MICROSCOPIC FOUNDATIONS OF NUCLEAR SUPERSYMMETRY

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We discuss a microscopic framework for phenomenological boson-fermion models of nuclear structure based on the U\((n/m)\) type of superalgebras. The generalized Dyson mapping of fermion collective superalgebras provides a basis to do so and to understand how collectivity selects the required preservation of boson plus fermion number as a good quantum number. We also consider the difference between dynamical and invariant supersymmetries based on possible supermultiplets of spectra of neighboring odd and even nuclei. We point out that different criteria exist for choosing the appropriate single particle transfer operators in the two cases and discuss a microscopically based method to construct these operators in the case of dynamical supersymmetry.

1. Dynamical vs invariant SUSY in nuclear structure

Supersymmetry (SUSY) was originally introduced into relativistic quantum field theory to exhibit and study invariance with respect to the exchange of bosons and fermions (see e.g. Weinberg’s recent monograph\(^1\) for an overview), but the notion has since been successfully exploited in a variety of quantum mechanical and quantum many-body systems (see e.g. the review by Cooper \textit{et al}\(^2\) and Junker’s book\(^3\) for various examples).

Unfortunately discussions of dynamical supersymmetry on the phenomenological nuclear structure level, and its relation to the notion of supersymmetry above, have not always clarified the distinction between them, nor that between the concepts of dynamical supersymmetry and what we will refer to as ‘invariant supersymmetry’ in the nuclear context. It is therefore not totally surprising to find that somewhat negative opinions such as the following one by ’t Hooft\(^4\) have been voiced: “At first sight, the fact that supersymmetric patterns were discovered in nuclear physics has little to do with the question of supersymmetry among elementary particles, but
it may indicate that, as the spectrum of particles is getting more and more complex, some supersymmetric patterns might easily arise, even if there is no ‘fundamental’ reason for their existence.”

What we argue below (see also Ref. 5 for a more complete presentation) is that the situation is much more positive than this and that the ‘fundamental reasons’ might be found (i) in the nature of interactions on the nucleon level which gives rise to collectivity of nuclear states, and (ii) in the utility of boson-fermion mappings which show that dynamical supersymmetry can arise in a fermion system without violation of the Pauli principle.

Dynamical supersymmetry on the phenomenological level concerns situations where states of a quantum system with even and odd fermion numbers can be unified in a single representation of a certain supergroup. In the nuclear physics context this possibility was realized by Iachello 6 in the phenomenological interacting boson-fermion model (IBFM) 7 and subsequently shown to be applicable to various pairs of nuclei (see Ref. 7 for an introduction and review of applications). Renewed interest in this possibility has been created by the experimental results of Metz, Jolie et al 8 where a quartet of nuclei has been found to fit into a single extended supersymmetric multiplet of $U(6/12) \otimes U(6/4)$ which takes both neutron ($\nu$) and proton ($\pi$) degrees of freedom explicitly into account.

Although there is no fundamental difference between even and odd nuclei (states of both of them are in principle eigenstates of the same Hamiltonian with different particle numbers), their actual properties differ substantially. Thus unification of spectra of even and odd nuclei into a single framework is indeed a challenging possibility, with the prospect of unveiling a basic underlying symmetry. The notion of supersymmetry has in this regard proved to be quite fruitful. It is applied on the phenomenological IBFM level, where boson degrees of freedom ($s$- and $d$-bosons) are introduced to describe collective monopole and quadrupole fermion pairs, while fermions represent only the single (odd) nucleon. Dynamical supersymmetry then arises when different boson-fermion interaction strengths are related in a special way.

Invariant supersymmetry in nuclear structure, as considered, e.g., by Jolos and von Brentano, 9 is much closer in spirit and detail to SUSY quantum mechanics. As is well known, 3 the simplest form of invariant supersymmetry in quantum mechanics can be formulated in terms of nilpotent operators (‘supercharges’) of the form $Q = B\alpha^\dagger$ and $Q^\dagger$ with the $B$’s and $\alpha$’s the usual boson and fermion operators, which are defined to be kinematically independent: $[B, \alpha] = [B, \alpha^\dagger] = 0$. The supersymmetric Hamiltonian
\{Q, Q^\dagger\} = B^\dagger B + \alpha^\dagger \alpha \text{ obviously has eigenstates } |n_B, n_\alpha\rangle \text{ and displays the hallmark supersymmetric spectrum of a unique ground state } |0, 0\rangle \text{ and a set of doubly degenerate excited states (this can be easily extended to models with more than one supercharge). An (approximate) invariant supersymmetry in nuclear physics, if verified by experimental data, would therefore imply not only the above-discussed unique classification of states in even and odd nuclei, but also the actual (approximate) degeneracy of some of these states. As present nuclear data indicate no such a degeneracy, an invariant supersymmetry between single nucleons and collective pairs seems to be broken into a particular dynamical supersymmetry, quite similarly to the familiar scenario considered in elementary particle physics.}

2. Dynamical SUSY: phenomenology vs microscopy

In applications of dynamical supersymmetry to nuclear spectra,\textsuperscript{6,7,8} the appropriate Hamiltonian is expressed in terms of the above odd generators $Q$ and $Q^\dagger$, as well as even generators of the type $\alpha^\dagger \alpha$ and $B^\dagger B$. This selects the $\text{U}(n/m)$-type of superalgebras to form an appropriate algebraic framework describing dynamics of the system. In contrast to the invariant supersymmetry, the nuclear interaction requires more general terms than those appearing in the typical SUSY Hamiltonian above. While the invariant supersymmetry is therefore broken, it nevertheless remains possible to classify states of some even and odd neighbouring nuclei in terms of representations of a supermultiplet. This is still a non-trivial property of the interactions, which must allow for expressing the nuclear Hamiltonian only in terms of Casimir invariants corresponding to a certain chain of (super)algebras that decompose the dynamical superalgebra $\text{U}(n/m)$.

It is clear that for the whole analysis to be feasible, a phenomenological IBFM Hamiltonian which reflects interactions among bosons and fermions is a necessary prerequisite. A crucial question thus appears whether the dynamical supersymmetry can be compatible with the Pauli principle on the microscopic level or, equivalently, whether dynamical supersymmetry can be an exact property of a fermion system. From the point of view that there are important Pauli corrections to the lowest order association between collective fermion pairs and IBM bosons,\textsuperscript{10} one might anticipate a negative answer to this question. Nevertheless, the implementation of appropriate boson-fermion mappings indeed reveals instances where this compatibility holds exactly—see also Refs.\textsuperscript{11,12,13}.

Apart from providing a concrete link between fermion dynamics and dy-
namical supersymmetry, the use of boson-fermion mappings also allows one to construct various transition operators appropriate to the boson-fermion description, including the important single-particle transfer operators. This is in contrast to the phenomenological situation where one is obliged to truncate an infinite series of combinations of boson and fermion operators with phenomenological parameters and terms only restricted by their tensor and particle number changing properties. It should be emphasized that the choice of these transition operators in phenomenological models such as the IBM or IBFM is not dictated by the Hamiltonian parameters in general, specifically also in the case of dynamical symmetry or supersymmetry. (See also Ref.14 for a discussion of this point.)

3. Example: dynamical SUSY in the seniority model

The SU(2) seniority model has been analysed exhaustively. Here we briefly discuss how the known results may be obtained from a boson-fermion mapping and interpreted from the point of view of dynamical supersymmetry.

The SU(2) model is defined by considering in a single-\( j \) shell the monopole pair creation operator \( S^\dagger = \sqrt{\Omega/2} (a_j^\dagger a_{j}^{(0)}) \) with \( \Omega = j + 1/2 \). It fulfills the commutation relation \([S, S^\dagger] = \Omega - n\), where \( n \) is the fermion number operator. The SU(2) algebra can be generalized to describe odd systems by constructing the superalgebra generated by the operators \( S^\dagger, S, \Omega - n, a_{jm}, \) and \( a_{jm}^\dagger \). The relevant commutation relation is \([a_{jm}^\dagger, S] = -a_{jm}\), with \( a_{jm} = (-1)^{j+m}a_{j,m}\), while the single-fermion operators obey the standard anticommutation relations. Clearly this is a rather trivial superalgebra as the elements of the odd sector (single fermion operators) anticommute only to the identity. Alternatively, by considering the commutator of single-fermion operators, the set of bi- and single-fermion operators may of course also be viewed as generators of a standard (orthogonal) algebra.

In the single-\( j \) shell we consider the pairing Hamiltonian \( H = -G S^\dagger S \) which has the energy spectrum \( E(n, v) = -\frac{1}{4} G(n-v)(2\Omega-n-v+2) \), with \( n \) the total number of fermions and the seniority quantum number \( v \) denoting the number of fermions not coupled to angular momentum zero. This Hamiltonian describes both the even and odd systems, and the spectra in both cases are given by the same expression with \( v \) even or odd, respectively.

We can apply to this model the general Dyson boson-fermion mapping derived in Ref.12 to find an equivalent description in the boson-fermion space. As explained in the work cited, the construction, which utilises supercoherent states, is a two-step process that requires application of a
certain similarity transformation, the general form of which is derived in Ref.\textsuperscript{5}. This procedure finally yields typical non-Hermitian Dyson structures that generalize the images obtained in the case when only the even sector of the collective superalgebra is considered. Ideal fermion operators $\alpha^\dagger_{jm}$ and $\alpha_{jm}$ are now introduced. They obey the standard fermion algebra and commute with the boson operators $B^\dagger$ and $B$. Ideal fermion pair operators $\Sigma^\dagger$ and $\Sigma$ are obtained from $S^\dagger$ and $S$ by replacing all $a$’s by $\alpha$’s.

The mapping obtained for the SU(2) case is

\begin{align*}
S^\dagger &\longleftrightarrow B^\dagger(\Omega - \aleph), \\
S &\longleftrightarrow B, \\
n &\longleftrightarrow 2N_B + N_F = \aleph + N_B, \\
a^\dagger_{jm} &\longleftrightarrow \alpha^\dagger_{jm} \frac{\Omega - \aleph}{\Omega - N_F} + B^\dagger \tilde{\alpha}_{jm} - \Sigma^\dagger \check{\alpha}_{jm} \frac{\Omega - \aleph}{(\Omega - N_F)(\Omega - N_F + 1)}, \\
a_{jm} &\longleftrightarrow \alpha_{jm} + \tilde{\alpha}^\dagger_{jm} \frac{1}{\Omega - N_F} + \Sigma^\dagger \alpha_{jm} \frac{1}{(\Omega - N_F)(\Omega - N_F + 1)}. \tag{4}
\end{align*}

Here $N_F$ is the number of ideal fermions, $N_B$ the number of bosons, and $\aleph = N_F + N_B$. Note that the finiteness of the original single-$j$ Hilbert space implies the necessity to cut off a spurious sector from the ideal boson-fermion space. In the present case, the physical subspace satisfies the conditions $N_F \leq \Omega$ and $N_B \leq \Omega - N_F/2$. We see that the single fermion images (4) and (5) are finite and contain terms changing the ideal fermion number by one only. Furthermore, they preserve exactly the anticommutation relations on the full ideal space, i.e., as operator identities. This property, guaranteed by the construction, ensures the exact preservation of the Pauli exclusion principle once the original fermion problem is mapped into the boson-fermion space.

The mapping above transforms the two-body Hamiltonian $H = -GS^\dagger S$ into a one- plus two-body boson-fermion Hamiltonian of the form $H_{BF} = -GN_B(\Omega - N_B + 1 - N_F)$. $H$ and $H_{BF}$ have exactly the same spectrum, $E(n, v) = E(N_B, N_F)$, which can be seen explicitly by equating particle numbers in the two formulations, $n = 2N_B + N_F$, and associating $v = N_F$. $H_{BF}$ can also be expressed in a form which stresses its dependence on the total number of bosons and fermions, $\aleph$, i.e., $H_{BF} = -G(\aleph - N_F)(\Omega + 1 - \aleph)$. Note that the boson-fermion interaction term, $GN_BN_F$, can be expressed in terms of the odd generators, $O^\dagger_m = \alpha^\dagger_{jm} B$ and $O_m = B^\dagger \alpha_{jm}$, of the U(1/2) superalgebra. Since the boson and ideal fermion number operators can be linked to even generators, it is possible to write $H_{BF}$ in yet another form.
in terms of both even generators and supergenerators of $U(1/2\Omega)$:

$$H_{BF} = -G \left[ N_B (\Omega - N_B + 1) + N_F - \sum_m O^\dagger_m O_m \right]. \quad (6)$$

4. Preservation of the total number of bosons and fermions

Above we have given an example of how the appropriate boson and fermion degrees of freedom might enter, without violation of the Pauli principle, in a dynamical supersymmetric description of the states of a fermion system. However, since the real fermion number maps onto the number of ideal fermions plus twice the number of bosons, see Eq. (3), it is clear that preservation of the real fermion number alone cannot explain the appropriateness of the labelling of dynamical SUSY states by the total number of ideal particles in the generalized IBFM description.

On the other hand, in the above example it can be seen explicitly that the boson-fermion Hamiltonian (6) does indeed preserve the total number $\aleph$ of ideal particles. Quite recently we have shown, see Ref. 5, that in the most general case this property follows from the fact that the original fermion Hamiltonian can be expressed in terms of operators belonging only to the collective algebra, i.e., operators $S^\dagger, S, n$ in the above example. It is of course well known that this situation occurs exactly in all the well-studied algebraic models and one therefore suspects that (approximate) decoupling of a subset of collective states for realistic interactions leads to mapped boson-fermion Hamiltonians that can indeed be represented by the $U(n/m)$-type of dynamical superalgebras.

Moreover, as the above general condition requires the numbers of ideal fermions and bosons, $N_F$ and $N_B$, to be conserved separately, the collectivity in the original fermionic problem selects also a class of appropriate dynamical symmetries on the mapped boson-fermion level. These symmetries must involve the decoupling of the boson and fermion degrees of freedom in the first step of the corresponding superalgebraic chain, i.e., they must be of the form $U(n/m) \supset U_B(n) \otimes U_F(m) \supset \ldots$, as verified by experimental data. 8 This positive finding has also a somewhat disappointing aspect, namely that from the purely spectroscopic viewpoint the nuclear supersymmetry seems to be limited just to the possibility to simultaneously describe neighboring even and odd nuclei by a boson-fermion (IBFM) Hamiltonian with the same set of parameters—a situation which cannot be regarded as very surprising.
5. Single particle transfer operators in nuclear SUSY

It is important to realize that the phenomenological SUSY analysis depends non-trivially on the choice of single-particle transfer operators and that this choice is not dictated by the Hamiltonian which exhibits dynamical supersymmetry in a given case. On the other hand, from the microscopic viewpoint it is clear that the appropriate transfer operators to be used in conjunction with states classified according to representations of a dynamical superalgebra are boson-fermion images of the original single-fermion operators. This is simply so because single-fermion operators are the physical operators associated with single nucleon transfer.

Even the simple SU(2) seniority model discussed above illustrates interesting consequences for the structure of transfer operators. As we see in Eqs. (4) and (5), these operators acquire terms which are responsible for the Pauli correlations between the even core and the odd particle. While, as discussed above, in the phenomenological supersymmetric models these terms are postulated or motivated semi-microscopically, the SUSY picture derived from the boson-fermion mapping yields transfer operators fixed by the mapping procedure itself. The image of the fermion annihilation operator (5) is, e.g., a combination of the ideal fermion annihilation operator and two corrective terms. It should be stressed here that although the apparent non-Hermicity of the Dyson images seems to obscure their use on the phenomenological level, it is in fact possible to calculate all the relevant single-particle (and collective-pair) transfer matrix elements with no explicit reference to the microscopic structure of the collective states involved. This was discussed by the present authors in detail in Ref.5. When the microscopic nucleon level interaction is more involved than the schematic pairing case studied above, the images should of course be generalized along the lines of the seniority mapping as employed by Navrátil and Dobeš15,16 and by Geyer and Morrison17.

On this issue we therefore disagree with the position of Barea et al18 who indicate that single particle transfer should “theoretically (be) described by the supersymmetric generators that change a boson into a fermion and vice versa,” i.e., by the supercharges \(Q\) and \(Q^\dagger\) considered earlier. The supercharges can only be the appropriate transfer operators if, for a given set of states, some of the (operator dependent) coefficients in expressions such as (4) and (5) are suppressed to the extent that only the supercharge components are effective. This special situation is of course quite interesting in itself, and should be further explored within the framework of invariant su-
persymmetry in the sense of ‘fingerprint’ degenerate spectra, as discussed, e.g., by Jolos and von Brentano. The realization of an invariant supersymmetry with supercharge transfer operators, which is only a hypothetical possibility at present, would indeed bring the nuclear SUSY very close to the original notion of supersymmetry in elementary particle physics.

6. Conclusions

We have shown how a microscopically based framework for dynamical supersymmetry in nuclei arises, taking care to distinguish this situation from invariant supersymmetry for which far less evidence exists. In particular, the microscopic origin has been discussed of the total number of ideal particles (fermions plus bosons) as a good quantum number in dynamical supersymmetry. Amongst other things, resolving this issue implies that the boson-fermion mapping facilitates the identification of dynamical supersymmetry in a fermion system which may be perfectly compatible with all Pauli restrictions. We have also discussed the basis for constructing single particle transfer operators for the respective manifestations of supersymmetry above, pointing to the way ahead for calculations of single particle strengths in dynamical supersymmetry.

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