Final state interactions in the $\eta d$ system

S. Wycech
Soltan Institute for Nuclear Studies, Warsaw, Poland

A.M. Green†
Department of Physics and Helsinki Institute of Physics
P.O. Box 64, FIN–00014 University of Helsinki, Finland
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Abstract

The $\eta$-deuteron scattering amplitude is calculated with a recent K-matrix model of $\eta N$ interactions. The existence of a narrow virtual state in the $\eta d$ system is inferred. This state couples strongly to the $\eta d$ channel but its effect on final state interactions in the $pn \rightarrow \eta d$ reaction is shown to be rather weak. Fairly large $\eta N$ scattering lengths are allowed by the experimental data.

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I. INTRODUCTION

Interactions of the $\eta$ meson with few-nucleon systems complement our knowledge of the $\eta$-nucleon interaction. An independent, although related, interest in these systems stems from the conjectured existence of $\eta$-nuclear quasi-bound states. Such states have been predicted by Haider and Liu [1], and Li et al. [2], when it was realised that the $\eta$-nucleon interaction is attractive. So far there has been no direct experimental verification of this hypothesis but, on the other hand, there is a mounting evidence that such states exist in the lightest few-nucleon systems.

The earliest measurement was done in the four-body case. It was found at SATURNE that the $pd \to \eta^3He$ production amplitude falls rapidly just above the $\eta$ production threshold [3], [4]. This led Wilkin to the suggestion that such a quasi-bound state has been produced [5]. A similar but weaker slope was found in the $dd \to \eta^3He$ amplitude [4], and this may indicate that the quasi-bound states in five-body systems are broader and located further away from the $\eta$ meson production threshold [6].

In more recent studies of three-body systems, very strong final state interactions (FSI) were found in the $pp \to pp\eta$ cross section in the threshold region [7], [8], [9]. Those led to a speculation that this system may form a quasi-borromean state, [10]. As the final state interactions in the three-body continuum are difficult to analyse, one finds perhaps more convincing evidence in the measurement performed at CELSIUS, [11]. There, the $\eta$-deuteron final states are observed in the scattering of a proton on a deuteron target. This method allows the determination of the $pn \to \eta d$ amplitude, which turns out to be a smooth function of energy but displays an enhancement close to the threshold. Much stronger enhancement is indicated by an earlier SATURNE measurements performed by neutron scattering on a proton target [12]. These data taken together indicate a very sharp threshold effect in the $\eta$-deuteron system. Earlier calculations performed by Ueda confirm such an effect [13], and indicate the existence of quasi-bound $\eta NN$ states.

The correlations seen in the three-body systems may be due to quasi-bound states or quasi-virtual states, i.e. three-body analogs of the deuteron or the virtual spin singlet state in the $NN$ system. However, there has been a serious difficulty in such an interpretation. The strength of the $\eta N$ interaction and the magnitude of the scattering length $a_{\eta N}$ were not large enough. The early values of Re $a_{\eta N}$ ranged between 0.25 fm and 0.5 fm, [1], [2], [3], [4], [5], [6], [7], although larger values have also been suggested [8], [9], [10], [11], [12], [13]. The lengths up to 0.5 fm are not large enough to support long lived structures in $\eta NN$ systems. Calculations have shown that the $\eta$-deuteron virtual state is only likely to be formed provided Re $a_{\eta N}$ exceeds a critical value of about 0.7 fm, [14]. For larger values of about 1.2 fm, one finds the $\eta d$ system likely to be quasi-bound [15]. Such states would be reflected in a large value of the $\eta d$ scattering length $A_{\eta d}$. Unfortunately, the calculations of $A_{\eta d}$ done in this critical region depend on details of the $\eta N$ interaction model. The significance of the scattering length is one factor, but another equally important one is the way the $\eta N$ scattering amplitude extrapolates to the region below the $\eta N$ threshold.

Recent phenomenological analyses of a four coupled-channel ($\eta N$, $\pi N$, $\gamma N$, $\pi \pi N$) K-matrix model have helped to resolve these difficulties. In Ref. [16] a large scattering length $a_{\eta N} = 0.75(4) + i 0.27(3)$ fm was found, and in addition the effective range expansion was shown to work well in a broad region of both positive and negative energies. A very similar
value of $a_{\eta N} = 0.72(3) + i0.26(2)\, fm$ was obtained in a dispersion theoretic analysis of reference [21]. However, further studies [27] of the K-matrix model done in the spirit of Ref. [26] have now revealed the existence of another solution, which is characterised by a small scattering length of $a_{\eta N} \approx 0.3 + i0.2\, fm$ and a very large effective range. It should be added that a low value of $a$ is also produced by the GW21 solution in the authors $N(1535)$ pole-position studies of Ref. [28]. In that work only the channels $\pi N$, $\eta N$ and $\pi\pi N$ were treated explicitly. However, when we use an extended version of the $K$-matrix model, in which both the $\gamma\pi$ and $\gamma\eta$ channels are included [29], then this GW21 solution disappears leaving only GW11 with its large scattering length $\text{Re}\, a_{\eta N} = 0.87\, fm$. One hopes that the few-body data are going to be selective on these possibilities. In addition, if $a_{\eta N}$ is large there should arise a rich spectroscopy of nuclear $\eta$ states. Forthcoming experiments may resolve this question, [30], although difficulties in the interpretation have been suggested, [31].

It seems that now there exist enough information to study the $\eta NN$ phenomena in an almost quantitative way, and some related problems are studied in this paper:

In section II, the $\eta$-deuteron scattering amplitude at low energies is calculated. A method of partial summation of the multiple scattering series used previously to calculate the eta-deuteron scattering length, [23], is extended to the case of the continuum. A strong cusp in the $\eta d$ scattering amplitude at the threshold is found and attributed to a virtual state. The width of this state is very small, in the 1 MeV range.

In section III, the final state interactions in the $\eta d$ system are discussed. We find the virtual state to couple rather weakly to this system. It produces rather limited effects at energies below the deuteron breakup. The results of Ueda, [13], who has predicted a broad and prominent peak above the $\eta d$ threshold, are not reproduced.

In section IV, we discuss the constraints induced upon the $\eta N$ scattering matrices by the $\eta NN$ data. Also, the nature of exotic $\eta NN$ states is discussed in this section.

Some parts of this research could be, and have been, done in an exact way by solving the three-body Faddeev equations [24], [25], [32]. The solution for $\eta d$ scattering is the simplest one that can be improved in this way. However, even for this simple question the general multiple channel analysis has never been fully implemented. It seems also to be premature, as many ingredients of the $\eta N$ interaction are not known. The advantage of the method presented here consists in the following:

1. It is numerically simple and transparent in its physical interpretation. Uncertainties in the two-body interactions and the meson formation process may be separated and discussed in simple terms.
2. This method is applicable to more complicated few nucleon systems.
3. It is very accurate numerically, at least in the calculation of the $\eta d$ scattering length.

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1. The authors wish to thank Bengt Karlsson for emphasising this point.
II. THE $\eta$ DEUTERON SYSTEM AT LOW ENERGIES

We discuss the $\eta d$ scattering at very low energies. This section contains three parts. First a simple method is introduced which allows us to find the scattering matrix $T_{\eta d}$. At very low energies, below deuteron breakup, $T_{\eta d}$ is dominated by a narrow virtual three-body state. Next, we study the off-shell extension of $T_{\eta d}$ in order to construct the $\eta d$ wave function. The latter is used to describe final state interactions in the $\eta d$ formation processes. In the third part the static nucleon approximation is discussed. Everywhere throughout this paper the dominance of S waves is assumed.

A. A Formula for the $\eta$-deuteron $T$ matrix

The purpose of this subsection is to derive a simple formula which relates the low-energy meson-deuteron scattering amplitude to a few multiple scattering integrals involving the meson-nucleon amplitudes. The formula is described in more detail in Ref. [23], where it was used to calculate the $\eta d$ scattering length. Its numerical precision was later confirmed by a direct solution of this problem in terms of the Faddeev equation [32]. In the limiting case of fixed nucleons an agreement with other calculations has also been established [38]. Here, this simple procedure is outlined and its extension to the elastic or inelastic continuum and to the off-shell region is discussed.

The starting point is a fixed nucleon approximation. Next, upon the solution of this simple limit a more realistic calculation is built up. It is achieved in a step-by-step summation of the multiple scattering series until the necessary precision is reached. The scattering length of a meson on a pair of fixed nucleons is given by

$$A_{\eta d} = \frac{2a_{\eta N}}{1 - a_{\eta N}/R_d},$$

where $a_{\eta N}$ is the meson-nucleon scattering length and $R_d$ is the inter-nucleon distance, [39]. Eq. (1) is obtained in a simple way by setting boundary conditions for the meson wave function $\psi$ at each scatterer as $\psi'/\psi = 1/a_{\eta N}$. An alternative way to obtain the same result is to sum the multiple scattering series, which in this case is a geometric series. With this expansion, the $1/R_d$ term stands for the meson propagator while $a_{\eta N} \frac{1}{R_d} a_{\eta N}$ is a double scattering amplitude etc. The main advantage of Eq. (1) is that it offers the solution even for a divergent multiple scattering series when $a_{\eta N}/R_d > 1$. The main weakness is that it is based on distinguishable nucleons. Also, important kinematical and dynamical effects of the three body propagation are missing and the actual value of $R_d$ is left undetermined. The method to be described below retains the main advantage, but removes the weak points and offers a correct value of $R_d$.

To improve Eq. (1) for the non-fixed nucleon situation one needs to introduce: the three-body propagator, the NN interactions and an off-shell $\hat{t}_{\eta N}$ scattering matrix. To achieve this, we study the multiple scattering expansion for meson-deuteron scattering. The basic $NN$ interactions are assumed to be known and described by the deuteron wave function $\phi_d$ and a complete off-shell $\hat{t}_{NN}$ scattering matrix. With $\hat{t}_{NN}$ given one finds a partial three-body propagator.
\[ G = G_0 + G_{NN}, \]  
(2)

where \( G_0 \) is the free three-body propagator and 

\[ G_{NN} = G_0 \hat{t}_{NN} G_0 \]  
(3)

constitutes that part of the three-body propagator which describes the \( NN \) interactions. Now, the series appropriate for the \( \eta \)-deuteron scattering matrix \( \hat{T}_{\eta d} \) follows from the Faddeev equations:

\[
\hat{T}_{\eta d} = \hat{t}_1 + \hat{t}_2 + \hat{t}_1 G_0 \hat{t}_2 + \hat{t}_2 G_0 \hat{t}_1 + \hat{t}_1 G_0 \hat{t}_2 G_0 \hat{t}_1 + (\hat{t}_1 + \hat{t}_2) G_{NN} (\hat{t}_1 + \hat{t}_2) + ..., 
\]  
(4)

where the nucleons are labeled 1, 2 and the \( \hat{t}_i \) are \( \eta \)-nucleon scattering matrices. This expansion is performed in momentum space. All quantities involved in Eq.(4) are operators and this is denoted by the hat (\( \hat{\ldots} \)) symbols. Thus, appropriate multifold integrations over the intermediate Jacobi momenta (\( \vec{p}, \vec{q} \)) must be performed. In full generality the kernel of \( \hat{T}_{\eta d} \) is a four point function \( T_{\eta d}(p, q, q', p'; E) \) and to find the \( \eta d \) scattering matrix \( T_{\eta d} \) a matrix element has to be calculated

\[
T_{\eta d}(p_i, E, p_f) = < \phi_d \psi_\eta(p_i) | \hat{T}_{\eta d} | \phi_d \psi_\eta(p_f) >, 
\]  
(5)

where the \( \psi_\eta \) are S-wave functions for the initial and final mesons of momenta \( p_i, p_f \). In this section we are mainly concerned with the on-shell value of this scattering matrix, which is denoted by \( T_{\eta d}(E) \) with \( E \) being the nonrelativistic energy of the \( \eta d \) system. The leading order (impulse approximation) matrix is

\[
T^0_{\eta d} = < \hat{T}^0_{\eta d} >= < \hat{t}_1 + \hat{t}_2 >, 
\]  
(6)

where \(< \ldots >\) denotes the average over the deuteron and mesonic wave functions as indicated in Eq.(5). The first partial sum of the series for \( T_{\eta d}(E) \) is obtained with the expression

\[
T^1_{\eta d} = \frac{< \hat{T}^0_{\eta d} >}{1 - \Sigma_1 - \Omega_1} 
\]  
(7)

which contains the double scattering terms

\[
\Sigma_1 = \frac{< \hat{T}^0_{\eta d} G_{NN} \hat{T}^0_{\eta d} >}{< \hat{T}^0_{\eta d} >} \]  
(8)

\[
\Omega_1 = \frac{< \hat{t}_1 G_0 \hat{t}_2 + \hat{t}_2 G_0 \hat{t}_1 >}{< \hat{T}^0_{\eta d} >}. 
\]  
(9)

The first term \( \Sigma_1 \) describes the double scattering with an intermediate \( NN \) interaction. It is essentially an "optical model " double scattering term. The second term \( \Omega_1 \) includes excitations of nucleons to free continuum states. The expression in Eq.(7) is analogous to the one in Eq. (1), since it contains terms linear in \(< \hat{t} >\) in the denominator. However,
it also now contains the prescription for how to calculate the effective distance $R_d$ and additionally contains sizable effects from the excitations. Again, the expansion parameter is $\langle \hat{t} \rangle / R_d$ and it does not need to be small to guarantee the success of Eq. (7), which is also applicable if the multiple scattering series is divergent. Higher orders of $\langle \hat{t} \rangle / R_d$ in the denominator of Eq. (7) are obtained by comparing higher orders in Eq. (4) with an expansion of Eq. (7) with respect to $\Omega_1$ and $\Sigma_1$. In this way the next order approximation is obtained as

$$T_{\eta d}^2 = \frac{\langle \hat{T}_{\eta d}^0 \rangle}{1 - \Sigma_1 - \Omega_1 - S_2}$$

where

$$S_2 = [\Sigma_2 - (\Sigma_1)^2] - [\Omega_2 - (\Omega_1)^2] - [\Delta_2 - \Omega_1 \Sigma_1], \quad \Sigma_2 = \frac{\langle \hat{T}_{\eta d}^0 G_{NN} \hat{T}_{\eta d}^0 G_{NN} \hat{T}_{\eta d}^0 \rangle}{\langle T_{\eta d}^0 \rangle},$$

$$\Omega_2 = \frac{\langle \hat{t}_1 G_0 \hat{t}_2 G_0 \hat{t}_1 + \hat{t}_2 G_0 \hat{t}_1 G_0 \hat{t}_2 \rangle}{\langle T_{\eta d}^0 \rangle}, \quad \Delta_2 = 2 \frac{\langle \hat{T}_{\eta d}^0 G_{NN} (\hat{t}_1 G_0 \hat{t}_2 + \hat{t}_2 G_0 \hat{t}_1) \rangle}{\langle T_{\eta d}^0 \rangle}.$$ 

One may continue this procedure to include higher powers of $\langle \hat{t} \rangle / R_d$ in the denominator, but for the present problem the precision of the order of 1% has been already reached. This was shown in calculations of the $\eta d$ scattering length, [23], [32], and optical model calculations for $\eta He$ systems, [3]. This fast convergence rate occurs as a result of an almost exact cancellation of the $\Sigma_2 - \Sigma_1$ and higher order $\Sigma_n$ terms. This property holds also in the scattering region.

B. Scattering amplitudes, off-shell extension, unitarity

In this subsection some details of the $\eta$ deuteron scattering are discussed, the $\eta N$ interaction is introduced and explicit expressions for the multiple scattering integrals are given. Next, the question of unitarity is discussed for energies below the deuteron disintegration threshold.

The scattering matrices $T_{\eta d}(E), t_{\eta N}(E)$ used for the series summation have the natural dimensions of length$^2$. To discuss the cross sections it is more convenient to use the scattering amplitudes with the dimension of a length. These are denoted by $A_{\eta d}(E), a_{\eta N}(E)$, where the convention Im $A \geq 0$ is used and the threshold values $A_{\eta d}, a_{\eta N}$ are the scattering lengths. The relations between scattering amplitudes and scattering matrices are

$$A_{\eta d}(E) = -(2\pi)^2 \mu_{\eta d} T_{\eta d}(E),$$

$$a_{\eta N}(E) = -(2\pi)^2 \mu_{\eta N} t_{\eta N}(E).$$

For the multiple scattering expansion, these relations introduce factors $\mu_{\eta d}/\mu_{\eta N}$ which reflect kinematical differences in the propagation of these two systems.
Now, some definitions used in this calculation are written down. The integrations in Eqs.(3)-(12) are performed in momentum space. To describe wave functions the Jacobi coordinates $\vec{q}_{NN}$ (the relative $NN$ momentum) and $\vec{p}_\eta$ (the $\eta - NN$ relative momentum) are used. However, to describe meson-nucleon interactions within the 3-body system one uses other pairs denoted by $(\vec{q}_{NN}, \vec{p}_N)$. A useful relation is $\vec{q}_{NN} = \vec{p}_N - \vec{p}_\eta/2$. It allows one to express the integration volumes $d\vec{p}_\eta d\vec{q}_{NN}$, $d\vec{p}_N d\vec{q}_{NN}$ also in terms of $d\vec{p}_N d\vec{p}_\eta$. The energy $E$ of the $\eta NN$ system is related to the on-shell momentum $p_\eta$ by $E = \vec{p}_\eta^2/2\mu_N + E_d$ where $E_d$ is the deuteron energy of $-2.2$MeV. For the free propagator one has now

$$G_0 = [E_{NN}(q_{NN}) + E_\eta(p_\eta) - E]^{-1}\delta(\vec{p}_\eta - \vec{p}_\eta')\delta(\vec{q}_{NN} - \vec{q}_{NN}').$$  \hspace{1cm} (15)

The meson-nucleon scattering matrices are $t_{\eta N}[q_{\eta N}, q_{NN}, E - E(p_N)]\delta(\vec{p}_N - \vec{p}_N')$, where the delta-function is used to conserve the spectator nucleon momentum and $E(p_N)$ accounts for the spectator nucleon recoil energy. A typical on-shell $a_{\eta N}(E)$ is given in Fig.1. The cusp in the real part at the $\eta N$ threshold indicates an attraction that is due partly to the $N(1535)$ resonance and partly to other attractive interactions. It is this cusp and the attraction related to it, that is responsible for the attraction in the few-body-$\eta$ systems, leading ultimately to the formation of virtual or quasibound states. The strength of the cusp i.e. the value of scattering length $a_{\eta N}$ is controversial. The one shown in Fig.1 comes from the phenomenological approach of Ref. [23], but a number of smaller as well as larger values have been predicted in the literature. For this particular solution one finds the effective range expansion

$$1/a_{\eta N}(E) = 1/a_{\eta N} + \frac{1}{2}a_{\eta N}^2 r_{\eta N}$$ \hspace{1cm} (16)

with parameters $a_{\eta N} = 0.75(4) + i 0.27(3)$ fm and $r_{\eta N} = -1.50(13) - i 0.24(4)$ fm to cover the broad energy range in Fig.1. Such a wide applicability of the effective range expansion indicates very short range $\eta N$ forces. The attitude of this paper is to keep $a_{\eta N}$ as a semi-free parameter, since it results from an interplay of the well established $N(1535)$ resonance and unknown ”background” interactions. On the other hand, the energy dependence of $a_{\eta N}(E)$ is determined mostly by the $N(1535)$ which generates a negative effective range. Thus the value of $r_{\eta N}$ given above is believed to be weakly dependent on the particular choice of $a_{\eta N}$. Notice that $a_{\eta N}(E)$ is small at energies far below the threshold. This allows a perturbative approach in this region.

Although $a_{\eta N}$ is ”known” only on the energy-shell one also needs an off-shell extension. A separable one is used here: $a_{\eta N} = v_\eta(q_{\eta N})a_{\eta N}(E)v_\eta(q_{\eta N}')$ with a Yamaguchi form factor $v_\eta = (1 + q_{\eta N}^2/\kappa_{\eta}^2)^{-1}$. The inverse range parameter $\kappa_{\eta}$ is apparently very large for two reasons. Firstly, there is no long ranged meson exchange within the $\eta N$ forces and, as said before, the effective range expansion converges rapidly. Secondly, the form factors in any $S$-channel resonance must correspond to the small sizes of these objects. It has to be stressed that in the $\eta d$ system at threshold one is close to a zero energy binding situation. The results are sensitive to every parameter involved including the uncertain $\kappa_{\eta}$. A tentative value $\kappa_{\eta} = 3.3$ fm, which corresponds to a natural radius of $N(1535)$, is used. The sensitivity to $\kappa_{\eta}$, if it arises, will be indicated. On the other hand, to simplify notation all formulas are given in the zero range $\eta N$ force limit. However, the actual calculations are performed without this restriction.
Within the three-body system, the $\eta N$ scattering matrix is to be averaged over some energy region, generated by the recoil of the spectator nucleon. So, the $\eta d$ scattering length given by the impulse approximation formula (6) and normalisations in Eqs. (13,14) becomes

$$A^0_{\eta d}(p_i, E, p_f) = 2\frac{\mu_{\eta d}}{\mu_{\eta N}} \left< \int d\vec{p}_d(\vec{p} - \vec{p}_i/2)a_{\eta N}(E - \frac{p^2}{2\mu_{N,\eta N}})\phi_d(\vec{p} - \vec{p}_f/2) \right>, \quad (17)$$

where $\vec{p}_i$ and $\vec{p}_f$ are the initial and final meson momenta. Eq.(17) has been obtained with initial and final plane waves. In order to extract the S wave contribution an average over directions of these momenta is to be performed and this is indicated by the brackets $<>$. For on-shell scattering conditions one has $p_i = p_f = p_\eta$. This formula is more general, however, and provides an off-shell extrapolation for $A^0_{\eta d}(p_i, E, p_f)$ to situations where these quantities are not related. The integration in Eq.(17) averages the $a_{\eta N}(E)$ amplitude over some region of recoil energies $p^2/2\mu_{N,\eta N}$. The extent of this region is determined by momenta involved in the deuteron wave function. At the threshold it starts with a negative $E_d$ and extends towards negative energies down to about $-20$ MeV. For positive energies, $E$ covers a range of about 20 MeV that includes the $\eta N$ threshold cusp.

For a limited region of $E$ it makes sense to introduce an effective amplitude $\bar{a}_{\eta N}(E)$ that has been averaged over these 20 MeV or so of the recoil energies. We use Eq.(17) to define such an average $\bar{a}_{\eta N}(E)$ and this value will be used to calculate higher order scattering terms. In this way the impulse approximation amplitude becomes

$$A^0_{\eta d}(p_i, E, p_f) = 2\frac{\mu_{\eta d}}{\mu_{\eta N}} \bar{a}_{\eta N}(E) \left< \int d\vec{p}_d(\vec{p} - \vec{p}_i/2)\phi_d(\vec{p} - \vec{p}_f/2) \right>. \quad (18)$$

It is expressed in terms of a deuteron formfactor

$$F_d(p_i, p_f) = \left< \int d\vec{p}_d(\vec{p} - \vec{p}_i/2)\phi_d(\vec{p} - \vec{p}_f/2) \right>, \quad (19)$$

which may be also presented in terms of space coordinates as

$$F_d(p_i, p_f) = \left< \int d\vec{r} \exp[i\vec{r}(\vec{p}_i/2 - \vec{p}_f/2)]\phi_d^2(r) \right> = \int d\vec{r} j_0(rp_i/2)\phi_d^2(r)j_0(rp_f/2). \quad (20)$$

When the multiple scattering series is summed in Eq.(10) one obtains the result

$$A_{\eta d}(p_i, E, p_f) = \frac{A^0_{\eta d}(p_i, E, p_f)}{1 - \Sigma_1 - \Omega_1 - S_2}. \quad (21)$$

The effect of free continuum $\eta NN$ states is, to the lowest order in $\bar{a}_{\eta N}$, given by $\Omega_1$. As the integrals in Eq.(8) are lengthy we reproduce this formula only in the limit of zero range $\eta N$ forces

$$\Omega_1(p_i, E, p_f) = \frac{\bar{a}_{\eta N}}{F_d(p_i, p_f)} \int \frac{d\vec{q}d\vec{q}'}{(2\pi)^2\mu_{\eta N}} \frac{\phi_d(\vec{q} - \vec{p}_i/2)\phi_d(\vec{q}' - \vec{p}_f/2)}{E_{NN}(\vec{q} - \vec{q}') + E_{\eta}(\vec{q} + \vec{q}' - E)}. \quad (22)$$

Now, this equation is used to define an inverse radius of propagation between two successive collisions.
Inverse radii analogous to $<\frac{1}{r}>_3B$ appear in many meson-deuteron models, in particular in pion-deuteron calculations. Later we use these to compare approximations and fix potential parameters.

To proceed any further, the nucleon-nucleon scattering matrix $t_{NN}(q, q', E)$ is needed. It may be expressed by the NN scattering amplitude via a relation analogous to Eq.(14). To generate this matrix a separable Yamaguchi potential $V_{NN} = v_{NN}(q)\lambda_{NN}v_{NN}(q')$ is used with form factors $v_{NN} = \kappa_{NN}^2/(q^2 + \kappa_{NN}^2)$. The two free parameters $\kappa_{NN}$ and $\lambda_{NN}$ are fixed to reproduce the deuteron binding and one of the following two parameters: the NN scattering length or the inverse deuteron radius $<\frac{1}{r}>$. The details and motivation for the latter choice are presented later. The $t_{NN}$ enters multiple scattering series of the $\Sigma_n$ and $\Delta_n$ type. As the relevant integrals are lengthy we reproduce only the leading term in the limit of zero range $\eta N$ forces and energy averaged $\bar{a}$.

$$\Sigma_1(p_i, E, p_f) \equiv 2\bar{a}_{\eta N} \frac{\mu_{\eta d}}{\mu_{\eta N}} \ll \frac{1}{r} >_3B.$$  

(23)

The unitarity condition for the on-shell scattering amplitude follows from Eq.(21). Let us concentrate on the energy region between the threshold and the deuteron breakup. At low energies $t_{NN}(E)$ is dominated by the deuteron pole and it may be presented as

$$t_{NN}(E) \approx \frac{N_d^2}{E_d + E_\eta(p) - E},$$

(27)

where $N_d$ is the deuteron wave function normalisation and $E_d$ is the deuteron binding energy. The absorptive part of $t_{NN}$ comes from this singularity. Below the breakup it guarantees the unitarity for the $\eta d$ scattering amplitude. To see that, one extracts the absorptive part of $t_{NN}(E)$ equal to $i\pi N_d^2\delta[E - E_\eta(p) - E_d]$ and performs the integration in Eq.(24) to obtain

$$Im \frac{\Sigma_1}{a_{\eta d}} = 2\frac{\mu_{\eta d}}{\mu_{\eta N}} F_d(p_\eta, p_\eta) p_\eta.$$  

(28)

The same procedure shows exact cancellation of the absorptive deuteron pole contributions to the triple scattering term $S_2$ in Eq.(18). In addition, an inspection of Eqs.(9), (12) shows
that there is no pole in the integrands of the $\Omega_n$ terms. In particular the singularity that enters $\Omega_1$ is related to the cut in $G_0$. This generates a branch point at the deuteron breakup threshold and absorptive contributions at higher energies. Altogether, one can present the meson-deuteron scattering amplitude in the form

$$A_{\eta d}(E) = \left[\frac{1}{K_{\eta d}(E)} - ip_\eta\right]^{-1}$$

(29)

as required by unitarity. Below the deuteron breakup threshold and for closed $\pi N$ channels the $K_{\eta d}(E)$ would be a real analytic function of energy. Above this threshold, the $\eta NN$ continuum induces an imaginary part to $K_{\eta d}(E)$. It comes mostly from $\Omega_1$, much less from $\Sigma_1$ and effects of higher orders are very small.

To describe final states in the $\eta d$ system a half-off-shell scattering amplitude is needed. It is obtained by relating $p_f$ and $E$, in the single and multiple scattering terms $F_d(p_i, p_f(E)), \Sigma_1(p_i, E, p_f(E)), \Omega_1(p_i, E, p_f(E))$ etc. In the region of our main interest: $E \approx E_d$ and $p_i$ in the 0 up to 1 fm$^{-1}$ range one finds a moderate (up to 25%) fall of $F_d(p_i, 0)$. On the other hand the other quantities $\Sigma_1(p_i, 0, E_d), \Omega_1(p_i, 0, E_d)$ are almost constant. This occurs since these functions are defined by the ratios of two functions that fall down with increasing $p_i$ in the same moderate way. The consequence is that the half-off shell amplitude may be presented as I square some round brackets in next eq.

$$A_{\eta d}[p_i, E, p_f(E)] = \frac{A_{\eta d}^0[p_i, E, p_f(E)]}{1 - \Sigma_1(E) - \Omega_1(E) - S_2(E)}.$$

(30)

For further use in the description of the $\eta d$ system in the process of $\eta d$ formation, Eq. (30) is extended into an operator form

$$\hat{A}_{\eta d} = \frac{\hat{A}_{\eta d}^0}{1 - \Sigma_1(E) - \Omega_1(E) - S_2(E)}.$$

(31)

The process of formation and final state interactions requires more elaborate calculations. Approximations are done, and to study the value of these we first discuss some standard approximations used in elastic meson-deuteron scattering.

**C. The static nucleon and deuteron dominance approximations**

In the literature these approximations appear in many forms. The common element is that intermediate states of the nucleon-nucleon pair are reduced to the state of the deuteron. Another, related approximation neglects the recoil energy of the spectator nucleon. Although these approximations are rather crude this method is useful to discuss limiting situations. Below, it is used to indicate a simple interpretation of the terms in our multiple scattering expansion.

The first double scattering contribution is the $\Omega_1$ term of Eq.(22). If the nucleon recoil $E_{NN}$ is neglected, one obtains at $E = 0$

$$\Omega_{\eta d}^{\text{static}} = \frac{\mu_{\eta d}}{\mu_{\eta N}} = \frac{1}{r},$$

(32)
where the relevant "inverse deuteron radius" is

\[ < \frac{1}{r} > = \int d\vec{r} \phi_d(r)^2. \] (33)

The \(< \frac{1}{r} >\) is a standard parameter in low energy pion deuteron scattering [35]. It describes the meson propagator at zero energy, averaged over the deuteron density. One finds a weak but significant dependence of this parameter on the NN potential used to calculate it. In particular \(< \frac{1}{r} > = 0.448/\text{fm}\) for the Paris NN potential [34], [35] and 0.463/\text{fm}\) for the Bonn potential [35], [36]. In this calculation the separable Yamaguchi potential is used which generates the Hulthen deuteron wave function. Two sets of potential parameters are tried. The standard choice for the potential strength and range reproduces the deuteron binding and \(pn\) scattering length (Yamaguchi I). These are obtained with \(\kappa_{NN} = 1.41 fm^{-1}\) and \(1/\lambda_{NN} = -0.52 fm\). However, this choice generates \(< \frac{1}{r} > = 0.559 fm^{-1}\), which is too large when compared to the superior calculations from the Bonn and Paris potentials. To improve the separable model its parameters are fixed to reproduce the deuteron binding and the average of \(< \frac{1}{r} >\) obtained with these two local potentials. This requires \(\kappa_{NN} = 0.91 fm^{-1}\) and \(1/\lambda_{NN} = -0.289 fm\) (Yamaguchi II). The \(np\) scattering length becomes 6.02\text{fm} instead of 5.40\text{fm} but, for this calculation, the deuteron radius is more important than the \(np\) scattering length. The results are given in Table 1, where the comparison with the exact three-body calculation is made. A factor of two difference is found. It comes from the large nucleon recoil entering the propagator in Eq.(22) and the deuteron binding. The latter is introduced into the three body calculation by the condition \(E = E_d\) (not \(E = 0\)) at the threshold.

Now, we proceed to study the other double scattering term \(\Sigma_1\). At very low energies the deuteron dominance is expressed via

\[ G_{NN}(q, E, q') \approx \frac{\phi_d(q)\phi_d(q')}{E - E_d - E_\eta(p)}. \] (34)

This formula greatly simplifies all the multiple scattering integrals. To present the argument, the threshold scattering (\(E = E_d\)) is considered. The dominant term \(\Sigma_1\) becomes

\[ \Sigma_1^{\text{static}} = 2\bar{a}_{\eta N} \frac{\mu_{\eta d}}{\mu_{\eta N}} \ll \frac{1}{r}, \] (35)

where another inverse deuteron radius is introduced

\[ \ll \frac{1}{r} = \int d\vec{r} d\vec{r}' \frac{\phi_d(r)^2\phi_d(r')^2}{|\vec{r}/2 - \vec{r}'/2|^2}. \] (36)

in close analogy to the more general three-body definition of Eq.(24). The factor of 2 in the denominator is essential, since \(\Sigma_1^{\text{static}}\) corresponds to the double scattering amplitude generated by an optical potential

\[ V^{\text{static}}(r) = -\frac{2\pi}{\mu_{\eta N}} [2\bar{a}_{\eta N}]^2 [2^3 \phi_d^2(2r)]. \] (37)

The strength of this potential \(2\bar{a}_{\eta N}\) is normalized to two nucleons. The profile \(\phi_d^2(2r)\) is "compressed" as it refers to the center of the deuteron, hence the argument \(2r\) and the
normalization factor $2^3$. $V_{\text{static}}$ gives the correct expression for the single scattering amplitude but misses badly the double scattering term. In the static approximation one obtains $\ll 1/r \gg = 0.83 \text{fm}^{-1}$ while the correct three-body value is $\ll 1/r \gg_{3B} = 0.32 \text{fm}^{-1}$ (Hulthén-Yamaguchi I wave function and zero range $\eta N$ force, see Table 1). This large difference is, again, related to the nucleon recoil in the three-body propagators. Now the propagator enters Eq. (24) twice and this makes the static approximation even worse than in the $<1/r>$ case.

For scattering from a system of $A$ particles, a "remedy" for this inconsistency has been found long ago by substituting the $AA$ factor by $A(A-1)$. For the deuteron this leads to a reduction of the double scattering term by one half. One can see in Table 1 that such a procedure brings the static optical potential model into reasonable consistency with the sum of $\Sigma_1$ and $\Omega_1$ terms.

D. The pionic channels

So far the effect of pionic channels has been hidden in the absorptive part of the elastic $\eta N$ scattering amplitudes. These describe processes related to a single nucleon. In addition there exist processes when an intermediate $\pi$ meson is exchanged between two nucleons. The effect of such intermediate states is expected to be small for two reasons. First, the transition from the $\eta$ to the $\pi$ meson involves a 300 MeV energy release. This energy is shared between the three participants $N, N$ and $\pi$. The bulk of the $NN$ interactions occur at fairly high energies where the $NN$ scattering amplitude is small. Second, at these energies the intermediate state propagator oscillates in space at distances much shorter than the deuteron radius. This suppresses the multiple scattering integrals. However, since the $NN\eta$ system is close to binding even small effects may matter. On general grounds one could expect that the $NN\pi$ channel which has a lower threshold would contribute some repulsion into the $NN\eta$ channel which has a higher threshold. That could reduce the chances for binding in the $NN\eta$ system.

With the explicit pions the series for the $\eta$-deuteron scattering matrix $\hat{T}_{\eta d}$ is supplemented by terms

$$\Delta \hat{T}_{\eta d} = \hat{h}_1 G_0 \hat{h}'_2 + \hat{h}_2 G_0 \hat{h}'_1 + (\hat{h}_1 + \hat{h}_2) G_{NN} (\hat{h}'_1 + \hat{h}'_2) + \ldots ,$$

(38)

where the scattering matrices $\hat{h}$ describe the $\eta N \rightarrow \pi N$ transitions and the $\hat{h}'$ describe $\pi N \rightarrow \eta N$ transitions. These are related to the scattering amplitudes via Eq. (14). However, the kinematic factor is now $\sqrt{\mu_{\eta N} \mu_{\pi N}}$. The small pion mass reduces the effect of these factors as the relevant ratio $\mu_{\pi N}/\mu_{\pi d}$ is close to unity.

The pion exchange scattering is described by

$$\Omega^\pi_1(p_i, E, p_f) = \frac{\bar{a}^2_{\eta N, \pi N}}{a_{\eta N} F_d(p_i, p_f)} I(m_\pi),$$

(39)

where $I(m_\pi)$ has the form of the integral in Eq. (22) but with the mass of the $\eta$ meson entering this integral being changed into the mass of the $\pi$ meson. The expression Eq. (39) corresponds to the free propagator part of the series in Eq. (38). The other terms are described by a formula similar to Eq. (24).
\[ \Sigma_1^\pi(p_i, E, p_f) = \frac{2\tilde{a}_{\eta N, \pi N}^2}{\tilde{a}_{\eta N} F_d(p_i, p_f)} I(m_\pi, m_\pi), \]  

where \( I(m_\pi, m_\pi) \) is given by the integral in Eq. (24) but with the mass of the \( \eta \) meson being replaced by the mass of the \( \pi \) meson.

A few numbers calculated at the \( \eta d \) threshold are now presented. These correspond to the older version of the \( \eta N \) model of Ref. [25]. The scattering parameters are: \( \tilde{a}_{\eta N} = 0.57 + i0.14 \text{fm} \) and \( \tilde{a}_{\eta N, \pi N} = 0.21 + i0.15 \text{fm} \), while the double scattering integrals are \( I(m_\pi) = (-3.5 + i14.9)10^{-2} \text{fm}^{-1} \) and \( I(m_\pi, m_\pi) = (6.89 - i3.93)10^{-3} \text{fm}^{-1} \). Thus the \( \Sigma_1^{\pi} \) is a negligible 1/pm correction to the multiple scattering. The other term \( \Omega_1^{\pi} \) is significant. Corresponding numbers for the elastic \( \eta N \) channel may be found in Ref. [23] and in Table I.

The calculation of \( A_{\eta d} \) without the \( \pi \) exchange effect and the precision obtained with the multiple scattering theory (MST) can be estimated by comparing with the solution of the Faddeev equations (F) obtained in Ref. [32] with the same input. For small \( a_{\eta N} \) these methods give essentially the same answer. At larger values, close to the binding situation, differences arise. These are given in Table II.

The MST method slightly underestimates the real parts and overestimates imaginary parts.

The dominant source of these differences is related to the use of energy averaged amplitudes \( \tilde{a}_{\eta N} \) in the second and higher orders scattering terms. No attempt is done here to correct for that. Also there are higher order effects and slight differences in the effective range corrections involved in these two models.

Our best result for the \( A_{\eta d} \) scattering length obtained with the elastic \( \eta N \) amplitude of Ref. [25] and the Yamaguchi II model for the NN interactions is \( A_{\eta d} = 2.31 + i1.73 \text{fm} \). This number should be compared with \( A_{\eta d} = 2.61 + i1.72 \text{fm} \) obtained in Ref. [32] from the Faddeev equations, using the same \( a_{\eta N} \) but a somewhat superior separable NN potential. The results given above do not include the pion exchange effects. If the pion exchange is introduced we obtain \( A_{\eta d} = 2.27 + i1.68 \text{fm} \). The change is rather small and corresponds to a repulsive effect. Qualitatively, such a behaviour has been predicted in Ref. [32] but the size of the effect obtained there is much larger and is not reproduced here. The difference may be related to the small values of \( \tilde{a}_{\eta N, \pi N} \) inherent to the K-matrix model of Ref. [27].

Eq. (39) indicates that \( \Omega_1^{\pi} \) is composed of three uncertain complex numbers and may be unstable with respect to the input values of \( \tilde{a}_{\eta N, \pi N} \) and \( \tilde{a}_{\eta N} \). Here, the averaged propagator \( I(m_\pi) \) has been calculated with non-relativistic kinematics as used in Ref. [32]. A relativistic expression for the pion energy would make the pion exchange effect even smaller.

Numerical results for the \( \eta d \) scattering amplitude \( A_{\eta d}(E) \) and the \( \eta d \) elastic cross section are given in Figs. 2a, 2b. The impulse approximation is valid at energies \( E > 15 \text{MeV} \) above the threshold. Multiple scattering effects enter at lower energies: in the intermediate region of \( E \approx 10 \text{MeV} \) it is \( \Omega_1 \) that dominates while for lower energies the main contributor is \( \Sigma_1 \). The real three-body exotics is seen only very close to the threshold for \( E < 1 \text{MeV} + E_d \). The effect is strong and very narrow. At present a direct way to see it experimentally is not available, and one has to study inelastic processes that involve the \( \eta d \) system in final states. One reaction of interest is \( pn \rightarrow d\eta \), with a related one being the \( pp \rightarrow pp\eta \). In the next section the deuteron reaction is discussed.
III. FINAL STATE INTERACTIONS IN THE $\eta D$ SYSTEM

The amplitude squared for the $pn \rightarrow \eta d$ process extracted from the CELSIUS experiment is given in Fig. 4. Three regions may be specified:

1. The ”asymptotic” region of $E > 20\text{MeV}$, where the final state interactions are small, can be described by the impulse approximation.

2. The ”interference” region of $E \approx 10\text{MeV}$, where the experimental amplitude indicates a shallow minimum.

3. The region of a ”virtual three body state” located below the deuteron breakup threshold.

In this section we find $F_{\eta d}$, the amplitude for the $pn \rightarrow d\eta$ reaction. It consists of several terms that differ in their physical content. In order to link this amplitude to the standard description of two-body final state interactions, it is presented in the form

$$F_{\eta d} = A^c[1 + \frac{A_{\eta d}(E)}{R_d(E)}]. \quad (41)$$

This expression consists of the on-shell $\eta$-deuteron scattering amplitude $A_{\eta d}(E)$ and a radius $R_d(E)$. For simple models, like the static nucleon approximation, a simple interpretation and a closed formula for $1/R_d(E)$ are found. The radius comes out as an interplay of other radii: the $\eta$ meson source radius, the $\eta N$ force range and the deuteron radius. For scattering on static as well as interactive NN pairs this radius makes sense only at very low energies. At energies exceeding about 10 MeV it becomes strongly energy dependent. The presentation of FSI in terms of Eq.(41) is still possible but the simple physical interpretation is lost.

The three regions indicated in Fig. 4 are specified for three different purposes. The main emphasis is put on the low energy enhancement and properties of the three-body state. There, it is the ratio $A_{\eta d}/R_d$ that represents the strength of the enhancement. In order to check on $A_{\eta d}$ one needs full control of $1/R_d$. The interference region indicates an energy dependent phase of $1/R_d(E)$ that together with the phase of $A_{\eta d}(E)$ produce a minimum. Finally the cross section at higher energies is used to fix $A^c$ which is an amplitude for coalescent formation of the deuteron. To calculate it, one needs a specific model of the meson formation. This question is discussed in the first subsection.

A. Meson formation amplitude

The formation of an $\eta$ meson in nucleon-nucleon collisions involves large momentum transfers and the mechanism of this process is uncertain. It is believed to be dominated by an intermediate $\rho$ meson exchange, although other mesons may also contribute [5], [40], [41], [42], [43]. In particular the recent calculation, [44], stresses the role of $\eta$ meson exchange. All models assume three stages for this process, namely, $NN \rightarrow NN(1535) \rightarrow NN\eta$, where the intermediate stage is dominated by the $N(1535)$ resonance. The first transition involves large momentum transfers, whereas the second is a low energy process.

This paper aims at the final state effects and the formation amplitude is described in a phenomenological way by a function of initial and final momenta $A^{form}$. What is essential for our study is to find a functional form for this amplitude and pinpoint the effects (if any)
that may generate a rapid energy and momentum dependence. Such a dependence would be equivalent to a long range space structure of the $\eta$ meson source and this structure is to be viewed on the scale of the deuteron radius.

The $\eta$-meson production model is illustrated in Fig. 3. It consists of a short ranged meson exchange part affected by initial NN interactions. As compared to the deuteron radius it is obviously a short range process, since it requires large initial NN c.m. momenta $Q_{NN} \approx 3.5 \text{ fm}^{-1}$ and comparable momentum transfers. This part of the production process may be described by an amplitude $(q_t^2 + \mu^2)^{-1}$, where $q_t$ is the momentum transfer and $\mu$ is the mass of the exchanged meson. Now, the momentum transfer may be expressed in terms of the initial and final momenta $q_t = -Q_{NN} - p_{1N}$, where the final nucleon momentum is $p_{1N} = \vec{q}_{NN} - \vec{p}_\eta/2$. For the final states of interest, $Q_{NN} \gg p_{1N}$ and this amplitude is almost constant. Projected into an $S$ wave it generates a momentum dependence $(Q_{NN}^2 + \mu^2 + p_{1N}^2)^{-1}$.

As might have been guessed already from the uncertainty principle, the range involved $1/\sqrt{Q_{NN}^2 + \mu^2}$ is less than 0.35 fm even for $\pi$-meson exchange. However, there is another part of the formation mechanism, related to the propagation of the $N(1535)$ resonance relative to the spectator nucleon. The propagation range is not negligible on the scale of the deuteron radius. It may be described by a function $G_r = \left[\Delta - i\Gamma(E)/2 + E_{rec}\right]^{-1}$, where $\Delta = E_r - M_N - M_\eta - E$, the nucleon recoil energy $E_{rec} = p_{1N}^2/2\mu_{\eta N,N}$, and $E_r - i\Gamma(E)/2$ are the resonance energy and width. It is the recoil energy that introduces a sizable momentum dependence in this propagator and generates a long range structure of the formation process. Altogether, the formation amplitude is assumed to be of the form

$$A_{\text{form}}(p_{1N}) = \frac{A_f 2\pi^2}{\kappa_s^2 + (p_{1N})^2}, \tag{42}$$

where $\kappa_s$ is the inverse radius of the $\eta$ formation source. The Fourier transform of this amplitude with respect to the nucleon momentum $\vec{p}_{1N}$ becomes

$$A_{\text{form}}(r) = \frac{A_f}{r} \exp(-\kappa_s r). \tag{43}$$

In general $\kappa_s$, given by the relation $\kappa_s^2 = 2\mu_{\eta N,N}[\Delta - i\Gamma(E)/2]$, is energy dependent but our main interest is the $\eta N$ threshold. The resonance position and half width of $1535 - i148/2$ MeV are taken from Ref. [25]. At the threshold, the decay of the $N(1535)$ into the $\eta N$ channel becomes negligible and the actual width there is about 60MeV. These parameters yield a complex value of $\kappa_s = 1.40 - i0.39 \text{ fm}^{-1}$. Strength of the meson formation amplitude $A_f$ is a free parameter. It is used to fix the calculated $pn \to \eta d$ cross section to the experimental one in the “asymptotic” region.

### B. Final state wave function

The full amplitude $F_{\eta d}$ for reaction $pn \to d\eta$ is given by an integral

$$F_{\eta d} = \int d\vec{q} d\vec{p} A_{\text{form}}(\vec{q} - \vec{p}/2) \Phi_{\eta d}^{-}(q, p), \tag{44}$$
where the nucleon momentum is expressed in terms of Jacobi NN and η momenta. In this equation \( \Phi_{\eta d} \) is a wave function for the final three-body system that fulfills the ingoing wave boundary condition. To find it, let us construct the multiple scattering series for the wave function similar to the series for the scattering matrix. From the Faddeev equations one obtains

\[
\Phi_{\eta d} = [1 + G_0 \hat{T}_{\eta d} + G_{NN} \hat{T}_{\eta d}] \phi_d \psi_\eta, \tag{45}
\]

where \( \hat{T}_{\eta d} \) is the scattering matrix operator given by the series in Eq. 4. Now the reaction amplitude contains three distinctly different terms

\[
F_{\eta d} = A^c + A^{ex} + A^{vs} \tag{46}
\]

that correspond to the three terms in the square bracket of Eq. (45). The first one is a direct, coalescence transition from the source to the deuteron:

\[
A^c = \int d\vec{q}d\vec{p} A^{form}(\vec{q} - \vec{p}/2)\phi_d(q)\psi_\eta(p). \tag{47}
\]

For an S-wave free meson final state of momentum \( \vec{p}_\eta \), and the specific choice of the formation amplitude in Eqs. (42,43) the coalescent amplitude is given by the integral

\[
A^c = \int d\vec{r} A^{form}(\vec{r})\phi_d(r)j_0(rp_\eta/2). \tag{48}
\]

The formation amplitude is short ranged in comparison to the deuteron radius. Thus the coalescent formation amplitude involves \( \phi_d(r \approx 0) \) and in the region of interest it depends weakly on the meson momentum.

The second term in Eq. (46) describes a part of the η meson final state interactions

\[
A^{ex} = \int d\vec{q}d\vec{p} A^{form} G_0(q,p) \hat{T}_{\eta d} \phi_d \psi_\eta. \tag{49}
\]

The superfix \( ex \) indicates that this term contains an η meson exchange diagram. Here the \( \eta NN \) system propagates freely from the formation source and this process is described by \( G_0 \). Next, the scattering series begins with the meson-spectator-nucleon collision. Thus the meson produced on one nucleon is exchanged and interacts with the other nucleon. After that the deuteron is built either in the coalescent manner or in subsequent NN interactions.

The last term \( (A^{us}) \) in Eq. (46) is our main interest. It describes the final state scattering process that begins with the NN interactions. These are the strongest interactions in the \( \eta NN \) system. For low energies the corresponding propagator \( G_{NN} \) is dominated by the state of the deuteron. The related amplitude is

\[
A^{us} = \int d\vec{q}d\vec{p} A^{form} G_{NN}(q,p) \hat{T}_{\eta d} \phi_d \psi_\eta. \tag{50}
\]

On top of the deuteron described by \( G_{NN} \) the system builds up the \( \eta NN \) virtual state \( (us) \) and this process is described by \( \hat{T}_{\eta d} \). It is to be seen as a narrow enhancement on the background of the standard coalescence term \( A^c \).

Before discussing the \( \eta NN \) collective state let us return to the meson exchange process and the related amplitude \( A^{ex} \) of Eq. (49). The final state scattering begins with the η-meson propagating from the source to the other nucleon and in leading order it is described
by $G_0 \hat{t}_{\eta N}$. To this order one can rearrange the scattering series in Eq. (4) and the final state wave function to the form

$$\Phi_{\eta d} = (1 + G_0 \hat{t}_{\eta N}) [1 + G_{NN} \hat{T}_{\eta d}] \phi_d \psi_\eta,$$

which reduces the FSI to the coalescent ($A^c$) and virtual state ($A^{vs}$) amplitudes. However, now the the $\eta$ formation amplitude is redefined to $A^{form} [1 + G_0 \hat{t}_{\eta N}]$. This formation process is of an extended range in space but is still much shorter than the deuteron radius. The initial meson exchange process involves nucleons correlated at short distances and thus requires large momenta and a large recoil. The relevant energies in the $\eta N$ system extend far below the threshold. There the $a_{\eta N}$ amplitude is not known. According to the model shown in Fig.1 it is expected to be real and small. Following this we terminate the exchange process at this level. The result of this analysis is that the unknown strength of the $\eta$ source is renormalized and its size may be somewhat larger than that predicted by the $N(1535)$ resonance model. In the numerical calculations a plausible value of $\kappa_s = 1.5 \text{ fm}^{-1}$ is used to describe the three factors that contribute to the size of the $\eta$ meson formation source.

The task now is to find the coupling of the virtual state to the final $\eta$ - deuteron channel, and again the method of Sect.II is used for this purpose. We begin with the simplest calculation.

### C. The deuteron dominance approximation

This approximation has been systematically used in all calculations of the $\eta$ final state interactions on light nuclei. In this section we discuss its applicability in the deuteron case. Ultimately it is shown that the use of static nucleons is a bad approximation. On the other hand, this method is useful to discuss some approximations. We take advantage of that.

At very low energies the deuteron dominance of $G_{NN}$ is expressed by Eq.(34). This formula inserted into Eq.(50) gives $A^{vs}$ in terms the scattering matrix $T_{\eta d}$. One finds

$$A^{vs} = \int d\vec{q} d\vec{p} \ A^{form}(\vec{q} - \vec{p}/2) \phi_d(q) \frac{T_{\eta d}(\vec{q}, E, p_\eta)}{E_d + E_\eta(p) - E}. \quad (52)$$

The right hand side of this equation expresses $A^{vs}$ in terms of the half-off-shell $T_{\eta d}(\vec{q}, E, p_\eta)$ scattering matrix. The problem that arises at this stage is the off-shell continuation of $T_{\eta d}$. As discussed in the previous section the main dependence of the off-shell extrapolation comes essentially from the numerator of Eq.(18) i.e. the deuteron formfactor of Eq.(19). Within this approximation it is easy to extend the scattering to the case of intermediate mesons which are described by plane waves. The virtual-state amplitude in Eq.(52) may be presented in the coordinate representation. With the use of the intermediate meson propagator

$$\int d\vec{p} \frac{\exp(i\vec{p} \vec{R})}{(2\pi)^2 \mu_{\eta d} [E_d + E_\eta(p) - E]} = \frac{\exp(i\eta \vec{R})}{\vec{R}} \quad (53)$$

and Eqs.(18,19) one obtains
$A^{\text{vs}} = \frac{A_{nd}(E)}{F_d(p_\eta, p_\eta)} \int d\vec{r} d\vec{r}' A^{\text{form}}(r') \phi_d(r') \frac{\exp[ip_\eta |\vec{r}/2 - \vec{r}'/2|]}{|\vec{r}/2 - \vec{r}'/2|} \phi_d^2(r) j_0(rp_\eta/2).$  

(54)

The basic amplitudes $A^c$ and $A^{\text{vs}}$ are now easy to calculate in two extreme cases: of a zero radius and a large radius meson source. For the zero radius source given by $A^{\text{form}}(r') = A^f \delta(\vec{r}')$ one obtains the coalescence amplitude

$$A^c = A^f \phi_d(r = 0),$$  

(55)

and the total amplitude which includes final state interactions

$$F_{nd} = A^c + A^{\text{vs}} = A^c[1 + \frac{A_{nd}(E)}{R_d(E)}],$$  

(56)

where the inverse interaction radius is defined as

$$\frac{1}{R_d(E)} = \frac{1}{F_d(p_\eta, p_\eta)} \int d\vec{r} \phi_d^2(r) \frac{\exp[ip_\eta r/2]}{r/2} j_0(rp_\eta/2).$$  

(57)

With the zero range source and static nucleons the radius $R_d(E)$ is due entirely to the deuteron size. For energies close to the threshold Eq. (57) relates it the previously defined static inverse radius

$$\frac{1}{R_d} = 2 < \frac{1}{r} > +ip_\eta + O(p^2_\eta).$$  

(58)

Numerically one obtains $\text{Re} 1/R_d = 1.12 \, fm^{-1}$ for the Hulthén I deuteron wave function. The expansion in Eq. (58) adds a simple imaginary part to $< \frac{1}{r} >$. For static nucleons this is an exact expression as may be easily checked with Eq. (20) for the deuteron formfactor. For the FSI in two-body systems described by a potential, such an imaginary term is due to the imaginary part of the outgoing wave $\exp(ipr)/r$. It gives rise to an interference effect in the square-bracketed term of Eq. (56). By splitting the FSI amplitude into real and imaginary parts one obtains

$$|F_{nd}|^2 = |A^c|^2 \left[1 + \text{Re}A_{nd} < \frac{1}{r} > -p_\eta \text{Im}A_{nd}]^2 + [\text{Im}A_{nd} < \frac{1}{r} > +p_\eta \text{Re}A_{nd}]^2\right].$$  

(59)

As the amplitudes in this equation are positive, one notices that, with increasing momentum $p_\eta$ and constant amplitudes, the first term in Eq. (59) decreases and the second term increases. This leads to a minimum in the amplitude. Such a minimum is indicated by the experimental data in Fig. 4 in the region above the enhancement, where the meson deuteron scattering amplitude becomes more stable.

The static nucleon approximation is convenient for the other simple limit of a large meson source. If $A^{\text{form}}(r') = A^f \phi_d(r)$, i.e. the source is as large as the deuteron, then the low energy expansion is given by formula (56)

$$\frac{1}{R_d} = \ll \frac{1}{r} \gg +ip_\eta + O(p^2_\eta).$$  

(60)

The radius $R_d$ given by this equation is sizably larger as one finds $\frac{1}{R_d} = 0.83 \, fm^{-1}$. The two examples given here indicate the sensitivity of the final state interactions to the source.
radius. For a more realistic source determined by the propagation of the \( N^* \) resonance via Eq. (33) we find \( \text{Re} \frac{1}{R_d} = 0.90 \text{fm}^{-1} \), at the threshold.

However, static nucleons are a rather poor approximation. The results are given in Table III, but as may be guessed from Table I the values of \( 1/R_d \) are likely to be overestimated by a factor of 2 to 3. To improve this situation it is necessary to analyze the dynamic deuteron formation via the \( NN \) interactions. This is studied in the next section.

D. A dynamic formation of the deuteron

The main approximation behind Eq. (34) is a reduction of three body propagator to a quasi two body propagator. On the other hand, a correct expression for the \( NN \) propagator, that follows from the definition in Eq. (3) is

\[
G_{NN}(q, q', E) = \frac{v_{NN}(q)t_{NN}[E - E_\eta(p)]v_{NN}(q')}{[E_{NN}(q) + E_\eta(p) - E][E_{NN}(q') + E_\eta(p) - E]}. \tag{61}
\]

For the threshold energy \( E = E_d \) the denominators in the three body propagator involved in this equation have no singularities. Together with the form factors \( v_{NN} \) these generate the deuteron wave function

\[
\phi_d(q) = \frac{N_d v_{NN}(q)}{E_{NN}(q) - E_d} \tag{62}
\]

provided the energies \( E_\eta(p) \) in the denominators are dropped. The normalization factor \( N_d \) results from the residue of \( t_{NN} \) at the deuteron pole. Thus the static nucleon approximation of Eq. (31) consists in dropping the \( E_\eta(p) \) in the propagator and reducing the \( t_{NN} \) to a bare singularity of Eq. (27). These two approximations, in part, tend to compensate each other. Roughly, the two energies in the denominators \( E_\eta(p) \) and \( E_{NN}(q) \) are comparable and the deuteron dominance overestimates \( G_{NN}^2 \) by a factor of \( 2^2 \). This happens because the deuteron is a weakly bound object and the binding energies in the denominators are small in comparison to the recoil energies. On the other hand by limiting \( t_{NN} \) to the pole term one neglects a sizable attraction that exists in the \( NN \) system at large negative energies. The pure pole term of Eq. (27) underestimates \( t_{NN} \) by a factor of \( 1/2 \). The net effect for \( G_{NN} \) is a reduction to about a half of its static nucleon values. The details depend on the intermediate momenta involved in this process.

In the dynamic approach the virtual state amplitude \( A^{vs} \) is given by an updated version of Eq. (52), where now the deuteron pole expression for \( G_{NN} \) given by Eq. (34) is substituted by the full expression of Eq. (61). In addition, the half-off-shell \( \eta \) deuteron scattering amplitude given by Eq. (30) is used. The full \( A^{vs} \) is given by an integral similar to the integral in Eq. (24) for the \( \Sigma_1 \). However, in Eqs. (24), (25) one of the deuteron wave functions \( \phi_d(\vec{q} - \frac{1}{2}\vec{p}) \) should be replaced by the source function \( A^{form}(\vec{q} - \frac{1}{2}\vec{p}) \). In the limit of the deuteron domination this procedure returns back to the simple equation (52). If one wishes to present \( A^{vs} \) in the form of Eq. (41), then the inverse radius \( \frac{1}{R_d} \) becomes a ratio of two integrals. At the threshold, the effects of the three body propagator and source size reduce this quantity to \( \frac{1}{R_d(0)} = 0.46 \). This number is more than a factor of 2 smaller than that
given in the zero range source, deuteron dominance model. Some other values calculated for two NN potentials and two source ranges are given in Table III. It is also found that the energy dependence of $R_d(E)$ is more moderate than that given by Eq. (57). This reflects the significance of the attraction in $t_{NN}$ at negative energies below the deuteron pole.

IV. RESULTS

The elastic $\eta d$ cross section is plotted in Fig. 2a for two values of the inverse scattering parameters $<\frac{1}{r}>$. These parameters differ by 15% only, but the effect on the cross section is significant. It reflects proximity of the $\eta d$ quasibound state. A similar sensitivity is displayed by the cross section on the value of the $\eta N$ scattering lengths. A few possibilities are plotted in Fig. 2b. These differ by the input scattering lengths $a_{\eta N}$ and effective ranges. Since the cross section is not measured, it cannot be a test for $a_{\eta N}$. The point of interest is the relation of the peak in the cross section to the position of the quasibound singularity in the complex $p_{\eta}$ plane. This singularity is found as the position of zero in the denominator of Eq. (21). An approximate solution for $a_{\eta N} = 0.75 + i0.27\text{fm}$ case is $p_{\eta v} = -0.09 - i0.09\text{fm}^{-1}$ and it is located in the third quadrant of the complex plane. This corresponds to a virtual-state, that is analogous to the spin singlet proton-neutron state. Coupling to the pionic channels pushes the singularity from the negative imaginary semi-axis into the third quadrant of the $p_{\eta}$ plane. This singularity is far from the region where quasi-bound $\eta d$ states might exist. Those would be analogous to the deuteron. To enter the quasi-bound state region with the model discussed here, in particular with the subthreshold amplitudes shown in Fig. 1, one would need $Re a_{\eta N} > 1.2\text{fm}$. The K-matrix model [29] based on two body data allows scattering lengths larger than 0.75 fm, up to about 1.05 fm. The corresponding $\eta - d$ amplitudes are given in Fig. 2b. However, as discussed below the large values close to 1 fm seem to be excluded by the $\eta d$ final state interactions.

In Fig. 4 the CELSIUS data [11] are plotted and compared with calculations. The cross section around the 10 MeV excess energy region is used to fix our normalisation constant $A'$. The choice of fixing point is not particularly relevant, provided it is done above the low few-MeV energy region. However, as is seen from the figure, our calculations are on the lower side of the experiment for $E > 15\text{MeV}$ and on the upper side for $E < 3\text{MeV}$. The 9 MeV point chosen helps to make a good overall fit to the data. At low energies the experimental energy resolution is vital. The experimental points refer to the middle of the corresponding energy bins. No attempt is made to fold the calculated cross section over the experimental energy resolution. At low energies and in the enhancement region this cross section is a check on the $\eta N$ model. The data and the results calculated with the Yamaguchi II NN potential and $\eta N$ interactions of Ref. [25] are consistent. The same data with the experimental errors enlarged by the beam energy uncertainty (0.3 - 0.5 MeV) could also accommodate the results obtained with the Yamaguchi I NN potential. Large values of $a_{\eta N}$ would not be allowed, at least with the subthreshold behavior generated by this K-matrix model.

The position of the minimum in the scaled cross section comes out as an interference effect. It depends on details of the energy dependent $\eta d$ scattering amplitude. However, with the present precision of the data and uncertainties of the input, it does not present a test for the consistency of the data and calculations.
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TABLE I. Inverse deuteron radii (propagator at zero energy averaged over deuteron wave functions) calculated with different deuteron wave functions and/or different propagators for the $\eta N N$ system. The $<>$ entries correspond to free continuum states: $<>$ static nucleons, $<>_{3B}$ 3-body propagator. The $\ll<>\gg$ entries correspond to intermediate NN interactions: $\ll<>\gg$ projection onto the deuteron state, $\ll<>_{3B}\gg$ complete NN interactions. The three-body entries do not include effects of a finite $\eta N$ force range. With $\kappa_{\eta N} = 3.3 fm^{-1}$ these should be reduced: $<>_{3B}$ by 18% and $\ll<>_{3B}\gg$ by 13%. All entries are in $fm^{-1}$

| Method                  | $<\frac{1}{R_d}>$ | $<\frac{1}{R_d}>_{3B}$ | $\ll<>\gg$ | $\ll<>_{3B}\gg$ |
|-------------------------|------------------|------------------------|------------|----------------|
| Paris                   | 0.449            | 0.197                  | 0.721      | 0.280          |
| Bonn                    | 0.463            | 0.254                  | 0.831      | 0.324          |
| Hulthén II              | 0.456            | 0.559                  |            |                |
| Hulthén I               | 0.559            |                       |            |                |

TABLE II. Comparison between the Multiple Scattering Theory (MST) used in this paper and the Faddeev(F) results for $A_{\eta d}$ – see Ref. [32]

| $\eta N$ (fm) | $A_{\eta d}$ (fm)(MST) | $A_{\eta d}$ (fm)(F) |
|---------------|------------------------|-----------------------|
| 0.44+i0.30    | 1.01+i1.50             | 1.03+i1.49            |
| 0.62+i0.30    | 1.65+i2.41             | 1.71+i2.34            |
| 0.888+i0.274  | 2.37+i5.79             | 2.65+i5.48            |

TABLE III. Inverse radii $<\frac{1}{R_d}>$ used for the description of $\eta d$ final state interactions at zero energies. All entries are in $fm^{-1}$

| Method                  | $\eta$ source         | Hulthén II | Hulthén I |
|-------------------------|-----------------------|------------|-----------|
| Deuteron Dominance      | Zero range            | 0.91       | 1.12      |
|                         | "                     | 0.75       | 0.90      |
| Deuteron Formation      | Zero range            | 0.504      | 0.555     |
|                         | "                     | 0.465      | 0.528     |
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FIG. 1. The $\eta N \rightarrow \eta N$ amplitude $a_{\eta N}$ in fm as a function of the center-of-mass energy $E_{C.M.}$ in MeV. Solid line for Re $a_{\eta N}$ and dotted line for Im $a_{\eta N}$.
FIG. 2. $|A_{\eta d}(E)|^2$ in fm$^2$ as a function of $E$(MeV) the $\eta - d$ kinetic energy in the center-of-mass system. Figure a) shows the results for Re $a_{\eta d}$=0.75 fm but with two different NN potentials: solid line – Yamaguchi II and the dashed line Yamaguchi I. Figure b) shows the results for Yamaguchi II and Re $a_{\eta N}$ = 0.75 fm (solid line), 0.87 fm (dotted line), 1.05 fm (dashed line) and 0.21 fm (long-dashed line) from $^{25}$ $^{29}$. 
FIG. 3. The $\eta$-formation reaction
FIG. 4. The reduced cross section \( \tau = \sigma(pn \rightarrow \eta d)/p_\eta(E) \) from Ref. [11] normalised to unity at 9 MeV. The theoretical equivalent \(|F_{\eta d}(E)|^2/|F_{\eta d}(9 \text{ MeV})|^2\) is calculated for \(\text{Re } a_{\eta N} = 0.75 \text{ fm} \) (solid line), 0.87 fm (dotted line), 1.05 fm (dashed line) and 0.21 fm (long-dashed line) from [25] [29].