Finite size effects in Derrida’s model multicriticities and limits of generalization for the Zamolodchikov’s C-theorem

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Abstract

Finite size effects in the multicritical point and boundaries between phases are calculated. There are anomalous large finite size effects on the boundary of ferromagnetic phase with paramagnetic or spin-glass. Multicritical point is not giving global minimum for the finite size corrections of free energy.

According to $C$-theorem of Zamolodchikov[1], one can introduce some function $c(g)$ for $2d$ conformal theories, which decreases along the renorm-group paths. Kutasov [2] has suggested to consider as a $c(g)$ partition for the models on the $2d$ gravitation. For some model he found, that partition has maximum at multicriticality points. The question arose, is it general property for any phase transition?

Independent of $C$ theorem some years ago I began to investigate another (similar!) problem [3]. If symmetry was broken spontaneously during the phase transition and system has chosen some vacuum from the set, it should spend some free energy-”cost of decision making”. To found it, we need in asymptotics for the $\ln Z(N,T)$, where $Z$ is a partition for a system of $N$ spins at temperature $T$. Let we can define in some manner

$$\lim_{N \to \infty} \ln Z(N,T) = F(N,T) + o(1)$$

(1)
Then at phase transition point $T_c$ function $F(N,T)$ probably has some jump, describing waste of free energy for vacuum choosing:

$$\lim_{\epsilon \to 0} F(N, T_c - \epsilon) - F(N, T_c - \epsilon) \neq 0$$  \hspace{1cm} (2)

If we have some expansion for the $F(N,T)$,

$$F(N,T) = f_0(T)N + ..f_1 \ln N + f_2(T) + o(1)$$  \hspace{1cm} (3)

then jump should be due to term $f_2(T)$ as a rule, sometimes due to $f_1(T)$. It was well known, that for the case of 2d Ising model on a periodic rigid lattice jump equals $\ln 2$. For the 2d Potts models with $Z(Q)$ symmetry and 3d Ising model Monte-Carlo simulations give [3], that jump for all this models on the rigid periodic lattices equals $\ln Q$. For the Ising model on the dynamical lattices jump equals $g \ln 2$ [4], where $g$ is a genus number of surface. Perhaps such proportionality between jump and $g$ absens for a nonunitar models[5].

A question arose to find this jumps for a general case of phase transitions. It is also interesting to calculate $F(N,T)$ at phase transition point with accuracy inclusively $O(1)$. Such problem was well known in statistical mechanics [6-7], where correction terms are defined by conformal symmetry.

We are going to calculate free energy of REM with finite size corrections, to understand jumps of $F$. It is mean-field like simple system, which could be solved with finite size corrections. REM [8] is a system, with the independent gaussian distribution of $2^N$ energy levels. For a single ferromagnetic energy level we have a distribution [9]

$$P_1(E_1) = \frac{1}{\sqrt{\pi N}} \exp(-(E_1 + J_0 N)^2/N)$$  \hspace{1cm} (4)

and for other $2^N - 1$ levels

$$P(E) = \frac{1}{\sqrt{\pi N}} \exp(-E^2/N)$$  \hspace{1cm} (5)

For the free energy we have an expression [9,10]

$$< \ln Z >= \Gamma'(1) + \int_{-\infty}^{\infty} ud[f(u + u_f)f(u)^M]$$  \hspace{1cm} (6)
where \( u_f = J_0 NB, \lambda = B\sqrt{N}, M = 2^N - 1 \) and

\[
f(u) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(x) \exp[-ux + \lambda^2 x^2 / 4] dx
\]

(7)

In this integral the loop over passes point 0 from right. Function \( f(u) \) is monotonic, like step. With exponential accuracy it equals 1 below 0, then become 0 above it (with the same accuracy). We need in three asymptotic regimes

\[
f(u) \approx \frac{1}{\sqrt{\pi x}} \Gamma(2u^2 / \lambda^2) \exp(-u^2 / \lambda^2), \lambda \ll u
\]

\[
f(u) \approx \frac{1}{\sqrt{\pi}} \int_{u/\lambda}^{\infty} dx \exp(-x^2), \lambda \ll |u| \ll \lambda
\]

\[
f(u) \approx 1 - \frac{1}{\sqrt{\pi}} \Gamma(-2u^2 / \lambda^2) \exp(-u^2 / \lambda^2), -\lambda^2 / 2 < u < -\lambda
\]

\[
f(u) \approx 1 - \exp(u + \lambda^2 / 4), -\lambda^2 < u < -\lambda^2 / 2
\]

(8)

As the \([f(u + u_f)f(u)]^M\) likes step function, its derivative is like \( \delta \) function with a coordinate of wall at some \(-u_0\). The vicinity of that point gives main contribution to the integral in (6) (bulk value is \( u_0 \)). Ferromagnetic phase appears, when wall of function \( f(u + u_f) \) is left, than wall of \( f(u)^M \). For the derivative we derive

\[
f'(u) = -\exp[u + \lambda^2 / 4] f(u + \lambda^2 / 2)
\]

(9)

For our convenient presentation of \( f(u) \) it is easy to calculate value of \( f(u), f'(u) \) at 2 points:

\[
f(0) = 1/2, \quad f'(0) = -\frac{1}{\lambda \sqrt{\pi}}
\]

\[
f(-\lambda^2 / 2) = 1 - 1/2 \exp[-\lambda^2 / 4], \quad f'(-\lambda^2 / 2) = -1/2 \exp[-\lambda^2 / 4]
\]

(10)

As was found in [8], in paramagnetic phase \( O(1) \) corrections to free energy disappear, situation is the same in the ferromagnetic phase [9]. In the case of SG phase there are corrections \(-B^2 / 2B_c \ln N \) [8].

Let us consider boundary SG-PM. In this case we can neglect by ferromagnetic level. We have \( U_0 = \lambda^2 / 2 \). The boundary described by a line

\[
0 < J_0 < \sqrt{\ln 2}
\]

\[
B = \sqrt{\ln 2}
\]

(11)
Near the $u = -u_0$ we consider an expansion by degrees of $u + u_0$ for $f(u)$ like

$$f(u) = 1 - a \exp[-\lambda^2/4 + b(u + u_0) + ..]$$

(12)

We found, that

$$f(u) = 1 - \frac{1}{2} \exp[u + \lambda^2/4][1 + O(\frac{1}{\lambda})]$$

(13)

Let us take $M$ power of (13)

$$\exp[-\exp(u + u_0 - \ln 2)] = \exp(-\phi)$$

(14)

Its solution

$$u = -u_0 + \ln 2 + \ln \phi$$

(15)

So (6) goes to

$$< \ln Z > = \Gamma'(1) - \int_{-\infty}^{\infty} u e^{-\phi} d\phi = \Gamma'(1) - u_0 - [\ln 2 + \Gamma'(1)] = 2N \ln 2 - \ln 2$$

(16)

Let us consider boundary between FM and SG phases. It is a line

$$J_0 = \sqrt{\ln 2}$$

$$\infty > B > \sqrt{\ln 2}$$

(17)

We using the property, that if there is only 1-st level, $< \ln Z > = J_0 NB :$

$$\Gamma'(1) + \int_{-\infty}^{\infty} u d[f(u + u_f)] = J_0 NB$$

(18)

Using this equality, after simple transformations we derive

$$< \ln Z > = \Gamma'(1) + \int_{-\infty}^{\infty} u d[f(u + u_f)f(u)^M]$$

$$= \Gamma'(1) + \int_{-\infty}^{\infty} u df(u + u_f) - \int_{-\infty}^{\infty} u d[f(u + u_f)[1 - f(u)^M]$$

$$= J_0 NB + \int_{-\infty}^{\infty} f(u + u_f)[1 - f(u)^M]du$$

(19)

We have a product of 2 monotonic functions, decreasing (one-to left, another- to right) far the point $u = -u_f$. Let us introduce $F(u)$

$$F'(u) = f(u + u_f)$$

(20)
At \( \lambda \ll |u| \ll \lambda^2 \) we derive
\[
F(u-u_f) = \int_0^{u/\lambda} dx \left[ \frac{\lambda}{\sqrt{\pi}} \int_x^\infty \exp[-t^2] dt - \frac{C}{\sqrt{\pi}} \exp(-u^2/\lambda^2) \right]
\]
(21)

After transformations \( < \ln Z > \) goes to
\[
< \ln Z > = \Gamma'(1) + \int_{-\infty}^{\infty} ud[f(u+u_f)f(u)]
\]
\[
= \Gamma'(1) + \int_{-\infty}^{\infty} ud[f(u+u_f)] - \int_{-\infty}^{\infty} ud[f(u+u_f)][1 - f(u)]
\]
\[
= J_0 NB + \int_{-\infty}^{\infty} [f(u+u_f)][1 - f(u)] du
\]
(22)

Let \( \Psi(u) = 1 - f(u) \). Then
\[
< \ln Z > = J_0 NB + \int_{-\infty}^{\infty} F'(u) \Psi(u) du
\]
\[
= J_0 NB + F(\infty) \Psi(\infty) - F(-\infty) \Psi(-\infty) - \int_{-\infty}^{\infty} F(u) \Psi'(u) du
\]
\[
= J_0 NB + \int_{-\infty}^{\infty} [F(\infty) - F(u)] \Psi'(u) du
\]
\[
= J_0 NB + F(\infty) - F(-u_f) - F'(-u_f) \int_{-\infty}^{\infty} (u + u_f) \Psi'(u) du
\]
(23)

We truncated expansion in degrees \( u + u_f \), because \( \Psi'(u) \) is similar to \( \delta \) function near \( -u_f \). Then we derive
\[
< \ln Z > = J_0 NB + \int_{-\infty}^{\infty} F'(u) \Psi(u) du
\]
\[
= J_0 NB + \int_{-\infty}^{\infty} dx \int_x^\infty \exp[-t^2] dt - C/2 - \frac{1}{2} \int_{-\infty}^{\infty} (u + u_f) \Psi'(u) du
\]
(24)

Let us take last integral using representation \( d\Psi(u) = -e^{-\phi} d\phi \). We have
\[
e^{-\phi} = \frac{A}{\lambda} \exp[-U^2/\lambda^2 + N \ln 2]
\]
\[
\ln \phi = -U^2/\lambda^2 + N \ln 2 + \ln A - \ln \lambda
\]
(25)

Let us consider expansion \( u = u_f + u_1 \). For the \( u_1 \) we derive an equation
\[
u_1 = (\ln \phi - \ln A + \ln \lambda) B/B_c, B_c = 2\sqrt{\ln 2}
\]
(26)

Eventually we have an expression
\[
< \ln Z > = J_0 NB + \frac{B\sqrt{N}}{\sqrt{\pi}} \int_0^{\infty} dx \int_x^\infty \exp[-t^2] dt - C/2 - [\ln N + \Gamma'(1) + \ln \frac{B\sqrt{\pi}}{\Gamma(B)}] B/(2B_c)
\]
(27)
We see the $\sqrt{N}$ order corrections, which are very strange.

For the boundary PM-FM we have an equation

$$J_0 B = B^2 / 4 + \ln 2$$
$$J_0 > \sqrt{\ln 2}$$

(28)

Everything is similar to previous section, only

$$\Psi(u) = \exp[-\exp(u + u_0)], u + u_0 = \ln \phi$$

(29)

For free energy we derive

$$< \ln Z > = J_0 N B + \frac{B \sqrt{N}}{\sqrt{\pi}} \int_0^\infty dx \int_x^\infty \exp[-t^2] dt - \Gamma(1)/2$$

(30)

So $\sqrt{N}$ corrections stay. At multicritical point everything is the same, as in previous section, only

$$\Psi(u) = \exp\{-\frac{1}{2}[\exp(-\lambda^2/4 + u + u_f)]\}$$

(31)

So

$$u + u_f = \ln \phi + \ln 2$$
$$< \ln Z > = J_0 N B + \frac{B \sqrt{N}}{\sqrt{\pi}} \int_0^\infty dx \int_x^\infty \exp[-t^2] dt - C/2 - (\Gamma(1) + \ln 2)/2$$

(32)

We see, that on FM-PM line $< \ln Z >$ is more, than at tricritical point. Perhaps it is connected with the lack of unitarity in spin glass models (or unhomogeneity [10]). This work was supported by German ministry of Science and Technology Grant 211 - 5231.

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