Pricing and Capital Allocation for Multiline Insurance Firms With Finite Assets in an Imperfect Market

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Abstract

We analyze multiline pricing and capital allocation in equilibrium no-arbitrage markets. Existing theories often assume a perfect complete market, but when pricing is linear, there is no diversification benefit from risk pooling and therefore no role for insurance companies. Instead of a perfect market, we assume a non-additive distortion pricing functional and the principle of equal priority of payments in default. Under these assumptions, we derive a canonical allocation of premium and margin, with properties that merit the name the natural allocation. The natural allocation gives non-negative margins to all independent lines for default-free insurance but can exhibit negative margins for low-risk lines under limited liability. We introduce novel conditional expectation measures of relative risk within a portfolio and use them to derive simple, intuitively appealing expressions for risk margins and capital allocations. We give a unique capital allocation consistent with our law invariant pricing functional. Such allocations produce returns that vary by line, in contrast to many other approaches. Our model provides a bridge between the theoretical perspective that there should be no compensation for bearing diversifiable risk and the empirical observation that more risky lines fetch higher margins relative to subjective expected values.

JEL Codes: G22, G10

1 Introduction

The complete perfect market paradigm dominates financial models of insurance pricing developed since the 1970s. These models imply the existence of an additive valuation rule, meaning there is no benefit from diversification and no role for insurer intermediaries. Moreover, they typically provide no compensation for bearing diversifiable risk. Milestones in this approach include the use of discounted cash flows, Myers and Cohn (1987) and Cummins (1990), and options pricing Doherty and Garven (1986) and Cummins (1988). Phillips, Cummins, and Allen (1998) and Myers and Read Jr. (2001) considered multiple line pricing, allowing for ex ante default rules, which Sherris (2006) and Ibragimov, Jaffee, and Walden (2010) extended to ex post rules. From Phillips’s contribution forward, there is a focus on what happens in default states to explain premiums by line. Aside from non-diversifiable risk
and tail-driven default, these models do not reflect the overall distribution of losses, or what we will call the shape of risk.

Holding capital in an intermediary insurance company introduces frictional costs, which Cummins (2000) explains are primarily driven by agency conflict, taxes, and regulation. Why would insureds incur the extra expense of buying through an insurer in a perfect market? Ibragimov, Jaffee, and Walden (2010) assume insureds do not have direct access to ultimate capital providers or that there are additional costs to do so. This somewhat unsatisfactory explanation was anticipated by Grundl and Schmeiser (2007), who pointed out that pricing does not depend on allocation in a perfect market. Bauer and Zanjani (2013) consider capital allocation in the context of allocating frictional costs.

The fundamental theorem of asset pricing states that in a perfect market the existence of an additive valuation rule is essentially equivalent to no arbitrage, Ross (1978), Dybvig and Ross (1989), Delbaen and Schachermayer (1994). The thought that insurance valuation rules must be additive has exerted a considerable influence on theory and practice, Borch (1982), Venter (1991). But while indubitably competitive, the insurance market is neither perfect nor complete.

Imperfect or incomplete pricing paradigms isolate different failures of the perfect complete model but tend to have very similar implications. They result in non-additive pricing functionals that are often additive on comonotonic risks, and that can be expressed as a worst-case expectation over a set of probability distribution outcomes. Wang (1996) and Wang, Young, and Panjer (1997) apply a non-additive functional using distorted probabilities to insurance pricing, leveraging diverse theoretical underpinnings including Huber (1981), Schmeidler (1986), Schmeidler (1989), Yaari (1987), and Denneberg (1994). Within this theory, distortion risk measures (DRM) occur repeatedly and in many guises. Kusuoka (2001) and Acerbi (2002) characterize DRMs as coherent, law invariant, and comonotonic additive functionals.

DRMs are easy to apply in practice and have many appealing properties. However, they are not additive and include transaction costs, via an implied bid-ask spread, Castagnoli, Maccheroni, and Marinacci (2004). We must ask whether they are consistent with arbitrage-free pricing. Fortunately, the answer is yes. The presence of transaction costs render apparent arbitrage opportunities impractical and so a non-additive pricing rule can still be arbitrage-free.

Results from Chateauneuf, Kast, and Lapied (1996), De Waegenaere (2000), and especially Castagnoli, Maccheroni, and Marinacci (2002) and De Waegenaere, Kast, and Lapied (2003) produce general equilibrium models that allow for non-additive prices, and show that DRM pricing—in the equivalent guise as a Choquet integral—is consistent with general equilibrium. As a result, it is legitimate to use DRMs to price risk transfer. They return a premium that charges for the shape of risk, even diversifiable risk. DRMs can be applied both to non-intermediated, direct-to-investor pricing and insurer intermediated pricing, where insurers sell the risk to investors.

Why is it reasonable to charge for the shape of risk, especially when it is diversifiable? There are two related reasons: ambiguity aversion and winner’s curse.
Insurance pricing is a horse lottery, not a roulette lottery; it relies on subjective probabilities Ancombe and Aumann (1963). Losses are ambiguous and investors are ambiguity averse. Zhang (2002) and Klibanoff, Marinacci, and Mukerji (2005) describe ambiguity relevant to insurance pricing. The latter paper has been applied in an insurance context by Robert and Therond (2014), Dietz and Walker (2017) and Jiang, Escobar-Anel, and Ren (2020). Epstein and Schneider (2008) can be applied equally to underwriters who often weight bad news more heavily than good. Premium ambiguity is confounded with risk because more risky classes of business tend to have more ambiguous premiums. The failure of the terrorism insurance market is a case in point.

The second reason is the winner’s curse: the winning bid is biased low when there are multiple quotes, Thaler (1988). The insurance market is very competitive and is characterized by big-data, predictive modeling pricing with heterogeneous insureds. Competing classification plans exacerbate winner’s curse. As a result, a margin over subjective expected loss, even when subjective probabilities are unbiased, is justified. Winner’s curse will be correlated to ambiguity: greater ambiguity will result in wider quote dispersion. The winner’s curse margin will appear in quotes but will not appear ex post in results. The winner’s curse is distinct from adverse selection, D’Arcy and Doherty (1990).

In this paper, we consider applying a DRM pricing functional to individual policies. We do this in two steps. First, we consider pricing without insurers and associated frictional costs. In this world, insureds contract directly with investor capital providers. The capital providers can be risk-neutral, but they are ambiguity averse. Uncertain subjective probabilities drive the market; there are no objective probabilities. Investors may be willing to write a roulette lottery at cost, but they are unwilling to write a horse lottery at cost because they are ambiguity averse and because they know their winning quotes will be biased below subjective expectation. The DRM incorporates a margin to allow for ambiguity aversion and help correct for the winner’s curse.

Non-additive DRMs create a benefit to independent insureds who pool their risks. Estimation risk, process risk, and entropy are all lower for the pool than for the individual risks, resulting in less ambiguous subjective probabilities. Pool pricing is closer to the subjective expected value, lowering the average premium for pool participants. These conclusions do not hold if the insureds are not independent, for example if the risk is driven by catastrophe events. Boonen, Tsanakas, and Wüthrich (2017) and Mildenhall (2017) discuss the important differences between the independent and dependent cases.

Pool members still have the problem of allocating any gains from diversification. We show there is a unique way to do this consistent with the value of insurance cash flows and an assumption of equal priority in default, but relying on no other inputs. We call our method the natural allocation. Our approach is similar to Ibragimov, Jaffee, and Walden (2010). We model fair value to insureds rather than marginal costs to the pool. The pool is a transparent, contractual pass-through.

Our method identifies the conditional expectation of policy losses given total pool loss as a controlling function, which we call $\kappa$. It is exquisitely sensitive to relative risk within a portfolio. We explain our allocation in reference to the relative consumption of more or less
ambiguous, and hence costly, capital layers.

Risk pools have a clear role in our model: they minimize ambiguity-based risk costs. Pools accumulate risk to create more credible pricing signals, credible being synonymous with stable and low volatility. Stable pools will be financed more cheaply by investors than more ambiguous pools, or than single risks. A similar data-centric role for insurance pools was suggested in Froot and O'Connell (2008).

A risk pool does not have to be structured as an insurance company intermediary. Intermediaries emerge as pool managers look to bolster their credibility with investors by putting skin in the game and assuming risk. This provides a solid rationale for the existence of insurers versus pools managed by non-risk bearing underwriting managers. Interestingly, non-risk-bearing managers are now quite common in the property catastrophe reinsurance market. There, credibility is enhanced by reliance on independent and highly regarded third-party catastrophe models. Such models do not exist for most lines. Generally, aggregated data is needed to price individual policies fairly.

Insurance company pools are successful if they lower the cost of pure risk transfer by an amount that more than offsets their frictional costs of holding capital. Individual policy pricing still requires an allocation of frictional costs. Since frictional costs are usually regarded as a flat tax on capital, this can be done via a capital allocation. We show there is a unique “natural” way to perform the allocation, giving a complete solution to pooled risk insurance pricing under a DRM. Like the premium allocation, the capital allocation only assumes aggregate pricing by a DRM and equal priority in default. It relies on the fact that DRMs are law invariant.

Although many of the results in the paper have been known, at least in principle, for many years, we believe it contains several noteworthy results. In particular, we highlight the following contributions.

- We analyze multiline pricing and capital allocation in equilibrium no-arbitrage markets. We address the allocation of risk measures from the insured’s perspective, as in Ibragimov, Jaffee, and Walden (2010), but generalize their work to pricing in imperfect and incomplete markets with non-additive functionals.
- We find there is a canonical allocation of premium and margin in a market where prices are determined by a distortion risk measure with a rule of equal priority in default. The allocation relies on no additional assumptions and we believe it merits the name the natural allocation. The natural allocation gives non-negative margins to all independent lines for default-free insurance but can exhibit negative margins for low-risk lines under limited liability. The natural allocation reveals subtle interactions between margin, default, and idiosyncratic risk.
- We introduce novel conditional expectation measures of relative risk within a portfolio and use them to derive simple, intuitively appealing expressions for risk margins and capital allocations consistent with law invariant DRM pricing functionals.
- We give a unique capital allocation consistent with our law invariant pricing functional. Such allocations produce returns that vary by line, in contrast to many other approaches.
- We illustrate the theory with examples. The examples elucidate the interplay of loss
and margin between lines at various levels of portfolio risk. We demonstrate how the
cost of the shape of risk reflects a complex interaction between the relative consumption
of low layer, certain, high loss ratio assets, and high layer, uncertain, low loss ratio
assets.

- In contrast with cost allocation-based approaches, we find that margin by line is more
driven by behavior in solvent states than in default states. Our results hold even when
there is no possibility of default.
- Our model provides a bridge between the theoretical perspective that there should be
no compensation for bearing diversifiable risk and the empirical observation that more
risky lines fetch higher margins relative to subjective expected values.

The remainder of the paper is structured as follows. Section 2 recalls the definition and
properties of a DRM. Section 3 states and proves canonical formulas giving the natural
allocation of loss and premium by policy under a DRM, and shows margins are non-negative
for independent risks. Section 4 discusses the $\kappa$ and two associated functions that control
the natural allocation and that are informative in their own right. Section 5 derives some
properties of the natural margin allocation by policy. Section 6 extends the natural premium
allocation to a natural allocation of capital. This allows frictional costs to be allocated.
Section 7 gives two examples, one simple and one more realistic. Finally, Section 8 offers
concluding remarks, discusses limitations, and makes suggestions for further research.

**Notation and Conventions**

We consider insurance written directly by investors or intermediated by an insurance company.
We distinguish, when necessary, by saying investor-written, or intermediary-written or
intermediated insurance. When irrelevant, we just say the insurance is written by a provider.
In either case, insurance consists of a pool of individual policies written for insureds. Policies
last one period, with premium collected at $t = 0$ and losses paid in full at $t = 1$. Line of
business or business unit or other grouping can be substituted for policy. We refer to the
components as lines or policies, whichever is most appropriate.

When insurance is supported by finite assets, the provider has limited liability.

When intermediated, the intermediary is a stock insurer. At $t = 0$ it sells its residual value
to investors to raise equity. At time one it pays claims up to the amount of assets available.
If assets are insufficient to pay claims it defaults. If there are excess assets they are returned
to investors.

The terminology describing risk measures is standard, and follows Föllmer and Schied (2011).
We work on a standard probability space, Svidland (2009), Appendix. It can be taken as
$\Omega = [0, 1]$, with the Borel sigma-algebra and $\mathbb{P}$ Lebesgue measure. The indicator function on
a set $A$ is $1_A$, meaning $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ otherwise.

Total insured loss, or total risk, is described by a random variable $X \geq 0$. $X$ reflects policy
limits but is not limited by provider assets. $X = \sum_t X_t$ describes the split of losses by policy.$F, S, f,$ and $q$ are the distribution, survival, density, and (lower) quantile functions of $X$. 
Subscripts are used to disambiguate, e.g., \( S_{X_i} \) is the survival function of \( X_i \). \( X \wedge a \) denotes \( \min(X,a) \) and \( X^+ = \max(X,0) \).

The letters \( S, P, M \) and \( Q \) refer to expected loss, premium, margin and equity, and \( a \) refers to assets. The value of survival function \( S(x) \) is the loss cost of the insurance paying \( 1_{\{X>x\}} \), so the two uses of \( S \) are consistent. Premium equals expected loss plus margin; assets equal premium plus equity. All these quantities are functions of assets underlying the insurance.

We use the actuarial sign convention: large positive values are bad. Our concern is with quantiles \( q(p) \) for \( p \) near 1. Distortions are usually reversed, with \( g(s) \) for small \( s = 1-p \) corresponding to bad outcomes. As far as possible we will use \( p \) in the context \( p \) close to 1 is bad and \( s \) when small \( s \) is bad.

Tail value at risk is defined for \( 0 \leq p < 1 \) by

\[
TVaR_p(X) = \frac{1}{1-p} \int_p^1 q(t) dt.
\]

Prices exclude all expenses. The risk free interest rate is zero. These are standard simplifying assumptions, e.g. Ibragimov, Jaffee, and Walden (2010).

## 2 Distortion Risk Measures and Pricing Functionals

We define DRMs and recall results describing their different representations. By De Waelegenaere, Kast, and Lapied (2003) DRMs are consistent with general equilibrium and so it makes sense to consider them as pricing functionals. The DRM is interpreted as the (ask) price for an investor-written risk transfer. The rest of the paper will explore multi-policy pricing implied by a DRM.

**Definition 1.** A distortion function is an increasing concave function \( g : [0,1] \to [0,1] \) satisfying \( g(0) = 0 \) and \( g(1) = 1 \).

A distortion risk measure \( \rho_g \) associated with a distortion \( g \) acts on a non-negative random variable \( X \) as

\[
\rho_g(X) = \int_0^\infty g(S(x)) dx.
\]  

The simplest distortion if the identity \( g(s) = s \). Then \( \rho_g(X) = \mathbb{E}[X] \) from the integration-by-parts identity

\[
\int_0^\infty S(x)dx = \int_0^\infty x dF(x).
\]

Other well-known distortions include the proportional hazard \( g(s) = s^r \) for \( 0 < r \leq 1 \) and the Wang transform \( g(s) = \Phi(\Phi^{-1}(s) + \lambda) \) for \( \lambda \geq 0 \), Wang (1995).

Since \( g \) is concave \( g(s) \geq 0g(0) + sg(1) = s \) for all \( 0 \leq s \leq 1 \), showing \( \rho_g \) adds a non-negative margin.
Going forward, \( g \) is a distortion and \( \rho \) is its associated distortion risk measure. We interpret \( \rho \) as a pricing functional and refer to \( \rho(X) \) as the price or premium for investor-written insurance on \( X \). When we price intermediated insurance we need to add frictional costs of holding capital. This is considered in section 6.2.

DRMs are translation invariant, monotonic, subadditive and positive homogeneous, and hence coherent, Acerbi (2002). In addition they are law invariant and comonotonic additive. In fact, all such functionals are DRMs. As well as having these properties, DRMs are powerful because we have a complete understanding of their representation and structure, which we summarize in the following theorem.

**Theorem 1.** Subject to \( \rho \) satisfying certain continuity assumptions, the following are equivalent.

1. \( \rho \) is a law invariant, coherent, comonotonic additive risk measure.
2. \( \rho = \rho_g \) for a concave distortion \( g \).
3. \( \rho \) has a representation as a weighted average of TVaRs for a measure \( \mu \) on \([0,1]\)
   \( \rho(X) = \int_0^1 \text{TVaR}_p(X) \mu(dp) \).
4. \( \rho(X) = \max_{Q \in \mathcal{Q}} \mathbb{E}_Q[X] \) where \( \mathcal{Q} \) is the set of (finitely) additive measures with \( Q(A) \leq g(P(A)) \) for all measurable \( A \).
5. \( \rho(X) = \max_{Z \in \mathcal{Z}} \mathbb{E}[XZ] \) where \( \mathcal{Z} \) is the set of positive functions on \( \Omega \) satisfying
   \( \int_p^1 q_Z(t) dt \leq g(1-p) \), and \( q_Z \) is the quantile function of \( Z \).

The theorem combines results from Föllmer and Schied (2011) (4.79, 4.80, 4.93, 4.94, 4.95), Delbaen (2000), Kusuoka (2001), and Carlier and Dana (2003). The theorem requires that \( \rho \) is continuous from above to rule out the possibility \( \rho = \sup \). In certain situations, the sup risk measure applied to an unbounded random variable can only be represented as a sup over a set of test measures and not a max. Note that the roles of from above and below are swapped from Föllmer and Schied (2011) because they use the asset, negative is bad, sign convention whereas we use the actuarial, positive is bad, convention.

The relationship between \( \mu \) and \( g \) is given by Föllmer and Schied (2011) 4.69 and 4.70. The elements of \( \mathcal{Z} \) are the Radon-Nikodym derivatives of the measures in \( \mathcal{Q} \).

### 3 Loss and Premium by Policy and Layer

This section introduces the idea of layer densities and proves that DRM premium can be allocated to policy in a natural and unique way.

#### 3.1 Layer Densities

Risk is often tranched into layers that are then insured and priced separately. Meyers (1996) describes layering in the context of liability increased limits factors and Culp and O’Donnell
Define a layer \( y \) excess \( x \) by its payout function \( I_{(x,x+y)}(X) := (X - x)^+ \wedge y \). The expected layer loss is

\[
\mathbb{E}[I_{(x,x+y)}(X)] = \int_x^{x+y} (t - x) dF(t) + yS(x + y) \\
= \int_x^{x+y} tdF(t) + tS(t)|_x^{x+y} \\
= \int_x^{x+y} S(t) \, dt.
\]

Based on this equation, Wang (1996) points out that \( S \) can be interpreted as the layer loss (net premium) density. Specifically, \( S \) is the layer loss density in the sense that \( S(x) = d/dx(\mathbb{E}[I_{(0,x)}(X)]) \) is the marginal rate of increase in expected losses in the layer at \( x \). We use density in this sense to define premium, margin and equity densities, in addition to loss density.

Clearly \( S(x) \) equals the expected loss to the cover \( 1_{\{X > x\}} \). By scaling, \( S(x)dx \) is the close to the expected loss for \( I_{(x,x+dx)} \) when \( dx \) is very small; Bodoff (2007) calls these infinitesimal layers.

Wang (1996) goes on to interpret

\[
\int_x^{x+y} g(S(t)) \, dt
\]

as the layer premium and hence \( g(S(x)) \) as the layer premium density. We write \( P(x) := g(S(x)) \) for the premium density.

We can decompose \( X \) into a sum of thin layers. All these layers are comonotonic with one another and with \( X \), resulting in an additive decomposition of \( \rho(X) \), since \( \rho \) is comonotonic additive. The decomposition mirrors the definition of \( \rho \) as an integral, eq. (1).

The amount of assets \( a \) available to pay claims determines the quality of insurance, and premium and expected losses are functions of \( a \). Premiums are well-known to be sensitive to the insurer’s asset resources and solvency, Phillips, Cummins, and Allen (1998). Assets may be infinite, implying unlimited cover. When \( a \) is finite there is usually some chance of default. Using the layer density view, define expected loss \( \bar{S} \) and premium \( \bar{P} \) functions as

\[
\bar{S}(a) := \mathbb{E}[X \wedge a] = \int_0^a S(x) \, dx \\
\bar{P}(a) := \rho(X \wedge a) = \int_0^\infty g(S_{X \wedge a}(x)) \, dx = \int_0^a g(S_X(x)) \, dx.
\]

Margin is \( \bar{M}(a) := \bar{P}(a) - \bar{S}(a) \) and margin density is \( M(a) = d\bar{M}(a)/da \). Assets are funded by premium and equity \( \bar{Q}(a) := a - \bar{P}(a) \). Again \( Q(a) = d\bar{Q}/da = 1 - P(a) \). Together \( S, M, \) and \( Q \) give the split of layer funding between expected loss, margin and equity. Layers up to \( a \) are, by definition, fully collateralized. Thus \( \rho(X \wedge a) \) is the premium for a defaultable cover on \( X \) supported by assets \( a \), whereas \( \rho(X) \) is the premium for an unlimited, default-free cover.
The layer density view is consistent with more standard approaches to pricing. If $X$ is a Bernoulli risk with $\Pr(X = 1) = s$ and expected loss cost $s$, then $\rho(X) = g(s)$ can be regarded as pricing a unit width layer with attachment probability $s$. In an intermediated context, the funding constraint requires layers to be fully collateralized by premium plus equity—without such funding the insurance would not be credible since the insurer has no other source of funds.

Given $g$ we can compute insurance market statistics for each layer. The loss, premium, margin, and equity densities are $s$, $g(s)$, $g(s) - s$ and $1 - g(s)$. The layer loss ratio is $s/g(s)$ and $(g(s) - s)/(1 - g(s))$ is the layer return on equity. These quantities are illustrated in fig. 1 for a typical distortion function. The corresponding statistics for ground-up covers can be computed by integrating densities.

![Insurance Statistics](image)

Figure 1: Relationship between distortion $g$ and insurance market statistics as a function of exceedance probability $s$.

For an investor-written risk we regard the margin as compensation for ambiguity aversion and associated winner’s curse drag. Both of these effects are correlated with risk, so the margin is hard to distinguish from a risk load, but the rationale is different. Again, recall, although $\rho$ is non-additive and appears to charge for diversifiable risk, De Waegenaere, Kast, and Lapied (2003) assures us the pricing is consistent with a general equilibrium.

The layer density is distinct from models that vary the volume of each line in a homogeneous portfolio model. Our portfolio is static. By varying assets we are implicitly varying the quality of insurance.

### 3.2 The Equal Priority Default Rule

If assets are finite and the provider has limited liability we need to to determine policy-level cash flows in default states before we can determine the fair market value of insurance. The most common way to do this is using equal priority in default.
Under limited liability, total losses are split between provider payments and provider default as
\[ X = X \land a + (X - a)^+. \]
Next, actual payments \( X \land a \) must be allocated to each policy.
\( X_i \) is the amount promised to \( i \) by their insurance contract. Promises are limited by policy provisions but are not limited by provider assets. At the policy level, equal priority implies the payments made to, and default borne by, policy \( i \) are split as
\[ X_i = X_i \frac{X \land a}{X} + X_i \frac{(X - a)^+}{X} = (\text{payments to policy } i) + (\text{default borne by policy } i). \]
Therefore the payments made to policy \( i \) are given by
\[ X_i(a) := X_i \frac{X \land a}{X} = \begin{cases} X_i & X \leq a \\ X_i a_X & X > a. \end{cases} \]
\( X_i(a) \) is the amount actually paid to policy \( i \). It depends on \( a, X \) and \( X_i \). The dependence on \( X \) is critical. It is responsible for almost all the theoretical complexity of insurance pricing.
It is worth reiterating that with this definition \( \sum_i X_i(a) = X \land a \).

**Example.** Here is an example illustrating the effect of equal priority. Consider a certain loss \( X_0 = 1000 \) and \( X_1 \) given by a lognormal with mean 1000 and coefficient of variation 2.0. Prudence requires losses be backed by assets equal to the 0.9 quantile. On a stand-alone basis \( X_0 \) is backed by \( a_0 = 1000 \) and is risk-free. \( X_1 \) is backed by \( a_1 = 2272 \) and the recovery is subject to a considerable haircut, since \( \mathbb{E}[X_1 \land 2272] = 732.3 \). If these risks are pooled, the pool must hold \( a = a_0 + a_1 \) for the same level of prudence. When \( X_1 \leq a_1 \) both lines are paid in full. But when \( X_1 > a_1 \), \( X_0 \) receives \( 1000(a/(1000 + X_1)) \) and \( X_1 \) receives the remaining \( X_1(a/(1000 + X_1)) \). Payment to both lines is pro rated down by the same factor \( a/(1000 + X_1) \)—hence the name equal priority. In the pooled case, the expected recovery to \( X_0 \) is 967.5 and 764.8 to \( X_1 \). Pooling and equal priority result in a transfer of 32.5 from \( X_0 \) to \( X_1 \). This example shows what can occur when a thin tailed line pools with a thick tailed one under a weak capital standard with equal priority. We shall see how pricing compensates for these loss payment transfers, with \( X_1 \) paying a positive margin and \( X_0 \) a negative one.

### 3.3 Expected Loss Payments at Different Asset Levels

Expected losses paid to policy \( i \) are \( \tilde{S}_i(a) := \mathbb{E}[X_i(a)] \). \( \tilde{S}_i(a) \) can be computed, conditioning on \( X \), as
\[ \tilde{S}_i(a) = \mathbb{E}[\mathbb{E}[X_i(a) \mid X]] = \mathbb{E}[X_i \mid X \leq a] F(a) + a \mathbb{E}\left[ \frac{X_i}{X} \mid X > a \right] S(a). \]
Because of its importance in allocating losses, define
\[ \alpha_i(a) := \mathbb{E}[X_i/X \mid X > a]. \]
The value \( \alpha_i(x) \) is the expected proportion of recoveries by line \( i \) in the layer at \( x \). Since total assets available to pay losses always equals the layer width, and the chance the layer attaches is \( S(x) \), it is intuitively clear \( \alpha_i(x)S(x) \) is the loss density for line \( i \), that is, the derivative of \( \tilde{S}_i(x) \) with respect to \( x \). We now show this rigorously.

**Proposition 1.** Expected losses to policy \( i \) under equal priority, when total losses are supported by assets \( a \), is given by

\[
\tilde{S}_i(a) = \mathbb{E}[X_i(a)] = \int_0^a \alpha_i(x)S(x)dx
\]

and so the policy loss density at \( x \) is \( S_i(x) := \alpha_i(x)S(x) \).

**Proof.** By the definition of conditional expectation, \( \alpha_i(a)S(a) = \mathbb{E}[(X_i/X)1_{X>a}] \). Conditioning on \( X \), using the tower property, and taking out the functions of \( X \) on the right shows

\[
\alpha_i(a)S(a) = \mathbb{E}[(X_i/X)1_{X>a} | X] = \int_a^\infty \mathbb{E}[X_i | X = x] \frac{f(x)}{x} dx
\]

and therefore

\[
\frac{d}{da}(\alpha_i(a)S(a)) = -\mathbb{E}[X_i | X = a] \frac{f(a)}{a}.
\]

Now we can use integration by parts to compute

\[
\int_0^a \alpha_i(x)S(x) dx = x\alpha_i(x)S(x) \bigg|_0^a + \int_0^a x \mathbb{E}[X_i | X = x] \frac{f(x)}{x} dx
\]

\[
= aa\alpha_i(a)S(a) + \mathbb{E}[X_i | X \leq a] F(a)
\]

by eq. (5). Therefore the policy \( i \) loss density in the asset layer at \( a \), i.e. the derivative of eq. (5) with respect to \( a \), is \( S_i(a) = \alpha_i(a)S(a) \) as required. \( \Box \)

To recap, eq. (7) gives a direct analog to eq. (2) for policy \( i \) losses. Note that \( S_i \) is not the survival function of \( X_i(a) \) nor of \( X_i \). Equation (7) is surprising because it gives a decomposition of \( S \) through the convolution of random variables:

\[
X \xrightarrow{=} \sum_i X_i \quad X \wedge a \xrightarrow{=} \sum_i X_i(a)
\]

\[
\mathbb{E}[X] \xrightarrow{=} \sum_i \mathbb{E}[X_i] \quad \mathbb{E}[X \wedge a] \xrightarrow{=} \sum_i \mathbb{E}[X_i(a)]
\]

\[
\int S \xrightarrow{=} \sum_i \int S_{X_i} \quad \int_0^a S \xrightarrow{=} \sum_i \int_0^a \alpha_i S
\]

\[
S \xrightarrow{\neq} \sum S_{X_i}, \quad S \xrightarrow{=} \sum \alpha_i S.
\]
3.4 Premiums at Different Asset Levels

Premium under $\rho$ is given by eq. (3). We can interpret $g(S(a))$ as the portfolio premium density in the layer at $a$. We now consider the premium and premium density for each policy.

Using integration by parts we can express the price of an unlimited cover on $X$ as

$$
\rho(X) = \int_0^\infty g(S(x)) \, dx = \int_0^\infty xg'(S(x))f(x) \, dx = E[\rho(X)]
$$

Equation (9) makes sense because a concave distortion is continuous on $(0, 1]$ and can have at most countably infinitely many points where it is not differentiable (it has a kink). In total these points have measure zero, Borwein and Vanderwerff (2010), and we can ignore them in the integral. For more details see Dhaene et al. (2012).

By Equation (9), and the properties of a distortion function, $g'(S(X))$ is the Radon-Nikodym derivative of a measure $Q$ with $\rho(X) = E_Q[X]$. In fact, $E_Q[Y] = E[Yg'(S(X))]$ for all random variables $Y$. In general, any non-negative function $Z$ (measure $Q$) with $E_Z = 1$ and $\rho(X) = E[XZ]$ is called a contact function (subgradient) for $\rho$ at $X$, see Shapiro, Dentcheva, and Ruszczyński (2009). Thus $g'(S(X))$ is a contact function for $\rho$ at $X$. The name subgradient comes from the fact that $\rho(X + Y) \geq E_Q[X + Y] = \rho(X) + E_Q[Y]$, by theorem 1. The set of subgradients is called the subdifferential of $\rho$ at $X$. If there is a unique subgradient then $\rho$ is differentiable. Delbaen (2000) Theorem 17 shows that subgradients are contact functions.

We can interpret $g'(S(X))$ as a state price density specific to the $X$, suggesting that $E[X_i g'(S(X))]$ gives the value of the cash flows to policy $i$. This motivates the following definition.

**Definition 2.** For $X = \sum_i X_i$ with $Q \in Q$ so that $\rho(X) = E_Q[X]$, the natural allocation premium to policy $X_j$ as part of the portfolio $X$ is $E_Q[X_j]$. It is denoted $\rho_X(X_j)$.

The natural allocation premium is a standard approach, appearing in Delbaen (2000), Venter, Major, and Kreps (2006) and Tsanakas and Barnett (2003) for example. It has many desirable properties. Delbaen shows it is a fair allocation in the sense of fuzzy games and that it has a directional derivative, marginal interpretation when $\rho$ is differentiable. It is consistent with Jouini and Kallal (2001) and Campi, Jouini, and Porte (2013), which show the rational price of $X$ in a market with frictions must be computed by state prices that are anti-comonotonic $X$. In our application the signs are reversed: $g'(S(X))$ and $X$ are comonotonic.

The choice $g'(S(X))$ is economically meaningful because it weights the largest outcomes of $X$ the most, which is appropriate from a social, regulatory and investor perspective. It is also the only choice of weights that works for all levels of assets. Since investors stand ready to write any layer at the price determined by $g$, their solution must work for all $a$.

However, there are two technical issues with the proposed natural allocation. First, unlike prior works, we are allocating the premium for $X \wedge a$, not $X$, a problem also considered in Major (2018). And second, $Q$ may not be unique. In general, uniqueness fails at capped
variables like $X \wedge a$. Both issues are surmountable for a DRM, resulting in a unique, well defined natural allocation. For a non-comonotonic additive risk measure this is not the case.

It is helpful to define the premium, risk adjusted, analog of the $\alpha_i$ as

$$\beta_i(a) := \mathbb{E}_Q[(X_i/X) \mid X > a].$$  \hfill (10)

$\beta_i(x)$ is the value of the recoveries paid to line $i$ by a policy paying 1 in states $\{X > a\}$, i.e. an allocation of the premium for $1_{X>a}$. By the properties of conditional expectations, we have

$$\beta_i(a) = \frac{\mathbb{E}[(X_i/X)Z \mid X > a]}{\mathbb{E}[Z \mid X > a]}.$$ \hfill (11)

The denominator equals $Q(X > a)/P(X > a)$. Remember that while $\mathbb{E}_Q[X] = \mathbb{E}[XZ]$, for conditional expectations $\mathbb{E}_Q[X \mid F] = \mathbb{E}[XZ \mid F]/\mathbb{E}[Z \mid F]$, see Föllmer and Schied (2011), Proposition A.12.

To compute $\alpha_i$ and $\beta_i$ we use a third function,

$$\kappa_i(x) := \mathbb{E}[X_i \mid X = x],$$ \hfill (12)

the conditional expectation of loss by policy, given the total loss. It is an important fact that the risk adjusted version of $\kappa$ is unchanged because DRMs are law invariant. With these preliminaries we can state the main theorem of this section.

**Theorem 2.** Let $Q \in \mathbb{Q}$ be the measure with Radon-Nikodym derivative $Z = g'(S_X(X))$.

1. $\mathbb{E}[X_i \mid X = x] = \mathbb{E}_Q[X_i \mid X = x]$.

2. $\beta_i$ can be computed from $\kappa_i$ as

$$\beta_i(a) = \frac{1}{Q(X > a)} \int_a^\infty \frac{\kappa_i(x)}{x} g'(S(x)) f(x) \, dx.$$ \hfill (13)

3. The natural allocation premium for policy $i$ under equal priority when total losses are supported by assets $a$, $\bar{P}_i(a) := \rho_{X\wedge a}(X_i(a))$, is given by

$$\bar{P}_i(a) = \mathbb{E}_Q[X_i \mid X \leq a](1 - g(S(a))) + a\mathbb{E}_Q[X_i/X \mid X > a]g(S(a)) \hfill (14)$$

$$= \mathbb{E}[X_i Z \mid X \leq a](1 - S(a)) + a\mathbb{E}[(X_i/X)Z \mid X > a]S(a).$$ \hfill (15)

4. The policy $i$ premium density is

$$P_i(a) = \beta_i(a)g(S(a)).$$ \hfill (16)

The Theorem offers two contributions. First, it shows we can replace $\mathbb{E}_Q[X_i \mid X]$ with $\mathbb{E}[X_i \mid X]$, which enables explicit calculation. There is no risk adjusted version of $\kappa_i$. Intuitively, a law invariant risk measure cannot change probabilities within an event defined by $X$: if it did then it would be distinguishing between events on information other than $S(X)$ whereas law invariance says this is all that can matter. And second, it identifies the
Proof. Part (1) follows in the same way as eq. (11).

Since $\rho$ is comonotonic additive $\rho(X) = \rho(X \land a) + \rho((X - a)⁺)$ and hence $\rho(X) = E_Q[X \land a] + E_Q[(X - a)⁺] \leq \rho(X \land a) + \rho((X - a)⁺) = \rho(X)$. But since $E_Q[X \land a] \leq \rho(X \land a)$ and $E_Q[(X - a)⁺] \leq \rho((X - a)⁺)$ it follows $E_Q[X \land a] = \rho(X \land a)$ and $E_Q[(X - a)⁺] = \rho((X - a)⁺)$. Therefore the contact functions for $X$ and $X \land a$ are the same and it is legitimate to assume $Z = g'(S(X))$ when allocating premium for $X \land a$.

To prove Part (2), note that by eq. (11) $\beta_i(a)g(S(a)) = E_Q[(X_i/X)1_{X>a}]$. Conditioning on $X$, using the tower property, and taking out the known functions of $X$ on the right, shows

$$
\beta_i(a)g(S(a)) = E[E[(X_i/X)g'(S(X))1_{X>a} \mid X]]
= E[E[(X_i \mid X/X)g'(S(X))1_{X>a}]]
= \int_a^\infty \frac{E[X_i \mid X = x]}{x}g'(S(x))f(x) \, dx.
$$

It follows from the definition of $X_i(a)$, eq. (4), and the fact $Z$ is a contact function for $X \land a$ that

$$
\bar{P}_i(a) = E[X_i(a)g'(S(X))]
= E[X_i g'(S(X))1_{X \leq a}] + E[a(X_i/X)g'(S(X))1_{X>a}]
= E[X_i g'(S(X)) \mid X \leq a](1 - g(S(a)) + \rho(aE[(X_i/X)g'(S(X)) \mid X > a]S(a))
= E_Q[X_i \mid X \leq a](1 - g(S(a)) + \rho(aE_Q[(X_i/X) \mid X > a]g(S(a))
$$

giving Part (3).

Rearranging eq. (13) and differentiating gives

$$
\frac{d}{da}(\beta_i(a)g(S(a))) = -\frac{E[X_i \mid X = a]}{a}g'(S(a))f(a).
$$

Now use integration by parts to compute

$$
\int_0^a \beta_i(x)g(S(x)) \, dx = x\beta_i(x)g(S(x))\bigg|_0^a + \int_0^a \frac{E[X_i \mid X = x]}{x}g'(S(x))f(x) \, dx
= a\beta_i(a)g(S(a)) + E_Q[X_i \mid X \leq a](1 - g(S(a))
= \bar{P}_i(a)
$$
by eq. (14). As a result, the policy \(i\) premium density in the asset layer at \(a\), i.e. the derivative of \(\bar{P}_i(a)\) with respect to \(a\), is \(P_i(a) = \beta_i(a)g(S(a))\), giving Part (4).

The proof writes the price of a limited liability cover as the price of default-free protection minus the value of the default put. This is the standard starting point for allocation in a perfect competitive market taken by Phillips, Cummins, and Allen (1998), Myers and Read Jr. (2001), Sherris (2006), and Ibragimov, Jaffee, and Walden (2010). They then allocate the default put rather than the value of insurance payments directly.

The problem that can occur when \(Q\) is not unique, but that can be circumvented when \(\rho\) is a DRM, can be illustrated as follows. Suppose \(\rho\) is given by \(p\text{-}TVaR\). The measure \(Q\) weights the worst \(1 - p\) proportion of outcomes of \(X\) by a factor of \((1 - p)^{-1}\) and ignores the others. Suppose \(a\) is chosen as \(p\text{-}VaR\) for a lower threshold \(p < p\). Let \(X_a = X \wedge a\) be capped insured losses and \(C = \{X_a = a\}\). By definition \(Pr(C) \geq 1 - p' > 1 - p\). Pick any \(A \subset C\) of measure \(1 - p\) so that \(\rho(X) = E[X | A]\). Let \(\psi\) be a measure preserving transformation of \(\Omega\) that acts non-trivially on \(C\) but trivially off \(C\). Then \(Q' = Q\psi\) will satisfy \(E_{Q'}[X_a] = E_Q[X_a\psi^{-1}] = \rho(X_a)\) but in general \(E_{Q'}[X] < \rho(X)\). The natural allocation with respect to \(Q'\) will be different from that for \(Q\). The theorem isolates a specific \(Q\) to obtain a unique answer. The same idea applies to \(Q\) from other, non- \(TVaR\), \(\rho\): you can always shuffle part of the contact function within \(C\) to generate non-unique allocations. See section 7.1 for an example.

To recap: the premium formulas eqs. (14) and (16) have been derived assuming capital is provided at a cost \(g\) and there is equal priority by line. They are computationally tractable and require no other assumptions. There is no need to assume the \(X_i\) are independent. They produce an entirely general, canonical determination of premium in the presence of shared costly capital. This result extends Grundl and Schmeiser (2007), who pointed out that with an additive pricing functional there is no need to allocate capital in order to price, to the situation of a non-additive DRM pricing functional.

The key formulas we have derived are summarized in table 1.

4 Properties of Alpha, Beta, and Kappa

In this section we explore properties of \(\alpha_i\), \(\beta_i\), and \(\kappa_i\), see eq. (6), eq. (10), and eq. (12), and show how they interact to determine premiums by line via the natural allocation.

For a measurable \(h\), \(E[X_i h(X)] = E[\kappa_i(X)h(X)]\) by the tower property. This simple observation results in huge simplifications. In general, \(E[X_i h(X)]\) requires knowing the full bivariate distribution of \(X_i\) and \(X\). Using \(\kappa_i\) reduces it to a one dimensional problem. This is true even if the \(X_i\) are correlated. The \(\kappa_i\) functions can be estimated from data using regression and they provide an alternative way to model correlations.

Despite their central role, the \(\kappa_i\) functions are probably unfamiliar so we begin by giving
| Quantity     | Loss                                                                 | Premium               |
|--------------|----------------------------------------------------------------------|-----------------------|
| Cash flow    | \( X_i(a) = X_i \frac{X \wedge a}{X} \)                            | \( N/a \)             |
| Measure      | Objective, \( S(x), f(x) \)                                         | Risk adjusted, \( Q, g(S(x)), g'(S(x))f(x) \) |
| Expectation  | \( \tilde{S}_i(a) = E[X_i(a)] \)                                    | \( \tilde{P}_i(a) = E_Q[X_i(a)] = E[X_i(a)g'(S(X))] \) |
| Conditioning | \( E[X_i \mid X \leq a]F(a) + aE[X_i/X \mid X > a]S(a) \)             | \( E_Q[X_i \mid X \leq a](1 - g(S(a))) + aE_Q[X_i/X \mid X > a]g(S(a)) \) |
| Share function | \( \alpha_i(x) = E[X_i/X \mid X > x] \)                               | \( \beta_i(x) = E_Q[X_i/X \mid X > x] \) |
| Derivative of share function | \( (\alpha_i S)'(x) = -E[X_i \mid X = x]f(x)/x = -\kappa_i(x)f(x)/x \) | \( (\beta_i g(S))'(x) = -E[X_i \mid X = x]g'(S(x))f(x)/x = -\kappa_i(x)g'(S(x))f(x)/x \) |
| Lee integral expectation | \( \int_0^a \alpha_i(x)S(x) \, dx \)                               | \( \int_0^a \beta_i(x)g(S(x)) \, dx \) |
| Outcome integral expectation | \( \int_0^a \kappa_i(x)f(x) \, dx + a\alpha_i(a)S(a) \)             | \( \int_0^a \kappa_i(x)g'(S(x))f(x) \, dx + a\beta_i(a)g(S(a)) \) |
| Scenario integral expectation | \( \int_{F(a)}^{F(a)} \kappa_i(q(p)) \, dp + a\alpha_i(a)S(a) \)    | \( \int_0^{1-g(S(a))} \kappa_i(q(1 - g^{-1}(1 - p))) \, dp + a\beta_i(a)g(S(a)) \) |

Table 1: Different ways of computing expected losses and the natural allocation.
several examples to illustrate how they behave. In general, they are non-linear and usually, but not always, increasing.

### 4.1 Examples of $\kappa$ functions

1. If $Y_i$ are independent and identically distributed and $X_n = Y_1 + \cdots + Y_n$ then $E[X_{m+n} \mid X_{m+n} = x] = mx/(m+n)$ for $m \geq 1, n \geq 0$. This is obvious when $m = 1$ because the functions $E[Y_i \mid X_n]$ are independent across $i = 1, \ldots, n$ and sum to $x$. The result follows because conditional expectations are linear. In this case $\kappa_i(x) = mx/(m+n)$ is a line through the origin.

2. If $X_i$ are multivariate normal then $\kappa_i$ are straight lines, given by the usual least-squares fits

   $$\kappa_i(x) = E[X_i] + \frac{\text{cov}(X_i, X)}{\text{var}(X)}(x - E[X]).$$

   This example is familiar from the securities market line and the CAPM analysis of stock returns. If $X_i$ are iid it reduces to the previous example because the slope is $1/n$.

3. If $X_i, i = 1, 2,$ are compound Poisson with the same severity distribution then $\kappa_i$ are again lines through the origin. Suppose $X_i$ has expected claim count $\lambda_i$. Write the conditional expectation as an integral, expand the density of the compound Poisson by conditioning on the claim count, and then swap the sum and integral to see that $\kappa_1(x) = E[X_1 \mid X_1+X_2 = x] = x E[N(\lambda_1)/(N(\lambda_1)+N(\lambda_2))]$ where $N(\lambda)$ are independent Poisson with mean $\lambda$. This example generalizes the iid case. Further conditioning on a common mixing variable extends the result to mixed Poisson frequencies where each aggregate can have a separate or shared mixing distribution. The common severity is essential. The result means that if a line of business is defined to be a group of policies that shares the same severity distribution, then premiums for policies within the line will have rates proportional to their expected claim counts.

4. A theorem of Efron says that if $X_i$ are independent and have log-concave densities then all $\kappa_i$ are non-decreasing, Saumard and Wellner (2014). The multivariate normal example is a special case of Efron’s theorem.

Denuit and Dhaene (2012) define an ex post risk sharing rule called the conditional mean risk allocation by taking $\kappa_i(x)$ to be the allocation to policy $i$ when $X = x$. A series of recent papers, see Denuit and Robert (2020) and references therein, considers the properties of the conditional mean risk allocation focusing on its use in peer-to-peer insurance and the case when $\kappa_i(x)$ is linear in $x$.

### 4.2 The Behavior of $\alpha_i, \beta_i$

By definition $\alpha_i(x)$ is the expected proportion of losses from policy $i$ in $1_{\{X>x\}}$ and $\beta_i(x)$ is the risk adjusted proportion. They are average proportions not proportions of the averages:
\( \alpha_i(x) = \mathbb{E}[X_i / X \mid X > x] \neq \mathbb{E}[X_i \mid X > x] / \mathbb{E}[X \mid X > x] \) because of Jensen’s inequality applied to the convex function \( x \mapsto 1/x \).

To better understand the shape of \( \alpha_i \) and \( \beta_i \) we can compute their derivatives. Differentiating \( \alpha_i(x)S(x) = \mathbb{E}[(X_i/X)1_{X>x}] \) and re-arranging gives

\[
\alpha'_i(x) = \left( \alpha_i(x) - \frac{\kappa_i(x)}{x} \right) \frac{f(x)}{S(x)}. \tag{17}
\]

The results for \( \beta_i \) are analogous. The function \( h(x) := f(x)/S(x) \) is called the hazard rate. If \( X \) models a lifetime, \( h \) is called the force of mortality. For thick right-skewed distributions \( h \) is typically an eventually decreasing function. For thin tailed distributions it is typically an eventually increasing function. It is constant for the exponential distribution.

The action of \( g \) is to make the right tail thicker and so to decrease the hazard rate. Since \( h(x) = -d/dx(\log(S(x))) \) it follows that

\[
S(x) = \exp \left( - \int_t^\infty h(s)ds \right).
\]

The integral is called the cumulative hazard function. From this formulation it is clear the proportional hazard \( g(s) = s^r, 0 < r \leq 1 \), acts on the hazard function as multiplication by \( r \), hence justifying its name.

Equation (17) shows that \( \alpha'_i(x) = 0 \) if \( f(x) = 0 \) and \( S(x) \) close to 1, which will occur in the extreme left tail when \( X \) includes some level of near certain losses. Then \( \alpha_i \) will be flat for small \( x \), while \( f(x) \approx 0 \). Flat behavior can also occur if \( \alpha_i(x) - \kappa_i(x)/x = 0 \), but that is an exceptional circumstance.

For thick tailed insurance distributions \( h(x) \) is eventually decreasing but remains strictly positive. If \( \kappa_i(x)/x \) is decreasing then \( \alpha'_i(x) < 0 \) because \( \alpha_i(x) \) is the probability weighted integral of \( \kappa_i(t)/t \) over \( t > x \), and so \( \alpha_i(x) < \kappa_i(x)/x \). Conversely if \( \kappa_i(x)/x \) is increasing \( \alpha'_i(x) > 0 \).

Since \( \sum_i \kappa_i(x) = x \) it follows that \( \sum_i \kappa'_i(x) = 1 \). It is typical for the thickest tail distribution, \( i \) say, to behave like \( \kappa_i(x) \approx x - \sum_{j \neq i} \mathbb{E}[X_j] \) for large \( x \). Then \( \kappa'_i(x) = 1 \) and the remaining \( \kappa_j(x) \approx \mathbb{E}[X_j] \) are almost constant for large \( x \). In that case \( \kappa_j(x)/x > \alpha_j(x) \) and so \( \alpha'_j(x) < 0 \) and \( \alpha'_i(x) > 0 \). To have two policies with \( \alpha_i \) increasing requires a very delicate balancing of the thickness of their tails with \( \kappa_i(x) \), growing with order \( x \). A compound Poisson with the same severity is an example.

## 5 Properties of the Natural Allocation

### 5.1 Aggregate Properties

We now explore margin, equity, and return in total and by policy. We begin by considering them in total.
By definition the average return with assets \( a \) is

\[
\bar{i}(a) := \frac{\bar{M}(a)}{\bar{Q}(a)}
\]  

(18)

where margin \( \bar{M} \) and equity \( \bar{Q} \) are defined in the paragraph following eq. (3).

Equation (18) has important implications. It tells us the investor priced expected return varies with the level of assets. For most distortions return decreases with increasing capital. In contrast, the standard RAROC models use a fixed average cost of capital, regardless of the overall asset level, Tasche (1999). CAPM or the Fama-French three factor model are often used to estimate the average return, with a typical range of 7 to 20 percent, Cummins and Phillips (2005). A common question of working actuaries performing capital allocation is about so-called excess capital, if the balance sheet contains more capital than is required by regulators, rating agencies, or managerial prudence. Our model suggests that higher layers of capital are cheaper, but not free, addressing this concern.

The varying returns in eq. (18) may seem inconsistent with Miller Modigliani. But that says the cost of funding a given amount of capital is independent of how it is split between debt and equity; it does not say the average cost is constant as the amount of capital varies.

5.2 No-Undercut and Positive Margin for Independent Risks

The natural allocation has two desirable properties. It is always less than the stand-alone premium, meaning it satisfies the no-undercut condition of Denault (2001), and it produces non-negative margins for independent risks.

**Proposition 2.** Let \(X = \sum^n_{i=1} X_i \), \(X_i \) non-negative and independent, and let \(g\) be a distortion. Then (1) the natural allocation is never greater than the stand-alone premium and (2) the natural allocation to every \(X_i\) contains a non-negative margin.

**Proof.** It is enough to prove for \(n = 2\) by considering \(X_1\) and \(X_2' = X_2 + \cdots + X_n\).

By theorem 1 we know that \(\rho(X) = \mathbb{E}_Q[X]\) where \(Q\) has Radon-Nikodym derivative \(g'(S_X(X))\).

By definition, the natural allocation is \(\bar{P}_1 = \mathbb{E}[X_1g'(S_X(X))].\) Therefore,

\[
\bar{P}_1 = \mathbb{E}[X_1g'(S_X(X))] \leq \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[X] = \rho(X_1)
\]

which shows Part (1), that the natural allocation is never greater than the stand-alone premium.

Let \(\tilde{X}_1 = X_1 + \mathbb{E}[X_2]\) and \(\tilde{X}_2 = X_2 - \mathbb{E}[X_2]\). Then by Rothschild and Stiglitz (1970) or Machina and Pratt (1997) \(\tilde{X}_1 + \tilde{X}_2 \succeq_2 \tilde{X}_1\), where \(\succeq_2\) denotes second order stochastic dominance. Bäuerle and Müller (2006) shows that DRMs respect second order stochastic dominance. Therefore

\[
\rho(\tilde{X}_1 + \tilde{X}_2) \geq \rho(\tilde{X}_1).
\]
By translation invariance $\rho(\tilde{X}_1) = \rho(X_1) + \mathbb{E}[X_2]$. Since $\tilde{X}_1 + \tilde{X}_2 = X_1 + X_2$ we conclude
\[
\rho(X_1 + X_2) \geq \rho(X_1) + \mathbb{E}[X_2].
\]
Combining these results we get
\[
\bar{P}_1 + \bar{P}_2 = \rho(X_1 + X_2) \geq \rho(X_1) + \mathbb{E}[X_2]
\implies \bar{P}_2 \geq \rho(X_1) - \bar{P}_1 + \mathbb{E}[X_2]
\]
as required for Part (2).

Part (1) is well known. The proof if Part (2) leverages the fact $\rho$ is translation invariant. If we add $X_2 = c$ to $X_1$ then its natural allocation is $c$. In a sense, this is the best case. Any non-constant independent variable, no matter how low its variance, must slightly increase risk. It does not make sense to grant the new variable a credit off expected loss, when we would not do so for a constant. A credit is possible for dependent variables, however.

Since $\bar{P}_i = \mathbb{E}[\kappa_i(X)g'(S(X))]$ we see the no-undercut condition holds if $\kappa_i(X)$ and $g'(S(X))$ are comonotonic, and hence if $\kappa_i$ is increasing, or if $\kappa_i(X)$ and $X$ are positively correlated (recall $\mathbb{E}[g'(S(X))] = 1$). Since $\sum \kappa_i(x) = x$ at least one $\kappa_i$, say $\kappa_i^*$, must be increasing. Policy $i^*$ is the capacity consuming line that will always have a positive margin. In this way $\kappa$ differentiates relative tail thickness.

### 5.3 Policy Level Properties, Varying with Asset Level

We start with a corollary of the results in section 3 which gives a nicely symmetric and computationally tractable expression for the natural margin allocation in the case of finite assets.

**Corollary 1.** The margin density for line $i$ at asset level $a$ is given by
\[
M_i(a) = \beta_i(a)g(S(a)) - \alpha_i(a)S(a). \tag{19}
\]

**Proof.** Using eqs. (7) and (16) we can compute margin $\bar{M}_i(a)$ in $\bar{P}_i(a)$ by line as
\[
\bar{M}_i(a) = \bar{P}_i(a) - \bar{L}_i(a)
= \int_0^a \beta_i(x)g(S(x)) - \alpha_i(x)S(x) \, dx. \tag{20}
\]

Differentiating we get the margin density for line $i$ at $a$ expressed in terms of $\alpha_i$ and $\beta_i$ as shown.

Margin in the current context is the cost of capital, thus eq. (19) is an important result. It allows us to compute economic value by line and to assess static portfolio performance by line—one of the motivations for performing capital allocation in the first place. In many ways it is also a good place to stop. Remember these results only assume we are using a distortion
risk measure and have equal priority in default. We are in a static model, so questions of portfolio homogeneity are irrelevant. We are not assuming $X_i$ are independent.

What does eq. (19) say about by margins by line? Since $g$ is increasing and concave $P(a) = g(S(a)) \geq S(a)$ for all $a \geq 0$. Thus all asset layers contain a non-negative total margin density. It is a different situation by line, where we can see

$$M_i(a) \geq 0 \iff \beta_i(a)g(S(a)) - \alpha_i(a)S(a) \geq 0 \iff \frac{\beta_i(a)}{\alpha_i(a)} \geq \frac{S(a)}{g(S(a))}.$$ 

The line layer margin density is positive when $\beta_i/\alpha_i$ is greater than the all-lines layer loss ratio. Since the loss ratio is $\leq 1$ there must be a positive layer margin density whenever $\beta_i(a)/\alpha_i(a) > 1$. But when $\beta_i(a)/\alpha_i(a) < 1$ it is possible the line has a negative margin density. How can that occur and why does it make sense? To explore this we look at the shape of $\alpha$ and $\beta$ in more detail.

It is important to remember why proposition 2 does not apply: it assumes unlimited cover, whereas here $a < \infty$. With finite capital there are potential transfers between lines caused by their behavior in default that overwhelm the positive margin implied by the proposition. Also note the proposition cannot be applied to $X \land a = \sum_i X_i(a)$ because the line payments are no longer independent.

In general we can make two predictions about margins.

**Prediction 1:** Lines where $\alpha_i(x)$ or $\kappa_i(x)/x$ increase with $x$ will have always have a positive margin.

**Prediction 2:** A log-concave (thin tailed) line aggregated with a non-log-concave (thick tailed) line can have a negative margin, especially for lower asset layers.

Prediction 1 follows because the risk adjustment puts more weight on $X_i/X$ for larger $X$ and so $\beta_i(x)/\alpha_i(x) > 1 > S(x)/g(S(x))$. Recall the risk adjustment is comonotonic with total losses $X$.

A thin tailed line aggregated with thick tailed lines will have $\alpha_i(x)$ decreasing with $x$. Now the risk adjustment will produce $\beta_i(x) < \alpha_i(x)$ and it is possible that $\beta_i(x)/\alpha_i(x) < S(x)/g(S(x))$. In most cases, $\alpha_i(x)$ approaches $E[X_i]/x$ and $\beta_i(x)/\alpha_i(x)$ increases with $x$, while the layer loss ratio decreases—and margin increases—and the thin line will eventually get a positive margin. Whether or not the thin line has a positive total margin $M_i(a) > 0$ depends on the particulars of the lines and the level of assets $a$. A negative margin is more likely for less well capitalized insurers, which makes sense because default states are more material and they have a lower overall dollar cost of capital. In the independent case, as $a \to \infty$ proposition 2 eventually guarantees positive margins for all lines.

These results are reasonable. Under limited liability, if assets and liabilities are pooled then the thick tailed line benefits from pooling with the thin one because pooling increases the assets available to pay losses when needed. Equal priority transfers wealth from thin to thick in states of the world where thick has a bad event, c.f., the example in section 3.2. But because thick dominates the total, the total losses are bad when thick is bad. The negative margin compensates the thin-tailed line for transfers.
Another interesting situation occurs for asset levels within attritional loss layers. Most realistic insured loss portfolios are quite skewed and never experience very low loss ratios. For low loss layers, $S(x)$ is close to 1 and the layer at $x$ is funded almost entirely by expected losses; the margin and equity density components are nearly zero. Since the sum of margin densities over component lines equals the total margin density, when the total is zero it necessarily follows that either all line margins are also zero or that some are positive and some are negative. For the reasons noted above, thin tailed lines get the negative margin as thick tailed lines compensate them for the improved cover the thick tail lines obtain by pooling.

In conclusion, the natural margin by line reflects the relative consumption of assets by layer, Mango (2005). Low layers are less ambiguous to the provider and have a lower margin relative to expected loss. Higher layers are more ambiguous and have lower loss ratios. High risk lines consume more higher layer assets and hence have a lower loss ratio. For independent lines with no default the margin is always positive. But there is a confounding effect when default is possible. Because more volatile lines are more likely to cause default, there is a wealth transfer to them. The natural premium allocation compensates low risk policies for this transfer, which can result in negative margins in some cases.

6 Equity Allocation by Policy

Although eq. (19) determines margin by line, we cannot compute return by line, or allocate frictional costs of capital, because we still lack an equity allocation, a problem we now address.

6.1 The Natural Allocation of Equity

Definition 3. The natural allocation of equity to line $i$ is given by

$$Q_i(a) = \frac{\beta_i(a)g(S(a)) - \alpha_i(x)S(a)}{g(S(a)) - S(a)} \times (1 - g(S(a))).$$

Why is this allocation natural? In total the layer return at $a$ is

$$\iota(a) := \frac{M(a)}{Q(a)} = \frac{P(a) - S(a)}{1 - P(a)} = \frac{g(S(a)) - S(a)}{1 - g(S(a))}.$$ 

We claim that for a law invariant pricing measure the layer return must be the same for all lines. Law invariance implies the risk measure is only concerned with the attachment probability of the layer at $a$, and not with the cause of loss within the layer. If return within a layer varied by line then the risk measure could not be law invariant.

We can now compute capital by layer by line, by solving for the unknown equity density $Q_i(a)$ via

$$\iota(a) = \frac{M(a)}{Q(a)} = \frac{M_i(a)}{Q_i(a)} \Rightarrow Q_i(a) = \frac{M_i(a)}{\iota(a)}.$$
Substituting for layer return and line margin gives eq. (21).

Since \(1 - g(S(a))\) is the proportion of capital in the layer at \(a\), eq. (21) says the allocation to line \(i\) is given by the nicely symmetric expression

\[
\frac{\beta_i(a)g(S(a)) - \alpha_i(x)S(a)}{g(S(a)) - S(a)}.
\]

(22)

To determine total capital by line we integrate the equity density

\[
\bar{Q}_i(a) := \int_0^a Q_i(x)dx.
\]

And finally we can determine the average return to line \(i\) at asset level \(a\)

\[
\bar{\iota}_i(a) = \frac{\bar{M}_i(a)}{\bar{Q}_i(a)}.
\]

(23)

The average return will generally vary by line and by asset level \(a\). Although the return within each layer is the same for all lines, the margin, the proportion of capital, and the proportion attributable to each line all vary by \(a\). Therefore average returns will vary by line and \(a\). This is in stark contrast to the standard industry approach, which uses the same return for each line and implicitly all \(a\). How these quantities vary by line is complicated. Academic approaches emphasized the possibility that returns vary by line, but struggled with parameterization, Myers and Cohn (1987).

Equation (23) shows the average return by line is an \(M_i\)-weighted harmonic mean of the layer returns given by the distortion \(g\), viz

\[
\frac{1}{\bar{\iota}_i(a)} = \int_0^a \frac{1}{\iota(x)M_i(a)} dx.
\]

The harmonic mean solves the problem that the return for lower layers of assets is potentially infinite (when \(g'(1) = 0\)). The infinities do not matter: at lower asset layers there is little or no equity and the layer is fully funded by the loss component of premium. When so funded, there is no margin and so the infinite return gets zero weight. In this instance, the sense of the problem dictates that \(0 \times \infty = 0\): with no initial capital there is no final capital regardless of the return.

6.2 Intermediated Pricing

An equity allocation to policy is needed to price intermediated insurance because of the frictional costs of holding capital in an insurance company.

The price of investor-written insurance is \(\rho(X)\). A cat bond transaction is an example of investor-written insurance. Following Myers and Read Jr. (2001) and Ibragimov, Jaffee, and
Walden (2010) we model frictional costs as a tax on equity at rate $\delta$. The density and limited price of intermediated insurance becomes

$$P^I_i(a) = P_i(a) + \delta Q_i(a)$$  \hspace{1cm} (24)

$$\bar{P}^I_i(a) = \bar{P}_i(a) + \delta \bar{Q}_i(a).$$  \hspace{1cm} (25)

The relative size of $M_i$ and $\delta Q_i$ is a topic for future research.

7 Examples

7.1 Example 1: A Simple Discrete Example

Consider a two line example where $X_1$ takes values 0, 9 and 10, and $X_2$ values 0, 1 and 90, the lines are independent and $X = X_1 + X_2$. Suppose the outcome probabilities are $1/2, 1/4,$ and $1/4$ respectively for each outcome and consider a risk measure given by the proportional hazard transform $g(s) = \sqrt{s}$. There are nine possible outcomes shown in table 2. The natural allocation appears to depend on the ordering of the two outcomes where $X = 10$. If these two rows are swapped the allocations are different, as shown in the last two rows of the table.

Table 2: Nine possible outcomes showing ambiguous ordering for $X = 10$. The natural allocation $E_Q[X_i]$ appears to depend on the ordering of outcomes 4 and 5. The next to last row shows $E_Q$ with these rows swapped.

| Outcome | $X_1$ | $X_2$ | $X$ | $P$ | $S(x)$ | $g(S)$ | $Q$ | $Z$ | $E[Z | X]$ | $\bar{Q}$ |
|---------|------|------|-----|-----|--------|--------|-----|-----|------------|--------|
| 1       | 0    | 0    | 0   | 4/16| 12/16  | 0.8660254| 0.1339746| 0.5358984| 0.5358984 | 0.1339746 |
| 2       | 0    | 1    | 1   | 2/16| 10/16  | 0.7905694| 0.0754599| 0.6036479| 0.6036479 | 0.0754599 |
| 3       | 9    | 0    | 9   | 2/16| 8/16   | 0.7071068| 0.08346263| 0.6670111| 0.6670111 | 0.08346263 |
| 4       | 9    | 1    | 10  | 1/16| 7/16   | 0.6614378| 0.0456685| 0.7307033| 0.7307033 | 0.04936326 |
| 5       | 10   | 0    | 10  | 2/16| 5/16   | 0.5590170| 0.1024208| 0.8193667| 0.8193667 | 0.09872652 |
| 6       | 10   | 1    | 11  | 1/16| 4/16   | 0.5   | 0.05901699| 0.9442719| 0.9442719 | 0.05901699 |
| 7       | 0    | 90   | 90  | 2/16| 2/16   | 0.3535534| 0.1464466| 1.171573| 1.171573 | 0.1461466 |
| 8       | 9    | 90   | 99  | 1/16| 1/16   | 0.25  | 0.1035534| 1.656854| 1.656854 | 0.1035534 |
| 9       | 10   | 90   | 100 | 1/16| 0      | 0     | 0.25  | 4    | 4     | 0.25    |

The last three columns of the table compute the measure $\bar{Q}$ corresponding to the unique $Q | X$. The $\bar{Q}$-expected values of $X_1$ and $X_2$ are 6.2048488 and 45.183836, respectively. Note these values for the natural allocation are different from the average of the the two orderings of rows 4 and 5.

Table 3 replaces $X_i$ with $\kappa_i(x) = E[X_i | X = x]$, resulting in one row per value of $X$, and uses theorem 2 to compute expectations. The results are the same as using $\bar{Q}$.
Table 3: Combining outcomes 4 and 5 and working with $E[X_i | X]$ resolves the ambiguity and produces the natural allocation.

| Outcome | $E[X_1 | X]$ | $E[X_2 | X]$ | $X$ | $P$ | $S(x)$ | $g(S)$ | $Q$ |
|---------|--------------|--------------|-----|-----|--------|--------|-----|
| 1       | 0            | 0            | 0   | 4/16| 12/16  | 0.8660254 | 0.1339746 |
| 2       | 0            | 1            | 1   | 2/16| 10/16  | 0.7905694 | 0.0754559 |
| 3       | 9            | 0            | 9   | 2/16| 8/16   | 0.7071068 | 0.0834626 |
| 4, 5    | 9            | 2/3          | 1/3 | 10  | 3/16   | 0.5590170 | 0.1480898 |
| 6       | 10           | 1            | 11  | 1/16| 4/16   | 0.5      | 0.0590169 |
| 7       | 0            | 90           | 90  | 2/16| 2/16   | 0.3535534 | 0.1464466 |
| 8       | 9            | 90           | 99  | 1/16| 1/16   | 0.25     | 0.1035534 |
| 9       | 10           | 90           | 100 | 1/16| 0      | 0        | 0.25  |

$E$ | $4.75$ | $22.75$ | $27.5$ |
$E_Q$ | $6.2048488$ | $45.183836$ | $51.38869$ |

Actuaries commonly perform this type of calculation, often with catastrophe model output. They make the simplifying assumption that $X_i = E[X_i | X]$ when all rows are distinct. However, the ordering problem illustrated does occur in real data, especially when limits and retentions are involved. Theorem 2 shows how to rigorously resolve the ordering problem to compute the unique natural allocation.

### 7.2 Example 2: Thick-tailed and Thin-tailed Lines

Example 2 is based on two distributions with mean 1. $X_1$ is thin tailed with a gamma distribution with coefficient of variation 0.25. $X_2$ is a translated lognormal $X_2 = 0.3 + 0.7X'_2$, where $X'_2$ has a coefficient of variation $1.25/0.7$, resulting in a coefficient of variation of 1.25 for $X_2$. Total assets are 12.5, corresponding to capital at a 563 year return period. The aggregate coefficient of variation is 0.637 in total. $X_1$ approximates a moderate limit book of commercial auto and $X_2$ a catastrophe exposed property book with a stable attritional loss component. In aggregate the portfolio would be considered as volatile. The distortion $g$ uses a Wang transform with $\lambda = 0.755$, producing a 10 percent return on assets. The natural allocation premium is 1.057 for $X_1$ and 1.889 for line (94.6 percent and 52.4 percent loss ratios), producing an overall 67.6 percent loss ratio, all without expenses. The profit is realistic for a gross portfolio with these characteristics.

Figure 2 illustrates the theory we have developed. We refer to the charts as $(r, c)$ for row $r = 1, 2, 3, 4$ and column $c = 1, 2, 3$, starting at the top left. The horizontal axis shows the asset level in all charts except $(3, 3)$ and $(4, 3)$, where it shows probability, and $(1, 3)$ where it shows loss. Blue represents the thin tailed line, orange thick tailed and green total. When both dashed and solid lines appear on the same plot, the solid represent risk-adjusted and dashed represent non-risk-adjusted functions. Here is the key.

- $(1, 1)$ shows density for $X_1, X_2$ and $X = X_1 + X_2$; the two lines are independent. Both lines have mean 1.
- $(1, 2)$: log density; comparing tail thickness.
- $(1, 3)$: the bivariate log-density. This plot illustrates where $(X_1, X_2)$ lives. The diagonal lines show $X = k$ for different $k$. These show that large values of $X$ correspond to large
values of $X_2$, with $X_1$ about average.

• (2, 1): the form of $\kappa_i$ is clear from looking at (1, 3). $\kappa_1$ peaks at $x = 2.15$ with maximum value 1.14. Thereafter it declines to 1.0. $\kappa_2$ is monotonically increasing.

• (2, 2): The $\alpha_i$ functions. For small $x$ the expected proportion of losses is approximately 50/50, since the means are equal. As $x$ increases $X_2$ dominates. The two functions sum to 1.

• (2, 3): The solid lines are $\beta_i$ and the dashed lines $\alpha_i$ from (2, 2). Since $\alpha_1$ decreases $\beta_1(x) \leq \alpha_1(x)$. This can lead to $X_1$ having a negative margin in low asset layers. $X_2$ is the opposite.

• (3, 1): illustrates premium and margin determination by asset layer for $X_1$ using eq. (7) and eq. (16). For low asset layers $\alpha_1(x)S(x) > \beta_1(x)g(S(x))$ (dashed above solid) corresponding to a negative margin. Beyond about $x = 1.38$ the lines cross and the margin is positive.

• (4, 1): shows the same thing for $X_2$. Since $\alpha_2$ is increasing, $\beta_2(x) > \alpha_2(x)$ for all $x$ and so all layers get a positive margin. The solid line $\beta_2gS$ is above the dashed $\alpha_2S$ line.

• (3, 2): the layer margin densities. For low asset layers premium is fully funded by loss with zero overall margin. $X_2$ requires a positive margin and $X_1$ a negative one, reflecting the benefit the thick line receives from pooling in low layers. The overall margin is always non-negative. Beyond $x = 1.38$, $X_1$’s margin is also positive.

• (4, 2): the cumulative margin in premium by asset level. Total margin is zero in low dollar-swapping layers and then increases. It is always non-negative. The curves in this plot are the integrals of those in (3, 2) from 0 to $x$.

• (3, 3): shows stand-alone loss $(1 - S(x), x) = (p, q(p))$ (dashed) and premium $(1 - g(S(x)), x) = (p, q(1 - g^{-1}(1 - p)))$ (solid, shifted left) for each line and total. The margin is the shaded area between the two. Each set of three lines (solid or dashed) does not add up vertically because of diversification. The same distortion $g$ is applied to each line to the stand-alone $S_X$. It is calibrated to produce a 10 percent return overall. On a stand-alone basis, calculating capital by line to the same return period as total, $X_1$ is priced to a 83.5 percent loss ratio and $X_2$ a 51.8 percent, producing an average 64.0 percent, vs. 67.6 percent on a combined basis. Returns are 28.7 percent and 9.6 percent respectively, averaging 10.9 percent, vs 10 percent on a combined basis.

• (4, 3): shows the natural allocation of loss and premium to each line. The total (green) is the same as (3, 3). For each $i$, dashed shows $(p, E[X_i \mid X = q(p))]$, i.e. the expected loss recovery conditioned on total losses $X = q(p)$, and solid shows $(p, E[X_i \mid X = q(1 - g^{-1}(1 - p))]$, i.e. the natural premium allocation (see the bottom row of table 1). Here the solid and dashed lines add up vertically: they are allocations of the total. Looking vertically above $p$, the shaded areas show how the total margin at that loss level is allocated between lines. $X_1$ mostly consumes assets at low layers, and the blue area is thicker for small $p$, corresponding to smaller total losses. For $p$ close to 1, large total losses, margin is dominated by $X_2$ and in fact $X_1$ gets a slight credit (dashed above solid). The change in shape of the shaded margin area for $X_1$ is particularly evident: it shows $X_1$ benefits from pooling and requires a lower overall margin. The natural allocation returns are 5.3, 10.6 and 10.0 percent. The overall premium to surplus leverage is 0.308 to 1; on an allocated basis it is 0.986 and 0.223 to 1 for each line.
There may appear to be a contradiction between figures (3, 2) and (4, 3) but it should be noted that a particular $p$ value in (4, 3) refers to different events on the dotted and solid lines.

Plots (3, 3) and (4, 3) explain why the thick line requires relatively more margin (0.698 out of a total 0.728): its shape does not change when it is pooled with $X_1$. In (3, 3) the green shaded area is essentially an upwards shift of the orange, and the orange areas in (3, 3) and (3, 4) are essentially the same. This means that adding $X_1$ has virtually no impact on the shape of $X_2$; it is like adding a constant, as discussed after the proof of proposition 2. This can also be seen in (4, 3) where the blue region is almost a straight line.

These shifts are illustrated in fig. 3. The left hand plot shows that the stand-alone margin area for $X_2$ shifted up by 1, the mean of $X_1$, lies almost exactly over the total margin area (orange over green). The right hand plot compares the stand-alone margin areas for each line with the natural margin allocation. $X_2$ is shifted up by 1 for clarity. Again, there is essentially no difference for $X_2$, especially in the expensive, large loss states where $p$ is close to 1. $X_1$ is completely transformed: its margin is much lower (smaller area) and it is concentrated in low $p$, small total loss events. $X_1$ actually gets a credit at large total losses because its losses will be close to the mean, and hence low ambiguity, whereas $X_2$’s loss will be large and more ambiguous.

8 Conclusions

We have explored how the shape of risk impacts the price of risk transfer in an imperfect, incomplete market—a holy grail for practicing actuaries. We provide a natural, assumption-free allocation of aggregate premium to policy, incorporating an allocation of capital consistent with law invariance in order to price individual policies in the presence of frictional costs. Premium by policy is determined by the relative consumption of low and high ambiguity assets in a complex, but intuitively reasonable manner. The margin by line is driven more by behavior in solvent states than in default states (default states are often the major focus of allocation methods). Premium is interpreted as the value of insurance cash flows under a risk-dependent state price density. This is in contrast to most other approaches that adopt a cost allocation perspective. The natural allocation is insured-centric, rather than insurer-centric.

The natural allocation depends on the fact that DRMs are law invariant and comonotonic additive. It does not apply to more general convex risk measures. Notwithstanding this limitation, it provides a useful and practical method that can be applied by an insurance company to understand how to share its diversification benefit between policies.

Further research is needed to determine how the shapes of the $X_i$ interact to determine the natural allocation, as well as the impact of different distortions $g$ and capitalization standards. The underlying distortion can be calibrated to market prices. Market calibration to cat bond data and standard intermediated insurance data would reveal how much of the cost of capital arises from frictional costs and how much from shape of risk. The work can also be extended
Figure 2: A thin tailed line combined with a thick tailed line. See text for a key to the graphs.
Figure 3: Impact of the natural allocation by line. Left: stand-alone margin area for thick line shifted up by mean of thin line lies almost exactly over the margin area for the total. Right: stand-alone vs. natural allocation margin areas, showing minimal impact for thick line but dramatic impact on thin.
to dynamic portfolios and then applied to questions of optimal risk pooling under costly capital.

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