Finite-horizon $H_{\infty}$ filtering for time-varying delay systems with randomly varying nonlinearities and sensor saturations

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This paper mainly focuses on the $H_{\infty}$ filtering problem for a class of discrete time-varying systems with delays and randomly varying nonlinearities and sensor saturations. Two sets of binary switching sequences taking values of 1 and 0 are introduced to account for the stochastic phenomena of nonlinearities and sensor saturations which occur and influence the dynamics of the system in a probabilistic way. To further reflect the realities of transmission failure in the measurement, missing observation case is also considered simultaneously. By appropriately constructing a time-varying Lyapunov function and utilizing the stochastic analysis technique, sufficient criteria are presented in terms of a set of recursive linear matrix inequalities (RLMIs) under which the filtering error dynamics achieves the prescribed $H_{\infty}$ performance over a finite horizon. Moreover, at each time point $k$, the time-varying filter parameters can be solved iteratively according to the explicit solutions of the RLMIs. Finally, a numerical simulation is exploited to demonstrate the effectiveness of the proposed filter design scheme.

Keywords: $H_{\infty}$ filtering; time-varying delayed systems; randomly varying sensor saturations; recursive linear matrix inequalities; finite horizon

1. Introduction

In practical engineering fields such as signal processing area, to carry out some specific design tasks, the state information or some combinations of the state information are needed to be known which however, are often unavailable. And this is one of the main backgrounds for investigating the estimation problems. Generally speaking, the aim of the estimation problem is to estimate certain system parameters or state variables by utilizing the accessible measurements, which might be with the existence of stochastic errors. In the literature, much work has been done on various estimation problems, and several filtering methodologies have been proposed (Ahmad & Namerikawa, 2013; Lu, Xie, Zhang, & Wang, 2007; Mohamed, Nahla, & Safya, 2013; Reif & Unbehauen, 1999). Among them, the Kalman filtering (Lu et al., 2007; Reif & Unbehauen, 1999; Xie, Soh, & de Souza, 1994) and the $H_{\infty}$ filtering (Dong, Wang, Ho, & Gao, 2011; Li, Lam, & Shu, 2010; Zhang, Chen, & Tseng, 2005; Zhang, Feng, & Duan, 2006) are two notable ones. The main idea for the Kalman filtering is to estimate the future values of the signal by utilizing the past/current observations. When employing the $H_{\infty}$ filtering, criteria are often presented in the form of Riccati difference equations (Gershon, Shaked, & Yaesh, 2001; Hung & Yang, 2003; Xie & de Souza, 1992; Zhang et al., 2006) or linear matrix inequalities (LMIs) (Li et al., 2010; Shen, Wang, & Liu, 2011; Shen, Wang, Shu, & Wei, 2010; Wang, Shen, & Liu, 2012).

It is well known that almost all the realistic systems are intrinsically time-varying, and frequently affected by the nonlinear exogenous disturbances and time delays, which markedly increase the difficulty when analyzing the system due to the complexity. In the past decades, much efforts have been devoted to the filtering and control problems for the discrete time-varying systems (Shen et al., 2011; Shen, Ding, & Wang, 2013). For instance, robust $H_{\infty}$ filtering problem has been investigated in Dong et al. (2011) for the Markovian jump time-varying systems. When dealing with the time-varying systems in practice, a fundamental issue arises naturally, that is, the state performance constrains are restricted only over a finite horizon instead of the infinite one. Such kind of finite-horizon filtering problem has attracted much attention in recent years, and it is desirable to develop effective and executable algorithms to determine the filter parameters. Motivated by the novel difference linear matrix inequality method proposed in Shaked and Suplin (2001), Gershon and Shaked (2008), Gershon, Shaked, and Yaesh (2005), a new and practical recursive linear matrix inequality (RLMI) approach has been firstly presented in Shen et al. (2010) where the available state estimates have also been utilized which might decrease the conservation of
the results obtained. On the other hand, time delays are ubiquitous in systems mainly due to the reasons such as finite capabilities of signal transmission among different parts of the systems. Numerous results pertaining to the filtering problem have been obtained in Lu et al. (2007), Chen and Zheng (2012), Wu and Wang (2009), Wei, Wang, and Shen (2013) for the delayed time-invariant systems. When referring to the time-varying delayed systems, relating results are relatively few (Basin, Shi, Alvarez, & Wang, 2009; Wei, Wang, & Shen, 2010), which might mainly due to the mathematical complexity. For example, in Basin et al. (2009), Wang, & Shen, 2010), which might mainly due to the saturation effect. Therefore, sensor saturation issue is currently changeable of the nonlinear disturbances in terms of their randomly or instruments abrade, this will result in the randomly occurring nonlinearities (RONs) and randomly occurring phenomena are considered, which are introduced to reflect the intricate working circumstances of the underlying system. In addition, randomly varying nonlinearities between the current and the delayed state nonlinearities are also involved, which together make the $H_\infty$ filtering problem hard to be analyzed, not to mention the design problem for the $H_\infty$ filter. And this is the main aim of this paper to shorten such a gap.

The rest of the paper is organized as follows. In Section 2, the discrete time-varying delayed system with randomly varying nonlinearities and sensor saturations is presented, and the $H_\infty$ filtering problem addressed is formulated. In Section 3, by resorting to the stochastic analysis techniques, sufficient conditions are established in the form of time-varying matrix inequalities under which the output estimation error is assured to meet the constraint of the given $H_\infty$ performance level. Furthermore, the parameters of the $H_\infty$ filter are designed according to the feasible solutions of a set of RLMI and a recursive filtering algorithm is developed. In Section 4, one illustrative example is given to demonstrate the effectiveness of the results derived. Finally, in Section 5 the conclusion is drawn.

**Notations:** The notations used here are fairly standard except where otherwise stated. $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. The set of all integers is represented by $\mathbb{Z}$ and $\mathbb{R}$ means the set of all real numbers. The interval $[m, n]$ with $m, n \in \mathbb{Z}$ and $m < n$ denotes the set of integer sequence $\{m, m + 1, \ldots, n\}$, and $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$ represents the set of real numbers between $a$ and $b$. The notation $X \succeq Y$ (respectively, $X \succ Y$), where $X$ and $Y$ are real symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $M^T$ represents the transpose of the matrix $M$ and $I$ is used to denote the identity matrix with compatible dimensions. $\text{diag} \{\cdots\}$ stands for a block-diagonal matrix. Moreover, $\text{Prob}\{X\}$ means the occurrence probability of the event $X$ and $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable $x$ with respect to the given probability measure $\text{Prob}$. The asterisk “*” in a matrix is used to denote a term that is induced by symmetry. Matrices, if they are not explicitly specified, are assumed to have compatible dimensions.

### 2. Problem formulation and preliminaries

Consider the following discrete time-varying delayed system defined on the finite horizon $k \in [0, N]$:

\[
\begin{align*}
x(k+1) &= A(k)x(k) + A_1(k)x(k-d) + B(k)v(k) + \alpha(k)f(k, x(k)) + (1 - \alpha(k))g(k, x(k-d)), \\
y(k) &= \psi(C(k)x(k)) + D(k)v(k), \\
z(k) &= M(k)x(k), \\
x(s) &= \phi(s), \quad s = -d, -d + 1, \ldots, 0,
\end{align*}
\]

where $x(k) \in \mathbb{R}^n$ is the state vector, $y(k) \in \mathbb{R}^m$ is the measured output vector, $z(k) \in \mathbb{R}^r$ is the signal to be estimated; $A(k), A_1(k), B(k), C(k), D(k)$ and $M(k)$ are known real...
time-varying matrices with appropriate dimensions; \( d > 0 \) is an integer representing the constant delay of the system; \( \phi(\cdot) \in \mathbb{R}^n \) is the initial state vector function defined on \([-d, 0] \); \( v(k) \in \mathbb{R}^l \) is the exogenous disturbance signal belonging to \( l_2[0, N] \) which denotes the space of square summable sequences with the norm

\[
\| v \|^2_{[0, N]} = \mathbb{E} \left\{ \sum_{k=0}^{N} \| v(k) \|^2 \right\}.
\]

The nonlinear functions \( f(\cdot, \cdot), g(\cdot, \cdot) : [0, N] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) are assumed to be continuous and satisfy the following sector-bounded conditions (Cao, Lin, & Chen, 2003; Wang, Liu, & Liu, 2008):

\[
\begin{align*}
[f(k, x) - U_1(k)x] & T f(k, x) - U_2(k)x] \leq 0, \quad (2) \\
g[k, x] - V_1(k)x] & T g[k, x] - V_2(k)x] \leq 0, \quad (3)
\end{align*}
\]

where \( k \in [0, N] \) and \( x \in \mathbb{R}^n \); \( U_1(k), U_2(k), V_1(k) \) and \( V_2(k) \) are known real matrices with appropriate dimensions, and \( U(k) = U_1(k) - U_2(k), V(k) = V_1(k) - V_2(k) \) are symmetric positive-definite matrices.

The random variable \( \alpha(k) \in \mathbb{R} \) takes values of 1 and 0 with

\[
\text{Prob}[\alpha(k) = 1] = \tilde{a}, \quad \text{Prob}[\alpha(k) = 0] = 1 - \tilde{a}, \quad (4)
\]

where constant \( \tilde{a} \in [0, 1] \) is known.

Remark 1 Most of realistic systems are subject to nonlinear disturbances which themselves might change abruptly due mainly to the reasons such as sudden changes in environment, random switching between subsystems, failure connection between part of the nodes of networks as well as asynchronous information transmission within networks. In other words, the nonlinear disturbances might occur in a probabilistic way. Such kind of phenomena has been firstly named RONs in Wang, Wang, and Liang (2009) to account for the probabilistic occurrence of different nonlinear functions. Here, illuminated by the ideas proposed in Wang et al. (2009), \( \alpha(k) \) is used just to account for the phenomena of randomly varying nonlinearities between the current state nonlinearities \( f(k, x(k)) \) and the delayed state nonlinearity \( g(k, x(k - d)) \).

The nonlinear function \( \psi(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is given as follows:

\[
\psi(C(k)x(k)) = \beta(k) \sigma(C(k)x(k)) + (1 - \beta(k)) \gamma(k) C(k)x(k), \quad (5)
\]

where \( \sigma(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m \) represents the sensor saturation function with the following form:

\[
\sigma(u) = [\sigma_1^T(u_1), \sigma_2^T(u_2), \ldots, \sigma_m^T(u_m)]^T, \quad (6)
\]

where \( u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m \), \( \sigma_i(u_i) = \text{sign}(u_i) \min\{u_{i \text{max}}, |u_i|\} \) for \( i = 1, 2, \ldots, m \) and \( u_{i \text{max}} \) denotes the \( i \)th element of the saturation vector \( u_{\text{max}} \).

In Equation (5), the random variables \( \beta(k) \in \mathbb{R} \) and \( \gamma(k) \in \mathbb{R} \) are Bernoulli distributed sequences taking values of 1 and 0 with

\[
\begin{align*}
\text{Prob}[\beta(k) = 1] &= \bar{\beta}, \quad \text{Prob}[\beta(k) = 0] = 1 - \bar{\beta}, \\
\text{Prob}[\gamma(k) = 1] &= \bar{\gamma}, \quad \text{Prob}[\gamma(k) = 0] = 1 - \bar{\gamma},
\end{align*}
\]

where \( \bar{\beta}, \bar{\gamma} \in [0, 1] \) are known constants. Here, \( \beta(k) \) is introduced to account for the phenomena of randomly varying sensor saturations caused by physical electron device constraints, while \( \gamma(k) \) is used to describe the probable data-missing phenomenon caused by mutative working conditions or fluctuant signal transmission channels when the sensors fail to work. Throughout this paper, we further assume that the stochastic variables \( \alpha(k), \beta(k) \) and \( \gamma(k) \) are mutually independent.

Remark 2 In recent years, the RONs have been extensively studied in Dong et al. (2011), Wang et al. (2010) and Wei, Wang, and Han (2013) for the Markovian jump systems and the complex networks. When such kind of phenomena occur in a sensor network, the notation of ROSSs has been firstly introduced in Wang et al. (2012) and Ding et al. (2012) by further considering the physical limitations of components. It should be noted that such kind of ideas have also been employed in earlier works such as Nahi (1969) and Chau, Qin, Sayed, Wahab, and Yang (2010), where a Markov chain has been proposed to capture the battery recovery (Chau et al., 2010), an the missing measurements (or uncertain observations) have been considered for the optimal estimation problems (Nahi, 1969).

Remark 3 The measurement output model given in Equations (1) and (5) is originated from Wang et al. (2012) where the \( H_\infty \) filtering problem has been investigated for the nonlinear sensor networks with time-invariant system matrices. As pointed out in Wang et al. (2012), the main advantage of such kind of measurement output equation is that it is capable of accounting for the phenomena of both ROSSs and missing measurements in a unified form. To be specific, if \( \beta(k) = 0 \) and \( \gamma(k) = 0 \), the output observer receives only the noise signal; if \( \beta(k) = 0 \) and \( \gamma(k) = 1 \), it means that the output observer works regularly; if \( \beta(k) = 1 \), whatever the value of \( \gamma(k) \) is, only saturated signals are received by the output observer. In practice, the case that the sensor saturation phenomenon and the data-missing phenomenon occur simultaneously does exist. At this time, we only consider the former one since the saturated signals can also be viewed as one special form of the data-missing phenomenon.

Illuminated by the analysis method used in Dong et al. (2011) and Yang and Li (2009), we assume that there exist diagonal matrices \( H_1(k) \) and \( H_2(k) \) such that \( 0 \leq H_1(k) < I \leq H_2(k) \), and the saturation function \( \sigma(C(k)x(k)) \) in
Equation (5) is rewritten as follows:
\[ \sigma(C(k)x(k)) = H_1(k)C(k)x(k) + h(C(k)x(k)), \]
where \( h(C(k)x(k)) \) is a nonlinear vector-valued function satisfying the following inequality:
\[ h^T(C(k)x(k))(h(C(k)x(k)) - H(k)C(k)x(k)) \leq 0 \]
with \( H(k) = H_2(k) - H_1(k) \).

According to the above discussions, system (1) can be rewritten as follows:
\[
\begin{align*}
&x(k + 1) = A(x)x(k) + A_1(k)x(k - d) + B(v)k\nu(k) \\
&\quad + \alpha(k)(x(k)) + (1 - \alpha(k))g(k,x(k - d)), \\
&y(k) = \beta(k)H_1(k)C(k)x(k) + \beta(k)h(C(k)x(k)) \\
&\quad + (1 - \beta(k))\gamma(C(k)x(k)) + D(k)v(k), \\
&z(k) = M(k)x(k), \\
&x(s) = \phi(s), \quad s = -d, -d + 1, \ldots, 0.
\end{align*}
\]

In this paper, we will design the following filter for the time-varying system (10):
\[
\begin{align*}
\hat{x}(k + 1) &= F_f(k)\hat{x}(k) + G_f(k)y(k), \\
\hat{z}(k) &= M_f(k)\hat{x}(k),
\end{align*}
\]
where \( \hat{x}(k) \in \mathbb{R}^n \) is the state vector of the filter, \( \hat{z}(k) \in \mathbb{R}^r \) is the estimate of \( z(k) \); \( F_f(k) \), \( G_f(k) \) and \( M_f(k) \) are time-varying filter matrices to be designed. Here, we take \( \hat{x}(k) \equiv 0 \) for \( k \leq 0 \), which will be used when designing the filter algorithm in the sequel.

For convenience of expression, we introduce the following notions:
\[
\begin{align*}
\tilde{A}(e(k)) &= (A(k) - \tilde{\beta}G_f(k)H_1(k)C(k) \\
&\quad - (1 - \tilde{\beta})\gamma G_f(k)(C(k))e(k) \\
&\quad + (B(k) - G_f(k)D(k))v(k) \\
&\quad + (A(k) - \tilde{\beta}G_f(k)H_1(k)C(k) \\
&\quad - (1 - \tilde{\beta})\gamma G_f(k)(C(k))e(k) \\
&\quad - F_f(k)\hat{x}(k) - \tilde{\beta}G_f(k)h(C(k)x(k)) \\
&\quad + \tilde{\alpha}f(k,x(k)) + (1 - \tilde{\alpha})g(k,x(k - d)),
\end{align*}
\]
\[
\begin{align*}
\tilde{B}(e(k)) &= -G_f(k)H_1(k)C(k)e(k) - G_f(k)H_1(k)C(k)\hat{x}(k) \\
&\quad - G_f(k)h(C(k)x(k)),
\end{align*}
\]
\[
\begin{align*}
\tilde{C}(e(k)) &= -G_f(k)C(k)e(k) - G_f(k)C(k)\hat{x}(k),
\end{align*}
\]
\[
\begin{align*}
\tilde{A}_1(e(k)) &= A_1(k)e(k - d) + A_1(k)\hat{x}(k - d),
\end{align*}
\]
\[
\begin{align*}
\tilde{F}(e(k)) &= f(k,x(k)) - g(k,x(k - d)),
\end{align*}
\]
\[
\begin{align*}
\tilde{H}(e(k)) &= M(k)e(k) + (M(k) - M_f(k))\hat{x}(k).
\end{align*}
\]

By letting \( e(k) = x(k) - \hat{x}(k) \) and \( \tilde{z}(k) = z(k) - \hat{z}(k) \), the error dynamics can be obtained as follows from Equations (10) and (11):
\[
\begin{align*}
e(k + 1) &= \tilde{A}(e(k)) + (\beta(k) - \tilde{\beta})\tilde{B}(e(k)) \\
&\quad + ((1 - \beta(k))\gamma(k) - (1 - \tilde{\beta})\tilde{\gamma}(e(k)) \\
&\quad + (\alpha(k) - \tilde{\alpha})\tilde{C}(e(k)) + \tilde{\omega}_f(e(k)),
\end{align*}
\]
\[
\begin{align*}
\tilde{z}(k) &= \tilde{H}(e(k)).
\end{align*}
\]

The filtering problem to be addressed is as follows: design the filter (11) such that the \( H_\infty \) performance constraint (13) is satisfied. More specially, for any nonzero exogenous disturbance \( v(k) \in l_2([0,N],\mathbb{R}^r) \), the estimation error \( \tilde{z}(k) \) satisfies the following inequality:
\[
\|\tilde{z}\|_{[0,N]}^2 \leq \gamma^2 \left\{ \|v\|_{[0,N]}^2 + \mathbb{E} \left\{ \sum_{k=-d}^{0} e^T(k)S(k)e(k) \right\} \right\}
\]
where \( \gamma > 0 \) is a given disturbance attenuation level and \( \{S(k)\}_{-d\leq k\leq 0} \) is a known positive-definite matrix sequence.

Remark 4 In recent years, the finite-horizon filtering problem has attracted much attention for its practicability. For example, the robust \( H_\infty \) filtering problem with error variance constraints has been investigated for the discrete linear time-varying systems in Hung and Yang (2003), where the conditions are in the form of forward recursive Riccati equations which are hard to be solved in practice. Recently, novel works have been done in Shen et al. (2010) and Wei et al. (2010), respectively, for the robust \( H_\infty \) finite-horizon filtering of systems with RONs and quantization effects and the error-constrained filtering of nonlinear delayed systems with non-Gaussian noises. It should be noted that the delay effects have not been considered in Shen et al. (2010), and in Wei et al. (2010) the phenomena of ROSSs and missing measurements have not been taken into account. By taking the phenomena of time delay, ROSSs and missing measurements together, it will be hard to analyze the \( H_\infty \) filtering problem, not to mention the design problem for the time-varying \( H_\infty \) filter, which mainly motivates the present work of this article.

3. Main results
In this section, in order to design the filter (11), we first give a sufficient criterion to guarantee that the error system (12) satisfies the \( H_\infty \) performance constraint (13) via the RLM1 approach, which is given by the following theorem.

**Theorem 1** Consider the error system (12) with known filter parameters \( \{F_f(k)\}_{0\leq k\leq N} \), \( \{G_f(k)\}_{0\leq k\leq N} \) and \( \{M_f(k)\}_{0\leq k\leq N} \). Let the disturbance attenuation level \( \gamma > 0 \) and the positive-definite matrix sequence \( \{S(k)\}_{-d\leq k\leq 0} \) be given, the estimation error \( \tilde{z}(k) \) satisfies the \( H_\infty \) performance constraint (13) if there exist four families of
positive scalars \( \{\epsilon_1(k)\}_{0\leq k \leq N}, \{\epsilon_2(k)\}_{0\leq k \leq N}, \{\epsilon_3(k)\}_{0\leq k \leq N}, \{\mu(k)\}_{0\leq k \leq N+1} \) and two families of positive-definite matrices \( \{P(k)\}_{0\leq k \leq N+1}, \{Q(k)\}_{-d+1\leq k \leq N+1} \) satisfying the following initial condition:

\[
\mathbb{E} \left\{ e^T(0)P(0)e(0) + \sum_{k=-d}^{-1} e^T(k)Q(k+1)e(k) \right\} + \mu(0) \leq \gamma^2 \mathbb{E} \left\{ \sum_{k=-d}^{0} e^T(k)S(k)e(k) \right\}
\]

(14)

and the RLMIs

\[
\begin{bmatrix}
\Xi_1(k) & \hat{\mathcal{A}}^T(k)P(k+1) & \hat{\mathcal{B}}^T(k)P(k+1) \\
* & -P(k+1) & 0 \\
* & * & -P(k+1) \\
* & * & * \\
* & * & * \\
* & * & * \\
\end{bmatrix}
\begin{bmatrix}
\hat{x}(k+1) \\
\hat{y}(k+1) \\
\hat{z}(k+1) \\
\end{bmatrix}
\leq 0
\]

(15)

for \( 0 \leq k \leq N \), where \( \hat{\alpha} = \alpha(1 - \tilde{\alpha}), \hat{\beta} = \beta(1 - \tilde{\beta}), \hat{\gamma} = (1 - \tilde{\beta})y - (1 - \tilde{\beta})^2\hat{\gamma}, \hat{\mu} = \hat{\gamma}(1 - \hat{\gamma})(1 - \hat{\beta}), \hat{\tilde{\beta}} = \tilde{\beta}/\gamma, \hat{\tilde{\mu}} = \tilde{\mu}/\gamma.\)

Proof Select a Lyapunov function for the time-varying system (12) as follows:

\[
V(k, e(k)) = e^T(k)P(k)e(k) + \sum_{s=k-d}^{k-1} e^T(s)Q(s+1)e(s) + \mu(k),
\]

(16)

where \( k = 0, 1, \ldots, N+1 \) and \( \{P(k)\}_{0\leq k \leq N+1}, \{Q(k)\}_{-d+1\leq k \leq N+1}, \{\mu(k)\}_{0\leq k \leq N+1} \) are the solutions of the RLMIs (15) with the initial condition (14).

It follows from Equation (12) that

\[
\mathbb{E}\left\{ e^T(k+1)P(k+1)e(k+1) \right\} = \mathbb{E}\left\{ \hat{\mathcal{A}}^T(e(k))P(k+1)e(k) \right\} + \mathbb{E}\left\{ \hat{\mathcal{B}}^T(e(k))P(k+1)e(k) \right\} + \mathbb{E}\left\{ \hat{\mathcal{C}}^T(e(k))P(k+1)e(k) \right\} \times P(k+1)e(k).
\]
where the independence properties of $\alpha(k)$, $\beta(k)$ and $\gamma(k)$ in conditions (4) and (7) are utilized. More specifically, to derive the second equality of Equation (17), the following facts have been used:

\[
\begin{align*}
\mathbb{E}[(\alpha(k) - \tilde{\alpha})^2] &= \alpha(1 - \tilde{\alpha}), \\
\mathbb{E}[(\beta(k) - \tilde{\beta})^2] &= \beta(1 - \tilde{\beta}), \\
\mathbb{E}[(1 - \beta(k))\gamma(k) - (1 - \tilde{\beta})\tilde{\gamma}]^2 &= (1 - \tilde{\beta})\tilde{\gamma}^2 - (1 - \tilde{\beta})^2\tilde{\gamma}^2, \\
\mathbb{E}[(\beta(k) - \tilde{\beta})(1 - \beta(k))\gamma(k) - (1 - \tilde{\beta})\tilde{\gamma}] &= \mathbb{E}[(\beta(k) - \tilde{\beta})(1 - \beta(k))\gamma(k) - \beta(k)\gamma(k) + \tilde{\beta}\gamma(k) - \gamma(k)(\tilde{\beta} - \beta(k)) + \tilde{\beta}(\gamma(k) - \tilde{\gamma}) + \gamma(k)(\tilde{\beta} - \beta(k))].
\end{align*}
\]

Define $\eta(k) = [e^T(k) \quad e^T(k - d) \quad v^T(k) \quad f^T(k, x(k)) \quad g^T(k, x(k - d)) \quad h^T(C(k)x(k)) \quad 1]^T$, we can easily calculate, in view of Equations (12), (16) and (17) that

\[
\begin{align*}
\mathbb{E}[V(k + 1, e(k + 1))] - \mathbb{E}[V(k, e(k))]
&= \mathbb{E}[\|\tilde{w}(k)\|^2] - \gamma^2 \mathbb{E}[\|v(k)\|^2]
\end{align*}
\]

where

\[
\Phi(k) = \begin{bmatrix}
\Phi_{11}(k) & 0 & 0 & 0 \\
* & -Q(k - d + 1) & 0 & 0 \\
* & * & -\gamma^2I & 0 \\
* & * & * & 0 \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
0 & 0 & M^T(k)(M(k) - M_f(k))\hat{x}(k) & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
* & * & * & * \\
* & * & * & \hat{y}(k)
\end{bmatrix}
\]

with $\Phi_{11}(k) = -P(k) + Q(k + 1) + M^T(k)M(k)$ and $\hat{y}(k) = \mu(k + 1) - \mu(k) + \tilde{x}^T(k)(M(k) - M_f(k))^2 \times (M(k) - M_f(k))\hat{x}(k)$.
by considering Equations (18)–(20) together, the inequality which guarantees the $H_\infty$ performance of the error system turns to:

$$\eta^T(k)\Phi_1(k)\eta(k) \leq 0, \quad \eta^T(k)\Phi_2(k)\eta(k) \leq 0, \quad (19)$$

where

$$\Phi_1(k) = \begin{bmatrix} \tilde{U}_1(k) & 0 & \tilde{U}_2(k) & 0 & \tilde{U}_1(k) \tilde{x}(k) \\
* & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & * & I & 0 \\
* & * & * & * & \tilde{x}^T(k) \tilde{U}_1(k) \tilde{x}(k) \end{bmatrix}$$

and

$$\Phi_2(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\
\tilde{V}_1(k) & 0 & \tilde{V}_2(k) & 0 & \tilde{V}_1(k) \tilde{x}(k-d) \\
* & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & * & I & 0 \\
* & * & * & * & \tilde{x}^T(k-d) \tilde{V}_1(k) \tilde{x}(k-d) \end{bmatrix}$$

Similarly, Equation (9) can be transformed to an equivalent matrix inequality given as follows:

$$\eta^T(k)\Phi_3(k)\eta(k) \leq 0, \quad (20)$$

where

$$\Phi_3(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{-C^T(k)H^T(k)}{2} & 0 \\
* & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 \\
* & * & * & * & I & -\frac{H(k)C(k)\tilde{x}(k)}{2} \end{bmatrix}$$

By considering Equations (18)–(20) together, the inequality which guarantees the $H_\infty$ performance of the error system turns to:

$$\mathbb{E}\{V(k+1,e(k+1))\} - \mathbb{E}\{V(k,e(k))\} + \mathbb{E}\{\|\tilde{z}(k)\|^2\} - \gamma^2 \mathbb{E}\{\|v(k)\|^2\} \leq 0, \quad (22)$$

Summing up both sides of Equation (22) with $k$ varying from 0 to $N$ yields

$$\mathbb{E}\{V(k+1,e(k+1))\} - \mathbb{E}\{V(k,e(k))\} + \mathbb{E}\{\|\tilde{z}(k)\|^2\} \leq \gamma^2 \mathbb{E}\{\|v(k)\|^2\} + \mu(0), \quad (23)$$

and the $H_\infty$ performance constraint (13) is assured by also considering the initial condition (14). The proof of this theorem is complete.

According to the $H_\infty$ performance analysis established in Theorem 1, the design problem of the $H_\infty$ filter for the stochastic system (1) is reduced to the one for finding feasible solutions to a set of RLMIs.

**Theorem 2** Let the disturbance attenuation level $\gamma > 0$ and the matrix sequence $\{S(k)\}_{-d+1 \leq k \leq 0}$ be given. The $H_\infty$ filtering problem is solvable for the discrete time-varying system (1) if there exist two families of positive-definite matrices $\{P(k)\}_{0 \leq k \leq N+1}$, $\{Q(k)\}_{-d+1 \leq k \leq N+1}$, two families of matrices $\{X(k)\}_{0 \leq k \leq N}$ and $\{Y(k)\}_{0 \leq k \leq N}$ and four families of positive scalars $\{\epsilon_1(k)\}_{0 \leq k \leq N}$, $\{\epsilon_2(k)\}_{0 \leq k \leq N}$, $\{\epsilon_3(k)\}_{0 \leq k \leq N}$, $\{\mu(k)\}_{0 \leq k \leq N+1}$ satisfying the initial condition (14) and the
According to Theorem 2, the following RLMIs algorithm is given for the $H_\infty$ filtering problem which is illuminated by the design ideas in Shen et al. (2010) and Dong et al. (2011). The $H_\infty$ filtering problem can be implemented recursively as follows:

**Step 1.** Let the $H_\infty$ performance index $\gamma$, the final time $N$, the initial matrix sequence $\{S(k)\}_{-d \leq k \leq 0}$ and the initial states $\{\phi(k)\}_{-d \leq k \leq 0}$ be given. Select appropriate positive-definite matrix $P(0)$, positive-definite matrix sequence $\{Q(s)\}_{-d+1 \leq s \leq 0}$ and positive scalar $\mu(0)$ satisfying the initial condition (14) and set $k = 0$.

**Step 2.** Solve the RLMIs (24) to obtain the positive-definite matrices $P(k+1)$, $Q(k+1)$, the positive scalars $\mu(k+1)$, $\varepsilon_1(k)$, $\varepsilon_2(k)$, $\varepsilon_3(k)$, the matrices $X(k)$, $Y(k)$ and the filter parameter matrix $M_f(k)$ by utilizing the known parameters $P(k)$, $Q(s)$, $\mu(k)$ and $\hat{x}(k)$.

**Step 3.** From Equation (25), derive the other two filter parameter matrices $F_f(k)$ and $G_f(k)$, then get the state estimate $\hat{x}(k+1)$ according to Equation (11).

**Step 4.** If $k < N$, set $k = k + 1$ and go to Step 2, else go to Step 5.

**Step 5.** Exit.

### 4. Numerical Example

In this section, we provide a numerical example to test the proposed design algorithm. Consider a discrete time-varying delayed system described by model (1) with the following time-varying parameters:

$$
A(k) = \begin{bmatrix} 0 & 0.1 \sin(k) \\ \sin(6k) & 0.2 \end{bmatrix},
A_1(k) = \begin{bmatrix} 0 & 0.12 \\ -0.12 & 0.1 \sin(6k) \end{bmatrix},
D(k) = 1,
B(k) = \begin{bmatrix} 0.15 \\ 0.3 \end{bmatrix},
C(k) = [0.12 \sin(6k) \ 0.1],
M(k) = [0.1 \ 0.1],
$$

and the nonlinear functions $f(\cdot, \cdot)$, $g(\cdot, \cdot)$ are the same as those in Shen et al. (2011) represented as follows:

$$
f(k, x(k)) = \begin{bmatrix} 0.1x_1(k) + 0.1x_2(k) + 0.25x_2(k) \sin(x_1(k)) \\ 0.5(x_1(k) + (1/3)x_2(k)) - 0.1x_1(k) + 0.3x_2(k) \end{bmatrix},
g(k, x(k-d)) = \begin{bmatrix} 0.3(x_1(k-d) + x_2(k-d)) + x_2^2(k-d) \\ +x_2^2(k-d) + 0.1x_1(k-d) + 0.1x_2(k-d) \\ 0.3x_1(k-d) + 0.3x_2(k-d) \end{bmatrix}.
$$
It has been shown in Shen et al. (2011) that the above nonlinearities satisfying conditions (2) and (3) with
\[
U_1(k) = V_1(k) = \begin{bmatrix} 0.1 & 0.05 \\ 0 & 0.1 \end{bmatrix},
\]
\[
U_2(k) = V_2(k) = \begin{bmatrix} -0.05 & 0 \\ -0.05 & 0.05 \end{bmatrix}.
\]
It is assumed that the saturation function is
\[
\sigma(u(k)) = \begin{cases} 
  u(k) & \text{if } -u_{\text{max}} < u(k) < u_{\text{max}}, \\
  u_{\text{max}} & \text{if } u(k) > u_{\text{max}}, \\
  -u_{\text{max}} & \text{if } u(k) < -u_{\text{max}},
\end{cases}
\]
where \( u(k) \) is the position value of target. In this example, the saturation value \( u_{\text{max}} \) is taken as 0.08, \( H_1(k) \) and \( H_2(k) \) are set as \( H_1(k) = 0.3(1 + |\tanh(k)|) \), \( H_2(k) \equiv 1 \).

In this simulation, we choose the exogenous disturbance input \( v(k) \) to be \( v(k) = \exp(-k/20) \times n(k) \), where \( n(k) \) is uniformly distributed over \([-0.05, 0.05]\). The random variables \( \alpha(k) \), \( \beta(k) \) and \( \gamma(k) \) are with the probabilities as \( \bar{\alpha} = 0.8 \), \( \bar{\beta} = 0.5 \), \( \bar{\gamma} = 0.6 \) and the constant time delay is set with \( d = 1 \). For illustration purposes, set \( N = 20 \), the \( H_\infty \) performance level \( \gamma = 0.5 \), \( S(k) = \text{diag}[20, 1] \) and the initial value \( x(-1) = [0.3 \ -0.2]^T \), \( x(0) = [0.2 \ 0]^T \). The initial condition (14) is satisfied with \( P(0) = I \), \( Q(0) = \text{diag}(2, 2) \), \( \mu(0) = 0.3 \). According to the given RLMI algorithm, the time-varying LMIs in Equation (24) can be solved recursively and the desired filter matrices \( F_f(k) \), \( G_f(k) \) and \( M_f(k) \) from time \( k = 0 \) to \( k = 6 \) are given in Table 1. It follows from Theorem 2 that the \( H_\infty \) filtering problem is solvable for the discrete time-varying system (1).

| \( k \) | 0    | 1    | 2    | 3    |
|---|---|---|---|---|
| \( F_f(k) \) | \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} | \begin{bmatrix} 0.0283 & 0.0654 \\ 0.0692 & 0.0195 \end{bmatrix} | \begin{bmatrix} 0.1794 & 0.1440 \\ 0.1702 & -0.3478 \end{bmatrix} | \begin{bmatrix} -0.0105 & -0.0629 \\ -0.0859 & 0.2099 \end{bmatrix} |
| \( G_f(k) \) | \begin{bmatrix} 0.0998 \\ 0.2057 \end{bmatrix} | \begin{bmatrix} 0.1029 \\ 0.2105 \end{bmatrix} | \begin{bmatrix} 0.1032 \\ 0.2209 \end{bmatrix} | \begin{bmatrix} 0.1235 \\ 0.2834 \end{bmatrix} |
| \( M_f(k) \) | \begin{bmatrix} 0 & 0 \\ 0.0524 & 0.1066 \end{bmatrix} | \begin{bmatrix} 0.0597 & 0.1066 \end{bmatrix} | \begin{bmatrix} 0.0073 & 0.0676 \end{bmatrix} |

| \( k \) | 4    | 5    | 6    | \ldots |
|---|---|---|---|---|
| \( F_f(k) \) | \begin{bmatrix} 0.0186 & 0.2248 \\ 0.3614 & -0.1823 \end{bmatrix} | \begin{bmatrix} -0.0021 & -0.0063 \\ -0.0105 & 0.0618 \end{bmatrix} | \begin{bmatrix} -0.2377 & 0.0155 \\ 0.0572 & -0.1260 \end{bmatrix} | \ldots |
| \( G_f(k) \) | \begin{bmatrix} 0.1103 \\ 0.2924 \end{bmatrix} | \begin{bmatrix} 0.1308 \\ 0.3489 \end{bmatrix} | \begin{bmatrix} 0.1405 \\ 0.3104 \end{bmatrix} | \ldots |
| \( M_f(k) \) | \begin{bmatrix} 0.0077 & 0.0681 \\ -0.0176 & -0.0417 \end{bmatrix} | \begin{bmatrix} -0.1006 & 0.0598 \end{bmatrix} | \ldots |

![Figure 1](image-url). Trajectories of the output \( z \) and its estimate \( \hat{z} \).
By employing the filter (11) with the parameters as given in Table 1, the estimation trajectories are shown in Figures 1 and 2. More specially, Figure 1 presents the output \( z(k) \) and its estimate \( \hat{z}(k) \) and Figure 2 draws the estimation error output \( \tilde{z}(k) \). The simulation results further demonstrate the effectiveness of the filter design scheme.

5. Conclusions
In this paper, we have investigated the finite horizon \( H_\infty \) filtering problem for the time-varying delayed system with incomplete information such as RONs, ROSSs as well as missing measurements. A time-varying filter has been designed for the system under consideration such that the filtering error system satisfies the \( H_\infty \) performance constraints on the finite horizon. By resorting to the stochastic analysis and matrix inequality techniques, sufficient conditions have been derived in the form of RLMIs which not only guarantee the error system to preserve the \( H_\infty \) performance but also give the explicit expressions of the desired filtering parameters. Simulation results further demonstrate the feasibility of the proposed filtering methods.

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References
Ahmad, H., & Namerikawa, T. (2013). Extended Kalman filter-based mobile robot localization with intermittent measurements. Systems Science & Control Engineering: An Open Access Journal, 1, 113–126.
Basin, M. V., Shi, P., Alvarez, D. C., & Wang, J. (2009). Central suboptimal \( H_\infty \) filter design for linear time-varying systems with state or measurement delay. Circuits, Systems and Signal Processing, 28(2), 305–330.
Boyd, S., El Ghaoui, L., Feron, E., & Balakrishnan, V. (1994). Linear matrix inequalities in system and control theory. Philadelphia, PA: SIAM.
Cao, Y-Y., Lin, Z., & Chen, B. M. (2003). An output feedback \( H_\infty \) controller design for linear systems subject to sensor nonlinearities. IEEE Transactions on Circuits and Systems – I, 50(7), 914–921.
Chau, C-K., Qin, F., Sayed, S., Wahab, M. H., & Yang, Y. (2010). Harnessing battery recovery effect in wireless sensor networks: Experiments and analysis. IEEE Journal on Selected Areas in Communications, 28(7), 1222–1232.
Chen, Y., & Zheng, W. (2012). Stochastic state estimation for neural networks with distributed delays and Markovian jump. Neural Networks, 25, 14–20.
Ding, D., Wang, Z., Shen, B., & Shu, H. (2012). \( H_\infty \) state estimation for discrete-time complex networks with randomly occurring sensor saturations and randomly varying sensor delays. IEEE Transactions on Neural Networks and Learning Systems, 23(5), 725–736.
Ding, D., Wang, Z., Ho, D. W. C., & Gao, H. (2011). Robust \( H_\infty \) filtering for Markovian jump systems with randomly occurring nonlinearities and sensor saturation: The finite-horizon case. IEEE Transactions on Signal Processing, 59(7), 3048–3057.
Gershon, E., & Shaked, U. (2008). \( H_\infty \) output-feedback control of discrete-time systems with state-multiplicative noise. Automatica, 44(2), 574–579.
Gershon, E., Shaked, U., & Yaesh, I. (2001). \( H_\infty \) control and filtering of discrete-time stochastic systems with multiplicative noise. Automatica, 37(3), 409–417.
Gershon, E., Shaked, U., & Yaesh, I. (2005). \( H_\infty \) control and estimation of state-multiplicative linear systems. New York, NY: Springer-Verlag.
Hung, Y. S., & Yang, F. (2003). Robust $H_{\infty}$ filtering with error variance constraints for discrete time-varying systems with uncertainty. *Automatica, 39*(7), 1185–1194.

Li, P., Lam, J., & Shu, Z. (2010). $H_{\infty}$ positive filtering for positive linear discrete-time systems: An augmentation approach. *IEEE Transactions on Automatic Control, 55*(10), 2337–2342.

Liu, J., Wang, J., & Yang, G. (2003). Reliable guaranteed variance filtering against sensor failures. *IEEE Transactions on Signal Processing, 51*(5), 1403–1411.

Lu, X., Xie, L., Zhang, H., & Wang, W. (2007). Robust Kalman filtering for discrete-time systems with measurement delay. *IEEE Transactions on Circuits and Systems – II, 54*(6), 522–526.

Mohamed, M., Nahla, K., & Safya, B. (2013). Design of a nonlinear observer for mechanical systems with unknown inputs. *Systems Science & Control Engineering: An Open Access Journal, 1*, 105–112.

Nahi, N. E. (1969). Optimal recursive estimation with uncertain observations. *IEEE Transactions on Information Theory, IT-15*(4), 457–462.

Reif, K., & Unbehauen, R. (1999). The extended Kalman filter as an exponential observer for nonlinear systems. *IEEE Transactions on Signal Processing, 47*(8), 2324–2328.

Shaked, U., & Suplin, V. (2001). A new bounded real lemma representation for the continuous-time case. *IEEE Transactions on Automatic Control, 46*(9), 1420–1426.

Shen, B., Ding, S. X., & Wang, Z. (2013). Finite-horizon $H_{\infty}$ fault estimation for linear discrete time-varying systems with delayed measurements. *Automatica, 49*(1), 293–296.

Shen, B., Wang, Z., & Liu, X. (2011). Bounded $H_{\infty}$ synchronization and state estimation for discrete time-varying stochastic complex networks over a finite horizon. *IEEE Transactions on Neural Networks, 22*(1), 145–157.

Shen, B., Wang, Z., Shu, H., & Wei, G. (2010). Robust $H_{\infty}$ finite-horizon filtering with randomly occurred nonlinearities and quantization effects. *Automatica, 46*(11), 1743–1751.

Shen, B., Wang, Z., Shu, H., & Wei, G. (2011). $H_{\infty}$ filtering for uncertain time-varying systems with multiple randomly occurred nonlinearities and successive packet dropouts. *International Journal of Robust and Nonlinear Control, 21*(14), 1693–1709.

Wang, Z., Dong, H., Shen, B., & Gao, H. (2013). Finite-horizon $H_{\infty}$ filtering with missing measurements and quantization effects. *IEEE Transactions on Automatic Control, 58*(7), 1707–1718.