UPPER TAILS FOR ARITHMETIC PROGRESSIONS IN RANDOM SUBSETS

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ABSTRACT

We study the upper tail of the number of arithmetic progressions of a given length in a random subset of \{1, \ldots, n\}, establishing exponential bounds which are best possible up to constant factors in the exponent. The proof also extends to Schur triples, and, more generally, to the number of edges in random induced subhypergraphs of ‘almost linear’ \(k\)-uniform hypergraphs.

1. Introduction

What is the (typical) behaviour of a given function depending on many independent random variables \(\xi_j\)? This fundamental concentration-of-measure question is of great interest in various areas of pure and applied mathematics, including functional analysis, statistical mechanics, and theoretical computer science. In applications, concentration inequalities are particularly important: these quantify random fluctuations of \(X = f(\xi_1, \ldots, \xi_n)\) by bounding the probability that \(X\) deviates significantly from its mean \(\mathbb{E}X\). During the last decades a wide variety of different methods for proving such inequalities have been developed (see, e.g., [26, 11, 4]), including martingale based methods [28, 25], Talagrand’s methodology [40], combinatorial approaches [22], and information theoretic methods [10, 3].

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Despite this large body of work, in concrete applications our understanding is often still far from satisfactory — even if we restrict our attention to the important case where $X$ is a sum of (dependent) indicator variables and $\xi_j \in \{0,1\}$. For example, in probabilistic combinatorics the random variable $X$ often counts objects, for instance the number of certain subgraphs in random graphs. Here Janson’s and Suen’s inequalities [17, 18, 24, 31] usually give sharp estimates for the lower tail $\mathbb{P}(X \leq (1-\varepsilon)EX)$. In contrast, obtaining tight estimates for $\mathbb{P}(X \geq (1+\varepsilon)EX)$ is more delicate, and this ‘upper tail problem’ is well-known to be a technical challenge (see, e.g., [21, 23]).

In fact, in many such counting problems each indicator variable depends only on a few $\xi_j$, in which case $X$ has a special structure: it is a low-degree polynomial of independent Bernoulli random variables. With this in mind, it is surprising that, despite intensive research of Kim and Vu [25, 43] and many others (see, e.g., [22, 37, 46, 26, 11, 4]), there is no concentration inequality that routinely gives the ‘correct’ upper tail behaviour in these basic situations. Consequently, the investigation of these and related problems is an important issue — not only from an applications point of view, but also as a question in concentration-of-measure.

In this context, Janson, Oleszkiewicz and Ruciński [20] developed in 2002 a moment-based method that, for subgraph counts in random graphs, gives estimates for $\mathbb{P}(X \geq (1+\varepsilon)EX)$ which are best possible up to logarithmic factors in the exponent. Subsequently, Janson and Ruciński [23] extended this technique so that it also gives comparable estimates for arithmetic progressions in random subsets. To be more concrete, given $k \geq 3$, let $X$ be the number of arithmetic progressions of length $k$ in $[n]_p$, the random subset of $[n] = \{1, \ldots, n\}$ where each element is included independently with probability $p$. In [23] it was shown that for essentially all $p$ and $\varepsilon > 0$ of interest we have

$$\exp\left(-C(\varepsilon, k)\sqrt{EX \log(1/p)}\right) \leq \mathbb{P}(X \geq (1+\varepsilon)EX) \leq \exp\left(-c(\varepsilon, k)\sqrt{EX}\right),$$

determining, as in [20], the upper tail up to a factor of $O(\log(1/p))$ in the exponent for constant $\varepsilon$. The problem of closing this logarithmic gap in the approach of Janson et al. [20, 23] has remained open for several years, and only very recently have there been some breakthroughs by Chatterjee [5] and DeMarco and Kahn [9, 8] for certain subgraph counts.

In this paper we solve the upper tail problem for a wide class of random variables, including arithmetic progressions and Schur triples, by establishing
upper and lower bounds which match up to constant factors in the exponent. For simplicity, we first consider the special case of arithmetic progressions (in Section 1.1 we turn to the general results). In particular, (2) below shows that
\[
\log \mathbb{P}(X \geq (1 + \varepsilon)E X) = -\Theta(\min\{EX, \sqrt{EX} \log(1/p)\})
\]
for constant \(\varepsilon\), closing the \(\log(1/p)\) gap that was present until now.

**Theorem 1:** Given \(k \geq 3\), let \(X = X_{k,n,p}\) be the number of arithmetic progressions of length \(k\) in \([n]_p\). Set \(\mu = EX\). There are \(n_0, b, B > 0\) (depending only on \(k\)) such that for all \(n \geq n_0\), \(p \in (0, 1)\) and \(\varepsilon > 0\) we have
\[
(2) \quad 1_{\{1 \leq (1+\varepsilon)\mu \leq X_{k,n,1}\}} \exp(-C(\varepsilon)\Phi) \leq \mathbb{P}(X \geq (1 + \varepsilon)\mu) \leq \exp(-c(\varepsilon)\Phi),
\]
with \(\Phi = \min\{\mu, \sqrt{\mu} \log(1/p)\}\), \(c(\varepsilon) = b \min\{\varepsilon^3, \varepsilon^{1/2}\}\) and \(C(\varepsilon) = B \max\{1, \varepsilon^2\}\).

Note that \(\mu = EX = \Theta(n^2 p^k)\), and that \(p\) and \(\varepsilon\) may depend on \(n\) (we do not assume \(n \geq n_0(\varepsilon)\), \(\varepsilon = \Theta(1)\) or \(p \geq n^{-2/k}\), which are common in this context). The additional condition \((1 + \varepsilon)\mu \leq X_{k,n,1}\) assumed for the lower bound is necessary (and also implies \(p \leq (1 + \varepsilon)^{-1/k} < 1\); otherwise \(X \geq (1 + \varepsilon)\mu\) is impossible. The condition \((1 + \varepsilon)\mu \geq 1\), which holds automatically under common assumptions such as \(\mu = \omega(1)\) or \(\mu \geq 1\), is natural; otherwise \(\mathbb{P}(X \geq (1 + \varepsilon)\mu) = \mathbb{P}(X \geq 1)\). The form of the exponent in (2) can be motivated as follows. Since an interval \([m] = \{1, \ldots, m\}\) contains \(\Theta(m^2)\) arithmetic progressions of length \(k\), for suitable \(m = \Theta(\sqrt{\mu})\) we have
\[
\mathbb{P}(X \geq 2\mu) \geq \mathbb{P}([m] \subseteq [n]_p) = p^{\Theta(\sqrt{\mu})} = e^{-\Theta(\sqrt{\mu} \log(1/p))}.
\]
Moreover, for small \(p\) (say, \(p = n^{-2/k}\)) we expect that \(X\) is approximately Poisson, which suggests
\[
\mathbb{P}(X \geq 2\mu) \approx e^{-\Theta(\mu)}.
\]
Theorem 1 essentially states that the larger of these bounds determines the decay of the upper tail for constant \(\varepsilon\).

A weakness of Theorem 1 is that is does not guarantee a similar dependence of \(c(\varepsilon)\) and \(C(\varepsilon)\) on \(\varepsilon\). Although results of this form (see, e.g., [5, 8, 9, 20]) are the widely accepted standard for the ‘infamous’ upper tail problem [21], here we go much further. Our next result establishes, over a wide range of the parameters, the dependence of the upper tail on \(\varepsilon\), up to constants (that are independent of \(\varepsilon\)). In the language of large deviations, (3) below determines, for \(p\) bounded away from one, the order of magnitude of the large deviation rate function \(\log \mathbb{P}(X \geq (1 + \varepsilon)EX)\) for all \(\varepsilon \geq n^{-\alpha}\) of interest.

**Theorem 2:** Given \(k \geq 3\), let \(X = X_{k,n,p}\) be the number of arithmetic progressions of length \(k\) in \([n]_p\). Set \(\mu = EX\) and \(\varphi(x) = (1 + x) \log(1 + x) - x\).
Given $\gamma \in (0, 1)$, there are $n_0, \alpha > 1/(6k)$ (depending only on $k$) and $c, C > 0$ (depending only on $\gamma, k$) such that for all $n \geq n_0$, $p \in (0, 1 - \gamma]$ and $\varepsilon \geq n^{-\alpha}$ satisfying $\Phi(\varepsilon) \geq 1$ we have
\begin{equation}
1_{\{1 \leq (1 + \varepsilon)\mu \leq X_{k,n,1}\}} \exp(-C\Phi(\varepsilon)) \leq \mathbb{P}(X \geq (1 + \varepsilon)\mu) \leq \exp(-c\Phi(\varepsilon)),
\end{equation}
where $\Phi(\varepsilon) = \min\{\varphi(\varepsilon)\mu^2/\text{Var } X, \sqrt{\varepsilon\mu}\log(1/p)\}$.

It is not hard to check that $\text{Var } X = \Theta(\mu(1 + np^{k-1}))$ for $p$ bounded away from one (see, e.g., Example 3.2 and Lemma 3.5 in [19]). Note that the condition $\Phi(\varepsilon) \geq 1$ is natural since our focus is on exponentially small probabilities. The function $\varphi(x)$ appears in standard Chernoff bounds; it satisfies $\varphi(x) = \Theta(x \log(1+ x))$ for $x \geq 0$, so that $\varphi(x) = \Theta(x^2)$ as $x \to 0$. The proof of Theorem 2 shows that the form of the exponent in (3) is determined by Normal approximation considerations (the $\varphi(\varepsilon)\mu^2/\text{Var } X$ term) and the interval clustering idea (the $\sqrt{\varepsilon\mu}\log(1/p)$ term). The sharp estimates of Theorem 2 are conceptually quite different from previous work on the upper tail problem. Indeed, somewhat related work for subgraph counts in the binomial random graph $G_{n,p}$ (which aims to determine the precise constants in the exponent as $n \to \infty$; see, e.g., [6, 7, 27]) focuses on the case where $\varepsilon$ is constant and $p$ is large (with $p = \Theta(1)$ or $p \geq n^{-\delta}$). In fact, for moderately large $p$, our next result completely resolves the qualitative behaviour of the upper tail.

**Theorem 3:** Given $k \geq 3$, let $X = X_{k,n,p}$ be the number of arithmetic progressions of length $k$ in $[n]_p$. Set $\mu = \mathbb{E}X$. Given $\gamma \in (0, 1)$, there are $n_0 > 0$ (depending only on $k$) and $c, C > 0$ (depending only on $\gamma, k$) such that for all $n \geq n_0$, $(\log n)^{1/(k-1)}n^{-1/(k-1)} \leq p \leq 1 - \gamma$ and $t \geq \sqrt{\text{Var } X}$ we have
\begin{equation}
1_{\{\mu + t \leq X_{k,n,1}\}} \exp(-C\Psi(t)) \leq \mathbb{P}(X \geq \mu + t) \leq \exp(-c\Psi(t)),
\end{equation}
where $\Psi(t) = \min\{t^2/\text{Var } X, \sqrt{t}\log(1/p)\}$.

Finally, as the reader can guess, in Theorems 2 and 3 various conditions (for $\varepsilon$ and $p$) are not best possible. However, for ease of exposition we defer more precise results to the next section, where we state our more general tail estimates (which include Theorems 1–3 as special cases or corollaries). Here we just mention that there is a tradeoff between $p$ and $t = \varepsilon\mu$ in Theorems 2 and 3. Indeed, Theorem 2 works for all $0 < p \leq 1 - \gamma$, but (3) is restricted to deviations of form $\varepsilon \geq n^{-\alpha}$ (for some fixed $\alpha > 0$). By contrast, Theorem 3 requires
\( n^{-1/(k-1)+o(1)} \leq p \leq 1 - \gamma \), but (4) applies to essentially all exponentially small deviations \( t > 0 \) (note that \( \Psi(t) \leq 1 \) for \( t \leq \sqrt{\text{Var} X} \)).

1.1. Counting edges of random induced subhypergraphs. In this section we present the main results of this paper, Theorems 4 and 6, which resolve the upper tail problem (up to constant factors in the exponent) for a large class of random variables, including arithmetic progressions and Schur triples. We shall phrase our results in the language of random induced subhypergraphs. More precisely, given a \( k \)-uniform hypergraph \( H \) with vertex set \( V(H) \), let \( V_p(H) \) be the random subset of \( V(H) \) where each vertex is included independently with probability \( p \). Define \( H_p = H[V_p(H)] \) and

\[
X = e(H_p),
\]

so that \( X \) counts the number of edges induced by \( V_p(H) \). Note that \( \mathbb{E} X = e(H)p^k \). Random variables of this form occur frequently in probabilistic combinatorics (see, e.g., [32, 21, 36, 13, 47, 34]), and, in the setting of Theorems 1–3, the edges of \( H = H_n \) are all \( k \)-subsets \( \{x_1, \ldots, x_k\} \subseteq [n] = V(H) \) forming an arithmetic progression of length \( k \). To state our results, we define

\[
\Delta_j(H) = \max_{S \subseteq V(H) : |S| = j} |\{f \in H : S \subseteq f\}|,
\]

which for \( j \in \{1, 2\} \) corresponds to the maximum degree and codegree of \( H \), respectively. The main examples of [23] concern \( k \)-uniform hypergraphs \( H = H_n \) with \( v(H) = n \) vertices and \( e(H) = \Theta(n^2) \) edges that are almost linear, i.e., with \( \Delta_2(H) = O(1) \), and satisfy property \( \mathcal{X}(H, D, (1 + \varepsilon)\mu) \) with \( D = \Theta(1) \), where

\[
\mathcal{X}(H, D, x) : \text{there exists } W \subseteq V(H) \text{ with } |W| \leq D \max\{\sqrt{x}, 1\}
\]

\[
\text{and } e(H[W]) \geq x.
\]

Note that \( H = H_n \) encoding \( k \)-term arithmetic progressions in \([n]\) is of this form (see also Remark 5 below). Under the aforementioned conditions, Janson and Ruciński [23] proved that the upper tail of \( X = e(H_p) \) is of type (1), leaving a \( \log(1/p) \) gap between the upper and lower bounds for constant \( \varepsilon \) (see Theorem 2.1 in [23] with \( q = 2 \)). The following theorem rectifies this issue, by closing the gap.

**Theorem 4:** Given \( k \geq 3 \), \( a > 0 \) and \( D \geq 1 \), suppose that \( H = H_n \) is a \( k \)-uniform hypergraph satisfying \( v(H) \leq Dn \), \( e(H) \geq an^2 \) and \( \Delta_2(H) \leq D \). Let \( X = e(H_p) \) and \( \mu = \mathbb{E} X \). There are \( n_0, b, B > 0 \) (depending only on
Let \( (k,a,D) \) such that for all \( n \geq n_0 \), \( p \in (0,1] \) and \( \varepsilon > 0 \) we have, with \( c(\varepsilon) = b \min\{\varepsilon^3, \varepsilon^{1/2}\} \),

\[
P(X \geq (1 + \varepsilon)\mu) \leq \exp\left(-c(\varepsilon) \min\{\mu, \sqrt{\mu}\log(e/p)\}\right).
\]

If, in addition, \( \mathcal{X}(H, D, (1 + \varepsilon)\mu) \) and \( (1 + \varepsilon)\mu \geq 1 \) hold, then we have, with \( C(\varepsilon) = B \max\{1, \varepsilon^2\} \),

\[
P(X \geq (1 + \varepsilon)\mu) \geq \exp\left(-C(\varepsilon) \min\{\mu, \sqrt{\mu}\log(1/p)\}\right).
\]

**Remark 5:** In many applications \( \mathcal{X}(H, D, x) \) holds automatically for all \( x \leq e(H) \). Indeed, often we consider sequences \( (H_n)_{n \in \mathbb{N}} \) of hypergraphs satisfying \( e(H_n \cap H_m) \geq \beta e(H_m) \) for all \( n \geq m \geq n_0 \), where \( \beta \in (0,1] \) and \( n_0 \geq 1 \) are constants (\( \beta = 1 \) for monotone sequences, where \( H_n \subseteq H_{n+1} \)). Then \( \mathcal{X}(H_n, D', x) \) follows (by increasing \( D \)) from \( v(H_m) \leq Dm \) and \( e(H_m) \geq am^2 \) for \( m = \min\{r, n\} \) and suitable \( r = \Theta(\max\{\sqrt{x}, 1\}) \).

Note that \( an^2 \leq e(H) \leq (v(H))^2 \Delta_2(H) \leq D^3n^2 \), so \( \mu = \Theta(n^2p^k) \). For (7), the necessary condition \( (1 + \varepsilon)\mu \leq e(H) \) usually entails \( \mathcal{X}(H, D, (1 + \varepsilon)\mu) \) by Remark 5, and, as discussed, \( (1 + \varepsilon)\mu \geq 1 \) is very natural (in fact, usually vacuous). The assumption \( k \geq 3 \) is also necessary. Indeed, for a concrete counterexample with \( k = 2 \), let \( H = H_n \) contain all pairs \( \{x,y\} \subseteq [n] \). Since \( |[n]_p| \) has a binomial distribution, using \( X = e(H_p) = \left(\begin{bmatrix} |[n]_p| \\ 2 \end{bmatrix}\right) \approx |[n]_p|^2/2 \) it is not difficult to see that \( \log P(X \geq (1 + \varepsilon)EX) = -\Theta(\sqrt{EX}) \) for constant \( \varepsilon \) (so there is no extra logarithmic factor).

Turning to applications, using Remark 5 it is easy to see that Theorem 4 applies to the number of arithmetic progressions of length \( k \) in \( [n]_p \), and so implies Theorem 1. The assumptions of Theorem 4 are also satisfied by Schur triples, which are classical objects in Number theory and Ramsey theory (see, e.g., [15, 35] and [14, 36]): in this case \( H = H_n \) contains all 3-element subsets \( \{x,y,z\} \subseteq [n] \) satisfying \( x + y = z \). A similar remark applies to the more general notion of \( \ell \)-sums (studied, e.g., in [2, 33]), where the 3-element subsets \( \{x,y,z\} \subseteq [n] \) satisfy \( x + y = \ell z \). Finally, the arguments in Section 2.1 of [23] reveal that Theorem 4 also applies to the number of integer solutions of certain homogeneous linear systems of equations with rank \( k - 2 \).

While results similar to Theorem 4 (with constants \( c, C \) depending on \( \varepsilon \)) are usually already considered satisfactory, in this paper we obtain much more precise estimates. Indeed, with Theorem 6 below we recover, in a very wide range,
the dependence of the upper tail on $t = \varepsilon \mu$ (up to constants). Theorem 6 looks hard to digest, so we will now spend some time motivating and explaining it. As a warm-up, let us first informally discuss the asymptotic form of its upper tail estimates for $X = e(H_p)$. In particular, since our focus is on exponentially decaying probabilities, in (9) and (10) below the multiplicative factors of $1 + n^{-1}$ and $d$ are usually negligible (i.e., can be removed by adjusting the constants $c, C$). Hence, assuming $n^{-2/k}(\log n)^{2/k} \leq p \leq 1/2$ and $t \geq \sqrt{\text{Var} X}$, say, via Remarks 7–8 the form of (9)–(10) eventually simplifies to

$$\log \mathbb{P}(X \geq \mu + t) = -\Theta\left(\min\left\{\frac{t^2}{\text{Var} X}, \sqrt{t} \log(1/p)\right\}\right).$$

With this in mind, Theorem 6 essentially states that the upper tail of $X = e(H_p)$ is either of sub-Gaussian type $\exp(-ct^2/\text{Var} X)$ or of ‘clustered’ type $\exp(-c\sqrt{t} \log(1/p))$, and that the transition between the two happens roughly for $t$ around $(\text{Var} X)^{2/3}$. In this context the upper bound (9) of Theorem 6 is very satisfactory. Namely, it holds via (a) for all $t > 0$ unless $p$ is close to $p_0 = n^{-1/(k-1)}$, in which case (9) still holds for $t \geq (\text{Var} X)^{2/3}(\log n)^{4/3}$ via (b).

In words, our upper bound (9) recovers the qualitative behaviour of the upper tail for all $t > 0$, unless $p$ is in a tiny exceptional interval around $p_0$ (where we basically only miss the sub-Gaussian regime).

**Theorem 6:** Given $k \geq 3$, $a > 0$ and $D \geq 1$, suppose that $\mathcal{H} = \mathcal{H}_n$ is a $k$-uniform hypergraph satisfying $v(\mathcal{H}) \leq Dn$, $e(\mathcal{H}) \geq an^2$ and $\Delta_2(\mathcal{H}) \leq D$. Let $X = e(H_p)$, $\mu = \mathbb{E} X$, $\Lambda = \mu(1+np^{k-1})$ and $\varphi(x) = (1+x)\log(1+x) - x$. Given $\gamma \in (0,1)$, there are $n_0 > 0$ (depending only on $k, a, D$) as well as $c, C, d > 0$ and $\lambda \geq 1$ (depending only on $\gamma, k, a, D$) such that for all $n \geq n_0$, $p \in (0,1]$ and $t > 0$ the following holds. If one of

- (a) $p \notin \left(n^{-1/(k-1)} - \gamma, n^{-1/(k-1)}(\log n)^{1/(k-1)}\right)$, or
- (b) $t \geq \gamma \min\left\{(\text{Var} X)^{2/3}, \mu^{2/3}\right\}(\log n)^{4/3}$, or
- (c) $t \geq \mu p^{(k-2)/3 - \gamma}$

holds, then we have the upper bound

$$\mathbb{P}(X \geq \mu + t) \leq (1 + n^{-1}) \exp\left(-c\min\{\varphi(t/\mu)\mu^2/\Lambda, \sqrt{t} \log(e/p)\}\right).$$

Furthermore, if one of

- (i) $p \leq n^{-2/(k+1/3)}$, or
- (ii) $t \geq \min\{(\text{Var} X)^{2/3}, \mu^{2/3}\}(\log n)^{2/3}$ and $p \leq n^{-1/(k-1)} \log n$, or
- (iii) $t \geq \min\{\sqrt{\text{Var} X}, \sqrt{\Lambda}\}$ and $\gamma n^{-1/(k-1)} \leq p \leq 1 - \gamma$
holds, then \( X(H, D, \min\{\lambda t, \mu + t\}) \) and \( \mu + t \geq 1 \) imply the lower bound

\[
P(X \geq \mu + t) \geq d \exp\left(-C \min\{\varphi(t/\mu)\mu^2/\Lambda, \sqrt{t} \log(1/p)\}\right).
\]

**Remark 7:** It is routine to check that \( \text{Var} X = \Theta((1-p)\Lambda) \), where the implicit constants depend only on \( k, a, D \) (analogously to, e.g., Example 3.2 and Lemma 3.5 in [19]). In particular, \( \Lambda = \Theta(\text{Var} X) \) holds whenever \( p \) is bounded away from one.

**Remark 8:** If \( p \geq \gamma n^{-2/k}(\log n)^{2/k} \) or \( t \leq \mu \), then (9)–(10) hold with \( \varphi(t/\mu)\mu^2/\Lambda \) replaced by \( t^2/\Lambda \).

In the above assumptions (a)–(c) and (i)–(iii), the use of \( \mu \) and \( \Lambda \) is convenient for applications (see, e.g., (11) below), while \( \text{Var} X \) seems more insightful from a conceptual point of view. In particular, since we are interested in exponentially small probabilities, by central limit theorem considerations a natural target assumption is \( t \geq \sqrt{\text{Var} X} \), say. We now discuss the lower bound (10) of Theorem 6, which tends to have fewer applications. Indeed, for our purposes (10) is mainly important from a concentration-of-measure perspective, since it rigorously proves that our upper bound (9) is sharp in a wide range. In view of (i)+(iii), our lower bound (10) only falls short of the target assumption \( t \geq \sqrt{\text{Var} X} \) for \( p \in (n^{-2/(k+1/3)}, n^{-1/(k-1)}) \), where \( t \geq (\text{Var} X)^{2/3}(\log n)^{2/3} \) suffices by (ii). Perhaps surprisingly, these gaps are solely due to lacking lower bounds of sub-Gaussian type (note that the variance undergoes a transition around \( p_0 = n^{-1/(k-1)} \) by Remark 7), which until now have been widely ignored in the upper tail literature (see, e.g., [43, 47]). Here our current approaches seem not strong enough to work for all relevant \( p \) and \( t \). We leave it as an open problem to develop a generic method for obtaining suitable sub-Gaussian type lower bounds (see Section 4.2). Finally, we also conjecture that the upper tail estimates (9)–(10) remain valid for all \( p \in (0, 1-\gamma] \) and \( t \geq \sqrt{\text{Var} X} \).

Turning to the remaining applications stated in the introduction, Theorem 3 for arithmetic progressions follows easily by combining (a)+(iii) of Theorem 6 with Remarks 5, 7 and 8. For Theorem 2 we use that, modulo obvious assumptions, the tail estimates (9)–(10) both apply if \( t > 0 \) satisfies, say,

\[
t \geq \begin{cases} 
0 & \text{if } 0 < p \leq n^{-2/(k+1/3)}, \\
\mu^{2/3} (\log n)^{4/3} & \text{if } n^{-2/(k+1/3)} < p < n^{-1/(k-1)} (\log n)^{1/(k-1)}, \\
\sqrt{\Lambda} & \text{if } n^{-1/(k-1)} (\log n)^{1/(k-1)} \leq p \leq 1 - \gamma,
\end{cases}
\]
(using (a)+(i) for $p \leq n^{-2/(k+1/3)}$, (b)+(ii) for larger $p < n^{-1/(k-1)}(\log n)^{1/(k-1)}$, and (a)+(iii) otherwise). As $\mu \geq an^2p^k$ and $\Lambda = \mu(1 + np^{k-1})$, a short calculation reveals that, say, $t \geq \mu n^{-1/(5k+1)}$ implies (11) for all $n \geq n_0(k,a)$ and $p \in (0,1]$. Hence, using Remarks 5 and 7, inequality (3) of Theorem 2 follows.

The proofs of the upper and lower bounds of Theorems 4 and 6 are based on completely different techniques. For the upper bounds (6) and (9), the most important ingredients are two new concentration inequalities of Chernoff-type, which we prove in Section 2. These allow us to combine and extend the combinatorial and probabilistic ideas used in the ‘deletion method’ and the ‘approximating by a disjoint subfamily’ technique of Janson and Ruciński [22] and Spencer [39, 21], respectively. The idea of applying the BK-inequality of van den Berg and Kesten [42] and Reimer [30] in the context of the ‘infamous’ upper tail problem [21] may perhaps also be of independent interest. For the lower bounds (7) and (10), we analyze three different mechanisms that yield deviations of $X = e(H_p)$, and with some care (using, e.g., Harris’ inequality [16] and the Paley–Zygmund inequality) we recover the correct dependence of the exponent on $t = \varepsilon \mu$.

The remainder of this paper is organized as follows. In Section 2 we introduce our new concentration inequalities, and in Section 3 we apply them (together with combinatorial arguments) to prove the upper bounds of Theorems 4 and 6. Finally, in Section 4 we establish the corresponding lower bounds (and also prove Remark 8).

2. Concentration inequalities

In this section we introduce our main probabilistic tools: two concentration inequalities which essentially state that Chernoff-type upper tail estimates hold whenever $X$ is bounded from above by a sum of random variables with ‘well-behaved dependencies’. They develop ideas of Janson and Ruciński [22], Erdős and Tetali [12], and Spencer [39], and seem of independent interest. On first reading of Theorem 9 it might be useful to consider the special case where there are independent random variables $(\xi_i)_{i \in A}$ such that each $Y_\alpha \in \{0,1\}$ with $\alpha \in I$ is a function of $(\xi_i)_{i \in \alpha}$. Then, defining $\alpha \sim \beta$ if $\alpha \cap \beta \neq \emptyset$, it is immediate that the independence assumption holds (as $\alpha \not\sim \beta$ implies that $Y_\alpha$ and $Y_\beta$ depend on disjoint sets of variables $\xi_i$). Now, consider $X = \sum_{\alpha \in I} Y_\alpha$ with $\mu = \mathbb{E}X$, $J = I$ and $C = \max_{\beta \in I} |\{\alpha \in I : \alpha \sim \beta\}|$. Then $X = Z_C$,
where $\max_{\beta \in J} \sum_{\alpha \in J: \alpha \sim \beta} Y_\alpha \leq C$ intuitively corresponds to a Lipschitz-like condition. With this in mind, part of the power of (12) is that the exponent scales with $1/C$ (instead of the usual $1/C^2$), and that the Lipschitz condition need not hold deterministically (it suffices if $X \leq Z_C$ or $X \approx Z_C$ holds off some exceptional event).

**Theorem 9:** Given a family of non-negative random variables $(Y_\alpha)_{\alpha \in I}$ with $\sum_{\alpha \in I} E[Y_\alpha] \leq \mu$, assume that $\sim$ is a symmetric relation on $I$ such that each $Y_\alpha$ with $\alpha \in I$ is independent of $\{Y_\beta : \beta \in I \text{ and } \beta \not\sim \alpha\}$. Let $Z_C = \max \sum_{\alpha \in J} Y_\alpha$, where the maximum is taken over all $J \subseteq I$ with $\max_{\beta \in J} \sum_{\alpha \in J: \alpha \sim \beta} Y_\alpha \leq C$.

Set $\phi(x) = (1+x) \log(1+x) - x$. Then for all $C, t > 0$ we have

$$\mathbb{P}(Z_C \geq \mu + t) \leq \exp \left( -\frac{\phi(t/\mu)\mu}{C} \right) = e^{-\mu/C} \cdot \left( \frac{\mu}{\mu + t} \right)^{(\mu+t)/C}$$

(12)

$$\leq \min \left\{ \exp \left( -\frac{t^2}{2C(\mu + t/3)} \right), \left( 1 + \frac{t}{2\mu} \right)^{-t/(2C)} \right\}.$$

**Remark 10:** Theorem 9 remains valid after weakening the independence assumption to a form of negative correlation: it suffices if $E(\prod_{i \in [s]} Y_{\alpha_i}) \leq \prod_{i \in [s]} E[Y_{\alpha_i}]$ for all $(\alpha_1, \ldots, \alpha_s) \in I^s$ satisfying $\alpha_i \not\sim \alpha_j$ for $i \neq j$.

Theorem 9 extends several upper tail inequalities discussed in the survey of Janson and Ruciński [21]. Indeed, consider $X = \sum_{\alpha \in I} Y_\alpha$ with $\mu = \mathbb{E}X$ and $J = I$. For independent $Y_\alpha \in [0, 1]$ we have $X = Z_1$ (note that $\alpha \sim \alpha$ for non-constant $Y_\alpha$), so that (12) reduces to the classical Chernoff bound; see, e.g., Theorem 2.1 in [19]. Similarly, for generic $Y_\alpha \in [0, 1]$ with dependency graph $\mathcal{G} = \mathcal{G}(I)$, where distinct $\alpha, \beta \in I = V(\mathcal{G})$ form an edge if $\alpha \sim \beta$ (cf. Section 2.6 in [21]), we have $X = Z_{\Delta_1(\mathcal{G})+1}$. Hence (12) improves Theorem 5 in [21], which is based on the ‘breaking into disjoint matchings’ technique of Rödl and Ruciński [32]. Furthermore, using $C = t/(2r)$ it is easy to see that Theorem 9 tightens Theorem 2.1 in [22], i.e., the basic theorem of the ‘deletion method’ of Janson and Ruciński. In addition, (12) extends Lemma 2 in [21], i.e., the main probabilistic ingredient of Spencer’s ‘approximating by a disjoint subfamily’ technique [39]. Theorem 9 is also related to a concentration inequality of Chatterjee [5]; our assumptions are less technical and subjectively easier to check (e.g., readily implying Proposition 4.1 in [5] via $C = 3\varepsilon \ell np$). Remark 10 is useful in the context of the uniform random graph $G_{n,m}$ (and
related uniform models). To illustrate this we consider \( Y_\alpha = 1_{\{\alpha \subseteq E(G_{n,m})\}} \) and set \( \alpha \sim \beta \) if \( \alpha \cap \beta \neq \emptyset \). In that case it is well-known (and not hard to check) that the negative correlation condition of Remark 10 holds, demonstrating that Theorem 9 applies to \( G_{n,m} \).

**Proof of Theorem 9.** The proof is based on a variant of the \( m \)-th factorial moment which ‘forces independence’. In fact, we closely follow Lemma 2.3 in [22] and Lemma 2.46 in [19], but differ in some important details. Assume that \( m \in \mathbb{N} \) satisfies \( 1 \leq m \leq \lceil (\mu + t) / C \rceil \). For all \( \mathcal{K} \subseteq \mathcal{I} \) and \( s \in \mathbb{N} \) with \( s \geq 1 \) we define

\[
M_s(\mathcal{K}) = \sum \bigstar_{(\alpha_1, \ldots, \alpha_s) \in \mathcal{K}^s} \prod_{i \in [s]} Y_{\alpha_i},
\]

where \( \sum \bigstar_{(\alpha_1, \ldots, \alpha_s) \in \mathcal{K}^s} \) denotes the sum over all tuples \( (\alpha_1, \ldots, \alpha_s) \in \mathcal{K}^s \) satisfying \( \alpha_i \not\sim \alpha_j \) for \( i \neq j \). The key point is that, by construction, the factors \( Y_{\alpha_i} \geq 0 \) in each term of \( M_s(\mathcal{K}) \) are independent. Hence

\[
\mathbb{E} M_m(\mathcal{I}) = \sum \bigstar_{(\alpha_1, \ldots, \alpha_m) \in \mathcal{I}^m} \prod_{i \in [m]} Y_{\alpha_i} = \sum \bigstar_{(\alpha_1, \ldots, \alpha_m) \in \mathcal{I}^m} \prod_{i \in [m]} \mathbb{E} Y_{\alpha_i} \leq \left( \sum_{\alpha \in \mathcal{I}} \mathbb{E} Y_{\alpha} \right)^m \leq \mu^m.
\]

(13)

Now assume that \( Z_C \geq \mu + t \) and \( Z_C = \sum_{\alpha \in \mathcal{J}} Y_\alpha \) hold. Note that, by construction, \( M_1(\mathcal{J}) = \sum_{\alpha \in \mathcal{J}} Y_\alpha = Z_C \geq \mu + t \). Furthermore, by choice of \( \mathcal{J} \) (see the definition of \( Z_C \)), for all \( (\alpha_1, \ldots, \alpha_s) \in \mathcal{J}^s \) we have

\[
\sum_{\alpha \in \mathcal{J} : \alpha \sim \alpha_i} Y_\alpha \leq \sum_{i \in [s]} \sum_{\alpha \in \mathcal{J} : \alpha \sim \alpha_i} Y_\alpha \leq C s.
\]

So, for all \( s \in \mathbb{N} \) with \( 1 \leq s < m \leq \lceil (\mu + t) / C \rceil \) it follows that

\[
M_{s + 1}(\mathcal{J}) = \sum \bigstar_{(\alpha_1, \ldots, \alpha_s) \in \mathcal{J}^s} \prod_{i \in [s]} Y_{\alpha_i} \cdot \left( \sum_{\alpha \in \mathcal{J}} Y_\alpha - \sum_{\alpha \in \mathcal{J} : \alpha \sim \alpha_i} Y_\alpha \right) \geq M_s(\mathcal{J}) \cdot (\mu + t - C s),
\]

(14)

which by induction yields \( M_m(\mathcal{J}) \geq \prod_{j=0}^{m-1} (\mu + t - C j) \).

Combining the above estimates for \( M_m(\mathcal{I}) \geq M_m(\mathcal{J}) \) and \( \mathbb{E} M_m(\mathcal{I}) \) with Markov’s inequality, we obtain

\[
\mathbb{P}(Z_C \geq \mu + t) \leq \mathbb{P}\left( M_m(\mathcal{I}) \geq \prod_{j=0}^{m-1} (\mu + t - C j) \right) \leq \prod_{j=0}^{m-1} \frac{\mu}{\mu + t - C j}.
\]

(15)
Set $m = \lceil t/C \rceil \geq 1$. If $\mu = 0$, then $P(Z_C \geq \mu + t) = 0$ by (15), and (12) is trivial, so we henceforth assume $\mu > 0$. For $0 \leq x \leq t/C$, the function $f(x) = \log(\mu/(\mu + t - Cx))$ is increasing and satisfies $f(x) \leq 0$. As $f(t/C) = 0$, it follows that $f(j) \leq \int_{\min}\{j+1,t/C\} f(x)dx$ for $0 \leq j \leq t/C$. We deduce
\[
\log P(Z_C \geq \mu + t) \leq \sum_{j=0}^{[t/C]-1} \log \left( \frac{\mu}{\mu + t - Cj} \right)
\leq \int_0^{t/C} \log \left( \frac{\mu}{\mu + t - Cx} \right) dx =: \Psi.
\]
Using $\log(a/b) = \log a - \log b$, integration yields $\Psi = -\varphi(t/\mu) \mu/C$. It is well-known that
\[
(16) \quad \varphi(x) \geq x^2/(2 + 2x/3)
\]
for $x \geq 0$ (see, e.g., the proof of Theorem 2.1 in [19]), so $\Psi \leq -t^2/(2C(\mu + t/3))$. Finally, for $u = t/(2C)$ we have $\Psi = \int_0^{t/C} f(x)dx \leq \int_0^u f(x)dx \leq uf(u)$, which establishes (12).

For all integers $x \geq 1$, by formally defining $xC = \mu + t$ and $m = x$ in the above proof (so that $\mu + t - Cj = C(x - j)$ holds), note that inequality (15) and Stirling’s formula imply
\[
P(Z_C \geq xC) \leq \left( \frac{\mu}{C} \right)^x /x! \leq \left( \frac{e\mu}{xC} \right)^x /\sqrt{2\pi x}.
\]
While this estimate is often weaker than (12), for $C = 1$ it extends, in the upper tail context, the so-called ‘disjointness lemma’ of Erdős and Tetali [12]; see, e.g., Lemma 8.4.1 in [1]. In the proof of Theorem 9, inequality (13) is the only step in which anything is assumed about the $Y_\alpha$, and independence is used in a limited way: $E(\prod Y_\alpha) \leq \prod E(Y_\alpha)$ suffices (in fact, replacing the assumption $\sum EY_\alpha \leq \mu$ with $\sum \lambda_\alpha \leq \mu$ and $\lambda_\alpha \geq 0$, it suffices if $E(\prod Y_\alpha) \leq \prod \lambda_\alpha$ holds). This suggests that the argument is rather robust, since, e.g., ad-hoc upper bounds for $E(\prod Y_\alpha)$ are enough to obtain tail inequalities; see the proof of Lemma 4.5 in [45]. Finally, in (14) there is also potential for relaxing $\max_{\beta \in J} \sum_{\alpha \in J: \alpha \sim \beta} Y_\alpha \leq C$ to an accumulative condition (e.g., replacing $Cs$ by $t/2$).

The following variant of Theorem 9 exploits the BK-inequality [42] to further relax the independence assumption. Clearly, two events $E_1$, $E_2$ depending on disjoint sets of independent random variables are independent. For our purposes
it intuitively suffices if, for each possible outcome \( \omega \in \Omega \), we can ‘certify’ the occurrence of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) by disjoint sets of variables (which may depend on \( \omega \)).

For \( \omega = (\omega_1, \ldots, \omega_M) \in \Omega = \Omega_1 \times \cdots \times \Omega_M \) and \( K \subseteq [M] = \{1, \ldots, M\} \) we write \( \omega|_K = (\omega_i)_{i \in K} \) and \( [\omega]|_K = \{\omega' \in \Omega : \omega'|_K = \omega|_K\} \). If \( [\omega]|_K \subseteq \mathcal{E} \), then \( \omega|_K \) is called a certificate for the occurrence of the event \( \mathcal{E} \) (in words, \( \mathcal{E} \) occurs on all sample points that agree with \( \omega \) restricted to \( K \)). Intuitively speaking, in Theorem 11 the random variable \( Z \) counts the maximum number of events that ‘occur disjointly’, i.e., have disjoint certificates. With this in mind, a key feature of inequalities (12) and (17) is that they are dimension-free: they do not involve the sizes of the certificates (in contrast to ‘certificate-based’ variants of Talagrand’s inequality such as Theorem 2 in [29]).

**Theorem 11:** Given a product space \( \Omega = \Omega_1 \times \cdots \times \Omega_M \), with finite \( \Omega_i \), let \((\mathcal{E}_\alpha)_{\alpha \in \mathcal{I}}\) be a family of events with \( \sum_{\alpha \in \mathcal{I}} \mathbb{P}(\mathcal{E}_\alpha) \leq \mu \). Let \( Z = \max |J| \), where the maximum is taken over all \( J \subseteq \mathcal{I} \) for which there are disjoint \( K_i \subseteq [M] \) satisfying \( [\omega]|_{K_i} \subseteq \mathcal{E}_{\alpha_i} \) for all \( \alpha_i \in J \). Then (12) and (17) hold with \( C = 1 \) and \( Z_1 = Z \).

**Remark 12:** Theorem 11 remains valid after weakening the product space assumption: restricting to increasing events \( \mathcal{E}_\alpha \subseteq \Omega = \{0, 1\}^M \), it suffices if \( \mathbb{P} \) satisfies the BK-inequality (18) for increasing events (in this case \( \square \) is associative, so we may replace \( Z \) by the maximum of \( |J| \) over all \( J \subseteq \mathcal{I} \) for which \( \square_{\alpha \in J} \mathcal{E}_\alpha \) holds).

The proof of Theorem 11 is based on the BK-inequality, which is a partial converse to Harris’ inequality [16]. Intuitively, \( \mathcal{A} \square \mathcal{B} \) means that the events \( \mathcal{A} \) and \( \mathcal{B} \) have disjoint certificates. Formally, we define

\[
\mathcal{A} \square \mathcal{B} = \{\omega \in \Omega : \exists \text{ disjoint } K, L \subseteq [M] \text{ such that } [\omega]|_K \subseteq \mathcal{A} \text{ and } [\omega]|_L \subseteq \mathcal{B}\},
\]

which need not be associative. The general BK-inequality of Reimer [30] states that for any product space \( \Omega = \Omega_1 \times \cdots \times \Omega_M \), with finite \( \Omega_i \), the following holds: for any two events \( \mathcal{A}, \mathcal{B} \subseteq \Omega \) we have

\[
\mathbb{P}(\mathcal{A} \square \mathcal{B}) \leq \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B}).
\]

**Proof of Theorem 11.** The proof uses a \( \square \)-based variant of the \( m \)-th moment (inspired by Theorem 9).
For all $m \in \mathbb{N}$ we define $D(\alpha_1, \ldots, \alpha_m) = (\cdots (E_{\alpha_1} \square E_{\alpha_2}) \square \cdots \square E_{\alpha_{m-1}}) \square E_{\alpha_m}$ and

$$M_m(K) = \sum_{(\alpha_1, \ldots, \alpha_m) \in K^m} 1_{\{D(\alpha_1, \ldots, \alpha_m)\}}.$$  

Using the BK-inequality (18) inductively, we obtain $P(D(\alpha_1, \ldots, \alpha_m)) \leq \prod_{i \in [m]} P(E_{\alpha_i})$. So, analogous to (13), we deduce $E M_m(I) \leq \mu^m$. Now assume that $Z \geq y$ and $Z = |J|$ hold. For each $m \leq \lceil y \rceil \leq |J|$, by definition of $Z$ we see that $D(\alpha_1, \ldots, \alpha_m)$ occurs for all $m$-element subsets $\{\alpha_1, \ldots, \alpha_m\} \subseteq J$. Hence

$$M_m(I) \geq M_m(J) \geq \binom{|J|}{m} m! \geq \prod_{j=0}^{m-1} (y - j).$$

Let $Z_1 = Z$ and $C = 1$. With $y = \mu + t$, the proof of Theorem 9 carries over unchanged from (15) onwards, and (12) follows. Similarly, with $y = x$, $m = x$ and $\mu + t = x$, (15) establishes (17). □

The sufficient condition of Remark 12 has recently been established in [41] for $P$ assigning equal probability to all $\omega \in \{0, 1\}^M$ with exactly $k$ ones. Hence Theorem 11 applies to $G_{n,m}$ and related uniform models.

3. Upper bounds

In this section we establish the upper bounds (6) and (9) of Theorems 4 and 6. The executive summary of our proof strategy is as follows: using combinatorial arguments we shall approximate $X = e(H_p)$ using several ‘well-behaved’ auxiliary random variables, which we in turn estimate by the concentration inequalities of Section 2. Of course, the actual details are much more involved, and our arguments in fact develop combinatorial and probabilistic ideas of the ‘deletion method’ [22] and the ‘approximating by a disjoint subfamily’ technique [39, 21]. We have added a substantial amount of informal discussion and motivation to the remainder of this section, in an attempt to make the underlying ideas and techniques more accessible (the actual proofs could be recorded in a much shorter way). For example, in order to milder some of the technical difficulties, we shall not only informally discuss the intriguing $\log(e/p)$ factors in the exponent, but also prove (6) using a simplified version of our arguments (instead of proving (6) and (9) in a unified way).
The remainder of this section is organized as follows. In Section 3.1 we motivate parts of our proof strategy, and illustrate how logarithmic terms arise in our tail estimates. In Section 3.2 we then present our basic proof framework, and establish the upper bound of Theorem 4. Finally, in Section 3.3 we refine the aforementioned framework, and prove the more involved upper bound of Theorem 6.

3.1. Warming up. The upper bounds of Theorems 4 and 6 involve exponentially small probabilities, so error probabilities of form $o(1)$ are too crude for our purposes (and the proofs require more care). In fact, the exponents in (6) and (9) are fairly involved, and both contain somewhat unusual $\log(e/p)$ terms. With these non-standard features in mind, the goals of this informal section are two-fold: (i) to motivate some details of our upcoming proof strategy, and (ii) to illustrate the way in which we eventually obtain the $\log(e/p)$ factors.

3.1.1. Motivation and preliminaries. Let us start with a basic estimate for the number of induced edges $X = e(H_p)$. For brevity we set

$$\Gamma_v(G) = \{f \in G : v \in f\},$$

so that $|\Gamma_v(H_p)|$ equals the degree of vertex $v$ in $H_p$. Clearly, for all $r > 0$ we have

$$P(X \geq \mu + t) \leq P(X \geq \mu + t \text{ and } \Delta_1(H_p) \leq r) + P(\Delta_1(H_p) > r)$$

$$\leq P(X \geq \mu + t \text{ and } \Delta_1(H_p) \leq r) + \sum_{v \in V(H)} P(|\Gamma_v(H_p)| > r).$$

(19)

A similar decomposition forms the basis of the inductive ‘deletion method’ of Janson and Ruciński [22]; see, e.g., Theorem 2.5 and Section 3 in [22]. The inductive approach of Kim and Vu [25] is also based on a related idea; see, e.g., Section 3.2 in [43].

One bottleneck of the above approach (19) is that it relies on a uniform upper bound on the degree of all vertices. We shall rectify this issue via the following sparsification strategy (which allows for some vertices with larger degrees): we first decrease the maximum degree of $H_p$ by removing some carefully chosen edges, and then estimate the number of remaining edges via the Chernoff-type tail inequality Theorem 9. In other words, our plan is to first apply further combinatorial arguments to $H_p$, before using any probabilistic tail estimates or induction. An embryonic version of this idea is contained in the ‘approximating
by a disjoint subfamily’ technique of Spencer [39, 21], but Janson and Ruciński argued in their upper tail survey [21] that this technique is ‘never better’ than the ‘deletion method’ [22] (see Remark 2 in Section 2.3.4 and Example 7 in Section 3.2 of [21]). In Sections 3.2–3.3 we shall, in some sense, crossbreed ideas of both approaches to go one step further.

3.1.2. Extra logarithmic factors in tail estimates? Let us illustrate how extra logarithmic factors can arise in our upper tail estimates. To this end we shall, in the context of Theorem 4, a heuristic look at the exponential decay of the degrees \(|\Gamma_v(\mathcal{H})|\). Here the key observation is that the dependencies among the edges in \(\Gamma_v(\mathcal{H}_p) \subseteq \Gamma_v(\mathcal{H})\) are severely limited by the codegree condition \(\Delta_2(\mathcal{H}) = O(1)\): for every \(e \in \Gamma_v(\mathcal{H})\) there are only at most \(k\Delta_2(\mathcal{H}) = O(1)\) edges \(f \in \Gamma_v(\mathcal{H})\) which intersect \(e \setminus \{v\}\), i.e., with \((f \cap e) \setminus \{v\} \neq \emptyset\) (because all such \(f\) contain \(v\) and at least one vertex from \(e \setminus \{v\}\)). As \(\mathcal{H}\) is \(k\)-uniform, it thus seems plausible that, conditioned on \(v \in V_p(\mathcal{H})\), the upper tail of \(|\Gamma_v(\mathcal{H}_p)|\) decays roughly like a binomial random variable \(Y \sim \text{Bin}(|\Gamma_v(\mathcal{H})|, p^{k-1})\). Note that for all positive integers \(x\), we have

\[
P(Y \geq x) \leq \left(\frac{|\Gamma_v(\mathcal{H})|}{x}\right) p^{(k-1)x} \leq \frac{(|\Gamma_v(\mathcal{H})| p^{k-1})^x}{x!} \leq \left(\frac{O(np^{k-1})}{x}\right)^x,
\]

where we used \(|\Gamma_v(\mathcal{H})| \leq |V(\mathcal{H})| \cdot \Delta_2(\mathcal{H}) = O(n)\) for the last inequality. As expected, the decay of \(|\Gamma_v(\mathcal{H}_p)|\) turns out to be very similar to (20). Indeed, ignoring a number of technicalities, we later approximately show (see (37) in the proof of Lemma 17) that for a certain range of \(x\) we have

\[
P(\Delta_1(\mathcal{H}_p) \geq x) \leq \sum_{v \in V(\mathcal{H})} P(|\Gamma_v(\mathcal{H}_p)| \geq x) \leq \left(\frac{O(np^{k-1})}{x}\right)^{\Theta(x)}.
\]

With this in mind, the basic idea for ‘extra’ logarithmic terms is simple: if \(x \gg ynp^{k-1}\) holds, then (21) suggests \(P(\Delta_1(\mathcal{H}_p) \geq x) \leq \exp(-\Theta(x \log y))\). In words, if the deviation \(x\) ‘overshoots’ the expectation \(|\Gamma_v(\mathcal{H})| p^{k-1} = O(np^{k-1})\) significantly, then we should win a logarithmic factor in the exponent.

In Sections 3.2–3.3 we shall exploit the aforementioned ‘overshooting’ phenomenon for a range of different degrees (to intuitively show that there are not too many vertices with high degrees). Of course, using this approach we shall eventually need to check a number of technical conditions such as \(np^{k-1}/x = O(p^{\Theta(1)})\): these are key for obtaining the \(\log(e/p)\) factors missing in previous work of Janson and Ruciński [23].
3.2. Basic proof framework. In this section we introduce our basic proof framework (for arbitrary hypergraphs $\mathcal{H}$), which seems of independent interest. In the combinatorial part we implement the sparsification idea mentioned in Section 3.1.1, and essentially show the number of induced edges $X = e(\mathcal{H}_p)$ can be estimated via two carefully defined auxiliary random variables $X_r = X_r(\mathcal{H}_p)$ and $M_r = M_r(\mathcal{H}_p)$. In the probabilistic part we systematically obtain upper tail estimates for $X_r$ and $M_r$, by exploiting the Chernoff-type concentration inequalities of Section 2. Finally, we demonstrate the applicability of this framework by proving the upper bound of Theorem 4.

Recall that our strategy is to decrease the maximum degree of $\mathcal{H}_p$ by removing edges. To estimate the upper tail of the remaining edges, we now introduce the following ‘smooth approximation’ of $X = e(\mathcal{H}_p)$:

$$X_r = \max \{ e(G) : G \subseteq \mathcal{H}_p \text{ and } \Delta_1(G) \leq r \}.$$  

In words, $X_r = X_r(\mathcal{H}_p)$ denotes the maximum number of edges in any subhypergraph $G \subseteq \mathcal{H}_p$ with maximum degree at most $r$. Via Theorem 9 this ‘bounded degree’ property eventually yields (23), i.e, a general upper tail estimate for $X_r$. For $\varepsilon = \Theta(1)$ and $k = \Theta(1)$, note that (23) yields $\mathbb{P}(X_r \geq (1 + \varepsilon/2)\mu) \leq \exp(-\Theta(\mu/r))$.

**Lemma 13:** Suppose that $\mathcal{H}$ satisfies $\max_{f \in \mathcal{H}} |f| \leq k$. Set $X = e(\mathcal{H}_p)$, $\mu = \mathbb{E}X$ and $\varphi(x) = (1 + x) \log(1 + x) - x$. Then, for all $p \in [0, 1]$ and $r, t > 0$ we have

$$\mathbb{P}(X_r \geq \mu + t/2) \leq \exp\left(-\frac{\varphi(t/\mu)\mu}{4kr}\right) \leq \exp\left(-\frac{\min\{t, t^2/\mu\}}{12kr}\right).$$

The main observation required to deduce Lemma 13 from Theorem 9 is that every edge $f \in G \subseteq \mathcal{H}$ is incident to at most $k\Delta_1(G)$ other edges of $G$. This allows us to bring the Lipschitz-like condition of Theorem 9 into play (with $C = kr$).

**Proof of Lemma 13.** Defining $Y_f = 1_{\{f \subseteq V_\mathcal{H}\}}$, we have $\sum_{f \in \mathcal{H}} \mathbb{E}Y_f = \mathbb{E}X = \mu$. Set $e \sim f$ if $e \cap f \neq \emptyset$. Hence, by the discussion preceding Theorem 9, the independence assumption of Theorem 9 holds (here the $\xi_i = 1_{\{i \in V_\mathcal{H}\}}$ are independent indicators, so $Y_f = \prod_{i \in f} \xi_i$). Observe that for all $f \in G \subseteq \mathcal{H}$ we have

$$\sum_{e \in G : e \sim f} Y_e \leq \sum_{v \in f} \sum_{e \in G : v \in e} Y_e \leq |f| \cdot \max_{v \in f} |\Gamma_v(G)| \leq k\Delta_1(G).$$
Hence, for $C = kr$ we deduce $X_r \leq Z_C$, where $Z_C$ is defined as in Theorem 9 with $I = H$. So, using (12),
\[
P(X_r \geq \mu + t/2) \leq P(Z_C \geq \mu + t/2) \leq \exp\left(-\frac{\varphi(t/(2\mu))\mu}{kr}\right),
\]
and it remains to rewrite this estimate. Since (16) implies (by distinguishing the cases $x \geq 1$ and $x \leq 1$) that
\[
\varphi(x) \geq \min\{x, x^2\}/3,
\]
we see that (23) follows if $\varphi(t/(2\mu)) \geq \varphi(t/\mu)/4$. To sum up, it suffices to prove that
\[
\varphi(x/2) \geq \varphi(x)/4
\]
for $x \geq 0$. To this end we consider $f(x) = \varphi(x/2) - \varphi(x)/4$. Now, for $x \geq 0$ we have $4f'(x) = \log((1+x/2)^2/(1+x)) \geq 0$, so that $f(x) \geq f(0) = 0$, completing the proof. 

Our sparsification strategy intuitively focuses on high-degree vertices (with degree at least $r$). To quantify the number of removed edges, we shall introduce the auxiliary variable $M_r = M_r(H_p)$, which essentially counts high-degree vertices with ‘disjoint certificates’ (in the sense of Section 2). More precisely, we call $S = (v, W)$ an $r$-star in $G$ if $W = \{f_1, \ldots, f_{\lceil r \rceil}\} \subseteq \Gamma_v(G)$ and $|W| = \lceil r \rceil$. We write $V(S) = \bigcup_{1 \leq i \leq \lceil r \rceil} f_i$, which contains all vertices of the $r$-star $S$. Note that $V(S) \subseteq V_{\lceil r \rceil}(H)$ implies $|\Gamma_v(H)| \geq \lceil r \rceil$, i.e., that vertex $v$ has degree at least $\lceil r \rceil$. Writing $T_r(G)$ for the collection of all $r$-stars $S = (v, W)$ in $G$, we define
\[
M_r(G) = \max\{|\mathcal{M}| : \mathcal{M} \subseteq T_r(G) \text{ and } V(S_1) \cap V(S_2) = \emptyset \text{ for all distinct } S_1, S_2 \in \mathcal{M}\}.
\]
In words, $M_r(H_p)$ denotes the size of the largest vertex disjoint collection of $r$-stars in $H_p$, i.e., $r$-star matching. (As indicated earlier, it might be useful to think of $M_r(H_p)$ as the maximum number of degree $\geq r$ vertices that ‘occur disjointly’.) For future reference we note the following basic relation between $\Delta_1(H_p)$ and $M_r(H_p)$.

**Lemma 14:** Given $H$, for all $p \in [0, 1]$ and $z > 0$ we have $P(\Delta_1(H_p) \geq z) = P(M_z(H_p) \geq 1)$. 

The following combinatorial lemma is at the heart of our basic sparsification strategy: it intuitively relates \( X = e(H_p) \) with the auxiliary random variables \( X_r \) and \( M_r(H_p) \). In fact, inequality (27) below is inspired by the main deterministic ingredient of the ‘approximating by a disjoint subfamily’ technique (see, e.g., Lemma 3 in [21], which is used to count vertices in an auxiliary graph with \( V(G) = H_p \)). While Spencer’s technique hinges on the fact that disjoint edges are nearly independent (see also [39, 12]), here one important conceptual difference is that we allow for dependencies, i.e., overlaps of the edges (via \( r \geq 2 \) in \( X_r \)). For our applications the crux of (27) is that \( X_r < (1 + \varepsilon/2) \mu \) and \( k \geq r \) \( M_r(H_p) \Delta_1(H_p) < \varepsilon \mu / 2 \) together imply \( X < (1 + \varepsilon) \mu \).

**Lemma 15:** Suppose that \( H \) satisfies \( \max_{f \in H} |f| \leq k \). Then, for all \( p \in [0, 1] \) and \( r > 0 \) we have

\[
X_r \leq X \leq X_r + \mathbb{1}_{\{\Delta_1(H_p) > r\}} k[r] M_r(H_p) \Delta_1(H_p).
\]

The proof idea is simple: if \( M \subseteq T_r(H_p) \) attains the maximum in the definition of \( M_r(H_p) \), then after removing all edges incident to some star \( S = (v, W) \in M \) we obtain a hypergraph \( G \) with maximum degree at most \( |r| - 1 \leq r \) (otherwise we could add another \( r \)-star to the vertex disjoint collection \( M \)), so \( e(G) \leq X_r \). Inequality (27) combines this observation with trivial estimates for the number of removed edges.

**Proof of Lemma 15.** The lower bound \( X = e(H_p) \geq X_r \) is immediate. For the upper bound, note that \( X = X_r \) whenever \( \Delta_1(H_p) \leq r \), so we may henceforth assume \( \Delta_1(H_p) > r \). We fix some \( M \subseteq T_r(H_p) \) which attains the maximum in (26), so \( M_r(H_p) = |M| \). We remove all edges from \( H_p \) which contain at least one vertex from (the edges of) some \( r \)-star \( S = (v, \{f_1, \ldots, f_{|r|}\}) \in M \), and denote the remaining hypergraph by \( G \). As every edge contains at most \( \max_{f \in H} |f| \leq k \) vertices, we removed at most \( e(H_p) - e(G) \leq |M| \cdot |r| k \cdot \Delta_1(H_p) \) edges from \( H_p \). Clearly \( \Delta_1(G) \leq |r| - 1 \leq r \), because otherwise we could add another \( r \)-star to \( M \) (contradicting maximality). Hence \( G \) contains at most \( e(G) \leq X_r \) edges, and (27) follows.

Next, we shall exploit the disjoint-like structure of \( M_r(H_p) \) via the BK-inequality based Theorem 11. This leads to (28), a generic upper tail estimate for the size of the largest \( r \)-star matching \( M_r(H_p) \). Note that \( \mathbb{P}(\Delta_1(H_p) \geq r) \leq \sum_{v \in V(H)} \mathbb{P}(|\Gamma_v(H_p)| \geq |r|) = \Phi_r \). In this paper we mainly have very unlikely
degrees in mind, where $\Phi_r \leq Q^{-r}$ for some $Q > 1$. Then the probability that at least $y$ of such high-degree vertices (with degree at least $r$) ‘occur disjointly’ is roughly at most $Q^{-ry}$ by (28) below.

**Lemma 16:** Given $\mathcal{H}$, for all $p \in [0, 1]$ and $y, r > 0$ we have

$$\mathbb{P}(M_r(\mathcal{H}_p) \geq y) \leq \frac{\Phi_r^y}{y!} \leq \frac{1}{\sqrt{2\pi y}} \left(\frac{e\Phi_r}{y}\right)^y,$$

where $\Phi_r = \sum_{v \in V(\mathcal{H})} \mathbb{P}(|\Gamma_v(\mathcal{H}_p)| \geq \lceil r \rceil)$.

The main idea is very intuitive: if $M \subseteq T_r(\mathcal{H}_p)$ attains the maximum in the definition of $M_r(\mathcal{H}_p)$, then $\mathcal{H}_p$ contains $|M|$ vertex disjoint stars $S_v = (v, W) \in M$, each of which ‘certifies’ that the corresponding vertex $v$ has degree at least $\lceil r \rceil$ in $\mathcal{H}_p$ (in the sense of Section 2). Hence $M_r(\mathcal{H}_p) = |M|$ events of form $E_v = \{|\Gamma_v(\mathcal{H}_p)| \geq \lceil r \rceil\}$ ‘occur disjointly’, which allows us to bring (17) of Theorem 11 into play (with $C = 1$).

**Proof of Lemma 16.** We claim that $M_r(\mathcal{H}_p) \leq Z$ for $Z = Z_1$ as defined in Theorem 11 with $I = V(\mathcal{H})$, where $E_v$ denotes the event that $|\Gamma_v(\mathcal{H}_p)| \geq \lceil r \rceil$. This claim implies $\mathbb{P}(M_r(\mathcal{H}_p) \geq y) \leq \mathbb{P}(Z \geq y) \leq \mathbb{P}(Z \geq \lceil y \rceil)$, and we then deduce (28) by applying (17) with $C = 1$.

To establish $M_r(\mathcal{H}_p) \leq Z$, we pick any $M \subseteq T_r(\mathcal{H}_p)$ which attains the maximum in (26), so that $M_r(\mathcal{H}_p) = |M|$. For every $r$-star $S_v = (v, \{f_1,v, \ldots, f_{\lceil r \rceil},v\}) \in M$ we know that $V(S_v) = \bigcup_{1 \leq i \leq \lceil r \rceil} f_i,v \subseteq V_p(\mathcal{H})$ holds, which in turn implies $E_v$. In other words, the presence of the vertices $V(S_v) \subseteq V_p(\mathcal{H})$ constitutes a certificate for the event $E_v$ (using the notation of Section 2, we have $[\omega]_{V(S_v)} \subseteq E_v$). By definition of $M_r(\mathcal{H}_p)$ these certificates $(V(S_v))_{S_v \in M}$ are all disjoint, so $Z \geq |M| = M_r(\mathcal{H}_p)$, as claimed.

To summarize our proof framework: Lemmas 13–16 apply to arbitrary hypergraphs $\mathcal{H}$ with $\max_{f \in \mathcal{H}} |f| \leq k$, and they basically reduce the upper tail problem for $X = e(\mathcal{H}_p)$ to the upper tail problem for the degrees of $\mathcal{H}_p$, i.e., to $\Phi_x = \sum_v \mathbb{P}(|\Gamma_v(\mathcal{H}_p)| \geq \lceil x \rceil)$; see also (29) below. (These ideas are developed further in [44].)

In general, by noting $\mathbb{P}(|\Gamma_v(\mathcal{H}_p)| \geq \lceil x \rceil) \leq \mathbb{P}(|\Gamma_v(\mathcal{H}_p)| \geq \lceil x \rceil \mid v \in V_p(\mathcal{H}))$ there is room for induction (on the number of vertices per edge), analogous to [22, 25]. However, for the purposes of Theorems 4 and 6 it seems easier to
exploit the codegree condition $\Delta_2(H) = O(1)$ more directly (see the proof of Lemma 17).

3.2.1. *Sketch of the upper bound of Theorem 4.* In this section we sketch the proof of the upper bound of Theorem 4, illustrating the discussed proof framework. As we shall see, the desired ‘overshooting’ phenomenon (which yields the extra $\log(e/p)$ factor in the exponent) arises naturally. First, using Lemma 15, for all $r, y, z > 0$ satisfying $\mathbb{1}_{\{y > 1\}} k [r] y z \leq \varepsilon \mu / 2$ we obtain

\begin{equation}
\mathbb{P}(X \geq (1 + \varepsilon) \mu) \\
\leq \mathbb{P}(X_r \geq (1 + \varepsilon/2) \mu) + \mathbb{P}(M_r(H_p) \geq y) + \mathbb{1}_{\{y > 1\}} \mathbb{P}(\Delta_1(H_p) \geq z).
\end{equation}

(To clarify: for the indicator $\mathbb{1}_{\{y > 1\}}$ we exploited that $M_r(H_p) < 1$ implies $M_r(H_p) = 0$, which in turn entails $\Delta_1(H_p) < r$.)

Turning to further estimates of the right-hand side of (29), for $\varepsilon = \Theta(1)$ Lemma 13 yields

$$
\mathbb{P}(X_r \geq (1 + \varepsilon/2) \mu) \leq \exp\left(-\Theta\left(\frac{\mu}{r}\right)\right).
$$

This suggests that, in order to ‘match’ the exponent of our target bound (6), we should pick

\begin{equation}
r = \Theta\left(\max\{1, \sqrt{\mu}/\log(e/p)\}\right).
\end{equation}

It later turns out, see (45), that this natural choice satisfies $np^{k-1}/r = o(p^{1/4})$ for $k \geq 3$ (this fails for $k = 2$). In view of (21), we thus expect to obtain an extra $\log(e/p)$ factor in the exponent for $x \geq r$:

\begin{equation}
\Phi_x = \sum_{v \in V(H)} \mathbb{P}(|\Gamma_v(H_p)| \geq \lceil x \rceil)
\end{equation}

\begin{equation}
\leq \left[\left(\frac{p}{e}\right)^{1/4}\right]^{\Theta(x)} = \exp\left(-\Theta(x \log(e/p))\right).
\end{equation}

By Lemma 16 it thus seems plausible that for $x \geq r$ we have

\begin{equation}
\mathbb{P}(M_x(H_p) \geq y) \leq \Phi_x^{[y]} \leq \exp\left(-\Theta(xy \log(e/p))\right).
\end{equation}

Combining our heuristic findings with Lemma 14, for $\varepsilon = \Theta(1)$ and $z \geq r$ we thus expect that

\begin{equation}
\mathbb{P}(X \geq (1 + \varepsilon) \mu) \leq \exp\left(-\Theta\left(\frac{\mu}{r}\right)\right) + \exp\left(-\Theta\left(r y \log(e/p)\right)\right)
\end{equation}

\begin{equation}
+ \mathbb{1}_{\{y > 1\}} \exp\left(-\Theta\left(z \log(e/p)\right)\right).
\end{equation}
To ‘match’ the exponent of our target bound (6), in view of (30) it seems natural to set \( y = z/r \) and \( z = \sqrt{\varepsilon \mu/(4k)} \), say. In fact, these choices also satisfy two further technical conditions used above, namely, that \( k[r]yz \leq 2kryz = 2kz^2 \leq \varepsilon \mu/2 \) holds, and that \( y > 1 \) implies \( z \geq r \). Hence, if \( r \) is chosen as in (30), then for \( \varepsilon = \Theta(1) \) and \( \mu \geq 1 \) we expect that

\[
\mathbb{P}(X \geq (1+\varepsilon)\mu) \leq \exp\left(-\Theta\left(\min\{\mu, \sqrt{\mu} \log(e/p)\}\right)\right),
\]

which ‘matches’ the target bound (6) of Theorem 4. With hindsight, the freedom that via \( M_r(H_p) \) we can pick \( z \gg r \) in (29) seems key for going beyond the more basic decomposition (19).

3.2.2. Proof of the upper bound of Theorem 4. In this section we follow our heuristic proof sketch, and establish the upper bound of Theorem 4. We start with the size of the largest \( r \)-star matching \( M_r(H_p) \), and make the upper tail estimate (32) rigorous via Lemma 17 below (its statement is formulated with an eye on the upcoming proof of Theorem 6, where the \( n^2(\max\{y, 1\})^{3/2} \geq 1 \) term facilitates union bound arguments). The technical assumption (35) intuitively ensures that vertices with degree at least \( r \) are sufficiently concentrated (recall that the expected degree should be \( O(np^{k-1}) \); see the discussion in Section 3.1.2). For example, \( r = C(1+np^{k-1}) \) satisfies (35) when \( np^{k-1} \geq \log n \) or \( np^{k-1} \leq n^{-\gamma} \) for \( C = C(\gamma, B, k, D) \) sufficiently large, but for \( np^{k-1} \approx 1 \) a somewhat larger choice of \( r \) seems necessary (unless we impose additional constraints on \( y \) in (36) below). By the heuristics of Section 3.2.1, for \( r \) as defined in (30) we expect that \( np^{k-1}/x \leq p^{1/4} \) holds in inequality (36), i.e., as in (32) we should gain an extra logarithmic factor in the exponent of the upper tail by ‘overshooting’.

**Lemma 17:** Given \( k \geq 2, a > 0 \) and \( D \geq 1 \), let \( H = H_n \) be a \( k \)-uniform hypergraph satisfying \( v(H) \leq Dn \) and \( \Delta_2(H) \leq D \). Then there are \( B, n_0 \geq 1 \) (depending on \( k, D \)), such that for all \( n \geq n_0, p \in [0, 1], r > 0 \) satisfying

\[
(Bnp^{k-1}/r)^r \leq n^{-8kD}
\]

the following holds. For all \( x \geq r \) and \( y > 0 \) we have

\[
\mathbb{P}(M_x(H_p) \geq y) \leq \frac{1}{n^2(\max\{y, 1\})^{3/2}} \left( \frac{np^{k-1}}{ex} \right)^{xy/(2kD)}.
\]
Our plan is to deduce Lemma 17 from inequality (28) of Lemma 16, and in view of the parameter \( \Phi_x = \sum_{v \in V(H)} \mathbb{P}(|\Gamma_v(H_p)| \geq \lfloor x \rfloor) \) we thus study the degrees \( |\Gamma_v(H_p)| \). Here our main observation is simple, namely, as discussed in Section 3.1.2, every edge \( e \in \Gamma_v(H) \) intersects at most \( k\Delta_2(H) = O(1) \) edges \( f \in \Gamma_v(H) \), which suggests that the dependencies between the edges in \( \Gamma_v(H_p) \) are extremely weak. It thus seems plausible that, conditioned on \( v \in V_p(H) \), the tails of \( |\Gamma_v(H_p)| \) are comparable to those of \( \text{Bin}(|\Gamma_v(H)|, p^{k-1}) \) with \( |\Gamma_v(H)|p^{k-1} = O(np^{k-1}) \); see also (20)–(21). This line of reasoning can easily be made rigorous via Theorem 9, but below we take a more direct combinatorial route (which suffices for our purposes).

**Proof of Lemma 17.** It suffices to prove that for all \( x \geq r \) and \( n \geq n_0(D) \) we have

\[
\Phi_x = \sum_{v \in V(H)} \mathbb{P}(|\Gamma_v(H_p)| \geq \lfloor x \rfloor) \leq \frac{1}{e n^2} \left(\frac{np^{k-1}}{ex}\right)^{x/(2kD)}.
\]

Indeed, since \( y > 0 \) implies \( \lceil y \rceil \geq \max\{y, 1\} \), by applying (28) of Lemma 16 it then follows that

\[
\mathbb{P}(M_x(H_p) \geq y) \leq \mathbb{P}(M_x(H_p) \geq \lceil y \rceil) \leq \frac{(e\Phi_x)[y]}{\sqrt{|y|} \cdot |y||y|} \leq \left(\frac{np^{k-1}}{ex}\right)^{xy/(2kD)}\frac{1}{n^2(\max\{y, 1\})^{3/2}}.
\]

In the remainder we verify inequality (37), by focusing on combinatorial implications of the degree event \( |\Gamma_v(H_p)| \geq \lfloor x \rfloor \). To this end we pick a subset \( W \subseteq \Gamma_v(H_p) \) of the edges which is size maximal subject to the restriction that all edges of \( W \) are vertex disjoint outside of the centre vertex \( v \), i.e., that all distinct edges \( f_i, f_j \in W \) satisfy \( (f_i \cap f_j) \setminus \{v\} = \emptyset \). Note that for every edge \( e \in \Gamma_v(H_p) \) there are a total of (including \( e \) itself) at most \( k\Delta_2(H) \leq kD = C \) edges \( f \in \Gamma_v(H_p) \) with \( (f \cap e) \setminus \{v\} \neq \emptyset \) (because all such edges \( f \) contain \( v \) and at least one vertex from \( e \setminus \{v\} \)). Hence, \( |\Gamma_v(H_p)| \geq \lfloor x \rfloor \) implies

\[
|W| \geq |\Gamma_v(H_p)|/C \geq x/C.
\]

Since the union of all edges in \( W \) contains exactly \( |\bigcup_{f \in W} f| = 1 + (k-1)|W| \) vertices, it follows that

\[
\mathbb{P}(|\Gamma_v(H_p)| \geq \lfloor x \rfloor) \leq \left(\frac{|\Gamma_v(H)|}{\lfloor x/C \rfloor}\right)^{1+(k-1)[x/C]}.
\]
Recalling $|\Gamma_v(\mathcal{H})| \leq |V(\mathcal{H})| \Delta_2(\mathcal{H}) \leq D^2 n, \binom{m}{z} \leq (em/z)^z$ and $p \leq 1$, we obtain
\begin{equation}
\mathbb{P}(|\Gamma_v(\mathcal{H}_p)| \geq \lceil x \rceil) \leq \binom{D^2 n}{x/C} (eD^2 C n p^{k-1})^{\lceil x/C \rceil} \leq (\frac{eD^2 C n p^{k-1}}{x})^{\lceil x/C \rceil}.
\end{equation}

Defining $B = e^3 D^4 C^2$, using $C = kD$, $x \geq r$, and the assumption (35) it follows that
\[ \mathbb{P}(|\Gamma_v(\mathcal{H}_p)| \geq \lceil x \rceil) \leq \left( \frac{B n p^{k-1}}{r} \cdot \frac{np^{k-1}}{ex} \right)^{x/(2kD)} \leq n^{-4} \cdot \left( \frac{np^{k-1}}{ex} \right)^{x/(2kD)}.
\]

Recalling $|V(\mathcal{H})| \leq Dn$, this readily establishes inequality (37) for $n \geq n_0(D)$, completing the proof.

For the interested reader we remark that from the above proof idea it, e.g., also directly follows that
\[ \mathbb{P}(\mathbb{M}_r(\mathcal{H}_p) \geq x) \leq \sum_{U \subseteq V(\mathcal{H})} \left[ \prod_{v \in U} \left( \frac{|\Gamma_v(\mathcal{H})|}{\lceil r/C \rceil} \right) \right] (1+(k-1)[r/C]) n p^{(1+(k-1)[r/C])}\n \]
which can alternatively be used to derive (36). We find our general BK-inequality based approach more informative and flexible (e.g., with respect to possible extensions and generalizations, see [44]).

We are now ready to prove the upper bound of Theorem 4. Below we shall first pick $r$ as in (30), and then closely mimic the heuristic considerations (33)–(34) of Section 3.2.1. Only afterwards we verify $np^{k-1}/r = O(p^{1/4})$, the technical condition (35), and the heuristic tail inequality (32).

**Proof of (6) of Theorem 4.** With foresight, we define
\begin{equation}
s = \log(e/p^\gamma), \quad \gamma = 1/4, \quad \text{and} \quad A = \max\{eB/\sqrt{a}, 16k^2 D / \gamma\},
\end{equation}
where $B = B(k, D) \geq 1$ is as in Lemma 17. Furthermore, analogous to our heuristic outline, we set
\begin{equation}
r = A \max\{1, \sqrt{\mu/s}\}, \quad z = \sqrt{\varepsilon \mu/(4k)}, \quad \text{and} \quad y = z/r,
\end{equation}
so that $k[r]yz \leq 2kz^2 = \varepsilon \mu/2$. Since $y > 1$ implies $z \geq r$, using inequality (29) and Lemma 14 we obtain
\begin{equation}
\mathbb{P}(X \geq (1+\varepsilon)\mu) \leq \mathbb{P}(X_r \geq \mu + \varepsilon \mu/2) + \mathbb{P}(\mathbb{M}_r(\mathcal{H}_p) \geq y) + \mathbb{1}_{\{z \geq r\}} \mathbb{P}(\mathbb{M}_z(\mathcal{H}_p) \geq 1).
\end{equation}
We defer the proof of the technical claim that for all for \( x \geq r \) and \( y > 0 \) we have

\[
\mathbb{P}(M_x(H_p) \geq y) \leq \exp\left(-\frac{xy s}{2kD}\right).
\]

Inserting (42) into (41), using Lemma 13, \( ry = z \) and the definitions of \( r, z \) from (40) we infer

\[
\mathbb{P}(X \geq (1 + \varepsilon)\mu) \leq \exp\left(-\frac{\min\{\varepsilon, \varepsilon^2\} \mu}{12kr} \right) + 2\exp\left(-\frac{zs}{2kD}\right)
= \exp\left(-\frac{\min\{\varepsilon, \varepsilon^2\} \min\{\mu, \sqrt{s}\}}{12kA} \right) + 2\exp\left(-\frac{\sqrt{\varepsilon \mu s}}{2kD\sqrt{4k}}\right).
\]

Noting \( s \geq \gamma \log(e/p) \) and \( \min\{\varepsilon, \varepsilon^2, \sqrt{\varepsilon}\} = \min\{\varepsilon^2, \varepsilon^{1/2}\} \), there is \( d = d(k, A, D, \gamma) > 0 \) such that

\[
\mathbb{P}(X \geq (1 + \varepsilon)\mu) \leq 3\exp\left(-d \min\{\varepsilon, \varepsilon^{1/2}\} \min\{\mu, \sqrt{\mu \log(e/p)}\}\right) =: 3\exp\left(-\Psi\right).
\]

We claim that (6) holds with \( c(\varepsilon) = b \min\{\varepsilon^3, \varepsilon^{1/2}\} \) and \( b = d/6 \). In the main case \( \Psi \geq 3 \) this is obvious (as \( 3e^{-5\Psi/6} \leq 1 \) and \( \min\{\varepsilon^2, \varepsilon^{1/2}\} \geq \min\{\varepsilon^3, \varepsilon^{1/2}\} \)). In the degenerate case \( 1 \leq \Psi/3 \), Markov’s inequality yields

\[
\mathbb{P}(X \geq (1 + \varepsilon)\mu) \leq \frac{1}{1 + \varepsilon} = 1 - \frac{\varepsilon}{1 + \varepsilon} \leq \exp\left(-\frac{\varepsilon}{1 + \varepsilon}\right) \leq \exp\left(-\frac{\varepsilon \Psi}{3(1 + \varepsilon)}\right),
\]

which, due to \( \varepsilon/(1 + \varepsilon) \cdot \min\{\varepsilon^2, \varepsilon^{1/2}\} \geq \min\{\varepsilon^3, \varepsilon^{1/2}\}/2 \), establishes the claim.

In the remainder we verify the claimed estimate (42). Our below proof is based on Lemma 17, which requires us to check the technical condition (35). Calculus shows that

\[
p^\gamma s = p^\gamma \log(e/p^\gamma) \leq 1.
\]

Using \( r \geq A\sqrt{\mu}/2, \mu = e(H)p^k \geq an^2p^k, \) and \( k \geq 3 \) (this is the only time \( k \geq 2 \) is not enough), we obtain

\[
\frac{np^{k-1}}{r} \leq \frac{np^{k-1}s}{A\sqrt{\mu}} \leq \frac{p^{(k-2)/2}s}{A\sqrt{\alpha}} \leq \frac{p^{1/2}s}{A\sqrt{\alpha}} = \frac{p^{2\gamma}s}{A\sqrt{\alpha}} \leq \frac{p^\gamma}{eB},
\]

which also implies \( r \geq eBnp^{k-1-\gamma} \). Observe that \( p \geq n^{-1/(2k)} \) implies \( r \geq n^{1/2} \), say, and that \( p \leq n^{-1/(2k)} \) implies \( p^\gamma \leq n^{-\gamma/(2k)} \). Using \( r \geq A \), for \( n \geq n_0(k, D) \)
we thus infer
\[
\left( Bnp^{k-1/r} \right)^r \leq \left( p^\gamma/e \right)^r \leq \min \{ e^{-r}, p^{\gamma A} \} = \begin{cases} 1 & \{ p > n^{-1/(2k)} \} e^{-n^{1/2}} + \{ p \leq n^{-1/(2k)} \} n^{-\gamma A/(2k)} \\ \leq n^{-8kD} \end{cases}
\]
establishing (35). As (45) and \( B \geq 1 \) imply \( np^{k-1/x} \leq e^{-s} \) for all \( x \geq r \), inequality (36) of Lemma 17 now readily establishes the technical estimate (42), completing the proof.

Since our proofs are based on applications of Theorems 9 and 11, using Remarks 10 and 12 it is not difficult to see that all arguments carry over (essentially unchanged) to the uniform model \( \mathcal{H}_m = \mathcal{H}[V_m(\mathcal{H})] \) with \( k \leq m \leq v(\mathcal{H}) \) and \( p = m/v(\mathcal{H}) \), say, where \( V_m(\mathcal{H}) \subseteq V(\mathcal{H}) \) with \( |V_m(\mathcal{H})| = m \) is chosen uniformly at random (note that \( e(\mathcal{H}_m) = 0 \) if \( m < k \)). A similar remark also applies to the weighted case, where \( X = \sum_{e \in \mathcal{H}_p} w_e \) for positive constants \( w_e \in [\tilde{a}, \tilde{D}] \), say. In both cases we leave the straightforward details to the interested reader (these variations also carry over to the upcoming proofs of Section 3.3).

### 3.3. Some refinements (proof of the upper bound of Theorem 6).

In this section we refine our basic proof framework, and establish the more precise upper bound (9) of Theorem 6. Recall that the exponent of (9) is essentially either of sub-Gaussian type \( \exp(-ct^2/\text{Var}X) \) or clustered type \( \exp(-c\sqrt{t}\log(1/p)) \); see also (8). Heuristically speaking, the corresponding phase transition near \( (\text{Var}X)^{2/3} \) causes some technical difficulties for the approach taken in Section 3.2 (for \( p \geq n^{-1/(k-1)+o(1)} \) it turns out that sharp tail estimates are easier when \( t \) is far away from \( (\text{Var}X)^{2/3} \)). Here one bottleneck is Lemma 15, which on an intuitive level only distinguishes between two ranges of the degrees: smaller and larger than \( r \). In this section we shall rectify this issue, by distinguishing between a wide range of different degrees.

More concretely, our refined sparsification strategy is to iteratively decrease the maximum degree of \( \mathcal{H}_p \), until we are able to bound the number of remaining edges by \( X_r \) as defined in (22). Using the convention \( \mathbb{N} = \{0,1,\ldots\} \), we shall eventually implement this strategy via \( T(\beta, \gamma, r, t) \), which is the event that

\[
M_{r_j}(\mathcal{H}_p) < \beta \sqrt{ts}/r_j \quad \text{for all } j \in \mathbb{N} \text{ with } r_j < \sqrt{t}/s,
\]
\begin{equation}
M_{r_j}(\mathcal{H}_p) < \beta \sqrt{t}/r_j \quad \text{for all } j \in \mathbb{N} \text{ with } r_j \geq \sqrt{t}/s,
\end{equation}

where we tacitly used the following convenient parametrization:
\begin{equation}
s = s(\gamma) = \log(e/p\gamma) \quad \text{and} \quad r_j = r_j(r) = 2^j r.
\end{equation}

(The intricate form of (46)–(47) is hard to digest on first sight; both events are based on a delicate interplay between the combinatorial and probabilistic estimates in the upcoming proofs of Lemmas 18 and 19.)

The following combinatorial lemma intuitively states that \( X \approx X_r \) whenever \( T(\beta, \gamma, r, t) \) holds.

**Lemma 18:** Given \( k \geq 1 \), suppose that \( \mathcal{H} \) satisfies \( \max_{f \in \mathcal{H}} |f| \leq k \). Then, for all \( \beta \in (0, 1/(32k)] \), \( r \geq 1 \) and \( \gamma, t > 0 \), the event \( T(\beta, \gamma, r, t) \) implies \( X_r \leq X \leq X_r + t/2 \).

The idea is to iterate the proof of Lemma 15: using the resulting hypergraph sequence \( \mathcal{H}_p = \mathcal{G}_J \supseteq \cdots \supseteq \mathcal{G}_0 \) we shall estimate \( X = e(\mathcal{H}_p) \) in terms of the step-wise differences: \( X = e(\mathcal{G}_0) + \sum_{0 \leq j < J} [e(\mathcal{G}_{j+1}) - e(\mathcal{G}_j)] \). The definition of \( T(\beta, \gamma, r, t) \) then ensures that \( \sum_{0 \leq j < J} [e(\mathcal{G}_{j+1}) - e(\mathcal{G}_j)] \leq t/2 \) and \( e(\mathcal{G}_0) \leq X_r \) hold.

**Proof of Lemma 18.** The lower bound \( X = e(\mathcal{H}_p) \geq X_r \) is trivial, so we henceforth focus on the upper bound. Let \( J \) be the smallest integer \( J \geq 0 \) with \( r_j \geq \sqrt{t} \). We now construct the sequence \( (\mathcal{G}_j)_{0 \leq j \leq J} \) with \( \mathcal{G}_J = \mathcal{H}_p \) and \( \Delta_1(\mathcal{G}_j) \leq \lfloor r_j \rfloor \). For \( \mathcal{G}_J = \mathcal{H}_p \), observe that (47) and \( \beta \leq 1 \leq s \) imply \( M_{r_j}(\mathcal{H}_p) < \beta \leq 1 \) for all \( r_j \geq \sqrt{t} \). Hence, since \( \Delta_1(\mathcal{H}_p) \geq \lceil r_j \rceil \) implies \( M_{r_j}(\mathcal{H}_p) \geq 1 \), it follows that \( \Delta_1(\mathcal{G}_J) = \Delta_1(\mathcal{H}_p) \leq \lfloor r_J \rfloor - 1 \leq \lceil r_J \rceil \). Given \( \mathcal{G}_{j+1} \) with \( 0 \leq j < J \), we fix some \( \mathcal{M} \subseteq T_{[r_j]}(\mathcal{G}_{j+1}) \) which attains the maximum in (26), so that \( |\mathcal{M}| = M_{r_j}(\mathcal{G}_{j+1}) \leq M_{r_j}(\mathcal{G}_J) = M_{r_j}(\mathcal{H}_p) \) by monotonicity. We remove all edges from \( \mathcal{G}_{j+1} \) which contain at least one vertex from some \( r_j \)-star \( S \in \mathcal{M} \), and denote the resulting hypergraph by \( \mathcal{G}_j \). Hence \( \Delta_1(\mathcal{G}_j) \leq \lfloor r_j \rfloor - 1 \leq \lceil r_j \rceil \), because otherwise we could add another \( r_j \)-star to \( \mathcal{M} \) (contradicting the maximality of \(|\mathcal{M}|\)).

Next we estimate \( X = e(\mathcal{H}_p) \) in terms of the hypergraph sequence \( (\mathcal{G}_j)_{0 \leq j \leq J} \). Since each \( r_j \)-star consists of \( \lceil r_j \rceil \) edges, for \( 0 \leq j < J \) it follows by construction and monotonicity (using \( M_{r_j}(\mathcal{G}_{j+1}) \leq M_{r_j}(\mathcal{H}_p), \lfloor r_j \rfloor \leq r_j + 1 \leq 2r_j \) and...
\[ \Delta_1(G_{j+1}) \leq r_{j+1} = 2r_j \]

\[ e(G_{j+1}) - e(G_j) \leq M_{r_j}(G_{j+1}) \cdot [r_j] \max_{f \in \mathcal{H}} |f| \cdot \Delta_1(G_{j+1}) \leq M_{r_j}(\mathcal{H}_p) \cdot 4kr_j^2. \]

Hence, using \( \mathcal{H}_p = G_J \), (46)–(47) and \( \max_{0 \leq j < J} r_j \leq \sqrt{t} \) we readily obtain

\[
X = e(G_J) \leq e(G_0) + 4k \sum_{0 \leq j < J} M_{r_j}(\mathcal{H}_p)r_j^2 \\
\leq e(G_0) + 4\beta k \sqrt{t} \left( \sum_{0 \leq j < J: r_j \leq \sqrt{t}/s} r_j + \sum_{0 \leq j < J: \sqrt{t}/s \leq r_j \leq \sqrt{t}} r_j \right).
\]

For any \( z > 0 \), in view of \( r_j = 2^j r \) it is easy to see that

\[ \sum_{j \in \mathbb{N}: r_j \leq z} r_j = z \sum_{j \in \mathbb{N}: r_j \leq z} r_j/z \leq z \sum_{j \in \mathbb{N}} 2^{-j} = 2z. \]

Thus, noting that \( \Delta_1(G_0) \leq \lfloor r_0 \rfloor \leq r \) implies \( e(G_0) \leq X_r \), using \( \beta \leq 1/(32k) \) it follows that

\[ X \leq e(G_0) + 16\beta kt \leq X_r + t/2, \]

completing the proof.

In view of Lemmas 13 and 18, we now focus on the probability of the event \( \neg \mathcal{T}(\beta, \gamma, r, t) \). Ignoring some technical assumptions (which are similar to those of Lemma 17), the following result essentially states that \( \mathbb{P}(\neg \mathcal{T}(\beta, \gamma, r, t)) \) is negligible for our purposes (the \( 1/n \) prefactor in (50) is ad-hoc, and eventually becomes the usually irrelevant \( n^{-1} \) term in (9) of Theorem 6).

**Lemma 19:** Given \( k \geq 3 \), \( a > 0 \) and \( D \geq 1 \), let \( \mathcal{H} = \mathcal{H}_n \) be a \( k \)-uniform hypergraph satisfying \( v(\mathcal{H}) \leq Dn \), \( e(\mathcal{H}) \geq an^2 \) and \( \Delta_2(\mathcal{H}) \leq D \). Set \( X = e(\mathcal{H}_p) \), \( \mu = E[X] \) and \( \varphi(x) = (1 + x) \log(1 + x) - x \). Then there are \( B, n_0 \geq 1 \) (depending on \( k, D \)), such that for all \( n \geq n_0 \), \( p \in (0, 1] \), \( \beta \in (0, 1] \), \( \gamma \in (0, 1/8] \), and \( r, t > 0 \) satisfying (35) we have

\[ \mathbb{P}(\neg \mathcal{T}(\beta, \gamma, r, t)) \leq \frac{1}{n} \exp \left( -\min\{a, \beta\} \frac{\min \{ \varphi(t/\mu) \mu^2 \Lambda, \sqrt{ts} \}}{2kD} \right). \]

The definition of \( \mathcal{T}(\beta, \gamma, r, t) \) is, in some sense, already a significant part of the proof. Indeed, writing \( C = 2kD \), our argument hinges on the fact that (36) of Lemma 17 yields, in our case, a bound of the form

\[ \mathbb{P}(M_{r_j}(\mathcal{H}_p) \geq y) \leq \frac{1}{n^2} \min \left\{ e^{-r_j y/C}, \left( \frac{np^{k-1}}{er_j} \right)^{r_j y/C} \right\}. \]
Hence $\mathbb{P}(M_{r_j}(\mathcal{H}_p) \geq \beta \sqrt{t/s}/r_j) \leq n^{-2} e^{-\beta \sqrt{t/s}/C}$. Furthermore, for $r_j \geq \sqrt{t/s}$ it turns out that usually $np^{k-1}/(er_j) \leq p^j/e = e^{-s}$ holds, so $\mathbb{P}(M_{r_j}(\mathcal{H}_p) \geq \beta \sqrt{t/r_j}) \leq n^{-2} e^{-\beta \sqrt{t/s}/C}$ by ‘overshooting’. Recalling (46)–(47), using a careful union bound argument this reasoning eventually establishes inequality (50).

Proof of Lemma 19. Let $C = 2kD$. We use $B = B(k, D) \geq 1$ as given by Lemma 17, so that (36) holds for all $x = r_j$ and $y > 0$. Note that (35) entails $r \geq Bnp^{k-1} \geq np^{k-1}$. With (36) in hand, we now estimate $\mathbb{P}(\mathcal{T}(\beta, \gamma, r, t))$ by a delicate union bound argument. With foresight, we first assume $r \geq a \Phi$, where

$$\Phi = \frac{\varphi(t/\mu)\mu}{np^{k-1}}.$$  

Note that $M_{r_0}(\mathcal{H}_p) = 0$ entails $M_{r_j}(\mathcal{H}_p) = 0$ for all $j \geq 0$, which in view of (46) and (47) implies $\mathcal{T}(\beta, \gamma, r, t)$. Hence, using $r_0 = r \geq \max\{np^{k-1}, a \Phi\}$ and (36), we infer

$$\mathbb{P}(\mathcal{T}(\beta, \gamma, r, t)) \leq \mathbb{P}(M_{r_0}(\mathcal{H}_p) > 0) = \mathbb{P}(M_r(\mathcal{H}_p) \geq 1)$$

$$\leq \frac{1}{n} \left( \frac{np^{k-1}}{er} \right)^{r/C} \leq \frac{1}{n} \exp\left(-a \Phi/C\right).$$

We henceforth assume $r < a \Phi$. Using Lemma 17, $r_j = 2^j r \geq np^{k-1}$ and $s \geq 1$, we infer for $n \geq n_0(\beta)$ that

$$\mathbb{P}((46) \text{ fails}) \leq \sum_{j \in \mathbb{N} : r_j \leq \sqrt{t/s}} \mathbb{P}(M_{r_j}(\mathcal{H}_p) \geq \lceil \beta \sqrt{t/s}/r_j \rceil)$$

$$\leq \frac{r_j^{3/2}}{n^2(\beta \sqrt{ts})^{3/2} \cdot \exp\left(-\beta \sqrt{ts}/C\right)}$$

$$\leq \frac{1}{2n} \exp\left(-\beta \sqrt{ts}/C\right),$$

where the last inequality follows analogously to (49). Observing that $M_{r_{j+1}}(\mathcal{H}_p) \geq 1$ implies $M_{r_j}(\mathcal{H}_p) \geq 1$, for $n \geq n_0(\beta)$ a similar argument (exploiting that $r_j \geq \sqrt{t}$ implies $\beta \sqrt{t}/r_j \leq \beta \leq 1$) yields

$$\mathbb{P}((47) \text{ fails}) \leq \sum_{j \in \mathbb{N} : \sqrt{t/s} \leq r_j \leq \max(2\sqrt{t}, r)} \mathbb{P}(M_{r_j}(\mathcal{H}_p) \geq \lceil \beta \sqrt{t}/r_j \rceil)$$

$$\leq \frac{1}{2n} \max_{j \in \mathbb{N} : r_j \geq \sqrt{t/s}} \left( \frac{np^{k-1}/er_j}{e \sqrt{t}} \right)^{\beta \sqrt{t}/C} \leq \frac{1}{2n} \left( \frac{np^{k-1}s}{e \sqrt{t}} \right)^{\beta \sqrt{t}/C}.$$
(To clarify: the condition \( r_j \leq \max\{2\sqrt{t}, r\} \) ensures that the considered range of \( r_j \) is non-empty.) In the following we exploit the assumption \( r < a\Phi \) to further estimate (54). Note that \( \log(1 + x) \leq x \) implies

\[
\varphi(x) = (1 + x) \log(1 + x) - x \leq x^2.
\]

In view of (51) and (55), using \( \Phi > r/a \geq np^{k-1}/a \) and \( \mu = e(H)p^k \geq an^2p^k \) we deduce

\[
t^2 \geq \varphi(t/\mu)\mu^2 = \Phi \mu np^{k-1} \geq n^4p^{3k-2}.
\]

Since \( k \geq 3 \) and \( \gamma \leq 1/8 \) (in fact, \( \gamma \leq (k - 2)/8 \) suffices), using (56) and (44) we obtain

\[
\frac{np^{k-1}s}{e\sqrt{t}} \leq \frac{p^{(k-2)/4}s}{e} \leq \frac{p^{(k-2)/4-\gamma}}{e} \leq \frac{p^{1/4-\gamma}}{e} \leq \frac{p^\gamma}{e} = e^{-s}.
\]

Now, inserting (57) into (54), in view of (53) we infer (for \( r < a\Phi \)) that

\[
\mathbb{P}(-T(\beta, \gamma, r, t)) = \mathbb{P}((46) \text{ or } (47) \text{ fails}) \leq \frac{1}{n} \exp\left(-\beta \sqrt{ts/C}\right),
\]

which together with (52), \( C = kD \) and \( \Phi \geq \varphi(t/\mu)\mu^2/\Lambda \) completes the proof of (50). \( \blacksquare \)

We are now ready to prove the upper bound of Theorem 6, and our main remaining task is to pick a suitable parameter \( r \). Here the technical condition (35) prevents the natural choice \( r = CA/\mu = \Theta(1 + np^{k-1}) \) when \( np^{k-1} \approx 1 \), which explains the more involved form of \( r \) in the next proof (this complication is only needed in the pedestrian case (iii) below).

Proof of (9) of Theorem 6. It suffices to consider the following three cases:

(i) \( p \geq \gamma n^{-1/(k-1)}(\log n)^1/(k-1) \),

(ii) \( p \leq n^{-1/(k-1)-\gamma} \),

(iii) \( t \geq \min\{\gamma \min\{\text{Var} \, X\}^{2/3}, \mu^{2/3}\}(\log n)^{4/3}, \mu p^{(k-2)/3-\gamma}\} \).

Of course, in all cases we may assume \( \gamma \leq 1/8 \) (decreasing \( \gamma \) yields less restrictive assumptions), and in case (iii) we may also assume \( n^{-1/(k-1)-\gamma} \leq p \leq n^{-1/(2k)} \), say (otherwise case (i) or (ii) applies). We start by introducing several parameters. By Remark 7 there is a constant \( b = b(k, a, D) \in (0, 1] \) such that for all \( p \in [0, 1/2] \) we have

\[
\text{Var} \, X \geq b\Lambda.
\]
Let $\beta = 1/(32k)$. Define $s = s(\gamma)$ as in (48), and set
\[
r = A\tilde{r}, \quad A = \max \left\{ \frac{3B}{\min\{1, a^{1/2}, b\}}, \frac{32k^2D}{\gamma^{k-1}}, \frac{24kD}{\min\{1, a^{1/2}, b\}^{3/2}} \right\},
\]
and
\[
\tilde{r} = \max \left\{ \frac{\Lambda}{\mu}, \frac{\varphi(t/\mu)\mu}{\sqrt{ts}} \right\},
\]
where $B = B(k, D)$ is as in Lemma 19. We defer the proof of the claim that $r$ satisfies the technical condition (35), and first apply Lemmas 13 and 18–19.

So, using the definition of $r$, it follows that
\[
P(X \geq \mu + t) \leq P(X_r \geq \mu + t/2) + P(\mathcal{T}(\beta, \gamma, r, t))
\]
\[
\leq \exp \left( -\frac{\varphi(t/\mu)\mu}{4kr} \right) + \frac{1}{n} \exp \left( -\min\{a, \beta\} \frac{\varphi(t/\mu)\mu^2}{2kD} \min \left\{ \frac{\varphi(t/\mu)\mu^2}{\Lambda}, \sqrt{ts} \right\} \right)
\]
\[
\leq (1 + n^{-1}) \exp \left( -\frac{\min\{a, \beta, 1\}}{4kA} \min \left\{ \frac{\varphi(t/\mu)\mu^2}{\Lambda}, \sqrt{ts} \right\} \right).
\]
Since $s = \log(e/p^\gamma) \geq \gamma \log(e/p)$, we have (9) with $c = \gamma \min\{a, \beta, 1\}/(4kA)$.

In the remainder we verify the technical condition (35). For later reference, note that
\[
\tilde{r} \geq \Lambda/\mu \geq \max\{np^{k-1}, 1\}.
\]
Recalling $r = A\tilde{r}$, in case (i) we have $r \geq Anp^{k-1} \geq A\gamma^{k-1} \log n$, and in case (ii) we have $np^{k-1} \leq n^{-(k-1)\gamma}$ and $r \geq A$. In both cases, using $r \geq \max\{eBnp^{k-1}, B\}$ and $r \geq A \geq 8kD/\gamma^{k-1}$ we infer that
\[
(Bnp^{k-1}/r)^r \leq \min\{e^{\tilde{r}}, (np^{k-1})^\tilde{r}\}
\]
\[
\leq \max\{n^{-A\gamma^{k-1}}, n^{-(k-1)\gamma}\} \leq n^{-8kD}.
\]
The remaining case (iii) requires somewhat tedious case distinctions. Recalling (24), it follows that
\[
\tilde{r} \geq \frac{\varphi(t/\mu)\mu}{\sqrt{ts}} \geq \min\{t^{1/2}, t^{3/2}/\mu\} \geq 1_{\{t \geq \mu\}} \frac{\mu^{1/2}}{3s} + 1_{\{t < \mu\}} \frac{t^{3/2}}{3\mu s}.
\]
With foresight, note that (44) and $p \geq n^{-(k-1)-\gamma}$ imply, for $n \geq n_0$, that
\[
s = \log(e/p^\gamma) \leq \min\{1 + \gamma \log(1/p), p^{-\gamma}\} \leq \min\{\log n, p^{-\gamma}\}.
\]
Using (58) and \( p = o(1) \) we have \( \text{Var} X \geq b \mu \), where \( b \in (0, 1] \). Combining this estimate with the assumed lower bound for \( t \) in the case (iii), using \( \mu = e(H)p^k \geq an^2p^k \) and (62) it follows that

\[
\frac{t^{3/2}}{\mu s} \geq \min \left\{ \frac{\gamma^{3/2}b(\log n)^2}{s}, \frac{\mu^{1/2}p^{(k-2)/2-3\gamma/2}}{s} \right\}
\]

\[
\geq \min \left\{ \gamma^{3/2}b \log n, a^{1/2}np^{k-1-\gamma/2} \right\}.
\]

Since \( k \geq 3 \) and \( \gamma \leq 1/8 \) imply \( 1 \geq p^{(k-2)/2-3\gamma/2} \), note that the final expression in (63) is also a lower bound for \( \mu^{1/2}/s \). In view of (61), we thus infer

\[
\tilde{r} \geq 3^{-1} \min\{a^{1/2}, b\} \cdot \min\{\gamma^{3/2} \log n, np^{k-1-\gamma/2}\}.
\]

If the minimum in (64) is attained by the \( \gamma^{3/2} \log n \) term, then \( r = A\tilde{r} \geq eB\tilde{r} \) and (59) imply \( (Bnp^{k-1}/r)^r \leq e^{-r} = e^{-A\tilde{r}} \), so that \( A\tilde{r} \geq 8kD \log n \) establishes (35). Otherwise the minimum in (64) is attained by the \( np^{k-1-\gamma/2} \) term, in which case \( r = A\tilde{r} \) implies \( (Bnp^{k-1}/r)^r \leq (p^{\gamma/2})^r \) by choice of \( A \).

Using \( p \leq n^{-1/(2k)} \) and \( r = A\tilde{r} \geq A \geq 32k^2D/\gamma \), this readily establishes (35), completing the proof.

4. Lower bounds

In this section we establish the lower bounds (7) and (10) of Theorem 4 and 6. The proofs are based on three different ‘configurations’ of the vertices in \( V_p(H) \), which each yield a distinct lower bound for the upper tail of \( X = e(H_p) \). The heuristic idea is that one of them should hopefully always approximate the most likely way to obtain \( X \approx (1 + \varepsilon)\mu \) or \( X \approx \mu + t \), respectively. In brief, we shall use configurations where many edges cluster on few vertices (Section 4.1), where many edges arise disjointly (Section 4.2), or where there are overall too many vertices (Section 4.3). Here one main novelty is on a conceptual level: in contrast to previous work we obtain, in a wide range, the correct dependence on \( t = \varepsilon \mu \).

4.1. Configurations with clustering. The first lower bound is based on property \( \mathcal{X}(H, D, x) \) defined in (5), which intuitively states that many edges can cluster on comparatively few vertices. In other words, enforcing \( W \subseteq V_p(H) \) for a reasonably small set of vertices \( W \) is enough to guarantee that the number of induced edges \( X = e(H_p) = e(H[V_p(H)]) \) is fairly large. A related approach was taken in [23] and [20] for arithmetic progressions and subgraphs, respectively.
Theorem 20: Given a hypergraph $\mathcal{H}$, set $X = e(\mathcal{H}_p)$ and $\mu = \mathbb{E}X$. For all $D \geq 1$, $p \in (0, 1]$ and $t \geq 0$ satisfying $\mathfrak{X}(\mathcal{H}, D, \mu + t)$ and $\mu + t \geq 1$ we have

$$
P(X \geq \mu + t) \geq \exp\left(-D\sqrt{\mu + t} \log(1/p)\right).$$

Proof. By $\mathfrak{X}(\mathcal{H}, D, \mu + t)$ there is $W \subseteq V(\mathcal{H})$ satisfying $|W| \leq D\sqrt{\mu + t}$ and $e(\mathcal{H}[W]) \geq \mu + t$. Hence

$$
P(X \geq \mu + t) \geq P(W \subseteq V_p(\mathcal{H})) = p^{|W|} \geq p^{D\sqrt{\mu + t}},$$

completing the proof. \qed

Using a new ‘local’ variant of the above argument we now improve the $\sqrt{\mu + t}$ in the exponent of (65) to $\sqrt{t}$, which is crucial when $t = o(\mu)$. The basic idea is to ‘create’ at least $\mu + t$ edges as follows: (i) first we use the above clustering construction to ‘locally’ enforce, say, $2t$ edges, and (ii) then we use correlation inequalities and a one-sided version of Chebyshev’s inequality to show that typically at least $\mu - t$ of the remaining $r = e(\mathcal{H}) - 2t$ edges are present in $\mathcal{H}_p$. (The crux is that the expected number of remaining edges is at least $rp^k = \mu - 2tp^k$.) This approach seems of independent interest, and a similar reasoning can, e.g., be used to refine the lower bounds for subgraph counts obtained by Janson, Oleszkiewicz and Ruciński [20].

Theorem 21: Given $k \geq 2$, $a > 0$ and $D \geq 1$, let $\mathcal{H} = \mathcal{H}_n$ be a $k$-uniform hypergraph satisfying $v(\mathcal{H}) \leq Dn$, $e(\mathcal{H}) \geq an^2$ and $\Delta_2(\mathcal{H}) \leq D$. Set $X = e(\mathcal{H}_p)$, $\mu = \mathbb{E}X$ and $\Lambda = \mu(1 + np^{k-1})$. Given $\alpha \in (0, 1)$, there are $n_0 > 0$ (depending only on $k, a, D$) and $c, \lambda \geq 1$ (depending only on $\alpha, k, a, D$) such that for all $n \geq n_0$, $p \in (0, 1 - \alpha]$ and $t \geq 1_{\{\mu \geq 1/2\}} \min\{\sqrt{\text{Var}X}, \sqrt{\Lambda}\}$ satisfying $\mathfrak{X}(\mathcal{H}, D, \min\{\lambda t, \mu + t\})$ and $\mu + t \geq 1$ we have

$$
P(X \geq \mu + t) \geq \exp\left(-c\sqrt{t} \log(1/p)\right).$$

We remark that the form of the somewhat strange-looking assumption $t \geq 1_{\{\mu \geq 1/2\}} \min\{\sqrt{\text{Var}X}, \sqrt{\Lambda}\}$ will be convenient later on. Before giving the proof of Theorem 21, let us informally discuss the structure of the argument. The clustering construction intuitively ‘marks’ a set of $2t$ edges in $\mathcal{H}$. Let $Z$ denote the number of ‘unmarked’ edges that occur in $\mathcal{H}_p$, so $\mathbb{E}Z = (e(\mathcal{H}) - 2t)p^k = \mu - 2tp^k$. The punchline is that the clustering construction (which enforces the $2t$ ‘marked’ edges) allows us to shift our focus from the unlikely event $X \geq \mathbb{E}X + t$ to the ‘typical’ event $Z \geq \mathbb{E}Z - t/2$. Indeed, it turns out that, using Harris’
inequality [16] and \( \mu = \mathbb{E}Z + 2tp^k \), for suitable \( W \subseteq V(\mathcal{H}) \) with \( |W| = O(\sqrt{t}) \) and \( e(\mathcal{H}[W]) \geq 2t \) we eventually arrive at
\[
\mathbb{P}(X \geq \mu + t) \geq \mathbb{P}(W \subseteq V_p(\mathcal{H})) \cdot \mathbb{P}(Z \geq \mu - t) \geq p^{\Theta(\sqrt{t})} \cdot \mathbb{P}(Z \geq \mathbb{E}Z - t + 2tp^k).
\]

It seems plausible that \( \text{Var } Z = O(\text{Var } X) \) holds. A folklore variant of the Paley–Zygmund inequality states that, given any random variable \( Y \geq 0 \), for all \( 0 \leq t < \mathbb{E}Y \) we have
\[
\mathbb{P}(Y \geq \mathbb{E}Y - t) \geq \frac{t^2}{\text{Var } Y + t^2}.
\]

So, assuming \( p \leq 1/2 \) (which implies \( 2p^k \leq 1/2 \) for \( k \geq 2 \)), for \( t \geq \sqrt{\text{Var } X} \) we should intuitively obtain
\[
\mathbb{P}(Z \geq \mathbb{E}Z - t + 2tp^k) \geq \mathbb{P}(Z \geq \mathbb{E}Z - t/2) \geq \Omega \left( \frac{t^2}{\text{Var } Z + t^2} \right) = \Omega(1).
\]

The proof below makes this reasoning rigorous, but there are a number of subtle issues (which make the details somewhat cumbersome). For example, the parameter \( t \) may be very small, so we cannot, as usual, ignore rounding issues. Furthermore, to allow for \( p \leq 1 - \alpha \) we need to plant \( \lambda t \) copies (instead of just \( 2t \) copies) for carefully chosen \( \lambda = \lambda(\alpha, k) > 0 \). In addition, the \( W \subseteq V_p(\mathcal{H}) \) based construction does not work if \( \lambda t \) is larger than the total number of edges \( e(\mathcal{H}) \), so we shall only enforce \( \min\{\lambda t, \mu + t\} \) copies.

**Proof of Theorem 21.** We defer the elementary proof of the fact that there is \( \lambda = \lambda(\alpha, k) > 0 \) satisfying
\[
\lambda t \geq 2.
\]

Defining \( x = \min\{\lambda t, \mu + t\} \), by \( X(\mathcal{H}, D, x) \) there is \( W \subseteq V(\mathcal{H}) \) satisfying \( |W| \leq D\sqrt{\lambda t} \) and \( e(\mathcal{H}[W]) \geq x \). To later avoid rounding issues, we pick \( \beta + 1 \in [\lambda/2, \lambda] \) such that \( (\beta + 1)t \) is an integer. Defining \( y = \min\{((\beta + 1)t, \mu + t\} \), note that there is \( G \subseteq \mathcal{H}[W] \) with \( e(G) = [y] \). Define \( Y = e(G[V_p(\mathcal{H})]) \). Clearly,
\[
\mathbb{P}(Y \geq y) = \mathbb{P}(Y \geq \min\{((\beta + 1)t, \mu + t\}) \geq \mathbb{P}(W \subseteq V_p(\mathcal{H})) = p^{|W|} \geq p^{D\sqrt{\lambda t}}.
\]

In the case \( \mu \leq \beta t \) we have \( \mu + t \leq y \), so that \( \mathbb{P}(X \geq \mu + t) \geq \mathbb{P}(X \geq y) \geq \mathbb{P}(Y \geq y) \) and (69) establish inequality (66) for any constant \( c \) satisfying \( c \geq D\sqrt{\lambda} \) (we defer the precise choice of \( c \)).
Henceforth we focus on the more interesting case $\mu > \beta t$. Define $Z = X - Y$. Since $Y \geq (\beta + 1)t$ and $Z \geq \mu - \beta t$ are both increasing events, using $X = Y + Z$, Harris’ inequality [16], and (69) we infer

$$\mathbb{P}(X \geq \mu + t) \geq \mathbb{P}(Y \geq (\beta + 1)t)$ and $Z \geq \mu - \beta t)$

$$\geq \mathbb{P}(Y \geq (\beta + 1)t)\mathbb{P}(Z \geq \mu - \beta t) \geq p^{D\sqrt{\lambda t}}\mathbb{P}(Z \geq \mu - \beta t).$$

(70)

We defer the proof of the conceptually straightforward (but slightly tedious) claim that

$$\mathbb{E} Y \leq (\beta - 1)t,$$

(71)

$$\text{Var} Z \leq Ct^2,$$

(72)

where $C = C(k, a, D, \lambda) \geq 1$. Using $\mathbb{E} Z - t = \mathbb{E} X - \mathbb{E} Y - t \geq \mu - \beta t$ and the Paley–Zygmund inequality (67), for $d = \log_{1-\alpha}(1/(C + 1)) > 0$ it follows (exploiting $1 - \alpha \geq p$ and $1 \leq \lambda t$) that

$$\mathbb{P}(Z \geq \mu - \beta t) \geq \mathbb{P}(Z \geq \mathbb{E} Z - t) \geq \frac{t^2}{\text{Var} Z + t^2} \geq \frac{1}{C + 1} = (1 - \alpha)^d \geq p^d \geq p^{d\sqrt{\lambda t}}.$$

(73)

Inserting (73) into (70) establishes inequality (66) with $c = D\sqrt{\lambda} + d\sqrt{\lambda}$.

It remains to prove the auxiliary claims (68) and (71)–(72). Let $\lambda = 4/(1 - (1 - \alpha)^k)$. Writing $Y_e = \mathbb{1}_{\{e \subseteq V_p(\mathcal{H})\}}$, note that Harris’ inequality yields $\mathbb{E}(Y_e Y_f) \geq \mathbb{E} Y_e \mathbb{E} Y_f$. As $\mathbb{E} Y_e^2 = \mathbb{E} Y_e$, we infer

$$\text{Var} X = \sum_{e, f \in \mathcal{H}} \left[\mathbb{E}(Y_e Y_f) - \mathbb{E} Y_e \mathbb{E} Y_f\right] \geq \sum_{e \in \mathcal{H}} (1 - \mathbb{E} Y_e) \mathbb{E} Y_e$$

$$\geq (1 - p^k)\mu \geq (1 - (1 - \alpha)^k)\mu = 4\mu/\lambda.$$

(74)

Observing $\Lambda \geq \mu$ and $t \geq 1 - \mu$, using the assumed lower bound for $t$ (and $\lambda \geq 4$) it follows that

$$\lambda t \geq \lambda \left(\mathbb{1}_{\{\mu \geq 1/2\}} \min\{\sqrt{4\mu/\lambda}, \sqrt{\mu}\} + \mathbb{1}_{\{\mu < 1/2\}} (1 - \mu)\right) \geq 2,$$

establishing the claimed inequality (68). Recall that we only need to prove (71)–(72) whenever $\mu > \beta t$. In this case $[y] = (\beta + 1)t$ holds by choice of $\beta$, so that $(1 - \alpha)^k = 1 - 4/\lambda$ and $\beta + 1 \geq \lambda/2$ imply

$$\mathbb{E} Y = [y]p^k \leq (\beta + 1)(1 - \alpha)^k t = [\beta + 1 - (\beta + 1)4/\lambda] t \leq (\beta - 1)t,$$
establishing the claimed inequality (71). To get a handle on \( \text{Var} Z \) in (72), note that \( Z \) is a restriction of \( X \) to a subset of the edges of \( \mathcal{H} \). So, with (74) and \( E(Y_\epsilon Y_f) - EY_\epsilon EY_f \geq 0 \) in mind, it is not difficult to see that \( \text{Var} Z \leq \text{Var} X \) holds. By Remark 7 there is a constant \( A = A(k, a, D) > 0 \) such that

\[
(75) \quad \text{Var} X \leq A\Lambda = A\mu(1 + np^{k-1}).
\]

Recalling \( \mu \geq an^2p^k \), it is easy to see that \( \mu < 1/2 \) implies \( p = O(n^{-2/k}) \) and \( \text{Var} X \leq B = B(A, k, a, D) > 0 \). Using (75) and the assumed lower bound for \( t \) in case of \( \mu \geq 1/2 \), it follows (exploiting \( 1 \leq \lambda t \)) that

\[
\text{Var} Z \leq \text{Var} X \leq \max\{B\lambda^2, A, 1\} t^2,
\]

completing the proof.

Using a variant of the above proof, it alternatively suffices to assume \( t \geq \max\{\sqrt{p \text{Var} X}, 1\} \), say. Furthermore, for \( p = o(1) \) and \( t = O(\mu) \) with \( t = \omega(1) \) we can easily improve the constant \( c \) by planting only \((1 + o(1))t \) edges (in some cases, this approach presumably yields the ‘optimal’ form of the exponent).

4.2. Configurations with many disjoint edges. The second lower bound is based on the heuristic that, for small \( p \), most edges of \( \mathcal{H}_p \) should arise disjointly. Exploiting the implied ‘approximate independence’ of the edges, we obtain the following Chernoff-like lower bound. In fact, (76) is of sub-Gaussian type since \( \mathbb{E}X = (1 + o(1)) \text{Var} X \) for the \( p \) under consideration.

**Theorem 22:** Given \( k \geq 3 \), \( a > 0 \) and \( D \geq 1 \), let \( \mathcal{H} = \mathcal{H}_n \) be a \( k \)-uniform hypergraph satisfying \( v(\mathcal{H}) \leq Dn \), \( e(\mathcal{H}) \geq an^2 \) and \( \Delta_2(\mathcal{H}) \leq D \). Set \( X = e(\mathcal{H}_p) \), \( \mu = \mathbb{E}X \) and \( \varphi(x) = (1 + x)\log(1 + x) - x \). There are \( n_0, c, d > 0 \) (depending only on \( k, a, D \)) such that for all \( n \geq n_0 \), \( 0 < p \leq n^{-2/(k+1/3)} \) and \( t \geq 0 \) satisfying \( 1 \leq \mu + t \leq 9 \max\{\mu, n^{1/(2k)}\} \) we have

\[
(76) \quad \mathbb{P}(X \geq \mu + t) \geq d \exp(-c\varphi(t/\mu)\mu) \geq d \exp(-ct^2/\mu).
\]

We have not tried to optimize \( p \leq n^{-2/(k+1/3)} \), but conjecture that this condition can be relaxed to \( p = O(n^{-1/(k-1)}) \). In fact, it would be interesting to have a general method which yields such Poisson-type lower bounds for the upper tail when \( \text{Var} X = (1 + o(1))\mathbb{E}X \) holds (for the lower tail this was very recently settled by Janson and Warnke [24]). In the proof of Theorem 22 we shall use the idea that, for small \( p \), most edges \( f \in \mathcal{H} \) should appear disjointly (and
thus nearly independently) in $\mathcal{H}_p$. The next lemma makes this more precise: it relates $\mathbb{P}(X = m)$ with $\mathbb{P}(\text{Bin}(e(\mathcal{H}), p^k) = m)$ over a convenient (but ad-hoc) range of $m$.

**Lemma 23:** Given $k \geq 3$, $a > 0$ and $D \geq 1$, let $\mathcal{H} = \mathcal{H}_n$ be a $k$-uniform hypergraph satisfying $v(\mathcal{H}) \leq Dn$, $e(\mathcal{H}) \geq an^2$ and $\Delta_2(\mathcal{H}) \leq D$. Set $X = e(\mathcal{H}_p)$ and $\mu = \mathbb{E}X$. There are $n_0, b > 0$ (depending only on $k, a, D$) such that for all $n \geq n_0$, $0 < p \leq n^{-2/(k+1/3)}$ and integers $0 \leq m \leq 99 \max\{\mu, n^{1/(2k)}\}$ we have

$$\mathbb{P}(X = m) \geq e^{-b\left(\frac{e(\mathcal{H})}{m}\right)}p^{km}(1 - p^k)^{e(\mathcal{H}) - m}. \tag{77}$$

With Lemma 23 in hand, the proof of Theorem 22 essentially reduces to folklore lower bounds for the binomial distribution (based on Stirling’s formula); we include the details in Appendix A for completeness (some minor care is needed when $t$ is small). A similar analysis can be used to tighten related results in the theory of random graphs due to DeMarco and Kahn [8] and Šileikis [38].

Let us informally discuss the strategy used in the proof of Lemma 23. For (77) the basic plan is to consider the event that $\mathcal{H}_p$ consists of exactly $m$ vertex disjoint edges. It turns out that, for small $m$, there are roughly $\binom{e(\mathcal{H})}{m}$ ways to select such edge collections, and with probability $p^{km}$ their $m$ disjoint edges are all present. Of course, we also need to take into account that all of the remaining $e(\mathcal{H}) - m$ edges are not present (to avoid overcounting). If these were independent events, then this would yield another factor of $(1 - p^k)^{e(\mathcal{H}) - m}$, and for small $p$ we expect that this is usually close to the truth. The proof below follows the discussed outline, dropping the (de facto redundant) disjointness condition. However, we need to deal with one subtle technicality that we ignored so far: given a collection of edges $\{f_1, \ldots, f_m\} \subseteq \mathcal{H}$, it can happen that the union of their vertex sets $\bigcup_{i \in [m]} f_i$ induces additional ‘extra’ edges from $\mathcal{H}$ (even if all the $f_i$ are vertex disjoint). In particular, for our construction this means that the second part is impossible: in this ‘bad’ case at least one of the remaining $e(\mathcal{H}) - m$ edges must occur. Luckily, such bad edge collections are rare for small $m$, so we can simply ignore them in our proof (see the definition of $\mathcal{S}_m$ below).

**Proof of Lemma 23.** Define

$$\mathcal{S}_m = \left\{ \mathcal{I} \subseteq \mathcal{H} : e(\mathcal{I}) = m, \text{ and there are no } g \in \mathcal{H} \setminus \mathcal{I} \text{ with } g \subseteq \bigcup_{f \in \mathcal{I}} f \right\}. \tag{78}$$
Recall that \( f \in \mathcal{H}_p \) if and only if \( f \subseteq V_p(\mathcal{H}) \). As the union of all edges in \( I \in \mathcal{G}_m \) contains at most \( km \) vertices, we have \( P(I \subseteq \mathcal{H}_p) \geq p^{km} \) (for disjoint edges this would hold with equality.) So, since the events \( \{I = \mathcal{H}_p\}_{I \in \mathcal{G}_m} \) are mutually exclusive, using \( P(I = \mathcal{H}_p) = P(I \subseteq \mathcal{H}_p) P(I = \mathcal{H}_p | I \subseteq \mathcal{H}_p) \) it follows that

\[
P(X = m) \geq \sum_{I \in \mathcal{G}_m} P(I \subseteq \mathcal{H}_p) P(I = \mathcal{H}_p | I \subseteq \mathcal{H}_p)
\]

\[
\geq |\mathcal{G}_m| p^{km} \min_{I \in \mathcal{G}_m} P(I = \mathcal{H}_p | I \subseteq \mathcal{H}_p).
\]

(79)

It remains to estimate \( |\mathcal{G}_m| \) and \( P(I = \mathcal{H}_p | I \subseteq \mathcal{H}_p) \) from below. We defer the routine proof of the auxiliary claim that there is \( \lambda = \lambda(k,a,D) > 0 \) such that for \( n \geq n_0(k,a,D) \) we have

\[
k^3 D^3 nm^2/e(\mathcal{H}) \leq 1/2 \quad \text{and} \quad \max \{ nm^3/e(\mathcal{H}), m^2p, nmp^{k-1} \} \leq \lambda.
\]

We bound \( |\mathcal{G}_m| \) from below by constructing certain edge-subsets \( I = \{f_1, \ldots, f_m\} \in \mathcal{G}_m \), counting the number of choices in each step. For \( 0 \leq j < m \) we iteratively select \( f_{j+1} \in \mathcal{H} \setminus (\mathcal{B}_{1,j+1} \cup \mathcal{B}_{2,j+1}) \), where

\[
\mathcal{B}_{x,j+1} = \left\{ f \in \mathcal{H} : \text{there is } g \in \mathcal{H} \text{ with } |g \cap \bigcup_{i \in [j]} f_i| \geq x \text{ and } |g \cap f| \geq 3 - x \right\}.
\]

Since \( \{f_1, \ldots, f_j\} \subseteq \mathcal{B}_{1,j+1} \) holds (consider \( g = f = f_i \)), all edges \( f_i \) are distinct (in fact, vertex disjoint). Next, aiming at a contradiction, suppose there is an edge \( g \in \mathcal{H} \setminus I \) and an index \( \ell \in [m] \) such that \( g \subseteq \bigcup_{i \in [\ell]} f_i \) and \( g \not\subseteq \bigcup_{i \in [\ell-1]} f_i \). If \( |g \cap \bigcup_{i \in [\ell-1]} f_i| = 1 \), then \( |g \cap f_\ell| = k - 1 \geq 2 \) implies \( f_\ell \in \mathcal{B}_{1,\ell} \). If \( |g \cap \bigcup_{i \in [\ell-1]} f_i| \geq 2 \), then \( |g \cap f_\ell| \geq 1 \) implies \( f_\ell \in \mathcal{B}_{2,\ell} \). Both conclusions contradict \( f_\ell \not\in \left( \mathcal{B}_{1,\ell} \cup \mathcal{B}_{2,\ell} \right) \), showing that all constructed sets \( I = \{f_1, \ldots, f_m\} \) indeed satisfy \( I \in \mathcal{G}_m \). Turning to the number of choices in the above greedy construction, note that \( |\mathcal{B}_{1,j+1}| \leq kj \cdot \Delta_1(\mathcal{H}) \cdot \binom{k}{2} \Delta_2(\mathcal{H}) \) and \( |\mathcal{B}_{2,j+1}| \leq \binom{k^2}{2} \Delta_2(\mathcal{H}) \cdot k \Delta_1(\mathcal{H}) \). Since \( \Delta_2(\mathcal{H}) \leq D \) and \( \Delta_1(\mathcal{H}) \leq v(\mathcal{H}) \Delta_2(\mathcal{H}) \leq D^2n \), we infer that for each edge \( f_{j+1} \) there are at least

\[
e(\mathcal{H}) - (|\mathcal{B}_{1,j+1}| + |\mathcal{B}_{2,j+1}|) \geq e(\mathcal{H}) - (k^3 D^3 nj/2 + k^3 D^2 n j^2/2) \geq e(\mathcal{H}) - k^3 D^3 nj^2
\]

choices. Recall that \( 1 - x \geq e^{-2x} \) if \( x \in [0,1/2] \). Since each edge-subset \( I \) can be generated in up to \( m! \) different ways by our greedy construction, using
\[ z^n/y! \geq \left( \frac{s}{y} \right) \] and (80) it follows for \( b = 8k^3D^3\lambda \) that, say,

\[ |\mathcal{S}_m| \geq \frac{\prod_{0 \leq j < m}(e(\mathcal{H}) - k^3D^3nj^2)}{m!} \geq \frac{e(\mathcal{H})^m}{m!} \left( 1 - \frac{k^3D^3nm^2}{e(\mathcal{H})} \right)^m \]

\[ \geq \left( \frac{e(\mathcal{H})}{m} \right) \exp(-2k^3D^3nm^3/e(\mathcal{H})) \geq \left( \frac{e(\mathcal{H})}{m} \right) e^{-b/4}. \]

Next, we estimate \( \mathbb{P}(\mathcal{I} = \mathcal{H}_p \mid \mathcal{I} \subseteq \mathcal{H}_p) \) for all \( \mathcal{I} \in \mathcal{S}_m \). Let \( \mathcal{F}_2 \) contain all \( g \in \mathcal{H} \setminus \mathcal{I} \) with \( 2 \leq |g \cap \bigcup_{f \in \mathcal{I}} f| < k \). Similarly, let \( \mathcal{F}_1 \) contain all \( g \in \mathcal{H} \setminus \mathcal{I} \) with \( |g \cap \bigcup_{f \in \mathcal{I}} f| = 1 \). Set \( \mathcal{F}_0 = \mathcal{H} \setminus (\mathcal{I} \cup \mathcal{F}_1 \cup \mathcal{F}_2) \), and note that by definition of \( \mathcal{S}_m \), see (78), all \( g \in \mathcal{F}_0 \) satisfy \( |g \cap \bigcup_{f \in \mathcal{I}} f| = 0 \). Since \( f \in \mathcal{H}_p \) if and only if \( f \subseteq V_p(\mathcal{H}) \), using Harris’ inequality [16] we deduce, say,

\[ \mathbb{P}(\mathcal{I} = \mathcal{H}_p \mid \mathcal{I} \subseteq \mathcal{H}_p) = \mathbb{P}\left( \bigcap_{g \in \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2} \{ g \not\subseteq V_p(\mathcal{H}) \} \bigcup_{f \in \mathcal{I}} f \subseteq V_p(\mathcal{H}) \right) \]

\[ \geq (1 - p^k)|\mathcal{F}_0|(1 - p^{-k-1})|\mathcal{F}_1|(1 - p)|\mathcal{F}_2|. \]

Note that \( |\mathcal{F}_1| \leq km \cdot \Delta_1(\mathcal{H}) \) and \( |\mathcal{F}_2| \leq \binom{km}{2} \cdot \Delta_2(\mathcal{H}) \). Since \( \Delta_1(\mathcal{H}) \leq D^2n \) and \( \Delta_2(\mathcal{H}) \leq D \), using (80) we infer, by choice of \( b = 8k^3D^3\lambda \), that

\[ |\mathcal{F}_1|p^{k-1} + |\mathcal{F}_2|p \leq kD^2nmp^{k-1} + k^2Dm^2p \leq (kD^2 + k^2D)\lambda \leq b/4. \]

Recalling \( 1 - x \geq e^{-2x} \) if \( x \in [0, 1/2] \), using \( p \leq 1/2 \) and \( |\mathcal{F}_0| < e(\mathcal{H}) - m \) we thus obtain

\[ \mathbb{P}(\mathcal{I} = \mathcal{H}_p \mid \mathcal{I} \subseteq \mathcal{H}_p) \geq (1 - p^k)|\mathcal{F}_0|e^{-2(|\mathcal{F}_1|p^{k-1} + |\mathcal{F}_2|p)} \geq (1 - p^k)e^{(\mathcal{H}) - m}e^{-b/2}, \]

which together with (79) and (81) establishes inequality (77), with room to spare.

In the remainder we sketch the verification of (80), using the convention that all implicit constants may depend on \( k, a, D \). Let \( \alpha = 2/(k + 1/3) \) and \( \beta = 2 - k\alpha = 2/(3k + 1) = \alpha/3 \), so that \( \mu = O(n^2p^k) \), \( p \leq n^{-\alpha} \) and \( 1/(2k) \leq \beta \) imply \( m = O(n^\beta) \). Using \( e(\mathcal{H}) = \Omega(n^2) \), \( p \leq n^{-\alpha} \) and \( \beta < 1/3 \) it now is routine to check that (80) holds for suitable \( \lambda > 0 \) (as \( 2\beta - 1 < 0 \) and \( \max\{3\beta - 1, 2\beta - \alpha, 1 + \beta - (k - 1)\alpha\} \leq 0 \) for \( k \geq 3 \)).

4.3. Configurations with too many vertices. Our third lower bound is based on the following heuristic: if \( V_p(\mathcal{H}) \) contains ‘too many vertices’ (more than expected), then it seems likely that the induced subgraph \( \mathcal{H}_p = \mathcal{H}[V_p(\mathcal{H})] \)
also contains ‘too many edges’ (more than the average number). For moderately large \( p \), this approach eventually yields the following lower bound of sub-Gaussian type (by Remark 7 we have \( \Lambda = \Theta(\text{Var} X) \), since \( p \) is bounded away from one).

**Theorem 24:** Given \( k \geq 2 \), \( a > 0 \) and \( D \geq 1 \), let \( \mathcal{H} = \mathcal{H}_n \) be a \( k \)-uniform hypergraph satisfying \( v(\mathcal{H}) \leq Dn \), \( e(\mathcal{H}) \geq an^2 \) and \( \Delta_2(\mathcal{H}) \leq D \). Set \( X = e(\mathcal{H}_p), \mu = \mathbb{E}X, \Lambda = \mu(1 + np^{k-1}) \) and \( \varphi(x) = (1 + x)\log(1 + x) - x \). Given \( \alpha \in (0, 1) \), there are \( n_0 > 0 \) (depending only on \( k, a, D \)) and \( \beta, c > 0 \) (depending only on \( \alpha, k, a, D \)) such that for all \( n \geq n_0 \), \( \alpha n^{-1/(k-1)} \leq p \leq 1 - \alpha \) and \( \min\{\sqrt{\Lambda}, \sqrt{\text{Var} X}\} \leq t \leq \beta \mu \) we have

\[
\mathbb{P}(X \geq \mu + t) \geq \exp\left(-c\varphi(t/\mu)\mu^2/\Lambda\right) \geq \exp\left(-ct^2/\Lambda\right).
\]

(82)

The key observation is that \( \mu^2/\Lambda = \Theta(np) \) for the relevant range of \( p \). With this in mind, the proof of Theorem 24 is based on the following two ideas:

(i) since \( V_p(\mathcal{H}) \sim \text{Bin}(v(\mathcal{H}), p) \) and \( v(\mathcal{H}) = \Theta(n) \), with probability at least \( \exp(-\Theta(\varepsilon^2 np)) = \exp(-\Theta((\varepsilon\mu)^2/\Lambda)) \) we have \( |V_p(\mathcal{H})| \geq (1 + \varepsilon)\mathbb{E}|V_p(\mathcal{H})| \), and

(ii) conditioning on \( |V_p(\mathcal{H})| \geq (1 + \varepsilon)\mathbb{E}|V_p(\mathcal{H})| \) intuitively increases the expected number \( e(\mathcal{H}_p) = e(\mathcal{H}[V_p(\mathcal{H})]) \) of induced edges, effectively turning the unlikely event \( X \geq \mu + t \) into a ‘typical’ one; see also (83) below. For the number of copies of \( H \) in the binomial random graph \( G_{n,p} \) an analogous reasoning (based on a deviation of the number of edges) applies for \( p = \Omega(n^{-1/m_2(H)}) \), where \( m_2(H) \) is the so-called 2-density of \( H \); for the lower tail this idea was used by Janson and Warnke [24].

We now informally discuss the high-level structure of the proof, which is similar to Theorem 21. Let \( \mu = \mathbb{E}X, \varepsilon = t/\mu \), and \( m = (1 + \varepsilon)\mathbb{E}|V_p(\mathcal{H})| \).

Applying (i) as outlined above, using monotonicity we expect that

\[
\mathbb{P}(X \geq \mu + t) \geq \mathbb{P}(|V_p(\mathcal{H})| \geq m) \cdot \mathbb{P}(X \geq \mu + t | |V_p(\mathcal{H})| \geq m) \\
\geq e^{-\Theta(t^2/\Lambda)} \cdot \mathbb{P}(X \geq \mu + t | |V_p(\mathcal{H})| = m).
\]

Thinking of the uniform random graph \( G_{n,m} \), using \( \mathbb{E}|V_p(\mathcal{H})| = v(\mathcal{H})p \) it seems plausible that \( \mathbb{E}(X | |V_p(\mathcal{H})| = m) \) is approximately \( e(\mathcal{H}) \cdot (m/v(\mathcal{H}))^k = (1 + \varepsilon)^k\mathbb{E}X \). Similarly, we expect \( \text{Var}(X | |V_p(\mathcal{H})| = m) = O(\text{Var} X) \) for \( \varepsilon = O(1) \). Noting \( t = \varepsilon\mathbb{E}X \) and \( (1 + \varepsilon)^k > 1 + 2\varepsilon \), we see that \( \mathbb{E}(X | |V_p(\mathcal{H})| = m) - t \) ought to be roughly at least \( (1 + \varepsilon)\mathbb{E}X = \mu + t \). To sum up, for \( t \geq \sqrt{\text{Var} X} \)
the Paley–Zygmund inequality (67) should yield
\[
P(X \geq \mu + t \mid |V_p(\mathcal{H})| = m) \geq P(X \geq \mathbb{E}(X) \mid |V_p(\mathcal{H})| = m) - t \mid |V_p(\mathcal{H})| = m) \\
\geq \Omega\left(\frac{t^2}{\text{Var} X + t^2}\right) = \Omega(1),
\]
and the following proof basically makes this rigorous (with some care about border cases).

**Proof of Theorem 24.** Let \( \varepsilon = t/\mu \), \( N = v(\mathcal{H}) \), and \( m = (1 + \varepsilon)Np \). Given \( 0 \leq j \leq N \), we henceforth write \( P_j(\cdot) = P(\cdot \mid |V_p(\mathcal{H})| = j) \) for brevity. We analogously use \( E_j(\cdot) \) and \( \text{Var}_j(\cdot) \), respectively. Note that, by monotonicity, we have
\[
P(X \geq \mu + t) \geq \sum_{j \geq m} P_j(X \geq \mu + t)P(|V_p(\mathcal{H})| = j) \\
\geq P_m(X \geq \mu + t)P(|V_p(\mathcal{H})| \geq m).
\]

It remains to estimate \( P_m(X \geq \mu + t) \) and \( P(|V_p(\mathcal{H})| \geq m) \) from below. We start by defining \( \beta = \beta(\alpha, k, a, D) \in (0, 1) \) in a somewhat technical way (that will be convenient in border cases). We use the convention that all implicit constants may depend on \( k, a, D \) (but not on \( \alpha \)). In particular, \( e(\mathcal{H}) = \Omega(n^2) \) and \( \Delta_2(\mathcal{H}) = O(1) \) imply \( v(\mathcal{H}) = \Omega(n) \), so that \( N = \Theta(n) \). Observing that \( \Lambda Np/\mu^2 = \Theta(1 + (np^{k-1})^{-1}) \) holds, we infer
\[
\varepsilon^2 Np = \Omega(\varepsilon^2 \mu^2 / \Lambda) \quad \text{and} \quad \varepsilon^2 Np = O((1 + \alpha^{-k}) \varepsilon^2 \mu^2 / \Lambda).
\]
Furthermore, by assumption and Remark 7 we have \( \varepsilon \mu = t \geq \min\{\sqrt{\Lambda}, \sqrt{\text{Var} X}\} = \Omega(\sqrt{\alpha \Lambda}) \), so that \( \varepsilon^2 Np = \Omega(\alpha) \) by (85). With \( \varepsilon \leq \beta \) in mind, we now pick \( \beta \in (0, \alpha/4] \) small enough such that
\[
\varepsilon Np = \varepsilon^2 Np/\varepsilon = \Omega(\alpha\beta^{-1}) \geq 2k^2 \quad \text{and} \quad Np = \Omega(\alpha\beta^{-2}) \geq 16\alpha^{-2}.
\]
Note that \( m = (1 + \varepsilon)Np \leq (1 + \alpha)(1 - \alpha)N < N \). So, since \( N = \Theta(n) \) and \( |V_p(\mathcal{H})| \sim \text{Bin}(N, p) \), for \( n \geq n_0(k, a, D) \) folklore estimates for binomial random variables yield
\[
P(|V_p(\mathcal{H})| \geq m) = P(|V_p(\mathcal{H})| \geq (1 + \varepsilon)Np) \geq d_1 \exp(-c_1 \varepsilon^2 Np)),
\]
where the constants \( c_1, d_1 > 0 \) depend only on \( \alpha, k, a, D \). (This can, e.g., be deduced analogous to the proof of Theorem 22 by means of Stirling’s formula. One minor difference in the estimates is perhaps that in (97) we can, e.g., via \( 1 - q = 1 - p \geq \alpha \) and \( j \leq 4T = 4\max\{\varepsilon Np, \sqrt{Np}\} \) here directly obtain
\(j^2/(1-q)N = O(\alpha^{-1}\epsilon^2 Np + \alpha^{-1}),\) say. To be pedantic, by choice of \(\beta\) in (86) we have also ensured that \(M \leq Np + 4T = Np(1 + 4 \max\{\epsilon, 1/\sqrt{Np}\}) \leq N(1 - \alpha)(1 + \alpha) < N\) holds.

Turning to \(\mathbb{P}_m(X \geq \mu + t),\) note that \(\epsilon \leq \beta \leq 1\) implies \(\varphi(\epsilon) = \Theta(\epsilon^2)\) via (24) and (55). So, in view of (85), (87) and \(\varepsilon = t/\mu,\) we see that (82) follows if \(\mathbb{P}_m(X \geq \mu + t) \geq \frac{(t/2)^2}{\text{Var}_m(X) + (t/2)^2} = \Omega\left(\frac{1}{\alpha^{-1} + 1}\right),\)

It is not difficult to see that the final expression of (89) is at most \(4^k \cdot O(\Lambda),\) so that Remark 7 and \(1 - p \geq \alpha\) imply \(\text{Var}_m(X) = O(\alpha^{-1} \min\{\Lambda, \text{Var} X\}),\) say. Using the assumed lower bounds for \(t,\) we now infer \(\text{Var}_m(X) = O(\alpha^{-1} \epsilon^2).\) Recalling (88), the Paley–Zygmund inequality (67) implies

\[\mathbb{P}_m(X \geq \mu + t) \geq \mathbb{P}_m(X \geq \mathbb{E}_m(X) - t/2) \geq \frac{(t/2)^2}{\text{Var}_m(X) + (t/2)^2} = \Omega\left(\frac{1}{\alpha^{-1} + 1}\right),\]

which, as discussed, completes the proof. \(\blacksquare\)
4.4. Proof of the lower bounds of Theorems 4 and 6 (and Remark 8). In this section we combine the previous estimates, and prove the lower bounds of Theorems 4 and 6 (as well as Remark 8). This is in principle straightforward but, at least as written here, requires several case distinctions (that are not very illuminating). Some complications are due to the fact that the results of Sections 4.1–4.3 are only valid in some range of the parameters (they need to be merged seamlessly), whereas others stem from the fact that our estimates are uniform (e.g., our $n_0$ does not depend on $\varepsilon$ or $\gamma$), from the fact that our assumptions are very weak (e.g., $p > 0$ instead of $p \geq n^{-2/k}$), or from the fact that the exponents are more involved than usual (e.g., (10) yields up to five different asymptotic expressions).

Proof of (7) of Theorem 4. The case $\sqrt{\mu} \log(1/p) \leq \mu$ is easy: then Theorem 20 implies

\begin{equation}
\mathbb{P}(X \geq (1 + \varepsilon)\mu) \geq \exp(-2D \max\{1, \sqrt{\varepsilon}\} \sqrt{\mu} \log(1/p)).
\end{equation}

In the remainder we may thus assume $\sqrt{\mu} \log(1/p) \geq \mu$, which for $n \geq n_0(k, a)$ implies $p \leq n^{-2/(k+1/3)}$, with room to spare. If $\varepsilon \mu \leq \max\{\mu, n^{1/(2k)}\}$, then Theorem 22 and $1 \leq 2 \max\{\mu, \varepsilon \mu\}$ (as $(1 + \varepsilon)\mu \geq 1$) yield

\begin{equation}
\mathbb{P}(X \geq (1 + \varepsilon)\mu) \geq \exp(-\log(1/d) - c\varepsilon^2 \mu) \\
\geq \exp(-2\max\{2\log(1/d), c\} \max\{1, \varepsilon^2\} \mu).
\end{equation}

It remains to consider the case $\varepsilon \mu \geq \max\{\mu, n^{1/(2k)}\}$. Since $p \log(1/p) \leq 1$ analogous to (44), using $\mu \leq D^3 n^2 p^k$ and $p \leq n^{-2/(k+1/3)}$ it follows for $n \geq n_0(k, D)$ that

\[
\sqrt{\mu} \log(1/p) \leq 1_{\{p \leq n^{-4/(k-2)}\}} D^{3/2} n p^{(k-2)/2} \cdot p \log(1/p) \\
+ 1_{\{p \geq n^{-4/(k-2)}\}} 4 D^{3/2} n p^{k/2} \log(n) \\
\leq n^{1/(2k)} \leq \varepsilon \mu.
\]

Since $\sqrt{1 + \varepsilon} \leq 2\varepsilon$ (as $\varepsilon \mu \geq \mu$ implies $\varepsilon \geq 1$), now Theorem 20 gives

\begin{equation}
\mathbb{P}(X \geq (1 + \varepsilon)\mu) \geq \exp(-D \sqrt{(1 + \varepsilon)\mu} \log(1/p)) \geq \exp(-2D\varepsilon^2 \mu).
\end{equation}

To sum up, (90)–(92) readily establish the lower bound (7), completing the proof. ■

Proof of (10) of Theorem 6 and Remark 8. Note that we may assume $\gamma \leq 1/2$ (since decreasing $\gamma$ yields less restrictive assumptions). We use the convention
that all implicit constants may depend on $k, a, D$ (not on $\gamma$), and tacitly assume $n \geq n_0(k, a, D)$ whenever necessary. With foresight, we start with some technical but useful auxiliary estimates. Recalling (24), for $t = \beta \mu$ we have

$$\varphi(t/\mu)\mu^2 \geq \min\{\beta, \beta^2\} \mu^2/3.$$  

Since $\mu = \Theta(n^2p^k)$ and $\Lambda = \mu(1 + np^{k-1})$, it follows for $t = \beta \mu$ that

$$\frac{\varphi(t/\mu)\mu^2}{\sqrt{t} \log(1/p)\Lambda} \geq \frac{\min\{\beta^{1/2}, \beta^{3/2}\} \mu^{1/2}}{3(1 + np^{k-1}) \log(1/p)}$$

(93)

$$= \min\{\beta^{1/2}, \beta^{3/2}\} \left(1_{\{p < n^{-1/(k-1)}\}} \frac{\Omega(np^{k/2})}{\log(1/p)} + 1_{\{p \geq n^{-1/(k-1)}\}} \frac{\Omega(1)}{p^{k/2-1} \log(1/p)} \right).$$

Analogously to (44), calculus yields $p^{1/2} \log(1/p) \in (0, 2]$ for $p \in (0, 1)$. Since $k \geq 3$ entails $p^{k-2/3} \leq p^{1/2}$, we see that $\gamma n^{-2/k}(\log n)^{2/k} \leq p \leq 1 - \gamma$ and $t \geq \mu$ imply $C_1 \sqrt{t} \log(1/p) \leq \varphi(t/\mu)\mu^2/\Lambda$, where $C_1 = C_1(\gamma, k, a, D) > 0$. Replacing $\log(1/p)$ with $\log(e/p)$ in (93), we similarly see that $C_2 \sqrt{t} \log(e/p) \leq \varphi(t/\mu)\mu^2/\Lambda$ for all $\gamma n^{-2/k}(\log n)^{2/k} \leq p \leq 1$ and $t \geq \mu$, where $C_2 = C_2(\gamma, k, a, D) > 0$. Since (24) and (55) imply $\varphi(t/\mu)\mu^2 = \Theta(t^2)$ for $t \leq \mu$, this completes the proof of Remark 8 (by adjusting the constants $n_0, c, C$).

We turn to (10) of Theorem 6, and start with case (iii), where $\gamma n^{-1/(k-1)} \leq p \leq 1 - \gamma$. Applying Theorem 21 and 24 (with $\alpha = \gamma$) there is $\beta = \beta(\gamma, k, a, D) > 0$ such that

$$\mathbb{P}(X \geq \mu + t) \geq \max\left\{\exp\left(-c_1 \sqrt{t} \log(1/p)\right), 1_{\{t \leq \beta \mu\}} \exp\left(-c_2 \varphi(t/\mu)\mu^2/\Lambda\right)\right\}.$$  

(94)

Proceeding as in the discussion following (93), for $t \geq \beta \mu$ we infer $A \sqrt{t} \log(1/p) \leq \varphi(t/\mu)\mu^2/\Lambda$, where $A = A(\beta, \gamma, a, k, D) > 0$. Replacing $c_2$ by $c_3 = \max\{c_2, c_1/A\}$ we thus can remove the indicator $1_{\{t \leq \beta \mu\}}$ in (94), establishing (10).

Next we consider case (ii) in the range $n^{-1/(k-1)} \leq p \leq n^{-1/(k-1)} \log n$. As in (58), by Remark 7 we have $\text{Var} X \geq b\Lambda \geq b\mu$, where $b = b(k, a, D) \in (0, 1]$. Since $\Lambda = O(\mu(\log n)^{k-1})$ and $\mu = \Omega(n^{(k-2)/(k-1)})$, it is easy to see that $t \geq b^{2/3} \mu^{2/3}(\log n)^{2/3} \geq \sqrt{\Lambda}$ holds. Hence, by case (iii) above there is nothing to show.

We now turn to case (i), where $p \leq n^{-2/(k+1/3)}$. If $\sqrt{t} \log(1/p) \leq \varphi(t/\mu)\mu^2/\Lambda$ holds, then using $\varphi(t/\mu)\mu^2 \leq t^2$, see (55), and $\Lambda = \Theta(\mu)$ we infer $t \geq \Lambda^{2/3}(\log(1/p))^{2/3} \geq 1_{\{\mu \geq 1/2\}} \sqrt{\Lambda}$, so Theorem 21 applies. Noting $\mu^2/\Lambda =$
\( \Theta(\mu) \), it thus remains to show that Theorem 22 applies when \( \varphi(t/\mu)\mu^2/\Lambda \leq \sqrt{t} \log(1/p) \). Aiming at a contradiction, we now assume that \( t \geq 8 \max\{\mu, n^{1/(2k)}\} \). Noting that \( \varphi(x) = (1 + x) \log(1 + x) - x \geq x(\log x)/2 \) for \( x \geq e^2 \approx 7.4 \), using \( \Lambda = \Theta(\mu) \) and

\[
1 \geq \frac{\varphi(t/\mu)\mu^2}{\sqrt{t} \log(1/p)\Lambda} \geq \frac{t^{1/2}\mu \log(t/\mu)}{2 \log(1/p)\Lambda} = n^{1/(4k)} \cdot \Omega\left(\frac{\log(t/\mu)}{\log(1/p)}\right).
\]

We now argue that the right hand side of (95) is \( \omega(1) \). Observe that \( p \leq n^{-2/(k-1)} \) implies \( t/\mu \geq \Omega(n^{1/(2k)}/(n^2p^k)) = \omega(p^{-1}) \), and that \( p \geq n^{-2/(k-1)} \) implies \( \log(t/\mu)/\log(1/p) \geq \Omega((\log n)^{-1}) \). In both cases we readily obtain a contradiction in (95) for large \( n \), which by our above discussion establishes (10).

Finally, by case (i) above it remains to verify case (ii) in the range \( n^{-2/(k+1/3)} \leq p \leq n^{-1/(k-1)} \). Note that \( \Lambda = \Theta(\mu) \), \( \Var X \geq b \Lambda \geq b \mu \), and \( \mu = \Omega(n^{2/(k+1)}) \) imply \( t \geq b^{2/3} \mu^{2/3}(\log n)^{2/3} \geq \sqrt{\Lambda} \) and \( \mu + t \geq 1 \), with room to spare. In case of \( t \leq \mu \), by (24) we have \( \varphi(t/\mu)\mu^2 \geq t^2/3 \), so that \( \Lambda = \Theta(\mu) \) yields

\[
\frac{\varphi(t/\mu)\mu^2}{\sqrt{t} \log(1/p)\Lambda} \geq \frac{t^{3/2}}{3 \log(1/p)\Lambda} \geq \frac{b \mu \log n}{3 \log(1/p)\Lambda} = \Omega(1).
\]

Using the discussion after (93) in case of \( t \geq \mu \), it thus follows (in both cases) that \( B \sqrt{t} \log(1/p) \leq \varphi(t/\mu)\mu^2/\Lambda \), where \( B = B(b, \gamma, k, a, D) > 0 \). Hence an application of Theorem 21 establishes (10).

Appendix A. Appendix

The following proof is based on Stirling’s approximation formula

\[
1 \leq x!/\left[ \sqrt{2\pi x} (x/e)^x \right] \leq e^{1/(12x)}.
\]

Some of the minor complications below stem from the fact that our assumption \( \mu + t \geq 1 \) is extremely weak.

Proof of Theorem 22. With foresight, let \( T = \max\{t, \sqrt{\mu}\} \), \( L = [\mu + T] \) and \( M = [\mu + 2T] \). Clearly,

\[
\mathbb{P}(X \geq \mu + t) \geq \mathbb{P}(X \geq \mu + T) \geq \sum_{m \in \mathbb{N}: L \leq m \leq M} \mathbb{P}(X = m).
\]

In view of Lemma 23, we now estimate the right hand side of (77). To avoid clutter, let \( N = e(\mathcal{H}) \) and \( q = p^k \). Recalling \( 1 - x \leq e^{-x} \), \( \mu = Nq > 0 \) and Stirling’s formula, standard (somewhat tedious but simple) calculations show
that for any $\mu + j \in \mathbb{N}$ satisfying $1 \leq \mu + j < N$ we have, say,

$$\binom{N}{\mu + j} q^{\mu + j} (1 - q)^{N - \mu - j} \geq \frac{\exp \left( - \frac{1}{12(\mu + j)} - \frac{1}{12(N - \mu - j)} \right)}{\sqrt{2\pi(\mu + j)(1 - \frac{j}{N}) (1 + \frac{j}{\mu})^{\mu + j} (1 - \frac{j}{N - \mu})^{N - \mu - j}}} \exp \left( - \frac{1}{6} - ((\mu + j) \log(1 + j/\mu) - j) - \frac{j^2}{(1 - q)N} \right).$$

Note that $(\mu + j) \log(1 + j/\mu) - j = \varphi(j/\mu)\mu$, and that $\varphi(j/\mu)$ is monotone increasing in $j \geq 0$. Since $\mu + t \geq 1$ implies $T \geq 1/2$, we deduce $M - \mu \leq 2T + 1 \leq 4T$. Since $N = e(H) \geq an^2$, from the proof of Lemma 23 it follows that $M \leq \mu + 4T = O(n^3)$ satisfies $M^2/N = o(1)$ and $M < N$. In particular, $q = p^k \leq 1/2$ implies $j^2/(1 - q)N \leq 2M^2/N = o(1)$. By combining (96) with Lemma 23 and (97), we now infer that, say,

$$\mathbb{P}(X \geq \mu + t) \geq \max\{T, 1\} \cdot \frac{\exp(-(b + 1) - \varphi(4T/\mu))}{\sqrt{2\pi M}}.$$ 

Noting that $\max\{T, 1\} = \max\{t, \sqrt{\mu}, 1\}$ and $M \leq 4\max\{t, \mu, 1\}$, we deduce $\max\{T, 1\}/\sqrt{M} \geq 1/\sqrt{A}$. Next we estimate $\varphi(4T/\mu)\mu$. If $T = \sqrt{\mu}$ holds, then $\varphi(4T/\mu)\mu \leq 16T^2/\mu = 16$ by (55), and if $T = t$ holds, then $\varphi(4T/\mu)\mu \leq 16\varphi(t/\mu)\mu$ by applying (25) twice. Combining our findings, it follows that, say,

$$\mathbb{P}(X \geq \mu + t) \geq e^{-(b + 17)}/\sqrt{32\pi} \cdot \exp\left( -\frac{1}{(t > \sqrt{\mu})} 16\varphi(t/\mu)\mu \right),$$

which together with (55) readily establishes (76) with $c = 16$ and $d = e^{-(b + 17)}/\sqrt{32\pi}$. 

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References

[1] N. Alon and J. H. Spencer, The probabilistic method, third ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2008.

[2] A. Baltz, P. Hegarty, J. Knape, U. Larsson and T. Schoen, The structure of maximum subsets of \{1, \ldots, n\} with no solutions to \(a + b = kc\), Electron. J. Combin. 12 (2005), Research Paper 19.

[3] S. Boucheron, G. Lugosi and P. Massart, Concentration inequalities using the entropy method, Ann. Probab. 31 (2003), 1583–1614.

[4] S. Boucheron, G. Lugosi and P. Massart, Concentration inequalities, Oxford University Press, Oxford, 2013, A nonasymptotic theory of independence, With a foreword by Michel Ledoux.

[5] S. Chatterjee, The missing log in large deviations for triangle counts, Random Structures Algorithms 40 (2012), 437–451.

[6] S. Chatterjee and P. S. Dey, Applications of Stein’s method for concentration inequalities, Ann. Probab. 38 (2010), 2443–2485.

[7] S. Chatterjee and S. R. S. Varadhan, The large deviation principle for the Erdős-Rényi random graph, European J. Combin. 32 (2011), 1000–1017.

[8] B. Demarco and J. Kahn, Tight upper tail bounds for cliques, Random Structures Algorithms 41 (2012), 469–487.

[9] B. DeMarco and J. Kahn, Upper tails for triangles, Random Structures Algorithms 40 (2012), 452–459.

[10] A. Dembo, Information inequalities and concentration of measure, Ann. Probab. 25 (1997), 927–939.

[11] D. P. Dubhashi and A. Panconesi, Concentration of measure for the analysis of randomized algorithms, Cambridge University Press, Cambridge, 2009.

[12] P. Erdős and P. Tetali, Representations of integers as the sum of \(k\) terms, Random Structures Algorithms 1 (1990), 245–261.

[13] E. Friedgut, V. Rödl and M. Schacht, Ramsey properties of random discrete structures, Random Structures Algorithms 37 (2010), 407–436.

[14] R. Graham, V. Rödl and A. Ruciński, On Schur properties of random subsets of integers, J. Number Theory 61 (1996), 388–408.

[15] B. Green, The Cameron-Erdős conjecture, Bull. London Math. Soc. 36 (2004), 769–778.

[16] T. E. Harris, A lower bound for the critical probability in a certain percolation process, Proc. Cambridge Philos. Soc. 56 (1960), 13–20.

[17] S. Janson, Poisson approximation for large deviations, Random Structures Algorithms 1 (1990), 221–229.

[18] S. Janson, New versions of Suen’s correlation inequality, in Proceedings of the Eighth International Conference “Random Structures and Algorithms” (Poznan, 1997), Vol. 13, 1998, pp. 467–483.

[19] S. Janson, T. Łuczak and A. Ruciński, Random graphs, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.

[20] S. Janson, K. Oleszkiewicz and A. Ruciński, Upper tails for subgraph counts in random graphs, Israel J. Math. 142 (2004), 61–92.
[21] S. Janson and A. Ruciński, *The infamous upper tail*, Random Structures Algorithms **20** (2002), 317–342. Probabilistic methods in combinatorial optimization.

[22] S. Janson and A. Ruciński, *The deletion method for upper tail estimates*, Combinatorica **24** (2004), 615–640.

[23] S. Janson and A. Ruciński, *Upper tails for counting objects in randomly induced subhypergraphs and rooted random graphs*, Ark. Mat. **49** (2011), 79–96.

[24] S. Janson and L. Warnke, *The lower tail: Poisson approximation revisited*, Random Structures Algorithms **48** (2016), 219–246.

[25] J. H. Kim and V. H. Vu, *Concentration of multivariate polynomials and its applications*, Combinatorica **20** (2000), 417–434.

[26] M. Ledoux, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs, Vol. 89, American Mathematical Society, Providence, RI, 2001.

[27] E. Lubetzky and Y. Zhao, *On replica symmetry of large deviations in random graphs*, Random Structures Algorithms **47** (2015), 109–146.

[28] C. McDiarmid, *On the method of bounded differences*, in *Surveys in combinatorics, 1989 (Norwich, 1989)*, London Math. Soc. Lecture Note Ser., Vol. 141, Cambridge Univ. Press, Cambridge, 1989, pp. 148–188.

[29] C. McDiarmid and B. Reed, *Concentration for self-bounding functions and an inequality of Talagrand*, Random Structures Algorithms **29** (2006), 549–557.

[30] D. Reimer, *Proof of the van den Berg-Kesten conjecture*, Combin. Probab. Comput. **9** (2000), 27–32.

[31] O. Riordan and L. Warnke, *The Janson inequalities for general up-sets*, Random Structures Algorithms **46** (2015), 391–395.

[32] V. Rödl and A. Ruciński, *Random graphs with monochromatic triangles in every edge coloring*, Random Structures Algorithms **5** (1994), 253–270.

[33] J. Rué and A. Zumalacárregui, *Threshold functions for systems of equations on random sets*, arXiv:1212.5496 (2012).

[34] W. Samotij, *Stability results for random discrete structures*, Random Structures Algorithms **44** (2014), 269–289.

[35] A. A. Sapozhenko, *The Cameron-Erdős conjecture*, Dokl. Akad. Nauk **393** (2003), 749–752.

[36] M. Schacht, *Extremal results for random discrete structures*, Ann. of Math. (2) **184** (2016), 333–365.

[37] W. Schudy and M. Sviridenko, *Concentration and moment inequalities for polynomials of independent random variables*, in *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms*, ACM, New York, 2012, pp. 437–446.

[38] M. Šileikis, *On the upper tail of counts of strictly balanced subgraphs*, Electron. J. Combin. **19** (2012), Paper 4, 14.

[39] J. Spencer, *Counting extensions*, J. Combin. Theory Ser. A **55** (1990), 247–255.

[40] M. Talagrand, *Concentration of measure and isoperimetric inequalities in product spaces*, Inst. Hautes Études Sci. Publ. Math. (1995), 73–205.

[41] J. van den Berg and J. Jonasson, *A BK inequality for randomly drawn subsets of fixed size*, Probab. Theory Related Fields **154** (2012), 835–844.
[42] J. van den Berg and H. Kesten, Inequalities with applications to percolation and reliability, J. Appl. Probab. 22 (1985), 556–569.

[43] V. H. Vu, Concentration of non-Lipschitz functions and applications, Random Structures Algorithms 20 (2002), 262–316, Probabilistic methods in combinatorial optimization.

[44] L. Warnke, On the missing log in upper tail estimates, arXiv:1612.08561 (2016).

[45] L. Warnke, When does the $K_4$-free process stop?, Random Structures Algorithms 44 (2014), 355–397.

[46] L. Warnke, On the method of typical bounded differences, Combin. Probab. Comput. 25 (2016), 269–299.

[47] G. Wolfovitz, A concentration result with application to subgraph count, Random Structures Algorithms 40 (2012), 254–267.