LIFTING PARTIAL ACTIONS: FROM GROUPS TO GROUPOIDS

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Abstract. In this paper we are interested in the study of the existence of connections between partial groupoid actions and partial group actions. Precisely, we prove that there exists a datum connecting a partial action of a connected groupoid and a partial action of any of its isotropy groups. Furthermore, it will be proved that under a suitable condition the partial skew groupoid ring corresponding to a partial action by a connected groupoid is isomorphic to a specific partial skew group ring. We also present a Morita theory and a Galois theory related to these partial actions as well as considerations about the strictness of the corresponding Morita contexts. Semisimplicity, separability and Frobenius properties of the corresponding partial skew groupoid rings are also considered.

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1. Introduction

This paper is motivated by the following three well known facts:

i) any groupoid is a disjoint union of its connected components;
ii) any groupoid action is uniquely determined by the respective actions of its connected components;
iii) any connected groupoid is equivalent to a group as categories.

Therefore it is natural to expect some kind of connection between groupoid actions and group actions and it is enough to investigate this in the connected case. Our aim
in this paper is to investigate the connections between such both concepts in the most general setting of partial actions.

Throughout this work, by ring we mean an associative and not necessarily unital ring.

According to [BP] a partial action of a groupoid \( \mathcal{G} \) on a ring \( A \) is a set \( \alpha = (A_g, \alpha_g)_{g \in \mathcal{G}} \) where, for all \( g \in \mathcal{G} \), \( A_g \) is an ideal of \( A_{t(g)} \) \( (t(g) = gg^{-1}) \), \( A_{t(g)} \) is an ideal of \( A \) and \( \alpha_g : A_{g^{-1}} \to A_g \) is a ring isomorphism, for which some appropriate conditions of compatibility hold.

In the case that \( \mathcal{G} \) is connected \( \alpha \) gives rise to a specific datum. To construct such a datum we start by fixing an object \( x \) of \( \mathcal{G} \) and a set \( \tau(x) \) whose elements are obtained picking up one unique morphism \( \tau_y : x \to y \) for each object \( y \) of \( \mathcal{G} \) (notice that \( \tau_y \) always exists for \( \mathcal{G} \) is connected). The datum arisen from \( \alpha \) is then the triple \( (A_\alpha, \alpha_{\tau(x)}, \alpha(x)) \) where \( A_\alpha \) denotes the set of the ideals \( A_y \) indexed by the objects of \( \mathcal{G} \), \( \alpha_{\tau(x)} \) denotes the set of the ring isomorphisms \( \alpha_{\tau_y} : A_{t(y)^{-1}} \to A_{t(y)} \) for all \( \tau_y \in \tau(x) \) and \( \alpha(x) \) denotes the partial action of the isotropy group \( \mathcal{G}(x) \) of \( \mathcal{G} \) associated to the object \( x \) on the ideal \( A_x \) of \( A \). This correspondence \( \alpha \mapsto (A_\alpha, \alpha_{\tau(x)}, \alpha(x)) \) gives rise to a functor from the category of all partial actions of \( \mathcal{G} \) on \( A \) to the category of all data of this type.

Our main purpose in this paper is indeed to show how to construct partial actions of \( \mathcal{G} \) on \( A \) coming from datum of the type above described, the so called lifted partial groupoid actions.

In the next section we recall the respective formal definitions of (connected) groupoid and partial groupoid action, and detailing the fact that every groupoid is a disjoint union of its connected components.

From the section 3 on we restrict our study to the partial actions by connected groupoids.

In section 3 we introduce the category \( \mathcal{D}_{\text{par}}(A) \) of all partial actions of \( \mathcal{G} \) on \( A \), the category \( \mathcal{D}_{\tau(x)}(A) \), whose objects are datum of the type above described, and establish the right conditions for the existence of a good correlation between them via appropriate functors. As consequence we show that any partial groupoid action is an extension of a lifted one (see Proposition 3.5 (i)). We also prove that the category \( \mathcal{D}_{\tau(x)}(A) \) is independent of the choice of the object \( x \) of \( \in \mathcal{G} \) as well as of its transversal \( \tau(x) \) (see Theorem 3.2). We exploit exhaustively such a correlation in the ensuing sections.

In section 4 we discuss the necessary and sufficient conditions for a lifted partial groupoid action to be globalizable. In particular we introduce the notion of a globalizable datum and describe in details the globalization of the corresponding lifted partial groupoid action (see Theorem 4.3).

Again according to [BP] any unital partial action \( \alpha \) of a groupoid \( \mathcal{G} \) on a ring \( A \) gives rise, as expected, to two new rings, namely, the ring \( A^\alpha \) of the invariants of \( A \) under \( \alpha \) and the partial skew groupoid ring \( A \star_\alpha \mathcal{G} \), as well as a canonical Morita context connecting them. Such a context keeps a close relation with the notion of Galois extension. In fact, the condition to ensure that \( A \) is a Galois extension of \( A^\alpha \) depends on the strictness of this context. Actually, by [BP] Theorem 5.3, such a strictness is equivalent to \( A \) being a Galois of \( A^\alpha \) and a specific map, called trace, from \( A \) to \( A^\alpha \) being surjective. Sections 5, 6 and 7 are dedicated to dealing with all these mentioned concepts.
Section 5 concerns to the partial skew groupoid ring $A \rtimes_\alpha \mathcal{G}$. We prove, under an appropriate condition, the existence of a ring isomorphism between $A \rtimes_\alpha \mathcal{G}$ and the partial skew group ring $(A \rtimes_{\alpha^*} \mathcal{G}_0^2) \star_{\theta} \mathcal{G}(x)$, where $\mathcal{G}_0$ denotes the set of the objects of $\mathcal{G}$, $\mathcal{G}_0^2 = \mathcal{G}_0 \times \mathcal{G}_0$ is a groupoid whose structure is standard and well known, $\alpha^*$ is a global action of $\mathcal{G}_0^2$ on $A$ and $\theta$ is a partial group action of $\mathcal{G}(x)$ on $A \rtimes_{\alpha^*} \mathcal{G}_0^2$ (see Theorem 5.4). Also, we specialize such condition to the case of lifted partial groupoid actions (see Proposition 5.5).

In sections 6, 7 and 8 we restrict our study only to the lifted partial groupoid actions. In section 6 we recall the notions of invariants and trace map and discuss their properties and correlations (see Theorem 6.4).

Section 7 is devoted to the Morita theory related to lifted partial groupoid actions. More specifically, we construct Morita contexts connecting rings of invariants and partial skew groupoid rings and make considerations on their strictness. In particular we show how closed are the corresponding Morita and Galois theories (see Theorem 7.6).

Finally we end this manuscript with the section 8, where we deal with ring theoretic properties such the semisimplicity, separability and Frobenius of the partial skew groupoid ring corresponding to a lifted partial groupoid action (see Theorems 8.2 and 8.4, and Corollary 8.3).

2. Preliminaries

As preliminaries we firstly recall from the literature the notions of groupoid, connected groupoid and partial group action and, in the sequel, we show how a groupoid $\mathcal{G}$ can be decomposed in a disjoint union of connected ones. Via such a decomposition it becomes clear that any partial action of $\mathcal{G}$ is univocally determined by the partial actions of its connected components, which reduces the study of partial groupoid actions to the connected case.

2.1. Groupoids. A groupoid $\mathcal{G}$ is a small category where every morphism is an isomorphism. As a small category $\mathcal{G}$ is composed by a set of morphisms and a set of objects. Furthermore, to each morphism $h$ of $\mathcal{G}$ correspond naturally the objects $s(h)$ and $t(h)$ of $\mathcal{G}$ called the source (or domain) and the target (or range) of $h$ respectively. In particular each object $x$ of $\mathcal{G}$ can be identified with its identity morphism, here also denoted by $x$. Hence, any element of $\mathcal{G}$ can be seen as a morphism of $\mathcal{G}$ and all such considerations can be summarized in the following presentation

$$\mathcal{G} \xleftrightarrow{s,t} \mathcal{G}_0$$

where $\mathcal{G}_0$ denotes the set of the objects of $\mathcal{G}$ and $s$ and $t$ the source and the target maps.

The composition in $\mathcal{G}$ is the map

$$m : \mathcal{G}_s \times_t \mathcal{G} \to \mathcal{G} \quad \text{denoted by} \quad m(g, h) = gh,$$

where

$$\mathcal{G}_s \times_t \mathcal{G} = \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = t(h)\}.$$
In particular it follows from this characterization of a groupoid that indeed

\( s(g) = g^{-1}g = t(g^{-1}) \) for all \( g \in \mathcal{G} \),

(1)

\( s(x) = t(x) = x = x^2 \) for all \( x \in \mathcal{G}_0 \),

(2)

\( t(g)g = g = gs(g) \) for all \( g \in \mathcal{G} \),

(3)

\( s(gh) = s(h) \) and \( t(gh) = t(g) \), for all \( (g, h) \in \mathcal{G}_s \times \mathcal{G}_t \).

(4)

Such above relations will be freely used along the text.

For each \( x \in \mathcal{G}_0 \) we set \( \mathcal{S}_x = \{ g \in \mathcal{G} : s(g) = x \} \), \( \mathcal{T}_x = \{ g \in \mathcal{G} : t(g) = x \} \) and \( \mathcal{G}(x) = \mathcal{S}_x \cap \mathcal{T}_x \). In particular \( \mathcal{G}(x) \) is a group, called the isotropy (or principal) group associated to \( x \).

2.2. Connected groupoids. Let \( \mathcal{G} \rightrightarrows \mathcal{G}_0 \) be as in the previous subsection. Any subgroupoid \( \mathcal{H} \) of \( \mathcal{G} \) is called connected if for every \( x, y \in \mathcal{H}_0 \) the set \( \mathcal{H}(x, y) \) of all morphisms from \( x \) to \( y \) is not empty. Such a notion of connectedness suggests a natural way to obtain a partition of \( \mathcal{G} \) in connected components via the following equivalence relation on \( \mathcal{G}_0 \): for any \( x, y \in \mathcal{G}_0 \)

\[ x \sim y \iff \mathcal{G}(x, y) \neq \emptyset, \]

that is, if and only if there exists \( g \in \mathcal{G} \) such that \( s(g) = x \) and \( t(g) = y \).

Every equivalence class \( X \in \mathcal{G}_0 / \sim \) determines a full connected subgroupoid \( \mathcal{G}_X \rightrightarrows X \) of \( \mathcal{G} \), which is also called a connected component of \( \mathcal{G} \). It is immediate to see that

\[ \mathcal{G} = \bigcup_{X \in \mathcal{G}_0 / \sim} \mathcal{G}_X \]

(where the symbol \( \bigcup \) denotes a disjoint union).

2.3. Partial groupoid action. According to \([BP]\), a partial action of a groupoid \( \mathcal{G} \) on a ring \( A \) is a pair \( \alpha = (A_g, \alpha_g)_{g \in \mathcal{G}} \) where, for each \( g \in \mathcal{G} \), \( A_{t(g)} \) is an ideal of \( A \), \( A_g \) is an ideal of \( A_{t(g)} \), \( \alpha_g : A_{g^{-1}} \to A_g \) is an isomorphism of rings, and the following conditions hold:

(5) \( \alpha_x \) is the identity map of \( A_x \),

(6) \( \alpha_g \alpha_h \leq \alpha_{gh} \),

for all \( x \in \mathcal{G}_0 \) and \( (g, h) \in \mathcal{G}_s \times \mathcal{G}_t \). We say that \( \alpha \) is global if \( \alpha_g \alpha_h = \alpha_{gh} \), for all \( (g, h) \in \mathcal{G}_s \times \mathcal{G}_t \).

The condition (6) above means that \( \alpha_{gh} \) is an extension of \( \alpha_g \alpha_h \), that is, the domain \( A_{(gh)^{-1}} \) of \( \alpha_{gh} \) contains the domain \( \alpha_{h}^{-1}(A_{g^{-1}} \cap A_h) \) of \( \alpha_g \alpha_h \), and both the maps coincide on this last set.

Notice also that \( \alpha \) induces by restriction a partial action \( \alpha_x = (A_g, \alpha_g)_{g \in \mathcal{G}(x)} \) of the group \( \mathcal{G}(x) \) on the ring \( A_x \), for all \( x \in \mathcal{G}_0 \).
Lemma 1.1 from [BP] gives some complementary properties of partial groupoid actions, enumerated below, that will also be useful in the sequel.

(7) \( \alpha \) is global if and only if \( A_g = A_{t(g)} \), for all \( g \in \mathcal{G} \),

(8) \( \alpha_{g^{-1}} = \alpha^{-1}_g \), for all \( g \in \mathcal{G} \),

(9) \( \alpha_g(A_g^{-1} \cap A_h) = A_g \cap A_{gh} \), for all \((g,h) \in G_s \times G_t\).

Remark 2.1. It is clear that given any two disjoint groupoids \( G'_s \Rightarrow G'_0 \) and \( G''_s \Rightarrow G''_0 \) one always can form the groupoid \( G = G'_s \cup G''_s \Rightarrow G'_0 \cup G''_0 \), whose composition is induced in an obvious way by the compositions in \( G'_s \) and \( G''_s \) respectively. Furthermore, it is also straightforward to check that partial actions of \( G \) on a ring \( A \) induce by restriction partial actions of \( G'_s \) and \( G''_s \) on \( A \) and, conversely, they are univocally determined by the partial actions of \( G'_s \) and \( G''_s \) on \( A \). Therefore, it follows from the subsection 2.2 that the study of partial groupoid actions reduces to the connected case.

3. Lifting partial group actions

From now on by \( G \Rightarrow G_0 \) we will always denote a connected groupoid and by \( A \) a ring on which \( G \) acts partially. Recalling notations,

\[ \mathcal{G}(x,y) = \{ g \in \mathcal{G} \mid s(g) = x \text{ and } t(g) = y \} = S_x \cap T_y \]

and, in particular,

\[ \mathcal{G}(x,x) = S_x \cap T_x = \mathcal{G}(x), \]

for all \( x, y \in \mathcal{G}_0 \). Furthermore, recall that the identity map of each object \( y \in \mathcal{G}_0 \) is also denoted by \( y \).

Hereafter, let \( x \in \mathcal{G}_0 \) be a fixed object of \( \mathcal{G} \). Now consider on \( S_x \) the following equivalence relation: \( g \equiv_x l \iff l^{-1}g \in \mathcal{G}(x) \iff t(g) = t(l) \). A transversal for \( \equiv_x \) such that \( \tau_x = x \) will be called a transversal for \( x \) and denoted by \( \tau(x) \), that is, \( \tau(x) = \{ \tau_y : y \in \mathcal{G}_0 \} \) where \( \tau_y \) is a chosen morphism in \( \mathcal{G}(x,y) \), for each \( y \in \mathcal{G}_0 \) with \( \tau_x = x \).

Observe that any traversal \( \tau(x) \) determines a natural map \( \pi \) from \( \mathcal{G} \) to \( \mathcal{G}(x) \) that associates to each element \( g \in \mathcal{G} \) a unique element \( g_x \in \mathcal{G}(x) \) via the following formula

\[ \pi(g) = g_x = \tau_{t(g)}^{-1} g \tau_{s(g)}, \]

illustrated by the following diagram

\[ \begin{array}{ccc}
  s(g) & \xrightarrow{g} & t(g) \\
  \downarrow{\tau_{s(g)}} & & \downarrow{\tau_{t(g)}} \\
  x & & x
\end{array} \]

Lemma 3.1. \( \pi \) is a groupoid epimorphism such that

\[ \pi(\tau(x)) = \{ x \} \text{ and } \pi(h) = h, \text{ for all } h \in \mathcal{G}(x). \]
Proof. Observe that \( \pi(G_0) = \{ x \} \). Indeed,
\[
y_x = \tau_{t(y)}^{-1} y \tau_s(y) = \tau_y^{-1} y \tau_y = \tau_x = x.
\]
Moreover, for all \( g, h \in G \) such that \( s(g) = t(h) \), we have
\[
(gh)_x = \tau_{t(gh)}^{-1} gh \tau_s(gh) = \tau_{t(g)}^{-1} \tau_{t(h)}^{-1} g \tau_s(g) \tau_{t(h)}^{-1} h \tau_s(h)
\]
\[
= \tau_{t(g)}^{-1} h \tau_s(h) = \tau_x.
\]
Also, for all \( y \in G_0 \),
\[
(t_y)_x = \tau_{t(t_y)}^{-1} t_y \tau_s(t_y) = \tau_{t_y}^{-1} t_y \tau_x = s(t_y) t_x = x^2 = x.
\]
Finally, if \( h \in G(x) \) then
\[
h_x = \tau_{t(h)}^{-1} h \tau_s(h) = \tau_x h x = h.
\]

3.1. The category \( G_{par}(A) \). We will denote by \( G_{par}(A) \) the category whose objects are partial actions of \( G \) on a fixed ring \( A \) and whose morphisms are defined as follows. Given \( \alpha = (A_g, \alpha_g)_{g \in G} \) and \( \alpha' = (A'_g, \alpha'_g)_{g \in G} \in G_{par}(A) \), a morphism \( \psi : \alpha \rightarrow \alpha' \) is a set of ring homomorphisms \( \psi = \{ \psi_y : A_y \rightarrow A'_y \}_{y \in G_0} \) such that
\begin{enumerate}
\item \( \psi_{t(g)}(A_g) \subseteq A'_g \),
\item \( \alpha'_g \circ \psi_s(g) = \psi_t(g) \circ \alpha_g \) in \( A^{-1}_g \),
\end{enumerate}
for all \( g \in G \).

Observe now that any pair \( (\alpha, \tau(x)) \), where \( \alpha = (A_g, \alpha_g)_{g \in G} \in G_{par}(A) \) and \( \tau(x) \) is a transversal for \( x \), determines the following datum:
\begin{itemize}
\item a set of ideals of \( A \): \( A_\alpha = \{ A_y \}_{y \in G_0} \),
\item a set of ring isomorphisms \( \alpha_{\tau(x)} = \{ \alpha_{\tau_y} : A_{\tau_y}^{-1} \rightarrow A_{\tau_y} \}_{y \in G_0} \) and
\item a partial group action \( \alpha_{\tau(x)} = (A_g, \alpha_g)_{g \in G(x)} \) of \( G(x) \) on \( A_x \).
\end{itemize}
Moreover, since \( t(\tau_y) = y \) (resp., \( t(\tau^{-1}_x) = x \)), then \( A_{\tau_y} \) (resp., \( A_{\tau^{-1}_x} \)) is an ideal of \( A_y \) (resp., \( A_x \)), for all \( y \in G_0 \). In particular, \( A_{x^{-1}} = A_x = A_{x^{-1}} \) and \( \alpha_{\tau_x} = \alpha_x \) is the identity map of \( A_x \) by (5).

Finally, observe that
\[
\alpha_{\tau_t(g)}(A_{t^{-1}_t(g)} \cap A_{t(g)} \cap A_{t(g)}^{-1}) \cap \alpha_{\tau_x} (A_{x^{-1}} \cap A_{\tau^{-1}_t(g)} \cap A_{t(g)} \cap A_{t(g)}^{-1}) \cap \alpha_{\tau_x} \cap A_g \cap A_{t(g)} \cap A_{t(g)}^{-1}
\]
which is an ideal of \( A_{t(g)} \), for all \( g \in G \).
This suggests the existence of a new category, named here a data category, which we will describe below.

3.2. The category \( \mathcal{D}_{\tau(x)}(A) \). To describe the data category \( \mathcal{D}_{\tau(x)}(A) \), with \( \tau(x) \) a fixed transversal, our inspiration is the datum we point out in the previous subsection. The objects of \( \mathcal{D}_{\tau(x)}(A) \) are given by datum of the following type:

(a) a set of ideals of \( A \): \( I = \{ I_y \}_{y \in G_0} \),

(b) a set of ring isomorphisms: \( \gamma_{\tau(y)} = \{ \gamma_{\tau(y)} : I_{\tau(y)}^{-1} \to I_{\tau(y)} \}_{y \in G_0} \) and

(c) a partial group action \( \gamma(x) = (I_y, \gamma_g)_{g \in \mathcal{G}(x)} \) of \( \mathcal{G}(x) \) on \( I_x \),

under the additional assumptions

(d) \( \gamma_x = \gamma_{\tau(x)} \) is the identity map of \( I_x = I_{\tau(x)} \),

(e) \( I_{\tau(y)} \) (resp., \( I_{\tau(y)}^{-1} \)) is an ideal of \( I_{\tau(y)} = I_y \) (resp., \( I_{\tau(y)}^{-1} = I_x \)), for all \( y \in G_0 \),

(f) \( \gamma_{\tau(y)}(I_{\tau(y)}^{-1} \cap \gamma_{\tau(x)}(I_{\tau(y)}^{-1} \cap I_{\tau^{-1}(y)}) \) is an ideal of \( I_{\tau(y)} \), for all \( g \in \mathcal{G} \).

For each such a datum we denote the corresponding object of \( \mathcal{D}_{\tau(x)}(A) \) by the triple \( (I, \gamma_{\tau(x)}, \gamma_{\tau(x)}(x)) \). The morphisms of \( \mathcal{D}_{\tau(x)}(A) \) are given by sets of ring homomorphisms denoted and described as follows:

\[
\mathcal{f} = \{ f_y \}_{y \in G_0} : (I, \gamma_{\tau(x)}, \gamma_{\tau(x)}(x)) \to (I', \gamma'_{\tau(x)}(x), \gamma'_{\tau(x)}(x))
\]

where

- for each \( y \in G_0 \), \( f_y : I_y \to I_y' \) is a ring homomorphism such that \( f_y(I_{\tau(y)}) \subseteq I_{\tau(y)}' \) (if \( y \neq x \)), \( f_x(I_{\tau(y)}^{-1}) \subseteq I_{\tau(y)}'^{-1} \) and \( \gamma_y f_x = f_y \gamma_{\tau(y)} \).

- \( f_x : \gamma(x) \to \gamma'(x) \) is a morphism of partial group actions, i.e., \( f_x : I_x \to I_x' \) satisfies \( f_x(I_h) \subseteq I_h' \), \( \gamma'_h f_x = f_x \gamma_h \) in \( I_{h^{-1}}' \), for all \( h \in \mathcal{G}(x) \).

Now, we prove that the category \( \mathcal{D}_{\tau(x)}(A) \) does not depend neither on the choice of the object \( x \in G_0 \), nor on the choice of the transversal \( \tau(x) \) of \( x \).

**Proposition 3.2.** \( \mathcal{D}_{\tau(x)}(A) \) and \( \mathcal{D}_{\lambda(z)}(A) \) are isomorphic as categories, for all \( x, z \in G_0 \) and their respective transversals \( \tau(x) \) and \( \lambda(z) \).

**Proof.** We start by defining the functor \( \mathcal{F}_{\tau(x)}^{\lambda(z)} : \mathcal{D}_{\tau(x)}(A) \to \mathcal{D}_{\lambda(z)}(A) \), its reverse is defined similarly.

Firstly the correspondence between the objects: to each \( \gamma = (I, \gamma_{\tau(x)}, \gamma_{\tau(x)}(x)) \in \mathcal{D}_{\tau(x)}(A) \) we associate \( \gamma' = (I', \gamma'_{\tau(x)}(x), \gamma'_{\tau(x)}(x)) \in \mathcal{D}_{\lambda(z)}(A) \) chosen in the following way:

- if \( I = \{ I_y \}_{y \in G_0} \) we take \( I' = \{ I_y' \}_{y \in G_0} \) such that \( I_y' = I_x \), \( I_z' = I_z \) and \( I_y = I_y' \) for all \( y \notin \{ x, z \} \),

- if \( \gamma_{\tau(x)} = \{ \gamma_{\tau(y)} : I_{\tau(y)} \to I_{\tau(y)} \}_{y \in G_0} \) we take \( \gamma'_{\lambda(z)} = \{ \gamma'_{\lambda_z} : I_{\lambda_z}'^{-1} \to I_{\lambda_z} \}_{z \in G_0} \) such that \( \gamma'_z = \gamma_{\lambda_z} \), \( \gamma'_{\lambda_z} = \gamma_{\tau(z)} = \gamma_x \), \( \gamma'_{\lambda_z} = \gamma'_{\lambda_z} = \gamma_{\tau(y)} \) and \( \gamma'_{\lambda_z} = \gamma_{\tau(y)} \) for all \( y \notin \{ z, x \} \),

- if \( \gamma(x) = (I_h, \gamma_h)_{h \in \mathcal{G}(x)} \) we take \( \gamma'_{\gamma(z)} = (I_h', \gamma_h')_{h \in \mathcal{G}(z)} \) such that \( I'_h = I_{\phi(i)}' \), \( \gamma'_h = \gamma_{\phi(i)} \) for all \( h \in \mathcal{G}(z) \), where \( \phi : \mathcal{G}(z) \to \mathcal{G}(x) \) is the group isomorphism defined by \( i \mapsto \tau_z^{-1} l \tau_z \).
Now the correspondence between the morphisms: for a morphism \((f_y, f^{(x)}): \gamma \to \delta\) in \(\mathcal{D}_{\tau(x)}(A)\) we associate the morphism \((f'_y, f'^{(x)}): \gamma' \to \delta'\) in \(\mathcal{D}_{\lambda(z)}(A)\), given by \(f'^{(x)} = f^{(x)}\), \(f'_y = f_z\) and \(f'_y = f_y\) for all \(y \notin \{x, z\}\).

This way we have got the functors \(F^{\lambda(z)}_{\tau(x)}\) and \(F^{\tau(x)}_{\lambda(z)}\). It is straightforward to check that the compositions \(F^{\tau(x)}_{\lambda(z)} \circ F^{\lambda(z)}_{\tau(x)}\) and \(F^{\lambda(z)}_{\tau(x)} \circ F^{\tau(x)}_{\lambda(z)}\) are respectively the identity functors of \(\mathcal{D}_{\tau(x)}(A)\) and \(\mathcal{D}_{\lambda(z)}(A)\).

\[\square\]

### 3.3. The functors \(F_{\tau(x)}\) and \(G_{\tau(x)}\)

Our purpose in this subsection is to find functors relating both the categories \(\mathcal{G}_{\text{par}}(A)\) and \(\mathcal{D}_{\tau(x)}(A)\). It is easy to see from the subsection 3.1 that the association \(F_{\tau(x)}: \mathcal{G}_{\text{par}}(A) \to \mathcal{D}_{\tau(x)}(A)\) given by

\[
\alpha \longmapsto (A\alpha, \alpha_{\tau(x)}, \alpha_{(x)}),
\alpha \longmapsto F_{\tau(x)}(\alpha) \xrightarrow{\psi} F_{\tau(x)}(\alpha'), \text{ where } F_{\tau(x)}(\psi) = \{\psi_y\}_{y \in \mathcal{G}_0},
\]

is indeed a functor.

For the reverse functor, given a triple \((I, \gamma_{\tau(x)}, \gamma_{(x)})\) we set \(\beta = (B_g, \beta_g)_{g \in \mathcal{G}}\), where each \(\beta_g\) is the following ring isomorphism

\[
\beta_g = \begin{cases} 
\text{the identity map of } I_y, & \text{if } g = y \in \mathcal{G}_0, \\
\gamma_{\tau(g)} \circ \gamma_{g_x} \circ \gamma_{\tau(g)}^{-1}, & \text{if } g \notin \mathcal{G}_0.
\end{cases}
\]

and each \(B_g\) is taken as the range of \(\beta_g\), for all \(g \in \mathcal{G}\).

In order to completely understand (12) it is convenient to recall how are defined the domain and the range of a composition of two partial bijections. Notice that the maps in (12) are partial bijections of \(A\). In general, if \(u\) and \(v\) are two given partial bijections of a non empty set, the domain and the range of the composition \(u \circ v\) are respectively defined as follows:

\[\text{dom}(u \circ v) = v^{-1}(\text{dom}(u) \cap \text{ran}(v)) \quad \text{and} \quad \text{ran}(u \circ v) = u(\text{dom}(u) \cap \text{ran}(v)).\]

In the specific case of (12) we have

\[
B_g = \begin{cases} 
I_y, & \text{if } g = y \in \mathcal{G}_0, \\
\gamma_{\tau(g)}(I_{\tau(g)}^{-1} \cap \gamma_{g_x}(I_{\tau(g)}^{-1} \cap I_{g_x}^{-1})) & \text{if } g \notin \mathcal{G}_0.
\end{cases}
\]

**Theorem 3.3.** The pair \(\beta = (B_g, \beta_g)_{g \in \mathcal{G}}\) constructed above is an object in the category \(\mathcal{G}_{\text{par}}(A)\).
Proof. Take \( g, h \notin \mathcal{G}_0 \) such that \( s(g) = t(h) \). Then, by restriction to the domain of \( \beta_y \beta_h \), we have the following inequalities

\[
\beta_y \beta_h = \gamma_{t(y)} \gamma_g \gamma_{s(y)}^{-1} \gamma_{t(h)} \gamma_h \gamma_{s(h)}^{-1} \\
\leq \gamma_{t(y)} \gamma_g \gamma_h \gamma_{s(h)}^{-1} \\
(\ast) \\
\leq \gamma_{t(y)} \gamma_g \gamma_{s(h)}^{-1} \\
(10) \\
= \gamma_{t(y)} \gamma_{(gh)} \gamma_{s((gh))}^{-1} \\
(4) \\
= \beta_{gh}.
\]

Notice that (\ast) holds for follows for \( \gamma \) is a partial group action of \( \mathcal{G}(x) \) over \( I_x \). Since \( \text{dom}(\beta_y \beta_h) \subseteq \text{dom}(\beta_{gh}) \), we obtain \( \beta_y \beta_h \leq \beta_{gh} \).

If \( h = y \in \mathcal{G}_0 \) (resp. \( g = y \in \mathcal{G}_0 \)) and \( s(g) = y \) (resp. \( t(h) = y \)) then \( \beta_y \beta_h = \beta_y = \beta_{gh} \) (resp. \( \beta_y \beta_h = \beta_h = \beta_{gh} \)). When \( g = h = y \in \mathcal{G}_0 \), we have \( \beta_y \beta_h = \beta_y = \beta_{gh} \).

By assumption, \( B_y = I_y \) is an ideal of \( A \) and \( B_y \) is an ideal of \( B_t(y) = I_t(y) \), for all \( y \in \mathcal{G}_0 \) and \( g \notin \mathcal{G}_0 \).

\[\square\]

Remark 3.4. It is clear from (12) and (13) that \( \beta_y = \gamma_y \) and \( B_y = I_y \) for all \( g \in \mathcal{G}(x) \), that is, \( \beta_y = \gamma_y \). Thence, any partial action \( \beta = (B_y, \beta_y)_{y \in \mathcal{G}} \) of \( \mathcal{G} \) on \( A \), constructed as above from a given datum \( \gamma = (I, \gamma_A(x), \gamma(x)) \), will be referred as a lifting partial groupoid action.

By Theorem 3.3, the association \( G_{\tau(x)} : \mathcal{D}_{\tau(x)}(A) \rightarrow \mathcal{G}_{\text{par}}(A) \) given by

\[
G_{\tau(x)}((I, \gamma_A(x), \gamma(x))) = \beta = (B_y, \beta_y)_{y \in \mathcal{G}}, \quad (\text{see } (12) \text{ and } (13))
\]

\[
G_{\tau(x)}((f_y, f^{(x)})_{y \in \mathcal{G}_0}) = \{f_y\}_{y \in \mathcal{G}_0},
\]

is a functor.

The next result establishes the relation between the functors \( F_{\tau(x)} \) and \( G_{\tau(x)} \).

Proposition 3.5. Let \( F_{\tau(x)} \) and \( G_{\tau(x)} \) be the functors constructed above.

(i) If \( \alpha = (A_g, \alpha_g)_{g \in \mathcal{G}} \in \mathcal{G}_{\text{par}}(A) \) and \( \beta = G_{\tau(x)} \circ F_{\tau(x)}(\alpha) \) then \( \beta \leq \alpha \), that is, \( \beta_g \leq \alpha_g \), for all \( g \in \mathcal{G} \). Moreover, \( \beta = \alpha \) if and only if \( A_g \subseteq A_{\tau(y)} \cap A_{\tau(g)} \), for all \( g \notin \mathcal{G}_0 \).

(ii) \( F_{\tau(x)} \) is a left inverse functor of \( G_{\tau(x)} \), that is, \( F_{\tau(x)} \circ G_{\tau(x)}(\gamma) = \gamma \), for all \( \gamma \in \mathcal{D}_{\tau(x)}(A) \).
Example 3.6. Let $A$ be a partial (not global) groupoid action which is described below:

If $g = y \in G_0$, then it is immediate that $\beta_g = \beta_y = \alpha_y = \alpha_g$ and the first part of (i) follows. Since

$$B_g = \alpha_{\tau(g)}(A_{\tau(g)}^{-1} \cap \alpha_g(A_{\tau(g)}^{-1} \cap A_{g^{-1}}))$$

for all $g \notin G_0$, we obtain the second part of (i). The item (ii) is easily checked using the definitions of $G_{\tau(x)}$ and $F_{\tau(x)}$. \hfill $\square$

A partial action $\alpha = (A_g, \alpha_g)_{g \in G}$ of $G$ on $A$ will be called $\tau(x)$-global if

$$A_{\tau_g^{-1}} = A_x \text{ and } A_{\tau_y} = A_y, \text{ for all } y \in G_0.$$ (14)

Clearly, any global action of $G$ on $A$ is $\tau(x)$-global. We present below an example of a partial (not global) groupoid action which is $\tau(x)$-global.

Example 3.6. Let $G = \{g, h, l, m, l^{-1}, m^{-1}\} \Rightarrow \{x, y\} = G_0$ be the groupoid given in Example 3.9 and $A = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4$, where $\mathbb{C}$ denotes the complex number field. Consider the partial action $(A_z, \alpha_z)_{z \in G}$ of $G$ on $A$ described below:

$$A_x = \mathbb{C}e_1 \oplus \mathbb{C}e_2 = A_l^{-1}, \quad A_y = \mathbb{C}e_3 \oplus \mathbb{C}e_4 = A_l,$$

$$A_g = \mathbb{C}e_1 = A_{g^{-1}} = A_{m^{-1}}, \quad A_m = A_h = \mathbb{C}e_3 = A_{h^{-1}},$$

and

$$\alpha_x = id_{A_x}, \quad \alpha_y = id_{A_y}, \quad \alpha_g : ae_1 \mapsto \overline{ae}_1, \quad \alpha_l : ae_3 \mapsto \overline{ae}_3, \quad \alpha_m : ae_1 \mapsto \overline{ae}_3,$$

$$\alpha_m^{-1} : ae_3 \mapsto \overline{ae}_1, \quad \alpha_l : ae_1 + be_2 \mapsto ae_3 + be_4, \quad \alpha_l^{-1} : ae_3 + be_4 \mapsto ae_1 + be_2,$$

where $\overline{a}$ denotes the conjugate of $a$, for all $a \in \mathbb{C}$. It is clear that $\alpha$ satisfies the required for the transversal $\tau(x) = \{\tau_x = x, \tau_y = l\}$ of $x$.

Remark 3.7. The notion of partial action $\tau(x)$-global indeed depends on the choice of the transversal $\tau(x)$. Notice that the partial action $\alpha$ of $G$ on $A$ given in the previous example is $\tau(x)$-global but not $\lambda(x)$-global, where $\lambda(x) = \{\lambda_x = x, \lambda_y = m\}$. 
The full subcategory of $\mathcal{D}_{\tau(x)}(A)$ with objects $(I, \gamma_{\tau(x)}, \gamma(x))$ satisfying $I_y = I_y$ and $I_y^{-1} = I_x$, for all $y \in \mathcal{G}_0$, will be denoted by $\mathcal{D}^{gl}_{\tau(x)}(A)$. Also, the full subcategory of $\mathcal{G}_{par}(A)$ whose objects are the $\tau(x)$-global actions of $\mathcal{G}$ on $A$ will be denoted by $\mathcal{G}^{gl}_{\tau(x)}(A)$.

**Corollary 3.8.** The categories $\mathcal{G}^{gl}_{\tau(x)}(A)$ and $\mathcal{D}^{gl}_{\tau(x)}(A)$ are isomorphic.

**Proof.** Let $\alpha = (A_g, \alpha_g)_{g \in \mathcal{G}} \in \mathcal{G}^{gl}_{\tau(x)}(A)$. Then

\[
A_g = \alpha_g(A_{g^{-1}} \cap A_{s(g)})
\]

(14) $\alpha_g(A_{g^{-1}} \cap A_{r(s(g)})$)

(9) $A_g \cap A_{g\tau s(g)}$

$\subset A_{t(g)} \cap A_{g\tau s(g)}$

$= A_{t(g)} \cap A_{g\tau s(g)},$

for all $g \in \mathcal{G}$. It follows from Proposition 3.5 (i) that $G_{\tau(x)} \circ F_{\tau(x)}(\alpha) = \alpha$ and whence $F_{\tau(x)}$ is a right inverse of $G_{\tau(x)}$. Using Proposition 3.5 (ii), we conclude the result. \[\square\]

We end this subsection with a pair of examples to illustrate the construction of a lifted partial groupoid action.

**Example 3.9.** Let $\mathcal{G} = \{g,h,l,m, l^{-1}, m^{-1}\} \Rightarrow \{x,y\} = \mathcal{G}_0$ the groupoid with the following composition rules

\[
g^2 = x, \quad h^2 = y, \quad lg = m = hl, \quad g \in \mathcal{G}(x), \quad h \in \mathcal{G}(y) \quad \text{and} \quad l, m \in \mathcal{G}(x, y).
\]

The diagram bellow illustrates the structure of $\mathcal{G}$:

```
x
\downarrow g
\quad l
\downarrow
y
```

```
x
\quad
\downarrow h

m
\quad
\downarrow
y
```

Fix a transversal $\tau(x) = \{\tau_x = x, \tau_y = l\}$ of $x$, a ring $A$ and $J, L, I$ ideals of $A$ such that $J, L \subset I$. Fix also ring automorphisms $\gamma$ and $\sigma$ of $I$ such that $\sigma(L) = L, \sigma|_L = \sigma^{-1}|_L$ and $\gamma(J) = J$. In all what follows id$_I$ denotes the identity map of $I$.

Now take the following datum $(I, \gamma_{\tau(x)}, \gamma(x))$, where

- $\{I_x = I_y = I\}$ is the set of the ideals of $A$ indexed by the elements of $\mathcal{G}_0$,
- $\gamma_{\tau(x)} = \{\gamma_{\tau_x} = \gamma_x = \text{id}_I, \gamma_{l} = \gamma_l = \gamma\}$ is the set of the ring isomorphisms,
- $\gamma(x) = \{(I_x = I, I_g = L = I_y^{-1}), \{\gamma_x = \text{id}_I, \gamma_g = \sigma = \gamma_{g^{-1}}\}\}$ is the partial action of $\mathcal{G}(x) = \{x, g\}$ on $I_x$.

Applying the functor $G_{\tau(x)}$ in the datum $(I, \gamma_{\tau(x)}, \gamma(x))$ we obtain the following partial action $\beta = (B_u, \beta_u)_{u \in \mathcal{G}}$ of $\mathcal{G}$ on $A$. Firstly, note that (10) implies that

\[
g_x = h_x = m_x = m_x^{-1} = g, \quad l_x = l_x^{-1} = x_x = y_x = x.
\]
From (12) and (13) it follows that
\[ \beta_x = \beta_y = id_I, \quad \beta_g = \sigma|L = \beta_g^{-1}, \quad \beta_h = \gamma \sigma \gamma^{-1}|_{\gamma(J \cap \sigma(J \cap L))} = \beta_h^{-1}, \]
\[ \beta_m = \gamma \sigma : \sigma(J \cap L) \rightarrow \gamma(J \cap L), \quad \beta_{m^{-1}} = \sigma \gamma^{-1} : \gamma(J \cap L) \rightarrow \sigma(J \cap L). \]

**Example 3.10.** Let \( A \) be a unital ring, \( m \) be a positive integer and \( \{e_i\}_{i=1}^m \) a set of orthogonal idempotents in the center of \( A \) whose sum is \( 1_A \). Fix \( i_0 \in \{1, \ldots, m\} \) and consider a group \( G \) acting on \( Ae_{i_0} \) by ring isomorphisms \( \sigma_g, g \in G \).

Denote by \( \Gamma_{\gamma(i_0)}^m \) the groupoid having as objects the set \{1, \ldots, m\} and morphisms \((g, i, j), \) where \( g \in G \) and \( i, j \in \{1, \ldots, m\} \). The source and target maps on \( \Gamma_{\gamma(i_0)}^m \) are \( s(g, i, j) = j \) and \( t(g, i, j) = i \), respectively. The composition is given by the rule \((g, i, j)(h, j, k) = (gh, i, k) \). Clearly, \( \Gamma_{\gamma(i_0)}^m \) is a connected groupoid. Fix the transversal \( \tau(i_0) = \{\tau_i = (e, i, i_0)\}_{1 \leq i \leq m} \), where \( e \) denotes the identity element of \( G \). Consider the following datum \((I, \gamma_{\tau(i_0)}, \gamma(i_0))\):

- \( \{I_i = I_{\tau_i} = Ae_i\}_{1 \leq i \leq m} \) and \( \{I_{\tau_i^{-1}} = Ae_{i_0}\}_{1 \leq i \leq m} \) is the set of the ideals of \( A \) indexed by the objects of \( \Gamma_{\gamma(i_0)}^m \),
- \( \gamma_{\tau(i_0)} = \{\tau_i = Ae_{e_i} \supset ae_i \mapsto ae_{i_0} \in Ae_i\} \) is the set of the ring isomorphisms,\n- \( \gamma(i_0) = \{\gamma_{(g, i_0, i_0)} = \sigma_g\} \) is the action of the group \( \Gamma_{\gamma(i_0)}^m = \{(g, i_0, i_0) \mid g \in G\} \) on \( Ae_{i_0} \).

Applying the functor \( G_{\tau(i_0)} \) in the datum \((I, \gamma_{\tau(i_0)}, \gamma(i_0))\) we obtain the following global action \( \beta \) of \( \Gamma_{\gamma(i_0)}^m \) on \( A \):

\[ \beta_{(g, i, j)} = \begin{cases} \text{the identity map of } Ae_i, & \text{if } (g, i, j) = (e, i, i), \\ \gamma_{\tau_i} \circ \sigma_g \circ \gamma_{\tau_i}^{-1} : Ae_j \rightarrow Ae_i, & \text{otherwise.} \end{cases} \]

4. **Globalization**

In this section we study the globalization problem for a lifted partial groupoid action. We start by recalling that a globalization of a partial action \( \alpha = (A_g, \alpha_g)_{g \in G} \) of a groupoid \( G \) on a ring \( A \) is a pair \((\hat{A}, \hat{\alpha})\), where \( \hat{A} \) is a ring and \( \hat{\alpha} = (\hat{A}_g, \hat{\alpha}_g)_{g \in G} \) is a global action of \( G \) on \( \hat{A} \), that satisfies the following properties:

(a) \( A_y \) is an ideal of \( \hat{A}_y \), for all \( y \in G_0 \),
(b) \( A_g = A_{t(g)} \cap \hat{\alpha}_g(A_{s(g)}) \),
(c) \( \hat{\alpha}_g(a) = \alpha_g(a) \), for all \( a \in A_{g^{-1}} \),
(d) \( \hat{A}_g = \sum_{(h) = (g)} \hat{\alpha}_h(A_{s(h)}) \), for all \( g \in G \).

When \( \alpha \) admits a globalization we say that \( \alpha \) is globalizable. The globalization of \( \alpha \) is unique, up to isomorphism; see [BP, Section 2] for details.

**Theorem 4.1** ([BP], Theorem 2.1). Let \( \alpha = (\{A_y\}_{g \in G}, \{\alpha_g\}_{g \in G}) \) be a partial action of a groupoid \( G \) on a ring \( A \) and suppose that \( A_y \) is a unital ring for each \( y \in G_0 \). Then, \( \alpha \) is globalizable if and only if each ideal \( A_g, g \in G \), is a unital ring.
In what follows in this section, $\mathcal{G}$ denotes a connected groupoid, $x \in \mathcal{G}_0$ is a fixed object of $\mathcal{G}$ and $\tau(x) = \{\tau_y : y \in \mathcal{G}_0\}$ is a transversal for $x$. We also fix a ring $A$, a datum $\gamma = (I, \gamma(x), \gamma(x)) \in \mathcal{D}_{\tau(x)}(A)$ and $\beta = (B_g, \beta_g)_{g \in \mathcal{G}}$ the corresponding lifted partial groupoid action, that is, $\beta$ is the partial action of $\mathcal{G}$ on $A$ given by (12) and (13).

The next result relates the existence of globalization for $\beta$ with the existence of globalization for $\gamma(x)$.

**Proposition 4.2.** The partial groupoid action $\beta$ of $\mathcal{G}$ on $A$ is globalizable if and only if the partial group action $\gamma(x)$ of $\mathcal{G}(x)$ on $I_x$ is globalizable and the ideals $I_y, I_{\gamma^{-1}}$ are unital rings, for all $y \in \mathcal{G}_0$.

*Proof.* Suppose that $\beta$ is globalizable. By Theorem 4.1, the ideal $B_g$ is a unital ring, for all $g \in \mathcal{G}$. Let $1_g$ be a central idempotent of $A$ such that $B_g = A1_g$, $g \in \mathcal{G}$. Then $B_h = A1_h = A1_x1_h = B_x1_h = I_x1_h$, for all $h \in \mathcal{G}(x)$. Thus, by Theorem 4.5 of [DE], the partial group action $\gamma(x)$ of $\mathcal{G}(x)$ on $I_x$ is globalizable. Also, $I_y = B_y = A1_y$ and $I_{\gamma^{-1}} = B_{\gamma^{-1}} = A1_{\gamma^{-1}}$, for all $y \in \mathcal{G}_0$.

Conversely, since $\gamma(x)$ is globalizable it follows from Theorem 4.5 of [DE] that $I_h$ is a unital ring for all $h \in \mathcal{G}(x)$. Hence there are central idempotents $1_h$ of $I_h$ such that $I_h = I_x1_h$. Consider also central idempotents $1_y, 1_{\gamma^{-1}}$ of $A$ such that $I_y = A1_y$ and $I_{\gamma^{-1}} = A1_{\gamma^{-1}}$, for all $y \in \mathcal{G}_0$. By (13), $B_g = \gamma_{\tau(g)}(1_{\gamma^{-1}} \gamma_{g^{-1}}(1_{\gamma^{-1}} \gamma_{g^{-1}}))$ for all $g \in \mathcal{G}$, $g \notin \mathcal{G}_0$. It is straightforward to check that $1_g = \gamma_{\tau(g)}(1_{\gamma^{-1}} \gamma_{g^{-1}}(1_{\gamma^{-1}} \gamma_{g^{-1}}))$ is a central idempotent of $A$ that satisfies $A1_g = B_g$, for all $g \in \mathcal{G}$, $g \notin \mathcal{G}_0$. If $g = y \in \mathcal{G}_0$ then $B_y = B_y = I_y = A1_y$. Hence, Theorem 4.1 implies that $\beta$ is globalizable. □

Now we present explicitly the globalization of $\beta$ for a specific type of datum $\gamma$. In order to do this we introduce the following definition. The datum $\gamma = (I, \gamma(x), \gamma(x)) \in \mathcal{D}_{\tau(x)}(A)$ will be called globalizable if

(a) $I_x$ is a unital ring and the partial group action $\gamma(x)$ of $\mathcal{G}(x)$ on $I_x$ admits a globalization $(J_x, \tilde{\gamma}(x))$,

(b) there exist a ring $B$, a family of ideals $\{J_y\}_{y \in \mathcal{G}_0}$ of $B$ and a family of isomorphisms $\{\tilde{\gamma}_y : J_y \to J_y\}_{y \in \mathcal{G}_0}$ of rings such that $I_y$ is an ideal of $J_y$ and $\tilde{\gamma}_y(a) = \gamma_y(a)$, for all $a \in I_{\gamma^{-1}}$, and $y \in \mathcal{G}_0$,

(c) $I_{\gamma^{-1}} = I_x$ and $I_{\gamma^{-1}} = I_y$, for all $y \in \mathcal{G}_0$.

We will denote: $\tilde{\gamma} = (J, \tilde{\gamma}_{\tau(x)}, \tilde{\gamma}(x))$, where $J = \{J_y\}_{y \in \mathcal{G}_0}$ and $\tilde{\gamma}_y = \{\tilde{\gamma}_y : J_y \to J_y\}_{y \in \mathcal{G}_0}$.

The triple $\tilde{\gamma}$ will be called a globalization of $\gamma$ in $B$.

Suppose that $\gamma$ is globalizable. Then $\gamma_{\tau^{-1}} : I_{\gamma^{-1}} = I_x \to I_{\gamma^{-1}} = I_y$ is an isomorphism and whence $I_x = I_{\gamma^{-1}}$ and $I_y = I_{\gamma^{-1}}$ are unital rings, for all $y \in \mathcal{G}_0$. Consequently, by Proposition 4.2, $\beta$ is globalizable. The next theorem describes the globalization of $\beta$ in the case that $\gamma$ is globalizable.

**Theorem 4.3.** If $\tilde{\gamma} = (J, \tilde{\gamma}_{\tau(x)}, \tilde{\gamma}(x))$ is a globalization of $\gamma$ in $B$ then the groupoid action $\tilde{\beta}$ of $\mathcal{G}$ on $B$ given by

$$\beta = (B_g, \beta_g)_{g \in \mathcal{G}}, \quad \tilde{\beta}_g = \tilde{\gamma}_{\tau(y)} \circ \tilde{\gamma}_{g^-1} \circ \tilde{\gamma}_{\tau(x)} : J_{\tau(y)} \to J_{\tau(x)},$$

$$B_g = J_{\tau(y)}$$
is a globalization of $\beta$.

Proof. Clearly $\tilde{\beta}$ is a global action of $\mathcal{G}$ on $B$ and $\tilde{\beta}_g(a) = \beta_g(a)$, for all $a \in B_{g^{-1}}$ and $g \in \mathcal{G}$. Notice that

$$B_{t(g)} \cap \tilde{\beta}_g(B_{s(g)}) = I_{t(g)} \cap \tilde{\beta}_g(I_{s(g)})$$

$$= I_{t(g)} \cap \tilde{\gamma}_{t(g)} \tilde{\gamma}_{x} \tilde{\gamma}_{s(g)}^{-1} (I_{s(g)})$$

$$= I_{t(g)} \cap \tilde{\gamma}_{t(g)} \tilde{\gamma}_{x} \tilde{\gamma}_{s(g)}^{-1} (I_{x})$$

$$= \tilde{\gamma}_{t(g)}(I_x) \cap \tilde{\gamma}_{t(g)} \tilde{\gamma}_{x} (I_x)$$

$$= \tilde{\gamma}_{t(g)}(I_x) \cap \tilde{\gamma}_{x} (I_x))$$

$$(\ast) \tilde{\gamma}_{t(g)}(I_x),$$

where $$(\ast)$$ holds because $(J_x, \tilde{\gamma}(x))$ is a globalization of the partial group action $\gamma(x)$ of $\mathcal{G}(x)$ on $I_x$. On the other hand,

$$B_g = \gamma_{\tau(g)}(I_{s(g)}^{-1} \cap \gamma_{x} (I_{s(g)}^{-1} \cap I_{x}^{-1}))$$

$$= \gamma_{\tau(g)}(I_x \cap \gamma_{x} (I_x \cap I_{x}^{-1}))$$

$$= \gamma_{\tau(g)}(I_{x})$$

$$= \tilde{\gamma}_{\tau(g)}(I_{x})$$

Hence, $B_g = B_{t(g)} \cap \tilde{\beta}_g(B_{s(g)})$. Note also that, for each $g \in \mathcal{G}$, we have

$$\sum_{t(h) = t(g)} \tilde{\beta}_h(B_{s(h)}) = \sum_{t(h) = t(g)} \tilde{\gamma}_{t(h)} \tilde{\gamma}_{x}^{-1} \tilde{\gamma}_{s(h)} (I_{s(h)})$$

$$= \tilde{\gamma}_{t(g)} \left( \sum_{t(h) = t(g)} \tilde{\gamma}_{x}^{-1} \tilde{\gamma}_{s(h)} (I_{x}) \right)$$

$$(\ast\ast) \tilde{\gamma}_{t(g)}(J_x)$$

$$= J_{t(g)}$$

$$= B_g.$$

In order to justify $$(\ast\ast)$$ note that the restriction of the map $\pi : \mathcal{G} \to \mathcal{G}(x)$ (see (10)) to $\tilde{\gamma}_{t(g)}$ is surjective. In fact, given $l \in \mathcal{G}(x)$ we have that $\pi(\tau_{l(g)})l = (\tau_{t(g)})_la = x = l$. Thus, $\sum_{t(h) = t(g)} \tilde{\gamma}_{h}(I_{x}) = \sum_{l \in \mathcal{G}(x)} \tilde{\gamma}_{l}(I_{x})$. Since $(J_x, \tilde{\gamma}(x))$ is the globalization of $\gamma(x)$ it follows that $J_x = \sum_{l \in \mathcal{G}(x)} \tilde{\gamma}_{l}(I_{x})$. \hfill $\Box$

Example 4.4. Let $\mathcal{G} = \{g, h, l, m, l^{-1}, m^{-1}\} \Rightarrow \{x, y\} = \mathcal{G}_0$ be the groupoid given in Example 3.9 and fix $\tau(x) = \{\tau_x = x, \tau_y = l\}$. Let $A$ be a ring, $\sigma, \gamma$ ring automorphisms of $A$ with $\sigma^{-1} = \sigma$ and $e \in A$ a central idempotent of $A$. Consider the datum $(I, \gamma_x(\gamma(x)), \gamma(x))$, where

- $I = \{I_x = Ae, I_y = A\gamma(e)\}$ is the set of the ideals of $A$, 
- $\sigma^{-1} = \sigma$ and $e \in A$ a central idempotent of $A$. Consider the datum $(I, \gamma_x(\gamma(x)), \gamma(x))$, where

- $I = \{I_x = Ae, I_y = A\gamma(e)\}$ is the set of the ideals of $A$, 

\( \gamma_{\tau(x)} = \{ \gamma_{\tau_x} = \gamma_x = id_{I_x}, \gamma_{\tau_y} = \gamma_l = \gamma |_{I_x} : I_x \to I_y \} \) is the set of ring isomorphisms,
\( \gamma(x) \) is the partial action of \( G(x) = \{ x, g \} \) on \( I_x \), given by \( \gamma_x = id_{I_x} \) and \( \gamma_g = \gamma_{g^{-1}} = \sigma |_{A\sigma(e)} : A\sigma(e) \to A\sigma(e) \).

The lifted partial groupoid action corresponding to this datum is the partial action \( \beta = (B_z, \beta_z)_{z \in G} \) of \( G \) on \( A \) given by
\[
\begin{align*}
\beta_x &= id_{I_x}, & \beta_y &= id_{I_y}, & \beta_g &= \gamma_g, & \beta_l &= \gamma_l, & \beta_m &= \gamma \sigma |_{A\sigma(e)} : A\sigma(e) \to A\gamma(e \sigma(e)), \\
\beta_h &= \gamma \sigma \gamma^{-1}, & \beta_{l^{-1}} &= \gamma_l^{-1}, & \beta_{m^{-1}} &= \sigma \gamma^{-1}. 
\end{align*}
\]

The datum \( (I, \gamma_{\tau(x)}, \gamma(x)) \) admits a globalization \( (J, \tilde{\gamma}_{\tau(x)}, \tilde{\gamma}(x)) \) in \( A \), where
\[
\tilde{\gamma}_{\tau(x)} = \{ \tilde{\gamma}_{\tau_x} = \tilde{\gamma}_x = id_A, \tilde{\gamma}_{\tau_y} = \gamma \} \text{ is the set of ring isomorphisms,}
\tilde{\gamma}(x) \text{ is the globalization of } \gamma(x), \text{ i.e. } \tilde{\gamma}(x) \text{ is the global action of } G(x) \text{ on } A \text{ given by } \tilde{\gamma}_x = id_A \text{ and } \tilde{\gamma}_g = \sigma.
\]

By Theorem 4.3, the global action \( \tilde{\beta} \) of \( G \) on \( A \) given by
\[
\tilde{\beta}_x = \tilde{\beta}_y = id_A, \quad \tilde{\beta}_g = \sigma, \quad \tilde{\beta}_h = \gamma \sigma \gamma^{-1}, \quad \tilde{\beta}_m = \gamma \sigma, \quad \tilde{\beta}_{l^{-1}} = \sigma \gamma^{-1}, \quad \tilde{\beta}_l = \gamma, \quad \tilde{\beta}_{l^{-1}} = \gamma^{-1},
\]
is the globalization of the partial action \( \beta \).

5. The Partial Skew Groupoid Ring

In this section the partial action \( \alpha = (A_g, \alpha_g)_{g \in G} \) of \( G \) on the ring \( A \) is assumed to be unital, that is, each \( A_g \) is unital with identity element denoted by \( 1_g \) which is a central idempotent of \( A \) and \( A_g = A1_g \). Also, \( x \in G_0 \) is a fixed object of \( G \) and \( \tau(x) = \{ \tau_y \mid y \in G_0 \} \) is a transversal for \( x \).

Suppose that \( G_0 \) is finite and \( \alpha \) is \( \tau(x) \)-global. Our main goal in this section is to prove that there exists a unital ring \( C \) and a partial group action \( \theta \) of \( G(x) \) on \( C \) such that \( A \ast_\alpha G \) is isomorphic to \( C \ast_\theta G(x) \).

We start by observing that \( G \) and \( G^2_0 \times G(x) \) are isomorphic as groupoids, where the groupoid structure in \( G^2_0 = G_0 \times G_0 \) is \( (y, z) = y \) and \( t(y, z) = z \), for all \( (y, z) \in G^2_0 \). By Lemma 3.1 the map \( f : G \to G^2_0 \times G(x) \) defined by \( g \mapsto ((s(g), t(g)), g_x) \), with \( g_x = \tau^{-1}_{t(g)} g \tau_{s(g)} \), for all \( g \in G \), is a groupoid epimorphism. Suppose that \( f(g) \) is an identity of \( G^2_0 \times G(x) \). Then \( f(g) = ((y, y), x) \) for some \( y \in G_0 \). Hence \( s(g) = t(g) = y \) and \( x = g_x = \tau^{-1}_y g \tau_y \) which implies that \( g = \tau_y \tau_y^{-1} = x \). This ensures that \( f \) is injective.

Lemma 5.1. If \( \alpha \) is \( \tau(x) \)-global then \( \alpha^* = (A^*_u, \alpha^*_u)_{u \in G^2_0} \), given by \( A^*_u = A_{t(u)} \) and \( \alpha^*_u = \alpha_{r(t(u))} \circ \alpha^{-1}_{s(t(u))} \), is a global action of \( G^2_0 \) on \( A \).

Proof. Indeed, for any identity \( e = (y, y) \) of \( G^2_0 \) we have that \( A^*_e = A_y \) and \( \alpha^*_e = \alpha_y = \alpha_{y^{-1}} \) is the identity map of \( A_{y^{-1}} = A_y \). Moreover, if \( u = (y, z) \) and \( v = (r, w) \) are elements in \( G^2_0 \) such that the product \( uv \) exists, then \( uv = (r, z) \) and \( \alpha^*_u \circ \alpha^*_v = \alpha_y \circ \alpha_{y^{-1}} = \alpha_{y^{-1}} = \alpha_{y^{-1}} \).
\( \square \)
Recalling from [BP, Section 3], that the partial skew groupoid ring $A \rtimes_{\alpha} \mathcal{G}$ is defined as the direct sum

$$A \rtimes_{\alpha} \mathcal{G} = \bigoplus_{g \in \mathcal{G}} A_g \delta_g$$

(where the $\delta_g$'s are placeholder symbols) with the usual addition and multiplication induced by the rule

$$(a \delta_g)(b \delta_h) = \begin{cases} 
  a \alpha_g(b_{1_{g^{-1}}} \delta_{gh}) & \text{if } s(g) = t(h) \\
  0 & \text{otherwise,}
\end{cases}$$

for all $g, h \in \mathcal{G}$, $a \in A_g$ and $b \in A_h$. Endowed with such a multiplication $A \rtimes_{\alpha} \mathcal{G}$ has a structure of an associative ring. If in addition the set $\mathcal{G}_0$ of the objects of $\mathcal{G}$ is finite then $A \rtimes_{\alpha} \mathcal{G}$ is also unital with identity element $1_{A \rtimes_{\alpha} \mathcal{G}} = \sum_{y \in \mathcal{G}_0} 1_y \delta_y$.

Thanks to Lemma 5.1 we can consider the corresponding (global) skew groupoid ring $C = A \rtimes_{\alpha^*} \mathcal{G}_0^2$. In the sequel we will also see, under the same conditions listed in Lemma 5.1, that the group $\mathcal{G}(x)$ acts on $C$ via an appropriate partial action $\theta$, which will allows us to consider the corresponding partial skew group ring $C \rtimes_{\theta} \mathcal{G}(x)$.

In order to construct $\theta$ we start by looking for a family of ideals in $C$. For this purpose set

$$(15) \quad C_{z,h} = \alpha_{\tau_s}(A_h), \text{ for all } z \in \mathcal{G}_0 \text{ and } h \in \mathcal{G}(x),$$

and take

$$(16) \quad C_h = \bigoplus_{u \in \mathcal{G}_0^2} C_{t(u),h} \delta_u.$$

**Lemma 5.2.** If $\alpha$ is $\tau(x)$-global then the following statements are true:

(i) $C_x = C$,

(ii) If $\mathcal{G}_0$ is finite then $C_h$ is a unital ideal of $C$, for all $h \in \mathcal{G}(x)$.

**Proof.** Firstly,

$$C = \bigoplus_{u \in \mathcal{G}_0^2} A_{t(u)} \delta_u = \bigoplus_{u \in \mathcal{G}_0^2} A_{\tau(u)} \delta_u = \bigoplus_{u \in \mathcal{G}_0^2} \alpha_{\tau(u)}(A_{-1_{\tau(u)}}) \delta_u = \bigoplus_{u \in \mathcal{G}_0^2} \alpha_{\tau(u)}(A_{\tau(x)}) \delta_u = C_x.$$
On the other hand, \( a \in \mathcal{G}_0 \) is finite, \( \sum_{z \in \mathcal{G}_0} \alpha_{\tau_z}(1_h) \delta_{(z,z)} \) is the identity element of \( C_h \), for all \( h \in \mathcal{G}(x) \). Indeed, for every \( a = \alpha_{\tau_w}(a_h) \delta_{(y,w)} \in C_w, h \delta_u \), with \( a_h \in A_h \), one has that

\[
a1'_h = \sum_{z \in \mathcal{G}_0} \alpha_{\tau_w}(a_h) \delta_{(y,w)} \alpha_{\tau_z}(1_h) \delta_{(z,z)}
= \alpha_{\tau_w}(a_h) \delta_{(y,w)} \alpha_{\tau_y}(1_h) \delta_{(y,w)}
= \alpha_{\tau_w}(a_h) \alpha_{\tau_y}(1_h) \delta_{(y,w)}
= \alpha_{\tau_w}(a_h) \delta_{(y,w)}
= a,
\]

where \((*)\) follows from Lemma 5.1. By similar arguments one gets \( 1'_h a = a \). Hence \( 1'_h \) is a central idempotent of \( C \) and it is easy to see that \( C_h = 1'_h C \).

For each \((z, h) \in \mathcal{G}_0 \times \mathcal{G}(x)\) define \( \theta_{z,h} : C_{z,h^{-1}} \to C_{z,h} \) by \( \alpha_{\tau_z}(a) \mapsto \alpha_{\tau_z}(\alpha_h(a)) \), for all \( a \in A_{h^{-1}} \). Clearly \( \theta_{z,h} \) is a bijection. Moreover, these maps induce the bijection \( \theta_h : C_{h^{-1}} \to C_h \), given by \( \theta_h(\alpha_{\tau_z(u)}(a)) \delta_{u} = \theta_h(u_h)(a) \delta_u \), for all \( a \in A_{h^{-1}} \) and \( u \in \mathcal{G}_0^2 \).

In all what follows in this section we will assume that \( \mathcal{G}_0 \) is finite.

**Lemma 5.3.** The pair \( \theta = (C_h, \theta_h)_{h \in \mathcal{G}(x)} \) is a partial action of \( \mathcal{G}(x) \) on \( C \).

**Proof.** We will proceed by steps.

**Step 1:** \( \theta_x \) is the identity map of \( C_x = C \).

It follows straightforward from the definition of \( \theta \).

**Step 2:** \( \theta_h \) is multiplicative, for all \( h \in \mathcal{G}(x) \).

For all \( a, b \in A_{h^{-1}} \),

\[
\theta_h((\alpha_{\tau_z}(a)) \delta_{(y,z)})(\alpha_{\tau_y}(b)) \delta_{(w,z)} = \theta_h(\alpha_{\tau_z}(a) \alpha_{\tau_y}(b)) \delta_{(w,z)}
= \theta_h(\alpha_{\tau_z}(a) \alpha_{\tau_y}(b)) \delta_{(w,z)}
= \theta_h(\alpha_{\tau_z}(ab)) \delta_{(w,z)}
= \alpha_{\tau_z}(\alpha_h(ab)) \delta_{(w,z)}
= \alpha_{\tau_z}(\alpha_h(a)) \alpha_{\tau_y}(\alpha_h(b)) \delta_{(w,z)}.
\]

On the other hand,

\[
\theta_h(\alpha_{\tau_z}(a)) \delta_{(y,z)} \theta_h(\alpha_{\tau_y}(b)) \delta_{(w,y)} = \alpha_{\tau_z}(\alpha_h(a)) \delta_{(y,z)} \alpha_{\tau_y}(\alpha_h(b)) \delta_{(w,y)}
= \alpha_{\tau_z}(\alpha_h(a)) \alpha_{\tau_y}(\alpha_h(b)) \delta_{(w,z)}
= \alpha_{\tau_z}(\alpha_h(a)) \alpha_{\tau_y}(\alpha_h(b)) \delta_{(w,z)}.
\]
Step 3: $\theta_{l^{-1}}(C_l \cap C_{h^{-1}}) \subseteq C_{(hl)^{-1}}$, for all $h, l \in \mathcal{G}(x)$.

$$\theta_{l^{-1}}(C_l \cap C_{h^{-1}}) = \bigoplus_{u \in \mathcal{G}_0^2} \theta_{l^{-1}}(\alpha_{\tau(u)}(A_l \cap A_{h^{-1}})) \delta_u$$

$$= \bigoplus_{u \in \mathcal{G}_0^2} \alpha_{\tau(u)}(\alpha_{l^{-1}}(A_l \cap A_{h^{-1}})) \delta_u$$

$$\subseteq \bigoplus_{u \in \mathcal{G}_0^2} \alpha_{\tau(u)}(A_{l^{-1}} \cap A_{l^{-1}h^{-1}}) \delta_u \quad \text{(for $\alpha_{(x)}$ is a partial group action)}$$

$$= C_{(hl)^{-1}}.$$

Step 4: $\theta_l(\theta_h(c)) = \theta_h(\theta_l(c))$, for all $c = \alpha_{\tau_s}(\alpha_{l^{-1}}(a)) \delta_{(y,z)} \in \theta_{l^{-1}}(C_l \cap C_{h^{-1}})$.

$$\theta_h(\theta_l(c)) = \theta_h(\theta_l(\alpha_{\tau_s}(\alpha_{l^{-1}}(a)) \delta_{(y,z)}))$$

$$= \theta_h(\alpha_{\tau_s}(a) \delta_{(y,z)})$$

$$= \alpha_{\tau_s}(\alpha_{hl}(a)) \delta_{(y,z)}.$$

On the other hand, since $\alpha_{hl} = \alpha_h \alpha_l$ in $\alpha_{l^{-1}}(A_l \cap A_{h^{-1}})$ we have

$$\theta_{hl}(c) = \alpha_{\tau_s}(\alpha_{hl}(\alpha_{l^{-1}}(a))) \delta_{(y,z)} = \alpha_{\tau_s}(\alpha_{hl}(a)) \delta_{(y,z)}.$$

Hence $\theta_{hl} = \theta_h \theta_l$ in $\theta_{l^{-1}}(C_l \cap C_{h^{-1}}). \square$

Now we will prove the main result of this section.

**Theorem 5.4.** If $\alpha$ is $\tau(x)$-global then

$$\varphi : A \star_{\alpha} \mathcal{G} \rightarrow (A \star_{\alpha^*} \mathcal{G}_0^2) \star_0 \mathcal{G}(x), \quad a \delta_g \mapsto a \delta_{(s(g),t(g))} \delta_{ga}$$

is a ring isomorphism.

**Proof.** We proceed again by steps.

**Step 1:** $\varphi$ is well defined.

First of all we have by Lemma 5.1 that $A_g \subseteq A_{l(g)} = A_{s(g),t(g)}$, for all $g \in \mathcal{G}$. Hence, we only need to show that $a \delta_{(s(g),t(g))} \in C_{ga}$, for all $a \in A_g$. Notice that

$$\alpha_{\tau_l(g)}(A_{gx}) = \alpha_{\tau_l(g)}(A_{\tau_l^{-1}(g)} \cap A_{x})$$

$$= \alpha_{\tau_l(g)}(A_{\tau_l^{-1}(g)} \cap A_{x})$$

$$\subseteq \bigoplus_{u \in \mathcal{G}_0^2} \alpha_{\tau(u)}(A_{l^{-1}} \cap A_{l^{-1}h^{-1}}) \delta_u \quad \text{(for $\alpha_{(x)}$ is a partial group action)}$$

$$= C_{(hl)^{-1}}.$$

Since $A_{g^{-1}} = A_{g^{-1}} \cap A_{s(g)} \subseteq A_{s(g)}$, we have $A_g = \alpha_g(A_{g^{-1}} \cap A_{s(g)}) = A_g \cap A_{g\tau_s(g)}$.

Hence $a \in A_g \subseteq A_{g\tau_s(g)} = \alpha_{\tau_s(g)}(A_{gx})$ which implies $a \delta_{(s(g),t(g))} \in C_{ga}$ by (15) and (16).

**Step 2:** $\varphi$ is a ring homomorphism.
For all $g, h \in G$ such that $s(g) = t(h), a \in A_g$ and $b \in A_h$ we have

\[
\varphi((a\delta_g)(b\delta_h)) = \varphi(a\alpha_g(b_1g^{-1})\delta_{gh}) \\
= a\alpha_g(b_1g^{-1})\delta(s(gh), t(gh))\delta_{gh} \\
= a\alpha_g(b_1g^{-1})\delta(s(h), t(g))\delta_{gh}.
\]

On the other hand

\[
\varphi(a\delta_g)\varphi(b\delta_h) = (a\delta(s(g), t(g))\delta_{gz})(b\delta(s(h), t(h))\delta_{hz}) \\
= (a\delta(s(g), t(g)))(\gamma_{gz} b\delta(s(h), t(h))^{g^{-1}})\delta_{gz}h_z.
\]

Since $b \in A_h \subseteq A_h \tau(s(h)) = \alpha_{\tau(h)}(A_{h_x})$ there is $b' \in A_{h_x}$ such that $b = \alpha_{\tau(h)}(b')$ and

\[
b\delta_{(s(h),t(h))}^{g^{-1}} = \alpha_{\tau(h)}(b')\delta_{s(h),t(h)} \sum_{g \in G} \alpha_{\tau_x}(1_{g^{-1}})\delta_{z,z} \\
= \alpha_{\tau(h)}(b')\delta_{s(h),t(h)} \alpha_{\tau_s(h)}(1_{g^{-1}})\delta_{s(h),s(h)} \\
= \alpha_{\tau(h)}(b')\alpha_{\tau(h)}(1_{g^{-1}})\alpha_{\tau(h)}(1_{g^{-1}})\delta_{s(h),t(h)} \\
= \alpha_{\tau(h)}(b')1_{g^{-1}}\delta_{s(h),t(h)}.
\]

Hence,

\[
\varphi((a\delta_g)(b\delta_h)) = (a\delta(s(g), t(g)))(\gamma_{gz} b\delta(s(h), t(h))^{g^{-1}})\delta_{gz}h_x \\
= (a\delta(s(g), t(g)))(\gamma_{gz} (\alpha_{\tau(h)}(b'1_{g^{-1}})\delta_{s(h),t(h)}))\delta_{gz}h_x \\
= (a\delta(s(g), t(g)))(\alpha_{\tau(h)}(1_{g^{-1}})\alpha_{\tau(h)}(1_{g^{-1}})\delta_{s(h),t(h)})\delta_{gz}h_x \\
= a\alpha_g\alpha_{\tau_x(g)}(1_{g^{-1}})\alpha_{\tau(h)}(b'1_{g^{-1}})\alpha_{\tau(h)}(1_{g^{-1}})\delta_{s(h),t(g)}\delta_{gz}h_x \\
= a\alpha_g\alpha_{\tau(h)}(b'1_{g^{-1}})\delta_{s(h),t(g)}\delta_{gz}h_x \\
= a\alpha_g(\alpha_{\tau(h)}(b'1_{g^{-1}})1_{\tau(h)}^{-1})\delta_{s(h),t(g)}\delta_{gz}h_x \\
= a\alpha_g(\alpha_{\tau(h)}(b'1_{g'^{-1}})1_{\tau(g)}^{-1})\delta_{s(h),t(g)}\delta_{gz}h_x \\
= a\alpha_g(\alpha_{\tau(g)}(b'1_{g^{-1}})1_{\tau(h)}^{-1})\delta_{s(h),t(g)}\delta_{gz}h_x \\
= a\alpha_g(\alpha_{\tau(g)}(b'1_{g'^{-1}})1_{\tau(g)}^{-1})\delta_{s(h),t(g)}\delta_{gz}h_x \\
= a\alpha_g(\alpha_{\tau(g)}(b'1_{g'^{-1}}))\delta_{s(h),t(g)}\delta_{gz}h_x \\
= a\alpha_g(\alpha_{\tau(h)}(b')1_{g^{-1}})\delta_{s(h),t(g)}\delta_{gz}h_x \\
= a\alpha_g(\alpha_{\tau(h)}(b')1_{g^{-1}})\delta_{s(h),t(g)}\delta_{gz}h_x \\
= \varphi((a\delta_g)(b\delta_h)).
\]

**Step 3:** $\varphi$ is injective.
Let \( v = \sum_{g \in G} a_g \delta_g \in A \star G \) such that \( \varphi(v) = 0 \). Then
\[
0 = \sum_{g \in G} a_g \delta_{(s(g), t(g))} \delta_{g'} = \sum_{h \in G} \sum_{\pi(g) = h} a_g \delta_{(s(g), t(g))} \delta_h
\]
which implies that \( \sum_{\pi(g) = h} a_g \delta_{(s(g), t(g))} = 0 \) for all \( h \in G(x) \) (notice that \( C \star_\theta G(x) \) is a direct sum). Finally, it is straightforward to check that for any \( h, g \in G(x) \) and any \( g, g' \in \pi^{-1}(h) \), \((s(g), t(g)) = (s(g'), t(g'))\) if and only if \( g = g' \). Therefore \((*)\) holds if and only if \( a_g = 0 \), for all \( g \in G \), and so \( v = 0 \).

**Step 4:** \( \varphi \) is surjective.

It is enough to check that given any element of the type \( \alpha_{\tau_x}(a) \delta_{(y, z)} \delta_h \), with \( h \in G(x) \) and \( a \in A_h \), there exists an element \( w \in A \star G \) such that \( \varphi(w) = \alpha_{\tau_x}(a) \delta_{(y, z)} \delta_h \). To do that observe firstly that
\[
\alpha_{\tau_x}(a) \in \alpha_{\tau_x}(A_h) = \alpha_{\tau_x}(A_h \cap A_z)
\]
by \((14)\)

\[
= A_{\tau_x} (A_h \cap A_{\tau_x}^{-1})
\]
by \((9)\)

\[
= A_{\tau_x} (A_h \cap A_z)
\]
by \((14)\)

\[
= A_{\tau_x} h \cap A_z
\]
by \((\star)\)

\[
= A_{\tau_x} h \quad \text{for} \quad A_{\tau_x} h \subseteq A_{(\tau_x) h} = A_{(\tau_x) h} = A_z.
\]

Therefore, for \( g = \tau_x h \tau_y^{-1} \) we have \( t(g) = t(\tau_x) = z \), \( s(g) = s(\tau_y^{-1}) = y \) and

\[
\alpha_{\tau_x}(a) \in A_{\tau_x} h = A_{(\tau_x) h} = \alpha_{(\tau_x) h} (A_h) \subseteq \alpha_{(\tau_x) h} (A_{h \tau_x^{-1}}) = A_{\tau_x} (A_{h \tau_x^{-1}}) = A_{\tau_x} h \tau_y^{-1} = A_y,
\]
where \((\ast)\) is ensured by:

\[
\alpha_{(\tau_x) h} (A_h) = \alpha_{(\tau_x) h} (A_h \cap A_x) \quad \text{by \((14)\)}
\]

\[
= \alpha_{(\tau_x) h} (A_h \cap A_{\tau_x^{-1}}) = A_{(\tau_x) h} h,
\]
and \((\ast\ast)\) by:

\[
A_h = \alpha_h (A_h^{-1} \cap A_x) \quad \text{by \((14)\)}
\]

\[
= \alpha_h (A_h^{-1} \cap A_{\tau_x^{-1}}) = A_h \cap A_{h \tau_x^{-1}} \subseteq A_{h \tau_x^{-1}}.
\]

Now, taking \( w = \alpha_{\tau_x}(a) \delta_g \) we are done. \(\Box\)

The next proposition characterizes when a lifted partial action is \( \tau(x) \)-global.

**Proposition 5.5.** A lifted partial action of \( G \) on \( A \) coming from a datum \((I, \gamma_{\tau(x)}, \gamma(x))\) in \( D_{\tau(x)}(A) \) is \( \tau(x) \)-global if and only if \( I_{\tau_y^{-1}} = I_x \) and \( I_{\tau_y} = I_y \) for all \( y \in G_0 \).

**Proof.** Along the proof we will freely use the formula \((13)\) without any mention to it. Let \( \beta = (B_y, \beta_y) \) be the lifted partial action coming from \((I, \gamma_{\tau(x)}, \gamma(x))\). If \( \beta \) satisfies \((14)\) then \( B_{\tau_y^{-1}} = B_x \), for all \( y \in G_0 \). Thus,

\[
B_{\tau_y^{-1}} = \gamma_{t(\tau_y^{-1})} \left(I_{\tau_y^{-1}} \cap \gamma_{(\tau_y^{-1}) s(x)} (I_{\tau_y^{-1}} \cap I_{(\tau_y^{-1}) s(x)})\right)
\]

\[
= \gamma_x (I_x \cap \gamma_x (I_{\tau_y^{-1}} \cap I_s)) = \gamma_x (I_{\tau_y^{-1}}^{-1}) = I_{\tau_y^{-1}}^{-1},
\]
for all $y \in G_0$ with $y \neq x$. Hence, $I_{\tau y}^{-1} = B_{\tau y}^{-1} = B_y = I_x$ for all $y \in G_0$. It is analogous to verify that $I_y = I_y$.

Conversely, if $I_{\tau y}^{-1} = I_x$ and $I_{\tau y} = I_y$ for all $y \in G_0$, then $B_y = \gamma_{\tau(y)}(I_{g_y})$ for all $g \in G$, $g \notin G_0$. Particularly, $B_{\tau y}^{-1} = I_x = B_x$ and $B_{\tau y} = I_y = B_y$ and whence $\beta$ satisfies (14).

Now we introduce a definition that will be useful in the rest of the paper. A lifted partial action $\beta = (B_g, \beta_g)_{g \in G}$ of $G$ on $A$ coming from a datum $(I, \gamma_{\tau(x)}, \gamma_x) \in D_{\tau(x)}(A)$ will be called $\gamma$-unital if $I_x$ is a unital ring and $\gamma_x$ is a unital partial group action.

**Remark 5.6.** Let $\beta$ be a lifted partial action of $G$ on $A$ which is $\tau(x)$-global and $\gamma$-unital. Then $\beta$ is unital, i.e. $B_g$ is a unital ring for all $g \in G$. Indeed, suppose that $I_x = A_{1x}$ and $I_h = A_{1h}$, with $1_x$ and $1_h$ central idempotents of $A$, for all $h \in G(x)$. Using that $\beta$ is $\tau$-global we obtain that $B_g = \gamma_{\tau(y)}(I_{g_y})$ and whence $B_g = A_{\gamma_{\tau(y)}(1_{g_y})}$, for all $g \in G$.

Thus $B_g = A_{1g}$, where $1_g = \gamma_{\tau(y)}(1_{g_y})$ is a central idempotent of $A$, for all $g \in G$.

Throughout the rest of this paper we will assume that $G$ is finite, $\beta = (B_g, \beta_g)_{g \in G}$ will denote a fixed partial action of $G$ on $A$ lifted from a given datum $(I, \gamma_{\tau(x)}, \gamma_x) \in D_{\tau(x)}(A)$. We will also assume that $\beta$ is $\tau(x)$-global, $\gamma$-unital and $A = \oplus_{y \in G_0} I_y = \oplus_{y \in G_0} B_g$. As in Remark 5.6, the unit of $B_g$ will be denoted by $1_g$. Precisely, $B_x = I_x = A_{1x}$, $B_h = I_h = A_{1h}$ and $B_g = A_{1g}$ where $1_g = \gamma_{\tau(y)}(1_{g_y})$, for all $h \in G(x)$ and $g \in G$.

6. The subring of invariants and the trace map

According to [BP] an element $a \in A$ is called invariant by $\beta$ if $\beta_g(a)_{1g^{-1}} = a_{1g}$, for all $g \in G$. We will denote by $A^\beta$ the set of all invariants of $A$. Clearly, $A^\beta$ is a subring of $A$.

**Proposition 6.1.** Let $b = \sum_{y \in G_0} b_y \in A$. Then $b \in A^\beta$ if and only if $b_x \in B_x^\gamma(x)$ and $b_y = \gamma_{\tau(y)}(b_x)$, for all $y \in G_0$.

**Proof.** If $b \in A^\beta$ then $b_x_{1h} = b_{1h} = \beta_h(b_{1h^{-1}}) = \gamma_h(b_{x_{1h^{-1}}})$ for all $h \in G(x)$. Hence $b_x \in B_x^\gamma(x)$. Also, $\gamma_{\tau_y}(b_x) = \gamma_{\tau_y} \gamma_x \gamma_{\tau_y}^{-1}(b_x)_{1x} = \beta_{\tau_y}(b_{1x^{-1}}) = b_{1x^{-1}} = b_y$, for all $y \in G_0$.

Conversely, let $b = \sum_{y \in G_0} \gamma_{\tau(y)}(b_x)$, for some $b_x \in B_x^\gamma(x)$. Then, for all $g \in G$,

$$\beta_g(b_{1g^{-1}}) = \gamma_{\tau(g)} \gamma_g \gamma_{\tau(g)}^{-1}(b_{1g^{-1}})$$

$$= \gamma_{\tau(g)} \gamma_g \gamma_{\tau(g)}^{-1}(\gamma_{\tau(g)}(b_x) \gamma_{\tau(g)}(1_{g^{-1}}))$$ (by assumption)

$$= \gamma_{\tau(g)}(b_x) \gamma_{\tau(g)}(1_{g^{-1}})$$

$$= \gamma_{\tau(g)}(b_x)$$

$$= b_{1g}.\quad \square$$

The $\beta$-trace map is defined as the map $t_\beta : A \to A$ given by $t_\beta(a) = \sum_{g \in G} \beta_g(a)_{1g^{-1}}$, for all $a \in A$. Similarly, the $\beta(x)$-trace map corresponding to the partial action $\beta(x)$ of
$G(x)$ over $B_x$ is defined as the map $t_{\beta(x)} : B_x \rightarrow B_x$ given by $t_{\beta(x)}(b) = \sum_{h \in G(x)} \beta_h(b1_{h^{-1}})$, for all $b \in B_x$. Notice that, by Remark 3.4, $t_{\beta(x)} = t_{\gamma(x)}$.

It follows from [BP, Lemma 4.2] (resp., [DFP, Lemma 2.1]) that $t_{\beta}(A) \subseteq A^\beta$ (resp., $t_{\beta(x)}(B_x) \subseteq B_x^{\beta(x)}$)

**Proposition 6.2.** Let $b_x \in B_x$ and $b_z = \gamma_{\tau_x}(b_x) \in B_z$. Then

$$t_{\beta}(b_z) = \sum_{y \in G_0} \gamma_{\tau_y}(t_{\beta(x)}(b_x)).$$

**Proof.** Indeed,

$$t_{\beta}(b_z) = \sum_{g \in G} \beta_g(b_z1_{g^{-1}})$$

$$= \sum_{g \in G} \gamma_{\tau_g}(g_{x}) \gamma_{\tau_g}^{-1}(\gamma_{\tau_x}(b_x)\gamma_{\tau_g}(1_{g^{-1}}))$$

$$= \sum_{g \in G} \gamma_{\tau_g}(g_{x}) b_x1_{g^{-1}}$$

$$= \sum_{l \in S_x} \gamma_{\tau_g}(l_{x}) b_x1_{l^{-1}}$$

$$= \sum_{l \in S_x} \gamma_{\tau_g}(l_{x}) \gamma_{\tau_y}(b_x1_{l^{-1}})$$

$$= \sum_{y \in G_0} \sum_{l \in G(x,y)} \gamma_{\tau_y}(l_{x}) \gamma_{\tau_h}(b_x1_{l^{-1}})$$

$$= \sum_{y \in G_0} \sum_{h \in G(x)} \gamma_{\tau_y}(\gamma_{\tau_h}(b_x1_{h^{-1}}))$$

$$= \sum_{y \in G_0} \gamma_{\tau_y}(t_{\beta(x)}(b_x)).$$

□

**Corollary 6.3.** Suppose that $a = \sum_{z \in G_0} b_z \in A$, with $b_z = \gamma_{\tau_z}(c_z)$ for some $c_z \in B_x$, for each $z \in G_0$. Then, $t_{\beta}(a) = \sum_{y \in G_0} \gamma_{\tau_y}(t_{\beta(x)}(c))$, where $c = \sum_{z \in G_0} c_z$. 

Proof. Indeed,

\[ t_\beta(a) = \sum_{z \in G_0} t_\beta(b_z) \]

Prop. 6.2 \[= \sum_{z \in G_0} \sum_{y \in G_0} \gamma_{\tau_y}(t_\beta(z)(c_z)) \]

\[= \sum_{y \in G_0} \gamma_{\tau_y}(t_\beta(z)(c)) \]

\[= \sum_{y \in G_0} \gamma_{\tau_y}(t_\beta(z)(c)) \]

\[= \sum_{y \in G_0} \gamma_{\tau_y}(t_\beta(z)(c)). \]

Theorem 6.4. \( t_\beta(A) = A_\beta \) if and only if \( t_\beta(B_x) = B_x^{\beta(x)} \).

Proof. Assume that \( t_\beta \) is onto. Take \( b_x \in B_x^{\beta(x)} \) and set \( b = \sum_{y \in G_0} \gamma_{\tau_y}(b_x) \) which lies in \( A_\beta \) by Proposition 6.1. Hence \( b = t_\beta(a) \) for some \( a \in A \). Since \( A = \bigoplus_{x \in G_0} B_x \) and \( \gamma_{\tau_x} \) is a ring isomorphism from \( B_x \) to \( B_x \) for each \( z \in G_0 \), it follows that \( a = \sum_{z \in G_0} \gamma_{\tau_z}(c_z) \), with \( c_z \in B_x \). Therefore \( \sum_{y \in G_0} \gamma_{\tau_y}(b_x) = b = t_\beta(a) \) and \( \gamma_{\tau_y}(t_\beta(z)(c)) \) with \( c = \sum_{z \in G_0} c_z \), which implies that \( b_x = t_\beta(z)(c) \in t_\beta(B_x) \).

Conversely, it is enough to observe that, by Propositions 6.1 and 6.2, and the assumption on \( \beta(x) \), each element in \( A_\beta \) is of the form \( b = \sum_{y \in G_0} \gamma_{\tau_y}(b_x) = t_\beta(a_x) \), with \( a_x \in B_x \) such that \( t_\beta(x)(a_x) = b_x \). \( \square \)

7. The related Morita theory

We start by recalling the definition of a Morita context. Given two unital rings \( R \) and \( S \), bimodules \( RS \) and \( SV_R \), and bimodule maps \( \mu : U \otimes_S V \rightarrow R \) and \( \nu : V \otimes_R U \rightarrow S \), the sixtuple \( (R, S, U, V, \mu, \nu) \) is called a Morita context (associated with \( RS \)) if the set

\[ (R \otimes S) = \left\{ \begin{pmatrix} r & u \\ v & s \end{pmatrix} \mid r \in R, s \in S, u \in U, v \in V \right\} \]

with the usual addition and multiplication given by the rule

\[ \begin{pmatrix} r & u \\ v & s \end{pmatrix} \begin{pmatrix} r' & u' \\ v' & s' \end{pmatrix} = \begin{pmatrix} rr' + \mu(u \otimes v') & ru' + us' \\ vr' + sv' & \nu(v \otimes u') + ss' \end{pmatrix} \]

is a unital ring, which is equivalent to say that the maps \( \mu \) and \( \nu \) satisfy the following two associativity conditions:

\[ uv(v \otimes u') = \mu(u \otimes v)u' \quad \text{and} \quad v\mu(u \otimes v') = \nu(v \otimes u)v'. \]

We say that this context is strict if the maps \( \mu \) and \( \nu \) are isomorphisms and, in this case, the categories \( \text{Mod}_R \) and \( \text{Mod}_S \) are equivalent via the mutually inverse equivalences \( V \otimes_R - : \text{Mod}_R \rightarrow \text{Mod}_S \) and \( U \otimes_S - : \text{Mod}_S \rightarrow \text{Mod}_R \). When it happens we also say that the rings \( R \) and \( S \) are Morita equivalent. If \( RS \) is faithfully projective, that is, \( RS \) is faithful, projective and finitely generated, then it is enough the surjectivity of \( \mu \) and
ν in order to ensure the strictness of the above context. Similar statements equally hold for right module categories.

7.1. About $B_x \ast \beta(x) \mathcal{G}(x)$ and $A \ast \beta \mathcal{G}$. In order to relate the partial skew groupoid ring $A \ast \beta \mathcal{G}$ and the partial skew group ring $B_x \ast \beta(x) \mathcal{G}(x)$ we firstly recall that

\[(18) \quad B_g = \gamma_{\tau(g)}(I_{g_x}) \quad \text{and} \quad \beta_g(\gamma_{\tau(g)}(a)) = \gamma_{\tau(g)}(\gamma_{g_x}(a)), \quad \text{for all} \ g \in \mathcal{G}, \ a \in I_{g^{-1}}.\]

Lemma 7.1. Let $R := A \ast \beta \mathcal{G}$ and $S := B_x \ast \beta(x) \mathcal{G}(x)$. Then

(i) $R1S = \sum_{g \in \mathcal{S}_x} B_g \delta_g$,

(ii) $1_SR = \sum_{g \in \mathcal{T}_x} B_g \delta_g$,

(iii) $1_SR1S = S$ and $R1SR = R$.

Proof. Let $g \in \mathcal{S}_x$ and $a \in B_g$. By (18), there exists $a' \in I_{g_x}$ such that $a = \gamma_{\tau(g)}(a')$ and whence

\[(a\delta_g)1_S = (\gamma_{\tau(g)}(a')\delta_g)(1_x\delta_x) = \gamma_{\tau(g)}(a')\beta_g(\gamma_{\tau(g)}(1_{g^{-1}}))\delta_g = \gamma_{\tau(g)}(a'1_{g_x})\delta_g = a\delta_g.\]

For $g \notin \mathcal{S}_x$ and $a \in B_g$, we have $(a\delta_g)1_S = 0$. Consequently, if $r = \sum_{g \in \mathcal{G}} a_g \delta_g \in R$ then $r1_S = \sum_{g \in \mathcal{S}_x} a_g \delta_g$ and (i) follows. The proof of (ii) is similar.

For (iii), it is clear that $R1SR \subseteq R$. For the reverse inclusion notice that

\[\gamma_{\tau(g)}(a')\delta_g = \gamma_{\tau(g)}(a')\gamma_{\tau(g)}(1_{g_x})\delta_g = (\gamma_{\tau(g)}(a')\delta_{\tau(g)})(1_{g_x}\delta_{\tau(g)^{-1}}),\]

for all $g \in \mathcal{G}$ and $a' \in I_{g_x}$. By (ii), $1_{g_x}\delta_{\tau(g)^{-1}} \in 1_SR$. Hence $\gamma_{\tau(g)}(a')\delta_g \in R1SR$. □

Fix the $(R,S)$-bimodule $U := R1S$ and the $(S,R)$-bimodule $V := 1_SR$. Define also the bimodule map $\mu : U \otimes_S V \to R$ (resp. $\nu : V \otimes_R U \to S$) given by $u \otimes v \mapsto uv$ (resp. $v \otimes u \mapsto vu$), for all $u \in U$, $v \in V$. With this notation we have the following result.

Theorem 7.2. The sixtuple $(A \ast \beta \mathcal{G}, B_x \ast \beta(x) \mathcal{G}(x), U, V, \mu, \nu)$ is a Morita context and $\mu, \nu$ are surjective maps.

Proof. It is straightforward to check that $\mu, \nu$ satisfy (17). By Lemma 7.1 (iii), $\mu$ and $\nu$ are surjective maps. □

7.2. About $A^\beta$ and $A \ast \beta \mathcal{G}$. According to [BP, § 4] $A$ is a $(A^\beta, A \ast \beta \mathcal{G})$-bimodule and a $(A \ast \beta \mathcal{G}, A^\beta)$-bimodule. The left and the right actions of $A^\beta$ on $A$ are given by the multiplication of $A$, and the left (resp., right) action of $A \ast \beta \mathcal{G}$ on $A$ is given by $a\delta_g \cdot b = a\beta_g(b1_{g^{-1}})$ (resp., $b \cdot a\delta_g = \beta_{g^{-1}}(ba)$), for all $a, b \in A$ and $g \in \mathcal{G}$.

Moreover, the maps

$\Gamma : A \otimes_{A \ast \beta \mathcal{G}} A \to A^\beta, \quad a \otimes b \mapsto t_\beta(ab)$
and
\[ \Gamma' : A \otimes A_\beta A \to A \star_\beta G, \quad a \otimes b \mapsto \sum_{g \in G} a \beta_g(b_1 g^{-1}) \delta_g, \]
are bimodule morphisms. By [BP, Proposition 4.4], the sixtuple \((A \star_\beta G, A^\beta, A, A, \Gamma, \Gamma')\) is a Morita context. Furthermore, the sixtuple \((B \star_{\beta(x)} G(x), B^\beta_{\beta(x)}, B_x, B_x, \Gamma_x, \Gamma'_x)\) is also a Morita context constructed in a similar way as the previous one for the partial group action of \(G(x)\) on \(B_x\) (see [AL, Theorem 1.5]).

**Corollary 7.3.** \(\Gamma\) is surjective if and only if \(\Gamma_x\) is surjective.

**Proof.** Since \(A\) (resp. \(B_x\)) is a unital ring, it is easy to see that \(\Gamma\) (resp. \(\Gamma_x\)) is surjective if and only if \(t_\beta\) (resp. \(t_{\beta(x)}\)) is surjective. Hence, the result follows from Theorem 6.4. \(\Box\)

**Lemma 7.4.** \(\Gamma'\) is surjective if and only if \(\Gamma'_x\) is surjective.

**Proof.** (\(\Rightarrow\)) Let \(b \delta_h \in B_x \star_{\beta(x)} G(x)\) with \(b \in B_h \subseteq B_x\). Consider \(b \delta_h\) as in element of \(A \star_\beta G\) and the ring isomorphism \(\varphi : A \star_\beta G \to (B \star_\beta G_0) \star G(x)\), \(a \delta_g \mapsto a \delta_{\gamma_{(g),t(g)}}\delta_g\), given in Theorem 5.4. Since \(\Gamma'\) is surjective, there exist \(a_i, b_i \in A\), \(1 \leq i \leq r\), such that
\[
\varphi \circ \Gamma' \left( \sum_{1 \leq i \leq r} a_i \otimes b_i \right) = \varphi(b \delta_h) = b \delta_{(x,x)} \delta_h.
\]
We can assume that \(a_i = \sum_{z \in G_0} \gamma_{\tau_i}(a'_{i,z})\) and \(b_i = \sum_{z \in G_0} \gamma_{\tau_i}(b'_{i,z})\) with \(a_i, b_i \in B_x\), for all \(1 \leq i \leq r\). Then
\[
\varphi \circ \Gamma' \left( \sum_{1 \leq i \leq r} a_i \otimes b_i \right) = \sum_{1 \leq i \leq r} \sum_{g \in G} a_i \beta_g(b_1 g^{-1}) \delta_{\gamma_{(g),t(g)}} \delta_g = \sum_{1 \leq i \leq r} \sum_{g \in G} a_i \gamma_{\tau_i(g)} \gamma_{g^{-1}} \gamma_{(t(g),s(g)}}(b'_{i,s(g)} g^{-1}) \delta_{\gamma_{(g),t(g)}} \delta_g.
\]
Hence,
\[
b \delta_{(x,x)} \delta_h = \sum_{1 \leq i \leq r} \sum_{g \in G} \gamma_{\tau_i(g)}(a'_{i,t(g)} \gamma_{g^{-1}} b'_{i,s(g)} g^{-1}) \delta_{\gamma_{(g),t(g)}} \delta_g.
\]
which implies that
\[
\sum_{1 \leq i \leq r} a'_{i,x} \gamma_{h}(b'_{i,x} g^{-1}) = b \quad \text{and} \quad \sum_{1 \leq i \leq r} a'_{i,x} \gamma_{l}(b'_{i,x} g^{-1}) = 0
\]
if \(l \neq h, l \in G\). Thus \(\Gamma'_x(\sum_{1 \leq i \leq r} a'_{i,x} \otimes b'_{i,x}) = b \delta_h\) and \(\Gamma'_x\) is surjective.

(\(\Leftarrow\)) Conversely it is enough to check that for any element of the form \(v = \gamma_{\tau_i}(a) \delta_{(y,z)} \delta_h\) in \((A \star_\beta G_0) \star G(x)\), with \(a \in A_h\) and \(y, z \in G_0\), there exists \(w \in A \otimes A\) such that
\( \phi \circ \Gamma'(w) = v \). Observe that \( a\delta_h \in B_x \ast \beta(x) \mathcal{G}(x) \) and whence there exist \( a_{x,i}, b_{x,i} \in B_x \) such that \( a\delta_h = \Gamma'(\sum_{1 \leq i \leq r} a_{x,i} \otimes b_{x,i}) = \sum_{t \in \mathcal{G}(x)} \sum_{1 \leq i \leq r} a_{x,i} \gamma_t(b_{x,i}1_{l-1}) \delta_l \). Thus

\[
\sum_{1 \leq i \leq r} a_{x,i} \gamma_t(b_{x,i}1_{h-1}) = a \quad \text{and} \quad \sum_{1 \leq i \leq r} a_{x,i} \gamma_t(b_{x,i}1_{l-1}) = 0 \quad \text{if} \ l \neq h, \ l \in \mathcal{G}(x). 
\]

Now setting \( a_i = \gamma_{\tau_z}(a_{x,i}) \) and \( b_i = \gamma_{\tau_y}(b_{x,i}) \) in \( A \) one has

\[
\varphi \circ \Gamma' \left( \sum_{1 \leq i \leq r} a_i \otimes b_i \right) = \sum_{1 \leq i \leq r} \sum_{g \in \mathcal{G}} a_i \beta_g(b_11_g^{-1}) \delta_{(s(g),t(g))} \delta_g \\
= \sum_{1 \leq i \leq r} \sum_{g \in \mathcal{G}} \gamma_{\tau_z}(a_{x,i}) \gamma_{\tau_t(g)} \gamma_{g} \gamma_{l}(b_{x,i}) \gamma_{\tau_s(g)}(1_{g^{-1}}) \delta_{(s(g),t(g))} \delta_g \\
= \sum_{1 \leq i \leq r} \sum_{g \in \mathcal{G}(y,z)} \gamma_{\tau_z}(a_{x,i}) \gamma_{g}(b_{x,i}1_{g^{-1}}) \delta_{(y,z)} \delta_g \\
= \sum_{g \in \mathcal{G}(y,z)} \gamma_{\tau_z}(a) \delta_{(y,z)} \delta_h.
\]

\[\square\]

**Theorem 7.5.** If \( A \) is a finitely generated projective (resp., \( B_x \)) left \( A^\beta \) (resp., \( B_x^\beta(x) \))-module then the following statements are equivalent:

(i) The Morita context \( (A^\beta, A \ast \beta \mathcal{G}, A, A, \Gamma, \Gamma') \) is strict.

(ii) The Morita context \( (B_x^\beta(x), B_x \ast \beta(x) \mathcal{G}(x), B_x, B_x, \Gamma_x, \Gamma'_x) \) is strict.

**Proof.** It follows from Lemmas 7.3 and 7.4. \[\square\]

### 7.3 Galois theory

The notion of partial Galois extension for partial groupoid actions was introduced in [BP] as a generalization of the classical one for group actions due to S.U. Chase, D.K. Harrison and A. Rosenberg in the global case [CHR] and to M. Dokuchaev, M. Ferrero and the second author in the partial case [DFP]. This notion in both cases is very closed related with the strictness of the Morita context associated to it as well ensured by [BP, Theorem 5.3] and [AL, Theorem 3.1].

We say that \( A \) is a \( \beta \)-partial Galois extension of its subring of invariants \( A^\beta \) (recalling that \( \mathcal{G} \) was assumed to be finite) if there exist elements \( a_i, b_i \in A, 1 \leq i \leq r \), such that

\[
\sum_{1 \leq i \leq r} a_i \beta_g(b_11_g^{-1}) = \delta_{y,g}1_y, \quad \text{for all} \ y \in \mathcal{G}_0, \ \text{where} \ \delta_{y,g} \ \text{denotes the Kronecker symbol.}
\]

This is equivalent to say that the map \( \Gamma' \) defined in the previous subsection is a ring isomorphism (cf. assertion (vi) in [BP, Theorem 5.3]). Furthermore, in this case \( A \) is also projective and finitely generated as a right \( A^\beta \)-module. (cf. assertion (ii) in [BP, Theorem 5.3]).

The following theorem shows how really closed are the Galois theory for partial connected groupoid actions and the one for partial group actions.

**Theorem 7.6.** The following statements are equivalent:
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(i) $A$ is a partial Galois extension of $A^\beta$ and $t_{\beta}$ is onto.
(ii) $(A^\beta, A \ast \beta G, A, A, \Gamma, \Gamma')$ is strict.
(iii) $(B_x^{\beta(x)}, B_x \ast \beta G(x), B_x, B_x, \Gamma_x, \Gamma'_x)$ is strict.
(iv) $B_x$ is a partial Galois extension of $B_x^{\beta(x)}$ and $t_{\beta(x)}$ is onto.

Proof. It is an immediate consequence of Theorem 6.4, Theorem 7.5, of [BP, Theorem 5.3] and [AL, Theorem 3.1].

8. Separability, Semisimplicity and Frobenius properties

In this section we analyze the properties of separability, Frobenius and semisimplicity concerning to the extensions $A \subset A \ast \beta G$ and $B_x \subset B_x \ast \beta G(x)$.

8.1. Separability and semisimplicity. A unital ring extension $R \subset S$ is called separable if the multiplication map $m : S \otimes_R S \to S$ is a splitting epimorphism of $S$-bimodules. This is equivalent to say that there exists an idempotent of separability of $S$ over $R$, i.e., an element $x \in S \otimes_R S$ such that $sx = xs$, for all $s \in S$, and $m(x) = 1_S$. A ring extension $R \subset S$ is left (right) semisimple if any left (right) $S$-submodule $N$ of a left (right) $S$-module $M$ having an $R$-complement in $M$, has an $S$-complement in $M$. Further details on separable and semisimple extensions can be seen in [DI], [HS] and [CIM].

Remark 8.1. It is well-known that any separable extension is a left (right) semisimple extension (see for instance [CIM]).

For the partial action $\beta$ of $G$ on $A$ we associate the maps, as introduced in [BPi], $t_{y,z} : A \to A$ and $t_z : A \to A$ given by

$$t_{y,z}(a) = \sum_{g \in G(y,z)} \beta_g(a_1g^{-1}), \quad t_z(a) = \sum_{y \in G_0} t_{y,z}(a), \quad y, z \in G_0, \quad a \in A.$$  

Notice that $\varphi : A \to A \ast \beta G, \quad a \mapsto \sum_{y \in \mathbb{G}_0}(a_1y)\delta_y$, is a monomorphism of rings and whence $A \ast \beta G$ is an extension of $A$. Moreover, $\varphi$ induces the following $(A, A)$-bimodule structure on $A \ast \beta G$:

$$(20) \quad a \cdot (a_2\delta_g) = aa_2\delta_g, \quad (a_2\delta_g) \cdot a = a_2\beta_g(a_1g^{-1})\delta_g, \quad g \in G, \quad a \in A.$$  

Now we can present the main result of this subsection.

Theorem 8.2. The following statements are equivalent:

(i) there is an element $a$ in the center $C(A)$ of $A$ such that $t_y(a) = 1_y$ for all $y \in G_0$,
(ii) there is an element $b$ in the center $C(B_x)$ of $B_x$ such that $t_{\beta(x)}(b) = 1_x$,
(iii) $B_x \subset B_x \ast \beta G(x)$ is separable,
(iv) $A \subset A \ast \beta G$ is separable.
Proof. (i) $\Rightarrow$ (ii) Let $a \in C(A)$ such that $t_y(a) = 1_y$ for all $y \in G_0$. Observe that $a = \sum_{z \in G_0} \gamma_z (a'_z)$ with $a'_z \in C(B_x)$ for each $z \in G_0$. Also,

$$t_{y,x}(a) = \sum_{g \in \mathcal{G}(y,x)} \gamma_{y,z} \gamma_{g,z}^{-1} (a \gamma_{g,y} (1_{g_x^{-1}}))$$

$$= \sum_{g \in \mathcal{G}(y,x)} \gamma_{g,z} \gamma_{g,z}^{-1} (a'_y 1_{g_x^{-1}})$$

$$= \sum_{g \in \mathcal{G}(y,x)} \gamma_{g,x} (a'_y 1_{g_x^{-1}})$$

$$= \sum_{h \in \mathcal{G}(x)} \gamma_{h} (a'_y 1_{h^{-1}}) \quad (h = g \tau_y)$$

$$= t_{\beta(x)} (a'_y).$$

Hence, $1_x = t_x(a) = \sum_{y \in G_0} t_{\beta(x)} (a'_y) = t_{\beta(x)} (b)$, where $b = \sum_{y \in G_0} a'_y \in C(B_x)$.

(ii) $\Rightarrow$ (i) Conversely, assume that there exists $b \in C(B_x)$ such that $t_{\beta(x)} (b) = 1_x$. Notice that $b \in C(A)$, $t_{y,z}(b) = 0$ if $y \neq x$ and

$$t_{x,z}(b) = \sum_{g \in \mathcal{G}(x,z)} \gamma_{x,z} \gamma_{g,z} (b 1_{g_x^{-1}}) = \gamma_{x,z} (t_{\beta(x)} (b)) = \gamma_{x,z} (1_x) = 1_z.$$

Consequently, $t_z(b) = 1_z$ for all $z \in G_0$.

Moreover, by [BLP, Theorem 3.1] we have that (ii) $\Leftrightarrow$ (iii) and [BPi, Theorem 4.1] implies that (i) $\Leftrightarrow$ (iv). \hfill $\square$

**Corollary 8.3.** If the equivalent statements of Theorem 8.2 hold then the ring extensions $B_x \subset B_x \ast_{\beta(x)} G(x)$ and $A \subset A \ast_{\beta} G$ are semisimple.

**Proof.** It follows from Remark 8.1 and Theorem 8.2. \hfill $\square$

### 8.2. Frobenius property.
A ring extension $R \subset S$ is called Frobenius if there exist an element $u = \sum_{i=1}^n s_{i1} \otimes s_{i2} \in S \otimes_R S$ and an $R$-bimodule map $\varepsilon : S \to R$ such that $x$ is $S$-central and $\sum_{i=1}^n \varepsilon(s_{i1}) s_{i2} = \sum_{i=1}^n s_{i1} \varepsilon(s_{i2}) = 1$. More details on Frobenius extension can be seen, for example, in [CIM].

**Theorem 8.4.** The extensions $A \subset A \ast_{\beta} G$ and $B_x \subset B_x \ast_{\beta(x)} G(x)$ are Frobenius.
Proof. Consider 

\[ u = \sum_{g \in G} 1_g \delta_g \otimes 1_g^{-1} \delta_g^{-1} \in A \ast \beta \mathcal{G} \otimes_A A \ast \beta \mathcal{G} \text{ and note that} \]

\[
(a_l \delta_l)u = \sum_{g \in G} a_l \delta_l (1_g 1^{-1}_g) \delta_g \otimes 1_g^{-1} \delta_g^{-1} = \sum_{h \in \mathcal{T}_l(l)} a_l 1_h \delta_h \otimes 1_h^{-1} \delta_h^{-1}
\]

for all \( l \in \mathcal{G} \) and \( a_l \in B_l \). Hence, \( u \) is \( A \ast \beta \mathcal{G} \)-central. Note that \( \varepsilon : A \ast \beta \mathcal{G} \rightarrow A \) given by

\[ \varepsilon(a_g \delta_g) = \delta_g \mathcal{G}_0 a_g = \begin{cases} a_g, & \text{if } g \in \mathcal{G}_0 \\ 0, & \text{otherwise} \end{cases} \]

is an \( R \)-bimodule map. Moreover,

\[
\sum_{g \in \mathcal{G}} \varepsilon(1_g \delta_g)1_g^{-1} \delta_g^{-1} = \sum_{g \in \mathcal{G}} 1_g \delta_g 1_g^{-1} \varepsilon(\delta_g^{-1}) = \sum_{y \in \mathcal{G}_0} 1_y \delta_y = 1.
\]

Thus \( A \subset A \ast \beta \mathcal{G} \) is a Frobenius extension. The second assertion is an immediate consequence of [BLP, Theorem 3.6].

\[ \square \]

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