Constant-factor approximations for Capacitated Arc Routing without triangle inequality

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Abstract
Given an undirected graph with edge costs and edge demands, the Capacitated Arc Routing problem (CARP) asks for minimum-cost routes for equal-capacity vehicles so as to satisfy all demands. Constant-factor polynomial-time approximation algorithms were proposed for CARP with triangle inequality, while CARP was claimed to be NP-hard to approximate within any constant factor in general. Correcting this claim, we show that any factor \( \alpha \) approximation for CARP with triangle inequality yields a factor \( \alpha \) approximation for the general CARP.

1 Introduction
Golden and Wong [5] introduced the Capacitated Arc Routing problem in order to model the search for minimum-cost routes for vehicles of equal capacity that satisfy all “customer” demands. Herein, “customers” are often the roads of a road network and, hence, are modeled as edges of a graph with corresponding integer demands. The vertices of the graph can be thought of as road intersections.

Capacitated Arc Routing Problem (CARP)

Instance: An undirected graph \( G = (V, E) \), a vehicle depot vertex \( v_0 \in V \), edge costs \( c(e) \geq 0 \) and edge demands \( d(e) \geq 0 \) for every \( e \in E \), and a vehicle capacity \( W \).

Task: Find a set \( C \) of cycles in \( G \), each corresponding to the route of one vehicle and each passing through the depot vertex \( v_0 \), and a serving function \( s: C \to 2^E \) such that
1. \( \sum_{C \in \mathcal{C}} \sum_{e \in C} c(e) \) is minimized,
2. each cycle \( C \in \mathcal{C} \) serves a subset \( s(C) \) of edges of \( C \) such that \( \sum_{e \in s(C)} d(e) \leq W \), and
3. each edge \( e \) with \( d(e) > 0 \) is served by exactly one cycle in \( C \).
Well-known special cases of CARP are the NP-hard Rural Postman Problem [7], where the vehicle capacity is unbounded, and the polynomial-time solvable Chinese Postman Problem [3, 4], where the vehicle capacity is unbounded and all edges have positive demand.

Jansen [6] and Wøhlk [11] gave polynomial-time factor \( (7/2 - 3/W) \) approximation algorithms for CARP when the edge cost function satisfies the triangle inequality. That is, for any two edges \( \{u, v\} \) and \( \{v, w\} \) there is an edge \( \{u, w\} \) such that
\[
c(\{u, w\}) \leq c(\{u, v\}) + c(\{v, w\}).
\]

Golden and Wong [5] and Wøhlk [11] claimed that CARP is NP-hard to approximate within any constant factor \( \alpha > 0 \). However, a recent arc routing survey [1] pointed out that the argument leading to this claim is erroneous, thus calling for an alternative proof for the inapproximability of CARP or for a constant-factor approximation. We find the latter by proving the following theorem:

**Theorem 1.** CARP is polynomial-time self-reducible, mapping any instance \( I \) to an instance \( I' \) in such a way that

i) \( I' \) satisfies the triangle inequality and

ii) a factor-\( \alpha \) approximate solution for \( I' \) is polynomial-time transformable into a factor-\( \alpha \) approximate solution for \( I \).

In terms of approximation-preserving polynomial-time reductions [9], we more specifically show a \((1, 1)\) L-reduction from CARP without triangle inequality to CARP with triangle inequality.

Theorem 1 and the factor \( (7/2 - 3/W) \) approximation given by Jansen [6] and Wøhlk [11] for CARP with triangle inequality then immediately yields the following corollary, which answers Challenge 6 of the above-mentioned arc routing survey [1].

**Corollary 1.** There is a polynomial-time factor \( (7/2 - 3/W) \) approximation for Capacitated Arc Routing, even if the edge cost function does not respect the triangle inequality.

Before proving Theorem 1, we quickly recall the erroneous argument [5, 11] for the approximation hardness of CARP without triangle inequality.

## 2 Erroneous argument towards approximation hardness

Golden and Wong [5] and Wøhlk [11] claim that CARP without triangle inequality is NP-hard to approximate within any constant factor \( \alpha > 0 \). Their claim is based on the fact that the Traveling Salesperson problem (TSP) without triangle inequality is NP-hard to approximate within any constant factor \( \alpha > 0 \) [8].
Figure 1: Counterexample to correctness of the canonical reduction from TSP. The left shows an instance of TSP along with an optimal tour (solid edges) of cost $3 + \ell$. On the right, the resulting CARP instance is shown (edge costs are as in their TSP counterparts, solid thick edges have positive demand) along with an optimal tour of cost six (solid and thick edges). Scaling up $\ell$ shows that the CARP tour may be arbitrarily less costly than the TSP tour.

They use the following polynomial-time transformation from TSP to CARP. Given a TSP instance, split each vertex of the input TSP graph and join them by an edge of demand one and cost zero and set the vehicle capacity $W$ to be at least the number of input vertices. The remaining edges in the CARP instance inherit their cost from the input TSP instance and have demand zero.

Clearly, any solution for the input TSP instance translates into a solution for CARP with the same cost. However, the reverse is not true: while TSP allows every vertex to be visited at most once, CARP imposes no such restrictions. Hence, in order to reach a positive-demand edge from another positive-demand edge, always a shortest path may be used in the optimal CARP tour. A counterexample to the correctness of the above reduction is given in Figure 1.

The reduction from TSP to CARP used by Golden and Wong [5] is correct when reducing from TSP with triangle inequality. In this case, however, TSP is factor-3/2 approximable using the algorithm by Christofides [2] and, thus, the presented reduction does not imply inapproximability for CARP.

3 Constant-factor approximations for CARP without triangle inequality

In the following, we show how to obtain constant-factor approximations for CARP without triangle inequality. Therein, we assume that the input graph is a complete graph, since missing edges can be simulated by edges of cost $\infty$. First we adjust the edge costs so that the triangle inequality is satisfied and then apply any constant-factor approximation algorithm for CARP with triangle inequality [6, 11].

In order to transform the edge costs so that the triangle inequality holds, we set the cost of each edge $\{u, v\}$ to the cost of a shortest path between $u$ and $v$. For zero-demand edges, this transformation is correct: since edges can be traversed more than once, instead of using a zero-demand edge, an optimal
solution can always use a shortest path between its endpoints. For edges with positive demand, the key idea is to visit these edges only once when serving them. Whenever the edge is traversed without serving it, an optimal solution can again use the shortest path between the endpoints. We now formalize this idea.

We start by formally describing the transformation of an edge cost function $c$ into a cost function $c\downarrow$ that satisfies the triangle inequality for zero-demand edges and then into a cost function $c\downarrow\uparrow$ fully satisfying the triangle inequality.

**Definition 1.** Let $(G, c, d, W)$ be a CARP instance. We define the following modified edge cost functions.

$$c\downarrow: E(G) \rightarrow \mathbb{N}, \{u,v\} \mapsto \begin{cases} c(\{u,v\}) & \text{if } d(e) > 0 \\ \text{dist}_c(u,v) & \text{otherwise} \end{cases}$$

$$c\downarrow\uparrow: E(G) \rightarrow \mathbb{N}, \{u,v\} \mapsto \text{dist}_c(u,v).$$

Herein, dist$_c$ is the cost of a shortest path between $u$ and $v$ with respect to the cost function $c$. Finally, we use

$$R := \{e \in E(G) \mid d(e) > 0 \land c\downarrow(e) \neq c\downarrow\uparrow(e)\}$$

to denote the set of positive-demand edges with costs exceeding the length of the shortest path between its endpoints and

$$r := \sum_{e \in R} (c\downarrow(e) - c\downarrow\uparrow(e))$$

to denote the total cost decrease of the edges in $R$ from $c\downarrow$ to $c\downarrow\uparrow$.

It is easy to verify that $c\downarrow\uparrow$ satisfies the triangle inequality. Moreover, since any solution contains each edge in $R$ at least once, the following observation immediately follows.

**Observation 1.** Let $(G, c, d, W)$ be a CARP instance. Any feasible solution to $(G, c\downarrow\uparrow, d, W)$ of cost $w$ has cost at most $w - r$ in $(G, c\downarrow\uparrow, d, W)$.

Enforcing the triangle inequality on all edges with zero demand does not change the cost of an optimal solution:

**Lemma 1.** Let $(G, c, d, W)$ be a CARP instance.

i) Any feasible solution for $(G, c, d, W)$ is a feasible solution of at most the same cost for $(G, c\downarrow\uparrow, d, W)$ and

ii) any feasible solution for $(G, c\downarrow\uparrow, d, W)$ can be transformed into a feasible solution of the same cost for $(G, c, d, W)$ in polynomial time.

**Proof.** (i) is trivial, since $c\downarrow\uparrow(e) \leq c(e)$ for all edges $e \in E(G)$.

(ii) Let $(C, s)$ be a feasible solution for $(G, c\downarrow\uparrow, d, W)$. We obtain a modified set $C'$ of cycles in polynomial time as follows. In each cycle $C \in C$, replace each edge $\{u, v\}$ with $d(\{u, v\}) = 0$ by a shortest path between $u$ and $v$ with respect to $c$. Then, $(C', s)$ is a feasible solution for $(G, c, d, W)$ since edges and vertices may be shared between cycles and may be used multiple times. Moreover, by choice of $c\downarrow\uparrow$, the cost of the cycles $C'$ with respect to $c$ is the same as that of $C$. \qed
If we enforce the triangle inequality for all input edges, then we may assume that an optimal solution uses every edge with positive demand and modified cost at most once:

**Lemma 2.** Let \((G, c, d, W)\) be a CARP instance. Any feasible solution for \((G, c^\triangledown, d, W)\) can be transformed into a feasible solution \((C, s)\) with the same cost in polynomial time such that every edge in \(R\) is contained in exactly one cycle \(C\) of \(C\) and is contained in \(C\) exactly once.

**Proof.** Observe that, for each edge \(e := \{u, v\} \in R\), the condition \(c^\triangledown(e) \neq c^\triangledown(e)\) implies that there is a shortest path \(p_e\) between \(u\) and \(v\) with respect to \(c\) that does not contain \(e\) but has the same cost as \(e\) with respect to \(c^\triangledown\).

Thus, in any cycle \(C \in C\) that does not serve \(e\), we simply replace any occurrence \(e\) by \(p_e\) without increasing the cost of \(C\) with respect to \(c^\triangledown\).

For the cycle \(C \in C\) that serves \(e\), we replace all but one occurrence of \(e\) by \(p_e\), again without increasing the cost of \(C\) with respect to \(c^\triangledown\).

Clearly, these replacements work in polynomial time. \(\square\)

We now prove Theorem 1.

**Theorem 1.** CARP is polynomial-time self-reducible, mapping any instance \(I\) to an instance \(I'\) in such a way that

i) \(I'\) satisfies the triangle inequality and

ii) a factor-\(\alpha\) approximate solution for \(I'\) is polynomial-time transformable into a factor-\(\alpha\) approximate solution for \(I\).

**Proof.** Let \(I := (G, c, d, W)\) be a CARP instance and let OPT denote the cost of an optimal solution. The edge cost functions \(c^\triangledown\) and \(c^\triangledown\) can clearly be computed from \(c\) in polynomial time. Thus, \(I' := (G, c^\triangledown, d, W)\) is polynomial-time computable and satisfies the triangle inequality.

Let \((C^*, s^*)\) be an optimal solution to \((G, c^\triangledown, d, W)\) and let its cost be OPT. By Lemma 2, we may assume that \((C^*, s^*)\) contains every edge of \(R\) exactly once. Hence, \((C^*, s^*)\) is a solution of cost OPT + \(r\) for \((G, c^\triangledown, d, W)\) and, by Lemma 1, can be transformed into a solution \((C', s')\) of cost OPT + \(r\) for \((G, c, d, W)\). Moreover, \((C', s')\) is an optimal solution for \((G, c, d, W)\) since, by Lemma 1 and Observation 1, a cheaper solution of cost less than OPT + \(r\) for \((G, c, d, W)\) would imply a solution of cost less than OPT for \((G, c^\triangledown, d, W)\). It follows that OPT \(\geq\) OPT + \(r\).

Now, assume that \((C, s)\) is a solution for \((G, c^\triangledown, d, W)\) of cost \(\alpha \cdot\) OPT. We transform it into a solution of cost \(\alpha \cdot\) OPT for \((G, c, d, W)\) in polynomial time. Lemma 2 allows us to assume that \((C, s)\) contains every edge of \(R\) exactly once, it follows that \((C, s)\) is a solution of cost \(\alpha \cdot\) OPT + \(r\) for \((G, c^\triangledown, d, W)\) and, by Lemma 1, is polynomial-time transformable into a solution of the same cost for \((G, c, d, W)\). Finally, since OPT + \(r \leq\) OPT, it follows that \((C, s)\) has cost at most \(\alpha \cdot\) OPT. \(\square\)
4 Conclusion

We have shown that the triangle inequality is is not necessary for finding good approximate solutions to Capacitated Arc Routing, since one can almost always replace an edge by a shortest path.

Our proof can be carried out analogously for variants of CARP on directed graphs: set the cost of any arc \((u, v)\) to the cost of a shortest directed path from \(u\) to \(v\). However, it does not work for graphs that have a mixture of directed and undirected edges, which also appear in applications [10]: it is not clear whether the cost of an undirected edge \(\{u, v\}\) should be set to the length of a shortest path from \(u\) to \(v\) or from \(v\) to \(u\). It would be interesting to show approximation results for this problem variant.

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