Local well-posedness of the capillary-gravity water waves with acute contact angles

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Abstract

We consider the two-dimensional capillary-gravity water waves problem where the free surface $\Gamma_t$ intersects the bottom $\Gamma_b$ at two contact points. In our previous works [52, 53], the local well-posedness for this problem has been proved with the contact angles less than $\pi/16$. In this paper, we study the case where the contact angles belong to $(0, \pi/2)$. It involves much worse singularities generated from corresponding elliptic systems, which have a strong influence on the regularities for the free surface and the velocity field. Combining the theory of singularity decompositions for elliptic problems with the structure of the water waves system, we obtain a priori energy estimates. Based on these estimates, we also prove the local well-posedness of the solutions in a geometric formulation.

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1 Introduction

We consider the irrotational incompressible capillary-gravity waterwaves problem in a two-dimensional domain \( \Omega_t \), where \( \Omega_t \) is a bounded domain with an upper free surface \( \Gamma_t \) and a fixed bottom \( \Gamma_b \). This moving domain contains two moving contact points \( p_l, p_r \) (left and right) with the contact angles \( \omega_l, \omega_r \in (0, \pi/2) \), which are the intersection points of \( \Gamma_t, \Gamma_b \):

\[ \Gamma_t \cap \Gamma_b = \{ p_l, p_r \}. \]

Moreover, the fixed bottom \( \Gamma_b \) is assumed to be smooth enough, and it becomes straight near the contact points \( p_i (i = l, r) \) for the sake of simplicity.

The water waves problem has been widely studied in centuries, see for example [63, 49]. This problem focuses on the motion of an ideal fluid and describes the evolution of the free surface \( \Gamma_t \) as well as the velocity field \( v \). Mathematically, it is described by Euler's equation with boundary conditions and initial conditions, and in our case, we also need some boundary conditions at contact points.

We express the water waves problem on the corner domain \( \Omega_t \) as the following system (WW) of velocity \( v \) and pressure \( P \):

\[
\begin{align*}
    \partial_t v + v \cdot \nabla v &= -\nabla P + g \quad \text{in} \quad \Omega_t, \\
    \text{div} v &= 0, \quad \text{curl} v = 0 \quad \text{in} \quad \Omega_t, \\
    P|_{\Gamma_t} &= \sigma \kappa, \\
    D_t = \partial_t + v \cdot \nabla \text{is tangent to} \quad \{(t, X)|X \in \partial \Omega_t\}, \\
    v \cdot n_b|_{\Gamma_b} &= 0, \\
    \beta_c v_i &= \sigma (\cos \omega_s - \cos \omega_i) \quad \text{at} \quad p_i (i = l, r).
\end{align*}
\]

Here (1.1) from (WW) is the Euler's equation where \( g = -ge_z \) is the vertical gravity vector; (1.2) describes the incompressibility and irrotationality; (1.3) is the condition of the pressure on the free surface in the case with surface tension, where \( \sigma \) is the coefficient of surface tension and \( \kappa \) is the mean curvature of \( \Gamma_t \) (see Section 1.2); (1.4) is the classical kinematic condition on the free surface \( \Gamma_t \) with \( D_t \) the material derivative; Meanwhile, (1.5) describes that the velocity along the fixed bottom \( \Gamma_b \) is always tangential, where \( n_b \) is the unit outward normal vector of \( \Gamma_b \). These equations and conditions are standard in water waves, see [45, 46].
In particular, (1.6) gives the conditions at contact points, which come from [58]. We denote by \( v_i \) the upward tangential components of the velocity at the corner points along \( \Gamma_b \):

\[
v_l = -v \cdot \tau_b \text{ at } p_l, \quad v_r = v \cdot \tau_b \text{ at } p_r.
\]

Here \( \omega_s \) is the stationary contact angle decided by the materials of the bottom and the fluid (see [76]), and \( \beta_c \) denotes the effective friction coefficient. This condition shows that the slip velocity is dominated by the unbalanced Young stress, and it is indeed an effective variation of Young’s law (1805) for stationary contact angles [76]. In fact, this kind of conditions are commonly seen, see [11, 12, 62, 27].

Before presenting our results, we recall briefly earlier works on the well-posedness of classical water waves problem, where one has smooth surfaces \( \Gamma_t \) satisfying \( \Gamma_t \cap \Gamma_b = \emptyset \).

We recall results on the local well-posedness. When the fluid is irrotational, some early works like Nalimov [55], Yosihara [74, 75] and Craig [22] established the two-dimensional local well-posedness with small initial data in Sobolev spaces. In late 1990s, Wu [68, 69] proved for the first time the local well-posedness with general initial data in Sobolev spaces and showed that the Taylor sign condition

\[
-\nabla_n \cdot P \geq c_0 > 0 \tag{1.7}
\]

held on \( \Gamma_t \) as long as \( \Gamma_t \) was not self-intersecting. Iguchi, Tanaka and Tani [38] and Iguchi [36] proved the local well-posedness in two-dimensional case respectively. Later on, Lannes [45] derived the local well-posedness of the gravity water waves under Zakharov formulation, which is convenient to link with approximate models. Alazard, Burq and Zuily in [2, 3, 4, 5] used paradifferential operators and Strichartz estimates to study the problem in a low-regularity space. On the other hand, when the fluid is rotational, Christodoulou and Lindblad [18] proved a priori estimates based on the geometry of the moving domain; Lindblad [50] obtained the existence of solutions using Nash-Moser iteration. In 2007, Coutand and Shkoller [20] used Lagrangian coordinates to show the local well-posedness. Shatah and Zeng [60, 61] adopted a geometric point of view to reformulate the problem and prove the local well-posedness, while Beyer and Günther [13, 14] used a similar geometric approach to study the irrotational flow. Zhang and Zhang [77] proved the local well-posedness for rotational flow using a framework of Clifford analysis introduced by Wu [69]. For more references, see [56, 57, 8, 9, 59, 77, 67, 7, 46, 54] e.t.c..

For the global well-posedness of small data, Wu [72] and Germain, Masmoudi and Shatah [28] proved the global three-dimensional existence of gravity water waves respectively using different approaches. One can check [71, 6, 39, 25, 33, 34, 35, 66, 73] e.t.c. and their references for more results on gravity or capillary-gravity water waves. Meanwhile, there are also some works concerning geometric singularities on the free surfaces. The authors in [16] proved the existence of a wave which is given initially as the graph of a function and then can overturn at a later time. Later on, the authors in [17] showed the existence for some “splash” or “splat” singularities. This result was extended to three-dimensional case and some other models in [21].

Compared to the rich literature on the well-posedness of classical water waves, the research on the well-posedness of water waves problem with non-smooth boundaries (we call it “non-smooth water waves”) just started several years ago and there are a lot of open questions. In general, there are two kinds of non-smooth water waves problems: The first kind of problems has contact points (or contact lines) between the free surface and the bottom, i.e. \( \Gamma_t \cap \Gamma_b \neq \emptyset \); The other kind contains crests or cusps on the free surface, i.e. the surface is Lipschitz. We would like to mention that in the case with large crest angles, the famous Stokes waves can be dated back to papers by G. Stokes [63, 64] which obtained traveling-wave solutions with limit crest angle \( 2\pi/3 \). Obviously, the main difference here compared to classical water waves lies in the corners on boundaries. As a result, the analysis involving the corners (i.e. domain singularities) becomes the key point in the non-smooth water waves.

Now, we are in a position to state an informal version of the main result. To begin with, we introduce the following compatibility conditions at \( t = 0 \)

\[
\beta_c \partial_t^k v_i(0) = \sigma \partial_t^k (\cos \omega_s - \cos \omega_0) \quad \text{at} \quad p_i \ (i = l, r), \quad k = 0, 1, 2, 3. \tag{1.8}
\]
These conditions are needed since we will reduce system (WW) into initial-boundary value problem (see (3.33), (5.30) and (6.44)).

**Theorem 1.1** Let the initial data belong to a suitable space and the initial contact angles $\omega_0 \in (0, \pi/2)$ for $i = l, r$. If the compatibility conditions (1.8) are satisfied, there exists a time interval depending on the initial data such that system (WW) is locally well-posed in a suitable space.

**Remark 1.1** The suitable space is defined by $\Sigma_h$ in Section 6.3 for some good unknowns. In fact, the proof of Theorem 1.1 is divided into three parts. In Section 4 and Section 5, we obtain a priori estimates (see Theorem 4.1 and Theorem 5.1). In the last section, we construct approximate solutions to prove the well-posedness based on the a priori estimates. The precise statement of local well-posedness for a geometric form of (WW) is given in Theorem 6.1 (Section 6.3).

**Remark 1.2** When there is no surface tension, the authors of [40, 19] studied the case with contact angles. They assume that the wall $\Gamma_b$ is vertical and the contact angle $\omega \in (0, \pi/4)$. Then, by a symmetric extension, they reduced the problem to the crest case with the crest angle less than $\pi/2$. As a result, [40, 19] do not contain the case of limit Stokes waves with the “$2\pi/3$” crest. In that case, one needs a contact angle of $\pi/3$ even with the help of a symmetric extension. In this paper, we only require that the contact angles $\omega_i \in (0, \pi/2)$, which brings us some useful experiences to deal with domain singularities in a more general case for water waves.

**Remark 1.3** By continuity argument of the contact angle in time similarly as in the last part of [52], one can see that as long as initial contact angles $\omega_0 \in (0, \pi/2)$, we will have $\omega_i(t) \in (0, \pi/2)$ in a short time interval.

We recall some works concerning the local well-posedness of non-smooth water waves. In the case where there are crests or cusps on the free surface (the “crest” case), Alazard, Burq and Zuily [5] study a special case (without surface tension) when the contact angle is equal to $\pi/2$ (the right angle), where they used symmetric and periodic extension to turn this problem into a classical smooth periodic case. A breakthrough in this subject is made by Kinsey and Wu, see [40, 70]. They focus on gravity water waves where the crest angle is less than $\pi/2$. The main difficulty is that the free surface is a non-$C^1$ interface with angled crest and the Taylor sign $-\nabla_n P$ degenerates at the crest point. To be more precise, they start with reducing the water waves problem into the following equation in [40, 70]:

$$\partial_t^2 u + a \nabla_n u = f(u, \partial_t u)$$

where $a = -\nabla_n P$ is the Taylor sign. When $a$ degenerates at the crest point, the above system will loss its hyperbolicity, and classical analysis does not apply any more. To solve the problem, Kinsey and Wu flatten the domain with a Riemann mapping, and some singular weights appear naturally in their equation. As a result, they introduce some weighted Sobolev spaces accordingly for energy estimates to deal with these singular weights. Based on these works, Agrawal [1] show that these singularities are “rigid”, which means that the angle of these crests can not change in time. Very recently, Córdoba, Enciso and Grubic [19] study a similar case with cusps and crests without gravity, where the angles of these crests are less than $\pi/2$ and change in time.

For the other case where there are contact angles (that is $\Gamma_t \cap \Gamma_b \neq \emptyset$), things become different from the “crest” case. First, the corners appear due to intersections of the free surface and the bottom (or wall); Second, there are different boundary conditions in corresponding elliptic systems compared to the crest case. In fact, various boundary conditions may have no big difference if we only focus on the elliptic theory, but there will be a series of consequences in water waves when boundary conditions change. For example, the evolution of the free surface is different from the crest case. Moreover, when there is surface tension, boundary conditions at the contact points as (1.6) are needed in order to close the system, and dissipations appear at the contact points too (See Section 4 and [52]).

In the case with contact angles, de Poyferré [23] prove a priori estimates in bounded n-dimensional corner domains without surface tension. The contact angle is assumed to be small to ensure sufficient
Sobolev regularity near the corner. Meanwhile, under a similar assumption of small contact angles, we obtain the local well-posedness in a two-dimensional corner domain (beach type) with surface tension, see [52, 53]. Meanwhile, we notice that [23] and [52, 53] use similar geometry formulations introduced in [60].

To explain why small contact angles are needed in [52, 53, 23], we look at a typical mixed-boundary system in water waves:

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in} \quad \Omega_t, \\
u|_{\Gamma_i} &= f, \quad \nabla_n u|_{\Gamma_b} = 0.
\end{aligned}
\]

In fact, the elliptic theory on corner domains is well known already, see for example [43, 30, 44].

Generally, one still has variational solution \( u \in H^1(\Omega_t) \) if the right-side functions lie in proper Sobolev spaces. If one seeks for \( H^2 \) and above regularities, singularity decompositions are needed naturally for the solution, which decompose the solution into a singular part \( u_s \) (i.e. not good enough) near the corners and a regular part \( u_r \):

\[ u = u_s + u_r \]

according to the required regularity.

For the mixed-boundary system above, the most singular part in \( u_s \) is like \( r^{\pi/2\omega} \), where \( r \) is the radius with respect to the corner point and \( \omega \) is the contact angle. Consequently, when \( \omega \) is small enough, the singular part \( u_s \) will be good enough so that we will have enough regularities from elliptic systems as in classical water waves to close the energy estimates. One can also find a singularity decomposition for \( v \) in Proposition 2.3 (Section 2).

In contrast, when the contact angle is larger or more general, the idea of taking small angles to improve regularities in [23, 52, 53] does not work any more. Meanwhile, there is no obvious weighted space to use due to the structure of the water waves problem.

We want to show here the main ingredients of this paper. Similarly as in our previous works [52, 53], we still adopt the geometric formulation from Shatah and Zeng [60, 61]. In fact, we rewrite (WW) into an equation for \( \mathcal{J} = \nabla \mathcal{H} \) with boundary conditions, where \( \mathcal{H} \) is the modified mean curvature on \( \Gamma_t \) and \( \mathcal{H} \) means the harmonic extension in \( \Omega_t \) (see Section 3, and we only present a simpler form for this equation here):

\[
D^2_t \mathcal{J} + \sigma A \mathcal{J} = R \quad \text{with} \quad A \mathcal{J} = \nabla \mathcal{H}(-\Delta \mathcal{H}, 3^\perp),
\]

(1.9)

where \( R \) is some remainder part.

The trouble here is that singularities from the domain \( \Omega_t \) affect directly the regularity of the solution to (WW), which means singularities always appear in related elliptic systems even if the boundary conditions are good enough in Sobolev spaces (see for example [30]). More precisely, the natural norm \( \|A^k f\|_{L^2(\Omega_t)} + \|f\|_{L^2(\Omega_t)} \) with \( k \geq 1 \) arising from this equation above is not equivalent with \( \|f\|_{H^{3k}(\Omega_t)} \) due to larger contact angles, and apparently “singular parts” are contained in this norm.

Due to this kind of singularities from elliptic systems, we only have limited regularities for some quantities (for example, the velocity \( v \)) in Sobolev spaces. Compared to [52, 53], we must make full use of the maximal regularity for each quantity (especially for \( v \)) while the contact angles \( \omega_i \in (0, \pi/2) \). To do this, some delicate estimates together with singular parts from singularity decompositions (see [51, 30]) are carefully used. Meanwhile, it is also very important to gain more information from the structure of (WW).

The main part of the (lower-order) energy functional is defined as

\[
E(t) = \|D_t \mathcal{J}\|_{L^2(\Omega_t)}^2 + \|\mathcal{J} \cdot n_t\|_{H^1(\Gamma_t)}^2,
\]

which gives us the following estimate (see Section 4.1)

\[
\|v\|_{H^2(\Omega_t)} + \|D_tv\|_{H^{3/2}(\Omega_t)} + \|D_t^2 v\|_{L^2(\Omega_t)} + \|\nabla v\|_{L^\infty(\Omega_t)} + \|\kappa\|_{H^2(\Gamma_t)} \leq P(E(t))
\]
with \( P(E(t)) \) the positive-coefficient polynomial of \( E(t) \). This means that the free surface \( \Gamma_t \) still has enough regularity and \( v \) is Lipschitz.

In our previous work [52, 53], one main part from the remainder term \( R \) in (1.9) is about the higher-order derivative terms of \( P_{v,v} \), where \( P_{v,v} \) is defined by an elliptic system with mixed-boundary conditions. According to the elliptic theory on corner domains (see for example [30]), the system of \( P_{v,v} \) only gives \( P_{v,v} \) limited regularity around \( H^2(\Omega_t) \), when the contact angles are less than \( \pi/2 \). To improve its regularity, we modify the definition of \( P_{v,v} \) in this paper to have a Neumann-boundary system, see (3.19). In fact, thanks to the elliptic theory, solutions of Neumann-boundary system (or Dirichlet system) may become more regular than solutions of mixed-boundary problem in Sobolev spaces, while the right-side data have the same regularity. As a result, we obtain a bit more regularity from the elliptic system of \( P_{v,v} \) (we have \( P_{v,v} \in H^3(\Omega_t) \) indeed), which is important in the energy estimates.

Moreover, another main part of \( R \) lies in the higher-order derivative terms of the velocity \( v \). Here, we point out that compared to [52, 53], \( v \) loses some regularity due to the existence of corners. Fortunately, \( D_t v \) and \( D_t^2 v \) have the same regularities as before. As a result, when we deal with \( v \), sometimes we need to use material derivatives \( D_t \) instead of spatial derivatives. Meanwhile, we also need to apply singularity decompositions to \( v \) and its potential \( \phi \) in the estimates, see for example Lemma 4.5.

For the higher-order energy estimates, we use the material derivative \( D_t \) instead of \( \mathcal{A}^{1/2} \) from [60]. The higher-order energy is defined as

\[
E_1(t) = \|\nabla_t D_t (3 \cdot n_t)\|_{L^2(\Gamma_t)}^2 + \|D_t^2 \mathfrak{J}\|_{L^2(\Omega_t)}^2.
\]

In fact, one will see that \( D_t \) is convenient to use when there are contact points. For example, taking \( D_t \) on elliptic systems does not violate boundary conditions, while it will change boundary conditions if one takes spatial derivatives. Moreover, one will find in the higher-order energy estimate that taking \( D_t \) leads to better regularities for some quantities than their own regularities (like \( D_t v \) for the velocity \( v \)). However, due to the singularities of the boundary (contact points), it is not as convenient as before to turn these regularities into spatial regularities, which explains the reason why we need to choose very carefully for “a suitable space” in Theorem 1.1. As long as the energy estimates for \( E(t), E_1(t) \) are finished, we use the equation of \( \mathfrak{J} \) to gain more spatial regularities.

In the end, we mention some other related works. Lannes and Métivier [48] studied the Green-Naghdi system in a beach-type domain, which is a shallow-water model of the water waves problem. Lannes [47] studied the floating-body problem and proposed a new formulation that can be easily generalized in order to take into account the presence of a floating body. Lannes and Iguchi [37] proved some sharp results for initial boundary value problem with a free boundary arising in wave-structure interaction, and it contains the floating problem in the shallow water regime. Besides, Guo and Tice [31] showed a priori estimates for the contact line problem in the case of the stokes equations. Later on, Tice and Zheng proved the local well-posedness of the contact line problem in two-dimensional Stokes flow, see [65]. In 2020, Guo and Tice [32] proved a priori estimates for the contact line problem for two-dimensional Navire-Stokes flow. For Darcy’s flow, one can see [41, 42].

### 1.1 Organization of the paper

In Section 2, we present various useful lemmas including singularity decompositions and estimates for elliptic systems. In Section 3, we derive the equation for the good unknown \( \mathfrak{J} \) from (WW) with modified curvature \( \mathcal{R} \) and modified pressure \( P_{v,v} \). The lower-order energy is constructed and the energy estimate is proved in Section 4, where estimates for various quantities like \( \Gamma_t, v, P_{v,v} \) are proved. Moreover, we consider the higher-order energy estimate using \( D_t \) in Section 5. In Section 6, we present the precise main theorem in our paper and show the local well-posedness.
1.2 Notations

- $X$ stands for a point in $\Omega_t \subset \mathbb{R}^2$. $p_l, p_r$ are the left and right contact points. $n_j(j = t, b)$ are the unit outward normal vectors on $\Gamma_j$, and $\tau_j$ are the corresponding unit tangential vectors obeying the right-hand rule with $n_j$.
- $\sigma$ is the surface tension coefficient. $\beta_j$ is the effective friction coefficient determined by interfacial widths, interactions between the fluid and the bottom, and the normal stress contributions.
- $\chi_\omega$ is a characteristic function of contact angles:

$$\chi_\omega(\theta) = \begin{cases} 
1, & \theta \in (\pi/3, \pi/2), \\
0, & \theta \in (0, \pi/3]. 
\end{cases}$$

- $\chi_i$ $(i = l, r)$ are cut-off functions near the corner points $p_i$:

$$\chi_i(X) = \begin{cases} 
1, & |X - p_i| \leq r_0, \ X \in \Omega_i; \\
0, & \text{otherwise}
\end{cases}$$

with some small $r_0 > 0$.
- $f|_i = \chi_i(f|_{p_l}) + \chi_i(f|_{p_r})$ stands for taking values of $f$ at the corner points.
- $S_{\Omega_t}$ are straightened sector of $\Omega_t$ with radius $r_0 > 0$ near the corner points $p_i$.
- $D_t = \partial_t + \nabla_t$ is the material derivative.
- $M^*$ denotes the transport of a matrix $M$.
- $w^\pm$ on $\Gamma$: $w \cdot n_i$ for a vector $w \in T_X\Gamma_i$.
- $w^\top$ on $\Gamma$: $(w \cdot \tau_t) \tau_t$. Sometimes we also use $w^\top$ on $\Gamma_b$ with a similar definition.
- $\Pi$ is the second fundamental form on $\Gamma_i$, where $\Pi(w) = \nabla_w h_t = T_X\Gamma_i$ for $w \in T_X\Gamma_i$.
- $\kappa = tr\Pi = \nabla_\tau n_i \cdot \tau_t$ is the mean curvature of the surface $\Gamma_i$.
- We define on $\Gamma_t$ that $Dw = Dw_t = (\nabla_\tau w)^\top = (\nabla_\tau w \cdot \tau_t)\tau_t$ for a vector $w \in T_X\Gamma_t$.
- $D^2f(\tau_1, \tau_2) = D^2f(\tau_1) - (\Pi(\tau_1) \cdot \tau_2)\nabla_{\tau_1} f$ for any two vector $\tau_1, \tau_2 \in T_X\Gamma_t$.
- $\Delta_{\Gamma_t}$ is the Beltrami-Laplace operator on $\Gamma_t$:

$$\Delta_{\Gamma_t} f = D^2 f(\tau_t, \tau_t) = \nabla_\tau \nabla_\tau f - (\nabla_\tau \tau_t)^\top f.$$ 

- $H(f)$ or $f_H$ is the harmonic extension for some function $f$ on $\Gamma_t$, which is defined by the elliptic system

$$\begin{cases} 
\Delta H(f) = 0 & \text{in } \Omega_t, \\
H(f)|_{\Gamma_t} = f, \quad \nabla_{n_b} H(f)|_{\Gamma_b} = 0.
\end{cases}$$

- $\mathcal{N} = \nabla_{n_b} H$ is the Dirichlet-Neumann operator on $\Gamma_t$.
- $\Delta^{-1}(h, g)$ stands for the solution $u$ to the system

$$\begin{cases} 
\Delta u = h & \text{in } \Omega_t, \\
u|_{\Gamma_t} = 0, \quad \nabla_{n_b} u|_{\Gamma_b} = g.
\end{cases}$$

- The Sobolev norm $H^s$ for the boundary $\Gamma_t$ or $\Gamma_b$ is defined by local coordinates and local graphs.
- $\dot{H}^{1/2}(\Gamma_j)$ $(j = t, b)$ (see [30]) is a subspace of $H^{1/2}(\Gamma_j)$ related to corner domains

$$\dot{H}^{1/2}(\Gamma_j) = \left\{ u \in H^{1/2}(\Gamma_j) \mid \rho_i^{-1/2} u \in L^2(\Gamma_j), \ i = l, r \right\}$$

where $H^{1/2}(\Gamma_j)$ is the closure of $\mathcal{D}(\Gamma_j)$ in $H^{1/2}(\Gamma_j)$, and $\rho_i = \rho_i(X)$ $(i = l, r)$ is the distance (arc length) between the point $X \in \Gamma_j$ and the end $p_i$. We define the norm

$$\|u\|_{\dot{H}^{1/2}(\Gamma_j)}^2 = \|u\|_{H^{1/2}(\Gamma_j)}^2 + \int_{\Gamma_j} \rho_i^{-1} |u|^2 dX + \int_{\Gamma_j} \rho_r^{-1} |u|^2 dX.$$
Lemma 2.3 (Hardy inequalities) Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) with a Lipschitz boundary. Moreover, one also has

\[
W^{s,p}(\Omega) \subseteq L^q(\Omega)
\]

for \( 1/q = 1/p - s/n \) and \( \Omega \) any bounded open subset of \( \mathbb{R}^n \) with a Lipschitz boundary. Moreover, one also has

\[
W^{s,p}(\Omega) \subseteq W^{t,q}(\bar{\Omega})
\]

for \( t \leq s, q \geq p \) such that \( s - n/p = t - n/q \).

Lemma 2.2 (Product estimates) (1) For functions \( f \in H^{1/2}(\Omega_1) \) and \( g \in H^1(\Omega_1) \cup L^\infty(\Omega_1) \), one has the following product estimate:

\[
\|fg\|_{H^{1/2}(\Omega_1)} \leq C\|f\|_{H^{1/2}(\Omega_1)}(\|g\|_{H^1(\Omega_1)} + \|g\|_{L^\infty(\Omega_1)})
\]

with a constant \( C \) independent of \( f, g \);

(2) For functions \( f, g \in L^\infty(\Gamma_1) \cup H^{1/2}(\Gamma_1) \), one has

\[
\|fg\|_{H^{1/2}(\Gamma_1)} \leq C(\|f\|_{L^\infty(\Gamma_1)}\|g\|_{H^{1/2}(\Gamma_1)} + \|f\|_{H^{1/2}(\Gamma_1)}\|g\|_{L^\infty(\Gamma_1)})
\]

with a constant \( C \) independent of \( f, g \).

Proof. (1) In fact, one firstly extends \( f, g \) to be defined on the full plane \( \mathbb{R}^2 \) with a control of their corresponding norms. Secondly, one can apply standard para-product analysis to prove the estimate on \( \mathbb{R}^2 \). The details are omitted here. (2) The proof is similar to the proof of (1). \( \square \)

We quote some Hardy inequalities here.

Lemma 2.3 (Hardy inequalities) (1) (Corollary 2.3[15]) Let \( f \in H^1(0,d) \cup C^0(0,d) \) with \( d > 0 \) and \( f(0) = 0 \). Then there exists a positive constant \( C = C(\epsilon, d) \) such that

\[
\int_0^d r^{-2\epsilon} |f(r)|^2 dr \leq C \int_0^d r^{-2\epsilon+2} |f'(r)|^2 dr
\]

for \( \epsilon \in (1/2, 1) \);

(2) (Fractional-order version, see [26, 29]) For a number \( \epsilon \in (0, 1) \) and any function \( f \in H^\epsilon(\mathbb{R}^+) \) with \( f(0) = 0 \), there exists a positive constant \( C = C(\epsilon) \) such that

\[
\int_{\mathbb{R}^+} r^{-2\epsilon} |f(r)|^2 dr \leq C \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{|f(r) - f(\rho)|^2}{|r - \rho|^{1+2\epsilon}} dr d\rho.
\]
Lemma 2.4 (Traces on $\Gamma_t$ or $\Gamma_b$, Theorem 5.3 [52]) The maps

$$u \mapsto \{u, \nabla_n u\}|_{\Gamma_j}, \quad \text{for} \quad j = t, b$$

have unique continuous extensions as operators from $H^s(\Omega_t)$ onto $\Pi_{i=1}^1 H^{s-i-1/2}(\Gamma_j)$ for $s > 3/2$.

Moreover, one has the estimate:

$$\|u\|_{H^{s-1/2}(\Gamma_j)} + \|\nabla_n u\|_{H^{s-1/2}(\Gamma_j)} \leq C(\|\Gamma_i\|_{H^{s-1/2}})\|u\|_{H^s(\Omega_t)}.$$

Next, we present some special trace theorems on corner domains involving $\tilde{H}^{1/2}(\Gamma_j)$ and $\tilde{H}^{-1/2}(\Gamma_j)$ ($j = t, b$).

Lemma 2.5 Assume that $u|_{\Gamma_t} = f$ with $f|_{p_i} = 0$ ($i = l, r$) for a function $u \in H^1(\Omega_t)$. Then one has $f \in \tilde{H}^{1/2}(\Gamma_t)$ and

$$\|f\|_{\tilde{H}^{1/2}(\Gamma_t)} \leq C(\|\Gamma_i\|_{H^{5/2}})\|u\|_{H^1(\Omega_t)}.$$

The case on $\Gamma_b$ holds similarly.

Proof. First, one has $f \in H^{1/2}(\Gamma_t)$ immediately by Lemma 2.4. Second, noticing $f|_{p_i} = 0$ and applying Lemma 2.3(2) with some straightening localizations near $p_i$, one can see that $f \in H^{1/2}(\Gamma_t)$ with the desired estimate. Moreover, a similar lemma can be found as Lemma 5.5 [52].

Lemma 2.6 (Lemma 5.6[52]) Let $u \in H^{1/2}(\Omega_t)$ ($j = t, b$), then $\nabla_{\tau_j} u$ belongs to $\tilde{H}^{-1/2}(\Gamma_j)$ and satisfies the estimate

$$\|\nabla_{\tau_j} u\|_{\tilde{H}^{-1/2}(\Gamma_j)} \leq C(\|\Gamma_j\|_{H^{5/2}})\|u\|_{H^{1/2}(\Omega_t)}.$$

We now recall some elliptic systems and estimates including singular decompositions in corner domains. First, for the mixed-boundary system

$$(\text{MBVP}) \quad \begin{cases}
\Delta u = h, & \text{in} \quad \Omega_t \\
u|_{\Gamma_t} = f, & \nabla_n u|_{\Gamma_b} = g,
\end{cases}$$

we quote directly the following variational result.

Lemma 2.7 (Lemma 5.9 [52]) For a given function $f \in H^{1/2}(\Gamma_t)$, the system

$$\begin{cases}
\Delta \mathcal{H}(f) = 0 & \text{in} \quad \Omega_t \\
\mathcal{H}(f)|_{\Gamma_t} = f, \quad \nabla_n \mathcal{H}(f)|_{\Gamma_b} = 0
\end{cases}$$

(2.11)

admits a unique solution $f_\mathcal{H} = \mathcal{H}(f) \in H^1(\Omega_t)$, and there holds

$$\|\mathcal{H}(f)\|_{H^1(\Omega_t)} \leq C(\|\Gamma_i\|_{H^{5/2}})\|f\|_{H^{1/2}(\Gamma_t)}.$$

The following proposition shows the existence and estimate for solutions in $H^2(\Omega_t)$. Notice that when the contact angles are below $\pi/2$, no singularity appears.

Proposition 2.1 Let $h \in L^2(\Omega_t)$, $f \in H^{3/2}(\Gamma_t)$, $g \in H^{1/2}(\Omega_b)$ and $\Gamma_t \in H^{5/2}$ be given in (MBVP). The contact angles $\omega_i \in (0, \pi/2)$. Then there exists a unique solution $u \in H^2(\Omega_t)$ to (MBVP). Moreover, one has

$$\|u\|_{H^2(\Omega_t)} \leq C(\|h\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\Gamma_t)} + \|g\|_{H^{1/2}(\Gamma_b)}).$$

with the constant $C = C(\|\Gamma_i\|_{H^{5/2}})$. 


As a result, the desired estimate in (2) can be proved by an interpolation.

Remark 2.1

the norm $H^4$. Similarly to the proof for Proposition 5.13 in [51], one has an estimate from Theorem 3.2.5 [10] and Proposition 5.1 [51], Theorem 4.3.1.4 [30].

Proposition 2.3

Let $C$ with the constant $\omega$ angles $(NBVP)$ such that $T_u$ and the regular part $u_{\delta}$ satisfying the compatibility condition since the most singular part for (NBVP) behaves like $r^{s}$ (up to an additive constant) variational solution $u_h$ and the regular part $u_{\delta}$ for a constant $s \in [0,1]$.

Proof. We only write a sketch for the proof here, since one can find similar details in our previous papers [51, 52, 53]. First, the existence of the variational solution $u \in H^1(\Omega_h)$ can be proved directly by a standard variation procedure based on Lemma 4.4.3.1 [30]. Note that here one doesn’t require that the contact angles are below $\pi/2$. Second, one needs to notice that, there is no singular part since the most singular part for (NBVP) behaves like $r^{s}$ near the corners (see for example [30]). Therefore, one can have directly $H^2$ estimate from Theorem 3.2.5 [10] and Proposition 5.1 [51], Theorem 4.3.1.4 [30]. Similarly to the proof for Proposition 5.13 [51], one has $H^3$ estimate without singular part. As a result, the desired estimate in (2) can be proved by an interpolation.

Remark 2.1 For the first estimate in Proposition 2.2, when one has additionally

$$\int_{\Omega_i} u dX = 0,$$

the norm $\|u\|_{L^2(\Omega_i)}$ can be deleted from the right side during the proof.

Moreover, we present the $H^4$ singularity decomposition and estimate.

Proposition 2.3 Let $h \in H^2(\Omega_h)$, $f \in H^{5/2}(\Gamma_0)$, $g \in H^{5/2}(\Gamma_b)$ and $\Gamma_i \in H^4$ in (NBVP). The contact angles $\omega_i \in (0, \pi/2)$. Then there exists a unique (up to an additive constant) solution $u \in H^3(\Omega_h)$ to (NBVP) such that

$$u = u_r + u_s \quad \text{with the singular part} \quad u_s = \chi_{\omega}(\omega_i) c_l S_l \circ T_l + \chi_{\omega}(\omega_r) c_r S_r \circ T_r,$$

and the regular part $u_r \in H^4(\Omega_h)$. Here the cut-off functions $\chi_{\omega}, \chi_i$ are defined in the notation part. $T_i \in H^4(\Omega_i)$ are boundary-straightening diffeomorphisms from $\Omega_i$ onto the sectors $S_i$, near the corner points $p_i$ (from [51]), and $S_i = r^{\pi/\omega_i} s(\theta)$ with $r$ the radius with respect to $p_i$ in $S_i$, and $s_i(\theta)$ some fixed sine or cosine functions.

Moreover, one has estimates for the singular coefficients $c_i$ ($i = l, r$) and the regular part $u_r$:

$$|c_l| + \|u_r\|_{H^4(\Omega_h)} \leq C \left( \|h\|_{H^3(\Omega_h)} + \|f\|_{H^{5/2}(\Gamma_0)} + \|g\|_{H^{5/2}(\Gamma_b)} + \|u\|_{L^2(\Omega_i)} \right)$$

with the constant $C = C(\|\Gamma_i\|_{H^4})$. 

Proof. This proposition is a direct conclusion from Proposition 5.1, Lemma 5.2 and Theorem 5.3 in [51].

Second, we consider the Neumann-boundary system:

\[(NBVP) \begin{cases}
\Delta u = h & \text{in } \Omega_h, \\
\nabla_m u|_{\Gamma_i} = f, \quad \nabla_m u|_{\Gamma_b} = g
\end{cases}\] (2.12)

satisfying the compatibility condition

$$\int_{\Omega_i} h dX = \int_{\Gamma_i} f ds + \int_{\Gamma_b} g ds.$$
In this section, we derive the equation for a good unknown with \( D \). The solution \( u \) to (NBVP) with the same right side will be in \( H^4(\Omega_t) \) with the corresponding estimate. Remark 2.2 A direct conclusion from this proposition is that when the contact angles \( \omega_i \in (0, \pi/3) \), the solution \( u \) to (NBVP) with the same right side will be in \( H^4(\Omega_t) \) with the corresponding estimate.

**Remark 2.3** Based on the proposition above and estimate (9.15) in [24], when the contact angles \( \omega_i \in (\pi/3, \pi/2) \), one can have a more delicate and also natural estimate for (NBVP) with \( h \in H^{1+\epsilon}(\Omega_t) \), \( f \in H^{3/2+\epsilon}(\Gamma_t) \), \( g \in H^{3/2+\epsilon}(\Gamma_b) \) and \( \Gamma_t \in H^4 \):

\[
\|u\|_{H^{3+\epsilon}(\Omega_t)} \leq C(\|\tau\|_{H^4} (\|h\|_{H^{1+\epsilon}(\Omega_t)} + \|f\|_{H^{3/2+\epsilon}(\Gamma_t)} + \|g\|_{H^{3/2+\epsilon}(\Gamma_b)} + \|u\|_{L^2(\Gamma_t)}) ,
\]

where \( \epsilon \in (0, \pi/\omega - 2) \subset (0, 1) \).

In the end, we recall some useful expressions and commutators from [60, 52, 53].

- \( D_t n_t \) and \( D_t \tau_t \). One has

\[
D_t n_t = -((\nabla v)^t n_t)^\top, \quad D_t \tau_t = (\nabla \tau_t v \cdot n_t)n_t. \tag{2.13}
\]

- \([D_t, H]\). One has for a smooth function \( f \) on \( \Gamma_t \) that

\[
[D_t, H]f = \Delta^{-1}(2\nabla v \cdot \nabla^2 f_H + \Delta v \cdot \nabla f_H, (\nabla N_b v - \nabla v N_b) \cdot \nabla f_H) \quad \text{in} \quad \Omega_t, \tag{2.14}
\]

where \( \Delta^{-1}(h, g) \) and \( H \) are defined in the notation part.

- \([D_t, N]\). One has

\[
[D_t, N]f = \nabla n_t \Delta^{-1}(2\nabla v \cdot \nabla^2 f_H + \Delta v \cdot \nabla f_H, (\nabla n_t v - \nabla v n_t) \cdot \nabla f_H)
\]

\[
- \nabla n_t v \cdot \nabla f_H - (\nabla f_H)^\top \cdot n_t \quad \text{on} \quad \Gamma_t. \tag{2.15}
\]

- \([D_t, \Delta_{\Gamma_t}]\). For a smooth function \( f \) on \( \Gamma_t \), there holds

\[
[D_t, \Delta_{\Gamma_t}]f = 2D^2 f(\tau_t, (\nabla \tau_t v)^\top) - (\nabla f)^\top \cdot \Delta_{\Gamma_t} v + \kappa(\nabla f)^\top \cdot n_t \quad \text{on} \quad \Gamma_t. \tag{2.16}
\]

- \([D_t, \Delta^{-1}]\). We have

\[
D_t \Delta^{-1}(h, g) = \Delta^{-1}(D_t h, D_t g) + \Delta^{-1}(h_1, g_1) \tag{2.17}
\]

with

\[
h_1 = 2\nabla v \cdot \nabla^2 \Delta^{-1}(h, g) + \Delta v \cdot \nabla \Delta^{-1}(h, g), \quad g_1 = (\nabla N_b v - \nabla v N_b) \cdot \nabla \Delta^{-1}(h, g).
\]

- \([D_t, \nabla \tau_t]\). Direct computations lead to

\[
[D_t, \nabla \tau_t] = (D_t \tau_t - \nabla \tau_t v) \cdot \nabla = (n_t \nabla \tau_t v \cdot n_t - \nabla \tau_t v) \cdot \nabla = -(\nabla \tau_t v \cdot \tau_t) \nabla \tau_t. \tag{2.18}
\]

### 3 Reformulation of the problem

In this section, we derive the equation for a good unknown \( \mathfrak{F} \), which is slightly different from the quantity \( J = \nabla \kappa \mathcal{H} \) introduced in [60] and used in our previous papers [52, 53].

To begin with, we define \( P_{v,v} \) by the following Neumann-boundary system:

\[
\begin{aligned}
\Delta P_{v,v} &= -\text{tr}(\nabla v \nabla v) \quad \text{in} \quad \Omega_t \\
\nabla n_t P_{v,v}|_{\Gamma_t} &= C_{v,v}(t), \quad \nabla n_b P_{v,v}|_{\Gamma_b} = v \cdot \nabla v n_b.
\end{aligned} \tag{3.19}
\]
where $C_{v,v}(t)$ satisfies
\[ |\Gamma_t| C_{v,v}(t) = - \int_{\Omega_t} tr(\nabla v \nabla v) dX - \int_{\Gamma_k} v \cdot \nabla v n_b ds. \]

Moreover, due to the non-uniqueness of the variational solution to this Neumann problem, we assume that $P_{v,v}$ satisfies
\[ \int_{\Omega_t} P_{v,v} dX = 0. \tag{3.20} \]
In this way, we will have a unique solution to (3.19) in Sobolev space.

Let the pressure $P$ be decomposed into
\[ P = \mathfrak{R}_H + P_{v,v}, \tag{3.21} \]
where $\mathfrak{R}$ is the modified curvature on $\Gamma_t$ defined by
\[ \mathfrak{R} = \sigma \kappa - P_{v,v} \quad \text{on} \quad \Gamma_t. \tag{3.22} \]

Now, we can define the new quantity
\[ \mathfrak{J} = \nabla \mathfrak{R}_H \]
and we are ready to derive its equation from (WW). In fact, since the derivation for the equation of $\mathfrak{J}$ is much similar to the derivation for $J$ in [52], many details are omitted here.

First, recall the equation for the curvature $\kappa$ from [52]:
\[ D_t \kappa = -\Delta_{\Gamma_t} v + 2 \Pi(\tau_t) \cdot \nabla_{\tau_t} v \quad \text{on} \quad \Gamma_t, \tag{3.24} \]
Applying $D_t$ to (3.24) and using (3.22), we have
\[ D_t^2 \mathfrak{R} = -\sigma \Delta_{\Gamma_t} D_t v \cdot n_t - 2 \Pi(\tau_t) \cdot \nabla_{\tau_t} J + \sigma R_1 - D_t^2 P_{v,v} \quad \text{on} \quad \Gamma_t \tag{3.25} \]
with
\[ R_1 = -[D_t, \Delta_{\Gamma_t}] v \cdot n_t - \Delta_{\Gamma_t} v \cdot D_t n_t + 2 \Pi(\tau_t) \cdot \nabla_{\tau_t} \nabla P_{v,v} - 2 \Pi(\tau_t) \cdot [D_t, \nabla_{\tau_t}] v - 2 D_t(\Pi(\tau_t)) \cdot \nabla_{\tau_t} v. \]
Moreover, direct computations almost the same as in [52] lead to
\[ D_t^2 \mathfrak{J} = \nabla H(D_t^2 \mathfrak{R}) + A_1 + A_2 + A_3 \tag{3.26} \]
where the remainder terms
\[ A_1 = \nabla[D_t, H] D_t \mathfrak{R} = \nabla \Delta^{-1} (2 \nabla v \cdot \nabla + \Delta v, (\nabla n_b v - \nabla v n_b) \cdot ) \]
\[ \left(D_t \mathfrak{J} - \Delta^{-1} (2 \nabla v \cdot \nabla \mathfrak{J} + \Delta v \cdot \mathfrak{J}, (\nabla n_b v - \nabla v n_b) \cdot ) \right) + (\nabla v)^* \mathfrak{J}, \tag{3.27} \]
\[ A_2 = \nabla D_t \Delta^{-1} (2 \nabla v \cdot \nabla \mathfrak{J} + \Delta v \cdot \mathfrak{J}, (\nabla n_b v - \nabla v n_b) \cdot ) \]
\[ = \nabla \Delta^{-1} (2 \nabla v \cdot \nabla^2 w_{A2} + \Delta v \cdot \nabla w_{A2}, (\nabla n_b v - \nabla v n_b) \cdot ) \cdot \nabla w_{A2} + \nabla \Delta^{-1} (h_{A2}, g_{A2}) \tag{3.28} \]
with
\[ w_{A2} = \Delta^{-1} (2 \nabla v \cdot \nabla \mathfrak{J} + \Delta v \cdot \mathfrak{J}, (\nabla n_b v - \nabla v n_b) \cdot ), \]
\[ h_{A2} = 2 \nabla v \cdot \left( \nabla D_t \mathfrak{J} - (\nabla v)^* \mathfrak{J} \right) + 2 (\nabla D_t v - (\nabla v)^* \nabla v) \cdot \nabla \mathfrak{J} + D_t \mathfrak{J} \cdot \Delta v \]
\[ + \mathfrak{J} \cdot (\Delta D_t v - \Delta v \cdot \nabla v - 2 \nabla v \cdot \nabla^2 v), \]
\[ g_{A2} = (\nabla n_b v - \nabla n_b v) \cdot \nabla v \cdot \mathfrak{J} + \nabla n_b D_t v \cdot \mathfrak{J} - (D_t v - \nabla v) \cdot \nabla n_b \cdot \mathfrak{J} \]
\[ + \nabla v ((\nabla v)^* \mathfrak{J})^\top \cdot \mathfrak{J} + (\nabla n_b v - \nabla v n_b) \cdot D_t \mathfrak{J}, \]
and
\[ A_3 = -2(\nabla v)^*D_t\mathcal{J} - (\nabla D_t v)^*\mathcal{J} - ((\nabla v)^2)^*\mathcal{J} + ((\nabla v)^*\nabla v)\mathcal{J}. \]  
(3.29)

Here we note that the leading-order terms in \( A_1, A_2 \) are like \( \mathcal{J}, D_t\mathcal{J}, \nabla v, \nabla D_t v \).

Substituting (3.25) into (3.26) and applying Euler’s equation, we can have after some more computations that
\[ D_t^2\mathcal{J} = \sigma\nabla\mathcal{H}(\Delta_{\Gamma_t}\mathcal{J}) + R_0 + D_t\nabla\mathcal{H}(D_tP_{v,v}). \]  
(3.30)

with
\[ R_0 = -\sigma\nabla\mathcal{H}(\mathcal{J} \cdot \Delta_{\Gamma_t} n_t) + \sigma\nabla\mathcal{H}(n_t \cdot \Delta_{\Gamma_t} \nabla P_{v,v}) + \sigma\nabla\mathcal{H}(R_1) + A_1 + A_2 + A_3 + [D_t, \nabla\mathcal{H}](D_tP_{v,v}). \]

Here, one observation is that since \( \nabla n_t P_{v,v}|_{\Gamma_t} = C_v(t) \) from (3.19), we find immediately in \( R_0 \) that
\[ n_t \cdot \Delta_{\Gamma_t} \nabla P_{v,v} = |n_t, \Delta_{\Gamma_t}| \cdot \nabla P_{v,v}. \]

Moreover, we use the following Hodge decomposition from [60, 52]:
\[ D_t\mathcal{J} = D_t\mathcal{J} + \nabla P_{3,v} \]  
(3.31)
such that \( D_t\mathcal{J} \) satisfies
\[ \nabla \cdot D_t\mathcal{J} = 0 \quad \text{on } \Omega_t, \quad \text{and } n_b : D_t\mathcal{J}|_{\Gamma_b} = 0, \]
and \( P_{3,v} \) satisfies the Neumann-boundary system
\[ \begin{cases} 
\Delta P_{3,v} = -tr(\nabla \mathcal{J} \nabla v) & \text{in } \Omega_t \\
\nabla n_b P_{3,v}|_{\Gamma_b} = C_{\mathcal{J},v}(t) - \nabla \tau_{\tau_v} R_{\tau_v} v \cdot n_t, & \nabla n_b P_{3,v}|_{\Gamma_b} = \mathcal{J} \cdot \nabla_v n_b 
\end{cases} \]  
(3.32)

with
\[ |\Gamma_t| C_{\mathcal{J},v}(t) = -\int_{\Omega_t} tr(\nabla \mathcal{J} \nabla v)dX + \int_{\Gamma_t} \nabla \tau_{\tau_v} R_{\tau_v} v \cdot n_t ds - \int_{\Gamma_b} \mathcal{J} \cdot \nabla_v n_b ds. \]

Here the assumption
\[ \int_{\Omega_t} P_{3,v} dX = 0 \]
holds similarly to guarantee the uniqueness as for \( P_{v,v} \) system.

This implies immediately
\[ D_t D_t\mathcal{J} = D_t(D_t\mathcal{J} + \nabla P_{3,v}) = D_t^2\mathcal{J} + D_t\nabla P_{3,v}. \]

As a result, we finally conclude the equation for \( \mathcal{J} \) in the following form based on (3.30):
\[ D_t\left[ D_t\mathcal{J} - \nabla\mathcal{H}(D_tP_{v,v} - v \cdot (\nabla P_{v,v}|_{\Gamma_c})) \right] + \sigma\mathcal{A}\mathcal{J} = \mathcal{R} \]  
(3.33)
where the third-order elliptic operator \( \mathcal{A} \) is defined in the same way as in [60, 52]:
\[ \mathcal{A}(w) = \nabla\mathcal{H}(\Delta_{\Gamma_t}(w|_{\Gamma_t})) \]
for any smooth-enough function \( w \) defined on \( \Omega_t \), and the remainder term \( \mathcal{R} \) is
\[ \mathcal{R} = R_0 + D_t\nabla P_{3,v} + D_t\nabla\mathcal{H}(v \cdot (\nabla P_{v,v}|_{\Gamma_c})). \]

Moreover, one can find the definition of \( \nabla P_{v,v}|_{\Gamma_c} \) in the notation part.

**Remark 3.1** Instead of (3.30), we write the equation of \( \mathcal{J} \) in a more complicated form (3.33). This happens because of technical reasons. In fact, \( D_t\mathcal{J} \) is more handy to use in the energy estimates, while as a price we have the remainder \( P_{3,v} \) part to deal with. Meanwhile, the part \( v \cdot (\nabla P_{v,v}|_{\Gamma_c}) \) is added to derive better estimate for \( D_tP_{v,v} \), see Lemma 4.7.
4 Lower-order energy estimates

The lower-order energy $E(t)$ is defined as

$$E(t) = \|\nabla_{\Gamma} J^{+} \|^2_{L^2(\Gamma_t)} + \|D_{\Gamma} J\|^2_{L^2(\Omega_t)} + \|\Gamma_t\|^2_{H^{5/2}} + \|v\|^2_{H^{3/2}(\Omega_t)},$$

where we write

$$E_i(t) = \|\Gamma_t\|^2_{H^{5/2}} + \|v\|^2_{H^{3/2}(\Omega_t)}.$$

Moreover, recalling from [52, 53] that our (WW) system has some dissipation at corner points, we have the following dissipation term at corner points

$$F(t) = \sum_{i=1, r} (\sin \omega_i) \nabla_{\Gamma_t} J^{+}_{|_{\Gamma_t}}.$$

**Theorem 4.1** Let the contact angles $\omega_i \in (0, \pi/2)$. Assume that $E(t)$, $\int_0^T F(t) dt$ are both bounded above in $[0, T]$ for some $T > 0$. Then the following a priori estimate holds for system (WW):

$$\sup_{0 \leq t \leq T} E(t) + \int_0^T F(t) dt \leq P(E(0)) + \int_0^T P(E(t)) dt,$$

where $P(\cdot)$ is some polynomial with positive constant coefficients depending on $\sigma, \beta, c$.

One can see immediately that, our energy is defined mainly in terms of $J$. As a result, we need to related $(v, P)$, $\Gamma_t$ and other quantities to $J$ firstly before we prove this energy estimate.

4.1 Dependence on $E(t)$

We show in this part that all the quantities related to our problem can be controlled by $E(t)$. To start with, the following proposition focuses on the estimates for $P_{v,v}, n_{\Gamma}, \mathfrak{R}_H, J$.

**Lemma 4.1** Assuming that $E(t) \in L^\infty[0, T]$ for some $T > 0$, one has the following estimates

$$\|P_{v,v}\|_{H^2(\Omega_t)} \leq P(E(t))$$

and

$$\|n_{\Gamma}\|_{H^2(\Gamma_t)} + \|J\|_{H^{3/2}(\Omega_t)} + \|\mathfrak{R}_H\|_{H^{5/2}(\Omega_t)} \leq P(E(t)).$$

**Proof.** (1) The first inequality. Applying Proposition 2.2 with an interpolation to $P_{v,v}$ system (3.19), we get

$$\|P_{v,v}\|_{H^2(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{5/2}}) \left( \|tr(\nabla v \nabla v)\|_{L^2(\Omega_t)} + \|\Gamma_t\|^{1/2} |C_{v,v}(t)| + \|v \cdot n_{\Gamma_b}\|_{H^{3/2}(\Gamma_b)} + \|P_{v,v}\|_{L^2(\Omega_t)} \right).$$

For the right-hand side, we have firstly by Lemma 2.1 that

$$\|tr(\nabla v \nabla v)\|_{L^2(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{5/2}}) \|v\|^2_{H^{3/2}(\Omega_t)}.$$

Second, it’s straightforward to find

$$|C_{v,v}(t)| \leq C\|v\|^2_{H^{3/2}(\Omega_t)}$$

and

$$\|P_{v,v}\|_{L^2(\Omega_t)} = \|P_{v,v} - \int_{\Omega_{\Gamma_t}} P_{v,v} dX\|_{L^2(\Omega_t)} \leq C\|\nabla P_{v,v}\|_{L^2(\Omega_t)}$$

$$\leq C(\|\Gamma_t\|_{H^{5/2}}) \left( \|tr(\nabla v \nabla v)\|_{L^2(\Omega_t)} + \|\Gamma_t\|^{1/2} |C_{v,v}(t)| + \|v \cdot n_{\Gamma_b}\|_{L^2(\Gamma_b)} \right).$$
As a result, the proof is finished thanks to the fact

\[ \|P_v\|_{H^2(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{5/2}})\|v\|_{H^{3/2}(\Omega_t)} \leq P(E(t)). \]

(2) The second inequality. In fact, the key point of the proof lies in the estimates for mean curvature \( \kappa \) and \( \mathcal{R} = \sigma \kappa - P_{v,v} \).

To begin with, one can see immediately that \( \mathcal{R} \) satisfies the system

\[
\begin{aligned}
\Delta \mathcal{R} = 0 & \quad \text{in } \Omega_t \\
\nabla_n \mathcal{R} |_{\Gamma_t} = 3^+ |_{\Gamma_t} & \in H^1(\Gamma_t), \\
\nabla_n \mathcal{R} |_{\Gamma_b} = 0.
\end{aligned}
\]

Applying Proposition 2.2(2), we find

\[
\|\mathcal{R}\|_{H^{5/2}(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{5/2}})(\|3^+\|_{H^1(\Gamma_t)} + \|\mathcal{R}\|_{L^2(\Omega_t)})
\]

\[
\leq C(\|\Gamma_t\|_{H^{5/2}})(\|\nabla_\Gamma 3^+\|_{L^2(\Gamma_t)} + \|\mathcal{R}\|_{L^2(\Omega_t)}),
\]

where interpolation for \( \mathcal{R} \) is applied to \( \|3^+\|_{L^2} \).

Moreover, from the definition of \( \mathcal{R} \) and Lemma 2.7 we have

\[
\|\mathcal{R}\|_{L^2(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{5/2}})(\|\kappa\|_{H^{1/2}(\Gamma_t)} + \|P_{v,v}\|_{H^{1/2}(\Gamma_t)}),
\]

which together with the previous inequality leads to

\[
\|\mathcal{R}\|_{H^{5/2}(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{5/2}})(1 + \|\nabla_\Gamma 3^+\|_{L^2(\Gamma_t)} + \|P_{v,v}\|_{H^{1/2}(\Omega_t)}).
\]

This implies immediately the desired \( H^{3/2} \) estimate for \( \mathcal{R} \).

For the estimate for \( n_t \), we find by Lemma 2.4, (4.35) and part (1) that

\[
\|\kappa\|_{H^1(\Gamma_t)} \leq \sigma^{-1}(\|\mathcal{R}\|_{H^1(\Gamma_t)} + \|P_{v,v}\|_{H^1(\Gamma_t)})
\]

\[
\leq C(\|\Gamma_t\|_{H^{5/2}})(1 + \|\nabla_\Gamma 3^+\|_{L^2(\Gamma_t)} + \|P_{v,v}\|_{H^{1/2}(\Omega_t)}) \leq P(E(t)).
\]

As a result, the proof is finished thanks to the fact

\[ \kappa = \nabla_\Gamma n_t \cdot \tau_t \quad \text{where } \nabla_\Gamma n_t \parallel \tau_t \quad \text{on } \Gamma_t. \]

\[ \square \]

**Lemma 4.2** Assuming that \( E(t) \in L^\infty[0,T] \) for some \( T > 0 \), one has the following estimates for \( v \):

\[ \|v\|_{H^2(\Omega_t)} + \|v^+\|_{H^{5/2}(\Gamma_t)} \leq P(E(t)). \]

Meanwhile, one also has

\[ \|D_tP_{v,v}\|_{H^1(\Omega_t)} + \|P_{3,v}\|_{H^{1/2}(\Omega_t)} \leq P(E(t)) \]

**Proof.** - \( H^1 \) estimate for \( P_{3,v} \). From system (3.32) of \( P_{3,v} \), we can have the following variational estimate by Proposition 2.2(1) and Remark 2.1:

\[
\|P_{3,v}\|_{H^1(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{5/2}})(\|\text{tr} (\nabla v)\|_{L^2(\Omega_t)} + |\Gamma_t|^{1/2} |C_{3,v}(t)| + \|\nabla_\tau v \cdot \nabla_\tau v \cdot n_t\|_{L^2(\Gamma_t)}
\]

\[ + \|3^+ \cdot \nabla_\tau n_t\|_{L^2(\Gamma_t)}, \]

which together with Lemma 4.1 leads to the desired estimate for \( P_{3,v} \).

- \( v^+ \) estimate. In fact, it is straightforward to get by Lemma 2.4 that

\[
\|D_t\kappa\|_{H^{1/2}(\Gamma_t)} \leq C(\|\Gamma_t\|_{H^{5/2}})D_t\kappa\|_{H^1(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{5/2}})(\|\nabla D_t\kappa\|_{L^2(\Omega_t)} + \|D_t\kappa\|_{L^2(\Gamma_t)}).
\]
Rewriting $\nabla \kappa_t$ in terms of $\mathfrak{J}, \nabla P_{v,v}$ leads to

$$
\|D_t \kappa\|_{H^{1/2}(\Gamma_t)} \leq C(\|\Gamma_t\|_{H^{5/2}})(\|D_t \mathfrak{J}\|_{L^2(\Omega_t)} + \|D_t \nabla P_{v,v}\|_{L^2(\Omega_t)} + \|\nabla v \cdot \mathfrak{J}\|_{L^2(\Omega_t)} + \|\nabla v \cdot \nabla P_{v,v}\|_{L^2(\Omega_t)} + \|D_t \kappa\|_{L^2(\Gamma_t)})
$$

\[\leq P(E(t))(1 + \|\nabla P_{3,v}\|_{L^2(\Omega_t)} + \|D_t P_{v,v}\|_{H^1(\Omega_t)} + \|D_t \kappa\|_{L^2(\Gamma_t)}).
\]

Applying Lemma 4.1, we have

$$
\|D_t \kappa\|_{H^{1/2}(\Gamma_t)} \leq P(E(t))(1 + \|\nabla P_{3,v}\|_{L^2(\Omega_t)} + \|D_t P_{v,v}\|_{H^1(\Omega_t)} + \|D_t \kappa\|_{L^2(\Gamma_t)}).
$$

On the other hand, we rewrite (3.23) into

$$
\Delta_t v^+ = -D_t \kappa - v^+ |\nabla \tau^+ n_t|^2 + (\nabla \tau^+ n_t) \cdot \tau_t - (\nabla \tau^+ v^+) \cdot \nabla n_t \cdot \tau_t,
$$

which together with the inequality above for $\|D_t \kappa\|_{H^{1/2}(\Gamma_t)}$ and the estimate for $P_{3,v}$ imply

$$
\|\Delta_t v^+\|_{H^{1/2}(\Gamma_t)} \leq C(\|\Gamma_t\|_{H^{5/2}}, \|n_t\|_{H^{5/2}(\Gamma_t)})(\|D_t \kappa\|_{H^{1/2}(\Gamma_t)} + \|v\|_{H^2(\Omega_t)})
$$

\[\leq P(E(t))(1 + \|\nabla P_{3,v}\|_{L^2(\Omega_t)} + \|D_t P_{v,v}\|_{H^1(\Omega_t)} + \|D_t \kappa\|_{L^2(\Gamma_t)} + \|v\|_{H^2(\Omega_t)})
(4.37)
\]

Here we use (3.24) to obtain

$$
\|D_t \kappa\|_{L^2(\Gamma_t)} \leq C(\|\Gamma_t\|_{H^{5/2}})\|v\|_{H^2(\Omega_t)}.
$$

As a result, the estimate for $v^+$ depends on the estimates for $\|D_t P_{v,v}\|_{H^1(\Omega_t)}, \|v\|_{H^2(\Omega_t)}$.

- $H^2$ estimate for $v$. Recalling system (4.42) for velocity potential $\phi$ with $v = \nabla \phi$, we have by Proposition 2.2 that

$$
\|v\|_{H^2(\Omega_t)} \leq \|\phi\|_{H^3(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{5/2}})(\|v^+\|_{H^{3/2}(\Gamma_t)} + \|\phi\|_{L^2(\Omega_t)})
$$

\[\leq C(\|\Gamma_t\|_{H^{5/2}})(\|v^+\|_{H^{3/2}(\Gamma_t)} + \|\phi\|_{L^2(\Omega_t)})
\]

\[\leq P(E(t))(1 + \|v^+\|_{H^{3/2}(\Gamma_t)}).
\]

Bringing the above estimate into (4.37) and applying interpolations to $\|v^+\|_{H^{3/2}(\Gamma_t)}$, we arrive at

$$
\|\Delta_t v^+\|_{H^{1/2}(\Gamma_t)} + \|v\|_{H^2(\Omega_t)} \leq P(E(t))(1 + \|D_t P_{v,v}\|_{H^1(\Omega_t)}).
(4.38)
$$

Therefore, it remains to deal with $\|D_t P_{v,v}\|_{H^1(\Omega_t)}$.

- $D_t P_{v,v}$ estimates. From the definition of $P_{v,v}$, we derive the system for $D_t P_{v,v}$:

$$
\begin{cases}
\Delta D_t P_{v,v} = -D_t tr(\nabla v \nabla v) + 2tr(\nabla v \nabla^2 P_{v,v}) & \text{in } \Omega_t \\
\nabla_n D_t P_{v,v}|_{\Gamma_t} = C_{v,v}(t) + \nabla_n v \cdot \nabla P_{v,v}|_{\Gamma_t}, & \nabla_n D_t P_{v,v}|_{\Gamma_b} = D_t(v \cdot \nabla n_b) + \nabla_n v \cdot \nabla P_{v,v}|_{\Gamma_b},
\end{cases}
$$

(4.39)

with

$$
\int_{\Omega_t} D_t P_{v,v} dX = 0.
$$

It is easy to see from Euler’s equation that

$$
D_t tr(\nabla v \nabla v) = -2tr(\nabla v \cdot \nabla v \nabla v) - 2tr(\nabla(\nabla P_{v,v} + \nabla R_H) \nabla v),
$$

(4.40)

and similar expression can be derived for $D_t(v \cdot \nabla n_b)$ on $\Gamma_b$.

Moreover, we have

$$
C_{v,v} = |\Gamma_t|^{-1} \int_{\Omega_t} 2tr(D_t \nabla v \nabla v)dX - |\Gamma_t|^{-2} \int_{\Gamma_t} tr(\nabla v \nabla v)dX,
$$

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where \( \frac{d}{dt} |\Gamma_t| = \int_{\Gamma_t} (v^+ + \nabla_v (v \cdot \tau_t)) ds \). So we obtain by Lemma 4.1, (4.35) and (4.36) that
\[
|C_{v,v} | \leq C(\|\Gamma_t\|_{H^{5/2}}, \|v\|_{H^{5/2}(\Omega_t)}, \|P_{v,v}\|_{H^2(\Omega_t)}, \|\mathcal{R}_H\|_{H^2(\Omega_t)}, \|\kappa\|_{L^2(\Gamma_t)}) \leq P(E(t)).
\] (4.41)

To prove \( H^1 \) estimate for \( D_t P_{v,v} \), we use the following form of variation equation:
\[
\int_{\Omega_t} |\nabla D_t P_{v,v}|^2 dX = - \int_{\Omega_t} (\Delta D_t P_{v,v}) D_t P_{v,v} dX + \int_{\Gamma_t} (\nabla_n D_t P_{v,v}) D_t P_{v,v} ds
\]
\[
+ \int_{\Gamma_h} (\nabla_{n_h} D_t P_{v,v}) D_t P_{v,v} ds.
\]
Substituting the expressions in (4.39) into the equality above, we can deal with the integrals one by one, where Lemma 2.1 is applied. For example, for the highest-order terms of \( v \) on the boundary, we use Green’s Formula and Lemma 2.1 to find
\[
\int_{\Gamma_t} \nabla_n v \cdot P_{v,v} D_t P_{v,v} ds + \int_{\Gamma_h} \nabla_{n_h} v \cdot P_{v,v} D_t P_{v,v} ds
\]
\[
= \int_{\Omega_t} tr(\nabla v \nabla^2 P_{v,v}) D_t P_{v,v} dX + \int_{\Omega_t} \nabla v \cdot \nabla P_{v,v} \nabla D_t P_{v,v} dX
\]
\[
\leq \|\nabla v\|_{L^4(\Omega_t)} \|\nabla^2 P_{v,v}\|_{L^4(\Omega_t)} \|D_t P_{v,v}\|_{L^4(\Omega_t)} + \|\nabla v\|_{L^4(\Omega_t)} \|\nabla P_{v,v}\|_{L^4(\Omega_t)} \|D_t P_{v,v}\|_{L^2(\Omega_t)}
\]
\[
\leq C(\|\Gamma_t\|_{H^{5/2}} \|v\|_{H^{5/2}(\Omega_t)} \|P_{v,v}\|_{H^2(\Omega_t)} \|D_t P_{v,v}\|_{H^1(\Omega_t)}).
\]
The other integrals above can be handled similarly and the details are omitted.

As a result, we obtain by Lemma 4.1 the following desired estimate:
\[
\|D_t P_{v,v}\|_{H^1(\Omega_t)} \leq P(E(t))(\|P_{v,v}\|_{H^2(\Omega_t)} + \|\mathcal{R}_H\|_{H^{5/2}(\Omega_t)} + |C_{v,v}|) \leq P(E(t)).
\]
Consequently, applying (4.38) leads to the estimates for \( v^+ \) and \( v \).

**Remark 4.1** We know immediately from the proof of Lemma 4.2 the following estimate
\[
\|D_t \kappa\|_{H^{1/2}(\Gamma_t)} \leq P(E(t)).
\]
Moreover, we also have
\[
\|D_t \mathcal{R}_H\|_{H^1(\Omega_t)} \leq \sigma \|D_t \kappa\|_{H^{1/2}(\Gamma_t)} + \|D_t P_{v,v}\|_{H^1(\Omega_t)} \leq P(E(t)).
\]

Based on these lemmas above, we are ready to show some higher-order estimates.

**Lemma 4.3** Let \( E(t) \in L^\infty[0,T] \) for some \( T > 0 \), then one has
\[
\|P_{v,v}\|_{H^{5/2}(\Omega_t)} + \|n_t\|_{H^3(\Gamma_t)} \leq P(E(t)).
\]

**Proof.** For \( P_{v,v} \), similar arguments as in the proof of Lemma 4.1 lead to
\[
\|P_{v,v}\|_{H^{5/2}(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{5/2}}) \|tr(\nabla v \nabla v)\|_{H^{1/2}(\Omega_t)} + \|\Gamma_t\|_{H^{1/2}} |C_{v,v}(t)| + \|v \cdot \nabla v n_t\|_{H^1(\Gamma_t)}
\]
\[
\leq P(E(t))(1 + \|v\|_{H^2(\Omega_t)}).
\]
Applying Lemma 4.2, we have the desired estimate.

Again, similar arguments as (4.36) in the proof of Lemma 4.1 lead directly to the estimate
\[
\|\kappa\|_{H^2(\Gamma_t)} \leq \sigma^{-1}(\|\mathcal{R}\|_{H^2(\Gamma_t)} + \|P_{v,v}\|_{H^2(\Gamma_t)}) \leq C(\|\Gamma_t\|_{H^{5/2}}(1 + \|\nabla v J\|_{L^2(\Gamma_t)} + \|P_{v,v}\|_{H^{5/2}(\Omega_t)}),
\]
which implies
\[
\|n_t\|_{H^3(\Gamma_t)} \leq P(E(t))(1 + \|P_{v,v}\|_{H^{5/2}(\Omega_t)}).
\]
Therefore, combining this with the estimate for \( P_{v,v} \), the proof is finished.

We also present more estimates for \( v \).
Lemma 4.4 Assuming that $E(t) \in L^\infty[0,T]$ for some $T > 0$, one has
\[\|D_t v\|_{H^{3/2}(\Omega)} + \|D^2_t v\|_{L^2(\Omega)} \leq P(E(t))\]

Proof. First, one recalls Euler’s equation to get that
\[\|D_t v\|_{H^{3/2}(\Omega)} \leq C(1 + \|\nabla P\|_{H^{3/2}(\Omega)} + \|\nabla P_{\omega,\nu}\|_{H^{3/2}(\Omega)}),\]
and applying Lemma 4.3 leads to the estimate for $D_t v$.
Second, taking $D_t$ on Euler’s equation leads to the following estimate:
\[\|D^2_t v\|_{L^2(\Omega)} \leq C(\|D_t \partial_3\|_{L^2(\Omega)} + \|\nabla P_{\omega,\nu}\|_{L^2(\Omega)} + \|D_t \nabla P_{\omega,\nu}\|_{L^2(\Omega)}).\]
Using Lemma 4.2 again, we can finish the estimate for $D^2_t v$.

We now consider the following Neumann system for velocity potential $\phi$:
\[
\begin{cases}
\Delta \phi = 0, & \text{on } \Omega \\
\nabla_n \phi|_{\Gamma_s} = v^s, & \nabla_n \phi|_{\Gamma_b} = 0,
\end{cases}
\tag{4.42}
\]
where in order to obtain the uniqueness we chose $\phi$ to satisfy
\[\int_{\Omega} \phi dX = 0\]
without loss of generality.
We show by singularity decompositions that $\nabla v$ lies in $L^\infty(\Omega_t)$, which is a key ingredient in the following estimates.

Lemma 4.5 Assume that $E(t) \in L^\infty[0,T]$ for some $T > 0$, then there exists a unique $\phi \in H^3(\Omega_t)$ to system (4.42) and one finds the following singular decomposition:
\[\phi = \phi_r + \phi_s\]
where the regular part $\phi_r \in H^4(\Omega_t)$, and the singular part is expressed in the same way as $u_s$ in Proposition 2.3. Moreover, one has
\[\|\nabla \phi\|_{L^\infty(\Omega_t)} \leq P(E(t)).\]

Proof. In fact, applying Proposition 2.3, we know immediately about the existence of $\phi \in H^3(\Omega_t)$ and the singular decomposition. Therefore, we have
\[v = \nabla \phi_r + \nabla \phi_s = v_r + v_s, \quad \text{with } v_r \in H^3(\Omega_t).\tag{4.43}\]
It only remains to show the estimate for $\nabla v$. Since we have
\[\phi_s = \chi_\omega(\omega_t) \chi_t c_{1}/r^{\pi/\omega_t} \circ T_t + \chi_\omega(\omega_r) \chi_r c_{r}/r^{\pi/\omega_r} \circ T_r\]
where the singular part exists when $\omega_t \in (\pi/3, \pi/2)$. In this case, we find by Proposition 2.3 that
\[\|\nabla^2 \phi_s\|_{L^\infty(\Omega_t)} \leq C(\|\Gamma_t\|_{H^4})(\|v_r^s\|_{H^{5/2}(\Gamma_s)} + \|\phi_r\|_{L^2(\Omega_t)}) (\|\chi_t^{r^\pi/\omega_t - 2} \circ T_t\|_{L^\infty(\Omega_t)})
+ \|\chi_r^{r^\pi/\omega_r - 2} \circ T_r\|_{L^\infty(\Omega_t)})\]
with $\pi/\omega_t - 2 \in (0,1)$. Consequently, we obtain
\[\|\nabla^2 \phi_s\|_{L^\infty(\Omega_t)} \leq C(\|\Gamma_t\|_{H^4})(\|v_r^s\|_{H^{5/2}(\Gamma_s)} + \|\phi_r\|_{L^2(\Omega_t)}) \leq C(\|\Gamma_t\|_{H^4}) \|v_r^s\|_{H^{5/2}(\Gamma_s)}\]
thanks to a direct variational estimate
\[ \| \phi \|^2_{L^2(\Omega)} \leq C(\| \Gamma_t \|^2_{H^{5/2}}) \| \nabla \phi \|^2_{L^2(\Gamma_t)} \leq C(\| \Gamma_t \|^2_{H^{5/2}}) \| v^\perp \|^2_{L^2(\Gamma_t)}. \]

Moreover, we also have by Proposition 2.3 the following estimate:
\[ \| \phi_r \|^2_{H^4(\Omega)} \leq C(\| \Gamma_t \|^2_{H^4}) \left( \| v^\perp \|_{H^{5/2}(\Gamma_t)} + \| \phi \|^2_{L^2(\Omega)} \right) \leq C(\| \Gamma_t \|^2_{H^4}) \| v^\perp \|^2_{H^{5/2}(\Gamma_t)}. \]

As a result, we find that
\[ \| \nabla v \|_{L^\infty(\Omega)} \leq \| \nabla^2 \phi_r \|_{L^\infty(\Omega)} + \| \nabla^2 \phi_s \|_{L^\infty(\Omega)} \leq C(\| \Gamma_t \|^2_{H^4}) \| v^\perp \|^2_{H^{5/2}(\Gamma_t)}, \]

which together with Lemma 4.2 lead to the desired estimate.

As long as we have \( \| \nabla v \|_{L^\infty(\Omega)} \) estimate, we are able to deal with more higher-order estimates.

**Lemma 4.6** Let \( E(t) \in L^\infty[0, T] \) for some \( T > 0 \), one has the following estimate:
\[ \| P_{\nu, v} \|^2_{H^1(\Omega)} + \| P_{3, v} \|^2_{H^{5/2}(\Omega)} + \| D_t P_{\nu, v} \|^2_{H^{5/2}(\Omega)} \leq P(E(t)) \]

**Proof.** - \( P_{\nu, v} \) estimate. In fact, similar arguments as in the proof of Lemma 4.1 show that
\[ \| P_{\nu, v} \|^2_{H^1(\Omega)} \leq P(E(t))(1 + \| v \|^2_{H^2(\Omega)})(1 + \| \nabla v \|^2_{L^\infty(\Omega)}). \]

Consequently, applying Lemma 4.2 and Lemma 4.5 leads to the desired estimate.

- Estimate for \( P_{3, v} \). Apply Proposition 2.2, we have
\[ \| P_{3, v} \|^2_{H^{5/2}(\Omega)} \leq C(\| \Gamma_t \|^2_{H^{5/2}}) \left( |\text{tr}(\nabla^2 v)|_{H^{1/2}(\Omega)} + |C_{3, v}(t)| + \| \nabla_n \cdot \nabla v \|_{H^1(\Gamma_t)} + \| P_{3, v} \|^2_{L^2(\Omega)} \right). \]

Using Lemma 2.1, Lemma 2.2, Lemma 4.1, Lemma 4.2 and Lemma 4.5, we can finish the estimate.

- \( H^{3/2} \) estimate for \( D_t P_{\nu, v} \). Following the proof of Lemma 4.2 and using Lemma 4.2 again to improve estimates for \( v \), one can easily see that we have the desired estimate.

We now give a high-order estimate for \( D_t P_{\nu, v} \).

**Lemma 4.7** Assuming \( E(t) \in L^\infty[0, T] \) for some \( T > 0 \), one has
\[ \| D_t P_{\nu, v} - v \cdot (\nabla P_{\nu, v} |_c) \|_{H^{5/2}(\Omega)} \leq P(E(t)). \]

**Proof.** We denote by
\[ w = D_t P_{\nu, v} - v \cdot (\nabla P_{\nu, v} |_c) \quad \text{with} \quad \nabla P_{\nu, v} |_c = \chi_l(\nabla P_{\nu, v} |_{p_l}) + \chi_r(\nabla P_{\nu, v} |_{p_r}). \]

A direct computation using (4.39) leads to
\[ \begin{cases} \Delta w = -\text{tr}D_t(\nabla v \cdot v) + [\Delta, v] \cdot (\nabla P_{\nu, v} - \nabla P_{\nu, v} |_c) - v \cdot \Delta(\nabla P_{\nu, v} |_c) & \text{in} \quad \Omega_t \\ \nabla n \cdot w |_{\Gamma_t} = C_{\nu, v}(t) + \nabla_n v \cdot (\nabla P_{\nu, v} - \nabla P_{\nu, v} |_c) - v \cdot \nabla_n(\nabla P_{\nu, v} |_c) |_{\Gamma_t}, \\ \nabla n \cdot w |_{\Gamma_b} = D(t)(v \cdot \nabla v |_{n_b} + \nabla v \cdot (\nabla P_{\nu, v} - \nabla P_{\nu, v} |_c) - v \cdot \nabla_n(\nabla P_{\nu, v} |_c) |_{\Gamma_b}. \end{cases} \]

Applying Proposition 2.2 to system (4.44), we have
\[ \| w \|^2_{H^{5/2}(\Omega)} \leq C(\| \Gamma_t \|^2_{H^{5/2}}) \left( \| \Delta w \|^2_{H^{1/2}(\Omega)} + \| \nabla n \cdot w \|^2_{H^{1}(\Gamma_t)} + \| \nabla n \cdot w \|^2_{H^{1}(\Gamma_b)} + \| w \|^2_{L^2(\Omega)} \right). \]
We need to deal with the terms on the right side above. First, thanks to (4.40), Lemma 4.2, Lemma 4.3 and Lemma 4.5, it is straightforward to check that

\[
\|D_t r(\nabla v \nabla v)\|_{H^{1/2}(\Omega_t)} + \|
abla (\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{H^{1/2}(\Omega_t)} + \|v \cdot \Delta (\nabla P_{v,v}|_c)\|_{H^{1/2}(\Omega_t)} \leq P(E(t)).
\]

Notice that here we need to use Lemma 2.2 for the product estimate of \(\|\nabla^2 R_{\Omega} v\|_{H^{1/2}(\Omega_t)}\) from \(D_t r(\nabla v \nabla v)\|_{H^{1/2}(\Omega_t)}\):

\[
\|\nabla^2 R_{\Omega} v\|_{H^{1/2}(\Omega_t)} \leq C\|\nabla^2 R_{\Omega} v\|_{H^{1/2}(\Omega_t)} (\|\nabla v\|_{H^1(\Omega_t)} + \|\nabla v\|_{L^\infty(\Omega_t)}) \leq P(E(t)).
\]

The estimate for \(\|\nabla^2 P_{v,v} \nabla v\|_{H^{1/2}(\Omega_t)}\) also follows in a similar way.

Second, for the boundary terms, we only need to take care of the following estimate from \(\|\nabla_n v \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{H^1(\Gamma_t)}\):

\[
\|\nabla_n v \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{L^2(\Gamma_t)} \\
\leq \|\nabla_n v \cdot \chi_t(\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{L^2(\Gamma_t)} + \|\nabla_n v \cdot (1 - \chi_t) \nabla P_{v,v}\|_{L^2(\Gamma_t)} (4.45) \\
\leq \|\nabla_n v \cdot \chi_t(\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{L^2(\Gamma_t)} + P(E(t)).
\]

Here the estimate for the second term in the equality above holds thanks to (4.43) and Proposition 2.3.

To handle the first term in (4.45), using a straightening diffeomorphism as \(T_r\) from Proposition 2.3 and (4.43), we know immediately

\[
\|\nabla_n v \cdot \chi_t(\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{L^2(\Gamma_t)} \\
\leq C(\|\nabla \nabla_n v \|_{H^{3/2}(\Omega_t)} (\|\nabla P_{v,v} - \nabla P_{v,v}|_c\|_{H^{1/2}(\Gamma_t)} + \|\nabla_n v \cdot (1 - \chi_t) \nabla P_{v,v}\|_{L^2(\Gamma_t)} (4.46) \\
\leq C(\|\nabla \nabla_n v \|_{H^{3/2}(\Omega_t)} (\|\nabla P_{v,v} - \nabla P_{v,v}|_c\|_{H^{1/2}(\Gamma_t)} + \|\nabla_n v \cdot \chi_t(\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{L^2(\Gamma_t)} + P(E(t)) (4.45).
\]

Moreover, applying Lemma 2.3 leads to

\[
\|r^{-3/4}(\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{L^2(0,\Gamma_0)} \leq C(\|\nabla P_{v,v} - \nabla P_{v,v}|_c\|_{H^{3/2}(0,\Gamma_0)} + \|\nabla_n v \cdot \chi_t(\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{L^2(\Gamma_t)} (4.47)
\]

so we obtain by an interpolation the following inequality

\[
\|r^{-3/4}(\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{L^\infty(0,\Gamma_0)} \leq C(\|\nabla P_{v,v} - \nabla P_{v,v}|_c\|_{H^{3/2}(0,\Gamma_0)} + \|\nabla_n v \cdot \chi_t(\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{L^2(\Gamma_t)} (4.48)
\]

As a result, combining this with (4.46), we conclude that

\[
\|\nabla_n v \cdot \chi_t(\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{L^2(\Gamma_t)} \leq P(E(t))(\|P_{v,v}\|_{H^3(\Omega_t)} + 1) \leq P(E(t)).
\]

Substituting this estimate into (4.45), we finally obtain

\[
\|\nabla_n v \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{H^1(\Gamma_t)} \leq P(E(t)).
\]

In the end, the estimate for \(\|\nabla_n v \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{H^1(\Gamma_t)}\) follows in a similar way, and the proof can be finished.

\[
\int_{\Omega_t} \nabla D_t P_{3,v} \cdot D_3 dX.
\]
Lemma 4.8 Let $E(t) \in L^2[0,T]$ for some $T > 0$. Then, we have the following estimate:

$$\left| \int_{\Omega_t} \nabla D_t \mathcal{P}_{3,v} \cdot D_t \mathcal{H} dX - \frac{d}{dt} \int_{\Omega_t} \nabla \mathcal{P}_{3,v} \cdot \nabla \mathcal{H} dX \right| \leq P(E(t)).$$

**Proof.** Recalling system (3.32), we get

$$\begin{align*}
\left\{ \begin{array}{l}
\Delta D_t \mathcal{P}_{3,v} = -D_t tr(\nabla \mathcal{H} \nabla v) + 2tr(\nabla \mathcal{H}^2 \mathcal{P}_{3,v}) \\
\nabla_n D_t \mathcal{P}_{3,v} \Gamma_t = C_{3,v} - D_t (\nabla \mathcal{H} \cdot \nabla v \cdot n_t) + \nabla_n \mathcal{V} \cdot \nabla \mathcal{P}_{3,v}, \\
\nabla_n D_t \mathcal{P}_{3,v} \Gamma_h = D_t (\mathcal{J} \cdot \nabla v n_h) + \nabla_n v \cdot \nabla \mathcal{P}_{3,v}.
\end{array} \right.
\end{align*} \quad (4.47)$$

Therefore, we have directly

$$\begin{align*}
\int_{\Omega_t} \nabla D_t \mathcal{P}_{3,v} \cdot D_t \mathcal{H} dX &= \int_{\Omega_t} \nabla D_t \mathcal{P}_{3,v} \cdot D_t \mathcal{H} dX \\
&= \int_{\Omega_t} \nabla D_t \mathcal{P}_{3,v} \cdot \nabla \mathcal{H} dX + \int_{\Omega_t} \nabla D_t \mathcal{P}_{3,v} \cdot \nabla v \cdot \nabla \mathcal{H} dX \\
&= \int_{\Omega_t} \nabla D_t \mathcal{P}_{3,v} \cdot \nabla D_t \mathcal{H} dX + \frac{d}{dt} \int_{\Omega_t} \nabla \mathcal{P}_{3,v} \cdot \nabla v \cdot \nabla \mathcal{H} dX - \int_{\Omega_t} \nabla \mathcal{P}_{3,v} \cdot D_t (\nabla v \cdot \nabla \mathcal{H}) dX \\
&\quad + \int_{\Omega_t} \nabla v \cdot \nabla \mathcal{P}_{3,v} \cdot \nabla v \cdot \nabla \mathcal{H} dX,
\end{align*}$$

which leads to the following equality:

$$\begin{align*}
\int_{\Omega_t} \nabla D_t \mathcal{P}_{3,v} \cdot D_t \mathcal{H} dX &= \int_{\Omega_t} \nabla D_t \mathcal{P}_{3,v} \cdot \nabla v \cdot \nabla \mathcal{H} dX \\
&= \int_{\Omega_t} \nabla D_t \mathcal{P}_{3,v} \cdot \nabla D_t \mathcal{H} dX - \int_{\Omega_t} \nabla \mathcal{P}_{3,v} \cdot D_t (\nabla v \cdot \nabla \mathcal{H}) dX + \int_{\Omega_t} \nabla v \cdot \nabla \mathcal{P}_{3,v} \cdot \nabla v \cdot \nabla \mathcal{H} dX.
\end{align*}$$

To finish the proof, the key lies in the analysis for the first integral. The remainder part is controlled by $P(E(t))$ thanks to Lemma 4.1, Lemma 4.2, Remark 4.1 and Lemma 4.6.

For the first integral on the right side of the equality above, we have by Green’s Formula that

$$\begin{align*}
\int_{\Omega_t} \nabla D_t \mathcal{P}_{3,v} \cdot \nabla D_t \mathcal{H} dX \\
= \int_{\Gamma_t} \nabla_n D_t \mathcal{P}_{3,v} D_t \mathcal{H} ds + \int_{\Gamma_b} \nabla_n D_t \mathcal{P}_{3,v} D_t \mathcal{H} ds - \int_{\Omega_t} \Delta D_t \mathcal{P}_{3,v} D_t \mathcal{H} dX \\
= \int_{\Gamma_t} \left[ C_{3,v} - D_t (\nabla \mathcal{H} \cdot \nabla v \cdot n_t) + \nabla_n v \cdot \nabla \mathcal{P}_{3,v} \right] D_t \mathcal{H} ds + \int_{\Gamma_b} D_t (\mathcal{J} \cdot \nabla v n_b) D_t \mathcal{H} ds \\
&\quad + \int_{\Omega_t} \nabla_n \mathcal{P}_{3,v} \cdot D_t \mathcal{H} ds + \int_{\Omega_t} D_t tr(\nabla \mathcal{H} \nabla v) D_t \mathcal{H} dX - \int_{\Omega_t} 2tr(\nabla v \nabla^2 \mathcal{P}_{3,v}) D_t \mathcal{H} dX \\
&= C_{3,v} \int_{\Gamma_t} D_t \mathcal{H} ds - \int_{\Gamma_t} (\nabla \mathcal{H} \cdot \nabla \mathcal{P}_{3,v} \cdot n_t) D_t \mathcal{H} ds + \int_{\Gamma_b} (\mathcal{J} \cdot \nabla v n_b) D_t \mathcal{H} ds \\
&\quad + \int_{\Omega_t} tr(\nabla D_t \mathcal{H} \nabla v) D_t \mathcal{H} dX + l.o.t.,
\end{align*}$$

where the remainder lower-order terms can be controlled by $P(E(t))$ in a similar way as before.

Now we deal with the terms in (4.48) one by one. To begin with, we write from the definition of $C_{3,v}$ in system (3.32) that

$$\begin{align*}
\frac{d}{dt} \langle \Gamma_t C_{3,v} \rangle &= -\int_{\Omega_t} D_t tr(\nabla \mathcal{H} \nabla v) dX + \int_{\Gamma_t} D_t (\nabla \mathcal{H} \cdot \nabla v \cdot n_t) ds - \int_{\Gamma_b} D_t (\mathcal{J} \cdot \nabla v n_b) ds \\
&= -\int_{\Omega_t} tr(\nabla D_t \mathcal{H} \nabla v) dX + \int_{\Gamma_t} \nabla \mathcal{H} \cdot \nabla \mathcal{P}_{3,v} \cdot n_t ds - \int_{\Gamma_b} D_t \mathcal{J} \cdot \nabla v n_b ds + l.o.t.,
\end{align*}$$
where the lower-order terms can be controlled again and hence the details are omitted. Applying Green’s Formula again and using the decompositions

\[
\begin{align*}
\nabla D_t \mathcal{R}_H &= (\nabla_{\tau_t} D_t \mathcal{R}) \tau_t + (\nabla_{n_t} D_t \mathcal{R}_H) n_t \quad \text{on} \quad \Gamma_t, \\
\nabla D_t \mathcal{R}_H &= (\nabla_{\tau_b} D_t \mathcal{R}_H) \tau_b + (\nabla_{n_b} D_t \mathcal{R}_H) n_b \quad \text{on} \quad \Gamma_b,
\end{align*}
\]

we find

\[
\frac{d}{dt}(\Gamma_t | C_{3,v})
\]

\[
= - \int_{\Gamma_t} D_t \mathcal{R}_H \cdot \nabla v \cdot n_t ds - \int_{\Gamma_t} D_t \mathcal{R}_H v \cdot n_t ds + \int_{\Gamma_t} D_t \mathcal{R}_H \cdot \nabla \tau_t v \cdot n_t ds - \int_{\Gamma_b} D_t \mathcal{R}_H \cdot \nabla v \cdot n_b ds + \text{l.o.t.}
\]

\[
= - \int_{\Gamma_t} \nabla D_t \mathcal{R}_H \cdot \nabla v \cdot n_t ds + \int_{\Gamma_t} \nabla D_t \mathcal{R}_H v \cdot n_t ds - \int_{\Gamma_b} \nabla D_t \mathcal{R}_H \cdot \nabla v \cdot n_b ds - \int_{\Gamma_b} \nabla D_t \mathcal{R}_H v \cdot n_b ds + \text{l.o.t.}
\]

\[
= - \int_{\Gamma_t} \nabla D_t \mathcal{R}_H \cdot \nabla v \cdot n_t ds + \int_{\Gamma_t} \nabla D_t \mathcal{R}_H v \cdot n_t ds - \int_{\Gamma_b} \nabla D_t \mathcal{R}_H \cdot \nabla v \cdot n_b ds - \int_{\Gamma_b} \nabla D_t \mathcal{R}_H v \cdot n_b ds + \text{l.o.t.}
\]

\[
= - \int_{\Gamma_t} \nabla D_t \mathcal{R}_H \cdot \nabla v \cdot n_t ds + \int_{\Gamma_b} \nabla D_t \mathcal{R}_H v \cdot n_b ds + \text{l.o.t.}
\]

(4.49)

Here, for the last three terms on \( \Gamma_b \), thanks to the assumption that \( n_b \) is constant near the contact points \( p_i \) and \( v \cdot n_b |_{\Gamma_b} = 0 \), we know that \( \nabla_{\tau_b} v \cdot n_b = -v \cdot \nabla_{\tau_b} n_b \) and \( \nabla_v n_b = (\nabla_v n_b \cdot \tau_b) \tau_b \) vanish near \( p_i \). Moreover, we also know from the definition of \( \mathcal{R}_H \) that

\[
\nabla_{n_b} D_t \mathcal{R}_H |_{\Gamma_b} = [\nabla_{n_b} D_t] \mathcal{R}_H = \nabla_{n_b} v \cdot \nabla \mathcal{R}_H.
\]

Consequently, applying Lemma 2.5 and Lemma 2.6, we have

\[
\int_{\Gamma_b} \nabla_{\tau_b} D_t \mathcal{R}_H \cdot \nabla_{\tau_b} v \cdot n_b ds + \int_{\Gamma_b} \nabla D_t \mathcal{R}_H \cdot \nabla_v n_b ds
\]

\[
\leq \| \nabla_{\tau_b} D_t \mathcal{R}_H \|_{H^{1/2}(\Omega_t)} \| v \cdot \nabla_{\tau_b} n_b \|_{H^{1/2}(\Gamma_b)} + \| \nabla_v n_b \cdot \tau_b \|_{H^{1/2}(\Gamma_b)}
\]

\[
\leq C (\| \Gamma_b \|_{H^{1/2}}) \| D_t \mathcal{R}_H \|_{H^1(\Omega_t)} \| v \cdot \nabla_{\tau_b} \mathcal{R}_H (n_b) \|_{H^{1/2}(\Omega_t)} + \| \nabla_v \mathcal{R}_H (n_b) \cdot \mathcal{R}_H (n_b) \|_{H^{1/2}(\Omega_t)} \leq P(E(t))
\]

and

\[
\int_{\Gamma_b} \nabla_{n_b} D_t \mathcal{R}_H \nabla_{n_b} v \cdot n_b ds \leq P(E(t)).
\]

For the first integral in (4.49), applying Green’s Formula again leads to

\[
\int_{\Gamma_t} \nabla_{n_b} D_t \mathcal{R}_H \nabla_{n_b} v \cdot n_b ds = \int_{\Omega_t} [\delta, D_t] \mathcal{R}_H \nabla_{\mathcal{R}_H (n_b)} v \cdot \mathcal{R}_H (n_b) ds - \int_{\Gamma_b} [\nabla_{n_b} D_t] \mathcal{R}_H \nabla_{\mathcal{R}_H (n_b)} v \cdot \mathcal{R}_H (n_b) ds,
\]

so this term is controlled by \( P(E(t)) \) again.

Therefore, summing up all these estimates above and going back to (4.49), we arrive at

\[
\frac{d}{dt}(\Gamma_t | C_{3,v}) = \text{l.o.t.}
\]

with the lower-order terms controlled by \( P(E(t)) \). This leads to the following estimate

\[
C_{3,v} \leq P(E(t)).
\]
For the moment, we can go back and deal with the other terms in (4.48). In fact, using similar arguments as above, we also conclude that
\[
\int_{\Gamma_b} (\nabla_{\tau_1} D_t \mathbb{R} \cdot \nabla_{\tau_1} v \cdot n_t) D_t \mathbb{R} H ds + \int_{\Gamma_b} (D_t \mathbb{J} \cdot \nabla_{\tau_1} n_b) D_t \mathbb{R} H ds - \int_{\Omega_t} \text{tr}(\nabla D_t \mathbb{J} \nabla v) D_t \mathbb{R} H dX \leq P(E(t))
\]
Therefore, the proof is finished.

\[\Box\]

4.2 Boundary terms at the contact points

**Lemma 4.9** We have the following equation on \(\Gamma_b\):
\[
D_t \mathbb{J} = \frac{\sigma^2}{\beta_c} (n_t \cdot \tau_b) \nabla \cdot (\nabla_{\tau_1} \mathbb{J}) + \mathbb{R} \cdot \nabla H \mathbb{J} + \mathbb{J} \cdot \nabla H \mathbb{R} - \nabla H D_t \mathbb{R} + R_{c,1},
\]
(4.50)
and there holds at the contact points \(p_i\) (\(i = l, r\)) that
\[
(D_t \mathbb{J})^\perp (\nabla_{\tau_1} \mathbb{J})^\perp |_{p_i} = -\frac{\sigma^2}{\beta_c} F_i(t) + R_{c,2,i},
\]
\[
(D_t \mathbb{J})^\perp (\nabla_{\tau_1} \mathbb{J})^\perp |_{p_r} = \frac{\sigma^2}{\beta_c} F_r(t) + R_{c,2,r},
\]
where
\[
R_{c,1} = r_c \tau_b - (\mathbb{J} \cdot D_t n_b) n_b, \quad R_{c,2,i} = (- r_c + \cot \omega (\mathbb{J} \cdot D_t n_b)) (\sin \omega) \nabla \cdot (\nabla_{\tau_1} \mathbb{J})^\perp |_{p_i}, \quad \text{and}
\]
\[
r_c = -\sigma \sin \omega (\nabla_{\tau_1}, v \cdot D_t n_b) + \sigma \tau_b \cdot D_t n_b \nabla_{\tau_1} n_b + \sigma \sin \omega (\nabla P_{v,v} \cdot \nabla_{\tau_1} n_b + [D_t, \nabla_{\tau_1} v] \cdot n_b)
\]
\[= -\beta_c D_t \nabla P_{v,v} \cdot \tau_b \]
satisfy
\[
|R_{c,1}| \leq P(E(t)), \quad |R_{c,2,i}| \leq P(E(t)) F(t)^{1/2}.
\]
**Proof.** The proof follows the proof of Lemma 7.1 in [52]. For the estimates, we apply Lemma 2.1, Lemma 4.3, Lemma 4.5 and Lemma 4.7. Here the differences compared to the proof of Lemma 7.1 in [52] lie in that we have different \(\mathbb{J}\) and
\[
\nabla_{\tau_1} \nabla P_{v,v} \cdot n_t = -\nabla P_{v,v} \cdot \nabla_{\tau_1} n_t
\]
in \(r_c\) due to the definition of \(P_{v,v}\).

\[\Box\]

4.3 Proof of Theorem 4.1

We are ready to show the a priori energy estimate in Theorem 4.1. In fact, taking \(L^2(\Omega_t)\) inner product with \(\mathbb{J} \cdot \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_{\xi}))\) on both sides of (3.33), we have
\[
\int_{\Omega_t} D_t \mathbb{J} \cdot \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_{\xi})) |D_t \mathbb{J} - \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_{\xi}))| dX
\]
\[+ \sigma \int_{\Omega_t} A \mathbb{J} \cdot \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_{\xi})) |D_t \mathbb{J} - \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_{\xi}))| dX = \int_{\Omega_t} R \cdot \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_{\xi})) |D_t \mathbb{J} - \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_{\xi}))| dX.
\]
(4.51)

We deal with these integrals above one by one. To get started, for the first term on the left side, we rewrite it as
\[
\int_{\Omega_t} D_t \mathbb{J} \cdot \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_{\xi})) |D_t \mathbb{J} - \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_{\xi}))| dX
\]
\[= \frac{1}{2} \frac{d}{dt} \int_{\Omega_t} |D_t \mathbb{J} - \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_{\xi}))|^2 dX.
\]
For the second term on the left side of (4.51), we have by Green’s Formula that
\[
\int_{\Omega_t} A_3 \cdot [D_t \tilde{J} - \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|c))] \, dX = - \int_{\Gamma_t} \Delta_{\Gamma} \tilde{J} \cdot n_t \, ds + \int_{\Gamma_t} \Delta_{\Gamma} \tilde{J} \cdot \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|c)) \cdot n_t \, ds. \tag{4.52}
\]

For the first term on the right side of (4.51), one deduces from Hodge decomposition (3.31) and integration by parts as in [53] that
\[
- \int_{\Gamma_t} \Delta_{\Gamma} \tilde{J} \cdot D_t \tilde{J} \cdot n_t \, ds
= \int_{\Gamma_t} \nabla_{\tau} \tilde{J} \cdot \nabla_{\tau} (D_t \tilde{J} \cdot n_t) \, ds - (D_t \tilde{J}) \cdot \nabla_{\tau} \tilde{J} \big|_{\partial \Gamma_t}^{p_t} - \int_{\Gamma_t} \Delta_{\Gamma} \tilde{J} \cdot (\nabla P_{3,v} \cdot n_t) \, ds
= \frac{1}{2} \frac{d}{dt} \int_{\Gamma_t} |\nabla_{\tau} \tilde{J}|^2 \, ds - \int_{\Gamma_t} \nabla_{\tau} \tilde{J} \cdot D_t \nabla_{\tau} (\tilde{J} \cdot n_t) \, ds + \int_{\Gamma_t} \Delta_{\Gamma} \tilde{J} (\nabla P_{3,v} \cdot n_t) \, ds
\leq P(E(t))(1 + F(t)^{1/2}),
\]
where Lemma 2.1 and Lemma 4.7 are used.

As a result, going back to (4.51), we summarise that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} |D_t \tilde{J} - \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|c))|^2 \, dX + \frac{\sigma}{2} \frac{d}{dt} \int_{\Gamma_t} |\nabla_{\tau} \tilde{J}|^2 \, ds + \frac{\sigma^3}{2 \beta_c} F(t)
\leq P(E(t)) + \sigma \int_{\Gamma_t} \nabla_{\tau} \tilde{J} \cdot D_t \nabla_{\tau} (\tilde{J} \cdot n_t) \, ds + \sigma \int_{\Gamma_t} \Delta_{\Gamma} \tilde{J} (\nabla P_{3,v} \cdot n_t) \, ds
+ \int_{\Omega_t} \mathcal{R} \cdot [D_t \tilde{J} - \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|c))] \, dX.
\]
Moreover, direct estimates similarly as in [53] lead to the following estimate
\[
\sigma \int_{\Gamma_t} \nabla_{\tau} \tilde{J} \cdot D_t \nabla_{\tau} (\tilde{J} \cdot n_t) \, ds + \sigma \int_{\Gamma_t} \Delta_{\Gamma} \tilde{J} (\nabla P_{3,v} \cdot n_t) \, ds
\leq P(E(t)) + \frac{\sigma^3}{4 \beta_c} F(t),
\]
so we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega_t} |D_t \tilde{J} - \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|c))|^2 \, dX \right) + \sigma \int_{\Gamma_t} |\nabla_{\tau} \tilde{J}|^2 \, ds + \frac{\sigma^3}{4 \beta_c} F(t)
\leq P(E(t)) + \int_{\Omega_t} \mathcal{R} \cdot [D_t \tilde{J} - \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|c))] \, dX. \tag{4.53}
\]

In order to finish the energy estimate, we still need to deal with the integral on the right side involving the remainder term \( \mathcal{R} \) in (3.33).

- Estimates for the part \( R_0 - \sigma \nabla H(R_1) \) in \( \mathcal{R} \).
Lemma 4.10 For the remainder term $R_0$ defined in (3.30), we have the estimate

$$\|R_0 - \sigma \nabla H(R_1)\|_{L^2(\Omega_t)} \leq P(E(t)).$$

**Proof.** Recalling from (3.30), we know that

$$R_0 - \sigma \nabla H(R_1) = -\sigma \nabla H(J \cdot \Delta^\Gamma, n_t) + \sigma \nabla H(n_t \cdot \Delta^\Gamma \nabla v, v) + [D_t, \nabla H](D_t P_{v,v}) + A_1 + A_2 + A_3$$

where $R_1$ and $A_1, A_2, A_3$ are defined in (3.25) and (3.27), (3.28), (3.26) respectively.

For the term $\nabla H(n_t \cdot \Delta^\Gamma \nabla v, v)$, we have

$$\| \nabla H(n_t \cdot \Delta^\Gamma \nabla v, v) \|_{L^2(\Omega_t)} = \| \nabla H([n_t, \Delta^\Gamma] \nabla v, v) \|_{L^2(\Omega_t)} \leq P(E(t)),$$

where the boundary condition $\nabla n_t P_{v,v}|_{\Gamma_t} = C_{v,v}(t)$ and Lemma 4.6 are applied.

For the term $[D_t, \nabla H](D_t P_{v,v})$, direct computations using (2.14) and similar arguments as in the proof of Lemma 4.2 lead to

$$\|[D_t, \nabla H](D_t P_{v,v})\|_{L^2(\Omega_t)} \leq \|\nabla v \cdot \nabla H(D_t P_{v,v})\|_{L^2(\Omega_t)} + \|[D_t, H](D_t P_{v,v})\|_{H^1(\Omega_t)} \leq P(E(t)).$$

The estimates for the other terms follow from lemmas in the previous section and can be done similarly as [53], so we omit the details here. □

- The part $\sigma \nabla H(R_1)$ in $\mathcal{R}$. In fact, the following integral from the right side of (4.53) is rewritten as

$$\int_{\Omega_t} \nabla H(R_1) \cdot [D_t \mathfrak{J} - \nabla H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_c))] \, dX$$

$$= \int_{\Gamma_t} R_1 D_t \mathfrak{J} \cdot n_t \, ds - \int_{\Gamma_t} R_1 \nabla n_t H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_c)) \, ds$$

$$= \int_{\Gamma_t} R_1 D_t \mathfrak{J} \cdot n_t \, ds + \int_{\Gamma_t} R_1 \nabla \mathfrak{J}_{P_{v,v}} \cdot n_t \, ds - \int_{\Gamma_t} R_1 \nabla n_t H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_c)) \, ds$$

On the other hand, noticing from (3.25) that $R_1$ contains terms like $\nabla^2 v, \nabla n_t, \kappa$ and $\nabla^2 P_{v,v}$ and using (4.43) and lemmas in the previous subsections, we have

$$\int_{\Gamma_t} R_1 \nabla n_t H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_c)) \, ds \leq \int_{\Gamma_t} |R_1| \, |\nabla n_t H(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_c))| \, ds \leq P(E(t)).$$

The details in the estimate above are omitted and we only note that for $\partial^2 v$ terms in $R_1$, we can have the following estimate thanks to (4.43):

$$\int_{\Gamma_t} |\nabla^2 v| \, ds \leq \int_{\Gamma_t} |\nabla^3 \phi_v| \, ds + \int_{\Gamma_t} |\nabla^3 \phi_s| \, ds \leq P(E(t)).$$

Similarly but more easily, we also obtain

$$\int_{\Gamma_t} R_1 \nabla P_{\mathfrak{J},v} \cdot n_t \, ds \leq P(E(t)).$$

Besides, for the part $\int_{\Gamma_t} R_1 D_t \mathfrak{J} \cdot n_t \, ds$, we put $D_t$ out of the integral:

$$\int_{\Gamma_t} R_1 D_t \mathfrak{J} \cdot n_t \, ds = \int_{\Gamma_t} R_1 D_t \mathfrak{J} \cdot (\mathfrak{J} - \mathfrak{J}|_c) \, ds + \int_{\Gamma_t} R_1 D_t (\mathfrak{J} - \mathfrak{J}|_c) \, ds$$

$$= \frac{d}{dt} \int_{\Gamma_t} R_1 (\mathfrak{J} - \mathfrak{J}|_c) \, ds - \int_{\Gamma_t} D_t R_1 (\mathfrak{J} - \mathfrak{J}|_c) \, ds - \int_{\Gamma_t} R_1 (\mathfrak{J} - \mathfrak{J}|_c) D_t \, ds + \int_{\Gamma_t} R_1 D_t (\mathfrak{J} - \mathfrak{J}|_c) \, ds.$$
Here we need to take care of the terms in \( \int_{\Gamma_t} D_t R_1 (\tilde{3} - \tilde{3}_c) ds \) and \( \int_{\Gamma_t} R_1 (\tilde{3} - \tilde{3}_c) D_t ds \). In fact, similarly to the analysis in (4.46), the key terms like \( \| \nabla^2 (\tilde{3} - \tilde{3}_c) \|_{L^2(\Gamma_t)} \) can be handled as follows:

\[
\| \nabla^2 (\tilde{3} - \tilde{3}_c) \|_{L^2(\Gamma_t)} \leq C \| \nabla_{\Gamma_t} \|_{H^5/2}} \| v^\delta (\partial^2 v \circ T_i^1) \|_{L^\infty(\Gamma_t)} \| r^{-\delta} (\tilde{3} - \tilde{3}_c) \circ T_i^1 \|_{L^2(\Gamma_t)} \leq P(E(t)),
\]

(4.54)

where the constant \( \delta \in (0, 1) \) is chosen to satisfy

\[
\delta + \pi/\omega - 3 > 0, \quad \text{when} \quad \omega_i \in (\pi/3, \pi/2).
\]

Consequently, we derive

\[
\int_{\Gamma_t} R_1 D_t \tilde{3} \cdot n_t ds = \frac{d}{dt} \int_{\Gamma_t} R_1 (\tilde{3} - \tilde{3}_c) \cdot n_t ds + \text{l.o.t.},
\]

where all the lower-order terms are controlled by \( P(E(t)) \).

As a result, we conclude that

\[
\int_{\Omega_t} \nabla \mathcal{H}(R_1) \cdot [D_t \tilde{3} - \nabla \mathcal{H}(D_t P_{v,v} - v \cdot (\nabla P_{v,v})_c))] dx = \frac{d}{dt} \int_{\Gamma_t} R_1 (\tilde{3} - \tilde{3}_c) \cdot n_t ds + \text{l.o.t.}
\]

with the lower-order terms controlled by \( P(E(t)) \).

- The terms \( D_t \nabla P_{3,v} + D_t \nabla R_v \cdot (\nabla P_{v,v})_c \) in \( \mathcal{R} \). First, one has directly from Lemma 4.8 that

\[
\int_{\Omega_t} \nabla D_t P_{3,v} \cdot D_t \tilde{3} dx = \frac{d}{dt} \int_{\Omega_t} \nabla P_{3,v} \cdot \nabla v \cdot \nabla \mathcal{H} dX + \text{l.o.t.}
\]

where the lower-order terms are all controlled by \( P(E(t)) \).

Second, for the integral

\[
\int_{\Omega_t} \nabla D_t P_{3,v} \cdot \nabla \mathcal{H}(D_t P_{v,v} - v \cdot (\nabla P_{v,v})_c)) dx,
\]

applying Green’s formula and similar calculations as in (4.48) show directly that it can be controlled by \( P(E(t)) \), and we omit the details.

Therefore, we conclude that

\[
\int_{\Omega_t} \nabla D_t P_{3,v} \cdot [D_t \tilde{3} - \nabla \mathcal{H}(D_t P_{v,v} - v \cdot (\nabla P_{v,v})_c))] dx = \frac{d}{dt} \int_{\Omega_t} \nabla P_{3,v} \cdot \nabla v \cdot \nabla \mathcal{H} dX + \text{l.o.t.},
\]

where all the lower-order terms are controlled by \( P(E(t)) \).

Summing up all these estimates related to \( \mathcal{R} \) above and going back to (4.53), we obtain the following estimate

\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega_t} |D_t \tilde{3} - \nabla \mathcal{H}(D_t P_{v,v} - v \cdot (\nabla P_{v,v})_c))|^2 dx + \sigma \int_{\Gamma_t} |\nabla_{\Gamma_t} \tilde{3}_c|^2 ds \right) + \frac{\sigma^3}{4\beta_c} F(t)
\]

\[
\leq P(E(t)) + \frac{d}{dt} \int_{\Gamma_t} R_1 (\tilde{3} - \tilde{3}_c) \cdot n_t ds + \frac{d}{dt} \int_{\Omega_t} \nabla P_{3,v} \cdot \nabla v \cdot \nabla \mathcal{H} dX.
\]

Integrating on both sides with respect to time on \([0, t]\), we have

\[
\frac{1}{2} \int_{\Omega_0} |D_t \tilde{3} - \nabla \mathcal{H}(D_t P_{v,v} - v \cdot (\nabla P_{v,v})_c))|^2 dx + \sigma \int_{\Gamma_0} |\nabla_{\Gamma_0} \tilde{3}_c|^2 ds + \frac{\sigma^3}{4\beta_c} \int_0^t F(t') dt'
\]

\[
\leq \frac{1}{2} \int_{\Omega_0} |D_t \tilde{3} - \nabla \mathcal{H}(D_t P_{v,v} - v \cdot (\nabla P_{v,v})_c))|_t' = 0|^2 dx + \sigma \int_{\Gamma_0} |\nabla_{\Gamma_0} \tilde{3}_c|^2 ds + \frac{\sigma^3}{4\beta_c} \int_0^t F(t') dt'
\]

\[
+ \int_0^t P(E(t')) dt' + \int_{\Gamma_t} R_1 (\tilde{3} - \tilde{3}_c) \cdot n_t ds + \int_{\Omega_t} \nabla P_{3,v} \cdot \nabla v \cdot \nabla \mathcal{H} dX |^t_0
\]

\[
\leq P(E(0)) + \int_0^t P(E(t')) dt' + \int_{\Gamma_t} R_1 (\tilde{3} - \tilde{3}_c) \cdot n_t ds + \int_{\Omega_t} \nabla P_{3,v} \cdot \nabla v \cdot \nabla \mathcal{H} dX |^t_0.
\]
Replacing the first integral with $\|\mathcal{D}_t \mathfrak{J}\|_{L^2(\Omega_t)}^2$, we arrive at the following inequality:

$$E(t) + \int_0^t F(t')dt' \leq P(E(0)) + \int_0^t P(E(t'))dt' + C \int_{\Gamma_t} R_1(\mathfrak{J} - \mathfrak{J}|_{\partial\Omega}) \cdot n_t ds \bigg|_0^t + C \int_{\Omega_t} \nabla P_3 \cdot \nabla \mathfrak{J} dX \bigg|_0^t + C \int_{\Omega_t} |\nabla \mathcal{H}(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_{\partial\Omega}))|^2 dX,$$

where the constant $C$ depends on $\sigma, \beta_c$.

We now deal with the last three integrals on the right side above one by one. First, for the terms in $R_1$, similarly as in (4.54), we can have by careful estimates with interpolations that

$$\|r^\delta (\partial^2 v) \circ T_{t}^{-1}\|_{L^\infty(0, r_0)} \leq P(E(t)) E(t)^{1/2}$$

and

$$\|r^{-\delta} (\mathfrak{J} - \mathfrak{J}|_{\partial\Omega}) \circ T_{t}^{-1}\|_{L^2(0, r_0)} \leq C(\|\Gamma_t\|_{H^{5/2}}) \|\mathfrak{J} - \mathfrak{J}|_{\partial\Omega}\|_{H^{4}(\Gamma_t)} \leq C(\|\Gamma_t\|_{H^{5/2}}) \|\mathfrak{J}\|_{H^{4+1/2}(\Omega_t)},$$

where $1 > \delta > 3 - \pi/\omega$ when $\omega \in (\pi/3, \pi/2)$.

Therefore, we can prove by interpolations that there exist $\delta_0 \in (1/2, 1)$ and $\epsilon$ small enough such that

$$\int_{\Gamma_t} R_1(\mathfrak{J} - \mathfrak{J}|_{\partial\Omega}) \cdot n_t ds \leq P(E(t)) E(t)^{\delta_0} \leq \epsilon E(t) + C_{\epsilon, \delta_0} P(E(t)).$$

Moreover, the following estimates can also be proved in a similar but easier way:

$$\int_{\Omega_t} \nabla P_3 \cdot \nabla \mathfrak{J} dX + \int_{\Omega_t} |\nabla \mathcal{H}(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_{\partial\Omega}))|^2 dX \leq \epsilon E(t) + C_{\epsilon, \delta_0} P(E(t)).$$

As a result, summing up the estimates above, we are able to conclude that

$$E(t) + \int_0^t F(t')dt' \leq P(E(0)) + \int_0^t P(E(t'))dt' + P(E(t)).$$

In the end, to close the energy estimates, we deal with $E_l(t)$. First, we show by direct calculations using Euler’s equation and lemmas from the previous section that

$$\|v\|_{H^k(\Omega_t)}^2 \leq C(T)\|v(0)\|_{H^k(\Omega_0)}^2 + C(T)\|\mathfrak{J}\|_{L^2([0,T], H^k(\Omega_t))}^2 + \int_0^t P(E(t')) dt'$$

for $k = 1, 2$. Then we take the square root on both sides of the inequalities above and apply an interpolation between $H^1(\Omega_t), H^2(\Omega_t)$ as well as Lemma 4.1 to obtain

$$\|v\|_{H^{3/2}(\Omega_t)} \leq C(T)\|v(0)\|_{H^{3/2}(\Omega_0)} + \int_0^t P(E(t')) dt'.$$

Second, the estimate for $\|\Gamma_t\|_{H^{5/2}}$ can be derived in a similar way as above. Consequently, we have

$$E_l(t) \leq P(E(0)) + \int_0^t P(E(t')) dt'.$$

Combining this estimate with the estimate above for $E(t)$, we finish the lower-order energy estimates.
5 Higher-order time-derivative energy estimates

In this section, we prove higher-order energy estimates with respect to $D_t$, which is needed in the local well-posedness part. To start with, we define the energy functional

$$E_1(t) = \|\nabla_{\tau_v} D_t \mathcal{J}^\perp\|^2_{L^2(\Gamma_t)} + \|D_t^2 \mathcal{J}\|^2_{L^2(\Omega_t)},$$

and the dissipation

$$F_1(t) = \sum_{i=l,r} |(\sin \omega_i) \nabla_{\tau_v} D_t \mathcal{J}^\perp|_{P_i}|^2.$$

The main result of this section is as follows:

**Theorem 5.1** Let the contact angles $\omega_i \in (0, \pi/2)$ and $E(t), \int_0^T F(t)dt, E_1(t), \int_0^T F_1(t)dt$ be bounded above in $[0, T]$ for some $T > 0$. Then the following higher-order a priori estimate holds

$$\sup_{0 \leq t \leq T} E_1(t) + \int_0^T F_1(t)dt \leq P(E_1(0)) + \int_0^T P(E_1(t))dt.$$

To prove this theorem, we begin with a higher-order equation for $\mathcal{J}$ and more delicate estimates involving $D_t$ based on Section 4.1. With these preparations, we are able to finish the energy estimate in the last subsection of this part.

5.1 The higher-order equation for $\mathcal{J}$

Firstly, we recall system (3.30) and rewrite it as follows:

$$D_t^2 \mathcal{J} = \sigma \nabla \mathcal{H}(\Delta_{\Gamma_t}, \mathcal{J}^\perp - h_v) + \bar{R}_0,$$

where

$$\bar{R}_0 = -\sigma \nabla \mathcal{H}(\mathcal{J} \cdot \Delta_{\Gamma_t} n_t) + \sigma \nabla \mathcal{H}([n_t, \Delta_{\Gamma_t}] \cdot \nabla P_{v,v}) + \sigma \nabla \mathcal{H}(R_1 + h_v) + A_1 + A_2 + A_3 + \nabla \mathcal{H}(D_t^2 P_{v,v}),$$

and $h_v$ contains all the second-order terms of $v$ in $R_1$ which come from the commutator $[D_t, n_t \Delta_{\Gamma_t}] \cdot v$. More precisely, we have

$$h_v = \Delta_{\Gamma_t} v \cdot D_t n_t + 2D^2 v \left(\nabla_{\tau_v} (\nabla_{\tau_v} v)^\top \right) \cdot n_t.$$

(5.2)

As a result, $R_1 + h_v$ only contains lower-order derivatives like $\partial_v, \partial n_t, \kappa$.

Acting $D_t$ on both sides of (3.30), we obtain

$$D_t (D_t^2 \mathcal{J}) + \nabla \mathcal{H}(\Delta_{\Gamma_t} D_t \mathcal{J}^\perp - D_t h_v) = \bar{R}_2,$$

(5.3)

with the right side

$$\bar{R}_2 = D_t \bar{R}_0 + \nabla_v [\nabla, D_t] \mathcal{H}(\Delta_{\Gamma_t} \mathcal{J}^\perp - h_v) + \nabla [\mathcal{H}, D_t] (\Delta_{\Gamma_t} \mathcal{J}^\perp - h_v) + \nabla \mathcal{H}[\Delta_{\Gamma_t}, D_t] \mathcal{J}^\perp + D_t (D_t \nabla P_{v,v} + \nabla P_{\nabla P_{v,v}} + \nabla P_{D_t \mathcal{J}^\perp}).$$

Here we use

$$-[A, D_t] = [\nabla, D_t] \mathcal{H} \Delta_{\Gamma_t} + \nabla [\mathcal{H}, D_t] \Delta_{\Gamma_t} + \nabla \mathcal{H}[\Delta_{\Gamma_t}, D_t]$$

with all these commutators from the end of Section 2, and

$$D_t^2 \mathcal{J} = D_t^2 \mathcal{J} + D_t \nabla P_{v,v} + \nabla P_{\nabla P_{v,v}} + \nabla P_{D_t \mathcal{J}^\perp},$$

(5.4)

where $P_{w,v}$ is defined in (1.10).
5.2 More preliminary estimates.

Based on Section 4.1, we are going to prove higher-order estimates for different quantities using our higher-order energy.

- Higher-order estimates for \( \mathcal{R}_H \) and \( D_t\mathcal{J} \). To get started, we recall system (4.34) to find

\[
\begin{align*}
\Delta D_t\mathcal{R}_H &= 2tr(\nabla v \nabla^2 \mathcal{R}_H) \quad \text{in } \Omega_t \\
\nabla_{n_t} D_t\mathcal{R}_H|_{\Gamma_t} &= D_t\mathcal{J}^\perp + \nabla_{n_t} v \cdot \nabla \mathcal{R}_H|_{\Gamma_t}, \\
\nabla_{n_h} D_t\mathcal{R}_H|_{\Gamma_h} &= \nabla_{n_h} v \cdot \nabla \mathcal{R}_H|_{\Gamma_h}.
\end{align*}
\]

Applying Proposition 2.2 and lemmas in Section 4.1, we have

\[
\| D_t\mathcal{R}_H \|_{H^2(\Omega_t)} \leq C(\| \mathcal{R}_t \|_{H^{5/2}(\Omega_t)})(\| 2tr(\nabla v \nabla^2 \mathcal{R}_H) \|_{L^2(\Omega_t)} + \| D_t\mathcal{J}^\perp + \nabla_{n_t} v \cdot \nabla \mathcal{R}_H \|_{H^{1/2}(\Gamma_t)}) \\
+ \| \nabla_{n_{\mathcal{R}}} \cdot \nabla \mathcal{R}_H \|_{H^{1/2}(\Gamma_h)} + \| D_t\mathcal{R}_H \|_{L^2(\Omega_t)}) \] (5.5)

Here we notice that the regularity of \( D_t\mathcal{R}_H \) is constrained due to the regularity of \( v \). To get a higher-order estimate, we need to get rid of the worst part.

Similarly as in Lemma 4.7, we consider a good quantity \( D_t\mathcal{R}_H - v \cdot (\nabla \mathcal{R}_H|_c) \) which satisfies

\[
\begin{align*}
\Delta(D_t\mathcal{R}_H - v \cdot (\nabla \mathcal{R}_H|_c)) &= 2tr(\nabla v \nabla^2 \mathcal{R}_H) - \Delta(v \cdot (\nabla \mathcal{R}_H|_c)) \quad \text{in } \Omega_t \\
\nabla_{n_t}(D_t\mathcal{R}_H - v \cdot (\nabla \mathcal{R}_H|_c)|_{\Gamma_t}) &= D_t\mathcal{J}^\perp + \nabla_{n_t} v \cdot (\nabla \mathcal{R}_H - \nabla \mathcal{R}_H|_c) - \nabla_{n_t} v \cdot (\nabla \mathcal{R}_H|_c)|_{\Gamma_h}, \\
\nabla_{n_h}(D_t\mathcal{R}_H - v \cdot (\nabla \mathcal{R}_H|_c)|_{\Gamma_h}) &= \nabla_{n_h} v \cdot (\nabla \mathcal{R}_H - \nabla \mathcal{R}_H|_c) - \nabla_{n_h} v \cdot (\nabla \mathcal{R}_H|_c)|_{\Gamma_h}.
\end{align*}
\]

As a result, similar arguments as in (4.54) applied for the part \( \nabla v \cdot (\nabla \mathcal{R}_H - \nabla \mathcal{R}_H|_c) \), we can have

\[
\| D_t\mathcal{R}_H - v \cdot (\nabla \mathcal{R}_H|_c) \|_{H^{5/2}(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}). \] (5.6)

where \( \| D_t\mathcal{J}^\perp \|_{L^2(\Gamma_t)} \) is handled by (5.5).

On the other hand, we deal with \( D_t\mathcal{J} \). First, we know directly from (5.5) that

\[
\| D_t\mathcal{J} \|_{H^1(\Omega_t)} \leq P(E(t))(1 + \| D_t\mathcal{J}^\perp \|_{H^{1/2}(\Gamma_t)}). \] (5.7)

Second, we notice that

\[
D_t\mathcal{J} = D_t\mathcal{R}_H = \nabla(D_t\mathcal{R}_H - v \cdot (\mathcal{J}|_c)) - \nabla v \cdot (\mathcal{J} - \mathcal{J}|_c).
\]

Similarly as in (4.46), we apply Lemma 2.2(1) and (4.43) to obtain the estimate

\[
\begin{align*}
\| \nabla v \cdot (\mathcal{J} - \mathcal{J}|_c) \|_{H^{3/2}(\Omega_t)} &\leq P(E(t))(1 + \| \partial \nabla v \cdot (\mathcal{J} - \mathcal{J}|_c) \|_{H^{1/2}(\Omega_t)}) \\
&\leq P(E(t))(1 + \sum_{i=t,r} (\| r^{\pi/\omega_i - 2} \|_{L^\infty(0,r_0)} + \| r^{\pi/\omega_i - 2} \|_{H^1(0,r_0)}) \| r^{-1}(\mathcal{J} - \mathcal{J}|_{\mathcal{S}_t}) \|_{T_t^{-1}H^{1/2}(S_t^{\mathcal{S}_t})} \]
&\leq P(E(t)),
\end{align*}
\]

while recall that \( \mathcal{S}_{t,i} \) are straightened sector of \( \Omega_t \) with radius \( r_0 \) near the corners.

Therefore, combining this estimate with (5.6), we have

\[
\| D_t\mathcal{J} \|_{H^{3/2}(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}) \] (5.8)

and also

\[
\| D_t\mathcal{R}_H \|_{H^2(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}) .
\]

- Estimates for \( D_tP_{v,v} \). Checking system (4.39) carefully and applying lemmas from Section 4.1, one has immediately

\[
\| D_tP_{v,v} \|_{H^{5/2}(\Omega_t)} \leq P(E(t)) ,
\] (5.9)
Lemma 5.1

Let

First, we deal with \( D \) and (5.5), (5.9) and (5.11) the estimate

Applying Lemma 4.2, Lemma 4.6, (5.7), and (5.8), we find

Moreover, a similar argument also leads to

- Estimates for \( D^2 v, D^3 v \). First, acting \( D_t \) on both sides of Euler’s equation leads to

Applying Lemma 4.2, Lemma 4.6, (5.7), and (5.8), we find

Next, we give the estimates of \( D^2 v \). Before that, we firstly prove the following lemma.

**Lemma 5.1** Let \( P_{3,v}, P_{V_{P_3,v,v}} \) and \( P_{D_3,v} \) be defined by (1.10). Then there hold

\[
\|D_t \nabla P_{3,v} + \nabla P_{V_{P_3,v,v}} + \nabla P_{D_3,v}\|_{H^1(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2})
\]

and

\[
\|D^2 \nabla P_{v,v}\|_{L^2(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}).
\]

**Proof.** First, we deal with \( D_t P_{3,v} \). Recalling system (4.47), we have by Proposition 2.2, (4.49), (5.5), (5.8) and lemmas from Section 4.1 that

\[
\|D_t P_{3,v}\|_{H^2(\Omega_t)} \\
\leq C(\|\Gamma\|_{H^2}) \left( \| - D_t tr(\nabla \nabla v \cdot \nabla) + 2tr(\nabla v \nabla^2 P_{3,v})\|_{L^2(\Omega_t)} + \|D_t(\nabla \nabla v \cdot \nabla) + \nabla n_{v,v} \cdot \nabla P_{3,v}\|_{H^{1/2}(\Gamma_b)} + \|D_t(\nabla v \cdot \nabla) \nabla v \cdot \nabla P_{3,v}\|_{H^{1/2}(\Gamma_1)} + \|D_t P_{3,v}\|_{L^2(\Omega_t)} \right) \\
\leq P(E(t))(1 + E_1(t)^{1/2}).
\]

Second, checking the definition (1.10) and applying Lemma 2.2 (1) and Lemma 4.6, we can have for \( P_{V_{P_3,v,v}} \) the following estimate immediately:

\[
\|P_{V_{P_3,v,v}}\|_{H^2(\Omega_t)} \leq P(E(t)).
\]

Moreover, a similar argument also leads to

\[
\|P_{D_3,v}\|_{H^2(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}).
\]

In the end, for the part \( D^2 P_{v,v} \), we deduce from (4.39) the following system:

\[
\begin{cases}
\Delta (D^2 P_{v,v}) = - tr D^2 _t (\nabla v \nabla v) + 2tr D_t (\nabla v \nabla^2 P_{v,v}) & \text{in } \Omega_t \\
\nabla n_{v} (D^2 P_{v,v}) |_{\Gamma_b} = C_{v,v}(t) + D_t (\nabla n_{v,v} \cdot \nabla P_{v,v}) + \nabla n_{v,v} \cdot \nabla D_t P_{v,v} |_{\Gamma_b} \\
\nabla n_{v} (D^2 P_{v,v}) |_{\Gamma_b} = D^2 _t (v \cdot \nabla n_{v}) + D_t (\nabla n_{v,v} \cdot \nabla P_{v,v}) + \nabla n_{v,v} \cdot \nabla D_t P_{v,v} |_{\Gamma_b}
\end{cases}
\]
with
\[ \int_{\Omega_t} D^2_t \Pi_{v,v} d\chi = 0. \]

Applying (5.16), (5.12) and checking term by term, we obtain the variational estimate
\[ \| D^2_t \Pi_{v,v} \|_{H^1(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}), \quad (5.15) \]
and the proof is finished.

Now, we are in a position to give the estimate for $D^3_t v$. In fact, acting $D^2_t$ on both sides of Euler’s equation, we derive by the previous lemma and lemmas from Section 4.1 that
\[ \| D^3_t v \|_{L^2(\Omega_t)} \leq \| D^2_t \phi \|_{L^2(\Omega_t)} + \| D^2_t \nabla \Pi_{v,v} \|_{L^2(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}). \quad (5.16) \]

- Estimate for $v^+$. We obtain by the definition of $\phi$ that
\[ \nabla (D^2_t \phi) = D^2_t \phi + \nabla v \cdot \nabla D_t \phi + D_t (\nabla v \cdot \nabla \phi), \]
and applying (3.23) and (5.5) implies
\[ \| D^2_t \phi \|_{H^1(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}). \quad (5.17) \]

Recalling (3.23) again, we know
\[ \sigma D_t \kappa = D_t (\phi + \Pi_{v,v}) = -\Delta_t v^+ - v^+ |\nabla v_t|^2 + \nabla v_t \cdot \nabla v \cdot v_t \quad \text{on} \; \Gamma_t, \]
so we obtain by (5.15) and (5.17) the estimate
\[ \| D_t \Delta_t v^+ \|_{L^2(\Gamma_t)} \leq P(E(t))(1 + E_1(t)^{1/2}). \quad (5.18) \]

- Some more higher-order estimates for $\phi$, $\Pi_{v,v}$ and $\phi$.

**Lemma 5.2** Assuming that $E(t), E_1(t) \in L^\infty[0, T]$ for some $T > 0$, we have the following estimate:
\[ \| \phi \|_{H^1(\Omega_t)} + \| D_t v \|_{H^1(\Omega_t)} + \| \nabla^2 \phi \|_{L^2(\Omega_t)} + \| D_t \nabla v \|_{L^2(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}). \]

**Proof.** (1) Higher regularity for $\phi$. To begin with, we use Euler’s equation to rewrite (3.25) as follows:
\[ D^2_t \phi = (\Delta_t^i J^i - h_v) + \left[ n_t, \Delta_t^i \right] (J + \nabla P_{v,v} + g) + 2\sigma \Pi(\tau) \cdot \nabla \tau \cdot J + \left( R_1 + h_v \right) - D^2_t \Pi_{v,v} \quad \text{on} \; \Gamma_t \]
where we notice that $\Delta_t (\nabla n_t P_{v,v})|_{\Gamma_t} = 0$ thanks to the definition of $P_{v,v}$ and $(R_1 + h_v)$ contains only $\partial v$ terms instead of $\partial^2 v$ terms.

Meanwhile, we know from (5.15) and (5.17) that
\[ \| D^2_t \phi \|_{H^{1/2}(\Gamma_t)} + \| D^2_t P_{v,v} \|_{H^{1/2}(\Gamma_t)} \leq P(E(t))(1 + E_1(t)^{1/2}), \]
so checking term by term in the equation above, we have immediately
\[ \| \Delta_t J^i - h_v \|_{H^{1/2}(\Gamma_t)} \leq P(E(t))(1 + E_1(t)^{1/2}). \quad (5.19) \]

On the other hand, we know from (4.43) that (when $\omega_i \in (\pi/3, \pi/2)$)
\[ h_v = h_{v,r} + h_{v,s} \quad \text{with} \; h_{v,r} \in H^{1/2}(\Gamma_t), \; h_s = \sum_i \left[ \chi_1 \omega_i r^{2\alpha_i} + \chi_2 \omega_i (\partial v \circ T^{-1}) \right] (\alpha_i - 2) \circ T_i, \]
where we note $\alpha_i = \pi/\omega_i - 1 \in (1, 2)$, $a_{i,k}$ ($i = l, r$) contain $n_l, \tau_l, \partial n_l, \partial \tau_l$ and the singular coefficient from (4.43). Moreover, $a_{i,3}(\partial v_r \circ T_i^{-1})$ is linear with respect to $\partial v_r \circ T_i^{-1}$, where we recall that $v_r = \nabla \phi_r \in H^3(\Omega)$ and $\|v_r\|_{H^3(\Omega)}$ is controlled by $P(E(t))$.

As a result, we find

$$2\alpha_i - 3 \in (-1, 1), \quad \alpha_i - 2 \in (-1, 0),$$

which implies immediately that

$$\|h_v\|_{L^p(\Gamma_t)} \leq P(E(t)), \quad \text{for } 1 < p < \min\{(2 - \alpha_i)^{-1}, |3 - 2\alpha_i|^{-1}\}.$$

Summing up the estimates above for $\|\Delta J^\perp - h_v\|_{H^{1/2}(\Gamma_t)}$ and $\|h_v\|_{L^p(\Gamma_t)}$, we obtain

$$\|\Delta J^\perp\|_{L^p(\Gamma_t)} \leq \|\Delta J^\perp - h_v\|_{L^p(\Gamma_t)} + \|h_v\|_{L^p(\Gamma_t)} \leq C(\|\Gamma_t\|_{L^p(\Omega)} \|J^\perp\|_{H^{1/2}(\Gamma_t)} + \|h_v\|_{L^p(\Gamma_t)} \leq P(E(t))(1 + E_1(t)^{1/2}),$$

and this leads to

$$\|\nabla J^\perp\|_{W^{1, p}(\Gamma_t)} \leq P(E(t))(1 + E_1(t)^{1/2}).$$

Applying Lemma 2.1, we finally show that

$$\|\nabla J^\perp\|_{L^\infty(\Gamma_t)} + \|J^\perp\|_{H^{3/2+\epsilon}(\Gamma_t)} \leq P(E(t))(1 + E_1(t)^{1/2}) \quad (5.20)$$

with $\epsilon = 1 - 1/p$.

Consequently, applying Proposition 2.2, we have the desired estimate for $\|\mathfrak{r}_h\|_{H^3(\Omega_t)}$. Notice that we have in fact the estimate by Remark 2.3:

$$\|\mathfrak{r}_h\|_{H^{5+\epsilon}(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}). \quad (5.21)$$

Moreover, we apply Lemma 2.4 to have $\mathfrak{r} = \sigma \mathcal{K} - P_{v,v} \in H^{5/2+\epsilon}(\Gamma_t)$ with the estimate

$$\|\mathfrak{r}\|_{H^{5/2+\epsilon}(\Gamma_t)} \leq P(E(t))(1 + E_1(t)^{1/2}).$$

(2) $H^2$ estimate for $D_t v$. Recalling the Euler’s equations and $P_{v,v}$ estimate from Lemma 4.6, we get

$$\|D_t v\|_{H^2(\Omega_t)} \leq C(\|\Gamma_t\|_{H^2(\Omega_t)} + \|\nabla P_{v,v}\|_{H^2(\Omega_t)} + \|g\|_{H^2(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}).$$

(3) $L^\infty$ estimates. Applying Remark 2.3 and (5.21) with the same $\epsilon = 1 - 1/p$ as above lead immediately to

$$\|\nabla^2 \mathfrak{r}_h\|_{L^\infty} \leq C(\|\Gamma_t\|_{H^5(\Omega_t)} \|\nabla^2 \mathfrak{r}_h\|_{H^{1+\epsilon}(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}).$$

Applying Remark 2.3 to $P_{v,v}$, we obtain

$$\|\nabla^2 P_{v,v}\|_{L^\infty(\Omega_t)} \leq C(\|\Gamma_t\|_{H^5(\Omega_t)} \|\nabla^2 P_{v,v}\|_{H^{1+\epsilon}(\Omega_t)} \leq C(\|\Gamma_t\|_{H^5(\Omega_t)} (\|\Gamma_t\|_{H^5(\Omega_t)} + \|\nabla P_{v,v}\|_{H^{1/2+\epsilon}(\Omega_t)} + \|v \cdot \nabla n_0\|_{H^{1/2+\epsilon}(\Omega_t)} + \|P_{v,v}\|_{L^2(\Omega_t)} \leq P(E(t))(1 + \|\nabla v\|_{H^{1+\epsilon}(\Omega_t)}^2).$$

Using (4.43) and a similar argument as in (1), we have

$$\|\nabla v\|_{H^{1+\epsilon}(\Omega_t)} \leq \|\nabla^2 \phi_s\|_{H^{1+\epsilon}(\Omega_t)} + \|\nabla^2 \phi_r\|_{H^2(\Omega_t)} \leq P(E(t)),$$

which implies

$$\|\nabla^2 P_{v,v}\|_{L^\infty(\Omega_t)} \leq P(E(t)).$$
In the end, apply Euler’s equation again leads to
\[
\|\nabla D_t v\|_{L^\infty(\Omega_t)} \leq \|\nabla^2 \mathcal{R}_H\|_{L^\infty(\Omega_t)} + \|\nabla^2 P_{v,v}\|_{L^\infty(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}),
\]
which finishes the proof. ■

- Estimate for $D_t^2 P_{v,v}$. Based the above estimates, we firstly improve the estimate for $D_t P_{v,v}$. Using system (4.44) and Lemma 5.2, we improve the estimate in Lemma 4.7 and get
\[
\|D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_c)\|_{H^3(\Omega_t)} \\
\leq C(\|\nabla c\|_{H^3}(\| - tr D_t (\nabla v \nabla v) + |\Delta, v| \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c) + v \cdot \Delta (\nabla P_{v,v}|_c))\|_{H^1(\Omega_t)} + C v(t) \\
+ \|\nabla n_t v \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c) - v \cdot \nabla n_t (\nabla P_{v,v}|_c)\|_{H^{3/2}(\Gamma_t)} + \|D_t (v \cdot \nabla n_b)|_{\Gamma_t}\|_{H^{3/2}(\Gamma_t)} \\
+ \|\nabla n_b^1 v \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c) - v \cdot \nabla n_b (\nabla P_{v,v}|_c)\|_{H^{3/2}(\Gamma_b)} \\
\leq P(E(t))(1 + E_1(t)^{1/2}).
\]
(5.22)

Here we use Lemma 2.2 (2) for the boundary terms like $\|\nabla n_t v \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c)\|_{H^{3/2}(\Gamma_t)}$, which are handled similarly as in the proof for (5.8).

Next, we derive the equation of $D_t(D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_c))$. To simplify the notation, we define
\[
\mathcal{P}_{t,1} = D_t P_{v,v} - v \cdot (\nabla P_{v,v}|_c),
\]
and we rewrite (5.22) as
\[
\|\mathcal{P}_{t,1}\|_{H^3(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}).
\]
(5.23)

Direct computations lead to the following system for $D_t \mathcal{P}_{t,1}$:
\[
\Delta(D_t \mathcal{P}_{t,1}) = -tr D_t^2(\nabla v \nabla v) + 2tr D_t(\nabla v \nabla (\nabla P_{v,v} - \nabla P_{v,v}|_c)) - D_t(v \cdot \Delta (\nabla P_{v,v}|_c)) \\
+ |D_t, \Delta|_{\|\cdot\|_{H^3(\Omega_t)}} in \Omega_t \\
\nabla n_t (D_t \mathcal{P}_{t,1})|_{\Gamma_t} = c''(v)(t) + D_t(\nabla n_t v \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c)) \cdot (\nabla n_t v \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c)) \\
+ \nabla n_b^1 \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c) \cdot (\nabla n_b^1 \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c)) \\
(5.24)
\]

Moreover, we define
\[
\mathcal{P}_{t,2} = D_t \mathcal{P}_{t,1} - v \cdot (\nabla \mathcal{P}_{t,1}|_c) = D_t(\nabla P_{v,v} - D_t(v \cdot (\nabla P_{v,v}|_c)) - v \cdot (\nabla \mathcal{P}_{t,1}|_c) \\
(5.25)
\]
and we modify this system above as in Lemma 4.7 into a new system for $\mathcal{P}_{t,2}$ below:
\[
\Delta \mathcal{P}_{t,2} = -tr D_t^2(\nabla v \nabla v) + 2tr D_t(\nabla v \nabla (\nabla P_{v,v} - \nabla P_{v,v}|_c)) - D_t(v \cdot \Delta (\nabla P_{v,v}|_c)) \\
+ |D_t, \Delta|_{\|\cdot\|_{H^3(\Omega_t)}} in \Omega_t \\
\nabla n_t \mathcal{P}_{t,2}|_{\Gamma_t} = c''(v)(t) + D_t(\nabla n_t v \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c)) \cdot (\nabla n_t v \cdot (\nabla P_{v,v} - \nabla P_{v,v}|_c)) \\
+ \nabla n_b^1 \cdot (\nabla \mathcal{P}_{t,1} - \nabla \mathcal{P}_{t,1}|_c) \cdot (\nabla n_b^1 \cdot (\nabla \mathcal{P}_{t,1} - \nabla \mathcal{P}_{t,1}|_c)) \\
(5.26)
\]

Thanks to Lemma 4.7, (5.12) and (5.23), it is straightforward to show that
\[
\|\mathcal{P}_{t,2}\|_{H^{3/2}(\Omega)} \leq P(E(t))(1 + E_1(t)^{1/2}).
\]
(5.27)

- The boundary condition for $D_t^3 P_{v,v}$ at corner points.
Lemma 5.3 We have at the contact points \( p_i (i = l, r) \) the following equations

\[
D_t^2 \tilde{J} = \frac{\sigma^2}{\beta_c} (\sin \omega_i) D_t (\nabla_{\tau_i} \tilde{J}^\perp) \tau_b + R_{c, h, i},
\]

(5.28)

where the remainder terms

\[
R_{c, h, i} = D_t R_{c1} + \frac{\sigma^2}{\beta_c} D_t (\tau_b \sin \omega_i) (\nabla_{\tau_i} \tilde{J}^\perp - \frac{\sigma^2}{\beta_c} (\sin \omega_i) D_t (\nabla_{\tau_i} n_t \cdot \tilde{J}) \tau_b) \bigg|_{p_i},
\]

with \( R_{c1} \) defined in Lemma 4.9. Moreover, there hold for \( i = l, r \) that

\[
|R_{c, h, i}| \leq P(E(t)) (1 + E_1(t)^{1/2}).
\]

Proof. First, Acting Lemma 5.3 with Consequently, for any \( t' \in [0, T] \) there holds

\[
D_t \tilde{J} = -((\nabla v)^\perp n_t)^\perp = -(n_t \cdot \nabla v) \tau_t = -((\nabla v)^\perp) \tau_t + (\nabla v n_t \cdot v) \tau_t.
\]

(5.29)

5.3 Proof of Theorem 5.1

At this moment, we are finally ready to prove the higher-order energy estimate in Theorem 5.1. To begin with, we rewrite equation (5.3) of \( D_t^2 \tilde{J} \) into:

\[
D_t (D_t^2 \tilde{J} - \nabla H(P_{t, 2})) + \nabla H(\Delta_{\Gamma_t} D_t \tilde{J}^\perp - D_t h_v) = R_2 + D_t D_t \nabla P_{3, v} + D_t \nabla P_{D, 3, v},
\]

(5.30)

where

\[
R_2 = D_t (\tilde{R}_0 - \nabla H(D_t^2 P_{\omega, v})) + [\nabla, D_t] H(\Delta_{\Gamma_t} \tilde{J}^\perp - h_v) + \nabla [H, D_t] (\Delta_{\Gamma_t} D_t \tilde{J}^\perp + \nabla H(\Delta_{\Gamma_t} D_t \tilde{J}^\perp) + D_t (\nabla P_{\omega, v} - D_t \nabla H[D_t (v \cdot (\nabla P_{\omega, v}|_c)) + v \cdot (\nabla P_{\omega, v}|_c)].
\]

In fact, this complicated form is used due to similar technical reasons as \((3.33), \) see Remark 3.1, where the terms \( \nabla H(P_{t, 2}) \), \( D_t h_v \) are added on the left side to improve the estimates.

Next, we apply \( L^2(\Omega_t) \) inner product with \( D_t^2 \tilde{J} - \nabla H(P_{t, 2}) \) on both sides of (5.30) to get

\[
\frac{1}{2} \frac{d}{dt} \|D_t^2 \tilde{J} - \nabla H(P_{t, 2})\|_{L^2(\Omega_t)}^2 - \int_{\Omega_t} \nabla H(\Delta_{\Gamma_t} D_t \tilde{J}^\perp - D_t h_v) \cdot (D_t^2 \tilde{J} - \nabla H(P_{t, 2})) dX = \int_{\Omega_t} R_2 \cdot (D_t^2 \tilde{J} - \nabla H(P_{t, 2})) dX + \int_{\Omega_t} (D_t D_t \nabla P_{3, v} + D_t \nabla P_{D, 3, v}) \cdot (D_t^2 \tilde{J} - \nabla H(P_{t, 2})) dX.
\]

Consequently, for any \( t' \in [0, T] \) there holds

\[
\frac{1}{2} \|D_t^2 \tilde{J}(t') - \nabla H(P_{t, 2})(t')\|_{L^2(\Omega_t)}^2 \left( \int_0^{t'} \int_{\Omega_t} \nabla H(\Delta_{\Gamma_t} D_t \tilde{J}^\perp - D_t h_v) \cdot (D_t^2 \tilde{J} - \nabla H(P_{t, 2})) dX dt \right)
\]

\[
= \frac{1}{2} \|D_t^2 \tilde{J}(0) - \nabla H(P_{t, 2})(0)\|_{L^2(\Omega_0)}^2 \left( \int_0^{t'} \int_{\Omega_t} R_2 \cdot (D_t^2 \tilde{J} - \nabla H(P_{t, 2})) dX dt \right)
\]

\[
+ \int_0^{t'} \int_{\Omega_t} (D_t D_t \nabla P_{3, v} + D_t \nabla P_{D, 3, v}) \cdot (D_t^2 \tilde{J} - \nabla H(P_{t, 2})) dX dt.
\]

(5.31)
5.3.1 Left side of (5.31).

In this subsection, we prove estimates for the second term on the left side of (5.31).

**Proposition 5.4** One has for some \( \delta_0 \in (0,1) \) the following estimate:

\[
- \int_{\Gamma_t} \nabla H(\Delta_t D_t \overrightarrow{3}^3 - D_t h_v) \cdot (D_t^2 \overrightarrow{3}^3 - \nabla H(\mathcal{P}_{t,2})) \, dX dt \\
\geq \frac{1}{2} \sup_{t \in [0,T]} \int_{\Gamma_t} |\nabla_\tau D_t \overrightarrow{3}^3|^2 \, ds + \frac{1}{4} \int_0^T F_1(t) \, dt - \int_0^T P(E(t)) (1 + E_1(t)^{3/2}) \, dt \\
- \sup_{t \in [0,T]} P(E(t)) (1 + E_1(t)^{\delta_0}) - P(E(0)) (1 + E_1(0)^{\delta_0}).
\]

**Proof.** Applying Green’s Formula, we have

\[
- \int_{\Gamma_t} \nabla H(\Delta_t D_t \overrightarrow{3}^3 - D_t h_v) \cdot (D_t^2 \overrightarrow{3}^3 - \nabla H(\mathcal{P}_{t,2})) \, dX \\
= - \int_{\Gamma_t} (\Delta_t D_t \overrightarrow{3}^3 - D_t h_v) (D_t^2 \overrightarrow{3}^3 - \nabla H(\mathcal{P}_{t,2})) \cdot n_t ds \\
= - \int_{\Gamma_t} (\Delta_t D_t \overrightarrow{3}^3 - D_t h_v) (D_t^2 \overrightarrow{3}^3 + D_t \nabla P_{3,v} + \nabla P_{\nabla P_{3,v}} + \nabla P_{D_t^2 \overrightarrow{3}^3}) \cdot n_t ds + \int_{\Gamma_t} D_t h_v D_t^2 \overrightarrow{3}^3 \cdot n_t ds + I_R,
\]

where

\[
I_R = - \int_{\Gamma_t} (\Delta_t D_t \overrightarrow{3}^3 - D_t h_v) (D_t \nabla P_{3,v} + \nabla P_{\nabla P_{3,v}} + \nabla P_{D_t^2 \overrightarrow{3}^3}) \cdot n_t ds \\
+ \int_{\Gamma_t} (\Delta_t D_t \overrightarrow{3}^3 - D_t h_v) \nabla_{n_t} H(\mathcal{P}_{t,2}) ds.
\]

We deal with these integrals above one by one. First, integrating by parts and taking one \( D_t \) out of the first integral in (5.32), we get

\[
- \int_{\Gamma_t} \Delta_t D_t \overrightarrow{3}^3 D_t^2 \overrightarrow{3}^3 \cdot n_t ds - \frac{1}{2} \int_{\Gamma_t} |\nabla_\tau D_t \overrightarrow{3}^3|^2 \, ds + \frac{1}{2} \int_{\Gamma_t} |\nabla_\tau D_t \overrightarrow{3}^3|^2 \, ds + \int_{\Gamma_t} \nabla_{n_t} D_t \overrightarrow{3}^3 \cdot \nabla_\tau \overrightarrow{3}^3 \cdot n_t ds + I_R \\
\geq \int_{\Gamma_t} \nabla_{n_t} D_t \overrightarrow{3}^3 \cdot \nabla_\tau \overrightarrow{3}^3 \cdot n_t ds.
\]
so one has by (5.21) and lemmas from Section 4.1 that

\[
\nabla_\tau D_t \mathfrak{J} \mathfrak{J} (D^2_t \mathfrak{J} \cdot n_t)_{p_1}^{p_1} \geq \frac{1}{2} F_1(t) - P(E(t))(1 + E_1(t)).
\]

On the other hand, since \( D_t ds = (v^+ \kappa + \nabla_\tau (v \cdot \tau)) ds \), we obtain directly the estimate:

\[
\frac{1}{2} \int_{\Gamma_t} |\nabla_\tau D_t \mathfrak{J}|^2 D_t ds \leq P(E(t))E_1(t).
\]

Moreover, a direct computation using (5.29) shows that

\[
[n_t, D^2_t] \cdot \mathfrak{J} = -D_t (D_t n_t) \cdot \mathfrak{J} - 2(D_t n_t) \cdot D_t \mathfrak{J} \nabla_\tau t (\nabla_\tau n_t \cdot v) t \cdot \mathfrak{J} = -D_t \left( - (\nabla_\tau v^\perp) t + (\nabla_\tau n_t \cdot v) t \right) \cdot \mathfrak{J} - 2 \left( - (\nabla_\tau v^\perp) t + (\nabla_\tau n_t \cdot v) t \right) \cdot D_t \mathfrak{J}.
\]

Therefore, thanks to (5.8), Lemma 5.2 and checking term by term, we have the following estimate for the last two integrals in (5.33):

\[
\int_{\Gamma_t} \nabla_\tau D_t \mathfrak{J} \mathfrak{J} \left[ \nabla_\tau, D_t \right] D_t \mathfrak{J} \mathfrak{J} ds + \int_{\Gamma_t} \nabla_\tau D_t \mathfrak{J} \mathfrak{J} \nabla_\tau \left( [n_t, D^2_t] \cdot \mathfrak{J} \right) ds \leq P(E(t))(1 + E_1(t)).
\]

As a result, the proof is finished as long as we have Lemma 5.5 and Lemma 5.6.

We deal with the remainder integrals in (5.32) in the following two lemmas.

**Lemma 5.5** One has for the second term of the right hand of (5.32) that

\[
\left| \int_0^T \int_{\Gamma_t} D_t h_\nu D^2_t \mathfrak{J} \cdot n_t ds \right| \leq P(E(0))(1 + E_1(0)^{\delta_0}) + \sup_{t \in [0,T]} P(E(t))(1 + E_1(t)^{\delta_0}) + \int_0^T P(E(t))(1 + E_1(t)^{3/2}) dt + \frac{1}{8} \int_0^T F_1(t).
\]

**Proof.** To begin with, one recalls the definition of \( h_\nu \) in (5.2) to obtain

\[
\int_{\Gamma_t} D_t h_\nu D^2_t \mathfrak{J} \cdot n_t ds = \int_{\Gamma_t} D_t (\Delta_{\Gamma_t} v \cdot D_t n_t) D^2_t \mathfrak{J} \cdot n_t ds + \int_{\Gamma_t} D_t [2D^2 v (\nabla_\tau, v^\perp) \cdot n_t] D^2_t \mathfrak{J} \cdot n_t ds.
\]

We only deal with the first integral in the integral above, since the second one can be handled in a similar way.

In fact, it is straightforward to see by (5.29) that

\[
\int_{\Gamma_t} D_t (\Delta_{\Gamma_t} v \cdot D_t n_t) D^2_t \mathfrak{J} \cdot n_t ds
\]
\[
= \int_{\Gamma_t} D_t [\Delta_{\Gamma_t} v \cdot \tau_t (-\nabla_\tau v^\perp + \nabla_\tau n_t \cdot v)] D^2_t \mathfrak{J} \cdot n_t ds
\]
\[
= \int_{\Gamma_t} D_t \Delta_{\Gamma_t} v \cdot \tau_t (-\nabla_\tau v^\perp + \nabla_\tau n_t \cdot v) D^2_t \mathfrak{J} \cdot n_t ds + \int_{\Gamma_t} \Delta_{\Gamma_t} v \cdot D_t [\tau_t (-\nabla_\tau v^\perp + \nabla_\tau n_t \cdot v)] D^2_t \mathfrak{J} \cdot n_t ds
\]
\[
\equiv I_1 + I_2.
\]

The estimates for \( I_1, I_2 \) are proved in the following lines.
- Estimates of $I_1$. First, direct computations lead to

$$I_1 = \int_{\Gamma_t} D_t \Delta_{\Gamma_t} v \cdot \tau_t ( - \nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v ) L^2 (3 - \tilde{3}) \cdot n_t \, ds$$

$$+ \int_{\Gamma_t} D_t \Delta_{\Gamma_t} v \cdot \tau_t (- \nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v ) L^2 (3 - \tilde{3}) \cdot n_t \, ds$$

$$- \frac{d}{dt} \int_{\Gamma_t} D_t \Delta_{\Gamma_t} v \cdot \tau_t (- \nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v ) L_t (3 - \tilde{3}) \cdot n_t \, ds$$

$$- \int_{\Gamma_t} D_t \Delta_{\Gamma_t} v \cdot D_t \tau_t (- \nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v ) L_t (3 - \tilde{3}) \cdot n_t \, ds$$

$$- \int_{\Gamma_t} D_t \Delta_{\Gamma_t} v \cdot \tau_t (- \nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v ) L_t (3 - \tilde{3}) \cdot D_t (n_t \, ds)$$

$$+ \int_{\Gamma_t} D_t \Delta_{\Gamma_t} v \cdot \tau_t (- \nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v ) L_t (3 - \tilde{3}) \cdot n_t \, ds$$

$$\triangleq \frac{d}{dt} \int_{\Gamma_t} D_t \Delta_{\Gamma_t} v \cdot \tau_t (- \nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v ) L_t (3 - \tilde{3}) \cdot n_t \, ds + \sum_{i=1}^4 I_{1i}.$$

We deal with $I_{1i}$ one by one. In fact, applying integration by parts, we have

$$I_{14} = D_t \nabla_{\tau_t} v \cdot \tau_t (- \nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v ) L^2 (3 \tilde{3}) \cdot n_t \bigg|_{p_t}^{p_t}$$

$$- \int_{\Gamma_t} D_t \nabla_{\tau_t} v \cdot \nabla_{\tau_t} \tau_t \big( - \nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v \big) L^2 (3 \tilde{3}) \cdot n_t \, ds$$

$$+ \int_{\Gamma_t} \{ D_t, \nabla_{\tau_t} \} \nabla_{\tau_t} v \cdot \tau_t (- \nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v ) L^2 (3 \tilde{3}) \cdot n_t \, ds$$

Using (5.11) and similar arguments as in (4.54) for $\partial^2 v$ and $v^\perp$ terms and applying Lemma 5.3 to $D^2 (3 \tilde{3}) \big|_{p_t}$, we derive

$$|I_{14}| \leq \frac{1}{16} F_1 (t) + P (E (t) (1 + E (t))).$$

For $I_{1i}$, since

$$D_t^2 \Delta_{\Gamma_t} v = D_t \nabla_{\tau_t} v \cdot \tau_t (- \nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v )$$

$$= \nabla_{\tau_t} \Delta_{\Gamma_t} v + [D_t, \nabla_{\tau_t}] \nabla_{\tau_t} v - D_t \Delta_{\Gamma_t} v$$

$$= \nabla_{\tau_t} \Delta_{\Gamma_t} v + D_t (\nabla_{\tau_t} \tau_t \cdot \nabla_{\tau_t} v )$$

$$\triangleq v_{u, 2} + l.o.t.$$ where

$$h_{v, 2} = \nabla_{\tau_t} \Delta_{\Gamma_t} v + 2 \nabla_{\tau_t} \tau_t \cdot \nabla_{\tau_t} v$$

contains higher-order terms of $v$, and the remainder part contains products like $\partial D_t \tau_t \partial^2 v$ and other lower-order terms and can be controlled by (5.8), (5.12), Lemma 5.2 and similar arguments as in (4.54).

Therefore, we arrive at

$$I_{11} = \int_{\Gamma_t} h_{v, 2} \cdot \tau_t (- \nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v ) L_t (3 - \tilde{3}) \cdot n_t \, ds + l.o.t..$$
To finish the estimate for $I_{11}$, we firstly take care of the first term $\nabla_{\tau} D_t^2 \nabla_{\tau} v$ in $h_{\nu,2}$. In fact, we use integration by parts on $\Gamma_t$ to obtain

\[
\int_{\Gamma_t} \nabla_{\tau} D_t^2 \nabla_{\tau} v \cdot \tau_t (-\nabla_{\tau} v^\perp + \nabla_{\tau} n_t \cdot v) D_t (3 - 3|c|) \cdot n_t ds
= - \int_{\Gamma_t} D_t^2 \nabla_{\tau} v \cdot \nabla_{\tau} (\tau_t (-\nabla_{\tau} v^\perp + \nabla_{\tau} n_t \cdot v)) D_t (3 - 3|c|) \cdot n_t ds
- \int_{\Gamma_t} D_t^2 \nabla_{\tau} v \cdot \tau_t (-\nabla_{\tau} v^\perp + \nabla_{\tau} n_t \cdot v) \nabla_{\tau} (D_t (3 - 3|c|) \cdot n_t) ds.
\]

Similar arguments as in (4.54) and applying (5.8), (5.12), we derive

\[
\int_{\Gamma_t} \nabla_{\tau} D_t^2 \nabla_{\tau} v \cdot \tau_t (-\nabla_{\tau} v^\perp + \nabla_{\tau} n_t \cdot v) D_t (3 - 3|c|) \cdot n_t ds \leq P(E(t))(1 + E_1(t)).
\]

Meanwhile, the other terms in $I_{11}$ can also be handled similarly, so we conclude

\[
|I_{11}| \leq P(E(t))(1 + E_1(t)).
\]

For $I_{12}$, we have

\[
I_{12} = - \int_{\Gamma_t} \left( [D_t, \nabla_{\tau}] \nabla_{\tau} v - D_t (\nabla_{\tau} \tau_t \cdot \nabla_{\tau} v) \right) \cdot D_t [\tau_t (-\nabla_{\tau} v^\perp + \nabla_{\tau} n_t \cdot v)] D_t (3 - 3|c|) \cdot n_t ds
- \int_{\Gamma_t} \nabla_{\tau} D_t \nabla_{\tau} v \cdot D_t [\tau_t (-\nabla_{\tau} v^\perp + \nabla_{\tau} n_t \cdot v)] D_t (3 - 3|c|) \cdot n_t ds,
\]

where the first integral can be handled as above. For the second integral, we know from Lemma 2.5 that

\[
D_t (3 - 3|c|) \in \dot{H}^{1/2}(\Gamma_t),
\]

so we apply Lemma 2.5, Lemma 2.6, (5.8) and Lemma 5.2 to find

\[
- \int_{\Gamma_t} \nabla_{\tau} D_t \nabla_{\tau} v \cdot D_t [\tau_t (-\nabla_{\tau} v^\perp + \nabla_{\tau} n_t \cdot v)] D_t (3 - 3|c|) \cdot n_t ds
\leq \|D_t [\tau_t (-\nabla_{\tau} v^\perp + \nabla_{\tau} n_t \cdot v)]\|_{L^\infty(\Gamma_t)} \|\nabla_{\tau} D_t \nabla_{\tau} v\|_{\dot{H}^{-1/2}(\Gamma_t)} \|\nabla_{\tau} (3 - 3|c|) \cdot n_t\|_{\dot{H}^{1/2}(\Gamma_t)}
\leq P(E(t))\|D_t \nabla_{\tau} v\|_{H^{1/2}(\Gamma_t)} \|D_t 3\|_{H^1(\Omega_t)}
\leq P(E(t))(1 + E_1(t)).
\]

As a result, we summarize that

\[
|I_{12}| \leq P(E(t))(1 + E_1(t)^{3/2}).
\]

Moreover, similar arguments as for $I_{12}$, we also have

\[
|I_{13}| \leq P(E(t))(1 + E_1(t)).
\]

Together with all these estimates above for $I_{11}$ to $I_{14}$, we can go back to $I_1$ expression and integrate on both sides with respect to time $t$ on $[0, T]$ to find

\[
\int_0^T I_1 dt \leq \int_{\Gamma_t} D_t \nabla_{\tau} v \cdot \tau_t (-\nabla_{\tau} v^\perp + \nabla_{\tau} n_t \cdot v) D_t (3 - 3|c|) \cdot n_t ds|_0^T
+ \int_0^T P(E(t))(1 + E_1(t)^{3/2}) dt + \frac{1}{16} \int_0^T F_1(t).
\]

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Moreover, we have for the first integral on the right side above the following estimate
\[
\int_{\Gamma_t} D_t \Delta_{\Gamma_t} v \cdot \tau_t (-\nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v) D_t (3 - 3|^c|) \cdot n_t ds
\]
\[
= \int_{\Gamma_t} \left( [D_t, \nabla_{\tau_t}] \nabla_{\tau_t} v - D_t (\nabla_{\tau_t} \tau_t \cdot \tau_t \nabla_{\tau_t} v) \right) \cdot \tau_t (-\nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v) D_t (3 - 3|^c|) \cdot n_t ds
\]
\[
+ \int_{\Gamma_t} \nabla_{\tau_t} D_t \nabla_{\tau_t} v \cdot \tau_t (-\nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v) D_t (3 - 3|^c|) \cdot n_t ds.
\]

Similar arguments as in (5.34) and checking carefully on the highest-order terms, we derive
\[
\int_{\Gamma_t} D_t \Delta_{\Gamma_t} v \cdot \tau_t (-\nabla_{\tau_t} v^\perp + \nabla_{\tau_t} n_t \cdot v) D_t (3 - 3|^c|) \cdot n_t ds \leq P(E(t))(1 + E_1(t)^{\delta_0})
\]
with the number \(\delta_0 \in (0, 1)\) as above.

Consequently, we have the following estimate for \(I_1\):
\[
\left| \int_0^T I_1 dt \right| \leq P(E(0))(1 + E_1(0)^{\delta_0}) + \sup_{t \in [0, T]} P(E(t))(1 + E_1(t)^{\delta_0})
\]
\[
+ \int_0^T P(E(t))(1 + E_1(t)^{3/2}) dt + \frac{1}{16} \int_0^T F_1(t).
\]

Estimates of \(I_2\). The integral \(I_2\) can be handled in the same way as \(I_1\). We simply conclude that
\[
\left| \int_0^T I_2 dt \right| \leq P(E(0))(1 + E_1(0)^{\delta_0}) + \sup_{t \in [0, T]} P(E(t))(1 + E_1(t)^{\delta_0})
\]
\[
+ \int_0^T P(E(t))(1 + E_1(t)^{3/2}) dt + \frac{1}{16} \int_0^T F_1(t).
\]

In the end, combing (5.35) with (5.36), the proof is finished.

Next, we deal with \(I_R\).

Lemma 5.6 One has
\[
|I_R| \leq \frac{1}{8} F_1(t) + P(E_1(t)).
\]

Proof. First, to simplify the notations in the first integral of \(I_R\), we denote by
\[
C_P = (D_t \nabla P_{3,v} + \nabla P_{\nabla P_{3,v}} + \nabla P_{D_t 3,v}) \cdot n_t.
\]

Thanks to the definition of \(P_{w,v}\) (see (1.10)), we have
\[
C_P = C_{3,v} + C_{\nabla P_{3,v}} + C_{D_t 3,v} - D_t n_t \cdot \nabla P_{3,v} - D_t ((3 \cdot \tau_t) \nabla_{\tau_t} v \cdot n_t) - (\nabla_{\tau_t} P_{3,v} + D_t 3 \cdot \tau_t) \nabla_{\tau_t} v \cdot n_t.
\]

By (5.5), (5.8), Lemma 5.2 and (5.27), we show immediately
\[
\|C_P\|_{H^1(\Gamma_t)} + \|\nabla_{n_t} H(P_{t,v})\|_{H^1(\Gamma_t)} \leq P(E(t))(1 + E_1(t)^{1/2}).
\]

Now we are ready to have estimate of \(I_R\). For the first integral in \(I_R\), we integrate by parts to derive
\[
\int_{\Gamma_t} \Delta_{\Gamma_t} D_t 3^\perp C_P ds = \nabla_{\tau_t} D_t 3^\perp C_P |_{P_t} - \int_{\Gamma_t} \nabla_{\tau_t} D_t 3^\perp \cdot \nabla_{\tau_t} C_P ds
\]
\[
\leq \frac{1}{8} F_1(t) + P(E(t))(1 + E_1(t))
\]
where (5.8), (5.11), (5.29), Lemma 5.3 and (5.38) are used.

Moreover, we have from (5.2) and (5.38) that

$$
\int_{\Gamma_t} D_t h_v C_P ds = \int_{\Gamma_t} [\Delta_{\Gamma_t} v \cdot D_t n_t + 2D^2 v(\tau_t, (\nabla_{\tau_t} v)^T) \cdot n_t] C_P ds
\leq P(E(t))(1 + E_1(t)),
$$

where similar analysis as in the proof of the previous lemma is applied.

On the other hand, the second integral in $I_R$ can also be handled by a similar argument as above. As a result, the proof is finished.

\[\boxdot\]

5.3.2 Right side of (5.31).

We firstly deal with the integral involving $R_2$.

**Lemma 5.7** One has for some $\delta_0 \in (0, 1)$ the following estimate:

$$
\left| \int_0^T \int_{\Omega_t} R_2 \cdot (D_t^2 \hat{\mathcal{J}} - \nabla \mathcal{H}(P_{t,2})) dX dt \right| \leq P(E(0))(1 + E_1(0)^{\delta_0}) + \sup_{t \in [0,T]} P(E(t))(1 + E_1(t)^{\delta_0})
\left. + \int_0^T P(E(t))(1 + E_1(t)^{3/2}) dt + \frac{1}{16} \int_0^T F_1(t). \right.
$$

**Proof.** Recalling from (5.30), we deal with the terms in $R_2$ one by one.

- Estimates of $D_t(\hat{R}_0 - \nabla \mathcal{H}(D_t^2 P_{t,v}))$. In fact, one knows directly from (5.1) that

$$
\hat{R}_0 - \nabla \mathcal{H}(D_t^2 P_{t,v}) = -\sigma \nabla \mathcal{H}(J \cdot \Delta \tau_t n_t) + \sigma \nabla \mathcal{H}([n_t, \Delta \tau_t] \cdot \nabla P_{t,v}) + \sigma \nabla \mathcal{H}(R_1 + h_v) + A_1 + A_2 + A_3
$$

where recall that $R_1 + h_v$ only contains lower-order derivatives like $\partial v, \partial n_t, \kappa$ and $A_1$ to $A_3$ are defined in (3.27)-(3.29).

As a result, checking term by term on $\hat{R}_0 - \nabla \mathcal{H}(D_t^2 P_{t,v})$, one finds that it contains $\partial D_t v, D_t J, \partial v$ and other lower-order terms, so acting $D_t$ on $\hat{R}_0 - \nabla \mathcal{H}(D_t^2 P_{t,v})$ and applying lemmas in Section 4.1 and Section 5.2 lead to the following estimate:

$$
\left\| D_t(\hat{R}_0 - \nabla \mathcal{H}(D_t^2 P_{t,v})) \right\|_{L^2(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}).
$$

- Estimates of $[\nabla, D_t] \mathcal{H}(\Delta \tau_t \hat{\mathcal{J}} - h_v)$. It is straightforward to show by (5.19) and Lemma 2.7 that

$$
\left\| [\nabla, D_t] \mathcal{H}(\Delta \tau_t \hat{\mathcal{J}} - h_v) \right\|_{L^2(\Omega_t)} \leq P(E(t))\|\Delta \tau_t \hat{\mathcal{J}} - h_v\|_{H^{1/2}(\Gamma_t)} \leq P(E(t))(1 + E_1(t)^{1/2}).
$$

- Estimates of $\nabla [\mathcal{H}, D_t](\Delta \tau_t \hat{\mathcal{J}} - h_v)$. Recalling (2.14) and using Lemma 2.7, (5.19) and applying variational estimates as in page 33 [52] imply that

$$
\left\| \nabla [\mathcal{H}, D_t](\Delta \tau_t \hat{\mathcal{J}} - h_v) \right\|_{L^2(\Omega_t)} \leq P(E(t))\|\mathcal{H}(\Delta \tau_t \hat{\mathcal{J}} - h_v)\|_{H^1(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}).
$$

- Estimates of $\nabla \mathcal{H}(\Delta \tau_t, D_t \hat{\mathcal{J}})$. To estimate this term, we use similar arguments as in the proof of Lemma 5.5.

First, using Green’s Formula, we obtain

$$
\int_{\Omega_t} \nabla \mathcal{H}(\Delta \tau_t, D_t \hat{\mathcal{J}}) \cdot D_t^2 \hat{\mathcal{J}} dX = \int_{\Gamma_t} [\Delta \tau_t, D_t \hat{\mathcal{J}}] \cdot D_t^2 \hat{\mathcal{J}} \cdot n_t ds
= \int_{\Gamma_t} (\nabla_{\tau_t} v \cdot \tau_t - (\nabla_{\tau_t} v \cdot \tau_t) \nabla_{\tau_t} v) \cdot D_t \hat{\mathcal{J}} \cdot n_t ds
= \int_{\Gamma_t} 2(\nabla_{\tau_t} v \cdot \tau_t) \nabla_{\tau_t} \hat{\mathcal{J}} \cdot n_t ds
= \int_{\Gamma_t} (\nabla_{\tau_t} v \cdot \tau_t + D_t(\nabla_{\tau_t} v \cdot \tau_t) - (\nabla_{\tau_t} v \cdot \tau_t)(\nabla_{\tau_t} v \cdot \tau_t)) \nabla_{\tau_t} \hat{\mathcal{J}} \cdot n_t ds.
$$
These two integrals can be handled in a similar way as before, and here we only give the details for the first part in the second integral above. Moreover, the analysis for \( \int_{\Omega} \nabla \mathcal{H}(\Delta_{\Gamma_t}, D_t\tau \cdot \nabla \mathcal{H}(\tau, v))dX \) can be done in a similar and easier way thanks to (5.27), and hence estimates for all these remainder parts are omitted.

Recalling (5.4) and the definition of \( C_P \), we rewrite the first part in the second integral above as follows:

\[
\int_{\Gamma_t} \nabla_{\tau_t}(\nabla_{\tau_t}v \cdot \tau_t)\nabla_{\tau_t}(\tau_t \cdot n)D_t^2 \cdot n_tds
= \int_{\Gamma_t} \nabla_{\tau_t}(\nabla_{\tau_t}v \cdot \tau_t)\nabla_{\tau_t}(\tau_t \cdot n)D_t^2 \cdot n_tds + \int_{\Gamma_t} \nabla_{\tau_t}(\nabla_{\tau_t}v \cdot \tau_t)\nabla_{\tau_t}(\tau_t \cdot n)C_Pds \equiv I_3 + I_4.
\]

Similarly as the estimates for \( I_1 \) in the proof of Lemma 5.5, one has for \( I_3 \) that

\[
I_3 = \int_{\Gamma_t} \nabla_{\tau_t}(\nabla_{\tau_t}v \cdot \tau_t)\nabla_{\tau_t}(\tau_t \cdot n)D_t^2 (\tau - \tau|_c) \cdot n_tds + \int_{\Gamma_t} \nabla_{\tau_t}(\nabla_{\tau_t}v \cdot \tau_t)\nabla_{\tau_t}(\tau_t \cdot n)D_t^2 (\tau|_c) \cdot n_tds
\leq \frac{1}{8} F_1(t) \bigg( 1 + \left( 1 + E_1(t) \right) + \frac{d}{dt} \int_{\Gamma_t} \nabla_{\tau_t}(\nabla_{\tau_t}v \cdot \tau_t)\nabla_{\tau_t}(\tau_t \cdot n)D_t (\tau - \tau|_c) \cdot n_tds
- \int_{\Gamma_t} D_t \nabla_{\tau_t}(\nabla_{\tau_t}v \cdot \tau_t)\nabla_{\tau_t}(\tau_t \cdot n)D_t (\tau - \tau|_c) \cdot n_tds
- \int_{\Gamma_t} \nabla_{\tau_t}(\nabla_{\tau_t}v \cdot \tau_t)D_t \nabla_{\tau_t}(\tau_t \cdot n)D_t (\tau - \tau|_c) \cdot n_tds
\leq \frac{1}{8} F_1(t) \bigg( 1 + \left( 1 + E_1(t) \right) + \frac{d}{dt} \int_{\Gamma_t} \nabla_{\tau_t}(\nabla_{\tau_t}v \cdot \tau_t)\nabla_{\tau_t}(\tau_t \cdot n)D_t (\tau - \tau|_c) \cdot n_tds,
\]

where we use (5.20), Lemma 5.2 and Lemma 5.3 to have

\[
\int_{\Gamma_t} \nabla_{\tau_t}(\nabla_{\tau_t}v \cdot \tau_t)D_t \nabla_{\tau_t}(\tau_t \cdot n)D_t (\tau - \tau|_c) \cdot n_tds
\leq P(E(t)) \bigg( 1 + \sum_i \left\| \nu^i (\nabla_{\tau_t}v \cdot \tau_t) \right\|_{L^2(\Gamma_t)} \bigg) \leq P(E(t)) \bigg( 1 + \sum_i \left\| \nabla_{\tau_t}v \cdot \tau_t \right\|_{L^2(\Gamma_t)} \bigg)
\]

with \( 1 > \delta > 3 - \pi/\omega_i \in (\pi/3, \pi/2) \) and

\[
\left\| D_t \nabla_{\tau_t}(\tau_t \cdot n) \right\|_{L^2(\Gamma_t)} \leq \left\| (\nabla_{\tau_t}v \cdot \tau_t)D_t (\tau_t \cdot n) \right\|_{L^2(\Gamma_t)} + \left\| \nabla_{\tau_t}D_t (\tau_t \cdot n) \right\|_{L^2(\Gamma_t)}
\leq P(E(t)) \left( 1 + \sum_i \left\| \nabla_{\tau_t}v \cdot \tau_t \right\|_{L^2(\Gamma_t)} \bigg)
\]

As a result, we have

\[
\left| \int_0^T I_3 dt \right| \leq P(E(0)) \left( 1 + E_1(0)^{\delta_0} \right) + \sup_{t \in [0,T]} \left( 1 + E_1(t)^{\delta_0} \right)
+ \int_0^T P(E(t))^2(1 + E_1(t))^{3/2} dt + \frac{1}{64} \int_0^T F_1(t).
\]

Since similar arguments can be applied to \( I_4 \) and other integrals, we conclude directly

\[
\left| \int_0^T \int_{\Omega_t} \nabla \mathcal{H}(\Delta_{\Gamma_t}, D_\tau \tau \cdot \nabla \mathcal{H}(\tau, v))dX dt \right| \leq P(E(0)) \left( 1 + E_1(0)^{\delta_0} \right) + \sup_{t \in [0,T]} \left( 1 + E_1(t)^{\delta_0} \right)
+ \int_0^T P(E(t))^2(1 + E_1(t))^{3/2} dt + \frac{1}{64} \int_0^T F_1(t).
\]

- Estimates of \( D_t(\nabla P_{\forall P_{\exists, \tau, v}}) \). Recalling the definition of \( P_{\exists, \tau, v} \) and \( P_{\forall P_{\exists, \tau, v}} \) by (1.10), one has firstly the estimate (5.13). Moreover, one can take \( D_t \) on the system of \( P_{\forall P_{\exists, \tau, v}} \) as in (4.39) to obtain the system for \( D_t P_{\forall P_{\exists, \tau, v}} \).
Consequently, checking term by term and applying Lemma 4.6, Lemma 5.1, one derives
\[ \|D_t(\nabla P \nabla P_{t, \nu})\|_{L^2(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}). \]

- Estimates of the last term in \( R_2 \): Thanks to (2.14), (5.12), (5.27) and Lemma 5.1, we have
\[ \|D_t \nabla H[D_t \left( \nu \cdot (\nabla P_{t, \nu}) \right) + v \cdot (\nabla P_{t, 1})] \|_{L^2(\Omega_t)} \leq P(E(t))(1 + E_1(t)^{1/2}). \]

In the end, summing up all these estimates above, the proof is finished.

At this moment, it remains to handle the last integral on the right side of (5.31). In fact, similar arguments as in the proof of Lemma 4.8, we can conclude the following estimate:
\[
\int_0^T \int_{\Omega_t} \left( D_t \nabla P_{J, \nu} + D_t \nabla P_{J, \nu} \cdot (D_t^2 \mathbf{3} - \nabla H(P_{t, 2})) \right) dX dt \\
\leq P(E(0))(1 + E_1(0)^{\delta_0}) + \sup_{t \in [0, T]} P(E(t))(1 + E_1(0)^{\delta_0}) + \int_0^T P(E(t))(1 + E_1(t)^{3/2}) dt + \frac{1}{16} \int_0^T F_1(t). 
\]

5.3.3 The end of the higher-order energy estimate.

Summing up all the estimates in the previous two subsections, we finally arrive at:
\[
\sup_{t \in [0, T]} \left( \|D_t^2 \mathbf{3} - \nabla H(P_{t, 2})\|_{L^2(\Omega_t)}^2 + \int_{\Gamma_t} |\nabla_{\tau} D_t \mathbf{3}|^2 \right) + \frac{1}{4} \int_0^T F_1(t) \\
\leq P(E(0))(1 + E_1(0)^{\delta_0}) + \sup_{t \in [0, T]} P(E(t))(1 + E_1(0)^{\delta_0}) + \int_0^T P(E(t))(1 + E_1(t)^{3/2}) dt.
\]

Thanks to (5.27) and Lemma 5.1, we have for a number \( \delta_1 > 0 \) small enough such that
\[
\sup_{t \in [0, T]} E_1(t) + \int_0^T F_1(t) \leq P(E(0))(1 + E_1(0)^{\delta_0}) + \delta_1 \sup_{t \in [0, T]} E_1(t) + \int_0^T P(E(t))(1 + E_1(t)^{3/2}) dt.
\]

Therefore, we’ve finished the proof of Theorem 5.1.

6 Well-posedness of system (WW)

In this section, we use Picard iteration to prove the existence of solutions to (WW). The main idea is the same as [53], although some necessary modifications are needed and presented here. As a result, we only show the sketch of the proof. For more details, see [53].

6.1 Definitions of surfaces and domains

Following [61, 53], we introduce a map \( \Phi_{S_t} \) on the boundary \( S_t = \Gamma_t \cup \Gamma_b \) to fix the moving domain \( \Omega_t \). To start with, we choose a reference domain \( \Omega_* \) with upper surface \( \Gamma_* \) and bottom \( \Gamma_{bs} \), which can be taken as the initial domain \( \Omega_0 \) without loss of generality. The contact points of \( \Omega_* \) are denoted by \( p_{i*} (i = l, r) \) and the other notations follow similarly.

we define a unit upward vector field \( \mu \in H^s(\Gamma_*, S^1) \) with some large \( s \) satisfying
\[
\mu \cdot n_{i*} \geq c_0 \quad \text{on} \quad \Gamma_{i*}, \quad \text{and} \quad \mu|_{p_{i*}} = -\tau_{bs}|_{p_{i*}}, \quad \mu|_{p_{r*}} = \tau_{bs}|_{p_{r*}}
\]

for some fixed constant \( c_0 \in (0, 1) \). Here we notice that the conditions above hold at \( p_{i*}, p_{r*} \).
Applying the Implicit Function Theorem, there exists a small constant \(d_0 > 0\) such that the map
\[
\Phi : \Gamma_{t^*} \times [-d_0, d_0] \to \mathbb{R}^2 \quad \text{with} \quad \Phi(p, d) = p + d \mu(p)
\]
is an \(H^s\) diffeomorphism from its domain to a neighborhood of \(\Gamma_s\).

Consequently, this map identifies each upper surface \(\Gamma_t\) near \(\Gamma_{t^*}\) with a unique function
\[
d_{\Gamma_t} : \Gamma_{t^*} \to \mathbb{R}
\]
and we can define the following map
\[
\Phi_{S_t} : \Gamma_{t^*} \to \Gamma_t \subset \mathbb{R}^2 \quad \text{with} \quad \Phi_{S_t}(p) = p + d_{\Gamma_t}(p)\mu(p).
\]

Meanwhile, we can use the function \(d_{\Gamma_t}(p)\) as the expression of the upper surface \(\Gamma_t\), and we have at the corner points that
\[
d_{\Gamma_t}(p_{i^*}) = p_i, \quad i = l, r.
\]

Moreover, \(\Phi_{S_t}\) can be extended to the entire boundary \(S_* = \Gamma_{t^*} \cup \Gamma_{b_*}\). Consequently, we obtain the map on \(S_*\):
\[
\Phi_{S_t} : S_* \to \Omega_t.
\]

Using the harmonic extension, we define the following map on \(\Omega_*\):
\[
T_{S_t} : \Omega_* \to \Omega \quad \text{with} \quad T_{S_t} = H_* \Phi_{S_t} - Id_{S_*} + Id.
\]

Here \(H_*(\Phi_{S_t} - Id_{S_*})\) is the harmonic extension of \(\Phi_{S_t} - Id_{S_*}\) satisfying
\[
\begin{cases}
\Delta H_* (\Phi_{S_t} - Id_{S_*}) = 0 & \text{in} \ \Omega_*, \\
H_* (\Phi_{S_t} - Id_{S_*}) = d_{\Gamma_t} \mu & \text{on} \ \Gamma_{t^*}, \quad H_* (\Phi_{S_t} - Id_{S_*}) = \Phi_{S_t} \mid_{\Gamma_*} - Id_{\Gamma_*}.
\end{cases}
\]

### 6.2 Recovery of the velocity

When the domain \(\Omega_t\) is defined by \(T_{S_t}\), we can define the velocity \(v\) by the free surface function \(d_{\Gamma_t}\). To begin with, the kinematic condition on \(\Gamma_t\) in (WW) is rewritten into
\[
\partial_t \Phi_{S_t} \cdot (n_t \circ \Phi_{S_t}) = (v \cdot N_t) \circ \Phi_{S_t} \quad \text{with} \quad \partial_t \Phi_{S_t} = (\partial_t d_{\Gamma_t}) \mu \quad \text{on} \ \Gamma_*.
\]

So we obtain
\[
\partial_t d_{\Gamma_t} = \frac{(v \cdot n_t) \circ \Phi_{S_t}}{\mu \cdot (n_t \circ \Phi_{S_t})} \quad \text{i.e.} \quad v \cdot n_t = (\partial_t d_{\Gamma_t} \mu) \circ \Phi_{S_t}^{-1} \cdot n_t. \quad (6.39)
\]

Due to the assumption that the velocity \(v\) is irrotational, we define \(v\) by
\[
v = \nabla \phi \quad (6.40)
\]
with \(\phi\) satisfying
\[
\begin{cases}
\Delta \phi = \xi \gamma & \text{in} \ \Omega_t, \\
\n_t \phi \big|_{\Gamma_t} = (\partial_t d_{\Gamma_t} \mu) \circ \Phi_{S_t}^{-1} \cdot n_t, \quad \n_b \phi \big|_{\Gamma_b} = 0,
\end{cases} \quad (6.41)
\]

where
\[
\gamma = |\Omega_t|^{-1} \quad \text{and} \quad \xi = \int_{\Gamma_t} v \cdot n_t \, ds = \int_{\Gamma_t} (\partial_t d_{\Gamma_t} \mu) \circ \Phi_{S_t}^{-1} \cdot n_t \, ds.
\]

Moreover, we define as in [53] that
\[
D_{t^*} = \partial_t + \nabla v^*.
\]

where \(v^* = D \Phi_{S_t}^{-1} (v^T \circ \Phi_{S_t} - \partial_t d_{\Gamma_t} \mu^T)\). A direct computation shows that
\[
(D_{t^*} f) \circ \Phi_{S_t} = D_{t^*} (f \circ \Phi_{S_t}),
\]
for a function \(f\) on \(\Gamma_t\).
6.3 The modified formulation and the precise form of the main theorem

Before we construct the approximate solutions, we derive a new equation modified from the previous sections, which turns out to be more convenient in this part.

First, we define \( \mathfrak{R}_a \) based on the definition of \( \mathfrak{R} \):

\[
\mathfrak{R}_a = \mathfrak{R} + \sigma a d_{\Gamma_t} \circ \Phi_{S_l}^{-1} - \sigma(\kappa + a d_{\Gamma_t} \circ \Phi_{S_l}^{-1}) - P_{e,v} \quad \text{on} \quad \Gamma_t
\]

for some constant \( a > 0 \), where \( \kappa \) can be expressed by \( d_{\Gamma_t} \) and \( P_{e,v} \) is defined by (3.19). So we have

\[
\mathfrak{J}_a = \nabla \mathcal{H}(\mathfrak{R}_a) = \mathfrak{J} + \sigma a \nabla \mathcal{H}(d_{\Gamma_t} \circ \Phi_{S_l}^{-1}).
\]

Applying (3.25) and using \( \mathfrak{R}_a \) instead of \( \mathfrak{R} \), we obtain

\[
D_t^2 \mathfrak{J}_a + \sigma(a - \Delta_{\Gamma_t}) \mathcal{N}(\mathfrak{R}_a) = R_{a,0},
\]

where \( R_{a,0} \) is defined by

\[
R_{a,0} = \sigma R_1 - D_t^2 P_{e,v} + \sigma[n_t, \Delta_{\Gamma_t}] \cdot \mathfrak{J} + \sigma \Delta_{\Gamma_t} \nabla P_{e,v} \cdot n_t + 2\sigma \nabla n_t \cdot \nabla \mathfrak{J} + \sigma a D_t^2(d_{\Gamma_t} \circ \Phi_{S_l}^{-1}) - \sigma^2 a \Delta_{\Gamma_t} N(d_{\Gamma_t} \circ \Phi_{S_l}^{-1}) + \sigma a \mathcal{N}(\mathfrak{R}_a).
\]

Acting \( \nabla \mathcal{H} \) on both sides of the above equation, we get the equation of \( \mathfrak{J}_a \):

\[
D_t^2 \mathfrak{J}_a + \sigma \nabla \mathcal{H}[(a - \Delta_{\Gamma_t}) \mathfrak{J}_a^\perp + h_v] = R_a,
\]

where

\[
R_a = \nabla \mathcal{H}(R_{a,0} + \sigma h_v) - [\nabla \mathcal{H}, D_t^2] \mathfrak{R}_a,
\]

and recall that \( h_v \) is defined in (5.2) and comes from \( R_1 \).

Meanwhile, the condition (4.50) at the contact points are rewritten as

\[
D_t \mathfrak{J}_a = \pm \frac{\sigma^2}{\rho c} \sin \omega_i (\nabla \mathfrak{J}_a)^\perp n_b + R_{c1,a} \quad \text{at} \quad p_t(i = l, r)
\]

with

\[
R_{c1,a} = R_{c1} + \sigma a D_t \nabla \mathcal{H}(d_{\Gamma_t} \circ \Phi_{S_l}^{-1}) + \sigma a \frac{\sigma^2}{\rho c} (n_b \cdot n_t)(\nabla n_t \nabla \mathcal{H}(d_{\Gamma_t} \circ \Phi_{S_l}^{-1}))^\perp.
\]

Next, we consider how to recover the free surface and the domain from \( \mathfrak{R}_a \), which is slightly different from [53]. In [53], we use the equation of \( \mathfrak{R}_a = \mathcal{N}(\kappa + a d_{\Gamma_t} \circ \Phi_{S_l}^{-1}) \) together with the boundary information \( d_i = d_{\Gamma_i}|_{\Gamma_t} \) of \( d_{\Gamma_t} \) to recover the free surface \( d_{\Gamma_t} \), so the system of \( (\mathfrak{R}_a, d_i, d_r) \) is needed; In this paper, we use the quantity \( \mathfrak{R}_a \) instead of \( \mathfrak{R}_a \), where an extra \( P_{e,v} \) is added here in (6.42). Moreover, we will need to use the equation and norms of \( \mathfrak{J}_a \) in the iteration scheme, where \( \mathfrak{R}_a \) can be retrieved. In fact, we have \( \mathfrak{J}_a^\perp = \mathfrak{J}_a \cdot n_t = \mathcal{N}(\mathfrak{R}_a) \) on \( \Gamma_t \). Therefore, to identify \( \mathfrak{R}_a \), we look at the Neumann-boundary elliptic system of \( \mathcal{H}(\mathfrak{R}_a) \) with the compatibility condition \( \int_{\Gamma_t} \mathfrak{J}_a^\perp d s = 0 \). We know immediately that there exists a unique solution \( \mathcal{H}(\mathfrak{R}_a) \) up to an additive constant to this system. As a result, as long as we have \( \mathfrak{J}_a \), we obtain \( \mathfrak{R}_a \). (One can also check Lemma 2.5 in [53].)

Consequently, to recover \( d_{\Gamma_t} \), which is the key to recover the water-waves system, we need the system of \( (\mathfrak{R}_a, P_{e,v}, d_i, d_r) \). As long as we have proved the existence of the solution to this system, we obtain immediately the following quantity

\[
\kappa + a d_{\Gamma_t} \circ \Phi_{S_l}^{-1} = \sigma^{-1}(\mathfrak{R}_a + P_{e,v}) \quad \text{with the boundary information} \ d_i, d_r.
\]

As a result, the the desired function \( d_{\Gamma_t} \) can be solved directly from these quantities above in a similar way as in Proposition 4.2 [53], and then we can finally recover our water-waves system (WW).
Based on the analysis above, we need to give the boundary condition of \( d_r \), which is deduced from (6.39) (for more details, see (4.27) in [53]). In fact, one has the following evolution equations for \( d_i(t) = d_{ri}(p_{i*}) \) (\( i = l, r \)):

\[
d''_i(t) = \mathfrak{B}_i, \quad i = l, r,
\]

where

\[
\mathfrak{B}_i = - \frac{1}{\mu \cdot (n_t \circ \Phi_{S\ell})} \left( \mu \cdot (n_t \circ \Phi_{S\ell}) \nabla_v \cdot \partial_t d_{\ell}, + \nabla_v \cdot \mu \cdot (n_t \circ \Phi_{S\ell}) \partial_t d_{\ell}, + \sigma K_{\ell} \circ \Phi_{S\ell} \right) + \sigma^2 aN(d_{\ell}, \circ \Phi_{S\ell}^{-1}) \circ \Phi_{S\ell} + (\nabla P_{v,v} + g) \circ \Phi_{S\ell} \cdot (n_t \circ \Phi_{S\ell}) \bigg|_{p_{i*}}.
\]

We rewrite the equation for \( P_{v,v} \) by (4.39):

\[
D_t P_{v,v} = \Delta_N^{-1}(h_p, f_p, g_p),
\]

where \( \Delta_N^{-1} \) means solving the Neumann-boundary system (4.39) with \( \int_{\Gamma_i} D_t P_{v,v} dX = 0 \), and the right-side functions are

\[
h_p = 2\tau(\nabla v \cdot \nabla v \nabla v) + 2\tau((\nabla P_{v,v} + \mathfrak{H}) \nabla v) + 2\tau(\nabla v \nabla v^2 P_{v,v}), \quad f_p = C'_v(t) + \nabla n_v \cdot \nabla P_{v,v}\bigg|_{\Gamma_i},
\]

and

\[
g_p = -(mJ + \nabla P_{v,v} + g) \cdot (\nabla n_b + (\nabla n_b)^*) \cdot v + v \cdot D_t(\nabla n_b) \cdot v + \nabla n_b \cdot \nabla P_{v,v}.
\]

As a result, we sum up the system of \( (\mathfrak{R}_a, P_{v,v}, d_l, d_r) \) as follows:

\[
\begin{align*}
D_t^2 \mathfrak{R}_a + \sigma(a - \Delta_{\Gamma_\ell})N(\mathfrak{R}_a) &= R_{a,0}, \\
D_t \mathfrak{R}_a &= \pm \sigma^2 \beta_{\ell}^{-1} \sin \omega(\nabla_v \mathfrak{R}_{a})^2 \mathfrak{R}_a + R_{c1,a}, \quad \text{at} \quad p_i(i = l, r), \\
D_t P_{v,v} &= \Delta_N^{-1}(h_p, f_p, g_p), \\
\frac{d^2}{dt^2}d_i(t) &= \mathfrak{B}_i, \quad i = l, r.
\end{align*}
\]

Based on these preparations above, we are finally ready to state our precise form of Theorem 1.1. We start with introducing the space \( \Sigma \) for given \( T, L > 0 \):

\[
\Sigma = \{ (\mathfrak{R}_a, P_{v,v}, d_l, d_r) | (\mathfrak{R}_a, P_{v,v}, d_l, d_r) \|_{\Sigma} \leq L \}
\]

where the norm

\[
\| (\mathfrak{R}_a, P_{v,v}, d_l, d_r) \|_{\Sigma} \triangleq \| d_t(\mathfrak{R}_a \circ \Phi_{S\ell}) \|_{C([0,T];L^2(\Omega))} + \| (\mathfrak{R}_a \circ \Phi_{S\ell}) \cdot n_{t*} \|_{C([0,T];H^1(\Gamma_\ell))}
\]

\[
+ \| P_{v,v} \circ \Phi_{S\ell} \|_{C([0,T];H^{5/2}(\Omega))} + \sum_{i=l,r} \left( \| d_i \|_{C([0,T])} + |d(d_i)/dt|_{C([0,T])} \right).
\]

Meanwhile, according to the higher-order energy \( E_h(t) \), we also define the space (for given \( L_1 > 0 \))

\[
\Sigma_h = \{ (\mathfrak{R}_a, P_{v,v}, d_l, d_r) | (\mathfrak{R}_a, P_{v,v}, d_l, d_r) \|_{\Sigma_h} \leq L_1 \}
\]

by the norm

\[
\| (\mathfrak{R}_a, P_{v,v}, d_l, d_r) \|_{\Sigma_h} \triangleq \| (\mathfrak{R}_a, P_{v,v}, d_l, d_r) \|_{\Sigma} + \| d_t(\mathfrak{R}_a \circ \Phi_{S\ell}) \cdot n_{t*} \|_{L^\infty([0,T];H^1(\Gamma_\ell))}
\]

\[
+ \| \partial_t^2(\mathfrak{R}_a \circ \Phi_{S\ell}) \|_{L^\infty([0,T];L^2(\Omega))}.
\]

The initial data is given by

\[
\begin{align*}
(\mathfrak{R}_a \circ \Phi_{S\ell}) \cdot n_{t*} \bigg|_{t=0} &= \bar{\mathfrak{R}}_{a,0}, \quad \partial_t(\mathfrak{R}_a \circ \Phi_{S\ell}) \bigg|_{t=0} = \bar{\mathfrak{R}}_{a,1}, \\
P_{v,v} \circ \Phi_{S\ell} \bigg|_{t=0} &= \bar{P}_{v,v}, \quad d_l(0) = d_{l,0}, \quad d_r(0) = d_{r,0}.
\end{align*}
\]

Now we can present the local well-posedness theorem.

**Theorem 6.1** Assume that the initial data \( (\mathfrak{R}_{a,0}, \mathfrak{R}_{a,1}, \bar{P}_{v,v,0}, d_{l,0}, d_{r,1}) \in H^1(\Gamma_\ell) \times L^2(\Omega_\ell) \times H^{5/2}(\Omega_\ell) \times \mathbb{R}^2 \) and initial contact angles \( \omega_{i0} \in (0, \pi/2) \) for \( i = l, r \). When the compatibility conditions (1.8) at \( t = 0 \) are satisfied for \( k = 0, 1, 2, 3 \), there exists a unique solution \( (\mathfrak{R}_a, P_{v,v}, d_l, d_r) \in \Sigma_h \) to system (6.44). Moreover, system (6.44) is locally well-posed with \( (\mathfrak{R}_a, P_{v,v}, d_l, d_r) \) depending continuously on the initial data in \( \Sigma \).
6.4 Iteration scheme

In this subsection, we present the iteration scheme. First of all, we set the initial boundary $S_0 = S_*$ without loss of generality. To simplify the notations, we denote by

$$D_{ts} = \partial_t + v_s^k \cdot \nabla, \quad D_t = \partial_t + v^k \cdot \nabla,$$

when no confusion will be made.

When we have $(\tilde{R}^k_{a,v}, P^{k+1}_{v,v}, d^{k+1}_i, d^{k+1}_r)$, the linear system of $(\tilde{R}^{k+1}_a, P^{k+1}_{v,v}, d^{k+1}_i, d^{k+1}_r)$ for the iteration scheme is set to be

$$
\begin{align*}
D_{ts}^2(\tilde{R}^{k+1}_a \circ \Phi^k_{S_i}) + \sigma (a - \Delta_{T_i}) N(\tilde{R}^{k+1}_a) \circ \Phi^k_{S_i} &= R^k_{a,0} \circ \Phi^k_{S_i}, \\
D_{ts}(\tilde{N}^{k+1}_a \circ \Phi^k_{S_i}) &= \pm \sigma^2 \beta c \sin \omega_i (\nabla_{T_i} \tilde{R}^{k+1}_a) \perp \Phi^k_{S_i} + R^k_{c,1,a} \circ \Phi^k_{S_i}, \quad p_{i,s}, \quad i = l, r, \\
D_{ts}(P^{k+1}_{v,v} \circ \Phi^k_{S_i}) &= \Delta^{-1}_{N}(h^k_{i_p}, f^k_{i_p}, g^k_{i_p}) \circ \Phi^k_{S_i}, \\
\frac{d^2}{d^2t}d^{k+1}_i(t) &= \Phi^k_{S_i}, \quad i = l, r.
\end{align*}
$$

(6.45)

Here we use the superscript $k$ on $R_{a,0}$ (for example) to denote that all the quantities there are obtained using $(\tilde{R}^k_{a,v}, P^{k}_{v,v}, d^{k}_i, d^{k}_r)$.

Moreover, we point out that the velocity $v$ in the definition of $R^k_{a,0}$ is given by

$$D_t v^k = -J^k - \nabla P_{v,v} - g,$$

while in the other quantities we use $v^k$ defined by (6.39)-(6.41). This happens due to the difference of regularities using these two definitions, which can be seen already in the previous sections.

Besides, the initial data is given by

$$
\begin{align*}
\left\{ \begin{array}{l}
(\tilde{\gamma}^{k+1}_a \circ \Phi^k_{S_i}) \cdot n^k_{i_0} |_{t=0} = \tilde{\gamma}^0_{a,0}, \\
P^{k+1}_{v,v} \circ \Phi^k_{S_i} |_{t=0} = P^0_{v,v}, \\
d^{k+1}_i(0) = d^0_i, \\
\frac{d}{dt}d^{k+1}_i(0) = d^1_i.
\end{array} \right.
\end{align*}
$$

(6.46)

As a result, a similar proof as the proof of Proposition 5.1 in [53], we show the existence of the solution $(\tilde{R}^{k+1}_a, P^{k+1}_{v,v}, d^{k+1}_i, d^{k+1}_r)$ to the linear system (6.45)-(6.46). The details for the proof are omitted.

**Proposition 6.1** Let $(\tilde{\gamma}^0_{a,0}, \tilde{\gamma}^0_{a,1}, P^0_{v,v,0}, d^0_0, d^1_0) \in H^1(\Gamma_*) \times L^2(\Omega_*) \times H^{5/2}(\Omega_*) \times \mathbb{R}^2$, $\omega_0 \in (0, \pi/2)$ and $(\tilde{R}^k_{a,v}, P^k_{v,v}, d^k_i, d^k_r)$ be given correspondingly. Moreover we assume that the conditions for the corner points from (6.45) hold at $t = 0$. Then there exists a small $T > 0$ such that the system (6.45)-(6.46) has a unique solution on $[0, T]$.

6.5 Uniform estimates

Now we are ready to give the uniform estimates for the linear system (6.45)-(6.46). To begin with, we define the energy functional for $k \in \mathbb{N}$ as below:

$$E_{k+1}^k(t) = E_{k+1}^k(t) + E_{k+1}^{k+1}(t)$$

where $E_{low}^{k+1} (t)$ and $E_{high}^{k+1} (t)$ are defined by

$$E_{low}^{k+1} (t) = a \| (\tilde{\gamma}^{k+1}_a) \|_{L^2(\Gamma_*)}^2 + \| \nabla_{T} (\tilde{\gamma}^{k+1}_a) \|_{L^2(\Gamma_*)}^2 + \| D_t (\tilde{\gamma}^{k+1}_a) \|_{L^2(\Gamma_*)}^2 + \| P_{v,v}^{k+1} \|_{H^{5/2}(\Omega_*)}^2$$

$$+ \sum_{i=l,r} \left( |d_i^{k+1}(t)|^2 + \frac{d}{dt} |d_i^{k+1}(t)|^2 \right),$$

and

$$E_{high}^{k+1} (t) = a \| D_t (\tilde{\gamma}^{k}_a) \|_{L^2(\Gamma_*)}^2 + \| \nabla_{T} D_t (\tilde{\gamma}^{k}_a) \|_{L^2(\Gamma_*)}^2 + \| D_t (\tilde{\gamma}^{k}_a) \|_{L^2(\Gamma_*)}^2.$$
Moreover, the dissipation $F^{k+1}(t)$ is

$$F^{k+1}(t) = F_{low}^{k+1}(t) + F_{high}^{k+1}(t),$$

where

$$F_{low}^{k+1}(t) = \sum_{i,l,r} |(\sin \omega_i^k) \nabla_{\tau} (\tilde{\mathcal{J}}^{k+1})_l^r |^2, \quad F_{high}^{k+1}(t) = \sum_{i,l,r} |(\sin \omega_i^k) \nabla_{\tau} D_t (\tilde{\mathcal{J}}^{k+1})_l^r |^2.$$ 

Meanwhile, we define some more auxiliary functions. Recalling from (5.25), here $\mathcal{P}^k_{t,1}$ and $\mathcal{P}^k_{t,2}$ are defined by

$$\mathcal{P}^k_{t,1} = D_t \Phi_{v,0}^k - \nu^k \cdot (\nabla \Phi_{v,0}^k|_{c}), \quad \mathcal{P}^k_{t,2} = D_t \mathcal{P}^k_{t,1} - \nu^k \cdot (\mathcal{P}^k_{t,1}|_{c}).$$

The following proposition is our main result on the uniform estimates:

**Proposition 6.2** Let $(\tilde{\mathcal{J}}_{a,0}, \tilde{\mathcal{J}}_{a,1}, \mathcal{P}_{v,0}, d_{i,0}, d_{i,1})$ and $(\mathcal{P}^k_{a,0}, \mathcal{P}^k_{a,1}, d_{i,k}^k, d_{i,k}^k)$ be given as in Proposition 6.1. Then there exists constants $T > 0$ small enough and $A > 0$ large enough such that when $a \geq A$, the inequality below holds

$$\sup_{t \in [0,T]} E^{k+1}(t) + \int_0^T F^{k+1}(t) dt \leq P(E(0)).$$

**Proof.** Since the main steps of the proof follow Theorem 4.1 and Theorem 5.1, we only present the sketch of the proof here.

First, we consider the basic energy estimates $E_{a,1}^{k+1}$, where we only focus on the estimates for $\mathcal{J}_{a}$ or $\tilde{\mathcal{J}}_{a}$ and the other estimates follow from Lemma 4.6 and [53].

To begin with, acting $(\Phi_{k}^2)^{-1}$ on both sides of (6.45), one has

$$D_t^2 \Phi_{a} + \sigma (a - \Delta \tau) \mathcal{N}(\Phi_{a}) = R_{a,0},$$

which implies that

$$D_t^2 \Phi_{a} + \sigma \nabla \mathcal{H} ((a - \Delta \tau)(\Phi_{a} + h_{v})) = \nabla \mathcal{H}(R_{a,0} + \sigma h_{v}) - [\nabla \mathcal{H}, D_t^2] \Phi_{a} + 1.$$  

(6.47)

Using the same arguments as in Section 4 and Section 5, we have

$$\|\nabla \mathcal{H}(R_{a,0} + \sigma h_{v}) - [\nabla \mathcal{H}, D_t^2] \Phi_{a} + 1\|_{L^2(\Omega)} \leq P(E(t))(1 + E_{low}^{k+1}(t)).$$

Next, we define $P_{f,k+1,v}$ on $\Omega$ by system (3.32). Taking the $L^2(\Omega)$ inner product of (6.47) with $D_t \Phi_{a} + \nabla P_{f,k+1,v}$, one has

$$\int_{\Omega} \left( D_t^2 \Phi_{a} + \nabla \mathcal{H} ((a - \Delta \tau)(\Phi_{a} + h_{v})) \right) \cdot (D_t \Phi_{a} + \nabla P_{f,k+1,v}) dX = \int_{\Omega} \left( \nabla \mathcal{H}(R_{a,0} + \sigma h_{v}) - [\nabla \mathcal{H}, D_t^2] \Phi_{a} + 1 \right) \cdot (D_t \Phi_{a} + \nabla P_{f,k+1,v}) dX.$$ 

Following the energy estimates in Section 4, there exists a constant $A$ large enough such that when $a \geq A$, one concludes

$$\sup_{t \in [0,T]} E_{low}^{k+1}(t) + \int_0^T F_{low}^{k+1}(t) \leq P(E(0)) + \int_0^T P(E(t)).$$

On the other hand, for the higher-order energy $E_{h}^{k+1}(t)$, we get similarly as (5.30) the equation

$$D_t [D_t^2 \Phi_{a} + \nabla \mathcal{H} (P_{f,2}) + [\nabla \mathcal{H}, D_t^2] \Phi_{a} + 1] + \sigma \nabla \mathcal{H} ((a - \Delta \tau) D_t (\Phi_{a} + h_{v})) = R_{a,1},$$

47
where $R_{a,1}^k$ is given by

$$R_{a,1}^k = D_t \nabla \mathcal{H}(R_{a,0}^k + \sigma h_{\bar{v}k}) - D_t \nabla \mathcal{H}(P_{t,2}^k) + \sigma [\nabla \mathcal{H}, D_t]\mathcal{H}((a - \Delta_{\Gamma_t})(\mathcal{R}_{a,1}^{k+1})^\perp + \sigma h_{\bar{v}k}) - \sigma \nabla \mathcal{H}[\Delta_{\Gamma_t}, D_t](\mathcal{R}_{a,1}^{k+1})^\perp.$$

Compared to the energy estimate of Theorem 5.1, the main difference lies in the term $D_t[(\nabla \mathcal{H}, D_t^2)R_{a,1}^k]$ which contains a term like $D_t(\partial D_t v^k)\partial \mathcal{H}(R_{a,0}^k)$. In Theorem 5.1, since $v$ satisfies Euler’s equation, we have the estimate for $D_t^2 \nabla v$. But here in the iteration scheme, $D_t^2 \partial D_t v^k$ acts like $\partial \partial D_t v^k$, which cannot be controlled by the energy. Therefore, we put $D_t[(\nabla \mathcal{H}, D_t^2)R_{a,1}^k]$ together with $D_t^2 R_{a,1}^k$ to go back to the form $D_t^2 \mathcal{R}_{a,1}^{k+1}$:

$$D_t^2 \mathcal{R}_{a,1}^{k+1} + [\nabla \mathcal{H}, D_t^2]R_{a,1}^k = \nabla \mathcal{H}(D_t^2 \mathcal{R}_{a,1}^{k+1}).$$

Taking the $L^2(\Omega_t^k)$ inner product of the above equation with $\nabla \mathcal{H}(D_t^2 \mathcal{R}_{a,1}^{k+1}) - \nabla \mathcal{H}(P_{t,2}^k)$, we derive

$$\int_{\Omega_t^k} D_t(\nabla \mathcal{H}(D_t^2 \mathcal{R}_{a,1}^{k+1}) - \nabla \mathcal{H}(P_{t,2}^k)) \cdot \left(\nabla \mathcal{H}(D_t^2 \mathcal{R}_{a,1}^{k+1}) - \nabla \mathcal{H}(P_{t,2}^k)\right) dX + \sigma \int_{\Omega_t^k} \nabla \mathcal{H}((a - \Delta_{\Gamma_t})D_t(\mathcal{R}_{a,0}^{k+1})^\perp - D_t h_{\bar{v}k}) \cdot \left(\nabla \mathcal{H}(D_t^2 \mathcal{R}_{a,1}^{k+1}) - \nabla \mathcal{H}(P_{t,2}^k)\right) dX = \int_{\Omega_t^k} R_{a,1}^k \cdot \left(\nabla \mathcal{H}(D_t^2 \mathcal{R}_{a,1}^{k+1}) - \nabla \mathcal{H}(P_{t,2}^k)\right) dX.$$

For the second integral on the left side of the above equation, we have by Green’s Formula that

$$\int_{\Omega_t^k} \nabla \mathcal{H}((a - \Delta_{\Gamma_t})D_t(\mathcal{R}_{a,0}^{k+1})^\perp + D_t h_{\bar{v}k}) \cdot \left(\nabla \mathcal{H}(D_t^2 \mathcal{R}_{a,1}^{k+1}) - \nabla \mathcal{H}(P_{t,2}^k)\right) dX = \int_{\Gamma_t^k} ((a - \Delta_{\Gamma_t})D_t(\mathcal{R}_{a,0}^{k+1})^\perp + D_t h_{\bar{v}k}) \cdot N(D_t^2 \mathcal{R}_{a,1}^{k+1} - P_{t,2}^k) ds + \int_{\Gamma_t^k} \nabla_{\Gamma_t^k} \mathcal{H}((a - \Delta_{\Gamma_t})D_t(\mathcal{R}_{a,0}^{k+1})^\perp + D_t h_{\bar{v}k}) \cdot \left(\nabla_{\Gamma_t^k} \mathcal{H}(D_t^2 \mathcal{R}_{a,1}^{k+1}) - \nabla_{\Gamma_t^k} \mathcal{H}(P_{t,2}^k)\right) dS,$$

and proceeding in a similar way as before, we obtain

$$\int_{\Omega_t^k} \nabla \mathcal{H}((a - \Delta_{\Gamma_t})D_t(\mathcal{R}_{a,0}^{k+1})^\perp + D_t h_{\bar{v}k}) \cdot \left(\nabla \mathcal{H}(D_t^2 \mathcal{R}_{a,1}^{k+1}) - \nabla \mathcal{H}(P_{t,2}^k)\right) dX = \frac{d}{dt} \left(\|D_t(\mathcal{R}_{a,0}^{k+1})^\perp\|^2_{L^2(\Gamma_t^k)} + \|\nabla_{\Gamma_t^k} D_t(\mathcal{R}_{a,0}^{k+1})^\perp\|^2_{L^2(\Gamma_t^k)}\right) + \nabla_{\Gamma_t^k} D_t(\mathcal{R}_{a,0}^{k+1})^\perp \cdot D_t^2(\mathcal{R}_{a,0}^{k+1})^\perp_{P_{t}},$$

$$+ \int_{\Gamma_t^k} (a - \Delta_{\Gamma_t})D_t(\mathcal{R}_{a,0}^{k+1})^\perp \cdot \left(\nabla_{\Gamma_t^k} \mathcal{H}(D_t^2 \mathcal{R}_{a,1}^{k+1}) - \nabla_{\Gamma_t^k} \mathcal{H}(P_{t,2}^k)\right) dS,$$

On one hand, since the commutator $[D_t, N]$ is already expressed in (2.15), one can conclude that

$$\|\nabla_{\Gamma_t^k}(D_t^2 \mathcal{R}_{a,1}^{k+1})\|_{H^1(\Gamma_t^k)} \leq P(E^{k+1}(t)).$$

Consequently, we have

$$\int_{\Gamma_t^k} (a - \Delta_{\Gamma_t})D_t(\mathcal{R}_{a,0}^{k+1})^\perp \cdot \left(\nabla_{\Gamma_t^k} \mathcal{H}(D_t^2 \mathcal{R}_{a,1}^{k+1}) - \nabla_{\Gamma_t^k} \mathcal{H}(P_{t,2}^k)\right) dS \leq \frac{1}{8} F_{k+1}(t) + P(E^{k+1}(t)).$$
On the other hand, using similar arguments as in Lemma 5.5, we get
\[
\int_0^T \int_{\Gamma^*_t} D_t h_{\tilde{v}^k} \cdot (D_t^2 (\delta_{a,k}^{k+1}) + [N, (D_t^2) \delta_{a,k}^{k+1} - N(P_{\Gamma^*_t})] ds dt
\]
\[
\leq P(E(0)) + \sup_{t \in [0,T]} \left( \frac{1}{8} E_{high}^{k+1}(t) + P(E_{low}^{k+1}(t)) \right) + \int_0^T P(E^k(t)) dt + \frac{1}{8} \int_0^T E_{high}^{k+1}(t) dt.
\]

In the end, using similar arguments as in Section 5.3, we can show that when \( T \) small enough, the following estimate holds
\[
\sup_{t \in [0,T]} E_{high}^{k+1}(t) + \int_0^T E_{high}^{k+1}(t) dt \leq P(E(0)) + \int_0^T P(E^k(t)) dt + \sup_{t \in [0,T]} P(E_{low}^{k+1}(t)).
\]

Therefore, by a bootstrap argument, we can prove the desired result.

\[\blacksquare\]

### 6.6 Cauchy sequence and going back to (WW)

In this part, we are finally in a position to prove that the sequence of \( (\tilde{R}^k_a, P^k_{\tilde{v},a}, d^k_i, d^k_r) \) is indeed a Cauchy sequence. In fact, we know from the previous subsection that
\[
\sup_{t \in [0,T]} E_{high}^{k+1}(t) + \int_0^T E_{high}^{k+1}(t) dt \leq C \quad \text{for all } k \in \mathbb{N},
\]
where \( C > 0 \) is a constant depending on \( E(0) \).

To simplify the notation, we denote by
\[
\tilde{f}^{k+1} = f^{k+1} \circ \Phi_{S^*_k}, \quad \delta_{\tilde{f}}^{k+1} = \delta^{k+1} - \delta^k,
\]
and
\[
(a - \Delta_{\Gamma^*_k}) g^k = h_{\tilde{v}^k} \quad \text{with} \quad g^k|_{p_i} = 0.
\]

Using (6.47) and rewriting it similarly as (3.33), we have the equation of \( \delta_{\tilde{f}}^{k} \):
\[
D_{t^*} (D_{t^*} \delta_{\tilde{f}}^{k} - \delta_{\Gamma^*_k}^{k+1} - \delta_{\Gamma^*_k}^{k+1} + \sigma (d_{t^*}^{k+1} - d_{t^*}^k)) = D_{R^k},
\]
where we note
\[
A(d_{t^*}) f = (N(a - \Delta_{\Gamma^*_k})(f \circ \Phi_{S^*_k}^{-1})) \circ \Phi_{S^*_k},
\]
and
\[
D_{R^k} = \left( (\partial_t + v^k \cdot \nabla)^2 - (\partial_t + v^k \cdot \nabla)^2 \right) \tilde{f}^k + \left( A(d_{t^*}) - A(d_{t^*}^{k+1}) \right) (\tilde{f}^k - f^k) + \delta_{\Gamma^*_k}^{k+1} (\delta_{\Gamma^*_k}^{k+1} + \sigma h_{\tilde{v}^{k+1}})
\]
\[\cdot \delta_{\Gamma^*_k}^{k+1} - D_{t^*} \delta_{\Gamma^*_k}^{k+1} + \sigma h_{\tilde{v}^{k+1}}.
\]

Besides, similar but simpler equations for \( (\delta_{\tilde{f}}^{k}, \delta_{\tilde{f}}^{k+1}, \delta_{\tilde{f}}^{k-1}) \) can be derived, and we omit the details here.

Moreover, we define the energy of the difference according to the definition of \( \Sigma \) as
\[
E_{\delta}^k(t) = \| \delta_{\tilde{f}}^{k} \|_{C([0,T]; L^2(\Omega, \Sigma))} + a \| \delta_{\tilde{f}}^{k} \cdot n_{t^*} \|_{C([0,T]; L^2(\Gamma^*_k, \Sigma))} + \| \delta_{\tilde{f}}^{k} \cdot n_{t^*} \|_{C([0,T]; H^1(\Gamma^*_k))}
\]
\[+ \| \delta_{\Gamma^*_k}^{k+1} \|_{C([0,T]; H^{3/2}(\Omega, \Sigma))} + \sum_{i=1, r} \left( \| \delta_{d_i}^{k} \|_{C([0,T])} + \| \delta_{d_i}^{k} \|_{C([0,T])} \right).
\]

Before we consider the convergence, we need to deal with \( D_{R^k} \) first.
Lemma 6.1 The right side of (6.49) satisfies the following estimate:

$$\|D_k^{\pm}\|_{L^2(\Omega)} \leq CE_0^k(t)$$

with the positive constant $C$ depending on $E(0)$.

**Proof.** Here we only give the outline of the proof, and one can see similar details in [53]. First, we consider the estimate for \((\partial_t + v^k \cdot \nabla)^2 - (\partial_t + v^{k-1} \cdot \nabla)^2\) as in Theorem 4.1 and Section 6.3 in [53] and applying (6.45) and (6.48), we can have

$$\|((\partial_t + v^k \cdot \nabla)^2 - (\partial_t + v^{k-1} \cdot \nabla)^2) \nabla\|_{L^2(\Omega)} \leq C\|\partial_t \delta_t \|_{L^2(\Gamma)} + C\|\partial_t \delta_{t+1} \|_{H^2(\Gamma)} \leq C(a^{-1}\|\delta_{t+1} \cdot n\|_{H^1(\Gamma)} + \|\partial_t \delta_{t+1} \|_{L^2(\Omega)}) \leq CE_0^k(t).$$

Second, for the term $(\mathcal{A}(d_{t+1}) - \mathcal{A}(d_t))(\delta_{t+1} - \delta_t)$, we notice by (5.19) that

$$\|\delta_{t+1} - \delta_t\|_{H^{n/2}(\Gamma)} \leq C.$$

As a result, by similar arguments as in the proof of Proposition 6.3 [53], we have

$$\|((\mathcal{A}(d_{t+1}) - \mathcal{A}(d_t))(\delta_{t+1} - \delta_t))\|_{L^2(\Omega)} \leq CE_0^k(t).$$

In the end, similar arguments as in Lemma 4.6 [53], we can have

$$\|\mathcal{A}(d_{t+1}) - \mathcal{A}(d_t)\|_{L^2(\Omega)} \leq CE_0^k(t).$$

Combining all these estimates above, the proof is finished.

Now, we are able to conclude about the convergence result.

**Proposition 6.3** The sequence $(\mathcal{A}_a, P_{v,v}^{k}, d_t^{k}, d_r^{k})$ is a Cauchy sequence.

**Proof.** We follow the steps in Theorem 4.1 and Section 6.3 in [53] to conclude that there exists a constant $T$ small enough and $A$ large enough such that when $a \geq A$, we have

$$E_0^k(t) \leq C \int_0^T E_0^k(t)dt.$$

As a result, this implies immediately that $(\mathcal{A}_a, P_{v,v}^{k}, d_t^{k}, d_r^{k})$ is convergent.

We are finally in a position to finish the proof for Theorem 6.1.

**Proof of Theorem 6.1.** We only present the sketch for the proof here. In fact, since we have proved in Proposition 6.3 that $(\mathcal{A}_a, P_{v,v}^{k}, d_t^{k}, d_r^{k}) \in \Sigma$ is a Cauchy sequence, we know immediately that there exists $(\mathcal{A}_a, P_{v,v}, d_t, d_r) \in \Sigma$ satisfying

$$(\mathcal{A}_a, P_{v,v}^{k}, d_t^{k}, d_r^{k}) \rightarrow (\mathcal{A}_a, P_{v,v}, d_t, d_r) \quad \text{in} \quad \Sigma.$$ 

As a result, one can show in a standard way that $(\mathcal{A}_a, P_{v,v}, d_t, d_r)$ satisfies system (6.44). Moreover, one also has $(\mathcal{A}_a, P_{v,v}, d_t, d_r) \in \Sigma$ in the proof of Proposition 6.3.

In the end, we go back to our water-waves system (WW). In fact, thanks to system (6.44) for $(\mathcal{A}_a, P_{v,v}, d_t, d_r) \in \Sigma$, we derive the mean curvature $\kappa$, which defines the free surface. Based on the knowledge of $\Gamma_t$, we also obtain $v$ by Section 6.2. Therefore, using similar arguments as in Section 6.4 [53] and thanks to discussions in Section 6.3, we can finally retrieve the solution $(v, P)$ to the water-waves system (WW).

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