ON COMMUTING MATRICES AND EXPONENTIALS

CLÉMENT DE SEGUINS PAZZIS

(Communicated by Gail R. Letzter)

Abstract. Let $A$ and $B$ be matrices of $M_n(\mathbb{C})$. We show that if $\exp(A)^k \exp(B)^l = \exp(kA + lB)$ for all integers $k$ and $l$, then $AB = BA$. We also show that if $\exp(A)^k \exp(B) = \exp(B)\exp(A)^k = \exp(kA + B)$ for every positive integer $k$, then the pair $(A, B)$ has property L of Motzkin and Taussky.

As a consequence, if $G$ is a subgroup of $(M_n(\mathbb{C}), +)$ and $M \mapsto \exp(M)$ is a homomorphism from $G$ to $(GL_n(\mathbb{C}), \times)$, then $G$ consists of commuting matrices. If $S$ is a subsemigroup of $(M_n(\mathbb{C}), +)$ and $M \mapsto \exp(M)$ is a homomorphism from $S$ to $(GL_n(\mathbb{C}), \times)$, then the linear subspace $\text{Span}(S)$ of $M_n(\mathbb{C})$ has property L of Motzkin and Taussky.

1. Introduction

1.1. Notation and definition.

i) We denote by $\mathbb{N}$ the set of non-negative integers.

ii) If $M \in M_n(\mathbb{C})$, we denote by $e^M$ or $\exp(M)$ its exponential, by $\text{Sp}(M)$ its set of eigenvalues.

iii) The $n \times n$ complex matrices $A, B$ are said to be simultaneously triangularizable if there exists an invertible matrix $P$ such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular.

iv) A pair $(A, B)$ of complex $n \times n$ matrices is said to have property L if for a special ordering $(\lambda_i)_{1 \leq i \leq n}, (\mu_i)_{1 \leq i \leq n}$ of the eigenvalues of $A, B$, the eigenvalues of $xA + yB$ are $(x\lambda_i + y\mu_i)_{1 \leq i \leq n}$ for all values of the complex numbers $x, y$.

1.2. The problem. It is well-known that the exponential is not a group homomorphism from $(M_n(\mathbb{C}), +)$ to $(GL_n(\mathbb{C}), \times)$ if $n \geq 2$. Nevertheless, when $A$ and $B$ are commuting matrices of $M_n(\mathbb{C})$, one has

$$e^{A+B} = e^A e^B = e^B e^A.$$  

However (1) is not a sufficient condition for the commutativity of $A$ with $B$, nor even for $A$ and $B$ to be simultaneously triangularizable. Still, if

$$\forall t \in \mathbb{R}, \ e^{tA} e^{tB} = e^{tB} e^{tA},$$  

or

$$\forall t \in \mathbb{R}, \ e^{t(A+B)} = e^{tA} e^{tB},$$  

Received by the editors December 30, 2010 and, in revised form, May 24, 2011 and July 16, 2011.

2010 Mathematics Subject Classification. Primary 15A16; Secondary 15A22.
Key words and phrases. Matrix pencils, commuting exponentials, property L.
then a power series expansion at \( t = 0 \) shows that \( AB = BA \). In the 1950’s, pairs of matrices \((A, B)\) of small size such that \( e^{A+B} = e^A e^B \) have been under extensive scrutiny \[3 \, 4 \, 6 \, 7 \, 9 \, 10\]. More recently, Wermuth \[16, 17\] and Schmoeger \[14, 15\] studied the problem of adding extra conditions on the matrices \( A \) and \( B \) for the commutativity of \( e^A \) with \( e^B \) to imply the commutativity of \( A \) with \( B \). A few years ago, Bourgeois (see \[1\]) investigated, for small \( n \), the pairs \((A, B) \in M_n(\mathbb{C})^2\) that satisfy
\[
\forall k \in \mathbb{N}, \quad e^{kA+B} = e^{kA} e^B = e^B e^{kA}.
\]

The main interest in this condition lies in the fact that, contrary to Conditions \((2)\) and \((3)\), it is not possible to use it to obtain information on \( A \) and \( B \) based only on the local behavior of the exponential around 0. Bourgeois showed that Condition \((4)\) implies that \( A \) and \( B \) are simultaneously triangularizable if \( n = 2 \), and produced a proof that this also holds when \( n = 3 \). This last result is however false, as the following counterexample, communicated to us by Jean-Louis Tu, shows: consider the matrices
\[
A_1 := 2i\pi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B_1 := 2i\pi \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & 1 & 0 \end{bmatrix}.
\]

Notice that \( A_1 \) and \( B_1 \) are not simultaneously triangularizable since they share no eigenvector (indeed, the eigenspaces of \( A_1 \) are the lines spanned by the three vectors of the canonical basis, and none of them is stabilized by \( B_1 \)). However, for every \( t \in \mathbb{C} \), a straightforward computation shows that the characteristic polynomial of \( tA_1 + B_1 \) is
\[
X \left( X - 2i\pi(t + 2) \right) \left( X - 2i\pi(2t + 3) \right).
\]

Then for every \( t \in \mathbb{N} \), the matrix \( tA_1 + B_1 \) has three distinct eigenvalues in \( 2i\pi \mathbb{Z} \), hence is diagonalizable with \( e^{tA_1+B_1} = I_3 \). In particular \( e^{B_1} = I_3 \), and on the other hand, \( e^{A_1} = I_3 \). This shows that Condition \((4)\) holds.

It then appears that one should strengthen Bourgeois’ condition as follows in order to obtain at least the simultaneous triangularizability of \( A \) and \( B \):
\[
\forall (k, l) \in \mathbb{Z}^2, \quad e^{kA+lB} = e^{kA} e^{lB}.
\]

Notice immediately that this condition implies that \( e^A \) and \( e^B \) commute. Indeed, if Condition \((5)\) holds, then
\[
e^B e^A = (e^{-A} e^{-B})^{-1} = (e^{-A-B})^{-1} = e^{A+B} = e^A e^B.
\]

Therefore Condition \((5)\) is equivalent to
\[
\forall (k, l) \in \mathbb{Z}^2, \quad e^{kA+lB} = e^{kA} e^{lB} = e^{lB} e^{kA}.
\]

Here is our main result.

**Theorem 1.** Let \((A, B) \in M_n(\mathbb{C})^2\) be such that for all \((k, l) \in \mathbb{Z}^2\), \( e^{kA+lB} = e^{kA} e^{lB} \). Then \( AB = BA \).

The following corollary is straightforward.

**Theorem 2.** Let \( G \) be a subgroup of \((M_n(\mathbb{C}), +)\) and assume that \( M \mapsto \exp(M) \) is a homomorphism from \((G, +)\) to \((\text{GL}_n(\mathbb{C}), \times)\). Then, for all \((A, B) \in G^2\), \( AB = BA \).
The key to the proof of Theorem 1 is Proposition 3.

**Proposition 3.** Let \((A, B) \in M_n(\mathbb{C})^2\). Assume that, for every \((k, l) \in \mathbb{Z}^2\), the matrix \(kA + lB\) is diagonalizable and \(\text{Sp}(kA + lB) \subset \mathbb{Z}\). Then \(AB = BA\).

For subsemigroups of \((M_n(\mathbb{C}), +)\), Theorem 2 surely fails. A very simple counterexample is indeed given by the semigroup generated by

\[ A := \begin{bmatrix} 0 & 0 \\ 0 & 2i\pi \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & 1 \\ 0 & 2i\pi \end{bmatrix}. \]

One may however wonder whether a subsemigroup \(S\) on which the exponential is a homomorphism must be simultaneously triangularizable. Obviously the additive semigroup generated by the matrices \(A_1\) and \(B_1\) above is a counterexample. Nevertheless, we will prove a weaker result, which rectifies and generalizes Bourgeois’ results [1].

**Proposition 4.** Let \((A, B) \in M_n(\mathbb{C})^2\) be such that \(\forall k \in \mathbb{N}, e^{kA+B} = e^{kA}e^B = e^Be^{kA}\). Then \((A, B)\) has property L.

Note that the converse is obviously false.

The proofs of Theorem 1 and of Proposition 4 have largely similar parts, so they will be tackled simultaneously. There are three main steps.

- We will prove Proposition 4 in the special case where \(\text{Sp}(A) \subset 2i\pi\mathbb{Z}\) and \(\text{Sp}(B) \subset 2i\pi\mathbb{Z}\). This will involve a study of the matrix pencil \(z \mapsto A + zB\). We will then easily derive Proposition 3 using a refinement of the Motzkin-Taussky theorem.
- We will handle the more general case \(\text{Sp}(A) \subset 2i\pi\mathbb{Z}\) and \(\text{Sp}(B) \subset 2i\pi\mathbb{Z}\) in Theorem 1 by using the Jordan-Chevalley decompositions of \(A\) and \(B\) together with Proposition 3.
- In the general case, we will use an induction to reduce the situation to the previous one, both for Theorem 1 and Proposition 4.

In the last section, we will prove a sort a generalized version of Proposition 4 for additive semigroups of matrices (see Theorem 11).

### 2. Additive Groups and Semigroups of Matrices with an Integral Spectrum

#### 2.1. Notation.

i) We denote by \(\Sigma_n\) the group of permutations of \(\{1, \ldots, n\}\), make it act on \(\mathbb{C}^n\) by \(\sigma.(z_1, \ldots, z_n) := (z_{\sigma(1)}, \ldots, z_{\sigma(n)})\), and consider the quotient set \(\mathbb{C}^n/\Sigma_n\). The class of a list \((z_1, \ldots, z_n) \in \mathbb{C}^n\) in \(\mathbb{C}^n/\Sigma_n\) will be denoted by \([z_1, \ldots, z_n]\).

ii) For \(M \in M_n(\mathbb{C})\), we denote by \(\chi_M(X) \in \mathbb{C}[X]\) its characteristic polynomial, and we set \(\text{OSp}(M) := [z_1, \ldots, z_n], \quad \text{where } \chi_M(X) = \prod_{k=1}^n (X - z_k)\).

iii) Given an integer \(N \geq 1\), we set \(U_N(z) := \{\zeta \in \mathbb{C} : \zeta^N = z\}\).
Therefore one has: for every integer $A, B \in M_n(\mathbb{C})^2$ has property L when there are $n$ linear forms $f_1, \ldots, f_n$ on $\mathbb{C}^2$ such that

$$\forall (x, y) \in \mathbb{C}^2, \quad \text{OSp}(xA + yB) = [f_k(x, y)]_{1 \leq k \leq n}.$$ 

Using the fact that the eigenvalues are continuous functions of the coefficients, it is obvious that a pair $(A, B) \in M_n(\mathbb{C})^2$ has property L if and only if there are affine maps $f_1, \ldots, f_n$ from $\mathbb{C}$ to $\mathbb{C}$ such that

$$\forall z \in \mathbb{C}, \quad \text{OSp}(A + zB) = [f_k(z)]_{1 \leq k \leq n}.$$ 

2.3. Property L for pairs of matrices with an integral spectrum. We denote by $\mathcal{K}(\mathbb{C})$ the quotient field of the integral domain $H(\mathbb{C})$ of entire functions (i.e. analytic functions from $\mathbb{C}$ to $\mathbb{C}$). Considering id$_\mathbb{C}$ as an element of $\mathcal{K}(\mathbb{C})$, we may view $A + \text{id}_\mathbb{C} B$ as a matrix of $M_n(\mathbb{K}(\mathbb{C}))$. We define the generic number $p$ of eigenvalues of the pencil $z \mapsto A + zB$ as the number of the distinct eigenvalues of $A + \text{id}_\mathbb{C} B$ in an algebraic closure of $\mathcal{K}(\mathbb{C})$. A complex number $z$ is called regular when $A + zB$ has exactly $p$ distinct eigenvalues, and exceptional otherwise. In a neighborhood of 0, the spectrum of $A + zB$ may be classically described with Puiseux series as follows (see [2, chapter 7]): there exists a radius $r > 0$, an integer $q \in \{1, \ldots, n\}$, positive integers $d_1, \ldots, d_q$ such that $n = d_1 + \cdots + d_q$, and analytic functions $f_1, \ldots, f_q$ defined on a neighborhood of 0 such that

$$\forall z \in \mathbb{C} \setminus \{0\}, \quad |z| < r \Rightarrow \chi_{A+zB}(X) = \prod_{k=1}^{q} \prod_{\zeta \in \mathbb{U}_{d_k}(z)} (X - f_k(\zeta)).$$ 

We may now prove the following result.

**Proposition 5.** Let $(A, B) \in M_n(\mathbb{C})^2$. Assume that $\text{Sp}(kA + B) \subset \mathbb{Z}$ for every $k \in \mathbb{N}$. Then $(A, B)$ has property L.

**Proof.** With the above notation, we prove that $f_1, \ldots, f_q$ are polynomial functions. For instance, consider $f_1$ and its power series expansion

$$f_1(z) = \sum_{j=0}^{+\infty} a_j z^j.$$ 

Set $N := d_1$ for convenience. Let $k_0$ be a positive integer such that $\frac{1}{k_0} < r$. For every integer $k \geq k_0$, $k f_1(k^{-\frac{1}{k}})$ is an eigenvalue of $kA + B$; hence it is an integer. Therefore one has: for every integer $k \geq k_0$,

$$\begin{aligned}
(k + 1) f_1 \left( \left( k + 1 \right)^{-\frac{1}{k}} \right) - k f_1 \left( k^{-\frac{1}{k}} \right) &\in \mathbb{Z}.
\end{aligned}$$

For every integer $k \geq k_0$, the following equality holds:

$$\begin{aligned}
(k + 1) f_1 \left( \left( k + 1 \right)^{-\frac{1}{k}} \right) - k f_1 \left( k^{-\frac{1}{k}} \right) &= a_0 + \sum_{j \in \mathbb{N} \setminus \{0, N\}} a_j \left( \left( k + 1 \right)^{-\frac{1}{k}} - k^{-\frac{1}{k}} \right).
\end{aligned}$$

Assume that $a_j \neq 0$ for some $j \geq 1$ with $j \neq N$, and define $s$ as the smallest such $j$. On the one hand, one has for every integer $j \in \mathbb{N}$,

$$\begin{aligned}
(k + 1)^{-\frac{1}{k}} - k^{-\frac{1}{k}} &= k^{1 - \frac{1}{k}} \left( \left( 1 + \frac{1}{k} \right)^{1 - \frac{1}{k}} - 1 \right) \sim_{k \to +\infty} k^{1 - \frac{1}{k}} \frac{1 - \frac{j}{N}}{k} = \frac{1 - \frac{j}{N}}{N} k^{-\frac{1}{k}}.
\end{aligned}$$
On the other hand, when $k \to +\infty$, one has

$$
\sum_{j=s+N+1}^{+\infty} a_j k^{1-\frac{j}{N}} = o(k^{-\frac{s}{N}})
$$

and

$$
\sum_{j=s+N+1}^{+\infty} a_j (k+1)^{1-\frac{j}{N}} = o(k^{-\frac{s}{N}}).
$$

It follows that

$$
\sum_{j \in \mathbb{N} \setminus \{0, N\}} a_j ((k+1)^{1-\frac{j}{N}} - k^{1-\frac{j}{N}}) \sim a_s \left(1 - \frac{s}{N}\right) k^{-\frac{s}{N}}.
$$

The sequence $\left((k+1)f_1((k+1)^{-\frac{j}{N}}) - kf_1(k^{-\frac{j}{N}}) - a_0\right)_{k \geq k_0}$ is discrete, converges to 0 and is not ultimately zero. This is a contradiction. Therefore $\forall j \in \mathbb{N} \setminus \{0, N\}$, $a_j = 0$. In the same way, one shows that, for every $k \in \{1, \ldots, q\}$, there exists a $b_k \in \mathbb{C}$ such that $f_k(z) = f_k(0) + b_k z^{d_k}$ in a neighborhood of 0. It follows that, in a neighborhood of 0,

$$
\chi_{A + zB}(X) = \prod_{k=1}^{q} (X - f_k(0) - b_k z)^{d_k}.
$$

Therefore we found affine maps $g_1, \ldots, g_n$ from $\mathbb{C}$ to $\mathbb{C}$ such that, in a neighborhood of 0,

$$
\chi_{A + zB}(X) = \prod_{k=1}^{n} (X - g_k(z)).
$$

The coefficients of these polynomials are polynomial functions of $z$ that coincide on a neighborhood of 0; therefore

$$
\forall z \in \mathbb{C}, \chi_{A + zB}(X) = \prod_{k=1}^{n} (X - g_k(z)).
$$

The pair $(A, B)$ has property L, and Proposition 5 is proven. \qed

2.4. **Commutativity for subgroups of diagonalizable matrices with an integral spectrum.** Given a matrix $M \in \mathfrak{M}_n(\mathbb{C})$ and an eigenvalue $\lambda$ of it, recall that the **eigenprojection** of $M$ associated to $\lambda$ is the projection onto $\text{Ker}(M - \lambda I_n)^n$ alongside $\text{im}(M - \lambda I_n)^n = \sum_{\mu \in \text{Sp}(M), \mu \neq \lambda} \text{Ker}(M - \mu I_n)^n$.

Here, we derive Proposition 5 from Proposition 5. We start by explaining how Kato’s proof [8, p. 85, Theorem 2.6] of the Motzkin-Taussky theorem [12] leads to the following refinement.

**Theorem 6 (Refined Motzkin-Taussky theorem).** Let $(A, B) \in \mathfrak{M}_n(\mathbb{C})^2$ be a pair of matrices which satisfies property L. Assume that $B$ is diagonalizable and that $A + z_0B$ is diagonalizable for every exceptional point $z_0$ of the matrix pencil $z \mapsto A + zB$. Then $AB = BA$.

**Proof.** We refer to the line of reasoning of [8, p. 85, Theorem 2.6] and explain how it may be adapted to prove Theorem 6. Denote by $p$ the generic number of eigenvalues of $z \mapsto A + zB$, and by $f_1, \ldots, f_p$ the $p$ distinct affine maps such that
∀z ∈ C, Sp(A + zB) = \{f_1(z), \ldots, f_p(z)\}. Denote by Ω the (open) set of regular points of z → A + zB, i.e.

Ω = C \ − \ \{z ∈ C : \exists(i,j) ∈ \{1, \ldots, p\}^2 : i ≠ j and f_i(z) = f_j(z)\}.

For z ∈ Ω and i ∈ \{1, \ldots, p\}, denote by Π_i(z) the eigenprojection of A + zB associated to the eigenvalue f_i(z). Then z → Π_i(z) is holomorphic on Ω for any i ∈ \{1, \ldots, p\} (see [S II.1.4]). Let z_0 ∈ C \ − \ Ω. Then A + z_0B is diagonalizable and hence [S p. 82, Theorem 2.3] shows that z_0 is a regular point for each map z → Π_i(z). We deduce that the functions (Π_i)_i≤p are restrictions of entire functions. Since B is diagonalizable, these functions are bounded at infinity (see the last paragraph of [S p. 85]) and Liouville’s theorem yields that they are constant. By a classical continuity argument (see [8, II.1.4, formula (1.16)]), we deduce that each eigenprojection of B is a sum of some projections chosen among the (Π_i(0))_i≤p. As B is diagonalizable, it is a linear combination of the (Π_i(0))_i≤p, which all commute with A + zB for any regular z. Therefore AB = BA. □

We now turn to the proof of Proposition 3.

Proof of Proposition 3. Let (A, B) ∈ M_n(C)^2. Assume that, for every (k, l) ∈ \mathbb{Z}^2, the matrix kA + lB is diagonalizable and Sp(kA + lB) ⊂ Z. Proposition 3 then shows that (A, B) has property L. For k ∈ \{1, \ldots, n\}, choose f_k : (y, z) → α_ky + β_kz such that

∀(y, z) ∈ C^2, OSp(yA + zB) = \{f_k(y, z)\}_1≤k≤n.

Since Sp(A) = \{α_1, \ldots, α_n\} and Sp(B) = \{β_1, \ldots, β_n\}, the families (α_k)_k≤n and (β_k)_k≤n are made of integers. It follows that the exceptional points of the matrix pencil z → A + zB are rational numbers. As the matrix A + \frac{k}{l}B = \frac{k}{l}(kA + lB) is diagonalizable for every (k, l) ∈ (\mathbb{Z} \ − \ \{0\}) \ × \ \mathbb{Z}, the refined Motzkin-Taussky theorem implies that AB = BA. □

We now deduce the following special case of Theorem 1.

Lemma 7. Let (A, B) ∈ M_n(C)^2 be such that for all (k, l) ∈ \mathbb{Z}^2, e^{kA+lB} = I_n. Then AB = BA.

Proof. Recall that the solutions of the equation e^M = I_n are the diagonalizable matrices M such that Sp(M) ⊂ 2iπZ (see [S Theorem 1.27]). In particular, for every (k, l) ∈ \mathbb{Z}^2, the matrix kA + lB is diagonalizable and Sp(kA + lB) ⊂ 2iπZ. Setting A′ := \frac{1}{2iπ} A and B′ := \frac{1}{2iπ} B, we deduce that (A′, B′) satisfies the assumptions of Proposition 3. It follows that A′B′ = B′A′, and hence AB = BA. □

3. The case Sp(A) ⊂ 2iπZ and Sp(B) ⊂ 2iπZ in Theorem 1

Proposition 8. Let (A, B) ∈ M_n(C)^2 be such that ∀(k, l) ∈ \mathbb{Z}^2, e^{kA+lB} = e^{kA}e^{lB}, Sp(A) ⊂ 2iπZ and Sp(B) ⊂ 2iπZ. Then AB = BA.

Proof. We consider the Jordan-Chevalley decompositions A = D + N and B = D′ + N′, where D and D′ are diagonalizable, N and N′ are nilpotent and DN = ND and D′N′ = N′D′. Clearly, for every integer k, kA = kD + kN (resp. kB = kD′ + kN′) is the Jordan-Chevalley decomposition of kA (resp. of kB), and Sp(kD) = Sp(kA) = k Sp(A) ⊂ 2iπZ (resp. Sp(kD′) = Sp(kB) = k Sp(B) ⊂ 2iπZ). This shows that

e^{kA} = e^{kN} and e^{kB} = e^{kN′}.  

Condition (6) may be written as

\[ \forall (k, l) \in \mathbb{Z}^2, \quad e^{kA+lB} = e^{kN}e^{lN'} = e^{lN'}e^{kN}. \]

Note in particular that \( e^N \) and \( e^{N'} \) commute. Since \( N \) is nilpotent, we have

\[ N = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} (e^N - I_n)^k. \]

That shows that \( N \) is a polynomial in \( e^N \). Similarly \( N' \) is a polynomial in \( e^{N'} \). Therefore,

\[ NN' = N'N. \]

The above condition yields

\[ \forall (k, l) \in \mathbb{Z}^2, \quad e^{kA+lB} = e^{kN+lN'}. \]

For any \((k, l) \in \mathbb{Z}^2, kN + lN' \) is nilpotent since \( N \) and \( N' \) commute. Hence \( kN + lN' \) is a polynomial in \( e^{kN+lN'} \). Since \( kA + lB \) commute with \( e^{kA+lB} \), it commutes with \( kN + lN' \). Therefore

\[ e^{kD+lD'} = e^{kA+lB}e^{-kN-lN'} = I_n. \]

In particular, this yields that \( kD + lD' \) is diagonalizable with \( \text{Sp}(kD + lD') \subset 2i\pi \mathbb{Z} \), and the Jordan-Chevalley decomposition of \( kA + lB \) is \( kA + lB = (kD + lD') + (kN + lN') \) as \( kN + lN' \) commute with \( kA + lB \).

By Lemma \[ \text{7} \] the matrices \( D \) and \( D' \) commute. In particular \((D, D')\) has properties \( L \), which yields affine maps \( f_1, \ldots, f_n \) from \( \mathbb{C} \) to \( \mathbb{C} \) such that

\[ \forall z \in \mathbb{C}, \quad \text{OSp}(D + zD') = \left[ f_k(z) \right]_{1 \leq k \leq n}. \]

The set

\[ E := \{ k \in \mathbb{Z} : \exists (i, j) \in \{1, \ldots, n\}^2 : f_i \neq f_j \text{ and } f_i(k) = f_j(k) \} \]

is clearly finite. We may choose two distinct elements \( a \) and \( b \) in \( \mathbb{Z} \setminus E \). The following equivalence holds:

\[ \forall (i, j) \in \{1, \ldots, n\}^2, \quad f_i(a) = f_j(a) \iff f_i = f_j \iff f_i(b) = f_j(b). \]

Since \( D \) and \( D' \) are simultaneously diagonalizable, it easily follows that \( D + aD' \) is a polynomial in \( D + bD' \) and conversely \( D + bD' \) is a polynomial in \( D + aD' \). Hence \( N + aN' \) and \( N + bN' \) both commute with \( D + aD' \) and \( D + bD' \). Since \( N + aN' \) and \( N + bN' \) commute with one another, we deduce that \( A + aB = (D + aD') + (N + aN') \) commutes with \( A + bB = (D + bD') + (N + bN') \). Since \( a \neq b \), we conclude that \( AB = BA \). \( \square \)

4. Proofs of Theorem \[ \text{1} \] and Proposition \[ \text{2} \]

**Definition 2.** Let \((A, B) \in M_n(\mathbb{C})^2 \).

i) \((A, B)\) is said to be decomposable if there exists a non-trivial decomposition \( \mathbb{C}^n = F \oplus G \) in which \( F \) and \( G \) are invariant linear subspaces for both \( A \) and \( B \).

ii) In the sequel, we consider, for \( k \in \mathbb{N} \setminus \{0\} \), the function

\[ \gamma_k : (\lambda, \mu) \in \text{Sp}(e^A) \times \text{Sp}(e^B) \mapsto \lambda^k \mu \in \mathbb{C}. \]

iii) For \( \lambda \in \mathbb{C} \), we denote by \( C_\lambda(M) \) the characteristic subspace of \( M \) with respect to \( \lambda \), i.e. \( C_\lambda(M) = \text{Ker}(M - \lambda I_n)^n \).
Lemma 9. Assume that A satisfies Condition
\[ \forall (\lambda, \mu) \in \text{Sp}(A)^2, \lambda - \mu \in 2i\pi \mathbb{Q} \Rightarrow \lambda - \mu \in 2i\pi \mathbb{Z}. \]
Then there exists \( k \in \mathbb{N} \setminus \{0\} \) such that \( \gamma_k \) is one-to-one.

Proof. Assume that for every \( k \in \mathbb{N} \setminus \{0\} \), there are distinct pairs \((\lambda, \mu)\) and \((\lambda', \mu')\) in \( \text{Sp}(e^A) \times \text{Sp}(e^B) \) such that \( \lambda^k \mu = (\lambda')^k \mu' \). Since \( \text{Sp}(e^A) \times \text{Sp}(e^B) \) is finite and \( \mathbb{N} \setminus \{0\} \) is infinite, we may then find distinct pairs \((\lambda, \mu)\) and \((\lambda', \mu')\) in \( \text{Sp}(e^A) \times \text{Sp}(e^B) \) and distinct non-zero integers \( a \) and \( b \) such that
\[ \lambda^a \mu = (\lambda')^a \mu' \quad \text{and} \quad \lambda^b \mu = (\lambda')^b \mu'. \]

All those eigenvalues are non-zero and \( \left( \frac{\lambda}{\lambda'} \right)^{a-b} = 1 \) with \( a \neq b \). It follows that \( \frac{\lambda}{\lambda'} \) is a root of unity. However \( \lambda = e^\alpha \) and \( \lambda' = e^\beta \) for some \((\alpha, \beta) \in \text{Sp}(A)^2\), which shows that \( (a-b)(\alpha - \beta) \in 2i\pi \mathbb{Z} \); hence \( \lambda = \lambda' \). It follows that \( \mu = \mu' \), in contradiction with \((\lambda, \mu) \neq (\lambda', \mu')\). \( \square \)

Lemma 10. Assume that \( \gamma_1 \) is one-to-one and that \((A, B)\) satisfies Equality \( (5) \) (resp. Equality \( (4) \)). Then the characteristic subspaces of \( e^A \) and \( e^B \) are stabilized by A and B.

Proof. Notice that \( A + B \) commutes with \( e^{A+B} \), hence commutes with \( e^A e^B \). It thus stabilizes the characteristic subspaces of \( e^A e^B \). Let us show that
\[ \forall \mu \in \text{Sp}(e^B), \ C_\mu(e^B) = \bigoplus_{\lambda \in \text{Sp}(e^A)} C_{\lambda \mu}(e^A e^B). \]

- Since \( e^B \) and \( e^A \) commute, \( e^A \) stabilizes the characteristic subspaces of \( e^B \). Considering the characteristic subspaces of the endomorphism of \( C_\mu(e^B) \) induced by \( e^A \), we find
\[ \forall \mu \in \text{Sp}(e^B), \ C_\mu(e^B) = \bigoplus_{\lambda \in \text{Sp}(e^A)} \left[ C_\lambda(e^A) \cap C_\mu(e^B) \right]. \]

- Let \((\lambda, \mu) \in \text{Sp}(e^A) \times \text{Sp}(e^B)\). Since \( e^A \) and \( e^B \) commute, they both stabilize \( C_\lambda(e^A) \cap C_\mu(e^B) \) and induce simultaneously triangularizable endomorphisms of \( C_\lambda(e^A) \cap C_\mu(e^B) \) each with a sole eigenvalue, respectively \( \lambda \) and \( \mu \): it follows that
\[ C_\lambda(e^A) \cap C_\mu(e^B) \subset C_{\lambda \mu}(e^A e^B). \]

- Finally, the application \((\lambda, \mu) \mapsto \lambda \mu \) is one-to-one on \( \text{Sp}(e^A) \times \text{Sp}(e^B) \). Therefore
\[ C_{\lambda \mu}(e^A e^B) \cap C_{\lambda' \mu'}(e^A e^B) = \{0\} \]
for all distinct pairs \((\lambda, \mu)\) and \((\lambda', \mu')\) in \( \text{Sp}(e^A) \times \text{Sp}(e^B) \).

One has
\[ \mathbb{C}^n = \bigoplus_{\mu \in \text{Sp}(e^B)} C_\mu(e^B) = \bigoplus_{\mu \in \text{Sp}(e^B)} \bigoplus_{\lambda \in \text{Sp}(e^A)} \left[ C_\lambda(e^A) \cap C_\mu(e^B) \right] \]
and \( \mathbb{C}^n \) is the sum of all the characteristic subspaces of \( e^A e^B \). We deduce that
\[ \forall (\lambda, \mu) \in \text{Sp}(e^A) \times \text{Sp}(e^B), \ C_{\lambda \mu}(e^A e^B) = C_\lambda(e^A) \cap C_\mu(e^B). \]

This gives Equality \( (3) \).

We deduce that \( A + B \) stabilizes every characteristic subspace of \( e^B \). However this is also true of \( B \) since it commutes with \( e^B \). Hence both \( A \) and \( B \) stabilize the characteristic subspaces of \( e^B \). Symmetrically, every characteristic subspace of \( e^A \) is stabilized by both \( A \) and \( B \). \( \square \)
Proof of Theorem 1 and Proposition 4. We use an induction on \( n \). Both Theorem 1 and Proposition 4 obviously hold for \( n = 1 \), so we fix \( n \geq 2 \) and assume that they hold for any pair \((A, B) \in M_k(\mathbb{C})^2\) with \( k \in \{1, \ldots, n - 1\} \). Let \((A, B) \in M_n(\mathbb{C})^2\) satisfy Equality (5) (resp. Equality (4)). Assume first that \((A, B)\) is decomposable. Then there exists \( p \in \{1, \ldots, n - 1\} \), a non-singular matrix \( P \in \text{GL}_n(\mathbb{C}) \) and square matrices \( A_1, B_1, A_2, B_2 \) respectively in \( M_p(\mathbb{C}) \), in \( M_p(\mathbb{C}) \), in \( M_{n-p}(\mathbb{C}) \) and in \( M_{n-p}(\mathbb{C}) \) such that

\[
A = P \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} P^{-1} \quad \text{and} \quad B = P \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} P^{-1}.
\]

Since the pair \((A, B)\) satisfies Equality (5) (resp. Equality (4)), it easily follows that this is also the case of \((A_1, B_1)\) and \((A_2, B_2)\); hence the induction hypothesis yields that \((A_1, B_1)\) and \((A_2, B_2)\) are commuting pairs (resp. have property L). Therefore \((A, B)\) is also a commuting pair (resp. has property L).

From that point on, we assume that \((A, B)\) is indecomposable. We may also assume that \( A \) satisfies Condition (7). Indeed, consider in general the finite set

\[
\mathcal{E} := \mathbb{Q} \cap \frac{1}{2i\pi} \{ \lambda - \mu \mid (\lambda, \mu) \in \text{Sp}(A)^2 \}.
\]

Since its elements are rational numbers, we may find some integer \( p > 0 \) such that \( p\mathcal{E} \subset \mathbb{Z} \). Replacing \( A \) with \( pA \), we notice that \((pA, B)\) still satisfies Equality (5) (resp. Equality (4)) and that it is a commuting pair (resp. satisfies property L) if and only if \((A, B)\) is a commuting pair (resp. satisfies property L).

Assume now that \( A \) satisfies Condition (7) as well as all the previous assumptions; i.e. \((A, B)\) is indecomposable and satisfies Equality (5) (resp. Equality (4)). By Lemma 9 we may choose \( k \in \mathbb{N} \setminus \{0\} \) such that \( \gamma_k \) is one-to-one. Replacing \( A \) with \( kA \), we lose no generality in assuming that \( \gamma_1 \) is one-to-one.

We can conclude: if \( e^B \) has several eigenvalues, Lemma 10 contradicts the assumption that \((A, B)\) is indecomposable. It follows that \( e^B \) has a sole eigenvalue, and for the same reason this is also true of \( e^A \). Choosing \( (\alpha, \beta) \in \mathbb{C}^2 \) such that \( \text{Sp}(e^A) = \{e^\alpha\} \) and \( \text{Sp}(e^B) = \{e^\beta\} \), we find that \( \exp(A - \alpha I_n) \) and \( \exp(B - \beta I_n) \) both have 1 as their sole eigenvalue. We deduce that \( \text{Sp}(A - \alpha I_n) \subset 2i\pi\mathbb{Z} \) and \( \text{Sp}(B - \beta I_n) \subset 2i\pi\mathbb{Z} \). Set \( A' := A - \alpha I_n \) and \( B' := B - \beta I_n \). We now conclude the proofs of Theorem 1 and Proposition 4 by considering the two cases separately.

• Case 1. \((A, B)\) satisfies Equality (5). The pair \((A', B')\) clearly satisfies Equality (5). Proposition 3 yields that \( A' \) commutes with \( B' \); hence \( AB = BA \).

• Case 2. \((A, B)\) satisfies Equality (4). The pair \((A', B')\) obviously satisfies Equality (4). The matrices \( e^{A'} \) and \( e^{B'} \) commute and are therefore simultaneously triangularizable (see Theorem 1.1.5)). Moreover, they have 1 as their sole eigenvalue. Therefore \( e^{kA' + B'} = (e^{A'})^k e^{B'} \) has 1 as sole eigenvalue for every \( k \in \mathbb{N} \). Proposition 3 shows that \( \left( \frac{1}{2i\pi} A', \frac{1}{2i\pi} B' \right) \) has property L, which clearly entails that \((A, B)\) has property L.

Thus Theorem 1 and Proposition 4 are proven. \( \square \)
5. ADDITIVE SEMIGROUPS ON WHICH THE EXPONENTIAL IS A HOMOMORPHISM

Notation 3. We denote by $\mathbb{Q}_+$ the set of non-negative rational numbers.

Definition 4. A linear subspace $V$ of $M_n(\mathbb{C})$ has property L when there are $n$ linear forms $f_1, \ldots, f_n$ on $V$ such that

$$\forall M \in V, \text{OSp}(M) = [f_k(M)]_{1 \leq k \leq n}.$$ 

In this short section, we prove the following result.

Theorem 11. Let $S$ be a subsemigroup of $(M_n(\mathbb{C}), +)$ and assume that $M \mapsto \exp(M)$ is a homomorphism from $(S, +)$ to $(\text{GL}_n(\mathbb{C}), \times)$. Then span$(S)$ has property L.

By Proposition 4, it suffices to establish the following lemma.

Lemma 12. Let $S$ be a subsemigroup of $(M_n(\mathbb{C}), +)$. Assume that every pair $(A, B) \in S^2$ has property L. Then the linear subspace span$(S)$ has property L.

Proof. Let $(A_1, \ldots, A_r)$ be a basis of span$(S)$ formed of elements of $S$. For every $j \in \{1, \ldots, r\}$, we choose a list $(a_1^{(j)}, \ldots, a_n^{(j)}) \in \mathbb{C}^n$ such that

$$\text{OSp}(A_j) = [a_k^{(j)}]_{1 \leq k \leq n}.$$ 

Since, for every $(p_1, \ldots, p_r) \in \mathbb{N}^r$, the pair $\left( \sum_{k=1}^{j-1} p_k A_k, A_j \right)$ has property L for every $j \in \{2, \ldots, r\}$, by induction we obtain a list $((\sigma_1, \ldots, \sigma_r) \in (\Sigma_n)^r$ such that

$$\text{OSp}\left( \sum_{j=1}^r p_j A_j \right) = \left[ \sum_{j=1}^r p_j a_{\sigma_j(k)}^{(j)} \right]_{1 \leq k \leq n}.$$ 

Multiplying by inverses of positive integers, we readily generalize this as follows: for every $(z_1, \ldots, z_r) \in (\mathbb{Q}_+)^r$, there exists a list $((\sigma_1, \ldots, \sigma_r) \in (\Sigma_n)^r$ such that

$$\text{OSp}\left( \sum_{j=1}^r z_j A_j \right) = \left[ \sum_{j=1}^r z_j a_{\sigma_j(k)}^{(j)} \right]_{1 \leq k \leq n}.$$ 

Now, we prove the following property, depending on $l \in \{0, \ldots, r\}$, by downward induction:

$$\mathcal{P}(l) : \text{For every } (z_1, \ldots, z_l) \in (\mathbb{Q}_+)^l, \text{ there exists a list } ((\sigma_1, \ldots, \sigma_r) \in (\Sigma_n)^r \text{ satisfying}$$

$$\forall (z_{l+1}, \ldots, z_r) \in \mathbb{C}^{r-l}, \text{OSp}\left( \sum_{j=1}^r z_j A_j \right) = \left[ \sum_{j=1}^r z_j a_{\sigma_j(k)}^{(j)} \right]_{1 \leq k \leq n}.$$ 

In particular, $\mathcal{P}(r)$ is precisely what we have just proven, whilst $\mathcal{P}(0)$ means that there exists a list $((\sigma_1, \ldots, \sigma_r) \in (\Sigma_n)^r \text{ such that, for every } (z_1, \ldots, z_r) \in \mathbb{C}^r,$

$$\text{OSp}\left( \sum_{j=1}^r z_j A_j \right) = \left[ \sum_{j=1}^r z_j a_{\sigma_j(k)}^{(j)} \right]_{1 \leq k \leq n}.$$ 

Therefore $\mathcal{P}(0)$ implies that span$(S)$ has property L.
Let \( l \in \{1, \ldots, r\} \) be such that \( \mathcal{P}(l) \) holds, and fix \((z_1, \ldots, z_{l-1}) \in (\mathbb{Q}_+)^{l-1}\). By \( \mathcal{P}(l) \), for every \( z_l \in \mathbb{Q}_+ \), we may choose a list \((\sigma_1^{z_l}, \ldots, \sigma_r^{z_l}) \in (\Sigma_n)^r \) such that
\[
\forall (z_{l+1}, \ldots, z_r) \in \mathbb{C}^{r-l}, \quad \text{OSp} \left( \sum_{j=1}^{r} z_j A_j \right) = \left[ \sum_{j=1}^{r} z_j a_{\sigma_j^{z_l}(k)}^{(j)} \right]_{1 \leq k \leq n}.
\]
Since \((\Sigma_n)^r\) is finite and \(\mathbb{Q}_+ \cap (0, 1)\) is infinite, some list \((\sigma_1, \ldots, \sigma_r) \in (\Sigma_n)^r\) equals \((\sigma_1^{z_1}, \ldots, \sigma_r^{z_1})\) for infinitely many values of \( z_l \) in \(\mathbb{Q}_+ \cap (0, 1)\). Fixing \((z_{l+1}, \ldots, z_r) \in \mathbb{C}^{r-l}\), we deduce the identity
\[
\forall z_l \in \mathbb{C}, \quad X \sum_{j=1}^{r} z_j A_j(X) = \prod_{k=1}^{n} \left( X - \sum_{j=1}^{r} z_j a_{\sigma_j^{z_l}(k)}^{(j)} \right)
\]
by remarking that, on both sides, the coefficients of the polynomials are polynomials in \( z_l \). Hence
\[
\forall (z_1, \ldots, z_r) \in \mathbb{C}^{r-l+1}, \quad \text{OSp} \left( \sum_{j=1}^{r} z_j A_j \right) = \left[ \sum_{j=1}^{r} z_j a_{\sigma_j^{z_l}(k)}^{(j)} \right]_{1 \leq k \leq n}.
\]
This proves that \( \mathcal{P}(l-1) \) holds.

\[ \square \]

Acknowledgement

The author would like to thank the referee for helping enhance the quality of this article in a very significant way.

References

[1] G. Bourgeois, On commuting exponentials in low dimensions, Linear Algebra Appl. 423 (2007) 277-286. MR2312407 (2008d:39030)
[2] G. Fischer. Plane Algebraic Curves, Student Mathematical Library, Volume 15, AMS, 2001. MR1836037 (2002g:14042)
[3] M. Fréchet, Les solutions non-commutables de l’équation matricielle \( e^{x+y} = e^x e^y \), Rend. Circ. Math. Palermo 2 (1952) 11-27. MR0049857 (14:237a)
[4] M. Fréchet, Les solutions non-commutables de l’équation matricielle \( e^{x+y} = e^x e^y \), Rectification, Rend. Circ. Math. Palermo 2 (1953) 71-72. MR0057836 (15:279h)
[5] N.J. Higham. Functions of Matrices. Theory and Computation, SIAM, 2008. MR2396439 (2009b:15001)
[6] C.W. Huff, On pairs of matrices (of order two) \( A, B \) satisfying the condition \( e^{A+B} = e^A e^B \neq e^B e^A \), Rend. Circ. Math. Palermo 2 (1953) 326-330. MR0062706 (16:4c)
[7] A.G. Kakar, Non-commuting solutions of the matrix equation \( \exp(X+Y) = \exp(X) \exp(Y) \), Rend. Circ. Math. Palermo 2 (1953) 331-345. MR0062708 (16:4e)
[8] T. Kato. Perturbation Theory for Linear Operators, Grundlehren der Mathematischen Wissenschaften, Second edition, Springer-Verlag, 1976. MR0407617 (53:11389)
[9] K. Morinaga, T. Nono, On the non-commutative solutions of the exponential equation \( e^{x+y} = e^{x+y} \), J. Sci. Hiroshima Univ. (A)17 (1954) 345-358. MR0066337 (16:558f)
[10] K. Morinaga, T. Nono, On the non-commutative solutions of the exponential equation \( e^{x+y} = e^{x+y} \), J. Sci. Hiroshima Univ. (A)18 (1954) 137-178. MR0072104 (17:228h)
[11] T.S. Motzkin, O. Taussky, Pairs of matrices with property L, Trans. Amer. Math. Soc. 73 (1952) 108-114. MR0049855 (14:236c)
[12] T.S. Motzkin, O. Taussky, Pairs of matrices with property L (II), Trans. Amer. Math. Soc. 80 (1955) 387-401. MR0067871 (19:242c)
[13] H. Radjavi, P. Rosenthal, Simultaneous Triangularization, Universitext, Springer-Verlag (2000). MR1736065 (2001e:47001)
[14] C. Schmoeger, Remarks on commuting exponentials in Banach algebras, Proc. Amer. Math. Soc. 127 (5) (1999) 1337-1338. MR1763911 (99h:46090)
[15] C. Schmoeger, Remarks on commuting exponentials in Banach algebras II, *Proc. Amer. Math. Soc.* 128 (11) (2000) 3405-3409. MR1691002 (2001b:46077)

[16] E.M.E. Wermuth, Two remarks on matrix exponentials, *Linear Algebra Appl.* 117 (1989) 127-132. MR993038 (90e:15019)

[17] E.M.E. Wermuth, A remark on commuting operator exponentials, *Proc. Amer. Math. Soc.* 125 (6) (1997) 1685-1688. MR1353407 (97g:39011)

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