Vacuum current and polarization induced by magnetic flux in a higher-dimensional cosmic string in the presence of a flat boundary

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July 14, 2020

Abstract

In this paper we analyze the vacuum bosonic current and polarization induced by a magnetic flux running along a higher dimensional cosmic string in the presence of a flat boundary orthogonal to the string. In our analysis we assume that the quantum field obeys Dirichlet or Neumann conditions on the flat boundary. In order to develop this analysis we calculate the corresponding Wightman function. As consequence of the boundary, the Wightman function is expressed in term of two contributions: The first one corresponds to the boundary-free cosmic string Wightman function, while the second one is induced by the boundary. The boundary-induced contributions have opposite signs for Dirichlet and Newman scalars. Because the analysis of vacuum current and polarization effects in the boundary-free cosmic string Wightman function, we are mainly interested in the calculations of the effects induced by the boundary. Regarding to the induced current, we show that, depending on the condition adopted, the boundary-induced azimuthal current can cancel or intensifies the total induced azimuthal current on the boundary; moreover, the boundary-induced azimuthal current is a periodic odd function of the magnetic flux. As to the vacuum expectation values of the field squared and the energy-momentum tensor, the boundary-induced contributions are even functions of magnetic flux. In particular, we consider some special cases of the boundary-induced part of the energy density and evaluate the normal vacuum force on the boundary.

PACS numbers: 98.80.Cq, 11.10.Gh, 11.27. + d

1 Introduction

In the context of grand unified theories, different types of topological defects can be produced in the early Universe as consequence of vacuum symmetry breaking phase transitions [1–3]. Depending on the topology of the vacuum manifold these defects can be: domain walls, cosmic strings, monopoles and texture. Among them cosmic strings have attracted considerable attention. Although the recent observational data on the cosmic microwave background have
ruled out cosmic strings as the primary source for large scale structure formation, they are still candidates for a variety of interesting physical phenomena like as the generation of gravitational waves [4], high energy cosmic rays [5], and gamma ray bursts [6]. Moreover, recently the cosmic strings have attracted a renewed interest partly because a variant of their formation mechanism is proposed in the framework of brane inflation [7–9].

The gravitational field produced by a cosmic string may be approximated by a planar angle deficit in the two-dimensional sub-space orthogonal to the string. The simplest theoretical model which describes a straight and infinitely long cosmic string is given by a delta-Dirac type distribution for the energy-momentum tensor along the linear defect. Also, this object can be described by a classical field theory, coupling the energy-momentum tensor associated with the Maxwell-Higgs system investigated by Nielsen and Olesen in [10] with the Einstein equations [11,12].

Although the geometry of the spacetime produced by an idealized cosmic string is locally flat, its conical structure alters the vacuum fluctuations associated with quantum fields. As a consequence, the vacuum expectation value (VEV) of physical observables like the energy-momentum tensor, \( \langle T_{\mu\nu} \rangle \), gets a nonzero value. The calculation of the VEV of physical observables associated with the scalar and fermionic fields in the cosmic string spacetime has been developed in [13–17] and [18–20], respectively. Furthermore, the presence of a magnetic flux running through the core of the string gives additional contributions to the VEVs associated with charged fields [21–27] as well as induces vacuum current densities, \( \langle j^\mu \rangle \). This phenomenon has been investigated for massless and massive scalar fields in [28] and [29], respectively. In these papers, the authors have shown that induced vacuum current densities along the azimuthal direction arise if the ratio of the magnetic flux by the quantum one has a nonzero fractional part. The induced bosonic current in higher-dimensional compactified cosmic string spacetime was calculated in [31]. Moreover, the calculation of induced fermionic currents in higher-dimensional cosmic string spacetime in the presence of a magnetic flux has been developed in [30].

Another type of vacuum polarization takes place when boundaries are present. In this sense, modifications on the vacuum expectation values of physical observable of the system take places due to imposition of boundary conditions on quantum fields. This is the well-known Casimir effects. The analysis of Casimir effects in the idealized cosmic string space-time have been developed for a scalar [33], vector [34] and fermionic fields [35], obeying boundary conditions on the cylindrical surfaces. Also the analysis of Casimir effect induced by a flat boundary orthogonal to the string was developed in [36].

In this paper we will continue in the same line of investigation, i.e., the investigation of the combined effect of the geometry and boundary condition on quantum vacuum. In this sense, we will analyze the physical system composed by a charged quantum bosonic fields propagating in a higher dimensional cosmic string having a magnetic flux running along its core and considering the presence of a flat hypersurface orthogonal to it. Two different boundaries conditions will be imposed to the field on the flat boundary: the Dirichlet and Newman conditions. In this way want to investigate the influence of the boundary on the induced vacuum current \( \langle j^\mu \rangle \), and on the vacuum expectation value (VEV) of the field squared, \( \langle | \varphi |^2 \rangle \), and the energy-momentum tensor, \( \langle T_{\mu\nu} \rangle \).

This paper is organized as follows: In Section 2 we present the background geometry of the spacetime that we want to work, and the explicit expression for the four-vector potential considered. Also we provide the complete set of normalized wave-function associated with a charged scalar quantum field obeying Dirichlet/Newman boundary condition on a flat plane orthogonal to the string, which presents a magnetic flux along its core. By using the mode

\[ \text{[37]} \]
summation formula, we calculate the positive frequency Wightman function. As we will see the corresponding Wightman functions are expressed in terms of two distinct contributions: The first one is the standard Wightman function associated with a charged massive scalar field in a higher dimensional cosmic string spacetime in absence of boundaries, and the second contribution is due to the boundary condition obeyed by the field. The first contribution is divergent at coincidence limit; as to the second one, it is finite in this limit for points away from the boundary. In Section 3 we investigate the vacuum bosonic current induced by the magnetic flux and boundary. There we will see that, depending on the boundary condition obeyed by the field on the flat boundary, the induced azimuthal current can decreases or increases the intensity of the total induced current. In Section 4 we calculate the contribution of the VEV of the field squared and the energy-momentum tensor induced by the boundary and magnetic flux. One of our main objective is to investigate how the presence of a magnetic flux modifies these quantities; moreover, we will analyze these observable in different regions of the space. Specifically for points close and far from the string and/or boundary. In this sense some asymptotic expressions will be explicitly provided. In Section 5 we summarize the most relevant results obtained. In this paper we will use the units $\hbar = G = c = 1$.

2 Wightman function

In this section we present the geometry background of the spacetime that we want to work. It corresponds to a generalization of a four-dimensional idealized cosmic string spacetime for higher dimensions. Considering $D \geq 4$ as the dimension of the spacetime, by using cylindrical coordinate system this $D$-dimensional conical space is given by the line element below,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - dr^2 - r^2d\phi^2 - dz^2 - \sum_{i=4}^{D-1}(dx^i)^2.$$  \hspace{1cm} (1)

In this coordinate system we are assuming: $r \geq 0$, $0 \leq \phi \leq 2\pi/q$ and $-\infty < (t, x^i) < +\infty$ for $i = 4, ..., D - 1$. The presence of the cosmic string is codified through the parameter $q > 1$. In a four dimensional spacetime, this parameter is related to the linear mass density of the string by $q^{-1} = 1 - 4\mu$. Because we want to investigate the influence of a flat boundary orthogonal to the string located at $z = 0$ on the quantum system, we will assume that the coordinate $z \geq 0$.

The quantum dynamics of a charged bosonic field with mass $m$ in a curved spacetime and in the presence of an electromagnetic potential vector, $A_\mu$, is governed by the equation below,

$$\left[ \frac{1}{\sqrt{|g|}} D_\mu \left( \sqrt{|g|} g^{\mu\nu} D_\nu \right) + m^2 + \xi R \right] \varphi_\sigma(x) = 0.$$ \hspace{1cm} (2)

In the above equation we have included the non-minimal coupling between the field with the geometry, given by $\xi R$, where $R$ represents the curvature scalar, and $\xi$ the non-minimal coupling. Moreover, $D_\mu = \partial_\mu + ieA_\mu$ and $g = \det(g_{\mu\nu})$. Considering a thin and infinitely straight cosmic string, we have that $R = 0$ for $r \neq 0$.

In our analysis we will assume that only the azimuthal component of the vector potential does not vanish, i.e., there is only $A_\phi = -q\Phi/2\pi$, being $\Phi$ the magnetic flux along the string.

In the spacetime defined by (1) and in the presence of the vector potential given above, the equation (2) becomes

$$\left[ \partial_t^2 - \partial_r^2 - \frac{1}{r} \partial_r - \frac{1}{r^2} (\partial_\phi + ieA_\phi)^2 - \partial_z^2 - \sum_{i=4}^{D-1} \partial_i^2 + m^2 \right] \varphi_\sigma(x) = 0.$$ \hspace{1cm} (3)
The positive energy solution of this equation can be obtained by considering the general expression,

$$\varphi_\sigma(x) = C_\sigma R(r)e^{-iEt+iqn\phi+i\vec{k}\cdot\vec{r}}g(z),$$

where \(\vec{r}\) represents the coordinates of the extra dimensions, \(\vec{k}\) the momentum along these directions and \(C_\sigma\) is a normalization constant. The unknown function \(g(z)\) will be specified by the boundary condition obeyed by the field on the boundary placed at \(z = 0\).

First of all we impose that \(g(z)\) satisfies the differential equation,

$$\frac{d^2 g(z)}{dz^2} = -k_z^2 g(z).$$

Accepting that, and substituting (4) into (3), the differential equation for the radial function \(R(r)\) becomes,

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \lambda^2 - \frac{q^2(n + \alpha)^2}{r^2}\right]R(r) = 0,$$

where

$$\lambda = \sqrt{E^2 - k_z^2 - m^2},$$

$$\alpha = -\frac{e\Phi}{2\pi}.$$

The solution of (6) regular at \(r = 0\), is:

$$R(r) = J_{q|n+\alpha|}(\lambda r),$$

being \(J_\nu(z)\) the Bessel function. As to the solution of (5), we have two possibilities:

- For the field obeying Dirichlet condition, \(g(z) = 0\) at \(z = 0\), we have
  $$g(z) = \sin(k_z z),$$

- For the field obeying Newman condition, \(\frac{dg(z)}{dz} = 0\) at \(z = 0\), we have
  $$g(z) = \cos(k_z z).$$

The solution (4) is then characterized by the set of quantum number, \(\sigma = \{\lambda, n, k_z, \vec{k}\}\). Its normalization constant \(C_\sigma\) can be obtained by the normalization condition

$$i \int d^Dx \sqrt{|g|} [\varphi_\sigma^*(x)\partial_t \varphi_\sigma(x) - \varphi_\sigma(x)\partial_t \varphi_\sigma^*(x)] = \delta_{\sigma,\sigma'},$$

where the delta symbol on the right-hand side is understood as Dirac delta function for the continuous quantum number, \(\lambda, k_z\) and \(\vec{k}\), and Kronecker delta for the discrete ones, \(n\). From (11) one finds

$$|C_\sigma| = \sqrt{\frac{2\gamma\lambda}{(2\pi)^{D-2}E}},$$

for both modes of wave-function \(g(z)\).

The properties of the vacuum state can be described in terms of the positive frequency Wightman function, \(W(x, x') = \langle 0 | \varphi(x)\varphi^*(x') | 0 \rangle\), where \(0\) represents the vacuum state. Having this function we can evaluate the induced bosonic current and the VEV of the field squared and energy-momentum tensor.
For the evaluation of the Wightman function, we adopt the mode sum formula

\[ W(x, x') = \sum_{\sigma} \varphi_{\sigma}(x)\varphi^*_{\sigma}(x') , \quad (13) \]

where we are using the compact notation for the sum defined as

\[ \sum_{\sigma} = \sum_{n=-\infty}^{+\infty} \int d\tilde{k} \int_{0}^{\infty} dk_z \int_{0}^{\infty} d\lambda . \quad (14) \]

The set \( \{\varphi_{\sigma}(x), \varphi^*_{\sigma}(x')\} \) represents a complete set of normalized mode functions satisfying the Dirichlet/Newman boundary condition.

Substituting (4) with (8) and (12) into (13), we obtain:

\[
W(x, x') = \frac{2q}{(2\pi)^{D-2}} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} d\lambda \lambda J_{q|n+\alpha|}(\lambda r)J_{q|n+\alpha|}(\lambda r')e^{iqn(\phi-\phi')} \\
\times \int_{0}^{\infty} dk_z g(z)g(z') \int d^{D-4}ke^{-i\vec{k} \cdot \vec{r}_s} \frac{e^{-iE(t-t')}}{\sqrt{\lambda^2 + k_z^2 + \vec{k}^2 + m^2}}. \quad (15)
\]

By adopting a Wick rotation, \( t \to -i\tau \) and using the identity below,

\[
e^{-E\Delta \tau} = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} ds e^{-E^2s^2-\Delta \tau^2/(4s^2)}, \quad (16)
\]

we obtain,

\[
W(x, x') = \frac{4q}{(2\pi)^{D-2}2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} ds \int_{0}^{\infty} d\lambda \lambda e^{-\lambda^2s^2} J_{q|n+\alpha|}(\lambda r)J_{q|n+\alpha|}(\lambda r')e^{iqn(\phi-\phi')} \\
\times \int_{0}^{\infty} dk_z e^{-k_z^2s^2} g(z)g(z') \int d^{D-4}ke^{-\vec{k}^2s^2} e^{i\vec{k} \cdot (\vec{r}_s - \vec{r}_s')} e^{-m^2s^2} e^{-\Delta \tau^2/(4s^2)}. \quad (17)
\]

For the integration over the quantum number \( \lambda \) we use the formula [38]

\[
\int_{0}^{\infty} d\lambda \lambda J_{\beta}(\lambda r)J_{\beta}(\lambda r')e^{-s^2\lambda^2} = \frac{1}{2s^2} \exp \left( -\frac{r^2 + r'^2}{4s^2} \right) I_{\beta} \left( \frac{rr'}{2s^2} \right). \quad (18)
\]

As to the integral in \( k_z \) we have,

\[
\int_{0}^{\infty} e^{-k_z^2s^2} g(z)g(z')dk_z = \frac{\sqrt{\pi}}{4s} \left( e^{-\frac{(z-z')^2}{4s^2}} + e^{-\frac{(z+z')^2}{4s^2}} \right). \quad (19)
\]

The negative/positive signal in the expression above refers to Dirichlet/Newman boundary condition.

So, the positive energy Wightman function given by (17) can be express in terms of two function, as exhibit below:

\[
W(x, x') = W_{ca}(x, x') \mp W_b(x, x') . \quad (20)
\]

The negative/positive signal corresponds to the Dirichlet/Newman condition.

Defining a new variable \( u = 1/2s^2 \), from (17) we find

\[
W_{ca}(x, x') = \frac{q}{2(2\pi)^{D/2}} \int_{0}^{\infty} du \, u^{D/2-2} \exp \left( -\frac{m^2}{4\pi u} - \frac{y^2}{4u} \right) J(\alpha, \Delta \phi, urr') , \quad (21)
\]
and
\[ W_b(x, x') = \frac{q}{2(2\pi)^{D/2}} \int_0^\infty du \frac{u^{D-2} e^{-\frac{m^2}{2u} - \frac{2}{\nu} V^2(+)}}{u^{D-2}} I(\alpha, \Delta \phi, urr') , \] (22)
with
\[ V^2(+) = r^2 + r'^2 + |\Delta r|^2 + (z + z')^2 - (\Delta t)^2 . \] (23)
Moreover, we have introduced the notation
\[ I(\alpha, \Delta \phi, urr') = \sum_{n=-\infty}^{\infty} e^{i\eta \Delta \phi} I_{q|n+\alpha|}(urr') . \] (24)

We can obtain a more convenient expressions for (21) and (22) writing the parameter \( \alpha \) defined in (7) in the form
\[ \alpha = n_0 + \alpha_0 \quad \text{with} \quad |\alpha_0| < 1/2, \] (25)
where \( n_0 \) is an integer number. Using the form of the summation over \( n \) given in [39], we get
\[ I(\alpha, \Delta \phi, x) = \frac{1}{q} \sum_k e^{i\alpha(2k\pi\Delta \phi)} e^{iq0\Delta \phi} \frac{e^{-iqn\Delta \phi}}{2\pi i} \sum_{j=\pm 1} j e^{j\pi \alpha_0} \]
\[ \times \int_0^\infty dy \frac{\cosh[q(1 - \alpha_0)] - \cosh(q\alpha_0 y)e^{-iq(\Delta \phi + j\pi)}}{\cosh(qy) - \cos(q(\Delta \phi + j\pi))} . \] (26)

As to the summation over \( k \) there exist the condition
\[ -\frac{q}{2} < q \frac{2\pi}{q} \Delta \phi \leq k \leq \frac{q}{2} \frac{2\pi}{q} \Delta \phi . \] (27)
Substituting (26) with \( x = urr' \) into the (21) and (22) we get:
\[ W_{cs}(x, x') = \frac{m^{D-2}}{(2\pi)^{D/2}} \left\{ \sum_k e^{i\alpha(2k\pi\Delta \phi)} f_{D-2} \left[ m \sqrt{V^2(-) - 2rr' \cos(2k\pi\Delta \phi)} \right] \right. 
\[ - \frac{qe^{-iqn\Delta \phi}}{2\pi i} \sum_{j=\pm 1} j e^{j\pi \alpha_0} \int_0^\infty dy \frac{\cosh[q(1 - \alpha_0)] - \cosh(q\alpha_0 y)e^{-iq(\Delta \phi + j\pi)}}{\cosh(qy) - \cos(q(\Delta \phi + j\pi))} \]
\[ \times f_{D-2} \left[ m \sqrt{V^2(+) + 2rr' \cosh y} \right] \} . \] (28)
and
\[ W_b(x, x') = \frac{m^{D-2}}{(2\pi)^{D/2}} \left\{ \sum_k e^{i\alpha(2k\pi\Delta \phi)} f_{D-2} \left[ m \sqrt{V^2(+)} - 2rr' \cos(2k\pi\Delta \phi) \right] \right. 
\[ - \frac{qe^{-iqn\Delta \phi}}{2\pi i} \sum_{j=\pm 1} j e^{j\pi \alpha_0} \int_0^\infty dy \frac{\cosh[q(1 - \alpha_0)] - \cosh(q\alpha_0 y)e^{-iq(\Delta \phi + j\pi)}}{\cosh(qy) - \cos(q(\Delta \phi + j\pi))} \]
\[ \times f_{D-2} \left[ m \sqrt{V^2(-) + 2rr' \cosh y} \right] \} , \] (29)

In the above equations we have used the definition
\[ f_\nu(x) = \frac{K_\nu(x)}{x^\nu} . \] (30)

At this point the results obtained deserve to be commented: The Wightman function [28] is divergent at the coincidence limit and its divergence comes from the term \( k = 0 \). As to [29], it is a consequence of the boundary condition imposed on the field. This function is finite at coincidence limit for points outside the boundary.
3 Current densities

The bosonic current density operator is given by

$$j_\mu(x) = ie[\phi^*(x)D_\mu\phi(x) - (D_\mu\phi)^*(x)] = ie[\phi^*(x)\partial_\mu\phi(x) \cdot \partial_\mu\phi(x)\phi(x)] - 2e^2A_\mu(x)|\phi(x)|^2.$$ (31)

Its vacuum expectation value (VEV) can be evaluated in terms of the positive frequency Wightman function as shown below:

$$\langle j_\mu(x) \rangle = ie \lim_{x' \to x}\{(\partial_\mu - \partial_\mu')W(x,x') + 2ieA_\mu W(x,x')\}. \quad (32)$$

However, for the case under consideration, the only nonzero component of the current density is the azimuthal one. In this way, we will focus only on the evaluation of this component.

So, specifically the azimuthal component, reads:

$$\langle j_\phi(x) \rangle = ie \lim_{x' \to x}\{(\partial_\phi - \partial_\phi')W(x,x') + 2iqA_\phi W(x,x')\}, \quad (33)$$

where we have substitute $A_\phi = q\alpha/e$.

Because the Wightman function (20) is expressed as the sum of two contributions, the azimuthal current can be expressed as:

$$\langle j_\phi(x) \rangle = \langle j_\phi(x) \rangle_{cs} \mp \langle j_\phi(x) \rangle_b. \quad (34)$$

The first contribution corresponds to the induced current in the cosmic string spacetime in absence of boundary while the second one is induced by the boundary. The latter can be negative or positive, depending on the boundary condition obeyed by the scalar field. Because the $\langle j_\phi(x) \rangle_{cs}$ has been calculated by many authors, here we are mainly interested in the analysis of $\langle j_\phi(x) \rangle_b$.

Returning to (22) and defining a dimensionless variable $w = ur^2$, from (33), we get:

$$\langle j_\phi(x) \rangle_b = -\frac{eq}{(2\pi)^{D/2}r^{D-2}} \int_0^\infty dw w^{D/2-2}e^{-w(1+2(z/r)^2)}e^{-m^2r^2/(2w)}I(q,\alpha, w), \quad (35)$$

where

$$I(q,\alpha, w) = \sum_{n=-\infty}^\infty q(n + \alpha)I_{q\alpha n}(w). \quad (36)$$

In [31] we have obtained a compact expression for the summation above:

$$I(q,\alpha, w) = \frac{2w}{q} \sum_{k=1}^{[q/2]} \sin(2k\pi/q) \sin(2k\pi\alpha_0)e^{w\cos(2k\pi/q)} + \frac{w}{\pi} \int_0^\infty dy \sinh y \frac{e^{-w\cosh y}g(q,\alpha_0, y)}{\cosh(qy) - \cos(\pi q)} \quad (37)$$

with

$$g(q,\alpha_0, y) = \sin(q\pi\alpha_0) \sinh[(1 - |\alpha_0|)qy] - \sinh(yq\alpha_0) \sin[(1 - |\alpha_0|)\pi q]. \quad (38)$$

In (37), the symbol $[q/2]$ represents the integer part of $q/2$, and the prime on the sign of the summation means that in the case $q = 2p$ the term $k = q/2$ should be taken with the coefficient 1/2.
Substituting (37) and (38) into (35), and using the integral representation below for the Macdonald function \[38\]

\[
K_{\nu}(x) = \frac{1}{2} \left(\frac{x}{2}\right)^{\nu} \int_{0}^{\infty} dt \frac{e^{-t-x^2/4}}{t^{\nu+1}},
\]

after some intermediate steps we obtain:

\[
\langle j^\phi(x) \rangle_b = \frac{4e m^D}{(2\pi)^z} \left[ \sum_{k=1}^{[q/2]} \sin(2k\pi/q) \sin(2k\pi\alpha_0) f_\nu(\gamma_{k,z}) \right.
\]
\[
+ \frac{q}{\pi} \int_{0}^{\infty} dy \left. \frac{g(q,\alpha_0,2y) \sinh(2y) \cosh(2qy) - \cos(q\pi) f_\nu(\gamma_{k,y})}{\cos(q\pi)} \right],
\]

where

\[
\gamma_{k,z} = 2mr \sqrt{s_k^2 + (z/r)^2} \quad \text{and} \quad \gamma_{y,z} = 2mr \sqrt{c_y^2 + (z/r)^2},
\]

with

\[
s_k = \sin(k\pi/q) \quad \text{and} \quad c_y = \cosh(y).
\]

The azimuthal current induced by the boundary is an odd function of \(\alpha_0\). In addition, we can note that \(\langle j^\phi(x) \rangle_b\) is finite in \(r = 0\) for points outside the boundary if \(q|\alpha_0| > 1\). In the case where \(q|\alpha_0| < 1\) the azimuthal current diverges and the leading divergence term is given by

\[
\langle j^\phi(x) \rangle_b \approx \frac{4e m^D q \sin(q\alpha_0)}{(2\pi)^z (2mr)^{2(1-q\alpha_0)} f_\nu(\gamma_{k,y})} \left[ 2^{3/2}mz \right],
\]

where we have used the notation (30). Moreover, for the case of the field obeying the Dirichlet boundary condition the total current, (34), vanishes on the boundary. Also we can see that the azimuthal current density goes exponentially to zero for large distance from the boundary, i.e. \(z/r \gg 1\), according with

\[
\langle j^\phi(x) \rangle_b \approx \frac{e}{z^{D-1}} \left(\frac{m}{4\pi}\right)^{D/2} \left[ \sum_{k=1}^{[q/2]} \sin(2k\pi/q) \sin(2k\pi\alpha_0)e^{-2mqz} \right.
\]
\[
+ \frac{q}{\pi} \int_{0}^{\infty} dy \left. \frac{g(q,\alpha_0,2y) \sinh(2y) \cosh(2qy) - \cos(q\pi) e^{-\gamma_{k,y}}}{\cos(q\pi)} \right] \left[ 1 + (rc_y/z)^2 \right]^{(D-1)/2}.
\]

The massless limit of \(\langle j^\phi(x) \rangle_b\) can be obtained by taking the asymptotic limit of modified Bessel function for small arguments \[40\]. In this limit, from (30) we have,

\[
f_\nu(z) = 2^{\nu-1/2} \frac{\Gamma(\nu)}{\pi^{1/2}}.
\]

So using the above result we obtain:

\[
\langle j^\phi(x) \rangle_b = \frac{e \Gamma(D/2)}{2D-1 \pi D/2 r^D} \left[ \sum_{k=1}^{[q/2]} \sin(2k\pi/q) \sin(2k\pi\alpha_0) \right.
\]
\[
\left. \frac{1}{s_k^2 + (z/r)^2} \right]^{D/2} \left. \frac{q}{\pi} \int_{0}^{\infty} dy \left. \frac{g(q,\alpha_0,2y) \sinh(2y) \cosh(2qy) - \cos(q\pi) \left[ c_y^2 + (z/r)^2 \right]^{D/2}}{\cos(q\pi)} \right].
\]

In the Fig. 1 we show the behavior of the VEV of the azimuthal current density as function of \(mr\) (left plot) and \(z/r\) (right plot). We note that as \(mr\) increase, \(\langle j^\phi(x) \rangle_b\) goes to zero. In other hand, for small values of \(mr\) the azimuthal current presents a divergence, as indicated in the Eq. (43). Also, as \(z/r\) increases \(\langle j^\phi(x) \rangle_b\) goes to zero, which agrees with the Eq. (44).
Figure 1: VEV of the azimuthal current density induced by the boundary as function of \( m r \) (left plot) and \( z/r \) (right plot) for different values of the parameter \( q \) and \( D = 4 \). In addition, in the left plot we consider \( mz = 0.5 \) and \( \alpha_0 = 0.4 \), while in the right one we have \( mr = 1 \) and \( \alpha_0 = 0.25 \).

4 Vacuum polarization

In this section we want to develop the calculations of two important characteristics of the vacuum state: the VEVs of the field squared, \( \langle |\varphi(x)|^2 \rangle \), and the energy-momentum tensor, \( \langle T_\mu\nu(x) \rangle \).

Let us begin with the VEV of the field squared.

4.1 Calculation of \( \langle |\varphi(x)|^2 \rangle \)

By taking into account the Eq. (20), the VEV of the field squared can be decomposed in the same way. Here, we are mainly interested in the effects induced by the boundary. So, we will consider only the analysis of the VEV of the field squared induced by it. This part of the field squared, can be obtained by taking the limit of coincidence in (29) as follows

\[
\langle |\varphi(x')|^2 \rangle_b = \lim_{x' \to x} W_b(x, x'),
\]

(47)

After taking the coincidence limit and solve the summation over \( j \), we can write the boundary-induced part of the field squared as

\[
\langle |\varphi(x)|^2 \rangle_b = \langle |\varphi(x)|^2 \rangle_b^{(M)} + \langle |\varphi(x)|^2 \rangle_b^{(q, \alpha_0)},
\]

(48)

where

\[
\langle |\varphi(x)|^2 \rangle_b^{(M)} = \frac{m^{D-2}}{(2\pi)^{D/2}} f_{D-2} \left( \frac{2mz}{2} \right).
\]

(49)

The above expression is the \( k = 0 \) term of (29) with the coefficient \( 1/2 \) and corresponds to the one for a boundary in Minkowski spacetime in the absence of the cosmic string and magnetic flux. Note that this contribution also is independent of the radial coordinate. The second term on the right-hand side of (48) is the contribution induced by the conical geometry, boundary and the magnetic flux. It is given by

\[
\langle |\varphi(x)|^2 \rangle_b^{(q, \alpha_0)} = \frac{2m^{D-2}}{(2\pi)^{D/2}} \left\{ \sum_{k=1}^{[q/2]} \cos(2k\pi\alpha_0) f_{D-2} \left( \gamma_{k,z} \right) \right. \\
- \frac{q}{\pi} \int_{0}^{\infty} dy \frac{h(q, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)} f_{D-2} \left( \gamma_{k,y} \right) \right\},
\]

(50)
with
\[ h(q, \alpha_0, 2y) = \cosh[2qy(1 - |\alpha_0|)] \sin(q\pi|\alpha_0|) + \cosh(2q\alpha_0 y) \sin[q\pi(1 - |\alpha_0|)] . \] (51)

In the interval where \(1 \leq q < 2\), the first term in the square bracket of the Eq. (50) is absent. Note that the field squared is an even function of \(\alpha_0\).

The boundary induced part of the field squared can be considered in special cases. In the regions near the boundary, \(m|z| \ll 1\) and \(|z| \ll r\), the leading contribution is due to the VEV (49) being given by
\[ \langle |\varphi(x)|^2 \rangle^{(M)}_b \approx \frac{\Gamma(D/2 - 1)}{(4\pi)^{D/2} |z|^{D-2}} . \] (52)

Considering a massless scalar field, by taking into account the Eq. (45), the field squared is given by
\[ \langle |\varphi(x)|^2 \rangle^{(q, \alpha_0)}_b = \frac{\Gamma(D/2 - 1)}{2^{D-1} \pi^{D/2} r^{D-2}} \left\{ \sum_{k=1}^{p} \frac{\cos(2k\pi \alpha_0)}{s_k^2 + (z/r)^2} \left[ \cosh(2qy) - \cos(q\pi) \right]^{1-D/2} \right\} . \] (53)

In the regime where \(z/r \gg 1\), we have
\[ \langle |\varphi(x)|^2 \rangle^{(q, \alpha_0)}_b \approx \frac{m^{D-3}}{(4\pi z)^{D/2}} \left\{ \sum_{k=1}^{p} \cos(2k\pi \alpha_0) e^{-2mz} - \frac{q}{\pi} \int_0^\infty dy h(q, \alpha_0, 2y) e^{-\gamma_{k,y}} \left[ \cosh(2qy) - \cos(q\pi) \right]^{1-D/2} \right\} . \] (54)

Figure 2: VEV of the field squared as function of \(z/r\) (left plot) and \(mr\) (right plot) considering different values of \(a_0\) and \(D = 4\). We also considered \(mr = 0.75\) in the left plot and \(mz = 0.5\) in the right one. And for both plots we have \(q = 1.5\).

The behavior of the boundary induced part of the field squared is shown in Fig. 2 as function of \(z/r\) (left plot) and as function of \(mr\) (right plot) considering different values of the magnetic flux and \(q = 1.5\).

4.2 Calculation of \( \langle T_{\mu\nu}(x) \rangle \)

Another quantity which characterizes the quantum state is the VEV of the energy momentum-tensor. Here, we are interested mainly in the boundary effects. So, as for the others physical observables in this paper, we will determine only the contribution of the energy-momentum...
tensor induced by the boundary. In order to evaluate this VEV, we use the following formula obtained in [41]

\[
\langle T_{\mu\nu}(x) \rangle_b = \lim_{x' \to x} \left( D_\mu D^*_{\nu} + D^*_{\mu} D_\nu \right) W_b(x, x') - 2 [\xi R_{\mu\nu} + \xi \nabla_\mu \nabla_\nu - (\xi - 1/4) g_{\mu\nu} \Box] \langle \varphi^2(x) \rangle_b.
\]

In the spacetime that we are considering here, the Ricci tensor for points outside the string vanishes.

We start considering the d'Alembertian operator of the field squared. This operator presents a dependence only on the radial and axial coordinates. As the field squared is decomposed into two contributions, we also have two contributions for the d'Alembartian operator of it. These contributions are given by

\[
\Box \langle \varphi^2(x) \rangle_b^{(M)} = -\frac{m^D}{(2\pi)^{D/2}} \left[ (2m z)^2 f_{\frac{D+2}{2}}(2m|z|) - f_{\frac{D}{2}}(2m|z|) \right],
\]

and

\[
\Box \langle \varphi^2(x) \rangle_b^{(q,\alpha_n)} = -\frac{8m^D}{(2\pi)^{D/2}} \left\{ \frac{q}{2\alpha_n} \sum_{k=1}^{[q/2]} \cos(k\pi\alpha_n) \left[ 4m^2 \left( r^2 s^4 + z^2 \right) f_{\frac{D+2}{2}}(\gamma_k z) - (2s^2 k + 1) f_{\frac{D}{2}}(\gamma_k z) \right] \\
- \frac{q}{\pi} \int_0^\infty \frac{dy}{\cosh(2qy) - \cos(q\pi)} \left[ 4m^2 \left( r^2 c^4 + z^2 \right) f_{\frac{D+2}{2}}(\gamma_y z) - (2c^2 y + 1) f_{\frac{D}{2}}(\gamma_y z) \right] \right\}.
\]

The above d'Alembertian operators are calculated from the Eqs. (49) and (50), respectively.

In the geometry under consideration, the differential operators \( \nabla_r \nabla_r, \nabla_\phi \nabla_\phi, \nabla_z \nabla_z \) and \( \nabla_\phi \nabla_z \) present contributions when acting on the VEV of the field squared. In particular, for the azimuthal contribution, we shall use the expression (22). After some intermediate steps, we arrive at the summation below

\[
S(q, \alpha, s) = \sum_{n=-\infty}^{\infty} q^2 (n + \alpha)^2 I_{q|n+\alpha|}(s),
\]

with \( s = u r' \). We can use the following differential operator obeyed by the modified Bessel function to evaluate the above summation:

\[
S(q, \alpha, s) = \left( s^2 \frac{d^2}{ds^2} + s \frac{d}{ds} - s^2 \right) \sum_{n=-\infty}^{\infty} I_{q|n+\alpha|}(s),
\]

with \( 31 \)

\[
\sum_{n=-\infty}^{\infty} I_{q|n+\alpha|}(s) = \frac{e^s}{q} + \frac{2^{[q/2]}}{q} \sum_{k=1}^{[q/2]} \cos(2k\pi\alpha_0) e^{s\cos(2k\pi/q)} - 2 \frac{h(q, \alpha_0, 2y) e^{-s \cosh(2y)}}{\cosh(2qy) - \cos(q\pi)}.
\]

After long but straightforward calculations, we can decompose the boundary induced part of the energy-momentum tensor as

\[
\langle T_{\mu\nu}^b(x) \rangle_b = \langle T_{\mu\nu}^b(x) \rangle_b^{(M)} + \langle T_{\mu\nu}^b(x) \rangle_b^{(q,\alpha_n)}.
\]

The first term in the r.h.s of the above equation is the VEV of the energy-momentum tensor induced by a boundary in Minkowski spacetime in the absence of the magnetic flux. This contribution is obtained taking the \( k = 0 \) term of (29) with the coefficient \( 1/2 \) along with the
Eqs. (49) and (56). Then, the only nonzero contributions of the energy-momentum tensor in Minkowski spacetime in the absence of the magnetic flux is written as

\[
\langle T^\nu_\mu(x) \rangle_b = -\frac{2mD}{(2\pi)^{D/2}} \left[ 4m^2 z^2 (4\xi - 1) f_{D+2}^2 (2m|z|) + 2(1 - 2\xi) f_{D}^2 (2m|z|) \right],
\]

(62)

for the components \( \mu = \nu = 0, 1, 2, 4, \ldots, D - 1 \). Note that for a massless scalar field, the above equation reduces to

\[
\langle T^\nu_\mu(x) \rangle_b = -\frac{4\Gamma(D/2)}{(4\pi)^{D/2}|z|} (D - 1)(\xi - \xi_D),
\]

(63)

where \( \xi_D = \frac{(D-2)}{4(D-1)} \). For a conformally coupled massless scalar field, \( \xi = \xi_D \), we have that (62) vanishes.

The components of the energy-momentum tensor induced by the boundary and the magnetic flux are given by the followings VEVs:

\[
\langle T^\nu_\mu(x) \rangle_b^{(q,a_0)} = -\frac{4mD}{(2\pi)^{D/2}} \left[ \frac{\lceil q/2 \rceil}{\sum_{k=1}^{\lceil q/2 \rceil}} \cos(2k\pi a_0) G^\nu_\mu(2mr, 2mz, s_k) \right.
\]

\[
-\frac{q}{\pi} \int_0^\infty dy \frac{h(q, a_0, 2y)}{\cosh(2qy) - \cos(q\pi)} G^\nu_\mu(2mr, 2mz, c_s),
\]

(64)

where we have defined the notation

\[
G^0_0(u, v, \omega) = (4\xi - 1)(\omega^4 u^2 + v^2) f_{D+2}^3 (\gamma) + [1 - (4\xi - 1)(1 + 2\omega^2)] f_{D}^2 (\gamma),
\]

\[
G^1_0(u, v, \omega) = (4\xi - 1)\omega^2 f_{D+2}^3 (\gamma) + 2[1 - 2\xi(1 + \omega^2)] f_{D}^2 (\gamma),
\]

\[
G^2_2(u, v, \omega) = [4\xi(\omega^4 u^2 + v^2) - \gamma^2] f_{D+2}^3 (\gamma) + 2[1 - 2\xi(1 + \omega^2)] f_{D}^2 (\gamma),
\]

\[
G^3_0(u, v, \omega) = (4\xi - 1)\omega^2[\omega^2 u^2 f_{D+2}^3 (\gamma) - 2 f_{D}^2 (\gamma)],
\]

\[
G^1_1(u, v, \omega) = -\langle 4\xi - 1 \rangle \omega^2 uv f_{D+2}^3 (\gamma).
\]

(65)

We note that the energy-momentum tensor is an even function of \( a_0 \). In the above notation, the indices 0, 1, 2, 3 correspond to the coordinates \( t, r, \varphi, z \). As a consequence of boost invariance of our system along the directions \( x^j, j = 4, \ldots, D - 1 \), we have the relation \( \langle T^j_j(x) \rangle_b = \langle T^0_0(x) \rangle_b \) for the components (no summation over \( j \)) with \( j = 4, \ldots, D - 1 \). From the above expressions, we note that \( \langle T^0_0(x) \rangle_b \neq \langle T^3_3(x) \rangle_b \). This is a consequence of the lost of invariance along the string axis due the presence of the boundary.

Considering a massless scalar field, the VEVs of the energy-momentum tensor induced by the boundary and the magnetic flux are given by

\[
\langle T^\nu_\mu(x) \rangle_b^{(q,a_0)} = -\frac{2\Gamma(D/2)}{(4\pi)^{D/2}} \left[ \frac{\lceil q/2 \rceil}{\sum_{k=1}^{\lceil q/2 \rceil}} \cos(2k\pi a_0) \mathcal{F}^\nu_\mu(r, z, s_k) \right.
\]

\[
-\frac{q}{\pi} \int_0^\infty dy \frac{h(q, a_0, 2y)}{\cosh(2qy) - \cos(q\pi)} \mathcal{F}^\nu_\mu(r, z, c_s),
\]

(66)

with the notation

\[
\mathcal{F}^0_0(r, z, \omega) = \{ (4\xi - 1)(D - 2)\omega^2 - 1 \} r^2 \omega^2 [(4\xi - 1)(D - 1 - 2\omega^2) + 1] z^2,
\]

\[
\mathcal{F}^1_0(r, z, \omega) = \{ 2 - 4\xi(1 + \omega^2) \} r^2 \omega^2 + \{ 4\xi(D - 1 - \omega^2) - D + 2 \} z^2,
\]

\[
\mathcal{F}^2_2(r, z, \omega) = \{ 4\xi(D - 1)\omega^2 - 1 - D + 2 \} r^2 \omega^2 + \{ 4\xi(D - 1 - \omega^2) - D + 2 \} z^2,
\]

\[
\mathcal{F}^3_0(r, z, \omega) = (4\xi - 1)\omega^2 [(D - 2)r^2 \omega^2 - 2z^2],
\]

\[
\mathcal{F}^1_1(r, z, \omega) = -D(4\xi - 1)rz \omega^2.
\]

(67)
The Fig. 3 shows the behavior of the VEV of the energy density induced by both the boundary, conical geometry and magnetic flux as function of $z/r$, for different values of $\alpha_0$. We note that the $\langle T^{\nu}_{\mu}(x) \rangle_b^{(q,\alpha_0)}$ depend crucially on the curvature coupling. In addition, similarly to the azimuthal current density, the energy density is finite at the origin for points outside the boundary if $q|\alpha_0| > 1$. In the case where $q|\alpha_0| < 1$, the energy density diverge on the origin as $\langle T^{\nu}_{\mu}(x) \rangle_b^{(q,\alpha_0)} \propto r^{-2(1-q|\alpha_0|)}$.

Figure 3: VEV of the energy-density induced by the boundary and the magnetic flux as function of $mz$, considering different values of the parameter $\alpha_0$ for a minimally (left plot) and conformally (right plot) coupled scalar field. In addition, the plots are considered for $D = 4$, $mr = 0.75$ and $q = 1.5$.

The energy-momentum tensor induced by the boundary presents a off-diagonal component. Consequently, from the covariant conservation condition, $\nabla^\mu \langle T^\mu_\nu(x) \rangle_b = 0$, we found the following non-trivial differential equations

$$\partial_r \left( r \langle T^1_1(x) \rangle_b \right) + r \partial_z \langle T^3_1(x) \rangle_b = \langle T^2_2(x) \rangle_b$$

and

$$\partial_z \langle T^3_3(x) \rangle_b = -\frac{1}{r} \partial_r \left( r \langle T^3_1(x) \rangle_b \right).$$

It is possible to check that the previous expressions found for the energy-momentum tensor obey the above relations. In addition, the VEV of the energy-momentum tensor obey the trace relation

$$\langle T^\mu_\mu(x) \rangle_b = 2(D-1)(\xi - \xi_D)\nabla^\mu \langle |\varphi(x)|^2 \rangle_b + 2m^2 \langle |\varphi(x)|^2 \rangle_b.$$  

Note that the energy-momentum tensor is traceless for a massless conformally coupled field ($\xi = \xi_D$).

The component $\langle T^3_3(x) \rangle_b$ at $z = 0$ determines the normal vacuum force on the boundary. This contribution is finite outside the string axis and is written as

$$\langle T^3_3(x) \rangle_b^{(z=0)} = \frac{4m^D(1-4\xi)}{(2\pi)^{D/2}} \left[ \sum_{k=1}^{[q/2]} s_k^2 \cos(2k\pi\alpha_0) G_3(2mr\alpha_0) \right]$$

where

$$G_3(v) = v^2 f_{D/2}(v) - 2f_D(v).$$

The effective pressure on the boundary, $P = \langle T^3_3(x) \rangle_b^{(z=0)}$, and presents a dependence on the curvature coupling parameter in the form of the factor $(1-4\xi)$.  

In the present paper, we have investigated the vacuum bosonic current and polarization associated with a quantum charged scalar field in a higher-dimensional cosmic string spacetime considering the presence of a flat boundary orthogonal to the string. We have also considered the presence of a magnetic flux running along the string axis and that the quantum field obeys the Dirichlet or Neumann boundary condition on the boundary. As the first step of our analysis, we evaluate the Wightman function and we found a closed form of it for general values of the parameter that codifies the presence of the conical defect, $q$. The Wightman function could be decomposed into two contributions, one in the spacetime time of a cosmic string in the absence of the planar boundary and another one induced by the boundary, Eqs. (28) and (29), respectively. As the vacuum induced observable in the absence of the boundary have been investigated in the literature by several authors, here we were concerned only in the analysis of the planar boundary effects.

The induced bosonic current was the first physical quantity that we have developed. The only nonzero component of the bosonic current is the azimuthal one and the contribution induced by the boundary is given by the Eq. (40) which can be positive or negative depending on the boundary condition obeyed by the scalar quantum field. In addition, the azimuthal current induced by the boundary is an odd function of $\alpha_0$ and is finite on the string for points outside the boundary if $q|\alpha_0| > 1$. For the case where $q|\alpha_0| < 1$ the azimuthal current is divergent on the string and the leading divergence term is given by (43). We also have calculated the boundary induced part of the azimuthal current considering large distances from the boundary and in the limit of a massless scalar field, expressed by Eqs. (44) and (46), respectively. In the Fig. 1 we have exhibited the behavior of the azimuthal current induced by the boundary as function of $mr$ and $z/r$, considering $D = 4$. In the addition, we have shown that depending on the boundary condition adopted, the total azimuthal current density can be canceled or intensified on the boundary.

Figure 4: VEV of the vacuum force on the boundary as function of $mr$ for different values of $\alpha_0$ and a minimally coupled scalar field considering $q = 1.5$ (left plot) and $q = 2.5$ (right plot). In both plots we also have $D = 4$. In Fig. 4 we show the behavior of the normal vacuum force (71) as function of $mr$ considering different values of the parameter $\alpha_0$ for $q = 1.5$ (left plot) and $q = 2.5$ (right plot). In the absence of the magnetic flux, $\alpha_0 = 0$, we have that the effective pressure is always positive. However, when we take into account the presence of a magnetic flux along the string axis, the effective pressure can assume positive or negative values. Although not exhibited in the graph, the pressure in the case of absence of magnetic flux is also positive for conformally coupled field.

5 Conclusions

In the present paper, we have investigated the vacuum bosonic current and polarization associated with a quantum charged scalar field in a higher-dimensional cosmic string spacetime considering the presence of a flat boundary orthogonal to the string. We have also considered the presence of a magnetic flux running along the string axis and that the quantum field obeys the Dirichlet or Neumann boundary condition on the boundary. As the first step of our analysis, we evaluate the Wightman function and we found a closed form of it for general values of the parameter that codifies the presence of the conical defect, $q$. The Wightman function could be decomposed into two contributions, one in the spacetime time of a cosmic string in the absence of the planar boundary and another one induced by the boundary, Eqs. (28) and (29), respectively. As the vacuum induced observable in the absence of the boundary have been investigated in the literature by several authors, here we were concerned only in the analysis of the planar boundary effects.

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Our next step was to develop the calculation of the vacuum polarization considering the VEV of the field squared and the energy momentum tensor. The field squared induced by the boundary could be decomposed into two contributions: one contribution induced by a planar boundary in Minkowski spacetime, Eq. (49), and another one induced by the conical defect and the magnetic flux, Eq. (50). The first is independent of the magnetic flux and the radial coordinate and the latter is an even function of $\alpha_0$. For the boundary induced part of the field squared we have considered some special cases. For the regions near the boundary the leading contribution comes from the Eq. (49) and is given by (52). In the Eq. (53) we have the boundary part of the azimuthal current induced by the conical defect and the magnetic flux considering a massless scalar field. Also, this part of the azimuthal current were considered taking into account large distances from the boundary, Eq. (54), which presents an exponential decay with $z/r$. The behavior of the field squared induced by the magnetic flux and the conical defect is shown in the Fig. 2 as function of $z/r$ and $mr$ considering $D = 4$.

We also developed the analysis of the another quantity that characterizes the quantum vacuum state: the VEV of the energy-momentum tensor. As the previous physical quantities, we developed only the analysis of the boundary effects of the energy-momentum tensor, which could be decomposed into a contribution due a flat boundary in Minkowsk spacetime, Eq. (62), and another one induced by the magnetic flux and the conical defect, Eq. (64). For the first, the only nonzero contributions are the components $\mu = \nu = 0, 1, 2, 4, ..., D - 1$. This part of the energy-momentum tensor also was calculated considering a massless scalar field, Eq. (63). The contribution of the energy-momentum tensor induced by the magnetic flux and the conical defect is an even function of $\alpha_0$. We have found the relation $\langle T_{\mu\nu}(x) \rangle_b = \langle T_{00}(x) \rangle_b$, for the components along the extra dimensions. This is a directly consequence of the boost invariance along these directions. However, this invariance is lost along the $z$-direction due the presence of the boundary, consequently, $\langle T_{00}(x) \rangle_b \neq \langle T_{33}(x) \rangle_b$. We also have considered the VEV of the energy-momentum tensor induced by the boundary for the case of a massless scalar field, which is given by the Eq. (66). In the Fig. 3 we have plotted the behavior of boundary part of the energy density induced by the magnetic flux and conical defect as function of $z/r$ considering $D = 4$, where we note that the energy density depends crucially of the curvature coupling. In addition, the energy density in finite on the string for points outside the boundary only if $q|\alpha_0| > 1$ and diverges as $\langle T_{\mu\nu}(x) \rangle_b \propto r^{-2(1-q|\alpha_0|)}$ at the origin if $q|\alpha_0| < 1$.

To finish our analysis, we have evaluate the normal vacuum force on the boundary, which is determined by the component $\langle T_{33}(x) \rangle_b$ at $z = 0$. This contribution is given by the Eq. (71) that presents a dependence on the curvature coupling through the factor $(1 - 4\xi)$. And interesting characteristic of the normal vacuum force is that in the absence of the magnetic flux, this force has always positive values for both minimally and conformally scalar fields. However, when the magnetic flux is present, the normal vacuum force can assume positive or negative values. The profile of the normal vacuum force on the boundary is shown in the Fig. 4 as function of $mr$ for $D = 4$.

**Acknowledgments**

E.R.B.M is partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico - Brasil (CNPq) under grant No 301.783/2019-3. E.A.F.B is grateful by the hospitality of the State University of the Tocantina Region of Maranhão - Brazil.

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