Complete proper holomorphic embeddings of strictly
pseudoconvex domains into balls

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Abstract We construct a complete proper holomorphic embedding from any
strictly pseudoconvex domain with $C^2$-boundary in $\mathbb{C}^n$ into the unit ball of $\mathbb{C}^N$, for $N$ large enough, thereby answering a question of Alarcón and Forstnerič [1].

Keywords complete map, proper holomorphic map, peak function.

MSC (2010): 32H35, 32H02, 32T40

1. Introduction

The question of existence of complete bounded submanifolds in $\mathbb{C}^n$ was raised by Yang in 1977 [16, 17], and even before, in 1965, Calabi conjectured the nonexistence of complete minimal surfaces in $\mathbb{R}^3$ with bounded projection into a straight line [3], which turned out to be false [11]. These inspired many results in complex analysis and minimal surface theory. For a survey of the results and references, see the introduction in [1] and the survey [2]. Most of the known results regarding Yang’s question hold for complex curves, including the first positive answer by Jones [10], whereas for higher dimensional submanifolds not much was known until recently: Globevnik [7] proved that for any $n, m, 1 \leq n < m$, there is a complete closed $n$-dimensional complex submanifold in the unit ball of $\mathbb{C}^m$, and therefore he completely answered Yang’s question. In his construction there is no control on the topology of the submanifolds.

Alarcón and Forstnerič [1] constructed a complete proper holomorphic immersion from any bordered Riemann surface into the unit ball in $\mathbb{C}^2$, and a complete proper holomorphic embedding into the unit ball in $\mathbb{C}^m$, $m \geq 3$. They used the method of exposing boundary points of a complex curve in $\mathbb{C}^2$ [6] together with the approximate solution to a Riemann-Hilbert boundary value problem. None of these is available in higher dimensions. They asked if there is a complete proper holomorphic immersion/embedding from the unit ball in $\mathbb{C}^n$ into the unit ball of a higher dimensional Euclidean space. The aim of this note is to give an affirmative answer to their question.

Let $B_m$ denote the open unit ball in $\mathbb{C}^m$. An embedding $f: D \to \mathbb{C}^m$ from an open subset $D \subset \mathbb{C}^n$ is complete if the induced Riemannian metric $f^* ds^2$ on $D$ obtained by pulling back the Euclidean metric $ds^2$ on $\mathbb{C}^m$ is a complete metric on $D$. The main result of this note is the following theorem:

**Theorem 1.1.** Let $D$ be a bounded strictly convex domain with $C^2$-boundary in $\mathbb{C}^n$. There exists a positive integer $s$ with the following property. For any positive integer $p$ and for any continuous map $h: D \to B_p$, which is an injective holomorphic immersion in $D$, there exists a holomorphic map $f: D \to \mathbb{C}^{2s}$, such that the map $(f, h): D \to B_{2s+p}$ is a complete proper holomorphic embedding.
The main ingredient in the proof are holomorphic peak functions, the idea which goes back to Hakim and Sibony [9] and Løw [12], and the construction of inner functions on the unit ball. More precisely, we refine the construction of Forstnerič [5] of a proper holomorphic map from a strictly convex domain with $C^2$-boundary in $\mathbb{C}^n$ into a unit ball of some Euclidean space; see also [14] where the author obtained in addition to the above, a proper holomorphic map into a higher dimensional unit ball, which extends continuously to the boundary. Note that recently Globevnik [8] proved that there are no complete proper holomorphic maps from the open unit disc in $\mathbb{C}$ to the open unit bidisc in $\mathbb{C}^2$ which extend continuously to the boundary.

By Fornaess’ embedding theorem [4] any bounded strictly pseudoconvex domain with $C^2$-boundary embeds properly holomorphically into a strictly convex domain in Euclidean space. Since the composition of a proper and a complete proper holomorphic embedding is a complete proper holomorphic embedding we have the following corollary.

**Corollary 1.2.** Let $D$ be a bounded strictly pseudoconvex domain with $C^2$-boundary in $\mathbb{C}^n$. For $N$ large enough there exists a complete proper holomorphic embedding $F: D \to \mathbb{B}_N$.

Note that one could also extend the construction in the proof of Theorem 1.1 to obtain the same result where the domain $D$ is strictly pseudoconvex using the arguments of Løw [13]. More precisely, we could use different holomorphic peak functions with estimates similar to Lemma 2.1 below, see [13, Lemma 2.7].

2. Proof of Theorem 1.1

Throughout this section, $D$ is a bounded strictly convex domain with $C^2$-boundary in $\mathbb{C}^n$. Let $S$ denote its boundary and $\nu(w)$ the outward unit normal to $S$ at the point $w \in S$. For $a \in \mathbb{C}^n$ and $r > 0$ let $\mathbb{B}(a, r)$ denote the open ball of radius $r$ centered at $a$ in $\mathbb{C}^n$. We denote by $\langle \cdot, \cdot \rangle$, $\| \cdot \|$, and $\text{dist}(\cdot, \cdot)$ the Hermitian inner product, norm, and distance in $\mathbb{C}^n$.

The following lemma is a slight generalization of [5, Lemma 5.1].

**Lemma 2.1.** There are constants $\alpha_1, \alpha_2, r_1 > 0$ such that the following hold:

\[
\Re\langle w - z, \nu(w) \rangle \geq \alpha_1\|z - w\|^2 \text{ for all } w \in S, z \in \overline{D} \text{ such that } \text{dist}(z, bD) < r_1,
\]

\[
\Re\langle w - z, \nu(w) \rangle \leq \alpha_2\|z - w\|^2 \text{ for all } z, w \in S.
\]

**Proof.** The existence of $\alpha_2 > 0$ satisfying the second estimate was already a part of [5, Lemma 5.1]. Let $\rho$ denote a $C^2$-defining function of $D$ such that $\{z: \rho(z) < 0\} = D$ and $\text{grad}\rho(z)$ does not vanish for any $z \in bD$. Then there exists $\gamma_1 > 0$ such that $\text{grad}\rho(z)$ does not vanish for any $z, \rho(z) \in [-\gamma_1, 0]$, and the proof of [5, Lemma 5.1] provides a constant $\alpha_1 > 0$ such that

\[
\Re\langle w - z, \nu(w) \rangle \geq \alpha_1\|z - w\|^2 \text{ for all } z, w \in \overline{D} \text{ such that } \rho(z) = \rho(w) \in [-\gamma_1, 0].
\]

We may assume that $\alpha_1 > 0$ is so small that $1 - \alpha_1(1 + 2diam\ D) > 0$, where $diam\ D$ denotes the diameter of $D$. Since the boundary $bD$ is of class $C^2$ we can choose $r_1, 0 < r_1 < \frac{1}{2}$, so small that

\[
\{z \in \overline{D}: \text{dist}(z, bD) < r_1\} \subset \{w - r\nu(w): w \in S, r \in [0, 2r_1]\} \cap \rho^{-1}([-\gamma_1, 0]),
\]

\[
\|\nu(w - r\nu(w)) - \nu(w)\| \leq \frac{1 - \alpha_1(1 + 2diam\ D)}{diam\ D} r \text{ for all } w \in S, r \in [0, 2r_1].
\]


By the choice of \( r_1 \), for any \( w \in S \) and \( z \notin \overline{D} \) such that \( \text{dist}(z, bD) < r_1 \) there is \( r \in [0, 2r_1] \) such that \( p(w - rv(w)) = p(z) \in [-\gamma_1, 0] \). Letting \( w' = w - rv(w) \) we get

\[
\mathcal{R}(w - z, \nu(w)) = \mathcal{R}(w' - z, \nu(w)) + r
\]

(2.4)

\[
\geq 2 \alpha_1 \|w' - z\|^2 - (1 - \alpha_1(1 + 2\text{diam } D))r + r.
\]

On the other hand, we have

\[
\|w - z\|^2 = \|w' + w' - z\|^2
\]

\[
\leq r^2 + \|w' - z\|^2 + 2r \|w' - z\|
\]

\[
\leq \|w' - z\|^2 + (1 + 2\text{diam } D)r.
\]

By (2.4) we obtain \( \mathcal{R}(w - z, \nu(w)) \geq 2 \alpha_1 \|w - z\|^2 \), which completes the proof. \( \square \)

For the convenience of the reader we recall the next covering lemma from [5], see also [15]:

**Lemma 2.2.** [5] Lemma 5.2] For every \( \lambda > 1 \) there exists an integer \( s > 0 \) with the following property: For each \( r > 0 \) there are \( s \) families of balls \( \mathcal{F}_1, \ldots, \mathcal{F}_s \),

\[
\mathcal{F}_i = \{ \mathcal{B}(z_{i,j}, \lambda r) : 1 \leq j \leq N_i \},
\]

with centers \( z_{i,j} \in S \), such that the balls in each family are pairwise disjoint, and

\[
(2.5)
\]

\[
S \subset \bigcup_{i=1}^{s} \bigcup_{j=1}^{N_i} \mathcal{B}(z_{i,j}, r).
\]

Let \( \alpha_1 \) and \( \alpha_2 \) be as in Lemma 2.1 and let

\[
(2.6)
\]

\[
\lambda = 4 \sqrt{\frac{\alpha_2}{\alpha_1}}.
\]

Note that our choice of the constant \( \lambda \) is different from the one in [5] (5.7)), because we need more precise estimates in the next lemma.

For the chosen \( \lambda \) we get a positive integer \( s \) satisfying the properties in Lemma 2.2.

Therefore, for any \( r > 0 \) we have \( s \) families of balls \( \mathcal{F}_1, \ldots, \mathcal{F}_s \), \( \mathcal{F}_i = \{ \mathcal{B}(z_{i,j}, \lambda r) : 1 \leq j \leq N_i \}, z_{i,j} \in S \), such that the balls in each \( \mathcal{F}_i \) are pairwise disjoint and balls with the same centers and radii \( r \) cover \( S \) (2.5).

For each \( 1 \leq i \leq s \) and \( 1 \leq j \leq N_i \) we define \( z_{i+s, j} = z_{i,j} \) and \( \mathcal{F}_{i+s} = \mathcal{F}_i \). Further, for \( m > 0, 1 \leq i \leq 2s \) and \( 1 \leq j \leq N_i \) we define

\[
(2.7)
\]

\[
\phi_{i,j}(z) = e^{-m(z_{i,j} - z, \nu(z_{i,j}))}, \quad z \in \overline{D}.
\]

By (2.1) we get the following estimates

\[
|\phi_{i,j}(z)| \leq e^{-\alpha_1 m\|z - z_{i,j}\|^2} \quad \text{for all } w \in S, z \in \overline{D} \text{ such that } \text{dist}(z, bD) < r_1,
\]

\[
|\phi_{i,j}(z)| \geq e^{-\alpha_2 m\|z - z_{i,j}\|^2} \quad \text{for each } z \in S.
\]

For given \( |\beta_{i,j}| \leq 1 \), let \( g_i \) be the entire function

\[
(2.9)
\]

\[
g_i(z) = \sum_{j=1}^{N_i} \beta_{i,j} \phi_{i,j}(z), \quad z \in \overline{D}.
\]
The next lemma is similar to [5] Lemma 5.3, with the following differences: The estimate in (b) holds on \( \mathcal{T} \cap B(z_{i,j}, \lambda r) \) whereas in [5] it holds on \( S \cap B(z_{i,j}, \lambda r) \). The growth of \( \eta \) in (c) is different since we chose different \( \lambda \), and the property (d) is added since it will be needed in the proof of Theorem 1.1.

**Lemma 2.3.** Let \( r_1, \lambda, s, F_i, g_i, \beta_{i,j}, \) and \( \phi_{i,j} \) be as above. For each sufficiently small \( \eta > 0 \) there are \( m, r > 0 \), \( 0 < \lambda r < r_1 \), such that for each \( i \), \( 1 \leq i \leq 2s \), the following hold for the family of balls \( F_i \) and for the functions \( g_i \):

(a) If a point \( z \in S \) lies in no ball in \( F_i \), then \( |g_i(z)| < \eta \).

(b) If \( z \in \mathcal{T} \cap B(z_{i,j}, \lambda r) \) for some \( j \), then \( |g_i(z) - \beta_{i,j} \phi_{i,j}(z)| < \eta \).

(c) If \( z \in S \cap B(z_{i,j}, r) \) for some \( j \), then \( |\phi_{i,j}(z)| \geq C \eta^{\frac{1}{2m}} \), where the constant \( C \) is independent of \( r, m \) and \( \eta \).

(d) If \( z \in \mathcal{T} \cap bB(z_{i,j}, \lambda r) \) for some \( j \), then \( |\phi_{i,j}(z)| < \eta^{\frac{1}{2}} \).

Moreover, we can choose \( r > 0 \) arbitrarily small and make \( m \geq 0 \) as large as we want.

**Proof.** Properties (a), (c) are proved the same way as in the proof of [5] Lemma 5.3. We recall some parts of the proof, because we need the right choices of constants in the proof of (d).

If \( z \in S \) lies in no ball in \( F_i \) then as in [5] we obtain \( |g_i(z)| < C_2 e^{-\beta} \), where \( \beta = 16 \alpha_2 m^2 \geq \frac{4}{3} \) and the constant \( C_2 \) does not depend on \( r, m \) or \( \eta \). Here \( \beta \) is slightly different than in [5] since we chose a different \( \lambda \). Given \( \eta > 0 \), we take \( m > 0 \) and \( r > 0 \) such that \( C_2 e^{-\beta} = \eta \). If \( \eta \leq C_2 e^{-\frac{3}{4}} \), then \( \beta \geq \frac{1}{4} \) as needed. Since

\[
(2.10) \quad mr^2 = \frac{1}{16\alpha_2} \ln \frac{C_2}{\eta}
\]

we can choose \( r > 0 \) arbitrarily small and make \( m \) as large as we want. This proves (a).

For the proof of (b) note that the second estimate in (2.8) holds also on \( B(z_{i,j}, \lambda r) \), and then the same proof as in [5] gives (b).

Take \( z \in S \cap B(z_{i,j}, r) \) and according to (2.8) and (2.10) we get

\[
|\phi_{i,j}(z)| \geq e^{-\alpha_2 m r^2} = C_2^{-\frac{1}{16\alpha_2}} \eta^{\frac{1}{16\alpha_2}},
\]

which proves (c).

To prove (d), denote by \( \pi: \mathcal{T} \cap B(z_{i,j}, \lambda r) \to S \cap B(z_{i,j}, \lambda r) \) the orthogonal projection to the boundary in the \( \nu(z_{i,j}) \) direction. By strict convexity, the map \( \pi \) is well defined for any \( r > 0 \) small enough, and for any \( z \in \mathcal{T} \cap B(z_{i,j}, \lambda r) \) there exists \( s(z) \geq 0 \) such that \( z = \pi(z) - s(z)\nu(z_{i,j}) \), i.e. \( s(z) = \Re(\pi(z) - z, \nu(z_{i,j})) \). By (2.7) we have

\[
(2.11) \quad |\phi_{i,j}(z)| = e^{-m \Re(\pi_{i,j}, \nu(z_{i,j}))} e^{-ms(z)}.
\]

Both factors on the right are not bigger than 1. We split \( \mathcal{T} \cap bB(z_{i,j}, \lambda r) \) into two parts in such a way that on each part one of the factors is small enough to obtain the estimate (d). Fix any \( \mu, \sqrt{\frac{3}{2}} \mu < \mu < \lambda \). Let \( S_1 = \mathcal{T} \cap bB(z_{i,j}, \lambda r) \cap \nu^{-1}(S \cap B(z_{i,j}, \mu r)) \) and \( S_2 = \mathcal{T} \cap bB(z_{i,j}, \lambda r) \setminus S_1 \). For \( z \in S_2 \) we have \( \|\pi(z) - z_{i,j}\| \geq \mu r \), thus we get

\[
|\phi_{i,j}(z)| \leq e^{-m \Re(\pi_{i,j}, \nu(z_{i,j}))} \leq e^{-\mu^2 \alpha_2 mr^2} \leq \left( \frac{\eta}{C_2} \right)^{\frac{1}{2\alpha_2}} < \eta^{\frac{1}{2}},
\]

which proves (d).
for each \( \eta > 0 \) small enough. For \( z \in S_1 \) we have \( \| \pi(z) - z_{i,j} \| < \mu r \), therefore Pythagorean theorem and (2.1) imply
\[
s(z) = \Re(\zeta_{i,j} - z_1) - \Re(\zeta_{i,j} - \pi(z), \nu(z_{i,j})) \geq \sqrt{\lambda^2 - \mu^2r - \alpha_2\mu^2r^2}.
\]
For any given \( \eta > 0 \), we have
\[
|\phi_{i,j}(z)| \leq e^{-\sqrt{\lambda^2 - \mu^2r + \alpha_2\mu^2r^2}} e^{\frac{1}{2}\sqrt{\lambda^2 - \mu^2r + \alpha_2\mu^2}(\frac{1}{16\sqrt{2}})\ln \frac{2\mu}{\eta} < \eta^3},
\]
where the last estimate holds for any \( r > 0 \) small enough. This proves (d). \( \square \)

The following lemma refines [5] Lemma 6.1. The main addition is part (e) which guarantees that we increase the induced distance between a given point in \( D \) and the boundary \( S \) by a certain amount. Notice that the condition (iii) is slightly different from [5] Lemma 6.1 (iii); it provides control of how much we gain in (e). We shall denote the induced distance by a map \( F \) on \( D \) by dist\( F \).

**Lemma 2.4.** Let \( D, S = bD \) and \( h \) be as in the statement of Theorem 1.1 and \( s \) as above. Then there is \( \epsilon_0 > 0 \) such that the following implication holds: If we are given
\[
(i) \text{ numbers } a \text{ and } \epsilon, 0 < \epsilon < \epsilon_0, \text{ such that } a - \epsilon \frac{1}{2} > \frac{1}{2} \text{ and } a + \epsilon < 1,
(ii) \text{ a compact subset } K \subset D,
(iii) \text{ a continuous map } f = (f_1, \ldots, f_{2s}) : \overline{D} \rightarrow \mathbb{C}^{2s}, \text{ holomorphic in } D, \text{ such that for the map } F = (f, h) \text{ we have } \|F(z)\| < a - \epsilon \frac{1}{2} \text{ for each } z \in S,
(iv) \text{ a point } p \in D \text{ and a number } \sigma > 0 \text{ such that } \text{dist}_{F}(p, S) > \sigma, \text{ and}
(v) \text{ a number } \delta > 0,
\]
then there exists an entire mapping \( G = (g_1, \ldots, g_{2s}, 0, \ldots, 0) : \mathbb{C}^n \rightarrow \mathbb{C}^{2s+p} \) satisfying the following properties:
\[
(a) \|G(z)\| \leq a + \epsilon \text{ for all } z \in S,
(b) \text{ if } \|F + G(z)\| \leq a - \epsilon \frac{1}{2} \text{ for some } z \in S, \text{ then } \|G(z)\| > \|F(z)\| + \epsilon \frac{1}{2},
(c) \|G(z)\| < \delta \text{ for all } z \in K,
(d) \|G(z)\|^2 < 1 - \|F(z)\| \text{ for all } z \in S,
(e) \text{ dist}_{F+G}(p, S) > \sigma + E\epsilon \frac{\delta}{\mu}, \text{ where the constant } E \text{ depends only on } \epsilon_0.
\]

Note that the fact that \( h \) is an injective holomorphic immersion implies that \( F \) and \( G \) are injective holomorphic immersions.

**Proof.** The proof of (a)-(d) follows the proof of [5] Lemma 6.1. To obtain (e), we need a slightly different condition (iii) and we need to choose different growth of \( \epsilon \) in (b). The main idea of the proof is the same but we need to repeat the construction to make the necessary adjustments for the second part of the proof.

Let \( \eta = \frac{1}{2\lambda r_0} \). Let \( r_1 > 0 \) be the number provided by Lemma 2.1 and \( \lambda > 0 \) defined by (2.6). By continuity of \( F \) on \( \overline{D} \), there is \( r_0, 0 < r_0 < r_1 \), such that for any \( z, w \in \overline{D} \) with \( \|z - w\| < 2\lambda r_0 \) we have
\[
(2.12) \quad |f_i(z) - f_i(w)| < \eta, \quad 1 \leq i \leq 2s, \quad \|F(z)\| - \|F(w)\| < \eta.
\]
Given \( r, 0 < r < r_0 \), to be chosen later we choose \( s \) families of balls \( \mathcal{F}_1, \ldots, \mathcal{F}_s \), \( \mathcal{F}_i = \{B(z_{i,j}, \lambda r) : 1 \leq j \leq N_i\} \), with centers \( z_{i,j} \in S \), such that the balls in each family are pairwise disjoint and the small balls also cover \( S \) (Lemma 2.2). Let \( z_{i+s,j} = z_{i,j} \) and \( \mathcal{F}_{i+s} = \mathcal{F}_i, 1 \leq i \leq s \).
We define the coefficients $\beta_{i,j}$ and $\beta_{i+s,j}$, $1 \leq i \leq s$, $1 \leq j \leq N_i$, (2.9), as follows:

$$f_i(z_{i,j})\beta_{i,j} + f_i(z_{i+s,j})\beta_{i+s,j} = 0,$$

(2.13)

$$|\beta_{i,j}|^2 + |\beta_{i+s,j}|^2 = \frac{a^2 - \|F(z_{i,j})\|^2}{2s}.$$

This implies that the vector $(\beta_{i,j}, \beta_{i+s,j})$ is perpendicular to the vector $(f_i(z_{i,j}), f_i(z_{i+s,j}))$ and $|\beta_{i,j}| < 1$, $|\beta_{i+s,j}| < 1$. We shall prove that the entire map $G = (g_1, \ldots, g_{2s}, 0, \ldots, 0)$, where $g_i$ are defined by (2.9) and satisfy Lemma 2.2, has the properties (a)-(e), provided that the constant $m > 0$ is chosen large enough and $r > 0$ is chosen small enough.

Part (a) is proved exactly as in the proof [5, Lemma 6.1] and will not be repeated. The proof of (b) is very similar but the choice of the constants is different, so for the sake of the reader we repeat the relevant parts. As in the proof of [5, Lemma 6.1] we obtain: For $D_i(z) = |f_i(z) + g_i(z)|^2 + |f_{i+s}(z) + g_{i+s}(z)|^2 - |f_i(z)|^2 - |f_{i+s}(z)|^2$ we have

$$D_i(z) = (|\beta_{i,j}|^2 + |\beta_{i+s,j}|^2) |\phi_{i,j}(z)|^2 + O(\epsilon), \text{ if } z \in B(z_{i,j}, \lambda r) \text{ for some } j,$$

(2.14)

$$D_i(z) = O(\epsilon), \quad z \text{ lies in no ball in } F_i,$$

and furthermore

$$\|F(z) + G(z)\| - \|F(z)\| \geq O(\epsilon).$$

(2.15)

Suppose $\|(F + G)(z)\| \leq a - \epsilon \frac{1}{7}$ for some $z \in S$. Choose a ball $B(z_{i,j}, r)$ containing $z$. Then we have

$$\|F(z_{i,j})\| \leq \|F(z)\| + \epsilon \leq a - \epsilon \frac{1}{7} + O(\epsilon) < a - \frac{1}{2} \epsilon \frac{1}{7},$$

for any $\epsilon \in (0, \epsilon_0)$, if $\epsilon_0 > 0$ is chosen small enough. Therefore, since $a \geq \frac{1}{7}$ we get

$$a^2 - \|F(z_{i,j})\|^2 \geq \frac{1}{7} \epsilon \frac{1}{7},$$

which implies by (2.13) that $|\beta_{i,j}|^2 + |\beta_{i+s,j}|^2 \geq \frac{1}{8s} \epsilon \frac{1}{7}$. By Lemma 2.3(e), we obtain $|\phi_{i,j}(z)|^2 \geq C_2 \eta \frac{1}{7}$, which by (2.14) leads to

$$\|F(z) + G(z)\|^2 - \|F(z)\|^2 = \sum_{i=1}^{s} D_i(z) \geq \frac{C^2}{8s} \eta \frac{1}{7} \epsilon \frac{1}{7} + O(\epsilon) \geq 2 \epsilon \frac{1}{7},$$

for any $\epsilon \in (0, \epsilon_0)$, if $\epsilon_0 > 0$ is chosen small enough. Then we get

$$\|F(z) + G(z)\| - \|F(z)\| = \frac{\|F(z) + G(z)\|^2 - \|F(z)\|^2}{\|F(z) + G(z)\| + \|F(z)\|} \geq \epsilon \frac{1}{7},$$

which proves (b).

Property (iv) implies that there exists a compact set $L \subset D$ such that

$$\text{dist}_F(p, bL) > \sigma.$$

(2.16)

By enlarging $K$ if necessary, we may assume that $L \Subset K$. The part (c) and (d) are proved exactly as in [5], and the constant $m$ has to be chosen large enough. Moreover, $\|G(z)\|$ can be made arbitrarily small for all $z \in K$. Furthermore, by taking $m$ even larger if necessary, we can assume that $r > 0$ is so small that

$$L \cap B(z_{i,j}, \lambda r) = \emptyset, \quad \text{for all } 1 \leq i \leq s, \ 1 \leq j \leq N_i.$$

Since uniform approximation of $F$ on the compact set $K$ implies $C^1$-approximation of $F$ on the relatively compact subset $L$ we get from (2.16) that

$$\text{dist}_{F_* + G}(p, bL) > \sigma,$$

(2.17)
if $\|G(z)\|$ is small enough for all $z \in K$.

To prove (e), we consider $\text{dist}_{F+G}(bL, S)$. Choose any path $\gamma$ in $\overline{D}$ from $S$ to $bL$. Denote its starting point by $q_1 \in S$ and its ending point by $q_2 \in bL$. Choose a ball $B(z_{i,j}, r)$ containing $q_1$. Since $L \cap B(z_{i,j}, \lambda r) = \emptyset$ the path $\gamma$ intersects $bB(z_{i,j}, \lambda r)$; let $q_3$ denote any intersection point. We have

\begin{equation}
\text{length}((F + G)(\gamma)) \geq \|(F + G)(q_1) - (F + G)(q_3)\| \geq \sqrt{|(f_i + g_i)(q_1) - (f_i + g_i)(q_3)|^2 + |(f_{i+s} + g_{i+s})(q_1) - (f_{i+s} + g_{i+s})(q_3)|^2}.
\end{equation}

Note that $|a + b|^2 \geq |a|^2 - 2|b|$ for each $a, b \in \mathbb{C}$ such that $|a| \leq 1$. By Lemma 2.3 and (2.12) we get

\begin{align*}
&|f_i + g_i)(q_1) - (f_i + g_i)(q_3)|^2 \\
&\quad = |\beta_{i,j}\phi_{i,j}(q_1) + (g_i(q_1) - \beta_{i,j}\phi_{i,j}(q_1)) + (f_i(q_1) - f_i(q_3)) \\
&\quad \quad - \beta_{i,j}\phi_{i,j}(q_3) - (g_i(q_3) - \beta_{i,j}\phi_{i,j}(q_3))|^2 \\
&\quad \geq C^2|\beta_{i,j}|^2\eta^\frac{1}{2} - 2(3\eta + \eta^\frac{1}{2}),
\end{align*}

and similar for the index $i + s$ instead of $i$. Therefore by (2.18), (2.13) and (iii) we get

\begin{equation}
\text{length}((F + G)(\gamma)) \geq \sqrt{C^2(|\beta_{i,j}|^2 + |\beta_{i+s,j}|^2)\eta^\frac{1}{2} - 4(3\eta + \eta^\frac{1}{2})}
\end{equation}

(2.19)

\begin{align*}
\geq &\sqrt{C^2(2ae^\frac{1}{2} - \epsilon)\eta^\frac{1}{2} + O(\eta^\frac{3}{2})} \\
\geq &\sqrt{2C_1\eta^\frac{1}{2} + O(\eta^\frac{3}{2})} \geq \sqrt{C_1\eta^\frac{1}{2}} = E\eta^\frac{1}{2},
\end{align*}

for any $\epsilon \in (0, \epsilon_0)$, provided $\epsilon_0 > 0$ is small enough; the constants $C_1$ and $E$ depend only on $\epsilon_0$. Since (2.19) holds for any path from $bL$ to $S$ we have $\text{dist}_{F+G}(bL, S) \geq E\eta^\frac{1}{2}$, and by (2.17) we get $\text{dist}_{F+G}(p, S) \geq \sigma + E\eta^\frac{1}{2}$, which proves (e). □

**Proof of Theorem 1.1.** Let $h : \overline{D} \to \mathbb{B}_p$ be as in the statement of Theorem 1.1. We shall construct the map $F$ inductively in a way similar to the proof of [5, Theorem 1.3].

Choose an increasing sequence $\{a_k\}_{k \geq 1}$ converging to 1, and a decreasing sequence $\{\epsilon_k\}_{k \geq 1}$ converging to 0, such that the following hold:

(i) $\sup_S \|h\|, \frac{1}{2} < a_1 - \epsilon_1^\frac{1}{2}$,

(ii) $\sum_{k=1}^{\infty} \epsilon_k^\frac{1}{2} < \infty$, $\sum_{k=1}^{\infty} \epsilon_k^\frac{5}{2} = \infty$,

(iii) $a_k + \epsilon_k < a_{k+1} - \epsilon_{k+1}^\frac{1}{2}$ for all $k \geq 1$.

We can obtain the two sequences as follows: First we choose $a_1, \frac{1}{2} < a_1 < 1$, so close to 1, and $\epsilon_1 > 0$ so close to 0 that (i) holds. Then we choose a decreasing sequence $\{\epsilon_k\}$ converging to 0 such that

\begin{align*}
3 \sum_{k=1}^{\infty} \epsilon_k^\frac{1}{2} &= 1 - a_1, \\
\sum_{k=1}^{\infty} \epsilon_k^\frac{5}{2} &= \infty,
\end{align*}
which implies property (ii). The sequence
\[ a_k = a_1 + 3 \sum_{l=1}^{k-1} \epsilon_l^2 + 2 \epsilon_k^2, \quad k \geq 2, \]
converges to 1 and satisfies (iii).

Let \( F_0 = (0, \ldots, 0, h) \) and fix any \( p \in D \). Since \( h \) is nonconstant we have \( \text{dist}_{F_0}(p, S) > 0 \). Let \( s \) be the number provided by Lemma 2.2 for \( \lambda \) defined by (2.6), Using Lemma 2.4 we will inductively construct a sequence of entire maps \( \{G_j : \mathbb{C}^n \to \mathbb{C}^{2s+p}\}_{j \geq 1} \), a sequence of injective holomorphic immersions \( F_k = F_0 + \sum_{j=1}^k G_j \), two increasing sequences of compact subsets \( \{K_k\}_{k \geq 1}, \{L_k\}_{k \geq 1} \) of \( D \) such that
\[ L_k \subset \hat{K}_k, \quad \text{and} \quad \bigcup_{k=1}^\infty K_k = \bigcup_{k=1}^\infty L_k = D, \]
a decreasing sequence \( \{\delta_k\}_{k \geq 1} \) converging to 0, \( 0 < \delta_k < \epsilon_k \), such that for every \( k \geq 1 \) the following properties hold:

(a) \( \|F_{k-1}(z)\| \geq \min_{w \in S} \|F_{k-1}(w)\| - \frac{1}{2^k} \) for each \( z \in \overline{D} \setminus K_k \),
(b) \( \|F_k(z)\| \leq a_k + \epsilon_k \) for each \( z \in \overline{D} \),
(c) if \( \|F_k(z)\| \leq a_k - \epsilon_k^2 \) for some \( z \in S \), then \( \|F_k(z)\| > \|F_{k-1}(z)\| + \epsilon_k^2 \),
(d) \( \|G_k(z)\| < \frac{\delta_k}{2^2} \) for each \( z \in K_k \),
(e) \( \|G_k(z)\|^2 < 1 - \min_{w \in S} \|F_k(w)\| \) for all \( z \in \overline{D} \),
(f) \( \text{dist}_{F_{k-1}}(p, bL_k) > \frac{1}{2} \text{dist}_{F_0}(p, bD) + E \sum_{j=1}^{k-1} \epsilon_j^{5/2} \),
(g) if \( F : D \to \mathbb{C}^{2s+p} \) is holomorphic and \( \|F(z) - F_{k-1}(z)\| < \delta_k \) for all \( z \in K_k \), then \( \text{dist}_F(p, bL_k) > \text{dist}_{F_{k-1}}(p, bL_k) - 1 \).

First choose \( L_1 \) such that (f) holds for \( k = 1 \), then choose \( K_1, L_1 \subset \hat{K}_1 \), such that (a) holds for \( k = 1 \). Since uniform approximation of \( F_0 \) on the compact set \( K_1 \) implies \( C^1 \)-approximation of \( F_0 \) on the relatively compact subset \( L_1 \), there is \( \delta_1, 0 < \delta_1 < \epsilon_1 \), satisfying (g). We apply Lemma 2.4 to \( F_0, a_1, \epsilon_1, \delta_1/2 \) to obtain an entire map \( G_1 \), which satisfies properties (b)-(e), and \( \text{dist}_{F_1}(p, bD) > \frac{1}{2} \text{dist}_{F_0}(p, bD) + E \epsilon_1^{5/2} \). We proceed similarly, taking (iii) into account and we obtain sequences \( G_k, K_k, L_k \) and \( \delta_k \), which satisfy (a)-(g).

Property (d) implies that the sequence \( F_k \) converges uniformly on compact sets in \( D \) to a holomorphic map \( F : D \to \mathbb{C}^{2s+p} \) and we get the estimate
\[
\|F_{k-1}(z) - F(z)\| \leq \|F_{k-1}(z) - F_k(z)\| + \|F_k(z) - F_{k+1}(z)\| + \cdots \leq \frac{\delta_k}{2^k} + \frac{\delta_{k+1}}{2^{k+1}} + \cdots \leq \delta_k \quad \text{for every} \quad z \in K_k
\]
(2.20)
This implies together with (f) and (g) that
\[
\text{dist}_F(p, bL_k) > \frac{1}{2} \text{dist}_{F_0}(p, bD) + E \sum_{j=1}^{k-1} \epsilon_j^{5/2} - 1.
\]
By (ii) the series \( \sum_j \epsilon_j^{5/2} \) diverges, which implies that the map \( F \) is complete. Property (b) and the maximum principle imply that \( F(D) \subset \mathbb{B}_{2s+p} \). Since the map \( h \) is an injective immersion on \( D \), \( F_0 = (0, \ldots, 0, h) \) and all last \( p \) components of the maps \( G_k \) are zero for
each \( k \), all the maps \( F_k \) and the limit map \( F \) are injective immersions. The fact that the map \( F \) is proper is proved as in [5], where we take into account that the series \( \sum_j e_j^2 \) is divergent by (ii). This completes the proof of Theorem 1.1. \( \square \)

Acknowledgements. The author is partially supported by the research program P1-0291 and the grant J1-5432 from ARRS, Republic of Slovenia.

Part of this work was done while the author was visiting University of Oslo. She would like to thank the Department of Mathematics for the hospitality and partial financial support.

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