GLOBAL WELL-POSEDNESS OF UNSTEADY MOTION OF VISCOUS INCOMPRESSIBLE CAPILLARY LIQUID BOUNDED BY A FREE SURFACE

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(Communicated by Matthias Hieber)

Abstract. In this paper, we prove the global well-posedness of free boundary problems of the Navier-Stokes equations in a bounded domain with surface tension. The velocity field is obtained in the $L^p$ in time $L^q$ in space maximal regularity class, $(2 < p < \infty, N < q < \infty, \text{and } 2/p + N/q < 1)$, under the assumption that the initial domain is close to a ball and initial data are sufficiently small. The essential point of our approach is to drive the exponential decay theorem in the $L^p$-$L^q$ framework for the linearized equations with the help of maximal $L^p$-$L^q$ regularity theory for the Stokes equations with free boundary conditions and spectral analysis of the Stokes operator and the Laplace-Beltrami operator.

1. Introduction and results. This paper deals with the global well-posedness of free boundary problems governing the motion of a finite isolated mass of a viscous incompressible capillary liquid in the $L^p$ in time and $L^q$ in space framework. Let $\Omega$ be a domain in the $N$-dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$), whose boundary consists of the compact hypersurface $\Gamma$. Our problem is to find a time dependent domain $\Omega_t$ ($t > 0$), the velocity vector field $v(x, t) = (v_1(x, t), \ldots, v_N(x, t))$, where $\top M$ denotes the transposed $M$, and the pressure field $p(x, t)$ defined for $x = (x_1, \ldots, x_N) \in \Omega_t$ ($t > 0$), which satisfy the initial boundary value problem for the Navier-Stokes equations:

$$\begin{cases}
\partial_t v + (v \cdot \nabla)v - \text{Div}(\mu D(v) - pI) = 0, & \text{in } \Omega_t \text{ for } t \in (0, T), \\
(\mu D(v) - pI)n_t = \sigma h(\Gamma_t)n_t, & \text{on } \Gamma_t \text{ for } t \in (0, T), \\
v|_{t=0} = v_0 & \text{in } \Omega_0 = \Omega,
\end{cases}$$

where $\Gamma_t$ is the boundary of $\Omega_t$, $\Gamma_0 = \Gamma$, and $T \in (0, \infty]$. As for the remaining notation in problem (1), $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $v_0 = (v_01, \ldots, v_{0N})$ is a given initial velocity, $D(v) = (\nabla v + \top \nabla v)$ the doubled strain tensor with $(i, j)^{th}$ component.

2000 Mathematics Subject Classification. Primary: 35R35; Secondary: 35Q30, 76D05, 76D03.

Key words and phrases. Navier-Stokes equations, free boundary problems, surface tension, global well-posedness.

Partially supported by JSPS Grant-in-aid for Scientific Research (A) 17H0109 and Top Global University Project. Adjunct faculty member in the Department of Mechanical Engineering and Materials Science, University of Pittsburgh.
\( D_{ij}(v) = \partial_i v_j + \partial_j v_i, \) the \( N \times N \) identity matrix, \( H(\Gamma_t) \) the \( N-1 \) times mean curvature of \( \Gamma_t \) given by \( H(\Gamma_t) u_t = \Delta_{\Gamma_t} x \ (x \in \Gamma_t) \), where \( \Delta_{\Gamma_t} \) is the Laplace-Beltrami operator on \( \Gamma_t \), \( V \) the velocity of the evolution of \( \Gamma_t \) in the direction of \( u_t \), and \( \mu \) and \( \sigma \) are positive constants representing the viscous coefficient and the coefficient of surface tension, respectively. Moreover, for any matrix field \( K \) with \( (i,j) \)^{th} component \( K_{ij} \), the quantity \( \text{Div} K \) is an \( N \)-vector with \( i^{th} \) component \( \sum_{j=1}^{N} \partial_j K_{ij} \) and for any vector of functions \( w = (w_1, \ldots, w_N) \) we set \( \text{div} w = \sum_{j=1}^{N} \partial_j w_j \) and \( (w \cdot \nabla)w \) is an \( N \)-vector with \( i^{th} \) component \( \sum_{j=1}^{N} w_j \partial_i w_i \).

In the case where \( \Omega \) is bounded (so called drop problem), the local well-posedness was proved by Solonnikov [26, 27] and references therein, Mogilevskii and Solonnikov [9], Padula and Solonnikov [13], and Schweizer [19]. On the other hand, in the case where \( \Omega \) is a layer (so called ocean problem), the local well-posedness was proved by T. Beale [3], Allain [2] and Tani [30].

Solonnikov [24] proved the global well-posedness for the drop problem in the Sobolev-Slobodcevich space \( W^{2+\ell,1+\ell/2}_2 \) for \( \ell \in (1/2,1) \) under the assumption that the initial state \( \Omega \) is close to a ball and initial data are small enough. The global well-posedness for the ocean problem was proved by Beale [4], Beale and Nishida [5, 12], Tani and Tanaka [31], and Kawashima and Hataya [6, 7] in the \( L_2 \) framework under the assumption that the initial state \( \Omega \) is very close to a flat space and initial data are small enough. Recently, Saito and Shibata [17, 18] proved the global well-posedness for the ocean problem without bottom in the \( H^{2,1}_{2,q,p} \) framework, where \( H^2_{2,q,p} = H^2_{q,p}((0,T), L_q(\Omega)) \cap L_p((0,T), H^2_q(\Omega)) \).

We remark that without surface tension case, the local well-posedness for any initial data and the global well-posedness for small initial data were proved by Solonnikov [25], and Mucha and Zajączkowski [10] in \( W^{2+\ell,1+\ell/2}_2 \) for \( \ell \in (1/2,1) \), by Abels [1] in \( H^{2,1}_{p,q} \) \( (N < p < \infty) \), and by Shibata and Shimizu [23], and Shibata [20] in \( H^{2,1}_{q,p} \) \((2 < p < \infty \) and \( N < q < \infty) \).

The purpose of this paper is to prove the global well-posedness of problem (1) under the assumption that the initial state \( \Omega \) is very close to a ball and the initial data are sufficiently small with the help of maximal \( H^2_{p,q} \) regularity theory for the Stokes equations and spectral analysis for the Stokes operator and the Laplace-Beltrami operator, which is an extension of Solonnikov’s result [24] from maximal regularity view point. As a related topic, the local and global well-posedness of the two phase viscous incomplessible flows separated by a sharp interface in a container has been extensively studied by Prüss and Simonett [14, 15] (cf. also [8], [16] and references therein) with the help of maximal \( L_p \) regularity theory for the Stokes equations and spectral analysis for the Laplace-Beltrami operator.

Now, we formulate the problem. Let \( B_R = \{ y \in \mathbb{R}^N \mid |y| < R \} \) and \( S_R = \{ y \in \mathbb{R}^N \mid |y| = R \} \). We assume that

(A1) \(|\Omega| = |B_R| = \frac{R^N \omega_N}{N}, \) where \(|D|\) denotes the Lebesgue measure of a Lebesgue measurable set \( D \) in \( \mathbb{R}^N \) and \( \omega_N \) is the area of \( S_1 \);

(A2) \( \int_{\Omega} x \, dx = 0; \)

(A3) \( \Gamma \) is a normal perturbation of \( S_R \) given by

\[
\Gamma = \{ x = y + \rho_0(y)(y/|y|) \mid y \in S_R \} = \{ x = (1 + R^{-1}\rho_0(y))y \mid y \in S_R \}
\]

with given small function \( \rho_0 \) defined on \( S_R \).
Let \( \Gamma_t \) be given by
\[
\Gamma_t = \{ x = y + \rho(y, t)(y/y) + \xi(t) \mid y \in S_R \}
= \{ x = (1 + R^{-1}\rho(y, t))y + \xi(t) \mid y \in S_R \}
\]
where \( \rho(y, t) \) is an unknown function with \( \rho(y, 0) = \rho_0(y) \) for \( y \in S_R \) and \( \xi(t) \) is the barycenter point of the domain \( \Omega_t \) defined by
\[
\xi(t) = \frac{1}{|\Omega_t|} \int_{\Omega_t} x \, dx,
\]
which is also unknown. It follows from the assumption (A2) that \( \xi(0) = 0 \). Moreover,
\[
\xi'(t) = \frac{1}{|\Omega_t|} \int_{\Omega_t} \mathbf{v}(x, t) \, dx.
\]
In fact, let \( w(\xi, t) \) be the Lagrange description of the velocity field \( \mathbf{v}(x, t) \), and then using the Lagrange transformation: \( x = \xi + \int_0^t w(\xi, s) \, ds \) (\( \xi \in \Omega \)), we have
\[
\xi'(t) = \frac{d}{dt} \left\{ \frac{1}{|\Omega_t|} \int_{\Omega_t} x \, dx \right\} = \frac{d}{dt} \left\{ \frac{1}{|\Omega_t|} \int_{\Omega_t} (\xi + \int_0^t w(\xi, t) \, dt) \, d\xi \right\}
= \frac{1}{|\Omega_t|} \int_{\Omega_t} w(\xi, t) \, d\xi = \frac{1}{|\Omega_t|} \int_{\Omega_t} \mathbf{v}(x, t) \, dx.
\]
Given a height function \( \rho(y, t) \), let \( H_\rho(y, t) \) be a solution to the Dirichlet problem:
\[
(1 - \Delta)H_\rho = 0 \quad \text{in } B_R, \quad H_\rho|_{S_R} = \rho.
\]
We introduce the Hanzawa transformation centered at \( \xi(t) \) defined by
\[
x = \mathbf{h}_\xi(y, t) := (1 + R^{-1}H_\rho(y, t))y + \xi(t) \quad \text{for } y \in B_R.
\]
Let \( H_{\rho_0}(y) \) be a solution to the Dirichlet problem: \( (1 - \Delta)H_{\rho_0} = 0 \) in \( B_R \) and \( H_{\rho_0}|_{S_R} = \rho_0 \), and then we set \( \mathbf{h}_{\xi_0}(y) := (1 + R^{-1}H_{\rho_0}(y))y \) for \( y \in B_R \).

In the following, we assume that
\[
\|H_\rho(\cdot, t)\|_{H_\infty(B_R)} \leq \epsilon_0 \quad \text{for any } t \in [0, T).
\]
Let \( 0 < \epsilon_0 < 1/4 \). Since \( |H_\rho(y, t) - H_\rho(y', t)| \leq \|\nabla H_\rho(\cdot, t)\|_{L_\infty(B_R)}|y - y'| \), we have
\[
|\mathbf{h}_\xi(y, t) - \mathbf{h}_\xi(y', t)| \geq (1 - 2\epsilon_0)|y - y'| \geq (1/2)|y - y'| \quad \text{for any } y, y' \in B_R,
\]
and therefore the Hanzawa transformation is a bijective map from \( B_R \) onto \( \Omega_t \), where we have set
\[
\Omega_t = \{ x = \mathbf{h}_\xi(y, t) = (1 + H_\rho(y, t))y + \xi(t) \mid y \in B_R \} \quad (t \in [0, T)).
\]
In our proof below, we will choose \( \epsilon_0 \in (0, 1/4) \) suitably small in such a way that several sufficient conditions to obtain our main result hold.

In the following, we set
\[
\mathbf{u}_0 = \mathbf{v}_0 \circ \mathbf{h}_{\xi_0},
\]
where \( \mathbf{v}_0 \) is an initial data for problem (1). Let \( \mathbf{v} \) and \( \mathbf{p} \) satisfy (1) and let
\[
\mathbf{u}(y, t) = \mathbf{v} \circ \mathbf{h}_\xi, \quad \mathbf{q}(y, t) = \mathbf{p} \circ \mathbf{h}_\xi - \frac{\sigma(N - 1)}{R},
\]
then $u, q$, and $\rho$ satisfy the following equations:

$$
\begin{align*}
\partial_t u - \text{Div}(\mu D(u) - qI) &= f(u, \rho) \quad \text{in} \ B_R \times (0, T), \\
\text{div} u &= f_d(u, \rho) = \text{div} f_d(u, \rho) \quad \text{in} \ B_R \times (0, T), \\
\rho_t - \omega \cdot \nabla u &= k_n(u, \rho) \quad \text{on} \ S_R \times (0, T), \\
\Pi_0(\mu D(u)\omega) &= \mu g^r(u, \rho) \quad \text{on} \ S_R \times (0, T), \\
\langle \mu D(u)\omega, \omega \rangle - q - \sigma B\rho &= g_n(u, \rho) \quad \text{on} \ S_R \times (0, T), \\
(u, \rho)|_{t=0} &= (u_0, \rho_0) \quad \text{in} \ B_R \times S_R,
\end{align*}
$$

(10)

where $\omega = y/|y| \in S_1$, $P_{u} = u - \frac{1}{|B_R|} \int_{B_R} u \, dy$, $B = \Delta_{S_R} + \frac{N - 1}{R^2}$, and $\Delta_{S_R}$ is the Laplace-Beltrami operator on $S_R$. Moreover, we set $\langle a, b \rangle = a \cdot b = \sum_{j=1}^{N} a_j b_j$ for any $N$ vectors $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$, and $\Pi_0 a = a - \langle a, \omega \rangle \omega$. In the equations (10), $f, f_d, f_d, k_n, g^r$ and $g_n$ denote some nonlinear functions with respect to $u$ and $\rho$, which will be given in Sect.2 below. We note that

$$
\Delta_{S_R} = R^{-2}\Delta_{S_1}, \quad B = R^{-2}(\Delta_{S_1} + (N - 1)),
$$

(11)

where $\Delta_{S_1}$ is the Laplace-Beltrami operator on $S_1$.

Before stating our main results, at this stage we explain the symbols used throughout the paper. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all natural numbers, real numbers, and complex numbers, respectively. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any multi-index $\kappa = (\kappa_1, \ldots, \kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1 + \cdots + \kappa_N$ and $\partial_x^{\kappa} = \partial_1^{\kappa_1} \cdots \partial_N^{\kappa_N}$ with $\partial_i = \partial/\partial x_i$. For any scalar function $f$ and vector valued function $g = (g_1, \ldots, g_k)$, we set

$$
\nabla f = (\partial_1 f, \ldots, \partial_N f), \quad \nabla^\ell f = (\partial_\alpha^f f \mid |\alpha| = \ell),
$$

$$
\nabla g = (\partial_i g_j \mid i = 1, \ldots, N, j = 1, \ldots, k), \quad \nabla^\ell g = (\nabla^\ell g_1, \ldots, \nabla^\ell g_k).
$$

For any domain $D$, let $L_q(D)$, $H^m_q(D)$ ($m \in \mathbb{N}$) and $B^s_{q,p}(D)$ ($s \in \mathbb{R} \setminus \mathbb{N}_0$) be the Lebesgue space, Sobolev space and Besov space, and let $\| \cdot \|_{L_q(D)}$, $\| \cdot \|_{H^m_q(D)}$ and $\| \cdot \|_{B^s_{q,p}(D)}$ denote their respective norms. We write $H^m_0(D) = L_0(D)$ and $B^s_{q,p}(D) = W^s_{q,p}(D)$. Let $X$ be a Banach space with norm $\| \cdot \|_X$. For an interval $J = (0, T)$ with $T \in (0, \infty]$, $L_p(J, X)$ and $H^m_p(J, X)$ ($m \in \mathbb{N}$) denote the $X$-valued Lebesgue space and $X$-valued Sobolev space, respectively, and $\| \cdot \|_{L_p(J, X)}$ and $\| \cdot \|_{H^m_p(J, X)}$ denote their respective norms. We set

$$
\|e^{\eta t}f\|_{L_p((a, b), X)} = \left( \int_a^b (e^{\eta t}\|f(t)\|_X)^p \, dt \right)^{1/p}
$$

for any real number $\eta$ and time interval $(a, b)$. Let $X^d = \{f = (f_1, \ldots, f_d) \mid f_i \in X \ (i = 1, \ldots, d)\}$, whose norm is written by $\| \cdot \|_X$ instead of $\| \cdot \|_{X^d}$ for short. In particular, vectors and vector valued functions are denoted with bold face letters. Let

$$
(f, g)_{B_R} = \int_{B_R} f(x) \overline{g(x)} \, dx, \quad (f, g)_{S_R} = \int_{S_R} f(x) \overline{g(x)} \, d\tau,
$$

$$
(f, g)_{B_R} = \int_{B_R} f(x) \cdot \overline{g(x)} \, dx, \quad (f, g)_{S_R} = \int_{S_R} f(x) \cdot \overline{g(x)} \, d\tau
$$

where $d\tau$ is the surface element of $S_R$ and $\bar{f}$ denotes the complex conjugate of $f$. The letters $C$ and $c$ denote generic positive constants and $C_{a,b,\ldots}$ denotes that the
following conditions for constant $C$ change from line to line.

Now, we state our main results. The assumptions (A1) and (A2) lead to the following conditions for $\rho_0$:

$$\frac{R^N \omega_N}{N} = \int_{\Omega} dx = \int_{[\omega]=1} \int_{0}^{R+\rho_0(R\omega)} r^{N-1} dr d\omega = \int_{[\omega]=1} \frac{(R + \rho_0(R\omega))^N}{N} d\omega;$$

$$0 = \int_{\Omega} x_i dx = \int_{[\omega]=1} \int_{0}^{R+\rho_0(R\omega)} \omega_i r^{N} dr d\omega = \int_{[\omega]=1} \frac{(R + \rho_0(R\omega))^{N+1}}{N+1} \omega_i d\omega,$$

$(i = 1, \ldots, N)$, and so, we have compatibility conditions for $\rho_0$ as follows:

$$\sum_{k=1}^{N} \lambda C_k \int_{S_R} (R^{-1} \rho_0(y))^k d\tau = 0,$$

$$(12)$$

$$\sum_{k=1}^{N+1} \lambda C_k \int_{S_R} \psi_i (R^{-1} \rho_0(y))^k d\tau = 0 \quad (i = 1, \ldots, N),$$

because $\int_{S_1} d\omega = \omega_N$ and $\int_{S_1} \omega_i d\omega = 0$. Here,

$$\lambda C_k = \frac{N!}{k!(N-k)!}, \quad \lambda C_k = \frac{(N+1)!}{k!(N+1-k)!}, \quad \ell! = 1 \cdot 2 \cdots \ell \quad (\ell \geq 1), \quad 0! = 1.$$

To obtain the decay properties of solutions, we introduce some orthogonal conditions. Let

$$\mathcal{R}_d = \{u | D(u) = 0\}.$$

We know that $\mathcal{R}_d$ consists of constant $N$-vectors and any linear combinations of $x_i e_j - x_j e_i$ $(i, j = 1, \ldots, N)$, where $e_i$ is the unit $N$-vector whose $i$th component is one. Note that $(x_i e_j - x_j e_i) \cdot x = 0$. Let $p_\ell = |B_R|^{-1} e_\ell$ $(\ell = 1, \ldots, N)$ and let $p_\ell$ $(\ell = N+1, \ldots, M)$ be some linear combinations of $x_i e_j - x_j e_i$ such that $(p_\ell, p_m)_{B_R} = \delta_{\ell,m}$, where $\delta_{\ell,m}$ are the Kronecker’s delta symbols, that is $\delta_{\ell,m} = 0$ and $\delta_{\ell,m} = 1$ for $\ell \neq m$. Since $(e_i, p_m)_{B_R} = 0$ for $\ell = 1, \ldots, N$ and $m = N+1, \ldots, M$, $\{p_\ell\}_{\ell=1}^M$ is the orthonormal basis of $\mathcal{R}_d$ with respect to the $L_2(B_R)$ inner-product.

We know that $B_{R} x_i = 0$ on $S_R$ for $i = 1, \ldots, N$. Thus, let $\varphi_1 = |S_R|^{-1}$ and let $\varphi_\ell$ $(i = 2, \ldots, N+1)$ be some linear combinations of $x_1, \ldots, x_N$ such that $(\varphi_1, \varphi_\ell)_{S_R} = \delta_{1,\ell}$. Since $(\varphi_1, \varphi_\ell)_{S_R} = 0$ for $i = 2, \ldots, N+1$, the $\{\varphi_\ell\}_{\ell=1}^{N+1}$ is the orthonormal basis of the subspace $\{\psi | B\psi = 0$ on $S_R \} \cup \mathbb{C}$ with respect to $L_2(S_R)$ inner-product.

Let

$$S_{p,q}((a, b)) = \{(u, q, \rho) | u \in H^1_p((a, b), L_q(B_R)^N) \cap L_p((a, b), H^2_q(B_R)^N), q \in L_p((a, b), H^1_q(B_R)), \rho \in H^1_p((a, b), W^2_{q-1/q}(S_R)) \cap L_p((a, b), W^3_{q-1/q}(S_R))\},$$

which is the solution class of problem (10), and then the main result of this paper is stated as follows.

**Main Theorem.** Let $p$ and $q$ be real numbers such that $2 < p < \infty$, $N < q < \infty$ and $2/p + N/q < 1$. Assume that (A1), (A2) and (A3) hold. Then, there exists a small number $\epsilon \in (0, 1)$ such that for any initial data $u_0 \in B^{2-2/p}_{q,p}(B_R)$ and $\rho_0 \in B^{3-1/p-1/q}_{q,p}(S_R)$ satisfying the smallness condition:

$$\|u_0\|_{B^{2-2/p}_{q,p}(B_R)} + \|\rho_0\|_{B^{3-1/p-1/q}_{q,p}(S_R)} \leq \epsilon,$$

(13)
the compatibility conditions:
\[
\text{div } \mathbf{u}_0 = f_d(\mathbf{u}_0, \rho_0) = \text{div } f_d(\mathbf{u}_0, \rho_0) \quad \text{in } B_R,
\]
\[
\Pi_0(\mu \mathbf{D}(\mathbf{u}_0)\omega) = \mu g'(\mathbf{u}_0, \rho_0) \quad \text{on } S_R,
\]
the condition (12), and the orthogonal condition:
\[
(\mathbf{v}_0, e_i)_\Omega = 0 \quad (i = 1, \ldots, N), \quad (\mathbf{v}_0, x_i e_j - x_j e_i)_\Omega = 0 \quad (i, j = 1, \ldots, N),
\]
where \(\mathbf{v}_0\) is given by the formula (8), problem (10) with \(T = \infty\) admits a unique solution \((\mathbf{u}, \mathbf{q}, \rho) \in \mathcal{S}_{p,q}(0, \infty)\) possessing the estimate:
\[
\|e^{\eta t}\partial_t \mathbf{u}\|_{L_p((0, \infty), L_q(B_R))} + \|e^{\eta t} \mathbf{q}\|_{L_p((0, \infty), H^2_q(B_R))} + \|e^{\eta t} \nabla \mathbf{q}\|_{L_p((0, \infty), L_q(B_R))} + \|e^{\eta t} \rho\|_{L_p((0, \infty), W^{2-1/q}_q(S_R))} \leq C \epsilon
\]
for some positive constants \(C\) and \(\eta\) independent of \(\epsilon\).

**Remark 1.** (1) When \(\epsilon_0 \in (0, 1/4)\), \(x = \mathbf{h}_z(y, t)\) is a bijective map from \(B_R\) onto \(\Omega_t\). Let \(y = \Psi(x, t)\) be the inverse map, and then \(\mathbf{v} = \mathbf{u} \circ \Psi\) and \(\mathbf{p} = (q + \sigma (N - 1) R^{-1}) \circ \Psi\) are unique solutions of the equations (1). Thus, by Theorem 1 we can find \(\Omega_t\), \(\mathbf{v}\) and \(\mathbf{p}\) satisfying problem (1) uniquely for any \(t \in (0, \infty)\).

(2) By a real interpolation theorem, we see that
\[
\mathbf{u} \in BUC((0, T), B_{q, p}^{2-2/p}(B_R)), \quad \mathbf{h}_z(y, t) \in BUC((0, T), B_{q, p}^{3-1/p}(B_R)),
\]
where \(BUC\) denotes the set of bounded uniform continuous functions. Thus, the condition \(2/p + N/q < 1\) guarantees that \(\mathbf{v} \in C^0([0, T], C^1(\overline{\Omega}))\). Therefore, we can solve the Cauchy problem:
\[
\frac{d}{dt} x = \mathbf{v}(x, t) \quad \text{for } t > 0, \quad x|_{t=t_0} = \xi \in \overline{\Omega}
\]
for some small interval \([t_0, t_1] \subset [0, T]\), where \(\overline{\Omega}\) means the closure of \(\Omega\). Let \(x = x(\xi, t)\) be a unique solution of the Cauchy problem (16). And then, the kinetic condition \(V_N = \mathbf{v} \cdot \mathbf{n}\) in problem (1) guarantees that the Lagrangean transformation:
\[
x = \xi + \int_0^t \mathbf{v}(x(\xi, s), s) \, ds
\]
gives \(C^1\) diffeomorphism from \(\Omega\) onto \(\Omega_t\) for any \(t \in [t_0, t_1]\). This fact will be used to find the conservation of mass, the conservation of momentum and the conservation of angular momentum, which plays an essential role to guarantees the exponential decay properties of \(\mathbf{u}\) and \(\rho\).

2. **Derivation of the equations (10).** Let \(\Phi(\cdot, t) = H_\rho(\cdot) + \xi(t)\), and then the Hanzawa transformation is rewritten as \(x = y + \Phi(\rho, y, t) + \xi(t)\). Let \(T \in (0, \infty]\) and let \(\mathbf{v}\) and \(\mathbf{p}\) satisfy the equations (1) for \(t \in (0, T)\). In the following, we assume that (6) holds with small constant \(\epsilon_0 > 0\). The \((i, j)\)th component of Jacobi matrix for the Hanzawa transform is
\[
\frac{\partial x_i}{\partial y_j} = \delta_{ij} + \frac{\partial \Phi_{p,i}}{\partial y_j} \left( \Phi_p = (\Phi_{p,1}, \ldots, \Phi_{p,N}) \right)
\]
and so under the assumption (6) there exist some \(C^\infty\) functions \(V_{ij}^0(k)\) defined for \(|k| \leq \kappa_0\) with some small \(\kappa_0 \in (0, 1)\) such that \(V_{ij}^0(0) = 0\) and
\[
\frac{\partial y_j}{\partial x_i} = (\delta_{ij} + V_{ij}^0(\nabla \Phi_p))
\]
where \( k = (k_1, \ldots, k_N) \) is the corresponding variables to \( \nabla \Phi \). Let \( V^0(\mathbf{k}) \) be an \( N \times N \) matrix whose \((i, j)^{th}\) component is \( V^0_{ij}(\mathbf{k}) \), and then

\[
\nabla_x = (I + V^0(\nabla \Phi)) \nabla_y, \quad \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i} + \sum_{k=1}^{N} V^0_{ik}(\nabla \Phi) \frac{\partial}{\partial y_k},
\]

with \( \nabla_z = \nabla (z_1, \ldots, z_N) \). Let \( V_D(u, \rho) \) be an \( N \times N \) matrix whose \((i, j)^{th}\) component is \( V_{D,ij}(u, \rho) \) with

\[
V_{D,ij}(u, \rho) = \sum_{k=1}^{N} (V^0_{ik}(\nabla \Phi) \partial_k u_j + V^0_{jk}(\nabla \Phi) \partial_k u_i),
\]

and then,

\[
D(v) = D(u) + V_D(u, \rho), \quad D_{ij}(v) := D_{ij}(u) + V_{D,ij}(u, \rho).
\]

where \( u = (u_1, \ldots, u_N) \) and \( v = (v_1, \ldots, v_N) \). Moreover,

\[
\partial_t v(x, t) = \partial_t u(y + \Phi_\rho(y, t) + \xi(t), t)) = \partial_t u + (\partial_t \Phi_\rho + \xi(t)) \cdot \nabla u.
\]

To represent \( \xi'(t) \), we introduce the Jacobian \( J(y, t) \) of the Hanzawa transformation (5), and then there exists a polynomial \( V^1(\mathbf{k}) \) with \( V^1(0) = 0 \) such that

\[
J(y, t) = 1 + V^1(\nabla \Phi_\rho).
\]

Since \( |\Omega| = |B_R| \) as follows from (A1), by (3) and (20),

\[
\xi'(t) = \frac{1}{|B_R|} \int_{B_R} u(1 + V^1(\nabla \Phi_\rho)) dy.
\]

Thus, by (17) and (19), the first equation in (1) is transformed to the equation:

\[
0 = \partial_t v - \mu \text{Div} D(v) + \nabla p = \partial_t u - \mu \text{Div} D(u) + (I + V^0(\nabla \Phi_\rho)) \nabla q + \tilde{f}(u, q, \rho)
\]

with

\[
\tilde{f}(u, q, \rho) = u \cdot (I + V^0(\nabla \Phi_\rho)) \nabla u + (\partial_t \Phi_\rho + \frac{1}{|B_R|} \int_{B_R} u(1 + V^1(\nabla \Phi_\rho)) dy) \cdot \nabla u - \mu \text{Div} D(u, \rho) - \mu V^0(\nabla \Phi_\rho) \nabla (D(u) + V_D(u, \rho)).
\]

Let \( (I + V^0(\nabla \Phi_\rho))^{-1} = I + V_{-1}(\nabla \Phi_\rho) \) with \( V_{-1}(\nabla \Phi_\rho) = \sum_{j=1}^{\infty} (-V^0(\nabla \Phi_\rho))^j \), and then,

\[
\partial_t u - \mu \text{Div} D(u) + \nabla q = f(u, \rho)
\]

with

\[
f(u, \rho) = -V_{-1}(\nabla \Phi_\rho)(\partial_t u - \mu \text{Div} (D(u))) - (I + V_{-1}(\nabla \Phi_\rho)) \tilde{f}(u, \rho).
\]

Next we consider \( \text{div} v = 0 \) in (1). By (17),

\[
\text{div} v = \text{div} u + V^0(\nabla \Phi_\rho) : \nabla u.
\]

with \( V^0(\nabla \Phi_\rho) : \nabla u = \sum_{j,k=1}^{N} V^0_{jk}(\nabla \Phi_\rho) \partial_j u_k \). To obtain another representation formula, we use the following integral formula:

\[
(\text{div} v, \varphi)_{\Omega} = -\langle v, \nabla \varphi \rangle_{\Omega} = -((I + V^1(\nabla \Phi_\rho)) u, (I + V^0(\nabla \Phi_\rho)) \nabla (\varphi \circ h_z))_{B_R}
\]

\[
= (\text{div} \{(1 + V^1(\nabla \Phi_\rho))^T (I + V^0(\nabla \Phi_\rho)) u\}, \varphi \circ h_z)_{B_R},
\]

for any \( \varphi \in C_0^\infty(\Omega) \), which yields that

\[
(1 + V^1(\nabla \Phi_\rho)) \text{div} v = \text{div} \{(1 + V^1(\nabla \Phi_\rho))^T (I + V^0(\nabla \Phi_\rho)) u\}.
\]
Combining (23) and (24), we have
\[
(1 + V^1(\nabla \Phi_y)) \text{div} \ v = \text{div} u + V^1(\nabla \Phi_y) \text{div} u + (1 + V^1(\nabla \Phi_y)) V^0(\nabla \Phi_y) : \nabla u \\
= \text{div} u + \text{div} (V^1(\nabla \Phi_y) u + (1 + V^1(\nabla \Phi_y)) V^0(\nabla \Phi_y) u)
\]
Thus, the divergence condition in (1) is transformed to
\[
\text{div} u = f_d(u, \rho) = \text{div} f_d(u, \rho)
\]
with
\[
f_d = -\{V^1(\nabla \Phi_y) \text{div} u + (1 + V^1(\nabla \Phi_y)) V^0(\nabla \Phi_y) : \nabla u\},
\]
\[
f_d = -\{V^1(\nabla \Phi_y) u + (1 + V^1(\nabla \Phi_y)) V^0(\nabla \Phi_y) u\}.
\]
Next, we consider the kinetic condition: \( V_N = v \cdot n_t \) on \( \Gamma_t \). For this purpose, we represent \( n_t \). By (2), the \( \Gamma_t \) is represented by \( x = (1 + R^{-1} \rho(y, t)) y + \xi(t) \) with \( y \in S_R \). Let \( S_{\rho} \) be parametrized locally by \( y = y(u) \) with \( u = (u_1, \ldots, u_{N-1}) \in U \subset \mathbb{R}^{N-1} \), and then \( \Gamma_t \) is parametrized by
\[
x = (1 + \eta(u, t)) y(u) + \xi(t) \quad \text{for} \quad u \in U \quad \text{with} \quad \eta(u, t) = R^{-1} \rho(y(u), t).
\]
Let \( n_t = \mu(y + \sum_{k=1}^{N-1} a_k \tau_k) \) with \( \tau_k = \partial y(u)/\partial u_k \) \((k = 1, \ldots, N - 1)\). Since \( |y|^2 = R^2 \), we have \( y \cdot \tau_k = 0 \) for \( k = 1, \ldots, N - 1 \). Since \( |n_t| = 1 \), we have
\[
1 = \mu^2(y + \sum_{k=1}^{N-1} a_k \tau_k) \cdot (y + \sum_{k=1}^{N-1} a_k \tau_k) = \mu^2(R^2 + \sum_{k=1}^{N-1} k \ell g_{k \ell} a_k a_\ell),
\]
where \( g_{k \ell} = \tau_k \cdot \tau_\ell \). Let \( G \) be the \( N \times N \) matrix whose \((k, \ell)\) element is \( g_{k \ell} \), which is the first fundamental form of \( S_{\rho} \), and let \( G^{-1} \) be the inverse matrix of \( G \). Let \( g^{ij} \) be the \((i, j)\)th component of \( G^{-1} \). Since \( \Gamma_t \) is given by (26), we have
\[
0 = n_t \cdot \frac{\partial x}{\partial u_k} = \mu(y + \sum_{k=1}^{N-1} a_k \tau_k) \cdot \left( \frac{\partial \eta}{\partial u_k} y + (1 + \eta) \tau_k \right)
\]
\[
= \mu \{ R^2 \frac{\partial \eta}{\partial u_k} + (1 + \eta) \sum_{k=1}^{N-1} g_{k \ell} a_\ell \},
\]
which leads to
\[
a_i = -\frac{R^2}{1 + \eta} \sum_{k=1}^{N-1} g^{ik} \frac{\partial \eta}{\partial u_k}, \quad (28)
\]
Combining (27) and (28) yields that
\[
\mu = R^{-1} \left( 1 + \frac{R^2}{(1 + \eta)^2} \sum_{k, \ell=1}^{N-1} g^k_\ell \frac{\partial \eta}{\partial u_k} \frac{\partial \eta}{\partial u_\ell} \right)^{-1/2} = R^{-1}(1 + V^2(\rho)) \quad (29)
\]
with
\[
V^2(\rho) = -\frac{1}{2} \int_0^1 \left( 1 + s \frac{R^2}{(1 + \eta)^2} \sum_{k, \ell=1}^{N-1} g^k_\ell \frac{\partial \eta}{\partial u_k} \frac{\partial \eta}{\partial u_\ell} \right)^{-3/2} ds \times \frac{R^2}{(1 + \eta)^2} \sum_{k, \ell=1}^{N-1} g^k_\ell \frac{\partial \eta}{\partial u_k} \frac{\partial \eta}{\partial u_\ell}, \quad (30)
\]
where \( \eta = R^{-1}\rho(y(u), t) \), provided that \( \epsilon_0 \) is sufficiently small in (6). Summing up, we have proved that

\[
\mathbf{n}_t - \omega = -\frac{R}{1 + \eta} \sum_{k, \ell=1}^{N-1} g^{k\ell} \frac{\partial \eta}{\partial u_\ell} \tau_k + V^2(\rho) \left( \omega - \frac{R}{1 + \eta} \sum_{k, \ell=1}^{N-1} g^{k\ell} \frac{\partial \eta}{\partial u_\ell} \tau_k \right)
\]  

(31)

holds in a local chart with \( \omega = y/R = y/|y| \in S_1 \) and \( \eta = R^{-1}\rho(y(u), t) \). In particular, we have

\[
< \mathbf{n}_t - \omega, \omega > = V^2(\rho).
\]  

(32)

Since

\[
V_N = \frac{\partial x}{\partial t}, \mathbf{n}_t > = \rho \omega + \frac{1}{|B_R|} \int_{B_R} u(1 + V^1(\nabla \Phi_\rho)) dy, \mathbf{n}_t >,
\]

as follows from the fact that \( |\Omega| = |B_R| \), (2), (3) and (20), the kinetic condition: \( V_N = \mathbf{v} \cdot \mathbf{n}_t \) leads to

\[
\rho_t - \left( u - \frac{1}{|B_R|} \int_{B_R} u dy \right) \cdot \omega = k_{in}(u, \rho)
\]

with

\[
k_{in}(u, \rho) = \left( \frac{1}{\omega, \mathbf{n}_t} - 1 \right) \left( u - \frac{1}{|B_R|} \int_{B_R} u dy \right) \cdot \omega - \frac{1}{\omega, \mathbf{n}_t} \left( u - \frac{1}{|B_R|} \int_{B_R} u dy \right) \cdot (\mathbf{n}_t - \omega) + \frac{1}{\omega, \mathbf{n}_t} \int_{B_R} u V^1(\nabla \Phi_\rho) dy \cdot \mathbf{n}_t
\]  

(33)

Next, we consider the boundary condition in (1). We use the following lemma due to Solomnikov [27, p.155]

**Lemma 2.1.** If \( \mathbf{n}_t \cdot \omega \neq 0 \), then for any vector \( \mathbf{d} \), \( \mathbf{d} = 0 \) is equivalent to

\[
\Pi_0 \Pi_1 \mathbf{d} = 0 \quad \text{and} \quad \omega \cdot \mathbf{d} = 0.
\]

Here, \( \Pi_0 \) and \( \Pi_1 \) are defined by

\[
\Pi_0 \mathbf{d} = \mathbf{d} - < \mathbf{d}, \mathbf{n}_t > \mathbf{n}_t, \quad \Pi_1 \mathbf{d} = \mathbf{d} - < \mathbf{d}, \omega > \omega.
\]

**Remark 2.** We can replace \( \omega \) by any unit vector \( \mathbf{n} \) with \( \mathbf{n}_t \cdot \mathbf{n} \neq 0 \). In this case, \( \Pi_0 \) is defined by

\[
\Pi_0 \mathbf{d} = \mathbf{d} - < \mathbf{d}, \mathbf{n} > \mathbf{n}.
\]

In view of Lemma 2.1, boundary conditions in (1) are equivalent to

\[
\Pi_0 \Pi_1 \mathbf{D}(\mathbf{v}) \mathbf{n}_t = 0 \quad \text{and} \quad < (\mu \mathbf{D}(\mathbf{v}) - \rho \mathbf{I}) \mathbf{n}_t - \sigma \mathbf{H}(\Gamma_1) \mathbf{n}_t, \omega > = 0.
\]  

(34)

Since

\[
\Pi_0 \Pi_1 \mathbf{d} = \Pi_0 (\Pi_1 - \Pi_0) \mathbf{d} + \Pi_0 \mathbf{d}
\]

\[
= < \mathbf{d}, \mathbf{n}_t > \{ (\omega - \mathbf{n}_t) - < \omega - \mathbf{n}_t, \omega > \} + \mathbf{d} - < \mathbf{d}, \omega > \omega,
\]

by (19) and (31) the condition: \( \Pi_0 \Pi_1 \mathbf{D}(\mathbf{v}) \mathbf{n}_t = 0 \) leads to \( \Pi_0 (\mu \mathbf{D}(\mathbf{v}) \omega) = g'(\mathbf{u}, \rho) \) with

\[
g'(\mathbf{u}, \rho) = < \mathbf{D}(\mathbf{u}) + V_{\mathbf{D}}(\mathbf{u}, \rho), \mathbf{n}_t > \Pi_0 (\mathbf{n}_t - \omega) + \Pi_0 (\mathbf{D}(\mathbf{u})(\mathbf{n}_t - \omega)) - \Pi_0 (V_{\mathbf{D}}(\mathbf{u}, \rho) \mathbf{n}_t).
\]  

(35)
Before turning to the second condition in (34), using the formula: $\mathcal{H}(\Gamma_t)n_t = \Delta_{\Gamma_t}x$ for $x \in \Gamma_t$ we calculate $\langle \mathcal{H}(\Gamma_t)n_t, \omega \rangle$. According to the parametrization of $\Gamma_t$ given above, we know that

$$\Delta_{\Gamma_t}f = \frac{1}{g_t} \sum_{i,j=1}^{N-1} \frac{\partial}{\partial u_i} \left( g_t g_t^{ij} \frac{\partial f}{\partial u_j} \right)$$

(36)

with $g_t = \sqrt{\det \mathbf{G}_t}$, where $\mathbf{G}_t = (g_{tij})$ is the first fundamental form of $\Gamma_t$ and and $\mathbf{G}_t^{-1} = (g_t^{ij})$. Thus, we have

$$\langle \Delta_{\Gamma_t}x, \omega \rangle = \frac{1}{g_t} \sum_{i,j=1}^{N-1} \frac{\partial}{\partial u_i} \{ g_t g_t^{ij} ((1 + \eta) \frac{\partial y}{\partial u_j} + \frac{\partial \eta}{\partial u_j} y) \}, \omega \rangle$$

$$= \Delta_{\Gamma_t}t + \langle (1 + \eta) \sum_{i,j=1}^{N-1} g_t^{ij} < \frac{\partial^2 y}{\partial u_i \partial u_j}, \omega \rangle, \omega \rangle$$

(37)

because $\langle \frac{\partial y}{\partial u_j}, \omega \rangle = 0$ and $\langle y, \omega \rangle = R$.

To proceed with the calculation in (37), using the parametrization in (26), we represent $\mathbf{G}_t$, $g_t$ and $\mathbf{G}_t^{-1}$. Let us denote $\mathbf{a'} \otimes \mathbf{b'}$ be the $(N-1) \times (N-1)$ matrix whose $(i,j)^{th}$ component is $a_i b_j$ for any $\mathbf{a'} = (a_1, \ldots, a_{N-1})$ and $\mathbf{b'} = (b_1, \ldots, b_{N-1}) \in \mathbb{R}^{N-1}$. We then have

$$\det (\mathbf{I} + \mathbf{a'} \otimes \mathbf{b'}) = 1 + \mathbf{a'} \cdot \mathbf{b'}, \quad (\mathbf{I} + \mathbf{a'} \otimes \mathbf{b'})^{-1} = \mathbf{I} - \frac{\mathbf{a'} \otimes \mathbf{b'}}{1 + \mathbf{a'} \cdot \mathbf{b'}}$$

(38)

From (26) it follows that

$$\frac{\partial x}{\partial u_i} = \{ (1 + \eta) \frac{\partial y}{\partial u_j} + \frac{\partial \rho}{\partial u_j} \omega \}.$$

(39)

Since the $(i,j)^{th}$ component of $\mathbf{G}_t$ is $\frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j}$, by (39) we have

$$\mathbf{G}_t = ((1 + \eta)^2 \mathbf{G} + (\nabla' \rho) \otimes (\nabla' \rho)) = (1 + \eta)^2 \mathbf{G}(\mathbf{I} + ((1 + \eta)^{-2} \mathbf{G}^{-1} \nabla' \rho) \otimes (\nabla' \rho)),$$

where $\nabla' \rho = \frac{1}{2} \frac{\partial \rho}{\partial u_1}, \ldots, \frac{\partial \rho}{\partial u_{N-1}}$. Thus, by (38)

$$\det \mathbf{G}_t = (1 + \eta)^{2(N-1)}(\det \mathbf{G})(1 + (1 + \eta)^{-2} (\mathbf{G}^{-1} \nabla' \rho) \cdot (\nabla' \rho)),$$

$$\mathbf{G}_t^{-1} = (1 + \eta)^{-2} (\mathbf{I} - \frac{(\mathbf{G}^{-1} \nabla' \rho) \otimes \nabla' \rho}{(1 + \eta)^2 + (\mathbf{G}^{-1} \nabla' \rho) \cdot (\nabla' \rho)}) \mathbf{G}^{-1},$$

and so

$$g_t = g + V^3(\rho), \quad \mathbf{G}_t^{-1} = (1 + \eta)^{-2} \mathbf{G}^{-1} + V^4(\rho)$$

(40)

with

$$V^3(\rho) = ((1 + \eta)^{N-1} - 1)g + \frac{1}{2} \int_0^1 (1 + \tau(1 + \eta)^{-2} (\mathbf{G}^{-1} \nabla' \rho) \cdot (\nabla' \rho))^{-1/2} d\tau \times (1 + \eta)^{-2} (\mathbf{G}^{-1} \nabla' \rho) \cdot (\nabla' \rho);$$

$$V^4(\rho) = -(1 + \eta)^{-2} \frac{(\mathbf{G}^{-1} \nabla' \rho) \otimes (\nabla' \rho)}{(1 + \eta)^2 + (\mathbf{G}^{-1} \nabla' \rho) \cdot (\nabla' \rho)} \mathbf{G}^{-1}$$

(41)
(\eta = R^{-1} \rho(y(u), t)). From the first formula in (40) we have \( g_{t}^{-1} = g^{-1} + V^5(\eta) \) with
\[
V^5(\rho) = \frac{-V^3(\rho)}{(g + V^3(\rho))g}.
\]

Let \( V^4_{ij}(\rho) \) be the \((i, j)^{th}\) component of \( \mathbf{V}^4(\rho) \), and then \( g_{t}^{ij} = (1 + \eta)^{-2} g^{ij} + V^4_{ij}(\rho) \). Moreover,
\[
\Delta_{\Gamma_{t}} f = \frac{1}{g_{t}} \sum_{i,j=1}^{N-1} \frac{\partial}{\partial u_i} \left( g_{t} g_{ij} \frac{\partial f}{\partial u_j} \right) = \Delta_{S_{\Gamma_{t}}} f + \Delta_{\Gamma_{t}} f \tag{42}
\]
with
\[
\Delta_{\Gamma_{t}} f = \frac{1}{g} \sum_{i,j=1}^{N-1} \frac{\partial}{\partial u_i} \left( g \{(1 + \eta)^{-2} - 1\} g^{ij} + V^4_{ij}(\rho) \right) \frac{\partial f}{\partial u_j} \\
+ \frac{1}{g} \sum_{i,j=1}^{N-1} \frac{\partial}{\partial u_i} \left( (V_3(\rho)((1 + \eta)^{-2} g^{ij} + V^4_{ij}(\rho)) \frac{\partial f}{\partial u_j} \right) \\
+ V^5(\rho) \sum_{i,j=1}^{N-1} \frac{\partial}{\partial u_i} \left( (g + V^3(\rho))((1 + \eta)^{-2} g^{ij} + V^4_{ij}(\rho)) \frac{\partial f}{\partial u_j} \right). \tag{43}
\]

Thus, we have \( \Delta_{\Gamma_{t}} \rho = \Delta_{S_{\Gamma_{t}}} \rho + \Delta_{\Gamma_{t}} \rho \) and
\[
(1 + \eta) \sum_{i,j=1}^{N-1} g_{t}^{ij} < \frac{\partial^2 y}{\partial u_i \partial u_j}, \omega > = R^{-1}(1 + R^{-1} \rho)^{-1} < \Delta_{S_{\Gamma_{t}}} \omega, \omega > \\
+ (1 + \eta) \sum_{i,j=1}^{N-1} V^4_{ij}(\rho) < \frac{\partial^2 y}{\partial u_i \partial u_j}, \omega > .
\]

Note that
\[
< \Delta_{S_{\Gamma_{t}}} \omega, \omega > = -(N - 1). \tag{44}
\]

In fact, using the polar coordinate: \( x = r\omega \in \mathbb{R}^N \) with \( r = |x| \), we have
\[
0 = \Delta x_{i} = \left( \frac{\partial^2}{\partial r^2} + \frac{N - 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S_{\Gamma_{t}}} \right) (r \omega_{i}) = \frac{1}{r} ((N - 1) \omega_{i} + \Delta_{S_{\Gamma_{t}}} \omega_{i}),
\]
and so, setting \( r = 1 \) yields (44). Moreover,
\[
R^{-1}(1 + R^{-1} \rho)^{-1} = R^{-1} - R^{-2} \rho + \frac{\rho^2}{R^2(R + \rho)}.
\]

Thus, recalling (37), we have
\[
< \mathcal{H}(\Gamma_{t}) n_{t}, \omega > = \left( \Delta_{S_{\Gamma_{t}}} + \frac{N - 1}{R^2} \right) \rho - \frac{N - 1}{R} + V^6(\rho) \tag{45}
\]
with
\[
V^6(\rho) = \Delta_{\Gamma_{t}} \rho - \frac{(N - 1) \rho^2}{R^2(R + \rho)} + (1 + \eta) \sum_{i,j=1}^{N-1} V^4_{ij}(\rho) < \frac{\partial^2 y}{\partial u_i \partial u_j}, \omega > .
\]

(\eta = R^{-1} \rho(y(u), t)).

Finally, by (19), (31) and (45),
\[
\mu < \mathbf{D}(u) \omega, \omega > - < n_{t}, \omega > \left( p - \frac{\sigma(N - 1)}{R} \right) - \sigma \left( \Delta_{S_{\Gamma_{t}}} + \frac{N - 1}{R^2} \right) \rho = \tilde{g}_{n}(u, \rho)
\]
on $S_R \times (0, T)$ with
\[
\tilde{g}_n(u, q, \rho) = -\langle \mu D(u)(n_t - \omega), \omega \rangle - \langle \mu V_D(u, \rho)n_t, \omega \rangle \\
+ \sigma V^\rho(\rho) + \sigma (N - 1) \frac{1}{R} < n_t - \omega, \omega >.
\]
Thus, we have
\[
\mu < D(u)\omega, \omega > - \left( p - \frac{\sigma(N - 1)}{R} \right) - \sigma \left( \Delta S_R + \frac{N - 1}{R^2} \right) \rho = g_n(u, \rho) \tag{46}
\]
on $S_R \times (0, T)$ with
\[
g_n(u, q, \rho) = \left(1 - \frac{1}{< n_t, \omega >}\right) \left( \mu < D(u)\omega, \omega > - \sigma \left( \Delta S_R + \frac{N - 1}{R^2} \right) \rho \right) \\
- \frac{1}{< n_t, \omega >} < \mu D(u)(n_t - \omega), \omega > + \mu V_D(u, \rho)n_t, \omega > \\
- \sigma V^\rho(\rho) - \frac{\sigma(N - 1)}{R} < n_t - \omega, \omega >.
\]

3. Decay estimate for the linearized equations. In this section and next section, we consider the following linear equations that is the linear part of problem (10):
\[
\begin{cases}
\partial_t u - \text{Div} (\mu D(u) - pI) = f & \text{in } B_R \times (0, T), \\
\text{div } u = f_d = \text{div } f_d & \text{in } B_R \times (0, T), \\
\partial_t \rho - \omega : P u = g & \text{on } S_R \times (0, T), \\
(\mu D(u) - pI)\omega - \sigma(B\rho)\omega = h & \text{on } S_R \times (0, T), \\
(u, \rho)|_{t=0} = (w_0, \rho_0) & \text{on } B_R \times S_R.
\end{cases}
\tag{47}
\]
Let $\iota$ be an extension map from $L_{1,loc}(B_R)$ into $L_{1,loc}(\mathbb{R}^N)$ having the following properties:
(e-1) For any $1 < q < \infty$ and $f \in H^1_q(B_R)$, $\iota f \in H^1_q(\mathbb{R}^N)$, $\iota f = f$ on $B_R$ and $\|\iota f\|_{H^1_q(\mathbb{R}^N)} \leq C_q \|f\|_{H^1_q(B_R)}$ for $i = 0, 1, 2, 3$;
(e-2) For any $1 < q < \infty$ and $f \in H^1_q(B_R)$, $\|\iota (\nabla f)\| W^{-1}_{q,1}(\mathbb{R}^N) \leq C_q \|f\|_{L_q(B_R)}$.
In the following, such $\iota$ is fixed. Let $W^1_q(B_R)$ be a space defined by
\[
W^{-1}_q(B_R) = \{ f \in L_{1,loc}(\mathbb{R}) \mid \| f\| W^{-1}_q(B_R) := \| \iota f\| W^{-1}_q(\mathbb{R}^N) < \infty \}.
\]
Let $S_{p,q}((a, b))$ be the solution class defined before Theorem 1 in Sect. 1, and let
\[
D_{p,q}((a, b)) = \{(f, f_d, f_d, g, h) \mid f \in L_p((a, b), L_q(B_R)^N), \\
f_d \in L_p((a, b), H^1_p(B_R)) \cap H^1_p((a, b), W^{-1}_q(B_R)), \\
f_d \in H^1_p((a, b), L_q(B_R)^N)), \\
g \in L_p((a, b), W^{2-1/q}(S_R)), \\
h \in H^1_p((a, b), W^{-1}_q(B_R)^N) \cap L_p((a, b), H^1(B_R)^N)\}
\]
We prove

**Theorem 3.1.** Let $1 < p, q < \infty$ and $T > 0$. Let $\{p_i\}_{i=1}^M$ and $\{q_j\}_{j=1}^{N+1}$ be the orthonormal basis of $\mathcal{R}_d = \{ u \mid D(u) = 0 \}$ and $\{\psi \mid B\psi = 0\} \cup \mathbb{C}$ given in Sect. 1. Then, for any initial data $w_0 \in B^2_{2/p}(B_R)$, $\rho_0 \in B^{3-1/p-1/q}_{q,p}(S_R)$, and right members $f, f_d, f_d, g, h \in D_{p,q}((0, T))$, satisfying the compatibility condition:
\[
\text{div } w_0 = f_d|_{t=0} = \text{div } f_d|_{t=0} \quad \text{in } B_R, \\
\Pi_0(\mu D(w_0)\omega) = \Pi_0(h|_{t=0}) \quad \text{on } S_R \quad \text{when } 2/p + 1/q < 1 \text{ in addition},
\tag{48}
\]
problem (47) admits a unique solution \((u, p, \rho) \in S_{p,q}(0, T)\) possessing the estimate:

\[
\mathcal{I}_{p,q}(u, p, \rho; \eta, (0, t)) \leq C \left\{ J_{p,q}(w_0, \rho_0, f, d, g, h; \eta, (0, t)) \right. \\
+ \sum_{l=1}^{M} \left( \int_{0}^{t} (e^{\eta s}|(u(\cdot, s), p_t)|^p) \, ds \right)^{1/p} \\
+ \sum_{j=1}^{N+1} \left( \int_{0}^{t} (e^{\eta s}|(\rho(\cdot, s), \varphi_j)|^p) \, ds \right)^{1/p} \right\}
\]

(49)

for any \(t \in (0, T]\) with some positive constants \(\eta\) and \(C\) independent of \(t\) and \(T\), where for any \(0 \leq a < b \leq \infty\) and \(t \in (a, b)\) we have set

\[
\mathcal{I}_{p,q}(u, p, \rho; \eta, (a, t)) = \|e^\eta \partial_t u\|_{L_p((a, t), L_q(B_R))} + \|e^\eta u\|_{L_p((a, t), H^2_q(B_R))} + \|e^\eta \nabla \rho\|_{L_p((a, t), L_q(B_R))} + \|e^\eta s\|_{L_p((a, t), W^{2-2/q}_q(S_R))};
\]

\[
J_{p,q}(w_0, \rho_0, f, d, g, h; \eta, (a, t)) = \|w_0\|_{B^{3/2-p}_{q, p}(B_R)} + \|\rho_0\|_{B^{3-1/q-1/4}_q(S_R)} + \|e^\eta f\|_{L_p((a, t), L_q(B_R))} + \|e^\eta g\|_{L_p((a, t), W^{2-2/q}_q(S_R))} + \|e^\eta h\|_{L_p((a, t), H^1_q(B_R))} + \max(1, (b - a)^{-1})\left( \|e^\eta \partial_d d\|_{L_p((a, t), W^{-1/q}_q(B_R))} + \|e^\eta \partial_d h\|_{L_p((a, t), L_q(B_R))} \right).
\]

To prove Theorem 3.1, we first consider the following shifted equations:

\[
\begin{aligned}
\partial_t u_1 + \lambda_0 u_1 - \text{Div} (\mu D(u_1) - p_1 I) = f & \quad \text{in} \ B_R \times (0, T), \\
\text{div} u_1 = d & \quad \text{in} \ B_R \times (0, T), \\
\partial_t p_1 + \lambda_0 p_1 - \omega \cdot P u_1 = g & \quad \text{on} \ S_R \times (0, T), \\
(\mu D(u_1) - p_1 I) \omega - \sigma(B p_1) \omega = h & \quad \text{on} \ S_R \times (0, T), \\
(u_1, p_1)|_{t=0} = (w_0, \rho_0) & \quad \text{on} \ B_R \times S_R.
\end{aligned}
\]

(50)

For the shifted equation (50), we have

**Theorem 3.2.** Let \(1 < p, q < \infty\) and \(T > 0\). Then, there exist positive constants \(\lambda_0 > 0\) and \(\eta_0\) such that if initial data \(w_0 \in B^{3/2-p}_{q, p}(B_R)\), \(\rho_0 \in B^{3-1/q-1/4}_q(S_R)\), and right members \((f, d, g, h) \in \mathcal{D}_{p,q}(0, T)) satisfy the compatibility condition (48), then problem (50) admits a unique solution \((u_1, p_1, \rho_1) \in S_{p,q}(0, T)) possessing the estimate:

\[
\mathcal{I}_{p,q}(u_1, p_1, \rho_1; \eta, (0, t)) \leq C J_{p,q}(w_0, \rho_0, f, d, g, h; \eta, (0, t)),
\]

(51)

for any \(t \in (0, T]\) and \(\eta \leq \eta_0\) with some positive constants \(C\) independent of \(t\) and \(T\), where

\[
\mathcal{I}_{p,q}(u, p, \rho; \eta, (a, b)), \quad J_{p,q}(w_0, \rho_0, f, d, g, h; \eta, (a, b))
\]

are the same norms as in Theorem 3.1.

**Proof.** Employing the argument in Shibata [20] and [21], we can prove the unique existence of \(u_1, p_1\) and \(\rho_1\) possessing the estimate (51) with the help of the \(R\) bounded solution operators for the generalized resolvent problem obtained by the Laplace transform of (50) with respect to time variable \(t\) and the Weis operator valued Fourier multiplier theorem, so we may omit the detailed proof. \(\square\)
We consider the solutions $u$, $p$ and $\rho$ of problem (47) of the form: $u = u_1 + v$, $p = p_1 + q$ and $\rho = \rho_1 + h$, where $u_1$, $p_1$ and $\rho_1$ are solutions of the shifted equations (50), and then $v$, $q$ and $h$ should satisfy the equations:

$$
\begin{align*}
\partial_t v - \text{Div} (\mu D(v) - q I) &= -\lambda_0 u_1, \quad \text{div } v = 0 \quad \text{in } B_R \times (0, T), \\
\partial_t h - \omega \cdot P v &= -\lambda_0 \rho_1 \quad \text{on } S_R \times (0, T), \\
(\mu D(v) - q I)\omega - \sigma(B h) \omega &= 0 \quad \text{on } S_R \times (0, T), \\
(v, h)|_{t=0} &= (0, 0) \quad \text{on } B_R \times S_R.
\end{align*}
$$

(52)

Let

$$
H^1_{q,0}(B_R) = \{ \varphi \in H^1_{q}(B_R) \mid \varphi|_{S_R} = 0 \},
$$

$$
J_q(B_R) = \{ u \in L^q(B_R)^N \mid (u, \nabla \varphi)_R = 0 \quad \text{for any } \varphi \in H^1_{q,0}(B_R) \}.
$$

(53)

Since $C_0^\infty(B_R)$ is dense in $H^1_{q,0}(B_R)$, the necessary and sufficient condition in order that $u \in J_q(B_R)$ is that $\text{div } u = 0$ in $B_R$. Let $\psi \in H^1_{q,0}(B_R)$ be a solution of the variational equation:

$$
(\nabla \psi, \nabla \varphi)_R = (u_1, \nabla \varphi)_R \quad \text{for any } \varphi \in H^1_{q,0}(B_R),
$$

(54)

and let $w = u_1 - \nabla \psi$. Then, $w \in J_q(B_R)$ and

$$
\|w\|_{L^q(B_R)} + \|\psi\|_{H^1_q(B_R)} \leq C\|u_1\|_{L^q(B_R)}.
$$

(55)

Using $w$ and $\psi$, we can rewrite the first equation in (52) as follows:

$$
\partial_t v - \text{Div} (\mu D(v) - (q + \lambda_0 \psi) I) = -\lambda_0 w, \quad \text{div } v = 0 \quad \text{in } B_R \times (0, T).
$$

Thus, in what follows we may assume that

$$
u_1 \in H^1_p((0, T), J_q(B_R)) \cap L^p((0, T), J_q(B_R) \cap H^2_q(B_R)^N).
$$

(56)

To handle problem (52), in view of Duhamel’s principle we consider the initial value problem:

$$
\begin{align*}
\partial_t v - \text{Div} (\mu D(v) - q I) &= 0, \quad \text{div } v = 0 \quad \text{in } B_R \times (0, \infty), \\
\partial_t h - \omega \cdot P v &= 0 \quad \text{on } S_R \times (0, \infty), \\
(\mu D(v) - q I)\omega - \sigma(B h) \omega &= 0 \quad \text{on } S_R \times (0, \infty), \\
(v, h)|_{t=0} &= (v_0, h_0) \quad \text{on } B_R \times S_R.
\end{align*}
$$

(57)

To handle equations (57) in the semigroup setting, we eliminate the pressure term $q$ by using a unique solution $K_1 = K_1(v)$ of the weak Dirichlet problem:

$$
(\nabla K_1, \nabla \varphi)_B = (\text{Div} (\mu D(v)) - \nabla \text{div } v, \nabla \varphi)_B \quad \text{for any } \varphi \in H^1_{q,0}(B_R)
$$

(58)

subject to $K_1 = \mu D(v) w, \omega > - \text{div } v$ on $S_R$, and a unique solution $K_2 = K_2(h)$ of the weak Dirichlet problem:

$$
(\nabla K_2, \nabla \varphi)_B = 0 \quad \text{for any } \varphi \in H^1_{q,0}(B_R),
$$

(59)

subject to $K_2 = -\sigma B h$ on $S_R$. In fact, given $v \in H^2_q(B_R)$, there exists a $K_1 = \mu \mu D(v)x, x > - \text{div } v + K'_1$, where $K'_1 \in H^1_{q,0}(B_R)$ is a unique solution of the variational equation:

$$
(\nabla K'_1, \nabla \varphi)_B = (\text{Div} (\mu D(v)) - \nabla \text{div } v - \nabla (\mu \mu D(v)x, x > - \text{div } v), \nabla \varphi)_B
$$

(59)

for any $\varphi \in H^1_{q,0}(B_R)$. In particular, we have

$$
\|K_1(v)\|_{H^1_q(B_R)} \leq c\|\nabla v\|_{H^1_q(B_R)}.
$$

(60)
On the other hand, to solve \((59)\), let \(H \in H^3_q(B_R)\) be a solution to the harmonic equation: \(\Delta H = 0\) in \(B_R\) subject to \(H|_{S_R} = -\sigma Bh\), which possesses the estimate:
\[
\|H\|_{H^3_q(B_R)} \leq C\|h\|_{W^{3-1/q}_q(S_R)}.
\]
And then, \(K_2\) is obtained by \(K_2 = H + K_3\), where \(K_3 \in H^{1}_q(\Omega)\) is a solution to the variational equation:
\[
(\nabla K_3, \nabla \varphi)_{B_R} = (\nabla H, \nabla \varphi)_{B_R} \quad \text{for any } \varphi \in H^{1}_q(\partial B_R).
\]
We also have the estimate:
\[
\|K_2(h)\|_{H^1_q(B_R)} \leq C\|h\|_{W^{3-1/q}_q(S_R)}, \quad (61)
\]
Instead of \((57)\), we consider the equation:
\[
\begin{cases}
\partial_t v - \text{Div} (\mu D(v) - (K_1(v) + K_2(h))I) = 0 & \text{in } B_R \times (0, \infty), \\
\partial_t h - \omega \cdot Fv = 0 & \text{on } S_R \times (0, \infty), \\
(\mu D(v) - (K_1(v) + K_2(h))I)\omega - \sigma(h)\omega = 0 & \text{on } S_R \times (0, \infty), \\
(v, h)|_{t=0} = (v_0, h_0) & \text{on } B_R \times S_R.
\end{cases} \quad (62)
\]
Note that \((\mu D(v) - (K_1(v) + K_2(h))I)\omega - \sigma(h)\omega = 0\) on \(S_R \times (0, \infty)\) is equivalent to
\[
\Pi_0(\mu D(v)\omega) = 0, \quad \text{div } v = 0 \quad \text{on } S_R \times (0, \infty). \quad (63)
\]
Moreover, \(\text{div } v = 0\) in \(B_R \times (0, T)\) provided that \(\text{div } v_0 = 0\) in \(B_R\). In fact, let \(t_0 \in (0, T)\), and for any \(\varphi \in C^\infty_0(B_R)\) let \(u\) be a unique solution to the reversed heat equations:
\[
\partial_t u + \Delta u = 0 \quad \text{in } B_R \times (0, t_0), \quad u|_{S_R} = 0, \quad u|_{t=t_0} = \varphi.
\]
Since \(u(\cdot, 0) \in H^1_q(\partial B_R)\), we have \((\text{div } u(\cdot, 0))_{B_R} = 0\) provided that \(\text{div } v_0 = 0\). Thus, noting that \(\partial_t u|_{S_R} = 0\), by \((58)\), \((59)\) and \((63)\),
\[
(\nabla u(\cdot, t), \nabla \varphi)_{B_R} = (\nabla (\cdot, t_0), \nabla u(\cdot, t_0))_{B_R} - (\nabla (\cdot, 0), \nabla u(\cdot, 0))_{B_R}
\]
\[
= \int_0^{t_0} \{(\text{div } u(\cdot, t), \nabla \varphi(\cdot, t))_{B_R} + (\nabla u(\cdot, t), \nabla \partial_t u(\cdot, t))_{B_R} \} dt
\]
\[
= \int_0^{t_0} \{(\text{div } v(\cdot, t), \nabla u(\cdot, t))_{B_R} - (\text{div } v, \partial_t u)_{B_R} \} dt
\]
\[
= \int_0^{t_0} \{(\nabla \text{div } v, \nabla u)_{B_R} - (\text{div } v, \partial_t u)_{B_R} \} dt
\]
\[
= -\int_0^{t_0} (\text{div } v, \partial_t u + \Delta u)_{B_R} dt = 0,
\]
which leads to \(\text{div } v = 0\) in \(B_R \times (0, T)\).

Let
\[
\dot{J}_q(B_R) = \{ f \in J_q(B_R) \mid (f, p)_{B_R} = 0 \ (\ell = 1, \ldots, M) \};
\]
\[
W^2_q(S_R) = \{ g \in W^2_q(S_R) \mid (g, \varphi)_{S_R} = 0 \ (j = 1, \ldots, N + 1) \};
\]
\[
\mathcal{H}_q = \{ F = (f, g) \mid f \in J_q(B_R), \ g \in W^{2-1/q}_q(S_R) \};
\]
Remark 3. Let $p$ be fixed. Moreover, since $\text{div} B \in \mathcal{A}$, the equations (57).

Theorem 3.1. Let $\mathcal{A}(v, h) = (\text{Div} (\mu D(v) - (K_1(v) + K_2(h))I) \omega - \sigma(Bh)\omega = 0)$ for $(v, h) \in \mathcal{D}_q$;

$$\|f, g\|_{\mathcal{H}_q} = \|f\|_{L_q(B_R)} + \|g\|_{W^{2-1/q}(S_R)};$$

$$\|v, h\|_{\mathcal{D}_q} = \|v\|_{H^2_q(B_R)} + \|h\|_{W^{3-1/q}(S_R)}.$$ We have

**Theorem 3.3.** Let $1 < q < \infty$. Then, the operator $\mathcal{A}$ generates a continuous analytic semigroup $\{T(t)\}_{t \geq 0}$ on $\mathcal{H}_q$, which is exponentially stable on $\mathcal{H}_q$, that is

$$\|T(t)(f, g)\|_{\mathcal{H}_q} \leq C e^{-nt}\|f, g\|_{\mathcal{H}_q} \quad (65)$$

for any $t > 0$ and $(f, g) \in \mathcal{H}_q$ with some positive constants $C$ and $\eta_1$.

**Remark 3.** Since $(v, h) = T(t)(v_0, h_0) \in \mathcal{D}_q$ for $t > 0$, we have $\text{div} v = 0$ for $t > 0$ in $B_R$. Thus, setting $q = K_1(v) + K_2(h)$, we see that $v$, $q$ and $h$ are solutions of the equations (57).

Postponing the proof of Theorem 3.3 to the next section, we continue to prove Theorem 3.1. Let

$$\tilde{u}_1 = u_1 - \sum_{\ell=1}^{M}(u_1(\cdot, p_{\ell})_{B_R} p_{\ell}, \tilde{\rho}_1 = \rho_1 - \sum_{j=1}^{N+1}(\rho_1(\cdot, \tilde{\varphi}_j)_{S_R} \tilde{\varphi}_j),$$

and then $(\tilde{u}_1, p_{\ell})_{B_R} = 0$ ($\ell = 1, \ldots, M$) and $(\tilde{\rho}_1, \tilde{\varphi}_j)_{S_R} = 0$ ($j = 1, \ldots, N + 1$). Moreover, since $\text{div} p_{\ell} = 0$, by (56) we have $\tilde{u}_1 \in L_p((0, T), \tilde{J}_q(B_R))$. Let

$$\tilde{v}, \tilde{h} = \int_0^s T(s-r)(-\lambda_0 \tilde{u}_1(\cdot, r), -\lambda_0 \tilde{\rho}_1(\cdot, r)) dr,$$

and then by the Duhamel principle, $\tilde{v}$ and $\tilde{h}$ satisfy the equations:

$$\begin{cases}
\partial_t \tilde{v} - \text{Div} (\mu D(\tilde{v}) - (K_1(\tilde{v}) + K_2(\tilde{h}))I) = -\lambda_0 \tilde{u}_1 \\
\partial_t \tilde{h} - \omega \cdot P\tilde{v} = -\lambda_0 \tilde{\rho}_1 \\
(\mu D(\tilde{v}) - (K_1(\tilde{v}) + K_2(\tilde{h}))I)\omega - \sigma(B\tilde{h})\omega = 0 \\
(\tilde{v}, \tilde{h})_{|t=0} = (0, 0)
\end{cases} \quad (66)$$

By (65),

$$\|\tilde{v}, \tilde{h}\|_{\mathcal{H}_q} \leq C \int_0^s e^{-\eta_1(s-r)}\|\tilde{u}_1(\cdot, r), \tilde{\rho}_1(\cdot, r)\|_{\mathcal{H}_q} dr$$

$$\leq C \left( \int_0^s e^{-\eta_1(s-r)} dr \right)^{1/p'} \left( \int_0^s e^{-\eta_1(s-r)}\|\tilde{u}_1(\cdot, r), \tilde{\rho}_1(\cdot, r)\|_{\mathcal{H}_q}^p dr \right)^{1/p}.$$

Choosing $\eta > 0$ smaller if necessary, we may assume that $0 < \eta p < \eta_1$ without loss of generality. Thus, using the inequality above, we have

$$\int_0^t (e^{\eta s}\|\tilde{v}, \tilde{h}\|_{\mathcal{H}_q})^p ds$$

$$\leq C \int_0^t \left( \int_0^s e^{\eta p s} e^{-\eta_1(s-r)}\|\tilde{u}_1(\cdot, r), \tilde{\rho}_1(\cdot, r)\|_{\mathcal{H}_q}^p dr \right) ds$$

$$= \int_0^t \left( \int_0^s e^{-(\eta - p \eta_1)(s-r)}(e^{\eta p r})\|\tilde{u}_1(\cdot, r), \tilde{\rho}_1(\cdot, r)\|_{\mathcal{H}_q}^p dr \right) ds$$
which, combined with (51), leads to

\[
\|e^{\eta r}(\hat{v}, \hat{h})\|_{L^p((0,t), H^s_\rho)} \leq C J_{p,q}(w_0, \rho_0, f, f_d, g, h; \eta, (0,t)).
\]  

(67)

Since \( \hat{v} \) and \( \hat{h} \) satisfy the shifted equations:

\[
\left\{\begin{align*}
\partial_t \hat{v} + \lambda_0 \hat{v} - \text{Div} \ (\mu D(\hat{v}) - (K_1(\hat{v}) + K_2(\hat{h}))I) &= -\lambda_0 \hat{u}_1 + \lambda_0 \hat{v} \quad \text{in } B_R \times (0,T), \\
\partial_t \hat{h} + \lambda_0 \hat{h} - \omega \cdot P\hat{v} &= -\lambda_0 \hat{\rho}_1 + \lambda_0 \hat{h} \quad \text{on } S_R \times (0,T), \\
\mu D(\hat{v}) - (K_1(\hat{v}) + K_2(\hat{h}))I \omega - \sigma(B\hat{h})\omega &= 0 \quad \text{on } S_R \times (0,T), \\
(\hat{v}, \hat{h})|_{t=0} &= (0,0) \quad \text{on } B_R \times S_R,
\end{align*}\right.
\]

by Theorem 3.2, (59) and (67),

\[
I_{p,q}(\hat{v}, \tilde{h}; \eta, (0,t)) \leq C J_{p,q}(w_0, \rho_0, f, f_d, g, h; \eta, (0,t)).
\]  

(68)

Let

\[
v = \hat{v} - \lambda_0 \sum_{\ell=1}^M \int_0^t (u_1(\cdot, s), p_\ell)_{B_R} ds p_\ell,
\]

\[
h = \hat{h} - \lambda_0 \sum_{j=1}^{N+1} \int_0^t (\rho_1(\cdot, s), \varphi_j)_{S_R} ds \varphi_j.
\]

Since \( D(p_\ell) = 0 \) and \( \text{div } p_\ell = 0 \), we have \( K_1(p_\ell) = 0 \). Since \( B\varphi_j = 0 \) on \( S_R \) for \( j = 2, \ldots, N + 1 \), we have \( K_2(\varphi_j) = 0 \) for \( j = 2, \ldots, N + 1 \). Since \( \varphi_1 = |S_1|^{-1} \) is a constant, \( B\varphi_1 = (N-1)R^{-2}\varphi_1 \) on \( S_R \), and so \( K_2(\varphi_1) = -\sigma(N-1)R^{-2}\varphi_1 \). Thus, by (66), we have

\[
\partial_t v - \text{Div} \ (\mu D(v) - (K_1(v) + K_2(h))I) = -\lambda_0 u_1, \quad \text{div } v = 0 \quad \text{in } B_R \times (0,T),
\]

\[
(\mu D(v) - (K_1(v) + K_2(h))I) \omega - \sigma(Bh)\omega = 0 \quad \text{on } S_R \times (0,T).
\]

Recall that \( p_\ell \cdot \omega|_{S_R} = 0 \) for \( \ell = 1, \ldots, M \). Moreover, recalling that \( p_\ell = |B_R|^{-1}e_\ell \ (\ell = 1, \ldots, N) \), we have

\[
Fp_\ell = |B_R|^{-1}(e_\ell - |B_R|^{-1} \int_{B_R} e_\ell dy) = 0,
\]

and therefore,

\[
\partial_t h - \omega \cdot P v = \partial_t \hat{h} - \omega \cdot P\hat{v} - \lambda_0 \sum_{j=1}^{N+1} (\rho_1(\cdot, t), \varphi_j)_{S_R} \varphi_j = -\lambda_0 \rho_1 \quad \text{on } S_R \times (0,T).
\]

Summing up, we have proved that \( v \) and \( h \) satisfy the equations (62).

By (68) and (49), we have

\[
\|e^{\eta r} \partial_s (v, h)\|_{L^p((0,t), L^q(B_R) \times W^{2-1/q}_q(S_R))} \leq C J_{p,q}(w_0, \rho_0, f, f_d, g, h; \eta, (0,t)).
\]  

(69)

To estimate \( \|e^{\eta r}(v, h)\|_{L^p((0,t), H^s_q(B_R) \times W^{3-1/q}_q(S_R))} \), we use the following lemma.
Lemma 3.4. Let $1 < q < \infty$. Let $u \in H^2_q(B_R)^N \cap J_q(B_R)$ and $\rho \in W^{2-1/q}_q(S_R)$ satisfy the equations:

$$
\begin{aligned}
-\text{Div} (\mu D(u) - (K_1(u) + K_2(\rho))I) &= f & \text{in } B_R, \\
\omega \cdot Pu &= g & \text{on } S_R, \\
(\mu D(u) - (K_1(u) + K_2(\rho))I)\omega - \sigma(B\rho)\omega &= 0 & \text{on } S_R.
\end{aligned}
$$

(70)

Then, there exists a constant $C > 0$ such that

$$
\|(u, \rho)\|_{S_q} \leq C \left\{ \|f, g\|_{L^q} + \sum_{\ell=1}^M \|(u, p_\ell)\|_{B_R} + \sum_{j=1}^{N+1} \|\rho, \varphi_j\|_{S_R} \right\}. 
$$

(71)

Postponing the proof of Lemma 3.4 to the next section, we continue to prove Theorem 3.1. By (52), $v$ and $h$ satisfy the elliptic equations:

$$
\begin{aligned}
-\text{Div} (\mu D(v) - (K_1(v) + K_2(h))I) &= -\lambda_0 u_1 - \partial_t v & \text{in } B_R, \\
\omega \cdot Pv &= \lambda_0 \rho_1 + \partial_t h & \text{on } S_R, \\
(\mu D(v) - (K_1(v) + K_2(h))I)\omega - \sigma(Bh)\omega &= 0 & \text{on } S_R,
\end{aligned}
$$

and therefore, applying Lemma 3.4 and using (69) and (51) yield that

$$
\|e^{nt}v\|_{L^p((0,t), H^2_q(B_R))} \leq C \left\{ J_{p,q}(w_0, r_0, f, f_0, f_0, g, h; \eta, (0, t)) \right\} + \sum_{\ell=1}^M \left( \int_0^t \left( e^{ns}\| (v(\cdot, s), p_\ell)\|_{B_R} \right) ds \right)^{1/p},
$$

(72)

where we have set $q = K_1(v) + K_2(h)$. Let $u = u_1 + v$, $p = p_1 + q$ and $\rho = \rho_1 + h$. By (50) and (52), $u$, $v$ and $\rho$ satisfy the equations (47). Since

$$
\begin{aligned}
\left( \int_0^t e^{ns}\| (v(\cdot, s), p_\ell)\|_{B_R} \right)^{1/p} \leq \left( \int_0^t e^{ns}\| (u(\cdot, s), p_\ell)\|_{B_R} \right)^{1/p} + CJ_n t, \\
\left( \int_0^t e^{ns}\| (h(\cdot, s), \varphi_j)\|_{S_R} \right)^{1/p} \leq \left( \int_0^t e^{ns}\| (\rho(\cdot, s), \varphi_j)\|_{S_R} \right)^{1/p} + CJ_n t,
\end{aligned}
$$

(73)

with $J_{n,t} = J_{p,q}(w_0, r_0, f, f_0, f_0, g, h; \eta, (0, t))$ as follows from (51), by (69), (72) and (73), we see that $u$, $v$ and $\rho$ satisfy the inequality (49).

4. Continuous analytic semigroup associated with (62). To treat the equations (57) and (62) in the semigroup setting, we consider the corresponding resolvent problems:

$$
\begin{aligned}
\lambda v - \text{Div} (\mu D(v) - (K_1(v) + K_2(h))I) &= f & \text{in } B_R, \\
\lambda h - \omega \cdot P v &= g & \text{on } S_R, \\
(\mu D(v) - (K_1(v) + K_2(h))I)\omega - \sigma(Bh)\omega &= 0 & \text{on } S_R.
\end{aligned}
$$

(74)

As was proved in Shibata [21], we have

Theorem 4.1. Let $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Then, there exists a positive constant $\lambda_0$ such that for any $\lambda \in \Sigma_{\varepsilon, \lambda_0}$, $f \in L^q(B_R)$ and $g \in W^{2-1/q}_q(S_R)$ problem
Lemma 4.3. \((74)\) admits unique solutions \(\mathbf{v} \in H_q^2(B_R)^N\) and \(h \in W_q^{3-1/q}(S_R)\) possessing the estimates:
\[
|\lambda|(|\mathbf{v}|_{L_q(B_R)} + \|h\|_{W_q^{3-1/q}(S_R)}) + \|\mathbf{v}\|_{H_q^2(B_R)} + \|h\|_{W_q^{3-1/q}(S_R)} \leq C(\|\mathbf{f}\|_{L_q(B_R)} + \|g\|_{W_q^{3-1/q}(S_R)}),
\]
(75)
Here and in the following, \(\Sigma_{\epsilon,\lambda_0}\) denotes the subset in \(\mathbb{C}\) defined by
\[
\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \mid \arg \lambda \leq \pi - \epsilon, \ |\lambda| \geq \lambda_0\}.
\]
Let \((\mathbf{v}, h) \in H_q^2(B_R)^N \times W_q^{3-1/q}(S_R)\) be solution of \((74)\). We see that
\[
\mathbf{v} \in J_q(B_1) \quad \text{provided that } \mathbf{f} \in J_q(B_1).
\]
(76)
Thus, from Theorem 4.1 it follows that the operator \(\mathcal{A}\) defined in \((64)\) generates a continuous analytic semigroup \(\{T(t)\}_{t \geq 0}\) on \(H_q\).

The assertion \((76)\) is proved as follows: For any \(\varphi \in H_{1,q}^1(B_R) \cap H_{q}^2(B_R)\), by \((58), (59),\) and \((63)\), we have
\[
0 = (\mathbf{f}, \nabla \varphi)_{B_R} = \lambda (\mathbf{v}, \nabla \varphi)_{B_R} - (\nabla \div \mathbf{v}, \nabla \varphi)_{B_R} = -(\div \mathbf{v}, \lambda \varphi - \Delta \varphi)_{B_R}.
\]
For any \(\lambda \in \Sigma_{\epsilon,\lambda_0}\) the operator \((\lambda - \Delta)\) is a bijection from \(H_{q}^2(B_R) \cap H_{1,q}^1(B_R)\) onto \(L_q(B_R)\), which follows from the following Lemma 4.2, and so \(\div \mathbf{v} = 0\) in \(B_R\).
Thus, \((\mathbf{v}, h) \in D_q\).

Lemma 4.2. Let \(1 < q < \infty\) and \(\lambda \in \mathbb{C} \setminus (-\infty, 0)\). Then, for any \(f \in L_q(B_R)\), the Dirichlet problem:
\[
(\lambda - \Delta)u = f \quad \text{in } B_R, \quad u|_{S_R} = 0,
\]
(77)
admits a unique solution \(u \in H_q^2(B_R)\).

Proof. For any large \(\lambda > 0\), we know the unique existence of solutions of the equations \((77)\). From the Riesz-Schauder theorem, especially the Fredholm alternative principle, we know that the uniqueness implies the existence. Thus, it suffices to prove the uniqueness. First we consider the case \(2 \leq q < \infty\). Let \(u \in H_q^2(B_R)\) satisfy the homogeneous equations:
\[
(\lambda - \Delta)u = 0 \quad \text{in } B_R, \quad u|_{S_R} = 0.
\]
(78)
Since \(u \in H_q^2(B_R)\), by the divergence theorem of Gauß
\[
0 = \lambda \|u\|_{L_2(B_R)}^2 + \|\nabla u\|_{L_2(B_R)}^2,
\]
and so, taking the real part and the imaginary part, we have
\[
(\text{Im} \lambda) \|u\|_{L_2(B_R)}^2 = 0, \quad (\text{Re} \lambda) \|u\|_{L_2(B_R)}^2 + \|\nabla u\|_{L_2(B_R)}^2 = 0.
\]
When \(\text{Im} \lambda \neq 0\), we have \(u = 0\). When \(\text{Re} \lambda \geq 0\), we have \(\nabla u = 0\), which, combined with \(u|_{S_R} = 0\), leads to \(u = 0\). Thus, we have the unique existence theorem for \(2 \leq q < \infty\). When \(1 < q < 2\), the uniqueness follows from the existence theorem for the problem with dual exponents \(2 < q' < \infty\), and so we also have the unique existence theorem for \(1 < q < 2\), which completes the proof of Lemma 4.2. \(\square\)

Let \(\mathcal{H}_q\) be the space defined in \((64)\). To prove the exponential stability of \(\{T(t)\}_{t \geq 0}\) on \(\mathcal{H}_q\), we start with

Lemma 4.3. Let \(1 < q < \infty\), let \(\lambda \in \mathbb{C} \setminus (-\infty, 0)\), and let \(\mathbf{v} \in H_q^2(B_R)^N\) and \(h \in W_q^{3-1/q}(S_R)\) be solution of the equations \((74)\). If \((\mathbf{f}, g) \in \mathcal{H}_q\), then \((\mathbf{v}, h)\) also belongs to \(\mathcal{H}_q\).
Proof. First we prove that $\mathbf{v} \in \mathcal{J}_q(B_R)$. By (76), we know that $\mathbf{v} \in J_q(B_1)$. Since $(\mathbf{f}, \mathbf{p}_\ell)_{B_R} = 0$ as follows from $\mathbf{f} \in J_q(B_1)$, we have

$$0 = (\mathbf{f}, \mathbf{p}_\ell)_{B_R} = (\lambda \mathbf{v} - \text{Div} (\mu (\mathbf{D}(\mathbf{v}) - (K_1(\mathbf{v}) + K_2(h))))_{B_R}$$

$$= \lambda (\mathbf{v}, \mathbf{p}_\ell)_{B_R} - \sigma (B_h, \omega \cdot \mathbf{p}_\ell)_{B_R} + \frac{\mu}{2} (\mathbf{D}(\mathbf{v}), \mathbf{D}(\mathbf{p}_\ell))_{B_R}$$

$$- ((K_1(\mathbf{v}) + K_2(h)), \text{div} \mathbf{p}_\ell)_{B_R}.$$  

We see that

$$(B_h, \omega \cdot \mathbf{p}_\ell)_{B_R} = 0 \quad (\ell = 1, \ldots, M).$$  

(79)

In fact, recalling that $\mathbf{p}_\ell$ consists of some linear combinations of $x_k e_j - x_j e_k$, for $\ell = N + 1, \ldots, M$, and therefore $(B_h, \omega \cdot \mathbf{p}_\ell)_{B_R} = 0$ for $\ell = N + 1, \ldots, M$.

Next, we prove that $h \in \tilde{W}_q^{3-1/q}(S_R)$. Since $(g, \varphi_j)_{S_R} = 0$, by the divergence theorem of Gauß,

$$0 = (g, \varphi_j)_{S_R} - (P\mathbf{v} \cdot \omega, \varphi_j)_{S_R} = \lambda (h, \varphi_j)_{S_R} - \int_{B_R} \text{div} ((P\mathbf{v})\varphi_j) \, dx.$$  

We have

$$\int_{B_R} \text{div} ((P\mathbf{v})\varphi_j) \, dx$$

$$= \int_{B_R} (\text{div} (P\mathbf{v})) \varphi_j \, dx + \sum_{\ell=1}^{N+1} \int_{B_R} \left( v_\ell - |B_R|^{-1} \int_{B_R} v_\ell \, dy \right) (\partial_\ell \varphi_j) \, dx = 0,$$  

(80)

because $\text{div} P\mathbf{v} = \text{div} \mathbf{v} = 0$, and $\partial_\ell \varphi_j$ are constants. Thus, $\lambda (h, \varphi_j)_{S_R} = 0$, which, combined with $\lambda \neq 0$, leads to $(h, \varphi_j)_{S_R} = 0$. This completes the proof of Lemma 4.3.

Combining Theorem 4.1 and Lemma 4.3, we have

**Corollary 4.1.** Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and let $\mathcal{H}_q$ and $\mathcal{D}_q$ be the spaces defined in (64). Then, there exists a positive constant $\lambda_0$ such that for any $\lambda \in \Sigma_{\epsilon, \lambda_0}$ and $(\mathbf{f}, g) \in \mathcal{H}_q$ problem (74) admits a unique solution $(\mathbf{v}, h) \in \mathcal{D}_q \cap \mathcal{H}_q$ possessing the estimates (75).

In view of Corollary 4.1, in order to prove Theorem 3.3 it suffices to prove

**Theorem 4.4.** Let $1 < q < \infty$ and let $\lambda_0$ be the same positive number as in Corollary 4.1. Let

$$Q_{\lambda_0} = \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \geq 0, \ |\lambda| \leq \lambda_0 \}.$$  

Then, for any $\lambda \in Q_{\lambda_0}$ and $(\mathbf{f}, g) \in \mathcal{H}_q$ problem (74) admits a unique solution $(\mathbf{v}, h) \in \mathcal{D}_q \cap \mathcal{H}_q$ possessing the estimate:

$$\| (\mathbf{v}, h) \|_{\mathcal{D}_q} \leq C \| (\mathbf{f}, g) \|_{\mathcal{H}_q}$$  

(81)

with some constant $C$ independent of $\lambda \in Q_{\lambda_0}$.
Proof. Let $\mathcal{A}$ be the operator defined in (64), and then problem (74) is written as

$$
\lambda(v, h) - \mathcal{A}(v, h) = (f, g).
$$

(82)

Let $\hat{D}_q = D_q \cap \hat{\mathcal{H}}_q$. First, we observe that

$$
\mathcal{A}D_q \subset \hat{\mathcal{H}}_q.
$$

(83)

In fact, for $(v, h) \in \hat{D}_q$, we set $\mathcal{A}(v, h) = (f, g)$. For any $\varphi \in H^1_{q, 0}(B_R)$,

$$(f, \nabla \varphi)_B = (\text{Div} (\mu \mathbf{D}(v) - (K_1(v) + K_2(h))I), \nabla \varphi)_B = (\nabla \text{div} v, \nabla \varphi)_B = 0,$$

because $\text{div} v = 0$, and so $(f, g) \in D_q(B_R)$.

Next, we observe that

$$(f, p_\ell)_B = (\text{Div} (\mu \mathbf{D}(v) - (K_1(v) + K_2(h))I), p_\ell)_B = \sigma(\mathcal{B}h, \omega \cdot p_\ell)_S - \frac{\mu}{2}(\mathbf{D}(v), \mathbf{D}(p_\ell))_B + (K_1(v) + K_2(h), \text{div} p_\ell)_B.$$  

Thus, by (79) and the facts that $\mathbf{D}(p_\ell) = 0$ and $\text{div} p_\ell = 0$, we have $(f, p_\ell)_B = 0$, and so $(f, g) \in \hat{D}_q(B_R)$.

Finally, by (80) we have

$$(g, \varphi_j)_S = (\omega \cdot P\mathbf{v}, \varphi_j)_S = \int_{B_R} \text{div} ((P\mathbf{v})\varphi_j) \, dx = 0,$$

and so $(f, g) \in W^{3-1/q}_q(S_R)$, which completes the proof of (83).

In view of Corollary 4.1, $(\lambda_0 I - A)^{-1}$ exists as a bounded linear operator from $\hat{\mathcal{H}}_q$ onto $\hat{D}_q$, and then, the equation (82) is rewritten as

$$(f, g) = (\lambda I - A)(v, h) = (\lambda - \lambda_0)(v, h) + (\lambda_0 I - A)(v, h),$$

If $(I + (\lambda - \lambda_0)(\lambda_0 I - A)^{-1})$ exists as a bounded linear operator from $\hat{\mathcal{H}}_q$ into itself, then we have

$$(v, h) = (\lambda_0 I - A)^{-1}(I + (\lambda - \lambda_0)(\lambda_0 I - A)^{-1})(f, g).$$

(84)

Thus, our task is to prove that the inverse operator $(I + (\lambda - \lambda_0)(\lambda_0 I - A)^{-1})$ exists. By Rellich compact embedding theorem $H^2_q(B_R)$ and $W^{3-1/q}_q(S_R)$ are compactly embedded into $L_q(B_R)$ and $W^{2-1/q}_q(S_R)$, respectively, and so $(\lambda_0 I - A)^{-1}$ is a compact operator from $\hat{\mathcal{H}}_q$ into itself. Thus, in view of Riesz-Schauder theory, in order to prove the existence of the inverse operator $(I + (\lambda - \lambda_0)(\lambda_0 I - A)^{-1})^{-1}$ it suffices to prove that the kernel of the map $I + (\lambda - \lambda_0)(\lambda_0 I - A)^{-1}$ is trivial. Thus, let $(f, g)$ be an element in $\hat{\mathcal{H}}_q$ such that

$$(I + (\lambda - \lambda_0)(\lambda_0 I - A)^{-1})(f, g) = (0, 0).$$

(85)

Our task is to prove that $(f, g) = (0, 0)$. Notice that $(f, g) \in \hat{D}_q$ because $(f, g) = -(-\lambda - \lambda_0)(\lambda_0 I - A)^{-1}(f, g)$. We then have $(\lambda_0 I - A)(f, g) = -(-\lambda - \lambda_0)(f, g)$, and so $(f, g)$ satisfies the homogeneous equation:

$$(\lambda I - A)(f, g) = (0, 0).$$

(86)

Namely, $(f, g) \in \hat{D}_q$ satisfies the homogeneous equations:

$$
\begin{cases}
\lambda f - \text{Div} (\mu \mathbf{D}(f) - (K_1(f) + K_2(g))I) = 0 & \text{in } B_R, \\
\lambda g - \omega \cdot Pf = 0 & \text{on } S_R, \\
(\mu \mathbf{D}(f) - (K_1(f) + K_2(g))I)\omega - \sigma(\mathbf{B}g)\omega = 0 & \text{on } S_R.
\end{cases}
$$

(87)
We first consider the case where $2 \leq q < \infty$. Since $(f, g) \in \hat{D}_q \subset \hat{D}_2$, by (87) and the divergence theorem of Gauss,

$$0 = (\lambda f - \text{Div} (\mu D(f) - (K_1(f) + K_2(g)) \mathbf{I}) f)_{B_R}$$

$$= \lambda \|f\|_{L_2(B_R)}^2 - \sigma (Bg, \omega f)_{S_R} + \frac{\mu}{2} \|D(f)\|_{L_2(B_R)}^2 - (K_1(f) + K_2(g), \text{div} f)_{B_R}.$$  

For $h \in H^2_q(S_R)$ and $g = (g_1, \ldots, g_N)$,

$$(Bh, P g \cdot \omega)_{S_R} = (Bh, g \cdot \omega)_{S_R}, \quad (88)$$

because

$$\sum_{j=1}^N |B_R|^{-1} \int_{B_R} g_t dy(Bh, \omega_t)_{S_R} = \sum_{j=1}^N |B_R|^{-1} \int_{B_R} g_t dy(h, R^{-2}(N - 1 + \Delta_S) \omega_t)_{S_R} = 0.$$  

Moreover, $\text{div} f = 0$, because $f \in J_q(B_R)$. Thus, noting that $\lambda g = Pf \cdot \omega$ on $S_R$, we have

$$\lambda \|f\|_{L_2(B_R)}^2 - \sigma \lambda (Bg, g)_{S_R} + \frac{\mu}{2} \|D(f)\|_{L_2(B_R)}^2 = 0. \quad (89)$$

To treat $(Bg, g)_{S_R}$, we use the following lemma.

**Lemma 4.5.** Let

$$\hat{H}^2_q(S_R) = \{ h \in H^2_q(S_R) \mid (h, 1)_{S_R} = 0, (h, x_j)_{S_R} = 0 \ (j = 1, \ldots, N) \}.$$  

Then,

$$- (Bh, h) \geq c \|h\|_{L_2(S_R)}^2 \quad (90)$$

for any $h \in \hat{H}^2_q(S_R)$ with some constant $c > 0$.

Postponing the proof of Lemma 4.5, we continue the proof of Theorem 4.4. Since $g \in \hat{W}^{3-1/q}_q(S_R) \subset \hat{H}^2_q(S_R)$, taking the real part of (89) we have

$$0 = \text{Re} \lambda \|f\|_{L_2(B_R)}^2 - \sigma (Bg, g)_{S_R} + \frac{\mu}{2} \|D(f)\|_{L_2(B_R)}^2$$

$$\geq \text{Re} \lambda \|f\|_{L_2(B_R)}^2 + c \sigma \|g\|_{L_2(S_R)}^2 + \frac{\mu}{2} \|D(f)\|_{L_2(B_R)}^2,$$

which, combined with $\text{Re} \lambda \geq 0$, leads to $D(f) = 0$. But, $(f, P_r)_{B_R^\ell} = 0$ for $\ell = 1, \ldots, M$, and so $f = 0$. Thus, by the first equation in (87), $\nabla (K_1(f) + K_2(g)) = 0$, and so $K_1(f) + K_2(g) = f_0$ with some constant $f_0$, which, combined with the third equation in (87), leads to $B g = -\sigma^{-1} f_0$ on $S_R$. Since $(g, 1)_{S_R} = |S_R|(g, \varphi_1)_{S_R} = 0$, we have

$$-\sigma^{-1}|S_R|f_0 = (Bg, 1)_{S_R} = (g, \Delta_{S_R} 1)_{S_R} + R^{-2}(N - 1)(g, 1)_{S_R} = 0,$$

and so $f_0 = 0$, which implies that $B g = 0$ on $S_R$. Recalling that $g \in \hat{W}^{3-1/q}_q(S_R) \subset \hat{H}^2_q(S_R)$, by Lemma 4.5 $g = 0$. Thus, we have $(f, g) = (0, 0)$, and so the formula (84) holds. Namely, problem (74) admits a unique solution $(v, h) \in \hat{D}_q$ possessing the estimate:

$$\|v, h\|_{\hat{D}_q} \leq C_\lambda \|f, g\|_{\hat{H}_q} \quad (91)$$

with some constant $C_\lambda$ depending on $\lambda$, when $2 \leq q < \infty$ and $\lambda \in Q_{\lambda_0}$.

Before considering the case where $1 < q < 2$, at this point we give a

**Proof of Lemma 4.5.** Let $\{\lambda_j\}_{j=1}^\infty$ be the set of all eigen-values of the Laplace-Beltrami operator $\Delta_{S_R}$ on $S_R$. We may assume that $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_j > \cdots \to -\infty$, and then $\lambda_1 = 0$ and $\lambda_2 = -(N - 1)R^{-2}$. Let $E_j$ be the eigen-space
corresponding to $\lambda_j$, and then the dimension of $E_j$ is finite (cf. Neri [11, Chapter III, Spherical Harmonics]). Let $d_j = \dim E_j$, and then $d_1 = 1$ and $d_2 = N$. In particular, $E_1 = \{ a \mid a \in \mathbb{C} \}$ and $E_2 = \{ a_1 x_1 + \cdots + a_N x_N \mid a_i \in \mathbb{C} (i = 1, \ldots, N) \}$. Let $\{ \varphi_{ij} \}_{j=1}^{d_i}$ be the orthogonal basis of $E_i$ in $L_2(S_R)$, and then for any $h \in \dot{H}_2^2(S_R)$ we have

$$h = \sum_{i=3}^{\infty} \sum_{j=1}^{d_i} a_{ij} \varphi_{ij} \quad (a_{ij} = (h, \varphi_{ij})_{S_R}),$$

because $(h, \varphi_{ij})_{S_R} = 0$ for $i = 1, 2$. Thus, we have

$$-(Bh, h)_{S_R} = \sum_{i=3}^{\infty} \sum_{j=1}^{d_i} |a_{ij}|^2 (-\lambda_i - (N - 1)R^{-1}) \| \varphi_{ij} \|^2_{L_2(S_R)}.$$

Since $-\lambda_i - (N - 1)R^{-1} \geq c$ with some positive constant $c$ for any $i \geq 2$, we have (90), which completes the proof of Lemma 4.5.

Next, we consider the case where $1 < q < 2$. Let $(f, g) \in \hat{D}_q$ satisfy the homogeneous equations (87). First, we prove that

$$(f, g)_{B_R} = 0 \quad \text{for any } g \in \dot{J}_q(B_R). \tag{92}$$

Let $(u, \rho) \in \hat{D}_q'$ be a solution of the equations:

$$\begin{cases}
\tilde{\lambda} u - \text{Div}(\mu D(u)) - (K_1(u) + K_2(\rho)) I = g & \text{in } B_R, \\
\bar{\lambda} \rho - \omega \cdot Pu = 0 & \text{on } S_R, \\
(\mu D(u) - (K_1(u) + K_2(\rho)) I) \omega - \sigma(B\rho) \omega = 0 & \text{on } S_R.
\end{cases} \tag{93}$$

Since $\lambda \in Q_{\lambda_0}$, $\bar{\lambda} \in Q_{\lambda_0}$, and moreover $2 < q' < \infty$, and so by the fact proved above we know the unique existence of $(u, \rho) \in \hat{D}_q'$. By (87), (93) and the divergence theorem of Gauß,

$$(f, g)_{B_R} = (f, \tilde{\lambda} u - \text{Div}(\mu D(u)) - (K_1(u) + K_2(\rho)) I)_{B_R}$$

$$= \lambda(f, u)_{B_R} - (f \cdot \omega, \sigma(B\rho) S_R) + \frac{\mu}{2} (D(f), D(u))_{B_R} - (\text{div } f, K_1(u) + K_2(\rho))_{B_R}. \tag{94}$$

Noting that $P \omega = \lambda g$ and $\text{div } f = 0$ and using (88), we have

$$(f, g)_{B_R} = \lambda(f, u)_{B_R} + \sigma \lambda \{(\nabla g \cdot \nabla \rho)_{S_R} - R^{-2}(N - 1)(g, \rho)_{S_R}\}$$

$$+ \frac{\mu}{2} (D(f), D(u))_{B_R}. \tag{95}$$

On the other hand, we have

$$0 = (\lambda f - \text{Div} (\mu D(f) - (K_1(f) + K_2(g)) I), u)_{B_R}$$

$$= \lambda(f, u)_{B_R} - (Bg \cdot \omega, u)_{S_R} + \frac{\mu}{2} (D(f), D(u))_{B_R} - (K_1(f) + K_2(g), \text{div } u)_{B_R}. \tag{96}$$

Noting that $P \omega = \lambda g$ and $\text{div } u = 0$ and using (88), we have

$$0 = \lambda(f, u)_{B_R} + \sigma \lambda \{(\nabla g \cdot \nabla \rho)_{S_R} - R^{-2}(N - 1)(g, \rho)_{S_R}\} + \frac{\mu}{2} (D(f), D(u))_{B_R},$$

which, combined with (95), leads to (92).

Next, we prove that $(f, g)_{B_R} = 0$ for any $g \in L_q'(B_R)^N$. Given any $g \in L_q'(B_R)^N$, let $\psi \in H^{1}_{q,0}(B_R)$ be a solution to the variational equation:

$$(\nabla \psi, \nabla \varphi)_{B_R} = (g, \nabla \varphi)_{B_R} \quad \text{for any } \varphi \in H^{1}_{q,0}(B_R).$$
Let \( h = g - \nabla \psi \), and then
\[
g = \nabla \psi + h - \sum_{j=1}^{M} (h, p_{\ell}) B_{\ell} p_{\ell} + \sum_{j=1}^{M} (h, p_{\ell}) B_{\ell} p_{\ell}.
\]
Since \( f \in J_{q}(B_{R}) \), we have \( (f, g)_{B_{R}} = (f, h - \sum_{j=1}^{M} (h, p_{\ell}) B_{\ell} p_{\ell}) \). Since \( \text{div} p_{\ell} = 0 \), \( h - \sum_{j=1}^{M} (h, p_{\ell}) B_{\ell} p_{\ell} \in J_{q}(B_{R}) \), and so by (92) \( (f, h - \sum_{j=1}^{M} (h, p_{\ell}) B_{\ell} p_{\ell})_{B_{R}} = 0 \), which leads to \( f, g \in L_{q}(B_{R}) \). Thus, we have \( f = 0 \). By the first equation in (87), \( \nabla (K_{1}(f) + K_{2}(g)) = 0 \) in \( B_{R} \), which leads to \( K_{1}(f) + K_{2}(g) = f_{0} \) with some constant \( f_{0} \). Thus, by the third equation in (87), we have \( \mathcal{B} g = -\sigma^{-1} f_{0} \) on \( S_{R} \). Since \( (g, 1)_{S_{R}} = 0 \), we have \( -\sigma^{-1} f_{0} |_{S_{R}} = (\mathcal{B} g, 1)_{S_{R}} = R^{-2}(N-1)\langle g, 1 \rangle_{S_{R}} = 0 \), which leads to \( f_{0} = 0 \). Thus, we have \( \mathcal{B} g = 0 \) on \( S_{R} \). By the hypoellipticity of the operator \( \Delta_{S_{R}} \), we see that \( g \in H^{2}_{2}(S_{R}) \), and \( g \in H^{2}_{2}(S_{R}) \), which, combined with Lemma 4.5, leads to \( g = 0 \). Thus, the formula (84) holds, and therefore problem (74) admits a unique solution \( (v, h) \in \tilde{D}_{q} \) possessing the estimate (91) when \( 1 < q < 2 \) and \( \lambda \in Q_{\lambda_{0}} \).

Finally, we prove that the constant in the estimate (91) is independent of \( \lambda \in Q_{\lambda_{0}} \). Let \( \lambda \in Q_{\lambda_{0}} \) and \( \mu \in \mathbb{C} \), and we consider the equation:
\[
(\mu I - A)(v, h) = (f, g).
\]
We write this equation as
\[
(f, g) = ((\mu - \lambda) I + (\lambda I - A))\cdot (v, h) = (I + (\mu - \lambda)(\lambda I - A)^{-1})(\lambda I - A)(v, h).
\]
Let us denote the operator norm of the bounded linear operator from \( \hat{H}_{q} \) into itself by \( \| \cdot \|_{\mathcal{L}(\hat{H}_{q})} \). Since \( \| (\mu - \lambda)(\lambda I - A)^{-1} \|_{\mathcal{L}(\hat{H}_{q})} \leq \| \mu - \lambda \| C_{\lambda} \) as follows from (91), choosing \( \mu \in \mathbb{C} \) in such a way that \( |\mu - \lambda| C_{\lambda} \leq 1/2 \), we see that the inverse operator \( (I + (\mu - \lambda)(\lambda I - A)^{-1})^{-1} \) exists as a bounded linear operator from \( \hat{H}_{q} \) into itself and
\[
\| (I + (\mu - \lambda)(\lambda I - A)^{-1})^{-1} \|_{\mathcal{L}(\hat{H}_{q})} \leq 2.
\]
Thus, \( (v, h) = (\lambda I - A)^{-1}(I + (\mu - \lambda)(\lambda I - A)^{-1})(f, g) \) belongs to \( \tilde{D}_{q} \) and solves the equation (95). Moreover, denoting the operator norm of bounded linear operators from \( \hat{H}_{q} \) into \( \tilde{D}_{q} \) by \( \| \cdot \|_{\mathcal{L}(\hat{H}_{q}, \tilde{D}_{q})} \), we have
\[
\| (v, h) \|_{\tilde{D}_{q}} \leq \| (\lambda I - A)^{-1} \|_{\mathcal{L}(\hat{H}_{q}, \tilde{D}_{q})} \| (I + (\mu - \lambda)(\lambda I - A)^{-1} \|_{\mathcal{L}(\hat{H}_{q})} \| (f, g) \|_{\hat{H}_{q}} \leq 2C_{\lambda} \| (f, g) \|_{\hat{H}_{q}},
\]
provided that \( |\mu - \lambda| \leq (2C_{\lambda})^{-1} \). Since \( Q_{\lambda_{0}} \) is a compact set, we have (81), which completes the proof of Theorem 4.4. \( \square \)

**Proof of Lemma 3.4.** Finally we prove Lemma 3.4. Let
\[
\hat{u} = u - \sum_{j=1}^{M} (u, p_{\ell}) B_{\ell} p_{\ell}, \quad \hat{\rho} = \rho - \sum_{j=1}^{N+1} (\rho, \varphi_{j}) S_{R} \varphi_{j}.
\]
Since \( D(p_{\ell}) = 0 \) and \( \text{div} p_{\ell} = 0 \), we have \( K_{1}(\hat{u}) = K_{1}(u) \). Since \( B \varphi_{j} = 0 \) on \( S_{R} \) for \( j = 2, \ldots, N+1 \), and \( \varphi_{1} = |S_{R}|^{-1} \) is constant, we see that
\[
K_{2}(\hat{\rho}) = K_{2}(\rho) + \sigma(\rho, \varphi_{1}) S_{R} R^{-2}(N-1)|S_{R}|^{-1},
\]
\[
\mathcal{B} \hat{\rho} = \mathcal{B} \rho - (\rho, \varphi_{1}) S_{R} (N-1)R^{-2}|S_{R}|^{-1}.
\]
Thus, by (83) and Theorem 4.4 with $\lambda = 0$, we have $\| (u, \rho) \|_{\mathcal{D}_q} \leq C \| (f, g) \|_{\mathcal{H}_q}$, which, combined with the estimate:

$$\| (u, \rho) \|_{\mathcal{D}_q} \leq \| (\tilde{u}, \tilde{\rho}) \|_{\mathcal{D}_q} + C \left\{ \sum_{i=1}^{M} | (u, p_i)_{B_R} | + \sum_{j=1}^{N+1} | (\rho, \varphi_j)_{S_R} | \right\},$$

leads to (71). This completes the proof of Lemma 3.4. 

5. Local well-posedness. In this section, we prove the local well-posedness of problem (10) in the time interval $(a, a + 1)$. Namely, we consider problem:

$$\begin{align*}
\partial_t u - \text{Div} (\mu D(u) - q I) &= f(u, \rho) \quad \text{in } B_R \times (a, a + 1), \\
\text{div} u &= f_d(u, \rho) = \text{div} f_d(u, \rho) \quad \text{in } B_R \times (a, a + 1), \\
\rho_t - \omega \cdot P u &= k \iota_{un}(u, \rho) \quad \text{on } S_R \times (a, a + 1), \\
\Pi_0 (\mu D(u) \omega) &= \mu g'(u, \rho) \quad \text{on } S_R \times (a, a + 1), \\
< \mu D(u) \omega, \omega > - q - \sigma \mathcal{B} \rho &= g_n(u, \rho) \quad \text{on } S_R \times (a, a + 1), \\
(u, \rho)_{t=a} &= (w_0, \rho_0) \quad \text{in } B_R \times S_R.
\end{align*}$$

(96)

Theorem 5.1. Let $N < q < \infty$, $2 < p < \infty$, $a \geq 0$ and $\epsilon_1 > 0$. Then, there exists a positive constants $\epsilon$ depending on $\epsilon_1$ such that if initial data $w_0 \in B^{2-2/q}_{q,p}(B_R)$ and $\rho_0 \in B^{3-1/p-1/q}_{q,p}(S_R)$ for problem (96) satisfying the condition:

$$\|w_0\|_{B^{2-2/q}_{q,p}(B_R)} + \|\rho_0\|_{B^{3-1/p-1/q}_{q,p}(S_R)} \leq \epsilon$$

(97)

and compatibility condition:

$$\text{div} w_0 = f_d(w_0, \rho_0) = \text{div} f_d(w_0, \rho_0) \quad \text{in } B_R,$$

$$\Pi_0 (\mu D(w_0) \omega) = \mu g'(w_0, \rho_0) \quad \text{on } S_R \text{ when } 2/p + 1/q < 1 \text{ in addition},$$

(98)

then problem (96) admits a unique solution $(u, q, \rho) \in \mathcal{S}_{p,q}((a, a + 1))$ possessing the estimate:

$$\mathcal{T}_{p,q}(u, q, \rho; 0, (a, a + 1)) + \sup_{a \leq t \leq a + 1} \| (u(\cdot, t))_{B^{2-2/q}_{q,p}(B_R)} + \|\rho(\cdot, t)\|_{W^{3-1/p-1/q}_{q,p}(S_R)} \leq \epsilon_1$$

where $\mathcal{T}_{p,q}(u, q, \rho; 0, (a, a + 1))$ is the norm given in Theorem 3.1 with $\eta = 0$ and $b = a + 1$.

To prove Theorem 5.1, we use the maximal $L_p$-$L_q$ regularity for the linearized equations:

$$\begin{align*}
\partial_t u - \text{Div} (\mu D(u) - q I) &= f \quad \text{in } B_R \times (a, a + 1), \\
\text{div} u &= f_d = \text{div} f_d \quad \text{in } B_R \times (a, a + 1), \\
\rho_t - \omega \cdot P u &= g \quad \text{on } S_R \times (a, a + 1), \\
(\mu D(u) - q I) \omega - \sigma \mathcal{B} \rho &= h \quad \text{on } S_R \times (a, a + 1), \\
(u, \rho)_{t=a} &= (w_0, \rho_0) \quad \text{in } B_R \times S_R.
\end{align*}$$

(99)

Using a result due to Shibata [21] we have

Theorem 5.2. Let $1 < p, q < \infty$. Then, for any initial data $w_0 \in B^{2-2/p}_{q,p}(B_R)$ and $\rho_0 \in B^{3-1/p-1/q}_{q,p}(S_R)$ and right members: $(f, f_d, f_d, g, h) \in \mathcal{D}_{p,q}((a, a + 1))$ satisfying compatibility conditions:

$$\text{div} w_0 = f_d|_{t=a} = \text{div} f_d|_{t=a} \text{ in } B_R,$$
embedding theorem, we have
\[ (99) \]
in Sect. 2. In the following, we assume that
\[ N < q < 1 \]
with some positive constant \( C \) independent of \( a \). Here, \( I_{p,q} \) and \( J_{p,q} \) are the same norms as in Theorem 3.1.

Proof. Since the coefficients of the partial differential operators appearing in problem \((99)\) are constant, setting \( \psi(\cdot, t) = u(\cdot, t + a) \), \( p(\cdot, t) = q(\cdot, t + a) \) and \( h(\cdot, t) = \rho(\cdot, t + a) \), we can reduce the problem to the case where \( a = 0 \), and so Theorem 5.2 follows from a result due to Shibata [21].

To prove Theorem 5.1, we start with the estimate of the nonlinear terms given in Sect. 2. In the following, we assume that \( N < q < \infty \), and then by the Sobolev embedding theorem, we have
\[
\| f \|_{H_{1}^{q}(B_{R})} \leq C \| f \|_{W_{1}^{q-1/2}(S_{R})} \leq C \| f \|_{W_{1}^{q-1/2}(S_{R})}.
\]
Moreover, we assume that \( H_{\rho} \) and \( \rho \) satisfy the smallness condition:
\[
\sup_{a < t < a + 1} \| H_{\rho}(\cdot, t) \|_{H_{1}^{2}(B_{R})} \leq \epsilon_{0}, \quad \sup_{a < t < a + 1} \| \rho(\cdot, t) \|_{B_{1}^{2}(S_{R})} \leq \epsilon_{0}
\]
with some small constant \( \epsilon_{0} \in (0, 1/4) \), which is determined in the course of the proof of Theorem 5.1 below.

Let \( f(u, \rho) \) be the nonlinear function given in (22). Using (100) and (101), we have
\[
\| \nabla \Phi_{\rho} \|_{L_{\infty}(B_{R})} \leq C \| H_{\rho} \|_{H_{1}^{2}(B_{R})} \leq C \| H_{\rho} \|_{H_{2}^{1}(B_{R})} \leq C \| \rho \|_{W_{q}^{2-1/q}(S_{R})}.
\]
Since \( \nabla \Phi_{\rho} = \sum_{j=1}^{\infty} \nabla \Phi_{j} \), we have \( \| \nabla \Phi_{\rho} \|_{L_{\infty}(B_{R})} \leq C \| \rho \|_{W_{q}^{2-1/q}(S_{R})} \), and so
\[
\| \nabla \Phi_{\rho}(\cdot, t) \|_{H_{1}^{2}(B_{R})} \leq \epsilon_{0}, \quad \| \rho(\cdot, t) \|_{B_{1}^{2}(S_{R})} \leq \epsilon_{0}.
\]
Moreover, using (100) and (101), we have
\[
\| \nabla \Phi_{\rho}(\cdot, t) \|_{H_{1}^{2}(B_{R})} \leq C \| \rho \|_{W_{q}^{2-1/q}(S_{R})},
\]
\[
\| \nabla \Phi_{\rho}(\cdot, t) \|_{H_{1}^{2}(B_{R})} \leq C \| \rho \|_{W_{q}^{2-1/q}(S_{R})}.
\]
Thus, we have
\[
\| f(u, \rho) \|_{L_{q}(B_{R})} \leq C \{ \| \partial_{u} u \|_{L_{q}(B_{R})} \| \rho \|_{W_{q}^{2-1/q}(S_{R})} + \| \partial_{\rho} \rho \|_{W_{q}^{2-1/q}(S_{R})} \| \nabla u \|_{L_{q}(B_{R})}.
\]
Let $f_d(u, \rho)$ and $f_d(u, \rho)$ be nonlinear functions defined in (25). By (100),
\[
\| (\nabla H_\rho) \nabla u \|_{H^2_\rho(B_R)} \leq C \| H_\rho \|_{H^2_\rho(B_R)} \| \nabla u \|_{H^1_\rho(B_R)} \\
\leq C \| \rho \|_{W^{2-1/q}_q(S_R)} \| \nabla u \|_{H^1_\rho(B_R)},
\]
and so, using (101) and (100), we have
\[
\| f_d(u, \rho) \|_{H^2_\rho(B_R)} \leq C \| \rho \|_{W^{2-1/q}_q(S_R)} \| \nabla u \|_{H^1_\rho(B_R)}.
\]
Here and in the following, we use the fact that $H_\rho$ is a solution of the Dirichlet problem (4) and satisfies the estimates:
\[
\| \partial_i^t H_\rho(\cdot, t) \|_{H^2_\rho(B_R)} \leq C \| \partial_i^t \rho(\cdot, t) \|_{W^{2-1/q}_q(S_R)} (i = 0, 1), \\
\| H_\rho(\cdot, t) \|_{H^2_\rho(B_R)} \leq C \| \rho(\cdot, t) \|_{W^{3-1/q}_q(S_R)}.
\]
To estimate $\| \partial_t f_d(u, \rho) \|_{W^{2-1}_q(B_R)}$, we use the following estimates proved in Shibata [22, Lemma 2.3].

**Lemma 5.3.** Let $N < q < \infty$. Then, we have
\[
\| \nabla u \|_{W^{2-1}_q(B_R)} \leq C \| u \|_{L_q(B_R)}, \\
\| |u|v \|_{W^{2-1}_q(B_R)} \leq C \| u \|_{W^{2-1}_q(B_R)} \| v \|_{L_q(B_R)}, \\
\| |u|v \|_{W^{2-1}_q(B_R)} \leq C \| u \|_{L_q(B_R)} \| v \|_{L_q(B_R)}.
\]
Using Lemma 5.3, we have
\[
\| \partial_t (\nabla H_\rho) \nabla u \|_{W^{2-1}_q(B_R)} \leq C \{ \| \partial_t \nabla H_\rho \|_{L_q(B_R)} \| \nabla u \|_{L_q(B_R)} \\
+ \| \nabla (\nabla H_\rho) \partial_t u \|_{W^{2-1}_q(B_R)} + \| (\nabla^2 H_\rho) \partial_t u \|_{W^{2-1}_q(B_R)} \} \\
\leq C \{ \| \partial_t \rho \|_{B^{2-1/q}_q(S_R)} \| \nabla u \|_{L_q(B_R)} + \| \rho \|_{B^{2-1/q}_q(S_R)} \| \partial_t u \|_{L_q(B_R)} \}.
\]
Thus, noting (101) and using (100) and Lemma 5.3, we have
\[
\| \partial_t f_d(u, \rho) \|_{W^{2-1}_q(B_R)} \leq C \{ \| \partial_t \rho \|_{B^{2-1/q}_q(S_R)} \| \nabla u \|_{L_q(B_R)} + \| \rho \|_{B^{2-1/q}_q(S_R)} \| \partial_t u \|_{L_q(B_R)} \}.
\]
Moreover, by (100)
\[
\| \partial_t (\nabla H_\rho) u \|_{L_q(B_R)} \leq C \{ \| \partial_t \rho \|_{W^{2-1/q}_q(S_R)} \| u \|_{L_q(B_R)} + \| \rho \|_{W^{2-1/q}_q(S_R)} \| \partial_t u \|_{L_q(B_R)} \}.
\]
Thus, noting (101) and using (100), we have
\[
\| \partial_t f_d(u, \rho) \|_{L_q(B_R)} \leq C \{ \| \partial_t \rho \|_{W^{2-1/q}_q(S_R)} \| u \|_{L_q(B_R)} + \| \rho \|_{W^{2-1/q}_q(S_R)} \| \partial_t u \|_{L_q(B_R)} \}.
\]
Let $k_{in}(u, \rho)$ be the nonlinear function given in (33). The $n_i$ is suitably extended to $B_R$ replacing $\rho$ and $\omega$ by $H_\rho$ and $x$. By (32), $< \omega, n_i > = 1 + V_2(\rho)$, and so $< \omega, n_i >^{-1} = 1 + \sum_{j=1}^{\infty} (-V_2(\rho))^j$. Thus, by (100) and (101) we have
\[
\| 1 - \frac{1}{< n_i, \omega >} \|_{W^{2-1/q}_q(S_R)} \leq C \| \rho \|_{W^{2-1/q}_q(S_R)} \\
\| 1 - \frac{1}{< n_i, \omega >} \|_{W^{2-1/q}_q(S_R)} \leq C \| \rho \|_{W^{2-1/q}_q(S_R)}.
\]
By (100),
\[
\| fg \|_{H^{2-1/q}_q(B_R)} \leq C \{ \| f \|_{H^{2-1/q}_q(B_R)} \| g \|_{H^{2-1/q}_q(B_R)} + \| f \|_{H^{2-1/q}_q(B_R)} \| g \|_{H^{2-1/q}_q(B_R)} \}.
\]
Thus, noting (101) and using (100), we have
\[
\|k_{in}(u, \rho)\|_{H^2_{(S_R)}} \leq C[\|\rho\|_{W^{3-1/q(S_R)}} + \|\rho\|_{W^{2-1/q(S_R)}} + \|u\|_{H^1_0(B_R)}].
\] (109)

Let \( g'(u, \rho) \) be the nonlinear function given in (35). Noting (101) and (108), and using (100), (103), (105), and Lemma 5.3, we have
\[
\|g'(u, \rho)\|_{H^1_0(B_R)} \leq C[\|\rho\|_{W^{2-1/q(S_R)}} + \|\rho\|_{W^{2-1/q(S_R)}} + \|u\|_{L_4(B_R)} + \|\rho\|_{W^{2-1/q(S_R)}}]\|\partial u\|_{L_4(B_R)).
\] (110)

Finally, we consider the nonlinear function \( g_n(u, \rho) \) appearing in (46). Let \( \langle u - \omega, \omega \rangle + V_n(\rho) \) be suitably extended to \( B_R \) in the definition of \( g_n(u, \rho) \) by using \( H_\rho \). By (100) and Lemma 5.3, we have
\[
\|g_n(u, \rho)\|_{H^1_0(B_R)} \leq C[\|\rho\|_{W^{3-1/q(S_R)}}\|\nabla u\|_{H^1_0(B_R)}]
\]
\[
\|\partial g_n(u, \rho)\|_{W^{-1}(B_R)} \leq C[\|\partial \rho\|_{W^{2-1/q(S_R)}}\|u\|_{L_4(B_R)} + \|\rho\|_{W^{2-1/q(S_R)}}\|\partial u\|_{L_4(B_R)}].
\] (111)

From now on, we prove Theorem 5.1. Let \( \epsilon_2 \) be a number such that \( 0 < \epsilon_2 \leq \min(\epsilon_0, \epsilon_1) \), and let \( \epsilon \) be a positive number. We choose \( \epsilon \) and \( \epsilon_2 \) small enough eventually, and therefore, we may assume that \( 0 < \epsilon, \epsilon_2 < 1 \), in the following. We assume that initial data \( w_0 \in B_{q,p}^{2-2/p}(B_R)^N \) and \( \rho_0 \in B_{q,p}^{3-1/p-1/q}(S_R) \) satisfy smallness condition (97) and compatibility condition (98). Let
\[
\mathcal{H}_{\epsilon_2} = \{(u, q, \rho) \in S_{p,q}(a, a + 1) | (u, \rho)|_{t=a} = (w_0, \rho_0) \text{ in } B_R \times S_R,
\]
\[
\mathcal{E}(u, q, \rho) = \mathcal{I}_{p,q}(u, q, \rho; 0, (a, a + 1))
\]
\[
+ \sup_{a < t < a + 1} \|u(\cdot, t)\|_{B_{q,p}^{2-2/p}(B_R)} + \|\rho(\cdot, t)\|_{B_{q,p}^{3-1/p-1/q}(S_R)} \leq \epsilon_2 \}. \] (112)

Here, \( \mathcal{I}_{p,q}(u, q, \rho; 0, (a, a + 1)) \) is the norm given in Theorem 3.1 with \( \eta = 0 \) and \( b = a + 1 \). And, in what follows, let \( \mathcal{J}_{p,q} \) be the norm given in Theorem 3.1 with \( \eta = 0 \) and \( b = a + 1 \), too. Let \( (u, q, \rho) \in \mathcal{H}_{\epsilon_2} \) and let \( (v, p, h) \) be solutions to the equations:
\[
\begin{aligned}
\partial_t v - \text{Div}(\mu D(v) - p I) &= f(u, \rho) \quad \text{in } B_R \times (a, a + 1), \\
\text{div } v &= f_\rho(u, \rho) \quad \text{in } B_R \times (a, a + 1), \\
h_t - \omega \cdot P v &= k_{in}(u, \rho) \quad \text{on } S_R \times (a, a + 1), \\
\Pi_0(\mu D(v)\omega) &= \mu g(u, \rho) \quad \text{on } S_R \times (a, a + 1), \\
<\mu D(v)\omega, \omega > - p - \sigma Bh &= g_n(u, \rho) \quad \text{on } S_R \times (a, a + 1), \\
(v, h)|_{t=a} &= (u_0, \rho_0) \quad \text{in } B_R \times S_R.
\end{aligned}
\] (113)

By Theorem 5.2, problem (113) admits a unique solution \((v, p, h) \in S_{p,q}(a, a + 1)\) possessing the estimate:
\[
\mathcal{I}_{p,q}(v, p, h; 0, (a, a + 1)) \leq C\mathcal{J}_{p,q}(w_0, \rho_0, f, f_\rho, f_\rho, k_{in}, g; 0, (a, a + 1)),
\]
with
\[ f = f(u, \rho), \quad f_d = f_d(u, \rho), \quad f_{\nu} = f_{\nu}(u, \rho), \]
\[ k_{in} = k_{in}(u, \rho), \quad g = (\mu g'(u, \rho), g_n(u, \rho)). \]

Choosing \( \epsilon_2 > 0 \) smaller if necessary, we may assume that (101) holds. Thus, by (97), (102), (104), (106), (107), (109), (110), (111), and (112), we have
\[ \mathcal{J}_{p,q}(w_0, \rho_0, f, f_d, f_{\nu}, k_{in}, g; 0, (a, a + 1)) \leq C(\epsilon + \epsilon_2^2), \]
and therefore,
\[ \mathcal{J}_{p,q}((0, \rho, v; h; 0, (a, a + 1)) \leq C(\epsilon + \epsilon_2^2). \]

To estimate \( \sup_{a < t < a + 1} \| v(\cdot, t) \|_{H^1_q(B_R)} \) and \( \sup_{a < t < a + 1} \| h(\cdot, t) \|_{W^{2,1/q}_q(S_R)} \), we use the following lemma.

**Lemma 5.4.** Let \( 1 < p, q < \infty \) and \( a \geq 0 \). Then, we have
\[
\sup_{a < t < a + 1} \| f(\cdot, t) \|_{B^{a-2/p}_q(B_R)} \leq C \| f(\cdot, 0) \|_{B^{a-2/p}_q(B_R)} + \| f \|_{L_p((a,a+1),H^1_q(B_R))} + \| \partial_t f \|_{L_p((a,a+1),L_q(B_R))}.
\]

\[
\sup_{a < t < a + 1} \| g(\cdot, t) \|_{B^{a-1/p-1/q}_q(S_R)} \leq C \| g(\cdot, 0) \|_{B^{a-1/p-1/q}_q(B_R)} + \| g \|_{L_p((a,a+1),W^{a-1/q}_q(S_R))} + \| \partial_t g \|_{L_p((a,a+1),W^{a-1/q}_q(S_R))}.
\]

Here, \( C \) is a positive constant independent of \( a \).

**Proof.** If we set \( \tilde{f}(\cdot, t) = f(\cdot, a + t) \) and \( \tilde{g}(\cdot, t) = g(\cdot, a + t) \), we can reduce the problem to the case \( a = 0 \). Thus, we prove Lemma 5.4 in the case \( a = 0 \). The first assertion was proved in Shibata [20, (3.20) in p.4139] and can be proved in the same way as in the proof of the second assertion below, and so we may omit its proof. We only prove the second assertion. Given function \( h \) defined on \( (0, 1) \), \( h_0 \) denotes the zero extension of \( h \) to \( (-\infty, 0) \), that is \( h_0(\cdot, t) = h(\cdot, t) \) for \( t \in (0, 1) \) and \( h_0(\cdot, t) = 0 \) for \( t \leq 0 \). Let
\[ [E_1 h](\cdot, t) = \begin{cases} h_0(\cdot, t) & t \leq 1, \\ h_0(\cdot, 2 - t) & t > 1. \end{cases} \]

If \( h \big|_{t=0} = 0 \), then
\[ \partial_t [E_1 h](\cdot, t) = \begin{cases} \partial_t h(\cdot, t) & 0 \leq t \leq 1, \\ -\partial_t h(\cdot, 2 - t) & 1 \leq t \leq 2, \\ 0 & t \notin (0, 2). \end{cases} \]

Let \( G \) be a function in \( B^{a-1/q}_q(\mathbb{R}^N) \) such that \( G|_{S_R} = g(\cdot, 0) \) and
\[ \| G \|_{B^{a-1/p}_q(\mathbb{R}^N)} \leq C \| g(\cdot, 0) \|_{B^{a-1/p-1/q}_q(S_R)}. \]

Let \( d = \mathcal{F}^{-1}[e^{-t\sqrt{1 + |\xi|^2}} \mathcal{F}[G](\xi)] \), where \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denotes the Fourier transform and the inverse Fourier transform on \( \mathbb{R}^N \), and then we have
\[
\| d \|_{L_p((0, \infty), H^1_q(\mathbb{R}^N))} + \| \partial_t d \|_{L_p((0, \infty), H^1_q(\mathbb{R}^N))} \leq C \| G \|_{B^{a-1/p}_q(\mathbb{R}^N)} \leq C \| g(\cdot, 0) \|_{B^{a-1/p-1/q}_q(S_R)}. \]

Noting that \( g = E_1(g - d) + d \) for \( 0 < t < 1 \) and using the interpolation relation:
\[ L_p((0, \infty), X_1) \cap H^1_q((0, \infty), X_0) \subset BUC((0, \infty), (X_0, X_1)_{1-1/p,p}), \]
where $BUC$ means the set of all uniformly bounded continuous functions, $X_0$ and $X_1$ are any Banach spaces such that $X_1$ is dense in $X_0$, and $1 < p < \infty$, we have

\[
\sup_{0 < t < 1} \|g(\cdot, t)\|_{B^p_{q,p}((0,1),H^\infty(\mathbb{R}^N))} \\
\leq \sup_{0 < t < \infty} \|E_1(g - d)\|_{B^p_{q,p}((0,1),H^\infty(\mathbb{R}^N))} + C \sup_{0 < t < \infty} \|d(\cdot, t)\|_{B^p_{q,p}((0,1),H^\infty(\mathbb{R}^N))} \\
\leq C\{\|E_1(g - d)\|_{L_p((0,1),W^3_{q,p}(S_R))} + \|\partial_t E_1(g - d)\|_{L_p((0,1),W^2_{q,p}(S_R))} + \|\partial_t d\|_{L_p((0,1),H^2_{q,p}(\mathbb{R}^N))}\}.
\]

Since $(g - d)|_{t = 0} = 0$ on $S_R$, by (116),

\[
\|\partial_t E_1(g - d)\|_{L_p((0,1),W^2_{q,p}(S_R))} \\
\leq C\{\|\partial_t g\|_{L_p((0,1),W^3_{q,p}(S_R))} + \|\partial_t d\|_{L_p((0,1),W^2_{q,p}(S_R))}\};
\]

\[
\|E_1(g - d)\|_{L_p((0,1),W^3_{q,p}(S_R))} \\
\leq C\{\|g\|_{L_p((0,1),W^3_{q,p}(S_R))} + \|d\|_{L_p((0,1),W^3_{q,p}(S_R))}\},
\]

which, combined with (117) and (118), leads to the second inequality in (115). This completes the proof of Lemma 5.4.

By Lemma 5.4, (97) and (114),

\[
\sup_{a < t < a + 1} \|v(\cdot, t)\|_{B^p_{q,p}((a,a+1),H^\infty(\mathbb{R}^N))} + \sup_{a < t < a + 1} \|h(\cdot, t)\|_{B^p_{q,p}((a,a+1),H^\infty(\mathbb{R}^N))} \\
\leq C\{\|w_0\|_{B^p_{q,p}((a,a+1),H^\infty(\mathbb{R}^N))} + \|\rho_0\|_{B^p_{q,p}((a,a+1),H^\infty(\mathbb{R}^N))} + \|v\|_{L_p((a,a+1),H^\infty(\mathbb{R}^N))} \\
+ \|h\|_{L_p((a,a+1),H^\infty(\mathbb{R}^N))} + \|\partial_t h\|_{L_p((a,a+1),H^\infty(\mathbb{R}^N))}\}.
\]

(119)

Choosing $\epsilon$ and $\epsilon_2$ in such a way that

\[
C\epsilon_2 \leq 1/2, \quad C\epsilon \leq \epsilon_2/2,
\]

we have

\[
\mathcal{E}(v, p, h) \leq C(\epsilon + \epsilon_2^2).
\]

Choosing $\epsilon$ and $\epsilon_2$ in such a way that

\[
C\epsilon_2 \leq 1/2, \quad C\epsilon \leq \epsilon_2/2,
\]

we have

\[
\mathcal{E}(v, p, h) \leq \epsilon_2.
\]

(121)

Let $\Psi$ be a map defined by $\Psi(u, q, \rho) = (v, p, h)$, and then by (121) $\Psi$ maps $\mathcal{H}_{\epsilon_2}$ into itself.

Analogously, we can prove that

\[
\mathcal{E}(\Psi(u_1, q_1, \rho_1) - \Psi(u_2, q_2, \rho_2)) \\
\leq C(\mathcal{E}(u_1, q_1, \rho_1) + \mathcal{E}(u_2, q_2, \rho_2))\mathcal{E}(u_1 - u_2, q_1 - q_2, \rho_1 - \rho_2) \\
\leq 2C(\epsilon + \epsilon_2^2)\mathcal{E}(u_1 - u_2, q_1 - q_2, \rho_1 - \rho_2)
\]

for any $(u_i, q_i, \rho_i) \in \mathcal{H}_{\epsilon_2}$ ($i = 1, 2$). Choosing $\epsilon$ and $\epsilon_2$ in such a way that

\[
2C\epsilon_2^2 \leq 1/4, \quad 2C\epsilon \leq 1/4,
\]

we have

\[
\mathcal{E}(\Psi(u_1, q_1, \rho_1) - \Psi(u_2, q_2, \rho_2)) \leq (1/2)\mathcal{E}(u_1 - u_2, q_1 - q_2, \rho_1 - \rho_2)
\]

for any $(u_i, q_i, \rho_i) \in \mathcal{H}_{\epsilon_2}$ ($i = 1, 2$), which shows that $\Psi$ is a contraction map on $\mathcal{H}_{\epsilon_2}$. Thus, there exists a unique element $(u, q, \rho) \in \mathcal{H}_{\epsilon_2}$ such that $\Psi(u, q, \rho) =
(u, q, \rho). In particular, (u, q, \rho) is a unique solution of the equations (99), which completes the proof of Theorem 5.1.

6. Global wellposedness. Assuming that the initial data u_0 \in B_{q,p}^{2-2/p}(B_R) and \rho_0 \in W^{3-1/p-1/q}(S_R) satisfy the smallness condition:

\|u_0\|_{H^2_q(B_R)} + \|\rho_0\|_{W^{3-1/p-1/q}(S_R)} \leq \epsilon

(122)

with small constant \epsilon > 0 as well as the compatibility condition (14), we shall prove Theorem 1, below. Since we choose \epsilon small enough eventually, we may assume that 0 < \epsilon < 1. Let T be a positive number such that problem (10) admits a unique solution (u, q, \rho) \in S_{p,q}(0, T), which satisfies the condition:

\sup_{0 < t < T} \|H_p(\cdot, t)\|_{H^2_q(B_R)} \leq \epsilon_0, \quad \sup_{0 < t < T} \|\rho(\cdot, t)\|_{W^{3-1/p-1/q}(S_R)} \leq \epsilon_0

(123)

where \epsilon_0 \in (0, 1/4) is the same constant as in (101) in Sect. 5. In view of Theorem 5.1, we may assume that \epsilon > 0 is small enough.

Let t \in (0, T] and let

\mathcal{E}_{t, \eta} = \sup_{0 < s < t} \left( \|\mathbf{u}(\cdot, s)\|_{B_{q,p}^{2-2/p}(B_R)} + \|\mathbf{u}(\cdot, s)\|_{B_{q,p}^{3-1/p-1/q}(S_R)} + \mathcal{I}_{p,q}(\mathbf{u}, q, \rho; \eta, (0, t)) \right)

where \mathcal{I}_{p,q}(\mathbf{u}, q, \rho; \eta, (0, t)) is the same norm as in Theorem 3.1 with some positive constant \eta and (a, b) = (0, t). In this case, T^{-1} \leq 1 and so

\mathcal{I}_{p,q}(\mathbf{u}_0, \rho_0, f, \mathbf{f}_d, \mathbf{f}_d, g, \mathbf{h}; \eta, (0, t)) = \|\mathbf{u}_0\|_{B_{q,p}^{2-2/p}(B_R)} + \|\rho_0\|_{B_{q,p}^{3-1/p-1/q}(S_R)}

+ \|e^{\eta s}f\|_{L_p((0,t), L_q(B_R))} + \|e^{\eta s}f_d\|_{L_p((0,t), H^2(B_R))} + \|e^{\eta s}g\|_{L_p((0,t), W^{3-1/q}(S_R))}

+ \|e^{\eta s}h\|_{L_p((0,t), H^2_q(B_R))} + \|e^{\eta s}\mathbf{f}_d\|_{L_p((0,t), W^{3-1/q}(S_R))} + \|e^{\eta s}\mathbf{f}_d\|_{L_p((0,t), L_q(B_R))}

+ \|e^{\eta s}\mathbf{f}_d\|_{L_p((0,t), W^{3-1/q}(B_R))}

in Theorem 3.1. Applying Theorem 3.1 to the equations (10) and using the formulas (102), (104), (106), (107), (110), and (111), we have

\mathcal{I}_{p,q}(\mathbf{u}, q, \rho; \eta, (0, t)) \leq C\{ \mathcal{E}_{t, \eta}^2 + \sum_{\ell=1}^{M} \left( \int_0^t (e^{\eta s}|(u(\cdot, s), p_\ell)|_{B_R})^p \, ds \right)^{1/p}

+ \sum_{j=1}^{N+1} \left( \int_0^t (e^{\eta s}|(\rho(\cdot, s), \varphi_j)|_{S_R})^p \, ds \right)^{1/p} \}.

(124)

Moreover, by Lemma 5.4 and (124), we have

\mathcal{E}_{t, \eta} \leq C\{ \mathcal{E}_{t, \eta}^2 + \sum_{\ell=1}^{M} \left( \int_0^t (e^{\eta s}|(u(\cdot, s), p_\ell)|_{B_R})^p \, ds \right)^{1/p}

+ \sum_{j=1}^{N+1} \left( \int_0^t (e^{\eta s}|(\rho(\cdot, s), \varphi_j)|_{S_R})^p \, ds \right)^{1/p} \}.

(125)

Our task is to prove that

\sum_{\ell=1}^{M} \left( \int_0^t (e^{\eta s}|(u(\cdot, s), p_\ell)|_{B_R})^p \, ds \right)^{1/p} + \sum_{j=1}^{N+1} \left( \int_0^t (e^{\eta s}|(\rho(\cdot, s), \varphi_j)|_{S_R})^p \, ds \right)^{1/p}

\leq C\mathcal{E}_{t, \eta}^2

(126)
with some constant $C > 0$ independent of $\varepsilon$ and $T > 0$. If we have (126), then combining (125) and (126) yields

$$E_{t, \eta} \leq M_1(\varepsilon + E^2_{t, \eta})$$

(127)

for any $t \in (0, T]$ with some constant $M_1$ independent of $\varepsilon$ and $T$, from which Theorem 1 follows. In fact, let $r_{\pm}(\varepsilon)$ be two roots of the quadratic equation: $M_1^{-1}x = \varepsilon + x^2$, that is $r_{\pm}(\varepsilon) = (2M_1)^{-1} \pm \sqrt{(2M_1)^{-2} - \varepsilon}$. We find a small positive number $\varepsilon_1 > 0$ such that $0 < r_-(\varepsilon) < r_+\varepsilon$ whenever $0 < \varepsilon < \varepsilon_1$. In this case, $r_-(\varepsilon) = M_1\varepsilon + O(\varepsilon^2)$ as $\varepsilon \to 0 + 0$. Since $E_{\eta, t} \to 0$ as $t \to 0$ and $E_{\eta, t}$ is a continuous function with respect to $t$, by (127) we have $E_{0, t} \leq r_-(\varepsilon)$ for any $t \in (0, T]$. In particular,

$$\|u(\cdot, T)\|_{B^{2-2/p}_p(B_R)} + \|\rho(\cdot, T)\|_{W^{3-1/p-1/q}_{q}(S_R)} \leq r_-(\varepsilon).$$

Thus, choosing $\varepsilon > 0$ small enough, by Theorem 5.1 we have the unique existence of solutions $(v, q, h) \in S_{p,q}((T, T + 1))$ of the equations:

$$\left\{ \begin{array}{l}
\partial_t v - \text{Div}(\mu D(v) - p I) = f(v, h) \\
\text{div } u = f_d(v, h) = \text{div } f_d(v, h) \\
\rho_t - \omega \cdot Pu = k_{in}(v, h) \\
\Pi_0(\mu D(v)\omega) = \mu g'(v, h) \\
< \mu D(v)\omega, \omega > - p - \sigma B\rho = g_n(v, h) \\
(v, h)|_{t=T} = (u(\cdot, T), \rho(\cdot, T))
\end{array} \right. \text{ in } B_R \times (T, T + 1),$$

on $S_R \times (T, T + 1)$.

which satisfies the condition:

$$\sup_{T < t < T + 1} \|H_k(\cdot, t)\|_{H^2_p(B_R)} \leq \varepsilon_0, \quad \sup_{T < t < T + 1} \|h(\cdot, t)\|_{W^{2-1/p-1/q}_{q}(S_R)} \leq \varepsilon_0.$$

Here $H_k$ is a solution to the Dirichlet problem:

$$(1 - \Delta)H_k = 0 \text{ in } B_R, \quad H_k|_{S_R} = R^{-1}h.$$  

Let

$$u_1 = \left\{ \begin{array}{ll}
u & (0 < t \leq T), \\
v & (T < t < T + 1),
\end{array} \right. \quad q_1 = \left\{ \begin{array}{ll}
q & (0 < t \leq T), \\
p & (T < t < T + 1),
\end{array} \right.$$  

$$\rho_1 = \left\{ \begin{array}{ll}
\rho & (0 < t \leq T), \\
h & (T < t < T + 1),
\end{array} \right.$$  

and then $(u_1, q_1, \rho_1)$ belongs to $S_{p,q}([0, T + 1])$ and satisfies the condition:

$$\sup_{0 < t < T + 1} \|H_{\rho_1}(\cdot, t)\|_{H^2_p(B_R)} \leq \varepsilon_0, \quad \sup_{0 < t < T + 1} \|\rho_1(\cdot, t)\|_{W^{2-1/p-1/q}_{q}(S_R)} \leq \varepsilon_0,$$

and the equations (10) in the time interval $(0, T + 1)$ instead of $(0, T)$. Here, $H_{\rho_1}$ is defined by $H_{\rho_1} = H_{\rho}$ for $t \in (0, T)$ and $H_{\rho_1} = H_k$ for $t \in (T, T + 1)$. Repeating this argument, we can prolong $u$, $q$ and $\rho$ to the time interval $(0, \infty)$. This completes the proof of Theorem 1.

From now on, we prove (126). Let $\Psi(x, t)$ be the inverse map of the Hanzawa transform $x = h_1(y, t)$ for each $t \in [0, T]$, which is the diffeomorphism from $B_R$ onto $\Omega_R$ of $W^2_2$ class for each $t \in [0, T]$. Let $v = u \circ \Psi$ and $p = (q + \sigma(N - 1)R^{-1}) \circ \Psi$, and then $v$ and $p$ satisfy the equations (1). As was mentioned in Remark 1 the condition $2/p + N/q < 1$ guarantees that for any $t_0 \in [0, T)$, there exists a small time interval $[t_0, t_1] \subset [0, T]$ and the velocity field $w(\xi, t)$ such that the map: $x = \xi + \int_{t_0}^t w(\xi, s) \, ds$.
is the \( C^1 \) diffeomorphism from \( \Omega = \Omega_0 \) onto \( \Omega_t \) and \( w(\xi, t) = v(x, t) \) for \( t \in [t_0, t_1] \). Let \( J \) be the Jacobian of this map, and then the condition: \( \text{div} \, v = 0 \) implies that \( J = 1 \). Thus, for any \( t_0 \in [0, T) \),
\[
\frac{d}{dt} |\Omega_t| = \frac{d}{dt} \int_{\Omega_t} = \frac{d}{dt} \int J \, d\xi = \int_{\Omega} \partial_t J \, d\xi = 0,
\]
which leads to the conservation of mass, that is
\[
|\Omega_t| = |\Omega| = |B_R| = R^N \omega_N / N
\]
(128)
where we have used the assumption (A1). Moreover, we have the conservation of momentum and the conservation of angular momentum, that is
\[
\int_{\Omega_t} v(x,t) \, dx = \int_{\Omega} v_0(x) \, dx,
\]
(129)
\[
\int_{\Omega_t} (x_i v_j(x, t) - x_j v_i(x, t)) \, dx = \int_{\Omega} (x_i v_j(x) - x_j v_i(x)) \, dx
\]
(130)
where we have set \( v = (v_1, \ldots, v_N) \) and \( v_0 = (v_{01}, \ldots, v_{0N}) \). In fact, using the Lagrange transform: \( x = \xi + w(\xi, t) \) mentioned above, for each \( t \in (0, T) \) we have
\[
\frac{d}{dt} \int_{\Omega_t} v(x,t) \, dx = \frac{d}{dt} \int_{\Omega} v(\xi + \int_0^t w(s,t) \, ds) \, d\xi
\]
\[
= \int_{\Omega_t} (v_t + v \cdot \nabla v) \, dx = \int_{\Omega_t} \text{Div} (\mu D(v) - \sigma I) \, dx
\]
\[
= \int_{\Gamma_t} (\mu D(v) - \sigma I) n_t \, d\tau = \sigma \int_{\Gamma_t} \Delta_G, x \, d\tau = 0,
\]
which leads to (129). Here, \( d\tau \) is the surface area of \( \Gamma_t \), and we used the relation: \( \mathcal{H}(\Gamma_t) = \Delta_G, x \) for \( x \in \Gamma_t \). Analogously,
\[
\frac{d}{dt} \int_{\Omega_t} (x_i v_j - x_j v_i) \, dx = \frac{d}{dt} \int_{\Omega} \{ (\xi_i + \int_0^t w_i \, ds) v_j - (\xi_j + \int_0^t w_j \, ds) v_i \} \, d\xi
\]
\[
= \int_{\Omega_t} \{ (w_i w_j - w_j w_i) + x_i (\partial_t v_j + v \cdot \nabla v_j) - x_j (\partial_t v_i + v \cdot \nabla v_i) \} \, d\xi
\]
\[
= \int_{\Omega_t} \{ \sum_{k=1}^N \partial_k (w_j D_{ik} - \delta_{jk} \sigma) - \sum_{k=1}^N \partial_k (w_i D_{jk} - \delta_{ik} \sigma) \} \, dx
\]
\[
= \int_{\Gamma_t} \{ \sum_{k=1}^N \nu_k (w_j D_{ik} - \delta_{jk} \sigma) - \sum_{k=1}^N \nu_k (w_i D_{jk} - \delta_{ik} \sigma) \} \, d\tau
\]
\[
- \int_{\Omega_t} (\mu D_{ij} - \delta_{ij} \sigma) \, dx,
\]
where we have set \( w = (w_1, \ldots, w_N) \), \( D_{ij} \) denotes the \((i, j)\)th component of \( D \), and \( n_t = (v_1, \ldots, v_N) \). Since \( \sum_{k=1}^N \nu_k (w_j D_{ik} - \delta_{jk} \sigma) = \sigma \mathcal{H}(\Gamma_t) v_i = \sigma \Delta_G, x_i \), we have
\[
\frac{d}{dt} \int_{\Omega_t} (x_i v_j - x_j v_i) \, dx = \sigma \int_{\Gamma_t} (x_i \Delta_G, x_j - x_j \Delta_G, x_i) \, d\tau
\]
\[
= -\sigma \int_{\Gamma_t} \{ \nabla_G, x_i \cdot \nabla_G, x_j - \nabla_G, x_j \cdot \nabla_G, x_i \} \, d\tau = 0,
\]
which leads to (130).
By the orthogonal condition (15),
\[(v, e_i)_{\Omega_t} = 0 \quad (i = 1, \ldots, N), \quad (v, x_i e_j - x_j e_i)_{\Omega_t} = 0 \quad (i, j = 1, \ldots, N), \quad (131)\]
and so, by the Hanzawa transform, we have
\[
0 = \int_{B_R} u(y, t) \cdot p_\ell(y + H_\rho(y, t)t + \xi(t)) (1 + V^1(\nabla \Phi_\rho)) dy
\]
\[
= (u, p_\ell)_{B_R} + \int_{B_R} u(y, t) \cdot p_\ell(y) V^1(\nabla \Phi_\rho) dy
\]
\[
+ \int_{B_R} u(y, t) \tilde{p_\ell}(H_\rho(y, t)y + \xi(t)) (1 + V^1(\nabla \Phi_\rho)) dy
\]
where \(\tilde{p_\ell} = 0\) for \(\ell = 1, \ldots, N\) and \(\tilde{p_\ell}\) are some matrices of first order polynomials for \(\ell = N + 1, \ldots, M\). Since
\[
\xi(t) = \xi(0) + \int_0^t \xi'(s) ds = \frac{1}{|\Omega|} \int_{\Omega_t} v(x, t) dx = \frac{1}{|\Omega|} \int_{B_R} u(y, t) (1 + V^1(\nabla \Phi_\rho)) dy
\]
as follows from (3), we have
\[
\xi(t) = \xi(0) + \int_0^t \xi'(s) ds = \frac{1}{|\Omega|} \int_0^t \int_{B_R} u(y, s) (1 + V^1(\nabla \Phi_\rho)) dy ds,
\]
and therefore, \(|\xi(t)| \leq C E_{T, \eta}\). Thus, noting (123), we have
\[
\left( \int_0^T \left| (e^{nt}|(u(\cdot, t), p_\ell)_{B_R}) \right|^p dt \right)^{1/p} \leq C \left\{ \sup_{0 < t < T} \| \rho(\cdot, t) \|_{W^{2-1/p}_2(S_R)} E_{T, \eta} + E^2_{T, \eta} \right\}
\]
\[(132)\]

Next, we consider \((\rho, \varphi_j)_{S_R}\). Using the Hanzawa transform (5), we can write the condition (128) as follows:
\[
\left| B_R \right| = |\Omega_t| = \int_{\Omega_t} dx = \int_{|\omega| = 1} d\omega \int_0^{R + \rho(R\omega, t)} s^{N-1} ds
\]
\[
= \frac{1}{N} \int_{|\omega| = 1} (R + \rho(R\omega, t))^N d\omega = \left| B_R \right| + \frac{1}{N} \sum_{j=1}^N N C_j R^{1-j} \int_{S_R} \rho(y, t) d\tau,
\]
which leads to
\[
(\rho, 1)_{S_R} = - \sum_{j=2}^N N C_j R^{1-j} \int_{S_R} \rho(y, t) d\tau. \quad (133)
\]
Since \(\xi(t) = |\Omega|^{-1} \int_{\Omega_t} dx\) and since \(|\Omega_t| = |\Omega| = |B_R|\) as follows from (128), we have
\[0 = \int_{\Omega_t} x dx - |\Omega| \xi(t) = \int_{\Omega_t} (x - \xi(t)) dx,
\]
which leads to
\[
0 = \int_{\Omega_t} (x_i - \xi_i(t)) dx = \int_{|\omega| = 1} d\omega \int_0^{R + \rho(R\omega, t)} (s \omega_i) s^{N-1} ds
\]
\[
= \frac{1}{N + 1} \int_{|\omega| = 1} \omega_i (R + \rho(R\omega, t))^{N+1} d\omega
\]
\[
= \frac{R^{N+1}}{N + 1} \int_{|\omega| = 1} \omega_i d\omega + R^N \int_{|\omega| = 1} \rho(R\omega, t) \omega_i d\omega
\]
\[ + \frac{1}{N+1} \sum_{k=2}^{N+1} N+1 \sum_{k=2}^{N+1} C_k R^{N+1-k} \int_{|\omega|=1} \rho(R\omega, t)^k \omega_i \, d\omega \]
\[ = (\rho, x_i)_{S_R} + \frac{1}{N+1} \sum_{k=2}^{N+1} N+1 \sum_{k=2}^{N+1} C_k R^{1-k}(\rho, x_i)_{S_R}. \]

Thus, we have
\[ \left( \int_0^T \left( \epsilon^\eta(t) \|(\cdot, t), (\cdot, t)_{S_R}\right)^p \right)^{1/p} \]
\[ \leq C \sup_{0 < s < T} \|\rho(s, t)\|_{W^{2-1/4}(S_R)} \|\rho\|_{L^p((0,T), L_q(S_R))} \leq C^2_{T, \eta}, \tag{134} \]

which, combined with (132), leads to (126). Since the uniqueness follows from the local well-posedness, we have completed the proof of Theorem 1.

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Received May 2016; revised August 2017.

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