The Effect Of Microscopic Correlations On The Information Geometric Complexity Of Gaussian Statistical Models

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We present an analytical computation of the asymptotic temporal behavior of the information geometric complexity (IGC) of finite-dimensional Gaussian statistical manifolds in the presence of microcorrelations (correlations between microvariables). We observe a power law decay of the IGC at a rate determined by the correlation coefficient. It is found that microcorrelations lead to the emergence of an asymptotic information geometric compression of the statistical macrostates explored by the system at a faster rate than that observed in absence of microcorrelations. This finding uncovers an important connection between (micro)-correlations and (macro)-complexity in Gaussian statistical dynamical systems.

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I. INTRODUCTION

The study of complexity $^1$ has created a new set of ideas on how very simple systems may give rise to very complex behaviors. In many cases, the ”laws of complexity” have been found to hold universally, independent of the details of the system’s constituents. Chaotic behavior is a particular case of complex behavior and it will be the object of the present work. In this article we make use of the so-called Entropic Dynamics (ED) $^2$ and Information Geometrodynamical Approach to Chaos (IGAC) $^3$. ED arises from the combination of inductive inference (Maximum Entropy Methods, $^3$ $^6$) and Information Geometry $^7$. ED is a theoretical framework whose objective - among others - is to derive dynamics from purely entropic arguments. The applicability of ED has been extended to temporally-complex (chaotic) dynamical systems on curved statistical manifolds $\mathcal{M}_S$ resulting in the information geometrodynamical approach to chaos (IGAC) $^3$. IGAC arises as a theoretical framework to study chaos in informational geodesic flows describing physical, biological or chemical systems. A geodesic on a curved statistical manifold $\mathcal{M}_S$ represents the maximum probability path a complex dynamical system explores in its evolution between initial and final macrostates. Each point of the geodesic is parametrized by the macroscopic dynamical variables $\{\Theta\}$ defining the macrostate of the system. Furthermore, each macrostate is in a one-to-one correspondence with the probability distribution $\{p(X|\Theta)\}$ representing the maximally probable description of the system being considered. The set of macrostates forms the parameter space $\mathcal{D}_\Theta$ while the set of probability distributions forms the statistical manifold $\mathcal{M}_S$. IGAC is the information geometric analogue of conventional geometrodynamical approaches $^8$ where the classical configuration space $\Gamma_E$ is being replaced by a statistical manifold $\mathcal{M}_S$. This procedure affords the possibility of considering chaotic dynamics arising from non conformally flat metrics (the Jacobi metric is always conformally flat, instead). It is an information geometric extension of the Jacobi geometrodynamics (the geometrization of a Hamiltonian system by transforming it to a geodesic flow $^1$). The reformulation of dynamics in terms of a geodesic problem allows the application of a wide range of well-known geometrical techniques in the investigation of the solution space and properties of the equation of motion. The power of the Jacobi reformulation is that all of the dynamical information is collected into a single geometric object in which all the available manifest symmetries are retained- the manifold on which geodesic flow is induced. For example, integrability of the system is connected with existence of Killing vectors and tensors on this manifold. The sensitive dependence of trajectories on initial conditions, which is a key ingredient of chaos, can be investigated from the equation of geodesic deviation. In the Riemannian $^8$ and Finslerian $^9$ (a Finsler metric is obtained from a Riemannian metric by relaxing the requirement that the metric be quadratic on each tangent space) geometrodynamical approach to chaos in classical Hamiltonian systems, an active field of research concerns the possibility of finding a rigorous relation among the sectional curvature, the Lyapunov exponents, and the Kolmogorov-Sinai dynamical entropy (i. e. the sum of positive Lyapunov exponents) $^{11}$.

Understanding the relationship between microscopic dynamics and experimentally observable macroscopic dynamics is a fundamental issue in physics $^{12}$ $^{13}$. An interesting manifestation of such a relationship appears in the study of the effects of microscopic external noise (noise imposed on the microscopic variables of the system) on the observed
collective motion (macroscopic variables) of a globally coupled map [16]. These effects are quantified in terms of the complexity of the collective motion. Furthermore, it turns out that noise at a microscopic level reduces the complexity of the macroscopic motion, which in turn is characterized by the number of effective degrees of freedom of the system.

In this article, using statistical inference and information geometric techniques, we investigate the macroscopic behavior of complex systems in terms of the underlying statistical structure of its microscopic degrees of freedom in the presence of correlations. We compute the asymptotic temporal behavior of the information geometric complexity of the maximum probability trajectories on finite-dimensional Gaussian statistical manifolds in the presence of microcorrelations. We observe a power law decay of the IGC at a rate determined by the correlation coefficient. The ratio between the IGC in the presence and in the absence of microcorrelations is explicitly computed. We conclude that microcorrelations lead to the emergence of an asymptotic information geometric compression of the explored statistical macrostates (on the configuration manifold of the model in its evolution between the initial and final macrostates) that is faster than that observed in absence of microcorrelations.

The layout of the article is as follows. In Section II, we briefly discuss Gaussian statistical models in absence and presence of microcorrelations. In Section III, we introduce the Gaussian statistical model being considered. We compute the Ricci scalar curvature and the geodesic trajectories of the system. In Section IV, we compute the asymptotic behavior of the dynamical IGC of the model. Our conclusions are presented in Section V.

II. ON GAUSSIAN STATISTICAL MODELS AND MICROCORRELATIONS

In this Section, we introduce the notion of Gaussian statistical models (manifolds) in the presence of correlations between the microscopic degrees of freedom (microvariables) of the system (microcorrelations).

A. Statistical Models in Absence of Microcorrelations

Consider a Gaussian statistical model whose microstates span a \( n \)-dimensional space labelled by the variables \( \{X\} = \{x_1, x_2, \ldots, x_n\} \) with \( x_j \in \mathbb{R}, \forall j = 1, \ldots, n \). We assume the only testable information pertaining to the quantities \( x_j \) consists of the expectation values \( \langle x_j \rangle \) and the variance \( \Delta x_j \). The set of these expected values define the \( 2n \)-dimensional space of macrostates of the system. A measure of distinguishability among the macrostates of the Gaussian model is achieved by assigning a probability distribution \( P(X|\Theta) \) to each \( 2n \)-dimensional macrostate \( \Theta \equiv \{(\langle x_j \rangle, \langle \Delta x_j \rangle)\}_{n \text{-pairs}} = \{(\langle x_j \rangle, \Delta x_j)\}_{n \text{-pairs}} \). The process of assigning a probability distribution to each state endows \( \mathcal{M}_S \) with a metric structure. Specifically, the Fisher-Rao information metric \( g_{\mu \nu} (\Theta) \) [7] is a measure of distinguishability among macrostates on the statistical manifold \( \mathcal{M}_S \),

\[
g_{\mu \nu} (\Theta) = \int dX P(X|\Theta) \partial_\mu \log P(X|\Theta) \partial_\nu \log P(X|\Theta) = 4 \int dX \partial_\mu \sqrt{P(X|\Theta)} \partial_\nu \sqrt{P(X|\Theta)}, \tag{1}
\]

with \( \mu, \nu = 1, \ldots, 2n \) and \( \partial_\mu = \frac{\partial}{\partial x_\mu} \). It assigns an information geometry to the space of states. The information metric \( g_{\mu \nu} (\Theta) \) is a symmetric, positive definite Riemannian metric. For the sake of completeness and in view of its potential relevance in the study of correlations, we point out that the Fisher-Rao metric satisfies the following two properties: 1) invariance under (invertible) transformations of microvariables \( \{x\} \in \mathcal{X}; 2) \) covariance under reparametrization of the statistical macrospace \( \{\theta\} \in \mathcal{D}_\theta \). The invariance of \( g_{\mu \nu} (\Theta) \) under reparametrization of the microspace \( \mathcal{X} \) implies [7],

\[
\mathcal{X} \subseteq \mathbb{R}^n \ni x \mapsto y \overset{\text{def}}{=} f(x) \in \mathcal{Y} \subseteq \mathbb{R}^n \implies p(x|\theta) \mapsto p'(y|\theta) = \left[ \frac{1}{\det F} p(x|\theta) \right]_{x=f^{-1}(y)} . \tag{2}
\]

The covariance under reparametrization of the parameter space \( \mathcal{D}_\theta \) (homeomorphic to \( \mathcal{M}_S \)) implies [7],

\[
\mathcal{D}_\theta \ni \theta \mapsto \theta' \overset{\text{def}}{=} f(\theta) \in \mathcal{D}_{\theta'} \implies g_{\mu \nu} (\theta) \mapsto g'_{\mu \nu} (\theta') = \left[ \frac{\partial \theta'^\alpha}{\partial \theta^\alpha} \frac{\partial \theta'^\beta}{\partial \theta^\beta} g_{\alpha \beta} (\theta) \right]_{\theta=f^{-1}(\theta')} , \tag{3}
\]

where

\[
g'_{\mu \nu} (\theta') = \int dx p'(x|\theta') \partial'_\mu \log p'(x|\theta') \partial'_\nu \log p'(x|\theta') , \tag{4}
\]
with \( \theta' = \frac{\partial}{\partial \theta} \) and \( p'(x|\theta') = p(x|\theta = f^{-1}(\theta')) \). Our 2n-dimensional Gaussian statistical model represents a macroscopic (probabilistic) description of a microscopic n-dimensional (microscopic) physical system evolving over a n-dimensional (micro) space. The variables \( \{X\} = \{x_1, x_2, ..., x_n\} \) label the n-dimensional space of microstates of the system. We assume that all information relevant to the dynamical evolution of the system is contained in the probability distributions. For this reason, no other information is required. Each macrostate may be thought of as a point of a 2n-dimensional statistical manifold with coordinates given by the numerical values of the expectations \( \langle \theta_j \rangle \) and \( \langle \sigma_j \rangle \). The available relevant information can be written in the form of the following 2n information constraint equations,

\[
\langle x_j \rangle = \int_{-\infty}^{+\infty} dx_j x_j P_j \left( x_j^{(1)} \theta_j, (2) \theta_j \right), \quad \Delta x_j = \int_{-\infty}^{+\infty} dx_j (x_j - \langle x_j \rangle)^2 P_j \left( x_j^{(1)} \theta_j, (2) \theta_j \right) \frac{d}{d \theta_j}. \tag{5}
\]

The probability distributions \( P_j \) in (5) are constrained by the conditions of normalization,

\[
\int_{-\infty}^{+\infty} dx_j P_j \left( x_j^{(1)} \theta_j, (2) \theta_j \right) = 1. \tag{6}
\]

Information theory identifies the Gaussian distribution as the maximum entropy distribution if only the expectation value and the variance are known [17]. Maximum relative Entropy methods [5,18,19] allow us to associate a probability distribution \( P(X|\Theta) \) to each point in the space of states \( \Theta \). The distribution that best reflects the information contained in the prior distribution \( m(X) \) updated by the information \( (\langle x_j \rangle, \Delta x_j) \) is obtained by maximizing the relative entropy,

\[
S(\Theta) = -\int d^\nu X P(X|\Theta) \log \left( \frac{P(X|\Theta)}{m(X)} \right), \tag{7}
\]

where \( m(X) \) is the prior probability distribution. As a working hypothesis, the prior \( m(X) \) is set to be uniform since we assume the lack of prior available information about the system [20]. We assume uncoupled constraints among microvariables \( x_j \). In other words, we assume that information about correlations between the microvariables need not to be tracked. Therefore, upon maximizing (7) given the constraints (5) and (6), we obtain

\[
P(X|\Theta) = \prod_{j=1}^{n} P_j \left( x_j^{(1)} \theta_j, (2) \theta_j \right) \tag{8}
\]

where

\[
P_j \left( x_j^{(1)} \theta_j, (2) \theta_j \right) = (2\pi \sigma_j^2)^{-\frac{1}{2}} \exp \left[ -\frac{(x_j - \mu_j)^2}{2\sigma_j^2} \right]. \tag{9}
\]

and, in standard notation for Gaussians, \( (1) \theta_j \equiv \langle x_j \rangle = \mu_j, (2) \theta_j \equiv \Delta x_j = \sigma_j \). The probability distribution (8) encodes the available information concerning the system. The statistical manifold \( \mathcal{M}_S \) associated to (8) is formally defined as follows,

\[
\mathcal{M}_S = \left\{ P(X|\Theta) = \prod_{j=1}^{n} P_j \left( x_j | \mu_j, \sigma_j \right) \right\}, \tag{10}
\]

where \( X \in \mathbb{R}^n \) and \( \Theta \) belongs to the 2n-dimensional parameter space \( \mathcal{D}_\Theta = [\mathcal{I}_\mu \times \mathcal{I}_\sigma]^n \). The parameter space \( \mathcal{D}_\Theta \) (homeomorphic to \( \mathcal{M}_S \)) is the direct product of the parameter subspaces \( \mathcal{I}_\mu \) and \( \mathcal{I}_\sigma \), where (in the Gaussian case, unless specified otherwise) \( \mathcal{I}_\mu = (-\infty, +\infty)_\mu \) and \( \mathcal{I}_\sigma = (0, +\infty)_\sigma \). The line element \( ds^2 = g_{\mu \nu} (\Theta) d\Theta^\mu d\Theta^\nu \) arising from (3) is [21],

\[
ds^2_{\mathcal{M}_S} \equiv \sum_{j=1}^{n} \left( \frac{1}{\sigma_j^2} d\mu_j^2 + \frac{2}{\sigma_j^2} d\sigma_j^2 \right), \tag{11}
\]

with \( \mu, \nu = 1, \ldots, 2n \).
B. Gaussian Statistical Models in Presence of Microcorrelations

Coupled constraints would lead to a "generalized" product rule and to a metric tensor with non-trivial off-diagonal elements (covariance terms). In presence of correlated degrees of freedom \( \{x_j\} \), the "generalized" product rule becomes,

\[
P_{\text{tot}}(x_1, ..., x_n) = \prod_{j=1}^{n} P_j(x_j) \xrightarrow{\text{correlations}} P'_{\text{tot}}(x_1, ..., x_n) \neq \prod_{j=1}^{n} P_j(x_j),
\]

where,

\[
P'_{\text{tot}}(x_1, ..., x_n) \equiv P_n(x_n|x_1, ..., x_{n-1}) P_{n-1}(x_{n-1}|x_1, ..., x_{n-2}) ... P_2(x_2|x_1) P_1(x_1).
\]

Correlations among the degrees of freedom may be introduced in terms of the following information-constraints,

\[
x_j = f_j(x_1, ..., x_{j-1}), \forall j = 2, ..., n.
\]

In such a case, we obtain

\[
P'_{\text{tot}}(x_1, ..., x_n) = \delta(x_n - f_n(x_1, ..., x_{n-1})) \delta(x_{n-1} - f_{n-1}(x_1, ..., x_{n-2})) ... \delta(x_2 - f_2(x_1)) P_1(x_1),
\]

where the \( j \)-th probability distribution \( P_j(x_j) \) is given by,

\[
P_j(x_j) = \int ... \int dx_{j-1} ... dx_1 dx_{j+1} ... dx_n P'_\text{tot}(x_1, ..., x_n).
\]

A formal manner in which correlations are introduced in probability theory is as follows. Given two arbitrary randomly distributed variables \( x_1 \) and \( x_2 \), consider the problem of finding a linear expression of the form \( c_1 + c_2 x_2 \), involving real constants \( c_1 \) and \( c_2 \) such that \( c_1 + c_2 x_2 \) is the best "mean square approximation" to \( x_1 \). The best approximation is such that

\[
\left\langle (x_1 - c_1 - c_2 x_2)^2 \right\rangle = \min_{c_1, c_2} \left\langle (x_1 - c_1 - c_2 x_2)^2 \right\rangle,
\]

where the minimum is taken with respect to all real constants \( c_1 \) and \( c_2 \). To solve this problem, let

\[
\mu_1 = \left\langle x_1 \right\rangle, \sigma_1^2 = \left\langle (x_1 - \left\langle x_1 \right\rangle)^2 \right\rangle, \mu_2 = \left\langle x_2 \right\rangle, \sigma_2^2 = \left\langle (x_2 - \left\langle x_2 \right\rangle)^2 \right\rangle
\]

and introduce the quantity \( r \),

\[
r = \frac{\left\langle (x_1 - \left\langle x_1 \right\rangle)(x_2 - \left\langle x_2 \right\rangle) \right\rangle}{\sigma_1 \sigma_2} = \frac{\left\langle x_1 x_2 \right\rangle - \mu_1 \mu_2}{\sigma_1 \sigma_2}.
\]

The quantity \( r \) is the so-called correlation coefficient of the random variables \( x_1 \) and \( x_2 \). For the sake of convenience, we may introduce the "normalized" random variables,

\[
\eta_1 = \frac{x_1 - \mu_1}{\sigma_1} \text{ and } \eta_2 = \frac{x_2 - \mu_2}{\sigma_2},
\]

The problem in (17) can now be reduced to,

\[
\min_{c_1, c_2} \left( \eta_1 - c_1 - c_2 \eta_2 \right)^2 = \min_{c_1, c_2} \left[ (1 - r^2) + c_1^2 + (r - c_2)^2 \right] = 1 - r^2 \geq 0.
\]

The minimum is achieved for \( c_1 = 0 \) and \( c_2 = r \), where \( r \) lies in the interval \(-1 \leq r \leq +1\).

In our work, correlations among the microscopic degrees of freedom of the system \( \{x_j\} \) (microcorrelations) are conventionally introduced by means of the correlation coefficients \( r_{ij}^{(\text{micro})} \),

\[
r_{ij}^{(\text{micro})} = r(x_i, x_j) = \frac{(x_i x_j) - \left\langle x_i \right\rangle \left\langle x_j \right\rangle}{\sigma_i \sigma_j}, \text{ with } \sigma_i = \sqrt{\left\langle (x_i - \left\langle x_i \right\rangle)^2 \right\rangle}.
\]

with \( r_{ij}^{(\text{micro})} \in (-1, 1) \) and \( i, j = 1, ..., n \). For the 2n-dimensional Gaussian statistical model in presence of microcorrelations, the system is described by the following probability distribution \( P(X|\Theta) \),

\[
P(X|\Theta) = \frac{1}{(2\pi)^n \det C(\Theta)^{\frac{1}{2}}} \exp \left[ -\frac{1}{2} (X - M)^t C^{-1}(\Theta) (X - M) \right] \prod_{j=1}^{n} (2\pi \sigma_j^2)^{-\frac{1}{2}} \exp \left[ -\frac{(x_j - \mu_j)^2}{2\sigma_j^2} \right],
\]

where \( X = (x_1, ..., x_n) \), \( M = (\mu_1, ..., \mu_n) \) and \( C(\Theta) \) is the \((2n \times 2n)\)-dimensional (non-singular) covariance matrix.
III. THE MODEL

In this Section we focus on microcorrelated Gaussian statistical models with $2n = 4$. For $n = 2$, (23) leads to the probability distribution $P (x, y | \mu_x, \sigma_x, \mu_y, \sigma_y)$ which takes the form,

$$
P (x, y | \mu_x, \sigma_x, \mu_y, \sigma_y) = \exp \left\{ -\frac{1}{2 (1 - r^2)} \left[ \frac{(x - \mu_x)^2 - 2r (x - \mu_x) (y - \mu_y)}{\sigma_x \sigma_y} + \frac{(y - \mu_y)^2}{\sigma_y^2} \right] \right\}, \quad (24)
$$

where $\sigma_x > 0$, $\sigma_y > 0$, $r \in (-1, +1)$. Substituting (21) in (11), the Fisher-Rao information metric $g_{\mu\nu} (\mu_x, \sigma_x, \mu_y, \sigma_y; r)$ becomes,

$$
g_{\mu\nu} (\mu_x, \sigma_x, \mu_y, \sigma_y; r) = \begin{pmatrix}
-\frac{1}{\sigma_x^2 (r^2 - 1)} & 0 & 0 & 0 \\
0 & -\frac{1}{\sigma_y^2 (r^2 - 1)} & 0 & 0 \\
0 & 0 & -\frac{r^2}{\sigma_x \sigma_y (r^2 - 1)} & 0 \\
0 & 0 & 0 & -\frac{2 r^2}{\sigma_y^2 (r^2 - 1)}
\end{pmatrix} \quad (25)
$$

The infinitesimal line element $ds_{M_S}^2$ relative to $g_{\mu\nu} (\mu_x, \sigma_x, \mu_y, \sigma_y; r)$ is given by,

$$
ds_{M_S}^2 = g_{11} (\sigma_x; r) \, d\mu_x^2 + g_{33} (\sigma_y; r) \, d\mu_y^2 + g_{22} (\sigma_x; r) \, da_x^2 + g_{44} (\sigma_y; r) \, da_y^2 + 2 g_{13} (\sigma_x, \sigma_y; r) \, d\mu_x \, d\mu_y + 2 g_{24} (\sigma_x, \sigma_y; r) \, d\sigma_x \, d\sigma_y, \quad (26)
$$

where,

$$
g_{11} (\sigma_x; r) = -\frac{1}{\sigma_x^2 (r^2 - 1)}; \quad g_{13} (\sigma_x, \sigma_y; r) = \frac{r}{\sigma_x \sigma_y (r^2 - 1)}; \quad g_{22} (\sigma_x; r) = -\frac{2 - r^2}{\sigma_x^2 (r^2 - 1)}, \quad (27)
$$

$$
g_{24} (\sigma_x, \sigma_y; r) = \frac{r^2}{\sigma_x \sigma_y (r^2 - 1)}; \quad g_{33} (\sigma_x, \sigma_y; r) = \frac{r}{\sigma_x \sigma_y (r^2 - 1)}; \quad g_{33} (\sigma_y; r) = -\frac{1}{\sigma_y^2 (r^2 - 1)}, \quad (27)
$$

$$
g_{42} (\sigma_x, \sigma_y; r) = \frac{r^2}{\sigma_x \sigma_y (r^2 - 1)}; \quad g_{44} (\sigma_y; r) = -\frac{2 - r^2}{\sigma_y^2 (r^2 - 1)}. \quad (27)
$$

The analytical study of the IGAC arising on a curved statistical manifold with infinitesimal line element given by $ds_{M_S}^2$ in (20) turns out to be rather difficult. Hence, as working hypothesis, we are going to assume two correlated Gaussian-distributed microvariables characterized by the same variance, that is we assume $\sigma_x = \sigma_y \equiv \sigma$. Thus, the simplified line element becomes,

$$
ds_{M_S}^2 = g_{11} (\sigma_x; r) \, d\mu_x^2 + g_{33} (\sigma_y; r) \, d\mu_y^2 + 2 g_{13} (\sigma_x; r) \, d\mu_x \, d\mu_y + [g_{22} (\sigma_x; r) + g_{44} (\sigma_y; r) + 2 g_{24} (\sigma_x, \sigma_y; r)] \, da^2. \quad (28)
$$

The new Fisher-Rao matrix $g_{\mu\nu} (\mu_x, \mu_y, \sigma; r)$ associated with line element $ds_{M_S}^2$ in (28) becomes,

$$
g_{\mu\nu} (\mu_x, \mu_y, \sigma; r) = \frac{1}{\sigma^2} \begin{pmatrix}
-\frac{1}{2 (r^2 - 1)} & \frac{r}{2 (r^2 - 1)} & 0 & 0 \\
\frac{r}{2 (r^2 - 1)} & -\frac{1}{2 (r^2 - 1)} & 0 & 0 \\
0 & 0 & -\frac{2 r^2}{4 (r^2 - 1)} & 0 \\
0 & 0 & 0 & 4
\end{pmatrix}. \quad (29)
$$

We will study the information dynamics on curved statistical manifolds $M_S^{(\text{correlations})}$ and $M_S^{(\text{no-correlations})}$ with infinitesimal line elements $(ds_{M_S}^2)^{\text{correlations}}$ and $(ds_{M_S}^2)^{\text{no-correlations}}$, respectively. The line element $(ds_{M_S}^2)^{\text{correlations}}$ is defined by,

$$
(ds_{M_S}^2)^{\text{correlations}} \overset{\text{def}}{=} \frac{1}{\sigma^2} \left( \frac{1}{1 - r^2} d\mu_x^2 + \frac{1}{1 - r^2} d\mu_y^2 - \frac{2 r}{1 - r^2} d\mu_x d\mu_y + 4 \sigma^2 \right), \quad (30)
$$

while $(ds_{M_S}^2)^{\text{no-correlations}}$ is obtained from $(ds_{M_S}^2)^{\text{correlations}}$ in the limit that $r$ approaches zero.
A. Information Geometry of The Model

Consider the information dynamics of the Model introduced in Section II. The Fisher-Rao line element \( ds_{M_S}^2 \) of such statistical model \( M_S^{(\text{correlations})} \) is given in (30). The inverse metric tensor \( g^{\mu\nu} (\mu_x, \mu_y, \sigma; r) \) is given by,

\[
g^{\mu\nu} (\mu_x, \mu_y, \sigma; r) = \sigma^2 \begin{pmatrix} \frac{4(r^2-1)}{r^2} & \frac{2r(r^2-1)}{r^2} & 0 \\ \frac{2r(r^2-1)}{r^2} & \frac{4(r^2-1)}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\] (31)

The metric tensor \( g_{\mu\nu} (\mu_x, \mu_y, \sigma; r) \) and its inverse \( g^{\mu\nu} (\mu_x, \mu_y, \sigma; r) \) are necessary to determine the Christoffel connection coefficients \( \Gamma_{ij}^k \) of the manifold \( M_S^{(\text{correlations})} \). Recall that the connection coefficients \( \Gamma_{ij}^k \) are defined as [23],

\[
\Gamma_{ij}^k = \frac{1}{2} g^{km} (\partial_i g_{mj} + \partial_j g_{im} - \partial_m g_{ij}).
\] (32)

In our case, the non-vanishing connection coefficients are given by,

\[
\Gamma_{11}^3 = -\frac{1}{4} \frac{1}{r^2 - 1} \frac{1}{\sigma}, \quad \Gamma_{12}^3 = \frac{1}{8} \frac{r}{(r^2 - 1)} \frac{1}{\sigma}, \quad \Gamma_{13}^3 = \Gamma_{31}^1 = -\frac{1}{\sigma}, \quad \Gamma_{33}^3 = -\frac{1}{\sigma}.
\] (33)

Once the non-vanishing components of \( \Gamma_{ij}^k \) are obtained, we compute the Ricci curvature tensor \( R_{ij} \) defined as [23],

\[
R_{ij} = \partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{jk}^k \Gamma_{im}^n - \Gamma_{ik}^n \Gamma_{jm}^k.
\] (34)

Substituting (33) in (34), we obtain the non-vanishing Ricci curvature tensor components \( R_{ij} \),

\[
R_{11} = \frac{1}{2} \frac{1}{(r^2 - 1)} \frac{1}{\sigma^2}, \quad R_{12} = R_{21} = -\frac{r}{4} \frac{1}{(r^2 - 1)} \frac{1}{\sigma^2}, \quad R_{22} = \frac{1}{2} \frac{1}{(r^2 - 1)} \frac{1}{\sigma^2}, \quad R_{33} = -\frac{2}{\sigma^2}.
\] (35)

Finally, we compute Ricci scalar curvature \( R_{M_S} (r) \),

\[
R_{M_S} (r) = R_{ij} g^{ij}.
\] (36)

Substituting (35) and (31) in (36), \( R_{M_S} (r) \) becomes,

\[
R_{M_S} (r) = g^{11} R_{11} + 2 g^{12} R_{12} + g^{22} R_{22} + g^{33} R_{33} = -\frac{3}{2}.
\] (37)

Therefore, we conclude that \( M_S^{(\text{correlations})} \) is a curved statistical manifold of constant negative curvature.

B. Information Dynamics on \( M_S \)

The information dynamics can be derived from a standard principle of least action of Jacobi type [2]. The geodesic equations for the macrovariables of the Gaussian ED model are given by nonlinear second order coupled ordinary differential equations,

\[
\frac{d^2 \Theta^\mu}{dt^2} + \Gamma_{\nu\rho}^\mu \frac{d \Theta^\nu}{dt} \frac{d \Theta^\rho}{dt} = 0.
\] (38)

The geodesic equations in (38) describe a reversible dynamics whose solution is the trajectory between an initial \( \Theta^{(\text{initial})} \) and a final macrostate \( \Theta^{(\text{final})} \). The trajectory can be equally well traversed in both directions. In the case
under consideration, substituting (29) in (38), the three geodesic equations become,

\[
0 = \frac{d^2 \mu_x (\tau)}{d\tau^2} - \frac{2}{\sigma (\tau)} \frac{d \mu_x (\tau)}{d\tau} \frac{d\sigma (\tau)}{d\tau},
\]

\[
0 = \frac{d^2 \mu_y (\tau)}{d\tau^2} - \frac{2}{\sigma (\tau)} \frac{d \mu_y (\tau)}{d\tau} \frac{d\sigma (\tau)}{d\tau},
\]

\[
0 = \frac{d^2 \sigma (\tau)}{d\tau^2} - \frac{1}{\sigma (\tau)} \left( \frac{d \sigma (\tau)}{d\tau} \right)^2 - \frac{1}{4} \frac{1}{1 - \sigma (\tau)} \left( \frac{d \mu_x (\tau)}{d\tau} \right)^2 - \frac{1}{4} \frac{1}{1 - \sigma (\tau)} \left( \frac{d \mu_y (\tau)}{d\tau} \right)^2 + \frac{r}{4 (\tau^2 - 1)} \frac{1}{\sigma (\tau)} \frac{d \mu_x (\tau)}{d\tau} \frac{d \mu_y (\tau)}{d\tau},
\]

Integration of the above coupled system of differential equations is non trivial. A detailed derivation of the geodesic paths is given in the Appendix. After integration of (39), the geodesic trajectories become,

\[
\mu_x (\tau; r) = - \frac{2 \sigma_0 A_1}{\sqrt{\mathcal{A}(r)}} \frac{1}{1 + \exp \left( 2 \sigma_0 \sqrt{\mathcal{A}(r) \tau} \right)}, \quad \mu_y (\tau; r) = - \frac{2 \sigma_0 A_2}{\sqrt{\mathcal{A}(r)}} \frac{1}{1 + \exp \left( 2 \sigma_0 \sqrt{\mathcal{A}(r) \tau} \right)},
\]

\[
\sigma (\tau; r) = 2 \sigma_0 \frac{\exp \left( \sigma_0 \sqrt{\mathcal{A}(r) \tau} \right)}{1 + \exp \left( 2 \sigma_0 \sqrt{\mathcal{A}(r) \tau} \right)},
\]

where,

\[
\mathcal{A}(r) \equiv \frac{A_1^2 + A_2^2 - r A_1 A_2}{4 (1 - r^2)}.
\]

Notice that for any real value of \(A_1, A_2\), \(0 \leq (A_1 - A_2)^2 = A_1^2 + A_2^2 - 2 A_1 A_2 \leq A_1^2 + A_2^2 - r A_1 A_2 \) and \(4 (1 - r^2) \geq 0\) for \(r \in (-1, 1)\). It then follows that, \(\mathcal{A}(r) \geq 0\). Note that \(\sigma (\tau; r) \in (0, +\infty)\) while \(\mu_x (\tau; r)\) and \(\mu_y (\tau; r) \in (-\infty, +\infty)\).

### IV. THE INFORMATION GEOMETRIC COMPLEXITY AND MICROCORRELATIONS

We recall that a suitable indicator of temporal complexity within the IGAC framework is provided by the information geometric entropy (IGE) \(S_{\mathcal{M}_s} (\tau)\) defined

\[
S_{\mathcal{M}_s} (\tau) \equiv \log \operatorname{vol} \left( \mathcal{D}_{\Theta}^{(\text{geodesic})} (\tau) \right).
\]

The information geometric complexity (IGC) is defined as the average dynamical statistical volume \(\operatorname{vol} \left[ \mathcal{D}_{\Theta}^{(\text{geodesic})} (\tau) \right]\) given by,

\[
\operatorname{vol} \left[ \mathcal{D}_{\Theta}^{(\text{geodesic})} (\tau) \right] \equiv \lim_{\tau \to \infty} \left( \frac{1}{\tau} \int_0^\tau \operatorname{vol} \left[ \mathcal{D}_{\Theta}^{(\text{geodesic})} (\tau') \right] d\tau' \right),
\]

where the "tilde" symbol denotes the operation of temporal average. The volume \(\operatorname{vol} \left[ \mathcal{D}_{\Theta}^{(\text{geodesic})} (\tau') \right]\) is given by,

\[
\operatorname{vol} \left[ \mathcal{D}_{\Theta}^{(\text{geodesic})} (\tau') \right] \equiv \int_{\mathcal{D}_{\Theta}^{(\text{geodesic})} (\tau')} \rho_{(\mathcal{M}_s, g)} (\theta^1, ..., \theta^N) d^N \Theta,
\]

where \(N\) is the dimensionality of the statistical manifold \(\mathcal{M}_s\) and \(\rho_{(\mathcal{M}_s, g)} (\theta^1, ..., \theta^N)\) is the so-called Fisher density and equals the square root of the determinant of the metric tensor \(g_{\mu \nu} (\Theta)\),

\[
\rho_{(\mathcal{M}_s, g)} (\theta^1, ..., \theta^N) \equiv \sqrt{g (\theta^1, ..., \theta^N)}.
\]
The integration space \( D_{\Theta}^{(\text{geodesic})} (\tau') \) in (46) is defined as follows,

\[
D_{\Theta}^{(\text{geodesic})} (\tau') \equiv \{ \Theta \equiv (\theta^1, ..., \theta^N) : \theta^k (0) \leq \theta^k (\tau') \},
\]

where \( k = 1, ..., N \) and \( \theta^k \equiv \theta^k (s) \) with \( 0 \leq s \leq \tau' \) such that \( \theta^k (s) \) satisfies (38). The integration space \( D_{\Theta}^{(\text{geodesic})} (\tau') \) in (46) is an \( N \)-dimensional subspace of the whole (permitted) parameter space \( D_{\Theta}^{(\text{tot})} \). The elements of \( D_{\Theta}^{(\text{geodesic})} (\tau') \) are the \( N \)-dimensional macrovariables \( \{ \Theta \} \) whose components \( \theta^k \) are bounded by specified limits of integration \( \theta^k (0) \) and \( \theta^k (\tau') \) with \( k = 1, ..., N \). The limits of integration are obtained via integration of the geodesic equations. Formally, the IGE \( S_M, (\tau) \) is defined in terms of a averaged parametric \((N + 1)\)-fold integral \((\tau \equiv \text{the parameter})\) over the multidimensional geodesic paths connecting \( \Theta (0) \) to \( \Theta (\tau) \). The quantity \( \text{vol} \left[ D_{\Theta}^{(\text{geodesic})} (\tau') \right] \) is the volume of the effective parameter space explored by the system at time \( \tau' \). The temporal average has been introduced in order to average out the possibly very complex fine details of the entropic dynamical description of the system on \( M_S \).

Thus, comparing the asymptotic expressions of the IGCs in the presence and absence of microcorrelations, we obtain,

\[
\frac{\text{vol} \left[ D_{\Theta}^{(\text{geodesic})} (\tau; r) \right]}{\text{vol} \left[ D_{\Theta}^{(\text{geodesic})} (\tau; 0) \right]} \bigg|_{\text{no-cor.}} = \frac{g^{\frac{4}{2}} (r) A^{\frac{4}{2}} (0)}{A^{\frac{4}{2}} (r) g^{\frac{4}{2}} (0)} = \frac{1}{2^2} \sqrt{\frac{4 (4 - r^2)}{(2 - 2 r^2)^2} \left( \frac{2 + r}{4 (1 - r^2)} \right)^{-\frac{2}{3}}} \equiv \mathcal{F}_{M_S} (r). \tag{51}
\]

Written alternatively,

\[
\text{vol} \left[ D_{\Theta}^{(\text{geodesic})} (\tau; r) \right] \bigg|_{\text{cor.}} = \left[ \frac{1}{2^2} \sqrt{\frac{4 (4 - r^2)}{(2 - 2 r^2)^2} \left( \frac{2 + r}{4 (1 - r^2)} \right)^{-\frac{2}{3}}} \right] \cdot \text{vol} \left[ D_{\Theta}^{(\text{geodesic})} (\tau; 0) \right]_{\text{no-cor.}}. \tag{52}
\]
We emphasize that $F_{M_S}(r)$ is a monotonically decreasing function of $r$, that is $F_{M_S}(r_1) \geq F_{M_S}(r_2)$ for any $r_1 \leq r_2$ with $r_1, r_2 \in (0, 1)$ and $0 \leq F_{M_S}(r) \leq 1$ for $r \in (0, 1)$. We observe an asymptotic power law decay of the IGC in (50) at a rate determined by the correlation coefficient $r$. The ratio between the IGC in the presence and in the absence of microcorrelations in (51) leads to conclude that microcorrelations cause an asymptotic information geometric compression of the explored statistical macrostates at a faster rate than the that observed in absence of microcorrelations. Our finding presented in (52) shows an important connection between (micro)-correlations and (macro)-complexity in Gaussian statistical models.

V. FINAL REMARKS

In this article, we presented an analytical computation of the asymptotic temporal behavior of the IGC for a finite-dimensional microcorrelated Gaussian statistical model. The ratio between the IGC in the presence and in absence of microcorrelations was explicitly computed. We observed a power law decay of the IGC at a rate determined by the correlation coefficient. Specifically, the presence of microcorrelations lead to the emergence of an asymptotic information geometric compression of the statistical macrostates explored by the system at a faster rate than that observed in absence of microcorrelations. This result constitute an important and explicit connection between (micro)-correlations and (macro)-complexity in statistical dynamical systems. The relevance of our finding is twofold: first, it provides a neat description of the effect of information encoded in microscopic variables on experimentally observable quantities defined in terms of dynamical macroscopic variables [27]; second, it clearly shows the change in behavior of the macroscopic complexity of a statistical model caused by the existence of correlations at the underlying microscopic level.

We are confident that this work constitutes an important preliminary step towards the computation of the asymptotic behavior of the dynamical complexity of microscopically correlated multidimensional Gaussian statistical models and other models of relevance in more realistic physical systems. In principle, our approach extends its application to arbitrary statistical models that may arise upon maximization of the logarithmic relative entropy subject to the selected relevant information constraints. In particular, our findings here presented could find practical applications in the statistical analysis of biological and social systems since Gaussian statistical models are of primary importance in statistical studies [25]. However, our ultimate hope is to extend this approach in the field of Quantum Information to better understand the connection between quantum correlations (entanglement) and quantum complexity [26–30].

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Appendix A: Integration of the Geodesic Equations

From the first and second Equations in (39), we obtain,

\[ \frac{\dot{\mu}_x(\tau)}{\mu_x(\tau)} = 2 \frac{\dot{\sigma}(\tau)}{\sigma(\tau)} \quad \text{and} \quad \frac{\dot{\mu}_y(\tau)}{\mu_y(\tau)} = 2 \frac{\dot{\sigma}(\tau)}{\sigma(\tau)}, \]  

(A1)

respectively. From (A1) it follows that,

\[ \dot{\mu}_x(\tau) = A_1 \sigma^2(\tau) \quad \text{and} \quad \dot{\mu}_y(\tau) = A_2 \sigma^2(\tau), \]  

(A2)

where $A_1$ and $A_2$ are real constants. Substituting (A2) in the third Equation of (39) we obtain,

\[ \ddot{\sigma}(\tau) \sigma(\tau) - \dot{\sigma}^2(\tau) + \frac{A_1^2 + A_2^2 - r A_1 A_2}{4 (1 - r^2)} \sigma^4(\tau) = 0. \]  

(A3)
Therefore, the coupled system of differential equations reduces to,
\[ \dot{\mu}_x (\tau) - A_1 \sigma^2 (\tau) = 0, \]
\[ \dot{\mu}_y (\tau) - A_2 \sigma^2 (\tau) = 0, \]
\[ \ddot{\sigma} (\tau) \sigma (\tau) - \dot{\sigma}^2 (\tau) + A (r) \sigma^4 (\tau) = 0, \]  
(A4)

where we recall that,
\[ A (r) \overset{\text{def}}{=} \frac{A_1^2 + A_2^2 - rA_1 A_2}{4 (1 - r^2)}. \]  
(A5)

We now proceed as follows: integrate the nonlinear differential equation \[ \ddot{\sigma} (\tau) \sigma (\tau) - \dot{\sigma}^2 (\tau) + A (r) \sigma^4 (\tau) = 0, \]  
(A4) and then calculate \( \mu_x (\tau) \) and \( \mu_y (\tau) \).

Letting \( y (\tau) \overset{\text{def}}{=} \sigma (\tau) \), the first nonlinear differential equation to integrate becomes,
\[ \ddot{y} (\tau) y (\tau) - \dot{y}^2 (\tau) + A (r) y^4 (\tau) = 0. \]  
(A6)

Performing the following change of variables,
\[ y (\tau) = \frac{dx (\tau)}{d\tau} = \dot{x} (\tau) \]  
(A7)

equation (A6) becomes
\[ \dot{x} \ddot{x} - \dot{x}^2 + A (r) \dot{x}^4 = 0. \]  
(A8)

Equation (A8) can be integrated as follows. Performing the following additional change of variables,
\[ \dot{x} = \frac{dx (\tau)}{d\tau} = z (x) \]  
(A9)
leads to
\[ \ddot{x} = zz' \text{ and, } \dot{z} = (z'' z + z'^2) z, \]  
(A10)

with \( z' = \frac{dx}{dx} \). Substituting (A9) and (A10) into (A8), we find
\[ z'' + A (r) z = 0. \]  
(A11)

Integration of (A11) yields
\[ z (x) = A_3 \cos \left( \sqrt{A (r)} x \right) + A_4 \sin \left( \sqrt{A (r)} x \right), \]  
(A12)

where \( A_3 \) and \( A_4 \) are \textit{real} constants. Recalling that \( \dot{x} = \frac{dx (\tau)}{d\tau} = z (x) \), we have
\[ \int A_3 \cos \left( \sqrt{A (r)} x \right) + A_4 \sin \left( \sqrt{A (r)} x \right) \]  
\[ = \int d\tau + A_5. \]  
(A13)

\textit{Reality} conditions imply that \( A_3 = 0 \) and without loss of generality we can set \( A_5 = 0 \). Integration of (A13) leads to,
\[ x (\tau) = \frac{1}{\sqrt{A (r)}} \text{arccos} \left( \frac{1 - \exp \left( 2A_4 \sqrt{A (r)} \tau \right)}{1 + \exp \left( 2A_4 \sqrt{A (r)} \tau \right)} \right). \]  
(A14)

Finally, recalling that \( y (\tau) = \frac{dx (\tau)}{d\tau} \) and \( y (\tau) \overset{\text{def}}{=} \sigma (\tau) \), we get
\[ \sigma (\tau) = 2A_4 \frac{\exp \left( A_4 \sqrt{A (r)} \tau \right)}{1 + \exp \left( 2A_4 \sqrt{A (r)} \tau \right)}. \]  
(A15)
Note that $\sigma(\tau)$ in (A15) satisfies the equation $\dot{\sigma}(\tau) \sigma(\tau) - \dot{\sigma}^2(\tau) + A(r)\sigma^4(\tau) = 0$. Once we have obtained $\sigma(\tau)$, we have

$$\mu_x(\tau) = \int A_1 \sigma^2(\tau) \, d\tau + A_6 \quad \text{and} \quad \mu_y(\tau) = \int A_2 \sigma^2(\tau) \, d\tau + A_7, \quad \text{(A16)}$$

where $A_6$ and $A_7$ are real constants. Integrating, we get

$$\mu_x(\tau) = -\frac{2A_1 A_2}{\sqrt{A(r)}} \frac{1}{1 + \exp\left(\frac{2A_1 \sqrt{A(r)} \tau}{A_1 \sqrt{A(r)}}\right)} + A_6, \quad \text{(A17)}$$

and

$$\mu_y(\tau) = -\frac{2A_1 A_2}{\sqrt{A(r)}} \frac{1}{1 + \exp\left(\frac{2A_2 \sqrt{A(r)} \tau}{A_2 \sqrt{A(r)}}\right)} + A_7. \quad \text{(A18)}$$

Assuming the following boundary conditions $\sigma(0) = \sigma_0 > 0$, $\mu_x(\tau_\infty) = \mu_y(\tau_\infty) = 0$, we find that $\sigma_0 = A_4$, $A_6 = A_7 = 0$. Finally, the geodesic trajectories become,

$$\mu_x(\tau; r) = -\frac{2\sigma_0 A_1}{\sqrt{A(r)}} \frac{1}{1 + \exp\left(\frac{2\sigma_0 \sqrt{A(r)} \tau}{A_1 \sqrt{A(r)}}\right)}, \quad \mu_y(\tau; r) = -\frac{2\sigma_0 A_2}{\sqrt{A(r)}} \frac{1}{1 + \exp\left(\frac{2\sigma_0 \sqrt{A(r)} \tau}{A_2 \sqrt{A(r)}}\right)},$$

$$\sigma(\tau; r) = \frac{2\sigma_0}{1 + \exp\left(\frac{2\sigma_0 \sqrt{A(r)} \tau}{A_0 \sqrt{A(r)}}\right)}. \quad \text{(A19)}$$

with $A(r)$ defined in (A3). $A_1$ and $A_2$ real constants and $\sigma_0 > 0$. Note that $\sigma(\tau; r) \in (0, +\infty)$ while $\mu_x(\tau; r)$ and $\mu_y(\tau; r) \in (−\infty, +\infty)$.

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