Jacobi structures on affine bundles

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\textbf{Abstract}

We study affine Jacobi structures (brackets) on an affine bundle \(\pi : A \to M\), i.e. Jacobi brackets that close on affine functions. We prove that if the rank of \(A\) is non-zero, there is a one-to-one correspondence between affine Jacobi structures on \(A\) and Lie algebroid structures on the vector bundle \(A^+=\bigcup_{p \in M} \text{Aff}(A_p, \mathbb{R})\) of affine functionals. In the case \(\text{rank } A = 0\), it is shown that there is a one-to-one correspondence between affine Jacobi structures on \(A\) and local Lie algebras on \(A^+\). Some examples and applications, also for the linear case, are discussed. For a special type of affine Jacobi structures which are canonically exhibited (strongly-affine or affine-homogeneous Jacobi structures) over a real vector space of finite dimension, we describe the leaves of its characteristic foliation as the orbits of an affine representation. These affine Jacobi structures can be viewed as an analog of the Kostant-Arnold-Liouville linear Poisson structure on the dual space of a real finite-dimensional Lie algebra.

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1 Introduction

The Lie algebroid structures draw more and more attention in the literature as structures generalizing the standard Cartan differential calculus on differentiable manifolds.

There is a well-known correspondence between linear Poisson structures on a vector bundle $\pi: A \to M$ and Lie algebroid structures on the dual vector bundle $A^*$, which shows that the theory of Lie algebroids is, in fact, the theory of linear Poisson brackets. This correspondence is built on the fact that sections of $A^*$ can be considered linear functions on $A$. It can be easily extended to the correspondence between affine Poisson structures on $A$ (whose brackets close on affine functions) and central extensions of Lie algebroids, i.e., Lie algebroid structures on $A^* \times \mathbb{R}$ with the property that the section $(0, 1)$ is a central element of the Lie algebroid bracket. More generally, a Lie algebroid structure on $A^* \times \mathbb{R}$ is uniquely represented by an affine Jacobi structure on $A$ (whose Jacobi brackets close on affine functions) as well as by a linear Poisson structure on $A \times \mathbb{R}$. Let us remark that, as we will show later, the parts $\Lambda$ and $E$ of an affine Jacobi tensor $(\Lambda, E)$ need not be affine themselves.

The presence of affine Poisson and Jacobi structures as counterparts for Lie algebroids justifies reconsideration of the mentioned relations in an affine setting, i.e., by admitting only affine bundle structure on $A$. The dual bundle is the vector bundle $A^+ = \text{Aff}(A, \mathbb{R})$ whose fiber over $p \in M$ consists of affine functions on the fiber $A_p$. It has a distinguished section corresponding to the constant function 1 on $A$. In this paper we will prove that there is one-to-one correspondence between affine Jacobi structures on $A$ (the Jacobi bracket of affine functions is an affine function) and Lie algebroids on $A^+$.

The standard definition of a Lie algebroid structure on a vector bundle $A$ consists of a Lie bracket defined on sections and an anchor map $\rho: A \to TM$. It is instructive to look at a Lie algebroid as a restriction to sections of the corresponding Schouten bracket $[\cdot,\cdot]_{SN}$ (which is, in fact, a graded Poisson bracket) on the graded algebra of multisections of $A$. The Schouten bracket is graded anticommutative, satisfies the graded Jacobi identity and the graded Leibniz rule. One can interpret a Poisson structure on $M$ as a canonical structure for the Schouten bracket of multivector fields on $M$, i.e., as an element $\Lambda$ of Lie degree $-1$ satisfying the master equation $[\Lambda, \Lambda]_{SN} = 0$. The analogy with the classical Yang-Baxter equation is not an accident, but the essence of the theory.

This point of view provides an easy passage to the theory of Jacobi structures and Jacobi algebroids (or generalized Lie algebroids in the sense of [17]). It is enough to replace the (graded) Leibniz rule by the generalized Leibniz rule which is valid for the first-order differential operators. The obtained bracket is called a Schouten-Jacobi bracket on $A$. Its restriction to sections of $A$ defines a Lie algebroid structure on $A$, but its restriction to sections and functions defines a Jacobi algebroid (structure on $A^*$) (generalized Lie algebroid in [17]).

A Jacobi structure on $M$ in this setting turns out to be a canonical structure (in the sense we explain later on) for the Schouten-Jacobi bracket of the first order polydifferential operators on $M$, i.e., skew-symmetric multidifferential operators.

The passage from Poisson structures and Lie algebroids to Jacobi structures and Jacobi algebroids is, essentially, the passage from derivations to first-order differential operators. The notion of a derivative depends on a reference frame (trivialization), but the notion of a first-order operator is not, and we make use of this difference. A similar situation we encounter in physics when we pass from a reference frame-dependent to frame-independent description of a physical system. This is why frame-independent formulations require affine bundles and Jacobi structures (algebroids). We refer here to [7, 8, 29, 32, 30] (time-dependent mechanics) and [34].
With a Lie algebroid structure on $A$ we associate the complete lift of multi-sections of $A$ to multivector fields on $A$. It is a homomorphism of the Lie algebroid Schouten bracket into the standard Schouten bracket and the complete lift of a canonical structure of $A$ is a linear Poisson structure on $A$.

Similarly, there is a complete lift (cf. [11]) of a canonical structure for a Jacobi algebroid $A$ to an affine Jacobi structure on $A$.

These remarks show that there is a need to look closer at affine Jacobi brackets on affine (and also linear) bundles as to those which are responsible for all these structures. This time, however, the structures have an affine flavor.

The aim of this paper is a study of affine Jacobi structures on affine and vector bundles and the corresponding Lie algebroids.

A linear (resp., affine) Poisson structure on a vector bundle $A$ can be characterized by its behaviour with respect to the graded algebra of polynomial functions on $A$ or with respect to the Liouville (called also Euler) vector field $\Delta_A$ on $A$. Recall that the Liouville vector field $\Delta_A$ is the generator of the one-parameter group of (positive) homoteties on $A$. For example, a Poisson structure is linear, i.e., linear functions are closed with respect to the Poisson bracket, if and only if one of the following sentences is satisfied

1. the corresponding tensor $\Lambda$ is homogeneous with respect to the Liouville vector field ($\mathcal{L}_{\Delta_A}\Lambda = -\Lambda$, where $\mathcal{L}$ is the Lie derivative operator on $A$).
2. the Hamiltonian vector field of a linear function is linear.

A Poisson structure is affine if and only $[Y,[X,\Lambda]] = 0$ for each pair of invariant vector fields $X,Y$ on $A$ (i.e., vertical lifts of sections of $A$). In this case we say that $\Lambda$ is affine homogeneous. This definition has its advantage, when comparing with the action of the Liouville vector field, that it can be used in non-commutative cases, i.e., for structures on Lie groups or Lie groupoids.

These characterizations cannot be extended to the case of Jacobi structures. In particular, a linear Jacobi structure may not be homogeneous and an affine Jacobi structure may not be affine homogeneous, so one has to find the proper notion of homogeneity in the affine case. In the paper we propose such notion and we establish relations between different concepts and specify the corresponding Lie algebroids.

In Section 2 some definitions and results about Jacobi structures, homogeneous multivectors in a vector bundle and Lie algebroids are recalled. In Section 3 we discuss affine Jacobi structures on an affine bundle in relation to linear Poisson structures on its vector hull and Lie algebroids on the vector dual bundle. In Section 4 we provide several examples. The most important ones are given by the canonical structures which induce triangular bialgebroid structures (Lie and Jacobi). In Section 5 we analyze the relation between homogeneous and linear Jacobi structures (previously, in Section 4, some results have been obtained). Moreover, we introduce the notion of affine-homogeneous Jacobi structures. We establish in Proposition 5.3 its relation to affine and strongly-affine Jacobi structures. We remark that an affine Jacobi structure is said to be strongly-affine if the hamiltonian vector fields of affine functions are affine. On the other hand, we prove that affine-homogeneous Jacobi structures on an affine bundle $A$ correspond to Lie algebroids on the vector dual $A^\ast$ which have an ideal of sections of the subbundle spanned by $1_A$. The Section 6 is devoted to the description of leaves of the characteristic foliation of a strongly-affine Jacobi structure on a vector space, as the orbits of an affine representation of a Lie group on the vector space. It can be viewed as a generalization of viewing symplectic leaves of the Kostant-Arnold- Liouville
2 Jacobi manifolds and Lie algebroids

A Jacobi manifold \[25\] is a differentiable manifold \(M\) endowed with a pair \((\Lambda, E)\), where \(\Lambda\) is a 2-vector and \(E\) is a vector field on \(M\) satisfying
\[
[\Lambda, \Lambda]_{SN} = -2E \wedge \Lambda, \quad [E, \Lambda]_{SN} = 0.
\]

Here \([\cdot, \cdot]_{SN}\) denotes the Schouten bracket. Note that we use the version of the Schouten-Nijenhuis bracket which gives a graded Lie algebra structure on multivector fields and which differs from the classical one \([8, 24]\) by signs. For this type of manifolds, a bracket of functions (the Jacobi bracket) is defined by
\[
\{f, g\}_{\Lambda, E} = \Lambda(df, dg) + fE(g) - gE(f),
\]
for all \(f, g \in C^\infty(M, \mathbb{R})\). This bracket is skew-symmetric, satisfies the Jacobi identity and it is a first-order differential operator on each of its arguments, with respect to the ordinary multiplication of functions.

We will often identify the Jacobi bracket with the first-order bidifferential operator \(\Lambda + I \wedge E\), where \(I\) is the identity on \(C^\infty(M, \mathbb{R})\). The space \(C^\infty(M, \mathbb{R})\) of \(C^\infty\) real valued functions on \(M\) endowed with the Jacobi bracket is a local Lie algebra on \(M\) (see \[21\]). Conversely, a local Lie algebra on \(C^\infty(M, \mathbb{R})\) defines a Jacobi structure on \(M\) (see \[6, 21\]). Note that Poisson manifolds \[21\] are Jacobi manifolds with \(E = 0\).

Other interesting examples of Jacobi manifolds are contact and locally conformal symplectic manifolds (see for example \[6, 21\]). We will often identify sections \(\mu\) of the dual bundle \(A^*\) with linear (along fibres) functions \(\iota_\mu\) on the vector bundle \(A\): \(\iota_\mu(X_p) = \langle \mu(p), X_p \rangle\). Note that if \(f : A \to \mathbb{R}\) is a smooth real function and \(\Delta_A\) is the Liouville vector field of \(A\) then
\[
f \text{ is linear } \iff \Delta_A(f) = f.
\]

\[2.1\]
We recall that $\Delta_A$ is the vector field on $A$ given by $\Delta_A = \sum_\alpha y^\alpha \frac{\partial}{\partial y^\alpha}$, for fibred coordinates $(x^i, y^\alpha)$.

On the other hand, a 2-vector $\Lambda$ on $A$ is linear if and only if the induced bracket is closed on linear functions, that is, $\langle \Lambda, d\mu \wedge df \rangle = \{\mu, f\}_A$ is again a linear function associated with an element $[\mu, f]_A$. The operation $[\mu, f]_A$ on sections of $A^*$ is called the bracket induced by $\Lambda$. If $\Lambda$ is a linear 2-vector field on $A$ and $f, g : A \to \mathbb{R}$ are basic functions then

$$\langle \Lambda, d\mu \wedge df \rangle \text{ is a basic function and } \langle \Lambda, df \wedge dg \rangle = 0. \quad (2.2)$$

Using the above facts, it is easy to prove that $\Lambda$ is linear if and only if it is a homogeneous bivector field on $A$ with respect to $\Delta_A$.

A Lie algebroid structure on a differentiable vector bundle $\pi : A \to M$ is a pair which consists of a Lie algebra structure $[\cdot, \cdot]_\Lambda$ on the space $\Gamma(A)$ of the global sections of $\pi : A \to M$ and a homomorphism of vector bundles $\rho : A \to TM$, the anchor map, such that if $\rho : \Gamma(A) \to \mathfrak{X}(M)$ also denotes the homomorphism of $C^\infty(M, \mathbb{R})$-modules induced by the anchor map, then

$$[[X, fY]] = f[X, Y] + \rho(X)(f)Y,$$

for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M, \mathbb{R})$. It follows that $\rho : (\Gamma(A), [\cdot, \cdot]) \to (\mathfrak{X}(M), [\cdot, \cdot])$ is a Lie algebra homomorphism. Note that the anchor is uniquely determined by the Lie algebroid bracket.

**Theorem 2.2** [2] There is a one-one correspondence between Lie algebroid brackets $[\cdot, \cdot]_\Lambda$ on the vector bundle $A$ and homogeneous (linear) Poisson structures $\Lambda$ on the dual bundle $A^*$ determined by

$$\iota_{[X, Y]_\Lambda} = \{\iota_X, \iota_Y\}_\Lambda = \Lambda(dx_X, dx_Y).$$

Every Poisson structure $\Lambda$ on $M$ determines a Lie algebroid bracket $[\cdot, \cdot]_\Lambda$ on $T^*M$ with the anchor $\#_\Lambda$ and the bracket $[\cdot, \cdot]_\Lambda$ defined by $[\alpha, \beta]_\Lambda = i_{\#_\Lambda(\alpha)}d\beta - i_{\#_\Lambda(\beta)}d\alpha + d(\Lambda(\alpha, \beta))$. Also every Jacobi manifold $(M, \Lambda, E)$ is associated with a Lie algebroid $(T^*M \times \mathbb{R}, [\cdot, \cdot]_{(\Lambda, E)}, \#_{(\Lambda, E)})$, where the Lie bracket $[\cdot, \cdot]_{(\Lambda, E)} : (\Omega^1(M) \times C^\infty(M, \mathbb{R}))^2 \to \Omega^1(M) \times C^\infty(M, \mathbb{R})$, and the anchor map $\#_{(\Lambda, E)} : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \to \mathfrak{X}(M)$ are defined by (see [20])

$$[[\alpha, \beta]_{(\Lambda, E)}] = (i_{\#_\Lambda(\alpha)}d\beta - i_{\#_\Lambda(\beta)}d\alpha + d(\Lambda(\alpha, \beta)) + fE \wedge \beta - gE \wedge \alpha - i_E (\alpha \wedge \beta),$$

$$\Lambda(\beta, \alpha) + \#_\Lambda(\alpha)(g) - \#_\Lambda(\beta)(f) + fE(g) - gE(f), \quad (2.3)$$

for $(\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$.

As it has been observed in [22], a Lie algebroid structure on a vector bundle $A$ can be identified with a Gerstenhaber algebra structure (in the terminology of [22]) on the exterior algebra of multisections of $A$, $\Gamma(\wedge A) = \oplus_{k \geq 0} \Gamma(\wedge^k A)$, which is just a graded Poisson bracket (Schouten bracket) on $\Gamma(\wedge A)$ of degree -1 (linear).

A Schouten bracket induces the well-known generalization of the standard Cartan calculus of differential forms and vector fields [20]. The exterior derivative $d : \Gamma(\wedge^k A) \to \Gamma(\wedge^{k+1} A)$ is defined by the standard formula

$$d\mu(X_1, \ldots, X_{k+1}) = \sum_i (-1)^{i+1}[X_i, \mu(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})]$$

$$+ \sum_{i<j} (-1)^{i+j}\mu([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}). \quad (2.4)$$
For $X \in \Gamma(A)$, the contraction $i_X: \Gamma(\wedge^p A) \to \Gamma(\wedge^{p-1} A)$ is defined in the standard way and the Lie differential operator $\mathcal{L}_X$ is defined by the graded commutator $\mathcal{L}_X = i_X \circ d + d \circ i_X$.

It is obvious that the notion of Schouten bracket extends naturally to more general gradings in the algebra. For a graded commutative algebra with unity 1, a natural generalization of a graded Poisson bracket is graded Jacobi bracket. The only difference is that we replace the Leibniz rule by the generalized Leibniz rule:

$$[a, bc] = [[a, b]c + (-1)^{(|a||+k)|}b[a, c]] - [a, 1]bc. \quad (2.5)$$

Graded Jacobi brackets on $\Gamma(\wedge^1 A)$ of degree $k = -1$ (linear) is called Schouten-Jacobi brackets. An element $X \in \Gamma(\wedge^2 A)$ is called a canonical structure with respect to the corresponding Schouten-Jacobi bracket $\{\cdot, \cdot\}$ if $\{X, X\} = 0$.

Since Schouten brackets on $\Gamma(\wedge^A)$ are just Lie algebroid structures on $A$ (see [22]), by a generalized Lie algebroid (or Jacobi algebroid) structure on $A$ we mean a Schouten-Jacobi bracket on $\Gamma(\wedge^A)$. The generalized Lie algebroids are in one-one correspondence with pairs consisting of a Lie algebroid $A$ and a 1-cocycle $\phi_0 \in \Gamma(A^*)$ relative to the Lie algebroid exterior derivative $d$, i.e., $d\phi_0 = 0$, (cf. [13] [22] [17]).

A canonical example of a Jacobi algebroid is $(TM, (0, 1))$, where $TM = TM \oplus \mathbb{R}$ is the Lie algebroid of first-order linear differential operators on $C^\infty(M, \mathbb{R})$ with the bracket

$$\{(X, f), (Y, g)\}_1 = ([X, Y], X(g) - Y(f)), \quad X, Y \in \mathfrak{x}(M), \quad f, g \in C^\infty(M, \mathbb{R}), \quad (2.6)$$

and the 1-cocycle $\phi_0 = (0, 1)$ is $\phi_0((X, f)) = f$. Note that we have the canonical decomposition $X = X_1 + I \wedge X_2$ of any tensor $X \in \Gamma(\wedge^k TM)$, where $X_1$ (resp. $X_2$) is a $k$-vector field (resp. $(k-1)$-vector field) and $I$ represents the identity operator on $C^\infty(M, \mathbb{R})$ which is a generating section of $\mathbb{R}$ in $TM \oplus \mathbb{R}$. A canonical structure with respect to the corresponding Schouten-Jacobi bracket on the Grassmann algebra $\Gamma(\wedge TM)$ of first-order polydifferential operators on $C^\infty(M, \mathbb{R})$, which we will denote by $\{\cdot, \cdot\}_1$, turns out to be a standard Jacobi structure. Indeed, it is easy to see that the Schouten-Jacobi bracket reads

$$[A_1 + I \wedge A_2, B_1 + I \wedge B_2]_1 = [A_1, B_1]_{SN} + (-1)^{a} I \wedge [A_1, B_2]_{SN} + I \wedge [A_2, B_1]_{SN} + a A_1 \wedge B_2 - (-1)^{a} b A_2 \wedge B_1 + (a-b) I \wedge A_2 \wedge B_2.$$

Hence, the bracket $\{\cdot, \cdot\}$ on $C^\infty(M, \mathbb{R})$ defined by a bilinear differential operator $\Lambda + I \wedge \Gamma \in \Gamma(\wedge^2 TM) = \Gamma(\wedge^2 TM) \oplus \Gamma(TM)$ is a Lie bracket (Jacobi bracket on $C^\infty(M, \mathbb{R})$) if and only if

$$[\Lambda + I \wedge \Gamma, \Lambda + I \wedge \Gamma]_1 = [\Lambda, \Lambda]_{SN} + 2I \wedge [\Gamma, \Lambda]_{SN} + 2\Lambda \wedge \Gamma = 0.$$

Thus, we get the conditions defining a Jacobi structure on $M$.

There is another approach to Lie algebroids. As it was shown in [13] [15], a Lie algebroid structure (or the corresponding Schouten bracket) is determined by the Lie algebroid lift $X \mapsto X^c$ which associates with $X \in \Gamma(\wedge^A)$ a multivector field $X^c$ on $A$. The complete lifts of Lie algebroids are described as follows. For a given Lie algebroid structure on a vector bundle $A$ over $M$ there is a unique complete lift of elements $X \in \Gamma(\wedge^k A)$ of the Grassmann algebra $\Gamma(\wedge A) = \oplus_k \Gamma(\wedge^k A)$ to multivector fields $X^c \in \Gamma(\wedge^k (\mathcal{T}A))$ on $A$, such that

$$f^c = \iota_{df}, \quad X^c(\iota_{d\mu}) = \iota_{\mathcal{L}_{X\mu}}, \quad (X \wedge Y)^c = X^c \wedge Y^v + X^v \wedge Y^c,$$

for $f \in C^\infty(M, \mathbb{R})$, $X, Y \in \Gamma(A)$ and $\mu \in \Gamma(A^*)$, where $X \mapsto X^v$ is the standard vertical lift of tensors from $\Gamma(\wedge A)$ to tensors from $\Gamma(\wedge \mathcal{T}A)$. Moreover, this complete lift is a homomorphism of the corresponding Schouten brackets:

$$[X, Y]^c = [X^c, Y^c]_{SN} \quad \text{and} \quad [[X, Y]^v = [X^v, Y^v]_{SN}.$$
A vector bundle \( A \) is a smooth bundle such that the fiber at the point \( x \in M \), \( A_x = \pi^{-1}(x) \), is an affine space modelled on the vector space \( V_x = (V(\pi))^{-1}(x) \), and we pass from one local trivialization to another using the group of affine transformations. If \( A \) is a line bundle then a Lie QD-algebroid is a local Lie algebra in the sense of Kirillov \( [21] \). If rank \( A > 1 \), a Lie QD-algebroid on \( M \) is just a Lie algebroid on \( M \) (for more details, see \( [9] \)).

3 Affine Jacobi structures on affine bundles

Let \( \pi : A \to M \) be an affine bundle over \( M \) of rank \( n \) modelled on a vector bundle \( V(\pi) : V(A) \to M \), that is, \( \pi : A \to M \) is a (locally trivial) smooth bundle such that the fiber at the point \( x \in M \), \( A_x = \pi^{-1}(x) \), is an affine space modelled on the vector space \( V_x = (V(\pi))^{-1}(x) \), and we pass from one local trivialization to another using the group of affine transformations. If \( p \in M \), we denote by \( \text{Aff}(A_p, \mathbb{R}) \) the vector space of affine functions from the fiber \( A_p \) of \( \pi : A \to M \) at \( p \) and by \( A^+ \) the vector bundle \( A^+ = \bigcup_{p \in M} \text{Aff}(A_p, \mathbb{R}) \to M \) of rank \( n + 1 \). Note that \( A^+ \) has a distinguished 1-section, \( 1 : M \to A^+ \), defined by the constant function 1 on \( A \), i.e., \( 1(p) = 1_{A_p} \in \text{Aff}(A_p, \mathbb{R}) \). We call \( A^\dagger = (A^+, 1) \) the (special) vector dual of the affine bundle \( A \). In general, by a special vector bundle (cf. \( [7, 8] \)) we mean a vector bundle with a distinguished nowhere-vanishing section, so that \( A \to A^\dagger \) gives rise to a (contravariant) functor from the category of affine bundles to the category of special vector bundles.

We also have a dual functor which assigns to every special vector bundle \((V, X)\) an affine bundle \((V, X)^\dagger\) which is the affine bundle defined as the 1-level set of the linear function \( \psi \in \text{Aff}(V, X) \) and dual to \( V \). It is easy to see that for an affine bundle \( A \) we have \( (A^\dagger)^\dagger \simeq A \), so that we can identify \( A \) with an affine subbundle of \( \hat{A} = (A^\dagger)^\dagger \) (in fact, \( A = \iota_1^{-1}(1) \)). Note that we have a full duality, since also \( (V, X) = (((V, X)^\dagger)^\dagger, \hat{1}) \).

Using this fact, one can prove that there is a one-to-one correspondence between affine functions on \( A \) and linear functions on \( \hat{A} \). In fact, if \( a : A \to \mathbb{R} \) is an affine function on each fiber of \( A \) then the corresponding linear function \( \hat{a} : \hat{A} \to \mathbb{R} \) on each fiber of \( \hat{A} \) is given by \( \hat{a}(\psi_p) = \psi_p(a|_{A_p}) \), for all \( \psi_p \in \hat{A}_p = (A^\dagger_p)^\dagger \). Note that \( \hat{a}|_{A} = a \).

Moreover, there is an obvious natural one-to-one correspondence between affine functions and sections of \( A^\dagger \) which associates with the section of \( A^\dagger \), \( \hat{a} \in \Gamma(A^\dagger) \), the affine function \( a : A \to \mathbb{R} \) and the linear map \( \iota_a : \hat{A} \to \mathbb{R} \) is just the function \( \hat{a} \).

**Definition 3.1** A Jacobi structure on a vector bundle (resp. affine bundle) is called linear (resp. affine) if the corresponding Jacobi bracket of linear functions is again a linear function (resp. the bracket of affine functions is an affine function).

Now, we consider an affine Jacobi structure \((A_A, E_A)\) on an affine bundle \( A \). Denote by \( \{\cdot, \cdot\}^{\text{(standard)}}_{(A_A, E_A)} \) the corresponding Jacobi bracket.

If the rank of \( A \) is zero, i.e., \( A = M \times \{x_0\} \), then \((A_A, E_A)\) induces a Jacobi structure \((A, E)\) over \( M \).

Moreover, \( A^\dagger = M \times \mathbb{R} \) and thus \( \Gamma(A^\dagger) \cong C^\infty(M, \mathbb{R}) \). Therefore, the Jacobi bracket \( \{\cdot, \cdot\}^{\text{(standard)}}_{(A, E)} \) induces a Lie QD-algebroid structure on \( A^\dagger \). Conversely, if \( \{\cdot, \cdot\} \) defines a Lie QD-algebroid structure on \( A^\dagger \) then...
we have that a local Lie algebra structure on the real line bundle $A^+ = M \times \mathbb{R} \to M$ or equivalently, a Jacobi structure on $M$, i.e., an affine Jacobi structure on $A$.

Now, we suppose that the rank of the affine bundle $A$ is non-zero. Then, we have the following result.

**Lemma 3.2** Let $f : A \to \mathbb{R}$ be a basic function.

(i) If $a : A \to \mathbb{R}$ is an affine function, then $\{f, a\}_{(A,A,E_A)} - f\{1, a\}_{(A,A,E_A)}$ is a basic function.

(ii) If $g : A \to \mathbb{R}$ is a basic function, then $\{f, g\}_{(A,A,E_A)}$ is a basic function and

$$\{f, g\}_{(A,A,E_A)} = f\{1, g\}_{(A,A,E_A)} + g\{1, f\}_{(A,A,E_A)}.$$  \hspace{1cm} (3.1)

**Proof.**- Let $p$ be a point of $M$. Since rank $A > 0$ one can choose an affine function $b : A \to \mathbb{R}$ such that the linear function associated with the affine function $b|_{A_p} : A_p \to \mathbb{R}$ is non-zero. Then,

$$\{bf, a\}_{(A,A,E_A)} = b\{f, a\}_{(A,A,E_A)} + f\{b, a\}_{(A,A,E_A)} - bf\{1, a\}_{(A,A,E_A)}. \hspace{1cm} (3.1)$$

Since $a$ is an affine function, then $\{bf, a\}_{(A,A,E_A)}$ and $f\{b, a\}_{(A,A,E_A)}$ are affine functions and therefore, from \[\text{[C.1]}\], we have that $b\{f, a\}_{(A,A,E_A)} - f\{1, a\}_{(A,A,E_A)}$ is affine, that is, $(\{f, a\}_{(A,A,E_A)} - f\{1, a\}_{(A,A,E_A)})|_{A_p}$ is a constant function. Therefore, $\{f, a\}_{(A,A,E_A)} - f\{1, a\}_{(A,A,E_A)}$ is a basic function, i.e., (i) holds.

If $g$ is a basic function then, $\{bf, g\}_{(A,A,E_A)}$ and $f\{b, g\}_{(A,A,E_A)}$ are affine functions. Moreover, using (i) for the affine function $a \equiv 1$ and the basic function $g$, we have that $\{1, g\}_{(A,A,E_A)}$ is a basic function. Thus, since

$$\{bf, g\}_{(A,A,E_A)} = b\{f, g\}_{(A,A,E_A)} + f\{b, g\}_{(A,A,E_A)} - bf\{1, g\}_{(A,A,E_A)},$$

the function $b\{f, g\}_{(A,A,E_A)}$ is affine. Consequently, $(\{f, g\}_{(A,A,E_A)})|_{A_p}$ is a constant function. This proves that $\{f, g\}_{(A,A,E_A)}$ is a basic function. Furthermore, from (i), we obtain that

$$\{g, bf\}_{(A,A,E_A)} - g\{1, bf\}_{(A,A,E_A)} = f(\{g, b\}_{(A,A,E_A)} - g\{1, b\}_{(A,A,E_A)})$$

for all $a : A \to \mathbb{R}$ affine functions and $f_M \in C^\infty(M, \mathbb{R})$.

**Theorem 3.3** Let $(A,A,E_A)$ be an affine Jacobi structure on an affine bundle $\pi : A \to M$ and assume that the rank of $A$ is $n$, $n > 0$. Then, the bracket $[\cdot, \cdot]^+ : \Gamma(A^+ \times \Gamma(A^+) \to \Gamma(A^+)$ and the map $\rho^+ : \Gamma(A^+) \to \mathfrak{X}(M)$ given in \[\text{[3.2]}\] define a Lie algebroid structure on $A^+$.\]
Proof.- Since \( \{\cdot, \cdot\}_{(A,A,E_A)} \) is skew-symmetric and it satisfies the Jacobi identity, one deduces easily that \( (\Gamma(A^+), [\cdot, \cdot]^+) \) is a Lie algebra. Moreover, using Lemma 3.2 and the fact that \( \{\cdot, \cdot\}_{(A,A,E_A)} \) is a first-order bi-differential operator, we obtain that \( \rho^+ : A^+ \to TM \) is a homomorphism of vector bundles.

Finally, from the fact that \( \{\cdot, \cdot\}_{(A,A,E_A)} \) is a first-order bidifferential operator, we conclude that \( ([\cdot, \cdot]^+, \rho^+) \) is a Lie algebroid structure on \( A^+ \).

\[ \square \]

Remark 3.4 Let \( (A, A, E_A) \) be an affine Jacobi structure on an affine bundle \( \pi : A \to M \) of rank \( n, n > 0 \), and \( ([\cdot, \cdot]^+, \rho^+) \) be the corresponding Lie algebroid structure on \( A^+ \). Denote by \( \tilde{A} \) the linear Poisson structure on \( \hat{A} = (A^+)^* \) induced by the Lie algebroid structure \( ([\cdot, \cdot]^+, \rho^+) \). Then, we have that

\[ \ell_{\hat{a}, \hat{b}}^+ = \{\hat{a}, \hat{b}\}_{\tilde{A}}, \quad (3.3) \]

for \( a, b : A \to \mathbb{R} \) affine functions on \( A \). On the other hand, if \( \Delta_{\hat{A}} \) is the Liouville vector field we deduce that \( (\Delta_{\hat{A}}(\ell_1))(A) = (\ell_1)_{|A} = 1 \), which implies that \( \Delta_{\hat{A}} \) is a transverse vector field of \( A \) as a submanifold of \( \hat{A} \). Thus, using \ref{221}, Proposition 2.3 in \ref{5} and since \( \hat{A} \) is a homogeneous Poisson structure on \( \hat{A} \), we obtain that there exists a Jacobi structure \( (N_A', E_A') \) on \( A \) such that

\[ \{a, b\}_{(N_A', E_A')} = \{\hat{a}, \hat{b}\}_{\tilde{A}}, \quad (3.4) \]

for \( a, b : A \to \mathbb{R} \) affine functions on \( A \). Therefore, from \ref{882}, \ref{883} and \ref{884}, we conclude that \( \{a, b\}_{(N_A', E_A')} = \{a, b\}_{(A,A,E_A)} \), i.e., \( (N_A', E_A') \) is just the affine Jacobi structure \( (A_A, E_A) \).

Now, we will prove the converse of Theorem \ref{883}.

Theorem 3.5 Let \( ([\cdot, \cdot]^+, \rho^+) \) be a Lie algebroid structure on \( \pi^+ : A^+ \to M \) and assume that the rank of \( A \) is \( > 0 \). Then, there exists a unique affine Jacobi structure \( (A_A, E_A) \) on \( A \) such that

\[ \{a, b\}_{(A_A, E_A)} = [\hat{a}, \hat{b}]^+, \quad \forall a, b : A \to \mathbb{R} \text{ affine functions.} \quad (3.5) \]

Proof.- The uniqueness is deduced from the fact that a Jacobi structure is characterized by the Jacobi bracket of linear functions and the Jacobi bracket of a linear function and the constant function 1. Thus, two Jacobi structures satisfying \ref{3.5} are equal.

Now, we will define a Jacobi structure on \( A \) which satisfies \ref{3.5}. Denote by \( \tilde{A} \) the linear Poisson structure on \( \hat{A} \) induced by the Lie algebroid structure \( ([\cdot, \cdot]^+, \rho^+) \). Then, proceeding as in Remark \ref{884}, we have that there exists a Jacobi structure \( (A_A, E_A) \) on \( A \) such that

\[ \{a, b\}_{(A_A, E_A)} = \{\hat{a}, \hat{b}\}_{\tilde{A}}, \quad (3.6) \]

for \( a, b : A \to \mathbb{R} \) affine functions on \( A \). In fact, if \( \Delta_{\hat{A}} \) is the Liouville vector field on \( \hat{A} \), \( E_A \) is the hamiltonian vector field of the linear function \( \ell_1 : \hat{A} \to \mathbb{R} \) with respect to \( \tilde{A} \) and \( \Lambda_{\hat{A}} \) is the 2-vector on \( \hat{A} \) given by

\[ \Lambda_{\hat{A}} = \tilde{A} - \Delta_{\hat{A}} \wedge E_A, \quad (3.7) \]

then, we obtain that \( (A_A, E_A) \) is a Jacobi structure on \( \hat{A} \) and, from \ref{886}, it follows that \( \Lambda_{\hat{A}} \) (respectively, \( E_A \)) is the restriction to \( A \) of the 2-vector \( \Lambda_{\hat{A}} \) (respectively, \( E_A \)). Moreover, using again \ref{886}, it follows that the Jacobi structure \( (A_A, E_A) \) is affine and, in addition, \ref{885} holds.

\[ \square \]
Remark 3.6 Let $\pi : A \to M$ be an affine bundle such that the dual vector bundle $\pi^+ : A^+ \to M$ carries a Lie algebroid structure $[[\cdot, \cdot]^+, \rho^+]$. Let $(x^l)_{l=1,\ldots,m}$ be local coordinates on an open subset $U_M$ of $M$. We can consider $\{c_0, e_1, \ldots, e_n\}$ a local basis of sections of $\pi^+ : A^+ \to M$ such that $e_0$ is the section of $A^+$ associated with the linear function $\iota_1 : \hat{A} \to \mathbb{R}$. Then, there is an open coordinate neighbourhood $U$ on $\hat{A}$ with coordinates $(x^1, \ldots, x^n, \tau_1, y^1, \ldots, y^n)$, where $y^\alpha : U \to \mathbb{R}$ is the linear map associated with $e_\alpha$. If the structure functions of $[\cdot, \cdot]^+$ and the components of the anchor map $\rho^+$ for these coordinates are $c_{\alpha \beta}^\gamma, \rho^\alpha \in C^\infty(U_M, \mathbb{R})$ then $(\Lambda_A, E_A)$ is given by

$$
\Lambda_A = \sum_{\alpha < \beta} \sum_{\gamma = 1, \ldots, n} (c^\gamma_{0 \alpha} y^\gamma y^\beta - c^\gamma_{0 \beta} y^\gamma y^\alpha + c^\gamma_{\alpha \beta} y^\gamma + c^0_{0 \alpha} y^\beta - c^0_{0 \beta} y^\alpha + c^0_{\alpha \beta}) \frac{\partial}{\partial y^\gamma} \land \frac{\partial}{\partial y^\alpha} + \sum_{l=1, \ldots, m} \sum_{\alpha = 1, \ldots, n} (\rho^\alpha_l - y^\alpha \rho^0_l) \frac{\partial}{\partial y^\alpha} \land \frac{\partial}{\partial x^l},
$$

$$
E_A = \sum_{\beta = 1, \ldots, n} \sum_{\gamma = 1, \ldots, n} (c^\gamma_{0 \beta} y^\gamma + c^0_{0 \beta}) \frac{\partial}{\partial y^\gamma} + \sum_{l=1, \ldots, m} \rho^0_l \frac{\partial}{\partial x^l}. \tag{3.8}
$$

Note that, in general, the local components of $\Lambda_A$ are not affine functions.

From Theorems 3.3 and 3.5 and taking into account that there exists a one-to-one correspondence between affine Jacobi structures on an affine bundle $A$ of rank zero and Lie QD-algebroid structures on $A^+$, we obtain that

Corollary 3.7 Let $\pi : A \to M$ be an affine bundle of rank $n$. Then:

(i) If $n > 0$, there is a one-to-one correspondence between affine Jacobi brackets on $\pi : A \to M$ and Lie algebroid structures on the vector bundle $A^+$ uniquely determined by the equation 3.8.

(ii) If $n = 0$, there is a one-to-one correspondence between affine Jacobi brackets on $\pi : A \to M$ and local Lie algebra structures on $A^+ = M \times \mathbb{R}$.

Using the equivalence between Lie algebroids and linear Poisson brackets we can formulate also a Poisson version of the above Corollary. First, we introduce the following definition.

Definition 3.8 Let $A$ be an affine bundle over $M$ of rank zero. A $k$-vector $\bar{P}_{\hat{A}-\{O\}}$ on $\hat{A} - \{O\} = M \times (\mathbb{R} - \{0\})$ is said to be homogeneous (sometimes called also linear) if

$$
\bar{P}_{\hat{A}-\{O\}}(d(tf_1), \ldots, d(tf_k)) = (th)_{\hat{A}-\{O\}},
$$

for $f_1, \ldots, f_k \in C^\infty(M, \mathbb{R})$, where $t$ is the usual coordinate on $\mathbb{R}$ and $h \in C^\infty(M, \mathbb{R})$.

Now, we deduce

Corollary 3.9 Let $\pi : A \to M$ be an affine bundle on $M$ of rank $n$. Then:

(i) If $n > 0$, there is a one-to-one correspondence between affine Jacobi brackets $\{\cdot, \cdot\}_{A, \hat{A}}$ on $A$ and homogeneous Poisson brackets $\{\cdot, \cdot\}_{\hat{A}}$ on the vector bundle $A$, uniquely determined by the equation

$$
\{a, b\}_{A, \hat{A}} = \{\bar{a}, \bar{b}\}_{\hat{A}}
$$

for $a, b : A \to \mathbb{R}$ affine functions on $A$. The Jacobi structure $(\Lambda_A, E_A)$ on $A$ is the restriction to $A$ of the Jacobi structure $(\Lambda_{\hat{A}} - \Lambda_{\hat{A}} \land \hat{E}_A - \hat{E}_A)$ on $\hat{A}$, where $\hat{E}_A$ is the hamiltonian vector field of the linear function $\iota_1 : \hat{A} \to \mathbb{R}$ with respect to $\Lambda_{\hat{A}}$.
(ii) If \( n = 0 \), there is a one-to-one correspondence between affine Jacobi brackets \( \{ \cdot, \cdot \}_{(A, E)} \) on \( A \) and linear Poisson tensors \( \tilde{A}_{\Lambda - \{O\}} \) on \( A - \{O\} = M \times (\mathbb{R} - \{0\}) \), uniquely determined by the equation
\[
\{tf, tg\}_{\tilde{A}_{\Lambda - \{O\}}} = \tilde{A}_{\Lambda - \{O\}}(d(tf), d(tg)) = t(\{f, g\}_{(\Lambda, E)})_{\tilde{A}_{\Lambda - \{O\}}}
\]
for all \( f, g \in C^\infty(M \times (\mathbb{R} - \{0\}, \mathbb{R})) \), where \( t \) is the usual coordinate on \( \mathbb{R} \) and \( (\Lambda, E) \) is the Jacobi structure on \( M \) induced by \( (A, E) \). The linear Poisson tensor \( \tilde{A}_{\Lambda - \{O\}} \) is given by
\[
\tilde{A}_{\Lambda - \{O\}} = \frac{1}{t} \Lambda + \frac{\partial}{\partial t} \wedge E.
\]

**Remark 3.10** Let \( \pi : A \to M \) be an affine bundle on \( M \) of rank \( n \), \( n > 0 \), and \( \hat{A} \) the dual space of \( A^+ \). If \( (TA = TA \oplus \mathbb{R}, (0,1)) \) is the Jacobi algebroid of first-order differential operators on \( A \), then \( P + I \wedge Q \in \Gamma(\wedge^k TA) \) is affine if \( (P + I \wedge Q)(a_1, \ldots, a_k) \) is an affine function, for all \( a_1, \ldots, a_k : A \to \mathbb{R} \) affine functions on \( A \).

On the other hand, a \( k \)-vector \( \bar{P} \in \Gamma(\wedge^k T\hat{A}) \) on \( \hat{A} \) is linear if \( \bar{P}(\bar{a}_1, \ldots, \bar{a}_k) \) is a linear function, for all \( \bar{a}_1, \ldots, \bar{a}_k \) linear functions on \( \hat{A} \).

Now, let \( \bar{P} \in \Gamma(\wedge^k T\hat{A}) \) be a linear \( k \)-vector on \( \hat{A} \). We consider the \( k \)-section \( P' \) and the \((k - 1)\)-section \( Q' \) on \( \hat{A} \) given by
\[
P' = \bar{P} - \Delta_{\hat{A}} \wedge i(dt_1)\bar{P}, \quad Q' = i(dt_1)\bar{P}.
\]

Then, the restrictions \( P \) and \( Q \) to \( A \) of \( P' \) and \( Q' \), respectively, are tangent to \( A \) and \( P + I \wedge Q \in \Gamma(\wedge^k TA) \) defines an affine first-order differential operator on \( A \). In fact, we have that this correspondence between linear \( k \)-vectors on \( \hat{A} \) and affine first-order \( k \)-differential operators is one-to-one and that
\[
(P + I \wedge Q)(a_1, \ldots, a_k) = \bar{P}(\bar{a}_1, \ldots, \bar{a}_k)\big|_{\Lambda A}.
\]

for all \( a_1, \ldots, a_k \) affine functions on \( A \). Here \( \bar{a}_i : \hat{A} \to \mathbb{R} \) denotes the linear function associated with \( a_i : A \to \mathbb{R} \). Moreover, if \( \bar{P}_1 \) (respectively, \( \bar{P}_2 \)) is a linear \( k_1 \)-vector (respectively, \( k_2 \)-vector) on \( \hat{A} \) and \( [\cdot, \cdot]_1 \) is a Jacobi-Schouten bracket on \( A \) (see Section 2) then
\[
[P_1 + I \wedge Q_1, P_2 + I \wedge Q_2]_1 = [\bar{P}_1, \bar{P}_2]_{SN},
\]
where \( P_1 + I \wedge Q_1 \) and \( P_2 + I \wedge Q_2 \) are the corresponding affine first-order differential operators associated with \( \bar{P}_1 \) and \( \bar{P}_2 \), respectively. The details and proofs of these results can be found in [10].

Using the above facts one may directly deduce the first part of Corollary 3.3.

In the case \( n = 0 \), if \( \bar{P}_{\Lambda - \{O\}} \) is a linear \( k \)-vector on \( \hat{A} - \{O\} = M \times (\mathbb{R} - \{0\}) \) then we can consider the \( k \)-vector \( P' \) and the \((k - 1)\)-vector \( Q' \) on \( \hat{A} - \{O\} \) given by
\[
P' = t\bar{P}_{\Lambda - \{O\}} - t\frac{\partial}{\partial t} \wedge i(dt)\bar{P}_{\Lambda - \{O\}}, \quad Q' = i(dt)\bar{P}_{\Lambda - \{O\}}.
\]

The restrictions \( P \) and \( Q \) to \( M \) of \( P' \) and \( Q' \), respectively, are tangent to \( M \) and \( P + I \wedge Q \in \Gamma(\wedge^k (TM \oplus \mathbb{R})) \) defines an affine first-order differential operator on \( M \). Moreover, we have that this correspondence between linear \( k \)-vectors on \( \hat{A} - \{O\} \) and affine first-order \( k \)-differential operators is bijective. Note that the relation between local Lie algebras on rank 1 vector bundles \( L \) and homogeneous Poisson brackets on \( L - \{0\} \) has been first established by C.-M. Marle [28].
4 Examples

In this section we present some examples and applications of the above section.

1.– Affine Poisson structures and special Lie algebroid structures. Let \((V, X)\) be a special vector bundle over a manifold \(M\). A special Lie algebroid (resp. QD-algebroid) structure on \((V, X)\) is a Lie algebroid (resp. QD-algebroid) structure \(([\cdot, \cdot], \rho)\) on \((V, X)\) for which the section \(X\) belongs to the center of the Lie algebra \((\Gamma(V), [, ,])\), that is, \([X, Y] = 0\), for all \(Y \in \Gamma(V)\).

Then, if \(A\) is an affine bundle with rank non-zero (resp. zero), one can deduce from Theorems 13 and 15 and Corollary 8.7 that there is a one-to-one correspondence between affine Poisson structures on \(A\) and special Lie algebroid (resp. QD-algebroid) structures on \(A^\perp = (A^+, \hat{1})\).

2.– Affine Jacobi structures on an affine space and Lie algebra structures. Let \(A\) be an affine space of finite dimension \(n > 0\) modeled on the space vector \(V\). Then, using Corollary 8.7 we deduce that there is a one-to-one correspondence between affine Jacobi structures on \(A\) and Lie algebra structures on \(A^+\).

In the particular case, when \(A\) is a vector space \(V\), we have a one-to-one correspondence between affine Jacobi structures on \(V\) and Lie algebra structures on \(V^* \times \mathbb{R}, V^*\) being the dual vector space of \(V\).

As a consequence of these facts and Example 1, we obtain a well-known result (see 2) which establishes a bijection between affine Poisson structures on the vector space \(V\) and central extensions of Lie algebra structures on \(V^*\).

3.– Affine Jacobi structures and triangular generalized Lie bialgebroids. We recall that a triangular generalized Lie bialgebroid is a triple \(((A, [, ,], \rho), \phi_0, P)\), where \(A\) is a vector bundle over \(M\), \(([, ,], \rho)\) is a Lie algebroid structure on \(A\), \(\phi_0 \in \Gamma(A^+)^1\) is a 1-cocycle and \(P \in \Gamma(\wedge^2 A)\) a bisection on \(A\) satisfying \([P, P] + 2P \wedge i(\phi_0)P = 0\) (see 17).

Assume that \(\phi_0\) is nowhere vanishing and consider the affine bundle \(A_{\phi_0} = (A^*, \phi_0)^\perp\). A direct computation proves that the (special) vector dual \(A_{\phi_0}^\perp\) of \(A_{\phi_0}\) is isomorphic to the dual bundle \(A^*\) of \(A\). Thus, the vector bundles \(\hat{A}_{\phi_0}\) and \(A\) are isomorphic.

Now, we consider the Poisson complete lift to \(A \cong \hat{A}_{\phi_0}\) of \(P\) and \(\phi_0\) given by

\[ \hat{P}_{\phi_0}^c = P^c - \kappa_{\phi_0} P^v + \Delta_A \wedge (i_{\phi_0} P)^v. \]

\(\hat{P}_{\phi_0}^c\) is a linear Poisson structure on \(A \cong \hat{A}_{\phi_0}\) (see 11). Next, we will see which is the affine Jacobi structure on \(A_{\phi_0}\) induced by \(\hat{P}_{\phi_0}^c\).

In view of Corollary 8.9 this structure is the restriction to \(A_{\phi_0}\) of the Jacobi structure \((\hat{P}_{\phi_0}^c - \Delta_A \wedge E_{\phi_0}, E_{\phi_0})\), where \(E_{\phi_0}\) is the Hamiltonian vector field of \(\kappa_{\phi_0}\). Using the identities \(\Delta_A(\kappa_{\mu}) = \kappa_{\mu}, i_{d\kappa_{\mu}} X^c = (\kappa_{\mu})^c X + \kappa_{\mu} (d\mu)\) and \(i_{d\kappa_{\mu}} X^v = (\kappa_{\mu})^v X\) which are valid for any 1-form \(\mu\), we get

\[ E_{\phi_0} = i_{d\kappa_{\phi_0}} (P^c - \kappa_{\phi_0} P^v + \Delta_A \wedge (i_{\phi_0} P)^v) = (i_{\phi_0} P)^c - \kappa_{\phi_0} (i_{\phi_0} P)^v + \kappa_{\phi_0} (i_{\phi_0} P)^v - (i_{\phi_0} i_{\phi_0} P)^v \Delta_A = (i_{\phi_0} P)^c. \]

Hence

\[ \hat{P}_{\phi_0}^c - \Delta_A \wedge E_{\phi_0} = P^c - \kappa_{\phi_0} P^v - \Delta_A \wedge ((i_{\phi_0} P)^c - (i_{\phi_0} P)^v), \]

and the Jacobi structure on \(A_{\phi_0}\) is the restriction of the Jacobi structure

\[ (P^c - \kappa_{\phi_0} P^v - \Delta_A \wedge ((i_{\phi_0} P)^c - (i_{\phi_0} P)^v), (i_{\phi_0} P)^c). \]
Note that $\iota_{\phi_0} = 1$ on $A_{\phi_0}$. Let now $I_0$ be a section of $A_{\phi_0}$. We have the decomposition $A = A_0 \oplus (I_0) \simeq A_0 \oplus \mathbb{R}$, where $A_0 = V(A_{\phi_0}) = \text{Ker}(\phi_0)$ is a 1-codimensional vector subbundle of $A$. Since $d\phi_0 = 0$, $A_0$ is a Lie subalgebroid. Using the canonical linear coordinate $s$ in the 1-dimensional subbundle $(I_0) \simeq \mathbb{R}$, we have that

$$(I_0)^v = \partial_s, \quad \iota_{\phi_0} = s, \quad Q_0^c = Q_0^c + s[I_0, Q_0]^v,$$

for $Q_0 \in \Gamma(\Lambda^1 A_0)$, where $c_0$ and $v_0$ denote the complete and vertical lift of the Lie algebroid $A_0$. Here, of course, we understand tensors on $A_0$ as tensors on $A \simeq A_0 \times \mathbb{R}$ in obvious way. Note that if we identify $A_{\phi_0}$ with $A_0$ via the translation by $I_0$, then the restriction of $Q_0^c$ to $A_{\phi_0}$ is tangent to $A_{\phi_0}$ and such a restriction is the complete lift of $Q_0$ with respect to the Lie affgebroid structure on $A_{\phi_0}$ in the terminology of [17]. A Lie affgebroid is a possible generalization of the notion of a Lie algebroid to affine bundles. The main motivation of the study of this concept was to create a geometrical model which would be a natural environment for a time-dependent version of Lagrange equations on Lie algebroids (cf. [17] [8] [29] [32]).

Now, we can decompose $P = \Lambda + I_0 \wedge E$, where $\Lambda \in \Gamma(\Lambda^2 A_0)$ and $E \in \Gamma(A_0)$, and we deduce

$$P^c = \Lambda^{c_0} + s[I_0, \Lambda]^{v_0} + I_0^c \wedge E^{v_0} + \partial_s \wedge (E^{c_0} + s[I_0, E]^{v_0}) \quad \text{and} \quad P^v = \Lambda^{v_0} + \partial_s \wedge E^{v_0}.$$

Thus, writing $\Delta_A = \Delta_{A_0} + s\partial_s$, we get finally the Jacobi structure on $A_{\phi_0}$ (identified with $A_0$ via the translation by $I_0$) in the form

$$(\Lambda^{c_0} - \Lambda^{v_0} + [I_0, \Lambda]^{v_0} + (I_0)^c \wedge E^{v_0} - \Delta_{A_0} \wedge (E^{c_0} - E^{v_0} + [I_0, E]^{v_0}), E^{v_0}).$$

In particular, if $\rho(I_0) = 0$ and $I_0$ is central, i.e., $(A, I_0)$ is a special Lie algebroid, then $(I_0)^c = 0$, so we end up with the Jacobi structure

$$(\Lambda^{c_0} - \Lambda^{v_0} - \Delta_{A_0} \wedge (E^{c_0} - E^{v_0}), E^{v_0}). \quad (4.1)$$

Now, we consider a particular example of triangular generalized Lie bialgebroid.

Let $(A, [\cdot, \cdot]_A, \rho_A)$ be a Lie algebroid over $M$ and let $(\Lambda, E) \in \Gamma(\Lambda^2 A) \oplus \Gamma(A)$ be a pair satisfying the following properties

$$[\Lambda, \Lambda]_A = -2\Lambda \wedge E, \quad [\Lambda, E]_A = 0.$$

Here $[\cdot, \cdot]_A$ denotes the Schouten bracket associated with the Lie algebroid $A$.

We will prove that, in such a case, it is possible to define an affine Jacobi structure over $A$. In fact, we can consider the Lie algebroid structure $([\cdot, \cdot]_{A_1}, \rho_{A_1})$ over $A_1 = A \oplus \mathbb{R}$ given as follows

$$[[X, f], (Y, g)]_{A_1} = ([X, Y]_A, \rho_A(X)(g) - \rho_A(Y)(f)), \quad \rho_{A_1}(X, f) = \rho_A(X),$$

for all $(X, f), (Y, g) \in \Gamma(A) \times C^\infty(M, \mathbb{R}) \cong \Gamma(A_1)$. The pair $(0, 1) \in \Gamma(A^* \oplus \mathbb{R}) \cong \Gamma(A^*_1)$ defines a 1-cocycle for this algebroid. Moreover, since $P = \Lambda + I \wedge E$ is a canonical structure for the corresponding Schouten-Jacobi bracket, $(A_1, (0, 1), (A, E))$ is a triangular generalized Lie bialgebroid (see [17]). In this case, $I_0 = (0, 1) \in \Gamma(A_1)$ is central and $\rho_{A_1}(I_0) = 0$. Thus, we have defined an affine Jacobi structure on $A$ given by

$$(A_A, E_A) = (A^c - A^v - \Delta_A \wedge (E^c - E^v), E^c), \quad (4.2)$$

where the complete lifts are lifts for the Lie algebroid $A$.
Now, if we consider the particular case when \((A, [\cdot, \cdot]_A, \rho_A) = (TM, [\cdot, \cdot]_{SN}, 1_{TM})\) and \((M, \Lambda, E)\) is a Jacobi manifold, then we have that the affine Jacobi structure \((\Lambda_{TM}, E_{TM})\) on \(TM\) (the affine tangent Jacobi structure on \(TM\)) is given by (see [12])

\[
\Lambda_{TM} = \Lambda^c - \Lambda^v - \Delta_{TM} \land (E^c - E^v), \quad E_{TM} = E^c,
\]

where \(\Delta_{TM}\) is the Liouville vector field on \(TM\), \(\Lambda^c\) (resp. \(E^c\)) and \(\Lambda^v\) (resp. \(E^v\)) are the complete and vertical lift of \(\Lambda\) (resp. \(E\)). This structure was first considered by Vaisman in [37]. If \(E = 0\) (that is, \((M, \Lambda)\) is a Poisson manifold) we obtain the affine tangent Poisson structure on \(TM\) given by \(\Lambda_{TM} = \Lambda^c - \Lambda^v\).

Note that the linear tangent Poisson structure on \(TM\) is \(\Lambda^c\).

4. Homogeneous Jacobi structures and Jacobi algebroids. A Jacobi structure \((\Lambda, E)\) on a manifold \(M\) is called homogeneous of degree \(k\) with respect to a vector field \(\Delta\) on \(M\) if \(\Lambda\) and \(E\) are homogeneous of degree \(k\) with respect to \(\Delta\). A homogeneous Jacobi structure of degree \(-1\) we will just call homogeneous. By a homogeneous Jacobi structure on a vector bundle \(A\), we will always understand a Jacobi structure which is homogeneous with respect to the Liouville vector field \(\Delta_A\). Then, we have the following characterizations.

**Theorem 4.1** Let \((\Lambda, E)\) be a Jacobi structure on a vector bundle \(A\). Then, the following are equivalent:

(a) \((\Lambda, E)\) is homogeneous;

(b) The Jacobi bracket \({\cdot, \cdot}\)\((\Lambda, E)\) is linear and affine and the bracket of a linear function and the constant function \(1\) is a basic function;

**Proof.** (a) \(\Rightarrow\) (b) If \(\mu, \nu\) are sections of \(A^*\) then, from [24], it follows that

\[
D(\{t_\mu, t_\nu\}_{(\Lambda, E)}) = \{D(t_\mu), t_\nu\}_{(\Lambda, E)} + \{t_\mu, D(t_\nu)\}_{(\Lambda, E)} = 0,
\]

for \(D = \Delta_A - I\), which implies that \(\{t_\mu, t_\nu\}_{(\Lambda, E)}\) is linear.

On the other hand, since \(E\) is homogeneous, we obtain that \(\Delta_A(E(t_\mu)) = 0\) and thus \(E(t_\mu) = \{1, t_\mu\}_{(\Lambda, E)}\) is a basic function. Consequently, using the results of [19], we deduce that \(\{\cdot, \cdot\}_{(\Lambda, E)}\) is affine.

(b) \(\Rightarrow\) (a) Let \(f\) be a basic function. If \(\mu\) is a section of \(A^*\) then \(\{1, f t_\mu\}_{(\Lambda, E)} = E(f t_\mu) = E(f t_\mu + f(1, t_\mu)_{(\Lambda, E)}\) is a basic function. Therefore, \(E(f) = 0\). Moreover, \(\Delta_A(E(t_\mu)) = \Delta_A(\{1, t_\mu\}_{(\Lambda, E)}) = 0\). Consequently, \(\Delta_A(E) = -E\).

Next, we will prove that \(\Lambda\) is linear. For \(\mu, \nu \in \Gamma(A^*)\), we have

\[
\Lambda(dt_\mu, dt_\nu) = \{t_\mu, t_\nu\}_{(\Lambda, E)} - t_\mu E(t_\nu) + t_\nu E(t_\mu)
\]

since \(E(t_\mu)\) and \(E(t_\nu)\) are basic functions, we conclude that \(\Lambda(dt_\mu, dt_\nu)\) is a linear function. This implies that \(\Lambda\) is homogeneous (see Section 2).

\(\square\)

Let \(\pi: A \to M\) be a vector bundle over the manifold \(M\) of rank \(n, n > 0\). Consider a homogeneous Jacobi structure \((\Lambda_A, E_A)\) on \(A\). Then \((\Lambda_A, E_A)\) is an affine Jacobi structure on \(A\) (see Theorem 11). Thus, from Theorem 3.3 we have that there exists a Lie algebroid structure on \(A^+ = A^* \oplus \mathbb{R}\). It is not difficult to show that such a Lie algebroid structure is given by

\[
[(\alpha, f), (\beta, g)]^+ = [(\alpha, \beta) + \rho_*(\alpha)(g) - \rho_*(\beta)(f) + (g\alpha - f\beta)(X_0)), \quad \rho^*(\alpha, f) = \rho_*(\alpha)
\]
for all \((\alpha, f), (\beta, g) \in \Gamma(A^*) \times C^\infty(M, \mathbb{R}) \cong \Gamma(A^+)\), where \([\cdot, \cdot], _\rho_x\) is a Lie algebroid structure on \(A^*\) and \(X_0\) is a 1-cocycle on \(A^*\).

Using the above fact and Corollary \ref{cor}, we conclude that there exists a one-to-one correspondence between homogeneous Jacobi structures on \(A\) and Jacobi algebroid structures on \(A^*\). This result was proved in \cite{ref}.

\section{Affine-homogeneous Jacobi structures on an affine bundle}

Let \(\pi : A \to M\) be a vector bundle over a manifold \(M\). We stress the existence of the canonical family \(\mathcal{V}(A) = \{X^v : X \in \Gamma(A)\}\) of vertical lifts of sections of \(A\). These are pair-wise commuting vector fields. They can be viewed as invariant vector fields on \(A\) when we view \(A\) as a commutative Lie groupoid. Any tensor \(Y\) on \(A\) is called invariant if it is invariant with respect to \(\mathcal{V}(A)\), i.e., the Lie derivative \(\mathcal{L}_X Y\) vanishes for every \(X \in \mathcal{V}(A)\). It is easy to see that the set of invariant vector fields coincides with \(\mathcal{V}(A)\).

It is worth noticing that linear tensor fields have a special property closely related to the fact that they live on vector bundles. On the zero-section of a vector bundle \(A\) we have the full decomposition of \(T_z A\) into the vertical and the horizontal parts. This, of course, makes sense for any contravariant tensor and we will say that a \(r\)-vector \(\Lambda\) on \(A\) is \textit{vertically vanishing on the zero-section} if its vertical part vanishes on the zero-section. This simply means that \(\Lambda(d\mu_1, \ldots, d\mu_r) \circ O = O\), for \(\mu_1, \ldots, \mu_r \in \Gamma(A^*)\), where \(O\) is the zero-section of \(A\). In fact, if \(\Lambda\) is a \(r\)-vector on \(A\) then one may define a section \(V_\Lambda\) of the vector bundle \(\wedge^r A \to M\) as follows. If \(\mu_1, \ldots, \mu_r \in \Gamma(A^*)\) then

\[V_\Lambda(\mu_1, \ldots, \mu_r) = \Lambda(d\mu_1, \ldots, d\mu_r) \circ O.\]

Now, a 2-vector \(\Lambda\) on \(A\) is \textit{affine} if \(\Lambda(da, db)\) is an affine function, for \(a, b : A \to \mathbb{R}\) affine functions on \(A\). It is easy to prove that \(\Lambda\) is affine if and only if \(\Lambda - V^v_\Lambda\) is a linear 2-vector on \(A\), where \(V^v_\Lambda\) is the standard vertical lift of \(V_\Lambda \in \Gamma(\wedge^2 A)\). Furthermore, we have

\textbf{Lemma 5.1} Let \(\Lambda\) be a 2-vector on a vector bundle \(A\). Then, \(\Lambda\) is affine if and only if \(\mathcal{L}_X \Lambda\) is an invariant bivector field for any invariant vector field \(X \in \mathcal{V}(A)\). Moreover, \(\Lambda\) is linear if and only if it is affine and vertically vanishes on the zero-section \(A\).

\textbf{Proof.} The proof is obvious and depends on the fact that if \(f : A \to \mathbb{R}\) is a smooth real function on \(A\) then \(f\) is affine if and only if \(X(f)\) is a basic function, for any invariant vector field \(X \in \mathcal{V}(A)\). In addition, \(f\) is linear if and only if \(X(f)\) is a basic function, for all \(X \in \mathcal{V}(A)\) and \(f \circ O\) identically vanishes. Using these facts and \cite{22}, we deduce the result. \(\square\)

We recall that a 2-vector \(\Lambda\) on a vector bundle is linear if and only if it is homogeneous (with respect to the Liouville vector field of \(A\)). On the other hand, a Jacobi structure \((\Lambda, E)\) on \(A\) is homogeneous if \(\Lambda\) and \(E\) are homogeneous (see Example 4 in Section \ref{section}). Next, using the definition of invariant tensor fields on \(A\), we will characterize homogeneous Jacobi structures.

\textbf{Theorem 5.2} Let \((\Lambda, E)\) be a Jacobi structure on a vector bundle \(A\). Then, the following sentences are equivalent:

(a) \((\Lambda, E)\) is homogeneous;
(b) $E \in \mathcal{V}(A)$ and there is a linear Poisson structure $\bar{\Lambda}$ such that

$$\Lambda = \bar{\Lambda} + E \wedge \Delta_A;$$

(c) $E \in \mathcal{V}(A)$, $\mathcal{L}_X \Lambda$ is an invariant bivector field for any invariant vector field $X \in \mathcal{V}(A)$ and $\Lambda$ vanishes vertically on the zero-section of $A$.

Proof. (a) $\Rightarrow$ (b) Let $f$ be a basic function. If $\mu$ is a section of $A^*$ then, from Theorem 4.1, we deduce that the function $\{1, f\mu\}_{(\Lambda, E)} = E(f\mu) = E(f)\mu + f\{1, \mu\}_{(\Lambda, E)}$ is a basic function. Thus, $E(f) = 0$. Therefore, since $\mathcal{E}(i_v)$ is a basic function, for all $v \in \Gamma(A^*)$, we conclude that $E \in \mathcal{V}(A)$.

Now, using that the Jacobi bracket $\{\cdot, \cdot\}_{(\Lambda, E)}$ is linear (see Theorem 4.1), we obtain that $\Lambda$ is linear. This implies that $\mathcal{L}_\Delta \Lambda = -\Lambda$ and, since that $E \in \mathcal{V}(A)$, it follows that $[\Lambda, \Lambda]|_\mathcal{SN} = 0$, that is, $\bar{\Lambda}$ is a Poisson structure on $A$. Finally, from (2.4) and using that $\Lambda$ is linear and the fact that $E \in \mathcal{V}(A)$, we have that $\bar{\Lambda}$ is a linear 2-vector on $A$.

(b) $\Rightarrow$ (c) It follows from Lemma 5.1.

(c) $\Rightarrow$ (a) If $E \in \mathcal{V}(A)$, it is clear that $E$ is homogeneous. Therefore, using again Lemma 5.1, we have that $\Lambda$ is linear and, consequently, $\Lambda$ is homogeneous.

$\square$

Note that the description of homogeneity of tensor fields given in (c) of the above theorem is a variant of the description of multiplicative tensors on Lie groups or Lie groupoids, here in the commutative case.

We can also try to define homogeneity for affine bundles. In order to do this we describe homoge-neous Jacobi structures, using the Schouten-Jacobi bracket $[\cdot, \cdot|_1$ on the Grassmann algebra of first-order polydifferential operators.

**Theorem 5.3** (a) A Jacobi structure $(\Lambda, E)$ on a manifold $M$ is homogeneous with respect to a vector field $\Delta$ if and only if

$$[\Delta - I, \Lambda + I \wedge E]|_1 = 0, \quad (5.1)$$

where $[\cdot, \cdot]|_1$ is the canonical Schouten-Jacobi bracket on the Grassmann algebra of first-order polydif-

erential operators on $M$.

(b) A Jacobi structure $(\Lambda, E)$ on a vector bundle $A$ is homogeneous if and only if the bivector field $\Lambda$ vertically vanishes on the zero-section of $A$ and

$$[(X_1)^\nu, [(X_2)^\nu, \Lambda + I \wedge E]|_1]_1 = 0, \text{ for all } X_1, X_2 \in \Gamma(A) \oplus C^\infty(M, \mathbb{R}). \quad (5.2)$$

Proof. (a) It is a direct consequence from $[\Delta - I, \Lambda + I \wedge E]|_1 = [\Delta, \Lambda]|_\mathcal{SN} + I \wedge [\Delta, E]|_\mathcal{SN} + \Lambda + I \wedge E$.

(b) Suppose that $\Lambda$ vertically vanishes on the zero-section of $A$ and that (5.2) holds. Then, since for any $X \in \Gamma(A)$, $[X^\nu, [(1_M)^\nu, \Lambda + I \wedge E]|_1]_1 = [E, X^\nu]|_\mathcal{SN} = 0$, the vector field $E$ is invariant, i.e., $E \in \mathcal{V}(A)$.

Hence, for any $X_1, X_2 \in \Gamma(A)$, $[X_1^\nu, [X_2^\nu, \Lambda + I \wedge E]|_1]_1 = [X_1^\nu, [X_2^\nu, \Lambda]|_\mathcal{SN}]_1 = 0$, i.e., $\Lambda$ is affine. Thus, from Lemma 5.1 we deduce that $\Lambda$ is linear.

Conversely, if $\Lambda$ is linear and $E$ is invariant, then for any $X \in \Gamma(A)$, $f \in C^\infty(M, \mathbb{R})$,

$$[X^\nu + f^\nu, \Lambda + I \wedge E]|_1 = [X^\nu, \Lambda]|_\mathcal{SN} + [f^\nu, \Lambda]|_\mathcal{SN} - f^\nu E.$$

But $[X^\nu, \Lambda]|_\mathcal{SN}$ and $([f^\nu, \Lambda]|_\mathcal{SN} - f^\nu E)$ are invariant tensors and the theorem follows. $\square$
Since on an affine bundle $A$ no Liouville vector field and no linear functions exist, we will use a concept of homogeneity suggested by Theorem 5.3. The concept of invariance is clear: the model vector bundle $V(A)$ acts on $A$ by translations, so we can lift vertically sections $X$ of $V(A)$ to vector fields $X^v$ on $A$. We lift vertically functions on $M$ to functions on $A$ in obvious way. Denote the vector space of first-order differential operators spanned by vertical lifts of both types by $\mathcal{V}_1(A)$. It is easy to see that $\mathcal{V}_1(A)$ is a maximal subalgebra in the Lie algebra of all first order differential operators on $A$. A first-order polydifferential operator $F$ on $A$ is called affine-invariant if it is invariant with respect to elements of $\mathcal{V}_1(A)$, i.e., $[D,F] = 0$ for any $D \in \mathcal{V}_1(A)$, where $[,]_1$ is the canonical Schouten-Jacobi bracket on the Grassmann algebra of first-order polydifferential operators on $A$ and affine-homogeneous if $[D,F]$ is affine-invariant for any $D \in \mathcal{V}_1(A)$. In other words, $F$ is affine-homogeneous if $[D_1,[D_2,F]]_1 = 0$ for all $D_1,D_2 \in \mathcal{V}_1(A)$. In particular, we propose the following

**Definition 5.4** A Jacobi structure $(\Lambda_A,E_A)$ on the affine bundle $A \to M$ is said to be affine-homogeneous if $[D_1,[D_2,\Lambda_A + I \wedge E_A]]_1 = 0$ for all $D_1,D_2 \in \mathcal{V}_1(A)$.

Although the Jacobi bracket associated with an affine Jacobi manifold $(A,\Lambda,E)$ is closed with respect to the affine functions, the hamiltonian vector field $X^A_{(\Lambda,E)}$ of an affine function $a : A \to \mathbb{R}$ is not, in general, affine, that is, if $b : A \to \mathbb{R}$ is an affine function then $X^A_{(\Lambda,E)}(b)$ is not, in general, an affine function.

In fact, if $A$ is the vector space $\mathbb{R}^3$ and we consider the Jacobi structure $(\Lambda,E)$ on $A$ given by

$$
\Lambda = x_1 x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} - x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3}, \quad E = x_1 \frac{\partial}{\partial x_2}, \quad (5.3)
$$

where $(x_1,x_2,x_3)$ are canonical coordinates in $\mathbb{R}^3$, then $(\Lambda,E)$ is an affine Jacobi structure over $\mathbb{R}^3$ and $X^A_{(\Lambda,E)}(x_3) = x_1 x_3$ is not an affine function. Note that, in the case of affine Poisson structures, the hamiltonian vector fields of affine functions are affine.

**Definition 5.5** An affine Jacobi structure $(\Lambda_A,E_A)$ over an affine bundle $\pi : A \to M$ is a strongly-affine Jacobi structure if the hamiltonian vector field $X^A_{(\Lambda_A,E_A)}$ is affine, for every affine function $a : A \to \mathbb{R}$.

The following result relates affine-homogeneous and strongly-affine Jacobi structures.

**Proposition 5.6** Let $(\Lambda_A,E_A)$ be a Jacobi structure over an affine bundle $\pi : A \to M$ of rank $n$, $n > 0$. Then, the following sentences are equivalent:

(i) $(\Lambda_A,E_A)$ is affine-homogeneous;

(ii) $E_A$ is affine-invariant and $\Lambda_A$ is affine;

(iii) $(\Lambda_A,E_A)$ is strongly-affine;

(iv) $(\Lambda_A,E_A)$ is affine and basic functions form an ideal in the algebra of affine functions with respect to the corresponding Jacobi bracket;

(v) $(\Lambda_A,E_A)$ is affine and there exists $\bar{X}_0 \in \Gamma(\bar{A})$ such that

$$
E_A(a) = -\bar{X}_0(\bar{a}) \circ \pi \quad (5.4)
$$

for all affine functions $a : A \to \mathbb{R}$, where $\bar{a} \in \Gamma(\bar{A}^+)$ is the section of $\bar{A}^+$ associated with the affine function $a$. 


Remark 5.7

If \( (\Lambda, E) \) is an affine function and, thus, \( \Lambda \) is an affine function. In fact, if \( p \) is a point of \( M \), we will show that \( \{ (a, f) \}_{(\Lambda, E)} \) is constant. Consequently, from (5.5) and (5.6), we deduce that \( \tilde{X}_0 : \Gamma(A^+) \to C^\infty(M, \mathbb{R}) \) is a section of the vector bundle \( \tilde{A} = (A^+)^* \).

Proof.- (i) \( \Leftrightarrow \) (ii) Proceeding as in the proof of Theorem 5.3 (b) we deduce that the sentences (i) and (ii) are equivalent.

(ii) \( \Rightarrow \) (iii) Let \( a \) be an affine function and consider the hamiltonian vector field \( X_a^{(\Lambda, E)} \) of \( a \) with respect to \( (\Lambda, E) \). If \( b : A \to \mathbb{R} \) is an affine function then

\[
X_a^{(\Lambda, E)}(b) = \Lambda_A(da, db) + aE_A(b).
\]

Now, the condition \([X, E_A] = 0\), for all \( X \in \mathcal{V}(A) \), implies that \( E_A \in \mathcal{V}(A) \) and, thus, \( E_A(b) \) is a basic function. Therefore, \( X_a^{(\Lambda, E)}(b) \) is an affine function.

(iii) \( \Rightarrow \) (iv) Let \( a : A \to \mathbb{R} \) be an affine function. We will prove that \( E_A(a) = \{ 1, a \}_{(\Lambda, E)} \) is a basic function. Indeed, if \( p \) is a point of \( M \), we will show that \( \{ (a, f) \}_{(\Lambda, E)} \) is constant. For this purpose, we consider an affine function \( b : A \to \mathbb{R} \) such that the restriction to \( V(A)_p \), \( b_{\mid V(A)_p} \), of the linear map \( b \) associated with \( b \) is not zero. Then, we have that \( X_a^{(\Lambda, E)}(b) = \{ a, b \}_{(\Lambda, E)} - bE_A(a) \) and \( \{ a, b \}_{(\Lambda, E)} \) are affine functions. Thus, \( bE_A(a) \) is an affine function and \( E_A(a) \) is constant.

Now, suppose that \( f \) is a basic function. Then,

\[
\{ a, fb \}_{(\Lambda, E)} = f\{ a, b \}_{(\Lambda, E)} + b\{ a, f \}_{(\Lambda, E)} - fb\{ a, 1 \}_{(\Lambda, E)}
\]

is an affine function. This implies that \( b\{ a, f \}_{(\Lambda, E)} \) is an affine function and, therefore, \( \{ (a, f) \}_{(\Lambda, E)} \mid_{A_p} \) is constant. Consequently, we have proved that \( \{ (a, f) \}_{(\Lambda, E)} \) is a basic function.

(iv) \( \Rightarrow \) (v) Let \( a : A \to \mathbb{R} \) be an affine function on \( A \). Then, the function \( E_A(a) = \{ 1, a \}_{(\Lambda, E)} \) is basic. Thus, there exists \( \tilde{X}_0(\tilde{a}) \in C^\infty(M, \mathbb{R}) \) such that

\[
E_A(a) = -\tilde{X}_0(\tilde{a}) \circ \pi.
\]

Next, we will prove that if \( f \) is a basic function then

\[
E_A(f) = \{ 1, f \}_{(\Lambda, E)} = 0.
\]

Suppose that \( p \) is a point of \( M \) and that \( a : A \to \mathbb{R} \) is an affine function such that the restriction to \( V(A)_p \), \( a_{\mid V(A)_p} \), of the linear map \( a \) associated with \( a \) is not zero. Then, \( \{ 1, fa \}_{(\Lambda, E)} = \{ 1, f \}_{(\Lambda, E)} a + f\{ 1, a \}_{(\Lambda, E)} \) is a basic function and therefore, \( \{ (1, f) \}_{(\Lambda, E)} \mid_{A_p} = 0 \). Consequently, from (5.5) and (5.6), we deduce that \( \tilde{X}_0 : \Gamma(A^+) \to C^\infty(M, \mathbb{R}) \) is a section of the vector bundle \( \tilde{A} = (A^+)^* \).

(v) \( \Rightarrow \) (ii) Let \( f \) be a basic function on \( A \). Using (5.4), it follows that \( aE_A(f) = 0 \), for any affine function \( a : A \to \mathbb{R} \) on \( A \). This implies that

\[
E_A(f) = 0.
\]

Consequently, from (5.3) and (5.7), we obtain that \( E_A \) is affine-invariant.

On the other hand, if \( a, b : A \to \mathbb{R} \) are affine functions then, \( \{ a, b \}_{(\Lambda, E)} = \Lambda_A(da, db) + aE_A(b) - bE_A(a) \) is an affine function and, thus, \( \Lambda_A(da, db) \) is again an affine function. This proves that \( \Lambda_A \) is affine. \( \square \)

Remark 5.7 If \( (\Lambda, E) \) is an affine-homogeneous Jacobi structure on an affine bundle \( A \) then, from Proposition 5.6, we can deduce that it is an affine Jacobi structure and that the local expressions of \( \Lambda_A \) and \( E_A \) are as in (5.3). Moreover, in this case, \( c^0_{\alpha} = 0 \), for all \( \alpha, \beta = 1, \ldots, n \).

Next, we will establish a one-to-one correspondence between strongly-affine Jacobi structures over an affine bundle \( \pi : A \to M \) and a particular class of Lie algebroid structures on \( A^+ \).
**Definition 5.8** An almost-special Lie algebroid structure on a special vector bundle \((V, X)\) is a Lie algebroid structure \(([\cdot, \cdot], \rho)\) on \(V\) such that the submodule of \(\Gamma(V)\) generated by the section \(X\) is an ideal of the Lie algebra \((\Gamma(V), [\cdot, \cdot])\).

**Remark 5.9** i) Every special vector bundle is of the form \((A^+, \tilde{1})\) for an affine bundle \(A^+\).

ii) A Lie algebroid structure \(([\cdot, \cdot]^+, \rho^+)\) on \(A^+\) is almost-special if there exists a map \(\tilde{X}_0 : \Gamma(A^+) \to C^\infty(M, \mathbb{R})\) such that

\[
[\tilde{1}, \tilde{a}]^+ = -\tilde{X}_0(\tilde{a})\tilde{1}, \quad \text{for all } \tilde{a} \in \Gamma(A^+).
\]

This type of Lie algebroids has the following properties.

**Proposition 5.10** Let \(\pi : A \to M\) be an affine bundle of rank \(n > 0\), and \(([\cdot, \cdot]^+, \rho^+)\) be an almost-special Lie algebroid structure over \(A^+\). Then:

1. \(\rho^+(\tilde{1}) = 0\),
2. \(\tilde{X}_0\) is a 1-cocycle of the Lie algebroid \((A^+, [\cdot, \cdot]^+, \rho^+)\).

**Proof.** Consider \(f_M \in C^\infty(M, \mathbb{R})\) and \(p \in M\). We will prove that \(\rho^+(\tilde{1})(f_M)(p) = 0\). In fact, if \(\tilde{a} \in \Gamma(A^+)\), then

\[
\tilde{X}_0(f_M\tilde{a})\tilde{1} = -[\tilde{1}, f_M\tilde{a}]^+ = -f_M[\tilde{1}, \tilde{a}]^+ - \rho^+(\tilde{1})(f_M)\tilde{a} = f_M\tilde{X}_0(\tilde{a})\tilde{1} - \rho^+(\tilde{1})(f_M)\tilde{a}.
\]

Thus,

\[
(f_M\tilde{X}_0(\tilde{a}) - \tilde{X}_0(f_M\tilde{a}))\tilde{1} - \rho^+(\tilde{1})(f_M)\tilde{a} = 0. \quad (5.9)
\]

If we consider \(a : A \to \mathbb{R}\) an affine function such that the associated linear function \(\tilde{a}\) satisfies \(\tilde{a}(p) \neq 0\), from (5.9), one can deduce that \(\rho^+(\tilde{1})(f_M)(p) = 0\). So, we have (i).

Substituting (i) in (5.9), we prove that \(\tilde{X}_0\) is \(C^\infty(M, \mathbb{R})\)-linear.

Finally, using the Jacobi identity of \(\mathbb{R}^+\) and (5.9), we obtain

\[
0 = [1, [\tilde{a}, \tilde{b}]^+]^+ - [\tilde{b}, [\tilde{1}, \tilde{a}]^+]^+ - [\tilde{b}, [\tilde{1}, \tilde{a}]^+]^+ = -\tilde{X}_0([\tilde{a}, \tilde{b}]^+)\tilde{1} - [\tilde{1}, \tilde{a}]^+\tilde{X}_0(\tilde{b})\tilde{1} + [\tilde{b}, \tilde{X}_0(\tilde{a})\tilde{1}]^+ = (-\tilde{X}_0([\tilde{a}, \tilde{b}]^+) + \rho^+(\tilde{a})(\tilde{X}_0(\tilde{b})) - \rho^+(\tilde{b})(\tilde{X}_0(\tilde{a}))\tilde{1}).
\]

Therefore, \(\tilde{X}_0\) is a 1-cocycle in \(A^+\).

Now, as a consequence of Corollary 3.7 and Propositions 5.9 and 5.10, we conclude that

**Corollary 5.11** There exists a one-to-one correspondence between strongly-affine Jacobi structures over an affine bundle \(\pi : A \to M\) of rank \(n > 0\) and almost-special Lie algebroid structures on \(A^+\).

Now, suppose that \(\pi : A \to M\) is a vector bundle and that \(A_1 = A \times \mathbb{R}\). Let \((\Lambda_A, E_A)\) be a strongly-affine Jacobi structure over \(A\) and \(([\cdot, \cdot]^+, \rho^+), \tilde{X}_0)\) be the associated almost-special Lie algebroid structure on \(A^+\). Under the identification between \(A^+\) and \(A_1^+\), the section \(\tilde{1}\) of \(A^+\) is the pair \((0, 1) \in \Gamma(A^+ \times C^\infty(M, \mathbb{R})) \cong \Gamma(A_1^+)\). Moreover, since \(\tilde{X}_0(\tilde{1}) = 0\) and \(\rho^+(\tilde{1}) = 0\) (see 5.8 and Proposition 5.10), we deduce that there exist maps

\[
[\cdot, \cdot] : \Gamma(A^+) \times \Gamma(A^+) \to \Gamma(A^+), \quad \rho^* : \Gamma(A^+) \to \mathcal{X}(M),
\]

\[
X_0 : \Gamma(A^+) \to C^\infty(M, \mathbb{R}), \quad P_0 : \Gamma(A^+) \times \Gamma(A^+) \to C^\infty(M, \mathbb{R})
\]
such that
\[
\rho^+(a',f_M) = \rho_*(a') + \rho_*(a')\left(\langle a',b'\rangle_x - P_0(a',b') - f_M X_0(b') + g_M X_0(a') - \rho_*(b')(f_M) + \rho_*(a')(g_M)\right),
\]
for all \((a',f_M), (b',g_M) \in \Gamma(A^*) \times C^\infty(M,\mathbb{R})\). A direct computation, using that \(\langle [\cdot,\cdot]^+, \rho^+ \rangle\) is a Lie algebroid structure, shows that \((A^*,[\cdot,\cdot]^+,\rho_*)\) is a Lie algebroid, that \(X_0\) defines a 1-cocycle in \((A^*, [\cdot,\cdot]^+, \rho_*)\) and that \(P_0 : \Gamma(A^*) \times \Gamma(A^*) \to C^\infty(M,\mathbb{R})\) is a skew-symmetric \(C^\infty(M,\mathbb{R})\)-bilinear mapping such that \(d_\ast P_0 = -X_0 \wedge P_0\), where \(d_\ast\) denotes the differential of the Lie algebroid \((A^*, [\cdot,\cdot]^+, \rho_*)\).

**Remark 5.12** Using that \(X_0\) is a 1-cocycle of the Lie algebroid \((A^*,[\cdot,\cdot]^+,\rho_*)\) and that \(d_\ast P_0 = -X_0 \wedge P_0\), we deduce the following facts:

(i) The map \(\nabla : \Gamma(A^*) \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})\) given by
\[
\nabla_a f_M = \rho_*(a')(f_M) + a'(X_0)f_M,
\]
for \(a' \in \Gamma(A^*)\) and \(f_M \in C^\infty(M,\mathbb{R})\), defines a representation of the Lie algebroid \((A^*,[\cdot,\cdot]^+,\rho_*)\) on the vector bundle \(M \times \mathbb{R} \to M\).

(ii) \(P_0 : \Gamma(A^*) \times \Gamma(A^*) \to C^\infty(M,\mathbb{R})\) is a 2-cocycle for the representation \(\nabla\).

Thus, in the terminology of Mackenzie [20] (see pag. 205-206 in [24]), the Lie algebroid \((A^*_1,[\cdot,\cdot]^+_1,\rho^+_1)\) is just the extension of the Lie algebroid \((A^*,[\cdot,\cdot]^+,\rho_*)\) by the vector bundle \(M \times \mathbb{R} \to M\) associated with the representation \(\nabla\) and the 2-cocycle \(P_0\).

Now, one can easily prove the following result.

**Proposition 5.13** Let \(\pi : A \to M\) be a vector bundle. Consider \([\cdot,\cdot]^+ : \Gamma(A^*_1) \times \Gamma(A^*_1) \to \Gamma(A^*_1)\) (resp. \(\rho^+ : \Gamma(A^*_1) \to C^\infty(M,\mathbb{R})\)) a bracket over \(A^*_1\) (resp. a mapping) given as in (5.10). Then, \((A^*_1,[\cdot,\cdot]^+_1,\rho^+_1)\) is an almost-special Lie algebroid if and only if the following sentences are satisfied:

(i) \((A^*,[\cdot,\cdot]^+,\rho_*)\) is a Lie algebroid,

(ii) \(X_0\) defines a 1-cocycle of \((A^*,[\cdot,\cdot]^+,\rho_*)\).

(iii) \(P_0 : \Gamma(A^*) \times \Gamma(A^*) \to C^\infty(M,\mathbb{R})\) is a skew-symmetric \(C^\infty(M,\mathbb{R})\)-bilinear mapping such that \(d_\ast P_0 = -X_0 \wedge P_0\), where \(d_\ast\) denotes the differential of the Lie algebroid \(A^*\).

Using this result, Corollary 5.11 and the local expression of \(\Lambda_A\) and \(E_A\), we have that the strongly-affine Jacobi structure \((\Lambda_A,E_A)\) over \(A\) associated with an almost-special Lie algebroid structure over \(A^*_1\) is given by
\[
\Lambda_A = \tilde{\Lambda}^*_A - P_0^v + \Delta_A \times X_0^v, \quad E_A = -X_0^v,
\]
where \(\tilde{\Lambda}^*_A\) is the linear Poisson structure over \(A\) induced by the Lie algebroid \((A^*,[\cdot,\cdot]^+,\rho_*)\), \(\Delta_A\) is the Liouville vector field of \(A\) and \(X_0^v\) (resp. \(P_0^v\)) is the vertical lift of \(X_0\) (resp. \(P_0\)).

**Corollary 5.14** There is a one-to-one correspondence between strongly-affine Jacobi structures on a vector bundle \(\pi : A \to M\) of rank \(n, n > 0\), and Jacobi algebroid structures \(([[\cdot,\cdot]^+,\rho_*,X_0]\), \(X_0\)) on \(A^*\) with a skew-symmetric \(C^\infty(M,\mathbb{R})\)-bilinear map \(P_0 : \Gamma(A^*) \times \Gamma(A^*) \to C^\infty(M,\mathbb{R})\) such that \(d_\ast P_0 = -X_0 \wedge P_0\), where \(d_\ast\) denotes the differential of \(A^*\).
Remark 5.15 If \( P_0 = 0 \) in Corollary 5.14, then we recover the one-to-one correspondence between linear-homogeneous Jacobi structures on a vector bundle \( \pi : A \to M \) and Jacobi algebroid structures on \( A^* \) (see Section 4, Example 4).

6 The characteristic foliation of a strongly-affine Jacobi structure on a vector space

Let \( \mathfrak{g} \) be a real vector space of finite dimension and \( \tilde{\mathfrak{g}} \) be a linear Poisson structure on \( \mathfrak{g} \). Then, \( \mathfrak{g}^* \) is a Lie algebra. Denote by \( G^+ \) a connected and simply connected Lie group with Lie algebra \( \mathfrak{g}^* \). Then, the leaves of the symplectic foliation associated with \( \tilde{\mathfrak{g}} \) are the orbits of the coadjoint representation associated with \( G^+ \). In this section we will obtain the corresponding result in the Jacobi setting.

First of all, we must replace the terms linear and Poisson by the terms affine and Jacobi, respectively. So, suppose that we have an affine Jacobi structure \((\Lambda_\mathfrak{g}, E_\mathfrak{g})\) on \( \mathfrak{g} \). Then, \((\mathfrak{g}^+ = \text{Aff}(\mathfrak{g}, \mathbb{R}) = \mathfrak{g}^* \times \mathbb{R}, [-, -]^+\)) is a Lie algebra such that the mapping

\[
-X(\Lambda_\mathfrak{g}, E_\mathfrak{g}) : \mathfrak{g}^+ \to \mathfrak{g}(\mathfrak{g}), \quad a \mapsto -X(\Lambda_\mathfrak{g}, E_\mathfrak{g})(a) = -X_a(\Lambda_\mathfrak{g}, E_\mathfrak{g})
\]

is a Lie algebra antihomomorphism. Denote by \( G^+ \) a connected and simply connected Lie group with Lie algebra \((\mathfrak{g}^+ + [-, -]^+)\). In general, there does not exist a global action of \( G^+ \) on \( \mathfrak{g} \) whose associated infinitesimal action to be \(-X(\Lambda_\mathfrak{g}, E_\mathfrak{g})\). In fact, in general, if \( a \in \mathfrak{g}^+ \) then \(-X_a(\Lambda_\mathfrak{g}, E_\mathfrak{g})\) is not complete (see, for instance, [23]).

If, "additionally", we suppose that \((\Lambda_\mathfrak{g}, E_\mathfrak{g})\) is strongly affine, then \(-X(\Lambda_\mathfrak{g}, E_\mathfrak{g}) : \mathfrak{g}^+ \to \text{Aff}(\mathfrak{g}, \mathfrak{g})\) defines an affine representation of \( \mathfrak{g}^+ \) on \( \mathfrak{g} \) in the sense of [23]. Therefore, using a result of Palais [24], one can prove that there is an affine representation \(\text{Coad} : G^+ \times \mathfrak{g} \to \mathfrak{g}\), such that the associated affine representation of \( \mathfrak{g}^* \) on \( \mathfrak{g} \), \(\text{coad} : \mathfrak{g}^+ \times \mathfrak{g} \to \mathfrak{g}\), is \(-X(\Lambda_\mathfrak{g}, E_\mathfrak{g})\). Consequently,

\[\text{Theorem 6.1} \quad \text{Let } (\Lambda_\mathfrak{g}, E_\mathfrak{g}) \text{ be a strongly-affine Jacobi structure over a real vector space } \mathfrak{g} \text{ of finite dimension. Then, the leaves of the characteristic foliation associated with the Jacobi structure } (\Lambda_\mathfrak{g}, E_\mathfrak{g}) \text{ are just the orbits of the affine representation } \text{Coad} : G^+ \times \mathfrak{g} \to \mathfrak{g}.\]

In the following, we will give an explicit description of the Lie group \( G^+ \) and the affine representation \(\text{Coad}\).

Let \((\Lambda_\mathfrak{g}, E_\mathfrak{g})\) be a strongly-affine Jacobi structure on the real vector space \( \mathfrak{g} \). Then, there exist a Lie algebra structure \([-, -]_s\), over the dual vector space \(\mathfrak{g}^*\), a 1-cocycle \(X_0 \in \mathfrak{g}\) of \((\mathfrak{g}^*, [\cdot, \cdot]_s)\) and a 2-section \(P_0 \in \Lambda^2 \mathfrak{g}\) such that

\[
d_*P_0 = -X_0 \wedge P_0,
\]

\[\text{d}_* \text{being the differential of } (\mathfrak{g}^*, [\cdot, \cdot]_s). \quad \text{Moreover, see } (5.11)\]

\[\Lambda_\mathfrak{g} = \Delta_\mathfrak{g} + \Delta_\mathfrak{g} \wedge X^*_0 - P^*_0, \quad E_\mathfrak{g} = -X^*_0
\]

where \(\Delta_\mathfrak{g}\) is the Lie-Poisson structure on \( \mathfrak{g} \) and \(\Delta_\mathfrak{g}\) is the radial vector field on \( \mathfrak{g} \).

Note that \(X^*_0\) (resp. \(P^*_0\)) is the constant vector field \(C_{P_0}\) (resp. the constant 2-vector \(C_0\)) over \( \mathfrak{g} \) defined by \(X_0 \in \mathfrak{g}\) (resp. \(P_0 \in \Lambda^2 \mathfrak{g}\)).
Let $G^*$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}^*$. Since $d_sX_0 = 0$, then there is a unique multiplicative function $\sigma_0 : G^* \rightarrow \mathbb{R}$ such that
\[ d\sigma_0(e) = X_0, \]  
\[(6.2)\]
e being the identity element of $G^*$. We recall that $\sigma_0 : G^* \rightarrow \mathbb{R}$ is multiplicative if $\sigma_0(gh) = \sigma_0(g) + \sigma_0(h)$, for $g, h \in G^*$.

On the other hand, $P_0 \in \wedge^2 \mathfrak{g}$ is a 2-cocycle in $\mathfrak{g}^*$ with respect to the representation $R_{X_0}$ of $\mathfrak{g}^*$ on $\mathbb{R}$ defined by
\[ R_{X_0} : \mathfrak{g}^* \times \mathbb{R} \rightarrow \mathbb{R}, \quad \alpha \in \mathfrak{g}^* \mapsto R_{X_0}(\alpha)(\lambda) = \lambda \alpha(X_0), \quad \forall \lambda \in \mathbb{R}. \]

In fact, the cohomology complex associated with this representation is defined as follows. The space of $k$-cochains $C^k(\mathfrak{g}^*, \mathbb{R})$ consists of skew-symmetric $k$-linear mappings $P : \mathfrak{g}^* \times \ldots \times \mathfrak{g}^* \rightarrow \mathbb{R}$ and the cohomology operator $d_{\ast X_0}$ is given by
\[ (d_{\ast X_0}P)(\alpha_1, \ldots, \alpha_{k+1}) = \sum_{i=1}^{k+1} (-1)^i R_{X_0}(\alpha_i)(P(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_{k+1})) + \sum_{i<j} (-1)^{i+j} P(\alpha_i, \alpha_j, \alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_{k+1}), \]
for all $\alpha_i \in \mathfrak{g}^*$. Then, $d_{\ast X_0}P = d_sP + X_0 \wedge P$ and, in particular, we have that $d_{\ast X_0}P_0 = 0$ (see (6.1)).

Denote by $Z^k_{R_{X_0}}(\mathfrak{g}^*, \mathbb{R})$ (resp. $B^k_{R_{X_0}}(\mathfrak{g}^*, \mathbb{R})$) the space of $k$-cocycles (resp. $k$-coboundaries) of the above complex and by $H^k_{R_{X_0}}(\mathfrak{g}^*, \mathbb{R})$ the corresponding cohomology group.

On the other hand, $R_{X_0}$ is just the infinitesimal representation associated with the linear representation $\phi_{\sigma_0}$ of $G^*$ on $\mathbb{R}$ given by
\[ \phi_{\sigma_0} : G^* \times \mathbb{R} \rightarrow \mathbb{R}, \quad (g, t) \mapsto te^{\sigma_0(g)}. \]

Now, we consider the cohomology complex over $G^*$ associated with $\phi_{\sigma_0}$. We recall that the space of $k$-cochains in this complex is the set $C^k(G^*, \mathbb{R})$ of differentiable mappings $\varphi : G^* \times \ldots \times G^* \rightarrow \mathbb{R}$. The cohomology operator is given by
\[ \partial_{\sigma_0}\varphi(g_1, g_2, \ldots, g_{k+1}) = e^{\sigma_0(g_1)}\varphi(g_2, \ldots, g_{k+1}) + \sum_{j=1}^{k} (-1)^j \varphi(g_1, \ldots, g_{j-1}, g_j, g_{j+1}, g_{j+2}, \ldots, g_{k+1}) + (-1)^{k+1}\varphi(g_1, \ldots, g_k), \]
for $\varphi \in C^k(G^*, \mathbb{R})$ and $g_1, \ldots, g_{k+1} \in G^*$. Denote by $H^k_{\phi_{\sigma_0}}(G^*, \mathbb{R}) = \frac{Z^k_{\phi_{\sigma_0}}(G^*, \mathbb{R})}{B^k_{\phi_{\sigma_0}}(G^*, \mathbb{R})}$ the cohomology $k$-group of $G$ associated with $\phi_{\sigma_0}$.

Then, there is an isomorphism between $H^2_{R_{X_0}}(\mathfrak{g}^*, \mathbb{R})$ and $H^2_{\phi_{\sigma_0}}(G^*, \mathbb{R})$ (see, for instance, [1]). In fact, this isomorphism is induced by the correspondence
\[ \Phi : Z^2_{\phi_{\sigma_0}}(G^*, \mathbb{R}) \rightarrow Z^2_{R_{X_0}}(\mathfrak{g}^*, \mathbb{R}), \]
\[ (6.3) \]
where
\[ \Phi(\varphi)(\xi, \eta) = \frac{d}{ds}|_{s=0} \frac{d}{dt}|_{t=0} (\varphi(exp(t\xi), exp(sn\eta)) - \varphi(exp(s\eta), exp(t\xi))) \]
\[ \text{exp} : \mathfrak{g}^* \rightarrow G^* \] being the exponential associated with $G^*$.

Therefore, for $P_0 \in Z^2_{R_{X_0}}(\mathfrak{g}^*, \mathbb{R})$, one may consider $\varphi_0 \in Z^2_{\phi_{\sigma_0}}(G^*, \mathbb{R})$ such that $P_0$ and $\varphi_0$ satisfy (6.3). Using [1] and [3], we prove the following result.
Theorem 6.2 The Lie group $G^+$ is isomorphic to the product $G^* \times \mathbb{R}$ and the multiplication in $G^+ = G^* \times \mathbb{R}$ is given by

$$(g_1,t_1)(g_2,t_2) = (g_1g_2,t_1 + e^{\alpha_0(g_1)}t_2 - \varphi_0(g_1,g_2)),$$

for all $(g_1,t_1), (g_2,t_2) \in G^+ = G^* \times \mathbb{R}$. In particular,

(i) If $\varphi_0 = 0$ (that is, the Jacobi structure $(\Lambda_0, E_0)$ is linear), $G^+$ is the semi-direct product $G^* \times \phi_{\alpha_0} \mathbb{R}$.

(ii) If $\sigma_0 = 0$ (that is, the Jacobi structure $(\Lambda_0, E_0)$ is Poisson), $G^+$ is a central extension of $G^*$.

Next, we will describe the affine representation $\text{Coad} : G^+ \times \mathfrak{g} \to \mathfrak{g}$.

Theorem 6.3 Let $\mathfrak{g}$ be a real vector space of finite dimension and $(\Lambda_0, E_0)$ be a strongly-affine Jacobi structure over $\mathfrak{g}$. Then, the leaves of the characteristic foliation associated with $(\Lambda_0, E_0)$ are the orbits of the affine representation $\text{Coad} : G^+ \times \mathfrak{g} \to \mathfrak{g}$ given by

$$\text{Coad}_{(g,t)}(X) = e^{\sigma_0(g)} \text{Coad}^G_X + tX_0 - e^{\sigma_0(g)} df_0^{\mathfrak{g}^{-1}}(e),$$

for $(g,t) \in G^+ = G^* \times \mathbb{R}$ and $X \in \mathfrak{g}$, where $\text{Coad}^G : G^* \times \mathfrak{g} \to \mathfrak{g}$ is the coadjoint representation associated with $G^*$ and for each $h \in G^*$, $f_0^h : G^* \to \mathbb{R}$ is the real function defined by $f_0^h(g') = \varphi_0(h,g') + \varphi_0(hg', h^{-1})$, for all $g' \in G$.

Proof.- Consider $(\Lambda_0, E_0)$ the Jacobi structure on $\hat{\mathfrak{g}} = (\mathfrak{g}^+)^* = \mathfrak{g} \times \mathbb{R}$ given by (see [841])

$$\Lambda_{\hat{\mathfrak{g}}} = \hat{\Lambda}^\mathfrak{g} - \Lambda^\mathfrak{g} \wedge X^\mathfrak{g}_{t(0,1)}, \quad E_{\hat{\mathfrak{g}}} = X^\mathfrak{g}_{t(0,1)},$$

where $\hat{\Lambda}^\mathfrak{g}$ is the linear Poisson structure on $\hat{\mathfrak{g}}$ associated with the Lie algebra $(\mathfrak{g}^+, [, , ]^+)$ and $X^\mathfrak{g}_{t(0,1)}$ is the Hamiltonian vector field with respect to $\hat{\Lambda}^\mathfrak{g}$ associated with the linear function $t(0,1) : \hat{\mathfrak{g}} \to \mathbb{R} \in \mathfrak{g}^+$ defined by $t(0,1) = (0,1) \in \mathfrak{g}^* \times \mathbb{R} = \mathfrak{g}^+$. Denote by $\hat{\alpha} : \hat{\mathfrak{g}} \to \mathbb{R}$ the linear function associated with $\alpha = (\alpha, \lambda) \in \mathfrak{g}^* \times \mathbb{R} = \mathfrak{g}^+$. Then, the Hamiltonian vector field with respect to $(\Lambda_{\hat{\mathfrak{g}}}, E_{\hat{\mathfrak{g}}})$ associated with $\hat{\alpha}$ is

$$X_{\hat{\alpha}}^{(\Lambda_{\hat{\mathfrak{g}}}, E_{\hat{\mathfrak{g}}})} = -\hat{\alpha}^{\text{Coad}} G^+ + X^\mathfrak{g}_{t(0,1)}(t_\hat{\alpha}) \Delta_{\hat{\mathfrak{g}}},$$

being the infinitesimal generator of $\alpha$ associated with the coadjoint representation $\text{Coad}^G$ of $G^+$. Consequently, since $\mathfrak{g} = t^{-1}_{(0,1)}(1)$, $\Lambda_{\mathfrak{g}} = \Lambda^\mathfrak{g} \mid \mathfrak{g}$ and $E_{\mathfrak{g}} = E_{\hat{\mathfrak{g}}} \mid \mathfrak{g}$, we have that

$$X_{\alpha}^{(\Lambda_0, E_0)} = (X^{(\Lambda_{\hat{\mathfrak{g}}}, E_{\hat{\mathfrak{g}}})})_\mid \mathfrak{g} = (-\alpha^{\text{Coad}} G^+ \mid \mathfrak{g} + X^\mathfrak{g}_{t(0,1)}(t_\alpha) \Delta_{\mathfrak{g}}),$$

for all $(g,t) \in G^+ = G^* \times \mathbb{R}$ and $(X, \lambda) \in \hat{\mathfrak{g}} = \mathfrak{g} \times \mathbb{R}$.

On the other hand, the coadjoint representation $\text{Coad}^G : G^+ \times \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ is given by

$$\text{Coad}^G_{(g,t)}(X, \lambda) = (\text{Coad}^G_g X + \lambda \epsilon^{-\sigma_0(g)}X_0 - \lambda df_0^{\mathfrak{g}^{-1}}(e), \lambda e^{-\sigma_0(g)}),$$

for all $(g,t) \in G^+ = G^* \times \mathbb{R}$ and $(X, \lambda) \in \hat{\mathfrak{g}} = \mathfrak{g} \times \mathbb{R}$.

From [825] and [840], we obtain that the action $\tilde{\text{Coad}} : G^+ \times \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ of $G^+$ on $\hat{\mathfrak{g}}$ defined by

$$\tilde{\text{Coad}}_{(g,t)}(X, \lambda) = e^{\sigma_0(g)} \text{Coad}^G_{(g,t)}(X, \lambda)$$

satisfies

$$\tilde{\text{Coad}}_{(g,t)}(\mathfrak{g}) = \mathfrak{g} \quad \text{and} \quad (\alpha^{\tilde{\text{Coad}}})_\mid \mathfrak{g} = -X_{\alpha}^{(\Lambda_0, E_0)}.$$
for all \( a \in \mathfrak{g}^+ \) and \((g, t) \in G^+\), \(a^\widetilde{\text{CoAd}}_g\) being the infinitesimal generator of \( a \) associated with the action \( \widetilde{\text{CoAd}} \). Consequently, the restriction to \( \mathfrak{g} \) of \( \widetilde{\text{CoAd}}_{(g, t)} \) is just \( \text{CoAd}_{(g, t)} \), for all \((g, t) \in G^+\). Using this fact, \( \Box_0 \) and \( \Box_1 \), we obtain \( \Box_2 \).

Finally, using the above Theorem, we describe the Jacobi structure on the leaves of the characteristic foliation of a strongly-affine Jacobi structure over a vector space.

**Theorem 6.4** Let \( \mathfrak{g} \) be a real vector space of finite dimension and \((\Lambda_\mathfrak{g}, E_\mathfrak{g})\) be a strongly-affine Jacobi structure over \( \mathfrak{g} \). Consider \( X \in \mathfrak{g} \) and \( L_X \) the leaf of the characteristic foliation over the point \( X \) associated with \((\Lambda_\mathfrak{g}, E_\mathfrak{g})\).

(i) If \( E_\mathfrak{g}(X) \notin \#_{\Lambda_\mathfrak{g}}(T_X^* \mathfrak{g}) \) and \( Y \in L_X \) then

\[
T_Y L_X = \left\{ \alpha_Y = \alpha^\text{CoAd}_\mathfrak{g} G^+(Y) + \alpha(X_0) \Delta_\mathfrak{g}(Y) + (i(\alpha) P_0)^x(Y) \right\}_{\alpha \in \mathfrak{g}^*}, X_0^\mathfrak{g}(Y) >
\]

and \((\Lambda_\mathfrak{g}, E_\mathfrak{g})\) induces a contact structure \( \eta_{L_X} \) on \( L_X \) given by

\[
\eta_{L_X}(Y)(\alpha_Y) = -\alpha(Y), \quad \eta_{L_X}(Y)(X_0^\mathfrak{g}(Y)) = -1,
\]

for all \( \alpha \in \mathfrak{g}^* \).

(ii) If \( E_\mathfrak{g}(X) \in \#_{\Lambda_\mathfrak{g}}(T_X^* \mathfrak{g}) \) and \( Y \in L_X \) then

\[
T_Y L_X = \left\{ \alpha_Y = \alpha^\text{CoAd}_\mathfrak{g} G^+(Y) + \alpha(X_0) \Delta_\mathfrak{g}(Y) + (i(\alpha) P_0)^x(Y) \right\}_{\alpha \in \mathfrak{g}^*} >
\]

and \((\Lambda_\mathfrak{g}, E_\mathfrak{g})\) induces a l.c.s. structure \((\Omega_{L_X}, \omega_{L_X})\) on \( L_X \) defined by

\[
\Omega_{L_X}(Y)(\alpha_Y, \beta_Y) = [\alpha, \beta]_+(Y) - P_0(\alpha, \beta), \quad \omega_{L_X}(Y)(\alpha_Y) = -\alpha(X_0),
\]

for all \( \alpha, \beta \in \mathfrak{g}^* \).

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