Families of explicit quasi-hyperbolic and hyperbolic surfaces

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Abstract
We construct explicit families of quasi-hyperbolic and hyperbolic surfaces parametrized by quasi-projective bases. The method we develop in this paper extends earlier works of Vojta and the first author for smooth surfaces to the case of singular surfaces, through the use of ramification indices on exceptional divisors. The novelty of the method allows us to obtain new results for the surface of cuboids, the generalized surfaces of cuboids, and other explicit families of Diophantine surfaces of general type. In particular, we produce new families of smooth complete intersection surfaces of multidegrees \((m_1, \ldots, m_n)\) in \(\mathbb{P}^{n+2}\) which are hyperbolic, for any \(n \geq 8\) and any degrees \(m_i \geq 2\). As far as we know, hyperbolic complete intersection surfaces were not known for low degrees in this generality. We also show similar results for complete intersection surfaces in \(\mathbb{P}^{n+2}\) for \(n = 4, 5, 6, 7\). These families give evidence for [6, Conjecture 0.18] in the case of surfaces.

1 Introduction

The purpose of this paper is to give an explicit method to find low genus curves in a wide range of algebraic surfaces. The method extends an earlier work of Vojta [19] for smooth surfaces, which has roots in the seminal work of Bogomolov [3] (see [7]), and the recent generalization of Vojta’s method by the first author [10]. The novelty of the method in the present article is that we include the case of singular surfaces by means of considering ramification indices on exceptional divisors. This is key for the new results we present below.

In addition, we show that the method allows us to test Brody hyperbolicity in this singular setting. In particular, we show new examples of families of quasi-hyperbolic and hyperbolic surfaces. This part is based on Nevanlinna theory (cf. [21]). We recall some definitions to be precise. An entire curve in a variety \(X\) is the image of a nonconstant holomorphic map \(\mathbb{C} \to X\). A surface \(X\) is said to be Brody hyperbolic (or simply hyperbolic, for short) if it has no entire curves. A surface \(X\) is said to be quasi-hyperbolic if all entire curves are contained...
in a proper Zariski closed subset of \(X\). Hence, when \(X\) is a smooth projective surface, we have that \(X\) is hyperbolic if it is in the sense of Kobayashi or in the sense of Brody; cf. [13].

A main motivation for us comes from describing the set of rational points of particular Diophantine varieties under the Bombieri-Lang conjecture. For instance, by finding all non-hyperbolic curves on suitable Diophantine surfaces, Vojta [19] shows that the “\(n\) squares problem” of Büchi follows from that conjecture, and later the first author shows that the analogous problem for arbitrary \(k\)-powers would also be a consequence of it [9].

Let us consider one example, which will be used in Sect. 2 to develop the ideas and computations around the method. In this example the singularities are rational double points of type A.

Let \(n \geq 3\), \(m \geq 2\) be integers. Let \(\{F_1, \ldots, F_{nm}\}\) and \(\{G_1, \ldots, G_{nm}\}\) be collections of distinct \(nm\) vertical and horizontal fibres of \(\mathbb{P}^1 \times \mathbb{P}^1\) respectively. Let us denote the elements of \(\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}^2\) by \((a, b)\). Then we have

\[
F_{km+1} + \cdots + F_{(k+1)m} + G_{km+1} + \cdots + G_{(k+1)m} = (m, m)
\]

in \(\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)\) for \(0 \leq k \leq n - 1\). These equations define a tower of \(n\) cyclic covers of degree \(m\)

\[
X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1.
\]

All \(X_k\) are normal projective surfaces with \(km^{k+1}\) singularities of type

\[
A_{m-1}: (0, 0) \in (z^m - xy) \subset \mathbb{C}^3.
\]

The surface \(X_n\) is simply connected. It also has ample canonical class, and so it is of general type.

Let \(n = 3\). We can take as a model of \(\mathbb{P}^1 \times \mathbb{P}^1\) the quadric

\[
(z_0 z_3 - z_1 z_2) \subset \mathbb{P}^3,
\]

and so the surface \(X_3\) can be presented as

\[
\prod_{i=1}^{m}(z_0 - a_i z_1 - b_i z_2 + a_i b_i z_3) = z_4^m, \quad \prod_{i=1}^{m}(z_0 - c_i z_1 - d_i z_2 + c_i d_i z_3) = z_5^m,
\]

\[
\prod_{i=1}^{m}(z_0 - e_i z_1 - f_i z_2 + e_i f_i z_3) = z_6^m, \quad z_0 z_3 - z_1 z_2 = 0
\]

in \(\mathbb{P}^6\) for distinct \(a_i, c_i, e_i \in \mathbb{C}\) and distinct \(b_i, d_i, f_i \in \mathbb{C}\). For \(m = 2\) and a specific choice of \(a_i, c_i, e_i, b_i, d_i, f_i\), the surface \(X_3\) is isomorphic to the surface of cuboids \(S\) defined by

\[
x_0^2 + x_1^2 + x_2^2 = x_3^2, \quad x_0^2 + x_1^2 = x_4^2, \quad x_0^2 + x_2^2 = x_5^2, \quad x_1^2 + x_2^2 = x_6^2
\]

in \(\mathbb{P}^6\) (see Sect. 3). A positive rational point in \(S\) would realize a perfect cuboid. It is unknown if there are any such points. This famous old problem goes back to Euler; cf. [16], [18], [2], [8]. According to the Bombieri-Lang conjecture, outside of a certain finite set of curves of geometric genus at most one, there can only be a finite number of solutions. Hence it is of interest to find all such curves. We prove the following (see Sects. 2 and 5).

**Theorem 1.1** The surfaces \(X_n\) are hyperbolic for any \(m > 2\).

Recall that the surface of cuboids \(S\) is the surface \(X_n\) with \(m = 2, n = 3\) and a particular choice of vertical and horizontal fibres. For \(S\), the results are less strong and involve further adaptations of the method. Details are worked out in Sect. 3.
Theorem 1.2 Let $S$ be the surface of cuboids. Let $S' \rightarrow S$ be its minimal resolution, and let $E$ be the sum of the 48 exceptional curves. We have:

(a) Every curve of geometric genus 0 or 1 must contain at least 2 of the 48 singularities of $S$.
(b) If $C \subset S$ is a curve which is smooth at the singular points of $S$, then $\deg(C) \leq 4g(C) + 44$.
(c) If $C \subset S'$ is a rational curve which is neither exceptional nor contained in $x_0x_1x_2x_3 = 0$, then $C \cdot E \geq 8$.

In Sect. 4 we develop the method for arbitrary cyclic quotient singularities, which is captured in the following particular example. Let us consider the lines $L_{t,u}$ for $(t,u) \in \mathbb{P}^1$. They are precisely the tangent lines to the conic $(y^2 - 4xz) \subset \mathbb{P}^2$. Let $[L_1, \ldots, L_d]$ be distinct lines such that $L_i = L_{t_i,u_i}$ for some $(t_i, u_i) \in \mathbb{P}^1$. Let us take positive integers $a_1, \ldots, a_d$ such that $\sum_{i=1}^d a_i = mR$ for some integers $m, R > 0$. Assume that $a_i < m$ and $\gcd(a_i, m) = 1$ for all $i$, and that $a_i + a_j$ is not divisible by $m$ for all $i \neq j$.

The method we develop in Sect. 4 allows us to prove the following.

Theorem 1.3 If $4m < d$, then the surface

$$(t_1^2x + t_1u_1y + u_1^2z)^{a_1} \cdots (t_d^2x + t_du_dy + u_d^2z)^{a_d} = w^m$$

in $\mathbb{P}(1, 1, 1, R)$ contains no curves with geometric genus $\leq 1$ apart from the $\mathbb{P}^1$s defined by $t_i^2x + t_iu_1y + u_1^2z = 0$. Its normalization is a simply connected normal projective surface with ample canonical class.

The Diophantine hypersurface $\left( \prod_{i=1}^{15} (i^2x + iy + z) = w^3 \right) \subset \mathbb{P}(1, 1, 1, 5)$ works as an example for Theorem 1.3.

Let $X$ be a smooth projective surface, and let $\omega \in H^0(X, \mathcal{L} \otimes S^r \Omega_X^1)$ for some line bundle $\mathcal{L}$, and some integer $r > 0$. Key in the above results is the notion of $\omega$-integral curves; see Definition 2.1. We construct surfaces via a composition of cyclic covers of the fixed surface $X$, which are branched along $\omega$-integral curves, and for which we know all $\omega$-integral curves on $X$. In Sect. 4.3 we prove the following general theorem.

Theorem 1.4 Assume we have the relations $m_i \mathcal{M}_i = \sum_{j=1}^{s_i} a_{i,j} D_{i,j}$ in $\text{Pic}(X)$, for some line bundles $\mathcal{M}_i$, with $0 < a_{i,j} < m_i$ and $\gcd(a_{i,j}, m_i) = 1$ for all $i, j$. Assume that the divisor $\sum_{i=1}^n \sum_{j=1}^{s_i} D_{i,j}$ has simple normal crossings with $D_{i,j}$ $\omega$-integral curves. Then a tower of $n$ cyclic covers of degree $m$

$$X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 := X$$

is defined, where all $X_k$ are normal projective surfaces with only cyclic quotient singularities. If

$$\sum_{i=1}^n \frac{1}{m_i} \left( \sum_{j=1}^{s_i} D_{i,j} \right) - \mathcal{L} \text{ is } \mathbb{Q}\text{-ample}, \text{ and } a_{i,j} \neq -a_{i,j'} \text{ (mod } m_i) \text{ for all } i, j \neq j',$$

then $X_n$ can have curves of geometric genus $\leq 1$ only in the set of preimages of $\omega$-integral curves in $X$.

From this theorem, and via a particular result of Vojta [20] in Nevanlinna theory for algebraic varieties, in Sect. 5 we obtain the following.
Theorem 1.5 Let us consider the hypothesis and notation as in Theorem 1.4. In addition, assume that all solutions to the differential equation given by $\omega = 0$ on $X$ are $\omega$-integral curves. Then an entire curve in $X_n$ must be contained in the set of preimages of $\omega$-integral curves in $X$. In particular, if the set of preimages of $\omega$-integral curves in $X$ does not contain curves of geometric genus $0$ or $1$, then $X_n$ is hyperbolic.

The following application of the method and its adaptations produces explicit families of smooth complete intersection surfaces which are hyperbolic and have arbitrary multidegrees, in particular, low multidegrees. We note that it is key for this application to work with singularities.

Theorem 1.6 There are explicit families of smooth complete intersections in $\mathbb{P}^{n+2}$, parametrized by a Zariski open subset of $\mathbb{A}^N$, which are hyperbolic for multidegrees

(a) $(m_1, \ldots, m_n)$ when $n \geq 8$ and $m_i \geq 2$, and $N = \sum_{i=1}^{n} m_i$ when $m_i > 2$, or $N = r + 2 \sum_{i=1}^{n-r-1} m_i$ where $m_1 = m_2 = \ldots = m_r = 2$ and $m_i > 2$ for $i \geq r$.

(b) $(m_1, \ldots, m_n)$ when $n \geq 5$ and $m_i \geq 3$, and $N = \sum_{i=1}^{n} m_i$.

(c) $(2, m_1, \ldots, m_{n-1})$ when $n \geq 6$, $m_i \geq 2$ for all $i < n - 1$, $m_{n-1} \geq 3$, and $N = r + 1 + 2 \sum_{i=0}^{n-r-1} m_i$ where $m_1 = m_2 = \ldots = m_{r-1} = 2$ and $m_i > 2$ for $i \geq r$.

(d) $(2, m_1, \ldots, m_{n-1})$ when $n \geq 4$ and $m_i \geq 3$, and $N = 2 \sum_{i=1}^{n-1} m_i$.

The explicit families are shown at the end of Sect. 5. This theorem gives evidence for [6, Conjecture 0.18] in the case of surfaces. Smooth complete intersection surfaces $X_{n,k} \subset \mathbb{P}^n$ of multidegree $(k, \ldots, k)$ and hyperbolic have been constructed in [9] for $k = 3$, $n \geq 6$; $k = 4, 5$, $n \geq 5$; $k \geq 6$, $n \geq 4$. We also point out that hyperbolic complete intersections of high multidegree have been constructed by Brotbek [4] (see also Xie [22] and Brotbek-Darondeau [5]), and, as far as we know, this theorem provides the first such hyperbolic families for low degrees.

2 The generalized surfaces of cuboids

Let $n \geq 3$, $m \geq 2$ be integers. Let $\{F_1, \ldots, F_{nm}\}$ and $\{G_1, \ldots, G_{nm}\}$ be collections of distinct $nm$ vertical and horizontal fibres of $\mathbb{P}^1 \times \mathbb{P}^1$ as in the introduction. Let us denote the elements of $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}^2$ by $(a, b)$. Then we have

$$F_{km+1} + \cdots + F_{(k+1)m} + G_{km+1} + \cdots + G_{(k+1)m} = (m, m)$$

for $0 \leq k \leq n - 1$. These expressions define a tower of $n$ cyclic covers of degree $m$

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 := \mathbb{P}^1 \times \mathbb{P}^1$$

inductively as follows: Let $f_{k+1}: X_{k+1} \to X_k$ be the cyclic cover defined by the equation of line bundles

$$g_k^*(F_{km+1} + \cdots + F_{(k+1)m} + G_{km+1} + \cdots + G_{(k+1)m}) \simeq g_k^*(1, 1)^{\otimes m},$$

where $g_k := f_1 \circ \cdots \circ f_k$. To be more precise, the surface $X_{k+1}$ is defined as $X_{k+1} := \text{Spec} \mathbb{C}[x_j \otimes_{j=0}^{m-1} g_k^*(1, 1)^{-j}]$, and so the finite morphism $f_{k+1}$ satisfies $f_{k+1*} \mathcal{O}_{X_{k+1}} = \bigoplus_{j=0}^{m-1} g_k^*(1, 1)^{-j}$; for details see e.g. [17, Section 1]. All $X_k$ are normal projective surfaces with $km^{k+1}$ singularities of type

$$A_{m-1} : (0, 0) \in (z^m - xy) \subset \mathbb{C}^3.$$
As shown in the introduction for \( n = 3 \), the surfaces \( X_k \) are complete intersections in \( \mathbb{P}^{k+3} \), and so they are simply connected.

Let \( D_k \subset X_{k-1} \) be the branch divisor of \( f_k : X_k \to X_{k-1} \), this is

\[
D_k := g_{k-1}^* \left( F_{km+1} + \cdots + F_{(k+1)m} + G_{km+1} + \cdots + G_{(k+1)m} \right).
\]

Hence it is a collection of \( 2m \) smooth curves which form a simple normal crossings divisor with \( m^{k+1} \) nodes. By the Riemann-Hurwitz formula, each smooth curve \( \Gamma \) in \( D_k \) satisfies

\[
2g(\Gamma) - 2 = m^{k-1}((m-1)(k-1) - 2),
\]

and \( \Gamma^2 = 0 \), where \( g(\Gamma) \) is the genus of \( \Gamma \). We have \( D_k^2 = 2m^{k+1} \), and \( D_k \cdot K_{X_{k-1}} = 2m(2g(\Gamma) - 2) \). We also have the formulas

\[
\chi(\mathcal{O}_{X_k}) = m \chi(\mathcal{O}_{X_{k-1}}) + \frac{(m-1)(2m-1)}{12m} D_k^2 + \frac{(m-1)}{4} D_k \cdot K_{X_{k-1}},
\]

and \( K_{X_k}^2 = mK_{X_{k-1}}^2 + \frac{(m-1)^2}{m} D_k^2 + 2(m-1)D_k \cdot K_{X_{k-1}} \), and so

\[
K_{X_k}^2 - 8\chi(\mathcal{O}_{X_k}) = -\frac{2k}{3} m^k (m^2 - 1) < 0.
\]

In fact \( \frac{K_{X_k}^2}{\chi(\mathcal{O}_{X_k})} \) approaches 8 as \( k \gg 0 \). We also have

\[
K_{X_k} \sim g_k^* \left( -2, -2 + \frac{(m-1)}{m} (m, m) k \right) = (k(m-1) - 2) g_k^* (1, 1),
\]

and so \( X_k \) has ample canonical class if and only if \( k(m-1) > 2 \). Hence for \( k = n \) the surface \( X_n \) is of general type.

Let \( \sigma_n : X'_n \to X_n \) be the minimal resolution of the singularities in \( X_n \), and let \( g'_n = \sigma_n \circ g_n \). Then

\[
g'_n^*(mn, mn) = mR + mE
\]

where \( R \) is the strict transform by \( \sigma_n \) of the branch divisor (in \( X_0 \)) of \( g_n \), and \( E \) is the (reduced) sum of the exceptional curves of \( \sigma_n \). That is a simple local toric computation; see e.g. [11, 2.1]. Therefore, we have

\[
g'_n^*(n, n) = R + E.
\]

We now recall the key general notion of \( \omega \)-integral curve (see [19, Definition 2.4], [10, Definition 3.2]).

**Definition 2.1** Let \( X \) be a smooth surface. Let \( r \geq 1 \) be an integer, let \( \mathcal{L} \) be an invertible sheaf on \( X \), and let \( \omega \in H^0(X, \mathcal{L} \otimes S^r \Omega^1_X) \). Let \( C \) be an irreducible curve on \( X \), and let \( \varphi_C : \tilde{C} \to X \) be the normalization of \( C \subset X \). The curve \( C \) is said to be \( \omega \)-integral if \( \varphi_C^* \omega \in H^0(\tilde{C}, \varphi_C^* \mathcal{L} \otimes S^r \Omega^1_{\tilde{C}}) \) is zero.

Let \( \omega \in H^0(X_0, (2, 2) \otimes S^2 \Omega^1_{X_0}) \) be the global section \( z_3^2dz_1dz_2 \). Consider the isomorphism \( h : \mathbb{P}^1 \times \mathbb{P}^1 \to X_0 \) given by \( h([x, y] \times [w, z]) = [xw, xz, yw, yz] \). The section \( \omega \) corresponds to the section \( y^2z^2dxdw \) under \( h \). Therefore, horizontal and vertical fibres are \( \omega \)-integral, and this is the complete set of \( \omega \)-integral curves by [10, 6.6]. In particular the branch loci of \( g_n : X_n \to X_0 \) is formed by \( \omega \)-integral curves.
Notation 2.2 Let $X, r, \mathcal{L}$, and $\omega$ be as in Definition 2.1. Let $Y$ be a smooth surface, and $\pi : Y \to X$ be a dominant morphism. We denote by $\pi^*\omega$ the image of $\omega$ under the natural pull-back of $dx$. Therefore we have the pull-back relations

\[ H^0(X', \mathcal{O}_{X'}(-(m - 1)R) \otimes g_n^*(2, 2) \otimes S^2\Omega^1_{X'}) \to H^0(Y, \pi^*(\mathcal{L}) \otimes S^2\Omega^1_Y) \]

where the $g_n$ which also has $\omega_n$ in our case is precisely $g_n^*\omega$. We now show the existence of a section $\omega''$ in a “more negative” sheaf, which also has $g_n^*\omega$ as image.

Lemma 2.3 Let $m > 2$. Then there is $\omega''$ in $H^0(X', \mathcal{O}_{X'}(-(m - 1)R - E) \otimes g_n^*(2, 2) \otimes S^2\Omega^1_{X'})$ whose image under the natural morphism

\[ H^0(X', \mathcal{O}_{X'}(-(m - 1)R - E) \otimes g_n^*(2, 2) \otimes S^2\Omega^1_{X'}) \to H^0(Y, \pi^*(\mathcal{L}) \otimes S^2\Omega^1_Y) \]

is $g_n^*\omega$.

Proof We only need to prove that $g_n^*\omega$ vanishes along the divisor $E$. This is a general toric local computation with differentials. So let us say that $\omega \in H^0(X_0, \mathcal{L} \otimes S^2\Omega^1_{X_0})$, where in our case $\mathcal{L} = (2, 2)$ and $r = 2$.

Let us consider one node $P$ of the branch divisor of $g_k$ for some $0 < k < n$. Take one preimage $Q$ of this node $P$ by the morphism $g_n$. At $Q$ we have a rational double point of type $A_{m-1}$, which is in particular a cyclic quotient singularity. The singularity and the map $g_n : X_n \to X_0$ is locally analytically isomorphic to $U = (\mathbb{C}^m - xy) \subset \mathbb{C}^2$ and the projection $g(x, y, z) = (x, y)$ respectively. Thus to compute the pull-back of $\omega$ under $g_n$, we use this local model. Moreover, this model has a toric description as follows. See e.g. [15].

Let $\sigma' : V \to U$ be the minimal resolution, and $g' : V \to \mathbb{C}^2$ the composition of $\sigma'$ with $g$. Let $E_0$ be the (reduced) preimage under $g'$ of $x = 0$, and let $E_m$ be the (reduced) preimage under $g'$ of $y = 0$. Let $E_1, E_2, \ldots, E_m$ be the chain of $\mathbb{P}^1$'s in $V$ which corresponds to the exceptional divisor of $\sigma'$. Hence $E_0, E_1, \ldots, E_{m-1}, E_m$ also form a chain. Let $u_i$ be the local coordinate defining $E_i$, so that at each node of $E_0, E_1, \ldots, E_m$, $E_m$ we have local coordinates $u_i, u_{i+1}$ for $V$. Then (see [15, Example 3.1]) we have that locally $g'$ is given by

\[ g'(u_i, u_{i+1}) = (u_i^{m-i}u_{i+1}^{m-i-1}, u_i^{m-i}u_{i+1}^{m-i-2}) \]

Therefore we have the pull-back relations

\[ dx = (m - i)u_i^{m-i-1}u_{i+1}^{m-i-1}du_i + (m - i - 1)u_i^{m-i}u_{i+1}^{m-i-2}du_{i+1} \]

and

\[ dy = iu_i^{m-i-1}u_{i+1}^{m-i}du_i + (i + 1)u_i^{m-i}u_{i+1}^{m-i}du_{i+1} \]

Via a local trivialization of $\mathcal{L}$, we can identify $\omega$ with a section of $\mathcal{L} \otimes \Omega^1_{\mathbb{C}^2}$ around $(0, 0)$ as

\[ \omega = a_0dx^{\otimes r} + a_1dx^{\otimes(r-1)} \otimes dy + \cdots + a_rdy^{\otimes r}, \]

where the $a_i$ are holomorphic around $(0, 0)$. Since $x = 0$ and $y = 0$ are $\omega$-integral curves, then as in [10, Theorem 3.87], we have that $a_0 = ya_0'$ and $a_r = xa_r'$. Therefore the pull-back of $a_0dx^{\otimes r}$ and $a_rdy^{\otimes r}$ vanish along the divisor $E_1 + \cdots + E_{m-1}$. On the other hand, the pull-back of $dx^{\otimes(r-k)} \otimes dy^{\otimes k}$ for $0 < k < r$ vanishes on $u_i = 0$ if and only if $i - 1 > 0$
or \( m - i - 1 > 0 \), and both inequalities hold because \( m > 2 \). Therefore the pull-back of \( \omega \) vanishes along \( E_0 + E_1 + \cdots + E_{m-1} + E_m \). Moreover, by [10, Theorem 3.87], it vanishes of order \( m - 1 \) along \( E_0 \) and \( E_m \).

\[ \square \]

**Remark 2.4** The previous lemma is valid for any tower of cyclic morphisms of order \( m > 2 \) branched along a simple normal crossings divisor which is formed by \( \omega \)-integral curves, so that the singularities are rational double points of type \( A_{m-1} \).

**Theorem 2.5** Let \( m > 2 \), and let \( C \subset X'_n \) be a curve of geometric genus \( g \) which is not an exceptional curve of \( \sigma_n \). If

\[
\frac{4g - 4}{n - 2} < g'_n(C) \cdot (1, 1),
\]

then \( g'_n(C) \) is an \( \omega \)-integral curve.

**Proof** By Lemma 2.3, we have

\[
\omega'' \in H^0(X'_n, \mathcal{O}_{X'_n}(-m - 1)R - E) \otimes g'^n(2, 2) \otimes S^2\Omega^1_{X'_n})
\]

whose image under the natural morphism

\[
H^0 \left( X'_n, \mathcal{O}_{X'_n}(-m - 1)R - E) \otimes g'^n(2, 2) \otimes S^2\Omega^1_{X'_n}) \right) \to H^0 \left( X'_n, \mathcal{O}_{X'_n}(-m - 1)R - E) \otimes g'^n(2, 2) \otimes S^2\Omega^1_{X'_n}) \right)
\]

is \( g'^n\omega \). Let \( \varphi_C : X'_n \to X'_n \) be the normalization of \( C \subset X'_n \). By Equality (2.1), we obtain that

\[
g'^n(-n, -n) - (m - 2)R = -(m - 1)R - E
\]

in \( \text{Pic}(X'_n) \). In this way

\[
\mathcal{O}_{X'_n}(-(m - 1)R - E) \otimes g'^n(2, 2) \simeq g'^n(2 - n, 2 - n) \otimes \mathcal{O}_{X'_n}(-(m - 2)R) =: \mathcal{L},
\]

and so \( \deg_C \left( \varphi_C^* \mathcal{L} \otimes S^2\Omega^1_C \right) \leq (2 - n, 2 - n) \cdot g'_n(C) + 2(2g - 2) < 0 \) by the projection formula and the hypothesis. Therefore \( H^0(C, \varphi_C^* \mathcal{L} \otimes S^2\Omega^1_C) = 0 \), and \( C \) is a \( \omega'' \)-integral curve. As in [10, Proposition 3.88], the \( \omega'' \)-integral curves in \( X'_n \) are also \( g'_n\omega \)-integral curves. On the other hand, by [10, Theorem 3.35] the \( g'_n \)-integral curves in \( X'_n \) are either the exceptional divisors of \( \sigma_n \) or curves \( C \subset X'_n \) such that \( g'_n(C) \) is \( \omega \)-integral in \( X_0 \).

\[ \square \]

**Corollary 2.6** There are no curves of geometric genus \( \leq 1 \) in \( X_n \) for any \( m > 2 \).

**Proof** By Theorem 2.5, if \( C \) is a curve in \( X'_n \) of geometric genus 0 or 1 which is not an exceptional curve of \( \sigma_n : X'_n \to X_n \), then \( g'_n(C) \) is \( \omega \)-integral. All \( \omega \)-integral curves are fibres of \( \mathbb{P}^1 \times \mathbb{P}^1 \). But, since \( n > 2 \) and \( m > 2 \), we have that the preimage of a fibre in \( \mathbb{P}^1 \times \mathbb{P}^1 \) has geometric genus bigger than 1. Therefore the only curves of geometric genus 0 or 1 in \( X'_n \) are the exceptional curves. As \( \sigma_n \) contracts them, \( X_n \) has no such curves.

\[ \square \]

**Remark 2.7** Theorem 2.5 is not true for \( m = 2 \), since otherwise we would obtain all curves with geometric genus \( \leq 1 \) from fibres, and that is not the case (see Sect. 3). The problem is that Lemma 2.3 does not work for \( m = 2 \), showing optimality in that sense. We will revisit this issue in Sect. 4.
Example 2.8  Under the Bombieri-Lang conjecture, Corollary 2.6 says that, for example, the complete intersection surface
\[
\prod_{i=1}^{m} (x_0 - i x_1 + i^2 x_2) = x_4^m \\ \prod_{i=1}^{m} (x_0 - (i + m)x_1 + (i + m)^2 x_2) = x_5^m \\ \prod_{i=1}^{m} (x_0 - (i - m)x_1 + (i - m)^2 x_2) = x_6^m \\
x_0 x_3 + x_2^3 = x_1 x_2
\]
in \( \mathbb{P}^6 \) can only have a finite number of points in \( \mathbb{P}^6(\mathbb{Q}) \). Here \( n = 3 \), and we have chosen a specific model and set of parameters in \( \mathbb{Z} \).

3 The surface of cuboids

We know that our proof of Theorem 2.5 does not apply to the surface of cuboids
\[
z_0 z_3 = z_4^2 \\ (z_0 - z_3)^2 + (z_1 + z_2)^2 = z_5^2 \\ (z_0 + z_3)^2 - (z_1 + z_2)^2 = z_6^2 \\
z_0 z_3 = z_1 z_2
\]
in \( \mathbb{P}^6 \), since here \( m = 2 \). The purpose of this section is to adapt the method to get some results on rational curves of this surface. We follow the notation of Sect. 2 for this particular example, and so the surface of cuboids is denoted by \( X_3 \). We recall that \( X_3 \) is isomorphic to the original surface of cuboids:
\[
x_0^2 + x_1^2 + x_2^2 = x_3^2, \\ x_0^2 + x_1^2 = x_4^2, \\ x_0^2 + x_2^2 = x_5^2, \\ x_1^2 + x_2^2 = x_6^2
\]
in \( \mathbb{P}^6 \), by the isomorphism \( z_0 = x_0 - i x_1, z_1 = x_1 - x_2, z_2 = x_3 + x_2, z_3 = x_0 + i x_1, z_4 = x_4, z_5 = 2x_5, z_6 = 2i x_6, \) where \( i = \sqrt{-1} \). This surface has been extensively studied, because it is related to the Perfect Cuboid Problem of Euler. Some references are \([2,8,16,18]\).

There are at least 92 curves of geometric genus zero or one in \( X_3 \). They are all smooth (see \([18]\)):

- The irreducible components of \( x_0 x_1 x_2 x_3 = 0 \), which are 32 rational curves;
- The irreducible components of \( x_4 x_5 x_6 = 0 \), which are 12 elliptic curves, corresponding to the pull-backs of the 6 horizontal fibres \( \{F_1, \ldots, F_6\} \), and the 6 vertical fibres \( \{G_1, \ldots, G_6\} \);
- The curve defined by the equations
  \[
  x_0 = x_1, \quad x_4 = x_5, \quad \sqrt{2} x_0 = x_6, \quad x_2^2 + x_6^2 = x_3^2, \quad 2x_5^2 + x_6^2 = 2x_3^2,
  \]
  and the curves obtained as orbits by applying the automorphisms of \( X_3 \). This gives us 48 elliptic curves.

It was proved by Stoll and Testa \([16]\) that every curve of geometric genus \( \leq 1 \) in \( X_3 \) of degree less than or equal to 4 belongs to this list, and they conjectured that these are all the curves with geometric genus \( \leq 1 \) in this surface.

Using the global section (appearing in Sect. 2)
\[
\omega \in H^0(X_0, (2, 2) \otimes S^2 \Omega^1_{X_0}),
\]
from a weaker version of Lemma 2.3 that works for the case \( m = 2 \), it is obtained the following (cf. Corollary 6.43 and Theorem 6.48 in \([10]\)):
Theorem 3.1 Let $C$ be an irreducible curve on $X_3$ with strict transform $C' \subseteq X'_3$. If
\[
\deg_C(\varphi_C^*(\mathcal{O}_{X_3}(-R + E) \otimes s_3^*(2, 2)) \otimes S^2\Omega^1_C) = -\deg(C) + (E.C') + 4g(C) - 4
\]
is negative, then $g'_n(C)$ is an $\omega$-integral curve.

This result is not enough to capture all the curves of geometric genus $\leq 1$ on $X_3$, but it allows us to give extra information about these curves, as the following corollaries show (cf. Proposition 1.11 and Corollary 1.13 in [10]):

Corollary 3.2 Every curve of geometric genus zero or one on $X_3$ contains at least two of the 48 singular points of $X_3$.

It was known by work of Freitag and Salvati Manni [8] (from [2]) that a curve of geometric genus zero or one must contain at least one of the 48 singular points of $X_3$.

Corollary 3.3 Let $C$ be an irreducible curve in $X_3$, smooth at the singularities of this surface ($C$ can have singularities outside of the 48 singular points of $X_3$). Then
\[
\deg(C) \leq 4g(C) + 44.
\]

This was also obtained by Kani for smooth curves [12], using different methods, and it improves a result of Freitag and Salvati Manni from [8]. In this work we want to improve these results, by using different global twisted differentials at the same time, in order to get better control on the exceptional divisors.

Recall the tower of cyclic covers of degree $m = 2$
\[
X_3 \rightarrow X_2 \rightarrow X_1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1.
\]
In this case $R$ consists of the horizontal fibres at the points
\[
[1 : 1], \ [1 : -1], \ [1 : i], \ [1 : -i], \ [1 : 0], \ [0 : 1],
\]
and the vertical fibres at the same points. We will denote the horizontal fibre at $[a : b]$ by $h_{b/a}$ and the vertical fibre at $[a : b]$ by $v_{b/a}$, with the convention that $1/0 = \infty$.

The image of $x_4 = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$ consists of $h_1 \cup h_{-1} \cup v_1 \cup v_{-1}$, the image of $x_5 = 0$ consists of $h_1 \cup h_{-1} \cup v_1 \cup v_{-1}$, the image of $x_6 = 0$ consists of $h_0 \cup h_\infty \cup v_0 \cup v_\infty$. Thus, each of these consists of two horizontal fibres and two vertical fibres, intersecting at 4 points. Over each of the 12 intersection points, there are 4 out of the 48 singular points of $X_3$, and every singular point of $X_3$ maps to one of these intersections.

The image of $x_0 = 0$ under $g_3$ is the curve $C_0 = \{xz + yw = 0\}$. Similarly, the image of $x_1 = 0$ is $C_1 = \{yw - xz = 0\}$, the image of $x_2 = 0$ is $C_2 = \{yz - xw = 0\}$, and the image of $x_3 = 0$ is $C_3 = \{xw + yz = 0\}$.

Consider the following global sections in $H^0(\mathbb{P}^1 \times \mathbb{P}^1, (3, 3) \otimes S^2\Omega^1_{\mathbb{P}^1 \times \mathbb{P}^1})$:
\[
\omega_0 = (xz - yw)y^2z^2dxdw \quad \omega_1 = (yw - xz)y^2z^2dxdw \\
\omega_2 = (yz - xw)y^2z^2dxdw \quad \omega_3 = (xw + yz)y^2z^2dxdw.
\]

For each $0 \leq i \leq 3$, the $\omega_i$-integral curves consist of the horizontal fibres, the vertical fibres, and $C_i$.

Let $E_i$ be the sum of the exceptional divisors from the 24 singular points of $X_3$ whose image belongs to $C_i$, and let $E'_i$ be the sum of the exceptional divisors from the 24 singular points whose image does not belong to $C_i$. We have the following version of Lemma 2.3 adapted to these new global sections.
Lemma 3.4 For each $0 \leq i \leq 3$, there is $\omega''_i$ in

$$H^0(X'_3, \mathcal{O}_{X'_3}(-R - E_i) \otimes g''_3(3, 3) \otimes S^2 \Omega^1_{X'_3})$$

whose image under the natural morphism

$$H^0 \left( X'_3, \mathcal{O}_{X'_3}(-R - E_i) \otimes g''_3(3, 3) \otimes S^2 \Omega^1_{X'_3} \right) \rightarrow H^0 \left( X'_3, g''_3(3, 3) \otimes S^2 \Omega^1_{X'_3} \right)$$

is $g''_3 \omega_i$.

Proof Fix $0 \leq i \leq 3$. We know that $g''_3 \omega_i$ vanishes along $R$. We will prove that $g''_3 \omega$ vanishes along the divisor $E_i$.

We consider a node $P$ of the branch divisor of $g_k$ for some $0 < k < 3$, and such that $P \in g'_k(C_i)$. At the preimage $Q$ of $P$, we have a singularity of type $A_4$. This singularity is locally analytically isomorphic to $(z^2 - xy) \subset \mathbb{C}^2$ and $g(x, y, z) = (x, y)$.

Let $E$ be the exceptional divisor at $Q$, let $u_1 = 0$ be the local coordinate defining it, and, as before, the local coordinates $u_0 = 0, u_2 = 0$ define the (reduced) preimages of $x = 0$ and $y = 0$ respectively. Then we have

$$dx = 2u_0u_1du_0 + u_0^2du_1, \quad dy = u_0^2du_1 + 2u_1u_2du_2.$$ 

Via a local trivialization of $g''_3(3, 3)$, we can identify $\omega_0$ with a section of $S^2 \Omega^1_{X'_3}$ around $(0, 0)$ as $\omega_0 = (x - y)\bar{\omega}$ with $\bar{\omega} = a_0dx \otimes dy + a_1dy \otimes dx + a_2dy \otimes dy$. Since $g''_3(x) = u_0^2u_1$, and $g''_3(y) = u_1u_2^2$, we obtain that the pull-back of $\omega_0$ vanishes along $E$ with order one. Doing this for every singular point contained in $C_0$, we obtain that $g''_3 \omega_0$ vanishes along the divisor $E_0$. A similar computation shows that for every $i$, the pull-back of $\omega_i$ vanishes along the divisor $E_i$.

Theorem 3.5 Let $C \subset X'_3$ be a curve of geometric genus $0$, which is not an exceptional curve of $\sigma_3$, and it is not in the pull-back of the $C_i$’s. Then for every $0 \leq i \leq 3$, we have $(C.E'_i) \geq 4$.

Proof From Proposition 3.88 in [10], we have that the $\omega''_i$-integral curves in $X'_4$ are among the pull-back of the horizontal fibres, the pull-back of the vertical fibres and the curves $C_i$. Let $C \subset X'_3$ be a curve of geometric genus zero. We have

$$\mathcal{O}_{X'_3}(-R - E_i) \otimes g''_3(3, 3) = \mathcal{O}_{X'_3}(-R - E_i + R + E) = \mathcal{O}_{X'_3}(E'_i),$$

thus if for some $0 \leq i \leq 3$ we have $(C.E_i) < 4$, then we obtain

$$\deg_C(\varphi''_C \mathcal{O}_{X'_3}(-R - E_i) \otimes \varphi''_C g''_3(3, 3) \otimes S^2 \Omega^1_{X'_3}) = (C.E'_i) - 4 < 0,$$

hence $C$ must be an $\omega''_i$-integral curve.

Corollary 3.6 Let $C \subset X'_3$ be a curve of geometric genus $0$, which is not an exceptional curve of $\sigma_3$, and not in the pull-back of the $C_i$’s. Then $(C.E) \geq 8$.

Proof We know that $(C.E'_i) \geq 4$ for each $0 \leq i \leq 3$. Since $E'_0 + E'_1 + E'_2 + E'_3 = 2E$, we obtain $(C.E) \geq 8$. 

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4 Low genus curves in cyclic covers

4.1 Local picture

We first recall the local picture of a cyclic cover, together with cyclic quotient singularities, and their minimal resolution.

Let \( 0 < q < m \) be integers with \( \gcd(q, m) = 1 \). Consider the action of \( \tau(x, y) = (\mu x, \mu^q y) \) on \( \mathbb{C}^2 \), where \( \mu \) is a primitive \( m \)-th root of 1. A cyclic quotient singularity \( \frac{1}{m}(1, q) \) is a germ at the origin of the quotient of \( \mathbb{C}^2 \) by \( \langle \tau \rangle \); cf. [1, III §5]. For us it will be convenient to use the following toric description. Consider the inclusions of rings

\[
\mathbb{C}[x^m, y^m, x^{m-q}y] \subset \mathbb{C}[x, y]^{\langle \tau \rangle} \subset \mathbb{C}[x, y].
\]

We note that

\[
\mathbb{C}[x^m, y^m, x^{m-q}y] \simeq \mathbb{C}[u, v, w]/(uv^m - q - w^m),
\]

where \( v = x^m, u = y^m, \) and \( w = x^{m-q}y \). The inclusions define morphisms between the corresponding spectrums of the rings, which translates into the maps

\[
\mathbb{C}^2 \xrightarrow{q} \mathbb{C}^2/\langle \tau \rangle \xrightarrow{\eta} (uv^m - w^m) \subset \mathbb{C}^3 \xrightarrow{r} \mathbb{C}^2
\]

where \( r(u, v, w) = (u, v) \) is the cyclic cover branch along \( \{uv^m - q = 0\} \) of degree \( m \), \( \eta \) is the normalization map, and \( q \) is the quotient map. As in [1, III §5], around \( (0, 0) \in \mathbb{C}^2 \) the local picture for any cyclic cover of degree \( m \) is given by

\[
\{ua^v = w^m\} \subset \mathbb{C}^3 \rightarrow \mathbb{C}^2,
\]

where \( \gcd(a, m) = \gcd(b, m) = 1 \), and \( q \) is such that \( aq + b \equiv 0 \) modulo \( m \).

Let \( \sigma: \tilde{Y} \rightarrow Y \) be the minimal resolution of \( Y := \frac{1}{m}(1, q) \). Figure 1 shows the exceptional curves \( E_i = \mathbb{P}^1 \) of \( \sigma \), for \( 1 \leq i \leq s \), and the strict transforms \( E_0 \) and \( E_{s+1} \) of \( (y = 0) \) and \( (x = 0) \) respectively.

The numbers \( E_i^2 = -b_i \) are computed using the Hirzebruch-Jung continued fraction

\[
\frac{m}{q} = b_1 - \frac{1}{b_2 - \frac{1}{\ddots - \frac{1}{b_s}}} =: [b_1, \ldots, b_s].
\]

The continued fraction \([b_1, \ldots, b_s]\) defines the sequence of integers

\[
0 = \beta_{s+1} < 1 = \beta_s < \cdots < q = \beta_1 < m = \beta_0
\]

where \( \beta_{i+1} = b_i \beta_i - \beta_{i-1} \). In this way, \( \frac{\beta_{i+1}}{\beta_i} = [b_i, \ldots, b_s] \). Partial fractions \( \frac{a_i}{\beta_i} = [b_1, \ldots, b_{i-1}] \) are computed through the sequences

\[
0 = a_0 < 1 = \alpha_1 < \cdots < q^{-1} = \alpha_s < m = \alpha_{s+1}.
\]

![Fig. 1 Exceptional divisors over \( \frac{1}{m}(1, q) \), \( E_0 \) and \( E_{s+1} \) ](image-url)
where \( \alpha_{i+1} = b_i \alpha_i - \alpha_{i-1} \) (\( q^{-1} \) is the integer such that \( 0 < q^{-1} < m \) and \( qq^{-1} \equiv 1 \) (mod \( m \))), and \( \gamma_0 = -1, \gamma_1 = 0, \gamma_{i+1} = b_i \gamma_i - \gamma_{i-1} \). We have \( \alpha_{i+1} \gamma_i - \alpha_i \gamma_{i+1} = -1, \beta_i = q \alpha_i - m \gamma_i \), and \( \frac{m}{q} = [b_s, \ldots, b_1] \). These numbers appear in the pull-back formulas

\[
g^* ((u = 0)) = \sum_{i=0}^{s+1} \beta_i E_i, \quad g^* ((v = 0)) = \sum_{i=0}^{s+1} \alpha_i E_i,
\]

where \( g' := \sigma \circ \eta \circ r \), and \( K_Y \equiv \sigma^*(K_Y) + \sum_{i=1}^s \left( -1 + \frac{\beta_i + \alpha_i}{m} \right) E_i \). The numbers \( d_i := -1 + \frac{\beta_i + \alpha_i}{m} \) are the discrepancies of \( E_i \). We have \( d_i \in [1, 0] \). Let \( u_i \) be a local coordinate defining \( E_i \), so that at each node of \( E_0, E_1, \ldots, E_s, E_{s+1} \) we have local coordinates \( u_i, u_{i+1} \) for \( Y \). Then (see [15]) we have that locally \( g' \) is given by

\[
g'(u_i, u_{i+1}) = \left( u_i^{\beta_i}, u_{i+1}^{\alpha_i} u_{i+1}^{\gamma_i} \right).
\]

Therefore we have the pull-back relations

\[
du = \beta_i u_i^{\beta_i - 1} u_{i+1}^{\beta_i + 1} du_i + \beta_{i+1} u_i^{\beta_i} u_{i+1}^{\beta_i + 1 - 1} du_{i+1}
\]

and

\[
dv = \alpha_i u_i^{\alpha_i - 1} u_{i+1}^{\alpha_i + 1} du_i + \alpha_{i+1} u_i^{\alpha_i} u_{i+1}^{\alpha_i + 1 - 1} du_{i+1}.
\]

### 4.2 Global picture

The following is taken from [17, Section 1]. Let \( X \) be a smooth projective surface over \( \mathbb{C} \), and let \( \sum_{j=1}^d D_j \) be a simple normal crossings divisor in \( X \), that is, the irreducible curves \( D_j \) are all smooth, and the singularities of the divisor are at most nodes. Let us assume the existence of a line bundle \( \mathcal{M} \) on \( X \) such that

\[
\mathcal{O}_X (a_1 D_1 + a_2 D_2 + \cdots + a_d D_d) \simeq \mathcal{M}^\otimes m
\]

for some integers \( 0 < a_j < m \) such that \( \gcd(a_j, m) = 1 \). With this data, one constructs a smooth projective surface \( Y' \) which represents the \( m \)-th root of \( D := \sum_{j=1}^d a_j D_j \) as follows. Let \( s \in H^0(X, \mathcal{O}_X(D)) \) be a section whose zero locus is \( D \). This section defines a structure of \( \mathcal{O}_X \)-algebra on \( \bigoplus_{i=0}^{m-1} \mathcal{M}^{-i} \) by means of the induced injection \( \mathcal{M}^{-m} \to \mathcal{O}_X(-D) \to \mathcal{O}_X \). Then we have the affine morphism \( f_0 : Y_0 \to X \), where \( Y_0 := \text{Spec}_X \left( \bigoplus_{i=0}^{m-1} \mathcal{M}^{-i} \right) \). The variety \( Y_0 \) might not be normal. To normalize it, we define the line bundles

\[
\mathcal{M}^{(i)} := \mathcal{M}^i \otimes \mathcal{O}_X \left( -\sum_{j=1}^d \left[ \frac{a_j}{m} \right] D_j \right)
\]

on \( X \) for \( 0 \leq i < m \). Then \( \eta : Y := \text{Spec}_X \left( \bigoplus_{i=0}^{m-1} \mathcal{M}^{-(i)} \right) \to Y_0 \) is the normalization of \( Y_0 \). Hence if \( f : Y \to X \) is the composition of \( \eta \) with \( f_0 \), then \( f_* \mathcal{O}_Y = \bigoplus_{i=0}^{m-1} \mathcal{M}^{-(i)} \).

We note that \( Y \) may have only cyclic quotient singularities over the nodes of \( \sum_{j=1}^d D_j \). More precisely, given a node in \( D_i \cap D_j \), we have one singularity in \( Y \) over that node (since \( \gcd(a_j, m) = 1 \) for all \( j \)), and it is of type \( \frac{1}{m}(1, q) \) where \( a_i q + a_j \equiv 0 \) modulo \( m \). Locally around that singularity, the map \( f : Y \to X \) is isomorphic to the local picture described in Sect. 4.1, i.e. it is \( \mathbb{C}^2 / \langle \tau \rangle \to \mathbb{C}^2 \) where \( \{u = 0\} = D_j \) and \( \{v = 0\} = D_j \).
Let $\sigma : Y' \to Y$ be the minimal resolution of the singularities in $Y$. The surface $Y'$ is a smooth (irreducible) projective surface. Let $f' : Y' \to X$ be the composition of $\sigma$ with $f$. Then $f'_* \mathcal{O}_Y = \bigoplus_{i=0}^{m-1} \mathcal{M}^{-(i)}$. Again, the local picture of $f'$ over a node of $D_i \cap D_j$ is as in Sect. 4.1, and so $f'$ is locally isomorphic to $\sigma \circ \eta \circ r$.

Now let us consider $\omega \in H^0(X, \mathcal{L} \otimes S^r \Omega_X^1)$ for some line bundle $\mathcal{L}$ on $X$, and some integer $r > 0$.

We are now going to prove a particular case of Theorem 1.4, which will give a geometric criterion for the low genus curves to be $\omega$-integral. Remark that, in this theorem, we need the assumption $a_i + a_j \not\equiv 0$ modulo $m$ to avoid the situation of cyclic quotient singularities of type $\frac{1}{m}(1, 1)$. As we saw in Sects. 2 and 3 for the $A_1$ rational double points, we cannot prove the existence of $\omega''$ when dealing with singularities of type $\frac{1}{m}(1, 1)$.

**Theorem 4.1** Assume that $D_j$ is $\omega$-integral (Definition 2.1) for all $j$. If $\sum_{j=1}^d D_j - m\mathcal{L}$ is ample and $a_j \not\equiv -a_j$ modulo $m$ for all $j \neq j'$, then $Y$ can have curves of geometric genus $\leq 1$ only in the set of preimages of $\omega$-integral curves in $X$.

**Proof** The proof follows the strategy of Sect. 2: Lemma 2.3, Theorem 2.5, and Corollary 2.6. As in Lemma 2.3, let us prove the existence of $\omega''$ in $H^0(Y', \mathcal{O}_{Y'}(-(m-1)R - E) \otimes f'^* \mathcal{L} \otimes S^r \Omega_{Y'}^1)$, where $R$ is the sum of the strict transforms of the $D_j$, and $E = \sum_k E_k$ is the sum of all exceptional curves of $\sigma$. This is a local computation, and so let $D_i = \{u = 0\}$ and $D_j = \{v = 0\}$ at a node of $D_i \cap D_j$ in $X$. We assume that the cyclic quotient singularity is $\frac{1}{m}(1, q)$ with continued fraction of length $s$. Following the proof of Lemma 2.3, we only need to check that the pull-back of $du^\otimes s \otimes dv^\otimes k$ by $f'$ vanishes on $E_l$, where $0 < k < r$ and $0 \leq l \leq s + 1$. According to the local computation in Sect. 4.1, this happens if and only if $\alpha_l > 1$ or $\beta_l > 1$. So assume that $\alpha_l \leq 1$ and $\beta_l \leq 1$. If $\alpha_l = 0$, then $l = 0$ and so $\beta_l = \beta_0 = m > 1$ a contradiction. The same for $\beta_l = 0$. If $\alpha_l = 1$, then $l = 1$ and $\beta_l = \beta_1 = q \geq 1$. Hence $q = 1$, but this singularity is $\frac{1}{m}(1, 1)$ and so the multiplicities $a_i, a_j$ of $D_i, D_j$ respectively must satisfy $a_i + a_j \equiv 0$ modulo $m$. But this is contrary to our assumptions. Same for $\beta_l = 1$. Therefore for any $l$ we have $\alpha_l > 1$ or $\beta_l > 1$.

On the other hand, we have the numerical equivalence

$$f'^* \left( \sum_{j=1}^d D_j \right) \equiv mR + m \sum_k (1 + d_k)E_k,$$

where $d_k$ is the discrepancy associated to $E_k$ (see the end of Sect. 4.1). Hence we obtain

$$\frac{1}{m} f'^* \left( - \sum_{j=1}^d D_j \right) - (m-2)R + \sum_k d_k E_k + f'^* \mathcal{L} \equiv -(m-1)R - E + f'^* \mathcal{L}.$$

We recall that $-1 < d_k < 0$ for all $k$.

Let $\mathcal{N} := -(m-1)R - E + f'^* \mathcal{L}$, let $C \subset Y'$ be a curve of geometric genus $g$, and not exceptional for $\sigma$. Let $\varphi_C : \tilde{C} \to Y'$ be the normalization of $C \subset Y'$. Then

$$\deg_{\tilde{C}} \left( \varphi_C^* \mathcal{N} \otimes S^r \Omega_{\tilde{C}}^1 \right) \leq \frac{1}{m} \left( - \sum_{j=1}^d D_j + m\mathcal{L} \right) \cdot f'(C) + r(2g-2).$$
by the projection formula. By our hypothesis we have
\[
\left(-\sum_{j=1}^{d} D_j + mL \right) \cdot f'(C) < 0,
\]
and so if \( g \leq 1 \), then \( \deg_C \left( \varphi_C^* N \otimes S^r \Omega_C^1 \right) = 0 \), and so the curve \( C \) is \( \omega'' \)-integral. As in Lemma 2.3, we conclude that \( f'(C) \) must be an \( \omega \)-integral curve in \( X \). Since \( \sigma: X' \rightarrow Y \) is a birational morphism contracting \( E \), we obtain that \( Y \) can have curves of geometric genus \( \leq 1 \) only in the set of preimages of \( \omega \)-integral curves in \( X \). \( \square \)

We finish this section with an explicit example, where \( X = \mathbb{P}^2 \) and \( D_j \) are lines. Let us consider the lines \( L_{t, u} = (t^2 x + t u y + u^2 z) \subset \mathbb{P}^2 \) for \( (t, u) \in \mathbb{P}^1 \). They are precisely the tangent lines to the conic \( (y^2 - 4xz) \subset \mathbb{P}^2 \). Let \( \{L_1, \ldots, L_d\} \) be distinct lines such that \( L_i = L_{t_i, u_i} \) for some \( (t_i, u_i) \in \mathbb{P}^1 \). Let us take positive integers \( a_1, \ldots, a_d \) such that \( \sum_{i=1}^{d} a_i = mR \) for some integers \( m, R > 0 \). Assume that \( a_i < m \) and \( \gcd(a_i, m) = 1 \) for all \( i \), and that \( a_i + a_j \) is not divisible by \( m \) for all \( i \neq j \).

**Corollary 4.2** If \( 4m < d \), then the surface
\[
(t_1^2 x + t_1 u_1 y + u_1^2 z)^{a_1} \cdots (t_d^2 x + t_d u_d y + u_d^2 z)^{a_d} = w^m
\]
in \( \mathbb{P}(1, 1, 1, R) \) contains no curves of geometric genus \( \leq 1 \) apart from the \( \mathbb{P}^1 \)'s defined by \( t_i^2 x + t_i u_i y + u_i^2 z = 0 \). Its normalization is a simply connected normal projective surface with canonical class.

**Proof** First we need to indicate \( \mathcal{L}, r \) and \( \omega \). By taking the differential of \( t_1^2 x + t_1 u_1 y + u_1^2 z \) for \( u = 1 \) and \( z = 1 \), we obtain the differential
\[
\omega = dx^2 - ydxdy + xdy^2
\]
which is a global section of \( H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4) \otimes S^2 \Omega_{\mathbb{P}^2}^1) \), and so \( \mathcal{L} := \mathcal{O}(4) \) and \( r = 2 \). Since we are considering this particular \( \omega \), we know that the lines \( (t_i^2 x + t_i u_i y + u_i^2 z) \) are all \( \omega \)-integral. By essentially [10, Theorem 3.76], these lines are all the \( \omega \)-integral curves together with the discriminant curve \( (y^2 - 4xz) \).

For the cyclic cover, we are considering \( D_j := L_j \) for all \( j \), and
\[
\mathcal{O}_{\mathbb{P}^2} \left( \sum_{j=1}^{d} a_j D_j \right) \simeq \mathcal{O}_{\mathbb{P}^2}(R)^d^{m},
\]
and so \( \mathcal{M} := \mathcal{O}_{\mathbb{P}^2}(R) \). Note that the variety \( Y_0 \) can be considered as
\[
Y_0 = \{(t_1^2 x + t_1 u_1 y + u_1^2 z)^{a_1} \cdots (t_d^2 x + t_d u_d y + u_d^2 z)^{a_d} = w^m \} \subset \mathbb{P}(1, 1, 1, R).
\]
We also have that \( \sum_{j=1}^{d} D_j - mL = \mathcal{O}(d - 4m) \), and so it is ample by assumption. Then we can apply Theorem 4.1, and we obtain that \( Y \) has curves of geometric genus \( \leq 1 \) only in the set of preimages of \( \omega \)-integral curves in \( \mathbb{P}^2 \). A simple calculation with Riemann-Hurwitz says that the only preimages of \( \omega \)-integral curves which give a curve of geometric genus 0 or 1 are the ramification curves, with genus 0 indeed. With the normalization map \( \eta: Y \rightarrow Y_0 \), we have no modifications on geometric genus of curves, so the same statement holds for \( Y_0 \). The claim on simply connectedness and ampleness of canonical class for \( Y \) follows from [17, Theorem 8.5] and the Canonical class formula [17, Proposition 1.4], which is generalized Riemann-Hurwitz. \( \square \)
4.3 Low genus curves in towers of cyclic covers

In this section we put all together to give the construction of a wide range of algebraic surfaces in which we can control curves of geometric genus \( \leq 1 \).

**Theorem 4.3** Let \( X \) be a smooth projective surface, and let \( \omega \in H^0(X, \mathcal{L} \otimes S^r \Omega^1_X) \). Assume we have the relations \( m_i \mathcal{M}_i = \sum_{j=1}^{s_i} a_{i,j} D_{i,j} \) in \( \text{Pic}(X) \), for some line bundles \( \mathcal{M}_i \), with \( 0 < a_{i,j} < m_i \) and \( \gcd(a_{i,j}, m_i) = 1 \) for all \( i, j \). Assume also that the divisor \( \sum_{i=1}^{n} \sum_{j=1}^{s_i} D_{i,j} \) has simple normal crossings, and \( D_{i,j} \) are \( \omega \)-integral curves. Then a tower of \( n \) cyclic covers of degree \( m_i \)

\[
X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 := X
\]

is defined, where all \( X_k \) are normal projective surfaces with only cyclic quotient singularities.

If

\[
\sum_{i=1}^{n} \frac{1}{m_i} \left( \sum_{j=1}^{s_i} D_{i,j} \right) - \mathcal{L} \text{ is } \mathbb{Q}\text{-ample, and } a_{i,j} \neq -a_{i,j'}(\text{mod } m_i) \text{ for all } i, j \neq j',
\]

then \( X_n \) can have curves of geometric genus \( \leq 1 \) only in the set of preimages of \( \omega \)-integral curves in \( X \).

**Proof** The expressions \( m_i \mathcal{M}_i = \sum_{j=1}^{s_i} a_{i,j} D_{i,j} \) in \( \text{Pic}(X) \) define a tower of \( n \) cyclic covers of degree \( m_i \)

\[
X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 := X
\]

inductively as follows: Let \( f_{k+1}: X_{k+1} \to X_k \) be the cyclic cover defined by

\[
g_k^* \mathcal{O}_X \left( \sum_{j=1}^{s_i} a_{k+1,j} D_{k+1,j} \right) \simeq g_k^* \mathcal{M}_i \otimes m_i,
\]

where \( g_k := f_1 \circ \cdots \circ f_k \) as done in Sect. 4.2. We note that if \( D'_{k+1,j} \) is the strict transform of \( D_{k+1,j} \) under \( g_k \), then \( g_k^* \mathcal{O}_X \left( \sum_{j=1}^{s_{k+1}} a_{k+1,j} D_{k+1,j} \right) = \mathcal{O}_{X_k} \left( \sum_{j=1}^{s_{k+1}} a_{k+1,j} D'_{k+1,j} \right) \), and \( \sum_{j=1}^{s_{k+1}} D'_{k+1,j} \) is a simple normal crossings divisor. All \( X_k \) are normal projective surfaces with cyclic quotient singularities. Let \( \sigma_k: X'_k \to X_k \) be the minimal resolution of all singularities in \( X_k \). The surface \( X'_k \) is a smooth (irreducible) projective surface. Let \( g'_k: X'_k \to X \) be the composition of \( \sigma_k \) with \( g_k \).

The rest of the proof is very similar to the proof of Theorem 4.1. We obtain the existence of \( \omega'' \) in \( H^0(X'_n, \mathcal{O}_{X'_n}(-R - E) \otimes g''_n^* \mathcal{L} \otimes S^r \Omega^1_{X'_n}) \), where \( R = \sum_{i=1}^{n} (m_i - 1) R_i \) and \( R_i \) is the sum of the strict transforms of the \( \sum_{j=1}^{s_i} D_{i,j} \), and \( E = \sum_{k=1}^{n} E_k \) is the sum of all exceptional curves of \( \sigma_n \). For that it is key the hypothesis \( a_{i,j} \neq -a_{i,j'}(\text{mod } m_i) \) for all \( i, j \neq j' \).

In this setting we have the numerical equivalence

\[
g''_n \left( \sum_{j=1}^{s_j} D_{i,j} \right) \equiv m_i R_i + m_i \sum_{k_i} (1 + d_{k_i}) E_{k_i},
\]
where the sum of exceptional curves runs over the singularities due to $\sum_{j=1}^{s_j} D_{i,j}$, and $d_{k_i}$ is the discrepancy associated to $E_{k_i}$. Thus,

$$-g'^* \left( \sum_{i=1}^{n} \frac{1}{m_i} \left( \sum_{j=1}^{s_j} D_{i,j} \right) \right) - \sum_{i=1}^{n} (m_i - 2) R_i + \sum_{k} d_k E_k + g'^* N$$

is numerically equivalent to $N := -R - E + g'^* L$.

Let $C \subset X'_n$ be a curve of geometric genus $g$, and not exceptional for $\sigma_n$. Let $\varphi_C : \tilde{C} \to X'_n$ be the normalization of $C \subset X'_n$. Then

$$\deg \tilde{C} (\varphi^*_C N \otimes S^r \Omega^1_{\tilde{C}}) \leq \left( -\sum_{i=1}^{n} \frac{1}{m_i} \left( \sum_{j=1}^{s_j} D_{i,j} \right) + L \right) \cdot g'_n(C) + r(2g - 2)$$

by the projection formula. By our hypothesis we have

$$\left( -\sum_{i=1}^{n} \frac{1}{m_i} \left( \sum_{j=1}^{s_j} D_{i,j} \right) + L \right) \cdot g'_n(C) < 0,$$

and so if $g \leq 1$, then $\deg \tilde{C} (\varphi^*_C N \otimes S^r \Omega^1_{\tilde{C}}) < 0$, and so the curve $C$ is $\omega''$-integral. Hence we conclude that $g'_n(C)$ must be an $\omega$-integral curve in $X$. Since $\sigma_n : X'_n \to X_n$ is a birational morphism contracting $E$, we obtain that $X_n$ can have curves of geometric genus $\leq 1$ only in the set of preimages of $\omega$-integral curves in $X$.

\section{5 Hyperbolicity}

By using certain results in Nevanlinna theory for complex varieties (cf. [19,20]), we will prove that if the normal projective surface $X_n$ in Theorem 4.3 contains no curves of geometric genus $\leq 1$ and $\omega = 0$ has only algebraic solutions, then in fact $X_n$ is hyperbolic, that is, the only holomorphic maps $f : \tilde{C} \to X_n$ are constant. These in practice produce many families of smooth projective surfaces of general type which are hyperbolic. For example, at the end of this section we prove existence of hyperbolic complete intersection surfaces of low degrees.

We recall that an entire curve in a variety $X$ is a nonconstant holomorphic map $\tilde{C} \to X$. We begin with some definitions in Nevanlinna theory (cf. [21, Section 11]). Let $X$ be a smooth projective surface, let $f : \tilde{C} \to X$ be an entire curve, and let $D$ be a divisor on $X$ whose support does not contain the image of $f$. We define the counting function of $D$ in $X$ to be

$$N_f(D, R) = \sum_{0 < |z| < R} \text{ord}_z f^* D \cdot \log \left( \frac{R}{|z|} \right) + \text{ord}_0 f^* D \cdot \log(R).$$

Let $\lambda$ be a Weil function for $D$. We define the proximity function for $f$ relative to $D$ to be

$$m_f(D, R) = \int_0^{2\pi} \lambda(f(Re^{i\theta})) \frac{d\theta}{2\pi}.$$

It is defined up to $O(1)$. We can now define the height of $f$ relative to $D$ by

$$T_{D,f}(R) = m_f(D, R) + N_f(D, R).$$

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The height of $f$ relative to an invertible sheaf $\mathcal{L}$ on $X$ is $T_{L,f}(R) = T_{D,f}(R) + O(1)$, where $D$ is any divisor such that $\mathcal{L} \simeq \mathcal{O}_X(D)$.

The following result is [20, Corollary 5.2] for $D = 0$ (see also [19, Proposition 6.1.1]). We use the notation $\log^+(a) = \max\{0, \log(a)\}$.

**Theorem 5.1** Let $X$ be a smooth complex projective variety, let $f : \mathbb{C} \to X$ be an entire curve, let $r$ be a positive integer, let $\omega$ be a global section of $\mathcal{L}^r \otimes S^r \Omega^1_{X/\mathbb{C}}$, and let $\mathcal{A}$ be a line bundle which is big on the Zariski closure of $f(\mathbb{C})$. If $f^* \omega \neq 0$, then

$$T_{L,f}(R) \leq_{exc} O\left(\log^+ T_{\mathcal{A},f}(R)\right) + o\left(\log(ab)\right),$$

where the notation $\leq_{exc}$ means that the inequality holds for all $R > 0$ outside of a set of finite Lebesgue measure.

The following will be the main tool to prove hyperbolicity for surfaces.

**Corollary 5.2** Let $X$ be a smooth projective surface, let $\mathcal{L}$ be a line bundle on $X$, let $r > 0$ be an integer, and let $\omega \in H^0(X, \mathcal{L}^r \otimes S^r \Omega^1_X)$. Assume that $f : \mathbb{C} \to X$ is an entire curve whose image is Zariski dense. Then $f^* \omega = 0$.

**Proof** By [21, Prop. 11.11], we have that $T_{O(1),f}(R) \leq CT_{L,f}(R) + O(1)$ for all $R > 0$, and a constant $C > 0$ depending on $O(1)$ and $\mathcal{L}$. Here $O(1)$ is the hyperplane line bundle given by some embedding of $X$ in a projective space. On the other hand, we have that there are constants $M > 0$ and $N$ such that $M \log(R) + N \leq T_{O(1),f}(R)$ for all $R > 0$, and so there are constants $M' > 0$ and $N'$ such that $M' \log(R) + N' \leq T_{L,f}(R)$.

Suppose by contradiction that $f^* \omega \neq 0$. From Theorem 5.1 we have that

$$T_{L,f}(R) \leq_{exc} S \log(T_{L,f}(R)) + \epsilon \log(R)$$

for a constant $S > 0$, a given $0 < \epsilon \leq M'/4$, and $R >> 0$ out of a set of finite Lebesgue measure. But $\log(T_{L,f}(R)) < T_{L,f}(R)/2S$ for $R >> 0$, and so by the inequality above we get $T_{L,f}(R) <_{exc} 2\epsilon \log(R) < M'/2 \log(R)$, for certain $R >> 0$. But this contradicts $M' \log(R) + N' \leq T_{L,f}(R)$. Therefore $f^* \omega = 0$. \hfill $\square$

The following theorem gives a criteria for hyperbolicity of the singular surfaces $X_n$ in Theorem 4.3.

**Theorem 5.3** Let us consider the hypothesis and the notation in Theorem 4.3. In addition, assume that all solutions to the differential equation given by $\omega = 0$ on $X$ are $\omega$-integral curves. Then an entire curve in $X_n$ must be contained in the set of preimages of $\omega$-integral curves in $X$.

In particular, if the set of preimages of $\omega$-integral curves in $X$ does not contain curves of geometric genus 0 or 1, then $X_n$ is hyperbolic.

**Proof** Let $\sigma_n : X'_n \to X_n$ be the minimal resolution of singularities of $X_n$. Consider an entire curve $f : \mathbb{C} \to X_n$. Then it has a lifting $f' : \mathbb{C} \to X'_n$. Assume that $f'(\mathbb{C})$ is Zariski dense in $X'_n$. We have a section

$$\omega'' \in H^0(X', \mathcal{O}_{X'_n}(-R - E) \otimes g''^* \mathcal{L} \otimes S^r \Omega^1_{X'_n}),$$

and a line bundle $\mathcal{N} := \mathcal{O}_{X'_n}(-R - E) \otimes g''^* \mathcal{L}$ which is numerically

$$\mathcal{N}^\vee \equiv g''^* \left(-\mathcal{L} + \sum_{i=1}^n \frac{1}{m_i} \left(\sum_{j=1}^{s_i} D_{i,j}\right) + \sum_{i=1}^n (m_i - 2) R_i - \sum_k d_k E_k, \right)$$

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where \(-1 < d_k < 0\) are the discrepancies of \(E_k\), and

\[-L + \sum_{i=1}^{n} \frac{1}{m_i} \left( \sum_{j=1}^{s_i} D_{i,j} \right)\]

is an ample divisor in \(X\) by hypothesis. Therefore \(\mathcal{N}^\vee\) is numerically the sum of the pull-back of an ample divisor plus an effective divisor. It is easy to see that \(\mathcal{N}^\vee\) is then big. Therefore by Corollary 5.2 we have that \(f^*\omega'' = 0\). But then, locally analytically \(f\) satisfies the differential equation given by \(g''^*\omega\), and then this gives a solution to the differential equation given by \(\omega\). By hypothesis we know that all solutions are given by \(\omega\)-integral curves in \(X\), and so this contradicts the Zariski density of \(f(\mathbb{C})\).

Therefore \(f(\mathbb{C})\) must be contained in an irreducible algebraic curve. But by Liouville’s theorem, the geometric genus of this irreducible curve must be less than or equal to 1. Moreover, by Theorem 4.3, all curves of geometric genus \(\leq 1\) are in the set of preimages of \(\omega\)-integral curves in \(X\).

The following corollaries give a proof of Theorem 1.6.

**Corollary 5.4** Let \(n \geq 4\), and let \(m_i \geq 3\) be \(n - 1\) integers. Let \(\{a_{i,j}\}\) and \(\{b_{i,j}\}\) be two collections of distinct \(\sum_{i=1}^{n-1} m_i\) complex numbers. Let \(\{G_i = G_i(z_0, z_1, z_2, z_3)\}_{i=1}^{n-1}\) be a collection of \(n - 1\) homogeneous polynomials of degree \(m_i\), such that \(z_0 - a_{i,j}z_1 - b_{i,j}z_2 + a_{i,j}b_{i,j}z_3\) does not divide \(G_i\). Then the complete intersection

\[
\prod_{j=1}^{m_i} (z_0 - a_{i,j}z_1 - b_{i,j}z_2 + a_{i,j}b_{i,j}z_3) + t_i G_i = z_3^{m_i + 1}, \quad z_0z_3 - z_1z_2 = 0
\]

for \(i = 1, \ldots, n - 1\) in \(\mathbb{P}^{n+2}\) is hyperbolic for sufficiently small \(t_i \in \mathbb{C}\). In particular, we obtain a family parametrized by a Zariski open subset of \(\mathbb{A}^N\) (given by the parameters \(a_{i,j}, b_{i,j}\)), where \(N = 2 \sum_{i=1}^{n-1} m_i\) satisfying the requirements of part (d) in Theorem 1.6.

**Proof** Let us evaluate the complete intersection

\[
\prod_{j=1}^{m_i} (z_0 - a_{i,j}z_1 - b_{i,j}z_2 + a_{i,j}b_{i,j}z_3) + t_i G_i = z_3^{m_i + 1}, \quad z_0z_3 - z_1z_2 = 0
\]

for \(i = 1, \ldots, n - 1\) in \(\mathbb{P}^{n+2}\) in \(t_i = 0\) for all \(i\). We denote this surface by \(X_{n-1}\), which is as in the construction of the generalized surfaces of cuboids but for not necessarily equal degrees \(m_i > 2\) (see Sect. 2). As in Corollary 2.6, this surface \(X_{n-1}\) has no curves of geometric genus \(\leq 1\). In fact, the construction satisfies the hypothesis in Theorem 5.3, and so \(X_{n-1}\) has no entire curve. We now consider an small deformation of the branched divisor. It gives a smooth branch divisor, and so a smooth surface \(X_{n-1}(t_1, \ldots, t_{n-1})\). At the same time, this gives a small perturbation of the hyperbolic surface \(X_{n-1}\), and it is known that hyperbolicity is preserved by small deformations (see [13, p.105 Corollary(3.6.8)]).

Next corollary is proved as the previous one, but we need to take care of the degrees equal to 2, which produce \(\Lambda_1\) singularities.

**Corollary 5.5** Let \(r \geq 1, s \geq 1\) be integers such that \(r + s \geq 5\), and let \(\{m_i \geq 3\}_{i=r+1}^{r+s+1}\) be integers. Let \(\{a_{i,j}, b_{i,j}\}_{i=r+1}^{r+s+1}\) be two collections of \(\sum_{i=r+1}^{r+s+1} m_i\) distinct complex numbers. Let \(a_i\) be a collection of \(r\) distinct complex numbers such that \(a_i \neq \pm a_j, b_{i,j} \neq \pm a_k\),
For $i = 1, \ldots, r + s$ in $\mathbb{P}^{r+s+3}$ is hyperbolic for sufficiently small $t_i \in \mathbb{C}$. In particular, we obtain a family parametrized by a Zariski open subset of $\mathbb{C}^N$ (through the parameters $a_k, b_i, c_i, j$) satisfying the requirements of part (c) in Theorem 1.6.

**Proof** Let us evaluate the complete intersection at $t_i = 0$, and denote this surface by $X_{r+s}$. Then $X_{r+s}$ is constructed from $X_0 := \mathbb{P}^1 \times \mathbb{P}^1$ as for the generalized cuboids but for distinct degrees 2 and $m_i$. As before, we consider the section $\omega \in H^0(X_0, (2, 2) \otimes S^2 \Omega^1_{X_0})$ defined by $z_2^2 d z_1 d z_2$. We recall that given the isomorphism $h : \mathbb{P}^1 \times \mathbb{P}^1 \to X_0, h([x, y] \times [w, z]) = [x w, x z, y w, y z]$, the section $\omega$ corresponds to the section $y^2 z^2 dx dw$ under $h$. The problem is that over multiplicities equal to 2, we obtain $A_1$ singularities and we cannot apply our results, like the key Lemma 2.3. Instead we consider another section so that we can use the result in Lemma 3.4. For that, let

$$\omega_0 \in H^0 \left( X_0, (4, 4) \otimes S^2 \Omega^1_{X_0} \right)$$

be defined by $(x^2 - w^2) dx dw$ for affine coordinates $x, w$. Then if $R$ is the strict transform of the branch divisor in $X_{r+s}$, and $E$ is the exceptional divisor of $X_{r+s} \to X_{r+s}$, then there is

$$\omega_0' \in H^0 \left( X_{r+s}, \mathcal{O}_{X_{r+s}}(-R - E) \otimes g^{r+s}_{r+s}(4, 4) \otimes S^2 \Omega^1_{X_{r+s}} \right)$$

corresponding to $g^{r+s}_{r+s} \omega_0$. At the same time we have

$$g^{r+s}_{r+s}(r + s, r + s) \equiv R + E,$$

and so $\mathcal{O}_{X_{r+s}}(-R - E) \otimes g^{r+s}_{r+s}(4, 4) \equiv g^{r+s}_{r+s}(-r - s + 4, -r - s + 4)$. But $r + s \geq 5$, and so we obtain as in Theorems 4.3 and 5.3 that the surfaces $X_{r+s}$ are hyperbolic. This is indeed because we know all $\omega_0$-integral curves (fibres and the two $(1, 1)$ extra curves), and so we can check all the pre-images. To avoid curves of geometric genus $\leq 1$, here we use that $s > 0, m_i \geq 3$, and $s + r \geq 5$. For example $s = 0$ would be a problem with the $(1, 1)$ curves.

The next corollary uses the example at the end of Sect. 4.

**Corollary 5.6** Let $n \geq 5$, and let $m_i \geq 3$ be $n$ integers. Let $[a_i, b_i]$ be a collection of distinct $\sum_{i=1}^n m_i$ points in $\mathbb{P}^1$. Let $\{G_i = G_i(x, y, z)\}_{i=1}^n$ be a collection of $n$ homogeneous polynomials of degree $m_i$, such that $a^2_j x + a_j b_j y + b^2_j z$ does not divide $G_i$ for $j = m_i - 1, m_i - 1 + 1, \ldots, m_i$ (where $m_0 := 1$). Then the complete intersection

$$\prod_{j=1}^{m_i} (a^2_j x + a_j b_j y + b^2_j z) + t_i G_i = w_i^{m_i}$$



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for $i = 1, \ldots, n$ in $\mathbb{P}^{n+2}$ is hyperbolic for sufficiently small $t_i \in \mathbb{C}$. In particular, we obtain a family parametrized by a Zariski open subset of $\mathbb{A}^n$ (through the parameters $[a_i, b_i]$) satisfying the requirements of part (b) in Theorem 1.6, and part (a) in Theorem 1.6 when all multiplicities are bigger than or equal to 3.

**Proof** Let us consider instead the situation in Corollary 4.2, but with $n \geq 5$ equations, $a_i = 1$ for all $i$, and $m_i \geq 3$ for all $i = 1, \ldots, n$. We take $X_0 := \mathbb{P}^2$, $\omega \in H^0(O_{\mathbb{P}^2}(4) \otimes S^2\Omega^1_{\mathbb{P}^2})$, and we construct the surface $X_n$ as in Theorem 4.3 from this data. As in Corollary 5.5, for $n \geq 5$ and $m_i \geq 3$, we construct smooth complete intersections in $\mathbb{P}^{n+2}$ of multidegree $(m_1, \ldots, m_n)$ which are hyperbolic, by Theorem 5.3. The proof follows the same strategy as Corollary 5.5, since we know all $\omega$-integral curves. □

**Corollary 5.7** Let $r \geq 1$, $s \geq 0$ be integers such that $r + s \geq 7$, and let $\{m_i \geq 3\}_{i=r+1}^{r+s}$ be integers. Let $\{b_{i,j}, c_{i,j}\}_{i=r+1,\ldots,r+s}$ be two collections of $\sum_{i=r+1}^{r+s} m_i$ distinct complex numbers. Let $a_i$ be a collection of $r$ distinct complex numbers such that $a_i \neq a_j \pm 1, b_{i,j} \neq a_k$, $c_{i,j} \neq a_k, b_{i,j} \neq a_k \pm 1$, and $c_{i,j} \neq a_k \pm 1$ for all $i, j, k$. Let $\{F_i = F_i(z_0, z_1, z_2, z_3)\}_{i=1}^{r+s}$ be a collection of $r$ homogeneous polynomials of degree 2, such that $z_0 - a_i z_1 - a_i z_2 + a_i^2 z_3$ and $z_0 - (a_i - 1) z_1 - (a_i + 1) z_2 + (a_i^2 - 1) z_3$ do not divide $F_i$. Let $\{G_i = G_i(z_0, z_1, z_2, z_3)\}_{i=r+1}^{r+s}$ be a collection of $s$ homogeneous polynomials of degree $m_i$, such that $z_0 - b_{i,j} z_1 - c_{i,j} z_2 + b_{i,j} c_{i,j} z_3$ does not divide $G_i$.

Then the complete intersection defined by

$$\prod_{j=1}^{m_i} (z_0 - b_{i,j} z_1 - c_{i,j} z_2 + b_{i,j} c_{i,j} z_3) + t_i F_i = z_{3+i}^{2}\quad \cap \quad \prod_{j=1}^{m_i} (z_0 - a_i z_1 - a_i z_2 + a_i^2 z_3) (z_0 - (a_i - 1) z_1 - (a_i + 1) z_2 + (a_i^2 - 1) z_3) + t_i G_i = z_{3+i}^{m_i}, \quad z_0 z_3 - z_1 z_2 = 0$$

for $i = 1, \ldots, r + s$ in $\mathbb{P}^{r+s+3}$ is hyperbolic for sufficiently small $t_i \in \mathbb{C}$. In particular, we obtain a family parametrized by a Zariski open subset of $\mathbb{A}^n$ (through the parameters $a_k, b_{i,j}, c_{i,j}$) satisfying the requirements of part (a) in Theorem 1.6, when some (or all) multiplicities are equal to 2.

**Proof** This is as in the proof of Corollary 5.5, but with $\omega_0 \in H^0(X_0, (6, 6) \otimes S^2\Omega^1_{X_0})$ defined by

$$(x - w)(x - w + 1)(x - w - 1)(x - w - 2)dx dw$$

for affine coordinates $x, w$. We note that each of the 4 nodes in the 4 fibres $(z_0 - a_i z_1 - a_i z_2 + a_i^2 z_3) (z_0 - (a_i - 1) z_1 - (a_i + 1) z_2 + (a_i^2 - 1) z_3) = 0$ in $X_0 = \mathbb{P}^1 \times \mathbb{P}^1$ belongs to one of the 4 lines $(x - w)(x - w + 1)(x - w - 1)(x - w - 2) = 0$. This gives that the pull-back of each of the 4 lines do not contain any curves of geometric genus $\leq 1$. □

The list of multidegrees not included in the previous corollaries is $(2, \ldots, 2)$ in $\mathbb{P}^9, \mathbb{P}^8$, and $\mathbb{P}^7; (m_1, m_2, m_3, m_4)$ for $m_i > 2$ and $(2, 2, 2, 2)$ in $\mathbb{P}^6$, for which the surface of cuboids is an example. We do not know existence of hyperbolic surfaces in those cases, except for the ones $(k, k, k, k) \subset \mathbb{P}^6$ for $k \geq 3$ [9]. This gives explicit evidence for [6, Conjecture 0.18] in the case of surfaces. We point out that hyperbolic complete intersections of high multidegree have been constructed by Brotbek [4] (see also Xie [22] and Brotbek-Darondeau [5])

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