Equally spaced collinear points in Euclidean Ramsey theory

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Abstract

It is proved that for \( k \geq 4 \), if the points of \( k \)-dimensional Euclidean space are coloured in red and blue, then there are either two red points distance one apart or \( k+3 \) blue collinear points with distance one between any two consecutive points. This result is new for \( 4 \leq k \leq 10 \).

1 Introduction

Let \( \mathbb{E}^k \) be the \( k \)-dimensional Euclidean space and let \( \ell_i \) denote the configuration of \( i \) collinear points with distance 1 between any two consecutive points. Say that two geometric configurations are congruent iff there exists an isometry (distance preserving bijection) between them. For \( d \in \mathbb{Z}^+ \), and geometric configurations \( F_1, F_2 \), let the notation \( \mathbb{E}^d \to (F_1, F_2) \) mean that for any red-blue coloring of \( \mathbb{E}^d \), either the red points contain a congruent copy of \( F_1 \), or the blue points contain a congruent copy of \( F_2 \).

It was asked by Erdős et al. \[5\] if \( \mathbb{E}^3 \to (\ell_2, \ell_5) \) or even if \( \mathbb{E}^2 \to (\ell_2, \ell_5) \). The result of Iván \[8\] implies the positive answer to the first question. Arman and Tsaturian \[1\] presented a simple proof of \( \mathbb{E}^3 \to (\ell_2, \ell_5) \) and proved a stronger result, namely that \( \mathbb{E}^3 \to (\ell_2, \ell_6) \). Tsaturian \[9\] proved that \( \mathbb{E}^2 \to (\ell_2, \ell_5) \).

Denote by \( m(k) \) the maximal number such that \( \mathbb{E}^k \to (\ell_2, \ell_{m(k)}) \), if it exists. Erdős and Graham \[3\] claimed that \( m(2) \) exists. The existence of \( m(k) \) for all \( k \) follows from a recent result by Conlon and Fox \[2\], who proved that

\[
(1 + o(1))1.2^k < m(k) < 10^{5k}.
\]

In this short note, it is proved that for all \( k \geq 4 \), \( m(k) \geq k + 3 \), which is better bound for small values of \( k \), i.e. \( k \leq 10 \). The techniques used here are not applicable when \( k \leq 3 \), so this note does not imply \( \mathbb{E}^2 \to (\ell_2, \ell_5) \) or \( \mathbb{E}^3 \to (\ell_2, \ell_6) \).

For a detailed overview of other results in Euclidean Ramsey theory, see Erdős et al. \[4\] \[5\] \[6\] and Graham’s survey \[7\].

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2 Main result

Theorem 2.1. For an integer \( k \geq 4 \), \( \mathbb{E}^k \to (\ell_2, \ell_{k+3}) \).

The following notation and preliminary lemmas are needed to prove Theorem 2.1. Denote by \( \Delta^k \) any set of \( k + 1 \) points in \( \mathbb{E}^k \) such that distance between any two points in \( \Delta^k \) is equal to one. In other words, \( \Delta^k \) is a vertex set of a unit regular \( k \)-dimensional simplex in \( \mathbb{E}^k \).

Lemma 2.2. Let \( k \geq 4 \) and let the Euclidean space \( \mathbb{E}^{k-1} \) be coloured in red and blue so that there are no two red points distance 1 apart. Let \( S^{k-2} \) be a \((k-2)\)-dimensional sphere of radius \( \frac{\sqrt{2}}{2} \) with the centre at point \( O \). Then there is a copy of \( \Delta^{k-2} \subset S^{k-2} \) all points of which are blue.

Proof. Assume the contrary, namely that there is no blue \( \Delta^{k-2} \subset S^{k-2} \). Since all points in \( \Delta^{k-2} \) are distance one to each other, it is equivalent to assume that any \( \Delta^{k-2} \subset S^{k-2} \) contains exactly one red point. The following claim is the main part of the proof.

Claim. There is an angle \( \theta > 0 \), such that if \( A \) is red point on \( S^{k-2} \) and \( B \) is antipodal point to \( A \), then all points \( C \) on \( S^{k-2} \), such that \( \angle COB = \theta \), are red.

Proof of the Claim. Let \( A \) and \( B \) be antipodal points on \( S^{k-2} \) and let \( A \) be red. Let \( X \) be the set of points in \( S^{k-2} \) that are at distance 1 to \( A \). Then \( X \) is a \((k-3)\)-dimensional sphere with radius \( \frac{\sqrt{2}}{2} \). Let \( A_1, A_2, \ldots, A_{k-2} \in X \) be such that \( \{A, A_1, A_2, \ldots, A_{k-2}\} \) is a copy of \( \Delta^{k-2} \). Since any simplex \( \Delta^{k-2} \) contains exactly one red point and point \( A \) is red, all points \( A_1, A_2, \ldots, A_{k-2} \) are blue.

Let \( A_{k-1} \) be the point symmetric to \( A \) through the hyperplane \( \pi \) spanned by points \( A_1, A_2, \ldots, A_{k-2}, O \). The point \( A_{k-1} \) belongs to \( S^{k-2} \) and is red, since \( \{A_1, A_2, \ldots, A_{k-1}\} \) is a copy of \( \Delta^{k-2} \). Let \( \theta = \angle A_{k-1}OB \), then \( \theta > 0 \), because \( \pi \) does not contain \( X \). When the points \( A_1, A_2, \ldots, A_{k-2} \) are rotated in \( S^{k-2} \), the point \( A_{k-1} \) spans the set of all points \( C \in S^{k-2} \), such that \( \angle COB = \theta \). This concludes the proof of the claim.

Let \( A \) be a red point on \( S^{k-2} \) and let \( B \) be the antipodal point to \( A \) on \( S^{k-2} \).

Let \( S_A^{k-3} \subset S^{k-2} \) be the set of all points \( C \), such that \( \angle COB = \theta \). By the Claim, all points of \( S_A^{k-3} \) are red. For a point \( C \in S_A^{k-3} \) let \( C_1 \) be the antipodal point on \( S^{k-2} \). Let \( S_C^{k-3} \subset S^{k-2} \) be the set of points \( D \), such that \( \angle DOC = \theta \). By the Claim, the set \( S_C^{k-3} \) contains only red points. For a positive angle \( \phi \), define a "hypercap" \( HC_A(2\phi) = \{D \in S^{k-2} : \angle DOA \leq \phi\} \). When \( C \) is rotated in \( S_A^{k-3} \), red hyper-circles \( S_C^{k-3} \) span the red hypercap \( HC_A(2\theta) \).

The argument in last paragraph shows that if \( A \) is a red point, then \( HC_A(2\theta) \) is red. By reapplying this statement to any point in \( HC_A(2\theta) \), it can be proved that the set \( HC_A(4\theta) \) is red, the set \( HC_A(8\theta) \) is red, and eventually the whole sphere \( S^{k-2} \) is red. Hence, \( S^{k-2} \) contains two red points distance 1 apart, which contradicts the assumption that \( S^{k-2} \) does not contain a blue \( \Delta^{k-2} \).

For a positive integer \( n \), denote by \([n]\) the set of all positive integers \( i \leq n \).
Lemma 2.3. Let $\mathbb{E}^k$ be coloured in red and blue so that there is no red $\ell_2$. If there exists an integer $d$, $2 \leq d \leq k+1$, and two red points distance $d$ apart, then there exists a blue $\ell_{k+3}$.

Proof. Let $A_0$ and $A_d$ be two red points distance $d$ apart. Assume that $A_0 = (\frac{1}{2},0,\cdots,0)$ and $A_d = (d + \frac{1}{2},0\cdots,0)$.

For $0 \leq j \leq k + 2$ define

$$S^{k-2}_j = \{(j, x_2, \ldots, x_k) : x_2^2 + \cdots + x_k^2 = \frac{3}{4}\}.$$

Note that $S^{k-2}_0$ and $S^{k-2}_1$ contain only blue points, since any point in $S^{k-2}_0$ or $S^{k-2}_1$ is distance one to $A_0$. For the same reason, sets $S^{k-2}_d$ and $S^{k-2}_{d+1}$ contain only blue points. Let $i \in [k + 2]$ be a number not equal to $1, d$ or $d + 1$. By Lemma 2.2 applied to the hyperspace $x_1 = i$ and $S^{k-2}_i$, there is a blue $\Delta^{k-2} \subset S^{k-2}_i$. Let $\Delta^{k-2} = \{A_i^1, A_i^2, \ldots, A_i^{k-1}\}$. For all $0 \leq j \leq k + 2$ and $s \in [k - 1]$, define

$$A_s^j = A_s^i + (j - i, 0, 0, \cdots, 0).$$

Let $C = [k + 2]\{d, d+1, i\}$. For each $j \in C$, the set $\{A_s^1, \ldots, A_s^{k-1}\}$ is a copy of $\Delta^{k-2}$, and therefore contains at most one red point. Since there are $k - 2$ possible choices for $j \in C$ and there are $k - 1$ possible choices for $s \in [k - 1]$, there is an $s \in [k - 1]$, such that for all $j \in C$, point $A_s^j$ is blue. Hence, points $A_0^j, A_1^j, \ldots, A_{k+2}^j$ are all blue and form a blue $\ell_{k+3}$. \qed

Proof of Theorem 2.1. Assume the contrary, that there is a colouring of $\mathbb{E}^k$ in red and blue, such that there is neither red $\ell_2$, nor blue $\ell_{k+3}$.

According to Lemma 2.3 there are no two red points distance $1, 2, \cdots, k+1$ apart. Let $A$ be a red point. Then for all $j \in [k+1]$, the sphere

$$S^{k-1}(j) = \{X \in \mathbb{E}^k : |XA| = j\}$$

contains only blue points. Let $S^{k-1}(k+2) = \{X \in \mathbb{E}^k : |XA| = k + 2\}$ and $S^{k-1}(k+3) = \{X \in \mathbb{E}^k : |XA| = k + 3\}$. There are two cases to consider.

If $S^{k-1}(k+2)$ contains only blue points, let $P_1$ and $P_2$ be two points on $S^{k-1}(k+2)$, such that $|P_1P_2| = \frac{k+2}{2}$. If $S^{k-1}(k+2)$ contains a red point $B$, let $P_1$ and $P_2$ be two points on $S^{k-1}(k+2)$, such that $|P_1P_2| = \frac{k+2}{4}$ and $|BP_1| = |BP_2| = 1$. In any case, both $P_1$ and $P_2$ are blue.

Let the lines $AP_1$ and $AP_2$ intersect hypersphere $S^{k-1}(k+3)$ at points $Q_1$ and $Q_2$ respectively. Then, $|Q_1Q_2| = 1$, so one of the points, say, $Q_1$, is blue. For all $j \in [k+3]$ the line $AQ_1$ intersects the sphere $S^{k-1}_j$ at a blue point, so the points of intersections form a blue $\ell_{k+3}$. \qed

3 Concluding remarks

The result of Conlon and Fox [2] (as well as the result of this note) implies that for any $k$, there is $n$ such that $\mathbb{E}^n \rightarrow (\ell_2, \ell_k)$. One of the results of Erdős
et al. [4] implies that for all \( n \), \( \mathbb{E}^n \not\rightarrow (\ell_6, \ell_6) \). This motivates the following question: what is the minimal \( s \) such that there exists \( k \) such that for all \( n \), \( \mathbb{E}^n \not\rightarrow (\ell_s, \ell_k) \)? We conjecture that \( s = 3 \):

**Conjecture 1.** There is an integer \( k \), such that for every integer \( n \)

\[ \mathbb{E}^n \not\rightarrow (\ell_3, \ell_k) \]

During the preparation of this note, the paper of Conlon and Fox [2] appeared, where the authors made a similar conjecture.

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