MOMENT INEQUALITIES FOR TRIGONOMETRIC POLYNOMIALS WITH SPECTRUM IN CURVED HYPERSURFACES

J. BOURGAIN

(0). Summary

In this note we develop further the technique from [B-G], based on the multi-linear restriction theory from [B-C-T], to establish some new inequalities on the distribution of trigonometric polynomials on the $n$-dimensional torus $T^n$, $n \geq 2$, of the form

$$f(x) = \sum_{\mathbf{z} \in \mathcal{E}} a_{\mathbf{z}} e^{2\pi i x \cdot \mathbf{z}}$$

where $\mathcal{E}$ stands for the set of $\mathbb{Z}^n$-points on some dilate $D.S$ of a fixed compact, smooth hypersurface $S$ in $\mathbb{R}^n$ with positive definite second fundamental form. More precisely, we prove that for $p \leq \frac{2n}{n-1}$ and any fixed $\varepsilon > 0$, the bound

$$\|f\|_{L^p(T^n)} \leq C_\varepsilon D^\varepsilon \|f\|_{L^2(T^n)}$$

(0.2) holds.

In particular, if $\Delta$ stands for the Laplacian on $T^n$ and

$$-\Delta f = Ef$$

(0.3)

we have that for $p \leq \frac{2n}{n-1}$, $n \geq 2$

$$\|f\|_{L^p(T^n)} \ll \varepsilon \|f\|_{L^2(T^n)}.$$ 

(0.4)

Recall that if $n = 2$, one has the inequality, for $f$ satisfying (0.3),

$$\|f\|_{L^4(T^2)} \leq C \|f\|_{L^2(T^2)}$$

(0.5)
due to Zygmund and Cook. For \( n = 3 \), arithmetical considerations permit to obtain a bound
\[
\| f \|_{L^4(T^3)} \ll \varepsilon \| f \|_{L^2(T^3)}
\]  
(0.6)

For \( n \geq 4 \), no estimate of the type (0.4) for some \( p > 2 \) seemed to be known. Recall also that it is conjectured that one has uniform bounds
\[
\| f \|_{L^q(T^n)} \leq C_q \| f \|_{L^2(T^n)} \text{ if } q \leq \frac{2n}{n-2}
\]  
(0.7)
and
\[
\| f \|_{L^q(T^n)} \leq C_q \varepsilon^{1/2} \left( \frac{n+2}{4} \right) \| f \|_{L^2(T^n)} \text{ if } q \geq \frac{2n}{n-2}
\]  
(0.8)
if \( f \) satisfies (0.3). The inequality (0.8) was proven in [B1] (using the Hardy-Littlewood circle method) under the assumption
\[
q > \frac{2(n+1)}{n-3}
\]  
(0.9)
(up to an \( \varepsilon \)-factor).

Another application of (0.2) relates to the periodic Schrödinger group \( e^{it\Delta} \). For \( n \geq 1 \), one has the Strichartz’ type inequality
\[
\| (e^{it\Delta} f)(x) \|_{L^q(T^{n+1})} \ll R^\varepsilon \| f \|_{L^2(T^n)}
\]  
(0.10)
for \( q \leq \frac{2(n+1)}{n} \) and \( f \) satisfying \( \text{supp} \hat{f} \subset \mathbb{Z}^n \cap B(0,R) \).

Combined with results from [B3], (0.10) implies that for \( q > \frac{2(n+3)}{n} \)
\[
\| (e^{it\Delta} f)(x) \|_{L^q(T^{n+1})} \leq C_q R^{\frac{n}{2} - \frac{n+2}{q}} \| f \|_{L^2(T^n)}
\]  
(0.11)
for \( f \) as above. Note that inequality (0.11) is optimal. This result is new (and of interest to the theory of the nonlinear Schrödinger equations with periodic boundary conditions) for \( n \geq 4 \). (See [B3] for more details).

More generally, fix a smooth function \( \psi : U \to \mathbb{R} \) on a neighborhood \( U \) of \( 0 \in \mathbb{R}^n \) such that \( D^2 \psi \) is positive definite. For \( q \leq \frac{2(n+1)}{n} \) and \( R \to \infty \),
\[
\left[ \int_{[0,1]^{n+1}} \left| \sum_{z \in \mathbb{Z}^n, |z| < R} a_z e^{2\pi i (x.z + R^2 t \psi(\frac{z}{R}))} \right|^q dx dt \right]^{1/q} 
\ll R^\varepsilon \left( \sum |a_z|^2 \right)^{\frac{1}{2}}.
\]  
(0.12)
Taking $\psi(x) = \alpha_1 x_1^2 + \cdots + \alpha_n x_n^2, \alpha_1, \ldots, \alpha_n > 0$, generalizes (0.10) to irrational tori (cf. [B]).

(1). Multilinear Estimates

Fix a smooth, compact hyper-surface $S$ in $\mathbb{R}^n$ with positive definite second fundamental form. For $x \in S$, denote $x' \in S^{(n-1)} = \{ |x| = 1 \}$ the normal vector at the point $x$ and let $\sim: S^{(n-1)} \to S$ be the Gauss map. Thus $x' = x$ for $x \in S$. Let $\sigma$ be the surface measure of $S$.

The estimates below depend on the multi-linear theory developed in [BCT] to bound oscillatory integral operators. We recall the following version for later use. Let

$$\phi(x, y) = x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n (\langle Ay, y \rangle + O(|y|^2))$$  \hspace{2cm} (1.1)

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{n-1}$ is restricted to a small neighborhood of 0 and $A$ is symmetric and definite (in particular, $A$ is non-degenerate).

Denote

$$Z(x, y) = \partial_{y_1} (\nabla_x \phi) \wedge \cdots \wedge \partial_{y_{n-1}} (\nabla_x \phi).$$  \hspace{2cm} (1.2)

Fix $2 \leq k \leq n$ and disjoint balls $U_1, \ldots, U_k \subset \mathbb{R}^{n-1}$ such that the transversality condition holds

$$|Z(x, y^{(1)}) \wedge \cdots \wedge Z(x, y^{(k)})| > c \text{ for all } x \text{ and } y^{(i)} \in U_i.$$  \hspace{2cm} (1.3)

Then

$$\left\| \left( \prod_{i=1}^{k} |Tf_i| \right)^\frac{1}{k} \right\|_{L^q(B_R)} \ll R^c \left( \prod_{i=1}^{k} \|f_i\|_2 \right)^\frac{1}{q}$$  \hspace{2cm} (1.4)

with $q = \frac{2k}{k-1}$, provided $\text{supp } f_i \subset U_i$.

(2). Preliminary Lemmas

We recall a few estimates from [B-G], §3.

Lemma 1.

Let $U_1, \ldots, U_n \subset S$ be small caps such that $|x'_1 \wedge \cdots \wedge x'_n| > c$ for $x_i \in U_i$.

Let $M$ be large and $\mathcal{D}_i \subset U_i (1 \leq i \leq n)$ discrete sets of $\frac{1}{M}$-separated points.
Let $B_M \subset \mathbb{R}^n$ be a ball of radius $M$. Then, for $q = \frac{2n}{n-1}$

$$\int_{B_M} \prod_{i=1}^{n} \left| \sum_{\xi \in D_i} a(\xi) e^{ix \cdot \xi} \right|^{q/n} \ll M^\varepsilon \prod_{i=1}^{n} \left[ \sum_{\xi \in D_i} |a(\xi)|^2 \right]^{q/n}$$

(2.1)

where $\int$ denotes the average.

**Proof.**

This is just a discretized version of (2.4) with $k = n$; our assumption ensures the required transversality condition (1.3).

We can assume $B_M$ centered at 0. Introduce functions $g_i$ on $U_i$ defined by

$$\begin{cases} g_i(\zeta) = a(\xi) \text{ if } |\zeta - \xi| < \frac{c}{M}, \xi \in D_i \\ g_i(\zeta) = 0 \text{ otherwise.} \end{cases}$$

(c > 0 a small constant). One may then replace $\sum_{\xi \in D_i} a(\xi) e^{ix \cdot \xi}$ by $c' M^{n-1} \int_S g_i(\zeta) e^{ix \cdot \zeta} \sigma(d\zeta)$ if $x \in B_M$. Hence

$$\int_{B_M} \prod_{i=1}^{n} \left| \sum_{\xi \in D_i} a(\xi) e^{ix \cdot \xi} \right|^{q/n} \ll$$

$$M^{(n-1)q} \int_{B_M} \prod_{i=1}^{n} \left| \int_S g_i(\zeta) e^{ix \cdot \zeta} \sigma(d\zeta) \right|^{q/n} dx \ll$$

$$M^{(n-1)q + \varepsilon} \prod_{i=1}^{n} \|g_i\|_{L^2(U_i)} \sim M^{n-1} \prod_{i=1}^{n} \left[ \sum_{\xi \in D_i} |a(\xi)|^2 \right]^{q/n}.$$  

(2.3)

Since $\int_{B_M}$ refers to the average, (2.1) follows, since $q = \frac{2n}{n-1}$.

**Lemma 2.**

Let $S \subset \mathbb{R}^n$ be as above and $2 \leq m \leq n$. Let $V$ be an $m$-dimensional subspace of $\mathbb{R}^n$, $P_1, \ldots, P_m \in S$ such that

$$P'_1, \ldots, P'_m \in V \text{ and } |P_1 \wedge \cdots \wedge P_m| > c$$

(2.4)

and $U_1, \ldots, U_m \subset S$ sufficiently small neighborhoods of $P_1, \ldots, P_m$.

Let $M$ be large and $D_i \subset U_i (1 \leq i \leq m)$ discrete sets of $\frac{1}{M}$-separated points $\xi \in S$ such that $\text{dist}(\xi', V) < \frac{1}{M}$. Let $g_i \in L^\infty(U_i)(1 \leq i \leq m)$. Then
letting \( q = \frac{2m}{m-1} \)

\[
\int_{B_M} \prod_{i=1}^m \left| \sum_{\zeta \in D_i} \left( \int_{|\zeta-\xi| < \frac{1}{M}} g_i(\zeta) e^{ix.\zeta} \sigma(d\zeta) \right) \right|^{q/m} dx \ll
\]

\[
M^\varepsilon \left\{ \int_{B_M} \prod_{i=1}^m \left[ \sum_{\zeta \in D_i} \left| \int_{|\zeta-\xi| < \frac{1}{M}} g_i(\zeta) e^{ix.\zeta} \sigma(d\zeta) \right|^2 \right]^{1/2m} \right\}^q .
\]

(2.5)

**Proof.**

Performing a rotation, we may assume \( V = [e_1, \ldots, e_m] \) and denote \( \tilde{V} \subset S \) the image of \( V \cap S^{(n-1)} \) under the Gauss map. Let again \( B_M \) be centered at 0. For each \( \xi \in \bigcup_{i=1}^m D_i \) there is by assumption some \( \hat{\xi} \in \tilde{V} \). \( |\xi - \hat{\xi}| < \frac{c}{M} \).

Write

\[
\int_{|\zeta-\xi| < \frac{1}{M}} g_i(\zeta) e^{ix.\zeta} \sigma(d\zeta) = e^{ix\hat{\xi}} \int_{|\zeta-\hat{\xi}| < \frac{1}{M}} g_i(\zeta) e^{ix.(\zeta-\hat{\xi})} \sigma(d\zeta). \quad (2.6)
\]

Since in the second factor of (2.6), \( |\zeta-\hat{\xi}| = o\left(\frac{1}{M}\right) \), we may view it as constant \( a(\xi) \) on \( B_M \subset \mathbb{R}^n \).

Thus we need to estimate

\[
\int_{B_M} \left\{ \prod_{i=1}^m \left| \sum_{\zeta \in D_i} e^{ix.\hat{\xi}} a(\zeta) \right|^{q/m} \right\} dx .
\]

Writing \( x = (u, v) \in B_M^{(m)} \times B_M^{(n-m)} \), (2.7) may be bounded by

\[
\max_{v \in B_M^{(n-m)}} \int_{B_M^{(m)}} \left\{ \prod_{i=1}^m \left| \sum_{\zeta \in D_i} e^{iu \cdot \pi_m(\hat{\xi})} a_v(\zeta) \right|^{q/m} \right\} du \quad (2.8)
\]

with \( a_v(\zeta) = e^{iv \cdot \hat{\xi}} a(\zeta) \).

Since \( S \) has positive definite second fundamental form, \( \pi_m(\tilde{V}) \subset V = [e_1, \ldots, e_m] \) is a hypersurface in \( V \) with same property and the normal vector at \( \pi_m(\hat{\xi}) = (\hat{\xi})' \in V \). Since (2.4), application of (2.1) with \( n \) replaced by \( m \) and \( D_i \) by \( \{ \pi_m \hat{\xi}; \xi \in D_i \} \) gives the estimate on (2.7)

\[
\ll M^\varepsilon \prod_{i=1}^m \left[ \sum_{\zeta \in D_i} |a(\zeta)|^2 \right]^{q/2m}
\]

and (2.5) follows.
Lemma 3. Let 
\[ p = \frac{2n}{n-1}. \]

Take \( K_n \gg K_{n-1} \gg \cdots \gg K_1 \gg 1 \). For \( 1 \leq j \leq n \), denote by \( \{ U^{(j)}_\alpha \} \) a partition of \( S \) in cells of size \( \frac{1}{K_j} \). Then, for \( R > K_n \) and \( g \in L^2(S) \),

\[
\left\| \int g(\xi)e^{ix.\xi}\sigma(d\xi) \right\|_{L^p(B_R)} \ll \varepsilon
\]

\[
C(K_n)R^\varepsilon \left[ \int_{S} |g(\xi)|^2\sigma(d\xi) \right]^{1/2} + \sum_{2 \leq j \leq n} C(K_{j-1})K_j^\varepsilon \left\{ \sum_{\alpha} \left\| \int_{U^{(j)}_\alpha} g(\xi)e^{ix.\xi}\sigma(d\xi) \right\|_{L^p(B_R)}^2 \right\}^{1/2}
\]

\[
+ \left\{ \sum_{\alpha} \left\| \int_{U^{(1)}_\alpha} g(\xi)e^{ix.\xi}\sigma(d\xi) \right\|_{L^p(B_R)}^2 \right\}^{1/2}
\]

(2.9)

where \( C(K) \) denotes some polynomial function of \( K \).

Proof. We follow the analysis from §3 in [B-G].

For \( x \in B_R \), let

\[
(2.10) = \int_{S} g(\xi)e^{ix.\xi}\sigma(d\xi)
\]

Start decomposing \( S = \bigcup_\alpha U_\alpha(\frac{1}{K_n}) \) in caps of size \( \frac{1}{K_n} \) and write

\[
(2.10) = \sum_{\alpha} \int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi) = \sum_{\alpha} c_\alpha(x).
\]

Fixing \( x \), there are 2 possibilities

(2.11) There are \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that

\[
|c_{\alpha_1}(x)|, \ldots, |c_{\alpha_n}(x)| > K_n^{-(n-1)} \max_{\alpha} |c_\alpha(x)|
\]

(2.12)

and

\[
|\xi_1 \wedge \cdots \wedge \xi_n| \gtrsim K_n^{-n} \text{ for } \xi_i \in U_{\alpha_i}. \quad (2.13)
\]

(2.14) The negation of (2.11), which implies that there is an \( (n-1) \)-dim subspace \( V_{n-1} \) such that

\[
|c_\alpha(x)| \leq K_n^{-(n-1)} \max_{\alpha} |c_\alpha(x)| \text{ if } \text{dist}(U_\alpha, \bar{V}_{n-1}) \gtrsim \frac{1}{K_n}.
\]
If (2.11), it follows from (2.12) that
\[
\left| \int_S g(\xi) e^{ix.\xi} \sigma(d\xi) \right| \leq K_n^{n-1} \max |c_\alpha(x)| \leq K_n^{2n-2} \left[ \prod_{i=1}^n |c_{\alpha_i}(x)| \right]^{\frac{1}{n}}
\]
and the corresponding contribution to the $L^p_{BR}$-norm of (4.1) is bounded by
\[
\int_{BR} (2.11) \left| \int_S g(\xi) e^{ix.\xi} \sigma(d\xi) \right|^p \leq K_n^{2p(n-1)} \sum_{\alpha_1, \ldots, \alpha_n} \int_{BR} \prod_{i=1}^n \left| \int_{U_{\alpha_i}(1/K_n)} g(\xi) e^{ix.\xi} \sigma(d\xi) \right|^\frac{p}{n}.
\]
(2.15)

In view of (2.13), the [BCT]-estimate (1.4) with $k = n$ applies to each (2.15) term. Thus
\[
\int_{BR} \prod_{i=1}^n \left| \int_{U_{\alpha_i}(1/K_n)} g(\xi) e^{ix.\xi} \sigma(d\xi) \right|^\frac{2}{n-1} dx \ll C(K_n) R^\varepsilon \left[ \int_S |g(\xi)|^2 \sigma(d\xi) \right]^{\frac{n-1}{n}}.
\]
(2.16)

Next consider the case (2.14). Thus
\[
|2.10| \leq \left| \int_{dist(\xi, \bar{V}_{n-1})} g(\xi) e^{ix.\xi} \sigma(d\xi) \right| + \max_\alpha \left| \int_{U_{\alpha}(1/K_n)} g(\xi) e^{ix.\xi} \sigma(d\xi) \right|
\]
\[
= (2.17) + (2.18)
\]
where $V_{n-1}$ depends on $x$.

Note however that, from its definition, we may view $|c_\alpha(x)|$ as ‘essentially’ constant on balls of size $K_n$. Making this claim rigorous requires some extra work and one replaces $|c_\alpha(x)|$ by a majorant $|c_\alpha| * \eta_{K_n}$, $\eta_{K_n}(x) = \frac{1}{K_n} \eta\left( \frac{x}{K_n} \right)$ and $\eta$ a suitable bump-function. We may then ensure that $|c_\alpha| * \eta_{K_n}$ is approximately constant at scale $K_n$. But we will not sidetrack the reader with these technicalities that may be found in [B-G], §2.

Thus, upon viewing the $|c_\alpha|$ approximatively constant at scale $K_n$, the bound (2.17) + (2.18) may clearly be considered valid on $B(\bar{x}, K_n)$ with the same linear space $V_{n-1}$.

Obviously
\[
(2.18) \leq \left( \sum_\alpha \left| \int_{U_{\alpha}(1/K_n)} g(\xi) e^{ix.\xi} \sigma(d\xi) \right|^p \right)^\frac{1}{p}
\]
and the corresponding $L^p_{BR}$-contribution is bounded by
\[
\left\{ \sum_{\alpha} \left\| \int_{U_{\alpha}(\frac{1}{K_n})} g(\xi) e^{ix\cdot\xi} \sigma(d\xi) \right\|_{L^p_{BR}}^2 \right\}^{1/2}. \tag{2.19}
\]

Consider the term (2.17). Proceeding similarly, write for $x \in B(\bar{x}, K_n)$
\[
\int_{\text{dist} (\xi, V_{n-1}) \leq \frac{1}{K_n}} g(\xi) e^{ix\cdot\xi} \sigma(d\xi) = \sum_{\alpha} \int_{U_{\alpha}(\frac{1}{K_{n-1}}) \cap \text{dist} (\xi, V_{n-1}) \leq \frac{1}{K_n}} g(\xi) e^{ix\cdot\xi} \sigma(d\xi) = \sum_{\alpha} c^{(n-1)}_{\alpha}(x). \tag{2.20}
\]

We distinguish the cases

(2.20) There are $\alpha_1, \ldots, \alpha_{n-1}$ such that
\[
|c^{(n-1)}_{\alpha_1}(x)|, \ldots, |c^{(n-1)}_{\alpha_{n-1}}(x)| > K_{n-1}^{-(n-2)} \max_{\alpha} |c^{(n-1)}_{\alpha}(x)| \tag{2.21}
\]
and
\[
|\xi_1' \land \ldots \land \xi_{n-1}'| \gtrsim K_{n-1}^{-(n-1)} \text{ for } \xi_i \in U_{\alpha_i}(\frac{1}{K_{n-1}}). \tag{2.22}
\]

(2.23) Negation of (2.20), implying that there is an $(n-2)$-dim subspace $V_{n-2} \subset V_{n-1}$ (depending on $x$) such that
\[
|c^{(n-1)}_{\alpha}(x)| < K_{n-1}^{-(n-2)} \max_{\alpha} |c^{(n-1)}_{\alpha}(x)| \text{ for dist } (U_{\alpha}, V_{n-2}) \gtrsim \frac{1}{K_{n-1}}.
\]

This space $V_{n-2}$ can then again be taken the same on a $K_{n-1}$-neighborhood of $x$.

We analyze the contribution of (2.20). By (2.21)
\[
|(2.19)| < K_{n-1}^{2n-4} \left[ \prod_{i=1}^{n-1} |c^{(n-1)}_{\alpha_i}(x)| \right]^{\frac{1}{n-1}} \tag{2.24}
\]
and hence
\[
\int_{\text{dist} (\xi, V_{n-1}) \leq \frac{1}{K_n}} g(\xi) e^{ix\cdot\xi} \sigma(d\xi) \leq
\]
\[
K_{n-1}^{p(2n-4)} \sum_{\alpha_1, \ldots, \alpha_{n-1}} \left\{ \prod_{i=1}^{n-1} \int_{U_{\alpha_i}(\frac{1}{K_{n-1}}) \cap \text{dist} (\xi, V_{n-1}) \leq \frac{1}{K_n}} g(\xi) e^{ix\cdot\xi} \sigma(d\xi) \right\}^{p/(n-1)}. \tag{2.25}
\]
We use the bound (2.5) to estimate the individual integrals

\[
(2.26) \quad \int_{B(\bar{x}, K_n)} \left\{ \prod_{i=1}^{n-1} \left| \int_{U_{\alpha_i} \left( \frac{1}{K_{n-1}} \right) \cap \text{dist} \left( \xi, \hat{V}_{n-1} \right) \leq \frac{1}{K_n} \right] g(\xi) e^{ix \cdot \xi} \sigma(d\xi) \right\}^{\frac{q}{n-1}} \text{ with } q = \frac{2(n - 1)}{n - 2}.
\]

Thus \( m = n - 1 \), \( V = V_{n-1} \) and \( P_i \) is the center of \( U_{\alpha_i} \left( \frac{1}{K_{n-1}} \right) \). Let \( M = K_n \) and \( D_i \) the centers of a cover of \( U_{\alpha_i} \left( \frac{1}{K_{n-1}} \right) \cap \text{dist} \left( \xi, \hat{V}_{n-1} \right) \leq \frac{1}{K_n} \) by caps \( U_{\alpha} \left( \frac{1}{K_n} \right) \).

By (2.5) we get an estimate

\[
(2.26) \ll K_n^C(K_{n-1}) \left\{ \int_{B(\bar{x}, K_n)} \prod_{i=1}^{n-1} \left[ \sum_{\alpha} \left| \int_{U_{\alpha} \left( \frac{1}{K_n} \right)} g(\xi) e^{ix \cdot \xi} \sigma(d\xi) \right|^2 \right]^{\frac{1}{n-1}} \right\}^q
\]

where in \( \sum^{(i)} \) the sum is over those \( \alpha \) such that \( U_{\alpha} \left( \frac{1}{K_n} \right) \subset U_{\alpha_i} \left( \frac{1}{K_{n-1}} \right) \) and \( U_{\alpha} \left( \frac{1}{K_n} \right) \cap \hat{V}_{n-1} \neq \phi \). Hence, we certainly have

\[
(2.26) \ll K_n^C(K_{n-1}) \left\{ \int_{B(\bar{x}, K_n)} \left[ \sum_{\alpha} \left| \int_{U_{\alpha} \left( \frac{1}{K_n} \right)} g(\xi) e^{ix \cdot \xi} \sigma(d\xi) \right|^2 \right]^{\frac{1}{2}} \right\}^q
\]

and therefore, since \( p < q \),

\[
(2.25) \ll K_n^C(K_{n-1}) \left\{ \int_{B(\bar{x}, K_n)} \left[ \sum_{\alpha} \left| \int_{U_{\alpha} \left( \frac{1}{K_n} \right)} g(\xi) e^{ix \cdot \xi} \sigma(d\xi) \right|^2 \right]^{p/2} \right\}.
\]

Hence the collected contribution over \( B_R \) of (2.28) is bounded by

\[
K_n^C(K_{n-1}) \left\{ \sum_{\alpha} \left\| \int_{U_{\alpha} \left( \frac{1}{K_n} \right)} g(\xi) e^{ix \cdot \xi} \sigma(d\xi) \right\|_{L^p(B_R)}^2 \right\}^{1/2}.
\]

Next, we analyze the contribution of (2.23) which is similar to that of (2.14) with \( n - 1 \) replaced by \( n - 2 \) and \( K_n \) by \( K_{n-1} \). The local estimate (2.27) becomes

\[
K_{n-1}^C(K_{n-2}) \left\{ \int_{B(\bar{x}, K_{n-1})} \prod_{i=1}^{n-2} \left[ \sum_{\alpha} \left| \int_{U_{\alpha} \left( \frac{1}{K_{n-1}} \right)} g(\xi) e^{ix \cdot \xi} \sigma(d\xi) \right|^2 \right]^{\frac{1}{2(n-2)}} \right\}^q
\]

with \( q = \frac{2(n - 2)}{n - 3} \) and where in \( \sum^{(i)} \) the sum is over those \( \alpha \) such that

\[
U_{\alpha} \left( \frac{1}{K_{n-1}} \right) \subset U_{\alpha_i} \left( \frac{1}{K_{n-2}} \right) \text{ and } U_{\alpha} \left( \frac{1}{K_{n-1}} \right) \cap \hat{V}_{n-2} \neq \phi.
\]
The collected contribution of (2.30) to the $L^p_{B_R}$-norm of (2.10) is bounded by
\[
K_n^{-1} C(K_n-2) \left\{ \sum_\alpha \left\| \int_{U_\alpha(\frac{1}{K_n-1})} g(\xi)e^{ix.\xi}\sigma(d\xi) \right\|_{L^p_{B_R}}^2 \right\}^{\frac{1}{2}}. \tag{3.31}
\]
The continuation of the process is now clear and leads to the bound (2.9). This proves Lemma 3.

Taking $K_j > K_1^{C/\varepsilon}$ in Lemma 3, we obtain

**Lemma 4.** Fix $\varepsilon > 0$. Let $K_1 \gg 1$ be large enough and assume $R > K_1^{C(\varepsilon)}$.

Then, with $p = \frac{2n}{n-1}$
\[
\left\| \int g(\xi)e^{ix.\xi}\sigma(d\xi) \right\|_{L^p_{B_R}} \leq R^\varepsilon \left[ \int_S |g(\xi)|^2\sigma(d\xi) \right]^{\frac{1}{2}}
+ \max_{K_1 < K < K_1^{C(\varepsilon)}} \left\{ K\varepsilon \sum_\alpha \left\| \int_{U_\alpha(\frac{1}{K})} g(\xi)e^{ix.\xi}\sigma(d\xi) \right\|_{L^p_{B_R}}^2 \right\}^{1/2}, \tag{2.32}
\]
with $\{U_\alpha(\frac{1}{K})\}$ a cover of $S$ by $\frac{1}{K}$-size caps.

The first term on the right side of (2.32) may be eliminated.

Observe first that since $|x| < R$, the left side may be replaced by
\[
\left\| \int G(\xi)e^{ix.\xi}\sigma(d\xi) \right\|_{L^p_{B_R}} \tag{2.33}
\]
where $G$ is a smoothing of $g$ at scale $\frac{1}{R}$.

Applying (2.32) with $g$ replaced by $G$, the first term on the right
\[
\left[ \int_S |G(\xi)|^2\sigma(d\xi) \right]^{\frac{1}{2}} \lesssim \left\{ \sum_\alpha \left\| \int_{U_\alpha(\frac{1}{R})} g(\xi)e^{ix.\xi}\sigma(d\xi) \right\|_{L^p_{B_R}}^2 \right\}^{\frac{1}{2}} \tag{2.34}
\]
and the other terms may be majorized by
\[
\left\| \int_{U_\alpha(\frac{1}{R})} G(\xi)e^{ix.\xi}\sigma(d\xi) \right\|_{L^p_{B_R}} \lesssim \left\| \int_{U_\alpha(\frac{1}{R})} g_1(\xi)e^{ix.\xi}\sigma(d\xi) \right\|_{L^p_{B_R}} \tag{2.35}
\]
for some $g_1 = \eta g$ with $\eta$ a smooth function.

Hence we obtain
Lemma 5. Fix $\varepsilon > 0$. Let $K_1 \gg 1$ be large enough and assume $R > K_1^{C(\varepsilon)}$. Then, with $p = \frac{2n}{n-1}$, we have

$$\left\| \int g(\xi) e^{ix.\xi} \sigma(d\xi) \right\|_{L^p(B_R)} < R^\varepsilon \left\{ \sum_{\alpha} \left( \int_{U_\alpha(\frac{1}{R})} g(\xi) e^{ix.\xi} \sigma(d\xi) \right)^2 \right\}^{\frac{1}{2}} + \max_{K_1 < K < K_1^{C(\varepsilon)}} \left\{ K^\varepsilon \sum_{\alpha} \left( \int_{U_\alpha(\frac{1}{K})} g(\xi) e^{ix.\xi} \sigma(d\xi) \right)^2 \right\}^{\frac{1}{2}}$$

(2.36)

where $L^p(R) = L^p(\omega(\frac{1}{R})dx)$ with $0 < \omega < 1$ some rapidly decaying function on $\mathbb{R}^n$.

In order to iterate (2.36), we rely on rescaling.

Parametrize $S$ (locally, after affine coordinate change) as

$$\begin{cases}
\xi_i = y_i (1 \leq i \leq n-1) \\
\xi_n = y_1^2 + \cdots + y_{n-1}^2 + O(|y|^3)
\end{cases}$$

(2.37)

with $y$ taken in a small neighborhood of 0.

Let $U(\rho)$ be a $\rho$-cap on $S$ and evaluate

$$\left\| \int_{U(\rho)} g(\xi) e^{ix.\xi} \sigma(d\xi) \right\|_{L^p(B_R)}.$$  

(2.38)

Thus in view of (2.37), (2.38) amounts to

$$\left\| \int_{B(a,\rho)} g(y) e^{i\varphi(x,y)} dy \right\|_{L^p(B_R)}$$

(2.39)

with

$$\varphi(x,y) = x_1y_1 + \cdots + x_{n-1}y_{n-1} + x_n(|y|^2 + O(|y|^3))$$

(2.40)

and $B(a,\rho) \subset \mathbb{R}^{n-1}$.

A shift $y \mapsto y - a$ and change of variables $x'_i = x_i + x_n(2a_i + \cdots) (1 \leq i < n)$ permits to set $a = 0$. By parabolic rescaling

$$y = \rho y' \text{ and } \rho x_i = x'_i (1 \leq i < n), \rho^2 x_n = x'_n$$

(2.41)
we obtain a new phase function $\psi(x', y')$ and (2.39) becomes

$$\rho^{n-1 - \frac{n+1}{p}} \left\| \int_{B(0,1)} g(a + \rho y') e^{i\psi(x',y')} dy' \right\|_{L^p(\Omega)}$$

(2.42)

where $\Omega = \{ |x_i'| < \rho R (1 \leq i < n), |x'_n| < \rho^2 R \}.$

Partition $\Omega = \bigcup \Omega_s$ in size-$\rho^2 R$ balls $\Omega_s$ and apply Lemma 5 on each $\Omega_s$ with $R$ replaced by $\rho^2 R$. Assuming

$$R > \rho^{-2} K_1^{C(\epsilon)}$$

(2.43)

(2.36) implies that

$$\left\| \int_{B(0,1)} g(a + \rho y') e^{i\psi(x',y')} dy' \right\|_{L^p(\Omega_s)} <$$

$$(\rho^2 R)^\varepsilon \left\{ \sum_{\alpha} \left\| \int_{U_{\alpha} \left( \frac{\rho^2 R}{R} \right)} g(a + \rho y') e^{i\psi(x',y')} dy' \right\|_{L^p \left( \omega \left( \frac{x' - b_s}{\rho^2 R} \right) dx' \right)}^2 \right\}^{\frac{1}{2}} +$$

$$\max_{K_1 < K < K_1^{C(\epsilon)}} \left\{ \sum_{\alpha} \left\| \int_{U_{\alpha} \left( \frac{1}{R} \right)} g(a + \rho y') e^{i\psi(x',y')} dy' \right\|_{L^p \left( \omega \left( \frac{x' - b_s}{\rho^2 R} \right) dx' \right)}^2 \right\}^{\frac{1}{2}}$$

(2.44)

with $b_s$ the center of $\Omega_s$.

Note that certainly

$$\sum_s \omega \left( \frac{x' - b_s}{\rho^2 R} \right) < \omega_1 \left( \frac{x}{R} \right).$$

Summing (2.44)$^p$ over $s$ and reversing the coordinate changes clearly implies that

$$(2.39), (2.42) <$$

$$(\rho^2 R)^\varepsilon \left\{ \sum_{\alpha} \left\| \int_{U_{\alpha} \left( \frac{\rho^2 R}{R} \right)} g(\phi) e^{i\phi(x,y)} dy \right\|_{L^p_R}^2 \right\}^{\frac{1}{2}} +$$

$$\max_{K_1 < K < K_1^{C(\epsilon)}} \left\{ K_1^{\varepsilon} \sum_{\alpha} \left\| \int_{U_{\alpha} \left( \frac{1}{R} \right)} g(\phi) e^{i\phi(x,y)} dy \right\|_{L^p_R}^2 \right\}^{\frac{1}{2}}$$

(2.45)

under the assumption (2.43).

Taking $R = \rho^{-2} K_2$ with $K_2 > K_1^{C(\epsilon)}$ in (2.45), we obtain
Lemma 6. Let $K_2 > K_1^{C(\varepsilon)}$. Then

$$
\left\| \int_{U(\rho)} g(\xi) e^{ix.\xi} \sigma(d\xi) \right\|_{L^p(B_{K_2\rho^{-2}})} 
\ll_{\varepsilon} \max_{K_1 < K < K_2} \left\{ K^\varepsilon \sum_{\alpha} \left\| \int_{U_{\alpha}(\frac{c}{K})} g(\xi) e^{ix.\xi} \sigma(d\xi) \right\|^2_{L^p_{(\rho^{-2})}} \right\}^{\frac{1}{2}}.
$$

(2.46)

If $R > K_2\rho^{-2}$, we can partition $B_R$ in cubes of size $K_2\rho^{-2}$ and apply (2.46) on each of them, with $g(\xi)$ replaced by $g(\xi) e^{i\alpha.\xi}$ for some $\alpha \in B_R$. Hence

Lemma 6'. Let $R > K_2\rho^{-2}, K_2 = K_1^{C(\varepsilon)}$. Then

$$
\left\| \int_{U(\rho)} g(\xi) e^{ix.\xi} \sigma(d\xi) \right\|_{L^p(B_R)} 
\ll_{\varepsilon} \max_{K_1 < K < K_2} \left\{ K^\varepsilon \sum_{\alpha} \left\| \int_{U_{\alpha}(\frac{c}{K})} g(\xi) e^{ix.\xi} \sigma(d\xi) \right\|^2_{L^p(R)} \right\}^{\frac{1}{2}}.
$$

(2.47)

It is now straightforward to iterate Lemma 6' and derive the following statement

Proposition 1. Let $0 < \delta \ll 1$ and $R > C(\varepsilon)\delta^{-2}$. Then, with $p = \frac{2n}{n-1}$

$$
\left\| \int g(\xi) e^{ix.\xi} \sigma(d\xi) \right\|_{L^p_{(R)}} 
\ll_{\varepsilon} \delta^{-\varepsilon} \left\{ \sum_{\alpha} \left\| \int_{U_{\alpha}(\delta)} g(\xi) e^{ix.\xi} \sigma(d\xi) \right\|^2_{L^p_{(R)}} \right\}^{\frac{1}{2}}.
$$

(2.48)

(3). $L^p$-bounds for certain exponential polynomials and applications

We fix a smooth compact hyper-surface $S$ in $\mathbb{R}^n$ with positive definite second fundamental form. We consider exponential polynomials with frequencies on some dilate $D.S$ of $S$.

Proposition 2. Let $0 < \rho < D$ and let $\mathcal{E}$ be a discrete set of points on the dilate $D.S$ that are mutually at least $\rho$ separated. Then, for $p = \frac{2n}{n-1}$ and any (fixed) $\varepsilon > 0$

$$
\left[ \int_{B_R} \left| \sum_{z \in \mathcal{E}} a_z e^{ix.z} \right|^p dx \right]^{\frac{1}{p}} 
\ll_{\varepsilon} \left( \frac{D}{\rho} \right)^{\varepsilon} \left( \sum_{z \in \mathcal{E}} |a_z|^2 \right)^{\frac{1}{2}}.
$$

(3.1)
provided
\[ R > C(\varepsilon)D\rho^{-2}. \] (3.2)

Proof.

By rescaling, we may clearly assume \( D = 1 \).

Let \( 0 < \tau < \rho/10 \) and let \( g \) be the function on \( S \) defined by
\[
g(\xi) = \frac{a_z}{\sigma(U(z, \tau))} \text{ if } \xi \in U(z, \tau) \\
= 0 \text{ otherwise} \tag{3.3}
\]

Here \( U(z, \tau) \subset S \) denotes a \( \tau \)-neighborhood of \( z \) on \( S \). Thus
\[
\int g(\xi) e^{ix.\xi} \sigma(d\xi) = \sum_{z \in \mathcal{E}} a_z \int_{U(z, \tau)} e^{ix.\xi} \sigma(d\xi). \tag{3.4}
\]

Applying (2.48) with \( \delta = \rho \), it follows from (3.3), (3.4) that
\[
\left\{ \int_{B_R} \left| \sum_{z \in \mathcal{E}} a_z \int_{U(z, \tau)} e^{ix.\xi} \sigma(d\xi) \right|^p dx \right\}^{\frac{1}{p}} \ll_{\varepsilon} \rho^{-\varepsilon} \left( \sum_{z} |a_z|^2 \right)^{1/2} \tag{3.5}
\]

letting \( \tau \to 0 \), (3.1) clearly follows.

Next, observe that if \( \mathcal{E} \) is contained in a lattice, then \( \sum_{z \in \mathcal{E}} a_z e^{ix.\xi} \) is a periodic function. Hence Proposition 2 implies

**Proposition 3.** Let \( S \) be as above and \( \mathcal{E} = \mathbb{Z}^n \cap DS, D \to \infty \).

Then, with \( p = \frac{2n}{n-1} \)
\[
\left[ \int_{\mathbb{T}^n} \left| \sum_{z \in \mathcal{E}} a_z e^{2\pi ix.z} \right|^p dx \right]^{\frac{1}{p}} \ll_{\varepsilon} D^{\varepsilon} \left( \sum_{z} |a_z|^2 \right)^{1/2} \tag{3.6}
\]

where \( \mathbb{T}^n \) stands for the \( n \)-dimensional torus.

**Corollary 4.** Let \( \varphi = \varphi_E, -\Delta \varphi_E = E\varphi_E \) be an eigenfunction of \( \mathbb{T}^n, n \geq 2 \).

Then for \( p = \frac{2n}{n-1} \) and any \( \varepsilon > 0 \), we have
\[
\| \varphi \|_{L^p(\mathbb{T}^n)} \leq C(\varepsilon) E^{\varepsilon} \| \varphi \|_{L^2(\mathbb{T}^n)}. \tag{3.7}
\]
Remark. Corollary 4 should be compared with the result from [B1]. It is conjectured that for eigenfunctions of $T^n$, $n \geq 2$, there is a uniform bound
\[ \|\varphi\|_p \leq C(p)\|\varphi\|_2 \text{ for } p < \frac{2n}{n-2}. \] (3.8)

If $n = 2$, (3.8) is known to hold for $p \leq 4$ (due to Zygmund-Cook) but for no exponent $p > 4$.

If $n = 3$, (3.7) is valid for $p \leq 4$. This is a consequence of the following observation. One clearly has the estimate
\[ \|\varphi\|_4 \leq K^{1/4}\|\varphi\|_2 \]
denoting
\[ K = \max_{\xi \in \mathbb{Z}^3} \left( \# \{(\xi_1, \xi_2) \in \mathbb{Z} \times \mathbb{Z}; |\xi_1|^2 = E = |\xi_2|^2 \text{ and } \xi_1 + \xi_2 = \xi \} \right). \]

Projecting on one of the coordinate planes reduces the issue to bounding the number $|\mathcal{E} \cap \mathbb{Z}^2|$ with $\mathcal{E} \subset \mathbb{R}^2$ some ellipse of size at most $E^{1/2}$. It is well known that
\[ |\mathcal{E} \cap \mathbb{Z}^2| \ll E^\varepsilon \] (3.9)
(cf. [B-R]) and hence $K \ll E^\varepsilon$.

For $n \geq 4$, no estimates of the type (3.7) for some $p > 2$, seemed to be previously known. Recall that for $n \geq 4$ and $R$ a large positive integer
\[ |RS^{(n-1)} \cap \mathbb{Z}^n| \sim R^{n-2}. \] (3.10)

Thus Corollary 4 provides for any $p = \frac{2n}{n-1}$ an explicit construction of an ‘almost’ $\Lambda_p$-set which is not a $\Lambda_q$-set for $q \geq \frac{2n}{n-2}$. No explicit constructions of proper $\Lambda_p$-sets for $2 < p < 4$ seem to be known and their existence results from probabilistic arguments (see [B2], [B4]).

In view of (3.10), Corollary 4 also provides explicit almost Euclidean subspaces of dimension $\sim N^{\frac{4}{p}-1}$ in $\ell_N^p$, for $p$ of the form $\frac{2n}{n-1}$, $n \geq 4$ (while their maximal dimension is $\sim N^{\frac{4}{p}}$ for $2 < p < \infty$). To be compared with the result from [G-L-R] on explicit almost Euclidean subspaces of $\ell_1^n$.

Returning to Proposition 3, we have more generally

**Proposition 3’.** Let $S$ be as in Proposition 3 and $T \in GL_n(\mathbb{R}), \|T\| > 1$, an arbitrary invertible linear transformation. Let $\mathcal{E} = \mathbb{Z}^n \cap T(S)$. Then, letting $p = \frac{2n}{n-1}$, we have the inequality
\[ \left[ \int_{\mathbb{T}^n} \left| \sum_{z \in \mathcal{E}} a_z e^{2\pi i x \cdot z} \right|^p dx \right]^{\frac{1}{p}} \ll \|T\|^\varepsilon \left( \sum_{x \in \mathcal{E}} |a_z|^2 \right)^{\frac{1}{2}}. \] (3.11)
Proof. Consider the set
\[ E' = \{ T^{-1}z; z \in E \} \subset S \]
which elements are at least \( \frac{1}{\|T\|} \)-separated. Applying Proposition 2 with \( D = 1 \) and \( \rho = \frac{1}{\|T\|} \), we obtain
\[
\lim_{R \to \infty} \int_{B_R} \left| \sum_{z \in E} a_z e^{2\pi i x \cdot T^{-1}z} \right|^p dx' \ll \|T\|^\varepsilon \left( \sum_{z \in E} |a_z|^2 \right)^{\frac{1}{2}}.
\] (3.12)

By change of variables \( x = (T^{-1})^* x' \), it follows that
\[
\lim_{R \to \infty} \left[ \int_{(T^{-1})^*(B_R)} \left| \sum_{z \in E} a_z e^{2\pi i x \cdot z} \right|^p dx \right]^{\frac{1}{p}} \ll \|T\|^\varepsilon \left( \sum_{z \in E} |a_z|^2 \right)^{\frac{1}{2}}
\] (3.13)
which, by periodicity, is equivalent to (3.11).

Take \( S = \{ (y, |y|^2); y \in \mathbb{R}^n, |y| < 1 \} \) the truncated paraboloid in \( \mathbb{R}^{n+1} \) and let \( T(x, t) = (Rx, R^2 t), R > 1 \). From Proposition 3', we immediately derive the following Strichartz’ type inequality for the periodic Schrödinger group \( e^{it\Delta} \).

**Corollary 5.** Denote \( \Delta \) the Laplacian on \( \mathbb{T}^n \). Then, for \( p = \frac{2(n+1)}{n} \), we have the inequality
\[
\|e^{it\Delta} f\|_{L^p(\mathbb{T}^n \times \mathbb{T})} \ll R^\varepsilon \|f\|_{L^2(\mathbb{T})}
\] (3.14)
assuming supp \( \hat{f} \subset B(0, R) \).

This bound should be compared with the following result established in [B3].

**Proposition 6.** Let \( f \in L^2(\mathbb{T}^n), \|f\|_2 = 1 \) and such that supp \( \hat{f} \subset B(0, R) \). Then, for \( \lambda > R^{\frac{2}{n}} \) and \( q > \frac{2(n+2)}{n} \), the following inequality holds
\[
\text{mes } \left[ \{(x, t) \in \mathbb{T}^{n+1}; |e^{it\Delta} f|(x) > \lambda \} \right] < C_q R^{\frac{2}{n}q-(n+2)\lambda^{-q}}.
\] (3.15)

Combining Corollary 5, Proposition 6, we obtain the following improvement over Proposition 3.110 in [B3].
Corollary 7. Let \( n \geq 4 \) (for \( n < 4 \), better result may be obtained by arithmetical means, cf. [B3]).

Let \( f \) be as in Proposition 6. Then, for \( q > \frac{2(n+3)}{n} \)

\[
\|e^{it\Delta}f\|_{L^q(T^{n+1})} < C_{q}R^{\frac{n+2}{2n}}
\]

(3.16) holds.

Note that (3.16) is optimal.

Proof.

Denote \( q_0 = \frac{2(n+1)}{n} \) and \( q_1 \) some exponent \( > \frac{2(n+2)}{n} \). Let \( F(x,t) = (e^{it\Delta}f)(x) \) and estimate for \( q > q_1 \)

\[
\int_{T^{n+1}} |F|^q \leq \int_{|F| > R^{\frac{n}{2}}} |F|^q + R^{\frac{n}{2}(q-q_0)} \int |F|^{q_0} < C_{q_1}R^{\frac{n}{2}q_1-(n+2)} + C_{q}R^{\frac{n}{2}(q-q_0)+\varepsilon}
\]

\[
C_{q_1} \frac{1}{q-q_1} R^{\frac{n}{2}q-(n+2)} + C_{\varepsilon}R^{\frac{n}{2}(q-q_0)+\varepsilon} < C_{q}R^{\frac{n}{2}q-(n+2)}
\]

for \( q \) as above.

Corollary 5 admits a generalization that we discuss next. Assume \( \psi : \cup \to \mathbb{R}, U \subset \mathbb{R}^n \) a neighborhood of 0, is a smooth function such that \( D^2\psi \) is positive (or negative) definite. Then one has

Proposition 8. Let \( p = \frac{2(n+1)}{n} \) and \( N \to \infty \). Then for all \( \varepsilon > 0 \),

\[
\left[ \int_{[0,1]^{n+1}} \left| \sum_{z \in \mathbb{Z}^n} a_z e^{2\pi i (x,z+N^2t\psi(z/N))} \right|^p dx dt \right]^{\frac{1}{p}} \ll N^\varepsilon \left( \sum |a_z|^2 \right)^{1/2}.
\]

(3.17)

Note that a coordinate change \( x \mapsto x + Nt\nabla\psi(0) \) permits to assume \( \psi(0) = \nabla\psi(0) = 0 \). Let \( S = \{(x, \psi(x), x \in U]\) and

\[
\mathcal{E} = \left\{ \left( \frac{z}{N}, \psi\left( \frac{z}{N} \right) \right) ; z \in \mathbb{Z}^n, \frac{z}{N} \in U \right\} \subset S.
\]
Application of Proposition 2 with $\rho \sim \frac{1}{N}$ implies that

$$\left[ \int_{[0,1]^{n+1}} \left| \sum_{z \in \mathbb{Z}^n, \frac{x}{N} \in U} a_z e^{2\pi i (Nz \cdot x + N^2 \psi(\frac{x}{N}) t)} \right|^p \, dx \, dt \right]^{\frac{1}{p}} \ll N^{\varepsilon} \left( \sum |a_z|^2 \right)^{1/2}$$

(3.18)

and (3.17) follows by exploiting periodicity in $x$. This proves Proposition 8.

Finally, observe that by taking $\psi(x) = \alpha_1 x_1^2 + \cdots + \alpha_n x_n^2$ with $\alpha_1, \ldots, \alpha_n > 0$, Corollary 5 generalizes to a Strichartz inequality for irrational tori, as considered in [B]. Applications to nonlinear Schrödinger type equations will not be discussed in this paper.

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