Revisiting Quantum Volume Operator

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Abstract

In this paper we introduce the n-dimensional hypersurface quantum volume operator by using the n-dimensional holonomy variation formula. Instead of trying to construct the n-dimensional hypersurface volume operator by using the n-1 dimensional hypersurfac volume operators, as it is usually done in 3d case, we introduce the n-dimensional volume operator directly. We use two facts - first, that the area of the n-dimensional hypersurface of the n+1 dimensional manifold \( S \) is the volume of the n dimensional induced metric and secondly that the holonomy variation formula (3) is valid for the n-dimensional hypersurface in the n+1 manifold with connection values in any Lie algebra.

1 Introduction

The area of the n-dimensional hypersurface of the n+1 dimensional manifold \( S \) is the volume of the n dimensional induced metric and can be written as:

\[
A(S) = \int_U d^n u \sqrt{n_\alpha(u)n_\beta(u)E^a_j E^b_j}
\]

(1)

, where \( a, b, j = 1, ..., (n + 1) \)

The main goal of this paper is to introduce the quantum volume (area) operator for the volume of the n-dimensional hypersurface in n+1 dimensional space. Instead of trying to create the n-dimensional volume operator by using n-1 dimensional area electric flux operators as it is usually done in LQG (8), (9) we propose to introduce the volume operator directly. We will illustrate first the idea on the 3 dimensional volume of the 4 dimensional space first before moving to the n-dimensional hypersurface.

The paper is organized as follows. In the next section we derive the 3d volume operator by using the new approach. In section we generalize this approach for the volume operator of the n-dimensional hypersurface. The discussion section concludes the paper.
Let us consider the 4-dimensional Lorentzian manifold with the Ashtekar complex variables \(A_{\mu}^{I,J}\) with the values in \(sl(2, C) \otimes C = sl(2, C) \oplus \overline{sl(2, C)}\), and the corresponding conjugate momentum variable - complex valued fluxes \(\Pi_{\mu}^{I,J}\), where \(I, J, \mu = 1, ..., 4\). We consider a curve \(\gamma\) intersecting the 3d hypersurface of the 4 dimensional manifold. The variation of the holonomy along this curve with respect to the connection according to (3) will have the following form:

\[
\frac{\delta}{\delta A_{\mu}^{I,J}(x)} U(A, \gamma) = \int ds \dot{\gamma}^\alpha(s) \delta^4(\gamma(s), x) [U(A, \gamma_1) X_{I,J} U(A, \gamma_2)]
\]  

(2)

, where \(U(A, \gamma)\) is a holonomy along the path \(\gamma(s)\), \(X_{I,J}\) are \(sl(2, C)\) algebra generators, \(I, J = 1, .., 4\). The electric field quantum operator inserts \(sl(2, C)\) algebra generator \(X_{I,J}\) when the curve \(\gamma(s)\) intersects the 3d hypersurface. We then introduce the volume grasp operator:

\[
\hat{\Pi}_{I,J} U(A, \gamma) = -i\hbar \int d\sigma^1 d\sigma^2 d\sigma^3 n_\alpha(\vec{\sigma}) \frac{\delta}{\delta A_{\alpha}^{I,J}(x(\vec{\sigma}))}
\]

(3)

, where \(n_\alpha(\vec{\sigma})\) is a 3-D hypersurface norm covector

\[
n_\alpha = \epsilon_{\alpha\beta\gamma\delta} \frac{\partial x^\beta}{\partial \sigma^1} \frac{\partial x^\gamma}{\partial \sigma^2} \frac{\partial x^\delta}{\partial \sigma^3}
\]

(4)

By substituting (2) into (3) we obtain:

\[
\hat{\Pi}_{I,J} U(A, \gamma) = -i\hbar \int d\gamma ds \int \Sigma d\sigma^1 d\sigma^2 d\sigma^3 \epsilon_{\alpha\beta\gamma\delta} \frac{\partial x^\beta}{\partial \sigma^1} \frac{\partial x^\gamma}{\partial \sigma^2} \frac{\partial x^\delta}{\partial \sigma^3} \times \delta^4(\gamma(s), x) [U(A, \gamma_1) X_{I,J} U(A, \gamma_2)]
\]

(5)

If we assume that the 4-dimensional manifold can be spanned by the curves \(\gamma\) and 3d hypersurfaces then we obtain:

\[
\hat{\Pi}_{I,J} U(A, \gamma) = \sum_{p \in (\Sigma \cap \gamma)} \pm i\hbar U(A, \gamma_1^p) X_{I,J} U(A, \gamma_2^p)
\]

(6)

We introduce the operator:

\[
\hat{\Pi}_2^{I,J}(V) = \sum_{I,J} \delta^{IJ} \hat{\Pi}_{I,J}(V) \hat{\Pi}_{I,J}(V)
\]

(7)

The operator acts by inserting the algebra generators \(X_I\), therefore:

\[
\hat{\Pi}_2^{I,J}(V) |V\rangle = -\hbar^2 \sum_{I,J} \delta^{IJ} (X_{I,J})(X^{I,J}) |V\rangle
\]

(8)

The \(\sum_{I,J} \delta^{IJ} (X_{I,J})(X^{I,J})\) is \(SL(2, C)\) Casimir. Using the principal series \(SL(2, C)\) representation with parameters \((n \in Z, \rho \in C)\) that includes also unitary representations (for \(\rho \in R\)) and all non-unitary ones (for \(\rho \in C\)) we know that the sum is equal
\[
\sum_{ij} \delta_{ij} (X_{ij})(X^{ij}) |V\rangle = -\frac{1}{2} (n^2 - \rho^2 - 4) |V\rangle \tag{9}
\]

from (8) and (9) we obtain:
\[
\hat{\Pi}_{ij}^2 (V) |V\rangle = \frac{\hbar^2}{2} \left( n^2 - \rho^2 - 4 \right) |V\rangle \tag{10}
\]

The volume operator is then defined as:
\[
\hat{V}(V) = \lim_{k \to \infty} \sum_k \sqrt{\hat{\Pi}_{ij}^2 (V_k)} = \lim_{k \to \infty} \hbar \sum_k \sqrt{\frac{1}{2} (n^2 - \rho^2 - 4)} \tag{11}
\]

3 N-Dim Volume Operator

The same formalism can be generalized to the volume of the n-dimensional hypersurface in the n+1 dimensional space. The difference is n-dimensional connection variables \( A_{\mu}^{ij} \) with the values in some Lie algebra \( so(n) \) or even in the algebra of non-compact group \( sl(n, \mathbb{C}) \) with the corresponding n-dimensional conjugate momentum. We would again consider a curve \( \gamma \) intersecting the n-dimensional hypersurface in the n+1-dimensional manifold. The the holonomy variation formula is valid for the n+1 dimensional manifold and any connection as mentioned in [3]:
\[
\frac{\delta}{\delta A_{\alpha}^{ij} (x)} U(A, \gamma) = \int ds \zeta^n (s) \delta^{n+1} (\gamma(s), x) [U(A, \gamma_1) X_{ij} U(A, \gamma_2)] \tag{12}
\]

, where \( X_{ij} \) are the generators of the connection \( A_{\mu}^{ij} \) algebra. By introducing the n-volume grasp operator:
\[
\hat{\Pi}_{ij} U(A, \gamma) = -i \hbar \int ds \int_{\Sigma} d\sigma^1 d\sigma^2 \cdots d\sigma^n n_\alpha (\vec{\sigma}) \frac{\partial}{\partial A_{ij}^\alpha (x(\vec{\sigma}))} \tag{13}
\]

, where \( n_\alpha (\vec{\sigma}) \) is the n-dimensional hypersurface norm covector
\[
n_\alpha = \epsilon_{\alpha \beta \gamma \cdots} \frac{\partial x^\beta}{\partial \sigma^1} \frac{\partial x^\gamma}{\partial \sigma^2} \cdots \frac{\partial x^n}{\partial \sigma^n} \tag{14}
\]

By substituting (12) into (13) we obtain:
\[
\hat{\Pi}_{ij} U(A, \gamma) = -i \hbar \int ds \int_{\Sigma} d\sigma^1 d\sigma^2 \cdots d\sigma^n \epsilon_{\alpha \beta \gamma \cdots} \frac{\partial x^\beta}{\partial \sigma^1} \frac{\partial x^\gamma}{\partial \sigma^2} \cdots \frac{\partial x^n}{\partial \sigma^n} \frac{\partial}{\partial s} \times
\]
\[
\delta^{n+1} (\gamma(s), x) [U(A, \gamma_1) X_{ij} U(A, \gamma_2)] \tag{15}
\]

by doing integration as in 4-dimensional case we obtain:
\[
\hat{\Pi}_{ij} U(A, \gamma) = \sum_{p \in (\Sigma \gamma)} \pm i \hbar U(A, \gamma_{1p}) X_{ij} U(A, \gamma_{2p}) \tag{16}
\]
The volume operator of the n-dimensional hypersurface is then defined similarly to the 3-dimensional case:

\[
\hat{V}(V) = \lim_{k \to \infty} h \sum_k \sqrt{-C(X_{IJ})} = \lim_{k \to \infty} \sum_k \sqrt{-C(X_{IJ})}
\]  

(17)

, where \(C(X_{IJ})\) is the Casimir of the connection algebra spanned by the algebra generators \(A_{IJ}\). Thus for the two dimensional spacelike hypersurface of the 3 dimensional space the volume (area) operator will have the form:

\[
\hat{V}(V) = \lim_{k \to \infty} h \sum_k \sqrt{-C(su(2))} = \lim_{k \to \infty} \sum_k \sqrt{j(j+1)}
\]  

(18)

for the two dimensional timelike hypersurface:

\[
\hat{V}(V) = \lim_{k \to \infty} h \sum_k \sqrt{-C(sl(2,R))}
\]  

(19)

the three dimensional hypersurface volume operator is:

\[
\hat{V}(V) = \lim_{k \to \infty} h \sum_k \sqrt{-C(sl(2,C))} = \lim_{k \to \infty} \sum_k \sqrt{\frac{1}{2} (n^2 - \rho^2 - 4)}
\]  

(20)

4 Discussion

In this paper we have suggested a new approach to the n-dimensional hypersurface quantum volume operator. Instead of deriving the 3d volume operator by using the 2-dimensional case grasp operators, we derived it directly by using the holonomy variation formula [3]. We also generalize this approach for the n-dimensional hypersurface of the \(n + 1\) dimensional manifold, with the connection values in any Lie algebra.

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