Enhanced Joint Sparsity via Iterative Support Detection

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Abstract

Compressed sensing (CS) \cite{1,2} demonstrates that sparse signals can be recovered from underdetermined linear measurements. The idea of iterative support detection (ISD, for short) method first proposed by Wang et. al \cite{3} has demonstrated its superior performance for the reconstruction of the single channel sparse signals. In this paper, we extend ISD from sparsity to the more general structured sparsity, by considering a specific case, i.e. joint sparsity based recovery problem where multiple signals share the same common sparse support sets, and they are measured through the same sensing matrix. While ISD can be applied to various existing models and algorithms of joint sparse recovery, we consider the popular $\ell_{2,1}$ convex model. Numerical tests show that ISD brings significant recovery enhancement for the plain $\ell_{2,1}$ model, and performs even better than the simultaneous orthogonal matching pursuit (SOMP) algorithm and $p$-threshold algorithm in both noiseless and noisy environments in our settings. More important, the original ISD paper shows that ISD fails to bring benefits for the plain $\ell_1$ model for the single channel sparse Bernoulli signals, where the nonzero components has the same amplitude, because the fast decaying property of the nonzeros is required for the performance improvement when threshold based support detection is adopted. However, as for the joint sparsity, we have found that ISD is still able to bring significant recovery improvement, even for the multi-channel sparse Bernoulli signals, partially because the joint sparsity structure can be naturally incorporated into the implementation of ISD and we will give some preliminary analysis on it.

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1. Introduction and Contributions

1.1. Brief Introduction to Joint Sparsity

Finding sparse solutions from underdetermined linear systems in the area of compressive sensing, pattern recognition and statistics in an efficient way, has become an active research topic in the last few years. Sparsity makes it possible for us to reconstruct high dimensional data with just few samples or measurements. Recent studies pay more attention to the additional information and knowledge about the potential structure of the solutions, for the sake of enhancing the recoverability by properly making use of this prior. While there exist some works on how to find out these potential structures \cite{4}, a certain ”group sparsity” structure \cite{5} is widely known. Making the solutions get together in a certain underlying property, where the components of solutions are likely to be either all zeros or all nonzeros in a group. Thus, decreasing the dispersion of solutions via making use of the prior of group sparsity structure, can facilitate recovering a much better solution.

Now, we focus on joint sparsity, which is an special case of the group sparsity, though the idea we presented in this paper can be applied to the more general cases. It addresses the joint reconstruction of $l$ jointly sparse signals, which share a common nonzero support, from their $m$ measurement vectors obtained with a common measurement matrix, and this kind problem is also called the multiple measurement vectors (MMV) problem \cite{6,7,8}. The joint sparsity can be formalized as follows: given the vectors $x_1, \cdots, x_l$, which share the sparsity pattern $M$, i.e., the entries of $x_1, \cdots, x_l$ are not zero on this pattern. We want to recovery the $x_i$ from the $l$ noisy underdetermined systems $B_i = Ax_i + e_i, i = 1, \cdots, l$, where the measurement matrix $A \in \mathbb{R}^{m \times n}$ is presumed and the $e_i$ are noise vectors. The noiseless data is $e_i = 0$, for all $i$, the following statement holds:

$$\min |M| \quad s.t. \quad B_i = Ax_i + e_i, i = 1, \cdots, l$$

reverses all $x_1, \cdots, x_l$ with rank $[x_1, \cdots, x_l] = K$ if and only if

$$|M| < \frac{\text{spark}(A) - 1 + K}{2}$$

(2)
where \(|M|\) and \(\text{spark}(A)\) are the cardinality of \(S\) and the smallest set of linearly dependent columns of \(A\), respectively. The threshold (2) significantly improved the well-known \(\text{spark}(A)/2\)-threshold for the single measurement vector (SMV) case \((l = 1)\). Since \((1)\) is NP-hard, this problem is usually relaxed with a convex form which is of computationally efficient algorithms at the expenses of more required measurements. Like \(\ell_1\)-norm being the convex relaxation of \(\ell_0\)-norm, the \(\ell_2,1\)-norm minimization is among them used for the replacement of \(|M|\) and has showed significant practical performance.

For the case of general grouping structures, the \(\ell_{2,1}\)-norm is defined as follows: Let the \(X\) be partitioned into \(n\) groups denoted as \(\{x_{g_i} \in \mathbb{R}^{l_i}, i = 1, \ldots, n\}\), where \(x_{g_i} \subseteq \{1, 2, \ldots, L = \sum l_i\}\) is an index set corresponding to the \(i\)-th group.

\[
\|X\|_{2,1} := \sum_{i=1}^{n} \|x_{g_i}\|_2
\]

(3)

While in many general cases the grouping information is unknown, the specific joint sparsity provides us with the group information. Therefore, for the joint sparsity, we are not focusing on the mining of the comprehensive grouping information, but rather on designing the efficient algorithms to solve the related models. While the \(\ell_{2,1}\)-norm minimization facilitates joint sparsity and results in a convex problem, the \(\ell_{2,1}\)-regularized problem is still generally considered difficult to solve due to the non-smoothness and the mixed-norm structure. The \(\ell_{2,1}\)-regularized problem considered as a second-order cone programming problem or a semidefinite programming problem, can be solved by standard algorithms, which is computationally expensive due to their failure to make use of the sparsity property of the solutions. Correspondingly, there are several efficient first-order algorithms existed, e.g., block-coordinate descent algorithms, a special projected gradient method, SpaRSA and an accelerated gradient method (SLEP), which all try to make use of the sparsity of the solutions in varied ways. Recently, the alternating direction method (ADM), which was developed in the 1970s, with roots in the 1950s, is proposed and extended for solving the \(\ell_{2,1}\)-regularized problem. ADM successfully applied to all kinds of convex or nonconvex sparse optimization problems, such as image processing with total variation regularized problems, tensor and matrix completion, \(\ell_1\)-regularized problem and matrix decomposition problems.
due to its being tailored for better exploiting of the sparsity of the solutions.

Despite the convexity and popularity of $\ell_{2,1}$ regularization, it has become well known that convex regularization methods can suffer from the bias issue that is inherited from the convexity of the penalty function $[25]$. This issue can deteriorate the power of accurately recovering the sparse signals. Therefore, we would like to turn to non-convex regularization methods and develop corresponding fast algorithms in order to further reduce its required measurements while still keeping the computational efficiency.

1.2. Our Contributions

In this paper, we consider the extension of convex joint sparsity model to the non-convex joint sparsity model, though this idea can be extended to other structured sparsity models. Specifically, we propose a new non-convex model based on weighted $\ell_{2,1}$-norm minimization convex model called weighted joint sparsity (WJS) model, for the purpose of reducing the required measurements for the same quality of signal reconstruction or improve the signal reconstruction quality given the same measurements with the very little expensive of extra computation, or even none. Motivated by the idea of iterative support detection method first proposed by Wang $[3]$ in compressive sensing for sparse signal channel signal reconstruction, we will present an alternative optimization algorithm, named iterative support detection based joint sparsity algorithm (ISDJS), to solve this non-convex model.

Besides, a novel point of this paper is that ISD is really suitable for joint sparsity or even more general structural sparsity. Specifically, for the single channel sparse signal, ISD works well depending on the assumption of the fast decaying property of the nonzero components of the underlying true sparse signal, as showed in $[3]$. In this paper, we show that ISD can naturally make use of the specific structure of the joint sparsity in the implementation of the support detection in order to obtain better recovery performance. Correspondingly, the assumption of fast decaying property is no longer required for these cases of multi-channel sparse signal recovery, because the joint sparsity structure is adopted in the support detection.

1.3. Outline

The paper is organized as follows. In section 2, we review a typical algorithm for $\ell_{2,1}$ regularized joint sparsity model. In section 3, we propose our new weighted joint sparsity model and the corresponding ISDJS algorithm.
based on the idea of iterative support detection. In section 4, we provide numerical experiments of the performance of our proposed algorithm for joint sparsity, and compare it with three state of the art algorithms, i.e., SOMP, YALL1 group and p-threshold algorithms, and demonstrate its advantages. Section 5 is devoted to the conclusion of this paper and discussions on some possible future research directions.

2. Brief Review of $\ell_{2,1}$ Regularized Model and Corresponding Algorithms

In this section, we briefly review a typical $\ell_{2,1}$ joint sparsity model, and an efficient algorithm, i.e. the group version of Your ALgorithms for $\ell_1$ (YALL1) minimization. Given by

$$\min_X \|X\|_{2,1} := \sum_{i=1}^{n} ||x_{g_i}||_2$$  \hspace{1cm} (4)

$$s.t. \quad AX = B$$

where $x_{g_i} \subseteq \{1, 2, \cdots, L \} \div \sum l_i$ denotes the unknown jointly sparse solutions of $n$ groups, $A \in R^{m \times n} (m < n)$ is the known measurement matrix, and $B \in R^{m \times l}$ is the known measured data.

As known, joint sparsity can be viewed as a special non-overlapping group sparsity structure with each group containing one row of the solution matrix. Further, we put $x_{g_i}$ substitute $x^i$, and the problem (4) is replaced by the following problem:

$$\min_X \|X\|_{2,1} := \sum_{i=1}^{n} ||x^i||_2$$  \hspace{1cm} (5)

$$s.t. \quad AX = B$$

where $x^i$ and $x_j$ denote the $i$-th row and $j$-th column of $X$, respectively.

While there have existed several algorithms for the above $\ell_{2,1}$ model, we will take the YALL1 group version as an example to demonstrate our new idea, though the ISD can be applied to other algorithms. YALL1 group, as a solver for group sparse reconstruction, is based on a variable splitting strategy and the classic alternating direction method (ADM). The convergence of
YALL1 group is guaranteed by the existing ADM theory. The algorithm in YALL1-group for (4) or (5) has the following form:

\[
\begin{align*}
X & \leftarrow (\beta_1 I + \beta_2 A^T A)^{-1}(\beta_1 Z - \Lambda_1 + \beta_2 A^T B + A^T \Lambda_2), \\
Z & \leftarrow \text{Shrink}(X + \frac{1}{\beta_1} \Lambda_1, \frac{1}{\beta_1})(\text{row-wise}), \\
\Lambda_1 & \leftarrow \Lambda_1 - \gamma_1 \beta_1(Z - X), \\
\Lambda_2 & \leftarrow \Lambda_2 - \gamma_2 \beta_2(A X - B).
\end{align*}
\]

(6)

Here \( \Lambda_1, \Lambda_2 \) are multipliers, \( \beta_1, \beta_2 \) are penalty parameters, \( \gamma_1, \gamma_2 \) are step lengths and the updating of \( Z \) via row-wise shrinkage represents

\[
z^i = \max\{|r^i|_2 - \frac{1}{\beta_1}, 0\} \frac{r^i}{|r^i|_2}, \text{ for } i = 1, \ldots, n
\]

(7)

where

\[
r^i := x^i + \frac{1}{\beta_1} \lambda_1^i
\]

(8)

Now, we will present the YALL1 group algorithm for solving the model (4) or (5) below for latter reference.

\begin{algorithm}
\textbf{Algorithm 1 The YALL1 Group Algorithm}
\begin{enumerate}
\item Initialize \( X \in \mathbb{R}^{n \times l}, \Lambda_1 \in \mathbb{R}^{n \times l}, \Lambda_2 \in \mathbb{R}^{m \times l} > 0,\) \( \beta_1, \beta_2 > 0 \) and \( \gamma_1, \gamma_2 > 0; \)
\item While stopping criterion is not met, do
(a) \( X \leftarrow (\beta_1 I + \beta_2 A^T A)^{-1}(\beta_1 Z - \Lambda_1 + \beta_2 A^T B + A^T \Lambda_2), \)
(b) \( Z \leftarrow \text{Shrink}(X + \frac{1}{\beta_1} \Lambda_1, \frac{1}{\beta_1})(\text{row-wise}), \)
(c) \( \Lambda_1 \leftarrow \Lambda_1 - \gamma_1 \beta_1(Z - X), \)
(d) \( \Lambda_2 \leftarrow \Lambda_2 - \gamma_2 \beta_2(A X - B). \)
\end{enumerate}
\end{algorithm}

3. Our Proposed Model and Corresponding Algorithm

3.1. The Weighted Joint Sparsity Model

The model (5) is based on the \( \ell_{2,1} \)-norm, which is a popular sparsity enforcement regularization due to its convexity. However, in general, the non-convex sparse regularization prefers an even more sparse solution and usually has a better theoretical property, such as the \( l_p \)-norm \( (0 \leq p < 1) \). A major difficulty with nonconvex formulations is that the global optimal solution cannot be still efficiently computed, and solving a given non-convex model using different algorithms may lead to different local solutions, whose
property are often hard to analyze [26]. Correspondingly, in this paper, we presented multi-stage convex relaxation algorithm, which is more tractable and corresponds to a non-convex sparse model based on the convex model (5), based on the iterative support detection (ISD) [3]. The solution by ISD has proved to be better than the pure \( \ell_1 \) solution in theory in the case of the single channel sparse signal reconstruction [3], when considering the specific feature of the underlying signal.

Now we introduce our new model based on the original joint sparsity model (5) as follows:

\[
\min_X \|X\|_{w,2,1} := \sum_{i=1}^{n} w_i \|x^i\|_2 \quad (WJS)
\]

\[s.t. \quad AX = B\]

where the weight parameter vector \( w = [w_1, w_2, \ldots, w_n] \) is dependent on the original true signals \( \hat{X} \). By comparing the original \( \ell_{2,1} \) model and our proposed new model, the main difference is that we introduce the weights \( w \). With regard to the plain \( \ell_{2,1} \) model, ones just solved a plain convex optimization minimization problem one time and take the solution as the final result. That is to say, the original \( \ell_{2,1} \) is only a special case of our weighted model, i.e. setting the weights \( w_i \) (\( i = 1, \ldots, n \)) is always ones. However, it is well known that by setting appropriate weights, which is usually depending on the true \( \hat{X} \), the reconstruction can be much better than the plain \( \ell_{2,1} \) model. Compared with the existing non-convex model like those based on the \( \ell_p \) model (\( 0 < p < 1 \)) which is mainly focus on cardinality, our model is more explicitly based on the our prior information about the true solution via proper settings of weights \( w \).

The idea of the weighted joint sparsity model is natural and there have existed some related work about it. The key component and also the difficult part is how to determine the weight whose appropriate value is depending on the correct knowledge of the true solution \( \hat{X} \), and this determines the performance of the weighted model. As it sounds like the puzzle “What came first: the chicken or the egg?”, one difficulty is that we do not know the original true solution \( \hat{X} \), and the other is that once we have some information about the true solution, we need to find a proper way to make use of it in order to obtain more appropriate updated weights. In this paper, we proposed an alternative optimization procedure repeatedly applying the following two steps to deal with the above two difficulties:
Step 1: we optimize \( x^i \) with \( w \) fixed (initially \( \vec{1} \)): this is a convex problem in terms of \( X \).

Step 2: we determine the value of \( w \) according to the current \( X \). The value of \( w \) will be used in the Step 1 of the next iteration.

In Step 2, we proposed to extract the information about the true solution from the rough intermediate estimates and present an efficient weighting scheme based on the idea of the iterative support detection [3]. A feature of this kind of weighting is that the weight value is either 1 or 0 and we will explain its advantages in the following parts. We can see that for our new WJS model, we get the final results from multi-stage process by solving a series of weighted joint sparsity problems with weights known. We start from solving the plain \( \ell_{2,1} \) model (5), then improve the solution gradually via a multistage procedure. At each iteration in the process, the weights will change according to the most recently recovered solution. This full procedure, Step 1 and Step 2 will be introduced in more details in next sections, where the iterative support detection will be also reviewed.

3.2. Step 1: Solving the WJS model given Weights

Now we consider how to solve the WJS model once the weights are given, i.e. how to realize Step 1. It suggests that solving problem (9) is equivalent to solving problem (5). Then YALL1 group algorithm can be adopted to solve the WJS model (9). We have the following algorithmic framework for the weighted version, based on the YALL1 group version for the plain \( \ell_{2,1} \) model. The iteration framework is

\[
\begin{aligned}
X &\leftarrow (\beta_1 I + \beta_2 A^T A)^{-1}(\beta_1 Z - \Lambda_1 + \beta_2 A^T B + A^T \Lambda_2), \\
Z &\leftarrow \text{Shrink}(X + \frac{1}{\beta_1} \Lambda_1, \frac{1}{\beta_1} w)\text{(row-wise)}, \\
\Lambda_1 &\leftarrow \Lambda_1 - \gamma_1 \beta_1 (Z - X), \\
\Lambda_2 &\leftarrow \Lambda_2 - \gamma_2 \beta_2 (AX - B). \\
\end{aligned}
\]

Here \( \Lambda_1, \Lambda_2 \) are multipliers, \( \beta_1, \beta_2 \) are penalty parameters, \( \gamma_1, \gamma_2 \) are step lengths. Now, we discuss the different results for different kinds of the weights. As special cases, when the \( w_i \) is 0, it’s easy to get

\[
z^i = r^i := x^i + \frac{1}{\beta_1} \lambda_1^i, \quad i = 1, \ldots, n, \tag{11}
\]

and when the \( w_i \) is 1, we can get

\[
z^i = \max\{||r^i||_2 - \frac{1}{\beta_1} r^i, 0\} ||r^i||_2, \quad i = 1, \ldots, n \tag{12}
\]
where

\[ r^i := x^i + \frac{1}{\beta_1} \lambda^i. \]  

(13)

In general, the updating of \( Z \) represents

\[ z^i = \max \left\{ ||r^i||_2 - \frac{w_i}{\beta_1}, 0 \right\} \frac{r^i}{||r^i||_2}, \quad i = 1, \ldots, n \]  

(14)

where

\[ r^i := x^i + \frac{1}{\beta_1} \lambda^i. \]  

(15)

The above procedure is summarized in the following Algorithm 2.

Notice that the above procedure is in fact a selective shrinkage procedure, if the weight value is either 1 or 0. The YALL1 group model obtains the sparse solution via the shrinkage operator. However, this kind of shrinkage has a disadvantage, i.e., it shrinks the true nonzero components as well, and reduces the sharpness of the solution or introduces bias to the final solution. By using 0-1 weights, some components will not be shrunk if we believe that they are unlikely to be zero. In such cases, their corresponding weights are set as 0. This is a natural settings and the advantages of the 0-1 weights over other kinds of weights have been demonstrated in either theoretic or practical point of view in [3] and will be presented in more details in next part. Here the weights are added on each groups. In other words, components within a group are associated with the same weight in order to respect the prior grouping information.

Algorithm 2 Solving the WJS model (The Inner Iteration)

1. Initialize \( X \in \mathbb{R}^{n \times l}, \Lambda_1 \in \mathbb{R}^{n \times l}, \Lambda_2 \in \mathbb{R}^{m \times l} > 0, \beta_1, \beta_2 > 0 \) and \( \gamma_1, \gamma_2 > 0; \)

2. While stopping criterion is not met, do

(a) \( X \leftarrow (\beta_1 I + \beta_2 A^T A)^{-1}(\beta_1 Z - \Lambda_1 + \beta_2 A^T B + A^T \Lambda_2), \)
(b) \( Z \leftarrow \text{Shrink} \left( X + \frac{1}{\beta_1} \Lambda_1; \frac{1}{\beta_1} w \right) \text{ (row-wise)}, \)
(c) \( \Lambda_1 \leftarrow \Lambda_1 - \gamma_1 \beta_1 (Z - X), \)
(d) \( \Lambda_2 \leftarrow \Lambda_2 - \gamma_2 \beta_2 (AX - B). \)

3.3. Step 2: Weights Determination Based on the Iterative Support Detection

We propose a way to determine the weights \( w \) in Step 2. The main idea is coming from the iterative support detection (ISD) proposed in [3]
for sparse signal reconstruction, and we will propose how we can apply the idea of ISD to joint sparsity problem. In addition, we will give some novel analysis of the advantages of the proposed $0 – 1$ weighting scheme by ISD, compared with other weighting alternatives. In addition, we would like to give some preliminary analysis about why threshold-support detection works well for joint sparse Bernoulli signals while it fails for the single channel sparse Bernoulli signal as showed in [3].

We first briefly review the idea of iterative support detection (ISD) in the compressive sensing for the single channel sparse signal reconstruction. Compressive sensing (CS) [27, 28] reconstructs a sparse unknown signal from a small set of linear projections. Let $\mathbf{\pi}$ denote a $k$-sparse signal, and let $b = A\mathbf{\pi}$ represent a set of $m$ linear projections of $\mathbf{\pi}$. The general optimization problem is the basis pursuit (BP) problem

$$\min_{\mathbf{u}} ||\mathbf{u}||_1 \quad s.t. \quad A\mathbf{u} = \mathbf{b}. \quad (16)$$

ISD alternatively calls its two components: support detection and signal reconstruction. Support detection identifies an index set $I$ from an incorrect reconstruction, due to the insufficient measurements for the $\ell_{2,1}$ model. $I$ contain some elements of $\text{supp}(\mathbf{\pi}) = \{i : u_i \neq 0\}$. After acquiring the support detection, the signal reconstruction models transforms

$$\min_{\mathbf{u}} ||\mathbf{u}_T||_1 \quad s.t. \quad A\mathbf{u} = \mathbf{b} \quad (17)$$

where $T = I^C$ and $||\mathbf{u}_T||_1 = \sum_{i \notin I} |u_i|$. Assuming a sparse original signal $\mathbf{\pi}$, if the support detection $I = \text{supp}(\mathbf{\pi})$, then the solution of (18) is certainly equal to true $\mathbf{\pi}$. But we should point out that even if $I$ contains enough, not necessarily the all, entries of $\text{supp}(\mathbf{\pi})$, it can also happens. When $I$ does not have enough of $\text{supp}(\mathbf{\pi})$ for an exact reconstruction, those entries of $\text{supp}(\mathbf{\pi})$ in $I$ will help (18) return a better solution in comparison to (17), and from this better solution, support detection will be able to identify more entries in $\text{supp}(\mathbf{\pi})$ and then yield an even better $I$. By this means, the two components of ISD work together to gradually recover $\text{supp}(\mathbf{\pi})$ and improve the reconstruction. It is clearly that the idea of ISD is a multi-stage process.

ISD needs the reliable support detection from inexact reconstruction, which can be obtained by taking advantages of the features and prior information about the original true signal $\mathbf{\pi}$ [3, 29]. The authors in [3] focused on the sparse or compressible signals with components having a fast decaying
distribution of nonzeros, and they performed the support detection by thresholding the solution of \((18)\), and called the corresponding ISD algorithm as \(\text{threshold-ISD}\). Notice that the fast decaying property is a mild assumption actually, because in practice the most natural signals satisfies this property under a appropriately basis.

As for the joint sparsity problems in this paper, the support detection method, \(\text{threshold-ISD}\), can also be applied in a similar way as follows. We present effective support detection strategies for joint sparse model, where we assume that the norms of each non-zeros rows has a fast decaying property. Our method are based on thresholding

\[
I^{(s+1)} := \{i : |t_i^{(s)}| > \epsilon^{(s)}\}, \quad s = 0, 1, 2, \cdots
\]  

where \(t_i^{(s)}\) is the 2-norm of the \(i\)-th row of \(X\). The weighting vector \(w^{(s+1)}\) whose elements equals to 0 if its corresponding position belongs to the support detection \(I^{(s+1)}\), or 1 otherwise. The support sets \(I^{(s)}\) are not necessarily increasing and nested, which means that all \(s\) may be not in \(I^{(s)} \subset I^{(s+1)}\). It is significant due to it is impossible to completely avoid wrong detections by setting \(\epsilon^{(s)}\), which based on \(t_i^{(i)}\) for \(i \leq s\). No matter how big a component of \(t_i^{(i)}\) is, it could nevertheless still be zero in the true solution. Because \(I^{(s)}\) is not required to be monotonic, support detection can remove previous wrong detections, which makes \(I^{(s)}\) less sensitive to \(\epsilon^{(s)}\).

After discussing the support sets \(I^{(s)}\), we study the choice of \(\epsilon^{(s)}\). There are many different rules for \(\epsilon^{(s)}\). We select a simple and well-understood rule, based on the locating “first significant jump” in the increasingly sorted sequence \(|t_{[i]}^{(s)}|\) (\(|t_{[i]}|\) denotes the \(i\)-th largest component of \(t\) by magnitude), as used in \(^3\). The rule aims to looking for the smallest \(i\) such that

\[
|t_{[i+1]}^{(s)}| - |t_{[i]}^{(s)}| > \tau^{(s)},
\]  

This amounts to checking up the increasing sequence \(|t_{[i]}^{(s+1)}|\) and looking for the value of the left of \((20)\). Further, we set \(\epsilon^{(s)} = |t_{[i]}^{(s)}|\). Numerical experiments in section 4 show the rule has access to detect lots of true nonzeros with few false signals. Moreover, there are several simple and heuristic methods has been adopted to define \(\tau^{(s)}\) for different kinds of sparse or compressible signals. For example, for the sparse Gaussian signals, one can set \(\tau^{(s)} = m^{-1}||t^{(s)}||_\infty\), where \(m\) is the number of measurements.

In addition, our \(0-1\) weights of ISDJS algorithm is a more explicit and straightforward way to make use of the correct information (find the
locations of components of large magnitude and set the corresponding weights as 0) and give up the rest too noisy information (for those components of small magnitudes, they are mostly overwhelmed by the reconstructed noise). Therefore there is very little meaning to set different weights according to their magnitudes. The practical performance of ISDJS algorithm is also mostly better than the iterative reweighted \( \ell_1 \) algorithm and the iterative reweighted \( l_2 \) algorithm \[30\], which has been demonstrated in \[3\].

For signals which decay slowly or have no decay at all, such as sparse Bernoulli signals, ISD does not work well in the cases of the single channel sparse signal reconstruction \[3\]. However, for joint sparsity cases, ISDJS algorithm still works well, surprisingly. We give a brief explanation as below.

Notice that while the implementation of support detection is the key component of our algorithm, it is very flexible and easy to incorporate the structure of the underlying signals. For example, in \[31\], we have compared ISD based algorithm with the recent \( \ell_0 \)-norm regularized based algorithm for sparse signal recovery under the wavelet frame transform. Due to the incorporation of the specific structure of the wavelet frame coefficients, ISD outperform the \( \ell_0 \)-norm based recovery algorithms, which fail to consider this structure. In this paper, when considering joint sparsity cases, the implementation of the support detection by \[18\] has in fact already take the grouping information into consideration, though it is very simple and natural. That is to say, this structural information enhances the performance of support detection. While the fast decaying property is still able to improve the performance of support detection, it is not necessarily required. This is a big difference with the original single channel sparse signal recovery problem.

3.4. Algorithmic framework

Now, we summarized the algorithm framework of the multi-stage convex relaxation for the weighted model based on ISD. The Algorithm 3 repeatedly performs above two steps: support detection to determine the 0 − 1 weights and solving the resulted weighted joint sparsity. Therefore, we called our new algorithm as ISDJS.

Note that Since \( T^{(0)} = \emptyset \), and \( T^{(0)} = \Omega \), weighted \( \ell_{2,1} \) model in Step 1 reduces to the plain \( \ell_{2,1} \) model in iteration 1. The weighted \( \ell_{2,1} \) model in 2(b) of Algorithm 3 can be solved by Algorithm 2. The support detection in 2(c) has been introduced in the section 3.3. While our new algorithm is a multistage procedure, its costing time is not necessarily several times of that of the original \( \ell_{2,1} \) model. The reason is due to the warm-starting, i.e.
Algorithm 3 The ISDJS Algorithm (The Outside Iteration)

Input: $A$ and $B$

1. Set the iteration number $s \leftarrow 0$ and
   initialize the set of detected entries $I^{(s)} \leftarrow \emptyset$;
2. While stopping condition is not met, do
   (a) $w^{(s)} \leftarrow (I^{(s)})^{C} := \{1, 2, \ldots, n\} \setminus I^{(s)}$;
   (b) $X^{(s)} \leftarrow$ solve problem (9) or (10) for $w = w^{(s)}$;
   (c) $w^{(s+1)} \leftarrow$ support detection using $X^{(s)}$ as the reference;
   (d) $s \leftarrow s + 1$.

The output of the current stage (outside iteration) is used to be the input of the next stage. Then in the next stage, we often just need to run only a few inner iterations to get a better updated solution. In addition, the number of the stages is often small, like around 4 in the under numerical experiments.

4. Numerical Experiments

In this section, we present several numerical results to evaluate the performance of our proposed ISDJS algorithm in comparison with the YALL1 group algorithm [3], SOMP algorithm [10, 32] and p-threshold algorithm [33], which are all state of the art algorithms for joint sparsity. Due to the running time of ISDJS being only reasonably larger than YALL1 group in most cases, or even smaller in some cases, as suggested by the cases of single channel sparse signals in [3] and our own experiences in joint sparsity, we mainly focus on the the recovery rate or accuracy in this paper, as we have known that SOMP is usually the fastest in most cases. In addition, considering that the number of the channel has a great influence on recovery rate of joint sparsity, we tested different settings of noise levels for ISDJS algorithm with different number of channels, and drew a comparison between ISDJS algorithm and other alternative algorithms performed on cases with different noise levels and different number of channels. The carefully design synthetic examples and a more realistic example from collaborative spectrum sensing [35], which aims at detecting spectrum holes (i.e., channels not used by any primal users), are both considered to demonstrate the performance of tested algorithms.
4.1. Parameter Settings of Algorithms

For SOMP and p-thresholding, we use their default parameter settings in the literature [10, 33]. The ISDJS algorithm and YALL1 algorithm are both terminated when

$$\frac{||t^{(k+1)} - t^{(k)}||}{||t^{(k+1)}||} \leq \epsilon.$$  \hspace{1cm} (20)

It means the relative change of two consecutive iterates becomes smaller than a prescribed tolerance $\epsilon$. We empirically turned the tolerance value, which is based on comparing consecutive support sets $I^{(s)}$. For YALL1 group algorithm, $\epsilon$ was set as $10^{-6}$ [2]. As for ISDJS, for the first few outside steps, we only want to get an rough estimate of the support information of $X$, so we just set a relatively loose tolerance, for example, $\epsilon = 10^{-2}$. But for the last step, the $\epsilon$ was also set as $10^{-6}$ for consistency with YALL1. In all tests, ISDJS algorithm was set to run no more than 5 outer iterations. We set the parameters of the YALL1 group and ISDJS algorithms referring to [3], as follows: $\beta_1 = 0.3/\text{mean}(\text{abs}(B))$, $\beta_2 = 3/\text{mean}(\text{abs}(B))$ and $\gamma_1 = \gamma_2 = 1.618$. $B$ is denoted by the MATLAB-type notation $\text{mean}(\text{abs}(B))$. Randomized partial Walsh-Hadamard transform matrices are utilized as the measurement matrices $A \in \mathbb{R}^{m \times n}$ in our examples, because these transform matrices are suitable for large-scale computation and have the property $AA^T = I$. Therefore, in this paper, we only take the Walsh-Hadamard transform matrix as the example to demonstrate the performance of ISDJS and other alternative algorithms. The other parameters are described below for different tests.

4.2. Parameter Settings of the Synthetic Signal Tests

Gaussian distribution has been commonly used in the synthetic signal recovery experiments due to its ability to mimic many real signals. Correspondingly, our jointly sparse solutions $X \in \mathbb{R}^{n \times l}$ are generated by randomly selecting $k$ rows to have iid random Gaussian entries and letting the other rows to be zero in the first test, and sparse Bernoulli signals (whose nonzeros are either 1 or $-1$) in the second test, respectively. While it has been shown in [3], threshold-ISD, applied to the plain $\ell_1$ model can not bring any recovery improvement for the single channel Bernoulli signals, things are quite different in the multi-channel cases, i.e., we will show that our algorithm can achieve much better recovery accuracy than the plain $\ell_{2,1}$ model, even for the sparse Bernoulli signals, as shown in Figure 9 and Figure 10.

Experimentally, the recoverability of our tested algorithms will all become better and better as the number of channels increases gradually, though to
different degrees \[34\]. Therefore, different channel numbers are tried in our experiments, for example, \(L = 1, 2, 4, 8, 16\), respectively. We also try the sparsity levels varying from \(k = 80\) to \(160\), but fix \(n = 1024\), \(m = 256\) in all figures excluding figure 1, 2, 6, 7. In the following figures the experimental results are usually an average of 100 runs due to the randomness inherent in the \(A\) and \(X\).

4.2.1. Test 1: Joint Sparse Gaussian Signals

We firstly perform a simple demo to show the thresholding based support detection used in ISDJS algorithm is efficient for sparse Gaussian signals. In order to more clearly see the effect, we generate a sparse signal \(\bar{X} \in \mathbb{R}^{600 \times L}\) with \(k = 30\) nonzero rows. With 600 dimensions, it is difficult to recover a signal with 30 nonzero rows from measurements of \(m\) (60-120) rows. However, the ISDJS algorithm returns an exact reconstruction and achieves very small relative error as iterative numbers increasing in a set of channels. The performance of the ISDJS algorithm in the first iteration and the fourth iteration for a set of channels are depicted in the figure \[1\] where we set \(\bar{t}\) (a vector of 2-norm of each row of \(\bar{X}\)) on behalf of the true signal \(\bar{X}\) and \(t\) (a vector of 2-norm of each row of \(X\) from ISDJS algorithm) on behalf of the recovered signal. To measure the accuracy of our support detection, we give the quadruplet ”(Total, Detected, Correct, False)” and ”Err” in the title of each subplot and in the table. They are defined as follows:

- (Total, Detected, Correct, False):
  - Total: the number of total nonzero rows of the true signal \(\bar{X}\);
  - Detected: the number of detected nonzero rows, equal to \(|I| = (Correct) + (False)\);
  - Correct: the number of correctly detected nonzero rows, i.e., \(|I \cap \{i : \bar{t}^i \neq 0\}|\);
  - False: the number of falsely detected nonzero rows, i.e., \(|I \cap \{i : \bar{t}^i = 0\}|\).

- Err: the relative error \(\|X - \bar{X}\|_2/\|\bar{X}\|_2\).

From the upper left subplot (a) in Figure \[1\] we can see that the solution of ISDJS algorithm in the first iteration, which is equivalent to the solution of YALL1 group algorithm, is not good. Nevertheless ISDJS algorithm in the fourth iteration could well match the true signal with a very small relative error. With increasing channels \(L\) and decreasing measurements \(m\), ISDJS algorithm always performs better in the fourth iteration than the first iteration, which shows the advantage of our proposed algorithm over YALL1 group algorithm. Notably, most of true nonzero rows with large magnitude
had been correctly detected in fourth iteration. ISDJS algorithm is insensitive to a small number of false detections and has an attractive self-correction capacity. What’s more, it is difficult to recover a signal with 30 nonzero rows from measurements of 60 rows, but ISDJS algorithm does well in subplot (e) in the given settings. It suggests ISD is greatly improved for Gaussian signal with the joint sparsity.

In order to better understand our proposed algorithm, we illustrate every outer iteration of ISDJS in Figure 2 by taking an example of $L = 4$. From the upper left subplot, it is clear that the ISDJS algorithm in first iteration (i.e. YALL1 group algorithm), finds very few positions of correct nonzero row and has a large relative error. However, a half positions of correct nonzero row could be found in the next iteration, although the relative error is improved a litter. In the third iteration, our algorithm has already correctly detected the most nonzero row. The fourth iteration is good enough to return a much better solution with a small relative error. In addition, Table 1 exhibits the portion of correctly detected nonzeros among all detections of each iteration, in cases of different channels for sparse Gaussian signal recovery, in order to fully show the performance of the thresholding based support detection.

Figure 3 shows the recovery performance of ISDJS algorithm for sparse Gaussian signals with different numbers of channel in a set of noise levels. As expected, the performance of the ISDJS algorithm becomes better as the number of channel increases even if the noise level is raised meanwhile. Note that the recoverability of ISDJS algorithm on the multichannel cases is much better than the single channel ($L=1$). Then the robustness of ISDJS to the noise is also demonstrated.

In Figure 4 and Figure 5, we compare the recovery rates and relative errors of ISDJS algorithm with YALL1 group algorithm, SOMP algorithm and p-threshold algorithm, for noiseless cases and noisy cases (0.5% Gaussian noise), respectively. We can see that the ISDJS algorithm outperforms the other tested algorithms in different channels. In particular, the plain $\ell_{2,1}$ model is worse for sparse Gaussian signals than the SOMP for 4 channels and 8 channels. However, when ISD is applied to the plain $\ell_{2,1}$ model, the resulted weighted model and correspondingly ISDJS algorithm behaviors better than SOMP.

4.2.2. Test 2: Joint Sparse Bernoulli Signals

Similar with Test 1, we depict a demo to show the support detection of the ISDJS algorithm for jointly sparse Bernoulli signals. As mentioned in

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threshold-ISD requires reliable support detection, which works well for signals with a fast decaying distribution of nonzero values, which include sparse Gaussian signals and certain power-law decaying signals. However, ISD does not work on signals that decay slowly or have no decay at all such as sparse single channel Bernoulli signals. As the upper left subplot (a) in Figure 6 shows, the support detection is poor and does not correctly detect the true nonzeros even in the fourth iteration. Specially the upper right subplot (b) displays the support detection finds abundant true nonzeros after 4 iterations just for $L = 2$, which means ISD work well for multichannel Bernoulli signals with joint sparse structure. In the following subplot (c), (d) and (e) of Figure 6, the ISDJS algorithm in the first iteration performs badly while in fourth iteration one detects correct nonzero positions by one hundred percent and achieves tiny relative error.

For $L = 4$, in the Figure 7, we illustrate the support detection works well on joint sparse Bernoulli signals in four iterations. The support detection performs better and better as more iteration numbers. It is possible to add more iterations but four iterations are good enough to let ISDJS algorithm return an exact solution with a very small relative error. In addition, we also fully exhibit the accuracy of the support detection in each iteration in different channels for joint sparse Bernoulli signals in Table 2.

In Figure 8, we show the recovery rates of ISDJS algorithm with different channels in a set of noise levels for Bernoulli sparse signals. Consistently, the ISDJS algorithm performs much better for multichannel Bernoulli signals than single channel Bernoulli signals. With increasing noise levels, the ISDJS algorithm is robust for multichannel Bernoulli signals with joint sparse structure. In Figure 9, in particular, the upper left subplot (a) presents the recovery rate of the ideal ISDJS algorithm (i.e. the support detection could find the correct position of all true nonzeros) in comparison with the other three algorithms for noiseless single channel Bernoulli signal. The performance of the ideal ISDJS algorithm is very well with the highest recovery rate. It suggests that the ISDJS algorithm could obtain a more exact solution with an accurate support detection, which is one of our researches in the future. Then, we present the recovery rate of ISDJS algorithm in contrast to the other three algorithms in different channels for noiseless Bernoulli signals in the following subplots. The upper right subplot (b) shows the four algorithms perform poor on single channel Bernoulli signals. However, the recoverability of ISDJS algorithm is dramatically improved on the multichannel Bernoulli sparse signals in latter subplots, although recoverability of
the other algorithms are increasingly enhanced as channel increases. Corresponding to the Figure 9, Figure 10 exhibits the relative error of the ideal ISDJS algorithm compared with the other three algorithms with 0.5\% noise for single channel Bernoulli signal in subplot (a) and the relative error of the ISDJS algorithm compared with the other three algorithms with 0.5\% noise in a set of channels in latter subplots (b), (c), (d) and (e). Particularly, with some noise, the advantage of ISDJS algorithm is more obvious in this case according to the smallest relative error in subplot (a). From the latter subplots in Figure 10, we observe that the recoverability of tested algorithms are increasingly enhanced as the number of channels increases, especially for ISDJS algorithm. Here we point out that the recoverability of our algorithm is always better than that of the YALL1 groups for the plain $\ell_{2,1}$ model.

The numerical experiments above fully attest that joint sparsity via the incorporation of the idea of ISD is practicable and valuable, and confirm that ISD can make significant improvement for multichannel joint sparse signal recovery even without the fast decaying property. In addition, as the above Figures showed, ISDJS requires reliable support detection, and we will develop more effective support detection schemes in the future work to optimize our proposed algorithm, beyond the simple thresholding method adopted in this paper.

4.3. An Example from Collaborative Spectrum Sensing

In order to better understand the advantages of ISDJS algorithm, we apply ISDJS algorithm to an example coming from a real application. It comes from collaborative spectrum sensing [35, 36, 37], which wants to detect spectrum holes (i.e., channels not used by any primal users), is the precondition for the implementation of cognitive radio (CR). The cognitive radio (CR) nodes must constantly sense the spectrum in order to detect the presence of the primary radio (PR) nodes and use the spectrum holes without causing harmful interference to the PRs. With a reliable manner, therefore sensing the spectrum is of critical significant and causes a major challenge in CR networks. Improving the ability of detecting complete spectrum usage in collaborative spectrum sensing is expected. Our example is coming from the research group of Prof. Zhu Han of ECE department of University of Houston.

We consider a cognitive radio network with $m$ CR nodes that locally monitor a subset of $n$ channels. A channel is either occupied by a PR or unoccupied, corresponding to the states 1 and 0, respectively. It is assumed
that the numbers of occupied channels is much smaller than $n$. We aim to recovering the occupied channels form the CR nodes observations. With frequency-selective filters, a CR takes few measurements that are linear combinations of multiple channels. The filter coefficients are designed to be random numbers to mix the different channel sensing information. Then, the outputs of filter are fused in the next step. We assume $p$ frequency selective filters in each CR node sending out $p$ reports regarding the $n$ channels. The sensing process at each CR can be represented by a $p \times n$ filter coefficients matrix $A$. Let an $n \times n$ diagonal matrix $H$ represent the states of all the channel sources using 0 and 1 as diagonal entries, indicating the unoccupied or occupied states, respectively. There are $s$ nonzero entries in the diagonal matrix $H$. In addition, channel gains are characterized by an $m \times n$ channel gain matrix $G$. Then, the fusion can be formulated as a $p \times m$ matrix as follows:

$$B_{p \times m} = A_{p \times n} H_{n \times n} (G_{m \times n})^T. \quad (21)$$

Now, we need a highly effective method for recovering:

$$X_{n \times m} = H_{n \times n} (G_{m \times n})^T \quad (22)$$

Here, we consider a 25 node cognitive radio network (i.e., $m = 25$), the number of channels is 256 (i.e., $n = 256$), the number of active PR nodes ranging from 1 to 25 on the given set of 256 channels, the measurement matrix $A$ is Gaussian random matrix, and the size is fixed as $p = 32$. Empirically, we set the $\tau^{(s)}(\text{mean}(\text{diff}(t^{(s)})))$ in MATLAB. In this test, we also give the comparison between ISDJS algorithm, YALL1 group algorithm, SOMP algorithm and p-threshold algorithm in a set of channels. Obviously, in Figure 11, our proposed ISDJS algorithm outperforms all the competing algorithms with different channels in collaborative spectrum sensing, with the highest recovery rate. The experimental results from collaborative spectrum sensing also demonstrate the effectiveness of the proposed algorithm.

5. Conclusion

In this paper we analyzed the enhanced recovery quality of joint sparsity via the incorporation of the idea of ISD [3]. In particular, while it is well known that SOMP is usually better than the plain $\ell_{2,1}$ model in noiseless cases, ISDJS which is kind of the combination of ISD and the $\ell_{2,1}$ model, behaviors even better than SOMP, at least in our settings. Moreover, it
has been observed that ISDJS algorithm is very robust to the noise in the multi-channel cases.

In this paper, we have observed that ISD is able to work with joint sparsity almost seamlessly, without other assumptions including the fast decaying property of nonzero components. Thus we expect that ISD can also work well with other kinds of structural sparsity models, which has attracted more and more attention because meaningful structures exist in many practical problems \[39\]. We will do more researches along this direction.

Finally, we need to point out that support detection, as a general idea for adaptive sparse signal recovery, is not limited to thresholding. Therefore, the future work covers studying other signal models and developing more effective support detection methods based on the specific property of the underlying sparse signals in many practical applications \[40\].

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| accuracy | 1  | 2  | 4  | 8  | 16 |
|----------|----|----|----|----|----|
| itr      |    |    |    |    |    |
| 1        | 0.74(0.18) | 0.92(0.29) | 0.93(0.79) | 1(0.96) | 1(0.97) |
| 2        | 0.84(0.24) | 0.98(0.61) | 0.99(0.96) | 1(0.97) | 1(0.98) |
| 3        | 0.91(0.33) | 0.99(0.85) | 1(0.97)    | 1(0.99) | 1(1.00) |
| 4        | 0.98(0.46) | 1(0.95)   | 1(0.97)    | 1(1.00) | 1(1.00) |

Table 1: the support detection is exhibited in each iteration of different channels for sparse Gaussian signals, where the accuracy is defined as Correct/Detection (Correct/Total).
Table 2: The support detection is exhibited in each iteration of different channels for sparse Bernoulli signals, where the accuracy is defined as Correct/Detection (Correct/Total).

| itr | 1       | 2       | 4       | 8       | 16      |
|-----|---------|---------|---------|---------|---------|
| 1   | 0.80(0.56) | 0.99(0.87) | 1(0.89) | 1(0.97) | 1(0.98) |
| 2   | 0.85(0.64) | 0.99(0.97) | 1(0.97) | 1(0.98) | 1(0.99) |
| 3   | 0.91(0.71) | 1(0.98)  | 1(1.00) | 1(1.00) | 1(1.00) |
| 4   | 0.99(0.75) | 1(0.98)  | 1(1.00) | 1(1.00) | 1(1.00) |
Figure 1: compare the true Gaussian signals and recovered signals from ISDJS algorithm in different channels, where the two parts in each subplot are the results in the first iteration and the fourth iteration from ISDJS algorithm, respectively. (a)L=1, m=120, (b)L=2, m=100, (c)L=4, m=80, (d)L=8, m=60, (e)L=16, m=60.
Figure 2: compare the true Gaussian signals and recovered signals from ISDJS algorithm in each iteration in 4-channel.
| Sparsity Level | Recovery Rate |
|---------------|---------------|
| L=1           |               |
| L=2           |               |
| L=4           |               |
| L=8           |               |
| L=16          |               |

Figure 3: compare the recovery rate of ISDJS algorithm with L=1, 2, 4, 8, 16 in different noise levels for Gaussian signals, (a) noiseless data, (b) 0.5% noise, (c) 1% noise, (d) 10% noise.
Figure 4: compare the recovery rate of four algorithms in different channels for noiseless Gaussian signals, (a) L=1, (b) L=2, (c) L=4, (d) L=8, (e) L=16.
Figure 5: compare relative error of four algorithms in different channels with 0.5% noise for Gaussian signals, (a)L=1, (b)L=2, (c)L=4, (d)L=8, (e)L=16.
Figure 6: compare the true Bernoulli signals and recovered signals from ISDJS algorithm in different channels, where the two components in each subplot is the result in the first iteration and the fourth iteration from ISDJS algorithm, respectively. (a) L=1, m=110, (b) L=2, m=90, (c) L=4, m=70, (d) L=8, m=60, (e) L=16, m=50.

| L-Iteration Number | Nonzeros | Relative error |
|---------------------|----------|----------------|
|                     | Total true | Detected | Correct | False |                     |
| 1-1                 | 30        | 2         | 1       | 1      | 7.69e-01             |
| 1-4                 | 30        | 20        | 13      | 7      | 6.46e-01             |
| 2-1                 | 30        | 14        | 14      | 0      | 1.82e-01             |
| 2-4                 | 30        | 30        | 30      | 0      | 5.51e-05             |
| 4-1                 | 30        | 28        | 28      | 0      | 4.37e-01             |
| 4-4                 | 30        | 30        | 30      | 0      | 6.69e-05             |
| 8-1                 | 30        | 29        | 29      | 0      | 4.55e-01             |
| 8-4                 | 30        | 30        | 30      | 0      | 6.70e-04             |
| 16-1                | 30        | 29        | 29      | 0      | 4.67e-01             |
| 16-4                | 30        | 30        | 30      | 0      | 5.48e-03             |
Figure 7: compare the true Bernoulli signals and recovered signals from ISDJS algorithm in each iteration in the 4-channel cases.
Figure 8: compare the recovery rate of ISDJS algorithm with \( L=1, 2, 4, 8, 16 \) in different noise levels for Bernoulli signals, (a)noiseless data, (b)0.5% noise, (c)1% noise, (d)10% noise.
Figure 9: compare the recovery rate of four algorithms in different channels for noiseless Bernoulli signals, (a) compare ideal ISDJS algorithm with other tested algorithms in $L=1$, (b) $L=1$, (c) $L=2$, (d) $L=4$, (e) $L=8$, (f) $L=16$. 
Figure 10: compare relative error of four algorithms in different channels with 0.5% noise for Bernoulli signals, (a) compare ideal ISDJS algorithm with other tested algorithms in $L=1$, (b) $L=1$, (c) $L=2$, (d) $L=4$, (e) $L=8$, (f) $L=16$. 
Figure 11: compare the recovery rate of four algorithms in different channels in the example from Cognitive Radio (CR), (a)\(L=1\), (b)\(L=2\), (c)\(L=4\), (d)\(L=8\), (e)\(L=16\).