Some Applications of Ricci Flow in Physics

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Abstract

I discuss certain applications of the Ricci ow in physics. I first review how it arises in the renormalization group (RG) ow of a nonlinear sigma model. I then review the concept of a Ricci soliton and recall how a soliton was used to discuss the RG ow of mass in 2-dimensions. I then present recent results obtained with Olynyk on the ow of mass in higher dimensions. The nal section discusses how Ricci ow may arise in general relativity, particularly for static metrics.

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I Introduction

The Ricci ow

\[
\frac{\partial g_{ij}}{\partial t} = 2R_{ij} \tag{I.1}
\]

was introduced by Hamilton [1] over 25 years ago. During the intervening time, it has been studied by mathematicians primarily as a tool for proving the geometrization conjecture for closed 3-manifolds, which includes as a special case the Poincaré conjecture that every closed, simply connected 3-manifold is a 3-sphere. This endeavor has now met with success [2,3,4].

The equation describes the deformation of a Riemannian metric \( g_{ij} \) with \( \text{time} \). The deformation is driven by Ricci curvature, so that parts of the manifold with greater Ricci curvature will undergo greater deformation. Regions with very little Ricci curvature will change only a little during the ow. Fixed points of the ow are the Ricci at manifolds

\[
R_{ij} = 0 : \tag{I.2}
\]

At about the time of Hamilton's work, equation (I.1) was already making its rst appearance in physics in Friedan's 1980 thesis [5] on the renormalization group ow of non-linear sigma models. Equation (I.1), with the coe cient 2 on the right-hand side replaced by a positive constant written as \( T \) by Friedan but now denoted by \( \beta \), was the renormalization group (RG) ow equation with the \( \beta \)-function truncated to leading order in \( \beta \).

The mathematical study of the Ricci ow has experienced tremendous growth in the aftermath of Perelman's breakthrough work [2] with its consequences for geometrization. It is a good time to look for physical applications. There are several possibilities, some but not all related to renormalization group ow. A brief list of Ricci ow questions and their possible physical applications would include:

- Are there classes of Lorentzian metrics to which Ricci ow can be applied? (I say "classes" in part because the unrestricted initial value problem would not be well-posed.) If so, this would surely nd application in general relativity, but what precisely are the useful applications?

- Is space stable under Ricci ow? If not, then mass would behave in unexpected ways under RG ow. This would have applications to tachyon condensation in string theory.
Can we do Ricci ow on manifolds with boundary? There is recent work on this in the context of an application to black hole thermodynamics [6]. As well, this question has some relevance to certain formulations of quasilocal mass.

Can Perelmam's entropy definition be generalized to full RG ow (of non-linear sigma models) to express this as a gradient ow on a space of coupling constants equipped with a positive definite metric? If so, then this would yield a C-function for this ow. See [7,8].

This list is not intended to be exhaustive by any means, but is already too large to be addressed in any reasonably complete fashion in the present article. I will confine myself to an introductory treatment of Ricci ow for physicists working in general relativity and quantum field theory, together with a report on some recent progress in two areas: the ow of mass and the ow of static metrics.

Section II has a brief discussion of Ricci ow as an approximation to RG ow for Sigma models. Section III introduces the important notion of self-similar Ricci ow solutions called Ricci solitons. Section IV reviews a discussion in [9], which uses a 2-dimensional Ricci soliton to illustrate the behaviour of mass under Ricci ow. Section V describes recent results that Ollynyk and I have obtained regarding the mass question in dimension 3 and greater, in spherical symmetry [10]. Section VI considers the Ricci ow of static spacetime metrics and describes a soliton recently found by Akbar and me [11] which can be interpreted as generating a nontrivial Lorentzian ow.

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I first came to appreciate the importance for physics of a firm mathematical foundation from Rafael Sorkin. Early on, together with Penrose, we formulated an argument for a positive mass theorem which drew on causal structure of Lorentzian manifolds but also relied heavily on the kinds of arguments that arise in both comparison geometry and differential topology. What I learned then has influenced me ever since, as will I hope be at least somewhat evident in what follows below. It is a pleasure to dedicate this article to Rafael on the occasion of his 60th birthday.
II Ricci ow and Sigma M odels

Let us start with the connection between Ricci ow and RG ow for string theory. More precisely, for bosonic nonlinear sigma models from 2-dimensional spacetime to a curved Riemannian target manifold. Consider the sigma model with only the coupling to the target space metric $g_{ij}$ (it is easy to add a dilaton). The action is

$$S = \sum_{\mathcal{M}} h_{ab}(\theta) g_{ij}(X) \frac{\partial X^i}{\partial \theta^a} \frac{\partial X^j}{\partial \theta^b} \sqrt{h}$$  \hspace{1cm} (II.1)

plus a boundary term if the manifold $\mathcal{M}$ is not closed. Here $\theta = (\theta^1; \theta^2)$ denotes coordinates on the worldsheet, $h_{ab}$ is any metric on that worldsheet, $X^i$ denotes the target manifold coordinates of the embedded (or immersed) worldsheet, and $h$ is a square distance (the string scale). Thinking of this as the starting point for a theory of quantum scalar fields $X^i$ in 2 dimensions, then $g_{ij}(X)$ represents the coupling constants of the theory and can be expected to renormalize. The renormalization group ow can be expressed perturbatively as a power series in $\alpha_0$:

$$\frac{\partial g_{ij}}{\partial \theta} = R_{ij} + \frac{\alpha_0}{2} R_{iklm} R_{jklm} + \cdots;$$  \hspace{1cm} (II.2)

where the ellipsis denotes a power series with terms of order $\alpha_0^2$ and higher.

Now say that $g(\theta)$ is a solution of this equation on a \textquotedblleft time\textquotedblright\ interval 2 $(T_1; T_2)$. Denote

$$^\wedge g_{ij}(\theta) = a(\theta^2; \theta^2) = (aT_1; aT_2);$$

$$g_{ij}(\theta) = a\left(\theta^2 = a; \theta^2 = a\right);$$  \hspace{1cm} (II.3)

for some constant $a > 0$. The RG ow equation becomes

$$\frac{\partial ^\wedge g_{ij}}{\partial ^\wedge \theta} = R_{ij} + \frac{\alpha_0}{2} R_{iklm} R_{jklm} + \cdots;$$  \hspace{1cm} (II.4)

The terms represented by the ellipsis have coefficients $a^{p-1} \Phi$ with $p > 3$. Thus, if we take $a = 0$, we obtain the Ricci ow equation (II.1).

However, this does not necessarily mean that solutions of (II.1) and (II.4) approach each other on arbitrary scales as $a$ becomes small. Even worse, if the original solution $g(\theta)$ was defined on a bounded interval 2 $(T_1; T_2)$, then rescaling shrinks this interval to $^\wedge 2 (aT_1; aT_2)$, which tends to zero width. To avoid this, we will primarily consider solutions $g$ defined on
a semi-infinite or in finite domain, so for any finite a the domain of \( ^{\infty} \) is also semi-infinite or in finite. Flows on such domains are classified as either ancient (\( 2 (1; 0) \)), immortal (\( 2 (0; 1) \)), or eternal (\( 2 (1; 1) \)). The ancient and eternal flows are of greatest interest. If an ancient or eternal Ricci flow stays close to a full RG flow as \( ^{\infty} \), then the sigma model is well-defined when the energy cutoff is removed.\(^2\)

By theorem (9.15) of [12], if an ancient or eternal solution \( \phi_{ij}(\hat{t}) \) is asymptotically at each \( \hat{t} \) (so the sectional curvatures are always bounded), then the scalar curvature obeys \( R = 0 \) for all \( \hat{t} \). That means the positive mass theorem is obeyed all along the \( \phi \), and so we learn that positive target manifold mass is a requirement for a well-defined sigma model at high energy.

This also provokes the question of how mass behaves under RG flow or at least, in the current context, under Ricci flow. This leads us next to a discussion of a special solution of the flow known as a Ricci soliton, and then to the question of whether at space is asymptotically stable with respect to Ricci flow.

### III Solitons

As a model for Ricci flow, consider the one-dimensional linear heat equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (III.1)
\]

for the temperature \( u(t; x) \).

For the moment, let \( x \) take values in \([0; \) ] and impose periodic boundary conditions \( u(t; 0) = u(t; ) = 0 \). Then there are separable solutions

\[
u_n(t; x) = T_n(t)X_n(x) = e^{n^2t} \sin nx \quad (III.2)
\]

\(^2\)Think of a quantum field theory characterized by \( k \) coupling constants \( c_i \), all of whose UV divergences can be traced back to just one particular Feynman graph. Impose a momentum cutoff \( \hat{c} \), so the graph takes a finite value \( (\hat{c}; : \cdot \cdot \cdot ; c_i) = C = \text{const}. The predictions of the theory will now depend on the cutoff \( \hat{c} \) as well as on the coupling constants \( c_i \). Now follow a curve on the level set \( C = \hat{c} \), along which the coupling constants \( c_i \) and the cutoff \( \hat{c} \) will vary but of course will not. This is an RG flow. If one can follow some such curve until \( \hat{c} = 1 \), introducing no new divergences in other graphs, this is cutoff removal. The limiting theory (defined by coupling constants \( c_i \)) obviously has no dependence and its Feynman diagrams have no UV divergences. This can work even when more than one Feynman graph is responsible for the divergences of the theory, provided there are sufficiently many coupling constants.
for $n$ a positive integer. Since the equation is linear, then arbitrary linear combinations

$$u(t; x) = \sum_{n=1}^{N} c_n e^{n^2 t} \sin nx \quad (III.3)$$

are also solutions, where the $c_n$ are constants. An elementary argument using the maximum principle shows that this is the general solution. The Fourier decomposition of the initial data $u(0; x)$ uniquely determines the constants $c_n$.

Separable solutions (III.3) are self-similar. Self-similarity means that they evolve in time purely by rescaling the amplitude. Now the general solution is not self-similar because it is not separable. Self-similar solutions arise only for very special initial data. Specifically, if the initial data contain precisely one Fourier mode, say the $n^\text{th}$ mode, then the particular solution evolving from that data is $u_n(t; x) = c_n e^{n^2 t} \sin nx$ and is self-similar and separable.

If the initial data contain many Fourier modes, the lowest of which is the $n^\text{th}$ mode, then at late times the higher modes $m > n$ are suppressed relative to this mode by a factor of $e^{(m^2 - n^2)t}$, so the solution tends to $e^{n^2 t} \sin nx$. Solutions always approach self-similarity at late times.

In Ricci ow, self-similar solutions exist and are called Ricci solitons. They are always either ancient, immortal, or eternal, and so are of interest for us. Of course, in contrast to the heat equation case, an arbitrary Ricci ow is not a sum of solitons, since Ricci ow is non-linear, but general ows som etimes asymptote to solitons in certain limits.

Steady solitons do not rescale under Ricci ow. Any Ricci- ow metric is an example of a steady soliton. There are also expanding and shrinking solitons, which do rescale. A simple example of a shrinking soliton is the round shrinking $n$-sphere

$$g_{ij}(\cdot) = 2(n - 1)(n^2) g_{ij}^{\text{can}}; \quad \text{const}; \quad (III.4)$$

where $g^{\text{can}}$ denotes the canonical sphere metric $d\theta^2 + \sin^2 \theta \, d\phi^2 + \cdots$. This solution exists for $2 > 1$; and is said to "extinguish" at $t = -1$. Like the lowest Fourier mode in heat ow, this soliton "attracts" in the sense that any slightly deformed sphere will approach the round shrinking sphere as $ow$ to extinction.

A more sophisticated example is the solve geometry expanding soliton (Bianchi type IV 1)

$$ds^2 = e^{2z} dx^2 + e^{2z} dy^2 + 4 \, dz^2 \quad (III.5)$$
This metric has a discrete group of isometries such that the quotient space is a smooth compact manifold \([13]\), but the soliton interpretation is valid only on \(<^3\), not on a quotient.

Why is this a Ricci soliton? To answer, we can directly compute the Ricci curvature of \([13]\), and we get \(R_{ij} = 2 \frac{\bar{z}^i}{\bar{z}^j}\). Also by direct computation, we see that \(\frac{\partial g_{ij}}{\partial \bar{z}} = 4 \frac{\bar{z}^i}{\bar{z}^j}\). Comparing these two results, we see that (1.1) is satisfied. So we have a Ricci ow, but is it self-similar? To answer this, we change coordinates and write the metric \([13]\) as

\[
\text{ds}^2 = g_{ij}(x)dx^i dx^j = 4 \ e^{2x} dx^2 + e^{2y} dy^2 + dz^2 = 4 \ g^0_{ij}(x)dx^i dx^j ;
\]

(III.6)

where we have performed a \(-\text{dependent} \text{diffeomorphism} \) taking \(x \mapsto x = e^{\bar{z}^i}, y \mapsto y = e^{\bar{z}^j}, \) and \(z \mapsto z = z\). Since the coefficients \(g^0_{ij}\) do not vary with \(z\), \(\text{ds}^2\) evolves self-similarly; i.e., it evolves only through the overall scale factor 4 in front of \(g^0\) and via the \(-\text{dependent} \text{diffeomorphism} \).

However, now we have from (III.6) that \(\frac{\partial g_{ij}}{\partial z} = g^0_{ij}\), while the Ricci curvature has not changed, and so now \(\frac{\partial g_{ij}}{\partial \bar{z}} = 2R_{ij}\). This shows that (1.1) is not geometric, in that it depends on the choice of coordinates along the ow. There should really be an extra Lie derivative term accounting for the fact that the coordinates can vary with \(z\). This is a new feature not seen in our scalar 1-dimensional heat equation analogy, and gives rise to the Hamilton-DeTurck equation

\[
\frac{\partial g_{ij}}{\partial \bar{z}} = 2R_{ij} + \frac{\partial}{\partial x} g_{ij} ;
\]

(III.7)

where the vector \(\partial x\) generates the diffeomorphism by which the coordinates vary along the ow and \(\frac{\partial}{\partial x} \) denotes the Lie derivative along \(X\). This also gives rise to the Ricci soliton equation

\[
R_{ij} \ \frac{1}{2} \frac{\partial}{\partial x} g_{ij} \ \text{c} \ g_{ij} = 0 ;
\]

(III.8)

where \(c\) is a constant. If \(g\) is a solution of this equation, then there is a Ricci ow given by \(a(\ ) \cdot g\), and so this ow is self-similar. For the solve geometry, the metric \(g^0\) is a solution of (III.8).

IV The Mass of a 2-Dimensional Soliton

We now begin our discussion of the behaviour of mass under Ricci ow. Before passing to the asymptotically at case, this section will discuss the
2-dimensional, rotationally symmetric, expanding soliton used in a study of mass under RG now [9]. This soliton is immortal but not eternal. It’s a gradient soliton, meaning that the vector field $X$ generating the soliton is the gradient of a scalar potential $X^i = g^{ij} r_j'$. The metric and potential function can be written as

$$ds^2 = f^2(r) dr^2 + r^2 d^2$$

where $f$ is given implicitly by

$$f(x) = \exp \frac{1}{1 \exp \frac{1}{f(x)}} = \frac{1}{f(x)}$$

Now take $[0; 2]$. For $0 < r < 0$, at any $r > 0$, we expect the mass to rescale with $r$, but this is not what happens. At any $r > 0$, the metric is asymptotic at large $r$ to the same cone, so the mass remains the same. But at any $r > 0$, there is no cone point (since $f(= -1) = 0$), so the manifold is smooth.

However, as $r \to 0$, the curvature within any $r$-distance of the origin dissipates and so the Grumov-Hausdorff limit is just a space. Thus, if the initial de Sitter angle is positive in this sense to a first order, and thus zero mass metric in the infinite time limit; if the initial de Sitter angle is negative (i.e., if $\gamma > 1$), the mass increases to zero in this sense. Either way, the $\omega$ is attracted to $0$ at space in the limit of large $r$, but at any finite $r$, the mass (= de Sitter angle) is unchanged from its initial value.

$^3$Smoothness at the origin during the $\omega$ is actually a boundary condition in posed at
V  Mass in Higher Dimensions

In dimension $n \geq 3$, we can define asymptotic atness by choosing suitable coordinates near infinity which will be Cartesian coordinates for a reference metric. An asymptotically at metric is one whose components differ from the reference by an element of some $H^k$ space, where the fall-off rate (not related to the previous section) is sufficient to give a well-defined ADM mass. For definitions and details, see [10].

Returning momentarily to the heat equation analogy, recall that the 1-dimensional heat equation on an infinite domain has a certain non-uniqueness. Namely, Tychono pointed out that

$$u(t;x) = \sum_{k=0}^{\infty} \frac{x^{2k} e^{1-t^2}}{(2k)!}$$

solves the 1-dimensional heat equation for $t \geq 0$ and $u(t;x) = 0$. Thus, we can add an arbitrary multiple of (V.1) to any solution to get a new solution for the same initial data. An extreme case is if the temperature is initially zero everywhere, but at any positive time the temperature distribution can be that described by (V.1), and so is large at large $x$.

This shows that it is necessary to impose fall-off conditions on solutions of the heat equation all along the $\omega$, not just on the initial temperature distribution, in order to have uniqueness. Remarkably, this problem does not seem to arise for Ricci $\omega$. At least, asymptotic atness of initial data is always preserved so long as the $\omega$ remains nonsingular [13, 14]. Not only that, but the mass does not change, just as we saw in the rotationally symmetric 2-dimensional example of the last section, and as we will now see in higher dimensions (without assuming any symmetry).

Because asymptotic atness is preserved, we can evaluate the mass at any time along the $\omega$. If the mass evaluated at two different values differs, then at some intermediate value the mass must have nonzero $\omega$ derivative. But the following simple computation shows that this cannot happen.

On an asymptotic end $\omega \ M$, let $ab$ be a fixed (constant along the $\omega$) Euclidean metric compatible with the derivative $\partial_a$ and let $\Sigma$ be a closed embedded hypersurface. Define the functional $E : S^2 \omega ! < acting

the origin, where the polar coordinate system breaks down. One can impose any $\omega$ direction at the origin instead. Then the $\omega$ limiting metric will be at cone with this $\omega$ direction. Therefore, this solution can describe transitions between two different cones.
on symmetric covariant 2-tensors by

\[ E(g) = \frac{1}{16} \sum_{ij} \partial \partial_i g_{ij} \partial \partial_j g_{ij} \, dA(g) \]  \quad (V.2)

where \( \partial \partial_i = \partial_i \partial_j \) and \( \partial \partial_j \) is the outward unit normal vector to the metric. If \( g \) belongs to a sequence of hyperspheres \( n \) whose interiors exhaust \( U \), then in the limit we get the ADM mass of \( g \):

\[ E^{ADM} = \lim_{n \to 1} E_n(g) \]  \quad (V.3)

Now this functional is linear in \( g \) so

\[ \frac{dE}{dt} = E \frac{\partial g}{\partial t} = \frac{1}{16} \sum_{ij} \partial \partial_i \partial g_{ij} \partial \partial_j g_{ij} \, dA(g) \]  \quad (V.4)

If the metric evolves by Ricci flow (I.1) then we get

\[
\begin{align*}
\frac{dE}{dt} &= E \frac{\partial g}{\partial t} = \frac{1}{8} \sum_{ij} \partial \partial_i R_{ij} \partial \partial_j g_{ij} \, dA(g) \\
&= \frac{1}{8} \sum_{ij} \partial \partial_i R_{ij} \, dA(g) \quad \left( \frac{1}{2} \partial R + O(1) = \partial O(r^{-1}) \right) ; \quad (V.5)
\end{align*}
\]

where we used the Bianchi identity to pass from one line to the next and used asymptotic flatness to treat derivatives compatible with \( g \) as equivalent near infinity (they differ by terms that fall off fast enough to ignore). We can write the asymptotic flatness condition in the succinct form \( \partial^k g = O(1=r^{n+k}) \) for \( k = 0,1,2,3 \). This gives immediately that \( r \partial R \partial^3 g = O(1=r^{n+1}) \) while \( \partial^3 r^{-1} \), so the integral falls off as claimed in the last step of (V.2). Hence the derivative of the mass is always zero and the mass is constant, just as in the special 2-dimensional example of the last section.

A nother feature of the 2-dimensional solution appears relevant in higher dimensions. For rotationally symmetric metrics, the ow converges to \( t \) space [10]. This result assumes the initial data for the ow are asymptotically at and contain no m in allhypersphere (if such a sphere were present, it might pinch off), but the data are otherwise quite arbitrary. Such a ow is then immortal and (Cheeger-Gromoll) converges to the at metric in the limit at of in time. Moreover, if the initial metric is not spherically symmetric but is in a certain sense close to at, the ow is also immortal and converges to a at metric [20]. (The notion of "close to at"
used in [20] is too restrictive to allow nonzero mass in 3 dimensions, but allows it in all higher dimensions.)

It may seem paradoxical that ADM mass is constant throughout the ow and yet the limiting metric is as. Certainly then \( \lim_{\lambda \to 1} E_{ADM}(g(\lambda)) \in E_{ADM}(g(1)) = 0 \). The resolution is that the notion of convergence of manifolds in the \( \lambda \to 1 \) limit involves picking a marked point (e.g., the origin of symmetry in the rotationally symmetric case) and examining the geometry of sequences of arbitrarily large metric balls about that point. Thus, the quasilocal mass of these balls is relevant. Ol'jnyk and I studied the decay of quasilocal mass in our rotationally symmetric ow [10] and found that within any ball of xed (proper) radius, the quasilocal mass evaporates at a rate \( 1 = \). The coefficient of this rate depends on the radius and vanishes at infinity fast enough that the ADM mass remains constant at any xite.

VI Static Lorentzian Metrics

The discussion of the previous section points to the likely importance that an understanding of the ow of quasilocal mass would have for the general question of stability of at space (and other asymptotically at xed points) when no symmetry is assumed. Specically, an understanding of the conditions under which the mass evaporates monotonically in is likely to be an important tool.

Conversely, the Ricci ow (and the closely related ow below) may lead to better understanding of quasilocal mass. Bartnik has proposed a denition of quasilocal mass; for details, see [14]. Say that \( B \) is a Riemannian manifold with boundary \( \partial B \). Embed \( B \) in an asymptotically at manifold \( (M;g) \) that has non-negative scalar curvature and contains no minimum other than possibly a so-called compact outermost horizon. Compute the ADM mass of \( (M;g) \). The minimum of the ADM mass over all such possible extensions \( (M;g) \) is the Bartnik quasilocal mass of \( B \).

Quasilocal mass is important only insofar as it provides a useful geometric characterization of bounded regions within a manifold. As such, the quasilocal mass must have useful properties, among which monotonicity (if \( B_1 \subset B_2 \), then \( m_{B_1} \leq m_{B_2} \)) and positivity are classic examples [15, 16]. Bartnik's mass has both of these. But is it nonzero in general? Yes, if the minimum in the denition were a genuine minimum. That is, if there were a specic asymptotically at but non at \( \mathbb{R}^0 \) extension \( (M;g) \) \( B \) whose ADM mass were that of any other extension, then this ADM mass and,

\[ 4 \text{This is the "spatial" version of the problem. There is also a "spacetime" version.} \]
thus, the Bartnik mass of $B$, would necessarily be $> 0$ (by the positive mass theorem).

Bartnik [14] conjectured that a mass-minimizing extension exists and outside of $B$ is a solution of the static Einstein equations. These equations are

\begin{align}
R_{ij} &= r_{ij} u r_{ij} ; \\
u &= 0 ;
\end{align}

where the second equation is merely an integrability condition for the first; it follows by applying a divergence to (VI.1) and using the contracted second Bianchi identity. Given a solution $(u; g)$ of the fixed point equations, then the metric

\begin{equation}
\begin{aligned}
ds^2 &= G dx^i dx^j = e^{2u} dt^2 + e^{\epsilon} g_{ij} dx^i dx^j \quad (\text{VI.3})
\end{aligned}
\end{equation}

is Ricci-flat and static.

But how does one produce a mass-minimizing metric on the extension of $B$? One suggestion is to extend $B$ by an arbitrary asymptotically flat metric obeying \"geom. metric boundary conditions\" (that the induced metric and mean curvature of $@B$ should match from both sides) and having $R = 0$, and then to ow this metric. The problem then is to nd a ow which exists subject to these boundary conditions, preserves $R = 0$ and asymptotic flatness, and whose fixed points are solutions of (VI.1). If this ow converges to a fixed point, then perhaps we would obtain sequences of metrics converging on a possible minimizer (though if the ow is similar to Ricci ow, we expect the ADM mass to remain constant along the sequence and \"jump\" at the limit).

Such a ow was studied by List [18] in his PhD dissertation under the direction of G. Huisken. The ow equations are

\begin{align}
\frac{\partial g_{ij}}{\partial t} &= 2 (R_{ij} r_{ij} u) ; \\
\frac{\partial u}{\partial t} &= u ;
\end{align}

For reasonable fall-off (say $u \sim r$, $\epsilon \sim 1$), ADM mass remains constant along this ow, just as it does for Ricci ow.

If $(u; g)$ satisfies the ow equations (VI.4, VI.5) then the metric

\begin{equation}
\begin{aligned}
ds^2 &= G dx^i dx^j = e^{2u} dt^2 + g_{ij} dx^i dx^j \quad (\text{VI.6})
\end{aligned}
\end{equation}
solves the Hamilton-DeTurck ow
\[ \frac{\partial g_{ij}}{\partial t} = 2R_{ij} + \frac{1}{2} \left( \nabla g_{ij} \right)^2 ; \quad (VI.7) \]
with diffeomorphism generated by \( ru \), where \( R \) refers to the metric \( g \) of \((VI.5)\). This opens up some possibilities. For example, if we take \( g_{ij} \) to have Lorentzian signature, then \((VI.8)\) is the Einstein equation in the presence of a free (massless) scalar field. Many solutions are known (e.g., \[21,22\]). Taken together with the \(+\) sign in \((VI.7)\), this produces a Lorentzian Ricci soliton.\({}^5\)

If \( g_{ij} \) has Euclidean signature, non-negative Ricci curvature (as implied by \((VI.8)\)) is an obstruction to nding nonconstant bounded-below (or above) solutions of \((VI.7)\) \[23\]. Then \( u \) will be unbounded above and below, which is not unacceptable, but a minor modification will circumvent this in any case. Specifically, Akbar and I \[11\] consider instead the Einstein-free scalar system \( w \) with a cosmological term
\[ R_{ij} = ru_{ir}j + g_{ij} ; \quad (VI.8) \]
where we will take \( \rho = \text{const} < 0 \) to make the Ricci curvature nonpositive. As before, \((VI.7)\) is an integrability condition for this equation as well. Then it's straightforward (see \[11\]) to show that

Lemma. The metric \((VI.8)\) constructed from any solution \((u; g_{ij})\) of \((VI.8)\) obeys
\[ R_{ij} - \frac{1}{2} \nabla g_{ij} = 0 ; \quad (VI.9) \]
where \( X = \frac{\partial}{\partial t} g_{ij} \frac{\partial}{\partial x^i} \), and is therefore a Ricci soliton.

One example is given by the metric
\[ g_{ij} dx^i dx^j = dx^2 + 1 + e^{2x} \ dy^2 + \sinh^2 y \ dz^2 ; \quad (VI.10) \]
on either \( M = \mathbb{S}^2 \) or \( M = H(g) \) with \( H(g) \) a compact Riemann surface of genus \( g > 1 \). The scalar field is
\[ u = x - \frac{1}{2} \log \left( 1 + e^{2x} \right) ; \quad (VI.11) \]
\({}^5\) It will be only a local soliton; the singularities of Einstein theory will be present of course; see \[23\] for a singularity theorem tailored to this.
Clearly this metric is complete and \( u \) is everywhere defined on \( M \). The corresponding soliton metric

\[
\text{ds}^2 = \frac{e^{2x}}{1 + e^{-2x}} \text{dt}^2 + \text{dx}^2 + 1 + e^{2x} \text{d}\theta^2 + \text{sinh}^2 \text{d}\varphi^2 \quad (\text{VI.12})
\]

solves (VI.9) with \( \alpha \equiv 1 \). It is complete and has bounded sectional curvature. This produces a nontrivial immortal Ricci ow on a nonsingular Lorentzian manifold.

On the issue of the behavior of mass under the ow (VI.4, VI.5) (particularly, in the limit of approach to fixed points), work continues.

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