An extension of the Derrida–Lebowitz–Speer–Spohn equation

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Received 19 April 2015, revised 17 September 2015
Accepted for publication 5 October 2015
Published 6 November 2015

Abstract
We show how the derivation of the Derrida–Lebowitz–Speer–Spohn equation can be prolonged to obtain a new equation, generalizing the models obtained in the paper by these authors. We then investigate its properties from both an analytical and numerical perspective. Specifically, a numerical method is presented to approximate solutions of the prolonged equation. Using this method, we investigate the relationship between the solutions of the prolonged equation and the Tracy–Widom GOE distribution.

Keywords: nonlinear diffusion, Derrida–Lebowitz–Speer–Spohn equation, Tracy–Widom distribution, KPZ universality
Mathematics Subject Classification: 35K35, 35K55, 82C22, 60K35

(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. The physics

In [10], Derrida, Lebowitz, Speer and Spohn proposed a simplified model to describe the low temperature Glauber dynamics of the North–East model in the presence of two phases with an anchored interface. It can be described as a Markov process \( \eta(t) \) on \( \{-1, 1\}^n \). The model has two parameters, \( \lambda_+, \lambda_- > 0 \). Informally, at time \( t \geq 0 \), each site \( x \in \{1, \ldots, n\} \) has an
The magnetization and its asymptotic behaviour

In [10], the authors are mainly interested by the stationary magnetization. It is the random variable

\[ M_n = \sum_{x=1}^{n} \eta_x, \]

where \( \eta \) has the invariant distribution of the Markov process. The variable can also be easily deduced from the particle system \( X \) in stationary regime
For the remainder of this paper, we set
\[ C = \frac{\sqrt{\lambda_-} - \sqrt{\lambda_+}}{\sqrt{\lambda_-} + \sqrt{\lambda_+}} \text{ and } n = \frac{\sqrt{\lambda_- \lambda_+}}{(\sqrt{\lambda_-} + \sqrt{\lambda_+})^2}. \]

(observe that \( 0 \leq C \leq \frac{1}{4} \) and \(-1 \leq \mu \leq 1\). The case \( \mu = 0 \) is called the unbiased (or symmetric) case. The case \( \mu \neq 0 \), the biased (or asymmetric) case. In [10], based on a non-rigorous approximation, the authors conjecture that if \( \mu = 0 \), a central limit theorem holds for \( M_n \), for any \( x \in \mathbb{R} \),
\[ \lim_{n \to \infty} \mathbb{P} \left( \frac{M_n}{((3/2)n)^{1/4}} \geq x \right) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{s^2}{2}} ds. \]

In the unbiased case, \( \mu = 0 \), [10] conjectures that a different scaling and weak limit appear. Namely, for all \( x \in \mathbb{R} \),
\[ \lim_{n \to \infty} \mathbb{P} \left( \frac{M_n - \mu n}{(\mu C n)^{1/3}} \geq x \right) = Q([x, \infty)), \]
for some probability measure \( Q \) on \( \mathbb{R} \) independent of \( (\lambda_+, \lambda_-) \). In the unbiased case, the scaling of the variance as \( \sqrt{n} \) suggests that the unbiased process falls into the universality class of Edwards–Wilkinson. In the biased case, the scaling \( n^{1/3} \) suggests the KPZ universality class, for a recent survey on the latter see [8].

1.3. A new PDE governing the equilibrium measure

In [10], the authors derive non-rigorously a PDE associated to the rescaled marginal of \( M_n \). In the present paper, we will revisit the computation of [10] and obtain a generalization of their PDE. To be more precise, we introduce the functions
\[
\begin{align*}
H_\mu(m) &= \mathbb{E} \left[ \eta_{n+1} | M_n = m \right], \\
U^\pm_\mu(m) &= \lambda_\pm \mathbb{E} \left[ K_n^\pm 1(M_n = m) \right], \\
W_\mu(m) &= \mathbb{P} (M_n = m),
\end{align*}
\]
where \( \mathbb{E} \) denotes the expectation with respect to the stationary measure \( \mathbb{P} \), \( I \) is the indicator function and \( K_n^\pm \) is the number of successive left neighbors of site \( n \) (including \( n \)) such that \( \eta_n = \pm 1 \). The stationarity of the process implies that \( U^\pm_\mu(m) = U^\pm_\mu(m) \) ([10, equation (4.18)]). We set their common value to be \( U(m) \).

We adopt the ansatz
\[
\begin{align*}
H_\mu(m) &= \mu + h \left( \varepsilon^3 n, \varepsilon (m - \mu n) \right), \\
U_\mu(m) &= u \left( \varepsilon^3 n, \varepsilon (m - \mu n) \right), \\
W_\mu(m) &= w \left( \varepsilon^3 n, \varepsilon (m - \mu n) \right).
\end{align*}
\]

Observe that the scaling between \( n \) and \( m \) is consistent with (1.2). Under a non-rigorous approximation, we show that under this scaling, as \( \varepsilon \to 0 \), \( w \) is governed by the following partial differential equation:
\[
\partial_t w - \mu C \left( \frac{4}{3} \partial^3 w - \partial_t \left( \frac{\partial_t w}{w} \right) \right) = \varepsilon \left( 2C^2 - 2C \right) \left( \partial^4 w - \partial_t^2 \left( \frac{\partial_t w}{w} \right) \right) \quad (1.3)
\]

(notice that \( 0 \leq C - 2C^2 \leq \frac{1}{4} \), which can also be written
\[
\partial_t w - \mu C \partial_x \left( w^{3/4} \partial_x^2 w^{1/4} \right) = \varepsilon \left( 2C^2 - 2C \right) \partial_x^2 \left( w \partial_x \log w \right).
\]

For \( \mu = 0 \), the leading term vanish and the presence of \( \varepsilon \) agrees with (1.1). The equation for \( v \) such that \( w = v^2 \) is somewhat simpler: it reads
\[
\partial_t v - \frac{4}{3} \mu C \partial_x^3 v = \varepsilon \left( 2C^2 - 2C \right) \left( \partial_x^4 v - \frac{\left( \partial_x^2 v \right)^2}{v} \right) \quad (1.4)
\]

or equivalently
\[
\partial_t v - \frac{4}{3} \mu C \partial_x^3 v = \varepsilon \left( 2C^2 - 2C \right) \frac{1}{v} \partial_x \left( v^2 \partial_x \left( \frac{\partial_x^3 v}{v} \right) \right).
\]

The equations (1.3) and (1.4) generalize the PDEs found in [10], which correspond to the cases \( \mu = 0 \) and \( C = \frac{1}{4} \), or \( \mu \neq 0 \) and \( \varepsilon = 0 \).

The equation obtained in the unbiased case \( \mu = 0 \), \( C = \frac{1}{4} \) has been the subject of intensive research in the PDE community, see [3, 7, 11, 13–15, 17–19, 22] and the references therein; we will come back to the results obtained in these papers in section 3.

In the biased case \( \mu \neq 0 \), the equation derived in the present paper, namely (1.4), corresponds to adding a right-hand side to the equation derived in [10]. This seems to reflect an important correction, since it has a dissipative behaviour, thus potentially giving (up to rescaling!) a trend towards a universal profile as \( t \to \infty \). We could not identify this profile in the case where the equation is set in \( \mathbb{R} \), which is the most interesting one, but it should be related to the crucial question of the asymptotic invariant measure of the random process under consideration.
1.4. A numerical investigation

We also perform a detailed numerical investigation of the solutions of (1.3). See figure 1 for an example solution. The numerical solution of (1.3) is difficult from two points of view.

1. The nonlinearities in (1.3) are singular so that the equation must be rewritten for numerical purposes.
2. If \( w > 0 \) initially, then we argue below that \( w \) should be positive for all time. The third-order linear term in (1.3) works to make the function vanish while the nonlinearities prevent this. There is a strong, non-trivial coupling between these terms and split-step methods, which are standard in the numerical solution of nonlinear dispersive equations, cannot be used. We are able to overcome these issues and simulate (1.3) for moderate times with a stable, highly accurate pseudospectral scheme.

We provide evidence that

\[
w(t, x) \sim \frac{1}{t^{1/3}} f \left( \frac{x}{t^{1/3}} \right),
\]

for some, yet unknown, function \( f \). The authors in [2] use Monte Carlo simulations to approximate the asymptotic equilibrium measure for the random process we consider. They show that it is approximated well with the Tracy–Widom (\( \beta = 1 \)) GOE distribution [25]. More precisely, they find that with their conjectured scaling and \( n = 100,000 \) there is a 7% error in the mean, 0.9% error in the variance and a supremum norm error on the order of 0.02 when compared with the Tracy–Widom GOE distribution.

Due to the high accuracy of our numerical method, and the fact that \( f \) in (1.5) is, in our computations, distinct from the Tracy–Widom GOE density, we raise the question of whether the asymptotic equilibrium measure for the random process could be something other than the Tracy–Widom GOE distribution.

1.5. An instructive analogy: the sum of independent random variables

In order to better understand the computations which will follow, let us start with the simple example of independent and identically distributed (iid) variables. Assume that \( (\eta_k)_{k \geq 1} \) is a sequence of iid variables on \( \{-1, 1\} \) with \( \nu = \mathbb{E}\eta_k \), i.e. \( \mathbb{P}(\eta_k = 1) = 1 - \mathbb{P}(\eta_k = -1) = (1 + \nu)/2 \). We set

\[
S_n = \sum_{k=1}^n \eta_k
\]

and \( P_n(m) = \mathbb{P}(S_n = m) \). We have the recursion

\[
P_{n+1}(m) = \frac{1 + \nu}{2} P_n(m - 1) + \frac{1 - \nu}{2} P_n(m + 1).
\]

The convergence of a properly rescaled version of \( P_n \) to the heat equation could be obtained heuristically as follows. For any \( \varepsilon > 0 \), we may define a function \( p_\varepsilon(t, x) \)
such that \( P_m(m) = p_\varepsilon(\varepsilon^2 n, \varepsilon(m - \nu n)) \). We may rewrite (1.6) as, if \( t = \varepsilon^2 n \) and \( x = \varepsilon(m - \nu n) \),
\[
p_m(t + \varepsilon^2, x - \nu\varepsilon) = \frac{1 + \nu}{2} p_m(t, x - \varepsilon) + \frac{1 - \nu}{2} p_m(t, x + \varepsilon).
\]

We now let \( \varepsilon \to 0 \). The central limit theorem implies notably that \( p_\varepsilon / \varepsilon \) converges to a probability density function \( p \). We expand in powers of \( \varepsilon \) the above identity. The first non-zero term is in \( \varepsilon^2 \), it gives the PDE
\[
\partial_t p = \theta \partial_x^2 p.
\]

With \( \theta = \left( 1 - \frac{\nu^2}{2} \right) \). We recognize the heat equation in one dimension. Also, for any \( \varepsilon, \varepsilon' > 0 \), we have that \( P_n(m) = p_\varepsilon(\varepsilon^2 n, \varepsilon(m - \nu n)) = p_{\varepsilon'}(\varepsilon'^2 n, \varepsilon'(m - \nu n)) \). Hence, for any \( s > 0 \), if we consider the case \( \varepsilon' = s \varepsilon \) and \( \varepsilon \to 0 \), we deduce that the probabilistically relevant solution of the PDE should also satisfy, \( p(t, x) = sp(s^2 t, sx) \). In other words, they should be of the form
\[
p(t, x) = g(x/\sqrt{t})/\sqrt{t},
\]
for a probability density function \( g \). It follows that \( g \) satisfies an ODE which we can of course explicitly solve in this simple case and retrieve the Gaussian density. As in [10], for the DLSS Markov process, we will follow a similar strategy.

1.6. Plan of the paper

The non-rigorous derivation of (1.3) and (1.4) is presented in section 2, while some properties of these equations are analyzed formally in section 3. Finally, we present a method for the numerical solution of (1.3) and a detailed analysis of the approximate solutions in section 4.

2. Derivation of the extended DLSS equation

2.1. Outline of the derivation

We will use the equations
\[
W_{n+1}(m) = \frac{1}{2}(1 + H_0(m - 1))W_n(m - 1) + \frac{1}{2}(1 - H_0(m + 1))W_n(m + 1),
\]
\[
H_0(m) = \frac{U_0(m + 1) - U_0(m - 1) + (\lambda_- - \lambda_+)W_0(m)}{U_0(m + 1) + (\lambda_- + \lambda_+)W_0(m),}
\]
\[
U_0(m) = \frac{\lambda_- W_0(m) + U_0(m + 1)(\lambda_+ W_0(m) + U_0(m - 1))}{U_0(m + 1) + (\lambda_- + \lambda_+)W_0(m) + U_0(m - 1)}.
\]

Which appear in [10] as (6.4), (6.9), and (6.10) respectively. These equations rely on the simplifying approximation that, given \( M_n, \eta_{h+1} \) is (approximately) independent of \( K^n_{h+1} \). This could be justified heuristically by observing that the DLSS Markov process now defined on \([-1, 1]^{n/2} \) instead of \([-1, 1]^n \) preserves the number of + and - sites and, given the number of + and - sites, the invariant probability measure is the uniform measure (see [10]). Hence, we may expect that when \( n \) and \( m \) are large, \( P(\eta_{h+1} = a; -k \leq \ell \leq k | M_n = m) \) could be approximated by
\[
\prod_{-k \leq \ell \leq k} P(\eta_{h+1} = a; -k \leq \ell \leq k | M_n = m).
\]

Our plan is now as follows
(1) Expand \( u \) in powers of \( \varepsilon \), with coefficients depending on \( w \) (section 2.2).
(2) Expand \( wh \) in powers of \( \varepsilon \), with coefficients depending on \( w \) (section 2.3).
(3) Find the equation satisfied by \( w \) (section 2.4).

2.2. Expansion of \( u \) in \( \varepsilon \)

We start with the ansatz
\[
    u = \alpha + \varepsilon \beta + \varepsilon^2 \gamma + \varepsilon^3 \delta
\]
and aim at determining \( \alpha, \beta, \gamma \) and \( \delta \) as functions of \( w \). First, expanding the left-hand side of (2.3) to order 3 gives
\[
    \text{lhs}(2.3) = \alpha + \varepsilon(\beta - \mu \partial_x \alpha) + \varepsilon^2 \left( \gamma - \mu \partial_x \beta + \frac{1}{2} \mu^2 \partial_x^2 \alpha \right)
    + \varepsilon^3 \left( \delta - \partial_x \alpha - \mu \partial_x \gamma + \frac{1}{2} \mu^2 \partial_x^2 \beta - \frac{1}{6} \mu^3 \partial_x^3 \alpha \right) + O(\varepsilon^4)
\]
(2.4)
while expanding the right-hand side of (2.3) to order 1 yields
\[
    \text{rhs}(2.3) = \frac{(\lambda_+ w + \alpha + \varepsilon \beta + \varepsilon \partial_x \alpha)(\lambda_+ w + \alpha + \varepsilon \beta - \varepsilon \partial_x \alpha)}{2\alpha + 2\varepsilon \beta + (\lambda_+ + \lambda_-)w} + O(\varepsilon^2)
\]
\[
= \frac{(\lambda_+ w + \alpha)(\lambda_+ w + \alpha)}{2\alpha + (\lambda_+ + \lambda_-)w}
+ \varepsilon \left[ \frac{\beta(2\alpha + (\lambda_+ + \lambda_-)w) + \partial_x \alpha(\lambda_+ - \lambda_-)w}{2\alpha + (\lambda_+ + \lambda_-)w} - \frac{2\beta(\lambda_+ w + \alpha)(\lambda_+ w + \alpha)}{(2\alpha + (\lambda_+ + \lambda_-)w)^2} \right] + O(\varepsilon^2).
\]

Identifying terms of order 0 and 1 in \( \varepsilon \) in the left- and right-hand sides of (2.3) leads to
\[
    \alpha = \sqrt{\lambda_+ \lambda_-}w \quad \text{and} \quad \beta = 0.
\]

Next, expand the right-hand side of (2.3) to order 3 in \( \varepsilon \), taking advantage of the fact that \( \beta = 0 \). This gives
\[
    \text{rhs}(2.3) = \frac{(\lambda_+ w + \alpha + \varepsilon^2 \gamma + \varepsilon^3 \delta + \varepsilon \partial_x \alpha + \varepsilon^3 \partial_x \gamma + \frac{\varepsilon^2}{2} \partial_x^2 \alpha + \frac{\varepsilon^3}{6} \partial_x^3 \alpha)}{2\alpha + 2\varepsilon^2 \gamma + 2\varepsilon^3 \delta + \varepsilon^3 \partial_x \alpha + \frac{\varepsilon^2}{2} \partial_x^2 \alpha - \frac{\varepsilon^3}{6} \partial_x^3 \alpha}
\]
\[
\times \frac{1}{2\alpha + 2\varepsilon^2 \gamma + 2\varepsilon^3 \delta + \varepsilon^3 \partial_x \alpha + (\lambda_+ + \lambda_-)w} + O(\varepsilon^4)
\]
\[
= A + B\varepsilon + C\varepsilon^2 + D\varepsilon^3 + O(\varepsilon^4),
\]
where $A$ and $B$ have already been determined, and

$$
C = -\left(\frac{\partial_\alpha^2}{2} + \gamma \left( (\lambda_+ + \lambda_-)w + 2\alpha \right) + \partial_\alpha^2 \left( \alpha + \frac{1}{2} (\lambda_+ + \lambda_-)w \right) \right)
$$

$$
= \frac{2\gamma + \partial_\alpha^2}{2\alpha + (\lambda_+ + \lambda_-)w} \left( \lambda_+ w + \alpha \right) \left( \lambda_- w + \alpha \right)
$$

$$
D = \delta - \frac{2\delta}{2\alpha + (\lambda_+ + \lambda_-)w} \left( \lambda_+ w + \alpha \right) \left( \lambda_- w + \alpha \right)
$$

$$
\quad - \frac{\partial_\alpha \left( \lambda_+ - \lambda_- \right)w \left( 2\gamma + \partial_\alpha^2 \right)}{2\alpha + (\lambda_+ + \lambda_-)w}
$$

$$
\quad + \frac{\left( \partial_\alpha \gamma \right) + \frac{1}{6} \partial_\alpha^2}{2\alpha + (\lambda_+ + \lambda_-)w} \left( \lambda_+ - \lambda_- \right)w
$$

Using the equality $\alpha = \sqrt{\lambda_+ \lambda_-}w$ as well as the definitions of $C$ and $\mu$ leads to the more simple formulas

$$
C = \sqrt{\frac{\lambda_+ \lambda_-}{2}} \left( \frac{1}{2} - C \right) \partial_\alpha^2 w - \sqrt{\lambda_+ \lambda_-} C \frac{(\partial_\alpha w)^2}{w} + (1 - 2C) \gamma
$$

$$
D = (1 - 2C) \delta + \mu C \sqrt{\frac{\lambda_+ \lambda_-}{2}} \frac{(\partial_\alpha w)^2}{w} - \mu \partial_\alpha \gamma - \frac{\mu}{6} \partial_\alpha^2 \alpha
$$

and identifying the terms of order 2 in $\varepsilon$ in the left- and right-hand sides of (2.3) gives, respectively

$$
\gamma = \sqrt{\frac{\lambda_+ \lambda_-}{2}} \left( \partial_\alpha^2 w - \frac{(\partial_\alpha w)^2}{w} \right)
$$

$$
\delta = \sqrt{\frac{\lambda_+ \lambda_-}{2}} \left( -\frac{1}{2C} \partial_\alpha w - \frac{\mu}{3} \partial_\alpha^3 w + \frac{\partial_\alpha w \partial_\alpha^2 w}{w} - \frac{\mu}{2} \frac{(\partial_\alpha w)^3}{w^2} \right)
$$

2.3. Expansion of $wh$ in $\varepsilon$

Expanding the right-hand side of (2.2) to order 3 in $\varepsilon$ gives

$$
\mu + h = \frac{(\lambda_- - \lambda_+)w + 2\varepsilon \partial_\alpha \alpha + 2\varepsilon^2 \partial_\gamma + \frac{1}{3} \varepsilon^3 \partial_\gamma^3 + \frac{1}{2} \varepsilon^2 \partial_\alpha^2 \alpha + (\lambda_+ + \lambda_-)w + O(\varepsilon^4)}{2\alpha + \varepsilon \gamma + 2\varepsilon^2 \delta + \varepsilon^2 \partial_\gamma \alpha + (\lambda_+ + \lambda_-)w + O(\varepsilon^4)}
$$

or, after replacing $\alpha, \gamma$ and $\delta$ by the formulas derived above

$$
wh = \varepsilon 2\partial_\alpha w + \varepsilon^2 \mu C \left( -2\partial_\alpha^2 w + \frac{(\partial_\alpha w)^2}{w} \right)
$$

$$
+ \varepsilon^3 \left( \mu \partial_\alpha w + C \left( \frac{4}{3} \partial_\alpha \alpha + 2\varepsilon \partial_\gamma + \frac{1}{3} \varepsilon^3 \partial_\gamma^3 + \frac{1}{2} \varepsilon^2 \partial_\gamma \partial_\alpha^2 \alpha + (\lambda_+ + \lambda_-)w + O(\varepsilon^4) \right) \right)
$$

(2.5)
2.4. Equation satisfied by $w$

The left-hand side of (2.1) reads, to order 4 in $\varepsilon$,
\[
\text{lhs}(2.1) = w + \varepsilon^3 \partial_t w - \mu \varepsilon \partial_t w + \frac{1}{2} \mu^2 \varepsilon^2 \partial_t^2 w - \frac{1}{6} \mu^3 \varepsilon^3 \partial_t^3 w + \frac{1}{24} \mu^4 \varepsilon^4 \partial_t^4 w - \mu \varepsilon \partial_t w + O(\varepsilon^5),
\]
while the right-hand side of (2.1) can be expanded as
\[
\text{rhs}(2.1) = w + \frac{\varepsilon^2}{2} \partial_t^2 w + \frac{\varepsilon^4}{24} \partial_t^4 w - \mu \varepsilon \partial_t w - \frac{\varepsilon^3}{6} \partial_t^3 w - \frac{\varepsilon^3}{6} \partial_t^3 (hw) + O(\varepsilon^5),
\]
which, with the help of (2.5), gives
\[
\text{rhs}(2.1) = w - \varepsilon \mu \partial_t w + \varepsilon^2 \left( \frac{1}{2} - 2C \right) \partial_t^2 w + \varepsilon^3 \left( -\frac{\mu}{6} + 2\mu C \right) \partial_t^3 w - \frac{1}{2} \mu^2 C \partial_t \left( \frac{(\partial_t w)^2}{w} \right) + \varepsilon^4 \left( -\frac{2}{3} \mu^2 C - \frac{5C}{3} + \frac{1}{24} \right) \partial_t^4 w - \mu \partial_t \partial_t w + C \left( 1 + \mu^2 + 2C \right) \partial_t \left( \frac{(\partial_t w)^2}{w} \right) + O(\varepsilon^5).
\]
Equating terms of order 4 and 5 on the left- and right-hand sides of (2.1) yields
\[
\partial_t w - \frac{4}{3} \mu C \partial_t^3 w + \mu C \partial_t \left( \frac{(\partial_t w)^2}{w} \right) = \varepsilon \left( 2C^2 - 2C \right) \partial_t^2 w - \frac{1}{2} \mu^2 C \partial_t \left( \frac{(\partial_t w)^2}{w} \right),
\]
which is the desired result.

3. A few properties of the equation

Our aim in this section is to establish heuristically some properties of (1.3) and (1.4) without developing a full theory. A rigorous proof of some of these heuristic properties seems out of reach for the moment, even for the simpler DLSS equation, for which important questions—in particular regarding zeros of $v$ or $w$—remain open in spite of intensive research.

In order to alleviate the notations, we denote in the following
\[
K = \frac{4}{3} \mu C \quad \text{and} \quad L = \varepsilon \left( 2C - 2C^2 \right)
\]
(notice that $L \geq 0$). We let the independent variables $(t, x)$ range over $\mathbb{R}_+ \times \mathbb{T}$ or $\mathbb{R}_+ \times \mathbb{R}$, the second case being physically more relevant.
The equation on \( w \) (taking values in \( \mathbb{R}_+ \)) reads now
\[
\partial_t w - K \left( \partial_t^3 w - \frac{3}{4} \partial_t \left( \frac{(\partial_t w)^2}{w} \right) \right) = -L \left( \partial_t^4 w - \partial_t \left( \frac{\partial_t^2 v}{v} \right) \right)
\] (3.1)
while the equation on \( v \) (taking values in \( \mathbb{R} \)) is given by
\[
\partial_t v - K \partial_t^3 v = -L \left( \partial_t^4 v - \frac{(\partial_t^2 v)^2}{v} \right).
\] (3.2)

### 3.1. Questions of sign

The physically relevant quantity is \( w \) (density of probability), which is non-negative. This allows to define (possibly ambiguously, in the presence of zeros) \( v \) by \( w = v^2 \).

Physical, non-vanishing solutions. For \( K = 0, L > 0 \), it is expected that solutions \( w \) become positive for positive time, and the same should hold for \( K = 0, L > 0 \).

Heuristically, this can be understood through the following argument, which was pointed out to us by Percy Deift. Assume that \( L > 0 \), \( v(t = 0) > 0 \), and assume that \( v \) vanishes at a later time, say at \( (t_0, x_0) \). Generically, it happens in such a way that \( \partial_t^2 v(t_0, x_0) > 0 \). But then, as \( t \to t_0 \), \( \frac{(\partial_t^2 v)^2}{v} \to \infty \), which implies \( \partial_t v(t_0, x_0) = \infty \), which contradicts the vanishing of \( v \) at \( (t_0, x_0) \).

Non-physical, vanishing solutions. Though physical solutions, as argued above, become positive for positive time, there exists more exotic solutions which have zeros for positive time. These more exotic solutions have been considered since the beginning of the theory, see [3, 10].

### 3.2. Symmetries

Space or time translations of course leave the equation invariant. A more interesting symmetry is given by
\[
w \mapsto \lambda w \quad \text{and} \quad v \mapsto \lambda v
\]
(where \( \lambda \) is non-negative). However, the probabilistic interpretation of the equation requires that \( w \) be the density of a probability measure, making its multiplication by a non-negative number physically irrelevant.

For \( K = 0 \) or \( L = 0 \), the equation has a scaling symmetry \( v \mapsto v(\lambda_1 t, \lambda x) \) and \( v \mapsto v(\lambda t, \lambda x) \) respectively, which is lost for general \( K \) and \( L \). The gradient flow structure noticed and exploited in [15, 22] for \( K = 0 \) is also lost if \( K \neq 0 \).

### 3.3. Lyapunov functions

On the one hand, it was first noticed in [3] that quantities of the type \( \int \lvert w \rvert^4 \) or \( \int \lvert (w^\beta) \rvert^2 \) are monotonic for solutions of (3.1) if \( K = 0 \). The range of \( \beta \) was later extended in [17]. On the other hand, (3.1) is simply Airy’s equation if \( L = 0 \), for which conserved quantities are well-known: \( \int v \) and all the \( L^2 \)-based Sobolev norms \( \int \lvert \partial_t^4 v \rvert^2 \). It is not surprising that Lyapunov functions for the general case \( K, L \neq 0 \) correspond to these quantities which are invariant or monotonic both if \( K = 0 \) and \( L = 0 \):
• The ‘mass’ \( \int w \, dx = \int v^2 \, dx \) of \( w \) is conserved: \( \frac{d}{dt} \int w \, dx = 0 \) (since \( w \) models a density of probability, the physical interpretation is clear).

• The ‘momentum’ \( \int \sqrt{w} \, dx = \int v \, dx \) of \( v \) is increasing: \( \frac{d}{dt} \int v \, dx = L \int \frac{(\partial_x v)^2}{v} \, dx \).

• The ‘Fisher information’ \( \int \frac{(\partial_x w)^2}{w} \, dx = \int (\partial_x v)^2 \, dx \) of \( w \) is decreasing:

\[
\frac{d}{dt} \int v_x^2 \, dx = L \int \partial_x^2 v \left( \partial_x v - \frac{(\partial_x v)^2}{v} \right) \, dx = -L \int \left( \partial_x^2 v - \frac{\partial_x^2 v \partial_x v}{v} \right)^2 \, dx.
\]

• Finally, the ‘mean’ \( \int wx \, dx = \int v^2 x \, dx \) varies according to

\[
\frac{d}{dt} \int v^2 x \, dx = \frac{3}{2} K \int (\partial_x v)^2 \, dx
\]

(so that \( \frac{d^2}{dt^2} \int v^2 x \, dx \) has the same sign as \( -K \), indicating the direction in which \( v \) has a tendency to drift).

3.4. Exact solutions

For \( K = 0 \), examples of exact solutions were first given in [3]; we show how some of these examples can be extended to the case \( K \neq 0 \).

Unfortunately, the physical relevance of these solutions is questionable, for two reasons: they have zeros for \( t > 0 \), which is not expected for physical solutions; and, when considered on \( \mathbb{R} \), correspond to \( w \) of infinite mass \( \int w = \infty \). Therefore, these explicit but exotic solutions probably do not say much about the large time behaviour of physical solutions. Nevertheless, we hope that they shed some light on the dynamics.

• First, it is immediate to check that \( v = \sin(x - Kt) \) is an exact traveling wave solution, which becomes stationary if \( K = 0 \).

• Similarly for \( v = \sinh(x + Kt) \).

• Next, it was already noticed that the Airy functions \( \text{Ai}(x) = \int_0^\infty \cos(tx + t^3/3) \, dt \) gives, if \( K = 0 \), the traveling wave \( v = \text{Ai}(x - 2Lt) \). It is also well-known that

\[
\frac{1}{t^{1/3}} \text{Ai} \left( \frac{x}{(-3Kt)^{1/3}} \right)
\]

is a solution of the Airy equation obtained if \( L = 0 \). For \( K, L \neq 0 \), we were able to find an exact solution based on the Airy function:

\[
v(t, x) = \frac{1}{t^{1/3}} \text{Ai} \left( \frac{x + 2L}{3K} \log t \right)
\]

(this formula can be checked directly using the fact that the Airy function \( \text{Ai} \) solves the ODE \( y'' = xy \)).

• This remains true if \( \text{Ai} \) is replaced by the Airy function of the second kind often denoted \( \text{Bi} \).
3.5. Asymptotic behaviour

3.5.1. The case $x \in T$. Without loss of generality, we assume here that $w_1$. For $L = 0$, there is no trend to equilibrium, and $w$ oscillates indefinitely. For $K = 0$, it was proved in [7, 11] that $w$ converges exponentially fast to the constant $w \equiv 1$ (see also [15] for a much more general framework). This remains true for $K, L \neq 0$: we claim that there exists $\mu > 0$ such that the Fisher information of a solution $w$ of (3.1) satisfies for $t \geq 0$

$$
\int \left| \partial_t \sqrt{w(t)} \right|^2 \mathrm{d}x \leq \left( \int \left| \partial_t \sqrt{w(t = 0)} \right|^2 \mathrm{d}x \right) e^{-\mu L t}
$$

Indeed, a small computation gives

$$
\frac{d}{dt} \int |\partial_t \sqrt{w(t)}|^2 \mathrm{d}x = L \int \left( \partial^2_t \left( \frac{\partial^2_w}{w} \right) \right) \left( 2 \frac{\partial^2_w}{w} - \left( \frac{\partial_w}{w} \right)^2 \right) \mathrm{d}x.
$$

The inequality (3.3) in [11] gives a majorization of the above right-hand side by $-\mu L \int |\partial_t \sqrt{w}|^2 \mathrm{d}x$, for a constant $\mu > 0$. This leads to the differential inequality

$$
\frac{d}{dt} \int |\partial_t \sqrt{w(t)}|^2 \mathrm{d}x \leq -\mu L \int |\partial_t \sqrt{w(t)}|^2 \mathrm{d}x,
$$

from which the desired result follows by Poincaré’s inequality.

3.5.2. The case $x \in \mathbb{R}, K = 0$. Still under the assumption that $\int w = 1$, it was established in [22] that the solution $w$ of (3.1) converges to a Gaussian:

$$
w \sim \frac{1}{\sqrt{\pi t^{1/4}}} e^{-\frac{x^2}{4t}} \quad \text{as } t \to \infty. \quad (3.3)
$$

3.5.3. The case $x \in \mathbb{R}, L = 0$. If $L = 0$, $v$ is simply a solution of the Airy equation, for which the asymptotics are

$$
v(t, x) \sim \frac{1}{t^{1/3}} \Re \left[ \hat{A}(\frac{x}{(3Kt)^{1/3}}) F\left( \frac{x}{t} \right) \right] \quad \text{as } t \to \infty,
$$

where $F$ is complex-valued and can be expressed in terms of the Fourier transform of the initial data, while the modified Airy function $\hat{A}$ is given by

$$
\hat{A}(z) = \int_0^\infty e^{i4\xi} + i^3 \frac{e^2}{3} \mathrm{d}\xi
$$

(this is classical, see for instance [16], equation (2.3)).

3.5.4. The case $x \in \mathbb{R}, K, L \neq 0$. This is the most interesting case, but it seems very difficult to analyze. It is argued heuristically in [2] that the invariant law for the random process that (3.2) is supposed to model should be given by a rescaling of the Tracy–Widom distribution $F_1$. This prediction is then confirmed numerically. For the equation (3.2), this line of reasoning would lead one to conjecture the following asymptotics:

$$
w(t, x) \sim \frac{2}{(6Kt)^{1/3}} F_1\left( \frac{2x}{(6Kt)^{1/3}} \right) \quad \text{as } t \to \infty.
$$

In section 4 we provide evidence that $F_1$ is not the correct candidate.
4. Moderate-time numerical simulation

Here we develop a stable numerical scheme to approximate solutions of the initial-value problem of (1.3). We investigate the limiting form of solutions for moderate times. We write the equation (1.4) for \( v = \sqrt{w} \) assuming \( w > 0 \) so that

\[
v = F(v) = F\left(\sqrt{w}\right),
\]

\[
w = 2\sqrt{w}F\left(\sqrt{w}\right) = \frac{8}{3}\mu C\sqrt{w}\partial^4_{\tau}\sqrt{w} + 2\varepsilon\left(2C^2 - 2C\right)\left(\sqrt{w}\partial^2_{\tau}\sqrt{w} - \left(\partial^2_{\tau}\sqrt{w}\right)^2\right).
\]

(4.1)

In this way of writing the equation, we have no division operations. While some issues could persist from performing the square-root, the smoothness and exponential decay of the solution make this operation accurate. We employ a standard technique to compute solutions of (4.1). Let \( \ell > 0 \) and consider (4.1) on the periodic interval \((\ell, \ell]\) with initial data \(w_0(x)\). We choose \(w_0(x)\) to be an exponentially decaying function defined on \(\mathbb{R}\) and \(\ell > 0\) sufficiently large so that \(|w_0(x)|\) is less than, say, \(10^{-16}\) outside \((-\ell, \ell]\). We also have to choose \(\ell\) sufficiently large so that the approximate solution remains zero (or approximately zero) near the boundary points \(\pm\ell\) for the largest \(t\) used in the computation.

From here, the problem fits into the classical theory for the numerical solution of time-dependent problems, see [6, section 9.6]. One uses the pseudospectral differentiation operator \(D_{n,\ell}\) to approximate the derivatives in the right-hand side. More precisely, the operator is described by the following schematic (FFT stands for the Fast Fourier transform):

\[
\begin{align*}
2^n \text{ sample points} & \rightarrow \left(f(x_1, \ell), f(x_2, \ell), \ldots, f(x_{2^n}, \ell)\right)^\top \\
\text{FFT} & \rightarrow \left(f_1, f_2, \ldots, f_{2^n}\right)^\top \\
\text{differentiate} & \rightarrow \tilde{f}_k = \frac{i k \pi}{\ell} f_k \\
\text{inverse FFT} & \rightarrow D_{n,\ell}\left(f(x_1, \ell), f(x_2, \ell), \ldots, f(x_{2^n}, \ell)\right)^\top \\
\end{align*}
\]

The end result of this is that if \(f\) is sufficiently smooth (and periodic) then

\[
D_{n,\ell}\left(f(x_1, \ell), f(x_2, \ell), \ldots, f(x_{2^n}, \ell)\right)^\top \approx \left(f'(x_1, \ell), f'(x_2, \ell), \ldots, f'(x_{2^n}, \ell)\right)^\top,
\]

is a good approximation. This allows us to accurately approximate the right-hand side of (4.1). We use the fourth-order Runge–Kutta method to time step the solution. Explicitly, given a time step \(h > 0\), the method is for \(m \geq 0\)
\[ w(t_0) = \left( w_0(x_{1,t}), w_0(x_{2,t}), \ldots, w_0(x_{2^\nu,t}) \right)^\top, \]
\[ w(t_{m+1}) = w(t_m) + \frac{h}{6} \left( k_1 + k_2 + k_3 + k_4 \right), \]
\[ t_0 = 0, \quad t_{m+1} = t_m + h, \]
\[ k_1 = F_{n,t}(w(t_m)), \]
\[ k_2 = F_{n,t}(w(t_m) + \frac{h}{2} k_1), \]
\[ k_3 = F_{n,t}(w(t_m) + \frac{h}{2} k_2), \]
\[ k_4 = F_{n,t}(w(t_m) + h k_3), \]
\[ F_{n,t}(w) = \frac{8}{3} \mu C \sqrt{w} D_{n,t}^4 \sqrt{w} + 2 \epsilon \left( 2C^2 - 2C \right) \left( \sqrt{w} D_{n,t}^4 \sqrt{w} - (D_{n,t}^4 \sqrt{w})^2 \right). \]

We highlight a numerical complication. Often, when time-stepping a time-evolution PDE with a high-order linear term, one wants to treat the linear term explicitly. This is the so-called method of exponential integrators, see [20], for example. But, it is clear that the linear terms in \((1.3)\), treated alone, will cause the solution to vanish, making it impossible to apply the nonlinear terms Thus, there must be a close interplay between the linear and nonlinear terms in \((4.1)\) and we cannot use exponential integrators. This forces a small time step. We are still able to perform simulations for moderate times but at a much higher computational cost.

### 4.1. The Gaussian limit

If \( \mu = 0 \) then the solution of \((4.1)\) should limit to the Gaussian similarity solution as was shown in [22]. To test our numerical scheme on this we do the following.

- Set \( w_0(x) = \frac{1}{(2\pi)^{1/4}} e^{-x^2/4}, \) i.e. we start with a non-standard Gaussian density.
- At each \( t_m \), approximate \( a_m = \int w(t_m, x) dx \), \( b_m = a_m^{-1} \int x w_0(t_m, x) dx \) with the trapezoidal rule.
- Define \( \bar{w}(t_m, x) = \frac{c_m^{1/2}}{a_m} \int w(t_m, c_m^{1/2} x + b_m) \). This is a probability density with mean zero and variance one.
- We monitor how close \( \bar{w}(t_m, x) \) is to a standard Gaussian density \( g(x) = (2\pi)^{-1/2} e^{-x^2/2} \) with an estimate of the supremum norm.

To be precise, we use \( \epsilon = 0.1, C = 0.2, n = 10 \) and \( h = 0.0025 \) and to get \( t = 1000 \) we require 400 000 time steps. See figure 2(a) for a demonstration of convergence to the Gaussian limit. We also run this same calculation with \( w_0(x) = e^{-x^2-e^{-x}} \) and show the results in figure 2(a). From (3.3), the variance \( c_m - b_m^2 \) should scale like \( t^{1/2} \) and we confirm this in figure 2(a).

### 4.2. The biased case

When \( \mu \neq 0 \), if the conjecture made in [2] is correct and carries through the formal derivation above, we should see \( \bar{w}(t_m, x) \) converge to the density for the Tracy–Widom (\( \beta = 1 \)) GOE distribution after it is normalized to mean zero and variance one and possibly reflected across \( x = 0 \) (due the sign of \( \mu \)). We call this normalized density \( f^I(x) \). It is easily computed once one can compute the Hastings–McLeod solution of the Painlevé II equation, see [23].
perform the same computations as in the previous section but now with multiple choices for initial data to examine the convergence deeper. We choose the following functions for initial data

\[
  w_0(x) = \frac{1}{(2\pi)^{1/2}} e^{-x^2/4}, \text{ (‘Gaussian’)}
\]

\[
  w_0(x) = \frac{1}{(2\pi)^{1/2}} e^{-(\alpha+2)x^2/4} + \frac{1}{(2\pi)^{1/2}} e^{-(\alpha-2)x^2/4}, \text{ (‘Mixed Gaussians’)}
\]

\[
  w_0(x) = f(x), \text{ (‘Tracy–Widom’)}.
\]

We use \( \mu = 1, \varepsilon = 0.1, C = 0.2, n = 10 \) and \( h = 0.0025 \) and to get to \( t = 7000 \) we require 2 800 000 time steps. We note that one choice of parameters is sufficient because scaling \( x \mapsto c_1x \) and \( t \mapsto c_2t \) for an appropriate choice of \( c_1 \) and \( c_2 \) allows one set all

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{(a) An estimate of the difference \( \sup |\hat{w}(t_m, x) - g(x)| \) for as \( t_m \) increases for both the skew initial condition \( w_0 = e^{-x^2} \) and the Gaussian initial condition \( w_0 = \frac{1}{(2\pi)^{1/2}} e^{-x^2/4} \). The difference stays small with for the Gaussian initial data and the difference decreases in time for the skew initial data. (b) The comparison of the scaling of the variance \( c_m - h_2 \) as a function of \( t \). The diamonds correspond to the least-squares fit with equation 0.420 577\( t^{0.478 72} \) which is close to expected \( t^{1/2} \) scaling. We note that the least-squares fit is only performed for \( t > 500 \).}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{(a) The evolution of the approximation of \( \hat{w}(t_m, x) \) as \( t_m \) increases from \( t_0 = 0 \) to \( t_m = 20 \) \((n = 8000)\) with the mixed Gaussians initial data. (b) The evolution of the approximation of \( w(t_m, x) \) as \( t_m \) increases from \( t_0 = 0 \) to \( t_m = 900 \) \((n = 360 000)\) with the mixed Gaussians initial data.}
\end{figure}
coefficients in (1.3) to one. In all cases we consider, our numerical method preserves the $L^1(\mathbb{R})$ norm of the solution to within $10^{-10}$. It is approximately conserved and this is a consistency check on the numerical method. In figures 3 and 4, we plot the evolution of the Mixed Gaussians initial data under the flow.

In figures 5(a) and (b) we plot the mean $b_m$ and the variance $\sigma_m^2 = c_m - b_m^2$ of the solution as a function of $t$ for each of the choices of initial data on log–log axes. Performing a least-squares fit we conjecture that $b_m \sim t^{1/3}$ and $\sigma_m^2 \sim t^{2/3}$ for large $t$. To see that $b_m \sim t^{1/3}$ we consider the Fisher information $I_m$ in figure 5(e) which is, as discussed above, the time derivative of the mean. It is clear here that $f_m \sim t^{-2/3}$ implying that $b_m \sim t^{1/3}$. The discrepancy in the exponent of our least-squares fit in figure 5(a) appears to be due to $O(1)$ or $O(\log t)$ terms that arise from integrating the Fisher information. This all means that we have a limiting form of

$$w(t, x) \sim \frac{1}{t^{1/3} \left( \frac{x}{t^{1/3}} \right)}$$

(4.2)
Finally, to see that \( f \), in our experiments, exists empirically but is distinct from \( f_1 \), we plot estimates of the difference \( \sup_x |\hat{f}_n(x) - f_1(x)| \) for a series of times for each initial condition in figures 5(c) and (d). From this it appears that an \( f \) in (4.2) exists but is differs from \( f_1(x) \) by approximately \( 2 \times 10^{-2} \). At this point, these results are intriguing but we cannot claim to refute or substantiate the conjecture in [2]. An additional intriguing detail is that the computations in [2, figure 3] appear to have densities that differ from a scaled Tracy–Widom GOE by \( 3 \times 10^{-2} \). Without accounting for the normalization of the mean and the variance, our computations cannot be compared qualitatively with these other than to say that the errors are on the same order of magnitude and are therefore consistent.

If the true limiting state of the system, after normalization, is \( f_1(x) \) the following reasons could explain our discrepancy:
• The periodic approximation excites an instability not a numerical instability that acts in \( O(1) \) time and is sufficient to eliminate convergence. This seems unlikely because as \( \ell \) and \( n \) are increased with \( h \) being decreased the solution does not appear to close in on \( f_\ell(x) \).
• The expansion in \( \varepsilon \) must be carried out to higher orders to achieve greater accuracy.

Acknowledgments

The authors are grateful to Percy Deift for very helpful discussions while this article was being prepared. P Germain is partially supported by NSF grant DMS-1101269, a start-up grant from the Courant Institute, and a Sloan fellowship. Ch Bordenave is partially supported by ANR-11-JS02-005-01. T Trogdon is partially supported by NSF grant DMS-1303018.

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