Moduli of sheaves: a modern primer

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1. Introduction

This paper is focused on results coming from the moduli spaces of sheaves, broadly understood, over the last 15 years or so. The subject has seen an interesting convergence of mathematical physics, derived categories, number theory, non-commutative algebra, and the theory of stacks. It has spun off results that inform our understanding of the Hodge conjecture, the Tate conjecture, local-to-global principles, the structural properties of division algebras, the $u$-invariants of quadratic forms over function fields, and the geometry of moduli spaces of K3 surfaces, among other things. All of these results turn out to be interconnected in fascinating ways. For example, the earliest results on derived categories of elliptic threefolds using “non-fine moduli spaces of sheaves” (as studied by Căldăraru [11]) are very closely related to subsequent attacks, nearly a decade later, on the Tate conjecture and the geometry of the Ogus space of supersingular K3 surfaces, where the non-fineness is turned around to spawn new surfaces from Brauer classes.

The most fascinating thing about these developments is that they rely heavily on porting chunks of the classical GIT-driven theory of moduli of sheaves into the domain of so-called “twisted sheaves”. Thus, for example, unirationality results for moduli spaces of vector bundles on curves, originally due to Serre, interact with the Graber-Harris-Starr theorem (and its positive characteristic refinement by Starr and de Jong) to produce similar kinds of estimates for function fields of surfaces over algebraically closed fields. Similarly, irreducibility results dating back to Gieseker, Li, and O’Grady in the classical case, themselves inspired by the work of Donaldson, Mrowka and Taubes in the analytic category, can be proven for twisted sheaves and combined with finite-field methods and the aforementioned results over algebraically closed fields to yield explicit bounds on the rank of division algebras over function fields of surfaces over finite fields.

Spending enough time with these ideas, one is naturally led to the observation that the central function of these moduli spaces is to create a flow of information through correspondences created by the universal sheaf. (This is laid most bare in the study of derived equivalences of twisted sheaves. Indeed, in some sense the derived equivalences are the richest shadows of underlying equivalences of various kinds of motives, something we will not discuss here at all.) It then becomes apparent that the successes arising from the theory of “twisted sheaves” are primarily psychological: by viewing the moduli stack as its own object (rather than something to be compressed into a scheme), one more easily retains information that can be flowed back to the land of varieties.

This hints that a more uniform theory – one that treats sheaves and twisted sheaves as objects of an identical flavor – may be useful for future study of these objects. Among other things, we endeavor to describe such a theory here.

1.1. The structure of this paper. This paper has two primary purposes: (1) describing some background and recent results, and (2) doing a thought experiment about the right way to set up a more coherent theory that incorporates both sheaves and twisted sheaves as equals. The first purpose is addressed in Part 1 and the second is attempted in Part 2.
Part 1 includes a brief tour of the classical theory of sheaves in Section 2, including key examples and a few remarks about stability, and some less classical examples in Section 3 showing the usefulness of “thinking in stacks” from the beginning. Part 1 concludes with an impressionistic catalogue of results proven using twisted sheaves in the last decade or so.

Part 2 attempts to build a theory where ambient space and moduli space are treated symmetrically, and universal sheaves always exist. After establishing some terminology in Section 5, we describe the resulting theory of moduli of sheaves in Section 6, and some brief case studies illustrating how it can be used in Section 7.

1.2. Background assumed of the reader. I assume that the reader is familiar with schemes, algebraic spaces, and stacks. I also assume that she has seen the Picard scheme and the Quot scheme. While it is almost certainly the case that a reader who knows about algebraic stacks has also seen moduli spaces of sheaves to some extent, in the modern world there are many students of stacks who have little or no exposure to the moduli theory of sheaves, classical or modern. Thus, I will tell some of that story from scratch here.

Part 1. Background

2. A mild approach to the classical theory

We start with a chunk of the classical theory, to give a reader the flavor of the geometry of the moduli spaces of sheaves. In Section 6 we will give a more rigorous development of the theory in a more general framework, so we content ourselves here to discuss the beautiful geometry without worrying too much about the details.

2.1. The Quot scheme. The first moduli problem that many people encounter is a moduli problem of diagrams of sheaves: the Quot functor. Fix a morphism \( f : X \to S \) of schemes that is locally of finite presentation and a quasi-coherent sheaf of finite presentation \( F \) on \( X \).

Definition 2.1.1. The quotient functor associated to \( F \) is the functor

\[
\text{Quot}_F : \text{Sch}_S \to \text{Set}
\]

that sends \( T \to S \) to the set of isomorphism classes of quotients

\[ F_T \to Q, \]

with \( Q \) a \( T \)-flat finitely presented quasi-coherent sheaf on \( X_T \) whose support is proper and of finite presentation over \( T \).

An isomorphism of quotients \( F \to Q_1 \) and \( F \to Q_2 \) is an isomorphism \( Q_1 \to Q_2 \) commuting with the maps from \( F \). One can check that if such an isomorphism exists then it is unique.

Example 2.1.2. The simplest example of \( \text{Quot}_F \) is the following: let \( X \to S \) be \( \text{Spec} \mathbb{Z} \to \text{Spec} \mathbb{Z} \) and let \( F \) be \( \mathcal{O}_{\text{Spec} \mathbb{Z}}^\oplus n \). Then \( \text{Quot}_F \) is the disjoint union of all of the Grassmannians of an \( n \)-dimensional vector space (over \( \text{Spec} \mathbb{Z} \)).

Example 2.1.3. The other standard example (and the original motivation for studying the Quot functor) is the following: suppose \( X \to S \) is separated and \( F = \mathcal{O}_X \). In this case, \( \text{Quot}_F \) is the Hilbert scheme of \( X \). Among many other incredibly complex components, it contains one component isomorphic to \( X \) itself: one can look at those quotients \( \mathcal{O}_{X_T} \to Q \) where \( Q \) is
the structure sheaf of a section $T \to X_T$ of the canonical projection $X_T \to T$. (Here we use the fact that $X \to S$ is separated to know that $Q$ has proper support.)

In particular, we see that $\text{Quot}_F$ can have arbitrary geometry: many components with arbitrary dimensions and Kodaira dimensions.

The fundamental theorem about $\text{Quot}_F$ is that it is representable. (Of course, the representing object depends heavily on the context; it may be an algebraic space.)

Theorem 2.1.4 (Grothendieck). If $X \to S$ is locally quasi-projective and locally of finite presentation over $S$, then $\text{Quot}_F$ is representable by a scheme that is locally of finite presentation and locally quasi-projective over $S$.

Much of the classical theory of moduli spaces of sheaves is built by bootstrapping from Theorem 2.1.4, as we will indicate below. The general idea is to rigidify a problem by adding some kind of additional structure (e.g., a basis for the space of sections of a large twist), and then realize the original as a quotient of a locally closed subscheme of some $\text{Quot}$ scheme.

One question that is naturally raised by the Quot functor, and more particularly by the Hilbert scheme, is the following.

Question 2.1.5. What happens if we try to make the Hilbert scheme without the quotient structure? I.e., why not take moduli of sheaves $L$ on $X$ that are invertible sheaves supported on closed subschemes of $X$? How does this moduli problem compare to the Hilbert scheme?

2.2. The Picard scheme. The next moduli space of sheaves that we learn about is the Picard scheme. Fix a proper morphism $f : X \to S$ of finite presentation between schemes.

Definition 2.2.6. The naive Picard functor of $X/S$ is the functor

$$\text{Pic}^{\text{naive}}_{X/S} : \text{Sch}_S \to \text{Set}$$

sending $T \to S$ to $\text{Pic}(X_T)$.

This can't possibly be represented by a scheme, which one can see by considering the identity map $S \to S$. The functor $T \mapsto \text{Pic}(T)$ has the property that for all $T$ and all $f \in \text{Pic}(T)$ there is a Zariski covering $U \to T$ such that $f$ maps to $0$ in $\text{Pic}(U)$. That is, the functor defined is not a sheaf (it is not even a separated presheaf in the Zariski topology) and thus cannot be a scheme.

The traditional way to fix this is to study the sheafification of $\text{Pic}^{\text{naive}}_{X/S}$. Once one has studied Grothendieck topologies, one recognizes that one is in fact studying the higher direct image $R^1f_*\mathbb{G}_m$ in the fppf topology. This functor is usually written $\text{Pic}^{\text{fppf}}_{X/S}$.

Theorem 2.2.7 (Grothendieck). If $X \to S$ is a proper morphism of schemes of finite presentation that is locally projective on $S$ and cohomologically flat in dimension $0$ then $\text{Pic}^{\text{fppf}}_{X/S}$ is representable by a scheme locally of finite presentation over $S$.

This is beautifully described in Chapter 8 of [9]. (The definition of “cohomologically flat in dimension $0$” is on page 206, in the paragraph preceding Theorem 7 of that chapter.)

Example 2.2.8. Everyone’s first example is the Picard scheme of $\mathbb{P}^1$. By cohomology and base change, for any scheme $T$ we have a canonical isomorphism

$$\text{Pic}(\mathbb{P}^1 \times T) \xrightarrow{\sim} \text{Pic}(\mathbb{P}^1) \times \text{Pic}(T) = \mathbb{Z} \times \text{Pic}(T).$$
Since the second factor is annihilated upon sheafification, we see that
\[ \text{Pic}_{\mathbb{P}^1/\text{Spec} \mathbb{Z}} \cong \mathbb{Z}_{\text{Spec} \mathbb{Z}}. \]
the constant group scheme with fiber \( \mathbb{Z} \).

In Section 3.1 we will elaborate on this example for non-split conics.

2.3. Sheaves on a curve. General locally free sheaves on a curve turn out to have nice moduli. Fix a smooth proper curve \( C \) over an algebraically closed field \( k \), and fix an invertible sheaf \( L \).

Definition 2.3.9. The stack of locally free sheaves on \( C \) of rank \( n \) and determinant \( L \), denoted \( \text{Sh}_{C/k}(n, L) \), is the stack whose objects over a \( k \)-scheme \( T \) are pairs \( (F, \varphi) \) where \( F \) is a locally free sheaf on \( C \times T \), of rank \( n \) and \( \varphi : \text{det}(F) \rightarrow L_T \) is an identification of the determinant of \( F \) with \( L \).

The primary geometric result about \( \text{Sh}_{C/k}(n, L) \) is the following.

Proposition 2.3.10. The stack \( \text{Sh}_{C/k}(n, L) \) is an integral algebraic stack that is an ascending union of open substacks \( \mathcal{U}_N \), each of which admits a surjection from an affine space \( \mathbb{A}^{n(N)} \). That is, \( \text{Sh}_{C/k}(n, L) \) is an ascending union of unirational open substacks.

There are several ways to prove Proposition 2.3.10. Here is one of them. First, let \( F \) be a locally free sheaf of rank \( n \) and determinant \( L \).

Lemma 2.3.11. For sufficiently large \( m \), a general map \( \mathcal{O}^{n-1} \rightarrow F(m) \) has invertible cokernel isomorphic to \( \text{det}(F(m)) \cong \text{det}(F)(nm) \).

Sketch of proof. This is a Bertini-type argument. Choose \( m \) large enough that for every point \( c \) of \( C \) the restriction map
\[ \Gamma(C, F(m)) \rightarrow F(m) \otimes \kappa(c) \]
is surjective. Let \( A \) be the affine space underlying \( \text{Hom}(\mathcal{O}^{n-1}, F(m)) \), and let
\[ \Phi : \mathcal{O}^{n-1} \rightarrow \text{pr}_2^* F(m) \]
on \( A \times C \) be the universal map. There is a well-defined closed locus \( Z \subset A \times C \) parametrizing points where the fiber of \( \Phi \) does not have full rank. As with the Bertini theorem, we want to show that the codimension of \( Z \) is at least 2, as then it cannot dominate \( A \), giving the desired map.

To show it has codimension at least 2, it suffices to show this in each fiber over a point \( c \in C \). By assumption, the universal map specializes to all maps \( \kappa(c)^{n-1} \rightarrow F(m) \otimes \kappa(c) \), so it suffices to show that the locus of maps in the linear space \( \text{Hom}(k^{n-1}, k^n) \) that have non-maximal rank is of codimension at least 2. One way to do this is to show that the determinantal variety cut out by the \( (n-1) \times (n-1) \)-minors of the \( (n-1) \times n \)-matrices over \( k \) is not a divisor (i.e., there are non-redundant relations coming from the determinants). Another quick and dirty way to see the desired inequality in this case is to note that any map of non-maximal rank must factor through the quotient by a line. The space of quotients has dimension \( n-2 \), while the maps from the quotient vector space have dimension \( n(n-2) \). Thus, the space of non-injective maps has dimension at most \( (n+1)(n-2) = n^2 - n - 2 \), but the ambient space has dimension \( n(n-1) = n^2 - n \). The codimension is thus at least 2, as desired.
The value of the cokernel arises from a computation of the determinant of the exact sequence of locally free sheaves resulting from a general map \( \mathcal{O}^{n-1} \to F(m) \).

Proof of Proposition 2.3.10. We first address the latter part of the Proposition, deferring algebraicity of the stack for a moment.

Given \( m \), define a functor
\[
e_m : \text{AffSch}_k \to \text{Set}
\]
on the category of affine \( k \)-schemes that sends \( T \) to
\[
\text{Ext}^1_{C \times T}(\text{pr}_1^* L(mn), \mathcal{O}^{n-1}).
\]
By cohomology and base change, for all sufficiently large \( m \), the functor \( e_m \) is representable by an affine space \( \mathbb{A}^N \). The identity map defines a universal extension
\[
0 \to \mathcal{O}^{n-1} \to \mathcal{F} \to \text{pr}_1^* L(mn) \to 0
\]
over \( C \times \mathbb{A}^N \). By taking determinants, this comes with an isomorphism
\[
\varphi : \text{det}(\mathcal{F}) \sim \to \text{pr}_1^* L(mn).
\]
Sending this universal extension to \( (\mathcal{F}(-m), \varphi(-mn)) \) defines a morphism
\[
\varepsilon_m : \mathbb{A}^N \to \text{Sh}_{C/k}^f(n, L).
\]
By Lemma 2.3.11, these maps surject onto \( \text{Sh}_{C/k}^f(n, L) \) (as \( m \) ranges over any given unbounded collection of integers \( m \) for which \( e_m \) is representable).

Let us briefly address algebraicity of the stack. First, we note that for large enough \( m \), the morphisms \( \varepsilon_m \) are smooth over a dense open subset of \( \mathbb{A}^N \). This is equivalent to the statement that the sections of \( F(m) \) lift under any infinitesimal deformation (for \( F \) in the image), and this follows precisely from the vanishing of \( H^1(C, F(m)) \), which happens over a dense open of \( \mathbb{A}^N \) for sufficiently large \( m \). Moreover, for any \( F \), there is some \( m \) such that the smooth locus of \( \varepsilon_m \) contains \( F \). The diagonal of \( \text{Sh}_{C/k}^f(n, L) \) is given by the Isom functor, which we know to be representable by Grothendieck’s work on cohomology and base change. We conclude that we have defined a representable smooth cover of \( \text{Sh}_{C/k}^f(n, L) \), showing that it is algebraic.

It is also possible to prove algebraicity using Artin’s theorem and studying the deformation theory of sheaves, which is particularly simple for locally free sheaves on a curve. In fact, there are never any obstructions, showing that the stack \( \text{Sh}_{C/k}^f(n, L) \) is itself smooth.

We have studied a completely general moduli problem, but the reader will usually encounter an open substack of \( \text{Sh}_{C/k}^f(n, L) \) in the literature: the stack of semistable sheaves. What’s more, the classical literature usually addresses the Geometric Invariant Theory (GIT) quotient of this stack. For the sake of quasi-completeness, we give the definitions and make a few remarks. We will not discuss GIT here at all. The reader is referred to [18] (or the original [31]) for further information.

Definition 2.3.12. A locally free sheaf \( V \) on a curve \( C \) is stable (respectively semistable) if for all nonzero subsheaves \( W \subset V \) we have the inequality
\[
\frac{\deg(W)}{\text{rk}(W)} < \frac{\deg(V)}{\text{rk}(V)}.
\]
(respectively,
\[
\frac{\deg(W)}{\text{rk}(W)} \leq \frac{\deg(V)}{\text{rk}(V)}.
\]

The quantity \( \frac{\deg(F)}{\text{rk}(F)} \) is usually called the \textit{slope}, written \( \mu(F) \).

After Mumford revived the theory of stable bundles, this was studied by Narasimhan and Seshadri \cite{32}, who were interested in unitary bundles over a curve. It turns out that stable sheaves of degree \( 0 \) and rank \( n \) over a curve correspond to irreducible unitary representations of the fundamental group of \( C \); this was the original target of study for Narasimhan and Seshadri.

Stable and semistable sheaves have a number of nice categorical properties. Stable sheaves are simple (that is, a map \( F \to G \) is either an isomorphism or \( 0 \)). Semistable sheaves \( H \) admit Jordan-Hölder filtrations
\[
H = H^0 \supset H^1 \supset \cdots \supset H^n = 0
\]
such that \( H^i/H^{i+1} \) is stable with the same slope as \( H \), and the associated graded \( \text{gr}(H^*) \) is uniquely determined by \( H \) (even though the filtration itself is not). We will write this common graded sheaf as \( \text{gr}(H) \) (omitting the filtration from the notation). This is all nicely summarized in \cite{18}.

The basic geometric result concerning stability is the following. We can form a stack of stable sheaves \( \text{Sh}^s_{\mathcal{C}/k}(n, L) \) (resp. semistable sheaves \( \text{Sh}^{ss}_{\mathcal{C}/k}(n, L) \)) inside \( \text{Sh}^{\text{lf}}_{\mathcal{C}/k}(n, L) \) by requiring that the geometric fibers in the family are stable (resp. semistable).

\textbf{Proposition 2.3.13.} \textit{The inclusions}
\[
\text{Sh}^s_{\mathcal{C}/k}(n, L) \to \text{Sh}^{ss}_{\mathcal{C}/k}(n, L) \to \text{Sh}^{\text{lf}}_{\mathcal{C}/k}(n, L)
\]
\textit{are open immersions.}

\textit{Idea of proof.} Everything in sight is locally of finite presentation, and the arrows in the diagram are of finite presentation, hence have constructible images. It thus suffices to show that the images are stable under generization. In other words, we may restrict ourselves to families of sheaves on \( \mathcal{C} \otimes R \) where \( R \) is a complete dvr. Given a locally free sheaf \( V \) on \( C \otimes R \) and a subsheaf \( W^0 \subset V^\eta \) of the generic fiber such that \( \mu(W^0) \geq \mu(V^\eta) \), then we can extend \( W^0 \) to a subsheaf \( W \subset V \) such that \( V/W \) is flat over \( R \). This means that the sheaf \( W \) is locally free on \( C \otimes R \). But any locally free sheaf has constant degree in fibers, so the specialization \( W_0 \) also satisfies \( \mu(W_0) \geq \mu(V_0) \). This shows the desired stability under generization (by showing the complement is closed under specialization). \( \square \)

One of the early successes of GIT was a description of a scheme-theoretic avatar of \( \text{Sh}^{ss}_{\mathcal{C}/k}(n, L) \).

\textbf{Theorem 2.3.14 (Mumford, modern interpretation).} \textit{There is a diagram}
\[
\begin{array}{ccc}
\text{Sh}^s_{\mathcal{C}/k}(n, L) & \longrightarrow & \text{Sh}^{ss}_{\mathcal{C}/k}(n, L) \\
\pi & & \pi' \\
\tilde{\text{Sh}}^s_{\mathcal{C}/k}(n, L) & \longrightarrow & \tilde{\text{Sh}}^{ss}_{\mathcal{C}/k}(n, L),
\end{array}
\]
\textit{in which}
\begin{enumerate}
\item \( \tilde{\text{Sh}}^{ss}_{\mathcal{C}/k}(n, L) \) is a locally factorial integral projective scheme;
\item \( \tilde{\text{Sh}}^s_{\mathcal{C}/k}(n, L) \) is smooth;
\end{enumerate}
(3) \( \widetilde{\text{Sh}}^s_{C/k}(n, L) \to \text{Sh}^\text{sa}_{C/k}(n, L) \) is an open immersion;

(4) \( \pi \) together with the scalar multiplication structure realize \( \text{Sh}^s_{C/k}(n, L) \) as a \( \mu_n \)-gerbe over \( \text{Sh}^\text{sa}_{C/k}(n, L) \).

Moreover,

(1) \( \pi' \) universal among all morphisms from \( \text{Sh}^\text{sa}_{C/k}(n, L) \) to schemes;

(2) \( \pi \) establishes a bijection between the geometric points of \( S \) and isomorphism classes of geometric points of \( \text{Sh}^s_{C/k}(n, L) \);

(3) given an algebraically closed field \( \kappa \) and two sheaves \( H, H' \in \text{Sh}^\text{sa}_{C/k}(n, L)(\kappa) \), we have that \( \pi(H) = \pi(H') \) if and only if \( \text{gr}(H) \cong \text{gr}(H') \).

Extremely vague idea of proof. The very vague idea of the proof is this: given locally free sheaves \( V \) on \( C \), rigidify them by adding bases for the space \( \Gamma(C, V(m)) \) (i.e., surjections \( \mathcal{O}^{\chi(C, V(m))} \to V(m) \) for \( m \gg 0 \)). One then shows that the rigidified problem lies inside an appropriate \( \text{Quot} \) scheme. This \( \text{Quot} \) scheme has an action of \( \text{PGL}_N \), and Geometric Invariant Theory precisely describes quotients for such actions (once those actions have been suitably linearized in a projective embedding). The key to this part of the argument is the link between stability as defined above and the kind of stability that arises in GIT (which is phrased in terms of stabilizers of points).

One of the most miraculous properties arising from stability is that \( \text{Sh}^s_{C/k}(n, L) \) is separated. That is, a family of stable sheaves parametrized by a dvr is uniquely determined by its generic fiber. This is very far from true in the absence of stability. For example, a generic extension of \( \mathcal{O}(-1) \) by \( \mathcal{O}(1) \) on a curve \( C \) (where \( \mathcal{O}(1) \) is a sufficiently ample invertible sheaf) is stable, but any such sheaf admits the direct sum \( \mathcal{O}(-1) \oplus \mathcal{O}(1) \) as a limit. (So, in this case, a stable sheaf in a constant family admits a non-stable limit!)

We see from the preceding material that \( \widetilde{\text{Sh}}^s_{C/k}(n, L) \) is a unirational projective variety, containing \( \text{Sh}^s_{C/k}(n, L) \) as a smooth open subvariety that carries a canonical Brauer class represented by \( \text{Sh}^s_{C/k}(n, L) \). This immediately gives us an interesting geometric picture.

Theorem 2.3.15 (Main result of [8]). The Brauer group of \( \widetilde{\text{Sh}}^s_{C/k}(n, L) \) is generated by the class \([\text{Sh}^s_{C/k}(n, L)]\), whose order equals \( \gcd(n, \deg(L)) \).

The most interesting open question about the variety \( \widetilde{\text{Sh}}^s_{C/k}(n, L) \) is this: is it rational? The answer is known to be “yes” when \( n \) and \( \deg(L) \) are coprime, but otherwise the question is mysterious.

2.4. Sheaves on a surface. Increasing the dimension of the ambient space vastly increases the complexity of the moduli space of sheaves. The space is almost never smooth. Fixing discrete invariants no longer guarantees irreducibility. The Kodaira dimension of the moduli space can be arbitrary. One cannot always form locally free limits of locally free sheaves. The book [18] is a wonderful reference for the geometry of these moduli spaces, especially their GIT quotients, for surfaces. We touch on some of the main themes in this section.

Fix a smooth projective surface \( X \) over an algebraically closed field \( k \); let \( \mathcal{O}(1) \) denote a fixed ample invertible sheaf. We first define the basic stacks that contain the moduli problems of interest. Fix a positive integer \( n \), an invertible sheaf \( L \), and an integer \( c \).
Definition 2.4.16. The stack of torsion free sheaves of rank $n$, determinant $L$, and second Chern class $c$, denoted $\text{Sh}_{X/k}^{\text{tf}}(n, L, c)$, is the stack whose objects over $T$ are pairs $(F, \varphi)$, where $F$ is a $T$-flat quasi-coherent sheaf $F$ of finite presentation, $\varphi : \det(F) \sim \to L_T$ is an isomorphism, and such that for each geometric point $t \to T$, the fiber $F_t$ is torsion free of rank $n$ and second Chern class $c$.

There is an open substack $\text{Sh}_{X/k}^{\text{lf}}(n, L, c)$ parametrizing those sheaves that are also locally free.

Since the degeneracy locus of a general map $\mathcal{O}^{n-1} \to \mathcal{O}^n$ has codimension 2, no analogue of the Serre trick will work to show that the stack $\text{Sh}_{X/k}^{\text{lf}}(n, L, c)$ is unirational. In fact, this is not true.

The notion of stability is also significantly more complicated for a surface. There are now several notions, and these notions depend upon a choice of ample class. Given a sheaf $G$, let $P_G$ denote its Hilbert polynomial (whose value at $n$ is $\chi(X, G(n))$). We can write

$$P_G(n) = \sum_{i=0}^d \alpha_i(G) \frac{n^i}{i!}$$

for some rational numbers $\alpha$, where $d$ is the dimension of the support of $G$.

Definition 2.4.17. Given a torsion free sheaf $F$ on a surface $X$, the slope of $F$ is

$$\mu(F) = \frac{\det(F) \cdot \mathcal{O}(1)}{\text{rk}(F)},$$

where $\cdot$ denotes the intersection product of invertible sheaves.

Given any sheaf $G$ on $X$ with support of dimension $d$, the reduced Hilbert polynomial of $G$, denoted $p_G$ is the polynomial $\frac{1}{\alpha_d} P_G$.

Recall that there is an ordering on real-valued polynomials: $f < g$, resp. $f \leq g$, if and only if for all $n$ sufficiently large $f(n) < g(n)$, resp. $f(n) \leq g(n)$. We will use this to define one of the notions of stability for sheaves on a surface.

Definition 2.4.18. A torsion free sheaf $F$ on $X$ is slope-stable, resp. slope-semistable, if for any non-zero subsheaf $G \subset F$ we have $\mu(G) < \mu(F)$, resp. $\mu(G) \leq \mu(F)$.

A sheaf $G$ on $X$ is Gieseker-stable, resp. Gieseker-semistable, if for any subsheaf $G \subset F$ we have $p_G < p_F$, resp. $p_G \leq p_F$.

Slope-stability is often called $\mu$-stability, and similarly for semistability. Gieseker-stability is usually simply called stability, and similary for semistability.

The following lemma gives the basic relation between these stability conditions.

Lemma 2.4.19. Any slope-stable sheaf is Gieseker-stable. Any torsion free Gieseker-semistable sheaf is slope-semistable.

Just as for curves, one has the following description of stable and semistable moduli. We restrict our attention to torsion free sheaves for the sake of simplicity.

Proposition 2.4.20. The (semi-)stability conditions define a chain of open immersions of finite type Artin stacks over $k$

$$\text{Sh}_{X/k}^{\mu-ss}(n, L, c) \hookrightarrow \text{Sh}_{X/k}^{\text{ss}}(n, L, c) \hookrightarrow \text{Sh}_{X/k}(n, L, c) \hookrightarrow \text{Sh}_{X/k}^{\mu}(n, L, c),$$
The essence of the moduli theory of sheaves on surfaces, as discovered by O’Grady, is in the hierarchy implicit in the second Chern class. If a moduli space contains both locally free and non-locally free (but torsion free) points, then forming reflexive hulls of torsion free points relates the boundary (the non-locally free locus) to locally free loci in moduli spaces of lower second Chern class. In this way, one gets a hierarchy of moduli spaces in which the boundary of a given space is made of spaces lower in the hierarchy. This is very similar to the moduli space of curves: compactifying it, one finds that the boundary is made of curves of lower genus, and this is what permitted Deligne and Mumford to give their beautiful proof of the irreducibility of $\mathcal{M}_g$, essentially by induction.
As O’Grady realized, a similar kind of “induction” gives us quite a bit of information about the moduli spaces of sheaves on $X$, asymptotically in the second Chern class. More precisely, he proved the following.

Theorem 2.4.23 (O’Grady). Given $n$ and $L$ and an integer $d$, there is an integer $C$ such that for all $c \geq C$, the projective scheme $\widetilde{\text{Sh}}_{X/k}^{ss}(n, L, c)$ is geometrically integral and lci with singular locus of codimension at least $d$.

Numerous authors have studied these moduli spaces in special cases, their Kodaira dimensions for surfaces of general type, etc. It is beyond the scope of this article to give a comprehensive guide to literature, but the reader will be well-served by consulting [18] (which has as comprehensive a reference list as I’ve ever seen).

### 2.5. Guiding principles.

As a conclusion to this highly selective whirlwind tour of the basics of moduli of semistable sheaves on curves and surfaces, let me state a few key points.

1. The geometry of moduli spaces of sheaves is not pathological for low-dimensional varieties.
2. In dimension 1, the moduli spaces are rationally connected.
3. In dimension 2, a phenomenon very similar to the behavior of moduli of curves is observed: a hierarchy indexed by a discrete invariant that inductively improves geometry.
4. (This one will be more evident below.) The classical moduli spaces (the coarse spaces or the GIT quotients) rarely carry universal sheaves, because the stable loci are often non-trivial gerbes over them.

Much of the recent successes achieved with twisted sheaves leverage these phenomena, after recognizing that they continue to hold in greater generality (for other kinds of ambient spaces, over more complex bases, without stability conditions, on the level of stacks of sheaves, etc.). As our thought experiment progresses in Section 6, we will come back to these principles, establishing appropriate analogues of them, and we will apply these analogues in Section 7 to give some concrete examples demonstrating how the modicum of geometry we can develop in a more general setting suffices to prove non-trivial results.

Before getting to that, however, we give a few examples showing foundational difficulties inherent in this entire formulation of the moduli theory of sheaves.

### 3. Some less classical examples

#### 3.1. A simple example.

A simple example is given by the Picard scheme. Let’s study a very special case. Let $X$ be the conic curve in $\mathbb{P}^2_\mathbb{R}$ defined by the equation $x^2 + y^2 + z^2 = 0$. This is a non-split conic. The Picard scheme of $X$ over $\text{Spec } \mathbb{R}$ is the constant group scheme $\mathbb{Z}_\mathbb{R}$. But not every section of $\mathbb{Z}$ corresponds to an invertible sheaf on $X$. For example, the section 1 cannot lift to an invertible sheaf, because any such sheaf would have degree 1, and thus $X$ would have an $\mathbb{R}$-point, which it does not.

How do we understand this failure? The classical answer goes as follows. Write

$$ f : X \to \text{Spec } \mathbb{R} $$

for the structure morphism. The Picard scheme represents the functor $\mathbb{R}^1 f_* \mathbb{G}_m$. The sequence of low-degree terms in the Leray spectral sequence for $\mathbb{G}_m$ relative to $f$ is

$$ 0 \to \text{Pic}(X) \to H^0(\text{Spec } \mathbb{R}, \mathbb{R}^1 f_* \mathbb{G}_m) \to H^2(\text{Spec } \mathbb{R}, \mathbb{G}_m) \to H^2(X, \mathbb{G}_m). $$
The second group is just the space of sections of $\text{Pic}_{X/R}$, and this sequence tells us that there is an obstruction in the Brauer group of $R$ to lifting a section of the Picard scheme to an invertible sheaf, and that this obstruction lies in the kernel of the restriction map $\text{Br}(R) \to \text{Br}(X)$. In the case of the non-split conic, this restriction map is the 0 map (i.e., the quaternions are split by $X$), and $1 \in \mathbb{Z}$ maps to the non-trivial element of $\text{Br}(R)$, giving the obstruction map. The identity map $\text{Pic}_{X/R} \to \text{Pic}_{X/R}$ is a section over $\text{Pic}_{X/R}$; there is an associated universal obstruction in $H^2(\text{Pic}_{X/R}, G_m)$.

There is another way to understand this failure that is more geometric. Instead of thinking about the Picard scheme, we could instead think about the Picard stack, which we will temporarily write as $\mathcal{P}$. This is the stack on $\text{Spec } R$ whose objects over $T$ are invertible sheaves on $X \times_{\text{Spec } R} T$. The classical formulation of Grothendieck says precisely that the Picard scheme represents the sheafification of $\mathcal{P}$. That is, there is a morphism $\mathcal{P} \to \text{Pic}_{X/R}$. The stack $\mathcal{P}$ has a special property: the automorphism sheaf of any object of $\mathcal{P}$ is canonically identified with $G_m$ (i.e., scalar multiplications in a sheaf). This makes $\mathcal{P} \to \text{Pic}_{X/R}$ a $G_m$-gerbe. By Giraud’s theory, $G_m$-gerbes over a space $Z$ are classified by $H^2(Z, G_m)$. The cohomology class corresponding to $\mathcal{P} \to \text{Pic}_{X/R}$ is precisely the universal obstruction – the obstruction to lifting a point of $\text{Pic}_{X/R}$ to an object of $\mathcal{P}$.

The advantage of the latter point of view is that we can then work with the universal sheaf $L$ on $X \times \mathcal{P}$. This universal sheaf has a special property: the canonical right action of the inertia stack of $\mathcal{P}$ on $L$ is via the formula $(\ell, \alpha) \mapsto \alpha^{-1} \ell$ on local sections. (Technical note: the inertial action is a right action, whereas we usually think of the $\mathcal{O}$-module structure on an invertible sheaf as a left action. We want the associated left action to be the usual scalar multiplication action, which means that the right action must be the inverse of this, resulting in the confusing sign.)

### 3.2. A more complex example

This example has to do with the Brauer group and Azumaya algebras. A reader totally unfamiliar with these things is encouraged to come back to this example later.

Fix an Azumaya algebra $\mathcal{A}$ on a scheme $X$, say of degree $n$ (so that $\mathcal{A}$ has rank $n^2$ as a sheaf of $\mathcal{O}_X$-modules). The sheaf $\mathcal{A}$ is a form of some matrix algebra $M_n(\mathcal{O}_X)$, where $n$ is a global section of the constant sheaf $\mathbb{Z}$ on $X$. The Skolem-Noether theorem tells us that the sheaf of automorphisms of $M_n(\mathcal{O}_X)$ is $\text{PGL}_n$, and descent theory gives us a class in $H^1(X, \text{PGL}_n)$ that represents $\mathcal{A}$. The classical exact sequences

$$1 \to G_m \to GL_n \to \text{PGL}_n \to 1$$

and

$$1 \to \mu_n \to SL_n \to \text{PGL}_n \to 1$$

yield a diagram of connecting maps

$$
\begin{array}{ccc}
\text{H}^2(X, G_m) & \to & \text{H}^2(X, \mu_n) \\
\downarrow & & \downarrow \\
\text{H}^1(X, \text{PGL}_n) & \to & \text{H}^2(X, \mu_n)
\end{array}
$$
giving us two abelian cohomology classes one can attach to $\mathcal{A}$. Write $[\mathcal{A}]$ for the image in $H^2(X, \mathbb{G}_m)$. We know from the theory of Giraud that the latter group parametrizes isomorphism classes of $\mathbb{G}_m$-gerbes $\mathcal{G} \to X$.

When $[\mathcal{A}] = 0$, the gerbe $\mathcal{G}$ is isomorphic to $\mathbb{B}\mathbb{G}_m \times X$, and the algebra $\mathcal{A}$ is isomorphic to $\mathcal{E}nd(V)$ for some locally free sheaf $V$ on $X$; moreover, $V$ is unique up to tensoring with an invertible sheaf on $X$. This gives a classification of all Azumaya algebras with trivial Brauer class. Moreover, this helps us understand their moduli: the moduli space is essentially the moduli space of locally free sheaves on $X$ modulo the tensoring action of the Picard scheme. When $X$ is a curve or surface, these spaces are rather well understood; they have well-known geometrically irreducible locally closed subspaces, etc.

But what about non-trivial classes? It turns out that there is always a locally free sheaf $V$ on the gerbe $\mathcal{G}$ such that the original Azumaya algebra $\mathcal{A}$ is isomorphic to $\mathcal{E}nd(V)$. This sheaf $V$ doesn't come from $X$, but it has identical formal properties. In particular, we can study the moduli of the Azumaya algebras $\mathcal{A}$ with the fixed class in terms of moduli of the sheaves $V$ that appear. When $X$ is a curve or surface, we again get strong structural results. In particular, there are always canonically defined geometrically integral locally closed subschemes of the moduli space. This has real consequences. For example, when working over a finite field, any geometrically integral scheme of finite type has a 0-cycle of degree 1. If $X$ is a curve, the moduli space is geometrically integral and rationally connected (if one fixes the determinant, an operation that is quite mysterious from the Azumaya algebra point of view), and then the Graber-Harris-Starr theorem shows that there is a rational point. These geometric statements ultimately tell us that if $X$ is a smooth projective surface over an algebraically closed or finite field and $\alpha$ is a Brauer class on $X$ of order $n$, then there is an Azumaya algebra on $X$ in the class $\alpha$ that has degree $n$.

The point of this example is to illustrate a particular workflow: an algebraic question can be transformed into a geometric question about a moduli space of sheaves on a gerbe. It turns out that if one places these gerbes and associated moduli gerbes of sheaves on an equal footing, then the resulting symmetry in the theory sheds light on several problems, coming from algebra and number theory. Among these are the Tate conjecture for K3 surfaces, the index-reduction problem for field extensions, and the finiteness of the $u$-invariant of a field of transcendence degree 1 over $\mathbb{Q}$. We will discuss some of these below.

### 3.3. A stop-gap solution: twisted sheaves

Faced with the kinds of difficulties described in the previous two sections, algebraic geometers have spent the last decade or so rewriting the theory in terms of “twisted sheaves”. The basic idea is to embrace the gerbes that appear naturally in the moduli problems (coming from their natural stacky structures). Given a variety $X$, then, the moduli “space” of sheaves on $X$ is really a gerbe $\mathcal{M} \rightarrow M$ over some other space (at least near the general point). The universal sheaf lives on $X \times \mathcal{M}$ and can then be used to compare $X$ and $\mathcal{M}$ (for example, using cohomology, $K$-theory, Chow theory, derived categories, motives, . . .).

In Section 4 we describe some of the results that this approach has yielded, due to many authors working in disparate areas of the subject over a number of years.

### 4. A catalog of results

In this section, we provide a brief catalog of some results proven about twisted sheaves and results that use twisted sheaves. The literature is rather vast, so this is only the tip of the
iceberg. The reader will profit from following the references in the works described below. In particular, I have given very short shrift to the massive amount of work done in the direction of mathematical physics (including the deep work of Block, Pantev, Ben-Bassat, Sawon, and many others).

The “theorems” stated here are occasionally a bit vague. They should be seen purely as signposts indicating an interesting paper that deserves a careful reading. I have kept this catalog purposely impressionistic in the hope that a reader might get intrigued by a theorem or two and follow the included references.

4.1. Categorical results.

Theorem 4.1.1 (Antieau, conjectured by Căldăraru, [6]). Given quasi-compact quasiseparated schemes \(X\) and \(Y\) over a commutative ring \(R\) and Brauer classes \(\alpha \in \text{Br}(X)\) and \(\beta \in \text{Br}(Y)\), if there is an equivalence \(\text{QCoh}(X, \alpha) \cong \text{QCoh}(Y, \beta)\) then there is an isomorphism \(f : X \to Y\) such that \(f^* \beta = \alpha\).

Theorem 4.1.2 (Căldăraru, [11]). Suppose \(X \to S\) is a generic elliptic threefold with relative Jacobian \(J \to S\), and let \(\overline{J} \to J\) be an analytic resolution of singularities. Let \(\alpha \in \text{Br}(\overline{J})\) be the universal obstruction to the Poincaré sheaf on \(X \times_S J\). Then there is an equivalence of categories \(D(X) \cong D(\overline{J}, \alpha)\) induced by a twisted sheaf on \(X \times_S \overline{J}\).

Theorem 4.1.3 (Lieblich-Olsson, [29]). In characteristic \(p\), if \(X\) is a K3 surfaces and \(Y\) is a variety such that \(D(X) \cong D(Y)\), then \(Y\) is isomorphic to a moduli space of stable sheaves on \(X\).

Theorem 4.1.4 (Huybrechts-Stellari, [19]). Given a K3 surface \(X\) over \(\mathbb{C}\) and a Brauer class \(\alpha\), the set of isomorphism classes of pairs \((Y, \beta)\) with \(Y\) a K3 surface and \(\beta \in \text{Br}(Y)\) such that \(D(X, \alpha) \cong D(Y, \beta)\) is finite.

4.2. Results related to the geometry of moduli spaces.

Theorem 4.2.5 (Yoshioka, [35]). The moduli space of stable twisted sheaves on a complex K3 surface \(X\) is an irreducible symplectic manifold deformation equivalent to a Hilbert scheme of points on \(X\).

Theorem 4.2.6 (Lieblich, [24]). The stack of semistable twisted sheaves with positive rank and fixed determinant on a curve is geometrically unirational. The stack of semistable twisted sheaves with positive rank, fixed determinant, and sufficiently large second Chern class on a surface is geometrically integral, and lci, with singular locus of high codimension.

Theorem 4.2.7 (Lieblich, [27]). The Ogus moduli space of supersingular K3 surfaces is naturally covered by rational curves. Moreover, a general point is verifiably contained in countably many pairwise distinct images of \(\mathbb{A}^1\).

4.3. Results related to non-commutative algebra.

Theorem 4.3.8 (Gabber). If \(X\) is a quasi-compact separated scheme with ample invertible sheaf then \(\text{Br}(X) = \text{Br}'(X)\).

Theorem 4.3.9 (de Jong, [13]). If \(K\) is a field of transcendence degree 2 over an algebraically closed field, then for all \(\alpha \in \text{Br}(K)\), we have \(\text{ind}(\alpha) = \text{per}(\alpha)\).

Theorem 4.3.10 (Lieblich, [26]). If \(K\) is a field of transcendence degree 2 over a finite field then for all \(\alpha \in \text{Br}(K)\) we have \(\text{ind}(\alpha) | \text{per}(\alpha)^2\)
Theorem 4.3.11 (Krashen-Lieblich, [21]). Given a field \( k \), a smooth proper geometrically connected curve \( X \) over \( k \), and a Brauer class \( \beta \in \text{Br}(k) \), the index of \( \beta \) restricted to \( k(X) \) can be computed in terms of the restriction of the universal obstruction over the moduli space of stable vector bundles on \( X \).

4.4. Results related to arithmetic.

Theorem 4.4.12 (Lieblich-Parimala-Suresh, [30]). If Colliot-Thélène’s conjecture on 0-cycles of degree 1 holds for geometrically rationally connected varieties, then any field \( K \) of transcendence degree 1 over a totally imaginary number field has finite u-invariant.

Theorem 4.4.13 (Lieblich-Maulik-Snowden, [28]). Given a finite field \( k \), the Tate conjecture for K3 surfaces over finite extensions of \( k \) is equivalent to the statement that for each finite extension \( L \) of \( k \), the set of isomorphism classes of K3 surfaces over \( L \) is finite.

Theorem 4.4.14 (Charles, [12]). The Tate conjecture holds for K3 surfaces over finite fields of characteristic at least 5.

Part 2. A thought experiment

In this part, we describe a uniform theory that erases the geometric distinction between the ambient space holding the sheaves and its moduli space: the theory of merbes.

5. Some terminology

We will work over the base site of schemes in the fppf topology. Given a stack \( Z \), we will let \( \iota(Z) : I(Z) \to Z \) denote the inertia stack of \( Z \), with its canonical projection map. Given a morphism of stacks \( p : X \to S \), we will let \( I(X/S) \) denote the kernel of the natural map \( I(X) \to p^* I(S) \). (Even though \( p^* \) is not unique, the subsheaf \( I(X/S) \subset I(X) \) is uniquely determined by any choice of \( p^* \).) We will let \( \text{Sh}(Z) \) denote the sheafification of \( Z \), which is a sheaf on schemes. We will write \( \text{Site}(Z) \) for the site of \( Z \) induced by the fppf topology on the base category.

Several categories will be important throughout this paper. Given a stack \( S \), we will let \( \mathbf{Sch}_S \) denote the category of schemes over \( S \); this is the total space of \( S \), viewed as a fibered category over \( \text{Spec} \mathbb{Z} \). We will let \( \text{Set} \) denote the category of sets and \( \text{Grpd} \) denote the (2-)category of groupoids. The notation \( \text{GrpSch}_S \) denotes the category of group schemes over a base \( S \).

We end this section by defining a convenient notational structure on the subgroups of the multiplicative group. This will be useful in our discussion of merbes starting in Section 5.1.

Definition 5.1. Let Level be the category with objects \( \mathbb{N} \cup \{\infty\} \), and with

\[
\text{Hom}(m, m') = \begin{cases} 
\{\emptyset\} & \text{if } m|m' \\
\emptyset & \text{otherwise}
\end{cases}
\]

(5.0.1)

By convention, we assume that every natural number divides \( \infty \).

Notation 5.2. The notation \( \mathbb{Z}/\infty \mathbb{Z} \) will mean simply \( \mathbb{Z} \).

Definition 5.3. Define the functor

\[ G : \text{Level} \to \text{GrpSch}_\mathbb{Z} \]
by letting \( G(m) = \mu_m \) for \( m < \infty \) and \( G(\infty) = \mathbb{G}_m \). Given a divisibility relation \( m|m' \), the arrow \( G(m) \to G(m') \) is the canonical inclusion morphism.

By Notation 5.2, for all \( m \) in Level we have that \( \mathbb{Z}/m\mathbb{Z} \) is naturally the dual group scheme of \( G(m) \).

5.1. The 2-category of merbes. In order to give a symmetric description of moduli problems attached to sheaves, we will need to work with \( \mathbb{G}_m \)-gerbes and \( \mu_n \)-gerbes. Rather than constantly refer to “\( \mathbb{G}_m \)-gerbes or \( \mu_n \)-gerbes”, we will phrase results using a theory we call merbes. We do this for two reasons: (1) when people see the term “\( \mathbb{G}_m \)-gerbe” or “\( \mu_n \)-gerbe”, they are primed to think of it as a relative term (i.e., they think of a gerbe over a space, as in Giraud’s original theory), and (2) many of our results work simultaneously for \( \mathbb{G}_m \)-gerbes and \( \mu_n \)-gerbes, or involve relationships among such gerbes, and we prefer to put them under a single unified umbrella.

Definition 5.1.4. A merbe over a stack \( S \) is a pair \((X \to S, i)\), where \( X \) is a stack with a morphism to \( S \) and \( i : G(m) \to I(X/S) \) is a monomorphism of sheaves of groups with central image for some \( m \). We will call \( m \) the level of the merbe. A morphism of merbes \((X, i) \to (Y, j)\) is a 1-morphism \( f : X \to Y \) over \( S \) such that the induced diagram of canonical maps

\[
\begin{array}{ccc}
G(m) & \longrightarrow & I(X/S) \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
f^*G(m')|_Y & \longrightarrow & f^*I(Y/S)
\end{array}
\]

commutes. Here we mean that the level of \( X \) (written \( m \)) is assumed to divide the level of \( Y \) (written \( m' \)), and the left vertical map is the one induced by the canonical inclusion \( G(m) \to G(m') \).

We will say that the merbe has a property of a morphism of stacks (e.g. “separated”) if the underlying morphism \( X \to S \) has that property. For example, if the underlying stack \( X \) is relatively algebraic over \( S \), we will call the merbe algebraic.

Notation 5.1.5. We will generally abuse notation and omit \( i \) from the notation, instead referring to \( G(m) \) as a subsheaf of \( I(X/S) \), and we will call it the level subgroup. Given a merbe \( X \), we will write \( G(X) \) for the level subgroup when we do not want or need to refer explicitly to its level. We will write \( \overline{X} \) for the rigidification \( X/G(X) \) in the sense of [4, 2, 3], and we will call it the rigidification of \( X \).

As we know from [loc. cit.], the morphism \( X \to \overline{X} \) is a \( G(X) \)-gerbe (in the fppf topology on \( \overline{X} \)).

Notation 5.1.6. We will write Merbe for the 2-category of merbes.

Sending a merbe to its level defines a fibered category

\[ \lambda : \text{Merbe} \to \text{Level}. \]

Lemma 5.1.7. The functor \( \lambda \) is co-Cartesian.

Proof. This means: given a merbe \( X \) of level \( m \) and a divisibility relation \( m|m' \), there is a merbe \( X' \) of level \( m' \) with a map \( X \to X' \) that is universal for morphisms from \( X \) to merbes of level \( m' \). To see this, let \( \overline{X} \) be the rigidification of \( X \) along \( G(m) \). Giraud’s construction [17]
gives rise to an extension of $X \to \overline{X}$ to a $G(m')$-gerbe $X' \to \overline{X}$. Any map of merbes $X \to Y$ with $Y$ of level $m'$ induces a morphism of rigidifications $\overline{X} \to \overline{Y}$, giving a map $X \to Y \times_{\overline{X}} \overline{Y}$ of gerbes over $\overline{X}$ that respects the map $G(m) \to G(m')$. The techniques of [17] show that this factors uniquely through $X'$, as desired.

Example 5.1.8. Every morphism of stacks $X \to S$ has an associated merbe for each level $m$, namely $X \times BG(m)$. Thus, for example, any morphism of schemes has a canonically associated merbe of level $m$.

Definition 5.1.9. Given a merbe $X \to S$ of level $m$ and a divisibility relation $m|m'$, we will let $\text{Can}_{m|m'}(X)$ denote the canonically associated merbe of level $m'$ arising from Lemma 5.1.7.

As an example, any stack is canonically a merbe of level 1. For any $m$, we have

$$\text{Can}_{1|m}(X) = X \times BG(m),$$

with the inclusion $G(m) \to I(X \times BG(m))$ arising from the canonical isomorphism

$$I(X \times BG(m)) \cong I(X) \times G(m)|_X.$$

Remark 5.1.10. Given a merbe $X$ of level $m$ and two divisibility relations $m|m'|m''$, there is a canonical isomorphism

$$\text{Can}_{m'|m''}(\text{Can}_{m|m'}(X)) \cong \text{Can}_{m|m''}(X)$$

compatible with the natural maps from $X$.

Definition 5.1.11. A merbe $X$ is splittable if there is an isomorphism

$$\text{Can}_{\lambda(X)|\infty}(X) \cong \text{Can}_{1|\infty}(\overline{X}).$$

In less technical terms, a merbe is splittable if the associated $G_m$-gerbe is trivializable.

There is one more natural operation one can perform on merbes.

Definition 5.1.12. Given a merbe $X \to S$ and a divisibility relation $m|\lambda(X)$, the contraction of $X$, denoted $\text{Contr}_{\lambda(X)/m}(X)$ is the rigidification of $X$ along the subgroup $G(\lambda(X)/m)$.

Thus, for example, we have $\overline{X} = \text{Contr}_{\lambda(X)/1}(X)$.

5.2. Sheaves on merbes. Fix a merbe $X \to S$. Since $G(m)$ is a subsheaf of $I(X/S)$ (and thus of $I(X)$), any abelian sheaf $F$ on $X$ admits a natural right action of $G(m)$. In particular, if $X$ is algebraic and $F$ is quasi-coherent $F$ breaks up as a direct sum of eigensheaves

$$F = \bigoplus_{n \in \mathbb{Z}/m\mathbb{Z}} F_n,$$

where the action $F_n \times G(m) \to F_n$ is described on local sections by

$$(f, \alpha) \mapsto \alpha^{-n}f.$$ (In other words, the left action canonically associated to the inertial action is multiplication by $n$th powers via the $\mathcal{O}_X$-module structure on $F$.)

Definition 5.2.13. A sheaf $F$ of $\mathcal{O}_X$-modules on the merbe $X$ will be called an $n$-sheaf if the right action $F \times G(m) \to F$ satisfies $(f, \alpha) \mapsto \alpha^{-n}f$ on local sections.

Notation 5.2.14. We will write $\text{Sh}^{(n)}(X)$ for the category of $n$-sheaves on $X$. 

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In popular terminology, \( n \)-sheaves are usually called “\( n \)-fold twisted sheaves” and 1-sheaves are usually called “twisted sheaves”.

Example 5.2.15. Given a morphism of stacks \( X \to S \) with canonical merbe \( \text{Can}_{1|m}(X) \), there is a canonical invertible \( n \)-sheaf for every \( n \). Indeed, via the canonical map
\[
\text{Can}_{1|m}(X) \to \text{Can}_{1|m}(S) = BG(m)
\]
we see that it suffices to demonstrate this for \( BG(m) \). Any character of \( G(m) \) induces a canonical invertible sheaf (but note that the equivalence of representations of \( G(m) \) and sheaves on \( BG(m) \) naturally involves right representations, so the corresponding left action requires composition with inversion). The \( n \)th power map
\[
G(m) \to^n G(m) \to G_m
\]
gives an \( n \)-sheaf.

Notation 5.2.16. The canonical invertible \( n \)-sheaf on \( \text{Can}(X) \) will be called \( \chi^{(n)}_X \).

The key to the symmetrization presented in this article is the following pair of mundane lemmas.

Lemma 5.2.17. Suppose \( X \to S \) is a morphism of stacks. For any \( n \) and \( m \), there is a canonical equivalence of categories
\[
\text{Mod}_{\mathcal{O}_X} \to \text{Sh}^{(n)}(\text{Can}_{1|m}(X))
\]
given by pulling back along the projection \( \text{Can}_{1|m}(X) \to X \) and tensoring with \( \chi^{(n)}_X \).

Proof. The inverse equivalence is given by tensoring with \( \chi^{(-n)}_X \) and pushing forward to \( X \). More details that these are inverse equivalences may be found in [25, 24].

Lemma 5.2.18. For any merbe \( X \) of level \( m \) over \( S \), any divisibility relation \( m|m' \), and any \( n \), the natural map
\[
X \to \text{Can}_{m|m'}(X)
\]
induces an equivalence
\[
\text{Sh}^{(n)}(\text{Can}_{m|m'}(X)) \to \text{Sh}^{(n)}(X)
\]
by pullback.

Proof. Let \( \overline{X} \) be the rigidification of \( X \) along \( G(m) \), so that \( X \to \overline{X} \) (respectively, \( \text{Can}_{m|m'}(X) \to \overline{X} \)) is a \( G(m) \)-gerbe (respectively, a \( G(m') \)-gerbe). Each of the the categories \( \text{Sh}^{(n)}(X) \) and \( \text{Sh}^{(n)}(\text{Can}_{m|m'}(X)) \) is the global sections of a stack on \( \overline{X} \), and pullback is a morphism between these stacks. Working locally on \( \overline{X} \), we see that it thus suffices to prove this when \( X = \overline{X} \times BG(m) \), in which case \( \text{Can}_{m|m'}(X) = \overline{X} \times BG(m') \). Since we are working on the big fpf site, and the stacks of \( n \)-sheaves are the restrictions of the corresponding stacks for \( BG(m) \) and \( BG(m') \) over \( \text{Spec} \mathbb{Z} \), we see that it suffices to prove the result for \( BG(m) \) and \( BG(m') \) over \( \mathbb{Z} \). Now, the quasi-coherent sheaves parametrized by \( T \) are precisely right representations of \( G(m) \) and \( G(m') \) over \( T \), and the \( n \)-sheaf condition just says that the module action factors through the map \( G(m) \to G_m \to G_m \), where the latter is the \( n \)th power map. Both stacks are identified with the stack of \( n \)-sheaves on \( BG_m \), completing the proof.
6. Moduli of sheaves: basics and examples

6.1. The basics. In this section, we introduce the basic moduli problems. Fix an algebraic morphism \( f : X \rightarrow S \). We will assume to start that \( f \) is separated and of finite presentation.

Definition 6.1.1. Given a morphism of stacks \( T \rightarrow S \), a family of sheaves on \( X/S \) parametrized by \( T \) is a quasi-coherent sheaf \( F \) on \( X \times_X T \) that is locally of finite presentation and flat over \( T \).

We will be mostly interested in families of sheaves with proper support.

Definition 6.1.2. A family of sheaves on \( X/S \) with proper support parametrized by \( T \) is a family of sheaves \( F \) on \( X/S \) parametrized by \( T \) such that the support of \( F \) is proper over \( T \).

Definition 6.1.3. Given \( n \), a family of \( n \)-sheaves parametrized by \( T \rightarrow S \) is a family of sheaves \( F \) parametrized by \( T \) such that \( F \) is an \( n \)-sheaf.

Definition 6.1.4. A family \( F \) of sheaves on \( X/S \) parametrized by \( T \rightarrow S \) is perfect if there is an fppf cover \( U = \{ \sqcup U_i \} \rightarrow X \) by a disjoint union of affine schemes such that for each \( i \) the restriction \( F|_{U_i} \) has a finite resolution by free \( O_U \) modules.

The key feature of a perfect sheaf \( F \) is its determinant \( \det(F) \), which is an invertible sheaf on \( X \) that is additive in \( F \) (that is, given a sequence of perfect sheaves \( 0 \rightarrow F \rightarrow F' \rightarrow F'' \rightarrow 0 \) we have \( \det(F') \cong \det(F) \otimes \det(F'') \)) and has value \( \bigwedge^r F \) when \( F \) is locally free of constant rank \( r \). The existence of the determinant follows from the work of Mumford and Knudsen [20].

It is relatively straightforward to show that the pullback of a family is a family, and that pullback respects perfection and determinants.

In the rest of the paper, I will relax the terminology and may refer more loosely to objects of this kind as “families of sheaves” (without always mentioning the original morphism or the base of the family).

The basic (almost totally vacuous) result about families of sheaves is the following.

Proposition 6.1.5. Given a morphism \( X \rightarrow S \), families of \( n \)-sheaves on \( X/S \) naturally form a morphism \( \text{Sh}^{(n)}_{X/S} \rightarrow S \) of level \( \infty \). If \( X \) is algebraic and of finite presentation over \( S \), the perfect families are parametrized by an open substack \( \text{Sh}^{(n)}_{X/S} \text{perf} \subset \text{Sh}^{(n)}_{X/S} \).

Proof. We know that fppf descent is effective for quasi-coherent sheaves of finite presentation on a stack. Since the condition of being an \( n \)-sheaf is local in the fppf topology, descent is also effective for \( n \)-sheaves. Thus, \( n \)-sheaves form a stack \( \text{Sh}^{(n)}_{X/S} \). Finally, the module structure gives a canonical identification of \( \mathbb{G}_m \) with a central subgroup of the inertia, making \( \text{Sh}^{(n)}_{X/S} \) a morphism of level \( \infty \), as desired.

To see the last statement, note that we may first assume \( S \) is affine, and that \( X \) admits an fppf cover by a finite list of affines \( U_1, \ldots, U_t \), each of finite presentation over \( S \). Suppose given an \( n \)-sheaf \( F \) on \( X_T \). We wish to establish that the locus in \( T \) over which \( F \) is perfect is open, and to do this we may work locally on \( T \) and thus assume that \( T \) is affine. By standard constructibility results, this then readily reduces to the case in which \( X \) itself is affine. For arguments of this kind, the reader is referred to [1], [23], or the beautiful appendix to [34]. \( \square \)

Write \( \text{Sh}^{(n)}_{X/S}(r) \) for the substack of sheaves of constant rank \( r \in \mathbb{N} \). Fix an invertible sheaf \( L \) on \( X \).
Definition 6.1.6. The stack of perfect $n$-sheaves with determinant $L$ is the stack of pairs $(F, \varphi)$, where $F$ is a perfect $n$-sheaf and $\varphi : \det(F) \to L$ is an isomorphism. We will write it as $\text{Sh}_{X/S}^{(n)\text{perf}}(r, L)$.

Lemma 6.1.7. The stack $\text{Sh}_{X/S}^{(n)\text{perf}}(r, L)$ is naturally a merbe of level $r$.

Proof. The merbe structure arises from scalar multiplications as before, but since they have to commute with the identification of the determinant with $L$, only multiplications by sections of $\mu_r$ give automorphisms. □

One of the great games of the moduli theory is to use the universal sheaf $F$ on $X \times \text{Sh}_{X/S}$ to push information between the two merbes. We will see examples of this later.

This theory is heavily driven by examples. We describe some of the basic examples in the following sections using the language we have been developing here.

6.2. Example: Almost Hilbert. As students, we all learn to compute the “Hilbert scheme of one point” on $X/S$ and to show that it is $X$ itself (when $X$ is separated!). After having done this, one invariably wonders what would happen were one to consider the “Almost Hilbert” scheme of one point: colloquially, the scheme parametrizing sheaves on $X$ of rank 1 on closed points. We briefly investigate this construction for merbes in this section as a first example.

Assumption 6.2.8. We assume throughout this section that $X \to S$ is an algebraic merbe that is locally of finite presentation whose rigidification $\overline{X} \to S$ is a separated algebraic space.

Definition 6.2.9. Given a merbe $X \to S$ satisfying Assumption 6.2.8, the almost Hilbert scheme of $X/S$, denoted $\text{AHilb}_{X/S}$ is the stack of families of 1-sheaves on $X/S$ satisfying the following condition: for any family $F$ on $X \times_S T$ parametrized by $T$, we have that $F$ is an invertible 1-sheaf supported on a closed submerbe $Z \subset X$ whose rigidification $\overline{Z} \subset \overline{X}$ is a section of the projection map $\overline{X} \times_S T \to T$.

Proposition 6.2.10. Given a merbe $X \to S$ satisfying Assumption 6.2.8, the stack $\text{AHilb}_{X/S}$ is naturally a merbe of level $\infty$ and $\text{AHilb}_{X/S} \cong \text{Can}_{\lambda(X)\mid\infty}(X)$.

Proof. Scalar multiplication defines the natural merbe structure on $\text{AHilb}_{X/S}$; the level is $\infty$. Sending a family to its support defines an $S$-morphism $\text{AHilb}_{X/S} \to \overline{X}$. Fix a section $\sigma : T \to \overline{X}$. The fiber $\text{AHilb}_{X/S} \times_{\overline{X}} T \to T$ is the $G_m$-gerbe of invertible 1-sheaves on $X \times_{\overline{X}} T$, which is itself a $G(X)$-gerbe over $T$. As shown in [22], the $G_m$-gerbe associated to $X \times_{\overline{X}} T$ is precisely the gerbe of invertible 1-sheaves. (In the classical language, this is the gerbe of invertible twisted sheaves.) Applying this to the universal point $\text{id} : \overline{X} \to \overline{X}$ establishes the desired isomorphism. □

6.3. Example: invertible 1-sheaves on an elliptic merbe. Fix an elliptic curve $E$ over a field $K$. There are many merbes of level $\infty$ with rigidification $E$. By Giraud’s theorem, they are parametrized by $H^2(E, G_m)$, which, by the Leray spectral sequence for $G_m$, fits into a split exact sequence

$$0 \to H^2(\text{Spec } K, G_m) \to H^2(E, G_m) \to H^1(\text{Spec } K, \text{Pic}_{E/K}) = H^1(\text{Spec } K, \text{Jac}(E)) \to 0$$
(the latter equality arising from the vanishing of $H^1(\text{Spec } K, \mathbb{Z})$). Among other things, this sequence tells us that any Brauer class on $E$ has an associated $\text{Jac}(E)$-torsor. How can we identify it?

Given a class $\alpha \in \text{Br}(E)$ such that $\alpha|_0 = 0 \in \text{Br}(K)$ (with 0 denoting the identity point of $E$), let $\mathcal{E}$ be an associated merbe (in classical notation, a $G_m$-gerbe associated to $\alpha$). Let $\mathcal{M}$ be the stack of invertible 1-sheaves on $\mathcal{E}$; $\mathcal{M}$ is itself a merbe of level $\infty$. Tensoring with invertible 0-sheaves on $E$ defines an action of $\text{Pic}_{E/K}$ on $\mathcal{M}$. The resulting action on $\mathcal{M}$ gives a torsor, and this is precisely the image of the coboundary map.

6.4. Example: sheaves on a curve. In this section we reprise a small chunk of the moduli theory of sheaves on a curve.

Definition 6.4.11. A merbe $C \to S$ is a smooth proper curve-merbe (among merbes) if its rigidification $\mathcal{C} \to S$ is a smooth proper relative curve with connected geometric fibers.

We start with a smooth proper curve-merbe $C \to S$. Fix a positive integer $n$ and an invertible $n$-sheaf $L$ on $C$. We will consider some of the structure of $\mathcal{M} := \text{Sh}_{C/S}^{(1)}(n, L)^{\text{lf}}$, where $\text{lf}$ stands for the locus parametrizing locally free sheaves.

Theorem 6.4.12. The merbe $\mathcal{M} \to S$ has type $n$, and its rigidification $\overline{\mathcal{M}}$ is an ascending union of open subspaces that have integral rational geometric fibers over $S$.

Idea of proof. Tsen’s theorem implies that there is an fppf cover $S' \to S$ such that $C \times_S S'$ is splittable. In this circumstance, by Lemma 5.2.17, there is an invertible sheaf $L'$ on $\mathcal{C} \times_S S'$ and an isomorphism

$$\text{Sh}_{C/S}^{(1)}(L) \times_S S' \cong \text{Can}_{1|n}(\text{Sh}_{C \times_S S'/S'}(L')).$$

We can then use the classical geometry of the stack of sheaves on a curve (described in Section 2.3) to deduce geometric properties of $\mathcal{M} \to S$. □

Corollary 6.4.13. If $C$ is a smooth proper curve-merbe over a finite extension $K$ of $k(t)$, with $k$ algebraically closed, and $L$ is an invertible $n$-sheaf with $n$ invertible in $K$, then $\mathcal{M} := \text{Sh}_{C/K}^{(1)}(n, L)^{\text{lf}}$ contains an object over $K$.

Proof. By Theorem 6.4.12, $\mathcal{M}$ is a merbe of level $n$ whose rigidification is geometrically rationnally connected. Since $n$ is invertible in $K$, the fiber of $\mathcal{M}$ is separably rationally connected. By [15], the rigidification of $\mathcal{M}$ has a $K$-rational point. But the obstruction to lifting from $\overline{\mathcal{M}}$ to $\mathcal{M}$ lies in $\text{Br}(K)$, which vanishes by Tsen’s theorem. □

6.5. Example: sheaves on a surface. It turns out that essentially all of the key properties enumerated in Section 2.5 of classical theory of sheaves on surfaces carry over to sheaves on smooth proper surface-merbes. This has some interesting consequences we will describe later.

Definition 6.5.14. A smooth proper surface-merbe is a merbe $X \to S$ whose rigidification is a relative proper smooth surface with connected geometric fibers.

The general theory is described in [24] (using the language of twisted sheaves). For the purposes of certain applications, let me isolate a much less precise theorem that suffices in applications.
Theorem 6.5.15. Suppose $X \to \text{Spec } k$ is a smooth proper surface-merbe over a field, $n$ is a natural number invertible in $k$, and $L$ is an invertible $n$-sheaf on $X$. Then the stack $\mathcal{M} := \text{Sh}^{(1)}_{X/k}(n, L)$ contains a geometrically integral locally closed substack, if $\mathcal{M}$ is non-empty.

Idea of proof. The proof is rather complex, but the basic idea is easy to convey. The reader is referred to [26] for a proof in the language of twisted sheaves. Roughly speaking, the idea is to organize a hierarchy of locally closed substacks of $\mathcal{M}$ using the second Chern class, and to show that the substacks get nicer and nicer as the second Chern class grows. Using deformation theory, one shows that the set of components must shrink as the second Chern class grows, and each component must get closer to being smooth. Eventually, all of the components coalesce and a geometrically integral locally closed substack results. The hierarchy in question arises from the taking the reflexive hull of a sheaf $\mathcal{F}$, and the “height” in the hierarchy is determined by the length of $\mathcal{F}^\vee \mathcal{F}$; sheaves with greater length have higher second Chern class.

Remark 6.5.16. The basic idea sketched above is a recurring theme in the theory of moduli. A propitious choice of compactification of a moduli problem leads to a boundary that can be understood using associated moduli spaces of a lower level in some natural hierarchy, and playing one level of the hierarchy off another leads to limiting theorems. The Deligne-Mumford proof of irreducibility of $\mathcal{M}_g$ [16] is essentially built from this idea: the boundary of $\mathcal{M}_g$ is stratified by pieces made out of lower-genus moduli spaces, and their argument essentially works by a subtle induction. Similarly, O’Grady’s proof of the asymptotic irreducibility of the moduli space of stable vector bundles with fixed Chern classes on a smooth projective surface [33] works by the same basic outline as above, using reflexive hulls to relate boundary strata of the moduli space to the open part of moduli spaces lower in the hierarchy. Finally, the work of de Jong, He, and Starr on higher rational connectedness [14] uses this same strategy. The hierarchy in that case is indexed by degrees of stable maps into a fibration $f : X \to C$, and the transition among strata is achieved by attaching vertical curves to the image (thus inserting the sections of a given degree into the boundary of the compactified space of higher-degree sections).

Corollary 6.5.17. Suppose $k$ is PAC. Given a proper smooth surface-merbe $X \to \text{Spec } k$, a natural number $n$ invertible in $k$, and an invertible $n$-sheaf $L$, there is an object of $\text{Sh}^{(1)}_{X/k}(n, L)$ over $k$, assuming it is non-empty.

6.6. Example: sheaves on a K3 merbe. While a lot more can be said for a general surface, the case of K3 surfaces is a very interesting special case. The material in this section is drawn from [28] and [12]. The latter contains a cleverer approach to handling the numerical properties of the resulting theory, giving stronger boundedness results (relevant in Section 7.2).

Definition 6.6.18. A K3 merbe is a proper smooth surface-merbe $X \to S$ such that each geometric fiber $X_s$ has rigidification isomorphic to a K3 surface.

Fix a K3 merbe $X$ over a field $k$ and a prime $\ell$ that is invertible in $k$. We assume for the sake of simplicity that $\lambda(X) = m < \infty$ is invertible in $k$. In this case, since $X$ itself is smooth and proper, the Chow theory $\text{CH}(X)$ and étale cohomology $H^i(X, \mathbb{Z}_\ell(i))$ behave well.
Definition 6.6.19. The *Mukai representation* attached to $X$ is the $\text{Gal}_k$-module

$$H(X_k, \mathbb{Z}_\ell) := H^0(X_k, \mathbb{Q}_\ell) \oplus H^2(X_k, \mathbb{Q}_\ell(1)) \oplus H^4(X_k, \mathbb{Q}_\ell(2)).$$

There is a Galois-invariant quadratic form with values in $\mathbb{Q}_\ell$ coming from the formula

$$(a, b, c) \cdot (a', b', c') = b b' - a c' - a' c,$$

with the products on the right side arising from the usual cup product in étale cohomology.

Suppose that $X$ fits into a sequence of K3 merbes $(X_n)$ with the following properties.

1. $X_n$ has level $\ell^n$.
2. For each $n$, the merbe $X_n$ is isomorphic to $\text{Contr}_{\ell^{n+1}/\ell^n}(X_{n+1})$ (in the notation of Definition 5.1.12).

For reasons that we will not go into here, such a sequence is the arithmetic analogue of a $B$-field from mathematical physics.

In this case, for each $n$ one can find a distinguished lattice $\Lambda_n \subset H(X_n, \mathbb{Q}_\ell)$ together with an additive map $v : K^{(1)}(X_n) \to \Lambda_n$ such that for all $E, F \in K^{(1)}(X_n)$ we have $\chi(E, F) = -v(E) \cdot v(F)$. Here, $K^{(1)}$ stands for the $K$-theory of 1-sheaves. This is called the $\ell$-adic Mukai-Chow lattice of $X_n$.

Remark 6.6.20. It is an open question to understand the existence of the $\ell$-adic Mukai-Chow lattice in the absence of the full sequence $(X_n)$. This has numerical implications for the study of K3 merbes over finite fields, and it is probably relevant to the existence of certain kinds of uniform bounds on the size of the transcendental quotient for Brauer groups of K3 surfaces over finite fields and number fields.

If $k$ is algebraically closed, then such a sequence always exists (for $\ell$ invertible in $k$).

The fascinating part of the theory of K3 merbes is that we can use the Mukai-Chow lattice to manufacture new K3 merbes. The key theorem is the following. (The list of names is in roughly chronological order; first, Mukai proved this for classical sheaves on K3 surfaces over $\mathbb{C}$, then Yoshioka extended it to twisted sheaves on K3 surfaces over $\mathbb{C}$, and finally Lieblich-Maulik-Snowden proved it for twisted sheaves on K3 surfaces over arbitrary fields. For a more detailed look at a larger class of moduli spaces, see [12]).

Theorem 6.6.21 (Mukai, Yoshioka, Lieblich-Maulik-Snowden). *Given $v \in \Lambda_n$ such that $\text{rk } v = \ell^n$ and $v^2 = 0$ (in the lattice structure on $\Lambda_n$), the stack $M$ of stable 1-sheaves $F$ on $X_n$ with $v(F) = v$ is a K3 merbe of level $\ell^n$. Moreover, the universal sheaf defines an equivalence of derived categories

$$D^{(-1)}(X_n) \cong D^{(1)}(M).$$

Finally if there is some $u \in \Lambda_n$ such that $v \cdot u$ is relatively prime to $\ell$, then $M$ is splittable."

In other words, the numerical properties of the lattice $\Lambda_n$ can produce equivalences of derived categories between various kinds of K3 merbes, and can also ensure that one side of each such equivalence is splittable. As we will see below, this is directly relevant to the Tate conjecture for K3 surfaces. In this connection, we recall the following theorem.

Theorem 6.6.22 (Huybrechts-Stellari, [19]). *Give a K3 merbe $X$ over $\mathbb{C}$ of level $\infty$, there are only finitely many K3 merbes $Y$ of level $\infty$ such that $D^{(1)}(X) \cong D^{(1)}(Y)$. 
Remark 6.6.23. By dualizing $X_n$ over its rigidification, the difference between $D^{(1)}$ and $D^{(-1)}$ becomes unimportant. The restriction to level $\infty$ is necessary because passing from a merbe $Z$ to $\text{Can}_{\lambda(Z)\lambda(Z)}(Z)$ for any $m$ always induces an equivalence of abelian categories of 1-sheaves, hence an equivalence of derived categories. So counting statements for derived equivalences should only be evaluated at level $\infty$.

7. Case studies

We conclude this tour with a couple of almost immediate consequences of the theory described above to two problems: the period-index problem for the Brauer group and the Tate conjecture for K3 surfaces.

7.1. Period-index results. Let $K$ be a field. Given a Brauer class $\alpha \in \text{Br}(K)$, there are two natural numbers one can produce: the period of $\alpha$, denoted $\text{per}(\alpha)$, and the index of $\alpha$, denoted $\text{ind}(\alpha)$. Using basic Galois cohomology, one can show that $\text{per}(\alpha) | \text{ind}(\alpha)$ and that both numbers have the set of prime factors. (Note: see [7] for the more interesting situation over a scheme larger than one point.) The basic period-index problem is to determine how large $e$ must be in order to ensure $\text{ind}(\alpha) | \text{per}(\alpha)$. A specific form of the basic question is due to Colliot-Thélène.

Question 7.1.1 (Colliot-Thélène). Suppose $K$ is a $C_d$-field. Is it always true that $\text{ind}(\alpha) | \text{per}(\alpha)^{d-1}$?

The key connection between this question and the theory developed here is the following: given a Brauer class $\alpha$ over a field $K$, there is an associated $\mathbb{G}_m$-gerbe $G \to \text{Spec} K$. The index of $\alpha$ divides a number $N$ if and only if there is a 1-sheaf of rank $N$ on $G$. When $K$ is the function field of a reasonable scheme $X$, the $\mathbb{G}_m$-gerbe $G$ extends to a reasonable merbe closely related to $X$. By studying the stack of 1-sheaves, we get a new merbe whose rational-point properties are closely bound to Question 7.1.1. (This is all explained in [25].)

For global function fields an affirmative answer is given by the Albert-Brauer-Hasse-Noether theorem (see [5] and [10] for the classical references). For function fields of surfaces over algebraically closed fields, the conjecture was proven by de Jong [13], using the deformation theory of Azumaya algebras. There is also a simple proof using the results of Section 6.4: fiber the surface $X$ over $\mathbb{P}^1$ and consider the generic fiber. The Brauer class gives a curve-merbe $C \to \text{Spec} k(t)$, and the associated merbe of 1-sheaves $M$ is filled with unirational opens. By Corollary 6.4.13, $M$ contains an object.

The first non-trivial class of $C_3$-fields for which we know the answer is function fields of surfaces over finite fields. It is an immediate consequence of the theory developed here.

Theorem 7.1.2 (Lieblich, [26]). If $K$ has transcendence degree 2 over a finite field, then

$$\text{ind}(\alpha) | \text{per}(\alpha)^2$$

for all $\alpha \in \text{Br}(K)$.

Idea of proof. Given a class $\alpha$ of period $\ell$, one associates a smooth proper geometrically connected surface-merbe $X$ of level $\ell$ over a finite field. The merbe $M$ of 1-sheaves of rank $\ell^2$ on $X$ contains a geometrically integral locally closed substack by Theorem 6.5.15 (a moderate amount of the work is devoted to non-emptiness, and is the explanation for the $\ell^2$). The Lang-Weil estimates then imply that there is an object of $M$ over a field extension prime to $\ell$. Standard Galois cohomology then tells us that the desired divisibility relation holds.

□
7.2. The Tate conjecture for K3 surfaces. We conclude with a few words on the Tate conjecture for K3 surfaces. This gives a somewhat better illustration of how the symmetric nature of the theory of merbes lets mathematical information flow.

Theorem 7.2.3 (Ogus, Nygaard, Maulik, Madapusi-Pera, Charles). The Tate conjecture holds for K3 surfaces over a finite field $k$ of characteristic at least 5.

Idea of proof. There are now many proofs of this in various forms. We comment on the proof of Charles [12], which builds on ideas developed in [28] that have already appeared in Section 6.6. The idea is this: first, the Tate conjecture for all K3 surfaces over all finite extensions of $k$ is equivalent to the statement that for each such extension there are only finitely many K3 surfaces. Second, one can verify the finiteness statement for K3 surfaces using an analogue of Zarhin’s trick, familiar from abelian varieties.

This equivalence arises almost directly from Theorem 6.6.21 and Theorem 6.6.22. We sketch the implication of Tate from finiteness: If $X$ has infinite Brauer group, one gets a sequence of K3 merbes $(X_n)$ as in Section 6.6, and thus by Theorem 6.6.21 one gets a sequence of K3 surfaces $M_n$ arising as rigidifications of merbes of stable 1-sheaves on each $X_n$, together with equivalences $D^{-1}(X_n) \cong D(M_n)$. But Theorem 6.6.22 says that for a fixed K3 surface, there are only finitely many such partners up to isomorphism. (One must do a little work to get this down from $C$ to a relevant statement for our sequence of equivalences, but it can be done.) If there are only finitely many K3 surfaces, then we end up with infinitely many partners for one of them. The conclusion is that no such sequence $(X_n)$ can exist, so the Brauer group must be finite.

To prove finiteness, Charles used inspiration from complex geometry (birational boundedness statements for holomorphic symplectic varieties) to prove a Zarhin-type statement for K3 surfaces using moduli spaces of stable twisted sheaves, which have properties in characteristic $p$ very similar to those described by Yoshioka over $C$ [35], as hinted at in Theorem 4.2.5. (The proof is quite subtle, as Charles can’t simply port over complex techniques to characteristic $p$; rather he needs to use a tiny chunk of the theory of canonical integral models of Shimura varieties and a relative Kuga-Satake map to get an appropriate replacement for a particular period map.)

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