Can rapidity become a gauge variable? Dirac Hamiltonian method and relativistic rotators

Łukasz Bratek

The Henryk Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, Radzikowskiego 152, PL-31342 Kraków, Poland

E-mail: lukasz.bratek@ifj.edu.pl

Received 9 January 2011, in final form 5 December 2011
Published 12 January 2012
Online at stacks.iop.org/JPhysA/45/055204

Abstract

The minimal Hamiltonian for a family of relativistic rotators is constructed by a direct application of the Dirac procedure for constrained systems. The Hamiltonian equations can be easily solved. It is found that the resulting motion is unique and qualitatively the same for all phenomenological rotators, only the relation between mass and spin is different. There is a critical point in the construction when such a relation cannot be established, implying that the number of primary constraints is greater. In that case the mass and the spin become unrelated, separately fixed parameters, and the corresponding Hamiltonian changes qualitatively. Furthermore, a genuine physical observable becomes a gauge variable. This paradoxical result is consistent with the fact already known at the Lagrangian level that the Hessian rank is lower than expected, and the equations of motion indeterminate on $\mathbb{R}^3 \times S^2$.

PACS numbers: 45.20.Jj, 45.10.-b, 03.30.+p, 45.50.-j

1. Historical background and motivation

A geometric model of a spinning particle with fixed mass and spin both in a Lagrangian and a Hamiltonian frame was proposed in [1]. A solution called general with a fixed frequency was presented. The Hamiltonian found therein becomes the basis for the quantization of the model. Later, a short paper was written in which a family of relativistic rotators was proposed in a Lagrangian frame [2]. The two papers converge on a final particular Lagrangian defining the same dynamical system with separately fixed Casimir mass and spin.

Next, it was shown in [3] that a single and quite arbitrary function of the time appears in the general solution and that the fact is associated with that the Hessian rank of the Lagrangian is only 4, not 5 as expected originally for that system. In the language of the Dirac formalism, this means that the secondary constraint is trivial $0 = 0$ in the free motion. However, a nontrivial secondary constraint will usually appear when an interaction term is added incapable of
removing the Hessian singularity. In particular, depending on the structure of the interaction, such a constraint could impose nonphysical limitations on the freedom in choosing initial data.

The arbitrariness in the frequency was also encountered in [4]; however, the authors did not try to understand the reason for the singularity and did not pay due attention to it. They minimally coupled the rotator with the electromagnetic field and a constraint indeed appeared. But this is what was to be expected on account of the fact that the particular form of the interaction was insufficient to remove the Hessian rank deficiency [5]. With this constraint, the particular motion in the uniform magnetic field discussed in [4] indeed was unique. But later, with the help of a toy model with the same kind of Hessian singularity, it was illustrated that this uniqueness was accidental and the constrained motion would be still non-unique in the general electromagnetic field [6].

In the meantime, a paper was published [7] in which a Hamiltonization scheme alternative to that suggested in [1] (and different from that we present later for all rotators) was proposed for the particular fundamental rotator. This scheme employed additional auxiliary variables not present in the original Lagrangian and resulted in a non-commutativity structure on which the authors focused. The motivation for introducing these variables was the author’s argumentation that the Lagrangian in the form presented by Staruszkiewicz with its complicated structure of time derivatives and the presence of nested square roots were not suitable for Hamiltonian analysis due to troubles expected to arise in trying to express velocities as functions of momenta. However, it is not clear what kind of troubles they meant precisely, since the Dirac Hamiltonian method, as we shall see, can be applied directly, without the necessity of introducing the additional variables. Also, the earlier Hessian deficiency result of [3] was probably not meant, since it was not referred to in their paper (nor the arbitrary frequency problem). The result of [3] says that even when the action integral has been expressed in terms of only the degrees of freedom on $\mathbb{R}^3 \times S^2$ specific of a rotator, the corresponding momenta could not be inverted for velocities. But this singularity is not a problem for the Dirac procedure at all; it even predicts the fourth primary constraint not present in the case of phenomenological rotators.

The issue of the indeterminate motion of the fundamental rotator was described exhaustively in [5]. Later, we aim to elucidate this issue at the Hamiltonian level. To this end, we carry out a Hamiltonization appropriate for all relativistic rotators by employing the Dirac method exactly as is described in his lectures on quantum mechanics [8]. We shall see the important difference between the motion of fundamental and phenomenological rotators. Moreover, the Dirac procedure will reveal, in the form of a paradox, another surprising nature of the indeterminate motion.

2. General remarks

Any isolated relativistic dynamical system has ten independent integrals of motion associated with Poincaré symmetries. The most important and meaningful functions of the integrals are Casimir invariants of the Poincaré group, defining the mass and the spin in a covariant manner. They are used along with other quantum numbers to identify quantum particles and other irreducible quantum states. The Casimir invariants of the Poincaré group are of primary importance for the quantum formalism. The Wigner irreducibility idea [9], which is something pertinent to relativistic quantum systems, can also be realized at the classical level by the requirement of the fixed mass and spin for classical relativistic systems. From this standpoint, it is convenient and physically desirable to divide classical relativistic dynamical systems into two classes: phenomenological and fundamental. Fundamental are those whose Casimir invariants are parameters, whereas phenomenological are those whose Casimir invariants
are mere integrals of motion. This important distinction between the two kinds of classical relativistic dynamical systems was suggested by Staruszkiewicz in [2].

Since their mass and spin are parameters, like for quantum particles, classical fundamental dynamical systems are natural candidates for relativistic particle models. The next stage toward the approximate description of real particles is the quantization of such models. To this end one needs a Hamiltonian. It is difficult to find a relativistic model in the standard Hamiltonian frame. The best suited frame for incorporating various kinds of symmetries is the Lagrangian frame (for example, a relativistic invariant action by construction leads to relativistically invariant equations of motion). It is also of primary importance that the mass and the spin are defined at the Lagrangian level, since the Casimir invariants are constructed from canonical momenta resulting from relativistic symmetries. From this standpoint, the Lagrangian level can be considered primary, whereas the Hamiltonian level, which gives a convenient way of dealing with the equations of motion, is secondary. However, the Hamiltonian frame is better suited for quantization.

It is important that one correctly finds the Hamiltonian corresponding to a given Lagrangian. One must be sure that the Lagrangian and the Hamiltonian frames are mutually invertible. This task is challenging when constraints are present but difficult to identify. Recall that according to Dirac the first step toward constructing a Hamiltonian is to acquire the knowledge about all primary constraints. Primary constraints follow from the definition of momenta. (Some of the constraints are imposed already at the stage of the Lagrangian construction as subsidiary conditions, cf the appendix.) Sometimes, it is difficult to detect all constraints as they can be overlooked. A good example is the fundamental relativistic rotator. When regarded as a non-degenerate rotator, one can easily detect only the constraints characteristic of all relativistic rotators, such as the reparametrization and projection invariance constraints. In finding primary constraints, it is helpful to know the rank of the Hessian matrix present in the Lagrangian equations of motion. The number of primary constraints can be deduced from the Hessian rank. Unfortunately, the task of the determination of the Hessian rank can be cumbersome and computationally challenging [10].

3. Relativistic rotators and the issue of constraints

According to Staruszkiewicz, a relativistic rotator is a dynamical system described in spacetime by a position $x$ and a single null direction $k$ and, additionally, by two parameters, $m$ (mass) and $\ell$ (length) [2]. This leads to the following most general form of action defining a family of relativistic rotators:

$$S = -m \int d\tau \sqrt{\dot{x} \cdot \dot{x} G(Q)}, \quad \text{where} \quad Q = \sqrt{-\ell^2 \frac{kk}{(kk)^2}}, \quad kk = 0,$$

(3.1)

with $G$ being some positive function enumerating the rotators. Here, $\tau$ is an arbitrary variable along the worldline.

Since $k$ is null, the first primary constraint is $\varphi_1 = kk \approx 0$. We know it prior to calculating momenta which read:

$$p = m\sqrt{G(Q)} \frac{\dot{x}}{\sqrt{\dot{x} \cdot \dot{x} G(Q)}} = \frac{m QG'(Q)}{2 \sqrt{G(Q)}} \sqrt{\dot{x} \cdot \dot{x} G(Q)} \frac{k}{kk}, \quad \chi = \frac{m QG'(Q)}{2 \sqrt{G(Q)}} \sqrt{\dot{x} \cdot \dot{x} G(Q)} \frac{k}{kk}.$$

The action is projection invariant—the transformation $k \to k_\lambda = e^\lambda k$, with the arbitrary function $\lambda$ being defined on the worldline, is a symmetry of the Lagrangian. Only the

1 We use $\approx$ to denote a weak equation in the sense of the definition given in [8, 11].

2 We assume the signature $+, - , - , -$ for the metric tensor. Correspondingly, we use definitions of momenta with a ‘minus’ sign: e.g. $p = -\partial_x L$. 

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null direction assigned to $k$ is physically relevant. The corresponding projection invariance constraint is $\phi_2 = \chi k$. Its presence is the consequence of the definition of a momentum $\chi$ and that $kk = 0$.

There is something to be cautious about. A physical state should be projection invariant since the arbitrary factor of $k$ is effectively absent from the action integral (3.1). The momentum $\chi$ is not a reparametrization invariant; it transforms as

$$\chi \rightarrow \chi_\lambda = e^{-\lambda} \left( \chi + \frac{m}{2} \frac{OG(Q)}{G(Q)} \sqrt{k k} k \right).$$

Thus, it is not a classical observable. For example, one could add to $\chi$ anything proportional to $k$ without altering the physical state, and the condition $k \chi = 0$ would be still satisfied. (This can be realized by adding to the Lagrangian a total derivative $\hat{\Sigma} kk + 2 \Sigma kk$ which vanishes on the constraint surface, while at the Hamiltonian level we have the constraint $k \chi = 0$ which defines $\chi$ to within the addition of a quantity proportional to $k$.) For that reason, it is justified to expect that momenta $\chi$ derived at the Lagrangian and at the Hamiltonian level will not be identical but will be equal to each other only to within an additive term proportional to $k$, and therefore some caution is required when the two levels (although equivalent) are compared. This remark can be generalized—the momenta defined for a dynamical system at the Lagrangian and at the Hamiltonian level might not be identical but might be equal to each other to within a gauge transformation.

The action integral (3.1) is also reparametrization invariant in accordance with the requirement of relativity. Related to this invariance is the fact that the Lagrangian is homogeneous of the first degree in the velocities $\dot{q} \partial q L \equiv L$. This tells us that the ordinary Hamiltonian known from elementary mechanics vanishes identically. The corresponding reparametrization invariance constraint is difficult to find for general $G$, but we are guaranteed of its existence. Differentiation of the homogeneity relation gives $\delta \dot{q} \partial q L + \dot{q} \partial q \partial p L = \partial p \partial q L$; hence $\dot{q} \partial q p_j = 0$, which means that the Jacobian $\partial q x$ of a map from velocities to momenta has a lower rank than the number of velocities and that there must exist at least one constraint involving momenta, distinct from constraints $\phi_1$ and $\phi_2$.

Above, we have detected the three constraints characteristic of all relativistic rotators by examining the general transformation properties of the action integral (3.1). With this number of constraints one could expect $8 - 3 = 5$ degrees of freedom uniquely defining the physical state of a rotator (here, 8 is the dimensionality of the full configuration space in the action integral). But the number of constraints can be greater, depending on the function $G$ in the action integral, which can be inferred from the behavior of Casimir invariants. In units of $m$ and $\ell$, the Casimir invariants of the Poincaré group are

$$C_M := \frac{pp}{m^2} > 0, \quad C_j := \frac{XX(pk)^2}{\frac{1}{2}m^4\ell^2} > 0. \quad (3.2)$$

3 We use the definition $C_j = \frac{ww}{\frac{1}{4}\ell^2}$ assuming $kk = 0$ and $k \chi = 0$, where

$$WW = \begin{vmatrix} pp & px & pk \\ xp & xx & xk \\ kp & mx & kk \end{vmatrix}$$

is the square of Pauli–Lubanski spin pseudovector. In the literature, one can often encounter incorrect spelling "Lubanski" instead of "Lubanski".

Historical note: Józef Kazimierz Lubanski defined the vector known as the Pauli–Lubanski (pseudo)vector [13]. It seems that the list of inventors of the vector should be extended and it would be more appropriate to refer to it as the Mathisson–Pauli–Lubanski spin pseudovector: Sometime in the 1960s, Weyssenhoff told his then PhD student Andrzej Białas that it was Mathisson who had explained to Lubanski how to construct, from the spin bivector $s^{\mu\nu}$, the object that is now known as the Pauli–Lubanski vector' [14].

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Using the definitions of momenta, the invariants can be expressed in terms of \( G(Q) \): \( C_M = G(Q) - QG'(Q) \) and \( C_J = G'(Q)^2 \), or alternatively as
\[
C_M \pm Q\sqrt{C_J} = G(Q), \quad \pm\sqrt{C_J} = G'(Q). \tag{3.3}
\]
Function \( G > 0 \) cannot be arbitrary, it should satisfy the following requirement:
\[
G(Q) > QG'(Q) \neq 0.
\]
The condition \( G(Q) > QG'(Q) \) is a consequence of the positivity of the square of mass, \( C_M > 0 \), whereas the condition \( G'(Q) \neq 0 \) indicates that we are excluding the case of the structureless point particle. A variation in \( Q \) will cause the corresponding variations to occur in the Casimir invariants: \( \delta C_M = -QG'(Q)\delta Q \) and \( \delta C_J = 2G'(Q)G''(Q)\delta Q \). Hence, if \( G''(Q) \neq 0 \), the Casimir invariants are functions of each other, and there is a \( G \)-dependent function \( F_G(C_M, C_J) = 0 \) involving momenta and positions through three scalars: \( pp, pk \) and \( \chi \). This gives us the third reparametrization invariance constraint
\[
\varphi_3 = F_G(C_M, C_J) \approx 0, \quad G''(Q) \neq 0,
\]
existence of which we already know. Thus, when \( G''(Q) \neq 0 \), we indeed have three constraints for eight degrees of freedom in agreement with the expectation that a rotator is a dynamical system with five (physical) degrees of freedom.

When \( G''(Q) \equiv 0 \), the situation is qualitatively different and somewhat degenerate. In that case, the constraint \( F_G(C_M, C_J) = 0 \) is no longer valid, since the necessary condition for the invertibility of \( C_J = G'(Q)^2 \) is broken. Instead, we must use \( C_M = G(0) > 0 \) and \( C_J = G'(0)^2 \) separately; only then relations (3.3) can be satisfied. Then, both the Casimir invariants are identically fixed, independently of the initial conditions, in which case the Casimir mass and spin are the parameters, not merely the integrals of motion. Without loss of generality, we can then put both the \( C_M \) and \( C_J \) equal to 1, since fixed scales can be absorbed by dimensional constants \( m \) and \( \ell \) which have yet been unspecified. This gives us \( G(Q) = 1 \pm Q \). This way we have arrived at the Lagrangians of two degenerate rotators
\[
S = -m \int \tau \sqrt{\dot{\chi}^2 - \ell^2 \frac{\dot{k}^2}{(k\chi)^2}}, \tag{3.4}
\]
Originally, the Lagrangians were found by imposing the condition of fixed mass and spin [1, 2], and later, by requiring that the Hessian for the rotator degrees of freedom on \( \mathbb{R}^3 \times S^2 \) should be singular [3, 5]. Now, we have obtained them as the critical case when the reparametrization constraint of phenomenological rotators is no longer valid in the form of \( \varphi_3 \). One can say, there is a transition in the number of constraints when function \( G(Q) \) becomes linear, in which case the reparametrization invariance constraint splits into two independent constraints. This phenomenon is consistent with the result of [3, 5] implying that there must be \( 8 - 4 = 4 \) primary constraints when \( G''(Q) \equiv 0 \) (and \( G'(Q) \neq 0 \)) on account of the fact that then the Hessian rank is 4. A similar argument shows that we have \( 8 - 5 = 3 \) primary constraints when \( G''(Q) \neq 0 \), since then the Hessian rank is 5 (the connection between the Hessian rank and the number of primary constraints is illustrated in a simple model in the appendix).

4. The Hamiltonian for phenomenological rotators

All three primary constraints found for phenomenological rotators are regular and independent of the constraint surface. This can be verified by ascertaining if the rank of the Jacobian of a
map from the phase space coordinates to constraints regarded as new coordinates is the same as the number of constraints. Instead, one can check that a $3 \times 3$ matrix with elements

$$J_m = \eta^{\mu\nu} \left( \alpha \frac{\partial \psi_m}{\partial p^\rho} \frac{\partial \psi_n}{\partial p^\sigma} + \beta \frac{\partial \psi_m}{\partial k^\rho} \frac{\partial \psi_n}{\partial k^\sigma} + \gamma \frac{\partial \psi_m}{\partial \chi^\rho} \frac{\partial \psi_n}{\partial \chi^\sigma} \right)$$  \hspace{1cm} (4.1)

is non-singular on the constraint surface (\(\alpha, \beta, \gamma\) are arbitrary functions introducing appropriate units). Its determinant is equal to $-16 \beta^3 \chi \mathcal{C}_J^2 (\mathcal{F}_G C_j)^2$ on the constraint surface and cannot be identically zero for the nonzero spin (if \(\mathcal{F}_G C_j \equiv 0\), then \(\mathcal{G}'' \equiv 0\), and we are excluding that case). This regularity result remains unaltered if the \(\psi_m\)'s are multiplied by any functions nonzero on the constraint surface. From a calculation carried out in [5] it follows that the Hessian rank is 5 when \(\mathcal{G}''(Q) \neq 0\). Since the number of velocities is 8, we can be sure to have found all independent primary constraints. Furthermore, \([\psi_1, \psi_2]_{PB} = 2\psi_1 \approx 0, [\psi_2, \psi_3]_{PB} = 0\) and

$$[\psi_3, \psi_1]_{PB} = \frac{16}{m^2} \rho^2 \mathcal{F}_G C_j \psi_2 \approx 0.$$  

This means that all primary constraints are of the first class. There are no secondary constraints. Thus, the total Hamiltonian, in accordance with the Dirac procedure [8, 11], is a linear combination of the primary constraints

$$H_T = u_1 k \ell + u_2 \ell \chi + u_3 \mathcal{F}_G (C_M, C_j), \quad \mathcal{G}''(Q) \neq 0,$$

with the arbitrary functions \(u_m\), where \(C_M\) and \(C_j\) must be expressed in terms of momenta using (3.2). Hamiltonian \(H_T\) is also the full Hamiltonian describing phenomenological rotators.

In what follows we shall examine the dynamics resulting from the above Hamiltonian. Let us consider the trajectory perceived in the CM frame. By definition, the trajectory is a projection of the worldline onto the subspace orthogonal to \(p\). By \(\perp\) we denote a projection operation: \(y \rightarrow y_\perp = y - \frac{y \cdot p}{p^2} p\) for any vector \(y\). From the Hamiltonian equations, \(\dot{y} = d_T y = [y, H_T]_{PB}\); hence, \(y\) must be a linear combination of \(p, k\) and \(\chi\). Vectors \(\dot{x}_\perp \equiv (d_T x)_\perp, \dot{x}_\perp \equiv (d_T (\dot{x}_\perp))_\perp\) and \(\dot{x}_\perp \equiv (d_T (\dot{x}_\perp))_\perp,\) being \(p\)-orthogonal linear combinations of \(k, \chi\) and \(p\), must be coplanar. Thus, the torsion scalar vanishes for this trajectory. The radius of curvature of the trajectory, \(\rho\), is constant and fixed by the initial conditions

$$\rho = \frac{\left( \dot{x}_\perp \times \ddot{x}_\perp \right)^{3/2}}{\sqrt{\dot{x}_\perp \cdot \ddot{x}_\perp}} = \frac{\ell \sqrt{\mathcal{C}_J}}{2 C_M},$$

where \(\dot{x}_\perp = 2 \frac{\omega m}{\ell} C_j F_G C_j \left( k - \frac{kp}{pk} p \right)\) and \(\ddot{x}_\perp = \frac{\omega}{\mathcal{C}_J} \left( \frac{\omega}{\ell} C_j F_G C_j \right)^2 \frac{kp}{pk} \chi - \frac{pp}{pk} k\) on account of the Hamiltonian equations. The trajectory is thus circular. The hyperbolic angle \(\psi\) between \(p\) and \(\dot{x}_\perp\) (called rapidity) describing the rotation speed with respect to the CM frame is (we may assume \(pk > 0\))

$$\tanh \psi = \frac{pk \dot{p} - pp k \ddot{p}}{pk p \dot{p}^2} = -\sqrt{pp \frac{n^2}{\dot{p}^2}}, \quad n \equiv \frac{pp k - pk p}{pk \sqrt{pp}}.$$  \hspace{1cm} (4.2)

The Hamiltonian equations of motion can be used to prove the identity

$$\tanh \psi = \rho \omega, \quad \omega \approx 2 \frac{\mathcal{C}_M \sqrt{\mathcal{C}_J} F_G C_j}{\ell C_M F_G C_M + C_j F_G C_j},$$  \hspace{1cm} (4.3)

which holds on the constraint surface, where we have introduced the angular velocity \(\omega\). Being independent of the arbitrary functions \(u_m\), \(\psi\) is a genuine gauge-invariant and Lorenz-invariant quantity with a definite meaning. In addition, \(\psi\) remains constant during motion and depends only on the initial conditions, similarly as any function of the Casimir mass and spin, since
Using the proportional, in which \( f \) constraint surface. Then of motion no longer contain arbitrary functions and read conclusion that the above gauge is always feasible. In this gauge, the Hamiltonian equations of motion no longer contain arbitrary functions \( u_m \) so as

\[
px = \sqrt{pp}, \quad pk = \sqrt{pp}, \quad p\chi = 0.
\]

With these supplementary conditions, the physical state will be unaltered and we are left only with five degrees of freedom as required for a rotator. The first gauge condition sets the arbitrary \( \tau \) variable to be the proper time in the CM frame. The second one sets the arbitrary scale of \( k \). The third one is also admissible since the constraint \( \varphi_2 = k\chi \approx 0 \) defines \( \chi \) to within an additive term proportional to \( k \). Recall that this gauge is consistent with the Lagrangian frame in which \( \chi \propto k \), that is, \( k\chi = 0 \)—had the scale of \( k \) be varied, the momentum \( \chi \) would have been changed by an additive term proportional to \( k \). With the above gauge conditions, the description of a physical state is naturally adapted to the CM frame, and is still relativistically covariant. These conditions now must be made consistent with the equations of motion. Irrespective of any gauge, we always have \([px, H_T]_{PB} \approx 0\), and \( px = \sqrt{pp} \) gives \( u_1 \), whereas we will have \([pk, H_T]_{PB} \approx 0\) and \([p\chi, H_T]_{PB} \approx 0\) holding only when

\[
u_1 = -\frac{1}{2} \left( \frac{px}{2pk\sqrt{\chi^2}} \right) \omega, \quad u_2 = \frac{p\chi}{\sqrt{-\chi^2}} \omega, \quad u_3 = \frac{m^2 \ell^2}{8pk\sqrt{-\chi^2}} \omega.
\]

In order to verify that the denominator in the definition of \( \omega (4.3) \) is nonzero (and so, the above \( u_m \) finite), consider a function \( u(x, y) = xF_x + yF_y \), and suppose that the constraint \( F(x, y) = 0 \) can be locally solved for \( x \), that is, \( x = f(y) \) and \( F(f(y), y) \equiv 0 \). In that case, there is a nonzero function \( s(x, y) \) such that \( F(x, y) = (x - f(y))s(x, y) \). Hence, \( u(f(y), y) = s(f(y), y) (f(y) - yf'(y)) \). Suppose for contradiction that \( u(x, y) = 0 \) on the constraint surface. Then \( f(y) = yf'(y) = 0 \), which implies that the Casimir invariants are proportional, \( C_j = \alpha C_M, \alpha > 0 \). Then, relations (3.3) imply \( G^2 - \alpha G + \alpha QG' = 0 \) and, on differentiation, \( G' (2G + \alpha Q) = 0 \). This gives us either \( G' = 0 \) or \( G = -\frac{\alpha}{2} Q^2 < 0 \). Both possibilities are in conflict with our assumptions concerning \( G \). This way we have come to the conclusion that the above gauge is always feasible. In this gauge, the Hamiltonian equations of motion no longer contain arbitrary functions and read

\[
\dot{x} = \frac{p}{\sqrt{pp}} + \frac{G}{\sqrt{-\chi^2}} \omega, \quad \dot{p} = 0, \quad \dot{k} = \omega \frac{p\chi - pk \chi}{\sqrt{-\chi^2}}\sqrt{pp}, \quad \dot{\chi} = \omega \left( \frac{pk - pk}{\sqrt{pp}} \frac{p\chi}{\sqrt{-\chi^2}} + \frac{pk - pk \chi}{pk \sqrt{-\chi^2}} \right).
\]

Using the CM gauge explicitly, the equations for the variable quantities reduce to

\[
\dot{x} = \frac{p}{\sqrt{pp}} + R\Omega n, \quad \dot{n} = -\Omega \frac{\chi}{\sqrt{-\chi^2}}, \quad \dot{\chi} = \Omega n, \quad k = \frac{p}{\sqrt{pp}} + n.
\]

Here, \( P, \Omega \) and \( R \) denote the constant values of the momentum \( p \) and of the scalars \( \omega \) and \( \rho \), respectively, which we have already demonstrated to be constants of motion. It is thus a simple matter to solve these equations, and the solution reads

\[
x(t) = \frac{p}{\sqrt{pp}} t + R\Omega \int n(t) \, dt, \quad \dot{n}(t) + \Omega^2 n(t) = 0, \quad Pn = 0, \quad nn = -1.
\]
This is the composition of two motions—an inertial motion of the CM frame and a uniform rotation with a constant frequency \( \Omega \) around the circle of radius \( R \) in the CM frame in a space-like plane perpendicular to both the space-like (Pauli–Lubański) spin pseudovector and the time-like momentum vector. It is interesting to see that all phenomenological rotators move in the same qualitative manner. Only the relation between constants \( \Omega \), \( R \), and Casimir mass or spin is different.

Finally, we give the Hamiltonian \( H_{CM} \) defining the family of phenomenological rotators in the CM gauge that leads to the Hamiltonian equations of motion (4.4)

\[
H_{CM} = \frac{m}{2} f'(C_J) \sqrt{f(C_J)} \left[ C_M f - f(C_J) \right] + 4 \left( \frac{p\chi}{m^2} \right)^2 - \frac{8pkp\chi}{m^4\ell^2} kk - \frac{8pkp\chi}{m^4\ell^2} k\chi ,
\]

where \( C_M \) and \( C_J \) must be expressed in terms of momenta using (3.2). Function \( f \) establishes the dependence of the Casimir mass on the Casimir spin for a given rotator in the form \( C_M = f(C_J) \), \( C_J > 0 \), and is related to function \( G \) in the action (3.1) in a complicated way, \( C_J = G^2(Q) \), where \( Q^2 = 4C_J f^2(C_J) \) (this can be shown by differentiating \( C_M \) and \( C_J \) expressed in terms of \( G \)). In this notation, \( \tanh \psi = \frac{C_J f(C_J) - f(C_J)}{C_J f(C_J) + f(C_J)} \), \( \omega = \frac{\tanh \psi}{2} \) and \( \rho = \frac{\tanh \psi}{2} f(C_J) \). The condition \( |\tanh \psi| < 1 \) imposes a limitation on admissible \( f \), related to that we have already had for \( G \): \( C_M > 0 \Rightarrow G(Q) = QG'(Q) > 0 \Rightarrow |\tanh \psi| = \left| \frac{GG'(Q)}{2G(Q)} \right| < 1 \).

5. The Hamiltonian for fundamental rotators

The complete set of primary constraints we have found for fundamental rotators are

\[
\tilde{\varphi}_1 = kk, \quad \tilde{\varphi}_2 = \chi k, \quad \tilde{\varphi}_3 = C_J - 1, \quad \tilde{\varphi}_4 = C_M - 1,
\]
or possibly their linear combinations, which are equivalent. Similar to the previous section, we check the correctness of the constraints. A determinant of a \((4 \times 4)\)-dimensional matrix with elements defined by analogy with (4.1) is equal to \( \frac{m^4\omega^6}{16} \alpha^3 \beta^3 \) on the constraint surface and cannot be zero, showing that the rank of the associated Jacobian matrix is equal to the number of constraints. The four constraints are therefore regular and independent. All the primary constraints are the first class constraints: \([\tilde{\varphi}_m, \tilde{\varphi}_m]_{PB} \approx 0 \). We are now prepared to write down the total first class Hamiltonian, which is a linear combination of the primary constraints. It is essentially different from that previously found for phenomenological rotators. It reads

\[
\tilde{H}_T = \tilde{u}_1 kk + \tilde{u}_2 k\chi + \tilde{u}_3 (C_J - 1) + \frac{1}{\tilde{u}_4} \tilde{u}_3 (C_M - 1).
\]

For the later convenience, we have defined the redundant auxiliary fields in the last term, which is quite permissible on account of the fact that the functions \( \tilde{u}_m \) are arbitrary, and \( \tilde{\varphi}_3 \) and \( \tilde{\varphi}_4 \) are dimensionless. Owing to the fact that we have managed to detect all primary constraints, it took only several lines to find the above Hamiltonian in the Dirac formalism. In [7], a Hamiltonian equivalent to that we just found was arrived at through conceptually quite a different and more intricate way, employing the additional auxiliary degrees of freedom not present in the original Lagrangian (3.4).

The Poisson brackets \([\tilde{H}_T, \tilde{\varphi}_m]_{PB} \) all vanish on the constraint surface: \([\tilde{H}_T, \tilde{\varphi}_m]_{PB} \approx 0 \) with the help of primary constraints. There are no further constraints—the Hamiltonian equations are already consistent. Functions \( \tilde{u}_m \) are completely arbitrary. All the functions have the nature of gauge variables in the Dirac formalism, since all the primary constraints are of the first class. Thus, there will be four arbitrary functions in the general solution for eight original degrees of freedom. This result is unusual as it means that only four degrees of freedom are physical, not five as expected for a rotator. The result is also consistent with the observation made in [3, 5] that the Lagrangian equations of motion on \( \mathbb{R}^3 \times \mathbb{S}^2 \) are linearly dependent and
that the general solution expressed in terms of only the coordinates on $\mathbb{R}^3 \times S^2$ contains a single arbitrary function, which is tantamount to the observation of the absence of (nontrivial) secondary constraints we have just made at the Hamiltonian level. Moreover, with the help of the Hamiltonian equations of motion corresponding to $\hat{H}_T$:

$$\dot{x} = \frac{2}{m} \tilde{u}_3 \left( n + \frac{1}{m} \frac{p \tilde{u}_4}{m} \right), \quad \dot{p} = 0,$$

$$\dot{k} = \tilde{u}_2 k - \tilde{u}_1 \frac{8 (pk)^2}{m^2 \ell^2} \chi, \quad \dot{\chi} = -2 \tilde{u}_1 k - \tilde{u}_2 \chi - \frac{2}{pk} \tilde{u}_3 p,$$

we come to the conclusion that the rapidity calculated according to definition (4.2) is directly related to the gauge variable $\tilde{u}_4$, namely $\tanh \psi$ is numerically equal to $\tilde{u}_4$ on the constraint surface ($\dagger$):

$$\tanh \psi \approx \tilde{u}_4.$$ Alternatively, in place of $\tilde{u}_4$ we can use the angular velocity $\tilde{\omega}$ of rotation perceived in the CM frame and corresponding to $\psi$ according to the formula $\frac{\tilde{\omega}}{m^2} \equiv \tilde{u}_4 \approx \tanh \psi$.

The rapidity is a quantity restricted to the constraint surface. It satisfies the gauge invariance condition—its Dirac bracket weakly vanishes with all first class constraints (here, Poisson brackets are equivalent to Dirac brackets on account of the fact that there is no second class constraints), that is, $[\tanh \psi, \tilde{u}_m]_{\text{PB}} = 0$ on the constraint surface (this can be verified with the help of definition (4.2) and the above Hamiltonian equations of motion used for calculating scalar products $pi$ and $k\tilde{x}$, whilst taking care that the constraints must not be used before working out the Poisson brackets). Thus, the rapidity satisfies the requirements for being a classical observable in the sense of the definition given in Henneaux and Teitelboim’s handbook on *Quantization of Gauge Systems* [12]. A physical state should not depend on gauge variables. Surely, $\psi$ defines a physical state and simultaneously, as we have seen, on the constraint surface it is numerically equal to a gauge variable. This way we have come across the apparent paradox that a genuine physical quantity $\psi$ turns out to be a genuine gauge variable! It should be stressed that $\psi$ is reparametrization invariant and projection invariant on the constraint surface; therefore, the arbitrariness of $\psi$ has nothing to do with the arbitrariness in choosing the time variable or the scale of $k$ and is the characteristic of the fundamental rotators only and originates from the Hessian rank deficiency discussed earlier.

Having said this, we could end this section. However, we shall find the motion of the system described by $\hat{H}_T$ in order to see its correspondence with the results of [3, 5] arrived at the Lagrangian level. Since $[p, \hat{H}_T]_{\text{PB}} = 0$ and $pk[k \wedge \chi, \hat{H}_T]_{\text{PB}} = \tilde{u}_3 k \wedge \tilde{p}$, there are two conserved vectors, the momentum $p$ and the spin vector $p \wedge k \wedge \chi$. We can make use of the apparent symmetries of the model to set three arbitrary functions from among $\tilde{u}_m$. Similar to phenomenological rotators, we impose three admissible gauge conditions naturally adapted to the CM frame (of which choice has been already justified in the previous section): the proper time condition $\dot{x}p \equiv m$, the projection condition $kp \equiv m$ and the orthogonality condition $p\chi = 0$. This requires the gauge variables to be set as follows:

$$\tilde{u}_1 = -\frac{m}{2 (pk)^2} \left( 1 + \frac{4 (pk)^2 (p \chi)^2}{m^2 \ell^2} \right) \tilde{u}_4, \quad \tilde{u}_2 = \frac{4 pk p \chi}{m^2 \ell^2} \tilde{u}_4, \quad \tilde{u}_3 = \frac{m}{2 \tilde{u}_4}.$$ In the CM gauge the Hamiltonian of fundamental rotators reads

$$\begin{align*}
\tilde{H}_{\text{CM}} & = \frac{m}{2} \left( \frac{pp}{m^2} - 1 \right) + \cdots \\
& - \frac{m\ell}{4} \tilde{\omega} \left[ \left( \frac{pp}{m^2} + \frac{4 (pk)^2 \chi \chi}{m^2 \ell^2} \right) + \frac{4 (p \chi)^2 - pp \chi \chi}{m^2 \ell^2} kk - \frac{8 pk p \chi}{m^2 \ell^2} \chi \chi \right],
\end{align*}$$

(5.1)
where we have expressed the function \( \tilde{u}_4 \) in terms of the arbitrary function \( \tilde{\omega} \) (as defined earlier). Recall that the constraints assumed for that Hamiltonian are \( kk = 0, k\chi = 0, pp = m^2 \) and \( (pk)^2 \chi\chi = \frac{1}{4}m^4\ell^2 \).

Any function of six independent phase-variable scalars \( kk, pk, k\chi, pp, p\chi \) and \( \chi\chi \) has its Poisson bracket with the Hamiltonian \( \tilde{H}_{CM} \) already vanishing on the constraint surface. The persistent arbitrariness present in the Hamiltonian through \( \tilde{\omega} \) cannot be therefore removed by using an argumentation from within the framework of the model. The Hamiltonian equations of motion corresponding to \( \tilde{H}_{CM} \) are

\[
\dot{x} = \frac{p}{m} + \frac{\ell}{2} \tilde{\omega} \left( \frac{m}{pk} k - \frac{p}{m} \right), \quad \dot{p} = 0,
\]

\[
\dot{k} = \tilde{\omega} \frac{2pk(p\chi k - pk\chi)}{m^3\ell}, \quad \dot{\chi} = \frac{m^2\ell}{2pk\tilde{\omega}} \left( \frac{m}{pk} k - \frac{p}{m} + \frac{4pkp\chi}{m^2\ell^2} (p\chi k - pk\chi) \right).
\]

By using the CM gauge explicitly, the above equations reduce to

\[
\dot{x}(t) = \frac{p}{m} + \frac{\ell}{2} \tilde{\omega}(t) \chi(t), \quad \dot{\chi}(t) = \frac{m\ell}{2} \tilde{\omega}(t) \chi(t),
\]

\[
n\chi = -1, \quad nP = 0, \quad P = \text{const.}, \quad PP = m^2,
\]

where we have shown explicitly dependence of \( t \), where \( t \) is the proper time measured in the CM frame. These equations can easily be integrated as

\[
x(t) = x(0) + \frac{p}{m} t + \frac{\ell}{2} \int \tilde{\omega}(t) \chi(t) \, dt,
\]

\[
\left( \frac{1}{\tilde{\omega}(t)} \frac{d}{dt} \right) n(t) = -n(t).
\]

It is now clear why \( \tilde{\omega}(t) \) has the interpretation of the frequency of rotation on the unit sphere (of null directions) perceived in the CM frame. These are the same solutions with the arbitrary frequency we arrived at in the Lagrangian frame in [3, 5] (shown explicitly therein). This shows that we have found the correct minimal Hamiltonian corresponding to the Lagrangian (3.4).

Finally, we stress again that the frequency \( \tilde{\omega}(t) \) in the above solution is a gauge variable—a completely arbitrary function of the proper time in the CM frame, whereas there was no such arbitrariness in the motion of phenomenological rotators for which the analogous frequency was a constant of motion fixed by the initial data (and, of course, gauge independent).

### 6. Summary and conclusions

According to Dirac, the physical state of a system should be independent of gauge variables, which can be arbitrary functions. This statement can be used as an argument for deciding whether a geometric particle model can be considered physical. We applied this idea to the family of relativistic rotators defined in [2]. To this end, we constructed minimal Hamiltonians for such systems using the Hamiltonian method for constrained systems suggested by Dirac [8, 11] and solved the resulting equations of motion in the gauge adapted to the center of momentum frame. It turned out that there are in fact two distinct kinds of rotators described by qualitatively different Hamiltonians, namely a continuous family of phenomenological rotators with unique and qualitatively the same motion, differing from each other in the spin–mass relation only, and a two-element family of fundamental rotators with separately fixed mass and spin and indeterminate motion. In the gauge, adapted to the center of momentum frame, the Hamiltonian for phenomenological rotators is unique, whereas the Hamiltonian for fundamental rotators is not unique and still contains a single gauge variable.
To see clearly the nature of the singularity in the motion of the fundamental rotator, we chose a classical observable being what particle physicists would call the rapidity with respect to the center of momentum frame, constructed for rotators as the hyperbolic angle between the 4-momentum and the 4-velocity. It provides a measure of the frequency of rotation perceived in the center of momentum frame. Beyond all question, this observable is a genuine physical quantity and as such should be independent of gauge variables. At the Hamiltonian level, this expectation is confirmed for phenomenological rotators—the rapidity is independent of gauge variables present in the Hamiltonian, and the Cauchy problem for the Hamiltonian equations of motion has a unique solution in the CM gauge. Surprisingly, this is quite different from the situation with the fundamental rotator [2] or, equivalently, the geometric model of the arbitrary spin massive particle [1]. At the Hamiltonian level, we confirmed for the fundamental rotator the result we obtained at the Lagrangian level in [3, 5], that the rapidity remains completely indeterminate and can be an arbitrary function of the time—the secondary or Hessian constraint present owing to the Hessian rank deficiency is absent (it is trivial $0 = 0$) in the free motion. Furthermore, according to the Dirac formalism, the rapidity of the fundamental rotator (or the associated frequency of rotation) should be regarded as a genuine gauge variable. Here, a paradox comes about—we have a physical degree of freedom that simultaneously is a gauge variable. Thus, we have another way, complementary to that outlined in [3] and [5], of seeing that the fundamental rotator is defective as a dynamical system. For physical reasons, it is not suitable for quantization, despite the fact that the minimal Hamiltonian is known (cf (5.1)) and the quantization procedure could in principle be applied. Although there is a possibility of setting the frequency at the level of the Hamiltonian (cf $\tilde{\omega}$ in (5.1)), which is permissible for gauge variables, this should not be done on account of the physical interpretation of $\tilde{\omega}$ (even when the frequency has been fixed ‘by hands’ in the Hamiltonian, the motion would be unstable on account of the fact that the null space of the Hessian on the rotator manifold $\mathbb{R}^3 \times S^2$ is nontrivial [5, 6], whereas a physical state should be stable).

There is also another importance of our paper. It can be regarded as an illustration that relativistic dynamical systems whose Casimir mass and spin are fixed parameters can have at least one primary constraint more than their counterparts, with Casimir invariants being ordinary constants of motion. This observation may be helpful in the context of finding a well-behaved fundamental dynamical system (if it exists).

Appendix. An example of a Lagrangian on the unit sphere illustrating the origin of constraints $\varphi_1$ and $\varphi_2$

In this example, we illustrate the mechanism of how constraints $\varphi_1$ and $\varphi_2$ we encountered for rotators come about in a simpler projection invariant model involving a null vector in Minkowski space. To this end, we consider a particle model described by the Lagrangian

$$L = -\frac{1}{2} \frac{\dot{q} \dot{q}}{(wq)^2},$$

with $w$ being a constant time-like vector, and assume that positions $q$ are constrained to a light cone $qq = 0$, then always $wq \neq 0$ and $\dot{q}$ is space-like: $\dot{q} \dot{q} < 0$. For simplicity, we deliberately break the reparametrization invariance. The Lagrangian is projection invariant; the transformation $q \to \lambda q$ is a symmetry when $qq = 0$. We see that there will be only two important degrees of freedom similar to a particle constrained to a unit sphere. To find the equations of motion in a covariant notation without introducing the internal coordinates on the sphere, we apply the Lagrange multiplier method and add to the Lagrangian the term $\frac{1}{2} \Lambda \dot{q} q$
vanishing on the light cone, with a function $\Lambda$ yet to be determined. This gives us the extended Lagrangian and the momentum $p$ conjugate to $q$:

$$L = -\frac{1}{2} \dot{q}^2 \left( wq \right)^2 + \frac{1}{2} \Lambda qq,$$

and the Lagrangian equations of motion

$$qq = 0, \quad \dot{p} = -\frac{\dot{q} q}{\left( wq \right)^3} w - \Lambda q, \quad \Rightarrow \quad \frac{w \dot{p}}{qw} + \frac{\dot{q} q}{\left( wq \right)^3} w w = -\Lambda.$$ 

The equations may be rewritten as

$$\dot{q} = -\frac{\dot{q} q}{\left( wq \right)^3} w - \Lambda q, \quad \Rightarrow \quad \frac{2 w \dot{q} q}{\left( wq \right)^3} \left( \dot{q} q - \frac{w \dot{q} q}{qw} q \right),$$

with the help of $\dot{q} q = -q \ddot{q}$ which holds on the light cone. We see that the Hessian has two independent eigenvectors $q$ and $w$ both to the eigenvalue 0. The Hessian rank is thus $4 - 2 = 2$, as it should be for a particle constrained to a sphere. The Hessian constraints are $qq = 0$ and $qq = 0$. They commute with taking the projection. The corresponding primary constraints are $pq = 0$ and $qq = 0$. These two constraints are the primary constraints used to construct the Hamiltonian in the Dirac formalism. Alternatively, we could have introduced the internal coordinates on a sphere and find the ordinary Hamiltonian (in that case the corresponding Hessian would be a non-singular $2 \times 2$ matrix, that is, with rank 2).

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