Discrete-layered damping model of multilayer plate with account of internal damping

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Abstract. Construction of discrete-layered damping model of multilayer plate in small displacement and deformations with account of internal damping of layers of Thompson-Kelvin-Voight model is presented. Based on derived equations, analytical solution is given to the static damping problem of simply supported single-layer rectangular plate subjected to uniformly distributed pressure, which is applied to one of its boundary planes. Convergence to the three-dimensional case is analysed for the obtained solution with respect to the dependence on dimension of mesh in the thickness direction of plate. For thin plates, dimension reduction of the formulated problem is set on the basis of simplifying hypothesis applied for each layer.

1. Introduction
Composite materials from high strength carbon or glass fibers, which has found so far widespread utilization as construction material in manufacture of different types of products, have relatively low impact toughness, internal friction parameters and absorption of impact energy. Thereby, one of solution in improvement of specified properties of structures is application of adhesive bonding of different composite materials. Structure of those materials consists traditional composites with carbon or glass fiber reinforcements and protecting layer with various types of reinforced or unreinforced elastomers, which are inherent impact toughness and high level of energy absorption. In particular, as a protecting layer, it is expedient to use composites from high-strength and high-modulus polyethylene fibers (HSMPF) with elastomer matrixes. HSMPF differ from other high-strength fibers not only by their high level of specific mechanical properties, but also by minimum coefficient of friction, positive effect of their deformation rates on strength, sharp increase of strength in below zero temperatures and other properties. It is worth to note that due to the poor adhesion almost many plastics HSMPF is mainly used in production of cords and ropes. However, recently, there have been developed technologies that enable, for example, by irradiation with cold plasma substantially improve the adhesion properties HSMPF fibers and composites to create on their basis [1–3 et al.]. Such composites can be used merely in production of various structural parts as construction material. In particular, mentioned above specific properties make possible to recommend them in production of fuel tanks of air vehicles using as matrix of resin or polyurethane. Having relatively high rigidity in tension, they retain the flexibility to bend. To ensure the fire safety in potential penetration of tank, it
is advised to use them in multilayer structure consecutively stacking unreinforced soft resin layers and rigid, in tension, elastomer matrix composite layers.

The application of elements with a multilayer structure of the described class in different structures requires the development of computational models and methods, taking into account the features of the structure and behavior of materials of the layers. Among these features, in particular, layered character of structure and relatively low stiffness in directions that do not coincide with the direction of the reinforcement can be emphasized. In addition, soft layers of such structural elements are also characterized by high values of internal friction, which induces the need accounting internal damping of material in steady dynamic damping process.

Definition of structural elements, in general, can be attributed to class of multilayered [4], consisting of rigid layers, reinforced in tangential directions and transversely soft core layers. Mechanics of deformation of multilayer structures is studied extensively in scientific literature review of which is contained in many papers [5-7 et al.]. Among them, in particular, we refer to monograph [8], in which the theoretical bases of construction of non-classical mathematical models for the dynamics of single-layer and multilayer plates and shells with variable thickness from traditional materials as well as composites with comprehensive accounting of effects of material and (or) geometrically nonlinearity during deformation, anisotropy and strength, viscoelastic properties, heterogeneity of elastic and strength parameters. For the all proposed models, numerical methods were developed on the basis of which a wide range of nonlinear deformation problems as well as stability and optimum design of single-layer or composite multilayer elements and spatial structures under impulsive loadings and set the application field of classical shell theory.

Present paper is devoted to further improvements of described research directions with the main goal being construction of discrete-structural model of deformation of multilayer plate in small displacements and deformations considering the internal damping of layer materials according to Thompson-Kelvin-Voight viscoelastic model.

2. Equations of multilayer plate theory in small displacements and deformations

Let’s consider plate consist of \( M = N - 1 \) layers (fig.1), space of which \( V^k \) is attributed to parametrization \( R^k = x_1 e_1 + x_2 e_2 + z_1 m \). \(-h^k \leq z^k \leq h^k\)

![Figure 1. Multilayer plate scheme.](image)

where layers in direction of unit vector \( m \) normal to the middle plane \( \sigma^k \) have thickness \( h^k \); \( x_1, x_2 \) – orthogonal Descartes coordinates on any plane \( \sigma \), adopted as base of parametrization. For the description of deformation mechanics of each \( [k] \)-th layer we will use Timoshenko kinematic model [8-10 et al.], employin for the displacement vectors \( U^k \) the following representation

\[
U^k = u^k + z^k \gamma^k = u^k e_1 + w^k m + z^k \left( \gamma^k e_1 + \gamma^k m \right); k = 1, N-1
\]
In small deformations, translations and rotations representation (1) corresponds to tangential deformation components \( \varepsilon_i^1 \), \( \gamma_{12}^1 \), transverse compression component \( \varepsilon_i^3 \) and transverse shear components \( \gamma_{13}^1 \), which are calculated by formulæ

\[
e_i^1 = e_i^1 + z_i^1 \gamma_{12}^1, \quad \gamma_{12}^1 = e_i^1 + z_i^1 \left( \Omega_{12}^1 + \Omega_{21}^1 \right); \quad \varepsilon_i^3 = \varepsilon_i^3 + \gamma_{13}^1, \quad \gamma_{13}^1 = 2 \left( \varepsilon_i^3 + z_i^3 \gamma_{13}^1 \right)
\]

(2)

where

\[
e_i^1 = u_i^1, \quad 2 \varepsilon_i^3 = w_i^1 + \gamma_{13}^1, \quad \Omega_{ij}^1 = \gamma_{ij}^1, \quad 2 \Omega_{ij}^3 = \gamma_{ij}^3
\]

(3)

Let’s assume that boundary sections of plate are formed by the motion of normal \( m \) along the coordinate lines \( x_i = x_i^0 \), \( x_i = x_i^0 + \Delta x_i \), at the points of middle planes \( \sigma_{ij} \) of each \( [k] \)-th layer and their contour lines, edge forces and moments are given as

\[
Q_i^k = Q_{ij}^1 e_i + Q_{ij}^2 e_j + Q_{ij}^3 m, \quad L_i^k = L_{ij}^1 e_i + L_{ij}^2 e_j + L_{ij}^3 m; \quad k = 1, N - 1
\]

(4)

Applied to planes \( \sigma_{ij} \) and vectors of surface loads and moments

\[
X_i^k = X_{ij}^1 e_i + X_{ij}^2 e_j + X_{ij}^3 m, \quad M_i^k = M_{ij}^1 e_i + M_{ij}^2 e_j + M_{ij}^3 m; \quad k = 1, N - 1
\]

(5)

are attributed to unit area of surfaces \( \sigma_{ij} \). Variation of work of these forces and moments with respect to corresponding displacements will be equal to

\[
\delta A = \sum_{k=1}^{N-1} \delta A_i^k = \sum_{k=1}^{N-1} \sum_{i=1}^{2} \left( \int_{x_i^0}^{x_i^0 + \Delta x_i} \left( Q_{ij}^1 \delta u_i^1 + Q_{ij}^2 \delta w_i^1 + L_{ij}^1 \delta \gamma_{ij}^1 + L_{ij}^3 \delta \gamma_{ij}^3 \right) dx_i \right) + \int_{x_i^0}^{x_i^0 + \Delta x_i} \left( X_{ij}^1 \delta u_i^1 + X_{ij}^2 \delta w_i^1 + M_{ij}^1 \delta \gamma_{ij}^1 + M_{ij}^3 \delta \gamma_{ij}^3 \right) dx_i
\]

(6)

If internal forces and moments are introduced

\[
T_{ij}^0 = \int_{-\eta_{ij}/2}^{\eta_{ij}/2} \sigma_{ij}^0 z_k dz_k, \quad M_{ij}^a = \int_{-\eta_{ij}/2}^{\eta_{ij}/2} \sigma_{ij}^a z_k dz_k
\]

(7)

applied to middle plane of layers, then for the variation of strain potential energy of plate by using relationship (2), (3), we can end up with the formula below

\[
\delta \Pi = \sum_{k=1}^{N-1} \delta \Pi_i^k = \sum_{k=1}^{N-1} \int_{x_i^0}^{x_i^0 + \Delta x_i} \left( T_{ij}^0 \delta e_i^0 + T_{ij}^3 \delta \gamma_{ij}^0 \right) + \int_{x_i^0}^{x_i^0 + \Delta x_i} \left( M_{ij}^1 \delta \gamma_{ij}^1 + M_{ij}^3 \delta \gamma_{ij}^3 \right) d\sigma
\]

(8)

For the variation of kinetic strain energy taking into account the energy of rotations, the following expression takes place

\[
\delta K = \sum_{k=1}^{N-1} \delta K_i^k = \sum_{k=1}^{N-1} \int_{x_i^0}^{x_i^0 + \Delta x_i} \left( \tilde{X}_{ij}^k \delta u_i^k + \tilde{X}_{ij}^3 \delta w_i^k + \tilde{Y}_{ij}^1 \delta \gamma_{ij}^1 + \tilde{Y}_{ij}^3 \delta \gamma_{ij}^3 \right) d\sigma
\]

(9)

where

\[
\tilde{X}_{ij}^k = \rho_p h_k [j] \tilde{w}_{ij}^k, \quad \tilde{X}_{ij}^3 = \rho_p h_k [j] \tilde{w}_{ij}^k, \quad \tilde{Y}_{ij}^1 = \frac{\rho_p h_k [j]}{12} \tilde{w}_{ij}^k, \quad \tilde{Y}_{ij}^3 = \frac{\rho_p h_k [j]}{12} \tilde{w}_{ij}^k
\]

(10)

Here \( \rho_p [k] \) – density of material of \([k]\)-th layer, dots above the functions show derivations with respect to time \( \tau \).

We introduce \( N \) vectors of displacements

\[
v_i^k = u_i^k e_i + w_i^k m; \quad k = 1, N
\]

(11)
of boundary planes points \( z_0 = -h_{[i]}/2, z_{[N-1]} = h_{[N-1]}/2 \), interlayer planes and set \( N - 2 \) conditions for interlayers of the plate according to displacement as follow

\[
U^{[k]}(z_0) = -h_{[k]}/2, \quad U^{[k]}(z_{[N-1]}) = h_{[k]}/2; \quad k = 1, N - 2
\]

Substituting representations (1) and (11) from the conditions (12), the following relations follow

\[
u^{[k]} = \frac{u^{[k]}_i + U^{[k]}_i}{2}, \quad w^{[k]} = \frac{w^{[k]}_i + w^{[k]}_j}{2}, \quad y^{[k]} = \frac{u^{[k]}_i - U^{[k]}_i}{h_{[k]}}, \quad \gamma^{[k]} = \frac{w^{[k]}_i - w^{[k]}_j}{h_{[k]}}
\]

and substitution of them into (3) we come to relations

\[
e^{[k]}_g = \frac{e^{(k+1)}_g + e^{(k)}_g}{2}, \quad \alpha^{[k]}_g = \frac{\alpha^{(k+1)}_g + \alpha^{(k)}_g}{2}
\]

\[
\Omega^{[k]}_g = \frac{\omega^{(k)}_g - \omega^{(k+1)}_g}{h_{[k]}}, \quad \Omega^{[k]}_g = \frac{\alpha^{(k)}_g - \alpha^{(k+1)}_g}{h_{[k]}}; \quad k = 1, N - 1
\]

in which the following kinematical relationships are introduced

\[
e^{(k)}_g = u^{(k)}_g; \quad \omega^{(k)}_g = w^{(k)}_g
\]

Using the relationships (13), (14), we can bring expressions (6), (8) and (9) into form

\[
\delta A = \sum_{k=1}^{N} \sum_{i=x}^{N_x} \int_{z_i}^{z_{i+1}} \left( Q^{[k]}_a \delta u^{[k]}_a + Q^{[k]}_a \delta v^{[k]}_a \right) dx + \sum_{k=1}^{N} \int_\sigma \left( X^{[k]}_a \delta u^{[k]}_a + X^{[k]}_a \delta v^{[k]}_a \right) dx dx_2 \]

\[
\delta A = \sum_{k=1}^{N} \sum_{i=x}^{N_x} \int_{z_i}^{z_{i+1}} \left( Q^{[k]}_a \delta u^{[k]}_a + Q^{[k]}_a \delta v^{[k]}_a \right) dx dx_2
\]

where

\[
Q^{[k]}_a = \frac{Q^{[k]}_a}{2} \frac{L^{[k]}_a}{h_{[k]}}, \quad Q^{[k]}_a = \frac{Q^{[k]}_a}{2} \frac{L^{[k]}_a}{h_{[N-1]}}
\]

\[
Q^{[k]}_a = \frac{Q^{[k+1]}_a + Q^{[k]}_a}{2} \frac{L^{[k]}_a}{h_{[k]}}, \quad k = 2, N - 1; i = 1, 2; \alpha = 1, 3
\]

\[
X^{[k]}_a = \frac{X^{[k]}_a}{2} \frac{M^{[k]}_a}{h_{[k]}}, \quad X^{[k]}_a = \frac{X^{[k]}_a}{2} \frac{M^{[k]}_a}{h_{[N-1]}}
\]

\[
X^{[k]}_a = \frac{X^{[k+1]}_a + X^{[k]}_a}{2} \frac{M^{[k]}_a}{h_{[k]}}, \quad \alpha = 1, 3; \quad k = 2, N - 1
\]

\[
S^{[k]}_a = \frac{T^{[k]}_a}{2} \frac{M^{[k]}_a}{h_{[k]}}, \quad S^{[k]}_a = \frac{T^{[k+1]}_a}{2} \frac{M^{[k]}_a}{h_{[N-1]}}
\]

\[
S^{[k]}_a = \frac{T^{[k]}_a + T^{[k+1]}_a}{2} \frac{M^{[k]}_a}{h_{[k]}}, \quad k = 2, N - 1; i = 1, 2; \alpha = 1, 3
\]

\[
N^{[k]}_a = \frac{T^{[k]}_a}{h_{[k]}}, \quad N^{[k]}_a = \frac{T^{[N-1]}_a}{h_{[N-1]}} \frac{T^{[k]}_a}{h_{[k]}}, \quad k = 2, N - 1; \alpha = 1, 3
\]
\[
\tilde{X}_a = \frac{\tilde{X}_a}{2} - \frac{\tilde{Y}_a}{h_t}, \quad \tilde{X}_a^{[i]} = \frac{\tilde{X}_a^{[i]}}{h_{t[x]}}, \quad \tilde{X}_a^{[i]} = \frac{\tilde{Y}_a^{[i]}}{h_{t[x]}}, \quad \alpha = 1,3; \quad k = 2, N - 1
\]

2.1. Equations of equilibrium

In static equilibrium state of the plate, variational principle of virtual displacements \(\delta T - \delta A = 0\) must be satisfied. The last takes the following form after substitution of expression (15) and doing standard some transformations

\[
\sum_{i=1}^{2} \sum_{k=1}^{N} \left( \int_{x_i}^{x_{i+1}} \left( S_1^{(k)} - Q_{ij}^{(k)} \right) \delta u_j^{(k)} + (S_{ij}^{(k)} - Q_{ij}^{(k)}) \delta w_j^{(k)} \right) dx_{i,j} - \int_{x_i}^{x_{i+1}} \left( f_1^{(k)} \delta u_1^{(k)} + f_2^{(k)} \delta w_1^{(k)} \right) dx_{i,j} = 0
\]

In view of the arbitrariness of variations \(\delta u_i^{(k)}, \delta w_i^{(k)}\) from (20) follows the system of \(3N\) differential equation of equilibrium

\[
f_1^{(k)} = S_{11}^{(k)} + S_{12}^{(k)} - N_{11}^{(k)} + X_1^{(k)} = 0; \quad \frac{1, 2}{1, 3}
\]

\[
f_3^{(k)} = S_{13}^{(k)} + S_{23}^{(k)} - N_{13}^{(k)} + X_3^{(k)} = 0
\]

for which in edge sections \(x_i = x_i^-, x_i = x_i^+\) boundary conditions are formulated as follow

\[
S_{ij}^{(k)} = Q_{ij}^{(k)} \text{ when } \delta u_i^{(k)} \neq 0, \quad S_{ij}^{(k)} = Q_{ij}^{(k)} \text{ when } \delta w_i^{(k)} \neq 0
\]

2.2. Equation of motion

From Hamilton-Ostrogradsky variational principle, system of differential equations of motion can be obtained as follow

\[
f_1^{(k)} = S_{11}^{(k)} + S_{12}^{(k)} - N_{11}^{(k)} + X_1^{(k)} - \tilde{X}_1^{(k)} = 0; \quad \frac{1, 2}{1, 3}
\]

\[
f_3^{(k)} = S_{13}^{(k)} + S_{23}^{(k)} - N_{13}^{(k)} + X_3^{(k)} - \tilde{X}_3^{(k)} = 0
\]

In it, in contrast to (21), inertial terms \(\tilde{X}_1^{(k)}, \tilde{X}_3^{(k)}\) appeared, which are defined by formulas (10), (19).

2.3. Kinematic relationships

We assume that materials of layers of shell are linear elastic and orthotropic, and orthotropy axes coincide with the selected coordinate system. For such a material, taking into account its viscoelastic properties, resulting stress and strain components in plate can be related by dependencies

\[
\sigma_{\alpha i}^{[k]} = G_{\alpha j}^{[k]} \dot{e}_j^{[k]}, \quad \sigma_{\alpha i}^{[k]} = G_{\alpha j}^{[k]} \dot{e}_j^{[k]}, \quad \sigma_{\alpha i} = \tilde{G}_{\alpha j}^{[k]} \dot{e}_j
\]

\[
\sigma_{\alpha i}^{[k]} = \tilde{G}_{\alpha j}^{[k]} \dot{e}_j
\]

Here \(\tilde{G}_{\alpha j}^{[k]}, \tilde{G}_{\alpha j}^{[k]}\) – differential operators, which takes form when Thompson-Kelvin-Voight model [11-14] is used

\[
\tilde{G}_{12} = G_{21} + \frac{\delta_{12}}{\rho \omega} \frac{\partial}{\partial \tau}, \quad \tilde{G}_{13} = G_{13} \left(1 + \frac{\delta_{13}}{\rho \omega} \frac{\partial}{\partial \tau}\right), \quad \tilde{G}_{23} = G_{23} \left(1 + \frac{\delta_{23}}{\rho \omega} \frac{\partial}{\partial \tau}\right)
\]
In this type of loading, the equations (22) for \( \omega = -i \) are applied to boundary planes \( \Delta^{[1]} \), \( \Delta^{[2]} \), \( \Delta^{[3]} \) respectively. In this type of loading, the equations (22) for \( \omega = -i \) are applied to boundary planes \( \Delta^{[1]} \), \( \Delta^{[2]} \), \( \Delta^{[3]} \) respectively.

Substituting expression (24) into (7) and using (2) we obtain two-dimensional elasticity relationships for forces and moments

\[
T_{\alpha \beta}^{[k]} = C_{\alpha \gamma}^{[k]} e_{\gamma 
\text{in which the following designations are introduced for the corresponding rigidities}
\]

\[
M_{\alpha 
\text{in this type of loading, the equations (22) for [k]-th layer, by using the relationships (26), (19), (13)-(15), which are constructed with respect to}
\]

\[
U^{[k]} = \begin{bmatrix} u^{[k]} 
\text{form}
\]

\[
f_1^{[k]} = \frac{h_{[k]} g_{11}^{[k]}}{6} u_{1,1}^{[k;1]} + \frac{h_{[k]} g_{11}^{[k]} + h_{[k-\ell]} g_{11}^{[k-1]}}{3} u_{1,1}^{[k;1]} + \frac{h_{[k-\ell]} g_{11}^{[k-1]}}{6} u_{1,1}^{[k;1]} + 
\]

\[
+ \frac{h_{[k]} G_{12}^{[k]}}{6} u_{1,2}^{[k;2]} + \frac{h_{[k]} G_{12}^{[k]} + h_{[k-\ell]} G_{12}^{[k-1]}}{3} u_{1,2}^{[k;2]} + \frac{h_{[k-\ell]} G_{12}^{[k-1]}}{6} u_{1,2}^{[k;2]} + 
\]

\[
+ \frac{h_{[k]} (g_{12}^{[k]} + G_{12}^{[k-1]})}{6} u_{1,2}^{[k;2]} + \frac{h_{[k]} (g_{12}^{[k]} + G_{12}^{[k-1]}) + h_{[k-\ell]} (g_{12}^{[k]} + g_{12}^{[k-1]})}{3} u_{1,2}^{[k;2]} + 
\]

\[
+ \frac{h_{[k]} (g_{12}^{[k]} + G_{12}^{[k-1]})}{6} u_{1,2}^{[k;2]} + \frac{h_{[k]} (g_{12}^{[k]} + G_{12}^{[k-1]})}{6} w_{1,1}^{[k;1]} + 
\]

\[
+ \frac{g_{31}^{[k]} - g_{31}^{[k-1]} - G_{13}^{[k]} + G_{13}^{[k-1]}}{2} w_{1,1}^{[k;1]} + \frac{G_{13}^{[k]} - G_{13}^{[k-1]}}{2} w_{1,1}^{[k;1]} + 
\]

\[
+ \frac{G_{13}^{[k]} - G_{13}^{[k-1]}}{2} u_{1,1}^{[k;1]} - \frac{h_{[k]} G_{p}^{[k]}}{6} u_{1,1}^{[k;1]} - \frac{h_{[k]} G_{p}^{[k]} + h_{[k-\ell]} G_{p}^{[k-1]}}{3} \overline{u}_{1}^{[k]} - \frac{h_{[k-\ell]} G_{p}^{[k-1]}}{6} \overline{u}_{1}^{[k]} = 0; 1, 2, 3
\]
algebraic equations of motion are obtained from (28) we get system of equations of motion.

In case of simply supported edge faces of plates solutions of equation (28) should be subject to boundary conditions:

\[ \text{at } x_i = 0, x_i = a \quad u^{(k)}_{i,1} = 0, \quad u^{(k)}_{i,2} = 0, \quad w^{(k)} = 0 \quad (29) \]

In accordance with the above assumption of monoharmonic loading, solution (28), satisfying the boundary conditions (29), will have the form

\[ u^{(k)}_1 = e^{j\omega t} \sum_{m=1,3,\ldots} u^{(k)}_{mn} \cos \lambda_m x \cdot \sin \lambda_n y, \]
\[ u^{(k)}_2 = e^{j\omega t} \sum_{m=1,3,\ldots} u^{(k)}_{2mn} \sin \lambda_m x \cdot \cos \lambda_n y, \]
\[ w^{(k)} = e^{j\omega t} \sum_{m=1,3,\ldots} w^{(k)}_{mn} \sin \lambda_m x \cdot \sin \lambda_n y \]

where \( \lambda_m = m\pi/a, \quad \lambda_n = n\pi/b \). In the virtue of (30) the pressure amplitude values represented as a Fourier series expansion

\[ p^{(k)} = p^{(h)} \sum_{m=1} f^{(h)}_{mn} \sin \lambda_m x \cdot \sin \lambda_n y, \quad f^{(h)}_{mn} = \frac{16}{\pi^2 mn} \]

Substituting (30) and (31) into equations of motion (28) we get system of 3N algebraic equations with respect to amplitude values \( \hat{u}^{(k)}_{mn} \) and \( \hat{w}^{(k)}_{mn} \)

\[ f^{(h)}_{1mn} = k^{[1]}_{11} \hat{u}^{(h)}_{1mn} + \left( k^{[1]}_{11} + k^{[1]}_{12} \right) \hat{u}^{(h)}_{2mn} + k^{[1]}_{13} \hat{w}^{(h)}_{2mn} + \]
\[ + k^{[1]}_{12} \hat{w}^{(h)}_{2mn} + \left( k^{[1]}_{12} + k^{[1]}_{13} \right) \hat{u}^{(h)}_{2mn} + k^{[1]}_{13} \hat{w}^{(h)}_{2mn} + \]
\[ + k^{[1]}_{13} \hat{w}^{(h)}_{2mn} \]

where \( i = 1, 2 \)
\[ f_3^{(k)} = k_3^{[4]} = k_3^{[1]} n_{i,m,n} + \left( \tilde{e}_3^{(k)} - k_3^{[1]} \tilde{u}_m^{(k)} \right) n_{i,m,n} + \]
\[ + k_3^{[2]} \tilde{u}^{(k)} + \left( \tilde{e}_3^{(k)} - k_3^{[1]} \tilde{u}_m^{(k)} \right) n_{i,m,n} + \]
\[ + k_3^{[3]} \tilde{u}^{(k)} + \left( \tilde{e}_3^{(k)} - k_3^{[1]} \tilde{u}_m^{(k)} \right) n_{i,m,n} = -b^{(k)} f_{mn} \]

where the following designations were introduced

\[ k_{11}^{[1]} = \frac{h_{[1]} p_{p}^{[1]} \omega^2}{6} - \frac{h_{[1]} \tilde{G}_{12}^{[1]} \lambda_m}{6} - \frac{h_{[1]} \tilde{G}_{13}^{[1]} \lambda_m^2}{6} \]
\[ k_{22}^{[1]} = \frac{h_{[1]} p_{p}^{[1]} \omega^2}{6} - \frac{h_{[1]} \tilde{G}_{12}^{[1]} \lambda_m}{6} - \frac{h_{[1]} \tilde{G}_{23}^{[1]} \lambda_m^2}{6} \]
\[ k_{33}^{[1]} = -k_{33}^{[1]} \]
\[ k_{12}^{[1]} = k_{21}^{[1]} \]
\[ k_{13}^{[1]} = k_{31}^{[1]} = \frac{h_{[1]} \left( \tilde{g}_{11}^{[1]} + \tilde{G}_{11}^{[1]} \right)}{3} \]
\[ k_{23}^{[1]} = \frac{h_{[1]} \left( \tilde{g}_{23}^{[1]} + \tilde{G}_{23}^{[1]} \right)}{3} \]
\[ k_{32}^{[1]} = -k_{32}^{[1]} \]

Note that under static loading of plate, the corresponding equilibrium equations for transverse bending of a multilayer plate are obtained from (32) and (33) at \( \omega = 0 \).

4. Results of numerical studies on the transverse bending of single-layer plate and their analysis

On the basis of the solution found, let’s take into consideration the problem of determining the parameters of stress-strain state of rectangular plate subjected to uniformly distributed static pressure acting on one of its boundary planes so that \( p^{(1)} = 1 \) kPa, and \( p^{(2)} = 0 \). We will assume that plate is made of steel and have the following values of parameters

\[ E_1 = E_2 = E_3 = E = 200 \text{ GPa}, \quad v_{12} = v_{13} = v_{23} = 0.3, \quad a = 480 \text{ mm}, \quad b = 560 \text{ mm}, \quad t = 3 \text{ mm}. \]

In order to approximate solutions based on the use of two-dimensional equations of the theory of plates to the solutions obtained by correspondingly use of three-dimensional equations of elasticity theory, let’s consider described above single-layer plate in the form of a multilayer plate from \( M \) layers with thicknesses \( t/M \) and having equal physical and mechanical properties. Obtained solution of the problem should be regarded as the use of differential forms of the finite element method [15], in which the displacements in the layers of the plate are approximated by linear shape functions.
As an illustration, in fig.2-3, distribution of deflection \( w \) and normal stress \( \sigma_z = \sigma_{11} \) through the plate thickness are introduced at the centre of the plate. Here and below dashed lines correspond to solutions, obtained at \( M = 1 \) (six mode variant of theory of plates and shells [16], corresponding to use in the plate Timoshenko model with transverse normal strains), results with dotted dash correspond to \( M = 5 \) and dotted lines to \( M = 10 \). These curves are compared with results, obtained on the basis of classical Kirchhoff-Love model for single-layer isotropic plate (continuous line), when the values of \( w \) and \( \sigma_z = \sigma_{11} \) are calculated by formula

\[
\begin{align*}
    w &= \sum_{m,n=1,3,..} w_{mn} = \sum_{m,n=1,3,..} \frac{12p^{(j)} f_{mn} (1-\nu^2)}{Et^3 (\lambda_m^2 + \lambda_n^2)^3}, \\
    \sigma_z &= \frac{Et}{2(1-\nu^2)} \sum_{m,n=1,3,..} (\lambda_m^2 + \nu \lambda_n^2) w_{mn}
\end{align*}
\]

![Figure 2. Relationship \( w(z) \) for steel plate](image2)

![Figure 3. Relationship \( \sigma_z(z) \) for steel plate](image3)

Analysing the demonstrated results, it is not difficult to see the main drawback of six-mode variant of theory, related with constant transverse normal strain of the plate, which cause [16] bending stiffness to be distorted, but, in numerical computation, cause to increase of the stiffness of discrete system: results at \( M = 1 \) significantly differ from the results obtained from the use of classical Kirchhoff-Love model. Results with tolerated accuracy can be obtained only increasing the number of layers in system (in this case at \( M \geq 10 \)) and they are being closer to 3D elasticity solutions when \( M > 10 \).

The analysis of the equations of equilibrium and motion of the plate showed that the described disadvantage of equations with small values of \( M \) is due to the significant difference between the elasticity relations (24), (25) from the relationships, corresponding to plane stress state of the plate. Conducted computing experiments have shown that well acceptable solutions to the problem for small values of \( M \) can be obtained by taking Poisson coefficients equal to zero. For an illustration, in fig. 4-5 relationships \( w = w(z) \), \( \sigma_z = \sigma_z(z) \) are shown at the center of the plate, which are calculated taking \( \nu_{13} = \nu_{23} = 0 \). It is obvious that for the all values of parameter \( M \) results are almost identical.

![Figure 4. Relationship \( w(z) \) for steel plate in case \( \nu_{13} = \nu_{23} = 0 \)](image4)

![Figure 5. Relationship \( \sigma_z(z) \) for steel plate in case of \( \nu_{13} = \nu_{23} = 0 \)](image5)

Finally, through the thickness variation of transverse normal stress \( \sigma_z = \sigma_{13} \) resulted at the centre of the plate is shown in fig.6., which is obtained by taking \( \nu_{13} = \nu_{23} = 0 \). Comparing the results,
presented in fig.5 and fig.6, it can be seen that, transverse normal stress $\sigma_{33}$ for the concerned thin plate is smaller than $\sigma_{11}$ by three orders of its magnitude.

![Figure 6](image.png)

**Figure 6.** Relationship $\sigma(z)$ for steel plate in case of $\nu_{13} = \nu_{23} = 0$

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