DONALDSON-THOMAS TRANSFORMATION OF DOUBLE BRUHAT CELLS IN
SEMISIMPLE LIE GROUPS

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Abstract. Double Bruhat cells $G^{u,v}$ were studied by Fomin and Zelevinsky [FZ98]. They provide important examples of cluster algebras [FZ00] and cluster Poisson varieties [FG06]. Cluster varieties produce examples of 3d Calabi-Yau categories with stability conditions, and their Donaldson-Thomas invariants, defined by Kontsevich and Soibelman [KS08], are encoded by a formal automorphism on the cluster variety known as the Donaldson-Thomas transformation. Goncharov and Shen conjectured in [GS16] that for any semisimple Lie group $G$, the Donaldson-Thomas transformation of the cluster Poisson variety $H\backslash G^{u,v}/H$ is a slight modification of Fomin and Zelevinsky’s twist map [FZ98]. In this paper we prove this conjecture, using crucially Fock and Goncharov’s cluster ensembles [FG03] and the amalgamation construction [FG06]. Our result, combined with the work of Gross, Hacking, Keel, and Kontsevich [GHKK14], proves the duality conjecture of Fock and Goncharov [FG03] in the case of $H\backslash G^{u,v}/H$.

1. Introduction

Cluster algebras were defined by Fomin and Zelevinsky in [FZ01]. Cluster varieties were introduced by Fock and Goncharov in [FG03]. They can be used to construct examples of 3d Calabi-Yau categories with stability conditions. One important object of study for such categories are their Donaldson-Thomas invariants, introduced by Kontsevich and Soibelman [KS08], which generalize geometric invariants of Calabi-Yau manifolds. For a 3d Calabi-Yau category with stability condition constructed from a cluster variety, its Donaldson-Thomas invariants are encoded by a single formal automorphism on the corresponding cluster.

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variety, which is also known as the Donaldson-Thomas transformation [KS08]. Keller [Kel13] gave a combinatorial characterization of a certain class of Donaldson-Thomas transformations based on quiver mutation. Goncharov and Shen gave an equivalent definition of the Donaldson-Thomas transformations using tropical points of cluster varieties in [GS16], which we use in this paper.

Double Bruhat cells have been an important family of examples in the study of cluster algebras and cluster Poisson varieties since the very beginning of the subject. On one hand, Berenstein, Fomin, and Zelevinsky [BFZ03] proved that the algebra of regular functions on double Bruhat cells in simply connected semisimple Lie groups are upper cluster algebras. On the other hand, Fock and Goncharov [FG06a] showed that double Bruhat cells in adjoint semisimple Lie groups are cluster Poisson varieties. Furthermore, Fock and Goncharov proved in the same paper that the Poisson structure in the biggest double Bruhat cell, which is a Zariski open subset of the Lie group, coincides with the Poisson-Lie structure defined by Drinfeld in [Dri83]. These two constructions can be combined into a cluster ensemble in the sense of Fock and Goncharov [FG03], which will play a central role in our construction of Donaldson-Thomas transformation on the double quotient $H\backslash G^{u,v}/H$.

Recall the flag variety $B$ associated to a semisimple Lie group $G$. Generic configurations of flags were studied by Fock and Goncharov [FG06b]. The cluster Donaldson-Thomas transformation of such configuration space was constructed by Goncharov and Shen in [GS16]. In this paper we make use of the study of quadruple of flags with certain degenerate conditions depending on a pair of Weyl group elements $(u,v)$, which we call $Conf^{u,v}(B)$, and show that such configuration space is isomorphic to the quotient $H\backslash G^{u,v}/H$ of double Bruhat cells. We relate the Donaldson-Thomas transformation on $H\backslash G^{u,v}/H$ to some explicit automorphism on the configuration space $Conf^{u,v}(B)$.

1.1. Main Result. Let $G$ be a semisimple Lie group. Fix a pair of opposite Borel subgroups $B_{+}$ of $G$ and let $H := B_{+} \cap B_{-}$ be the corresponding maximal torus. Then with respect to these two Borel subgroups, $G$ admits two Bruhat decompositions

$$G = \bigsqcup_{w \in W} B_{+} w B_{+} = \bigsqcup_{w \in W} B_{-} w B_{-},$$

where $W$ denotes the Weyl group with respect to the maximal torus $H$. For a pair of Weyl group elements $(u,v)$, we define the double Bruhat cell $G^{u,v}$ to be the intersection

$$G^{u,v} := B_{+} u B_{+} \cap B_{-} v B_{-}.$$ 

Let $B$ be the flag variety associated to a semisimple Lie group $G$. For any two Borel subgroups $B = xB_{+}x^{-1}$ and $B' = yB_{+}y^{-1}$ we write $B \xrightarrow{w} B'$ if $x^{-1} y \in B_{+} w B_{+}$, and write $B \xrightarrow{B'} B'$ if $B$ and $B'$ are opposite Borel subgroups. Then the configuration space $Conf^{u,v}(B)$ is defined to be the configuration space of quadruple of Borel subgroups $(B_{1}, B_{2}, B_{3}, B_{4})$ satisfying the relative condition

$$
\begin{array}{ccc}
B_{1} & \xrightarrow{u} & B_{4} \\
\downarrow & & \downarrow \\
B_{3} & \xrightarrow{v^*} & B_{2}
\end{array}
$$

with $v^* := w_{0} v w_{0}$, modulo the diagonal adjoint action by $G$. As it turns out, $Conf^{u,v}(B)$ is naturally isomorphic to the double quotient of double Bruhat cells $H\backslash G^{u,v}/H$ (Proposition 2.24):

$$i : Conf^{u,v}(B) \xrightarrow{\cong} H\backslash G^{u,v}/H.$$ 

We consider the following three maps of the quotient $H\backslash G^{u,v}/H$ of double Bruhat cell and the configuration space $Conf^{u,v}(B)$.

1. The twist map

$$\eta : H\backslash G^{u,v}/H \to H\backslash G^{u,v}/H,$$

$$H\backslash x/H \to H \left( \left[ x^{-1} \right]_{-}^{-1} \left[ x^{-1} x v^{-1} x v^{-1} \right]_{+} \right)^{t} / H,$$

where $t$ is the twist element of $Conf^{u,v}(B)$. 

where \( x = [x]-[x]_0[x]_+ \) denotes the Gaussian decomposition and \( \varpi \) denotes the lift of a Weyl group element \( w \) defined by Equation (2.8) (see also [FZ98], [BFZ03], and [GS16]).

(2) The map
\[
\eta : \text{Conf}^{u,v}(B) \to \text{Conf}^{u,v}(B)
\]
\[
[B_1, B_2, B_3, B_4] \mapsto [B'_3, B'_5, B'_6]
\]
where \( \ast \) denotes an involution on \( G \) defined by (2.9) and the two new Borel subgroups \( B_5 \) and \( B_6 \) are uniquely determined by the following relative configurations with \( u^c := w_0u^{-1} \).

(3) The composition
\[
\chi \circ p \circ \psi \circ s : H \setminus G^{u,v}/H \longrightarrow H \setminus G^{u,v}/H,
\]
where the maps are drawn from the following diagram where \( (A^{u,v}, \chi^{u,v}, p) \) is the cluster ensemble associated to the pair of Weyl group elements \( (u, v) \) (see Section 2.5):

\[
\begin{array}{ccc}
G^{u,v} & \overset{\psi}{\longrightarrow} & A^{u,v} \\
\downarrow{s} & & \downarrow{p} \\
H \setminus G^{u,v}/H & \overset{\chi}{\longrightarrow} & H \setminus G^{u,v}/H
\end{array}
\]

In particular,
- \( s \) is an arbitrary section of the natural projection \( G^{u,v}_{sc} \to H \setminus G^{u,v}/H \); the resulting composition is independent of such choice;
- \( \psi : G^{u,v}_{sc} \to A^{u,v} \) comes from the result of Berenstein, Fomin, and Zelevinsky saying that \( \mathcal{O}(G^{u,v}_{sc}) \) is an upper cluster algebra [BFZ03];
- \( p : A^{u,v} \to \chi^{u,v} \) comes from Fock and Goncharov’s theory of cluster ensemble [FG03];
- \( \chi : \chi^{u,v} \to H \setminus G^{u,v}/H \) is given by Fock and Goncharov’s amalgamation map [FG06a].

Let \( G_{ad} := G/\text{Center}(G) \) be the adjoint group associated to \( G \). Fock and Goncharov showed in [FG06a] that the double Bruhat cells \( G^{u,v}_{ad} \) as well as their double quotients \( H \setminus G^{u,v}/H \) are cluster Poisson varieties. We drop the subscript in the double quotient because forms of semisimple Lie groups with the same Lie algebra differ only by center elements, which are contained in the maximal torus \( H \). Goncharov and Shen conjectured in [GS16] that the Donaldson-Thomas transformation on \( H \setminus G^{u,v}/H \) is a slight modification of Fomin and Zelevinsky’s twist map. The precise statement is the following, which is the main result of this paper.

**Theorem 1.1.** Let \( G \) be a semisimple Lie group and let \( (u, v) \) be a pair of Weyl group elements.

(a) The maps (1), (2), and (3) defined above are rationally equivalent. Precisely:

(i) the isomorphism \( i \) between \( H \setminus G^{u,v}/H \) and \( \text{Conf}^{u,v}(B) \) intertwines the maps (1) and (2);

(ii) the maps (1) and (3) are rationally equivalent.

(b) The Donaldson-Thomas transformation of the cluster Poisson variety \( H \setminus G^{u,v}/H \) is a cluster transformation. It is given by either of the maps (1), (2), and (3), which agree by part (a).

Notice that the expression of the twist map in (1) above, modulo the double quotients, differs from Fomin and Zelevinsky’s original version of twist map (Definition 1.5 in [FZ98]) only by an anti-automorphism \( x \mapsto x^\ast \) on \( G \), which is defined on the generators by
\[
e^{\ast}_{\pm \alpha} = e^{\pm \alpha} \quad \text{and} \quad h^\ast = h^{-1} \quad \forall h \in H.
\]
We show in Subsection 3.3 that the anti-automorphism \( x \mapsto x^t \) coincides with the involution \( i_X \) introduced by Goncharov and Shen, and hence Fomin and Zelevinsky’s original version of twist map in turn coincides with Goncharov and Shen’s involution \( D_X \), which is defined as \( D_X := i_X \circ \text{DT} \) (see Section 1.6 in [GS16]).

The special case of double Bruhat cells in \( GL_n \) was solved in another paper of the author [Wen16a]; however, the bipartite graph method, which was also used in the computation of Donaldson-Thomas transformation of Grassmannian [Wen16b], does not apply in the general case of double Bruhat cells in semisimple Lie groups. The following is an important application of our main result. We proved in our main theorem that the Donaldson-Thomas transformation of the cluster Poisson variety \( H \backslash G^{u,v}/H \) is a cluster transformation. Combined with the work of Gross, Hacking, Keel, and Kontsevich (Theorem 0.10 of [GHKK14]), our result proves the Fock and Goncharov’s conjecture [FG03] in the case of \( H \backslash G^{u,v}/H \).

1.2. Structure of the Paper. We divide the rest of the paper into two sections. Section 2 contains all the preliminaries necessary for our proof of the main theorem. Subsection 2.1 introduces double Bruhat cells \( G^{u,v} \) and related structures, most of which are similar to the original work of Fomin and Zelevinsky [FZ98] and that of Fock and Goncharov [FG06a]. Subsection 2.2 describes a link between double Bruhat cells and configurations of quadruples of Borel subgroups, which is the key to proving the cluster nature of our candidate map for Donaldson-Thomas transformation; such link and the proof of cluster nature are both credit to Shen. Subsections 2.3 and 2.4 review Fock and Goncharov’s theory of cluster ensemble and tropicalization, the main source of reference of which is [FG03]. Subsection 2.5 focuses on the cluster structures related to the main object of our study, namely the double Bruhat cells \( G^{u,v} \), the main resources of reference of which are [FG06a] and [BFZ03].

Section 3 is the proof of our main theorem itself. Subsection 3.1 uses cluster ensemble to rewrite the twist map on \( H \backslash G^{u,v}/H \). Subsection 3.2 constructs the cluster Donaldson-Thomas transformation on \( H \backslash G^{u,v}/H \) and proves that it satisfies the two defining properties of the cluster Donaldson-Thomas transformation, the latter of which is due to a private conversation with Shen. Subsection 3.3 summarizes the relation among all the maps we have discussed thusfar in the paper, which include our version of the twist map, Fomin and Zelevinsky’s version of the twist map, the cluster Donaldson-Thomas transformation on \( H \backslash G^{u,v}/H \) and so on.

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2. Preliminaries

2.1. Structures in Semisimple Lie Groups. This subsection serves as a brief introduction to the structure theory of semisimple Lie groups with focus on structures that will play important roles in our paper. Please see [Hum75] or other standard Lie group textbook for more details.

Let \( G \) be a semisimple Lie group and let \( \mathfrak{g} \) be its Lie algebra. Fix a pair of opposite Borel subgroups \( B_\pm \) in \( G \). Then \( H := B_+ \cap B_- \) is a maximal torus in \( G \). The Weyl group \( W \) of \( G \) is defined to be the quotient \( N_G/H \). It is known that for any Borel subgroup \( B \) of \( G \), there is a Bruhat decomposition \( G = \bigcup_{w \in W} B w B \). Taking the two Bruhat decomposition corresponding to the pair of opposite Borel subgroups \( B_\pm \) and intersecting them, we obtain our object of interest.

**Definition 2.1.** For a pair of Weyl group elements \( (u, v) \in W \times W \), the double Bruhat cell \( G^{u,v} \) is defined to be

\[
G^{u,v} := (B_+ u B_+) \cap (B_- v B_-).
\]
It is known that the commutator subgroup of a Borel subgroup is a maximal unipotent subgroup. We will denote the commutator subgroup of each of the pair of opposite subgroups by $N_\pm := [B_\pm, B_\pm]$.

**Definition 2.2.** The subset of Gaussian decomposable elements in $G$ is defined to be the Zariski open subset $G_0 := N_- H N_+$. A Gaussian decomposition of an element $x \in G_0$ is the factorization

$$x = [x] - [x]_0 [x]_+$$

where $[x]_\pm \in N_\pm$ and $[x]_0 \in H$.

**Proposition 2.3.** The Gaussian decomposition of a Gaussian decomposable element is unique.

**Proof.** It follows from the standard facts that $N_- H = B_-, H N_+ = B_+$, and $N_\pm \cap B_\mp = \{ e \}$.

The adjoint action of $H$ on $\mathfrak{g}$ admits a root space decomposition, and the choice of the pair of opposite Borel subgroups $B_\pm$ defines a subset of simple roots $\Pi$ in the root system. The root system spans a lattice called the root lattice $Q$. There is a dual notion called simple coroots, which can be identified with elements inside the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We will denote the simple coroot dual to $\alpha$ as $H_\alpha$. The Cartan matrix of $G$ can then be defined as

$$C_{\alpha\beta} := \langle H_\alpha, \beta \rangle.$$

The Cartan matrix $C_{\alpha\beta}$ of a semisimple Lie group $G$ is known to have 2 along the diagonal and non-positive integers entries elsewhere. In particular, the Cartan matrix $C_{\alpha\beta}$ matrix is invertible and symmetrizable, i.e., there exists a diagonal matrix $D := \text{diag}(D_\alpha)$ with integer entries such that

$$C_{\alpha\beta} := D_\alpha C_{\alpha\beta}$$

is a symmetric matrix.

The Lie algebra $\mathfrak{g}$ can be generated by the Chevalley generators $E_{\pm\alpha}$ and $H_\alpha$; the relations among the Chevalley generators are

$$[H_\alpha, H_\beta] = 0,$$

$$[H_\alpha, E_{\pm\beta}] = \pm C_{\alpha\beta} E_{\pm\beta},$$

$$[E_{\pm\alpha}, E_{\pm\beta}] = \pm \delta_{\alpha\beta} H_\alpha,$$

$$(\text{ad}_{E_{\pm\alpha}})^{1-C_{\alpha\beta}} E_{\pm\beta} = 0 \quad \text{for } \alpha \neq \beta.$$  

The simple coroots $H_\alpha$ are also cocharacters of $H$, and hence they define group homomorphisms $C^* \to H$; we will denote such homomorphisms by

$$a \mapsto a^{H_\alpha}$$

for any $a \in C^*$. Using the exponential map $\exp : \mathfrak{g} \to G$ we also define a group homomorphisms $e_{\pm\alpha} : \mathbb{C} \to G$ by

$$e_{\pm\alpha}(t) := \exp (t E_{\pm\alpha}).$$

Particularly when $t = 1$ we will omit the argument and simply write $e_{\pm\alpha}$. The arguments $t$ in $e_{\pm\alpha}(t)$ are known as Lusztig coordinates, which can be used to define coordinate system on double Bruhat cells as well (see [FZ98] for details).

It is known that a semisimple Lie group $G$ is generated by elements of the form $e_{\pm\alpha}(t)$ and $a^{H_\alpha}$. We can then define an anti-involution $t$ on $G$ called transposition, which acts on the generators by

$$(e_{\pm\alpha}(p))^t = e_{\mp\alpha}(p) \quad \text{and} \quad (a^{H_\alpha})^t = a^{H_\alpha}.$$  

It is not hard to verify on generators that transposition commutes with taking inverse, which is another commonly seen anti-involution on a semisimple Lie group $G$. Note that if $G$ is the semisimple Lie group $\text{SL}_n$, the transposition anti-involution we defined above is indeed the transposition of matrices.

The set of simple roots also defines a Coxeter generating set $S = \{s_\alpha\}$ for the Weyl group $W$. The braid relations among these Coxeter generators are the following:

$$s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta \quad \text{if } C_{\alpha\beta} C_{\beta\alpha} = 1;$$

$$s_\alpha s_\beta s_\alpha s_\beta = s_\beta s_\alpha s_\beta s_\alpha \quad \text{if } C_{\alpha\beta} C_{\beta\alpha} = 2;$$

$$s_\alpha s_\beta s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha \quad \text{if } C_{\alpha\beta} C_{\beta\alpha} = 3.$$
Definition 2.6. For an element \( w \) in the Weyl group \( W \), a reduced word of \( w \) (with respect to the choice of Coxeter generating set \( S \)) is a sequence of simple roots \( i = (\alpha(1), \alpha(2), \ldots, \alpha(l)) \) which is the shortest among all sequences satisfying \( s_{\alpha(1)} s_{\alpha(2)} \cdots s_{\alpha(l)} = w \). Entries of a reduced word are called letters, and the number \( l \) is called the length of \( w \).

One important fact about reduced words is that any two reduced words of the same Weyl group element can be obtained from each other via a finite sequence of braid relations. It is also known that there exists a unique longest element inside the Weyl group \( W \) of any semisimple Lie group \( G \), and we will denote this longest element by \( w_0 \). It follows easily from the uniqueness that \( w_0^{-1} = w_0 \). Conjugation by any lift of \( w_0 \) swaps the pair of opposite Borel subgroups, i.e., \( w_0 B_\pm w_0 = B_\mp \).

Definition 2.7. For a pair of Weyl group elements \((u, v) \in W \times W\), a reduced word of \((u, v)\) is a sequence \( i = (\alpha(1), \alpha(2), \ldots, \alpha(l)) \) in which every letter is either a simple root or the opposite of a simple root, satisfying the conditions that if we pick out the subsequence consisting of simple roots we get back a reduced word of \( u \), and if we pick out the subsequence consisting of the opposite simple roots and then drop the minus sign in front of every letter we get back a reduced word of \( v \). The number \( l \) is again called the length of the pair \((u, v)\).

Generally speaking, the Weyl group \( W \) does not live inside the Lie group \( G \). However, one can define a lift of each Coxeter generator \( s_\alpha \) by

\[
\overline{s}_\alpha := e_\alpha^{-1} e_{-\alpha}^* e_\alpha^{-1} e_{-\alpha}^* e_\alpha.
\]

It is not hard to verify that these lifts of the Coxeter generators satisfy the braid relations 2.5. Thus by using a reduced word one can define a lift \( \overline{w} \) for any Weyl group element \( w \), which is independent of the choice of the reduced word (see also [FZ98] and [GS16]).

Conjugation by the longest element \( w_0 \) defines an involution \( w^* := w_0 w w_0 \) on the Weyl group \( W \). We can lift such involution up to the semisimple Lie group \( G \) by defining

\[
x^* := \overline{w_0} (x^{-1})^t \overline{w_0}^{-1}.
\]

Since \( w_0 B_\pm w_0 = B_\mp \) and \( B_\pm \) is invariant under the involution \( * \), i.e., \( B_\pm = B_\mp \). We call the involution \( * \) on \( G \) a lift of the involution \( * \) on \( W \) because of the following proposition.

Proposition 2.10. For a Weyl group element \( w \), \( \overline{w^*} = \overline{w}\).

Proof. We only need to show it for the Coxeter generators. Let \( s_\alpha \) be a Coxeter generator. From the definition of the lift \( \overline{s}_\alpha \) we see that \( \overline{s}_\alpha^* = \overline{s}_\alpha^{-1} \). By a length argument we also know that \( s_\beta^* \) is also a Coxeter generator, say \( s_\beta \). Then it follows that \( w_0 s_\beta = s_\beta w_0 \). Since \( l(w_0 s_\alpha) = l(s_\beta w_0) = l(w_0) - 1 \), it follows that

\[
\overline{w_0 s_\alpha} = \overline{s_\beta} \overline{w_0} \overline{s_\beta}^{-1} = \overline{s_\beta} \overline{w_0} \overline{s_\beta}^{-1} = \overline{s_\beta}.
\]

Therefore we know that

\[
\overline{s}_\alpha^* = \overline{w_0} \overline{s}_\alpha \overline{w_0}^{-1} = \overline{s}_\beta.
\]

Each simple root \( \alpha \) also defines a group homomorphism \( \varphi_\alpha : SL_2 \to G \), which maps the generators as follows:

\[
\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto e_\alpha(t), \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto e_{-\alpha}(t), \quad \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto a^{H_\alpha}.
\]

Using this group homomorphism, the lift \( \overline{s}_\alpha \) can be alternatively defined as

\[
\overline{s}_\alpha := \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The following identities are credit to Fomin and Zelevinsky [FZ98], which can be easily verified within \( SL_2 \) and then mapped over to \( G \) under \( \varphi_\alpha \):

\[
e_\alpha(p) e_{-\alpha}(q) = e_{-\alpha} \left( \frac{q}{1 + pq} \right) (1 + pq)^{H_\alpha} e_\alpha \left( \frac{p}{1 + pq} \right); \tag{2.11}
\]

\[
e_\alpha(t) \overline{s}_\alpha = e_{-\alpha} (t^{-1}) t^{H_\alpha} e_\alpha (-t^{-1}); \tag{2.12}
\]
s\alpha^{-1} e_{-\alpha}(t) = e_{-\alpha} \left(-t^{-1}\right) t^{H_\alpha} e_{\alpha} \left(t^{-1}\right).

In general there are more than one Lie group associated to the same semisimple Lie algebra \( \mathfrak{g} \). Among such a family of Lie groups, there is a simply connected one \( G_{sc} \) and a centerless one \( G_{ad} \) (which is also known as the adjoint form and hence the subscript), and they are unique up to isomorphism. Any Lie group \( G \) associated to the Lie algebra \( \mathfrak{g} \) is a quotient of \( G_{sc} \) by some subgroup of the center \( C(G_{sc}) \), and \( G_{ad} \cong G/C(G) \). For the rest of this subsection we will discuss some structures special to each of these two objects.

Let’s start with the simply connected case \( G_{sc} \). One important fact from representation theory is that the finite dimensional irreducible representations of a simply connected semisimple Lie group \( G_{sc} \) are classified by dominant weights, which form a cone inside the weight lattice \( P \). The weight lattice \( P \) can be identified with the lattice of characters of the maximal torus \( H \), and contains the root lattice \( Q \) as a sublattice. The dual lattice of \( P \) is spanned by the simple coroots \( H_\alpha \). Hence by dualizing \( \{ H_\alpha \} \) we obtain a basis \( \{ \omega_\alpha \} \) of \( P \), and we call the weights \( \omega_\alpha \) fundamental weights. The fundamental weights are dominant, and their convex hull is precisely the cone of dominant weights. Since fundamental weights are elements in the lattice of characters of the maximal torus \( H \), they define group homomorphisms \( H \rightarrow \mathbb{C}^* \) which we will denote as \( h \mapsto h^{\omega_\alpha} \).

Further, it is known that for each fundamental weight \( \omega_\alpha \), there is a regular function \( \Delta_\alpha \) on \( G_{sc} \) uniquely determined by the equation

\[ \Delta_\alpha(x) = [x]^{\omega_\alpha}_0 \]

when restricted to the subset of Gaussian decomposable elements of \( G_{sc} \) (see also [Hum75] Section 31.4). In particular, if we consider \( \mathbb{C}[G_{sc}] \) as a representation of \( G_{sc} \) under the natural action \( (x.f)(y) := f(x^{-1}y) \), then \( \Delta_\alpha \) is a highest weight vector of weight \( \omega_\alpha \) (see [FZ98] Proposition 2.2). Such regular functions \( \Delta_\alpha \) are called generalized minors, and they are indeed the principal minors in the case of \( SL_n \).

**Proposition 2.14.** \( \Delta_\alpha(x) = \Delta_\alpha(x^t) \). If \( \beta \neq \alpha \), then \( \Delta_\alpha(\pi^{-1}_\beta x) = \Delta_\alpha(x) = \Delta_\alpha(x\pi_\beta) \).

**Proof.** The first part follows from the fact that transposition swaps \( N_\pm \) while leaving \( H \) invariant. To show the second part, recall that \( \pi_\beta = \beta^{-1} e_{-\beta} \beta^{-1} \); therefore

\[ \Delta_\alpha(\pi^{-1}_\beta x) = \Delta_\alpha(\beta^{-1} e_{-\beta} x) \]

But then since \( \Delta_\alpha \) is a highest weight vector of weight \( \omega_\alpha \), it is invariant under the action of \( e_{-\beta}^{-1} \). Therefore we can conclude that

\[ \Delta_\alpha(\pi^{-1}_\beta x) = \Delta_\alpha(\beta^{-1} e_{-\beta} x) = \Delta_\alpha(e_{-\beta} x) = \Delta_\alpha(x) \]

The other equality can be obtained from this one by taking transposition. \( \square \)

Now let’s turn to the adjoint form \( G_{ad} \). Since the Cartan matrix \( C_{\alpha\beta} \) is invertible, and we can use it to construct another basis \( \{ H^\alpha \} \) of the Cartan subalgebra \( \mathfrak{h} \), which is defined by

\[ H_\alpha = \sum_{\beta} C_{\alpha\beta} H^\beta. \]

Replacing \( H_\alpha \) with \( H^\alpha \) we can rewrite the relations among the Chevalley generators as

\[ [H^\alpha, H^\beta] = 0, \]

\[ [H^\alpha, E_{\pm\beta}] = \pm \delta_{\alpha\beta} E_{\pm\beta}, \]

\[ [E_{\pm\alpha}, E_{\pm\beta}] = \pm \delta_{\alpha\beta} C_{\alpha\gamma} H^\gamma, \]

\[ (\text{ad}_{E_{\pm\alpha}})^{1-C_{\alpha\beta}} E_{\pm\beta} = 0 \quad \text{for } \alpha \neq \beta. \]

It turns out that \( H^\alpha \) are cocharacters of the maximal torus \( H \) of \( G_{ad} \), and hence we can extend our earlier notation \( a \mapsto a^{H^\alpha} \) to \( a \mapsto a^{H^\alpha}. \) In particular the following statement is true.

**Proposition 2.15.** If \( \beta \neq \alpha \), then \( e_{\pm\beta} a^{H^\alpha} = a^{H^\alpha} e_{\pm\beta} \).

**Proof.** This follows the fact that \( [H^\alpha, E_{\pm\beta}] = 0 \) whenever \( \beta \neq \alpha. \) \( \square \)
Note that $H^*$ are generally not cocharacters of the maximal torus $H$ of a non-adjoint-form semisimple Lie group $G$. Thus for a non-adjoint-form semisimple Lie group $G$, $a^{H^*}$ is only well-defined up to a center element; however, since we will consider the double quotient $H \backslash G^{u,v}/H$, this center element ambiguity will be gone because the center $C(G)$ is contained in the maximal torus $H$. The basis $\{H^*\}$ of $\mathfrak{h}$ was first introduced by Fock and Goncharov [FG06a] to give the cluster Poisson structure on the double Bruhat cell $G^{u,v}_{ad}$, which will play an important role in our paper.

2.2. Configuration of Quadruples of Borel Subgroups. In this subsection we will look at the double quotients of double Bruhat cells $H \backslash G^{u,v}/H$ from a different angle. Recall that a semisimple Lie group $G$ acts transitively on the space of its Borel subgroups $\mathcal{B}$ by conjugation, with the stable subgroup of the point $B$ being the Borel subgroup $B$ itself. Therefore we can identify $\mathcal{B}$ with either of the quotients $B_- \backslash G$ or $G/B_+$ (the choice of left vs. right quotients will be clear later). For notation simplicity, we will not distinguish cosets in $B_- \backslash G$, cosets in $G/B_+$, and Borel subgroups of $G$ in this paper, and hence phrases like “the Borel subgroup $xB_+$” should make sense tautologically.

Definition 2.16. Let $B_1 = g_1 B_+$ and $B_2 = g_2 B_+$ be two Borel subgroups; we define a map

$$d_+: \mathcal{B} \times \mathcal{B} \to W$$

$$(B_1, B_2) \mapsto w$$

if $g_1^{-1} g_2 \in B_+ w B_+$. Analogously if $B_1 = B_- g_1$ and $B_2 = B_- g_2$ and $g_1 g_2^{-1} \in B_- w B_-$ then we define

$$d_-(B_1, B_2) := w.$$

Our first observation on these two maps is the anti-symmetry on their arguments: $d_\pm(B_1, B_2) = w$ if and only if $d_\pm(B_2, B_1) = w^{-1}$. But there are more symmetry to these two maps, as we will see in the following several propositions.

Proposition 2.17. The maps $d_\pm$ are $G$-equivariant, i.e., $d_\pm(B_1, B_2) = d_\pm(x B_1 x^{-1}, x B_2 x^{-1})$ for any $x \in G$.

Proof. Just note that the $x$ in the first argument becomes $x^{-1}$ after inverting and cancels with the $x$ in the second argument. □

Recall that we have an involutive automorphism $*$ defined on $G$, which can be extended naturally to the space of Borel subgroups $\mathcal{B}$. Note that $B_+^* = B_-^*$; further we have the following observation.

Proposition 2.18. $d_\pm(B_1, B_2) = w$ if and only if $d_\pm(B_1^*, B_2^*) = w^*$.

Proof. It follows from the fact that $*$ is an automorphism and $B^*_\pm = B_\pm$. □

Proposition 2.19. Let $B_1$ and $B_2$ be two Borel subgroups. Then $d_+(B_1, B_2) = w$ if and only if $d_-(B_1, B_2) = w^*$.

Proof. Let’s show one direction only, for the other direction is completely analogous. Suppose $B_1 = x B_+ x^{-1}$ and $B_2 = y B_+ y^{-1}$. Then $d_+(B_1, B_2) = w$ means that $x^{-1} y \in B_+ w B_+$. On the other hand we know that $B_1 = x w_0 B_- w_0 x^{-1}$ and $B_2 = y w_0 B_- w_0 y^{-1}$; therefore we know that

$$\overline{w_0 x^{-1} y \overline{w_0^{-1}}} \in w_0 B_+ w B_+ w_0 = B_- w^* B_-,$$

which implies $d_-(B_1, B_2) = w^*$. □

Since $d_+$ already contains the information of $d_-$, we introduce the following more concise notation: we write $B_1 \xrightarrow{w} B_2$ to mean $d_+(B_1, B_2) = w$; then $d_-(B_1, B_2) = w$ can be denoted as $B_1 \xrightarrow{w^*} B_2$; moreover, since $w_0^* = w_0^{-1} = w_0$, we further simplify $B_1 \xrightarrow{w_0} B_2$ to $B_1 \xrightarrow{w_0} B_2$ without the arrow or the argument $w_0$.

Proposition 2.20. Let $B_1$ and $B_2$ be two Borel subgroups. Then the followings are equivalent:

1. $B_1$ and $B_2$ are opposite Borel subgroups;
2. $B_1 \longrightarrow B_2$;
3. there exists an element $g \in G$ such that $B_1 = g B_+ g^{-1}$ and $B_2 = g B_- g^{-1}$. 
Lemma 2.22. Let \(\Box\) which implies (2).

and hence \(x\quad\) for \(w\quad\) and \(y\quad\) due to the identification \(B \cong G/B_+\). But then this implies that \(x^{-1}y \in B_+w_0B_+\), which implies (2).

(2) \implies (3). Suppose \(B_1 = xB_+x^{-1}\) and \(B_2 = yB_+y^{-1}\). Then by assumption \(xB_+x^{-1}\) is opposite to \(yB_+y^{-1}\) as Borel subgroups. But this then implies that

\[x^{-1}yB_+y^{-1}x = B_- = w_0B_+w_0,\]

and hence \(x^{-1}yB_+ = w_0B_+\) due to the identification \(B \cong G/B_+\). But then this implies that \(x^{-1}y \in B_+w_0B_+\), which implies (2).

Further the choice of \(g\) in (3) is unique up to a right multiple of an element from \(H\).

Proof. (1) \implies (2). Suppose \(B_1 = xB_+x^{-1}\) and \(B_2 = yB_+y^{-1}\). Then by assumption \(xB_+x^{-1}\) is opposite to \(yB_+y^{-1}\) as Borel subgroups. But then this implies that

\[x^{-1}yB_+y^{-1}x = B_- = w_0B_+w_0,\]

and hence \(x^{-1}yB_+ = w_0B_+\) due to the identification \(B \cong G/B_+\). But then this implies that \(x^{-1}y \in B_+w_0B_+\), which implies (2).

(2) \implies (3). Suppose \(B_1 = xB_+x^{-1}\) and \(B_2 = yB_+y^{-1}\). Then by assumption \(x^{-1}y \in B_+w_0B_+\). Thus we can find \(b\) and \(b'\) from \(B_+\) such that \(x^{-1}y = bw_0b'\). Let \(g := xb\); then

\[gB_+g^{-1} = xB_+x^{-1} = B_1\quad\text{and}\quad gB_-g^{-1} = xbw_0B_+w_0b^{-1}x^{-1} = yB_+y^{-1} = B_2.\]

(3) \implies (1) is trivial since \(gB_+g^{-1}\) and \(gB_-g^{-1}\) are obviously opposite Borel subgroups. For the remark on the uniqueness of \(g\), note that if \(gB_+ = g' B_+\) and \(B_-g^{-1} = B_-g'^{-1}\), then \(g^{-1}g'\) is in both \(B_+\) and \(B_-\); since \(B_+ \cap B_- = H\), it follows that \(g\) and \(g'\) can only differ by a right multiple of an element from \(H\). \(\square\)

Proposition 2.21. Suppose \(l(uv) = l(u) + l(v)\). Then \(B_1 \xrightarrow{uv} B_2\) if and only if there exists a Borel subgroup \(B_3\) such that \(B_1 \xrightarrow{u} B_3\) and \(B_3 \xrightarrow{v} B_2\). In particular, such a Borel subgroup \(B_3\) is unique.

Proof. The existence part follows from the general fact about semisimple Lie groups that

\[(B_1uB_+)(B_1vB_+) = B_+uvB_+\]

whenever \(l(uv) = l(u) + l(v)\) (see for example [Hum75] Section 29.3 Lemma A). The unique part follows from the following lemma, which also holds for all semisimple Lie groups. \(\square\)

Lemma 2.22. Let \(B\) be a Borel subgroup of \(G\). If \(x \in BwB\) where \(w = s_{\alpha(1)}s_{\alpha(2)}\ldots s_{\alpha(l)}\) is a reduced word for \(w\), then there exists \(x_k \in Bs_{\alpha(k)}B\) such that \(x = x_1x_2\ldots x_l\); further if \(x = x_1'x_2'\ldots x_l'\) is another such factorization then \(x_k^{-1}x_k' \in B\) for all \(1 \leq k \leq l - 1\) and \(x_k'x_k^{-1} \in B\) for all \(2 \leq k \leq l\).

Proof. The existence part is essentially the same as the existence part of the above proposition, so it suffices to show the uniqueness part. We will do an induction on \(l\). There is nothing to show for the case \(l = 1\). Suppose \(l > 1\). Let \(x = yx_l = y'x_l'\) where both \(y\) and \(y'\) are in \(Bs_{\alpha(1)}\ldots s_{\alpha(l-1)}B\) and both \(x_1\) and \(x_l'\) are in \(Bs_{\alpha(l)}B\). Then from the fact that \((Bs_{\alpha(l)}B)^2 \subset B \cup Bs_{\alpha(l)}B\) we know that \(x_l'x_l^{-1}\) is in either \(B\) or \(Bs_{\alpha(l)}B\). To rule out the latter possibility, note that if \(x_l'x_l^{-1} \in Bs_{\alpha(l)}B\) then \(y'y_l'x_l^{-1} = xx_l^{-1} = y\) is in both Bruhat cells \(Bs_{\alpha(1)}\ldots s_{\alpha(l-1)}B\) and \(BwB\), which is a contradiction. Thus \(x_l'x_l^{-1} \in B\).

The fact that \(x_l'x_l^{-1} \in B\) implies that \(y^{-1}y' = x_l'x_l^{-1}x_l'x_l^{-1} = x_lx_l^{-1} \in B\). Thus \(y\) and \(y'\) can only differ by a right multiple of \(B\). This difference can be absorbed into the right ambiguity of \(x_{l-1}\), and hence without loss of generality one can assume that \(y = y'\), and the proof is finished by induction. \(\square\)

After all the basic facts and notations, we are ready to link \(B\) to double Bruhat cells \(G^{u,v}\).

Definition 2.23. For a pair of Weyl group elements \((u, v)\) we define the configuration space of Borel subgroups \(\text{Conf}^{u,v}(B)\) to be the quotient space of quadruples of Borel subgroups \((B_1, B_2, B_3, B_4)\) satisfying the following relation

\[B_1 \xrightarrow{u} B_4\]

\[B_3 \xrightarrow{v} B_2\]

modulo the diagonal action by \(G\), i.e., \((gB_1g^{-1}, gB_2g^{-1}, gB_3g^{-1}, gB_4g^{-1}) \sim (B_1, B_2, B_3, B_4)\). We also call diagrams like the one above square diagrams.

As it turns out, such configuration space \(\text{Conf}^{u,v}(B)\) is just our old friend \(H \backslash G^{u,v}/H\) in disguise.

Proposition 2.24. There is a natural isomorphism \(i : \text{Conf}^{u,v}(B) \xrightarrow{\cong} H \backslash G^{u,v}/H\).
Proof. By Proposition 2.20, any element \([B_1, B_2, B_3, B_4]\) in Conf\(_{u,v}(B)\) can be represented by the square diagram

\[
\begin{array}{c}
B_+ \\ \downarrow u \downarrow \\
\mid \mid \\
B_- \\ \uparrow v \uparrow \\
\end{array}
\rightarrow x
\]

for some \(x \in G\), and the choice of \(x\) is unique up to a left multiple and a right multiple by elements in \(H\). Note that by definition of Conf\(_{u,v}(B)\), \(x \in B_+uB_+ \cap B_-vB_-\). Thus the map

\[
i : [B_1, B_2, B_3, B_4] \mapsto H \setminus x/H
\]

is a well-defined map from Conf\(_{u,v}(B)\) to \(H \setminus G_{u,v}/H\), and it is not hard to see that it is indeed an isomorphism. \(\square\)

Let \(w^c := w_0w^{-1}\) for any Weyl group element \(w\); by computation it is not hard to see that \(w_0 = w^c w = w^*w^c\) and \(l(w_0) = l(w^c) + l(w) = l(w^*) + l(w^c)\). But then Proposition 2.21 tells us that we can find two new Borel subgroups \(B_5\) and \(B_6\) to put into the middle of the two vertical edges in the above diagram, forming the following hexagon diagram.

\[
\begin{array}{c}
B_6 \\ \uparrow u^c \uparrow \\
\mid \mid \\
B_1 \\ \downarrow u \downarrow \\
\end{array}
\rightarrow
\begin{array}{c}
B_3 \\ \downarrow u \downarrow \\
\mid \mid \\
B_4 \\ \uparrow u \uparrow \\
\end{array}
\rightarrow
\begin{array}{c}
B_2 \\ \downarrow v \downarrow \\
\mid \mid \\
B_5 \\ \uparrow v \uparrow \\
\end{array}
\rightarrow
\begin{array}{c}
B_4 \\ \uparrow v \uparrow \\
\mid \mid \\
B_5 \\ \downarrow v \downarrow \\
\end{array}
\rightarrow
\begin{array}{c}
B_6 \\ \uparrow u^c \uparrow \\
\mid \mid \\
B_1 \\ \downarrow u \downarrow \\
\end{array}
\]

Note that if we take out the "square" with vertices \(B_3, B_4, B_5,\) and \(B_6,\) and apply the involution \(*\), we get another square diagram of a quadruple representing a point in Conf\(_{u,v}(B)\).

\[
\begin{array}{c}
B_5^* \\ \downarrow u \downarrow \\
\mid \mid \\
B_6^* \\ \uparrow u \uparrow \\
\end{array}
\rightarrow
\begin{array}{c}
B_3^* \\ \downarrow u \downarrow \\
\mid \mid \\
B_4^* \\ \uparrow u \uparrow \\
\end{array}
\]

This observation gives rise to a map

\[
\eta : \text{Conf}_{u,v}(B) \to \text{Conf}_{u,v}(B) \\
[B_1, B_2, B_3, B_4] \mapsto [B_3^*, B_4^*, B_5^*, B_6^*].
\]

Now we are ready to prove part (a)(i) of our main theorem 1.1.

Proposition 2.25. Via the identification Conf\(_{u,v}(B) \cong H \setminus G_{u,v}/H\), the map \(\eta\) can be expressed as the following map on \(H \setminus G_{u,v}/H\) as well:

\[
\eta : H \setminus x/H \mapsto H \setminus \left(\left[\frac{u^{-1}x}{xu^{-1}}\right]_{+}^{-1} \frac{v^{-1}xu^{-1}}{xv^{-1}} \frac{v^{-1}}{x} + 1\right)^t/H.
\]

Proof. Recall that \(H \setminus x/H\) corresponds to a configuration that can be represented as

\[
\begin{array}{c}
B_+ \\ \downarrow u \downarrow \\
\mid \mid \\
B_- \\ \uparrow v \uparrow \\
\end{array}
\rightarrow x
\]

It is not hard to see that

\[
B_5 := xu^{-1}B_+ \quad \text{and} \quad B_6 := B_-v^{-1}
\]
will fit into the hexagon diagram

Thus by definition the \( \eta \) map maps the configuration \([B_+, B_- x^{-1}, B_-, x B_+]\) to

\[
\begin{pmatrix}
B_+ & u^* & B_-
\end{pmatrix}^{-1}
\]

\[
x v^{-1} B_+ \quad v
\]

To compute the corresponding image of \( \eta \) in \( H \setminus G^{u,v}/H \) we need to rewrite the quadruple of Borel subgroups \((B_-, x B_+, x v^{-1} B_+, B_+ x^{-1})\) as \((y B_+, B_- z^{-1}, B_+ y^{-1}, z B_+)\) for some elements \(y\) and \(z\) in \(G\). Following the guideline we have in the proof of Proposition 2.20 we can easily compute

\[
y = \left[ x v^{-1} \right]^{-1} \pi_1^{-1} \quad \text{and} \quad z = \pi \left[ \pi^{-1} x \right]^{-1} \pi_0^{-1}.
\]

Thus the corresponding image of \( \eta \) is

\[
H \setminus g x H = H \setminus (y^{-1} x)^* H = H \setminus \left( [\pi^{-1} x]^{-1} \pi^{-1} x v^{-1} \left[ x v^{-1} \right]^{-1} \right)^t / H.
\]

2.3. Cluster Varieties and Amalgamation. We will give a brief review of Fock and Goncharov’s theory of cluster ensemble and amalgamation. We will mainly follow the coordinate description presented in [FG03] and [FG06a].

**Definition 2.26.** A **seed** \( i \) is a quadruple \((I, I_0, \epsilon, d)\) satisfying the following properties:

1. \( I \) is a finite set;
2. \( I_0 \subset I \);
3. \( \epsilon = (\epsilon_{ab})\) is a \( \mathbb{Q} \)-coefficient matrix in which \( \epsilon_{ab} \) is an integer unless \( a, b \in I_0 \);
4. \( d = (d_a)\) is a \(|I|\)-tuple of positive integers such that \( \epsilon_{ab} := \epsilon_{ab} d_b \) is a skew-symmetric matrix.

In the special case where \( \epsilon_{ab} \) is itself skew-symmetric, the data of a seed defined as above is equivalent to the data of a quiver with vertex set \( I \) and exchange matrix \( \epsilon_{ab} \). Thus by extending the terminology from quivers to seeds, we call elements of \( I \) **vertices**, call elements of \( I_0 \) **frozen vertices**, and call \( \epsilon \) the **exchange matrix**.

**Definition 2.27.** Let \( i = (I, I_0, \epsilon, d) \) be a seed and let \( c \) be a non-frozen vertex. Then the **mutation** of \( i \) at \( c \), which we will denote as \( \mu_c \), gives rise to new seed \( i' = (I', I_0', \epsilon', d') \) where \( I' = I, I_0' = I_0, d' = d \), and

\[
\epsilon'_{ab} = \begin{cases} 
-\epsilon_{ab} & \text{if } c \in \{a, b\} \\
\epsilon_{ab} & \text{if } \epsilon_{ac} \epsilon_{cb} \leq 0 \text{ and } c \notin \{a, b\} \\
\epsilon_{ab} + |\epsilon_{ac}| \epsilon_{cb} & \text{if } \epsilon_{ac} \epsilon_{cb} > 0 \text{ and } c \notin \{a, b\}.
\end{cases}
\]

(2.28)

It is not hard to check that mutating twice at the same vertex gives back the original seed. In fact, if we start with a skew-symmetric matrix \( \epsilon \) so the data of a seed can be translated into a quiver, then seed mutation precisely corresponds to quiver mutation.

Starting with an initial seed \( i_0 \), we say that a seed \( i \) is **mutation equivalent** to \( i_0 \) if there is a sequence of seed mutations that turns \( i_0 \) into \( i \); we denote the set of all seeds mutation equivalent to \( i_0 \) by \([i_0]_0 \). To each seed \( i \) in \([i_0]_0 \) we associate two split algebraic tori \( A_i = (\mathbb{C}^*)^{\lvert I \rvert} \) and \( A'_i = (\mathbb{C}^*)^{|I|} \), which are equipped with
canonical coordinates \((A_a)\) and \((X_a)\) indexed by the set \(I\). These two split algebraic tori are linked by a map \(p_i : A_i \to X_i\) given by
\[
p_i^0(X_a) = \prod_{j \in I} A_{b}^{c_{ab}}.
\]
The split algebraic tori \(A_i\) and \(X_i\) are called a seed \(A\)-torus and a seed \(X\)-torus respectively.

A seed mutation \(\mu_i : i \to i'\) gives rise to birational equivalences between the corresponding seed tori, which by an abuse of notation we also denote both as \(\mu_i\); in terms of the canonical coordinates \((A'_a)\) and \((X'_a)\) they can be expressed as
\[
\mu^*_c(A'_a) = \begin{cases} A_c^{-1} \left( \prod_{c, b > 0} A'^{c}_{b} + \prod_{c, b < 0} A'^{-c}_{b} \right) & \text{if } a = c, \\ A_a & \text{if } a \neq c, \end{cases}
\]
and
\[
\mu^*_c(X'_a) = \begin{cases} X^{-1}_c \left( 1 + X^{-\text{sign}(\epsilon_{ac})}_{a} \right)^{-\epsilon_{ac}} & \text{if } a = c, \\ X_a & \text{if } a \neq c. \end{cases}
\]

These two birational equivalences are called cluster \(A\)-mutation and cluster \(X\)-mutation respectively. One important feature about cluster mutations is that they commute with the respective \(p\) maps.

\[
\begin{array}{ccc}
A_i \xrightarrow{\mu_i} A_{i'} \\
p_i \downarrow & & \downarrow p_{i'} \\
X_i \xrightarrow{\mu_i} X_{i'}
\end{array}
\]

Besides cluster mutations between seed tori we also care about cluster isomorphisms induced by seed isomorphisms. A seed isomorphism \(\sigma : i \to Y\) is a bijection \(\sigma : I \to I'\) that fixes the subset \(I_0 \subseteq I \cap I'\) such that \(\epsilon_{\sigma(i)\sigma(j)} = \epsilon_{ij}\). Given a seed isomorphism \(\sigma : i \to i'\) between two seeds in \([i_0]\), we obtain isomorphisms on the corresponding seed tori, which by an abuse of notation we also denote by \(\sigma\):
\[
\sigma^*(A'_{\sigma(a)}) = A_a \quad \text{and} \quad \sigma^*(X'_{\sigma(a)}) = X_a.
\]
We call these isomorphisms cluster isomorphisms. It is not hard to see that cluster isomorphisms also commute with the \(p\) maps.

\[
\begin{array}{ccc}
A_i \xrightarrow{\sigma} A_{i'} \\
p_i \downarrow & & \downarrow p_{i'} \\
X_i \xrightarrow{\sigma} X_{i'}
\end{array}
\]

Compositions of seed mutations and seed isomorphisms are called seed transformations, and compositions of cluster mutations and cluster isomorphisms are called cluster transformations. A seed transformation \(i \to i\) is called trivial if it induces identity maps on the corresponding seed \(A\)-torus \(A_i\) and seed \(X\)-torus \(X_i\).

**Definition 2.29.** By gluing the seed tori via cluster mutations we obtain the corresponding cluster varieties, which will be denoted as \(A_{[i_0]}\) and \(X_{[i_0]}\) respectively. Cluster transformations can be seen as automorphisms on these cluster varieties. Since the maps \(p_i\) commute with cluster mutations, they naturally glue into a map \(p : A_{[i_0]} \to X_{[i_0]}\) of cluster varieties. The triple \((A_{[i_0]}, X_{[i_0]}, p)\) associated to a mutation equivalent family of seeds \([i_0]\) is called a cluster ensemble.

Cluster ensemble connect the theory of cluster algebras with the Poisson geometry: on the one hand, the coordinate rings on cluster \(A\)-varieties are examples of Fomin and Zelevinsky’s upper cluster algebras [BFZ03], and on the other hand, cluster \(X\)-varieties carry natural Poisson variety structures given by
\[
\{X_a, X_b\} = \epsilon_{ab} X_a X_b.
\]
Thus a cluster \(X\)-variety is also known as a cluster Poisson variety. More details are available in [FG03].

Our next goal of this subsection is to describe a process of constructing cluster ensembles from smaller pieces known as amalgamation; it was first introduced by Fock and Goncharov in [FG06a] when they studied the Poisson structure on double Bruhat cells.
Definition 2.30. Let \( \{i^s\} = \{(I^s, I_0^s, e^s, d^s)\} \) be a finite collection of seeds, together with a collection of injective maps \( i^s: I^s \to K \) for some finite set \( K \) satisfying the following conditions:

1. The images of \( i^s \) cover \( K \);
2. \( (i^s)^{-1}(i^t(I^t)) \subset I_0^s \) for any \( s \neq t \);
3. If \( i^s(a) = i^t(b) \) then \( d_a^s = d_b^t \).

Then the amalgamation of such collection of seeds is defined to be a new seed \( (K, K_0, \epsilon, d) \) where

\[
\epsilon_{ab} := \sum_{i^s(a^s) = a} \epsilon_{a^s b^s}, \quad \quad d_a := d_a^s \quad \text{for any } a^s \text{ with } i^s(a^s) = a,
\]

and \( K_0 := \left( \bigcup_s i^s(I^s_0) \right) \setminus L. \)

The set \( L \) in the last line can be any subset of the set

\[ \{ a \in K \mid \text{both } \epsilon_{ab} \text{ and } \epsilon_{ba} \text{ are integers for all } b \in K \}. \]

In particular, elements of the set \( L \) are called defrosted vertices.

Observe that if \( a = i^s(a^s) \) for some non-frozen vertex \( a^s \in I^s \setminus I_0^s \), then \( a \) cannot possibly lie inside the image of any other \( i^t \). Therefore mutation at \( a \) after amalgamation will give the same seed as amalgamation after mutation at \( a^s \). This shows that amalgamation commutes with mutation at non-frozen vertices ([FG06a] Lemma 2.2).

If the seed \( k \) is the amalgamation of seeds \( i^s \), then on the seed torus level we can induce two amalgamation maps:

\[
\Delta : \mathcal{A}_k \to \prod_s \mathcal{A}_{i^s} \quad \text{and} \quad m : \prod_s \mathcal{A}_{i^s} \to \mathcal{A}_k,
\]

whose pull-backs are

\[
\Delta^*: (A_{a^s}) = A_{i^s(a^s)} \quad \text{and} \quad m^*: (X_a) = \prod_{i^s(a^s) = a} X_{a^s}.
\]

One should think of \( \Delta \) as some sort of diagonal embedding and \( m \) as some sort of multiplication. This point will become much clearer when we construct the cluster structures on double Bruhat cells of semisimple Lie groups.

Proposition 2.31. The map \( p_k : \mathcal{A}_k \to \mathcal{X}_k \) can be factored as the composition

\[
\mathcal{A}_k \xrightarrow{\Delta} \prod_s \mathcal{A}_{i^s} \xrightarrow{\prod_s p_{i^s}} \prod_s \mathcal{X}_{i^s} \xrightarrow{m} \mathcal{X}_k
\]

Proof. This can be verified by direction computation using the definitions of the \( p \) maps and amalgamation maps \( \Delta \) and \( m. \)

Remark 2.32. (Important!) There is a reduced version of a seed \( \mathcal{X} \)-torus, which is obtained as the image of \( \mathcal{X}_k \) under the projection to the coordinates corresponding to non-frozen vertices (and hence is isomorphic to \( (\mathbb{C}^*)^{I \setminus I_0} \)). If we need to distinguish the two seed \( \mathcal{X} \)-tori, we will denote the reduced one as \( \mathcal{X}'_k. \) One can view the reduced seed \( \mathcal{X}' \)-torus \( \mathcal{X}'_k \) as a seed \( \mathcal{X} \)-torus for the seed \( (I \setminus I_0, \emptyset, \xi, d) \) where \( \xi \) and \( d \) are obtained by deleting the data entries of \( \epsilon \) and \( d \) that involve \( I_0 \) respectively. By composing the \( p \) map \( p_1 : \mathcal{A}_1 \to \mathcal{X}_1 \) and the projection map \( \mathcal{X}_1 \to \mathcal{X}'_1 \) we obtain another \( p \) map, which by an abuse of notation we also denote as \( p_1 : \mathcal{A}_1 \to \mathcal{X}'_1 \). In fact, the reduced \( p \) map makes more sense since it is guaranteed to be algebraic, whereas the unreduced one may have fractional exponents.

In addition, by looking at the formulas for cluster \( \mathcal{X} \)-mutation we see that the frozen coordinates never get into the formula of unfrozen ones; therefore we can conclude that the projection \( \mathcal{X}_1 \to \mathcal{X}'_1 \) commutes with cluster transformation. In particular, the reduced seed \( \mathcal{X}' \)-tori also glue together into a reduced cluster
The reduced version is actually more useful and more relevant to our story, but the unreduced one also makes our life easier as it allows us to use amalgamation to define the cluster structure of double Bruhat cells. We therefore decide to include both in this section and use both later when we define the cluster structures on double Bruhat cells; however, after we finish the construction of the cluster structures we will drop the underline notation and use $X_{[a]}$ to denote the reduced cluster $X$-variety.

2.4. Tropicalization. One important feature of the theory of cluster ensemble is that all the maps present in the construction (cluster transformation and the $p$ map) are positive, which enables us to tropicalize a cluster ensemble. For the rest of this subsection we will make this statement precise, and use the tropical language to define cluster Donaldson-Thomas transformation.

Let’s start with the definition of tropicalization. Consider a split algebraic torus $X$. The semiring of positive rational functions on $X$, which we denote as $P(X)$, is the semiring consisting of elements in the form $f/g$ where $f$ and $g$ are linear combinations of characters on $X$ with positive integral coefficients. A rational map $\phi : X \rightarrow Y$ between two split algebraic tori is said to be positive if it induces a semiring homomorphism $\phi^* : P(Y) \rightarrow P(X)$. It then follows that composition of positive rational maps is still a positive rational map.

One typical example of a positive rational map is a cocharacter $\chi$ of a split algebraic torus $X$: the induced map $\chi^*$ pulls back an element $f/g \in P(X)$ to $\frac{f(\chi)}{g(\chi)}$ in $P(C^*)$, where $\langle f, \chi \rangle$ and $\langle g, \chi \rangle$ are understood as linear extensions of the canonical pairing between characters and cocharacters with values in powers of $z$. We will denote the lattice of cocharacters of a split algebraic torus $X$ by $X^t$ for reasons that will become clear in a moment.

Note that $P(C^*)$ is the semiring of rational functions in a single variable $z$ with positive integral coefficients. Thus if we let $Z^t$ be the semiring $(\mathbb{Z}, \max, +)$, then there is a semiring homomorphism $\deg_z : P(C^*) \rightarrow Z^t$ defined by $f(z)/g(z) \mapsto \deg_z f - \deg_z g$. Therefore a cocharacter $\chi$ on $X$ gives rise to a natural semiring homomorphism

$$\deg_z \langle \cdot, \chi \rangle : P(X) \rightarrow Z^t$$

**Proposition 2.33.** The map $\chi \mapsto \deg_z \langle \cdot, \chi \rangle$ is a bijection between the lattice of cocharacters and set of semiring homomorphisms from $P(X)$ to $Z^t$.

**Proof.** Note that $P(X)$ is a free commutative semiring generated by any basis of the lattice of characters, and in particular any choice of coordinates $(X_i)_{i=1}^r$. Therefore to define a semiring homomorphism from $P(X)$ to $Z^t$ we just need to assign to each $X_i$ some integer $a_i$. But for any such $r$-tuple $(a_i)$ there exists a unique cocharacter $\chi$ such that $\langle X_i, \chi \rangle = z^{a_i}$. Therefore $\chi \mapsto \deg_z \langle \cdot, \chi \rangle$ is indeed a bijection. $\square$

**Corollary 2.34.** A positive rational map $\phi : X \rightarrow Y$ between split algebraic tori gives rise to a natural map $\phi^* : X^t \rightarrow Y^t$ between the respective lattice of cocharacters.

**Proof.** Note that $\phi$ induces a semiring homomorphism $\phi^* : P(Y) \rightarrow P(X)$. Therefore for any cocharacter $\chi$ of $X$, the map $f \mapsto \deg_z \langle \phi^* f, \chi \rangle$ is a semiring homomorphism from $P(Y) \rightarrow Z^t$. By the above proposition there is a unique cocharacter $\eta$ of $Y$ representing this semiring homomorphism, and we assign $\phi^*(\chi) = \eta$. $\square$

We also want to give an explicit way to compute the induced map $\phi^*$. Fix two coordinate charts $(X_i)$ on $X$ and $(Y_j)$ on $Y$. Then $(X_i)$ gives rise to a basis $\{\chi_i\}$ of the lattice of cocharacters $X^t$, which is defined by

$$\chi_i^*(X_k) := \begin{cases} z & \text{if } k = i; \\ 1 & \text{if } k \neq i. \end{cases}$$

This basis allows us to write each cocharacter $\chi$ of $X$ as a linear combination $\sum x_i \chi_i$. It is not hard to see that

$$x_i = \deg_z \langle X_i, \chi \rangle.$$
(1) replace addition in $q(X_1, \ldots, X_r)$ by taking maximum;
(2) replace multiplication in $q(X_1, \ldots, X_r)$ by addition;
(3) replace division in $q(X_1, \ldots, X_r)$ by subtraction;
(4) replace every constant by zero;
(5) replace $X_i$ by $x_i$.

It is not hard to see that, given a positive rational map $\phi : \mathcal{X} \to \mathcal{Y}$, the induced map $\phi'$ maps $\sum x_i\chi_i$ to $\sum y_i\eta_j$ where

\begin{equation}
(2.35) \quad y_j := (\phi^*(Y_j))(x_i).
\end{equation}

Now we are ready to define tropicalization.

**Definition 2.36.** The *tropicalization* of a split algebraic torus $\mathcal{X}$ is defined to be its lattice of cocharacters $\mathcal{X}^\vee$ (and hence the notation). For a positive rational map $\phi : \mathcal{X} \to \mathcal{Y}$ between split algebraic tori, the *tropicalization* of $\phi$ is defined to be the map $\phi' : \mathcal{X}^\vee \to \mathcal{Y}^\vee$. The basis $\{\chi_i\}$ of $\mathcal{X}^\vee$ corresponding to a coordinate system $(X_i)$ on $\mathcal{X}$ is called the *basic laminations* associated to $(X_i)$.

Now let’s go back to the cluster varieties $\mathcal{A}_{\mathbf{i}k|l}$ and $\mathcal{X}_{\mathbf{i}k|l}$. Since both cluster varieties are obtained by gluing seed tori via positive birational equivalences, we can tropicalize everything and obtain two new glued objects which we call *tropicalized cluster varieties* and denote as $\mathcal{A}_{\mathbf{i}k|l}^t$ and $\mathcal{X}_{\mathbf{i}k|l}^t$.

Since each seed $\mathcal{X}$-torus $\mathcal{X}_i$ is given a split algebraic torus, it has a set of basic laminations associated to the canonical coordinates $(X_i)$; we will call this set of basic laminations the *positive basic $\mathcal{X}$-laminations* and denote them as $l_i^+$. Note that $\{-l_i^+\}$ is also a set of basic laminations on $\mathcal{X}_i$, which will be called the *negative basic $\mathcal{X}$-laminations* and denote them as $l_i^-$. With all the terminologies developed, we can now state the definition of Goncharov and Shen’s cluster Donaldson-Thomas transformation as follows.

**Definition 2.37** (Definition 2.15 in [GS16]). A *cluster Donaldson-Thomas transformation* (of a seed $\mathcal{X}$-torus $\mathcal{X}_i$) is a cluster transformation $\mathcal{D}T : \mathcal{X}_i \to \mathcal{X}_i$ whose tropicalization $\mathcal{D}T^t : \mathcal{X}_i^t \to \mathcal{X}_i^t$ maps each positive basic $\mathcal{X}$-laminations $l_i^+$ to its corresponding negative basic $\mathcal{X}$-laminations $l_i^-$. Goncharov and Shen proved that a cluster Donaldson-Thomas transformation enjoys the following properties.

**Theorem 2.38** (Goncharov-Shen, Theorem 2.16 in [GS16]). A *cluster Donaldson-Thomas transformation* $\mathcal{D}T : \mathcal{X}_i \to \mathcal{X}_i$ is unique if it exists. If $i'$ is another seed in $\mathbf{i}$ (the collection of seeds mutation equivalent to $i$) and $\tau : \mathcal{X}_i \to \mathcal{X}_{i'}$ is a cluster transformation, then the conjugate $\tau\mathcal{D}T\tau^{-1}$ is the cluster Donaldson-Thomas transformation of $\mathcal{X}_i$. Therefore it makes sense to say that the cluster Donaldson-Thomas transformation $\mathcal{D}T$ exists on a cluster $\mathcal{X}$-variety without referring to any one specific seed $\mathcal{X}$-torus.

From our discussion on tropicalization above, we can translate the definition of a cluster Donaldson-Thomas transformation into the following equivalent one, which we will use to prove our main theorem.

**Proposition 2.39**. A *cluster transformation* $\mathcal{D}T : \mathcal{X}_{\mathbf{i}k|l} \to \mathcal{X}_{\mathbf{i}k|l}$ is a cluster Donaldson-Thomas transformation if and only if on one (any hence any) seed $\mathcal{X}$-torus $\mathcal{X}_i$ with cluster coordinates $(X_i)$, we have

$\deg_{\mathcal{X}_i} \mathcal{D}T^*(X_j) = -\delta_{ij}$

where $\delta_{ij}$ denotes the Kronecker delta.

**Proof.** From Equation (2.35) we see that $\mathcal{D}T^t(l_i^+) = l_i^-$ for all $i$ if and only if $\deg_{\mathcal{X}_i} \mathcal{D}T^*(X_j) = -\delta_{ij}$. \hfill $\square$

2.5. **Double Bruhat Cells as Cluster Varieties.** Recall that there are a pair of semisimple Lie groups $G_{sc}$ and $G_{ad}$ associated to each semisimple Lie algebra $\mathfrak{g}$, with $G_{sc}$ being simply connected and $G_{ad} \cong G_{sc} / C(G_{sc})$ being centerless. In this subsection we will describe a reduced cluster ensemble $(\mathcal{A}^{u,v}, \mathcal{X}^{u,v}, p)$ (Remark 2.32) together with two rational maps

$\psi : G_{sc}^{u,v} \to \mathcal{A}^{u,v}$ and $\chi : \mathcal{X}^{u,v} \to H \setminus G_{sc}^{u,v}$

for any given pair of Weyl group elements $(u,v)$. The first map $\psi$ can be obtained from a cluster algebra result of Berenstein, Fomin, and Zelevinsky [BFZ03], whereas the second map $\chi$ is the amalgamation map.
introduced by Fock and Goncharov in [FG06a]. Unfortunately the seed data used in the two references differ by a sign; we choose to follow Fock and Goncharov’s treatment and later will comment on its relation to Berenstein, Fomin, and Zelevinsky’s result.

The amalgamation that produces the cluster varieties \( \mathcal{A}^{u,v} \) and \( \mathcal{A}'^{u,v} \) comes from the spelling of a (equivalently any) reduced word of the pair of Weyl group elements \((u, v)\). To describe it more precisely, we start with the building block pieces, namely seeds that correspond to letters. Recall that the spelling alphabet is \(-\Pi \cup \Pi\) where \(\Pi\) is the set of simple roots, so the letters naturally come in two types: the ones that are simple roots and the ones that are opposite to simple roots. We will describe the seed data associated to each of these two types.

Let \(\alpha\) be a simple root. We define a new set \(\Pi^\alpha := (\Pi \setminus \{\alpha\}) \cup \{\alpha_-, \alpha_+\}\). One should think of \(\Pi^\alpha\) as almost the same as the set of simple roots \(\Pi\) except the simple root \(\alpha\) splits into two copies \(\alpha_-\) and \(\alpha_+\).

Now we define the seed associated to the simple root \(\alpha\) to be \(i^\alpha := (\Pi^\alpha, \Pi^\alpha, e^\alpha, d^\alpha)\) where

\[
e^\alpha_{ab} := \begin{cases} 
\pm 1 & \text{if } a = \alpha_\pm \text{ and } b = \alpha_\mp; \\
\pm C_{\beta\alpha}/2 & \text{if } a = \alpha_\pm \text{ and } b = \beta \text{ that is neither of the two split copies of } \alpha; \\
\pm C_{\alpha\beta}/2 & \text{if } a = \beta \text{ that is neither of the two split copies of } \alpha \text{ and } b = \alpha_\mp; \\
0 & \text{otherwise,}
\end{cases}
\]

\[
d^\alpha_{ab} := \begin{cases} 
D_\alpha & \text{if } a = \alpha_\pm; \\
D_\beta & \text{if } a = \beta \text{ that is neither of the two split copies of } \alpha.
\end{cases}
\]

Here \(D\) is the diagonal matrix that symmetrize the Cartan matrix \(C\). In contrast, we define the seed associated to \(-\alpha\) to be \(i^{-\alpha} := (\Pi^{-\alpha}, \Pi^{-\alpha}, e^{-\alpha}, d^{-\alpha})\) where

\[
\Pi^{-\alpha} := \Pi^\alpha, \quad d^{-\alpha} := d^\alpha, \quad \text{and} \quad e^{-\alpha} := -e^\alpha.
\]

It is straightforward to verify that \(i^\alpha\) and \(i^{-\alpha}\) are seeds. Note that all vertices of these two seeds are frozen, so there is no mutation available. Before we start amalgamation, we would like to introduce two families rational maps that will go along with amalgamation:

\[
\psi^{\pm\alpha}: G_{sc} \rightarrow \mathcal{A}_{i^{\pm\alpha}} \quad \text{and} \quad \chi^{\pm\alpha}: \mathcal{X}_{i^{\pm\alpha}} \rightarrow G_{ad}.
\]

The rational maps \(\psi^{\pm\alpha}\) are defined by the pull-backs

\[
\psi^{-\alpha*}(A_a) = \begin{cases} 
\Delta_\alpha & \text{if } a = \alpha_-; \\
\Delta_\alpha (\bar{\sigma}_a^{-1} \cdot ) & \text{if } a = \alpha_+; \\
\Delta_\beta & \text{if } a = \beta \text{ that is neither of the two split copies of } \alpha;
\end{cases}
\]

\[
\psi^{\alpha*}(A_a) = \begin{cases} 
\Delta_\alpha (\cdot \bar{\sigma}_a) & \text{if } a = \alpha_-; \\
\Delta_\alpha & \text{if } a = \alpha_+; \\
\Delta_\beta & \text{if } a = \beta \text{ that is neither of the two split copies of } \alpha.
\end{cases}
\]

The maps \(\chi^{\pm\alpha}\) are defined by

\[
\chi^{\pm\alpha}(X_a) \mapsto \left( X_{\alpha_-}^{H^\alpha} e_{\pm\alpha} X_{\alpha_+}^{H^\alpha} \prod_{\beta \neq \alpha} X_{\beta}^{H^\beta} \right),
\]

where the product on the right hand side takes place inside the Lie group \(G_{ad}\).

Now we are ready for amalgamation. Fix a pair of Weyl group elements \((u, v)\) and a reduced word \(i := (\alpha(1), \ldots, \alpha(l))\) of the pair \((u, v)\). The family of seeds that we are amalgamating is

\[
\left\{i^{\alpha(k)}\right\}_{k=1}^{l} := \left\{\left(\Pi^{\alpha(k)}, \Pi^{\alpha(k)}, e^{\alpha(k)}, d^{\alpha(k)}\right)\right\}_{k=1}^{l},
\]

one for each letter \(\alpha(k)\) in the reduced word. To define the amalgamation, we also need a finite set \(K\) and a collection of injective maps \(i^k: \Pi^{\alpha(k)} \rightarrow K\). Let \(n_\alpha\) be the total number of times that letters \(\pm \alpha\) appear in the reduced word \(i\); then we define

\[
K := \left\{\binom{n}{i} \mid \alpha \in \Pi, 0 \leq i \leq n_\alpha\right\}.
\]

Further we define

\[
|\alpha(k)| = \begin{cases} 
\alpha(k) & \text{if } \alpha(k) \text{ is a simple root,} \\
-\alpha(k) & \text{if } \alpha(k) \text{ is opposite to a simple root,}
\end{cases}
\]
which we then use to define the injective maps $i^k : \Pi^{\alpha(k)} \to K$ as follows:

$$i^k(a) = \begin{cases} 
(\alpha(k)_i) & \text{if } a = |\alpha(k)|_i\text{ and }\alpha(k) \text{ is the }i\text{th appearance of letters }\pm|\alpha(k)|; \\
(\alpha(k)_i^{-1}) & \text{if } a = |\alpha(k)|_i\text{ and }\alpha(k) \text{ is the }i\text{th appearance of letters }\pm|\alpha(k)|; \\
(\beta_j) & \text{if } a = \beta \text{ that is neither of the two split copies of }|\alpha(k)| \text{ and there have been }j\text{ numbers of }\pm\beta \text{ appearing before }\alpha(k). 
\end{cases}$$

**Example 2.40.** Although the amalgamation data above is heavy on notations, the idea is intuitive. One should think of $(\alpha_i)$ as the space before the first appearance of $\pm\alpha$, the gap between every two appearances of $\pm\alpha$, or the space after the last appearance of $\pm\alpha$. Then the injective map $i^k$ basically replaces the letter $\alpha(k)$ with the seed $i\alpha(k)$, and the gluing connects to the left via $(\Pi \setminus \{|\alpha(k)|\}) \cup \{|\alpha(k)|_+\}$ and connects to the right via $(\Pi \setminus \{|\alpha(k)|\}) \cup \{|\alpha(k)|_-\}$.

To better convey the idea, consider a rank 3 semisimple group $G$ whose simple roots are $\{\alpha, \beta, \gamma\}$. Let $i := (\alpha, -\beta, -\alpha, \gamma, \beta, -\beta)$ be a reduced word of a pair of Weyl group elements. By writing different letters (disregard the signs) on different horizontal lines, the elements of $K$ then become very clear, as illustrated below.

```
  \alpha_0
  \alpha_1
  \alpha_2

-\alpha

  \beta_0
  \beta_1

  \beta_2

  \beta_3

  \gamma_0
  \gamma_1
```

On the other hand, if we also draw the seed associated to each letter the same way, i.e.,

```
  \alpha_*
  \pm \alpha 
  \alpha_+

  \beta

  \beta_*
  \beta_+

  \gamma

  \gamma_*
  \gamma_+
```

then the injective maps $i^k$ are just placing pieces of building blocks of $i$ into the right positions. We will call the diagram where we put letters associated to different simple roots (disregarding the sign) on different horizontal lines a **string diagram**. In a string diagram we call the letters **nodes** and horizontal lines cut out by nodes **strings**. We say a string is on **level** $\alpha$ if it is on the horizontal line corresponding to the simple root $\alpha$. In particular, we say that a string is **closed** if it is cut out by letters on both ends, and otherwise we say that it is **open**. Obviously a string is open if and only if it is at either end of the string diagram.

**Proposition 2.41.** The set $K$ and injective maps $i^k$ satisfy the conditions for amalgamation (Definition 2.30).

**Proof.** (1) and (2) are obvious, and (3) holds since the data of $d^{\alpha(k)}$ for every seed $i^{\alpha(k)}$ is identical to the diagonal matrix $D$ which symmetrizes the Cartan matrix $C$. \qed

The amalgamated seed obtained this way is not very interesting unless we defrost some of its vertices. As it turns out, the right choice of the set of defrosted vertices is

$$L := \{ \langle \alpha \rangle \mid 0 < i < n_\alpha \} .$$

Of course, we need to show the following statement which is demanded by Definition 2.30.
Proposition 2.42. In the exchange matrix $\epsilon$ of the amalgamated seed, any entry involving elements of $L$ is an integer.

Proof. The only possible non-integer entry of $\epsilon$ always comes from entries $\pm C_{\alpha \beta}/2$ of the exchange matrices of the seeds for individual letters. Notice that if $0 < i < n_\alpha$, then $\binom{i}{1}$ is a gap between two appearances of $\pm \alpha$. Thus any entry of $\epsilon$ involving $\binom{i}{1}$ will have contributions from the both ends of this gap, which will always have the same absolute value with denominator 2 according to the construction. Therefore any entry of $\epsilon$ involving $\binom{i}{1}$ has to be an integer. 

Remark 2.43. By now it should be clear that vertices of the amalgamated seed are in bijection with strings in the corresponding string diagram, under which the frozen vertices are in bijection with the open strings.

Remark 2.44. The seed data obtained from our amalgamation construction differs from the seed constructed by Berenstein, Fomin, and Zelevinsky in [BFZ03] by a minus sign. This is okay since the cluster $A$-mutation formula is invariant under $\epsilon_{ab} \mapsto -\epsilon_{ab}$, and hence Fomin and Zelevinskys upper cluster algebra structure on $\mathbb{C}[G_{sc}^{u,v}]$ is still applicable to our story.

Now we have the complete set of data for the amalgamated seed $(K,K_0,\epsilon,d)$, which by an abuse of notation we will also denote as $i$. Recall that amalgamation can also be lifted to the seed torus level, and hence we have maps

$$\Delta : \mathcal{A}_i \to \prod_{k=1}^l \mathcal{A}_{\psi^{(k)}} \quad \text{and} \quad m : \prod_{k=1}^l \mathcal{X}_{\psi^{(k)}} \to \mathcal{X}_i.$$ 

Our next goal is to find maps to complete the following commutative diagram.

$$
\begin{array}{ccc}
G_{sc} & \xrightarrow{\Delta} & \prod_{k=1}^l G_{sc} \\
\updownarrow \psi_i & & \updownarrow \prod_{k=1}^l \psi_{\alpha^{(k)}} \\
\mathcal{A}_i & \xrightarrow{\Delta} & \prod_{k=1}^l \mathcal{A}_{\psi^{(k)}} \\
\end{array}
\begin{array}{ccc}
\prod_{k=1}^l \mathcal{X}_{\psi^{(k)}} & \xrightarrow{m} & \mathcal{X}_i \\
\downarrow \chi_i & & \downarrow \chi_i \\
\prod_{k=1}^l G_{ad} & \xrightarrow{m} & G_{ad} \\
\end{array}
$$

Among all the new maps we need to define, the easiest one is $m : \prod_{k=1}^l G_{ad} \to G_{ad}$; as the notation suggested, it is just multiplication in $G_{ad}$ following the order $1 \leq k \leq l$.

To define $\Delta : G_{sc} \to \prod_{k=1}^l G_{sc}$, however, requires a bit more work. Recall the choice of reduced word $i = (\alpha(1), \ldots, \alpha(l))$ of our pair of Weyl group elements $(u,v)$. For every $1 \leq k \leq l$ we define two new Weyl group elements $u_{<k}$ and $v_{>k}$ as following: pick out all the entries of $i$ before $\alpha(k)$ (not including $\alpha(k)$) that are opposite to simple roots, and then multiply the corresponding simple reflections to get $u_{<k}$; similarly, pick out all the entries of $i$ after $\alpha(k)$ (not including $\alpha(k)$) that are simple roots, and then multiply the corresponding simple reflections to get $v_{>k}$. The map $\Delta$ is then defined to be

$$\Delta : x \mapsto \left( u_{<k}^{-1} x(v_{>k})^{-1} \right)_{k=1}^l.$$

Now comes the most difficult part, which is completing the square with the remaining yet-to-define map $\psi_i$ and $\chi_i$.

Proposition 2.45. There exists unique maps $\psi_i$ and $\chi_i$ that fit into the commutative diagrams above and make them commute.

Proof. Let’s first consider the commutative diagram on the left. On the one hand, by following the top arrow and then the right arrow of the square we get a rational map $G_{sc} \dashrightarrow \prod_k A_{\psi^{(k)}}$. On the other hand, the bottom arrow is some sort of diagonal embedding and is injective. Therefore all we need to show is that the image of $G_{sc} \dashrightarrow \prod_k A_{\psi^{(k)}}$ lies inside the image of the bottom arrow. In other words, we need to show that if the vertex $a$ of $\psi^{(k)}$ is glued to vertex $b$ of $\psi^{(k+1)}$ during amalgamation, then

$$A_a^{\alpha(k)} \left( \psi^{\alpha(k)} \left( u_{<k}^{-1} x(v_{>k})^{-1} \right) \right) = A_b^{\alpha(k+1)} \left( \psi^{\alpha(k+1)} \left( u_{<k+1}^{-1} x(v_{>k+1})^{-1} \right) \right).$$

There are a few possible cases to analyze, but the arguments are all analogous. Thus without loss of generality let’s assume that $\alpha(k) = \alpha$ and $\alpha(k+1) = \beta$ are both simple roots and $\alpha \neq \beta$. If $a = \beta$ and $b = \beta_-$, then
we see right away from the definition of $\psi^{\alpha(k+1)}$ that both sides of the above equation are exactly the same. If $a = b = \gamma$ for some $\gamma$ other than $\beta$, we then know that the left hand side of the desired equality above is $\Delta_\gamma \left( \frac{u_{<k}}{u_{<k+1}} x(v_{>k})^{-1} \right)$ whereas the right hand side is $\Delta_\gamma \left( \frac{u_{<k}}{u_{<k+1}} x(v_{>k})^{-1} s_\beta \right)$. Since we have assumed that both $\alpha$ and $\beta$ are simple roots, it follows that $u_{<k} = u_{<k+1}$ and $v_{>k} = s_\beta v_{>k+1}$. Therefore we have

$$\Delta_\gamma \left( \frac{u_{<k}}{u_{<k+1}} x(v_{>k})^{-1} \right) = \Delta_\gamma \left( \frac{u_{<k}}{u_{<k+1}} x(v_{>k})^{-1} s_\beta \right),$$

where the last equality is due to Proposition 2.14.

Now let’s turn to the commutative diagram on the right. Observe that the top arrow is surjective. So to define $\chi_1 : X_1 \to G_{ad}$, we can first lift the input to $\prod_k X_{\psi(e)}$ (2.9) and then follow the left arrow and the bottom arrow to arrive at $G_{ad}$. Then all we need to do is the verify that such map is well-defined, i.e., the final output does not depend on the lift. But this is obvious since

$$X_a^H X_b^H = (X_a X_b)^H$$

and Proposition 2.15 tells us that whenever $\beta \neq \alpha$, $e_{\pm, \beta} X^H = X^H e_{\pm, \beta}$. □

Note that the only frozen vertices of the amalgamated seed are of the form $(\alpha)_{a}$ and $(\alpha)_{b}$. According to the map $\chi_1$ we constructed above, the coordinates corresponding to these frozen vertices are equivalent to multiplication by elements of the maximal torus $H$ on the left or on the right. Therefore the map $\chi_1$ can be passed to a map

$$\chi_1 : X_1 \to H \backslash G / H.$$ 

Note that we don’t keep the subscript of $G$ any more because the double quotient of any Lie group $G$ with the same Lie algebra by its maximal torus on both sides is the same variety.

At this point it is natural to ask the question: what if we start with a different reduced word? Is there any relation between the resulting seed $A$-tori and the resulting seed $X$-tori? For the rest of this subsection we will explore the relations among these seed tori, which ultimately will give us the reduced cluster ensemble $\Delta_{\beta}(\alpha, \beta, \gamma_1, \gamma_2)$ and $\Delta_{\gamma_1}(\alpha, \beta, \gamma_2)$ (2.10). According to Proposition 2.15, a composition of $3$ mutations if $C_{\alpha, \beta} C_{\beta, \alpha} = 1$, a composition of $3$ mutations if $C_{\alpha, \beta} C_{\beta, \alpha} = 2$, and a composition of $10$ mutations if $C_{\alpha, \beta} C_{\beta, \alpha} = 3$. 

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Proof. Since both the amalgamation and the two moves as described above are local operations, without loss of generality we may assume that there are no other letters present in the reduced words other than the ones involved in the moves. Let’s consider move (1) first. If $\alpha \neq \beta$, then the string diagram representing move (1) will look like the following.

$$
\begin{array}{c}
\beta_0
\end{array}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
\gamma
\end{array}
\sim
\begin{array}{c}
\beta_0
\end{array}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
\gamma
\end{array}
\begin{array}{c}
\beta
\end{array}

Thus the only pieces of seed data that may be affected by such a move are the entries $\epsilon_{\alpha,\beta}(\gamma)$, $\epsilon_{\alpha,\beta}(\gamma)$, $\epsilon_{\gamma,\beta}(\gamma)$, and $\epsilon_{\gamma,\alpha}(\gamma)$. But the amalgamation construction tells us that they all vanish before and after the move. Thus move (1) does not induce any change on the seed data if $\alpha \neq \beta$.

If we are applying move (1) in the special case $(\alpha, -\alpha) \sim (-\alpha, \alpha)$, then the string diagram will look like the following ($\beta$ can be any simple root other than $\alpha$).

$$
\begin{array}{c}
\beta_0
\end{array}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
\gamma
\end{array}
\sim
\begin{array}{c}
\beta_0
\end{array}
\begin{array}{c}
-\alpha
\end{array}
\begin{array}{c}
\gamma
\end{array}
\begin{array}{c}
-\alpha
\end{array}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
\gamma
\end{array}
\begin{array}{c}
\beta
\end{array}
$$

If we use the unprime notation to denote the exchange matrix of the left and the prime notation to denote that of the right, then we see that the entries of the exchange matrix that got changed under such a move are

$$
\epsilon_{\alpha,\beta}(\gamma) = C_{\beta\alpha}, \quad \epsilon_{\alpha,\beta}(\gamma) = -C_{\alpha\beta},
$$

$$
\epsilon_{\gamma,\beta}(\gamma) = C_{\alpha\beta}, \quad \epsilon_{\gamma,\beta}(\gamma) = -C_{\alpha\beta},
$$

$$
\epsilon_{\alpha,\gamma}(\gamma) = C_{\alpha\beta}, \quad \epsilon_{\alpha,\gamma}(\gamma) = -C_{\alpha\beta},
$$

which change to

$$
\epsilon'_{\alpha,\beta}(\gamma) = -C_{\beta\alpha}, \quad \epsilon'_{\alpha,\beta}(\gamma) = C_{\alpha\beta},
$$

$$
\epsilon'_{\gamma,\beta}(\gamma) = -C_{\alpha\beta}, \quad \epsilon'_{\gamma,\beta}(\gamma) = C_{\alpha\beta},
$$

$$
\epsilon'_{\alpha,\gamma}(\gamma) = -C_{\alpha\beta}, \quad \epsilon'_{\alpha,\gamma}(\gamma) = C_{\alpha\beta},
$$

One can verify easily that such change is exactly the same as a mutation at the vertex $(\gamma)$.

Now let’s consider move (2). Due to symmetry, we will only prove for the case where all the letters are simple roots; the case where the letters are opposite to simple roots is completely analogous. Let’s start with the simplest case where $C_{\alpha\beta}C_{\beta\alpha} = 1$. Then move (2) says that $(\alpha, \beta, \alpha) \sim (\beta, \alpha, \beta)$. The following is the corresponding string diagram.

$$
\begin{array}{c}
1
\end{array}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
2
\end{array}
\sim
\begin{array}{c}
1
\end{array}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
2
\end{array}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
\beta
\end{array}
\begin{array}{c}
4
\end{array}
\begin{array}{c}
\beta
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
\beta
\end{array}
\begin{array}{c}
4
\end{array}
$$

To avoid possible confusion we rename the vertices (gaps) of the seeds with numbers. Note that the vertex 0 in either picture only has non-vanishing exchange matrix entries with the other 4 vertices that are present but nothing else. We claim that the move (2) in this case induces a single seed mutation at the vertex 0. In fact since $C_{\alpha\beta}C_{\beta\alpha} = 1$ we can present the each seed data with a quiver, and it is obvious from the quiver presentation that such a move is indeed a seed (quiver) mutation. (We use the convention that dashed arrows represent half weight exchange matrix in either direction).
Next let’s consider the case $C_{\alpha \beta} C_{\beta \alpha} = 2$, for which move (2) says $(\alpha, \beta, \alpha, \beta) \sim (\beta, \alpha, \beta, \alpha)$. Without loss of generality let’s assume that $C_{\alpha \beta} = -2$ and $C_{\beta \alpha} = -1$. The corresponding string diagram is the following.

\[
\begin{array}{c}
\alpha \quad \alpha \\
\beta \quad \beta \\
\sim \\
\alpha \quad \alpha \\
\beta \quad \beta 
\end{array}
\]

Unfortunately in this case the exchange matrix is not antisymmetric, so we cannot use an ordinary quiver to present the seed data. One way to go around is to introduce the following new notation to replace arrows in a quiver

\[
\begin{array}{c}
a \bullet \quad \longrightarrow \quad \bullet \ b \\
\text{means} \quad \epsilon_{ab} = 1 \quad \text{and} \quad \epsilon_{ba} = -2,
\end{array}
\]

\[
\begin{array}{c}
a \bullet \quad \longrightarrow \quad \bullet \ b \\
\text{means} \quad \epsilon_{ab} = 2 \quad \text{and} \quad \epsilon_{ba} = -1;
\end{array}
\]

we call the resulting picture a quasi-quiver, which is a useful tool to make the mutation computation more illustrative. Translating the picture on the left and the picture on the right into the language of quasi-quiver (again we will use dashed line to represent half weight), we get

\[
\begin{array}{c}
\begin{array}{c}
a \\
b
\end{array}
\end{array}
\quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
a \\
b
\end{array}
\end{array}
\]

We claim that either seed (quasi-quiver) can be obtained from the other via a sequence of 3 mutations at the vertices labeled $a$ and $b$. The following is the quasi-quiver illustration of such mutation sequence.

\[
\begin{array}{c}
\begin{array}{c}
a \\
b
\end{array}
\end{array}
\quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
a \\
b
\end{array}
\end{array}
\]

The case $C_{\alpha \beta} C_{\beta \alpha} = 3$ can be done similarly. To save space, we will put the quasi-quiver demonstration of one possible sequence of 10 mutations corresponding move (2) in Appendix A.

The last two propositions together implies the following statement, which allows us to glue the seed tori $A_i$ and $X_i$ into cluster varieties $A_{u,v}$ and $X_{u,v}$.

**Corollary 2.48.** Any two seeds associated to reduced words of the same pair of Weyl group elements $(u,v)$ are mutation equivalent.

After obtaining the cluster varieties $A_{u,v}$ and $X_{u,v}$, the next natural question to ask is whether the following two diagrams commute, where the map $\mu$ is the cluster transformation induced by a sequence of
move (1) and move (2) that transforms the reduced word \( i \) into the reduced word \( i' \) of the pair of Weyl group elements \((u, v)\).

Proposition 2.49. The two diagrams above commute, and therefore they pass to maps \( \psi : G^{sc}_{u,v} \to H\backslash G/H \) and \( \chi : \mathcal{X}^{u,v} \to H\backslash G/H \). In particular \( \psi \) restricts to a birational equivalence \( \psi : G^{sc}_{u,v} \to \mathcal{A}^{u,v} \), and \( \chi \) has image in \( H\backslash G_{u,v}/H \).

Proof. The commutativity of these two diagrams follows from a collection of identities corresponding move (1) and move (2), which have been laid out by Fomin and Zelevinsky on the \( A \) side [FZ98] and Fock and Goncharov on the \( X \) side [FG06a]. We will simply quote some of these identities from their papers (in particular we omit the formulas for the \( C_{\alpha \beta} \) case) in Appendix B; readers who are interested in these formulas should go to the respective paper to find them.

The fact that \( \psi \) restricts to a birational equivalence \( \psi : G^{sc}_{u,v} \to \mathcal{A}^{u,v} \) was proved by Berenstein, Fomin, and Zelevinsky in [BFZ03]: in their paper they showed that the coordinate ring \( \mathbb{C}[G^{sc}_{u,v}] \) is isomorphic to the upper cluster algebra generated by the seed data associated to any reduced word of the pair \((u, v)\), which is exactly the coordinate ring of our cluster variety \( \mathcal{A}^{u,v} \).

The fact that the image of \( \chi \) lies inside \( H\backslash G_{u,v}/H \) follows from the fact that \( e_{\pm \alpha} \in B_{\pm} \cap B_{\mp} s_\alpha B_{\mp} \) and \( B_{\pm} u B_{\pm} v B_{\pm} = B_{\pm} u v B_{\pm} \) if \( l(u) + l(v) = l(uv) \) ([Hum75] Section 29.3 Lemma A). \( \square \)

Now we finally obtained what we want at the beginning of this subsection, namely the reduced cluster ensemble \((\mathcal{A}^{u,v}, \mathcal{X}^{u,v}, \rho)\) and maps

\[ \psi : G^{sc}_{u,v} \to \mathcal{A}^{u,v} \quad \text{and} \quad \chi : \mathcal{X}^{u,v} \to H\backslash G^{u,v}/H. \]

These structures enable us to rewrite the twist map \( \eta \) in a new way, as we will soon see in the next section.

Remark 2.50. Since from now on we will only focus on the reduced cluster ensemble \((\mathcal{A}^{u,v}, \mathcal{X}^{u,v}, \rho)\), we will drop the underline notation and just write the cluster ensemble as \((\mathcal{A}^{u,v},\mathcal{X}^{u,v},\rho)\).

3. Proof of Our Main Results

3.1. Rewriting the Twist Map in Cluster Language. So far we have seen the twist map (a slight modification from Fomin and Zelevinsky’s original one in [FZ98]) in two different forms: on the one hand, we can write it using Gaussian decomposition

\[ \eta : H\backslash G^{u,v}/H \to H\backslash G^{u,v}/H \]

on the other hand, if we identify the double quotient \( H\backslash G^{u,v}/H \) with the configuration space \( \text{Conf}^{u,v}(\mathcal{B}) \), the twist map can be rewritten as

\[ \eta : \text{Conf}^{u,v}(\mathcal{B}) \to \text{Conf}^{u,v}(\mathcal{B}) \]

\[ [B_1, B_2, B_3, B_4] \mapsto [B_3, B_4, B_5, B_6] \]
where the six Borel subgroups fit into the following hexagon diagram (Proposition 2.25).

In this subsection, we will rewrite the twist map $\eta$ once again using the cluster ensemble $(A_{u,v}, X_{u,v}, p)$ and the maps $\psi$ and $\chi$ we constructed in last subsection and prove part (a).(ii) of our main theorem; we copy down that part of the statement as follows.

**Proposition 3.1.** The twist map $\eta : H \backslash G^{u,v} / H \rightarrow H \backslash G^{u,v} / H$ is rationally equivalent to the following composition of maps:

\[
\begin{array}{c}
G_{sc}^{u,v} \xrightarrow{\psi} A_{u,v} \\
H \backslash G^{u,v} / H \xrightarrow{p} H \backslash G^{u,v} / H
\end{array}
\]

where $G_{sc}^{u,v} \rightarrow H \backslash G^{u,v} / H$ is the quotient map $G_{sc}^{u,v} \rightarrow H \backslash G_{sc}^{u,v} / H \cong H \backslash G^{u,v} / H$, and we go against this map by taking a lift at the very first step. In particular, the resulting map does not depend on the lift that we take.

The key to prove this proposition is the following two lemmas.

**Lemma 3.2.** If $x \in B_+ \cap B_- v B_-$ and $(\alpha(1), \ldots, \alpha(n))$ is a reduced word of $v$, then

\[
[x v^{-1}]^t = \prod_{i=1}^n e_{\alpha(i)}(t_i)
\]

where

\[
t_i := \frac{\prod_{\beta \neq \alpha(i)} (\Delta_\beta (x(v_{>i}^{-1}))^{-C_{\beta \alpha(i)}})}{\Delta_{\alpha(i)} (x(v_{>i}^{-1})) \Delta_{\alpha(i)} (x(v_{>i-1}^{-1}))}.
\]

**Proof.** We will prove by induction on $n$. There is nothing to show when $n = 0$. For $n > 0$, we may assume without loss of generality that

\[
x = \left( \prod_{\beta} a_\beta^{H_\beta} \right) e_{\alpha(1)}(p_1) \ldots e_{\alpha(n)}(p_n).
\]

Then it is obvious that for any simple root $\beta$,

\[
\Delta_\beta(x) = a_\beta.
\]

Next let’s consider $x(v_{>n-1}^{-1})^{-1} = x\pi(\alpha(n))$. By using the identity

\[
e_{\beta}(p)\pi(\beta) = e_{-\beta}(p^{-1})p^{H_\beta} e_{\beta}(-p^{-1})
\]

we get

\[
x\pi(\alpha(n)) = \left( \prod_{\beta} a_\beta^{H_\beta} \right) e_{\alpha(1)}(p_1) \ldots e_{\alpha(n-1)}(p_{n-1}) e_{-\alpha(n)}(p_n^{-1}) p_n^{H_\alpha(n)} e_{\alpha(n)}(-p_n^{-1}).
\]

Now our mission is to move the factor $e_{-\alpha(n)}(p_n^{-1}) p_n^{H_\alpha(n)}$ all the way to the front so that we can use induction. The way to do this is to use the fact that $e_{\beta}(p)$ commutes with $e_{-\alpha}(q)$ whenever $\beta \neq \alpha$, plus the
following identities (the first one is identity (2.11) and the second one comes from the Lie algebra identity $[H_n, E_\beta] = C_{\alpha\beta} E_\beta$):

\[
e^\beta(q)e^-\beta(p) = e^-\beta \left( \frac{p}{1 + pq} \right) (1 + pq)^{H_\beta} e^\beta \left( \frac{q}{1 + pq} \right); \]
\[
e^\beta(q)a^{H_n} = a^{H_n} e^\beta \left( a^{-C_{\alpha\beta}} q \right).
\]

Combining these facts we see that

\[
e^\beta(q)e^-\alpha(n)(p-1)p^{H_n} = \begin{cases} e^-\alpha(n)(p^{-1})p^{H_n} e^\beta(\cdots), & \text{if } \beta \neq \alpha(n); \\
e^-\alpha(n) \left( \frac{1}{p^2 - q} \right) (p + q)^{H_n} e^\beta(\cdots), & \text{if } \beta = \alpha(n).
\end{cases}
\]

Using the above identity recursively, we get

\[
x^\alpha(n) = e^-\alpha(n) \left( \prod_{\alpha(i_k) = \alpha(n)} a^{-C_{\beta(i_k)}} \right) \left( \sum_{\alpha(i_k) = \alpha(n)} p_{i_k} \right)^{H_n} \left( \prod_{\beta} a^{H_\beta} \right) e^\alpha(\cdots) \cdots e^\alpha(n-1)(\cdots)e^\alpha(n)(-p_n^{-1}).
\]

Thus it follows that

\[
\Delta_\alpha(n) \left( x^\alpha(n) \right) = a_\alpha(n) \sum_{\alpha(i_k) = \alpha(n)} p_{i_k}.
\]

Note that if we define $t_n := \prod_{\beta \neq \alpha(n)} a^{-C_{\beta(n)}} \sum_{\alpha(i_k) = \alpha(n)} p_{i_k}$, then it follows that

\[
t_n = \frac{\prod_{\beta \neq \alpha(n)} a^{-C_{\beta(n)}}}{a_\alpha(n) \sum_{\alpha(i_k) = \alpha(n)} p_{i_k}} = \frac{\prod_{\beta \neq \alpha(n)} \left( x^{\alpha(n)} \right)^{-C_{\beta(n)}}}{\Delta_\alpha(n)(x) \Delta_\alpha(n) \left( x^\alpha(n) \right)}.
\]

On the other hand, if we define $\ell' := e^-\alpha(n)(-t_n) x^\alpha(n) e^\alpha(n)(p_n^{-1})$

\[
\ell' = \left( \sum_{\alpha(i_k) = \alpha(n)} p_{i_k} \right)^{H_n} \left( \prod_{\beta} a^{H_\beta} \right) e^\alpha(\cdots) \cdots e^{\alpha(n-1)}(\cdots),
\]

then it follows that $x' \in B_+ \cap B_- \ell' B_-$ and

\[
\left[ x^{\ell'} \right] = e^-\alpha(n)(t_n) \left[ x^{\ell'} \right]_-
\]

Note that $l(\ell') = l(\ell) - 1$; hence we can use induction to finish the proof. The only remaining thing one needs to realize is that

\[
\Delta_\beta \left( x^{\ell'}(\ell')^{-1} \right) = \Delta_\beta \left( e^-\alpha(n)(-t_n) x^\alpha(n) e^\alpha(n)(p_n^{-1}) (\ell')^{-1} \right) = \Delta_\beta \left( x^{\ell'}(\ell')^{-1} \right).
\]

The last equality holds because $v_{\ell'}(\alpha(n)) > 0$ and hence

\[
e^{\alpha(n)}(p_n^{-1})(\ell')^{-1} = (\ell')^{-1} n_+
\]

for some unipotent element $n_+ \in N_+$. \hfill \Box

Analogously, one can also prove the following lemma.

**Lemma 3.3.** If $x \in B_+ u B_+ \cap B_-$ and $(\alpha(1), \ldots, \alpha(m))$ is a reduced word of $u$, then

\[
\left[ \eta^{-1} x \right] = \prod_{i=1}^{m} e^-\alpha(i)(t_i)
\]

where

\[
t_i := \frac{\prod_{\beta \neq \alpha(i)} \left( \Delta_\beta \left( \eta^{-1} x \right) \right)^{-C_{\beta(i)}}}{\Delta_\alpha(i) \left( \eta^{-1} x \right) \Delta_\alpha(i) \left( \eta^{i+1} x \right)}.
\]
Proof of Proposition 3.1. Since every seed torus is a Zariski open subset of the corresponding cluster variety and we only care about rational equivalence, we can reduced Proposition 3.1 to proving that the twist map is rationally equivalent to the composition

\[
G_{sc}^{u,v} = \frac{\psi_i}{\pi} \rightarrow A_i
\]

\[
H \backslash G_{sc}^{u,v} / H \xrightarrow{\chi_i} X_i \xrightarrow{\pi} H \backslash G_{sc}^{u,v} / H
\]

for some nice reduced word \( i \) of the pair of Weyl group elements \((u, v)\). Our choice of reduced word for this proof is any reduced word \( i := (\alpha(1), \ldots, \alpha(n), \alpha(n+1), \ldots, \alpha(l)) \) satisfies the fact that \((\alpha(1), \ldots, \alpha(n))\) is a reduced word of \( v \) (so all the letters \( \alpha(1), \ldots, \alpha(n) \) are simple roots) and \(-(\alpha(n+1), \ldots, \alpha(l))\) is a reduced word of \( u \) (so all the letters \( \alpha(n+1), \ldots, \alpha(l) \) are opposite to simple roots). In other words, our reduced word \( i \) can be broken down into two parts: the \( v \) part and the \( u \) part, as depicted below

\[
i := (\alpha(1), \ldots, \alpha(n), \alpha(n+1), \ldots, \alpha(l)).
\]

Let’s ignore the lifting at the very first step for now. Suppose we start with an element \( x \in G_{sc}^{u,v} \). As the space of Gaussian decomposable elements \( G_0 \) is dense in \( G_{sc}^{u,v} \) (see [FZ98] Proposition 2.14), we may further assume that \( x \) is Gaussian decomposable, i.e., \( x = [x] - [x_0]_0) \). In particular, from the assumption that \( x \in G_{sc}^{u,v} \) we know that \([x] - [x_0]_0 \in B_+ \cap B_- \) and \([x_0]_0 \in B_+ \cap B_- \). Now the twist map \( \eta \) maps \( H \backslash x / H \) to

\[
\eta(x) = H \left( \left[ u^{-1} x \right]_0^{-1} \right) / H.
\]

By applying the two lemmas we had above, we see that the element in the middle can be rewritten as

\[
\left( \prod_{i=1}^{n} e_{\alpha(i)}(t_i) \right) \left[ x_0 \right]_0^{-1} \left( \prod_{i=n+1}^{l} e_{\alpha(i)}(t_i) \right)
\]

where

\[
t_i := \begin{cases} 
\prod_{\beta \neq \alpha(i)} \Delta_\beta \left( [x_0]_0 x + (v_{>i})^{-1} \right)^{-C_\beta \alpha(i)} & \text{if } 1 \leq i \leq n; \\
\Delta_\alpha(i) \left( [x_0]_0 x + (v_{>i})^{-1} \right) \Delta_\alpha(i) \left( [x_0]_0 x + (v_{>i-1})^{-1} \right) & \text{if } n + 1 \leq i \leq l.
\end{cases}
\]

But then since \( \Delta_\gamma \left( [x_0]_0 x + (v_{>i})^{-1} \right) = \Delta_\beta \left( x(v_{>i})^{-1} \right) \) and \( \Delta_\gamma \left( \frac{1}{u_{<i}} \right) = \Delta_\gamma \left( \frac{1}{u_{<i}} \right) \) for any \( \gamma \), we can rewrite the \( t_i \)'s as

\[
t_i := \begin{cases} 
\prod_{\beta \neq \alpha(i)} \Delta_\beta \left( x(v_{>i})^{-1} \right)^{-C_\beta \alpha(i)} & \text{if } 1 \leq i \leq n; \\
\Delta_\alpha(i) \left( x(v_{>i})^{-1} \right) \Delta_\alpha(i) \left( x(v_{>i-1})^{-1} \right) & \text{if } n + 1 \leq i \leq l.
\end{cases}
\]

Note that every generalized minor factor present in the above expression is a cluster coordinate of the seed torus \( A_i \)!. This should be the first hint why the twist map should be translatable into cluster language.
To put things into the right places, we need the following additional identities:

\[ e_{\alpha}(t) = t^{\alpha^{\circ}} e_{\alpha} t^{-\alpha^{\circ}}, \quad e_{-\alpha}(t) = t^{-\alpha^{\circ}} e_{-\alpha} t^{\alpha^{\circ}}. \]

Note that we have secretly moved from the \( G_{ad} \) territory into the \( G_{ad} \) territory: the right hand side of either identities above obviously lives in \( G_{ad} \). This is okay because what we care at the end is the image of \( \eta \) in the double quotient \( H \backslash G^{\alpha,\circ} / H \), and it’s completely fine to project \( \left( \prod_{i=1}^{n} e_{\alpha(i)}(t_i) \right) [x]_0^{-1} \left( \prod_{i=n+1}^{l} e_{\alpha(i)}(t_i) \right) \) into \( G_{ad} \) first before taking the double quotient. We now make the bold claim that

\[
H \backslash \left( \prod_{i=1}^{n} e_{\alpha(i)}(t_i) \right) [x]_0^{-1} \left( \prod_{i=n+1}^{l} e_{\alpha(i)}(t_i) \right) / H = H \backslash (\chi_3 \circ p_k \circ \psi_1(x)) / H.
\]

To show this, we break the argument down into three cases.

1. The first case is for vertices between two occurrences of the same simple root, i.e., vertices that are strictly contained in the \( \nu \) part of the seed \( i \). The string diagram looks like the following, and we suppose \( b \) is the vertex that we are interested in.

\[
\begin{array}{ccccccc}
  \alpha(i) & b & \alpha(j) & c \\
  & & & & & \\
  \vdots & & & & & \\
  d & \alpha(k) & e & \ldots & f & \alpha(m) & g
\end{array}
\]

From the way we constructed the seed data, we know that \( \epsilon_{bc} = \epsilon_{bf} = 0 \), and the cluster \( \chi \)-coordinate

\[
X_b \left( p_k \circ \psi_1(x) \right) = A_{\alpha(i)}(\psi_1(x)) \left( A_{\beta}(\psi_1(x)) \right)^{C_{\alpha(m)\alpha(j)}} \ldots \\
A_{\gamma}(\psi_1(x)) \left( A_{\delta}(\psi_1(x)) \right)^{C_{\alpha(m)\alpha(j)}} \ldots \\
= \Delta_{\alpha(i)} \left( x(v_{>1-1})^{-1} \right) \left( \Delta_{\alpha(m)} \left( x(v_{>m-1})^{-1} \right) \right)^{C_{\alpha(m)\alpha(j)}} \ldots \\
= \Delta_{\alpha(j)} \left( x(v_{>j-1})^{-1} \right) \left( \Delta_{\alpha(k)} \left( x(v_{>k-1})^{-1} \right) \right)^{C_{\alpha(k)\alpha(i)}} \ldots \\
= \Delta_{\alpha(i)} \left( x(v_{>i-1})^{-1} \right) \left( \prod_{\beta \neq \alpha(j)} \Delta_{\beta} \left( x(v_{\beta-1})^{-1} \right) \right)^{C_{\beta\alpha(j)}} \ldots \\
= \Delta_{\alpha(j)} \left( x(v_{>j-1})^{-1} \right) \left( \prod_{\beta \neq \alpha(i)} \Delta_{\beta} \left( x(v_{\beta-1})^{-1} \right) \right)^{C_{\beta\alpha(i)}} \ldots \\
= \Delta_{\alpha(i)} \left( x(v_{>i-1})^{-1} \right) \left( x(v_{>1})^{-1} \right)^{C_{\beta\alpha(i)}} \ldots \\
= \delta_{i}^{-1} t_{j}.
\]

On the other hand, we also know that inside the product \( \left( \prod_{k=1}^{n} e_{\alpha(k)}(t_k) \right) [x]_0^{-1} \left( \prod_{k=n+1}^{l} e_{\alpha(k)}(t_k) \right) \), the two factors corresponding to the letters \( \alpha(i) \) and \( \alpha(j) \) are

\[
e_{\alpha(i)}(t_i) = t_i^{H_{\alpha}^{(i)}} e_{\alpha(i)} t_i^{-H_{\alpha}^{(i)}} \quad \text{and} \quad e_{\alpha(j)}(t_j) = t_j^{H_{\alpha}^{(j)}} e_{\alpha(j)} t_j^{-H_{\alpha}^{(j)}}.
\]

Since there are no other letter that is the simple root \( \alpha(i) = \alpha(j) = \alpha \) between the \( i \)th and the \( j \)th place, we know that the \( \chi \)-variable \( X_b \) is exactly the \( H_{\alpha} \) part of whatever that lies between the two letters, i.e.,

\[
\ldots e_{\alpha(i)}(t_i) \ldots e_{\alpha(j)}(t_j) \ldots = \ldots e_{\alpha} X_b^{H_{\alpha}} \ldots e_{\alpha} \ldots
\]

2. By a completely symmetric argument, we can also take care of the case of vertices between two occurrences of the same letter that is opposite to a simple root, i.e., vertices that are strictly contained in the \( \mu \) part of the seed \( i \).
(3) So the remaining case is for the vertices that lie between the \( v \) part and the \( u \) part of the seed. Consider the vertex \( a \) in the string diagram below, where \( \alpha(i) = -\alpha(j) = \alpha \) is a simple root.

\[ \alpha(i) \quad \alpha(j) \]

By an analysis similar to what we have done in part (1), we know that the cluster \( X' \)-coordinate

\[ X_a (p_i \circ \psi_i(x)) = \frac{\prod_{\beta \neq \alpha} (\Delta_{\beta}(x))^{-C_{\beta \alpha}}}{t_i t_j (\Delta_{\alpha}(x))^2} = t_i^{-1} t_j^{-1} \prod_{\beta} (\Delta_{\beta}(x))^{-C_{\beta \alpha}}. \]

This is perfect because inside the product \( \prod_{k=1}^{n} e_{\alpha(k)}(t_k) \), we know that by adding an extra map \( \chi \) to the replacement of \( x \)

Theorem 1.6). In particular, since the twist map \( \eta \) is dominant, so is the map \( \chi \). But then since \( X'_i \) and

\[ \eta \]

In conclusion, by applying \( \chi \) to the image \( p_i \circ \psi_i(x) \), we have successfully recovered our desired image \( H \backslash \left( \prod_{\beta} (\Delta_{\beta}(x))^{-C_{\beta \alpha}} \right)^{H^\alpha} \).

So far we have shown that if we start with an element \( x \) in \( G_{sc}^{u,v} \), then

\[ H \backslash \left( \prod_{\beta} (\Delta_{\beta}(x))^{-C_{\beta \alpha}} \right)^{H^\alpha} \]

to finish the proof of the proposition, we also need to show that the image does not depend on the lift from \( H \backslash G^{u,v} / H \) to \( G_{sc}^{u,v} \) in the first place. Suppose instead of \( x \), we start with \( h x \) for some element \( h \in H \). Then

\[ H \backslash \left( \prod_{\beta} (\Delta_{\beta}(x))^{-C_{\beta \alpha}} \right)^{H^\alpha} \]

But by the definition of the Weyl group \( W := N_G H / H \), we know that \( \pi^{-1} \) normalizes \( H \); therefore \( \pi^{-1} h = h' \pi^{-1} \) for some \( h' \in H \) and we can simplify the above equality to

\[ H \backslash \left( \prod_{\beta} (\Delta_{\beta}(x))^{-C_{\beta \alpha}} \right)^{H^\alpha} \]

A similar argument can also be applied to the replacement of \( x \) by \( h' x \) for some \( h \in H \). This finishes our proof of Proposition 3.1.

\[ \square \]

**Corollary 3.4.** For any reduced word \( i \) of a pair of Weyl group elements \((u,v)\), there is a rational map \( \psi_i : H \backslash G^{u,v} / H \rightarrow X_i \) that makes the following diagram commute.

\[
\begin{array}{ccc}
G_{sc}^{u,v} & \xrightarrow{\psi_i} & A_i \\
\downarrow & & \downarrow \\
H \backslash G^{u,v} / H & \xrightarrow{\psi_i} & X_i
\end{array}
\]

In particular, these rational maps \( \psi_i \) can be glued into a rational map \( \psi : H \backslash G^{u,v} / H \rightarrow X^{u,v} \).

**Proof:** To show that such a rational map exists, it suffices to show its well-defined-ness. From Proposition 3.1 we know that by adding an extra map \( \chi_i : X_i \rightarrow H \backslash G^{u,v} / H \) to the lower right hand corner we get the twist map \( \eta \), which is a birational equivalence since it only differs from Fomin and Zelevinsky’s twist map by an anti-involution and Fomin and Zelevinsky proved that their twist map is a birational involution ([FZ98] Theorem 1.6). In particular, since the twist map \( \eta \) is dominant, so is the map \( \chi_i \). But then since \( X_i \) and
$H\backslash G_{sc}^{u,v} / H$ are algebraic varieties of the same dimension, the fibers of $\chi_i$ can only be finite sets of points. But the fibers of $G_{sc}^{u,v} \to H \backslash G_{sc}^{u,v} / H$ are algebraic tori, which are irreducible; therefore the image of each such fiber must lie entirely within a single point, and this proves the well-defined-ness of the rational map $\psi_1 : H \backslash G_{sc}^{u,v} / H \dashrightarrow X_i$.

The gluability of $\psi_1$ directly follows from the gluabilities of $\psi_1 : G_{sc}^{u,v} \dashrightarrow A_i$ and $p_k : A_i \to X_i$. $\square$

3.2. Construction of the Donaldson-Thomas Cluster Transformation. Now we have got two maps between the space $H \setminus G_{sc}^{u,v} / H$ and the cluster Poisson variety $\mathcal{X}_{sc}^{u,v}$, namely

$$\psi : H \setminus G_{sc}^{u,v} / H \dashrightarrow \mathcal{X}_{sc}^{u,v} \quad \text{and} \quad \chi : \mathcal{X}_{sc}^{u,v} \to H \setminus G_{sc}^{u,v} / H.$$ In last subsection we have proved that the composition $\chi \circ \psi$ is rationally equivalent to the twist map $\eta$ on the double quotient $H \setminus G_{sc}^{u,v} / H$, which has been interpreted in many different ways. In this subsection we will consider the composition in the opposite direction, i.e.,

$$\text{DT} := \psi \circ \chi,$$

and show that DT is in fact the Donaldson-Thomas cluster transformation of the cluster Poisson variety $\mathcal{X}_{sc}^{u,v}$.

Following our subscript convention, we define $\text{DT}_1 := \psi_1 \circ \chi_1$ to be the restriction of DT to the seed torus $X_i$ for any reduced word $i$ of the pair of Weyl group elements $(u,v)$. According to Theorem 2.38 and Proposition 2.39, in order to achieve our goal, it suffices to show two following things for some (equivalently any) reduced word $i$ of $(u,v)$:

- $\text{DT}_1$ satisfy the condition that $\deg_{\chi_i} \text{DT}_1 (X_a) = -\delta_{ab}$;
- $\text{DT}_1 : X_i \dashrightarrow X_i$ is a cluster transformation.

Again, proving these two statements comes down to choosing the right reduced word $i$. In contrast to our choice in the last subsection, this time our choice of reduced word has the $u$ part comes before the $v$ part:

$$i := (\alpha(1), \ldots, \alpha(n), \alpha(n+1), \ldots, \alpha(l)),$$

in other words, the first $n$ letters are opposite to simple roots, which will give a reduced word of $u$ if you switch their signs, and the last $l - n$ letters are simple roots, which gives a reduced word of $v$.

Fix one such reduced word $i$. Let $(X_a)$ be a generic point in the seed torus $X_i$ and let $x$ be an element in $G_{sc}^{u,v}$ such that

$$H \setminus x / H = \chi_i(X_a).$$

To compute $\text{DT}_1(X_a)$, we need to first compute the image $\psi_1(x)$, whose coordinates are generalized minors of the form

$$\Delta_{a_1} \left( \prod_{i=1}^n x_{i}^{-1} \right) \quad \text{or} \quad \Delta_{a} \left( \prod_{i=n+1}^l x_{i}^{-1} \right),$$

depending the corresponding vertex in the seed.

The good news is that we sort of already know the Gaussian decompositon of $x$: from the definition $x := \chi_1(x)$ we know that we can write $x$ as a product of $e_{\alpha(i)}$ and $X^{H_{\omega}}_{\beta}$, and since the $u$ part of $i$ comes before the $v$ part, the first half of such product is in $B_-$ while the second half is in $B_+$. But then since $N_{\pm}$ is normal in $B_\pm$, we know that for any $n \in N_{\pm}$ and $h \in H$, there always exists $n_\pm'$ such that

$$hn_\pm = n_\pm' h \quad \text{and} \quad n_\pm h = hn_\pm'.$$

Using these two identities we can conclude that

$$\Delta_{a} (x) = \Delta_{a} \left( \prod_{\beta, i} X^{H_{\omega_{\beta}}}_{\beta i} \right) = \prod_{\beta, i} X^{H_{\beta} \omega_{\beta}}_{\beta i} = \prod_{\beta, i} X^{C^{-1}_{\beta a}}_{\beta i},$$

where $C^{-1}$ denotes the inverse of the Cartan matrix $C$.

A couple of the bad things happen here. One is that although the Cartan matrix $C$ has integer entries, it is generally not true for $C^{-1}$ to have integer entries, and we may run into trouble of taking fraction power of an algebraic variable. However, this may not be as bad as we imagine after all, because at the end of the day what we need is not individual generalized minors but some of their ratios, and hopefully these ratios
are algebraic over the cluster variables. The other bad thing is that $\Delta_\alpha(x)$ is not even on the list of the

generalized minors we are trying to compute!

Was our effort in vain? Of course not. We can modify it slightly to compute the generalized minors we
actually need. The key is again the identities (2.12) and (2.13), which we will write them down again:

$$e_\alpha(t)\sigma_\alpha = e_{-\alpha} \left( t^{e_\alpha} \right) t^{H_\alpha} e_\alpha \left( t^{-1} \right); \quad \sigma_\alpha^{-1} e_{-\alpha}(t) = e_{-\alpha} \left( t^{e_\alpha} \right) t^{H_\alpha} e_\alpha \left( t^{-1} \right).$$

From these two identities we see that multiplying $\sigma_{\alpha(i)}$ on the right of $x := \chi_i(X_a)$, assuming that $v$
is non-trivial so that $\alpha(i)$ is a simple root, changes the very last factor $e_{\alpha(i)}$ to $e_{-\alpha(i)} e_{\alpha(i)}^{-1}$; now if we were to
calculate $\Delta_\alpha(x \sigma_\alpha)$, we just need to move the factor $e_{-\alpha(i)}$ all the way through the $B_+$ part of the product
$\chi_i(X_a)$ into the $N_-$ part by using the identity (2.11)

$$e_{\alpha}(p)e_{\alpha}(q) = e_{\alpha} \left( \frac{q}{1 + pq} \right) \left( 1 + pq \right) e_\alpha \left( \frac{p}{1 + pq} \right),$$

and then pick up whatever is left in the $H$ part and compute the generalized minors. Variation of this
observation is also true when we multiply $\sigma_{\alpha(i-1)}, \sigma_{\alpha(i-2)}, \ldots$ on the right and $\sigma_{\alpha(1)}^{-1}, \sigma_{\alpha(2)}^{-1}, \ldots$ on the left of

$x$.

But things can still get very messy as we keep applying these identities. Fortunately we don’t actually
need the explicit expression of $DT^*_i(X_a)$ but just to verify the identity $\deg_{\chi_i} DT^*_i(X_a) = -\delta_{ab}$. This allows us to
go around the difficulty by only considering the leading power of each variable.

To better record the data of the leading power of each cluster $X$-variable, we introduce the following
notations replacing the strings and the nodes in the string diagram, each of which carries a monomial inside

and represents an element in $G_{ad}$:

$$\begin{align*}
\Pi_\alpha x_{i_0}^α & = e_\alpha \left( c \prod_a X_a^{p_a} + \text{terms of lower powers} \right), \\
\Pi_\alpha x_{i_0}^\beta & = e_{-\alpha} \left( c \prod_a X_a^{p_a} + \text{terms of lower powers} \right), \\
\Pi_\alpha x_{i_0}^\beta & = \left( c \prod_a X_a^{p_a} + \text{terms of lower powers} \right)^{H_\alpha}.
\end{align*}$$

We impose the convention that an empty figure means the monomial inside is 1. In particular, we don’t
draw empty circles since it is just the identity element.

**Example 3.5.** Consider a rank 2 root system with simple roots $\alpha$ and $\beta$. Then the image of the amalgamation map
$\chi_i : X_1 \rightarrow H \backslash G^{\alpha,\beta} / H$ corresponding to the reduced word $i := (-\alpha, -\beta, \alpha, \beta)$ can be represented by
our notation above as

Using such notation, we see that to compute the leading powers of cluster $X$-variables in certain generali-
zed minors, all we need to do is to multiply the corresponding lifts of Weyl group elements on the two sides, and
then move figures around so that all triangles of the form $\bigtriangleup$ are on the left of circles and all triangles of
the form $\bigtriangleup$ are on the right of circles, and at the end whatever monomials are left in the circles in the
middle will give us the leading powers of the cluster $X$-variables.
This may sound more difficult than it really is; in fact, there are many identities we can use to help us achieve the goal.

**Proposition 3.6.** The following identities hold (for notation simplicity we only include one cluster $X$-variable $X$ here).

(1) Neighboring circles on the same level can be merged into one single circle with the respective monomials multiplied. Different figures on different levels commute with each other. Circles on different levels also commute with each other.

(2) $X^p X^d = X^d X^{p-d}$.

(3) $X^d X^p = X^{p-d} X^d$.

(4) Define $d := \max\{0, d\}$; then $X^p X^q = X^{q-(p+d)} X^{2(p+q)} C_{\alpha\beta} X^{p-(p+d)}$.

Proof. (1) follows from the fact that $[E_{\pm\alpha}, E_{\mp\beta}] = [H^\alpha, E_{\pm\beta}] = [H^\alpha, H^\beta] = 0$ whenever $\alpha \neq \beta$; (2) and (3) follows from the commutator relation $[H^\alpha, E_{\pm\alpha}] = \pm E_{\pm\alpha}$; (4), (5), and (6) follow from identities (2.11), (2.12), and (2.13) respectively. □

This may still sound difficult, but there are further simplifications. For example, since we require our reduced word $i$ to have its $u$ part before its $v$ part, we know that starting from the very beginning we already have the figures more or less in the order that we want, namely triangles of the form $\triangle$ are on the left of triangles of the form $\triangle$; there are some circles scattering in the mix, but we can use identities (2) and (3) from Proposition 3.6 to move the circles to the middle. Notice that all triangles in the image of $\chi_i$ are empty by definition, and after using identities (2) and (3) from Proposition 3.6, all triangles are filled with monomials with non-positive exponents.
Suppose our reduced word \( i := (\alpha(1), \ldots, \alpha(n), \alpha(n+1), \ldots, \alpha(l)) \) has non-trivial \( v \) part. According to (5) of Proposition 3.6, multiplying \( s_{\alpha(l)} \) on the right of \( x \) changes the right most triangle \( \triangle \) to \( \bigtriangleup \). So if we want to compute a generalized minor of \( x s_{\alpha(l)} \), all we need to do then is just move the triangle \( \triangle \) to the left until it passes the collection of circles in the middle, and then evaluate the generalized minor based on the circles in the middle.

**Remark 3.7.** We claim that we may even drop the last triangle of the form \( \triangle \) when applying (5) of Proposition 3.6: since \( (\alpha(n+1), \ldots, \alpha(l)) \) is a reduced word of \( v \), for any \( m > n \) we know that \( s_{\alpha(m)} \cdots s_{\alpha(l-1)} \) is guaranteed to map \( e_{\alpha(l)} \) to some unipotent element \( n_+ \in N_+ \), and hence dropping the last triangle of the form \( \triangle \) will not affect our computation of generalized minor whatsoever. A similar argument can be applied to (6) of Proposition 3.6 as well; in conclusion, we may effectively replace (5) and (6) of Proposition 3.6 by

\[
(5') \quad \begin{array}{c}
\xymatrix{X^p \ar[r] & \bigtriangleup} \\
\xymatrix{X^{pC_{\alpha}}} \ar[r] & \alpha \ar[l] \\
\xymatrix{X^{2p}} \ar[r] & \beta \ar[l]
\end{array}
\]

\[
(6') \quad \begin{array}{c}
\xymatrix{X^p \ar[r] & \bigtriangleup} \\
\xymatrix{X^{pC_{\alpha}}} \ar[r] & \alpha \ar[l] \\
\xymatrix{X^{2p}} \ar[r] & \beta \ar[l]
\end{array}
\]

Let’s use (5’) and (1) - (4) of Proposition 3.6 to compute \( x^{v^{-1}} \). As a convention, we will use a dashed vertical line to keep track of the separation between the \( u \) part and the \( v \) part of the reduced word. If \( X \) is a cluster \( X \)-variable not from the \( v \) part of the reduced word, it is not hard to see that

\[
\begin{array}{c}
\xymatrix{\ldots \bigtriangleup \ar[r] & X \ar[r] & \ldots \bigtriangleup} \\
\xymatrix{\ldots \bigtriangleup \ar[r] & x \ar[r] & \ldots \bigtriangleup} \\
\xymatrix{\ldots \bigtriangleup \ar[r] & x^{-1} \ar[r] & \ldots \bigtriangleup}
\end{array}
\]

Things get a bit trickier when \( X \) is a cluster \( X \)-variable from the \( v \) part of the reduced word. Consider the following identities (we will always assume that the top level is level \( \alpha \) and the bottom level is level \( \beta \) unless
otherwise specified):

\[ \ldots \triangle \ldots \triangle \ldots \triangle \quad \ldots \quad X \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad v^{-T} \]

\[ = \ldots \quad X \quad \ldots \quad X^{-1} \quad \ldots \quad X^{-1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad v^{-T} \]

\[ = \ldots \quad \lhd X^{-1} \quad \ldots \quad \ldots \quad X \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad v^T \]

\[ = \ldots \quad \lhd X^{-1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad v^T \]

We may want to pause here and make a few observations before proceeding any further.

Remark 3.8. First, we claim that from the last line and on, we always have the pattern

\[ \ldots \quad X^d \quad \ldots \quad X^{-d} \quad \ldots \quad X^{-d} \quad \ldots \]

on every level (say \( \alpha \)) in the \( v \) part of the reduced word for some integer \( d \) (such integer may differ for different level); this is because when multiplied another \( \zeta_\alpha \) on the right of the last \( \triangle \), it turns to

\[ \lhd X^d \]

\[ \lhd X^{-2d} \]

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with some additional $X_{-dC_{\alpha\beta}}$ on other levels $\beta$. But then because (4) of Proposition 3.6 says that

$$X^d X^{-d} = X^d X^{-d}$$

we can move $X^d$ through all $X^{-d}$ on level $\alpha$ without changing anything; lastly the circles on the far right, while moving towards the middle, change the remaining triangles in the $v$ part of the reduced word uniformly through each level; hence at the end of the day we have

$$\cdots X^d \cdots X^{-d} \cdots X^{-d} = \cdots X^{-1} \cdots X^{-1} \cdots X^p$$

Remark 3.9. Second, we also notice that if $X$ is a cluster $X$-variable in the $v$ part on the level $\alpha$ to begin with, then every triangle from the $v$ part of the reduced word will no longer carry any factor of $X$ after it is moved into the $u$ part, except those that are on the right of $X$ on the same level to begin with (which we color red in the picture below):

$$\cdots \cdots X \cdots \cdots v^{-1} = \cdots X^{-1} \cdots X^{-1} \cdots X^p$$

This can be seen as a direct consequence of our first observation.

Unfortunately we don’t have well-formed formula to compute the exponents $p$; however, we can still formulate the following proposition that we help us prove statements about such exponents.

**Proposition 3.10.** Let $X$ be a cluster $X$-variable originally on level $\alpha$ strictly contained in the $v$ part. Let $p_\beta$ be the exponents of $X$ in the circle on level $\beta$ in the schematic picture of $x(v_{>k})^{-1}$ (where circles have all been moved to the middle in between the $u$ part and the $v$ part). If in addition $\alpha(k)$ is still a simple root (a letter in the $v$ part) and $\alpha(k) \neq \alpha$, then

$$\sum_{\beta} q_\beta (C^{-1})_{\beta\alpha} = \sum_{\beta} p_\beta (C^{-1})_{\beta\alpha},$$

where $q_\beta$ are the exponents of $X$ in the circle on level $\beta$ in the schematic picture of $x(v_{>k-1})^{-1}$, and $C^{-1}$ is the inverse of the Cartan matrix $C$.

**Proof.** Suppose $\alpha(k) = \gamma \neq \alpha$ is a simple root. Then by applying (5') and Remark 3.8 we know that $q_\beta = p_\beta - C_{\gamma\beta}p_{\gamma}$. But then

$$\sum_{\beta} q_\beta (C^{-1})_{\beta\alpha} = \sum_{\beta} (p_\beta - C_{\gamma\beta}p_{\gamma}) (C^{-1})_{\beta\alpha} = \sum_{\beta} p_\beta (C^{-1})_{\beta\alpha},$$

where in the last equality we have used the fact that $\sum_{\beta} C_{\gamma\beta} (C^{-1})_{\beta\alpha} = \delta_{\gamma\alpha} = 0$. \qed

Now we finally have accomplished something: the general minors of $xv^{-1}$ do come up when computing $DT$. To finish what we have started, we now need to multiply some part of the reduced word of $u$ on the left to get $u_{<k}^{-1}xv^{-1}$. This means we start with whatever we get at the end when computing $xv^{-1}$, and then start to change the triangles from the left using (6') and move them all the way to the right. We will now draw two dashed lines in the schematic picture, with one representing the separation between triangles.
that are originally in the \( u \) part of the reduced word and those that have come across from the \( v \) part of the reduced word, and the other one representing the original separation between the \( u \) part and the \( v \) part of the reduced word, as drawn below.

Suppose \( X \) is a cluster \( \mathcal{X} \)-variable on level \( \alpha \) in the \( v \) part of the reduced word. Then from our discussion on \( x^{v-1} \) we know that in the schematic diagram of \( x^{u^{-1}} \), only the triangles that are originally on level \( \alpha \) to the right of \( X \) (and hence from the \( v \) part) will carry a factor of \( X^{-1} \) in them.

Therefore when we apply (4) and (6') over and over again to compute \( \frac{u_{<k}^{-1}}{x^{v^{-1}}} \), no leading power of \( X \) is going to change except when the triangles originally from the \( u \) part finally arrive in the \( v \) part.
If $X$ is a cluster $\mathcal{X}$-variable on level $\alpha$ in between the $u$ part and the $v$ part of the reduced word, then we can also see easily that

\[
\overline{u_{<k}^{-1}} \cdots \bigtriangleup \cdots \bigtriangleup \cdots X^{-1} \cdots X^{-1} \cdots X
\]

Lastly, if $X$ is a cluster $\mathcal{X}$-variable on level $\alpha$ in the $u$ part to begin with, then in the schematic diagram of $xv^{-1}$ we will have the following (the dashed circle represents where $X$ was originally).

\[
\cdots \bigtriangleup \cdots \bigtriangleup \cdots X^{-1} \cdots X^{-1} \cdots X^{-1} \cdots X
\]

By going over similar computations we did before Remarks 3.8 and 3.9, we see that we will arrive at similar conclusion as Remarks 3.8 and 3.9 again. Therefore we can conclude that

\[
\overline{u_{<k}^{-1}xv^{-1}} = \cdots X^{-p_\alpha} \cdots X^{-p_\alpha} \cdots X^{p_\alpha} \cdots X^{-p_\alpha} \cdots X^{-p_\beta} \cdots X^{p_\beta} \cdots
\]

for some integers $p_\alpha$ and $p_\beta$.

Remark 3.11. Through all the computations we have done, we see that when we apply (4) of Proposition 3.6, the factor of circles never appear. Therefore one can effectively replace (4) by

\[
(4') \bigtriangleup X^p \bigtriangleup X^q = \bigtriangleup X^q \bigtriangleup X^p
\]
when doing computation with reduced words whose \( u \) part is strictly before the \( v \) part.

By using the same technique and a symmetric argument one can also compute \( \pi^{-1}x(v_{>k})^{-1} \). We will leave the details as exercise for the readers.

Now we are ready to prove the following theorem, which is the first half of what we need to prove that \( DT \) is the cluster Donaldson-Thomas transformation on \( X^{u,v} \).

**Proposition 3.12.** For our choice of the reduced word \( i \) (whose \( u \) part comes before its \( v \) part), \( \deg_{X_a} DT^*_i (X_a) = -\delta_{ab} \) for any cluster \( X \)-variables \( X_a \) and \( X_b \).

**Proof.** In the proof we will mainly focus on two cases depending on where \( X_a \) is: (i) \( X_a \) is strictly contained in the \( u \) part, and (ii) \( X_a \) is in between the \( u \) part and the \( v \) part; the case where \( X_a \) is strictly contained in the \( v \) part can be proved by a symmetric argument of case (i).

Let’s consider case (i) first. Suppose the vertex (string) \( a \) is on level \( \alpha \) cut out by nodes \( \alpha(i) \) and \( \alpha(j) \) as below.

\[
\begin{array}{c}
\alpha(i) \\
a \\
\alpha(j)
\end{array}
\]

From earlier computation we learn that

\[
DT^*_i (X_a) = \frac{\Delta_\alpha \left( \frac{u_{<j+1}^{-1}x^{-1}}{x^{-1}} \right) \prod_{\beta \neq \alpha} \left( \Delta_\beta \left( \frac{u_{<j}^{-1}x^{-1}}{x^{-1}} \right) \right)^{-C_{\beta \alpha}}}{\Delta_\alpha \left( \frac{u_{<j}^{-1}x^{-1}}{x^{-1}} \right) \prod_{\beta \neq \alpha} \left( \Delta_\beta \left( \frac{u_{<j+1}^{-1}x^{-1}}{x^{-1}} \right) \right)^{-C_{\beta \alpha}}}.
\]

If \( b \) is a vertex strictly contained in the \( v \) part of the reduced word or in between the \( u \) part and the \( v \) part, then it is not hard to see from the schematic picture that multiplying \( u_{<j}^{-1} \) and \( u_{<j+1}^{-1} \) on the left of \( x^{-1} \) yield the same factors of \( X_b \) in the middle, and hence \( \deg_{X_b} DT^*_i (X_a) = 0 \).

This reduces the case to the situation where \( b \) is a vertex strictly contained in the \( u \) part of the reduced word. Under such circumstance, it is helpful to rewrite \( DT^*_i (X_a) \) as

\[
(3.13) \quad DT^*_i (X_a) = \frac{\Delta_\alpha \left( \frac{u_{<j}^{-1}x^{-1}}{x^{-1}} \right) \prod_{\beta} \left( \Delta_\beta \left( \frac{u_{<j+1}^{-1}x^{-1}}{x^{-1}} \right) \right)^{C_{\beta \alpha}}}{\Delta_\alpha \left( \frac{u_{<j+1}^{-1}x^{-1}}{x^{-1}} \right) \prod_{\beta} \left( \Delta_\beta \left( \frac{u_{<j}^{-1}x^{-1}}{x^{-1}} \right) \right)^{C_{\beta \alpha}}}.
\]

The last expression may still look complicated, but it is actually easy to compute, especially if we only care about the leading power: note that if you try to compute \( \prod_{\beta} \left( \Delta_\beta \left( n_- \left( \prod_{\mu} t_{\mu}^{H_{\nu}} \right) n_+ \right) \right)^{C_{\beta \alpha}} \) for some \( n_\pm \in N \) and \( t_{\mu} \in \mathbb{C}^* \), all you need is just to realize

\[
\sum_{\beta} C_{\beta \alpha} \langle H^\mu, \omega_\beta \rangle = \sum_{\beta, \nu} C_{\beta \alpha} (C^{-1})_{\mu \nu} \langle H_\nu, \omega_\beta \rangle = \delta_{\alpha \mu},
\]

which implies immediately that

\[
\prod_{\beta} \left( \Delta_\beta \left( n_- \left( \prod_{\mu} t_{\mu}^{H_{\nu}} \right) n_+ \right) \right)^{C_{\beta \alpha}} = t_\alpha.
\]

Let’s first consider the subcase \( b \neq a \) with \( b \) being a vertex strictly contained in the \( u \) part. By Remark 3.8, we can assume that \( u_{<j}^{-1}x^{-1} \) and \( u_{<j+1}^{-1}x^{-1} \) are represented by the following schematic pictures respectively:

\[
\begin{align*}
 \frac{u_{<j}^{-1}x^{-1}}{x^{-1}} &= \frac{X_b^{-p_\alpha}}{} \ldots \frac{X_b^{-p_\alpha}}{} \ldots \frac{X_b^{p_\alpha}}{} \ldots, \\
 \frac{u_{<j+1}^{-1}x^{-1}}{x^{-1}} &= \ldots \frac{X_b^{-p_\beta}}{} \ldots \frac{X_b^{-p_\beta}}{} \ldots
\end{align*}
\]
Then to prove \( \deg_X D^*_I (X_a) = 0 \), all we need to show is the following equality:

\[
p_\alpha - \left( \sum_\beta p_\beta H^\beta, \omega_\alpha \right) = q_\alpha - \left( \sum_\beta q_\beta H^\beta, \omega_\alpha \right),
\]

which is equivalent to showing

\[
p_\alpha - \sum_\beta p_\beta (C^{-1})_{\beta \alpha} = q_\alpha - \sum_\beta q_\beta (C^{-1})_{\beta \alpha},
\]

where \( C^{-1} \) is the inverse of the Cartan matrix \( C \). To see this, we need the schematic pictures of \( u_{<i+1}^{-1}xv^{-1} \) and \( u_{<i}^{-1}xv^{-1} \), which are the following.

By renaming the exponents of \( u_{<i+1}^{-1}xv^{-1} \) as \( p'_\beta \) and the exponents of \( u_{<i}^{-1}xv^{-1} \) as \( q'_\beta \), we see that

\[
\sum_\beta p'_\beta (C^{-1})_{\beta \alpha} = \sum_\beta (p_\beta - C_{\alpha \beta}p_a) (C^{-1})_{\beta \alpha} = -p_\alpha + \sum_\beta p_\beta (C^{-1})_{\beta \alpha},
\]

\[
\sum_\beta q'_\beta (C^{-1})_{\beta \alpha} = \sum_\beta (q_\beta - C_{\alpha \beta}q_a) (C^{-1})_{\beta \alpha} = -q_\alpha + \sum_\beta q_\beta (C^{-1})_{\beta \alpha}.
\]

The problem now reduces to showing that

\[
\sum_\beta p'_\beta (C^{-1})_{\beta \alpha} = \sum_\beta q'_\beta (C^{-1})_{\beta \alpha},
\]

which is a direct consequence of (a symmetric version of) Proposition 3.10.

This leaves us with one remaining vertex, namely \( b = a \). In this subcase, the schematic pictures of \( u_{<i}^{-1}xv^{-1} \) and \( u_{<i+1}^{-1}xv^{-1} \) are the following:

Then according to Equation (3.13), we can easily compute

\[
\deg_X D^*_I (X_a) = \langle H^a, \omega_\alpha \rangle - 1 - \left( -H^a - \sum_{\beta \neq \alpha} C_{\alpha \beta} H^\beta, \omega_\alpha \right) - 1
\]

\[
= \left( \sum_\beta C_{\alpha \beta} H^\beta, \omega_\alpha \right) - 2
\]

\[
= -1.
\]
which concludes the proof of case (i).

Let’s now turn to case (ii). Again suppose the vertex (string) $a$ is on level $\alpha$ cut out by nodes $\alpha(i)$ and $\alpha(j)$, but this time with $-\alpha(i) = \alpha(j) = \alpha$. Then we know that

$$
\text{deg}_{X_a} \text{DT}^*_i (X_a) = \left( \frac{\prod_{\beta \neq \alpha} \left( \Delta_{\beta} \left( \pi^{-1} x_{(v_j)^{-1}} \right) \right)}{\Delta_{\alpha} \left( \pi^{-1} x_{(v_j)^{-1}} \right) \prod_{\beta \neq \alpha} \left( \Delta_{\beta} \left( \pi^{-1} x_{(v_j)^{-1}} \right) \right)} \right) \left( \frac{\prod_{\beta \neq \alpha} \left( \Delta_{\beta} \left( \pi^{-1} x_{(v_j)^{-1}} \right) \right)}{\Delta_{\alpha} \left( \pi^{-1} x_{(v_j)^{-1}} \right) \prod_{\beta \neq \alpha} \left( \Delta_{\beta} \left( \pi^{-1} x_{(v_j)^{-1}} \right) \right)} \right) C_{\beta \alpha}.
$$

This time the easy ones are the vertices $b$ that are originally in between the $u$ part and the $v$ part, for which there is at most one on each level. From the analysis in schematic pictures we know that they do not change as we move triangles across. Suppose $b$ is on level $\beta$ to begin with. Then by simple computation we get that

$$
\text{deg}_{X_b} \text{DT}^*_i (X_a) = 2 \langle H^\beta, \omega_\alpha \rangle + \delta_{\alpha \beta} - 2 \langle H^\beta, \omega_\alpha \rangle - 2\delta_{\alpha \beta} = -\delta_{\alpha \beta} = -\delta_{ab}.
$$

Therefore to finish the proof, we only need to show that $\text{deg}_{X_b} \text{DT}^*_i (X_a) = 0$ for any vertex $b$ that is strictly contained in either the $u$ part or the $v$ part. Due to the symmetry of the arguments, we will only consider the subcase where $b$ is a vertex strictly contained in the $u$ part of the reduced word. Without further due, we lay down the schematic pictures of $\pi^{-1} x_{(v_j)^{-1}}$, $\pi^{-1} x_{(v_j)^{-1}}$, and $\pi^{-1} x_{(v_j)^{-1}}$ as below:

Based on these schematic pictures, we can go straight into the computation of the degree of $X_b$ in $\text{DT}^*_i (X_a)$:

$$
\text{deg}_{X_b} \text{DT}^*_i (X_a) = \sum_{\beta} p_\beta H^\beta, \omega_\alpha \quad \text{and} \quad \sum_{\beta} (p_\beta - C_{\alpha \beta} p_\alpha) H^\beta, \omega_\alpha - p_\alpha
$$

$$
- 2 \left( \sum_{\beta} (p_\beta - C_{\alpha \beta} p_\alpha) H^\beta, \omega_\alpha \right) - p_\alpha - (-p_\alpha)
$$

$$
= \sum_{\beta} C_{\alpha \beta} p_\alpha H^\beta, \omega_\alpha - p_\alpha
$$

$$
= 0.
$$

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This finally completes the whole proof of Proposition 3.12.

Now we are half way through proving that DT is the cluster Donaldson-Thomas transformation on $\mathcal{X}^{u,v}$. The remaining part of this subsection will be devoted to proving the other claim we need, i.e., DT is a cluster transformation.

Recall from Proposition 2.25 that we can identify the twist map $\eta$ on double Bruhat cells $H \setminus G^{u,v}/H$ with the map $\eta : [B_1, B_2, B_3, B_4] \to [B_3^*, B_4^*, B_5^*, B_6^*]$ on $\text{Conf}^{u,v}(\mathcal{B})$, where the six Borel subgroups can be fit in the following hexagon diagram.

The key to show that $\psi \circ \chi$ is a cluster transformation is to break $\eta$ down to a composition of a series of small “clockwise tilting” on the square diagram. To be more precise, let

$$i := (-\alpha(1), \ldots, -\alpha(m), \beta(1), \ldots, \beta(n))$$

again be a reduced word for the pair of Weyl group elements $(u, v)$ whose $u$ part comes before its $v$ part (for notation simplicity later we have put in a minus sign for the $u$ part so now all $\alpha(i)$ are simple roots, and we use $\beta(j)$ for the $v$ part to distinguish it from the $u$ part). From the way the reduced word $i$ is structured, we see that if $x = \chi_i(X_u)$, then $x$ can be written as a product $x_- x_+$ where $x_\pm \in B_\pm$ (please be aware that the notation is different from Gaussian decomposition $x = [x_-] [x_0] [x_+]$). Fix a choice of such factorization $x = x_- x_+$. Then we know that the point $[x_+^{-1}B_+, B_- x_+^{-1}, B_-, B_+]$ in $\text{Conf}^{u,v}(\mathcal{B})$ corresponds to the equivalence class $H \setminus x/H$ in $H \setminus G^{u,v}/H$. We can further represent such a point by the square diagram below.

We now initiate a sequence of tilting on the edge $x_-^{-1}B_+ \longrightarrow B_-$ with respect to the reduced word $(\alpha(1), \ldots, \alpha(m))$ of $u$. First set $B_{u(0)} := x_+^{-1}B_+$ and $B_{u(m)} := B_+$; by using Proposition 2.21 we can find a sequence of Borel subgroups $\left(B_{u(k)}^{(k)}\right)$ such that for $1 \leq k \leq m$,

$$B_{u(k-1)}^{(k-1)} \overset{s_{\alpha(k)}}{\longrightarrow} B_{u(k)}^{(k)}.$$

Next set $B_{u(0)^*}^* := B_-$ and again use Proposition 2.21 to find a sequence of Borel subgroups $\left(B_{u(0)^*}^{(k)}\right)$ such that for $1 \leq k \leq m$,

$$B_{u(0)^*}^{(k-1)} \overset{s_{\alpha(k)}}{\longrightarrow} B_{u(0)^*}^{(k)}.$$
We can view these two sequences as a sequence of tilting of the edge $x_{-1}B_+ - B_-$. This can be seen more intuitively on the original square diagram.

We can do the same construction for $v$, by finding two sequences $(B^{(k)}_{u^*})_{k=1}^l$ and $(B^{(k)}_{v^*})_{k=1}^l$ to fit into the square diagram in an analogous way. To save time, we will just draw the resulting tilting diagram.

One can see that we are tilting the right vertical edge $B_+ - B_- x_{+1}$ clockwise a bit at a time, going from index $l$, $l-1$, and so on, till we get to $B^{(0)}_{v^*} - B_-$. Note that by the time we finish both tilting sequences, we can apply $*$ to the final square diagram and obtain $\eta[x_{-1}B_+, B_- x_{+1}^{-1}, B_-, B_+] = \eta_{\mathcal{A}}[x_{-1}B_+, B_- x_{+1}^{-1}, B_-, B_+]$
\[
[B^*, B_+^*, (B_u^{(0)})^*, (B_u^{(m)})^*].
\]

\[
\begin{array}{ccc}
B^* & \xrightarrow{u} & (B_u^{(m)})^* \\
& \downarrow & \\
(B_u^{(0)})^* & \xrightarrow{v^*} & B_+^*
\end{array}
\]

Our next mission is to figure out how to realize such a tilting in terms of operations on the element \(x\) in \(G^{u,v}\). By viewing \(x\) as a transformation on pairs of opposite Borel subgroups, we can break down the square diagram of the quadruple \([x^{-1}B_+, x_+, B_-, B_+, B_+]\) into a two-step process: first taking the opposite pair \(x^{-1}B_+ \to B_-\) to the opposite pair \(B_+ \to B_-\) and then to the opposite pair \(B_+ \to B_-x_+^{-1}\). We can describe such transformation by the below diagram.

\[
\begin{array}{ccc}
x^{-1}B_+ & \xrightarrow{u} & B_+ \\
& \searrow & \nearrow \\
B_- & \xrightarrow{v^*} & B_-x_+^{-1}
\end{array}
\]

In the diagram above, note that the middle opposite pair \(B_+ \to B_-\) corresponds to the diagonal in the tilting diagram that separates the two tilting sequence, so we can expect that the tilting sequence for \(u\) will take place inside the triangle on the left, whereas the tilting sequence will take place inside the triangle on the right. This diagram also links to the string diagram we have used to produce the seed corresponding to the reduced word \(i\), which in turn can be used to prove the cluster nature of \(\text{DT}_i\), as we will see near the end of this subsection.

**Remark 3.14.** A sidetrack: you may have wondered why we have used triangles of the form \(\triangle\) in the \(u\) part and triangles of the form \(\triangledown\) in the \(v\) part when computing the leading power of \(\text{DT}_i^* (X_u)\); well, the above diagram is why.

**Proposition 3.15.** Based on the representative \((x^{-1}B_+, B_-x_+^{-1}, B_-, B_+)\), the following identity holds for \(0 \leq k \leq m\):

\[
B_u^{(k)} \quad \frac{(e_{\alpha(k)}e^{-1}_{-\alpha(k)} \cdots e_{\alpha(1)}e^{-1}_{-\alpha(1)}x_+^{-1})^{-1} B_+}{(e_{\alpha(k)}e^{-1}_{-\alpha(k)} \cdots e_{\alpha(1)}e^{-1}_{-\alpha(1)}x_-)}.
\]

**Proof.** We just need to verify the defining conditions for \(B_u^{(k)}\) and \(B_u^{(k)}^*\), and the key facts are

\[
e_{\alpha(k)} \in B_+ \cap B_- s_{\alpha(k)} B_+ \quad \text{and} \quad e_{-\alpha(k)} \in B_+ s_{\alpha(k)} B_+ \cap B_-
\]

Let’s first look at \(B_u^{(k)}\). Obviously \(B_u^{(0)} = x^{-1}B_+\), and since

\[
(e_{\alpha(k-1)}e^{-1}_{-\alpha(k-1)} \cdots e_{\alpha(1)}e^{-1}_{-\alpha(1)}x_+^{-1})(e_{\alpha(k)}e^{-1}_{-\alpha(k)} \cdots e_{\alpha(1)}e^{-1}_{-\alpha(1)}x_-^{-1})^{-1} = e_{-\alpha(k)}e^{-1}_{-\alpha(k)} \in B_+ s_{\alpha(k)} B_+,
\]

we know that \((e_{\alpha(k-1)}e^{-1}_{-\alpha(k-1)} \cdots e_{\alpha(1)}e^{-1}_{-\alpha(1)}x_-^{-1})^{-1} B_+ \xrightarrow{s_{\alpha(k)}} (e_{\alpha(k)}e^{-1}_{-\alpha(k)} \cdots e_{\alpha(1)}e^{-1}_{-\alpha(1)}x_-^{-1})^{-1} B_+\). Thus we only need to show that \((e_{\alpha(m)}e^{-1}_{-\alpha(m)} \cdots e_{\alpha(1)}e^{-1}_{-\alpha(1)}x_-^{-1})^{-1} B_+ = B_+\), which is equivalent to showing that \(e_{\alpha(m)}e^{-1}_{-\alpha(m)} \cdots e_{\alpha(1)}e^{-1}_{-\alpha(1)}x_-^{-1}\) is an element of \(B_+\). But if we look at the \(u\) part of the string diagram associated to \(i\) (which is a string diagram associated to the the reduced word \((-\alpha(1), \ldots, -\alpha(m))\) and can
be used to give a parametrization of $x_-$, we see that multiplying $e_{\alpha(1)}e_{-\alpha(1)}^{-1}$ on the left of $x_-$ turns the left most node from $-\alpha(1)$ to $\alpha(1)$; then we can move this node to the right of $x_-$ using the move (1) of Proposition 2.46; similar arguments apply to $e_{\alpha(2)}e_{-\alpha(2)}^{-1}$ and so on. Thus at the end we will obtain a string diagram with only nodes that are simple roots, and this shows that $e_{\alpha(m)}e_{-\alpha(m)}^{-1}\cdots e_{\alpha(1)}e_{-\alpha(1)}^{-1}x_-$ is an element of $B_+$.

As for $B_{+}^{(k)}$, we see that $B_{+}^{(0)} = B_+ - x_-$, and since

$$
\left( e_{\alpha(k-1)}^{-1}e_{-\alpha(k-1)}^{-1} \ldots e_{\alpha(1)}^{-1}e_{-\alpha(1)}^{-1}x_- \right) \left( e_{\alpha(k)}^{-1}e_{-\alpha(k)}^{-1} \ldots e_{\alpha(1)}^{-1}e_{-\alpha(1)}^{-1}x_- \right)^{-1} = e_{-\alpha(k)}e_{\alpha(k)}^{-1} \in B_- s_{\alpha(k)}B_-, 
$$

we know that

$$
\left( e_{\alpha(k-1)}^{-1}e_{-\alpha(k-1)}^{-1} \ldots e_{\alpha(1)}^{-1}e_{-\alpha(1)}^{-1}x_- \right)^{-1} B_- s_{\alpha(k)} \left( e_{\alpha(k)}^{-1}e_{-\alpha(k)}^{-1} \ldots e_{\alpha(1)}^{-1}e_{-\alpha(1)}^{-1}x_- \right)^{-1} B_-. 
$$

Thus we only need to show that

$$
B_+ \left( e_{\alpha(m)}^{-1}e_{-\alpha(m)}^{-1} \ldots e_{\alpha(1)}^{-1}e_{-\alpha(1)}^{-1}x_- \right) \xrightarrow{u^c} x_-^{-1} B_+.
$$

But this is equivalent to showing that

$$
\overline{u} e_{\alpha(m)}^{-1}e_{-\alpha(m)}^{-1} \cdots e_{\alpha(1)}^{-1}e_{-\alpha(1)}^{-1} \in B_+.
$$

for which it suffices to show that

$$
\overline{u} e_{\alpha(m)}^{-1}e_{-\alpha(m)}^{-1} \cdots e_{\alpha(1)}^{-1}e_{-\alpha(1)}^{-1} \in B_+.
$$

To show this, we recall that $\overline{u} := e_{\alpha}^{-1}e_{-\alpha}^{1}$, thus

$$
\overline{u} e_{\alpha(m)}^{-1}e_{-\alpha(m)}^{-1} \cdots e_{\alpha(1)}^{-1}e_{-\alpha(1)}^{-1} = \overline{u} e_{\alpha(m)}^{-1} e_{\alpha(m-1)}^{-1} \cdots e_{\alpha(1)}^{-1} e_{-\alpha(1)}^{-1}.
$$

But then since $(\alpha(1), \ldots, \alpha(m))$ is a reduced word of $u$, $s_{\alpha(1)} \cdots s_{\alpha(m-1)}$ maps the simple root $\alpha_{\alpha(m)}$ to a positive root, which implies that

$$
\overline{u} e_{\alpha(m)}^{-1} e_{\alpha(m-1)}^{-1} \cdots e_{\alpha(1)}^{-1} e_{-\alpha(1)}^{-1} = \overline{u} e_{\alpha(m)}^{-1} e_{\alpha(m-1)}^{-1} \cdots e_{\alpha(1)}^{-1} e_{-\alpha(1)}^{-1}.
$$

for some $b \in B_+$. The proof is then finished by induction on $m$. 

By a completely analogous proof one can also show the following proposition.

**Proposition 3.16.** Based on the representative $(x_-^{-1}B_+, B_+^{-1}, B_-, B_+)$, the following identity holds for $0 \leq k \leq l$:

$$
B_{+}^{(k)} = \left( x_+ e_{\beta(1)}^{-1} e_{-\beta(1)} \cdots e_{\beta(k+1)}^{-1} e_{-\beta(k+1)} \right) B_+ B_{+}^{(k)} = B_{-}^{(k)} B_+^{-1} x_+ e_{\beta(1)}^{-1} e_{-\beta(1)} \cdots e_{\beta(k+1)}^{-1} e_{-\beta(k+1)}^{-1}.
$$

Our last two propositions show that, in order to reflect the two tilting sequences in terms of $x_-$, all we need to do is to multiply $e_{\alpha(m)}e_{-\alpha(m)}^{-1} \cdots e_{\alpha(1)}e_{-\alpha(1)}^{-1}$ on the left and $e_{\beta(1)}^{-1} e_{-\beta(1)} \cdots e_{\beta(k+1)}^{-1} e_{-\beta(k+1)}$ on the right, which is similar to what we have done when computing the general minors in earlier discussion (turning triangles up-side-down and moving them across). With these two results in our pockets, we are ready to prove our last proposition.

**Proposition 3.17.** DT := $\psi \circ \chi$ is a cluster transformation.

**Proof.** Let $H \setminus x/H := \chi_1(X_f)$ and consider the representative $[x_-^{-1}B_+, x_+B_-, B_-, B_+]$ corresponding to $H \setminus x/H$ in Conf$^u$,$v$$(B)$. We learned from Propositions 2.25 and 3.1 that the composition $\chi \circ \psi$ is the same
as the map $\eta$ on $\text{Conf}^{u,v}(B)$, and hence $\chi \circ \psi(H\backslash x/H)$ will correspond to the equivalence class of the configuration represented by the following square diagrams

\[
\begin{pmatrix}
 x^{-1}B_+ \xrightarrow{u} B_+ \\
 B_- \xrightarrow{v^*} x_+B_-
\end{pmatrix}
\]

\[
\eta
\]

\[
\begin{pmatrix}
 B_- \xrightarrow{u} B_- \left( e_{\alpha(m)}e_{-\alpha(m)}^{-1} \cdots e_{\alpha(1)}e_{-\alpha(1)}^{-1}x_\alpha \right)^* \\
 B_+ \xrightarrow{v^*} B_+
\end{pmatrix}
\]

\[
\begin{pmatrix}
 x_+e_{\beta(1)}^{-1}e_{-\beta(1)}^{-1} \cdots e_{\beta(1)}^{-1}e_{-\beta(1)}^{-1} \\
 x_+e_{\beta(1)}^{-1}e_{-\beta(1)}^{-1} \cdots e_{\beta(1)}^{-1}e_{-\beta(1)}^{-1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
 B_- \xrightarrow{v^*} B_- \left( e_{\alpha(m)}e_{-\alpha(m)}^{-1} \cdots e_{\alpha(1)}e_{-\alpha(1)}^{-1}x_\alpha \right)^* \\
 B_+ \xrightarrow{v^*} B_+
\end{pmatrix}
\]

Note that the last result corresponds to the element

\[
H\left( e_{\alpha(m)}e_{-\alpha(m)}^{-1} \cdots e_{\alpha(1)}e_{-\alpha(1)}^{-1}x_\alpha e_{\beta(1)}^{-1}e_{-\beta(1)}^{-1} \cdots e_{\beta(1)}^{-1}e_{-\beta(1)}^{-1} \right)^* / H
\]

in $H\backslash G^{u,v}/H$.

If we take a look at the string diagram associated to the reduced word $i$, as we have argued in the proof of Proposition 3.15, each time when we multiply $e_{\alpha(k)}e_{-\alpha(k)}^{-1}$ on the left, all we is doing is change the left most node from an opposite simple root to the corresponding simple root, and then move it to the middle (right after the last letter of $u$). Since $H\backslash G^{u,v}/H$ only corresponds to the non-frozen part of the seed $i$, changing the left most from an opposite simple root to a simple root does nothing to the quiver, and moving it to the middle is just a sequence of seed mutations, which gives rise to a corresponding sequence of cluster mutation on the cluster variety $\mathcal{X}^{u,v}$. Similar argument also applies to each time we multiply $e_{\beta(k)}^{-1}e_{-\beta(k)}$ on the right. Then taking transposition flips the string diagram horizontally while simultaneously changing the nodes from the simple roots to opposite simple roots and vice versa. Thus transposition gives rise to a seed isomorphism which in turn produces a cluster isomorphism on $\mathcal{X}^{u,v}$. Lastly we need to use move (1) of Proposition 2.46 again to restore to the original layout with $u$ part on the left and $v$ part on the right, which is again another sequence of cluster mutations. The following picture summarizes the idea

\[
\begin{pmatrix}
 u \\
 v
\end{pmatrix}
\]

\[
\begin{pmatrix}
 u^1 \\
 v^1
\end{pmatrix}
\]

Multiplying $e_{\alpha(m)}e_{-\alpha(m)}^{-1} \cdots e_{\alpha(1)}e_{-\alpha(1)}^{-1}$ on the left and $e_{\beta(1)}^{-1}e_{-\beta(1)}^{-1} \cdots e_{\beta(1)}^{-1}e_{-\beta(1)}^{-1}$ on the right, which is a sequence of cluster mutations.
Transposition, which is a cluster isomorphism.

Restoring the original layout, which is another sequence of cluster mutations.

Combining these observations we see that $\psi \circ \chi$ is indeed a composition of cluster mutations and cluster isomorphisms, which is by definition a cluster transformation. □

Remark 3.18. Shen also pointed out that if we single out either the $u$ triangles or the $v$ triangles from the above diagram, the sequence of seed mutations also defines a reddening sequence in the sense of Keller [Kel13].

3.3. Relation to Fomin and Zelevinsky’s Twist Map. If we compare our version of the twist map to the original one defined by Fomin and Zelevinsky (Definition 1.5 of [FZ98]), we see that they by an anti-automorphism $x \mapsto x^\iota$ on $G$, which is uniquely defined by (Equation (2.2) of [FZ98]):

$$e_{\pm i}^\iota = e_{\pm i}$$
$$a^\iota = a^{-1} \quad \forall a \in H.$$

As it turns out, such anti-involution also has a cluster counterpart, which is the involution $i_X$ introduced by Fock and Goncharov in [FG03]. We shall explain their relation in more details now.

Given any seed $i = (I, I_0, \epsilon, d)$, we can define a new seed $i^\iota = (I^\iota, I_0^\iota, \epsilon^\iota, d^\iota)$ by setting

$$I^\iota := I, \quad I_0^\iota := I_0, \quad \epsilon^\iota := -\epsilon, \quad \text{and} \quad d^\iota = d.$$

From the identification $I^\iota = I$ we get a natural correspondence between coordinates $(X_a)$ of $X_i$ and $(X_a^\iota)$ of $X_i^\iota$. Then the involution $i_X$ is the map from the seed torus $X_i$ to the seed torus $X_i^\iota$ defined by

$$i_X^\iota(X_a^\iota) = X_a^{-1}.$$

It is not hard to see that the involutions $i_X$ commute with cluster mutations as below.

$$X_i - \mu_k \Rightarrow X_i'$$
$$i_X \downarrow \quad \downarrow i_X$$
$$X_i^\iota - \mu_k \Rightarrow X_i^\iota'$$

Thus the involutions $i_X$ can be glued into a single involution $i_X : \mathcal{X}^{u,v} \rightarrow \mathcal{X}^{u,v}$.

The proposition below shows that in the case of double Bruhat cells $G^{u,v}$, the involution $i_X$ is essentially the same as the anti-automorphism $x \mapsto x^\iota$.

Proposition 3.19. Let $i$ be any reduced word for a pair of Weyl group elements $(u, v)$. Then $(u^{-1}, v^{-1})$ is also a pair of Weyl group elements and $i^\iota$ is a reduced word for it. Further the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{X}^{u,v} & \xrightarrow{\chi} & H \backslash G^{u,v}/H \\
\downarrow i_X & & \downarrow i \\
\mathcal{X}^{u,v} & \xrightarrow{\chi} & H \backslash G^{u^{-1},v^{-1}}/H
\end{array}$$
Proof. We claim that if $i$ is the seed associated to the reduced word $i$, then $i^\circ$ is the seed associated to the reduced word $i'$ which is obtained by reversing the order of the letters. Such a claim can be seen obviously by a horizontal flip of the picture of seed $i$. Note that such a flip also reverse the order of multiplication for the map $\psi$ and the cluster involution $i_X$ maps $X_a$ to $X_a^{-1}$, which is precisely what the definition of the anti-involution $\iota$ says. □

As stated in Conjecture 3.12 of [GS16], Goncharov and Shen conjectured that the composition $D_X := i_X \circ DT$ is an involution. In the case of $H \backslash G^{u,v}/H$, we see that $D_X$ is precisely Fomin and Zelevinsky’s twist map $T$ (passed to the double quotients), which is indeed a biregular involution (Theorem 1.6 of [FZ98]). To summarize, we put everything into the following commutative diagram.

\[
\begin{array}{cccccc}
G_{sc}^{u,v} & \xrightarrow{\psi} & A^{u,v} \\
\downarrow{\psi} & & \downarrow{\eta} \\
\chi & H \backslash G^{u,v}/H & \xrightarrow{T} & H \backslash G^{u,v}/H \\
\downarrow{\chi} & & \downarrow{\chi} \\
\chi & H \backslash G^{u^{-1},v^{-1}}/H & \xrightarrow{i_X} & H \backslash G^{u^{-1},v^{-1}}/H \\
\end{array}
\]

Appendix

A. A Sequence of 10 Mutations Corresponding to Move (2) in the $G_2$ Case. This case says that $(\alpha, \beta, \alpha, \beta, \alpha, \beta) \sim (\beta, \alpha, \beta, \alpha, \beta, \alpha)$. Without loss of generality let’s assume that $C_{\alpha\beta} = -1$ and $C_{\beta\alpha} = -3$. The following pictures are the string diagrams and their corresponding quasi-quivers together with the sequence of seed mutations that transform one into the other. For more details and deeper reasons for why this is true, please see Fock and Goncharov’s paper on amalgamation ([FG06a] Section 3.7).
\[\begin{align*}
\mu_b & \downarrow \\
\mu_a & \downarrow \\
\mu_d & \downarrow
\end{align*}\]
## B. Cluster Identities on Double Bruhat Cells

The following table contains identities corresponding to move (1) and move (2) in the cases of $C_{\alpha \beta}C_{\beta \alpha} = 1$ and $C_{\alpha \beta}C_{\beta \alpha} = 2$.

| $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|----------|----------|----------|----------|
| $\alpha$ | $\beta$ | $\alpha$ | $\beta$ |
| $\beta$ | $\alpha$ | $\beta$ | $\alpha$ |
| $\alpha$ | $\beta$, $\alpha$, $\beta$ | $\alpha$, $\beta$, $\alpha$, $\beta$ | $\beta$, $\alpha$, $\beta$, $\alpha$, $\beta$ |

with $C_{\alpha \beta} = -2$ and $C_{\beta \alpha} = -1$

| $\Delta$ | $\Delta'$ |
|----------|----------|
| $\Delta_{\alpha}(\Delta_{\alpha}^{-1}x_{\beta}) = \frac{\Delta_{\alpha}(\Delta_{\alpha}^{-1}x_{\beta})}{\Delta_{\alpha}(x_{\alpha})}$ | $\epsilon_{\alpha} t_{H^\alpha} = (1 + t_{H^\alpha})^2 \epsilon_{\alpha} + \epsilon_{\beta} t_{H^\beta}$ |

where $y_1, y_2, \ldots, y_6$ are functions of $t_{\alpha}$ and $t_{\beta}$ as described below.

\begin{align*}
y_1 &= \frac{1 + t_{\beta} + 2t_{\alpha}t_{\beta} + t_{\alpha}^2 t_{\beta}}{1 + t_{\beta} + t_{\alpha}t_{\beta}} \\
y_3 &= \frac{t_{\alpha}}{1 + t_{\beta} + 2t_{\alpha}t_{\beta} + t_{\alpha}^2 t_{\beta}} \\
y_5 &= 1 + t_{\beta} + t_{\alpha}t_{\beta}
\end{align*}
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