Blow up limits of the fractional Laplacian and their applications to the fractional Nirenberg problem

Xusheng Du, Tianling Jin, Jingang Xiong, Hui Yang

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Abstract

We show a convergence result of the fractional Laplacian for sequences of nonnegative functions without uniform boundedness near infinity. As an application, we construct a sequence of solutions to the fractional Nirenberg problem that blows up in the region where the prescribed functions are negative. This is a different phenomenon from the classical Nirenberg problem.

Keywords: Fractional Laplacian, fractional Nirenberg problem, blow up phenomena

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1 Introduction

Let \( n \geq 1 \) be an integer. In the blow up analysis for equations with the Laplacian operator \( \Delta \), the following trivial fact plays an essential role:

"If \( u_i \to u \) in \( C^2_{\text{loc}}(\mathbb{R}^n) \) as \( i \to \infty \), then \( \Delta u_i \to \Delta u \) in \( C^0_{\text{loc}}(\mathbb{R}^n) \)."

In this paper, we first show a similar but different property for the fractional Laplacian operator.

Let \( \sigma \in (0, 1) \). The fractional Laplacian \( (-\Delta)^\sigma \) is defined by

\[
(-\Delta)^\sigma u(x) = c_{n,\sigma} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} dy
\]

with \( c_{n,\sigma} = \frac{2^{2\sigma} \Gamma(\frac{n+2\sigma}{2})}{\pi^{n/2} \Gamma(1-\sigma)} \) and the gamma function \( \Gamma \). Define

\[
L_{\sigma}(\mathbb{R}^n) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2\sigma}} dx < \infty \right\}
\]

Then \( (-\Delta)^\sigma u(x) \) is well-defined if \( u \in L_{\sigma}(\mathbb{R}^n) \) and \( u \) is \( C^{2\sigma+\alpha} \) near \( x \) for some \( \alpha > 0 \).

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Theorem 1.1. Let \( n \geq 1, \sigma \in (0, 1) \) and \( \alpha > 0 \). Suppose that \( \{ R_i \} \) is a sequence of positive numbers converging to \( +\infty \), and \( \{ u_i \} \subset L^p(\mathbb{R}^n) \cap C^{2\sigma+\alpha}(B_{R_i}) \) is a sequence of nonnegative functions. If \( \{ u_i \} \) converges in \( C^{2\sigma+\alpha}_{\text{loc}}(\mathbb{R}^n) \) to a function \( u \in L^p(\mathbb{R}^n) \), and \( \{ (-\Delta)^\sigma u_i \} \) converges pointwisely in \( \mathbb{R}^n \), then there exists a constant \( b \geq 0 \) such that
\[
\lim_{i \to \infty} (-\Delta)^\sigma u_i(x) = (-\Delta)^\sigma u(x) - b \quad \forall x \in \mathbb{R}^n.
\] (1)
Moreover,
\[
b = c_{n,\sigma} \lim_{R \to \infty} \lim_{i \to \infty} \int_{B_R^c} \frac{u_i(x)}{|x|^{n+2\sigma}} \, dx \quad \text{(the limit exists and is finite)}.
\]

Note that Theorem 1.1 does not assume uniform boundedness of \( \{ u_i \} \) near infinity in any topology, but rather their non-negativity instead. The following example shows that the constant \( b \) in (1) can be positive.

Theorem 1.2. Let \( n \geq 1 \) and \( \sigma \in (0, 1) \). There exists a sequence of nonnegative smooth functions \( \{ v_j \} \) with compact support in \( \mathbb{R}^n \) such that \( v_j \) converges to \( 1 \) in \( C^2(\mathbb{R}^n) \), and
\[
\lim_{j \to \infty} (-\Delta)^\sigma v_j(x) = -1 \quad \forall x \in \mathbb{R}^n.
\]

The nonzero constant \( b \) in Theorem 1.1 will lead to some different blow up phenomena for equations with the fractional Laplacian, compared with those involving the Laplacian.

In [4], Gidas-Spruck used blow up analysis, together with Liouville theorems that were proved by them earlier in [3], to derive a priori estimate of positive solutions to general elliptic equations including
\[
-\Delta u = K(x) u^p,
\] (2)
where \( p \) is a subcritical Sobolev exponent, i.e., \( 1 < p < \infty \) if \( n = 1, 2, \) or \( 1 < p < \frac{n+2}{n-2} \) if \( n \geq 3 \).

When \( n \geq 3 \) and \( p = \frac{n+2}{n-2} \), which is the critical Sobolev case, the equation (2) is the one for the Nirenberg problem, that is the problem of finding a metric conformal to the flat metric on \( \mathbb{R}^n \) such that \( K(x) \) is the scalar curvature of the new metric. One main ingredient in establishing the existence of solutions to the Nirenberg problem is to obtain a priori estimates for the solutions of (2), which have been achieved in Chang-Gursky-Yang [1], Li [8] and Schoen-Zhang [10] for positive functions \( K \), and in Chen-Li [2] and Lin [9] for \( K \) changing signs. In particular, when \( K \) is allowed to change signs, one important step is to estimate the solutions in the region where \( K \) is negative. It has been shown in Chen-Li [2] and Lin [9] that the solutions of (2) in \( B_2 \) are uniformly bounded in \( B_1 \) provided that \( K \in C^\alpha(B_2) \) for some \( \alpha \in (0, 1) \) and \( K(x) \leq -c_0 \) in \( B_2 \) for some \( c_0 > 0 \). The uniform bound depends only on \( n, c_0 \) and \( |K|_{C^\alpha(B_2)} \).

On the contrary, we will show that such a priori estimates do not hold for the fractional Nirenberg problem. The fractional Nirenberg problem is equivalent to studying the equations
\[
(-\Delta)^\sigma u = K(x) u^p \quad \text{in} \ \mathbb{R}^n.
\] (3)
We will construct a sequence of positive smooth solutions to (3) that blows up in the region where \( K \) is negative. Note that a priori estimates of the fractional equation (3) for positive functions \( K \) have been derived in Jin-Li-Xiong [5–7]. More generally, blow up can occur for the equation (3) with any \( p \in \mathbb{R} \).
Theorem 1.3. Let \( n \geq 1, \sigma \in (0, 1), p \in \mathbb{R} \) and \( q > -2\sigma \). There exist two positive constants \( c \) and \( C \) depending only on \( n, \sigma, p \) and \( q \), a family of functions \( \{K_\lambda\}_{\lambda \geq 1} \subset C^\infty(\mathbb{R}^n) \) satisfying

\[-C \leq K_\lambda(x) \leq -c, \quad c \leq |\nabla K_\lambda(x)| \leq C \quad \text{and} \quad |\nabla^2 K_\lambda(x)| \leq C \quad \forall x \in B_2 \quad \text{and} \quad \forall \lambda \geq 1,
\]

and a family of positive functions \( \{u_\lambda\}_{\lambda \geq 1} \subset C^\infty(\mathbb{R}^n) \) satisfying

\[(-\Delta)^\sigma u_\lambda = K_\lambda(x) u_\lambda^p \quad \text{in} \quad \mathbb{R}^n, \quad \lim_{|x| \to \infty} |x|^q u_\lambda(x) = 1,
\]

and

\[\min_{\overline{B_1}} u_\lambda \to +\infty \quad \text{as} \quad \lambda \to +\infty.\]

The failure of the compactness of solutions to (3) in the region where \( K \) is negative is caused by Theorem 1.1 and Theorem 1.2 that the blow up limit equation for the fractional equation (3) in the region \( \{K(x) \leq -c_0\} \) is

\[(-\Delta)^\sigma u - b = -u^p \quad \text{in} \quad \mathbb{R}^n \]

(4)

with a constant \( b \geq 0 \), and \( b \) can be positive. If \( b > 0 \), then the equation (4) clearly has the positive constant \( b^{\frac{1}{p}} \) as a solution. Such nonzero constant solutions of (4) cannot be ruled out if one does not know a priori uniform boundedness in any topology.

This paper is organized as follows. In Section 2, we show the convergence for the fractional Laplacian in Theorem 1.1 and give the example in Theorem 1.2. In Section 3, we construct blow up solutions for the fractional Nirenberg problem stated in Theorem 1.3.

2 Convergence for the fractional Laplacian

Proof of Theorem 1.1. Our proof is inspired by that of [7, Proposition 2.9] on an integral equation. Fix \( x \in \mathbb{R}^n \). Let \( R \gg |x| + 1 \). Then for all large \( i \), we have

\[(-\Delta)^\sigma u(x) - (-\Delta)^\sigma u_i(x) = c_{n,\sigma} \int_{B_R} \frac{(u - u_i)(x) - (u - u_i)(y)}{|x - y|^{n+2\sigma}} \, dy
\]

\[+ c_{n,\sigma} \int_{B_R^c} \frac{(u - u_i)(x) - u(y)}{|x - y|^{n+2\sigma}} \, dy
\]

\[+ c_{n,\sigma} \int_{B_R^c} u_i(y) \, dy
\]

\[=: A_i(x, R) + E_i(x, R) + F_i(x, R).
\]

By the assumptions, \( \{u_i\} \) converges to \( u \) in \( C^{2\sigma+\alpha}(B_{2R}) \). Hence,

\[\lim_{i \to \infty} A_i(x, R) = 0 \quad \text{and} \quad \lim_{i \to \infty} E_i(x, R) = -c_{n,\sigma} \int_{B_R} \frac{u(y)}{|x - y|^{n+2\sigma}} \, dy.
\]

(6)

Since \( u \in L_\sigma(\mathbb{R}^n) \), we have

\[\lim_{R \to +\infty} \lim_{i \to \infty} E_i(x, R) = 0.
\]

(7)
Since the sequence \( \{(-\Delta)^\sigma u_i(x)\} \) converges, it follows from (5) that
\[
F(x, R) := \lim_{i \to \infty} F_i(x, R) \quad \exists \text{ and is finite.}
\]

Sending \( i \to \infty \) in (5), and using (6)-(7), we obtain
\[
\lim_{R \to +\infty} \left| (-\Delta)^\sigma u(x) - \lim_{i \to \infty} (-\Delta)^\sigma u_i(x) - F(x, R) \right| = 0. \tag{8}
\]

Since \( u_i \) is nonnegative on \( \mathbb{R}^n \), we have
\[
\left( \frac{R}{R + |x|} \right)^{n+2\sigma} \int_{B_R^c} \frac{u_i(y)}{|y|^{n+2\sigma}} \, dy \leq \int_{B_R^c} \frac{u_i(y)}{|x - y|^{n+2\sigma}} \, dy \leq \left( \frac{R}{R - |x|} \right)^{n+2\sigma} \int_{B_R^c} \frac{u_i(y)}{|y|^{n+2\sigma}} \, dy.
\]

Sending \( i \to \infty \), it follows that
\[
\left( \frac{R}{R + |x|} \right)^{n+2\sigma} F(0, R) \leq F(x, R) \leq \left( \frac{R}{R - |x|} \right)^{n+2\sigma} F(0, R). \tag{9}
\]

Notice that \( F_i(x, R) \) is nonnegative and non-increasing in \( R \), so \( F(x, R) \). Hence, \( \lim_{R \to \infty} F(x, R) \) exists, and is nonnegative and finite. By sending \( R \) to \( \infty \) in (9), we obtain
\[
\lim_{R \to \infty} F(x, R) = \lim_{R \to \infty} F(0, R) =: b \geq 0.
\]

Then the conclusion follows from (8). \( \square \)

The proof of Theorem 1.2 can be illustrated by the following simple functions. Let \( \lambda > 0 \). Define
\[
W_\lambda(x) = \begin{cases} 
\lambda & \text{if } x \in B_3, \\
\lambda + \lambda^2 & \text{if } x \in B_3^c.
\end{cases}
\]

For \( j \geq 1 \), let
\[
R_j = j^{\frac{1}{2\sigma}},
\]
and
\[
V_j(x) = j^{-1}W_j \left( j^{-\frac{1}{2\sigma}} x \right) = \begin{cases} 
1 & \text{if } y \in B_3R_j, \\
1 + j & \text{if } y \in B_3^cR_j.
\end{cases}
\]

Then \( V_j \) converges to the constant function 1 in \( C^2_{\text{loc}}(\mathbb{R}^n) \). However, for any \( x \in \mathbb{R}^n \),
\[
\lim_{j \to \infty} (-\Delta)^\sigma V_j(x) = -c_{n,\sigma} \lim_{j \to \infty} \int_{B_3} \frac{dy}{\left| j^{-\frac{1}{2\sigma}} x - y \right|^{n+2\sigma}} = -\frac{c_{n,\sigma} |S^{n-1}|}{2\sigma 3^{2\sigma}}.
\]

A mollification of \( V_j \) by smoothing it out and cutting it off at infinity will give the example for Theorem 1.2. The details are as follows.
Proof of Theorem 1.2. Let \( \eta : \mathbb{R} \to [0, 1] \) be a \( C^\infty \) cut-off function such that \( \eta(t) \equiv 0 \) if \( t \leq 0 \), \( \eta(t) \equiv 1 \) if \( t \geq 1 \), and \( 0 \leq \eta(t) \leq 1 \) if \( 0 \leq t \leq 1 \). Let \( \psi(x) := \eta(|x| - 3) \) and \( \varphi(x) := \eta(|x| - 6) \).

For \( \lambda \geq 1 \), define

\[
 w_\lambda(x) = \begin{cases} 
  \lambda & \text{if } x \in B_3, \\
  \lambda + \lambda^2 \psi(x) & \text{if } x \in B_4 \setminus B_3, \\
  \lambda + \lambda^2 & \text{if } x \in B_6 \setminus B_4, \\
  (1 - \varphi(x)) (\lambda + \lambda^2) & \text{if } x \in B_6^c.
\end{cases}
\]

Then \( w_\lambda \in C^\infty_c(\mathbb{R}^n) \), \( w_\lambda \geq 0 \) in \( \mathbb{R}^n \), and \( \text{supp}(w_\lambda) \subset \overline{B_7} \) for any \( \lambda \geq 1 \).

Define

\[
 f_\lambda(x) = \lambda^{-2} (-\Delta)^\sigma w_\lambda(x), \quad x \in \mathbb{R}^n.
\]

Then for \( x \in B_2 \),

\[
 f_\lambda(x) = -c_{n,\sigma} \int_{B_4 \setminus B_3} \frac{\psi(y)}{|x - y|^{n+2\sigma}} \, dy - c_{n,\sigma} \int_{B_6 \setminus B_4} \frac{dy}{|x - y|^{n+2\sigma}} + \lambda^{-1} c_{n,\sigma} \int_{B_6^c} \frac{\varphi(y)}{|x - y|^{n+2\sigma}} \, dy - c_{n,\sigma} \int_{B_6^c} \frac{1 - \varphi(y)}{|x - y|^{n+2\sigma}} \, dy
\]

\[
 - c_{n,\sigma} \left( \int_{B_4 \setminus B_3} \frac{\psi(y)}{|x - y|^{n+2\sigma}} \, dy + \int_{B_6 \setminus B_4} \frac{dy}{|x - y|^{n+2\sigma}} + \int_{B_6^c} \frac{1 - \varphi(y)}{|x - y|^{n+2\sigma}} \, dy \right)
\]

uniformly as \( \lambda \) tends to \( \infty \).

Let

\[
 \beta = \left( - \lim_{\lambda \to \infty} f_\lambda(0) \right)^{-\frac{1}{2\sigma}}.
\]

For \( j \geq 1 \), let

\[
 R_j = \beta^{-\frac{1}{2\sigma}} \quad \text{and} \quad v_j(x) = \beta^{-\frac{1}{2\sigma}} w_j \left( \beta^{-\frac{1}{2\sigma}} x \right).
\]

Then \( v_j \in C^\infty_c(\mathbb{R}^n) \), \( v_j = 1 \) in \( B_{R_j} \), \( v_j \geq 0 \) in \( \mathbb{R}^n \), and

\[
 (-\Delta)^\sigma v_j(x) = \beta^{2\sigma} f_j \left( \beta^{-\frac{1}{2\sigma}} x \right) \quad \text{for } x \in B_{R_j}.
\]

It follows that \( v_j \) converges to the constant function 1 in \( C^2_{\text{loc}}(\mathbb{R}^n) \) as \( j \to \infty \), and

\[
 \lim_{j \to \infty} (-\Delta)^\sigma v_j(x) = \beta^{2\sigma} \lim_{j \to \infty} f_j(\beta^{-\frac{1}{2\sigma}} x) = -1 \quad \forall \, x \in \mathbb{R}^n.
\]

The proof is completed. \( \square \)

3 Blow up phenomena for a fractional Nirenberg problem when the prescribed functions are negative

Our example to Theorem 1.3 is inspired by the following simple functions. Let \( \lambda > 0 \). Define

\[
 U_\lambda(x) = \begin{cases} 
  \lambda & \text{if } x \in B_3, \\
  \lambda + \lambda^2 & \text{if } x \in B_3^c.
\end{cases}
\]
Then for $x \in B_2$,
\[
\frac{(-\Delta)^\sigma U(x)}{U(x)^p} = -c_{n,\sigma} \int_{B_3^c} \frac{dy}{|x-y|^{n+2\sigma}} =: K(x).
\]
It is clear that $K$ is smooth in $B_2$, and bounded from above and below by two negative constants in $B_2$. Nevertheless,
\[
\min_{x \in \mathcal{B}_1} U(x) \to +\infty \quad \text{as} \quad \lambda \to +\infty.
\]
A similar mollification of $U$ by smoothing it out and cutting it off at infinity will give the example for Theorem 1.3. The details are as follows.

**Proof of Theorem 1.3.** Let $\eta : \mathbb{R} \to [0, 1]$ be a $C^\infty$ cut-off function such that $\eta(t) \equiv 0$ if $t \leq 0$, $\eta(t) \equiv 1$ if $t \geq 1$, and $0 \leq \eta(t) \leq 1$ if $0 \leq t \leq 1$. For $\lambda \geq 1$, we define
\[
u_{\lambda}(x) = \begin{cases} 
\lambda & \text{if } x \in B_3, \\
\lambda + \lambda^p \psi(x) & \text{if } x \in B_4 \setminus B_3, \\
\lambda + \lambda^p & \text{if } x \in B_R \setminus B_4,
\end{cases}
\]
where $R = R(n, \sigma, p, q, \lambda) > 9$ is a large constant to be specified later, $\psi(x) := \eta(|x| - 3)$ and $\varphi(x) := \eta(|x| - R)$. Since $q > -2\sigma$, $u_{\lambda} \in C^\infty(\mathbb{R}^n \cap L_\sigma(\mathbb{R}^n)$ and $u_{\lambda} > 0$ in $\mathbb{R}^n$. Define
\[
K_{\lambda}(x) = \frac{(-\Delta)^\sigma u_{\lambda}(x)}{u_{\lambda}(x)^p}, \quad x \in \mathbb{R}^n.
\]
Then $K_{\lambda} \in C^\infty(\mathbb{R}^n)$.

**Step 1.** There exist two positive constants $c_1$ and $c_2$ depending only on $n$ and $\sigma$ such that
\[
-c_1 \leq K_{\lambda}(x) \leq -c_2 \quad \forall \ x \in B_2, \ \lambda \geq 1. \tag{10}
\]
Indeed, for $x \in B_2$, we have
\[
c_{n,\sigma}^{-1} K_{\lambda}(x) = -\int_{B_3^c} \frac{\psi(y)}{|x-y|^{n+2\sigma}} dy - \int_{B_4^c} \frac{dy}{|x-y|^{n+2\sigma}} + \int_{B_R^c} \frac{\lambda^{1-p} \varphi(y) + \varphi(y) - 1 - \lambda^{-p} \varphi(y)|y|^{-q}}{|x-y|^{n+2\sigma}} dy \tag{11}
\]
\[
=: \sum_{i=1}^{3} I_i.
\]
For convenience, denote
\[
\gamma_{n,\sigma} := \int_{B_3^c} \frac{dy}{|y|^{n+2\sigma}} = \frac{|\mathbb{S}^{n-1}|}{2\sigma}, \quad \tilde{\gamma}_{n,\sigma,q} := \int_{B_4^c} \frac{dy}{|y|^{n+2\sigma+q}} = \frac{|\mathbb{S}^{n-1}|}{2\sigma + q}.
\]
It is clear that
\[
0 \geq I_1 \geq -\int_{B_2(x)^c} \frac{dy}{|x-y|^{n+2\sigma}} = -\int_{B_1^c} \frac{dy}{|y|^{n+2\sigma}} = -\gamma_{n,\sigma}.
\]
Similarly, we obtain
\[ -\gamma_{n, \sigma} \leq I_2 \leq - \int_{B_{R-2} \setminus B_{\delta}} \frac{dy}{|y|^{n+2\sigma}} = -\gamma_{n, \sigma} \left( 6^{-2\sigma} - \frac{(R-2)^{-2\sigma}}{2} \right). \]

Since \( R > 9 \), we have \(|x - y| \geq |y|/2\) for all \( x \in B_2 \) and \( y \in B_{3R} \). Thus,
\[ |I_3| \leq 2^n + 2^{2\sigma} \int_{B_{3R}} \left( \lambda^{1-p} + 1 + \lambda^{-p} |y|^{-q} \right) dy \]
\[ = 2^n + 2^{2\sigma} \gamma_{n, \sigma} \left( 1 - \frac{1}{2} \right) R^{-2\sigma} + 2^{n+2\sigma} \gamma_{n, \sigma} \sigma \lambda^{-p} R^{-2\sigma - q}. \]

By choosing \( R = R(n, \sigma, p, q, \lambda) > 9 \) sufficiently large such that
\[\gamma_{n, \sigma} (R-2)^{-2\sigma} + 2^{n+2\sigma} \gamma_{n, \sigma} (1^{-p} + 1) R^{-2\sigma} + 2^{n+2\sigma} \gamma_{n, \sigma} \sigma \lambda^{-p} R^{-2\sigma - q} \leq \frac{\gamma_{n, \sigma} 6^{-2\sigma}}{2},\]
we have for all \( x \in B_2 \) and \( \lambda \geq 1 \) that
\[ -3c_{n, \sigma} \gamma_{n, \sigma} \leq K_{\lambda}(x) \leq -\frac{c_{n, \sigma} \gamma_{n, \sigma} 6^{-2\sigma}}{2}. \]

**Step 1** is proved.

**Step 2.** Moreover, by differentiating (11), we can similarly obtain
\[ |\nabla K_{\lambda}(x)| + |\nabla^2 K_{\lambda}(x)| + |\nabla^3 K_{\lambda}(x)| \leq c_{3} \quad \forall x \in B_2, \quad \lambda \geq 1, \]
for some constant \( c_{3} > 0 \) depending only on \( n \) and \( \sigma \).

**Step 3.** There exists a constant \( c_{4} > 0 \) depending only on \( n \) and \( \sigma \) such that
\[ \nabla^2 K_{\lambda}(0) \leq -c_{4} \mathbf{I}_n \quad \forall \lambda \geq 1, \]
where \( \mathbf{I}_n \) is the \( n \times n \) identity matrix.

Indeed, for \( y \in B_{3R} \), we have
\[ \frac{\partial^2}{\partial x_i \partial x_j} (|x - y|^{-n-2\sigma}) (0) = (n + 2\sigma) |y|^{-n-2\sigma - 4} \left[ (n + 2\sigma + 2) y_i y_j - \delta_{ij} |y|^2 \right]. \]

It follows from (11) that
\[ \left[ (n + 2\sigma) c_{n, \sigma} \right]^{-1} \frac{\partial^2 K_{\lambda}}{\partial x_i \partial x_j} (0) \]
\[ = - \int_{B_3 \setminus B_{\delta}} \frac{\psi(y)}{|y|^{n+2\sigma+4}} \left[ (n + 2\sigma + 2) y_i y_j - \delta_{ij} |y|^2 \right] dy \]
\[ - \int_{B_{3R} \setminus B_3} \frac{1}{|y|^{n+2\sigma+4}} \left[ (n + 2\sigma + 2) y_i y_j - \delta_{ij} |y|^2 \right] dy \]
\[ + \lambda^{1-p} \int_{B_{3R}} \frac{\varphi(y)}{|y|^{n+2\sigma+4}} \left[ (n + 2\sigma + 2) y_i y_j - \delta_{ij} |y|^2 \right] dy \]
Using the polar coordinate, and the radial symmetry of $\psi$ and $\varphi$, we have

$$E_{1,ii} = -\int_{B_4 \setminus B_3} \frac{\psi(y)}{|y|^{n + 2\sigma + 4}} \left[ \left( \frac{n + 2\sigma + 2}{n} \right) |y|^2 - |y|^2 \right] dy \leq 0.$$ 

Similarly, $E_{4,ii} \leq 0$, $E_{2,ii} = -\frac{2(\sigma + 1)}{n} \int_{B_4 \setminus B_3} \frac{dy}{|y|^{n + 2\sigma + 2}} = -|B_1| \left( 4^{-\sigma - 2} - R^{-2\sigma - 2} \right)$, and

$$E_{3,ii} \leq \lambda^{1-p} \frac{2(\sigma + 1)}{n} \int_{B_4} \frac{dy}{|y|^{n + 2\sigma + 2}} = \lambda^{1-p} |B_1| R^{-2\sigma - 2}.$$ 

Consequently, we obtain

$$\frac{\partial^2 K_\lambda}{\partial x_i^2}(0) \leq -(n + 2\sigma) c_{n,\sigma} |B_1| \left( 4^{-\sigma - 2} - (1 + \lambda^{1-p}) R^{-2\sigma - 2} \right).$$

We take $R = R(n, \sigma, p, q, \lambda) > 9$ sufficiently large such that

$$4^{-\sigma - 2} - (1 + \lambda^{1-p}) R^{-2\sigma - 2} \geq 4^{-2\sigma - 3}.$$ 

Then for all $\lambda \geq 1$ and $1 \leq i \leq n$,

$$\frac{\partial^2 K_\lambda}{\partial x_i^2}(0) \leq -(n + 2\sigma) c_{n,\sigma} |B_1| 4^{-2\sigma - 3}.$$ 

By the symmetry of the regions in the above integrals, we know that

$$\frac{\partial^2 K_\lambda}{\partial x_i \partial x_j}(0) = 0 \quad \text{if } i \neq j.$$ 

This proves Step 3.

**Step 4. Construct the desired functions.** Notice that $u_\lambda$ is radially symmetric about the origin, so is $K_\lambda$. Hence,

$$\nabla K_\lambda(0) = 0. \quad (14)$$

Then it follows from (12), (13) and (14) that there exist two constants $\delta_0 \in (0, 1/4)$ and $c_5 > 0$ depending only on $n$ and $\sigma$ such that

$$|\nabla K_\lambda(x)| \geq c_5 \quad \forall x \in B_{2\delta_0} (4\delta_0 e_1), \ \lambda \geq 1, \quad (15)$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. 

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Let
\[ \tilde{u}_\lambda(x) = \delta_0^q u_\lambda(\delta_0(x + 4e_1)) \quad \text{and} \quad \tilde{K}_\lambda(x) = \delta_0^{q+2\sigma-pq} K_\lambda(\delta_0(x + 4e_1)). \]
Then,
\[ (-\Delta)^q \tilde{u}_\lambda = \tilde{K}_\lambda(x) \tilde{u}_\lambda \quad \text{in} \quad \mathbb{R}^n. \]
It follows from (10), (12) and (15) that
\[ -C \leq \tilde{K}_\lambda(x) \leq -c, \quad c \leq |\nabla \tilde{K}_\lambda(x)| \leq C \quad \text{and} \quad |\nabla^2 \tilde{K}_\lambda(x)| \leq C \quad \forall \, x \in B_2 \quad \text{and} \quad \forall \, \lambda \geq 1 \]
for some positive constants \( c \) and \( C \) depending only on \( n, \sigma, p \) and \( q \). Meanwhile, it follows from the definition of \( u_\lambda \) that
\[ \lim_{|x| \to \infty} |x|^q \tilde{u}_\lambda(x) = 1, \]
and
\[ \min_{\mathbb{R}^n_1} \tilde{u}_\lambda = \delta_0^q \lambda \to +\infty \quad \text{as} \quad \lambda \to +\infty. \]
The proof of Theorem 1.3 is completed.

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X. Du, T. Jin, H. Yang
Department of Mathematics, The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong
Emails: xduah@connect.ust.hk, tianlingjin@ust.hk, mahuiyang@ust.hk

J. Xiong
School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, MOE
Beijing Normal University, Beijing 100875, China
Email: jx@bnu.edu.cn