On distance matrices of distance-regular graphs

Hui Zhou
Department of Mathematics, Southern University of Science and Technology,
Shenzhen, 518055, P. R. China
School of Mathematical Sciences, Peking University, Beijing, 100871, P. R. China
zhouhpku17@pku.edu.cn

Rongquan Feng
School of Mathematical Sciences, Peking University, Beijing, 100871, P. R. China
fengrq@math.pku.edu.cn

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Abstract
In this paper, we give a characterization of distance matrices of distance-regular graphs to be invertible.

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1 Introduction
The study on graphs and matrices is an important topic in algebraic graph theory. The determinant and inverse of the distance matrix of a graph is of great interest. This kind of research is initialed by Graham and Pollak [7] on the determinant of the distance matrix of a tree. Later, Graham, Hoffman and Hosoya [5] gave an attractive theorem expressing the determinant of the distance matrix of a graph through that of its blocks, and Graham and Lovasz [6] calculated the inverse of the distance matrix of a tree.
Recent years, a lot of research has been done on the determinant and the inverse of the distance matrix of a graph, such as: the block graph \([3]\), the odd-cycle-clique graph \([9]\), the bi-block graph \([10]\), the multi-block graph \([11]\), the mixed block graph \([12]\), the weighted tree \([1]\), the bidirected tree \([2]\), the arc-weighted tree \([13]\), the cactoid digraph \([8]\), the weighted cactoid digraph \([14]\), etc.

Distance-regular graphs are very important in algebraic graph theory, and they have very beautiful combinatorial properties. In this paper, we first give the inverse of the distance matrix of a strongly-regular graph by some tricky calculations in Section 3. Then by the theory of linear systems, we give characterizations of distance matrices of distance-regular graphs to be invertible in Section 5, and by using this result we give the inverse of the distance matrix of a strongly-regular graph in a systematic method.

## 2 Preliminaries

Let \(j\) be an appropriate size column vector whose entries are ones, let \(J\) be an appropriate size matrix whose entries are ones, let \(0\) be an appropriate size matrix whose entries are zeroes, and let \(I\) be the identity matrix with an appropriate size. For any matrix \(A\) and any column vector \(\alpha\), we use \(A^T\) and \(\alpha^T\) to denote their transposes. Let \(A = (a_{ij})\) be an \(n \times n\) matrix. We denote its determinant by \(\det(A)\) and \(a_{ij}\) the \((i, j)\)-entry of \(A\).

Let \(G\) be a connected graph. We use \(V(G)\) and \(E(G)\) to denote the vertex set and edge set of \(G\), respectively. The distance \(\partial_G(u, v)\) from vertex \(u\) to vertex \(v\) in \(G\) is the length (number of edges) of the shortest path from \(u\) to \(v\) in \(G\). The distance matrix \(D\) of \(G\) is a \(|V(G)| \times |V(G)|\) square matrix whose \((u, v)\)-entry is the distance \(\partial_G(u, v)\), that is \(D = (\partial_G(u, v))_{u,v \in V(G)}\). Let \(v\) be a vertex of \(G\). For any integer \(i \geq 0\), we use \(G_i(v)\) to denote the set of vertices \(w\) satisfying \(\partial_G(v, w) = i\). The degree of \(v\) is the number of adjacent vertices, i.e., \(\deg_G(v) = |G_1(v)|\). The graph \(G\) is called regular with degree \(k\), or \(k\)-regular, if for any vertex \(u\) of \(G\), the degree \(\deg_G(u) = k\). A cut vertex of \(G\) is a vertex whose deletion results in a disconnected graph. A block of \(G\) is a connected subgraph on at least two vertices such that it has no cut vertices and is maximal with respect to this property.

A complete graph is a graph, in which every pair of vertices are adjacent, and an empty graph is a graph with no edges. Let \(G\) be a regular graph, which is neither complete nor empty. Then \(G\) is said to be strongly-regular.
with parameters \((n, k, a, c)\), if it is a \(k\)-regular graph with \(n\) vertices, every pair of adjacent vertices has \(a\) common neighbours, and every pair of distinct non-adjacent vertices has \(c\) common neighbours.

A connected regular graph \(G\) with degree \(k \geq 1\) and diameter \(d \geq 1\) is called a distance-regular graph if there are natural numbers

\[
b_0 = k, b_1, \ldots, b_{d-1}, c_1 = 1, c_2, \ldots, c_d,
\]

such that for each pair of vertices \(u\) and \(v\) with \(\partial_G(u, v) = j\) we have:

1. the number of vertices in \(G_1(u) \cap G_{j-1}(v)\) is \(c_j\) \((1 \leq j \leq d)\);
2. the number of vertices in \(G_1(u) \cap G_{j+1}(v)\) is \(b_j\) \((0 \leq j \leq d - 1)\).

The array \(\{b_0 = k, b_1, \ldots, b_{d-1}; c_1 = 1, c_2, \ldots, c_d\}\) is called the intersection array of \(G\), and it is denoted by \(\iota(G)\). For \(1 \leq i \leq d - 1\), let \(a_i = k - b_i - c_i\), and let \(a_d = k - c_d\). The intersection array is also written as

\[
\iota(G) = \begin{bmatrix}
* & c_1 = 1 & c_2 & \cdots & c_{d-1} & c_d \\
0 & a_0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\
0 & b_0 & b_1 & b_2 & \cdots & b_{d-1} & * 
\end{bmatrix}.
\]

Let \(G\) be a distance-regular graph with valency \(k\) and diameter \(d\). Let \(A\) be the adjacency matrix of \(G\), and let \(D\) be the distance matrix of \(G\). The \(i\)-distance matrices \(A_i\) \((0 \leq i \leq d)\) of \(G\) are defined as follows:

\[
(A_i)_{uv} = 1, \text{ if } \partial_G(u, v) = i; (A_i)_{uv} = 0, \text{ otherwise.}
\]

We have

\[
A_0 = I, A_1 = A, A_j = k_j \text{ and } A_0 + A_1 + A_2 + \ldots + A_d = J.
\]

By definition, we know \(D = \sum_{i=0}^{d} iA_i = A_1 + 2A_2 + \ldots + dA_d\).

### 3 Strongly-regular graphs

By definitions, we know that a strongly-regular graph is a distance-regular graph with diameter two. In this section, we study the distance matrices of strongly-regular graphs.
Let $G$ be a strongly-regular graph with parameters $(n, k, a, c)$. Let $D$ be the distance matrix of $G$, and let $A$ be the adjacency matrix of $G$. Then

$$D = 2(J - I) - A$$

(this equation also holds for complete graphs). By Lemma 2.5 in [4], $(A^2)_{uv}$ is the number of walks of length two connecting $u$ and $v$. By counting walks of length two, we have

$$A^2 + (c - a)A + (c - k)I = cJ.$$  

Thus $D$ is a polynomial of $A$ with degree two and

$$D = \frac{2}{c}A^2 + (1 - \frac{2a}{c})A - \frac{2k}{c}I = \alpha(A)$$

where $\alpha(x) = \frac{2}{c}x^2 + (1 - \frac{2a}{c})x - \frac{2k}{c} = -x - 2 + \frac{2}{c}[x^2 + (c - a)x + (c - k)]$. Since $D = A + 2A_2$, we get

$$A_2 = \frac{1}{c}(A^2 - aA - kI).$$

The eigenvalues of $A$ are $k$ of multiplicity 1, $\theta$ of multiplicity $m_\theta$ and $\tau$ of multiplicity $m_\tau$ where $\theta$ and $\tau$ are the roots of $x^2 + (c - a)x + (c - k) = 0$ and the multiplicities satisfying $m_\theta + m_\tau = n - 1$ and $k + \theta m_\theta + \tau m_\tau = 0$ (since the trace is zero). By calculations, we find

$$\theta = \frac{1}{2} \left[ (a - c) + \sqrt{(a - c)^2 + 4(k - c)} \right],$$

$$\tau = \frac{1}{2} \left[ (a - c) - \sqrt{(a - c)^2 + 4(k - c)} \right],$$

$$m_\theta = \frac{1}{2} \left[ (n - 1) - \frac{2k + (n - 1)(a - c)}{\sqrt{(a - c)^2 + 4(k - c)}} \right],$$

$$m_\tau = \frac{1}{2} \left[ (n - 1) + \frac{2k + (n - 1)(a - c)}{\sqrt{(a - c)^2 + 4(k - c)}} \right].$$

Thus the eigenvalues of $D$ are $\alpha(k) = 2n - k - 2$ of multiplicity 1, $\alpha(\theta) = -\theta - 2$ of multiplicity $m_\theta$ and $\alpha(\tau) = -\tau - 2$ of multiplicity $m_\tau$. The determinant of $D$ is $\det(D) = (2n - k - 2)(-1)^{n-1}(\theta + 2)^{m_\theta}(\tau + 2)^{m_\tau}$. The distance matrix $D$ is not invertible if and only if $(\theta + 2)(\tau + 2) = 2a + 4 - k - c = 0$, i.e.,

$$k + c = 2a + 4.$$  

(3)

Note that $AJ = JA = kJ$, we have
\[ J(A - kI) = 0. \]

Since \( J \) is a polynomial of \( A \) with degree two, this means

\[
\beta(A) = cJ(A - kI) = A^3 + (c - a - k)A^2 + (c - k - kc + ka)A - k(c - k)I
\]

is the minimal polynomial of \( A \). When \( k \neq c \), we have

\[ A^{-1} = \frac{1}{k(c-k)} \left[ A^2 + (c - k - a)A + (c - k + ka - kc)I \right]. \]

Note that the condition \( k = c \) holds if and only if the graph \( G \) is the complete multipartite graph \( K_{m|b} \) where \( m \geq 2, b \geq 2 \) and \( mb = n \). Since \( Aj = kj \), by Equation (2), we have

\[ k(k - a - 1) = c(n - k - 1). \quad (4) \]

Suppose \( k + c \neq 2a + 4 \), then \( D \) is invertible. We now find the inverse of \( D \). By Equation (1), we have \( c(2n - k - 2) = k(2k + c - 2a - 2) \). Let

\[ \lambda = k + c - 2a - 4, \]
\[ \mu = 2k + c - 2a - 2, \]
\[ \delta = 2k + c - 2a - 4, \]
\[ f = \frac{\delta}{k\lambda \mu}. \]

(Note that \( k \geq a + 1 \) and \( c \geq 1 \), so \( 2k + c - 2a - 2 > 0 \). When \( k = a + 1 \), then the graph is the complete graph. So for non-complete graphs, we have \( k \geq a + 2 \), and in this case \( 2k + c - 2a - 4 > 0 \) which implies \( f \neq 0 \).) By Equation (1), we have

\[
\lambda f c DJ = \frac{c\delta}{k\mu} (2J - 2I - A)J = \frac{c\delta}{k\mu} (2n - k - 2)J = \delta J.
\]

By Equation (2), we have

\[
D[A - (2 + a - c)I] = 2(k + c - a - 2)J - \lambda I - [A^2 + (c - a)A] + (c - k)I = \delta J - \lambda I.
\]
Hence
\[ \lambda fcDJ + D[(2 + a - c)I - A] = \lambda I \]
which implies
\[ D^{-1} = fcJ + \frac{(2 + a - c)I - A}{\lambda} = \frac{c\delta}{k\lambda\mu}J + \frac{2 + a - c}{\lambda}I - \frac{1}{\lambda}A. \] (5)

By the above discussion, we have the following result.

**Theorem 1** Let \( G \) be a strongly-regular graph with parameters \((n, k, a, c)\). Let \( D \) be the distance matrix of \( G \), and let \( A \) be the adjacency matrix of \( G \). Then \( D \) is invertible if and only if
\[ k + c \neq 2a + 4, \] (6)
and when \( D \) is invertible the inverse is
\[ D^{-1} = \frac{(2 + a - c)I - A}{k + c - 2a - 4} + \frac{c(2k + c - 2a - 4)}{k(k + c - 2a - 4)(2k + c - 2a - 2)} J. \] (7)

Now we give an example. Let \( G \) be the cycle of length 5. Then \( G \) is a strongly-regular graph with parameter \((n, k, a, c) = (5, 2, 0, 1)\). We have \( k + c \neq 2a + 4 \) and \((\lambda, \mu, \delta, f) = (-1, 3, 1, -\frac{1}{6})\). The adjacency matrix of \( G \) is
\[ A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \]
and the distance matrix of \( G \) is
\[ D = 2(J - I) - A = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 \end{pmatrix}. \]
The inverse of $D$ is
\[
D^{-1} = \frac{1}{6} \begin{pmatrix}
-7 & 5 & -1 & -1 & 5 \\
5 & -7 & 5 & -1 & -1 \\
-1 & 5 & -7 & 5 & -1 \\
-1 & -1 & 5 & -7 & 5 \\
5 & -1 & -1 & 5 & -7
\end{pmatrix}
\]
\[
= -\frac{1}{6}J + A - I = fcJ + \frac{(2 + a - c)I - A}{\lambda}.
\]
This coincides with the formula in Theorem 1.

4 Distance-regular graphs

Let $G$ be a distance-regular graph with valency $k$ and diameter $d$. Suppose the intersection array of $G$ is $\iota(G) = \{b_0 = k, b_1, \ldots, b_{d-1}; c_1 = 1, c_2, \ldots, c_d\}$.

Let $A$ be the adjacency matrix of $G$, and let $D$ be the distance matrix of $G$. For $1 \leq i \leq d - 1$, let $a_i = k - b_i - c_i$, and let $a_d = k - c_d$. By Lemma 20.6 in [4], we have $AA_0 = A$,
\[
AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \quad (1 \leq i \leq d - 1)
\]
and $AA_d = b_{d-1}A_{d-1} + a_dA_d$.

By convention, we let $b_d = c_0 = 0$. By simple calculation, we have
\[
AD = \sum_{i=1}^{d-1} i(b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}) + d(b_{d-1}A_{d-1} + a_dA_d)
= b_0A_0 + \sum_{i=1}^{d-1} [(i - 1)c_i + ia_i + (i + 1)b_i]A_i + [(d - 1)c_d + da_d]A_d
= b_0A_0 + \sum_{i=1}^{d-1} [ki + (b_i - c_i)]A_i + (kd - c_d)A_d
= kD + \sum_{i=0}^{d} (b_i - c_i)A_i.
\]
Additionally, we let \( a_0 = 0, c_{d+1} = b_{d+1} = 0 \) and \( b_{-1} = c_{-1} = 0 \). For \(-1 \leq i \leq d+1\), we let \( \delta_i = b_i - c_i \). Then we have \( \delta_{-1} = \delta_{d+1} = 0 \) and \( AD = kD + \sum_{i=0}^{d} \delta_i A_i \). If we suppose \( A_{-1} = A_{d+1} = 0 \), then we have

\[
AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \text{ for } 0 \leq i \leq d.
\]

Let \( X_0 = D = \sum_{i=0}^{d} i A_i \) and \( X_1 = \sum_{i=0}^{d} \delta_i A_i \). We have \( AX_0 = kX_0 + X_1 \). Similarly, we have

\[
AX_1 = \sum_{i=0}^{d} \delta_i(b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1})
= \sum_{i=0}^{d} (\delta_{i-1}c_i + \delta_i a_i + \delta_{i+1}b_i)A_i
= k \sum_{i=0}^{d} \delta_i A_i + \sum_{i=0}^{d} ((\delta_{i+1} - \delta_i)b_i - (\delta_i - \delta_{i-1})c_i)A_i.
\]

For \( 0 \leq i \leq d \), let \( \zeta_i = (\delta_{i+1} - \delta_i)b_i - (\delta_i - \delta_{i-1})c_i \). Let \( X_2 = \sum_{i=0}^{d} \zeta_i A_i \). Then we have \( AX_1 = kX_1 + X_2 \). So \( A^2X_0 = k^2X_0 + 2kX_1 + X_2 \). Suppose \( \zeta_{-1} = \zeta_{d+1} = 0 \). By calculation, we have

\[
AX_2 = \sum_{i=0}^{d} \zeta_i(b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1})
= \sum_{i=0}^{d} (\zeta_{i-1}c_i + \zeta_i a_i + \zeta_{i+1}b_i)A_i
= k \sum_{i=0}^{d} \zeta_i A_i + \sum_{i=0}^{d} ((\zeta_{i+1} - \zeta_i)b_i - (\zeta_i - \zeta_{i-1})c_i)A_i.
\]

For \( 0 \leq i \leq d \), let \( \phi_i = (\zeta_{i+1} - \zeta_i)b_i - (\zeta_i - \zeta_{i-1})c_i \). Let \( X_3 = \sum_{i=0}^{d} \phi_i A_i \). Then we have \( AX_2 = kX_2 + X_3 \). So \( A^3X_0 = k^3X_0 + 3k^2X_1 + 3kX_2 + X_3 \).

We can calculate \( AX_i \) and \( A^iX_0 \) for \( 0 \leq i \leq d \) in a similar way. These properties between \( X_i \)'s can be extended to a general case in the next section.
5 Linear systems

By the properties of $i$-distance matrices ($0 \leq i \leq d$) of distance-regular graphs, we consider the following matrices $A_0, A_1, \ldots, A_d$ satisfying properties the same as $i$-distance matrices of distance-regular graphs. Then we analyze these matrices by linear systems to give characterizations of the matrix $X$ (corresponding to the distance matrix of a distance-regular graph) to be invertible. First, we give some notations in the following hypothesis.

**Hypothesis 1** Let $d \geq 2$ and $k \geq 3$.

1. Let $i = \begin{pmatrix} c_{-1} = 0 & c_0 = 0 & c_1 = 1 & c_2 & \cdots & c_{d-1} & c_d & c_{d+1} = 0 \\ a_{-1} & a_0 = 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d & a_{d+1} \\ b_{-1} = 0 & b_0 = k & b_1 & b_2 & \cdots & b_{d-1} & b_d = 0 & b_{d+1} = 0 \end{pmatrix}$ be an array satisfying $k = c_i + a_i + b_i$ for $-1 \leq i \leq d + 1$ and $1 = c_1 \leq c_2 \leq \cdots \leq c_d \leq k$, $k = b_0 \geq b_1 \geq \cdots \geq b_{d-1} \geq 1$.

2. Let $A_{-1} = 0, A_0, A_1, \ldots, A_d, A_{d+1} = 0$ be a series of matrices satisfying $A_0 = I$, $A_1 = A$, $A_j = kj$, the minimal polynomial of $A$ is of degree at least $d + 1$, $J = A_0 + A_1 + \ldots + A_d$ and

\[ AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \text{ for } 0 \leq i \leq d. \]  \hspace{1cm} (9)

3. Let $\{x_{i,j} \mid 0 \leq i \leq d, -1 \leq j \leq d+1\}$ be a series of numbers satisfying

\[ x_{i,-1} = x_{i,d+1} = 0 \text{ for } 0 \leq i \leq d, \text{ and } \]

\[ x_{i+1,j} = (x_{i,j+1} - x_{i,j})b_j - (x_{i,j} - x_{i,j-1})c_j \]

for $0 \leq i \leq d - 1$ and $0 \leq j \leq d$.

4. Let $X_i = \sum_{j=0}^{d} x_{i,j}A_j$ for $0 \leq i \leq d$.

5. Let $X = X_0 = \sum_{j=0}^{d} x_{0,j}A_j$.

6. Let $Q = (x_{i,j}')_{0 \leq i \leq d, 0 \leq j \leq d}$ where $x_{i,j}' = x_{j,i}$ for $0 \leq i \leq d$ and $0 \leq j \leq d$. 


Let \( \alpha_{h,i} = \sum_{j=0}^{i} \binom{i}{j} k^{i-j} x_{j,h} \) for \( 0 \leq i \leq d \) and \( 0 \leq h \leq d \).

Note that, by Proposition 2.6 in [4], here we suppose in (2) of the above hypothesis that the minimal polynomial of \( A \) is of degree at least \( d + 1 \).

**Lemma 1** We use notations as in Hypothesis [7]. Then \( A_i \) is a polynomial in \( A \) of degree \( i \) for \( 0 \leq i \leq d \), \( J \) is a polynomial \( f(A) \) in \( A \) of degree \( d \), and the minimal polynomial of \( A \) is \( \mu(A) = \left( \prod_{i=1}^{d} c_i \right) f(A)(A - kI) \) of degree \( d + 1 \).

**Proof.** By Equation (9), we can find the polynomial of \( A_i \) in \( A \) recursively for \( 2 \leq i \leq d \). So \( J \) is a polynomial in \( A \) of degree \( d \). Since \( AJ = kJ \), we have \( AJ = kJ \). Hence \( f(A)(A - kI) = J(A - kI) = 0 \). Note the degree of \( f(A)(A - kI) \) is \( d + 1 \). Thus the minimal polynomial of \( A \) is obtained. \( \blacksquare \)

By Equation (9) and Lemma 1, we list several \( A_i \)'s as follows.

\[
\begin{align*}
    f_2(A) &= c_2 A^2 = A^2 - a_1 A - kI, \\
    f_3(A) &= c_2 c_3 A^3 = A^3 - (a_1 + a_2) A^2 + (a_1 a_2 - b_1 c_2 - k) A + k a_2 I, \\
    f_4(A) &= c_2 c_3 c_4 A^4 = A^4 - (a_1 + a_2 + a_3) A^3 + (a_1 a_2 + a_1 a_3 + a_2 a_3 - b_1 c_2 - b_2 c_3 - k) A^2 + (k a_2 + k a_3 + b_1 c_2 a_3 + a_1 b_2 c_3 - a_1 a_2 a_3) A + (k b_2 c_3 - k a_2 a_3) I.
\end{align*}
\]

We pick the item \( f_d(A) = \left( \prod_{i=1}^{d} c_i \right) A_d \) in this list. For example, when \( d = 2 \), then \( f_3(A) \) is the minimal polynomial of \( A \); when \( d = 3 \), then \( f_4(A) \) is the minimal polynomial of \( A \). We may try to prove that \( f_{d+1}(A) \) is the minimal polynomial of \( A \) when the diameter is \( d \). It is easy to check that \( x = k \) is a root of \( f_{d+1}(x) \). So all the other eigenvalues of \( A \) are the roots of \( \beta(x) = \frac{f_{d+1}(x)}{x-k} \). By Theorem 20.7 in [4], a distance-regular graph with diameter \( d \) has just \( d + 1 \) distinct eigenvalues which are \( x = k \) and the roots of \( \beta(x) \).

Using notations as in Section 4 and Hypothesis 1 we have

\[
\begin{align*}
    x_{0,i} &= i, \; x_{1,i} = \delta_i, \; x_{2,i} = \zeta_i \text{ and } x_{3,i} = \phi_i \text{ for } 0 \leq i \leq d; \\
    X_0 &= D = \sum_{i=0}^{d} i A_i, \; X_1 = \sum_{i=0}^{d} \delta_i A_i, \; X_2 = \sum_{i=0}^{d} \zeta_i A_i, \; X_3 = \sum_{i=0}^{d} \phi_i A_i; \\
    AX_0 &= kX_0 + X_1, \; AX_1 = kX_1 + X_2, \; AX_2 = kX_2 + X_3; \\
    A^2 X_0 &= k^2 X_0 + 2k X_1 + X_2 \text{ and } A^3 X_0 = k^3 X_0 + 3k^2 X_1 + 3k X_2 + X_3.
\end{align*}
\]
Generally, we have the following result.

**Lemma 2** We use notations as in Hypothesis 1. Then

1. $AX_i = kX_i + X_{i+1}$ for $0 \leq i \leq d - 1$,

2. $A^iX = \sum_{j=0}^{i} \binom{i}{j} k^{i-j}X_j = \sum_{h=0}^{d} \alpha_{h,i}A_h$ for $0 \leq i \leq d$.

**Proof.**  (1) Let $0 \leq i \leq d - 1$. Then

$$AX_i = \sum_{j=0}^{d} x_{i,j}AA_j$$

$$= \sum_{j=0}^{d} x_{i,j}(b_{j-1}A_{j-1} + a_jA_j + c_{j+1}A_{j+1})$$

$$= \sum_{j=0}^{d} (x_{i,j-1}c_j + x_{i,j}a_j + x_{i,j+1}b_j)A_j$$

$$= \sum_{j=0}^{d} [kx_{i,j} + (x_{i,j+1} - x_{i,j})b_j - (x_{i,j} - x_{i,j-1})c_j]A_j$$

$$= \sum_{j=0}^{d} kx_{i,j}A_j + \sum_{j=0}^{d} x_{i+1,j}A_j$$

$$= kX_i + X_{i+1}.$$ 

(2) We prove the first equality by induction on $i$. Suppose for $i < d$, we have $A^iX = \sum_{j=0}^{i} \binom{i}{j} k^{i-j}X_j$. Then
\[A^{i+1}X = \sum_{j=0}^{i} \binom{i}{j} k^{i-j} (kX_j + X_{j+1})\]

\[= \binom{i}{0} k^{i+1}X_0 + \sum_{j=0}^{i-1} \left[ \binom{i}{j} + \binom{i}{j+1} \right] k^{i-j}X_{j+1} + \binom{i}{i} k^0 X_{i+1}\]

\[= \binom{i+1}{0} k^{i+1}X_0 + \sum_{j=0}^{i-1} \binom{i+1}{j+1} k^{(i+1)-(j+1)}X_{j+1} + \binom{i+1}{i+1} k^0 X_{i+1}\]

\[= \sum_{j=0}^{i+1} \binom{i+1}{j} k^{i+1-j} X_j.\]

Let \(0 \leq i \leq d\). We have

\[A^i X = \sum_{j=0}^{i} \binom{i}{j} k^{i-j} X_j = \sum_{h=0}^{d} \sum_{j=0}^{i} \binom{i}{j} k^{i-j} x_{j,h} A_h = \sum_{h=0}^{d} \alpha_{h,i} A_h.\]

By Hypothesis \(\Box\) \(X\) is a polynomial of \(A\). If the matrix \(X\) is invertible, by Lemma \(\Box\), the inverse of \(X\) is a polynomial in \(A\) of degree at most \(d\). Thus the matrix \(X\) is invertible if and only if there exist \(y_0, y_1, y_2, \ldots, y_d\) such that the following equation holds

\[\sum_{i=0}^{d} y_i A^i X = I.\]  \((10)\)

In this case, the inverse is \(X^{-1} = \sum_{i=0}^{d} y_i A^i\). We consider Equation \((10)\). Let

\[z_j = \sum_{i=j}^{d} \binom{i}{j} k^{i-j} y_i\]  \((11)\)

for \(0 \leq j \leq d\). By Möbius inversion formula, we have

\[y_i = \sum_{j=i}^{d} (-1)^{j-i} \binom{j}{i} k^{j-i} z_j\] for \(0 \leq i \leq d\).
Lemma 3 We use notations as in Hypothesis 1. Suppose \( y_0, y_1, y_2, \ldots, y_d \) is a solution of Equation (10), and we let \( z_0, z_1, z_2, \ldots, z_d \) be defined as in Equation (11). For \( 0 \leq h \leq d \), let

\[
L_h = \sum_{i=0}^{d} y_i \alpha_{h,i}, \quad (12)
\]

\[
M_h = \sum_{j=0}^{d} z_j x_{j,h}. \quad (13)
\]

Then

\[
\sum_{i=0}^{d} y_i A^i X = \sum_{h=0}^{d} L_h A_h = \sum_{h=0}^{d} M_h A_h. \quad (14)
\]

Proof. We have

\[
\sum_{i=0}^{d} y_i A^i X = \sum_{i=0}^{d} y_i \sum_{h=0}^{d} \alpha_{h,i} A_h = \sum_{h=0}^{d} \left( \sum_{i=0}^{d} y_i \alpha_{h,i} \right) A_h
\]

\[
= \sum_{h=0}^{d} \left( \sum_{i=0}^{d} y_i \sum_{j=0}^{i} \binom{i}{j} k^{i-j} x_{j,h} \right) A_h
\]

\[
= \sum_{h=0}^{d} \sum_{j=0}^{d} \left( \sum_{i=j}^{d} \binom{i}{j} k^{i-j} y_i \right) x_{j,h} A_h = \sum_{h=0}^{d} \left( \sum_{j=0}^{d} z_j x_{j,h} \right) A_h.
\]

By Equation (14) we consider the following linear systems. We denote \( L \) the following system of linear equations

\[
\begin{align*}
L_0 &= 1, \\
L_i &= 0 \quad (1 \leq i \leq d)
\end{align*}
\]

in \( y_0, y_1, y_2, \ldots, y_d \), so the coefficient matrix of \( L \) is \( P = (\alpha_{h,i})_{0 \leq h \leq d, 0 \leq i \leq d} \). We denote \( M \) the following system of linear equations

\[
\begin{align*}
M_0 &= 1, \\
M_i &= 0 \quad (1 \leq i \leq d)
\end{align*}
\]

in \( z_0, z_1, z_2, \ldots, z_d \), so the coefficient matrix of \( M \) is \( Q \).
By the above discussion and the theory of linear systems, we have the following result.

**Theorem 2** We use notations as in Hypothesis 1. The following statements are equivalent.

1. The matrix $X$ is invertible.
2. There exist $y_0, y_1, y_2, \ldots, y_d$ such that Equation (10) holds.
3. The system of linear equations $L$ has a solution.
4. The system of linear equations $M$ has a solution.
5. The matrix $P$ is invertible.
6. The matrix $Q$ is invertible.
7. The determinant $\det(P) \neq 0$.
8. The determinant $\det(Q) \neq 0$.

Suppose $\det(Q) \neq 0$. Then by Theorem 2, the matrix $X$ is invertible. By Cramer’s rule, we have a solution of $M$, that is $z_j = \frac{\det(Q_j)}{\det(Q)}$ for $0 \leq j \leq d$, where $Q_j$ is the matrix obtained from $Q$ by replacing its $j$-th column by $[1, 0, 0, \ldots, 0]^T$. So

$$y_i = \sum_{j=i}^{d} (-1)^{j-i} \binom{j}{i} k^{j-i} \frac{\det(Q_j)}{\det(Q)} \text{ for } 0 \leq i \leq d. \quad (15)$$

Thus

$$\sum_{i=0}^{d} \sum_{j=i}^{d} (-1)^{j-i} \binom{j}{i} k^{j-i} \det(Q_j) A^i X = \det(Q) I.$$ 

The inverse of $X$ is $X^{-1} = \sum_{i=0}^{d} y_i A^i = \sum_{i=0}^{d} \left( \sum_{j=i}^{d} (-1)^{j-i} \binom{j}{i} k^{j-i} \frac{\det(Q_j)}{\det(Q)} \right) A^i$. Thus we have the following result.
Theorem 3 Using notations as in Hypothesis \textbf{I}. Let $Q_j$ be the matrix obtained from $Q$ by replacing its $j$-th column by $[1, 0, 0, \ldots, 0]^T$ for $0 \leq j \leq d$. Then $X$ is invertible if and only if $\det(Q) \neq 0$. Furthermore, suppose $\det(Q) \neq 0$. Let $y_i$ ($0 \leq i \leq d$) be defined as in Equation (15). Then the inverse of $X$ is

$$X^{-1} = \sum_{i=0}^{d} y_i A^i = \sum_{i=0}^{d} \left( \sum_{j=i}^{d} (-1)^{j-i} \binom{j}{i} k^j - i \frac{\det(Q_j)}{\det(Q)} \right) A^i. $$

By Theorem 3, we will calculate $\det(Q)$ and the inverse matrix (when it exists) of the distance matrix of a strongly-regular graph systematically in the next section.

6 Strongly-regular graphs revisited

A strongly-regular graph with parameter $(n, k, a, c)$ is a distance-regular graph with intersection array \{\begin{align*} b_0 &= k, & b_1 &= k - a - 1; & c_1 &= 1, & c_2 &= c \end{align*}\}. Let $D$ be the distance matrix of this distance-regular graph. Let $X_0 = D = \sum_{i=0}^{2} i A_i$. Then $x_{0,j} = j$ for $j = 0, 1, 2$. By simple calculation, we get $x_{i,j}$ for $i = 1, 2$ and $j = 0, 1, 2$, and so

$$Q = \begin{pmatrix} 0 & k & -k(a + 2) \\ 1 & k - a - 2 & (a + 2 - c - k)(k - a - 1) + (a + 2) \\ 2 & -c & c(k + c - a - 2) \end{pmatrix}. $$

Hence

$$\det(Q) = k(4 + 2a - c - k)(2k + c - 2a - 2),$$

$$\det(Q_0) = c(4 + 2a - c - k),$$

$$\det(Q_1) = -4a - 2a^2 + 4c + 3ac - c^2 + 6k + 4ak - 3ck - 2k^2,$$

$$\det(Q_2) = 4 + 2a - c - 2k.$$

Since

$$\det(Q)y_0 = \det(Q_0) - k \det(Q_1) + k^2 \det(Q_2),$$

$$\det(Q)y_1 = \det(Q_1) - 2k \det(Q_2),$$

$$\det(Q)y_2 = \det(Q_2),$$
we have
\[ \det(Q)[y_0 - (c - k)y_2] = k(2 + a - c)(2 + 2a - c - 2k), \]
\[ \det(Q)[y_1 - (c - a)y_2] = k(c + 2k - 2a - 2). \]

So
\[
\begin{align*}
\det(Q)[y_0 I + y_1 A + y_2 A^2] &= \det(Q)[y_0 - (c - k)y_2]I + \det(Q)[y_1 - (c - a)y_2]A + \det(Q)y_2 cJ \\
&= k(2 + a - c)(2 + 2a - c - 2k)I + k(c + 2k - 2a - 2)A + \det(Q_2)cJ.
\end{align*}
\]

By the above calculation, we know
\[ \det(Q) \neq 0 \text{ if and only if } 4 + 2a - c - k \neq 0. \]

Suppose \( \det(Q) \neq 0 \), i.e., \( 4 + 2a - c - k \neq 0 \). Then the inverse of the distance matrix is
\[
D^{-1} = \frac{k(2 + a - c)(2 + 2a - c - 2k)I + k(c + 2k - 2a - 2)A + \det(Q_2)cJ}{\det(Q)}
\]
\[
= \frac{(2 + a - c)I - A}{k + c - 2a - 4} + \frac{c(2k + c - 2a - 4)}{k(k + c - 2a - 4)(2k + c - 2a - 2)}J.
\]

This coincides with Equation (7) in Section 3.

\section{The determinant of Q}

In this section, we consider the determinant of \( Q \). In Section 6 we have
\[ \det(Q) = c_2 \alpha(k) \alpha(\theta) \alpha(\tau). \] (16)

When the diameter is three, by calculation we will give a similar formula in the following. Hence we guess the determinant of \( Q \) has a formula related to the eigenvalues of the distance matrix of the distance-regular graph.

We suppose the diameter is three, i.e., \( d = 3 \). Let \( X_0 = D = \sum_{i=0}^{3} iA_i \). Then \( x_{0,j} = j \) for \( j = 0, 1, 2, 3 \). By simple calculation, we get \( x_{i,j} \) for \( i = 1, 2, 3 \) and \( j = 0, 1, 2, 3 \), and so

\[ \text{...} \]
where

\[ Q = \begin{pmatrix} 0 & k & (b_1 - 1 - k)k & x_{3,0} \\ 1 & b_1 - 1 & 1 - b_1^2 + b_1 b_2 - b_1 c_2 + k & x_{3,1} \\ 2 & b_2 - c_2 & -b_2^2 - c_2 + b_1 c_2 + c_2^2 - b_2 c_3 & x_{3,2} \\ 3 & -c_3 & c_3 (b_2 - c_2 + c_3) & x_{3,3} \end{pmatrix}, \]

so that

\[ x_{3,0} = k (1 - b_1^2 + b_1 b_2 - b_1 c_2 + 2k - b_1 k + k^2), \]
\[ x_{3,1} = -1 - b_1 + b_1^3 - b_1 b_2 - b_1 b_2^2 - b_1 b_2^2 + b_1^2 c_2 + b_1 c_2^2 - b_1 b_2 c_3 - 2k - k^2, \]
\[ x_{3,2} = b_2^3 + c_2 - b_1^2 c_2 + b_2 c_2 + b_2 c_2^2 + c_2 b_2 - 2b_1 c_2^2 - b_2 c_2^2 - c_2^2 + 2b_2^2 c_3 + b_2 c_3^2 + c_2 k, \]
\[ x_{3,3} = c_3 (-b_2^2 - c_2 + b_1 c_2 + c_2^2 - 2b_2 c_3 + c_2 - c_2^2). \]

Note that \( D = \alpha(A) = \frac{1}{c_2 c_3} [3 A^3 + (2c_3 - 3a_1 - 3a_2) A^2 + (3a_1 a_2 + c_2 c_3 - 2a_1 c_3 - 3b_1 c_2 - 3k) A + (3ka_2 - 2kc_3) I]. \) Then

\[ \alpha(k) = \frac{k}{c_2 c_3} (c_2 c_3 + 2b_1 c_3 + 3b_1 b_2). \]

Let \( f_4(x) \) be defined after Lemma 1, i.e., \( f_4(x) \) is the minimal polynomial of \( A \), and let \( \theta, \tau \) and \( \eta \) be the roots of

\[ \beta(x) = \frac{f_4(x)}{x - k} = x^3 + (c_3 - a_1 - a_2) x^2 + (a_1 a_2 + c_2 c_3 - a_1 c_3 - b_1 c_2 - k) x + (a_2 a_3 - b_2 c_3). \]

We know \( \alpha(x) = \frac{3}{c_2 c_3} \beta(x) - \frac{1}{c_2} r(x) \) where

\[ r(x) = x^2 + (2c_2 - a_1) x + (3c_2 - k). \]

Then \( \alpha(y) = -\frac{1}{c_2} r(y) \) for any root \( y \) of \( \beta(x) \). Let

\[ \pi = -3b_2 + 3b_1^2 b_2 + 3b_2^2 - 6b_1 b_2^2 + 6c_2 - 6b_1 b_2 c_2 - 3c_2^2 - 2c_3 + 2b_2 b_2 c_2 - 7b_1 b_2 c_3 - 3c_2 c_3 + 5b_1 c_2 c_3 - 2b_2 c_2 c_3 + 2c_2 c_3 + 2c_3^2 - 2b_2 c_2 - 2c_1 c_2 k - 2b_1 k - 2b_2^2 k - 5b_2 k + 3b_1 b_2 k + 2b_2 k + 7c_2 k.
\]

By calculation, we have

\[ \text{det}(Q) = -k (c_2 c_3 + 2b_1 c_3 + 3b_1 b_2) \pi = c_2^2 c_3 \alpha(k) \alpha(\theta) \alpha(\tau) \alpha(\eta). \] (17)

By Equation (16) and Equation (17), we may guess

\[ \text{det}(Q) = \left( \prod_{i=1}^{d} c_i^{d+1-i} \right) \left( \prod_{\lambda \text{ is an eigenvalue of } D} \lambda \right). \] (18)
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