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On the number of linearly independent admissible solutions to linear differential and linear difference equations. (English) Zbl 07456131
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Authors’ abstract: A classical theorem of Frei states that if $A_p$ is the last transcendental function in the sequence $A_0, \ldots, A_{n-1}$ of entire functions, then each solution base of the differential equation $f^{(n)} + A_{n-1}f^{(n-1)} + \cdots + A_1f' + A_0f = 0$ contains at least $n-p$ entire functions of infinite order. Here, the transcendental coefficient $A_p$ dominates the growth of the polynomial coefficients $A_{p+1}, \ldots, A_{n-1}$. By expressing the dominance of $A_p$ in different ways and allowing the coefficients $A_{p+1}, \ldots, A_{n-1}$ to be transcendental, we show that the conclusion of Frei’s theorem still holds along with an additional estimation on the asymptotic lower bound for the growth of solutions. At times, these new refined results give a larger number of linearly independent solutions of infinite order than the original theorem of Frei. For such solutions, we show that 0 is the only possible finite deficient value. Previously, this property has been known to hold for so-called admissible solutions and is commonly cited as Wittich’s theorem. Analogous results are discussed for linear differential equations in the unit disc, as well as for complex difference and complex $q$-difference equations.

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MSC:
34M03 Linear ordinary differential equations and systems in the complex domain
34M10 Oscillation, growth of solutions to ordinary differential equations in the complex domain
30D35 Value distribution of meromorphic functions of one complex variable, Nevanlinna theory

Keywords:
deficient values; entire functions; Frei’s theorem; linear difference equations; linear differential equations; Wittich’s theorem

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