Quantitative Strong Unique Continuation for Elliptic Operators - Application to an Inverse Spectral Problem

Mourad Choulli

Abstract. Based on the three-ball inequality and the doubling inequality established in [23], we quantify the strong unique continuation established by Koch and Tataru [21] for elliptic operators with unbounded lower-order coefficients. We also obtain a quantitative strong unique continuation for eigenfunctions that we use to prove that two Dirichlet-Laplace-Beltrami operators are gauge equivalent whenever their corresponding metrics coincide in the vicinity of the boundary and their boundary spectral data coincide on a subset of positive measure.

1. Quantitative Strong Unique Continuation

Let $\Omega$ be a Lipschitz domain of $\mathbb{R}^n$, $n \geq 3$, with boundary $\Gamma$, and $A = (a^{k\ell})$ be a symmetric matrix with Lipschitz continuous components so that

$$\kappa^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq \kappa|\xi|^2 \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n,$$

$$|a^{k\ell}(x) - a^{k\ell}(y)| \leq \kappa|x - y|, \quad x, y \in \overline{\Omega}, \quad 1 \leq k, \ell \leq n,$$

where $\kappa > 1$ and $\kappa > 0$ are constants.

Let $V \in L^{n/2}(\Omega; \mathbb{R})$ and $B, W \in L^p(\Omega, \mathbb{R}^n)$, where $p > n$ is fixed, and

$$\mathcal{E}u := \text{div}(A\nabla u + uB) + W \cdot \nabla u + Vu.$$

The following notation will be used in the rest of this text.

$$\Omega_\rho = \{x \in \Omega; \text{dist}(x, \Gamma) > 4\rho\}, \quad \rho > 0.$$

We assume that there exists $\rho_0 > 0$ such that $\Omega_{\rho_0}$ is connected for all $0 < \rho < \rho_0$, and the same condition holds when we deal with $\Omega \times (0,1)$ instead of $\Omega$.

We say that $u \in H^1(\Omega)$ satisfies $\mathcal{E}u = 0$ in $\Omega$ if

$$\int_{\Omega} \left[-(A\nabla u + uB) \cdot \nabla v + (W \cdot \nabla u + Vu)v\right]dx = 0, \quad v \in C^\infty_0(\Omega).$$

Recall that $x_0 \in \Omega$ is a zero of infinite order of $u \in L^2_{\text{loc}}(\Omega)$ if for all $N \in \mathbb{N}$ and a sufficiently small $r > 0$ we have

$$\int_{B(x_0,r)} u^2 dx = O(r^N).$$

Koch and Tataru [21] proved that if $u \in H^1(\Omega)$ satisfies $\mathcal{E}u = 0$ in $\Omega$ and has a zero of infinite order in $\Omega$, then $u = 0$. In other words, all zeros of $u \in H^1(\Omega) \setminus \{0\}$

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satisfying $\mathcal{E}u = 0$ in $\Omega$ are of finite order. Our objective is then to quantify the order of the zeros of the solutions of $\mathcal{E}u = 0$ in $\Omega$. Let $\Omega_0 \subset \Omega$. Let $\mathcal{C}$ be the set of all positive constants that depend on $n$, $\Omega_0$, $\kappa$, $\varkappa$, $B$, $W$ and $V$. When $\mathcal{N}$ is an arbitrary quantity, we write $\mathcal{c} \in \mathcal{C}_0$ to denote that $\mathcal{c}$ is a positive constant depending on $n$, $\Omega$, $\Omega_0$, $\kappa$, $\varkappa$, $B$, $W$, $V$ and $\mathcal{N}$.

Set

$$\mathcal{S}_0 = \{ u \in H^1(\Omega) \setminus \{0\}; \mathcal{E}u = 0 \text{ in } \Omega \}.$$

1.1. Statement of the main theorem.

**Theorem 1.1.** There exist $\rho_* \in \mathcal{C}$, $0 < \rho_* \leq \rho_0$, $\tau \in \mathcal{C}$ with $\tau < 1/4$ so that, for all $0 < \rho < \rho_*$, we find $C < 1$, $c > 1$ and $a > 1$ in $\mathcal{C}_0$ with the property that if $u \in \mathcal{S}_0$ then

$$C^b_r \left( \frac{\|u\|_{L^2(\Omega_0)}}{\|u\|_{L^2(\Omega)}} \right)^{b_r} \leq \frac{\|u\|_{L^2(B(x,r))}}{\|u\|_{L^2(\Omega)}}, \quad x \in \Omega_\rho, \quad 0 < r < \tau \rho,$$

where $b_r = \ln(\tau \rho/r) + a$.

With the assumptions and notations of theorem 1.1, if

$$q(u) = \left| \ln \left( \frac{\|u\|_{L^2(\Omega_0)}}{\|u\|_{L^2(\Omega)}} \right) \right|$$

then (1.1) gives for $x \in \Omega_\rho$ and $0 < r < \tau \rho$

$$\left[ C(\tau \rho)^{-q(u)} e^{-aq(u)} \|u\|_{L^2(\Omega)} \right]^{c+q(u)} \leq \|u\|_{L^2(B(x,r))}.$$

This inequality clearly quantifies strong unique continuation.

Davey and Zhu [13] considered the case $B = 0$, $W \in L^p$, $p > n$ and $V \in L^q$, $q > n/2$. The inequalities established in [13] are with constants depending explicitly on the norms of $W$ and $V$. The result in [13] was improved in [12] in the case $B = 0$, $W = 0$ and $V \in L^q$, $q > n/2$.

As a byproduct of Theorem 1.1 we obtain the following interpolation inequality.

**Corollary 1.1.** Let $\omega \in \Omega$. Then there exist $C > 0$ and $0 < t < 1$ in $\mathcal{C}_0$ such that for any $u \in \mathcal{S}_0$ we have

$$C \|u\|_{L^2(\Omega_0)} \leq \|u\|_{L^2(\Omega)}^{1-t} \|u\|_{L^2(\omega)}^t.$$

Henceforth, $\tau$ is as in Theorem 1.1.

1.2. Quantitative unique continuation from a set of positive measure. Let

$$\mathcal{S}_1 = \{ u \in H^1(\Omega); \|u\|_{L^2(\Omega)} = 1 \text{ and } \mathcal{E}u = 0 \text{ in } \Omega \}.$$

**Proposition 1.1.** ([23, Proposition 4]) Let $m > 0$, $\rho_1 > 0$ and $0 < \sigma < 1/n$. There exists $\varepsilon_0 = \varepsilon_0(m, \rho, \sigma)$ such that if $u \in \mathcal{S}_1$ and $E \subset \Omega_{\rho_1}$ verify $|E| > m$ and $\|E^{-1/2}u\|_{L^2(E)} < \varepsilon_0$ then we find $B(z, \tau\rho_1)$ with $z \in \Omega_{\rho_1}$ for which we have

$$\|u\|_{L^2(B(z,\tau\rho_1))} \leq |\ln |E|^{-1/2}u\|_{L^2(E)}|^{-\zeta},$$

where $\zeta \in \mathcal{C}$. 
Under the assumptions and the notations of Proposition 1.1, we apply (1.2) with $u \in \mathcal{H}_1$ and $\omega = B(z, \tau \rho_1)$ to obtain that if $||E||^{-1/2}u||_{L^2(\Omega)} < \epsilon_0$ then we have the following variant of the inequality of [23, Theorem 1].

\[(1.4) \quad ||u||_{L^2(\Omega_0)} \leq C \ln ||E||^{-1/2}u||_{L^2(\Omega)}^{-1},\]

where $C \in \mathcal{C}_\rho, m, \sigma, \rho_1, m$ and $\sigma$ being as in Proposition 1.1, and $t$ is as in Corollary 1.1.

1.3. Strong unique continuation for eigenfunctions. Let $\lambda > 0$ and $u \in H^1(\Omega)$ satisfies $(\sigma + \lambda)u = 0$ in $\Omega$. Then $v = e^{\sqrt{\lambda}u}$ is a solution of the equation

\[(\sigma + \partial_t^2)v = 0 \quad \text{in} \quad \Omega \times (0, 1).\]

By applying (1.1) with $\Omega$, $\Omega_0$ and $\sigma$ replaced by $\Omega \times (0, 1)$, $\Omega_0 \times (1/4, 3/4)$ and $\sigma + \partial_t^2$, respectively, we obtain

\[(1.5) \quad Cr^C\left(\frac{||v||_{L^2(\Omega_0 \times (1/4, 3/4))}}{||v||_{L^2(\Omega \times (0, 1))}}\right)^{b_r} \leq \frac{||v||_{L^2(B(x, t), r)}}{||v||_{L^2(\Omega \times (0, 1))}} \quad (x, t) \in (\Omega \times (0, 1))_{\rho}, \quad 0 < r < \tau \rho.\]

Here and henceforth, $C < 1, c > 1$ are generic constants belonging to $\mathcal{C}_\rho$ and $b_r$ is the same as in Theorem 1.1 when $\Omega$ and $\Omega_0$ are replaced by $\Omega \times (0, 1)$ and $\Omega_0 \times (1/4, 3/4)$, respectively.

Next, assume that $\rho < \min(1/8, \rho_0)$, where $\rho_0$ is as in Theorem 1.1. Let $(x, t) \in \Omega_\rho \times (4\rho, 1 - 4\rho) \subset (\Omega \times (0, 1))_\rho$. As $B((x, t), r) \subset B(x, r) \times (0, 1)$, (1.5) implies

\[(1.6) \quad Cr^C\left(\frac{||v||_{L^2(\Omega_0 \times (1/4, 3/4))}}{||v||_{L^2(\Omega \times (0, 1))}}\right)^{b_r} \leq \frac{||v||_{L^2(B(x, t), r)}}{||v||_{L^2(\Omega \times (0, 1))}}, \quad x \in \Omega_\rho, \quad 0 < r < \tau \rho.\]

Hence

\[(1.7) \quad C r^C e^{-3b_r/4} \sqrt{n} \left(\frac{||u||_{L^2(\Omega_0)}^r}{||u||_{L^2(\Omega)}}\right)^{b_r} \leq \frac{||u||_{L^2(B(x, t))}}{||u||_{L^2(\Omega)}}, \quad x \in \Omega_\rho, \quad 0 < r < \tau \rho.\]

Let $\omega \in \Omega$. The following inequality then follows easily from (1.2)

\[(1.8) \quad ||u||_{L^2(\Omega_0)} \leq Ce^{3\sqrt{n}/4}||u||_{L^2(\Omega)}^{-1} ||u||_{L^2(\omega)},\]

where $C > 0$ and $0 < t < 1$ belong to $\mathcal{C}_\omega$.

This type of estimate is well known for an eigenfunction $\varphi$ of the Laplace-Beltrami operator on a compact Riemannian manifold $M$ without boundary or a compact Riemannian manifold $M$ with boundary and $u \in H^1_0(M)$. Precisely, we have an estimate of the form

\[||u||_{L^2(M)} \leq C e^{c \sqrt{n}} ||u||_{L^2(\omega)},\]

where the constants $C'$ and $c'$ depend only on $\dim(M)$, $\omega$ and the metric on $M$ (e.g. [22, Theorem 1.10] and the references therein).

Next, suppose that $\rho_1$, given by Proposition 1.1 and corresponding to $\Omega \times (0, 1)$ in place of $\Omega$, is sufficiently small such that $E \times J \subset (\Omega \times (1/4, 3/4))_{\rho_1}$ and $|E| > 2m$ (so $|E||J| > m$). Moreover, assume $||u||_{L^2(\Omega)} = 1$ and let $v = \xi \lambda e^{\sqrt{\lambda}u}$, where $\xi = ||u||_{L^2((0, 1))}^{-1}$. Then it follows from (1.4) that

\[(1.9) \quad C ||v||_{L^2(\Omega_0 \times (1/4, 3/4))} \leq \left|\ln \left(||E||^{-1/2}v||_{L^2(E \times J)}^{-1}\right)\right|^{1},\]
where \( C \in \mathcal{C}_{p_1,m,\sigma}, p_1, m \) and \( \sigma \) being as in Proposition 1.1 when \( \Omega \) and \( \Omega_0 \) are replaced by \( \Omega \times (0,1) \) and \( \Omega_0 \times (1/4,3/4) \), respectively and \( t \) and \( \zeta \) are as in (1.4) with \( \Omega \) and \( \Omega_0 \) are replaced by \( \Omega \times (0,1) \) and \( \Omega_0 \times (1/4,3/4) \), respectively. Therefore, we have

\[
(1.10) \quad C\|u\|_{L^2(\Omega_0)} \leq \varsigma_1 e^{3\sqrt{\lambda}/4} \ln \left( \varsigma_2 e^{-\sqrt{\lambda}/2} \|u\|_{L^2(E)} \right)^{-\frac{1}{16}},
\]

where \( C, t \) and \( \zeta \) are the same as in (1.9), and \( \varsigma_1 = \varsigma_2 \|e^{t\sqrt{\lambda}}\|_{L^2(I)} \).

Now, consider the bilinear form

\[
\mathfrak{B}(u,v) = \int_{\Omega} [(A\nabla u + uB) \cdot \nabla v - (W \cdot \nabla u + Vu)v] \, dx, \quad u, v \in H_0^1(\Omega).
\]

We can proceed as [10, Subsection 1.3] to prove that \( a \) is continuous and coercive. Therefore, the operator \( \mathfrak{A} : H_0^1(\Omega) \to H^{-1}(\Omega) \) defined by

\[
\langle \mathfrak{A}u, v \rangle = \mathfrak{B}(u,v), \quad u, v \in H_0^1(\Omega),
\]

is bounded, where \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( H_0^1(\Omega) \) and \( H^{-1}(\Omega) \).

Under the hypothesis \( W = B \), \( \mathfrak{A} \) is self-adjoint and therefore, by [24, Theorem 3.37, page 49], \( \mathfrak{A} \) is diagonalizable, which means that the spectrum of \( \mathfrak{A} \) consists of a nondecreasing real-valued sequence \( \{ \lambda_j \} \) converging to \( \infty \):

\[-\infty < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots\]

and the exists an orthonormal basis \( \{ \phi_j \} \) of \( L^2(\Omega) \) consisting of eigenfunctions. That is, for all \( j \geq 1 \)

\[
\mathfrak{B}(\phi_j, v) = \lambda_j \langle \phi_j, v \rangle, \quad u, v \in H_0^1(\Omega),
\]

where \( \langle \cdot, \cdot \rangle \) is the usual scalar product of \( L^2(\Omega) \).

Fix \( \lambda^* \in \mathbb{R} \) so that \( \lambda_1 + \lambda^* > 0 \). By taking in (1.7), (1.8) and (1.10), in which we substitute \( \mathcal{E} \) by \( \mathcal{E} - \lambda^* \) and \( u = \phi_j, \ j \geq 1 \), we obtain, where for simplicity \( \lambda_j + \lambda^* \) is replaced by \( \lambda_j \),

\[
C e^{-3\sqrt{\lambda}/4} \|\phi_j\|_{L^2(\Omega_0)} \leq \|\phi_j\|_{L^2(B(x,r))}, \quad x \in \Omega_0, \ 0 \leq r < \tau \rho, \ j \geq 1,
\]

\[
\|\phi_j\|_{L^2(\Omega_0)} \leq C e^{3\sqrt{\lambda}/4} \|\phi_j\|_{L^2(\Omega)}, \quad j \geq 1,
\]

\[
C \|u\|_{L^2(\Omega_0)} \leq \varsigma_1 e^{3\sqrt{\lambda}/4} \ln \left( \varsigma_2 e^{-\sqrt{\lambda}/2} \|u\|_{L^2(E)} \right)^{-1}, \quad j \geq 1,
\]

where the above constants are the same as in (1.7), (1.8) and (1.10), respectively, and do not depend on \( j \).

1.4. Determining a metric tensor from a partial spectral boundary data.

In this subsection, \( \Omega \) is of class \( C^\infty \) and \( g = (g_{k\ell}) \) is a \( C^\infty \) Riemannian metric on \( \overline{\Omega} \). Denote by \( (g^{k\ell}) \) the inverse matrix of \( (g_{k\ell}) \) and recall that the Laplace-Beltrami operator associated to \( g \) is given by

\[
\Delta_g = \frac{1}{\sqrt{|g|}} \sum_{k,\ell=1}^n \frac{\partial}{\partial x_k} \left( \sqrt{|g|} g^{k\ell} \frac{\partial}{\partial x_\ell} \right),
\]

where \( |g| \) denotes the determinant of \( g \).

Let \( dV = \sqrt{|g|} dx^1 \ldots dx^n \) and define the unbounded operator

\[
A_g : L^2(\Omega, dV) \to L^2(\Omega, dV)
\]
acting as follows

\[ A_g = -\Delta_g \quad \text{with} \quad D(A_g) = H^1_0(\Omega) \cap H^2(\Omega). \]

As \( A_g \) is self-adjoint operator with compact resolvent, its spectrum is reduced to a sequence of eigenvalues:

\[ 0 < \lambda_1^g < \lambda_2^g \leq \ldots \lambda_k^g \leq \ldots \quad \text{and} \quad \lambda_k^g \to \infty \text{ as } k \to \infty. \]

Moreover, there exists \( (\phi_k^g) \), \( k \geq 1 \), an orthonormal basis of \( L^2(\Omega, dV) \) consisting of eigenfunctions, each \( \phi_k^g \) being an eigenfunction for \( \lambda_k^g \), that is for all \( k \geq 1 \), \( \phi_k^g \in H^1_0(\Omega) \cap H^2(\Omega) \) and

\[ -\Delta_g \phi_k^g = \lambda_k^g \phi_k^g. \]

Note that, according to the elliptic regularity, \( \phi_k^g \in C^\infty(\overline{\Omega}) \) for all \( k \geq 1 \).

For convenience, we use the following notation

\[ \psi_k^g = \partial_\nu \phi_k^g, \quad k \geq 1, \]

where \( \nu \) is the unit normal exterior vector field on \( \Gamma \) with respect to \( g \).

For \( \chi \in C^\infty(\overline{\Omega}) \) so that \( \chi > 0 \), define the operator

\[ A_\chi^g : L^2(\Omega, \chi^{-2}dV) \to L^2(\Omega, \chi^{-2}dV) \]

by

\[ A_\chi^g = \chi A_g \chi^{-1}, \quad D(A_\chi^g) = D(A_g). \]

In the following, \( g_1 \) and \( g_2 \) denote two metric tensors on \( \overline{\Omega} \). We say that the operators \( A_{g_1} \) and \( A_{g_2} \) are gauge equivalent if \( A_{g_2} = A_{g_1}^h \), for some \( \chi \in C^\infty(\overline{\Omega}) \) verifying \( \chi > 0 \). It is important to note that if \( A_{g_1} \) and \( A_{g_2} \) are gauge equivalent then \( (g_1^{\chi f}) = (g_2^{\chi f}) \) (see [17, 2.2.9]). Moreover, if \( A_{g_1} \) and \( A_{g_2} \) are gauge equivalent then \( (\lambda_k^{g_1}) = (\lambda_k^{g_2}) \) ([17, (2.54)]) and \( (\phi_k^{g_1}) = (\chi \phi_k^{g_2}) \) ([17, (2.55)]).

The sequence \( (\lambda_k^g, \psi_k^g) \) will be called in the following the spectral boundary data of \( A_g \).

Fix \( \tilde{x} \in \Gamma = \partial \Omega \) and let \( \Sigma \) be a measurable subset of \( \Gamma \) so that

\[ |\Sigma \cap B(r)| > 0, \quad 0 < r \leq r_0, \]

for some \( r_0 > 0 \), where \( B(r) = B(\tilde{x}, r) \). Assume in addition that \( \Omega \) is chosen so that \( \Gamma \) has a (smooth) connected neighborhood \( N \) in \( \overline{\Omega} \).

Pick \( \lambda > 0 \) and let \( u \in W^{2,\infty}(N) \) be a solution of the following BVP

\[ (1.11) \quad (\Delta_g + \lambda) u = 0 \text{ in } N \quad u = 0 \text{ on } \Gamma. \]

Paraphrasing the proof [8, Theorem 2.3], we find that there exists \( r_1 \leq r_0 \) such that

\[ (1.12) \quad \sup_{B(r_1/2) \cap N} (|u| + |\nabla u|) \leq C e^{\lambda r_1} \left( \sup_{B(r_1) \cap \Sigma} |\partial_\nu u| \right)^\alpha \left( \sup_{B(r_1) \cap N} (|u| + |\nabla u|) \right)^{1-\alpha}, \]

where the constants \( C > 0 \) and \( 0 < \alpha < 1 \) only depends on \( n \), \( N \) and \( \Sigma \).

Let \( N_0 \subset B(r_1/2) \cap N \) be arbitrarily fixed. Then (1.12) together with (1.8) imply

\[ (1.13) \quad \|u\|_{L^2(N_0)} \leq C e^{\lambda r_1 + 3\sqrt{\lambda}/4} \|\partial_\nu u\|_{L^\infty(\Sigma)}^{\alpha} \|u\|_{W^{1,\infty}(N)}^{1-\alpha}. \]

The partial spectral boundary data of \( A_g \) will consist of \( (\lambda_k^g, \psi_k^g|_{\Sigma}) \).
Theorem 1.2. Assume that \( g_1 = g_2 \) in \( N \). If \( A_{g_1} \) and \( A_{g_2} \) have the same partial boundary spectral data then \( A_{g_1} \) and \( A_{g_2} \) are gauge equivalent.

Proof. For all \( k \geq 1 \), we verify that \( u_k = \phi_k^{g_1} - \phi_k^{g_2} \) is a solution of (1.11) with \( \lambda = \lambda_k^{g_1} \). We then apply (1.13) to deduce that \( u_k = 0 \) in \( N_0 \) and therefore \( u = 0 \) in \( N \) by the unique continuation for \( \Delta_g + \lambda_k \). We have in particular \( \psi_k^{g_1} = \psi_k^{g_2} \) and therefore \( A_{g_1} \) and \( A_{g_2} \) admit the same spectral boundary data. We invoke [17, Theorem 3.3] to complete the proof. \( \square \)

We point out that when \( \Sigma \) is an arbitrary nonempty open set of \( \Gamma \), a logarithmic stability inequality has recently been established in [14] in the two-dimensional case. The problem of determining the potential in a Schrödinger operator or the potential and magnetic field appearing in a magnetic Schrödinger operator has been widely studied. We cite only a few recent works [1, 2, 3, 4, 5, 6, 7, 11, 15, 16, 17, 18, 19, 20, 25, 26].

2. Proof of Theorem 1.1

We use the following result which is a special case of [23, Theorem 3].

Theorem 2.1. (three-ball inequality) There exist \( 0 < \alpha < 1 \), \( \tau < 1/4 \), \( \rho_* \in \mathcal{C} \), \( \rho_* \leq \rho_0 \), and \( C \in \mathcal{C} \) such that for all \( 0 < \rho < \rho_* \), \( x \in \Omega_\rho \), \( r = \tau \rho/4 \) and \( u \in H^1(\Omega) \) satisfying \( \partial u = 0 \) in \( \Omega \) we have

\[
||u||_{L^2(B(x,2r))} \leq C\|u\|_{L^2(\Omega)}^{1-\alpha} \|u\|_{L^2(B(x,r))}^\alpha.
\]

Proof of Theorem 1.1. In this proof \( C \geq 1 \) and \( 0 < \alpha < 1 \) are generic constants belonging to \( \mathcal{C} \).

Let \( \rho < \rho_* \), where \( \rho_* \) is as in Theorem 2.1, \( r = \tau \rho/4 \). Let \( Q \) be the smallest closed cube containing \( \overline{\Omega_\rho} \). If \( d = \text{diam}(\Omega) \), then \( |Q| = d^n \). We divide \( Q \) into \( (d/(\sqrt{n} \rho) + 1)^n := m_\rho \) closed subcubes, where \( d/(\sqrt{n} \rho) \) is the integer part of \( d/(\sqrt{n} \rho) \). Let \( (Q_j)_{1 \leq j \leq m_\rho} \) be the family of these cubes. Note that, for each \( j \), \( |Q_j| < (\sqrt{n} \rho)^n \) and therefore \( Q_j \) is contained in a ball \( B_j \) of radius \( \rho \). Define

\[
I_\rho = \{ j \in \{1, \ldots, m_\rho\}; Q_j \cap \overline{\Omega_\rho} \neq \emptyset \} \quad \text{and} \quad Q^\rho = \bigcup_{j \in I_\rho} Q_j.
\]

In particular, \( Q^\rho \subset \Omega_{3\rho}/4 \) and, since \( \overline{\Omega_\rho} \) is connected then so is \( Q^\rho \). Let \( x,y \in \Omega_\rho \) and \( \psi : [0,1] \to Q^\rho \) be a path joining \( x \) to \( y \) such that \( \psi \) is constant on each \( Q_j, j \in I_\rho \), and the length of \( \psi \) restricted to each \( Q_j, j \in I_\rho \), is less than \( \sqrt{n} \rho \). Consequently, the length of \( \psi \), hereafter denoted \( \ell(\psi) \), does not exceed \( \sqrt{n} \rho m_\rho \). Let \( t_0 = 0 \) and define the sequence \( (t_k) \) as follows

\[
t_{k+1} = \inf\{ t \in [t_k,1]; \psi(t) \notin B(\psi(t_k), \rho) \}, \quad k \geq 0.
\]

Then \( |\psi(t_{k+1}) - \psi(t_k)| = \rho \). Thus, there exists a positive integer \( N_\rho \) so that \( \psi(1) \in B(\psi(t_{N_\rho}), \rho) \). As \( \sqrt{n} \rho N_\rho \leq \ell(\psi) \leq \sqrt{n} \rho m_\rho \), we obtain \( N_\rho \leq m_\rho \).

Let \( x_j = \psi(t_j), j = 0, \ldots, N_\rho \) and \( x_{N_\rho+1} = y \). Clearly, \( B(x_j,3\rho) \subset \Omega, j = 0, \ldots, N_\rho + 1, \) and \( B(x_{j+1}, \rho) \subset B(x_j,2\rho), j = 0, \ldots, N_\rho \).

We verify that

\[
N := N_\rho \leq N_\rho := c/r^n.
\]

Here and henceforth, \( c = c(n,d,\rho_0) > 0 \) is a generic constant.

Let \( u \in \mathcal{A}_0 \). It follows from (2.1)

\[
||u||_{L^2(B(x_j,2r))} \leq C\|u\|_{L^2(\Omega)}^{1-\alpha} \|u\|_{L^2(B(x_j,r))}^\alpha, \quad 0 \leq j \leq N_\rho + 1.
\]
Let $I_j = \|v\|_{L^2(B(x_j, r))}$, $0 \leq j \leq N + 1$, where $v = u/\|u\|_{L^2(\Omega)}$. Since $B(x_{j+1}, r) \subset B(x_j, 2r)$, $1 \leq j \leq N$, estimate (2.2) implies

$$I_{j+1} \leq CI_j^\alpha, \quad 0 \leq j \leq N. \tag{2.3}$$

As $C^{1+\alpha+...+\alpha N+1} \leq C^{1/(1-\alpha)}$, an induction argument shows that (2.3) yields

$$I_{N+1} \leq C^{1/(1-\alpha)I_0^{N+1}}, \tag{2.4}$$

which is combined with $I_0 \leq 1$, gives

$$\|v\|_{L^2(B(y, r))} \leq C\|v\|^{\alpha_{1/n}_{0}}_{L^2(B(x, r))}, \tag{2.5}$$

where we used that $N + 1 \leq \epsilon/r^n$.

Replacing $\rho_\ast$ by $2\rho_\ast$, we can rewrite (2.5) in the following form

$$\|v\|_{L^2(B(y, \rho_\ast/8))} \leq C\|v\|^{\alpha_{1/n}_{0}}_{L^2(B(x, \rho_\ast/8))}, \quad x, y \in \Omega_{\rho_\ast/2}. \tag{2.6}$$

Furthermore, by reducing $\rho_\ast$ if necessary, we assume that $\Omega_{(1+\tau/16)\rho_\ast/2} \supseteq \Omega_0$. Let $M := \|v\|_{L^2(\Omega_0)} = \|u\|_{L^2(\Omega_0)}/\|u\|_{L^2(\Omega)} (< 1)$. Then we have

$$M \leq \|v\|_{L^2(\Omega_{(1+\tau/16)\rho_\ast/2})}^2 = \sum_j \|v\|_{L^2(Q_j \cap \Omega_{(1+\tau/16)\rho_\ast/2})}^2,$$

where $Q_j$ are defined as before so that $|Q_j| = [\tau\rho/(8\sqrt{m})]^n$. Hence, there exists $Q_k$ so that

$$\|v\|_{L^2(Q_k)} \geq cM\rho^{-n},$$

As $Q_k \subset B(z, \tau\rho/8)$, for some $z \in \Omega_{\rho_\ast/2}$, we obtain

$$\|v\|_{B(z, \tau\rho/8)} \geq cM\rho^{-n}. \tag{2.7}$$

In the rest of this proof $C_\rho \in \mathbb{C}$ is a generic constant. Let $\alpha = 1/\alpha_{1/n}_{0}$. In light of (2.6), we obtain from (2.7)

$$C_\rho M^\alpha \leq \|v\|_{L^2(B(x, \tau\rho/8))}, \quad x \in \Omega_{\rho_\ast/2}. \tag{2.8}$$

Let

$$\mathcal{Q}_\rho(v) = \max_{x \in \Omega_{\rho_\ast/2}} \left(\left\|v\right\|_{L^2(B(x, \rho_\ast))}^{\beta} \right).$$

Then (2.8) implies

$$\mathcal{Q}_\rho(v) \leq C_\rho M^{-\beta\alpha}. \tag{2.9}$$

From now on, we suppose that $C_\rho > 1$. By applying [23, Proposition 2] (doubling inequality) we deduce

$$\|v\|_{L^2(B(x, 2s))} \leq C_\rho M^{-\beta\alpha}\|v\|_{L^2(B(x, s))}, \quad x \in \Omega_\rho, \quad 0 < s < \tau\rho. \tag{2.10}$$

Let $m$ be the nonnegative integer satisfying $2^{m-1}s < \tau\rho$ and $2^ms \geq \tau\rho$ Iterating (2.9) $(m - 1)$ times and using that $B(x, \tau\rho) \subset B(x, 2^ms)$, we obtain

$$\|v\|_{L^2(B(x, \tau\rho))} \leq C_\rho M^{-m\beta\alpha}\|v\|_{L^2(B(x, s))}, \quad x \in \Omega_\rho, \quad 0 < s < \tau\rho. \tag{2.11}$$

By noting that $\Omega_\rho = \Omega_{(2\rho)/2}$, an combination of (2.8) and (2.11) imply

$$C_\rho \leq C_\rho M^{-m\beta\alpha+S\alpha}\|v\|_{L^2(B(x, s))}, \quad x \in \Omega_\rho, \quad 0 < s < \tau\rho.$$

Replacing $\alpha + \alpha\beta$ by $\alpha$ and using that $C_\rho > 1$ and $M^{-1} > 1$, we obtain from (2.11)

$$C_\rho^{-1} \leq s^{-\alpha\beta}M^{-(\ln(\tau\rho/s)+\alpha}\|v\|_{L^2(B(x, s))}}, \quad x \in \Omega_\rho, \quad 0 < s < \tau\rho.$$
where \( c_\rho > 1 \) belongs to \( \mathcal{C}_\rho' \). In other words, we have
\[
C_\rho^{-1} \left\| u \right\|_{L^2(\Omega)} \leq s^{-c_\rho} \left\| u \right\|_{L^2(\Omega)} \leq \left\| u \right\|_{L^2(B(x,s))}, \quad x \in \Omega_\rho, \quad 0 < s < \tau \rho.
\]
The proof is then complete. \( \square \)

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Université de Lorraine

Email address: mourad.choulli@univ-lorraine.fr