ACYCLIC ORIENTATIONS AND SPANNING TREES.

BENJAMIN IRIARTE

Abstract. We introduce polytopal cell complexes associated with partial acyclic orientations of a simple graph, which generalize acyclic orientations. Using the theory of cellular resolutions, two of these polytopal cell complexes are observed to minimally resolve certain special combinatorial polynomial ideals related to acyclic orientations. These ideals are explicitly found to be Alexander dual, which relative to comparable results in the literature, generalizes in a cleaner and more illuminating way the well-known duality between permutohedron and tree ideals. The combinatorics underlying these results naturally leads to a canonical way to represent rooted spanning forests of a labelled simple graph as non-crossing trees, and these representations are observed to carry a plethora of information about generalized tree ideals and acyclic orientations of a graph, and about non-crossing partitions of a totally ordered set. A small sample of the enumerative and structural consequences of collecting and organizing this information are studied in detail. Applications of this combinatorial miscellanea are then introduced and explored, namely: Stochastic processes on state space equal to the set of all acyclic orientations of a simple graph, including irreducible Markov chains, which exhibit stationary distributions ranging from linear extensions-based to uniform; a surprising formula for the expected number of acyclic orientations of a random graph; and a purely algebraic presentation of the main problem in bootstrap percolation, likely making it tractable to explore the set of all percolating sets of a graph with a computer.

1. Introduction.

This article is a sequel to Iriarte G. (2014), focusing instead on the structural and enumerative properties of acyclic orientations. We introduce a number of novel perspectives, results and resources for the study and discovery of fundamental properties of acyclic orientations, and their generalization, partial acyclic orientations, of a simple graph; these include polytopal cell complexes and polynomial ideals [Miller and Sturmfels (2005)], graphical zonotopes [Postnikov (2009), Beck and Robins (2007)], and Markov chains [Lovász (1993), Aldous and Fill (2002)], among others. We adopt an original approach to the well-known connection between labelled trees, parking functions, non-crossing partitions, and graph orientations. This is the viewpoint of non-crossing trees, not properly treated or even reported in the...
literature, and which we exploit to obtain new results about these objects. Non-crossing trees are, in part, motivated by the techniques of Fink and Iriarte G. (2010), but owe their existence to the (subtleties of the) Alexander duality (Miller (1998)) between two special polynomials ideals defined during Section 3 of the present writing. The perspectives presented here complement those of previous key studies, including but not exhausting, those found in Chebikin and Pylyavskyy (2005), Postnikov and Shapiro (2004), Dochtermann and Sanyal (2012), Manjunath et al. (2012), Mohammadi and Shokrieh (2013), Stanley (1997) and Stanley (1998), and the references therein. The present work is, in fact, an evident seed for future research more than a conclusive exposition of the topic, and the number of (sometimes quite provocative) open problems and directions for future research should gradually become clear.

In principle, an inconvenient aspect of acyclic orientations of a simple graph is their apparent but, nevertheless, artificial relation to bijective labellings of the vertex set with a totally ordered set. This viewpoint was exploited during the author’s previous article on this subject. Conceivably, adopting a perspective different to that of bijective labellings seems equally fated to illuminate the study of acyclic orientations of a simple graph, and this is what we pursue in this writing. One example of how we apply ideas developed during this sequel is the construction of a random walk on a certain simple connected regular graph with vertex set equal to the set of all acyclic orientations of any fixed simple graph, which therefore exhibits a unique uniform stationary distribution. The importance and applicability of such constructions is evidently exemplified in Broder (1989), Aldous (1990) and Kelner and Madry (2009), and formalized in Lovász and Winkler (1995b) and Lovász and Winkler (1995). Many other fine works have made use of similar ideas to solve different combinatorial algorithmic problems.

Understanding acyclic orientations of a simple graph from their grounds usually entails making precise connections of their theory with the theory of spanning trees, much better understood; this is also the case in the present article. The particular connection between these sets of combinatorial objects that we choose to follow, developed here for the first time, is far from obvious and will be presented later in Section 4, where it sprouts naturally from the constructions of Sections 2-3.

Let us describe in fair detail the contents of the different sections of the article.

In Section 2 we introduce, again for the first time in the literature, an elegant inequality description of a well-known polytope related to the acyclic orientations of a fixed simple (connected) graph on vertex set \([n]\), \(n \in \mathbb{P}\); it can be found in Subsection 2.1. The above description has the form predicted in Postnikov (2009) for the generalized permutohedra. This polytope of partial acyclic orientations has a Minkowski sum decomposition whose summands appear also as summands in Postnikov’s expression of the graph associahedron as a sum of simplices. A first step along this road from the polytope of partial acyclic orientations to the graph associahedron of graph tubings (Devadoss (2009)) leads us to consider, in the case of connected graphs, the Minkowski sum of the former polytope with an \(p\) dimensional simplex. The construction of Cayley’s trick applied to this case serves us to discover one more polytopal cell complex associated to the graph, a complex pivotal in the study of certain “artinianizations” of the ideals defined in Section 3 and (therefore) instrumental in the search for minimal free resolutions of these
ideals \cite{BayerSturmfels1998}, and whose combinatorial dual is precisely the totally non-negative part of the graphical arrangement \cite{Stanley2004}.

In Section 3, this clean geometrical duality of polytopal cell complexes manifests itself as an algebraic duality between two polynomial ideals associated to a fixed simple connected graph, defined therein; one of these ideals is motivated by the role of acyclic orientations in the graphical zonotope, and the other by the inequality description of the polytope of Subsection 2.1. The proof of this Alexander duality, found early in the section, contains the stepping stones for Section 4. We regard some of the results contained in this section as being “close siblings” to those found in \cite{DochtermannSanyal2012}, \cite{Manjunathetal2012} and \cite{MohammadiShokrieh2013}, yet our modus operandi aims to fix a necessarily problematic (at least for our purposes) aspect of these other works: The generalization of the duality between the (standard) permutohedron and tree ideals implicit in them is by no means self-evident nor truly discussed, and it does not follow from a clean geometrical duality generalizing the picture of the permutahedron and the barycentric subdivision of the simplex; as such, these other perspectives do not yield the algorithmic consequences that we need later on in Sections 4-5.

Section 4 introduces non-crossing trees of a simple graph, certain pictorial representations of labelled rooted trees reminiscent of \cite{FinkIriarte2010}. There is one non-crossing tree per each rooted spanning forest of the graph. In Subsection 4.1, we explain how each non-crossing tree naturally encodes a uniquely determined standard monomial of the generalized tree ideal, defined in Section 3, and (therefore) a uniquely determined orientation of the graph with no directed cycles. Among these orientations supported on non-crossing trees, we find the acyclic orientations of the graph, which spring up, again naturally, from non-crossing trees satisfying a certain efficiency condition. In Subsection 4.2, we adopt “the other” point of view on non-crossing trees, and observe how we then obtain chains of the non-crossing partitions lattice. These two points of view are combined to produce a coherent picture of the combinatorial objects involved in this work.

Section 5 contains applications of the ideas developed in Sections 2-4 to algorithmic/computational problems involving (mostly random) acyclic orientations. Subsection 5.1 presents five different stochastic processes on state spaces equal to the set of all acyclic orientations of a simple graph, and whose stationary distributions range from dependent on the number of linear extensions \cite{Iriarte2014} to uniform. In order of appearance, these are the Card-Shuffling Markov chain, the Edge-Label Reversal and the Sliding-$p$ stochastic processes, the Cover-Reversal random walk, and the Interval-Reversal random walk. The Card-Shuffling Markov chain had also been previously discovered as a hyperplane walk in \cite{AthanasiadisDiaconis2010}, and the Cover-Reversal random walk is grounded on the work of \cite{SavageZhang1998} and of Section 2 of the present writing. This subsection culminates with the presentation of the Interval-Reversal random walk, an irreducible reversible Markov chain with uniform stationary distribution on the acyclic orientations of a simple graph, never presented before in the literature, and motivated by a close inspection of Section 2 here. Subsection 5.2 presents a surprising expression for the expected number of acyclic orientations of an Erdős-Rényi random graph in terms of parking functions, a consequence of the study of non-crossing trees in Section 4. Subsection 5.3 introduces a commutative-algebraic
approach to determining all percolating sets in \textit{k-bootstrap percolation} on any simple graph \cite[e.g.,][]{Balogh2009}; this direction could yield good fruits if further explored in the future.

\textbf{Acknowledgements}: I would like to specially thank my advisor Richard P. Stanley and Jacob Fox, whose support and always useful advice made it possible to write this work.

2. Polytopal complexes for acyclic orientations.

2.1. A Classical Polytope.

\textbf{Definition 2.1.} Let $G = G(V, E)$ be a simple graph and let:

$$\overline{E} := \{(u, v) \in V^2 : \{u, v\} \in E\}.$$

An orientation $O$ of $G$ is a function $O : E \to E \cup \overline{E}$ such that for all $e = \{u, v\} \in E$, we have that $O(e) \in \{e, (u, v), (v, u)\}$. We will let $O_{\text{trivial}}$ be the identity map $E \to E$.

\textbf{Definition 2.2.} For a simple graph $G = G(V, E)$, a partition $\Sigma$ of the set $V$ is said to be a connected partition of $G$ if $G[\sigma]$ is connected for all $\sigma \in \Sigma$, where $G[\sigma]$ denotes the induced subgraph of $G$ on vertex set $\sigma$.

\textbf{Definition 2.3.} Let $G = G(V, E)$ be a simple graph and $\Sigma$ a connected partition of $G$. Then, the $\Sigma$-partition graph $G^\Sigma = G^\Sigma(\Sigma, E^\Sigma)$ of $G$ is the graph such that, for $\sigma, \rho \in \Sigma$ with $\sigma \neq \rho$, $(\sigma, \rho) \in E^\Sigma$ if and only if there exists $u \in \sigma$ and $v \in \rho$ with $\{u, v\} \in E$.

\textbf{Definition 2.4.} Let $G = G(V, E)$ be a simple graph. An orientation $O$ of $G$ is said to be a partial acyclic orientation (p.a.o.) of $G$ if $O$ can be obtained in the following way:

There exists a connected partition $\Sigma$ of $G$ and an acyclic orientation $O^\Sigma$ of the $\Sigma$-partition graph $G^\Sigma$ of $G$ such that, for all $e = \{u, v\} \in E$:

1. If $e \subseteq \sigma \in \Sigma$, then $O(e) = e$.

2. If $u \in \sigma$ and $v \in \rho$ for some $\sigma, \rho \in \Sigma$ with $\sigma \neq \rho$, and if $O^\Sigma(\{\sigma, \rho\}) = (\sigma, \rho)$, then $O(e) = (u, v)$.

We will also consider two functions, $\dim_G$ and $J_G$, associated to the set of p.a.o.’s of $G$. To define them, let $O$ be a p.a.o. of $G$ with associated connected partition $\Sigma$.

The first function, $\dim_G$, maps from the set of all p.a.o.’s of $G$ to $\mathbb{N}$, and is given as:

$$\dim_G(O) = |V| - |\Sigma|.$$

The second function, $J_G$, has also domain the p.a.o.’s of $G$, but it maps to the set of finite distributive lattices:

$$J_G(O) = J(O^\Sigma),$$

where $J(O^\Sigma)$ is the poset of order ideals of $O^\Sigma$.

\textbf{Remark 2.5.} For a p.a.o. $O$ of $G$ as in Definition 2.4, we will often identify $O$ with its induced partially ordered set $(V, \leq_O)$, where for all $u, v \in V$ we have that $u <_O v$ if and only if $u \in \sigma$ and $v \in \rho$ for some $\sigma, \rho \in \Sigma$ with $\sigma \neq \rho$, and there exist $\sigma_0, \sigma_1, \ldots, \sigma_k \in \Sigma$ with $\sigma_0 = \sigma$ and $\sigma_k = \rho$ such that $(\sigma_{i-1}, \sigma_i) \in O^\Sigma[E^\Sigma]$ for all $i \in [k]$. 
Lemma 2.6. In Definition 2.4, if \( O \) is a p.a.o. of \( G \), then \( \dim_G(O) \) is equal to \( |V| - 1(J_G(O)) \), where \( 1(\cdot) \) denotes the length function for graded posets.

**Proof.** Let \( \Sigma \) be the connected partition of \( G \) associated to \( O \). The result follows since then \( 1(J(O^Z)) = |\Sigma| \).

Lemma 2.7. In Definition 2.4, consider a p.a.o. \( O \) of \( G \), and for \( I \in J_G(O) \), let \( I' = \bigcup_{\sigma \in I} \sigma \). If we let \( P \) be the poset of all \( I' \) with \( I \in J_G(O) \), ordered by inclusion of sets, then \( P \cong J_G(O) \).

**Proof.** This is straightforward, since for \( I_1, I_2 \in J_G(O) \), both \( I_1 \cap I_2 \in J_G(O) \) and \( I_1 \cup I_2 \in J_G(O) \).

Remark 2.8. Naturally, in Lemma 2.7 and in subsequent writing, for \( O \) a p.a.o. of \( G \), \( J_G(O) \) denotes the ground set of \( J_G(O) \).

Remark 2.9. In fact, following Lemma 2.7, in Definition 2.4 we will regard \( J_G(\cdot) \) as a collection of subsets of \( V \) ordered by inclusion.

Lemma 2.10. In Definition 2.4 and Remark 2.8, the map \( J_G \) is an injective map.

**Proof.** Let \( O_1 \) and \( O_2 \) be p.a.o.’s of \( G \) such that \( J_G(O_1) = J_G(O_2) \). Hence, \( J_G(O_1) = J_G(O_2) \). Considering a maximal chain \( \emptyset = \sigma_0 \subseteq \cdots \subseteq \sigma_k = V \) of this poset, we observe that \( \Sigma = \{ \sigma_i \setminus \sigma_{i-1} \}_{i \in [k]} \) is the connected partition of \( G \) associated to both \( O_1 \) and \( O_2 \). The poset of join-irreducibles of \( J_G(O_1) = J_G(O_2) \) determines a unique acyclic orientation \( O^Z \) of the \( \Sigma \)-partition graph \( G^Z \), and so both \( O_1 \) and \( O_2 \) are obtained from \( O^Z \) as in Definition 2.4. Clearly then \( O_1 = O_2 \). □

Definition 2.11. Consider a simple graph \( G = (V, E) \). We will define an abstract cell complex \( Z_G = (Z_G, \leq, \dim_z) \), with underlying set of faces \( Z_G \) ordered by \( \leq_z \), and with dimension function \( \dim_z \), in the following manner:

1. \( Z_G \) is the set of p.a.o.’s of \( G \).
2. For \( O_1, O_2 \) p.a.o.’s of \( G \), \( O_1 \leq_z O_2 \) if and only if \( J_G(O_2) \subseteq J_G(O_1) \).
3. For \( O \) a p.a.o. of \( G \), \( \dim_z(O) = \dim_G(O) \).

Example 2.12. In Figure 1 we present two examples of p.a.o.’s, \( O_1 \) and \( O_2 \), of a graph \( G \) on vertex set \([15] = \{ 1, 2, \ldots, 15 \} \), such that \( O_2 \leq_z O_1 \). Figure 1A shows a connected simple graph \( G = G([15], E) \). Figure 1B presents a particular p.a.o. \( O_1 \) of \( G \), with associated connected partition \( \Sigma_1 \) (each of its blocks represented by closed blue regions), and Figure 1C the \( \Sigma_1 \)-partition graph \( G^{\Sigma_1} \) and its acyclic orientation \( O^{\Sigma_1} \). Similarly, Figure 1D shows another p.a.o. \( O_2 \) of \( G \), with associated connected partition \( \Sigma_2 \) (blocks represented by closed blue regions), and Figure 1E the \( \Sigma_2 \)-partition graph \( G^{\Sigma_2} \) and its acyclic orientation \( O^{\Sigma_2} \). Table 1F then offers complete calculations of \( J_G(O_1), J_G(O_2), \dim_G(O_1) = \dim_z(O_1) \) and \( \dim_G(O_2) = \dim_z(O_2) \). Note that since \( J_G(O_1) \not\subseteq J_G(O_2) \), then \( O_2 <_z O_1 \).

Lemma 2.13. Let \( G = G([n], E) \) be a simple graph, and let \( a, b \in \mathbb{R} \) and \( c \in \mathbb{R}_+ \). Consider the function \( F : 2^{[n]} \to \mathbb{R} \) given by \( F(\sigma) = a + b|\sigma| + c|E(G(\sigma))|, \sigma \in [n] \). Then, for all \( \sigma, \rho \subseteq [n] \):

\[
F(\sigma) + F(\rho) \leq F(\sigma \cap \rho) + F(\sigma \cup \rho).
\]

Equality holds if and only if \( \sigma \cap \rho \) and \( \rho \setminus \sigma \) are completely non-adjacent sets in \( G \), i.e. if and only if \( \{i, j\} \in E : i \in \sigma \setminus \rho \) and \( j \in \rho \setminus \sigma \) = \( \emptyset \).
In standard combinatorial theory terminology, in Lemma 2.13, we say that the function $F$ is lower semi-modular \[ \text{Crapo and Rota (1970)}. \]

**Theorem 2.15.** Let $G = G([n], E)$ be a simple graph with abstract cell complex $\mathcal{Z}_G$, as in Definition 2.11. Then, the face complex of the polytope:

\[
Z_G := \left\{ x \in \mathbb{R}^{[n]} : \sum_{i \in [n]} x_i = n + |E| \text{ and } \sum_{i \in \sigma} x_i \geq |\sigma| + |E(G[\sigma])| \text{ for all } \sigma \subseteq [n] \right\},
\]

is a polytopal complex realization of $\mathcal{Z}_G$.

**Proof.** Per Lemma 2.10 and for the sake of clarity, in this proof we will think of elements of $\mathcal{Z}_G$ as their images under $J_G$. 

| p.a.o. | $J_G$ | $\dim Z_G$ or $\dim z$ |
|--------|--------|--------------------------|
| $O_1$  | $\emptyset, \{2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ | 10 |
| $O_2$  | $\emptyset, \{2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, \{6\}$ | 9 |

**Figure 1.** Examples of p.a.o.’s and the order relation $\leq_z$ of Definition 2.11.
To begin, an easy verification shows that the point $\frac{1}{2} \cdot d_e + 1$ lives inside $\mathcal{Z}_G$, so $\mathcal{Z}_G$ is non-empty. Also, $\mathcal{Z}_G$ is bounded.

Now, consider a (relatively open) non-empty face $F$ of $\mathcal{Z}_G$, and let $C_F$ be the collection of all $\sigma \subseteq [n]$ such that $\sum_{i \in \sigma} y_i = |\sigma| + |E(G[\sigma])|$ if $y \in F$.

A first key step in the proof will be to establish that $C_F \subseteq \mathcal{Z}_G$. We will do this in a series of sub-steps. Let $C_F$ be the poset on ground set $C_F$ ordered by inclusion.

**Claim i** $C_F$ is closed under intersections and unions, so $C_F$ is a distributive lattice.

Let $y \in F$. By definition, both $\emptyset$ and $[n]$ belong to $C_F$. Let us now take $\sigma, \rho \in C_F$ and let us assume that $\sigma \not\subseteq \rho, \rho \not\subseteq \sigma$. Then, $\sum_{i \in \sigma} y_i = |\sigma| + |E(G[\sigma])|$ and $\sum_{j \in \rho} y_j = |\rho| + |E(G[\rho])|$, so:

$$|\sigma \cup \rho| + |E(G[\sigma \cup \rho])| \leq \left( \sum_{i \in \sigma} y_i \right) + \left( \sum_{j \in \rho} y_j \right) - \sum_{k \in \sigma \cap \rho} y_k$$

$$\leq |\sigma \cup \rho| + |E(G[\sigma])| + |E(G[\rho])|$$

In particular, $|E(G[\sigma \cap \rho])| + |E(G[\sigma \cup \rho])| \leq |E(G[\sigma])| + |E(G[\rho])|$. However, per Lemma 2.13:

$$|E(G[\sigma \cap \rho])| + |E(G[\sigma \cup \rho])| = |E(G[\sigma])| + |E(G[\rho])|.$$  

This implies that $\sigma \cap \rho \in C_F$ and $\sigma \cup \rho \in C_F$.

**Claim ii** Let $\emptyset = \sigma_0 \subseteq \sigma_1 \subseteq \cdots \subseteq \sigma_k = [n]$ be a maximal chain in $C_F$. Then, $G[\sigma_i \setminus \sigma_{i-1}]$ is connected for all $i \in [k]$.

Let $i \in [k]$ and suppose that $G[\sigma_i \setminus \sigma_{i-1}]$ is disconnected. Let $\rho_1$ and $\rho_2$ be two completely non-adjacent sets of $G[\sigma_i \setminus \sigma_{i-1}]$ such that $\rho_1 \cup \rho_2 = \sigma_i \setminus \sigma_{i-1}$. Then Lemma 2.13 shows:

$$|E(G[\sigma_{i-1} \cup \rho_1])| + |E(G[\sigma_{i-1} \cup \rho_2])| = |E(G[\sigma_i])| + |E(G[\sigma_{i-1}])|.$$  

Also, for $y \in F$:

$$|\sigma_{i-1}| + |\sigma_i| + |E(G[\sigma_{i-1} \cup \rho_1])| + |E(G[\sigma_{i-1} \cup \rho_2])|$$

$$= |\sigma_{i-1} \cup \rho_1| + |E(G[\sigma_{i-1} \cup \rho_1])| + |\sigma_{i-1} \cup \rho_2| + |E(G[\sigma_{i-1} \cup \rho_2])|$$

$$\leq \sum_{j \in \sigma_{i-1} \cup \rho_1} y_j + \sum_{k \in \sigma_{i-1} \cup \rho_2} y_k = \sum_{j \in \sigma_{i-1}} y_j + \sum_{k \in \sigma_i} y_k = |\sigma_{i-1}| + |\sigma_i| + |E(G[\sigma_{i-1}])| + |E(G[\sigma_i])|.$$  

This implies that $\sigma_{i-1} \cup \rho_1 \in C_F$ and $\sigma_{i-1} \cup \rho_2 \in C_F$, contradicting the choice of a maximal chain in $C_F$.

**Claim iii** For a chain as in **Claim ii**, suppose that there exist $l, m \in \sigma_i \setminus \sigma_{i-1}$ with $m \neq l, i \in [k]$. If $\sigma \in C_F$, then either $\{m, l\} \cap \sigma = \emptyset$ or $\{m, l\} \subseteq \sigma$.

Suppose on the contrary that for some $\sigma \in C_F$, $m \in \sigma$ but $l \notin \sigma$. Then, $(\sigma \cap \sigma_i) \cup \sigma_{i-1} \in C_F$ per **Claim i** and $\sigma_{i-1} \subseteq (\sigma \cap \sigma_i) \cup \sigma_{i-1} \subseteq \sigma$, which contradicts the choice of maximal chain.

**Claim iv** Per **Claim iii**, every $\sigma \in C_F$ is a disjoint union of elements of the connected partition $\Sigma := \{\sigma_i \setminus \sigma_{i-1}\}_{i \in [k]}$ of $G$. Consider the acyclic orientation $O^\Sigma$ of $G^\Sigma = G^\Sigma(\Sigma, E^\Sigma)$ such that for $e = \{\sigma_i \setminus \sigma_{i-1}, \sigma_j \setminus \sigma_{j-1}\} \in E^\Sigma$ and $i < j,$
Claim from Claim iv let $O$ be the p.a.o. of $G$ obtained from $O^\Sigma$. If $\sigma \in C_F$, then $\sigma \in J_G(O)$.

This is essentially a corollary to the proof of Claim iv Consider a maximal chain of $C_F$ that contains $\sigma$. Then, clearly $\sigma \in J_G(O)$.

For the first part, it remains to prove that if $\sigma \in J_G(O)$, then $\sigma \in C_F$. This is easy to establish by considering a point $y \in F$. For $\rho \in \Sigma$, note that $\sum_{i \in \rho} y_i = |\rho| + |E(G[\rho])| + |O[E] \cap ([n] \setminus \rho \times \rho)|$. But then:

$$\sum_{i \in \sigma} y_i = \sum_{\rho \in \Sigma, \rho \subseteq \sigma} |\rho| + |E(G[\rho])| + |O[E] \cap ([n] \setminus \rho \times \rho)|$$

$$= |\sigma| + |E(G[\sigma])|,$$

since $\sigma \in J_G(O)$. Hence, $\sigma \in C_F$ and $C_F = J_G(O) \in \mathcal{Z}_G$.

For the second part, let us take a $J_G(O) \in \mathcal{Z}_G$ for some p.a.o. $O$ of $G$, and we want to prove that $J_G(O) = C_F$ for some (relatively-open) non-empty face $F$ of $\mathcal{Z}_G$. The first part gives us a clear hint of how to proceed here. Let us consider the point $y \in \mathbb{R}^n$ given by $y_i = |O[E] \cap ([n] \setminus \{i\} \times \{i\})| + \frac{1}{2} |O[E] \cap \{i \in E : i \in e\}| + 1$ for all $i \in [n]$. Since $O$ is a p.a.o. of $G$, then for $\sigma \in J_G(O)$, we have that $\sum_{i \in \sigma} y_i = |\sigma| + |E(G[\sigma])|$, as $O[E] \cap ([n] \setminus [\sigma \times \sigma] \cup \{i, j : i \in [n], j \in [n] \setminus [\sigma]\}) = \emptyset$. On the contrary, if $\sigma \notin J_G(O)$, the later set is non-empty and $\sum_{i \in \sigma} y_i > |\sigma| + |E(G[\sigma])|$. Therefore, $J_G(O) = C_F$ for some face $F$ of $\mathcal{Z}_G$.

We have now established how $\mathcal{Z}_G$ corresponds to the set of (relatively-open) non-empty faces of $\mathcal{Z}_G$. Naturally, $\leq_z$ corresponds to face containment and $\dim_z$ to affine dimension, and the correctness of these two is an immediate consequence of our correspondence and of basic properties of the inequality description of a polytope.

\begin{corollary}
Let $G = G([n], E)$ be a simple graph with graphical zonotope $\mathcal{Z}_G^{\text{central}}$ and degree vector $d_G$, where (see Definition 2.19 for notation):

$$\mathcal{Z}_G^{\text{central}} := \sum_{\{i,j\} \in E} [e_i - e_j, e_j - e_i].$$

Then:

$$\mathcal{Z}_G = \frac{1}{2} \cdot \mathcal{Z}_G^{\text{central}} + \frac{1}{2} \cdot d_G + 1. \tag{2.2}$$

\end{corollary}

\begin{proof}
As it is known from Iriarte G. (2014), the vertices of $\mathcal{Z}_G^{\text{central}}$ are given by all points of the form $x_O = (\text{indeg}_{3(i,0)} - \text{outdeg}_{3(i,0)})_{i \in [n]}$, where $O$ is an acyclic orientation of $G$. Hence, translating $\frac{1}{2} \cdot \mathcal{Z}_G^{\text{central}}$ by $\frac{1}{2} \cdot d_G + 1$, we obtain that the vertices of the new polytope are given by all vectors of the form $y_O = (\text{indeg}_{3(i,0)} + 1)_{i \in [n]}$, where $O$ is an acyclic orientation of $G$, but these are precisely the vertices of $\mathcal{Z}_G$.
\end{proof}
Definition 2.17. Let $G = G([n], E)$ be a simple graph with graphical zonotope $Z_G$. From Corollary 2.16 we will call the polytope $Z_G$ the clean graphical zonotope of $G$.

2.2. One More Degree of Freedom.

Definition 2.18. Let $G = G([n], E)$ be a connected graph. Define $\mathcal{Y}_G = (\mathcal{Y}_G, \preceq_Y, \dim_Y)$ to be the abstract cell complex with underlying set of cells $\mathcal{Y}_G$, order relation $\preceq_Y$, and dimension map $\dim_Y$, given by:

1. $\mathcal{Y}_G = \mathcal{Y}_G \cup (2^{[n]} \setminus \{\emptyset, [n]\})$, where:
   \[
   \mathcal{Y}_G := \{(\sigma, O) : \emptyset \neq \sigma \subseteq [n], \text{ and } O \text{ is a p.a.o. of } G[\sigma]\}.
   \]

2. For $A, B \in \mathcal{Y}_G$ with $A \neq B$, we have that $A \preceq_Y B$ if and only if one of the following holds:
   a. If $A, B \in 2^{[n]} \setminus \{\emptyset, [n]\}$, then $A \subseteq B$.
   b. If $A \in 2^{[n]} \setminus \{\emptyset, [n]\}$ and $B = (\sigma, O) \in \mathcal{Y}_G$, then $A \subseteq [n] \setminus \sigma$.
   c. If $A = (\sigma_0, O_0), B = (\sigma_1, O_1) \in \mathcal{Y}_G$, then $J_{G[\sigma_1]}(O_1) \subseteq J_{G[\sigma_0]}(O_0)$.

3. For $A \in \mathcal{Y}_G$:
   a. If $A \in 2^{[n]} \setminus \{\emptyset, [n]\}$, then $\dim_Y(A) = |A| - 1$.
   b. If $A = (\sigma, O) \in \mathcal{Y}_G$, then $\dim_Y(A) = |[n] \setminus \sigma| + \dim_Y(\sigma)(O)$.

Definition 2.19. Let $S$ and $T$ be non-empty subsets of $\mathbb{R}^n$. The join of $S$ and $T$ is the set:

\[ [S, T] := \{x \in \mathbb{R}^n : x = \alpha s + (1 - \alpha)t \text{ for some } \alpha \in [0, 1], s \in S \text{ and } t \in T \}. \]

The strict join of $S$ and $T$ is the set:

\[ (S, T) := \{x \in \mathbb{R}^n : x = \alpha s + (1 - \alpha)t \text{ for some } \alpha \in (0, 1), s \in S \text{ and } t \in T \}. \]

Proposition 2.20. Let $P$ and $Q$ be $(n-1)$-dimensional polytopes in $\mathbb{R}^n$ such that $\text{aff}(P)$ and $\text{aff}(Q)$ are parallel and disjoint affine hyperplanes. Consider an open segment $(x, z)$ with $x \in P$, $z \in Q$, and let $y \in [P, Q]$. Then, the following are true:

1. $y \in \text{int}\langle P, Q \rangle$ if and only if there exist $p^* \in \text{relint}\langle P \rangle$ and $q^* \in Q$ such that $y \in (p^*, q^*)$.
2. $(x, z) \subseteq \partial\langle P, Q \rangle$ if and only if there exist $p^* \in \text{relint}\langle P \rangle$ and $q^* \in \text{relint}(Q)$ such that $y \in (p^*, q^*)$.
3. $\pi_{\text{aff}(P)} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the projection operator onto the affine hyperplane containing $P$. If $\pi_{\text{aff}(P)}((x, z)) \cap \text{relint}\langle P \rangle \cap \text{relint}\langle \pi_{\text{aff}(P)}[Q] \rangle \neq \emptyset$, then $(x, z) \subseteq \text{int}\langle P, Q \rangle$.

Proof. We will obtain these results in order.
\((p_1, q) \cap B_y \in \text{int}(\langle P, Q \rangle)\). Since \(y_2 := y + (y-y_1) \in B_y \subseteq \text{int}(\langle P, Q \rangle)\), there exist \(p_2 \in P\) and \(q_2 \in Q\) such that \(y_2 = (p_2, q_2) \cap B_y\). But then, there exist \(p^* \in (p_1, p_2) \subseteq \text{relint}(P)\) and \(q^* \in (q, q_2) \subseteq Q\) such that \(y = (p^*, q^*) \cap B_y\), as we wanted. If \(g^* \in \partial \langle Q \rangle\), we can now repeat an analogous construction starting from \(q^*\) and \(p^*\) to find \(p^{**} \in \text{relint}(P)\) and \(q^{**} \in \text{relint}(Q)\) such that \(y \in (p^{**}, q^{**})\).

**ii** This is a consequence of \[\] and not easy to prove without it. We prove the second statement, which is equivalent to the first. For the “if” direction, suppose that for some \(p \in \text{relint}(P)\) and \(\epsilon > 0\), \(z + \epsilon (x-p) \in Q\). Take some \(y \in (x, z)\) and consider the line containing both \(z + \epsilon (x-p)\) and \(y\). For a sufficiently small \(\epsilon\), this line intersects \(\text{aff}(P)\) in some \(p_1 \in \text{relint}(P)\). But then, for a small open ball \(B_{p_1} \subseteq \text{relint}(P)\) centered at \(p_1\) and with \(\text{aff}(B_{p_1}) = \text{aff}(P)\), the open set \((B_{p_1}, z)\) contains \(y\) and lies completely inside \(\text{int}(\langle P, Q \rangle)\), so \(y \in \text{int}(\langle P, Q \rangle)\).

For the “only if” direction, suppose that \((x, z) \subseteq \text{int}(\langle P, Q \rangle)\) and take \(y \in (x, z)\). If \(x \in \text{relint}(P)\), then we are done since \(Q\) is also \((n-1)\)-dimensional. If \(x \in \partial \langle P \rangle\), from \[\] take \(p \in \text{relint}(P)\), \(p \neq x\), and \(q \in Q\) with \(y \in (p, q)\). But then, \(z + \epsilon (y-p) = q \in Q\) for some \(\epsilon > 0\).

**iii** Take \(p \in \pi_{\text{aff}P}([x, z]) \cap \text{relint}(P) \cap \text{relint}(\pi_{\text{aff}P}[Q])\) and let \(p \neq 0\) be a normal to \(\text{aff}(P)\). Then, for some \(y \in (x, z)\) and real number \(\alpha \neq 0\), \(y \in (p, p+\alpha p)\) and \(p+\alpha p \in \text{relint}(Q)\), so \[\] shows that \(y \in \text{int}(\langle P, Q \rangle)\).

Clearly then \((x, z) \subseteq \text{int}(\langle P, Q \rangle)\).

**Definition 2.21.** Let \(G = G([n], E)\) be a simple graph, and let \(O\) be a p.a.o. of \(G\) with connected partition \(\Sigma\) and acyclic orientation \(O^\Sigma\) of \(G^\Sigma\). Let us write \(\Sigma^{\min}\) for the set of elements of \(\Sigma\) that are minimal in \((\Sigma, \leq_{O^\Sigma})\), and for \(i \in [n]\) with \(i \in \rho \in \Sigma\), let:

\[
I^O_G(i, O^\Sigma) = \{\sigma \in \Sigma : \sigma \leq_{O^\Sigma} \rho\}, \quad \text{and} \quad I^O_G(i, O) = \{j \in [n] : j \in \sigma \in \Sigma \text{ and } \sigma \geq O^\Sigma \rho\}.
\]

With this notation, we now define certain functions associated to \(O\) and \(G\), called height and depth:

\[
\text{height}^O_G : [n] \to \mathbb{Q}, \\
\text{depth}^O_G : [n] \to \mathbb{Q}, \\
\text{height}^O_G(i) = \frac{1}{n - |\Sigma^{\min} \cap I^O_G(i, O^\Sigma)|}, \\
\text{depth}^O_G(i) = \sum_{j \in I^O_G(i, O)} \text{height}^O_G(j).
\]

**Example 2.22.** Figure \[\text{Example 2.22}\] exemplifies Definition \[\text{Definition 2.21}\] on a particular graph \(G\) on vertex set \([14] = \{1, 2, \ldots, 14\}\), with given p.a.o. \(O\). Since both \(\text{height}^O_G\) and \(\text{depth}^O_G\) are constant within each element/block of the connected partition \(\Sigma = \{\sigma_1 = \{7\}, \sigma_2 = \{1, 2\}, \sigma_3 = \{6, 10, 14\}, \ldots, \sigma_6 = \{3, 4, 8\}\}\) associated to \(O\), then we present only that common value for each block in the figure.

**Proposition 2.23.** In Definition \[\text{Definition 2.21}\] let \(\sigma \in J_G(O)\) and let \(\rho \subseteq [n]\) intersect every element of \(\Sigma^{\min}\) in exactly one point and contain only minimal elements of \(O\). Then:

\[
1 \geq \sum_{i \in \rho \cap \sigma} \text{depth}^O_G(i) \geq \frac{|\sigma|}{n}.
\]
Moreover, if $G$ is connected, then $\sum_{i \in \rho \cap \sigma} \text{depth}_{\sigma}^G(i) > \frac{|\sigma|}{\pi}$ if and only if $\sigma \neq [n]$, and whenever this holds, $\sum_{i \in \rho \cap \sigma} \text{depth}_{\sigma}^G(i) - \frac{|\sigma|}{\pi} > \frac{1}{\pi^2}$.

**Remark 2.24.** Figure 2A shows one such choice of a set $\rho$ in Proposition 2.23 that works for Example 2.22 (in red).

**Proof.** The verification is actually a simple double-counting argument using the fact that $\sigma$ is an order ideal, so we omit it. When $G$ is connected, if $\sigma \neq [n]$, then there must exist $i \in [n]\setminus \sigma$ that is strictly greater in $O$ than some element of $\sigma$ (and hence strictly greater than some element of $\rho$), again since $\sigma$ is an order ideal. Clearly, we must have $\text{height}_{\sigma}^G(i) > \frac{1}{\pi^2}$.

**Figure 2.** Visual aids/guides to the proofs of Proposition 2.23 (A) and Proposition 2.20 (B). A also offers an example for Definition 2.21.

**Theorem 2.25.** Let $G = G([n], E)$ be a connected simple graph with abstract cell complex $\mathcal{Y}_G$ as in Definition 2.18. For $N > 0, N \neq n + |E|$, consider the $(n - 1)$-dimensional simplex $N\Delta = \text{conv}(N e_1, N e_2, \ldots, N e_n)$ in $\mathbb{R}^n$. If we let $\mathcal{Y}_G$ be the polytopal complex obtained from the join $[Z_G, N\Delta]$ after removing the (open) $n$-dimensional cell and the (relatively open) $(n - 1)$-dimensional cell corresponding to $N\Delta$, then $\mathcal{Y}_G$ is a polytopal complex realization of $\mathcal{Y}_G$.

**Proof.** Let the faces of $\mathcal{Y}_G$ obtained from $2^{[n]} \setminus ([n], O)$ correspond to the faces of $\partial \langle N\Delta \rangle$ in the natural way. Also, let the faces of $\mathcal{Y}_G$ of the form $([n], O)$ correspond to the faces of $Z_G$ as in Theorem 2.15. The result is clearly true for the restriction to this two sub-complexes, so we will concentrate our efforts on the remaining cases.

First, for the sake of having a lighter notation during the proof, we will let $\hat{\rho} = [n] \setminus \rho$ for any set $\rho \subseteq [n]$.
A (relatively open) cell of \( \mathcal{Y}_G \setminus (\mathcal{Z}_G \cup \partial \langle N\Delta \rangle) \) can only be obtained as the strict join of a cell of \( \partial \langle Z_G \rangle \) and a cell of \( \partial \langle N\Delta \rangle \), so let us adopt some conventions to refer to this objects.

**Convention 2.26.** During the course of the proof, we will let \( S \) (or \( S_0 \)) denote a generic non-empty relatively open cell of \( N\Delta \) obtained from \( \rho \subseteq [n] \) (resp. \( \rho_0 \)), and \( F \) (or \( F_0 \)) a generic relatively open cell of \( Z_G \) with p.a.o. \( O \) of \( G \), associated connected partition \( \Sigma \) of \( G \), and acyclic orientation \( O^\Sigma \) of \( G^\Sigma \) yielding \( O \) (resp. \( O_0, \Sigma_0, O_0^{\Sigma_0} \)).

We argue that we will be done if we can prove the following claim:

**Claim i a)** \((F, S)\) is a cell of \( \mathcal{Y}_G \) if and only if \( b) \rho \neq [n] \) and \( \rho \) is a non-empty union of elements from the set \( \{G \in \Sigma : \sigma \text{ is maximal in } \langle \Sigma, \leq_{\sigma} \rangle \} \). This equivalence is established, then we will let \((F, S)\) correspond to the pair \((\hat{\rho}, O_\rho) \in \mathcal{Z}_G \), where \( O_\rho \) denotes the restriction of \( O \) to \( E(G[\hat{\rho}]) \).

Indeed, assume that Claim i holds. Then, under the stated correspondence of ground sets of cells, all elements of \( \mathcal{Z}_G \) are uniquely accounted for as cells of \( \mathcal{Y}_G \). This is true for \( Z_G \) clearly, and for the remaining cases since for any choice of \( \sigma_1 \subseteq [n], \sigma_1 \neq \emptyset, \) and of p.a.o. \( O_1 \) of \( G[\sigma_1] \), we can always extend uniquely \( O_1 \) to a p.a.o. of \( G \) in which all the elements of \( \sigma_1 \) are maximal.

Secondly, we verify that \( \leq_\rho \) corresponds to face containment in \( \mathcal{Y}_G \). Suppose that \((F_0, S_0) \) and \((F, S)\) are cells of \( \mathcal{Y}_G \). Then, \((F_0, S_0) \subseteq (F, S)\) if and only if \( F_0 \subseteq F \) and \( S_0 \subseteq S \), and if and only if \( J(G[O]) \subseteq J(G[O_0]) \) and \( \rho_0 \subseteq \rho \). Now, assuming Claim i the last statement is true if and only if \( J(G^{[\hat{\rho}]}(O|_\rho) \subseteq J(G^{[\hat{\rho}]}(O_0|_{\rho_0}) \subseteq J(G^{[\hat{\rho}]}(O_0|_{\rho_0}) \subseteq J(G^{[\hat{\rho}]}(O_0|_{\rho_0}) \subseteq J(G^{[\hat{\rho}]}(O_0) \subseteq J(G^{[\hat{\rho}]}(O_0) \subseteq J(G^{[\hat{\rho}]}(O_0)

The analogous verification pertaining to faces in \( \mathcal{Z}_G \) of the form \([\{n\}, O]\), or corresponding to Definition 2.26 is now a straightforward application of the same ideas, so we omit it here.

The correctness of \( \dim_\rho \) will be established in Claim i.3 so indeed if Claim i holds, the statement of the Theorem follows.

Let us now begin with our proof of Claim i which consists of three main steps.

**Claim i.1** Let \( F \) and \( S \) satisfy the conditions of Claim i.b). Then:

\((F, S) \subseteq \partial \langle [Z_G, N\Delta] \rangle .\)

Let \( x \in F \) and \( z \in S \). We must have that \( O \neq O_{\max} \) here. Now, since \( G \) is connected, there exists \( \sigma \in \Sigma \) that is minimal but not maximal in \( \langle \Sigma, \leq_{\sigma} \rangle \). Hence, \( \sigma \cap \rho = \emptyset \) and moreover, \( \sigma \in J(G[O]) \). But then, by the inequality description of \( Z_G \), for any \( p \in \text{relint}(\langle Z_G \rangle), \sum_{x \in \sigma} p_\sigma > |\sigma| + |E(G[\sigma])| = \sum_{x \in \sigma} x_\sigma \), and \( x - p \) must have a negative entry in \( \sigma \).

Therefore, \( z + \varepsilon(x - p) \notin N\Delta \) for all \( \varepsilon > 0 \) and Proposition 2.20[i] shows that \((x, z) \subseteq \partial \langle [Z_G, N\Delta] \rangle\).

**Claim i.2** Let \( F_0, O_0, \Sigma_0, O_0^{\Sigma_0}, S_0, \rho_0 \) be as in Convention 2.26. Then, there exist \( F, O, \Sigma, O^\Sigma, S, \rho \) also as in Convention 2.26 such that \( \rho \) is a union of elements of the set \( \{\sigma \in \Sigma : \sigma \text{ is maximal in } \langle \Sigma, \leq_{\sigma} \rangle \} \) and \((F_0, S_0) \subseteq (F, S)\).
(See Figures 3A and 3B for a particular example of the objects and setting considered during this proof) Let:

\[ \Sigma_{0,\rho_0} := \{ \sigma \in \Sigma_0 : \text{If } \rho \in \Sigma_0 \text{ and } \rho \leq_{G}^{\rho} \sigma, \text{ then } \rho \cap \rho_0 = \emptyset \} \cdot \]

Then, define:

\[ \sigma_0 := \bigcup_{\sigma \in \Sigma_{0,\rho_0}} \sigma. \]

If \( G[\hat{\sigma}_i] = G[\sigma_1] + \cdots + G[\sigma_k] \) is the decomposition of \( G[\hat{\sigma}_0] \) into its connected components, we will let \( \Sigma = \Sigma_{0,\rho_0} \cup \{ \sigma_1, \ldots, \sigma_k \} \). We will use the acyclic orientation \( O^\Sigma \) of \( G^\Sigma \) obtained from the two conditions 1) \( O^\Sigma|_{\Sigma_{0,\rho_0}} = O^{\Sigma_0}|_{\Sigma_{0,\rho_0}} \) and 2) \( \sigma_1, \ldots, \sigma_k \) are maximal in \( (\Sigma, \leq_{O^\Sigma}) \). The p.a.o. \( O \) is now obtained from \( O^\Sigma \), and let \( F \) be associated to \( O \) and \( S \) be obtained from \( \sigma_0 = \sigma_1 \cup \cdots \cup \sigma_k \). We now prove that \( (F_0, S_0) \subseteq (F, S) \).

Since \( (F_0, S_0) \subseteq (F, S) \), it is enough to find \( x \in F_0 \) and \( z \in S_0 \) such that \( (x, z) \in (F, S) \), so this is precisely what we will do. To begin, we note that for \( i \in [k] \), the restriction \( O_i := O_0|_{o_i} \) is a p.a.o. of \( G_i := G[\sigma_i] \), so we will let \( \Sigma_i \) be the connected partition of \( G_i \) and \( O_i^{\Sigma_i} \) the acyclic orientation of \( G_i^{\Sigma_i} \) associated to \( O_i \); moreover, we note that \( \rho_0 \) intersects every element of \( \Sigma_i \) minimal in \( (\Sigma_i, \leq_{O_i^{\Sigma_i}}) \). Hence, let us select \( \varrho_0 \leq \rho_0 \) so that for every \( i \in [k] \), \( \varrho_0 \) intersects every element of \( \Sigma_i \) minimal in \( (\Sigma_i, \leq_{O_i^{\Sigma_i}}) \) in exactly one point and so that \( \varrho_0 \cap \sigma_i \) contains only minimal elements in \( O_i \). Now, take any \( x \in F_0 \) and let:

\[ z = N \frac{1}{k} \sum_{i \in [k]} \sum_{j \in \varrho_i, \cap \sigma_i} \text{depth}^{G_i}(j) \cdot e_j \in S_0. \]

We will make use of the technique of Proposition 2.20 to prove that \( (x, z) \in (F, S) \), so for that we need to consider a point in \( S \), which we select as:

\[ s = N \frac{1}{k} \sum_{i \in [k]} \frac{|\varrho_i|}{|\sigma_i|} \sum_{j \in \varrho_i} e_j \in S. \]

For \( i \in [k] \), if we consider a \( \varrho_i \in J_{G_i}(O_i) \) with \( \varrho_i \neq \sigma_i \), Proposition 2.23 gives us:

\[ \sum_{j \in \varrho_i} (z - s)_j = N \frac{1}{k} \left( \sum_{j \in \varrho_i \cap \varrho_0} \text{depth}^{G_i}(j) \right) - N \frac{1}{k} \cdot \frac{|\varrho_i|}{|\sigma_i|} \]

\[ > N \frac{1}{k} \left( \frac{|\varrho_i|}{|\sigma_i|} + \frac{1}{|\sigma_i|^2} - \frac{|\varrho_i|}{|\sigma_i|} \right) = N \frac{1}{k \cdot |\sigma_i|^2} \]

\[ > 0. \]

Hence, for a sufficiently small \( \varepsilon > 0 \), \( x + \varepsilon (z - s) \in F \), so for each \( y \in (x, z) \) we can find \( x' \in F \) and \( s' \in S \) such that \( y \in (x', z') \). That implies \( (F_0, S_0) \subseteq (F, S) \).

**Claim 1.3** Let both \( F, S \) and \( F_0, S_0 \) satisfy the conditions of Claim 1b. Then, \( (F, S) \cap (F_0, S_0) \neq \emptyset \) if and only if \( F = F_0 \) and \( S = S_0 \). Moreover, \( (F, S) \) is a face of \( \mathcal{Y}_G \) and \( \dim_{\mathcal{Y}}(\langle F, S \rangle) = |p| + \dim_{\mathcal{Y}}(O_s) \) (similarly for \( (F_0, S_0) \)).

Let \( \alpha \in (0, 1) \) and consider the polytope \( P_\alpha = \{ x \in \mathbb{R}^{|m|} : \sum_{i \in [m]} x_i = \alpha(n + |E|) + (1 - \alpha)N \} \cap [Z_G, N \Delta] \). Every \( x \in P_\alpha \) satisfies the inequalities
\[ \sum_{i \in [n]} x_i = (1 - \alpha)N + \alpha(n + |E|) \] and \[ \sum_{\sigma \in \sigma} x_i = \alpha(|\sigma| + |E(G[\sigma])|) \] for all \( \sigma, n, \sigma \neq \emptyset \). Per Claim 1.1 and Claim 1.2 the set \( (F, S) \cap P_{\alpha} \) can be characterized by the condition that it contains all the points \( x \in P_{\alpha} \) which, among those inequalities, satisfy the and only the following equalities:

\[
\sum_{i \in [n]} x_i = (1 - \alpha)N + \alpha(n + |E|) \quad \text{and} \quad \sum_{\sigma \in \sigma} x_i = \alpha(|\sigma| + |E(G[\sigma])|),
\]

for all \( \sigma \in J_{G[\rho]}(O_{[\rho]}), \sigma \neq \emptyset \).

This observation proves the first statement.

For the second statement, we assume without loss of generality that \( N > n + |E| \) and select generic coefficients \( \beta_\sigma \in \mathbb{R}_+ \) with \( \sigma \in J_{G[\rho]}(O_{[\rho]}, \sigma) \), such that:

\[
\sum_{\sigma \in J_{G[\rho]}(O_{[\rho]}, \sigma)} \beta_\sigma(|\sigma| + |E(G[\sigma])|) = N - (n + |E|).
\]

The linear functional,

\[
f := \sum_{i \in [n]} e_i^* + \sum_{\sigma \in J_{G[\rho]}(O_{[\rho]}, \sigma)} \beta_\sigma \cdot \sum_{j \in \sigma} e_j^*,
\]

satisfies that, for \( x \in P_{\alpha} \),

\[
f(x) \geq (1 - \alpha)N + \alpha(n + |E|) + \alpha(N - (n + |E|)) = N.
\]

By the proof of the first claim, this inequality is tight if and only if \( x \in (X, S) \cap P_{\alpha} = (X, S) \cap P_{\alpha} \). Moreover, since this minimum is independent of \( \alpha \), the linear functional \( f \) is minimized in \( [G, N\Delta] \) exactly at \( (X, S) \). If \( N < n + |E| \), we must select negative coefficients and consider instead the maximum of the linear functional in question, analogously.
For the third statement, we simply note that an open ball in the affine space determined by all \(x \in \mathbb{R}^n\) satisfying Equalities 2.3-2.4 can be easily (but tediously) found inside \((F,S)\). Hence, \(\dim_{\mathbb{R}} \langle (F,S) \rangle = |p| + \dim_{\mathbb{R}[\rho]} |O|\).

\[\blacksquare\]

**Definition 2.27.** Let \(G = G([n], E)\) be a connected simple graph. Let \(\mathcal{X}_G^* = (\mathcal{X}_G, \leq_x, \dim_x)\) be the abstract cell complex dual to \(\mathcal{X}_G\) in Definition 2.18. Hence, for all \((o_0, o_0), (\sigma_1, O_1) \in \overline{\mathcal{X}_G}\):

1. \((\sigma_0, o_0) \leq_x (\sigma_1, O_1)\) if and only if \(J_G[\sigma_0](o_0) \subseteq J_G[\sigma_1](O_1)\), and
2. \(\dim_x (\sigma_0, o_0) = |\sigma_0| - 1 - \dim_{\mathbb{R}[\sigma_0]}(O_0)\).

**Theorem 2.28.** Let \(G = G([n], E)\) be a connected simple graph with abstract cell complex \(\mathcal{X}_G^*\) as in Definition 2.27. Then, the polytopal complex \(\mathcal{X}_G\) obtained from all faces of the intersection \(\mathcal{A}_G \cap \Delta\) inside \(\mathbb{R}^n\) is a polytopal complex realization of \(\mathcal{X}_G^*\), where \(\mathcal{A}_G\) is the graphical arrangement of \(G\) and \(\Delta = \text{conv} \{e_1, e_2, \ldots, e_n\}\):

\(\mathcal{A}_G := \{x \in \mathbb{R}^n: x_i - x_j = 0, \forall \{i, j\} \in E\}\).

**Proof.** From Theorems 2.15, 2.25, and letting \(N \to \infty\) in Equation 2.3, we know that the relatively open cone \(C^+_{(\sigma, O)}\) in the totally non-negative part of the normal fan of the polytope \(Z_G, \eta \Delta\) that corresponds to a cell \((\sigma, O) \in \overline{\mathcal{X}_G}\), is given by:

\[C^+_{(\sigma, O)} = \text{span}_{\mathbb{R}_+} \left\{ \sum_{i \in p} e_i : p \in J_G[\sigma](O) \setminus \{\emptyset\} \right\}.\]

Hence, since the affine dimension of the corresponding dual cell in \(\mathcal{Y}_G\) is \([n] - |\sigma| + \dim_{\mathbb{R}^n}[\sigma](O)\), then \(\dim_{\mathbb{R}^n}(C^+_{(\sigma, O)}) = n - |[n] - |\sigma| + \dim_{\mathbb{R}^n}[\sigma](O)\) and so \(\dim_{\mathbb{R}^n}(C^+_{(\sigma, O)} \cap \Delta) = |\sigma| - 1 - \dim_{\mathbb{R}^n}[\sigma](O)\), since \(C^+_{(\sigma, O)} \subseteq \text{span}_{\mathbb{R}_+} \{e_1, \ldots, e_n\}\).

Tangentially, we can also express \(C^+_{(\sigma, O)}\) more compactly by means of its positive basis as:

\[C^+_{(\sigma, O)} = \text{span}_{\mathbb{R}_+} \left\{ \sum_{i \in p} e_i : p \in J_G[\sigma](O) \setminus \{\emptyset\} \text{ and } G[p] \text{ is connected} \right\}.\]

Now, the intersection, \(\mathcal{A}^+_{G[\sigma]} = \mathcal{A}_G \cap \{x \in \mathbb{R}^n: x_i = 0 \text{ if } i \in [n] \setminus \sigma, \text{ and } x_j > 0 \text{ if } j \in \sigma\}\), is as suggested by our choice of notation, equal to the totally positive part of the graphical arrangement of \(G[\sigma]\), regarded here \(\mathbb{R}^n\) as a subspace of \(\mathbb{R}^n\). Per Theorem 2.15, since \(\mathcal{A}^+_{G[\sigma]}\) is precisely the normal fan of \(Z_G[\sigma]\), and \(\mathcal{A}^+_{G[\sigma]}\) the totally positive part of this fan, we know that the relatively open cones of \(\mathcal{A}^+_{G[\sigma]}\) correspond to the p.a.o.’s of \(G[\sigma]\). From the description of the cells of \(Z_G[\sigma]\), the cone \(C^+_{(\sigma, O)}\) is exactly the cone in \(\mathcal{A}^+_{G[\sigma]}\) normal to the cell of \(Z_G[\sigma]\) corresponding to \(O\). This establishes the correspondence between cells of \(\mathcal{A}_G \cap \Delta\) and elements of \(\mathcal{X}_G^*\), since we can go both ways in this discussion.

Using the same lens to regard cells of \(\mathcal{A}_G \cap \Delta\), the correctness of Definition 2.27 now follows from the analogous verification done in Theorem 2.25 by a standard result on normal fans of polytopes, namely, the duality of face containment.

\[\blacksquare\]

3. Two ideals for acyclic orientations.

**Definition 3.1.** Let \(G = G([n], E)\) be a simple graph.
For an orientation \( O \) of \( G \) and for every \( i \in [n] \), let:

\[
\begin{align*}
\text{indeg}_{(G,O)}(i) & := |\{(j,i) \in O[E] : j \in [n]\}|, \\
\text{outdeg}_{(G,O)}(i) & := |\{(i,j) \in O[E] : j \in [n]\}|, \\
\text{nod}_{(G,O)}(i) & := |\{e \in O[E] : \text{either } e = (j,i) \text{ or } e = (i,j), j \in [n]\}|,
\end{align*}
\]

where we denote the respective associated vectors in \( \mathbb{R}^n \) as \( \text{indeg}_{(G,O)} \), \( \text{outdeg}_{(G,O)} \) and \( \text{nod}_{(G,O)} \).

(2) For \( \sigma \subseteq [n] \) with \( \sigma \neq \emptyset \), define \( 1_{\sigma} := \sum_{i \in \sigma} e_i \in \mathbb{R}^n \), further writing \( 1 := 1_{[n]} \). Let now, for every \( i \in [n] \):

\[
\begin{align*}
\text{inof}_{(G,\sigma)}(i) & := \begin{cases} 
|\{(i,j) \in E : j \in \sigma\}| & \text{if } i \in \sigma, \\
0 & \text{otherwise},
\end{cases} \\
\text{outof}_{(G,\sigma)}(i) & := \begin{cases} 
|\{(i,j) \in E : j \in [n]\setminus\sigma\}| & \text{if } i \in \sigma, \\
0 & \text{otherwise},
\end{cases}
\end{align*}
\]

and denote the respective associated vectors of \( \mathbb{R}^n \) as \( \text{inof}_{(G,\sigma)} \) and \( \text{outof}_{(G,\sigma)} \).

**Remark 3.2.** During this section, we will follow the notation and definitions of Miller and Sturmfels (2005), Chapters 1, 4, 5, 6 and 8, in particular, those pertaining to labelled polytopal cell complexes. We refer the reader to this standard reference on the subject for further details. Some key conventions worth mentioning here are:

1. The letter \( k \) will denote an infinite field.
2. For \( a := (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n \), \( m^a := \langle x_i^{a_i} : i \in [n] \rangle \) is the ideal of \( k[x_1, \ldots, x_n] \) associated to \( a \).

**Definition 3.3.** Let \( G = G([n], E) \) be a connected simple graph. The ideal \( A_G \) of acyclic orientations of \( G \) is the monomial ideal of \( k[x_1, \ldots, x_n] \) minimally generated as:

\[
A_G := \left< x^{\text{indeg}_{(G,O)}+1} \right> := \left< \prod_{i \in [n]} x_i^{\text{indeg}_{(G,O)}(i)+1} : O \text{ is an acyclic orientation of } G \right>.
\]

**Definition 3.4.** Let \( G = G([n], E) \) be a connected simple graph. The tree ideal \( T_G \) of \( G \) is the monomial ideal of \( k[x_1, \ldots, x_n] \) minimally generated as:

\[
T_G := \left< x^{\text{outof}_{(G,\sigma)}+1} \right> := \left< \prod_{\sigma \in \sigma} x_i^{\text{outof}_{(G,\sigma)}(i)+1} : \sigma \in 2^{[n]} \setminus \{\emptyset\} \text{ and } G[\sigma] \text{ is connected} \right>.
\]

**Definition 3.5.** Given two vectors \( a, b \in \mathbb{N}^n \) with \( b \preceq a \) (\( b_i \leq a_i \) for all \( i \in [n] \)), let \( a \setminus b \) be the vector whose \( i \)-th coordinate is:

\[
a_i \setminus b_i = \begin{cases} 
 a_i + 1 - b_i & \text{if } b_i > 0, \\
0 & \text{if } b_i = 0.
\end{cases}
\]

If \( I \) is a monomial ideal whose minimal generators all divide \( x^a \), then the Alexander dual of \( I \) with respect to \( a \) is:

\[
I^{[a]} := \bigcap \{ m^{a \setminus b} : x^b \text{ is a minimal generator of } I \}.
\]
Theorem 3.6. Let \( G = G([n], E) \) be a simple connected graph. Then, the ideals \( A_G \) and \( T_G \) of Definitions 3.3 and 3.4 are Alexander dual to each other with respect to \( d_G + 1 \), so \( A_G^{[d_G + 1]} = T_G \) and \( T_G^{[d_G + 1]} = A_G \).

**Proof.** It is enough to prove one of these two equalities, so we will prove that \( A_G^{[d_G + 1]} = T_G \). Take some \( \sigma \in 2^n \setminus \{\emptyset\} \) such that \( G[\sigma] \) is connected and consider the minimal generator of \( T_G \) given by \( x^{\text{outof}(G, \sigma) + 1}_\sigma = \prod_{i \in \sigma} x_i^{\text{outof}(G, \sigma)(i) + 1} \). We will verify that \( x^{\text{outof}(G, \sigma) + 1}_\sigma \in m^{d_G + 1} \) for every minimal generator \( x^\sigma \) of \( A_G \).

Select an acyclic orientation \( O \) of \( G \) and let \( x^{\text{indeg}(G, O) + 1} = \prod_{i \in \sigma} x_i^{\text{indeg}(G, O)(i) + 1} \) be the minimal generator of \( A_G \) associated to \( O \). If we take \( m \in \sigma \) to be maximal in \( ([n], \leq) \) among all elements of \( \sigma \), so that \( i \geq m \) and \( i \in \sigma \) imply \( i = m \), then \( (d_G(m) + 1) \setminus \{\text{indeg}(G, O)(m) + 1\} = \text{outdeg}(G, O)(m) + 1 \leq |N_G(m)\setminus \sigma| + 1 = \text{outof}(G, \sigma)(m) + 1 \). Hence,

\[
x^{\text{outof}(G, \sigma) + 1}_\sigma \in \left( \prod_{i \in \sigma} x_i^{d_G(m) + 1} \right) \subseteq \left( \prod_{i \in \sigma} x_i^{(d_G(m) + 1) \setminus \{\text{indeg}(G, O)(m) + 1\}} \right)
\subseteq m^{d_G + 1} \setminus \{\text{indeg}(G, O) + 1\}.
\]

This proves that \( T_G \subseteq A_G^{[d_G + 1]} \).

Now, consider a monomial \( x^b \notin T_G \) with \( 0 < b \) (so \( b_i > 0 \) for some \( i \in [n] \)). Then, for every \( \sigma \in 2^n \setminus \{\emptyset\} \) there exists \( i \in \sigma \) such that \( b_i < \text{outof}(G, \sigma)(i) + 1 \), noting here that the condition on \( G[\sigma] \) being connected can be dropped. Hence, consider a bijective labeling \( f : [n] \rightarrow [n] \) of the vertices of \( G \) such that \( b_{f^{-1}(i)} < \text{outof}(G, f^{-1}(i), (f^{-1}(i)) + 1 \) for all \( i \in [n] \). If we let \( O \) be the acyclic orientation of \( G \) such that for every edge \( e = \{i, j\} \in E \), \( O(e) = \{i, j\} \) if and only if \( f(i) < f(j) \), then for all \( i \in [n] \), \( b_{f^{-1}(i)} < \text{outof}(G, f^{-1}(i), (f^{-1}(i)) + 1 = \text{outdeg}(G, O)(f^{-1}(i)) + 1 = (d_G(f^{-1}(i)) + 1) \setminus \{\text{indeg}(G, O)(f^{-1}(i)) + 1\} \). This shows that \( x^b \notin A_G^{[d_G + 1]} \), therefore \( T_G = A_G^{[d_G + 1]} \).

\[\square\]

**Corollary 3.7.** Let \( G = G([n], E) \) be a simple connected graph. Then:

\[ A_G = \bigcap \{ m^{\text{inof}(G, \sigma) + 1}_\sigma : \sigma \in 2^n \setminus \{\emptyset\} \text{ and } G[\sigma] \text{ is connected} \}, \]

is the irreducible decomposition of \( A_G \). Also:

\[ T_G = \bigcap \{ m^{\text{outof}(G, \sigma) + 1}_\sigma : O \text{ is an acyclic orientation of } G \}, \]

is the irreducible decomposition of \( T_G \).

**Definition 3.8.** For a simple connected graph \( G = G([n], E) \), consider the polytopal complexes \( Z_G, Y_G \) and \( X_G \), which respectively realize the abstract cell complexes \( \mathcal{Z}_G, \mathcal{Y}_G \) and \( \mathcal{X}_G \) of Definitions 2.11, 2.18 and 2.27. We will let \( Z_G = (Z_G, \ell_z) \), \( Y_G = (Y_G, \ell_y) \) and \( X_G = (X_G, \ell_x) \) be the \( n^n \)-labelled cell complexes with underlying polytopal complexes given by \( Z_G, Y_G \) and \( X_G \), respectively, and face labelling functions \( \ell_z, \ell_y, \ell_x \), defined according to:

(1) \( Z_G \): For a face \( F \) of \( Z_G \) corresponding to \( O \in \mathcal{Z}_G \):

\[ \ell_z(F)_i = \text{nod}(O, i) + 1, \quad i \in [n]. \]

(2) \( Y_G \):

---
(a) For a face $F$ of $Y_G$ corresponding to $(\sigma, O) \in \mathcal{H}_G \subseteq \mathcal{H}$:

$$
\ell_y(F)_i = \begin{cases} 
\text{nod}_{G[\sigma],O}(i) + 1 & \text{if } i \in \sigma, \\
\text{outdeg}_{G[\sigma],O}(i) + 2 & \text{otherwise.}
\end{cases}
$$

(b) For a face $F$ of $Y_G$ corresponding to $\sigma \in 2^{[n]} \setminus \{[n], \emptyset\} \subseteq \mathcal{H}_G$:

$$
\ell_y(F)_i = \begin{cases} 
\text{deg}(i) + 2 & \text{if } i \in \sigma, \\
0 & \text{otherwise.}
\end{cases}
$$

(3) $X_G$: For a face $F$ of $X_G$ corresponding to $(\sigma, O) \in \mathcal{H}_G^*$:

$$
\ell_x(F)_i = \begin{cases} 
\text{outdeg}_{G[\sigma],O}(i) + \text{outof}_{G,\sigma}(i) + 1 & \text{if } i \in \sigma, \\
0 & \text{otherwise.}
\end{cases}
$$

Lemma 3.9. Let $G = G([n], E)$ be a simple connected graph. Then, for any face $F$ of $Z_G$ with vertices $v_1, \ldots, v_k$, we have that:

$$
\chi^\ell_x(F) = \text{LCM}\left\{\chi^\ell_x(v_i)\right\}_{i \in [k]},
$$

where LCM stands for “least common multiple”.

Proof. Let $F$ be a face of $Z_G$ with corresponding p.a.o. $O$ of $G$ and connected partition $\Sigma$. Every acyclic orientation of $G$ that corresponds to a vertex of $F$ is obtained by 1) selecting an acyclic orientation for each of the $G[\sigma]$ with $\sigma \in \Sigma$, and then by 2) combining those $|\Sigma|$ acyclic orientations with $O[E] \cap \overline{E}$. For a fixed vertex $i \in \sigma$ with $\sigma \in \Sigma$, it is possible to select an acyclic orientation of $G[\sigma]$ in which $i$ is maximal and then to extend this to an acyclic orientation of $G$ that refines $O$, so if vertex $v_j$ of $F$ corresponds to one such orientation, then $\ell_x(v_j)_i = \text{nod}_{G[\sigma],O}(i) + 1$. On the other hand, clearly $\ell_x(v_j)_i \leq \text{nod}_{G[\sigma],O}(i) + 1$ for all vertices $v_j$ of $F$. Hence, $\chi^\ell_x(F) = \text{LCM}\left\{\chi^\ell_x(v_i)\right\}_{i \in [k]}$.

Corollary 3.10. Similarly, for $G$ as in Lemma 3.9 and for any face $F$ of $Y_G$ with vertices $v_1, \ldots, v_k$, we have that:

$$
\chi^\ell_y(F) = \text{LCM}\left\{\chi^\ell_y(v_i)\right\}_{i \in [k]},
$$

where LCM stands for “least common multiple”.

Proof. If $F$ is a face of $Y_G$ inside the simplex $N\Delta$, then this is immediate. If $F$ corresponds to some $(\sigma, O)$, then this is a consequence of the proof of Lemma 3.9 since the vertices of $F$ are all the $N \cdot e_i$ with $i \in [n] \setminus \sigma$, and all the vertices of $Y_G$ that correspond to acyclic orientations of $G$ whose restrictions to $G[\sigma]$ refine $O$ and in which all edges of $G$ connecting $\sigma$ with $[n] \setminus \sigma$ are directed out of $\sigma$.

Proposition 3.11. Let $G = G([n], E)$ be a simple connected graph. The cellular free complex $F_{Y_G}$ supported on $Y_G$ is a minimal free resolution of the artinian quotient $k[x_1, \ldots, x_n]/(A_G + m^{d_G+2})$.

Proof. Without loss of generality, we assume here that $N > n + |E|$. From standard results in topological combinatorics it is easy to see that for $b \in \mathbb{N}^{[n]}$, the closed faces of $Y_G$ that are contained in the closed cone $C_{\leq b} = \{v \in \mathbb{R}^{[n]} : v \leq b\}$ form a contractible polytopal complex, whenever this cone contains at least one face of $Y_G$. 

Now, suppose that $b_i \leq d_G(i) + 1$ for all $i \in [n]$. Then, the complex of faces of $Y_G$ in the cone $C_{\leq b}$ coincides with $Y_{G, \leq b}$, so the later is contractible and acyclic if non-empty. On the contrary, let $U_b$ be the set of all $i$ such that $b_i \geq d_G(i) + 2$, and let $D_b = [n] \setminus U_b$. Consider the vector $a \in \mathbb{R}^{|n|}$ such that:

$$a_i = \begin{cases} N & \text{if } i \in U_b, \\ b_i & \text{if } i \in D_b. \end{cases}$$

Then, the set of faces of $Y_G$ in the cone $C_{\leq a}$ coincides with $Y_{G, \leq b}$, so again the later is contractible and acyclic if non-empty. This shows that $F_{Y_G}$ supports a cellular resolution of $k[x_1, \ldots, x_n]/(A_G + m^{d_G+2})$.

To prove that this resolution is minimal, it suffices to check that whenever $F_0$ and $F_1$ are closed faces of $Y_G$ such that $F_0 \subseteq F_1$, then $\ell_y(F_0) < \ell_y(F_1)$. There are three cases to study:

1. $F_0$ and $F_1$ correspond respectively to $\sigma_0, \sigma_1 \in 2^{[n]} \setminus \emptyset \subseteq \mathcal{G}$;
   Then, $\sigma_0 \subseteq \sigma_1$ and for $i \in \sigma_1 \setminus \sigma_0$, $\ell_y(F_0)_i = 0 < d_G(i) + 2 = \ell_y(F_1)_i$.
2. $F_0$ corresponds to $\sigma_0 \in 2^{[n]} \setminus \emptyset \subseteq \mathcal{G}$ and $F_1$ to $(\sigma_1, O_1) \in \mathcal{G}$:
   Then, $\sigma_0 \subseteq [n] \setminus \sigma_1$ and for $i \in \sigma_1$, $\ell_y(F_0)_i = 0 < 1 < nod(\sigma_1, \sigma_0)(i) + 1 = \ell_y(F_1)_i$.
3. $F_0$ and $F_1$ correspond respectively to $(\sigma_0, O_0), (\sigma_1, O_1) \in \mathcal{G}$:
   Therefore, $J_{G[\sigma_0]}(O_1) \subseteq J_{G[\sigma_0]}(O_0)$, so if $\sigma_i \subseteq \sigma_0$, then for $i \in \sigma_0 \setminus \sigma_1$, $\ell_y(F_0)_i = nod(\sigma_1, \sigma_0)(i) + 1 < d_G(i) + 2 = \ell_y(F_1)_i$; and 2) if $\sigma = \sigma_0 = \sigma_1$, then letting $\Sigma_0$ and $\Sigma_1$ be the connected partitions of $G[\sigma]$ corresponding respectively to $O_0$ and $O_1$, we observe that $\Sigma_0$ is a strict refinement of $\Sigma_1$, so there exist $q_0 \in \Sigma_0$ and $q_1 \in \Sigma_1$ such that $q_0 \subseteq q_1$ and such that for some $i \in p_1 \setminus p_0$, we have that $\ell_y(F_0)_i = nod(\sigma_1, \sigma_0)(i) + 1 < nod(\sigma_1, \sigma_0)(i) + 1 = \ell_y(F_1)_i$, since $G[p_1]$ is connected (so there is an edge directed out of $i$ in $O_0$ which was not directed in $O_1$).

$\Box$

**Proposition 3.12.** For $G$ as in Proposition 3.11, the cellular free complex $F_{Z_G} = F_{Y_G, \leq d_G+1}$ supported on $Z_G$ gives a minimal free resolution of the quotient ring: $k[x_1, \ldots, x_n]/A_G$.

**Proof.** This follows from the proof of Proposition 3.11 since $Z_G = Y_G, \leq d_G+1$.

$\Box$

**Corollary 3.13.** For $G$ as in Proposition 3.11, let $Y_G^{\text{col}} = d_G + 2 - Y_G$. Then, the cocellular free complex $F_{Y_G^{\text{col}}, \leq d_G+1}$ supported on $Y_G^{\text{col}}$ is a minimal cocellular resolution of the monomial ideal $T_G$.

**Proposition 3.14.** For $G$ as in Proposition 3.11, the cellular free complex $F_{X_G}$ supported on $X_G$ is a minimal cellular resolution of the monomial ideal $T_G$.

**Proof.** This is now a consequence of Corollary 3.13 since the underlying polytopal complex of $Y_G^{\text{col}}, \leq d_G+1$ is combinatorially dual to the underlying complex of $X_G$, and cells from both complexes dual to each other have equal labels: If a face $F_y$ of $\mathcal{G}$
and a face \( F_x \) of \( X_G \) both correspond to \((\sigma, O) \in \mathcal{F}_G\), then,
\[
d_G(i) + 2 - \ell_y(F_y)_i = \begin{cases} 
  d_G(i) + 2 - (\text{nod}_{G[\{r\}, O]}(i) + 1) & \text{if } i \in \sigma, \\
  d_G(i) + 2 - (d_G(i) + 2) & \text{otherwise}, \\
  \text{outdeg}_{G[\{r\}, O]}(i) + \text{outof}_{G[\{r\}, \sigma]}(i) + 1 & \text{if } i \in \sigma, \\
  0 & \text{otherwise}.
\end{cases}
\]
\[
= \ell_x(F_x).
\]

The following is, in reality, a well-known result about Betti numbers of monomial quotients with a given cellular resolution, and not a definition. We present it here as a definition given its immediate connection to the topology of cellular complexes, clearly central for the results of this section.

**Definition 3.15.** If \( F_X \) is a cellular resolution of the monomial quotient \( S/I \), then the Betti numbers of \( I \) are the numbers calculated, for all \( i \geq 1 \), as:
\[
\beta_{i, b}(I) = \dim_k \tilde{H}_{i-1}(X_X b; k),
\]
where \( \tilde{H}_a \) stands for the reduced homology functor.

**Lemma 3.16.** For a simple connected graph \( G = G([n], E) \), the Betti numbers of the ideals \( A_G \) and \( T_G \) satisfy that, for all \( i \geq 0 \):
\[
\sum_{b \in [n]} \beta_{i, b}(A_G) = \# \text{ p.a.o.'s of } G \text{ on } n - i \text{ connected parts},
\]
\[
\sum_{b \in [n]} \beta_{i, b}(T_G) = \# \text{ of pairs } (O, \sigma) : O \text{ is a p.a.o. of } G[\sigma] \text{ on } i + 1 \text{ connected parts}.
\]

**Proof.** These results are clear from our choice of minimal cellular resolutions for these ideals, since \( i \)-th syzygies of each ideal correspond to \( i \)-dimensional faces of the respective geometrical complex.

\[\square\]

4. Non-crossing trees.

In this section we investigate, for a simple graph \( G = G([n], E) \), a useful and novel unifying relation between the standard monomials of \( T_G \), the rooted spanning forests of \( G \), and the maximal chains of the poset of non-crossing partitions. We show that, arguably, the phenomenology that binds these objects together and which has been hitherto discovered in the literature, is largely due to the existence of a simple canonical way to represent rooted spanning forests of a graph on vertex set \([n]\) as non-crossing spanning trees.

An analogous extension of the theory presented here to a more general poset of non-crossing partitions associated to \( G \), and the consideration of the equally arbitrary non-nesting trees and their connection to the Catalan arrangement, will not be discussed here, and will be the subject of a future writing by the author.

**Definition 4.1.** For a simple graph \( G = G(V, E) \), we will let \( G_r \) denote the graph on vertex set \( V \cup \{r\} \) and with edge set \( E \cup \{\{r, v\} : v \in V\} \), so \( G_r \) is the graph obtained from \( G \) by adding a new vertex \( r \) and connecting it to all other vertices in \( G \) (e.g. Figures 5A, 5B).
Definition 4.2. A planar depiction \((D, p)\) of a finite acyclic di-graph \(T = T(V, E)\) is a finite union of closed curves \(D \subseteq \mathbb{R}^2\) and a bijection \(p : V \rightarrow \{0, 1, 2, \ldots, |V| - 1\}\) (called a depiction function) such that:

1) \(p\) is order-reversing, so if \(e \in E\) and \(e = (u, v)\), then \(p(v) < p(u)\).

2) There exist strictly increasing and continuous real functions \(f\) and \(g\) such that \(f(0) = g(0) = 0\), and \(D\) is the image under \((f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) of the following union of semicircles:

\[
\bigcup_{(u,v) \in E} \left\{ (x, y) \in \mathbb{R}^2 : y = \pm \sqrt{\frac{(p(u) - p(v))^2}{2} - \left(x - \frac{p(u) + p(v)}{2}\right)^2} \right\}.
\]

A planar depiction \((D, p)\) of \(T\) is said to be non-crossing if for all \((x, y) \in D\) with \(y > 0\), a sufficiently small neighborhood of \((x, y)\) in \(D\) is homeomorphic to the real line.

Lemma 4.3. In Definition 4.2, the property of being a non-crossing planar depiction is independent of the choice of functions \(f\) and \(g\), and only depends on \(p\) and \(T\). In other words, any two planar depictions \((D_1, p)\) and \((D_2, p)\) of \(T\) are either both non-crossing or both crossing.

Example 4.4. Figure 4A shows a particular acyclic directed graph \(T = T(V, E)\) with \(|V| = 7\), and a choice of depiction function \(p : V \rightarrow \{0, 1, 2, 3, 4, 5, 6\}\) (in blue). With this choice of \(p\), Figure 4B then presents the set \(D\) obtained by taking \(f(x) = x\) and \(g(x) = \frac{1}{2}x\) in Definition 4.2. There are five crossings in \(D\), each marked with a square; these crossings are the points \((x, y) \in D\), \(y > 0\), that are locally non-homeomorphic to the real line.

Definition 4.5. A non-crossing tree is a non-crossing planar depiction of a rooted tree \(T = T(V, E)\). Vaguely, \(T\) is obtained from an acyclic connected simple graph on vertex set \(V\) by orienting all of its edges towards a distinguished vertex of \(T\), called the root of \(T\) (e.g. Figure 5C).

Remark 4.6. In Definition 4.5, for one such non-crossing tree \((D, p)\) of \(T\), if \(r\) is the root of \(T\), then necessarily \(p(r) = 0\).

Theorem 4.7. Let \(G = G([n], E)\) be a simple graph, and consider a spanning tree \(T\) of \(G\) rooted at \(r\). Then, there exists a unique depiction function \(p\) as in Definition 4.2 such that:

i) For all edges \((i, k)\) and \((j, k)\) of \(T\), \(p(i) > p(j)\) if \(i < j\).

ii) Any planar depiction \((D, p)\) of \(T\) is a non-crossing tree.

Proof. For any two \(i, j \in [n]\) with \(i \neq j\), consider the directed paths from \(i\) and \(j\) to the root \(r\) of \(T\). These paths meet initially at a unique vertex \(r_{ij}\) of \(T\). Let us say that \(i \prec_T j\) if either 1) \(r_{ij} = i\) or if 2) there exist edges \((i, j, r_{ij})\) in the path from \(i\) to \(r_{ij}\) and \((j, r_{ij})\) in the path from \(j\) to \(r_{ij}\) such that \(i_j > j_i\).

Firstly, we verify that the relation \(\leq_T\) is a total order on the set \([n]\) of vertices of \(G\). This is true since for \(i \prec_T j \prec_T k\) with \(i, j, k \in [n]\):

- a. If \(r_{ij} = i\), then either \(r_{ik} = i\) or \(r_{ik} = r_{jk}\) and in the later case \(i_k = j_k > k_j = k_i\).
- b. If \(r_{jk} = j\), then \(r_{ij} = r_{ik}\) and \(i_k = j_i > j_k = k_i\).
- c. If \(r_{ij} \neq i, r_{jk} \neq j\) and \(r_{ij} = r_{jk}\), then \(i_k = i_j > j_i = j_k > k_j = k_i\).
\[T = T(V, E) : \]
\[p : V \to \{0, 1, \ldots, 6\}\]

\[D, f(x) = x, g(x) = \frac{1}{2}x\]
\[p : V \to \{0, 1, \ldots, 6\}\]

Five Crossings:  

(B)

**Figure 4.** Example of a planar depiction, according to Definition 4.2

d. If \(r_{ij} \neq i, r_{jk} \neq j\) and \(r_{ij} \prec_T r_{jk}\), then \(i_k = i_j > j_i = k_i\).

e. If \(r_{ij} \neq i, r_{jk} \neq j\) and \(r_{jk} \prec_T r_{ij}\), then \(i_k = j_k > k_j = k_i\).

Let \(f : [n] \to [n]\) be the unique linear extension of this chain poset \(([n], \leq_T)\) and define \(p\) by requiring that \(p(r) = 0\) and \(p(i) = f(i)\) for all \(i \in [n]\). Clearly then \(p\) satisfies Condition 6.

We now want to check that any depiction \((D, p)\) of \(T\) is non-crossing. Suppose on the contrary that one such depiction is crossing. If that is the case, then there exist edges \((j, i)\) and \((m, k)\) in \(T\) such that \(p(i) < p(k) < p(j) < p(m)\), and hence \(j_m = j_k < k_j = m_j\), a contradiction. This proves 6.

To prove that \(p\) is the unique bijection \([n] \cup \{r\} \to \{0, 1, 2, \ldots, n\}\) satisfying 6 and let us suppose that another depiction function \(q\) works as well. Since \(q\) is order-reversing, then for any \(i, j \in [n]\) with \(i \neq j\) and \(r_{ij} = i\), we must have that \(q(i) < q(j)\). If instead \(r_{ij} \neq i, j\) and \(i_j > j_i\), then Condition 6 and transitivity imply that \(q(i_j) < q(j_i) < q(j)\), and then Condition 6 shows that \(q(i_j) < q(i) < q(j_i) < q(j)\) since in any planar depiction of \(T\) using \(q\), the depiction of the path from \(i\) to \(r_{ij}\) (or to \(i_j\) does not cross the depiction of the path from \(j\) to \(r_{ij}\) (or to \(j_i\)). Hence, \(q(i) < q(j)\). This shows that \(q = p\) from 1 and 2 above.

\[\square\]

**Example 4.8.** Figures 5A–5E offer an example of the unique depiction function \(p\) of Theorem 4.7. For the graph \(G = G([7], E)\) of Figure 5A, we calculate \(G_r\) in Figure 5B. We then select a particular spanning tree \(T\) of \(G_r\) (Figure 5C in red, left diagram) and root it at \(r\) (Figure 5C right diagram). Next, we present an inductive construction of the depiction function \(p\) of Theorem 4.7 associated to \(T\). Figure 5D exhibits an inductive calculation from \(T\) of a certain special diagram \(D\) (in red), and the final output of this calculation is fully illustrated in 5D. This final output of 5D shows a non-crossing tree from which \(p\) can be instantly read off (table). At every step of the construction, we aim to respect both Conditions 6 and 11 of Theorem 4.7 and this is seen to imply the uniqueness of \(p\) for this example. In fact, it is not difficult to observe that the analogous inductive process can be readily applied to any other example, from which Theorem 4.7 follows.
4.1. Standard monomials of $T_G$.

Definition 4.9. Let $G = G([n], E)$ be a simple graph and let $x^n$ be a standard monomial of the ideal $T_G$. From $a$, let us define a bijection $f_a : \{0, 1, \ldots, n\} \to [n] \sqcup \{r\}$ and an $r$-rooted spanning tree $T_a$ of $G_r$ recursively as follows:

(1) The edge set $E(T_a)$ of $T_a$ will be constructed one edge at a time. Similarly, a set $K$ will contain at each step the set of values in $\{0, 1, \ldots, n\}$ for which $f_a$ has already been defined.

(2) Initially, set $E(T_a) = \emptyset$, $f_a(0) = r$, $i = 0$, and $K = \{0\}$.

Since $f_a(k)$ has been defined for all $k \in R$, let us also denote this partially-defined function by $f_a$ (which should not cause any confusion).

Step $i$:

(3) Let $(k, j)$ be the lexicographically-maximal pair among all pairs such that:

a) $k \in K$.

b) $j \in [n] \setminus f_a[K]$, and

c) for $t_0 < \cdots < t_m$, $f_a^{-1} [ N_{G_r}(j) \cap f_a[K] ]$, we have $k = l_m$.

(4) From this pair $(k, j)$, set $f_a(i) = j$ and $E(T_a) = E(T_a) \cup \{(j, f_a(k))\}$.

(5) $K = K \cup \{i\}$.

(6) $i = i + 1$.

Figure 5. Fully worked example illustrating the central dogma of Section 4. Theorems 4.13 and 4.22 are dwelled on in tables E.i and E.ii, respectively.
ACYCLIC ORIENTATIONS AND SPANNING TREES.

(7) Go back to (3) if \( i \leq n \), otherwise stop.

**Proposition 4.10.** In Definition 4.9 both \( f_n \) and \( T_n \) are well-defined. Furthermore, if we set \( p_n = f_n^{-1} \), then \( p_n \) is the unique function of Theorem 4.7 such that any planar depiction \((D, p_n)\) of \( T_n \) is a non-crossing tree.

**Proof.** If the condition of Definition 4.9.3 can be attained at each step of the recursion, that is, if for all \( i \in [n] \) we are able to find at least one such pair of \( k \) and \( j \) for which \( k = l_{a_i} \), then it is clear that \( f_n \) is a bijection and \( T_n \) (with edge set \( E(T_n) \)) is a spanning \( r \)-rooted tree of \( G_r \). It then follows easily that \( p_n \) is order-reversing. Now suppose that we are at the \( i \)-th step of the recursion, say the first step of the recursion, that is, if for all \( i \leq n \), so that \( K = \{0, 1, \ldots, i-1\} \). Since for \( \emptyset \neq \sigma = [n] \backslash f_n[K] \) we have that \( x_{\text{outof}(G, \sigma)}^{-1} \in T_G \), then there must exist at least one \( j \in \sigma \) such that \( a_j \leq \text{outof}(G, \sigma)(j) \). Therefore, if we write \( l_0 < \cdots < l_m = f_n^{-1} \left[ N_{G_r}(j) \cap f_n[K] \right] \) and observe that in fact \( m = \text{outof}(G, \sigma)(j) \), it follows that \( k = l_{a_i} \) is defined correctly for this choice of \( j \).

Let us now establish the non-crossing condition given the choice of depiction \( p_n = f_n^{-1} \). Notably, the recursive definition of \( f_n \) is tailored at making this verification rather simple. Indeed, suppose that there exists a first step of the recursion, say the \( i \)-th step, \( i \leq n \), where a pair of crossing curves will be formed in any depiction \((D, p_n)\) of \( T_n \), and let \((k, j)\) be the lexicographically-maximal pair found in this step. Let also \((k_0, j_0)\) be the optimal pair found at the \( i_0 \)-th step with \( i_0 < i \), such that the curves representing the edges \((j_0, f_n(k_0))\) and \((j, f_n(k))\) cross in all \( p_n \)-depictions of \( T_n \). Then, \( k_0 < k < i_0 < i \). This implies that the pair \((k, j)\) is lexicographically-larger than \((k_0, j_0)\) and that, during the \( i_0 \)-th step, the condition of Definition 4.9.3 is also attained for \((k, j)\), so that \( k = l_{a_i} \). Contradiction.

It remains to prove that \( p_n \) satisfies Condition 6 of Theorem 4.7, but this follows immediately from the choice of lexicographically-maximal pairs at each step of the recursion.

\[ \square \]

**Definition 4.11.** Let \( G = G([n], E) \) be a simple graph, \( T \) an \( r \)-rooted spanning tree of \( G_r \), and \( p \) the unique depiction function of Theorem 4.7 associated to \( T \). Let us associate with \( T \) a vector \( b(T) \in \mathbb{N}^{|[n]|} \) in the following way:

For all \( i \in [n] \) and unique directed edge \((i, i_r)\) in \( T \), let \( b(T)_i = |\{j \in N_{G_r}(i) : p(j) < p(i_r)\}| \).

**Proposition 4.12.** In Definition 4.11, the monomial \( x^{b(T)} \) is a standard monomial of the ideal \( T_G \).

**Proof.** Consider the bijective function \( f : [n] \to [n] \) given by \( f(i) = n + 1 - p(i) \) for all \( i \in [n] \). Clearly then \( b(T)_{f^{-1}(i)} = \text{outof}(G, f^{-1}(i)) < \text{outof}(G, i) + 1 \), and we are exactly in the situation of the second part of the proof of Theorem 3.6 so we obtain that \( x^{b(T)} \not\in A_G^{d_G+1} = T_G \).

\[ \square \]

**Theorem 4.13.** Let \( G = G([n], E) \) be a simple graph, \( x^a \) a standard monomial of \( T_G \), and \( T \) an \( r \)-rooted spanning tree of \( G_r \). Then, using the notation and functions from Definitions 4.7, 4.12 and Proposition 4.10, we have that \( b(T_a) = a \) and \( T_{b(T)} = T \). Hence, the non-crossing trees obtained from the spanning trees of \( G_r \) interolate in a bijection between rooted spanning forests of \( G \) and standard monomials of \( T_G \), in such a way that every non-crossing tree naturally corresponds to a uniquely
determined object from each of these two sets of combinatorial objects associated to $G$.

**Proof.** This is now a straightforward application of the recursive definition of $f_n$ (or of $f_{b(T)}$). For the first equality, let us suppose that during the $i$-th step of the recursion to define $f_n$, so $K = \{0, 1, \ldots, i-1\}$ and $i \leq n$, we find a lexicographically-maximal pair $(k, j)$ with $k = l_{a_j}$, where $\{l_0 < \cdots < l_m\} = f_n^{-1}[N_G(j) \cap f_n[K]]$. Then:

$$b_{(T_n)_j} = |\{\ell \in N_{G_r}(j) : p_n(\ell) < p_n(j_r)\}| = |\{\ell \in N_{G_r}(j) : f_n^{-1}(\ell) < f_n^{-1}(f_n(k))\}|$$

$$= |\{\ell \in N_{G_r}(j) : f_n^{-1}(\ell) < l_{a_j}\}|$$

$$= |\{\ell \in N_{G_r}(j) : l_{a_j} - 1 \leq i - 1\}| = a_j.$$

This proves the first equality.

For the second equality, we use induction on $N$ to prove that $f_{b(T)}(N) = p^{-1}(N)$ for all $N \in \{0, 1, \ldots, n\}$, and then to argue that during step $N \geq 1$ of the recursion to define $f_{b(T)}$, $N \leq n$, the edge that will be added to the set $E(T_{b(T)})$ is an edge of $T$. Initially, when $N = 0$, we have $f_{b(T)}(0) = p^{-1}(0) = r$ and $E(T_{b(T)}) = \emptyset$. Suppose that the result is true for all $N < i$, $i \in [n]$, and let us consider the $i$-th step of the recursion, so that $K = \{0, 1, \ldots, i-1\}$. By induction, if $j \in [n] \setminus f_{b(T)}[K]$ and $\{l_0 < \cdots < l_m\} = f_{b(T)}^{-1}[N_{G_r}(j) \cap f_{b(T)}[K]]$, since $f_{b(T)}(k) = p^{-1}(k)$ for all $k \in K$, we have that when $b_{(T_n)_j} \leq m$:

$$l_{b(T)_j} = \left|\{\ell \in N_{G_r}(j) : p(\ell) < p(j_r)\}\right| = p(\ell) \quad ((j, j_r) \in E(T), \text{ definition of } b(T))$$

Hence, the choice of lexicographically-maximal pair $(k, j)$ necessarily corresponds to an edge of $T$, that is, $(j, f_{b(T)}(k)) \in E(T)$. Letting $s := p^{-1}(i)$ and $(s, s_r) \in E(T)$, that maximal pair selected from $T$ is easily seen to be $(p(s_r), s)$, again by the inductive step and the conditions satisfied by $p$ and $T$ from Theorem 4.7.

**Example 4.14.** Figure 5Ei presents the standard monomial of $T_G$ that corresponds to the spanning $T$ tree of $G_r$ in Example 4.8. For example, to calculate $(a)_4 = a_4$, we find cusp 4 (in black) in Figure 5Dv. To the left of cusp 4 in this diagram, there is exactly one adjacent cusp to 4 through a red arc. This is cusp 5 (in black), so we say that $5 = 4_r$. There is exactly one cusp in the diagram strictly to the left of 5 that is adjacent to 4, that is $r$. Therefore, $a_4 = 1$, as in Definition 4.11.

**Proposition 4.15.** Let $G = G([n], E)$ be a simple graph. Then, there exists a bijection between the following sets:

1. The set of acyclic orientations of $G$.
2. The set of $r$-rooted spanning trees $T$ of $G_r$ such that if $p$ is the depiction function for $T$ of Theorem 4.7, then for all $(i, i_r) \in E(T)$ and $j \in [n]$ with $p(i_r) < p(j) < p(i)$, we have that $(i, j) \notin E$.

Moreover, if $T$ (with depiction function $p$) corresponds to an acyclic orientation $O$ of $G$ under this bijection, then the function $f : [n] \to [n]$ given by $f(m) = \cdots$
Let us first show that the maximal (by divisibility) standard monomials of $T_G$ are in bijection with the acyclic orientations of $G$. Let $a \in \mathbb{N}^{|n|}$ be such that $x^a \notin T_G$ but $x^{a+e_i} \in T_G$ for all $i \in [n]$. From the Alexander duality of $A_G$ and $T_G$, consider an acyclic orientation $O$ of $G$ such that $a_i \leq \text{outdeg}_{(G,O)}(i)$ for all $i \in [n]$. Since $a_i + 1 \geq \text{outdeg}_{(G,O)}(i) + 1$ for all $i$, then it must be the case that $a_i = \text{outdeg}_{(G,O)}(i)$, so $a = \text{outdeg}_{(G,O)}$. It is well-known and not difficult to prove that the out-degree (or in-degree) sequences uniquely determine the acyclic orientations of a simple graph, so this establishes that the maximal standard monomials of $T_G$ are in bijection with the (out-degree sequences of the) acyclic orientations of $G$.

Now, given an $r$-rooted spanning tree $T$ of $G_r$ with depiction function $p$ as in Theorem 4.7, let us define an orientation $O$ (not necessarily a p.a.o.) of $G$ associated to $T$. For all $e = (i, j) \in E$, let:

$$O(e) = \begin{cases} (i, j) & \text{if } p(j) \leq p(i_r), \text{where } (i, i_r) \in E(T), \\ e & \text{otherwise.} \end{cases}$$

Consider the out-degree sequence $\text{outdeg}_{(G,O)}$ associated to the orientation $O$, i.e.

$$\text{outdeg}_{(G,O)}(i) = |\{j \in [n] : (i, j) \in O[E]\}|$$

for all $i \in [n]$. We then note that $b(T)_i = \text{outdeg}_{(G,O)}(i)$ for all $i$, so $b(T) = \text{outdeg}_{(G,O)}$. However, the out-degree sequence $\text{outdeg}_{(G,O)}$ corresponds to an acyclic orientation of $G$ if and only if $T$ satisfies that for all $(i, i_r) \in E(T)$ and $j \in [n]$ with $p(i_r) < p(j) < p(i)$, we have that $\{i, j\} \notin E$, since we require that all edges of $E$ get oriented (or get mapped to directed edges) through $O$. This proves the main statement.

That $f$ is a linear extension when $O$ is an acyclic orientation follows since then, for $(i, j) \in O[E]$, necessarily $p(j) \leq p(i_r) < p(i)$ by the definition of $O$ from $T$ and $p$; likewise if $(i, i_r) \in E(T)$ with $i, i_r \in [n]$, then $i_r$ covers $i$ in $O$ since $p(i_r) \geq p(j)$ for all $(i, j) \in E(T)$ and $p$ is order-reversing.

\[\square\]

**Example 4.16.** Figure 6 illustrates both the statement and proof of Proposition 4.15. Firstly, we show an acyclic orientation $O$ of a graph $G = G([7], E)$ (Fig. 6 left). Then, we select a particular special spanning tree of $G_r$ (Fig. 6 in red), and calculate the non-crossing tree representation of this spanning tree (Fig. 6 below). Arcs of this lower diagram represent edges of $G_r$. To each cusp $i$ (in black) of the diagram with $i \in [7] = \{1, 2, \ldots, 7\}$, there is a unique adjacent red arc to the left, and we let $i_r$ (in black) be the other cusp adjacent to the same red arc, e.g. for $i = 5$ we have $1 = 5_r$. Let us orient from right to left every arc of the diagram adjacent to cusp $i$ if the other cusp adjacent to the arc is either $i_r$ or lies to the left of $i_r$, e.g. the arcs from 5 to 1, 5 to 4, 5 to 3, and 5 to r, get all oriented from right to left. Doing this for all $i$, we obtain an orientation of (some of the arcs of) the diagram, and hence an orientation of $G_r$. In our example, this orientation yields an acyclic orientation of $G_r$, and all edges are assigned an orientation; however, this might not be the case for several other choices of spanning tree of $G_r$! Moreover, the restriction of this acyclic orientation to the edges of $G$ is precisely $O$, and this is the bijection of Proposition 4.15.
4.2. Non-crossing partitions.

Definition 4.17. A non-crossing partition of the totally ordered set \([0, n] = \{0, 1, \ldots, n\}\) is a set partition \(\pi\) of \([0, n]\) in which every block is non-empty and such that there does not exist integers \(i < j < k < l\) and blocks \(B \neq B'\) of \(\pi\) with \(i, k \in B\) and \(j, l \in B'\).

The set of all non-crossing partitions of \([0, n]\) ordered by refinement \((\leq_{\text{ref}})\) forms a graded lattice of length \(n\), and we will denote this lattice of non-crossing partitions of \([0, n]\) by \(\text{NC}([0, n])\).

Definition 4.18. Consider a maximal chain \(C = \{\pi_0 \leq_{\text{ref}} \pi_1 \leq_{\text{ref}} \ldots \leq_{\text{ref}} \pi_n\}\) of \(\text{NC}([0, n])\). For each \(i \in [n]\), there exists a unique element \(\bar{i} \in [n]\) such that \(\bar{i}\) is the minimal element of its block in \(\pi_{i-1}\) but \(\bar{i}\) is not the minimal element of its block in \(\pi_i\). Let then \(\bar{i} \neq \bar{i}\) be the element of the block of \(\bar{i}\) in \(\pi_i\) that immediately precedes \(\bar{i}\).

With this notation, we define a bijection \(\rho_C : [n] \sqcup \{r\} \to [0, n]\) and an \(r\)-rooted tree \(T_C\) of \((K[n])_r \cong K_{[n], r}\) associated to the chain \(C\) in the following way:

\[
\begin{align*}
(p_C): & \quad p_C(r) = 0, \\
& \quad p_C(i) = \bar{i}, \text{ for all } i \in [n]. \\
(T_C): & \quad E(T_C) = \{(p_C^{-1}(i), p_C^{-1}(\bar{i})) : i \in [n]\}.
\end{align*}
\]

Proposition 4.19. In Definition 4.18, both \(\rho_C\) and \(T_C\) are well-defined and moreover, \(\rho_C\) is the function of Theorem 4.7 such that any planar depiction \((D, \rho_C)\) of \(T_C\) is a non-crossing tree.

Proof. That \(\rho_C\) is well-defined is a consequence of the fact that taking the union of two disjoint blocks in a partition of \([0, n]\) will make exactly one minimal element of these blocks non-minimal in the newly formed block. Hence, in a maximal chain of \(\text{NC}([0, n])\), every non-zero element of \([0, n]\) stops being minimal in its own block at
exactly one cover relation in the chain, and every cover relation in the chain gives rise to one such element. That \( T_c \) is an \( r \)-rooted spanning tree of \((K_{[0,n]}\) comes from observing that, since \( p_c \) is well-defined, the di-graph \( p_c \circ T_c \) on vertex set \([0,n] \) and edge set \((\tilde{i}, \tilde{j}) \) for all \( i \in [n] \), is a 0-rooted spanning tree of \( K_{[0,n]} \). This is true because for every \( i \in [n] \), there exists exactly one edge in \( p_c \circ T_c \) of the form \((i,j) \) with \( j < i \), and these are all the edges of \( p_c \circ T_c \).

To verify that \( p_c \) and \( T_c \) satisfy Condition 1 of Theorem 4.17, suppose on the contrary that there are edges \((i,k),(j,k) \) in \( E(T_c) \) with \( i < j \) and \( p_c(i) < p_c(j) \). This means that \( i \) was minimal in its block in \( \pi_{i-1} \) but not in \( \pi_i \), and that both \( \tilde{i} = p_c(k) \) lied in the same block of \( \pi_{i-1} \). Similarly, \( j \) was minimal in \( \pi_{j-1} \) but not in \( \pi_j \), where it was immediately preceded by \( \tilde{j} = p_c(k) = \tilde{i} \). Since \( j > i \), all three \( i, j \) and \( \tilde{i} \) belonged to the same block of \( \pi_i \). This means that \( \pi_{i-1} \) and \( \pi_{j-1} \) are crossing, and let us assume \( \{i, \tilde{i}\} \) and \( \{j, \tilde{j}\} \) belong to different blocks of \( \pi_i \). In both cases, we observe that \((\tilde{i}, \tilde{j}) \) is an edge of \( T_c \). Since \( \tilde{j} \) does not immediately precede \( \tilde{i} \) in \( \pi_i \), contradiction.

To verify the non-crossing condition, note that if there is a crossing in a depiction \((D, p_c) \) of \( T_c \), then there is a smallest \( i \in [n] \) such that there exists \( j < i \) with either \( j < \tilde{i} < j < \tilde{i} \) or \( j < \tilde{i} < j < \tilde{i} \). In both cases, we observe that \((\tilde{i}, \tilde{j}) \) is an edge of \( T_c \).

\( \square \)

**Definition 4.20.** Let \( G = G([n], E) \) be a simple graph, and let \( T \) be an \( r \)-rooted spanning tree of \( G_r \). Suppose that \( p \) is the depiction function of Theorem 4.17 such that any depiction \((D, p) \) of \( T \) is non-crossing.

From \( T \) and \( p \), let us form a chain \( C_T = \{\pi_0 \prec_{\text{ref}} \pi_1 \prec_{\text{ref}} \cdots \prec_{\text{ref}} \pi_n\} \) of partitions on the set \([0,n] \) in the following way:

1. Let \( \pi_0 = \{\{0\}, \{1\}, \ldots, \{n-1\}, \{n\}\} \), and
2. for each \( i \in [n] \), let \( \pi_i \) be obtained from \( \pi_{i-1} \) by taking the union of the block that contains \( p(i) \) and the block that contains \( p(i_r) \), where \((i, i_r) \) is an edge of \( T \).

**Proposition 4.21.** In Definition 4.20, \( C_T \) is a well-defined and moreover, it is a maximal chain of partitions in \( \text{NC}([0,n]) \).

**Proof.** That \( C_T \) is a well-defined (maximal) chain of partitions of \([0,n] \) is a consequence of \( p \) being a bijection \([n] \cup r \to [0,n] \) and of \( T \) being a spanning tree of \( G_r \). We can think of the procedure of Definition 4.20 as that of beginning with an independent set of vertices \([n] \cup r \), and then adding one edge of \( T \) at a time until we form \( T \), keeping track at each step of the connected components of the graph so far formed (and mapping those connected components through \( p \)): there are \( n \) such steps and at each step we add a different edge of \( T \). In fact, since \( T \) is rooted and \( p \) is order-reversing, if for some \( i \in [n] \) we consider the edges \((1, 1_r), \ldots, (i, i_r) \) of \( T \) that have been added up to the \( i \)-th step in this process (so that the graph in consideration is a rooted forest), we see that if two numbers \( k < l \) in the set \([0,n] \) belong to the same block \( B \) of \( \pi_i \), then either \((p^{-1}(l'), p^{-1}(k)) \) is an edge of \( T \) for some \( l' \in B \) with \( k < l' \leq l \) and \( p^{-1}(l') \leq i \), or there exist \( k', l' \in B \) with \( k' < k < l' \leq l \) such that \((p^{-1}(l'), p^{-1}(k')) \) is an edge of \( T \) and \( p^{-1}(l') \leq i \).

Suppose now that some of the partitions in \( C_T \) are crossing, and let us assume that \( i \) is minimal such that \( \pi_i \) is crossing. Hence, the block \( B_i \) in \( \pi_i \) that contains both \( p(i) \) and \( p(i_r) \) crosses with another block \( B_j \) of \( \pi_i \), so there exist two consecutive elements \( i_1 < i_2 \) of \( B_i \) and two consecutive elements \( j_1 < j_2 \) of \( B_j \) such that
either \( a \) \( i_1 < j_1 < i_2 < j_2 \) or \( b \) \( j_1 < i_1 < j_2 < i_2 \). In \( \pi_{r-1} \), \( i_1 \) and \( i_2 \) belong to different blocks \( B_{i_1} \) and \( B_{i_2} \) respectively, and \( B_i = B_{i_1} \cup B_{i_2} \). Moreover, since \( i \) was chosen minimally, if \( a \) holds above then \( B_{i_2} \subseteq (j_1, j_2) \) and \( B_{i_1} \cap (j_1, j_2) = \emptyset \), and if \( b \) holds then \( B_{i_1} \subseteq (j_1, j_2) \) and \( B_{i_2} \cap (j_1, j_2) = \emptyset \). As \( p \) is order-reversing, so \( p(i_{r}) < p(i) \), we see that \( p(i_{r}) \in B_{i_1} \) and \( p(i) \in B_{i_2} \), and then that \( i_1 < p(i) \) and \( p(i_{r}) < i_2 \). These last two inequalities imply that \( p(i_{r}) \leq i_1 < i_2 \leq p(i) \). Also, since \( p \) satisfies Condition \( 1 \) of Theorem 4.17, we observe that necessarily \( i_1 = p(i_{r}) \).

Otherwise, as both \( i_1 \) and \( p(i_{r}) \) belong to the same block \( B_{i_1} \) of \( \pi_{r-1} \) and \( p(i_{r}) \leq i_1 \), then either \( (p^{-1}(l), i_{r}) \) is an edge of \( T \) for some \( l \in B_{i_1} \) with \( p(i_{r}) < l \leq i_1 \) and \( p^{-1}(l) < i \) (which cannot hold since \( i_1 < p(i) \)), or there exist \( k, l \in B_{i_1} \) with \( k < p(i_{r}) < l \leq i_1 < p(i) \) such that \( (p^{-1}(l), p^{-1}(k)) \) is an edge of \( T \) (which cannot hold because that edge crosses \( (i, i_{r}) \) in any depiction \((D, p)\) of \( T \)). More easily, since \( i_2 \leq p(i) \) and there are no edges of the form \((i, l)\) in \( T \) except for \((i, i_{r})\), we must in fact have that \( i_2 = p(i) \). It is now clear that if \( a \) or \( b \) holds above with \( i_1 = p(i_{r}) \) and \( i_2 = p(i) \), then in any depiction \((D, p)\) of \( T \) we may find an edge of \( T \) that crosses \((i, i_{r})\), which is impossible.

\( \Box \)

**Theorem 4.22.** Let \( K_{[n]} \) be the complete graph on \([n] \), \( T \) be an \( r \)-rooted spanning tree of \( (K_{[n]}), \) and \( C = \{ \pi_0 <_{ref} \pi_1 <_{ref} \ldots <_{ref} \pi_n \} \) a maximal chain of \( NC((0, n)) \). Then, using the notation and functions of Definitions 4.18,4.20 we have that \( T_{(C_T)} = T \) and \( C_{(T_C)} = C \). Hence, the non-crossing trees obtained from the spanning trees of \((K_{[n]}), \) interpolate in a bijection between rooted spanning forests of \( K_{[n]} \) and maximal chains of the non-crossing partitions lattice \( NC((0, n)) \): Every non-crossing tree corresponds bijectively to an element of each of these two combinatorial sets.

**Proof.** This is clear from the proofs of Propositions 4.19,4.21 through the following simple observations.

Firstly, the edges of \( T_C \) correspond to the cover relations in \( C \) so that an edge \((i, i_{r})\) with \( i \in [n] \) exists in \( T_C \) for every minimal element \( p_C(i) \) in its block of \( \pi_{r-1} \) that stops being minimal in its block of \( \pi_i \); the number \( i_{r} \) is then recollected by requiring that \( p_C(i_{r}) \) is the immediate predecessor of \( p_C(i) \) in the newly formed block of \( \pi_i \). Nextly, for all \( i \in [n] \), the \( i \)-th cover relation in \( C_{(T_C)} \) corresponds to taking the union of the block that contains \( p_C(i) \) and \( p_C(i_{r}) \). Therefore, \( C = C_{(T_C)} \).

Secondly, the \( i \)-th cover relation in \( C_{(T_C)} \), \( i \in [n] \), corresponds to taking the union of the (disjoint) blocks that contain \( p(i) \) and \( p(i_{r}) \), where \((i, i_{r})\) is an edge of \( T \) (and from the second part of the proof of Proposition 4.21) \( p(i) \) was minimal in its initial block and \( p(i_{r}) \) immediately precedes \( p(i) \) in the newly formed block). But then, the edges of \( T_{(C_{T_C})} \) are given by all the \((i, i_{r})\). Hence, \( T_{(C_{T_C})} = T \).

\( \Box \)

**Example 4.23.** Table 5Eii shows an example of the bijection of Theorem 4.22 presenting the maximal chain of \( NC((0, 7)) \) corresponding to the spanning tree \( T \) of \( G_r \) of Example 4.8 (top to bottom of table, blocks separated by commas). Let us discuss how this list of non-crossing partitions can be calculated from Figure 5Dv. We will inductively define a set of graphs \( G_0, G_1, \ldots, G_7 \), each on vertex set \([0, 7] = \{0, 1, \ldots, 7\}\) and with edge sets \( E_0, E_1, \ldots, E_7 \), respectively. Initially, \( G_0 \) has no edges, so \( E_0 = \emptyset \). Suppose then that we have defined \( G_i \) and \( E_i \) with \( i \leq 7 \), and that we want to define \( G_i \) and \( E_i \). We find cusp \( i \) (in black) in Figure 5Dv
and note that, to this cusp, there is exactly one red arc adjacent to the left. This arc is also adjacent to cusp \( i_e \) (in black). Let us then read off the blue labellings of cusps \( i \) and \( i_e \) in Figure 5D, and say that these are \( p(i) \) and \( p(i_e) \). Then, writing \( e := \{p(i), p(i_e)\} \), we let \( E_i = E_{i-1} \cup \{e\} \) and update \( G_i \) accordingly. We stop when \( G_7 \) is defined. Notably, \( G_7 \) is a spanning tree. Non-crossing partitions of Table 5E are then, in order, given by the connected components of the spanning forests \( G_0, G_1, \ldots, G_7 \).

**Corollary 4.24** (Germain Kreweras). The number of maximal chains in \( \text{NC}([0,n]) \) is \((n+1)^{n-1}\).

**Corollary 4.25.** We have that:

\[
(n+1)^{n-1} = \sum_{\{B_1, \ldots, B_m\} \in \text{NC}([n])} \left( |B_1|! \cdot |B_2|! \cdots |B_m|! \right).
\]

Therefore, using Speicher’s exponential formula for \( \text{NC}([n]) \) \([\text{Speicher [1994]}]\), we obtain the classic result:

\[
\sum_{n=1}^{\infty} n^{n-1} \frac{x^n}{n!} = \left( \frac{x}{e^x} \right)^{(-1)}.
\]

**Proof.** For each \( \{B_1, \ldots, B_m\} \in \text{NC}([n]) \), where \( b_1 < b_2 < \cdots < b_m \) are respectively the minimal elements of \( B_1, B_2, \ldots, B_m \), and for each bijection \( f : [n] \to [n] \) such that \( f \) is strictly decreasing on each block \( B_i, i \in [m] \), we can define an \( r \)-rooted spanning tree \( T \) of \( (K_{[n]}, \rho) \) by taking \( E(T) = \{(f(i), r) : i \in B_1\} \cup \{(f(i), f(b_k - 1)) : i \in B_k \text{ with } k > 1\} \). If we let \( p(r) = 0 \) and \( p(i) = f^{-1}(i) \) for all \( i \in [n] \) above, we can readily check that \( p \) is the depiction function of Theorem 4.7 associated to \( T \).

Conversely, given an \( r \)-rooted spanning tree \( T \) with depiction function \( p \) as in Theorem 4.7, the partition \( \cup_{k \in [n]} \{p(i) \in [n] : (i, k) \in T\} \) is an element of \( \text{NC}([n]) \).

Hence, since given a partition \( \{B_1, \ldots, B_m\} \in \text{NC}([n]) \), there are \((|B_1|! \cdot |B_2|! \cdots |B_m|!\) choices for \( f \) above, the result follows.

\[ \square \]

5. Applications.

5.1. Random Acyclic Orientations of a Simple Graph: Markov Chains.

**Definition 5.1.** Let \( G = (V, E) \) be a connected (finite) simple graph. A simple random walk on \( G \) is a Markov chain \( (v_t)_{t=0,1,2,\ldots} \) obtained by selecting an initial vertex \( v_0 \in V \), and then for all \( t \geq 1 \), selecting \( v_t \in V \) from a uniform distribution on the set \( N_G(v_{t-1}) \). If \( P \) is the Markov transition matrix for a simple random walk on \( G \), then for \( u, v \in V \):

\[
(P)_{uv} = p_{uv} = \begin{cases} 
\frac{1}{\deg(u)} & \text{if } v \in N_G(u), \\
0 & \text{otherwise}.
\end{cases}
\]

**Theorem 5.2.** The Markov chain of Definition 5.1 is always irreducible. Furthermore, it is aperiodic if and only if \( G \) is not bipartite.
If for all \( w \in V \), we let \( \pi_w := \frac{d_G(w)}{2|E|} \), then for any pair of vertices \( u, v \in V \), we have that:

\[
\pi_u p_{uv} = \pi_u p_{vu}.
\]

Consequently, since \( \sum_{v \in V} \pi_v = 1 \), random walks on \( G \) are reversible and they have a unique stationary distribution given by \( \pi = (\pi_v)_{v \in V} \), so that:

\[
(5.1) \quad \pi_v = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \Pr[v_t = v], \text{ for all } v \in V.
\]

Moreover, if \( G \) is not bipartite, then:

\[
(5.2) \quad \pi_v = \lim_{t \to \infty} \Pr[v_t = v], \text{ for all } v \in V.
\]

**Definition 5.3** (Card-Shuffling Markov Chain, see also [Athanasiadis and Diaconis (2010)]). Let \( G = G(V, E) \) be a simple graph with \( |V| = n \geq 3 \), and select an arbitrary bijective map \( f_0 : V \to [n] \) (regarded as a labelling of \( V \)). Let us consider a sequence \( (f_t)_{t=0,1,2,...} \) of bijective maps \( V \to [n] \) such that for \( t \geq 1 \), \( f_t \) is obtained from \( f_{t-1} \) through the following random process: Let \( v_t \in V \) be chosen uniformly at random, and let,

\[
f_t(v) = \begin{cases} 
    n & \text{if } v = v_t, \\
    f_{t-1}(v) - 1 & \text{if } f_{t-1}(v) > f_{t-1}(v_t), \\
    f_{t-1}(v) & \text{otherwise}.
\end{cases}
\]

Consider now the sequence of acyclic orientations \( (O_t)_{t=0,1,2,...} \) of \( G \) induced by the labelings \( (f_t)_{t=0,1,2,...} \) so that for all \( e = \{u,v\} \in E \) and \( t \geq 0 \), we have that \( O_t(e) = (u,v) \) if and only if \( f_t(u) < f_t(v) \). The sequence \( (O_t)_{t=0,1,2,...} \) is called the Card-shuffling (CS) Markov chain on the set of acyclic orientations of \( G \).

Equivalently, we can define this Markov chain by selecting an arbitrary acyclic orientation \( O_0 \) of \( G \), and then for each \( t \geq 1 \), letting \( O_t \) be obtained from \( O_{t-1} \) by selecting \( v_t \in V \) uniformly at random and taking, for all \( e \in E \):

\[
O_t(e) = \begin{cases} 
    O_{t-1}(e) & \text{if } v_t \notin e, \\
    (v,v_t) & \text{if } e = \{v,v_t\}.
\end{cases}
\]

**Theorem 5.4.** The Card-Shuffling Markov chain of \( G \) in Definition 5.3 is a well-defined, irreducible and aperiodic Markov chain on state space equal to the set of all acyclic orientations of \( G \); its unique stationary distribution \( \pi^{CS} \) is given by:

\[
\pi^{CS}_O = \frac{e(O)}{n}, \text{ for all acyclic orientations } O \text{ of } G,
\]

where \( e(\cdot) \) denotes the number of linear extensions of the induced poset \( (V, \leq_O) \).

**Proof.** If we consider instead the Markov chain \( (f_t)_{t=0,1,2,...} \), whose set of states is the set of all bijections \( V \to [n] \), it is not difficult to observe that this Markov chain is irreducible and aperiodic (see below), and hence that it has a unique stationary distribution \( \pi \) satisfying Equations 5.2. By the symmetry of the set of all bijective labelings \( V \to [n] \), or simply by direct inspection of the stationary equations for this Markov chain (since every state can be accessed in one step from exactly \( n \) different states and each one of these transitions occurs with probability \( \frac{1}{n} \)), we obtain that \( \pi_f = \frac{1}{n} \) for all bijective maps \( f : V \to [n] \). Hence, by the aforementioned construction of the Card-Shuffling (CS) Markov chain of \( G \) from bijective labelings of \( V \), we must have that this CS chain is also irreducible (since each labelling is accessible from every other labelling, hence each acyclic orientation...
from every other acyclic orientation), aperiodic (since both \( \Pr[f_t = f_{t-1}] > 0 \) and \( \Pr[O_t = O_{t-1} | O_{t-1}] > 0 \) for all \( t \geq 1 \)), and has a unique stationary distribution \( \pi_{CS}^O \), necessarily then given by \( \pi_{CS}^O = \frac{e(O)}{n!} \) for every acyclic orientation \( O \) of \( G \), from Equations 5.1.

**Definition 5.5** (Edge-Label-Reversal Stochastic Process). Let \( G = G(V,E) \) be a connected simple graph with \( |V| = n \), and select an arbitrary bijective map \( f_0 : V \to [n] \) (regarded as a labelling of \( V \)). Let us consider a sequence \( (f_t)_{t=0,1,2,...} \) of bijective maps \( V \to [n] \) such that for \( t \geq 1 \), \( f_t \) is obtained from \( f_{t-1} \) through the following random process: Let \( e_t = \{u_t,v_t\} \in E \) be chosen uniformly at random from this set, and let,

\[
\begin{align*}
    f_t(v) &= \begin{cases} 
    f_{t-1}(u_t) & \text{if } v = v_t, \\
    f_{t-1}(v_t) & \text{if } v = u_t, \\
    f_{t-1}(w) & \text{otherwise.}
    \end{cases}
\end{align*}
\]

Consider now the sequence of acyclic orientations \( (O_t)_{t=0,1,2,...} \) of \( G \) induced by the labellings \( (f_t)_{t=0,1,2,...} \); so that for all \( e = \{u,v\} \in E \) and \( t \geq 0 \), we have that \( O_t(e) = \{u,v\} \) if and only if \( f_t(u) < f_t(v) \). The sequence \( (O_t)_{t=0,1,2,...} \) is called the Edge-Label-Reversal (ELR) stochastic process on the set of acyclic orientations of \( G \).

**Theorem 5.6.** The Edge-Label-Reversal stochastic process of \( G \) in Definition 5.5 satisfies that, for every acyclic orientation \( O \) of \( G \):

\[
\pi_{CS}^{O} = \pi_{O}^{ELR} := \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \Pr[O_t = O] = \frac{e(O)}{n!},
\]

where \( e(O) \) denotes the number of linear extensions of the induced poset \((V,\leq_O)\), and this result holds independently of the initial choice of \( O_0 \).

**Proof.** Consider the simple graph \( H \) on vertex set equal to the set of all bijective maps \( V \to [n] \), and where two maps \( f \) and \( g \) are connected by an edge if and only if there exists \( \{u,v\} \in E \) such that \( f(u) = g(v), f(v) = g(u), \) and \( f(w) = g(w) \) for all \( w \in V \setminus \{u,v\} \). Since \( G \) is connected, a standard result in the algebraic theory of the symmetric group shows that \( H \) is connected, e.g. consider a spanning tree \( T \) of \( G \); then, any permutation in \( \mathcal{S}_V \) can be written as a product of transpositions of the form \( uv \) with \( \{u,v\} \in E(T) \). Moreover, by considering the parity of permutations in \( \mathcal{S}_V \), we observe that \( H \) is bipartite. Now, the sequence \( (f_t)_{t=0,1,2,...} \) of Definition 5.5 is precisely a simple random walk on \( H \), and the degree of each bijective map \( f : V \to [n] \) in \( H \) is clearly \( |E| \), so the stationary distribution for this Markov chain is uniform. Necessarily then, the result follows from the construction of \( (O_t)_{t=0,1,2,...} \) and Equations 5.1.

**Definition 5.7** (Sliding-(n+1) Stochastic Process). Let \( G = G(V,E) \) be a connected simple graph with \( |V| = n \), and consider the graph \( G_r \). Let us select an arbitrary bijective map \( f_0 : V \cup \{r\} \to [n+1] \), which we regard as a labelling of the vertices of \( G_r \), and define a sequence \( (f_t)_{t=0,1,2,...} \) of bijective maps \( V \cup \{r\} \to [n+1] \) such that for \( t \geq 1 \), \( f_t \) is obtained from \( f_{t-1} \) through the following random process: Let \( v_{t-1} \in V \cup \{r\} \) be such that \( f_{t-1}(v_{t-1}) = n+1 \), and select \( v_t \in N_{G_r}(v_{t-1}) \).
uniformly at random from this set. Then,

\[ f_t(v) = \begin{cases} 
  n + 1 & \text{if } v = v_t, \\
  f_{t-1}(v) & \text{if } v = v_{t-1}, \\
  f_{t-1}(v) & \text{otherwise.}
\end{cases} \]

Consider now the sequence of acyclic orientations \((O_t)_{t=0,1,2,...}\) of \(G_r\) induced by the labellings \((f_t)_{t=0,1,2,...}\), so that for all \(e = \{u,v\} \in E(G_r)\) and \(t \geq 0\), we have that \(O_t(e) = \{u,v\}\) if and only if \(f_t(u) < f_t(v)\). The sequence \((O_t)_{t=0,1,2,...}\) is called the Sliding-\((n+1)\) (SL) stochastic process on the set of acyclic orientations of \(G_r\).

**Theorem 5.8.** The Sliding-\((n+1)\) stochastic process of \(G_r\) of Definition 5.7 satisfies that, if \(S_r\) is the set of all acyclic orientations of \(G_r\) whose unique maximal element is \(r\), then \(\sum_{t=1}^\infty \Pr[O_t \in S_r] = \infty\) and for every \(O \in S_r\):

\[ (\pi^{sl}_O)_t = \pi^{sl}_O := \lim_{N \to \infty} \frac{\sum_{t=1}^N \Pr[O_t = O]}{\sum_{t=1}^N \Pr[O_t = O]} = \frac{e(O|_V)}{n!}, \]

where \(O|_V\) is the restriction of \(O\) to \(E\) (hence an acyclic orientation of \(G\)) and \(e(O|_V)\) denotes the number of linear extensions of the induced poset \((V, \leq_O|_V)\).

These results hold independently of the initial choice of \(O_0\).

**Proof.** Consider the simple graph \(H\) on vertex set equal to the set of all bijective maps \(V \uplus \{r\} \to [n+1]\), and where two maps \(f\) and \(g\) are connected by an edge if and only if there exists \( \{u, v\} \in E(G_r)\) such that \(f(u) = g(v) = n + 1\), \(f(v) = g(u)\), and \(f(w) = g(w)\) for all \(w \in V \setminus \{u,v\}\). If two bijective maps \(f, g : V \uplus \{r\} \to [n+1]\) differ only in one edge of \(G_r\), so that \(f(u) = g(v) \neq n + 1\) and \(f(v) = g(u) \neq n + 1\) for some \( \{u, v\} \in E(G_r)\), but \(f(w) = g(w)\) for all \(w \in V \setminus \{u,v\}\), then we can easily but somewhat tediously show that \(f\) and \(g\) belong to the same connected component of \(H\), making use of the facts that vertex \(r\) is adjacent to all other vertices of \(G_r\) and that \(G\) is connected. But then, the proof of Theorem 5.6 shows that \(H\) is a connected graph. Now, the sequence \((f_t)_{t=0,1,2,...}\) of Definition 5.7 is a simple random walk on \(H\), and the degree of a bijective map \(f : V \to [n]\) in \(H\) is clearly \(d_{G_r}(v_f)\), where \(v_f \in V\) depends on \(f\) and is such that \(f(v_f) = n + 1\), so the stationary distribution \(\pi\) for this Markov chain satisfies that \(\pi_f = c \cdot d_{G_r}(v_f)\), for some fixed normalization constant \(c \in \mathbb{R}_+\). The vertices of \(H\) that induce acyclic orientations of \(G_r\) from the set \(S_r\) are exactly the bijective maps \(f : V \uplus \{r\} \to [n+1]\) such that \(f(r) = n + 1\), and for these we have that \(\pi_f = c \cdot n\). The result then follows from the construction of \((O_t)_{t=0,1,2,...}\) and from Equations 5.1.

**Definition 5.9** (Cover-Reversal Random Walk). Let \(G = G(V, E)\) be a simple graph with \(|V| = n\), and select an arbitrary acyclic orientation \(O_0\) of \(G\). Let us consider a sequence \((O_t)_{t=0,1,2,...}\) of acyclic orientations of \(G\) such that for \(t \geq 1\), \(O_t\) is obtained from \(O_{t-1}\) through the following random process: (Let \(u, v\) be selected uniformly at random from the set,

\[ \text{Cov}(O_{t-1}) := \{ e \in O_{t-1}[E] : e \text{ represents a cover relation in } (V, \leq_{O_{t-1}}) \}, \]

and for all \(e \in E\), let,

\[ O_t(e) = \begin{cases} 
  (v, u) & \text{if } e = \{u,v\}, \\
  O_{t-1}(e) & \text{otherwise.}
\end{cases} \]
The sequence \( (O_i)_{i=0,1,2,...} \) is called the Cover-Reversal (CR) random walk on the set of acyclic orientations of \( G \).

**Theorem 5.10.** The Cover-Reversal random walk in \( G \) of Definition 5.2 is a simple 2-period random walk on the 1-skeleton of the clean graphical zonotope \( Z_G \) of Theorem 2.15 (hence, on a particular simple connected bipartite graph on vertex set equal to the set of all acyclic orientations of \( G \)), and its stationary distribution \( \pi^{\text{CR}} \) satisfies that, for every acyclic orientation \( O \) of \( G \):

\[
\pi^{\text{CR}}_O = c \cdot |\text{Cov}(O)|,
\]

where \( c \in \mathbb{R}_+ \) is a normalization constant independent of \( O \).

**Proof.** From the proof of Theorem 2.15, the edges of \( Z_G \) are in bijection with the set of all p.a.o.’s \( O \) of \( G \) such that if \( \Sigma \) is the connected partition associated to \( O \), then \( |\Sigma| = n - 1 \). Hence, the edges of \( Z_G \) are in bijection with the set of all pairs of the form \( (e, O) \), where \( e \in E \) and \( O \) is an acyclic orientation of the graph \( G/e \), obtained from \( G \) by contraction of the edge \( e \). The two vertices of \( Z_G \) adjacent to an edge corresponding to a \((e, O)\) with \( e = \{u, v\} \) are, respectively, obtained from the acyclic orientations \( O_1 \) and \( O_2 \) of \( G \) such that \( O_1(e) = (u, v) \), \( O_2(e) = (v, u) \), and such that \( O_1|_{E \setminus e} = O_2|_{E \setminus e} \) are naturally induced by \( O \) (e.g. see Definition 2.4). Necessarily then, both \( (u, v) \) and \( (v, u) \) correspond respectively to cover relations in the posets \( (V, \leq_{O_1}) \) and \( (V, \leq_{O_2}) \), since otherwise the orientation \( O \) of \( G/e \) would not be acyclic.

On the other hand, given an acyclic orientation \( O_1 \) of \( G \) and an edge \( (u, v) \in O_1[E] \) such that \( v \) covers \( u \) in \( (V, \leq_{O_1}) \), then, reversing the orientation of (only) that edge in \( O_1 \) yields a new acyclic orientation \( O_2 \) of \( G \), so \( (v, u) \in O_2[E] \). Otherwise, using a directed cycle formed by edges from \( O_2[E] \), which must then include the edge \( (v, u) \), we observe that the relation \( u \leq_{O_1} v \) is a consequence of other order relations in \( (V, \leq_{O_1}) \) and \( v \) does not cover \( u \) there. This is a contradiction, and it furthermore implies that both \( O_1 \) and \( O_2 \) naturally induce a well-defined acyclic orientation \( O \) of \( G/\{u, v\} \).

Hence, the Cover-Reversal random walk of \( G \) corresponds to a simple random walk on the 1-skeleton of \( Z_G \) (or of \( Z_G^{\text{clean}} \)) and the result follows now from Theorem 5.2, since this graph is connected and bipartite, clearly.

\( \square \)

**Remark 5.11.** Variants of the Cover-Reversal random walk on \( G \), obtained for example by flipping biased coins at each step, can be used to obtain stochastic processes that converge to a uniform distribution on the set of acyclic orientations of \( G \). However, these variants are clearly not very illuminating or efficient.

**Definition 5.12 (Interval-Reversal Random Walk).** Let \( G = (V, E) \) be a simple graph with \( |V| = n \), and select an arbitrary acyclic orientation \( O_0 \) of \( G \). Let us consider a sequence \( (O_i)_{i=0,1,2,...} \) of acyclic orientations of \( G \) such that for \( t \geq 1 \), \( O_t \) is obtained from \( O_{t-1} \) through the following random process: Let \( \{(u, v) \in E \} \) be selected uniformly at random from this set, with \( (u, v) \in O_{t-1}[E] \), and for all \( e = \{x, y\} \in E \) with \( (x, y) \in O_{t-1}[E] \), let,

\[
O_t(e) = \begin{cases} (y, x) & \text{if } u \leq_{O_{t-1}} x < y < v \leq_{O_{t-1}} v, \\
(x, y) = O_{t-1}(e) & \text{otherwise.}
\end{cases}
\]

The sequence \( (O_i)_{i=0,1,2,...} \) is called the Interval-Reversal (IR) random walk on the set of acyclic orientations of \( G \).
Lemma 5.13. Let \( G = G(V, E) \) be a simple graph and let \( O \) be any given acyclic orientation of \( G \). For an arbitrary edge \( \{u, v\} \in E \), say with \( (u, v) \in O[E] \), let us define a new orientation \( O_{(u,v)} \) of \( G \) by requiring that, for all \( e = \{x, y\} \in E \) with \( (x, y) \in O[E] \):

\[
O_{(u,v)}(e) = \begin{cases} 
(x, y) & \text{if } u \leq_O x <_O y \leq_O v, \\
(y, x) & \text{otherwise}.
\end{cases}
\]

Then, \( O_{(u,v)} \) is also an acyclic orientation of \( G \) and, furthermore, \( (O_{(u,v)})^{-1}_{(u,v)} = O \).

Additionally, for any choice of \( e_1, e_2 \in E \), we have that \( O_{e_1} = O_{e_2} \) if and only if \( e_1 = e_2 \).

**Proof.** Suppose on the contrary that \( O_{(u,v)} \) is not an acyclic orientation of \( G \). Then, there exists at least one directed cycle \( C \subseteq O_{(u,v)}[E] \) that has the following form:

For \( E^O_{(u,v)} = \{\{x, y\} \in E : u \leq_O x <_O y \leq_O v\} \), there exists \( k \in \mathbb{P} \) and pairwise disjoint non-empty sets,

\[
P_1, Q_1, P_2, Q_2, \ldots, P_k, Q_k \subseteq O_{(u,v)}[E] \quad \text{with} \quad C = \bigcup_{i=1}^k (P_i \cup Q_i),
\]

such that for all \( i \in [k] \),

\[
P_i = \{(p^i_{j-1}, p^i_j)\}_{j=1, \ldots, |P_i|} \subseteq O_{(u,v)}[E^O_{(u,v)}],
\]

\[
Q_i = \{(q^i_{j-1}, q^i_j)\}_{j=1, \ldots, |Q_i|} \subseteq O_{(u,v)}[E \setminus (O_{(u,v)}[E^O_{(u,v)}])],
\]

\[
p^i_{|P_i|} = q^i_{|Q_i|} = p^{i+1}_0, \quad \text{where } p^{i+1}_0 := p^0_0.
\]

This is true simply because any directed cycle in \( O_{(u,v)}[E] \) must necessarily involve edges from both \( E^O_{(u,v)} \) and \( E \setminus E^O_{(u,v)} \). Since 1) \( O \) and \( O_{(u,v)} \) agree on \( E \setminus E^O_{(u,v)} \); 2) \( u \leq_O p^0_1 \leq_O p^0_2 \leq_O v \), and 3) \( q^1_0 = p^1_1 \), \( q^1_{|Q_1|} = p^2_0 \), then \( u \leq_O q^0_0 \leq_O q^0_1 \leq_O \cdots \leq_O q^1_{|Q_1|} \leq_O y \), so in particular \( \{p^0_0, q^1_0\} \) is an acyclic orientation of \( G \).

To prove that \( (O_{(u,v)})^{-1}_{(u,v)} = O \), it suffices to check that if for some \( \{x, y\} \in E \) with \( (y, x) \in O_{(u,v)}[E] \), then in fact \( u \leq_O x <_O y \leq_O v \).

Somewhat analogously with the previous argument, suppose on the contrary that there exists some \( \{x, y\} \in E \) with \( (y, x) \in O_{(u,v)}[E] \) for which the condition fails to hold. Then, inside any directed path \( P = \{(p_{j-1}, p_j)\}_{j=1, \ldots, |P|} \subseteq O_{(u,v)}[E] \) such that \( (y, x) \in P, \ p_0 = v, \) and \( p_{|P|} = u \), there must exist a maximal (by containment) sub-path \( Q = \{(q_{j-1}, q_j)\}_{j=1, \ldots, |Q|} \subseteq P \) such that \( (y, x) \in Q \subseteq O_{(u,v)}[E \setminus (O_{(u,v)}[E^O_{(u,v)}])] \). Necessarily then, \( u \leq_O q_0 <_O q_{|Q|} \leq_O v \), so \( u \leq_O q_0 \leq_O y <_O x \leq_O q_{|Q|} \leq_O v \), and hence \( \{y, x\} \in E^O_{(u,v)} \). This is a contradiction.

The last statement is a simple consequence of observing that, for every choice of \( \{u, v\} \in E \), \( u \) and \( v \) determine a unique interval inside each of the posets \( (V, \leq_O) \), where \( O \) is an acyclic orientation of \( G \): A non-empty closed interval of a finite poset is uniquely determined by its maximal and minimal elements.

□

**Proposition 5.14.** In Lemma 5.13, consider the simple graph \( AO^\text{int}_G \) on vertex set equal to the set of all acyclic orientations of \( G \), and in which two acyclic orientations \( O_1 \) and \( O_2 \) of \( G \) are connected by an edge, if and only if there exists \( \{u, v\} \in E \) such that \( (O_1)_{(u,v)} = O_2 \). Then, \( AO^\text{int}_G \) is an \( |E| \)-regular connected graph.
Proof. Firstly, let us note that \( AO_G^{\text{inter}} \) is indeed a well-defined simple graph (so it does not have loops or multiple edges) per the three main statements of Lemma 5.13. Now, we point out that \( AO_G^{\text{inter}} \) contains as a spanning sub-graph the 1-skeleton of the (clean) graphical zonotope \( Z_G \) since, colloquially, all \textit{cover-reversals} are also \textit{interval-reversals}. Hence, since the later graph has been observed to be connected in the proof of Theorem 5.10, then \( AO_G^{\text{inter}} \) is also connected. Every vertex of this graph must have degree \(|E|\), clearly.

\[
\pi_{\text{IR}}^O = \frac{1}{|\chi_G(-1)|},
\]

where \(|\chi_G(-1)|\) is the number of acyclic orientations of \( G \) [Stanley (1973)].

Proof. That the Interval-Reversal random walk of \( G \) corresponds to a simple random walk on \( AO_G^{\text{inter}} \) is a direct consequence of Lemma 5.13. That \( AO_G^{\text{inter}} \) is connected and \(|E|\)-regular is the content of Proposition 5.14, so we can now rely on Theorem 5.2 to obtain the result.

5.2. Acyclic Orientations of a Random Graph.

This short subsection is aimed at proving a surprising formula for the expected number of acyclic orientations of an Erdős-Rényi random graph from \( G_{n,p} \), with.

\[
\beta_{\text{IR}}^{\text{inter}} = \frac{1}{|\chi_G(-1)|},
\]

where \(|\chi_G(-1)|\) is the number of acyclic orientations of \( G \) [Stanley (1973)].
As we wanted.

**Definition 5.16.** Let \( n \in \mathbb{P} \). A parking function of \([n]\) is a vector \( a \in \mathbb{N}^{[n]} \) such that for any \( r \in \mathbb{S}_n \) with \( a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots \leq a_{\sigma(n)} \), we have that \( a_{\sigma(i)} < i \) for all \( i \in [n] \). The set of all parking functions of \([n]\) will be denoted by \( \text{Park}_{[n]} \).

For \( a \in \mathbb{N}^{[n]} \), let us write \( \text{Area}(a) := a_1 + a_2 + \cdots + a_n \) and \( \text{supp}(a) := \{ i \in [n] : a_i > 0 \} \).

**Theorem 5.17.** Let \( n \in \mathbb{P} \), \( p \in (0, 1) \), and \( G \sim G_{[n],p} \). Write \( q := 1 - p \). If \( |\chi_G(-1)| \) is the number of acyclic orientations of \( G \), and we let \( S := k[x_1, \ldots, x_n]/T_{K_{[n]}} \), where as usual \( K_{[n]} \) denotes the complete graph on vertex set \([n]\), then we have:

\[
(5.3) \quad \mathbb{E}[|\chi_G(-1)|] = q^{\binom{n}{2}} \cdot \sum_{a \in \text{Park}_{[n]}} \left( \frac{1}{q} \right)^{\text{Area}(a)} |p|^{\text{supp}(a)}.
\]

**Proof.** We make use of Proposition 4.15. In general, for any simple graph \( H \) on vertex set \([n]\) (as \( G \) here and the complete graph \( K_{[n]} \)), we will let \( T^H \) be the set of all \( r \)-rooted spanning trees of \( H \). Now, for \( T \in T^R_{K_{[n]}} \), we will say that \( T \) is **useful** if \( T \in T^G \) and its unique depiction function \( p \) of Theorem 4.7 satisfies the conditions of Proposition 4.15. Then:

\[
\mathbb{E}[|\chi_G(-1)|] = \sum_{T \in T^R_{K_{[n]}}} \text{Pr}[T \text{ is useful}]
\]

\[
= \sum_{T \in T^R_{K_{[n]}}} \left( \text{Pr}[T \in T^G] \right) \cdot \left( \text{Pr}[\{ i, j \} \notin E(G_r) \text{ for all } i, j \in [n], \quad (i, i_r) \in E(T), \quad p(i_r) < p(j) < p(i) ] \right)
\]

\[
= \sum_{T \in T^R_{K_{[n]}}} \frac{p^n}{p^{d_T(r)}} \cdot \prod_{i \in [n]} \text{Pr}[\{ i, j \} \notin E(G_r) \text{ for all } j \in [n], p(i_r) < p(j) < p(i) ]
\]

\[
= q^{\binom{n}{2}} \cdot \sum_{T \in T^R_{K_{[n]}}} \left( \frac{1}{q} \right)^{\sum_{i \in [n]} p(i_r)} p^{[i \in [n]: a_i > 0]} |p|^{\text{supp}(a)}
\]

as we wanted.

\( \Box \)
5.3. **k-Neighbor Bootstrap Percolation.**

**Definition 5.18.** Let $G = G([n], E)$ be a finite simple graph, $k \in \mathbb{P}$, and $A \subseteq [n]$. The $k$-neighbor bootstrap percolation on $G$ with initial set $A$, is the process 
\[ A_t = A_{t-1} \cup \{i \in [n] : |N_G(i) \cap A_{t-1}| \geq k\} \] 
for all $t \geq 1$. The closure of $A$ is the set $cl(A) := \cup_{t \geq 0} A_t$, and we say that $A$ percolates in $G$ if $cl(A) = [n]$.

**Question 5.19.** Given a graph $G$ as in Definition 5.18, what is minimal size $|A|$ of $A \subseteq [n]$ such that $A$ percolates in $G$?

**Definition 5.20.** For fixed $G$ and $k$ as in Definition 5.18, let $C_{(G,k)} := \{\sigma \subseteq [n] : \operatorname{outof}_{G,\sigma}(i) < k \text{ for all } i \in \sigma\}$. The $k$-bootstrap percolation ideal $B_{C_{(G,k)}}$ of $G$ is the square-free monomial ideal of $k[x_1,\ldots,x_n]$ generated as:
\[ B_{C_{(G,k)}} = \left\langle \prod_{i \in \sigma} x_i : \sigma \in C_{(G,k)} \right\rangle. \]

**Proposition 5.21.** In Definitions 5.18, 5.20 the function that associates to each standard monomial $x^b \not\in B_{C_{(G,k)}}$, $b \in \mathbb{N}^n$, the set of vertices \( \{i \in [n] : b_i = 0\} \) of $G$, restricts to a bijection between the set of all square-free standard monomials of $B_{C_{(G,k)}}$ and the set of all $A \subseteq [n]$ such that $A$ percolates in $G$.

Colloquially, the percolating sets of $G$ are in bijection with the supporting sets of standard monomials of the ideal $B_{C_{(G,k)}}$.

**Proof.** Let $A \subseteq [n]$ be such that $cl(A) \subseteq [n]$, and consider the set $\sigma := [n] \setminus cl(A)$. Necessarily, every element of $\sigma$ must have fewer than $k$ neighbors inside $cl(A)$, so $\operatorname{outof}_{G,\sigma}(i) < k$, for all $i \in \sigma$. This implies that $\sigma \in C_{(G,k)}$, and $x^\sigma := \prod_{i \in \sigma} x_i \in B_{C_{(G,k)}}$. But then, since $\sigma \subseteq \hat{A} := [n] \setminus A$, we have that $x^\sigma|x^{\hat{A}} := \prod_{i \in \hat{A}} x_i$, so $x^{\hat{A}} \in B_{C_{(G,k)}}$ as well.

On the contrary, if $x^{\hat{A}} \in B_{C_{(G,k)}}$ for some $A \subseteq [n]$, there must exist some $\sigma \in C_{(G,k)}$ such that $x^\sigma|x^{\hat{A}}$. Necessarily then, $cl(A) \subseteq [n]\setminus \sigma$, since it is never possible to percolate the elements of $\sigma$ during a $k$-bootstrap percolation on $G$ from an initial set disjoint from $\sigma$, as $A$ here.

**References**

D. Aldous and J. Fill. Reversible markov chains and random walks on graphs, 2002.

D. J. Aldous. The random walk construction of uniform spanning trees and uniform labelled trees. *SIAM Journal on Discrete Mathematics*, 3(4):450–465, 1990.

D. Armstrong, B. Rhoades, and N. Williams. Rational associahedra and noncrossing partitions. *arXiv preprint arXiv:1305.7286*, 2013.

C. A. Athanasiadis and P. Diaconis. Functions of random walks on hyperplane arrangements. *Advances in Applied Mathematics*, 45(3):410–437, 2010.

J. Balogh, B. Bollobás, and R. Morris. Bootstrap percolation in three dimensions. *The Annals of Probability*, pages 1329–1380, 2009.

D. Bayer and B. Sturmfels. Cellular resolutions of monomial modules. In *J. reine angew. Math.* Citeseer, 1998.

M. Beck and S. Robins. *Computing the continuous discretely: Integer-point enumeration in polyhedra*. Springer, 2007.
A. Broder. Generating random spanning trees. In *Foundations of Computer Science, 1989., 30th Annual Symposium on*, pages 442–447. IEEE, 1989.

D. Chebikin and P. Pylyavskyy. A family of bijections between g-parking functions and spanning trees. *Journal of Combinatorial Theory, Series A*, 110(1):31–41, 2005.

H. H. Crapo and G.-C. Rota. On foundations of combinatorial theory. 2. combinatorial geometries. *Studies in Applied Mathematics*, 49(2):109, 1970.

S. L. Devadoss. A realization of graph associahedra. *Discrete Mathematics*, 309(1):271–276, 2009.

A. Dochtermann and R. Sanyal. Laplacian ideals, arrangements, and resolutions. *Journal of Algebraic Combinatorics*, pages 1–18, 2012.

A. Dochtermann and R. Sanyal. Laplacian ideals, arrangements, and resolutions. *Journal of Algebraic Combinatorics*, pages 1–18, 2012.

B. Iriarte G. Graph orientations and linear extensions. *arXiv preprint arXiv:1405.4880*, 2014.

J. A. Kelner and A. Madry. Faster generation of random spanning trees. In *Foundations of Computer Science, 2009. FOCS’09. 50th Annual IEEE Symposium on*, pages 13–21. IEEE, 2009.

L. Lovász. Random walks on graphs: A survey. *Combinatorics, Paul Erdos is eighty*, 2(1):1–46, 1993.

L. Lovász and P. Winkler. Efficient stopping rules for markov chains. In *Proceedings of the twenty-seventh annual ACM symposium on Theory of computing*, pages 76–82. ACM, 1995a.

L. Lovász and P. Winkler. Exact mixing in an unknown markov chain. *the electronic journal of combinatorics*, 2(R15):2, 1995b.

M. Manjunath, F.-O. Schreyer, and J. Wilmes. Minimal free resolutions of the g-parking function ideal and the toppling ideal. *arXiv preprint arXiv:1210.7569*, 2012.

E. Miller. Alexander duality for monomial ideals and their resolutions. *arXiv preprint math/9812095*, 1998.

E. Miller and B. Sturmfels. *Combinatorial commutative algebra*, volume 227. Springer, 2005.

F. Mohammadi and F. Shokrieh. Divisors on graphs, connected flags, and syzygies. *International Mathematics Research Notices*, page rnt186, 2013.

A. Postnikov. Permutohedra, associahedra, and beyond. *International Mathematics Research Notices*, 2009(6):1026–1106, 2009.

A. Postnikov and B. Shapiro. Trees, parking functions, syzygies, and deformations of monomial ideals. *Transactions of the American Mathematical Society*, 356(8):3109–3142, 2004.

C. Reidys. Acyclic orientations of random graphs. *Advances in Applied Mathematics*, 21(2):181–192, 1998.

R. Sanyal. *Constructions and obstructions for extremal polytopes*. PhD thesis, PhD thesis, Technische Universität Berlin, 2008.

C. D. Savage and C.-Q. Zhang. The connectivity of acyclic orientation graphs. *Discrete mathematics*, 184(1):281–287, 1998.

R. Speicher. Multiplicative functions on the lattice of non-crossing partitions and free convolution. *Mathematische Annalen*, 298(1):611–628, 1994.
R. P. Stanley. Acyclic orientations of graphs. *Discrete Mathematics*, 5(2):171–178, 1973.

R. P. Stanley. Parking functions and noncrossing partitions. *Electron. J. Combin*, 4(2):R20, 1997.

R. P. Stanley. Hyperplane arrangements, parking functions and tree inversions. In *Mathematical Essays in Honor of Gian-Carlo Rota*, pages 359–375. Springer, 1998.

R. P. Stanley. *Enumerative Combinatorics, Vol. 2*: Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2001.

R. P. Stanley. An introduction to hyperplane arrangements. In *Lecture notes, IAS/Park City Mathematics Institute*. Citeseer, 2004.

R. P. Stanley. *Enumerative Combinatorics, Vol. 1*: Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011.