(Supersymmetric) Kac-Moody Gauge Fields in 3+1 Dimensions

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Abstract

Lagrangians for gauge fields and matter fields can be constructed from the infinite dimensional Kac-Moody algebra and group. A continuum regularization is used to obtain such generic lagrangians, which contain new nonlinear and asymmetric interactions not present in gauge theories based on compact Lie groups. This technique is applied to deriving the Yang-Mills and Chern-Simons lagrangians for the Kac-Moody case. The extension of this method to $D = 4, N = (\frac{1}{2}, 0)$ supersymmetric Kac-Moody gauge fields is also made.

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1 Introduction

Gauge fields and matter fields having a local gauge invariance given by the Kac-Moody group have been written down in [1]. It was shown in [2] that $\hat{U}(1)$ Kac-Moody fermions exhibit a maximum limiting temperature and it was further shown in [3] that in one loop the $\hat{U}(1)$ Kac-Moody gauge field in $D=3+1$ is renormalizable as well as asymptotically free.

The lagrangian for Kac-Moody gauge fields was motivated in [1] using ideas from lattice gauge theory. The result was stated for the bosonic sector without any derivation, and the regularization required for the Kac-Moody gauge group was only addressed for the case of Kac-Moody fermions. In this paper we show how to derive the result directly from the continuum formulation and it will then become clear how the lagrangian for Kac-Moody gauge fields depends crucially on using a regularization.

A general feature of Kac-Moody gauge fields is that the central extension yields two sets of spacetime gauge fields, namely one set being in general a non-Abelian bulk gauge field $A_\mu$ defined on a $D = d + 1$-dimensional manifold and a $U(1)$ boundary gauge field $B_i$ defined on a $d$-dimensional manifold. The central extension couples these two fields in a gauge and supersymmetric invariant manner.

In Section 2 Kac-Moody gauge fields and field tensor, gauge transformations, the regularization as well as the generalization of the Wilson line to a Wilson cylinder are discussed. In Section 3 the Yang-Mills lagrangian for the Kac-Moody case is derived and in Section 4 the lagrangian for the Chern-Simons theory is obtained. In Section 5 the results are generalized to the case of supersymmetric Kac-Moody gauge fields and in Section 6 some conclusions are drawn.

2 Kac-Moody Gauge Fields

Let $Q_a$ be the generators of the Kac-Moody algebra based on underlying compact Lie group $G$ satisfying the commutation equations ($' = \partial/\partial \sigma$)

$$[Q_a(\sigma),Q_b(\sigma')] = iC_{abc}\delta(\sigma - \sigma')Q_c(\sigma) + ik\delta^{ab}\delta'(\sigma - \sigma')$$ (2.1)

where $\sigma, \sigma' \in S^1$, and $k = \text{integer}/2\pi$. For $k = 0$ we recover the loop group algebra.
Gauge transformations $\Phi(x)$ are given by finite elements of the Kac-Moody group $\hat{G}$. For $f \equiv \int_0^R d\sigma$ (that is $S^1$ has radius $R/2\pi$) we have

$$\Phi(x) = \exp\{i\Lambda(x) + i \int \phi^a(x, \sigma) Q_a(\sigma)\}$$

(2.2)

with $x \in M^3$ being an element of 3-dimensional Minkowski spacetime.

Define the Kac-Moody gauge field by

$$A_i(x) = eB_i(x) + g \int A^a_i(x, \sigma) Q_a(\sigma)$$

(2.3)

Note $B_i(x)$ is a $U(1)$ gauge field due to the central extension defined on $M^3$ and $A^a_i(x, \sigma)$ is a non-Abelian gauge field defined on spacetime $M^3 \times S^1$. We will refer to the fields on $M^3$ as boundary fields and fields on $M^3 \times S^1$ as bulk fields. On manifold $M^d \times S^1$ the coupling constant $g$ has mass dimension $(d-2)/2$ and $e$ has mass dimension of $(d-3)/2$. For simplicity, from now on we will set all couplings to unity. The indices $i, j, k$ etc run through 1,2,3 and indices $\mu, \nu$ etc run through 1,2,3,4 where $x_4 = \sigma$.

Gauge transformations are given by

$$A_i^\Phi(x) = \Phi(x) A_i(x) \Phi^\dagger(x) + i\Phi(x) \partial_i \Phi^\dagger(x)$$

(2.4)

To obtain the gauge transformation in terms of $B_i$ and $A^a_i$ note that [1] we have

$$\Phi(x) Q_a(\sigma) \Phi^\dagger(x) = \rho^T_{ab}(x, \sigma) Q_b(\sigma) + k e^T_{ab}(x, \sigma) \phi_b'(x, \sigma)$$

(2.5)

Let $T^a$ be the generators of $G$; define matrix $\phi(x, \sigma) = \exp\{\phi_a(x, \sigma) T^a\} \in G$; the vierbien on $G$ is given by $e_{ab}(x, \sigma) T^b = -i \phi^\dagger \partial_a \phi$ and the adjoint matrix by $\phi^\dagger T_a \phi = \rho_{ab} T^b$.

Using the functional differential realization of the Kac-Moody generators [1] we have

$$-i\partial_i \Phi(x) \Phi^\dagger(x) = \int \rho_{ab}(x, \sigma) \partial_a \phi_a(x, \sigma) Q_b(\sigma)$$

$$-k \int G_{ab}(x, \sigma) \partial_a \phi_a(x, \sigma) \phi_b'(x, \sigma)$$

(2.6)

where $G_{ab}$ is given in [1].

Hence from the equations given above we obtain for $A_i = A^a_i T^a$,
\[ A_i^\phi(x, \sigma) = \phi A_i \phi^\dagger(x, \sigma) \] (2.7)

which is the usual gauge transformation for compact Lie groups; the non-trivial transformation due to the central extension is given by [1]

\[ B_i^\Omega = B_i - \partial_i \Lambda + k \int e_{ab} \phi_b^' A_i^a - k \int G_{ab} \partial_i \phi_a \phi_b^' \] (2.8)

Define the Yang-Mills field tensor for the Kac-Moody case by

\[ F_{ij} = \partial_i A_j - \partial_j A_i + i[A_i, A_j] \] (2.9)

and which transforms covariantly under Kac-Moody gauge transformations. Introduce a new dynamical nonlinear scalar field \( \Omega \in \hat{G} \) defined by

\[ \Omega(x) = \exp\{i\alpha(x) + i \int \omega^a(x, \sigma) Q_a(\sigma)\} \] (2.10)

Define

\[ F_i^\Omega(x) = \Omega(x) F_{ij}(x) \Omega^\dagger(x) \] (2.11)

In components for, \( f_{ij} = \partial_i B_j - \partial_j B_i \), we have

\[ F_{ij}(x) = f_{ij} + k \int A_i^a A_j^a + k \int F_{ij}^a(x, \sigma) Q_a(\sigma) \] (2.12)

with \( F_{ij} = \partial_i A_j - \partial_j A_i + i[A_i, A_j] \).

We also have, from (2.11) and (2.12) that

\[ F_i^\Omega = \Gamma_{ij}(x) + \int F_i^a(x, \sigma) Q_b(\sigma) \rho^a_{ab}(\sigma) \] (2.13)

where

\[ \Gamma_{ij}(x) = f_{ij} + k \int A_i^a A_j^a + k \int F_i^a A_j^a \] (2.14)

It follows from (2.1) that for \( \omega = \exp\{i\omega^a(x, \sigma) T^a\} \) we have \( (\partial_4 = \partial/\partial\sigma) \)

\[ A_4(x, \sigma) = -i \omega^j \partial_4 \omega \] (2.15)

Note \( F_i^\Omega \) is gauge invariant since a gauge transformation \( \Phi \) can be undone by a change of variables.
\[ \Omega(x) \rightarrow \Omega(x)\Phi^\dagger(x) \] (2.16)

Then (2.1) implies from above (for two cocycle \( \omega_2 \)) that

\[ \alpha(x) \rightarrow \alpha(x) - \beta(x) + \omega_2(\omega, \phi) \] (2.17)

\[ \omega(x, \sigma) \rightarrow \omega(x, \sigma)\phi^\dagger(x, \sigma) \] (2.18)

Hence from (2.18) we have under gauge transformations

\[ A_4(x, \sigma) \rightarrow \phi A_4\phi^\dagger + i\phi \partial_4\phi^\dagger \] (2.19)

We see that the only way that \( \Omega \) couples to the other fields is through the combination given by \( A_4 \). In fact we see from the gauge transformation given above that \( A_\mu = (A_i, A_4) \) forms a 4-dimensional vector field \( \in M^3 \times S^1 \).

### 3 Regularization; Wilson Tube

One would naively expect that the trace of Wilson loops should constitute all the gauge invariant quantities. This does not hold for the Kac-Moody gauge group since the trace gives a divergent result and is due to the fact that the trace of any product of \( Q^a(\sigma) \)'s is infinity. It is essential that the trace be regulated; to do so introduce the operator \( L \) which is a generalization of the Virasoro operator and is defined by

\[ L = \int Q^a(\sigma)f(\sigma - \sigma')Q^a(\sigma') \] (3.1)

where \( f > 0 \) and choose normalization \( tre^{-\beta L} = 1 \). The partition function \( tre^{-\beta L} \) yields a two-dimensional chiral version of the WZW-model and has been studied in [6].

For \( \beta > 0 \) we have

\[ tre^{-\beta L}Q^{a_1}(\sigma_1)Q^{a_2}(\sigma_2)\ldots Q^{a_n}(\sigma_n) < \infty \] (3.2)

The trace is performed over an irrep of the Kac-Moody algebra. The details of the regulator \( L \) are unimportant in the continuum theory; however this is not so for lattice Kac-Moody gauge fields. For the purpose of deriving the continuum lagrangian we will only need that
\[ tre^{-\beta L}Q^a(\sigma) = 0 \] (3.3)

and

\[ \lim_{\beta \to 0} tre^{-\beta L}Q^a(\sigma)Q^b(\sigma') = \frac{1}{\beta} \delta^{ab} H_{\sigma - \sigma'} \] (3.4)

The regulator does not (cannot) commute with gauge transformations. Hence define the gauge invariant regulated Wilson tube starting at \( x \) and ending at \( y \in M^3 \) by

\[ W = tre^{-\beta L}\Omega(x)Pe^i \int_x^y dz^i A_i\Omega^\dagger(y) \] (3.5)

There is another class of gauge invariant operators given by

\[ V_{i..j} = tre^{-\beta L}\Omega(x)Pe^i \int_x^y dz^i D_i..D_j\Omega^\dagger(y) \] (3.6)

where

\[ D_i = -i\partial_i + A_i(y) \] (3.7)

The operator \( V \) is given by insertions at \( y \). For \( x = y \) we have a marked torus on which both \( W \) and \( V \) are defined.

We analyze the \( \hat{U}(1) \) case. In terms of Fourier modes of \( Q(\sigma) \) let

\[ L = \sum_{n=1}^{\infty} f_n Q_{-n}Q_n \] (3.8)

To perform the trace we use the irrep of \( \hat{U}(1) \) with \( Q_0|vac >= 0 \). The correlator for arbitrary \( \beta \) is given by \[ \text{[6]} \]

\[ tre^{-\beta L}Q_{-n}Q_n = k|n| \frac{e^{-\beta |n| f_n}}{1 - e^{-\beta |n| f_n}} \] (3.9)

\[ \sim \lim_{\beta \to 0} \frac{k}{f_n \beta} \] (3.10)

Hence we have from (3.4) and above

\[ f_n = \frac{k}{H_n} \] (3.11)
Consider a circle lying in the $(1, 2)$-plane with radius $\bar{R}$ and marked at $x_0$. $W$ is defined on the marked Wilson torus and is given by

$$W = t r e^{-\beta L} \Omega(x_0) P e^{i \oint dz^i A_i} \Omega^\dagger(x_0)$$  \hspace{1cm} (3.12)$$

$$= e^{i \Delta t r e^{-\beta L} e^{i \int M(\sigma)Q(\sigma)}}$$  \hspace{1cm} (3.13)$$

$$= e^{i \Delta e^F}$$  \hspace{1cm} (3.14)$$

where the magnetic flux at point $\sigma$ is given by

$$M(\sigma) = \oint dz^i A_i(z, \sigma)$$  \hspace{1cm} (3.15)$$

The phase $\Delta$ depends on $x_0$, and in cylindrical coordinates $A_i = (A_r, A_\theta, A_z)$ is given by

$$\Delta = \oint dz^i B_i + k \int M(\sigma)\omega'(x_0, \sigma)$$

$$+ ik(\frac{\bar{R}}{2\pi})^2 \int_0^\infty \int_0^{2\pi} d\sigma \int_0^{2\pi} d\nu A_\theta(\nu, \sigma) \int_0^{2\pi} d\nu A'_\theta(\bar{\nu}, \sigma)$$  \hspace{1cm} (3.16)$$

and $[6]$

$$F = -\frac{k}{2} \sum_{n=1}^{\infty} n \tanh(\frac{\beta n f_n}{2}) |M_n|^2$$  \hspace{1cm} (3.17)$$

$$\sim \lim_{\beta \to 0} -\frac{k}{2\beta} \sum_n \frac{1}{f_n} |M_n|^2$$  \hspace{1cm} (3.18)$$

Note the integration over the gauge field still has to be performed. Unlike the usual Abelian field for which $F$ is linear in the gauge field, for the $\hat{U}(1)$ case the dependence is quadratic; this probably means that $W$ may not be the right order parameter for indicating the onset of confinement.

4 Yang-Mills Lagrangian

As a warm-up for the supersymmetric case we derive the lagrangian for the pure gauge field. Define the lagrangian by
\[ \mathcal{L}_{KM} = \frac{1}{4e^2} \sum_{ij} t r e^{-\beta L} \Omega(x) F_{ij}^2(x) \Omega^1(x) + \mathcal{L}' \] (4.1)

\[ = \frac{1}{4e^2} \sum_{ij} t r e^{-\beta L} (\Gamma_{ij} + \int F_{ij}^{ab} \rho_{ab}^T Q_b)^2 + \mathcal{L}' \] (4.2)

\[ = -\frac{1}{4e^2} \sum_{ij} \{ \Gamma_{ij}^2 + \frac{1}{\beta} \int F_{ij}^a \rho_{ac}^T (x, \sigma) H_{\sigma - \sigma'} F_{ij}^b \rho_{bc}^T (x, \sigma') \} + \mathcal{L}' \] (4.3)

where equation (3.4) has been used to perform the trace. The piece \( \mathcal{L}' \) of the lagrangian is required for giving a kinetic term to \( A_i \) in the \( \sigma \) direction as well as for the dynamics of the \( \Omega \) field.

From eqn (2.15) we have

\[ \omega(x, \sigma) = Pe^\int_0^\sigma A_4^a T^a \] (4.4)

and hence for \( A_4^{\alpha\beta} = A_4^a C^a_{\alpha\beta} \) we have

\[ \rho(x, \sigma) = Pe^{-\int_0^\sigma A_4} \] (4.5)

Defining \( g^2 = e^2 \beta \) we have

\[ \mathcal{L}_{KM} = -\frac{1}{4e^2} \sum_{ij} \Gamma_{ij}^2 - \frac{1}{4g^2} \sum_{ij} \int F_{ij}(x, \sigma) Pe^{-\int_\sigma^{\sigma'} A_4 H_{\sigma - \sigma'} F_{ij}(x, \sigma')} + \mathcal{L}' \] (4.6)

We now derive \( \mathcal{L}' \). We have the natural gauge invariant interaction (the coefficient is fixed by the requirement that one recover full 3+1 symmetry for \( k = 0 \))

\[ \mathcal{L}' = -\frac{1}{2g^2} \sum_i \int F_{i4}(x, \sigma) Pe^{-\int_\sigma^{\sigma'} A_4 H_{\sigma - \sigma'} F_{i4}(x, \sigma')} \] (4.7)

The lagrangian \( \mathcal{L}_{KM} \) was derived in [2] using lattice theory arguments and without the explicit use of a regulator. The lagrangian (even for the \( \hat{U}(1) \) case) is asymmetric, nonlocal and nonlinear. The function \( H_\sigma \) at this point is arbitrary; we will fix it later by demanding renormalizability.

Note if we consider the special case of \( H_\sigma = \delta(\sigma) \) (which is one-loop renormalizable [3]) we have
\[ L_{KM}(x) = -\frac{1}{4e^2} \Gamma^2_{ij}(x) - \frac{1}{4g^2} \int d\sigma F^2_{\mu\nu}(x, \sigma) \]  

(4.8)  

\[ L_{central} + L_{YM} \]  

(4.9)

Note all the derivations so far hold in arbitrary dimensions; it is only when supersymmetry is required that the choice of dimensions becomes restricted. The theory is invariant under the \( d = 3 \) Lorentz group \( SL(2, R) \); for \( k = 0 \) the bulk fields yield a Yang-Mills theory with the full \( SL(2, C) \) symmetry of \( D = 4 \). This pattern of enhanced symmetry will also be seen to hold for the supersymmetric case.

We now rewrite \( L' \) in a 3-dimensional form which has a supersymmetric generalization. For this we need the (suitably regularized) Virasoro operator

\[ L_0 = \frac{1}{C_{Adj} + 2k} : Q^2(\sigma) : \]  

(4.10)

with

\[ [L_0, Q_a(\sigma)] = iQ'_a(\sigma) \]  

(4.11)

We then have the manifestly gauge invariant expression

\[ L'(x) = \frac{1}{2g^2} \sum_i tr e^{-\beta L} [A_i^\Omega(x), L_0]^2 \]  

(4.12)

Note \( L' \) is decoupled from the central extension due to the commutator. We hence have

\[ L_{KM} = -\frac{1}{e^2} tr e^{-\beta L} \{ \Omega, F^2_{ij} \Omega^i - 2[A_i^\Omega, L_0]^2 \} \]  

(4.13)

We use the background field method to quantize the theory. Consider the break-up

\[ A_i = C_i + a_i \]  

(4.14)

where \( C_i \) is a classical field and \( a_i \) the quantum field. We use a generalization of the 't Hooft gauge given by

\[ \partial_i a_i + i[C_i, a_i] = 0 \]  

(4.15)
This gauge is invariant under background Kac-Moody gauge transformations on $C$. In components, for $a_i = b_i + \int a^\alpha Q^\alpha$ we have from the equation above [3]

\[
q^\alpha(x, \sigma) \equiv \partial_i a^\alpha_i - C_{\alpha\beta\gamma} C_i^\beta a_i^\gamma = 0 \tag{4.16}
\]

\[
t(x) \equiv \partial_i b_i + k \int C_i^\alpha a_i^\alpha = 0 \tag{4.17}
\]

The quantum field theory is given by

\[
Z = \int DaDbe^{S[C+a]+S_{FP}} \prod_{x,\sigma,\alpha} \delta(q^\alpha(x, \sigma))\delta(t(x)) \tag{4.18}
\]

where $S_{FP}$ is the Faddeev-Popov ghosts.

It has been shown in [3] that for $\hat{U}(1)$ $S_{FP} = 0$. A one-loop calculation for the $\hat{U}(1)$ case [3] shows that the theory is renormalizable in D=3+1 if we take

\[
H_\sigma = \sum_{n=-\infty}^{\infty} e^{in\sigma}(1 + a|n|) \tag{4.19}
\]

The coupling constant in the theory is $\lambda = \frac{kq}{e}$ and which has a mass dimension of $(d-2)/2$; to make this dimensionless in 3+1 we define $\bar{\lambda} = \frac{\lambda}{\sqrt{R}}$ where $R$ is the radius of $S^1$. The one loop beta function for $\hat{U}(1)$ is given by [3]

\[
\beta = -\frac{1}{4a^2}\bar{\lambda}^3 \tag{4.20}
\]

We see that the $\hat{U}(1)$ theory is asymptotically free!

The limit of $a \to 0$ is not uniform. For $a = 0$ we have $H_\sigma = \delta(\sigma)$; the theory is still renormalizable and $\beta = 0$.

If one takes limit $R \to \infty$, we see that (for $d > 2$) $\bar{\lambda} \to 0$; in effect this sets $k = 0$ and the Kac-Moody algebra reduces to the loop group algebra.

5 Kac-Moody Chern-Simons Lagrangian

We consider the usual Chern-Simons lagrangian to be the dimensional reduction of the theory defined on $M^3 \times S^1$ to $M^3$. Define the generalization
of the Chern-Simons lagrangian with Kac-Moody gauge symmetry by

\[ \mathcal{L}_{CS} = i\lambda \epsilon^{ijk} tr e^{-\beta L} (F_{ij} A_k^\Omega - i \frac{2}{3} A_i^\Omega A_j^\Omega A_k^\Omega) \]

(5.1)

If one takes the bulk fields to be \( \sigma \)-independent \( \mathcal{L} \) reduces to a \( U(N) \) Chern-Simons theory in \( d = 3 \).

For the \( U(N) \) case it is known that for the theory to be gauge invariant, the coupling constant \( \lambda \) is an integer (upto a constant). For the Kac-Moody case there is no such restriction on \( \lambda \) since gauge invariance is obtained explicitly by a coupling to the regulator field \( \Omega \); however, if one requires that the dimensionally reduced theory be the usual Chern-Simons case \( \lambda \) has to be similarly restricted.

We work out the case of \( \hat{U}(1) \) using a simpler lagrangian consisting of only the first piece of \( \mathcal{L}_{CS} \) given in (5.1) since this is gauge invariant and reduces to the usual case. We then have

\[ \mathcal{L}_{CS} = i\lambda \epsilon^{ijk} tr e^{-\beta L} \Omega F_{ij}(A_k + i \partial_k)\Omega^\dagger + \mathcal{L}' \]

(5.2)

Note that unlike the case of (supersymmetric) Yang-Mills, the central extension (phase) of the \( \Omega \)-field couples to the theory due to the derivative acting on it; we need to have a kinetic term in the \( \sigma \) direction and \( \mathcal{L}' \) given in eqn (5.12) is added. Working out the trace we obtain, for \( \Omega = e^{i\alpha(x)} + \int \omega(x,\sigma)Q(\sigma) \) the following

\[ \mathcal{L}_{CS}(x) = i\lambda \epsilon^{ijk} \left\{ \int F_{ij}(x,\sigma)H_{\sigma-\sigma'} A_k(x,\sigma') + \Gamma_{ij}(x)(B_k - \partial_k \alpha + k \int \omega' A_k - \frac{k}{2} \int \omega' \partial_k \omega)(x) \right\} + \mathcal{L}'(x) \]

(5.3)

with the kinetic term given by

\[ \mathcal{L}'(x) = -\frac{1}{2g^2} \int F_{ij}(x,\sigma)H_{\sigma-\sigma'} F_{ij}(x,\sigma') \]

(5.4)

Since the kinetic piece \( \mathcal{L}' \) doesn’t couple to the central extension there is no kinetic term for the \( \alpha \) variable and it appears as a Lagrange multiplier; integration over it yields the constraint

\[ \epsilon^{ijk} \partial_i \Gamma_{jk} = 0 \]

(5.5)
If we had used the lagrangian given by (5.1) we would have obtained
\[ L_{CS} \rightarrow L_{CS} + \frac{i}{3} \lambda e^{ijk}(f^\Omega_{ij} - \Gamma_{ij})B_k^\Omega \]

Both \( \hat{U}(1) \) lagrangians given in equations (5.2) and (5.6) yield the same Chern-Simons lagrangian on dimensional reduction. Note that the lagrangian used for the \( \hat{U}(1) \) case in (5.2) can also be used for the non-Abelian case as it is fully gauge-invariant; of course this theory may not have much to do with non-Abelian Chern-Simons as we have an extra \( \Omega \)-field in order to achieve gauge invariance.

We have obtained a generalization of the \( U(1) \) Chern-Simons lagrangian to \( D = 3 + 1 \)-dimensions with an extra parameter \( k \). We need to check whether this lagrangian is renormalizable, and whether the new nonlinearities yield any new physics such as new classical solutions. A supersymmetric generalization of \( L_{CS} \) can also be made based on the results of the next section.

6 Supersymmetric Kac-Moody Gauge Fields

We obtain the action for the supersymmetric Kac-Moody gauge fields in \( d \)-dimensions. Note that it is necessary that supersymmetric Yang-Mills exists in both \( d \) and \( D = d + 1 \) dimensions for the theory to be defined without the introduction of extra fields. The reason being, as we have seen above, that the theory consists of a bulk field \( A_\mu \) on a \( D \)-dimensional manifold \( M_d \times S^1 \) and a boundary field \( B_i \) in \( d \) dimensions. The supersymmetry of the lagrangian will be limited to the supersymmetry of the \( d \) dimensional theory but we expect in the limit of \( k = 0 \) that we will have (enhanced) supersymmetry \( D = 4, N = 1 \) for the bulk fields.

Supersymmetric Yang-Mills exists only in 2, 3, 4, 6 and 10 dimensions\([8,9]\); hence we can at most expect to have supersymmetric Kac-Moody gauge fields in \( d = 2 \) and 3 dimensions.

We start in \( d = 3 \) with two supercharges \( Q_\alpha \), that is with \( N = 1, d = 3 \)\([9,11,12]\). Superspace in \( d = 3 \) is the extension of space \( x \) to superspace consisting of \( (x, \theta) \) where \( \theta_\alpha \) is an anticommuting two component Majorana spinor. Relevant formulae of supersymmetry in \( d = 3 \) and 4 are briefly reviewed in the Appendix. We choose pure imaginary Majorana \( \gamma^i \) matrices.
with \( \{ \gamma^i, \gamma^j \} = \eta^{ij} \) where the metric has signature \((+,-,-)\); \( \bar{\theta} = \theta^T \gamma^0 \).

The \( d = 3 \) real scalar multiplet superfield is dimensionless and unconstrained, and is given by (all superfields will be denoted by boldface).

\[
\mathbf{s}(x, \theta) = A(x) + \bar{\theta} \psi(x) + \frac{1}{2} \bar{\theta} \theta F(x) \quad (6.1)
\]

Note \( A \) is a real scalar field, \( \psi \) is a Majorana spinor and \( F \) a real auxiliary field. For the Kac-Moody case introduce an (infinite) collection of \( d = 3 \) real scalar superfields \( \mathbf{s}^a(x, \sigma, \theta) \) labeled by \( \sigma \) and define

\[
\mathbf{S}(x, \theta) = \mathbf{s}(x, \theta) + \int \mathbf{s}^a(x, \sigma, \theta) Q_a(\sigma) \quad (6.2)
\]

The Kac-Moody vector gauge field is described by a real spinor superfield \( \mathbf{V}_\alpha \); in the Wess-Zumino gauge

\[
\mathbf{V}_\alpha(x, \theta) = i \mathcal{A}_i(\gamma^i \theta)_\alpha + \frac{1}{2} \bar{\theta} \theta \Psi_\alpha \quad (6.3)
\]

where Kac-Moody fermions are given by

\[
\Psi_\alpha(x) = \chi_\alpha(x) + \int \psi^a_\alpha(x, \sigma) Q_a(\sigma) \quad (6.4)
\]

The gauge covariant field strength is given, in the Wess-Zumino gauge, by the real spinor superfield

\[
\mathbf{W}_\alpha(x, \theta) = \Psi_\alpha(x) + i \mathcal{F}_i(\gamma^i \theta)_\alpha - \frac{i}{2} \bar{\theta} \theta \mathcal{D}_i \gamma^i \Psi_\alpha \quad (6.5)
\]

where \( \mathcal{F}_i = \frac{1}{2} \epsilon^{ijk} \mathcal{F}_{jk} \) and

\[
\mathcal{D}_i \Psi = \partial_i \Psi + i [\mathcal{A}_i, \Psi] \quad (6.6)
\]

Note that unlike the case in \( D = 4 \) both \( \mathbf{V}_\alpha \) and \( \mathbf{W}_\alpha \) contain only physical degrees of freedom. Supergauge transformations in \( d=3 \) are given by \( e^{i \mathbf{s}(x, \theta) T^a} \) and Kac-Moody supergauge transformations are given by \( e^{i \mathbf{S}(x, \theta)} \).

Supersymmetry transformations in \( d = 3 \) are specified by a (real) Majorana spinor \( \epsilon \) and given by

\[
\delta \mathcal{A}_i = i \bar{\epsilon} \gamma_i \Psi \quad (6.7)
\]

\[
\delta \Psi = i \mathcal{F}_i \gamma^i \epsilon \quad (6.8)
\]
The bulk fields $\psi, A_i$ transform as components of a 3-dimensional field for each $\sigma$ whereas the $U(1)$ boundary field $\chi$ transforms nontrivially (see Appendix).

Let the regulator superfield be defined by

$$\Omega(x, \theta) = e^{i\alpha(x, \theta) + i \int \omega^a(x, \sigma, \theta) Q_a(\sigma)}$$  

(6.9)

The regulator superfield is equivalent to local supergauge transformations in $D = 3 + 1$, and is given by Lie group-valued local superfield as

$$g(x, \sigma, \theta) = e^{i\omega^a(x, \sigma, \theta) T_a}$$  

(6.10)

Note that both $\Omega$ and $g$ are well defined as elements of the Kac-Moody and compact Lie group respectively since in $d = 3$ the scalar superfield is real; in $D = 4$ supergauge transformations in full generality require the complexification of the group.

Let Lie algebra-valued superfield be defined by

$$s(x, \sigma, \theta) = -ig^\dagger \frac{\partial}{\partial \sigma} g$$  

(6.11)

$$= A_4(x, \sigma) + \bar{\theta} \xi(x, \sigma) + \frac{1}{2} \bar{\theta} \theta D(x, \sigma)$$  

(6.12)

Superfield $s^a(x, \sigma, \theta)$ can be considered to be an infinite collection of $d = 3$ real superfields or equivalently to be a single $D = 4$ real chiral superfield.

Note under ordinary (not super) gauge transformations in $D = 4$, we have $g \rightarrow g \phi^\dagger(x, \sigma)$ and which in turn yields

$$A_4 \rightarrow \phi A_4 \phi^\dagger + i \phi \partial_4 \phi^\dagger$$  

(6.13)

$$\xi \rightarrow \phi \xi \phi^\dagger$$  

(6.14)

$$D \rightarrow \phi D \phi^\dagger$$  

(6.15)

We see that superfield $s(x, \sigma, \theta)$ has a bosonic component which transforms like the $A_4$ component of a $D = 3 + 1$ gauge field and the fermionic and scalar fields transform like matter fields. Hence we can in principle have gauge invariant couplings of these fields with the other bulk fields in the system.
Since $s$ is a real scalar superfield under $d = 3$ supersymmetry transformations we have [11],[12]

$$\begin{align*}
\delta A_4 &= i\bar{\epsilon}\xi \\
\delta D &= \bar{\epsilon}\gamma^i\xi \\
\delta\xi_\alpha &= i\partial_i A_4(\gamma^i\epsilon)_\alpha + D\epsilon_\alpha
\end{align*}$$

(6.16)

(6.17)

(6.18)

We also have from eqns (6.7) and (6.8)

$$\begin{align*}
\delta A_i &= i\bar{\epsilon}\gamma^i\psi \\
\delta\psi_\alpha &= iF_i(\gamma^i\epsilon)_\alpha
\end{align*}$$

(6.19)

(6.20)

We check that fields $(A_\mu, \psi, \xi, D)$ with $d = 3, N = 1$ supersymmetry transformation properties given above can be combined to yield the supersymmetry transformation of $D = 4, N = (\frac{1}{2}, 0)$. It can be shown that for four component Majorana spinor given by

$$\lambda = \frac{1}{2} \left( \begin{array}{c} \psi + i\xi \\ \psi - i\xi \end{array} \right)$$

(6.21)

and with $N = (\frac{1}{2}, 0)$ supersymmetry (real) parameter given by

$$\alpha = \left( \begin{array}{c} \epsilon \\ \epsilon \end{array} \right)$$

(6.22)

the transformations (6.16) to (6.20) for the $d = 3$ case combine to yield the $D = 4, N = (\frac{1}{2}, 0)$ supersymmetry transformations. As is expected, in the WZ-gauge all the derivatives are covariantized [13], and the supersymmetry transformation [13],[14] is given by (see Appendix)

$$\begin{align*}
\delta A_\nu &= \bar{\alpha}\Gamma_\nu\lambda \\
\delta D &= \bar{\alpha}\Gamma_5\Gamma_\nu D^\nu\lambda \\
\delta\bar{\lambda} &= -\bar{\alpha}\Sigma_{\mu\nu}F^{\mu\nu} + i\bar{\alpha}\Gamma_5 D
\end{align*}$$

(6.23)

(6.24)

(6.25)

We see from above that the $D = 4, N = (\frac{1}{2}, 0)$ superalgebra is a tensor product of the expected $d = 3, N = 1$ subalgebra times the subalgebra
consisting of the supersymmetry transformation of the \(d = 3, N = 1\) scalar superfield \((A_4, \xi, D)\).

The full \(D = 4, N = 1 = (\frac{1}{2}, \frac{3}{2})\) superalgebra with four supercharges \([13]\) has an arbitrary Majorana fermion for \(\alpha\) given by a two-component complex spinor \(\zeta_\alpha\) such that

\[
\alpha = \begin{pmatrix} \zeta \\ \zeta^* \end{pmatrix}
\]  

(6.26)

Similar to the pure bosonic case, we expect that for the supersymmetric case the bulk fields will yield a \(D = 4, N = 1\) supersymmetric Yang-Mills lagrangian and the boundary fields will couple to the bulk fields, and will yield a \(d = 3, N = 1\) supersymmetric lagrangian. The boundary fields have the required degrees of freedom; the bulk fields \((\psi, A_i)\) together with the regulator (bulk) fields \((\xi, A_4, D)\) have the exact field content required to make up a \(D = 3 + 1, N = 1\) hermitian vector multiplet.

In analogy with the bosonic case we define the \(d = 3\) supersymmetric lagrangian by

\[
\mathcal{L}_{SKM}(x) = -\frac{1}{e^2} \int d^2 \theta \text{re}^{-\beta_L} \{ \Omega \tilde{W} \tilde{W}^\dagger \Omega - 2[\tilde{V}^\Omega, L_0][V^\Omega, L_0] \}
\]  

(6.27)

Note by construction the lagrangian is supergauge invariant.

To compute \(\mathcal{L}_{SKM}\) note from the Kac-Moody algebra we have

\[
\Omega W^a \Omega^i = U_\alpha(x, \theta) + \int W^a_{\rho \sigma}(\omega) Q_b
\]  

(6.28)

Note \(U_\alpha(x, \theta)\) is the supersymmetric generalization of the bosonic \(\Gamma_{ij}\)-term and is given by

\[
U_\alpha(x, \theta) = w_\alpha(x, \theta) + k C_\alpha(x, \theta)
\]  

(6.29)

where the central terms are given by

\[
C_\alpha = -\frac{i}{2} \epsilon^{ijk} \int A_j^{a} A_k^{a} (\gamma^i \theta)_\alpha - \frac{1}{2} \theta \theta \int A_i^{a} (\gamma^i \psi^a)_\alpha + \int W^a_{\alpha} s^a
\]  

(6.30)

and the \(d = 3\) spinor superfield is given by
\[ w_\alpha = \chi_\alpha(x) + i\epsilon_{ijk}(\gamma^i\theta)_\alpha \partial^j B^k(x) - \frac{i}{2} \bar{\theta} \partial_i \gamma^i \chi_\alpha(x) \quad (6.31) \]

To simplify the expression for the lagrangian we use a regularization which is local, that is

\[ \lim_{\beta \to 0} tr e^{-\beta L} Q^a(\sigma) Q^b(\sigma') = \frac{1}{\beta} \delta^{ab} \delta(\sigma - \sigma') \quad (6.32) \]

We hence obtain the action

\[ S = -\frac{1}{e^2} \int d^3x d^2 \theta U(x, \theta) \bar{U}(x, \theta) \]

\[ - \frac{1}{4g^2} \int d^3x d\sigma \{ F^2_{\mu\nu} + i \bar{\lambda} \Gamma^\mu D_\mu \lambda + \frac{1}{2} D^2 \} \quad (6.33) \]

\[ = S_{\text{central}} + S_{\text{SYM}} \]

The full supersymmetric theory consists of a \( D = 4, N = 1 \) supersymmetric Yang-Mills theory given by \( S_{\text{SYM}} \) coupled to a \( U(1) \) boundary field with \( d = 3, N = 1 \) supersymmetry. The complete action \( S \) has \( D = 4, N = (\frac{1}{2}, 0) \) supersymmetry.

For the case of \( k = 0 \) we obtain a higher \( D = 4, N = 1 \) supersymmetry for the Yang-Mills bulk fields whereas the (decoupled) boundary fields continue to have \( d = 3, N = 1 \) supersymmetry.

A consequence of our derivation, in particular the use of the regulator field \( \Omega \), is that for the \( D = 4, N = 1 \) Super Yang-Mills fields we can choose the \textit{superaxial} gauge given by \( s^a(x, \sigma, \theta) = 0 \), or in components

\[ A_4 = \xi_\alpha = D = 0 \quad (6.35) \]

### 7 Conclusions

We have generalized of the group of gauge symmetry to that of infinite dimensional groups. The key feature of regularizing the trace yields many new features in the lagrangian including new nonlinear, nonlocal and asymmetric interactions. Kac-Moody (super)gauge transformations act nontrivially
on the $d$ dimensional boundary fields, whereas on the $D = d + 1$ bulk fields these are simply the usual $D$ dimensional local (super)gauge transformations.

The cases of Yang-Mills, Chern-Simons and supersymmetric gauge fields all have new features.

The supersymmetric generalization for the loop group case of $k = 0$ is interesting in its own right since we can set the radius of the extra dimension to infinity. This then provides a way of obtaining a $D = 4, N = 1$ supersymmetric Yang-Mills theory starting from a $d = 3$ dimensional supersymmetric theory.

In the Kaluza-Klein approach one starts in $D = d + 1$ dimensions and reduces the theory to $d$ dimensions by compactification. The loop group approach is the inverse of the Kaluza-Klein approach since we started with a $d$ dimensional theory and 'lifted' it to $D = d + 1$. It was assumed that the system has the infinite dimensional loop group as its gauge symmetry; the continuous space $S^1$ underlying the loop group was then interpreted as an (extra) space dimension; the extra degrees of freedom in a (super) gauge transformation provided the extra fields required for increasing the dimension from $d$ to $D = d + 1$.

Loop group supergauge transformations in $d$ dimensions become the usual local supergauge transformations in $D = d + 1$. This programme was carried out in arbitrary dimensions for the bosonic case. However for the supersymmetric case one increased dimension from 3 to 3+1 (without adding extra fields by hand) since both of these dimensions allow for the existence of supersymmetric gauge fields.

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Appendix
A  Supersymmetry in d=3

Superspace in $d = 3$ is given by $(x, \theta)$, where $\theta_\alpha$ is a real two-component Majoran spinor; we have pure imaginary Majorana gamma-matrices given by

$$
\begin{align*}
\gamma^0 &\equiv \gamma^1 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
\gamma^2 &\equiv \gamma^3 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\gamma^3 &\equiv \gamma^1 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
$$

(A.1)

The supercharge $Q_\alpha$ and superderivative $D_\alpha$ are defined by

$$
Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + i(\gamma^i \theta)_{\alpha} \partial_i \\
D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i(\gamma^i \theta)_{\alpha} \partial_i
$$

(A.2)

(A.3)

Supersymmetry transformations are given by real two-component spinor $\epsilon_\alpha$; for a real scalar superfield we have

$$
\delta s = \bar{\epsilon} Q s
$$

(A.4)

with $\bar{\epsilon} = \epsilon^T \gamma^0$. Eqns (6.16)-(6.18) are obtained from the transformation above.

Kac-Moody superfield transformations given in (6.7) and (6.8) yield (6.19) and (6.20) for the bulk fields; for the boundary fields this yields

$$
\begin{align*}
\delta \chi_\alpha &= \frac{i}{2} \epsilon^{ijk} (f_{ij} + k \int A'_j A_j)(\gamma_k \epsilon)_\alpha \\
\delta B_i &= \bar{\epsilon} \gamma_i \chi
\end{align*}
$$

(A.5)

(A.6)

The pure imaginary Majorana representation of the $D = 4$ gamma matrices is given by [13]

$$
\Gamma_i = \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \Gamma_4 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\Gamma_5
$$

(A.7)

For $\Sigma_{\mu\nu} = \frac{1}{2}[\Gamma_{\mu}, \Gamma_\nu]$ we have
\[
\Sigma_{ij} = \frac{1}{2} \left( \begin{array}{cc} 0 & [\gamma_i, \gamma_j] \\ [\gamma_i, \gamma_j] & 0 \end{array} \right)
\]
(A.8)

\[
\Sigma_{i0} = \left( \begin{array}{cc} 0 & -\gamma_i \\ \gamma_i & 0 \end{array} \right)
\]
(A.9)

The charge conjugation matrix \( C \) is given by

\[
CT_{\mu}C^{-1} = -\Gamma^T_{\mu}
\]
(A.10)

\[
C^T = -C
\]
(A.11)

For the Majorana realization we have

\[
C = \left( \begin{array}{cc} \gamma_0 & 0 \\ 0 & \gamma_0 \end{array} \right)
\]
(A.12)

Let \( \psi_\alpha \) and \( \xi_\alpha \) be two-component real spinors; \( D = 4 \) Majorana fermion \( \lambda \) is given by

\[
\lambda = \frac{1}{2} \left( \begin{array}{c} \psi + i\xi \\ \psi - i\xi \end{array} \right)
\]
(A.13)

We then have \( (\Gamma_0 \equiv \Gamma_1) \)

\[
\bar{\lambda} = \lambda^\dagger \Gamma_0 = \lambda^T C
\]
(A.14)

\[
\bar{\lambda} = (\bar{\psi} + i\bar{\xi}, \bar{\psi} - i\bar{\xi})
\]
(A.15)

The spinor field-tensor \( W_\alpha \) is given by

\[
W_\alpha = \frac{1}{2} D^\beta D_\beta V_\alpha + \frac{i}{2} [V^\beta, D_\beta V_\alpha] + \frac{1}{6} [V^\beta, \{V_\beta, V_\alpha\}]
\]
(A.16)

Super gauge transformations are given by

\[
g = e^{i{s^\alpha(x,\theta) T^\alpha}}
\]
(A.17)

and
$V_\alpha \rightarrow g(V_\alpha + iD_\alpha)g^\dagger$  \hspace{1cm} (A.18)

$W_\alpha \rightarrow gW_\alpha g^\dagger$  \hspace{1cm} (A.19)

Let $W_\alpha$ be given in an arbitrary gauge; then we have

$$W_\alpha^{WZ} = g_{WZ} W_\alpha g_{WZ}^{\dagger}$$  \hspace{1cm} (A.20)

and similarly for $V_\alpha$. We see from above that we can always work in the Wess-Zumino gauge as long as we are computing gauge-invariant quantities; in our case since the regulator field ensures manifest gauge-invariance we can work in the WZ-gauge without any loss of generality.
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