GORENSTEIN WEAK DIMENSION OF A COHERENT POWER SERIES RINGS

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Abstract. We compute the Gorenstein weak dimension of a coherent power series rings over a commutative rings and we show that, in general, Gwdim (R) ≤ 1 does not imply that R is an arithmetical ring.

1. Introduction

Throughout this paper, all rings are commutative with identity element, and all modules are unital.

Let R be a ring, and let M be an R-module. As usual we use pd_R(M), id_R(M) and fd_R(M) to denote, respectively, the classical projective dimension, injective dimension and flat dimension of M. By gldim (R) and wdim (R) we denote, respectively, the classical global dimension and weak dimension of R.

For a two-sided Noetherian ring R, Auslander and Bridger [1] introduced the G-dimension, Gdim_R(M), for every finitely generated R-module M. They showed that there is an inequality Gdim_R(M) ≤ pd_R(M) for all finite R-modules M, and equality holds if pd_R(M) is finite.

Several decades later, Enochs and Jenda [9, 10] defined the notion of Gorenstein projective dimension (G-projective dimension for short), as an extension of G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (G-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas [11] introduced the Gorenstein flat dimension. Some references are [3, 6, 7, 10, 11, 14].

Recently in [4], the authors started the study of the notions global Gorenstein dimensions of ring R, which are denoted by GPD(R), GID(R), and Gwdim(R) and defined as follows:

(1) GPD(R) = sup{Gpd_R(M) | M be an R-module}
(2) GID(R) = sup{Gid_R(M) | M be an R-module}
(3) Gwdim(R) = sup{Gfd_R(M) | M be an R-module}

They proved that, for any ring R, Gwdim(R) ≤ GID(R) = GPD(R) (4 Theorems 2.1 and 2.11). So, according to the terminology of the classical theory of homological dimensions of rings, the common value of GPD(R) and GID(R) is called Gorenstein global dimension of R, and denoted by Ggldim(R). They also proved that the Gorenstein global and weak dimensions are refinement of the classical global and weak dimensions of rings. That means Ggldim(R) ≤ gldim(R) (resp.

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Gwdim \((R) \leq \text{wdim} \((R))\), and equality holds if \text{wdim} \((R))\) is finite \([4, \text{Propositions 2.12}]\).

In \([15]\) Jøndrup and Small gave a connection between a weak dimension of a coherent power series ring over a commutative ring \(R\) and the weak dimension of \(R, \text{see also \([13, \text{Theorem 8.1.1}]\). In the following we recall this result:

**Theorem 1.1.** Let \(R\) be a ring, and let \(x\) be an indeterminate over \(R\). If \(R[[x]]\) is a coherent ring, then \text{wdim} \((R[[x]]) = \text{wdim} \((R) + 1\).

In this paper, we give an extension of Theorem 1.1 to the Gorenstein weak dimension.

We know that if \text{wdim} \((R) \leq 1\), then \(R\) is an arithmetical ring (see for instance \([2]\)). Now it is natural to ask the following question: "Does Gwdim \((R) \leq 1\) imply that \(R\) is an arithmetical ring?" In Theorem \([2.12]\) we give a negative answer to this question. More precisely, we prove: Let \((R, m)\) be a local quasi-Frobenius ring which is not a field. Then Gwdim \((R[[X]]) = 1\) but \(R[[X]]\) is not an arithmetical ring.

2. Gorenstein weak dimension

First we recall the notion of strongly Gorenstein projective module which is introduced in \([3]\).

**Definition 2.1.** A module \(M\) is said to be strongly Gorenstein projective (SG-projective for short), if there exists an exact sequence of the form:

\[ P = \cdots \rightarrow P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \rightarrow \cdots \]

where \(P\) is a projective \(R\)-module and \(f\) is an endomorphism of \(P\), such that \(M \cong \text{Im} (f)\) and such that \(\text{Hom}(-, Q)\) leaves the sequence \(P\) exact whenever \(Q\) is a projective module.

These strongly Gorenstein projective modules has a simple characterization, and they are used to characterize the Gorenstein projective modules. We recall the following two results which are \([3, \text{Propositions 2.9}]\) and \([3, \text{Theorem 2.7}]\):

**Proposition 2.2.** A module \(M\) is strongly Gorenstein projective if, and only if, there exists a short exact sequence of modules:

\[ 0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0 \]

where \(P\) is projective and \(\text{Ext}(M, Q) = 0\) for any projective module \(Q\).

**Theorem 2.3.** A module is Gorenstein projective if, and only if, it is a direct summand of a strongly Gorenstein projective module.

**Lemma 2.4.** Let \(R\) be a ring and let \(X\) be an indeterminate over \(R\) and \(M\) an \(R[[X]]\)-module. Then \(X\) is a nonzero divisor on \(M\) if and only if \(\text{Tor}_{R[[X]]}(M, R) = 0\).

**Proof.** Let \(\varphi_X : M \rightarrow M\) be the homomorphism of \(R[[X]]\)-module such that \(\varphi_X(m) = Xm\) for every \(m \in M\). Consider the short exact sequence of \(R[[X]]\)-modules

\[ (\star) \quad 0 \rightarrow R[[X]] \xrightarrow{\text{H}X} R[[X]] \rightarrow R \cong R[[X]]/XR[[X]] \rightarrow 0 \]
where $\mu_X$ is the multiplication by $X$. The following sequence is induced from $(\ast)$

$$0 \rightarrow \text{Tor}_{R[[X]]}(M, R) \rightarrow R[[X]] \otimes_{R[[X]]} M \xrightarrow{1 \otimes \mu_X} R[[X]] \otimes_{R[[X]]} M \rightarrow R \otimes_{R[[X]]} M \rightarrow 0$$

By [17 Theorem 8.13] the $R[[X]]$-morphism $1_M \otimes \mu_X$ is multiplication by $X$. So, in the following diagram all squares are commutative

$$\begin{array}{ccc}
0 & \rightarrow & \text{Tor}_{R[[X]]}(M, R) \\
\downarrow & & \downarrow \\
R[[X]] \otimes_{R[[X]]} M & \xrightarrow{1 \otimes \mu_X} & R[[X]] \otimes_{R[[X]]} M \\
\downarrow & & \downarrow \\
M & \xrightarrow{\varphi_X} & M/\text{Ker} \varphi_X \\
\downarrow & & \downarrow \\
G/P & \rightarrow & M/\text{Ker} \varphi_X
\end{array}$$

Therefore, $\text{Ker} \varphi_X \cong \text{Tor}_{R[[X]]}(M, R)$ and hence $X$ is a nonzero divisor on $M$ if, and only if, $\text{Tor}_{R[[X]]}(M, R) = 0$. □

**Lemma 2.5.** Let $R$ be a ring and $X$ an indeterminate over $R$ such that $R[[X]]$ is coherent. If $M$ is a finitely presented $R[[X]]$-module such that $X$ is a nonzero divisor on $M$ then $\text{Gpd} R[[X]](M) \leq \text{Gpd} R(M/\text{XM})$.

**Proof.** First note that $R$ is a coherent ring by [13 Theorem 4.1.1(1)]. In addition, $X$ is contained in the Jacobson radical of $R[[X]]$. Let $M$ be a finitely presented $R[[X]]$-module $M$ over which $X$ is a nonzero divisor and put $n = \text{Gpd} R(M/\text{XM})$. We may assume that $n$ is finite.

The proof will be by induction on $n$.

If $M/\text{XM}$ is a Gorenstein projective $R$-module, then by [8 Proposition 10.2.6 (1)⇒(10)], the proof is the same as the one of [8 Corollary 1.4.6] (note that $X$ is an element of the Jacobson radical of $R[[X]]$) and so we may use the Nakayama’s Lemma in the proof of [8 Corollary 1.4.6]. In the original proof we use the **Local condition**.

Now, assume that $n > 0$ and consider the short exact sequence of $R[[X]]$-modules

$$0 \rightarrow G \rightarrow P \rightarrow M \rightarrow 0$$

where $P$ is a finitely presented projective $R[[X]]$-module. Using [13 Theorem 2.5.1], $G$ is also finitely presented since $R[[X]]$ is coherent. From Lemma 2.4, we have $\text{Tor}_{R[[X]]}(M, R) = 0$ since $X$ is a nonzero divisor on $M$. In addition, $\text{Tor}_{R[[X]]}(P, R) = 0$ since $P$ is a projective $R[[X]]$-module. Therefore, $\text{Tor}_{R[[X]]}(G, R) = 0$ (since $\text{fd} R[[X]] R \leq 1$). So, by Lemma 2.4, $X$ is a nonzero divisor on $G$. On the other hand, if we tensor the short exact sequence above with $- \otimes R[[X]] R$ we obtain a short exact sequence

$$0 \rightarrow G/\text{XM} \rightarrow P/\text{XM} \rightarrow M/\text{XM} \rightarrow 0$$

(note that $M \otimes R[[X]] R \cong M/\text{XM}$). Therefore, by the hypothesis condition of induction, $\text{Gpd}_R(G/\text{XM}) \leq \text{Gpd}_R(G/\text{XM}) \leq n - 1$. Thus, $\text{Gpd}_R(M) \leq \text{Gpd}_R(G/\text{XM}) + 1 \leq n$, as desired. □

**Definition 2.6** [18 and 12]. Let $R$ be a ring and let $M$ be an $R$-module.

1. We say that $M$ has $FP$-injective dimension at most $n$ (for some $n \geq 0$), denoted by $\text{FP-id}_R(M) \leq n$, if $\text{Ext}_R^{n+1}(P, M) = 0$ for every finitely presented $R$-module $P$.

2. A ring $R$ is said to be $n-FC$, if it is coherent and it has self-$FP$-injective at most at $n$ (i.e., $\text{FP-id}_R(R) \leq n$).

A ring is called $FC$ ring if it is 0-FC.

Using [5 Theorems 6 and 7], we deduce the following Lemma.
Lemma 2.7. Let $R$ be a coherent ring and let $n \geq 0$ be an integer. The following are equivalent:

1. $R$ is $n-\text{FC}$;
2. $\text{Gwdim}(R) \leq n$;
3. $\text{Gpd}_R(M) \leq n$ for every finitely presented $R$-module $M$.

Remark 2.8. (1) By Lemma 2.7 the Gorenstein weak dimension of a coherent ring $R$ is also determined by the formula:

$$\text{Gwdim}(R) = \sup \{ \text{Gpd}_R(M) \mid M \text{ is a finitely presented } R\text{-module} \}.$$ 

(2) In Lemma 2.7, the case $n = 0$ (i.e., if $R$ is FC) does not need the coherence condition (see [15, Theorem 6]).

Lemma 2.9. Let $R$ be a coherent ring and let $X$ be an indeterminate over $R$. Then, $\text{Gwdim}(R[[X]]) \geq \text{Gwdim}(R) + 1$.

Proof. By [13, Theorem 4.1.1(1)], $R \cong R[[X]]/XR[[X]]$ is coherent since it is a finitely presented $R$-module (from the short exact sequence $0 \to R[[X]] \xrightarrow{X} R[[X]] \to R \to 0$).

We may assume that $\text{Gwdim}(R[[X]]) = n < \infty$. Using [13, Theorem 1.3.3] and [14, Proposition 2.27], we have $\text{Gpd}_R(R[[X]])(R) = \text{pd}_{R[[X]]}(R) = 1$. Thus, by Lemma 2.7 $\text{Gwdim}(R[[X]]) = n \geq 1$ since $R$ is a finitely presented $R[[X]]$-module.

Now, let $M$ be a finitely presented $R$-module. Then, by [13, Theorem 2.1.8], $M$ is a finitely presented $R[[X]]$-module (since $R \cong R[[X]]/XR[[X]]$). Thus, by [13, Theorem 1.3.5] and Lemma 2.7, $\text{Ext}^n_R(M, R) = \text{Ext}^n_{R[[X]]}(M, R[[X]]) = 0$. Therefore, $R$ is $(n-1)-\text{FC}$. Hence, by Lemma 2.7 $\text{Gwdim}(R) \leq n-1$. Therefore $\text{Gwdim}(R) \leq \text{Gwdim}(R[[X]]) - 1$, as desired. \qed

Now we are ready to present our main result of this paper.

Theorem 2.10. Let $R$ be a ring and let $x$ be an indeterminate over $R$. If $R[[x]]$ is a coherent ring, then $\text{Gwdim}(R[[x]]) = \text{Gwdim}(R) + 1$.

Proof. If $\text{Gwdim}(R) = \infty$, then by Lemma 2.9 we have the desired equality. Otherwise we put $\text{Gwdim}(R) = n$. By Lemma 2.9 it is enough to show that $\text{Gwdim}(R[[X]]) \leq \text{Gwdim}(R) + 1$. Let $M$ be a finitely presented $R[[X]]$-module and consider a short exact sequence of $R[[X]]$-modules

$$(*) \quad 0 \to K \to P \to M \to 0,$$

where $P$ is a finitely generated projective $R[[X]]$-module. Then, by [13, Theorem 2.5.1], $K$ is also finitely presented since $R[[X]]$ is coherent. Thus $K/XK$ is also finitely presented $R$-module (by [13, Theorem 2.1.8]). On the other hand, from the short sequence $(*)$ we have $\text{Tor}_R(R[[X]])(K, R) = \text{Tor}_R^2(R[[X]])(M, R) = 0$ since $\text{fd}_R(R[[X]])(R) \leq 1$. So, from Lemma 2.4 $X$ is a nonzero divisor on $K$. Then, by Lemma 2.5 and Lemma 2.7 $\text{Gpd}_R(R[[X]])(K) \leq \text{Gpd}_R(K/XK) \leq n$. Then, $\text{Gpd}_R(R[[X]])(M) \leq n + 1$. Consequently, by Lemma 2.7 $\text{Gwdim}(R[[X]]) \leq n + 1 = \text{Gwdim}(R) + 1$. \qed

Recall that a ring is called quasi-Frobenius, if it is Noetherian and self-injective (see [16]).
Proposition 2.11. Let $R$ be a quasi-Frobenius ring. Then,

\[ \text{Gwdim} (R[[X]]) = \text{Ggldim} (R[[X]]) = 1 \]

Proof. Let $R$ be a quasi-Frobenius ring. Then, from [4, Proposition 2.8 and Theorem 2.9], $\text{Gwdim} (R) = \text{Ggldim} (R) = 0$. Thus, from Theorem 2.10, $\text{Gwdim} (R[[X]]) = 1$. On the other hand, $R$ is Noetherian and so $R[[X]]$ is also Noetherian. Therefore, by [4, Theorem 2.9], $\text{Gwdim} (R[[X]]) = 1$.

Recall that a ring $R$ is called an arithmetical ring if every finitely generated ideal is locally principal. If $\text{wdim} (R) \leq 1$, then $R$ is an arithmetical ring (see for instance [2]). So we lead to ask the following question: If $\text{Gwdim} (R) \leq 1$, then is $R$ arithmetical ring?

The following result shows that the above question is false in general.

Theorem 2.12. Let $(R, \mathfrak{m})$ be a local quasi-Frobenius ring which is not a field. Then the following statements hold:

1. \( \text{Gwdim} (R[[X]]) = 1 \).
2. \( R[[X]] \) is not an arithmetical ring.

Proof. (1) We have $\text{Gwdim} (R[[X]]) = 1$ by Proposition 2.11.

2) We claim that $\text{Gwdim} (R[[X]])$ is not an arithmetical ring. Deny. Let $a$ be a non-zero non-invertible element of $R$ and let $I := aR[[X]] + XR[[X]]$. Then $I = PR[[X]]$ for some $P := \sum_i a_i X^i \in R[[X]]$ (where $a_i \in R$), since $R[[X]]$ is a local arithmetical ring.

Since $P \in I = aR[[X]] + XR[[X]]$, we have $P = aQ_1 + XQ_2$ for some $Q_1 := \sum_i c_i X^i$, $Q_2 \in R[[X]]$. Hence, $a_0 = ac_0$.

On the other hand, we have $a = PQ$ for some $Q = \sum_i b_i X^i \in R[[X]]$ (where $b_i \in R$) since $a \in I = PR[[X]]$. Hence, $a = a_0 b_0$. We claim that $b_0 \in M$. If this is not the case, then $b_0$ is invertible in $R$ and so $Q$ is invertible in $R[[X]]; hence, we may assume that $P = a$ (since $aR[[X]] = PR([[X]] = PR[[X]] = I)$. But, $X \in I = aR[[X]]$ implies that $X = a \sum d_i X^i$ for some $d_i \in R$. Hence, $1 = ad_1$ and so $a$ is invertible in $R$, a contradiction. Therefore, $b_0 \in M$.

Therefore, $a = ab_0 c_0$ since $a_0 = ac_0$ and $a = a_0 b_0$ and so $a(1 - b_0 c_0) = 0$. But $1 - b_0 c_0$ is invertible in $R$ since $b_0 c_0 \in M$ (since $b_0 \in M$); hence $a = 0$, a contradiction. Hence, $R[[X]]$ is not an arithmetical ring, as desired.

In the rest of this paper, We compare the small finitistic Gorenstein projective dimension of the base ring $R$,

\[ \text{fGPD} (R) = \sup \left\{ \text{Gpd}_R (M) \mid M \text{ is an } R - \text{module with finite Gorenstein projective dimension and } M \text{ admits a finite projective resolution} \right\}. \]

with the usual small finitistic projective dimension, $\text{fPD} (R)$ (see [13]).

It is clear that if $R$ is coherent we have

\[ \text{fGPD} (R) = \sup \{ \text{Gpd}_R M \mid M \text{ is a finitely presented and } \text{Gpd}_R M < \infty \} \leq \text{Gwdim} (R), \]

with equality if $\text{Gwdim} (R)$ is finite (by Lemma 2.7).

In the proof of the next Theorem, we use the proofs of [14, Theorems 2.10 and 2.28].

Theorem 2.13. For any coherent ring $R$ there is an equality $\text{fGPD} (R) = \text{fPD} (R)$. 

Recall that a right co-proper projective resolution of an \(R\)-module \(M\) is an exact sequence \(X = 0 \to M \to X^0 \to X^1 \to \ldots\) with \(X^i\) is projective for each \(i \geq 0\) such that \(\text{Hom}_R(X, P)\) is exact for every projective module \(P\).

**Lemma 2.14.** Let \(R\) be a ring. Then, every finitely generated \(G\)-projective \(R\)-module admits a right co-proper resolution of finitely generated free \(R\)-module.

**Proof.** Let \(M\) be a finitely generated \(G\)-projective \(R\)-module. By [14, Proposition 2.4], there is an exact sequence of \(R\)-modules
\[
0 \to M \to L \to M' \to 0
\]
where \(L\) is a free \(R\)-module and \(M'\) a \(G\)-projective \(R\)-module. We identify \(M\) to a submodule of \(L\) and we assume that \(L\) admits basis \(\{x_k, k \in K\}\). Since \(M\) is finitely generated and each generator of \(M\) is a finite linear combination of finite subset of \(\{x_k, k \in K\}\), we consider \(L_0\) a finitely generated free direct summand of \(L\) which contains \(M\) and a free \(R\)-module \(L_1\) such that \(L = L_0 \oplus L_1\). Then, we have the following commutative diagram
\[
\begin{array}{c}
0 \\
\downarrow \\
0 \to M \to L \\
\downarrow \\
| \\
L_0 \to L_0/M \\
\downarrow \\
| \\
0 \to M' \to 0 \\
\downarrow \\
L_1 \\
\downarrow \\
0
\end{array}
\]

From [17, Exercise 2.7 page 29] the diagram above can be completed as
\[
\begin{array}{c}
0 \\
\downarrow \\
0 \to M \\
\downarrow \\
| \\
L_0 \to L_0/M \\
\downarrow \\
| \\
0 \to M' \to 0 \\
\downarrow \\
L_1 = L_1 \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\]

In the right vertical exact sequence \(L_1\) is a projective module and so \(M' \cong L_0/M \oplus L_1\). Then, from [14, Theorem 2.5], \(L_0/M\) is a \(G\)-projective \(R\)-module. In addition, by [11, Theorem 2.20], for every projective \(R\)-module \(F\), the short sequence
\[
0 \to \text{Hom}_R(L_0/M, F) \to \text{Hom}_R(L_0, F) \to \text{Hom}_R(M, F) \to 0
\]
is exact (since \(\text{Ext}(L_0/M, F) = 0\)). On the other hand, it is clear that \(L_0/M\) is finitely generated. Then, by repeating this procedure, \(M\) admits a right co-proper resolution of finitely generated free \(R\)-modules. \(\square\)

**Proof of Theorem 2.13.** Clearly \(\text{fPD}(R) \leq \text{fGPD}(R)\) by [14, Proposition 2.27]. In first we claim that \(\text{fGPD}(R) \leq \text{fPD}(R) + 1\). So, let \(M\) be a finitely presented module with \(0 < \text{Gpd}_R(M) = n < \infty\). We may pick an exact sequence, \(0 \to \ldots\).
Let $R$ be a coherent ring and let $x$ be a nonzero divisor in $R$ contained in the intersection of the maximal ideals of $R$. Then:

1. $\text{fGPD}(R) = \text{fGPD}(R/xR) + 1$

2. If $\text{Gwdim}(R) < \infty$, then $\text{Gwdim}(R) = \text{Gwdim}(R/xR) + 1$
Proof. The first equality follows from Theorem 2.13 and [13, Corollary 3.1.4]. Now assume that $\text{Gwdim}(R) = n$ is finite. We claim that $\text{Gwdim}(R/\mathfrak{x}R)$ is finite. In first see, by [13, Theorem 4.1.1(1)], that $R/\mathfrak{x}R$ is also coherent since $R/\mathfrak{x}R$ is a finitely presented $R$-module (from the short exact sequence $0 \to R \xrightarrow{\mathfrak{x}} R \to R/\mathfrak{x}R \to 0$). Using [13, Theorem 1.3.3] and [14, Proposition 2.27], we have

$$\text{Gpd}_R(R/\mathfrak{x}R) = \text{pd}_R(R/\mathfrak{x}R) = 1$$

Then, from Lemma 2.7 $\text{Gwdim}(R) = n \geq 1$ since $R/\mathfrak{x}R$ is a finitely presented $R$-module. Now, let $M$ be a finitely presented $R/\mathfrak{x}R$-module. Then, by [13, Theorem 2.1.8], $M$ is a finitely presented $R$-module. Thus, by [13, Theorem 1.3.5] and Lemma 2.7 $\text{Ext}^n_R(M, R/\mathfrak{x}R) = \text{Ext}^{n+1}_R(M, R) = 0$. Therefore, $R/\mathfrak{x}R$ is $(n - 1) - FC$. Hence, by Lemma 2.7 $\text{Gwdim}(R/\mathfrak{x}R) \leq n - 1 < \infty$. So, by Theorem 2.13 and [13, Corollary 3.1.4], we have

$$\text{Gwdim}(R) = f\text{GPD}(R) = f\text{GPD}(R/\mathfrak{x}R) + 1 = \text{Gwdim}(R/\mathfrak{x}R) + 1.$$

Now the assertion holds. $\square$

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