WEAK FUNCTIONAL ITÔ CALCULUS AND APPLICATIONS

DORIVAL LEÃO, ALBERTO OHASHI, AND ALEXANDRE B. SIMAS

Abstract. In this work, we develop weak differential representations for processes adapted to a multi-dimensional Brownian motion filtration. The theory extends the pathwise functional Itô calculus developed by Dupire [14], Cont and Fournie [8] and the Sobolev-type differential structure introduced by Peng and Song [42]. The main conceptual difference between our approach and the usual pathwise calculus lies on the development of an intrinsic differential theory based on processes rather than on smooth functionals. The weak functional calculus is applied to (a) Optimal stopping problems and (b) Itô formulas for path-dependent functionals with rough dependence w.r.t the noise. We formulate a stochastic weak solution concept based on approximating strong solutions. For that kind of solution, we prove the Snell envelope associated to a general continuous adapted process is the unique stochastic weak solution of variational inequalities with general terminal conditions. This provides the non-Markovian/non-semimartingale variational representation counterpart of the classical characterization of the Snell envelope given by El Karoui et.al [20] in terms of reflected BSDEs. Finally, we present novel path-dependent Itô formulas for adapted processes with rough dependence w.r.t Brownian motion under $p$-variation regularity assumptions. The local-time representations for path-dependent functionals are then expressed in terms of suitable pathwise 2D Young integrals.

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1. Introduction

A classical result in stochastic analysis is the unique decomposition of special semimartingales $X$ into the following form

$$X(t) = X(0) + M(t) + V(t) \text{ a.s.; } 0 \leq t \leq T,$$

where $M$ is an $\mathbb{F}$-local martingale and $V$ is an $\mathbb{F}$-predictable bounded variation process where $\mathbb{F} := \{\mathcal{F}_t; 0 \leq t \leq T\}$ is a complete filtration satisfying the usual conditions. In some particular cases, one can identify the pair $(M, V)$ in terms of an underlying differential structure. For instance, if $X = f(Y)$ is a smooth transformation of a semimartingale $Y$ then the classical Itô formula allows us to retrieve $(M, V)$ in terms of the first and second order derivative operator of $f$.

In stochastic control problems, feasible differential representations for value processes are typically obtained by PDE methods based on viscosity solutions as long as the underlying state process is Markovian. Backward SDEs (BSDEs for short) and their associated nonlinear Feynman-Kac representations can also be used in some particular non-Markovian cases. See e.g [47] and other references therein. In fully non-Markovian systems, feasible representations are much more involved. In this case, the value function is not a deterministic function of the current value of the underlying controlled process but rather a functional depending on the whole path so that the classical PDE techniques are not available in this more general context.

A more general picture was recently given by Dupire [14], who introduced suitable differential operators acting on functional representations of the underlying driving noise. One important result in his insightful construction is an Itô-type formula for smooth functionals in terms of the so-called horizontal, vertical and second-order vertical derivative operators given, respectively, by $(\nabla^h, \nabla^v, \nabla^{v,(2)})$. In order to work with such operators, one needs to extend the functionals from the path space of continuous functions $C([0, t]; \mathbb{R}^p)$ to the space of càdlàg paths $D([0, t]; \mathbb{R}^p)$. See also Cont and Fournie [8] for further details. The work of Dupire has been attracted the attention of many authors in the study of pathwise path-dependent PDEs associated to functional Itô formulas. Path dependent PDEs naturally arise from BSDEs in terms of Feynman-Kac representations in a non-Markovian setup. See Peng and Wang [41] for details. Very recently, the issue of providing a suitable notion of viscosity solutions for path-dependent PDEs has attracted a great interest. See e.g the works [16, 17, 18, 43, 11, 23] for different notions of pathwise viscosity solutions for quasi-linear and fully nonlinear path-dependent PDEs based on the functional Itô calculus.

In spite of deep results obtained by the pathwise functional stochastic calculus, there are some drawbacks for applications: (i) Strong and viscosity solutions for pathwise path-dependent PDEs as introduced by [16, 17, 18, 41, 43, 11, 23] are not amenable to a concrete numerical analysis where the space variable now lives in an infinite-dimensional space. (ii) Many functionals founded in practice exhibit terminal conditions which are not uniformly continuous (or Lipschitz) w.r.t. the driving noise. This is a very strong common assumption imposed on the terminal variables in those notions of pathwise solutions. (iii) Many important adapted processes do not admit $(\nabla^h, \nabla^v, \nabla^{v,(2)})$ in the usual functional stochastic calculus sense so that many important adapted processes are not represented in terms of these operators.

In view of the fact that many important (if not most) path-dependent adapted processes lacks the aforementioned regularity properties, it is natural and necessary to construct a theory of weak functional stochastic calculus. Likewise, in view of the intrinsic infinite-dimensionality of adapted processes, it is important to construct it in such way that one can perform a concrete numerical analysis for the resulting differential representations. This article tries to fill this gap by developing a flexible and feasible differential structure which are applicable to a wide range class of adapted processes.

We should mention two works which are very much related to the present article. Motivated by the study of viscosity-type solutions for path-dependent PDEs, Ekren, Touzi and Zhang [15] have introduced a weak concept of functional calculus based on Itô-type representations indexed by a family of
measures and under continuity assumptions w.r.t. the noisy. In Peng and Song [42], a notion of weak functional stochastic analysis was established while our investigation of the problem was in the early stage. In [42], G-Sobolev-type spaces are introduced in order to deal with G-BSDEs and their associated G-path dependent PDEs. Both works are restricted to Itô-type representations. The approach taken by [42] is very close in spirit to the classical Malliavin calculus where the weak differential structure is defined in terms of the completion of the set of cylindrical functionals w.r.t a suitable norm. The extended domain of the differential operators in [42] is the set of square-integrable G-Itô processes. For instance, the structure presented in [42] does not apply to generic Brownian semimartingales and many other important path-dependent processes with unbounded variation components.

1.1. Contributions of the current paper. The main idea of the standard pathwise functional calculus is to represent a given process $X$ adapted to e.g. a Brownian filtration (henceforth abbreviated by Wiener functional) in terms of $(\nabla^h F(B), \nabla^h F(B), \nabla^v(B))$ for a fixed version of smooth functional $F$ acting on the “vector bundle” $\cup_{t \in [0, T]} D([0, t]; \mathbb{R}^p)$, where $B$ is a $p$-dimensional Brownian motion. Once there exists a smooth version $\hat{F}$ at hand then we shall compose it with Brownian paths to get deep differential representations for $X$. This is the framework taken by the works e.g. [14, 8, 11, 35, 29]. See also [11] for a similar framework based on regularization ideas.

In the present paper, we develop an intrinsic differential theory for adapted processes $X$ without relying on a prioriy regularity assumptions on their functional representations $F_t : C([0, t]; \mathbb{R}^p) \rightarrow \mathbb{R}$. This point of view is radically different from the pure pathwise functional stochastic calculus and it allows us to characterize Wiener functionals in terms of differential representations with martingale and orthogonality constraints but under rather weak regularity assumptions. We provide differential representations for processes which are not smooth in the functional pathwise sense but they satisfy weaker regularity conditions w.r.t the underlying Brownian motion.

The starting point of the theory is inspired by Leão and Ohashi [32] who introduced a discrete structure for Wiener functionals by projecting them into smooth bounded variation processes adapted to a suitable jumping filtration $\{F^k; k \geq 1\}$ in the spirit of [27]. In this article, we continue the analysis developed by [32]. We demonstrate that the pathwise functional Itô calculus introduced by Dupire [14], Cont and Fournie [8] and the functional Sobolev structure introduced by Ekren, Touzi and Zhang [17] and Peng and Song [42] (under the Wiener measure) are particular cases of our approach under some mild integrability assumptions.

Our theory goes beyond semimartigale processes where bounded variation components are replaced by suitable functional space-time local time integrals interpreted in a weak sense. Conceptually, our construction can also be interpreted as a measure theoretic approach to the pathwise functional stochastic calculus. In order to illustrate the theory of this article, we present two applications: (a) Variational inequalities for general continuous adapted processes; (b) Itô-type formulas for path-dependent functionals with rough dependence w.r.t. the underlying Brownian motion under $p$-variation assumptions.

As we already mentioned, Peng and Yong [42] and Ekren, Touzi and Zhang [17] have introduced weak concepts of functional stochastic calculus. In one hand, we point out that the structure introduced in the present work allow us to work with more general differential representations than [42, 17] under mild orthogonality constraints. On the other hand, we observe that the results of this article are still restricted to the Wiener measure case. A more general framework based on $G$-expectation and functionals of Lévy-type noises are treated in the accompanying papers [33, 35] having the structural results presented in this article as the starting point.

1.2. Applications. The connection between optimal stopping and variational inequalities for semilinear parabolic PDEs is by now well-understood in the classical Markovian setup. See e.g. the books [47, 49] and other references therein. The general non-Markovian/non-semimartingale case is much less understood due to the lack of natural differential representations for path-dependent functionals without a priori smoothness assumptions. In this article, we study such connections in a rather general framework by using our notion of weak functional calculus.
To our best knowledge, only few articles have been made some progress in this direction. Chang, Pang and Yong [7] provide a backward stochastic partial variational inequality for optimal stopping induced by SDEs with random coefficients. Ekren [19] proves that the value function defined via second order reflected BSDE is the unique viscosity solution (in the sense of [17, 18]) of a variational inequality provided that the terminal variable is uniformly continuous w.r.t the noise.

We apply weak functional calculus developed in the first part of this article to establish novel differential representations for the Snell envelope. We formulate a suitable notion of stochastic solution of variational inequalities based on approximating strong solutions computed in terms of jumping filtrations (see e.g [27]) that we call stochastic weak solution. In contrast to Ekren [19], our definition is not inspired by the viscosity solution concept of ([10, 17, 18]). Instead, our notion is similar in nature to the concept of good solutions which turns out to be equivalent to viscosity-type solutions in the finite-dimensional case as demonstrated by Cosso and Russo [11] in the context of nonlinear Kolmogorov equations.

We prove the Snell envelope $S$ of an arbitrary continuous adapted process $X$ is the unique stochastic weak solution of the following obstacle problem

\begin{equation}
\begin{cases}
\max\{US, X - S\} = 0 \text{ Leb} \times \mathbb{P} - a.s \\
S(T) = X(T)
\end{cases}
\end{equation}

where $US$ is a suitable weak derivative operator acting intrinsically on $S$ rather than on functional representations[1]. This provides the non-Markovian/non-semimartingale variational representation counterpart of the classical characterization of the Snell envelope given by El Karoui et.al [20] in terms of reflected BSDEs. In the general case, (1.1) provides a variational inequality for the Snell envelope where usual PDEs are replaced by random differential representations due to the path-dependence of the underlying adapted process.

In the last part of the article, we apply our abstract results to derive novel path-dependent Itô formulas for Wiener functionals with rough dependence w.r.t Brownian paths. Under finite $p$-variation assumptions on space and time variables, we are able to prove differential representations for path-dependent functionals $\{F_t : 0 \leq t \leq T\}$ with rough dependence w.r.t Brownian paths $B_s = \{B(r); 0 \leq r \leq s\}$ as follows

\begin{equation}
F_t(B_t) = F_0(B_0) + \int_0^t \mathcal{D}F_s(B_s)dB(s) + \int_0^t \mathcal{D}^{F,h}F_s(B_s)ds - \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} \partial_x F_s(t(B_s, x))d\mathbb{P}(s),
\end{equation}

for $0 \leq t \leq T$. Here $(\mathcal{D}F(B), \mathcal{D}^{F,h}F(B))$ is similar in nature to $(\nabla^{v}F(B), \nabla^{h}F(B))$ in the pathwise functional Itô calculus. The $d_{(s,x)}\ell^p(s)$-integral is the pathwise 2D Young integral (see [10]) composed with the Brownian local-time $\{\ell^p(s); 0 \leq s \leq T, x \in \mathbb{R}\}$ and, roughly speaking, a space derivative of $F$ composed with a suitable “terminal value modification” $t(B_t, x)$ of the Brownian paths. One typical example of Wiener functionals which follows our functional Itô formula may heavily depend on trajectories of Brownian local-times $(x, t) \mapsto \ell^p(t)$ such as

\begin{equation}
\int_{-\infty}^{B(t)} H((\ell^p(t))_{x \in \mathbb{R}}, y)dy,
\end{equation}

where $(y, t) \mapsto H((\ell^p(t))_{x \in \mathbb{R}}, y)$ is a smooth functional in the sense of the $p$-variation norm on $\mathbb{R} \times [0, T]$. Formula (1.2) can be viewed as a path-dependent extension of the Föllmer-Protter-Shiryaev formula [22] and, more generally, the $p$-variation case treated by Feng and Zao [22] (see Th. 6 in [22] in the Young case).

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1In the Markovian case $U$ can be identified in terms of the Dynkin operator $\partial_t + A$ associated to an underlying Markovian diffusion with infinitesimal generator $A$. 
From the numerical point of view, the differential structure of this article is amenable to a concrete numerical analysis by means of Monte Carlo methods as demonstrated by [33] in the pure martingale case. In particular, we believe that concrete numerical schemes for our weak derivative operators can be fairly constructed in more general cases by following the steps indicated in the articles [32] and [33]. To be more precise, all the approximations proved in this article are smooth functionals of i.i.d random variables with explicit and simulatable parameters based on the work of Burq and Jones [4].

This work is organized as follows. The next section introduces some necessary notations and we present the basic underlying discrete structure of this article. In Section 3, abstract differential representations for adapted processes are provided. In Section 4, we study weak infinitesimal generators. In order to compare our approach with other methods, simple examples are described and discussed in Section 5. In Section 6, we illustrate the theory with obstacle problems for continuous adapted processes. Section 7 presents a comparison with the pathwise functional stochastic calculus and we apply the theory to functional Itô formulas. Section 8 presents local time representations for path dependent functionals.

2. Preliminares

Throughout this article, we are going to fix a $p$-dimensional Brownian motion $B = \{B^1, \ldots, B^p\}$ on the usual stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ is the space $C([0, +\infty); \mathbb{R}^p) := \{ f : [0, +\infty) \rightarrow \mathbb{R}^p : \text{continuous} \}$, $\mathbb{P}$ is the Wiener measure on $\Omega$ such that $\mathbb{P}(B(0) = 0) = 1$ and $\mathcal{F} := (\mathcal{F}_t)_{t \geq 0}$ is the usual $\mathbb{P}$-augmentation of the natural filtration generated by the Brownian motion. In the sequel, we briefly recall the random skeleton introduced by Leão and Ohashi [32]. The weak functional stochastic calculus developed in this paper will be constructed from this skeleton. We refer the reader to this work for more details and all unexplained arguments.

The basic structure of the skeleton is reminiscent from the classical work of F.Knight [29]. He introduced a one-dimensional $\epsilon_k \mathbb{Z}$-valued simple symmetric random walk using a single Brownian motion starting with a given sequence of positive numbers $\{\epsilon_k; k \geq 1\}$ such that $\epsilon_k \downarrow 0$ as $k \rightarrow \infty$. For a fixed positive integer $k \geq 1$ and for each $j = 1, \ldots, p$, we define $T_{0,j} := 0$ a.s. and we set

\begin{equation}
T_n^{k,j} := \inf \{ T_{n-1}^{k,j} < t < \infty; |B^j(t) - B^j(T_{n-1}^{k,j})| = \epsilon_k \}, \quad n \geq 1.
\end{equation}

For each $j \in \{1, \ldots, p\}$, the family $(T_n^{k,j})_{n \geq 0}$ is a sequence of $\mathbb{F}$-stopping times where the increments $\{T_n^{k,j} - T_{n-1}^{k,j}; n \geq 1\}$ is an i.i.d sequence with the same distribution as $T_1^{k,j}$. From this family of stopping times, we define $A^k := (A^{k,1}, \ldots, A^{k,p})$ as the $p$-dimensional step process given componentwise by

\begin{equation}
A^{k,j}(t) := \sum_{n=1}^{\infty} \epsilon_k \eta_n^{k,j} \mathbb{1}_{\{T_n^{k,j} \leq t\}}, \quad t \geq 0,
\end{equation}

where

\begin{equation}
\eta_n^{k,j} := \begin{cases} 
1; & \text{if } B^j(T_n^{k,j}) - B^j(T_{n-1}^{k,j}) = \epsilon_k \text{ and } T_n^{k,j} < \infty \smallskip 
-1; & \text{if } B^j(T_n^{k,j}) - B^j(T_{n-1}^{k,j}) = -\epsilon_k \text{ and } T_n^{k,j} < \infty \smallskip 
0; & \text{if } T_n^{k,j} = \infty,
\end{cases}
\end{equation}

for $k, n \geq 1$ and $j = 1, \ldots, p$. Let $\mathcal{F}_{t,j} := \{ \mathcal{F}_{t,j}^k; t \geq 0 \}$ be the natural filtration generated by $\{A^{k,j}(t); t \geq 0\}$. One should notice that $\mathcal{F}_{t,j}$ is a jumping filtration (see [27]) in the sense that

\begin{equation}
\mathcal{F}_{t,j}^{k,j} = \left\{ \bigcup_{\ell=0}^{\infty} D_\ell \cap \{ T_{\ell,j}^{k,j} \leq t < T_{\ell+1,j}^{k,j} \}; D_\ell \in \mathcal{F}_{t,j}^{k,j} \text{ for } \ell \geq 0 \right\}, \quad t \geq 0,
\end{equation}

where $\mathcal{F}_{0,j}^{k,j} = \{\Omega, \emptyset\}$ and $\mathcal{F}_{T_{n,j}^{k,j}}^{k,j} = \sigma(T_1^{k,j}, \ldots, T_{n,j}^{k,j}, \eta_1^{k,j}, \ldots, \eta_{n,j}^{k,j})$ for $m \geq 1$ and $j = 1, \ldots, p$. Then
such that \( \varepsilon \). We stress that all results of this article hold true for a generic sequence \( T \). increments of the stopping times

\[ T^k_{m} \leq t < T^k_{m+1} \]

for each \( m \geq 0 \) and \( j = 1, \ldots, p \). With a slight abuse of notation, we write \( \mathcal{F}^{k,j} \) to denote its \( \mathbb{P}\)-augmentation satisfying the usual conditions.

The multi-dimensional filtration generated by \( A^k \) is naturally characterized as follows. Let \( \mathbb{F}^k := \{ \mathcal{F}^k_t; 0 \leq t < \infty \} \) be the filtration given by \( \mathcal{F}^k_t := \mathcal{F}^{k,1}_t \otimes \mathcal{F}^{k,2}_t \otimes \cdots \otimes \mathcal{F}^{k,p}_t \) for \( t \geq 0 \). Let \( \mathcal{T}^k := \{ T^k_m; m \geq 0 \} \) be the order statistics obtained from the family of random variables \( \{ T^k_j; \ell \geq 0; j = 1, \ldots, p \} \). That is, we set \( T^k_0 := 0 \),

\[ T^k_1 := \inf_{1 \leq j \leq p} \{ T^k_{1,j} \} , \quad T^k_n := \inf_{m \geq 1} \{ T^k_{m,j} ; T^k_{m,j} \geq T^k_{n-1} \} \]

for \( n \geq 1 \). In this case, \( \mathcal{T}^k \) is the partition generated by all stopping times defined in \( \mathcal{T} \). The finite-dimensional distribution of \( B^k \) is absolutely continuous for each \( j = 1, \ldots, p \) and therefore the elements of \( \mathcal{T}^k \) are almost surely distinct for every \( k \geq 1 \). By construction,

\[ \mathcal{F}^k_{T^k_n} = \mathcal{F}^k_t \text{ a.s on } \{ T^k_n \leq t < T^k_{n+1} \}; \ k \geq 1, n \geq 0, \]

and \( \mathcal{T}^k \) is an exhaustive sequence of \( \mathbb{F}^k \)-stopping times. We also have,

\[ \sup_{0 \leq t \leq T} \| B^j(t) - A^{k,j}(t) \|_{\infty} \leq \varepsilon_k, \quad k \geq 1, \ j = 1, \ldots, p, \]

for every \( 0 < T < \infty \), where \( \| \cdot \|_{\infty} \) denotes the usual norm on the space \( L^\infty(\mathbb{P}) \). See Lemma 2.1 in \([32]\) for details.

**Remark 2.1.** Similar to the univariate case of Lemma 2.2 in \([32]\), one can easily check that \( \lim_{k \to \infty} \mathbb{F}^k = \mathbb{F} \) in the sense of weak convergence of filtrations (see e.g. \([10]\)).

In order to simplify notation, from now on we are going to fix \( \varepsilon_k = 2^{-k}; k \geq 1 \).

**Remark 2.2.** We stress that all results of this article hold true for a generic sequence \( \{ \varepsilon_k; k \geq 1 \} \) such that \( \varepsilon_k \downarrow 0 \) as \( k \to \infty \). The reason for this invariance is essentially because for any \( \varepsilon_k \), the increments of the stopping times \( T^k_n - T^k_{n-1} \) constitute an i.i.d sequence of random variables with mean \( \varepsilon_k^2/k \); \( k \geq 1 \) for each \( j = 1, \ldots, p \). Moreover, the strong Markov property yields the independence between \( T^k_n - T^k_{n-1} \) and \( \mathcal{F}^j_{T^k_{n-1}} \) for each \( j = 1, \ldots, p \) and \( k, n \geq 1 \).

Throughout this paper, we employ the following terminology.

**Definition 2.1.** We say that a real-valued process \( X \) is a **Wiener functional** if it has càdlàg paths, it is adapted w.r.t to the Brownian filtration and \( \mathbb{E}|X(T^n_{k})| < \infty \) for every \( k, n \geq 1 \).

Throughout this article, we are going to fix a terminal time \( 0 < T < \infty \). The asymptotic limits of this article will be based on the weak topology \( \sigma(\mathcal{B}^p, \mathcal{M}^p) \) of the space \( \mathcal{B}^p(\mathbb{F}) \) constituted of Wiener functionals \( X \) such that

\[ \mathbb{E}|X^*(T)|^p < \infty, \]

where \( X^*(T) := \sup_{0 \leq t \leq T} |X(t)| \) and \( 1 \leq p, q < \infty \) with \( 1/p + 1/q = 1 \). We refer the reader to e.g. \([12, 32]\) for more details on this topology. We also denote \( \mathcal{H}^p(\mathbb{F}) \) as the linear space of \( p \)-integrable \( \mathbb{F} \)-martingales starting at zero. We will see that the \( \sigma(\mathcal{B}^p, \mathcal{M}^p) \)-topology will be quite natural for the asymptotic limits of this article. The starting point of our construction towards a weak version of functional Itô calculus is the study of the following class of processes.

**Definition 2.2.** A family \( \mathcal{Y} = \{ X^k; k \geq 1 \} \) in \( \mathcal{B}^p(\mathbb{F}) \) is called a **good approximating sequence** for a Wiener functional \( X \) if for each \( k \geq 1 \), \( X^k \) is an \( \mathbb{F}^k \)-adapted process with càdlàg paths admitting a representation
processes: approximating sequence by GAS.

Remark 2.3. There exists a typo in equation (2.8) in [32], where \( \delta^k_X(t) \) is actually given by (2.7) above.
**Functional GAS.** Another example of a GAS can be constructed starting with a fixed functional representation. If $X$ is a Wiener functional, then there exists a functional $F = \{F_t; 0 \leq t \leq T\}$ defined on $\Lambda$ which realizes

\begin{equation}
X(t) = F_t(B_t); \quad 0 \leq t \leq T.
\end{equation}

Then, we shall define the following sequence of processes $F := \{F^k; k \geq 1\}$

\begin{equation}
F^k(t) := \sum_{\ell=0}^{\infty} F_{T^k}(A^k_{\ell}) \mathbb{1}_{(T^k_{\ell} \leq t < T^k_{\ell+1})}, \quad 0 \leq t \leq T.
\end{equation}

The reader should not confuse $F$ with $F^k$ because $\{F^k(t); 0 \leq t \leq T\}$ is a pure jump process while $\{F_t(A^k_\ell); 0 \leq t \leq T\}$ does not necessarily has this property. Under continuity assumptions in the sense of pathwise functional calculus, one can show that $\lim_{k \to \infty} F^k = F(B)$ weakly in $\mathcal{B}^2(\mathcal{F})$ (see Lemma 7.2) so that $F = \{F^k; k \geq 1\}$ is a GAS for the Wiener functional $F(B)$.

In some applications, it is natural to mix different GAS. One typical example is the following one.

**Mixed GAS.** Let $X$ be a bounded positive Wiener functional $X$. Let $S$ be the Snell envelope of $X$ given by

\begin{equation}
S(t) = \text{ess sup}_{\tau \geq t} \mathbb{E}[X(\tau) \mid \mathcal{F}_t], \quad 0 \leq t \leq T,
\end{equation}

where ess sup in (2.11) stands for the essential supremum of the family $\mathbb{E}[X(\tau) \mid \mathcal{F}_t]$ over the set of all $\mathcal{F}$-stopping times $\tau$ such that $t \leq \tau \leq T$ a.s. For a given $n \geq 0$, let $T^k_n$ be the class of all $\mathcal{F}^k$-stopping times of the form $\tau \land T$ where $[[\tau, \tau]] \subset \bigcup_{\ell \geq n}[[T^k_\ell, T^k_{\ell+1}]]$. For a given GAS $\{X^k; k \geq 1\}$ for $X$, let us define

\begin{equation}
S^k(T^k_n) := \text{ess sup}_{\tau \in T^k_n} \mathbb{E}[X^k(\tau) \mid \mathcal{F}^k_{T^k_n}]; \quad n \geq 0,
\end{equation}

and we set $S^k(t) := \sum_{n=0}^{\infty} S^k(T^k_n) \mathbb{1}_{(T^k_n \leq t < T^k_{n+1})}; \quad 0 \leq t \leq T$.

Then $\{S^k; k \geq 1\}$ is an $\mathcal{F}^k$-adapted pure jump process satisfying (2.10). Under mild regularity assumptions on $X$ (e.g. continuity), one can find a GAS $\{X^k; k \geq 1\}$ for $X$ such that (see Lemma 6.3) $\lim_{k \to \infty} S^k = S$ strongly in $\mathcal{B}^2(\mathcal{F})$.

**Remark 2.5.** In general, for a given Wiener functional $X$, one has plenty of choices for GAS. If one has an explicit functional representation $F \in \Lambda$ for $X = F(B)$ at hand, the functional GAS (2.9) is the natural one. In the pure martingale case, the canonical embedding $\delta^k X$ in (2.7) proves to be highly flexible and computationally efficient. See [33] for more details. Of course, one can also consider Euler-Maruyama discretization schemes as possible examples of GAS.

2.3. **Discrete differential structure.** In this section, we present the basic discrete differential structure which is the basis for the development of the weak functional stochastic calculus of this article. In the sequel, we fix a GAS $Y = \{X^k; k \geq 1\}$ for a Wiener functional $X$ (see Remark 2.8). At first, Doob-Meyer decomposition yields a unique $\mathcal{F}^k$-predictable process $N^{Y,k}$ with locally integrable variation such that

\begin{equation}
X^k(t) - X^k(0) - N^{Y,k}(t) =: M^{Y,k}(t), \quad 0 \leq t \leq T,
\end{equation}

is an $\mathcal{F}^k$-local martingale. By the very definition, $X^k$ is a pure-jump $\mathcal{F}^k$-semimartingale and hence, we shall write it as
(2.13) \[ X^k(t) = X^k(0) + \sum_{j=1}^{p} \int_0^t D^{Y,k,j} X(u)dA^{k,j}(u) \]

where

(2.14) \[ D^{Y,k,j} X := \sum_{\ell=1}^{\infty} \Delta X^k(T^{k,j}_\ell) \mathbb{1}_{[T^{k,j}_\ell,T^{k,j}_{\ell+1})}, \quad Y = \{X^k; k \geq 1\}, \]

and the integral in (2.13) is interpreted in the Lebesgue-Stieltjes sense.

**Explicit Doob-Meyer Decomposition.** Let us now provide an explicit characterization of the elements \((M^{Y,k}, N^{Y,k})\) in (2.12) for a given GAS \(Y\). The strategy is essentially the same as given by [32] for the particular case of the canonical GAS. For sake of clarity, we give the details here. In this article, we need the flexibility to work with different GAS in such way that they provide the same asymptotic behavior.

At first, let us identify the martingale component. By the very definition, for each \(n, k \geq 1\)

\[
[[T^k_n, T^k_{n+1})] \subset \bigcup_{j=1}^{p} \bigcup_{\ell=1}^{\infty} [T^{k,j}_\ell,T^{k,j}_{\ell+1}),
\]

and hence if

(2.15) \[ E \sum_{n=1}^{m} |\Delta X^k(T^k_n)|^2 < \infty \quad \forall m, k \geq 1, \]

then the \(F^k\)-optional stochastic integral

\[
\int_0^t D^{Y,k,j} X(s)dA^{k,j}(s) := \int_0^t D^{Y,k,j} X(s)dA^{k,j}(s) - \left( \int_0^t D^{Y,k,j} X(s)dA^{k,j}(s) \right)^{p,k}(t),
\]

is a well-defined \(F^k\)-local martingale, where \(\int_0^t D^{Y,k,j} X(s)dA^{k,j}(s)\) is interpreted in the Lebesgue-Stieltjes sense. The stochastic integral \(\int\) is the \(F^k\)-optional integral as introduced by Meyer and Dellacherie [12]. We refer the reader to the works [12, 26] for a detailed exposition of stochastic integrals w.r.t optional integrands. By setting

(2.16) \[ D^{Y,k,j} X(t) := \sum_{\ell=1}^{\infty} D^{Y,k,j} X(T^{k,j}_\ell) \mathbb{1}_{[T^{k,j}_\ell,T^{k,j}_{\ell+1})}; 0 \leq t \leq T; \ j = 1, \ldots, p, \]

we then have

\[ M^{Y,k}(\cdot) = \sum_{j=1}^{p} \int_0^t D^{Y,k,j} X(s)dA^{k,j}(s) \text{ and } N^{Y,k}(\cdot) = \sum_{j=1}^{p} \left( \int_0^t D^{Y,k,j} X(s)dA^{k,j}(s) \right)^{p,k}(\cdot). \]

**Remark 2.6.** Of course, \(D^{Y,k,j} X\) is the unique \(F^k\)-optional process which represents the martingale \(M^{Y,k,j}\) as an optional stochastic integral with respect to the martingale \(A^{k,j}\). Moreover, we recall that there is no \(F^k\)-predictable representation w.r.t \(A^k\) since \(F^k\) is not a completely continuous filtration for \(j = 1, \ldots, p\). See Remark 2.3 and Lemma 2.1 in [32] and Th.13.42 in [26] for more details.

Let us now characterize the non-martingale predictable component in decomposition (2.12). In the sequel, \(T^{k,j}_m\) is the left-hand side limit of the filtration \(F^k\) at the stopping time \(T^{k,j}_m\).
Lemma 2.1. Let $\mathcal{Y} = \{X^k; k \geq 1\}$ be a GAS for a Wiener functional $X$. The $\mathbb{R}^k$-dual predictable projection of $X^k - X^k(0)$ is given by the absolutely continuous process

$$\sum_{j=1}^{p} \int_0^t U^{Y,k,j} X(s) d\langle A^{k,j}, A^{k,j} \rangle(s), \quad 0 \leq t \leq T,$$

where $U^{Y,k,j} X := \mathbb{E}[A^{k,j}][\mathcal{D}^{Y,k,j} X/\Delta A^{k,j}]|_{\mathcal{P}^k}$. Here $\mathbb{E}[A^{k,j}][\cdot|\mathcal{P}^k]$ denotes the conditional expectation with respect to $\mathcal{P}^k$ under the Doléans measure generated by $[A^{k,j}, A^{k,j}]$, for every $j = 1, \ldots, p$. Moreover, for each $j = 1, \ldots, p$ the following representation holds

$$U^{Y,k,j} X(T_m^k) \mathbb{I}_{\{T_m^k \leq T\}} = \sum_{n=1}^{\infty} \mathbb{E} \left[ \frac{\Delta X^k(T_m^k)}{2^{-2k}} | X^{k}_{T_{n-1}}; T_n^k \mathbb{I}_{\{T_m^k \leq T_n^k\}} \right] \mathbb{I}_{\{T_m^k \leq T_n^k\}} \mathbb{I}_{\{T_n^k \leq T\}};$$

for $m \geq 1$.

Proof. The argument follows the same lines of the proof of Lemma 2.3 in [32], so we omit the details.

Remark 2.7. The process $U^{Y,k,j} X; j = 1, \ldots, p$ is reminiscent from the so-called “Laplacian approximation method” introduced by Meyer [36] in the context of the French school of “general theory of processes”. In Section 4 we interpret this operator in terms of functional representations. In particular, we shall split it into two component related to horizontal and second-order vertical perturbations which are similar in nature to the standard pathwise functional stochastic calculus.

Summing up the above results, we arrive at the following representation for a given GAS $\mathcal{Y} = \{X^k; k \geq 1\}$ which will be the basis for differential representations of Wiener functionals.

Proposition 2.1. Let $\mathcal{Y} = \{X^k; k \geq 1\}$ be a GAS for a Wiener functional $X$ which satisfies the quadratic growth condition (2.14). Then the $\mathbb{R}^k$-special semimartingale decomposition of $X^k$ is given by

$$X^k(t) = X^k(0) + \sum_{j=1}^{p} \int_0^t \mathbb{D}^{Y,k,j} X(s) dA^{k,j}(s) + \sum_{j=1}^{p} \int_0^t U^{Y,k,j} X(s) d\langle A^{k,j}, A^{k,j} \rangle(s), \quad 0 \leq t \leq T.$$

Remark 2.8. We stress that all results in section 2 hold when $\{X^k; k \geq 1\}$ is a pure jump process of the form (2.4) but it is not necessarily a GAS w.r.t some Wiener functional. We decide to maintain such assumption in this section to keep the notation simple.

### 3. Weak Differential Structure of Wiener Functionals

At first, let us present two concepts which were already introduced in [32] in the context of the particular case of a canonical GAS of type (2.7).

**Definition 3.1.** Let $\mathcal{Y} = \{X^k; k \geq 1\}$ be a GAS for a Wiener functional $X$. We say that $X$ has $\mathcal{Y}$-finite energy along the filtration family $(\mathbb{F}^k)_{k \geq 1}$ if

$$\mathcal{E}^2\mathcal{Y}(X) := \sup_{k \geq 1} \mathbb{E} \sum_{n \geq 1} |\Delta X^k(T_n^k)|^2 \mathbb{I}_{\{T_n^k \leq T\}} < \infty.$$

**Definition 3.2.** Let $\mathcal{Y} = \{X^k; k \geq 1\}$ be a GAS for a Wiener functional $X$. We say that $X$ admits the $\mathcal{Y}$-covariation w.r.t to $j$-th component of the Brownian motion $B$ if the limit

$$(X, B^j)^\mathcal{Y}(t) := \lim_{k \to \infty} [X^k, A^{k,j}](t)$$

exists weakly in $L^1(\mathbb{P})$ for each $t \in [0, T]$. 
In the remainder of this article, if \( \mathcal{Y} = \{ \delta^k X; k \geq 1 \} \) is the canonical GAS for a given Wiener functional \( X \), then the operators \( D^{k,j}X \) and \( U^{k,j}X \) will be denoted by \( D^{k,j}X \) and \( U^{k,j}X \), respectively. Similarly, \( \mathcal{E}^2 \mathcal{Y}(X), \langle X, B^j \rangle^{\mathcal{Y}} \) will be denoted by \( \mathcal{E}^2(X) \) and \( \langle X, B^j \rangle^\delta \), respectively.

Leão and Ohashi [32] have studied the finite energy and covariation concepts w.r.t the canonical GAS in the unidimensional case. In particular, they have shown a close relation between these two concepts and the existence of orthogonal decompositions. One fundamental concept introduced in [32] is the following notion of orthogonality

**Definition 3.3.** We say that a Wiener functional \( X \) is orthogonal to Brownian motion when \( \langle X, B^j \rangle^\delta = 0 \) for \( j = 1, \ldots, p \).

**Remark 3.1.** In general, the bracket \( \langle \cdot, \cdot \rangle^\delta \) does not coincide with usual brackets [\( X, B^j \)] computed in terms of limits over refining partitions like the ones used in weak Dirichlet processes and regularization procedures. However, if \( X \) is a continuous square-integrable semimartingale then \( \langle X, B^j \rangle^\delta = [X, B^j]; \ j = 1, \ldots, p \). See Proposition 3.1 below.

A straightforward multi-dimensional extension of Theorems 3.1 and 4.1 in [32] is the following result.

**Theorem 3.1.** Let \( X \) be a Wiener functional admitting the canonical GAS \( \{ \delta^k X; k \geq 1 \} \). Then \( H^j = \lim_{k \to \infty} D^{k,j}X \) exists weakly in \( L^2(\mathbb{P} \times \text{Leb}) \) for each \( j = 1, \ldots, p \) if, and only if, there exists a unique \( M \in H^2(\mathbb{R}) \) such that

\[
(3.1) \quad N := X - X(0) - M
\]

is orthogonal to Brownian motion. Moreover, \( N = \lim_{k \to \infty} \sum_{j=1}^{p} \int U^{k,j}X(s)d\langle A^{k,j}, A^{k,j} \rangle(s) \) weakly in \( B^2(\mathbb{R}) \) and \( M = \sum_{j=1}^{p} \int H^jdB^j \).

**Proof.** The proof is omitted since it is a straightforward multi-dimensional adaptation of the arguments given in the proofs of Theorems 3.1 and 4.1 in [32]. We left the details to the reader.

Theorem 3.1 suggests the following definition.

**Definition 3.4.** We say that \( X \) is a **weakly differentiable** Wiener functional when there exists \( M \in H^2(\mathbb{R}) \) such that (3.1) holds.

Let \( \mathcal{W} \) be the space of weakly differentiable Wiener functionals. For a given \( X \in \mathcal{W} \), we write

\[
\mathcal{D}^jX := \lim_{k \to \infty} D^{k,j}X; \ j = 1, \ldots, p
\]

and we set \( \mathcal{D}X := (\mathcal{D}^1X, \ldots, \mathcal{D}^pX) \). Then any \( X \in \mathcal{W} \) admits a unique differential decomposition

\[
X = X(0) + \sum_{j=1}^{p} \int \mathcal{D}^jX(s)dB^j(s) + \mathcal{I}^\perp X,
\]

where the orthogonal integral operator is explicitly characterized in terms of

\[
\mathcal{I}^\perp X := \lim_{k \to \infty} \sum_{j=1}^{p} \int U^{k,j}X(s)d\langle A^{k,j}, A^{k,j} \rangle(s) \text{ weakly in } B^2(\mathbb{R}).
\]

Shortly, the nomenclature weak differentiable will be justified in Sections 5 and 7 where we compare \( \langle \mathcal{D}X, \mathcal{I}^\perp X \rangle \) with the pathwise functional derivatives given by [14, 8] for a fixed functional representation \( F(B) = X \). In these sections, we also compare our framework with different versions of weak derivatives given by [17, 15, 32].

The localized version of the linear space \( \mathcal{W} \) will be denoted by \( \mathcal{W}_{\text{loc}} \). In the sequel, we provide the multi-dimensional extension of Lemma 3.1 in [32]. In the general case, it appears an additional term as follows.
Lemma 3.1. If $X$ is a Wiener functional then there exists a positive constant $C$ such that

$$
\mathbb{E} \sum_{n=1}^{\infty} |\Delta^k X(T_n^k)|^2 \mathbb{I}_{(T_n^k \leq T)} \leq C \mathbb{E} \sum_{n=1}^{\infty} |X(T_n^k) - X(T_{n-1}^k)|^2 \mathbb{I}_{(T_n^k \leq T)}
$$

(3.3)

$$
+ C \mathbb{E} \sum_{j=1}^{p} \sum_{\ell=1}^{\infty} |X(T_{\ell}^{k,j}) - X(T_{\ell-1}^{k,j})|^2 \mathbb{I}_{(T_{\ell}^{k,j} \leq T)}
$$

for every $k \geq 1$.

Proof. By the very definition, $\mathcal{F}_{T_{\ell}^{k,j}} = \sigma(A^k; s \wedge T_{\ell}^{k,j}; s \geq 0, m = 1, \ldots, p)$ and

$$
[\delta^k X, \delta^k X](T) = \sum_{n=1}^{\infty} |\Delta^k X(T_n^k)|^2 \mathbb{I}_{(T_n^k \leq T)} = \sum_{j=1}^{p} \sum_{\ell=1}^{\infty} |\Delta^k X(T_{\ell}^{k,j})|^2 \mathbb{I}_{(T_{\ell}^{k,j} \leq T)}.
$$

For a given $j = 1, \ldots, p$ and $\ell \geq 0$, let us define $\tau_{\ell}^{k,j} := \max\{T_n^k; T_n^k < T_{\ell}^{k,j}\}$. Of course, we shall write

$$
\tau_{\ell}^{k,j} = \sum_{n=1}^{\infty} T_n^k \mathbb{I}_{(T_n^k = T_{\ell}^{k,j})},
$$

so one can readily see that

(3.4) $\mathcal{F}_{T_{\ell}^{k,j}} = \mathcal{F}_{\tau_{\ell}^{k,j}} \vee \sigma(\xi_{\ell}^{k,j}, \eta_{\ell}^{k,j})$, where $\xi_{\ell}^{k,j} = T_{\ell}^{k,j} - T_{\ell-1}^{k,j}$.

In the sequel, we are going to fix $j = 1, \ldots, p$. Let us consider $W(\cdot) := B(T_{\ell}^{k,j} + \cdot) - B(T_{\ell-1}^{k,j})$ and let $\mathbb{F}^W = \{\mathcal{F}_t^W; t \geq 0\}$ be the natural filtration generated by $W$. By the strong Markov property, $W$ is a $p$-dimensional $\mathbb{F}^W$-Brownian motion which is independent from $\mathcal{F}_{T_{\ell-1}^{k,j}}$. In other words, $\mathcal{F}_t^W$ is independent from $\mathcal{F}_{T_{\ell-1}^{k,j}}$ for every $t > 0$. Therefore

(3.5) $\sigma(\xi_{\ell}^{k,j}, \eta_{\ell}^{k,j})$ is independent from $\mathcal{F}_{T_{\ell-1}^{k,j}}$

By applying (3.4) and (3.5), we conclude that

(3.6) $\mathbb{E} \left[ X(T_{\ell}^{k,j}) \mathcal{I}_{\mathcal{F}_{T_{\ell}^{k,j}}} \right] = \mathbb{E} \left[ X(T_{\ell}^{k,j}) \mathcal{I}_{\mathcal{F}_{\tau_{\ell}^{k,j}}} \right] = \mathbb{E} \left[ X(T_{\ell}^{k,j}) \mathcal{I}_{\mathcal{F}_{\tau_{\ell}^{k,j}}} \right].$

Jensen inequality and identity (3.6) yield

$$
|\Delta^k X(T_{\ell}^{k,j})|^2 = |\delta^k X(T_{\ell}^{k,j}) - \delta^k X(\tau_{\ell}^{k,j})|^2 = \mathbb{E} \left[ X(T_{\ell}^{k,j}) \mathcal{I}_{\mathcal{F}_{\tau_{\ell}^{k,j}}} \right] - \mathbb{E} \left[ X(\tau_{\ell}^{k,j}) \mathcal{I}_{\mathcal{F}_{\tau_{\ell}^{k,j}}} \right]^2 \leq 2\mathbb{E} \left[ X(T_{\ell}^{k,j}) - X(\tau_{\ell}^{k,j}) \right]^2 \mathcal{I}_{\mathcal{F}_{\tau_{\ell}^{k,j}}} + 2\mathbb{E} \left[ X(\tau_{\ell}^{k,j}) - X(\tau_{\ell}^{k,j}) \right]^2 \mathcal{I}_{\mathcal{F}_{\tau_{\ell}^{k,j}}}.
$$

Since $|X(T_{\ell}^{k,j}) - X(\tau_{\ell}^{k,j})|^2 \leq 2|X(T_{\ell-1}^{k,j}) - X(T_{\ell}^{k,j})|^2 + 2|X(T_{\ell}^{k,j}) - X(\tau_{\ell}^{k,j})|^2$, we then have

$$
\mathbb{E}[\delta^k X, \delta^k X](T) \leq 2 \sum_{j=1}^{p} \sum_{\ell=1}^{\infty} \mathbb{E} \left[ X(T_{\ell}^{k,j}) - X(T_{\ell-1}^{k,j}) \right]^2 \mathbb{I}_{(T_{\ell}^{k,j} \leq T)} + 4 \sum_{j=1}^{p} \sum_{\ell=1}^{\infty} \mathbb{E} \left[ X(T_{\ell}^{k,j}) - X(T_{\ell-1}^{k,j}) \right]^2 \mathbb{I}_{(T_{\ell}^{k,j} \leq T)}
$$
\[
+ 4 \sum_{j=1}^{p} \sum_{k=1}^{\infty} \mathbb{E} \left( X(T_{t,j}^{k}) - X(T_{t,j}^{k-1}) \right)^2 \mathbb{1}_{(T_{t,j}^{k} \leq T)}
\leq 0 \sum_{j=1}^{p} \sum_{k=1}^{\infty} \mathbb{E} \left( X(T_{t,j}^{k}) - X(T_{t,j}^{k-1}) \right)^2 \mathbb{1}_{(T_{t,j}^{k} \leq T)}
+ 4 \sum_{n=1}^{\infty} \mathbb{E} \left( X(T_{n}^{k}) - X(T_{n-1}^{k}) \right)^2 \mathbb{1}_{(T_{n}^{k} \leq T)}.
\]

The next result shows that any continuous strong Dirichlet process (see e.g. Bertoin [2]) is weakly differentiable.

**Proposition 3.1.** Let \( X = X(0) + M + N \) be a strong Dirichlet process with continuous paths with canonical decomposition \((M, N)\) where \( M = \sum_{j=1}^{p} \int H^{j} dB^{j} \in H^{2}(\mathbb{P}) \) and \( N \) is a zero energy process. Then it is weakly differentiable and \( DX = (H^{1}, \ldots, H^{p}) \). In particular, the differential decomposition \((3.1)\) and the Dirichlet decomposition coincide.

**Proof.** Let \( X = X(0) + M + N \) be the Dirichlet decomposition of \( X \) where by Brownian motion predictable representation, we can select \( M = \sum_{j=1}^{p} M^{j} \) where \( M^{j} = \int H^{j} dB^{j}; \ j = 1, \ldots, p \) for \( H^{j} \in L^{2}(\mathbb{P} \times \text{Leb}) \). By definition, \( \delta^{k} X = X(0) + \delta^{k} M + \delta^{k} N \) and from Remark \(2.3\) we know that \( \lim_{k \to \infty} \delta^{k} X = X \) in \( B^{2}(\mathbb{P}) \). By applying Lemma 3.5 in \([32]\) for each \( j = 1, \ldots, p \), we have

\[
\langle X, B^{j} \rangle^{\delta}(t) = \lim_{k \to \infty} [\delta^{k} M, A^{k,j}](t) + \lim_{k \to \infty} [\delta^{k} N, A^{k,j}](t)
= [M^{j}, B^{j}](t) + \lim_{k \to \infty} [\delta^{k} N, A^{k,j}](t); \ 0 \leq t \leq T,
\]

weakly in \( L^{1}(\mathbb{P}) \). Lemma \(5.1\) and the assumption that \( X \) is a strong Dirichlet process yield

\[
\mathbb{E} \sum_{n=1}^{\infty} (\Delta \delta^{k} X(T_{n}^{k}))^{2} \mathbb{1}_{(T_{n}^{k} \leq T)} < \infty.
\]

Then \( X \) has finite energy along the canonical GAS. By applying Kunita-Watanabe inequality, we get for each \( t \in [0, T] \)

\[
\mathbb{E}[|\delta^{k} N, A^{k,j}](t)| \leq (\mathbb{E}[\delta^{k} N, \delta^{k} N](t))^{1/2} \times (\mathbb{E}[A^{k,j}, A^{k,j}](t))^{1/2}
\leq C (\mathbb{E}[\delta^{k} N, \delta^{k} N](t))^{1/2} \to 0
\]
as \( k \to \infty \), where \( C = (\max_{1 \leq j \leq p} \sup_{n \geq 1} \mathbb{E}[A^{k,j}, A^{k,j}](T))^{1/2} < \infty \). In \(3.8\), \( \lim_{k \to \infty} \mathbb{E}[\delta^{k} N, \delta^{k} N](t) = 0 \) due to \(3.3\) and the fact that \( N \) has zero energy in the sense of \(2\). From \(3.7\) and \(3.8\), we have \( \langle X, B^{j} \rangle^{\delta} = [M^{j}, B^{j}]; j = 1, \ldots, p \). The limit \(3.8\) shows that \( N \) is orthogonal to Brownian motion in the sense of Definition \(3.3\). Hence, Theorem \(3.1\) allows us to conclude that \((M, N)\) coincides with \(3.1\) and \( DX = (H^{1}, \ldots, H^{p}) \).

An immediate consequence of Proposition \(3.1\) is the following result.

**Corollary 3.1.** Any square-integrable continuous semimartingale is weakly differentiable and it satisfies the differential representation \((3.2)\).

Section \(8.2\) provides a nontrivial example of weakly differentiable process in terms of pathwise 2D Young integrals when a given Wiener functional \( X \) admits a rough dependence w.r.t Brownian motion in the sense of \(p\)-variation. So the space \( \mathcal{W} \) is actually much larger than the space of semimartingales.
4. Weak Derivatives from General GAS

In concrete examples, it is highly desirable to compute weak derivatives not directly on the canonical GAS but rather in different GAS $\mathcal{Y} = \{X^k; k \geq 1\}$. In fact, shortly it will become clear that applications to functional Itô calculus, BSDE and optimal stopping problems require this type of flexibility. In this section, we give some criteria to compute $\mathcal{D}X$ based on distinct GAS for $X$.

Essentially, we would like to prove a similar result as stated in Theorem 3.1 for limits $\lim_{k \to \infty} \mathcal{D}^{Y,k,j}X$ where $\mathcal{Y}$ is a general GAS for a given Wiener functional $X$. What we are going to show is that GAS share differentiability in some sense. In the sequel, we set

$$\mathcal{D}^{Y,k,j}X := (\mathcal{D}^{Y,k,j,1}X, \ldots, \mathcal{D}^{Y,k,j,p}X),$$

and $\mathcal{D}^{Y}X := (\mathcal{D}^{Y,1}X, \ldots, \mathcal{D}^{Y,p}X)$ where

$$(4.1) \quad \mathcal{D}^{Y,j}X := \lim_{k \to \infty} \mathcal{D}^{Y,k,j}X; \quad j = 1, \ldots, p$$

whenever the right-hand side limit in (4.1) exists weakly in $L^2(\mathbb{P} \times \text{Leb})$. Of course, $\mathcal{D}X = \mathcal{D}^{Y}X$ if $\mathcal{Y}$ is the canonical GAS for $X$. However, in general, they may not coincide. See Remark 4.1. In fact, the following result holds true.

**Proposition 4.1.** Let $X$ be a Wiener functional admitting a GAS $\mathcal{Y} = \{X^k; k \geq 1\}$. Then $\mathcal{E}^{Y,2}(X) < \infty$ and $(X, B^j)^{\mathcal{Y}}$ exists for $j = 1, \ldots, p$ if, and only if, $\mathcal{D}^{Y,j}X; \quad j = 1, \ldots, p$ exists. Moreover, in this case $M = \lim_{k \to \infty} M^{Y,k}$ exists weakly in $\mathcal{B}^2(\mathbb{F})$ and $[M, B^j] = \int \mathcal{D}^{Y,j}Xds$ for $j = 1, \ldots, p$.

Even though a generic GAS does not share projection properties like the canonical GAS, the proof of Proposition 4.1 follows essentially the same lines of the proof of Th. 4.1 in [32]. For sake of clarity, we give the details here.

**Lemma 4.1.** Let $\mathcal{Y}$ be a GAS for a Wiener functional $X$. Then $\{\|\mathcal{D}^{Y,k}X\|_{\mathbb{P};k \geq 1}\}$ is bounded in $L^2(\mathbb{P} \times \text{Leb})$ if, and only if, $X$ has $\mathcal{Y}$-finite energy.

**Proof.** Let $\mathcal{Y} = \{X^k; k \geq 1\}$ be a GAS for $X$. By the very definition,

$$\mathcal{E}^{Y,2}(X) = \mathcal{E}^{Y,2}(X) < \infty$$

for every $k \geq 1$. We recall that $T_{n+1}^k - T_n^k$ is independent from $\mathcal{F}_{T_{n+1}^k}^k$ and $\mathbb{E}[T_{n+1}^k - T_n^k] = 2^{-2k}$ for every $k, n \geq 1$ and $j = 1, \ldots, p$. Then the second component in the right-hand side of (4.2) is bounded. This completes the proof.

**Lemma 4.2.** Let $\mathcal{Y}$ be a GAS for a Wiener functional $X$. If $\mathcal{E}^{Y,2}(X) < \infty$ then $\{M^{Y,k}; k \geq 1\}$ is $\mathcal{B}^2(\mathbb{F})$-weakly relatively sequentially compact where all limit points are $\mathbb{F}$-square-integrable martingales.

**Proof.** Assume that $\mathcal{E}^{Y,2}(X) < \infty$ for a GAS $\mathcal{Y} = \{X^k; k \geq 1\}$ of $X$. Let

$$(4.3) \quad Z^{k,j}(t) := \mathbb{E}[M^{Y,k}(T)|\mathcal{F}_t]; \quad 0 \leq t \leq T,$$

where $X^k = X(0) + M^{Y,k} + N^{Y,k}$ is the $\mathbb{R}^k$-special decomposition of $X^k$. Burkholder-Davis-Gundy inequality implies that $\{Z^{k,j}; k \geq 1\}$ is bounded in $\mathcal{H}^2(\mathbb{F})$ and hence it is weakly relatively sequentially compact on the space $\mathcal{H}^2(\mathbb{F})$. Then any sequence in $\{Z^{k,j}; k \geq 1\}$ admits a weakly convergent subsequence in $\mathcal{H}^2(\mathbb{F})$. With a slight abuse of notation, let us denote by $\{Z^{k,j}; k \geq 1\}$ this convergent subsequence and $Z$ its respective limit point in $\mathcal{H}^2(\mathbb{F})$. Let us fix an arbitrary $\mathbb{F}$-stopping time $T$. Since $\mathcal{Y}$ is a GAS, we can safely repeat the same steps given in the proofs of Lemma 3.2 and Prop. 3.1 in [32] to conclude that
for each $F$-measurable $g \in L^\infty(\mathbb{P})$. This shows that $\lim_{k \to \infty} (M^{Y,k}, C) = (Z, C)$ for every $C \in \Lambda^\infty$, i.e., $\lim_{k \to \infty} M^{Y,k} = Z$ in $\sigma(B^1(\mathbb{F}), \Lambda^\infty)$-topology. Since $\Delta X^k = \Delta M^{Y,k}$ for every $k \geq 1$ and $\mathcal{C}^2\mathcal{Y}(X) < \infty$, we then have the family $\{\sup_{0 \leq t \leq T} |M^{Y,k}(t); k \geq 1]\}$ is uniformly integrable. Remark 3.2 in [32] allows us to conclude that $M^{Y,k} \to Z$ in $\sigma(B^1(\mathbb{F}), \Lambda^\infty)$ as $k \to \infty$. It remains to check that $M^{Y,k} \to Z$ weakly in $B^2(\mathbb{F})$ as $k \to \infty$, but this can be easily checked by repeating the same argument given in the proof of Prop. 3.1 in [32]. This concludes the proof.

Lemma 4.3. Let $X$ be a $\mathcal{Y}$-finite energy Wiener functional with a GAS $\mathcal{Y} = \{X^k; k \geq 1\}$. Let $(M^{Y,k}, N^{Y,k})$ be the $\mathbb{F}^k$-special semimartingale decomposition of $X^k$. Let $\{M^{Y,k}_i; i \geq 1\}$ be a $B^2(\mathbb{F})$-weak convergent subsequence obtained from Lemma 4.2 such that $\lim_{k \to \infty} M^{Y,k}_i = Z$ where $Z \in H^2(\mathbb{P})$. If $W \in H^2(\mathbb{F})$ then

$$\lim_{i \to \infty} [M^{Y,k_i}, \delta^k W](t) = [Z, W](t) \quad \text{weakly in} \quad L^1(\mathbb{P})$$

for each $t \in [0, T]$. In particular, if $(X, B^i)^\mathcal{Y}; j = 1, \ldots, p$ exists for some finite energy GAS $\mathcal{Y}$, then $M = \lim_{k \to \infty} M^{Y,k}$ exists weakly in $B^2(\mathbb{F})$ and $\langle X, B^i \rangle^\mathcal{Y} = [M, B^i]; j = 1, \ldots, p$.

Proof. With a slight abuse of notation, let $Z^{k,\mathcal{Y}}$ be the $\mathbb{F}$-martingale convergence subsequence obtained from Lemma 4.2 in [32], where $Z \in H^2(\mathbb{F})$. By construction, $\lim_{k \to \infty} Z^{k,\mathcal{Y}} = \lim_{k \to \infty} M^{Y,k} = Z$ weakly in $B^2(\mathbb{F})$. Now the argument follows the same lines of the proof of Lemma 3.5 in [32] together with the Brownian motion representation property. We left the details to the reader.

Proof of Proposition 4.1 In the sequel, $C$ is a constant which may defer from line to line. Let us fix $t$ and $g \in L^\infty$. By the very definition,

$$E g \int_0^t \mathbb{D}^{Y,k,j} X(s)ds = E g \sum_{n=1}^\infty \mathbb{D}^{Y,k,j} X(T_{k-1}^j)(T_{k-1}^j - T_{n-1}^j)I_{\{T_{n-1}^j \leq t\}}$$

$$= E g \sum_{n=1}^\infty \mathbb{D}^{Y,k,j} X(T_{k-1}^j)(T_{k-1}^j - t)I_{\{T_{n-1}^j < t \leq T_{k-1}^j\}}$$

(4.4)

Since $T_{k-1}^j - T_{n-1}^j$ is independent from $T_{T_{k-1}^j}$ and $\mathbb{E}(T_{k-1}^j - T_{n-1}^j) = 2^{-2k}$, we shall estimate

$$|I_{k-1}^{j,2}(t)| \leq C 2^{-k} \sum_{n=1}^\infty E |\Delta X^k(T_{n-1}^j)|I_{\{T_{n-1}^j \leq t \leq T_{k-1}^j\}} \to 0$$

as $k \to \infty$. We shall write

$$I_{k-1}^{j,1}(t) = E \sum_{n=1}^\infty g_{n,j} \mathbb{D}^{Y,k,j} X(T_{k-1}^j)(T_{k-1}^j - T_{n-1}^j)I_{\{T_{n-1}^j \leq t\}}$$

$$- E g \sum_{n=1}^\infty \Delta X^k(T_{n-1}^j) \Delta A^{Y,j}(T_{n-1}^j)I_{\{T_{n-1}^j \leq t\}}$$

(4.5)

\footnote{See Section 3.1 in Leão and Ohashi [32] for the definition of this weak topology.}
where \( g_n^{k,j} := \mathbb{E}[g|\mathcal{F}_{T_n}^{k,j}] - \mathbb{E}[g|\mathcal{F}_{T_n-1}^{k,j}] \); \( k, n \geq 1 \) and \( j = 1, \ldots, p \). From Lemma 4.1 in [32], we know that \( \sup_{n \geq 1} |g_n^{k,j}|_{L^2(T_n^{k,j})} \to 0 \) in probability as \( k \to \infty \) for each \( j = 1, \ldots, p \) and the sequence \( \{(\sum_{n=1}^{\infty} \mathbb{D}^{k,j}X(T_n^{k,j})(T_n^{k,j} - T_{n-1}^{k,j})I_{T_n^{k,j} \leq t})^{1/2}; k \geq 1 \} \) is bounded in probability for each \( j = 1, \ldots, p \). Then by applying Cauchy-Schwarz inequality and using the fact that \( X \) has \( \mathcal{Y} \)-finite energy, we get \( |t_1^{k,j}X(t)| \to 0 \) as \( k \to \infty \). Therefore, we arrive at the following conclusion

\[
\lim_{k \to \infty} \mathbb{E}g \int_0^t \mathcal{D}^{k,j}X(s)ds \quad \text{exists} \iff \lim_{k \to \infty} \mathbb{E}g[X^{k, A^{k,j}}](t) \quad \text{exists}; \quad j = 1, \ldots, p.
\]

By Lemma 4.1, the set \( \{\mathcal{D}^{k,j}X; k \geq 1\} \) is \( L^2(\mathbb{P} \times \mathbb{L}e) \)-weakly relatively compact for every \( j = 1, \ldots, p \). In this case, the existence of \( \langle X, B^j \rangle^\mathcal{Y}; \quad j = 1, \ldots, p \) and (4.6) allow us to conclude that \( \mathcal{D}^{k,j}X \) exists. Reciprocally, if \( \mathcal{D}^{k,j}X \) exists then Lemma 4.1 and relation (4.6) yield the existence of \( \langle X, B^j \rangle^\mathcal{Y}; \quad j = 1, \ldots, p \). The conclusion that \( M = \lim_{k \to \infty} M^{k,j}X \) exists weakly in \( \mathcal{B}^2(\mathbb{F}) \) and \( \frac{d[M, B^j](t)}{dt} = \mathcal{D}^{k,j}X(t) \) for \( j = 1, \ldots, p \) follows from (4.3), (4.5) and Lemma 4.3

\section*{Corollary 4.1}

Let \( X \) be a Wiener functional admitting a GAS \( \mathcal{Y} \) such that \( \mathcal{D}^{k,j}X \) exists for \( j = 1, \ldots, p \). Then \( X \) admits a decomposition

\[
X(t) = X(0) + \sum_{j=1}^p \int_0^t \mathcal{D}^{k,j}X(s)d\mathcal{B}^j(s) + V
\]

where \( \langle V, B^j \rangle^\mathcal{Y} = 0; \quad j = 1, \ldots, p \) for a GAS \( \mathcal{R} = \{X^k - \delta^k M; k \geq 1\} \) w.r.t \( V \), where \( M = \sum_{j=1}^p \int \mathcal{D}^{k,j}X(s)d\mathcal{B}^j(s) \).

\section*{Proof.}
The existence of (4.7) is a direct consequence of Proposition 4.1. Property \( \langle V, B^j \rangle^\mathcal{Y} = 0; \quad j = 1, \ldots, p \) is a direct consequence of Lemma 4.3

\section*{Remark 4.1}

There exists one delicate point when computing \( \mathcal{D}^{k,j}X \) rather than \( \mathcal{D}X \) for a non-canonical GAS \( \mathcal{Y} \): It is not clear whether \( \mathcal{D}X = \mathcal{D}^{k,j}X \) whenever \( \mathcal{D}^{k,j}X \) exists. The non-martingale sequence \( \{\mathcal{N}^{k,j}, k \geq 1\} \) which converges to \( V \) in (4.7) may lack suitable compactness properties in such way that \( \mathcal{D}X \) and \( \mathcal{D}^{k,j}X \) may not coincide. Nevertheless, we are able to provide readable compactness conditions on a GAS \( \mathcal{Y} \) in such way that \( \mathcal{D}^{k,j}X = \mathcal{D}X \) when \( X \) is a square-integrable continuous semimartingale. In the most general case, we are not able to provide a sharp criteria but in typical examples, one can easily check that \( \mathcal{D}^{k,j}X \) exists for some GAS and \( \lim_{k \to \infty}(\mathcal{D}^{k,j}X - \mathcal{D}^{k,j}X) = 0; \quad j = 1, \ldots, p \) weakly in \( L^2(\mathbb{P} \times \mathbb{L}e) \). See e.g (3.3) in Lemma 5.1 in Section 6.2 and Theorem 4.7 for concrete examples.

\subsection*{4.1 Weak infinitesimal generator}

In this section, we characterize an important subset of \( W \) constituted by weakly differentiable Wiener functionals with absolutely continuous orthogonal component \( \mathcal{Z} \). In order to keep the whole information concentrated only at the hitting times, let us now define a stepwise predictable version of \( U^{k,j}X \) as follows. If \( \mathcal{Y} \) is a GAS for \( X \), then we set

\[
\mathcal{U}^{k,j}X := \sum_{t=0}^\infty \mathcal{U}^{k,j}X(T_t^{k,j})I_{T_t^{k,j} < t \leq T_{t+1}^{k,j}}; 0 \leq t \leq T; \quad j = 1, \ldots, p.
\]

We say that \( \mathcal{U}^{k,j}X \in L^1(\mathbb{P} \times \mathbb{L}e) \) exists if

\[
\mathcal{U}^{k,j}X := \lim_{k \to \infty} \sum_{j=1}^p \mathcal{U}^{k,j}X \quad \text{weakly in } L^1(\mathbb{P} \times \mathbb{L}e).
\]

In the remainder of this article, if \( \mathcal{Y} \) is the canonical GAS of \( X \), then we write \( UX \).

In the sequel, we say that \( X \in \mathcal{B}^2(\mathbb{F}) \) is a square-integrable Itô process, if there exist \( \mathbb{F} \)-adapted processes \( H^1, \ldots, H^p, V \) such that
For a given $j$ (4.12) 

$g$ other. It is sufficient to check for a given $N$ that 

$$
\text{from Proposition 2.1, we already know that}
$$

$N$ weakly in $L$. Moreover, $D$ and (4.10). Then 

$\text{Proof.}$ Let $X \in B^2(\mathbb{F})$ be a Wiener functional admitting a GAS $\mathcal{Y} = \{X^k; k \geq 1\}$ satisfying (4.9) and (4.10). Then $D^{\mathcal{Y},j}X; j = 1, \ldots, p$ and $\sum_{j=1}^p D^{\mathcal{Y},k}X$ are weakly relatively sequentially compact sequences in $L^2(\mathbb{P} \times \mathbb{F})$ and $L^1(\mathbb{P} \times \mathbb{F})$, respectively. Then we shall extract common weakly convergent subsequences. With a slight abuse of notation, we still denote them by $D^{\mathcal{Y},j}X; j = 1, \ldots, p$ and $\sum_{j=1}^p \mathcal{Y}^{\mathcal{Y},k}X$. By applying Proposition 4.1 and Corollary 4.1 along the convergent subsequence $\{\mathcal{Y}^{\mathcal{Y},k}X; k \geq 1\}$, there exists adapted $H \in L^2(\mathbb{P} \times \mathbb{F}); j = 1, \ldots, p$ and there exists $N \in B^2(\mathbb{F})$ such that

$$
X(t) = X(0) + \sum_{j=1}^p \int_0^t H^j(s)dB^j(s) + \int_0^t UX(s)ds; 0 \leq t \leq T,
$$

where

$$
\mathbb{E} \int_0^T |H^j(s)|^2ds < \infty, \quad \mathbb{E} \left( \int_0^T |V(s)|ds \right)^2 < \infty
$$

for every $j = 1, \ldots, p$. The following result characterizes the class of Itô processes in terms of $(D, UX)$.

**Theorem 4.1.** If there exists a GAS $\mathcal{Y}$ for $X \in B^2(\mathbb{F})$ such that

$$
\sup_{k \geq 1} \mathbb{E} \int_0^T \|D^{\mathcal{Y},k}X(s)\|_{L^2}^2 ds < \infty
$$

and

$$\{\mathbb{Y}^{\mathcal{Y},j}X; 1 \leq j \leq p, k \geq 1\}$$

is uniformly integrable in $L^1(\mathbb{P} \times \mathbb{F})$, then $X$ is an Itô process. In particular, any square-integrable Itô process $X$ is uniquely written in the following differential form

$$
X(t) = X(0) + \sum_{j=1}^p \int_0^t D^{\mathcal{Y},j}X(s)dB^j(s) + \int_0^t UX(s)ds; 0 \leq t \leq T,
$$

where $D^{\mathcal{Y}}X = UX$. Moreover, $D^{\mathcal{Y}}X = DX$ and $UX = UX$ for any GAS $\mathcal{Y}$ satisfying (4.9) and (4.10).

**Proof.** Let $X \in B^2(\mathbb{F})$ be a Wiener functional admitting a GAS $\mathcal{Y} = \{X^k; k \geq 1\}$ satisfying (4.9) and (4.10). Then $\sum_{j=1}^p \mathcal{Y}^{\mathcal{Y},k}X$ weakly relatively sequentially compact sequences in $L^2(\mathbb{P} \times \mathbb{F})$ and $L^1(\mathbb{P} \times \mathbb{F})$, respectively. Then we shall extract common weakly convergent subsequences. With a slight abuse of notation, we still denote them by $\sum_{j=1}^p \mathcal{Y}^{\mathcal{Y},k}X; j = 1, \ldots, p$ and $\sum_{j=1}^p \mathcal{Y}^{\mathcal{Y},k}X$. By applying Proposition 4.1 and Corollary 4.1 along the convergent subsequence $\{\sum_{j=1}^p \mathcal{Y}^{\mathcal{Y},k}X; k \geq 1\}$, there exists adapted $H \in L^2(\mathbb{P} \times \mathbb{F}); j = 1, \ldots, p$ and there exists $N \in B^2(\mathbb{F})$ such that

$$
X(t) = X(0) + \sum_{j=1}^p \int_0^t H^j(s)dB^j(s) + N(t); 0 \leq t \leq T.
$$

We claim that $N(t) = \int_0^t \gamma(s)ds; 0 \leq t \leq T$ where $\gamma := \lim_{k \to \infty} \sum_{j=1}^p \mathcal{Y}^{\mathcal{Y},k}X$ weakly in $L^1(\mathbb{P} \times \mathbb{F})$. From Proposition 2.1 we already know that

$$
\lim_{k \to \infty} \sum_{j=1}^p \int_0^t \mathcal{Y}^{\mathcal{Y},k}X(s)d(A^k, A^{k,j})(s) = N(t)
$$

weakly in $B^2(\mathbb{F})$. By construction, $N \in \mathbb{F}$-adapted and it has càdlàg paths. Hence, in order to show that $N$ and $\int \gamma(s)ds$ are indistinguishable, one only has to check they are modifications from each other. It is sufficient to check for a given $g \in L^\infty(\mathcal{F}_T - \text{measurable})$ and $0 \leq t \leq T$,

$$
\mathbb{E}gN(t) = \mathbb{E}g \int_0^t \gamma(s)ds.
$$

For a given $j = 1, \ldots, p$, we set

$$
(4.12) \quad \mathbb{E}gN(t) = \mathbb{E}g \int_0^t \gamma(s)ds.
$$
\[ \alpha_{k,j}(g)(s) := \alpha_{k}(g)(T_{n}^{k,j}) \quad \text{on } \{ T_{n}^{k,j} \leq s < T_{n+1}^{k,j} \} \]
\[ \beta_{k,j}(g)(s) := \beta_{k}(g)(T_{n}^{k,j}) \quad \text{on } \{ T_{n}^{k,j} < s \leq T_{n+1}^{k,j} \}; \ n \geq 1. \]

In fact, we shall apply the \( \mathbb{F}^{k} \)-dual predictable projection and Remark 4.1 in [32] to get the following identity

\[ \mathbb{E}\int_{0}^{t} U^{Y,j,k} X(s) d(A^{k,j}, A^{k,j})(s) = \sum_{j=1}^{p} \mathbb{E}\int_{0}^{t} U^{Y,j,k} X(s) d[A^{k,j}, A^{k,j}](s) \]
\[ = \mathbb{E}\int_{0}^{t} \{ U^{Y,j,k} X(s) \} d(\mathbb{F}^{k}) \]
\[ = \mathbb{E}\int_{0}^{t} \{ U^{Y,j,k} X(s) \} d(A^{k,j}, A^{k,j})(s) \]  
(4.13)

for each \( j = 1, \ldots, p \). Since \( U^{Y,j,k} X \) is \( \mathbb{F}^{k} \)-predictable, we shall use \( \mathbb{F}^{k} \)-dual predictable projection componentwise and the existence of \( \gamma \) to get

\[ \sum_{j=1}^{p} \mathbb{E}\int_{0}^{t} \{ U^{Y,j,k} X(s) \} d(A^{k,j}, A^{k,j})(s) \]
\[ = \mathbb{E}\int_{0}^{t} \{ U^{Y,j,k} X(s) \} d(A^{k,j}, A^{k,j})(s) \]
\[ \rightarrow \mathbb{E}\int_{0}^{t} \gamma(s) ds \]
as \( k \to \infty \). Since \( \{ U^{Y,j,k} X; k \geq 1, 1 \leq j \leq p \} \) is uniformly integrable, then the second component in (4.13) vanishes and hence (4.12) holds.

Now let us check the second part of the theorem. Let us assume that \( X \) is a square-integrable Itô process of the form (4.8). Let us now check that \( (D \gamma, UX) \) exists. From Proposition 3.1, we already know that \( X \in \mathbb{F} \) and \( DX = (H^{1}, \ldots, H^{p}) \) in \( L^{2}(\mathbb{P} \times \text{Leb}) \).

**Claim** \( \hat{U}X = V \) and \( \mathcal{I}X = \int UX ds \). In the sequel, \( C \) is a generic constant which may differ from line to line. To shorten notation, let us denote \( M := \sum_{j=1}^{p} \int H^{j} dB^{j} \) and \( Y := \int V(s) ds \). The \( \mathbb{F}^{k} \)-semimartingale decomposition based on the canonical GAS is

\[ \delta^{k} X(t) = X(0) + \sum_{j=1}^{p} \int_{0}^{t} D^{k,j} X(s) dA^{k,j}(s) + \sum_{j=1}^{p} \int_{0}^{t} U^{k,j} X(s) d(A^{k,j}, A^{k,j})(s); 0 \leq t \leq T. \]

Proposition 3.1 yields

\[ \sum_{j=1}^{p} \int_{0}^{t} D^{k,j} X(s) dA^{k,j}(s) \rightarrow M \quad \text{and} \quad \sum_{j=1}^{p} \int_{0}^{t} U^{k,j} X(s) d(A^{k,j}, A^{k,j})(s) \rightarrow Y \]

weakly in \( \mathbb{B}^{2}(\mathbb{F}) \) as \( k \to \infty \). In particular, by taking \( g \in L^{\infty}(\mathcal{F}_{T}) \) and \( t \in [0, T] \), we shall consider the bounded linear functional \( S = g \mathbb{1}_{[0,t]} \in \mathbb{M}^{2} \) to get

\[ \mathbb{E}g \int_{0}^{t} \sum_{j=1}^{p} \int_{0}^{t} U^{k,j} X(s) d(A^{k,j}, A^{k,j})(s) \rightarrow \mathbb{E}g \int_{0}^{t} V(s) ds \]
as \( k \to \infty \) for each \( g \in L^{\infty}(\mathcal{F}_{T}) \) and \( t \in [0, T] \). By using (4.13) based on the canonical GAS, we have
\[
\mathbb{E} g \int_0^t U^{k,j} X(s) d(A^{k,j}, A^{k,j})(s) = \mathbb{E} \int_0^t p^{k,j}(g)(s) U^{k,j} X(s) ds
\]

(4.14)

\[
+ \mathbb{E} \sum_{\ell = 0}^{\infty} \int_{T^{k,j}_{\ell+1}}^{T^{k,j}_{\ell+1}} p^{k,j}(g)(s) U^{k,j} X(s) ds \mathbb{I}_{(T^{k,j}_{\ell+1} < t \leq T^{k,j}_{\ell+1})},
\]

In the sequel, we fix \( t \in [0, T] \) and \( g \in L^\infty(\mathcal{F}_T) \). Let \( \tau^{k,j}_{t-} := \max \{ T^{k,j}_n; T^{k,j}_n < t \} \) and \( \tau^{k,j}_{t+} := \min \{ T^{k,j}_n; T^{k,j}_n \geq t \} \). Cauchy-Schwartz inequality yields

\[
\mathbb{E} \sum_{\ell = 0}^{\infty} \int_{T^{k,j}_{\ell+1}}^{T^{k,j}_{\ell+1}} p^{k,j}(g)(s) U^{k,j} X(s) ds \mathbb{I}_{(T^{k,j}_{\ell+1} < t \leq T^{k,j}_{\ell+1})} \leq \mathbb{E} \int_{\tau^{k,j}_{t-}}^{\tau^{k,j}_{t+}} p^{k,j}(g)(s) U^{k,j} X(s) ds \leq \mathbb{E} \int_{\tau^{k,j}_{t-}}^{\tau^{k,j}_{t+}} p^{k,j}(g)(s) U^{k,j} X(s) ds \leq \mathbb{E} \int_{\tau^{k,j}_{t-}}^{\tau^{k,j}_{t+}} U^{k,j} X(s) ds \leq C \mathbb{E} \int_{\tau^{k,j}_{t-}}^{\tau^{k,j}_{t+}} U^{k,j} X(s) ds
\]

(4.15)

where in \( (4.13) \) we have used the fact that \( \tau^{k,j}_{t+} \) is independent from \( U^{k,j} X(\tau^{k,j}_{t+}) \). One can easily check that \( \mathbb{E} |\tau^{k,j}_{t+} - \tau^{k,j}_{t-}|^2 = 2^{-4k} \mathbb{E} \tau^2 \) for \( \tau = \inf \{ s > 0; |W(s)| = 1 \} \) and \( W \) is a standard Brownian motion. From Lemma \( [3] \), Jensen inequality and the path continuity of \( X \), we have

\[
\mathbb{E} [U^{k,j} X(\tau^{k,j}_{t+})]^2 \mathbb{E} |\tau^{k,j}_{t+} - \tau^{k,j}_{t-}|^2 \leq C \mathbb{E} [\Delta \delta^k X(\tau^{k,j}_{t+})]^2 2^{-4k} \mathbb{E} |\tau^{k,j}_{t+} - \tau^{k,j}_{t-}|^2 \leq C \mathbb{E} [\Delta \delta^k X(\tau^{k,j}_{t+})]^2 \to 0 \quad \text{as} \quad k \to \infty.
\]

By summing up \( (4.14), (4.15) \) and \( (4.16) \), we have

\[
\mathbb{E} \int_0^t U^{k,j} X(s) ds = \mathbb{E} g \int_0^t \sum_{j=1}^p U^{k,j} X(s) ds \to \mathbb{E} g \int_0^t V(s) ds
\]

as \( k \to \infty \). Since \( g \in L^\infty(\mathcal{F}_T) \) and \( t \in [0, T] \) are arbitrary, then \( (4.17) \) allows us to state the \( UX = V \).

Lastly, if \( Y \) is a GAS for a square-integrable Itô process \( X \) such that \( Y \) satisfies \( (4.9) \) and \( (4.10) \), then from Lemmas \( [21] \) and \( [22] \) we know that \( \{Y^{k,j}; k \geq 1\} \) and \( \sum_{j=1}^p Y^{k,j, X} \) are weakly sequentially compact w.r.t to \( B^2(\mathbb{P}) \) and \( L^1(\mathbb{P} \times \text{Leb}) \), respectively. By applying Proposition \( [1] \) and the same argument given in \( (4.12) \) on any convergent subsequence, we conclude from the uniqueness of the semimartingale decomposition that \( U^X = UX \) and \( D^X = DX \). This concludes the proof. \( \square \)

The differential operator \( UX \) basically describes the mean of any square-integrable Itô process in an infinitesimal time interval

\[
\mathbb{E} X(t) \sim \mathbb{E} X(0) + tUX(t) \quad \text{for small} \ t > 0.
\]

(4.18)

Of course, when \( X(\cdot) = g(\cdot, Y(\cdot)) \) is a smooth transformation of a Markovian diffusion \( Y \), then \( UX \) coincides with the usual Dynkin operator \( \delta_t + \mathcal{A} \) applied to \( g \) at \( Y \), where \( \mathcal{A} \) is the infinitesimal generator of \( Y \). Relation \( (4.18) \) justifies the following definition.

Definition 4.1. A Wiener functional \( X \in W \) admits a weak infinitesimal generator when \( UX \) exists.

In the remainder of this article, we denote \( W_G \) the subspace of \( W \) given by Itô processes of the form \( (4.8) \). The localized version of \( W_G \) will be denoted by \( W_G^{loc} \).
5. Illustrative Examples

Let us start to compare the differential structure introduced in Section 3 with the standard functional Itô calculus given by Dupire [14] and Cont and Fournie [14]. We also discuss some canonical examples to compare our framework with Ekren, Touzi and Zhang [18] and Peng and Wong [42]. In the operational Itô calculus given by Dupire [14] and Cont and Fournie [14]. We also discuss some canonical examples to compare our framework with Ekren, Touzi and Zhang [18] and Peng and Wong [42]. In the section, we focus the discussion in the simplest possible setting. Our definition of weak derivative allows us to conclude that

Example: Let \( X(t) = F_t(B_t) \) be a smooth function of the \( p \)-dimensional Brownian motion \( B \), where \( F \) is given by \( F_t(x) = f(t, x(t)) \); \( x \in \Lambda \) and \( f \in C^{1,2}_b([0, T] \times \mathbb{R}^p) \). Then Theorem 4.1 yields

\[
\mathcal{D}^i X(t) = \nabla^{i,v} F_t(B_t); 1 \leq i \leq p,
\]

\[
T^i X(t) = \int^t_0 \nabla^h F_s(B_s) ds + \frac{1}{2} \sum^p_{j=1} \int^t_0 \nabla^{j,v,(2)} F_s(B_s) ds; \ 0 \leq t \leq T.
\]

In particular, \( UX = \nabla^K F(B) + \frac{1}{2} \sum^p_{j=1} \nabla^{j,v,(2)} F(B) \).

Example: Backward SDE. Let us consider a well-posed Backward SDE:

\[
Y(t) = \xi + \int^T_t f(r, Y(r), Z(r)) dr - \sum^p_{j=1} \int^T_t Z^j(r) dB^j(r), \quad 0 \leq t \leq T,
\]

where \( f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^p \to \mathbb{R} \) is a suitable generator of the Backward SDE and \( \xi \in L^2(\mathcal{F}_T) \). A strong solution of the Backward SDE (5.1) is an \((\mathbb{R} \times \mathbb{R}^p)\)-adapted process \((Y, Z)\) which satisfies (5.1) almost surely. For a given \( \xi \in L^2(\mathcal{F}_T) \) and under suitable technical assumptions on \( f \), it is well known (see e.g. [40]) that there exist a unique solution \((Y, Z)\). Then, one can easily check that \( Y \) is a square-integrable Itô process and by Theorem 4.1, we arrive at the following result.

Corollary 5.1. Let \( \xi \in L^2(\mathcal{F}_T) \) be a fixed terminal condition. A square-integrable Itô process \( Y \) is a solution of the Backward SDE (5.1) if, and only if, the pair \((Y, DY)\) is a solution of the following random PDE

\[
\begin{cases}
UX(t) + f(t, Y(t), DY(t)) = 0, \quad 0 \leq t \leq T, \\
Y(T) = \xi.
\end{cases}
\]

Proof. The proof is a straightforward consequence of the definition of the BSDE solution, so we omit the details. \( \square \)

Remark 5.1. By comparing Theorem 4.1 with the arXiv paper of Peng and Wong [42], the uniqueness of the Itô decomposition of \( X \) allows us to conclude that \( DX = D_x F(B) \) and \( UX = AF(B) \) for any functional representation \( F(B) \) of \( X \). Here, \( (D_x F(B), AF(B)) \) are the differential operators introduced in [42] by means of abstract completion of the cylindrical functionals. In case when \( X \) admits a weak infinitesimal generator, we stress that we can perform a concrete numerical analysis for the pair \((DX, UX)\) so that the random PDE (5.2) is amenable to Monte Carlo schemes. See [33] for concrete implementations of these ideas in the zero generator case for the BSDE (5.2). As far as the weak infinitesimal generator is concerned, similar Monte Carlo schemes can be constructed based on the explicit characterization (2.17).

Now, let us give a simple example which illustrates the generality of our weak differential structure. Let \( H^{1,2}_{loc}(\mathbb{R}) \) be the space of absolutely continuous functions with weak derivatives \( \nabla f \in L^2_{loc}(\mathbb{R}) \). In
the sequel, \( \{\xi^x(t); (x, t) \in \mathbb{R} \times [0, T]\} \) denotes the usual Brownian local-time and the \( d_x \xi^x(t) \)-integration below is considered in \( L^2(\mathbb{P}) \)-sense (see e.g. \[15\]). The reader is urged to compare the next example with the classical case \( f(B(t)) \) where \( f \in C^1(\mathbb{R}) \). See also Remark 3.4 in Cont and Fournie \[8\] in relation to the pathwise functional calculus.

**Proposition 5.1.** Let \( B \) be a one-dimensional Brownian motion and let \( f \in H^{1,2}_{\text{loc}}(\mathbb{R}) \). If \( X = f(B) \), then \( X \in \mathcal{W}_{\text{loc}} \) where

\[
\mathcal{D}X = \nabla f(B), \quad T^\perp X = \int_{-\infty}^{+\infty} \nabla f(x) ds \xi^x.
\]

In general, the class of functionals \( F_t(\eta) = f(\eta(t)); \eta \in \Lambda \) with \( f \in H^{1,2}_{\text{loc}}(\mathbb{R}) \) is not differentiable in the sense of Dupire \[14\], Cont and Fournie \[8\] and Ekren, Touzi and Zhang \[17\] \[13\]. Likewise, \( X = f(B) \) with \( f \in H^{1,2}_{\text{loc}}(\mathbb{R}) \) is not differentiable in the sense of Peng and Wong \[12\].

**Proof.** Indeed, let us consider \( f(x) = \max(0, x); x \in \mathbb{R} \) and take \( F_t(\eta) = f(\eta(t)); \eta \in \Lambda \). Then the right-hand vertical derivative \( \nabla^+ f_t(\eta) \) equals to 1 while the left-hand vertical derivative \( \nabla^- f_t(\eta) \) equals to 0 whenever \( \eta(t) = 0 \). Then, \( F_t(\eta) = f(\eta(t)); \eta \in \Lambda \) is not vertically differentiable on \( \Lambda \) in the sense of Dupire \[14\] and Cont and Fournie \[8\] so that \( \nabla^+ F_t(B_t) \) is not well-defined. We claim that \( F_t(B_t) = f(B(t)); 0 \leq t < T \) is locally weakly differentiable if \( f \in H^{1,2}_{\text{loc}}(\mathbb{R}) \).

For a given \( m \geq 1 \), let \( S_m := \inf\{t > 0; |B(t)| \geq 2^m\} \). One should notice that \( S_m \) is an \( \mathbb{R}^k \)-stopping time for every \( k \geq 1 \). With a slightly abuse of notation, we are going to denote \( B \) and \( A^k \) as, respectively, the stopped Brownian motion and the random walk \((2.2)\) at \( S_m \). Then \( f(B) \in \mathcal{B}^2(\mathbb{F}) \). We claim that \( f(B) \) has \( \mathcal{V} \)-finite energy w.r.t the canonical GAS \( \mathcal{V} = \{f(A^k); k \geq 1\} \). We obviously have \( f(A^k) \to f(B) \) strongly in \( \mathcal{B}^2(\mathbb{F}) \) as \( k \to \infty \). In the sequel, we need to introduce some additional notation. Let \( \eta^k(t, j2^{-k}) \) be \( 2^{-k} \times \) number of crossings made by \( A^k \) across the interval \(((j-1)2^{-k}, j2^{-k})\) in \([0, t]\). Let us denote

\[
\Lambda^k(t) := \sum_{j \in \mathbb{Z}} \eta^k(t, j2^{-k}) \mathbb{1}_{((j-1)2^{-k}, j2^{-k})}; \quad t \in [0, T].
\]

Let \( \nabla^k f(\cdot) := \sum_{j \in \mathbb{Z}} \frac{\Delta^k}{2^k} \mathbb{1}_{((j-1)2^{-k}, j2^{-k})} \) where \( \Delta^k := f(j2^{-k}) - f((j - 1)2^{-k}); j \in \mathbb{Z} \). By the very definition,

\[
[f(A^k), f(A^k)](T) = \int_{-2^m}^{2^m} |\nabla^k f(x)|^2 \Lambda^k(T, x) dx \ a.s
\]

for every \( k \geq 1 \). By assumption, \( f \in H^{1,2}_{\text{loc}}(\mathbb{R}) \) so that we shall use [Prop 1.50 in \[24\]] and [Th. 1 in \[1\]] to get

\[
\sup_{k \geq 1} \mathbb{E}[f(A^k), f(A^k)](T) \leq \sup_{r \geq 1} \int_{-2^m}^{2^m} |\nabla^r f(x)|^2 dx \sup_{r \geq 1} \sup_{x \in [-2^m, 2^m]} \Lambda^r(T, x) < \infty.
\]

This shows that \( f(B) \) has \( \mathcal{V} \)-finite energy. Now, let us fix an arbitrary \( t \in [0, T] \). By the very definition

\[
[f(A^k), A^k](t) = \int_{-2^m}^{2^m} \nabla^k f(x) \Lambda^k(t, x) dx \ a.s \ ; \ k \geq 1.
\]

Lebesgue differentiation theorem and Trotter’s theorem (see e.g Th. 6.19 in \[14\]) yield the pointwise convergence \( \lim_{k \to \infty} \nabla^k f(x) \Lambda^k(t, x) = \nabla f(x) \xi^x(t) \) a.s for a.a \( x \) w.r.t Lebesgue measure. Theorem 1 in \[1\] and Prop. 1.50 in \[24\] yield

\[
\mathbb{E} \int_{-2^m}^{2^m} |\nabla^k f(x) \Lambda^k(t, x)|^2 dx \leq \sup_{r \geq 1} \int_{-2^m}^{2^m} |\nabla^r f(x)|^2 dx \times \sup_{r \geq 1} \sup_{x \in [-2^m, 2^m]} |\Lambda^r(t, x)|^2 < \infty.
\]
Then
\begin{equation}
[f(A^k), A^k](t) = \int_{-2^m}^{2^m} \nabla f(x) \Lambda^k(t, x) dx \to \int_0^t \nabla f(B(s)) ds,
\end{equation}
weakly in $L^1(\mathbb{P})$ as $k \to \infty$. By applying Proposition \[11\] on the canonical GAS $\mathcal{Y}$ and Theorem \ref{thm:existence}, we conclude $f(B) \in \mathcal{W}_{loc}$ where $\mathcal{D}X = \nabla f(B)$.

Now, if for example $f$ is Lipschitz but not continuously differentiable then $f(B)$ is a Dirichlet process (see \cite{dupire1994}) and it may not be a semimartingale because the orthogonal component w.r.t Brownian motion is given by the local-time integral process $\int \nabla f(x) d\mathcal{L}^x(t)$ (see e.g \cite{contfournie2001}). In particular, $\int_{-2^m}^{2^m} \nabla f(x) dx \mathcal{L}^x(t) = \mathcal{I}^X(\cdot)$ on $[0, S_m]$ for each $m \geq 1$. In this case, in general $f(B)$ does not belong to the closure of cylindrical functionals w.r.t $\| \cdot \|_{\mathcal{W}}$ in $S^2_p(0, T)$. See Peng and Wong \cite{pengwong2000} for the definitions of these spaces. Lastly, if one fixes the Wiener measure on the set $\mathcal{P}_{\mathbb{R}}^0$, (see Section 2.3 in \cite{pengwong2000}), then we see that $f(B)$ is not differentiable in the sense of Ekren, Touzi and Zhang \cite{ekren2014}.

**Remark 5.2.** In contrast to Dupire \cite{dupire1994}, Cont and Fournie \cite{contfournie2001} and Peng and Wong \cite{pengwong2000}, the weak differential structure introduced in this article yields a sound interpretation of Wiener functionals when their weak infinitesimal generators do not exist. For any $f \in H^{2,2}_{loc}$, the existence of $\mathcal{D}f(B)$ does not require pointwise existence of $\nabla f$ as in the usual functional calculus \cite{dupire1994} \cite{contfournie2001}. Moreover, the existence of $\mathcal{I}^X f(B)$ does not require existence of $\mathcal{D}f$ as in \cite{pengwong2000}. While these examples are essentially trivial, it is nonetheless instructive to illustrate our results in this simplest possible setting.

**Example:** *The running maximum.* Let $F_t(\eta) = \max_{0 \leq s \leq t} \eta(s)$ for $\eta \in \Lambda$ and $p = 1$. It is well known that $F$ is not differentiable in the sense of Dupire’s pathwise calculus (see e.g example 9 in \cite{contfournie2001}). It is not differentiable in the sense of Ekren, Touzi and Zhang \cite{ekren2014} \cite{ekren2015} (see Example 2.10 in \cite{ekren2015}). Now let us consider the $X(t) = \sup_{0 \leq s \leq t} B(s); 0 \leq t \leq T$. Then $X$ is not differentiable in the sense of Peng and Wong \cite{pengwong2000} because it is not an Itô process. In contrast, $X$ is weakly differentiable because it is a square-integrable semimartingale. Since $X$ has increasing paths, we actually have $\mathcal{D}X = 0$.

**Example:** Let $p = 1$ and $Z(t) = F_t([Y, Y]_t]; 0 \leq t \leq T$ where $Y$ is a continuous locally-square-integrable Itô process and $\{F_t; 0 \leq t \leq T\}$ is any functional such that $Z$ is a locally-square integrable semimartingale. Then $Z \in \mathcal{W}_{loc}$, i.e., $Z$ satisfies the equation \ref{eq:variational} where $(\mathcal{D}Z, \mathcal{I}^Z)$ exists. But in general, horizontal derivatives in the sense of Cont and Fournie \cite{contfournie2001} does not. To see this, let $Z(t) = \int_0^t J_s([Y, Y]_s); 0 \leq t \leq T$ for some functional $\{J_t; 0 \leq t \leq T\}$. This process may be represented by a functional

$H_t(x_t, v_t) = \int_0^t G_s(x_s) v(s) ds, 0 \leq t \leq T,$

where $(x, v) \in \Lambda \times \Lambda$ and $v_t$ is a càdlàg non-negative path for each $t \in [0, T]$. Then $Z \in \mathcal{W}_{\mathcal{U}}_{loc}$ i.e., $\mathcal{U}Z$ exists but in general $\nabla^h H_t(x_t, v_t)$ does not. Indeed, take $G_s(x_s) = g(x(s))$ where $g$ is locally bounded but not continuous. In general, $\nabla^h H_t(B_t, A_t)$ is not well-defined for non-negative adapted processes of the form $A(t) = d[Y, Y]_t; 0 \leq t \leq T$ and $g$ locally bounded. The reader is urged to compare with example 3 in \cite{contfournie2001}.

**Remark 5.3.** *One typical obstruction to weak differentiability is the infinite energy property which can be founded in some classes of Gaussian processes like the fractional Brownian motions with exponents $0 < H < 1/2$ and non-path dependent functionals of the form $f(B)$ where $f \in H^{1,p}_{loc}(\mathbb{R}^d)$ for $1 \leq p < 2$.***

6. **Variational Inequalities in Non-Markovian Optimal Stopping Problems**

In this section, we illustrate the differential structure developed in the last section with variational inequalities in a very general setting. At first, let us introduce a subclass of GAS.
Definition 6.1. We say that a Wiener functional $X$ admits a strong GAS $\{X^k; k \geq 1\}$ if it satisfies (2.6) and $X^k \rightarrow X$ strongly in $B^2(F)$ as $k \rightarrow \infty$.

For example, if a Wiener functional $X$ has continuous paths, then the canonical GAS $\{\delta^k X : k \geq 1\}$ is a strong one. Let us now introduce the following concept of solution which will play a key role in this section.

Definition 6.2. For a given non-negative Wiener functional $X$, we say that $Y$ is a stochastic weak solution of the following variational inequality

\begin{equation}
\max \{UY; X - Y\} = 0 \quad X(T) = Y(T)
\end{equation}

if there exist strong non-negative GAS $Y = \{Y^k; k \geq 1\}$ and $X = \{X^k; k \geq 1\}$ for $Y$ and $X$, respectively, which solves the following obstacle problem

\begin{equation}
\max \{U^{Y^k}Y^k; X^k - Y^k\} = 0 \quad X^k(T) = Y^k(T),
\end{equation}

up to evanescent sets for every $k \geq 1$.

Remark 6.1. Our definition of stochastic weak solution based on limits of approximating strong solutions is similar in nature to the concept of strong-viscosity solution introduced by Cosso and Russo [11] in the context of path-dependent Kolmogorov equations.

Remark 6.2. It is essential in the definition of stochastic weak solution that (6.2) holds up to evanescent sets. In this case, we have

\begin{equation}
\max \{U^{Y^k}Y(T^n)k; X^k(T^n)k - Y^k(T^n)k\} = 0 \quad X^k(T) = Y^k(T) \text{ a.e. for } n \geq 0
\end{equation}

almost surely for every $k \geq 1$. This requirement is motivated by concrete applications to optimal stopping time problems, where the controller typically observes just a single trajectory.

The remainder of this section is devoted to the characterization of the Snell envelope of a non-negative Wiener functional $X$ as the unique stochastic weak solution of the obstacle problem (6.1). Throughout this section, $X$ is a bounded non-negative Wiener functional with continuous paths. For simplicity of exposition, in this section we restrict the discussion to the unidimensional case, but all the results in this section can be extended to the multi-dimensional case without any significant difficulty.

For $n \geq 0$, we denote by $ST_{k,n}(F)$ the class of all $F$-stopping times $\tau$ such that $T^n_k \leq \tau \leq T$ a.s. To shorten notation, we set $ST_{k,n} := ST_{k,n}(F)$ and $ST_0 := ST_{k,0; k \geq 1}$. In order to characterize the infinitesimal behavior of the Snell envelope $S$ (see (2.11)) of $X$

\begin{equation}
S(t) = \operatorname{ess sup}_{\tau \leq t \leq T} \mathbb{E} [X(\tau) \mid \mathcal{F}_t], \quad 0 \leq t \leq T, \quad \tau \in ST_0,
\end{equation}

we make use of the example given in (2.11). We recall that we shall embed $S$ into the $\mathbb{F}^k$-filtration through a given family of pure jump processes $\{X^k; k \geq 1\} \subset B^2(F)$ of the form (2.6) as follows

\begin{equation}
S^k(T^n_k) = \operatorname{ess sup}_{\tau \leq t \leq T} \mathbb{E} [X^k(\tau) \mid \mathcal{F}^k_{T^n_k}], \quad n \geq 0,
\end{equation}

and we set

\begin{equation}
S^k(t) = \sum_{n=0}^{\infty} S^k(T^n_k) \mathbb{1}_{\{T^n_k \leq t < T^n_{k+1}\}}, \quad 0 \leq t \leq T.
\end{equation}
We recall that the elements of $I^k_n$ are $\mathbb{F}^k$-stopping times having the form $\tau \land T$ where $[\tau, \tau]$ $\subset \cup_{j \geq n}[[T^n_j, T^{k+1}_j]]$ for $n \geq 0$. In order to simplify notation, we do not emphasize the dependence of $S^k$ on $\{X^k; k \geq 1\}$, but we stress here that $S^k$ heavily depends on $X^k$.

The first step towards the solution of (6.1) is to solve (6.2). Indeed, for a given Wiener functional $X$ equipped with a GAS $X = \{X^k; k \geq 1\}$, the natural candidate for $Y^k$ to solve (6.2) is $S^k$. In the sequel, we provide some technical results for $S = \{S^k; k \geq 1\}$.

**Lemma 6.1.** Let $\{X^k; k \geq 1\}$ be a sequence of non-negative pure jump process of the form (2.6) and let $\{S^k; k \geq 1\}$ be the associated process given by (6.3). Let $ST_{I, T}(\mathbb{F}^k)$ be the set of all $\mathbb{F}^k$-stopping times $\tau$ such that $0 \leq \tau \leq T$ a.s. Then $S^k$ is a càdlàg version of the Snell envelope $\text{ess sup}_{\tau \in ST_{I, T}(\mathbb{F}^k)} \mathbb{E}[X^k(\tau) | F^k_t]; 0 \leq t \leq T$.

**Proof.** At first, we claim that for each $t \in [0, T]$ and $\ell \geq 0$,

$$
(6.4) \quad \text{ess sup}_{\tau \in ST_{I, T}(\mathbb{F}^k)} \mathbb{E}\left[X^k(\tau) 1_{(T^k_\tau = 0)} | F^k_t\right] = \text{ess sup}_{\tau \in I^k_t} \mathbb{E}[X^k(\tau) 1_{(T^k_\tau = 0)} | F^k_t],
$$

where $\eta^k := \max\{T^n_k; T^{k+1}_n \leq t\}$. Let us fix $t \in [0, T]$ and $\ell \geq 0$. Since $X^k$ is a pure jump process of the form (2.6), then

$$
\mathbb{E}[X^k(\tau) 1_{(T^k_\tau = 0)} | F^k_t] ; \tau \in ST_{I, T}(\mathbb{F}^k) \}
$$

and hence

$$
\text{ess sup}_{\tau \in ST_{I, T}(\mathbb{F}^k)} \mathbb{E}\left[X^k(\tau) 1_{(T^k_\tau = 0)} | F^k_t\right] \leq \text{ess sup}_{\tau \in I^k_t} \mathbb{E}[X^k(\tau) 1_{(T^k_\tau = 0)} | F^k_t].
$$

Now if $\tau \in ST_{I, T}(\mathbb{F}^k)$ then we can find $\bar{\tau} \in I^k_t$ such that

$$
X^k(\tau) 1_{(T^k_\tau = 0)} = X^k(\bar{\tau}) 1_{(T^k_{\bar{\tau}} = 0)}.
$$

Indeed, we can take $\bar{\tau} = \sum_{r = 0}^{\infty} T^n_r \mathbb{1}_{(\tau = T^n_r)} \in I^k_t$ as follows

$$
\{\bar{\tau} = T^n_{k+m} := T^n_{k+m} \leq \tau < T^n_{k+1+m} ; m \geq 0 \mathbb{1}_{(\tau = T^n_r)} = 0 \text{ a.s} \text{ for } 0 \leq r \leq \ell - 1.
$$

This proves (6.4). Lastly,

$$
(6.5) \quad \text{ess sup}_{\tau \in ST_{I, T}(\mathbb{F}^k)} \mathbb{E}[X^k(\tau) | F^k_{T^k_\tau}] = \text{ess sup}_{\tau \in ST_{I, T}(\mathbb{F}^k)} \mathbb{E}[X^k(\tau) \sum_{\ell = 0}^{\infty} 1_{(T^k_\tau = 0)} | F^k_{T^k_\tau}] = \text{ess sup}_{\tau \in ST_{I, T}(\mathbb{F}^k)} \sum_{\ell = 0}^{\infty} \mathbb{E}[X^k(\tau) | F^k_{T^k_\tau}] \mathbb{1}_{(T^k_\tau = 0)}
$$

$$
(6.6) \quad = \sum_{\ell = 0}^{\infty} \text{ess sup}_{\tau \in ST_{I, T}(\mathbb{F}^k)} \left( \mathbb{E}[X^k(\tau) | F^k_{T^k_\tau}] \mathbb{1}_{(T^k_\tau = 0)} \right)
$$

$$
(6.7) \quad = \sum_{\ell = 0}^{\infty} S^k(T^n_\tau) \mathbb{1}_{(T^n_\tau \leq T^n_{\ell+1} < T^n_{\ell})} = S^k(t), \quad 0 \leq t \leq T.
$$

Relations (6.5) and (6.7) can be justified by noticing that $\{T^n_k = \eta^k\} \in F^k_\tau \cap \{T^k_n \leq \tau < T^k_{n+1}\} = F^k_{T^k_\tau} \cap \{T^k_n \leq \tau < T^k_{n+1}\}$ for every $n \geq 0$ and $1 = \sum_{n=0}^{\infty} \mathbb{1}_{(\eta^k = T^n_k)}$ a.s (disjoint). Relation (6.6) is due to (6.4). This concludes the proof. \qed
An immediate consequence of Lemma 6.1 is
\[ S^k(T) = X^k(T) \text{ a.s for each } k \geq 1, \]
for every family \( \{X^k; k \geq 1\} \) of pure jump non-negative processes of the form (2.6).

**Lemma 6.2.** Let \( X^k \) be a nonnegative pure jump process of the form (2.6). For each \( n \geq 0 \) and \( k \geq 1 \),
\[ \text{ess sup}_{\tau \in T^k_n} \mathbb{E}[X^k(\tau)|\mathcal{F}^k_{T^k_n}] = \text{ess sup}_{\tau \in ST_{k,n}} \mathbb{E}[X^k(\tau)|\mathcal{F}^k_{T^k_n}] \text{ a.s} \]

**Proof.** Take \( n \geq 0 \). Of course, \( \text{ess sup}_{\tau \in T^k_n} \mathbb{E}[X^k(\tau)|\mathcal{F}^k_{T^k_n}] \leq \text{ess sup}_{\tau \in ST_{k,n}} \mathbb{E}[X^k(\tau)|\mathcal{F}^k_{T^k_n}] \text{ a.s.} \) In the sequel, \( ST_{k,n} \) denotes the set of all \( \mathbb{F} \)-stopping times \( \eta \) such that \( T^k_n \leq \eta \leq T \text{ a.s and } \{\eta, \eta] \subset \cup_{j \geq n} \lbrack T^k_j, T^k_{j+1} \rbrack \}. \) Let us take an arbitrary \( \tau \in ST_{k,n} \). We shall write
\[ 1 = \sum_{m=n}^{\infty} \mathbb{I}_{\{T^k_m \leq \tau < T^k_{m+1}\}} \text{ a.s.} \]
Based in this partition, let us define
\[ \tilde{\tau} := T^k_m \text{ on } \{T^k_m \leq \tau < T^k_{m+1}\}, \]
so that \( \{\tilde{\tau} = T^k_m\} = \{T^k_m \leq \tau < T^k_{m+1}\}; m \geq n. \) Then, we have
\[ \tilde{\tau} = \sum_{m=n}^{\infty} T^k_m \mathbb{I}_{\{T^k_m = \tau\}} \in ST_{k,n} \]
where \( \tilde{\tau} \leq T \text{ a.s.} \) More importantly,
\[ \mathbb{E}[X^k(\tau)|\mathcal{F}^k_{T^k_n}] = \mathbb{E}[X^k(\tilde{\tau})|\mathcal{F}^k_{T^k_n}] \leq \text{ess sup}_{\tau \in ST_{k,n}} \mathbb{E}[X^k(\tau)|\mathcal{F}^k_{T^k_n}] \]
so that
\[ \text{ess sup}_{\tau \in ST_{k,n}} \mathbb{E}[X^k(\tau)|\mathcal{F}^k_{T^k_n}] = \text{ess sup}_{\tau \in ST_{k,n}} \mathbb{E}[X^k(\tau)|\mathcal{F}^k_{T^k_n}], \quad n \geq 0. \]
We claim for each \((j, n) \in \mathbb{N}^2\) with \( j > n \) there exists \( t^k_{n,j} \in \mathbb{N}^k_n\) such that
\[ \text{ess sup}_{\tau \in ST_{k,n}} \mathbb{E}[X^k(\tau \wedge T^k_{n,j})|\mathcal{F}^k_{T^k_n}] = \mathbb{E}[X^k(t^k_{n,j})|\mathcal{F}^k_{T^k_n}], \]
In order to construct such \( t^k_{n,j} \in \mathbb{N}^k_n\), we need to make use of a backward induction argument. In fact, what we are going to show is the existence of a finite sequence \( \{t^k_{n,j}; n \leq \ell \leq j\} \subset \mathbb{N}^k_n\) which realizes
\[ \text{ess sup}_{\tau \in ST_{k,\ell}} \mathbb{E}[X^k(\tau \wedge T^k_{\ell,j})|\mathcal{F}^k_{T^k_{\ell}}] = \mathbb{E}[X^k(t^k_{\ell,j})|\mathcal{F}^k_{T^k_{\ell}}], \quad n \leq \ell \leq j. \]
Let us take \( n < j \). In the sequel, the induction variable will be denoted by \( \ell \). At first, we notice for each \( \ell \in \{n, n+1, \ldots, j\} \), the family of random variables \( \{\mathbb{E}[X^k(\tau \wedge T^k_{\ell,j})|\mathcal{F}^k_{T^k_{\ell}}]; \tau \in ST_{k,\ell}\} \) satisfies the lattice property and hence
\[ \text{ess sup}_{\tau \in ST_{k,\ell}} \mathbb{E}[X^k(\tau \wedge T^k_{\ell,j})|\mathcal{F}^k_{T^k_{\ell}}] = \mathbb{E} \left[ \text{ess sup}_{\tau \in ST_{k,\ell}} \mathbb{E}[X^k(\tau \wedge T^k_{\ell,j})|\mathcal{F}^k_{T^k_{\ell}}] \right]. \]
For $\ell = j$, identity (6.12) clearly holds. Indeed, 
\[
\text{ess sup}_{\tau \in ST_{k,j}} \mathbb{E}[X^k(\tau \wedge T_j^k) | \mathcal{F}_{T_j^k}] = X^k(T_j^k) = \mathbb{E}[X^k(T_j^k) | \mathcal{F}_{T_j^k}],
\]
so we shall take $\tau_{j,j} := T_j^k$. If $\ell = j - 1$, then the dynamic programming principle based on the Brownian filtration yields 
\[
(6.14) \quad \text{ess sup}_{\tau \in ST_{k,j-1}} \mathbb{E}[X^k(\tau \wedge T_j^k) | \mathcal{F}_{T_j^k}] = \max \left\{ X^k(T_{j-1}^k), \mathbb{E}[X^k(\tau_{j,j}) | \mathcal{F}_{T_j^k}] \right\}.
\]
We recall that $\mathcal{F}_{T_{j-1}^k} = \sigma \{ \mathcal{F}_{T_{j-1}^k} : (T_j^k - T_{j-1}^k), \eta_j^k \}$ and the strong Markov property yields $(T_j^k - T_{j-1}^k)$ and $\eta_j^k$ are independent from $\mathcal{F}_{T_{j-1}^k}$. Hence, 
\[
\int_H \mathbb{E}[X^k(\tau_{j,j}) | \sigma \{ \mathcal{F}_{T_{j-1}^k} : (T_j^k - T_{j-1}^k), \eta_j^k \})] d\mathbb{P} = \int_H \mathbb{E}[X^k(\tau_{j,j}) | \mathcal{F}_{T_j^k}] d\mathbb{P}; \forall H \in \mathcal{F}_{T_{j-1}^k}
\]
and then 
\[
(6.15) \quad \mathbb{E}[X^k(\tau_{j,j}) | \mathcal{F}_{T_{j-1}^k}] = \mathbb{E}[X^k(\tau_{j,j}) | \mathcal{F}_{T_j^k}] .
\]
By applying (6.14) and (6.15), we arrive at the following identity 
\[
\text{ess sup}_{\tau \in ST_{k,j-1}} \mathbb{E}[X^k(\tau \wedge T_j^k) | \mathcal{F}_{T_j^k}] = \max \left\{ X^k(T_{j-1}^k), \mathbb{E}[X^k(\tau_{j,j}) | \mathcal{F}_{T_j^k}] \right\}.
\]
In this case, we define 
\[
\tau_{j,k-1} := \tau_{j,j} \mathbb{1}_{\{X^k(T_{j-1}^k) < \mathbb{E}[X^k(\tau_{j,j}) | \mathcal{F}_{T_j^k} \} \}} + T_{j-1}^k \mathbb{1}_{\{X^k(T_{j-1}^k) \geq \mathbb{E}[X^k(\tau_{j,j}) | \mathcal{F}_{T_j^k} \} \}} \in I^k.
\]
With this definition at hand, we have 
\[
\text{ess sup}_{\tau \in ST_{k,j-1}} \mathbb{E}[X^k(\tau \wedge T_j^k) | \mathcal{F}_{T_j^k}] = \mathbb{E}[X^k(\tau_{j,j}) | \mathcal{F}_{T_j^k}],
\]
and hence (6.13) allows us to conclude that (6.12) holds for $\ell = j - 1$. By backward induction and applying (6.13), we clearly see that we shall define $\tau_{j,k-1}$ for $\ell = j - 2, j - 3, \ldots, n$ in such way that (6.12) holds. This allows us to conclude that (6.11) holds. From (6.11), if $\tau \in ST_{k,n}$ then we have 
\[
\mathbb{E} \left[ X^k(\tau \wedge T_j^k) | \mathcal{F}_{T_j^k} \right] \leq \mathbb{E} \left[ X^k(\tau_{j,j}) | \mathcal{F}_{T_j^k} \right] \leq \text{ess sup}_{\tau \in I^k} \mathbb{E} \left[ X^k(\tau) | \mathcal{F}_{T_j^k} \right]
\]
for every $j > n$. Hence, 
\[
(6.16) \quad \text{ess sup}_{\tau \in I^k} \mathbb{E} \left[ X^k(\tau) | \mathcal{F}_{T_j^k} \right] \geq \lim_{j \to \infty} \mathbb{E} \left[ X^k(\tau \wedge T_j^k) | \mathcal{F}_{T_j^k} \right] = \mathbb{E} \left[ X^k(\tau) | \mathcal{F}_{T_j^k} \right],
\]
for every $\tau \in ST_{k,n}$. By (6.10) and (6.16), we conclude that 
\[
\text{ess sup}_{\tau \in I^k} \mathbb{E}[X^k(\tau) | \mathcal{F}_{T_j^k}] \geq \text{ess sup}_{\tau \in ST_{k,n}} \mathbb{E}[X^k(\tau) | \mathcal{F}_{T_j^k}].
\]
In the sequel, for a given family $X = \{X^k; k \geq 1\} \subset \mathcal{B}^2(F)$ of non-negative pure jump processes of the form (2.6), we decompose $X^k$ and $S = \{S^k; k \geq 1\}$ given by (6.3) as follows 
\[
X^k(t) = X^k(0) + M^{X^k}(t) + N^{X^k}(t); \quad S^k(t) = S^k(0) + M^{S^k}(t) + N^{S^k}(t), 0 \leq t \leq T,
\]
and
where $N^S_k$ has non increasing and continuous paths due to Lemma 6.1

**Lemma 6.3.** Let $X$ be a bounded non-negative Wiener functional with continuous paths admitting a strong nonnegative GAS $X = \{X^k; k \geq 1\}$ and let $S$ be the Snell envelope of $X$. Then $S = \{S^k; k \geq 1\}$ is a strong GAS for $S$. Moreover, $S$ has $S$-finite energy.

**Proof.** In the present setting, it is well known that the Snell envelope $S$ has continuous paths and hence from Remark 6.3 we know that $\lim_{k \to \infty} \delta^k S = S$ strongly in $\mathbb{B}^2(\mathcal{F})$, where $\{\delta^k S; k \geq 1\}$ is the canonical GAS for $S$. Then, it is enough to prove that $(S^k - \delta^k S)$ vanishes strongly in $\mathbb{B}^2(\mathcal{F})$. One can easily check that $\{E [X(\tau) \mid F_{T^k}]; \tau \in ST_{k,n}\}$ has the lattice property. Therefore, the tower property yields

$$\delta^k S(t) = \sum_{n=1}^{\infty} E \left[ S(T^k_n) \mid F_{T^k_n} \right] \mathbf{1}_{\{T^k_n \leq t < T^k_{n+1}\}} = \sum_{n=0}^{\infty} \sup_{\tau \in ST_{k,n}} E \left[ X(\tau) \mid F_{T^k_n} \right] \mathbf{1}_{\{T^k_n \leq t < T^k_{n+1}\}},$$

for $0 \leq t \leq T$. On the other hand, from Lemma 6.2 we have

$$S^k(t) = \sum_{n=0}^{\infty} \sup_{\tau \in ST_{k,n}} E \left[ X^k(\tau) \mid F_{T^k_n} \right] \mathbf{1}_{\{T^k_n \leq t < T^k_{n+1}\}},$$

$$= \sum_{n=0}^{\infty} \sup_{\tau \in ST_{k,n}} E \left[ X^k(\tau) \mid F_{T^k_n} \right] \mathbf{1}_{\{T^k_n \leq t < T^k_{n+1}\}}$$

for $0 \leq t \leq T$. We shall use triangle inequality, the definition of the essential supremum and the non-negativeness of $X^k$ and $X$ to get

$$E[X^k(\tau) \mid F_{T^k_n}] \leq E[|X^k(\tau) - X(\tau)| \mid F_{T^k_n}] + E[X(\tau) \mid F_{T^k_n}],$$

and then

$$\sup_{\tau \in ST_{k,n}} E[X^k(\tau) \mid F_{T^k_n}] \leq \sup_{\tau \in ST_{k,n}} E[|X^k(\tau) - X(\tau)| \mid F_{T^k_n}] + \sup_{\tau \in ST_{k,n}} E[X(\tau) \mid F_{T^k_n}].$$

Similarly,

$$\sup_{\tau \in ST_{k,n}} E[X(\tau) \mid F_{T^k_n}] \leq \sup_{\tau \in ST_{k,n}} E[|X(\tau) - X^k(\tau)| \mid F_{T^k_n}] + \sup_{\tau \in ST_{k,n}} E[X^k(\tau) \mid F_{T^k_n}].$$

This allows us to conclude that

$$|S^k(t) - \delta^k S(t)| = \sum_{n=0}^{\infty} \sup_{\tau \in ST_{k,n}} E \left[ X^k(\tau) \mid F_{T^k_n} \right] - \sup_{\tau \in ST_{k,n}} E \left[ X(\tau) \mid F_{T^k_n} \right] \mathbf{1}_{\{T^k_n \leq t < T^k_{n+1}\}}$$

$$\leq \sum_{n=0}^{\infty} E \left[ \sup_{0 \leq s \leq t} |X(s) - X^k(s)| \mid F_{T^k_n} \right] \mathbf{1}_{\{T^k_n \leq t < T^k_{n+1}\}} \leq 0 \leq t \leq T.$$ 

By applying Doob maximal and Jensen inequalities, we can find a positive constant $C$ such that

$$E \sup_{0 \leq t \leq T} |S^k(t) - \delta^k S(t)|^2 \leq C E \sup_{0 \leq t \leq T} |X(t) - X^k(t)|^2 \mathbf{1}_{\{T^k_n \leq t \leq T\}}$$

$$\leq C E \sup_{0 \leq t \leq T} |X(t) - X^k(t)|^2 \to 0$$
as $k \to \infty$. Let us now check that the Snell envelope has $S$-finite energy. From Lemma 6.4, we know that $S^k$ is a pure jump positive square-integrable supermartingale and the sequence $\{S^k; k \geq 1\}$ is bounded in $B^2(\mathcal{F})$. Then, from (12), pp. 202 we have

$$\sup_{k \geq 1} E[|X^{S^k}(T)|^2] = \sup_{k \geq 1} 2E[|S^k(t)|^2] < \infty.$$  

A straightforward application of Doob, Burkholder-Davis-Gundy and triangle inequalities yield

$$\sup_{k \geq 1} E[|S^k|, S^k(T)] = \sup_{k \geq 1} E[M^{S^k}, M^{S^k}](T) \leq C \sup_{k \geq 1} E[|M^{S^k}(t)|^2] < \infty,$$

for some positive constant $C$. This concludes the proof.

For a given $X^k$ of the form (2.6), let $S^k$ be the pure jump process given by (3). Let us define

$$L^{k}_n := \inf\{s \geq t; X^k(s) = S^k(s)\}; \quad 0 \leq t \leq T.$$

The next two results are well known for deterministic times but they also hold for the random times $\{T^k; j \geq 0\}$ by following well-known arguments in the standard optimal stopping theory (see e.g. 31), then we omit the details. To keep notation simple, in the remainder of this section we set $L^{k}_n := L^{k}_T$ for $n \geq 0$.

**Lemma 6.4.** Let $X = \{X^k; k \geq 1\} \subset B^2(\mathcal{F})$ be a sequence of pure jump processes of the form (2.6). Let $S = \{S^k; k \geq 1\}$ be the associated pure jump Snell envelope. For $n \geq 0$, let

$$L^{k}_n := \inf\{T^k_n \geq T^k_n; X^k(T^k_n) = S^k(T^k_n)\} \wedge T.$$

Then $L^{k}_n \in \mathbb{I}_{T^k_n}$ and $N^{S^k}(L^{k}_n) = N^{S^k}(T^k_n \wedge T)$ a.s for each $k \geq 1$ and $n \geq 0$.

An immediate consequence of Lemma 6.4 is the following.

**Lemma 6.5.** Let $\{X^k; k \geq 1\} \subset B^2(\mathcal{F})$ be a sequence of pure jump processes of the form (2.6) and let $S = \{S^k; k \geq 1\}$ be a sequence of $\mathbb{P}^k$-supermartingales with the form (2.6) with $\mathbb{P}^k$-special semimartingale decomposition given by $S^k = \tilde{S}^k(0) + M^{S^k} + N^{S^k}$. Then, $\tilde{S}^k$ is the pure jump Snell envelope of $X^k$ if, and only if, the following identities hold

1. $\tilde{S}^k(T^k_n) \geq X^k(T^k_n)$ a.s;
2. $\tilde{S}^k(T) = X^k(T)$ a.s;
3. For every $n \geq 0$, $N^{S^k}(T^k_n) = N^{\tilde{S}^k}(L^k_n)$, where $L^k_n = \inf\{T^k_n \geq T^k_n; X^k(T^k_n) = \tilde{S}^k(T^k_n)\} \wedge T$.

The following simple lemma is very useful for the approach taken in this work.

**Lemma 6.6.** For a given $k, n \geq 1$, let $Y$ be a $\mathcal{F}^k_{T^k_n}$-measurable and integrable random variable and let $R^k(t) = Y \mathbb{1}_{(T^k_n \leq t)}; \quad 0 \leq t \leq \infty$. Then $R^k$ is an $\mathbb{P}^k$-supermartingale (martingale) if, and only if, $E[Y | \mathcal{F}^k_{T^k_n}] \leq (\geq) 0$ a.s. In particular, a pure jump process $Y^k$ of class D with form (2.6) is an $\mathbb{R}^k$-supermartingale (martingale) if, and only if, $E[\Delta Y^k(T^k_n) | \mathcal{F}^k_{T^k_n}] \leq (\geq) 0$ a.s on $\{T^k_n \leq T\}$ for each $n \geq 1$.

Proof. The first part is a direct consequence of the Lemma in [13]. The last one is a direct consequence of the definitions together with the first part, so we omit the details.

In general, the Snell envelope $S$ associated to a given Wiener functional $X$ does not admit $(DS, US)$. Hence, in order to deal with weak solutions in the sense of Definition 6.4, we need to introduce some notation. For a given sequence $X = \{X^k; k \geq 1\}$ of pure jump process of the form (2.6) (not necessarily GAS), we write $U^Y$ and $D^Y$ to denote their discrete derivative and weak infinitesimal generator defined in [2.14] and Lemma 2.4 respectively. In other words,
\[ M^{X,k}(t) = \int_0^t D Y^k(s) d A^k(s); \quad N^{X,k}(t) = \int_0^t U Y^k(s) d (A^k, A^k)(s); 0 \leq t \leq T, \]

where \( Y^k = Y^k(0) + M^{X,k} + N^{X,k} \) is the associated \( \mathbb{F}^k \)-special semimartingale decomposition. This definition is perfectly consistent by Remark [7.3]. In particular, \( U^{X,k} Y = U Y^k \) and \( D^{X,k} Y = D Y^k \) for every GAS \( X = \{ Y^k; k \geq 1 \} \) w.r.t a given Wiener functional \( Y \).

**Corollary 6.1.** A pure jump process \( Y^k \) of the form \( \langle 2.6 \rangle \) is an \( \mathbb{F}^k \)-supermartingale (martingale) if, and only if, the associated discrete weak infinitesimal generator \( U Y^k \) satisfies the following relation

\[ (6.17) \quad U Y^k(\cdot) \leq (\cdot) 0 \text{ up to an evanescent set.} \]

**Proof.** From Lemma 6.6, we know that if \( \{ Y^k; k \geq 1 \} \) is an \( \mathbb{F}^k \)-supermartingale then \( \{ (\omega, t) \in \Omega \times [0, T]; U Y^k(\omega, t) > 0 \} \subset \bigcup_{n \geq 1} \{ |T^k_n, T^k_n] \} \) up to an evanescent set. From Lemma 2.4, we know that \( U Y^k = E_{[\mathcal{A}^k]}(|\mathcal{D} Y^k| \mathcal{P}^k) \) where \( \{ \mathcal{D} Y^k / \Delta A^k \neq 0 \} \subset \bigcup_{n \geq 1} \{ |T^k_n, T^k_n] \} \). By integrating the positive part \( (U Y^k)^+ \) of \( U Y^k \) w.r.t the Doléans measure \( \mu_{[\mathcal{A}^k]} \) generated by \( [A^k, A^k] \), we see that \( (U Y^k)^+ = \max\{0, U Y^k\} \) must be indistinguishable from zero w.r.t \( \mu_{[\mathcal{A}^k]} \), i.e.,

\[ (6.18) \quad E \int_0^T \mathbb{1}_{\{U Y^k > 0\}}(u)d[A^k, A^k](u) = 2^{-2k}E \sum_{n=1}^\infty \mathbb{1}_{\{U Y^k > 0\}}(T^k_n) \mathbb{1}_{\{T^k_n \leq T\}} = 0 \]

In other words, there exists \( \Omega^* \) with \( \mathbb{P}(\Omega^*) = 1 \) such that \( U Y^k \leq 0 \) on \( \bigcup_{n=0}^\infty \{ |T^k_n, T^k_n]\} \cap (\Omega^* \times [0, T]) \). Since \( U Y^k = E_{[\mathcal{A}^k]}(|\mathcal{D} Y^k| \mathcal{P}^k) \), then one can always modify (if necessary) \( U Y^k \) as follows

\[ U Y^k(\omega, t) := \begin{cases} U Y^k(\omega, t); & \text{if } (\omega, t) \in \bigcup_{n=0}^\infty \{ |T^k_n, T^k_n]\} \cap (\Omega^* \times [0, T]) \\ 0; & \text{otherwise}. \end{cases} \]

Then \( \bar{U} Y^k = U Y^k a.s \) w.r.t \( \mu_{[\mathcal{A}^k]} \) and this modification is non-positive up to an evanescent set. The same argument applies to the martingale case. This shows that \( (6.17) \) holds. The proof that \( (6.17) \) implies that \( Y^k \) is an \( \mathbb{F}^k \)-supermartingale (martingale) is a trivial consequence of Doob-Meyer decomposition. This concludes the proof.

In the sequel, we prove a key result to solve the variational inequality. The inequalities in \( (6.19) \) trivially hold up to \( \text{Leb} \times \mathbb{P} \)-null sets due to the fact that \( N^{S,k} \) has non-increasing paths, \( d(A^k, A^k)(t)/dt \geq 0 \) a.s (see Lemma 2.4 in [32]) and Lemma 6.4. The inequalities up to evanescent sets that deserve a more refined study.

**Proposition 6.1.** Let \( X^k \in \mathcal{B}^2(\mathcal{F}) \) be a pure jump process of the form \( \langle 2.6 \rangle \) and let \( S^k \) be the associated pure jump Snell envelope. Then the discrete weak infinitesimal generator \( U S^k \) satisfies

\[ (6.19) \quad U S^k \leq 0 \quad \text{and} \quad \mathbb{1}_{\{T^k_n \leq s\}} U S^k = 0 \quad \text{up to an evanescent set for each } n \geq 0, k \geq 1. \quad \text{Moreover, we have that} \]

\[ (6.20) \quad |U S^k(T^k_{n+1})| \leq |U X^k(T^k_{n+1})| \quad \text{a.s for each } n \geq 0, k \geq 1. \]

**Proof.** From Corollary 6.1, we already know that \( U S^k \leq 0 \) up to an evanescent set. Let us fix \( n \geq 0 \) and \( k \geq 1 \). Let \( S^k = S^k(0) + M^{S,k} + N^{S,k} \) be the \( \mathbb{F}^k \)-special semimartingale decomposition of \( S^k \). Let us introduce \( W^k(u) := (S^k(u \wedge L^k_n) - S^k(T^k_n)) \mathbb{1}_{\{T^k_n \leq u\}} \). From Lemma 6.5, we know that \( N^{S,k}(T^k_n) = N^{S,k}(L^k_n) \) a.s and hence the non-increasing paths of \( N^{S,k} \) yields
$W^k(u) = (S^k(u \wedge L^k_n) - S^k(T^k_n)) \mathbb{1}_{|[T^k_n, T^k_j]|}(u) = (M^{S,k}(u \wedge L^k_n) - M^{S,k}(T^k_n)) \mathbb{1}_{|[T^k_n, T^k_j]|}(u)$ a.s

for $0 \leq u \leq T$. Moreover, $W^k$ is a pure jump process of class $D$ of the form (2.6) whose the first jumping time is $T^k_{n+1}$. More importantly,

$$E \left[ \Delta W^k(T^k_j) \mid \mathcal{F}^k_{T^k_j-} \right] = E \left[ (M^{S,k}(T^k_j \wedge L^k_n) - M^{S,k}(T^k_{j-1} \wedge L^k_n)) \mid \mathcal{F}^k_{T^k_j-} \right] = 0, \text{ a.s.}$$

for $j \geq 1$. Lemma 6.6 allows us to conclude that $W^k$ is an $\mathbb{F}^k$-martingale. By writing

$$W^k(t) = \left( \int_{T^k_n}^t DS^k(u) dA^k(u \wedge L^k_n) \right) \mathbb{1}_{|[T^k_n, T^k_j]|}(t); \ 0 \leq t \leq T,$$

in the Lebesgue-Stieltjes sense and using the fact that the stochastic interval $|[T^k_n, L^k_n]|$ is $\mathbb{P}^k$-predictable, we obtain that

$$E \int_0^T \mathbb{1}_{(C)}(u) dW^k(u) = E \int_0^T \mathbb{1}_{(C)}(u) \mathbb{1}_{|[T^k_n, T^k_j]|}(u) DS^k(u) dA^k(u \wedge L^k_n) =$$

$$E \int_0^T \mathbb{1}_{(C)}(u) \mathbb{1}_{|[T^k_n, L^k_n]|}(u) US^k(u) d(A^k, A^k)(u) =$$

$$E \int_0^T \mathbb{1}_{(C)}(u) \mathbb{1}_{|[T^k_n, L^k_n]|}(u) US^k(u) d(A^k, A^k)(u),$$

for every $C \in \mathcal{P}^k$. Now for a given $G \in \mathcal{F}^k_{T^k_j-}$ with $j \geq 1$, we recall there exists an $\mathbb{P}^k$-predictable process $H$ such that $H(T^k_j) = \mathbb{1}_G$ and it is null outside the stochastic interval $|[T^k_{j-1}, T^k_j]|$ (see [3], Th. 31, pp. 307). The martingale property of $W^k$ and (6.21) yield

$$E \left[ \mathbb{1}_G \Delta W^k(T^k_j) \right] = E \int_0^T H(u) dW^k(u)$$

$$= E \int_0^T H(u) \mathbb{1}_{|[T^k_n, L^k_n]|}(u) US^k(u) d(A^k, A^k)(u)$$

$$= E \left[ \mathbb{1}_G \mathbb{1}_{|[T^k_n, L^k_n]|}(T^k_j) US^k(T^k_j) 2^{-2k} \right]$$

$$= 0$$

for each $j \geq 1$. Since $\mathbb{1}_{|[T^k_n, L^k_n]|}$ is $\mathbb{P}^k$-predictable, then (6.22) allows us to conclude that

$$\mathbb{1}_{|[T^k_n, L^k_n]|}(T^k_j) US^k(T^k_j) = 0 \ a.s \ j, n \geq 0.$$

From identity (6.23), we shall define the following $\mathbb{P}$-null sets

$$O^k_{n,j} = \{ \mathbb{1}_{|[T^k_n, L^k_n]|}(T^k_j) US^k(T^k_j) \neq 0 \}; \ j, n \geq 0,$$

and consequently $\cup_{n \geq 0} \cup_{j \geq 0} O^k_{n,j}$ is also a $\mathbb{P}$-null set. By arguing just like in the proof of Corollary 6.1 we shall conclude that $US^k$ satisfies (6.19).

Now, from (6.23), Lemma 2.3 in [32] and (26), Corollary 3.5, pp. 82) we have

$$-US^k(T^k_{n+1}) = -US^k(T^k_n) \mathbb{1}_{(c^k_n < T^k_{n+1})} - US^k(T^k_{n+1}) \mathbb{1}_{(c^k_n \geq T^k_{n+1})} = -US^k(T^k_n) \mathbb{1}_{(c^k_n < T^k_{n+1})}$$

$$= E \left[ S^k(T^k_n) - S^k(T^k_{n+1}) \right] \mathbb{1}_{(c^k_n < T^k_{n+1})}$$

$$= E \left[ S^k(T^k_n) - S^k(T^k_{n+1}) \right] \mathbb{1}_{(c^k_n < T^k_{n+1})} \mathbb{1}_{(c^k_n < T^k_{n+1})} \mathcal{F}^k_{T^k_n-}.$$
which in turn implies an evanescent set. Therefore, for each where \( \tilde{S} \) be the pure jump Snell envelope of \( X \) up to an evanescent set for every \( S \). Let us assume the existence of \( X \) for \( \{ S, k \} \) (Uniqueness). Let us now state the main result of this section.

Proof. We split the proof into two steps. (Existence) For a given \( X \) satisfying the assumptions of the Theorem we let \( S \) be the associated Snell envelope of \( X \). Let \( \{ \delta^k X; k \geq 1 \} \) be the canonical GAS for \( X \) and let \( S = \{ S^k; k \geq 1 \} \) where \( S^k \) is the associated pure jump Snell envelope of \( \delta^k X \) for \( k \geq 1 \). From Lemma we know that \( \{ \delta^k X; k \geq 1 \} \) and \( S \) are two strong GAS for \( X \) and \( S \), respectively. From Lemmas (i), (ii), (iii) and Proposition we get

\[
\max \left\{ U_{S^k} S; \delta^k X - S^k \right\} = 0
\]

\[
\delta^k X(T) = S^k(T),
\]

up to an evanescent set for each \( k \geq 1 \). This proves the existence.

(Uniqueness). Let us assume the existence of \( X = \{ X^k; k \geq 1 \} \) and \( \tilde{S} = \{ \tilde{S}^k; k \geq 1 \} \) two strong non-negative GAS such that \( \lim_{k \to \infty} X^k = X \) and \( \lim_{k \to \infty} \tilde{S}^k = Z \) strongly in \( B^2(\mathbb{F}) \) for some \( Z \in B^1(\mathbb{F}) \). In addition, they satisfy the obstacle problem

\[
\max \left\{ U_{\tilde{S}^k} \tilde{S}; X^k - \tilde{S}^k \right\} = 0
\]

\[
X^k(T) = \tilde{S}^k(T),
\]

(6.24)

up to an evanescent set for every \( k \geq 1 \). We claim that \( Z = S \). In fact, from (6.24), we have both \( \tilde{S}^k \geq X^k \) and \( U_{\tilde{S}^k} S \leq 0 \) up to evanescent sets for each \( k \geq 1 \). Doob-Meyer decomposition yields \( \tilde{S}^k \) is an \( \mathbb{R}^k \)-supermartingale which dominates \( X^k \). Also, from (6.24) we have \( U_{\tilde{S}^k} S(X^k - \tilde{S}^k) = 0 \) up to an evanescent set. Therefore, for each \( n \geq 0 \), we have \( U_{\tilde{S}^k} S \mathbb{1}_{|T^k_n \leq \tilde{T}^k_n|} = 0 \) up to an evanescent set, which in turn implies

\[
N_{\tilde{S}^k}(\tilde{T}^k_n) - N_{\tilde{S}^k}(T^k_n) = \int_{T^k_n}^{\tilde{T}^k_n} U_{\tilde{S}^k} S(u)(A^k, A^k)(u) = 0 \ a.s \ n \geq 0,
\]

where \( \tilde{T}^k_n \) is the stopping time given in Lemma Also, from this Lemma, we shall state \( \tilde{S}^k \) must be the pure jump Snell envelope of \( X^k \). Lemma yields \( S = Z \) and we shall conclude the proof. \( \square \)
Under regularity assumptions, we shall obtain a strong solution. At first, let us present an immediate consequence of our previous results.

**Lemma 6.7.** Let \( X \) be a square-integrable Itô process \( X(0) + \int_0^t \mathcal{D}X(s)dB(s) + \int_0^t \mathcal{U}X(s)ds; 0 \leq t \leq T \). Let \( S = \{ S^k; k \geq 1 \} \) be the strong GAS for \( S \) given by (6.3) based on the canonical GAS \( \{ \delta^k X; k \geq 1 \} \). Then \( \{ U^{S,k} S; k \geq 1 \} \) is uniformly integrable in \( L^1(\text{Leb} \times \mathbb{P}) \).

**Proof.** From (6.20), we have \( | U^{S,k} S | \leq | U^k X | \) for every \( k \geq 1 \). An inspection in the proof of Theorem 4.1 allows us to state that if \( X \) is a square-integrable Itô process then \( \{ U^k X; k \geq 1 \} \) is a convergent sequence in \( L^1(\text{Leb} \times \mathbb{P}) \) (see (3.17)). This conclude the proof. \( \square \)

The next corollary is a simple consequence of our results and it recovers a well-known result when \( \mu = 0 \). It gives a simple view of the variational inequalities when \( X \) is regular. It gives a simple view of the variational inequalities when \( X \) is an Itô process.

**Corollary 6.2.** Let \( S \) be the Snell envelope associated to a square-integrable Itô process \( X \). Let \( S = \{ S^k; k \geq 1 \} \) be the pure jump Snell envelope family of the canonical GAS \( \{ \delta^k X; k \geq 1 \} \) of \( X \). Then \( S \) is the unique Itô process which solves

\[
\max \{ US; X - S \} = 0 \quad \text{Leb} \times \mathbb{P} \ a.s.
\]

Moreover, \( S \) is the weak stochastic solution of (6.25) and \( U^k S = US \).

**Proof.** Let \( X = X(0) + \int_0^T \mathcal{D}X(s)dB(s) + \int_0^T \mathcal{U}X(s)ds; 0 \leq t \leq T \) be a square-integrable Itô process. Then \( S \) is a square-integrable continuous semimartingale and hence Corollary 3.1 yields \( S \in W \). It follows from Lemma 2 in [16] that for each \( h > 0 \)

\[
(6.26) \quad | \mathcal{T}^I S(t + h) - \mathcal{T}^I S(t) | \leq \int_t^{t+h} | \mathcal{U}X(s) | \ ds \ a.s. \ 0 \leq t < h \leq T.
\]

Then, \( \mathcal{T}^I S \) has absolutely continuous paths a.s and there exists an \( \mathbb{F} \)-adapted process \( b \) such that \( \mathcal{T}^I S(t) = \int_0^T b(s)ds; 0 \leq t \leq T \). The inequality (6.20) also implies that \( E \int_0^T | b(s)|ds |^2 < \infty \). Theorem 4.1 yields the existence of \( US \). The fact that \( S \) is the unique solution (6.25) is a straightforward consequence of the Snell envelope property (see e.g Th.2.3.9 in [31]). From Lemma 6.7, we know that \( S \) has \( \mathbf{S} \)-finite energy. In addition, Lemma 6.7 yields \( \{ U^{S,k} S; k \geq 1 \} \) is uniformly integrable. Therefore, we shall apply Theorem 4.1 to get \( U^k S = US \) a.s \( \text{Leb} \times \mathbb{P} \). By considering the canonical GAS \( \{ \delta^k X; k \geq 1 \} \) for \( X \) and the associated family of pure jump Snell envelopes \( \{ S^k; k \geq 1 \} \), we apply Lemmas 6.1, 6.2, 6.3, 6.5 and 6.4 to conclude that \( S \) is the stochastic weak solution of (6.25). \( \square \)

Let us give an example of a fully non-Markovian optimal stopping problem.

### 6.1. A non-Markovian example: Optimal stopping with fractional Brownian motion

Let \[ X_H(t) = e^{-rt} f(B_H(t)); 0 \leq t \leq T, \]

where \( r \) is a positive constant, \( f \in C_c^\infty(\mathbb{R}) \) is a non-negative real-valued function and \( B_H \) is the fractional Brownian motion (henceforth abbreviated by FBM) with exponent \( 0 < H < 1 \). For simplicity, we take \( 1/2 \leq H < 1 \) since the argument for \( 0 < H < 1/2 \) is entirely analogous. At first, we need to choose a GAS for the FBM. In principle, any representation of the FBM can be used by replacing the underlying Brownian motion by (2.2). In the sequel, for simplicity of exposition, we choose the FBM representation as described in Corollary 5 in [6]. Let \( S_H \) be the Snell envelope associated to \( X_H \). One natural GAS for \( X_H \) can be constructed as follows. Let \( B^k_H \) be a pure jump process defined by

\[
B^k_H(T_n^k) := \frac{\sin \pi(H - k)}{\pi} \int_0^{\infty} x^{\pi - H} Y_{H,k}(x, T_n^k)dx \quad \text{on} \{ T_n^k \leq t < T_{n+1} \}, n \geq 0
\]
where \( Y_{H,k}(x, u) := \int_0^u W_{H,k}(x, t) \ell^H - \frac{1}{2} dl \) and

\[
W_{H,k}(x, u) := \int_0^u e^{-x(u-s)} s^{H} dA^k(s); \ x \in \mathbb{R}, 0 \leq u \leq T.
\]

The integral in (6.27) is interpreted in the Lebesgue-Stieltjes sense. By using (2.5), one can easily check that \( \{ B^k_H; k \geq 1 \} \) is indeed a strong GAS for \( B^k_H \) so that \( X^k_H(T_n) := e^{-r\tau} f(B^k_H(T_n)) \) on \( \{ T_n \leq t < T_{n+1} \}; n \geq 0, k \geq 1 \) is a strong GAS for \( X^k_H \). The discrete weak infinitesimal generator \( U^{S,k}S_H \) along the GAS \( S^k_H; k \geq 1 \) at the stopping times \( \{ T_n^k; n \geq 1 \} \) is

\[
U^{S,k}S_H(T_n^k) = \frac{1}{2 \pi^{2k}} \mathbb{E}[S^k_H(T_n^k) - S^k_H(T_{n-1}) | \mathcal{F}_{T_n^k}]; n \geq 1,
\]

where \( S^k_H(T_n^k) := \text{ess sup}_{x \in \mathbb{R}} \mathbb{E}[e^{-r\tau} f(B^k_H(\tau)) | \mathcal{F}_{T_n^k}] \) on \( \{ T_n^k \leq t < T_{n+1}^k \}; n \geq 0 \). Then \( S^k_H \) is the unique solution of

\[
\max \{ U^{S,k}S_H; X^k_H - S^k_H \} = 0 \quad \quad X^k_H(T) = S^k_H(T),
\]

up to an evanescent set for each \( k \geq 1 \). The solution of (6.28) on \( \{ T_n^k; n \geq 1 \} \) allows us to reconstruct the Snell envelope of the GAS \( \{ X^k_H; k \geq 1 \} \) by using similar Monte Carlo-type schemes as employed by [33] for stochastic derivatives.

7. Functional Stochastic Calculus

In this section, we interpret \( (\mathbb{D}^{Y,k,1}X, \ldots, \mathbb{D}^{Y,k,p}X) \) and \( (\mathbb{U}^{Y,k,1}X, \ldots, \mathbb{U}^{Y,k,p}X) \) in terms of functionals acting on the “vector bundle” \( \Lambda = \bigcup_{0 \leq t \leq T} D([0, t]; \mathbb{R}^p) \), for a given GAS \( \mathcal{Y} \). For simplicity of exposition, we only consider the unidimensional case \( p = 1 \). The general multi-dimensional case easily follows from this case by noticing that

\[
\bigcup_{r=0}^{\infty} [ T^k_{t}, T^k_{t} ] = \bigcup_{r=1}^{p} \bigcup_{r=0}^{\infty} [ T^{k,j}_{t}, T^{k,j}_{t} ].
\]

In this section, we connect the pathwise functional calculus (see [14, 33]) to our weak differential structure. In the sequel, we recall the following notation: if \( x \) is a càdlàg path, then \( x_t = \{ x(r); 0 \leq r \leq t \} \) where \( x(r) \) is the value of \( x \) at time \( r \in [0, T] \). This notation is extended to stochastic processes.

7.1. Splitting weak infinitesimal generators. Let \( X \) be a Wiener functional and let \( \mathcal{Y} = \{ X^k; k \geq 1 \} \) be a GAS for \( X \). By Doob’s Theorem, there exist functionals \( \hat{F} = \{ \hat{F}_t; 0 \leq t \leq T \} \) and \( F^k = \{ F^k_t; 0 \leq t \leq T \} \) defined on \( \tilde{\Lambda} \) and \( \Lambda \), respectively, such that

\[
\hat{F}_t(B_t) = X(t), \quad F^k_t(A^k_t) = X^k(t); 0 \leq t \leq T.
\]

In the remainder of the paper, when we write \( F(B) \) for a given family \( F_t : D([0, t]; \mathbb{R}) \rightarrow \mathbb{R}; 0 \leq t \leq T \) it is implicitly assumed that we are fixing a functional \( \{ F_t; 0 \leq t \leq T \} \) which is consistent to \( \{ \hat{F}_t; 0 \leq t \leq T \} \) in the sense that

\[
F_t(x_t) = \hat{F}_t(x_t) \text{ for every } x \in \tilde{\Lambda}.
\]

Since we are dealing only with adapted processes, throughout this section we assume that all functionals are non-anticipative in the following sense. In the sequel, \( (\mathcal{B}_t)_{t \geq 0} \) is the canonical filtration on \( D([0, t]; \mathbb{R}) \); \( t \geq 0 \) and \( \mathbb{H}[0, T] \) is the Borel-sigma algebra on \( [0, T] \).

**Definition 7.1.** A non-anticipative functional \( \{ F_t; 0 \leq t \leq T \} \) is a family of mappings such that

\[
F_t : D([0, t]; \mathbb{R}) \rightarrow \mathbb{R}
\]
is \( \mathcal{B}_t \)-measurable for each \( t \in [0, T] \).
Lemma 7.1. For any GAS $\mathcal{Y} = \{X^k; k \geq 1\}$, let $F$ and $F^k$ be fixed functional representations of $X$ and $X^k$, respectively, which realize (7.1). Let $(L^k; \mathcal{Y} F(B), U^k; \mathcal{Y} F(B))$ be the operators defined in (2.16) and (2.17), respectively. We now want to split $U^k; \mathcal{Y} F(B)$ into two components which encode different modes of regularity. In the sequel, it is convenient to introduce the following perturbation scheme

$$A^k_{t-}i^{2-k}(u) = A^k(u), \quad 0 \leq u < t \quad \text{and} \quad A^k_{t-}i^{2-k}(t) = A^k(t-) + i^{2-k},$$

for $i = -1, 0, 1$ and to shorten notation, we denote $A^k_{t-} = A^k_{t-0}$ for $0 \leq t \leq T$. Since $X^k$ is a pure jump process, then

$$F^k_t(A^k_t) = \sum_{\ell = 0}^{\infty} F^k_{t-}\left[A^k_{\tau_{n-1}}(A^k_{\tau_{n-1}})\right] \mathbb{1}(T^k_{\tau_{n-1}} \leq t \leq T_{\tau_{n-1}}), \quad 0 \leq t \leq T. \quad (7.2)$$

A simple calculation based on the i.i.d. family $\{B(T^k_n) - B(T^k_{n-1}); n, k \geq 1\}$ of Bernoulli variables with parameter 1/2 yields the following splitting.

Lemma 7.1. For any GAS $\mathcal{Y}$ for $X$ with a functional representation (7.1), we have

$$U^k; \mathcal{Y} F(B) = D^k; \mathcal{Y} F(B) + \frac{1}{2} D^k; \mathcal{Y}^2 F(B)$$

where

$$D^k; \mathcal{Y} F(B) := \sum_{n=1}^{\infty} \frac{1}{2^{2k}} [F^k_t(A^k_{\tau_{n-1}}) \mathbb{1}(T^k_{\tau_{n-1}} \leq t \leq T_{\tau_{n-1}}) - F^k_t(T^k_{\tau_{n-1}})(A^k_{\tau_{n-1}}) \mathbb{1}(T^k_{\tau_{n-1}} \leq t \leq T_{\tau_{n-1}})]$$

and

$$D^k; \mathcal{Y}^2 F(B) := \frac{1}{2^{2k}} [F^k_t(A^k_{\tau_{n-1}}) + 2F^k_t(A^k_{\tau_{n-1}}) - 2F^k_t(A^k_{\tau_{n-1}})] \quad 0 \leq t \leq T \quad (7.3)$$

are $\mathcal{F}^k$-predictable processes.

Proof. The proof is a simple computation as follows: For a given $k \geq 1$,

$$U^k; \mathcal{Y} F(B) = \frac{1}{2^{2k}} \sum_{n=1}^{\infty} \mathbb{E} [F^k_t(A^k_{\tau_{n-1}}) - F^k_t(T^k_{\tau_{n-1}})(A^k_{\tau_{n-1}})] \mathbb{1}(T^k_{\tau_{n-1}} \leq t \leq T_{\tau_{n-1}}) = \frac{1}{2^{2k}} \sum_{n=1}^{\infty} \mathbb{E} [F^k_t(A^k_{\tau_{n-1}}) - F^k_t(T^k_{\tau_{n-1}})(A^k_{\tau_{n-1}})] \mathbb{1}(T^k_{\tau_{n-1}} \leq t \leq T_{\tau_{n-1}}) = \frac{1}{2^{2k}} \sum_{n=1}^{\infty} \mathbb{E} [F^k_t(A^k_{\tau_{n-1}} - T^k_{\tau_{n-1}})(A^k_{\tau_{n-1}})] \mathbb{1}(T^k_{\tau_{n-1}} \leq t \leq T_{\tau_{n-1}}) \quad (7.4)$$

Since $F^k_t(A^k_{\tau_{n-1}}) - F^k_t(T^k_{\tau_{n-1}})(A^k_{\tau_{n-1}}) \mathbb{1}(T^k_{\tau_{n-1}} \leq t \leq T_{\tau_{n-1}})$ is $\mathcal{F}^k$-predictable for each $n \geq 1$, then $D^k; \mathcal{Y} F(B)$ is $\mathcal{F}^k$-predictable. The predictability of $U^k; \mathcal{Y} F(B)$ yields $D^k; \mathcal{Y}^2 F(B)$ is predictable as well.
If \( \mathcal{Y} \) is the canonical GAS for a given Wiener functional \( X = F(B) \), then we write \( \mathcal{D}^{k,h} F(B) \) and \( \mathcal{D}^{k,2} F(B) \) so that

\[
U^k F(B) = \mathcal{D}^{k,h} F(B) + \frac{1}{2} \mathcal{D}^{k,2} F(B).
\]

**Remark 7.1.** The operators \( \mathcal{D}^{Y,k,h} \) and \( \mathcal{D}^{Y,k,2} \) encode variation in the same spirit of the horizontal and second order vertical derivatives used in the pathwise functional calculus, but with one fundamental difference: In contrast to the pathwise calculus where shifts are deterministic, the increments in the operators \( \{7.3\} \) and \( \{7.2\} \) are driven by the stopping times \( \{T_n^k; n \geq 1\} \) and the Bernoulli variables \( \{B(T_n^k) - B(T_{n-1}^k); n \geq 1\} \).

**Example:** For a given \( F \in \Lambda \), let \( F = \{F^k; k \geq 1\} \) be the sequence of pure jump processes given in \( \{2.2\} \). Under mild conditions (see Lemma \( \{7.2\} \), this is a GAS for \( F(B) \). Since for any functional representation \( \{F^k; k \geq 1\} \) for \( F \) which satisfies \( \{7.1\} \) and \( \{7.2\} \), we have \( F_{T_n^k}^k(A^k) = F_{T_n^k}^k(A_{T_{n-1}^k}^k) \), then from the computation in the proof of Lemma \( \{7.1\} \) we readily see that

\[
\mathcal{D}^{F,k,h} F_t(B_t) = \sum_{n=1}^{\infty} \frac{1}{2} \mathbb{E} \left[ F_t(A^k_{T_n^k}) - F_t(A_{T_{n-1}^k}^k) \mathbb{1}_{\{T_n^k - t \leq T\}} \right]
\]

and

\[
\mathcal{D}^{F,k,2} F_t(B_t) = \frac{1}{2} \mathbb{E} \left[ F_t(A^k_{T_n^k}) + F_t(A_{T_{n-1}^k}^k) - 2 F_t(A_{T_n^k}^k) \right]; 0 \leq t \leq T.
\]

**Corollary 7.1.** Let \( \mathcal{Y} \) be a GAS for \( X \) with functional representations given by \( \{7.1\} \) and \( \{7.2\} \). Under the assumption of Proposition \( \{2.7\} \), Lemma \( \{7.1\} \) yields the following functional representation of the GAS \( \mathcal{Y} \): If

\[
E \int_0^T |\mathcal{D}^{Y,k,2} F_s(B_s)| d(A^k)(s) < \infty; \quad k \geq 1,
\]

then

\[
F_t^k(A^k) = F_0^k(A^k) + \int_0^t \mathcal{D}^{Y,k} F_s(B_s) dA^k(s) + \int_0^t \left( \mathcal{D}^{Y,k,h} F_s(B_s) + \frac{1}{2} \mathcal{D}^{Y,k,2} F_s(B_s) \right) d(A^k)(s)
\]

for \( 0 \leq t \leq T, \quad k \geq 1. \)

It is convenient to concentrate the whole information on the hitting times as follows. Let us define

\[
\mathbb{D}^{Y,k,h} F(B) := \sum_{n=1}^{\infty} \mathcal{D}^{Y,k,h} F_{T_n^k}(B_{T_n^k}) \mathbb{1}_{[T_n^k, T_{n+1}^k]}; \quad \mathbb{D}^{Y,k,2} F(B) := \sum_{n=1}^{\infty} \mathcal{D}^{Y,k,2} F_{T_n^k}(B_{T_n^k}) \mathbb{1}_{[T_n^k, T_{n+1}^k]};
\]

so that \( \mathbb{D}^{Y,k} F(B) = \mathbb{D}^{Y,k,h} F(B) + \frac{1}{2} \mathbb{D}^{Y,k,2} F(B) \).

Let \( \mathcal{W}^{1,2} \) be the set of Wiener functionals \( F(B) \in \mathbb{W} \) such that there exists a \( \mathcal{Y} \)-finite energy GAS such that both

\[
\mathcal{D}^{Y,h} F(B) := \lim_{k \to \infty} \mathbb{D}^{Y,k,h} F(B), \quad \mathcal{D}^{Y,2} F(B) := \lim_{k \to \infty} \mathbb{D}^{Y,k,2} F(B)
\]

exist weakly in \( L^1(\mathbb{P} \times \text{Leb}) \). Theorem \( \{4.1\} \) and the definition of \( \mathcal{W}^{1,2} \) lead to

\[
\mathcal{W}^{1,2} \subset \mathbb{W} \subset \mathcal{W}
\]

and the following result.
Proposition 7.1. If $F(B) \in \mathcal{W}^{1,2}$ then the resulting semimartingale decomposition is given by

\[
F(t) = F(0) + \int_0^t \mathcal{D}F_s(B_s) dB(s) + \int_0^t \mathcal{U}F_s(B_s) ds; \quad 0 \leq t \leq T,
\]

where

\[
\mathcal{U}F(B) = \mathcal{D}_{\ref{sec:Comparison}} F(B) + \frac{1}{2} \mathcal{D}_{\ref{sec:Comparison}}^2 F(B),
\]

for every $\mathcal{Y}$-finite energy GAS of $F(B)$ such that both $\mathcal{D}_{\ref{sec:Comparison}} F(B)$ and $\mathcal{D}_{\ref{sec:Comparison}}^2 F(B)$ exist.

Remark 7.2. (i) It is not difficult to see the splitting ($\mathcal{D}_{\ref{sec:Comparison}} F(B), \mathcal{D}_{\ref{sec:Comparison}}^2 F(B)$) really depends on the GAS. It may happen that a given process $X = F(B)$ may admit two GAS such that both $\mathcal{D}_{\ref{sec:Comparison}} F(B)$ and $\mathcal{D}_{\ref{sec:Comparison}}^2 F(B)$ exist but they are not indistinguishable. A similar remark holds for $\mathcal{D}_{\ref{sec:Comparison}}^2 F(B)$ and $\mathcal{D}_{\ref{sec:Comparison}}^2 F(B)$. Examples of this fact can be easily constructed from the cases discussed by Oberhauser [37].

(ii) We do not require $\mathcal{D}(DF(B)) = \mathcal{D}_{\ref{sec:Comparison}}^2 F(B)$ for a GAS $\mathcal{Y}$ w.r.t $F(B)$, although this is true in typical cases. See Proposition 7.2.

Let us continue to investigate the relation between the weak differential structure developed in Section 3 and the standard pathwise Itô calculus introduced by Dupire [14].

7.2. Comparison with the pathwise functional Itô calculus. In the sequel, $\mathcal{W}^{1,2}_{\text{loc}}$ and $L^2_{\text{loc}}(\mathbb{P} \times \text{Leb})$ denote the localization of the linear spaces $\mathcal{W}^{1,2}$ and $L^2(\mathbb{P} \times \text{Leb})$, respectively. In the sequel, we recall that $\nabla^h$, $\nabla^v$ and $\nabla^{v,(2)}$ denote the horizontal, vertical and the second order derivative, respectively, in the sense of pathwise functional calculus. We refer the reader to Dupire [14] and Cont and Fournie [8] for details about these operators. For reader’s convenience, let us recall some basic objects of the functional calculus. A natural metric on $\Lambda$ is given by

\[
d_{\infty}(t, w; (s, v)) := |t - s| + \sup_{0 \leq u \leq T} |w_{t,T-u}(u) - v_{s,T-u}(u)|;
\]

for $(w, v)$ in $\Lambda \times \Lambda$. Here and in the sequel, we denote

\[
w_{t,\gamma}(u) := w(u); 0 \leq u \leq t \quad \text{and} \quad w_{t,\gamma}(u) := w(t); t < u \leq t + \gamma,
\]

\[
w^h_t(u) := w(u); 0 \leq u < t \quad \text{and} \quad w^h_t(t) := w(t) + h;
\]

and

\[
w^h_{t-}(u) := w(u); 0 \leq u < t \quad \text{and} \quad w^h_{t-}(t) := w(t) + h;
\]

for $h \in \mathbb{R}, \gamma > 0$ and $w \in \Lambda$. In the sequel, continuity of functionals is defined as follows (see e.g [8]):

Definition 7.2. A functional $F = \{F_t; 0 \leq t \leq T\}$ is said to be continuous at $x \in \Lambda$ if $\forall \varepsilon > 0$, there exists $\delta(x, \varepsilon) = \delta > 0$ such that if $y \in \Lambda$ satisfies $d_{\infty}(x, y) < \delta$ then $|F_t(x) - F_t(y)| < \varepsilon$. We say that $F$ is continuous if it is continuous for each $x \in \Lambda$. We say that $F$ is uniformly continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|F_t(x) - F_t(y)| < \varepsilon$ whenever $d_{\infty}(t, x; (s, y)) < \delta$.

The following assumptions will be sufficient for the construction of a GAS which connects with the pathwise functional calculus introduced by Dupire.

(H1) $F = \{F_t; 0 \leq t \leq T\}$ is a continuous functional.

We also recall a suitable notion of “boundedness-preserving” property as follows.

Definition 7.3. We say that a functional $F$ is a boundedness-preserving functional if for every compact subset $K$ of $\mathbb{R}$, there exist $C_K > 0$ such that $|F(x)| \leq C_K$ for every $x \in D([0, T]; K)$. 


Such notion will be important for the construction of natural GAS starting with a given functional $F$, so we consider the following assumption

(H2) $F = \{F_t; 0 \leq t \leq T\}$ is a boundedness-preserving functional.

For a given functional $F \in \Lambda$, let us now recall the sequence $\mathcal{F} = \{F^k; k \geq 1\}$ introduced in (2.10) given by

$$F^k(t) = \sum_{\ell=0}^{\infty} F_{T^k}\left(\mathbb{A}^k_{\ell} \mathbb{I}_{\{T^k_{\ell} \leq t < T^k_{\ell+1}\}}\right), \quad 0 \leq t \leq T.$$

**Lemma 7.2.** Let $F$ be a representative functional of a Wiener functional $X$ which satisfies (H1-H2). Then $\mathcal{F} = \{F^k; k \geq 1\}$ is a GAS of $F(B)$.

**Proof.** We only need to prove that $F^k \to F(B)$ weakly in $\mathcal{B}^2(\mathbb{F})$ as $k \to \infty$. By assumption (H2) holds, so by stopping, we shall assume that $\sup_0 \sup_{0 \leq t \leq T} [F^k(t)]$ and $\sup_0 \sup_{0 \leq t \leq T} |F_t(B_t)|$ are bounded. Then $\{\sup_0 \sup_{0 \leq t \leq T} |F^k(t)|; k \geq 1\}$ is a sequence of uniformly integrable random variables. From (2.5) and (H1), for any bounded $F$-stopping time $S$ we have

$$F^k(S) = \sum_{n=0}^{\infty} F_{T^k}\left(\mathbb{A}^k_n \mathbb{I}_{\{T^k_n \leq S < T^k_{n+1}\}}\right) \to F_S(B_S)$$

strongly in $L^1$ as $k \to \infty$. Then, from Remark 3.2 in [32] we know that $\lim_{k \to \infty} F^k = F(B)$ weakly in $\mathcal{B}^1(\mathbb{F})$. Since $\{F^k; k \geq 1\}$ is bounded in $\mathcal{B}^2(\mathbb{F})$, then we shall proceed similar to the argument made in [equation (3.6) in [32]] to conclude that we do have weak convergence in $\mathcal{B}^2(\mathbb{F})$. 

Let $C^{1,2}_b(\Lambda)$ be the set of functionals satisfying (H1-H2) and which are pathwise horizontally and twice vertically differentiable in $\Lambda$ such that all these derivatives are continuous functionals and preserve boundedness, such that $\nabla v(2)F$ is uniformly continuous.

**Proposition 7.2.** Let $F \in C^{1,2}_b(\Lambda)$ and let $\mathcal{F} = \{F^k(t); 0 \leq t \leq T\}$ be the GAS of $F(B)$ given by (2.7). Assume that $\mathcal{D}^{F,k,h}F(B)$ and $\mathcal{D}^{F,k,2}F(B)$ are locally bounded in $L^2(\mathbb{F} \times \text{Leb})$. Then $F(B) \in \mathcal{W}^{1,2}_{\text{loc}}$ and

$$\nabla v F(B) = \mathcal{D} F(B), \quad \nabla h F(B) = \mathcal{D}^{F,h} F(B) \quad \text{and} \quad \nabla v(2) F(B) = \mathcal{D}^{F,2} F(B).$$

**Proof.** By assumption $F \in C^{1,2}_b(\Lambda)$ and from Theorem 1 in [14], we know that $F(B)$ is a locally-square integrable Itô process. By a routine localization procedure, we shall assume that it is actually square-integrable and Theorem [13] says that $F(B)$ satisfies $\mathcal{H}$. Therefore, by uniqueness of the special semimartingale decomposition, we must have

$$\nabla v F(B) = \nabla v F(B) \quad \text{and} \quad \nabla h F(B) = \nabla h F(B) + \nabla v(2) F(B).$$

Let us now check that $\mathcal{D}^{F,k,h} F(B) = \nabla h F(B)$ and $\mathcal{D}^{F,k,2} F(B) = \nabla v(2) F(B)$. By linearity, it is sufficient to show that $\mathcal{D}^{F,2} F(B)$ and $\nabla v(2) F(B)$ are indistinguishable. To keep notation simple, we set $\varphi(u) := F_{T^k}\left(A^k_{\ell} \mathbb{I}_{\{T^k_{\ell} \leq u < T^k_{\ell+1}\}}\right)$; $u \in \mathbb{R}$. By assumption $F \in C^{1,2}_b(\Lambda)$ so that $\varphi \in C^2(\mathbb{R})$. For a given $h > 0$, the usual mean value theorem provides a bounded random function $u \mapsto \zeta(\varphi, h, u)$ ($0 < \zeta(\varphi, h, u) < 1$ a.s $\forall h > 0$, $u \in \mathbb{R}$) such that

$$(7.8) \quad \frac{1}{h^2}(\varphi(h) + \varphi(-h) - 2\varphi(0)) = \frac{1}{h} \int_0^h \Delta \varphi((u-h) + \zeta(\varphi, h, u)h) \, du \text{ a.s}$$

To shorten notation, let us define the random paths $a^k_n(u) := A^k_{T^{-}\Delta} + \zeta(2^{-k}, u)$ for $u \in \mathbb{R}; k, n \geq 1$. From (7.8), we shall write
\[ D^{r, k, 2}F_t(B_t) = \sum_{n=1}^{\infty} \frac{1}{2^k} \int_{0}^{2^{-k}} \nabla v^{(2)} F_{T^k_n}(a_n^k(u))du 1_{\{T^k_n < t \leq T^k_{n+1}\}}; \quad 0 \leq t \leq T. \]

Let us fix \( t \in (0, T] \). Let \( \tau^k_t = \max\{T^k_t; T^k_t < t\} \) and \( \tau^k_{t-} = \min\{T^k_t; T^k_t \geq t\} \) already introduced in the proof of Theorem 4.1. We know that

\[ \lim_{k \to \infty} \mathbb{E}[r^k_{t+} - r^k_{t-}] = 0 \]

On the one hand, Lebesgue differentiation theorem and the fact that \( u \mapsto \nabla v^{(2)} F_t(B_t^u) \) is continuous yield the following a.s. convergence \( \frac{1}{2^k} \int_{0}^{2^{-k}} \nabla v^{(2)} F_t(B_t^u)du \to \nabla v^{(2)} F_t(B_t) \) as \( k \to \infty \). On the other hand, for a given \( \frac{2}{\varepsilon} > 0 \), let \( \delta > 0 \) be the constant such that \( |\nabla v^{(2)} F_t(x_r) - \nabla v^{(2)} F_t(y_u)| < \frac{\varepsilon}{2} \) whenever \( d_\infty((r, x); (s, y)) < \delta \). Now we observe that \( (u - 2^{-k}) + \zeta(n, 2^{-k}, u)2^{-k} \to u \) a.s as \( k \to \infty \) uniformly on compact subsets of \( \mathbb{R} \). Then we shall take advantage of (2.5) and (7.9) (possibly by taking a subsequence for an almost sure convergence) to state that for a given \( \omega \in \Omega \), we shall find \( k \geq 1 \) large enough such that

\[ \left| D^{r, k, 2}F_t(B_t)(\omega) - \frac{1}{2^{-k}} \int_{0}^{2^{-k}} \nabla v^{(2)} F_t(B_t^u(\omega))du \right| < \frac{\varepsilon}{2}. \]

Triangle inequality yields \( \lim_{k \to \infty} D^{r, k, 2}F_t(B_t)(\omega) = \nabla v^{(2)} F_t(B_t(\omega)) \) for almost all \( (\omega, t) \in \Omega \times [0, T] \) and therefore, \( \lim_{k \to \infty} D^{r, k, 2}F(B) = \nabla v^{(2)} F(B) \) weakly in \( L^2(\mathbb{P} \times \text{Leb}) \) if, and only if, the boundedness assumption \( \sup_{k \geq 1} \mathbb{E} \int_{0}^{T} |D^{r, k, 2}F_t(B_t)|^2dt < \infty \) holds. This concludes the proof.

An obvious consequence of the previous results is the following corollary.

**Corollary 7.2.** Let \( F \in C^{1, 2}_0(\Lambda) \) be a non-anticipative functional such that \( F(B) \in \mathcal{B}^2(\mathbb{P}) \) is a smooth strong pathwise solution of

\[ \begin{cases} \nabla h F_t(B_t) + \frac{1}{2} \nabla^2 v^{(2)} F_t(B_t) + f(t, F_t(B_t), \nabla v F_t(B_t)) = 0; & 0 \leq t \leq T, \\
F_t(B_T) = \xi \in L^2(\mathbb{P}), \end{cases} \]

then it is a solution of the path-dependent PDE (5.3) where \( \mathcal{U}F(B) = \nabla h F(B) + \frac{1}{2} \nabla^2 v^{(2)} F(B) \) and \( \mathcal{D}F(B) = \nabla^2 F(B) \).

8. Local Time Representation for Path-Dependent Functionals

In this section, we explore weak differentiability when there is no weak infinitesimal generator in the sense of Section 4.1, so we go back to the largest space \( W \). The first step towards a local time representation for a path-dependent Wiener functional is to represent the second-order derivative operator (8.3) into a first order one combined with suitable occupation times of the pure jump process \( A_k \). To this end, let us introduce a pathwise modification of the terminal values of a given path as follows. In the sequel, for a given \( s \in [0, T] \), let us define

\[ D([0, s]; \mathbb{R}) \times \mathbb{R} \to D([0, s]; \mathbb{R}) \]

where we set

\[ \mathbf{t}(\eta, s, x)(u) := \begin{cases} \eta(u); & \text{if } 0 \leq u < s \\
x; & \text{if } u = s. \end{cases} \]

Based on this operation, we shall define the following \( \mathbb{P}^k \)-adapted pathwise “terminal value” modification of \( A_k \) as follows.
for $0 \leq s \leq T$, $j \in \mathbb{Z}$ and $i = -1, 0, 1$. With these objects at hand, if $F(B)$ admits a GAS $\mathcal{Y}$ with a given functional representation (7.1), then we define

$$ (8.1) \quad \nabla^{\mathcal{Y},k,v} F_s(B_s) := \sum_{j \in \mathbb{Z}} \frac{F^k_s(t(A^k_s, j2^{-k})) - F^k_s(t(A^k_s, (j-1)2^{-k}))}{2^{-k}} \mathbb{I}_{(12^{-k}, j2^{-k})}; \, k \geq 1, $$

for $0 \leq s \leq T$. For instance, if we take the associated functional GAS $\mathcal{F} = \{F^k; k \geq 1\}$, then

$$ (8.3) \quad \nabla^{\mathcal{F},k,v} F_s(B_s) := \sum_{j \in \mathbb{Z}} \frac{F^k_s(t(A^k_s, j2^{-k})) - F^k_s(t(A^k_s, (j-1)2^{-k}))}{2^{-k}} \mathbb{I}_{(12^{-k}, j2^{-k})}; \, k \geq 1, $$

for $0 \leq s \leq T$. If the GAS $\mathcal{Y}$ for a given Wiener functional $F(B)$ is the canonical one, then we write $\nabla^{\mathcal{Y},v} F(B)$.

Remark 8.1. The operator $\nabla^{\mathcal{Y},k,v}$ is reminiscent from the vertical derivative introduced by Dupire [14] but with one fundamental difference: We are localizing the terminal values on $\mathbb{R}$ by a refining partition and, more importantly, the past of the Brownian motion up to these terminal values is replaced by the stepwise martingale $A^k$. In doing this, we can take advantage of the weak differential structure induced by $(A^k, F^k)$.

In order to switch second-order derivative $\mathcal{D}^{\mathcal{Y},k,2}$ into (8.1), the following natural notion of occupation times plays a key role

$$ (8.2) \quad \ell^{k,x}(t) := \frac{1}{2^{k}} \int_0^t \mathbb{I}_{|A^k(s-x)| < 2^{-k}} d(A^k(s)); \, k \geq 1, \, x \in \mathbb{R}, \, 0 \leq t \leq T. $$

Instead of Lebesgue measure, the occupation time (8.2) is computed by a different clock $\langle A^k \rangle$ which has absolutely continuous paths. See Lemma 2.4 in [32] for more details.

In order to express our preliminary functional Itô formula, we need a natural notion of integration w.r.t the occupation times $\ell^k$. If $H = \{H(t,x); (t,x) \in [0,T] \times \mathbb{R}\}$ is a simple random field of the form

$$ H(t) = \sum_{j \in \mathbb{Z}} \alpha_j(t) \mathbb{I}_{(12^{-k}, j2^{-k})} $$

for some measurable process $\alpha_j : \Omega \times [0,T] \to \mathbb{R}$ then integration w.r.t $\{\ell^{k,x}(t); x \in \mathbb{R}, t \in [0,T]\}$ is naturally defined by

$$ (8.3) \quad \int_0^t \int_{\mathbb{R}} H(s,x) d(\ell^k,x)(s) := \sum_{j \in \mathbb{Z}} \int_0^t \alpha_j(s) \left[ d_x \ell^{k,j2^{-k}}(s) - d_x \ell^{k,(j-1)2^{-k}}(s) \right] $$

for $0 \leq t \leq T$, whenever the right-hand side of (8.3) is finite a.s.

Proposition 8.1. Let $X = F(B)$ be a Wiener functional satisfying the assumption of Proposition 2.1 and let $\mathcal{Y} = \{X^k; k \geq 1\}$ be a GAS for $F(B)$ with a functional representation (7.2) satisfying (7.5). Then

$$ (8.4) \quad \frac{1}{2} \int_0^t \mathcal{D}^{\mathcal{Y},k,2} F_s(B_s) d(A^k)(s) = -\frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} \nabla^{\mathcal{Y},k,v} F_s(B_s, x) d(\ell^{k,x})(s); \, 0 \leq t \leq T, $$
and hence, the following decomposition holds

\[
F_k^x(A^k_{\cdot}) = F_0^x(B_0) + \int_0^t \mathbb{D}^{Y,k} F_s(B_s) d A^k(s) + \int_0^t \mathbb{D}^{Y,k,k} F_s(B_s) d (A^k)(s) - \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} \nabla^{Y,k,v} F_s(B_s, x) d(s,x) \mathbb{I}^{k,v}(s); \quad 0 \leq t \leq T,
\]
for each \(k \geq 1\).

**Proof.** Let us fix \(k \geq 1\). By the very definition, for a given \(j \in \mathbb{Z}\) and \(s \in [0,T]\), \(t(A^k_s, (j + i)2^{-k}) = A^k_{s+2^{-k}}\) on \(\{A^k(s-) = j2^{-k}\}\) for each \(i = -1,0,1\). To shorten notation, let us introduce the following notation

\[
\Delta F_{s,j,k} := F_s^k(t(A^k_s, (j + 1)2^{-k})) - F_s^k(t(A^k_s, j2^{-k})), \quad b^{k,j}(s) := \frac{1}{2-k} \mathbb{I}(\|A^k(s-) - j2^{-k}\| < 2^{-k})
\]
for \(k \geq 1\), \(j \in \mathbb{Z}\), \(0 \leq s \leq T\). By the very definition, we shall write

\[
\mathbb{D}^{Y,k,2} F_s(B_s) = \sum_{j \in \mathbb{Z}} \left( \frac{\Delta F_{s,j,k} - \Delta F_{s,j-1,k}}{2-k} \right) \frac{1}{2-k} \mathbb{I}\{A^k(s-) = j2^{-k}\} a.s; \quad 0 \leq s \leq T.
\]

For a given positive integer \(m \geq 1\), let \(S^k_m := \inf\{0 \leq s \leq T; |A^k(s)| > 2^m\}\). Then

\[
\mathbb{D}^{Y,k,2} F_s(B_s) = \sum_{j=-2^m}^{2^k+m-1} \left( \frac{\Delta F_{s,j,k} - \Delta F_{s,j-1,k}}{2-k} \right) \frac{1}{2-k} \mathbb{I}\{A^k(s-) = j2^{-k}\} a.s; \quad 0 \leq s < S^k_m.
\]

For a given \(0 \leq s < S^k_m\), we perform a pathwise summation by parts on the set \([-2^m, 2^m] \cap 2^{2-k}\) as follows

\[
\frac{1}{2} \sum_{j=-2^k+m-1}^{2^k+m-1} \left( \frac{\Delta F_{s,j+k,j} - \Delta F_{s,j,k}}{2-k} \right) b^{k,j+1}(s) = \frac{1}{2} \left[ \frac{\Delta F_{s,2^k+m-1} - \Delta F_{s,2^k+1}}{2-k} \right] b^{k,2^k+m-1}(s) - \frac{1}{2} \sum_{j=-2^k+m-1}^{2^k+m-1} \frac{\Delta F_{s,j,k}}{2-k} [b^{k,j+1}(s) - b^{k,j}(s)] a.s,
\]

where \(b^{k,2^k+m}(s,\omega) = b^{k,2^k+m-1}(s,\omega) = 0\) if \(0 \leq s < S^k_m(\omega)\). Then

\[
\frac{1}{2} \sum_{j=-2^k+m-1}^{2^k+m-1} \left( \frac{\Delta F_{s,j+1,k} - \Delta F_{s,j,k}}{2-k} \right) b^{k,j+1}(s) = \frac{1}{2} \sum_{j=-2^k+m-1}^{2^k+m-1} \frac{\Delta F_{s,j,k}}{2-k} [b^{k,j+1}(s) - b^{k,j}(s)] a.s \quad \text{for } 0 \leq s < S^k_m.
\]

For a given \(0 \leq t \leq T\), \((8.7)\) yields

\[
\frac{1}{2} \int_{[0, S^k_m \wedge t]} \mathbb{D}^{Y,k,2} F_s(B_s) d(A^k)(s) = \frac{1}{2} \int_{[0, S^k_m \wedge t]} \int_{-\infty}^{+\infty} \nabla^{Y,k,v} F_s(B_s, x) d(s,x) \mathbb{I}^{k,v}(s) a.s
\]
for each \(m \geq 1\). By taking \(m \to \infty\) in \((8.8)\), we conclude the proof. \(\square\)

A straightforward consequence of Proposition \((8.1)\) is the following Tanaka-Meyer-Wang type representation for the occupation time \(f_k^x\). In the sequel, for a given \(x \in \mathbb{R}\), let \(j_k(x)\) be the unique integer such that \((j_k(x) - 1)2^{-k} < x \leq j_k(x)2^{-k}\). Moreover, we set \(f_k^x(y) := |y - j_k(x)2^{-k}|; y \in \mathbb{R}\) and

\[
\mathbb{D}^k f_k^x(B) := \sum_{n=1}^{\infty} \frac{f_k^x(B(T_n^k)) - f_k^x(B(T_{n-1}^k))}{B(T_n^k) - B(T_{n-1}^k)} \mathbb{I}([T_n^k, T_{n+1}^k[).
\]
A simple computation based on formulas (8.4) and (8.5) yields

**Corollary 8.1.** For every $x \in \mathbb{R}$ and $k \geq 1$, the following representation holds

$$\ell^k_j(x)^{2^{-k}}(t) = |A^k(t) - j_k(x)|^{2^{-k}} - |A^k(0) - j_k(x)|^{2^{-k}} - \int_0^t \mathbb{D}^k f^k_s(B) ds; \quad 0 \leq t \leq T.$$  

Weakly differentiable Wiener functionals can also be characterized in terms of horizontal perturbations and local-time space-time integrals. The following corollary reveals a broad connection between local-times and weak functional differentiability of path-dependent functionals.

**Corollary 8.2.** Let $X \in \mathcal{W}$ be a weakly differentiable Wiener functional with orthogonal decomposition

$$X(t) = X(0) + \int_0^t \mathcal{D}X(s) dB(s) + \mathcal{I}^X(X(t)) \cdot 0; \leq t \leq T.$$  

Assume that a functional representation $F(B) = X$ equipped with the canonical GAS satisfies the assumptions in Proposition 8.1. Then, the orthogonal integral operator is characterized as follows

$$I^X(t) = \lim_{k \to \infty} \left[ \int_0^t \mathbb{D}^{k,h} F_s(B_s) ds - \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} \nabla^{k,v} F_s(B_s,x) d_{(s,x)} \mathbb{L}^{k,x}(s) \right]$$

weakly in $B^2(\mathbb{F})$.

**Proof.** If $X \in \mathcal{W}$, then (3.2) together with representation (8.5) yield the result.

A direct computation of the limit (8.9) is actually rather difficult without knowing a priori the probabilistic structure of the Wiener functional, i.e., semimartingale, weak Dirichlet structures, etc. In fact, for a given functional representation $F(B) = X$, one has to play with functional versions of $\delta^k F(B)$ which is not trivial to work unless one has smooth regular conditional probabilities at hand. In the next sections, we show how to overcome this inherent difficulty provided that the underlying functional is smooth in the sense of $p$-variation.

### 8.1. Space-time local-time integration in the pathwise 2D Young sense.

In this section, we provide readable conditions on a Wiener functional $F(B) \in \mathcal{W}$ in such way that the orthogonal integral operator (8.9) is characterized in terms of limits along a functional GAS $F$ as follows

$$I^X(t) = \int_0^t \mathbb{D}^{k,h} F_s(B_s) ds - \frac{1}{2} \lim_{k \to \infty} \int_0^t \int_{-\infty}^{+\infty} \nabla^{k,v} F_s(B_s,x) d_{(s,x)} \mathbb{L}^{k,x}(s); \quad 0 \leq t \leq T,$$

where the limit in the right-hand side of (8.10) will be interpreted as a 2D Young integral (39). There are some possibilities for the characterization of the above limit. In this section, we focus our attention to Young’s approach and the analysis of other possible characterizations depending on the smoothness of $F$ will be discussed elsewhere.

The precise identification of the limit of local-time integrals in the right-hand side of (8.10) is not a trivial task. In terms of Young integration theory (see e.g [49, 25]), the first obstacle towards the asymptotic limit (8.10) is to handle two-dimensional variation of $(L^{k,x}(t), \ell^x(t))$ over $\mathbb{R} \times [0,T]$ simultaneously with the infinitesimal behavior of the past of a Brownian path composed with the functional $F$. To prove convergence of the local-time integrals in (8.10), sharp maximal estimates on the number of crossings and the 2D Young integrals play key roles (see [38, 39]) for the obtention of the limit.

We fix once and for all, a functional GAS $F = \{F_k; k \geq 1\}$ of the form (2.4). In the sequel, $d_{(s,x)}$ and $d_x$ are computed in the sense of a 2D and 1D Young integral, respectively. See the seminal L.C Young’s articles [18, 49] for further details. In this section, it will be more convenient to work with
processes defined on the whole period $[0, +\infty)$ but keeping in mind that we are just interested on the bounded interval $[0, T]$. For this reason, all processes are assumed to be defined over $[0, +\infty)$ but they vanish after the finite time $T$.

In the sequel, it will be more convenient to work with the following “clock” modification of $L^K$ as follows

$$L^K(t) := \sum_{j \in \mathbb{Z}} I_{[2^{-j}2^{-k}, 2^{-k+1})}(t),$$

where

$$I_{[2^{-j}2^{-k}, 2^{-k+1})}(t) := \frac{1}{2^{-k}} \int_0^t \mathbb{1}_{(|A^K(s) - x| < 2^{-k})} d[A^K, A^K](s); \quad k \geq 1, \; x \in \mathbb{R}, \; 0 \leq t < +\infty.$$

For each $k \geq 1$ and $x \in \mathbb{R}$, let $j_k(x)$ be the unique integer such that $(j_k(x) - 1)2^{-k} < x \leq j_k(x)2^{-k}$ and $N^k(t) := \max\{n; T^k_n \leq t\}$ is the length of the embedded random walk until time $t$. By the very definition, $L^K(t) = l^K_{j_k(x)2^{-k}}(t); \; (x, t) \in \mathbb{R} \times [0, +\infty)$. More precisely,

$$L^K(x) = 2^{-k} \# \{n \in \{1, \ldots, N^k(t) - 1\}; A^K(T^k_n) = j_k(x)2^{-k}\}$$

where

$$u(j_k(x)2^{-k}, k, t) := \# \{n \in \{1, \ldots, N^k(t) - 1\}; A^K(T^k_{n-1}) = (j_k(x) - 1)2^{-k}, A^K(T^k_n) = j_k(x)2^{-k}\};$$

$$d(j_k(x)2^{-k}, k, t) := \# \{n \in \{1, \ldots, N^k(t) - 1\}; A^K(T^k_{n-1}) = (j_k(x) + 1)2^{-k}, A^K(T^k_n) = j_k(x)2^{-k}\};$$

for $x \in \mathbb{R}, k \geq 1, 0 \leq t < +\infty$.

In the sequel, we denote $\{\ell^2(t); (x, t) \in \mathbb{R} \times [0, +\infty)\}$ as the standard Brownian local-time, i.e., it is the unique jointly continuous random field which realizes

$$\int_{\mathbb{R}} f(x) \ell^2(t) dx = \int_0^t f(B(s)) ds \; \forall t > 0 \text{ and measurable } f : \mathbb{R} \to \mathbb{R}.$$

In the reminder of this section, we are going to denote the usual $p$-variation norm over a compact set $[a, b]$ by $\| \cdot \|_{[a, b]; p}$ where $1 \leq p < \infty$. We refer the reader to Young [48, 49] for further details.

In the sequel, we state a result which plays a key role for establishing the existence of the 2D Young integral. In the remainder of this section, we set $I_m := [-2^m, 2^m]$ for a positive integer $m \geq 1$.

**Lemma 8.1.** For each $m \geq 1$, the following properties hold:

(i) $L^k, x(t) \to \ell^2(t)$ a.s uniformly in $(x, t) \in I_m \times [0, T]$ as $k \to +\infty$.

(ii) $\sup_{k \geq 1} \mathbb{E} \sup_{x \in I_m} \|L^k, x(\cdot)\|_{[0, T]; 1}^p < +\infty$ and $\sup_{k \geq 1} \sup_{x \in I_m} \|L^k, x(\cdot)\|_{[0, T]; 1} < +\infty$ a.s for every $p \geq 1$.

(iii) $\sup_{k \geq 1} \mathbb{E} \sup_{t \in [0, T]} \|L^k(t)\|_{I_m; 2+\delta}^{2+\delta} < +\infty$ and $\sup_{k \geq 1} \sup_{t \in [0, T]} \|L^k(t)\|_{I_m; 2+\delta}^{2+\delta} < a.s$ for every $\delta > 0$.

**Proof.** The component $2^{-k}u(j_k(x)2^{-k}, k, t) \to \frac{1}{2} \ell^2(t)$ a.s uniformly over $(x, t) \in I_m \times [0, T]$ due to the classical Th. 4.1 in [29]. By writing $2^{-k}d(j_k(x)2^{-k}, k, t) = 2^{-k}d(j_k(x)2^{-k}, k, t) - 2^{-k}d(j_k(x) - 1)2^{-k}, k, t) + 2^{-k}d(j_k(x) - 1)2^{-k}, k, t) - \frac{1}{2} \ell^2(t)$ and using Lemma 6.23 in [43], we conclude that item (i) holds. The proof of (ii) is quite simple because $L^k, x(\cdot)$ has increasing paths a.s for each $k \geq 1$ and $x \in I_m$. In fact, Th.1 in [1] yields
for every $p \geq 1$. This shows that (ii) holds. For the proof of item (iii), it will be sufficient to check only for the upcrossing component of $L^k$. Let us fix $M, \delta > 0$. For a given partition $\Pi = \{x_i\}_{i=0}^N$ of the interval $I_m$, let us define the following subset $\Lambda(\Pi, k) := \{x_i \in \Pi; (j_k(x_i) - j_k(x_{i-1}))2^{-k} > 0\}$. We notice that $\# \Lambda(\Pi, k) \leq 2^{2k+m}$ for every partition $\Pi$ of $I_m$. To shorten notation, let us write $U_p^k(t, x) := 2^{-k}U^k(j_k(x)2^{-k}, t)$; $(x, t) \in I_m \times [0, T]$. We readily see that

$$
\sum_{x_i \in \Pi} |U_p^k(t, x_i) - U_p^k(t, x_{i-1})|^{2+\delta} \leq \sum_{x_i \in \Lambda(\Pi, k)} |U_p^k(t, x_i) - U_p^k(t, x_{i-1})|^{2+\delta} \ a.s.,
$$

for every partition $\Pi$ of $I_m$. By writing $|U_p^k(t, x_i) - U_p^k(t, x_{i-1})| = |U_p^k(t, x_i) - 1/2\ell^x_i(t) + 1/2\ell^x_i(t) - 1/2\ell^x_i(t) - 1/2\ell^x_i-1(t) + 1/2\ell^x_i-1(t) - U_p^k(t, x_{i-1})|$; $x_i \in \Lambda(\Pi, k)$ and applying the standard inequality $|\alpha - \beta|^{2+\delta} \leq 2^{1+\delta}(|\alpha|^{2+\delta} + |\beta|^{2+\delta})$; $\alpha, \beta \in \mathbb{R}$, the bound (8.11) yields

$$
\sup_{0 \leq t \leq T} ||U_p^k(t)||_{l,m,2+\delta}^{2+\delta} \leq 2^{1+\delta} \sup_{0 \leq t \leq T} ||1/2\ell^x_i(t)||_{l,m,2+\delta}^{2+\delta} + 2^{1+\delta} \sup_{0 \leq t \leq T} \sum_{x_i \in \Lambda(\Pi, k)} |U_p^k(t, x_i) - 1/2\ell^x_i(t)|^{2+\delta}
$$

for every $k \geq 1$.

An inspection in the proof of Lemma 2.1 in [21] yields $\sup_{0 \leq t \leq T} ||\ell^x_i(t)||_{l,m,2+\delta}^{2+\delta} < \infty$. By applying Th. 1.4 and Remark 1.7.1 in [25], we get for every $k$ larger than a positive random number, the following bound

$$
\sup_{0 \leq t \leq T} \sum_{x_i \in \Lambda(\Pi, k)} |U_p^k(t, x_i) - 1/2\ell^x_i(t)|^{2+\delta} \leq \sup_{0 \leq t \leq T} \sup_{x \in I_m} |U_p^k(x, t) - 1/2\ell^x(t)|^{2+\delta} 2^{k+m}
$$

$$
\leq 2^{k+m-2^{1+\delta}} (\log(2))^{2+\delta} \left(M + \sup_{0 \leq t \leq T} \sqrt{\ell^x(t)}\right)^{2+\delta}
$$

$$
\leq (C + \ell^x(T)^{1+\delta})2^{-\frac{1}{2}2^{1+\delta}},
$$

where $C$ is a positive constant which only depends on $(M, n)$ and $\ell^x(t) := \sup_{x \in I_m} \ell^x(t)$; $0 \leq t \leq T$. The last term in (3.12) can be treated similarly. By using the fact that $2^{-\frac{1}{2}2^{1+\delta}k+\delta} = O(1)$ and $\ell^x(T) < \infty$ a.s., we conclude that $\sup_{k \geq 1} \sup_{t \in [0, T]} ||L^k_i(t)||_{l,m,2+\delta}^{2+\delta} < \infty$ a.s for every $\delta > 0$. It remains to check the $L^{2+\delta}$ bound in (iii) which is more delicate than the related almost sure bound. We refer the reader to Corollary 2.1 in [39] for a detailed proof of this bound.

In order to get an explicit limit for the space-time local time integrals in terms of a 2D Young integral, we need to assume some minimal smoothness in the sense of pathwise calculus. More precisely, let $C^1$ be the set of non-anticipative functionals $\{F_t; 0 \leq t \leq T\}$ such that $x \mapsto F_t(x(c_x, x))$ is $C^1(U; \mathbb{R})$ for each $c \in \Lambda$, $s \in [0, T]$ and a bounded open set $U \subset \mathbb{R}$.

**Remark 8.2.** Any functional in $C^1$ admits a representation $F_t(x) = \int_{c_x}^t H_i(c_x, y)\,dy$ for some functional $H$ such that $y \mapsto H_i(c_x, y)$ is continuous for every $t \in [0, T]$ and $c \in \Lambda$. Moreover,

$$
\nabla^v F_t(B_t) = \partial_x F_t(t(B_t, B(t))) = H_t(B_t, B(t)); \ 0 \leq t \leq T,
$$

where $\nabla^v$ denotes the pathwise vertical derivative.
Assumption (L1): For every càdlàg function \( c \), the map \( c \mapsto \partial_x F_s(t(c_s, x)) \) is continuous uniformly in the time variable \( s \), i.e., for every compact set \( K \subset \mathbb{R} \) and \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\sup_{0 \leq s \leq T} |c(s) - d(s)| < \delta \implies \sup_{0 \leq s \leq T} \sup_{x \in K} |\partial_x F_s(t(c_s, x)) - \partial_x F_s(t(d_s, x))| < \varepsilon.
\]

Remark 8.3. One should notice that this implies, in particular, that if \( c^k \) is a sequence of càdlàg functions such that \( \sup_{0 \leq t \leq T} |c^k(t) - d(t)| \to 0 \) as \( k \to \infty \), then \( \sup_{0 \leq s \leq T} \sup_{x \in K} |\partial_x F_s(t(c^k_s, x)) - \partial_x F_s(t(d_s, x))| \to 0 \), as \( k \to \infty \) for every compact set \( K \subset \mathbb{R} \).

The following lemma is straightforward in view of the definitions. We left the details of the proof to the reader.

Lemma 8.2. If \( F \in C^1 \) satisfies (L1), then \( \nabla^{F,v,k} F_s(B_{s}, x) \to \partial_x F_s(t(B_s, x)) \) a.s uniformly in \((x, s) \in I_m \times [0, T]\) for every \( m \geq 1 \).

Let us now introduce additional hypotheses in order to work with 2D-Young integrals. We refer the reader to Young [49] for further background. In the sequel, \( \Delta_h(t_i, x_j) := h(t_i, x_j) - h(t_{i-1}, x_j) \) denotes the first difference operator acting on the variable \( t \) of a given function \( h : [0, T] \times [-L, L] \to \mathbb{R} \).

Assumption (L2.1): Assume for every \( L > 0 \), there exists a positive constant \( M \) such that
\[
|\Delta_i \Delta_j \partial_x F_i(t(B_{t_i, x_j}))| \leq M |t_i - t_{i-1}|^{\frac{1}{p}} |x_j - x_{j-1}|^\frac{1}{q} \quad a.s
\]
for every partition \( \Pi = \{ t_i \}_{i=0}^N \times \{ x_j \}_{j=0}^{N'} \) of \([0, T] \times [-L, L]\), where \( q_1, q_2 > 1 \). There exists \( \alpha \in (0, 1), \delta > 0, p \geq 1 \) such that \( \min\{\alpha + \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{2\rho}, \alpha + \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{2\rho} \} > 1, \frac{1}{p} + \frac{1}{2\rho} > 1 \) and
\[
\sup_{0 \leq t \leq T} \|\partial_x F_i(t(B_{t, \cdot}))\|_{[-L, L]^p} \in L^\infty(\mathbb{P}),
\]
for every \( L > 0 \).

Assumption (L2.2): In addition to assumption (L2.1), let us assume \( \forall L > 0 \), there exists \( M > 0 \) such that
\[
\sup_{k \geq 1} |\Delta_i \partial_x F_i(t(B_{t_i, \cdot}))| \leq M |t_i - t_{i-1}|^{\frac{1}{p}} |x_j - x_{j-1}|^\frac{1}{q} \quad a.s
\]
for every partition \( \Pi = \{ t_i \}_{i=0}^N \times \{ x_j \}_{j=0}^{N'} \) of \([0, T] \times [-L, L]\), and
\[
\sup_{k \geq 1} \sup_{0 \leq t_i \leq T} \|\nabla^{F,v,k} F_i(B_{t_i, \cdot})\|_{[-L, L]^p} \in L^\infty(\mathbb{P}).
\]

Remark 8.4. In the language of rough path theory, assumption (8.13) precisely says that if \( q = q_1 = q_2 \) then \( \partial_x F_i(t(B_{t, x})) \) admits a 2D-control \( \omega([t_1, t_2] \times [x_1, x_2]) = |t_1 - t_2|^{\frac{1}{p}} |x_1 - x_2|^{\frac{1}{q}} \), so that (8.13) trivially implies that \( t(x) \mapsto \partial_x F_i(t(B_{t, x})) \) has \( q \)-joint variation in the sense of [25]. If the local time \( \ell^p(t); -L \leq x \leq L, 0 \leq t \leq T \) admits joint 2 + \( \delta \)-variation over compact sets \([-L, L] \times [0, T] \) a.s then assumptions (L2.1-L2.2) can be weakened to more general types of controls. At this stage, we only know that the Brownian local time admits (1, 2 + \( \delta \))-bivariation for every \( \delta > 0 \) which is weaker than joint variation. See Young [49] (page 583), Feng and Zao [21], Ohashi and Simas [35] and Friz and Victoir [25] for more details. However, assumptions (L2.1-L2.2) are sufficiently rich to accommodate a large class of examples. In order to extend Theorem 8.7 towards more general types of 2D-controls, more refined estimates on the joint variation of \( \{(L^{k,q}(t), \ell^p(t)); (x, t) \in \mathbb{R} \times \mathbb{R}_+ \} \) are required.
We are now in position to state our first approximation result which makes heavily use of the pathwise 2D Young integral in the context of the so-called \((p, q)\)-bivariation rather than joint variation. We refer the reader to the seminal Young article \([33]\) (section 6) for further details.

**Proposition 8.2.** Let \(F \in C^1\) satisfy assumptions (L1, L2.1, L2.2). Then, for every \(t \in [0, T]\) and \(m \geq 1\), we have

\[
\lim_{k \to \infty} \int_0^t \int_{-2m}^{2m} \nabla F(x) F_t(B_s, x) d(r, x) L^{k,x}(r) = \int_0^t \int_{-2m}^{2m} \partial_x F_t(t(B_s, x)) d(r, x) \ell^x(r) \quad a.s.,
\]

where the right-hand side of (8.17) is interpreted as the pathwise 2D Young integral.

**Proof.** Let us fix \(t \in [0, T]\) and a positive integer \(m\). Let \(\{0 = s_0 < s_1 < \ldots < s_p = t\}\) and \(\{-2m = x_0 < x_1 < \ldots < x_l = 2m\}\) be partitions of \([0, t]\) and \([-2m, 2m]\), respectively. Similar to identity (4.5) in \([21]\), we shall write

\[
\sum_{i=0}^{l-1} \sum_{j=0}^{p-1} \partial_x F_j(t(B_{s_j}, x_i)) \Delta_i \Delta_j \ell^{x,i+1}(s_j + 1) = \sum_{i=1}^l \sum_{j=1}^p \ell^x(s_j) \Delta_i \Delta_j \partial_x F_j(t(B_{s_j}, x_i)) - \sum_{i=1}^l \ell^x(t) \Delta_i F_i(t(B_t, x_i))
\]

From Lemmas 2.1, 2.2 in \([21]\), we know that \(\{\ell^x(s); 0 \leq s \leq T; x \in I_m\}\) has \((1, 2 + \delta)\)-bivariations a.s for every \(\delta > 0\) (See Young \([49]\), p. 583 for further details). Then by applying Th. 6.3 in \([49]\) and the one dimensional existence theorem in \([48]\) together with L2.1, we have

\[
\int_0^t \int_{-2m}^{2m} \partial_x F_s(t(B_s, x)) d(s, x) \ell^x(s) = \int_0^t \int_{-2m}^{2m} \ell^x(s) d(s, x) \partial_x F_s(t(B_s, x))
\]

\[
- \int_{-2m}^{2m} \ell^x(t) d_x F_t(t(B_t, x)) \quad a.s.
\]

We do the same argument to write

\[
\int_0^t \int_{-2m}^{2m} \nabla F(x) F_s(B_s, x) d(s, x) L^{k,x}(s) = \int_0^t \int_{-2m}^{2m} L^{k,x}(s) d(s, x) \nabla F(x) F_s(B_s, x)
\]

\[
- \int_{-2m}^{2m} L^{k,x}(t) d_x \nabla F(x) F_t(B_t, x) \quad a.s. 0 \leq t \leq T; k \geq 1.
\]

Now we apply (L1-L2.1-L2.2), Lemma \([31]\) together with Th. 6.3, 6.4 in \([49]\) and the term by term integration theorem in \([38]\) to conclude that

\[
\lim_{k \to \infty} \int_0^t \int_{-2m}^{2m} L^{k,x}(s) d(s, x) \nabla F(x) F_s(B_s, x) = \int_0^t \int_{-2m}^{2m} \partial_x F_s(t(B_s, x)) d(s, x) \ell^x(s) \quad a.s.
\]

\[
\lim_{k \to \infty} \int_{-2m}^{2m} L^{k,x}(t) d_x \nabla F(x) F_t(B_t, x) = \int_{-2m}^{2m} \ell^x(t) d_x F_t(t(B_t, x)) \quad a.s.
\]

up to some vanishing conditions on the boundaries \(t = 0\) and \(x = -2m\). They clearly vanish for \(t = 0\). For \(x = -2m\) we have to work a little. In fact, we will enlarge our domain in \(x\), from \([-2m, 2m]\) to \([-2m-1, 2m]\), and define for all functions with \(-2m - 1 \leq x < -2m\), the value 0, that is, for the functions \(L^{k,x}(s), \ell^x(s), \partial_x F_s(t(B_s, x))\) and \(\nabla F(x) F_s(B_s, x)\) we put the value 0, whenever
which only depend on the constants of assumption 

\[ L^{k,-2m-1}(s) = 0 \] and \[ \ell^{2m-1}(s) = 0 \] for all \( s \). Thus, we can apply Theorems 6.3 and 6.4 in [49] on the interval \([0, t] \times [-2m - 1, 2m]\). This concludes the proof.

Next, we need to translate convergence \([8.17]\) into \( L^1 \) convergence. The maximal inequality derived by Ohashi and Simas [35] (see Th. 1.3 and Corollary 1.1) based on bivariations plays a crucial role in the proof of the following lemma. See Remark 5.4.

**Proposition 8.3.** Assume that \( F \in C^1 \) satisfies assumptions \((L1-L2.1-L2.2)\). Then, for every non-negative random variable \( J \), we have

\[
\lim_{k \to \infty} \int_0^J \int_{-2m}^{2m} \nabla F_{v,k} t F_{x,v,k} \mathcal{L}^{k}(x) d(r,x) L^{k,x}(r) = \int_0^J \int_{-2m}^{2m} \partial_x F_t(t,B_r,r) d(r,x) \ell(t)
\]

strongly in \( L^1(\mathbb{P}) \) for every \( m \geq 1 \).

**Proof.** Let us fix a positive random variable \( J \). In view of \([8.18]\) and Proposition 8.2, it is sufficient to check that \( \int_0^J \int_{-2m}^{2m} L^{k,x}(s) d(s,x) \nabla F_{v,k} F_{v,k,x}(s,x) \) is uniformly integrable. By the classical Young inequality based on simple functions, the following bound holds

\[
I_s^k \leq C_1 \left( L^{k,-2m}(J) + \| L^k(J) \|_{[-2m,2m]} \right) a.s.
\]

where \( C_1 \) is a constant depending on \( \delta \) and \([8.19]\). From Th. 1 in [1] and item (ii) in Lemma 8.1, we conclude that \( \{ I_s^k; k \geq 1 \} \) is uniformly integrable.

A direct application of Cor. 1.1 in [35] together with \((L2.2)\) yield

\[
\int_0^J \int_{-2m}^{2m} L^{k,x}(s) d(s,x) \nabla F_{v,k} F_{v,k,x}(s,x) \leq K_0 L^{k,2m}(T) + K \| L^k \|_{[t_m,1]} \| L^k \|_{\text{space},2+\delta}^\alpha
\]

where

\[
\| L^k \|_{\text{space},2+\delta} := \sup_{(x,y) \in [0,T]^2} \| L^k(t) - L^k(s) \|_{L^m,2+\delta},
\]

\[
\| L^k \|_{[t_m,1]} := \sup_{(x,y) \in [t_m,1]} \| L^k(t) - L^k(t) \|_{0,1}.\]

Here \( K_0 \) is a constant which comes from assumption \([8.13]\) and \( K, K_1, K_2 \) are positive constants which only depend on the constants of assumption \((L2.1, L2.2)\) namely \( \alpha, q_1, q_2, \delta, T \land J, m \). From Lemma 8.1, we have \( \sup_{k \geq 1} \mathbb{E} \| L^k \|_{\text{space},2+\delta}^\alpha < \infty \), and \( \sup_{k \geq 1} \| L^k \|_{[t_m,1]} < \infty \). Again, Th. 1 in [1] yields the uniform integrability of \( \{ L^{k,2m}(T); k \geq 1 \} \). So we only need to check uniform integrability of \( \{ L^{k,2m}(T); k \geq 1 \} \). For \( \beta > 1 \), we apply Holder inequality to get

\[
\mathbb{E} \| L^k \|_{[t_m,1]} \| L^k \|_{\text{space},2+\delta}^\beta \leq \left( \sup_{r \geq 1} \mathbb{E} \| L^k \|_{\text{space},2+\delta}^\beta \right) \frac{1}{p} \left( \sup_{r \geq 1} \mathbb{E} \| L^k \|_{[t_m,1]} \| L^k \|_{\text{space},2+\delta}^\alpha \right)^{1/q}; \ k \geq 1
\]

where \( p = \frac{1}{1-\alpha} > 1 \), \( q = \frac{p}{p-1} = \frac{1}{1-\alpha} \), and \( \alpha \in (0, 1) \). By applying Lemma 8.1, we conclude that \( \{ I_s^k; k \geq 1 \} \) is uniformly integrable. Finally, Proposition 8.2 allows us to conclude that \([8.19]\) holds.

\[ \square \]

**Lemma 8.3.** For every \( \mathbb{F} \)-stopping time \( S \), there exists a sequence of non-negative random variables \( \{ J_k; k \geq 1 \} \) such that \( J_k \) is an \( \mathbb{F}^k \)-predictable stopping time for each \( k \geq 1 \) and \( \lim_{k \to \infty} J_k = S \) a.s.
Proof. We can repeat the same steps of the proof of Lemma 2.2 in [22], to get for a given $\mathbb{F}$-predictable set $O$, a sequence $O_k$ of $\mathbb{F}^k$-predictable set such that $\mathbb{P}[\pi(O) - \pi(O_k)] \to 0$ as $k \to \infty$, where $\pi$ is the usual projection of $\mathbb{R}^+ \times \Omega$ onto $\Omega$. Apply the usual Section Theorem (see e.g [21]) on each $O_k$ and on the graph $[[S, S]]$ as in [Lemma 3.3; [22]] to get a sequence $J^k$ of $\mathbb{F}^k$-predictable stopping times such that $\lim_{k \to \infty} J^k = J$ a.s on $\{J < \infty\}$. By defining, $J^k := J^k$ on $\{J < \infty\}$ and $J^k := +\infty$ on $\{J = +\infty\}$, we get the desired sequence.

Now we are in position to state the main result of this section.

**Theorem 8.1.** Let $F \in \mathcal{C}^1$ be a non-anticipative functional such that $\mathcal{D}^k F(B) = \mathcal{D} F(B)$ for the associated functional GAS $F$. If $\mathcal{D}^k F(B)$ exist, $F$ satisfies assumptions (L1, L2.1, L2.2) and (7.3) holds for $F$, then the differential representation of $F(B) \in \mathcal{W}$ is given by

$$F_t(B_t) = F_0(B_0) + \int_0^t \mathcal{D} s(B_s)dB(s) + \int_0^t \mathcal{D} s F_s(B_s)ds - \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} \partial_s F_s(t(B_s, x))d(s,x)\ell^x(s)$$

for $0 \leq t \leq T$.

Proof. By assumption, $F(B) \in \mathcal{W}$, $\mathcal{D}^k F(B) = \mathcal{D} F(B)$ and (7.3) holds. The combination of Corollaries [4.4 4.5] and Propositions [4.1 and 4.2] yield

$$F(B) = F_0(B_0) + \int \mathcal{D} s(B_s)dB(s) + \mathcal{I} F(B)$$

where

$$\mathcal{I} F(B)(\cdot) = \lim_{k \to \infty} \left[ \int_0^t \mathcal{D}^k F_s(B_s) d(A^k)(s) - \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} \nabla^x F_s(B_s, x) d(s,x)\Delta^x(s) \right]$$

weakly in $\mathcal{B}^1(F)$. By using Lemma [8.3] and the existence of $\mathcal{D}^k F(B)$, we can argue in the same way as in the proof of the convergence [4.1 4.2] in Theorem [4.1] to conclude that

$$\lim_{k \to \infty} \int_0^t \mathcal{D}^k F_s(B_s) d(A^k)(s) = \int_0^t \mathcal{D} s F_s(B_s)dt$$

weakly in $\mathcal{B}^1(F)$. Then $R(\cdot) := \lim_{k \to \infty} \int_0^{+\infty} \nabla^x F_s(B_s, x) d(s,x)\Delta^x(s)$ weakly in $\mathcal{B}^1(F)$. We claim that

$$\mathcal{I} F(B)(\cdot) = \lim_{k \to \infty} \left[ \int_0^t \mathcal{D}^k F_s(B_s) d(A^k)(s) - \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} \nabla^x F_s(B_s, x) d(s,x)\Delta^x(s) \right]$$

Let us consider the stopping time

$$S_m := \inf\{0 \leq t < \infty; |B(t)| \geq 2^m\}; \ m \geq 1.$$  

One should notice that $S_m$ is an $\mathbb{F}^k$-stopping time for every $k \geq 1$. Then, by the very definition $R(- S_m) = \lim_{k \to \infty} \int_0^{2^m} \mathcal{D}^k F_s(B_s, x) d(s,x)\Delta^k(s)\Delta^k(s - S_m)$ weakly in $\mathcal{B}^1(\mathbb{F})$. For a given arbitrary $F$-stopping time $J$ (bounded or not), let $J^k$ be a sequence of $F$-stopping times from Lemma [8.3] such that $J^k$ is an $\mathbb{F}^k$-stopping time for each $k \geq 1$ and $\lim_{k \to \infty} J^k = J$ a.s. Initially, we set $k > \ell > m$ and let denote $A^{k,m}$ by the stopped process $A^k$ at $S_m$.

By taking the functional GAS for $F(B)$, we readily see that $\frac{\Delta F_{s,j,k} - b^k(j - b^k(j - 1))}{2^{-k}}$ is $\mathbb{F}^k$-predictable for each $j \in \mathbb{Z}$ (recall the notation in [8.3]). Then, we shall use $\mathbb{F}^k$-dual predictable projection on the stochastic set $[[0, J^k]]$ (see Th. 5.26 in [21]) to get

$$\mathbb{E} \int_0^{J^k} \frac{\Delta F_{s,j,k} - b^k(j - b^k(j - 1))}{2^{-k}} d(A^{k,m}, A^{k,m})(s) = \mathbb{E} \int_0^{J^k} \frac{\Delta F_{s,j,k} - b^k(j - b^k(j - 1))}{2^{-k}} d(A^{k,m}, A^{k,m})(s);$$
for \( j \in \mathbb{Z} \), so that
\[
\mathbb{E} \int_0^f \int_{-2^m}^{2^m} \nabla F_s \cdot F_s(B, x) d_s \mathbb{P}^x(s \wedge S_m) = \mathbb{E} \int_0^f \int_{-2^m}^{2^m} \nabla F_s \cdot F_s(B, x) d_s \mathbb{P}^x(s \wedge S_m),
\]

The fact that \( S_m < \infty \) a.s, Proposition \( \text{(8.23)} \) and Lemma \( \text{(8.24)} \) yield

\[
\mathbb{E} \int_0^f \int_{-2^m}^{2^m} \partial_s F_s(t(B, x)) d_s \mathbb{P}^x(s \wedge S_m) = \mathbb{E} R(J \wedge S_m; \ m \geq 1),
\]

for every \( \ell \geq 1 \). Since \( R \in \mathcal{B}^1(\mathbb{F}) \) has continuous paths, we actually have

\[
\lim_{\ell \to \infty} \mathbb{E} R(J \wedge S_m) = \mathbb{E} (J \wedge S_m),
\]

for every \( \mathcal{F} \)-stopping time \( J \). We also know that the 2D Young integral \( \int_0^f \int_{-2^m}^{2^m} \partial_s F_s(t(B, x)) d_s \mathbb{P}^x(s \wedge S_m) \) has continuous paths for every \( m \geq 1 \). Moreover, by applying the inequality given in \([35]_; \text{Corollary 1.1}\) together with \( \text{(L2.1)} \), we readily see that it belongs to \( \mathcal{B}^1(\mathbb{F}) \). The bounded convergence theorem and relations \( \text{(8.25)} \) and \( \text{(8.26)} \) yield

\[
\mathbb{E} R(J \wedge S_m) = \mathbb{E} \left( \int_0^f \int_{-2^m}^{2^m} \partial_s F_s(t(B, x)) d_s \mathbb{P}^x(s) \right),
\]

for every \( m \geq 1 \). We shall write, \( \mathbb{E} R(J \wedge S_m) = \mathbb{E} (S_m) 1_{(J = +\infty)} + \mathbb{E} R(J \wedge S_m) 1_{(J < +\infty)} \). The path continuity of \( R \in \mathcal{B}^1(\mathbb{F}) \) and the fact that all processes are assumed to be null after time \( 0 < T < \infty \) yield

\[
\lim_{m \to \infty} \mathbb{E} R(J \wedge S_m) = \mathbb{E} (J) 1_{(J < +\infty)}.
\]

By using the fact that the Brownian local time has compact support, we do the same argument to get

\[
\lim_{m \to \infty} \mathbb{E} \left( \int_0^f \int_{-2^m}^{2^m} \partial_s F_s(t(B, x)) d_s \mathbb{P}^x(s) \right) = \mathbb{E} \left( \int_0^f \int_{-\infty}^{+\infty} \partial_s F_s(t(B, x)) d_s \mathbb{P}^x(s) 1_{(J < +\infty)} \right)
\]

Summing up \( \text{(8.27)} \), \( \text{(8.28)} \) and \( \text{(8.29)} \), we conclude that

\[
\mathbb{E} R(J) 1_{(J < +\infty)} = \mathbb{E} \left( \int_0^f \int_{-\infty}^{+\infty} \partial_s F_s(t(B, x)) d_s \mathbb{P}^x(s) 1_{(J < +\infty)} \right).
\]

Lastly, from Corollary 4.13 in \([26]\), we shall conclude that both \( R(\cdot) \) and \( \int_0^{+\infty} \partial_s F_s(t(B, x)) d_s \mathbb{P}^x(s) 1_{(J < +\infty)} \) are indistinguishable.

\[
\square
\]

8.2. Example. Let us give a concrete example of Wiener functional which satisfies the assumptions of Theorem \( \text{(8.4)} \). Let \( \varphi: \mathbb{R}^2 \to \mathbb{R} \) be a function which satisfies the following hypotheses:

\( \text{(E1)} \) For every compact set \( K \subset \mathbb{R} \), there exist constants \( M_1 \) and \( M_2 \) such that for every \( a, z \in K \),
\[
|\varphi(a, x) - \varphi(a, y)| \leq M_1 |x - y|^\gamma_1,
\]
and
\[
|\varphi(c, z) - \varphi(d, z)| \leq M_2 |c - d|^\gamma_2,
\]
where \( \gamma_1 \in \left( \frac{1 + \delta}{2\delta}, 1 \right], \gamma_2 \in (0, 1] \) and \( \delta > 0 \).
(E2) For every compact set $V_1 \subset \mathbb{R}$ there exists a compact set $V_2$ such that \( \{ x; \varphi(a, x) \neq 0 \} \subset V_2 \) for every $a \in V_1$

(E3) For every continuous path $c \in C([0, T]; \mathbb{R})$, $\int_{[0, T] \times \mathbb{R}} |\varphi(c(s), y)| ds dy < \infty$.

For such a kernel $\varphi$, we consider

\[
F_t(c_t) := \int_{-\infty}^{c(t)} \int_0^t \varphi(c(s), y) ds dy; \quad 0 \leq t \leq T.
\]

if $c \in \Lambda$. By the very definition, $F_t(c(t, x)) = \int_{-\infty}^{c(t)} \int_0^t \varphi(c(s), y) ds dy$; $0 \leq t \leq T, x \in \mathbb{R}, c \in \Lambda$.

Remark 8.5. A simple computation reveals that if $F$ satisfies (E1-E2-E3), then it is vertically and horizontally differentiable in the sense of pathwise functional calculus, where

\[
\nabla^v F_t(B_t) = \int_0^t \varphi(B(s), B(t)) ds, \quad \nabla^h F_t(B_t) = \int_{-\infty}^{B(t)} \varphi(B(t), y) dy; \quad 0 \leq t \leq T.
\]

In general, $F$ is not pathwise twice vertically differentiable due to the Hölder condition (E1).

Let us now show that in fact $F \in W_{loc}$ is weakly differentiable where $\nabla^v F(B) = D F(B)$ and $\nabla^h F(B) = D^{F,h} F(B)$.

Lemma 8.4. If $\varphi$ satisfies assumptions (E1, E2), then $F$ is a GAS for $F(B)$ and $D^{F,h} F(B)$ exists locally, i.e., there exists an exhaustive sequence of $\mathbb{F}$-stopping times $\{ S_m; m \geq 1 \}$ such that

\[
D^{F,h} F(B) = \int_{-\infty}^{B(-)} \varphi(B(\cdot), y) dy \quad \text{on} \quad \| [0, S_m] \|
\]

every $m \geq 1$.

Proof. By considering $S_m = \inf\{ t > 0; |B(t)| \geq 2^m \}$, both $A^k$ and $B$ are bounded on the stochastic interval $[0, S_m]$ for $m \geq 1$. We recall that $S_m$ if $\mathbb{F}$-stopping time for every $k > m$. To shorten notation, with a slight abuse of notation, let us write the stopped processes $A^{k,S_m}$ and $B^{S_m}$ as $A^k$ and $B$, respectively. The fact that $F$ is a GAS is a straightforward consequence of (E1,E2). By the very definition,

\[
D^{F,k,h} F_t(B_t) = \sum_{n=1}^{\infty} \int_{-\infty}^{A^k(T_{n-1}^k)} \varphi(A^k(T_{n-1}^k), y) dy(T_n^k - T_{n-1}^k) \frac{1}{2^{2k}} 1_{\{ T_n^k < t \leq T_{n+1}^k \}}; \quad 0 \leq t \leq T.
\]

Let us fix an arbitrary $g \in L^\infty(\mathbb{F})$ and $t \in [0, T]$. The $F^k$-dual predictable projection yields

\[
\mathbb{E}g \int_0^t D^{F,k,h} F_s(B_s) ds = \mathbb{E} \int_0^t p^k(g)(s) D^{F,k,h} F_s(B_s) ds,
\]

where $p^k(g)$ is the $F^k$-predictable projection of the bounded measurable process $g$. Moreover,

\[
p^k(g)(s) D^{F,k,h} F_s(B_s) = \mathbb{E}[g | F_{T_n^k - T_{n-1}^k}] \int_{-\infty}^{A^k(T_{n-1}^k)} \varphi(A^k(T_{n-1}^k), y) dy(T_n^k - T_{n-1}^k) \frac{1}{2^{2k}} 1_{\{ T_n^k < s \leq T_{n+1}^k \}}
\]

where $T_n^k - T_{n-1}^k$ is independent from $\mathbb{E}[g | F_{T_n^k - T_{n-1}^k}] \int_{-\infty}^{A^k(T_{n-1}^k)} \varphi(A^k(T_{n-1}^k), y) dy 1_{\{ T_n^k < s \leq T_{n+1}^k \}}$ for every $k$, $n \geq 1$. Then,
\[ \mathbb{E}^{p,k}(g(s)) = \mathbb{E} \sum_{n=1}^{\infty} \mathbb{E}[g|F_{\eta_n}^k] \int_{-2^n}^{2^n} \varphi(A^k(T_{n-1}^k),y)dy \mathbb{I}_{\{T^k_n < s \leq T^k_{n+1}\}}; \ 0 \leq s \leq t. \]

Now, assumption (E2), the weak convergence of \( \lim_{k \to \infty} \mathbb{P}^k = \mathbb{P} \) and Dominated Convergence Theorem yield
\[ \sum_{n=1}^{\infty} \mathbb{E}[g|F_{\eta_n}^k] \int_{-\infty}^{\infty} \varphi(A^k(T_{n-1}^k),y)dy \mathbb{I}_{\{T^k_n < s \leq T^k_{n+1}\}} \rightarrow \mathbb{E}[g|\mathcal{F}_s] \int_{-\infty}^{\infty} \varphi(B(s),y)dy \]
strongly in \( L^1(\mathbb{P}) \) as \( k \to \infty \) for every \( s \in [0,t] \) and also
\[ \lim_{k \to \infty} \int_0^t \mathbb{E}^{p,k}(g(s))ds = \lim_{k \to \infty} \mathbb{E} \int_0^t \mathcal{D}^{F,k,h}F_s(B_s)ds = \mathbb{E} \int_0^t \mathbb{E}[g|\mathcal{F}_s] \int_{-\infty}^{\infty} \varphi(B(s),y)dyds = \mathbb{E} \int_0^t \int_{-\infty}^{\infty} \varphi(B(s),y)dyds \]
That is, \( \lim_{k \to \infty} \mathcal{D}^{F,k,h}F(B) = \int_{-\infty}^{\infty} \varphi(B(),y)dy \) weakly in \( L^1(\mathbb{P} \times \text{Leb}) \). It remains to check that it converges weakly in \( L^2(\mathbb{P} \times \text{Leb}) \). By compactness, it is sufficient to check \( \{\mathcal{D}^{F,k,h}F(B); k \geq 1\} \) is bounded in \( L^2(\mathbb{P} \times \text{Leb}) \). Let \( W \) be an \( F \)-Brownian motion independent from \( B \) and we set \( \tau := \inf\{t > 0; |W(t)| = 1\} \). We recall that \( T^k_n - T^k_{n-1} = 2^{-2k}\tau \) (in law) for each \( k \geq 1 \). Then (E2) yields the existence of a positive constant \( C \) such that
\[ \mathbb{E} [\mathcal{D}^{F,k,h}F_s(B_s)]^2 = \mathbb{E} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \varphi(A^k(T_{n-1}^k),y)dy^2 \mathbb{I}_{\{T^k_n < s \leq T^k_{n+1}\}} \leq C \]
for every \( k \geq 1 \) and \( s \in [0,T] \). Hence, we do have convergence weakly in \( L^2(\mathbb{P} \times \text{Leb}) \). By usual localization arguments, there exists \( \mathcal{D}^{F,h}F(B) \in L^2_{\text{loc}}(\mathbb{P} \times \text{Leb}) \) which realizes (E2).

Lemma 8.5. If \( \varphi \) satisfies assumptions (E1,E2), then
\[ (8.33) \quad \mathcal{D}^F F_t(B_t) = \int_0^t \varphi(B(s),B(t))ds; \ 0 \leq t \leq T. \]

Proof. By localization as in Lemma 8.4, we may assume that both \( A^k \) and \( B \) live in the compact set \( I_m = [-2^m,2^m] \) for every \( k \geq 1 \). At first, in order to take advantage of the smoothness of \( x \mapsto F_t(x^2) \), let us denote \( \eta^k_n := A^k(T_{n-1}^k) + \Delta A^k(T_{n-1}^k) \) and we notice that we shall write \( A^k_{T^k_n} = t(A^k_{T^k_n-\tau},\eta^k_n) \), so that
\[ \mathcal{D}^{F,k}F_s(B_t) = \sum_{n=1}^{\infty} \left( F_{T^k_n}(t(A^k_{T^k_n-\tau},\eta^k_n)) - F_{T^k_n}(A^k_{T^k_n-\tau}) \right) \frac{1}{\Delta A^k(T_{n-1}^k)} \mathbb{I}_{\{T^k_n < s \leq T^k_{n+1}\}} \]
\[ + \sum_{n=1}^{\infty} \left( F_{T^k_n}(A^k_{T^k_n-\tau}) - F_{T^k_{n-1}}(A^k_{T^k_{n-1}-\tau}) \right) \frac{1}{\Delta A^k(T_{n}^k)} \mathbb{I}_{\{T^k_{n-1} < s < T^k_n\}} \]
\[ =: I^{k,1}(t) + I^{k,2}(t); \ 0 \leq t \leq T. \]
By the very definition, \( F \in C^1 \) so we shall apply the usual mean value theorem on \( \mathbb{R} \) to get the existence of a family of random variables \( 0 < \gamma_{x,n} < 1 \) a.s.

Throughout this proof, let us denote $a_{k,n} := A^k(T_{n-1}^k) + \gamma_{k,n}A^k(T_n^k); k,n \geq 1$. In fact, triangle inequality and (E1) yield
\[
\left| I^{k,1}(t) - \int_0^t \varphi(B(s),B(t))ds \right| \leq \int_0^t \left| \varphi(A_{T_n^k}^k(s),a_{k,n}) - \varphi(B(s),a_{k,n}) \right| ds + \int_0^t \left| \varphi(B(s),a_{k,n}) - \varphi(B(s),B(t)) \right| ds.
\]

(8.34)
\[
\leq M_2 \int_0^t \left| A_{T_n^k}^k(s) - B(s) \right|\,ds + M_1 \int_0^t \left| a_{k,n} - B(t) \right|\,ds
\]
on $\{T_n^k \leq t < T_{n+1}^k\}$. We have sup$_{0 \leq u \leq |A_{T_n^k}^k(u) - B(u)|} \leq 22^{-k}$ on $\{T_n^k \leq t < T_{n+1}^k\}, |\Delta A^k(T_n^k)| \leq 2^{-k}$ a.s and $0 < \gamma_{k,n} < 1$ a.s for every $k,n \geq 1$. From (8.34), we conclude that $\lim_{k \to \infty} I^{k,1}(t) = \int_0^t \varphi(B(s),B(t))ds$ a.s for every $t \in [0,T]$. Moreover, there exists a constant $C > 0$ such that sup$_{n \geq 1} \mathbb{E} \int_0^T |I^{k,1}(t)|^2\,dt \leq TC$ and hence $\lim_{k \to \infty} I^{k,1}(\cdot) = \int_0^T \varphi(B(s),B(\cdot))\,ds$ weakly in $L^2(\mathbb{P} \times \text{Leb})$. Now, by the very definition
\[
\mathbb{E} \int_0^T |I^{k,2}(t)|^2\,dt = 2^{-2k}\mathbb{E} \int_0^T \mathbb{D}^{F,k,h}F_s(B_s)^2\,ds \leq C2^{-2k} \to 0
\]
as $k \to \infty$ due to Lemma 8.4. \qed

Lemma 8.6. If $\varphi$ satisfies (E1, E2), then $F(B) \in \mathcal{W}_{loc}$ where $DF(B)$ is given by (8.25).

Proof. Throughout this proof, $C$ is a constant which may defer from line to line. By localization as in Lemma 8.4 we may assume that both $A^k$ and $B$ live in the compact set $I_m = [-2^m, 2^m]$ for every $k \geq 1$. We claim that
\[
\lim_{k \to \infty} \mathbb{E} \int_0^T \left| \mathbb{D}^{F,k}F_s(B_s) - \mathbb{D}^kF_s(B_s) \right|^2\,ds = 0
\]
From the strong Markov property of the Brownian motion,
\[
\sigma(\xi^k_n, \eta^k_n) \text{ is independent from } \mathcal{F}_{T_n^k-1},
\]
where $\xi^k_n := T_n^k - T_{n-1}^k$ and $\eta^k_n$ follows (2.3) for $k,n \geq 1$. By noticing that for every $k \geq 1$,
\[
\mathcal{F}_{T_n^k} = \sigma(\xi^k_1, \xi^k_2, \ldots, \xi^k_n, \eta^k_1, \eta^k_2, \ldots, \eta^k_n); n \geq 1
\]
then, we shall write
\[
\mathbb{D}^{F,k}F_s(B_s) - \mathbb{D}^kF_s(B_s) = \mathbb{I}_{(\Delta A^k(T_{n}^k) > 0)}2^{2k}\mathbb{E} \left[ \int_{B(T_{n}^k-1)}^{B(T_{n}^k)} q_k^k(s,y)\,dsdy + \int_{B(T_{n}^k-1)}^{B(T_{n}^k)} q_k^k(s,y)\,dsdy | \mathcal{F}_{T_n^k} \right]
\]
\[
\mathbb{I}_{(\Delta A^k(T_{n}^k) < 0)}2^{2k}\mathbb{E} \left[ \int_{B(T_{n}^k)}^{B(T_{n}^k-1)} q_k^k(s,y)\,dsdy - \int_{B(T_{n}^k)}^{B(T_{n}^k-1)} q_k^k(s,y)\,dsdy | \mathcal{F}_{T_n^k} \right] = J_{n,k}^{1,1} + J_{n,k}^{1,2}
\]
on $\{T_n^k \leq t < T_{n+1}^k\}$, where $q_k^k(s,y) := \varphi(A^k(s,y) - \varphi(B(s),y); y \in I_m, s \in [0,T]$. From (8.36) and the fact that $\mathbb{E}(T_{n+1}^k - T_n^k) = 2^{-2k}$, we have
\[ \mathbb{E} \int_0^T |\mathbb{D}^{F,k}(B_s) - \mathbb{D}^k F_i(B_s)|^2 ds \leq 2^{-2k} \mathbb{E} \sum_{n=1}^{\infty} |J_n^{k,1} + J_n^{k,2}|^2 \mathbb{1}_{\{T_n^k \leq T\}} \]

From (E1), we can find positive constants \( C > 0 \) and \( \gamma_2 \in (0, 1] \) such that
\[ |J_n^{k,1}(n)|^2 \leq \mathbb{1}_{\{\Delta A^k(T_n^k) > 0\}} \left( C 2^{-2k\gamma_2} + C 2^{2k(1-\gamma_2)}(T_n^k - T_{n-1}^k)^2 \right). \]

Then
\[ 2^{-2k} \mathbb{E} \sum_{n=1}^{\infty} |J_n^{k,1}|^2 \mathbb{1}_{\{T_n^k \leq T\}} \leq C 2^{-2k\gamma_2} \mathbb{E} [A^k, A^k](T) + C 2^{-2k\gamma_2} \mathbb{E} \sum_{n=1}^{\infty} (T_n^k - T_{n-1}^k)^2 \mathbb{1}_{\{T_n^k \leq T\}} \leq C 2^{-2k\gamma_2} + C 2^{-2k\gamma_2} \mathbb{E} \max_{n \geq 1} |T_n^k - T_{n-1}^k| \mathbb{1}_{\{T_n^k \leq T\}} \to 0 \]
as \( k \to \infty \). The component \( J_n^{k,2} \) can be treated similarly. This shows that (8.33) holds. From (8.33), we then have
\[ \lim_{k \to \infty} \mathbb{D}^k F(B) = \lim_{k \to \infty} \left( \mathbb{D}^k F(B) - \mathbb{D}^{F,k} F(B) \right) + \lim_{k \to \infty} \mathbb{D}^{F,k} F(B) = \mathbb{D}^F F(B) \]
weakly in \( L^2(P \times \text{Leb}) \). This concludes the proof. \( \square \)

**Proposition 8.4.** If the non-anticipative functional \( F \) in (8.21) satisfies (E1, E2, E3), then \( F(B) \in \mathcal{W}_{\text{loc}} \) and the following decomposition holds
\[
F_t(B_t) = \int_0^t \int_0^s \varphi(B(r), B(s)) dr dB(s) + \int_0^t \int_{-\infty}^B \varphi(B(s), y) dy ds
\]
(8.37)

\[ - \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} \int_0^s \varphi(B(r), x) dr dB(s,x) \ell(x) |s|; 0 \leq t \leq T. \]

**Proof.** At first, let us check if \( \varphi \) satisfies (E1) then \( F \) satisfies the assumptions (L1, L2.1, L2.2). By the very definition, \( F \in C^1 \). Let us begin by checking assumption (L1). Notice that \( \partial_x F_t(t(c, x)) = \int_0^\gamma \varphi(c(s), x) ds \). Therefore, for càdlàg paths \( c \) and \( d \), we easily get
\[ |\partial_x F_t(t(c, x)) - \partial_x F_t(t(d, x))| \leq M_2 T \sup_{0 \leq t \leq T} |c(t) - d(t)|^{\gamma_2}, \]
which clearly implies assumption (L1). Let us now check assumption (L2.1). Holder continuity yields
\[ |\Delta_j \Delta_j \partial_x F_t(t(B_t, x_j))| \leq M_1 (t_j - t_{j-1}) \alpha_j (x_j - x_{j-1})^{\gamma_2}, \]
and hence (8.13) holds.

By making the change of variables \( z = y - \Delta_j k(x_j) 2^{-k} \) with \( \Delta_j k(x_j) = (j_k(x_j) - j_k(x_{j-1})) \), we similarly have
\[ |\Delta_j \Delta_j \nabla^{F,v,k} F_t(B_t, x_j)| \leq M_1 (t_j - t_{j-1}) \alpha_j (x_j - x_{j-1})^{\gamma_2}, \]
This shows (8.13). For every \( L > 0 \), we clearly have
\[ \sup_{0 \leq t \leq T} \| \partial_x F_t(t(B_t, \cdot)) \|_{[-L, L], p} \leq M_1 2L \text{ a.s.} \]
where \( p = \frac{1}{\gamma_2} \). Similarly, the usual mean value theorem yields
\[ \sup_{0 \leq t \leq T} \sup_{0 \leq t \leq T} \| \nabla^{F,v,k} F_t(B_t, \cdot) \|_{[-L, L], p} \leq M_1 2L \text{ a.s.} \]
This shows (8.13) and (8.14). Now, we shall take \( q_1 = 1 \), \( q_2 = p \) and if \( \alpha \in (0, 1) \) and \( \delta > 0 \), we obviously have
\[ \min \{ \alpha + \frac{1}{q_1}, \frac{1}{q_2} \} > 1 \text{ and } \frac{1}{p} + \frac{1}{2\bar{\alpha}} > 1. \]
Summing up the previous steps, we shall use Lemmas 8.3, 8.5, 8.6 and Theorem 8.1 to conclude (8.37) on the stochastic interval $[0, S_m]$ where $S_m$ as given in the proof of Lemma 8.3. Finally, by taking $m \to \infty$ and using (E3), we arrive at (8.37). \qed

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Departamento de Matemática Aplicada e Estatística. Universidade de São Paulo, 13560-970, São Carlos - SP, Brazil.
E-mail address: leao@icmc.usp.br

Departamento de Matemática, Universidade Federal da Paraíba, 13560-970, João Pessoa - Paraíba, Brazil.
E-mail address: alberto.ohashi@pq.cnpq.br; ohashi@mat.ufpb.br

Departamento de Matemática, Universidade Federal da Paraíba, 13560-970, João Pessoa - Paraíba, Brazil.
E-mail address: alexandre@mat.ufpb.br