PERFECT MONOIDS WITH ZERO AND CATEGORIES OF $S$-ACTS

JOSEF DVOŘÁK AND JAN ŽEMLIČKA

Abstract. In this paper, we study the relationship between the two main categories of $S$-acts for a monoid $S$ with zero from the viewpoint of existence of projective covers and the equivalence is proven. Furthermore, monoids with zeros over which all compact acts are cyclic are characterized.

1. Introduction

The usefulness of the notion of projective cover within the context of module theory has been confirmed in countless occasions since the publication of the founding works of Bass [2], who coined the term, and Eilenberg [4], who effectively considered the notion for the first time. Together with the idea of projective cover, the closely related notion of a perfect ring, for which projective covers exist in the corresponding module category, appears. The definition of both terms uses a purely categorial language, yet structural and homological characterizations can be given, e.g.,

**Theorem 1.** [2] The following conditions are equivalent for a ring $R$

1. $R$ is left perfect
2. $R$ satisfies d.c.c. on principal right ideals
3. the class of projective $R$-modules coincides with the class of flat $R$-modules.

The corresponding notion within the branch of monoids and acts turned out to be similarly fruitful with applications to category theory and topological monoids (see [6]). Note that the Theorem 1 has its counterpart stated for monoids:

**Theorem 2.** [5, 6, 8] The following conditions are equivalent for a monoid $S$

1. $S$ is left perfect
2. $S$ satisfies the minimum condition on principal right ideals and each left $S$-act satisfies the a.c.c for cyclic subacts
3. the class of projective $S$-acts coincides with the class of strongly flat left $S$-acts.

The previous result as well as other results have been formulated and considered within the context of the category $S - \text{Act}$ (see below), but for a monoid with zero, the monograph [9] introduces another natural category, $S - \text{Act}_0$, which turns out to possess notably different categorial properties regarding e.g. its extensivity or compactness of objects (cf. [3]), hence the question of relationship of these two categories from the viewpoint of perfectness arises naturally and the aim of the present paper is an investigation on this topic.

The question of perfectness appears to be related to the problem over which monoids $S$ (or rings) there exists a non-cyclic act such that the corresponding covariant Hom-functor from a category of $S$-acts (or $S$-modules) commutes with coproducts (such monoid is then called non-steady). It is known that non-steady monoids are necessarily non-perfect in the category $S - \text{Act}$ (as well as in the case of modules).

The main tool of the paper is the functor $F : S - \text{Act} \to S - \text{Act}_0$ gluing all zero elements to one using Rees factor. It allows translating the properties of $S - \text{Act}$ to the category $S - \text{Act}_0$. Namely, Theorem 18 shows that the left perfectness of categories $S - \text{Act}$ and $S - \text{Act}_0$ coincide and a monoid with zero is left perfect if and only if it is left 0-perfect and it is left 0-steady if and only if it satisfies the ascending chain condition on cyclic subacts by Theorem 22.

2010 Mathematics Subject Classification. 20M50 (20M30).

Key words and phrases. act over a monoid, perfect monoid, steady monoid.

This work is part of the project SVV-2020-260589.
2. Preliminaries

Before we begin the exposition, let us recall some necessary terminology and notations.

Let $S = (S, 1)$ be a monoid and $A$ a nonempty set. If there exists a mapping $\cdot : S \times A \to A$ satisfying the following two conditions: $1 \cdot a = a$ and $(s_1 \cdot s_2) \cdot a = s_1 \cdot (s_2 \cdot a)$ then $A$ is said to be a left $S$-act and it is denoted $A^S$. A mapping $f : S^A \to S$ is a homomorphism of $S$-acts (an $S$-homomorphism) provided $f(sa) = sf(a)$ holds for all pairs $s \in S, a \in A$. In compliance with [9, Example I.6.5.] we denote by $S \rightarrow S$ the category of all left $S$-acts with homomorphisms of $S$-acts and $S \rightarrow \operatorname{Act}$ the category $S \rightarrow \operatorname{Act}$ enriched by an initial object $\emptyset$. Let the monoid $S$ contain a (necessarily unique) zero element $0$, which satisfies $0 \cdot a = 0 = 0 \cdot s$ for all $s \in S$. Then the category of all left $S$-acts $A$ with a unique zero element $\theta_A = 0A$ and homomorphisms of $S$-acts compatible with zero as morphisms will be denoted $S \rightarrow \operatorname{Act}_0$. Observe that $\theta := \{0\}$ is the initial object of the category $S \rightarrow \operatorname{Act}_0$ (but not of the category $S \rightarrow \operatorname{Act}$).

Recall that both of the categories $S \rightarrow \operatorname{Act}$ and $S \rightarrow \operatorname{Act}_0$ are complete and cocomplete [9, Remarks II.2.11, Remark II.2.22]. In particular, the coproduct of a system of objects $(A_i, i \in I)$ is

(i) $\bigcup_{i \in I} A_i = \bigcup_{i \in I} A_i$ in $S \rightarrow \operatorname{Act}$ by [9, Proposition II.1.8] and

(ii) $\prod_{i \in I} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \exists j : a_i = 0 \forall i \neq j \right\}$ in $S \rightarrow \operatorname{Act}_0$ by [9, Remark II.1.16].

Recall that for a subset $B$ of an act $A$ the Rees congruence $\rho_B$ on $A$ is defined by setting $a_1a_2$ if $a_1 = a_2$ or $a_1, a_2 \in B$ and the corresponding factor act is denoted by $A/B$ (cf. [9, Definition 4.20])

3. The functor $F : S \rightarrow \operatorname{Act} \to S \rightarrow \operatorname{Act}_0$

Throughout the paper, all monoids are considered to contain the zero element $0$, in particular, $S$ denotes a monoid $(S, 1)$ with the zero element $1$ and we suppose that $0 \neq 1$.

Let $A$ be a left $S$-act. Since $S \rightarrow A = z$ for each $z \in S \rightarrow 0A = \{0a \mid a \in A\}$ we say that $0A$ is a set of zero elements. Observe that $0A$ can contain more than one element in general and notice that while a morphism $\alpha : C \to D$ in the category $S \rightarrow \operatorname{Act}_0$ is required to preserve the unique zero, i.e., $\alpha(0C) = 0D$, the category $S \rightarrow \operatorname{Act}$ is less restrictive: for a morphism $\beta : A \to B$ the image of a zero element of $A$ from the set $0A$ is some zero element of $B$, in other words $\beta(0A) \subseteq 0B$. This leads to the following idea:

Define the functor $F$ from the category $S \rightarrow \operatorname{Act}$ to the category $S \rightarrow \operatorname{Act}_0$ as follows:

- for an object $A \in S \rightarrow \operatorname{Act}$, let $F(A) = A/0A$, i.e., the $S$-act obtained by gluing all zeroes of $A$ together or, in other words the image of the natural projection onto the Rees factor $\pi_{0A} : A \to A/0A$
- for a morphism $\alpha : A \to B$ define $F(\alpha)$ in the natural way so that the following square commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\pi_{0A}} & F(A) \\
\downarrow{\alpha} & & \downarrow{F(\alpha)} \\
B & \xrightarrow{\pi_{0B}} & F(B)
\end{array}
\]

The morphism $F(\alpha)$ can be obtained from the Homomorphism Theorem [9, Theorem 4.21], since $\ker \pi_{0A} \subseteq \ker \pi_{0B}$. The explicit formula for $F(\alpha)$ is then:

\[
F(\alpha)([a]) = [\alpha(a)] \quad \text{for} \quad a \not\in 0A.
\]

Now we formulate the key categorical observation on $F$: for the definition of a reflective subcategory we refer, e.g., to [11, Definition 4.16].

**Proposition 3.** The category $S \rightarrow \operatorname{Act}_0$ is a reflective subcategory of the category $S \rightarrow \operatorname{Act}$ via the reflector $F$.

**Proof.** Firstly, we show that $S \rightarrow \operatorname{Act}_0$ is a full subcategory of $S \rightarrow \operatorname{Act}$, i.e., $\operatorname{Mor}_{S \rightarrow \operatorname{Act}}(A, B) = \operatorname{Mor}_{S \rightarrow \operatorname{Act}_0}(A, B)$. For $A, B \in S \rightarrow \operatorname{Act}_0$ consider an $f \in \operatorname{Mor}_{S \rightarrow \operatorname{Act}}(A, B)$. Both $A, B$ being objects of $S \rightarrow \operatorname{Act}_0$ have their respective unique zeros $\theta_A, \theta_B$. Let $f(\theta_A) = b \in B$. Then

\[
f(\theta_A) = f(0\theta_A) = 0f(\theta_A) = \theta_B = 0\theta_B
\]
so $f$ preserves zero and as a consequence $f \in \text{Mor}_{S-\text{Act}_0}(A, B)$. The reverse inclusion of morphism sets is clear.

Let now be $A \in S-\text{Act}$, $X \in S-\text{Act}_0$ and $f : A \to X$ a morphism in $S-\text{Act}$. We claim that $F(f)$ is the unique morphism in $\text{Mor}_{S-\text{Act}_0}(F(A), X)$ that makes the following square commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\pi_{0A}} & F(A) \\
f \downarrow & & \downarrow F(f) \\
X & \xrightarrow{F(X)} & X
\end{array}
$$

Indeed, if $\beta : F(A) \to X$ satisfies $\beta \pi_{0A} = F(f) \pi_{0A}$, then $\beta = F(f)$, since $\pi_{0A}$ is surjective. $\square$

**Example 4.** (1) Let $S$ be an arbitrary non-trivial monoid and consider $A_1 = \{\theta\}$ and $A_1 = A_1 \coprod A_1 = \{\theta_1, \theta_2\}$ are two acts in $S-\text{Act}$. Then $F(A_1) = F(A_2)$.

Note that the functor $F$ is not faithful since $|\text{Hom}(A_1, A_2)| = 2$, while by applying $F$, we get $|\text{Hom}(F(A_1), F(A_2))| = |\text{Hom}(A_1, A_1)| = 1$.

(2) The functor $F$ is not left-exact (i.e. it does not preserve finite limits): consider the monoid $S = (\mathbb{Z}/2, \cdot)$ and the $S$-act $A = \{\theta_A, a\}$ with Cayley graph (omitting unit loops)

$$a \xrightarrow{0} \theta_A.$$

Put $B = A \cup \theta_S$, an object of $S-\text{Act}$ with two zeros. Then $F(B \coprod B)$ has 6 elements, while $F(B \coprod F(B)) = A \coprod A$ is a 4-element act.

The previous examples show that $F(A)$ cannot be considered in a reasonable way an analogy of localization or completion of $A$.

**Lemma 5.** The functor $F$ preserves coproducts.

*Proof.* Since $F$ is a reflector, hence a left adjoint (of the embedding functor $S-\text{Act}_0 \hookrightarrow S-\text{Act}$), it preserves colimits by the dual assertion of [10, Theorem 1, page 114]. $\square$

Recall that an act $P$ is projective, if for any pair of acts $A, B$, a homomorphism $\alpha : P \to B$ and an epimorphism $\pi : A \to B$, there exists a morphism $\overline{\alpha} : P \to A$ in $C$ such that $\alpha = \pi \overline{\alpha}$.

**Lemma 6.** Let $P \in S-\text{Act}$ be projective. Then $F(P)$ is projective in $S-\text{Act}_0$.

*Proof.* Let the projective situation in $S-\text{Act}_0$ be given:

$$
\begin{array}{ccc}
F(P) & \xrightarrow{f} & A \\
\pi \downarrow & & \downarrow \pi \\
B & & \\
\end{array}
$$

Since $S-\text{Act}_0$ is a subcategory of $S-\text{Act}$ and we have $\pi_0 P : P \to F(P)$, the projectivity of $P$ provides a morphism $\alpha : P \to B$ in $S-\text{Act}$ such that $\pi \alpha = f \pi_0 P$; furthermore, $\ker \pi_0 P \subseteq \ker \alpha$, hence $\alpha$ factorizes through $\pi_0 P$ via some $\alpha' : F(P) \to B$:

$$
\begin{array}{ccc}
P & \xrightarrow{\pi_0 P} & F(P) \\
\alpha \downarrow & & \downarrow \alpha' \\
B & \xrightarrow{\pi_0 P} & B \\
\pi \downarrow & & \downarrow \pi \\
A & & \\
\end{array}
$$

In total: $\pi \alpha' \pi_0 P = \pi \alpha = f \pi_0 P$ and since $\pi_0 P$ is an epimorphism, we get $\pi \alpha' = f$. $\square$

Let us observe that the description of projectivity in $S-\text{Act}_0$ works similarly as in $S-\text{Act}$ [9 Theorem III.17.8].

**Lemma 7.** For an indecomposable projective act $A$ in $S-\text{Act}_0$ there exists an idempotent $e \in S$ such that $A \cong Se$.

*Proof.* We follow the arguments of the proof of [9 Proposition III.17.7].

By [3 Lemma 4.4] there exist a retraction $p : S \to A$ and a coretraction $i : A \to S$ such that $pi = id_A$. If we put $e = ip(1)$ it is easy to see that $e = ip(1) = ip(e) = e^2$ and $A \cong i(A) = Se$. $\square$
Proposition 8. An act $A$ is projective in $S - \text{Act}_0$ if and only if there exist idempotents $e_i$, $i \in I$ such that $A = \bigsqcup_{i \in I} S e_i$.

Proof. We follow the arguments of the proof of [3] Theorem III.17.8.

By [3] Theorem 4.3, $A$ is a projective act if and only if it is isomorphic to a direct sum of indecomposable projective acts. Since every indecomposable projective act is isomorphic to $Se_f$ for some idempotent $e$ by Lemma 4 it remains to observe that for each act $Se_f$, where $f$ is an idempotent, the inclusion morphism $i : Se_f \rightarrow S$ forms a coretraction and the projection $p : S \rightarrow Se_f$ given by the rule $p(s) = se$ forms a retraction and since $S$ is projective, $Se_f$ is projective, too.  

Now, we show that locally cyclic acts contains only one zero-element.

Lemma 9. Any cyclic $S$-act $A$ contains a unique zero element $\theta_A$.

Proof. Since the act $A$ is cyclic, there exists a $g \in G$ for which $A = Sg$. Let $\theta$ be a zero element of $A$. Then there exists an $s \in S$ such that $\theta = s \cdot a$, and so $\theta = 0\theta = 0sa = 0a$. Thus $0A = \{\theta\}$.  

Recall that an $S$-act is locally cyclic, if for any pair of elements $a_1, a_2 \in A$ there exists a $b \in A$ with $a_i \in b$ for $i = 1, 2$.

Corollary 10. If $A$ is a locally cyclic $S$-act, then it contains a unique zero element $\theta_A$, the morphism $\pi_{0A}$ is bijective, and we can assume $F(A) = A$.

For any act $A \in S - \text{Act}$ we can consider the one-element $S$-act $S\theta$ being adjoined, $A \cup S\theta \simeq A \bigsqcup S\theta$. Therefore define a property $P$ of an $S$-act $A \in S - \text{Act}$ to hold up to zeros in the case $A \simeq A' \cup \bigcup_{i \in I} S\theta$, $A'$ cannot be decomposed as $A'' \cup S\theta$ and it has the property $P$. Call then $A'$ the substantial summand of $A$. Finally, a subact $B$ of $A$ is said to be superfluous if $B \cup C \neq A$ for each proper subact $C$ of $A$.

Note that in $S - \text{Act}_0$ the adjunction of $S\theta$ is trivial, since $A \bigcup S\theta \simeq A$, and let us list now some elementary properties of zero elements and substantial summands.

Lemma 11. Let $A \in S - \text{Act}$.

1. If $\emptyset \neq C \subseteq 0A$, then $C = \bigcup_{c \in C} \{c\} \cong \bigsqcup_{c \in C} \theta$ is a subact of $A$.
2. If $B$ is a subact of $A$ satisfying $A = B \cup 0A$, then $A \cong B \bigsqcup (0A \setminus B) \cong B \bigsqcup (\bigsqcup_{c \in 0A \setminus B} \theta)$.
3. $A$ contains a substantial summand.
4. If $A$ is indecomposable, then it is the substantial summand of itself and $0A$ is a superfluous subact of $A$.

Proof. (1) It is clear as $Sc = c = 0c$ for all $c \in 0A$.

(2) Since $A = B \cup (0A \setminus B)$ and $(0A \setminus B) \subseteq 0A$, the claim follows from (1).

(3) By [3] Theorem I.5.10 there exists, up to a permutation, a unique decomposition $A = \bigsqcup_{i \in I} A_i$ of $A$ into indecomposable summands. If we put $B = \bigcup\{A_i \mid A_i \not\subseteq 0A\}$ and $C = \bigcup\{A_i \mid A_i \subseteq 0A\}$, then $A = B \cup C \cong B \bigsqcup (\bigsqcup_{c \in C} \theta)$ by (1) and (2), hence $B$ is the substantial summand of $A$ by the uniqueness of the decomposition.

(4) If $B$ is a subact of an indecomposable act $A$ such that $B \cup 0A = A$, then $A \cong B \bigsqcup (\bigsqcup_{c \in 0A \setminus B} \theta)$ by (2), hence we have $0A \subseteq B$ and $B = A$.  

Lemma 12. If $A \in S - \text{Act}$ such that $F(A)$ is a nonzero cyclic $S$-act, then $A$ is, up to zeros, cyclic.

Proof. As $F(A) = \pi_{0A}(A)$ is cyclic, there exists $g \in A$ such that $F(A) = Sg \cup 0A$ by the definition of the Rees factor. Since $A \cong Sg \bigsqcup (\bigsqcup_{c \in 0A \setminus Sg} \theta)$ by Lemma 11.2, $A$ is, up to zeros, cyclic.  

Note that the image nor the preimage under $F$ of an indecomposable act may not be indecomposable, as the following examples illustrate:

Example 13. (1) Consider the monoid $S$ from Example [3]2 and the $S$-act $A = \{\theta_A, a, b\}$ with Cayley graph (omitting unit loops)
Then $A$ is indecomposable in $S - \overrightarrow{\text{Act}}$, but $F(A) = A$ is decomposable in $S - \overrightarrow{\text{Act}}_0$.

(2) For any indecomposable $A \in S - \overrightarrow{\text{Act}}_0$ and a nonempty index set $I$, the act $B = A \bigcup_{i \in I} (\theta_i) \in S - \overrightarrow{\text{Act}}$ is decomposable with $F(B) \simeq F(A)$ indecomposable.

Recall that projective objects of both categories $S - \overrightarrow{\text{Act}}$ and $S - \overrightarrow{\text{Act}}_0$ are isomorphic to coproducts (in the respective category) $\coprod_{i \in I} Se_i$ of cyclic $S$-acts of the form $Se_i$ with $e_i \in S$ idempotents by [9, Proposition 17.8] and Proposition 8.

Example 14. The functor $F$ is not bijective on the class of projective objects of $S - \overrightarrow{\text{Act}}$ for any monoid $S$, as there exists a non-projective $A \in S - \overrightarrow{\text{Act}}$ with $F(A)$ projective: consider the coproduct $A = S_1 \cup S_2$, where $S_i \cong S$. Then $F(S_1 \cup S_2)$ is a projective in $S - \overrightarrow{\text{Act}}$ while $F(F(S_1 \cup S_2)) = F(S_1 \cup S_2)$ is projective in $S - \overrightarrow{\text{Act}}_0$ by Lemma 5. In particular, $A = \{(a, b) \in \mathbb{Z}^2 \mid a = 0 \lor b = 0\}$ is not projective in $S - \overrightarrow{\text{Act}}$ and $F(A) \cong A$ is projective in $S - \overrightarrow{\text{Act}}_0$.

4. Perfect monoids

Recall that for an act $A$, a pair $(C, f)$ is a cover provided $f : C \rightarrow A$ is an epimorphism, and for any proper subact $C' \subset P$ the restriction $f|_{C'} : C' \rightarrow A$ is not an epimorphism in the corresponding category. A cover $(P, f)$ is called projective in case $P$ is projective (cf. [9, chapter 17]). Note that a projective cover is maximal among all covers.

Lemma 15. Let $(P, f)$ be a projective cover of $A$ in the category $S - \overrightarrow{\text{Act}}$. Then $(F(P), F(f))$ is a projective cover of $F(A)$ in the category $S - \overrightarrow{\text{Act}}_0$.

Proof. By Lemma 6, $F(P)$ is projective. Let $Q \subset F(P)$ be a subact and put $\hat{Q} = \pi_{0P}^{-1}(Q)$. Then $0P \subset \hat{Q} \subset P$, hence $f(\hat{Q}) \neq A$ by the hypothesis and $0A = 0f(P) = f(0P) \subset f(\hat{Q})$, as $f$ is surjective. It implies that $\pi_{0A}(\hat{Q}) \neq \pi_{0F}(A) = F(A)$, thus

$$F(f)(\hat{Q}) = F(f)(\pi_{0F}(\hat{Q})) = \pi_{0A}f(\hat{Q}) \neq F(A).$$

In analogy with module categories, call a monoid left perfect (left 0-perfect) if each $A \in S - \overrightarrow{\text{Act}}$ $(A \in S - \overrightarrow{\text{Act}}_0)$ has a projective cover (cf. [3][6][8]). Let us recall a characterization of left-perfect monoids:

Theorem 16. [6, 1.1] A monoid $S$ is left-perfect if and only if each cyclic $S$-act has a projective cover and every locally cyclic $S$-act is cyclic.

Proposition 17. If a monoid $S$ is left 0-perfect, then it is left perfect.

Proof. Suppose that $S$ is left 0-perfect and let us prove the two conditions from Theorem 16.

First suppose that $A \in S - \overrightarrow{\text{Act}}$ is a locally cyclic act. Then $A$ contains a unique zero $\theta_A$ and $A \cong \pi_{0A}(A)$ can be considered an act of the category $S - \overrightarrow{\text{Act}}_0$ by Corollary 10. Let $f : P \rightarrow A$ be a projective cover in $S - \overrightarrow{\text{Act}}_0$. We show that $P$ is indecomposable.

Applying Proposition 8 assume to the contrary that $P = P_1 \coprod P_2$ is a non-trivial decomposition, where $P_1 = Se$ is cyclic. Since $f(P) = A = f(P_1) \cup f(P_2)$, there exists $y \in A \setminus f(P_1)$ and there exists $z \in f(P_2)$ such that $f(e), y \in Sz \subseteq f(P_2)$. Hence $f(P_1) \subseteq Sz \subseteq f(P_2)$, and so $A = f(P_1) \cup f(P_2) = f(P_2)$, a contradiction.

Since $P$ is indecomposable, $P \cong Se$ for an idempotent $e \in S$ by Lemma 7, which implies that $A$ is cyclic. Furthermore, as $Se$ is a projective act also in the category $S - \overrightarrow{\text{Act}}$ by [9, Proposition 17.8], the morphism $f$ constitutes a projective cover in $S - \overrightarrow{\text{Act}}_0$.

Theorem 18. A monoid is left perfect if and only if it is left 0-perfect.

Proof. The direct implication follows from Lemma 15 and the reverse one is proven by Proposition 17.

□
Example 19. By [6], the examples of monoids which are left perfect (the argument does not require the zero) comprise: monoid of square matrices over a division ring, and finite monoids. By Theorem [17] the former is also left-0-perfect, while the latter in case it contains a zero element.

On the other hand, in the case of another class of perfect monoids (without zero) mentioned in [6], groups, the presented result cannot be employed, as adding 0 to a group may in general change the situation notably (see Example 22 below).

5. Steady monoids

An $S$-act $A$ is called hollow if each of its proper subacts is superfluous (cf. [7] Definition 3.1]). It is easy to see that hollow acts are indecomposable in both categories $S - \text{Act}$ and $S - \text{Act}_0$ (see [7] Theorem 3.4 and [3] Propositions 5.6 and 6.6).

In compliance with [3] call an act $C \in S - \text{Act}$ ($\in S - \text{Act}_0$, resp.) compact, if the corresponding covariant Hom-functor commutes with coproducts, i.e. for any family $(A_i, i \in I)$ of $S$-acts in the given category, for the natural functor $\text{Hom}(C, -) : S - \text{Act} \to \text{Set}$ ($S - \text{Act}_0 \to \text{Set}$, resp.) we have a surjective natural morphism

$$\text{Hom}(C, \bigsqcup_{i \in I} A_i) \to \prod_{i \in I} \text{Hom}(C, A_i) \to 0.$$ 

Recall that an act in the category $S - \text{Act}$ is compact if and only if it is hollow by [3] Proposition 6.6. It is easy to see that cyclic acts are compact and we say that a monoid $S$ is left steady (resp. left 0-steady) if every compact act in the category $S - \text{Act}$ (resp. $S - \text{Act}_0$) is cyclic (see [3] 6.2]).

Lemma 20. Let $A$ be an act in $S - \text{Act}$ such that $0A$ is superfluous in $A$. Then $A$ is hollow in the category $S - \text{Act}$ if and only if $F(A)$ is hollow in the category $S - \text{Act}_0$.

Proof. Let $A$ be hollow and $F(A) = B_1 \cup B_2$ for subacts $B_i$, $i = 1, 2$. Then $A = \pi_{0A}^{-1}(B_1) \cup \pi_{0A}^{-1}(B_2)$, hence there exists $i$ such that $A = \pi_{0A}^{-1}(B_i)$ and so $F(A) = \pi_{0A}(A) = B_i$. Thus $F(A)$ is hollow.

Conversely, suppose that $A = B_1 \cup B_2$ for subacts $B_i$ of $A$ and $i = 1, 2$. Then $F(A) = \pi_{0A}(B_1) \cup \pi_{0A}(B_2)$ and so there exists $i$ for which $F(A) = \pi_{0A}(B_i)$. It implies that $B_i \cup 0A = A$, thus $B_i = A$ since $0A$ is superfluous in $A$. □

Recall a description of the monoid structure via a property of hollow acts, which is employed in the next result:

Lemma 21. [7] Lemma 3.8| A monoid $S$ satisfies the ascending chain condition on cyclic subacts of an arbitrary $S$-act if and only if every hollow act in $S - \text{Act}$ is cyclic.

Theorem 22. A monoid $S$ is left 0-steady if and only if it satisfies the ascending chain condition on cyclic subacts.

Proof. By Lemma 21 it is enough to prove that $S$ is left 0-steady if and only if every hollow $S$-act in $S - \text{Act}$ is cyclic.

Let $S$ be left 0-steady and let $A$ be a hollow $S$-act in $S - \text{Act}$. Since $A$ is indecomposable, $0A$ is superfluous by Lemma 11. Applying Lemma 20 we obtain that $F(A)$ is hollow in the category $S - \text{Act}_0$, which implies that $F(A)$ is compact in $S - \text{Act}_0$ by [3] Proposition 6.6. Thus $F(A) = \pi_{0A}(A)$ is cyclic by the hypothesis and by Lemma 12 we get $A = Sa \cup 0A$. Finally, since $0A$ is superfluous, $A$ is cyclic.

Conversely, suppose that $A$ is a compact act in the category $S - \text{Act}_0$. Then it is hollow by [3] Proposition 6.6], and so indecomposable. Now, it follows from Lemmas 11 and 20 that $A \cong F(A)$ is hollow in $S - \text{Act}$, hence it is cyclic by the hypothesis. □

We conclude the paper by an example.

Example 23. Any group $G$ is right steady by [3] Example 6.7(1)], however 0-steadiness of a monoid $G_0$ obtained from $G$ by adding a zero element depends on the structure of subgroups by the last theorem. In particular $\mathbb{Q}^*$ is steady, while $\mathbb{Q}$ is not 0-steady.
REFERENCES

[1] Adámek, J., Herrlich, H., Strecker, G.E., Abstract and Concrete Categories, John Wiley and Sons, Inc, 1990, [http://katmat.math.uni-bremen.de/acc/acc.pdf].

[2] H. Bass (1960). Finitistic dimension and a homological generalization of semi-primary rings. Transactions of the American Mathematical Society, 95(3), 466.

[3] J. Dvořák, J. Žemlička: Compact objects in categories of $S$-acts, submitted, 2021, [arXiv:2009.12301]

[4] S. Eilenberg (1956). Homological Dimension and Syzygies. The Annals of Mathematics, 64(2), 328–336.

[5] J. Fountain (1976). Perfect semigroups. Proceedings of the Edinburgh Mathematical Society, 20(2), 87–93.

[6] J. Isbell : Perfect monoids. Semigroup Forum (1971) 2, 95–118.

[7] R. Khosravi, M. Roueentan, Co-uniform and hollow $S$-acts over monoids, arXiv: 1908.04559v1

[8] M. Kilp, Perfect monoids revisited. Semigroup Forum (1996) 53, 225–229.

[9] M. Kilp, U. Knauer, A.V. Mikhalev, Monoids, acts and categories, de Gruyter, Expositions in Mathematics 29, Walter de Gruyter, Berlin 2000.

[10] S. Mac Lane, Categories for the Working Mathematician. Springer New York, 1971.

Email address: pepa.dvorak@post.cz

CTU in Prague, FEE, Department of Mathematics, Technická 2, 166 27 Prague 6 & MFF UK, Department of Algebra, Sokolovská 83, 186 75 Praha 8, Czechia

Email address: zemlicka@karlin.mff.cuni.cz

Department of Algebra, Charles University in Prague, Faculty of Mathematics and Physics Sokolovská 83, 186 75 Praha 8, Czechia