Abstract. It is known that the long line supports $2^{\omega_1}$ many non-diffeomorphic differential structures. We show that the long plane supports a similar number of exotic differential structures, i.e., structures which are not merely diffeomorphic to the product of two structures on the factor spaces.

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1. Introduction

In this paper, by a differential structure we mean a $C^r$ differential structure for any $r \geq 1$. Recall, [3], that every $C^r$ structure contains a $C^s$ structure for any $s$ satisfying $r < s \leq \infty$. Hence we are not concerned about the value of $r$.

It is well-known that euclidean space, $\mathbb{R}^n$, possesses a unique differential structure up to diffeomorphism for $n \neq 4$ but $\mathbb{R}^4$ has $\epsilon$ mutually non-diffeomorphic differential structures; see [2, page 95] for early details and [5] for a recent survey. Thus $\mathbb{R}^4$ possesses many exotic differential structures, i.e., differential structures which are not diffeomorphic to the 4-fold product of $\mathbb{R}$ with the usual structure (or the 2-fold product of $\mathbb{R}^2$ with the usual structure). Of course exotic differential structures were discovered more than half a century ago by Milnor in [4] where there is given the first construction of a differential structure on the 7-sphere $S^7$ which is not diffeomorphic to the usual (product) differential structure inherited from $\mathbb{R}^8$. The existence of two mutually non-diffeomorphic differential structures on a manifold is not possible for metrisable manifolds in dimension up to 3, [5]: this result is due to Radó in dimensions 1 and 2 and Moise in dimension 3.

On the other hand in [8] it is shown that when we relax the metrisability condition then even in dimension 1 there are $2^{\omega_1}$ mutually non-diffeomorphic differential structures (on the long ray, hence also the long line $L$). As a result there are also $2^{\omega_1}$ mutually non-diffeomorphic differential structures on the long plane $L^2$. In this paper we address the question: does the long plane support differential structures which are not diffeomorphic to any product structure? Our answer is "yes."

As usual we denote by $\omega_1$ the set of countable ordinals with the order topology. Let $\mathbb{L}_{\geq 0}$ denote the closed long ray, ie the set $\omega_1 \times [0, 1)$ with the lexicographic order topology, and let $L$ denote the long line which is obtained from two copies of the closed long ray with their initial points identified to 0. The (open) long ray is the 1-manifold $L_+ = \mathbb{L}_{\geq 0} - \{(0, 0)\}$. Identify $\alpha \in \omega_1$ with $(\alpha, 0) \in \mathbb{L}_{\geq 0}$. We will exhibit non-product differential structures on $L^2_+$. As in [8] similar structures may then be deduced on $\mathbb{L}^2$.

The following result is well-known and is found in many books introducing Set Theory but we include it for completeness. Note that it does not matter whether we are considering $C$ and $D$ as subsets of $\omega_1$ or $\mathbb{L}^2$.

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Lemma 1. If $C,D \subset \omega_1$ are closed unbounded subsets then $C \cap D$ is also closed and unbounded.

Proposition 2. Suppose that $D$ is a differential structure on $\mathbb{L}_+$ and $\alpha \in \mathbb{L}_+$. Then $((\alpha, \omega_1), D((\alpha, \omega_1))$ is diffeomorphic to $(\mathbb{L}_+, D)$. 

Proof. Choose $\beta, \gamma \in \mathbb{L}_+$ such that $\alpha < \beta < \gamma$. Because $\mathbb{R}$ has a unique differential structure up to diffeomorphism and $(0, \gamma) \subset \mathbb{L}_+$ is homeomorphic to $\mathbb{R}$ we may choose a diffeomorphism $g : ((0, \gamma), D((0, \gamma))) \to ((0, 3), U)$, where $U$ is the usual differential structure on $\mathbb{R}$ restricted to $(0, 3)$. Furthermore we may assume that $g(\alpha) = 1$ and $g(\beta) = 2$. Next let $\theta : (0, 3) \to (1, 3)$ be a diffeomorphism (relative to $U$) such that $\theta(t) = t$ for all $t \in [2, 3)$. For example set $\theta(t) = \begin{cases} t + \sqrt{e^{1/t}} & \text{if } t < 2 \\ t & \text{if } t \geq 2 \end{cases}$. Now define $h : \mathbb{L}_+ \to (\alpha, \omega_1)$ by $h(t) = \begin{cases} g^{-1} \theta g(t) & \text{if } t < \gamma \\ t & \text{if } t > \beta \end{cases}$. Then $h$ is a diffeomorphism with respect to the structure $D$. \qed

Recall the following result from [1, Corollary 2.6].

Proposition 3. Suppose that $h : \mathbb{L}_+^2 \to \mathbb{L}_+^2$ is an orientation-preserving homeomorphism. Then 
\[ \{ \alpha \in \omega_1 \mid h(\mathbb{L}_+ \times \{ \alpha \}) = \mathbb{L}_+ \times \{ \alpha \} \} \] is a closed unbounded set.

Corollary 4. Suppose that $F$ is a differential structure on $\mathbb{L}_+^2$ and that $F$ is diffeomorphic to the product of two structures. Then 
\[ \{ \alpha \in \omega_1 \mid \mathbb{L}_+ \times \{ \alpha \} \} \text{ is a differentiable submanifold of } (\mathbb{L}_+^2, F) \]

and 
\[ \{ \alpha \in \omega_1 \mid \{ \alpha \} \times \mathbb{L}_+ \} \text{ is a differentiable submanifold of } (\mathbb{L}_+^2, F) \]

are closed unbounded subsets of $\omega_1$. 

Proof. There are two differential structures, say $D, E$, on $\mathbb{L}_+$ and a diffeomorphism $h : (\mathbb{L}_+, D) \times (\mathbb{L}_+, E) \to (\mathbb{L}_+^2, F)$. Interchanging the roles of $D$ and $E$ if necessary we may assume $h$ preserves orientation. By Proposition 3, $S = \{ \alpha \in \omega_1 \mid h(\mathbb{L}_+ \times \{ \alpha \}) = \mathbb{L}_+ \times \{ \alpha \} \}$ is a closed unbounded set. Thus $\mathbb{L}_+ \times \{ \alpha \}$ is a differentiable submanifold of $(\mathbb{L}_+^2, F)$ for each $\alpha \in S$. Interchanging the coordinates leads to the other half. \qed

We also require the following folklore result, cf [7, Theorem 1] and [6, Theorem 3 page 46].

Proposition 5. Let $M \subset \mathbb{R}^2$ be a compact topological manifold with boundary, $K \subset M$ a compact subset which contains the boundary of $M$ and suppose that $h : M \to M$ is a homeomorphism which is a diffeomorphism on a neighbourhood of $K$. Then $h$ can be approximated arbitrarily closely by a homeomorphism which is a diffeomorphism on $M$ and agrees with $h$ on a neighbourhood of $K$.

2. Exotic Differential Structures on $\mathbb{L}_+^2$ 

We now present a method of constructing from two differential structures on the long ray a differential structure on $\mathbb{L}_+^2$ which is not diffeomorphic to the product of any two differential structures on $\mathbb{L}_+$. The construction allows us to verify that there are $2^{\aleph_1}$ many non-diffeomorphic such structures.

We require an auxiliary shearing homeomorphism $\sigma : [0, 5]^2 \to [0, 5]^2$. The homeomorphism $\sigma$ is the identity except in the rectangle $(3, 4) \times (1, 4)$, does not change the first coordinate and maps the straight line segment $[3, 4] \times \{2\}$ onto the line segments $\{(x, 3 - 2|x - 3\frac{1}{2}|) : 3 \leq x \leq 4\}$. The notation $I_\alpha = (0, \alpha + 1)$, $\mathcal{I}_\alpha = [0, \alpha + 1]$, $O_\alpha = I_\alpha^2 \subset \mathbb{L}_+^2$ and $\mathcal{O}_\alpha = \mathcal{I}_\alpha^2 \subset \mathbb{L}_{\geq 0}^2$ is fixed throughout this section.
Begin with two differential structures \(D\) and \(E\) on \(\mathbb{L}_+\); for example any of those in [8] will do. For each \(\alpha \in \omega_1 \setminus \{0\}\) choose order-preserving homeomorphisms \(\psi_\alpha, \chi_\alpha : \mathcal{T}_\alpha \to [0,5]\) so that \(\psi_\alpha(\alpha) = \chi_\alpha(\alpha) = 2\), and that \((I_\alpha, \psi_\alpha) \in D\) and \((I_\alpha, \chi_\alpha) \in E\).

For each \(\alpha \in \omega_1 \setminus \{0\}\) we will construct by induction on \(\alpha\) a homeomorphism \(\varphi_\alpha : \overline{O_\alpha} \to [0,5]^2\) in such a way that \(\{(O_\alpha, \varphi_\alpha) / \alpha \in \omega_1 \setminus \{0\}\}\) is a basis for a differential structure on \(\mathbb{L}_+\), i.e., for each \(\alpha, \beta \in \omega_1 \setminus \{0\}\) the maps \(\varphi_\alpha \varphi_\beta^{-1}\) and \(\varphi_\beta \varphi_\alpha^{-1}\) are smooth where defined within \((0,5)^2\). The induction includes the further condition:

- The homeomorphisms \(\psi_\alpha \times \chi_\alpha\) and \(\varphi_\alpha\) agree on neighbourhoods of \(\overline{O_\alpha} \setminus O_\alpha\) and of \([0, \alpha) \times [\alpha, \alpha + 1]\) as well as on a neighbourhood in \(\overline{O_\alpha} \setminus [0, \alpha]^2\) of \([\alpha] \times [0, \alpha]\).

\[\begin{array}{c}
\hspace{1cm} \alpha + 1 \\
\alpha \\
\hspace{1cm} \alpha \\
0 \\
\end{array}\]

\[\begin{array}{c}
\hspace{1cm} \overline{O_\alpha} \\
\end{array}\]

\[\text{Figure 1. Where } \varphi_\alpha = \psi_\alpha \times \chi_\alpha.\]

**Definition of } \varphi_1:-\text{ Set } \varphi_1 = \sigma(\psi_1 \times \chi_1).\)**

**Definition of } \varphi_{\alpha+1} \text{ given } \varphi_\alpha:-\text{ Define}\)**

\[
\varphi_{\alpha+1}(z) = \begin{cases} 
(\psi_{\alpha+1} \times \chi_{\alpha+1})(\psi_\alpha \times \chi_\alpha)^{-1}\varphi_\alpha(z) & \text{if } z \in \overline{O_\alpha}; \\
\sigma(\psi_{\alpha+1} \times \chi_{\alpha+1})(z) & \text{if } z \in \overline{O_{\alpha+1}} - O_{\alpha}.
\end{cases}
\]

It is easily checked that the inductive conditions are satisfied.

**Definition of } \varphi_\alpha, \text{ where } \alpha \text{ is a limit ordinal, given } \varphi_\beta \text{ for all } \beta \in \omega_1 \setminus \{0\} \text{ with } \beta < \alpha:-\text{ Firstly choose some metric } d \text{ on } \overline{O_\alpha} \text{ compatible with the topology. Next choose an increasing sequence } \langle \alpha_n \rangle \text{ from } \omega_1 \setminus \{0\} \text{ converging to } \alpha; \text{ set } \alpha_0 = 0. \text{ Somewhat as in the previous case we would like to let } \varphi_\alpha \text{ be } (\psi_\alpha \times \chi_\alpha)(\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}\varphi_{\alpha_n} \text{ on } \overline{O_{\alpha_n}} \text{ and be } \sigma(\psi_\alpha \times \chi_\alpha) \text{ outside the union of these common domains but this would work only if all maps of the form } (\psi_{\alpha_m} \times \chi_{\alpha_m})^{-1}\varphi_{\alpha_m} \text{ and } (\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}\varphi_{\alpha_n} \text{ agree on } \overline{O_{\alpha_{\min(m,n)}}}. \text{ We modify these maps inductively so that they do agree, at least on enough of } \overline{O_{\alpha_{\min(m,n)}}}. \text{ To effect this we construct a sequence of homeomorphisms } \langle h_n : [0,3]^2 \to [0,3]^2 \rangle, \text{ where } n \geq 1. \text{ We demand the following properties:}\)**

- \(h_n : (0,3)^2 \to (0,3)^2\) is a diffeomorphism;
- \(h_n\) is the identity on a neighbourhood of \(((0,3] \times [2,3]) \cup ([2,3] \times [0,3])\);
- \(h_n = (\psi_{\alpha_n} \times \chi_{\alpha_n})(\psi_{\alpha_{n-1}} \times \chi_{\alpha_{n-1}})^{-1}\varphi_{\alpha_{n-1}} \varphi_{\alpha_n}^{-1}\) on a neighbourhood of \(\varphi_{\alpha_n}([0, \alpha_{n-1}^2])\) when \(n > 1\);
- \(h_n = (\psi_{\alpha_n} \times \chi_{\alpha_n})\varphi_{\alpha_n}^{-1}\) on a neighbourhood of \(\varphi_{\alpha_n}([0, \alpha_{n-1}) \times \alpha_{n-1})\) when \(n > 1\);
for $n > 1$, on $\varphi_{\alpha_m}(\alpha_{n-1}, \alpha_n] \times [0, \alpha_{n-2}])$, $h_n$ is sufficiently close to $(\psi_{\alpha_n} \times \chi_{\alpha_n})\varphi^{-1}_{\alpha_n}$ that for any $(x, y) \in \varphi_{\alpha_m}([\alpha_{n-1}, \alpha_n] \times [0, \alpha_{n-2}])$, we have

$$d\left((\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}(x, y), (\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}h_n(x, y)\right) < \frac{1}{n}.$$ 

To achieve this we use uniform continuity of $(\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}$.

Notice that the conditions on $h_n$ are mutually consistent by the inductively assumed conditions on $\varphi_{\alpha_m}$ and $h_{n-1}$.

![Figure 2. Constraints on $h_n$.](image)

Let $h_1 : [0, 3]^2 \rightarrow [0, 3]^2$ be the identity. Suppose given $n > 1$ such that $h_{n-1}$ has been defined. By Proposition 5 there is a homeomorphism $h_n : [0, 3]^2 \rightarrow [0, 3]^2$ satisfying the requirements.

Now define $\varphi_{\alpha}$ by

$$\varphi_{\alpha}(x) = \begin{cases} (\psi_{\alpha_n} \times \chi_{\alpha_n})(\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}h_n\varphi_{\alpha_n}(x) & \text{if } x \in [0, \alpha_n] \text{ for some } n \\ \sigma(\psi_{\alpha} \times \chi_{\alpha})(x) & \text{if } x \in \overline{O_{\alpha}} \setminus (0, \alpha)^2 \end{cases}.$$ 

The function $\varphi_{\alpha}$ is well-defined because if $m < n$ then $(\psi_{\alpha_m} \times \chi_{\alpha_m})^{-1}h_m\varphi_{\alpha_m}$ and $(\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}h_n\varphi_{\alpha_n}$ agree on $[0, \alpha_m]^2$. It is easily verified that $\varphi_{\alpha}$ is a homeomorphism, the main problem being to verify continuity on $\{\alpha\} \times [0, \alpha]$. It is here that we require precision in the approximation of the homeomorphism $(\psi_{\alpha_n} \times \chi_{\alpha_n})\varphi^{-1}_{\alpha_n}$ by the diffeomorphism as required in the last inductive assumption for $h_n$. The approximation must improve as $n$ increases so that any sequence $(x_n, y_n)$ in $[0, \alpha)^2$ converging to $(\alpha, y) \in \{\alpha\} \times [0, \alpha]$ is mapped by $(\psi_{\alpha_m} \times \chi_{\alpha_m})^{-1}h_m\varphi_{\alpha_m}$ (m increasing with $n$) to a sequence which also converges to $(\alpha, y)$. Then $\varphi_{\alpha}(x_n, y_n) \rightarrow (\psi_{\alpha} \times \chi_{\alpha})(\alpha, y) = \varphi(\alpha, y)$ as $\sigma$ is the identity on $\{2\} \times [0, 5]$.

Suppose $\beta < \alpha$. Then the coordinate transformation between $\varphi_{\alpha}$ and $\varphi_{\beta}$ is smooth on $\varphi_{\beta}(O_{\beta+1})$ because

$$\varphi_{\alpha}\varphi_{\beta}^{-1} = (\psi_{\alpha} \times \chi_{\alpha})(\psi_{\alpha_n} \times \chi_{\alpha_n})^{-1}h_n\varphi_{\alpha_n}\varphi_{\beta}^{-1}$$

is a composition of coordinate transition functions together with the diffeomorphism $h_n$ and hence is smooth, where $n$ is chosen so that $\alpha_n > \beta$. Similarly its inverse is smooth.

The remaining condition demanded of $\varphi_{\alpha}$ is also satisfied.
Thus we have constructed a basis \( \{ (O_\alpha, \varphi_\alpha) / \alpha \in \omega_1 \setminus \{0\} \} \) for a differential structure on \( \mathbb{L}_+^2 \). Call this structure \( \mathcal{F} \).

**Claim 6.** The differential structure \( \mathcal{F} \) is not diffeomorphic to a product of structures on \( \mathbb{L}_+^2 \).

**Proof.** Let \( \alpha < \omega_1 \) be any non-zero ordinal. We first show that \( \mathbb{L}_+ \times \{ \alpha \} \) is not a smooth submanifold of \( (\mathbb{L}_+^2, \mathcal{F}) \). Suppose instead that \( \mathbb{L}_+ \times \{ \alpha \} \) is a smooth submanifold of \( (\mathbb{L}_+^2, \mathcal{F}) \). Then \( (\alpha, \alpha + 1) \times \{ \alpha \} \) is also a smooth submanifold of \( (\mathbb{L}_+^2, \mathcal{F}) \), so there is a chart \( ((\alpha, \alpha + 1) \times (0, \alpha + 1), \varphi) \in \mathcal{F} \) such that \( \varphi^{-1}(\mathbb{R} \times \{0\}) = (\alpha, \alpha + 1) \times \{ \alpha \} \). It follows that \( \varphi_\alpha \varphi^{-1}(\mathbb{R} \times \{0\}) = \varphi_\alpha((\alpha, \alpha + 1) \times \{ \alpha \}) \) is a smooth submanifold of \( \mathbb{R}^2 \) with the usual differential structure. However, for \( t \in (\alpha, \alpha + 1) \) we have \( \varphi_\alpha(t, \alpha) = \sigma(\psi_\alpha(t), 2) \), and hence
\[
\varphi_\alpha((\alpha, \alpha + 1) \times \{ \alpha \}) = ((2, 3] \cup [4, 5]) \times \{2\} \cup \left\{ \left(x, 3 - 2 \left\lfloor x - 3 \frac{1}{2} \right\rfloor : 3 \leq x \leq 4 \right) \right\}.
\]

As this last set is not a smooth submanifold of \( \mathbb{R}^2 \), it follows that \( (0, \omega_1) \times \{ \alpha \} \) is not a smooth submanifold of \( (\mathbb{L}_+^2, \mathcal{F}) \).

The claim now follows from Lemma 1 and Corollary 4 because \( \omega_1 \setminus \{0\} \) is closed and unbounded. \( \square \)

We now address the question: how many exotic differential structures does \( \mathbb{L}_+^2 \) support? The argument presented in [8, p.156] that \( \mathbb{L}_+ \) supports no more than \( 2^{\aleph_1} \) many mutually non-diffeomorphic differential structures applies as well to \( \mathbb{L}_+^2 \). On the other hand [8, Theorem 5.2] exhibits exactly \( 2^{\aleph_1} \) many mutually non-diffeomorphic differential structures on \( \mathbb{L}_+^2 \). Thus we might expect to find \( 2^{\aleph_1} \) many mutually non-diffeomorphic exotic differential structures on \( \mathbb{L}_+^2 \), and this is indeed the case.

Let \( \mathcal{D} \) be any differential structure on \( \mathbb{L}_+^2 \). Apply the construction above with \( \mathcal{E} = \mathcal{D} \) and denote the resulting exotic differential structure on \( \mathbb{L}_+^2 \) by \( \hat{\mathcal{D}} \).

**Theorem 7.** There are \( 2^{\aleph_1} \) mutually non-diffeomorphic exotic differential structures on \( \mathbb{L}_+^2 \).

**Proof.** Suppose given differential structures \( \mathcal{D} \) and \( \mathcal{E} \) on \( \mathbb{L}_+^2 \) and an orientation-preserving diffeomorphism \( h : (\mathbb{L}_+^2, \hat{\mathcal{D}}) \to (\mathbb{L}_+^2, \hat{\mathcal{E}}) \). By Proposition 3, for a closed unbounded set of \( \alpha \in \omega_1 \), the map \( h \) restricts to a homeomorphism taking \( \{ \alpha \} \times (\alpha, \omega_1) \) to itself. Now \( \hat{\mathcal{D}} \{ \alpha \} \times (\alpha, \omega_1) \) is the same as \( \mathcal{D} \times \mathcal{D} \{ \alpha \} \times (\alpha, \omega_1) \) with the same for \( \mathcal{E} \) so, using Proposition 2 and denoting “is diffeomorphic to” by \( \approx \), we have
\[
(\mathbb{L}_+, \mathcal{D}) \approx (\{ \alpha \} \times (\alpha, \omega_1), \mathcal{D} \times \mathcal{D} \{ \alpha \} \times (\alpha, \omega_1)) \approx (\{ \alpha \} \times (\alpha, \omega_1), \hat{\mathcal{D}} \{ \alpha \} \times (\alpha, \omega_1)) \approx (\{ \alpha \} \times (\alpha, \omega_1), \hat{\mathcal{E}} \{ \alpha \} \times (\alpha, \omega_1)) \approx (\{ \alpha \} \times (\alpha, \omega_1), \mathcal{E} \times \mathcal{E} \{ \alpha \} \times (\alpha, \omega_1)) \approx ((\alpha, \omega_1), \mathcal{E}) \approx (\mathbb{L}_+, \mathcal{E}).
\]

It follows that for any differential structure \( \mathcal{D} \) on \( \mathbb{L}_+^2 \) there can be at most one equivalence class of structures, represented say by \( \mathcal{E} \), such that \( (\mathbb{L}_+^2, \hat{\mathcal{D}}) \) is diffeomorphic to \( (\mathbb{L}_+^2, \hat{\mathcal{E}}) \) but \( (\mathbb{L}_+^2, \mathcal{D}) \) is not diffeomorphic to \( (\mathbb{L}_+^2, \mathcal{E}) \). Indeed, if \( \mathcal{D}, \mathcal{E} \) and \( \mathcal{F} \) are three differential structures on \( \mathbb{L}_+^2 \) and \( (\mathbb{L}_+^2, \hat{\mathcal{D}}) \) is diffeomorphic to both \( (\mathbb{L}_+^2, \hat{\mathcal{E}}) \) and \( (\mathbb{L}_+^2, \hat{\mathcal{F}}) \) but \( (\mathbb{L}_+^2, \mathcal{D}) \) is not diffeomorphic to either \( (\mathbb{L}_+^2, \mathcal{E}) \) or \( (\mathbb{L}_+^2, \mathcal{F}) \), then diffeomorphisms \( g : (\mathbb{L}_+^2, \hat{\mathcal{D}}) \to (\mathbb{L}_+^2, \hat{\mathcal{E}}) \) and \( h : (\mathbb{L}_+^2, \hat{\mathcal{D}}) \to (\mathbb{L}_+^2, \hat{\mathcal{F}}) \) must reverse
orientation. In that case the diffeomorphism \( hg^{-1} : (L^2_+, \mathcal{E}) \rightarrow (L^2_+, \bar{\mathcal{F}}) \) preserves orientation and hence \((L_+, \mathcal{E})\) is diffeomorphic to \((L_+, \bar{\mathcal{F}})\) by what we have already shown.

It now follows from [8, Theorem 5.2] that there are \(2^{\aleph_1}\) mutually non-diffeomorphic exotic differential structures on \(L^2_+\).

As a complement to Theorem 7 we have the following.

**Theorem 8.** There are \(2^{\aleph_1}\) mutually non-diffeomorphic product differential structures on \(L^2_+\).

**Proof.** It suffices to show that if \(D\) and \(E\) are two differential structures on \(L\) such that \((L^2, D \times D)\) is diffeomorphic to \((L^2, E \times E)\) then \((L, D)\) is diffeomorphic to \((L, E)\). However it is easy to show that the homeomorphism \((x, x) \mapsto x\) from the diagonal \(\Delta\) of \(L^2\) to \(L\) is a diffeomorphism from \((\Delta, D \times D|\Delta)\) to \((L, D)\).

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