Exact solutions for a class of integrable Hénon-Heiles-type systems

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We study the exact solutions of a class of integrable Hénon-Heiles-type systems (according to the analysis of Bountis et al. (1982)). These solutions are expressed in terms of two-dimensional Kleinian functions. Special periodic solutions are expressed in terms of the well-known Weierstrass function. We extend some of our results to a generalized Hénon-Heiles-type system with \((n + 1)\) degrees of freedom.
I. INTRODUCTION

The famous Hénon-Heiles potential (for details, see Section 2) was introduced to model the motion of a star within a galaxy. But it has much larger connotations. Mathematically speaking, the model was constructed by adding two terms of third degree to the potential of a planar oscillator. Such a mathematical model also appears by expanding the potential corresponding to an integrable system (resulting via some canonical transformations applied to the motion of three particles on a circle under exponentially decreasing forces) to third degree terms (Boccaletti and Puacce 1996; Anisiu and Pal 1999).

This problem originates in a question largely discussed in the sixties and approached much earlier: the existence of a third isolating integral of the motion (besides the integrals of energy and angular momentum). For terminology, significance, and endeavours, see (Hénon and Heiles 1964 and the references therein).

Starting with 1957, G. Contopoulos searched for and found some cases of two potentials when such an integral (in the classical notation) exists. Quoting him, numerical results provide ample evidence of the existence of $I_3$ in quite general potential fields. As concrete exemplifications from astronomy, but not only: slightly elliptical stellar clusters or galaxies, Schmidt’s model of the Galaxy, noncentral gravitational field of the Earth; also, systems where the unperturbed frequency ratio is rational, or potential fields deprived of a symmetry plane.

Besides these situations, there exist problems of statistical mechanics, celestial mechanics, and quantum mechanics that join a Hénon-Heiles-type model.

In 1964 M. Hénon and C. Heiles simplified the initial problem and approached it numerically via Poincaré sections. Their paper and their numerical experiments had a great echo, benefitting today of more than 600 citations in the mathematical, physical and astronomical literature. The model bears their names, and the generalizations are called Hénon-Heiles-type models.

As a matter of fact, in many Hénon-Heiles-type models (and earlier models) the integral of angular momentum does not hold. Such models were also tackled via the tools of the theory of dynamical systems. Only one example at hand: the study of the collision and escape dynamics in the associated two-body problem (see Section II).

Coming back to the Hénon-Heiles-type model detailed in Section II in 1982, T. Bountis et al. used Painlevé analysis to identify all integrable cases. They identified three such situations, ruled by the values of the free parameters of the model; out of them case (ii) is the most general (see Section II).

The present paper intends to point out exact solutions for the class of Hénon-Heiles-type systems described by the case (ii) above. Section II establishes the basic equations for the problem to be investigated. We start from cylindrical coordinates (in the motion reduced to a meridian plane), but then we work with a generalized Hénon-Heiles-type system in configuration-momentum coordinates. In Section III we derive the Lax representation for the corresponding system.

Section IV provides exact quasi-periodic solutions for the system under consideration. These orbits are expressed in terms of Kleinian hyperelliptic functions.

In Section V we search for elliptic periodic solutions of our system. To this end in view, we resort to the method proposed in (1957), which allows the construction of periodic solutions in a straightforward way, applying the spectral theory for the Schrödinger equation to elliptic potentials. In this way we get the corresponding solution of the system under consideration.

Section VI tackles a generalized Hénon-Heiles-type system with $(n + 1)$ degrees of freedom. We apply the methods presented in the previous sections to such a system, and point out the exact solutions in this case.

The final Section VII summarizes the main results obtained in the paper and formulates some conclusions.

II. BASIC EQUATIONS

We consider here a model for the dynamics of a point mass (a star) within the gravitational field ruled by the potential $V_g$ of an axially symmetric galaxy. Let the mass of the star be $m$, and let the cylindrical coordinates we shall use be $(r, \phi, z)$. The axis $Oz$ is the axis of symmetry, $z$ is the distance of the star from the reference plane, $r := \sqrt{x^2 + y^2}$ is the distance between the star and the axis $Oz$ and $\phi := \arctan z$ is the polar angle. Here the $Oxyz$-plane is a reference plane (in rectangular coordinates) that defines the coordinate $r$ and the angle $\phi$.

The Hamiltonian of the model can be written as

$$H(p_r, p_z, p_\phi, r, \phi) = \frac{1}{2mr} (p_r^2 + p_z^2) + \frac{1}{2m} \dot{p}_\phi^2 + V_g(r, z),$$

where $p_r = mr\dot{r}$ and $p_z = mz\dot{z}$ are the linear momenta in the $r$ and $z$ directions respectively, $p_\phi = mr\dot{\phi}$ is the angular momentum around the symmetry axis, whereas $` = \frac{d}{dt}$ is the derivative with respect to the time $t$. 
We have two integrals of motion, the total energy, \( E \), and the angular momentum, \( l \), respectively:

\[
E = V_g(r, z) + \frac{m}{2} (\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2),
\]

\[
l = mr\dot{\phi}.
\]

With the help of the second integral \( l \), we reduce the dynamics of the star to the meridian plane \((r, z)\):

\[
\dot{r} = -\frac{\partial V(r, z)}{\partial r}, \quad \dot{z} = -\frac{\partial V(r, z)}{\partial z},
\]

\[
V(r, z) := V_g(r, z) + \frac{m}{2} r^2 \dot{\phi}^2 = V_g(r, z) + \frac{l^2}{2mr^2}.
\]

This problem originates in a question discussed in the sixties: the existence of a third isolating integral of the motion \((I_3)\), see the introductory section. Starting with 1957, G. Contopoulos \([4]\) found some cases of potentials \(V(r, z)\) when such integral \(I_3\) exists.

In 1964 M. Hénon and C. Heiles \([3]\) simplified the problem, canceling all terms of order \(\geq 4\) in the potential \(V(r, z)\).

The simplest such potential can be read as

\[
V(r, z) = Cr^3 + \frac{1}{2} r z^2 + \frac{1}{2} (Ar^2 + Bz^2),
\]

where \(A, B\) and \(C\) are free parameters.

Using Painlevé analysis, in 1982 T. Bountis, H. Segur and F. Vivaldi \([13]\) extracted all integrable cases of \((3)\), see also \([14, 17, 18, 19]\):

(i) \(A = B, \ C = \frac{1}{3}\) \hspace{1cm} (4)

(ii) \(C = 1, \ A\ and \ B\ are\ arbitrary\) \hspace{1cm} (5)

(iii) \(16A = B, \ C = \frac{16}{3}\) \hspace{1cm} (6)

In the case (i), the equations of motion decoupled in \((z + r), (z - r)\) coordinates and the general solution can be expressed via elliptic functions, see for example \([20]\) and references therein. The question about the effective exact solution in the case (iii) is still open \([20]\). For the case (ii) effective solutions are obtained in \([21, 22, 23, 24]\). In this paper we shall focus our attention to the Hénon-Heiles-type systems related to second case (ii).

Next we will use \(\tilde{q}_1\) and \(\tilde{q}_2\) instead of \(r\) and \(z\). We consider a generalized Hénon-Heiles-type system with two degrees of freedom \([25, 26, 27, 28, 29]\):

\[
\dot{q}_1 + 3\tilde{q}_1^2 + \frac{1}{2}\tilde{q}_2^2 + a_0 q_1 - \frac{a_1}{4} = 0,
\]

\[
\dot{q}_2 + q_1 q_2 - \frac{a_4}{4\tilde{q}_2^2} + a_0 q_2 = 0.
\]

Its Hamiltonian is

\[
H_0 = \frac{1}{2} (p_1^2 + p_2^2) + q_1^3 + \frac{1}{2} q_1 q_2^2 + \frac{a_4}{8q_2} + \frac{a_0}{2} \left( q_1^2 + \frac{1}{4} q_2^2 \right) - \frac{a_1}{4} q_1,
\]

where \(q_1, q_2, p_1, p_2\) are the canonical coordinates and momenta and \(a_0, a_1, a_4\) are free constant parameters. Moreover \(H_0\) is related to the Hamiltonian

\[
H_H = \frac{1}{2} (p_1^2 + p_2^2) + q_1^3 + \frac{1}{2} q_1 q_2^2 + \frac{a_4}{8q_2} + \frac{1}{2} (Aq_1^2 + Bq_2^2),
\]

through the map

\[
q_1 = \tilde{q}_1 + \frac{A}{2} - 2B, \quad q_2 = \tilde{q}_2,
\]

\[
a_0 = -2A + 12B, \quad a_1 = -A^2 + 16AB - 48B^2.
\]

The function \(H_H\) is the Hamiltonian of a classical integrable Hénon-Heiles system with the additional term \(a_4/8q_2^2\). The corresponding equations are equivalent to the ordinary differential equation for travelling wave solutions of the fifth-order flow in the Korteweg-de Vries (KdV) hierarchy \([25]\).
III. LAX REPRESENTATION

Next we will derive \((2 \times 2)\) matrix Lax representation for the generalized Hénon-Heiles system \((9)\). The Lax representation has the form \([30, 31]\)

\[
\dot{L} = [M(t, \lambda), L(t, \lambda)], \quad L = \begin{pmatrix} V & U \\ W & -V \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ Q & 0 \end{pmatrix}
\] (12)

where \(U, W, Q\) are \([32, 33, 34]\) see also \([35]\):

\[
U(t, \lambda) = F(t, \lambda) = \lambda^2 + \frac{1}{2} q_1 \lambda - \frac{1}{16} q_2^2, \\
V(t, \lambda) = -\frac{1}{2} \dot{F}(t, \lambda) = -\frac{1}{4} p_1 \lambda + \frac{1}{16} q_2 p_2, \\
W(t, \lambda) = -\frac{1}{2} \ddot{F} + QF = \lambda^3 - (\frac{1}{2} q_1 + \frac{1}{4} a_0) \lambda^2 \\
+ \left( \frac{1}{4} q_1 + \frac{1}{q_2} \right)^2 - \frac{1}{16} a_1 + \frac{1}{8} a_0 q_1 \lambda + \frac{1}{16} p_2^2 + \frac{1}{64} q_2^2, \\
Q(t, \lambda) = \lambda - q_1 - \frac{1}{4} a_0.
\]

The corresponding algebraic curve is \(\det(L(t, \lambda) - \lambda^2 I) = 0\)

\[
\nu^2 = 4 \lambda^5 - a_0 \lambda^4 - \frac{1}{4} a_1 \lambda^3 + \frac{1}{2} H_0 \lambda^2 + \frac{1}{8} H_1 \lambda - \frac{1}{256} a_4.
\] (13)

It is easy to derive a second integral \(H_1:\)

\[
H_1 = p_2 q_1 - p_1 p_2 q_2 - \frac{1}{2} q_1^2 q_2 - \frac{1}{8} q_2^4 + a_4 q_1 q_2^2 - \frac{a_0}{4} q_1 q_2^2 + \frac{a_1}{8} q_2^2.
\]

IV. EXACT QUASI-PERIODIC SOLUTIONS

In this section we give the trajectories of the system under consideration in terms of Kleinian hyperelliptic functions (see, e.g., \([36, 37, 38, 39]\)), being associated with the real algebraic curve of genus two \((13)\), which can be also written in the form

\[
\nu^2 = 4 \prod_{i=0}^4 (\lambda - \lambda_i) = 4 \lambda^5 + \sum_{k=0}^4 \alpha_k \lambda^k,
\] (14)

where \(\lambda_i \neq \lambda_i\) are branching points and

\[
\alpha_4 = -a_0, \quad \alpha_3 = -\frac{1}{4} a_1, \quad \alpha_2 = \frac{1}{2} H_0, \\
\alpha_1 = \frac{1}{8} H_1, \quad \alpha_0 = \frac{1}{256} a_4.
\] (15)

At all real branching points the closed intervals \([\lambda_{2i-1}, \lambda_{2i}]\), \(i = 1, 2\) will be referred further as \textit{lacunae} \([40, 41]\). Let us equip the curve with a homology basis \((a_1, a_2; b_1, b_2) \in H_1(K, \mathbb{Z})\) and fix the basis in the space of holomorphic differentials.

The exact integration of the system \((7)\) and \((8)\) reduces to the solution of the Jacobi inversion problem in the following form

\[
\lambda^2 - \wp_{22}(u) \lambda - \wp_{12}(u) = 0,
\] (16)

that is, the pair \((\mu_1, \mu_2)\) is the pair of roots of \((16)\). So we have

\[
\wp_{22}(u) = \mu_1 + \mu_2, \quad \wp_{12}(u) = -\mu_1 \mu_2.
\] (17)
Let us introduce finally the Baker-Akhiezer function, which, in the framework of the formalism developed, is expressible in terms of the Kleinian $\sigma$-function as follows [38]:\[
\Psi(\lambda, u) = \frac{\sigma \left( \int_{x}^{\lambda} \frac{du}{\sigma(u)} \right)}{\sigma(u)} \exp \left\{ \int_{x}^{\lambda} \frac{d\tau}{\sigma(\tau)} \right\},
\]

where $\lambda$ is arbitrary and $u$ is the Abel image of arbitrary point $(\nu_1, \mu_1) \times (\nu_2, \mu_2) \in K \times K$. It is straightforward to show by the direct calculation, based on the relations for three and four-index $\psi$-functions [38], that $\Psi(\lambda, u)$ satisfies the Schrödinger equation\[
\left( \frac{d^2}{du^2} - 2\psi_{22}(u) \right) \Psi(\lambda, u) = \left( \lambda + \frac{1}{4}a_4 \right) \Psi(\lambda, u)
\]

for all $(\nu, \mu)$. The solutions of (17), (18) have the following form in terms of Kleinian functions $\psi_{22}(u), \psi_{12}(u)$ [32, 33, 35, 42].\[
q_1 = -2\psi_{22}(u), \quad q_2^2 = 16\psi_{12}(u).
\]

V. ELLIPTIC PERIODIC SOLUTIONS

In this section we follow the method proposed in [14, 15, 16, 33], which allows us to construct periodic solutions of (17), (18) in a straightforward way based on the application of spectral theory for the Schrödinger equation with elliptic potentials [14, 15, 16, 44, 17, 48, 50, 52, 53, 54, 55, 56]. We start with the equation (19) for Baker function $\Psi(\lambda, u)$. We assume, without loss of generality, that the associated curve has the property $a_4 = 0$. To make this assumption applicable to the initial curve of the system (17), (18) being derived from the Lax representation, we undertake the shift of the spectral parameter\[
\lambda \rightarrow \lambda - a_0/20,
\]

Consider genus 2 Lamé potential $u = 6\psi(t + \omega')$ and construct the associated curve\[
\nu^2 = 4(\lambda^2 - 3g_2)(\lambda + 3c_1)(\lambda + 3c_2)(\lambda + 3c_3),
\]

The Hermite polynomial $F_3(\psi(t), \lambda)$, depending on the argument $t + \omega'$, [45, 57] associated to the Lamé potential $6\psi(t + \omega')$, which is already normalized has the form\[
F_3(\psi(t + \omega'), \lambda) = \lambda^2 - 3\psi(t + \omega')\lambda + 9\psi^2(t + \omega') - \frac{9}{4}g_2.
\]

Using explicit expression for Hermite polynomial (22) we obtain the following simple solutions for the system (19):\[
q_1 = -6\psi(t + \omega'), \quad q_2^2 = -2^4 \cdot 3^3 \cdot \psi(t + \omega')^2 + 2^2 \cdot 3^2 g_2,
\]

where $a_0 = 0, a_1 = 3 \cdot 4 \cdot 7 \cdot g_2, a_4 = -4^4 \cdot 3^4 \cdot g_2 g_3$. More general solution with $a_0 \neq 0$ can be written using the shift (20) in (22). Then we have\[
q_1 = -6\psi(t + \omega') - \frac{1}{5}a_0, \quad q_2^2 = -4^2 \cdot 3^2 \psi(t + \omega')^2 - \frac{12}{5}\psi(t + \omega')a_0 - \frac{1}{25}a_0^2 + 36g_2,
\]

where $a_1 = 3 \cdot 4 \cdot 7 \cdot g_2 - 2a_0^2/5$ and\[
a_4 = \frac{1}{3125}a_0^5 - \frac{84}{125}a_0^3g_2 - \frac{432}{25}a_0^2g_3 + \frac{1728}{5}a_0g_2^2 + 20736g_2g_3.
\]

After changing the variables\[
a_0 = 2^2 \cdot 5 \cdot \lambda, \quad \nu = 2^8 \cdot a_4
\]

we obtain the curve of genus 2\[
\nu^2 = 4(\lambda^2 - 3g_2)(\lambda^3 - 9\lambda g_2 + 27g_3),
\]
where $g_2,g_3$ are elliptic invariants (see for example \textsuperscript{43}). The last expression is the another form of \textsuperscript{21}, were $e_j, j = 1,2,3, e_3 \leq e_2 \leq e_1$ are real roots of equation $4\lambda^3 - g_2\lambda - g_3$. The Weierstrass function $\wp = \wp(t + \omega')$ shifted by half period $\omega'$ is related to sn Jacobian elliptic function with modulus $k$ by \textsuperscript{58}:

\[
\wp(t + \omega'; g_2, g_3) = \alpha^2 k^2 \sin^2(\alpha t, k) + e_3,
\]

where $\alpha = \sqrt{e_1 - e_3}$. Using wave height $\alpha$ and modulus $k = \sqrt{(e_2 - e_3)/(e_1 - e_3)}$ we have the following relations (see for example \textsuperscript{45}):

\[
e_1 = 2 - k^2, \quad e_2 = 2k^2 - 1, \quad e_3 = -(1 + k^2),
\]

\[
g_2 = 2(e_1^2 + e_2^2 + e_3^2) = 12(1 - k^2 + k^4),
\]

\[
g_3 = 4e_1 e_2 e_3 = 4(k^2 + 1)(2 - k^2)(1 - 2k^2).
\]

From Lax representation for polynomial $F$ we have the following nonlinear differential equation with spectral parameter $\lambda$

\[
\frac{1}{2} F \ddot{\Phi} - \frac{1}{4} F^2 - (u(t) + \lambda) F^2 + \frac{1}{4} \nu^2(\lambda) = 0,
\]

with eigenvalue equations

\[
\nu^2(\lambda) = 4\lambda^5 - 21\lambda^3 g_2 + 27\lambda g_2^2 + 27\lambda^2 g_3 - 81g_2g_3 = 0.
\]

\section{(n + 1)-DEGREES-OF-FREEDOM GENERALIZED HÉNON-HEILES- TYPE SYSTEM}

We consider a generalized Hénon-Heiles-type system with $n + 1$ degrees of freedom \textsuperscript{26, 28, 32, 33}:

\[
\dot{q}_0 + 3q_0^2 +\frac{1}{2} \sum_{j=1}^{n} q_j^2 + a_0q_0 = 0,
\]

\[
\ddot{q}_j + q_0q_j - \frac{\sigma_j^2}{q_j^2} - a_jq_j = 0.
\]

where $j = 1, \ldots, n$. Its Hamiltonian is

\[
H = \frac{1}{2} (p_0^2 + \sum_{j=1}^{n} p_j^2) + q_0^3 + \frac{1}{2} q_0 \sum_{j=1}^{n} q_j^2 + \frac{1}{2} a_0q_0^2 - \frac{1}{2} \sum_{j=1}^{n} \left( a_jq_j^2 - \frac{\sigma_j^2}{q_j^2} \right),
\]

where $q_0, q_j, p_0, p_j, j = 1, \ldots, n$, are the canonical coordinates and momenta, respectively, and $a_0, \sigma_j, a_j, j = 1, \ldots, n$, are free constant parameters. The function $H$ for $n = 1$ is the Hamiltonian of a classical integrable Hénon-Heiles system with the additional term $\sigma_j^2/q_j^2$.

Next we will present $(2 \times 2)$ matrix Lax representation for the generalized Hénon-Heiles system \textsuperscript{34}. The Lax representation has the form

\[
\dot{L} = [M(\lambda), L(\lambda)], \quad L = \begin{pmatrix} V & U \\ W & -V \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ Q & 0 \end{pmatrix}
\]

where $U, W, Q$ are \textsuperscript{32, 33, 35}:

\[
U(t, \lambda) = F(t, \lambda) = a(\lambda) \left( \lambda + \frac{1}{2} q_0 + \frac{1}{4} a_0 - \frac{1}{16} \sum_{j=1}^{n} \frac{q_j^2}{\lambda - a_j} \right),
\]

\[
V = -\frac{1}{2} \frac{\dot{F}}{a(\lambda)} + \frac{1}{4} \frac{p_0}{a_0} + \frac{1}{16} \sum_{j=1}^{n} \frac{q_j p_j}{\lambda - a_j},
\]

\[
W = -\frac{1}{2} \frac{\dot{F}}{a(\lambda)} + QF = a(\lambda) \left( \lambda^2 - \frac{1}{2} q_0 \lambda + \frac{1}{4} a_0 \lambda \right) +
\]

\[
+ a(\lambda) \left( \frac{1}{4} q_0^2 + \frac{1}{16} \sum_{j=1}^{n} q_j^2 \right) +
\]
where

\[ a(\lambda) = \prod_{i=1}^{n} (\lambda - a_i), \quad n \geq 2. \]

Further we will collect the well known facts from finite zone inverse scattering transform method \[14, 40, 41, 60, 61, 62, 63\] useful for our construction of exact solutions of generalized Hénon-Heiles-type system. Here we follow the Krichever construction \[62\]. The Baker–Akhiezer (BA)-function \( \psi(t, \lambda) \) of a nonspecial divisor \( D \) of degree \( n + 1 \) is given explicitly by

\[ \psi(t, \lambda) = C(P) \exp(i\Omega_1 t) \frac{\theta(A(P) + Z)}{\theta(Z)}, \]

where \( \theta(v|B) \) is a Riemann theta function, \( B = (B)_{ij} = \int_{b_i} \omega_j \) is a Riemann matrix, \( \omega_1, \omega_2, \ldots, \omega_{n+1} \) are normalized differentials of first kind, \( A(P) \) is Abel map \( A_k(P) = \int_{p_0}^{P_k} \omega_k \), \( q_0 \) is arbitrary point on real hyperelliptic Riemann surface \[39\] denoted by \( K \), \( \Omega_1 \) is the normalized differential of second kind with main parts at \( \infty \), \( d_k, Z = t u + Z_0, Z_0 = -A(D) - K_0, K_0 \) is the so called Riemann constant vector, the factor \( C(P) \) gives the normalization in \[37\].

The BA-function of nonspecial divisor \( D \) is the solution of

\[ \left( \frac{d^2}{dt^2} - u(t) \right) \psi(t, \lambda) = \lambda \psi(t, \lambda), \]

where

\[ u(t) = 2 \frac{d^2}{dt^2} \log \theta(Z) + \text{const.} \]

By the Riemann-Roch theorem there exists an nonspecial divisor \( D^* \) and unique abelian differential \( \Omega \) such that \( (\Omega) = D + D^* - 2\infty \) and \( \Omega = (1 + O(k^{-2}) dk \) at \( \infty \). There exists a unique function \( \psi^*(t, \lambda) \) called dual BA–function of \( D^* \). The BA-function \( \psi^*(t, \lambda) \) of a nonspecial divisor \( D^* \) of degree \( n + 1 \) is given explicitly by

\[ \psi^*(t, \lambda) = C^*(P) \exp(-i\Omega_1 t) \frac{\theta(A(P) - Z)}{\theta(Z)}, \]

The factor \( C^*(P) \) is fixed by the normalization.

Let \( u(t) \) be a real finite-gap potential such that the Hill’s operator has only finite number \( n \) eigenfunctions defined by

\[ \psi_i = \alpha_i \psi(t, p_i), \quad \tilde{\psi}_i = \beta_i \psi^*(t, p_i), \]

\[ \alpha_i \beta_i = \text{Res}_{p=p_i} \Omega E, \quad i = 1, \ldots, n \]
then the following expansion of \( u(t) \) in terms of squared eigenfunctions have the form

\[
    u(t) = -\sum_{i=1}^{n} \alpha_i \beta_i \psi(t, p_i) \psi^\tau(t, p_i) + \text{const.} \quad (42)
\]

Let us construct the meromorphic differential

\[
    \hat{\Omega} = E \psi(t, P) \psi^\tau(t, P). \quad (43)
\]

A straightforward calculations give us

\[
    \text{Res}_{\hat{\Omega}} = -u(t),
\]

\[
    \text{Res}_{\hat{\Omega}} = -\psi(t, p_i) \psi^\tau(t, p_i) \alpha_i \beta_i,
\]

where \( \alpha_i \beta_i = \text{Res}_{\hat{\Omega}} E \) and \( \text{Res}_{\hat{\Omega}} = - \text{Res}_{\hat{\Omega}} \).

In particular case when \( K \) is hyperelliptic Riemann surface \( \mathbb{CP}^1 \) or in another form \( \nu^2 = 4 \prod_{i=1}^{2n+3} (\lambda - \lambda_i) = R(\lambda) \).

The points of \( K \) are pairs \( P = (\lambda, R) \) and \( \lambda(P) \) is the value of the natural projection \( P \to \lambda(P) \) of \( K \) to the complex projective line \( \mathbb{CP}^1 \).

For given nonspecial divisor \( D \), there is an unique Baker-Akhiezer (BA) function \( \Psi(t, \lambda) \), such that

(i) the divisor of the poles of \( \Psi \) is \( D \),

(ii) \( \Psi \) is meromorphic on \( K \setminus \infty \)

(iii) when \( P \to \infty \)

\[
    \Psi(t, P) \exp(-kt) = 1 + \sum_{s=1}^{\infty} m_s(t) k^{-s},
\]

is holomorphic and \( k = \sqrt{\lambda(P)} \) is a local parameter near \( P = \infty \).

There is a unique function \( u(x) \) such that

\[
    \tilde{\Psi} - u(t) \Psi = \lambda(P) \Psi,
\]

where \( \Psi \) is a BA function. Inserting expansion (44) into (46), we obtain

\[
    \tilde{\Psi} - 2\dot{m}(t)\Psi - \lambda(P) \Psi = \exp(kt) O(k^{-1}),
\]

and due to the uniqueness of \( \Psi \), we prove (46), with \( u(x) = 2\dot{m}(t) \).

By the Riemann-Roch theorem, there exists a unique differential \( \hat{\Omega} \) and a nonspecial divisor \( D^\tau \) of degree \( n \) such that the zeros of \( \hat{\Omega} \) are \( D + D^\tau \) and the expansion at \( P = \infty \), \( \hat{\Omega}(P) = (1 + O(k^{-2})) \).

For given nonspecial divisor \( D^\tau \), there exists a unique dual Baker-Akhiezer (BA) function such that

(i) the divisor of the poles of \( \Psi \) is \( D^\tau \),

(ii) \( \Psi \) is meromorphic on \( K \setminus \infty \)

(iii) when \( P \to \infty \)

\[
    \Psi^\tau(t, P) \exp(-kt) = 1 + \sum_{s=1}^{\infty} \tilde{m}_s(t) k^{-s},
\]

Fix \( \tau \) to be the hyperelliptic involution \( P = (\lambda, R) \to (\lambda, -R) \), then we have \( D^\tau = \tau D \), \( \Psi^\tau(x, P) = \Psi(x, \tau P) \). Let \( \sum_{i=1}^{n} \mu_i(0) \) be the \( \lambda \)-projection of \( D \), and \( \sum_{i=1}^{n} \mu_i(t) \) be the \( \lambda \)-projection of the zero divisor of \( \Psi(x, P) \). The function \( \Psi(t, P) \Psi^\tau(t, P) \) is meromorphic on \( \mathbb{CP}^1 \) and the following identity takes place

\[
    \Psi(x, P) \Psi^\tau(t, P) = \frac{F(t, \lambda)}{F(0, \lambda)} \quad (49)
\]

where \( F(t, \lambda) = \prod_{i=1}^{n} (\lambda - \mu_i(t)) \) and \( \mu_j(t) \) satisfies the following system of differential equations (Kovalevski–Dubrovin equations (61))

\[
    \frac{d}{dt} \mu_j(t) = 2 \frac{\sqrt{R(\mu_j)}}{\prod_{i \neq k} (\mu_j(t) - \mu_k(t))}. \quad (50)
\]
with initial conditions
\[ \mu_j(0) \in [\lambda_{2i-1}, \lambda_{2i}]. \] (51)

Equations (50) are first written by Sonya Kovalevski for genus \( n = 2 \) in relation to the integrable case of the Kovalevski top and for Korteweg–de Vries hierarchy of equations by Dubrovin for general \( n \) ([61] and references therein). These equations are useful for numerical calculation of the polynomial \( F(t, \lambda) \).

Introduce the Wronskian
\[ \{ \Psi(t, P), \Psi^\tau(t, P) \} = \Psi(t, P)\Psi^\tau(t, P) - \Psi(t, P)\dot{\Psi}^\tau(t, P) \]
\[ = \frac{2\sqrt{R(\lambda)}}{\prod_{i=1}^{n}(\lambda - \mu_i(0))}. \] (52)

and the differential \( \hat{\Omega} \) is given explicitly by
\[ \hat{\Omega}(P) = \frac{1}{2} \frac{\prod_{i=1}^{n}(\lambda - \mu_i(0))}{\sqrt{\nu}} d\lambda. \] (53)

We assume that \( E(P) \) is a meromorphic function on \( K \) with \( n + 1 \) simple poles \( \infty, p_1, \ldots, p_n \) and at \( P \to \infty, E(P) = k + \ldots \), and \( \hat{E}(P) \) is meromorphic function with \( n + 1 \) simple poles \( q_0, q_1, \ldots, q_n \) and at \( P \to \infty, \hat{E}(P) = k^{-1} + \ldots \). We also suppose that the divisors of poles of \( E(P) \) and \( \hat{E}(P) \) are different from \( D, D^\tau \).

\[ \psi(t, \lambda) = \sqrt{F(t, \lambda)} \exp \left( i \int_0^t \frac{\nu(\lambda)}{F(t', \lambda)} dt' \right). \] (54)

A brief computation reveals that the BA function solves Hill’s equation
\[ \left( \frac{d^2}{dt^2} - u(t) \right) \psi = \lambda \psi, \] (55)

and for dual BA function \( \psi^\tau \) we have
\[ \psi^\tau(t, \lambda) = \sqrt{F(t, \lambda)} \exp \left( -i \int_0^t \frac{\nu(\lambda)}{F(t', \lambda)} dt' \right). \]

The solutions of the system with Hamiltonian (54) in terms of Novikov polynomials \( F(t, \lambda) \) of degree \( n + 1 \) in spectral parameter \( \lambda \) (61), for special points \( a_i, i = 1, \ldots, n \) in closed intervals \( [\lambda_{2i-1}, \lambda_{2i}], i = 1, \ldots, n \) the functions are given by (32, 33, 35)
\[ q_0 = -u(t), \quad q_i^2 = 16 \frac{F(t, a_i)}{\prod_{k \neq i}(a_i - a_k)}, \quad i = 1, \ldots, n. \] (56)

\( u(t) \) is the famous Its-Matveev formulae (39), (14, 63) and the points \( a_i \) lie in the lacunae \( [\lambda_{2i-1}, \lambda_{2i}], i = 1, \ldots, n \) for generalized multidimensional Hénon-Heiles system and are branch points in the case of the multidimensional Hénon-Heiles system (54) with \( E_j^2 = 0, \quad j = 1, \ldots, n \). The solution is real under the choice of the arbitrary constants \( a_i, i = 1, \ldots, n \) in such a way, that the constants \( a_i, i = 1, \ldots, n \) lie in different lacunae of Riemann surface \( K \). Then the constants \( \zeta_i \) are given as
\[ \zeta_i^2 = \text{const} \cdot \frac{\nu(a_i)^2}{\left( \prod_{k \neq i}(a_i - a_k) \right)^2}, \]

where \( i = 1, \ldots, n \) and \( \nu \) is the coordinate of the curve (50). The constants \( \zeta_i \) are fixed by the initial conditions.

Suppose that \( F(t, \lambda) = F(t, \lambda), \) where \( F \) is Hermite polynomial associated with Lamé potential \( u = (n + 1)(n + 2)p(t + \omega') \). Then the finite and real solution of the system (52), (33) is given by (50) with the Hermite polynomial depending on the argument \( t + \omega' \) (the shift in \( \omega' \) provides the holomorphy of the solution). Consider the potential \( 12p(t) \) and construct the associated curve (57)
\[ \nu^2 = 4\lambda \prod_{i=1}^{3}(\lambda^2 - 6e_i \lambda + 45e_i^2 - 15g_2), \] (57)
with branch points given by

\[
\begin{align*}
\lambda_0 &= 0, \quad \lambda_{1,2} = 3 \left(1 + k^2 \pm 2 \sqrt{4 - 7k^2 + 4k^4}\right), \\
\lambda_{3,4} &= 3 \left(1 - 2k^2 \pm 2 \sqrt{4 - k^2 + k^4}\right), \\
\lambda_{5,6} &= 3 \left(k^2 - 2 \pm 2 \sqrt{1 - k^2 + 4k^4}\right).
\end{align*}
\]

(58)

or in another form

\[
\frac{\nu^2}{4} = \lambda^2 - \frac{63}{2} \lambda^5 \gamma_2 - \frac{297}{2} \lambda^4 \gamma_3 + \frac{4185}{16} \lambda^3 \gamma_2^2 + \frac{18225}{8} \lambda^2 \gamma_2 \gamma_3 \\
+ \left(\frac{91125}{16} \gamma_3^2 - \frac{3375}{16} \gamma_2^3\right) \lambda.
\]

(59)

The Hermite polynomial \( \mathcal{F}(\nu(t), \lambda) \) associated with the Lamé potential \( 12 \nu(t) \) has the form

\[
\mathcal{F}(\nu(t + \omega'), \lambda) = \lambda^3 - 6 \nu(t + \omega') \lambda^2 - 3 \cdot 5(g_2 - 3 \nu(t + \omega')^2) \lambda \\
- \frac{3^2 \cdot 5^2}{4} (4 \nu(t + \omega') - g_2 \nu(t + \omega') - g_3).
\]

(60)

Then the finite and real solution of the system (32), (33) is given in Table I, see for example [59].

Next we list the periodic solutions of the system (32), (33) for convenience we present solutions in the following form

\[
C = \text{const} \cdot \frac{\nu(a_j)}{(a_j - a_i)^2}, \quad C_2 = \text{const} \cdot \frac{\nu(a_j)}{(a_j - a_i)^2},
\]

(61)

where \( i = 1, \ldots, n \) and \( \nu \) is the coordinate of the curve (57).

Next we list the periodic solutions of the system (32), (33) for \( n = 2 \) and for genus 3, where \( C_1 = 0 \) and \( C_2 = 0 \). The \((2n + 3)\) Lamé polynomials of order \( n + 1 \) are solutions of

\[
\frac{d^2 E_i}{dt^2} + ((n + 1)(n + 2)k^2 \nu^2(a_t) - \lambda_i) E_i = 0.
\]

(62)

For \( n = 2 \) we introduce the following eigenfunctions \( E_i^{(3)} \), \( i = 1, \ldots 7 \) and eigenvalues \( \lambda_i \) given in Table I, \( \lambda_1 < \lambda_2 < \cdots < \lambda_7 \) and the results are collected in Table I, see for example [59].

| \( i \) | \( E_i^{(3)} = \text{sn}(a_t, k)(\nu^2(a_t, k) + C_1^{(3)}) \) | \( \lambda_i^{(3)} = 7 - 5k^2 - 2\sqrt{4 - 7k^2 + 4k^4} \) |
| --- | --- | --- |
| 1 | \( E_1^{(3)} = \text{sn}(a_t, k)(\nu^2(a_t, k) + C_1^{(3)}) \) | \( \lambda_1^{(3)} = 7 - 5k^2 - 2\sqrt{4 - 7k^2 + 4k^4} \) |
| 2 | \( E_2^{(3)} = \text{cn}(a_t, k)(\nu^2(a_t, k) + C_2^{(3)}) \) | \( \lambda_2^{(3)} = 7 - 2k^2 - 2\sqrt{4 - k^2 + k^4} \) |
| 3 | \( E_3^{(3)} = \text{dn}(a_t, k)(\nu^2(a_t, k) + C_3^{(3)}) \) | \( \lambda_3^{(3)} = 5(2 - k^2) - 2\sqrt{1 - k^2 + 4k^4} \) |
| 4 | \( E_4^{(3)} = \text{sn}(a_t, k)\text{cn}(a_t, k)\nu^2(a_t, k) \) | \( \lambda_4^{(3)} = 4(2 - k^2) \) |
| 5 | \( E_5^{(3)} = \text{sn}(a_t, k)(\nu^2(a_t, k) + C_5^{(3)}) \) | \( \lambda_5^{(3)} = 7 - 5k^2 + 2\sqrt{4 - 7k^2 + 4k^4} \) |
| 6 | \( E_6^{(3)} = \text{cn}(a_t, k)(\nu^2(a_t, k) + C_6^{(3)}) \) | \( \lambda_6^{(3)} = 7 - 2k^2 + 2\sqrt{4 - k^2 + k^4} \) |
| 7 | \( E_7^{(3)} = \text{dn}(a_t, k)(\nu^2(a_t, k) + C_7^{(3)}) \) | \( \lambda_7^{(3)} = 5(2 - k^2) + 2\sqrt{1 - k^2 + 4k^4} \) |

TABLE I.

For convenience we present solutions in the following form

\[
q_0 = 12\tilde{C}_0 k^2 \nu^2(a_t), \quad q_1^{(i)} = \tilde{C}_1^{(i)} E_1^{(3)}, \quad q_2^{(j)} = \tilde{C}_2^{(j)} E_j^{(3)}, \quad i \neq j = 1, \ldots 7
\]

(63)

where the constants \( \tilde{C}_0, \tilde{C}_1^{(i)}, \tilde{C}_2^{(j)} \) are fixed by initial conditions. The same procedure is possible for general \( n \).

The Hermite polynomial \( \mathcal{F}(\nu(t), \lambda) \) associated with the Lamé potential can be written in a different form, useful for applications:

\[
\mathcal{F}(\nu(t), \lambda) = \sum_{k=0}^{n} A_k(\lambda)\nu(t)^{n-k}.
\]

(64)
For example for the genus 4 Lamé potential $20\wp(t)$ we have

$$A_0 = 11025, \quad A_1 = -1575\lambda, \quad A_2 = 135\lambda^2 - \frac{6615}{2}g_2$$

$$A_3 = -10\lambda^3 + \frac{1855}{4}\lambda g_2 - 2450g_3$$

$$A_4 = \lambda^4 - \frac{113}{2}\lambda^2g_2 + \frac{3969}{16}g_2^2 + \frac{1925}{4}\lambda g_3.$$  

or in explicit form we have the Hermite polynomial $F(\wp(t + \omega'), \lambda)$ associated to the Lamé potential $20\wp(t + \omega')$ can be written as

$$F(\wp(t + \omega'), \lambda) = 11025\wp(t + \omega')^4 - 1575\wp(t + \omega')^3\lambda$$

$$+ (135\lambda^2 - \frac{6615}{2}g_2)\wp(t + \omega')^2$$

$$+ (-10\lambda^3 + \frac{1855}{4}\lambda g_2 - 2450g_3)\wp(t + \omega')$$

$$+ \lambda^4 - \frac{113}{2}\lambda^2g_2 + \frac{3969}{16}g_2^2 + \frac{195}{4}\lambda g_3.$$  

(65)

For $n = 3$ and genus four, the Lamé curve have the following form

$$\nu^2 = 4 \left( \prod_{\ell=1}^{3} (\lambda^2 + 10c_1\lambda - 35c_1^2 - 7g_2) \right) (\lambda^3 - 52\lambda g_2 + 3610g_3).$$

or in another form convenient for practical use we have

$$\frac{\nu^2}{4} = \lambda^9 - \frac{231}{2}\lambda^7 g_2 + \frac{2145}{2}g_3 \lambda^6 + \frac{63129}{16}\lambda^5 g_2^2 - \frac{518505}{8}g_2 g_3 \lambda^4$$

$$+ \left( \frac{563227}{16}g_3^3 + \frac{4549125}{16}g_3^2 \right) \lambda^3 + \frac{991515}{2}g_3 g_2^2 \lambda^2$$

$$+ \left( \frac{361779}{4}g_2^4 - \frac{5273625}{4}g_2 g_3^2 \right) \lambda - 972405g_3 g_2^3 - 1500625g_3^3.$$  

(66)

VII. SUMMARY AND CONCLUSIONS

We approached the most general class among the three classes of integrable Henon-Heiles-type systems identified in [13].

We provided exact quasi-periodic solutions for this class of systems, expressing these orbits via Kleinian hyper-elliptic functions.

Applying the spectral theory for the Schrodinger equation to elliptic potentials, we pointed out elliptic periodic solutions.

To obtain these results, we resorted to various mathematical methods, characteristic to both physics and astronomy.

We applied these methods and results to a generalized Henon-Heiles-type system with $(n + 1)$ degrees of freedom, pointing out the exact solutions in this case.
By emphasizing the exact solutions for the integrable class of systems under consideration, our paper contributes to a better understanding of the generalized Henon-Heiles-type problem.
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