Direct computation of harmonic moments for
tomographic reconstruction

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Abstract. A novel algorithm to compute harmonic moments of a density function from
its projections is presented for tomographic reconstruction. For projection \( p(r, \theta) \), we define
harmonic moments of projection by
\[
\int_0^\pi \int_{-\infty}^{\infty} p(r, \theta)(re^{i\theta})^n drd\theta,
\]
and show that it coincides with the harmonic moments of the density function except a constant. Furthermore, we show that the harmonic moment of projection of order \( n \) can be exactly computed by using \( n+1 \) projection
directions, which leads to an efficient algorithm to reconstruct the vertices of a polygon from
projections.

1. Introduction

Moment problems in which a function is reconstructed from its moments have many applications
in science and engineering [1]. Harmonic or complex moments of a function have attracted
great attention in image processing due to their rotational invariance [7] and in tomographic
reconstruction [8, 5] due to their strong connections to features of the function in hand. Milanfar
et al. [8] formulated a ‘shape-from-moments’ problem in which the vertices of a polygon of
constant density are reconstructed from projection data. Generalizing Davis’ result [2] that
vertices of a triangle are uniquely determined by a finite number of its moments, they showed
using Prony’s method [11] that the vertices of a simply connected \( n \)-gon are uniquely determined
by its harmonic moments up to order \( 2n - 3 \). Golub [5] showed that this problem amounts to a
generalized eigenvalue problem, which can be solved by the QZ algorithm. They also analyzed
the sensitivity of the generalized eigenvalues with respect to perturbations in the harmonic
moments. Djafari [3] extended these results to reconstruction of a polyhedron in 3D space.

In conventional methods for the shape-from-moments problem, computation of harmonic
moments of the density function from its projections consists of two steps: first one computes
geometric moments of the density function from its projection, then computes the harmonic
moments from geometric moments. In this paper, we show a simple relationship between the
harmonic moments of the density function and projection data. As a result, the harmonic
moments are directly computed from projection data without using geometric moments, which
provides us with an efficient method to reconstruct vertices of a polygon from projections.

This paper is organized as follows. In section 2, a shape-from-moments problem is formulated.
In section 3, after a conventional method to compute the harmonic moments is reviewed, we
define a concept of a harmonic moment of projection, and show two theorems which allow one to compute the harmonic moments directly from projection data. Section 4 is devoted to numerical simulations.

2. Problem formulation

Let \( P \) be a polygon whose vertices are \( z_1, z_2, \ldots, z_N \) in a 2D plane. Let \( f(x, y) \) be the indicator function which is 1 in \( P \) and 0 elsewhere:

\[
f(x, y) = \begin{cases} 
1, & (x, y)^T \in P \\
0, & (x, y)^T \notin P 
\end{cases}
\] (1)

Let \( p(r, \theta) \) be the projection of \( f(x, y) \) along a line crossing a point \((r, \theta)\) in the polar coordinates as shown in Fig.1. Using the Dirac delta function, \( p(r, \theta) \) is written as

\[
p(r, \theta) = \int \int_{\mathbb{R}^2} f(x, y) \delta(r - x \cos \theta - y \sin \theta) dx dy.
\] (2)

In this paper, we consider the problem of reconstructing the positions \( z_1, z_2, \ldots, z_N \) as well as the number \( N \) of vertices of the polygon \( P \) from projection data.

The vertex positions of a polygon are related to harmonic moments of \( f(x, y) \) defined by

\[
c_n = \int \int_{\mathbb{R}^2} f(x, y)(x + iy)^n dx dy = \int \int_P (x + iy)^n dx dy,
\] (3)

or to complex moments of \( f(x, y) \) defined by

\[
\tau_n \equiv \frac{c_{n-2}}{n(n-1)} \quad (n \geq 2), \quad \tau_0 = \tau_1 = 0.
\] (4)

\( c_n \) and \( \tau_n \) are called harmonic/complex moments of order \( n \), respectively. Milanfar [8] showed that the following relationship between the vertices of a polygon and the complex moments holds:

\[
\tau_n = \sum_{k=1}^{N} q_k z_k^n \quad (n \geq 0),
\] (5)
where $q_k$ is defined by

$$q_k = \frac{i}{2} \left( \frac{z_{k-1} - z_k}{z_{k-1} - z_k} - \frac{z_k - z_{k+1}}{z_k - z_{k+1}} \right) = \sin(\theta_{k-1} - \theta_k) e^{-i(\theta_{k-1} - \theta_k)}$$

and is determined by the external angle at the $k$-th vertex as shown in Fig.2. To determine $z_k$ and $q_k$ in Eq. (5) from $\tau_n (n = 0, 1, \ldots, 2N - 1)$ appears in many inverse problems such as electroencephalography (EEG) inversion [4, 9], magnetoencephalography (MEG) inversion [10], locating zeros of an analytic function [6], and tomographic reconstruction of a polygonal shape from moments [8, 5]. Since algebraic solution to this problem has been already proposed [8, 5] summarized in Appendix, the vertices of a polygon as well as the angles between sides are directly reconstructed from complex moments. Therefore, what is important in the problem of shape-from-moments is computing the harmonic moments of the density function from projection data.

3. Computation of harmonic moments

3.1. Conventional method [8]

In the previous papers [8, 3], the harmonic moments were computed via geometric moments as follows. By expanding the weighting term of the harmonic moment, we have

$$\int \int_{\mathbb{R}^2} f(x, y)(x + iy)^n \, dx \, dy = \sum_{k=0}^{n} \binom{n}{k} i^k \int \int_{\mathbb{R}^2} f(x, y)x^{n-k}y^k \, dx \, dy,$$

where $\binom{n}{k}$ is the binomial coefficient. The terms, in the right side of Eq. (7),

$$\int \int_{\mathbb{R}^2} f(x, y)x^{n-k}y^k \, dx \, dy \quad (k = 0, 1, \ldots, n)$$

are called geometric moments of order $n$. Thus, the harmonic moments of order $n$ can be obtained by a linear combination of geometric moments of order $n$.

Now, let us consider the line integral $\int_{-\infty}^{\infty} p(r, \theta) r^n \, dr$. Then, by definition of projection, it holds that

$$\int_{-\infty}^{\infty} p(r, \theta) r^n \, dr = \int_{-\infty}^{\infty} \left[ \int \int_{\mathbb{R}^2} f(x, y) \delta(r - x \cos \theta - y \sin \theta) \, dx \, dy \right] r^n \, dr$$

$$= \int \int_{\mathbb{R}^2} f(x, y)(x \cos \theta + y \sin \theta)^n \, dx \, dy$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k} \theta \sin^k \theta \int \int_{\mathbb{R}^2} f(x, y)x^{n-k}y^k \, dx \, dy,$$

where in the last row the binomial theorem is used. Thus, selecting $n + 1$ projection directions $\theta_0, \theta_1, \ldots, \theta_n$ gives the following linear equations for the geometric moments of order $n$

$$\begin{pmatrix}
\int_{-\infty}^{\infty} p(r, \theta_0) r^n \, dr \\
\int_{-\infty}^{\infty} p(r, \theta_1) r^n \, dr \\
\vdots \\
\int_{-\infty}^{\infty} p(r, \theta_n) r^n \, dr \\
\end{pmatrix} = \begin{pmatrix}
\cos^n \theta_0 & n \cos^{n-1} \theta_0 \sin \theta_0 & \cdots & \sin^n \theta_0 \\
\cos^n \theta_1 & n \cos^{n-1} \theta_1 \sin \theta_1 & \cdots & \sin^n \theta_1 \\
\vdots & \vdots & \ddots & \vdots \\
\cos^n \theta_n & n \cos^{n-1} \theta_n \sin \theta_n & \cdots & \sin^n \theta_n \\
\end{pmatrix} \begin{pmatrix}
\int \int_{\mathbb{R}^2} f(x, y)x^{n-k}y^k \, dx \, dy \\
\int \int_{\mathbb{R}^2} f(x, y)x^{n-k-1}y^k \, dx \, dy \\
\vdots \\
\int \int_{\mathbb{R}^2} f(x, y)y^{n-k} \, dx \, dy \\
\end{pmatrix}.$$
Therefore, the geometric moments are obtained by solving Eq. (10).

From the above discussion, the conventional algorithm to compute harmonic moments can be summarized as follows:

For each order \( n \),
1) compute \( \int_{-\infty}^{\infty} p(r, \theta) r^n dr \) for \( n + 1 \) directions \( \theta_0, \theta_1, \ldots, \theta_n \);
2) solve the linear equations (10) for the geometric moments \( \int \int_{\mathbb{R}^2} f(x, y)x^{n-k}y^k dxdy \) \( (k = 0, 1, \ldots, n) \);
3) compute the linear combination in Eq. (7).

In other words, in order to obtain the harmonic moments, one has to solve Eq. (10) for each order. In the next section, we show a simpler relationship between the harmonic moments of the density function and projection data.

3.2. Direct method
First, we define harmonic moments of a projection.

**Definition** For projection \( p(r, \theta) \), the harmonic moment of projection of order \( n \) is defined by

\[
\int_0^\pi \int_{-\infty}^{\infty} p(r, \theta)(re^{i\theta})^n drd\theta. \tag{11}
\]

The following theorem holds.

**Theorem 1** The harmonic moment of the density function \( f(x, y) \) is the harmonic moment of projection \( p(r, \theta) \) multiplied by \( \frac{2^n}{\pi} \):

\[
\int \int_{\mathbb{R}^2} f(x, y)(x+iy)^n dxdy = \frac{2^n}{\pi} \int_0^\pi \int_{-\infty}^{\infty} p(r, \theta)(re^{i\theta})^n drd\theta. \tag{12}
\]

(proof) By definition of the harmonic moment of \( p(r, \theta) \), we have

\[
\int_0^\pi \int_{-\infty}^{\infty} p(r, \theta)(re^{i\theta})^n drd\theta
= \int \int_{\mathbb{R}^2} f(x, y) \left( \int_0^\pi \int_{-\infty}^{\infty} \delta(r - x \cos \theta - y \sin \theta) r^n dr e^{i\theta} d\theta \right) dxdy
= \int \int_{\mathbb{R}^2} f(x, y) \left( \int_0^\pi (x \cos \theta + y \sin \theta)^n e^{i\theta} d\theta \right) dxdy. \tag{13}
\]

Now, by expanding \( (x \cos \theta + y \sin \theta)^n \) via the binomial theorem, the integral with respect to \( \theta \) in Eq. (13) is rewritten as

\[
\int_0^\pi (x \cos \theta + y \sin \theta)^n e^{i\theta} d\theta
= \int_0^\pi \left( \frac{x + iy}{2} e^{-i\theta} + \frac{x - iy}{2} e^{i\theta} \right)^n e^{i\theta} d\theta
= \int_0^\pi \left( \frac{x + iy}{2} \right)^n + \left( \frac{x + iy}{2} \right)^{n-1} \left( \frac{x - iy}{2} \right) e^{2i\theta} + \cdots + \left( \frac{x - iy}{2} \right)^n e^{2i\theta} d\theta \tag{14}
\]
Since
\[ \int_{0}^{\pi} e^{2i\theta} d\theta = \cdots = \int_{0}^{\pi} e^{2in\theta} d\theta = 0, \]
all terms in Eq. (14) except the first one vanish thereby giving
\[ \int_{0}^{\pi} (x \cos \theta + y \sin \theta)^{n} e^{in\theta} d\theta = \pi \left( \frac{x + iy}{2} \right)^{n}. \]
Thus, we have
\[ \int_{0}^{\pi} \int_{-\infty}^{\infty} p(r, \theta)(re^{i\theta})^{n} drd\theta = \frac{\pi}{2^{n}} \int \int_{\mathbb{R}^{2}} f(x, y)(x + iy)^{n} dxdy, \]
which completes the proof. □

The next theorem is efficient from a computational viewpoint.

**Theorem 2** The \( n \)-th order harmonic moment of \( f(x, y) \) can be exactly and directly computed from \( n + 1 \) projections as
\[ \int \int_{\mathbb{R}^{2}} f(x, y)(x + iy)^{n} dxdy = 2^{n} \sum_{k=0}^{n} \int_{-\infty}^{\infty} p(r, \theta_{k})(re^{i\theta_{k}})^{n} dr \]
where \( \theta_{k} = \frac{k}{M} \pi. \)

(proof) Let us consider \( M \) projection directions whose angles \( \theta_{1}, \theta_{2}, \ldots, \theta_{M} \) are given by \( \theta_{k} = \frac{k}{M} \pi, \) and consider
\[ I_{M} = \sum_{k=0}^{M-1} \int_{-\infty}^{\infty} p(r, \theta_{k})(re^{i\theta_{k}})^{n} dr \]
By definition of projection,
\[ I_{M} = \sum_{k=0}^{M-1} \int_{-\infty}^{\infty} p(r, \theta_{k})r^{n}e^{in\theta_{k}}dr \]
\[ = \int \int_{\mathbb{R}^{2}} f(x, y) \left( \sum_{k=0}^{M-1} \int_{-\infty}^{\infty} \delta(r - x \cos \theta_{k} - y \sin \theta_{k})r^{n} dr e^{i\theta_{k}} \frac{\pi}{M} \right) dxdy \]
\[ = \int \int_{\mathbb{R}^{2}} f(x, y) \left( \sum_{k=0}^{M-1} (x \cos \theta_{k} + y \sin \theta_{k})^{n} e^{in\theta_{k}} \frac{\pi}{M} \right) dxdy \]
The binomial theorem gives
\[ \sum_{k=0}^{M-1} (x \cos \theta_{k} + y \sin \theta_{k})^{n} e^{in\theta_{k}} \frac{\pi}{M} \]
\[ = \sum_{k=0}^{M-1} \left( \frac{x + iy}{2} e^{-i\theta_{k}} + \frac{x - iy}{2} e^{i\theta_{k}} \right)^{n} e^{i\theta_{k}} \frac{\pi}{M} \]
\[ = \sum_{k=0}^{M-1} \left( \left( \frac{x + iy}{2} \right)^{n} + \binom{n}{1} \left( \frac{x + iy}{2} \right)^{n-1} \left( \frac{x - iy}{2} \right) e^{2i\theta_{k}} + \cdots + \left( \frac{x - iy}{2} \right)^{n} e^{2in\theta_{k}} \right) \frac{\pi}{M}. \]
Here, if \( M > n \), all the terms but the first one vanish since

\[
\sum_{k=0}^{M-1} e^{2i\theta_k} = \sum_{k=0}^{M-1} e^{2ik\frac{\pi}{M}} = \frac{1 - e^{2iM\frac{\pi}{M}}}{1 - e^{2i\frac{\pi}{M}}} = 0, \quad (22)
\]

\[
\vdots
\]

\[
\sum_{k=0}^{M-1} e^{2in\theta_k} = \sum_{k=0}^{M-1} e^{2ink\frac{\pi}{M}} = \frac{1 - e^{2inM\frac{\pi}{M}}}{1 - e^{2in\frac{\pi}{M}}} = 0. \quad (23)
\]

Therefore, when \( M > n \), we have

\[
\sum_{k=0}^{M-1} \int_{-\infty}^{\infty} p(r, \theta_k) r^n e^{in\theta_k} dr \frac{\pi}{M} = \frac{\pi}{2n} \int_{\mathbb{R}^2} f(x, y)(x + iy)^n dxdy. \quad (24)
\]

Since Eq. (24) holds for \( M = n + 1 (> n) \), we obtain Theorem 2.

Although the following proposition has been already proved in [8], we mention it here again as the corollary of Theorem 2:

**Corollary 3** The minimum number of projection directions required for reconstruction of a polygon with \( N \) vertices is \( 2N - 2 \).

(proof) Since the algebraic reconstruction of \( N \) vertices using Eq. (5) requires \( \tau_0 \) through \( \tau_{2N-1} \), one needs to compute the harmonic moments up to order \( 2N - 3 \). Thus from Theorem 2, the minimum number of projections is \( 2N - 2 \). □

The difference between the conventional methods and ours is summarized as follows. In the previous algorithms, in order to compute a single quantity, the harmonic moment of order \( n \), they have to compute the \( n + 1 \) intermediate quantities, the geometric moments, by solving the linear equation (10). On the contrary, our method can directly compute the harmonic moment of order \( n \) just from the weighted integral of projection data as in (12)/(18). This has brought us a simpler expression of the harmonic moments of the density function in terms of projection.

### 4. Numerical simulations

The first polygon we examine is a triangle whose vertices are \((0.5, 1), (-0.5, 0.5), (-0.5, -0.5)\). Since \( N = 3 \) in this case, Corollary 3 guarantees that \( 2N - 2 = 4 \) projections are enough for reconstruction. We take 100 sampling points along the \( r \)-axis. In order to verify Theorem 2 and Corollary 3, no noise is added first.

Fig. 3 shows the reconstruction results by using 3 and 4 projection directions, respectively. The vertices reconstructed from 4 projections (depicted by black dots) well coincide with the true vertices, whereas those reconstructed from 3 projections (depicted by the blue dots) do not. Fig. 4 shows the correspondence between the number of projections and the geometric mean of relative errors of three vertices positions. One observes that the vertices are accurately reconstructed when the number of projections is greater than 4, as predicted by Theorem 2 and Corollary 3.
3
2
1
0
-1
-2
-3

-2
-1
0
1
2
3

Figure 3. Reconstruction of a triangle (noiseless case) The estimated vertices denoted by black dots using 4 projection directions well coincide with the true vertices, whereas the results denoted by the blue dots using 3 projection directions do not, which supports Theorem 2 and Corollary 3 that $2N - 2 = 4$ projection directions are enough to reconstruct a triangle $N = 3$.

Next, we add a 5% Gaussian noise to each projection data, and reconstruct a square ($N = 4$) with vertices at $(\pm 1, \pm 1), (\pm 1, \mp 1)$. We use 20 sampling points in the $r$-axis direction. Fig. 5 shows the localization result, when $2N - 2 = 6$ projection directions are used. Even under noisy condition, the four vertices are well reconstructed.

However, in the realistic situation, we do not know a priori the number of vertices. In that case, we assume a number of $N'$ vertices greater than $N$. Fig. 6 shows the estimated $|q_k|$ when assuming $N' = 5$. One observes that $|q_k|$ is much smaller than the other four values $|q_k|$ ($k = 1, 2, 3, 4$). The external angle at the fifth vertex obtained from $|q_5|$ is 0.7 or 179.3 degree, whereas the other four external angles are 91.1, 91.6, 89.6, 86.5 degree. Since the sum of those four angles is about 360 degree, one can judge that the fifth vertex is ‘spurious’ and there are four substantial vertices.

5. Conclusion
In this paper, we showed that the harmonic moments of a 2D density function can be obtained from harmonic moments of its projection. As a result, we can compute the harmonic moments of the density function directly from projection data without computing geometric moments of the density function. Furthermore, we show that the harmonic moments of the density function can be obtained from discretized harmonic moments of projection, which allows us to compute the harmonic moments exactly from a finite number of projection data.
Figure 5. Reconstruction of a square \((N = 4)\) assuming that there are \(N' = 4\) vertices under 5\% Gaussian noise

Figure 6. Estimation of the number of vertices: the fifth \(|q_k|\) is much smaller than the other four values when assuming \(N' = 5\), from which one can reasonably judge that there are substantially four vertices.

Appendix: algebraic solution for Eq. (5)

\(z_k\) and \(q_k\) in Eq. (5) are obtained from \(\tau_n\) \((n = 0, 1, \ldots, 2N - 1)\) as follows [5]. Let \(H_0\) and \(H_1\) be the Hankel matrices

\[
H_0 = \begin{pmatrix}
\tau_0 & \tau_1 & \cdots & \tau_{N-1} \\
\tau_1 & \cdots & \tau_{N} & \\
\vdots & \vdots & \vdots & \\
\tau_{N-1} & \tau_{N} & \cdots & \tau_{2N-2}
\end{pmatrix}
\quad H_1 = \begin{pmatrix}
\tau_1 & \tau_2 & \cdots & \tau_{N} \\
\tau_2 & \cdots & \tau_{N+1} & \\
\vdots & \vdots & \vdots & \\
\tau_{N} & \tau_{N+1} & \cdots & \tau_{2N-1}
\end{pmatrix}.
\]

These are simultaneously diagonalized by the Vandermonde matrix as

\[
H_0 = \begin{pmatrix}
1 & \cdots & 1 \\
z_1^{N-1} & \cdots & z_N^{N-1}
\end{pmatrix}
\begin{pmatrix}
q_1 \\
q_N
\end{pmatrix}
\begin{pmatrix}
1 & \cdots & 1 \\
z_1^{N-1} & \cdots & z_N^{N-1}
\end{pmatrix}
\quad H_1 = \begin{pmatrix}
1 & \cdots & 1 \\
z_1 & \cdots & z_N
\end{pmatrix}
\begin{pmatrix}
q_1 \\
q_N
\end{pmatrix}
\begin{pmatrix}
1 & \cdots & 1 \\
z_1 & \cdots & z_N
\end{pmatrix}^T.
\]

(26)
Hence, $H_0^{-1}H_1$ is diagonalized as follows:

$$H_0^{-1}H_1 = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ z_1^{N-1} & \cdots & z_N^{N-1} \end{pmatrix}^{-T} \begin{pmatrix} z_1 \\ \vdots \\ z_N \\ \vdots \\ z_1^{N-1} \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ z_1^{N-1} & \cdots & z_N^{N-1} \end{pmatrix}^T \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ z_1^{N-1} & \cdots & z_N^{N-1} \end{pmatrix}.$$ (28)

Let

$$\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ z_1^{N-1} & \cdots & z_N^{N-1} \end{pmatrix}^{-T} = (w_1 \ w_2 \cdots \ w_N),$$ (29)

then Eq. (28) is written as

$$H_0^{-1}H_1 w_k = z_k w_k.$$ (30)

This means that $z_1, z_2, \ldots, z_N$ are the eigenvalues of the matrix $H_0^{-1}H_1$. Since Eq. (30) can be rewritten as

$$H_1 w_k = z_k H_0 w_k,$$ (31)

we can also say that $z_k$ is obtained as the solutions to the generalized eigenvalue problem (31). An identical relation and procedure can be also obtained between $z_k$ and $H_1 H_0^{-1}$ [4]. Once $z_k$ is obtained, $q_k$ is determined by the inversion of Eq. (5) for $l = 0, 1, \cdots, N - 1$ as

$$\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ z_1^{N-1} & \cdots & z_N^{N-1} \end{pmatrix}^{-1} \begin{pmatrix} \tau_0 \\ \vdots \\ \tau_{N-1} \end{pmatrix}.$$ (32)

In this way, given $N$, the vertex positions $z_k$ and quantities corresponding to external angles $q_k$ are algebraically obtained from $\tau_k (k = 0, 1, \cdots, 2N - 1)$.

Since $N$ is not known a priori, we estimate $N'$ vertices where $N'$ is greater than $N$. As numerically shown in section 4, $|q_k|$ for ‘spurious’ vertices are usually much smaller than those of the true ones, from which $N$ can be estimated.

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