Hamiltonian Algebroids and deformations of complex structures on Riemann curves
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Abstract

Starting with a Lie algebroid $A$ over a space $M$ we lift its action to the canonical transformations on the affine bundle $\mathcal{R}$ over the cotangent bundle $T^*M$. Such lifts are classified by the first cohomology $H^1(A)$. The resulting object is a Hamiltonian algebroid $A^H$ over $\mathcal{R}$ with the anchor map from $\Gamma(A^H)$ to Hamiltonians of canonical transformations. Hamiltonian algebroids generalize Lie algebras of canonical transformations. We prove that the BRST operator for $A^H$ is cubic in the ghost fields as in the Lie algebra case. The Poisson sigma model is a natural example of this construction. Canonical transformations of its phase space define a Hamiltonian algebroid with the Lie brackets related to the Poisson structure on the target space. We apply this scheme to analyze the symmetries of generalized deformations of complex structures on Riemann curves $\Sigma_{g,n}$ of genus $g$ with $n$ marked points. We endow the space of local $GL(N, \mathbb{C})$-opers with the Adler-Gelfand-Dikii (AGD) Poisson brackets. Its allows us to define a Hamiltonian algebroid over the phase space of $W_N$-gravity on $\Sigma_{g,n}$. The sections of the algebroid are Volterra operators on $\Sigma_{g,n}$ with the Lie brackets coming from the AGD bivector. The symplectic reduction defines the finite-dimensional moduli space of $W_N$-gravity and in particular the moduli space of the complex structures $\overline{\partial}$ on $\Sigma_{g,n}$ deformed by the Volterra operators.

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1 Introduction

Lie groups by no means exhaust symmetries in gauge theories. Their importance is related to the natural geometric structures defined by a group action in accordance with the Erlanger program of F. Klein. The first class constraints in Hamiltonian systems generate the canonical transformations of the phase space which generalize the Lie group actions [1]. There exists a powerful approach to treat such types of structures. It is the BRST method that is applicable in Hamiltonian and Lagrangian forms [2]. The BRST operator corresponding to arbitrary first class constraints acquires the most general form. An intermediate step in this direction is the canonical transformations generated by the quasigroups [3,4]. The BRST operator for the quasigroup action has the same form as for the Lie group case.

Here we consider the quasigroup symmetries constructed by means of special transformations of the "coordinate space" $M$. These transformations along with the coordinate space $M$ define the Lie groupoids, or their infinitesimal version - the Lie algebroids $A$ [5,6]. We lift the algebroid action from $M$ to the cotangent bundle $T^*M$, or, more generally, to the principle homogeneous space $R$ over the cotangent bundle $T^*M$. We call this bundle the Hamiltonian algebroid $A^H$ related to the Lie algebroid $A$. The Hamiltonian algebroid is an analog of the Lie algebra of symplectic vector fields with respect to the canonical symplectic structure on $R$ or $T^*M$. The lifts from $M$ to $R$'s are classified by the first cohomology group $H^1(A)$. We prove that the BRST operator of $A^H$ has the same structure as for the Lie algebras transformations.
The general example of this construction is the Poisson sigma-model \[7, 8\]. The Lie brackets of the Hamiltonian algebroid over the phase space of the Poisson sigma-model are defined by the Poisson bivector on the target space \( M \).

Our main interest lies in topological field theories, where the factorization with respect to the canonical gauge transformations may lead to generalized deformations of corresponding moduli spaces. We apply this scheme to analyze the moduli space of deformations of the complex structures on Riemann curves of genus \( g \) with \( n \) marked points \( \Sigma_{g,n} \) by differential operators of finite order, or equivalently by the Volterra operators.

To define the deformations we start with the space \( M_N \) of \( \text{SL}(N, \mathbb{C}) \)-opers over \( \Sigma_{g,n} \) \[9, 10\], and define a Lie algebroid \( \mathcal{A}_N \) over \( M_N \). The Lie brackets on the space of sections \( \Gamma(\mathcal{A}_N) \) are derived from the Adler-Gelfand-Dikii brackets for the local opers over a punctured disk \( D^* \) \[11, 12\]. In this way the set of the local opers serves as the target space of the Poisson sigma-model. The space \( M_2 \) of \( \text{SL}(2, \mathbb{C}) \)-opers is the space of the projective structures on \( \Sigma_{g,n} \) and the Lie algebroid \( \mathcal{A}_2 \) leads to the Lie algebra of vector fields on \( \Sigma_{g,n} \). The case \( N > 2 \) is more subtle and we deal with a genuine Lie algebroids since differential operators of order greater than one do not form a Lie algebra with respect to the standard commutator. The AGD brackets define a new commutator on \( \Gamma(\mathcal{A}_N) \) that depends on the projective structure and higher spin fields. In other words, for \( N > 2 \) we deal with the structure functions rather than with the structure constants.

The space \( M_N \) of \( \text{SL}(N, \mathbb{C}) \)-opers can be considered as a configuration space of \( W_N \)-gravity \[13, 14, 15\]. The whole phase space \( \mathcal{R}_N \) of \( W_N \)-gravity is the affinization of the cotangent bundle \( T^* M_N \). Its sections define the deformations of the operator \( \bar{\partial} \) by the Volterra operators. The canonical transformations of \( \mathcal{R}_N \) are sections of the Hamiltonian algebroid \( \mathcal{A}_N^H \) over \( \mathcal{R}_N \). The symplectic quotient of the phase space is the so-called \( W_N \)-geometry of \( \Sigma_{g,n} \). Roughly speaking, this space is a combination of the moduli of generalized complex structures and the spin \( 2 \ldots, \text{spin } N \) fields as the dual variables. To define the \( W_N \)-geometry we construct the BRST operator for the Hamiltonian algebroid. As it follows from the general construction, it has the same structure as in the Lie algebra case. We consider in detail the simplest nontrivial case \( N = 3 \). It is possible in this case to describe explicitly the sections of the algebroid as the second order differential operators, instead of Volterra operators. This algebroid is generalization of the Lie algebra vector fields on \( \Sigma_{g,n} \). It should be noted that the BRST operator for the \( W_3 \)-algebras was constructed in \[18\]. But here we construct the BRST operator for the different object - the algebroid symmetries of \( W_3 \)-gravity. Recently, another BRST description of \( W \)-symmetries was proposed in Ref.[19]. We explain our formulae and the origin of the algebroid by the special gauge procedure of the \( \text{SL}(N, \mathbb{C}) \) Chern-Simons theory using an approach developed in Ref.[14].

The paper is organized as follows. In the next section we define the general Hamiltonian algebroids, their cohomolgies and the BRST construction. We also introduce a special class of Hamiltonian algebroids related to Lie algebroids and prove that the BRST has the Lie algebraic form. In Section 3 we treat the Poisson sigma model and its symmetries as a Hamiltonian algebroid related to a Lie algebroid. In Section 4 we consider two examples of our construction when the algebroids coincide with Lie algebras. Namely, we analyze the moduli space of flat \( \text{SL}(N, \mathbb{C}) \)-bundles and the moduli of projective structures on \( \Sigma_{g,n} \). A nontrivial example of this construction is \( W_3 \)-gravity. It is considered in detail in Section 5. The general case \( W_N \) is analyzed in Section 6.
2 Hamiltonian algebroids and groupoids

2.1 Lie algebroids and groupoids

We start with a brief description of Lie algebroids and Lie groupoids. Details of this theory can be found in [4, 5, 6].

Definition 2.1 A Lie algebroid over a differential manifold $M$ is a vector bundle $A \to M$ with a Lie algebra structure on the space of its sections $\Gamma(A)$ defined by the Lie brackets $[\varepsilon_1, \varepsilon_2]$, $\varepsilon_1, \varepsilon_2 \in \Gamma(A)$ and a bundle map (the anchor) $\delta: A \to TM$, satisfying the following conditions:

(i) For any $\varepsilon_1, \varepsilon_2 \in \Gamma(A)$

$$[\delta \varepsilon_1, \delta \varepsilon_2] = \delta [\varepsilon_1, \varepsilon_2],$$

(ii) For any $\varepsilon_1, \varepsilon_2 \in \Gamma(A)$ and $f \in C^\infty(M)$

$$[\varepsilon_1, f \varepsilon_2] = f [\varepsilon_1, \varepsilon_2] + (\delta \varepsilon_1) f \varepsilon_2.$$  

In other words, the anchor defines a representation of $\Gamma(A)$ in the Lie algebra of vector fields on $M$. The second condition is the Leibnitz rule with respect to the multiplication of the sections by smooth functions.

Let $\{e^j(x)\}$ be a basis of local sections $\Gamma(A)$. Then the brackets are defined by the structure functions $f^{jk}_i(x)$ of the algebroid

$$[e^j, e^k] = f^{jk}_i(x)e^i, \quad x \in M.$$  

Using the Jacobi identity for the anchor action, we find

$$C^{n}_{j,k,m} \delta e_n = 0,$$

where

$$C^{n}_{jkm} = f^{jk}_i(x)f^{im}_n(x) + \delta e_m f^{jk}_n(x) + c.p.(j, k, m).$$

Thus, (2.4) implies the anomalous Jacobi identity (AJI)

$$f^{jk}_i(x)f^{im}_n(x) + \delta e_m f^{jk}_n(x) + c.p.(j, k, m) = 0.$$

Here are some examples of Lie algebroids.

1) If the anchor is trivial, then $A$ is just a bundle of Lie algebras.

2) Consider an integrable system that has the Lax representation

$$\partial_t \mathcal{L} = [\mathcal{L}, \mathcal{M}].$$

The Lax operator $\mathcal{L}$ belongs to some subvariety $M$ of an ambient space $\mathcal{R}$. In many cases it is a Lie coalgebra. The second operator $\mathcal{M}$ defines the tangent vector field to $M$. The operators $\mathcal{M}$ are sections of the Lie algebroid $\mathcal{A}_M$ over $M$ with the anchor determining by the Lax equation. In the similar way the dressing transformations are sections of the algebroid over $M$ [20].

3) Lie algebroids can be constructed from Lie algebras. Let $G$ be a Lie group that act on a space $S$ and $P$ is subgroup of $G$. Consider the set of orbits $M = S/P$. For $x \in M$ we have the decomposition the tangent space

$$T_x S = T_x M \oplus T_x \mathcal{O}_P,$$

1The sums over repeated indices are understood throughout the paper.
where $O_P$ is the orbit of $P$ containing $x$. Let $\epsilon$ be an element of the Lie algebra $\mathfrak{g}$ and $\text{Pr}_P$ be the projection on the second term in (2.7). Impose the following condition on the vector field $\delta_\epsilon$

$$\text{Pr}_P \delta_\epsilon(x) = 0$$

(2.8)

The subspace of the Lie algebra $\mathfrak{g}$ that satisfy this condition is the set of the sections of the Lie algebroid $\mathcal{A}_M(\mathfrak{g})$. The anchor is defined by the first term in (2.7). The commutators on the sections is the commutators of $\mathfrak{g}$.

In a generic case a Lie algebroid can be integrated to a global object - the Lie groupoid [3, 4, 5, 6].

**Definition 2.2** A Lie groupoid $G$ over a manifold $M$ is a pair of differential manifolds $(G, M)$, two differential mappings $l, r : G \to M$ and a partially defined binary operation (a product) $(g, h) \mapsto g \cdot h$ satisfying the following conditions:

(i) It is defined only when $l(g) = r(h)$.

(ii) It is associative: $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ whenever the products are defined.

(iii) For any $g \in G$ there exist the left and right identity elements $l_g$ and $r_g$ such that $l_g \cdot g = g$ and $g \cdot r_g = g$.

(iv) Each $g$ has an inverse $g^{-1}$ such that $g \cdot g^{-1} = l_g$ and $g^{-1} \cdot g = r_g$.

We denote an element of $g \in G$ by the triple $\langle\langle x | g | y \rangle\rangle$, where $x = l(g)$, $y = r(g)$. Then the product $g \cdot h$ is

$$g \cdot h \rightarrow \langle\langle x | g \cdot h | z \rangle\rangle = \langle\langle x | g | y \rangle\rangle + \langle\langle y | h | z \rangle\rangle.$$  

An orbit of the groupoid in the base $M$ is defined as an equivalence $x \sim y$ if $x = l(g)$, $y = r(g)$. The isotropy subgroup $G_x$ for $x \in M$ is defined as

$$G_x = \{g \in G \mid l(g) = x = r(g)\} \sim \{\langle\langle x | g | x \rangle\rangle\}.$$  

The Lie algebroid is a local version of the Lie groupoid. The anchor is determined in terms of the multiplication law. Details can be found in [3].

### 2.2 Lie algebroid representations and Lie algebroid cohomology

The definition of algebroids representations is rather evident:

**Definition 2.3** A vector bundle representation (VBR) $(\rho, \mathcal{M})$ of the Lie algebroid $\mathcal{A}$ over $M$ is a vector bundle $\mathcal{M}$ over $M$ and a map $\rho$ from $\mathcal{A}$ to the bundle of differential operators on $\mathcal{M}$ of the order less or equal to $1$, $\text{Diff}^{\leq 1}(\mathcal{M}, \mathcal{M})$, such that:

(i) the symbol of $\rho(\epsilon)$ is a scalar equal to the anchor of $\epsilon$:

$$\text{Symb}(\rho(\epsilon)) = \text{Id}_\mathcal{M} \delta_\epsilon,$$

(ii) for any $\epsilon_1, \epsilon_2 \in \Gamma(\mathcal{A})$

$$[\rho(\epsilon_1), \rho(\epsilon_2)] = \rho([\epsilon_1 \epsilon_2]),$$

where the l.h.s. denotes the commutator of differential operators.

For example, the trivial bundle is a VBR representation (the map $\rho$ is the anchor map $\delta$),
Consider a small disk $U_\alpha \subset M$ with local coordinates $x = (x_1, \ldots, x_a, \ldots)$. Then the anchor can be written as
\[
\delta e_j = b^j_a(x) \frac{\partial}{\partial x_a} = \langle b^j \big| \delta \rangle \partial_{x_a}^2 \tag{2.10}
\]
Let $w$ be a section of the tangent bundle $TM$. Then the VBR on $TM$ takes the form
\[
\rho_{e_j} w = \langle b^j \big| \delta \rangle w - \langle \delta \big| b^j \rangle w \tag{2.11}
\]
Similarly, the VBR the action of $\rho$ on a section $p$ of $T^*M$ is
\[
\rho_{e_j} p = \frac{\delta}{\delta x} \langle p | b^j(x) \rangle. \tag{2.12}
\]
We drop a more general definition of the sheaf representation.

Now we define cohomology groups of algebroids. First, we consider the case of contractible base $M$. Let $\mathcal{A}^*$ be a bundle over $M$ dual to $\mathcal{A}$. Consider the bundle of graded commutative algebras $\wedge^\bullet \mathcal{A}^*$. The space $\Gamma(M, \wedge^\bullet \mathcal{A}^*)$ is generated by the sections $\eta_k$: $\langle \eta_j | e^k \rangle = \delta^k_j$. It is a graded algebra
\[
\Gamma(M, \wedge^\bullet \mathcal{A}^*) = \oplus \mathcal{A}_n^*, \quad \mathcal{A}_n^* = \{ c_n(x) = \frac{1}{n!} e^{j_1, \ldots, j_n}(x) \eta_{j_1} \ldots \eta_{j_n}, \; x \in M \}. \]

Define the Cartan-Eilenberg operator “dual” to the brackets $[,]$
\[
s^c_n(x; e^1, \ldots, e^n, e^{n+1}) = (-1)^{i-1} \delta_{e_i} c_n(x; e^1, \ldots, e^i, e^n) - \sum_{j<i} (-1)^{i+j} c_n(x; e^i, e^j, \ldots, e^j, \ldots, e^i, \ldots, e^n). \tag{2.13}
\]
It follows from (2.1) and AJI (2.6) that $s^2 = 0$. Thus, $s$ determines a complex of bundles $\mathcal{A}^* \rightarrow \wedge^2 \mathcal{A}^* \rightarrow \cdots$.

The cohomology groups of this complex are called the cohomology groups of algebroid with trivial coefficients. This complex is a part of the BRST complex derived below.

The action of the coboundary operator $s$ takes the following form on the lower cochains:
\[
s^c(x; \varepsilon) = \delta^c c(x), \tag{2.14}
\]
\[
s^c(x; \varepsilon_1, \varepsilon_2) = \delta^c_1 c(x; \varepsilon_2) - \delta^c_2 c(x; \varepsilon_1) - c(x; [\varepsilon_1, \varepsilon_2]), \tag{2.15}
\]
\[
s^c(x; \varepsilon_1, \varepsilon_2, \varepsilon_3) = \delta^c_1 c(x; \varepsilon_2, \varepsilon_3) - \delta^c_2 c(x; \varepsilon_1, \varepsilon_3) \tag{2.16}
\]
\[
+ \delta^c_{13} c(x; \varepsilon_1, \varepsilon_2) - c(x; [\varepsilon_1, \varepsilon_2], \varepsilon_3) + c(x; [\varepsilon_1, \varepsilon_3], \varepsilon_2) - c(x; [\varepsilon_2, \varepsilon_3], \varepsilon_1).
\]

It follows from (2.14) that $H^0(\mathcal{A}, M)$ is isomorphic to the invariants in the space $C^\infty(M)$. The next cohomology group $H^1(\mathcal{A}, M)$ is responsible for the shift of the anchor action:
\[
\delta^c_1 f(x) = \delta^c_1 f(x) + c(x; \varepsilon), \; sc(x; \varepsilon) = 0. \tag{2.17}
\]
If $c(x; \varepsilon)$ is a cocycle (see (2.15)), then this action is consistent with the defining anchor property (2.1). The action (2.17) on $\Psi = \exp f(x)$ takes the form
\[
\delta^c_1 \Psi = (\delta^c_1 + c(x; \varepsilon)) \Psi(x). \tag{2.18}
\]

The brackets $\langle \rangle$ mean summations over all indices, taking a traces, integrations, etc.
This formula defines a “new” structure of VBR on the trivial line bundle. Let \( \tilde{M} = M/G \) be the set of orbits of the groupoid \( G \) on its base \( M \). The condition

\[
\hat{\delta}_e \Psi = 0 \tag{2.19}
\]

defines a linear bundle \( \mathcal{L}(\tilde{M}) \) over \( \tilde{M} \).

Two-cocycles \( c(x; \varepsilon_1, \varepsilon_2) \) allow to construct the central extensions of brackets on \( \Gamma(A) \)

\[
[(\varepsilon_1, k_1), (\varepsilon_2, k_2)]_{c.e.} = (\varepsilon_1, \varepsilon_2), c(x; \varepsilon_1, \varepsilon_2)) \tag{2.20}
\]

The cocycle condition (2.16) means that the new brackets \( , \) \( \varepsilon \), satisfies AJI (2.6). The exact cocycles leads to the splitted extensions.

If \( M \) is not contractible the definition of cohomology group is more complicated. We sketch the Čech version of it. Choose an acyclic covering \( U_\alpha \). Consider the Čech complex with coefficients in \( \Lambda^*(A^*) \) corresponding to this covering:

\[
\bigoplus \Gamma(U_\alpha, \Lambda^*(A^*)) \xrightarrow{d} \bigoplus \Gamma(U_{\alpha \beta}, \Lambda^*(A^*)) \xrightarrow{d} \cdots
\]

The Čech differential \( d \) commutes with the Cartan-Eilenberg operator \( s \), and cohomology of algebroid are cohomology of normalization of this bicomplex:

\[
\bigoplus \Gamma(U_\alpha, A^*_0) \xrightarrow{d,s} \bigoplus \Gamma(U_{\alpha \beta}, A^*_0) \oplus \bigoplus \Gamma(U_\alpha, A^*_1) \xrightarrow{d} \bigoplus \Gamma(U_{\alpha \beta}, A^*_1) \oplus \bigoplus \Gamma(U_\alpha, A^*_2) \xrightarrow{d} \cdots
\]

The cochains \( c_{ij} \in \bigoplus_{\alpha_1 \alpha_2 \cdots \alpha_j} \Gamma(U_{\alpha_1 \alpha_2 \cdots \alpha_j}, A^*_i) \) are bigraded. The differential maps \( c_{ij} \) to \((-1)^j d c_{ij} + sc_{ij} \), has type \((i, j + 1)\) for \((-1)^j d c_{ij} \) and \((i + 1, j)\) for \( sc_{ij} \).

Again, the group \( H^0(A, M) \) is isomorphic to the invariants in the whole space \( C^\infty(M) \).

Consider the next group \( H^1(A, M) \). It has two components

\( (c_\alpha(x, \varepsilon), c_{\alpha \beta}(x)) \). They are characterized by the following conditions (see (2.15))

\[
c_\alpha(x; [\varepsilon_1, \varepsilon_2]) = \delta_{\varepsilon_1} c_\alpha(x; \varepsilon_2) - \delta_{\varepsilon_2} c_\alpha(x; \varepsilon_1),
\]

\[
\delta_{\varepsilon} c_{\alpha \beta}(x) = -c_\alpha(x; \varepsilon) + c_\beta(x; \varepsilon), \tag{2.21}
\]

\[
c_{\alpha \gamma}(x) = c_{\alpha \beta}(x) + c_{\beta \gamma}(x). \tag{2.22}
\]

While the first component \( c_\alpha(x, \varepsilon) \) comes from the algebroid action on \( U_\alpha \) and define the action of the algebroid on the trivial bundle (2.18), the second component determines a line bundle \( \mathcal{L} \) on \( M \) by the transition functions \( \exp(c_{\alpha \beta}) \). The condition (2.21) shows that the actions on the restriction to \( U_{\alpha \beta} \) are compatible.

The continuation of the central extension (2.20) from \( U_\alpha \) on \( M \) is defined now by \( H^2(A, M) \). There is an obstacle to this continuations in \( H^3(A, M) \). We do not dwell on this point.
Let $\mathcal{R}$ be a Poisson manifold. Any smooth function $h \in C^\infty(\mathcal{R})$ gives rise to a vector field
\[ \delta_h x = \{x, h\}. \]

The space $C^\infty(\mathcal{R})$ has the structure of a Lie algebra with respect to the Poisson brackets. In what follows we assume that $\mathcal{R}$ is a symplectic manifold with the symplectic form $\omega$. In this case the Poisson brackets
\[ \{h_1, h_2\} = -i_{h_1} dh_2. \]

are defined by the internal derivation $i_{h}$ of the symplectic form $i_{h}\omega = dh$.

Let $\mathcal{A}^H$ be a vector bundle over $\mathcal{R}$ and assume that the space of sections $\Gamma(\mathcal{A}^H)$ is equipped by the Lie brackets $\{\epsilon_1, \epsilon_2\}.$

**Definition 2.4** $\mathcal{A}^H$ is a Hamiltonian algebroid over a Poisson manifold $\mathcal{R}$ if there exists a bundle map from $\mathcal{A}^H$ to the Lie algebra on $C^\infty(\mathcal{R})$:
\[ \epsilon \mapsto h_{\epsilon}, \text{ (i.e. } f \mapsto f_{h_{\epsilon}} \text{ for } f \in C^\infty(\mathcal{R}) \text{) satisfying the following conditions:} \]

(i) For any $\epsilon_1, \epsilon_2 \in \Gamma(\mathcal{A}^H)$
\[ \{h_{\epsilon_1}, h_{\epsilon_2}\} = h_{\{\epsilon_1, \epsilon_2\}}. \]  (2.23)

(ii) For any $\epsilon_1, \epsilon_2 \in \Gamma(\mathcal{A}^H)$ and $f \in C^\infty(\mathcal{R})$
\[ [\epsilon_1, f_{\epsilon_2}] = f[\epsilon_1, \epsilon_2] + \{h_{\epsilon_1}, f\}_{\epsilon_2}. \]

The both conditions are similar to the defining properties of the Lie algebroids (2.1),(2.2).

**Remark 2.1** In contrast with the Lie algebroids with the anchor $\delta_{\epsilon}$, that is a bundle map: $f\epsilon \mapsto f\delta_{\epsilon}$, for the Hamiltonian algebroids one has the map to the first order differential operators with respect to $f$
\[ f\epsilon \mapsto f\delta_{h_{\epsilon}} + h_{\epsilon}d_{\epsilon}. \]

Let $f_{i}^{jk}$ be structure functions of a Hamiltonian algebroid and
\[ C_{n}^{j,k,m} = f_{i}^{jk}(x)f_{n}^{im}(x) + \{h_{e_{m}}, f_{j,k}^{m}(x)\} + \text{c.p.}(j, k, m). \]

Then the Jacobi identity for the Poisson brackets implies
\[ C_{n}^{j,k,m}h_{e_{n}} = 0. \]  (2.24)

This identity is similar to (2.4) for Lie algebroids. But now one can add to $C_{n}^{j,k,m}$ the term proportional to $E_{[i]n}^{j,k,m}h_{e_{i}}$ without the breaking (2.24) (here $[,]$ means the antisymmetrization). Thus, the Jacobi identity for the Poisson algebra of Hamiltonians leads to following identity for the structure functions
\[ f_{i}^{jk}(x)f_{n}^{im}(x) + \{h_{e_{m}}, f_{j,k}^{m}(x)\} + E_{[i]n}^{j,k,m}h_{e_{i}} + \text{c.p.}(j, k, m) = 0. \]  (2.25)

This structure arises in the Hamiltonian systems with the first class constraints [3] and leads to the so-called open algebra of an arbitrary rank (see [1, 2]).

The important particular case
\[ f_{i}^{jk}(x)f_{n}^{im}(x) + \{H_{e_{m}}, f_{j,k}^{m}(x)\} + \text{c.p.}(j, k, m) = 0 \]  (2.26)
In other words, for any pair \( x, v \) \( x \in \mathbb{R}^3 \) Higgs bundles. \( \nabla T \) for \( R^\alpha \) (the antiHiggs field) is \( \text{sl}(N, \mathbb{C}) \) is determined by the operators the prequantization of the affinization \( R \) contrast with \( E \) linear space of sections in a trivialization of \( M \).

The set \( \mathcal{R} \) is an affinization over \( M \) (a principle homogeneous space over \( M \)) \( \mathcal{R}/M \) if the action of \( M \) on \( \mathcal{R} \) is transitive and exact.

In other words, for any pair \( x_1, x_2 \in \mathcal{R} \) there exists \( v \in M \) such that \( x_1 + v = x_2 \), and \( x_1 + v \neq x_2 \) if \( v \neq 0 \).

This construction is generalized on bundles. Let \( E \) be a bundle over \( M \) and \( \Gamma(U, E) \) be the linear space of sections in a trivialization of \( E \) over some disk \( U \).

**Definition 2.5** An affinization \( \mathcal{R}/E \) of \( E \) is a bundle over \( M \) with the space of local sections \( \Gamma(U, \mathcal{R}) \) defined as the affinization over \( \Gamma(U, E) \).

Two affinizations \( \mathcal{R}_1/E \) and \( \mathcal{R}_2/E \) are equivalent if there exists a bundle map compatible with the action of the corresponding vector bundles. It can be proved that non-equivalent affinizations are classified by \( H^1(M, \Gamma(E)) \).

Let \( E = T^*M \). Consider a linear bundle \( \mathcal{L} \) over \( M \). The space of connections \( \text{Conn}_M(\mathcal{L}) \) can be identified with the space of sections \( \mathcal{R}/T^*M \). In fact, for any connection \( \nabla_x, x \in U \subset M \) one can define another connection \( \nabla_x + \xi, \xi \in \Gamma(T^*M) \). The affinization \( \mathcal{R}/T^*M \) can be classified by the first Chern class \( c_1(\mathcal{L}) \). The trivial bundles correspond to \( T^*M \).

The affinization \( \mathcal{R}/T^*M \) is the symplectic space with the canonical form \( \langle dp \wedge dx \rangle \). In contrast with \( T^*M \) this form is not exact, since \( pdx \) is defined only locally. In the similar way as for \( T^*M \), the space of square integrable sections \( L^2(\Gamma(\mathcal{L})) \) plays the role of the Hilbert space in the prequantization of the affinization \( \mathcal{R}/T^*M \). For \( f \in \mathcal{R} \) define the Hamiltonian vector field \( \alpha_f \) and the covariant derivative \( \nabla(f)_x = i_{\alpha_f} \nabla_x \) along \( \alpha_f \). Then the prequantization of \( \mathcal{R}/T^*M \) is determined by the operators
\[
\rho(f) = \frac{1}{i} \nabla(f)_x + f
\]
acting on the space \( L^2(\Gamma(\mathcal{L})) \). In particular, \( \rho(p) = \frac{1}{i} \frac{\delta}{\delta x}, \rho(x) = x \).

The basic example, though for infinite-dimensional spaces, is the affinizations over the antiHiggs bundles. The antiHiggs bundle \( \mathcal{H}_N(\Sigma) \) is a cotangent bundle to the space of connections \( \nabla^{(1,0)} = \partial + A \) in a vector bundle of rank \( N \) over a Riemann curve \( \Sigma \). The cotangent vector (the antiHiggs field) is \( \text{sl}(N, \mathbb{C}) \) valued \( (0, 1) \)-form \( \Phi \). The symplectic form on \( \mathcal{H}_N(\Sigma) \) is \( -\int_\Sigma \text{tr}(d\Phi \wedge dA) \). The affinizations \( \mathcal{R}^{(\kappa)}/\mathcal{H}_N(\Sigma) \) are the space of connections.

---

\(^3\)We use the antiHiggs bundles instead of the standard Higgs bundles for reasons, that will become clear in Sect. 4.
\((\kappa\partial + \bar{A}, \partial + A)\) with the symplectic form \(\int_{\Sigma} \text{tr}(dA \wedge d\bar{A})\), where \(\kappa\) parameterizes the affinizations. The elements of the space \(\text{Conn}_{(\partial + A)}(\mathcal{L})\) giving rise to \(\mathcal{R}_{\text{SL}(N)}/H_\Sigma(\Sigma)\) are

\[
\nabla \Psi = \frac{\delta \Psi}{\delta \bar{A}} + \kappa \bar{A} \Psi .
\]

(2.27)

2.5 Hamiltonian algebroids related to Lie algebroids

Now we are ready to introduce an important subclass of Hamiltonian algebroids. They are extensions of the Lie algebroids and share with them SAJI (2.26).

**Lemma 2.1** The anchor action (2.10) of the Lie algebroid \(\mathcal{A}\) over \(M\) can be lifted to the Hamiltonian action on \(\mathcal{R}/T^*_M\) such that it defines the Hamiltonian algebroid \(\mathcal{A}^H\) over \(\mathcal{R}\). The equivalence classes of these lifts are isomorphic to \(H^1(\mathcal{A}, M)\).

**Proof.** Consider a small disk \(U_\alpha \subset M\). The anchor (2.17) has the form

\[
\delta_{e^j} = \langle b^j | \frac{\delta}{\delta x} \rangle + c(x; e^j).
\]

(2.28)

Next, continue the action on \(\mathcal{R}/T^*_M\). We represent the affinization as the space \(\text{Conn} \mathcal{L}(M) = \{\nabla^p = \frac{\delta}{\delta x} + p, \ x \in U_\alpha, \ p \in T^*_M\}\). Since \(\mathcal{L}\) on \(U_\alpha\) is trivialized we can identify the connections with one-forms \(p\). Let \(w \in TM\) and

\[
\nabla^p_w \Psi := i_w \nabla^p \Psi = \langle w | \frac{\delta \Psi}{\delta x} \rangle + \langle w | p \rangle \Psi
\]

be the covariant derivative along \(w\). To lift the action we use the Leibniz rule for the anchor action on the covariant derivatives:

\[
\hat{\delta}_{e^j} (\nabla^p)_\alpha \Psi = \hat{\delta}_{e^j} (\nabla^p \Psi) - \nabla^p_{\frac{\delta}{\delta x}} \hat{\delta}_{e^j} \Psi - \nabla^p_{\delta_{e^j} \alpha} \Psi.
\]

It follows from (2.12) and (2.28) that

\[
\hat{\delta}_{e^j} p = -\frac{\delta}{\delta x} \langle p | b^j(x) \rangle - \frac{\delta}{\delta x} c(x; e^j).
\]

(2.29)

Note that the second term is responsible for the pass from \(T^*M\) to the affinization \(\mathcal{R}\), otherwise \(p\) is transformed as a cotangent vector (see (2.10)).

The vector fields (2.28) are hamiltonian with respect to the canonical symplectic form \(\langle dp | dx \rangle\) on \(\mathcal{R}\). The corresponding Hamiltonians have the linear dependence on "momenta":

\[
h^j = \langle p | b^j(x) \rangle + c^j(x).
\]

(2.30)

Note that \(h^j\) satisfies the Hamiltonian algebroid property (2.23), since \(sc^j(x) = 0\) (2.15).

We have constructed the Hamiltonians locally and want to prove that this definition is compatible with gluing \(U_\alpha\) and \(U_\beta\). Note, that when we glue \(\mathcal{R}_{|U_\alpha}\) and \(\mathcal{R}_{|U_\beta}\) we shift fibers by \(\frac{\delta c_{\alpha\beta}}{\delta x} : p_\alpha = p_\beta + \frac{\delta c_{\alpha\beta}}{\delta x}\). Indeed, we glue the bundle \(\mathcal{L}(M)\) restricted on \(U_{\alpha\beta}\) by multiplication on \(\exp c_{\alpha\beta}(x)\). The connections are transformed by adding the logarithmic derivative of the transition functions. On the other hand, \(\delta_{e^j} c_{\alpha\beta}(x) = -c_\alpha(x; \varepsilon) + c_\beta(x; \varepsilon)\) (see (2.21)). So

\[
h^j_\alpha = \langle p_\alpha | b^j(x) \rangle + c^j_\alpha(x) = \langle p_\beta + \frac{\delta c_{\alpha\beta}}{\delta x} | b^j(x) \rangle - \delta_{e^j} c_{\alpha\beta}(x) + c^j_\beta(x; \varepsilon)
\]
= \langle p_\beta | b^j(x) \rangle + c^j_\beta(x) = h^j_\beta, \\
and the Hamiltonians become defined globally.

The exact cocycle $c^j_\alpha(x) = \delta_{\beta j} f_\alpha(x)$ shifts the momenta $h^j = \langle p_\alpha + \frac{\delta f_\alpha(x)}{\delta x} | b^j(x) \rangle$.

We rewrite the canonical transformations in the form
\[ \hat{\delta} e^j \Phi(p, x) = \delta e^j \Phi(p, x) + \langle f^j | \frac{\delta \Phi(p, x)}{\delta p} \rangle, \quad f^j = -\langle p | \frac{\delta b^j(x)}{\delta x} \rangle - \frac{\delta c^j(x)}{\delta x}. \]

Thus, all nonequivalent lifts of anchors from $M$ to $R/T^*M$ are in one-to-one correspondence with $\mathcal{H}^1(A, M)$. Thereby, we have constructed the Hamiltonian algebroid $A^H$ over the principle homogeneous space $R$. It has the same fibers and the same structure functions $f^{jk}_i(x)$ as the underlying Lie algebroid $A$ over $M$ and the bundle map $e^j \to h^j$ (2.30). $\Box$

Now investigate AJI (2.25) in this particular case.

**Lemma 2.2** The Hamiltonian algebroids $A^H$ have the SAJI (2.26).

**Proof.**

First note that the Lie algebroids we started with have the SAJI (2.6). The Hamiltonian algebroids $A^H$ have the same structure functions $f^{jk}_i(x)$ depending on coordinates on $M$ only. Consider the general AJI (2.25). It follows from (2.30) that
\[ \{ h^j, f^{nk}_i(x) \} = \langle b^j | \frac{\delta f^{nk}_i(x)}{\delta x} \rangle = \delta_{\beta j} f^{nk}_i(x). \]

The sum of the first two terms in (2.25) coincides with the SAJI (2.26) in the underlying Lie algebroid $A$, and therefore vanishes. $\Box$

### 2.6 Reduced phase space and its BRST description

In what follows we shall consider Hamiltonian algebroids related to Lie algebroids. Let $e^j$ be a basis of sections in $\Gamma(A^H)$. Then the Hamiltonians (2.30) can be represented in the form
\[ h^j = \langle e^j | F(x) \rangle, \quad \text{where } F(x) \in \Gamma((A^H)^*) \text{ defines the moment map} \]
\[ m : R \to \Gamma((A^H)^*), \quad m(x) = F(x). \]

The coadjoint action $\text{ad}^*_\varepsilon$ in $\Gamma((A^H)^*)$ is defined in the usual way
\[ \langle | \varepsilon, e^j | F(x) \rangle = \langle e^j | \text{ad}^*_\varepsilon F(x) \rangle. \]

One can fix a moment $F(x) = m_0$ in $\Gamma((A^H)^*)$. The reduced phase space is defined as the quotient
\[ R^{red} = \{ x \in R | (F(x) = m_0) / G_0 \}, \]
where $G_0$ is generated by the transformations $\text{ad}^*_\varepsilon$ such that $\text{ad}^*_\varepsilon m_0 = 0$. In other words, $R^{red}$ is the set of orbits of $G_0$ on the constraint surface $F(x) = m_0$. The symplectic form $\omega$ being restricted on $R^{red}$ is non-degenerate.

The BRST approach allows us to go around the reduction procedure by introducing additional fields (the ghosts). We shall construct the BRST complex for $A^H$ in a similar way as the Cartan-Eilenberg complex for the Lie algebroid $A$. The BRST complex is endowed with a Poisson structure and it allows us to define the nilpotent BRST operator.
Consider the dual bundle \((A^H)^*\). Its sections \(\eta \in \Gamma((A^H)^*)\) are the anti-commuting (odd) fields called the ghosts. Let \(h_{ij} = \langle \eta_i | F(x) \rangle\), where \(\{\eta_i\}\) is a basis in \(\Gamma((A^H)^*)\) and \(F(x) = 0\) are the moment constraints, generating the canonical algebroid action on \(\mathcal{R}\). Introduce another type of odd variables (the ghost momenta) \(\mathcal{P}^j, \ j = 1, 2, \ldots\) dual to the ghosts \(\eta_k, \ k = 1, 2, \ldots\) \(\mathcal{P} \in \Gamma(A^H)\). We attribute the ghost number one to the ghost fields \(\text{gh}(\eta) = 1\), minus one to the ghost momenta \(\text{gh}(\mathcal{P}) = -1\) and \(\text{gh}(x) = 0\) for \(x \in \mathcal{R}\). Introduce the Poisson brackets in addition to the non-degenerate Poisson structure on \(\mathcal{R}\)

\[
\{\eta_j, \mathcal{P}^k\} = \delta_j^k, \ \{\eta_j, x\} = \{\mathcal{P}_k, x\} = 0.
\]

Thus, all fields are incorporated in the graded Poisson superalgebra

\[
\mathcal{BFV} = (\Gamma(\wedge^*(A^H)^* \oplus A^H)) \otimes \mathcal{C}^\infty(\mathcal{R}) = \Gamma(\wedge^*(A^H)^*) \otimes \Gamma(\wedge^*A^H) \otimes \mathcal{C}^\infty(\mathcal{R}).
\]

(the Batalin-Fradkin-Volkovitsky (BFV) algebra).

There exists a nilpotent operator \(Q\) on the BFV algebra \(Q^2 = 0, \ \text{gh}(Q) = 1\) (the BRST operator) transforming \(\mathcal{BFV}\) into the BRST complex. The cohomology of the BRST complex give rise to the structure of the classical reduced phase space \(\mathcal{R}^{red}\). In some cases \(H^j(Q) = 0, \ j > 0\) and \(H^0(Q) = \) classical observables.

Represent the action of \(Q\) as the Poisson brackets:

\[
Q\psi = \{\psi, \Omega\}, \ \psi, \Omega \in \mathcal{BFV}.
\]

Due to the Jacobi identity for the Poisson brackets the nilpotency of \(Q\) is equivalent to \(\{\Omega, \Omega\} = 0\). Since \(\Omega\) is odd, the brackets are symmetric. For generic Hamiltonian algebroid \(\Omega\) can be represented as the expansion \([1]\)

\[
\Omega = h_\eta + \frac{1}{2} \langle [\eta, \eta']|\mathcal{P}\rangle + ..., \ (h_\eta = \langle \eta|F\rangle),
\]

where the higher order terms in \(\mathcal{P}\) are omitted. The highest order of \(\mathcal{P}\) in \(\Omega\) is called the rank of the BRST operator \(Q\). If \(A\) is a Lie algebra defined along with its canonical action on \(\mathcal{R}\) then \(Q\) has the rank one or less. In this case the BRST operator \(Q\) is the extension of the Cartan-Eilenberg operator giving rise to the cohomology of \(A\) with coefficients in \(\mathcal{C}^\infty(\mathcal{R})\). Due to the Jacobi identity the first two terms in the previous expression provide the nilpotency of \(Q\). It turns out that for the Hamiltonian algebroids \(A^H\) \(\Omega\) has the same structure as for the Lie algebras though the Jacobi identity has additional terms.

**Theorem 2.1** The BRST operator \(Q\) for the Hamiltonian algebroid \(A^H\) has the rank one:

\[
\Omega = \langle \eta|F\rangle + \frac{1}{2} \langle [\eta, \eta']|\mathcal{P}\rangle.
\]

**Proof.**

Straightforward calculations show that

\[
\{\Omega, \Omega\} = \{h_\eta_1, h_\eta_2\} + \frac{1}{2} \langle [\eta_2, \eta_2']|F\rangle - \frac{1}{2} \langle [\eta_1, \eta_1']|F\rangle
\]

\[
+ \frac{1}{2} \langle h_\eta_1, [\eta_2, \eta_2']|\mathcal{P}_2\rangle - \frac{1}{2} \langle h_\eta_2, [\eta_1, \eta_1']|\mathcal{P}_1\rangle + \frac{1}{4} \langle [\eta_1, \eta_1']|\mathcal{P}_1\rangle, [\eta_2, \eta_2']|\mathcal{P}_2\rangle\rangle.
\]

The sum of the first three terms vanishes due to (2.23). The sum of the rest terms is the left hand side of the SAJI (2.26). The additional dangerous term may come from the Poisson brackets of the structure functions \(\{[\eta_1, \eta_1'], [\eta_2, \eta_2']\}\). In fact, these brackets vanish because the structure functions do not depend on the ghost momenta. Thus, the SAJI leads to the desired identity \(\{\Omega, \Omega\} = 0\). □
3 Hamiltonian algebroids and Poisson sigma-model

3.1 Cotangent bundles to Poisson manifolds as Lie algebroids

Let $M$ be a Poisson manifold with Poisson bivector $\pi = \pi(\varepsilon, \varepsilon')$, where $\varepsilon, \varepsilon'$ are section of the bundle $T^*M$. It is a skewsymmetric tensor, with the vanishing Schouten brackets (the Jacobi identity) $[\pi, \pi]_S = 0$. It means in local coordinates $x = (x_1, \ldots, x_n)$

$$\partial_i \pi^{jk}(x) \pi^{im}(x) + \text{c.p.}(j, k, m) = 0.$$  \hfill (3.1)

The Poisson brackets are defined on the space of smooth functions $\mathcal{H}(M)$

$$\{f(x), g(x)\} := \langle df | \pi | dg \rangle, \quad df, dg \in T^*_x M.$$  \hfill (3.2)

The Poisson bivector gives rise to the map $V^\pi : T^*M \to TM$, $V^\pi_\varepsilon = \langle \varepsilon | \pi | \partial \rangle$. \hfill (3.3)

In this way we obtain a map from the space of smooth functionals $\mathcal{H}(M)$ to the space of the Hamiltonian vector fields $\Gamma^H(TM) = \{V^\pi_\varepsilon\}$

$$f \to V_f = \langle df | \pi | \partial \rangle.$$  \hfill (3.4)

The Poisson brackets can be rewritten as $\{f(x), g(x)\} = -i_{V^\pi g} dg$.

One can define the brackets on the one-forms $\varepsilon, \varepsilon' \in \Gamma(T^*M)$

$$[\varepsilon, \varepsilon'] = d\langle \varepsilon | \pi(x) | \varepsilon' \rangle + \langle d\varepsilon | \pi | \varepsilon' \rangle + \langle \varepsilon | \pi | d\varepsilon' \rangle.$$  \hfill (3.5)

**Lemma 3.1** $T^*M$ is a Lie algebroid $A$ over the Poisson manifold $M$ with the Lie brackets (3.5) and the anchor (3.2).

**Proof.**

It follows from the Jacobi identity (3.1), that the brackets (3.5) is the Lie brackets and the commutator of the vector fields satisfies (2.1)

$$[V_\varepsilon, V_{\varepsilon'}] = V_{[\varepsilon, \varepsilon']}.$$  \hfill (3.6)

The property (2.2) follows from the definition of Lie brackets (3.5). \hfill \Box

The structure functions of $T^*M$ is defined by the Poisson bivector

$$f^{jk}_i(x) = \partial_i \pi^{jk}(x).$$

This type of Lie algebroids was introduced in Ref. [21],[22].

**Remark 3.1** A linear space $M$ with the linear Poisson brackets $\pi^{jk}(x) = f^{jk}_i x^i$ can be identified with a Lie coalgebra $\mathfrak{g}^*$. Then the Hamiltonian vector field coming from the anchor $V^\pi$ is just the coadjoint action

$$V^\pi_\varepsilon \sim \text{ad}^*_\varepsilon.$$  \hfill (3.7)

Consider a punctured disk $D^* = |z| \leq 1$ and the space of the meromorphic maps $M = \{X : D^* \to M, \ X = X((z, z^{-1}))\}$. Define the Lie algebroid $\mathcal{A}_M$ over the space $M$ with the brackets (3.5) on the sections of $X^*(T^*M)$. For simplicity we do not change the notion of the anchor action

$$\delta_\varepsilon X = \pi(X) | \varepsilon \rangle.$$  \hfill (3.8)
3.2 Poisson sigma-model and Hamiltonian algebroids

The Poisson sigma-model is a way to construct a Hamiltonian algebroid related to the Lie algebroid $A_M$. The manifold $M$ serves as the target space for the Poisson sigma model. The space-time is the disk $D^*$. Consider the one-form $\xi$ on $D^*$ taking values in the pull-back by $X$ of the cotangent bundle $T^*M$, or the affinization over $T^*M$. Endow the space of fields $(X, \xi)$ with the canonical symplectic form

$$\omega = \frac{1}{2\pi} \oint \langle DX \wedge D\xi \rangle. \quad (3.9)$$

The canonical transformations of $\omega$ is represented by (3.8) and according with (2.29) by

$$\hat{\delta}_\varepsilon \xi = \delta \frac{\delta}{\delta X} c(X, \varepsilon) + \langle \varepsilon| \frac{\delta}{\delta X} \pi | \xi \rangle = -\bar{\partial} \varepsilon + \langle \varepsilon| \frac{\delta}{\delta X} \pi | \xi \rangle, \quad (3.10)$$

where $c(X, \varepsilon)$ is the anti-holomorphic one-cocycle from $H^1(A, \mathcal{M})$

$$c(X, \varepsilon) = -\frac{1}{2\pi} \oint \langle \varepsilon| \bar{\partial} X \rangle. \quad (3.11)$$

These transformations are generated by the first class constraints

$$F := \bar{\partial} X + \pi(X)|\xi \rangle = 0. \quad (3.12)$$

The action (3.10) amounts to the lift of the anchor action from $M$ to the affinization $\mathcal{R}/T^*M$ over $T^*M$ by means of the cocycle (3.11) in accordance with Lemma 2.1. The canonical transformations of a smooth functionals on $\mathcal{R}$ are the Hamiltonian transformation

$$\hat{\delta}_\varepsilon f(X, \xi) = \{ h_\varepsilon, f(X, \xi) \}. \quad (3.13)$$

Here the Poisson brackets are inverse to the symplectic form $\omega$ (3.9) and

$$h_\varepsilon = \oint \langle \varepsilon| F \rangle. \quad (3.14)$$

(see (2.30)). Again, due to the cocycle property, $\{ h_\varepsilon, h_{\varepsilon'} \} = h_{[\varepsilon, \varepsilon']}. $

Summarizing, we have defined the symplectic manifold $\mathcal{R}\{\xi, X\}$ and the Hamiltonian algebroid $A^H_{\mathcal{R}}$ over $\mathcal{R}$ with the sections $\varepsilon \in \Gamma(A^H_{\mathcal{R}})$ and the anchor (3.13),(3.14).

Following our approach we interpret the constraints (3.12) as the consistency conditions for a linear system. Let $\psi_1, \psi_2$ be sections of $X^*(T^*M)$ and $X^*(TM)$ correspondingly, and $B$ is a family of the linear maps $X^*(T^*M)$ to $X^*(TM)$

$$B(X) = \lambda + \pi(X), \quad B(X)\psi_1 = \psi_2. \quad (3.15)$$

Consider the linear system

$$B(X)\psi_1 = 0. \quad (3.16)$$

The second equation is

$$A^*\psi_1 = 0, \quad (3.17)$$

where $A$ is the linear map $A : X^*(T^*M) \rightarrow X^*(T^*M)$

$$A = -\bar{\partial} + \frac{\delta}{\delta X} \pi | \xi \rangle, \quad (3.18)$$

and $A^* : X^*(TM) \rightarrow X^*(TM)$

$$A^* = -\bar{\partial} - \frac{\delta}{\delta X} \pi | \xi \rangle. \quad (3.19)$$
Lemma 3.2 Let the Poisson bivector satisfies the non-degeneracy condition: the matrix $a^j_i = \left( \frac{\partial}{\partial X^i} \pi^j_m \right)$ is non-degenerate on $M$ for some $m$. Then the constraints (3.12) are the consistency conditions for (3.16) and (3.17).

Proof. The consistency condition of these equations is the operator equation

$$BA - A^*B = 0.$$ 

After substitution in it the expressions for $A, A^*, B$ and applying the Jacobi identity (3.1) one comes to the equality

$$(\bar{\partial}X^j + \pi^j_s \xi_s) \partial_i \pi^{jm}(\psi_1)_m = 0.$$ 

The later is equivalent to the constraint equation (3.12) if $\pi$ is non-degenerate in the above sense. $\Box$

3.3 BRST construction

Consider a smooth functional $\Psi(X)$ on $M$ annihilating by the anchor action

$$\hat{\delta}_\varepsilon \Psi(X) := \frac{1}{2\pi} \oint \langle \varepsilon|\pi(X)d\Psi(X) \rangle + \left( \frac{1}{2\pi} \oint \langle \varepsilon|\bar{\partial}X \rangle \right) \Psi(X) = 0. \quad (3.19)$$

Let $G$ be the Lie groupoid corresponding to the Lie algebroid $A_M$. Consider the space of orbits $\tilde{M} = M/G$ and the line bundle $L(\tilde{M})$ over $\tilde{M}$ with the space of sections $\Gamma(L) = \{\Psi(X)\}$ (3.19). The Hilbert space $L^2(\Gamma(L))$ arises in the prequantization of the symplectic quotient $\mathcal{R}^{red} = \mathcal{R}/G^H = \{F = 0\}/G^H$, where $G^H$ is the Hamiltonian groupoid.

The quantization of $\mathcal{R}^{red}$ can be performed by the BRST technique. The classical BRST complex is the set of fields

$$\bigwedge^\bullet \left( \Gamma(X^*(TM)) \oplus \Gamma(X^*(T^*M)) \right) \otimes C^\infty(\mathcal{R}),$$

where the first component is the space of sections of the anticommuting variables $\eta$ dual to the gauge generators $\varepsilon$, the second component is the space of their momenta $\mathcal{P}$. Theorem 2.1 states that the BRST operator has the rank one

$$\Omega = \frac{1}{2\pi} \oint \langle \eta|F \rangle + \frac{1}{\pi} \oint \langle [\eta, \eta]|\mathcal{P} \rangle.$$

It means that the nonlinear deformation of the Poisson bivector on $M$ does not affect the Lie algebraic form of $\Omega$. This form of $\Omega$ for the Poisson sigma model was established in [8].

4 Two examples of Hamiltonian algebroids with Lie algebra symmetries

In this section we consider two examples, where the spaces of sections of the Hamiltonian algebroids are just Lie algebras of hamiltonian vector fields and therefore the symmetries are the standard Lie symmetries. Nevertheless, they are in much the same as in the algebroid cases. In both examples we cast the known constructions in the form suitable to our approach.

Let $\Sigma_{g,n}$ be a Riemann curve of genus $g$ with $n$ marked points. The first examples is the moduli space of flat bundles over $\Sigma_{g,n}$. It will become clear later, that it is an universal system containing hidden algebroid symmetries. The second example is the projective structures ($W_2$-structures) on $\Sigma_{g,n}$. Their generalization is the $W_N$-structures, where the symmetries are defined by a nontrivial Hamiltonian algebroid, will be considered in next Sections.
4.1 Flat bundles with the regular singularities

Consider a $\text{SL}(N, \mathbb{C})$ holomorphic bundle $E$ over $\Sigma_{g,n}$. Let $D^*$ be a small punctured disk embedded to $\Sigma_{g,n}$ with a local coordinate $z$. Locally the derivatives $d' : E \to E \otimes \Omega^{1,0}(\Sigma_{g,n})$, $d'' : E \to E \otimes \Omega^{0,1}(\Sigma_{g,n})$ take the form

$$d' = \partial + A, \quad d'' = \tilde{\partial}. \quad (4.1)$$

Let $M_{\text{SL}_N}(D^*) = \{d' = k\partial + A\}$ be the set of derivatives restricted to $D^*$. This set has the structure of the affine Lie coalgebra $\hat{L}^*(\text{sl}(N, \mathbb{C}))$ with the Lie-Poisson brackets on the space of smooth functionals

$$\{f(A), g(A)\} = \oint \text{tr}([df(A), dg(A)]A) + \oint \text{tr}(df\partial(dg)), \quad (4.2)$$

where $df(A) \in L(\text{sl}(N, \mathbb{C}))$ is the variation of $f(A)$. Thereby $M_{\text{SL}_N}(D^*)$ can be considered as the base of the Lie algebroid $\mathcal{A}_{\text{SL}_N}(D^*)$ (see Remark 3.1). The space of sections of the algebroid is the Lie algebra $\mathcal{G}_{\text{SL}_N}(D^*) = L(\text{sl}(N, \mathbb{C}))$ of the gauge transformations

$$\delta_{\varepsilon}A = \partial\varepsilon + [A, \varepsilon]. \quad (4.3)$$

Though the Poisson structure is defined only on $D^*$ the algebroid can be defined over $\Sigma_{g,n}$, since the gauge algebra and the anchor action (4.2) are well defined globally. We denote by $\mathcal{G}_{\text{SL}_N}$ the algebra of the smooth global gauge transformations. To come to the global description we assume that $A$ has first order holomorphic poles at the marked points

$$A|_{z \to x_a} = \frac{A_a}{z - x_a}. \quad (4.4)$$

In addition, we consider a collection $P$ of $n$ elements from the Lie coalgebra

$$P = \{p = (p_1, \ldots, p_a, \ldots, p_n), \quad p_a \in \text{sl}^*(N, \mathbb{C})\}.$$ 

endowed with the Lie-Poisson structure $\{f(p_a), g(p_b)\} = \delta_{ab}\text{tr}([df, dg]p_a)$. We assume that the gauge transformations at the marked points are nontrivial

$$\varepsilon|_{z \to x_a} = r_a + O(z - x_a), \quad r_a \neq 0. \quad (4.5)$$

The gauge algebra $\mathcal{G}_{\text{SL}_N}$ acts on $P$ by the evaluation maps

$$\delta_{\varepsilon}p_a = [p_a, r_a], \quad \varepsilon \in \mathcal{G}_{\text{SL}_N}. \quad (4.6)$$

In this way we define a trivial Lie algebroid $\mathcal{A}_{\text{SL}_N} = \mathcal{G}_{\text{SL}_N} \times M_{\text{SL}_N}$ over $M_{\text{SL}_N} = \{d', P\}$ with the anchor map (4.2), (4.5).

The cohomology $H^i(\mathcal{A}_{\text{SL}_N}) = H^i(\mathcal{G}_{\text{SL}_N}, M)$ are the standard cohomology of the gauge algebra $\mathcal{G}_{\text{SL}_N}$ with the cochains taking values in functionals on $M$. There is a nontrivial one-cocycle

$$c(A, p; \varepsilon) = \int_{\Sigma_{g,n}} \text{tr}\varepsilon \left( \partial A - 2\pi i \sum_{a=1}^n \delta(x_a)p_a \right) = \langle \varepsilon|\partial A \rangle - 2\pi i \sum_{a=1}^n \text{tr}(r_a \cdot p_a) \quad (4.7)$$

representing an element of $H^1(\mathcal{A}_{\text{SL}_N})$. This cocycle provides a nontrivial extension of the anchor action (see (2.17))

$$\delta_{\varepsilon}f(A, p) = \langle \varepsilon|\partial A - \partial(df(A)) + [df(A), A] \rangle - 2\pi i \sum_{a=1}^n \text{tr}(r_a p_a). \quad (4.8)$$

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Next consider $2g$ contours $\gamma_\alpha$, ($\alpha = 1, \ldots, 2g$) generating $\pi_1(\Sigma_g)$. The contours determine the 2-cocycles
\[
c_\alpha(\varepsilon_1, \varepsilon_2) = \int_{\gamma_\alpha} \text{tr}(\varepsilon_1 \partial \varepsilon_2).
\] (4.8)
The cocycles (4.8) lead to $2g$ central extensions $\mathcal{G}_{SL_N}$ of $\mathcal{G}_{SL_N}$
\[
\mathcal{G}_{SL_N} = \mathcal{G}_{SL_N} \oplus_{\alpha=1}^{2g} \mathbb{C} \Lambda_\alpha,
\]
\[
[(\varepsilon_1, \sum_\alpha k_{1,\alpha}), (\varepsilon_2, \sum_\alpha k_{2,\alpha})]_{c.e.} = \left( [\varepsilon_1, \varepsilon_2], \sum_\alpha c_\alpha(\varepsilon_1, \varepsilon_2) \right).
\]

To define the corresponding Poisson sigma-model we consider the cotangent bundle $T^*E$. The conjugate to $\partial + A$ variables are the one-forms $\bar{\Phi} \in \Omega^{(0,1)}(\Sigma_{g,n}, \mathfrak{sl}(N, \mathbb{C}))$ – the anti-Higgs field. In fact, we shall consider the affinization $\mathcal{R}^0_{SL_N} = \text{Aff}(T^*E)$ over $T^*E$ provided by the cocycle (4.6). We have already mentioned that the role of momenta plays by the holomorphic $\omega$ bundles $T_\ast$.$\mathcal{O}_N$ form is $\omega_a$ leads to the Hamiltonian $\Omega_a = \{ p_a = g_a^{-1} p_a^{(0)} g_a \mid p_a^{(0)} = \text{diag}(\lambda_{a,1}, \ldots, \lambda_{a,N}), \lambda_{a,j} \neq \lambda_{a,k}, g_a \in \mathfrak{sl}(N, \mathbb{C}) \}$.

The orbits are the symplectic quotient $\mathcal{O}_a \sim \mathfrak{sl}(N, \mathbb{C}) \setminus T^*G_a$ with respect to the action $g_a \to f_a g_a$, $f_a \in \mathfrak{sl}(N, \mathbb{C})$. The form $\omega_a$ coincides on $\mathcal{O}_a$ with the Kirillov-Kostant form $\omega_a = \text{Dtr}(p_a^{(0)} Dg_a g_a^{-1})$. The orbits $\mathcal{O}_a$ are affinizations $\text{Aff}(T^*\mathcal{Fl}_a(N))$ over the cotangent bundles $T^*\mathcal{Fl}_a(N)$ to the flag varieties $\mathcal{Fl}_a(N)$.

Eventually we come to the symplectic manifold
\[
\mathcal{R}_{SL_N} = (\mathcal{R}^0_{SL_N}; \mathcal{O}_1, \ldots, \mathcal{O}_n) \sim \text{Aff}(T^*E); \text{Aff}(T^*\mathcal{Fl}_1), \ldots, \text{Aff}(T^*\mathcal{Fl}_n),
\]
\[
\omega = \omega^0 + \sum_{a=1}^n \omega_a = \langle DA \wedge D\bar{A} \rangle + \sum_{a=1}^n \text{Dtr}(p_a^{(0)} \wedge Dg_a g_a^{-1}).
\] (4.10)

According to (2.30) the lift of the anchor (4.2) to $\mathcal{R}_N$, defined by the cocycle $c(A; \varepsilon)$ (4.6) leads to the Hamiltonian
\[
h_\varepsilon = \langle \varepsilon | F(A, \bar{A}) - 2\pi i \sum_{a=1}^n \delta(x_a) p_a \rangle, \quad F(A, \bar{A}) = \bar{\partial}A - \partial \bar{A} + [\bar{A}, A],
\]
The Hamiltonian generates the canonical vector fields (4.2) (4.5) and
\[ \hat{\delta}_\varepsilon \vec{A} = \partial \varepsilon + [\vec{A}, \varepsilon], \quad \hat{\delta}_\varepsilon g_a = g_a r_a, \]
(see (2.29)). The global version of this transformations is the gauge group \( G_{SL_N} \) acting on \( \mathcal{R}_{SL_N} \). The flatness condition
\[ m := F(A, \vec{A}) - 2\pi i \sum_{a=1}^n \delta(x_a)p_a = 0 \quad (4.11) \]
is the moment constraint with respect to this action. This equation means that the residues \( A_a \) of \( A \) in the marked points (4.3) coincide with \( p_a \). The flatness is the compatibility condition for the linear system
\[ \begin{cases} (\partial + A)\psi = 0, \\ (\bar{\partial} + \vec{A})\psi = 0, \end{cases} \quad (4.12) \]
where \( \psi \in \Omega^0(\Sigma_{g,n}, \text{End}E) \). The second equation describes the deformation of the holomorphic structure of the bundle \( E \) (4.1).

The moduli space \( \mathcal{M}^\text{flat}_N \) of flat \( SL(N, \mathbb{C}) \)-bundles is the symplectic quotient \( \mathcal{R}_{SL_N} / G_{SL_N} \). It has dimension
\[ \dim \mathcal{M}^\text{flat}_N = 2(N^2 - 1)(g - 1) + N(N - 1)n, \quad (4.13) \]
where the last term is the contribution of the coadjoint orbits \( O_a \). Let \( \tilde{M}_{SL_N} = M_{SL_N} / G_{SL_N} \) be the set of the gauge orbits acting on the base space \( M_{SL_N} = \{ d', P \} \). Consider smooth functionals \( \Psi(A, p) \) annihilated by the anchor action \( \hat{\delta}_\varepsilon \Psi(A, p) = 0 \). These functionals generate the space of sections of the linear bundle \( L(\tilde{\mathcal{M}}_{SL_N}) \) we discussed before (2.19). It is the determinant bundle \( \text{det}(\partial + A) \) [24, 25]. The prequantization of \( \mathcal{M}^\text{flat}_N \) is defined in the Hilbert space \( L^2(\Gamma(L(\tilde{\mathcal{M}}_{SL_N}))) \).

On the other hand \( \mathcal{M}^\text{flat}_N \) can be described by the cohomology \( H^k(Q) \) of the BRST operator \( Q \) which we are going to define. Let \( \eta \) be the dual to \( \varepsilon \) fields (the ghosts) and \( P \) are their momenta \( P \in \Omega^{1,1}(\Sigma_{g,n}, \text{End}E) \). Consider the algebra
\[ C^\infty(\mathcal{R}_N) \otimes \wedge^* (\mathcal{G}_{SL_N} \oplus \mathcal{G}^*_{SL_N}). \]
Then the BRST operator \( Q \) acts on functionals on this algebra as
\[ Q \Psi(A, \vec{A}, \eta, P) = \{ \Omega, \Psi(A, \vec{A}, \eta, P) \}, \]
where
\[ \Omega = \langle \eta \vert F(A, \vec{A}) \rangle + \frac{1}{2} \langle [\eta, \eta'] \vert P \rangle, \]
where \( \text{res}A \vert_{x_a} = p_a \).

### 4.2 Projective structures on \( \Sigma_{g,n} \)

Consider the space \( M_2 \) of projective connections on \( \Sigma_{g,n} \). Locally on a punctured disk \( D^* \) the space \( M_2 \) is represented by the set \( M_2(D^*) \) of the second order differential operators \( \partial^2 - T \). The later is the Poisson manifold with the linear brackets
\[ \{ T(z), T(w) \} = \left( \frac{1}{2} \partial^3 + 2T\partial + \partial T \right) \delta(z - w), \quad (4.14) \]
where \( \delta(z - w) = \sum_{k \in \mathbb{Z}} z^k w^{-k-1} \) is the delta function on the contour \( |z| = 1 \).
The dual space to $M_2(D^*)$ is the Virasoro algebra $Vir = \mathcal{G}_1(D^*) = \{\varepsilon, c\}$ ($\varepsilon = \varepsilon(z, \bar{z}) \frac{\partial}{\partial z}$). The commutation relations can be read off from the Poisson brackets (see (3.5))

$$[\varepsilon_1, \varepsilon_2] = (\varepsilon_1 \partial \varepsilon_2 - \varepsilon_2 \partial \varepsilon_1, \frac{1}{2} \int \varepsilon_1 \partial^2 \varepsilon_2).$$ (4.15)

The coadjoint action of $Vir$ on $M_2(D^*)$

$$\delta \varepsilon T(z, \bar{z}) = -\varepsilon \partial T - 2T \partial \varepsilon - \frac{1}{2} \partial^2 \varepsilon.$$ (4.16)

defines the anchor in the trivial algebroid $A_2(D^*) = \mathcal{G}_1(D^*) \oplus M_2(D^*)$.

The algebroid (4.15), (4.16) can be defined globally over the space $M_2$. The section of the algebroid are the chiral vector fields $\varepsilon \sim \varepsilon(z, \bar{z}) \frac{\partial}{\partial z}$ on $\Sigma_{g,n}$. Its central extension is defined by the contour in $D^*$ (4.15). We call this algebra $\mathcal{G}_1$. We include in the definition the contribution of the marked points. Assume that the projective connections $T$ have poles at the marked points $x_a, (a = 1, \ldots, n)$ up to the second order:

$$T|_{z \to x_a} \sim \frac{T_{-2}^a}{(z - x_a)^2} + \frac{T_{-1}^a}{(z - x_a)} + \cdots,$$ (4.17)

and the vector fields have the first order holomorphic nulls at the marked points

$$\varepsilon|_{z \to x_a} = r_a(z - x_a) + o(z - x_a), \quad r_a \neq 0.$$ (4.18)

We denote this trivial algebroid bundle $A_2$.

Consider the cohomology $H^*(A_2) \sim H^*(\mathcal{G}_1, M_2)$. Due to (4.16) and (4.18) $\delta \varepsilon T_{-2}^a = 0$ and thereby $T_{-2}^a$ in (4.17) represents an element from $H^0(A_2)$.

The anchor action (4.16) can be extended by the one-cocycle $c(T; \varepsilon)$ representing a nontrivial element $c(\varepsilon; T)$ of $H^1(A_2)$

$$c(T; \varepsilon) = \int_{\Sigma_{g,n}} \varepsilon \bar{\partial} T = \langle \varepsilon \bar{\partial} T \rangle,$$ (4.19)

$$\hat{\delta} \varepsilon f(T) = \langle \delta \varepsilon T | df(T) \rangle + c(T; \varepsilon).$$ (4.20)

The contribution of the marked point in (4.19) is $2\pi i r_a T_{-2}^a$.

There exist $2g$ nontrivial two-cocycles defined by the integrals over non contractible contours $\gamma_a$:

$$c_a(\varepsilon_1, \varepsilon_2) = \int_{\gamma_a} \varepsilon_1 \partial^2 \varepsilon_2.$$ (4.21)

The cocycles give rise to the central extension $\hat{\mathcal{G}}_1$ of the Lie algebra of the first order differential operators on $\Sigma_g$.

The affinization $\mathcal{R}_2$ over the cotangent bundle $T^*M_2$ has the Darboux coordinates $T$ and $\mu$, where $\mu \in \Omega^{(-1,1)}(\Sigma_{g,n})$ is the Beltrami differential. The anchor (4.16) is lifted to $\mathcal{R}_2$ as

$$\delta \varepsilon \mu = -\varepsilon \partial \mu + \mu \partial \varepsilon + \bar{\partial} \varepsilon,$$ (4.22)

where the last term occurs due to the cocycle (4.19). We specify the dependence of $\mu$ on the positions of the marked points in the following way. Let $U_a$ be neighborhoods of the marked points $x_a, (a = 1, \ldots, n)$ such that $U_a \cap U_b = \emptyset$ for $a \neq b$. Define a smooth function $\chi_a(z, \bar{z})$

$$\chi_a(z, \bar{z}) = \begin{cases} 1, & z \in U_a, \ U_a' \supset U_a \\ 0, & z \in \Sigma_g \setminus U_a'. \end{cases}$$ (4.23)
Due to (4.21) at the neighborhoods of the marked points $\mu$ is defined up to the term $\bar{\partial}(z - x_a)\chi(z, \bar{\zeta})$. Then $\mu$ can be represented as

$$
\mu = \sum_{a=1}^{n} [t_{0,a}^{(1)} + t_{1,a}^{(1)}(z - x_a) + \ldots ] \mu_0, \quad \mu_0^a = \bar{\partial}\chi_a(z, \bar{\zeta}), \quad (t_{0,a} = x_a - x_0^a),
$$

(4.23)

where only $t_{0,a}$ cannot be removed by the gauge transformations (4.18), (4.21). The symplectic form on $\mathcal{R}_2$ is

$$
\omega = \int_{\Sigma_{g,n}} dT \wedge d\mu = \langle dT \wedge d\mu \rangle.
$$

For rational curves $\Sigma_{0,n}$ it takes the form

$$
\omega =dT_{-2}^a \wedge dt_{1,a} + dT_{-1}^a \wedge dt_{0,a}.
$$

(4.24)

**Remark 4.1** The space $\mathcal{R}_2$ is the classical phase space of the $2+1$-gravity on $\Sigma_{g,n} \times I$ [16]. In fact, $\mu$ is related to the conformal class of metrics on $\Sigma_{g,n}$ and plays the role of a coordinate, while $T$ is a momentum. In our construction $\mu$ and $T$ interchange their roles.

The Hamiltonian of the canonical transformations (4.16), (4.21) has the form

$$
h_\varepsilon = \langle \varepsilon | F(T, \mu) \rangle,
$$

(4.25)

$$
F(T, \mu) = (\bar{\partial} + \mu \partial + 2\partial \mu)T - \frac{1}{2} \partial^3 \mu.
$$

The Hamiltonian defines the moment map $m : \mathcal{R}_2 \to \mathcal{G}_1^*$

$$
m = (\bar{\partial} + \mu \partial + 2\partial \mu)T - \frac{1}{2} \partial^3 \mu,
$$

(4.26)

where $\mathcal{G}_1^*$ is the dual to $\mathcal{G}_1$ space of distributions of $(2,1)$-forms on $\Sigma_{g,n}$. As it follows from (4.18) in the neighborhoods of the marked points the elements $y \in \mathcal{G}_1^*$ takes the form

$$
y \sim b_{1,a} \partial \delta(x_a) + b_{2,a} \partial^2 \delta(x_a) + \ldots.
$$

(4.27)

We put $m$ equal

$$
m = -\sum_{a=1}^{n} T_{-2}^a \partial \delta(x_a), \quad (m = F(T, \mu)).
$$

(4.28)

The algebra $\mathcal{G}_1$ preserves $m : \text{ad}_\varepsilon^* m = m$ for any $\varepsilon$. Thus, in contrast with the previous example, we have trivial coadjoint orbits at the marked points. Since $T_{-2}^a$ are fixed, the dynamical parameters are $(t_{0,a}, T_{-1})$ that contribute in the symplectic structure (4.24). Let $\psi$ be a $(-\frac{1}{2}, 0)$ differential. Then (4.28) is the compatibility condition for the linear system

$$
\begin{cases}
(\partial^2 - T)\psi = 0, \\
(\bar{\partial} + \mu \partial - \frac{1}{2} \partial^3 - T)\psi = 0.
\end{cases}
$$

(4.29)

It follows from the second equation that the Beltrami differential $\mu$ provides the deformation of complex structure on $\Sigma_{g,n}$. Note, that we started from the first equation defining the projective connection and $\bar{\partial}\psi = 0$ on $M_2$. The last equation is modified after the passage from $M_2$ to $\mathcal{R}_2$.

The moduli space $\mathcal{W}_2$ of projective structure on $\Sigma_{g,n}$ is the symplectic quotient of $\mathcal{R}_2$ with respect to the action of $G_1$, where $G_1$ is the group corresponding to the algebra $\mathcal{G}_1$

$$
\mathcal{W}_2 = \mathcal{R}_2/G_1 = \{ F(T, \mu) - m = 0 \}/G_1.
$$
It has dimension $6(g-1)+2n$. To quantize $W_2$ one can consider the quotient space $\tilde{M}_2 = M_2 / G_1$. The space of sections of the linear bundle $\mathcal{L}(M_2)$ is defined as the space of functionals $\{\Psi(T)\}$ on $M_2$ that satisfy the invariance condition $\delta_2 \Psi(T) = 0$. The linear bundle $\mathcal{L}(\tilde{M}_2)$ is the determinant line bundle $\text{det}(\partial^2 - T)$ considered in [26, 27].

The tangent space $T_m \mathcal{W}_2$ to $\mathcal{W}_2$ is isomorphic to the cohomology $H^0$ of the BRST complex. It is generated by the fields $T, \mu \in R_2$, the ghosts fields $\eta$ dual to the vector fields $\varepsilon$ acting via the anchor (4.16),(4.21) on $R_2$ and the ghosts momenta $\mathcal{P}$. The BRST operator $Q$ is defined by $\Omega$

$$\Omega = \int_{\Sigma_{g,n}} \eta F(T, \mu) + \frac{1}{2} \int_{\Sigma_{g,n}} [\eta, \eta'] \mathcal{P}.$$

The first term is just the Hamiltonian (4.25), where the vector fields are replaced by the ghosts.

5 Hamiltonian algebroid structure in $\mathcal{W}_3$-gravity

Now consider the concrete example of the general construction with nontrivial algebroid structure. It is the $\mathcal{W}_N$-structures on $\Sigma_{g,n}$ [13, 14, 15] which generalize the projective structures described in previous Section. In this Section we consider in details the $\mathcal{W}_3$-structures.

5.1 $SL(N, \mathbb{C})$-opers

Opers are $G$-bundles over Riemann curves with additional structures [9, 10]. Let $E_N$ be a $SL(N, \mathbb{C})$-bundle over $\Sigma_{g,n}$. It is a $SL(N, \mathbb{C})$-oper if there exists a flag filtration $E_N \supset \ldots \supset E_1 \supset E_0 = 0$ and a covariant derivative, that acts as $\nabla E_j \subset E_j \otimes \Omega^{1,0}(\Sigma_{g,n})$. Moreover, $\nabla$ induces an isomorphism $E_j / E_{j-1} \rightarrow E_{j+1} / E_j \otimes \Omega^{1,0}(\Sigma_{g,n})$. It means that locally

$$\nabla = \partial - \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ W_N & W_{N-1} & \ldots & W_2 & 0 \end{pmatrix}.$$ 

(5.1)

In other word we define the $N$-order differential operator on $\Sigma_{g,n}$

$$L_N = \partial^N - W_2 \partial^{N-2} \ldots - W_N : \Omega^{(\frac{N}{2}-\frac{1}{2},0)}(\Sigma_{g,n}) \rightarrow \Omega^{(\frac{N+1}{2},0)}(\Sigma_{g,n})$$

(5.2)

with vanishing subprincipal symbol. The $GL(N, \mathbb{C})$-opers come from the $GL(N, \mathbb{C})$-bundles and have the additional term $-W_1 \partial^{N-1}$ in (5.2). We assume that in neighborhoods of the marked points the coefficients $W_j$ behave as

$$W_j|_{z \rightarrow x_a} \sim W_{a-j}(j)(z - x_a)^j + W_{a-j}(j-1)(z - x_a)^{j-1} + \ldots.$$ 

(5.3)

In this section we consider $SL(3, \mathbb{C})$-opers and postpone the general case to next Section. It is possible to choose $E_1 = \Omega^{-1,0}(\Sigma_{g,n})$. Instead of (5.1) we have

$$\nabla = \partial - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ W & T & 0 \end{pmatrix},$$ 

(5.4)
and the third order differential operator

\[ L_3 = \partial^3 - T \partial - W : \Omega^{(-1,0)}(\Sigma_{g,n}) \to \Omega^{(2,0)}(\Sigma_{g,n}). \]  

(5.5)

According with (5.3)

\[ T|_{z=x_a} \sim \frac{T^a_{-2}}{(z-x_a)^2} + \frac{T^a_{-1}}{(z-x_a)} + \ldots. \]  

(5.6)

\[ W|_{z=x_a} \sim \frac{W^a_{-3}}{(z-x_a)^3} + \frac{W^a_{-2}}{(z-x_a)^2} + \frac{W^a_{-1}}{(z-x_a)} + \ldots. \]  

(5.7)

5.2 Local Lie algebroid over SL(3, C)-opers

Consider the set \( M^3 = \{L_3\} \) of SL(3, C)-opers on a punctured disk \( D^* \). This set is a Poisson manifold with respect to the AGD brackets

\[ \{T(z), T(w)\} = (-2\kappa \delta^3 + 2T(z)\partial + \partial T(z) ) \delta(z - w), \]  

(5.8)

\[ \{T(z), W(w)\} = (\partial^4 - T(z) \partial^2 + 3W(z) \partial - \partial W(z) ) \delta(z - w), \]  

(5.9)

\[ \{W(z), W(w)\} = \]  

(5.10)

\[ + \left( \frac{2}{3} \partial^5 - \frac{4}{3} T(z) \partial^3 - 2\partial T(z) \partial^2 + \left( \frac{2}{3} T(z)^2 - 2\partial^2 T(z) + 2\partial W(z) \right) \right) \delta(z - w). \]  

It means that \( M_3(D^*) \) is a base of the local Lie algebroid \( A_3(D^*) \sim T^* M_3(D^*) \). To define the space of its sections we consider the dual space \( M^*_3(D^*) \) of the space of second order differential operators on \( D^* \) with a central extensions

\[ M^*_3(D^*) = \{(\varepsilon^{(1)}) \frac{d}{dz} + \varepsilon^{(2)} \frac{d^2}{dz^2}, c\}. \]  

It comes from the pairing

\[ \oint_{|z|=1} (\varepsilon^{(1)} T + \varepsilon^{(2)} W) \]  

(5.11)

and the central element \( c \) is dual to the highest order coefficient in(5.6) which we put equal to 1. The Lie brackets on \( T^* M_3(D^*) \) are determined by means of the AGD Poisson structure following (3.5)

\[ [\varepsilon^{(1)}, \varepsilon^{(1)}] = \left( \varepsilon^{(1)} \partial \varepsilon^{(1)} - \varepsilon^{(2)} \partial \varepsilon^{(1)} \right) \frac{d}{dz}, -2 \oint \varepsilon^{(1)} \partial^3 \varepsilon^{(1)} \right) \].  

(5.12)

\[ [\varepsilon^{(1)}, \varepsilon^{(2)}] = \left( -\varepsilon^{(2)} \partial^2 \varepsilon^{(1)} \right) \frac{d}{dz} + \left( -2\varepsilon^{(2)} \partial \varepsilon^{(1)} + \varepsilon^{(1)} \partial \varepsilon^{(2)} \right) \frac{d^2}{dz^2}, \oint \varepsilon^{(1)} \partial^4 \varepsilon^{(2)} \right) \].  

(5.13)

\[ [\varepsilon^{(2)}, \varepsilon^{(2)}] = \left( \frac{2}{3} \partial (\partial^2 - T) \varepsilon^{(2)} \right) \frac{d}{dz} + \left( -\varepsilon^{(2)} \partial^2 (\partial^2 - T) \varepsilon^{(2)} \right) \frac{d^2}{dz^2} + \left( \frac{2}{3} \oint \varepsilon^{(2)} \partial^5 \varepsilon^{(2)} \right) \].  

(5.14)
Note, that the brackets (5.12) are the Virasoro brackets and the whole set of the commutation relations is their generalization on the second order differential operators.

According with (3.3) the anchor action in $A_3(D^*)$ has the form

$$
\delta_{\varepsilon(1)} T = -2\partial^3 \varepsilon^{(1)} + 2T \partial \varepsilon^{(1)} + \partial T \varepsilon^{(1)},
$$

(5.15)

$$
\delta_{\varepsilon(1)} W = -\partial^4 \varepsilon^{(1)} + 3W \partial \varepsilon^{(1)} + \partial W \varepsilon^{(1)} + T \partial^2 \varepsilon^{(1)},
$$

(5.16)

$$
\delta_{\varepsilon(2)} T = \partial^4 \varepsilon^{(2)} - T \partial^2 \varepsilon^{(2)} + (3W - 2\partial T) \partial \varepsilon^{(2)} + (2\partial W - \partial^2 T) \varepsilon^{(2)},
$$

(5.17)

$$
\delta_{\varepsilon(2)} W = \frac{2}{3} \partial^5 \varepsilon^{(2)} - \frac{4}{3} T \partial^3 \varepsilon^{(2)} - 2\partial T \partial^2 \varepsilon^{(2)} + \left(\frac{2}{3} T^2 - 2\partial^2 T + 2\partial W) \partial \varepsilon^{(2)} + (\partial^2 W - \frac{2}{3} \partial^3 T + \frac{2}{3} T \partial T) \varepsilon^{(2)}\right).
$$

(5.18)

Thereby, we come to the Lie algebroid $A_3(D^*)$ over $M_3(D^*)$. This algebroid is nontrivial since the structure functions in (5.14) depend on the projective connection $\nabla$.

The SAJI (2.6) in $A_3(D^*)$ takes the form

$$
[\varepsilon_1^{(2)}, \varepsilon_2^{(2)}, \varepsilon_3^{(2)}]^{(1)} = (\varepsilon_1^{(2)} \varepsilon_2^{(2)} - \varepsilon_3^{(2)} \varepsilon_1^{(2)}) \delta_{\varepsilon_3^{(2)}} T + \text{c.p.}(1, 2, 3) = 0,
$$

(5.19)

$$
[[\varepsilon_1^{(2)}, \varepsilon_2^{(2)}, \varepsilon_3^{(2)}]]^{(1)} = (\varepsilon_1^{(2)} \varepsilon_2^{(2)} - \varepsilon_3^{(2)} \varepsilon_1^{(2)}) \delta_{\varepsilon_3^{(2)}} T = 0.
$$

(5.20)

The brackets here correspond to the product of structure functions in the left hand side of (2.6) and the superscript (1) corresponds to the $D^1$ component. For the rest brackets the Jacobi identity is the standard one.

The origin of the brackets and the anchor representations follow from the matrix description of $\text{SL}(3, \mathbb{C})$-opers (5.4). Consider the set $G_3(D^*)$ of automorphisms of the bundle $E_3$ over $D^*$

$$
A \rightarrow f^{-1} \partial f - f^{-1} A f,
$$

(5.21)

that preserve the $\text{SL}(3, \mathbb{C})$-oper structure

$$
f^{-1} \partial f - f^{-1} \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
W & T & 0
\end{pmatrix} f = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
W' & T' & 0
\end{pmatrix}.
$$

(5.22)

It is clear that $G_3(D^*)$ is the Lie groupoid over $M_3(D^*) = \{W, T\}$ with $l(f) = (W, T)$, $r(f) = (W', T')$, $f \rightarrow << W, T | f | W', T' >>$. The left identity map is the $\text{SL}(3, \mathbb{C})$ subgroup of $G_3$

$$
P \exp(- \int_{x_0}^z A(W, T)) \cdot C \cdot P \exp(\int_{x_0}^z A(W, T))
$$

where $C$ is an arbitrary matrix from $\text{SL}(3, \mathbb{C})$ and $A(W, T)$ has the oper structure (5.4). The right identity map has the same form with $(W, T)$ replaced by $(W', T')$.

The local version of (5.22) takes the form

$$
\partial X - \left[\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
W & T & 0
\end{pmatrix}, X\right] = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\delta W & \delta T & 0
\end{pmatrix}.
$$

(5.23)
The space of sections $G$ as in the previous examples, we can define the global algebroid $A_3$ over the space of opers $M_3$. The matrix elements $x_{j,k} \in \Omega^{j-k,0}(\Sigma_{g,n})$ depend on two arbitrary fields $x_{23} = \varepsilon^{(1)}$, $x_{13} = \varepsilon^{(2)}$. The solution takes the form

$$X = \begin{pmatrix} x_{11} & x_{12} & \varepsilon^{(2)} \\ x_{21} & x_{22} & \varepsilon^{(1)} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

where

$$x_{11} = \frac{2}{3}(\partial^2 - T)\varepsilon^{(2)} - \partial\varepsilon^{(1)}, \quad x_{12} = \varepsilon^{(1)} - \partial\varepsilon^{(2)},$$

$$x_{21} = \frac{2}{3}(\partial^2 - T)\varepsilon^{(2)} - \partial^2\varepsilon^{(1)} + W\varepsilon^{(2)}, \quad x_{22} = -\frac{1}{3}(\partial^2 - T)\varepsilon^{(2)},$$

$$x_{31} = \frac{2}{3}(\partial^2 - T)\varepsilon^{(2)} - \partial^2\varepsilon^{(1)} - \partial W\varepsilon^{(2)} + W\varepsilon^{(1)},$$

$$x_{32} = -\frac{1}{3}(\partial^2 - T)\varepsilon^{(2)} + W\varepsilon^{(2)} + T\varepsilon^{(1)},$$

$$x_{33} = -\frac{1}{3}(\partial^2 - T)\varepsilon^{(2)} + \partial\varepsilon^{(1)}.$$  

The matrix elements of the commutator $[X_1, X_2]_{13}$, $[X_1, X_2]_{23}$ give rise to the brackets (5.12), (5.13), (5.14). Simultaneously, from (5.23) one obtain the anchor action (5.15)–(5.18).

### 5.3 Lie algebroid over $SL(3, \mathbb{C})$-opers

As in the previous examples, we can define the global algebroid $A_3$ over the space of opers $M_3$. The space of sections $G_3 \sim \Gamma(A_3) = \{\varepsilon^{(1)}, \varepsilon^{(2)}\}$ are the second order differential operators on $\Sigma_{g,n}$. We assume that $\varepsilon^{(1)}$, $\varepsilon^{(2)}$ vanish holomorphically at the marked points as

$$\varepsilon^{(1)} \sim r^{(1)}_a(z - x_a) + o(z - x_a), \quad \varepsilon^{(2)} \sim r^{(2)}_a(z - x_a)^2 + o(z - x_a)^2, \quad r^{(j)} \neq 0.$$  

Note that these asymptotics are consistent with the Lie brackets and with asymptotics of $T$ (5.6) and $W$ (5.7).

Consider the cohomology of $A_3$. There exists a nontrivial cocycle corresponding to $H^1(A_3)$ with two components

$$c^{(1)} = \int_{\Sigma_{g,n}} \varepsilon^{(1)} \partial T, \quad c^{(2)} = \int_{\Sigma_{g,n}} \varepsilon^{(2)} \partial W.$$  

It follows from (5.6), (5.7) and (5.25) that the contributions from the marked points are equal

$$c^{(1)} \to \sum_{a=1}^n r^{(1)}_a T_{-2,a}, \quad c^{(2)} \to \sum_{a=1}^n r^{(2)}_a W_{-3,a}.$$  

The cocycle leads to the shift of the anchor action

$$\hat{\delta}_{\varepsilon^{(j)}} f(W, T) = \langle \delta_{\varepsilon^{(j)}} W | \frac{\delta f}{\delta W} \rangle + \langle \delta_{\varepsilon^{(j)}} T | \frac{\delta f}{\delta T} \rangle + c^{(j)}.$$  

There exists the 2g central extensions $c_\alpha$ of $G_3$, provided by the nontrivial cocycles from $H^2(A_3, M_3)$. They are the non-contractible contour integrals $\gamma_\alpha$

$$c_\alpha(\varepsilon^{(j)}_1, \varepsilon^{(k)}_2) = \oint_{\gamma_\alpha} \lambda(\varepsilon^{(j)}_1, \varepsilon^{(k)}_2), \quad (j, k = 1, 2),$$  

(5.27)
where
\[ \lambda(\varphi^{(1)}_1, \varphi^{(1)}_2) = -2\varphi^{(1)}_1 \partial \varphi^{(1)}_2, \quad \lambda(\varphi^{(1)}_1, \varphi^{(2)}_2) = \varphi^{(1)}_1 \partial \varphi^{(2)}_2, \]
\[ \lambda(\varphi^{(2)}_1, \varphi^{(2)}_2) = \frac{2}{3} \varphi^{(2)}_1 \partial \varphi^{(2)}_2. \]

It can be proved that \( \rho^{2,1} \) and that \( \rho \) are not exact. The proof is based on the matrix realization of \( \Gamma(A_3) \) (5.24) and the two-cocycle (4.8) of \( A_{SL}N \). These cocycles allow us to construct the extended brackets:
\[ [(\varphi^{(j)}_1, \sum_\alpha k^{(j)}_\alpha), (\varphi^{(m)}_2, \sum_\alpha k^{(m)}_\alpha)]_{\text{c.e.}} = ([(\varphi^{(j)}_1, \varphi^{(m)}_2), \sum_\alpha c_\alpha (\varphi^{(j)}_1, \varphi^{(m)}_2))]. \]

5.4 Hamiltonian algebroid over \( W_3 \)-gravity

Let \( R_3 = \text{Aff} T^* M_3 \) be the affinization over the cotangent bundle \( T^* M_3 \) to the space of \( SL(3, \mathbb{C}) \)-opers \( M_3 \). The dual fields are the Beltrami differentials \( \mu \) and the differentials \( \rho \in \Omega^{(2,1)}(\Sigma_{g,n}) \). We assume that near the marked points \( \rho \) has the form
\[ \rho \big|_{z \to x_a} \sim (t^{(2)}_{a,0} + t^{(2)}_{a,1}(z - x_a^0)) \delta \chi_a(z, \bar{z}), \]
and \( \mu \) as before satisfies (4.23). The space \( R_3 \) is the classical phase space for the \( W_3 \)-gravity [13, 14, 15]. The symplectic form on \( R_3 \) has the canonical form
\[ \omega = \int_{\Sigma_{g,n}} DT \wedge D\mu + DW \wedge D\rho. \]

According to the general theory the anchor (5.15)–(5.18) can be lifted from \( M_3 \) to \( R_3 \). This lift is nontrivial owing to the cocycle (5.26). It follows from (2.29) that the anchor action on \( \mu \) and \( \rho \) takes the form
\[ \delta_{\varphi^{(1)}} \mu = -\bar{\delta}_{\varphi^{(1)}} \mu - \bar{\rho} \bar{\varphi}^{(1)} - \rho \varphi^{(1)} - \rho^2 \varphi^{(1)}, \]
\[ \delta_{\varphi^{(1)}} \rho = -2 \rho \varphi^{(1)} + \rho \varphi^{(1)}, \]
\[ \delta_{\varphi^{(2)}} \mu = \rho \varphi^{(2)} - \frac{2}{3} \left( (\varphi(\partial^2 - T) \rho) \varphi^{(2)} - (\varphi(\partial^2 - T) \varphi^{(2)}) \rho \right), \]
\[ \delta_{\varphi^{(2)}} \rho = -\bar{\rho} \varphi^{(2)} + (\rho \varphi^{(2)} - \rho^2 \varphi^{(2)}) + 2 \rho \varphi^{(2)} - \mu \varphi^{(2)}. \]

There are two Hamiltonians, defining by the anchor
\[ h^{(1)} = \int_{\Sigma_{g,n}} (\mu \delta_{\varphi^{(1)}} T + \rho \delta_{\varphi^{(1)}} W) + c^{(1)}, \quad h^{(2)} = \int_{\Sigma_{g,n}} (\mu \delta_{\varphi^{(2)}} T + \rho \delta_{\varphi^{(2)}} W) + c^{(2)}. \]

After the integration by parts they take the form
\[ h^{(1)} = \int_{\Sigma_{g,n}} \varphi^{(1)} F^{(1)}, \quad h^{(2)} = \int_{\Sigma_{g,n}} \varphi^{(2)} F^{(2)}, \]
where \( F^{(1)} \in \Omega^{(2,1)}(\Sigma_{g,n}), F^2 \in \Omega^{(3,1)}(\Sigma_{g,n}) \)
\[ F^{(1)} = -\partial T - \partial^4 \rho + T \partial^2 \rho - (3W - 2\partial T) \partial \rho - \]
\[ -(2\partial W - \partial^2 T) \rho + 2 \partial^3 \mu - 2 \partial T \mu - \partial T \mu, \]
\[ F^{(2)} = -\partial W - \frac{2}{5} \partial^5 \rho + \frac{4}{3} T \partial^3 \rho + 2 \partial T \partial^2 \rho + \left( -\frac{2}{3} T^2 + 2 \partial^2 T - 2 \partial W \right) \partial \rho + \]

\[ + (-\partial^2 W + \frac{2}{3} \partial^3 T - \frac{2}{3} T \partial T) \rho + \partial^4 \mu - 3 W \partial \mu - \partial W \mu - T \partial^2 \mu. \]

The Hamiltonians carry out the moment map

\[ m = (m^{(1)} = F^{(1)}, m^{(2)} = F^{(2)}) : \mathcal{R}_3 \to \mathcal{G}_3^*. \]

The elements of the dual space \( \mathcal{G}_3^* \) are singular at the marked points. In addition to \( y \) \((4.27)\)

\[ v \sim c_{1,a} \partial^2 \delta(x_a) + c_{2,a} \partial^3 \delta(x_a) + \ldots. \]

Let \( m^{(1)} \) is defined as in \((4.28)\) and

\[ m^{(2)} = \sum_{a=1}^{n} W_{\alpha}^{a} \partial^2 \delta(x_{a}). \]

Then the coadjoint action of \( G_2 \) preserve \( m = (m^{(1)}, m^{(2)}) \). The moduli space \( \mathcal{W}_3 \) of the \( W_3 \)-gravity (\( W_3 \)-geometry) is the symplectic quotient with respect to the groupoid action

\[ \mathcal{W}_3 = \mathcal{R}_3/\mathcal{G}_2 = \left\{ F^1 = m^{(1)}, F^2 = m^{(2)} \right\}/G_2. \]

It has dimension \( \dim \mathcal{W}_3 = 16(g - 1) + 6n \). The term \( 6n \) comes from the coefficients \( T_{a_1}^a, W_{a_1}^a, W_{a_2}^a \), and the dual to them \( t_{a_1}^{(1)}, t_{a_1}^{(2)}, t_{a_2}^{(2)} \), \( (a = 1, \ldots, n) \) in \((4.23)\) and \((5.28)\).

The moment equations \( F^{(1)} = m^{(1)}, F^{(2)} = m^{(2)} \) are the consistency conditions for the linear system

\[
\left\{
\begin{array}{c}
(\partial^3 - T \partial - W) \psi(z, \bar{z}) = 0,
(\partial + (\mu - \partial \rho) \partial + \rho \partial^2 + \frac{2}{3} (\partial^2 - T) \rho - \partial \mu) \psi(z, \bar{z}) = 0,
\end{array}\right.
\]

where \( \psi(z, \bar{z}) \in \Omega^{-1,0}(\Sigma_{g,n}) \). We will prove this statement below. The last equation represents the deformation of the antiholomorphic operator \( \partial \) (or more general \( \partial + \mu \partial \) as in \((4.29)\)) by the second order differential operator \( \partial^2 \). The left hand side is the exact form of the deformed operator when it acts on \( \Omega^{-1,0}(\Sigma_{g,n}) \). This deformation cannot be supported by the structure of a Lie algebra and one leaves with the algebroid symmetries.

The prequantization of \( \mathcal{W}_3 \) can be realized in the space of sections of a linear bundle \( \mathcal{L} \) over the space of orbits \( \mathcal{M}_3 \sim M_3/G_3 \). The sections are functionals \( \Psi(T, W) \) on \( M_3 \) such that \( \delta_{\varepsilon(j)} \Psi(T, W) = 0, \ (j = 1, 2) \). The bundle \( \mathcal{L} \) can be identified with the determinant bundle \( \det(\partial^3 - T \partial - W) \).

Instead of the symplectic reduction one can apply the BRST construction. The cohomology of the moduli space \( \mathcal{W}_3 \) are isomorphic to \( H^j(Q) \). To construct the BRST complex we introduce the ghosts fields \( \eta^{(1)}, \eta^{(2)} \) and their momenta \( P^{(1)}, P^{(2)} \). Then it follows from Theorem 2.1 that for

\[ \Omega = \sum_{j=1,2} h^{(j)}(\eta^{(j)}) + \frac{1}{2} \sum_{j,k,l=1,2} \int_{\Sigma_{g,n}} (\eta^{(j)}, \eta^{(k)}) P^{(l)} \]

the operator \( Q F = \{ F, \Omega \} \) is nilpotent and define the BRST cohomology in the complex

\[ \bigwedge^{\bullet} (\mathcal{G}_3 \oplus \mathcal{G}_3^*) \otimes C^\infty(\mathcal{R}_3). \]
5.5 Chern-Simons derivation

We follow here the derivation of $W$-gravity proposed in Ref.[14]. We only add in the construction a contribution of the Wilson lines due to the presence of the marked points on $\Sigma_{g,n}$.

Consider the Chern-Simons functional on $\Sigma_{g,n} \oplus \mathbb{R}^+$

$$S = \int_{\Sigma_{g,n} \oplus \mathbb{R}^+} \text{tr}(A dA + \frac{2}{3} A^3) + \sum_{a=1}^{n} \int_{\mathbb{R}^+} \text{tr}(p_{a} g_{a}^{-1} \partial_{t} g_{a}) \ , \ (A = (A, \bar{A}, A_t)) .$$

Introduce $n$ Wilson lines $W_a(A_t)$ along the time directions and located at the marked points

$$W_a(A_t) = P \exp \left( p_a \int A_t \right), \ a = 1, \ldots, n .$$

In the hamiltonian picture the components $A, \bar{A}, p = (p_1, \ldots, p_n), g = (g_1, \ldots, g_n)$ are elements of the phase space with the symplectic form (4.10) while $A_t$ is the Lagrange multiplier for the first class constraints (4.11).

The phase space $\mathcal{R}_3$ can be derived from the Chern-Simons phase space. The flatness condition (4.11) generates the gauge transformations

$$A \rightarrow f^{-1} \partial f - f^{-1} A f \ , \ \bar{A} \rightarrow f^{-1} \bar{\partial} f - f^{-1} \bar{A} f \ , \ p_a \rightarrow f_a^{-1} p_a f_a \ , \ g_a \rightarrow g_a f_a . \quad (5.36)$$

The result of the gauge fixing with respect to the whole gauge group $G_{\text{SL}(3, \mathbb{C})}$ is the moduli space $\mathcal{M}_3^{\text{flat}}$ of the flat $\text{SL}(3, \mathbb{C})$ bundles over $\Sigma_{g,n}$.

Let $P$ be the maximal parabolic subgroup of $\text{SL}(3, \mathbb{C})$ of the form

$$P = \left( \begin{array}{ccc} * & * & 0 \\ * & * & 0 \\ * & * & * \end{array} \right) ,$$

and $G_P$ be the corresponding gauge group. First, we partly fix the gauge with respect to $G_P$. A generic connection $\nabla$ can be gauge transformed by $f \in G_P$ to the form (5.4). It follows from (4.11) that $A$ has simple poles at the marked points. To come to $M_3$ one should respect the behavior of the matrix elements at the marked points (5.6), (5.7). For this purpose we use an additional singular gauge transform by the diagonal matrix

$$h = \prod_{a=1}^{n} \chi_a(z, \bar{z}) \text{diag}(z - x_a, 1, (z - x_a)^{-1}) .$$

The resulting gauge group we denote $G_{(P,h)}$.

The form of $\bar{A}$ can be read off from (4.11)

$$\bar{A} = \left( \begin{array}{ccc} a_{11} & a_{12} & -\mu \\ a_{21} & a_{22} & -\rho \\ a_{31} & a_{32} & a_{33} \end{array} \right) , \quad (5.37)$$

$$a_{11} = -\frac{2}{3} (\partial^2 - T) \rho + \partial \rho , \ a_{12} = -\mu + \partial \rho ,$$

$$a_{21} = -\frac{2}{3} \partial (\partial^2 - T) \rho + \partial^2 \mu - W \rho , \ a_{22} = \frac{1}{3} (\partial^2 - T) \rho ,$$

$$a_{31} = -\frac{2}{3} \partial^2 (\partial^2 - T) \rho + \partial^3 \mu - \partial (W \rho) - W \mu ,$$

27
\[a_{32} = -\frac{1}{3} \partial(\partial^2 - T)\rho + \partial^2 \mu - W\rho - T\mu, \quad a_{33} = \frac{1}{3}(\partial^2 - T)\rho - \partial \mu.\]

The flatness (4.11) for the special choice \(A\) (5.4) and \(\bar{A}\) (5.37) gives rise to the moment constraints \(F^{(2)} = 0, \ F^{(1)} = 0.\) Namely, one has \(F(A, \bar{A})|_{(3,1)} = F^{(2)} (5.33), \ F(A, \bar{A})|_{(2,1)} = F^{(1)} (5.34),\) while the other matrix elements of \(F(A, \bar{A})\) vanish identically. At the same time, the matrix linear system (4.12) coincides with (5.35). In this way, we come to the matrix description of the moduli space \(W_3.\)

The cocycles \(c_\alpha(\varepsilon^{(j)}_1, \varepsilon^{(k)}_2) (5.27)\) can be derived from the two-cocycle (4.8) of \(\mathcal{A}_{SL_3}.\) Substituting in (4.8) the matrix realization of \(\Gamma(\mathcal{A}_3) (5.24),\) one come to (5.27).

The groupoid action on \(A, \bar{A}\) plays the role of the rest gauge transformations that complete the \(G_P\) action to the \(G_{SL_3}\) action. The algebroid symmetry arises in this theory as a result of the partial gauge fixing by \(G_{(P,\hbar)}\). Thus we come to the following diagram

\[
\begin{array}{c}
\mathcal{R}_{SL_3} \\
\downarrow \quad G_{p,\hbar} \\
\mathcal{R}_3 \\
\downarrow \\
\mathcal{M}_{SL_3}^{flat} \\
\downarrow \\
W_3 \\
\end{array}
\]

The tangent space to \(\mathcal{M}_{SL_3}^{flat}\) at the point \(A = 0, \bar{A} = 0, p_a = 0, g_a = id\) coincides with the tangent space to \(W_3\) at the point \(W = 0, T = 0, \mu = 0, \rho = 0.\) Their dimension is \(16(g-1) + 6n.\) But their global structure is different and the diagram cannot be closed by the horizontal isomorphisms. The interrelations between \(\mathcal{M}_{SL_3}^{flat}\) and \(W_3\) were analyzed in [28, 29].

6 \(W_N\)-gravity and general deformations of complex structures

In this section we present the general deformation of complex structures by the Volterra operators. To construct the Lie algebroid over the space of \(GL(N, \mathbb{C})\)-opers we use another form of the pairing. Instead of the differential operators and the pairing (5.11) we consider the Volterra operators that come from the pairing (6.4). We start with the description of the local AGD algebroid following Ref. [12] and then give its global version. The passage from the Lie algebroid to the corresponding Hamiltonian algebroid allows us to describe the deformations of the complex structures.

6.1 Local AGD algebroid

Consider the set \(B = \Psi DO(D^*)\) of pseudo-differential symbols on the disk \(D^*.\) It is a ring of formal Laurent series

\[B = B((\partial^{-1})) = \{B_{r,N}(D^*), \ r, N \in \mathbb{Z} \} = \{X(z, \partial)\}, \]

\[X(z, \partial) = \sum_{k=-\infty}^{r} a_k(z)\partial^k, \quad (a_k(z) \in \Omega^{-N-k}(D^*). \quad (6.1)\]

The multiplication on \(B\) is defined as the non-commutative multiplication of their symbols

\[X(z, \lambda) \circ Y(z, \lambda) = \sum_{k \geq 0} \frac{1}{k!} \frac{\partial^k}{\partial \lambda^k} X(z, \lambda) \frac{\partial^k}{\partial z^k} Y(z, \lambda). \quad (6.2)\]
In what follows we drop the multiplication symbol $\circ$.

Note that $B_{r,N}(D^*) \in \Psi DO(D^*)$ can be considered as the formal map of the sheaves

$$B_{r,N}(D^*) : \Omega^{\frac{N+1}{2}} \rightarrow \Omega^{-\frac{N+1}{2}}.$$  

(6.3)

Let $L_N = \partial^N + W_1 \partial^{N-1} + \ldots + W_N$ be $\text{GL}(N,\mathbb{C})$-oper on $D^*$. Then we have

$$L_N X : \Omega^{-\frac{N+1}{2}} \rightarrow \Omega^{-\frac{N+1}{2}}, \quad L_N X : \Omega^{\frac{N+1}{2}} \rightarrow \Omega^{\frac{N+1}{2}},$$

where the product is defined as (6.2). There is the functional on $\Omega^{-\frac{N+1}{2}}$

$$\langle L_N X \rangle = \frac{1}{2\pi} \int_{|z|=1} \text{Res}(L_N X)dz,$$  

(6.4)

where $\text{Res} = a_{-1}$. The important property is that $\langle L_N X \rangle = \langle XL_N \rangle$.

The AGD brackets on the space $M_G^G(D^*)$ of $\text{GL}(N,\mathbb{C})$-opers $L_N$ are defined as follows. The space of section of $T^*M_N^G(D^*)$ can be identified with the quotient space of the Volterra operators

$$\Gamma(T^*M_N^G(D^*)) = B_{0,N}(D^*)/B_{-N-2,N}(D^*)$$  

(6.5)

For $L_N$ and $X \in \Gamma(T^*M_N^G(D^*))$ define the functional $l_X = \langle L_N X \rangle$. In particular, $W_j(z) = \langle L_N \delta(z) \partial^{-j-1} \rangle$. The AGD brackets have the form

$$\{l_X, l_Y\} = \langle L_N X (L_N Y)_+ - XL_N (YL_N)_+ \rangle,$$  

(6.6)

where $X_+ = \sum_{k=0}^N a_k \partial^k$ is the differential part of $X$.

Using the general prescription we define the Lie brackets (3.5) in the space of sections of $T^*M_N^G(D^*)$

$$[X,Y] = X(L_N Y)_+ - (YL_N)_+ X + (XL_N)_+ Y - Y(L_N X)_+.$$  

(6.7)

and the anchor map

$$\delta_Y L_N = (L_N Y)_+ L_N - L_N (YL_N)_+.$$  

(6.8)

We can rewrite (6.6) in the form of the "Poisson-Lie brackets"

$$\{l_X, l_Y\} = \frac{1}{2}\langle [X,Y] L_N \rangle.$$

The coefficient 1/2 arises from the quadratic form of the Poisson bivector.

In this way we have constructed the local Lie algebroid $A_N^G(D^*)$ over the space of the local $\text{GL}(N,\mathbb{C})$ opers $M_N^G(D^*)$.

To be the $\text{SL}(N,\mathbb{C})$-oper $L_N = \sum_{k=0}^N a_k \partial^k$ should satisfy the second class constraints

$$W_1(z) = 0, \quad W_1 = \langle L_N, \delta(z) \partial^{-N} \rangle.$$  

since $\{W_1(z), W_1(w)\} \sim \delta(z - w)$. The functional $W_1(z)$ generates the vector field on $M_N^G(D^*)$

$$\delta_Y L_N = \{W_1, L_N\} = \frac{1}{2}\langle [\delta(z) \partial^{-N}, Y] L_N \rangle = \langle \delta(z) \partial^{-N} \delta_Y L_N \rangle.$$  

If $\delta_Y L_N = 0$ the transformed oper is the $\text{SL}(N,\mathbb{C})$-oper. It means that $\text{Res}[Y, L_N] = 0$. In this way the brackets (6.6) is the generalization of (4.14) ($\mathbb{N} = 2$) and (5.8)-(5.10) ($\mathbb{N} = 3$) on arbitrary $\mathbb{N}$. The relations between the Poisson manifolds $M_N^G(D^*)$ and $M_N(D^*)$ were discussed also in Ref. [30].
6.2 Poisson sigma-model

Here we construct the Poisson sigma-model with the target space $M_N^G(D^*)$ following 3.2. We modify the notion of $\Psi DO(D^*)$ assuming that instead of (6.3) we have

$$B_{(r,N,1)}(D^*) = B_{r,N}(D^*) \otimes \bar{K}(D^*), \quad B_{(r,N,1)}(D^*) : \Omega^{(\frac{N+1}{2},1)} \to \Omega^{(\frac{N-1}{2},1)}.$$

(6.9)

The affinization $\mathcal{R}_N = Aff T^*A_N(D^*)$ of $T^*M_N^G(D^*)$ is defined by the pair of fields $(L_N, \xi)$, where $\xi \in B_{(0,N,1)}/B_{(-N-2,1)}$ is the dual field to an oper with respect to the integral over $D^*$. The space $\mathcal{R}_N$ is the phase space of the Poisson sigma model with the canonical symplectic form

$$\omega = \int_{D^*} \text{Res}(DL_N \wedge D\xi).$$

(6.10)

The one-cocycle $c(L,Y) \in H^1(A_N(D^*))$

$$c(L,Y) = \int_{D^*} \text{Res}(Y \bar{\partial} L_N), \quad (Y \in B_{0,N})$$

(6.11)

provides the lift of the anchor action (6.8) on $\mathcal{R}_N$

$$\delta_Y \xi = -\bar{\partial} Y + Y(L_N \xi)_+ - (\xi L_N)_+ Y + (Y L_N)_+ \xi - \xi(L_N Y)_+. \quad (6.12)$$

Along with (6.8) this action defines the canonical transformations of (6.10). They are generated by the Hamiltonian

$$h_Y = \int_{D^*} \text{Res}(\xi \delta_Y L_N) + c(L,Y) = \int_{D^*} \text{Res}(Y(\bar{\partial} L_N - (L_N \xi)_+ L_N + L_N(\xi L_N)_+)) . \quad (6.13)$$

This expression yields the anchor map of the Hamiltonian algebroid $A_N^H(D^*)$ related to $A_N(D^*)$. The canonical transformations by $B_{0,N}(D^*)$

$$\delta_Y L_N = \{h_Y, L_N\}, \quad \delta_Y \xi = \{h_Y, \xi\}, \quad (Y \in B_{0,N}(D^*))$$

corresponding to the form (6.10) are generated by the first class constraints

$$F := \bar{\partial} L_N - (L_N \xi)_+ L_N + L_N(\xi L_N)_+ = 0 . \quad (6.14)$$

Define the operator

$$A = \bar{\partial} - (L_N \xi)_+ : \Omega^{(\frac{N+1}{2},0)}(D^*) \to \Omega^{(\frac{N+1}{2},1)}(D^*)$$

(6.15)

and the dual operator

$$A^* = \bar{\partial} + (\xi L_N)_+ : \Omega^{(-\frac{N+1}{2},0)}(D^*) \to \Omega^{(-\frac{N+1}{2},1)}(D^*).$$

Let $\psi = (\psi^-, \psi^+), \psi^- \in \Omega^{(-\frac{N+1}{2},0)}(D^*), \psi^+ \in \Omega^{(\frac{N+1}{2},0)}(D^*)$. Following Lemma 3.2 we conclude that the quadratic constraints (6.14) are equivalent to the linear problem

$$\begin{cases}
L_N \psi^-(z, \bar{z}) = 0,
\bigl(\bar{\partial} - (L_N \xi)_+\bigr) \psi^-(z, \bar{z}) = 0 .
\end{cases} \quad (6.16)$$

The second equation can be replaced on

$$\bigl(\bar{\partial} + (\xi L_N)_+\bigr) \psi^+(z, \bar{z}) = 0 . \quad (6.17)$$

In this way the local oper $L_N$ along with the dual element $\xi$ defines the deformation of the complex structure on the disk $D^*$. The second equation (6.16) is equivalent to the deformed holomorphy condition for the sections $\Omega^{(-\frac{N+1}{2},0)}$.

Let $G^H_N(D^*)$ be the Hamiltonian Lie groupoid corresponding to $A_N^H(D^*)$. The reduced phase space is the symplectic quotient

$$\mathcal{R}_N^{red} = \mathcal{R}/G^H_N(D^*) = \{F = 0\}/G^H_N(D^*).$$

6.3 Global algebroid

As we have noted in previous Section the SL($N, \mathbb{C}$)-opers are well defined on the curve $\Sigma_{g,n}$.

The global algebroid $A_N$ over the space of GL($N, \mathbb{C}$)-opers $G^N_{\Sigma}$ on $\Sigma_{g,n}$ is independent on the choice of $D^*$. The space of its sections $G_N \sim \Gamma(A_N)$ is the space of the Volterra operators on $\Sigma_{g,n}$ with smooth coefficients (see (6.5)). Near a marked point $x_a$ with a local coordinate $z$ for $X \in G_N$ we have

$$X = \epsilon^{(1)} \partial_z^{-1} + \epsilon^{(2)} \partial_z^{-2} + \ldots + \epsilon^{(N)} \partial_z^{-N} + \epsilon^{(N+1)} \partial_z^{-N-1},$$

where

$$\epsilon^{(j)} \sim r_a^{(j)} (z - x_a)^j + o(z - x_a)^j, \quad r_a^{(j)} \neq 0.$$

The anchor action of $X$ is defined as before by (6.8).

The one-cocycle representing $H^1(A_N)$ comes from the integration over $\Sigma_{g,n}$

$$c = \int_{\Sigma_{g,n}} X \bar{\partial} L_N.$$

Let

$$B_{(r,N,1)}(\Sigma_{g,n}) = B_{r,N}(\Sigma_{g,n}) \otimes \bar{K}(\Sigma_{g,n}),$$

$$B_{(r,N,1)}(\Sigma_{g,n}) : \Gamma(\Omega^{(N+1,1)}(\Sigma_{g,n})) \to \Gamma(\Omega^{(-N-1,1)}(\Sigma_{g,n})).$$

The affinization $R_N = Aff T^*A_N(\Sigma_{g,n})$ of $T^*M_N(\Sigma_{g,n})$ is defined by the fields $\xi$ from the quotient $\xi \in B_{(0,N,1)}(\Sigma_{g,n})/B_{(-N-2,N,1)}(\Sigma_{g,n})$. Near the marked points

$$\xi = \sum_{j=1}^{N+1} \nu_j \partial^{-j}, \quad (\nu_{N+1} = 1),$$

$$\nu_j|_{z \to x_a} = (t_{a,0}^{(j)} + \ldots + t_{a,j-1}^{(j)}(z - x_a)^{j-1}) \partial \chi_a(z, \bar{z}), \quad (\nu_j \in \Omega^{(j-1,1)}(\Sigma_{g,n})).$$

The symplectic form on $R_N$ is

$$\omega = \int_{\Sigma_{g,n}} \text{Res} (D L_N \wedge D \xi).$$

The anchor action on $M^G_N$ (6.8) can be lifted from $M^G_N$ to $R_N$ as canonical transformations of $\omega$. They are generated by the Hamiltonians

$$h_Y = \int_{\Sigma_{g,n}} \text{Res} (\xi \delta_Y L_N) + c(Y, L_N).$$

In addition to (6.8) we have the action on the dual variables are

$$\delta_Y \xi = \{h_Y, \xi\}.$$

(6.18)

The Hamiltonians can be represented in the form

$$h_Y = \int_{\Sigma_{g,n}} \text{Res} (Y F(L_N, \xi)), $$

where $F(L_N, \xi)$ is defined by (6.14). They lead to the moment map

$$m : R_N \to G^*_N, \quad m = F(L_N, \xi).$$
We take
\[ m = \sum_{a=1}^{n} \sum_{j=1}^{N} W_{-j}^a (j) \partial^{j-1} \big|_{z=x_a}, \]
where \( W_{-j}^a (k) \) are defined in (5.3). The moment equation \( F = m \) is equivalent to the linear problem (6.16), (6.17) on \( \Sigma_{g,n} \). In this way we come to the deformed \( \bar{\partial} \) operator \( \bar{\partial} - (L_N \xi)_+ \) (\( \bar{\partial} + (\xi L_N)_+ \)) acting on the space \( \Omega^{(-N+1/2,0)}(\Sigma_{g,n}) \) (\( \Omega^{(-N+1/2,0)}(\Sigma_{g,n}) \)).

The moduli space of the generalized complex structures is a part of the symplectic quotient \( \mathcal{W}_N \sim \mathcal{R}_N // G_N \), where \( G_N \) is the corresponding Hamiltonian groupoid. The cohomology of \( \mathcal{W}_N \) are defined by the BRST operator
\[ \Omega = h_\eta + \frac{1}{2} \int_{\Sigma_{g,n}} \text{Res} \left( (|\eta, \eta'| \mathcal{P}) \right), \]
where \( \eta \) is the ghost field corresponding to the gauge field \( Y \) and \( \mathcal{P} \) is its momenta.

7 Concluding Remarks

Let summarize the results and discuss some open problems.

(i) The Hamiltonian algebroid is a bundle over a Poisson manifold with a Lie algebra structure on its sections and the anchor map to the Hamiltonian vector fields. The special kind of the Hamiltonian algebroids are defined over affine space of cotangent bundles. They are lift of the Lie algebroids defined over the base of the cotangent bundles. The lifts are classified by the first cohomology of the Lie algebroids. The Hamiltonian algebroids of this type are most closed to the Lie algebras of Hamiltonian vector fields and has the same structure of the BRST operator.

(ii) The Lie algebroid over the space of SL(3, \( \mathbb{C} \))-opers on a Riemann curve with marked points has the space of the second order differential operators as the space of its sections. It contains the Lie subalgebra of the first order differential operators. After change the behavior of their coefficients at the marked points this subalgebra coincides with the Krichever-Novikov algebra [31]. It will be interesting to lift this correspondence to the second order differential operators. Another open question is the structure of opers and Lie algebroids defined on Riemann curves with double marked points.

(iii) In the limit \( N \to \infty \) the structure of the strongly homotopy Lie algebras should be recovered. In our approach this limit looks obscure.

(iv) The Chern-Simons derivation of the Hamiltonian algebroid in \( W_N \)-gravity explain the origin of the algebroid symmetry as a result of the two step gauge fixing. It will be plausible to have the same universal construction for an arbitrary Poisson sigma-model.

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