Fusion rules and singular vectors of the osp(1|2) current algebra

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ABSTRACT

The fusion of Verma modules of the osp(1|2) current algebra is studied. In the framework of an isotopic formalism, the singular vector decoupling conditions are analyzed. The fusion rules corresponding to the admissible representations of the osp(1|2) algebra are determined. A relation between the characters of these last representations and those corresponding to the minimal superconformal models is found. A series of equations that relate the descendants of the highest weight vectors resulting from a fusion of Verma modules are obtained. Solving these equations the singular vectors of the theory can be determined.

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1. Introduction

The models endowed with a current algebra symmetry based on an affine Lie algebra form a distinguished class in Conformal Field Theory (CFT). Indeed, in recent years a rich variety of results has been obtained for this class of models [1]. Most of these results correspond to the so-called integrable representations, which appear when the level $k$ of the affine symmetry is a positive integer. It turns out, however, that a larger class of representations and levels is needed in order to relate the model based on the current algebra to other CFT’s that do not enjoy this symmetry. Of particular relevance are the admissible representations [2] which, for example, in the case of the $sl(2)$ algebra, are those needed to construct the minimal Virasoro models by means of the hamiltonian reduction procedure [3]. These admissible representations, which in the $sl(2)$ case appear when the level and the isospin are rational numbers, are also necessary in the light-cone analysis of the two dimensional quantum gravity coupled to minimal matter [4] and in the study of the non-critical string theories by means of the topological coset models [5,6].

It is therefore interesting to extend the concepts and techniques of CFT to include general types of levels and representations. In the $sl(2)$ case, this problem has been addressed following different approaches [7-14]. The key point in most of these analysis is the introduction of an isotopic variable $x$ to represent the $sl(2)$ symmetry [15]. The primary fields of the theory (and thus its conformal blocks) depend both on the spacetime coordinate $z$ and on the “internal” coordinate $x$. Within this isotopic formalism it has been possible [12] to give a precise definition of the fusion of primary fields. This has allowed to develop an efficient algorithm to compute the singular vectors of the $sl(2)$ affine algebra. Moreover, in refs. [11, 16] it has been shown that the primary fields corresponding to the admissible representations close a well-defined fusion algebra. By using a free field realization of the current algebra (supplemented with the fractional calculus technique [17]), one can represent [13] the conformal blocks for the correlators of primary fields.
which carry quantum numbers of \(sl(2)\) admissible representations (\(i.e.\) with fractional isospins). When the isotopic and space-time coordinates are identified, the quantum hamiltonian reduction is implemented and one passes from the blocks of the \(sl(2)\) theory to those of the minimal Virasoro models [10].

In this paper we shall study the \(osp(1|2)\) affine Lie superalgebra [18, 19]. The rôle of this superalgebra, which is a graded version of \(sl(2)\) [20], in connection with the \(N=1\) superconformal symmetry is well-known. In fact, by means of the hamiltonian reduction method, the \(osp(1|2)\) CFT can be related to the minimal \(N=1\) superconformal models [21]. The \(osp(1|2)\) algebra also appears in the light-cone quantization of two-dimensional supergravity [22] and the corresponding topological coset, \(i.e.\) the \(osp(1|2)/osp(1|2)\) model, can be used to describe the non-critical Ramond-Neveu-Schwarz superstrings [23, 24].

For integer level and integer or half-integer isospins, the \(osp(1|2)\) conformal blocks have been studied in ref. [25]. For more general representations and levels one has to work with an isotopic representation which, in addition to a bosonic internal coordinate that also appears in the \(sl(2)\) case, requires the introduction of an internal Grassmann odd coordinate \(\theta\). Following the approach of ref. [12], we shall be able to characterize, in this isotopic formalism, the fusion of \(osp(1|2)\) primary fields. This fusion is greatly constrained when there are singular vectors in any of the Verma modules associated to the primary fields participating in the fusion. Actually, we shall find a set of polynomial relations for the isospins of these primary fields that encode the decoupling of the singular vectors in the highest weight module in which they are originated. These relations are enough to determine the fusion rules satisfied by the admissible representations. In analogy with what happens with the \(sl(2)\) algebra [11, 16], we shall find two types of fusion rules which cannot be simultaneously satisfied. In these fusion rules the operator algebra closes when the primary fields are restricted to belong to a conformal grid, which constitutes a new similarity between this kind of representations and those corresponding to the minimal \(N=1\) superconformal models.
The descendants of the highest weight vectors resulting from a fusion of primary fields satisfy the so-called descent equations [12]. These equations can be reformulated, by means of the Sugawara construction of the energy-momentum tensor, in such a way that they can be put in a triangular form, which provides a method to compute the descendant vectors. As was the case for the $sl(2)$ algebra [12], the solution of the descent equations will allow us to develop an algorithm for the computation of the singular vectors of the algebra.

The organization of this paper is the following. In section 2 we give the basic definitions and set up the formalism for the fusion of osp(1|2) Verma modules. The decoupling conditions induced in the fusion by the presence of a singular vector are worked out in section 3. In section 4 we restrict ourselves to the case of admissible representations. The fusion rules corresponding to these representations are found as a consequence of the decoupling conditions obtained in section 3. The descent equations are derived in section 5. The solution of these equations and their truncation are also analyzed in this section. The Sugawara recursion relations are obtained in section 6, where it is also shown how to use them to calculate singular vectors of the algebra. In section 7 we rederive the recursion relations of section 6 from the Knizhnik-Zamolodchikov equation. The results obtained in the paper are summarized in section 8, where some possible lines of future work are mentioned.

The paper ends with three appendices. In appendix A, the calculation of the characters of the osp(1|2) affine algebra is reviewed and a relation between the characters of the osp(1|2) admissible representations and those of the minimal superconformal models is obtained. In appendix B, the two simplest singular vectors of the algebra are computed by using the fusion formalism developed in the main text. Finally, in appendix C, the equivalence of two vectors, needed in the derivation of section 7, is proved.
2. Fusion of Verma modules

The osp(1|2) current algebra (which we shall denote simply by $\mathcal{A}$) is generated by the currents $J_a^n$ ($a = 0, \pm$, $n \in \mathbb{Z}$) and $j_\alpha^n$ ($\alpha = \pm$, $n \in \mathbb{Z}$), together with a central element $k$. The $J_a^n$ ($j_\alpha^n$) currents are bosonic (fermionic), i.e. Grassmann even (odd). The generator $k$ commutes with all the other elements of $\mathcal{A}$ and therefore we shall regard it as a c-number (the level of $\mathcal{A}$). Notice that the modes $n$ of the fermionic currents $j_\alpha^n$ run over the integers, which, properly speaking, means that we are considering the Ramond sector of the osp(1|2) affine superalgebra. The non-vanishing (anti)commutators of $\mathcal{A}$ are:

\[
\begin{align*}
[#_n^0, J_m^0] &= \pm J^\pm_{n+m} & [#_n^0, J_m^0] &= \frac{k}{2} n \delta_{n+m} \\
[#_n^+, J_m^-] &= kn\delta_{n+m} + 2J^0_{n+m} \\
[#_n^0, J_m^\pm] &= \pm \frac{1}{2} J_m^{\pm+n} & [#_n^\pm, J_m^\pm] &= 0 \\
[#_n^\pm, J_m^\mp] &= -j_{n+m}^\pm & \{ j_n^\pm, j_m^\pm \} &= \pm 2J^\pm_{n+m} \\
\{ j_n^+, j_m^- \} &= 2kn\delta_{n+m} + 2J^0_{n+m}.
\end{align*}
\] (2.1)

It is interesting to point out that this algebra is doubly graded. Let us denote by $\overline{d}$ and $\underline{d}$ the corresponding gradations. They are defined by:

\[
\begin{align*}
\overline{d}(J_a^n) &= a & \overline{d}(j_\alpha^n) &= \frac{\alpha}{2} & \overline{d}(k) &= 0 \\
\underline{d}(J_a^n) &= n & \underline{d}(j_\alpha^n) &= n & \underline{d}(k) &= 0. 
\end{align*}
\] (2.2)

From the commutation relations (2.1) it follows that $\overline{d}$ can be represented as $ad(J^0_0)$. For this reason we shall refer to $\overline{d}$ as the $J^0_0$-gradation. The so-called principal gradation [26] $d$ is obtained by combining $\overline{d}$ and $\underline{d}$ in the form:

\[
d = 2\overline{d} + \underline{d}.
\] (2.3)

Using eqs. (2.2) and (2.3) one can immediately obtain the $d$-grading of the different
generators of $\mathcal{A}$:

$$d(J^n_a) = 2n + a \quad d(j^n_\alpha) = 2n + \frac{\alpha}{2} \quad d(k) = 0 . \quad (2.4)$$

With respect to $d$ the algebra $\mathcal{A}$ splits as:

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_0 \oplus \mathcal{A}_+ , \quad (2.5)$$

where $\mathcal{A}_-$, $\mathcal{A}_0$ and $\mathcal{A}_+$ are the subspaces of $\mathcal{A}$ spanned by the elements that have, respectively, $d < 0$, $d = 0$ and $d > 0$. These elements are easy to identify from eq. (2.4) and so, for example, $\mathcal{A}_0$ is generated by $J_0^0$ and $k$, whereas $\mathcal{A}_+$ is the subspace spanned by $J_n^- (n \geq 1)$, $J_n^0 (n \geq 1)$, $J_n^+ (n \geq 0)$, $j_n^- (n \geq 1)$ and $j_n^+ (n \geq 0)$.

From the mode operators $J_n^a$ and $j_n^\alpha$ one can define the coordinate-dependent fields $J^a(z)$ and $j^\alpha(z)$ as:

$$J^a(z) = \sum_{n=-\infty}^{+\infty} J_n^a z^{-n-1} \quad j^\alpha(z) = \sum_{n=-\infty}^{+\infty} j_n^\alpha z^{-n-1} . \quad (2.6)$$

As it is well-known, one can construct a Conformal Field Theory associated to the currents $J^a(z)$ and $j^\alpha(z)$. The key ingredient in this construction is the Sugawara prescription for the energy-momentum tensor $T(z)$ of the theory as a quadratic expression in the currents. In our case, taking into account the form of the quadratic Casimir invariant of osp(1|2), one has:

$$T(z) = \frac{1}{2k+3} : \left[ 2(J^0(z))^2 + J^+(z) J^-(z) + J^-(z) J^+(z) - \frac{1}{2} j^+(z) j^-(z) + \frac{1}{2} j^-(z) j^+(z) \right] : . \quad (2.7)$$

In eq. (2.7) the double dot $: \:$ denotes normal-ordering. The energy-momentum tensor defined in eq. (2.7) is such that the currents are primary dimension-one
operators with respect to $T(z)$. Its mode expansion is:

$$T(z) = \sum_{n=-\infty}^{+\infty} L_n z^{-n-2},$$  \hspace{1cm} (2.8)

where the $L_n$'s are the generators of the Virasoro algebra. The commutators of these operators with the currents are:

$$[L_n, J^a_m] = -m J^a_{n+m} \quad \quad [L_n, j^a_m] = -m j^a_{n+m},$$  \hspace{1cm} (2.9)

from which it follows that $d$ can be represented as $-ad(L_0)$, and therefore we shall call $d$ the $L_0$-gradation. The algebra satisfied by the $L_n$'s is the Virasoro algebra, i.e.:

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} (m^3 - m) \delta_{n+m,0},$$  \hspace{1cm} (2.10)

where the central charge $c$ is related to the level $k$ by means of the expression:

$$c = \frac{2k}{2k+3}.$$  \hspace{1cm} (2.11)

Instead of the level $k$ we shall frequently use in what follows the quantity $t$, which in terms of the former is given by:

$$t = 2k + 3.$$  \hspace{1cm} (2.12)

The Verma modules associated to $\mathcal{A}$ are constructed by acting with elements of the universal enveloping algebra of $\mathcal{A}_-$ (denoted by $U(\mathcal{A}_-)$) on a highest weight vector $|j, t\rangle$. The latter is annihilated by the elements of $\mathcal{A}_+$, i.e.:

$$J^a_n |j, t\rangle = j^a_n |j, t\rangle = 0, \quad \forall \ (J^a_n, j^a_n) \in \mathcal{A}_+.$$  \hspace{1cm} (2.13)

On the contrary, $J^0_0$ and $L_0$ act diagonally on $|j, t\rangle$:

$$J^0_0 |j, t\rangle = j |j, t\rangle \quad \quad L_0 |j, t\rangle = h_j |j, t\rangle .$$  \hspace{1cm} (2.14)

From the Sugawara expression for $L_0$ (see eqs. (2.7) and (2.8)), one can easily get
the $L_0$ eigenvalue corresponding to $|j,t>$, namely:

$$h_j = \frac{j(2j + 1)}{2k + 3} = \frac{j(2j + 1)}{t}.$$  \hspace{1cm} (2.15)

In order to completely characterize the highest weight vector $|j,t>$ we must specify its Grassmann parity, which we shall denote by $p(j) : |j,t>$ is bosonic (fermionic) if $p(j) = 0$ ($p(j) = 1$). The Verma module whose highest weight vector is $|j,t>$ will be denoted by $V^{(j,t)}$. Any element in $V^{(j,t)}$ is of the form $u_- |j,t>$, where $u_- \in \mathcal{A}_-$. The gradations $d$ and $\overline{d}$ of $\mathcal{A}$ induce a doubly graded structure in $V^{(j,t)}$. Actually, if we denote by $n$ and $m$ the eigenvalues of $-d$ and $-\overline{d}$ respectively, one can decompose $V^{(j,t)}$ as:

$$V^{(j,t)} = \bigoplus_{(n,m)} V^{(j,t)}_{n,m}. \hspace{1cm} (2.16)$$

Notice that, according to the Poincaré-Birkhoff-Witt theorem, $U(\mathcal{A}_-)$ is generated by monomials and thus we can consider a basis of $V^{(j,t)}$ constituted by vectors of the form:

$$|\{m_i^a\}; j> = \prod_{i=0}^{+\infty} (j^-_i)^{2m^-_i} \prod_{i=1}^{+\infty} (J^0_{-i})^{m^0_i} \prod_{i=1}^{+\infty} (j^+_i)^{2m^+_i} |j,t>.$$  \hspace{1cm} (2.17)

In eq. (2.17), the numbers $m_i^\pm$ are integers or half-integers whereas the $m_i^a$’s are always integers ($m_i^a \geq 0$). It is important to point out that the vectors defined in eq. (2.17) are homogeneous, i.e. they have well-defined $J^0_0$ and $L_0$ eigenvalues. Actually, the $L_0$ eigenvalue for the vector (2.17) is $h_j + n$, where $n$ is given by:

$$n = 2 \sum_{i=0}^{+\infty} im^-_i + \sum_{i=1}^{+\infty} im^0_i + 2 \sum_{i=1}^{+\infty} im^+_i,$$ \hspace{1cm} (2.18)

while, if we define $m^\pm$ as:

$$m^+ = \sum_{i=1}^{+\infty} m^+_i, \quad m^- = \sum_{i=0}^{+\infty} m^-_i.$$ \hspace{1cm} (2.19)
the value of $m$ for the vector $|\{m_i^a\};j\rangle$ is simply:

$$m = m^- - m^+.$$  

(2.20)

Notice that the vectors (2.17) have also a well-defined Grassmann parity. Actually, by inspecting eq. (2.17) one easily concludes that if $m$ is integer (half-integer), $|\{m_i^a\};j\rangle$ and the highest weight vector $|j,t\rangle$ have the same (opposite) statistics.

The algebra (2.1) is endowed with a linear anti-automorphism $\sigma$ defined as:

$$\sigma(J_n^a) = J_{-n}^a, \quad \sigma(j_n^\alpha) = j_{-n}^\alpha, \quad \sigma(k) = k.$$  

(2.21)

Using $\sigma$ one can define an inner product for the elements of $V^{(j,t)}$. In fact, given two arbitrary elements $u$ and $v$ of $V^{(j,t)}$, they can be represented in the form:

$$u = u_- |j,t\rangle \quad v = v_- |j,t\rangle \quad u_-, v_- \in U(\mathcal{A}_-).$$  

(2.22)

As one can naturally extend $\sigma$ to $U(\mathcal{A})$, it makes sense to consider $\sigma(u_-)$. The inner product $\langle u|v \rangle$ is obtained by acting with $\sigma(u_-)v_-$ on the highest weight vector $|j,t\rangle$ and taking the projection of the result on the subspace $V^{(j,t)}_{0,0}$. If we define $\langle j,t|j,t\rangle = 1$, the above definition reduces to:

$$\langle u|v \rangle \equiv \langle j,t|\sigma(u_-)v_-|j,t \rangle.$$  

(2.23)

Eq. (2.23) defines a bilinear form on $V^{(j,t)}$, which is usually called the contravariant (or Shapovalov) form. Let us now introduce the states:

$$e^{xj_0^- + \theta j_\theta^-} |j,t\rangle,$$  

(2.24)

where $x$ is a complex number and $\theta$ a Grassmann variable ($\theta^2 = 0$). On the states (2.24), the zero-mode currents $J_0^a$ and $j_0^\alpha$, which generate the finite $osp(1|2)$
superalgebra, act as certain differential operators. Actually one can prove that:

\[
\begin{align*}
J_0^a e^{xJ_0^a + \theta j_0^a} | j, t > &= D_j^a e^{xJ_0^a + \theta j_0^a} | j, t > \quad \text{(2.25)} \\
\bar{J}_0^\alpha e^{x\bar{J}_0^\alpha - \theta j_0^\alpha} | j, t > &= d_j^\alpha e^{x\bar{J}_0^\alpha - \theta j_0^\alpha} | j, t > ,
\end{align*}
\]

where \( D_j^a \) and \( d_j^\alpha \) are given by:

\[
\begin{align*}
D_j^0 &= -x \partial_x - \frac{1}{2} \theta \partial_\theta + j \\
D_j^+ &= -x^2 \partial_x + 2 j x - \theta x \partial_\theta \\
D_j^- &= \partial_x \\
d_j^+ &= x \partial_\theta + \theta x \partial_x - 2 j \theta \\
d_j^- &= \partial_\theta + \theta \partial_x. 
\end{align*}
\]

Eq. (2.25) is a consequence of the highest weight conditions (2.13) and of the following conjugation formulas for the currents:

\[
\begin{align*}
e^{xJ_0^- + \theta j_0^-} J_n^0 e^{-xJ_0^- - \theta j_0^-} &= J_n^0 + x J_n^- + \frac{\theta}{2} j_n^- \\
e^{xJ_0^- + \theta j_0^-} J_n^+ e^{-xJ_0^- - \theta j_0^-} &= J_n^+ - 2 x J_n^- - x^2 J_n^- + \theta (j_n^+ - x j_n^-) \\
e^{xJ_0^- + \theta j_0^-} J_n^- e^{-xJ_0^- - \theta j_0^-} &= J_n^- \\
e^{xJ_0^- + \theta j_0^-} j_n^+ e^{-xJ_0^- - \theta j_0^-} &= j_n^+ - x j_n^- + 2 \theta (j_n^0 + x J_n^-) \\
e^{xJ_0^- + \theta j_0^-} j_n^- e^{-xJ_0^- - \theta j_0^-} &= j_n^- - 2 \theta J_n^- .
\end{align*}
\]

One can easily prove eq. (2.27) by using the defining relations (2.1) of the algebra and the Baker-Campbell-Hausdorff formula:

\[
e^A B e^{-A} = \sum_{p=0}^{\infty} \frac{1}{p!} [A, [A, \cdots, [A, B] \cdots] .
\]

The primary fields are fundamental objects in any Conformal Field Theory. The holomorphic part of one of such fields depends on the coordinate \( z \). If we
have some internal symmetry $G$ in our theory, these primary fields are associated to representations of $G$ and, in general, will have several components, which keep track of the representation space of $G$. In our case, the internal symmetry is the osp(1|2) zero-mode superalgebra which, as we have seen in eq. (2.25), is represented by differential operators in the isotopic variables $x$ and $\theta$. It is thus natural to think that the primary fields should depend also on these variables. We shall denote by $\phi_j(z, x, \theta)$ the primary field corresponding to the isospin $j$ representation.

The basic property that characterizes the primary fields in Conformal Field Theory is the fact that their insertion at the origin of coordinates $z = 0$ creates a highest weight state from the vacuum of the theory. As $L_{-1}$ is the translation operator in the variable $z$, the change of the insertion point is equivalent to the action of the operator $e^{zL_{-1}}$ on the $z = 0$ state. Moreover, the $x$ and $\theta$ dependence of the states can be naturally introduced as in (2.24). All these considerations lead us to define the action of $\phi_j(z, x, \theta)$ on the vacuum $|\Omega>$ of the theory as:

$$\phi_j(z, x, \theta)|\Omega> \equiv e^{zL_{-1} + xJ_0^- + \theta j_0^-} |j, t> .$$

We shall assume that the vacuum state $|\Omega>$ is bosonic, which means that the Grassmann parity of $\phi_j(z, x, \theta)$ is equal to $p(j)$.

The commutators of the currents with the fields $\phi_j(z, x, \theta)$ is given by the standard expression:

$$[J_\alpha^n , \phi_j(z, x, \theta)] = z^n D^n_\alpha \phi_j(z, x, \theta)$$
$$[J^\alpha_n , \phi_j(z, x, \theta)] = z^n d^n_\alpha \phi_j(z, x, \theta),$$

where we have taken into account the implementation of the osp(1|2) finite superalgebra by the differential operators (2.26). In order to write the second equation in (2.30), we have supposed that $\phi_j(z, x, \theta)$ has bosonic statistics. If this were not the case, one should substitute the commutator with $j^n_\alpha$ by the corresponding anticommutator. Moreover, the action of the Virasoro modes $L_n$ on the primary
fields $\phi_j(z, x, \theta)$ is determined by their conformal weight $h_j$:

\[
[ L_n, \phi_j(z, x, \theta) ] = \left( (n + 1) h_j z^n + z^{n+1} \partial_z \right) \phi_j(z, x, \theta).
\]  

(2.31)

Let us now consider the state obtained by acting with two primary fields, one of which is located at the origin of coordinates, on the vacuum $| \Omega >$. This state is:

\[
\phi_{j_2}(z, x, \theta) \phi_{j_1}(0, 0, 0) | \Omega > \equiv \phi_{j_2}(z, x, \theta) | j_1, t >.
\]  

(2.32)

The primary fields of the conformal field theory close operator product algebras when they are multiplied. For this reason, one can decompose the state (2.32) as:

\[
\phi_{j_2}(z, x, \theta) \phi_{j_1}(0, 0, 0) | \Omega > = \sum_{j_3} | j_3, t, z, x, \theta >.
\]  

(2.33)

In eq. (2.33), $| j_3, t, z, x, \theta >$ is a coordinate-dependent state of $V^{(j_3,t)}$. When a state of isospin $j_3$ is included in the right-hand side of eq. (2.33) we will say that the Verma module $V^{(j_3,t)}$ appears in the fusion of $V^{(j_1,t)}$ and $V^{(j_2,t)}$. The doubly graded decomposition (2.16) implies the following expansion of $| j_3, t, z, x, \theta >$:

\[
| j_3, t, z, x, \theta > = \sum_{n,m} | j_3, t, z, x, \theta, n, m >,
\]  

(2.34)

where the vectors appearing in the right-hand side of (2.34) have well-defined $L_0$ and $J_0^0$ eigenvalues. It is not difficult to obtain the coordinate dependence of the states resulting from the fusion (2.33). We shall verify in a moment that this dependence is fixed by the covariance constraints satisfied by the highest weight states. Indeed, the conditions (2.14) for the vector $| j_1, t >$ read:

\[
(L_0 - h_1) | j_1, t > = 0 \\
(J_0^0 - j_1) | j_1, t > = 0,
\]  

(2.35)
where $h_i \equiv h_{j_i}$. Multiplying by $\phi_{j_2}(z, x, \theta)$ the two equations in (2.35), one gets:

$$\phi_{j_2}(z, x, \theta) (L_0 - h_1) |j_1, t > = 0$$
$$\phi_{j_2}(z, x, \theta) (J_0^0 - j_1) |j_1, t > = 0.$$  (2.36)

Moreover, the commutation relations (2.30) and (2.31) imply that:

$$L_0 \phi_{j_2}(z, x, \theta) = \phi_{j_2}(z, x, \theta) L_0 + \left( z \partial_z + h_2 \right) \phi_{j_2}(z, x, \theta)$$
$$J_0^0 \phi_{j_2}(z, x, \theta) = \phi_{j_2}(z, x, \theta) J_0^0 + D_{j_2}^0 \phi_{j_2}(z, x, \theta).$$  (2.37)

Using eq. (2.37), the constraints (2.36) are converted into:

$$\left( L_0 - h_1 - h_2 - z \partial_z \right) \phi_{j_2}(z, x, \theta) |j_1, t > = 0$$
$$\left( J_0^0 - j_1 - D_{j_2}^0 \right) \phi_{j_2}(z, x, \theta) |j_1, t > = 0.$$  (2.38)

Substituting in (2.38) the expansions (2.33) and (2.34), one can obtain constraints projected on a given subspace $V^{(j_3, t)}_{(n,m)}$. As $L_0$ and $J_0^0$ act diagonally on the vectors of $V^{(j_3, t)}_{(n,m)}$ with eigenvalues $h_3 + n$ and $j_3 - m$ respectively, the constraints on a given $(j_3, n, m)$ sector read:

$$\left( n + h_3 - h_1 - h_2 - z \partial_z \right) |j_3, t, z, x, \theta, n, m > = 0$$
$$\left( j_3 - j_1 - j_2 - m + x \partial_x + \frac{1}{2} \theta \partial_\theta \right) |j_3, t, z, x, \theta, n, m > = 0.$$  (2.39)

Let us now assume the following general power dependence for $|j_3, t, z, x, \theta, n, m >$:

$$|j_3, t, z, x, \theta, n, m > = \theta^{\Delta_m} x^A z^B |n, m >_{j_3},$$  (2.40)

where $|n, m >_{j_3}$ is an element of $V^{(j_3, t)}_{(n,m)}$ and, due to the Grassmann nature of the variable $\theta$, $\Delta_m$ can only take the values 0 and 1. The constraints (2.39) allow to determine the exponents in (2.40). In fact, the substitution of the ansatz (2.40)
in the $L_0$-condition (i.e. the first equation in (2.39)), yields the value of the $z$ exponent $A$:

$$A = n + h_3 - h_1 - h_2 .$$

(2.41)

In the same way, the second equation in (2.39) implies that:

$$B = m + j_1 + j_2 - j_3 - \frac{\Delta m}{2} .$$

(2.42)

It remains to determine $\Delta m$. This can be done by examining the Grassmann parity of the fusion relations. Let us, first of all, introduce the function $\epsilon(m)$, defined for an integer or half-integer variable $m$ as follows:

$$\epsilon(m) \equiv 2(m - [m]) = \begin{cases} 
0 & \text{if } m \in \mathbb{Z} \\
1 & \text{if } m \in \mathbb{Z} + \frac{1}{2} .
\end{cases}$$

(2.43)

In eq. (2.43), $[m]$ is the integer part of $m$ for any $m \in \mathbb{Z}/2$, i.e. $[m] = m$ when $m \in \mathbb{Z}$ and $[m] = m - \frac{1}{2}$ when $m \in \mathbb{Z} + \frac{1}{2}$. The function $\epsilon(m)$ will be frequently used through this paper. Let us list some of its (obvious) properties:

$$\epsilon(m \pm 1) = \epsilon(m) \quad \epsilon(m \pm \frac{1}{2}) = 1 - \epsilon(m)$$

$$\epsilon(m)^2 = \epsilon(m) \quad \epsilon(m) \epsilon(m \pm \frac{1}{2}) = 0 .$$

(2.44)

Coming back to the evaluation of $\Delta m$, let us first notice that the Grassmann parity of the vector $|n, m >_{j_3}$ is $\epsilon(m) + p(j_3) \mod (2)$. Moreover, it is also evident from our previous equations that $\phi_{j_2}(z, x, \theta) | j_1, t >$ and $\theta^{\Delta m} | n, m >_{j_3}$ must have the same Grassmann parity. Therefore one must have:

$$p(j_1) + p(j_2) = \Delta m + \epsilon(m) + p(j_3) \mod (2) .$$

(2.45)

From eq. (2.45) one can easily find a closed expression for $\Delta m$ as a function of $m$ and of the parities $p(j_i)$ of the highest weight vectors participating in the fusion.
One gets:

\[ \Delta_m = \epsilon (m + \frac{p(j_1) + p(j_2) - p(j_3)}{2}) . \]  

(2.46)

Therefore, the splitting (2.34) can be written as:

\[ |j_3, t, z, x, \theta > = \sum_{n, m} \theta^{\Delta_m} z^{h_3 - h_1 - h_2 + n} x^{j_1 + j_2 - j_3 + m - \frac{\Delta_m}{2}} | n, m >_{j_3} . \]  

(2.47)

Eqs. (2.46) and (2.47) will be of great importance in our analysis of the fusion of Verma modules. It is interesting in what follows to obtain the range of the numbers \( n \) and \( m \) in (2.47). First of all, it is clear from its definition that \( n \) is a non-negative integer. Moreover, after analyzing the form of the basis elements (2.17), one easily concludes that \( m \) is an integer or half-integer number which is always greater or equal to \(-n\). Notice finally that the expansion (2.47) is the counterpart in our formalism of the operator product expansion of the products of primary fields in CFT. It is worth to mention that, in general, the right-hand side of eq. (2.47) is not only singular in the variable \( z \), but also in the isotopic coordinate \( x \).

### 3. Decoupling of singular vectors

For generic values of the isospins \( j_i \), the fusion of the Verma modules \( V^{(j_1, t)} \) and \( V^{(j_2, t)} \) in \( V^{(j_3, t)} \) is not restricted by any constraint. However, if one of the modules \( V^{(j, t)} \) is reducible, the situation changes completely. Indeed, if \( V^{(j, t)} \) is reducible, it contains an unique maximal submodule and we must formulate our Conformal Field Theory in the module obtained by taking the quotient of \( V^{(j, t)} \) by its maximal proper submodule. In this quotient module, the vectors belonging to the submodule vanish and, as we shall verify below, this condition implies that the fusion of the corresponding primary fields is not possible unless their isospins satisfy non-trivial polynomial relations.

A Verma module \( V^{(j, t)} \) is irreducible if and only if it contains no singular vectors. These are vectors of \( V^{(j, t)} \) which are annihilated by \( A_+ \) and have vanishing
projection on $V^{(j,t)}_{0,0}$. It is easy to prove that if $V^{(j,t)}$ has a singular vector, the contravariant form on $V^{(j,t)}$ is degenerate. Indeed, if $u$ is a singular vector and $v = v_- | j, t >$ is an arbitrary vector of $V^{(j,t)}$ ($v_- \in U(A_-)$), their inner product $< v | u > = < j, t | \sigma(v_-)u >$ vanishes since $\sigma(v_-) \in U(A_+)$. Therefore one can use the determinant of the contravariant form in the different subspaces $V^{(j,t)}_{n,m}$ to locate the singular vectors. For bosonic affine algebras this problem has been addressed in ref. [27]. The case of the osp(1|2) affine superalgebra has been studied in refs. [2, 28]. Let us briefly review the osp(1|2) results. For a given value of $t$, the singular vectors appear in Verma modules with highest weight vectors whose isospins belong to a discrete set labelled by two integers $r$ and $s$. These isospins are of the form:

$$4jr_s + 1 = r - st,$$  

(3.1)

where $r + s$ is odd and either $r > 0$ and $s \geq 0$ or $r < 0$ and $s < 0$. The $L_0$ and $J_{0}^0$ grades of the $V^{(j_{r,s},t)}$ subspace to which the singular vectors belong are $n = rs/2$ and $m = r/2$ respectively.

Once one has determined under which conditions singular vectors exist, one can try to find their explicit expressions. For bosonic affine algebras these expressions have been found by Malikov, Feigin and Fuks (MFF) in ref. [7]. These authors have found an equation giving the singular vectors in terms of monomials involving complex exponents of the generators. The corresponding analysis for osp(1|2) has been performed in refs. [2, 28]. In general, the singular vector of the $V^{(j_{r,s},t)}$ osp(1|2) module is given by:

$$\chi_{r,s}^\pm > = F^\pm(r,s,t) | j_{r,s}, t >,$$  

(3.2)

where the $\pm$ index refers to the two possible signs of $r$ and $F^\pm(r,s,t)$ is an element of $U(A_-)$. For $r > 0$ and $s \geq 0$, $F^+(r,s,t)$ is given by:

$$F^+(r,s,t) = j_0^- (J_0^-)^{r-1+s-t \over 2} (J_+^+)^{r+s-1 \over 2} j_0^- (J_0^-)^{r-1+s-2 \over 2} \cdots \times \cdots (J_+^-)^{r-(s-1) \over 2} j_0^- (J_0^-)^{r-1+s-1 \over 2},$$  

(3.3)
while, on the other hand, for $r < 0$ and $s < 0$ the corresponding operator $F^-(r, s, t)$ is:

$$
F^-(r, s, t) = \left( J_{-1}^+ \right)^{-\frac{r+1+(s+1)t}{2}} j_0^- \left( J_0^- \right)^{-\frac{r+1+(s+2)t}{2}} ( J_{-1}^+ )^{-\frac{r+1(s+3)t}{2}} \cdots \times
\times \cdots \left( J_0^- \right)^{-\frac{r+1+2t}{2}} ( J_{-1}^+ )^{-\frac{r}{2}} \cdot \cdots \cdot \left( J_{-1}^+ \right)^{-\frac{r}{2}} j_0^- ( J_0^- )^{-\frac{r+1}{2}} ( J_{-1}^+ )^{-\frac{r-1}{2}} j_0^- ( J_0^- )^{-\frac{r-2}{2}} ( J_{-1}^+ )^{-\frac{r-3}{2}} \cdots . \quad (3.4)
$$

It is far from obvious that the expressions (3.3) and (3.4) define an element of $U(A_-)$. In order to check this fact one must use some identities for products of operators that involve general complex powers. An example of such an identity is the following:

$$
A B^\gamma = \sum_{i=0}^{\infty} \begin{pmatrix} \gamma \\ i \end{pmatrix} B^{\gamma-i} \left[ \cdots [A, \underbrace{B}, B], \cdots, B \right], \quad (3.5)
$$

where $\gamma$ is not necessarily a non-negative integer. In eq. (3.5), the $i = 0$ term should be understood as $B^\gamma A$. Eq. (3.5) can be regarded as the analytical continuation of the case in which $\gamma \in \mathbb{Z}_+$. In this case, only a finite number of terms contribute to the right-hand side of (3.5) and this identity is easily proved by induction.

Although we shall not reproduce here the proof [2, 28] that $F^\pm(r, s, t) \in U(A_-)$, we shall give some arguments in support of this result. First of all, let us notice that, as can be checked by an elementary calculation, the sum of the exponents of $J_{-1}^+$ in (3.3) and (3.4) is $rs/2$, which is always integer and non-negative since $r + s$ is odd. This property is crucial if we want to end with a non-negative integer power of $J_{-1}^+$ after applying (3.5) to the operators (3.3) and (3.4). In the same way, one can verify that the sum of the exponents of $J_0^-$ in $F^\pm(r, s, t)$ is $(r+1)(s+1)/2$, which, again, always belongs to $\mathbb{Z}_+$. Finally, it is interesting to point out that there are $\pm(s+1)$ fermionic currents $j_0^-$ in $F^\pm(r, s, t)$, which implies that the operator $F^\pm(r, s, t)$ is bosonic (fermionic) if $r$ is even (odd).

Let us now study the consequences, for the fusion of Verma modules, of the existence of singular vectors. Let us assume that the isospin $j_1$ is of the form (3.1)
for some \( r = r_1 \) and \( s = s_1 \), i.e. that \( j_1 = j_{r_1 s_1} \). In the quotient module of \( V^{(j_1, t)} \) one should have:

\[
F^\pm (r_1, s_1, t) \mid j_1 , t > = 0 . \tag{3.6}
\]

Let us see how eq. (3.6) restricts the fusion of \( V^{(j_1, t)} \) with another module \( V^{(j_2, t)} \). Multiplying eq. (3.6) by the primary field \( \phi_{j_2} (z, x, \theta) \), one obviously gets:

\[
\phi_{j_2} (z, x, \theta) F^\pm (r_1, s_1, t) \mid j_1 , t > = 0 . \tag{3.7}
\]

In order to derive from eq. (3.7) a constraint for \( \phi_{j_2} (z, x, \theta) \mid j_1 , t > \), one should exchange the order of \( \phi_{j_2} (z, x, \theta) \) and \( F^\pm (r_1, s_1, t) \) in this equation. As \( F^\pm (r_1, s_1, t) \) is an element of \( U(\mathcal{A}_-) \), this can be done by using eq. (2.30). However, we are interested in keeping the factorized form (3.3) and (3.4) of \( F^\pm (r_1, s_1, t) \). Therefore we shall keep the non-integer exponents in the expression of the singular vectors. The commutation of \( \phi_{j_2} (z, x, \theta) \) with the \( j_0^- \) terms in (3.3) and (3.4) is easy to calculate. In fact, if the field \( \phi_{j_2} (z, x, \theta) \) is bosonic, eq. (2.30) yields:

\[
\phi_{j_2} (z, x, \theta) j_0^- = j_0^- \phi_{j_2} (z, x, \theta) + [ \phi_{j_2} (z, x, \theta), j_0^- ] =
\]

\[
= ( j_0^- - d_{j_2}^- ) \phi_{j_2} (z, x, \theta). \tag{3.8}
\]

When \( p(j_2) = 1 \) a similar calculation, using the anticommutator of \( \phi_{j_2} (z, x, \theta) \) and \( j_0^- \), can be performed. The final result differs from (3.8) in a global sign and, therefore, one can write in general:

\[
\phi_{j_2} (z, x, \theta) j_0^- = (-1)^{p(j_2)} ( j_0^- - d_{j_2}^- ) \phi_{j_2} (z, x, \theta). \tag{3.9}
\]

The commutation of \( \phi_{j_2} (z, x, \theta) \) with the \( J_0^- \) and \( J_{-1}^+ \) factors of \( F^\pm (r_1, s_1, t) \) is more delicate since these currents have non-integer powers in (3.3) and (3.4). We shall use, to deal with this case, the expression (3.5). Let us consider the
commutation of $J_0^-$ and $\phi_{j^2}(z, x, \theta)$ first. The iterated commutators one has to compute in this case are:

$$
\left[ \ldots \left[ \phi_{j^2}(z, x, \theta), J_{0^-} \right], J_{0^-}, \ldots, J_{0^-} \right] = [-D_{j^2}^-]^i \phi_{j^2}(z, x, \theta),
$$

(3.10)

and, therefore, one has:

$$
\phi_{j^2}(z, x, \theta)(J_0^-)^\gamma = \sum_{i=0}^{\infty} \binom{\gamma}{i} (J_0^-)^{\gamma-i} [-D_{j^2}^-]^i \phi_{j^2}(z, x, \theta).
$$

(3.11)

The right-hand side of (3.11) can be taken as the expansion of $(J_0^- - D_{j^2}^-)^\gamma$. Therefore we shall rewrite (3.11) as:

$$
\phi_{j^2}(z, x, \theta)(J_0^-)^\gamma = (J_0^- - D_{j^2}^-)^\gamma \phi_{j^2}(z, x, \theta).
$$

(3.12)

The same procedure can be applied to $J_{+1}^-$, with the result:

$$
\phi_{j^2}(z, x, \theta)(J_{+1}^-)^\gamma = (J_{+1}^- - z^{-1}D_{j^2}^+)^\gamma \phi_{j^2}(z, x, \theta).
$$

(3.13)

It is clear that, with this procedure, one ends up with the following exchange relation:

$$
\phi_{j^2}(z, x, \theta) F^\pm(r_1, s_1, t) = \tilde{F}^\pm_{j^2}(r_1, s_1, t) \phi_{j^2}(z, x, \theta),
$$

(3.14)

where the operator $\tilde{F}^\pm_{j^2}(r_1, s_1, t)$ is obtained by changing $J_{0^-}^0 \to (-1)^{p(j^2)}(J_{0^-}^0 - d_{j^2}^-)$, $J_{0^-}^0 \to J_{0^-}^0 - D_{j^2}^-$ and $J_{+1}^+ \to J_{+1}^+ - z^{-1}D_{j^2}^+$ in $F^\pm(r_1, s_1, t)$. Using eq. (3.14) in (3.7) one arrives at:

$$
\tilde{F}^\pm_{j^2}(r_1, s_1, t) \phi_{j^2}(z, x, \theta) \mid_{j^1, t} = 0.
$$

(3.15)

Notice that the operators $J_{-n}^a - z^{-n}D_{j^2}^a$ and $j_{-n}^a - z^{-n}d_{j^2}^a$ close the same algebra as the currents $J_{-n}^a$ and $j_{-n}^a$ when $J_{-n}^a$ and $j_{-n}^a$ belong to $\mathcal{A}_-$. For this
reason $\tilde{F}_{j_2}^{\pm}(r_1, s_1, t)$ can be arranged, by using the analytically continued commutation relations (3.5), as a polynomial expression in the operators $J_n^a - z^{-n}D_{j_2}^a$ and $J_n^a - z^{-n}d_{j_2}^a$ with exponents which are non-negative integers. The proof of this fact is the same that serves to demonstrate that $F^{\pm}(r_1, s_1, t) \in U(A_-)$. Therefore the operator $\tilde{F}_{j_2}^{\pm}(r_1, s_1, t)$ is unambiguously defined and it makes sense to project eq. (3.15) on the state $|j_3, t>$. After doing this projection, only the part of $\tilde{F}_{j_2}^{\pm}(r_1, s_1, t)$ containing the derivatives survives and, eliminating some global factor, one ends up with an equation of the type:

$$\hat{F}_{j_2}^{\pm}(r_1, s_1, t) \langle j_3, t| \phi_{j_2}(z, x, \theta) | j_1, t > = 0$$  \hspace{1cm} (3.16)$$

where $\hat{F}_{j_2}^{\pm}(r_1, s_1, t)$ is the differential operator obtained from $F^{\pm}(r_1, s_1, t)$ by making the substitutions $j_0^- \rightarrow d_{j_2}^-$, $J_0^- \rightarrow D_{j_2}^-$ and $J_1^+ \rightarrow D_{j_2}^+$. The form of the matrix element appearing in the left-hand side of eq. (3.16) can be obtained from the formalism of fusion of Verma modules developed in section 2. Indeed, in order to get the value of $\langle j_3, t| \phi_{j_2}(z, x, \theta) | j_1, t >$, one only needs to project the right-hand side of eq. (2.47) on the $n = m = 0$ sector. The result is:

$$\langle j_3, t| \phi_{j_2}(z, x, \theta) | j_1, t > = C_{123} \theta^{\delta_j} z^{h_3 - h_1 - h_2} x^{j_1 + j_2 - j_3 - \frac{1}{2}}$$  \hspace{1cm} (3.17)$$

where $C_{123}$ is a constant and $\delta_j$ is $\Delta_m$ for $m = 0$, i.e. (see eq. (2.46)):

$$\delta_j \equiv \epsilon \left( \frac{p(j_1) + p(j_2) - p(j_3)}{2} \right)$$  \hspace{1cm} (3.18)$$

Notice that $\delta_j$ takes the value zero if $p(j_3) = p(j_1) + p(j_2) \mod (2)$ and is equal to one when the parity of $|j_3, t>$ is not the sum (modulo two) of those of $|j_1, t>$ and $|j_2, t>$.

The expressions we have for the operators $\hat{F}_{j_2}^{\pm}(r_1, s_1, t)$ involve powers of derivatives with exponents that do not belong to $\mathbb{Z}_+$. We have already argued that, after using the analytically continued commutators, one gets perfectly well-defined differential operators $\tilde{F}_{j_2}^{\pm}(r_1, s_1, t)$. It is, however, more interesting for our purposes
to keep the non-integer powers in $\hat{F}^\pm_{j_2}(r_1,s_1,t)$ and to evaluate the action of this operator on the matrix element (3.17) by means of the fractional calculus techniques [17]. This method has been recently used in connection with the free field representation of the $sl(2)$ current algebra. Basically, one only needs to evaluate fractional derivatives on functions that are powers of the variables. In our case, the general power appearing in (3.16) is of the form $\theta^\gamma x^\lambda$ with $\gamma = 0,1$. The derivatives we need are:

\begin{equation}
(\partial_x)^n \theta^\gamma x^\lambda = \frac{\lambda!}{(\lambda - n)!} \theta^\gamma x^{\lambda-n}
\end{equation}

\begin{equation}
[D_{j_2}^+]^n \theta^\gamma x^\lambda = \frac{(2j_2 - \lambda - \gamma)!}{(2j_2 - \lambda - \gamma - n)!} \theta^\gamma x^{\lambda+n}
\end{equation}

\begin{equation}
[d_{j_2}^- [D_{j_2}^-]^n \theta^\gamma x^\lambda = \frac{\lambda!}{(\lambda - n - 1 + \gamma)!} \theta^{1-\gamma} x^{\lambda-n-1+\gamma}.
\end{equation}

The results displayed in eq. (3.19) can be proved by a direct calculation when $n \in \mathbb{Z}_+$ and then they can be analytically continued for complex $n$. The use of the derivation rules (3.19) allows to compute the left-hand side of eq. (3.16). In fact, one can prove that:

\begin{equation}
\hat{F}^\pm_{j_2}(r_1,s_1,t) \theta^{\delta_j} x^{-\frac{j_2}{2}} = f_{r_1,s_1}^\pm (t) \theta^{\epsilon(\frac{z_1+1+\delta_j}{2})} x^{-\frac{z_1}{2}-\frac{1}{2}\epsilon(\frac{z_1+1+\delta_j}{2})},
\end{equation}

where $f_{r_1,s_1}^\pm (t)$ is a numerical factor and $j$ is the following combination of the isospins:

\begin{equation}
j = j_1 + j_2 - j_3.
\end{equation}

If $V(j_3,t)$ appears in the fusion of $V(j_1,t)$ and $V(j_2,t)$, the matrix element (3.17) must be non-vanishing. This can only occur when the constant $C_{123}$ is different from zero and, in this case, the fulfillment of eq. (3.16) requires the vanishing of the factors $f_{r_1,s_1}^\pm (t)$. The explicit form of these factors can be obtained by collecting
the factorials resulting from the fractional derivations. For $r_1 > 0$ and $s_1 \geq 0$, one has:

$$f_{r_1, s_1}^+ (t) = \prod_{n=0}^{s_1} \frac{\left[ j + \frac{1}{2} \left( n t - \epsilon \left( \frac{n}{2} \right) - (-1)^n \delta_j \right) \right]!}{\left[ j + \frac{1}{2} \left( (s_1 - n) t - r_1 - 1 + \epsilon \left( \frac{n}{2} \right) + (-1)^n \delta_j \right) \right]!} \times$$

$$\times \prod_{m=1}^{s_1} \frac{\left[ 2j_2 - j + \frac{1}{2} \left( (m - 1 - s_1) t + r_1 - \epsilon \left( \frac{m}{2} \right) - (-1)^m \delta_j \right) \right]!}{\left[ 2j_2 - j - \frac{1}{2} \left( mt + \epsilon \left( \frac{m}{2} \right) + (-1)^m \delta_j \right) \right]!} ,$$

(3.22)

while if $r_1 < 0$ and $s_1 < 0$, a similar calculation allows to get $f_{r_1, s_1}^- (t)$:

$$f_{r_1, s_1}^- (t) = \prod_{n=0}^{-s_1-2} \frac{\left[ j - \frac{1}{2} \left( r_1 + \epsilon \left( \frac{n}{2} \right) - (s_1 + 1 + n) t + (-1)^n \delta_j \right) \right]!}{\left[ j - \frac{1}{2} \left( (n + 1) t + \epsilon \left( \frac{n+1}{2} \right) + (-1)^n \delta_j \right) \right]!} \times$$

$$\times \prod_{m=0}^{-s_1-1} \frac{\left[ 2j_2 - j - \frac{1}{2} \left( \epsilon \left( \frac{m}{2} \right) - mt + (-1)^m \delta_j \right) \right]!}{\left[ 2j_2 - j + \frac{1}{2} \left( r_1 - \epsilon \left( \frac{m}{2} \right) - (s_1 + 1 + m) t - (-1)^m \delta_j \right) \right]!} .$$

(3.23)

The expressions (3.22) and (3.23) can be simplified by dividing the factorials appearing in their numerators by those of their denominators. In general, if $A$ and $B$ are complex numbers such that $A - B \in \mathbb{Z}_+$, one has:

$$\frac{A!}{B!} = \prod_{i=0}^{A-B-1} (A - i) .$$

(3.24)

It can be verified that the products appearing in (3.22) and (3.23) can be rearranged in such a way that eq. (3.24) can be applied to all the quotients of factorials in these equations. In fact, one can prove that the first product in the
expression of $f_{r_1,s_1}^+(t)$ can be written as:

$$
\prod_{n=0}^{r_1-1} \prod_{m=0}^{s_1} \left( j - \frac{n}{2} + \frac{m}{2} t \right),
$$

(3.25)

whereas the second product in (3.22) can be put as:

$$
\prod_{n=1}^{r_1} \prod_{m=1}^{s_1} \left( 2j_2 - j + \frac{n}{2} - \frac{m}{2} t \right),
$$

(3.26)

and, therefore, $f_{r_1,s_1}^+(t)$ is given by:

$$
\prod_{n=0}^{r_1-1} \prod_{m=0}^{s_1} \left( j_1 + j_2 - j_3 - \frac{n}{2} + \frac{m}{2} t \right) \prod_{n=1}^{r_1} \prod_{m=1}^{s_1} \left( j_2 - j_1 + j_3 + \frac{n}{2} - \frac{m}{2} t \right).
$$

(3.27)

Similarly, the expression (3.23) for $f_{r_1,s_1}^-(t)$ can be shown to be equivalent to:

$$
\prod_{n=1}^{-r_1} \prod_{m=1}^{-s_1-1} \left( j_1 + j_2 - j_3 + \frac{n}{2} - \frac{m}{2} t \right) \prod_{n=0}^{-r_1-1} \prod_{m=0}^{-s_1-1} \left( j_2 - j_1 + j_3 - \frac{n}{2} + \frac{m}{2} t \right).
$$

(3.28)

For a given value of $t$, the vanishing of $f_{r_1,s_1}^\pm(t)$ imposes non-trivial polynomial conditions to the isospins $j_1$, $j_2$ and $j_3$. These conditions must be required in order to have the representation of isospin $j_3$ in the fusion of those with isospins $j_1$ and $j_2$. Notice that these conditions are induced by the reducibility of $V^{(j_1,t)}$. One could similarly find the relations that must be imposed when some of the other two modules has singular vectors. As we shall see in next section, for some types of representations these conditions are strong enough to determine the selection rules of the operator algebra of the theory. Before finishing this section, it is interesting to point out the great similarity between the functions (3.27) and (3.28) and those corresponding to the $sl(2)$ current algebra. The main difference between these two
cases is the appearance in the osp(1|2) result of the parameter $\delta^j$, which takes into account the relative Grassmann parity of the highest weight vectors participating in the fusion. We shall analyze in next section the implications of this statistics dependence in the fusion structure of the algebra.

4. Fusion rules for admissible representations

In this section we particularize our analysis to the so-called admissible representations of osp(1|2) [2, 28]. These representations occur for rational values of the parameter $t$. In fact, we shall assume in this section that $t$ is given by:

$$t = \frac{p}{p'},$$

where $p$ and $p'$ are coprime positive integers such that $p + p'$ is even and $p$ and $\frac{p + p'}{2}$ are relatively prime [28]. For this value of the level $t$, the representations with isospins given by eq. (3.1) with $r \neq 0 \mod (p)$ are completely degenerate. The proper maximal submodule of $V^{(j_r,s,t)}$, in this case, is generated by two singular vectors which give rise to a double line embedding diagram, very similar to the one appearing in the minimal models of the (super)Virasoro algebra. The admissible representations correspond to the case in which we restrict $r$ and $s$ to take values in the grid $1 \leq r \leq p - 1$, $0 \leq s \leq p' - 1$. It was shown in ref. [2] that the characters of these representations form a representation of the modular group. In appendix A we recall the calculation of these characters and study their relation with the ones corresponding to the minimal supersymmetric models. The results presented in this appendix generalize the relation, discovered in ref. [29], between the $sl(2)$ admissible representations and the minimal Virasoro models.

In this section we are going to prove, using the singular vector decoupling conditions found in section 3, that the primary fields corresponding to the admissible representations close a well-defined fusion algebra. Our result is very similar to the one established in refs. [11, 16] for the $sl(2)$ current algebra. We will find two
types of fusion rules which cannot be satisfied simultaneously. In order to derive this result, let us consider the fusion of two isospins $j_1$ and $j_2$ given by:

$$
4j_1 + 1 = r_1 - s_1 t \\
4j_2 + 1 = r_2 - s_2 t \\
1 \leq r_1, r_2 \leq p - 1 \\
0 \leq s_1, s_2 \leq p' - 1 .
$$

(4.2)

As $j_1 = j_{r_1, s_1}$ is of the form (3.1) with $r_1 > 0$ and $s_1 \geq 0$, the Verma module $V^{(j_1, t)}$ will have a singular vector of the type (3.3). Therefore, one should impose the singular vector decoupling condition $f_{r_1, s_1}^+ (t) = 0$. Moreover, the isospin $j_1$ can also be written as $j_{r_1 - p, s_1 - p'}$ and, when $r_1$ and $s_1$ belong to the grid of the admissible representations (see eq. (4.2)), one has that $r_1 - p < 0$ and $s_1 - p' < 0$. Therefore the module $V^{(j_1, t)}$ also possesses a singular vector of the type (3.4), whose decoupling condition requires the vanishing of $f_{r_1 - p, s_1 - p'}^- (t)$. In conclusion, one has the following two conditions:

$$
f_{r_1, s_1}^+ (t) = f_{r_1 - p, s_1 - p'}^- (t) = 0 .
$$

(4.3)

The explicit expression of $f_{r_1, s_1}^+ (t)$ is given in eq. (3.27), whereas, using eq. (3.28), the condition $f_{r_1 - p, s_1 - p'}^- (t) = 0$ takes the form:

$$
\prod_{n=1}^{p-r_1} \prod_{m=1}^{p'-s_1-1} \left( j_1 + j_2 - j_3 + \frac{n}{2} - \frac{m}{2} t \right) \prod_{n=0}^{p-r_1-1} \prod_{m=0}^{p'-s_1-1} \left( j_2 - j_1 + j_3 - \frac{n}{2} + \frac{m}{2} t \right) = 0 .
$$

(4.4)

We shall prove below that eq. (4.3) forces $j_3$ to take values corresponding to admissible representations. There are, however, two possibilities of satisfying (4.3) that we shall discuss separately in two subsections.
4.1. Fusion rule I

Let us first consider the equation $f_{r_1,s_1}^+ (t) = 0$. As the expression (3.27) of $f_{r_1,s_1}^+ (t)$ is completely factorized, it follows that $f_{r_1,s_1}^+ (t)$ vanishes if and only if, at least, one of its factors is zero. In eq. (3.27) there are two different products. Suppose that one of the factors in the first of these products vanishes. If this occurs, the isospin $j_3$ can be written as:

$$j_3 = j_1 + j_2 - \frac{n}{2} + \frac{m}{2} t \quad \text{with} \quad 0 \leq n \leq r_1 - 1 \quad 0 \leq m \leq s_1 . \quad (4.5)$$

Similar considerations can be applied to the $f_{r_1-p,s_1-p'}^+ (t) = 0$ condition (see eq. (4.4)). Suppose that one of the factors in the first product of the left-hand side of (4.4) vanishes. If this were the case, $j_3$ would be given by:

$$j_3 = j_1 + j_2 + \frac{\bar{n}}{2} - \frac{\bar{m}}{2} t \quad \text{with} \quad 1 \leq \bar{n} \leq p - r_1 \quad 1 \leq \bar{m} \leq p' - s_1 - 1 . \quad (4.6)$$

In eqs. (4.5) and (4.6), $n, m, \bar{n}$ and $\bar{m}$ are integers that can take values in the range indicated in these equations. If eqs. (4.5) and (4.6) are simultaneously satisfied, after subtracting them, one gets:

$$(m + \bar{m}) t = \bar{n} + n . \quad (4.7)$$

It is easy to see that eq. (4.7) cannot be satisfied. Indeed, from eqs. (4.5) and (4.6) it follows that $1 \leq m + \bar{m} \leq p' - 1$ and thus $(m + \bar{m}) t \notin \mathbb{Z}$, in flagrant contradiction with the right-hand side of (4.7). We thus conclude that the first product in (4.4) cannot vanish if eq. (4.5) holds. In order to satisfy simultaneously eqs. (4.5) and (4.4), the only remaining possibility is the cancellation of one of the factors in the second product of $f_{r_1-p,s_1-p'}^+ (t)$. In this case, $j_3$ would be:

$$j_3 = j_1 - j_2 + \frac{n'}{2} - \frac{m'}{2} t \quad \text{with} \quad 0 \leq n' \leq p - r_1 - 1 \quad 0 \leq m' \leq p' - s_1 - 1 . \quad (4.8)$$

Subtracting, as before, the two expressions for $j_3$ (eqs. (4.5) and (4.8)) one gets
the following parametrization for $j_2$:

$$4j_2 + 1 = n + n' + 1 - (m + m')t . \quad (4.9)$$

Notice that, as $1 \leq n + n' + 1 \leq p - 1$ and $0 \leq m + m' \leq p' - 1$, the numbers $n + n' + 1$ and $m + m'$ are in the grid of the admissible representations and, thus one gets the following parametrization for $r_2$ and $s_2$:

$$r_2 = n + n' + 1 \quad s_2 = m + m' . \quad (4.10)$$

Using these expressions for $r_2$ and $s_2$ in (4.5) and (4.8), one obtains that $j_3$ can be written as:

$$4j_3 + 1 = r_3 - s_3 t , \quad (4.11)$$

where $r_3$ and $s_3$ can be written in terms of $r_2$ and $s_2$ as:

$$r_3 = r_1 + r_2 - 2n - 1 = r_1 - r_2 + 2n' + 1$$
$$s_3 = s_1 + s_2 - 2m = s_1 - s_2 + 2m' . \quad (4.12)$$

As a consistency check, notice that from (4.12) it follows that $r_3 + s_3$ is odd if $r_1 + s_1 , \ r_2 + s_2 \in 2\mathbb{Z} + 1$. Moreover, by varying $n, m, n'$ and $m'$ within the range displayed in eqs. (4.5) and (4.8), one can obtain the range of allowed values for $r_3$ and $s_3$. After a straightforward calculation we get:

$$|r_1 - r_2| + 1 \leq r_3 \leq \min (r_1 + r_2 - 1, 2p - r_1 - r_2 - 1)$$
$$|s_1 - s_2| \leq s_3 \leq \min (s_1 + s_2, 2p' - s_1 - s_2 - 2) . \quad (4.13)$$

It is interesting to point out that, from our previous equations, it follows that $r_3$ and $s_3$ are always in the range allowed to the admissible representations. To establish this fact, let us notice that, eliminating $r_2$ and $s_2$ in (4.12) by means of (4.10), one gets that $r_3 = r_1 + n' - n$ and $s_3 = s_1 + m' - m$. If we freely vary
In order to completely identify the representations resulting from the fusion of \( j_1 \) and \( j_2 \), one should determine the parity of their highest weight vectors. In general, for an admissible representation with isospin \( j_{r,s} \), let us define the following quantity:

\[
\lambda_{r,s} = \frac{r + s - 1}{4} = j_{r,s} + \frac{s}{4}(1 + t). \tag{4.14}
\]

Notice that, since \( r + s \) is always odd, \( 2\lambda_{r,s} \in \mathbb{Z} \). It turns out that the Grassmann parity of the representations resulting from the fusion is determined by the difference:

\[
\Delta \lambda = \lambda_{r_1,s_1} + \lambda_{r_2,s_2} - \lambda_{r_3,s_3}. \tag{4.15}
\]

Using eqs. (4.12) and (4.14) one can evaluate \( \Delta \lambda \) with the result:

\[
\Delta \lambda = \frac{n + m}{2}. \tag{4.16}
\]

Recall (see eq. (3.27)) that \( n + m \in 2\mathbb{Z} + \delta_j \). Therefore, it follows that if \( \delta_j = 0 \) \( (\delta_j = 1) \), i.e. if \( p(j_3) = p(j_1) + p(j_2) \ mod(2) \) \( (p(j_3) = p(j_1) + p(j_2) + 1 \ mod(2)) \), then \( \Delta \lambda \in \mathbb{Z} \) \( (\Delta \lambda \in \mathbb{Z} + \frac{1}{2}) \). Thus we can write:

\[
p(j_3) = p(j_1) + p(j_2) + 2\Delta \lambda \ mod(2). \tag{4.17}
\]

To finish this subsection, let us write the fusion rules we have found in a more convenient form. We shall denote by \( [r, s] \) the admissible representation with isospin \( j_{r,s} \). With this notation, it follows from (4.13) that the fusion rules can be
written as:
\[
\begin{bmatrix} r_1, s_1 \end{bmatrix} \times \begin{bmatrix} r_2, s_2 \end{bmatrix} = \sum_{r_3 = |r_1 - r_2| + 1}^{\min(r_1 + r_2 - 1, 2p - r_1 - r_2 - 1)} \sum_{s_3 = |s_1 - s_2|}^{\min(s_1 + s_2, 2p' - s_1 - s_2 - 2)} \begin{bmatrix} r_3, s_3 \end{bmatrix}.
\]
(4.18)

One must keep in mind when using eq. (4.18) that, as can be seen from eq. (4.12), \( r_3 \) and \( s_3 \) jump in the sums (4.18) in steps of two units.

4.2. Fusion rule II

The possibility studied in section 4.1 is not the only way to fulfill the decoupling conditions (4.3). Indeed, one could satisfy the equation \( f^+_{r_1, s_1}(t) = 0 \) by requiring that one of the factors appearing in the second product of (3.27) vanishes. Following the same steps as in section 4.1, one can prove that this requirement is incompatible with the vanishing of one of the factors of the second product in (4.4). Therefore, one of the factors in the first product of \( f^-_{r_1 - p, s_1 - p'}(t) \) must vanish and, in conclusion, we must have:

\[
\begin{align*}
 j_3 &= j_1 - j_2 - \frac{n}{2} + \frac{m}{2}t \quad \text{with} \quad 1 \leq n \leq r_1 \quad 1 \leq m \leq s_1 \\
 j_3 &= j_1 + j_2 + \frac{n'}{2} - \frac{m'}{2}t \quad \text{with} \quad 1 \leq n' \leq p - r_1 \quad 1 \leq m' \leq p' - s_1 - 1.
\end{align*}
\]
(4.19)

Subtracting the two equations in (4.19) we can get the value of \( 4j_2 + 1 \). After adding \( p - p't = 0 \) to the right-hand side of the resulting equation, one gets:

\[
4j_2 + 1 = (p - n - n' + 1) - (p' - m - m')t.
\]
(4.20)

From eq. (4.20), one is tempted to identify \( r_2 \) and \( s_2 \) with:

\[
 r_2 = p - n - n' + 1 \quad s_2 = p' - m - m'.
\]
(4.21)

Varying \( n, m, n' \) and \( m' \) in the range written in (4.19), one gets that \( 1 \leq r_2 \leq p - 1 \) and \( 1 \leq s_2 \leq p' - 2 \). Notice that these values of \( r_2 \) and \( s_2 \) belong to the grid of the
admissible representations. Moreover, the range of $s_2$ reveals that only for $s_2 > 0$ and $p' > 2$ will this solution of eq. (4.3) take place. On the other hand, adding the two expressions of $j_3$ in (4.19), we arrive at:

\[ 4j_3 + 1 = r_1 + n' - n - (s_1 + m' - m)t , \]  

(4.22)

which suggests the following identification of $r_3$ and $s_3$:

\[ r_3 = r_1 + n' - n \quad s_3 = s_1 + m' - m . \]  

(4.23)

This identification can be confirmed by evaluating the range of the possible values of $r_3$ and $s_3$, which, after taking eq. (4.19) into account, is $1 \leq r_3 \leq p - 1$, $1 \leq s_3 \leq p' - 2$. This result shows that $s_3$ cannot be zero. Using eq. (4.21) in (4.23) one can eliminate one of the two integer indices in the right-hand side of (4.23) in favor of $r_2$ and $s_2$:

\[ r_3 = p + r_1 - r_2 + 1 - 2n = -p + r_1 + r_2 + 2n' - 1 \]
\[ s_3 = p' + s_1 - s_2 - 2m = -p' + s_1 + s_2 + 2m' . \]  

(4.24)

Notice that, again, $r_3 + s_3 \in 2\mathbb{Z} + 1$, as a consequence of the fact that $r_1 + s_1$ and $r_2 + s_2$ are odd and that $p + p'$ is even. It is now straightforward to get the range of variation of $r_3$ and $s_3$ for fixed $r_1$, $s_1$, $r_2$ and $s_2$. The result is:

\[ |p - r_1 - r_2| + 1 \leq r_3 \leq p - 1 - |r_1 - r_2| \]
\[ |p' - s_1 - s_2 - 1| + 1 \leq s_3 \leq p' - 2 - |s_1 - s_2| . \]  

(4.25)

Therefore, we can write the following fusion rule:

\[ [r_1, s_1] \times [r_2, s_2] = \sum_{r_3=|p-r_1-r_2|+1}^{p-1-|r_1-r_2|} \sum_{s_3=|p'-s_1-s_2-1|+1}^{p'-2-|s_1-s_2|} [r_3, s_3] , \]  

(4.26)

where, as in eq. (4.18), $r_3$ and $s_3$ jump in steps of two units.
Let us now determine the statistics of the \([r_3, s_3]\) representation. As in the fusion rule of section 4.1, the relevant quantity to consider is \(\Delta \lambda\) (defined as in eqs. (4.14) and (4.15)). An elementary calculation shows that in this case:

\[
\Delta \lambda = -\frac{n' + m'}{2} + \frac{p + p'}{4},
\]

and, therefore, \(p(j_3)\) is given by:

\[
p(j_3) = p(j_1) + p(j_2) + 2\Delta \lambda + \frac{p + p'}{2} \text{ mod}(2).
\]

Let us point out before finishing this section that the conditions (4.19) are incompatible with equations (4.5) and (4.8). This means that, as anticipated above, both sets of fusion rules cannot be satisfied simultaneously.

5. The descent equations

Let us continue elaborating the formalism for the fusion of Verma modules that was introduced in section 2. Our objective will be the computation of the vectors \(|n, m >_{j_3}\) appearing in the expansion (2.47). We shall verify that, generically, the determination of the vectors \(|n, m >_{j_3}\) can be performed once the action of the elements of \(\mathcal{A}_+\) on them is known. The equations encoding this action will be called the descent equations, following the denomination introduced in ref. [12] for the \(sl(2)\) current algebra. Our derivation of these equations starts with the highest weight conditions for the vector \(|j_1, t >:\)

\[
J^\alpha_p |j_1, t > = j^\alpha_p |j_1, t > = 0 \quad \forall \ (J^\alpha_p, j^\alpha_p) \in \mathcal{A}_+.
\]

Multiplying eq. (5.1) by \(\phi_{j_2}(z, x, \theta)\) and commuting this field with the currents, one gets:

\[
(J^\alpha_p - z^p D^\alpha_{j_2}) \phi_{j_2}(z, x, \theta) |j_1, t > = 0
\]

\[
(j^\alpha_p - z^p d^\alpha_{j_2}) \phi_{j_2}(z, x, \theta) |j_1, t > = 0.
\]

Substituting in this equation the expansions (2.33) and (2.47), one can determine the result of applying the currents of \(\mathcal{A}_+\) to the states resulting from the fusion
of \( V(j_1,t) \) and \( V(j_2,t) \). Let us detail this determination for the bosonic currents. Introducing the decompositions (2.33) and (2.47) in the first equation (5.2) and projecting on a given isospin \( j_3 \), one arrives at:

\[
\sum_{n,m} \theta^{\Delta_m} z^{h_3-h_1-h_2+n} x^{j_1+j_2-j_3+m-\frac{\Delta_m}{2}} J_p^a | n, m >_{j_3} = \sum_{n,m} z^{h_3-h_1-h_2+n+p} D_{j_2}^a (\theta^{\Delta_m} x^{j_1+j_2-j_3+m-\frac{\Delta_m}{2}}) | n, m >_{j_3}.
\]

(5.3)

In (5.3) the left-hand side contains \( J_p^a | n, m >_{j_3} \), while in the right-hand side the derivative \( D_{j_2}^a \) only acts on the \( \theta \) and \( x \) variables. Using eq. (2.26) and comparing the terms in both sides of eq. (5.3) with the same powers of \( \theta \) and \( x \), one can extract the value of \( J_p^a | n, m >_{j_3} \). Let us express this result in terms of the following combination of the isospins:

\[
i_\pm = -j_3 + j_1 \pm j_2.
\]

(5.4)

With this definition, the descent equations for the bosonic currents read:

\[
J_p^+ | n, m >_{j_3} = (-i_- + 1 - m - \Delta_m \frac{1}{2}) | n - p, m - 1 >_{j_3} \quad (p \geq 0)
\]

\[
J_p^0 | n, m >_{j_3} = -\left( \frac{i_+ + i_-}{2} + m \right) | n - p, m >_{j_3} \quad (p \geq 1)
\]

\[
J_p^- | n, m >_{j_3} = (i_+ + 1 + m - \Delta_m \frac{1}{2}) | n - p, m + 1 >_{j_3} \quad (p \geq 1).
\]

(5.5)

For the fermionic currents of \( A_+ \) one proceeds similarly. One must be specially careful in this case with the signs and with the powers of the Grassmann variable \( \theta \). The final result is:

\[
j_p^+ | n, m >_{j_3} = (-1)^{\Delta_m} [1 + (i_- + m - \frac{3}{2}) \Delta_m] | n - p, m - \frac{1}{2} >_{j_3} \quad (p \geq 0)
\]

\[
j_p^- | n, m >_{j_3} = (-1)^{\Delta_m} [1 + (i_+ + m - \frac{1}{2}) \Delta_m] | n - p, m + \frac{1}{2} >_{j_3} \quad (p \geq 1).
\]

(5.6)

Notice the remarkable fact that the coefficients multiplying the vectors appear-
ing in the right-hand side of eqs. (5.5) and (5.6) are independent of the current mode $p$. On the other hand, as a check of the correctness of eqs. (5.5) and (5.6), one can easily verify that the matrix elements of the currents of $\mathcal{A}_+$ displayed in these equations are compatible with the (anti)commutation relations of the algebra (eq. (2.1)).

Let us now see how the descent equations can be used to determine the vectors $|n, m >_{j_3}$. In general, these vectors belong to the subspace $V_{n,m}^{(j_3,t)}$. Therefore, they can be represented as a linear combination of the elements of a basis of $V_{n,m}^{(j_3,t)}$. One of such a basis was described in section 2 (see eq. (2.17)). In terms of the vectors (2.17) one can write:

$$|n, m >_{j_3} = \sum_{\{m_i^a\}} C_{\{m_i^a\}} | \{m_i^a\} ; j_3 > ,$$  \hspace{1cm} (5.7)

where $C_{\{m_i^a\}}$ are some constants and only those vectors with $L_0$ and $J_0^0$ grades $n$ and $m$ respectively enter the sum (5.7) (recall that for a given sequence $\{m_i^a\}$ the values of $n$ and $m$ are given in eqs. (2.18)-(2.20)). It is clear that the determination of $|n, m >_{j_3}$ is equivalent to obtaining the constants $C_{\{m_i^a\}}$. The latter can be determined with the help of the inner product defined in (2.23). Indeed, let us suppose that we multiply both sides of (5.7) by another basis vector $| \{\overline{m}_i^a\} ; j_3 > \in V_{n,m}^{(j_3,t)}$. Doing this we would get:

$$< \{\overline{m}_i^a\} ; j_3 | n, m >_{j_3} = \sum_{\{m_i^a\}} C_{\{m_i^a\}} < \{\overline{m}_i^a\} ; j_3 | \{m_i^a\} ; j_3 > .$$  \hspace{1cm} (5.8)

We shall regard eq. (5.8) as a linear system of equations whose unknowns are the constants $C_{\{m_i^a\}}$. The inner products appearing in the right-hand side of eq. (5.8) can be computed from the explicit expression of the vectors $| \{m_i^a\} ; j_3 >$ (see eq. (2.17)). Moreover, the products $< \{\overline{m}_i^a\} ; j_3 | n, m >_{j_3}$ can be evaluated with the help of the descent equations. Indeed, the anti-automorphism $\sigma$, appearing in (2.23), transforms the elements of $\mathcal{A}_-$ appearing in the definition (2.17) in currents...
of $A_+$, whose action on $|n,m \rangle_{j_3}$ is given by eqs. (5.5) and (5.6). An explicit calculation gives the following result:

$$
< \{ m_i^a \}; j_3 | n,m \rangle_{j_3} = \left( m^+ - \frac{i_+ + i_-}{2} \right)^{m_0} \prod_{i=1}^{[m^+ + \Delta m^+]} \left( -i_+ + m^+ + \frac{\Delta m^+}{2} - i \right) \times \\
\times \prod_{i=1}^{[m^- + \Delta m^-]} \left( -i_- - m - \frac{\Delta m^-}{2} + i \right).
$$

(5.9)

If the contravariant form at grades $(n,m)$ is non-degenerate, the system of equations (5.8) can be solved for $C_{\{ m_i^a \}}$ and, therefore, the vector $|n,m \rangle_{j_3}$ can be determined. In particular, if for a given value of $(n,m)$ all the products in the left-hand side of eq. (5.8) are zero and the contravariant form in non-degenerate, it follows from (5.8) that the vector $|n,m \rangle_{j_3}$ must vanish. We are now going to see that this actually happens for some particular values of $i_\pm$. In fact, we shall prove that when $i_- (i_+)$ is integer or half-integer, those vectors $|n,m \rangle_{j_3}$ with a grade $m$ greater (smaller) that a certain value vanish. This truncation of the descent equations will be very important in what follows and, for this reason, we are going to describe it in detail.

Let us first consider the case in which $m > 0$. Let us split $m^-$ in the upper limit of the last product in (5.9) as $m^- = m + m^+$ (see eq. (2.20)). In general, if $A$ and $B$ are integers or half-integers, the integer part of their sum $[A + B]$ can be related to $[A]$ and $[B]$ by means of the equation:

$$
[A + B] = [A] + [B] + \epsilon(A)\epsilon(B).
$$

(5.10)

In particular, taking in (5.10) $A = m + \frac{\Delta m}{2}$ and $B = m^+$, one has:

$$
[m^- + \frac{\Delta m}{2}] = [m + \frac{\Delta m}{2}] + [m^+] + \epsilon(m^+)\epsilon(m + \frac{\Delta m}{2}).
$$

(5.11)

Therefore, as $m^+ \geq 0$, it is clear that there exists a factor in $< \{ m_i^a \}; j_3 | n,m \rangle_{j_3}$.
equal to:

\[
\prod_{i=1}^{[m+\Delta m]} \left( -i_- - m - \frac{\Delta m}{2} + i \right). \tag{5.12}
\]

In general, a product of the form \( \prod_{i=1}^{N} (A+i) \) is zero if and only if the first (last) factor \( A+1 \) (\( A+N \)) is non-positive (non-negative) and the last factor, i.e. \( A+N \), is integer. Applying this result to eq. (5.12), it follows that the product (5.12), vanishes if the following three conditions are satisfied:

\[
m \geq -i_- + 1 - \frac{\Delta m}{2}, \quad i_- \leq -\frac{1}{2} \epsilon(m + \frac{\Delta m}{2}), \quad i_- \in \mathbb{Z} + \frac{1}{2} \epsilon(m + \frac{\Delta m}{2}). \tag{5.13}
\]

On the other hand, from eq. (2.46) it follows that:

\[
\epsilon(m + \frac{\Delta m}{2}) = \delta_j. \tag{5.14}
\]

This means that the truncation conditions (5.13) can be simplified and put as:

\[
\begin{cases}
m \geq -i_- + 1 - \frac{\Delta m}{2} \\
i_- \leq -\frac{\delta_j}{2} \\
i_- \in \mathbb{Z} + \frac{\delta_j}{2}
\end{cases} \quad \Rightarrow \quad |n, m >_{j_3} = 0. \tag{5.15}
\]

Eq. (5.15) implies that when \( i_- \) is a non-positive integer or half-integer (depending on the value of \( \delta_j \), i.e. on the parity of \( |j_3, t > \) all the vectors \( |n, m >_{j_3} \) vanish when \( m > 0 \) is large enough.

In the case \( m < 0 \) one can similarly demonstrate the existence of a truncation that eliminates vectors with large absolute value of \( m \). We could prove this by studying eq. (5.9) for \( m < 0 \). It is however simpler to introduce a new basis for
\(V^{(j,t)}, \) constituted by the vectors:

\[
\langle \{m^a_i\}; j > = \prod_{i=1}^{+\infty} (j^+_i)^{2m^+_i} \prod_{i=1}^{+\infty} (j^0_i)^{m^0_i} \prod_{i=0}^{+\infty} (j^-_i)^{2m^-_i} | j, t > . \quad (5.16)
\]

The inner products of these basis vectors and \( | n, m >_{j3} \) can be easily computed by using the descent equations. The result is:

\[
< \{m^a_i\}; j3 | n, m >_{j3} = (-m^- - i_+ + i_-) m_0 \prod_{i=1}^{[m^+ + \frac{\Delta m}{2}]} ( - i_- - m^- - \frac{\Delta m^-}{2} + i ) \times \prod_{i=1}^{[m^+ + \frac{\Delta m}{2}]} ( - i_+ - m + \frac{\Delta m}{2} - i ) . \quad (5.17)
\]

If, as before, when \( m < 0 \) we split the upper limit of the last factor in (5.17) as \( m^+ + \frac{\Delta m}{2} = -m + \frac{\Delta m}{2} + m^- \), we find that the scalar product (5.17) contains a factor:

\[
\prod_{i=1}^{[-m^+ + \frac{\Delta m}{2}]} ( - i_+ - m + \frac{\Delta m}{2} - i ) . \quad (5.18)
\]

After analyzing the situations in which this product vanishes, one arrives at the following truncation conditions for \( m < 0 \):

\[
m \leq - i_+ - 1 + \frac{\Delta m}{2} \\
i_+ \geq \frac{\delta_j}{2} \\
i_+ \in \mathbb{Z} + \frac{\delta_j}{2}
\]

\[
\Rightarrow \quad | n, m >_{j3} = 0 . \quad (5.19)
\]

Notice that in eq. (5.19) \( i_+ \) is a non-negative integer or half-integer (depending again on \( \delta_j \)). It is also interesting to point out that, when the conditions in (5.19) are satisfied, there are no singular terms in the variable \( x \) in the expansion (2.47).
We finish this section by recalling that the truncations (5.15) and (5.19) are only valid for the grades \(n\) and \(m\) in which the contravariant form is non-degenerate. Therefore, in order to apply these equations, one must be sure that there are not singular vectors in the corresponding subspace \(V^{(j_3,t)}_{n,m}\).

6. The Sugawara recursion relations and the singular vectors

The Sugawara expression of the energy-momentum tensor (eq. (2.7)) can be used to obtain a set of recursion relations among the vectors \(|n, m >_{j_3}\). In some cases, these relations, together with the truncation conditions (5.15) and (5.19), will allow us to write a finite system of linear equations, whose resolution provides a very efficient way of solving the descent equations. On the other hand, as it was the case for the \(sl(2)\) current algebra [12], the fusion formalism can be used to obtain the explicit form of the singular vectors of the \(osp(1|2)\) affine algebra. The basic tools in the computation of singular vectors will be precisely the truncation equations of section 5 and the Sugawara recursion relations which we are now going to derive.

Our starting point will be the expression of the Virasoro generators \(L_n\) in terms of the currents \(J^a_n\) and \(j^a_n\). This expression is readily obtained by substituting the mode expansions (2.6) in the Sugawara equation (2.7) and by identifying the result with eq. (2.8). One gets:

\[
L_n = \frac{1}{t} \sum_{p=-\infty}^{+\infty} \left[ 2J^0_{n-p}J^0_p + J^+_{n-p}J^-_p + J^-_{n-p}J^+_p - \frac{1}{2} j^+_n j^-_n + \frac{1}{2} j^-_n j^+_n \right].
\]

(6.1)

If, in particular, we put \(n = 0\) in (6.1), we obtain:

\[
\frac{t}{2} L_0 - J^0_0 \left( J^0_0 + \frac{1}{2} \right) = \sum_{p=1}^{+\infty} \left[ 2J^0_{-p}J^0_p + J^+_{-p}J^-_p - \frac{1}{2} j^+_p j^-_p \right] + \sum_{p=0}^{+\infty} \left[ J^-_p J^+_p + \frac{1}{2} j^-_p j^+_p \right].
\]

(6.2)
Let us apply both sides of eq. (6.2) to the vector $|n, m >_{j_{3}}$. From the action of $L_{0}$ and $J_{0}^{0}$ on $|n, m >_{j_{3}}$,

$$L_{0} |n, m >_{j_{3}} = (h_{3} + n) |n, m >_{j_{3}} = \left( \frac{2j_{3}(j_{3} + \frac{1}{2})}{t} + n \right) |n, m >_{j_{3}}$$  \hfill (6.3)

one finds the action of the left-hand side of (6.2):

$$\left[ \frac{t}{2} L_{0} - J_{0}^{0} (J_{0}^{0} + \frac{1}{2}) \right] |n, m >_{j_{3}} = |n, m >_{j_{3}},$$  \hfill (6.4)

where we have defined:

$$|n, m >_{j_{3}} \equiv \left[ \frac{t}{2} n + m(2j_{3} - m + \frac{1}{2}) \right] |n, m >_{j_{3}}.$$  \hfill (6.5)

On the other hand, as a consequence of the normal ordering, in the right-hand side of eq. (6.2) the currents of $A_{+}$ are to the right of those belonging to $A_{-}$. Therefore, we can use the descent equations (5.5) and (5.6) to evaluate the action of the right-hand side of (6.2) on $|n, m >_{j_{3}}$. Hence, taking eq. (6.4) into account, we get:

$$|n, m >_{j_{3}} = \left( i_{+} + 1 + m - \frac{\Delta m}{2} \right) \sum_{p=1}^{n} J_{-p}^{+} |n - p, m + 1 >_{j_{3}} -$$

$$\left( i_{+} + i_{-} + 2m \right) \sum_{p=1}^{n} J_{-p}^{0} |n - p, m >_{j_{3}} +$$

$$\left( -i_{-} + 1 - m - \frac{\Delta m}{2} \right) \sum_{p=0}^{n} J_{-p}^{-} |n - p, m - 1 >_{j_{3}} -$$

$$\frac{1}{2} (-1)^{\Delta m} \left[ 1 + (i_{+} + m - \frac{1}{2}) \Delta m \right] \sum_{p=1}^{n} J_{-p}^{+} |n - p, m + \frac{1}{2} >_{j_{3}} +$$

$$\frac{1}{2} (-1)^{\Delta m} \left[ 1 + (i_{-} + m - \frac{3}{2}) \Delta m \right] \sum_{p=0}^{n} J_{-p}^{-} |n - p, m - \frac{1}{2} >_{j_{3}}.$$

(6.6)

Taking different values of $n$ and $m$ in eq. (6.6), one obtains the announced Sugawara recursion relations. In fact, eq. (6.6) has a triangular structure with respect
to a partial ordering for the couples \((n, m)\) that, according to ref. [12], we define as follows. Given two couples \((n, m)\) and \((n', m')\) we will say that \((n, m) \leq (n', m')\) if and only if \(n \leq n'\) and \(n + m \leq n' + m'\). With respect to this ordering, all the terms in the right-hand side of (6.6) precede to the one in the left-hand side. Notice that the first couple in this ordering is \((n, m) = (0, 0)\). Actually, for \(n = m = 0\), eq. (6.6) is a trivial identity of the type \(0 = 0\) (recall that \(n \geq 0\) and \(m\) cannot be less than \(-n\)). Therefore, it is possible to solve eq. (6.6) iteratively, starting from the trivial \(n = m = 0\) equation and considering values of \((n, m)\) in increasing order.

Let us now study a property of the fusion states that will allow us to use them in the computation of the singular vectors. First of all, let us rewrite the descent equations (5.5) and (5.6) in the more condensed form:

\[
J^a_p | n, m >_{j_3} = \mu^a_j(m) | n - p, m - a >_{j_3}
\]

\[
j^a_p | n, m >_{j_3} = \rho^a_j(m) | n - p, m - \frac{\alpha}{2} >_{j_3} \quad \forall (J^a_p, j^a_p) \in A_+ , \quad (6.7)
\]

where the coefficients \(\mu^a_j(m)\) and \(\rho^a_j(m)\) can be read from (5.5) and (5.6). The vectors \(| n, m >_{j_3}\) satisfy:

\[
J^a_p | n, m >_{j_3} = [\frac{t}{2} n + m (2j_3 - m + \frac{1}{2})] \mu^a_j(m) | n - p, m - a >_{j_3}
\]

\[
j^a_p | n, m >_{j_3} = [\frac{t}{2} n + m (2j_3 - m + \frac{1}{2})] \rho^a_j(m) | n - p, m - \frac{\alpha}{2} >_{j_3} \quad \forall (J^a_p, j^a_p) \in A_+ . \quad (6.8)
\]

The proof of this equation is immediate if we use the relation between the \(| n, m >_{j_3}\) and \(| n, m >_{j_3}\) vectors (eq. (6.5)). It turns out, however, that it can be proved acting with the currents of \(A_+\) on the right-hand side of eq. (6.6). In order to verify this statement one must use the (anti)commutators of the algebra, together with the descent equations (5.5) and (5.6). In fact, to prove (6.8) it is enough to check the cases \(J^a_p = J^{-}_1\) and \(j^a_p = j^+_0\), since the result for the other currents
can be proved by using the algebra relations (2.1). This implies that $|n, m >_{j_3}$ would satisfy (6.8) if we had defined it by means of eq. (6.6) instead of using eq. (6.5). Another interesting observation is that the prefactor $\frac{j_3}{2} n + m (2j_3 - m + \frac{1}{2})$ appearing in the right-hand side of eq. (6.8) vanishes for $j_3 = j_{r,s}, n = \frac{r s}{2}$ and $m = \frac{r}{2}$, where $r$ and $s$ are integers such that $rs \geq 0$, $r + s \in \mathbb{Z}$ and $r \neq 0$ (this last condition eliminates the trivial solution $n = m = 0$). Notice that these are precisely the isospins and grades where the singular vectors are located. Therefore, as the vector $|rs/2, r/2 >_{j_{r,s}}$ satisfies:

$$J^a_p |rs/2, r/2 >_{j_{r,s}} = j^a_p |rs/2, r/2 >_{j_{r,s}} = 0, \quad \forall \ (J^a_p, j^a_p) \in A_+,$$

(6.9)

one is tempted to identify this vector with the corresponding singular vector $|\chi_{r,s} >$:

$$|\chi_{r,s} > \sim |rs/2, r/2 >_{j_{r,s}}.$$

(6.10)

Obviously, the identification (6.10) only makes sense when the vector $|rs/2, r/2 >_{j_{r,s}}$ is not identically zero. In order to have a non-trivial result we should find some criteria to discard a priori the $|rs/2, r/2 >_{j_{r,s}} = 0$ solution of the descent equations. In fact, for general values of $i_{\pm}$, it is easy to convince oneself that the recursion relations do not give a result in which $|rs/2, r/2 >_{j_{r,s}}$ is identically zero. For this reason, it is clear that $i_{\pm}$ must satisfy some non-trivial conditions in order to get a vanishing result for the vector (6.10). The crucial observation [12] to determine these conditions is that singular vectors do vanish in the quotient of the Verma module by its maximal proper submodule. The singular vector decoupling conditions studied in section 3 are precisely the requirements one has to impose to pass from the Verma module to the corresponding quotient. Thus we expect that these conditions are precisely the ones that $i_{\pm}$ must satisfy in order to get $|rs/2, r/2 >_{j_{r,s}} = 0$ as the solution of the recursion relations (6.6). It is thus clear that, in order to obtain a non-trivial result for the singular vectors, one must be sure that the decoupling conditions are not satisfied. Notice that eq.
(6.9) is satisfied independently of the decoupling conditions and, therefore, even when the latter are not satisfied, eq. (6.9) still holds.

Let us show how our previous considerations can be applied in practice to the determination of the singular vectors corresponding to the isopins \( j_3 = j_{r,s} \) with \( r > 0 \) and \( s \geq 0 \). The decoupling conditions of these singular vectors in the module \( V(j_3,t) \) take the form:

\[
g_{r,s}^+ (i_+, i_-) = 0 ,
\]

(6.11)

where the function \( g_{r,s}^+ (i_+, i_-) \) is given by:

\[
g_{r,s}^+ (i_+, i_-) = \prod_{i=0}^{r-1} \prod_{l=0}^{s} \left( -i_- - \frac{i}{2} + \frac{l}{2} t \right) \prod_{i=1}^{r} \prod_{l=1}^{s} \left( i_+ + \frac{i}{2} - \frac{l}{2} t \right) . \tag{6.12}
\]

The form (6.12) of \( g_{r,s}^+ (i_+, i_-) \) can be obtained from the function \( f_{r_1,s_1}^+ (t) \) written in eq. (3.27) by exchanging \( j_1 \leftrightarrow j_3 \) and substituting \( r_1 \) and \( s_1 \) by \( r \) and \( s \) respectively.

Let us now discuss the election of the isospins \( j_1 \) and \( j_2 \), i.e. of \( i_\pm \). In principle, \( j_1 \) and \( j_2 \) can be arbitrarily chosen. However, as we have just argued, we must require the condition \( g_{r,s}^+ (i_+, i_-) \neq 0 \) in order to avoid having a trivial result. Moreover, we can make use of the truncation conditions of section 5 in order to deal with the minimum number of intermediate vectors in the recursion relation, which will make the singular vector determination procedure more efficient. The truncation conditions of eqs. (5.15) and (5.19) determine the highest and lowest values of the \( J_0^0 \) grade \( m \). Since the singular vector is located at \( m = r/2 \), we shall require that:

\[
| n , m >_{j_{r,s}} = 0 \quad \text{for} \quad m \geq \frac{r + 1}{2} . \tag{6.13}
\]

Notice that for \( m \geq \frac{r + 1}{2} \) the contravariant form is non-degenerate in \( V_{n,m}^{(j_{r,s},t)} \), which ensures the validity of the implication written in eq. (5.15). Actually, in
view of this equation, one must have:

$$\frac{r + 1}{2} = -i_- + 1 - \frac{1}{2} \Delta_{++} . \quad (6.14)$$

Eq. (6.14) can be solved for $i_-$ for different values, modulo two, of $r$ and $\delta_j$. One can easily check that the result is $i_- = -\frac{r}{2}$ if $r + \delta_j = 0 \mod (2)$ and $i_- = -\frac{r-1}{2}$ if $r + \delta_j = 1 \mod (2)$, or in a more compact form:

$$i_- = -\frac{r}{2} \epsilon \left( \frac{r + \delta_j + 1}{2} \right) - \frac{r - 1}{2} \epsilon \left( \frac{r + \delta_j}{2} \right) . \quad (6.15)$$

Of these two solutions, one can discard one of them by looking at the value of the function $g_{r,s}^+(i_+, i_-)$. Indeed, it can be verified by direct substitution that:

$$g_{r,s}^+(i_+, -\frac{r - 1}{2}) = 0 \quad \text{if} \quad r + \delta_j = 1 \mod (2) , \quad (6.16)$$

and, therefore, in order to have a non-trivial singular vector we shall take $r + \delta_j = 0 \mod (2)$, i.e.:

$$\delta_j = \epsilon \left( \frac{r}{2} \right) . \quad (6.17)$$

Eq. (6.17) fixes the value of the parameter $\delta_j$, which determines the relative Grassmann parity of the modules involved in the fusion. For the value (6.17) of $\delta_j$, eq. (6.15) gives the value we must take for $i_-$, namely:

$$i_- = -\frac{r}{2} . \quad (6.18)$$

Let us now choose $i_+$. It can be verified by inspection that, for a generic value of $t$, the factor of $g_{r,s}^+(i_+, i_-)$ depending on $i_+$ never vanishes. Therefore, we have in this case a larger freedom in the election of $i_+$. Notice that, according to eq. (5.19), $i_+$ determines the lower value in the range of variation of $m$. We shall choose the smallest possible value of $i_+$ which, according to eq. (5.19), gives
rise to a value of $m$ that reduces maximally the range of values that $m$ can take. Therefore, taking eq. (5.19) into account, we put:

$$i_+ = \frac{\delta_j}{2} = \frac{1}{2} \epsilon \left(\frac{r}{2}\right). \quad (6.19)$$

For this value of $i_+$ the vectors $|n, m >_{jr,s}$ are zero if $m < -\delta_j/2$ and thus, in order to obtain the expression of the singular vector for the isospin $j_{r,s}$, which we shall denote by $|\lambda_{r,s} >$, we must vary $n$ and $m$ in the recursion relations within the intervals:

$$0 \leq n \leq \frac{rs}{2} \quad -\frac{1}{2} \epsilon \left(\frac{r}{2}\right) \leq m \leq \frac{r}{2}. \quad (6.20)$$

We have now all the ingredients needed for the computation of the singular vectors. In fact, as for the value of $\delta_j$ written in eq. (6.17) $\Delta \frac{r}{2} = 0$, one can write:

$$|\lambda_{r,s} > = -\frac{1}{2} \left( r + \epsilon \left(\frac{r}{2}\right) \right) \sum_{p=1}^{r} J^0_{-p} \left| \frac{rs}{2} - p, \frac{r}{2} >_{jr,s} \right. +$$

$$+ \sum_{p=0}^{\frac{rs}{2}} J^-_{-p} \left| \frac{rs}{2} - p, \frac{r}{2} - 1 >_{jr,s} \right. + \frac{1}{2} \sum_{p=0}^{\frac{rs}{2}} J^-_{-p} \left| \frac{rs}{2} - p, \frac{r-1}{2} >_{jr,s} \right.,$$

$$|\lambda_{r,s} >, \quad (6.21)$$

where we have used the values of $i_\pm$ of eqs. (6.18) and (6.19). The vectors appearing in the right-hand side of eq. (6.21) are computed from the recursion relations (6.6) using the values of $i_\pm$ and $\delta_j$ previously determined. In appendix B we present the detailed computation of the vectors $|\lambda_{1,2} >$ and $|\lambda_{2,1} >$. For more general values of $r$ and $s$ the calculation, although more involved, follows the same lines.
7. The Knizhnik-Zamolodchikov equation

The Sugawara recursion relations obtained in the previous section can be alternatively derived from the existence of a mixed Virasoro-Kac-Moody singular vector. This vector is the same that gives rise to the well-known Knizhnik-Zamolodchikov equation [30], which plays a fundamental role in the determination of the correlation functions of the theory. In this section we shall present this alternative derivation of the descent equations following the same method used in ref. [12] for the $sl(2)$ algebra.

It follows from the $L_{-1}$ expression (i.e. from eq. (6.1) with $n = -1$) and from the highest weight conditions (2.13) for the vector $| j_2, t >$ that:

$$
\left[ \frac{t}{2} L_{-1} - J_{-1}^+ J_0^- - J_{-1}^0 J_0^+ + \frac{1}{2} j_{-1}^+ j_0^- \right] | j_2, t > = 0 .
$$
(7.1)

As $J_0^0 | j_2, t > = j_2 | j_2, t >$, one can obtain from (7.1) the following equation:

$$
e^{zL_{-1} + xJ_0^- + \theta_j} \phi_{j_1} (-z, -x, -\theta) \left[ \frac{t}{2} L_{-1} - J_{-1}^+ J_0^- - 2 j_2 J_{-1}^0 + \frac{1}{2} j_{-1}^+ j_0^- \right] | j_2, t > = 0 .
$$
(7.2)

In the derivation of eq. (7.2) from (7.1) we have multiplied the latter by $e^{zL_{-1} + xJ_0^- + \theta_j} \phi_{j_1} (-z, -x, -\theta)$. Moreover, commuting $\phi_{j_1} (-z, -x, -\theta)$ with the currents in the left-hand side of eq. (7.2), one finds:

$$
e^{zL_{-1} + xJ_0^- + \theta_j} G_{j_1,j_2}(z, x, \theta) \phi_{j_1} (-z, -x, -\theta) | j_2, t > = 0 ,
$$
(7.3)

where $G_{j_1,j_2}(z, x, \theta)$ is an operator whose explicit form can be obtained from eq. (2.30). Using this last equation one gets:

$$
G_{j_1,j_2}(z, x, \theta) = \frac{t}{2} (L_{-1} + \partial_z) - (J_{-1}^+ + z^{-1} \tilde{D}_{j_1}^+) (J_0^- - \tilde{D}_{j_1}^-) - 2 j_2 (J_{-1}^0 + z^{-1} \tilde{D}_{j_1}^0) + \frac{1}{2} (J_{-1}^+ + z^{-1} \tilde{d}_{j_1}^+) (J_0^- - \tilde{d}_{j_1}^-) ,
$$
(7.4)

where the operators $\tilde{D}_{j_1}^a$ and $\tilde{d}_{j_1}^\alpha$ are obtained from those of eq. (2.26) by changing
\( x \rightarrow -x \) and \( \theta \rightarrow -\theta \), i.e. they are given by:

\[
\begin{align*}
\tilde{D}_j^0 &= -x \partial_x - \frac{1}{2} \theta \partial_\theta + j \\
\tilde{D}_j^+ &= x^2 \partial_x - 2 j x + \theta x \partial_\theta \\
\tilde{D}_j^- &= -\partial_x \\
\tilde{d}_j^+ &= x \partial_\theta - \theta x \partial_x + 2 j \theta \\
\tilde{d}_j^- &= -\partial_\theta + \theta \partial_x \, .
\end{align*}
\] (7.5)

On the other hand, it is proved in appendix C that the two vectors \(|\Lambda\rangle\) and \(|\widetilde{\Lambda}\rangle\) defined as:

\[
|\Lambda\rangle = \phi_{j_2}(z, x, \theta) \mid j_1, t >
\]

\[
|\widetilde{\Lambda}\rangle = e^{zL-1 + x\tilde{J}_0^- + \theta\tilde{j}_0^-} \phi_{j_1}(-z, -x, -\theta) \mid j_2, t > ,
\]

satisfy the same set of defining constraints and, therefore, they can be identified. It is interesting at this point to notice that the vector \(|\widetilde{\Lambda}\rangle\) can be generated in the left-hand side of (7.3) by inserting the exponential \(e^{zL-1 + x\tilde{J}_0^- + \theta\tilde{j}_0^-}\) and its inverse. Using the \(|\Lambda\rangle \equiv |\widetilde{\Lambda}\rangle\) identification, one arrives at:

\[
e^{zL-1 + x\tilde{J}_0^- + \theta\tilde{j}_0^-} G_{j_1, j_2}(z, x, \theta) e^{-zL-1 - x\tilde{J}_0^- - \theta\tilde{j}_0^-} \phi_{j_2}(z, x, \theta) \mid j_1, t > = 0 .
\] (7.7)

To proceed further with the calculation one has to conjugate the operator \(G_{j_1, j_2}(z, x, \theta)\). The conjugation of the currents with \(e^{x\tilde{J}_0^- + \theta\tilde{j}_0^-}\) was given in eq. (2.27). Moreover, the behaviour of the derivatives (7.5) under conjugation is easy to obtain from their explicit expressions. After a simple calculation one gets:

\[
\begin{align*}
e^{x\tilde{J}_0^- + \theta\tilde{j}_0^-} \tilde{D}_j^0 e^{-x\tilde{J}_0^- - \theta\tilde{j}_0^-} &= \tilde{D}_j^0 + x\tilde{J}_0^- + \frac{1}{2} \theta \tilde{j}_0^- \\
e^{x\tilde{J}_0^- + \theta\tilde{j}_0^-} \tilde{D}_j^+ e^{-x\tilde{J}_0^- - \theta\tilde{j}_0^-} &= \tilde{D}_j^+ - x^2\tilde{J}_0^- - \theta x \tilde{j}_0^- \\
e^{x\tilde{J}_0^- + \theta\tilde{j}_0^-} \tilde{D}_j^- e^{-x\tilde{J}_0^- - \theta\tilde{j}_0^-} &= \tilde{D}_j^- + \tilde{J}_0^- \\
e^{x\tilde{J}_0^- + \theta\tilde{j}_0^-} \tilde{d}_j^+ e^{-x\tilde{J}_0^- - \theta\tilde{j}_0^-} &= \tilde{d}_j^+ + 2\theta x \tilde{J}_0^- - x\tilde{j}_0^- \\
e^{x\tilde{J}_0^- + \theta\tilde{j}_0^-} \tilde{d}_j^- e^{-x\tilde{J}_0^- - \theta\tilde{j}_0^-} &= \tilde{d}_j^- + \tilde{j}_0^- - 2\theta \tilde{J}_0^- .
\end{align*}
\] (7.8)
In eq. (7.7) one must also conjugate with the operator \( e^{zL_{-1}} \). This conjugation does not affect the derivatives (7.5) and the zero-mode currents \( J_0^a \) and \( j_0^\alpha \). This result is due to the fact (see eq. (2.9)) that \( [L_{-1}, J_0^a] = [L_{-1}, j_0^\alpha] = 0 \). Moreover, one can easily establish that:

\[
e^{zL_{-1}} (L_{-1} + \partial_z) e^{-zL_{-1}} = \partial_z
\]

\[
e^{zL_{-1}} J_{-1}^a e^{-zL_{-1}} = \sum_{p=1}^{+\infty} J_{-p}^a z^{p-1}
\]

\[
e^{zL_{-1}} j_{-1}^\alpha e^{-zL_{-1}} = \sum_{p=1}^{+\infty} j_{-p}^\alpha z^{p-1}.
\]

(7.9)

On the other hand, let us define the “negative” part of the currents as:

\[
\tilde{J}^+(z) \equiv \sum_{p=1}^{+\infty} J_{-p}^+ z^{p-1} \quad \quad \quad \quad \quad \quad \quad \tilde{j}^+(z) \equiv \sum_{p=1}^{+\infty} j_{-p}^+ z^{p-1}
\]

\[
\tilde{J}^-(z) \equiv \sum_{p=0}^{+\infty} J_{-p}^- z^{p-1} \quad \quad \quad \quad \quad \quad \quad \tilde{j}^-(z) \equiv \sum_{p=0}^{+\infty} j_{-p}^- z^{p-1}
\]

(7.10)

\[
\tilde{J}^0(z) \equiv \sum_{p=1}^{+\infty} J_{-p}^0 z^{p-1}.
\]

Using these results, one can rewrite eq. (7.7) as:

\[
\hat{G}_{j_1,j_2} (z, x, \theta) \phi_{j_2} (z, x, \theta) \mid j_1, t > = 0,
\]

(7.11)

where \( \hat{G}_{j_1,j_2} (z, x, \theta) \) is the operator:

\[
\hat{G}_{j_1,j_2} (z, x, \theta) = \frac{t}{2} \partial_z + z^{-1} \left[ D_{j_2}^+ D_{j_2}^- - 2j_1 D_{j_2}^0 - \frac{1}{2} d_{j_2}^+ d_{j_2}^- \right] - \tilde{J}^+ D_{j_2}^- - 2\tilde{J}^0 D_{j_2}^0 - \tilde{J}^- D_{j_2}^+ + \frac{1}{2} \tilde{j}^+ d_{j_2}^- - \frac{1}{2} \tilde{j}^- d_{j_2}^+.
\]

(7.12)

Notice that the derivatives (2.26) (and not those defined in eq. (7.5)) appear in \( \hat{G}_{j_1,j_2} (z, x, \theta) \). If we now substitute the expansion of \( \phi_{j_2} (z, x, \theta) \mid j_1, t > \) written
in eqs. (2.33) and (2.47) and the expression (2.26) of the derivatives, one can easily obtain the recursion relations (6.6) from eq. (7.11). This is the result we wanted to demonstrate.

8. Conclusions and outlook

Let us now summarize our main results. We have been able to set up a formalism in which the fusion of general osp(1|2) Verma module can be properly defined. The isotopic dependence of the primary fields, together with the differential realization (2.26) of the finite algebra, are the crucial ingredients of our approach. Using the \((x, \theta)\) dependence of the three point function (eq. (3.17)), we have obtained a set of singular vector decoupling conditions (eqs. (3.27) and (3.28)) that encode the constraint induced in the coupling of three Verma modules when one of them is reducible. From these singular vector decoupling conditions, we have found two sets of fusion rules that the product of primary operators corresponding to admissible representations must obey. Moreover, we have obtained the action of the currents of \(\mathcal{A}_+\) on the fusion states \(|n, m >_{j_3}\) (i.e. the descent equations (5.5) and (5.6)). These \(|n, m >_{j_3}\) vectors satisfy a set of recursion relations (eq. (6.6)) that can be derived either from the Sugawara energy-momentum tensor (as in section 6) or, as was demonstrated in section 7, from the Knizhnik-Zamolodchikov equation.

At first sight our results are very similar to the ones found for the \(sl(2)\) current algebra [11, 12]. It is interesting to point out, however, the relevant rôle played in our osp(1|2) case by the Grassmann parity of the representations. Indeed, the relative Grasssmann parity of the highest weight vectors involved in the fusion appears explicitly in the decoupling conditions (3.27) and (3.28), in the descent equations (5.5) and (5.6), and in the recursion relations (6.6). As a consequence, the parity of the \([r_3, s_3]\) representation resulting from the fusion of the two admissible representations \([r_1, s_1]\) and \([r_2, s_2]\) is fixed by eqs. (4.17) and (4.28) and, to obtain the singular vectors of the algebra, the parameter \(\delta_j\) must be carefully adjusted as in eq. (6.17).
In order to construct a well-defined osp(1|2) CFT for general isospins and levels, one should be able to define the conformal blocks of the model. For the admissible osp(1|2) representations, one expects that this could be done in the framework of a free field realization. This is actually what occurs in the $sl(2)$ theory [13, 31]. In the osp(1|2) case we have at our disposal all the elements needed to define a free field realization of the conformal blocks for the correlators of fields associated to admissible representations. In fact, in the free field realization of osp(1|2), there exist two screening fields, one of them is local in the free fields [21], while the other is non-local [25]. For general admissible representations these two screening operators would be needed to represent the blocks, while, as shown in ref. [25], when $k \in \mathbb{Z}_+$ and the isospins are integer or half-integer, only the local screening is necessary. In order to deal with the expectation values of non-local powers of the fields that would appear in the correlators of the general case, one can make use of the fractional calculus techniques [17], conveniently adapted, as in eq. (3.19), to include fermionic variables.

One would expect to find in this free field approach a more concrete operator implementation of the quantum hamiltonian reduction. The two fusion rules found in section 4 should also appear in this formalism, as it occurs for the $sl(2)$ case [31]. Moreover, it should be possible to characterize completely the intermediate channels corresponding to both set of fusion rules. In the approach of ref. [31], the second $sl(2)$ fusion rule is obtained by the overscreening mechanism, which is a consequence of the freedom that exists, when the level is rational, in the election of the number of screening operators for a given correlator. The Grassmann parity assignments of eqs. (4.17) and (4.28), together with the fact that the two osp(1|2) screening operators are fermionic, are a hint which seems to indicate that, in our osp(1|2) case, the different screening prescriptions could generate both types of fusion rules. In order to get a definitive conclusion on this question more work is needed. We expect to report on this matter and on other related subjects in a near future.
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APPENDIX A

In this appendix we shall study the characters of the \( \text{osp}(1|2) \) current algebra. For an irreducible Verma module whose highest weight vector has isospin \( j \), the character \( \lambda_j(z, \tau) \) is defined as:

\[
\lambda_j(z, \tau) = \text{Tr}_j \left[ q^{L_0 - \frac{c}{24}} w^{J_0^3} \right],
\]

(A.1)

where \( q \) and \( w \) are two variables related to the modular parameter \( \tau \) and to the coordinate \( z \) by means of the expressions:

\[
q = e^{2\pi i \tau}, \quad w = e^{2\pi i z}.
\]

(A.2)

The trace in (A.1) must be taken over the module \( V^{(j,t)} \). Its explicit expression can be obtained by evaluating the action of the operator \( q^{L_0 - \frac{c}{24}} w^{J_0^3} \) on the states \( \{|m_i^a\} : j \rangle \), defined in section 2, that span \( V^{(j,t)} \). Since \( L_0 \) and \( J_0^3 \) act diagonally on these states, the trace (A.1) can be easily computed. One gets the following expression for \( \lambda_j(z, \tau) \):

\[
\lambda_j(z, \tau) = q^{h_j - \frac{c}{12}} w^j \frac{\prod_{n=1}^{\infty} (1 + q^n w^{\frac{1}{2}}) \prod_{n=0}^{\infty} (1 + q^n w^{-\frac{1}{2}})}{\prod_{n=1}^{\infty} (1 - q^n) \prod_{n=1}^{\infty} (1 - q^n w) \prod_{n=0}^{\infty} (1 - q^n w^{-1})}.
\]

(A.3)

By using the identities:

\[
\frac{\prod_{n=1}^{\infty} (1 + q^n w^{\frac{1}{2}})}{\prod_{n=1}^{\infty} (1 - q^n w)} = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n w^{\frac{1}{2}}) (1 - q^{2n-1} w)}
\]

(A.4)

and

\[
\frac{\prod_{n=0}^{\infty} (1 + q^n w^{-\frac{1}{2}})}{\prod_{n=0}^{\infty} (1 - q^n w^{-1})} = \frac{1}{\prod_{n=1}^{\infty} (1 - q^{n-1} w^{-\frac{1}{2}}) (1 - q^{2n-1} w^{-1})},
\]

one obtains the final expression for the character.
one can reexpress \( \lambda_j(z, \tau) \) as:

\[
\lambda_j(z, \tau) = \frac{q^{\frac{2(j+\frac{1}{2})^2}{2k+3}} w^{j+\frac{1}{2}}}{\Pi(z, \tau)},
\]

where the function \( \Pi(z, \tau) \) appearing in the denominator of eq. (A.5) has the following infinite product representation:

\[
\Pi(z, \tau) \equiv q^{\frac{1}{24}} w^{\frac{1}{4}} \prod_{n=1}^{+\infty} (1-q^n) (1-w^{\frac{1}{2}} q^n) (1-w^{-\frac{1}{2}} q^n-1) (1-wq^{2n-1}) (1-w^{-1} q^{2n-1}).
\]

In the derivation of eq. (A.5) one must use that:

\[
h_j - c - \frac{1}{24} = \frac{2(j+\frac{1}{4})^2}{2k+3},
\]

which can be easily checked by a direct calculation from eqs. (2.11) and (2.15). It is convenient in what follows to rewrite \( \Pi(z, \tau) \) as an infinite sum. To achieve this objective it is enough to use the Watson quintuple product identity, which reads:

\[
\prod_{n=1}^{+\infty} (1-q^n) (1-w^{\frac{1}{2}} q^n) (1-w^{-\frac{1}{2}} q^n-1) (1-wq^{2n-1}) (1-w^{-1} q^{2n-1}) = \sum_{m=-\infty}^{+\infty} (w^{3m} - w^{-3m-1}) q^{\frac{3m^2+2m}{2}}.
\]

Indeed, as a consequence of the identity (A.8), one can immediately verify that \( \Pi(z, \tau) \) can be written as:

\[
\Pi(z, \tau) = q^{\frac{1}{24}} w^{\frac{1}{4}} \sum_{m \in \mathbb{Z}} (w^{3m} - w^{-\frac{3m+1}{2}}) q^{\frac{3m^2+2m}{2}}.
\]

The main consequence of eq. (A.9) is the fact that \( \Pi(z, \tau) \) can be put as a difference
of two classical theta functions. In general, the latter are defined as:

\[
\Theta_{r,s}(z, \tau) = \sum_{m \in \mathbb{Z}} q^{s(m+\frac{r}{2})^2} w^{s(m+\frac{r}{2})}.
\]

(A.10)

From this definition it follows by inspecting the right-hand side of eq. (A.9) that:

\[
\Pi(z, \tau) = \Theta_{1,3}\left(\frac{z}{2}, \frac{\tau}{2}\right) - \Theta_{-1,3}\left(\frac{z}{2}, \frac{\tau}{2}\right).
\]

(A.11)

Let us now consider the case of admissible representations of the osp(1|2) current algebra. As was discussed in the main text, these representations appear when the level \(k\) is such that \(2k + 3 = \frac{p}{p'}\), where \(p\) and \(p'\) are two coprime integers and \(p + p' \in 2\mathbb{Z}\). The isospins \(j_{r,s}\) corresponding to these representations are labelled by two integers \(r\) and \(s\) which take values in the grid \(1 \leq r \leq p - 1\), \(0 \leq s \leq p' - 1\) with \(r + s \in 2\mathbb{Z} + 1\). The actual values of \(j_{r,s}\) are:

\[
4j_{r,s} + 1 = r - s \frac{p}{p'}.
\]

(A.12)

When \(j = j_{r,s}\), the Verma module \(V^{(j,t)}\) is not irreducible. The irreducible highest weight module for these isospins is obtained by taking the quotient of \(V^{(j,t)}\) by its maximum proper submodule. In fact, as we have discussed in section 4, for \(j = j_{r,s}\) the module \(V^{(j,t)}\) has two singular vectors with \(J_0^0\) eigenvalues \(j_{r,s} - \frac{r}{2}\) and \(j_{r,s} - \frac{r + 1}{2}\). These two vectors generate the maximum proper submodule of \(V^{(j,t)}\), which can be represented by means of the following embedding diagram:

\[
\begin{align*}
&\quad a(0) \quad \xrightarrow{\gamma} \\
&\quad \quad \quad \quad \quad \quad \quad b(0) \quad \rightarrow \quad a(1) \quad \rightarrow \quad b(1) \quad \rightarrow \quad a(2) \quad \rightarrow \quad \cdots \\
&\quad \quad \quad \quad \quad \quad \quad b(-1) \quad \rightarrow \quad a(-1) \quad \rightarrow \quad b(-2) \quad \rightarrow \quad a(-2) \quad \rightarrow \quad \cdots \\
\end{align*}
\]

where \(a(l)\) and \(b(l)\) are given by:

\[
a(l) \equiv j_{r-2lp,s} = \frac{r - 1}{4} - \frac{s}{4} p + \frac{l}{2},
\]

\[
b(l) \equiv j_{-r-2lp,s} = \frac{r - 1}{4} - \frac{s}{4} p + \frac{l}{2} - \frac{r}{2}.
\]

(A.13)
Each node in the above diagram represents a Verma module with $a(l)$ or $b(l)$ as the isospin of its highest weight state. An arrow connecting two spaces $E \rightarrow F$ means that the module $F$ is contained in the module $E$. The character of the irreducible module with isospin $j = j_{r,s}$ is constructed as an alternating sum of the form:

$$
\chi_{j_{r,s}}(z, \tau) = \sum_{l=-\infty}^{+\infty} \lambda_{a(l)}(z, \tau) - \sum_{l=-\infty}^{+\infty} \lambda_{b(l)}(z, \tau).
$$

(A.14)

Using eqs. (A.5) and (A.13) in the right-hand side of eq. (A.14), it is straightforward to prove that $\chi_{j_{r,s}}(z, \tau)$ can be written as a quotient of differences of theta functions. Actually, defining the constants $b_{\pm}$ and $a$ as:

$$
b_{\pm} = \pm p'r - ps
\quad
a = pp',
$$

(A.15)

the characters $\chi_{j_{r,s}}(z, \tau)$ can be put in the form:

$$
\chi_{j_{r,s}}(z, \tau) = \frac{\Theta_{b_{+},a}(\frac{z}{2p}, \frac{\tau}{2}) - \Theta_{b_{-},a}(\frac{z}{2p}, \frac{\tau}{2})}{\Pi(z, \tau)}.
$$

(A.16)

We are interested in analyzing the $z = 0$ behaviour of $\chi_{j_{r,s}}(z, \tau)$. It is easy to demonstrate that the denominator function $\Pi(\tau, z)$ vanishes linearly when $z \rightarrow 0$. A simple calculation shows that:

$$
\Pi(z, \tau) = i\pi z q^{\frac{1}{12}} \sum_{m \in \mathbb{Z}} (6m + 1) q^{\frac{3m^2 + m}{2}} + o(z^2).
$$

(A.17)

We shall see below that, in general, the numerator of the right-hand side of eq. (A.16) is non-vanishing. This fact implies that $\chi_{j_{r,s}}(z, \tau)$ will, in general, develop a simple pole in $z$ in the $z \rightarrow 0$ limit. The situation is very similar to the one found in ref. [29] for the admissible representations of the $sl(2)$ current algebra. In this latter case, a relation between the residues of the $sl(2)$ characters at the $z = 0$ pole and the Virasoro characters for the $c < 1$ minimal models was found. In
our case, one would expect to find the characters of the minimal supersymmetric models in the residue of \( \chi_{j,r,s}(z, \tau) \) at \( z = 0 \). This is actually what happens, as we shall shortly prove. First of all, let us rewrite the \( z \to 0 \) expansion (A.17) in a more convenient form. With this purpose in mind, let us recall the infinite product representation of the Dedekind \( \eta \)-function:

\[
\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).
\]  

(A.18)

Moreover, an identity due to Gordon [32] allows to express the sum appearing in the right-hand side of eq. (A.17) as an infinite product. In terms of \( \eta(\tau) \) and the Jacobi theta function \( \theta_2(0, \tau) \), the Gordon identity can be written as:

\[
q^{\frac{1}{24}} \sum_{m \in \mathbb{Z}} (6m + 1) q^{\frac{3m^2 + m}{2}} = 2 \left[ \frac{\eta(\tau)}{\theta_2(0, \tau)} \right]^4.
\]

(A.19)

The infinite product representation of \( \theta_2(0, \tau) \) can be obtained from its relation with the Dedekind function, namely:

\[
\frac{\theta_2(0, \tau)}{\eta(\tau)} = 2 q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 + q^n)^2.
\]

(A.20)

Using eq. (A.19), one can write the \( z \to 0 \) expansion of \( \Pi(z, \tau) \) as follows:

\[
\Pi(z, \tau) = 2i\pi z \left[ \frac{\eta(\tau)}{\theta_2(0, \tau)} \right]^4 + o(z^2).
\]

(A.21)

Let us now consider the difference of theta functions appearing in the numerator of the right-hand side of eq. (A.16). Using the definition of the functions \( \Theta_{b_{\pm,a}} \)}
(see eq. (A.10)), one can write this difference as:

$$\Theta_{b+,a}(\frac{z}{2p}, \frac{\tau}{2}) - \Theta_{b-,a}(\frac{z}{2p'}, \frac{\tau}{2}) =$$

$$= w_{4p'}^{b_+} q^{\frac{(b_+)^2}{8a}} \sum_{m \in \mathbb{Z}} q^{\frac{am^2+mb_+}{2}} \left( w^{\frac{am}{2p'}} - q^{\frac{z}{2}}(r+2p'tm) w^{-\frac{am}{2p'}-\frac{z}{2}} \right).$$

(A.22)

Taking $z = 0$ in eq. (A.22), one gets:

$$\Theta_{b+,a}(0, \frac{\tau}{2}) - \Theta_{b-,a}(0, \frac{\tau}{2}) = \sum_{m \in \mathbb{Z}} \left[ q^{\frac{(\lambda+2pp'm)^2}{8pp'}} - q^{\frac{(\bar{\lambda}+2pp'm)^2}{8pp'}} \right],$$

(A.23)

where $\lambda$ and $\bar{\lambda}$ are given by:

$$\lambda = ps - p'r \quad \bar{\lambda} = ps + p'r .$$

(A.24)

When $s \neq 0$, the right-hand side of eq. (A.23) is non-vanishing and, therefore, it makes sense to consider the residue of $\chi_{j_{r,s}}(z, \tau)$ at the point $z = 0$. Let us define in this case the following quantity:

$$\hat{\chi}_{r,s}(\tau) \equiv \left[ \frac{2\eta(\tau)}{\theta_2(0, \tau)} \right]^{\frac{1}{2}} \left[ \eta(\tau) \right]^{\frac{1}{2}} \lim_{z \to 0} \left\{ i\pi z \chi_{j_{r,s}}(z, \tau) \right\} .$$

(A.25)

As a consequence of our previous results (eqs. (A.21) and (A.23)), $\hat{\chi}_{r,s}(\tau)$ is equal to:

$$\hat{\chi}_{r,s}(\tau) = \left[ \frac{\theta_2(0, \tau)}{2\eta(\tau)} \right]^{\frac{1}{2}} \frac{\Theta_{b+,a}(0, \frac{\tau}{2}) - \Theta_{b-,a}(0, \frac{\tau}{2})}{\eta(\tau)} .$$

(A.26)

It is interesting to point out that for $1 \leq r \leq p - 1 \ , 1 \leq s \leq p' - 1$ and $r + s \in 2\mathbb{Z} + 1$, the functions of $\tau$ appearing in the right-hand side of eq. (A.26) are precisely the characters of the minimal supersymmetric models, with central charge $c = \frac{3}{2} \left( 1 - \frac{2(p-p')^2}{pp'} \right)$, in the Ramond sector. This is precisely the result we were looking for.
APPENDIX B

In this appendix we shall illustrate the algorithm to compute singular vectors described in section 6 by performing the explicit calculation of the simplest non-trivial cases. Let us, first of all, consider the case \( r = 1 \) and \( s = 2 \). The corresponding isospin is (see eq. (3.1)) \( j_{1,2} = -t/2 \). According to our general prescription we must take \( i_- = -1/2 \) and \( i_+ = 1/2 \) in the descent equations. For these values of \( i_\pm \), the numbers \( n \) and \( m \) appearing in the Sugawara recursion relations are restricted to the ranges \( 0 \leq n \leq 1 \) and \( -1/2 \leq m \leq 1/2 \) (see eq. (6.20)) and the vector \( |\lambda_{1,2}\rangle \) is given by:

\[
|\lambda_{1,2}\rangle = \frac{1}{2} J^-_{-1} |0, 0\rangle_{j_{1,2}} + J^0_{-1} |0, \frac{1}{2}\rangle_{j_{1,2}} + J^-_0 |1, -\frac{1}{2}\rangle_{j_{1,2}} + \frac{1}{2} J^0_0 |1, 0\rangle_{j_{1,2}}.
\]  
(B.1)

In order to obtain the vectors \( |n, m\rangle_{j_{1,2}} \) appearing in the right-hand side of eq. (B.1), one must solve the recursion relations (6.6). When \( j_3 = j_{1,2} = -t/2 \), eq. (6.6) for \( m = 0, \pm 1/2 \) gives rise to the following equations:

\[
\frac{1}{2} [(n + 1) t - 1] |n, -\frac{1}{2}\rangle_{j_{1,2}} = \sum_{p=1}^{n} J^+_{-p} |n - p, \frac{1}{2}\rangle_{j_{1,2}} + \sum_{p=1}^{n} J^0_{-p} |n - p, -\frac{1}{2}\rangle_{j_{1,2}} - \frac{1}{2} \sum_{p=1}^{n} J^-_{-p} |n - p, 0\rangle_{j_{1,2}}
\]

\[
\frac{t}{2} n |n, 0\rangle_{j_{1,2}} = \frac{1}{2} \sum_{p=1}^{n} J^+_{-p} |n - p, \frac{1}{2}\rangle_{j_{1,2}} + \frac{1}{2} \sum_{p=0}^{n} J^-_{-p} |n - p, -\frac{1}{2}\rangle_{j_{1,2}}
\]

\[
\frac{t}{2} (n - 1) |n, \frac{1}{2}\rangle_{j_{1,2}} = -\sum_{p=1}^{n} J^0_{-p} |n - p, \frac{1}{2}\rangle_{j_{1,2}} + \sum_{p=0}^{n} J^-_{-p} |n - p, -\frac{1}{2}\rangle_{j_{1,2}} + \frac{1}{2} \sum_{p=0}^{n} J^0_{-p} |n - p, 0\rangle_{j_{1,2}}.
\]  
(B.2)
It is a simple exercise to solve (B.2) recursively. One gets:

\begin{align*}
|0, \frac{1}{2} >_{j_1,2} &= -\frac{1}{t} j_0^- |0, 0 >_{j_1,2} \\
|1, -\frac{1}{2} >_{j_1,2} &= \frac{1}{(1-2t)} \left[ \frac{2}{t} j_1^+ j_0^- + j_0^+ \right] |0, 0 >_{j_1,2} \\
|0, 0 >_{j_1,2} &= -\frac{1}{t^2} \left[ j_1^+ j_0^- + \frac{1}{2} \left( 2 j_0^- J_{-1}^+ j_0^- + t j_0^- J_{-1}^+ \right) \right] |0, 0 >_{j_1,2} .
\end{align*}

(B.3)

Taking \(|0, 0 >_{j_1,2} = |j_{1,2}, t >\) and substituting eq. (B.3) in eq. (B.1), we arrive at:

\begin{align*}
|\lambda_{1,2} > &= -\frac{1}{t^2} \left[ J_0^- J_{-1}^+ j_0^- + (1-t) j_0^+ j_0^- + \frac{1}{2} \left( 2 j_0^- J_{-1}^+ j_0^- + t j_0^- J_{-1}^+ \right) \right] |j_{1,2}, t > .
\end{align*}

(B.4)

It can be checked directly that the vector (B.4) satisfies all the conditions required to an osp(1|2) singular vector. Moreover, up to a constant, the vector (B.4) coincides with the one obtained from the MFF expression (eq. (3.3)).

In the same way, one could work out the \(r = 2, s = 1\) case. We shall limit ourselves here to write the final result for the corresponding vector \(|\lambda_{2,1} >:\)

\begin{align*}
|\lambda_{2,1} > &= \frac{4}{t^2 - 1} \left[ J_{-1}^+ (J_0^-)^2 - j_{-1}^- j_0^- J_0^- - (t+1) J_{-1}^+ J_{-1}^- + \frac{1}{2} (t+1) j_{-1}^- j_0^- - \frac{1}{4} (t^2 - 1) J_{-1}^- \right] |j_{2,1}, t > .
\end{align*}

(B.5)
APPENDIX C

In our derivation of the descent equations from the Knizhnik-Zamolodchikov equations in section 7 we have used the fact that the vectors $|\Lambda>$ and $|\tilde{\Lambda}>$, defined in eq. (7.6), can be identified. In this appendix we shall verify that both states satisfy the same set of covariance constraints and, therefore, they should be considered as identical.

The non-trivial constraints satisfied by the state $|\Lambda>$ can be obtained from those verified by the highest weight $|j_1, t>$. The latter are determined from the left ideal of $A$ annihilating $|j_1, t>$, namely:

\[
\begin{aligned}
(J^0_0 - j_1 | j_1, t > &= (L_0 - j_1 | j_1, t > = 0 \\
J^\alpha_n | j_1, t > &= j^\alpha_n | j_1, t > = 0, \quad \forall (J^\alpha_n, j^\alpha_n) \in A_+ .
\end{aligned}
\]  

Multiplying by $\phi_{j_2}(z, x, \theta)$ the equations in (C.1) and commuting it with the operators that multiply the state $|j_1, t>$ in the constraints (C.1), we get the following conditions satisfied by $|\Lambda>$:

\[
\begin{aligned}
(J^0_0 - D^0_{j_2} - j_1 | \Lambda > &= (L_0 - z\partial_z - h_1 - h_2 | \Lambda > = 0 \\
(J^\alpha_n - z^n D^\alpha_{j_2} | \Lambda > &= (j^\alpha_n - z^n d^\alpha_{j_2} | \Lambda > = 0, \quad \forall (J^\alpha_n, j^\alpha_n) \in A_+ .
\end{aligned}
\]

We are going to prove that $|\tilde{\Lambda}>$ also satisfies eq. (C.2). Actually, the constraints verified by $|\tilde{\Lambda}>$ can be derived following steps similar to those employed to obtain (C.2). One starts, in this case, from the constraints satisfied by $|j_2, t>$ and multiplies them by $\phi_{j_1}(-z, -x, -\theta)$. After commuting this field with the operators that annihilate $|j_2, t>$ and performing a conjugation with $e^{zL_{-1} + xJ_0^0 + \theta j_0}$,
one gets:

\[ e^{zL_{-1} + xJ^0_0 + \theta j^0_0} (J^0_0 - \tilde{D}^0_{j_1} - j_2) e^{-zL_{-1} + xJ^0_0 - \theta j^0_0} |\tilde{\Lambda} > = 0 \]
\[ e^{zL_{-1} + xJ^0_0 + \theta j^0_0} (L_0 - z\partial_z - h_1 - h_2) e^{-zL_{-1} + xJ^0_0 - \theta j^0_0} |\tilde{\Lambda} > = 0 \]
\[ e^{zL_{-1} + xJ^a_n + \theta j^a_n} (J^a_n - (-1)^n z^n \tilde{D}^a_{j_1}) e^{-zL_{-1} + xJ^0_0 - \theta j^0_0} |\tilde{\Lambda} > = 0 \]
\[ e^{zL_{-1} + xJ^a_n + \theta j^a_n} (J^a_n - (-1)^n z^n \tilde{d}^a_{j_1}) e^{-zL_{-1} + xJ^0_0 - \theta j^0_0} |\tilde{\Lambda} > = 0 \]

\[ \forall (J^a_n, j^a_n) \in \mathcal{A}_+ . \] (C.3)

In eq. (C.3), the operators \( \tilde{D}^a_{j_1} \) and \( \tilde{d}^a_{j_1} \) are the ones defined in section 7 (see eq. (7.5)). We claim that eq. (C.3) implies eq. (C.2) with \( |\Lambda > \) substituted by \( |\tilde{\Lambda} > \).

Let us consider, first of all, the \( J^0_0 \) constraint. Using the conjugation properties of \( J^0_0 \) (section 2, eq. (2.27)) and \( \tilde{D}^0_{j_1} \) (section 7, eq. (7.8)), one can convert the first eq. in (C.3) into:

\[ (J^0_0 - \tilde{D}^0_{j_1} - j_2) |\tilde{\Lambda} > = 0 . \] (C.4)

As \( \tilde{D}^0_{j_1} + j_2 = D^0_{j_2} + j_1 \), eq. (C.4) can be written as:

\[ (J^0_0 - D^0_{j_2} - j_1) |\tilde{\Lambda} > = 0 , \] (C.5)

which is the first equation (C.2) with \( |\tilde{\Lambda} > \) instead of \( |\Lambda > \), as claimed. Similarly we can demonstrate that the \( L_0 \) constraints satisfied by \( |\Lambda > \) and \( |\tilde{\Lambda} > \) are the same. Indeed, as \( L_0 - z\partial_z \) is invariant under conjugation, one has:

\[ (L_0 - z\partial_z - h_1 - h_2) |\tilde{\Lambda} > = 0 . \] (C.6)

Let us consider from now on the constraints induced by the elements of \( \mathcal{A}_+ \). First of all, we notice that, when \( n \geq 0 \), the result of conjugating with \( L_{-1} \) the
currents $J_n^a$ and $j_n^\alpha$ is:

\[ e^{zL_{-1}} J_n^a e^{-zL_{-1}} = \sum_{p=0}^{n} (-1)^p z^p \binom{n}{p} J_{n-p}^a \]

\[ e^{zL_{-1}} j_n^\alpha e^{-zL_{-1}} = \sum_{p=0}^{n} (-1)^p z^p \binom{n}{p} j_{n-p}^\alpha . \]  

(C.7)

Using eqs. (C.7), (2.27) and (7.8), the $J_n^-$ constraint in (C.3) can be written as:

\[ \sum_{p=0}^{n-1} (-1)^p z^p \binom{n}{p} J_{n-p}^- | \tilde{\Lambda} > + (-1)^n z^n D_{j_2}^- | \tilde{\Lambda} > = 0 . \]  

(C.8)

Putting $n = 1$ in eq. (C.8), we get $(J_1^- - zD_{j_2}^-) | \tilde{\Lambda} > = 0$, which implies that $| \tilde{\Lambda} >$ and $| \Lambda >$ satisfy the same $J_n^-$ constraint for $n = 1$. It is not difficult to generalize this result for an arbitrary value of $n$. Let us apply the induction method and suppose that $| \tilde{\Lambda} >$ satisfies:

\[ J_{n-p}^- | \tilde{\Lambda} > = z^{n-p} D_{j_2}^- | \tilde{\Lambda} > , \quad \text{for } 1 < p < n . \]  

(C.9)

Making use of eq. (C.9) in eq. (C.8) one gets:

\[ \left[ \sum_{p=1}^{n-1} (-1)^p \binom{n}{p} \right] z^n D_{j_2}^- | \tilde{\Lambda} > + J_n^- | \tilde{\Lambda} > + (-1)^n z^n D_{j_2}^- | \tilde{\Lambda} > = 0 . \]  

(C.10)

As the sum in $p$ in eq. (C.10) is $-1 - (-1)^n$, it follows that $| \tilde{\Lambda} >$ verifies:

\[ (J_n^- - z^n D_{j_2}^-) | \tilde{\Lambda} > = 0 , \quad \text{for } n \geq 1 , \]  

(C.11)

which is the result we wanted to demonstrate.
One can proceed similarly with the other currents. To illustrate the procedure let us consider in detail the case of $j^-_n$. Using eqs. (C.7), (2.27) and (7.8) in eq.(C.3), it is straightforward to prove that:

$$ \sum_{p=0}^{n-1} (-1)^p z^p \binom{n}{p} (j^-_{n-p} - 2\theta J^-_{n-p}) |\Lambda > - (-1)^n z^n \tilde{d}^-_{j_1} |\Lambda >= 0. \quad (C.12) $$

For $n = 1$ eq. (C.12) reduces to:

$$ (j^-_1 - 2\theta J^-_1 + zd^-_{j_1}) |\Lambda >= 0, \quad \quad (C.13) $$

which, taking into account that $J^-_1 |\Lambda >= z D^-_{j_2} |\Lambda >$ (see eq. (C.11)), is equivalent to:

$$ (j^-_1 - 2\theta z D^-_{j_2} + zd^-_{j_1}) |\Lambda >= 0 . \quad (C.14) $$

An easy calculation, using the explicit expressions of the differential operators, shows that $2\theta D^-_{j_2} - \tilde{d}^-_{j_1} = d^-_{j_2}$. Substituting this result in eq. (C.14), one gets:

$$ (j^-_1 - zd_{j_2}) |\Lambda >= 0 , \quad (C.15) $$

which means that |$\Lambda >$ and |$\Lambda >$ satisfy the same $j^-_1$ constraint. To extend this result for arbitrary $n$, we proceed, as before, by induction. Assuming that the following equation holds:

$$ j^-_{n-p} |\Lambda > = z^{n-p} d^-_{j_2} |\Lambda > \quad \text{for } 1 < p < n , \quad (C.16) $$

together with eq. (C.11), the general $j^-_n$ constraint (eq. (C.12)) can be reduced to the form:

$$ j^-_n |\Lambda > - (-1)^n z^n (\tilde{d}^-_{j_2} - 2\theta D^-_{j_2} + \tilde{d}^-_{j_1}) |\Lambda > - z^n d^-_{j_1} |\Lambda >= 0 . \quad (C.17) $$
As $\tilde{d}^+_{j_2} - 2\theta D^+_{j_2} + \tilde{d}^-_{j_1} = 0$, it follows from eq. (C.17) that:

$$(j_n^- - z^n d^-_{j_2}) |\tilde{\Lambda} > = 0, \quad (n \geq 1).$$

(C.18)

Using similar arguments one can demonstrate that $|\tilde{\Lambda} >$ satisfies:

$$(J^0_n - z^n D^0_{j_2}) |\tilde{\Lambda} > = 0 \quad (n \geq 1)$$

$$(J^+_n - z^n D^+_{j_2}) |\tilde{\Lambda} > = 0 \quad (n \geq 0)$$

$$(j^+_n - z^n d^+_j) |\tilde{\Lambda} > = 0 \quad (n \geq 0).$$

(C.19)

This completes the proof of the equivalence of $|\Lambda >$ and $|\tilde{\Lambda} >$.

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