Palatini–Lovelock–Cartan gravity—Bianchi identities for stringy fluxes

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Abstract
A Palatini-type action for Einstein and Gauss–Bonnet gravity with non-trivial torsion is proposed. A three-form flux is incorporated via a deformation of the Riemann tensor, and consistency of the Palatini variational principle requires the flux to be covariantly constant and to satisfy a Jacobi identity. Studying gravity actions of third order in the curvature leads to a conjecture about general Palatini–Lovelock–Cartan gravity. We point out potential relations to string-theoretic Bianchi identities and, using the Schouten–Nijenhuis bracket, derive a set of Bianchi identities for the non-geometric $Q$- and $R$-fluxes which include derivative and curvature terms. Finally, the problem of relating torsional gravity to higher order corrections of the bosonic string-effective action is revisited.

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1. Introduction
One of the most distinctive and generic features of string theory and supergravity is that at the massless level, gravity is extended by an axionic anti-symmetric two-form field and the dilaton. In the case of string theory, this is a direct consequence of quantizing the string excitations. The Kalb–Ramond field and its contribution to the low-energy effective action have been a subject of study throughout the history of string theory. In particular, the application of T-duality [1, 2] led to a picture where the three-form flux of the Kalb–Ramond field was dualized not only to geometric flux, but also to so-called non-geometric fluxes. Here one distinguishes the still locally geometric $Q$-flux3 from the so-called $R$-flux, which is not even locally geometric. The former gave rise to the notion of $T$-folds, whereas the latter was argued to be related to a nonassociative geometry [4–7]. Furthermore, assuming that T-duality is not a symmetry only
of certain solutions of string theory but is a symmetry of the theory itself led to the idea of double-field theory \[8, 9\]. In this approach, one doubles the coordinates and formulates an effective action which is invariant under the T-duality group \(O(D, D)\) in \(D\) dimensions. This formalism was very successful for the action at leading order in a derivative expansion, but was shown to become more involved at next-to-leading order \[10\].

The purpose of this paper is to approach the question about the nature of the Kalb–Ramond field and its T-dual incarnations from a more gravity-based direction. Long ago it was realized \[11\] that at leading order in the string tension \(\alpha'\), i.e. at the two-derivative level, the low-energy effective action for the graviton and the Kalb–Ramond field \(B\) follows from Einstein gravity with a connection whose torsion is equal to \(H = dB\)—also known as Einstein–Cartan theory. However, already at second order in \(\alpha'\), this geometric picture was shown to break down. In particular, for the vanishing \(H\)-flux, the on-shell string scattering amplitudes of the graviton are known to be consistent with the ghost-free Gauss–Bonnet action. But for the non-vanishing flux, in \[12\] it was shown that the latter action with the torsional connection is not consistent with the string equations of motion.

In the work mentioned above, the choice of connection was put in by hand and did not follow dynamically from an equation of motion. For the Einstein gravity case, there exists an action which remedies this point, the so-called Einstein–Palatini action, whose Lagrangian density is considered to be a functional both of the metric and the connection \[13\] (see also, for instance, \[14\]). The field equations following from the variation with respect to the connection then imply that the latter has to be Lévi–Civitá. In \[15\], (see also \[16, 17\]), it was shown that this relation extends to the Gauss–Bonnet and, in fact, to all higher order Lovelock gravity actions \[18\]. (See also \[19\] for earlier work on higher order gravity actions.) This remarkable result relies on the fact that for the Lovelock-type combination of higher order curvature terms, the Bianchi identities for the curvature tensor lead to non-trivial cancellations. In \[20\], it was pointed out that the Lovelock actions are also singled out by the fact that they lead to consistent truncations. This means that the equations of motion in the Palatini approach are equivalent to those resulting from the variation of the Lovelock action with the Lévi–Civitá connection inserted by hand.

In view of this situation, we ask the question whether the Palatini formalism can be generalized to include the case where the torsion does not vanish and is identified with the field strength of the Kalb–Ramond field. More concretely, we seek a Palatini-type torsional Einstein–Hilbert, Gauss–Bonnet or even general Lovelock action—a Palatini–Lovelock–Cartan action—which on-shell reduces to the corresponding action with torsion\(^4\). Deforming the curvature tensor by Lagrangian multipliers of the form \(\text{three-form} \times \text{torsion}\), we find that this is indeed possible. And, as mentioned above, since the curvature Bianchi identities were playing the key role in the non-torsional case, one could expect that the Bianchi identity for the Kalb–Ramond field will be equally important. This is true but, as we will deduce in detail, the latter Bianchi identity turns out not to be sufficient. Instead, our computation points toward the stronger conditions that the three-form is covariantly constant and satisfies a Jacobi identity.

We observe that these conditions are not unfamiliar with string theory. They can be considered as the basic two requirements guaranteeing that also all Bianchi identities in the geometric as well as non-geometric T-dual descriptions are satisfied. As a new result, we derive Bianchi identities for the non-geometric \(Q\)- and \(R\)-fluxes not just for constant fluxes but with curvature terms and covariant derivatives of the fluxes included. For this purpose, we exploit the definition of the \(R\)-flux in terms of the Schouten–Nijenhuis bracket of a bi-vector field.

\(^4\) For an approach to Lovelock–Cartan theory, see \[21\].
Moreover, we revisit the question whether the Gauss–Bonnet action with torsion does reproduce the on-shell string scattering amplitudes for the graviton and the Kalb–Ramond field (for the vanishing dilaton). Recall that in [12] a negative answer was given. There, of course, only the Bianchi identity \( dH = 0 \) was taken into account to relate the various possible diffeomorphism-invariant combinations in the action. We show that if one uses in addition the two stronger conditions we derived from the Palatini approach, then the pure Gauss–Bonnet action with torsion remains to be in conflict with the string constraints. Nevertheless, we show that one can write the string corrections in a Gauss–Bonnet-type form.

This paper is organized as follows: in section 2, we review some aspects of the Einstein–Palatini action and generalize it to include torsion. In section 3, we analyze the Palatini approach to the Gauss–Bonnet action with torsion and obtain two new restrictions. In section 4, we revisit the Gauss–Bonnet action in view of the new conditions, and show that this formalism is consistent also for the third-order Lovelock action. The section closes with our central result, namely a conjecture about Palatini–Lovelock–Cartan gravity. In section 5, we connect our findings to the string theory action, and section 6 contains a detailed analysis of the Bianchi identities for three-form fluxes in string theory. Readers mainly interested in these results may move directly to this section. In section 7, we conclude and discuss some open questions.

2. Einstein–Palatini gravity with torsion

In this section, we deform Einstein gravity by a completely anti-symmetric three-index object \( \eta_{abc}(x) = \partial_a \rho_{bc}(x) + \partial_b \rho_{ca}(x) + \partial_c \rho_{ab}(x) \). Our aim is to obtain a theory with torsion, whose field equations are isomorphic to the leading-order string equations of motion for the metric and Kalb–Ramond field in the case of a constant dilaton.

Our approach is to work with the Einstein–Palatini formalism in which the metric \( g \) and the connection \( \Gamma \) are treated as independent fields. By solving the field equation for \( \Gamma \), one is led to a metric-compatible and torsion-free connection and thus recovers the Einstein–Hilbert theory. In the following, we recall this formalism for usual Einstein gravity and introduce our basic conventions.

**Einstein–Palatini action.** The Einstein–Palatini action in \( n \) spacetime dimensions is given by

\[
S = \frac{1}{2\kappa^2} \int d^n x \sqrt{-g} g^{ab} R_{ab}(\Gamma),
\]

where the Lagrangian density \( \mathcal{L}(g^{ab}, \Gamma^{abc}) \) is a functional of the inverse metric \( g^{ab} \) and the connection \( \Gamma^{abc} \). The Riemann curvature tensor, and therefore also the Ricci tensor \( R_{ab}(\Gamma) \), only depends on the connection \( \Gamma \).

\[
R_{abcd} = \partial_d \Gamma^a_{cb} - \partial_c \Gamma^a_{db} + \Gamma^a_{cm} \Gamma^m_{db} - \Gamma^a_{dm} \Gamma^m_{cb}. \tag{2}
\]

The variation of the Riemann curvature tensor with respect to the connection is given by the Palatini formula

\[
\delta R_{abcd} = \nabla_c (\delta \Gamma^a_{db}) - \nabla_d (\delta \Gamma^a_{cb}) + T^m_{cd} (\delta \Gamma^a_{mb}), \tag{3}
\]

where \( T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} \) denotes the torsion tensor. The covariant derivative \( \nabla_c \) appearing in (3) acts on \((1, 1)\)-tensors as follows:

\[
\nabla_c t^a_{cb} = \partial_c t^a_{cb} - \Gamma^a_{cm} t^m_{cb} + \Gamma^a_{cm} t^m_{cb}, \tag{4}
\]

with a straightforward generalization to \((p, q)\)-tensors.
Furthermore, as will become clear in the following, we are interested in metric-compatible connections which satisfy
\[ \nabla_{ab} c = 0. \]
These can be expressed in terms of the Lévi–Civitá connection \( \Gamma^a_{bc} \) and the contorsion \( K^a_{bc} \) as follows:

\[ \Gamma^a_{bc} = \hat{\Gamma}^a_{bc} + K^a_{bc}, \]

where the former, when expressed in terms of the metric, is given by

\[ \hat{\Gamma}^a_{bc} = \frac{1}{2} g^{am} (\partial_b g_{mc} + \partial_c g_{bm} - \partial_m g_{bc}). \]

The contorsion can be written using the torsion tensor as

\[ K^a_{bc} = \frac{1}{2} (T^a_{bc} + T^b_{ac} + T^c_{ab}). \]

Finally, we note that in the case of a metric-compatible connection, we can apply the following formula for any vector \( A^a \):

\[ \sqrt{-g} \nabla_a A = \partial_m (\sqrt{-g} A^m) + \sqrt{-g} T^m_{nm} A^m. \]

Let us point out that in order to keep the length of the expressions to a bearable level, throughout this paper we present formulas only up to terms which vanish for metric-compatible connections. Furthermore, we assume that the contorsion \( K^a_{bc} \) is totally anti-symmetric, which also implies that the trace part of the torsion \( T^a_{ab} \) vanishes. (In \cite{22}, it has been argued that the latter condition is equivalent to a gauge fixing.) Choosing then for instance \( A^a = B^{a_{m_1...m_n}}_{c_{1...m_n}} \) in equation (8), we can perform an integration by parts to obtain the relation

\[ \int d^dx \sqrt{-g} \nabla_a B^{a_{m_1...m_n}} C_{c_{1...m_n}} = - \int d^dx \sqrt{-g} \nabla_a B^{a_{m_1...m_n}} C_{c_{1...m_n}}. \]

The above formulas now allow us to easily compute the variation of the Einstein–Palatini action (1) with respect to the connection \( \Gamma^a_{bc} \). Up to terms proportional to \( \nabla_a g_{bc} \) and \( T^a_{ab} \) which vanish with our assumptions, we obtain

\[ T_{\lambda \nu \mu} = 0. \]

We can therefore conclude that a metric-compatible connection with vanishing torsion is a solution to the equations of motions. This connection is uniquely given by the Lévi–Civitá connection (6). Inserting this result into the Einstein–Palatini action (1), we arrive at the usual Einstein–Hilbert form of general relativity. The resulting equations of motion for the metric are the same as those arising from first varying the Einstein–Palatini action (1) and then inserting the Lévi–Civitá connection. This non-trivial feature is called a consistent truncation (see e.g. \cite{20}) and can be visualized by the following commuting diagram:

\[ \begin{array}{ccc} \mathcal{L}(g, \Gamma) & \xrightarrow{\text{trunc.}} & \mathcal{L}(g) \\ \downarrow & & \downarrow \\ \text{EOM} & \xrightarrow{\text{trunc.}} & \text{EOM} \end{array} \]

\[ \text{Einstein–Palatini with torsion.} \quad \text{Motivated by string theory, we now consider a theory which in addition to the metric contains a Kalb–Ramond field with field strength } \eta_{abc}. \text{ Our goal is to find a deformation of the Einstein–Palatini action so that the torsionful connection} \]

\[ \Gamma^a_{bc} = \hat{\Gamma}^a_{bc} + C \eta^a_{bc}. \]
is a solution to the equations of motion for $\Gamma$, and that on-shell the action reduces to the Einstein–Hilbert case for the torsional connection (12). A guess for an action which satisfies these conditions is as follows:

$$S = \frac{1}{2\kappa^2} \int d^nx \sqrt{-g} \left( R_{\mu
u} - C \eta_{\mu\nu} \eta_{ab} \eta^{ab} \right).$$

(13)

Note that, since the three-tensor $\eta_{abc}$ is coupled to the torsion tensor, the deformation is diffeomorphism invariant.

Performing the variation of (13) with respect to the connection, we obtain an additional contribution to (10) due to the deformation. In particular, with our assumptions above, the field equations for $\Gamma$ read

$$T_{\lambda\nu}^{\mu} - 2C \eta_{\lambda\nu}^{\mu} = 0.$$ 

(14)

Recalling (7), we see that this is indeed solved by (12). Moreover, the last term in action (13) ensures that, after employing (14), we are left with the Einstein–Hilbert action with the torsional connection (12).

The Riemann curvature tensor for connection (12) can be expressed in quantities involving the Lévi–Cività connection and $\eta_{abc}$ as follows:

$$R_{\mu\nu}^{\rho\lambda}(\Gamma) = \hat{R}_{\mu\nu}^{\rho\lambda}(\hat{\Gamma}) + C \left( \hat{\nabla}_{\rho} \eta_{\mu\nu}^{\lambda} - \hat{\nabla}_{\lambda} \eta_{\mu\nu}^{\rho} \right) - C^2 \left( \eta_{\mu\rho}^{\lambda} \eta_{\nu\lambda}^{\mu} - \eta_{\mu\lambda}^{\rho} \eta_{\nu\lambda}^{\mu} \right),$$

(15)

where $\hat{R}_{\mu\nu}^{\rho\lambda}(\hat{\Gamma})$ and $\hat{\nabla}$ denote the curvature tensor and covariant derivative for the Lévi–Cività connection $\hat{\Gamma}$, respectively. For the Ricci tensor and scalar, we find

$$R_{\mu\nu}^{\rho}(\Gamma) = \hat{R}_{\mu\nu}^{\rho}(\hat{\Gamma}) - C \hat{\nabla}_{\mu} \eta_{\nu}^{\rho} - C^2 \eta_{\eta}^{\rho} \eta_{\mu\nu},$$

(16)

so that the truncated action (13) becomes

$$S = \frac{1}{2\kappa^2} \int d^nx \sqrt{-g/2} \left( \hat{R} - C^2 \eta_{abc} \eta^{abc} \right).$$

(17)

Note that, by varying the action with respect to the inverse metric and Kalb–Ramond field $\beta$ (where $\eta = d\beta$), one can show that (17) is a consistent truncation of the Palatini–Einstein–Cartan action (13). Furthermore, we observe that for $C^2 = 1/12$, expression (17) is identical to the leading-order string-theoretic gravity action for the constant dilaton,

$$S_{\text{string}} = \frac{1}{2\kappa^2} \int d^nx \sqrt{-g} \left( \hat{R} - \frac{1}{12} H_{abc} H^{abc} \right).$$

(18)

Of course, it is well known that at leading order in $\alpha'$, the three-form flux can be interpreted as the torsion of the underlying Riemannian geometry [11]. An interesting question is whether this is also true at higher orders in $\alpha'$, which has been analyzed at next-to-leading order in [12]. However, as far as we know, the analysis has never been performed in conjunction with the Palatini formalism. In the following section, we study the Palatini formalism at higher order.

### 3. Palatini action for Gauss–Bonnet gravity

The aim of this section is to mimic the setup and analysis of the previous discussion for gravity theories quadratic in the curvature. We consider Gauss–Bonnet gravity for its compatibility with the Palatini approach, which also provides a consistent truncation in the torsion-free case. These properties stem from those of Lovelock gravity which will be discussed in section 4.
The torsion-free case. The Gauss–Bonnet action can be written as
\[ S_{\text{GB}} = \frac{\alpha}{2\kappa^2} \int d^4x \sqrt{-g} \left( R_{abcd} R^{abcd} - R_{ab} R^{ab} + 2 R_{ab} \tilde{R}^{ab} - \tilde{R}_{ab} \tilde{R}^{ab} + R^2 \right), \] (19)
where \( R_{ab} = R^m_{\ a mb} \) and \( \tilde{R}_{ab} = g^{mu} R_{amb} \). Note that for a generic connection, the Riemann curvature tensor is anti-symmetric only in the last two indices, and that the two different types of contractions for the Ricci tensor give different contributions to the equations of motion. However, for a metric-compatible connection also the first two indices of the Riemann tensor will be anti-symmetric.

Let us review the Palatini formalism of the Gauss–Bonnet action, for which we will employ the following notation. The variation of (19) with respect to \( \Gamma^\mu_{\ \nu\kappa} \) will be written as
\[ \delta \xi S_{\text{GB}} = \frac{\alpha'}{2\kappa^2} \int d^4x \sqrt{-g} \left[ \Sigma^{\mu}_{\ \nu\kappa} + g^{\nu\mu} \Sigma^{\lambda}_{\ \nu\kappa} \right] F_{\mu\nu\kappa}, \] (20)
and the contribution of the five terms in (19) to \( F_{\mu\nu\kappa} \), up to terms vanishing for \( \nabla_a g_{bc} = 0 \) and \( T^a_{\ ab} = 0 \), is listed below,
\[ +R_{abcd} R^{abcd} : \left[ \nabla^m (R_{vm\mu\lambda} - R_{vm\mu\lambda}) + T^m_{\ mn} R_{mn\mu\lambda} \right], \]
\[ -R_{ab} \tilde{R}^{ab} : \left[ \nabla_{\ [a} R_{b] \kappa} - g_{\ [a} \nabla \tilde{R}_{b] \kappa} - T_{\ [a}^{\ \ \ b} R_{b] \kappa} \right], \]
\[ -\tilde{R}_{ab} \tilde{R}^{ab} : \left[ \nabla_{\ [a} \tilde{R}_{b] \kappa} - g_{\ [a} \nabla \tilde{R}_{b] \kappa} - T_{\ [a}^{\ \ \ b} \tilde{R}_{b] \kappa} \right], \]
\[ +2R_{ab} \tilde{R}^{ab} : \left[ -\nabla_{\ [a} R_{b] \kappa} + g_{\ [a} \nabla \tilde{R}_{b] \kappa} + T_{\ [a}^{\ \ \ b} \tilde{R}_{b] \kappa} \right. \]
\[ \left. -\nabla_{\ [a} \tilde{R}_{b] \kappa} + g_{\ [a} \nabla \tilde{R}_{b] \kappa} + T_{\ [a}^{\ \ \ b} \tilde{R}_{b] \kappa} \right], \]
\[ +R^2 : \left[ g_{\ [a} \nabla \tilde{R} - g_{\ [a} \nabla \tilde{R} + T_{\ [a}^{\ \ \ b} \tilde{R} \right]. \] (21)
The equation of motion for the connection is the sum of these five contributions being equal to zero. To show that this field equation is satisfied for the Lévi–Civitá connection, we first realize that in this case, \( \tilde{R}_{ab} = -R_{ba} \). Next, we recall the second Bianchi identity for the curvature \( R_{abcd} \) (with vanishing torsion) which reads
\[ \nabla_{\ [a} R_{b] \kappa} = 0. \] (22)
Note that we underline the indices which are anti-symmetrized. We can then utilize
\[ \nabla^m \tilde{R}_{vm\mu\lambda} = -\nabla_{\ [a} \tilde{R}_{b] \kappa} + \nabla_{\ [a} \tilde{R}_{b] \kappa}, \]
\[ \nabla^m \tilde{R}_{\mu\kappa} = \frac{1}{2} \nabla \tilde{R}, \] (23)
to show that all terms in (21) which involve a covariant derivative cancel. The torsion terms vanish since we consider the Lévi–Civitá connection. Therefore, after performing the variation of the Gauss–Bonnet action with respect to the connection, the resulting equation of motion is satisfied for the Lévi–Civitá connection.

We also remark that the next-to-leading-order correction for the bosonic-string action for the vanishing three-form flux and dilaton can be cast into the Gauss–Bonnet form. In fact, this form of the action is singled out as it is explicitly ghost free [23].

The torsional case. Our objection is to analyze whether the above result can be generalized to the torsionful connection (12). Since terms of the form \( \nabla, R_{abcd} \), again have to be canceled, similarly to the analysis in section 2, we expect a deformation of the curvature tensors and scalars to involve the three-index object \( \eta_{abc} \).

However, in the present situation, the computation will become more involved. For instance, in the case of non-vanishing torsion, the first and second Bianchi identities read
\[ R^i_{\ [bc]df} = \nabla_{[a} T^{i\ df}_{\ c]ef} + T^i_{\ ab} T^m_{\ cde}, \] (24)
\[ \nabla_{[a} R^m_{\ abc]} = T^p_{\ [ab} R^m_{\ cp]}, \] (25)
Therefore, anticipating the final result, we again make the assumptions that the full connection is metric compatible, implying for instance that $R_{abcd} = -R_{bacd}$, as well as that the torsion tensor is completely anti-symmetric. Of course, the whole analysis can be performed without these assumptions, but then the underlying structure is less obvious.

Let us now consider the first term in (21). We employ (25) to write

$$\nabla^m R_{\mu
u\lambda\kappa} = -\nabla_\mu R_{\nu\lambda\kappa} + \nabla_\nu R_{\mu\lambda\kappa} - T_{\mu\nu}^{\rho} R_{\rho\lambda\kappa} + T_{\mu\nu}^{\rho} R_{\rho\lambda\kappa} - T_{\mu\nu}^{\rho} R_{\rho\lambda\kappa},$$  

(26)

where we remind the reader that all quantities involve the torsionful connection. For the first of these expressions, this reads

$$T_{\mu}^{\rho\nu\lambda} R_{\nu\lambda\kappa} = \frac{1}{2} T_{\mu}^{\rho\nu\lambda} R_{\nu\lambda\kappa} + \frac{1}{2} T_{\mu}^{\rho\nu\lambda} \left[ \nabla_\nu T_{\rho\lambda\kappa} + 2 \nabla_\nu T_{\rho\lambda\kappa} + 3 \nabla_\nu T_{\rho\lambda\kappa} \right] + \frac{1}{2} T_{\mu}^{\rho\nu\lambda} \left[ T_{\rho\kappa\mu} T_{\nu}^{\kappa\rho} - T_{\kappa\mu\rho} T_{\nu}^{\kappa\rho} \right],$$  

(27)

and similarly for the second one. Together with $T_{\mu}^{\rho\nu\lambda} R_{\nu\lambda\kappa}$, in the first line of (21), we then have three terms of the form

$$-T_{\mu}^{\rho\nu\lambda} R_{\nu\lambda\kappa} - T_{\mu}^{\rho\nu\lambda} R_{\nu\lambda\kappa} - T_{\mu}^{\rho\nu\lambda} R_{\nu\lambda\kappa},$$  

(28)

which need to be canceled by a deformation of the Gauss–Bonnet action. This suggests the $\eta$-dependent correction to the curvature tensor to be

$$R_{\mu
u\lambda\kappa}^\eta = R_{\mu
u\lambda\kappa}^{\eta}(\Gamma) + (\Delta R)^{\eta\mu
u\lambda\kappa},$$  

(29)

where the deformation reads

$$(\Delta R)^{\eta\mu
u\lambda\kappa} = -C \eta_{\mu\nu\lambda\kappa} (T_{\mu\nu} + T_{\mu\nu}^{\rho} + T_{\mu\nu}^{\rho}) - 6 C^2 \eta_{\mu\nu\lambda\kappa} \eta_{\mu\nu\lambda\kappa}.$$  

(30)

Note that the sum over the three torsion terms in (30) is not the contorsion, as the sign of the second term is different. Furthermore, we again included a correction of second order in $\eta$ so that on-shell we obtain the Gauss–Bonnet action with the torsional connection (12).

Considering then terms in (21) which are linear both in torsion and Ricci tensors, we realize that it is necessary to define the deformed Ricci tensor and scalar as

$$\tilde{R}_{\mu\nu} = R_{\mu\nu}(\Gamma) + \frac{1}{2} (\Delta R)_{\mu\nu},$$  

$$\tilde{\mathcal{R}} = R(\Gamma) + \frac{1}{2} (\Delta R),$$  

(31)

where again $\tilde{R}_{\mu\nu} = R_{\mu\nu}^{\eta}$ and $\tilde{\mathcal{R}} = \tilde{R}_{\mu\nu}^{\eta} R_{\mu\nu}^{\eta}$. Note that we introduced relative factors of 1/2 and 1/3 between the curvature tensor and the deformation by hand.

We now consider a generalized second-order Lovelock action where the above-mentioned deformation is included. In analogy to (19), we write

$$S = \frac{\alpha'}{2k^2} \int d^n x \sqrt{-g} (\hat{R}_{\mu\nu\lambda\kappa} R_{\mu\nu\lambda\kappa} - \hat{R}_{\mu\nu} \hat{R}_{\mu\nu} + 2 \hat{R}_{\mu\nu} \hat{R}_{\mu\nu} - \hat{R}_{\mu\nu} \hat{R}_{\mu\nu} + \hat{R}^2),$$  

(32)

and for the variation with respect to $T_{\mu\nu\lambda\kappa}$, we again use the notation introduced in equation (20). The contribution of the various terms in (32) then reads

$$+ \hat{R}_{\mu\nu\lambda\kappa} R_{\mu\nu\lambda\kappa} = \left[ \nabla^m (R_{\mu\nu\lambda\kappa} - R_{\nu\lambda\mu\kappa}) + T_{\mu\nu}^{\rho} R_{\rho\lambda\kappa} + 2 C \eta_{\mu\nu\lambda\kappa} R_{\rho\lambda\kappa} \right] + 2 C \eta_{\mu\nu\lambda\kappa} R_{\rho\lambda\kappa} + 2 C \eta_{\mu\nu\lambda\kappa} R_{\rho\lambda\kappa},$$  

(33)
and the equation of motion for the connection is again the sum of these terms being equal to zero. To make contact with the solution at linear order in curvature, we require that the field equation for $\Gamma$ be satisfied for $T_{abc} = 2C\eta_{abc}$. One could in principle allow for $\alpha'$ corrections to this relation, but this would induce corrections at order $(\alpha')^2$ in the equation of motion, which could then only be canceled by including additional higher order terms. We want to avoid such tuning between different terms in the $\alpha'$ expansion, so that we require that the relation $T_{abc} = 2C\eta_{abc}$ hold at each order separately.

Since in this case the connection is metric compatible, we can employ the relations

$$R_{abcd} = -R_{bacd}, \quad \tilde{R}_{ab} = -R_{ab}. \quad (34)$$

In addition to relations (26) and (27), we contract indices in (26) to obtain

$$\nabla^m R_{mc} = \frac{1}{2} \nabla^m R + T_{\mu
u\lambda}^m R_{\mu
u\lambda} + \frac{1}{2} T^{\mu
u\lambda p} R_{\mu
u\lambda p}, \quad (35)$$

which we use to cancel all terms of the form $\nabla R_{\mu\nu\lambda}$ and $\eta^{**} R_{\mu\nu\lambda}$ in (33). In fact, the deformations to the curvature tensors in (30) and (31) are designed to precisely do this. The only remaining terms in the field equation for the connection $\Gamma$ are of the schematic form $\nabla(\eta^{**} \eta_{\mu\nu\lambda})$ and $(\eta^{**})^3$, which can be summarized as

$$F_{\mu\nu\lambda} = +2C\eta^{mn}(-\tilde{\nabla}_m \eta_{\mu\nu\lambda} + 2\tilde{\nabla}_n \eta_{\mu\nu\lambda} + 3\tilde{\nabla}_\nu \eta_{\mu\lambda n})$$

$$-\eta_{\mu\nu\lambda} \eta_{\alpha\beta\gamma} - 2\tilde{\nabla}_p \eta_{\mu\nu\lambda} + 3\tilde{\nabla}_\nu \eta_{\mu\lambda n})$$

$$+ g_{\mu\nu}(\eta^{mn} \tilde{\nabla}_p \eta_{\mu\nu\lambda} + \eta^{pmn} \tilde{\nabla}_\nu \eta_{\mu\lambda p} - \eta^{pmn} \tilde{\nabla}_\lambda \eta_{\mu\nu p})$$

$$- g_{\nu\lambda}(\eta^{mn} \tilde{\nabla}_p \eta_{\mu\nu\lambda} + \eta^{pmn} \tilde{\nabla}_\nu \eta_{\mu\lambda p} - \eta^{pmn} \tilde{\nabla}_\lambda \eta_{\mu\nu p}) + 4\eta_{\mu\lambda\nu} \tilde{\nabla}_p \eta_{\nu\lambda p}$$

$$+ 2C\tilde{\nabla}^m \{ + \eta_{\mu\nu\lambda}(\eta_{\mu\nu\lambda} \eta_{\mu\nu\lambda}) - \eta_{\mu\nu\lambda} \eta_{\mu\nu\lambda}\}. \quad (36)$$

Since this expression is neither symmetric nor cyclic in the three indices $\{\mu, \nu, \lambda\}$, it seems hopeless to find a diffeomorphism-invariant additional deformation to the curvature tensors to cancel it. Therefore, we suspect that (36) must vanish. Note that we have not yet employed a Bianchi identity for the three-index object $\eta_{abc}$. Thus, one might suspect that, for instance, the Bianchi identity for the $H$-flux, that is, $dH = 0$, or in the components

$$0 = \tilde{\nabla} [\eta_{abc}], \quad (37)$$

makes $F_{\mu\nu\lambda}$ vanishing. However, using (37) we can rewrite (36) as

$$F_{\mu\nu\lambda} = +4C\eta^{mn}(-\tilde{\nabla}_m \eta_{\mu\nu\lambda} - \eta_{\mu\nu\lambda} \tilde{\nabla}_m \eta_{\nu\lambda})$$

$$+ g_{\nu\lambda}(\eta^{mn} \tilde{\nabla}_p \eta_{\mu\nu\lambda} - \eta^{mn} \tilde{\nabla}_\lambda \eta_{\mu\nu p})$$

$$- g_{\nu\lambda}(\eta^{mn} \tilde{\nabla}_p \eta_{\mu\nu\lambda} - \eta^{mn} \tilde{\nabla}_\lambda \eta_{\mu\nu p}) + 4\eta_{\mu\lambda\nu} \tilde{\nabla}_p \eta_{\nu\lambda p}$$

$$+ 2C\tilde{\nabla}^m \{ + \eta_{\mu\nu\lambda}(\eta_{\mu\nu\lambda} \eta_{\mu\nu\lambda}) - \eta_{\mu\nu\lambda} \eta_{\mu\nu\lambda}\}. \quad (38)$$

and we see that $F_{\mu\nu\lambda}$ does not vanish. The only reasonable and general (i.e. Bianchi-type) conditions to guarantee $F_{\nu\pi\lambda} = 0$ are

$$\tilde{\nabla}_a \eta_{b\pi\lambda} = 0, \quad \eta_{ab\nu} \eta_{\nu\lambda} = 0, \quad (39)$$

that is, the three-index object is covariantly constant and satisfies a Jacobi identity. Note, furthermore, that due to the latter relation, we have $\nabla_a \eta_{b\pi\lambda} = \tilde{\nabla}_a \eta_{b\pi\lambda} = 0$. In section 6, we will point out and elaborate on a possible origin of relations (39) in the context of string theory. Inserting then the connection into the deformed Gauss–Bonnet action (32), we obtain the usual Gauss–Bonnet action with torsion

$$S = \frac{\alpha'}{2\kappa^2} \int d^4x \sqrt{-g}\left[R(\Gamma)_{abcd}R(\Gamma)^{cdab} - 4R(\Gamma)_{ab}R(\Gamma)^{ab} + R(\Gamma)^2\right]. \quad (40)$$
A quite tedious computation shows that this action is indeed a consistent truncation of (32) with respect to variation of both the metric and the Kalb–Ramond field. We emphasize that this is not a trivial result and a number of cancellations occur in the course of the computation.

To conclude, we want to stress that we introduced deformations (30) and (31) in order to cancel terms (28). Moreover, the emergence of the conditions (39) to solve the equations of motion becomes conspicuous in this approach. However, the deformations are not unique, and in the following section we show that there exists a different and more elegant deformation.

4. Palatini–Lovelock–Cartan actions

In this section, we show that taking the two conditions (39) on the three-form into account from the very beginning, an even simpler Palatini-type Gauss–Bonnet action can be found. And, as we show, this structure generalizes to higher order Lovelock actions [18].

The Lovelock action. Let us recall a result of Exirifard and Sheikh–Jabbari [15]. These authors have studied higher curvature actions which are consistent with the Palatini variational principle, and have found that precisely for Lovelock actions, the equation of motion is consistently satisfied by the Lévi–Civitá connection. Also, requiring a consistent truncation for the Palatini approach singles out these actions [20].

The Lovelock action at order \( k \) in the curvature tensor can be expressed in the following way:

\[
S_{\text{Love}}^{(k)} = \frac{(\alpha')^{k-1}}{4n^k k!} \int \left( e^a \wedge e^b \wedge \ldots \wedge e^k \wedge (\Omega_{a_1 b_1} \wedge \ldots \wedge \Omega_{a_n b_n}) \right)\]

\[
= \frac{(\alpha')^{k-1}}{4n^k k!} \int d^n x \sqrt{-g} g^{a_1 b_1} \ldots g^{a_k b_k} \prod_{j=1}^k \left( R_{a_2 a_1 b_k b_1} - \tilde{R}_{a_2 a_1 b_k b_1} + \tilde{R}_{cd} R^{cd}_{a_2 a_1 b_k b_1} \right),
\]

(41)

where \( \sqrt{\alpha'} \) is a fundamental length scale. Here, \( \{e^a\} \) denotes a set of vielbeins and \( \Omega_{ab} \) is the curvature two-form,

\[
\Omega_{ab} = \frac{1}{2} R_{abcd} e^d \wedge e^c \wedge e^b \wedge e^a,
\]

(42)

with \( \omega^a_{\ b} = \Gamma^a_{\ cd} e^c \) being the connection one-form. For \( k = 1 \), we obtain from (41) the Einstein–Palatini action (1), and for \( k = 2 \) the Palatini–Gauss–Bonnet action (19).

Assuming constraints (39) to be valid, we now show that the deformation of the curvature tensor employed already for the Einstein–Palatini case (13) is also a viable solution at higher orders. In particular, we define a deformed Riemann curvature tensor as

\[
R_{abcd} = R_{abcd}(\Gamma) + C \eta^a_{\ mn} T^m_{\ cd} - 2C^2 \eta^m_{\ mn} R^m_{\ cd},
\]

(43)

and do not impose relative factors between the curvature tensors and the deformation as we did in section 3. Since the case \( k = 1 \) was considered already in section 2, we consider the case \( k = 2 \).

The second-order case. The Lovelock action at second order in the curvature can be obtained from (41) and reads

\[
S_{\text{Love}}^{(2)} = \frac{\alpha'}{2n^2} \int d^n x \sqrt{-g} (R_{abcd} R^{cdab} - R_{ab} R^{ab} + 2 R_{ab} \tilde{R}^{ab} - \tilde{R}_{cd} R^{cd}_{ab} - \tilde{R}_{ab} R^{ab} + R^2),
\]

(44)
where again \( \mathcal{R}_{ab} = \mathcal{R}_{\mu}^{\phantom{\mu} \nu \rho \lambda} \), \( \tilde{\mathcal{R}}_{ab} = \mathcal{R}_{\nu}^{\phantom{\nu} \mu \rho \lambda} \) and \( \mathcal{R} = \mathcal{R}_{ab}^{\phantom{ab}ab} \). Employing then the notation introduced in (20), the variation of action (44) with respect to the connection can be written as

\[
F_{\mu \nu \lambda} = -\nabla^m \mathcal{R}_{m \nu \lambda} + \nabla^m \mathcal{R}_{m \lambda \nu} + T_{\nu \mu}^{\phantom{\nu \mu} \lambda} \mathcal{R}_{m \mu \lambda} + C \eta^{\mu \nu \lambda}_m \left( \mathcal{R}_{\nu \lambda \mu} - \mathcal{R}_{\nu \mu \lambda} \right) \\
+ \nabla_\mu \left( \mathcal{R}_{\nu \lambda \mu} - \tilde{\mathcal{R}}_{\nu \lambda \mu} \right) - g_{\mu \nu} \nabla^m \left( \mathcal{R}_{m \lambda \mu} - \tilde{\mathcal{R}}_{m \lambda \mu} \right) \\
+ T_{\nu \mu}^{\phantom{\nu \mu} \lambda} \left( \mathcal{R}_{m \lambda \mu} - \tilde{\mathcal{R}}_{m \lambda \mu} \right) - T_{\lambda \mu}^{\phantom{\lambda \mu} \nu} \left( \mathcal{R}_{m \mu \lambda} - \tilde{\mathcal{R}}_{m \mu \lambda} \right) \\
+ 2C \left[ \eta^{\mu \nu \lambda}_m \left( \mathcal{R}_{\mu \lambda \nu} - \tilde{\mathcal{R}}_{\mu \lambda \nu} \right) + \eta^{\nu \mu \lambda}_m \left( \mathcal{R}_{\nu \mu \lambda} - \tilde{\mathcal{R}}_{\nu \mu \lambda} \right) \right] \\
+ g_{\mu \nu} \nabla_\lambda \mathcal{R} - g_{\nu \lambda} \nabla_\mu \mathcal{R} + \left( T_{\nu \mu \lambda} + 2C \eta^{\mu \nu \lambda} \right) \mathcal{R}.
\]

(45)

Next, similarly as for the analysis at linear order in the curvature, we require that the field equation for the connection be solved by a metric-compatible connection with \( T_{abc} = 2C \eta_{abc} \).

In this case, we can replace \( \mathcal{R}_{\mu}^{\phantom{\mu} \nu \rho \lambda} \) by \( \mathcal{R}_{\nu}^{\phantom{\nu} \mu \rho \lambda} \) in (45), as can be seen from (43), leading to

\[
F_{\mu \nu \lambda} = +2\nabla^m \mathcal{R}_{m \nu \lambda} + 2\nabla^m \mathcal{R}_{m \lambda \nu} - g_{\mu \nu} \left( 2\nabla^m \mathcal{R}_{m \lambda \nu} + \nabla_\nu \mathcal{R} \right) + g_{\nu \lambda} \left( 2\nabla^m \mathcal{R}_{m \nu \lambda} - \nabla_\mu \mathcal{R} \right) \\
+ 4C \eta^{\mu \nu \lambda}_m \mathcal{R} + 4C \eta^{\nu \mu \lambda}_m \mathcal{R} + 2C \eta^{\nu \mu \lambda}_m \mathcal{R} + 2C \eta^{\mu \nu \lambda}_m \mathcal{R}.
\]

(46)

Furthermore, making use of the first constraint in (39), from \( \left[ \nabla_m, \nabla_n \right] \eta_{abc} = 0 \), we can derive

\[
0 = R^m_{\nu \mu \lambda} \eta_{abc} + R^m_{\nu \lambda \nu} \eta_{apc} + R^m_{\mu \nu \lambda} \eta_{bap}.
\]

(47)

Employing then (39) as well as (47), we observe that the Bianchi identities shown in (24) and (25) simplify to

\[
R^m_{[bcd]} = 0, \quad \nabla_{[a} R^m_{b]c} = 0.
\]

(48)

Finally, using the second of these equations, we see that the first two lines in (46) vanish, and using the first Bianchi identity together with (47), the third line in (46) vanishes. We therefore arrive at

\[
F_{\mu \nu \lambda} = 0,
\]

(49)

that is, the variation of action (44) with respect to the connection vanishes for a metric-compatible connection with torsion \( T_{abc} = 2C \eta_{abc} \). Furthermore, the same computation can be performed with a more general deformed curvature tensor of the form

\[
\mathcal{R}^{a}_{\nu \rho \lambda} = R^a_{\nu \rho \lambda} \Gamma + C \eta^{a}_{bmn} T^{b}_{mn} - (2C^2 - A) \eta^{a}_{bmn} \eta^{b}_{mn}.
\]

(50)

In this case, consistency with the leading-order string action (18) yields the relation \( C^2 - A = 1/12 \).

**The third-order case.** After having studied the second-order Lovelock action, we now consider the case of \( k = 3 \). From (41), we find the third-order Lovelock action as

\[
S^{(3)}_{\text{Lovelock}} = \frac{2}{2k^2} \int d^4x \sqrt{-g} \left[ R^{abcd} R_{cdfe} R^{ef}_{ab} + 4 R^{ab}_{ce} R^{cd}_{ef} R^{ef}_{ab} \\
+ 3 R^{abcd} R_{cdfe} R^{ef}_{cb} - 3 R^{abcd} R_{cdab} R^{ef}_{b} \\
- 3 R^{abcd} R_{cdbe} R^{ef}_{a} + 3 R^{abcd} R_{cdab} R^{ef}_{b} \\
+ 3 R^{abcd} R_{be} R^{ef}_{cd} - 6 R^{abcd} R_{bc} R^{ef}_{cd} + 3 R^{abcd} R_{bc} R^{ef}_{a} \\
+ R^{ab}_{bc} R^{ce}_{a} - 3 R^{ab}_{bc} R^{ce}_{a} + 3 R^{ab}_{bc} R^{ce}_{a} - R^{ab}_{bc} R^{ce}_{a} \\
+ R \times \text{(lower order Lovelock terms)} \right].
\]

(51)
Emplopying again our previous notation, the variation of the above action with respect to the connection leads to

\[
F_{\mu \nu \lambda} = \frac{1}{2} C \left[ 3 \eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn} + 12 \eta_{\mu}^{ mn} \eta_{\rho \lambda}^{ pq mn} R_{\rho \lambda}^{ pq mn} \right] + 6 \eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn} + 6 \eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn} + 6 \eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn}
\]

where the terms arising from the last line in (51) cancel due to our analysis for the lower order Lovelock actions. After a tedious calculation and employing the following four relations

\[
\eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn} = -2 \eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn} + 6 \eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn}
\]

we can bring (52) into the form

\[
F_{\mu \nu \lambda} = \frac{1}{2} C \left[ 3 \eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn} + 12 \eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn} \right] + 6 \eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn} + 6 \eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn} + 6 \eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn} + 6 \eta_{\mu}^{ mn} \eta_{\rho}^{ pq} R_{\rho \lambda}^{ pq mn}
\]

Since all terms written out explicitly are anti-symmetric under the exchange \( \mu \leftrightarrow \nu \), we conclude that

\[
F_{\mu \nu \lambda} = 0.
\]

Therefore, also for the third-order Palatini–Lovelock–Cartan action, the torsionful connection with \( T^{\alpha}_{\eta \nu} = 2 C \eta_{\nu}^{\eta \nu} \) is a solution to the field equation of \( \Gamma \). Let us emphasize that this result is obtained through a long and non-trivial computation.

A conjecture for Palatini–Lovelock–Cartan gravity. The above observations and results lead us to the following conjecture.

**Conjecture.** The field equation of the connection of the Palatini–Lovelock–Cartan action

\[
S_{\text{Love}}^{(k)} = \frac{(\alpha')^{k-1}}{4 k^2 k!} \int d^n x \sqrt{-g} \left( g^{ab} \partial_1 \ldots \partial_k b_1 \ldots b_k \right) \prod_{j=1}^{k} R_{a_{2j-1} b_{2j-1} a_{2j} b_{2j}}
\]

with the deformed curvature tensor

\[
R^{a}_{bcd} \equiv R^{a}_{bcd} (\Gamma) + C \eta^{mn}_{bmn} T^{m}_{cd} - 2 C^2 \eta^{mn}_{bmn} T^{m}_{cd}
\]

is satisfied for a metric-compatible connection with \( T_{abc} = 2 C \eta_{abc} \), if the three-form \( \eta_{abc} \) is covariantly constant and satisfies the Jacobi identity \( \eta^{[a}_{1} \eta^{b}_{2} \eta^{c]}_{3} = 0 \). Inserting the solution into the Lovelock–Cartan action results in a consistent truncation.
5. Relation to bosonic string theory

Our analysis so far was based on a rather formal question, namely whether the Palatini variational principle can be generalized to torsional Gauss–Bonnet gravity. In this section, we study whether the Gauss–Bonnet action with torsion can be related to the effective action at second order in the curvature of the bosonic string for the constant dilaton. This was already analyzed a long time ago in [12] with a negative answer. However, in view of the new Bianchi-type identities shown in (39), it seems worthwhile to redo their analysis.

Let us recall parts of the results of [12]. Using equation (15) and invoking the Bianchi identities for the curvature tensor with Lévi–Civita connection as well as for the $H$-flux (37), it was shown that any action quadratic in the torsional curvature can be expressed by ten independent combinations,

$$S = \frac{\alpha'}{2k^2} \int d^3 x \sqrt{-g} \left[ \sum_{i=1}^{3} f_i T_i + \sum_{j=1}^{4} a_j T_{a_j} + \sum_{k=1}^{3} u_k T_{u_k} \right].$$  \hspace{1cm} (56)

The above terms read

$$T_1 = \tilde{R}_{abcd} \tilde{R}^{abcd}, \quad T_2 = \tilde{R}_{abcd} \eta^{d}{}_{m} \eta^{b}{}_{m}, \quad T_3 = \tilde{R}(\eta^{abc} \eta^{abc}),$$

$$T_4 = \tilde{R}_{ab} \tilde{R}^{ab}, \quad T_5 = \tilde{R}^{2},$$

$$T_6 = (\nabla_m \eta^{ab}) (\nabla_m \eta^{cib}),$$

where we employed the same notation as in [12]. As can be seen for instance from the redefinition of the metric, the Kalb–Ramond field and the dilaton (see e.g. [26–28]), the on-shell string scattering amplitudes impose four relations among the coefficients of these ten terms. They are given by

$$f_1 = 1, \quad f_2 = -1, \quad f_3 = 1/27, \quad a_1 + \frac{1}{4} a_4 + \frac{1}{144} u_1 = -\frac{1}{8}.$$  \hspace{1cm} (58)

The two conditions in (39) allow for a reduction to seven terms, where $T_{a_5}$ vanishes identically and where we can relate

$$2T_{f_5} = T_{a_5}, \quad 2T_{f_2} = T_{a_5}.$$  \hspace{1cm} (59)

Thus, relations (58) can be simplified to

$$f_1 = 1, \quad a_1 + \frac{1}{4} a_4 + \frac{1}{144} u_1 = -\frac{11}{32}.$$  \hspace{1cm} (60)

Now, using (15), we expand the curvature terms in the truncated action (40) to obtain

$$S = \frac{\alpha'}{2k^2} \int d^3 x \sqrt{-g} (T_{f_1} - 3C^4 T_{a_5} + C^4 T_{a_5} - 2C^2 T_{a_5} + 6C^2 T_{a_5} - 4T_{a_5} + T_{a_5}).$$  \hspace{1cm} (61)

As the second relation in (60) is off by 1/12, the pure torsional Gauss–Bonnet action is not consistent with the effective action of the bosonic string. Therefore, even with the stronger Bianchi-type identities, the negative result of [12] still holds.

However, it is possible to add an additional correction to the Gauss–Bonnet action to make it consistent. Defining the tensor

$$P_{abcd} = \frac{1}{6} \eta^{d}{}_{m} \eta^{cm}$$  \hspace{1cm} (62)

and contracting indices as for the curvature tensor, we obtain the effective action of the bosonic string as

$$S_{\text{string}} = \frac{\alpha'}{2k^2} \int d^3 x \sqrt{-g} \left[ (R_{abcd} R^{abcd} - 4R_{ab} R_{ba} + R^2) + [P_{abcd} P^{abcd} - 4P_{ab} P^{ab} + P^2] \right].$$  \hspace{1cm} (63)
where we used the short-hand notation $R_{abcd} = R(\Gamma)_{abcd}$. Note that this is still compatible with the Palatini formalism since we included terms only depending on $\eta_{abc}$. Furthermore, as can be checked, the additional term already included in the off-shell action (32) leads to a consistent truncation as well.

### 6. Bianchi identities

We have seen that our Palatini approach to the Gauss–Bonnet action with torsion imposes conditions on the three-form flux which go beyond the expected Bianchi identity for the $H$-flux,

$$\hat{\nabla}_{[a} H_{bcd]} = 0.$$ (64)

In particular, in (39) we observed that the three-form flux should be covariantly constant and should satisfy a Jacobi identity. Such conditions are not completely unfamiliar, since similar constraints appear for certain exact solutions to the string equations of motion, namely for parallelizable manifolds such as WZW models [29].

Moreover, the relations we found resemble Bianchi identities not for the $H$-flux itself but for the other three T-dual counterparts: geometric flux $f$ and non-geometric fluxes $Q$ and $R$. In [30], the form of these Bianchi identities was derived as Jacobi identities of generalized gauge transformations on a flat geometry,

$$H_{[ijkl} f^{k]l}^{ef} = 0,$$

$$H_{[ijkl} Q_{jkl}^{[^i} f^{^i]} + f^{^i}_{[ij} f^{k]l} = 0,$$

$$H_{ijkl} R^{cd} + f^{e}_{[ij} Q_{k]l}^{^e} - 4 f^{^e}_{[ij} f^{^i}_{k]l} = 0,$$ (65)

$$f^{[i}_{[ij} R_{kl]}^{^i} + Q_{^i}^{^i}_{[ij} Q_{kl]}^{^i} = 0,$$

$$Q_{[ij}^{[i} R_{kl]}^{^i} = 0.$$

Note that due to their origin, the left-hand sides of all these relations can be written as sums over terms of the form $\eta_{[a}^m \eta_{bc]}^n$.

For non-constant fluxes on general manifolds, one expects corrections to the above Bianchi identities containing derivative and curvature terms. It is the purpose of this section to derive these corrections which to large extent are unknown.

**Bianchi identity for $Q$- and $R$-fluxes.** Let us propose a possible way to derive the Bianchi identities for the non-geometric $Q$- and $R$-fluxes, which should be related to the last three relations in (65). The $R$-flux can be described as an anti-symmetric three-vector field, and so we expect defining relations to be determined by the use of a graded extension of the Lie bracket of vector fields.

For this purpose, we introduce the Schouten–Nijenhuis bracket $[\ , \ ]_{SN}$, which for the functions $f$, $g$ and ordinary vector fields $X$, $Y$ is defined by

$$[f, g]_{SN} = 0, \quad [X, f]_{SN} = X(f), \quad [X, Y]_{SN} = [X, Y].$$ (66)

with $[\ , \ ]$ denoting the Lie bracket. However, it extends uniquely to arbitrary alternating multi-vector fields by the following relations:

$$[X, Y \wedge Z]_{SN} = [X, Y]_{SN} \wedge Z + (-1)^{(|X| - 1)|Y|} Y \wedge [X, Z]_{SN},$$

$$[X, Y]_{SN} = -(-1)^{(|X| - 1)(|Y| - 1)} [Y, X]_{SN}.$$ (67)
with $|X|$ denoting the degree of $X$. Thus, the degree of the resulting multi-vector field is $|[X, Y]_{SN}| = |X| + |Y| - 1$. These properties define a so-called Gerstenhaber algebra. In addition, the graded Jacobi identity is satisfied,

$$[X, [Y, Z]_{SN}]_{SN} = [[X, Y]_{SN}, Z]_{SN} + (-1)^{(|X|-1)(|Y|-1)}[X, [Y, Z]_{SN}]_{SN}. \quad (68)$$

The algebraic properties of this algebra can now be used to derive Bianchi identities for the fluxes. For a basis of vector fields $\{e_a\}$, the geometric flux $f^{ab}$ is given by

$$[e_a, e_b]_{SN} = [e_a, e_b] = f^{c}_{ab} e_c, \quad (69)$$

which we assume to be vanishing in the following. In this case and for the vanishing $H$-flux, it has been proposed in [31–34] that the $R$-flux can be written in terms of a bi-vector field $\beta := \frac{1}{2} \beta^{ab} e_a \wedge e_b$ as

$$R = [\beta, \beta]_{SN} = \beta^{[am} \partial_m \beta^{bc]} e_a \wedge e_b = \beta^{[am} \tilde{\nabla}_m \beta^{bc]} e_a \wedge e_b, \quad (70)$$

where the Lévi–Civitá connection drops out. Using then (68), one finds the trivial relation

$$0 = [\beta, [\beta, \beta]_{SN}]_{SN} = [\beta, R]_{SN}, \quad (71)$$

and evaluating then the right-hand side of (71), we obtain

$$\beta^{[am} \partial_m R^{bc]} = 2 \beta^{[am} m Q_m^{bc]}, \quad (72)$$

where we employed the usual definition of the non-geometric $Q$-flux $\tilde{Q}_a^{bc} = \partial_a \beta^{bc}$. Note that (72) contains the partial derivative $\partial_m$ rather than the covariant derivative $\tilde{\nabla}_m$. However, the relation can be covariantized as

$$\beta^{[am} \tilde{\nabla}_m R^{bc]} = \frac{3}{2} \beta^{[am} m Q_m^{bc]}, \quad (73)$$

with $Q_a^{bc} = \tilde{\nabla}_a \beta^{bc}$. The terms containing the Lévi–Civitá connection cancel due to the symmetry of the Christoffel symbols.

As will be discussed elsewhere [35], this computation can be generalized. For instance, the identity

$$[\beta, [\beta, e_p]_{SN}]_{SN} = \frac{1}{2} [R, e_p]_{SN} = 0 \quad (74)$$

leads to a Bianchi identity for the $Q$-flux, that is,

$$\beta^{[am} \partial_m \tilde{Q}_p^{bc]} - \frac{1}{3!} \partial_p R^{[abc]} = \tilde{Q}_p^{m[a} \tilde{Q}_m^{bc]}. \quad (75)$$

Directly covariantizing this identity, one finds an extra curvature term so that the Bianchi identity for the $Q$-flux reads

$$\beta^{[am} \tilde{\nabla}_m \tilde{Q}_p^{bc]} - \frac{1}{3!} \tilde{\nabla}_p R^{[abc]} - 2 \beta^{[am} 3 \beta^{[m} \tilde{\nabla}_p \tilde{R}_m^{abc]} = Q_p^{m[a} Q_m^{bc]}. \quad (76)$$

The definition of the $Q$-flux leads to another trivial identity

$$\tilde{\nabla}_a Q_{p[bc]}^{[ed]} = -\tilde{R}^{[ed]} \tilde{m} [\tilde{Q}_m^{[a}], \tilde{Q}_m^{bc]}, \quad (77)$$

which can be used to exchange the indices $(m \leftrightarrow p)$ in the first term in (76). Therefore, using the Lévi–Civitá connection, we have the following six Bianchi identities written in a geometric

5 This is consistent with the $R$-flux Bianchi identity recently derived in [34] in the double-field theory approach, if one sets there the derivative with respect to the ‘winding’ coordinate $\partial_\phi$ to zero.
basis (i.e. for vanishing geometric flux):
\begin{align*}
\hat{\nabla}_{[a} H_{bcd]} &= 0, \\
\hat{\nabla}_{[a} R^{mn}_{\,bcd]} &= 0, \\
\hat{R}^{[abcdef]} &= 0, \\
\hat{\nabla}_{[a} Q_{bc]} &= \frac{1}{3!} \hat{\nabla}_p R_{[abc]}^{\,\,} = Q_p \cdot Q_{m}^{\,} . \\
\beta^{[a[m} \hat{\nabla}_n R_{bcd]}^{\,} &= \frac{1}{2} R^{[a[m} Q_{n}^{\,} .
\end{align*}

The claim is that, except for the second line in (78), the remaining five Bianchi identities are precisely in one-to-one correspondence with the five Bianchi identities in (65). The inclusion of geometric flux will be discussed in [35]. Note that the building blocks of these Bianchi identities are covariant derivatives of the fluxes and quadratic expressions of the type appearing in a Jacobi identity.

**Remark.** Let us remark the following. In the formalism of double-field theory, T-duality is not only a symmetry for special solutions of string theory but a symmetry of the underlying equations of motion. It is thus tempting to speculate that the two rather strong conditions (39) can be considered as universal, that is, if they are satisfied, all Bianchi identities for $H$-, $f$-, $Q$- and $R$-fluxes are fulfilled. If this line of thought is correct, it has the following implication for the usual theory formulated with only $H$-flux:

the Bianchi identities for $f$-, $Q$- and $R$-fluxes should be a consequence of the exact string equations of motion (at all orders of $\alpha'$) for the metric and the Kalb–Ramond field.

This means that there should not exist solutions violating any of the Bianchi identities in the other T-dual frames. In fact, all exact solutions we are familiar with, such as Calabi–Yau manifolds or WZW models, indeed satisfy these two stronger conditions.

### 7. Conclusion

In this paper, we have considered Einstein–Hilbert, Gauss–Bonnet and higher order Lovelock actions within the Palatini formalism, where torsion is identified dynamically with a three-form flux. For the Einstein–Hilbert action, this was straightforward, whereas for the Gauss–Bonnet case, the computation became more involved. In particular, for the consistency of the Palatini-variational principle, the three-form had to be covariantly constant and had to satisfy a Jacobi identity. These conditions are stronger than the Bianchi identity for the $H$-flux, but we argued for a conceivable connection to Bianchi identities for T-dual fluxes. More concretely, in double-field theory, our new restrictions can be regarded as the universal conditions guaranteeing that all Bianchi identities are satisfied. In this respect, we derived Bianchi identities for the non-geometric $Q$- and $R$-fluxes including a covariant derivative and curvature terms.

We also showed that the no-go theorem for relating torsional Gauss–Bonnet gravity to the second-order corrections of the bosonic string still applies, even if one imposes the stronger conditions on the three-form flux. However, we presented a form of the next-to-leading-order bosonic string action that contained a sum of two Gauss–Bonnet terms, one being the torsional Gauss–Bonnet gravity action and the other a pure Kalb–Ramond field-dependent contribution.

Clearly, there are a number of open questions. For instance, it would be desirable to have a proof for our conjecture that the Palatini variational principle carries over to all higher order Lovelock actions. The relation to string theory also demands the dilaton to be considered.
Furthermore, more evidence for our observation that in a T-duality covariant theory one needs to require the three-form to be covariantly constant and to satisfy a Jacobi identity is needed. Finally, a very interesting question is whether for the case of $R$-flux there is a relation to the nonassociativity observed in [4–7].

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