Location of Poles for the Hastings-McLeod Solution to the Second Painlevé Equation

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Abstract
We show the well-known Hastings-McLeod solution to the second Painlevé equation is pole-free in the region \( \arg x \in \left[ -\frac{\pi}{3}, \frac{\pi}{3} \right] \cup \left[ \frac{2\pi}{3}, \frac{4\pi}{3} \right] \), which proves an important special case of a general conjecture concerning pole distributions of Painlevé transcendents proposed by Novokshenov. Our strategy is to construct explicit quasi-solutions approximating the Hastings-McLeod solution in different regions of the complex plane, and estimate the errors rigorously. The main idea is very similar to the one used to prove Dubrovin's conjecture for the first Painlevé equation, but there are various technical improvements.

1 Introduction

Painlevé equations are six second order nonlinear ordinary differential equations first studied by Painlevé and his colleagues around 1900. They are well known for the so-called Painlevé property, i.e., their solutions are free from movable branch points; see [26 §32.2]. Here ‘movable’ means the location of the singularities (which in general can be poles, essential singularities or branch points) of the solutions depend on the constants of integration associated with the initial or boundary conditions of the differential equations. The solutions of these equations, often called the Painlevé transcendents [26], in general cannot be represented in terms of elementary functions or known classical special functions. They play important roles in both pure and applied mathematics, and are widely thought of as the nonlinear counterparts of the classical special functions.

For the first two Painlevé equations

\[
\begin{align*}
\text{PI} & \quad y'' = 6y^2 + x, \\
\text{PII} & \quad y'' = 2y^3 + xy + \alpha,
\end{align*}
\]

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all solutions are meromorphic in the complex plane with $x = \infty$ being the only essential singularity. The locations of the movable poles for the Painlevé transcendents are crucial for understanding a number of problems arising from mathematical physics; cf. [2, 13, 22, 23]. In the pioneering works [4, 5], Boutroux established the “deformed” elliptic function approximations in appropriate sectors near infinity, which leads to the degeneration of lattices of poles along the critical rays

$$\Gamma_k := \left\{ x \mid \arg x = \frac{2k\pi i}{N} \right\}, \quad k = 0, 1, \ldots, N - 1,$$

(1.3)

where

$$N = \begin{cases} 5, & \text{for PI}, \\ 6, & \text{for PII}. \end{cases}$$

(1.4)

This means generally the lines of poles are smooth curves which tend to one of the rays $\Gamma_k$ near infinity. Furthermore, Boutroux also showed the existence of solutions which have no lines of poles near infinity near $n$ ($n = 1, 2, 3$) of the critical rays $\Gamma_k$, which are called $n$-truncated solutions.

An interesting feature of the 2- or 3-truncated solutions is, as confirmed by numerical studies in [15, 16, 24], that the distributions of poles near infinity characterize the global behavior of the poles. More precisely, let $\Xi_k$ be the sector bounded by two consecutive critical rays:

$$\Xi_k := \left\{ x \mid 2k\pi i < \arg x < 2(k + 1)\pi i \right\}, \quad k = 0, 1, \ldots, N - 1.$$

(1.5)

The following conjecture was made in [25] by Novokshenov:

**Conjecture 1.1.** If the 2- or 3-truncated solution of Painlevé equation has no pole at infinity in a sector $\Xi_k$, then it has no poles in the whole sector $\Xi_k$.

For the 3-truncated solutions of PI, a special case of this conjecture is known as Dubrovin’s conjecture, which appeared in [13] with connections to the critical behavior of the nonlinear Schrödinger equation. It was proved recently in [10] with a technique developed in [9]; see also [21, 22, 23] for partial results.

In this paper, we will further improve the technique in [10] and give an analytic proof of Conjecture [1.1] in the context of a special 2-truncated solution of PII, namely, the Hastings-McLeod solution [18]. This solution might be the most famous one among the Painlevé transcendents, due to its frequent appearances in applications, especially in mathematical physics. For instance, the cumulative distribution function of the celebrated Tracy-Widom distribution [27, 28] admits an integral representation involving the Hastings-McLeod solution. More recent applications related to random matrix models, non-intersecting Brownian motions can be found in [3, 11, 12, 14]. Our main result is stated in the next section.
2 Statement of results

The Hastings-McLeod solution $y_{HM}$ is a special solution of (2.1) with $\alpha = 0$, i.e., it satisfies the equation

$$y'' = 2y^3 + xy. \tag{2.1}$$

The solution $y_{HM}$ is known to be pole-free on the real axis ([19]), and has the following asymptotics:

$$y_{HM}(x) \sim \begin{cases} 
\text{Ai}(x), & \text{as } x \to +\infty, \\
\sqrt{-x/2}, & \text{as } x \to -\infty,
\end{cases} \tag{2.2}$$

where $\text{Ai}(x)$ denotes the usual Airy function [26]. A plot of $y_{HM}(x)$ for real $x$ is shown in the left picture of Figure 1. The locations of poles for $y_{HM}$ is illustrated in the right picture of Figure 1, which is taken from [24]. The six dashed lines are the critical lines defined in (1.3), and it is clear from the picture that all the poles are located in the sectors $\Xi_1 \cup \Xi_4$, which is consistent with Conjecture 1.1.

Our main result is stated as follows.

**Theorem 2.1.** The Hastings-McLeod solution $y_{HM}$ of the second Painlevé equation (2.1) is pole-free in the region $\arg x \in [-\pi/3, \pi/3] \cup [2\pi/3, 4\pi/3]$.

For $|x|$ large enough, a partial result was shown in [19] via the Riemann-Hilbert approach; see also [17] Theorems 11.1 and 11.7]. A more recent progress toward this result was obtained by Bertola in [1], where he showed that $y_{HM}$ is pole-free in the sector $\arg x \in $.

![Figure 1: The Hastings-MeLeod solution (left) and its pole distribution (right).](image-url)
His proof is based on the representation of the Hastings-McLeod solution in terms of the second logarithmic derivative of the Fredholm determinant of a certain integral operator and an operator-norm estimate. In contrast, our method is based on a direct analysis of (2.1), and it can be applied to other equations including the general PII equation (1.2) with \( \alpha \neq 0 \).

3 Strategy of proof

Although our method works for both of the two sectors \( \arg x \in [-\pi/3, \pi/3] \) and \( \arg x \in [2\pi/3, 4\pi/3] \), we shall focus on the sector \( \arg x \in [2\pi/3, 4\pi/3] \) and briefly mention the ideas of proof for the sector \( \arg x \in [-\pi/3, \pi/3] \), as the desired result about this sector was already shown in [1]; see Section 8 below.

We first note that \( y_{HM}(\bar{z}) \) is also a solution to (2.1). This, together with the fact that \( y_{HM} \) is real on the real line and uniqueness of the solution, implies that

\[
y_{HM}(z) = y_{HM}(\bar{z}).
\]

Therefore, for \( \arg x \in [2\pi/3, 4\pi/3] \), it is sufficient to prove the following result:

**Theorem 3.1.** The Hastings-McLeod solution \( y_{HM} \) is pole-free in the region

\[
\Omega := \left\{ x \in \mathbb{C} \mid 2\pi/3 \leq \arg x \leq \pi \right\}.
\]

As mentioned before, we will use the same idea as in [10] to prove the Theorem. To be precise, we will analyze \( y_{HM} \) in two regions

\[
\Omega_0 := \left\{ x \in \mathbb{C} \mid |x| \geq \frac{3^{4/3}}{2}, \ 2\pi/3 \leq \arg x \leq \pi \right\}
\]

and

\[
\Omega_2 := \left\{ x \in \mathbb{C} \mid |x| \leq 9/4, \ 2\pi/3 \leq \arg x \leq \pi \right\}.
\]

In each region we will construct an explicit quasi-solution consisting of polynomials and exponential functions, and show that the difference between \( y_{HM} \) and the quasi-solution is small in a suitable norm. This shows that \( y_{HM} \) is pole-free in both \( \Omega_0 \) and \( \Omega_2 \), and hence in \( \Omega \subseteq \Omega_0 \cup \Omega_2 \).

The main challenge of the proof is to find an effective quasi-solution approximation of the Hastings-McLeod solution which has sufficient accuracy for both small and large \( |x| \). This requires comprehensive knowledge of the asymptotics of the solution near infinity. To this end, we mention the following asymptotics of Hastings-McLeod solution relevant to our proof (see [17, Theorem 11.7] and [19]).

**Proposition 3.2.** Let \( y_{HM} \) be the Hastings-McLeod solution of the second Painlevé equation (2.1), then

\[
y_{HM}(x) = \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} \left( 1 + \mathcal{O}(x^{-3/4}) \right)
\]
as $x \to +\infty$ and $\arg x = 0$;

$$y_{HM}(x) = \sqrt{-x/2} \left( 1 + O((x)^{-3/2}) \right) + c_-(x)^{-1/4} e^{-2\sqrt{x}/3} \left( 1 + O(x^{-1/4}) \right)$$

as $x \to \infty$ and $\arg x \in \left[ \frac{2\pi}{3}, \frac{4\pi}{3} \right]$, where

$$c_- = \frac{i2^{-7/4}}{\sqrt{\pi}}.$$  

The constant $c_-$ is the so-called quasi-linear Stokes’ multiplier, which reflects the quasi-linear Stokes phenomenon for the second Painlevé transcendent; see [7, 19, 20] for more details. We emphasize two features of the asymptotics in Proposition 3.2:

- The asymptotics (3.5) is valid along the critical line $\arg x = 2\pi/3$ (i.e., the boundary of the relevant sector), where the asymptotics is oscillatory.

- The Hastings-McLeod solution $y_{HM}$ is characterized by either of the two asymptotic relations (3.4) and (3.5). Indeed, it suffices to specify the asymptotics just along the boundary rays; see [17, Chapter 11].

As we shall see later, the construction of quasi-solutions in the regions away from the origin is based on these asymptotic behaviors.

The rest of this paper is organized as follows. Analysis of $y_{HM}$ in $\Omega_0$ is accomplished in Section 4. To get a concrete estimate of the initial values of $y_{HM}$ at 0, we will also need to study $y_{HM}$ along $[0, +\infty)$ before we are able to construct a quasi-solution in $\Omega_2$, which is carried out in Sections 5 and 6. Analysis in $\Omega_2$ is accomplished in Section 7. We conclude this paper with the proofs of our main results in Section 8.

4 Analysis of $y_{HM}$ in the region $\Omega_0$

We start with the analysis of the PII equation (2.1) in the region $|x| \geq 3^{4/3}/2$, $2\pi/3 \leq \arg x \leq \pi$. Our goal in this section is to prove the following result:

Proposition 4.1. The Hastings-McLeod solution $y_{HM}$ is pole-free in the region $\Omega_0$, where $\Omega_0$ is defined in (3.2).

As mentioned before, we will prove the above result by constructing a pole-free quasi-solution to PII, and showing that the difference between the quasi-solution and the Hastings-McLeod solution is bounded. Our construction of this quasi-solution is motivated by asymptotic expansions with exponential sums studied in [8], which suggests that we make the following change of variables:

$$t = \frac{2}{3} \sqrt{2} (-x)^{3/2}; \quad y(x) = \frac{\sqrt{3I}}{2} h(t).$$

(4.1)
This brings (2.1) into the normalized form
\[ h''(t) + \frac{h'(t)}{t} + \frac{h(t)}{2} - \frac{h(t)}{9t^2} - \frac{1}{2} h(t)^3 = 0, \quad (4.2) \]
and the region of interest in Proposition 4.1 corresponds to
\[ \Omega_1 := \left\{ t \in \mathbb{C} \mid |t| \geq 3, \ -\pi/2 \leq \arg t \leq 0 \right\}, \quad (4.3) \]
in the new variable \( t \).

Let \( h_{HM} \) denote the solution of (4.2) corresponding to the Hastings-McLeod solution. In view of (3.5), one naturally expects to have the decomposition
\[ h_{HM} = h_p + h_e, \quad (4.4) \]
where \( h_p \) is a solution of (4.2) with pure power series behavior near \(-i\infty\) (i.e., with zero quasi-linear Stokes’ multiplier), and \( h_e \) is exponentially small near \(-i\infty\). This is also consistent with the fact the Painlevé equations admit a one-parameter family of solution represented by the sum
\[ y = \text{(power series)} + \text{(exponential terms)}, \]
which was found by Boutroux [4, 5], and particularly this includes \( y_{HM} \) as a special case of PII. Since we only need to prove \( h_{HM} \) is pole-free in \( \Omega_1 \), we do not need to consider full expansions. Instead, we will only show the existence of a decomposition (4.4) with
\[ h_p \sim 1 - \frac{1}{9t^2} \quad \text{and} \quad h_e \sim \frac{\sqrt{2} \tilde{c} e^{-t}}{\sqrt{t}}, \]
where \( \tilde{c} \) is a constant related to the quasi-linear Stokes multiplier \( c_- \); see (4.14) below.

4.1 Existence and uniqueness of the power series solution \( h_p \)
Recall the asymptotics of \( y_{HM} \) in (3.5), by (4.1), this corresponds to a solution \( h \sim 1 \) of (4.2) in \( \Omega_1 \). Formal asymptotic analysis of (4.2) indicates that there should exist a solution \( h(t) \sim 1 - \frac{1}{9t^2} \). We thus substitute
\[ h(t) = 1 - \frac{1}{9t^2} + \frac{h_1(t)}{\sqrt{t}} \]
into (4.2), and get the equation
\[ h''_1(t) - h_1(t) \]
\[ = \frac{73}{162t^{7/2}} - \frac{1}{1458t^{11/2}} + \left( -\frac{17}{36t^2} + \frac{1}{54t^4} \right) h_1(t) + \left( \frac{3}{2\sqrt{t}} - \frac{1}{6t^{5/2}} \right) h_1(t)^2 + \frac{h_1(t)^3}{2t} \]
\[ =: R_h(t, h(t)). \quad (4.5) \]
Inverting the differential operator on the left side of (4.5), we get the integral equation

\[ h_1(t) = T_1(h(t)) := L_1(R_h(t, h(t))) \]

\[ := \frac{1}{2} \left( e^t \int_{-\infty}^{t} e^{-s} R_h(s, h(s)) ds - e^{-t} \int_{-\infty}^{t} e^{s} R_h(s, h(s)) ds \right). \tag{4.6} \]

We intend to prove existence of a solution \( h_p \) by showing that \( T_1 \) is a contractive map in a suitable Banach space. The expressions of \( R_1 \) and \( L \) indicate that it is necessary to estimate generalized exponential integrals in the complex plane. For this purpose, we introduce the following inequalities, which will also be used later:

**Lemma 4.2.** Assume \( f \) is analytic in the right half plane with \( |f(s)| \leq c/|s|^n \) where \( c > 0 \), \( n > 1 \), and \( \Re s \geq 0 \). For \( t \in \Omega_1 \), we have the estimates

\[ \left| \int_{+\infty}^{t} e^{-ms} f(s) ds \right| \leq \frac{ce^{-m \Re t}}{m |t|^n}, \quad m > 0, \tag{4.7} \]

\[ \left| \int_{+\infty}^{t} e^{-ms} f(s) ds \right| \leq \frac{ce^{-m \Re t}}{(n-1)|t|^{n-1}}, \quad m \geq 0, \tag{4.8} \]

\[ \left| \int_{-\infty}^{t} e^{ms} f(s) ds \right| \leq \frac{e^{\sqrt{\pi} \Gamma \left( \frac{n}{2} - \frac{1}{2} \right)} e^{m \Re t}}{2 \Gamma \left( \frac{n}{2} \right) |t|^{n-1}}, \quad m \geq 0. \tag{4.9} \]

**Proof.** We write \( t = a + bi \) where \( a \geq 0 \) and \( b \leq 0 \). To prove the first inequality, we note that since \( f \) is analytic with at least \( t^{-n} \) decay in the right half plane, we can rotate the integration path to a horizontal one, namely \( s = a + u + bi \) with \( u \) ranging from \( \infty \) to \( 0 \). Then by direct calculations we have

\[ \left| \int_{+\infty}^{t} e^{-ms} f(s) ds \right| \leq ce^{-ma} \left( \int_{0}^{+\infty} \frac{e^{-mu}}{((a+u)^2 + b^2)^{n/2}} du \right) \leq \frac{ce^{-ma}}{(a^2 + b^2)^{n/2}} \left( \int_{0}^{+\infty} e^{-mu} du \right) = \frac{ce^{-ma}}{m |t|^n}. \]

Alternatively, we can also rotate the integration path to a radial one, which gives

\[ \left| \int_{+\infty}^{t} e^{-ms} f(s) ds \right| \leq ce^{-ma} \left( \int_{|t|}^{+\infty} \frac{1}{|s|^{n}} |d|s| \right) = \frac{ce^{-m \Re t}}{(n-1)|t|^{n-1}}. \]

To prove the last inequality, we rotate the contour to a vertical one, namely \( s = a + ui \) with \( u \) ranging from \( -\infty \) to \( 0 \). By direct calculations we have

\[ \left| \int_{-\infty}^{t} e^{ms} f(s) ds \right| \leq ce^{ma} \left( \int_{-\infty}^{0} \frac{1}{(a^2 + (b + u)^2)^{n/2}} du \right) \leq \frac{ce^{ma}}{(a^2 + b^2 + u^2)^{n/2}} \left( \int_{-\infty}^{0} du \right) \]

\[ \leq ce^{ma} \left( \frac{1}{|t|^{n-1}} \int_{-\infty}^{0} \frac{1}{(1 + v^2)^{n/2}} dv \right) = \frac{c\sqrt{\pi} \Gamma \left( \frac{n}{2} - \frac{1}{2} \right) e^{ma}}{2 \Gamma \left( \frac{n}{2} \right) |t|^{n-1}}. \tag{4.10} \]

This completes the proof of the lemma. \( \square \)

Now we are ready to prove the main result below.
Proposition 4.3. There is a unique solution of equation (4.5) satisfying

\[ |h_1(t)| \leq \frac{6}{5} |t|^{7/2} \]  

in \( \Omega_1 \).

Proof. We will prove the proposition using the contraction mapping principle in the Banach space \( S_1 \) of analytic functions in the interior of \( \Omega_1 \), continuous up to the boundary, equipped with the weighted norm

\[ ||f||_1 = \sup_{t \in \Omega_1} \left| t^{7/2} f(t) \right| . \]

We now show that the operator \( T_1 \) (see (4.5) and (4.6)) is a contractive map in a ball of size \( \frac{6}{5} \) of \( S_1 \). Since \( T_1 \) clearly preserves analyticity and continuity, we only need to show two statements, namely,

(i) if \( ||f||_1 \leq \frac{6}{5} \), then \( ||T_1(f)||_1 \leq \frac{6}{5} \);

(ii) \( ||T_1(f_1) - T_1(f_2)||_1 \leq \lambda ||f_1 - f_2||_1 \) for some \( \lambda < 1 \).

These follow from direct calculations and elementary estimates using Lemma 4.2.

Proof of statement (i): We now estimate \( L_1(R_h(t,f(t))) \) in (4.6), assuming \( ||f||_1 \leq \frac{6}{5} \). The term \( \frac{73}{162} \) in (4.5) needs special care due to its slow decay, and we estimate it using (4.7), (4.9), and integration by parts:

\[ e^t \int_{+\infty}^{t} 73e^{-s} \frac{1}{162s^{7/2}} ds \leq \frac{73}{162} |t|^{7/2}, \]

\[ e^{-t} \int_{-\infty}^{t} 73e^{s} \frac{1}{162s^{7/2}} ds \leq \frac{73}{162} |t|^{7/2} + \frac{7e^{-t}}{2} \int_{-\infty}^{t} 73e^{s} \frac{1}{162s^{9/2}} ds \leq \frac{73 (7 \sqrt{\pi} \Gamma (\frac{9}{4}) + 4 \Gamma (\frac{7}{4}))}{648 |t|^{7/2} \Gamma (\frac{9}{4})} \leq \frac{1}{40}. \]

The rest of the terms in (4.5) are estimated by adding absolute values of all monomials in \( 1/\sqrt{t} \) and using (4.8) and (4.9). In summary, we have

\[ \left| t^{7/2} L_1 \left( \frac{73}{162t^{7/2}} - \frac{1}{1458t^{11/2}} \right) \right| \leq \frac{73}{162} + \frac{511 \sqrt{\pi} \Gamma (\frac{9}{4})}{1296 \Gamma (\frac{7}{4})} \leq \frac{1}{13122} + \frac{\sqrt{\pi} \Gamma (\frac{7}{4})}{5832 \Gamma (\frac{9}{4})} \leq \frac{1}{10}, \]

\[ \left| t^{7/2} L_1 \left( \frac{17}{36t^2} + \frac{1}{54t^4} \right) f(t) \right| \leq \frac{6}{5} \left( \frac{1}{|t|^3} \left( \frac{\sqrt{\pi} \Gamma (\frac{13}{4})}{216 \Gamma (\frac{11}{4})} + \frac{1}{702} \right) + \frac{1}{|t|} \left( \frac{17 \sqrt{\pi} \Gamma (\frac{7}{4})}{144 \Gamma (\frac{9}{4})} + \frac{17}{324} \right) \right) < \frac{1}{100}. \]
Adding up the above bounds we see that
\[
\left| f(t) \right| \leq \frac{3}{2\sqrt{t}} \left( \frac{3}{64^{1/2}} \right) f(t^2) \leq \frac{36}{25} \left( \frac{1}{102} + \frac{\sqrt{\pi} \Gamma \left( \frac{17}{4} \right)}{24\Gamma \left( \frac{13}{4} \right)} \right) \left( \frac{1}{102} + \frac{3\sqrt{\pi} \Gamma \left( \frac{13}{4} \right)}{8\Gamma \left( \frac{13}{4} \right)} \right) \] < \frac{3}{100},
\]
and
\[
\left| f(t) \right| \leq \frac{36}{25} \left( \frac{1}{102} + \frac{\sqrt{\pi} \Gamma \left( \frac{17}{4} \right)}{24\Gamma \left( \frac{13}{4} \right)} \right) \left( \frac{1}{102} + \frac{3\sqrt{\pi} \Gamma \left( \frac{13}{4} \right)}{8\Gamma \left( \frac{13}{4} \right)} \right) \] < 10^{-4}.
\]
Adding up the above bounds we see that
\[
\left| f(t) \right| < \frac{41}{40} + \frac{41}{500} + \frac{3}{100} + 10^{-4} = 1.1371 < \frac{6}{5}.
\]

**Proof of statement (ii):** We only need to do similar estimates for nonlinear terms in (4.5) using (4.8) and (4.9), as well as the simple facts that
\[
|f_1^2 - f_2^2| \leq 2||f_1 - f_2||_1 \frac{6}{5|t|^7}, \quad |f_1^3 - f_2^3| \leq 3||f_1 - f_2||_1 \frac{36}{25|t|^{21/2}}.
\]

Straightforward calculations give us
\[
\left| t^{7/2} L_1 \left( \frac{-17}{36t^2} + \frac{1}{54t^2} \right) (f_1(t) - f_2(t)) \right| \leq \frac{1}{|t|^3} \left( \frac{\sqrt{\pi} \Gamma \left( \frac{17}{4} \right)}{216\Gamma \left( \frac{13}{4} \right)} + \frac{17}{702} \right) + \frac{1}{|t|^3} \left( \frac{17\sqrt{\pi} \Gamma \left( \frac{21}{4} \right)}{144\Gamma \left( \frac{17}{4} \right)} + \frac{17}{324} \right) < \frac{7}{100}||f_1 - f_2||_1, \]
\[
\left| t^{7/2} L_1 \left( \frac{3}{2\sqrt{t}} - \frac{1}{64^{1/2}} \right) (f_1^3(t) - f_2^3(t)) \right| \leq \frac{6}{5} \left( \frac{1}{|t|^3} \left( \frac{1}{51} + \frac{\sqrt{\pi} \Gamma \left( \frac{17}{4} \right)}{12\Gamma \left( \frac{13}{4} \right)} \right) + \frac{1}{|t|^3} \left( \frac{3}{13} + \frac{3\sqrt{\pi} \Gamma \left( \frac{13}{4} \right)}{4\Gamma \left( \frac{13}{4} \right)} \right) \right) < \frac{1}{20}||f_1 - f_2||_1, \]
and
\[
\left| t^{7/2} L_1 \left( \frac{1}{2t} \right) (f_1(t) - f_2(t)) \right| \leq \frac{36}{25|t|^7} \left( \frac{1}{14} + \frac{\sqrt{\pi} \Gamma \left( \frac{21}{4} \right)}{8\Gamma \left( \frac{21}{4} \right)} \right) < 3 \cdot 10^{-4}||f_1 - f_2||_1.
\]
Adding up the above bounds we see that
\[
||T_1(f_1) - T_1(f_2)||_1 < \left( \frac{7}{100} + \frac{1}{20} + 3 \cdot 10^{-4} \right)||f_1 - f_2||_1 < \frac{7}{50}||f_1 - f_2||_1.
\]
This completes the proof of Proposition 4.3. \(\square\)

Now we define
\[
h_p(t) := 1 - \frac{1}{9t^2} + \frac{h_2(t)}{t^4}, \quad h_2(t) := t^{7/2} h_1(t).
\]
It follows from Proposition 4.3 that \(|h_2(t)| \leq \frac{6}{5}\). Clearly \(h_p\) is pole-free in \(\Omega_1\).
4.2 Existence and uniqueness of the exponential correction \( h_e \)

To analyze the exponential part of \( h_{HM} \), we see from (3.5) that

\[
y_{HM} \sim \sqrt{-x/2} + e_{-}(-x)^{-1/4}e^{-2\sqrt[4]{2}(-x)^{3/2}}
\]  

(4.13)

for \( x \to (-1 + \sqrt{3})\infty \), which means by (4.1),

\[
h_{HM} \sim 1 + \sqrt{2}\tilde{c}e^{-t} \quad \tilde{c} = \frac{2^{3/4}e_{-}}{\sqrt{3}} = \frac{i}{2\sqrt{3}\pi},
\]  

(4.14)

as \( t \to -i\infty \).

Thus we write

\[
h = h_{p} + \frac{\tilde{c}e^{-t}}{\sqrt{t}}h_{3},
\]

and substitute this expression into (4.2), which gives the equation

\[
h''_{3}(t) - 2h'_{3}(t) = \left(\frac{3}{2}h_{p}(t)^{2} - \frac{5}{36t^{2}} - \frac{3}{2}\right)h_{3}(t) + \frac{3\tilde{c}e^{-t}h_{p}(t)h_{3}(t)^{2}}{2\sqrt{t}} + \frac{\tilde{c}^{2}e^{-2t}h_{3}(t)^{3}}{2t}
\]  

(4.15)

with

\[
h_{3} \sim \sqrt{2} \quad \text{as} \quad t \to -i\infty.
\]

Based on the first few terms of the asymptotic expansion of \( h_{3} \), we construct a quasi-solution

\[
h_{a}(t) = \frac{\tilde{c}^{3}e^{-3t}}{2t^{3/2}} + \frac{\tilde{c}^{2}e^{-2t}}{\sqrt{2}t} - \frac{41\tilde{c}e^{-t}}{36t^{3/2}} + \frac{\tilde{c}e^{-t}}{\sqrt{t}} - \frac{17}{36\sqrt{2}t} + \sqrt{2},
\]  

(4.16)

and our goal is to show that there exists a solution to (4.15) of the form

\[
h_{3} = h_{a} + \delta_{1},
\]  

(4.17)

where \( \delta_{1} \) is small in a suitable norm. The equation for \( \delta_{1} \) can be found by substituting (4.17) into (4.15), which gives

\[
\delta''_{1}(t) - 2\delta'_{1}(t) = -R_{1}(t) + \delta_{1}(t) \left(\frac{3\tilde{c}^{2}e^{-2t}h_{a}(t)^{2}}{2t} + \frac{3\tilde{c}e^{-t}h_{p}(t)h_{a}(t)}{\sqrt{t}} + \frac{3}{2}h_{p}(t)^{2} - \frac{5}{36t^{2}} - \frac{3}{2}\right)
\]

\[
+ \delta_{1}(t)^{2} \left(\frac{3\tilde{c}^{2}e^{-2t}h_{a}(t)}{2t} + \frac{3\tilde{c}e^{-t}h_{p}(t)}{2\sqrt{t}}\right) + \frac{\tilde{c}^{2}e^{-2t}\delta_{1}(t)^{3}}{2t} =: R_{d}(\delta_{1}(t), t),
\]  

(4.18)

where

\[
R_{1}(t) = h''_{a}(t) - 2h'_{a}(t) - \left(\frac{3}{2}h_{p}(t)^{2} - \frac{5}{36t^{2}} - \frac{3}{2}\right)h_{a}(t)
\]

\[
- \frac{3\tilde{c}e^{-t}h_{p}(t)h_{a}(t)}{2\sqrt{t}} - \frac{\tilde{c}^{2}e^{-2t}h_{a}(t)^{3}}{2t},
\]  

(4.19)
We obtain the following integral equation by inverting the operator on the left side of (4.18):

$$\delta_1(t) = \mathcal{T}_2(\delta_1(t)) := \mathcal{L}_2(R_d(\delta_1(t), t)) := \int_{\infty}^{t} e^{2u} \int_{\infty}^{u} e^{-2s} R_d(\delta_1(s), s) dsdu. \quad (4.20)$$

To estimate $\mathcal{L}_2$, we introduce a lemma similar to Lemma 4.2.

**Lemma 4.4.** Assume $f$ is analytic in the right half plane with $|f(t)| \leq c/|t|^n$ where $c > 0$ and $\text{Re} \ t \geq 0$. For $n > 1$ and $m > 0$, we have the estimate

$$|\mathcal{L}_2(e^{-mt} f(t))| \leq \frac{ce^{-m \text{Re} \ t}}{m(m + 2)|t|^n}. \quad (4.21)$$

For $n > 2$ and $m \geq 0$, we have the estimate

$$|\mathcal{L}_2(e^{-mt} f(t))| \leq \frac{ce^{-m \text{Re} \ t}}{(n - 1)(n - 2)|t|^{n-2}}. \quad (4.22)$$

**Proof.** Since $f$ is analytic, we can deform the integration paths into horizontal ones as in Lemma 4.2. Denoting $t = a + bi$, where $a \geq 0$ and $b \leq 0$, we have

$$|\mathcal{L}_2(e^{-mt} f(t))| \leq \left| \int_{\infty}^{a} e^{2u} \int_{\infty}^{u} e^{-(m+2)s} f(s+bi) dsdu \right| \leq \frac{c}{|t|^n} \left| \int_{\infty}^{a} e^{2u} \int_{\infty}^{u} e^{-(m+2)s} dsdu \right| = \frac{ce^{-ma}}{m(m + 2)|t|^n}.$$

Alternatively, by deforming integration paths into radial ones, we have

$$\mathcal{L}_2(e^{-mt} f(t)) \leq ce^{-ma} \int_{\infty}^{r|t|} \int_{\infty}^{r|u|} |f(s)| dsdu \leq \frac{ce^{-ma}}{(n - 1)(n - 2)|t|^{n-2}},$$

which is (4.22). \qed

We are then ready to prove

**Proposition 4.5.** There is a unique solution of equation (4.18) satisfying

$$|\delta_1(t)| \leq \frac{5}{2|t|^2}$$

in $\Omega_1$.

**Proof.** The strategy is the same as in the proof of Proposition 4.3. We consider the Banach space $\mathcal{S}_2$ of analytic functions in $\Omega_1$, continuous up to the boundary, equipped with the weighted norm

$$||f||_2 = \sup_{t \in \Omega_1} |t^2 f(t)|.$$

We will prove that the operator $\mathcal{T}_2$ in (4.20) is contractive in a ball of size $\frac{5}{2}$ of $\mathcal{S}_2$ with the aids of Lemmas 4.2 and 4.4, by showing

(i) if $||f||_2 \leq \frac{5}{2}$, then $||\mathcal{T}_2(f)||_2 \leq \frac{5}{2}$,

(ii) $||\mathcal{T}_2(f_1) - \mathcal{T}_2(f_2)||_2 \leq \lambda ||f_1 - f_2||_2$ for some $\lambda < 1$. 

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Proof of statement (i): We first estimate $R_1$ in (4.19). Substituting the expression

$$h_p(t) = 1 - \frac{1}{9t^2} + \frac{h_2(t)}{t^4}$$

with $h_a$ defined in (4.16) into (4.19), we get an expression of the form

$$R_1(t) = \sum_{k=4}^{11} \sum_{m=0}^{11} \frac{c_{k,m}e^{-mt}}{tk/2} + h_2(t) \sum_{k=8}^{15} \sum_{m=0}^{7} \frac{\hat{c}_{k,m}e^{-mt}}{tk/2} + h_2(t)^2 \sum_{k=16}^{19} \sum_{m=0}^{3} \frac{c_{k,m}e^{-mt}}{tk/2},$$

where $c_{k,m}$ and $\hat{c}_{k,m}$ are constants that can be written down explicitly, and in fact most of them are zero or very small. For our purpose, it suffices to write out the terms with $k \leq 7$ and estimate the rest crudely. Elementary calculations show that

$$R_1(t) = \tilde{R}_{1,1}(t) + \tilde{R}_{1,2}(t),$$

where

$$\tilde{R}_{1,1}(t) = \frac{-15c_6 e^{-6t}}{2\sqrt{2}t^3} + \frac{29c_5 e^{-5t}}{2t^{5/2}} + \frac{15c_4 e^{-4t}}{6\sqrt{2}t^3} + \frac{6\sqrt{2}c_4 e^{-4t}}{2t^2} + \frac{359c_3 e^{-3t}}{24t^{5/2}} + \frac{343c_2 e^{-2t}}{288\sqrt{2}t^3} + \frac{47c_2 e^{-2t}}{3\sqrt{2}t^2} - \frac{9409c e^{-t}}{1728t^{5/2}} + \frac{27c e^{-7t}}{8t^{7/2}} + \frac{33c e^{-5t}}{4t^{7/2}} - \frac{1927c^3 e^{-3t}}{1728t^{7/2}} - \frac{1609c e^{-t}}{324t^{7/2}} - \frac{1513}{1296\sqrt{2}t^3}, \quad (4.23)$$

and

$$\left| \tilde{R}_{1,2}(t) \right| = \sum_{k=8}^{11} \sum_{m=0}^{11} \frac{|c_{k,m}|}{tk/2} + h_2(t) \sum_{k=8}^{15} \sum_{m=0}^{7} \frac{|\hat{c}_{k,m}|}{tk/2} + h_2(t)^2 \sum_{k=16}^{19} \sum_{m=0}^{3} \frac{|c_{k,m}|}{tk/2} \leq \sum_{k=8}^{11} \sum_{m=0}^{11} \frac{|c_{k,m}|}{10|t|^{9/2}} + 6 \sum_{k=8}^{15} \sum_{m=0}^{7} \frac{|\hat{c}_{k,m}|}{10|t|^{13/2}} + \frac{1}{20|t|^{15/2}} + \frac{1}{2|t|^{19/2}} + \frac{1}{4|t|^{9/2}} + \frac{3}{10|t|^{3/2}} + \frac{1}{5|t|^{3/2}} + \frac{1}{2|t|^{3/2}} + \frac{3}{2|t|^{3/2}} + \frac{26}{5|t|^{1/2}}. \quad (4.24)$$

To estimate $\mathcal{L}_2\left(\tilde{R}_{1,1}\right)$, we take the absolute value of each term in $\tilde{R}_{1,1}$, applying $\mathcal{L}_2$, and then adding them up. The last term $-\frac{1513}{1296\sqrt{2}t^3}$ in $\tilde{R}_{1,1}$ is special, and we use (4.7) to estimate the inner integral and a radial path for the outer integral, which gives

$$\left| \mathcal{L}_2\left(-\frac{1513}{1296\sqrt{2}t^3}\right) \right| \leq \frac{1513}{2592\sqrt{2}} \int_{\infty}^{\infty} \frac{1}{\rho^3} d\rho \leq \frac{1513}{5184\sqrt{2}} \frac{1}{|t|^2}. \quad (4.25)$$

For the other terms in $\mathcal{L}_2\left(\tilde{R}_{1,1}\right)$, we simply use (4.21), which together with (4.25) implies

$$\left| \mathcal{L}_2\left(\tilde{R}_{1,1}(t)\right) \right| < \frac{8}{25|t|^{5/2}} + \frac{7}{25|t|^{7/2}} + \frac{1}{250|t|^{3/2}} + \frac{1}{4|t|^2}. \quad (4.26)$$
To estimate $\mathcal{L}_2 \left( \tilde{R}_{1,2} \right)$, we use (4.22) and obtain

$$
\left| \mathcal{L}_2 \left( \tilde{R}_{1,2}(t) \right) \right| < \frac{26}{175|t|^{5/2}} + \frac{8}{105|t|^{7/2}} + \frac{2}{495|t|^{9/2}} + \frac{2}{715|t|^{11/2}} + \frac{2}{195|t|^{13/2}} + \frac{2}{255|t|^{15/2}} + \frac{1}{70|t|^2} + \frac{31}{420|t|^6} + \frac{1}{150|t|^5} + \frac{1}{20|t|^4} + \frac{1}{8|t|^3} + \frac{13}{15|t|^2}. 
$$

(4.27)

Combining (4.26)–(4.27) and the fact that $|t| \geq 3$, we get

$$
| \mathcal{L}_2 \left( R_1(t) \right) | < \frac{8}{5|t|^2}. 
$$

(4.28)

Now we assume $|f(t)| \leq \frac{5}{2|t|^7}$. We estimate the linear term in (4.18) in a similar way. Direct calculations using the definitions of $h_p$ and $h_a$ show that

$$
\frac{3c^2e^{-2t}h_a(t)^2}{2t} + \frac{3c^2e^{-t}h_p(t)h_a(t)}{\sqrt{t}} + \frac{3}{2}h_p(t)^2 - \frac{5}{36t^2} \left( \frac{3}{2} \right) = \tilde{R}_{1,3}(t) + \tilde{R}_{1,4}(t), 
$$

(4.29)

where

$$
\tilde{R}_{1,3}(t) = \frac{9c^3e^{-3t}}{\sqrt{2t^{3/2}}} + \frac{6c^2e^{-2t}}{t} - \frac{17c^2e^{-t}}{12\sqrt{2t^{3/2}}} + \frac{3\sqrt{2}c^2e^{-t}}{\sqrt{t}},
$$

and

$$
\tilde{R}_{1,4}(t) = \sum_{k=4}^{8} \sum_{m=0}^{8} \frac{\hat{b}_{k,m}e^{-mt}}{tk^{2}} + h_2(t) \sum_{k=8}^{12} \sum_{m=0}^{4} \frac{\hat{b}_{k,m}e^{-mt}}{tk^{2}} + \frac{3h_2(t)^2}{2t^8}
$$

with $\hat{b}_{k,m}$ being certain constants. As before, after applying $\mathcal{L}_2$, we estimate $\tilde{R}_{1,3}$ using (4.21), and $\tilde{R}_{1,4}$ by taking $|h_2(t)| \leq 6/5$ and using (4.22). This gives

$$
\left\| t^2 \mathcal{L}_2 \left( \tilde{R}_{1,3}(t)f(t) \right) \right\| \leq \frac{5}{2} \left( \frac{3c^3}{5\sqrt{2}|t|^{3/2}} + \frac{3c^2}{4|t|} + \frac{\sqrt{2}c}{36|t|^{3/2}} + \frac{17c}{36|t|^{3/2}} \right),
$$

(4.30)

and

$$
\left\| t^2 \mathcal{L}_2 \left( \tilde{R}_{1,4}(t)f(t) \right) \right\| \leq \frac{5}{2} \left( \frac{3}{2000|t|^{3/2}} + \frac{1}{250|t|^3} + \frac{7}{1000|t|^7/2} + \frac{3}{100|t|^6} + \frac{1}{50|t|^4} + \frac{1}{250|t|^5} + \frac{19}{1000|t|^2} + \frac{7}{10000|t|} + \frac{13}{1000\sqrt{|t|}} + \frac{11}{100} \right). 
$$

(4.31)

Thus,

$$
\left\| t^2 \mathcal{L}_2 \left( \left( \tilde{R}_{1,3}(t) + \tilde{R}_{1,4}(t) \right)f(t) \right) \right\| < \frac{3}{10} \frac{5}{2} = \frac{3}{4}. 
$$

(4.32)
The quadratic part of (4.20) is also estimated using (4.22). We have
\[
\left| t^2 \mathcal{L}_2 \left( \frac{3\hat{e}e^{-2th_a(t)}}{2t} + \frac{3\hat{e}e^{-t}h_p(t)}{2\sqrt{t}} \right) f(t)^2 \right| \\
\leq \frac{25}{4} \left( \frac{12|\hat{c}|}{325t^{9/2}} + \frac{2|\hat{c}|}{297|t|^{5/2}} + \frac{41|\hat{c}|^3}{594|t|^{3/2}} + \frac{|\hat{c}|^5}{33|t|^{5/2}} + \frac{17|\hat{c}|^2}{480\sqrt{2}|t|^2} + \frac{3|\hat{c}|^4}{40\sqrt{2}|t|^2} + \frac{2|\hat{c}|^3}{21|t|^{3/2}} + \frac{|\hat{c}|^2}{4\sqrt{2}|t|} + \frac{6|\hat{c}|}{35\sqrt{|t|}} \right) \\
< \frac{1}{50} \frac{25}{4} = \frac{1}{8}.
\] (4.33)

Finally we estimate the cubic term of (4.20) by (4.22):
\[
\left| t^2 \mathcal{L}_2 \left( \frac{\hat{e}e^{-2t}f(t)^3}{2t} \right) \right| \leq \frac{1}{120\pi t^3} \left( \frac{5}{2} \right)^3 < 3 \cdot 10^{-4}.
\] (4.34)

Therefore, combing the results in (4.28), (4.32), (4.33), and (4.34), we have
\[
\|f(t)\|_2 < \frac{5}{2} \Rightarrow \|\mathcal{L}_2(f(t))\|_2 < \frac{5}{2}.
\]

**Proof of statement (ii):** To estimate the linear part of (4.20), we still use (4.30) and (4.31), which gives
\[
\left| t^2 \mathcal{L}_2 \left( \left( \tilde{R}_{1.3}(t) + \tilde{R}_{1.4}(t) \right) (f_1(t) - f_2(t)) \right) \right| < \frac{3}{10} \|f_1 - f_2\|_2.
\] (4.35)

For the nonlinear parts, we use
\[
|f_1^2(t) - f_2^2(t)| \leq 2|f_1 - f_2|_2 \frac{5}{2|t|^1}, \quad |f_1^3(t) - f_2^3(t)| \leq 3|f_1 - f_2|_2 \frac{25}{4|t|^6},
\]
which gives us
\[
\left| t^2 \mathcal{L}_2 \left( \left( \frac{3\hat{e}e^{-2t}h_a(t)}{2t} + \frac{3\hat{e}e^{-t}h_p(t)}{2\sqrt{t}} \right) (f_1^2(t) - f_2^2(t)) \right) \right| \\
< 5|f_1 - f_2|_2 \left( \frac{12|\hat{c}|}{325t^{9/2}} + \frac{2|\hat{c}|}{297|t|^{5/2}} + \frac{41|\hat{c}|^3}{594|t|^{3/2}} + \frac{|\hat{c}|^5}{33|t|^{5/2}} + \frac{17|\hat{c}|^2}{480\sqrt{2}|t|^2} + \frac{3|\hat{c}|^4}{40\sqrt{2}|t|^2} + \frac{2|\hat{c}|^3}{21|t|^{3/2}} + \frac{|\hat{c}|^2}{4\sqrt{2}|t|} + \frac{6|\hat{c}|}{35\sqrt{|t|}} \right) \\
< \frac{5}{50} \|f_1(t) - f_2(t)\|_2 = \frac{1}{10} \|f_1(t) - f_2(t)\|_2,
\] (4.36)

and
\[
\left| t^2 \mathcal{L}_2 \left( \left( \frac{\hat{e}e^{-2t}}{2t} \right) (f_1^3(t) - f_2^3(t)) \right) \right| \\
\leq \frac{1}{240\pi |t|^3} \frac{25}{4} \|f_1(t) - f_2(t)\|_2 < 4 \cdot 10^{-4} \|f_1(t) - f_2(t)\|_2.
\] (4.37)
Combining the results in (4.35), (4.36), and (4.37), we get
\[ \|L_2(f_1(t)) - L_2(f_2(t))\|_2 < \frac{1}{2} \|f_1(t) - f_2(t)\|_2. \]
The conclusion of the proposition then follows from the contraction mapping principle. \[\square\]

4.3 Proof of Proposition 4.1

Proof. We define
\[ h_e(t) = \frac{\tilde{c} e^{-t}}{\sqrt{t}} (h_0(t) + \delta_1(t)). \]
By Proposition 4.5, it is clear that \( h_e \) is pole-free in \( \Omega_1 \) and \( h_e(t) \sim \frac{\sqrt{2\tilde{c}} e^{-t}}{\sqrt{t}} \) for large \( |t| \). We have now obtained a solution with the decomposition \( h = h_p + h_e \), which implies that \( h \) has the asymptotic behavior (4.14), corresponding to the asymptotics of the Hastings-McLeod solution in (4.13). Since it is known [17] that the Hastings-McLeod solution is the only solution having this asymptotic behavior (see also the comments after Proposition 3.2), we see that
\[ h_{HM} = h_p + h_e, \]
which implies by Propositions 4.3 and 4.5 that \( h_{HM} \) is pole-free in \( \Omega_1 \). By (4.1), this means \( y_{HM} \) is pole-free in \( \Omega_0 \). \[\square\]

5 Analysis of the \( y_{HM} \) for \( x \geq 3 \)

Proposition 4.1 implies that \( y_{HM} \) stays close to its truncated asymptotic expansion in \( \Omega_0 \). However, when \( |x| \) becomes small, no asymptotic expansion can provide sufficient information about \( y_{HM} \). Instead, we will have to reply on other methods, such as Taylor series and/or fitting numerical data, to build quasi-solutions with controlled error bounds. This requires knowledge of the initial value of \( y_{HM} \) at a finite point. Although our previous result \( h = h_p + h_e \) can provide initial conditions of \( y_{HM} \), say, at \( x = -\frac{3^{4/3}}{2} \), the error bound is much larger than \( 10^{-3} \), which is not sufficient. Instead, we will obtain an accurate initial condition at 0 using the asymptotic expansion of \( y_{HM} \) at \( +\infty \). On account of (3.4), we expect the asymptotic expansion to provide very accurate information of \( y_{HM} \) even for relatively small \( |x| \). Since the exponent in (3.4) is different from that of (4.13), we need to use a different change of variable, namely,
\[ t = \frac{2}{3} x^{3/2}, \quad y(x) = \left( \frac{2}{3} \right)^{1/6} t^{1/3} h(t) \]
(5.1)
to bring (2.1) to the normalized form
\[ h''(t) + \frac{h'(t)}{t} - h(t) - \frac{h(t)}{9t^2} - \frac{4}{3} h(t)^3 = 0. \]
(5.2)
Substituting \( h(t) = \frac{e^{-t}}{2\sqrt{\pi t}} h_4(t) \) into (5.2), we get

\[
h''_4(t) - 2h'_4(t) + \frac{5h_4(t)}{36t^2} - \frac{e^{-2t}h_4(t)^3}{3\pi t} = 0. \tag{5.3}
\]

Based on asymptotic analysis of (5.3), we construct a quasi-solution given by

\[
h_4(t) = 1 - \frac{85085}{2239488t^3} + \frac{385}{10368t^2} - \frac{5}{72t} + \frac{e^{-2t}}{24\pi t}. \tag{5.4}
\]

Now we insert \( h_4(t) = h_b(t) + \delta^2(t) \) into (5.3), and get the equation

\[
\delta''_2(t) - 2\delta'_2(t) = \delta_2(t) \left( \frac{e^{-2t}h_b(t)^2}{\pi t} - \frac{5}{36t^2} \right) + \frac{e^{-2t}h_b(t)\delta_2(t)^2}{\pi t} + \frac{e^{-2t}\delta_2(t)^3}{3\pi t} - R_2(t) =: R_s(\delta_2(t), t), \tag{5.5}
\]

where

\[
R_2(t) = h''_b(t) - 2h'_b(t) + \frac{5h_b(t)}{36t^2} - \frac{e^{-2t}h_b(t)^3}{3\pi t}.
\]

We write (5.5) in integral form

\[
\delta_2(t) = \mathcal{T}_3(\delta_2(t)) := \mathcal{L}_2(R_s(\delta_2(t), t)), \tag{5.7}
\]

where \( \mathcal{L}_2 \) is the same as defined in (4.20).

The main result of this section is the following:

**Proposition 5.1.** There is a unique solution of equation (5.5) satisfying

\[
|\delta_2(t)| \leq \frac{1}{80t^2}, \quad |\delta'_2(t)| \leq \frac{11}{1000t^2}, \tag{5.8}
\]

for \( t \geq 2\sqrt{3} \).

**Proof.** The proof is very similar to that of Proposition 4.5 but simpler since there is no power series part and \( t \) is real positive. We consider the Banach space \( \mathcal{S}_3 \) of continuous functions in \([2\sqrt{3}, +\infty)\) equipped with the weighted norm

\[
||f||_3 = \sup_{t \geq 2\sqrt{3}} |t^2 f(t)|.
\]

We will prove that the operator \( \mathcal{T}_3 \) in (5.7) is contractive in a ball of size \( \frac{1}{80} \) of \( \mathcal{S}_3 \) using Lemmas 4.2 and 4.4 by showing

(i) if \( ||f||_3 \leq \frac{1}{80} \), then \( ||\mathcal{T}_3(f)||_3 \leq \frac{1}{80} \),

(ii) \( ||\mathcal{T}_3(f_1) - \mathcal{T}_3(f_2)||_3 \leq \lambda||f_1 - f_2||_3 \) for some \( \lambda < 1 \).

Note that the estimates (4.7), (4.8), (4.21) and (4.22) are obviously true for continuous functions on the real line.
Proof of statement (i): We first estimate $R_2$ in (5.6). Substituting the definition of $h_b$ given in (5.4) into (5.6), we get an expression of the form

$$R_2(t) = \tilde{R}_{2,1}(t) + \tilde{R}_{2,2}(t),$$

where

$$\tilde{R}_{2,1}(t) = \frac{163e^{-2t}}{3456\pi^3 t^3} + \frac{5e^{-4t}}{864\pi^3 t^3} - \frac{e^{-6t}}{576\pi^3 t^3} - \frac{e^{-4t}}{24\pi^2 t^2} + \frac{23e^{-2t}}{72\pi t^2},$$

and

$$|\tilde{R}_{2,2}(t)| = \left| \sum_{k=4}^{10} \sum_{m=0}^{8} \frac{d_{k,m}e^{-mt}}{t^k} \right| \leq \sum_{k=4}^{10} \sum_{m=0}^{8} \frac{|d_{k,m}|e^{-2\sqrt{3}m}}{t^k} \leq \frac{12}{25t^5} + 10^{-6} \left( \frac{3}{500t^6} + \frac{1}{50t^9} + \frac{1}{20t^8} + \frac{3}{5t^7} + \frac{1}{2t^7} + 25 \right).$$

By (4.21), we obtain the estimate

$$\left| t^2 \mathcal{L}_2 \left( \tilde{R}_{2,1}(t) \right) \right| \leq \frac{e^{-4t}}{576\pi^2} + \frac{163e^{-2t}}{27648\pi t} + \frac{5e^{-4t}}{20736\pi^2 t} + \frac{e^{-6t}}{27648\pi^3 t} + \frac{23e^{-2t}}{576\pi},$$

and from (4.22), we have the estimate

$$\left| t^2 \mathcal{L}_2 \left( \tilde{R}_{2,2}(t) \right) \right| < \frac{1}{25t} + 10^{-7} \left( \frac{1}{1200t^6} + \frac{1}{280t^5} + \frac{1}{84t^4} + \frac{1}{5t^3} + \frac{1}{2t^2} + 25 \right).$$

Combining (5.11) and (5.12) and note that $t \geq 2\sqrt{3}$, we get

$$|t^2 \mathcal{L}_2(R_2(t))| < \frac{3}{250}.$$  \hspace{1cm} (5.13)

Now we assume $||f(t)||_3 \leq 1/80$. To analyze the linear term in (5.7), we simply use the definition of $h_b$ in (5.4) to obtain

$$\frac{e^{-2th_b(t)}^2}{\pi t} - \frac{5}{36t^2} = \tilde{R}_{2,3}(t) + \tilde{R}_{2,4}(t),$$

where

$$\tilde{R}_{2,3}(t) = -\frac{5}{36t^2} - \frac{5e^{-2t}}{36\pi^2 t^2} + \frac{e^{-4t}}{12\pi^2 t^2} + \frac{e^{-2t}}{\pi t},$$

and

$$|\tilde{R}_{2,4}(t)| = \left| \sum_{k=3}^{7} \sum_{m=2}^{4} \frac{\tilde{d}_{k,m}e^{-mt}}{t^k} \right| < 10^{-6} \left( \frac{1}{2t^7} + \frac{1}{t^6} + \frac{11}{5t^5} + \frac{51}{2t^4} + \frac{25}{t^3} \right).$$

For the term $-\frac{5}{36t^2}$ in (5.14), we use (4.7) once and (4.8) once to obtain

$$\left| t^2 \mathcal{L}_2 \left( \left( \frac{5}{36t^2} f(t) \right) \right) \right| \leq \frac{1}{80} \frac{5}{216t^4}.$$
Using (4.21) to estimate the rest of the terms in (5.14), we obtain
\[
|t^2 L_2 \left( \tilde{R}_{2,3}(t) f(t) \right)| \leq \frac{1}{80} \left( \frac{5e^{-2t}}{288\pi t^2} + \frac{e^{-4t}}{288\pi^2 t^2} + \frac{e^{-2t}}{8\pi t} + \frac{5}{216t} \right). \tag{5.15}
\]
By (4.22), we obtain the estimate
\[
|t^2 L_2 \left( \tilde{R}_{2,3}(t) f(t) \right)| < 10^{-6} \left( \frac{1}{112t^5} + \frac{1}{12t} + \frac{11}{150t^3} + \frac{51}{40t^2} + \frac{25}{12t} \right). \tag{5.16}
\]
Combining (5.15) and (5.16), we get
\[
\left| t^2 L_2 \left( \frac{e^{-2t}h_b(t)^2}{\pi t} - \frac{5}{36t^2} \right) f(t) \right| < \frac{1}{80} \frac{1}{125} = 10^{-4}. \tag{5.17}
\]
The quadratic and cubic terms of (5.7) can be estimated using (4.22). We have
\[
\left| t^2 L_2 \left( \frac{e^{-2t}h_b(t)^2}{\pi t} f(t)^2 \right) \right| < 10^{-6} \left( \frac{e^{-2t}}{20t^4} + \frac{7e^{-2t}}{100t^3} + \frac{e^{-2t}}{5t^2} + \frac{e^{-4t}}{25t^2} + \frac{5e^{-2t}}{t} \right) < 2 \cdot 10^{-9} \tag{5.18}
\]
and
\[
\left| t^2 L_2 \left( \frac{e^{-2t}f(t)^3}{3t^3} \right) \right| \leq \frac{e^{-2t}}{46080000\pi t^3} < 10^{-12}. \tag{5.19}
\]
Combining (5.13), (5.17), (5.18), and (5.19), we find
\[
|f(t)|_3 \leq \frac{1}{80} \Rightarrow ||T_3(f(t))||_3 \leq \frac{1}{80}.
\]

**Proof of statement (ii):** The estimate for the linear part of (5.7) is very similar to (5.17), which gives
\[
\left| t^2 L_2 \left( \frac{e^{-2t}h_b(t)^2}{\pi t} - \frac{5}{36t^2} \right) (f_1(t) - f_2(t)) \right| < \frac{1}{125} ||f_1(t) - f_2(t)||_3. \tag{5.20}
\]
For the nonlinear parts, we use
\[
|f_1^2 - f_2^2| \leq 2||f_1 - f_2||_3 \frac{1}{80t^4}, \quad |f_1^3 - f_2^3| \leq 3||f_1 - f_2||_3 \frac{1}{6400t^6},
\]
and (4.22) to obtain the estimates
\[
\left| t^2 L_2 \left( \frac{e^{-2t}h_b(t)}{\pi t} \right) (f_1(t)^2 - f_2(t)^2) \right| < 10^{-5} \left( \frac{4e^{-2t}}{5t^4} + \frac{e^{-2t}}{t^3} + \frac{3e^{-2t}}{5t^2} + \frac{80e^{-2t}}{t} \right) ||f_1(t) - f_2(t)||_3 \leq 10^{-6} ||f_1(t) - f_2(t)||_3, \tag{5.21}
\]
and
\[
\left| t^2 L_2 \left( \frac{e^{-2t}}{3\pi t} \right) (f_1(t)^3 - f_2(t)^3) \right| \leq \frac{e^{-2t}}{192000\pi t^3} < 10^{-10} ||f_1(t) - f_2(t)||_3. \tag{5.22}
\]
Combining (5.20), (5.21) and (5.22), we see that
\[ \| \mathcal{L}_2(f_1(t)) - \mathcal{L}_2(f_2(t)) \|_3 < \frac{1}{120} \| f_1(t) - f_2(t) \|_3. \]

To estimate \( \delta_2'(t) \), we first differentiate (5.7) once to get
\[ \delta_2'(t) = \mathcal{L}_2'(R_2(\delta_2(t), t)) = e^{2t} \int_{\infty}^{t} e^{-2s} R_2(\delta_2(s), s) ds. \]

The integral can be estimated by using (4.7) and (4.8). In particular, from (5.9) and (5.10), we get
\[ |t^2 \mathcal{L}_2'(R_2(t))| < \frac{e^{-4t}}{144\pi^2} + \frac{163e^{-2t}}{13824\pi^2 t} + \frac{5e^{-4t}}{5184\pi^2 t^2} + \frac{e^{-6t}}{4608\pi^3 t^3} + \frac{23e^{-2t}}{288\pi} + \frac{3}{25t^2} + 10^{-7} \left( \frac{1}{150t^7} + \frac{1}{40t^6} + \frac{1}{14t^5} + \frac{1}{t^4} + \frac{2}{t^3} + \frac{50}{t} \right) < \frac{21}{2000}. \tag{5.23} \]

The rest can be estimated crudely. First we take absolute value of each term in (5.4) to get a bound
\[ |h_b| < \frac{21}{20}. \]

Thus, by (4.8), we have
\[ |\delta_2(t) \left( \frac{e^{-2t}h_b(t)^2}{\pi t} - \frac{5}{36t^2} \right) + \frac{e^{-2t}h_b(t)\delta_2(t)^2}{3\pi t} + \frac{e^{-2t}\delta_2(t)^3}{3\pi t}| < 3 \cdot 10^{-7} \frac{e^{-2t}}{t^7} + \frac{6 \cdot 10^{-5} e^{-2t}}{t^5} + \frac{1}{576t^4} + \frac{e^{-2t}}{200t^3}. \]

Thus, by (4.8), we have
\[ |t^2 \mathcal{L}_2'(R_2(\delta_2(t), t))| < 10^{-7} \frac{e^{-2t}}{2t^4} + \frac{3 \cdot 10^{-5} e^{-2t}}{2t^2} + \frac{1}{1728t} + \frac{e^{-2t}}{400} < 18 \cdot 10^{-5}. \tag{5.24} \]

Therefore, from (5.23) and (5.24), we have
\[ |t^2 \mathcal{L}_2'(R_2(\delta_2(t), t))| < \frac{21}{2000} + 18 \cdot 10^{-5} < \frac{11}{1000}. \]

This completes the proof of Proposition 5.1. □

As a consequence of the above proposition, it follows

**Corollary 5.2.** For \( y_{HM} \) we have the estimates
\[ \left| y_{HM}(3) - \frac{4}{607} \right| < 8 \cdot 10^{-6}, \quad \left| y'_{HM}(3) + \frac{64}{5375} \right| < 24 \cdot 10^{-6}. \]
Proof. We first note that since \( y_{HM} \) is the unique solution satisfying (3.4), Proposition 5.1 and (5.1) implies

\[
y_{HM}(x) = \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} \left( h_b \left( \frac{2}{3}x^{3/2} \right) + \delta_2 \left( \frac{2}{3}x^{3/2} \right) \right).
\]

Thus, it follows that

\[
\left| y_{HM}(x) - \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} h_b \left( \frac{2}{3}x^{3/2} \right) \right| \leq \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} \left| \delta_2 \left( \frac{2}{3}x^{3/2} \right) \right| \leq \frac{9e^{-\frac{2}{3}x^{3/2}}}{640\sqrt{\pi}x^{13/4}}, \tag{5.25}
\]

and

\[
\left| y'_{HM}(x) - \left( \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} h_b \left( \frac{2}{3}x^{3/2} \right) \right)' \right| \leq \frac{e^{-\frac{2}{3}x^{3/2}}}{8\sqrt{\pi}x^{5/4}} \left( 4|\delta_2 \left( \frac{2}{3}x^{3/2} \right) | + \left| \delta_2' \left( \frac{2}{3}x^{3/2} \right) \right| \right) \leq \frac{9e^{-\frac{2}{3}x^{3/2}}}{64000\sqrt{\pi}x^{17/4}}. \tag{5.26}
\]

Plugging the expression (5.4) and the value \( x = 3 \) into (5.25) and (5.26), we get

\[
\left| y_{HM}(3) - \frac{4}{607} \right| \leq \left| y_{HM}(3) - \frac{e^{-\frac{2}{3}9^{3/2}}}{2\sqrt{\pi}9^{1/4}} h_b \left( \frac{2}{3}9^{3/2} \right) \right| + \left| \frac{4}{607} - \frac{e^{-\frac{2}{3}9^{3/2}}}{2\sqrt{\pi}9^{1/4}} h_b \left( \frac{2}{3}9^{3/2} \right) \right| < 8 \cdot 10^{-6},
\]

and

\[
\left| y'_{HM}(3) + \frac{64}{5375} \right| \leq \left| y'_{HM}(3) - \left( \frac{e^{-\frac{2}{3}9^{3/2}}}{2\sqrt{\pi}9^{1/4}} h_b \left( \frac{2}{3}9^{3/2} \right) \right)' \right| \bigg|_{x=3} + \left| \frac{64}{5375} + \left( \frac{e^{-\frac{2}{3}9^{3/2}}}{2\sqrt{\pi}9^{1/4}} h_b \left( \frac{2}{3}9^{3/2} \right) \right)' \right| \bigg|_{x=3} < 24 \cdot 10^{-6},
\]

as desired. \(\square\)

6 Analysis of \( y_{HM} \) for \( 0 \leq x \leq 3 \)

When studying \( y_{HM} \) near the origin, it is more convenient to use equation (2.1) directly, without any change of variable. In this region, the quasi-solution is found by numerically solving (2.1) using the initial conditions given in Corollary 5.2 and fitting the data using a
polynomial under the maximum norm (similar to Chebyshev polynomials). As a result, we have

\[ y_a(x) = \frac{t^{11}}{1929701} - \frac{t^{10}}{625758} - \frac{t^9}{192428} + \frac{t^8}{27779} - \frac{t^7}{23450} - \frac{13t^6}{44056} + \frac{90t^5}{64211} - \frac{19t^4}{21788} - \frac{125t^3}{9667} + \frac{1535t^2}{28314} - \frac{2759t}{28279} + \frac{1413}{19685}, \quad (6.1) \]

where \( t = x - \frac{3}{2} \).

Plugging \( y_a \) into (2.1), we get a remainder

\[ R_3(x) = y''_a(x) - 2y_a(x)^3 - xy_a(x), \]

which is also a polynomial. We first need to show that this remainder is small for \( 0 \leq x \leq 3 \).

**Remark 6.1.** Estimating a real polynomial \( P(x) \) on an interval \([a, b]\) rigorously and with good accuracy is elementary. Here we use the simple method in [9, 10]. We choose a suitable partition of \([a, b]\),

\[ \Pi = \{x_0, x_1, ..., x_{n-1}, x_n\}, \]

where \( x_0 = a, x_n = b \), then write

\[ x = (x_i + x_{i-1})/2 + u \]

on each subinterval \([x_{i-1}, x_i]\) for \( i = 1, ..., n \), and re-expand \( P \) to obtain a polynomial in \( u \). The polynomial in \( u \) is estimated by taking the extremum of the cubic sub-polynomial and bounding the rest terms by the sum of their absolute values. To be precise, if

\[ P(x) = \sum_{k=0}^{n} c_k \left(x - \frac{x_i + x_{i-1}}{2}\right)^k, \]

then we have

\[ \left| P\left(\frac{x_i + x_{i-1}}{2} + u\right) - \sum_{k=0}^{3} c_k u^k \right| \leq \sum_{k=4}^{n} |c_k| \left|\frac{x_i - x_{i-1}}{2}\right|^k, \quad |u| \leq \frac{|x_i - x_{i-1}|}{2}. \]

This technique can also be used to show \( P_1(x) \leq P_2(x) \) as it is equivalent to showing \( P_1(x) - P_2(x) \leq 0 \).

We choose the partition \( \{0, 1/4, 3/5, 6/5, 9/5, 12/5, 14/5, 3\} \). By Remark 6.1, it is easy to show that

\[ |R_3(x)| < 18 \cdot 10^{-6}. \quad (6.2) \]

Plugging

\[ y_{HM} = y_a + \delta_3 \]
into (2.1), we get
\[
\delta''_3(x) = \delta_3(x) \left( 6y_a(x)^2 + x \right) + 6\delta_3(x)^2 y_a(x) + 2\delta_3(x)^3 - R_3(x) =: F_1(\delta_3(x), x). \tag{6.3}
\]
Equation (6.3), together with the initial conditions \(\delta_3(3) = y_H M(3) - y_a(3), \quad \delta'_3(3) = y_H M'(3) - y_a'(3)\) guarantee that \(\delta_3 = y_H M - y_a\). By Corollary 5.2 and direct calculation, we have
\[
|\delta_3(3)| < 85 \cdot 10^{-7}, \quad |\delta'_3(3)| < 241 \cdot 10^{-7}. \tag{6.4}
\]
It is possible to show that \(\delta_3\) is small by constructing approximate Green’s functions and using the contraction mapping principle as in [9, 10]. However, here we will use a simpler method relying only on elementary inequalities. The idea is to construct an explicit function satisfying a “stronger” ODE, so that \(\delta_3\) must be bounded by that function. To be precise, we have the following lemma.

**Lemma 6.1** (Global existence of ODE solutions). Consider the equation
\[
\quad u''(x) = F(u(x), x)
\]
on the interval \([a, b]\) with initial conditions
\[
\quad u(a) = \alpha, \quad u'(a) = \tilde{\alpha},
\]
where \(F\) is Lipschitz continuous in \(B(\alpha, \beta) \times [a, b]\) (here \(B(\alpha, \beta)\) denotes the closed ball centered at \(\alpha\) with radius \(\beta\) in the complex plane). Suppose there exists an integrable function \(G\) such that
\[
\quad G(t, x) \geq |F(t, x)|
\]
in \(B(\alpha, \beta) \times [a, b]\) and \(G(t, x)\) is increasing in \(t\) (i.e., if \(|t_1| \geq |t_2|\), then \(G(t_1, x) \geq G(t_2, x)\)). Furthermore, suppose there exists a function \(y\) such that
\[
\quad y(a) > |\alpha|, \quad y'(a) \geq |\tilde{\alpha}|, \quad y''(x) \geq G(y(x), x).
\]
Then there exists a unique solution \(u(x)\) on \([a, b]\) such that
\[
\quad |u(x)| < y(x) \quad \text{and} \quad |u'(x)| < y'(x)
\]
for all \(x \in [a, b]\). Furthermore, if the conditions \(y(a) > |u(a)|, \quad y'(a) \geq |u'(a)|\) are replaced with \(y(b) > |u(b)|, \quad y'(b) \leq -|u'(b)|\), the conclusions still hold.

**Proof.** Local existence and uniqueness of a solution \(u\) near \(x = a\) follows from Picard’s Existence Theorem (cf. [6]). We define
\[
\quad u_1(x) = |\alpha| + |\tilde{\alpha}| (x - a) + \int_a^x \int_a^t G(u(s), s)dsdt.
\]
By straightforward calculation, we have

\[
|u(x)| = \left| \alpha + \tilde{\alpha}(x-a) + \int_a^x \int_a^t F(u(s),s)dsdt \right| \\
\leq |\alpha| + |\tilde{\alpha}|(x-a) + \int_a^x \int_a^t |F(u(s),s)|dsdt \leq u_1(x).
\]

Let

\[
y_1(x) = y(a) + y'(a)(x-a) + \int_a^x \int_a^t G(y(s),s)dsdt.
\]

By direct calculation, we see that

\[
y_1(a) = y(a), \quad y'_1(a) = y'(a), \quad y''_1(x) \leq y''(x),
\]

which means \(y_1(x) \leq y(x)\). Let \(b_1 = \sup_{x\in[a,b]}\{y(x) > |u(x)|, x \in [a,c]\}\). For \(a \leq x < b\), we have

\[
\int_a^x \int_a^t G(u(s),s)dsdt \leq \int_a^x \int_a^t G(y(s),s)dsdt,
\]

because \(G(t,x)\) is increasing in \(|t|\). Since \(y(a) > |u(a)|\), using definitions of \(u_1\) and \(y_1\), we see that

\[
u_1(x) < y_1(x)
\]

for all \(a \leq x < b\). Now suppose \(b_1 < b\). For every \(x < b\), Picard’s Existence Theorem implies that the solution \(u\) exists in \([x, \min(x + \beta/\sup |F|, b)]\). Thus the solution \(u\) exists for all \(x < \min(b_1 + \beta/\sup |F|, b)\). Then by continuity, we would have \(u_1(x) < y_1(x)\) for \(x \in [a, b_1 + \delta]\) and for some \(\delta > 0\), which would contradict with the definition of \(b_1\). Hence,

\[
|u(x)| \leq u_1(x) < y_1(x) \leq y(x)
\]

for all \(x \in [a, b]\).

To estimate \(u'\), note that

\[
|u'(x)| = \left| \tilde{\alpha} + \int_a^x F(u(s),s)ds \right| \leq |\tilde{\alpha}| + \int_a^x G(u(s),s)ds \\
\leq y'(a) + \int_a^x G(y(s),s)ds \leq y'(x),
\]

where we integrated the known inequality \(y''(x) \geq G(y(x),x)\) in the last step.

Finally, if conditions \(y(a) > |u(a)|, y'(a) \geq |u'(a)|\) are replaced with \(y(b) > |u(b)|, y'(b) \leq -|u'(b)|\), we can simply make the change of variable \(x \rightarrow -x\) and apply the previous result.

Now we are ready to prove the following result:
Proposition 6.2. We have the estimates
\[
\left| y_{HM}(0) - \frac{98}{267} \right| < 11 \cdot 10^{-4}, \quad \left| y_{HM}(0) + \frac{153}{518} \right| < 12 \cdot 10^{-4}.
\]

Proof. Clearly \( F_1 \) in (6.3) is Lipschitz continuous in every bounded region. Let
\[
G(t, x) = |t| \left( 6y_a(x)^2 + x \right) + 6y_a(x)|t|^2 + 2|t|^3 + 18 \cdot 10^{-6}.
\]
Since \( y_a \) is a polynomial, by using Remark 6.1 with partition \( \{0, 1, 2, 3\} \), we easily see that \( y_a \) is positive on \([0, 3]\). Therefore \( G(t, x) \) is increasing in \(|t|\). By (6.2), we see that
\[
G(t, x) \geq |F_1(t, x)|, \quad 0 \leq x \leq 3.
\]
Now we set
\[
y_1(x) = \frac{s^8}{55140149} - \frac{s^7}{15591646} - \frac{s^6}{35492113} + \frac{s^5}{526490476} + \frac{s^4}{144870} - \frac{3s^3}{63886} + \frac{7s^2}{46477} - \frac{43s}{172565} + \frac{5}{29696},
\]
where \( s = x - \frac{3}{2} \).

By direct calculation, we see
\[
y_1(3) > 9 \cdot 10^{-7} > |\delta_3(3)| \quad \text{and} \quad y_1'(3) < -245 \cdot 10^{-6} < -|\delta_3'(3)|.
\]
Since \( y''(x) - G(y_1(x), x) \) is a polynomial, using Remark 6.1 with partition \( \{0, 1, 2, 3\} \), we see that \( y''(x) - G(y_1(x), x) \geq 0 \). Therefore, by Lemma 6.1, we have
\[
|\delta_3(x)| \leq y_1(x) \quad \text{and} \quad |\delta_3'(x)| \leq y_1'(x).
\]
In particular, one has
\[
\left| y_{HM}(0) - \frac{98}{267} \right| \leq \left| y_a(0) - \frac{98}{267} \right| + |\delta_3(0)| \leq \left| y_a(0) - \frac{98}{267} \right| + y_1(0) < 11 \cdot 10^{-4},
\]
and
\[
\left| y_{HM}'(0) + \frac{153}{518} \right| \leq \left| y_a'(0) + \frac{153}{518} \right| + |\delta_3'(0)| \leq \left| y_a'(0) + \frac{153}{518} \right| + |y_1'(0)| < 12 \cdot 10^{-4}.
\]

7 Analysis of \( y_{HM} \) in the finite domain \( \Omega_2 \)

In this section we will show that \( y_{HM} \) is pole-free in \( \Omega_2 \), where \( \Omega_2 \) is defined in (3.3). This result, combined with Proposition 4.1, will be sufficient to prove Theorem 3.1. Our strategy is still to construct a quasi-solution and to use Lemma 6.1 to bound the error. However, since \( \Omega_2 \) is a sector in the complex plane, we first make a remark about estimating complex rational functions.
Remark 7.1. Assume $\Omega_p$ is a closed polygonal domain in the complex plane and 

$$F(z) = \frac{P(z)}{Q(z)},$$

where $P, Q$ are complex polynomials and $Q$ has no zero in $\Omega_p$. To estimate the modulus of $F$, we note that since $F$ is analytic in $\Omega_p$, by the maximum modulus principle, it is sufficient to estimate its modulus along the boundary $\partial \Omega_p$. Since $\Omega_p$ is a polygon, its boundary consists of line segments. On each line segment $[z_1, z_2]$, we have $z = z_1 + z_2 t$, $0 \leq t \leq 1$, and we note that 

$$\left| \frac{P(z_1 + z_2 t)}{Q(z_1 + z_2 t)} \right| \leq M \iff |P(z_1 + z_2 t)|^2 \leq M^2 |Q(z_1 + z_2 t)|^2. \quad (7.1)$$

Since $|P(z_1 + z_2 t)|^2$ and $|Q(z_1 + z_2 t)|^2$ are both real polynomials in $t$, (7.1) could be proved using the method in Remark 6.1.

In view of Remark 7.1, we consider the right triangular domain $\tilde{\Omega}_2$ with vertices at $0, -\frac{9}{2\sqrt{3}}, -\frac{9}{4\sqrt{3}} + \frac{9}{4}i$. It is easily seen that $\Omega_2 \subseteq \tilde{\Omega}_2$, as illustrated in figure 2.

In order to find a quasi-solution in the complex region $\tilde{\Omega}_2$, we solve (2.1) numerically using initial conditions given in Proposition 6.2 and use least-squares polynomial approximations, which results in the quasi-solution

$$y_b(x) = \frac{x^{15}}{13206825} + \frac{x^{14}}{717099} + \frac{x^{13}}{81755} + \frac{x^{12}}{15201} + \frac{11x^{11}}{47200} + \frac{13x^{10}}{24088} + \frac{39x^9}{53333} + \frac{18x^8}{18x^7} + \frac{17x^6}{93x^6} + \frac{224x^5}{360x^4} + \frac{203x^4}{30615} + \frac{36911}{10806} + \frac{68889}{35396} - \frac{98}{267}. \quad (7.2)$$

As usual, we need to estimate the remainder

$$R_4(x) = y_b''(x) - 2y_b(x)^3 - xy_b(x).$$
We note that by symmetry every polynomial or rational function $F(z)$ with real coefficients only needs to be estimated on two edges

$$\Sigma_1 : z = \frac{9}{4} \left( -\frac{2}{\sqrt{3}} + \left( \frac{1}{\sqrt{3}} + i \right) t \right) \quad \text{and} \quad \Sigma_2 : z = \frac{9}{4} \left( -\frac{1}{\sqrt{3}} + i \right) t,$$

where $0 \leq t \leq 1$. This is because by the maximum modulus principle $F$ only need to be estimated on the boundary of $\tilde{\Omega}_2 \cup \tilde{\Omega}_5$, where $\tilde{\Omega}_5$ is complex conjugate of $\tilde{\Omega}_2$, namely the reflection of $\tilde{\Omega}_2$ with respect to the real axis. However, since $|F(z)| = |F(\tilde{z})|$, it is sufficient to estimate $F$ on the two edges in the upper half plane.

We take the partition \{0, 2/5, 4/5, 14/15, 1\} for the edge $\Sigma_1$, and the partition \{0, 1/3, 3/4, 14/15, 1\} for the edge $\Sigma_2$. Using Remark 7.1, we see that

$$|R_4(x)|^2 < 3 \cdot 10^{-5} \quad \text{on} \quad \Gamma_{1,2} \Rightarrow |R_4(x)| < \frac{3}{500} \quad \text{on} \quad \tilde{\Omega}_2.$$

Now we are ready to show the following

**Proposition 7.1.** The Hastings-solution $y_{HM}$ satisfies the bound

$$|y_{HM}(x) - y_b(x)| < \frac{6}{5}$$

in $\Omega_2$. In particular, $y_{HM}$ is pole-free in $\Omega_2$.

**Proof.** We substitute $y_{HM} = y_b + \delta_4$ into (2.31) and get

$$\delta_4''(x) = (6y_b(x)^2 + x) \delta_4(x) + 6y_b(x)\delta_4'(x)^2 + 2\delta_4(x)^3 - R_4(x) =: F_2(\delta_4(x), x) \quad (7.3)$$

From (7.2), it is clear that $y_{b}(0) = \frac{98}{207}$ and $y'_b(0) = -\frac{153}{518}$. This, together with Proposition 6.2 implies that

$$|\delta_4(0)| < 11 \cdot 10^{-4}, \quad |\delta_4'(0)| < 12 \cdot 10^{-4}. \quad (7.4)$$

Now we will use Lemma 6.1 to estimate $\delta_4(x)$ along radial paths $x = re^{i\theta} \ (0 \leq r \leq \frac{2}{3})$ for all fixed $\frac{2\pi}{3} \leq \theta \leq \pi$. With the change of variable $\tilde{\delta}_4(r) = \delta_4(re^{i\theta})$, equation (7.3) becomes

$$\tilde{\delta}_4''(r) = e^{2i\theta} F_2(\tilde{\delta}_4(r), re^{i\theta}) =: \tilde{F}_2(\tilde{\delta}_4(r), r). \quad (7.5)$$

To find a suitable $G$ required by Lemma 6.1 we first estimate relevant terms in $\tilde{F}_2$. We take the same partition \{0, 1/2, 14/15, 1\} for both $\Sigma_1$ and $\Sigma_2$. Using Remark 7.1 we see that

$$|y_b(x)|^2 < \frac{83}{50} \quad \text{on} \quad \Sigma_{1,2} \Rightarrow 6|y_b(x)| < 8 \quad \text{on} \quad \tilde{\Omega}_2 \quad (7.6)$$

Similarly, we take the same partition \{0, 1/2, 3/4, 1\} for both $\Sigma_1$ and $\Sigma_2$ and use Remark 7.1, which gives

$$|6y_b(x)^2 + x|^2 - \frac{144}{25} |x - 1|^2 < 0 \quad \text{on} \quad \Sigma_{1,2}$$

$$\Rightarrow \left| \frac{6y_b(x)^2 + x}{x - 1} \right| < \frac{12}{5} \quad \text{on} \quad \tilde{\Omega}_2 \Rightarrow |6y_b(x)^2 + x| < \frac{12}{5} |x - 1| \quad \text{on} \quad \tilde{\Omega}_2. \quad (7.7)$$
Now we define
\[ G_2(t, r) = \frac{12}{5}(r + 1)|t| + 8|t|^2 + 2|t|^3 + \frac{3}{500}. \]

Obviously \( G_2(t, r) \) is increasing in \(|t|\). By (7.3), (7.5), (7.6) and (7.7), it is clear that \( G_2(t, r) \geq |\tilde{F}_2(t, r)| \). Let
\[ y_2(r) = \frac{400t^{10}}{11977} + \frac{265t^9}{11857} - \frac{855t^8}{10951} - \frac{149t^7}{6608} + \frac{293t^6}{2551} + \frac{1013t^5}{14669} + \frac{128t^4}{6441} + \frac{424t^3}{13159} + \frac{549t^2}{7508} + \frac{267}{9871}, \quad (7.8) \]
where \( t = r - 1 \).

By direct calculation and (7.4,) we have
\[ y_2(0) > 11 \cdot 10^{-4} > |\tilde{\delta}_4(0)| \quad \text{and} \quad y'_2(0) > 12 \cdot 10^{-4} > |\tilde{\delta}'_4(0)|. \]

Since \( y'_2(r) - G_2(y_2(r), r) \) is a polynomial, we take the partition \( \{0, 1/5, 1/2, 1, 3/2, 9/5, 2, 9/4\} \) and use Remark 6.1 to get
\[ y''_2(r) - G_2(y_2(r), r) > 0 \text{ for } r \in [0, 9/4]. \]

By Lemma 6.1 we have \( |\delta_4(r)| = |\tilde{\delta}_4(r)| \leq y_2(r) \) for \( r \in [0, 9/4] \). By taking the partition \( \{0, 1, 9/4\} \) and using Remark 6.1 we see that \( y_2(r) < 6/5 \). Thus \( |y_{HM} - y_b| < 6/5. \)

8 Proofs of Theorems 3.1 and 2.1

Proof of Theorem 3.1. The theorem simply follows from Proposition 4.1 and Proposition 7.1. Note that \( \Omega \subseteq \Omega_0 \cup \Omega_2 \), since \( \frac{4\sqrt{3}}{2} < \frac{9}{4} \).

Proof of Theorem 2.1. By the argument at the beginning of Section 3 and Theorem 3.1, it follows that \( y_{HM} \) is pole-free in the sector \( \arg x \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right] \). For the region \( \arg x \in [-\pi/3, \pi/3] \), it is already known that \( y_{HM} \) is pole-free (see [1]).

Remark 8.1. Our method also applies in the region \( \arg x \in [-\pi/3, \pi/3] \). In fact, we only need to change the definition of \( S_3 \) to include the sector \([-\pi/3, \pi/3]\) in complex plane in Proposition 5.1 and to construct a quasi-solution similar to \( y_b \) in (7.2) for \( \{x \in \mathbb{C} : |x| \leq 3, -\pi/3 \leq \arg x \leq \pi/3\} \). The details are left to the interested readers.

Furthermore, we would like to mention that one can also adopt the method to prove other cases of Conjecture 1.1. However, since there are infinitely many 2-truncated solutions of PI and PII with significantly different asymptotic behaviors, it is not clear to us at the moment how to find a uniform proof that works for the general case.
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