Holographic principle and dark energy

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Abstract

We discuss the relationship between holographic entropy bounds and gravitating systems. In order to obtain a holographic energy density, we introduce the Bekenstein-Hawking entropy $S_{BH}$ and its corresponding energy $E_{BH}$ using the Friedman equation. We show that the holographic energy bound proposed by Cohen et al comes from the Bekenstein-Hawking bound for a weakly gravitating system. Also we find that the holographic energy density with the future event horizon deriving an accelerating universe could be given by vacuum fluctuations of the energy density.

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1 Introduction

Supernova (SN Ia) observations suggest that our universe is accelerating and the dark energy contributes $\Omega_{DE} \simeq 0.60 - 0.70$ to the critical density of the present universe [1]. Also, cosmic microwave background (CMB) observations [2] imply that the standard cosmology is given by the inflation and FRW universe [3]. A typical candidate for the dark energy is the cosmological constant. Recently Cohen et al showed that in quantum field theory, a short distance cutoff (UV cutoff: $\Lambda$) is related to a long distance cutoff (IR cutoff: $L_\Lambda$) due to the limit set by forming a black hole [4]. In other words, if $\rho_\Lambda$ is the quantum zero-point energy density caused by a UV cutoff $\Lambda$, the total energy of the system with size $L_\Lambda$ should not exceed the mass of the same size-black hole: $L_\Lambda^3 \rho_\Lambda \leq L_\Lambda M_p^2$ with the Planck mass of $M_p^2 = 1/G$. The largest $L_\Lambda$ is chosen as the one saturating this inequality and its holographic energy density is then given by $\rho_\Lambda = 3c^2 M_p^2 / 8 \pi L_\Lambda^2$ with a numerical factor $3c^2$. Taking $L_\Lambda$ as the size of the present universe, the resulting energy is comparable to the present dark energy [5]. Even though this holographic approach leads to the data, this description is incomplete because it fails to explain the dark energy-dominated present universe [6]. In order to resolve this situation, one is forced to introduce another candidates for IR cutoff. One is the particle horizon $R_H$ which was used in the holographic description of cosmology by Fischler and Susskind [8]. This gives $\rho_\Lambda \sim a^{-2(1+1/c)}$ which implies $\omega_H > -1/3$ [9]. This corresponds to a decelerating universe and unfortunately is not our case. In order to find an accelerating universe, we need the future event horizon $R_h$. With $L_\Lambda = R_h$ one finds $\rho_\Lambda \sim a^{-2(1-1/c)}$ to describe the dark energy with $\omega_h < -1/3$. This is close enough to $-1$ to agree with the data [1]. However, this relation seems to be rather ad hoc chosen and one has to justify whether or not $\rho_\Lambda = 3c^2 M_p^2 / 8 \pi L_\Lambda^2$ is appropriate to describe the present universe.

On the other hand, the implications of the cosmic holographic principle have been investigated in the literature [10, 11, 8, 12, 13]. However, these focused on the decelerating universe, especially for a radiation-dominated universe.

In this letter we will clarify how the cosmic holographic principle could be used for obtaining the holographic energy density. This together with the future event horizon is a candidate for the dark energy to derive an accelerating universe. Further we wish to seek the origin of the holographic energy density.
2 Cosmic holographic bounds

We briefly review the cosmic holographic bounds for our purpose. Let us start an $(n+1)$-dimensional Friedman-Robertson-Walker (FRW) metric
\[ ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2d\Omega^2_{n-1} \right], \]
where $a$ is the scale factor of the universe and $d\Omega^2_{n-1}$ denotes the line element of an $(n-1)$-dimensional unit sphere. Here $k = -1, 0, 1$ represent that the universe is open, flat, closed, respectively. A cosmological evolution is determined by the two Friedman equations
\[ H^2 = \frac{16\pi G_{n+1} E}{n(n-1)V} - \frac{k}{a^2}, \]
\[ \dot{H} = -\frac{8\pi G_{n+1}}{n-1} \left( \frac{E}{V} + p \right) + \frac{k}{a^2}, \]
where $H$ represents the Hubble parameter with the definition $H = \dot{a}/a$ and the overdot stands for derivative with respect to the cosmic time $t$, $E$ is the total energy of matter filling the universe, and $p$ is its pressure. $V$ is the volume of the universe, $V = a^n \Sigma_k^n$ with $\Sigma_k^n$ being the volume of an $n$-dimensional space with a curvature constant $k$, and $G_{n+1}$ is the Newton constant in $(n+1)$ dimensions. Here we assume the equation of state: $p = \omega \rho$, $\rho = E/V$. First of all, we introduce two entropies for the holographic description of a universe \[14, 15\]:
\[ S_{BV} = \frac{2\pi}{n} E a, \quad S_{BH} = (n-1) \frac{V}{4G_{n+1}a}, \]
where $S_{BV}$ and $S_{BH}$ are called the Bekenstein-Verlinde entropy and Bekenstein-Hawking entropy, respectively. Then, the first Friedman equation can be rewritten as
\[ (Ha)^2 = 2 \frac{S_{BV}}{S_{BH}} - k. \]
We define a quantity $E_{BH}$ which corresponds to energy needed to form a universe-sized black hole by analogy with $S_{BV}$: $S_{BH} = (n-1)V/4G_{n+1}a \equiv 2\pi E_{BH}a/n$. Using these, for $Ha \leq \sqrt{2-k}$, one finds the Bekenstein-Hawking bound for a weakly self-gravitating system as
\[ E \leq E_{BH} \leftrightarrow S_{BV} \leq S_{BH}, \]
while for $Ha \geq \sqrt{2-k}$, one finds the cosmic holographic bound for a strongly self-gravitating system as
\[ E \geq E_{BH} \leftrightarrow S_{BV} \geq S_{BH}. \] (6)

3 Holographic energy bounds

First we study how the gravitational holography goes well with a (3+1)-dimensional effective theory. For convenience we choose the volume of the system as \( V_{A} = 4\pi L_{A}^{3}/3 \sim L_{A}^{3} \). For an effective quantum field theory in a box of volume \( V_{A} \) with a UV cutoff \( \Lambda_{1} \), its entropy scales extensively as

\[ S_{\Lambda} \sim L_{A}^{3}\Lambda^{3}. \] (7)

However, the Bekenstein postulated that the maximum entropy in a box of volume \( V_{A} \) behaves non-extensively, growing only as the enclosed area \( A_{\Lambda} \) of the box. We call it the gravitational holography. The Bekenstein entropy bound is satisfied in the effective theory if

\[ S_{\Lambda} \sim L_{A}^{3}\Lambda^{3} \leq S_{BH} \equiv \frac{2}{3}\pi M_{p}^{2}L_{A}^{2} \sim M_{p}^{2}L_{A}^{2}, \] (8)

where \( S_{BH} \) is the closest one which comes to the usual form of Bekenstein-Hawking entropy \( A_{\Lambda}/4G = \pi M_{p}^{2}L_{A}^{2} \) for a black hole of radius \( L_{A} \). Thus the IR cutoff cannot be chosen independently of the UV cutoff \( \Lambda \), if we introduce the gravitational holography. It scales like \( L_{A} \sim \Lambda^{-3} \). This bound is suitable for the system with a relatively low energy density. However, an effective theory that can saturate the inequality of Eq.(8) includes many states with the Schwarzschild radius \( L_{S} = 2GM_{S} \) much larger than the size of a box \( L_{A} \) \( (L_{S} > L_{A}) \). To avoid this difficulty, one proposes a rather strong constraint on the IR cutoff which excludes all states that lie within the Schwarzschild radius. Then one finds cases with \( L_{S} < L_{A} \). Since the maximum energy density \( \rho_{\Lambda} \) in the effective theory is \( \Lambda^{4} \), the total energy scales as \( E_{A} = V\rho_{\Lambda} \sim L_{A}^{3}\Lambda^{4} \). As a result, the constraint on the IR cutoff is given by

\[ E_{A} \sim L_{A}^{3}\Lambda^{4} \leq M_{S} \sim L_{A}M_{p}^{2}, \] (9)

where the IR cutoff scales as \( L_{A} \sim \Lambda^{-2} \). This bound is more restrictive than the Bekenstein bound in Eq.(8). By definition, the two scales \( \Lambda \) and \( L_{A} \) are independent to each other initially. To reconcile the breakdown of the quantum field theory to describe a black hole, one proposes a relationship between UV and IR cutoffs. An effective field theory could then describe a system including even a black hole.

\(^{1}\)Precisely, \( M_{\Lambda} \) is more suitable for an UV cutoff than \( \Lambda \), but we here use the latter instead of \( M_{\Lambda} \) for convenience.
Now we explain the bound of Eq.(9) within our framework. We wish to interpret it in view of cosmic holographic bounds. Here \( k = 0 \) and a physical scale \( a \sim L \). From Eq.(4), for \( L \leq \sqrt{2}/H \), one finds the holographic bound for a weakly self-gravitating system as \( E \sim L^3 \Lambda^4 \leq E_{\text{BH}} \equiv 2LM_p^2 \sim LM_p^2 \). Also this inequality is derived from the Bekenstein-Hawking entropy bound: \( S_{\text{BV}} \leq S_{\text{BH}} \). Here \( S_{\text{BV}} \) is not really as an entropy but rather as the energy measured with respect to an appropriately chosen conformal time coordinate \[14\]. Also the role of \( S_{\text{BH}} \) is not to serve a bound on the total entropy, but rather on a sub-extensive component of the entropy that is associated with the Casimir energy \( E_c \) of the CFT. Consequently, the bound of Eq.(9) comes from the Bekenstein-Hawking bound for a weakly gravitating system. Furthermore, if \( L = \sqrt{2}/H \), one finds the saturation which states that \( S_{\text{BV}} = S_{\text{BH}} \leftrightarrow E = E_{\text{BH}} \). We remind the reader that \( E_{\text{BH}} \) is an energy to form a universe-sized black hole. The universe is in a weakly self-gravitating phase when its total energy \( E \) is less than \( E_{\text{BH}} \), and in a strongly gravitating phase for \( E > E_{\text{BH}} \). We emphasize that comparing with the Bekenstein bound in Eq.(8), the Bekenstein-Hawking bound in Eq.(9) comes out only when taking the Friedman equation (dynamics) into account \[15\]. Hence the Bekenstein-Hawking bound is more suitable for cosmology than the Bekenstein bound. Up to now, we consider the cosmic holographic bounds for the decelerating universe which includes either a weakly gravitating system or a strongly gravitating one.

4 Holographic dark energy

In order to describe the dark energy, we have to choose a candidate. There are many candidates. In this work we choose the holographic energy to describe the dark energy. We take the largest \( L_\Lambda \) as the one saturating the inequality of Eq.(21). Then we find a relation for the cosmological energy density (cosmological constant): \( \rho_\Lambda = 3c^2 M_p^2 / 8\pi L_\Lambda^2 \) with a numerical constant \( 3c^2 \). In the case of \( c = 1 \), it corresponds to a variant of the cosmological constant because the conventional form is usually given by \( \rho_\Lambda \sim \Lambda^4 = 1/L^4 \) with \( \Lambda \sim 1/L \) in the de Sitter thermodynamics \[16\] \[17\] \[18\].

Here three choices are possible for \( L_\Lambda \) \[9\]. If one chooses IR cutoff as the size of our universe \( (L_\Lambda = d_H = 1/H) \), the resulting energy is comparable to the present dark energy \[5\]. Even though this holographic approach leads to the data, this description is incomplete because it fails to explain the present universe of dark-energy dominated phase with \( \omega = p/\rho \leq -0.78 \) \[6\] \[7\]. In this case the Friedman equation including a matter of \( \rho_m \) is given by \( \rho_m = 3(1 - c^2)M_p^2 H^2 / 8\pi \), which leads to the dark energy with \( \omega = 0 \). However, the accelerating universe requires \( \omega < -1/3 \) and thus this case is excluded. In
order to resolve this situation, one is forced to introduce the particle horizon $L_\Lambda = R_H = a \int_0^t (dt/a) = a \int_0^a (da/Ha^2)$ which was used in the holographic description of cosmology by Fischler and Susskind \[8\]. In this case, the Friedman equation of $H^2 = 8\pi \rho_\Lambda / 3M_p^2$ leads to an integral equation $HR_H = c$. Finally it takes the form of a differential equation

$$c \frac{d}{da} \left( \frac{H^{-1}}{a} \right) = \frac{1}{Ha^2}. \tag{10}$$

It gives $\rho_\Lambda \sim a^{-2(1+1/c)}$, which implies $\omega_H = -1/3(1 - 2/c) > -1/3$. This is still a decelerating phase because the comoving Hubble scale of $H^{-1}/a$ is increasing with time, as is in the radiation/matter-dominated universes. In order to find an accelerating universe which satisfies

$$\ddot{a} > 0 \iff \frac{d}{dt} \left( \frac{H^{-1}}{a} \right) < 0 \iff \omega < -\frac{1}{3}, \tag{11}$$

we need a shrinking Hubble scale, as was shown in the inflationary universe. It means that the changing rate of $H^{-1}/a$ with respect to $a$ is always negative for an accelerating universe. For this purpose, we introduce the future event horizon $L_\Lambda = R_h = a \int_t^\infty (dt/a) = a \int_0^\infty (da/Ha^2)$ for an observer \[9\ \[19\]. Using an integral form of Friedman equation of $HR_h = c$, one finds a promising differential equation

$$c \frac{d}{da} \left( \frac{H^{-1}}{a} \right) = -\frac{1}{Ha^2}. \tag{12}$$

This leads to $\rho_\Lambda \sim a^{-2(1-1/c)}$ with $\omega_h = -1/3(1 + 2/c) < -1/3$ which is close enough to $-1$ to agree with the data. For $c = 1$, we recover a case of cosmological constant with $\omega_h = 1$ exactly. We note that the Friedman equation with the holographic energy density $\rho_\Lambda$ takes the form $H = c/L_\Lambda$, whereas the Friedman equation with the conventional form $\tilde{\rho}_\Lambda$ is given by $H = \sqrt{8\pi / 3M_p^2/L_\Lambda^2}$. Hence the above result using the holographic energy density is no longer valid for the de Sitter thermal energy density.

At this stage we mention that $L_\Lambda = R_h$ seems to be rather ad hoc chosen and one thus requires establishing a close connection between the holographic energy density and dark energy. Actually the important fact to remark is that the holographic energy density $\rho_\Lambda = 3c^2M_p^2/8\pi L_\Lambda^2$ is originally derived for a decelerating phase due to radiation/matter-dominated universes. However, the dark energy usually derives an accelerating universe. There exists a difference between decelerating universe and accelerating universe. The key point is the existence of the future event horizon in the accelerating universe. Therefore it is not guaranteed that $\rho_\Lambda$ is applicable even for an accelerating universe.
5 Holographic dark energy and vacuum fluctuations

If the cosmological constant arises due to the energy density of the vacuum, one needs to investigate the structure of quantum vacuum at large cosmological scales. The renormalization group approach shows that the energy density depends on the scale at which it is probed. Suppose that the universe has endowed us the two independent length scales, \( L_p \sim 1/M_p \) and \( \tilde{L}_\Lambda \sim 1/\tilde{\Lambda} \) \([20, 21]\). Then we construct two energy scales: the Planck energy density of \( \rho_p = M_p^4 = 1/L_p^4 \) and the de Sitter thermal energy density of \( \tilde{\rho}_\Lambda = \tilde{\Lambda}^4 = 1/\tilde{L}_\Lambda^4 \). Thus \( L_p \) determines the highest possible energy density in the universe, whereas \( \tilde{L}_\Lambda \) determines the lowest possible energy density. In this picture, observation requires enormous fine tuning as \((L_p/\tilde{L}_\Lambda)^2 \leq 10^{-123}\). As the energy density of normal matter/radiation drop below \( L_\Lambda \), the thermal ambience of the de Sitter phase will remain constant and provide the vacuum noise. Then the dark energy may be the given by the geometric mean of two scales in the universe: \( \rho_{DE} = \sqrt{\rho_p \tilde{\rho}_\Lambda} = M_p^2/\tilde{L}_\Lambda^2 \) which looks like the holographic energy density \( \rho_\Lambda = 3c^2 M_p^2/8\pi \tilde{L}_\Lambda^2 \). On the other hand, the hierarchy of the two scales has the pattern

\[
\rho_{\text{vac}} = \frac{1}{L_p^4} + \frac{1}{L_p^4} \left( \frac{L_p}{L_\Lambda} \right)^2 + \frac{1}{L_p^4} \left( \frac{L_p}{L_\Lambda} \right)^4 + \cdots , \tag{13}
\]

where the first term is the bulk energy density that needs to be renormalized away, the second term is due to the vacuum fluctuations, and the third one is the de Sitter thermal energy density.

We will show that the holographic energy density \( \rho_\Lambda = 3c^2 M_p^2/8\pi \tilde{L}_\Lambda^2 \) could be generated by the vacuum fluctuations of the energy density. If the accelerating universe has the future event horizon (the cosmological horizon) that blocks information, the natural scale is given by the size of the horizon \( R_h \). The operator \( \mathcal{H}(<R_h) \), corresponding to the total energy inside a region bounded by a cosmological horizon, will exhibit fluctuations \( \Delta E \), because the vacuum state is not an eigenstate of \( \mathcal{H}(<R_h) \). The corresponding fluctuation in terms of the energy density is given by

\[
\Delta \rho \sim \frac{\Delta E}{R_h^3} \equiv f(L_p, R_h), \tag{14}
\]

where \( f \) is a function to be determined. In order that \( \Delta \rho \sim M_p^2/R_h^2 \), it is necessary to have \( (\Delta E)^2 \sim R_h^3/L_p^4 \). This means that the square of energy fluctuations should scale as the enclosed surface of the accelerating universe. Actually a calculation \([22]\) showed that for \( R_h \gg L_p \), \( (\Delta E)^2 = C_1 R_h^2/L_p^4 \) where \( C_1 \) depends on the UV cutoff. Hence we roughly prove that \( \rho_\Lambda \sim \Delta \rho \). This means that the holographic energy density deriving
an accelerating universe with the future event horizon could be given by the vacuum fluctuations of the energy density.

6 Conclusion

We show that the holographic energy bound \( \rho_\Lambda = 3c^2 M_p^2 / 8\pi R_h^2 \) proposed by Cohen et al can be derived from the cosmic holographic bound, the Bekenstein-Hawking bound for a weakly gravitating system. If the IR cutoff is chosen by the future event horizon, then the holographic energy density can derive an accelerating universe. In this case the holographic energy density could be given by vacuum fluctuations of the energy density.

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