A Family of Energy Stable, Skew-Symmetric Finite Difference Schemes on Collocated Grids
A Simple Way to Avoid Odd–Even Decoupling

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Abstract A simple scheme for incompressible, constant density flows is presented, which avoids odd-even decoupling for the Laplacian on collocated grids. Energy stability is implied by guaranteeing strict energy conservation. Momentum is conserved. Arbitrary order in space and time can easily be obtained. These conservation properties also hold on transformed grids.

Keywords Incompressible flows · Skew-symmetric schemes · Energy-stable schemes · Collocated grids · High order

Mathematics Subject Classification 76D05 · 35L65

1 Introduction

The odd-even decoupling is one of the central issues when simulating incompressible flows. It refers to the fact that the Laplace operator calculated from a discrete gradient and a discrete divergence operator decomposes into two Laplace operators living on super-grids, when central derivative operators are used. If, for example (in one dimension), the gradient at grid position \( i \) is defined by \((Gp)_i = (p_{i+1} - p_{i-1})/(2\Delta x)\) and the divergence by \((Du)_i = (u_{i+1} - u_{i-1})/(2\Delta x)\), the implied Laplace operator is \((DGp)_i = (p_{i-2} - 2p_i + p_{i+2})/(2\Delta x)^2\). Thus, only every second point on the grid is connected. This decoupling leads to severe convergence problems.

This problem can be avoided simply by using staggered grids, where the velocity components and the pressure are given at different points. This allows using central differences without further modifications. Central differences have a purely imaginary spectrum and are therefore a good base to build energy conserving schemes. Indeed the majority of incompress-
ible, energy conserving schemes are defined on staggered grids, like the classical scheme of Harlow and Welch [10], which is energy conserving [15,28] when using an implicit midpoint rule for time stepping.

The crucial term for discrete energy conservation is the nonlinear momentum transport. In fact, this nonlinearity in general breaks the conservation of kinetic energy for high order derivatives, as the product rule is not fulfilled in the discrete setting. This can be circumvented by the use of a skew-symmetric discretization, introduced by [6] and [24] for compressible flows. A proper skew-symmetric transport term conserves the kinetic energy. The concept was used for incompressible flows in [15], where also a fourth order scheme was presented, though strict conservation is not preserved on general grids. A fourth order skew symmetric scheme on arbitrary grids is achieved by [28]. A scheme in non-primitive variables of second order accuracy is [31]. The kinetic energy conservation is a secondary conservation, in a recent review [16] more related aspects can be found. All the mentioned schemes for incompressible flows use staggered grids.

Often collocated schemes are preferred, due to the simpler treatment of curvilinear coordinates and the fact that all variables are defined on prescribed boundaries. Collocated schemes have to avoid the previously described odd-even decoupling in some way. Frequently the Rhie-Chow interpolation [21] is used, where a small regularization term is added, which suppresses the decoupling. Other methods are for example the use of non-symmetric stencils as in [5] or operator splitting [4]. See [30] for a discussion of the different methods.

Here, a different approach is presented. It is based on the observation that asymmetric spatial derivatives do not necessarily imply that the resulting scheme violates the energy conservation. The discretization is obtained by using the skew-symmetric form of the transport term [15,28]. The resulting approach is elegant due to its simplicity. The scheme presented here is constructed analogous to a scheme for compressible flows by the author [19]. Since for compressible flows an equation of motion for the pressure can be formulated, odd-even decoupling is a minor issue. The new scheme has some resemblance with the one presented in [5], as also for non-symmetric derivatives a (decreasing) energy estimate can be derived in a similar manner. However, the energy estimate is done by tedious index calculations and is restricted to first order.

An alternative scheme which similarly builds on the skew-symmetric form, but instead uses a special interpolation procedure to avoid decoupling on collocated grids was recently proposed by [26]. This also works on unstructured grids.

The time stepping has to be adopted as well to obtain a conserved kinetic in the fully discrete setting. Time stepping schemes with second order accuracy were presented by [9] and [28]. Higher order was given for incompressible flow by [22] and for compressible flows by [2]. Both works build on Gauss collocation Runge–Kutta methods, and the compressible time stepping reduces to the incompressible when the density is set constant. If a standard time stepping is used a strictly reduced energy is expected in the stability region of the time stepper.

Most interesting for the focus of the present paper is [7], which is devoted to kinetic energy conservation of collocated schemes of high order and/or curvilinear grids. While the analysis and the numerical examples shed light on the typical problems, the conclusion seems to be too far reaching, as it seemingly rules out the possibility of a high order scheme on collocated grids which respects the solenoidal condition and conserves momentum and energy. This is exactly what is presented in the following.

A first version of the following scheme was presented by the author in [17]. Here additionally high order time integration schemes, conservation on arbitrary grids, treatment of boundary conditions and simulations on transformed and non-periodic grids are included.
1.1 Skew-Symmetric Schemes

Skew-symmetric schemes are schemes which conserve the kinetic energy by design. To illustrate the basic idea, consider an equation of motion for the vector quantity $\mathbf{u}$

$$\partial_t \mathbf{u} = A \mathbf{u},$$

where $A$ is a skew-symmetric matrix, meaning $A^T = -A$. The change of the norm or the kinetic energy can be derived as

$$u^T \partial_t u = \frac{1}{2} \partial_t u^T u$$

$$= -u^T Au = 0.$$  \hfill (1)

The last steps follows from $z = u^T Au = z^T = (u^T A^T u) = -u^T Au = -z$, so that $z = 0$. In other words quadratic forms of skew-symmetric matrices always vanish, as the terms $A_{i,j} u_i u_j$ and $A_{j,i} u_j u_i = -A_{i,j} u_j u_i$ cancel pairwise.

To utilize this concept for the numerical evaluation of the Navier–Stokes equation, the momentum equation is discretized in such a way that the nonlinear transport term is skew-symmetric, directly implying the conservation of the kinetic energy by this term. Numerical damping and the artificial change of kinetic energy are basically different perspectives on the same phenomenon. Thus, a conserved kinetic energy can be seen as zero numerical damping, or as an energy stable scheme. The physical damping of course reduces the kinetic energy. In reality it is converted to heat or to internal energy, which is usually not accounted for in the description of incompressible flows, because the change in temperature is typically very small.

In the following it will be shown that a skew-symmetric discretization can easily be constructed even with asymmetrical, i.e. not skew-symmetric derivatives. Thus, it is possible to construct schemes without numerical damping with asymmetrical stencils. The asymmetrical form in turn allows to avoid the odd-even decoupling on collocated grids. The main discussion in this work is restricted to the periodic case. Boundary conditions are discussed in sec. 4.

2 Derivation of the Scheme

The starting point is the standard Navier–Stokes or momentum equation with a constant density $\rho \equiv 1$

$$\partial_t u_\alpha + \partial_{x_\beta} u_\beta u_\alpha + \partial_{x_\alpha} p = \nu \Delta u_\alpha.$$  \hfill (2)

Here, $u_\alpha$ are the components of the velocity field, $p$ is the pressure and $\nu$ the kinematic viscosity. $\Delta$ denotes the Laplace operator. Greek letters mark the space directions $\alpha, \beta = 1, 2, 3$. The summing convention is assumed. The Navier–Stokes equation is complemented by the continuity equation or solenoidal condition

$$\partial_{x_\alpha} u_\alpha = 0,$$  \hfill (3)

to describe incompressible flow. To satisfy this algebraic condition the pressure has to be determined accordingly. The defining pressure Poisson equation is derived below by applying the solenoidal condition (3) on the momentum Eq. (2), see Eq. 14.
The Navier–Stokes equation can be rewritten with the help of the continuity equation as

$$\partial_t u_\alpha + u_\beta \partial_{x_\beta} u_\alpha + \partial_{x_\alpha} p = \nu \Delta u_\alpha.$$  

(4)

Summing eqs. (2) and (4) leads to the skew-symmetric form of the Navier–Stokes equation:

$$\partial_t u_\alpha + \frac{1}{2} \left( \partial_{x_\beta} u_\beta \cdot + u_\beta \partial_{x_\beta} \right) u_\alpha + \partial_{x_\alpha} p = \nu \Delta u_\alpha$$  

(5)

The derivatives in the parenthesis are understood to act on the velocity to the right of it, which is denoted by a dot.

The discretization can be done by replacing the functions $u_\alpha(x, y, z)$ and $p(x, y, z)$ by discrete values at $(x_i, y_j, z_k) = (i \cdot \Delta x, j \cdot \Delta y, k \cdot \Delta z)$, where equidistant spacing is assumed for now; and further by replacing the derivatives in (3) and (5) by derivative matrices. Later two different derivatives have to be utilized. In anticipation of these findings we insert the different derivatives in accordance with the use of the gradient ($G_\alpha$) or divergence($D_\alpha$). The semi-discrete equations are

$$D_\alpha u_\alpha = 0,$$  

(6)

$$\partial_t u_\alpha + \frac{1}{2} (D_\beta U_\beta + U_\beta G_\beta) u_\alpha + G_\alpha p = \nu L u_\alpha.$$  

(7)

The derivatives are assumed to have stencils only in the discretized direction, as usual. They can be written as standard one-dimensional matrices with the help of the Kronecker product. In two dimensions the divergence would be $D_x \equiv D_1 = I^2 \otimes d_1$ and $D_y \equiv D_2 = d_2 \otimes I^1$. Here, $d_\alpha$ is a one dimensional discretization of the divergence derivative and $I^\alpha$ is the unity matrix in the given direction. All fields are sorted accordingly in one dimensional vectors. Capital letters $U_\alpha$ mark pointwise multiplication, which can be represented by a diagonal matrix with the corresponding field on the diagonal. The abbreviation for the transport term

$$D^u = \frac{1}{2} (D_\beta U_\beta + U_\beta G_\beta)$$  

(8)

will be used. The symmetry of the transport term is found to be

$$D^{u^T} = \frac{1}{2} (U_\beta D^T_\beta + G^T_\beta U_\beta) \equiv -D^u,$$  

(9)

which is skew-symmetric, provided that

$$D^T_\alpha = -G_\alpha.$$  

(10)

It is further assumed that the stencil is the same for every grid point. The kinetic energy is conserved; multiplying (7) by $u_\alpha^T$ gives

$$u_\alpha^T \partial_t u_\alpha + u_\alpha^T D^u u_\alpha + u_\alpha^T G_\alpha p = \nu u_\alpha^T L u_\alpha.$$  

(11)

The contribution by skew-symmetric transport term is zero by construction $u_\alpha^T D^u u_\alpha = -u_\alpha^T D^u u_\alpha = 0$. The pressure work is

$$u_\alpha^T G_\alpha p = -p^T D_\alpha u_\alpha = 0,$$  

(12)

it vanishes for incompressible flows due to the solenoidal condition. Thus, the change of kinetic energy

$$\frac{1}{2} \partial_t u_\alpha^T u_\alpha = \nu u_\alpha^T L u_\alpha$$  

(13)
is given by the physical friction alone. This term is usually discretized in a symmetric fashion and negative semi-definite, so that it can only reduce the kinetic energy.

The pressure Poisson equation is derived by applying the divergence on (7), yielding with the help of (6)

$$D_\alpha G_\beta p = D_\alpha \left(- D^\alpha u_\alpha + \nu L u_\alpha\right) \quad (14)$$

We now come to the central point of this work. The restriction on the derivatives (10) is sufficient to conserve energy. This restriction does not imply that any of the derivatives has to be central. A decoupling of implied Laplace operator $$\hat{L} = D_\alpha G_\alpha$$ can simply be avoided by using asymmetric matrixes. As an example the first order left sided derivative for the divergence $$(D u)_i = (u_i - u_{i-1})/\Delta x$$, and $$(G u)_i = (u_{i+1} - u_i)/\Delta x$$ for the gradient is, in principle a valid choice and obviously leads to the standard second derivative for the Laplacian $$(D G p)_i = (p_{i-1} - 2p_i + p_{i+1})/(\Delta x)^2$$. Higher order derivatives can be used, as long as they are asymmetric. This discretization should be clearly distinguished from an upwind scheme. First, upwind stencils are changing with the flow direction, which is not the case in the present scheme. Secondly the asymmetrical stencil does not imply numerical damping as derived before. While both terms in the transport (8) separately do change the kinetic energy, their contributions exactly cancel in combination.

The conservation of momentum is less obvious, as (7) is not in divergence form. It is checked by summing over (7), which is a discrete analog of the integral. The sum is represented by a vector 1 where all components are one:

$$\partial_t 1^T u_\alpha + \frac{1}{2} 1^T (D_\beta U_\beta + U_\beta G_\beta) u_\alpha + 1^T G_\alpha p = \nu 1^T L u_\alpha. \quad (15)$$

Since the stencil is assumed to be the same for every grid point and periodic for now, we trivially have the telescoping sum property, i.e. the sum over the columns of the derivatives vanish. Thus $$1^T D_\alpha = 1^T G_\alpha = 1^T L = 0$$. The only remaining term is

$$1^T U_\beta G_\beta u_\alpha = u_\alpha^T G_\beta^T u_\beta, \quad (16)$$

which reduces with (10) to the solenoidal equation (6) and is therefore zero. Momentum is thus conserved

$$\partial_t 1^T u_\alpha = 0.$$

### 3 Transformed Grids

The presented scheme can be extended to curvilinear grids. This is done in the same manner as in the compressible scheme by the author [18,19]. It is found that the crucial point is, whether momentum and energy conservation strictly hold, since both build on the solenoidal condition in different ways, and thereby strongly restrict the discretization, while in the compressible case an extra energy equation is available. As will be seen, the conservation demands that the divergence of the metric terms vanishes. A proper definition of the metric terms especially in three dimensions leads to strict conservation of both, momentum and energy.

The transformation is given as $$x_\alpha = x_\alpha(\xi_1, \xi_2, \xi_3)$$. Following [11], one can define the local base vector as

$$e_\alpha = \partial_{\xi_\alpha} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad (17)$$
and obtain two equivalent forms of the Nabla operator
\[ \nabla \varphi = \frac{1}{J} \sum_{\alpha} \partial_{\xi_{\alpha}} m_{\alpha} \varphi = \frac{1}{J} \sum_{\alpha} m_{\alpha} \partial_{\xi_{\alpha}} \varphi. \] (18)

The first form is called the \textit{conservative form}, while the second is called the \textit{non-conservative form}. The Jacobian is \( J = (e_1 \times e_2) \cdot e_3 \). Vector valued metric factors are introduced, which are calculated as
\[ m_{\alpha} = (e_{\beta} \times e_{\gamma})_{\alpha, \beta, \gamma} \text{ cyclic}. \] (19)

Since both forms (18) contain the components of the derivatives in physical space, we can use these to discretize the divergence as
\[ D_\beta = \tilde{D}_{\alpha} M_{\alpha, \beta} \] (20)
and the gradient as
\[ G_\beta = M_{\alpha, \beta} \tilde{G}_{\alpha}. \] (21)

where \( M_{\alpha, \beta} \) is the pointwise multiplication by the \( \beta \) component of the metric vector and \( \tilde{D}_\beta \) and \( \tilde{G}_\beta \) are operators in the computational or \( \xi \)-space. As the discretization is chosen to be equidistant in the computational space, the same derivatives as used previously in the Cartesian case can be used.

The discrete, transformed Navier–Stokes equation is
\[ \frac{1}{J} \tilde{D}_\beta M_{\alpha, \beta} u_{\alpha} = 0, \] (22)
\[ J \partial_t u_{\alpha} + \frac{1}{2} \left( \tilde{D}_\gamma M_{\beta, \gamma} U_{\beta} + U_{\beta} M_{\beta, \gamma} \tilde{G}_{\gamma} \right) u_{\alpha} + M_{\alpha, \gamma} \tilde{G}_{\gamma} p = \nu \tilde{L} u_{\alpha}. \] (23)

One can simplify the equation by defining effective convective velocities
\[ \tilde{u}_\beta = M_{\alpha, \beta} u_{\alpha} \] (24)
to obtain
\[ \tilde{D}_\beta \tilde{u}_\beta = 0 \] (25)
\[ J \partial_t u_{\alpha} + D^a u_{\alpha} + \tilde{G}_{\alpha} p = \nu \tilde{L} u_{\alpha}. \] (26)

The factor \( J \) was dropped in the continuity equation. The transport term has the same structure as on the Cartesian grid.

\[ D^a = \frac{1}{2} \left( \tilde{D}_\gamma \tilde{U}_{\gamma} + \tilde{U}_{\gamma} \tilde{G}_{\gamma} \right). \] (27)

and the pressure gradient is \( \tilde{G}_{\alpha} = M_{\alpha, \gamma} \tilde{G}_{\gamma} \). The friction term can be discretized in the same way as the Laplacian of the pressure equation, see below.

The derivatives in the computational space satisfy, as before in the physical space,
\[ \tilde{D}_\beta = -\tilde{G}_{\beta}^T, \] (28)
which is again the key for the energy conservation.

The pressure Poisson equation becomes
\[ \bar{L} \bar{p} = \bar{D}_\beta M_{\alpha,\beta} \bar{G}_\gamma \bar{p} = \bar{D}_\beta M_{\alpha,\beta} \left( -D^\alpha u_\alpha + v \bar{L} u_\alpha \right) / J. \] (29)

A factor of \( J \) is canceled on both sides. A discrete approximation of the Laplacian is thus \( \Delta^{\text{disc}} = (1/J) \bar{L} \). The friction term in the source term must not be omitted, as could be done in the continuous case. Due to the metric factors the divergence operator does only approximately commute with the Laplacian.

In two dimensions, which will be considered in the numerical examples, we have
\[ \bar{u}_1 = \bar{u} = uy_\eta - vx_\eta \]
and
\[ \bar{u}_2 = \bar{v} = -uy_\xi + vx_\xi \]
and the pressure gradient is
\[ \bar{G}_x p = (y_\eta \bar{G}_1 - y_\xi \bar{G}_2) \bar{p} \]
\[ \bar{G}_y p = (-x_\eta \bar{G}_1 + x_\xi \bar{G}_2) \bar{p}. \]

In the following we derive the conservation properties. For the energy conservation we multiply (23) with \( u^T_\alpha \). The transport term is still skew-symmetric, thus it drops out as before. Assuming a vanishing physical friction \( v = 0 \), we arrive at
\[ \partial_t \left( u^T_\alpha J u_\alpha \right) / 2 = -u^T_\alpha M_{\alpha,\gamma} \bar{G}_\gamma p \]
\[ = -p^T \bar{G}_\gamma^T M_{\alpha,\gamma} u_\alpha \]
\[ = p^T \bar{D}_\gamma M_{\alpha,\gamma} u_\alpha = 0 \] (34)
with the help of the continuity equation. This is the energy conservation on the transformed grid. No numerical friction is introduced. A non-zero friction would create the extra term
\[ v u^T_\alpha \bar{L} u_\alpha = v u^T_\alpha \bar{D}_\beta M_{\alpha,\beta} \bar{G}_\gamma u_\alpha \]
\[ = - (M_{\alpha,\beta} \bar{D}_\beta u_\alpha)^T \frac{1}{J} (M_{\alpha,\gamma} \bar{G}_\gamma u_\alpha), \] (35)
which is due to its symmetry obviously negative semi-definit, if we assume the same structure for the friction Laplacian as in the the pressure Poisson equation.

The momentum conservation is checked by summing over (23), using the telescoping sum property
\[ \partial_t \left( 1^T J u_\alpha \right) = -\frac{1}{2} u^T_\beta M_{\beta,\gamma} \bar{G}_\gamma u_\alpha - 1^T M_{\alpha,\gamma} \bar{G}_\gamma p \]
\[ = + \frac{1}{2} u_\alpha \bar{D}_\gamma M_{\beta,\gamma} u_\beta + p^T \bar{G}_\gamma m_{\alpha,\gamma} \]
\[ = -p^T \bar{D}_\gamma m_{\alpha,\gamma}. \] (37)
Thus, we have to evaluate the divergence of the metric factors.
In two dimensions the metric factors are \( \mathbf{m}_1 = (y_\eta, -x_\eta, 0) \) and \( \mathbf{m}_2 = (-y_\xi, x_\xi, 0)^T \)
\[ \bar{D}_\gamma m_{\alpha,\gamma} = \begin{pmatrix} \bar{D}_\xi y_\eta - \bar{D}_\eta y_\xi \\ -\bar{D}_\xi x_\eta + \bar{D}_\eta x_\xi \end{pmatrix}, \]
which is zero when the metric factors are calculated with the derivative of the divergence:
\[ y_\xi = \bar{D}_\xi y \quad y_\eta = \bar{D}_\eta y. \] (39)
If the metric factors were calculated by the gradient, then the energy conservation would be only approximate.

For the general three dimensional case we obtain

\[
\bar{D}_\gamma^T m_{\alpha,\gamma} = \left( \bar{D}_\xi (y_\eta z_\xi - y_\xi z_\eta) + \bar{D}_\eta (y_\xi z_\eta - y_\eta z_\xi) + \bar{D}_\zeta (y_\xi z_\eta - y_\eta z_\xi) \right) 
\]

In the analytical case this can be shown to be zero by using the product rule on all terms. As the product rule does not hold in the discrete, this form would break the strict conservation of momentum. This problem is easily circumvented by the nice trick of Thomas and Lombard [25]: The metric factors above can be rewritten analytically as

\[
y_\eta z_\xi - y_\xi z_\eta = (y_\eta z_\xi - y_\zeta z_\eta) \\
y_\xi z_\eta - y_\eta z_\xi = (y_\xi z_\eta - y_\zeta z_\xi) \\
y_\xi z_\eta - y_\eta z_\xi = (y_\zeta z_\eta - y_\zeta z_\xi) 
\]

If the discretization builds on this form the divergence of the metric vanishes for any discretization. This rewriting also improves the quality of the simulation strongly, as reported in [29].

The proper determination of the metric factors is discussed in [11]. It is concluded, that the same numerical derivative as for the scheme should be used, as this leads to a zero derivative of constant fields on transformed grids. Here, however, two different, but by (28) related derivatives are used, and the derivative of the divergence is chosen to determine the metric factors. This choice seems to be most reasonable: By this all divergence operators, which are in conservative form, vanish for constant fields. All gradient operators, which are in non-conservative form, obtain by this a telescoping some property as just shown, thus become conservative. Thus, both derivatives are conservative and vanish for constant fields.

**4 Non-periodic Boundaries**

For the sake of completeness a procedure to introduce boundary conditions (BC) for the presented scheme is discussed. The treatment of boundaries is not different to other collocated finite difference schemes, so that different methods could be used. Here, the approach of [8] for collocated grids is adopted and is found to work well.

Boundary conditions modify the discrete momentum equation and also the pressure Poisson equation has to be changed. The need and the form of the BC conditions for the Poisson’s equation is subject to a lively debate, see [20] and [23]. In [8] it is argued that correct pressure boundary conditions are automatically created from the discrete Navier–Stokes equation, which includes the BC, together with the discrete continuity equation. The behavior of the pressure at the boundary will be examined in the numerical example.

In the following it is assumed, that the velocities are prescribed at the boundary. All derivative stencils are modified at the boundary to contain only internal points, i.e. one sided derivatives are used instead of ghost points. Thus, the divergence can be calculated without difficulties as before (22), as

\[
0 = \frac{1}{J} \bar{D}_\mu M_{\alpha,\mu} u_{\alpha} 
\]

To distinguish boundary points from internal point a projector \( B \) is introduced, which is one for internal points and zero for boundary points. Likewise \( (1 - B) \) is one for the boundary.
point and zero otherwise. By this the divergence is split into contributions from internal and from boundary points. As \(1 = B + (1 - B)\), the temporal change of the divergence is

\[
\tilde{D}_{\beta} M_{\alpha, \beta}(B) \partial_t u_{\alpha} = -\tilde{D}_{\beta} M_{\alpha, \beta}(1 - B) \partial_t u_{\alpha} = -\tilde{D}_{\beta} M_{\alpha, \beta} \partial_t u_{\alpha} |_{\partial V}
\]  

(42)

the internal points are described by the Navier–Stokes equation while the boundary values \(\partial_t u_{\alpha} |_{\partial V}\) are assumed to be known. Using the momentum equation (26) in the left hand side of (42) leads to

\[
\tilde{L}^B p = \tilde{D}_{\beta} M_{\alpha, \beta} M_{\alpha, \gamma} J B \tilde{G}_{\gamma, \gamma} p = -\tilde{D}_{\beta} M_{\alpha, \beta} \left[ B(D^B u_{\alpha} - \nu(1/J)\tilde{L}u_{\alpha})/J \right]
\]

\[
-\tilde{D}_{\beta} M_{\alpha, \beta} \partial_t u_{\alpha} |_{\partial V}
\]  

(43)

This is [8, eq. 18] for a general stencil. For a wall the last term will simply be zero. The Laplace operator \(\tilde{L}^B p\) in (43) is changed at the boundary in comparison with (29). As the pressure gradient terms at the boundary are missing the resulting operator is not a Laplace operator at the boundary. Instead it can be interpreted as a Neumann BC for the pressure [8]. This allows to calculate a unique and solenoidal solution for the velocity which exactly fulfills the BC. However the pressure will be found to be ambiguous, as the pressure derivative at the boundary is projected out in the above expression. This will further be discussed by the example of the respective numerical test case.

As the modified stencils for the non-periodic boundaries necessarily breaks both, the telescoping sum property and the skew symmetry, the momentum and the energy conservation do not hold at the boundary. This is expected, as boundaries are in general sources of momentum and energy. The source term can be interpreted as the flux at the boundary. This numerical flux depends on the number of terms of the modified stencil, which break the symmetry. It is, without providing a formal proof, expected to be consistent with the analytical flux, i.e. if the flow field is constant at the boundary the numerical flux reduces to the analytical boundary flux.

5 Time Stepping

To keep the strict conservation the time discretization has to respect the momentum and energy conservation. The implicit midpoint rule is an adequate, second order choice [28]. This implicit midpoint rule belongs to the class of Gauss collocation Runge–Kutta methods, which all conserve quadratic invariants. The general theory can be found in [14]. The corresponding fourth order rule is defined by the Butcher table

\[
\begin{array}{ccc|ccc}
\frac{1}{2} - \frac{\sqrt{3}}{6} & 1 & 1 & \frac{1}{4} & -\frac{\sqrt{3}}{6} \\
\frac{1}{2} + \frac{\sqrt{3}}{6} & 1 & 1 & \frac{1}{4} & \frac{\sqrt{3}}{6} \\
1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{array}
\]

In [1,2] it is shown how to use these schemes for the skew-symmetric, compressible Euler equations. For the incompressible case these schemes can be used without adaptation. This was already reported in [22]. Also even higher order Gauss collocation Runge–Kutta methods can be used, but might be of little practical interest. The pressure has to be determined for each implicit Runge–Kutta step. Splitting schemes violate the strict conservation. However, splitting gives in our experience well working an stable schemes. For the sake of completeness the proof of energy conservation is sketched. The time stepping is
where \( k_{\pm} = \text{rhs}(u_{\pm}) \) is the right hand side of the momentum equation, i.e. all terms but the time derivative and \( a = \sqrt{3}/6 \). The kinetic energy at \( t^{n+1} \) is calculated with (44)

\[
(u_{n+1}^\alpha)^T u_{n+1}^\alpha = (u_n^\alpha)^T u_n^\alpha + \Delta t (\frac{1}{4}k_n^- + \frac{1}{4}a) k_n^- + \frac{(\Delta t)^2}{4} (k_n^- + k_n^+)^T (k_n^- + k_n^+)^T
\]

expressing \( u_n^\alpha \) in the term linear in \( \Delta t \) by (45) in front of \( k_n^- \) and by (46) in front of \( k_n^+ \) all quadratic terms cancel:

\[
(u_n^{n+1})^T u_n^{n+1} = (u_n^\alpha)^T u_n^\alpha + \frac{\Delta t}{2} ((u_n^-)^T k_n^- + (u_n^+)^T k_n^+).
\]

The term linear in \( \Delta t \) vanishes as shown in the previous section, since the combinations are evaluated at the same time. The momentum conservation is right away obtained from summing (44).

In the numerical examples the equation is solved by a fixed point iteration. For not too large \( \Delta t \) it converges rapidly, for large time steps more sophisticated methods should be used. The energy conservation is expected to be good after a few iteration steps, as reported in [2]. The disadvantage is, that the pressure also has to be updated in each step. If an iterative pressure solver is used, one can save computational time by controlling the residuum of the pressure equation by the residuum of the fixed point iteration. A higher error of the pressure is accepted for the first time iteration steps. In first tries it was found for a multigrid method, that only in the range of 30–50% more iteration steps were needed for the pressure solver for the consistent implicit midpoint rule, compared with the corresponding fraction step method. This seems acceptable if a strictly conserved energy is important. However, for any explicit time stepping strict conservation is recovered in the limit of small time steps and for a finite \( \Delta t \) the the error is expected to be small for high order methods.

### 6 Numerical Examples

Three cases are presented to prove the principle soundness of the approach. The first two cases are periodic, the third case is an example of a non-periodic situation. All cases are two-dimensional. The first one is a vortex pair on a transformed grid. The second case consists of three merging vortices on an Cartesian grid. The third case is a vortex pair impinging on a wall. Third, fifth and seventh order stencils are used\(^1\). The third order divergence is given by \((0, -1/3, -1/2, 1, -1/6)\) and the fifth order divergence is given by the stencil \((0, 3, -30, -20, 60, -15, 2)/60\) and the seventh order divergence is given by \((0, -4, 42, -252, -105, 420, -126, 28, -3)/429\), the gradient is obtained by transposing.

\(^1\) Strictly one sided first order derivatives led to a quick dissolution of the flow structures.
For the non-periodic case the fifth order divergence is modified at the left boundary

\[
\begin{pmatrix}
-137/60 & 5 & -5/3 & -5/4 & 1/5 \\
-1/5 & -13/12 & 2 & -1 & 1/3 & -1/20 \\
& & & & & \\
& & & & & \\
& & & & & \\
\end{pmatrix}
\] (49)

and at the right boundary

\[
\begin{pmatrix}
\ldots \\
1/30 & 1/4 & -1 & 1/3 & 1/2 & -1/20 \\
1/20 & -1/3 & 1 & -2/13 & 1/5 \\
-1/5 & 5/4 & -10/3 & 5 & -5 & 137/60 \\
\end{pmatrix}
\] (50)

The stencil for the gradient is obtained from it by exchanging left and right boundary, arranging the elements in inverse order and invert the sign.

Time stepping algorithms of second and fourth order are used. The pressure equation can, for the given problem size, still be solved by a pivoted LU decomposition. The start solution is made divergence-free as usual by an initial Chorin projection step [3]. The implicit time stepping is solved by a fix point iteration, as discussed before.

6.1 Vortex Pair on a Transformed Grid

The quality of the scheme on distorted grids is examined by the transport of a vortex pair on a strongly distorted grid. The initial vortex pair is given by

\[
\omega = \alpha \left[ e^{-(x-x_1)^2+(y-y_1)^2}/\beta^2} - e^{-(x-x_2)^2+(y-y_2)^2}/\beta^2 \right]
\] (51)

with \(x_{1,2} = (L/2, L/2), y_{1,2} = (0.45 \cdot L, 0.55 \cdot L)\). The amplitude is \(\alpha = 10\) and the width is \(\beta = L/14\). The reference size and the size of the computational space is \(L = 2\pi\). The transformed grid is given by \(x = \xi + k \sin(\xi + \eta)\) and \(y = \eta + k \sin(\xi + \eta)\), with \(k = 0.2\). The resolution is \(N_\xi, N_\eta = 48, 49\). The time integration is the fourth order Gauss method, with \(\Delta t = 4 \cdot 10^{-3}\).

The vortices are allowed to travel once through the full domain, for the given situation \(T = 11.68\), crossing the lines of strong distortion. It is expected that the higher order derivatives reduce grid effects, as for a perfect derivative the grid transformation is analytically exact. For the third order derivative not only the structure is distorted, but the vortex pair is artificially deflected. This is evident by comparing with the reference solution obtained on a fine, Cartesian grid of 512\(^2\) points. The error clearly reduces when using higher order derivatives.

The spatial convergence of the solution on the transformed grid is checked by comparing it with the interpolated reference solution. The solutions are calculated for the time \(T = 11.68\) with the fourth order Gauss collocation RK method with a very fine time step of \(\Delta t = T/2000\). The reference solution with a seventh order derivative is compared with transformed grid calculations with a third and fifth order derivative for \(N = 64, 96, 128, 160\). The formal truncation error is found, see Fig. 1.

The conservation properties are shown in Fig. 2. Change of total momentum and energy are zero within the numerical accuracy for the frictionless case. Friction reduces the kinetic energy as expected.

In summary a solution with a strictly solenoidal solution with perfect conservation of momentum and energy and a high order truncation error is found.
6.2 Three Vortices Test Case

In this test case two out of three vortices are merging, see [13]. A periodic and square domain with a size of $2\pi$ is used. The vorticity is given by $\omega = \text{rot}(u) = \omega_0 + \sum_k \alpha \exp(r_k^2/\beta^2)$ with $r_k^2 = (x - x_k)^2 + (y - y_k)^2$. The vortices are located at $x_k = \pi(3/4, 5/4, 5/4)$ and $y_k = [1, 1, 1 + 1/(2\sqrt{2})]\pi$. Further $\beta = 1/\pi$ and $\alpha = \pi$. From this the stream function is $\Delta \Psi = \omega$ and finally $(u, v) = (-\Psi_y, \Psi_x)$. The Poisson-equation for the stream-function is only solvable if the integral condition $\sum \omega = \sum (-u_x + v_x) = 0$ is fulfilled. This determines $\omega_0 \approx -0.038$. The time integration is the second order Gauss scheme, i.e. the implicit midpoint rule, with $\Delta t = 2.5 \cdot 10^{-3}$.
The kinetic energy is reduced only by physical friction. Strict conservation of the kinetic energy is found for $\nu = 0$. The relative change (inset) is of the order of $10^{-15}$. The divergence is numerically zero, the momentum is conserved.

The level lines in Fig. 3 are chosen in the same manner as in [13]. Fine structures emerge after short time in the vortex merging process. The comparison with the cited work is challenging, as it is a comparison with a Fourier spectral scheme. However, we find apparently very good agreement with figures presented in that work.

To see in more detail how well the different scales are treated the energy spectrum is calculated, see Fig. 3 (last plot). The same scaling $\sim k^{-4}$ as in the reference is found, see [13, Fig. 3a]. Further the same structures at the same locations as in the reference is found.

To test the order of the scheme convergence test are performed, see Fig. 4. The spatial convergence is tested by performing simulating the above described test case for $T = 1$ with a sixth-order Gauss-collocation scheme in time with a time step of $\Delta t = 0.005 \cdot 64/N$ for the resolution $N^2$ with $N = 64, 128, 192, 256, 320$. The reference solution is created by a simulation with $N = 1024$ with a seventh order derivative. The convergence order of three and five in accordance with the formal order of the used derivatives is found. The temporal discretization-order is verified with fixed spatial resolution of $N = 256$, where the time step is varied as $\Delta t = 0.0025, 0.005, 0.01, 0.02, 0.04$, the simulation time is $T = 5$. The reference solution is obtained with a time step $\Delta t = 0.001$, with the sixth-order Gauss-collocation method.

6.3 Dipol-Wall Collision

A dipol-wall collision leads to the creation of secondary structures and a strongly enhanced vorticity at the wall. It is thus a good test case for wall boundary conditions and is therefore used to evaluate the proposed boundary treatment of Sect. 4. The case follows strictly [12] to allow a detailed comparison. Further the behavior of the pressure close to the wall will be discussed.

The domain consists of a square domain $[-1, 1] \times [-1, 1]$. The $x$ direction is periodic, the boundaries at $y = \pm 1$ are non-slip walls.

The start conditions are created from two single vortices of the form

$$\omega_0 = \omega_e \left(1 - \left(r/r_0\right)^2\right) \exp \left(-\left(r/r_0\right)^2\right)$$  (52)

$^2$ The there depicted time is measured in the vortex turn-over time leading to a factor of four.
Fig. 3 Three merging vortices at times $t = 0, 5, 10, 15$ and $t = 20$ starting from upper left, upper right. The plots show $\text{rot}(u) - \omega_s$, with $\omega_s = -0.038$. The negative values are dashed contour lines from $(-\pi, -\pi/100)$ and the positive values are solid lines from $(\pi/100, \pi)$. The maximal negative contour line is $-1.0577$. Last plot: The energy spectrum at two different times shows a cascade $\sim k^{-4}$. 
Fig. 4 The convergence of the spatial and the temporal discretization show the expected order in space (left) and time (right).

Fig. 5 Left: Vortex pair impinging on a wall at $t = 0$ and $t = 0.4$, vorticity contour lines are $-270, -210, \ldots, 270$. Right: detail at $t = 0.4$, vorticity contour lines are at $-270, -250, \ldots, 270$ with the distance from the vortex core $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ and a vortex size $r_0$.

As in the reference $r_0 = 0.1$ and the centers are at $(x_0, y_0) = (\pm0.1, 0)$. The strength is $\omega_v = \pm299.528385375226$, and is chosen such that at $y = \pm1$ the velocity is numerically zero. The viscosity is $\nu = 1/Re = 1/1000$.

A non-uniform grid of $512 \times 512$ points is used. A fine resolution in the center and at the walls is created by the transformation $x_i = \xi_i - \alpha \sin(\pi \cdot [\xi_i - 0.1 \sin(\pi \cdot \xi_i)])$ and $y_j = \tanh((\eta_j - 1)/\lambda)/\tanh(1/\lambda)$. The computational space is equidistant $\xi = [-1 + \Delta \xi, 1]$ and $\eta = [-1, 1]$. The parameter $\lambda = 0.9146$, $\alpha = 0.2320$ lead to the minimum square cell of the size $2/1024$ at $(0, \pm1)$. The time marching is done with the implicit midpoint rule, with $\Delta t = 2 \cdot 10^{-4}$.

Figure 5 shows the impinging vortex. The vortex pair propagates southwards, a much weaker pair is created which moves north. At about $t = 0.3$ the main pair impinges on the south wall. Strong vorticity with opposite sign is created at the wall in a thin layer, see Fig. 7. Comparing with [12, fig. 5f], all features of the vorticity are reproduced and at the correct location.
Remarkably the pressure shows strong wiggles at the boundary, see Fig. 6. This is not an odd-even decoupling, but an artifact of the boundary treatment: the pressure derivative is smooth with the exception of the very boundary point itself, see Fig. 6, right. By construction the pressure derivative at the boundary is not evaluated, as the values at the boundary are overwritten by the boundary condition. Thus the flow solution stays smooth. The pressure derivative at the boundary can be changed, without modifying the solution, which can be used to construct a smooth pressure. In Fig. 6 an interpolated pressure derivative is shown, from which by integration (inverting the numerical derivative) the pressure is then calculated. It is found to be smooth. This interpolation procedure could be added to the pressure Poisson equation, removing the ambiguity from the pressure equation. Similar the reconstructed pressure in the boundary layer is smooth, see inset of Fig. 7.

7 Conclusion

A novel approach to avoid the odd-even decoupling in the simulation of incompressible flows is presented. It builds on the combination of asymmetric derivatives which are found to be momentum and energy conserving if combined appropriately. High order discretization in space and time can be utilized. The scheme works on Cartesian and transformed, structured grids. Numerical simulations of two-dimensional test configurations are presented, where up
to seventh order in space and fourth order in space was used. The energy and momentum conservation was numerically verified.

In contrast to statements of [7], strict conservation and a high discretization order was possible, also on transformed grids. The discrepancy might be explained as follows: in the cited paper the source of the energy error is identified as arising from interpolation error of the nonlinear transport terms and from the non-central weights for the interpolation for a higher order. As no interpolation is needed for FD only the non-interpolated nonlinear transport has to be energy conserving, which is readily achieved by the skew symmetric form. The transformed grid is not treated by a change of weights, which would correspond to a changed stencil in FD but by a grid-mapping, which allows to use the same derivatives as in the euclidean case. The advantage is illustrated in [27]. There the convection problem is solved by two different methods. The method A, which can be interpreted as building on a grid transformation, is found superior to a method B, which is changing the stencil. A detailed comparison of the new scheme with [7] is desirable.

Important points which still need to be discussed are an efficient method to calculate a consistent pressure within the time step for reasonable cost. A paper about a suitable pressure correction scheme is in progress.

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