GB-KMV: An Augmented KMV Sketch for Approximate Containment Similarity Search

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Abstract—In this paper, we study the problem of approximate containment similarity search. Given two records \( Q \) and \( X \), the containment similarity between \( Q \) and \( X \) with respect to \( Q \) is \(|Q \cap X| / |Q|\). Given a query record \( Q \) and a set of records \( S \), the containment similarity search finds a set of records from \( S \) whose containment similarity regarding \( Q \) is not less than the given threshold. This problem has many important applications in commercial and scientific fields such as record matching and domain search. Existing solution relies on the asymmetric LSH method by transforming the containment similarity to well-studied Jaccard similarity. In this paper, we use a inherently different framework by transforming the containment similarity to set intersection. We propose a novel augmented KMV sketch technique, namely GB-KMV, which is data-dependent and can achieve a much better trade-off between the sketch size and the accuracy. We provide a set of theoretical analysis to underpin the proposed augmented KMV sketch technique, and show that it outperforms the state-of-the-art technique LSH-E in terms of estimation accuracy under practical assumption. Our comprehensive experiments on real-life datasets verify that GB-KMV is superior to LSH-E in terms of the space-accuracy trade-off, time-accuracy trade-off, and the sketch construction time. For instance, with similar estimation accuracy (F-1 score), GB-KMV is over 100 times faster than LSH-E on several real-life datasets.

I. INTRODUCTION

In many applications such as information retrieval, data cleaning, machine learning and user recommendation, an object (e.g., document, image, web page and user) is described by a set of elements (e.g., words, q-grams, and items). One of the most critical components in these applications is to define the set similarity between two objects and develop corresponding similarity query processing techniques. Given two records (objects) \( X \) and \( Y \), a variety of similarity functions/metrics have been identified in the literature for different scenarios (e.g., [23], [55]). Many indexing techniques have been developed to support efficient exact and approximate lookups and joins based on these similarity functions.

Many of the set similarity functions studied are symmetric functions, i.e., \( f(X, Y) = f(Y, X) \), including widely used Jaccard similarity and Cosine similarity. In recent years, much research attention has been given to the asymmetric set similarity functions, which are more appropriate in some applications. Containment similarity (a.k.a., Jaccard containment similarity) is one of the representative asymmetric set similarity functions, where the similarity between two records \( X \) and \( Y \) is defined as \( f(X, Y) = |X \cap Y| / |X| \) in which \(|X \cap Y|\) and \(|X|\) are intersection size of \( X \) and \( Y \) and the size of \( X \), respectively.

Compared with symmetric similarity such as Jaccard similarity, containment similarity gives special consideration on the query size, which makes it more suitable in some applications. As shown in [55], containment similarity is useful in record matching application. Given two text descriptions of two restaurants \( X \) and \( Y \) which are represented by two “set of words” records: \{five, guys, burgers, and, fries, downtown, brooklyn, new, york\} and \{five, kitchen, berkeley\} respectively. Suppose query \( Q \) is \{five, guys\}, we have that the Jaccard similarity of \( Q \) and \( X \) (resp. \( Y \)) is \( \frac{2}{5} = 0.44 \) (\( \frac{1}{5} = 0.20 \)). Note the Jaccard similarity is \( f(Q, X) = 1 / |Q| \times |X| \). Based on the Jaccard similarity, record \( Y \) matches better to query \( Q \), but intuitively \( X \) should be a better choice. This is because the Jaccard similarity unnessesarily favors the short records. On the other hand, the containment similarity will lead to the desired order with \( f(Q, X) = \frac{1}{2} = 1.0 \) and \( f(Q, Y) = \frac{1}{5} = 0.5 \). Containment similarity search can also support online error-tolerant search for matching user queries against addresses (map service) and products (product search). This is because the regular keyword search is usually based on the containment search, and containment similarity search provides a natural error-tolerant alternative [5]. In [44], Zhu et al. show that containment similarity search is essential in domain search which enables users to effectively search Open Data.

The containment similarity is also of interest to applications of computing the fraction of values of one column that are contained in another column. In a dataset, the discovery of all inclusion dependencies is a crucial part of data profiling efforts. It has many applications such as foreign-key detection and data integration(e.g., [22], [31], [8], [33], [30]).

Challenges. The problem of containment similarity search has been intensively studied in the literature in recent years (e.g., [5], [55], [44]). The key challenges of this problem come from the following three aspects: (i) The number of elements (i.e., vocabulary size) may be very large. For instance, the vocabulary will blow up quickly when the higher-order shingles are used [55]. Moreover, query and record may contain many elements. To deal with the sheer volume of the data, it is desirable to use sketch technique to provide effectively and efficiently approximate solutions. (ii) The data distribution (e.g., record size and element frequency) in real-life application may be highly skewed. This may lead to poor performance in practice for data independent sketch methods. (iii) A subtle difficulty of the approximate solution comes from the asymmetric property of the containment similarity. It is shown in [34] that there cannot exist any locality sensitive hashing (LSH) function family for containment similarity search.

To handle the large scale data and provide quick response, most existing solutions for containment similarity search seek to the approximate solutions. Although the use of LSH is restricted, the novel asymmetric LSH method has been designed in [34] to address the issue by padding techniques. Some enhancements of asymmetric LSH techniques are proposed in the following works by introducing different functions...
(e.g., [35]). Observe that the performance of the existing solutions are sensitive to the skewness of the record size, Zhu et. al propose a partition-based method based on Minhash LSH function. By using optimal partition strategy based on the size distribution of the records, the new approach can achieve much better time-accuracy trade-off.

We notice that all existing approximate solutions rely on the LSH functions by transforming the containment similarity to well-studied Jaccard similarity. That is,

$$\frac{|Q \cap X|}{|Q|} = \frac{|Q \cap X|}{|Q \cup X|} \times \frac{|Q \cup X|}{|Q|} \times 1$$

As the size of query is usually readily available, the estimation error come from the computation of Jaccard similarity and union size of $Q$ and $X$. Note that although the union size can be derived from jaccard similarity [44], the large variance caused by the combination of two estimations remains. This motivates us to use a different framework by transforming the containment similarity to set intersection size estimation, and the error is only contributed by the estimation of $|Q \cap X|$. The well-known KMV sketch [11] has been widely used to estimate the set intersection size, which can be immediately applied to our problem. However, this method is data-independent and hence cannot handle the skewed distributions of records size and element frequency, which is common in real-life applications. Intuitively, the record with larger size and the element with high-frequency should be allocated more resources. In this paper, we theoretically show that the existing KMV-sketch technique cannot consider these two perspectives by simple heuristics, e.g., explicitly allocating more resource to record with large size. Consequently, we develop an augmented KMV sketch to exploit both record size distribution and the element frequency distribution for better space-accuracy and time-accuracy trade-offs. Two technique are proposed: (i) we impose a global threshold to KMV sketch, namely G-KMV sketch, to achieve better estimate accuracy. As discussed in Section IV-A(2), this technique cannot be extended to the Minhash LSH. (ii) we introduce an extra buffer for each record to take advantage of the skewness of the element frequency. A cost model is proposed to carefully choose the buffer size to optimize the accuracy for the given total space budget and data distribution.

**Contributions.** Our principle contributions are summarized as follows.

- We propose a new augmented KMV sketch technique, namely GB-KMV, for the problem of approximate containment similarity search. By imposing a global threshold and an extra buffer for KMV sketches of the records, we significantly enhance the performance as the new method can better exploit the data distributions.

- We provide theoretical underpinnings to justify the design of GB-KMV method. We also theoretically show that GB-KMV outperforms the state-of-the-art technique LSH-E in terms of accuracy under realistic assumption on data distributions.

- Our comprehensive experiments on real-life set-valued data from various applications demonstrate the effectiveness and efficiency of our proposed method.

**Road Map.** The rest of the paper is organized as follows. Section II presents the preliminaries. Section III introduces the

### Table I. The Summary of Notations

| Notation | Definition |
|----------|------------|
| $S$      | a collection of records |
| $X, Q$   | record, query record |
| $e, q$   | record size of $X$, query size of $Q$ |
| $J(Q, X), s$ | Jaccard similarity between query $Q$ and set $X$ |
| $C(Q, X), t$ | Containment similarity of query $Q$ in set $X$ |
| $s^*$    | Jaccard similarity threshold |
| $E_X$    | the KMV signature (i.e., hash values) of record $X$ |
| $H_X$    | all hash values of the elements in record $X$ |
| $h_r$    | the buffer of record $X$ |
| $t^*$    | containment similarity threshold |
| $b$      | sketch space budget, measured by the number of signatures (i.e., hash values or elements) |
| $r$      | the global threshold for hash value |
| $m$      | the buffer size(with bit unit) of GB-KMV sketch |
| $n$      | number of distinct elements in dataset $S$ |

Existing solutions. Our approach, GB-KMV sketch, is devised in Section IV. Extensive experiments are reported in Section V followed by the related work in Section VI. Section VII concludes the paper.

### II. Preliminaries

In this section, we first formally present the problem of containment similarity search, then introduce some preliminary knowledge. In Table I we summarize the important mathematical notations appearing throughout this paper.

**A. Problem Definition**

In this paper, the element universe is $E = \{e_1, e_2, ..., e_n\}$. Let $S$ be a collection of records (sets) $\{X_1, X_2, ..., X_m\}$, where $X_i$ ($1 \leq i \leq m$) is a set of elements from $E$.

Before giving the definition of containment similarity, we first introduce the Jaccard similarity.

**Definition 1 (Jaccard Similarity).** Given two records $X$ and $Y$ from $S$, the Jaccard similarity between $X$ and $Y$ is defined as the size of the intersection divided by the size of the union, which is expressed as

$$J(X, Y) = \frac{|X \cap Y|}{|X \cup Y|}$$  \hspace{1cm} (1)

Similar to the Jaccard similarity, the containment similarity (a.k.a a Jaccard containment similarity) is defined as follows.

**Definition 2 (Containment Similarity).** Given two records $X$ and $Y$ from $S$, the containment similarity of $X$ in $Y$, denoted by $C(X, Y)$ is the size of the intersection divided by record size $|X|$, which is formally defined as

$$C(X, Y) = \frac{|X \cap Y|}{|X|}$$  \hspace{1cm} (2)

Note that by replacing the union size $|X \cup Y|$ in Equation I with size $|X|$, we get the containment similarity. It is easy to see that Jaccard similarity is symmetric while containment similarity is asymmetric.

In this paper, we focus on the problem of containment similarity search which is to look up a set of records whose containment similarity towards a given query record is not smaller than a given threshold. The formal definition is as follows.

**Definition 3 (Containment Similarity Search).** Given a query $Q$, and a threshold $t^* \in [0, 1]$ on the containment
| id | record         | C(Q,X_i) |
|----|----------------|-----------|
| X₁ | \{e₁,e₂,e₃,e₄,e₇\} | 0.67      |
| X₂ | \{e₂,e₃,e₅\}    | 0.5       |
| X₃ | \{e₂,e₄,e₅\}    | 0.5       |
| X₄ | \{e₁,e₂,e₆,e₁₀\}| 0.33      |
| Q  | \{e₁,e₂,e₃,e₇,e₉\} | 0.33     |

Fig. 1. A four-record dataset and a query \( Q \); \( C(Q,X_i) \) is the containment similarity of \( Q \) in \( X_i \).

Next, we give an example to show the problem of containment similarity search.

**Example 1.** Fig. 1 shows a dataset with four records \( \{X₁, X₂, X₃, X₄\} \) and the element universe is \( E = \{e₁,e₂,...,e₁₀\} \). Given a query \( Q = \{e₁,e₂,e₃,e₅,e₇,e₉\} \) and a containment similarity threshold \( t^* = 0.5 \), the records satisfying \( C(Q,X_i) \geq t^* \) are \( X₁, X₂ \).

**Problem Statement.** In this paper, we investigate the problem of approximate containment similarity search. For the dataset \( S \) with a large number of records, we aim to build a synopsis of the dataset such that (i) can efficiently support containment similarity search with high accuracy, (ii) can handle large size records, and (iii) has a compact index size.

### B. Minwise Hashing

Minwise Hashing is proposed by Broder in [13], [14] for estimating the Jaccard similarity of two records \( X \) and \( Y \). Let \( h \) be a hash function that maps the elements of \( X \) and \( Y \) to distinct integers, and define \( h_{min}(X) \) and \( h_{min}(Y) \) to be the minimum hash value of a record \( X \) and \( Y \), respectively. Assuming no hash collision, Broder [13] showed that the Jaccard similarity of \( X \) and \( Y \) is the probability of two minimum hash values being equal: \( Pr[h_{min}(X) = h_{min}(Y)] = J(X,Y) \). Applying such \( k \) different independent hash functions \( h₁, h₂, ..., hₖ \) to a record \( X(Y, \text{resp.}) \), the MinHash signature of \( X(Y, \text{resp.}) \) is to keep \( k \) values of \( h_{min}(X)(h_{min}(Y), \text{resp.}) \) for \( k \) functions. Let \( n_i, i = 1, 2, ..., k \) be the indicator function such that

\[
n_i := \begin{cases} 1 & \text{if } h_i^{min}(X) = h_i^{min}(Y), \\ 0 & \text{otherwise.} \end{cases}
\]

then the Jaccard similarity between record \( X \) and \( Y \) can be estimated as

\[
\hat{s} = \hat{J}(X,Y) = \frac{1}{k} \sum_{i=1}^{k} n_i
\]

Let \( s = J(X,Y) \) be the Jaccard similarity of set \( X \) and \( Y \), then the expectation of \( \hat{J} \) is

\[
E(\hat{s}) = s
\]

and the variance of \( \hat{s} \) is

\[
V ar(\hat{s}) = \frac{s(1-s)}{k}
\]

### C. KMOV Sketch

The \( k \) minimum values (KMOV) technique introduced by Bayer et. al in [11] is to estimate the number of distinct elements in a large dataset. Given a no-collision hash function \( h \) which maps elements to range \([0,1]\), a KMOV synopsis of a record \( X \), denoted by \( L_X \), is to keep \( k \) minimum hash values of \( X \). Then the number of distinct elements \( |X| \) can be estimated by \( |X| = \frac{1}{U(k)} \) where \( U(k) \) is \( k \)-th smallest hash value. By \( h(X) \), we denote hash values of all elements in the record \( X \).

In [11], Bayer et. al also methodically analyse the problem of distinct element estimation under multi-set operation. As for union operation, consider two records \( X \) and \( Y \) with corresponding KMOV synopses \( L_X \) and \( L_Y \) of size \( k_X \) and \( k_Y \), respectively. In [11], \( L_X \oplus L_Y \) represents the set consisting of the \( k \) smallest hash values in \( L_X \cup L_Y \) where

\[
k = \min(k_X, k_Y)
\]

Then the KMOV synopses of \( X \cup Y \) is \( L = L_X \oplus L_Y \). An unbiased estimator for the number of distinct elements in \( X \cup Y \), denoted by \( \hat{D}_∪ \) is as follows.

\[
\hat{D}_∪ = \frac{k-1}{U(k)}
\]

For intersection operation, the KMOV synopses is \( L = L_X \cap L_Y \) where \( k = \min(k_X, k_Y) \). Let \( \bar{K}_∩ = \{v \in L : v \in L_X \cap L_Y\} \), i.e., \( \bar{K}_∩ \) is the number of common distinct hash values of \( L_X \) and \( L_Y \) within \( L \). Then the number of distinct elements in \( X \cap Y \), denoted by \( \hat{D}_∩ \), can be estimated as follows.

\[
\hat{D}_∩ = \frac{\bar{K}_∩ \times (k-1)}{U(k)}
\]

The variance of \( \hat{D}_∩ \), as shown in [11], is

\[
V ar(\hat{D}_∩) = \frac{D∩(2D∪ - k^2 - kD∩ + k + D∪)}{k(k-2)}
\]

### III. EXISTING SOLUTIONS

In this section, we present the state-of-the-art technique for the approximate containment similarity search, followed by theoretical analysis on the limits of the existing solution.

#### A. LSH Ensemble Method

LSH Ensemble technique, LSH-E for short, is proposed by Zhu et. al in [44] to tackle the problem of approximate containment similarity search. The key idea is : (1) transform the containment similarity search to the well-studied Jaccard similarity search; and (2) partition the data by length and then apply the LSH forest [9] technique for each individual partition.

### Similarity Transformation.

Given a record \( X \) with size \( x = |X| \), a query \( Q \) with size \( q = |Q| \), containment similarity \( t = C(Q,X) \) and Jaccard similarity \( s = J(Q,X) \). The transformation back and forth are as follows.

\[
s = \frac{t}{q + 1 - t}, \quad t = \frac{(\frac{s}{q} + 1)s}{1 + s}
\]

Given the containment similarity search threshold as \( t^* \) for the query \( q \), we may come up with its corresponding Jaccard similarity.
similarity threshold $s^*$ by Equation 12. A straightforward solution is to apply the existing approximate Jaccard similarity search technique for each individual record $x \in D$ with the Jaccard similarity threshold $s^*$ (e.g., compute Jaccard similarity between the query $Q$ and a set $X$ based on their MinHash signatures). In order to take advantages of the efficient indexing techniques (e.g., LSH forest 9), LSH-E will partition the dataset $S$.

**Data Partition.** By partitioning the dataset $S$ according to the record size, $LSH-E$ can replace $x$ in Equation 12 with its upper bound $u$ (i.e., the largest record size in the partition) as an approximation. That is, for the given containment similarity $t^*$ we have

$$s^* = \frac{t^*}{\frac{t}{q} + 1 - t^*} \quad (13)$$

The use of upper bound $u$ will lead to false positives. In 44, an optimal partition method is designed to minimize the total number of false positives brought by the use of upper bound in each partition. By assuming that the record size distribution follows the power-law distribution and similarity values are uniformly distributed, it is shown that the optimal partition can be achieved by ensuring each partition has the equal number of records (i.e., equal-depth partition).

**Containment Similarity Search.** For each partition $S_i$ of the data, $LSH-E$ applies the dynamic LSH technique (e.g., LSH forest 9). Particularly, the records in $S_i$ are indexed by a MinHash LSH with parameter $(b, r)$ where $b$ is the number of bands used by the LSH index and $r$ is the number of hash values in each band. For the given query $Q$, the $b$ and $r$ values are carefully chosen by considering their corresponding number of false positives and false negatives regarding the existing records. Then the candidate records in each partition can be retrieved from the MinHash index according to the corresponding Jaccard similarity thresholds obtained by Equation 13. The union of the candidate records from all partitions will be returned as the result of the containment similarity search.

**B. Analysis**

One of the $LSH-E$’s advantages is that it converts the containment similarity problem to Jaccard similarity search problem which can be solved by the mature and efficient MinHash LSH method. Also, $LSH-E$ carefully considers the record size distribution and partitions the records by record size. In this sense, we say $LSH-E$ is a data-dependent method and it is reported that $LSH-E$ significantly outperforms existing asymmetric LSH based solutions 34, 35 (i.e., data-independent methods) as $LSH-E$ can exploit the information of data distribution by partitioning the dataset. However, this benefit is offset by the fact that the upper bound will bring extra false positives, in addition to the error from the MinHash technique.

Below we theoretically analyse the performance of $LSH-E$ by studying the expectation and variance of its estimator.

Using the notations same as above, let $s = J(Q, X)$ be the Jaccard similarity between query $Q$ and set $X$ and $t = C(Q, X)$ be the containment similarity of $Q$ in $X$. By Equation 5 given the MinHash signature of query $Q$ and $X$ respectively, an unbiased estimator $\hat{s}$ of Jaccard similarity $s = J(Q, X)$ is the ratio of collisions in the signature, and the variance of $\hat{s}$ is $Var[\hat{s}] = \frac{s(1-s)}{k}$ where $k$ is signature size of each record. Then by transformation Equation 12 the estimator $\hat{t}$ of containment similarity $t = C(Q, X)$ by MinHash LSH is

$$\hat{t} = \frac{\frac{u}{q} + 1}{1 + \frac{u}{s}} \quad (14)$$

where $q = |Q|$ and $x = |X|$. The estimator $\hat{t}'$ of containment similarity $t = C(Q, X)$ by LSH-E is

$$\hat{t}' = \frac{\frac{u}{q} + 1}{1 + \frac{u}{s}} \quad (15)$$

where $q = |Q|$ and $u$ is the upper bound of $|X|$.

Next, we use Taylor expansions to approximate the expectation and variance of a function with one random variable 26. We first give a lemma.

**Lemma 1.** Given a random variable $X$ with expectation $E[X]$ and variance $Var[X]$, the expectation of $f(X)$ can be approximated as

$$E[f(X)] = f(E[X]) + \frac{f''(E[X])}{2} Var[X] \quad (16)$$

and the variance of $f(X)$ can be approximated as

$$Var[f(X)] = [f'(E[X])]^2 Var[X] - \frac{[f''(E[X])]^2}{4} Var^2[X] \quad (17)$$

According to Equation 14 let $\hat{t} = f(\hat{s}) = \frac{1}{\frac{u}{s} + 1}$ where $\alpha = \frac{u}{s} + 1$. We can see that the estimator $\hat{t}$ is a function of $\hat{s}$, and $f'(\hat{s}) = \frac{-1}{(1+s)^2}$ and $f''(\hat{s}) = 2\alpha \frac{1}{(1+s)^3}$. Then based on Lemma 1 the expectation and variance of $\hat{t}$ are approximated as

$$E[\hat{t}] \approx t(1 - \frac{1 - s}{k(1+s)^2}) \quad (18)$$

$$Var[\hat{t}] \approx \frac{D_1^2(1-s)[k(1+s)^2 - s(1-s)]}{q^2 k^2 s(1+s)^4} \quad (19)$$

Similarly, the expectation and variance of $LSH-E$ estimator $\hat{t}'$ can be approximated as

$$E[\hat{t}'] \approx t' \frac{u + q}{x + q}(1 - \frac{1 - s}{k(1+s)^2}) \quad (20)$$

$$Var[\hat{t}'] \approx \frac{u + q}{x + q} \frac{D_2^2(1-s)[k(1+s)^2 - s(1-s)]}{q^2 k^2 s(1+s)^4} \quad (21)$$

The computation details are in technique report 41. Since $u$ is the upper bound of $x$, the variance of $LSH-E$ estimator $Var[\hat{t}']$ is larger than that of MinHash LSH estimator. Also, by Equation 13 and Equation 20 we can see that both estimators are biased and $LSH-E$ method is quite sensitive to the setting of the upper bound $u$ by Equation 20. Because the presence of upper bound $u$ will enlarge the estimator off true value, $LSH-E$ method favours recall while the precision will be deteriorated. The larger the upper bound $u$ is, the worse the precision will be. Our empirical study shows that $LSH-E$ cannot achieve a good trade-off between accuracy and space, compared with our proposed method.
that in this paper. Detailed algorithms and theoretical analysis will apply the existing set similarity join/search indexing technique to achieve better space-accuracy trade-off for a p-

Fig. 2. The KMV sketch of the dataset in Example[1] each signature consists of element-hash value pairs. \( k_i \) is the signature size of \( X_i \).

IV. OUR APPROACH

In this section, we introduce an augmented KMV sketch technique to achieve better space-accuracy trade-off for approximate containment similarity search. Section [IV-A] briefly introduces the motivation and main technique of our method, namely GB-KMV. The detailed implementation is presented in Section [IV-B] followed by extensive theoretical analysis in Section [IV-C].

A. Motivation and Techniques

The key idea of our method is to propose a data-dependent indexing technique such that we can exploit the distribution of the data (i.e., record size distribution and element frequency distribution) for better performance of containment similarity search. We augment the existing KMV technique by introducing a global threshold for sample size allocation and a buffer for frequent elements, namely GB-KMV, to achieve better trade-off between synopses size and accuracy. Then we apply the existing set similarity join/search indexing technique to speed up the containment similarity search.

Below we outline the motivation of the key techniques used in this paper. Detailed algorithms and theoretical analysis will be introduced in Section [IV-B] and [IV-C] respectively.

(1) Directly Apply KMV Sketch

Given a query \( Q \) and a threshold \( t^* \) on containment similarity, the goal is to find record \( X \) from dataset \( S \) such that

\[
\frac{|Q \cap X|}{|Q|} \geq t^*,
\]

Applying some simple transformation to Equation (22) we get

\[
|Q \cap X| \geq t^*|Q|,
\]

Let \( \theta = t^*|Q| \), then the containment similarity search problem is converted into finding record \( X \) whose intersection size with the query \( Q \) is not smaller than \( \theta \), i.e., \( |Q \cap X| \geq \theta \).

Therefore, we can directly apply the KMV method introduced in Section [II-C]. Given KMV signatures of a record \( X \) and a query \( Q \), we can estimate their intersection size \((|Q \cap X|)\) according to Equation (10) Then the containment similarity of \( Q \) in \( X \) is immediately available given the query size \( |Q| \). Below, we show an example on how to apply KMV method to containment similarity search.

Example 2. Fig. 2 shows the KMV sketch on dataset in Example[1]. Given KMV signature of \( Q \) \( \mathcal{L}_Q = \{(e_5,0.10),(e_2,0.24),(e_7,0.33),(e_9,0.56)\} \) and \( X_1 \) \( \mathcal{L}_{X_1} = \{(e_2,0.24),(e_7,0.33),(e_4,0.47)\} \), we have \( k = \min \{k_Q,k_1\} = 3 \), then the size-k KMV synopses of \( Q \cup X_1 \) is \( \mathcal{L} = \mathcal{L}_Q \cup \mathcal{L}_{X_1} = \{(e_5,0.10),(e_2,0.24),(e_7,0.33),(e_9,0.56)\} \), the \( k \)-th smallest hash value \( V(k) \) is 0.33 and the size of intersection of \( Q \cap \mathcal{L} \) and \( \mathcal{L}_{X_1} \) within \( \mathcal{L} \) is \( K_0 = |\{v : v \in \mathcal{L}_Q \cap \mathcal{L}_{X_1}, v \in \mathcal{L}\}| = 2 \). Then the intersection size of \( Q \) and \( X_1 \) is estimated as \( \hat{D} = \frac{k}{k} \times \frac{1}{\hat{V}(k)} = \frac{3}{3} \times \frac{1}{0.33} = 4.04 \), and the containment similarity is \( \hat{t} = \frac{\hat{D}}{|Q|} = 0.67 \). Then \( X_1 \) is returned if the given containment similarity threshold \( t^* \) is 0.5.

Remark 1. In [44], the size of the query is approximated by MinHash signature of \( Q \), where KMV sketch can also serve for the same purpose. But the exact query size is used their implementation for performance evaluation. In practice, the query size is readily available, we assume query size is given throughout the paper.

Optimization of KMV Sketch. Given a space budget \( b \), we can keep size-\( k \) KMV signatures (i.e., \( k \) minimal hash values) for each record \( X_i \) with \( \sum_{i=0}^n k_i = b \). A natural question is how to allocate the resource (e.g., setting of \( k \) values) to achieve the best overall estimation accuracy. Intuitively, more resources should be allocated to records with more frequent elements or larger record size, i.e., larger \( k_i \) for record with larger size. However, Theorem 1 (Section IV-C2) suggests that, the optimal resource allocation strategy in terms of estimation variance is to use the same size of signature for each record. This is because the minimal of two \( k \)-values is used in Equation (8) and hence the best solution is to evenly allocate the resource. Thus, we have that KMV sketch based method for approximate containment similarity search. For the given budget \( b \), we keep \( k_i = \lceil \frac{b}{|X_i|} \rceil \) minimal hash values for each record \( X_i \).

(2) Impose a Global Threshold to KMV Sketch (G-KMV)

The above analysis on optimal KMV sketch suggests an equal size allocation strategy, that is, each record is associated with the same size signature. Intuitively we should assign more resources (i.e., signature size) to the records with large size because they are more likely to appear in the results. However, the estimate accuracy of KMV for two sets size intersection is determined by the sketch with smaller size since we choose \( k = \min(k_1,k_2) \) for KMV signatures of \( X_1 \) and \( X_2 \) for \( D_1 \) and \( D_2 \) in Equation (2) thus it is useless to give more resource to one of the records. We further explain the reason behind with the following example.

Before we introduce the global threshold to KMV sketch, consider the KMV sketch shown in the Fig. 2.

Example 3. Suppose we have \( \mathcal{L}_Q = \{(e_5,0.10),(e_2,0.24),(e_7,0.33),(e_9,0.56)\} \) and \( \mathcal{L}_{X_3} = \{(e_3,0.10),(e_2,0.24)\} \). Although there are four hash values in \( \mathcal{L}_Q \cup \mathcal{L}_{X_3} = \{(e_5,0.10),(e_2,0.24),(e_7,0.33),(e_9,0.56)\} \), we can only consider \( k = \min \{k_Q,k_{X_3}\} = 2 \) smallest values of \( \mathcal{L}_Q \cup \mathcal{L}_{X_3} \) by Equation (8) which is \( \{(e_5,0.10),(e_2,0.24)\} \), and the \( k \)-th (\( k = 2 \)) minimum hash value used in Equation (8) is 0.24. We cannot use \( k = 4 \) (i.e., \( U(k)=0.56 \)) to estimate \( |Q \cup X_3| \) because the 4-th smallest hash value in \( \mathcal{L}_Q \cup \mathcal{L}_{X_3} \) may not be the 4-th smallest hash value in \( h(Q \cup X_3) \), because the unseen 3-rd smallest hash value of \( X_3 \) might be \( e_4 \) for example, which is smaller than 0.56. Recall that \( h(Q \cup X_3) \) denote the hash values of all elements in \( Q \cup X_3 \).

Nevertheless, if we know that all the hash values smaller than a global threshold, say 0.6, are kept for every record, we can safely use the 4-th hash value of \( \mathcal{L}_Q \cup \mathcal{L}_{X_3} \) (i.e., 0.56) for the estimation. This is because we can ensure the 4-th smallest
Meanwhile, we have

\[ k = |L_Q \cup L_X| \]  

(24)

Meanwhile, we have \( K = |L_Q \cap L_X| \). Let \( U(k) \) be the k-th minimal hash value in \( L_Q \cup L_X \), then the overlap size of \( Q \) and \( X \) can be estimated as

\[ \hat{\delta}_{G-KMV}^{(k)} = \frac{K - k - 1}{k} U(k) \]  

(25)

Then the containment similarity of \( Q \) in \( X \) is

\[ \hat{C} = \frac{\hat{\delta}_{G-KMV}^{(k)}}{q} \]  

(26)

where \( q \) is the query size. We remark that, as a by-product, the global threshold favours the record with large size because all elements with hash value smaller than \( \tau \) are kept for each record.

Below is an example on how to compute the containment similarity based on G-KMV sketch.

**Example 4.** Fig. 3 shows the G-KMV sketch of dataset in Example 1 with hash value threshold \( \tau = 0.5 \). Given the signature of \( Q(L_Q) = \{(e_5,0.10),(e_2,0.24),(e_7,0.33)\} \) and \( X_1(L_{X_1}) = \{(e_2,0.24),(e_7,0.33),(e_4,0.47)\} \), the G-KMV sketch of \( Q \cup X_1 \) is \( \hat{C} = \frac{3}{q} \times \hat{\delta}_{G-KMV}^{(k)} = \frac{2}{2} \times \frac{3}{0.47} = 3.19 \), and the containment similarity is \( \hat{C} = \frac{3}{0.53} \). Then \( X_1 \) is returned if the given containment similarity threshold \( \hat{C} \) is 0.5.

**Correctness of G-KMV sketch.** Theorem 2 in Section IV-C3 shows the correctness of the G-KMV sketch.

**Comparison with KMV.** In Theorem 3 (Section IV-C4), we theoretically show that G-KMV can achieve better accuracy compared with KMV.

**Remark 2.** Note that the global threshold technique cannot be applied to MinHash based techniques. In minHash LSH, the k minimum hash values are corresponding to k different independent hash functions, while in KMV sketch, the k-value sketch is obtained under one hash function. Thus we can only impose this global threshold on the same hash function for the KMV sketch based method.

(3) Use Buffer for KMV Sketch (GB-KMV)

In addition to the skewness of the record size, it is also worthwhile to exploit the skewness of the element frequency. Intuitively, more resource should be assigned to high-frequency elements because they are more likely to appear in the records. However, due to the nature of the hash function used by KMV sketch, the hash value of an element is independent to its frequency; that is, all elements have the same opportunity contributing to the KMV sketch.

One possible solution is to divide the elements into multiple disjoint groups according to their frequency (e.g., low-frequency and high-frequency ones), and then apply KMV sketch for each individual group. The intersection size between two records \( Q \) and \( X \) can be computed within each group and then sum up together. However, our initial experiments suggest that this will lead to poor accuracy because of the summation of the intersection size estimations. In Theorem 4 (Section IV-C5), our theoretical analysis suggests that the combination of estimated results are very likely to make the overall accuracy worse.

To avoid combining multiple estimation results, we use a bitmap buffer with size \( r \) for each record to exactly keep track of the r most frequent elements, denoted by \( E_r \). Then we apply G-KMV technique to the remaining elements, resulting in a new augmented sketch, namely GB-KMV. Now we can estimate \( |Q \cap X| \) by combining the intersection of their bitmap buffers (exact solution) and KMV sketches (estimated solution).

As shown in Fig. 4, suppose we have \( E_H = \{e_1,e_2\} \) and the global threshold for hash value is \( \tau = 0.5 \), then the sketch of each record consists of two parts \( L_H \) and \( L_{G-KMV} \); that is, for each record we use bitmap to keep the elements corresponding to high-frequency elements \( E_H = \{e_1,e_2\} \), then we store the left elements with hash value less than \( \tau = 0.5 \).

**Example 5.** Given the signature of \( Q(L_Q) = \{e_1,e_2\} \cup \{(e_5,0.10),(e_7,0.33)\} \) and \( X_1(L_{X_1}) = \{e_1,e_2\} \cup \{(e_7,0.33),(e_4,0.47)\} \), the intersection of high-frequency part is \( L_Q \cap L_{X_1} = \{e_1,e_2\} \) with intersection size as 2; next we consider the G-KMV part. Similar to Example 4 we compute the intersection of \( L_{G-KMV} \) part. The KMV sketch is \( L' = L_Q \cup L_{X_1} = \{(e_5,0.10),(e_7,0.33),(e_4,0.47)\} \). According to Equation 24 the k-th(k = 3) smallest hash value is \( U(k) = 0.47 \), and the size of intersection of \( L_Q \) and \( L_{X_1} \) within \( L' \) is \( K = |\{v : v \in L_Q \cap L_{X_1}, v \in L'\}| = 1 \). Then the intersection size of \( Q \) and \( X_1 \) in \( L_{G-KMV} \) part is
estimated as $D_\cap = \frac{k_\tau}{k} \times \frac{k_\tau - 1}{U_3} = \frac{1}{3} + \frac{0.27}{4} = 1.4$; together with the High-frequency part, the intersection size of $Q$ and $X_1$ is estimated as $2 + 1.4 = 3.4$ and the containment similarity is $l = D_\cap = 0.53$. Then $X_1$ is returned if the given containment similarity threshold $t^*$ is 0.5.

**Optimal Buffer Size.** The key challenge is how to set the size of bitmap buffer for the best expected performance of GB-KMV sketch. In Section IV-C6 we provide a theoretical analysis, which is verified in our performance evaluation.

**Comparison with G-KMV.** As the G-KMV is a special case of GB-KMV with buffer size 0 and we carefully choose the buffer size with our cost model, the accuracy of GB-KMV is not worse than G-KMV.

**Comparison with LSH-E.** Through theoretical analysis, we show that the performance (i.e., the variance of the estimator) of GB-KMV can always outperform that of LSH-E in Theorem 5 (Section IV-C7).

### B. Implementation of GB-KMV

In this section, we introduce the technique details of our proposed GB-KMV method. We first show how to build GB-KMV sketch on the dataset $S$ and then present the containment similarity search algorithm.

**GB-KMV Sketch Construction.** For each record $X \in S$, its GB-KMV sketch consists of two components: (1) a buffer which exactly keeps high-frequency elements, denoted by $H_X$; and (2) a G-KMV sketch, which is a KMV sketch with a global threshold value, denoted by $L_X$.

**Algorithm 1: GB-KMV Index Construction**

**Input** : $S$: dataset; $b$: space budget; 
$h$: a hash function; $\tau$: buffer size

**Output** : $L_S$, the GB-KMV index of dataset $S$

1. Compute buffer size $\tau$ based on distribution statistics of $S$ and the space budget $b$;
2. $E_H \leftarrow \text{Top } \tau \text{ most frequent elements; } E_K \leftarrow E \setminus E_H$;
3. $\tau \leftarrow$ compute the global threshold for hash values;
4. for each record $X \in S$ do
   5. $H_X \leftarrow \text{elements of } X \text{ in } E_H$;
   6. $L_X \leftarrow \text{hash values of elements } \{e\} \text{ of } X \text{ with } h(e) \leq \tau$;

Algorithm 1 illustrates the construction of GB-KMV sketch. Let the element universe be $E = \{e_1, e_2, ..., e_n\}$ and each element is associated with its frequency in dataset $S$. Line 1 calculates a buffer size $\tau$ for all records according to the skewness of record size and elements as well as the space budget $b$ in terms of elements. Details will be introduced in Section IV-C6.

We use $E_H$ to denote the set of top-$\tau$ most frequent elements (Line 1), and they will be kept in the buffer of each record. Let $E_K$ denote the remaining elements. Line 4 identifies maximal possible global threshold $\tau$ for elements in $E_K$ such that the total size of GB-KMV sketch meets the space budget $b$. For each record $X$, let $n_X$ denote the number of elements in $E_K$ with hash values less than $\tau$, we have $\sum_{X \in S} (\frac{1}{\tau} + n_X) \leq b$. Then Lines 1-4 build the buffer $H_X$ and G-KMV sketch $L_X$ for every record $X \in S$. In section 2 we will show the correctness of our sketch in Theorem 2.

**Containment Similarity Search.** Given the GB-KMV sketch of the query record $Q$ and the dataset $S$, we can conduct approximate similarity search as illustrated in Algorithm 2. Given a query $Q$ with size $q$ and the similarity threshold $t^*$, let $\theta = t^* \times g$ (Lines 1-2). With GB-KMV sketch $\{H_Q, L_Q\}$, we can calculate the containment similarity based on

$$|Q \cap X| = |H_Q \cap H_X| + \hat{D}_{\cap}^{GKMV}$$

(27)

where $\hat{D}_{\cap}^{GKMV}$ is the estimation of overlap size of $Q$ and $X$ which is calculated by Equation 25 in Section IV-A.

Note that $|H_Q \cap H_X|$ is the number of common elements of $Q$ and $X$ in $E_H$.

**Algorithm 2: Containment Similarity Search**

**Input** : $Q$, a query set
$t^*$, containment similarity threshold

**Output** : $R$: records $\{X\}$ with $C(Q, X) \geq t^*$

1. $q \leftarrow |Q|$;
2. $\theta \leftarrow t^* \times g$;
3. for each record $X \in S$ do
   4. $|Q \cap X| \leftarrow |H_Q \cap H_X| + \hat{D}_{\cap}^{GKMV}$;
   5. if $|Q \cap X| \geq \theta$ then
      6. $S_{\text{candidate}} = S_{\text{candidate}} \cup X$;

7. return $S_{\text{candidate}}$

**Implementation of Containment Similarity Search.** In our implementation, we use a bitmap with size $r$ to keep the elements in buffer where each bit is reserved for one frequent element. We can use bitwise intersection operator to efficiently compute $|H_Q \cap H_X|$ in Line 2 of Algorithm 2. Note that the estimator of overlap size by G-KMV method in Equation 25 is $\hat{D}_{\cap}^{GKMV} = \frac{b}{k} \times \frac{k - 1}{U_3}$. As to the computation of $|Q \cap X|$, we apply some transformation to $|L_Q \cap L_X| + \hat{D}_{\cap}^{GKMV} \geq \theta$. Then we get $K_\cap \geq o$ where $o = U_3 (\theta - a_1)$ and $a_1 = |H_Q \cap H_X|$. Since $K_\cap$ is the overlap size, then we make use of the PJoin* [40] to speed up the search. Note that in order to make the PJoin* which is designed for similarity join problem to be applicable to the similarity search problem, we partition the dataset $S$ by record size, and in each partition we search for the records which satisfy $K_\cap \geq o$, where overlap size is modified by the lower bound in corresponding partition.

**Remark 3.** Note that the size-aware overlap set similarity joins algorithm in [25] can not be applied to our GB-KMV method, because we need to online construct c-subset inverted list for each incoming query, which results in very inefficient performance.

**Processing Dynamic Data.** Note that our algorithm can be modified to process dynamic data. Particularly, when new records come, we compute the new global threshold $\tau$ under the fixed space budget by Line 1 of Algorithm 1 and with the new global threshold, we maintain the sketch of each record as shown in Line 1 of Algorithm 1.

C. Theoretical Analysis

In this section, we provide theoretical underpinnings of the claims and observations in this paper.

**1) Background.** We need some reasonable assumptions on the record size distribution, element frequency distribution and query work-load for a comprehensive analysis. Following are three popular assumptions widely used in the literature (e.g., [6, 25, 27, 18, 16, 44, 34]):
The element frequency in the dataset follows the power-law distribution, with \( p_1(x) = c_1 x^{-\alpha_1} \).

The record size in the dataset follows the power-law distribution, with \( p_2(x) = c_2 x^{-\alpha_2} \).

The query \( k \) from dataset \( k \) is set

\[ \text{Rank the sketch} \ L \text{ of} \ M \text{V sketch of} \ (k_1, k_2, \ldots, k_m) \text{ and set} \ X, \text{ with size-} k \text{ sketch method} \text{ is to keep the}\ \min \{k_q, k_i\} \text{ minimal hash values for each set} \ X_i, \text{ respectively.} \]

Proof: Given a query \( Q \) and dataset \( S = \{X_1, \ldots, X_m\} \), an optimal signature scheme for containment similarity search is to minimize the average variance between \( Q \) and \( X_i, i = 1, \ldots, m \). Considering the query \( Q \) and set \( X_i \) with size-\( k \), K MV sketch \( L_{X_i} \) and size-\( k \) sketch \( X \) set \( X \), respectively, the sketch size is \( k = \min \{k_q, k_i\} \) according to Equation (8). By Lemma 2, an optimal signature scheme is to maximize the total \( k \) value (say \( T \)), then we have the following optimization goal,

\[
\max T = \sum_{i=1}^{m} \min\{k_q, k_i\}
\]

s.t. \( b = \sum_{i=1}^{m} k_i, \ k_i > 0, i = 1, 2, \ldots, m \)

Rank the \( k_i \) by increasing order, w.l.o.g., let \( k_1, k_2, \ldots, k_m \) be the sketch size sequence after reordering. Let \( k_1 \) be the first in the sequence such that \( k_i = k_q \), then we have \( T = k_1 + \ldots + k_i + (m-1)k_q = b - \sum_{i=1}^{m} (k_i - k_q) \). In order to maximize \( T \), we set \( k_i = k_q, i = 1, \ldots, m \). Then by \( b = \sum_{i=1}^{m} k_i \), we have \( k_i = \ldots + k_i + k_q = (m-1)k_q \). So \( k_i \) can be computed given the index space budget \( b \), we can get that all the \( k_i \) are equal and \( k_1 = \left\lceil \frac{b}{m} \right\rceil \).

3) Correctness of GKMV Sketch: In this section, we show that the G-KMV sketch is a valid K MV sketch.

Theorem 2. Given two records \( X \) and \( Y \), let \( L_X \) and \( L_Y \) be the G-KMV sketch of \( X \) and \( Y \), respectively. Let \( k = |L_X \cap L_Y| \), then the size-\( k \) K MV synopses of \( X \cap Y \) is \( L \). Let \( v_k \) be the \( k \)-th smallest hash value in \( L \).

Proof: We show that the above \( L \) is \( L \cap L_X \cup L_Y \) is a valid K MV sketch of \( X \cup Y \). Let \( k = |L_X \cup L_Y| \) and \( v_k \) is the \( k \)-th smallest hash value in \( L_X \cup L_Y \). In order to prove that \( L_X \cup L_Y \) is valid, we show that \( v_k \) corresponds to the element with the \( k \)-th minimal hash value in \( X \cup Y \). If not, there should exist an element \( e \) such that \( h(e') < v_k, e' \in X \cup Y \) and \( h(e') \notin L_X \cup L_Y \). Note that \( v_k \leq \tau \), then \( h(e') \leq \tau \), thus \( h(e') \) is included in \( L_X \cup L_Y \), which contradicts to the above statement.

4) G-KMV: A Better K MV Sketch: In this part, we show that by imposing a global threshold to K MV sketch, we can achieve better accuracy. Let \( L_X \) and \( L_Y \) be the K MV sketch of \( X \) and \( Y \), respectively. Let \( k_1 = |L_X \cap L_Y| \) and \( k_2 = |L_X \cup L_Y| \), then the sketch size \( k \) value can be set by Equation (8). Similarly, let \( L_X \) and \( L_Y \) be the G-KMV sketch of \( X \) and \( Y \), respectively, and the sketch size \( k \) value can be set by Equation (24).

Theorem 3. With the fixed index space budget, for containment similarity search the G-KMV sketch method is better than K MV method in terms of accuracy when the power-law exponent of element frequency \( \alpha_1 \leq 3.4 \).

Proof: Let \( x = |X|, j = 1, 2, \ldots, m \) be the size and \( j \) be the signature size of record \( X_j \). The frequency of element \( e_i \) set to be \( f_i \). The index space budget \( b \).

For K MV sketch based method, by Theorem 1, the optimal signature scheme is \( k = \min (k_j, k_i) = \left\lfloor \frac{b}{m} \right\rfloor \) given the index space budget \( b \), then the average \( k \) value for all pairs of sets is

\[
\bar{k}_{K MV} = \frac{1}{m^2} \sum_{i=1}^{m} \sum_{l=1}^{m} \min(k_j, k_i) = \left\lfloor \frac{b}{m} \right\rfloor \tag{28}
\]

For G-KMV sketch based method, let \( \tau \) be the hash value threshold. The probability that hash value \( h(e_i) \) is included in signature \( L_{GK MV} \) is \( \Pr[h(e_i) \in L_{GK MV}] = \tau x_j \) where \( f_i \) is the frequency of element \( e_i \) and \( N = \sum_{i=1}^{n} f_i \) is the total number of elements. The size of \( L_{GK MV} \) can be computed by \( l_{j} = \sum_{i=1}^{n} \Pr[h(e_i) \in L_{GK MV}] = \tau x_j \) then the total index space is \( b = \sum_{j=1}^{n} l_j = \sum_{j=1}^{n} \tau x_j = \tau N \). Then the hash value threshold \( \tau = \frac{b}{N} \). Next we compute average sketch size \( k \) value of G-KMV method. The intersection size of \( L_{X_j} \) and \( L_{Y_i} \)

\[
|L_{X_j} \cap L_{Y_i}| = \sum_{i=1}^{n} \tau f_i x_j * \tau f_i x_i = \tau^2 x_j x_i f_n^2 \tag{29}
\]

where \( f_n = \frac{\sum_{i=1}^{n} f_i^2}{N} \). The \( k \) value of G-KMV method according to Equation (24) is

\[
|L_{X_j} \cup L_{Y_i}| = \tau x_j + \tau x_i - \tau^2 x_j x_i f_n^2 \tag{30}
\]

Then the average \( k \) value for all pairs of sets is

\[
\bar{k}_{GK MV} = \frac{1}{m^2} \sum_{j=1}^{m} \sum_{l=1}^{m} |L_{X_j} \cup L_{Y_i}| = \frac{2b}{m} - \frac{b^2}{m^2} f_n^2 \tag{31}
\]

Let \( \bar{k}_{GK MV} \geq \bar{k}_{K MV} \), we get \( \alpha_1 \in (0, 0.5] \cap (1 + \frac{1}{m}) - \sqrt{1 + \frac{1}{m} + \frac{1}{2}} + \sqrt{(1 + \frac{1}{m} + \frac{1}{2})^2} \). Note that for the common setting \( \frac{1}{m} \leq 1 \), we can get \( \alpha_1 \leq 3.4 \). The result makes sense since the power-law(Zipf’s law) exponent of element frequency is usually less than 3.4 for real datasets.
5) Partition of KMV Sketch Is Not Promising: In this part, we show that it is difficult to improve the performance of KMV by dividing elements to multiple groups according to their frequency and apply KMV estimation individually. W.l.o.g., we consider dividing elements into two groups.

We divide the sorted element universe $\mathcal{E}$ into two disjoint parts $\mathcal{E}_{H1}$ and $\mathcal{E}_{H2}$. Let $X$ and $Y$ be two sets from dataset $S$ with KMV sketch $\mathcal{L}_X$ and $\mathcal{L}_Y$ respectively. Let $k_x = |\mathcal{L}_X|$ and $k_y = |\mathcal{L}_Y|$. The estimator of containment similarity is $\hat{C} = \frac{D_{\alpha}}{k_x}$, where $D_{\alpha}$ is the estimator of intersection size $D_\cap$ and $q$ is the query size ($x$ or $y$).

Corresponding to $\mathcal{E}_{H1}$ and $\mathcal{E}_{H2}$, we divide $X(Y, \text{resp})$ to two parts $X_1$ and $X_2(Y_1 and Y_2, \text{resp})$. We have $X_1 \cap X_2 = \Phi$ and $Y_1 \cap Y_2 = \Phi$. Also, let $D_{\alpha} = |X \cap Y|, D_{\alpha} = |X \cup Y|$, we have $D_{\alpha} = |X_1 \cap Y_1| + |X_2 \cap Y_2|$ and $D_{\alpha} = |X_1 \cap Y_1| + |X_2 \cap Y_2|$ since $\mathcal{E}_{H1}$ and $\mathcal{E}_{H2}$ are disjoint. For simplicity, let $D_{\alpha} = |X_1 \cap Y_1|, D_{\alpha} = |X_1 \cup Y_1|, D_{\alpha} = |X_2 \cap Y_2|$ and $D_{\alpha} = |X_2 \cup Y_2|$. For $X_1, X_2, Y_1$ and $Y_2$, the KMV sketches are $\mathcal{L}_X, \mathcal{L}_Y, \mathcal{L}_Y, \mathcal{L}_Y$ with size $k_x, k_x, k_y, k_y$ and $k_y$, respectively. Based on this, we give another estimator as $C' = \frac{D_{\alpha} + D_{\alpha}}{q}$, where $D_{\alpha}$ is the estimator of intersection size $D_{\alpha}$, resp). Next, we compare the variance of $C$ and $C'$.

**Theorem 4.** After dividing the element universe into two groups and applying KMV sketch in each group, with the same index space budget, the variance of $C'$ is larger than that of $C$.

**Proof:** Recall the KMV sketch, we have $E(C') = E(D_{\alpha}) + E(D_{\alpha}) = D_{\alpha} + D_{\alpha} = D_{\alpha}$. Because of the two disjoint element groups, $D_{\alpha}$ and $D_{\alpha}$ are independent. Thus the variance $Var(C') = Var(D_{\alpha}) = \frac{Var(D_{\alpha})}{q^2}$. Next, we will show

$$Var(D_{\alpha}) + Var(D_{\alpha}) \geq Var(C')$$

Consider the KMV sketch for set $X$ and $Y$, the sketch size according to Equation 8 is $k = \min\{k_x, k_y\}$. Similarly, for $X_1$ and $Y_1$, we have the sketch size $k_1 = \min\{k_{x_1}, k_{y_1}\}$; for $X_2$ and $Y_2$, we have the sketch size $k_2 = \min\{k_{x_2}, k_{y_2}\}$. Since the index is fixed, we have $k_x = k_{x_1} + k_{y_2}$ and $k_y = k_{y_1} + k_{y_2}$. Then, $k_1 + k_2 = \min\{k_{x_1}, k_{y_1}\} + \min\{k_{x_2}, k_{y_2}\} \leq \min\{k_{x_1}, k_{y_1}\} = k$.

Let $\Delta = Var(D_{\alpha}) + Var(D_{\alpha}) - Var(C')$, after some calculation, we have $\Delta = \frac{k_1}{k_1} + \frac{k_2}{k_2} - \frac{k_1}{k_2} + \frac{k_2}{k_1} + \frac{k_1 k_2}{k_1}$. Next we show that $\frac{k_1}{k_2} + \frac{k_2}{k_1} \geq \frac{k_1 + k_2}{k_1 k_2} \geq 0$.

Let $k_1 = \frac{1}{\alpha}k$ and $k_2 = \frac{1}{\beta}k$, where $\alpha + 1 = 1$ and $\alpha, \beta > 1$. Then we have $\frac{k_1}{k_2} + \frac{k_2}{k_1} = \alpha \frac{k_1}{k_2} + \beta \frac{k_2}{k_1} = (\alpha - 1)k_1^2 + (\beta - 1)k_2^2 + 2k_1 k_2 \geq 2\sqrt{\alpha - 1 + \beta - 1} - 2k_1 k_2 \geq 2(\sqrt{\alpha + 1, \beta + 1} - 1 - 1)D_{\alpha} D_{\alpha}$. Since $\alpha + 1, \beta - 1 = 1$, we get $\sqrt{\alpha - 1 + \beta - 1} - 1 - 1 = \sqrt{\alpha + 1, \beta + 1} - 1 \geq 0$, thus $\frac{k_1}{k_2} + \frac{k_2}{k_1} \geq 0$.

Let $\Delta_1 = \frac{k_1 k_2}{k_1} + \frac{k_1 k_2}{k_2}$, after some computation, we have $\Delta_1 = \frac{k_1 k_2}{k_1 k_2}$. As for the numerator(upper) of $\Delta_1$, consider the two parts after dividing the element universe, if the union size in one part, say $D_{\alpha_2}$, is larger, meanwhile the corresponding intersection size $D_{\alpha_2}$ is larger, we have $\Delta_1 \geq 0$. This case can be realized since one of the two groups divided from element universe is made of high-frequency elements, which will result in large intersection size and large union size under the proper choice of $k, k_1, k_2$.

6) Optimal Buffer Size $r$: In this part, we show how to find optimal buffer size $r$ by analyzing the variance for GB-KMV method. Given the space budget $b$, we first show that the variance for GB-KMV sketch is a function of $f(r, \alpha_1, \alpha_2, b)$ and then we give a method to appropriately choose $r$. Below are some notations first.

Given two sets $X$ and $Y$ with G-KMV sketch $\mathcal{L}_X$ and $\mathcal{L}_Y$, respectively, the containment similarity of $Q$ in $X$ is computed by Equation 26 as $G_{GB-KMV} = \frac{D_{GB-KMV} |}{q}$, where $D_{GB-KMV} = \frac{q}{k + \frac{k_1 - 1}{(\alpha)}}$ is the overlap size.

As for the GB-KMV method of set $X$ and $Y$ with sketch $\mathcal{H}_X \cup \mathcal{L}_X$ and $\mathcal{H}_Y \cup \mathcal{L}_Y$, respectively, the containment similarity of $Q$ in $X$ is computed by Equation 27 as $G_{GB-KMV} = \frac{|Q \cap \mathcal{H}_X \cup \mathcal{L}_X|}{q}$, where $|Q \cap \mathcal{H}_X|$ is the number of common elements in $\mathcal{E}_H$ part. It is easy to verify that $G_{GB-KMV}$ is an unbiased estimator. Also, the variance of GB-KMV method estimator is $\text{Var}[^{\hat{C}_{GB-KMV}}] = \frac{\text{Var}[D_{GB-KMV}]}{q^2}$, where $\text{Var}[D_{GB-KMV}]$ corresponds to the variance of the G-KMV sketch in the GB-KMV sketch.

Next, with the same space budget $b$, we compute the average variance of GB-KMV method.

Consider the GB-KMV index construction which is introduced in Section 8.3 by Algorithm 1. Let $N$ be the total number of elements and $b$ the space budget in terms of elements for index construction. Assume that we keep $r$ high-frequency elements by bitmap in the buffer, which have $N_1 = \sum_{j=1} \sum_h |\mathcal{H}_X| = \sum_{i=1} \hat{f}_i$ elements and occupy $T_1 = m*r/32$ index space. Then the total number of elements left for G-KMV sketch is $N_2 = N - N_1$ and the index space for G-KMV sketch is $T_2 = b - T_1$.

Given two sets $X_j$ and $X_l$, the variance of overlap size estimator in Equation 11 is as follows

$$\text{Var}[^{\hat{D}_j}] = \frac{D_{\alpha}(kD_{\beta} - k^2 - D_{\beta} + k + D_{\beta})}{k(k - 2)}$$

where $D_{\alpha} = |X_j \cup X_l|, D_{\beta} = |X_j \cap X_l|$ and $k$ is the sketch size. Since the variance is concerned with the union size $D_{\alpha}$, the intersection size $D_{\beta}$ and the signature size $k$, first calculate these three formulas, then compute the variance.

Consider the two sets $X_j, X_l$ from dataset $S$ with G-KMV sketch $\mathcal{H}_X \cup \mathcal{L}_X$, and $\mathcal{H}_Y \cup \mathcal{L}_Y$, respectively. The element $e_i$ is associated with frequency $f_i$, and the probability of element $e_i$ appearing in record $x_j$ is $Pr[h(e_i) \in \mathcal{L}_X] = \frac{f_i}{\hat{f}_j}$. Given a hash value threshold $r$, the G-KMV signature size of set $X_j$ is computed as $k_j = \tau(x_j - |\mathcal{H}_X|)$. The total index space in G-KMV sketch is $\sum_{j=1}^{N_1} k_j = T_2 - b - T_1 = b - \frac{m*r}{32} + m$, then we get $\tau = \frac{r - m*r/32 + m}{N_2}$.

Similar to Equation 29, 30, the sketch size $k$ value for G-KMV sketch is $\tau(x_j + x_l) - \tau \times x_j x_l (f_{x}^2 - f_{x}^2)$ where
of $r$ larger than four. According to Abel’s impossibility theorem \cite{29}, there is no algebraic solution, thus we try to give the numerical solution.

Recall that we use bitmap to keep the $r$ high-frequency elements, given the space budget $b$, the element frequency and record size distribution with power-law exponent $\alpha_1$ and $\alpha_2$ respectively, the optimization goal max$_r V_{GB-KMV}$ can be considered as a function max$_r f(r; b, \alpha_1, \alpha_2)$. Given a dataset $\mathcal{S}$ and the space budget $b$, we can get the power-law exponent $\alpha_1, \alpha_2$. Then we assign $8, 16, 24, \ldots$ to $r$ and calculate the $f(r, b, \alpha_1, \alpha_2)$. In this way, we can give a good guide to the choice of $r$.

7) GB-KMV Sketch provides Better Accuracy than LSH-E Method: In Section \[11-13\] we have shown that the variance of LSH-E estimator(\cite{21}) is larger than that of MinHash LSH estimator(\cite{19}). Note that G-KMV sketch is a special case of GB-KMV sketch when the buffer size $r = 0$. By choosing an optimal buffer size $r$ in \[IV-C\] it can guarantee that the performance of GB-KMV is not worse than G-KMV. Below, we show that G-KMV outperforms MinHash LSH in terms of estimate accuracy.

**Theorem 5.** The variance of G-KMV sketch is smaller than that of minHash LSH method given the same sketch size.

**Proof:** Suppose that the minHash LSH method uses $k'$ hash functions to the dataset, then the total sketch size is $T = nk'$. Let $\tau$ be the global threshold of G-KMV method, we have $\tau = \frac{mk'}{N}$ where $N$ is the total number of elements in the dataset.

We first consider the G-KMV sketch method. Similar to Equation \[29, 30\] the intersection size of $X_j$ and $X_i$ is $D_{ij} = \sum_{l=1}^{n} \frac{x_l}{1-\tau}$ and the union size is $D_i = x_i + x_j - x_i x_j \sum_{l=1}^{n} \frac{x_l}{1-\tau}$. Then by Equation \[11\] the variance of the G-KMV method to estimate the containment similarity of $X_j$ in $X_i$ can be rewritten as

$$V_{G-KMV} = \frac{(x_j + x_i)x_j x_i}{k x_j^2} F_1 + \frac{(x_j x_i)^2}{k x_j^2} F_2 + \frac{x_j x_i}{x_j^2} F_4$$

where $F_1 = f_{n^2}$, $F_2 = -f_{n^2}$, $F_4 = -f_{n^2}$ and $k = \frac{1}{N}(x_j + x_i) - \frac{1}{(x_j + x_i)^2} x_j x_i f_{n^2}$. Let $\Delta Var = Var[\hat{C}_{GB-KMV}] - Var[\hat{C}_{G-KMV}]$, then for all pairs of $X_j$ and $X_i$, the average of $\Delta Var$ is $V_\Delta = \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{m} \Delta Var$. Moreover, we can rewrite $V_\Delta$ as $V_\Delta = L_1 F_1 + L_2 F_2 + L_3 F_3$.

Eventually, in order to find the optimal $r$, i.e., the number of high-frequency elements in GB-KMV method, we give the optimization goal as max$_r V_{GB-KMV} = f(r, \alpha_1, \alpha_2, b)$, s.t. $V_\Delta < 0$.

In order to compute the above optimization problem, we try to extract the roots of the first derivative function of $f(r, \alpha_1, \alpha_2, b)$ with respect to $r$. However, the derivative function is a polynomial function with degree...
Note that \(x_d(x_1, \text{resp.})\) is the largest (smallest, resp.) set size and \(d\) is the distinct number of elements.

Next we take into account the minHash LSH method.

Given two sets \(X_j\) and \(X_i\), by Equation \(19\) the variance of minHash LSH method to estimate the containment similarity of \(X_j\) to \(X_i\) is
\[
V_{minH} = \frac{1}{k^2} \left[ \sigma_1^2 f_{n_2}^2 + \sigma_2(f_{n_2})^2 + 3 \sigma(f_{n_2})^3 + 4 \sigma(f_{n_2})^4 \right]
\]
where \(\sigma_1 = x_i + x_j^2, \sigma_2 = -4x_i^2, \sigma_3 = 5x_i^2 + x_j, \sigma_4 = -2x_i x_j x_j^2 \). Then the average variance
\[
V_2 = \frac{1}{m^2} \sum_{j=1}^{m} \sum_{i=1}^{m} V_{minH}(i,j)
\]
and
\[
A_4 = -2 \left( \frac{(a_2 - 1)^2 (x_i^3 - 1) - 1}{2 + x_i^2 x_i^2 - x_i^2} \right)^2 - 2 \left( \frac{(a_2 - 1)^2 (x_i^3 - 1) - 1}{2 + x_i^2 x_i^2 - x_i^2} \right) + 3 \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)
\]
Similarly, we can get that \(V'_1 < V'_2\).

Remark 4. We have illustrated that the variance of GB-KMV is smaller than that of LSH-E. Then by Chebyshev’s inequality, i.e., \(\Pr(|X-\mu| > \sigma) \leq \frac{1}{\sigma^2}\) where \(\mu\) is the expectation, \(\sigma\) is the standard deviation and \(\epsilon > 1\) is a constant, we consider the probability that values lie outside the interval \([\mu - \delta, \mu + \delta]\), that is, values deviating from the expectation. By Theorem 5, we get that the standard deviation \(\delta_1\) of GB-KMV is smaller than \(\delta_2\) of LSH-E, then with the same interval \([\mu - \delta, \mu + \delta]\), the constant \(\epsilon_1\) for GB-KMV is larger than \(\epsilon_2\) for LSH-E, thus the probability that values lie outside the interval for GB-KMV is smaller than that for LSH-E, which means that the result of GB-KMV is more concentrated around the expected value than that of LSH-E.

V. PERFORMANCE STUDIES

In this section, we empirically evaluate the performance of our proposed GB-KMV method and compare LSH Ensemble [44] as baseline. We also compare our approximate GB-KMV method with the exact containment similarity search method. All experiments are conducted on PCs with Intel Xeon 2 \(\times\) 2.3GHz CPU and 128GB RAM running Debian Linux, and the source code of GB-KMV is made available [1].

A. Experimental Setup

![Fig. 5. Effect of Buffer Size](image)

**Approximate Algorithms.** In the experiments, the approximate algorithms evaluated are as follows.

- **GB-KMV.** Our approach proposed in Section IV-B
- **LSH-E.** The state-of-the-art approximante containment similarity search method proposed in [44].

The above two algorithms are implemented in Go programming language. We get the source code of LSH-E from [44]. For LSH-E, we follow the parameter setting from [44].

**Exact Algorithms.** To further evaluate the proposed methods, we also compare our approximate method GB-KMV with the following two exact containment similarity search methods.

- **PPjoin.** We extend the prefix-filtering based method from [40] to tackle the containment similarity search problem.
- **FrequentSet.** The state-of-the-art exact containment similarity search method proposed in [5].
### TABLE II. CHARACTERISTICS OF DATASETS

| Dataset     | Abbrev | Type      | Record | #Records | AvgLength | #DistinctEle | α₁-eleFreq | α₂-recSize |
|-------------|--------|-----------|--------|----------|-----------|--------------|-------------|------------|
| Netflix [4] | NETFLIX | Rating    | Movie  | 480,189  | 209.25    | 11,770       | 1.14        | 4.95       |
| Delicious [4] | DELIC | Folksonomy | User   | 833,081  | 98.42     | 4,512,099    | 1.14        | 3.05       |
| CaOpenData [4] | COD | Folksonomy | User   | 65,553   | 6284      | 111,011,807  | 1.09        | 1.81       |
| Enron [13]  | ENRON  | Text      | Email  | 517,431  | 133.37    | 1,113,219    | 1.16        | 3.10       |
| Reuters [4]  | REUTERS | Folksonomy | User   | 833,081  | 77.6      | 285,906      | 1.32        | 6.01       |
| Webspam [13] | WEBSFAM | Text      | Text   | 350,000  | 3728      | 16,669,143   | 1.33        | 9.34       |
| WDC Web Table [44] | WDC | Text      | Text   | 262,893,406 | 29.2     | 111,562,175  | 1.08        | 2.4        |

#### Remark 5.
A novel size-aware overlap set similarity join algorithm has been recently proposed in [13]. Although the containment similarity search relies on the set overlap, their technique cannot be trivially applied because we need to construct c-subset inverted lists for each possible query size. In particular, in the size-aware overlap set similarity join algorithm, it is required to build the c-subset inverted list for the given overlap threshold c. In our GB-KMV method, the threshold c corresponds to |Q| * t*, where |Q| is the query size and t* is the similarity threshold, thus with different query size |Q|, we need to build different |Q| * t*-subset inverted lists, which is very inefficient.

#### Datasets.
We deployed 7 real-life datasets with different data properties. Note that the records with size less than 10 are discarded from dataset. We also remove the stop words (e.g., "the") from the dataset. Table I shows the detailed characteristics of the 7 datasets. Each dataset is illustrated with the dataset type, the representations of record, the number of records in the dataset, the average record length, and the number of distinct elements in the dataset. We also report the power-law exponent α₁ and α₂ (skewness) of the record size and element frequency of the dataset respectively. Note that we make use of the framework in [18] to quantify the power-law exponent. The dataset Canadian Open Data appears in the state-of-the-art algorithm LSH-E [44].

#### Settings.
We borrow the idea from the evaluation of LSH-E in [44] to use \( F_0 \) score (\( α=1, 0.5 \)) to evaluate the accuracy of the containment similarity search. Given a query \( Q \) randomly selected from the dataset \( S \) and a containment similarity threshold \( t^* \), we define \( T = \{ (X, t): t \in S \} \) as the ground truth set and \( A \) as the collection of records returned by some search algorithms. The precision and recall to evaluate the experiment accuracy are Precision = \( |T \cap A| / |A| \) and Recall = \( |T \cap A| / |T| \) respectively. The \( F_0 \) score is defined as follows.

\[
F_0 = \frac{1 + \alpha^2 \cdot \text{Precision} \cdot \text{Recall}}{\alpha^2 \cdot \text{Precision} + \text{Recall}}
\]

Note that we use \( F_0, 0.5 \) score because LSH-E favours recall in [44]. We use the datasets from Table I to evaluate the performance of our algorithm, and we randomly choose 200 queries from the dataset.

As to the default values, the similarity threshold is set as \( t^* = 0.5 \). In the experiments, we use the ratio of space budget to the total dataset size to measure the space used. For our GB-KMV method, it is set to 10%. For LSH-E method, we use the same default values in [44], where the signature size of each record is 256 and the number of partition is 32. By varying the number of hash functions, we change the space used in LSH-E.

#### B. Performance Tuning

As shown in Section IV-C6 we can use the variance estimation function to identify a good buffer size \( r \) for GB-KMV method based on the skewness of record size and element frequency, as well as the space budget. In Fig. 5 we use NETFLIX and ENRON to evaluate the goodness of the function by comparing the trend of the variance and the estimation accuracy. By varying the buffer size \( r \), Fig. 5 reports the estimated variance (right side y axis) based on the variance function in Section IV-C6 as well as the \( F_1 \) score (left side y axis) of the corresponding GB-KMV sketch with buffer size \( r \). Fig. 5(a) shows that the best buffer size for variance estimation (prefer small value) is around 400, while the GB-KMV method achieves the best \( F_1 \) score (prefer large value) with buffer size around 380. They respectively become 220 and 230 in Fig. 5(b). This suggests that our variance estimation function is quite reliable to identify a good buffer size. In the following experiments, GB-KMV method will use buffer size suggested by this system, instead of manually tuning.
We also compare the performance of KMV, G-KMV, and GB-KMV methods in Fig. 5 to evaluate the effectiveness of using global threshold and the buffer on 7 datasets. It is shown that the use of new KMV estimator with global threshold (i.e., GB-KMV) can significantly improve the search accuracy. By using a buffer whose size is suggested by the system, we can further enhance the performance under the same space budget. In the following experiments, we use GB-KMV for the performance comparison with the state-of-the-art technique LSH-E.

### C. Space v.s. Accuracy

An important measurement for sketch technique is the space-accuracy trade-off between the space and accuracy. We evaluate the space-accuracy trade-offs of GB-KMV method and LSH-E method in Figs. 7, 8, 9, and 10. By varying the space usage on five datasets NETFLIX, DELIC, COD, ENRON, REUTERS, WEbspam and WDC. We use $F_1$ score, $F_{0.5}$ score, precision and recall to measure the accuracy. By changing the number of hash functions, we tune the space used in LSH-E. It is reported that our GB-KMV can beat the LSH-E in terms of space-accuracy trade-off with a big margin under all settings.

We also plot the distribution of accuracy (i.e., min, max and average value) to compare our GB-KMV method with LSH-E in Fig. 14.

Meanwhile, by changing the similarity threshold, $F_1$ score is reported in Fig. 15 on dataset NETFLIX and COD. We can see that with various similarity thresholds, our GB-KMV always outperforms LSH-E.

We also evaluate the space-accuracy trade-offs on synthetic datasets with 100K records in Fig. 16 where the record size and the element frequency follow the zipf distribution. We can see that on datasets with different record size and element frequency skewness, GB-KMV consistently outperforms LSH-E in terms of space-accuracy trade-off.

### D. Time v.s. Accuracy

Another important measurement for the sketch technique is the time-accuracy trade-off between time and accuracy. Hopefully, the sketch should be able to quickly complete the search with a good accuracy. We tune the index size of GB-KMV and LSH-E on four datasets COD, NETFLIX, DELIC and ENRON where the accuracy is measured by $F_1$ score. It is shown that with the similar accuracy ($F_1$ score), GB-KMV is significantly faster than LSH-E. For datasets COD, DELIC and ENRON, GB-KMV can be 100 times faster than LSH-E with the same $F_1$ score. It is observed that the accuracy ($F_1$ score) improvement of LSH-E algorithm is very slow compared with GB-KMV method. This is because the LSH-E method favours recall and the precision performance is quite poor even for a large number of hash functions, resulting in a poor $F_1$ score which considers both precision and recall.
Comparison with Exact Algorithms. We also compare the LSH-E much less time than GB-KMV of E. Sketch Construction Time

Accuracy versus Space on WDC

| Dataset   | GB-KMV | LSH-E |
|-----------|--------|-------|
| NETFLIX   | 10     | 117   |
| DELIC     | 10     | 211   |
| COD       | 10     | 4     |
| ENRON     | 10     | 185   |
| REUTERS   | 10     | 129   |
| WEBSPAM   | 10     | 7     |
| WDC       | 10     | 109   |

**TABLE III.** The space usage(%) 

E. Sketch Construction Time

In this part, we compare the sketch construction time of GB-KMV and LSH-E on different datasets under default settings. As expected, GB-KMV uses much less sketch construction time than that of LSH-E since GB-KMV sketch need only one hash function, while LSH-E needs multiple for a decent accuracy. Note that, for the internet scale dataset WDC, the index construction time for GB-KMV is around 10 minutes, while for LSH-E it is above 60 minutes. We also give the space usage of the two methods on each dataset in Table III. The space usage of GB-KMV is 10% as mentioned in Settings. For LSH-E in some dataset, the space is over 100% because there are many records with size less than the number of hash functions 256.

F. Supplementary Experiment

Evaluation on Uniform Distribution. In Theorem we have theoretically shown that when the dataset follows uniform distribution (i.e., \( \alpha_1 = 0 \) and \( \alpha_2 = 0 \)), our GB-KMV method can outperform the LSH-E method. In this part, we experimentally illustrate the performance on dataset with uniform distribution. We generate 100k records where the record size is uniformly distributed between 10 and 5000, and each element is randomly chosen from 100,000 distinct elements. Fig. [9] (a) illustrates the time-accuracy trade-off of GB-KMV and LSH-E on the synthetic dataset with 100K records. It is reported that, to achieve the same accuracy (F1 score), GB-KMV consumes much less time than LSH-E.

Comparison with Exact Algorithms. We also compare the running time of our proposed method GB-KMV with two exact containment similarity search methods PPJoin* [40] and FreqSet [5]. Experiments are conducted on the dataset WebSpam, which consists of 350,000 records and has the average length around 3,700. We partition the data into 5 groups based on their record size with boundaries increasing from 1,000 to 5,000. As expected, Fig. [19] (b) shows that the running time of our approximate algorithm is not sensitive to the growth of the record size because a fixed number of samples are used for a given budget. GB-KMV outperforms two exact algorithm by a big margin, especially when the record size is large, with a decent accuracy (i.e., with F1 score and recall always larger than 0.8 and 0.9 under all settings).

G. Discussion Summary

In the accuracy comparison between GB-KMV and LSH-E, it is remarkable to see that the accuracy (i.e., F1 score) is very low on some datasets. We give some discussions as follows.

First we should point out that in [44], the accuracy of LSH-E is only evaluated on only one dataset COD, in which both our GB-KMV method and LSH-E can achieve decent accuracy performance with F1 score above 0.5.

As mentioned in [44-A] the LSH-E method first transforms the containment similarity to Jaccard similarity, then in order to make use of the efficient index techniques, LSH-E partitions the dataset and uses the upper bound to approximate the record size in each partition, which can favour recall but result in extra false positives as analysed in section [44-B]. However, the LSH-E method does not provide a partition scheme associated with different data distribution, and the algorithm setting (e.g., 256 hash functions and 32 partitions) can not perform well in some dataset.

VI. RELATED WORK

In this Section, we review two closely related categories of work on set containment similarity search.
on tree index structure. In [25], Deng et al. in [24] develop a partition-filter based method which uses partition and enumeration techniques to search for exact similar records. Deng et al. in [24] propose an efficient candidate verification algorithm which significantly improves the efficiency compared with the other prefix filter algorithms. Wang et al. [36] consider the relations among records in query processing to improve the performance. Deng et al. in [25] present an efficient similarity search method where each object is a collection of sets. For partition-based method, in [24], Arasu et al. devise a two-level algorithm which uses partition and enumeration techniques to search for exact similar records. Deng et al. in [24] develop a partition-based method which can effectively prune the candidate size at the cost higher filtering cost. In [43], Zhang et al. propose an efficient framework for exact set similarity search based on tree index structure. In [25], Deng et al. present a size-aware algorithm which divides all the sets into small and large ones by size and processes them separately. Regarding exact containment similarity search, Agrawal et al. in [5] build the inverted lists on the token-sets and considered the string transformation.

Approximate Set Similarity Queries. The approximate set similarity queries mostly adopt the Locality Sensitive Hashing (LSH) [28] techniques. For Jaccard similarity, MinHash [14] is used for approximate similarity search. Asymmetric minwise hashing is a technique for approximate containment similarity search [5]. This method makes use of vector transformation by padding some values into sets, which makes all sets in the index have same cardinality as the largest set. After the transformation, the near neighbours with respect to Jaccard similarity of the transformed sets are the same as near neighbours in containment similarity of the original sets. Thus, MinHash LSH can be used to index the transformed sets, such that the sets with larger containment similarity scores can be returned with higher probability. In [43], they show that asymmetric minwise hashing is advantageous in containment similarity search over datasets such as news articles and emails, while Zhu et. al in [44] finds that for datasets which are very skewed in set size distribution, asymmetric minwise hashing will reduce the recall.

The KMV sketch technique has been widely used to esti-
In [21], Dahlgaard et al. develop a new sketch method which has the alignment property and same concentration bounds as MinHash.

VII. CONCLUSION

In this paper, we study the problem of approximate containment similarity search. The existing solutions to this problem are based on the MinHash LSH technique. We develop an augmented KMV sketch technique, namely GB-KMV, which is data-dependent and can effectively exploit the distributions of record size and element frequency. We provide thorough theoretical analysis to justify the design of GB-KMV, and show that the proposed method can outperform the state-of-the-art technique in terms of space-accuracy trade-off. Extensive experiments on real-life set-valued datasets from a variety of applications demonstrate the superior performance of GB-KMV method compared with the state-of-the-art technique.

RECOMMENDATIONS

In this paper, we study the problem of approximate containment similarity search. The existing solutions to this problem are based on the MinHash LSH technique. We develop an augmented KMV sketch technique, namely GB-KMV, which is data-dependent and can effectively exploit the distributions of record size and element frequency. We provide thorough theoretical analysis to justify the design of GB-KMV, and show that the proposed method can outperform the state-of-the-art technique in terms of space-accuracy trade-off. Extensive experiments on real-life set-valued datasets from a variety of applications demonstrate the superior performance of GB-KMV method compared with the state-of-the-art technique.

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Fig. 17. Time versus Accuracy

Fig. 18. Sketch Construction Time

Fig. 19. Supplementary experiments
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