Unified View on Lévy White Noises: General Integrability Conditions and Applications to Linear SPDE

July 31, 2018

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Abstract

There exists several ways of constructing Lévy white noise, for instance are as a generalized random process in the sense of I.M. Gelfand and N.Y. Vilenkin, or as an independently scattered random measure introduced by B.S. Rajput and J. Rosinski. In this article, we unify those two approaches by extending the Lévy white noise $\dot{X}$, defined as a generalized random process, to an independently scattered random measure. We are then able to give general integrability conditions for Lévy white noises, thereby maximally extending their domain of definition. Based on this connection, we provide new criteria for the practical determination of this domain of definition, including specific results for the subfamilies of Gaussian, symmetric-$\alpha$-stable, Laplace, and compound Poisson noises. We also apply our results to formulate a general criterion for the existence of generalized solutions of linear stochastic partial differential equations driven by a Lévy white noise.

\textit{MSC 2010 subject classifications:} 60G20, 60G57, 60G51, 60H15, 60H40

1 Introduction

1.1 Three Constructions for the Lévy White Noise

This paper is dedicated to the study of $d$-dimensional Lévy white noises. These entities have been defined in several ways and are available in the literature as follows.

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Partially supported by the Swiss National Foundation for Scientific Research and the European Research Council under Grant H2020-ERC (ERC grant agreement No 692726 - GlobalBioIm).
• Lévy white noise as a generalized random process: A generalized random process can be a priori observed though test functions $\varphi$ living in the space $\mathcal{D}(\mathbb{R}^d)$ of compactly supported and smooth functions. The construction of Lévy white noises as generalized random processes is established by I.M. Gelfand and N.Y. Vilenkin in [23, Chapter III]. It means that the Lévy white noise $\dot{X}$ is defined by the collection of random variables $\langle \dot{X}, \varphi \rangle$ with $\varphi \in \mathcal{D}(\mathbb{R}^d)$, as presented in Section 2.

The distributional point of view of Gelfand and Vilenkin offers the advantage of allowing a proper definition of Lévy white noise as a valid (random) generalized function. It can then be used as the driving term of a linear (stochastic) partial differential equation of the form $L_s = \dot{X}$.

• Lévy white noise as an independently scattered random measure: In this framework, the Lévy white noise is specified as a random measure; that is, a collection of random variables $\langle \dot{X}, 1_A \rangle$ indexed by Borelian subsets $A \subset \mathbb{R}^d$ with finite Lebesgue measure. Independently scattered random measures are investigated by B.S. Rajput and J. Rosinski in [33]. We provide a recap in Section 3.

• Lévy white noise and Itô-type stochastic integrals: This approach relies on the Lévy-Itô decomposition of Lévy processes. The starting point is the $d$-parameter Lévy sheet $X = (X_t)_{t \in \mathbb{R}^d}$ that generalizes Lévy processes in higher dimensions. The Itô theory of stochastic integration relies on the Lévy-Itô decomposition of a Lévy sheet (see Theorem 4.3). It gives a meaning to quantities of the form $\int_{\mathbb{R}^d} f(s) dX_s$ for well-suited deterministic functions $f$. In particular, with $f = 1_{[0,t]}$, one recovers the original Lévy sheet $\int_{\mathbb{R}^d} 1_{[0,t]}(s) dX_s = X_t$.

Strictly speaking, the Lévy white noise is not constructed but rather only suggested in the notation $dX$. It can nevertheless be defined as the partial derivative (in the sense of generalized functions) $\dot{X} = \frac{\partial^{d}}{\partial t^{d}} X$ of $X$. This construction was for instance presented in [14] and is reviewed in Section 4.

It can be argued that the specification of the Lévy white noise as an independently scattered random measure is more informative than its description as a generalized random process. Indeed, random measures are random generalized functions, while the converse in generally not true. However, the framework of generalized random processes is especially adapted for the study of linear stochastic partial differential equation. In particular, the (weak) derivative together with the integration (when it is well-defined) of a generalized random process is still a generalized random process, what is false in general for independently scattered random measures.

Despite these differences in formalisms, these three constructions are deeply interconnected and implicitly specify the same mathematical object, although observing it from different perspectives. They range from more to less general, in the sense that measures generalize functions, and generalized functions generalize measures [38, Chapter 1]. This remains valid for the random functions, measures, and generalized functions. Each approach brings its own advantages. It is therefore interesting to precisely connect them in order to benefit from their strengths when applied to different contexts.
1.2 The Domain of Definition of the Lévy White Noise

Since we are interested in the study of linear stochastic partial differential equations driven by Lévy white noise, we start from a Lévy white noise defined as a generalized random process. Any Lévy white noise \( \dot{X} \) can \textit{a priori} be observed through a test function \( \varphi \) living in the space of compactly supported and smooth functions. However, it is often desirable in practice to extend the definition of the family of random variables \( \langle \dot{X}, f \rangle \) to functions \( f \) that are possibly neither smooth nor compactly supported. Hereafter, we provide some motivations in that direction.

- **Expansion of Lévy white noise into orthonormal bases:** Consider an orthonormal basis \( (f_n) \) of \( L^2(\mathbb{R}^d) \) and a Lévy white noise \( \dot{X} \). We would like to characterize when it is reasonable to consider the family of coefficients \( \langle \dot{X}, f_n \rangle \). As a motivational example, we mention our recent works [19] where we use the Daubechies wavelets coefficients of a Lévy white noise to accurately estimate its regularity. Daubechies wavelets are compactly supported, but have a limited smoothness [17]. We shall see that the expansion on any Daubechies wavelet basis is possible for any Lévy white noise. More generally, we are interested in considering bases with elements that are neither compactly supported nor smooth.

- **Localizing the probability law of the Lévy white noise:** The domain of definition of Lévy white noise coincides with the domain of continuity of its characteristic functional. There are strong connections between the continuity properties of the characteristic functional and the localization of the process in appropriate Sobolev spaces, as exploited in [21, Proposition 4]. The more we can extend the domain of definition, the more we learn about the regularity of the Lévy white noise.

- **Construction of solutions of linear SDEs driven by Lévy white noise:** By extending the domain of definition of the Lévy white noise, one weakens the conditions on a differential operator \( L \) such that the stochastic differential equation \( Ls = \dot{X} \) admits a solution \( s \) as a generalized random process. Indeed, we have formally that, for \( \varphi \in D(\mathbb{R}^d) \),

\[
\langle s, \varphi \rangle = \langle L^{-1} \dot{X}, \varphi \rangle = \langle \dot{X}, (L^{-1})^* \{ \varphi \} \rangle,
\]

where \((L^{-1})^*\) is the adjoint of \(L^{-1}\). We therefore see that \((L^{-1})^* \{ \varphi \}\) must belong to the domain of definition of \( \dot{X} \) in order to give a meaning to (1.1).

The previous examples show the interest of extending the domain of definition of the Lévy white noise. We also want to go further and identify the broadest set of test functions such that the random variable \( \langle \dot{X}, f \rangle \) is well-defined. It turns out that the theory of Rajput and Rosinski is especially relevant to achieving this goal.

1.3 Contributions and Outline

The primary contributions of this paper are as follows.
• The connection of the Lévy white noise in $\mathcal{D}'(\mathbb{R}^d)$ with the theory of independently scattered random measures investigated by B.S. Rajput and J. Rosinski [33]. We rely on the work of those authors to identify in full generality the domain of definition of the Lévy white noise, along with the domain of definition with finite $p$-th moment (Section 3).

• The interpretation of this identification in terms of stochastic integration (Section 4). We show that there exists an integral representation for functions that are part of the domain of definition, and that, in general, this representation strictly includes the usual space of Itô-integrable functions associated to a Lévy white noise.

• The specification of practical criteria for the identification of the domain of definition of large classes of Lévy white noise (Section 5). This includes Gaussian, $\alpha$-stable, Laplace, and compound Poisson white noise.

• The application of our results for solving non-Gaussian linear stochastic partial differential equations (Section 6).

2 Generalized Random Processes and Lévy White Noises

The theory of generalized random processes was initiated in the 50’s independently by I.M. Gelfand [22] and K. Itô [26]. It has the advantage of allowing the construction of a broad class of random processes, including many instances that do not admit a pointwise representation. Generalized random processes are typically used as a general framework for the scaling limits of statistical models in conformal field theory [1, 2], where the continuous-domain limit fields are typically too irregular to admit a pointwise representation [9, 11]. The framework also lends itself to the construction of the $d$-dimensional Gaussian white noise, as is exploited in white noise analysis [24, 25] or for more general classes of Gaussian processes [30, 39, 47]. More generally, one can describe $d$-dimensional Lévy white noises—including Gaussian ones—as random elements in the space of generalized functions [23]. We can then study linear stochastic partial differential equations driven by Lévy white noise, whose solutions are defined as generalized random processes. This framework has been applied in signal processing in order to specify stochastic models for sparse signals [18, 43].

2.1 Generalized Random Processes

We denote by $\mathcal{D}(\mathbb{R}^d)$ the space of infinitely differentiable and compactly supported functions, associated with its usual nuclear topology [11]. In particular, $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$ as $n \to \infty$ if there exists a compact $K \subset \mathbb{R}^d$ that contains the support of all the $\varphi_n$ and if $\|D^\alpha\{\varphi_n - \varphi\}\|_\infty \to \infty$ for every multi-index $\alpha \in \mathbb{N}^d$. The topological dual of $\mathcal{D}(\mathbb{R}^d)$ is the space of generalized functions $\mathcal{D}'(\mathbb{R}^d)$ (often called distributions). We define a structure of measurability on $\mathcal{D}'(\mathbb{R}^d)$ by considering the cylindrical $\sigma$-field $\mathcal{C}$; that is, the $\sigma$-field generated by the cylinders

$$\left\{ u \in \mathcal{D}'(\mathbb{R}^d), \; \langle u, \varphi_1 \rangle, \ldots, \langle u, \varphi_N \rangle \in B \right\}$$
with \( N \geq 1, \varphi_1, \ldots, \varphi_N \in \mathcal{D}(\mathbb{R}^d) \), and \( B \) a Borel measurable set of \( \mathbb{R}^N \).

Let \((\Omega, \mathcal{F}, \mathcal{P})\) be a complete probability space. A random variable is a measurable function from \( \Omega \) to \( \mathbb{R} \). The space of random variables, \( \mathcal{L}^0(\Omega) \), is a Fréchet space when endowed with the convergence in probability. We also define \( \mathcal{L}^p(\Omega) \), the space of random variables with finite \( p \)-th moment, for \( 0 < p < \infty \). The space \( \mathcal{L}^p(\Omega) \) is a quasi-Banach space when \( 0 < p < 1 \), and a Banach space otherwise.

**Definition 2.1.** A generalized random process is a linear and continuous function \( s \) from \( \mathcal{D}(\mathbb{R}^d) \) to \( \mathcal{L}^0(\Omega) \). The linearity means that, for every \( \varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^d) \) and \( \lambda \in \mathbb{R} \),

\[
s(\varphi_1 + \lambda \varphi_2) = s(\varphi_1) + \lambda s(\varphi_2) \quad \text{almost surely.}
\]

The continuity means that if \( \varphi_n \to \varphi \) in \( \mathcal{D}(\mathbb{R}^d) \), then \( s(\varphi_n) \) converges to \( s(\varphi) \) in probability.

Due to the nuclear structure on \( \mathcal{D}(\mathbb{R}^d) \), a generalized random process has a version that is a measurable function from \((\Omega, \mathcal{F})\) to \((\mathcal{D}'(\mathbb{R}^d), \mathcal{C})\) (see \[17, Corollary 4.2\]). In other words, a generalized random process is a random generalized function. We therefore write \( s(\varphi) = \langle s, \varphi \rangle \) where \( s \) is a generalized random process and \( \varphi \in \mathcal{D}(\mathbb{R}^d) \) a test function.

**Definition 2.2.** The probability law of a generalized random process \( s \) is the probability measure on \( \mathcal{D}'(\mathbb{R}^d) \) defined by

\[
\mathcal{P}_s(B) = \mathcal{P}(s \in B) = \mathcal{P} \{ \omega \in \Omega, \langle s, \cdot \rangle(\omega) \in B \}
\]

for \( B \) in the cylindrical \( \sigma \)-field \( \mathcal{C} \).

Note that \( \{ s \in B \} \) is a measurable set of \( \Omega \) because \( s : \Omega \to \mathbb{R} \) is measurable by definition.

**Definition 2.3.** The characteristic functional of a generalized random process \( s \) is the Fourier transform of its probability law; that is, the functional \( \hat{\mathcal{P}}_s : \mathcal{D}(\mathbb{R}^d) \to \mathbb{C} \) defined by

\[
\hat{\mathcal{P}}_s(\varphi) = \int_{\mathcal{D}'(\mathbb{R}^d)} e^{i\langle u, \varphi \rangle} d\mathcal{P}_s(u) = \mathbb{E} \left[ e^{i\langle s, \varphi \rangle} \right].
\]

The characteristic functional characterizes the law of \( s \) in the sense that two random processes are equal in law if and only if they have the same characteristic functional.

**2.2 Lévy White Noises, Lévy Exponents, and Lévy Triplets**

One of the advantages of the theory of generalized random processes is that it allows to properly define processes that do not admit a pointwise representation. The typical example is the Gaussian white noise. Following \[23\], we define the Lévy white noise from its characteristic functional.

A random variable \( Y \) is said to be infinitely divisible if it can be decomposed as \( Y = Y_1 + \ldots + Y_N \) with \( (Y_1, \ldots, Y_N) \) i.i.d. for every \( N \). The characteristic function of an infinitely divisible random variable cannot vanish and there exists a unique continuous function \( \psi \) such that \( \hat{\mathcal{P}}_Y(\xi) = \exp(\psi(\xi)) \) (see \[36, Theorem 8.1\]). The log-characteristic function \( \psi \) of an infinitely divisible random variable is called a Lévy exponent (or a characteristic exponent). The complete family of Lévy exponents is specified by the Lévy-Khintchine theorem.
Theorem 2.4 (Theorem 8.1, [36]). A function $\psi : \mathbb{R} \to \mathbb{C}$ is a Lévy exponent if and only if it can be written as
\[
\psi(\xi) = i\gamma\xi - \frac{\sigma^2\xi^2}{2} + \int_{\mathbb{R}} (e^{i\xi x} - 1 - i\xi x 1_{|x| \leq 1}) \nu(dx),
\]
for every $\xi \in \mathbb{R}$, with $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, and $\nu$ a Lévy measure, that is a measure on $\mathbb{R}$ with $\int_{\mathbb{R}} (1 + x^2)\nu(dx) < \infty$ and $\nu(0) = 0$. The triplet $(\gamma, \sigma^2, \nu)$ is uniquely determined and called the Lévy triplet associated to $\psi$.

Definition 2.5. A Lévy white noise $\dot{X}$ is a generalized random process with characteristic functional of the form
\[
\hat{\mathbb{P}}_X(\phi) = \exp\left(\int_{\mathbb{R}^d} \psi(\phi(t))dt\right)
\]
for every $\phi \in \mathcal{D}(\mathbb{R}^d)$, where $\psi$ is a Lévy exponent.

The existence of a Lévy white noises as generalized random processes is proved in [23] (see also [18, Theorem 2]). A Lévy white noise $\dot{X}$ is stationary, meaning that $\dot{X}(\cdot - t_0)$ and $\dot{X}$ has the same law for every $t_0 \in \mathbb{R}^d$ (here, $\dot{X}(\cdot - t_0)$ is an abuse of notation, and is defined by $\langle \dot{X}(\cdot - t_0), \phi \rangle := \langle \dot{X}, \phi(\cdot + t_0) \rangle$ for any $\phi \in \mathcal{D}(\mathbb{R}^d)$). Moreover, $\dot{X}$ is independent at every point in the sense that $\langle \dot{X}, \phi_1 \rangle$ and $\langle \dot{X}, \phi_2 \rangle$ are independent when $\phi_1$ and $\phi_2$ have disjoint supports.

The random variable $\langle \dot{X}, \phi \rangle$ is a priori well-defined for $\phi \in \mathcal{D}(\mathbb{R}^d)$. Its characteristic function $\Phi_{\langle \cdot, \phi \rangle} : \mathbb{R} \to \mathbb{C}$ is given, for every $\xi \in \mathbb{R}$, by
\[
\Phi_{\langle \cdot, \phi \rangle}(\xi) = \mathbb{E}\left[e^{i\xi \langle \dot{X}, \phi \rangle}\right] = \exp\left(\int_{\mathbb{R}^d} \psi(\xi \phi(t))dt\right).
\]
However, we can reasonably extend the domain of definition of the noise to broader classes of functions.

We illustrate this idea on the Gaussian white noise $\dot{X}_{\text{Gauss}}$. Its characteristic functional is [23]
\[
\hat{\mathbb{P}}_{X_{\text{Gauss}}}(\phi) = \exp(-\sigma^2\|\phi\|_2^2/2).
\]
For each $\phi \in \mathcal{D}(\mathbb{R}^d)$, $\langle \dot{X}_{\text{Gauss}}, \phi \rangle$ is a centered normal random variable with variance $\sigma^2\|\phi\|_2^2$. One sees easily that $\langle \dot{X}_{\text{Gauss}}, f \rangle$ can be extended to every function $f \in L^2(\mathbb{R}^d)$. To do so, consider a sequence $(\phi_n)$ of functions in $\mathcal{D}(\mathbb{R}^d)$ converging to $f \in L^2(\mathbb{R}^d)$ for the usual Hilbert topology of $L^2(\mathbb{R}^d)$. Then, for every $n, m \in \mathbb{N}$, we have
\[
\mathbb{E}[\langle \dot{X}_{\text{Gauss}}, \phi_n \rangle - \langle \dot{X}_{\text{Gauss}}, \phi_m \rangle]^2 = \mathbb{E}[\langle \dot{X}_{\text{Gauss}}, \phi_n - \phi_m \rangle^2] = \sigma^2\|\phi_n - \phi_m\|_2^2. \tag{2.2}
\]
The sequence $(\phi_n)$ being convergent, it is a Cauchy sequence of $L^2(\mathbb{R}^d)$. Then, (2.2) implies that $(\langle \dot{X}_{\text{Gauss}}, \phi_n \rangle)$ is itself a Cauchy sequence in the complete space $L^2(\Omega)$, and hence is convergent in this space. One readily shows that the limit does not depend on the sequence $(\phi_n)$ and we denote it by $\langle \dot{X}_{\text{Gauss}}, f \rangle$. Then, as will be made more rigorous in the sequel, the linear and continuous functional $\dot{X}_{\text{Gauss}}$ initially defined from $\mathcal{D}(\mathbb{R}^d)$ to $L^0(\Omega)$ is actually a linear and continuous functional from $L^2(\mathbb{R}^d)$ to $L^2(\Omega)$. 

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Defining a Gaussian white noise only for smooth and rapidly decaying test functions appears highly conservative. As we shall see, this occurs for any Lévy white noise. The goal of this paper is precisely to identify the domain of definition general Lévy white noise; that is, the largest possible space of test functions that we can apply to the white noise. The identification of the domain has already been done in [33] in the context of independently scattered random measures, and the unification of Lévy white noise with this notion in Section 3.1 allows us to rely on this work. Moreover, we are aiming at practical criteria to identify this domain of definition for specific Lévy white noises.

3 Integrability Conditions with respect to Lévy White Noise

In this section, a Lévy white noise is understood as a generalized random process in the sense of Gelfand. In section 3.1 we connect this construction with the framework of independent scattered random measures of Rajput and Rosinski [33]. Then, in sections 3.2 and 3.3 we use this connection and results obtained in [33] to extend the domain of definition of Lévy white noise.

3.1 Lévy White Noises as Independent Scattered Random Measures

A random measure is a random process whose test functions are indicator functions. It is very popular for stochastic integration, the integral being defined for simple functions (i.e., linear combinations of indicator functions), and extended by a limit argument. Essentially, a random measure is independently scattered when two indicator functions with disjoint supports define independent random variables. For a proper definition, see [33, Section 1].

We show in this section that a Lévy white noise is an example of an independently scattered random measure. We first identify integrability conditions for test functions of the form $\mathbf{1}_A$ with $A$ a Borel set with finite Lebesgue measure.

**Definition 3.1.** Let $A \subset \mathbb{R}^d$ be a Borel measurable set. Let $\theta \in \mathcal{D}(\mathbb{R}^d)$, $\theta \geq 0$ and $\int_{\mathbb{R}^d} \theta(t) \, dt = 1$. For $n \in \mathbb{N}$, and $t \in \mathbb{R}^d$, let $\theta_n(t) = n^d \theta(nt)$ be a mollified version of the Dirac impulse. Then, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we define

$$\left\langle \dot{X}, \varphi \mathbf{1}_A \right\rangle := \lim_{n \to +\infty} \left\langle \dot{X}, \varphi (\theta_n \ast \mathbf{1}_A) \right\rangle,$$

where the limit is in probability.

**Proposition 3.2.** Definition 3.1 is well posed. In particular the limit exists and does not depend on the choice of the mollifier $\theta$. In addition, the characteristic function of the random variable $\left\langle \dot{X}, \varphi \mathbf{1}_A \right\rangle$ is given by

$$\Phi_{\left\langle \dot{X}, \varphi \mathbf{1}_A \right\rangle}(\xi) = \mathbb{E} \left[ \exp \left( i \xi \left\langle \dot{X}, \varphi \mathbf{1}_A \right\rangle \right) \right] = \exp \left( \int_A \psi(\xi \varphi(t)) \, dt \right), \quad \text{for all } \xi \in \mathbb{R},$$

where $\psi$ is a suitable function.
where $\psi$ is the Lévy exponent of $X$. Also, for any disjoint Borel measurable sets $A, B \subset \mathbb{R}^d$, 
\[
\langle \dot{X}, \varphi 1_{A \cup B} \rangle = \langle \dot{X}, \varphi 1_A \rangle + \langle \dot{X}, \varphi 1_B \rangle \quad \text{almost surely and} \quad \langle \dot{X}, \varphi 1_A \rangle \text{ and } \langle \dot{X}, \varphi 1_B \rangle \text{ are independent.}
\]

Proof. We first remark that for any $n \in \mathbb{N}$, the function $\varphi(\theta_n * 1_A)$ is in $\mathcal{D}(\mathbb{R}^d)$, therefore the random variables $Y_n = \langle \dot{X}, \varphi(\theta_n * 1_A) \rangle$ are well defined. It suffices to show that the sequence $(Y_n)_{n \in \mathbb{N}}$ is Cauchy in probability. By linearity, it is enough to show that this sequence is Cauchy in law. Let $n, m \in \mathbb{N}$. By definition, we know that for any $\xi \in \mathbb{R}$,
\[
\mathbb{E} \left[ \exp \left( i\xi \langle \dot{X}, \varphi((\theta_n - \theta_m) * 1_A) \rangle \right) \right] = \exp \left( \int_{\mathbb{R}^d} \psi(\xi \varphi(t) ((\theta_n - \theta_m) * 1_A(t))) \, dt \right),
\]
where
\[
\psi(\xi) = i\gamma \xi - \frac{\sigma^2 \xi^2}{2} + \int_{|x|<1} (e^{i\xi x} - 1 - i\xi x) \nu(dx) + \int_{|x|>1} (e^{i\xi x} - 1) \nu(dx).
\]

We treat each of the four terms of the Lévy exponent separately. Since $\varphi \in \mathcal{D}(\mathbb{R}^d)$, there is a compact $K$ such that $\text{supp} \varphi =: K$. Then,
\[
\left| \int_{\mathbb{R}^d} i\gamma \xi \varphi(t) ((\theta_n - \theta_m) * 1_A(t)) \, dt \right| \leq |\gamma \xi| \|\varphi\|_{\infty} \|((\theta_n - \theta_m) * 1_A)\|_{L^1(K)}.
\]

It is well known that for $p \geq 1$ and $f \in L^p(K)$, $(\theta_n - \theta_m)^{\ast} f \to 0$ in $L^p(K)$ as $n, m \to +\infty$. Therefore, since $1_A \in L^1(K)$, $\|((\theta_n - \theta_m) * 1_A)\|_{L^1(K)} \to 0$ as $n, m \to +\infty$. Similarly,
\[
\left| \int_{\mathbb{R}^d} \frac{\sigma^2 \xi^2 \varphi(t)^2 ((\theta_n - \theta_m) * 1_A(t))^2}{2} \, dt \right| \leq \frac{1}{2} \sigma^2 \|\varphi\|_{\infty}^2 \|((\theta_n - \theta_m) * 1_A)^2\|_{L^2(K)}.
\]

Since $1_A \in L^2(K)$, $\|((\theta_n - \theta_m) * 1_A)^2\|_{L^2(K)} \to 0$ as $n, m \to +\infty$. Then, by [27, Lemma 5.14],
\[
|e^{i\xi x} - 1 - i\xi x| \leq \frac{1}{2} |\xi x|^2,
\]
and
\[
\int_{\mathbb{R}^d} \int_{|x| \leq 1} |e^{i\xi((\theta_n - \theta_m) * 1_A(t))} - 1 - i\xi ((\theta_n - \theta_m) * 1_A(t))| \nu(dx) \, dt
\]
\[
\leq \frac{1}{2} |\xi|^2 \|\varphi\|_{\infty} \left( \int_{|x| \leq 1} x^2 \nu(dx) \right) \|((\theta_n - \theta_m) * 1_A)^2\|_{L^2(K)} \, dt,
\]
and we conclude as for (3.1). The last term represents the compound Poisson part of the Lévy-Itô decomposition of the Lévy white noise. It corresponds to the characteristic function of the random variable $M_{n,m} := \int_{\mathbb{R}^d} \int_{\mathbb{R}} z \varphi(t)(\theta_n - \theta_m) * 1_A(t)J(dt, dz)$ where $J$ is a Poisson random measure on $\mathbb{R}^d \times \mathbb{R}$ with intensity measure $dt 1_{|z| > 1} \nu(dz)$, and we know that
\[
M_{n,m} = \sum_{i \geq 1} Z_i \varphi(T_i) ((\theta_n - \theta_m) * 1_A(T_i))\,.
\]
for some random space-time points \((Z_i, T_i)_{i \geq 1}\), and the sum above has finitely many terms (independently of \(m, n\)) almost surely due to the compactness of the support of \(\varphi\). Indeed, with \(K = \text{supp} \varphi\), we have

\[
\mathbb{E} [J (K \times \mathbb{R})] = \int_{K \times \mathbb{R}} dt \mathbf{1}_{|z| > 1} \nu(dz) = \text{Leb}_d(K) \int_{|z| > 1} \nu(dz) < +\infty ,
\]

since \(\nu\) is a Lévy measure, and \(J (K \times \mathbb{R})\) is the random variables that counts the number of points \(T_i\) that fall in the support of \(\varphi\). By Lebesgue’s differentiation theorem (see [48, Chapter 7, Exercise 2]), \((\theta_n - \theta_m) * \mathbf{1}_A(t) \to 0\) as \(n, m \to +\infty\) for all \(t \in K \setminus H\), where \(H\) is a subset of \(\mathbb{R}^d\) such that the Lebesgue measure is \(\text{Leb}_d(H) = 0\). The random times \(T_i\) have an absolutely continuous law. Indeed, for any Borel set \(B \subset \mathbb{R}^d\),

\[
\mathbb{P}(T_i \in B) \leq \mathbb{P}(J(B \times \mathbb{R}) \geq 1) \leq \text{Leb}_d(B) \int_{|z| > 1} \nu(dz) .
\]

Therefore, for all \(i \geq 1\), \(\mathbb{P}(T_i \in H) = 0\) and \(M_{n,m} \to 0\) as \(n, m \to +\infty\) almost surely, hence also in law. Therefore \((Y_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in law, hence in probability, and there exists a random variable \(Y\) such that \(Y_n \to Y\) in probability as \(n \to +\infty\). By checking the convergence of each term of the decomposition of the Lévy exponent, it is easy to see that for all \(\xi \in \mathbb{R}\),

\[
\int_{\mathbb{R}^d} \psi(\xi \varphi(t) \theta_n * \mathbf{1}_A(t)) \, dt \to \int_A \psi(\xi \varphi(t)) \, dt , \quad \text{as} \ n \to +\infty ,
\]

hence

\[
\mathbb{E} [e^{itY}] = \exp \left( \int_A \psi(\xi \varphi(t)) \, dt \right) .
\]

If \(\tilde{\theta}\) is another mollifier, and \((\tilde{Y}_n)_{n \in \mathbb{N}}\) and \(\tilde{Y}\) are the associated sequence and limit, it is easy to see by linearity of \(\dot{X}\) that \(Y_n - \tilde{Y}_n \to 0\) in probability as \(n \to +\infty\). If \(A\) and \(B\) are disjoint Borel measurable sets of \(\mathbb{R}^d\), we observe that \(\theta_n * \mathbf{1}_{A \cup B} = \theta_n * \mathbf{1}_A + \theta_n * \mathbf{1}_B\), which proves the decomposition \(\langle \dot{X}, \varphi \mathbf{1}_{A \cup B} \rangle = \langle \dot{X}, \varphi \mathbf{1}_A \rangle + \langle \dot{X}, \varphi \mathbf{1}_B \rangle\) at the limit when \(n \to +\infty\). Independence comes from the factorisation of the characteristic function of these random variables.

From the previous definition, it is straightforward to define the random variables \(\langle \dot{X}, \mathbf{1}_A \rangle\) for any bounded borel set \(A\). Indeed, it suffices to choose any \(\varphi \in \mathcal{D}(\mathbb{R}^d)\) such that \(\varphi|_A = 1\) and set \(\langle \dot{X}, \mathbf{1}_A \rangle := \langle \dot{X}, \varphi \mathbf{1}_A \rangle\). This definition does not depend on the choice of \(\varphi\). Indeed, if \(\varphi\) and \(\psi\) are two such functions, by linearity of the noise and from the expression of the characteristic function, we get that \(\langle \dot{X}, (\varphi - \psi) \mathbf{1}_A \rangle = 0\) almost surely. In this particular case, we observe that the characteristic function of the noise takes the particular form

\[
\mathbb{E} \left[ \exp \left( i \xi \langle \dot{X}, \mathbf{1}_A \rangle \right) \right] = \exp \left( \text{Leb}_d(A) \psi(\xi) \right) . \quad (3.2)
\]

We now extend the definition to a Borel set with finite measure (but not necessarily bounded).
**Definition 3.3.** Let $A \subset \mathbb{R}^d$ be a Borel measurable set such that $\text{Leb}_d(A) < +\infty$. For $n \in \mathbb{N}^*$, let $A_n = A \cap [-n, n]^d$. Then we define

$$
\langle \dot{X}, 1_A \rangle := \lim_{n \to +\infty} \langle \dot{X}, 1_{A_n} \rangle,
$$

where the limit is in probability.

**Proposition 3.4.** Definition 3.3 is well posed. In addition, the characteristic function of the random variable $\langle \dot{X}, 1_A \rangle$ is given by

$$
E \left[ \exp \left( i\xi \langle \dot{X}, 1_A \rangle \right) \right] = \exp \left( \text{Leb}_d(A) \psi(\xi) \right), \quad \text{for all } \xi \in \mathbb{R},
$$

where $\psi$ is the Lévy exponent of $\dot{X}$. Also, for any disjoint Borel measurable sets $A, B \subset \mathbb{R}^d$, such that $\text{Leb}_d(A), \text{Leb}_d(B) < +\infty$, $\langle \dot{X}, 1_{A \cup B} \rangle = \langle \dot{X}, 1_A \rangle + \langle \dot{X}, 1_B \rangle$ almost surely and $\langle \dot{X}, 1_A \rangle$ and $\langle \dot{X}, 1_B \rangle$ are independent.

**Proof.** From the expression of the characteristic function in (3.2) and the fact that $A_n$ is a bounded Borel set, it is easy to see that the sequence $\left( \langle \dot{X}, 1_{A_n} \rangle \right)_{n \in \mathbb{N}}$ is Cauchy in law, and therefore converges in probability. The expression of the characteristic function follows from the fact that $\text{Leb}_d(A) < +\infty$ and the application of the dominated convergence theorem. The last statement comes directly from an application of Proposition 3.2 and the expression of the characteristic functions. 

Let $\mathcal{L}(\mathbb{R}^d)$ be the $\delta$-ring of Borel-measurable subsets of $\mathbb{R}^d$ finite Lebesgue measure. Since $\bigcup_{n \in \mathbb{N}} [-n, n]^d = \mathbb{R}^d$, condition (1.4) of [33] is satisfied. By the two previous propositions, we have defined random set function as an extension of the Lévy white noise $\dot{X}$ on $\mathcal{L}(\mathbb{R}^d)$. We still refer to this set function as Lévy white noise.

**Theorem 3.5.** The extension of the Lévy white noise $\dot{X}$ is an independently scattered random measure in the sense of [33].

**Proof.** The definition we refer to can be found in [33, p.455]. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of disjoint sets in $\mathcal{L}(\mathbb{R}^d)$. Let $k \in \mathbb{N}$ and $i_1 < \cdots < i_k \in \mathbb{N}$. We show that the random variables $\langle \dot{X}, 1_{A_{i_j}} \rangle$, $1 \leq j \leq k$ are independent. By linearity of the noise, this fact is an immediate consequence of [33] and the $\sigma$-additivity of Lebesgue measure. This proves that $\langle \dot{X}, 1_{A_n} \rangle$, $n \in \mathbb{N}$ is a sequence of independent random variables. If in addition $\sum_{n \in \mathbb{N}} \text{Leb}_d(A_n) < +\infty$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{L}(\mathbb{R}^d)$ and we need to show that

$$
\langle \dot{X}, 1_{\bigcup_{n \in \mathbb{N}} A_n} \rangle = \sum_{n \in \mathbb{N}} \langle \dot{X}, 1_{A_n} \rangle,
$$

A $\delta$-ring is a collection of sets that is closed under finite union, countable intersection, and relative complementation [7, Definition 1.2.13]. It appears in measure theory, especially when one wants to avoid sets with infinite measure.
where the series converges almost surely. From the second statement of Proposition 3.4, it is easy to see that for any \( k \in \mathbb{N} \),

\[
\left< \dot{X}, \mathbf{1}_{\bigcup_{n=1}^{k} A_n} \right> = \sum_{n=1}^{k} \left< \dot{X}, \mathbf{1}_{A_n} \right>.
\]  

(3.4)

From the expression of the characteristic function of the left-hand side of (3.4) and by linearity, we see that

\[
\left< \dot{X}, \mathbf{1}_{\bigcup_{n=1}^{k} A_n} \right> \to \left< \dot{X}, \mathbf{1}_{\bigcup_{n \in \mathbb{N}} A_n} \right>
\]

in probability as \( k \to +\infty \). Therefore the right-hand side of (3.4) is a sum of independent random variables that converges in probability. By [12, Theorem 5.3.4], the sum converges almost surely, which concludes the proof.

### 3.2 Extension of the Domain of Definition

Having connected Lévy white noises in \( \mathcal{D}'(\mathbb{R}^d) \) with independently scattered random measures, it is then possible to construct a stochastic integral of non-random functions. This is done in [33] and we simply restate the main definitions and theorems for the convenience of the reader. For any simple function

\[
f = \sum_{i=1}^{n} x_i \mathbf{1}_{A_i}
\]

where \( \text{Leb}_d(A_i) < +\infty \) for any \( 1 \leq i \leq n \), we can define for any Borel set \( A \),

\[
\left< \dot{X}, f \mathbf{1}_A \right> := \sum_{i=1}^{n} x_i \left< \dot{X}, \mathbf{1}_{A_i \cap A} \right>.
\]

**Definition 3.6.** We say that a Borel-measurable function \( f : \mathbb{R}^d \to \mathbb{R} \) is \( \dot{X} \)-integrable if there exists a sequence of simple function \( (f_n)_{n \in \mathbb{N}} \) such that \( f_n \to f \) almost everywhere for the Lebesgue measure as \( n \to +\infty \), and for any Borel set \( A \), the sequence \( \left( \left< \dot{X}, f_n \mathbf{1}_A \right> \right)_{n \in \mathbb{N}} \) converges in probability as \( n \to +\infty \). In this case we define for any Borel set \( A \),

\[
\left< \dot{X}, f \mathbf{1}_A \right> := \lim_{n \to +\infty} \left< \dot{X}, f_n \mathbf{1}_A \right>.
\]

We denote by \( L(\dot{X}) \) the set of all \( \dot{X} \)-integrable functions.

Definition 3.6 identifies the class of measurable test functions \( f \) such that \( \left< \dot{X}, f \right> \) is well-defined. Then, we have a characterization of \( \dot{X} \)-integrable functions.

**Proposition 3.7.** [33, Theorem 2.7] Let \( \dot{X} \) be a Lévy white noise with characteristic triplet \( (\gamma, \sigma, \nu) \), and \( f : \mathbb{R}^d \to \mathbb{R} \) be a Borel-measurable function. Then \( f \in L(\dot{X}) \) if and only if the following conditions are satisfied:

1. \( \int_{\mathbb{R}^d} \left|sf(s) + \left( \int_{\mathbb{R}} x f(s) \left( \mathbf{1}_{|xf(s)| \leq 1} - \mathbf{1}_{|x| \leq 1} \right) \nu(dx) \right) \right| ds < +\infty \),
2. \( \sigma f \in L^2(\mathbb{R}^d) \),
3. \( \int_{\mathbb{R}^d \times \mathbb{R}} (|xf(s)|^2 \wedge 1) ds \nu(dx) < +\infty \).
**Definition 3.8.** We set
\[
\Psi(\xi) = \left| \gamma \xi + \int_{\mathbb{R}} x \xi \left( 1_{|\xi| \leq 1} - 1_{|\xi| \leq 1} \right) \nu(dx) \right| + \sigma^2 \xi^2 + \int_{\mathbb{R}} (1 \wedge (x^2 \xi^2)) \nu(dx).
\]

We call \( \Psi \) the Rajput-Rosinski exponent of \( \dot{X} \).

Then, \( f \in L(\dot{X}) \) if and only if \( \Psi(f) := \int_{\mathbb{R}^d} \Psi(f(t)) dt < \infty \). It turns out that the space of \( \dot{X} \)-integrable functions has a rich structure of generalized Orlicz spaces. The definition and first properties of those spaces are recalled in Appendix A. We refer to [34] for an in-depth exposition, with a special emphasis on the Chapter X.

**Proposition 3.9.** The Rajput-Rosinski exponent \( \Psi \) of a Lévy white noise \( \dot{X} \) is a \( \Delta_2 \)-regular \( \varphi \)-function in the sense of Definition A.1 (see Appendix A). Therefore, \( L(\dot{X}) = L^\Psi(\mathbb{R}^d) \) is a generalized Orlicz space. It is therefore a complete linear metric space equipped with the \( F \)-norm
\[
\|f\|_\Psi := \inf\{\lambda > 0, \Psi(f/\lambda) \leq \lambda\}.
\]
The space \( D(\mathbb{R}^d) \) is dense in \( L(\dot{X}) \) and the convergence of a sequence of functions \( f_k \) in \( L(\dot{X}) \) to 0 is equivalent to \( \Psi(f_n) \to 0 \) as \( n \to \infty \).

**Proof.** Rajput and Rosinski have shown that \( \Psi \) is a \( \varphi \)-function in [33, Lemma 3.1]. By definition, \( L(\dot{X}) = L^\Psi(\mathbb{R}^d) \) is therefore a generalized Orlicz space in the sense of Definition A.2. Except for the density, the properties of the space \( L(\dot{X}) \) are then derived from Proposition A.3. For the density, Proposition A.3 implies that simple functions are dense in \( L(\dot{X}) \). It suffices therefore to remark that a simple function can be easily approximated by a function in \( D(\mathbb{R}^d) \) for the topology of \( L(\dot{X}) \) by taking a regularized version of the simple function.

The connection between Lévy white noises in \( D'(\mathbb{R}^d) \) and independently scattered random measures established in Theorem 3.5 allows us to apply [33, Theorem 3.3] to the Lévy white noise and leads to the following result.

**Theorem 3.10.** Let \( \dot{X} \) be a Lévy white noise. Then, the functional
\[
\dot{X} : L(\dot{X}) \to L^0(\Omega)
\]
\[
f \mapsto \langle \dot{X}, f \rangle
\]
is linear and continuous. In other words, \( \dot{X} \) is a random linear functional on \( L(\dot{X}) \).

Theorem 3.10 gives general conditions on test functions for integrability with respect to a given Lévy white noise. It therefore specifies the domain of definition of \( \dot{X} \); that is, the broadest class of test functions on which \( \dot{X} \) is a random linear functional. Once the random variable \( \langle \dot{X}, f \rangle \) is well-defined, it is important to identify its characteristic function. The following result is the last part of [33, Theorem 2.7].

**Proposition 3.11.** Consider a white noise \( \dot{X} \) with Lévy exponent \( \psi \). For a fixed \( f \in L(\dot{X}) \), the characteristic function of \( \langle \dot{X}, f \rangle \) is
\[
\Phi_{\langle \dot{X}, f \rangle}(\xi) = \exp \left( i \int_{\mathbb{R}^d} \psi(\xi f(t)) dt \right).
\]
More generally, one uses the previous results to extend the domain of continuity and positive-definiteness of the characteristic functional of a Lévy white noise.

**Proposition 3.12.** For any Lévy white noise \( \dot{X} \), the characteristic functional \( \widehat{\mathcal{P}}_{\dot{X}} \) is well-defined, continuous, and positive-definite over \( L(\dot{X}) \), and is given by

\[
\widehat{\mathcal{P}}_{\dot{X}}(f) = \exp \left( \int_{\mathbb{R}^d} \psi(f(t))dt \right).
\]

**Proof.** The characteristic functional \( \varphi \mapsto \widehat{\mathcal{P}}_{\dot{X}}(\varphi) = \mathbb{E}[e^{i\langle \dot{X}, \varphi \rangle}] \) is continuous over \( \mathcal{D}(\mathbb{R}^d) \).

For any \( f \in L(\dot{X}) \), we know that \( \langle \dot{X}, f \rangle \) is a well-defined random variable according to Proposition 3.11, and that its characteristic function is \( \xi \mapsto \mathbb{E}[e^{i\xi f}] = \exp \left( \int_{\mathbb{R}^d} \psi(\xi f(t))dt \right) \).

We can therefore extend \( \widehat{\mathcal{P}}_{\dot{X}} \) to \( L(\dot{X}) \) by setting

\[
\widehat{\mathcal{P}}_{\dot{X}}(f) = \mathbb{E}[e^{i\langle \dot{X}, f \rangle}] = \exp \left( \int_{\mathbb{R}^d} \psi(f(t))dt \right).
\]

**Positive-definiteness.** Let \( N \geq 1 \), \( a_n \in \mathbb{C} \), \( f_n \in L(\dot{X}) \), \( n = 1, \cdots, N \). The space \( \mathcal{D}(\mathbb{R}^d) \) is dense in \( L(\dot{X}) \) (Proposition 3.9), hence there exists for \( N \) sequences \( (\varphi_k^n)_{k \in \mathbb{N}} \) such that \( \varphi_k^n \to f_n \) in \( L(\dot{X}) \) for \( n = 1, \cdots, N \). From Theorem 3.16, we know that \( f \mapsto \langle \dot{X}, f \rangle \) is continuous from \( L(\dot{X}) \) to \( L^0(\Omega) \). In particular, we have \( \mathbb{E}[e^{i\langle \dot{X}, \varphi_k^n - \varphi_k^j \rangle}] \to \mathbb{E}[e^{i\langle \dot{X}, f - f_j \rangle}] \) for every \( 1 \leq i, j \leq N \). Finally, we have that

\[
\sum_{1 \leq i, j \leq N} a_i a_j \widehat{\mathcal{P}}_{\dot{X}}(f_i - f_j) = \sum_{1 \leq i, j \leq N} a_i a_j \mathbb{E}[e^{i\langle \dot{X}, f_i - f_j \rangle}] \\
= \lim_{k \to \infty} \sum_{1 \leq i, j \leq N} a_i a_j \mathbb{E}[e^{i\langle \dot{X}, \varphi_k^i - \varphi_k^j \rangle}] \\
= \lim_{k \to \infty} \sum_{1 \leq i, j \leq N} a_i a_j \widehat{\mathcal{P}}_{\dot{X}}(\varphi_i - \varphi_j) \\
\geq 0
\]

since \( \widehat{\mathcal{P}}_{\dot{X}} \) is positive-definite over \( \mathcal{D}(\mathbb{R}^d) \).

**Continuity.** Using the Lévy-Khintchine representation (2.1) of \( \psi \) with Lévy triplet \( (\gamma, \sigma^2, \nu) \), we have

\[
|\psi(\xi)| = \left| i\gamma \xi + i \int_{\mathbb{R}} x \xi \left( 1_{|x| \leq 1} - 1_{|x| \leq 1} \right) \nu(dx) + \sigma^2 \xi^2 + \int_{\mathbb{R}} (e^{ix\xi} - 1 - ix\xi 1_{|x| \leq 1}) \nu(dx) \right| \\
\leq \left| \gamma \xi + \int_{\mathbb{R}} x \xi \left( 1_{|x| \leq 1} - 1_{|x| \leq 1} \right) \nu(dx) \right| + \sigma^2 \xi^2 + 2 \int_{\mathbb{R}} 1 \land (x^2 \xi^2) \nu(dx) \\
\leq 2\Psi(\xi), \quad (3.5)
\]

where we used the triangular inequality and the relation \( |e^{iy} - 1 - iy 1_{|y| \leq 1}| \leq 2(1 \land y^2) \) applied to \( y = x \xi \). Applying (3.5) to \( \xi = f(t) \) and integrating over \( \mathbb{R}^d \), we have for every \( f \in L(\dot{X}) \),

\[
|\log \widehat{\mathcal{P}}_{\dot{X}}(f)| \leq \int_{\mathbb{R}^d} |\psi(f(t))|dt \leq 2\|f\|_{\Psi}.
\]

Hence \( \widehat{\mathcal{P}}_{\dot{X}} \) is continuous at 0. The functional \( \widehat{\mathcal{P}}_{\dot{X}} \) is positive-definite and continuous at 0, and therefore continuous [46, Section IV.1.2, Proposition 1.1]. \( \square \)
3.3 Moments of \( \langle \dot{X}, f \rangle \)

We say that a generalized random process \( s \) has \textit{finite} \( p \)-th \textit{moments} when \( E[|\langle \dot{X}, \varphi \rangle|^p] < \infty \) for any test function \( \varphi \in D(\mathbb{R}^d) \). Consider the \( d \)-dimensional Lévy white noise \( \dot{X} \) and the infinitely divisible random variable \( Y \) with common Lévy exponent. For \( p > 0 \), we have the equivalence [43]

\[
E[|Y|^p] < \infty \iff \forall \varphi \in D(\mathbb{R}^d), E[|\langle \dot{X}, \varphi \rangle|^p] < \infty.
\]

In particular, if \( \dot{X} \) has finite \( p \)-th moments, the random variables \( \langle \dot{X}, \varphi \rangle \) have all a finite \( p \)-th moment. This is not anymore the case in general for \( \langle \dot{X}, f \rangle \) when \( f \in L(\dot{X}) \).

**Definition 3.13.** If \( \dot{X} \) as finite \( p \)-th-moments for some \( p > 0 \), we set

\[
L^p(\dot{X}) = \{ f \in L(\dot{X}), E[|\langle \dot{X}, f \rangle|^p] < \infty \}.
\]

**Proposition 3.14** (Existence of moments, Theorem 3.3 in [33]). Let \( 0 \leq p \). For a white noise \( \dot{X} \) and \( f \in L(\dot{X}) \), we have

\[
f \in L^p(\dot{X}) \iff \int_{\mathbb{R}^d} \int_{\mathbb{R}} |x\varphi(t)|^p \mathbb{1}_{|x\varphi(t)|>1} \nu(dx) dt < \infty.
\]

**Definition 3.15.** For \( p > 0 \), the \( p \)-th \textit{order Rajput-Rosinski exponent} of the Lévy white noise \( \dot{X} \) is defined as

\[
\Psi_p(\xi) = |\gamma \xi + \int_{\mathbb{R}} x \xi \left( \mathbb{1}_{|x| \leq 1} - \mathbb{1}_{|x| \leq 1} \right) \nu(dx) + \sigma^2 \xi^2 + \int_{\mathbb{R}} \left( |x\varphi(t)|^p \mathbb{1}_{|x\varphi(t)|>1} + |x\varphi(t)|^2 \mathbb{1}_{|x\varphi(t)| \leq 1} \right) \nu(dx). \quad (3.6)
\]

Then, \( f \in L^p(\dot{X}) \) if and only if \( \Psi_p(f) := \int_{\mathbb{R}^d} \Psi_p(f(t)) dt < \infty \). Again, according to [33], \( \Psi_p \) is a \( \varphi \)-function for every \( p > 0 \). Its means that \( L^p(\dot{X}) \) is a generalized Orlicz space associated to \( \Psi_p \). Proposition [33] is therefore valid for \( L^p(\dot{X}) \) instead of \( L(\dot{X}) \).

The space \( L^p(\dot{X}) \) is the largest space of test functions such that \( \langle \dot{X}, f \rangle \) is well-defined and has a finite \( p \)-th-moment. Of course, \( L^p(\dot{X}) \subseteq L(\dot{X}) \), and we know from Theorem [3.10] that \( \dot{X} \) is a random linear map from \( L^p(\dot{X}) \) to \( L^0(\Omega) \). We have actually a stronger result, still taken from [33].

**Theorem 3.16** (Theorem 3.3 in [33]). Let \( \dot{X} \) be a Lévy white noise. Then, the functional

\[
\dot{X} : L^p(\dot{X}) \to L^p(\Omega)
\]

\[
f \mapsto \langle \dot{X}, f \rangle
\]

is linear and continuous. In other words, \( \dot{X} \) is an \( L^p(\Omega) \)-valued random linear functional on \( L^p(\dot{X}) \).
4 Representation of the Lévy White Noise as a Stochastic Integral

The definition of Lévy white noise we gave is in terms of its characteristic functional. It is an abstract definition that relies on the Minlos-Bochner theorem, which is not constructive. We give here an example of a construction of Lévy white noise as the derivative (in the sense of Schwartz’s theory of generalized functions) of a Lévy process (or a Lévy field in the case of dimension $d \geq 2$). We recall the definition of a Lévy field. A general presentation of the theory of multiparameter Lévy fields can be found in [3]; see also [15].

4.1 From Lévy Sheets to Lévy White Noises

Let $(X_t)_{t \in \mathbb{R}_+^d}$ be a $d$-parameter random field. We define the $2^d$ relations $\mathcal{R} = (\mathcal{R}_1, ..., \mathcal{R}_d)$, where $\mathcal{R}_i$ is either $\leq$ or $> \geq$, and $a \mathcal{R} b$ if and only if $a_i \mathcal{R}_i b_i$ for all $1 \leq i \leq d$. For $a \leq b \in \mathbb{R}_+^d$, we define the box $(a, b) = \{ t \in \mathbb{R}_d^d : a < t \leq b \}$, and the increment $\Delta^b_a X$ of $X$ over the box $(a, b)$ by

$$\Delta^b_a X = \sum_{\varepsilon \in \{0, 1\}^d} (-1)^{|\varepsilon|} X_{c_\varepsilon(a, b)},$$

where for any $\varepsilon \in \{0, 1\}^d$, we write $|\varepsilon| = \sum_{i=1}^d \varepsilon_i$ and $c_\varepsilon(a, b) \in \mathbb{R}_+^d$ is defined by $c_\varepsilon(a, b)_i = a_i \mathbb{1}_{\{\varepsilon_i = 1\}} + b_i \mathbb{1}_{\{\varepsilon_i = 0\}}$, for all $1 \leq i \leq d$. The next definition is a generalization of the càdlàg property to processes indexed by $\mathbb{R}_+^d$.

**Definition 4.1.** Using the terminology in [3] and [40], we say that $X$ is lamp (for limit along monotone paths) if we have the following:

1. For all $2^d$ relations $\mathcal{R}$, $\lim_{u \to t, t \mathcal{R} u} X_u$ exists.
2. If $\mathcal{R} = (\leq, ..., \leq)$ then $X_t = \lim_{u \to t, t \mathcal{R} u} X_u$.
3. $X_t = 0$ if $t_i = 0$ for some $1 \leq i \leq d$.

We are now ready to give the definition of a Lévy field in $\mathbb{R}_+^d$.

**Definition 4.2.** $X = (X_t)_{t \in \mathbb{R}_+^d}$ is a $d$-parameter Lévy field if it has the following properties:

1. $X$ is continuous in probability.
2. $X$ is lamp almost surely.
3. For any sequence of disjoint boxes $(a_k, b_k]$, $1 \leq k \leq n$, the random variables $\Delta^b_{a_k} X$ are independent.
4. Given two boxes $(a, b]$ and $(c, d]$ in $\mathbb{R}_+^d$ such that $(a, b] + t = (c, d]$ for some $t \in \mathbb{R}_d$, the increments $\Delta^b_{a} X$ and $\Delta^d_{c} X$ are identically distributed.

The jump $\Delta_t X$ of $X$ at time $t$ is defined by $\Delta_t X = \lim_{u \to t, u < t} \Delta^t_u X$. 

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This definition coincides with the notion of Lévy process when \( d = 1 \). In addition, for all \( t = (t_1, \ldots, t_d) \in \mathbb{R}_+^d \), and for all \( 1 \leq i \leq d \), the process \( X_i^{:t} = X_{(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_d)} \) is a Lévy process (the notation here means that it is the process in one parameter obtained by fixing all the coordinates of \( t \) except the \( i \)-th).

The Brownian sheet is an example of such a \( d \)-parameter Lévy field. It is the analog in this framework of Brownian motion and further properties of this field are detailed in [13], [16], [28] or [10].

We can now state the multidimensional analog of the Lévy-Itô decomposition, taken from [3, Theorem 4.6] particularized to the case of stationary increments (see also [15]).

**Theorem 4.3.** Let \( X \) be a \( d \)-parameter Lévy field with characteristic triplet \((\gamma, \sigma, \nu)\). The following holds:

(i) The jump measure \( J_X \) defined on \( \mathbb{R}_+^d \times (\mathbb{R} \setminus \{0\}) \) by \( J_X(B) = \# \{ (t, \Delta_tX) \in B \} \), for \( B \) in the Borel \( \sigma \)-algebra of \( \mathbb{R}_+^d \times (\mathbb{R} \setminus \{0\}) \), is a Poisson random measure with intensity \( \text{Leb}_d \times \nu \), and \( \nu \) is a Lévy measure. In particular,

\[
\int_{\mathbb{R}} (x^2 \wedge 1) \nu(dx) < +\infty.
\]

(ii) For all \( t \in \mathbb{R}_+^d \), we have the decomposition

\[
X_t = \gamma \text{Leb}_d([0, t]) + \sigma W_t + \int_{[0,t]} \int_{|x|>1} x J_X(ds, dx) + \int_{[0,t]} \int_{|x|\leq 1} x \tilde{J}_X(ds, dx),
\]

where \( W \) is a Brownian sheet, \( \tilde{J}_X = J_X - \text{Leb}_d \times \nu \) is the compensated jump measure, and the equality holds almost surely. In addition, the terms of the decomposition are independent random fields.

If \( X \) is a \( d \)-parameter Lévy field, by the lamp property of its sample paths, it is locally bounded and defines almost surely an element of \( \mathcal{D}'(\mathbb{R}^d) \) via the \( L^2 \)-inner product. We have restricted our study to the case where the parameter space is \( \mathbb{R}_+^d \), but we can easily generalize this notion of Lévy field on \( \mathbb{R}^d \) by taking independent copies of \( X \) on each \( 2^d \) orthants of \( \mathbb{R}^d \). We say this random field is a Lévy field on \( \mathbb{R}^d \).

**Proposition 4.4.** Let \( X \) be a \( d \)-parameter Lévy field on \( \mathbb{R}^d \) with characteristic triplet \((\gamma, \sigma, \nu)\). The \( d \)-th cross-derivative of \( X \) in the sense of Schwartz distributions is a Lévy white noise: for \( \omega \in \Omega \) and \( \varphi \in \mathcal{D}(\mathbb{R}^d) \),

\[
\left\langle \dot{X}, \varphi \right\rangle(\omega) := (-1)^d \left\langle X, \varphi^{(1d)} \right\rangle(\omega) := (-1)^d \int_{\mathbb{R}^d} X_s(\omega) \varphi^{(1d)}(s) ds,
\]

where \( \varphi^{(1d)} = \frac{\partial^d}{\partial t_1 \cdots \partial t_d} \varphi \).
4.2 Comparison with Itô-type Integration

According to [14, Proposition 3.17], Proposition 4.4 is coherent with the definition of Lévy white noise given in Definition 2.5. In particular, it can integrate not only functions in \( \mathcal{D}(\mathbb{R}^d) \), but also functions in \( L(\dot{X}) \). This particular version of Lévy white noise allows us to derive a stochastic integral representation of the noise. More precisely, we define two stochastic integral operators \( I \) and \( \tilde{I} \), and we show that, on their domain of definition, they agree with the Lévy white noise \( \dot{X} \) defined above.

**Definition 4.5.** We say a Borel measurable function \( f : \mathbb{R}^d \to \mathbb{R} \) is Itô \( \dot{X} \)-integrable if the following exists:

\[
I(f) := \int_{\mathbb{R}^d} \gamma f(s) \, ds + \int_{\mathbb{R}^d} \sigma f(s) \, dW_s
+ \int_{\mathbb{R}^d} \int_{|x| \leq 1} x f(s) \tilde{J}_X(ds, dx) + \int_{\mathbb{R}^d} \int_{|x| > 1} x f(s) J_X(ds, dx),
\]

where the first integral is in the \( L^1(\mathbb{R}^d) \) sense, the second is a Wiener integral, and the others are Poisson integrals in the sense of [27, Lemma 12.13]. We denote by \( L(I) \) the space of Itô \( \dot{X} \)-integrable functions.

We will see in the following that this stochastic integral representation is not always well defined for \( f \in L(\dot{X}) \). That is why we introduce an other operator \( \tilde{I} \).

**Definition 4.6.** We say a Borel measurable function \( f : \mathbb{R}^d \to \mathbb{R} \) is Poisson \( \dot{X} \)-integrable if the following integrals exist:

\[
\tilde{I}(f) := \int_{\mathbb{R}^d} \left( \gamma f(s) + \int_{\mathbb{R}} x f(s) \left( 1_{|xf(s)| \leq 1} - 1_{|x| \leq 1} \right) \nu(dx) \right) \, ds + \int_{\mathbb{R}^d} \sigma f(s) \, dW_s
+ \int_{\mathbb{R}^d \times \mathbb{R}} x f(s) 1_{|xf(s)| \leq 1} \tilde{J}_X(ds, dx) + \int_{\mathbb{R}^d \times \mathbb{R}} x f(s) 1_{|xf(s)| > 1} J_X(ds, dx),
\]

(4.1)

where the first integral is an \( L^1(\mathbb{R}^d) \) integral, the second is a Wiener integral, and the last two are compensated Poisson and Poisson stochastic integrals as defined in [27, Chapter 12]. We denote by \( L(\tilde{I}) \) the space of Poisson \( \dot{X} \)-integrable functions.

We can in fact characterize the domains of the operators \( I \) and \( \tilde{I} \).

**Proposition 4.7.** The following holds:

(i) \( f \in L(I) \) if and only if \( \gamma f \in L^1(\mathbb{R}^d) \), \( \sigma f \in L^2(\mathbb{R}^d) \), and

\[
\int_{\mathbb{R}^d} \int_{|x| > 1} (|xf(s)| \wedge 1) \, ds \, \nu(dx)
+ \int_{\mathbb{R}^d} \int_{|x| \leq 1} (|xf(s)|^2 \wedge |xf(s)|) \, ds \, \nu(dx) < +\infty.
\]

(4.2)

(ii) \( L(I) \subset L(\tilde{I}) \), and for all \( f \in L(I) \), \( I(f) = \tilde{I}(f) \) almost surely.
(iii) \( L(\tilde{I}) = L(\tilde{X}) \) and for all \( f \in L(\tilde{I}) \), \( \tilde{I}(f) = \left< \tilde{X}, f \right> \) almost surely.

**Proof.** We first prove (i). The deterministic and Wiener integral exist under the well known conditions \( \gamma \varphi \in L^1(\mathbb{R}^d) \), \( \alpha \varphi \in L^2(\mathbb{R}^d) \). By [27, Lemma 12.13], the compensated Poisson and Poisson integrals exist if and only if (4.2) is satisfied. To verify the equality \( L(\tilde{X}) = L(\tilde{I}) \), we can use the existence criterions of the different terms in (4.1) (see [27, Lemma 12.13] for the Poisson and compensated Poisson integrals) to see that a function \( f \) is \( \tilde{X} \)-integrable if and only if it is Poisson \( \tilde{X} \)-integrable. Then, let \( f \in L(I) \). Condition (ii) of Proposition 3.7 is satisfied. Moreover, we can see that

\[
\int_{\mathbb{R}^d} \int_{|x|>1} (|xf(s)| \wedge 1) \, ds \, \nu(dx) + \int_{\mathbb{R}^d} \int_{|x| \leq 1} (|xf(s)|^2 \wedge |xf(s)|) \, ds \, \nu(dx) \geq \int_{\mathbb{R}^d \times \mathbb{R}} (|xf(s)|^2 \wedge 1) \, ds \, \nu(dx),
\]

therefore condition (iii) of Proposition 3.7 is satisfied. Then,

\[
\int_{\mathbb{R}^d} \int_{|x|>1} |xf(s)| \, \mathbb{1}_{|xf(s)| \leq 1} - \mathbb{1}_{|x| \leq 1} \, \nu(dx) \, ds = \int_{\mathbb{R}^d} \int_{|x|>1} |xf(s)| \, \mathbb{1}_{|xf(s)| \leq 1} \, \nu(dx) \, ds + \int_{\mathbb{R}^d} \int_{|x| \leq 1} |xf(s)| \, \mathbb{1}_{|xf(s)|>1} \nu(dx) \, ds < +\infty,
\]

therefore (i) of Proposition 3.7 is satisfied, and the inclusion \( L(I) \subset L(\tilde{X}) = L(\tilde{I}) \) is satisfied. Then, we can assume without loss of generality that \( \gamma = \sigma = 0 \).

\[
I(f) = \int_{\mathbb{R}^d} \int_{|x| \leq 1} xf(s) \tilde{X}(dx, ds) + \int_{\mathbb{R}^d} \int_{|x|>1} xf(s) J_X(dx, ds)
\]

\[
= \int_{\mathbb{R}^d} \int_{|x| \leq 1} xf(s) \mathbb{1}_{|xf(s)| \leq 1} \tilde{X}(dx, ds) + \int_{\mathbb{R}^d} \int_{|x| < 1} xf(s) \mathbb{1}_{|xf(s)| > 1} \tilde{X}(dx, ds)
\]

\[
+ \int_{\mathbb{R}^d} \int_{|x|>1} xf(s) \mathbb{1}_{|xf(s)| \leq 1} J_X(dx, ds) + \int_{\mathbb{R}^d} \int_{|x|>1} xf(s) \mathbb{1}_{|xf(s)|>1} J_X(dx, ds).
\]

By (4.3), we can write

\[
\int_{\mathbb{R}^d} \int_{|x| \leq 1} xf(s) \mathbb{1}_{|xf(s)|>1} \tilde{X}(dx, ds) = \int_{\mathbb{R}^d} \int_{|x| \leq 1} xf(s) \mathbb{1}_{|xf(s)|>1} J_X(dx, ds)
\]

\[
- \int_{\mathbb{R}^d} \int_{|x| \leq 1} xf(s) \mathbb{1}_{|xf(s)|>1} \nu(dx) \, ds,
\]

and

\[
\int_{\mathbb{R}^d} \int_{|x|>1} xf(s) \mathbb{1}_{|xf(s)| \leq 1} J_X(dx, ds) = \int_{\mathbb{R}^d} \int_{|x| > 1} xf(s) \mathbb{1}_{|xf(s)| \leq 1} \tilde{X}(dx, ds)
\]

\[
+ \int_{\mathbb{R}^d} \int_{|x|>1} xf(s) \mathbb{1}_{|xf(s)| \leq 1} \nu(dx) \, ds.
\]

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Recombining (4.5) and (4.6) in (4.4), we get

\[
I(f) = \int_{\mathbb{R}^d} x f(s) 1_{|x| \leq 1} J_X(dx, ds) + \int_{\mathbb{R}^d} x f(s) 1_{|x| > 1} J_X(dx, ds) + \int_{\mathbb{R}^d} x f(s) \left(1_{|x| \leq 1} - 1_{|x| \leq 1}\right) \nu(dx) ds = \tilde{I}(f).
\]

We finally show that for \( f \in L(\tilde{I}) \), \( \left\langle \hat{X}, f \right\rangle = \tilde{I}(f) \). For \( f \in \mathcal{D}(\mathbb{R}^d) \), we can use Lemma 3.6 to deduce that \( \left\langle \hat{X}, f \right\rangle = I(f) \), and since \( \mathcal{D}(\mathbb{R}^d) \subset L(\mathcal{I}) \), we get \( \left\langle \hat{X}, f \right\rangle = \tilde{I}(f) \). Then let \( A \subset \mathbb{R}^d \) be a Borel set such that \( \text{Leb}_d(A) < +\infty \). Let \( (\theta_n)_{n \geq 1} \) be a sequence of mollifier as in Definition 3.1. Since for any \( n \in \mathbb{N} \), \( \left\langle \hat{X}, f(\theta_n \ast 1_n) \right\rangle = \tilde{I}(f(\theta_n \ast 1_n)) \), and since \( \left\langle \hat{X}, f(\theta_n \ast 1_n) \right\rangle \to \left\langle \hat{X}, f1_n \right\rangle \) in probability as \( n \to +\infty \), it suffices to show that \( \tilde{I}(f(\theta_n \ast 1_n)) \to \tilde{I}(f1_n) \) in probability as \( n \to +\infty \). In fact, one easily checks that \( f(\theta_n \ast 1_n) \) and \( f1_n \in L(\mathcal{I}) \). Therefore it is enough to show that \( \tilde{I}(f(\theta_n \ast 1_n)) \to \tilde{I}(f1_n) \) in probability as \( n \to +\infty \), and this is obtained using the linearity of \( I \) and the convergence in law of each part of the decomposition of the Lévy exponent as in the proof of Proposition 3.2. The same reasoning works to show \( \left\langle \hat{X}, 1_n \right\rangle = \tilde{I}(1_n) \). To extend the result to simple functions, the problem we have is that each term of the decomposition of \( \tilde{I} \) is not linear (although we will see that \( \tilde{I} \) is linear). Let \( \alpha > 0 \). Then,

\[
\int_{\mathbb{R}^d \times \mathbb{R}} x 1_{s \in A} 1_{|x| \leq 1} \tilde{J}_X(ds, dx) + \int_{\mathbb{R}^d \times \mathbb{R}} x 1_{s \in A} 1_{|x| > 1} J_X(ds, dx)
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{R}} x 1_{s \in A} 1_{|x| \leq \alpha} \tilde{J}_X(ds, dx) + \int_{\mathbb{R}^d \times \mathbb{R}} x 1_{s \in A} 1_{|x| > \alpha} J_X(ds, dx)
\]

\[
+ \int_{\mathbb{R}^d \times \mathbb{R}} x 1_{s \in A} \left(1_{|x| \leq \alpha} - 1_{|x| \leq 1}\right) \nu(dx)
\]

\[
:= I_M^\alpha(A) + I_P^\alpha(A) + D^\alpha(A).
\]

Let \( f = \sum_{i=1}^n y_i 1_{A_i} \) be a simple function, where for all \( 1 \leq i \leq n \), \( \text{Leb}_d(A_i) < +\infty \) and \( |y_i| > 0 \). Then,

\[
\left\langle \hat{X}, f \right\rangle = \sum_{i=1}^n y_i \left\langle \hat{X}, 1_{A_i} \right\rangle
\]

\[
= \sum_{i=1}^n y_i \int_{A_i \cap \mathbb{R}^d} \gamma ds + \int_{\mathbb{R}^d} \sigma 1_{A_i}(s) dW_s + I_M^1(A_i) + I_P^1(A_i)
\]

\[
= \int_{\mathbb{R}^d} \gamma f(s) ds + \int_{\mathbb{R}^d} \sigma f(s) dW_s + \sum_{i=1}^n y_i \left(I_M^{\{|y_i|^{-1}(A_i)} + I_P^{\{|y_i|^{-1}(A_i)} + D^{\{|y_i|^{-1}(A_i)} \right)
\]

\[
= \tilde{I}(f).
\]

Let \( f \in L(\hat{X}) \). By definition, there is a sequence of simple functions \( f_n \) such that \( f_n \to f \) almost everywhere as \( n \to +\infty \), and \( \langle \hat{X}, f_n \rangle \to \langle \hat{X}, f \rangle \) in probability as
as $n \to +\infty$. The proof of Theorem 2.7 shows that the sequence $(f_n)_{n \geq 1}$ can be chosen such that for any $n \in \mathbb{N}, |f_n| \leq |f|$. We only need to show that $\bar{I}(f_n) \to \bar{I}(f)$ in probability as $n \to +\infty$. We show the convergence in probability of each part of the decomposition of the stochastic integral. First we deal with the Gaussian part. By classical properties of Wiener stochastic integration, we get

$$
\mathbb{E} \left[ \left( \int_{\mathbb{R}^d} \sigma f_n(s) \, dW_s - \int_{\mathbb{R}^d} \sigma f(s) \, dW_s \right)^2 \right] = \int_{\mathbb{R}^d} \sigma^2 (f_n(s) - f(s))^2 \, ds \to 0 \quad \text{as } n \to +\infty,
$$

by the dominated convergence theorem. We deduce that $\int_{\mathbb{R}^d} \sigma f_n(s) \, dW_s \to \int_{\mathbb{R}^d} \sigma f(s) \, dW_s$ in $L^2(\Omega)$ as $n \to +\infty$, which implies the convergence in probability. Then, we show the convergence of the compensated Poisson term.

$$
I_1 := \int_{\mathbb{R}^d \times \mathbb{R}} x f_n(s) \mathbb{1}_{|xf_n(s)| \leq 1} \tilde{J}_X(ds, dx) - \int_{\mathbb{R}^d \times \mathbb{R}} x f(s) \mathbb{1}_{|xf(s)| \leq 1} \tilde{J}_X(ds, dx)
$$

$$
= \int_{\mathbb{R}^d \times \mathbb{R}} x (f_n(s) - f(s)) \mathbb{1}_{|xf(s)| \leq 1} \tilde{J}_X(ds, dx)
$$

$$
+ \int_{\mathbb{R}^d \times \mathbb{R}} x f_n(s) \mathbb{1}_{|xf_n(s)| \leq 1, |xf(s)| > 1} \tilde{J}_X(ds, dx).
$$

Each of these two integrals exist since

$$
\int_{\mathbb{R}^d \times \mathbb{R}} \left| x (f_n(s) - f(s)) \mathbb{1}_{|xf(s)| \leq 1} \right|^2 \, ds \nu(dx)
\leq 4 \int_{\mathbb{R}^d \times \mathbb{R}} \left| x f(s) \mathbb{1}_{|xf(s)| \leq 1} \right|^2 \, ds \nu(dx) < +\infty,
$$

and

$$
\int_{\mathbb{R}^d \times \mathbb{R}} \left| x f_n(s) \mathbb{1}_{|xf_n(s)| \leq 1, |xf(s)| > 1} \right|^2 \, ds \nu(dx)
\leq \int_{\mathbb{R}^d \times \mathbb{R}} \left| x f_n(s) \mathbb{1}_{|xf(s)| \leq 1, |xf(s)| > 1} \right|^2 \, ds \nu(dx) < +\infty.
$$

Furthermore, since these two integrals are compensated Poisson integrals over disjoint subsets of $\mathbb{R}^d \times \mathbb{R}$, they are independent and their mean is zero. Then,

$$
\mathbb{E} \left( J_1^2 \right) = \int_{\mathbb{R}^d \times \mathbb{R}} |x (f_n(s) - f(s))|^2 \mathbb{1}_{|xf(s)| \leq 1} \, ds \nu(dx) + \int_{\mathbb{R}^d \times \mathbb{R}} |x f_n(s)|^2 \mathbb{1}_{|xf_n(s)| \leq 1, |xf(s)| > 1} \, ds \nu(dx).
$$

Then, $|x (f_n(s) - f(s))|^2 \mathbb{1}_{|xf(s)| \leq 1} \leq 4 |xf(s)|^2 \mathbb{1}_{|xf(s)| \leq 1}$, so by dominated convergence theorem we get the convergence to zero of the first integral. Similarly,

$$
|xf_n(s)|^2 \mathbb{1}_{|xf_n(s)| \leq 1, |xf(s)| > 1} \leq \mathbb{1}_{|xf(s)| > 1},
$$

so by dominated convergence again, the second integral converges to zero. We deduce that

$$
\int_{\mathbb{R}^d \times \mathbb{R}} x f_n(s) \mathbb{1}_{|xf_n(s)| \leq 1} \tilde{J}_X(ds, dx) \to \int_{\mathbb{R}^d \times \mathbb{R}} x f(s) \mathbb{1}_{|xf(s)| \leq 1} \tilde{J}_X(ds, dx),
$$

$$
\int_{\mathbb{R}^d \times \mathbb{R}} x f_n(s) \mathbb{1}_{|xf_n(s)| \leq 1, |xf(s)| > 1} \tilde{J}_X(ds, dx) \to \int_{\mathbb{R}^d \times \mathbb{R}} x f(s) \mathbb{1}_{|xf(s)| \leq 1, |xf(s)| > 1} \tilde{J}_X(ds, dx),
$$

$$
\int_{\mathbb{R}^d \times \mathbb{R}} x f_n(s) \mathbb{1}_{|xf_n(s)| > 1} \tilde{J}_X(ds, dx) \to 0.
$$

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in $L^2(\Omega)$ as $n \to +\infty$, which implies the convergence in probability. The treatment of the compound Poisson term goes as follows:

$$\int_{\mathbb{R}^d \times \mathbb{R}} x f_n(s) \mathbb{1}_{|x f_n(s)| > 1} J_X(ds, dx) = \sum_{i \geq 1} X_i f_n(T_i) \mathbb{1}_{|X_i f_n(T_i)| > 1},$$

where the sum is finite almost surely. Then, $1_{|x f_n(s)| > 1} \leq 1_{|x f(s)| > 1}$, therefore $J_X(\{|x f_n(s)| > 1\})$ is an almost surely finite random variable, and does not depend on $n$. Therefore, since $f_n \to f$ almost everywhere, and since the law of $T_i$ is absolutely continuous with respect to Lebesgue measure, we deduce that

$$\sum_{i \geq 1} X_i f_n(T_i) \mathbb{1}_{|X_i f_n(T_i)| > 1} \to \sum_{i \geq 1} X_i f(T_i) \mathbb{1}_{|X_i f(T_i)| > 1} = \int_{\mathbb{R}^d \times \mathbb{R}} x f(s) \mathbb{1}_{|x f(s)| > 1} J_X(ds, dx),$$

almost surely as $n \to +\infty$, which implies the convergence in probability. We then have the following:

$$\left< \dot{X}, f_n \right> = \dot{I}(f_n) =: U(f_n) + \dot{I}_W(f_n) + \dot{I}_M(f_n) + \dot{I}_P(f_n) \quad \text{a.s.}$$

Also, we proved that $\dot{I}_W(f_n) + \dot{I}_M(f_n) + \dot{I}_P(f_n) \to \dot{I}_W(f) + \dot{I}_M(f) + \dot{I}_P(f)$ in probability as $n \to +\infty$. Also, $\left< \dot{X}, f_n \right>$ converges in probability to $\left< \dot{X}, f \right>$ as $n \to +\infty$, hence also in law. From these facts, we deduce that the deterministic part of the decomposition $U(f_n)$ converges as $n \to +\infty$, and from the expression of the characteristic function,

$$U(f_n) \to U(f) := \int_{\mathbb{R}^d} \gamma f(s) + \left( \int_{\mathbb{R}} x f(s) \left( \mathbb{1}_{|x f(s)| \leq 1} - \mathbb{1}_{|x| \leq 1} \right) \nu(dx) \right) ds \quad \text{as } n \to +\infty.$$

This concludes the proof. \qed

In general the inclusion $L(I) \subset L(\dot{X})$ is strict. For example, we can consider the case of an $\alpha$-stable white noise $\dot{X}_\alpha$ on $\mathbb{R}^d$, $\alpha \in (0, 2)$, that is a Lévy white noise on $\mathbb{R}^d$ with characteristic triplet $(0, 0, \nu_\alpha(dx))$, where $\nu_\alpha(dx) = \frac{dx}{|x|^{d+\alpha}}$. Then, we have that $L(\dot{X}_\alpha) = L^\alpha(\mathbb{R}^d)$ (see Proposition 5.11). On the other hand, $\varphi \in \dot{L}(I)$ if and only if

$$\int_{\mathbb{R}^d} \int_{|x| > 1} |x \varphi(s)| \wedge 1 \, ds \, \nu_\alpha(dx) + \int_{\mathbb{R}^d} \int_{|x| \leq 1} |x \varphi(s)|^2 \wedge |x \varphi(s)| \, ds \, \nu_\alpha(dx) < +\infty. \quad (4.7)$$

Replacing $\nu_\alpha(dx)$ by $\frac{1}{|x|^{d+\alpha}} \, dx$ yields when $\alpha \neq 1$

$$\frac{2}{\alpha} \text{Leb}_d \left( \{ s \in \mathbb{R}^d : |\varphi(s)| > 1 \} \right) + \int_{\mathbb{R}^d} |\varphi(s)|^\alpha \left( \frac{2}{\alpha} \mathbb{1}_{|\varphi(s)| \leq 1} + \frac{2}{2 - \alpha} \mathbb{1}_{|\varphi(s)| > 1} \right) ds + \frac{2}{1 - \alpha} \int_{\mathbb{R}^d} |\varphi(s)|^\alpha - |\varphi(s)| \, ds + \frac{2}{2 - \alpha} \int_{\mathbb{R}^d} |\varphi(x)|^2 \mathbb{1}_{|\varphi(x)| \leq 1} \, ds < +\infty.$$
For $\alpha = 1$, (4.7) is equivalent to
\[
\int_{\mathbb{R}^d} |\varphi(s)| \, ds + \int_{\mathbb{R}^d} |\varphi(s)| \ln (|\varphi(s)|) \, ds < +\infty.
\]

The integral representation via the operator $\tilde{I}$ is adaptive in the sense that the way we split the parts in the decomposition depends on the function $f$. Even if $\tilde{I}$ is linear, the different parts of the decomposition are not, which make the representation quite unusual. However, the traditional Lévy-Itô decomposition appears to be insufficient, since it requires an $L^1$-type constraint on $f$ that is artificial. This is better illustrated by the case of symmetric compound Poisson white noise. We shall see in Section 5.4.4 that $L(X_{\text{Poisson}}) = L^{0,2}(\mathbb{R}^d)$ (the space of measurable functions that are asymptotically in $L^2$; see Definition 5.4). However, due to [27, Lemma 12.13], we easily see that $L(I) = L^{0,2}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) = L^{1,2}(\mathbb{R}^d)$.

5 Practical Determination of the Domain of Definition

We provide here several criteria for the practical identification of the domain of definition of a Lévy white noise. We apply our result to the Gaussian, SaS, compound Poisson, and generalized Laplace noises. The results presented here are new for the two latter classes of noise to the best of our knowledge. Similar considerations are given for the domain of finite $p$th moments for $0 < p \leq 2$. In the rest of the paper, we shall consider the spaces $L^p(\dot{X})$ for $0 \leq p \leq 2$, with the convention that $L^0(\dot{X}) = L(\dot{X})$ is the domain of definition of the Lévy white noise $\dot{X}$.

5.1 Basic Properties

Proposition 5.1. Let $\dot{X}$ be a Lévy white noise and $p \geq 0$.

- Linearity: for $f, g \in L^p(\dot{X})$ and $\lambda \in \mathbb{R}$, $f + \lambda g \in L^p(\dot{X})$.

- Invariances: for $f \in L^p(\dot{X})$ and $H : \mathbb{R}^d \to \mathbb{R}^d$, a $C_1$-diffeomorphism, we have
  \[
t \mapsto f(H(t)) \in L^p(\dot{X}).
  \]

In particular, the translations $f(\cdot - t_0)$, rescalings $f(b \cdot)$, and rotations $f(R \cdot)$ of $f$, with $t_0 \in \mathbb{R}^d$, $b \neq 0$, and $R \in \text{SO}(d)$ an rotation matrix, are in $L^p(\dot{X})$.

Proof. The linearity is already known since $L^p(\dot{X})$ is a vectorial space by Proposition 3.9. For the invariance, we simply remark that, by the substitution $u = H(t)$,
\[
\int_{\mathbb{R}^d} \Psi_p(f(H(t))) \, dt = \int_{\mathbb{R}^d} |\det J_{H^{-1}}(u)| \Psi_p(f(u)) \, du
\]
with $J_H$ the invertible Jacobian matrix of $H$. By assumption on $H$, $|\det J_{H^{-1}}(u)|$ is bounded above and below by finite strictly positive constants, implying the result. \qed
For $\dot{X}$ a Lévy noise, the rescaling $\dot{X}(. / b)$ of a factor $b \neq 0$ is the generalized random process defined by $\langle \dot{X}(. / b), \varphi \rangle = \langle \dot{X}, b^d \varphi(b) \rangle$, for $\varphi \in \mathcal{D}(\mathbb{R}^d)$. We see easily that $\dot{X}(. / b)$ is itself a Lévy white noise. Similarly, for $a \neq 0$, the generalized random process $a \dot{X}$ is still a Lévy white noise. We say that two generalized random processes $s_1$ and $s_2$ independent if their finite-dimensional marginals are independent. By linearity, this is equivalent to the relation

$$
\widehat{\mathcal{P}}_{s_1 + s_2}(\varphi) = \widehat{\mathcal{P}}_{s_1}(\varphi) \widehat{\mathcal{P}}_{s_2}(\varphi), \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d).
$$

If $\dot{X}_1$ and $\dot{X}_2$ are two independent Lévy white noises, then $\dot{X}_1 + \dot{X}_2$ is also a Lévy white noise.

**Proposition 5.2.** Let $\dot{X}$ be a Lévy white noise and $p \geq 0$. Then we have, for $a$ and $b$ nonzero, and $t_0 \in \mathbb{R}^d$,

$$
L^p(\dot{X}) = L^p(a \dot{X}) = L^p(\dot{X}(. / b)) = L^p(\dot{X}(\cdot - t_0))
$$

If $X_1$ and $\dot{X}_2$ are two independent Lévy white noises, then

$$
L^p(\dot{X}_1) \cap L^p(\dot{X}_2) \subseteq L^p(\dot{X}_1 + \dot{X}_2).
$$

Moreover, if at least one of the two Lévy white noises is symmetric, then (5.1) is an equality.

**Proof.** We have $\langle \dot{X}(. / b), f \rangle = \langle \dot{X}, b^d f(b) \rangle$, so that $f \in L^p(\dot{X}(. / b))$ if and only if $b^d f(b) \in L^p(\dot{X})$. Then, $L^p(\dot{X})$ being a linear space that is invariant by rescaling (Proposition 5.1), the latter condition is equivalent to $f \in L^p(\dot{X})$, hence $L^p(\dot{X}(. / b)) = L^p(\dot{X})$. We proceed similarly for $L^p(a \dot{X})$ and $L^p(\dot{X}(\cdot - t_0))$.

For $i = 1, 2$, the Lévy triplet of $\dot{X}_i$ (\(\dot{X}_i\), respectively) is denoted by $(\gamma_i, \sigma_i^2, \nu_i)$ $(\gamma, \sigma^2, \nu)$, respectively, and the corresponding exponent is $\Psi_p, i$ $(\Psi_p, \text{ respectively})$. If $\dot{X}_1$ and $\dot{X}_2$ are independent, we have the relations

$$
\gamma = \gamma_1 + \gamma_2, \quad \sigma^2 = \sigma_1^2 + \sigma_2^2, \quad \nu = \nu_1 + \nu_2.
$$

Therefore, by the triangular inequality, we have

$$
\Psi_p(\xi) = \left| (\gamma_1 + \gamma_2) \xi + \int_{\mathbb{R}} x \xi \left( \mathbb{1}_{|x| \leq 1} - \mathbb{1}_{|x| \leq 1} \right) (\nu_1 + \nu_2) (dx) \right|

+ (\sigma_1^2 + \sigma_2^2) \xi^2 + \int_{\mathbb{R}} (|\xi x|^p \land |\xi x|^2) (\nu_1 + \nu_2) (dx)

\leq \Psi_{p,1}(\xi) + \Psi_{p,2}(\xi),
$$

which proves (5.1). When one of the noise is symmetric, for instance $\dot{X}_1$, the latter inequality is an equality since $\gamma_1 \xi + \int_{\mathbb{R}} x \xi \left( \mathbb{1}_{|x| \leq 1} - \mathbb{1}_{|x| \leq 1} \right) \nu_1 (dx) = 0$ and (5.1) is an equality.

In general, (5.1) is only an inclusion. Consider for instance the case where $\dot{X}_1$ and $\dot{X}_2$ have Lévy triplet $(1, 1, 0)$ and $(-1, 0, 0)$ respectively, meaning that $\dot{X}_1$ is a Gaussian white noise with drift $\gamma = 1$ and $\dot{X}_2$ a pure drift $\gamma = -1$. Then, $\dot{X}_1$ and $\dot{X}_2$ are clearly
independent, and \( \dot{X}_1 + \dot{X}_2 \) is a Gaussian white noise with no drift. Therefore, \( L^p(\dot{X}_1 + \dot{X}_2) = L^2(\mathbb{R}^d) \) but \( L^p(\dot{X}_1) \cap L^p(\dot{X}_2) = L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) (see Section 5.4.1 for more details on the determination of those domains).

For \( \gamma \in \mathbb{R} \) and \( \nu \) a Lévy measure, we set

\[
m_{\gamma,\nu}(\xi) = \left| \gamma \xi + \int_{\mathbb{R}} x \xi \left( 1_{|x| \leq 1} - 1_{|x| \leq 1} \right) \nu(dx) \right|.
\]

The next result is taken from [33].

**Proposition 5.3** (Reduction to the symmetric case without Gaussian part). Let \((\gamma, \sigma^2, \nu)\) be a Lévy triplet. We also denote by \( \nu_{\text{sym}} \) the symmetrization of \( \nu \). We consider the following Lévy white noises:

- \( \dot{X} \) with Lévy triplet \((\gamma, \sigma^2, \nu)\),
- \( \dot{X}_2 \) with Lévy triplet \((\gamma, 0, \nu)\),
- \( \dot{X}_{\text{sym}} \) with Lévy triplet \((0, \sigma^2, \nu_{\text{sym}})\).

Then, we have the following relations for \( p \geq 0 \):

- If \( \sigma^2 \neq 0 \), then
  \[
  L^p(\dot{X}) = L^2(\mathbb{R}^d) \cap L^p(\dot{X}_2). 
  \]
  \[
  \tag{5.2}
  \]
- In any case,
  \[
  L^p(\dot{X}) = L^p(\dot{X}_{\text{sym}}) \cap \{ f \in L(\dot{X}), \int_{\mathbb{R}^d} m_{\gamma,\nu}(f(t))dt < \infty \}. 
  \]
  \[
  \tag{5.3}
  \]

**Proof.** We can decompose \( \dot{X} = \dot{X}_2 + \dot{X}_{\text{Gauss}} \), where \( \dot{X}_2 \) and \( \dot{X}_{\text{Gauss}} \) are independent with respective Lévy triplets \((\gamma, 0, \nu)\) and \((0, \sigma^2, 0)\). Then, \( \dot{X}_{\text{Gauss}} \) is a Gaussian white noise, for which \( L^p(\dot{X}_{\text{Gauss}}) = L^2(\mathbb{R}^d) \). We apply (5.1) with equality (\( \dot{X}_{\text{Gauss}} \) being symmetric) to obtain (5.2). Finally, (5.3) is a reformulation of [33, Proposition 2.9].

Based on Proposition 5.3, we restrict our attention to symmetric Lévy white noises without Gaussian part. We first reduce to the case \( \sigma^2 = 0 \) thanks to (5.2). The only remaining part to deduce the general case from the symmetric one is the identification of functions \( f \) satisfying \( \int_{\mathbb{R}^d} m_{\gamma,\nu}(f(t))dt < \infty \). Primarily, for non-symmetric noise, this usually relies on \( L^1 \)-type conditions, but we leave this topic open for further investigations.

### 5.2 The spaces \( L^{p_0,p_\infty}(\mathbb{R}^d) \)

We introduce the family of function spaces that generalize the \( L^p \)-spaces for \( 0 < p < \infty \). They will be identified later on as the domain of definition of important classes of Lévy white noises. We first give some notations. For \( 0 \leq p_0, p_\infty < \infty \), we set

\[
p_{p_0,p_\infty}(\xi) := |\xi|^{p_0} 1_{|\xi| > 1} + |\xi|^{p_\infty} 1_{|\xi| \leq 1},
\]

\[
p_{\log,p_\infty}(\xi) := (1 + \log|\xi|) 1_{|\xi| > 1} + |\xi|^{p_\infty} 1_{|\xi| \leq 1}.
\]

with the convention that \( 0^0 = 1 \).
**Definition 5.4.** For $0 \leq p_0, p_\infty < \infty$, we define

$$L^{p_0,p_\infty}(\mathbb{R}^d) = \left\{ f \text{ measurable, } \rho_{p_0,p_\infty}(f) := \int_{\mathbb{R}^d} \rho_{p_0,p_\infty}(f(t)) \, dt < \infty \right\},$$

$$L^{\log,p_\infty}(\mathbb{R}^d) = \left\{ f \text{ measurable, } \rho_{\log,p_\infty}(f) := \int_{\mathbb{R}^d} \rho_{\log,p_\infty}(f(t)) \, dt < \infty \right\}.$$

For $p > 0$, we have $L^{p,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$. Roughly speaking, $p_0$ measures the local integrability of a function, while $p_\infty$ indicates the asymptotic one. This is illustrated by the following example. For $\alpha, \beta > 0$, the function $f(t) = |t|^{-\alpha} \mathbf{1}_{|t| < 1} + |t|^{-\beta} \mathbf{1}_{|t| \geq 1}$ is such that

$$\rho_{p_0,p_\infty}(f) = \int_{\mathbb{R}^d} (|f(t)|^{p_0} \mathbf{1}_{|f(t)| > 1} + |f(t)|^{p_\infty} \mathbf{1}_{|f(t)| \leq 1}) \, dt$$

$$= \int_{|t| < 1} |t|^{-p_0 \alpha} \, dt + \int_{|t| \geq 1} |t|^{-p_\infty \beta} \, dt.$$

Therefore, $f$ is in $L^{p_0,p_\infty}(\mathbb{R}^d)$ if and only if

$$\alpha < \frac{d}{p_0} \text{ and } \beta > \frac{d}{p_\infty}.$$  

The first inequality effectively refers to the integrability of $f$ at the origin (or local integrability), while the second covers its asymptotic integrability.

As we did in Sections 3.2 and 3.3 with the spaces $L(X)$ and $L^p(X)$, we rely on generalized Orlicz spaces [34, Chapter X] to identify the structure of the spaces $L^{p_0,p_\infty}(\mathbb{R}^d)$.

**Proposition 5.5.** We fix $p_0 \geq 0$ and $p_\infty > 0$. The functions $\rho_{p_0,p_\infty}$ and $\rho_{\log,p_\infty}$ are $\Delta_2$-regular $\varphi$-functions.

**Proof.** To simplify the notation, we write $\rho = \rho_{p_0,p_\infty}$ in this proof. The function $\rho$ is continuous, non-decreasing, symmetric, and vanishes at the origin (since $p_\infty \neq 0$). It is therefore a $\varphi$-function.

Then, we have the following decomposition

$$\rho(2\xi) = 2^{p_0} |\xi|^{p_0} \mathbf{1}_{|\xi| > 1} + 2^{p_0} |\xi|^{p_0} \mathbf{1}_{1/2 < |\xi| \leq 1} + 2^{p_\infty} |\xi|^{p_\infty} \mathbf{1}_{|\xi| \leq 1/2}.$$  

For $1/2 \leq |\xi| \leq 1$, we have that $|\xi|^{p_0-p_\infty} \leq \max(2^{p_\infty-p_0}, 1)$. Therefore, we have that

$$\rho(2\xi) \leq 2^{p_0} |\xi|^{p_0} \mathbf{1}_{|\xi| > 1} + 2^{p_0} \max(2^{p_\infty-p_0}, 1) |\xi|^{p_\infty} \mathbf{1}_{1/2 < |\xi| \leq 1} + 2^{p_\infty} |\xi|^{p_\infty} \mathbf{1}_{|\xi| \leq 1/2}$$

$$\leq \max(2^{p_0}, 2^{p_\infty}) \rho(\xi).$$

Therefore, $\rho$ is $\Delta_2$-regular. The proof for $\rho_{\log,p_\infty}$ is very similar. \hfill \Box

Proposition 5.5 coupled with Proposition A.3 allow us to identify the structure of the spaces $L^{p_0,p_\infty}(\mathbb{R}^d)$ and $L^{\log,p_\infty}(\mathbb{R}^d)$.
Proposition 5.6. We fix $p_0 \geq 0$ and $p_\infty > 0$. Then, $L^{p_0,p_\infty}(\mathbb{R}^d) = L^{p_0,0}(\mathbb{R}^d)$ is a generalized Orlicz space associated to the $\varphi$-function $\rho_{p_0,p_\infty}$. It is in particular a complete linear metric space for the $F$-norm
\[ \|f\|_{\rho_{p_0,p_\infty}} := \inf \{ \lambda > 0, \rho_{p_0,p_\infty}(f/\lambda) \leq \lambda \} \].

Finally, simple functions are dense in $L^{p_0,p_\infty}(\mathbb{R}^d)$. The same conclusions occur for $L^{\log,p_\infty}(\mathbb{R}^d)$. The following embeddings are easily deduced by bounding the $F$-norm of the considered function spaces.

Proposition 5.7. We fix $p_0 \geq 0$ and $p_\infty > 0$.

1. If $0 \leq p_1 \leq p_2 < \infty$, we have the embedding
\[ L^{p_2,p_\infty}(\mathbb{R}^d) \subseteq L^{p_1,p_\infty}(\mathbb{R}^d) \].

2. If $0 < p_1 \leq p_2 < \infty$, we have the embedding
\[ L^{p_0,p_1}(\mathbb{R}^d) \subseteq L^{p_0,p_2}(\mathbb{R}^d) \].

3. Conditions 1., 2., and 3. remain true by changing $p_0$ by log and we have the embeddings, for any $0 < p_0 \leq 2$ and $0 \leq p_\infty \leq 2$.
\[ L^{p_0,p_\infty}(\mathbb{R}^d) \subseteq L^{\log,p_\infty}(\mathbb{R}^d) \subseteq L^{0,p_\infty}(\mathbb{R}^d) \].

In Propositions 5.6 and 5.7, we restricted ourselves to the case when $p_\infty \neq 0$. The reason is that $\rho_{p_0,0}(0) \neq 0$, so that $\rho_{p_0,0}$ is not a $\varphi$-function. Therefore, we do not define a generalized Orlicz space in the sense of Rao and Ren [34]. The space $L^{p_0,0}(\mathbb{R}^d)$ can be described as follows: It is the space of functions in $L^{p_0}(\mathbb{R}^d)$ whose support has a finite Lebesgue measure. We do not specify any topological structure on those spaces, since they will not appear as the domain of definition of any Lévy noise. However, the space $L^{2,0}(\mathbb{R}^d)$ will play a role as a common subspace to all the domains of definition of the Lévy white noises (see Proposition 5.10).

5.3 Criteria for the Determination of the Domain of Definition

In this section, we consider a symmetric white noise $\hat{X}$ without Gaussian part and with symmetric Lévy measure $\nu$. In particular, for $p \geq 0$, the function $\Psi_p$ exponent in (3.6) simply becomes
\[ \Psi_p(\xi) = |\xi|^2 \int_{|x| \leq 1/|\xi|} |x|^2 \nu(dx) + |\xi|^p \int_{|x| > 1/|\xi|} |x|^p \nu(dx). \] (5.4)

The first criterion is applicable as soon as we are able to estimate the behavior of the function $\Psi_p$ at the origin and/or at infinity.
Proposition 5.8 (Criteria for the determination of the domain of definition). Let \( \hat{X} \) be a symmetric Lévy white noise without Gaussian part and \( 0 \leq p \leq 2 \).

1. Assume that \( \Psi_p(\xi) \leq C\rho_{p_0,p_\infty}(\xi) \) for some constant \( C > 0 \) and every \( \xi \), then we have the embedding

\[
L^{p_0,p_\infty}(\mathbb{R}^d) \subseteq L^p(\hat{X}). \tag{5.5}
\]

2. Assume that \( \rho_{p_0,p_\infty}(\xi) \leq C\Psi_p(\xi) \) for some constant \( C > 0 \) and every \( \xi \), then we have the embedding

\[
L^p(\hat{X}) \subseteq L^{p_0,p_\infty}(\mathbb{R}^d). \tag{5.6}
\]

3. Assume that \( \Psi_p(\xi) \sim A|\xi|^{p_\infty} \) and \( \Psi_p(\xi) \sim B|\xi|^{p_0} \), then

\[
L^p(\hat{X}) = L^{p_0,p_\infty}(\mathbb{R}^d). \tag{5.7}
\]

4. The same holds with \( L^{\log,p_\infty}(\mathbb{R}^d) \) instead of \( L^{p_0,p_\infty}(\mathbb{R}^d) \) if we replace \( |\xi|^{p_0} \) by \( \log|\xi| \).

Proof. The condition \( \Psi_p(\xi) \leq C\rho_{p_0,p_\infty}(\xi) \) implies that, for any function \( f \in L^{p_0,p_\infty}(\mathbb{R}^d) \), we have

\[
\|f\|_{\Psi_p} = \int_{\mathbb{R}^d} \Psi_p(f(t))dt \leq C \int_{\mathbb{R}^d} \rho_{p_0,p_\infty}(f(t))dt = C\|f\|_{p_0,p_\infty}.
\]

Therefore, the identity map is continuous from \( L^{p_0,p_\infty}(\mathbb{R}^d) \) to \( L^p(\hat{X}) \) proving (5.5). The proof of (5.6) is very similar. For the last point, we remark that the two functions \( \Psi_p \) and \( \rho_{p_0,p_\infty} \) do not vanish for \( \xi \neq 0 \), are continuous, and are equivalent at 0 and infinity. Hence, there exists two constants such that

\[
C_1\rho_{p_0,p_\infty}(\xi) \leq \Psi_p(\xi) \leq C_2\rho_{p_0,p_\infty}(\xi).
\]

We then apply (5.5) and (5.6) to obtain (5.7). \( \square \)

Note that the local integrability of test functions (parameter \( p_0 \)) is linked with the asymptotic behavior of \( \Psi_p \), while the asymptotic integrability (parameter \( p_\infty \)) is linked to the behavior of \( \Psi_p \) at 0.

If we know that the Lévy measure has some finite moments, then we obtain new information on the domain of definition of the Lévy white noise. For \( p, q \geq 0 \), we set

\[
m_{p,q}(\nu) := \int_{\mathbb{R}} \rho_{p,q}(x)\nu(dx) = \int_{|t| > 1} |t|^p \nu(dt) + \int_{|t| \leq 1} |t|^q \nu(dt), \tag{5.8}
\]

called the generalized moments of \( \nu \). Then, \( \nu \) being a Lévy measure, we have that \( m_{0,2}(\nu) < \infty \).

Consider the Lévy process \( X \), together with its corresponding Lévy white noise \( \hat{X} \), with Lévy triplet \((0, 0, \nu)\). Then, \( \hat{X} \) has finite \( p \)th moments if and only if \( m_{p,2}(\nu) < \infty \) [36 Theorem 25.3]. The quantity

\[
\beta_0 := \sup\{0 \leq p \leq 2, m_{p,2}(\nu) < \infty\}
\]

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is called the *Pruitt index* and was introduced in [32] to study the asymptotic behavior of Lévy processes. It measures the growth rate of $X$ at infinity [8, Section 5.3] and is therefore strongly related with the required rate of decay for the control of the Besov regularity of the Lévy white noise $X$ [19, Theorem 3].

In contrast, the *Blumenthal-Getoor index*, defined as

$$\beta_\infty := \inf \{ 0 \leq q \leq 2, \ m_{0,q}(\nu) < \infty \},$$

relies on the local regularity of $X$ and $\dot{X}$. This can be formulated in terms of the strong variation of $X$ [8, Section 5.4] or the local Besov regularity of $X$ (see [32] and of $\dot{X}$ (see [19, Corollary 3]).

In accordance with the previous remarks, the generalized moments of a Lévy measure $\nu$ have important interpretations for the local and asymptotic behaviors of the Lévy white noise and the corresponding Lévy process.

**Proposition 5.9.** Let $w$ be a symmetric Lévy noise without Gaussian part and with Lévy measure $\nu$.

- We assume that $m_{p,2}(\nu) < \infty$ for some $0 \leq p \leq 2$. Then, we have, for any $\xi \in \mathbb{R}$, that
  $$m_{p,2}(\nu) \rho_{p,2}(\xi) \leq \Psi_p(\xi) \leq m_{p,2}(\nu) \rho_{2,p}(\xi). \quad (5.9)$$

- We assume that $m_{p,2}(\nu) < \infty$ for some $p \geq 2$. Then, we have, for any $\xi \in \mathbb{R}$, that
  $$m_{p,2}(\nu) \rho_{2,p}(\xi) \leq \Psi_p(\xi) \leq m_{p,2}(\nu) \rho_{2,p}(\xi). \quad (5.10)$$

- For $p > 2$, we condense (5.9) and (5.10) as
  $$m_{p,2}(\nu) \rho_{\min(p,2),\max(p,2)}(\xi) \leq \Psi_p(\xi) \leq m_{p,2}(\nu) \rho_{\max(p,2),\min(p,2)}(\xi).$$

- If $m_{p_0,p_0}(\nu) < \infty$ for some $0 \leq p_0 \leq 2$, $0 < p_\infty < \infty$ and if $p \leq p_0, p_\infty$, then
  $$\Psi_p(\xi) \leq m_{\min(p_\infty,2),p_0}(\nu) \rho_{p_0,\min(p_\infty,2)}(\xi). \quad (5.11)$$

**Proof.** All the inequalities will be obtained by exploiting the position of $|x|$, $|\xi|$, or $|x\xi|$ with respect to 1. We first show (5.9), the proof for (5.10) being very similar. We start by proving the upper bound of (5.9). We first assume that $|\xi| \leq 1$. Then, using (5.3), we decompose $\Psi_p$ as

$$\Psi_p(\xi) = \int_{|x| \leq 1} |x\xi|^2 \nu(dx) + \int_{1 < |x| \leq \frac{1}{|\xi|}} |x\xi|^2 \nu(dx) + \int_{|x| > \frac{1}{|\xi|}} |x\xi|^p \nu(dx). \quad (5.12)$$

Since $p \leq 2$, we have that

$$\Psi_p(\xi) \leq \int_{|x| \leq 1} |x|^2 |\xi|^2 \nu(dx) + \int_{1 < |x| \leq \frac{1}{|\xi|}} |x\xi|^p \nu(dx) + \int_{|x| > \frac{1}{|\xi|}} |x\xi|^p \nu(dx)$$

$$= \left( \int_{|x| \leq 1} |x|^2 \nu(dx) + \int_{1 < |x|} |x|^p \nu(dx) \right) |\xi|^p$$

$$= m_{p,2}(\nu) |\xi|^p. \quad (5.13)$$
Assume now that $|\xi| > 1$. Then, we use the decomposition

$$
\Psi_p(\xi) = \int_{|x| \leq \frac{1}{|\xi|}} |x\xi|^2 \nu(dx) + \int_{\frac{1}{|\xi|} < |x| \leq 1} |x\xi|^p \nu(dx) + \int_{|x| > 1} |x\xi|^p \nu(dx).
$$

(5.14)

Again, due to $p \leq 2$, we have that

$$
\Psi_p(\xi) \leq \int_{|x| \leq \frac{1}{|\xi|}} |x\xi|^2 \nu(dx) + \int_{\frac{1}{|\xi|} < |x| \leq 1} |x\xi|^2 \nu(dx) + \int_{|x| > 1} |x|^p |\xi|^2 \nu(dx)
$$

$$
= \left(\int_{|x| \leq 1} |x|^2 \nu(dx) + \int_{1 < |x| \leq \frac{1}{|\xi|}} |x|^p \nu(dx) \right) |\xi|^2
$$

$$
= m_{p,2}(\nu) |\xi|^2.
$$

(5.15)

Combining (5.13) and (5.15), we deduce that $\Psi_p(\xi) \leq m_{p,2}(\nu) \rho_{2,p}(\xi)$.

For the lower bound in (5.9), we first assume that $|\xi| \leq 1$. Then, starting from (5.12), we have that

$$
\Psi_p(\xi) \geq \int_{|x| \leq \frac{1}{|\xi|}} |x|^2 |\xi|^2 \nu(dx) + \int_{\frac{1}{|\xi|} < |x| \leq 1} |x|^p |\xi|^2 \nu(dx) + \int_{|x| > 1} |x|^p |\xi|^2 \nu(dx)
$$

$$
= m_{p,2}(\nu) |\xi|^2.
$$

(5.16)

And finally, when $|\xi| > 1$, we have, using (5.14), that

$$
\Psi_p(\xi) \geq \int_{|x| \leq \frac{1}{|\xi|}} |x|^2 |\xi|^p \nu(dx) + \int_{\frac{1}{|\xi|} < |x| \leq 1} |x|^2 |\xi|^p \nu(dx) + \int_{|x| > \frac{1}{|\xi|}} |x|^p |\xi|^p \nu(dx)
$$

$$
= m_{p,2}(\nu) |\xi|^p.
$$

(5.17)

With (5.16) and (5.17), we deduce that $\Psi_p(\xi) \geq m_{p,2}(\nu) \rho_{p,2}(\xi)$ and (5.9) is proved.

Equation (5.11) is proved using the same principle. Assume that $|\xi| \leq 1$ and $p \leq p_\infty \leq 2$. Then, using (5.12), we deduce that

$$
\Psi_p(\xi) \leq \int_{|x| \leq \frac{1}{|\xi|}} |x|^2 |\xi|^{p_\infty} \nu(dx) + \int_{\frac{1}{|\xi|} < |x| \leq 1} |x\xi|^p \nu(dx) + \int_{|x| > \frac{1}{|\xi|}} |x\xi|^p \nu(dx)
$$

$$
= m_{p_\infty,2}(\nu) |\xi|^p.
$$

If now $p_\infty > 2$, we have, still for $|\xi| \leq 1$, that

$$
\Psi_p(\xi) \leq \int_{|x| \leq \frac{1}{|\xi|}} |x|^2 |\xi|^2 \nu(dx) + \int_{\frac{1}{|\xi|} < |x| \leq 1} |x|^{p_\infty} |\xi|^2 \nu(dx) + \int_{|x| > \frac{1}{|\xi|}} |\xi|^2 \nu(dx)
$$

$$
= m_{2,2}(\nu) |\xi|^2.
$$

We deduce that $\Psi_p(\xi) \leq m_{\min(p_\infty,2),2}(\nu) |\xi|^{\min(p_\infty,2)}$. 

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When $|\xi| > 1$, $p \leq p_0 \leq 2$, and $p < p_\infty$, we have using (5.14) that
\[
\Psi_p(\xi) \geq \int_{|x| \leq |\xi|^p} |x|^p \nu(dx) + \int_{|x| < |\xi|} |x|^p \nu(dx) + \int_{|x| > 1} |x|^{|\min(p,\xi)|} |\xi|^p \nu(dx)
= m_{\min(p,\xi),p_0}(\nu) |\xi|^p_0.
\]

Remarking that $m_{\min(p,\xi),2}(\nu) \leq m_{\min(p,\xi),p_0}(\nu)$ and combining the bounds for $|\xi| \leq 1$ and $|\xi| > 1$, we deduce (5.11). \hfill \square

**Proposition 5.10.** For any Lévy noise, we have
\[
L^{2,0}(\mathbb{R}^d) \subseteq L(\dot{X}) \subseteq L^{0,2}(\mathbb{R}^d), \quad (5.18)
\]
Let $0 < p \leq 2$. For any symmetric Lévy white noise such that $m_{p,2}(\nu) < \infty$, we have
\[
L^{2,p}(\mathbb{R}^d) \subseteq L^p(\dot{X}) \subseteq L^{p,2}(\mathbb{R}^d). \quad (5.19)
\]
Let $p \geq 2$. For any symmetric Lévy white noise such that $m_{p,2}(\nu) < \infty$, we have
\[
L^{p,2}(\mathbb{R}^d) \subseteq L^p(\dot{X}) \subseteq L^{2,p}(\mathbb{R}^d). \quad (5.20)
\]
For $p > 0$, assuming that $m_{p,2}(\nu) < \infty$, we condense (5.19) and (5.20) as
\[
L^{\max(p,2),\min(p,2)}(\mathbb{R}^d) \subseteq L^p(\dot{X}) \subseteq L^{\min(p,2),\max(p,2)}(\mathbb{R}^d). \quad (5.21)
\]
In particular, for any symmetric finite-variance Lévy noise
\[
L^2(\dot{X}) = L^2(\mathbb{R}^d). \quad (5.22)
\]
For any symmetric Lévy white noise without Gaussian part such that $m_{p,\infty,p_0}(\nu) < \infty$, with $0 \leq p \leq p_0, p_\infty \leq 2$, we have
\[
L^{p_0,p_\infty}(\mathbb{R}^d) \subseteq L^p(\dot{X}). \quad (5.23)
\]
**Proof.** When $w$ is symmetric without Gaussian part, (5.18), (5.19), and (5.22) are directly deduced from (5.9) by taking $p = 0$, $p$ general, and $p = 2$, respectively. Adding a Gaussian part does not change the conclusions since $L^{2,p}(\mathbb{R}^d) \subseteq L^p(\dot{X}_{\text{Gauss}}) = L^2(\mathbb{R}^d) \subseteq L^{p,2}(\mathbb{R}^d)$ for all $0 \leq p \leq 2$ and thanks to (5.2).

We now consider a general Lévy noise $\dot{X}$ with Lévy triplet $(\gamma, \sigma^2, \nu)$ and $w_{\text{sym}}$ its symmetric version with triplet $(0, \sigma^2, \nu_{\text{sym}})$. We already now that $L^{2,0}(\mathbb{R}^d) \subseteq L(\dot{X}_{\text{sym}}) \subseteq L^{0,2}(\mathbb{R}^d)$. Moreover, from (5.3), we know that
\[
L(\dot{X}) = L(\dot{X}_{\text{sym}}) \cap \{ f \in L(\dot{X}), \int_{\mathbb{R}^d} m_{\gamma,\nu}(f(t))dt < \infty \}. \quad (5.24)
\]
First, we have that $L(\dot{X}) \subseteq L(\dot{X}_{\text{sym}}) \subseteq L^{0,2}(\mathbb{R}^d)$. Second, due to (5.24), it is sufficient to prove that
\[
L^{2,0}(\mathbb{R}^d) \subseteq \{ f \in L(\dot{X}), \int_{\mathbb{R}^d} m_{\gamma,\nu}(f(t))dt < \infty \}
\]

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to deduce that $L^{2,0}(\mathbb{R}^d) \subseteq L(\dot{X})$. We remark that, for $|\xi| \leq 1$,

$$m_{\gamma,\nu}(\xi) = \left| \gamma \xi + \int_{1 \leq |x| \leq \frac{1}{|\xi|}} \xi t \nu(dx) \right| \leq |\gamma| + \int_{1 \leq |x| \leq \frac{1}{|\xi|}} \nu(dx)$$

and that, for $|\xi| > 1$,

$$m_{\gamma,\nu}(\xi) = \left| \gamma \xi + \int_{|\xi| \leq |x| \leq 1} \xi t \nu(dx) \right| \leq |\gamma| + \int_{|\xi| \leq |x| \leq 1} |\xi| t^2 \nu(dx) \leq \left( |\gamma| + \int_{|t| \leq 1} t^2 \nu(dx) \right) \xi^2.$$ 

Therefore, we have $m_{\gamma,\nu}(\xi) \leq C \rho_{2,0}(\xi)$ for some constant $C$, which implies that $L^{2,0}(\mathbb{R}^d)$ is included into $\{ f \in L(\dot{X}), \int_{\mathbb{R}^d} m_{\gamma,\nu}(f(t)) dt < \infty \}$, as expected. Finally, (5.23) is a direct consequence of (5.11).

Remarks.

• The embeddings (5.18) inform on the extreme cases. In particular, a function in $L^{2,0}(\mathbb{R}^d)$—the space of functions in $L^2(\mathbb{R}^d)$ whose support has a finite Lebesgue measure—can be applied to any Lévy noise. This includes in particular all the indicator functions $\mathbb{1}_B$ with $B$ a Borel set with finite Lebesgue measure, or the Daubechies wavelets that are compactly supported and in $L^2(\mathbb{R}^d)$. Remarkably, finite-variance compound Poisson noises reach the largest possible domain of definition $L^{0,2}(\mathbb{R}^d)$ (see below).

• Moreover, (5.23) is particularly important as it gives the implication of having finite moments of the form $\int_{|t| > 1} |t|^{p_\infty} \nu(dx) < \infty$ and $\int_{|t| \leq 1} |t|^{p_0} \nu(dx) < \infty$. This result will play a crucial role when identifying compatibility condition between a whitening operator and a Lévy white noise in Section 6.

• The embeddings (5.21) are useful to understand the finiteness of the moments of $\langle w, f \rangle$ for a Lévy white noise with finite $p$th-moments. In particular, a test function $f$ that is bounded with compact support is in the domain of definition of any noise and $\langle w, f \rangle$ has a finite $p$th-moment as soon as $w$ has.

### 5.4 Examples

In this section, we consider subfamilies of infinitely divisible laws that define important classes of Lévy white noises. For these different classes, we specify the domain of definition $L(\dot{X})$ and the domains $L^p(\dot{X})$ of the considered Lévy white noises.

The function $\mathbb{1}_{[0,1]^d} \in L^{2,0}(\mathbb{R}^d)$ is in the domain of definition of every Lévy white noise. Moreover, according to Proposition 3.11, a Lévy white noise with Lévy exponent $\psi$ is such that

$$\Phi_{\langle X, \mathbb{1}_{[0,1]^d} \rangle}(\xi) = \exp \left( \int_{\mathbb{R}^d} \psi(\xi \mathbb{1}_{[0,1]^d}(t)) dt \right) = \exp(\psi(\xi))$$

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since $\psi(0) = 0$. Therefore, the Lévy exponent of the Lévy white noise is also the Lévy exponent of the random variable $\langle \dot{X}, 1_{[0,1]^d} \rangle$. We take the convention that the terminology for the law of this random variable is inherited by the Lévy white noise. For instance, a white noise is said to be Gaussian if the random variable $\langle \dot{X}, 1_{[0,1]^d} \rangle$ is Gaussian.

We illustrate how to deduce the domain of definitions of Gaussian, SoS, generalized Laplace, and compound Poisson white noises.

5.4.1 Gaussian White Noises and Pure Drift White Noises

The Gaussian white noise of variance $\sigma^2$ is characterized by the Lévy triplet $(0, \sigma^2, 0)$. With Proposition [3.7] we directly obtain that, for every $0 \leq p \leq 2$,

$$L^p(\dot{X}_{\text{Gauss}}) = L^2(\mathbb{R}^d).$$

Based on these considerations and Proposition [5.3] we shall consider Lévy triplets with $\sigma^2 = 0$ from now.

Similarly, the pure drift white noise $\dot{X}_{\text{drift}}$ with mean $\gamma$ is defined from its triplet $(\gamma, 0, 0)$. We have in that case that $\dot{X}_{\text{drift}} = \gamma$ almost surely and $\dot{X}_{\text{drift}}$ is a constant—therefore non stochastic—process. Then, we have

$$L^p(\dot{X}_{\text{drift}}) = L^1(\mathbb{R}^d)$$

for every $0 \leq p \leq 2$.

If now $\dot{X} = \dot{X}_{\text{Gauss}} + \dot{X}_{\text{drift}}$ has Lévy triplet $(\gamma, \sigma^2, 0)$ with $\gamma$ and $\sigma^2 \neq 0$, then the domain is, due to (5.1) with equality,

$$L^p(\dot{X}) = L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d).$$

5.4.2 Non-Gaussian SoS White Noises

Stable random variables are an important subclass of infinitely divisible random variables. Extensive details on SoS random variables and random processes can be found in [35].

The extension of the symmetric $\alpha$-stable noise to an independently scattered random measure is in fact an $\alpha$-stable random measure in the sense of [35]. The identification of the space of deterministic integrable functions has already been carried out in this context in [35], and we merely re-state the result and prove it within our framework.

We fix $0 < \alpha < 2$. A random variable is symmetric-$\alpha$-stable (SoS) if its characteristic function can be written as $e^{-|\xi|^\alpha}$ for some $\gamma > 0$. For simplicity, we should only consider $\gamma = 1$ thereafter, since a different $\gamma$ will not change the domain of definition according to Proposition [5.2]. A SoS white noise $\dot{X}_\alpha$ is a Lévy white noise such that $\langle \dot{X}_\alpha, 1_{[0,1]^d} \rangle$ is a SoS random variable. Its characteristic functional is given for $\varphi \in \mathcal{D}(\mathbb{R}^d)$ by [43, Section 4.2.2]

$$\hat{\mathcal{P}}_{\dot{X}_\alpha}(\varphi) = \exp(-\|\varphi\|_\alpha).$$

**Proposition 5.11.** Let $0 < \alpha < 2$. Then, for every $0 \leq p < \alpha$, we have

$$L^p(\dot{X}_\alpha) = L^\alpha(\mathbb{R}^d).$$

For $p \geq \alpha$, we have $L^p(\dot{X}_\alpha) = \{0\}$. 

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Proof. The Lévy measure of $\dot{X}_\alpha$ is $\nu(dx) = \frac{C_\alpha}{|x|^{\alpha+1}}dx$ with $C_\alpha$ a constant. A non-trivial SoS random variable has an infinite $p$th-moment for $p \geq \alpha$, and for every $f \in L(\dot{X}_\alpha)$, $\langle \dot{X}, f \rangle$ is a SoS random variable. Hence $L^p(\dot{X}) = \{0\}$ for $p \geq \alpha$. The case of interest is therefore $0 \leq p < \alpha$. Then, from (5.4),

$$\Psi_p(\xi) = 2C_\alpha \int_0^{1/|\xi|} \frac{\xi^2}{x^{\alpha+1}}dx + 2C_\alpha \int_{1/|\xi|}^{\infty} \frac{|\xi|^p}{x^{\alpha+1-p}}dx$$

$$= 2C_\alpha |\xi|^{\alpha} \left( \int_0^{1} \frac{dy}{y^{\alpha-1}} + \int_{1}^{\infty} \frac{dy}{y^{\alpha+1-p}} \right)$$

$$= \left( \frac{2(2-p)C_\alpha}{(2-\alpha)(\alpha-p)} \right) |\xi|^\alpha.$$

Finally, the result follows from Proposition 5.8. \ \qed

5.4.3 Generalized Laplace White Noises

Our goal is to study the Laplace white noise, for which $\langle \dot{X}, 1_{[0,1]} \rangle$ follows a Laplace law. It requires to introduce the family of generalized Laplace laws. We follow here the terminology of [29, Section 4.1.1] and consider only the symmetric case.

A random variable $Y$ is called a generalized Laplace random variable if its characteristic function can be written as

$$\Phi(\xi) = \frac{1}{(1 + \frac{1}{2}\sigma^2 \xi^2)^\tau} = \exp \left( -\tau \log(1 + \frac{1}{2}\sigma^2 \xi^2) \right),$$

with $\tau > 0$ the shape parameter and $\sigma^2$ the scaling parameter. We denote this situation by $Y \sim GL(\sigma, \tau)$. Note that the variance of $Y$ is $\tau \sigma^2$. When $\tau = 1$, we recover the traditional Laplace law. The generalized Laplace laws are infinitely divisible [29, Section 2.4.1] and associated with the Lévy triplet $(0, 0, \nu_{\tau, \sigma^2})$ with [29, Proposition 2.4.2]

$$\nu_{\tau, \sigma^2}(dx) = \tau \frac{1}{|x|} e^{-2|x|/\sigma^2} dx.$$

Definition 5.12. We say that a Lévy white noise $\dot{X}_{Laplace}$ is a generalized Laplace white noise if $\langle \dot{X}_{Laplace}, 1_{[0,1]} \rangle \sim GL(\sigma, \tau)$ for some $\tau, \sigma^2 > 0$. We call $\tau$ and $\sigma^2$ respectively the shape parameter and the scaling parameter of $\dot{X}_{Laplace}$. When $\tau = 1$, we simply say that $\dot{X}_{Laplace}$ is a Laplace white noise.

For the best of our knowledge, general integrability conditions for (generalized) Laplace noise has not been investigated in the literature. Proposition 5.13 provides such conditions.

Proposition 5.13. For every generalized Laplace white noise $\dot{X}_{Laplace}$, we have

$$L(\dot{X}_{Laplace}) = L^{log,2}(\mathbb{R}^d). \quad (5.25)$$

Moreover, for $0 < p \leq 2$, we have

$$L^p(\dot{X}_{Laplace}) = L^{p,2}(\mathbb{R}^d). \quad (5.26)$$
Proof. Let $0 \leq p \leq 2$. We start from (5.4) and write

$$\Psi_p(\xi) = \xi^2 \int_{|x| \leq 1/|\xi|} x^2 \nu_{\tau, \sigma^2}(dx) + |\xi|^p \int_{|x| > 1/|\xi|} |x|^p \nu_{\tau, \sigma^2}(dx) := \Psi_{p,1}(\xi) + \Psi_{p,2}(\xi).$$

Without loss of generality, we consider the case $\sigma^2 = 2$ and $\tau = 1$, in which case $\nu_{1,2}(dx) = e^{-|x|}dx$. Then, by integration by parts, we have

$$\Psi_{p,1}(\xi) = 2|\xi|^2 \int_0^{1/|\xi|} x e^{-x} dx = 2|\xi|^2 \left(1 - e^{-1/|\xi|}(1 + \frac{1}{|\xi|})\right).$$

Hence, we have $\Psi_{p,1}(\xi) \to 2$ and $\Psi_{p,1}(\xi) \sim 2|\xi|^2$.

For $\Psi_{p,2}(\xi) = |\xi|^p \int_{|x| > 1/|\xi|} |x|^p \nu_{\tau, \sigma^2}(dx)$, we shall distinguish between $p = 0$ and $p > 0$.

For $p > 0$, the function $x^{p-1}e^{-x}$ is integrable over $\mathbb{R}$, so that $\Psi_{p,2}(\xi) \sim \left(\int_{\mathbb{R}} x^{p-1}e^{-x} dx\right) |\xi|^p$.

For $p = 0$, the function $x^{-1}e^{-x}$ is not anymore integrable around 0. Using the equivalence $x^{-1}e^{-x} \sim x^{-1}$, we deduce that

$$\Psi_{p,2}(\xi) = 2 \int_{1/|\xi|}^{\infty} x^{-1}e^{-x} dx \sim 2 \int_{1/|\xi|}^{1} x^{-1}e^{-x} dx \sim 2 \int_{1/|\xi|}^{1} x^{-1} dx = 2 \log |\xi|.$$

Moreover, since $p \leq 2$, we have, again by integration by parts,

$$\Psi_{p,2}(\xi) = 2 \int_{|x| > 1} (x|\xi|)^p e^{-\frac{x}{x}} dx \leq 2 \int_{|x| > 1} (x|\xi|)^2 e^{-\frac{x}{x}} dx = 2|\xi|(1 + |\xi|)e^{-1/|\xi|},$$

implying that $\Psi_{p,2}(\xi) \sim o(|\xi|^2)$. By combining the results on $\Psi_{p,1}$ and $\Psi_{p,2}$, we obtain that

- for $0 \leq p \leq 2$, $\Psi_p(\xi) \sim 2|\xi|^2$;
- for $0 < p \leq 2$, $\Psi_p(\xi) \sim \left(\int_{\mathbb{R}} x^{p-1}e^{-x} dx\right) |\xi|^p$;
- for $p = 0$, $\Psi_0(\xi) = \Psi(\xi) \sim 2 \log |\xi|$.

We apply now Proposition 5.8 to deduce (5.25) and (5.26). □

5.4.4 Compound Poisson White Noises

Definition 5.14. A Lévy white noise is a compound Poisson noise if its Lévy triplet has the form $(0, 0, \nu)$ and if its Lévy measure satisfies

$$\lambda := \int_{\mathbb{R}} \nu(dx) < \infty.$$

In that case, $\nu = \lambda \mathbb{P}$ with $\mathbb{P}$ a probability measure.
One can represent a compound Poisson noise $\dot{X}_{\text{Poisson}}$ as [42, Theorem 1]:

$$
\dot{X}_{\text{Poisson}} = \sum_{n \geq 0} a_n \delta(\cdot - t_n)
$$

with $\delta$ the Dirac distribution, $(a_n)$ i.i.d. random variables with probability law $\mathbb{P}$, and $(t_n)$ independent of $(a_n)$ such that, for every Borel set $B \in \mathbb{R}^d$, the random number of elements $t_n$ in $B$ follows a Poisson law with rate $\lambda \text{Leb}(B)$. Then, $\dot{X}_{\text{Poisson}}$ has a finite variance if and only if $\mathbb{P}$ has. In that case, it has a zero mean if and only if $\mathbb{P}$ has.

The integration with respect to compound Poisson noise is treated for instance in [27, Chapter 12]. However, it relies on $L^1$-type conditions, inherited from the fact that the stochastic integration follows the Lévy-Itô decomposition (see Section 4.1 for more details). We provide more general integrability conditions in Proposition 5.15, that are not sensitive to $L^1$-type integrability.

**Proposition 5.15.** If $\dot{X}_{\text{Poisson}}$ is a symmetric compound Poisson white noise with finite variance, then

$$
L^p(\dot{X}_{\text{Poisson}}) = L^{p,2}(\mathbb{R}^d).
$$

for every $0 \leq p \leq 2$.

**Proof.** First, $L^p(\dot{X}_{\text{Poisson}}) \subseteq L^{p,2}(\mathbb{R}^d)$ as for any symmetric Lévy white noise, according to (5.19). Moreover, for a compound Poisson noise with finite variance, we have for every $q \in [0, 2]$ that $\int_{\mathbb{R}} |x|^q \mathbb{P}(dx) < \infty$. Therefore, we have

$$
\Psi_p(\xi) = \lambda \int_{\mathbb{R}} (|x\xi|^p \wedge |x\xi|^2) \mathbb{P}(dx)
\leq \lambda \min \left( |\xi|^p \int_{\mathbb{R}} |x|^p \mathbb{P}(dx), |\xi|^2 \int_{\mathbb{R}} |x|^2 \mathbb{P}(dx) \right)
\leq C(|\xi|^p \wedge |\xi|^2) = \rho_{p,2}(\xi),
$$

so that $\|f\|_{\Psi_p} \leq C\|f\|_{p,2}$. This means that $L^{p,2}(\mathbb{R}^d) \subseteq L^p(\dot{X}_{\text{Poisson}})$, finishing the proof. \qed

We summarize the results of this section in Table 1.

6 Application to Linear Stochastic Partial Differential Equations

Until now, we restricted ourself to the study of Lévy white noise. In this section, we see how to apply our results to solve linear stochastic differential equations of the form

$$
Ls = \dot{X}
$$

with $\dot{X}$ a Lévy white noise and $L$ a differential operator. We give new conditions of compatibility between the operator $L$ and the white noise $\dot{X}$ such that the process $s$ exists as a generalized random process. Our results extend our previous works [18, 43, 44].
Table 1: Definition Domains of some Lévy White Noises

| White noise         | Parameters | $\Phi_{(X,\mathbb{H}_{[0,1]})}(\xi)$ | $L(\hat{X})$ | $L^p(\hat{X})$ |
|---------------------|------------|--------------------------------------|---------------|----------------|
| Gaussian            | $\sigma^2 > 0$ | $e^{-\sigma^2 \xi^2}$ | $L^2(\mathbb{R}^d)$ | $L^2(\mathbb{R}^d)$ |
| Pure drift          | $\gamma \in \mathbb{R}$ | $e^{i\gamma \xi}$ | $L^1(\mathbb{R}^d)$ | $L^1(\mathbb{R}^d)$ |
| SoS                 | $0 < \alpha < 2$ | $e^{-|\xi|^{\alpha}}$ | $L^{\alpha}(\mathbb{R}^d)$ | $\begin{cases} L^{\alpha}(\mathbb{R}^d) & \text{if } p < \alpha \\ \{0\} & \text{if } p \geq \alpha \end{cases}$ |
| generalized         | $\sigma^2 > 0$ | $\frac{1}{(1+\sigma^2 \xi/2)^\tau}$ | $L^{log,2}(\mathbb{R}^d)$ | $L^{p,2}(\mathbb{R}^d)$ |
| Laplace             | $\tau > 0$ | $\lambda > 0$ | $L^{0,2}(\mathbb{R}^d)$ | $L^{p,2}(\mathbb{R}^d)$ |
| symmetric finite-variance | $\mathbb{P}$ | $e^{\lambda(\hat{\mathbb{P}}(\xi)-1)}$ | |
| compound Poisson    |            |                                     |               |                |

**Theorem 6.1.** We consider a Lévy white noise $\hat{X}$. We assume that $T$ is a continuous and linear operator from $\mathcal{D}(\mathbb{R}^d)$ to $L(\hat{X})$. Then, the mapping

$$s : \mathcal{D}(\mathbb{R}^d) \rightarrow L^0(\Omega)$$

$$\varphi \mapsto \langle s, \varphi \rangle := \langle \hat{X}, T\{\varphi\} \rangle$$

(6.1)

specifies a generalized random process in the sense of Definition 2.1.

If moreover there exists an operator $L$ such that $TL^*\varphi = \varphi$ for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$ (left-inverse property), then we have that

$$Ls = w.$$ (6.2)

**Proof.** The mapping $\varphi \mapsto \langle \hat{X}, T\{\varphi\} \rangle$ is well-defined for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$ because $T\{\varphi\} \in L(\hat{X})$ by assumption. It is actually the composition of the operator $T$ with the random linear functional $\hat{X}$. This two mappings being linear, the composition is linear. Moreover, $T$ is continuous from $\mathcal{D}(\mathbb{R}^d)$ to $L(\hat{X})$ by assumption and $\hat{X}$ is continuous from $(\hat{X})$ to $L^0(\Omega)$ according to Theorem 1.10. Therefore, $s$ is linear and continuous mapping from $\mathcal{D}(\mathbb{R}^d)$ to $L(\hat{X})$ and therefore a valid generalized random process in $\mathcal{D}'(\mathbb{R}^d)$.

If now $TL^*\{\varphi\} = \varphi$ for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$, then the process $Ls$, defined as

$$Ls : \varphi \mapsto \langle s, L^*\varphi \rangle,$$ (6.3)

satisfies the relation

$$\langle Ls, \varphi \rangle = \langle \hat{X}, TL^*\varphi \rangle = \langle \hat{X}, \varphi \rangle$$ (6.4)

for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Equivalently, we have shown that $Ls = \hat{X}$ as elements of $\mathcal{D}'(\mathbb{R}^d)$, as expected. 

We justify shortly the assumptions of Theorem 6.1. Many differential operators admit a natural inverse $L^{-1}$, that is typically defined using the Green’s function of $L$. A solution of (6) can therefore be formally written as $s = L^{-1}w$; that is,

$$\langle s, \varphi \rangle = \langle L^{-1}\hat{X}, \varphi \rangle = \langle \hat{X}, (L^*)^{-1}\{\varphi\} \rangle.$$ (6.5)
In order to be valid, (6.5) should at least be meaningful for any \( \varphi \in \mathcal{D}(\mathbb{R}^d) \). It means in particular that \( (L^*)^{-1}\{\varphi\} \) should be in domain of definition of the Lévy white noise \( \dot{X} \). However, for many differential operators, including the derivative, the natural inverse operator to \( L^* \) exists but is not stable in the sense that it is not continuous from \( \mathcal{D}(\mathbb{R}^d) \) to any domain of definition \( L(\dot{X}) \). In that case, it is required to correct \( (L^*)^{-1} \) in order to make it stable. The role of the corrected version of \( (L^*)^{-1} \) is played by \( T \). Thanks to (6.4), we moreover see that we only require that \( T \) is a left-inverse of \( L^* \).

**Definition 6.2.** A generalized random process constructed according to Theorem 6.1 is called a **generalized Lévy process** (or **generalized Lévy field** when \( d \geq 2 \)). The operator \( L \) is the whitening operator of \( s \) and \( \dot{X} \) the underlying Lévy white noise.

The following result links the stability properties of the corrected left-inverse operator \( T \) with the finiteness of the generalized moments of the Lévy measure of \( \dot{X} \).

**Proposition 6.3.** We consider a symmetric Lévy white noise without Gaussian part \( \dot{X} \) and a linear, continuous, and shift-invariant operator \( L \). We assume that, for \( 0 \leq p_0, p_\infty \leq 2 \), we have

\[
\begin{align*}
\bullet \ m_{p_\infty,p_0}(\nu) &= \int_{\mathbb{R}} \rho_{p_\infty,p_0}(t)\nu(dt) < \infty, \text{ and} \\
\bullet \ \text{the adjoint operator } L^* \ &\text{admits a left-inverse } T \text{ that maps continuously } \mathcal{D}(\mathbb{R}^d) \text{ to } L^{p_0,p_\infty}(\mathbb{R}^d).
\end{align*}
\]

Then, there exists a generalized Lévy process \( s \) such that \( Ls = \dot{X} \).

**Proof.** Applying (5.23) with \( p = 0 \), the condition \( \int_{\mathbb{R}} \rho_{p_\infty,p_0}(t)\nu(dt) < \infty \) ensures that \( L^{p_0,p_\infty}(\mathbb{R}^d) \subset L(\dot{X}) \). This embedding and the assumption on \( T \) imply that \( T \) maps continuously \( \mathcal{D}(\mathbb{R}^d) \) to \( L(\dot{X}) \), and Theorem 6.1 applies.

We recall that the condition \( m_{p_\infty,p_0}(\nu) < \infty \) introduced in (5.8) is connected with the local properties (parameter \( p_0 \)) and the asymptotic properties (parameter \( p_\infty \)) of the Lévy white noise.

**Comparison with previous works.** Theorem 6.1 and Proposition 6.3 can be compared with other conditions of compatibility between the whitening operator \( L \) and the Lévy white noise \( \dot{X} \). The results are reformulated with our notation.

- First of all, we differentiate between two types of solutions of the linear SDE \( Ls = w \). We say that \( s \) is a **generalized solution** if \( Ls = w \) almost surely. In contrast, a generalized random process \( s \) such that \( Ls = w \) in law (that is, \( \mathcal{P}_{Ls} = \mathcal{P}_w \)) is called a **solution in law**. In our previous works, we constructed solutions in law of (6) essentially relying on the Minlos-Bochner theorem for the construction of \( s \). One important contribution of this paper is to construct generalized solutions, what requires the identification of the domain of definition of the Lévy white noise \( X \) to be as general as possible.
Throughout the paper, we have considered Lévy white noise as random elements in $\mathcal{D}'(\mathbb{R}^d)$. This is in line with the original work of Gelfand and Vilenkin [23]. It can be of interest, however, to restrict to the class of tempered Lévy white noise, that is, to consider Lévy white noise, and by extension generalized Lévy processes, as random elements in the space $S'(\mathbb{R}^d)$. The construction of generalized Lévy processes in $S'(\mathbb{R}^d)$ instead of $\mathcal{D}'(\mathbb{R}^d)$ can be found in [43]; see also [5] for a recent exposition of the main results in this framework. For a comparison between the two constructions, we refer to [18, Chapter 3].

Not every Lévy white noise is tempered. Actually, a Lévy white noise is tempered if and only if it has finite $p$th moments for some $p > 0$ (see [14]). The main point is that the construction is really analogous, since the spaces $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ are nuclear. Among the consequences, we mention that Theorem 6.1 and Proposition 6.3 remain valid when replacing $\mathcal{D}(\mathbb{R}^d)$ by $S(\mathbb{R}^d)$. In that case, the processes $\hat{X}$ and $s$ are both located in $S'(\mathbb{R}^d)$. In what follows, when comparing these two results with previous contributions, we consider the version of the result for tempered generalized random processes.

For $1 \leq p \leq 2$, the Lévy exponent $\psi$ is $p$-admissible if $|\psi(\xi)| + |\xi| |\psi'(\xi)| \leq C |\xi|^p$. Note that the derivative $\psi'(\xi)$ is well-defined as soon as the first moment of the underlying infinitely divisible random variable is finite, what we assume now. This notion was introduced in [43] together with the following compatibility condition: if $\psi$ is $p$-admissible and $T$ continuously map $S(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, then there exists a solution in law of $L_s = w$ with characteristic functional

$$\hat{\mathcal{P}}_s : \varphi \mapsto \hat{\mathcal{P}}_{\hat{X}}(T\{\varphi\}). \quad (6.6)$$

A sufficient condition for the $p$-admissible is that $\int_{\mathbb{R}} |t|^p \nu(dt) < \infty$. Therefore, (6.6) is a valid characteristic functional as soon as $\int_{\mathbb{R}} |t|^p \nu(dt) < \infty$ and $T$ maps continuously $S(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for some $1 \leq p \leq 2$. We recover this result by selecting $p_0 = p_\infty = p$ in Proposition 6.3. Actually, Proposition 6.3 extends this criterion in three ways. First, we can distinguish between the behavior of $\nu$ around 0 and at $\infty$. Second, we do not restrict to the case $p \geq 1$ (this second improvement was already achieved in our work [20] thanks to a relaxation of the $p$-admissibility). Finally, as we have said already, we specify generalized solutions, and not only solutions in law, of $L_s = \hat{X}$.

In our work with A. Amini, we have shown that the characteristic functional (6.6) specifies a generalized Lévy process if $\int_{\mathbb{R}} \rho_{p_\infty,p_0}(t)\nu(dt)$ and $T$ maps continuously $S(\mathbb{R}^d)$ to $L^{p_0,p_\infty}(\mathbb{R}^d)$ for $0 < p_\infty \leq p_0 \leq 2$ [18, Theorem 5]. When $p_\infty \leq p_0$, we have that

$$\max(|\xi|^{p_0}, |\xi|^{p_\infty}) \leq \rho_{p_0,p_\infty}(\xi) \leq |\xi|^{p_0} + |\xi|^{p_\infty}.$$ 

Therefore, $L^{p_0,p_\infty}(\mathbb{R}^d) = L^{p_0}(\mathbb{R}^d) \cap L^{p_\infty}(\mathbb{R}^d)$ and we recover our previous result (at least for symmetric Lévy white noise without Gaussian part). Moreover, Proposition 6.3 is a improvement, since one can consider $p_\infty > p_0$. In that case, $L^{p_0,p_\infty}(\mathbb{R}^d)$ contains but is strictly bigger than $L^{p_0}(\mathbb{R}^d) \cap L^{p_\infty}(\mathbb{R}^d)$ and the requirement on $T$ is less restrictive.
Combining (5.23) and Proposition 3.12, we generalize [4, Theorem 2] again by considering the case $p_\infty > p_0$: we are able to specify a larger domain of definition and of continuity than $L^{p_0}(\mathbb{R}^d) \cap L^{p_\infty}(\mathbb{R}^d)$ in that case.

A Generalized Orlicz Spaces

Definition A.1. We say that $\rho : \mathbb{R} \to \mathbb{R}^+$ is a $\varphi$-function if $\rho(0) = 0$ and $\rho$ is symmetric, continuous, and nondecreasing on $\mathbb{R}^+$. The $\varphi$-function $\rho$ is $\Delta_2$-regular if

$$\rho(2\xi) \leq M\rho(\xi)$$

for some $M, \xi_0 > 0$, and every $\xi \geq \xi_0$.

Definition A.2. Let $\rho$ be a $\varphi$-function. For $f : \mathbb{R}^d \to \mathbb{R}$, we set

$$\rho(f) := \int_{\mathbb{R}^d} \rho(f(t))dt.$$ 

The generalized Orlicz space associated to $\rho$ is

$$L^\rho(\mathbb{R}^d) := \{ f \text{ measurable}, \exists \lambda > 0, \rho(f/\lambda) < \infty \}.$$

Orlicz spaces were introduced in [6] as natural generalizations of $L^p$-spaces for $p \geq 1$. A systematic study with important extensions was done by J. Musielak [31]. The initial theory deals with Banach spaces, excluding for instance the $L^p$-spaces with $0 < p < 1$. Definition A.2 generalizes the Orlicz spaces in two ways: One does not require that $\rho$ is convex, neither that $\rho(\xi) \to \infty$ as $\xi \to \infty$. The need for a non-locally convex framework (related to non-convex $\varphi$-function) is notable in stochastic integration. It was initiated by K. Urbanik and W.A. Woyczyns [35]. It is at the heart of the study of the structure developed by Rajput and Rosinski. We follow here the exposition of M.M. Rao and Z.D. Ren in [34, Chapter X]. Proposition A.3 summarizes the results on generalized Orlicz spaces.

Proposition A.3. If $\rho$ is a $\Delta_2$-regular $\varphi$-function, then we have

$$L^\rho(\mathbb{R}^d) = \{ f \text{ measurable}, \forall \lambda > 0, \rho(f/\lambda) < \infty \} = \{ f \text{ measurable}, \rho(f) < \infty \}.$$

The space $L^\rho(\mathbb{R}^d)$ is a complete linear metric space for the F-norm

$$\|f\|_\rho := \inf\{\lambda > 0, \rho(f/\lambda) \leq \lambda\}$$

on which simple functions are dense. Moreover, we have the equivalence, for any sequence of elements $f_k \in L^\rho(\mathbb{R}^d)$,

$$\|f_k\|_\rho \underset{k\to\infty}{\to} 0 \Leftrightarrow \rho(f_k) \underset{k\to\infty}{\to} 0.$$

Acknowledgements

The authors are warmly grateful to Prof. Robert Dalang and Prof. Michael Unser for fruitful discussions. The precious comments they provided were of critical help in the preparation phase of the current manuscript.
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