Invariants of $D(q, p)$ singularities

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INTRODUCTION

The study of the geometry of non-isolated hypersurface singularities was begun by Siersma and his students ([16],[17],[14],[15]). The basic examples of such functions defining these singularities are the $A(d)$ singularities and the $D(q, p)$ singularities. The $A(d)$ singularities, up to analytic equivalence, are the product of a Morse function and the zero map, while the simplest $D(q, p)$ singularity is the Whitney umbrella. These are the basic examples, because they correspond to stable germs of functions in the study of germs of functions with isolated singularities. Given a germ of a function which defines a non-isolated hypersurface singularity at the origin, which in the appropriate sense, has finite codimension in the set of such germs, the singularity type of such germs away from the origin is $A(d)$ or $D(q, p)$. However, some of the basic invariants of the germs of type $D(q, p)$ have not been calculated yet. In this note we calculate the homotopy type of the Milnor fiber of germs of type $D(q, p)$, as well as their Lê numbers. The calculation of the Lê numbers involves the use of an incidence variety which may be useful for studying germs of finite codimension. The calculation shows that the set of symmetric matrices of kernel rank $\geq 1$ is an example of a hypersurface singularity with a Whitney stratification (given by the rank of the matrices) in which only one singular stratum gives a component of top dimension of the singular set of the conormal.

The results of this paper can be applied to the study of any function in which the generic singularity type is $A(d)$ or $D(p, q)$, thus to all the germs which are finitely determined in the sense of Pellikaan. This is because in the case of non-isolated singularities, the geometry of many strata may contribute to invariants at the origin. We describe such an application, which appeared in the recent paper of de Bobadilla and Gaffney ([2]), which computes the Euler invariant at the origin of the zero set of a function which has only singularities of type $A(k)$ or $D(q, p)$ off the origin.
1. The Milnor fiber and the Lê numbers of the $D(q,p)$ singularities

If $f$ has a non-isolated singularity of type $D(p(p+1)/2, p)$, then it is known that by a change of coordinates, $f$ has the normal form

$$f(x, y) = \left( \sum_{i \leq j} x_{i,j} y_{i} y_{j} \right) + y_{p+1}^2 + \ldots + y_{p+k}^2 = [y]^t [X] [y] + y_{p+1}^2 + \ldots + y_{p+k}^2.$$

Here $[X]$ is a symmetric matrix with diagonal entries $x_{i,i}$ and off diagonal entries $1/2 x_{i,j}$ and $n$, the dimension of the domain of $f$ is $p + k$, while the number of generators of the ideal $I = (y)$ which defines the singular locus of $f$ is $p + k$, while $q$ is the dimension of the singular set, and $n = p + k + q$. The smallest $q$ can be is $p(p+1)/2$. If $q > p(p+1)/2$, then the additional coordinates do not appear in the normal form. For the purposes of this paper, this normal form can be taken as a definition of this type of germ.

**Theorem (1.1)** Suppose $f : \mathbb{C}^n \to \mathbb{C}$ has a non-isolated singularity of type $D(p(p+1)/2, p)$, $n = p(p+1)/2 + p$. The homotopy type of the Milnor fiber of $f$ is that of the $S^{2p-1}$ sphere.

**Proof.** We use the technique that appears in the preprint of Fernandez de Bobadilla ([1]). We may choose coordinates so that $f$ is in normal form. Consider the set defined by $f = 1$. Since $f$ is a homogeneous polynomial its Milnor fiber is diffeomorphic to the set $M$ defined by $f = 1$. In turn, $M$ is the total space of the fibration $p$ defined by restricting the projection to $y$ to $M$. This is easy to check, as the differential of the map with components $(y, f)$ has maximal rank as long as $y \neq 0$, and this holds at all points of $M$. The base of the fibration is $(\mathbb{C}^p - 0)$, since $y \neq 0$. The fiber of $p$ over $(b)$ in $\mathbb{C}^p$ consists of the affine hyperplane in $\mathbb{C}^{p+1/2}$ defined by setting $y$ equal to $(b)$ in the equation defining $M$. Hence $M$ has a contractible fiber, and has the homotopy type of the base which is the $S^{2p-1}$ sphere.

**Corollary (1.2)** Suppose $f : \mathbb{C}^n \to \mathbb{C}$ has a non-isolated singularity of type $D(q,p)$. The homotopy type of the Milnor fiber of $f$ is that of the $S^{p+n-q-1}$ sphere.

**Proof.** If $n = q + p$, then the Milnor fiber is just the product of the Milnor fiber of the $f$ of Theorem 1.1 with $\mathbb{C}^{q-p(p+1)/2}$, so the homotopy type doesn’t change. If $n = q + p + k$, then the normal form for $f$ has $k$ square terms, so the Milnor fiber is the Milnor fiber of the case $n = q + p$ suspended $k$ times, which gives a $S^{p+n-q-1}$.

(Fernandez de Bobadilla has advised me by private coorespondence that he also has done this calculation.)

Now we turn to the Lê numbers.
The Lê numbers were introduced by Massey, ([11]) as a way of relating the Milnor fiber of a function with non-isolated singularities to the singularities of the function. For a survey of their properties and more details on their calculation, see [12]. They can be calculated in two ways. Let \( f : \mathbb{C}^n \to \mathbb{C}, 0 \), \( J(f) \) the jacobian ideal of \( f \). Consider the blowup \( B_{J(f)}(\mathbb{C}^n) \) of \( \mathbb{C}^n \) along \( J(f) \) with exceptional divisor \( E \). Since \( B_{J(f)}(\mathbb{C}^n) \) lies in \( \mathbb{C}^n \times \mathbb{P}^{n-1} \), we can intersect both \( E \) and \( B_{J(f)}(\mathbb{C}^n) \) with \( \mathbb{C}^n \times H \) where \( H \) is a linear subspace of \( \mathbb{P}^{n-1} \). We call \( H \) a plane on \( B_{J(f)}(\mathbb{C}^n) \). If \( H_i \) is a generic plane of codimension \( i \), then the projection of \( B_{J(f)}(\mathbb{C}^n) \cap H_i \) to \( \mathbb{C}^n \) is called the relative polar variety of \( f \) of codimension \( i \), denoted \( \Gamma_i(f) \). The projection of \( E \cap H_i \) to \( \mathbb{C}^n \) is a cycle called the Lê cycle of \( f \) of codimension \( i + 1 \). Sometimes we also refer to the cycles which are the components of the Lê cycle of \( f \) of codimension \( i + 1 \) as Lê cycles as well. The \( j \)-th cycle of codimension \( i \) is then denoted \( \Lambda_{i+1,j}(f) \). The Lê number of codimension \( i + 1 \) is the sum of the products of the multiplicity of the underlying sets with the degree of the component cycle.

From this description it is clear that the Lê cycles can also be constructed by looking at the intersection of \( \Gamma_i(f) \) with \( S(f) \) and calculating the degree of each component of codimension 1 in the polar variety.

The theory of integral closure is tied up with the Lê cycles, so we briefly discuss it.

Integral dependence is used in local analytic geometry to relate inequalities between analytic functions, stratification conditions, algebra and analytic invariants. The basic source for this is the work [8] of Monique Lejeune–Jalabert and Teissier. (See also [4], [7].)

Let \((X,0) \subset (\mathbb{C}^N,0)\) be a reduced analytic space germ. Let \( I \) be an ideal in the local ring \( \mathcal{O}_{X,0} \) of \( X \) at 0, and \( f \) an element in this ring. Then \( f \) is integrally dependent on \( I \) if one of the following equivalent conditions obtain:

(i) There exists a positive integer \( k \) and elements \( a_j \) in \( I \), so that \( f \) satisfies the relation \( f^k + a_1 f^{k-1} + \ldots + a_{k-1} f + a_k = 0 \) in \( \mathcal{O}_{X,0} \).

(ii) There exists a neighborhood \( U \) of 0 in \( \mathbb{C}^N \), a positive real number \( C \), representatives of the space germ \( X \), the function germ \( f \), and generators \( g_1, \ldots, g_m \) of \( I \) on \( U \), which we identify with the corresponding germs, so that for all \( x \) in \( X \) the following equality obtains:

\[ |f(x)| \leq C \max\{|g_1(x)|, \ldots, |g_m(x)|\}. \]

(iii) For all analytic path germs \( \phi : (\mathbb{C},0) \to (X,0) \) the pull–back \( \phi^* f \) is contained in the ideal generated by \( \phi^*(I) \) in the local ring of \( \mathbb{C} \) at 0.

If we consider the normalization \( \tilde{B} \) of the blowup \( B \) of \( X \) along the ideal \( I \) we get another equivalent condition for integral dependence. Denote the pull–back of the exceptional divisor \( D \) of \( B \) to \( \tilde{B} \) by \( \tilde{D} \).

(iv) For any component \( C \) of the underlying set of \( \tilde{D} \), the order of vanishing of the pullback of \( f \) to \( \tilde{B} \) along \( C \) is no smaller than the order of the divisor \( \tilde{D} \) along \( C \).
The elements \( f \) in \( \mathcal{O}_{X,0} \) that are integrally dependent on \( I \) form the ideal \( \bar{I} \), the integral closure of \( I \). Often we are only interested in the properties of the integral closure of an ideal \( I \); so we may replace \( I \) by an ideal \( J \) contained in \( I \) with the same integral closure as \( I \). Such an ideal \( J \) is called a reduction of \( I \).

It is easy to see that \( J \) is a reduction of \( I \) iff there exists a finite map \( \text{Bl}_1 X \to \text{Bl}_J X \).

Now we begin the calculation of the Lê numbers.

**Lemma (1.3)** Suppose \( f: \mathbb{C}^n \to \mathbb{C} \) has a non-isolated singularity of type \( D(p(p+1)/2, p) \), \( n = p(p+1)/2 + p \). Then \( \lambda^{p(p+1)/2-i}(f,0) = 0 \) for \( i > p > 1 \).

**Proof.** Let \( J \) denote the ideal generated by \((y_1^2, \ldots, y_p^2)\). Then \( J \) is a reduction of \( I^2 \).

To check this just use the curve criterion—given \( y_i \) and \( y_j \), and a curve \( \phi(t) \), then the order of \( y_i y_j \circ \phi(t) \), denoted \( o(y_i y_j, \phi) \), is greater than or equal to \( \min\{o(y_i^2, \phi), o(y_j^2, \phi)\} \). Since the generators of \( I^2 \) not in \( J \) are in the integral closure of \( J \) all of \( I^2 \) is in the integral closure of \( J \).

Denote the partial derivatives of \( f \) with respect to the \( y \) or \( x \) variables by \( J_y(f) \) or \( J_x(f) \). Then from the last paragraph it follows that \( J_y(f) + J \) is a reduction of \( J(f) \). Now this implies that \( B_{J_y(f)}(\mathbb{C}^n) \) is finite over \( B_{J_y(f)+J}(\mathbb{C}^n) \); since the second blow-up has fiber over 0 of dimension at most the number of generators of \( J_y(f) + J \) less 1, which is \( 2p - 1 \), it follows the same holds for the first blow-up. Now the Lê cycles are the projection of the intersection of the exceptional divisor of \( B_{J_y(f)}(\mathbb{C}^n) \) with \( \mathbb{C}^n \times H \) where \( H \) is a generic plane in \( \mathbb{P}^{n-1} \). So if the codimension of \( H \) is \( 2p \) or more it follows that \( \mathbb{C}^n \times H \) will miss \( E \), hence the Lê cycles of codimension \( 2p + 1 \) and more must be empty. These are exactly the Lê cycles of dimension \( n - p - i = p(p+1)/2 - i, i > p > 1 \).

**Corollary (1.4)** Suppose \( f: \mathbb{C}^n \to \mathbb{C} \) has a non-isolated singularity of type \( D(q,p) \). Then \( \lambda^{q-i}(f,0) = 0 \) for \( q \geq i > p > 1 \).

**Proof.** The proof follows the same lines as above noting that \( J(f) \) has a reduction with \( n + p - q \) elements.

In particular, Corollary 1.4 shows \( \lambda^0(f,0) = 0 \), for \( p > 1 \), so there is no component of the exceptional divisor of \( J(f) \) over the origin. In fact, we can describe the components of the exceptional divisor of \( J(f) \) completely.

**Lemma (1.5)** Suppose \( f \) has a non-isolated singularity of type \( D(q,p) \). Then \( E \) the exceptional divisor of \( B_{J(f)}(\mathbb{C}^n) \) has only two components, one of which surjects to \( V(I) \), and the other to \( V(I) \cap \det[X] \).

**Proof.** Since \( E \) projects to \( V(J(f)) = V(I) \), at least one component of \( E \) surjects onto \( V(I) \), which is smooth, hence irreducible. Since on \( V(I) \), \( f \) has generically an \( A(q) \) singularity, only one component surjects.

At the generic point \( V(I) \cap \det[X] \), \( f \) has the type of a \( D(q,1) \) (a Whitney umbrella with additional square terms) so again there exists a unique component
of $E$ which surjects to $V(I) \cap \det[X]$. (That there is exactly one component follows from Proposition 2.2 of [5].)

Now we proceed by induction on $p$, suppose $p > 1$ and suppose that there exists $V$, a component of $E$, which maps into $\Sigma_2$, the sets of points on $V(I)$ where the kernel rank of $[X]$ is two or more. Suppose at the generic point of the image of $V$ the kernel rank of $[X]$ is $j$, where $j > 1$, since he kernel rank of $[X]$ is two or more. Then since the germs of $f$ along points of constant rank are analytically equivalent, $V$ must surject onto all points of kernel rank $\geq j$.

Picking a suitable transversal $H$ to $\Sigma_j$ at the generic point of complementary dimension, the germ of $f|H$ has a singularity of type $D(j(j+1)/2,j)$, where the dimension of $H$ is $j(j+1)/2 + n - q = n - \dim(\Sigma_j)$. Since a component of the exceptional divisor maps to points of kernel rank $j$, the generic fiber dimension over a point of kernel rank $j$ is just the dimension of $V$ less the codimension of the points of kernel rank $j$. Generically, the fiber of the component will be the fiber of $B_{J(f|H)}(H)$ over the origin, hence the dimension of this fiber is $(n-1) - \dim(\Sigma_j)$, which implies that it is a component of the exceptional divisor of $J(f|H)$. But this contradicts the fact proved in Corollary 1.4, that for $j > 1$ there is no component of the exceptional divisor that projects to the origin.

There are two types of Lê cycles, fixed and moving cycles. The fixed cycles are the images of the components of the exceptional divisor. Since the $A_f$ condition holds generically between the underlying sets of the fixed cycles and the open stratum, a dimension count shows that the corresponding component of the exceptional divisor is the conormal of the underlying set. Thus the moving cycles, obtained by intersecting these conormals with $C^n \times H$ where $H$ is a generic plane in $P^{n-1}$ whose codimension is greater than the generic fiber dimension of the conormal over its image, are just the polar varieties of the underlying sets of the Lê cycles. With this observation we can now calculate the Lê numbers of $f$. We do this in the simple case of Theorem 1.1, then extend to the general case, which is mostly re-labeling.

**Theorem (1.6)** Suppose $f : C^n \to C$ has a non-isolated singularity of type $D(p(p+1)/2,p)$, $n = p(p+1)/2 + p$. Then the Lê numbers of $f$ are

$$\lambda^{n-p-i}(f) = 2^i \binom{p}{p-i}, 0 \leq i \leq p,$$

$$\lambda^{n-p-i}(f) = 0, i > p > 1.$$ 

**Proof.** The second equation is lemma 1.3. The only fixed Lê cycles are $[V(I)]$ and $2[V(I) \cap \det[X]]$. (The coefficients are because a Morse singularity has Milnor number 1, and a Whitney umbrella has $\lambda^0 = 2$ and these are the types we get working at a generic point of the underlying set of each cycle, and slicing by a generic transverse plane of complementary dimension.) So $\lambda^{n-p}(f) = 1$. Since $V(I)$ is smooth, it has no polar varieties except itself; while the multiplicity at 0 of $V(I) \cap \det[X]$ is $p$, hence $\lambda^{n-p-1}(f) = 2p.$
As usual we assume $f$ in normal form. To compute the remaining Lè numbers, we first consider the relative polar varieties of $f$. The underlying set of the Lè cycle of dimension $n - p - i$ is gotten by intersecting a relative polar variety of dimension $n - p - i + 1$ with $S(f)$. So consider the set of $p + i - 1$ linear combinations of the partial derivatives of $f$, $i > 1$. By imposing generic conditions and simplifying our expressions, we can assume they have the form

$$
\frac{\partial f}{\partial y_l} + q_l(y), p_j(y), 1 \leq l \leq p, 1 \leq j < i
$$

where $q_l$ and $p_j$ are quadratic forms in $y$, no $q_i$ in the span of the $p_j$.

The relative polar variety of $f$ of codimension $p + i - 1$ is the closure of the set of points where these expressions vanish where $y \neq 0$. Since all of our expressions are homogeneous, it follows that the underlying sets of the Lè cycles have a $\mathbb{C}^*$ action hence are cones. When we projectivize them we get sets of dimension $n - p - i - 1$ in $\mathbb{P}^{p(p-1)/2}$. We can compute the multiplicity of the Lè cycle at the origin by computing the multiplicity of this projective set, ie. its intersection number with a projective space of complementary dimension.

We can slightly change our normal form so that we can rewrite the first $p$ equations as

$$
2[X][y] = [q(y)].
$$

At this point we are going to make a construction which will allow us to find equations for a projective set with the same multiplicity as the projectivization of the underlying set of our Lè cycle.

Consider the map $G$ from $\mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^n$ given by

$$
G(x, y, \lambda) = (x, \lambda y).
$$

Notice that $G$ maps $\mathbb{C}^n \times 0 \rightarrow S(f)$. Pullback the $p + i - 1$ equations by $G$. We get:

$$
2[X][\lambda y] = \lambda^2[q(y)]
$$

$$
\lambda^2 p(y) = 0.
$$

The points on the set defined by these equations which map to the relative polar variety are the closure of the subset of points where $\lambda, y \neq 0$. These points satisfy the equations:

$$
2[X][y] = \lambda[q(y)]
$$

$$
p(y) = 0.
$$

So the points that map to the Lè cycle are defined by the closure of the points in the above set where $y \neq 0$, and $\lambda = 0$.

This gives the equations:

$$
[X][y] = [0]
$$
\( \mathbf{p}(\mathbf{y}) = 0. \)

The set defined by these equations is a union of components where one component is \( \mathbb{C}^{p(p+1)/2} \times \mathbb{P}^q \times 0 \), and the others map to the underlying set of the Lê cycle. In the components that map to the Lê cycle, the points where \( \mathbf{y} \neq 0 \) are dense; further if \( (\mathbf{x}, \mathbf{y}) \) satisfies the equations and \( \mathbf{y} \neq 0 \), then \( (\mathbf{x}, t\mathbf{y}) \) satisfies the equations for all \( t \). Further if \( (\mathbf{x}) \) is a point on the Lê cycle, then we may assume that if \( (\mathbf{x}) \) is generic, then the kernel rank of the matrix corresponding to \( (\mathbf{x}) \) is 1 hence the \( (\mathbf{x}, ty) \) is the whole inverse image of such a \( (\mathbf{x}) \).

These equations define a variety on \( \mathbb{C}^{p(p+1)/2} \times \mathbb{P}^q \), which maps by projection to \( \Lambda^{n-p-i}(f) \subset \mathbb{C}^{p(p+1)/2} \).

By the observation above, the map is generically 1-1.

Now, consider in \( \mathbb{P}^{p(p+1)/2} \times \mathbb{P}^q \) the set of points defined by \( [\mathbf{X}][\mathbf{y}] = 0 \), and \( \{p_j(y) = 0\} \). To calculate the multiplicity at the origin of \( \Lambda^{n-p-i}(f) \), it suffices to intersect this last set with \( n - p - i - 1 \) hyperplanes. The number of points realized in \( \mathbb{P}^{p(p+1)/2} \times \mathbb{P}^q \) will be the number of lines in the intersection of \( \Lambda^{n-p-i}(f) \) with the corresponding \( n - p - i - 1 \) hyperplanes in \( \mathbb{C}^n \), and each line counts one toward the multiplicity.

To calculate the number of points, we use the following result from Fulton ([3] p146 eg. 8.4.2)

Given \( H_1, \ldots, H_{n+m} \) hypersurfaces in \( \mathbb{P}^n \times \mathbb{P}^m \) of bidegree \((a_i, b_i)\), then

\[
\int [H_1] \ldots [H_{n+m}] = \sum a_1 \ldots a_{n} b_{j_1} \ldots b_{j_m}
\]

where \( i_1 < i_2 < \ldots < i_n \) and \( j_1 < j_2 < \ldots < j_m \).

In our situation the bi-degree of the equations coming from the matrix are \((1, 1)\), while the bidegree of the linear forms is \((1, 0)\), and the bidegree of the \( p_j \) is \((0, 2)\). The non-zero terms of the sum are gotten by using for the \( j_1, \ldots, j_{p-1} \) all \( i - 1 \) terms of bi-degree \((0, 2)\), no linear terms and \( p - i \) terms drawn from the matrix equation. The \( i \) part is determined by what remains. The number of such terms is \( \binom{p}{p-i} \), while the product of the bidegrees is always \( 2^{i-1} \). So the multiplicity of the underlying set of \( \Lambda^{n-p-i}(f) \) is \( 2^{i-1} \binom{p}{p-i} \). Since the multiplicity of the cycle is \( 2 \), we get \( \lambda^{n-p-i}(f) = 2^i \binom{p}{p-i} \), which finishes the proof.

It is interesting to note that the map induced by projection from \( \mathbb{C}^{p(p+1)/2} \times \mathbb{P}^q \) to \( \mathbb{C}^{p(p+1)/2} \) when restricted to the incidence variety, \( V = \{(x, t)|X|l = 0\} \) is a resolution of the variety in \( \mathbb{C}^{p(p+1)/2} \) of symmetric matrices of kernel rank \( \geq 1 \).

**Corollary (1.7)** Suppose \( f : \mathbb{C}^n \to \mathbb{C} \) has a non-isolated singularity of type \( D(q, p) \). Then the Lê numbers of \( f \) are

\[
\lambda^{q-i}(f) = 2^i \binom{p}{p-i}, 0 \leq i \leq p,
\]

\[
\lambda^{q-i}(f) = 0, q \geq i > p > 1.
\]
Proof. Let \( q_1 = q - (p(p + 1)/2), k = n - q - p \). Then \( q_1 \) is the number of coordinates not appearing in the normal form for \( f \), and \( k \) is the number of coordinates which appear as square terms. Assume first \( q_1 = k = 0 \). Then we are in the situation of theorem 1.6; rewrite \( n - p - i \) as \( (p(p+1)/2) - i \). Now keep \( k = 0 \), and increase \( q_1 \). The effect on the Lê cycles is to multiply them by \( C^{q_1} \), hence the dimensions of the cycles are shifted up by \( q_1 \), so \( n - p - i \) becomes \( (p(p+1)/2) - i + q - 1 = q - i \). Now let \( k \) increase. The effect of adding disjoint square terms to the normal form is to leave the Lê cycles unchanged, so \( n - p - i \) can again be replaced by \( q - i \).

Massey showed that the alternating sum of the Lê numbers of \( f \) is the reduced Euler characteristic of the Milnor fiber of \( f \). It is a pleasant exercise to recover this result for the \( D(q,p) \) singularities. On the one hand, by Corollary 1.2, the reduced Euler characteristic is \((-1)^{p+n-q-1}(2-1)^p\); expanding \((2-1)^p\), we get

\[
(-1)^{p+n-q-1} \sum_{i=0}^{p} (-1)^{p-i} 2^i \binom{p}{p-i} = (-1)^{p+n-q-1} \sum_{i=0}^{p} (-1)^{i-p} 2^i \binom{p}{p-i}
\]

\[
= \sum_{i=0}^{p} (-1)^{(n-1)-(q-i)} \lambda^{q-i}(f).
\]

The computations in Theorem 1.6 also compute the polar multiplicities at the origin of the set of \( p \times p \) symmetric matrices of kernel rank \( \geq 1 \).

Corollary (1.8) The polar multiplicities at the zero matrix of the set of \( p \times p \) symmetric matrices of kernel rank \( \geq 1 \) are given by:

\[
m^{p(p+1)/2-i-1} = 2^i \binom{p}{p-i-1}, 0 \leq i < p
\]

\[
m^{p(p+1)/2-i-1} = 0, i \geq p
\]

Proof. The multiplicities are half the corresponding Lê numbers, since the Lê cycles have multiplicity 2.

These computations show another reason why the \( D(q,p) \) singularities are interesting. If we put a Whitney stratification on \( S(f) \) such that the images of the components of the exceptional divisor are a union of strata, then we must include many strata which themselves do not correspond to components of the exceptional divisor. These strata are not seen by the exceptional divisor.

Corollary 1.8 shows that the set of \( p \times p \) symmetric matrices of kernel rank \( \geq 1 \) are also interesting—there are many strata in a minimal Whitney stratification which do not correspond to components of the singular set of the conormal variety of the matrices of kernel rank \( \geq 1 \). (In fact, a \( Z \)-open dense set of the singular set of the conormal variety are those points which map to matrices of kernel rank 2.)
This is easy to see as this set is a hypersurface, hence by it is known by [7] that every component of the conormal modification over the matrices of kernel rank $\geq 2$ has dimension $p(p - 3)/2$, yet Corollary 1.8 shows that only over the matrices of kernel rank 2 can the fiber dimension of the conormal be large enough for this to be true.

Now we apply the material of this paper to the calculation of the Euler obstruction of the $D(q, p)$ and of the $p$ by $p$ symmetric matrices.

The Euler obstruction is an idea introduced by MacPherson ([10]) as a key step in developing the notion of the Chern class for singular spaces. Lê and Teissier showed that the Euler obstruction of a complex analytic germ at a point $x$ can be computed as the alternating sum of polar multiplicities of $X$ at $x$ ([9]). We use this result in our first computation.

**Proposition (1.9)** The Euler obstruction of the set of $p \times p$ matrices of kernel rank $\geq 1$ is 0 for $p$ even and 1 for $p$ odd.

**Proof.** Let $\Sigma_1(p)$ denote the set of symmetric $p \times p$ matrices of kernel rank $\geq 1$. We know that the reduced Euler characteristic of the Milnor fiber $M$ of a $D(p(p + 1)/2, p)$ singularity satisfies

$$\tilde{\chi}(M) = (-1)^{2p-1} = -1 = \sum_{i=0}^{n-1} (-1)^{n-1-i} \lambda^i(f)$$

$$= (-1)^{p-1} + \sum_{i=n-2p-1}^{n-p-1} (-1)^{n-1-i} 2m^i(\Sigma_1, 0).$$

Now there are two cases. If $p$ is even, then we have

$$0 = \sum_{i=n-2p-1}^{n-p-1} (-1)^{n-1-i} m^i(\Sigma_1, 0).$$

So the Euler obstruction is 0.

If $p$ is odd, then we have

$$-1 = \sum_{i=n-2p-1}^{n-p-1} (-1)^{n-1-i} m^i(\Sigma_1, 0).$$

$$1 = \sum_{i=n-2p-1}^{n-p-1} (-1)^{n-i} m^i(\Sigma_1, 0) = Eu(\Sigma_1, 0).$$

where the last equality is Cor 5.1.2 [9]. (The sum alternates, but the coefficient of the top dimensional term is positive.)

In [2] a formula is proved for the Euler obstruction for $X$ a hypersurface in terms of the Lê numbers of $f$, where $f$ defines $X$. Using this formula we can easily find the Euler obstruction of a $D(q, p)$ singularity. The formula is:

$$Eu(X, 0) = \chi(L, X, 0) + \sum_{i>0, j} (-1)^{n-i} b(|\Lambda^i_{j,F}(f)|, X) Eu(|\Lambda^i_{j,F}(f)|, 0).$$

Here $\Lambda^i_{j,F}(f)$ refers to the $j$-th component of the fixed part of the Lê cycle of $f$ of dimension $i$, while $b(|\Lambda^i_{j,F}(f)|, X)$ is the number of spheres in the homotopy
type of the complex link of $|\Lambda_{i,j,F}(f)| \cap H$ at $z_{i,j}$, where $z_{i,j}$ a generic point of $\Lambda_{i,j,F}(f)$ and $H$ is a plane of dimension complementary to $|\Lambda_{i,j,F}(f)|$ and transverse to it, while $\chi(L,X,0)$ denotes the Euler characteristic of the complex link of $X$ at 0. By a result of Massey’s ([13]), this is just $1 + (-1)^{n-m} m(\Gamma^1(f))$ as $m(\Gamma^1(f))$ is the number of spheres in the complex link of $X$ at the origin. Using this formula we can show:

**Proposition (1.10)** if $f: \mathbb{C}^n, 0 \to 0$ has a singularity of type $D(q,p)$ at the origin, $p > 1$, $X = f^{-1}(0)$, then $\text{Eu}(X) = 1 + (-1)^{n-q}$ if $p$ is even, and 1 if $p$ is odd.

**Proof.** We only have 2 fixed Lê cycles, $S(f)$ and $\Sigma_1$. Further, the relative polar curve is empty for $p > 1$ by Cor. 1.7. So we get:

$$\text{Eu}(X) = 1 + (-1)^{n-q} + (-1)^{n-q-1} \text{Eu}(\Sigma_1, 0).$$

The second term is the contribution of $S(f)$, while the third term is the contribution of $\Sigma_1$. Now the result follows by applying proposition 1.9.

The development of these ideas is continued in [2] where the Euler obstruction is computed for those germs which have $A(d)$ or $D(q,p)$ singularities except at the origin. The results of this paper play a key role in applying the formula of [2] to these examples.

**References**

[1] J. Fernandez de Bobadilla, *Answers to some Equisingularity Questions* preprint 2004.
[2] J. Fernandez de Bobadilla and T. Gaffney, *The Lê numbers of the square of a function and their applications*, preprint 2005.
[3] W. Fulton, “Intersection Theory,” Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge · Band 2, Springer–Verlag, Berlin, 1984.
[4] T. Gaffney, *Integral closure of modules and Whitney equisingularity*, Invent. Math. 107 (1992), 301–22.
[5] T. Gaffney, *The multiplicity of pairs of modules and hypersurface singularities*, Real and Complex Singularities (Sao Carlos, 2004) Trends in Mathematics, Birkhauser 2006, 143–168.
[6] T. Gaffney and R. Gassler, *Segre numbers and hypersurface singularities*, J. Algebraic Geom. 8 (1999), 695–736.
[7] T. Gaffney and D. Massey, *Trends in equisingularity*, London Math. Soc. Lecture Note Ser. 263 (1999), 207–248.
[8] M. Lejeune-Jalabert and B. Teissier, *Clôture integrale des ideaux et equisingularité, chapitre 1* Publ. Inst. Fourier (1974).
[9] D. T. Lê and B. Teissier, *Variétés polaires locales et classes de Chern des variétés singulières*, Ann. of Math. (2) 114 (1981), no. 3, 457–491.
[10] R. D. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math. (2) 100 (1974), 423–432.
[11] D. Massey, *The L varieties. I* Invent. Math. 99 (1990), no. 2, 357–376.
[12] D. Massey *Lê Cycles and Hypersurface Singularities*, Springer Lecture Notes in Mathematics 1615, (1995).
[13] D. Massey, *Numerical invariants of perverse sheaves*, Duke Math. J. 73 (1994), no. 2, 307–369.

[14] R. Pellikaan, *Hypersurface singularities and resolutions of Jacobi modules*, Thesis, Rijkuniversiteit Utrecht, 1985.

[15] R. Pellikaan, *Finite determinacy of functions with non-isolated singularities*, Proc. London Math. Soc. vol. 57, pp. 1-26, 1988.

[16] D. Siersma, *Isolated line singularities*, Singularities, Part 2 (Arcata, Calif., 1981), 485–496, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983.

[17] D. Siersma, *Singularities with critical locus a 1-dimensional complete intersection and transversal type A₁*, Topology Appl. 27 (1987), no. 1, 51–73.