The second Betti number of doubly weighted homology groups of some pre Lie superalgebra

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1 Introduction

There is a notion of (weighted) (co)homology group theory of pre Lie superalgebras like those of Lie algebras. In [2], we introduced the notion of doubly weighted homology groups for doubly weighted (say \( (w, h) \)) pre Lie superalgebras. The pre Lie superalgebra we handle in this paper is the exterior algebra of polynomial coefficient multi-vector fields on \( n \)-plane with the super bracket is the Schouten bracket. Each generator there is written as

\[
x_1^{b_1} \cdots x_n^{b_n} \frac{\partial}{\partial x_{a_1}} \wedge \cdots \frac{\partial}{\partial x_{a_m}} \quad (= x^\beta \partial_\alpha)
\]

where

\[
\beta = (b_1, \ldots, b_n), \quad (b_i \geq 0 \text{ for } \forall i) \quad \text{and} \quad \alpha = (a_1, \ldots, a_m), \quad (1 \leq a_1 < \cdots < a_m \leq n).
\]

Then the first weight is \( m-1 \) and the double weight is \( (m-1, \sum_{i=1}^{n} b_i - 1) \) by definition. We often denote \(|\alpha| = m = \text{length of } \alpha\) and \(|\beta| = \sum_{i=1}^{n} b_i\).

In [3], we have proven a result:

(1) The Euler number is 0 for all doubly weighted homology groups.

In [2], we have proven several results:

(2) Each Betti number is 0 for \((w, h)\)-doubly weighted homology groups if \( w \neq h \).

(3) The first Betti number is 0 for all doubly weighted homology groups.

(4) The second Betti number is 0 for doubly weighted homology groups if \( w = h = 0 \).

In this paper, we have a main result below:

(5) The second Betti number is 0 for every doubly weighted homology groups with \( w = h \).

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Thus, combining (2) in [3] and (5), we conclude that (6) The second Betti number is 0 for all doubly weighted homology groups.

2 Preliminaries

We recall the notions and notations which we need here quickly. If it is not enough, refer to [3].

First we recall the definition of pre Lie superalgebra.

**Definition 1 (pre Lie superalgebra)** Suppose \( g \) is graded by \( \mathbb{Z} \) as \( g = \sum_{j \in \mathbb{Z}} g_j \) and has a bilinear operation \( [\cdot, \cdot] \) satisfying

\[
\begin{align*}
(2.1) & \quad [g_l, g_j] \subset g_{l+j} \\
(2.2) & \quad [X, Y] = (-1)^{1+xy}[Y, X] \quad \text{where } X \in g_x \text{ and } Y \in g_y \\
(2.3) & \quad (-1)^{xz}[X, Y, Z] + (-1)^{yz}[Y, Z, X] + (-1)^{zx}[Z, X, Y] = 0 \quad \text{(Jacobi identity)}.
\end{align*}
\]

Then we call \( g \) a (or \( \mathbb{Z} \)-graded) Lie superalgebra.

In the usual Lie algebra homology theory, \( m \)-th chain space is the exterior algebra \( \Lambda^m g \) of \( g \) and the boundary operator is essentially \( X \wedge Y \mapsto [X, Y] \).

In pre Lie superalgebras, by “super” skew-symmetry of bracket operation, \( m \)-th chain space \( C_m \) is defined as follows: \( C_m \) is the quotient of the tensor space \( \otimes^m g \) of \( g \) by the 2-sided ideal generated by

\[
X \otimes Y + (-1)^{xy}Y \otimes X \quad \text{where } X \in g_x, Y \in g_y,
\]

and we denote the equivalence class of \( X \otimes Y \) by \( X \Delta Y \).

Since \( X_{odd} \Delta Y_{odd} = Y_{odd} \Delta X_{odd} \) and \( X_{even} \Delta Y_{even} = -Y_{even} \Delta X_{even} \) hold, \( \Delta^m g_i \) is a symmetric algebra for odd \( i \) and is a skew-symmetric algebra for even \( i \) in the usual sense.

We introduce a recursive formula of the boundary operator using the left action.

\[
(2.5) \quad \partial(A_0 \Delta A_1 \Delta \cdots \Delta A_m) = -A_0 \Delta \partial(A_1 \Delta \cdots \Delta A_m) + A_0 \cdot (A_1 \Delta \cdots \Delta A_m)
\]

where

\[
(2.6) \quad A_0 \cdot (A_1 \Delta \cdots \Delta A_m) = [A_0, A_1] \Delta (A_2 \Delta \cdots \Delta A_m) + (-1)^{a_0 a_1} A_1 \Delta (A_0 \cdot (A_2 \Delta \cdots \Delta A_m))
\]

\[
(2.7) \quad = \sum_{i=1}^m (-1)^{a_0 \sum_{j<i} a_j} A_1 \Delta \cdots \Delta [A_0, A_i] \Delta \cdots \Delta A_m
\]

for each homogeneous elements \( A_i \in g_{a_i} \). In lower degree, the boundary operator is given as bellows:

\[
(2.8) \quad \partial(A \Delta B) = [A, B]
\]

\[
(2.9) \quad \partial(A \Delta B \Delta C) = -A \Delta [B, C] + [A, B] \Delta C + (-1)^{ab} B \Delta [A, C]
\]
for each homogeneous elements \( A \in \mathfrak{g}_a, B \in \mathfrak{g}_b, C \in \mathfrak{g}_c \).

**Example 2.1** A prototype of pre Lie superalgebra is the exterior algebra of the sections of exterior power of tangent bundle of a differentiable manifold \( M \) of dimension \( n \),

\[
(2.10) \quad \mathfrak{g} = \sum_{i=1}^{n} \Lambda^i T(M) = \sum_{i=0}^{n-1} \mathfrak{g}_i, \quad \text{where} \quad \mathfrak{g}_i = \Lambda^{i+1} T(M)
\]

with the Schouten bracket.

There are several ways defining the Schouten bracket, namely, axiomatic explanation, sophisticated one using Clifford algebra or more direct ones (cf. [1]). In the context of Lie algebra homology theory, we introduce the Schouten bracket as follows:

**Definition 2 (Schouten bracket)** For \( A \in \Lambda^a T(M) \) and \( B \in \Lambda^b T(M) \), define a binary operation by

\[
(2.11) \quad (-1)^{a+1}[A, B]_S = \partial_0(A \wedge B) - (\partial_0 A) \wedge B - (-1)^{a} A \wedge \partial_0 B,
\]

where \( \partial_0 \) is the boundary operator in the context of Lie algebra homology of vector fields.

In some sense, the Schouten bracket measures gap of the boundary operator \( \partial_0 \) from the derivation. Hereafter, we denote \([A, B]_S\) by \([A, B]\) simply.

The first chain space is \( C_1 = \mathfrak{g} = \sum_{p=1}^{n} \Lambda^p T(M) \). The second chain space is

\[
C_2 = g \Delta g = \sum_{1 \leq p \leq q \leq n} \Lambda^p T(M) \Delta \Lambda^q T(M) = \Lambda^1 T(M) \Delta \Lambda^1 T(M) + \Lambda^1 T(M) \Delta \Lambda^2 T(M) + \cdots + \Lambda^2 T(M) \Delta \Lambda^2 T(M) + \Lambda^2 T(M) \Delta \Lambda^3 T(M) + \cdots
\]

**Remark 2.1** Let \( \pi \in \Lambda^2 T(M) \). Then \( \pi \Delta \pi \in \Lambda^2 T(M) \Delta \Lambda^2 T(M) \subset C_2 \) and \( \partial(\pi \Delta \pi) = [\pi, \pi] \in C_1 \).

Thus, \( \pi \in \Lambda^2 T(M) \) is Poisson if and only if \( \partial(\pi \Delta \pi) = 0 \), and we express it by \( \pi \in \sqrt{\ker(\partial)} \)
symbolically. It will be interesting to study \( \sqrt{\ker(\partial)} \) and also interesting to study specific properties of Poisson structures in \( \sqrt{\ker(\partial) \wedge (C_3)} \), which come from the boundary image of the third chain space \( C_3 \).

### 2.1 First weight

**Definition 3** We say a non-zero element in \( \mathfrak{g}_{i_1} \Delta \cdots \Delta \mathfrak{g}_{i_m} \) has the (first) weight \( i_1 + \cdots + i_m \). Define the subspace of \( C_m \) by \( C_{m,w} = \sum_{i_1 \leq \cdots \leq i_m} \mathfrak{g}_{i_1} \Delta \cdots \Delta \mathfrak{g}_{i_m} \), which is the direct sum of different types of spaces

but the same weight \( w \).

**Proposition 2.1** The (first) weight \( w \) is preserved by \( \partial \), i.e., we have \( \partial(C_{m,w}) \subset C_{m-1,w} \). Thus, we have for a fixed \( w \), \( w \)-weighted homology groups

\[
H_{m,w}(\mathfrak{g}, \mathbb{R}) = \ker(\partial : C_{m,w} \to C_{m-1,w}) / \partial(C_{m+1,w}).
\]
2.2 Double weight

**Definition 4 (Double-weight)** Assume that each subspace \( g_i \) of a given pre Lie superalgebra \( g \) is direct decomposed by subspaces \( g_{i,j} \) as \( g_i = \sum_j g_{i,j} \) and satisfies

\[
X, Y \in g_{l_1+l_2,i_1+j_1} \quad \text{for each} \ X \in g_{i_1,j_1}, \ Y \in g_{i_2,j_2}.
\]

(2.12)

Thus, we have \( \Delta X \). We say such pre Lie superalgebras are double-weighted.

We may define double-weighted \( m \)-th chain space by

\[
C_{m,w,h} = \sum_{i_1 \leq \cdots \leq i_m, \sum_{i=1}^m i_i = w, \sum_{i=1}^m h_i = h} g_{i_1,h_1} \Delta \cdots \Delta g_{i_m,h_m}
\]

**Proposition 2.2** The double-weight \( (w, h) \) is preserved by \( \partial \), i.e., we have \( \partial(C_{m,w,h}) \subset C_{m-1,w,h} \). Thus, we have \((w, h)\)-weighted homology groups

\[
H_{m,w,h}(g, \mathbb{R}) = \ker(\partial : C_{m,w,h} \to C_{m-1,w,h})/\partial(C_{m+1,w,h})
\]

As we explained in Introduction, we consider the Euclidean space \( M = \mathbb{R}^n \) with the Cartesian coordinates \( x_1, \ldots, x_n \). Then, we get a pre Lie super subalgebra consisting of multi vector fields of polynomial coefficients. We define

\[
g_{i,j} = \mathcal{X}_{j+1}^i(\mathbb{R}^n) = \{(i+1)\text{-multi vector fields with } (j+1)\text{-homogeneous polynomials}\}.
\]

We see easily that \( [g_{i_1,j_1}, g_{i_2,j_2}] \subset g_{i_1+i_2,j_1+j_2} \) and so we get a double-weighted pre Lie superalgebra.

3 2nd Betti number for general \( w = h \)

2-chain space \( C_{2,w,h} \) consists of \( \mathcal{X}_b \Delta \mathcal{X}_b \) with \( a_1 + a_2 = 2 + w \) and \( b_1 + b_2 = 2 + w \). Since \( \Delta \) has a super symmetric property, \( \mathcal{X}_b_1 \Delta \mathcal{X}_b_2 = \mathcal{X}_b_2 \Delta \mathcal{X}_b_1 \) as spaces, so we may express each subspace \( \mathcal{X}_{b_1}^{a_1} \Delta \mathcal{X}_{b_2}^{a_2} = \mathcal{X}_{b_2}^{a_2} \Delta \mathcal{X}_{b_1}^{a_1} \) by \( \mathcal{Y}_b^{a_1} := \mathcal{X}_{b_1}^{a_1} \Delta \mathcal{X}_{2+w-b_1}^{a_1} \) with \( 1 \leq a_1 \leq 1 + w/2 \).

\( C_{2,w,w} \) has two expressions depending on \( w \)’s parity.

If \( w = 2\Omega + 1 \), then

\[
C_{2,w,w} = \sum_{a_1=1, b_1=0}^{\Omega+1} \sum_{a_2=0}^{+w} \mathcal{X}_{b_1}^{a_1} \Delta \mathcal{X}_{2+w-b_1}^{a_2}
\]

If \( w = 2\Omega \), then

\[
C_{2,w,w} = \sum_{a_1=1}^{\Omega} \sum_{b_1=0}^{+w} \mathcal{X}_{b_1}^{a_1} \Delta \mathcal{X}_{2+w-b_1}^{a_1} + \sum_{b_1=0}^{+w} \mathcal{X}_{b_1}^{\Omega+1} \Delta \mathcal{X}_{\Omega+1}^{\Omega+1}
\]

To understand how \( C_{2,w,w} \) is decomposed, we plot the point \((a_1, b_1)\) of \( \mathcal{Y}_b^{a_1} \) on the 2-plane. The next are examples of \( w = 3, 4 \).
Definition 5 We define the type of single 2-chain \( x^{B_1} \partial_{A_1} \Delta x^{B_2} \partial_{A_2} \) by

- type TR if \( |A_1| + |B_1| < |A_2| + |B_2| \) or \( |A_1| + |B_1| = |A_2| + |B_2| \) and \( |A_1| \leq |A_2| \)
- type TL otherwise.

For each \( x^{B_1} \partial_{A_1} \Delta x^{B_2} \partial_{A_2} \), we define a 3-chain by

\[
\begin{align*}
(3.1) & \quad \sum_{\ell=1}^{n} \partial_{\ell} \Delta( x^{B_1} \partial_{A_1} ) \Delta(x_{\ell} x^{B_2} \partial_{A_2} ) & \text{if type TR} \\
(3.2) & \quad \sum_{\ell=1}^{n} \partial_{\ell} \Delta(x_{\ell} x^{B_1} \partial_{A_1} ) \Delta( x^{B_2} \partial_{A_2} ) & \text{if type TL}
\end{align*}
\]

and extend linearly on \( C_{2,w,w} \) and denote it by \( \Phi \). Define a linear map \( \phi : C_{1,w,w} \to C_{2,w,w} \) by

\[
(3.3) \quad \phi(U) = \sum_{\ell=1}^{n} \partial_{\ell} \Delta(x_{\ell} U) .
\]

Lemma 3.1 Consider \( x^{B_1} \partial_{A_1} \Delta x^{B_2} \partial_{A_2} \).

If TR, then

\[
(\partial \Phi + \phi \partial)( x^{B_1} \partial_{A_1} \Delta x^{B_2} \partial_{A_2} ) = - \sum_{\ell=1}^{n} \partial_{\ell} \Delta [ x^{B_1} \partial_{A_1} , x_{\ell} ] x^{B_2} \partial_{A_2} + \sum_{\ell=1}^{n} [ \partial_{\ell} x^{B_1} \partial_{A_1} ] \Delta( x_{\ell} x^{B_2} \partial_{A_2} ) + (n + |B_2|) (x^{B_1} \partial_{A_1} ) \Delta( x^{B_2} \partial_{A_2} ) .
\]

If TL, then

\[
(\partial \Phi + \phi \partial)( x^{B_1} \partial_{A_1} \Delta x^{B_2} \partial_{A_2} ) = -(-1)^{|A_1|(|A_2|+1)} \sum_{\ell=1}^{n} \partial_{\ell} \Delta [ x_{\ell} , x^{B_2} \partial_{A_2} ] x^{B_1} \partial_{A_1} + (n + |B_1|) (x^{B_1} \partial_{A_1} ) \Delta( x^{B_2} \partial_{A_2} ) + \sum_{\ell=1}^{n} (x_{\ell} x^{B_1} \partial_{A_1} ) \Delta[ \partial_{\ell} x^{B_2} \partial_{A_2} ] .
\]

For an element in \( \Psi_{b}^{w} = x_{b}^{w} \Delta x_{2+w-b}^{2} \), when we want only to indicate its type, we sometimes use the symbol \( \text{Term}_{t}^{d} \), for this purpose. Using this abbreviation and denoting \( \partial \Phi + \phi \partial \) by \( \Psi \), the above become more simpler.
If $TR$, then
\begin{equation}
Ψ( x^{B_1} \partial_{A_1} \Delta x^{B_2} \partial_{A_2} ) = (n + |B_2|)(x^{B_1} \partial_{A_1}) \Delta(x^{B_2} \partial_{A_2}) + \text{Term}_{|B_1|-1}^{|A_1|} + \text{Term}_0^1
\end{equation}

If $TL$, then
\begin{equation}
Ψ( x^{B_1} \partial_{A_1} \Delta x^{B_2} \partial_{A_2} ) = (n + |B_1|)(x^{B_1} \partial_{A_1}) \Delta(x^{B_2} \partial_{A_2}) + \text{Term}_{|B_1|+1}^{|A_1|} + \text{Term}_0^1
\end{equation}

**Proof:** When $TR$,
\[
\partial(Φ( x^{B_1} \partial_{A_1} \Delta x^{B_2} \partial_{A_2} )) = \partial(\sum_{\ell=1}^n \partial_\ell \Delta( x^{B_1} \partial_{A_1}) \Delta(x_\ell \ x^{B_2} \partial_{A_2}))
\]
\[
= -\sum_{\ell=1}^n \partial_\ell \Delta[x^{B_1} \partial_{A_1}, x_\ell x^{B_2} \partial_{A_2}] + \sum_{\ell=1}^n [\partial_\ell, x^{B_1} \partial_{A_1}] \Delta(x_\ell x^{B_2} \partial_{A_2}) + \sum_{\ell=1}^n (x^{B_1} \partial_{A_1}) \Delta[\partial_\ell, x_\ell x^{B_2} \partial_{A_2}]
\]
\[
= -φ(\partial( x^{B_1} \partial_{A_1} \Delta x^{B_2} \partial_{A_2} )) - \sum_{\ell=1}^n \partial_\ell \Delta[x^{B_1} \partial_{A_1}, x_\ell x^{B_2} \partial_{A_2}]
\]
\[
+ \sum_{\ell=1}^n [\partial_\ell, x^{B_1} \partial_{A_1}] \Delta(x_\ell x^{B_2} \partial_{A_2}) + (n + |B_2|)(x^{B_1} \partial_{A_1}) \Delta(x^{B_2} \partial_{A_2}).
\]

When $TL$,
\[
\partial(Φ( x^{B_1} \partial_{A_1} \Delta x^{B_2} \partial_{A_2} )) = \partial(\sum_{\ell=1}^n \partial_\ell \Delta(x_\ell \ x^{B_2} \partial_{A_2}))
\]
\[
= -\sum_{\ell=1}^n \partial_\ell \Delta[x_\ell x^{B_1} \partial_{A_1}, x^{B_2} \partial_{A_2}] + \sum_{\ell=1}^n [\partial_\ell, x_\ell x^{B_1} \partial_{A_1}] \Delta(x^{B_2} \partial_{A_2}) + \sum_{\ell=1}^n (x_\ell x^{B_1} \partial_{A_1}) \Delta[\partial_\ell, x^{B_2} \partial_{A_2}]
\]
\[
= -φ(\partial( x^{B_1} \partial_{A_1} \Delta x^{B_2} \partial_{A_2} )) - (-1)^{|A_1|(|A_2|+1)} \sum_{\ell=1}^n \partial_\ell \Delta[x_\ell, x^{B_2} \partial_{A_2}]x^{B_1} \partial_{A_1}
\]
\[
+ (n + |B_1|)(x^{B_1} \partial_{A_1}) \Delta(x^{B_2} \partial_{A_2}) + \sum_{\ell=1}^n (x_\ell x^{B_1} \partial_{A_1}) \Delta[\partial_\ell, x^{B_2} \partial_{A_2}].
\]

\[\blacksquare\]

**Remark 3.1** (3.4) and (3.5) say that $x^{B_1} \partial_{A_1} \Delta x^{B_2} \partial_{A_2}$ is one step descending to $\text{Term}_{|B_1|-1}^{|A_1|}$ if $TR$ type and one step ascending to $\text{Term}_{|B_1|+1}^{|A_1|}$ if $TL$ type by the map $Ψ := \partial \circ Φ + φ \circ \partial$ modulo $\text{Term}_0^1$. Hereafter, sometimes or somewhere we express $Ψ(U)$ by omitting $\text{Term}_0^1$ but never forget its contribution. Actually, we see next interesting property.

**Lemma 3.2** If $U ∈ \Psi_0^1$ then it satisfies $Ψ(U) = (n + w + 1)U$.

**Proof:** We may express $U = \sum_{i, β, G_i^β} \partial_i ΔG_i^β \partial_β$ with $|β| = 1 + w, |G_i^β| = 2 + w$.

\[
\partial Φ(U) = \sum_{i, β, G_i^β} \sum_{\ell} \partial(\partial_\ell \Delta \partial_i x_\ell G_i^β \partial_β) = \sum_{i, β, G_i^β} \sum_{\ell} (-\partial_\ell Δ[\partial_\ell, x_\ell G_i^β \partial_β] + \partial_\ell Δ[\partial_\ell, x_\ell G_i^β \partial_β])
\]
\[-\phi \partial U - \sum_{\ell} \sum_{i,\beta, G^{i,\beta}} \partial \ell \Delta[\partial_i, x_\ell] G^{i,\beta} \partial \ell + (n + w + 2)U = -\phi \partial U + (n + w + 1)U , \]
\[
\Psi(U) = (n + w + 1)U .
\]

To know the type of $\Psi_{b_1}^{a_1}$ is easy as follows:

- *TL* type iff $a_1 + b_1 > 2 + w$, we express a general element there by $\xi_{b_1}^{a_1}$.
- *TR* type iff $a_1 + b_1 \leq 2 + w$, we express a general element there by $\eta_{b_1}^{a_1}$.

We divide $C_{2,w,w}$ into the two subspaces

\[
W_{[TR]} := \sum_{a_1 + b_1 \leq 2 + w} \Psi_{b_1}^{a_1} \text{ of type } TR, \quad \text{and } W_{[TL]} := \sum_{a_1 + b_1 > 2 + w} \Psi_{b_1}^{a_1} \text{ of type } TL .
\]

**Proposition 3.1** The subspace $W_{[TR]}$ is invariant under $\Psi(\partial \Phi + \phi \circ \partial)$.
The subspace $W_{[TL]}$ is invariant under $\Psi(\partial \Phi + \phi \circ \partial)$ modulo $\text{Term}_0^1$.

**Definition 6** Let

\[
TK^{[\ell]} := \sum_{a_1 + b_1 = \ell + 2 + w} \Psi_{b_1}^{a_1}, \quad TK^{[\ell, \lambda]} := \sum_{a_1 + b_1 = \ell + 2 + w} \Psi_{b_1}^{a_1} \subset TK^{[\ell]}, \quad UT^{[\ell]} := \sum_{a_1 + b_1 > \ell + 2 + w} \Psi_{b_1}^{a_1} .
\]

We easily see

**Proposition 3.2** $UT^{[\ell]} = TK^{[\ell + 1]} + UT^{[\ell + 1]}$ for $\ell \geq 0$ and $W_{[TL]} = UT^{[0]}$.

\[
W_{[TL]} = \sum_{\ell} \Omega_\ell^{[\ell]}, \quad \text{where } \Omega_\ell := \begin{cases} 
\Omega + 1 & \text{if } w = 2\Omega + 1 \\
\Omega & \text{if } w = 2\Omega 
\end{cases} .
\]

\[
\Psi(UT^{[\ell]}) = UT^{[\ell]}, \quad (\text{in precise } \Psi(UT^{[\ell]}) \subset UT^{[\ell]} + \Psi_{0}^{1}) .
\]

\[
\Psi(TK^{[\ell]}) = TK^{[\ell]} + TK^{[\ell + 1]}, \quad (\text{in precise } \Psi(TK^{[\ell]}) \subset TK^{[\ell]} + TK^{[\ell + 1]} + \Psi_{0}^{1}) .
\]

Our main result is the following:

**Theorem 3.1** The second Betti number is zero for $\{C_{\bullet,w,w}\}$.

In order to prove the theorem above, we follow three steps:

1. We reduce our discussion from $W_{[TL]} + W_{[TR]}$ to $W_{[TR]}$.
2. We decompose $W_{[TR]} = \sum_{a_1 + b_1 \leq 2 + w, b_1 > 1 + \Omega_\varepsilon} \Psi_{b_1}^{a_1} + \sum_{a_1 + b_1 \leq 2 + w, b_1 \leq 1 + \Omega_\varepsilon} \Psi_{b_1}^{a_1}$.

We reduce our discussion to the rectangular region $\sum_{a_1 + b_1 \leq 2 + w, b_1 \leq 1 + \Omega_\varepsilon} \Psi_{b_1}^{a_1}$.
3. We finish our discussion on the rectangular region.

3.1 **TL**

We prepare one of two key lemmas:

**Lemma 3.3** Let $a$ and $s$ satisfy with $1 \leq a \leq s \leq \Omega_e$.

Take

$$U = u + u' \quad \text{where} \quad u \in TK^{[a,s]} \setminus TK^{[a,s+1]} \quad \text{and} \quad u' \in UT^{[a]} + W_{[TR]}.$$  \hspace{1cm}(3.6)

Then

$$\Psi(U) = cU + U' \quad \exists c \neq 0, \quad U' \in TK^{[a,1+s]} + UT^{[a]} + W_{[TR]}.$$  \hspace{1cm}(3.7)

**Proof:** In general, we handle an element $U$ belong to $TK^{[a]} + UT^{[a]} + W_{TR}$, usually we write as follows: $U = u_1 + u_2 + u_3 \in TK^{[a]} + UT^{[a]} + W_{TR}$ or $U = u_1 + u_2 + u_3$ where $u_1 \in TK^{[1]}$, $u_2 \in UT^{[1]}$, $u_3 \in W_{[TR]}$. Sometimes or somewhere in this article, we impolitely write $U = u_1 + UT^{[a]} + W_{[TR]}$ where $u_1 \in TK^{[a]}$ for instance when we do not refer $u_2$ and $u_3$ later in precise.

Take

$$U = \sum_{s \leq t \leq \Omega_e} \xi_{w+2+a-t}^s + u' \quad \text{where} \quad \xi_{w+2+a-s}^s \neq 0.$$  \hspace{1cm}

Then

$$\Psi(U) = \Psi(\sum_{t=s}^{\Omega_e} \xi_{w+2+a-t}^t) + \Psi(u') = \sum_{t=s}^{\Omega_e} ((n + w + 2 + a - t) \xi_{w+2+a-t}^t + \text{Term}_{w+3+a-t}^t) + \Psi(u')$$

$$= \sum_{t=s}^{\Omega_e} (n + w + 2 + a - t) \xi_{w+2+a-t}^t + UT^{[a]} + \Psi(u')$$

$$= (n + w + 2 + a - s) \sum_{t=s}^{\Omega_e} \xi_{w+2+a-t}^t + \sum_{t=s}^{\Omega_e} (s - t) \xi_{w+2+a-t}^t + UT^{[a]} + \Psi(u')$$

$$= (n + w + 2 + a - s) (\sum_{t=s}^{\Omega_e} \xi_{w+2+a-t}^t + \sum_{t=s}^{\Omega_e} (s - t) \xi_{w+2+a-t}^t)$$

$$- (n + w + 2 + a - s) u' + UT^{[a]} + \Psi(u')$$

$$= (n + w + 2 + a - s) U + \sum_{t=1+s}^{\Omega_e} (s - t) \xi_{w+2+a-t}^t + UT^{[a]} + W_{[TR]}$$

$$= (n + w + 2 + a - s) U + U'$$

where

$$U' = \sum_{t=s+1}^{\Omega_e} (s - t) \xi_{w+2+a-t}^t + UT^{[a]} + W_{[TR]}.$$  \hspace{1cm}
**Corollary 3.4** Let $U \neq 0$ and $U \in TK^{[a]}$. Then we have some non-zero numbers $\{c_i\}$ so that

$$(\Psi + c_1) \circ \cdots \circ (\Psi + c_\ell) U \in UT^{[a]}.$$ 

**Proposition 3.3** Take $U \in C_{2,w,w} = W_{[TL]} + W_{[TR]}$. We reduce $U$ to $W_{[TR]}$ in the following sense.

$$(\Psi + c_1) \circ \cdots \circ (\Psi + c_\ell) U \in W_{[TR]},$$

for some non-zero numbers $\{c_i\}$.

**Proof:** $U = U_0 + U_1$ with $U_0 \in W_{[TL]}, U_1 \in W_{[TR]}$. We may assume $U_0 \neq 0$. Apply Proposition 3.4 several times, we see that $(\Psi + c_1) \circ \cdots \circ (\Psi + c_\ell) U_0 \in W_{[TR]}$ for some non-zero numbers $\{c_i\}$, and $(\Psi + c_1) \circ \cdots \circ (\Psi + c_\ell) U_1 \in W_{[TR]}$ because $W_{[TR]}$ is invariant under the action of $\Psi$. Thus, we see $(\Psi + c_1) \circ \cdots \circ (\Psi + c_\ell) U \in W_{[TR]}$. 

3.2 $TR$

Here we study $\Psi(U)$ of $U \in W_{[TR]}$. For that purpose, we divide $W_{[TR]}$ into two parts, one is roof part $\sum_{a+b=p, a+b, \Omega} \Psi_{[b]}^{a}$ and the other is rectangular basic part $\sum_{b_1, b, \Omega} \Psi_{[b]}^{a}$. We define three subspaces:

$$TK_{[p]} := \sum_{a+b=p, a+b, \Omega} \Psi_{[b]}^{a}, \quad TK_{[p,s]} := \sum_{a+b=p, a+b, \Omega} \Psi_{[b]}^{a}, \quad DT_{[p]} := \sum_{a+b<p} \Psi_{[b]}^{a}.$$ 

**Proposition 3.4** $W_{TR} = DT_{[2+w+1]} = \sum_{p \leq 2+w} TK_{[p]}$, $DT_{[p+1]} = TK_{[p]} + DT_{[p]}$, $\Psi(TK_{[p]}) = TK_{[p]} + TK_{[p-1]}$, $\Psi(DT_{[p]}) = DT_{[p]}$.

Making use of descending property of $TR$ for $\Psi$, we prepare another key lemma like Lemma 3.3.

**Lemma 3.5** Take $U \in TK_{[p,s]} \setminus TK_{[p,s-1]}$. Then $(\Psi + c) U \in TK_{[p,s-1]} + DT_{[p]}$ for some $c \neq 0$.

**Proof:**

$$U = \sum_{a+b=p, a+b, \Omega} \eta_{[b]}^{a}, \quad (\eta_{[p,s-1]}^{a} \neq 0).$$

$$\Psi(U) = \sum_{a+b=p, a+b, \Omega} \Psi(\eta_{[b]}^{a}) = \sum_{a+b, \Omega} ((n + 2 + w - p + a)\eta_{[p,a]}^{a} + \text{Term}_{[p,a]}^{a} + DT_{[p]})$$

$$= (n + 2 + w - p + s) \sum_{a+b, \Omega} \eta_{[p,a]}^{a} + \sum_{a+b, \Omega} (a-s)\eta_{[p,a]}^{a} + DT_{[p]}$$

$$= (n + 2 + w - p + s) U + \sum_{a+b, \Omega} (a-s)\eta_{[p,a]}^{a} + DT_{[p]}.$$ 

\[\blacksquare\]
**Proposition 3.5** Take a $U \in W_{[TR]}$. We reduce $U$ to the rectangular part in the following sense.

$$(\Psi + c_1) \circ \cdots \circ (\Psi + c_k)U \in \text{Rectangular part of } W_{[TR]}$$

for some non-zero numbers $\{c_i\}$.

**Proof:** Making use of descending property of $TR$, applying Lemma 3.5 several times along

$$(\Psi + c_1) \circ \cdots \circ (\Psi + c_k)U \in \text{Rectangular part of } W_{[TR]}$$

for $p = w + 2, \ldots, 2 + \Omega_e$.

We divide the rectangular region horizontally, i.e.,

$$\text{SkyH}_{[b]} := \sum_{1 \leq a \leq \Omega + 1} \Psi^a_b, \quad \text{SkyB}_{[b]} := \sum_{b' < b} \text{SkyH}_{[b']}.$$  

$\Psi(\text{SkyH}_{[b]}) \subset \text{SkyH}_{[b]} + \text{SkyH}_{[b-1]}$ and $\text{SkyB}_{[b]}$ is invariant by $\Psi = \partial \circ \Phi + \phi \circ \partial$. We have an easy lemma.

**Lemma 3.6** Take $U \in \text{SkyB}_{[b+1]} \setminus \text{SkyB}_{[b]}$. Then

$$\Psi(U) = (n + w + 2 - b)U + U' \quad \text{where } U' \in \text{SkyB}_{[b]}.$$  

**Proof:** $U = \sum_{a=1}^{\Omega+1} \eta^a_b$ and

$$\Psi(U) = \sum_{a=1}^{\Omega+1} \Psi(\eta^a_b) = \sum_{a=1}^{\Omega+1} ((n + w + 2 - b)\eta^a_b + \text{Term}^a_{b-1}) = (n + w + 2 - b)\sum_{a=1}^{\Omega+1} \eta^a_b + \text{SkyB}_{[b]}$$

$$= (n + w + 2 - b)U + U', \quad U' \in \text{SkyB}_{[b]}.$$  

**Proposition 3.6** Take $U$ from the rectangle region of $W_{[TR]}$ defined by

$$\sum_{1 \leq a \leq \Omega + 1, 0 \leq b_1 \leq 1 + \Omega_e} \Psi^a_{b_1}.$$  

Then we reduce $U$ to the null in the following sense.

$$(\Psi + c_1) \circ \cdots \circ (\Psi + c_k)U = 0$$

for some non-zero numbers $\{c_i\}$.

**Proof:** Starting $U \in \text{SkyB}_{[2+\Omega_e]}$ and applying the last easy lemma recursively, we have

$U^{(k)} = 0$ or reach the bottom $U^{(k)} \in \text{SkyH}_{[0]}$ where $U^{(k)} := (\Psi + c_1) \circ \cdots \circ (\Psi + c_k)U$ for some nonzero finite sequence $(c_i)$. If $U^{(k)} = 0$ the discussion finishes. If $U^{(k)} \neq 0$ we remember a contribution of $\Psi_l^1$ for
the action of \( \Psi \) and we have

\[(\Psi + c_{k+1})U^{(k)} = 0 + U^{(k+1)} \text{ where } U^{(k+1)} \in \mathfrak{p}_0^1.\]

Using Lemma 3.2, we know \( \Psi(U^{(k+1)}) = (n + w + 1)U^{(k+1)}. \) Thus, we get

\[(\Psi + c_1) \circ \cdots \circ (\Psi + c_k) \circ (\Psi + c_{k+1}) \circ (\Psi - (n + w + 1))U = 0.\]

\[\blacksquare\]

**Proof of Theorem:** Take \( U \in C_{2,w,w}. \) Combining Propositions 3.3, 3.5 and 3.6, we see that

\[(\Psi + c_1) \circ \cdots \circ (\Psi + c_m)U = 0\]

for some non-zero numbers \( \{c_i\}. \) Now express the polynomial \((t + c_1) \cdots (t + c_m)\) as \(c_1 \cdots c_m + t g(t)\) for some polynomial \(g(t)\) of one variable \(t.\)

Assume \( U \) is a cycle. Then \( \Psi(U) = (\partial \circ \Phi + \phi \circ \partial)U = (\partial \Phi)U \) and

\[0 = (\Psi + c_1) \circ \cdots \circ (\Psi + c_m)U = c_1 \cdots c_m U + \partial \circ \Phi \circ g(\partial \circ \Phi)U.\]

This implies \( U \) is exact. \[\blacksquare\]

**Remark 3.2** We can define \( \Phi \) for each \( m \)-chain for \( m \geq 2 \), and we may say \( \phi = \Phi \) for 1-chains. Thus, it seems to be interesting to study geometry and combinatorics of \( \Psi := \partial \circ \Phi + \Phi \circ \partial \) for \( m \)-chains with \( m > 2.\)

**References**

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