is an open subset of $\mathbb{R}^n$, $\Gamma$ is relatively closed in $\Omega$ and $h \geq 0$. We say that $\Gamma$ is an

\[\text{This definition is equivalent to [Whi16] 2.1 by [Whi16] 8.1.}\]
An \((m,h)\) set can be roughly described as a submanifold with no boundary and mean curvature bounded by \(h\) in the viscosity sense. This class contains all \(m\) dimensional varifolds \(V\) in \(\Omega\) such that \(\|\delta V\| \leq h\|V\|\), see \([Whi16, 2.8]\). Since, as proved in \([Whi16]\), the class of \((m,h)\) sets plays a fundamental role in important questions raised in differential geometry, it is of considerable interest to understand which kind of structural and regularity properties are shared between \((m,h)\) sets and their variational counterparts, i.e. varifolds. This is somehow analogous to ask in PDE’s theory, which properties of variational-type solutions can be extended to viscosity-type solutions. In this direction \([Whi16]\) already contains several significant contributions. Moreover an Allard-type regularity theorem is obtained in \([Sav18]\) for \((m,0)\) sets that are graphs of continuous functions. In this paper we provide answers to the following questions: \(^3\)

1 Problem. Is it possible to introduce meaningful notions of mean curvature and second fundamental form for \((m,h)\) sets? This question is of interest also in case of those varifolds that are \((m,h)\) sets, for which it is a difficult problem to understand how to introduce a notion of second fundamental form.

2 Problem. A basic fact in differential geometry asserts that the area of the spherical image of a smooth variety can be computed in terms of the curvature (see \((1)\)). Is it possible to make a similar assertion for \((m,h)\) sets? This question is open even for those varifolds that are \((m,h)\) sets.

3 Problem. Almgren in \([Alm86]\) proved that if \(\Gamma\) is an \((m,m)\) set in \(\mathbb{R}^n\), is it true that \(H^m(\Gamma) \geq H^m(S^m)\) with equality if and only if \(\Gamma\) is the round sphere? Second fundamental form and Coarea formula for the spherical image.

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\(^2\)In this paper we adopt the terminology in \([Alm86, Appendix C]\) for varifolds; in particular note that the variation function \(h(V, \cdot)\) differs from the one adopted in Allard’s paper \([All72, 4.2]\) by a sign.

\(^3\)Problems \(^1\), \(^2\) and \(^3\) can be analogously formulated for rectifiable varifolds of locally bounded first variation and possibly unbounded mean curvature. As we point out later in the introduction only for Problem \(^1\), and only for integral varifolds, a partial answer follows from deep results of Schätzle \([Sch02, Sch09]\) and Menne \([Men13]\).
of the spherical image of $M$ can be expressed in terms of the curvature in the following way: if $B$ is an $\mathcal{H}^{n-1}$ measurable subset of $\mathcal{N}(M)$ then

$$\int_{S^{n-1}} \mathcal{H}^0 \{ a : (a,u) \in B \} \, d\mathcal{H}^{n-1} u = \int_M \int_{\{\eta(z,\eta) \in B\}} |\text{discr} Q_M| d\mathcal{H}^{n-m-1} d\mathcal{H}^m z. \tag{1}$$

Smoothness of $M$ readily reduces the proof of this result to an application of Federer’s Coarea formula. On the other hand it is well known that integral varifolds $V$ with $\|\delta V\| \leq h\|V\|$ (in particular $(m,h)$ sets) can fail to be $C^1$ submanifolds almost everywhere and the singular set can be extremely complicated, as the example in [Bra78, 6.1] shows. Therefore it is a non-trivial question to understand how to introduce a second fundamental form for these varifolds. Up to now, the best result in this direction follows from the $C^2$ rectifiability results proved by Schätzle [Sch04, 5.1]-[Sch09, 3.1] and Menne [Men16a, 4.8]. In fact, these results imply the existence of an approximate second fundamental form, see [2.1], whose associated approximate mean curvature coincides with the variational mean curvature of the varifold. However, a Coarea-type formula as in (1) does not directly follow from $C^2$ rectifiability and a deeper analysis seems to be necessary. In this paper we provide an approach to solve Problems 1 and 2 for all $(m,h)$ sets (which are unions of countably many sets of finite $\mathcal{H}^m$ measure) which is essentially independent of $C^2$ rectifiability results. Our starting point is the notion of curvature introduced in the series of papers [Sta79, HLW04] and [San17b], where for an arbitrary closed set $\Gamma$ a notion of unit normal bundle $N(\Gamma)$, which is a countably $\mathcal{H}^{n-1}$, $n-1$ rectifiable subset of $\mathbb{R}^n \times S^{n-1}$, and second fundamental form

$$Q_{\Gamma}(z, \eta) : T_{\Gamma}(z, \eta) \times T_{\Gamma}(z, \eta) \to \mathbb{R},$$

which is a symmetric bilinear form associated to $\mathcal{H}^{n-1}$ a.e. $(z,\eta) \in N(\Gamma)$, are defined. A summary of the relevant definitions for the present work is provided in [2.2, 2.3]. This concept has already found several applications in stochastic geometry; however this is the first time that it is applied in variational problems. However, even if it is possible for arbitrary closed sets to prove Coarea-type formulas integrating suitable curvature functions on the normal bundle $N(\Gamma)$, see [San17b, 4.11(3), 5.6], a Coarea formula as in (1) cannot be proved for arbitrary closed sets. In fact, what turns out to be true is that such a formula holds whenever $\Gamma$ is a closed subset of $\mathbb{R}^n$ whose normal bundle $N(\Gamma)$ satisfies the following property:

$$\mathcal{H}^{n-1}(N(\Gamma)|S) = 0 \quad \text{for all } S \subseteq \Gamma \text{ such that } \mathcal{H}^m(\Gamma^{(m)} \cap S) = 0,$$

It is worth to mention that a second fundamental form is introduced in a variational way by Hutchinson in [Hut86], defining a special class of integral varifolds known as curvature varifolds (see also [Man96] for further extensions and [Men16b, 15.6] for a characterization in terms of weakly differentiable functions). However not every integral varifolds such that $\|\delta V\| \leq \kappa\|V\|$ (or even $\|\delta V\| = 0$) is a curvature varifolds.
where \( \Gamma^{(m)} = \Gamma \cap \{ a : 0 < \mathcal{H}^{n-m-1}(N(\Gamma,a)) < \infty \} \); see \(\text{3.3}\). We call such a property \(m\) dimensional Lusin (N) condition, in analogy with the terminology used in classical works on the Area formula for mappings. It follows from a recent result of Schneider \(\text{Sch15}\) that a typical (in the sense of Baire category) compact convex hypersurface in \(\mathbb{R}^n\) (which is known to be \(C^1\)) does not possess the \(n - 1\) dimensional Lusin (N) condition. It requires some technical effort to prove that all \((m,h)\) sets which are countable unions of sets of finite \(\mathcal{H}^m\) measure satisfy this property. The main result of section \(\text{3}\) can be summarized in the following theorem.

1.2 Theorem (Coarea formula for the spherical image map of \((m,h)\) sets). Suppose \(1 \leq m \leq n - 1, \ 0 \leq h < \infty, \ \Gamma\) is an \((m,h)\) subset of \(\mathbb{R}^n\) that is a countable union of sets with finite \(\mathcal{H}^m\) measure. Then \(N(\Gamma)\) satisfies the \(m\) dimensional Lusin (N) condition and

\[
\int_{S^{n-1}} \mathcal{H}^0\{ a : (a,u) \in B \} \, d\mathcal{H}^{n-1}u = \int_{\Gamma} \int_{\{\eta: (z,\eta) \in B\}} |\text{discr} \, Q_{\Gamma}| \, d\mathcal{H}^{n-m-1} \, d\mathcal{H}^mz.
\]

whenever \(B \subseteq N(\Gamma)\) is \(\mathcal{H}^{n-1}\) measurable. Moreover,

\[
\dim T_{\Gamma}(z,\eta) = m, \quad \text{trace} \, Q_{\Gamma}(z,\eta) \leq h
\]

for \(\mathcal{H}^{n-1}\) a.e. \((z,\eta) \in N(\Gamma)\).

The main tool to prove the Lusin (N) condition is the Barrier principle \(\text{[Whi16, 7.1]}\), which allows to prove that for \(L^1\) a.e. \(r \in (0,r_0)\) the sum of the first \(m\) approximate principal curvatures of the level set \(S(\Gamma,r)\) of the distance function \(d\delta_{\Gamma}\) from \(\Gamma\), is bounded from above by \(h\) at \(\mathcal{H}^{n-1}\) almost every point of suitable subset of \(S(A,r)\). In turn, this estimate implies that the approximate jacobian of the nearest point projection \(\xi_{\Gamma}\) onto \(\Gamma\) must be positive on this subset. Since there exists a bi-lipschitzian correspondence between the level sets \(S(\Gamma,r)\) and the unit normal bundle \(N(\Gamma)\), see \(\text{[San17b]}\ 3.18, 4.3\), the positivity of the approximate jacobian of \(\xi_{\Gamma}\) readily implies the Lusin (N) condition by \(\text{3.3}\). Moreover, the relation between the approximate principal curvatures of \(S(\Gamma,r)\) and the principal curvatures of \(\Gamma\) (i.e. the eigenvalues of \(Q_{\Gamma}\)) given in \(\text{2}\) allows to establish that \(\text{trace} \, Q_{\Gamma}(z,\eta) \leq h\) at \(\mathcal{H}^{n-1}\) almost every \((z,\eta) \in N(\Gamma)\). We remark that for arbitrary closed sets both the aforementioned bilipschitzian correspondence and the relation between principal curvatures in \(\text{2}\) are new facts, established in \(\text{[San17b]}\).

Theorem \(\text{1.2}\) clearly shows that \(Q_{\Gamma}\) and \(\text{trace} \, Q_{\Gamma}\) naturally describe key geometric properties of general \((m,h)\) sets, thus providing natural notions of second fundamental form and mean curvature for this class of varieties. If we consider those varifolds \(V\) that are also \((m,h)\) sets, it is important to understand if the first variation function \(h(V,\cdot)\) of \(V\) agrees with \(\text{trace} \, Q_{\text{opt}} \|V\|\) in order to regard \(Q_{\text{opt}} \|V\|\) as a suitable notion of second fundamental form for \(V\). In this
paper, on the basis of the locality theorem of Schätzle [Sch09], we provide an answer for integral varifolds.

1.3 Theorem (Second fundamental for integral varifolds of bounded mean curvature). Suppose $1 \leq m \leq n - 1$, $V \in V_m(\mathbb{R}^n)$ is an integral varifold such that the total variation $\|\delta V\|$ of the first variation $\delta V$ is absolutely continuous with respect to the variation measure $\|V\|$ and the first variation function $h(V, \cdot) \in L^\infty(\|V\|, \mathbb{R}^n)$. Then

$$\text{trace } Q_{spt \|V\|}(z, \eta) = h(V, z) \cdot \eta \quad \text{for } H^{n-1} \text{ a.e. } (z, \eta) \in N(\text{spt } \|V\|).$$

We finally remark that a similar Coarea-type formula has been announced in [Men12b] for $m$ dimensional integral varifolds $V$ in $\mathbb{R}^{m+1}$ with $\|\delta V\|$ absolutely continuous with respect to $\|V\|$, $h(V, \cdot) \in L^m(\|V\|, \mathbb{R}^{m+1})$ and $m \geq 2$.

Almgren sphere theorem for $(m,m)$ sets. Our extension of Almgren result reads as follows.

1.4 Theorem (Sphere theorem for $(m,m)$ sets). If $1 \leq m \leq n - 1$ and $\Gamma$ is a non-empty compact $(m,m)$ subset of $\mathbb{R}^n$ then

$$\mathcal{H}^m(\Gamma) \geq \mathcal{H}^m(S^m).$$

Moreover if $\Gamma = \text{spt}(\mathcal{H}^m, \Gamma)$ and $\mathcal{H}^m(\Gamma) = \mathcal{H}^m(S^m)$ then there exists an isometric injection $f : \mathbb{R}^{m+1} \to \mathbb{R}^n$ such that

$$\Gamma = f(S^m).$$

The proof of this theorem, as well as the content of section 3, owes much to Almgren’s ideas developed in [Alm86] and to some new insights contained in a proof of Almgren’s result due to Ulrich Menne (see [Men12a]). On the other hand, critical difficulties in extending Almgren’s result to the non-variational setting of $(m,m)$ sets are both the lack of those variational structures that are naturally associated with varifolds (e.g. first variation and first variation function), and the lack of a powerful regularity theory (Allard’s theorem). Therefore the proof of 1.4 presents aspects that are new even for the varifold’s case treated by Almgren. We describe now the main steps of this proof. For the inequality case we use compactness of $\Gamma$ to see that for each $\eta \in S^{n-1}$ there exists an $(n-1)$ dimensional plane $\pi$ perpendicular to $\eta$ such that $\Gamma$ lies on one side of $\pi$ and touches $\pi$ at least in one point. This is the Alexandrov technique of moving planes and can be precisely stated saying that the projection onto $S^{n-1}$ of the contact set

$$C = (\Gamma \times S^{n-1}) \cap \{(z, \eta) : (w - z) \cdot \eta \leq 0 \text{ for every } w \in \Gamma\} \subseteq N(\Gamma)$$

equals $S^{n-1}$. Then the estimate $\text{trace } Q_\Gamma \leq m$ in 1.2 and the more elementary fact that $Q_\Gamma$ has a sign when restricted on $C$ allows to obtain the estimate $\mathcal{H}^m(\Gamma) \geq \mathcal{H}^m(S^m)$. This crucial quantitative estimate is obtained working
directly on the projection of the contact set of \( C \) on \( \Gamma \), combining the Coarea formula \[1.2\] and the Barrier principle of White \[Whi16, 7.1\], and with no structural or smoothness assumptions at the touching points (see the introduction of \[DM19\] for further comments about the role of regularity in arguments based on Alexandrov moving plane technique and Maximum principles). This argument originates from the approach to Almgren’s theorem developed in \[Men12a\]. We discuss now the proof of the equality case. In the varifold case Almgren, after have proved that \( V \) must be a unit density varifold with mean curvature of constant length equal to \( m \) and \( \text{spt} \| V \| \) must be contained in the boundary of its convex hull, he reduces the proof to the codimension 1 case and makes use of the weak\(^*\) convergence of the blow up of a varifold to a stationary cone, in combination with a strong barrier principle based on first variation computations (see \[Alm86\] Appendix C (12)), to show that at each point of \( \text{spt} \| V \| \) there exists a unique unit-density varifold-tangent, which corresponds to the unique supporting hyperplane of the convex hull of \( \text{spt} \| V \| \). Then he concludes the proof employing Allard regularity theorem to prove that \( \text{spt} \| V \| \) coincides with the boundary of a smooth convex body, which must be the sphere because of the constant length of the mean curvature. In following this strategy, we need first to replace weak\(^*\) convergence with the Kuratowski converge of the blow up of an \((m,h)\) to an \((m,0)\) set (see \[Whi16\] 3.2) and using the strong barrier principle in \[Whi16, 7.3\] to conclude that at each point of \( \Gamma \) the tangent cone of \( \Gamma \) is the unique supporting hyperplane of the convex hull of \( \Gamma \). This implies that \( \Gamma \) actually coincides with the boundary of its convex hull and it is a \( C^1 \) hypersurface. At this point we cannot conclude using Allard’s regularity theorem, since such a theorem is not available in our context. Therefore to conclude the proof we use an idea that we have learned from \[Men12a\]. We apply the barrier principle \[Whi16, 7.1\] in combination with a result of Federer \[Fed69, 3.1.23\] to gain some further regularity for \( \Gamma \), namely it is a \( C^{1,1} \) hypersurface. At this point the conclusion can be easily deduced from a straightforward computation.

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2 Preliminaries

The notation and the terminology used without comments agree with \[Fed69\] pp. 669–676. For varifolds our terminology is based on \[Alm86\] Appendix C. The closure and the boundary in \( \mathbb{R}^n \) of a set \( A \) are denoted by \( \overline{A} \) and \( \partial A \) and, if \( \lambda > 0 \) and \( x \in \mathbb{R}^n \) then \( \lambda(A - x) = \{\lambda(y - x) : y \in A\} \). The gradient of a function \( f \) is denoted by \( \nabla f \). The symbol \( \cdot \) denotes the standard inner product of \( \mathbb{R}^n \). If \( T \) is a linear subspace of \( \mathbb{R}^n \), then \( T^\perp : \mathbb{R}^n \to \mathbb{R}^n \) is the orthogonal
projection onto $T$ and $T^\perp = \mathbb{R}^n \cap \{v : v \cdot u = 0 \text{ for } u \in T\}$. If $X$ and $Y$ are sets and $Z \subseteq X \times Y$ we define
\[
Z|S = Z \cap \{(x,y) : x \in S\} \quad \text{for } S \subseteq X,
\]
\[
Z(x) = Y \cap \{y : (x,y) \in Z\} \quad \text{for } x \in X.
\]
The maps $p, q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are
\[
p(x,v) = x, \quad q(x,v) = v.
\]
If $A \subseteq \mathbb{R}^n$ and $m \geq 1$ is an integer, we say that $A$ is countably $(\mathcal{H}^m, m)$ rectifiable of class 2 if $A$ can be $\mathcal{H}^m$ almost covered by the union of countably many $m$ dimensional submanifolds of class 2 of $\mathbb{R}^n$; we omit the prefix “countably” when $\mathcal{H}^m(A) < \infty$. If $X$ and $Y$ are metric spaces and $f : X \to Y$ is a function such that $f$ and $f^{-1}$ are Lipschitzian functions, then we say that $f$ is a bi-Lipschitzian homeomorphism.

**Approximate second fundamental form**

In this paper we employ weak notions of second fundamental form and mean curvature that can be naturally associated to each set $A \subseteq \mathbb{R}^n$ at those points $a \in \mathbb{R}^n$ where $A$ is approximately differentiable of order 2 in the sense of [San17]. In order to keep this preliminary section relatively short we directly refer to [San17b, 2.8-2.12], where relevant definitions and remarks about the theory developed in [San17] are summarized. On the basis of [San17b, 2.8-2.9] we can introduce the following definitions.

**2.1 Definition.** The *approximate second fundamental form of $A$ at $a$* is
\[
ap b_A(a) = ap D^2 A(a)| ap \Tan(A,a) \times ap \Tan(A,a)
\]
and the associated *approximate mean curvature of $A$ at $a$* is
\[
ap h_A(a) = \text{trace}( ap b_A(a)).
\]
If $A$ is an $m$ dimensional submanifold of class 2 then these notions agree with the classical notions from differential geometry, see [San17b, 2.10].

**Curvature for arbitrary closed sets**

Besides the concept of approximate second fundamental form, in this paper we make use of a more general notion of second fundamental form introduced in [San17b] that can be associated to arbitrary closed sets. The theory of curvature for arbitrary closed sets has been developed in [Sta79], [HLW04], [San17b] and here we summarize those concepts that are relevant for our purpose in the present paper.

Suppose $A$ is a closed subset of $\mathbb{R}^n$. 
2.2. (see [San17b, 2.1, 3.1, 3.3]) The distance function to $A$ is denoted by $\delta_A$ and $S(A, r) = \{x : \delta_A(x) = r\}$. It follows from [San17b, 2.2] that if $r > 0$ then $\mathcal{H}^{n-1}(S(A, r) \cap K) < \infty$ whenever $K \subseteq \mathbb{R}^n$ is compact and $S(A, r)$ is countably ($\mathcal{H}^{n-1}, n - 1$) rectifiable of class 2.

If $U$ is the set of all $x \in \mathbb{R}^n$ such that there exists a unique $a \in A$ with $|x - a| = \delta_A(x)$, we define the nearest point projection onto $A$ as the map $\xi_A$ characterised by the requirement

$$|x - \xi_A(x)| = \delta_A(x) \quad \text{for } x \in U.$$ 

Let $U(A) = \text{dmm} \xi_A \sim A$. The functions $\nu_A$ and $\psi_A$ are defined by

$$\nu_A(z) = \delta_A(z)^{-1}(z - \xi_A(z)) \quad \text{and} \quad \psi_A(z) = (\xi_A(z), \nu_A(z)),$$

whenever $z \in U(A)$.

2.3. (San17b 3.7, 3.14]) We define the Borel function $\rho(A, \cdot)$ setting

$$\rho(A, x) = \sup\{t : \delta_A(\xi_A(x) + t(x - \xi_A(x))) = t\delta_A(x)\} \quad \text{for } x \in U(A),$$

and we say that $x \in U(A)$ is a regular point of $\xi_A$ if and only if $\xi_A$ is approximately differentiable at $x$ with symmetric approximate differential and ap $\lim inf_{y \to x} \rho(A, y) \geq \rho(A, x) > 0$. The set of regular points of $\xi_A$ is denoted by $R(A)$.

2.4. (San17b 4.1, 4.4, 4.7, 4.9]) The generalized unit normal bundle of $A$ is defined as

$$N(A) = (A \times S^{n-1}) \cap \{(a, u) : \delta_A(a + su) = s \text{ for some } s > 0\}$$

and $N(A, a) = \{v : (a, v) \in N(A)\}$ for $a \in A$.

If $x \in R(A)$ then we say that $\psi_A(x)$ is a regular point of $N(A)$. We denote the set of all regular points of $N(A)$ by $R(N(A))$. For every $(a, u) \in R(N(A))$ we define

$$T_A(a, u) = \text{im ap} D \xi_A(x) \quad \text{and} \quad Q_A(a, u)(\tau, \tau_1) = \tau \cdot \text{ap} D \nu_A(x)(v_1),$$

where $x$ is a regular point of $\xi_A$ such that $\psi_A(x) = (a, u)$, $\tau \in T_A(a, u)$, $\tau_1 \in T_A(a, u)$ and $v_1 \in \mathbb{R}^n$ such that $\text{ap} D \xi_A(x)(v_1) = \tau_1$. We say that $Q_A(a, u)$ is the second fundamental form of $A$ at $a$ in the direction $u$.

If $(a, u) \in R(N(A))$ we define the principal curvatures of $A$ at $(a, u)$,

$$\kappa_{A, 1}(a, u) \leq \ldots \leq \kappa_{A, n-1}(a, u),$$

so that $\kappa_{A, m+1}(a, u) = \infty$, $\kappa_{A, 1}(a, u), \ldots, \kappa_{A, m}(a, u)$ are the eigenvalues of $Q_A(a, u)$ and $m = \dim T_A(a, u)$. Moreover

$$\chi_{A, 1}(x) \leq \ldots \leq \chi_{A, n-1}(x)$$

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*See [San17b, 2.4, 2.6] for the definition of approximate limit and approximate differentiability.*
are the eigenvalues of $\text{ap} D\nu_A(x)\{v : v \cdot \nu_A(x) = 0\}$ for $x \in R(A)$.

It follows from [San17b, 4.10] that if $r > 0$ and $x \in S(A, r) \cap R(A)$ then
\begin{equation}
\kappa_{A,i}(\psi_A(x)) = \chi_{A,i}(x)(1-r\chi_{A,i}(x))^{-1} \quad \text{for } i = 1, \ldots, n-1.
\end{equation}

2.5. ([San17b 5.1, 5.2]) For each $a \in A$ we define the closed convex subset
$$\text{Dis}(A, a) = \{v : |v| = \delta_A(a + v)\}$$
and we notice that $N(A, a) = \{v/|v| : 0 \neq v \in \text{Dis}(A,a)\}$. For every integer $0 \leq m \leq n$ we define the $m$-th stratum of $A$ by
$$A^{(m)} = A \cap \{a : \dim \text{Dis}(A, a) = n - m\};$$
this is a Borel set which is countably $m$ rectifiable and countably $(\mathcal{H}^m, m)$ rectifiable of class 2; see [MST 4.12].

If $a \in A^{(m)}$ then $\mathcal{H}^{n-m-1}(N(A, a) \cap V) > 0$ whenever $V$ is an open subset of $\mathbb{R}^n$ such that $V \cap N(A, a) \neq \emptyset$. In fact, noting that $\mathcal{H}^{n-m}(U \cap \text{Dis}(A, a)) > 0$ whenever $U$ is open and $U \cap \text{Dis}(A, a) \neq \emptyset$, the assertion follows applying Coarea formula [Fed69 3.2.22(3)].

2.6. ([San17b 6.1-6.2]) If $A \subseteq \mathbb{R}^n$ is a closed set, $1 \leq m \leq n - 1$ and $S \subseteq A$ is $\mathcal{H}^m$ measurable and $(\mathcal{H}^m, m)$ rectifiable of class 2 then there exists $R \subseteq S$ such that $\mathcal{H}^m(S \sim R) = 0$,
$$\text{apTan}(S, a) = T_A(a, u) \in G(n, m), \quad \text{ap b}_S(a)(\tau, v) \cdot u = -Q_A(a, u)(\tau, v)$$
for every $\tau, v \in T_A(a, u)$ and for $\mathcal{H}^{n-1}$ a.e. $(a, u) \in N(A)R$.

It is in general not possible to replace $N(A)\mid R$ with $N(A)\mid S$ in the conclusion, even if $S$ is the boundary of a $C^{1,\alpha}$ convex set $A$; see the example in [San17b 6.3].

Level sets of distance function

We conclude this preliminary section providing a structural result for the level sets of the distance function from an arbitrary closed set, which is sufficient for the purpose of the present work. Other structural results are available, in particular we refer to [RZ12] and references therein.

2.7 Theorem (Gariepy-Pepe). Suppose $A$ is a closed subset of $\mathbb{R}^n$, $r > 0$, $x \in S(A, r)$, $\delta_A$ is differentiable at $x$ and $T = \{v : v \cdot \nabla \delta_A(x) = 0\}$.

Then there exists an open neighborhood $V$ of $x$ and a Lipschitzian function $f : T \to T^{\perp}$ such that $f$ is differentiable at $T_2(x)$ with $Df(T_2(x)) = 0$ and
$$V \cap S(A, r) = V \cap \{\chi + f(\chi) : \chi \in T\}.$$

Proof. The arguments in the proof of [GP72 Theorem 1] prove the statement with the exception of the differentiability properties of $f$, which can be easily deduced, noting that $\text{Tan}(S(A, r), x) \subseteq T$. □

\footnote{In fact the following statement follows from the definition of tangent cone (see [Fed69 3.1.21]). If $T \in G(n,n-1)$, $\alpha \in T$, $f : T \to T^\perp$ is continuous at $\alpha$, $a = \alpha + f(\alpha)$, $A = \{\chi + f(\chi) : \chi \in T\}$ and $\text{Tan}(A, a) \subseteq T$ then $f$ is differentiable at $\alpha$ with $Df(\alpha) = 0$.}
The next result is a refinement of [San17b 3.13].

2.8 Lemma. If $A \subseteq \mathbb{R}^n$ is a closed set then for $\mathcal{L}^1$ a.e. $r > 0$ and for $\mathcal{H}^{n-1}$ a.e. $x \in S(A, r)$,

$$\tan(S(A, r), x) = \text{ap} \tan(S(A, r), x) = \{v : v \cdot \nu_A(x) = 0\}$$

and, if $T = \tan(S(A, r), x)$, there exists an open neighborhood $V$ of $x$ and a Lipschitzian function $f : T \to T^\perp$ such that $f$ is pointwise differentiable of order 2 at $T(x)$,

$$D^2 f(T(x))(u, v) \cdot \nu_A(x) = -\text{ap} D \nu_A(x)(u) \cdot v \quad \text{for } u, v \in T$$

and $V \cap S(A, r) = V \cap \{\chi + f(\chi) : \chi \in T\}$.

Proof. Since $\delta_A$ is differentiable at $\mathcal{L}^n$ a.e. $x \in \mathbb{R}^n$, it follows from [Fed59 4.8(3)] and Coarea formula that $\nu_A(x) = \nabla \delta_A(x)$ for $\mathcal{H}^{n-1}$ a.e. $x \in S(A, r)$ and for $\mathcal{L}^1$ a.e. $r > 0$. Henceforth, it follows from [Men16a 3.14] and [2.7] that for $\mathcal{L}^1$ a.e. $r > 0$ the level set $S(A, r)$ is pointwise differentiable of order 1 at $\mathcal{H}^{n-1}$ a.e. $x \in S(A, r)$ with

$$\tan(S(A, r), x) = \{v : v \cdot \nu_A(x) = 0\}.$$

Noting [San17b 2.16], we can argue as in the first paragraph of [San17b 3.13] to infer that for all $\mathcal{L}^1$ a.e. $r > 0$ and for $\mathcal{H}^{n-1}$ a.e. $x \in S(A, r)$ there exists $s > 0$ such that

$$U(x + s\nu_A(x), s) \cap S(A, r) = \emptyset;$$

therefore, since it is obvious that for every $x \in S(A, r)$ there exists $a \in A$ such that $|x - a| = r$ and $U(a, r) \cap S(A, r) = \emptyset$, it follows that

$$\limsup_{t \to 0} t^{-2} \sup \{\delta_{\tan(S(A, r), x)}(z - x) : z \in U(x, t) \cap S(A, r)\} < \infty$$

for $\mathcal{H}^{n-1}$ a.e. $x \in S(A, r)$ and for $\mathcal{L}^1$ a.e. $r > 0$. It follows that $S(A, r)$ is pointwise differentiable of order $(1, 1)$ at $\mathcal{H}^{n-1}$ a.e. $x \in S(A, r)$ and for $\mathcal{L}^1$ a.e. $r > 0$ (see [Men16a 3.3]) and we employ [Men16a 5.7(3)] to conclude that $S(A, r)$ is pointwise differentiable of order 2 at $\mathcal{H}^{n-1}$ a.e. $x \in S(A, r)$ and for $\mathcal{L}^1$ a.e. $r > 0$. Now the conclusion can be easily deduced with the help of [2.7] [Men16a 3.14, footnote of 3.12] and [San17b 3.13].

3 Area formula for the spherical image

We introduce now the key concept of Lusin (N) condition for the generalized unit normal bundle.

3.1 Definition. Suppose $A \subseteq \mathbb{R}^n$ is a closed set, $\Omega \subseteq \mathbb{R}^n$ is an open set and $1 \leq m < n$ is an integer. We say that $N(A)$ satisfies the $m$ dimensional Lusin (N) condition in $\Omega$ if and only if (see [2.4 2.5])

$$\mathcal{H}^{n-1}(N(A)|S) = 0, \quad \text{whenever } S \subseteq A \cap \Omega \text{ such that } \mathcal{H}^{m}(A^{(m)} \cap S) = 0.$$
3.2 Remark. If $N(A)$ satisfies the $m$ dimensional Lusin (N) condition in $\Omega$ then it follows from [San17b, 6.1] and [MS17, 4.12] that
\[
\dim T_A(a, u) = m \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } (a, u) \in N(A)|\Omega.
\]

The following coarea-type formula is a crucial consequence of the Lusin (N) condition.

3.3 Theorem. Suppose $1 \leq m < n$ is an integer, $\Omega \subseteq \mathbb{R}^n$ is open, $A \subseteq \mathbb{R}^n$ is closed and $N(A)$ satisfies the $m$ dimensional Lusin (N) condition in $\Omega$.

Then for every $\mathcal{H}^{n-1}$ measurable set $B \subseteq N(A)|\Omega$,
\[
\int_{S^{n-1}} \mathcal{H}^0 \{ a : (a, u) \in B \} \, d\mathcal{H}^{n-1}u = \int_A \int_{B(z)} | \text{discr } Q_A | d\mathcal{H}^{n-m-1}d\mathcal{H}^mz.
\]

Proof. It follows from 3.2 that for $\mathcal{H}^{n-1}$ a.e. $(a, u) \in N(A)|\Omega$,
\[
\kappa_{A,m+1}(a, u) = \infty \quad \text{and} \quad \text{discr } Q_A(a, u) = \prod_{i=1}^m \kappa_{A,i}(a, u).
\]

Therefore we use [San17b, 4.11(3), 5.6] to compute
\[
\int_{S^{n-1}} \mathcal{H}^0 \{ a : (a, v) \in B \} \, d\mathcal{H}^{n-1}v
\]
\[
= \int_B \prod_{i=1}^{n-1} | \kappa_{A,i}(a, u)| (1 + \kappa_{A,i}(a, u)^2)^{-1/2} \mathcal{H}^{n-1}(a, u)
\]
\[
= \int_{B|A^{(m)}} | \text{discr } Q_A(a, u)| \prod_{i=1}^m (1 + \kappa_{A,i}(a, u)^2)^{-1/2} \mathcal{H}^{n-1}(a, u)
\]
\[
= \int_{A^{(m)}} \int_{B(z)} | \text{discr } Q_A | d\mathcal{H}^{n-m-1}d\mathcal{H}^mz,
\]
whenever $B \subseteq N(A)|\Omega$ is $\mathcal{H}^{n-1}$ measurable.

We point out a simple and very useful generalization of the barrier principle in [Whi16, 7.1].

3.4 Lemma. Suppose $1 \leq m < n$ are integers, $T \in G(n, n-1)$, $\eta \in T^\perp$, $f : T \to T^\perp$ is pointwise differentiable of order 2 at 0 such that $f(0) = 0$ and $D f(0) = 0$, $h \geq 0$, $\Omega$ is an open subset of $\mathbb{R}^n$ and $\Gamma$ is an $(m, h)$ subset of $\Omega$ such that $0 \in \Gamma$ and
\[
\Gamma \cap V \subseteq \{ z : z \cdot \eta \leq f(T(z)) \cdot \eta \}
\]
for some open neighbourhood $V$ of 0. Then, denoting by $\chi_1 \geq \ldots \geq \chi_{n-1}$ the eigenvalues of $D^2 f(0) \cdot \eta$, it follows that
\[
\chi_1 + \ldots + \chi_m \geq -h.
\]
Proof. Fix \( \epsilon > 0 \). We define
\[
\psi(\chi) = \left( \frac{1}{2} D^2 f(0)(\chi, \chi) \cdot \eta + \epsilon |\chi|^2 \right) \eta \quad \text{for } \chi \in T,
\]
and we select \( r > 0 \) such that \( f(\chi) \cdot \eta \leq \psi(\chi) \cdot \eta \) for \( \chi \in U(0, r) \cap T \). By [Whi16, 7.1], if \( \kappa_1 \leq \ldots \leq \kappa_{n-1} \) are the principal curvatures at 0 of \( M \) with respect to the unit normal that points into \( \{ z : z \cdot \eta \leq \psi(T_{\eta}(z)) \cdot \eta \} \), then
\[
\kappa_1 + \ldots + \kappa_m \leq h.
\]
Since a standard and straightforward computation shows that \( \kappa_i = -\chi_i - \epsilon \) for \( i = 1, \ldots, n-1 \), we obtain the conclusion letting \( \epsilon \to 0 \).

Finally the following immediate consequence of Federer’s Coarea formula is needed.

**3.5 Lemma.** Suppose \( 0 \leq \mu \leq m \) are integers, \( W \) is a \((H^m, m)\) rectifiable and \( H^m \) measurable subset of \( \mathbb{R}^n \), \( S \subseteq \mathbb{R}^n \) is a countable union of sets with finite \( H^m \) measure and \( f : W \to \mathbb{R}^n \) is a Lipschitzian map such that
\[
H^m(W \cap \{ w : \Lambda_{\mu} \left( (H^m \setminus W, m) \right. \text{ap} D f(w) \} \} = 0, \]
\[
H^\mu(S \cap \{ z : H^m-f^{-1}(z) > 0 \}) = 0.
\]
Then \( H^m(f^{-1}[S]) = 0 \).

**Proof.** Firstly we reduce the problem to the case \( H^\mu(S) < \infty \); then, by [Fed69] 2.1.4, 2.10.26, to the case of a Borel subset \( S \) of \( \mathbb{R}^n \). Now the conclusion comes from the coarea formula in [Fed78, p. 300]. \( \square \)

In the proof of the next result it is convenient to introduce the following Borel sets (see [San17b, 3.9]).

**3.6 Definition.** If \( A \subseteq \mathbb{R}^n \) is closed and \( \lambda \geq 1 \) we define (see [2.3])
\[
A_\lambda = \{ x : \rho(A, x) \geq \lambda \}.
\]

**3.7 Theorem.** Suppose \( 1 \leq m \leq n-1 \), \( \Omega \) is an open subset of \( \mathbb{R}^n \), \( 0 \leq h < \infty \), \( \Gamma \) is an \((m, h)\) subset of \( \Omega \) that is a countable union of sets with finite \( H^m \) measure and \( A = \Gamma \).

Then the following two statements hold:

(1) \( N(A) \) satisfies the \( m \) dimensional Lusin (N) condition in \( \Omega \);

(2) for \( H^{n-1} \) a.e. \( (z, \eta) \in N(A)|\Omega \),
\[
\dim T_A(z, \eta) = m \quad \text{and} \quad \text{trace} Q_A(z, \eta) \leq h.
\]
Proof. Let \( \tau > 2m \).

Claim 1: If \( 0 \leq h < \frac{m}{3(2m-1)r} \) and \( x \in S(A, r) \cap R(A) \cap A_\tau \cap \xi_A^{-1}(\Gamma) \) (see (2.3)) such that \( \Theta = S(A, r) \) and the conclusions of (2.8) are satisfied, then

\[
\sum_{i=1}^{m} \chi_{A,i}(x) \leq h, \quad \| \Lambda_m ((S(A, r), n-1) \text{ ap } D \xi_A(x)) \| > 0.
\]

Noting that \( \xi_A|A_{2m} \) is approximately differentiable at \( x \), we employ [San17b, 3.9, 3.11(3)(6)] and [Fed69, 3.2.16] to conclude that

\[
(3) \quad \chi_{A,i}(x) \geq -(2m-1)r^{-1} \quad \text{for } i = 1, \ldots, n-1
\]

\[
(4) \quad \text{ap } D \xi_A(x) | \text{Tan}(S(A, r), x) = (S(A, r), n-1) \text{ ap } D \xi_A(x).
\]

We assume \( \xi_A(x) = 0 \) and we notice that \( T_\xi(x) = 0 \) and \( \nu_A(x) = r^{-1}x \). We choose \( f, V \) and \( T \) as in (2.8) and \( 0 < s < r/2 \) such that \( U(x, s) \subseteq V \). Then we define \( g(\xi) = f(\xi) - x \) for \( \xi \in T \),

\[
U = T_\xi(U(x, s) \cap \{ \chi + f(\chi) : \chi \in T \}), \quad W = \{ y - x : y \in T_\xi^{-1}(U) \cap U(x, s) \}.
\]

It follows that \( W \) is an open neighbourhood of 0 and

\[
(5) \quad W \cap A \subseteq \{ z : z \bullet \nu_A(x) \leq g(T_\xi(z)) \bullet \nu_A(x) \}.
\]

If (5) did not hold then there would be \( y \in U(x, s) \cap T_\xi^{-1}[U] \) such that \( y - x \in A \) and \( y \bullet \nu_A(x) \not> f(T_\xi(y)) \bullet \nu_A(x) \); noting that

\[
T_\xi(y) + f(T_\xi(y)) \in U(x, s) \cap S(A, r), \quad |T_\xi(y) + f(T_\xi(y)) - y| < r,
\]

we would conclude

\[
|T_\xi(y) + f(T_\xi(y)) - (y - x)| = r - (y - f(T_\xi(y))) \bullet \nu_A(x) < r,
\]

which is a contradiction. Since \( -\chi_{A,1}(x), \ldots, -\chi_{A,n-1}(x) \) are the eigenvalues of \( D^2g(0) \bullet \nu_A(x) \), we may apply (5.4) to infer that

\[
(6) \quad \chi_{A,1}(x) + \ldots + \chi_{A,m}(x) \leq h
\]

and combining (4) and (6) it follows that

\[
\chi_{A,j}(x) \leq \frac{4m - 3}{6m - 3} r^{-1} < r^{-1} \quad \text{for } j = 1, \ldots, m.
\]

Since it follows by (4) and [San17b, 3.6] that \( 1 - r\chi_{A,i}(x) \) are the eigenvalues of \( (S(A, r), n-1) \text{ ap } D \xi_A(x) \) for \( i = 1, \ldots, n-1 \), we get that

\[
\| \Lambda_m ((S(A, r), n-1) \text{ ap } D \xi_A(x)) \| \geq \prod_{i=1}^{m} (1 - \chi_{A,i}(x)r) > 0.
\]
Claim 2: for $\mathcal{H}^{n-1}$ a.e. $x \in S(A, r) \cap A_r \cap \xi^{-1}_A(\Gamma)$ and for $L^1$ a.e. $0 < r < \frac{m}{3(2m-1)} h$ the conclusion of Claim 1 holds. This is immediate since

$$\Theta^{n-1}(\mathcal{H}^{n-1} \setminus S(A, r) \sim A_r, x) = 0$$

for $\mathcal{H}^{n-1}$ a.e. $x \in S(A, r) \cap A_r$ and for every $r > 0$ by [San17b, 2.14(1)] and [Fed69, 2.10.19(4)], and $\mathcal{H}^{n-1}(S(A, r) \sim R(A)) = 0$ for $L^1$ a.e. $r > 0$ by [San17b, 3.16].

Claim 3: $N(A)$ satisfies the $m$ dimensional Lusin $(N)$ condition in $\Omega$. Let $S \subseteq \Gamma$ such that $\mathcal{H}^m(S \cap A^{(m)}) = 0$. For $r > 0$ it follows from [San17b, 3.17, 3.18(1), 4.3] that $\psi_A|A_r \cap S(A, r)$ is a bi-Lipschitz homeomorphism and

$$\psi_A(\xi^{-1}_A(x) \cap A_r \cap S(A, r)) \subseteq N(A, x) \text{ for } x \in A;$$

then we apply [San17b, 5.2] to get

$$A \cap \{x: \mathcal{H}^{m-1}(\xi^{-1}_A \{x\} \cap A_r \cap S(A, r)) > 0\} \subseteq \bigcup_{i=0}^m A^{(i)} \text{ for every } r > 0.$$

Since $\mathcal{H}^m(A^{(i)}) = 0$ for $i = 0, \ldots, m - 1$ (see 2.5), it follows

$$\mathcal{H}^m(S \cap \{x: \mathcal{H}^{m-1}(\xi^{-1}_A \{x\} \cap A_r \cap S(A, r)) > 0\}) = 0 \text{ for every } r > 0.$$

Noting Claim 2 and [San17b, 3.11(1)], we can apply [San17b, 4.3] with $W$ and $f$ replaced by $S(A, r) \cap A_r \cap \xi^{-1}_A(\Gamma)$ and $\psi_A|S(A, r) \cap A_r \cap \xi^{-1}_A(\Gamma)$ to infer that

$$\mathcal{H}^{n-1}(\xi^{-1}_A(S) \cap S(A, r) \cap A_r) = 0 \text{ for } L^1 \text{ a.e. } 0 < r < \frac{m}{3(2m-1)} h^{-1}.$$

We notice that $N(A)|S = \bigcup_{r > 0} \psi_A(S(A, r) \cap A_r \cap \xi^{-1}_A(S))$ by [San17b, 4.3] and $\psi_A(S(A, r) \cap A_r) \subseteq \psi_A(S(A, s) \cap A_r)$ if $s < r$ by [San17b, 3.18(2)]. Henceforth, it follows that

$$\mathcal{H}^{n-1}(N(A)|S) = 0.$$

Claim 4: for $\mathcal{H}^{n-1}$ a.e. $(z, \eta) \in N(A)|\Omega$,

$$\dim T_A(z, \eta) = m, \quad \text{trace } Q_A(z, \eta) \leq h.$$

By Claim 3, 3.2 Claim 2 and (2) it follows that

$$\dim T_A(z, \eta) = m \text{ for } \mathcal{H}^{n-1} \text{ a.e. } (z, \eta) \in N(A)|\Omega,$$

(7)

$$\sum_{i=1}^m \frac{\kappa_{A,i}(\psi_A(x))}{1 + r \kappa_{A,i}(\psi_A(x))} \leq h$$

for $\mathcal{H}^{n-1}$ a.e. $x \in S(A, r) \cap A_r \cap \xi^{-1}_A(\Gamma)$ and for $L^1$ a.e. $0 < r < \frac{m}{3(2m-1)} h^{-1}$. We choose a positive sequence $r_i \rightarrow 0$ such that if $M_i$ is the set of of points $x \in S(A, r_i) \cap A_r \cap \xi^{-1}_A(\Gamma)$ satisfying (7) with $r$ replaced by $r_i$, then

$$\mathcal{H}^{n-1}((S(A, r_i) \cap A_r \cap \xi^{-1}_A(\Gamma)) \sim M_i) = 0 \text{ for every } i \geq 1.$$
It follows that
\[ \text{trace } Q_A(z, \eta) \leq h \quad \text{if } (z, \eta) \in \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \psi_A(M_j) \]
and the inclusion
\[ (N(A)|\Omega) \sim \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \psi_A(M_j) \subseteq \bigcup_{i=1}^{\infty} \left( \psi_A(S(A, r_i) \cap A \cap \xi^{-1}(\Gamma)) \sim \psi_A(M_i) \right) \]
readily implies
\[ \mathcal{H}^{n-1}((N(A)|\Omega) \sim \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \psi_A(M_j)) = 0. \]

3.8 Corollary. Suppose \( 1 \leq m \leq n-1, \Omega \) is an open subset of \( \mathbb{R}^n \), \( 0 \leq h < \infty \), \( \Gamma \) is an \((m, h)\) subset of \( \Omega \) such that \( \mathcal{H}^m(\Gamma \cap K) < \infty \) for every compact set \( K \subseteq \Omega \) and \( A = \Gamma \). Then
\[ \text{ap Tan}(A^{(m)}|z, \eta) = T_A(z, \eta) \in G(n, m), \quad \text{ap } b_{A^{(m)}}(z) \cdot \eta = -Q_A(z, \eta) \]
for \( \mathcal{H}^{n-1} \) a.e. \((z, \eta) \in N(A)|\Omega\). Furthermore, if \( \Gamma = \text{spt} \| V \| \) for an integral varifold \( V \in V_m(\Omega) \) such that \( \| \delta V \| \) is absolutely continuous with respect to \( \| V \| \) and \( h(V, \cdot) \in L^\infty(\| V \|, \mathbb{R}^n) \), then
\[ \text{trace } Q_A(z, \eta) = -\text{ap } h_{A^{(m)}}(z) \cdot \eta = h(V, z) \cdot \eta \]
for \( \mathcal{H}^{n-1} \) a.e. \((z, \eta) \in N(A)|\Omega\).

Proof. If \( K \subseteq \Omega \) is compact then \( A^{(m)} \cap K \) is \((\mathcal{H}^m, m)\) rectifiable of class 2 by [MS17, 4.12]. Henceforth the conclusion follows from [Scho99, 4.12]. \( \square \)

3.9 Remark. Note that if \( V \in V_m(\Omega) \) is an integral varifold such that \( \| \delta V \| \) is absolutely continuous with respect to \( \| V \| \) and \( h(V, \cdot) \in L^\infty(\| V \|, \mathbb{R}^n) \), then it follows from [Men13, Theorem 1] and [Alm86, Appendix C(8)] that \( \text{spt} \| V \| \) is \((\mathcal{H}^m, m)\) rectifiable of class 2. In view of [Scho99, 4.13], if \( \Gamma = \text{spt} \| V \| \) in \( \mathcal{H}^{n-1} \), then we can replace \( A^{(m)} \) with \( \Gamma \) in the conclusion of the theorem; i.e.
\[ \text{ap Tan}(\Gamma, z) = T_A(z, \eta), \quad \text{ap } b_{\Gamma}(z) \cdot \eta = -Q_A(z, \eta) \]
for \( \mathcal{H}^{n-1} \) a.e. \((z, \eta) \in N(A)|\Omega\).

3.10 Remark. As pointed out in [Scho99, 2.19] the second fundamental form \( Q_\Gamma \) of an arbitrary closed set \( \Gamma \) when restricted over \( \Gamma^{(m)} \) may not be fully described by \( \text{ap } b_{\Gamma^{(m)}} \). In a certain sense [Scho99, 4.3] draws an interesting analogy with the theory of functions of bounded variation. In fact, it is well known that the total differential of a \( BV \) function is not equal to the approximate gradient, unless the function belongs to the Sobolev space. Following this analogy, \((m, h)\) sets correspond to Sobolev functions.
4 Sphere theorem

The following lemma will be useful in the proof of the rigidity theorem.

4.1 Lemma. Let $1 \leq m \leq n$ be integers and let $B$ be an $m$ dimensional submanifold of class 1 in $\mathbb{R}^n$. If $0 < \lambda < 1$ and $\varphi : B \rightarrow \mathbb{R}^n$ is a Lipschitzian map such that

$$\| D(\varphi - 1_B)(b) \| \leq \lambda \quad \text{for } \mathcal{H}^m \quad \text{a.e. } b \in B,$$

then for each $b \in B$ there exists an open neighbourhood $V$ of $b$ such that $\varphi|V \cap B$ is a bi-Lipschitzian homeomorphism.

Proof. First we prove the following claim. If $U$ is an open convex subset of $\mathbb{R}^m$, $0 \leq M < \infty$ and $g : U \rightarrow \mathbb{R}^n$ is a Lipschitzian map such that $\| D g(x) \| \leq M$ for $\mathcal{L}^m$ a.e. $x \in U$, then $\text{Lip} g \leq M$. In fact, if $a \in U$ and $r > 0$ such that $U(a, r) \subseteq U$ then Coarea formula [Fed69, 3.2.22(3)] and the fundamental theorem of calculus [Fed69, 2.9.20(1)] imply that for $\mathcal{H}^{m-1}$ a.e. $v \in \mathbb{S}^{n-1}$,

$$|g(a + tv) - g(a)| \leq Mt \quad \text{for } 0 < t < r;$$

since $g$ is continuous,

$$\limsup_{x \rightarrow a} \frac{|g(x) - g(a)|}{|x - a|} \leq M \quad \text{for } a \in U$$

and $\text{Lip} g \leq M$ by [Fed69, 2.2.7].

Now we fix $b \in B$ and $\sqrt{\lambda} < t < 1$, we define $A = \text{Tan}(B, b) + b$ and we select $r > 0$ and diffeomorphism $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of class 1 as in [Fed69, 3.1.23]. In particular we have that $f(U(b, tr)) \cap A = f(U(b, tr) \cap B)$ and

$$\| D((\varphi - 1_B) \circ f^{-1})(x) \| \leq \lambda t^{-1} \quad \text{for } \mathcal{H}^m \quad \text{a.e. } x \in f(U(b, tr) \cap B).$$

It follows that if $U$ is a convex subset of $f(U(b, tr) \cap B)$ such that $f(b) \in U$ and $U$ is relatively open in $A$ then

$$(8) \quad \text{Lip}[(\varphi - 1_B) \circ f^{-1}]|U| \leq \lambda t^{-1}. $$

Therefore one uses $(8)$ and $\text{Lip} f \leq t^{-1}$ to conclude

$$|\varphi(c) - \varphi(d)| \geq |c - d|(1 - \lambda t^{-2}) \quad \text{for } c, d \in f^{-1}(U).$$

4.2 Theorem. If $1 \leq m \leq n - 1$ and $\Gamma$ is a non-empty compact $(m, m)$ subset of $\mathbb{R}^n$ then

$$\mathcal{H}^m(\Gamma) \geq \mathcal{H}^m(\mathbb{S}^m).$$

Moreover if $\Gamma = \text{spt}(\mathcal{H}^m, \Gamma)$ and $\mathcal{H}^m(\Gamma) = \mathcal{H}^m(\mathbb{S}^m)$ then there exists an isometric injection $f : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^n$ such that

$$\Gamma = f(\mathbb{S}^m).$$
Proof. We assume $\mathcal{H}^m(\Gamma) < \infty$. We define
\[ C = (\Gamma \times \mathbb{R}^n) \cap \{(a, \eta) : \eta \bullet (w - a) \geq 0 \text{ for every } w \in \Gamma \} \]
and we notice that $C$ is a closed subset of $\Gamma \times \mathbb{R}^n$, $C(a)$ is a closed convex cone containing 0 for every $a \in \Gamma$ and, since $\Gamma$ is compact, for every $\eta \in S^{n-1}$ there exists $a \in \Gamma$ such that $\inf_{w \in \Gamma} (w \bullet \eta) = (a \bullet \eta)$; in other words,
\[ q(C \cap (\Gamma \times S^{n-1})) = S^{n-1}. \]
Moreover we let $B = \{(a, -\eta) : (a, \eta) \in C, |\eta| = 1\}$ and we notice that
(9) \[ B \subseteq N(\Gamma) \text{ and } q[B] = S^{n-1}. \]

We define $X \subseteq \Gamma^{(m)}$ as the set of $a \in \Gamma^{(m)}$ such that the following conditions are satisfied:

(i) $\Gamma^{(m)}$ is approximately differentiable of order 2 at $a$ with
\[ \text{ap Tan}(\Gamma^{(m)}, a) \in G(n, m), \]
(ii) $\text{ap b}_{\Gamma^{(m)}}(a) \bullet \eta = -Q_{\Gamma}(a, \eta)$ for $\mathcal{H}^{n-m-1}$ a.e. $\eta \in N(\Gamma, a)$,
(iii) trace $Q_{\Gamma}(a, \eta) \leq m$ for $\mathcal{H}^{n-m-1}$ a.e. $\eta \in N(\Gamma, a)$.

Since $\Gamma^{(m)}$ is $(\mathcal{H}^m, m)$ rectifiable of class 2 by [MS17, 4.12], it follows from [San17, 3.23] that condition (i) is satisfied $\mathcal{H}^m$ almost everywhere on $\Gamma^{(m)}$. Moreover, noting 3.7 and 3.8, we infer that
(10) \[ \text{ap b}_{\Gamma^{(m)}}(a) \bullet \eta = -Q_{\Gamma}(a, \eta) \text{ and } \text{trace} Q_{\Gamma}(a, \eta) \leq m \]
for $\mathcal{H}^{n-1}$ a.e. $(a, \eta) \in N(\Gamma)$ and we apply [Fed69, 2.10.25], with $X$ and $f$ replaced by $N(\Gamma)(\Gamma^{(m)})$ and $p$, to conclude that (10) holds for $\mathcal{H}^m$ a.e. $a \in \Gamma^{(m)}$ and for $\mathcal{H}^{n-m-1}$ a.e. $\eta \in N(\Gamma, a)$. Henceforth,
(11) \[ \mathcal{H}^m(\Gamma^{(m)} \sim X) = 0. \]

Furthermore we notice that if $a \in X$ then
\[ \text{ap h}_{\Gamma^{(m)}}(a) \bullet \eta \geq -m \text{ for } \mathcal{H}^{n-m-1} \text{ a.e. } \eta \in N(\Gamma, a), \]
whence we readily infer from 2.5 that
(12) \[ \text{ap h}_{\Gamma^{(m)}}(a) \bullet \eta \geq -m \text{ for all } \eta \in N(\Gamma, a). \]

If $a \in X$ we define
\[ g(a) = \xi_{C(a)}(\text{ap h}_{\Gamma^{(m)}}(a)) \]

---

A subset $C$ of $\mathbb{R}^n$ is a cone if and only if $\lambda c \in C$ whenever $0 < \lambda < \infty$ and $c \in C$. 

17
(notice \( C(a) \subseteq \text{ap } \text{Nor}(\Gamma^{(m)}), a \in G(n, n-m) \) and \( \text{ap } h_{\Gamma^{(m)}}(a) \in \text{ap } \text{Nor}(\Gamma^{(m)}), g(a)) \), we infer from [MS17, 3.9(3)] that \( \text{ap } h_{\Gamma^{(m)}}(a) - g(a) \in \text{Nor}(C(a), g(a)) \) and
\[
(\text{ap } h_{\Gamma^{(m)}}(a) - g(a)) \cdot (\eta - g(a)) \leq 0 \quad \text{for every } \eta \in C(a)
\]
and, noting that \( 0 \in C(a) \) and \( 2g(a) \in C(a) \), we conclude that
\[
(\text{ap } h_{\Gamma^{(m)}}(a) - g(a)) \cdot g(a) = 0,
\]
\[
(\text{ap } h_{\Gamma^{(m)}}(a) - g(a)) \cdot \eta \leq 0 \quad \text{for every } \eta \in C(a).
\]
Since \( -g(a)/|g(a)| \in N(\Gamma, a) \) when \( g(a) \neq 0 \) by (9), we obtain from (12) that
\[
|g(a)| = \text{ap } h_{\Gamma^{(m)}}(a) \cdot (g(a)/|g(a)|) \leq m.
\]
Moreover it follows from [San17, 4.12(3)],
\[
\text{ap } b_{\Gamma^{(m)}}(a) \cdot \eta \geq 0 \quad \text{for } a \in X \text{ and } \eta \in C(a),
\]
whence we deduce
\[
(14) \quad g(a) \cdot \eta \geq \text{ap } h_{\Gamma^{(m)}}(a) \cdot \eta \geq 0 \quad \text{for } a \in X \text{ and } \eta \in C(a),
\]
and, employing the classical inequality relating the arithmetic and geometric means of a family of non negative numbers,
\[
(15) \quad 0 \leq \text{discr}(\text{ap } b_{\Gamma^{(m)}}(a) \cdot \eta) \leq m^{-m}(\text{ap } h_{\Gamma^{(m)}}(a) \cdot \eta)^m \leq m^{-m}(g(a) \cdot \eta)^m
\]
for every \( a \in X \) and \( \eta \in C(a) \).

If \( T \in G(n, m) \) and \( v \in T^\perp \) we define
\[
D(T, v) = T^\perp \cap \{ u : u \cdot v \geq 0 \}.
\]
We readily infer that there exists \( 0 < \gamma(n, m) < \infty \) such that
\[
\gamma(n, m)|v|^m = \int_{D(T, v) \cap \mathbb{S}^{n-1}} (\eta \cdot v)^m dX^{n-m-1}\eta
\]
for every \( T \in G(n, m) \) and \( v \in T^\perp \). It follows from (14),
\[
(16) \quad C(a) \subseteq D(\text{ap } \text{Tan}(\Gamma^{(m)}, a), g(a)) \quad \text{for every } a \in X;
\]

\(^8\)If \( a_1, \ldots, a_m \) are non negative real numbers,
\[
a_1a_2 \cdots a_m \leq \left( \frac{a_1 + a_2 + \cdots + a_m}{m} \right)^m
\]
with equality only if \( a_1 = a_2 = \ldots = a_m \).
noting (9), (11), (15) and (13), we apply Coarea formula 3.3 to estimate
\[ H^{n-1}(S^{n-1}) \]
\[ \leq \int_{S^{n-1}} H^0 \{ a : (a, \eta) \in B \} d.H^{n-1}\eta \]
\[ = \int_{\Gamma} B(a) |\text{discr} Q_{\Gamma}(a, \eta)| d.H^{n-m-1}\eta d.H^m a \]
\[ = \int_{\Gamma(m)} \int_{C(a) \cap S^{n-1}} \text{discr} (\text{ap} b_{\Gamma(m)}(a) \bullet \eta) d.H^{n-m-1}\eta d.H^m a \]
\[ \leq m^{-m} \int_{\Gamma(m)} \int_{C(a) \cap S^{n-1}} (g(a) \bullet \eta)^m d.H^{n-m-1}\eta d.H^m a \]
\[ \leq m^{-m} \gamma(n, m) \int_{\Gamma(m)} |g(a)|^m d.H^m a \]
\[ \leq \gamma(n, m) H^m(\Gamma^{(m)}) \]
\[ \leq \gamma(n, m) H^m(\Gamma) . \]

Suppose \( T \in \mathbf{G}(n, m+1) \) and \( \Sigma = T \cap S^{n-1} \). We observe that if \( a \in \Sigma \) then
\[ D(\text{Tan}(\Sigma, a), -a) = \{ \eta : \eta \bullet (w - a) \geq 0 \text{ for every } w \in \Sigma \}, \]
\[ b_{\Sigma}(a)(u, v) = -a(u \bullet v) \quad \text{for } u, v \in \text{Tan}(\Sigma, a), \quad h_{\Sigma}(a) = -ma; \]
moreover \( \Sigma = \Sigma^{(m)} \) and
\[ H^0 \{ a : \eta \in D(\text{Tan}(\Sigma, a), -a) \} = 1 \quad \text{for every } \eta \in S^{n-1} \sim T^\perp. \]

Then we infer that inequalities (I)-(V) in the previous estimate are actually equalities when \( \Gamma = \Sigma \), and we conclude that
\[ H^{n-1}(S^{n-1}) = \gamma(n, m) H^m(\Sigma) = \gamma(n, m) H^m(S^m). \]

From this equation we finally obtain that
\[ H^m(\Gamma) \geq H^m(S^m) \]
and the proof of the first part of the theorem is concluded.

We now assume \( H^m(\Gamma) = H^m(S^m) \) and \( \text{spt}(H^m \downarrow \Gamma) = \Gamma \) and we prove that \( \Gamma = f(S^m) \) for some isometric injection \( f : R^{m+1} \rightarrow R^n \).

Firstly, noting that inequalities (I)-(V) are equalities, we infer that
\[ H^m(\Gamma \sim \Gamma^{(m)}) = 0 \quad \text{(by (V))} \]

19
and the following equalities hold for $\mathcal{H}^m$ a.e. $a \in \Gamma$,

(17) $|g(a)| = m$ (by (IV) and (13)), $\dim \text{ap} \text{Tan}(\Gamma, a) = m$,

(18) $D(\text{ap} \text{Tan}(\Gamma, a), g(a)) \cap S^{n-1} = C(a) \cap S^{n-1}$ (by (III) and (16)),

(19) $g(a) = \text{ap} h_{\Gamma}(a)$, $|\text{ap} h_{\Gamma}(a)| = m$,

\[
\text{dis} (\text{ap} b_{\Gamma}(a) \bullet \eta) = m^{-m}(\text{ap} h_{\Gamma}(a) \bullet \eta)^m \text{ for } \eta \in C(a) \quad (\text{by (II) and (15)}),
\]

(20) $\text{ap} b_{\Gamma}(a)(u, v) = m^{-1}(u \bullet v) \text{ap} h_{\Gamma}(a) \text{ for } u, v \in \text{ap} \text{Tan}(\Gamma, a)$.

Let $A$ be the convex hull of $\Gamma$ and let $B$ be the relative boundary of $A$. Note

that $A - a \subseteq \{v : v \bullet \eta \geq 0\}$ for every $a \in \Gamma$ and $\eta \in C(a)$. Then it follows from

(17) and (18) that $\dim \{u + \lambda g(a) : u \in \text{ap} \text{Tan}(\Gamma, a), \lambda \geq 0\} = m + 1$,

$A - a \subseteq \{u + \lambda g(a) : u \in \text{ap} \text{Tan}(\Gamma, a), \lambda \geq 0\}$

for $\mathcal{H}^m$ a.e. $a \in \Gamma$, whence we deduce that $\dim A \leq m + 1$. If $\dim A = m$ then we could apply [San17b, 2.10] with $M$ replaced by the relative interior of $A$ to infer that $\text{ap} h_{\Gamma}(x) = 0$ for $\mathcal{H}^m$ a.e. $x \in \Gamma$. Since this contradicts (19) we have proved that $\dim A = m + 1$. Then we notice that $\mathcal{H}^m(\Gamma \sim B) = 0$ and, since $\Gamma = \text{spt}(\mathcal{H}^m \downarrow \Gamma)$, it follows that $\Gamma \subseteq B$.

At this point it is not restrictive to assume $m = n - 1$ in the sequel.

Now we prove that if $x \in \Gamma$ then $\text{Tan}(\Gamma, x)$ is the unique supporting hyperplane of $A$ at $x$. We fix $x \in \Gamma$. By [Sch14, 1.3.2] there exists a closed halfspace $H$ of $\mathbb{R}^{m+1}$ such that $0 \in \partial H$ and $A - x \subseteq H$. By [AF09, Theorem 1.1.7] we choose a sequence $\lambda_i$ converging to $+\infty$ and a closed set $Z$ in $\mathbb{R}^{m+1}$ such that

(see [AF09, 1.1.1])

\[
\lambda_i (\Gamma - x) \to Z \quad \text{as } i \to \infty \text{ in the sense of Kuratowski.}
\]

Then we notice that $0 \in Z \subseteq H$, $Z$ is an $(m, 0)$ subset of $\mathbb{R}^{m+1}$ by [Whi16, 1.6, 3.2] and $\partial H \subseteq Z$ by [Whi16, 7.3]. Henceforth we have the following inclusions

$\partial H \subseteq Z \subseteq \text{Tan} (\Gamma, x) \subseteq \text{Tan} (\partial A, x) \subseteq \text{Tan} (A, x) \subseteq H$

and one may infer from [GH14, 5.7] that $\text{Tan} (A, x) = H$ and $\text{Tan} (\Gamma, x) = \partial H$.

Next we check that

$\partial A = \Gamma$.

Let $x \in \Gamma$. Then there exist an $m$ dimensional plane $T$, an open neighborhood $W$ of $x$ and a convex Lipschitzian function $f : U \to T^\perp$ defined on a relatively open convex subset $U$ of $T$ containing $T(x)$, such that

$W \cap \partial A = \{\chi + f(\chi) : \chi \in U\}$.
Since Lip \( f < \infty \) it follows that \( \text{Tan}(\partial A, y) \cap T^\perp = \{0\} \) for \( y \in W \cap \partial A \) and, since we have proved in the previous paragraph that \( \text{Tan}(\Gamma, y) \) is an \( m \) dimensional plane for every \( y \in \Gamma \), we employ \cite[first paragraph p. 234]{Fed69} to conclude

\[
T = T_s(\text{Tan}(\Gamma, y)) = \text{Tan}(T_s(W \cap \Gamma), T_s(y)) \quad \text{for every } y \in W \cap \Gamma.
\]

Noting that \( T_s(W \cap \Gamma) \) is relatively closed in \( U \), we infer\footnote{Suppose \( C \subseteq U \subseteq \mathbb{R}^n \). \( U \) is open and \( C \) is relatively closed in \( U \). If \( \text{Tan}(C, x) = \mathbb{R}^m \) for every \( x \in C \) then \( C = U \). In fact, if there was \( y \in U \setminus C \) and if \( t = \sup \{ s : \text{U}(y, s) \cap C = \varnothing \} \) then \( t > 0 \), \( \text{U}(y, t) \cap C = \varnothing \), \( B(y, t) \cap C \neq \varnothing \) and \( y - x \in \text{Nor}(C, x) \) for every \( x \in B(y, t) \cap C \). This is clearly a contradiction.} that \( T_s(W \cap \Gamma) = U \) and \( W \cap \partial A = W \cap \Gamma \). Since \( x \) is arbitrarily chosen in \( \Gamma \), it follows that \( \partial A = \Gamma \).

We combine the assertions of the previous two paragraphs with \cite[2.2.4]{Sch14} to conclude that \( \partial A \) is an \( m \) dimensional submanifold of class 1 in \( \mathbb{R}^{m+1} \). Moreover, it is well known that \( \text{dmn} \xi_A = \mathbb{R}^{m+1} \), \( \text{Lip} \xi_A \leq 1 \) (see \cite[1.2]{Sch14}) and \( \{ x : \delta_A(x) < r \} \) is an open convex set whose boundary \( S(A, r) \) is an \( m \) dimensional submanifold of class \( C^{1,1} \) for \( r > 0 \) (see \cite[4.8]{Fed59}). Let \( 0 < r < m^{-1} \) and \( \xi = \xi_A \big| S(A, r) \). For \( \mathcal{H}^m \) a.e. \( x \in S(A, r) \) we apply the barrier principle \cite[3.4]{San17} with \( T, \eta \) and \( f \) replaced by \( \{ v : v \cdot \nu_A(x) = 0 \}, \nu_A(x) \) and a concave function whose graph corresponds to \( S(A, r) \) in a neighborhood of \( x \), to infer (see \cite{San17}) that

\[
\chi_{A, i}(x) \geq 0 \quad \text{for } i = 1, \ldots, m, \quad \sum_{i=1}^m \chi_{A, i}(x) \leq m,
\]

and we combine these inequalities to conclude that \( \chi_{A, i}(x) \leq m \) for \( i = 1, \ldots, m \). Therefore \( \| D(\xi - 1_{S(A, r)})(x) \| \leq mr < 1 \) for \( \mathcal{H}^m \) a.e. \( x \in S(A, r) \) and, noting that \( \xi \) is univalent by \cite[4.8]{Fed59}, we apply \cite[4.1]{Fed59} to conclude that the function \( \xi^{-1} : \partial A \to S(A, r) \) is a locally Lipschitzian map and the unit normal vector field on \( \partial A \),

\[
\eta = \nu_A \circ \xi^{-1},
\]

is locally Lipschitzian. Combining \cite[3.25]{San17} with \cite{Fed69} and \cite{Fed69}, we infer for \( \mathcal{H}^m \) a.e. \( x \in \Gamma \) and for \( u, v \in \text{Tan}(\Gamma, x) \) that

\[
\begin{align*}
D \eta(x)(u) \cdot v &= -ap b_T(x)(u, v) \cdot \eta(x) \\
&= -m^{-1}(ap h_T(x) \cdot \eta(x))(u \cdot v) \\
&= u \cdot v,
\end{align*}
\]

whence we conclude that \( D(\eta - 1_{\Gamma})(x) = 0 \) for \( \mathcal{H}^m \) a.e. \( x \in \Gamma \). Therefore there exists \( a \in \mathbb{R}^{m+1} \) such that

\[
\eta(z) = z - a \quad \text{for every } z \in \Gamma
\]

and, since \( |\eta(z)| = 1 \) for \( z \in \Gamma \), we conclude that

\[
\Gamma = \partial B(a, 1).
\]
4.3 Remark. If $V$ is a varifold as in [Alm86, Theorem 1] and if we additionally assume that $V$ is integral then Brakke perpendicularity theorem [Bra78, 5.8] implies that $|h(V,x)| \leq m$, whence we deduce by [Whi16, 2.8] that $\text{spt} \|V\|$ is an $(m,m)$ subset of $\mathbb{R}^n$.

Without assuming $V$ to be integral, is it true that $\text{spt} \|V\|$ is an $(m,m)$ subset of $\mathbb{R}^n$?

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