AN EXPLICIT EMBEDDING OF THE TITS GROUP IN THE COMPACT REAL FORM OF $E_6$

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Abstract. We construct a set of $27 \times 27$ unitary matrices which give an explicit embedding of the Tits group in the compact real form of the Lie group of type $E_6$. A subset gives an embedding of $\text{PSL}_2(25)$ in $F_4$.

1. Introduction

It is well-known that the Tits simple group $^{2}F_4(2)'$ embeds in $E_6(\mathbb{C})$, but the published proof of this fact by Cohen and Wales [1] does not lend itself very easily to obtaining an explicit embedding. On the other hand, the 27-dimensional representation of $^{2}F_4(2)'$, written over $\mathbb{Q}(i)$, was constructed by Simon Nickerson in his PhD thesis [3], and is available from [4]. The task then is to determine the hermitian form and the cubic form preserved by this representation. By character theory, both forms are unique up to scalar multiplication, so they must be the ones which define the compact real form of $E_6$. One would then want to find a basis with respect to which this is the standard copy of $E_6$.

2. Strategy

The best subgroup to use to exhibit the forms would seem to be the maximal subgroup $5^2:4A_4$. This acts on the 27-space as the direct sum of the 3-dimensional deleted permutation representation of the quotient $A_4$, and a monomial representation formed from the 24 non-trivial representations of $5^2$. Inside this subgroup is an element of order 12, whose normalizer is $D_{24}$. Thus one can generate $^{2}F_4(2)'$ with the subgroup $5^2:4A_4$ and an involution in the outer half of this $D_{24}$. When correctly scaled, the 24 non-trivial eigenspaces of the $5^2$, together with the 3 non-trivial eigenspaces of $5^2:Q_8$, form an orthonormal basis. Moreover, the terms of the cubic form are severely restricted by the action of the $5^2$.

Date: First draft 08/08/2012; this version 21/08/2012.
3. Computing a good basis

We begin by taking Nickerson’s representation, which is written over \( \mathbb{Z}[\frac{1}{2}, i] \), and reduce all the matrix entries modulo 41 (so that we have a field with 4th and 5th roots of unity, but no cube roots). We take \( i = 9 \), and note that the primitive 5th roots of unity are \(-4, 16, 18, 10\). We find the subgroup \( 5^2:4A_4 \) using the words in the standard generators \( a, b \) given in [4], that is \( a \) and

\[
e = b^{(abab)^3}.
\]

Now we find a subgroup \( 4A_4 \) as the centralizer of the involution \( (ac)^6 \): this subgroup may be generated by \( ac \) and

\[
d = (ac)^6)^5.
\]

We next find two elements

\[
\begin{align*}
f_1 &= (a(ac)^6)^2, \\
f_2 &= (f_1)^{ac}
\end{align*}
\]

of order 5, generating the \( 5^2 \), and look for a non-trivial eigenspace, for example as the common nullspace of \( f_1 + 4 \) and \( f_2 + 4 \). Similarly we find a non-trivial eigenvector of \( 5^2:Q_8 \) in the simultaneous nullspace of \( f_1 - 1, f_2 - 1 \) and \( d - 1 \). The images of our eigenvectors under the group \( 4A_4 \) give us our 27 basis vectors (up to signs).

4. Computing good generators

We next look for elements in the centraliser in the Tits group of the involution \( (ac)^6 \). Using Bray’s algorithm, we quickly find the element

\[
e = ((ac)^6(ab^2)^{-1}(ac)^6(ab^2))^4,
\]

which lies in the outer half of the involution centralizer of shape

\[
2^{1+2+1+2+2}S_3.
\]

We want to find an element in this involution centralizer which inverts \( (ac)^4 \), so we apply the formula:

\[
e' = e(ac)^8e(ac)^4e
\]

and find that in fact \( e' \) is an involution inverting \( ac \). From now on we use exclusively the new generators

\[
f_1, f_2, d, ac, e'
\]

for \( ^2F_4(2)' \).
5. Adjusting the basis

Our next task is to adjust the scalars so that vectors in both parts of the basis have the same norm. To do this, we write $e'$ with respect to the new basis, and inspect the $3 \times 24$ and $24 \times 3$ blocks where the two parts of the basis interact. We find that in each block the non-zero entries are a particular scalar multiple of $\pm 1, \pm 9$, but they are different multiples in the two blocks. So we find a suitable scalar to multiply one of the orbits of basis vectors by, to make these multiples the same.

At this stage, the top row of the matrix contains 19 non-zero entries, which are 25 (twice), 33 (five times), and 8, 10, 31 (four times each), which are $-8$ times 2, $\pm 1, \pm 9$. It is possible to adjust some of the basis vectors by factors of $\pm 9$ so that the entries become $-8$ times 2 (twice), times 1 (nine times) and times $-1$ (eight times). Other cosmetic changes, such as re-ordering the coordinates, can be done according to one’s personal taste.

6. Lifting to complex numbers

Noticing that $-8 \equiv 1/5 \mod 41$ we see that this gives an obvious way to lift to the complex numbers, in such a way that the top row of the matrix of $e'$ has two entries $2/5$ and 17 entries $\pm 1/5$, and therefore has norm 1. We take

$$z = e^{2\pi i/5}$$

as the lift of 16, so that 10, 37, 18 lift to $z^2, z^3, z^4$ respectively, and also 7 lifts to

$$\sigma = (1 - \sqrt{5})/2 = -z - z^4$$

and 35 lifts to

$$\tau = (1 + \sqrt{5})/2 = -z^2 - z^3.$$  

Then $f_1$ and $f_2$ act diagonally as follows:

$$f_1 = \text{diag}(1, 1, 1; z, z^2, z^3, z^4; 1, 1, 1, 1; z^3, z, z^4, z^2; z^3, z, z^4, z^2; z^2, z^4, z, z^3),$$

$$f_2 = \text{diag}(1, 1, 1; z, z^2, z^3, z^4; z^3, z, z^4, z^2; z^3, z, z^4, z^2; z^2, z^4, z, z^3; z^2, z, z^4, z^2; 1, 1, 1, 1).$$

The elements $d$ and $ac$ permute the six 4-spaces which are the fixed spaces of one of the cyclic subgroups of $5^2$, which we have delimited by
semicolons above. Writing

\[ I = \begin{pmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 \\ . & . & . & 1 \end{pmatrix}, \quad J = \begin{pmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{pmatrix}, \]

\[ K = \begin{pmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \end{pmatrix}, \quad L = \begin{pmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{pmatrix}, \]

we have

\[ d = \begin{pmatrix} 1 & . & . & . \\ . & -1 & . & . \\ . & . & -1 \\ . & . & . & -1 \end{pmatrix} \]

\[ d = \begin{pmatrix} -I & . & . & . \\ . & -J & . & . \\ . & . & -iI & . \\ . & . & . & -iK \end{pmatrix} \]

\[ ac = \begin{pmatrix} . & 1 & . & . \\ . & 1 & . & . \\ . & . & K & . \\ K & . & . & . \\ . & . & . & K \\ . & . & . & . \\ . & . & K & . \\ . & . & . & . \\ . & . & . & K \end{pmatrix} \]

Finally, in order to write down \( e' \), we define the matrices

\[ A = \begin{pmatrix} 0 & \tau & \sigma & 0 \\ \tau & 0 & 0 & \sigma \\ \sigma & 0 & 0 & \tau \\ 0 & \sigma & \tau & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & \tau & \sigma & -1 \\ \tau & -1 & -1 & \sigma \\ \sigma & -1 & -1 & \tau \\ -1 & \sigma & \tau & -1 \end{pmatrix}, \]

\[ C = \begin{pmatrix} 1 & \sigma & \tau & 1 \\ \sigma & 1 & 1 & \tau \\ \tau & 1 & 1 & \sigma \\ 1 & \tau & \sigma & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} \tau & 1 & 1 & \sigma \\ 1 & \sigma & \tau & 1 \\ 1 & \tau & \sigma & 1 \\ \sigma & 1 & 1 & \tau \end{pmatrix}, \]
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$$F = \begin{pmatrix} -1 & \sigma & \tau & -1 \\ \sigma & -1 & -1 & \tau \\ \tau & -1 & -1 & \sigma \\ -1 & \tau & \sigma & -1 \end{pmatrix}, \quad G = \begin{pmatrix} \tau & 0 & 0 & \sigma \\ 0 & \sigma & \tau & 0 \\ 0 & \tau & \sigma & 0 \\ \sigma & 0 & 0 & \tau \end{pmatrix},$$

and then we have

$$e' = \frac{1}{5} \begin{pmatrix} 2 & 2 & 1 & 0 & -1 & 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & -1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & A & B & A & D & E & F \\ -1 & 0 & 1 & B & C & D & G & F & G \\ 1 & 0 & -1 & A & D & E & F & A & B \\ 1 & -1 & 0 & D & G & F & G & B & C \\ -1 & 1 & 0 & E & F & A & B & A & D \\ 0 & 1 & -1 & F & G & B & C & D & G \end{pmatrix}.$$  

7. THE PERMUTATION REPRESENTATION ON 2304 POINTS

The generators we have chosen have the property that the subset

$$\{f_1, f_2, ac, e'\}$$

generates the subgroup $\text{PSL}_2(25)$ of index 2304 in $^2F_4(2)'$. It is easy to see that this subgroup fixes the vector

$$(1, 1, 1; 0^{24}),$$

and hence we obtain an orbit of 2304 images of this vector, on which $^2F_4(2)'$ acts. This can be used to provide a (computer-aided) proof that our group really is the Tits group.

8. THE PERMUTATION REPRESENTATION ON 1755 POINTS

A calculation with the Meataxe, or GAP, shows that the 1-space spanned by the vector

$$(0, 0, 0; 0, 0, 0, 0; i, 1, -1, -i; 0^{16})$$

has 1755 images under $^2F_4(2)'$. The point stabilizer is $2^{1+4+4}:5:4$, namely the centralizer of the involution $d$, and the quotient of order 4 (represented for example by the powers of $(ac)^3$) multiplies the given vector by powers of $i$. 
9. The cubic form

Dickson’s cubic form \([2]\) for \(E_6\) has 45 terms in the 27 variables, and if \(uvw\) is one of the terms, then the eigenvalues \(\lambda_1, \lambda_2, \lambda_3\) of \(f_1\) (or \(f_2\)) on \(u, v, w\) respectively must satisfy
\[\lambda_1\lambda_2\lambda_3 = 1.\]

Hence any pair \(u, v\) of the last 24 basis vectors lies in at most one triple of the cubic form. Now a small calculation with the Meataxe finds the (unique) 27-dimensional quotient of the symmetric square of this 27-dimensional representation of \(2F_4(2)'\), from which one can easily read off which pairs of basis vectors actually occur.

It turns out that only terms involving one vector from each of the three blocks of eight actually occur, as well as terms involving one of the first three basis vectors. Labelling the coordinates
\[(-3, -2, -1; 1, 2, 3, 4; \ldots; 21, 22, 23, 24)\]
for convenience, the triples which occur are images under the monomial group \(5^2:4A_4\) of
\[(-3, -2, -1), (-3, 1, 4), (1, 9, 17), (1, 10, 24).\]

It remains to determine the signs of these terms. One may be specified arbitrarily, and then the rest can be read off from the above Meataxe calculation. They can be taken to be + on \((-3, -2, -1)\) and \((1, 10, 24)\), and – on \((-3, 1, 4)\) and \((1, 9, 17)\).

One can now identify the 27 coordinates (with some sign changes) with those given in Dickson’s original construction \([2]\), or other sources, such as \([5]\). One possible labelling of our 27 coordinates in the notation of \([5]\) is as follows:
\[(0, 0'', 0'; -4, 1, 8, -5; 7, 6, 3, 2; -4', 6', 3', -5'; 1', 7', 2', 8'; -4'', 7'', 2'', -5''; 6'', 1'', 8'', 3'').\]

The signs indicate that the first and fourth coordinate in each block of 8 must be negated.

10. Embedding \(\text{PSL}_2(25)\) in \(F_4\)

Finally, the fixed vector \((1, 1, 1; 0^{24})\) of the subgroup \(\text{PSL}_2(25)\) has as its support the three coordinates corresponding to one of the terms of Dickson’s cubic form, and therefore can be taken as the identity element of the exceptional Jordan algebra. Hence we obtain as a by-product an explicit embedding of \(\text{PSL}_2(25)\) in the compact real form of the Lie group of type \(F_4\).
I would like to thank Marek Mitros for asking the question how to find an explicit embedding of $2F_4(2)'$ in the compact real form of $E_6$, and for uncovering some typographical errors in an earlier draft which, if left uncorrected, would have rendered the results incorrect and therefore useless.

REFERENCES

[1] Arjeh Cohen and David Wales, On finite subgroups of $F_4(\mathbb{C})$ and $E_6(\mathbb{C})$, \textit{Proc. London Math. Soc.} \textbf{74} (1997), 105–150.

[2] L. E. Dickson, A class of groups in an arbitrary realm connected with the configuration of the 27 lines on a cubic surface, \textit{Quart. J. Pure Appl. Math.} \textbf{33} (1901), 145–173.

[3] S. J. Nickerson, \textit{An Atlas of characteristic zero representations}, PhD thesis, Birmingham, 2005.

[4] Robert Wilson, Peter Walsh, Jonathan Tripp, Ibrahim Suleiman, Richard Parker, Simon Norton, Simon Nickerson, Stephen Linton, John Bray, Richard Barraclough and Rachel Abbott, \textit{Atlas of Group Representations}, \url{http://brauer.maths.qmul.ac.uk/Atlas/v3/}, 2005–.

[5] R. A. Wilson, \textit{The finite simple groups}, Springer GTM 251, 2009.

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