THE BOREL COHOMOLOGY OF FREE ITERATED LOOP SPACES

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ABSTRACT. We compute the SO(n+1)-equivariant mod 2 Borel cohomology of the free iterated loop space $Z^{S^n}$ when $n$ is odd and $Z$ is a product of mod 2 Eilenberg Mac Lane spaces. When $n = 1$, this recovers Ottosen and Bökstedt’s computation for the free loop space. The highlight of our computation is a construction of cohomology classes using an $O(n)$-equivariant evaluation map and a pushforward map.

We then reinterpret our computation as giving a presentation of the zeroth derived functor of the Borel cohomology of $Z^{S^n}$ for arbitrary $Z$. We also include an appendix where we give formulas for computing the zeroth derived functor of the cohomology of mapping spaces, and study the dependence of such derived functors on the Steenrod operations.

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1. Introduction

Given spaces $Z, Y$, Jean Lannes found a way to approximate the mod $p$ cohomology of their mapping space $H^*(Z^Y)$, using the structure of $H^*(Y), H^*(Z)$ as objects in $\mathcal{K}$, where $\mathcal{K}$ is the category of unstable algebras over the Steenrod algebra $\mathcal{A}$.

Lannes considered a functor, called division by $H^*(Y)$ and denoted $(\cdot) : H^*(Y)$, obtained as the left adjoint to the endofunctor $(-) \otimes H^*(Y)$ on $\mathcal{K}$. This left adjoint exists if $H^*(Y)$ is finite type, meaning its cohomology is finite dimensional in each degree [Sch94]. We assume throughout this paper that all algebras in $\mathcal{K}$ are finite type, and will use $hS$ to denote the homotopy category of spaces with finite type $\mathbb{F}_p$-cohomology.\footnote{The finite type assumptions are not essential, but are the price of working with cohomology instead of homology. Many results here have an analog for unstable coalgebras over the dual Steenrod algebra without the finite type assumption.}

For any space $Z$, one obtains a natural transformation $H^*(Z) : H^*(Y) \rightarrow H^*(Z^Y)$ as follows: the evaluation map $ev : Y \times Z^Y \rightarrow Z$ gives a map in cohomology that is adjoint to a map

$$\theta : (H^*(Z) : H^*(Y)) \rightarrow H^*(Z^Y)$$

which for formal reasons is an isomorphism when $Z$ is a product of Eilenberg Mac Lane spaces of the form $K(\mathbb{Z}/p, m)$. Moreover, $H^*(Z) : H^*(Y)$’s universal property allows it to be computed in terms of generators and relations (which are spelled out in Appendix A).

In [BO99; Ott03], Ottosen and Bökstedt construct an equivariant generalization of Lannes’ construction when $Y = S^1$ is equipped with the natural $SO(2)$ action by rotation. They produce an endofunctor of $\mathcal{K}$, called $\ell$ which is equipped with a natural transformation $\eta : \ell(H^*(Z)) \rightarrow H^*(Z^{S^1}_{hSO(2)})$ that is an isomorphism when $H^*(Z)$ is a polynomial algebra. $Z^{S^1}_{hSO(2)}$ denotes the homotopy quotient of $Z^{S^1}$ by the natural right action of $SO(2)$. So $\ell$ approximates the Borel cohomology of free loop space, analogously to how $(\cdot) : H^*(S^1)$ approximates its ordinary cohomology. Moreover, $\ell(A)$ is defined in terms of generators and relations of $A$, so it is also computable.

This paper grew out of attempting to understand their construction and the extent to which it generalizes.

Let $EM_{\mathbb{F}_p}$ be the category of $\mathbb{F}_p$-EM-spaces: that is generalized Eilenberg Mac Lane spaces whose homotopy groups are finite dimensional $\mathbb{F}_p$-vector spaces. Moreover, from now on, let $p = 2$, so that all cohomology is mod 2 cohomology.
For a space $Z$, the mapping space $Z^{S^n}$ has a natural right $\text{SO}(n+1)$-action coming from the standard action of $\text{SO}(n+1)$ on $S^n$. Our main result is a computation of the cohomology of $Z_{h\text{SO}(n+1)}^{S^n}$ when $n$ is odd and $Z \in EM_{F_2}$.

**Notation 1.1.** $F_n : h\mathcal{S} \to h\mathcal{S}$ denotes the functor $Z \mapsto Z_{h\text{SO}(n+1)}^{S^n}$ and $L_n : \mathcal{K} \to \mathcal{K}$ denotes the functor $A \mapsto A : H^*(S^n)$.

We study the Serre spectral sequence for the Borel fibration

$$Z^{S^n} \to F_n(Z) \to \text{BSO}(n + 1)$$

when $Z \in EM_{F_2}$, which we call the **main spectral sequence**. Its $E_2$-term is the tensor product of $H^*(\text{BSO}(n + 1)) = F_2[w_2, \ldots, w_{n+1}]$ and $H^*(Z^{S^n})$. Observe that the existence of the spectral sequence ensures that $H^*(F_n(Z))$ is finite type.

As discussed above, there is a natural transformation $\theta_n : L_n(H^*(Z)) \to H^*(Z^{S^n})$, which is an isomorphism since $Z \in EM_{F_2}$. In Section 2 we explain how $H^*(Z) : H^*(S^n)$ is generated by elements called $a, da$, where $a$ ranges over $H^*(Z)$. In the theorem below, we regard $\theta_n(a), \theta_n(da)$ as elements in $E_2^{0,*}$ of the main spectral sequence. From now on we assume $n$ is odd.

**Theorem A.** For $Z$ an $F_2$-EM space, there are no differentials in the main spectral sequence until the $E_{n+1}$ page. On the $E_{n+1}$ page, the differentials are determined by $d_{n+1}(\theta_n(a)) = w_{n+1}\theta_n(da)$ and $d_{n+1}(\theta_n(da)) = 0$ for $a \in H^*(Z)$. Moreover, $E_{n+2} = E_\infty$.

The $E_{n+2}$-page of the main spectral sequence is generated as an algebra by the $w_i$, along with the classes $\theta_n(da), \theta_n(Sq_i a), \theta_n(Sq_i a + ada)$, for $a \in H^*(Z)$. The $w_i$ are pulled back from $\text{BSO}(n + 1)$, and the rest of the generators lie on the vertical edge of the spectral sequence. We show that the spectral sequence degenerates at the $E_{n+2}$-page by constructing classes in $H^*(F_n(Z))$, and showing they are detected in the fibre $H^*(Z^{S^n})$ by the generating elements.

Consider the functors $H^1(Z), H^1(F_n(Z)), H^1(Z^{S^n}) : h\mathcal{S}^{op} \to \text{Ab}$.

**Theorem B.** There are natural transformations

$$\delta : H^1(Z) \to H^{1-n}(F_n(Z))$$

$$\phi_i : H^1(Z) \to H^{2j-i}(F_n(Z))$$

such that if $q^* : H^*(F_n(Z)) \to H^*(Z^{S^n})$ is induced from the inclusion of the fibre, then $q^*(\delta(a)) = \theta_n(da), q^*(\phi_i(a)) = \theta_n(Sq_i a)$ for $i < n$, and $q^*(\phi_n(a)) = \theta_n(Sq_n a + ada)$. When $Z \in EM_{F_2}$, the images of $\phi_i, \delta$ along with $w_i$ generate $H^*(F_n(Z))$ as an algebra.

The construction of the natural transformations $\phi_i$ in the theorem above is interesting. The group $O(n)$ is an extension $C_2$ by $\text{SO}(n)$. If we give $\mathbb{Z}^2$ the structure of an
O(n)-space via the swap action of $C_2$ where SO(n) acts trivially, the map $Z^{S^n} \rightarrow Z^2$ given by evaluation at antipodal points is O(n)-equivariant.

Given $x \in H^*(Z)$, there is a total power operation $P(x) \in H^*(Z_{hC_2}^2)$, and a class $t \in H^*(Z_{hC_2}^2)$ that is the pullback of the generator of $t \in H^*(BC_2)$. $\phi_i(x)$ is then defined as the image of $t^{n-i}P(x)$ via the maps

$$H^*(Z_{hC_2}^2) \otimes H^*(BSO(n)) \cong H^*(Z_{hO(n)}^2) \rightarrow H^*(Z_{hO(n)}^S) \rightarrow H^{*-n}(F_n(Z))$$

where the first map is the evaluation and the second is the pushforward in cohomology of the fibration $Z_{hO(n)}^S \rightarrow F_n(Z)$ with fibre $\mathbb{RP}^n = SO(n+1)/O(n)$.

Having understood the main spectral sequence, we completely compute $grH^*(F_n(Z))$, which is the associated graded with respect to the filtration in the main spectral sequence for $Z \in EM_{F_2}$. We repackage this computation as giving a presentation of the 0th nonabelian derived functor of the functor $Z \mapsto grH^*(F_n(Z))$. This is explained in detail in section 2, but we outline it here.

If we are generally trying to compute some functor $F : hS_{op} \rightarrow D$ we can restrict it to the full subcategory $EM_{F_2}$. The cohomology functor $H^* : hS_{op} \rightarrow \mathcal{K}$ identifies $EM_{F_2}$ as the full subcategory of free unstable algebras over the Steenrod algebra.

We can then form the left Kan extension $\overline{F}$ below, which is called the zeroth derived functor of $F$:

$$
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\overline{F}} & D \\
\downarrow^{F} & & \\
EM_{F_2}^{op} & \\
\end{array}
$$

Under mild hypotheses on $D$, this left Kan extension always exists, and can be computed by taking a canonical free resolution of $A \in \mathcal{K}$, applying $F$, and taking the colimit.

The canonical free resolution can be realized on the level of spaces before applying the functor $H^*$. This observation leads to a natural transformation

$$\eta : \overline{F}(H^*(Z)) \rightarrow F(Z)$$

which can be thought of as realizing $\overline{F}$ as the best approximation of $F$ factoring through the functor $H^*$. In particular, $\eta$ is an isomorphism for $Z \in EM_{F_2}$, and is the structure map for the left Kan extension $\overline{F}$ when restricted to $EM_{F_2}$.

In the case of interest, $F$ is the functor sending $Z$ to $grH^*(F_n(Z))$, where we view $grH^*(F_n(Z))$ as an object of a category $\mathcal{K}^{gr}$ of graded algebras over the Steenrod algebra.
We construct a functor \( \mathcal{L}_n : \mathcal{K} \to \mathcal{K}_{gr} \) that takes \( A \in \mathcal{K} \) to an object in \( \mathcal{K}_{gr} \) that is presented in terms of generators and relations of \( A \) (see Definition 2.7). Then we construct a natural transformation \( \eta_n : \mathcal{L}_n H^* (Z) \to gr H^* (F_n (Z)) \), and show

**Theorem C.** The natural transformation \( \eta_n \) realizes \( \mathcal{L}_n \) as the zeroth derived functor of \( Z \mapsto gr H^* (F_n (Z)) \).

In particular, \( \eta_n \) gives a functorial computation of \( gr H^* (F_n (Z)) \) for \( Z \in EM_{\mathbb{F}_2} \). The generators of \( \mathcal{L}_n \) correspond exactly to the images of the natural transformations \( \phi_i, \delta \), along with the \( w_i \) in the associated graded of \( H^* (F_n (Z)) \).

Although we compute the zeroth derived functor of \( gr H^* (F_n (Z)) \), Theorem B suggests it is possible to compute the zeroth derived functor of the functor \( H^* (F_n (Z)) \). These zeroth derived functors are like higher dimensional analogs of \( \ell \), and can be thought of as a version of negative cyclic homology for the free iterated loop space that is constructed using Steenrod algebraic data. In Appendix B, we make partial progress on presenting this zeroth derived functor by computing some of the relations among classes in \( H^* (F_n (Z)) \).

**Outline of paper.** This paper is organized as follows:

In Section 2, we study zeroth derived functors, and explain how Lannes’ division functor gives an example. We give a presentation of division by \( H^* (S^n) \), which is a special case of a result in Appendix A, and also define \( \mathcal{L}_n \).

In Section 3, we explain some facts about \( SO(n + 1) \)-spaces relevant to the main results of the paper.

In Section 4, we construct classes of \( H^* (F_n (Z)) \) by producing the natural transformations \( \delta, \phi_i \), and show some relations that hold among these classes. Using these relations, we construct the natural transformation \( \eta_n \).

In Section 5, we use the results of Section 4 to completely describe the main spectral sequence.

In Section 6, we show via an algebraic computation that the map \( \eta_n : \mathcal{L}_n (H^* (Z)) \to H^* (F_n (Z)) \) is an isomorphism when \( Z \in EM_{\mathbb{F}_2} \).

In Section 7 we raise questions related to our work, and give some commentary.

We include two appendices. In Appendix A we explain in depth how to compute Lannes’ division functors \( (-) : M \) in general. We also examine which operations are actually required to compute the underlying \( \mathbb{F}_2 \)-vector space of the functor \( (-) : M \).

In Appendix B, we collect and prove some relations that hold among classes in the image of the natural transformations \( \delta, \phi_i \). A complete set of such relations would allow one to present the zeroth derived functor of \( Z \mapsto H^* (Z_{\text{h}SO(n + 1)}) \) in terms of \( \phi_i, \delta \), and \( w_i \).

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2. Cohomology and Derived Functors

In this section, we explain in more detail the construction and universal properties of the zeroth derived functor of a functor out of $hS^{op}$. We explain Lannes’ division functors and how they give an example of a zeroth derived functor. Finally, we give a definition of $\mathcal{L}_n$, and compute the zeroth derived functor of $Z \mapsto H^*(Z^{S^n})$.

Let $\mathcal{U}$ denote the symmetric monoidal category of unstable modules over the Steenrod algebra (see [Sch94]), so that objects in $K$, the category of unstable algebras over the Steenrod algebra, is the category of commutative monoids in $\mathcal{U}$ satisfying $Sq_0(x) = x^2$, where $Sq_i = Sq_i^{|x|^{-i}}$. As always, we assume that our modules and algebras are finite type.

Lemma 2.1. Let $F : EM_{\mathbb{F}_p}^{op} \to D$ be a functor, and suppose $D$ has reflexive coequalizers. Then the left Kan extension $\overline{F}$ as below exists. Moreover, a functor $\tilde{F}$ equipped with a natural transformation $\eta : H^*(\tilde{F}) \to F$ is the left Kan extension iff $\eta$ is a natural isomorphism and $\tilde{F}$ preserves reflexive coequalizers.

\[
\begin{array}{c}
\mathcal{K} \\
\uparrow H^* \\
EM_{\mathbb{F}_p}^{op} \\
\downarrow F \\
\overline{F} \quad \quad \quad \quad \quad \quad D
\end{array}
\]

Proof. It is easy to see that the category $\mathcal{K}$ is the same as finite presheaves out of $EM_{\mathbb{F}_p}^{op}$ that preserve products. Indeed, such a presheaf is determined by its value on $K(\mathbb{Z}/2\mathbb{Z}, m)$, which corresponds to the $m^{th}$-graded piece of an unstable algebra, and the functoriality corresponds to the Steenrod operations. This allows us to identify $\mathcal{K}$ with the free cocompletion of $EM_{\mathbb{F}_p}^{op}$ with respect to reflexive coequalizers. It follows that we can form an extension $\overline{F}$ that preserves reflexive coequalizers and is a left Kan extension. Moreover, $\overline{F}$ restricts to $F$ on $EM_{\mathbb{F}_p}$.

For the last statement, $\eta$ identifies $\tilde{F}$ with $\overline{F}$ on $EM_{\mathbb{F}_p}$, but since both functors preserve reflexive coequalizers, they must agree. \qed

Given a functor $F : hS^{op} \to D$, we can restrict it to $EM_{\mathbb{F}_p}^{op}$, and then construct the zeroth derived functor, which is the left Kan extension $\overline{F} : \mathcal{K} \to D$ that exists
by the above lemma. Next we show that the natural transformation \( \eta \) from the Kan extension naturally extends to the category \( hS^{op} \).

**Lemma 2.2.** If \( F : hS^{op} \to D \) is a functor, and \( \overline{F} : K \to D \), with the natural transformation \( \eta \), is its zeroth derived functor, then there is a unique natural transformation \( \eta' \) from \( \overline{F} \circ \text{H}^* \) to \( F \) as functors \( hS^{op} \to K \) extending \( \eta \) on the subcategory \( EM^{op}_{F_p} \).

**Proof.** Given a space \( X \), we can consider \( RX = \Omega^\infty(HF_2 \otimes \Sigma^\infty X) \), which is the Eilenberg Mac Lane space whose homotopy groups are the homology of \( X \). Since \( R \) comes from an adjunction, it is part of a monad, and gives a canonical resolution \( R^*(X) \) of \( X \) in terms of Eilenberg Mac Lane spaces. Let \( \bar{R} \) denote the diagram of just the first two steps in this resolution, which is a reflexive equalizer. Applying \( F \), we obtain a map from \( \overline{F} \text{colim}(\overline{R}(X)) \to F(X) \). The colimit \( \text{colim}F(\overline{R}(X)) \) is equal to \( \text{colim}(\overline{F} \circ \text{H}^*(\bar{R}(X))) \) since \( R \) takes values in \( EM_{F_p} \). The colimit, being a reflexive coequalizer, is preserved by \( \overline{F} \), so we get a map \( \overline{F} \text{colim}(\text{H}^*(\bar{R}(X))) \to F(X) \). The colimit in the domain is \( \text{H}^*(X) \), since \( \text{H}^*(\bar{R}(X)) \) is the canonical presentation of it as an object in \( K \). Thus this is a natural transformation \( \overline{F} \text{H}^*(X) \to F(X) \), and it is unique because any natural transformation extending \( \eta \) must factor through the colimit. \( \square \)

Now we recall Lannes’ division functor, and how it gives an example of a zeroth derived functor. Given \( A \in K \), the functor \((-) \otimes A\) has a left adjoint \( [\text{Sch94}] \), denoted \((-) : A \), and called division by \( A \). In Appendix A, we give a presentation that computes division by \( A \).

The following results are formal and well known. By convention here, \( V_i \) is in cohomological degree \(-i\).

**Lemma 2.3.** Suppose \( V_* \) is a graded \( \mathbb{F}_2\)-vector space and let \( K(V_*) \) denote the Eilenberg-Mac Lane space with homotopy groups \( V_* \) (truncated to be in nonnegative degrees). There is a natural homotopy equivalence \( K(V_*) \cong K(V_* \otimes \text{H}^*(Y)) \).

**Proposition 2.4.** Let \( Y \in hS^{op} \). There is a natural map \( \theta_n : (\text{H}^*(Z) : \text{H}^*(Y)) \to \text{H}^*(Z^Y) \) that is an isomorphism when \( Z \in EM_{F_p} \).

**Corollary 2.5.** The natural transformation \( \theta_n \) realizes \((-) : \text{H}^*(Y) \) as the zeroth derived functor of \( Z \mapsto \text{H}^*(Z^Y) \).

**Proof.** By Proposition 2.4, \( \theta_n \) is an isomorphism when \( Z \in EM_{F_p} \), and since \((-) : \text{H}^*(Y) \) is a left adjoint, it preserves all colimits, in particular reflexive coequalizers. Thus by Lemma 2.1, it is the zeroth derived functor. \( \square \)

Our next goal will be to define the functor \( \bar{\ell}_n \) as a certain zeroth derived functor. First we will need to define some of the relevant categories.
We define $\mathcal{U}^{\text{fil}}$ to be category $\text{Fun}(\mathbb{N}^{gr}, \mathcal{U})$ of filtered objects of $\mathcal{U}$. In other words, an object of $\mathcal{U}$ is a sequence $M_i$ of objects in $U$ with maps $M_i \to M_{i-1}$ (which are not required to be monomorphisms). Similarly, $\mathcal{U}^{gr}$ denotes the category of graded objects in $U$, or $\text{Fun}(\{\mathbb{N}\}, \mathcal{U})$. Both $\mathcal{U}^{\text{fil}}$ and $\mathcal{U}^{gr}$ are symmetric monoidal via Day convolution.

$\mathcal{K}^{\text{fil}}$ and $\mathcal{K}^{gr}$ denote the category of commutative monoid objects in $\mathcal{U}^{\text{fil}}$ and $\mathcal{U}^{gr}$ respectively, satisfying the following algebra instability conditions:

- For $A_s \in \mathcal{U}^{\text{fil}}$ and $x \in A_n$ we require $Sq_0(x) = x^2$ where we take the element $x^2$ as being in $A_n$ via the filtration map $A_{2n} \to A_n$.
- For $A_s \in \mathcal{U}^{gr}$ we require $Sq_0(x) = x^2$ for $x \in A_0$.

There is a lax symmetric monoidal functor $\text{gr} : \mathcal{U}^{\text{fil}} \to \mathcal{U}^{gr}$ sending a filtered object to its associated graded object. Because this is compatible with the instability conditions, it induces a functor $\text{gr} : \mathcal{K}^{\text{fil}} \to \mathcal{K}^{gr}$.

**Construction 2.6.** We refine the functor $H^*(F_n(Z))$, from a functor $\mathcal{K} \to \mathcal{K}$ to a functor $\mathcal{K} \to \mathcal{K}^{\text{fil}}$, by equipping $H^*(F_n(Z)) = H^*(Z_{hSO(n+1)}^g)$ with the filtration induced from the skeletal filtration of $BSO(n+1)$. This is the filtration arising in the main spectral sequence.

Our goal is to compute the zeroth derived functor of $\text{gr}H^*(F_n(Z))$, where $H^*(F_n(Z))$ is equipped with the filtration above. To do so, we write down a presentation for a functor that we later prove to be this zeroth derived functor.

**Definition 2.7.** $\tilde{t}_n(A)$ is the graded Steenrod algebra multiplicatively generated as an object in $\mathcal{K}^{gr}$ by classes $\tilde{t}_i(a), \tilde{S}_i(a)$ for $a \in A$ in graded degree $0$, and $\tilde{w}_i$ in graded degree $i$, with Steenrod degree $2|a| - i, |a| - n$, and $0$ respectively, with the following relations:

1. $\tilde{t}_{i+k}(a + b) = \tilde{t}_i(a) + \tilde{t}_i(b)$
2. $\tilde{S}_{i+k}(a + b) = \tilde{S}_i(a) + \tilde{S}_i(b)$
3. $\tilde{S}_i(ab) \tilde{t}_j(c) + \tilde{S}_i(bc) \tilde{t}_j(a) + \tilde{S}_i(ca) \tilde{t}_j(b) = 0$
4. $\tilde{w}_{n+1} \tilde{S}_i = 0$
5. $\tilde{w}_i \tilde{S}_i = \tilde{S}_i \tilde{w}_i$
6. $\tilde{S}_k(ab) = \sum_{i+j=k} \tilde{t}_i(a) \tilde{t}_j(b)$
7. $\tilde{S}_k \tilde{S}_i(a) = \sum_{j \geq 0} \sum_{i} (k + |a| - j) \tilde{S}_{n-i+k+2j}(S^i a) + \tilde{S}_k \tilde{S}_j = \tilde{S}_{n-i+k+2j}(S^i a) + \tilde{S}_k \tilde{S}_j = \tilde{S}_{n-i+k+2j}(S^i a) + \tilde{S}_k \tilde{S}_j$
8. $\tilde{S}_k \tilde{S}_i(a) = \tilde{S}_i \tilde{S}_k(a)$
9. $\tilde{S}_k(\tilde{t}_i(a)) = \tilde{S}_k(\tilde{S}_i(a)) = 0$ for $i < 0$
10. $\tilde{S}_k \tilde{S}_i(a) = \tilde{S}_i \tilde{S}_k(a) = \tilde{t}_i(a)^2$
11. $\tilde{S}_k(\tilde{t}_i a) = \sum_{j=0}^{(2|a|-i-k)/2} \tilde{t}_{2|a|-i-j} \tilde{S}_{2j-2i+k} + \tilde{S}_{k,n} \tilde{t}_i(a) \tilde{S}_j(S^i a)$ for $k > i$. 

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\( \tilde{\phi}_0(Sq_j a) = \tilde{\phi}_j(a)^2 + \delta_{j,n} \tilde{\phi}(a^2 Sq_n a) \)

where \( \tilde{\phi}_m(a) = 0 \) for \( m > n \).

In section 4, we produce a natural transformation \( \eta_n \) realizing \( \ell_n \) as the desired zeroth derived functor.

Finally, we explain how to compute \( A : H^*(S^n) \), which we denote \( L_n(A) \) for \( A \in \mathcal{K} \).
The result is a special case of a more general computation of division functors in the appendix.

**Proposition 2.8.** As a commutative \( \mathcal{A} \)-algebra, \( L_n(A) = A : H^*(S^n) \) is generated by generators \( da \) in degree \( |a| - n \) for each \( a \in A \), along with the relations:

1. \( d(a + b) = da + db \)
2. \( d(ab) = d(a)b + ad(b) \)
3. \( (da)^2 = d(Sq_n a) \)
4. \( d(Sq_n a) = 0 \) for all \( n > i \geq 0 \).

The action of the Steenrod algebra is determined by \( Sq^i(da) = d(Sq^i a) \) and the Cartan formula, and the universal map \( A \to L_n(A) \otimes H^*(S^n) \) sends \( a \mapsto a \otimes 1 + da \otimes y \) where \( y \) is the nontrivial element of \( H^n(S^n) \).

**Proof.** Consider the functor \( UF \) taking a graded \( \mathbb{F}_2 \)-vector space \( V \) to the free object on \( V \) in \( \mathcal{K} \). Let \( N = H^*(S^n) \). Then \( \text{Hom}(UF(V_*) : N, M) = \text{Hom}(UF(V_*), N \otimes M) = \text{Hom}_{\mathbb{F}_2}(UV_*, N \otimes M) = \text{Hom}(UF(V_* \otimes N^*), M) \), so by the Yoneda Lemma, there is an isomorphism \( UF(V_*) : N \cong UF(V_* \otimes N^*) \).

There is a unique graded basis \( 1, d \) for \( N^* \), and so the generators in \( UF(V_* \otimes N^*) \) of the form \( v \otimes 1 \) we call \( v \), and the ones of the form \( v \otimes d \) we call \( dv \). Moreover, following the isomorphisms show that the coevaluation is given by \( v \mapsto v \otimes 1 + dv \otimes y \). More generally, given \( a \in A \in \mathcal{K} \), giving a class \( a \in A_n \) is the same as a map \( a : UF(\Sigma^n \mathbb{F}_2) \to A_n \), and we define \( a \) and \( da \) to be the images of the corresponding classes after applying \( (\cdot) : N \). Clearly relation (1) holds.

Applying \( Sq^i \) on the coevaluation map, we see \( Sq^i a \mapsto Sq^i a \otimes 1 + Sq^i(da) \otimes y \) so we must have \( dSq^i a = Sq^i da \). From this it follows that relations (3) and (4) hold. Furthermore, applying the formula for the coevaluation on a product, we find \( ac \mapsto (a \otimes 1 + da \otimes y)(c \otimes 1 + dc \otimes y) \), yielding relation (2).

Now an arbitrary algebra \( A \) can be presented canonically as a pushout

\[
\begin{array}{ccc}
UF(A \otimes A) \otimes UF(\bigoplus_0^\infty \Sigma^i A) & \longrightarrow & UF(A) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & M
\end{array}
\]
The vertical nonzero map is given by \([a] \mapsto a\) and the horizontal one sends \([a \otimes b]\) to \([a][b] - [ab]\), and \(a \in \Sigma^i M\) to \(Sq^i[a] - [Sq^ia]\).

Applying \((-) : N\), since \((-) : N\) preserves pushouts, we get a pushout square

\[
\begin{array}{ccc}
UF(A \otimes A \otimes N^*) \otimes UF(\bigoplus_0^\infty \Sigma^i A \otimes N^*) & \longrightarrow & UF(A \otimes N^*) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A : N
\end{array}
\]

This gives a presentation for \(A : N\) as the quotient of \(UF(A \otimes N^*)\) by relation (2), and the additional relation \(Sq^i(da) = dSq^i(a)\).

From this we can extract a presentation for \(A : N\) as an algebra. The fact that \(Sq^i(da) = dSq^i(a)\) simplifies things: The fact that \(Sq^i(a) = 0\) for \(i < 0\) and \(Sq_0(a) = a^2\) reduces to relations (3) and (4). This presentation then reduces to the presentation of an algebra: \(a\) and \(da\) are generators for \(a \in A\) with relations (1) - (4).

\[\square\]

3. \(SO(n + 1)\) actions

In this section, we collect relevant facts about spaces with \(SO(n + 1)\)-actions, to set up our understanding of the \(SO(n + 1)\) action on the mapping space from \(S^n\). Many of the results here are well known, and the reader may wish to read later sections first. Throughout this section, \(X\) denotes a left or right \(SO(n + 1)\) space.

**Pushforward.** An important construction we use later is that of the pushforward on cohomology. It makes sense in great generality, but we are interested in it for a fibration \(f : E \to B\) whose fibre \(F\) is a compact manifold of dimension \(n\). The pushforward, denoted \(f_* : H^*(E; H^n(F)) \to H^{*-n}(B)\) can be defined in two equivalent ways [CK07].

The first description we need is via the normal bundle \(\nu\) of the fibration \(E \to B\). This normal bundle \(\nu\) is a virtual bundle of dimension \(-n\) on \(E\), whose fibre is the negative of the vectors tangent to the fibre. Then there is a Gysin map \(\Sigma^\infty B \to E^\nu\), where \(E^\nu\) is the Thom spectrum of \(\nu\). We can use the composite \(H^*(E) \cong H^{*,-n}(E^\nu) \to H^{*,-n}(B)\), where the first map is the Thom isomorphism, to obtain the pushforward.

The second description works when the fibre \(F\) is connected, and is defined via the Serre spectral sequence of the fibration. The \(E_2\) term is \(H^p(B; H^q(F))\). \(H^n(F)\) is \(\mathbb{Z}/2\mathbb{Z}\) (with trivial coefficients) by our connectedness assumption, and \(H^i(F) = 0\) for \(i > n\), so the \(q^{th}\) column of the \(E_\infty\) page of the spectral sequence is a quotient of \(H^*(E)\). The \(E_\infty\) page injects into the \(E_2\) page, so the composite \(H^*(E) \to E^{*,-n,n}_\infty \to E^{*,-n,n}_2 = H^{*-n}(B)\) gives the desired map.

The pushforward satisfies some nice properties. What we need are
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(1) \( f_\ast \) is a map of \( H^\ast(B) \) modules.
(2) \( f_\ast \) is natural with respect to pullback fibrations.
(3) \( f_\ast \) is functorial.

We are primarily interested in the following situation. Suppose \( G, H \) are compact Lie groups. If \( X \) is a \( G \)-space and \( H \subset G \) is a subgroup, then there is a fibration \( X_{hH} \to X_{hG} \) with fibre \( G/H \). The pushforward of this fibration is denoted \( \tau^G_H \).

Coactions and comparisons. The group \( O(n) \) sits inside \( SO(n+1) \) as the subgroup fixing (as a set) any given 1-dimensional subspace of the \( n+1 \)-dimensional real vector space \( SO(n+1) \) naturally acts on. First we compare the \( O(n) \) quotient and the \( SO(n+1) \) quotient.

**Lemma 3.1.** The map \( X_{hO(n)} \to X_{hSO(n+1)} \) realizes \( H^\ast(X_{hO(n)}) \) as
\[
H^\ast(BO(n)) \otimes H^\ast(BSO(n+1)) H^\ast(X_{hSO(n+1)}),
\]
which is a free module over \( H^\ast(X_{hSO(n+1)}) \) generated by the classes \( 1, w_1, \ldots, w_n \) (alternatively \( 1, w_1, \ldots, w^n_1 \)). Moreover, \( w_i \) in \( H^\ast(X_{hSO(n+1)}) \) is sent to \( w_i + w_{i-1}w_1 \) in \( H^\ast(X_{hO(n)}) \).

**Proof.** First consider the case when \( X \) is a point. In this case, the map is the fibration \( BO(n) \to BSO(n+1) \), given by taking an \( n \)-plane bundle and adding a copy of its determinant bundle. By the Whitney sum formula, we get that \( \sum_{i=0}^{n+1} w_i \) in \( H^\ast(BO(n)) \) pulls back to \( \left( \sum_{i=0}^{n} w_i \right)(1+w_1) \). Thus \( w_i \) pulls back to \( w_i + w_1w_{i-1} \), where we interpret \( w_i \) to be 0 when it is 0 for the universal bundle. The fibre of the map is \( \mathbb{R}P^n \), where the map to \( BO(n) \) is the orthogonal complement of the tautological bundle. Thus by the Whitney sum formula, its total Stiefel-Whitney class is \( \frac{1}{1+t} = 1+t+\cdots+t^n \). Thus \( w_i \) get sent to a basis of \( H^\ast(\mathbb{R}P^n) \). Now considering the Serre spectral sequence of the fibration \( \mathbb{R}P^n \to BO(n) \to BSO(n+1) \), and comparing Poincare polynomials, there is no room for differentials, so the spectral sequence degenerates at \( E_2 \), and the \( w_i \) generate \( H^\ast(BO(n)) \) freely. The basis \( w_i \) is related to the classes \( w^n_1 \) via a triangular matrix with diagonal entries 1, so both families work as a basis.

For the general case, consider the Eilenberg-Moore spectral sequence of the pullback square
\[
\begin{array}{ccc}
X_{hO(n)} & \longrightarrow & X_{hSO(n+1)} \\
\downarrow & & \downarrow \\
BO(n) & \longrightarrow & BSO(n+1)
\end{array}
\]
Since $H^*(BO(n))$ is a free module, the $E_2$-term is concentrated in one line and is $H^*(BO(n)) \otimes_{H^*BSO(n+1)} H^*(X_{hSO(n+1)})$. Thus the spectral sequence collapses at $E_2$, and gives the result we want.

Next, we study the left action of $SO(n+1)$ on $S^n$. Let $y$ denote the generator of $H^*(S^n)$, and $x_i$ the odd degree generators of $H^*(SO(n+1))$. The following lemmas are well known.

**Lemma 3.2.** The Serre spectral sequence of the fibration $SO(n) \to SO(n+1) \to S^n$ has no differentials. Moreover the class $y$ pulls back to $x_{i2^p}$ where $i2^p = n$.

**Lemma 3.3.** In the Serre spectral sequence of the path space fibration of $SO(n+1) \to \cdot \to BSO(n+1)$, the class $x_{i2^p}$ transgresses to hit $w_{i2^p+1}$.

**Lemma 3.4.** The coaction of $H^*(SO(n+1))$ on $H^*(S^n)$ is given by $y \mapsto y \otimes 1 + 1 \otimes x_{i2^p}$ where $i2^p = n$.

**Proof.** Since the action map $SO(n+1) \times S^n \to S^n$ is unital and $H^*(S^n)$ is 2-dimensional, $y$ has to get sent to $y \otimes 1 + 1 \otimes c$ for some class $c$. The class $c$ is then the image of $y$ in the composite $SO(n+1) \to SO(n+1) \times S^n \to S^n$. But the composite map is the fibration in Lemma 3.2, so we are done by that lemma. □

Here is another related spectral sequence we use later. It can also be used to give an alternate proof that $y$ pulls back to $x_{i2^p}$ in Lemma 3.2.

**Lemma 3.5.** The Serre spectral sequence of the fibration $SO(n) \to S^{n-1} \to BSO(n-1)$ has differentials sending $x_{i2^p}$ to $w_{i2^p+1}$ for $i < n$.

**Proof.** This follows by considering the comparison map from this spectral sequence to the one in Lemma 3.3. □

**The free iterated loop space.** We now compute the coaction and evaluation map relevant to the mapping space $Z^{S^n}$ where $Z$ is a space. Recall that given a class $a$ in $H^*(Z)$, there are corresponding classes $\theta_n(a), \theta_n(da)$ in $H^*(Z^{S^n})$.

The following lemma follows from the definition of $\theta_n$:

**Lemma 3.6.** The coevaluation map $H^*(Z) \to H^*(S^n \times Z^{S^n})$ on cohomology sends the class $a$ to $\theta_n(a) \otimes 1 + \theta_n(da) \otimes y$.

**Proposition 3.7.** The coaction $H^*(Z^{S^n}) \to H^*(Z^{S^n}) \otimes H^*(SO(n+1))$ sends $\theta_n(da) \mapsto \theta_n(da) \otimes 1, \theta_n(a) \mapsto \theta_n(a) \otimes 1 + \theta_n(da) \otimes x_{i2^p}$ where $i2^p = n$.

**Proof.** Consider the commutative diagram:

$$
\begin{array}{ccc}
Z^{S^n} \times S^n & \xrightarrow{ev} & Z \\
\alpha_L \uparrow & & \uparrow ev \\
Z^{S^n} \times SO(n+1) \times S^n & \xrightarrow{\alpha_R} & Z^{S^n} \times S^n
\end{array}
$$
Where $\alpha_L$ is the left action on $S^n$ and $\alpha_R$ is the right action on $Z^{S^n}$. We can pull back the class $a \in H^*(Z)$ to $H^*(Z^{S^n} \times \text{SO}(n + 1) \times S^n)$ along the top corner, which by Lemma 3.6 and Lemma 3.4 is given by $a \mapsto \theta_n(a) \otimes 1 + \theta_n(da) \otimes y \mapsto \theta_n(a) \otimes 1 \otimes 1 + \theta_n(da) \otimes y \otimes 1 + \theta_n(da) \otimes 1 \otimes x_i^{2n}$. Thus by commutativity, this is where $\alpha_R^*$ sends the class $\theta_n(a) \otimes 1 + \theta_n(da) \otimes y$, giving the result.

□

Next we end with some computations that are needed later. $\text{SO}(n)^2_h \text{SO}(n-1)$ denotes the quotient by the diagonal action, and $\text{SO}(n)^2_{h,\text{SO}(n-1) \times C_2}$ denotes the further quotient by the swap action.

Lemma 3.8. The Serre spectral sequence of $\text{SO}(n) \to \text{SO}(n)^2_{h,\text{SO}(n-1)} \to S^{n-1}$ degenerates at $E_2$, as does the Eilenberg Moore spectral sequence of the product of $\text{SO}(n)/\text{SO}(n-1)$ over $B\text{SO}(n-1)$. Moreover, the map $\text{SO}(n)^2 \to \text{SO}(n)^2_{h,\text{SO}(n-1)}$ is injective on cohomology.

Proof. For the first and last statement, one considers the map of spectral sequences coming from the following map of fibrations:

$$
\begin{array}{ccc}
\text{SO}(n) & \longrightarrow & \text{SO}(n)^2_{h,\text{SO}(n-1)} \\
\uparrow & & \uparrow \\
\text{SO}(n) & \longrightarrow & \text{SO}(n)^2 \longrightarrow \text{SO}(n).
\end{array}
$$

Then one uses the injectivity of the map on cohomology and the lack of differentials in the target to conclude. For the second statement, since the map $S^{n-1} \to B\text{SO}(n-1)$ is injective on trivial on cohomology by Lemma 3.3, the $E_2$ term is $H^*(S^n) \otimes H^*(S^n) \otimes \text{Tor}^*(H^*(\text{BSO}(n)))$, which cannot have any differentials, as it is the same dimension as the cohomology of $\text{SO}(n)^2_{h,\text{SO}(n-1)}$. □

4. Equivariant Evaluations

This section is the heart of this work. Our goal here is to construct the natural transformation $\eta_n$ used in Theorem C, which is done by constructing and studying the natural transformation $\phi_i, \delta$.

Recall that $n \in \mathbb{N}$ is odd.

Construction and identification of the classes. Here for an arbitrary space $Z$, we study classes in the cohomology of $F_n(Z)$. From the discussion in the introduction, when $Z \in EM_{F_2}$, the cohomology should be generated by the classes $w_i$ coming from $B\text{SO}(n+1)$, along with classes that under the map $Z^{S^n} \to F_n(Z)$ pullback to $\text{Sq}^i a, i < n, \text{Sq}^n a + ada, da$ where $a$ is an arbitrary cohomology class in $H^*(Z)$. Our goal is to construct classes $\phi_i(a), \delta(a)$ in $F_n(Z)$ that pull back to these (Proposition 4.5).
Our construction uses the equivariant evaluation maps. The first of these, \( ev_0 \) is the map \( ZS^n \to Z \) given by evaluating at a basepoint of \( S^n \). Since \( SO(n) \subset SO(n+1) \) is the stabilizer of a point in \( S^n \), this map is \( SO(n) \)-equivariant, where we have given \( Z \) a trivial action.

The second is the map \( ev_1 : ZS^n \to Z^2 \) given by evaluating at two antipodal points. The stabilizer of these points on \( S^n \) is \( O(n) \subset SO(n+1) \), so this map is \( O(n) \)-equivariant, where we equip \( Z^2 \) with the ‘swap’ \( O(n) \) action that factors through the quotient \( C_2 = O(n)/SO(n) \).

The notation \( ev_0, ev_1 \) is used both to refer to these maps as well as the induced maps on homotopy quotients.

Let \( \pi \) denote the projection map \( Z^2_{hO(n)} = (Z^2 \times BSO(n))_{hC_2} \to Z^2_{hC_2} \)

Recall that for \( a \in H^*(Z) \), there is a total power operation class \( P(a) \) in \( H^*(Z^2_{hC_2}) \) with the property that its pullback to \( Z^2 \) is \( a \otimes a \), and its pullback to the fixed points \( Z \times \mathbb{R}^\infty \) is \( \sum_i t^i S_i(a) \).

**Construction 4.1.** Given a class \( a \in H^*(Z) \), we define \( \delta(a) \) to be the image of \( a \) in the composite

\[
H^*(Z \times BSO(n)) = H^*(Z_{hSO(n)}) \xrightarrow{ev_0} H^*(Z_{hSO(n)}) \xrightarrow{\tau_{SO(n+1)}} H^{*-n}(F_n(Z)).
\]

For \( 0 \leq i \leq n \) we define \( \phi_i \) to be the image of \( t^{n-i}P(a) \) in the composite

\[
H^*(Z_{hC_2}) \xrightarrow{\pi^*} H^*(Z^2_{hO(n)}) \xrightarrow{ev_1} H^*(Z_{hSO(n)}) \xrightarrow{\tau_{O(n)}} H^*(F_n(Z)).
\]

First we show that \( \delta(a) \) has the desired image in \( H^*(ZS^n) \).

**Lemma 4.2.** The image of \( \delta(a) \) via the map \( H^*(F_n(Z)) \to H^*(ZS^n) \) is the class \( \theta_n(d a) \).

**Proof.** The key observation is that the homotopy quotient of the action map \( \alpha : ZS^n \times SO(n+1) \to ZS^n \) by \( SO(n+1) \) is exactly the quotient map we want to understand \( ZS^n \to F_n(Z) \). Then we can consider the commutative diagram

\[
\begin{array}{cccc}
H^*(Z_{hSO(n)}) & \xrightarrow{ev_0} & H^*(Z_{hSO(n)}) & \xrightarrow{\tau_{SO(n+1)}} H^*(F_n(Z)) \\
\downarrow \alpha^* & & \downarrow \alpha^* & \\
H^*(Z) & \xrightarrow{ev^*} & H^*(Z_{hSO(n)} \times S^n) & \xrightarrow{\tau_{SO(n+1)}} H^*(Z_{hSO(n)}) \times H^*(Z_{hSO(n)}).
\end{array}
\]

By the definition of \( \delta(a) \), we are trying to understand the image of \( a \) from the top left corner to the bottom right corner. But following the lower part of the diagram, and using Lemma 3.6 this gives \( a \mapsto \theta_n(a) \otimes 1 + \theta_n(da) \otimes y \mapsto \theta_n(da) \). □
The same basic strategy in Lemma 4.2 is used to prove the analogous result for \( \phi_i(a) \), except more effort is required in making it succeed. In particular, it is harder to make the analog of the commutative square on the left.

We now construct a map involved in the analogous diagram, which uses the fact that \( n \) is odd. Since \( n \) is odd, \( O(n) = SO(n) \times C_2 \), where the generator of \( C_2 \) is the negative of the identity in \( SO(n+1) \).

**Construction 4.3.** Suppose that \( Y \) carries an \( SO(n+1) \)-action. Then we can put a \( SO(n) \times C_2 = O(n) \) action on \( Y^2 \), where \( SO(n) \) acts diagonally on each factor, and \( C_2 \) swaps the two factors. Then there is a natural \( O(n) \)-equivariant map

\[
f_Y : SO(n+1) \times Y_0 \to Y^2
\]

given by \( (\alpha, -\alpha) \) where \( \alpha \) denotes the action map, and \( -\alpha \) denotes the action but composed with the action of \( -1 \subset SO(n+1) \). The corresponding map after applying \( (-)_{hO(n)} \) is given the same name.

To see why \( \phi_i(a) \) have the correct images in \( H^*(Z_{Sn}) \), we have to contemplate the diagram below, where \( Z' \) temporarily denotes \( Z_{Sn}^0 \times SO(n+1) \), and we have identified \( SO(n+1)/O(n) = \mathbb{RP}^n \).

\[
\begin{align*}
H^*(Z_{2hO(n)}) & \xrightarrow{ev_1^*} H^*(Z_{hO(n)}) \xrightarrow{\tau_{SO(n+1)}} H^*(F_n(Z)) \\
& \downarrow{(ev_0^*)} \downarrow{\alpha^*} \downarrow{\alpha^*} \\
H^*((Z_{Sn})^2_{hO(n)}) & \xrightarrow{f_{Z_{Sn}}^*} H^*(Z_0^{Sn} \times \mathbb{RP}^n) \xrightarrow{\tau_{O(n)}} H^*(Z_0^{Sn}) \\
& \downarrow{(\alpha^2)^*} \downarrow{1 \times \alpha^*} \\
H^*((Z')^2_{hO(n)}) & \xrightarrow{f_{Z'}^*} H^*(Z'_0 \times \mathbb{RP}^n)
\end{align*}
\]

Computing the image of \( \phi_i(a) \) in \( H^*(Z_{Sn}) \) is image of the class \( t^{n-i}P(a) \) in the diagram above from \( H^*(Z^2)_{hO(n)} \) to \( H^*(Z_0^{Sn}) \). The pushforward from \( H^*(Z_0^{Sn} \times \mathbb{RP}^n) \) to \( H^*(Z_0^{Sn}) \) extracts the coefficient of \( t^n \) so is easy to understand, and we can focus our efforts on computing the map from \( H^*(Z_0^{Sn}) \) to \( H^*(Z_0^{Sn} \times \mathbb{RP}^n) \).

However, the arrow indicated \( 1 \times \alpha^* \) is an injection, so it suffices to compute the image in \( H^*(Z'_0 \times \mathbb{RP}^n) \).

To do this, we need to understand the cohomology of \( (Z')^2_{hO(n)} \). It admits a projection map to \( (Z_{Sn})^2_{hC_2} \), and the pullback of a class \( P(a) \) along this map is given the same name.
Furthermore there is a projection map to $H^*(SO(n + 1)^2_{hO(n)})$ (where $-1$ acts by swapping the factors). We construct a class $P(y) \in H^*(SO(n + 1)^2_{hO(n)})$ whose pullback to $(Z')^2_{hO(n)}$ is given the same name.

**Lemma 4.4.** There is a unique class $P(y) \in H^*(SO(n + 1)^2_{hO(n)})$ that pulls back to the class $y \otimes y$ in $H^*(SO(n + 1)^2_{hSO(n)})$, and to 0 in $H^*(SO(n)^2_{hO(n)})$.

**Proof.** Recall that $y$ usually refers to class in the mod 2 cohomology of $H^*(S^n)$. Here, we have two projections $\pi_1, \pi_2$ from $SO(n + 1)^2_{hSO(n)}$ to $S^n$, and the pullback of $y$ along these projections are $y \otimes 1$ and $1 \otimes y$ respectively.

Now consider the following map of fibrations:

$$
\begin{array}{ccc}
SO(n + 1)^2 & \rightarrow & SO(n + 1)^2_{hC_2} \\
\downarrow & & \downarrow \\
SO(n + 1)^2_{hSO(n)} & \rightarrow & SO(n + 1)^2_{hO(n)} \\
\end{array}
\rightarrow

BC_2

The left and right vertical maps are injective on cohomology, so since all the classes of the form $a \otimes a$ of $H^*(SO(n + 1)^2 \otimes H^0(BC_2))$ survive to $E_\infty$ (because of $P(a)$), the same is true of classes in $H^*(SO(n + 1)^2)$ that pullback to ones of the form $a \otimes a$, for example $y \otimes y$. By looking at the relative Serre spectral sequences with respect to $SO(n) < SO(n + 1)$, this class is seen to be unique as described. $\square$

**Proposition 4.5.** There are natural transformations

\[ \delta : H^j(Z) \rightarrow H^{j-n}(F_n(Z)) \]

\[ \phi_i : H^j(Z) \rightarrow H^{2j-i}(F_n(Z)) \]

such that if $q^* : H^*(F_n(Z)) \rightarrow H^*(Z^{S^n})$ is the natural transformation induced from the quotient map, then $q^*(\delta(a)) = \theta_n(da)$, $q^*(\phi_i(a)) = \theta_n(Sq_i a)$ for $i < n$, and $q^*(\phi_n(a)) = \theta_n(Sq_n a + ada)$.

**Proof.** The case of $\delta(a)$ was treated in Lemma 4.2.

For $\phi_i(a)$, we first claim that the class $P(a) \in H^*(Z^2_{hC_2})$ is sent to the sum of the three terms $P(\theta_n(a)) + \tau_{SO(n)}^{O(n)}(a) \otimes \theta_n(da) \otimes 1 \otimes y + P(\theta_n(da))P(y)$ in $(Z')^2_{hO(n)}$.

To do this, we observe the map $Z^2_{hO(n)} \rightarrow Z^2_{hC_2}$ factors through the map $(Z^{S^n} \times S^n)^2_{hC_2} \rightarrow Z^2_{hC_2}$ given by evaluation on each factor. Via the latter map, by Lemma 3.6 $P(a)$ pulls back to $P(\theta_n(a) \otimes 1 + \theta_n(da) \otimes y)$. $P(\theta_n(a) \otimes 1 + \theta_n(da) \otimes y)$ is equal to $P(\theta_n(a)) + P(\theta_n(da))P(y) + \tau^2_{C_2}(\theta_n(a) \otimes \theta_n(da) \otimes 1 \otimes y)$, and so pulling back to $Z^2_{hO(n)}$ yields the claim.

Next we study each of the three terms. We can compute the image of $P(\theta_n(a))$ in $Z^0_{S^n} \otimes \mathbb{R}P^n$ by examining the commutative square:
$P(\theta_n(a))$ is pulled back from the class with the same name in $H^*((Z^S_n)^2_{hC_2}\times\text{BSO}(n))$. That class pulls back to $\sum_i Sq_i \theta_n(a)t^i$ in $H^*(Z^S_n\times\mathbb{R}P^n)$ essentially by definition of the Steenrod operations.

For $\tau^{O(n)}_{SO(n)}\theta_n(a)\otimes\theta_n(da)\otimes 1\otimes y$, by the naturality of the pushforward, its image in $H^*(Z^S_n\times\mathbb{R}P^n)$ can be obtained by pulling back the class $\theta_n(a)\otimes\theta_n(da)\otimes 1\otimes y \in H^*((Z^S_n\times S^n)^2)$ along the diagonal to $Z^S_n\times S^n$, and then pushing forward to $Z^S_n\times\mathbb{R}P^n$. Pulling back yields $\theta_n(ada)\otimes y$, and the pushing forward is a product with the pushforward map $S^n\rightarrow\mathbb{R}P^n$, giving the class $\theta_n(ada)\otimes t^i$.

Finally, we examine the class $P(y)$ using the fact that it is pulled back from $(\text{SO}(n))^2_{hO(n)}$ and the commutative diagram

\[
\begin{array}{ccc}
Z' \times \mathbb{R}P^n & \rightarrow & Z^2_{hO(n)} \\
\downarrow & & \downarrow \\
Z^S_0 \times \mathbb{R}P^n & \rightarrow & (Z^S_0)^2_{hC_2}\times\text{BSO}(n)
\end{array}
\]

Then, for degree reasons, the class $P(y)$ pulls back along the bottom map to $\sum c_i \otimes c_i$, where $c_i$ are classes of degree $i > 1$.

Putting it all together, the class $P(\theta_n(a)) \in H^*(Z^2_{hC_2})$ is sent to $\sum c_i^i \otimes t^i + \sum_i \theta_n(Sq_ia)\otimes t^i + \theta_n(ada)\otimes t^n$ in $H^*(Z'\times\mathbb{R}P^n)$ where $c_i$ are classes of positive degree. Because in the commutative diagram (1) above, the map indicated by $\rightarrow$ is injective, this means that $P(\theta_n(a))$ is sent to $\sum_n \theta_n(Sq_ia)\otimes t^i + \theta_n(ada)\otimes t^n$ in $H^*(Z^S_n\times\mathbb{R}P^n)$. Since the pushforward map $\tau^{O(n)}_{SO(n+1)} : H^*(Z^S_0\times\mathbb{R}P^n) \rightarrow H^*(Z^S_n)$ extracts the power of $t^n$, $t^i P(a)$ (and thus $\phi_i(a)$) is sent to $\theta_n(Sq_{n-i}a)$ for $i \neq 0$ and $\theta_n(Sq_n a - ada)$ for $i = 0$. 

\textbf{Verification of the relations.} Now that we have constructed the classes that survive to $E_\infty$, it remains to verify a few crucial relations among the real classes. Specifically, we show $w_{n+1}\delta(a) = 0$ and compute the action of the Steenrod algebra on $\phi_i$ and $\delta$.

To begin verifying the relations, we study the following diagram, where $\text{SO}(n)$ acts trivially on the spaces in the left side of the diagram:
The only square that does not obviously commute is the top left one. But the corresponding maps of spaces commutes up to homotopy since evaluation at any two points give homotopic maps as $S^n$ is connected.

**Lemma 4.6.** The class $a \otimes b \in H^*(Z^2) \to H^*(Z^2_{SO(n)})$ maps to $\sum_0^n t^i q^*(w_i \delta(ab))$ in $H^*(Z^2_{SO(n)})$ in diagram [2], where $q : Z^2_{SO(n)} \to F_n(Z)$ is the quotient map, $q^*(w_1) = 0, q^*(w_0) = 1$. This implies $w_{n+1} \delta(a) = 0$.

**Proof.** Moving the class $a \otimes b$ through the top row of the diagram gives $\delta(ab)$. $H^*(Z^2_{SO(n)})$ is a free module over $H^*(F_n(Z))$ with basis $t^i$, $0 \leq i \leq n$ (see Lemma 3.1), and $\tau^O_{SO(n+1)}$ extracts the coefficient of $t^n$. Thus the $t^n$ coefficient of the image of $a \otimes b$ is $\delta(ab)$. Now in $H^*(Z^2_{SO(n)})$, $\tau^O_{SO(n)} a \otimes b$ is killed by multiplication by $b$, so the same is true of the image of $a \otimes b$. But if $q$ is the map $Z^2_{SO(n)} \to F_n(Z)$, then $t^{n+1} = t^{n-1} q^* w_2 + t^{n-2} q^* w_3 + \cdots + q^* w_{n+1}$. Thus if $t(t^n \delta(ab) + \sum_0^{n-1} c_i t^i) = 0$, we must have that $c_i = \delta(ab) q^* w_{n-i} (c_{n-i} = 0)$, and $q^* w_{n+1} \delta(ab) = 0$, giving the result. 

Next, we look at the Steenrod action on $\delta(a), \phi_i(a)$. There is only one step in the construction of $\delta(a), \phi_i(a)$ that does not necessarily commute with the Steenrod action, namely the pushforward map. Nevertheless, the interaction of the pushforward with the Steenrod squares is understandable via the Stiefel-Whitney classes of the normal bundle of the fibration.

**Lemma 4.7.** Let $f : A \to B$ be a fibration whose fibres are manifolds of dimension $n$, let $\nu$ denote the normal bundle of $f$, $w_\nu$ its total Stiefel-Whitney class, and $\tau_f$ the pushforward map of $f$. Then $Sq(\tau_f(a)) = w_\nu, Sq(a)$.

**Proof.** Recall that the pushforward is the composite of the Thom isomorphism of $\nu$ with the the Gysin map $H^*(A^\nu) \to H^*(B)$ in cohomology. Since the latter map comes from a map of spectra, it commutes with Steenrod operations, and the former is given by multiplication by the Thom class $u$. $Squ = w_\nu, u$ essentially by definition.
of the $w_i$ [MS74], so $Sq(ua) = SquSqa = w_i u Sqa$. Then composing with the Gysin map gives the result. 

Lemma 4.8. The relations

\[ Sq\phi_{n-k}(a) = \sum_{j \geq 0} \sum_{i} \binom{k + |a| - j}{i - 2j} \phi_{n-i-k+2j}(Sq^i a) + \delta_{k0} \sum_{2j < i} \delta(Sq^j a \times Sq^{i-j} a) \]

\[ Sq^i \delta(a) = \delta(Sq^i a) \]

hold.

Proof. For $\delta(a)$, by Lemma 4.7, we have that $Sq \delta(a) = w_\nu \delta(Sq(a))$. However, $w_\nu$ is easy to compute: $\nu$ is the pullback of the normal bundle of the fibration $\text{BSO}(n) \to \text{BSO}(n+1)$. But $-\nu$ is the bundle that gives the tangent of the fibre, which is the tautological bundle with Stiefel Whitney classes $w_i$. Thus $w_\nu = (\sum w_i)^{-1}$. Putting this together gives relation (7).

For $\phi_i(a)$, because we started with the class $t^i P(a)$, it suffices to understand the action of the Steenrod squares on that. This is given by the Nishida relations (see [Nis68]):

\[ Sq(t^i P(a)) = \sum_{i > 0} \left( \sum_{j \geq 0} \binom{k + |a| - j}{i - 2j} t^{i+k-2j} P(Sq^j a) + \delta_{k0} \sum_{2j < i} \tau^{'i,j}(Sq^j a \otimes Sq^{i-j} a) \right) \]

which gives the desired result. □

Now, we can construct the natural transformation $\eta_n$.

Proposition 4.9. Let $Z$ be a space. For $a \in H^*(Z)$, the map sending $\overline{w}_i, \overline{\delta}(a), \overline{\phi}_i(a)$ to the image of $w_i, \delta(a), \phi_i(a)$ in the associated graded, with $w_i$ in grading $i$ and $\delta(a), \phi_i(a)$ in grading $0$, defines a natural transformation $\eta_n : L_n(H^*(Z)) \to grH^*(F_n(Z))$.

Proof. Because $\overline{L}_n$ is presented by generators and relations, it suffices to check that all of the generating relations among $\overline{w}_i, \overline{\delta}(a), \overline{\phi}_i(a)$ are satisfied in the declared images. We do this in the same order that the relations are listed in Definition 2.7.

(1) This is linearity of $Sq_i$.
(2) This is linearity of $d$.
(3) This follows from the Liebniz rule for $d$.
(4) This follows since the $d_{n+1}$ differential applied to $a$ is $w_{n+1} da$.
(5) We can calculate $d(a) Sq_i(b) = d(a Sq_i b)$ for $i < n$ and $d(a) (Sq_n(b) - b db) = d(ab) db$.
(6) This is essentially the Cartan formula.
(7) Lemma 4.8.
(8) Lemma 4.8, or alternatively the fact that $Sq_i$ commutes with $d$. 
(9) This is the instability condition.
(10) This is the compatibility of $Sq^0$ with the algebra structure.
(11) This is the Adem relations.
(12) This follows from the fact that $Sq^0(Sq^i a) = (Sq^i a)^2$.

\[\square\]

5. The Main Spectral Sequence

In this section, we use the results of section 4 to prove Theorem A and Theorem B. Namely when $Z \in EM_{F_2}$, we completely describe the main spectral sequence and show that the natural transformations $\delta, \phi_i$ constructed in section 4 generate for $H^*(F_n(Z))$. Recall that we can consider $\theta_n(a)$ and $\theta_n(da)$ as elements of $E_0^{0,*}$.

**Proposition 5.1.** For $Z$ an $F_2$-EM space, there are no differentials in the main spectral sequence until the $E_{n+1}$ page. On the $E_{n+1}$ page, the differential is given by $d_{n+1}(\theta_n(a)) = w_{n+1}\theta_n(da)$ for $a \in H^*(Z^{S^n})$.

**Proof.** The cohomology of $H^*(Z^{S^n})$ is generated by classes of the form $\theta_n(a)$ and $\theta_n(da)$, where $a$ is a class in $H^*(Z)$ pulled back via evaluation at a point. By the Liebniz rule, it suffices to just understand differentials on $\theta_n(a)$, and $\theta_n(da)$ for these classes. But by naturality of the differential and $\theta$, and the fact that $a$ is pulled back from an Eilenberg-Mac Lane space, we just need to prove the assertions when $Z = K(\mathbb{Z}/2, m)$ and the cohomology of the fibre is a free unstable algebra on $a$ and $da$. By Lemma 4.2, the class $\theta_n(da)$ survives to $E_{\infty}$, so the unstable algebra generated by $\theta_n(da)$ cannot have any differentials by the universality argument along with the Liebniz rule. We know that $w_{n+1}\theta_n(da) = 0$, so that class must get killed in the spectral sequence, and so for degree reasons, the only class that can kill $w_{n+1}\theta_n(da)$ is $\theta_n(a)$, giving the desired $E_{n+1}$ differential. This also shows that $\theta_n(a)$ cannot have any differentials before the $E_{n+1}$ page, and hence that nothing else can either. \[\square\]

Below is a picture of part of the main spectral sequence, where $a \in H^m(Z)$. 

Remark 5.1.1. Note that the $E_{n+1}$ differential commutes with the Steenrod algebra action on the columns of the $E_2$-page of the sequence.

Proposition 5.2. The classes $\theta_n(da), \theta_n(Sq_i a), \theta_n(Sq_n a + ada)$ for $a \in H^*(Z)$ along with the $w_i$ generate the $E_{n+2}$ page of the main spectral sequence as an algebra for $Z \in EM_{F_2}$.

Proof. Since the differential for the $E_{n+1}$ page is $d_{n+1}(\theta_n(a)) = w_{n+1}\theta_n(da)$, it suffices to show that $\ker(d) \subset H^*(Z^{S^n})$ is generated by $\theta_n(Sq_i a), \theta_n(Sq_n a + ada), \theta_n(da)$. This in turn is true if and only if the cohomology of $d$ acting on $H^*(Z^{S^n}) = L_n(H^*(Z))$ is generated by $Sq_i a, Sq_n a + ada$ for $i < n$.

Write $Z = \prod_{s \in S_m} K(Z/2, m)$, where $S_*$ is a finite nonnegatively graded set. Then $L_n(H^*(Z))$ with its differential is a tensor product of $L_n(H^*(K(Z/2, m)))$ over the set $S$. Thus we can reduce to the case when $Z = K(Z/2, m)$. Denote $Sq_I = Sq_{a_1} Sq_{a_2} \ldots Sq_{a_k}$ an admissible sequence of excess $< m$ and $i$ the generator in degree $m$. These $Sq_I$ generate $H^*(Z)$ as a polynomial algebra.

In order for $I = (a_1, \ldots, a_k)$ to be admissible of excess $< m$, the $a_i$ must be an increasing sequence of positive numbers less than $m$. Let $A_m$ be the set of admissible sequences of Steenrod operations of excess $< m$.

Observe that $L_n(H^*(Z))$ is generated as an algebra by

- $Sq_I$ with leading term $Sq_i$ for $i \neq n$ and $Sq_I \in A_m$
- $Sq_n Sq_I + Sq_I d Sq_I$ for $Sq_n Sq_I \in A_m$
- $d Sq_I$ for $I \in A_{m-1}$
Indeed, these generators are obtained by adding decomposable elements to the usual set of generators. Next, observe that the generators listed above that are nonzero are actually a set of free generators in the sense that the only relations among them are that \( x^2 = x \) when \( x \) is a generator of degree 0.

\[ d(Sq_{l}) = 0 \] when the first term is \( Sq_{i} \), with \( i < n \), \( d(Sq_{l}) = Sq_{l-n}dI \) where \( I - n \) is the sequence where \( n \) is subtracted from every term in \( I \). Thus by pairing up each generator \( x \) with \( dx \) we have decomposed \( H^{*}(X) \) as a tensor product of differential graded algebras. By the Kunneth formula, the cohomology of the differential graded algebra is generated by

\[ (G1) \quad Sq_{0}Sq_{l}(t) = (Sq_{l}t)^{2} \] where \( Sq_{l} \) has leading term \( Sq_{i} \) for \( i > n \).

\[ (G2) \quad Sq_{l}t \] with leading term \( Sq_{i} \) for \( i < n \).

\[ (G3) \quad Sq_{n}Sq_{l} + Sq_{l}dSq_{l} \] for \( Sq_{n}, Sq_{l} \in A_{m} \).

The only relation among the generators is when \( m = n \), where one has the relation \( Sq_{0}t = (Sq_{n}t + ldI)^{2} \), and otherwise, the classes above freely generate the cohomology in the sense above.

**Theorem A.** For \( Z \) an \( F_{2} \)-EM space, there are no differentials in the main spectral sequence until the \( E_{n+1} \) page. On the \( E_{n+1} \) page, the differentials are determined by \( d_{n+1}(\theta_{n}(a)) = w_{n+1}\theta_{n}(da) \) and \( d_{n+1}(\theta_{n}(da)) = 0 \) for \( a \in L_{n}(H^{*}(Z)) \). Moreover, \( E_{n+2} = E_{\infty} \).

**Proof.** This follows immediately from Proposition 4.5, 5.1, and 5.2.

**Theorem B.** There are natural transformations

\[ \delta : H^{j}(Z) \to H^{j-n}(F_{n}(Z)) \]

\[ \phi_{i} : H^{j}(Z) \to H^{2j-i}(F_{n}(Z)) \]

such that if \( q^{*} : H^{*}(F_{n}(Z)) \to H^{*}(Z^S) \) is induced from the inclusion of the fibre, then \( q^{*}(\delta(a)) = \theta_{n}(da) \), \( q^{*}(\phi_{i}(a)) = \theta_{n}(Sq_{n}a) \) for \( i < n \), and \( q^{*}(\phi_{n}(a)) = \theta_{n}(Sq_{n}a + ada) \).

When \( Z \in EM_{F_{2}} \), the images of \( \phi_{i}, \delta \) along with \( w_{i} \) generate \( H^{*}(F_{n}(Z)) \) as an algebra.

**Proof.** Everything but the last statement is in Proposition 4.5. For the last statement, by Proposition 5.2 we learn that the image of the natural transformations generate the associated graded algebra, and hence also generate \( H^{*}(F_{n}(Z)) \) itself.

**Remark 5.2.1.** Theorem B says that when \( Z \in EM_{F_{2}} \), then all of the classes in \( H^{*}(F_{n}(Z)) \) come from \( w_{i}, \delta, \phi_{i} \). The origin of \( \delta, \phi_{i} \) are the equivariant evaluation maps \( ev_{0}, ev_{1} \), so it’s surprising that these essentially account for the entire cohomology of \( F_{n}(Z) \). Note that by Lemma 3.1, for any \( SO(n+1) \)-space, \( H^{*}(X_{hSO(n)}) \) is a free module over \( H^{*}(X_{hSO(n+1)}) \) and the pushforward is a surjective module map. Thus the pushforward is not the surprising part of the construction of \( \phi_{i} \), but rather the evaluation map is.
6. DGA computations

The goal of this section is to prove Theorem C. The work done here is completely algebraic: we show that when \( Z \in EM_{F_2} \) that the natural transformation \( \eta_n \) is an isomorphism, and both sides are defined algebraically. Our strategy is to break up both sides into manageable pieces, and identify the pieces.

First we define functors \( E_{n+1}, E_\infty \) that describe algebraically the \( E_{n+1} \)-page and \( E_\infty \)-page of the main spectral sequence for \( Z \in EM_{F_2} \).

**Definition 6.1.** Given \( A \in K \), \( \overline{E}_{n+1}(A) \) is defined as the bigraded algebra given by \( L_n(A) \otimes H^*(BSO(n+1)) \) where the first tensor factor is in bidegrees \((0,*)\) and the second is in bidegree \((*,0)\). The Steenrod square \( Sq^i \) acts in the \((0,*)\) direction, so that \( \overline{E}_{n+1} \) can be viewed as an object of \( K^{gr} \).

**Definition 6.2.** The functor \( E_\infty : K \to K^{gr} \) defined to be the cohomology of \( \overline{E}_{n+1}(A) \) with respect to the differential \( d \).

\( E_\infty \) of course depends on \( n \), but we suppress the dependence in the notation.

**Lemma 6.3.** When \( Z \in EM_{F_2} \), \( E_\infty \) can be identified with the \( E_\infty \)-term of the main spectral sequence.

**Proof.** Recall that for \( Z \in EM_{F_2} \), the \( E_{n+1} \) page of the spectral sequence is the tensor product of \( H^*(BSO(n+1)) \) and \( H^*(F_n(Z)) = L_n(H^*(Z)) \) by Proposition 2.4, and by Theorem A the result immediately follows. \( \square \)

**Definition 6.4.** Define the natural transformation \( \overline{\eta}_n : \overline{\ell}_n \to \overline{E}_\infty \) which sends \( \delta(a) \mapsto da \), \( \phi_i(a) \mapsto Sq^i a \) for \( i < n \), and \( \phi_n(a) \mapsto Sq^i a + ada \). These images in \( L_n \) clearly are in the kernel of \( d \) and hence in \( \overline{E}_\infty \), and all the relations of \( \overline{\ell}_n \) are satisfied.

**Remark 6.4.1.** Definition 6.4 is compatible with our previous definition of \( \overline{\eta}_n \) when \( A = H^*(Z) \) for \( Z \in EM_{F_2} \).

In order to study \( \overline{\ell}_n \), we break it up into smaller pieces which contain essentially all of its information.

**Definition 6.5.** \( \overline{\ell}'_n \) is defined to be the quotient of \( \overline{\ell}_n \) by \( \overline{w}_i, i \leq n+1 \). \( \overline{\ell}'_n/\delta \) is defined to be the quotient of \( \overline{\ell}'_n \) by \( \overline{\delta}(a) \) for all \( a \).

Note that \( \overline{\ell}'_n \) and \( \overline{\ell}'_n/\delta \) live entirely in grading zero, so can be viewed as objects in \( K \).

**Lemma 6.6.** Lemma 2.7 gives a presentation for \( \overline{\ell}'_n \) in \( K \) if the \( \overline{w}_i \) generators as well as relation (4) are removed. Furthermore, \( \overline{\ell}'_n/\delta(A) \) is generated by \( \overline{\phi}_i(a) \) for \( a \in A \) modulo the following relations:
(1) \( \bar{\phi}(a + b) = \bar{\phi}(a) + \bar{\phi}(b) \)
(2) \( \bar{\phi}_k(ab) = \sum_{i+j=k} \bar{\phi}_i(a) \bar{\phi}_j(b) \)
(3) \( Sq^j \bar{\phi}_{n-k}(a) = \sum_j (k + |a| - j) \bar{\phi}_{n-i-k+2j}(Sq^j a) \)
(4) \( Sq_i(\bar{\phi}_j(a)) = 0 \) for \( i < 0 \)
(5) \( Sq_0(\bar{\phi}_j(a)) = \bar{\phi}_j(a)^2 \)
(6) \( \bar{\phi}_k(Sq_a a) = \sum_{j=0}^{[2|a|-i-k]/2} (|a| - i - j - 1) \bar{\phi}_{2j-2|a|+2i+k}(Sq^j a) \) for \( k > i \).
(7) \( \bar{\phi}_0(Sq_a a) = \bar{\phi}_j(a)^2 \)

with \( \bar{\phi}_m(a) = 0 \) for \( m > n \).

**Proof.** The first statement is clear since the \( w_i, i \leq n \) are not involved in the relations, and the only relation on \( w_{n+1} \) is \( w_{n+1} \bar{\delta}(a) = 0 \).

The second statement is also clear since \( \ell'_n/\bar{\delta} \) is obtained from \( \ell'_n \) by adding more relations, namely killing the \( \bar{\delta} \) classes.

We break up \( \overline{E}_\infty \) into smaller pieces.

**Definition 6.7.** \( E'_\infty \) is the quotient of \( \overline{E}_\infty \) by the \( w_i \). \( E'_\infty/d \) is the quotient of \( E'_\infty \) by elements of the form \( da \).

The natural map \( \ell'_n \to \overline{E}_\infty \to E'_\infty \) sends the \( w_i \) to 0, so factors as a map \( \eta'_n : \ell'_n \to E'_\infty \). Further quotienting the map to \( E'_\infty/d \) factors makes the map factor through \( \ell'_n/\bar{\delta} \), giving a map we can call \( \eta'_n/d \)

Note that \( E'_\infty(A) \), \( E'_\infty(A)/d \) can also be described as follows: Consider \( L_n(A) \) as a differential graded algebra, where the differential is \( a \to da \). Then \( E'_\infty(A) \) is the kernel of \( d \) and \( E'_\infty(A)/\bar{\delta} \) is the cohomology of \( d \).

Finally, we need an algebraic version of the pushforward map \( \tau_{SO(n+1)} \), and a ring map from \( \ell'_n \) into a subalgebra of \( L_n \) that is an algebraic version of the quotient map in cohomology.

**Definition 6.8.** The map \( \tau : L_n(A) \to \ell'_n(A) \) is an \( \mathbb{F}_2 \)-linear map that sends \( adb_1 \ldots db_n \) to \( \delta(a)\delta(b_1) \ldots \delta(b_n) \). The map \( i : \ell'_n(A) \to L_n(A) \) is a ring map sending \( \delta(a) \) to \( da \), \( \phi_i(a) \) to \( Sq_a a \) for \( i < n \) and \( \phi_n(a) \) to \( Sq_n a + ada \).

**Proposition 6.9.** The maps \( i, \tau \) are well-defined, and the differential \( d \) on \( L_n \) factors as \( i \circ \eta'_n \circ \tau \). Moreover, \( \tau \circ \eta'_n = 0 \) and \( \tau \) fits in an exact sequence \( L_n(A) \xrightarrow{\tau} \ell'_n(A) \to \ell'_n/d(A) \to 0 \).

**Proof.** To check \( \tau \) is well defined, it suffices to check the ideal generated by relations (1), (2), (3), (4) from Proposition 2.8 is sent to 0. Relation (1) is easy and relation (3), (4) follows from the fact that \( \ell'_n \in \mathcal{K} \). To check (2), if we have an element of the form \( adb_1 \ldots db_n(d(xy) - d(x)y + d(y)x) \), it is sent to \( \delta(b_1) \ldots \delta(b_n)(\delta(ay)\delta(x) + \delta(xy)\delta(a) + \delta(ax)\delta(y)) = 0 \).
To check \( i \) is well defined, we must check that the relations for \( \ell'_n \) hold in \( L_n(A) \). To see this, by naturality, it suffices to prove the relations hold in the universal cases. But the universal cases are \( F_2 \)-EM spaces, where we have an identification of \( L_n(H^*(Z)) \) with \( H^*(Z^{\infty}) \) which agrees with our defined map.

\( \Box \)

**Lemma 6.10.** The natural map \( \overline{\eta}_n : \overline{\ell}_n(A) \to \overline{E}_\infty(A) \) is an isomorphism iff \( \eta'_n/d(A) \) is an isomorphism.

**Proof.** We observe from the presentation that \( \overline{\ell}_n(A) \) decomposes as a direct sum
\[
\bigoplus_{\alpha_i \geq 0} w_2^{\alpha_2} \cdots w_n^{\alpha_n} \ell'_n(A) \oplus \bigoplus_{\beta_i \geq 0, \beta_{n+1} > 0} w_2^{\beta_2} \cdots w_{n+1}^{\beta_{n+1}} \ell'_n/\delta(A)
\]

Similarly, \( \overline{E}_\infty \) decomposes as
\[
\bigoplus_{\alpha_i \geq 0} w_2^{\alpha_2} \cdots w_n^{\alpha_n} E'_\infty \oplus \bigoplus_{\beta_i \geq 0, \beta_{n+1} > 0} w_2^{\beta_2} \cdots w_{n+1}^{\beta_{n+1}} E'_\infty/d
\]

via this decomposition the map \( \overline{\ell}_n(A) \to \overline{E}_\infty(A) \) splits as a sum of copies of \( \eta'_n/d(A) \) and \( \eta'_n(A) \) accordingly. Thus it suffices to prove that \( \eta'_n/d(A) \) and \( \eta'_n(A) \) are isomorphism.

We now show that \( \eta'_n/d \) being an isomorphism implies \( \eta'_n \) is too. \( E'_\infty \) is generated by lifts of \( E'_\infty/d \) as well as the elements \( da \). Because \( \eta'_n/d \) is surjective, and \( da \) is hit by \( \delta(a) \), the map is surjective. For injectivity, suppose \( a \) is in the kernel of \( \eta'_n \). Then because \( \eta'_n/d \) is injective, we can assume \( a \) is in the kernel of \( d \). But by Proposition 6.9, this means \( a = \tau(b) \) where \( db = 0 \). By surjectivity of \( \eta'_n/d \) this means that \( b \) is the sum of \( dc \) and a class in the image of \( \eta'_n \). But \( \tau(dc) = 0 \) and \( \tau \circ \eta'_n = 0 \), which shows \( a = 0 \).

\( \Box \)

The next lemma is the key to proving that \( \overline{\eta}_n \) is an isomorphism.

**Lemma 6.11.** For \( Z \in EM_{F_2} \), \( \eta'_n/d \) is an isomorphism.

**Proof.** We will use notation from Proposition 5.2. Write \( Z = \prod_{s \in S_m} K(\Z/2, m) \), where \( S_s \) is a finite nonnegatively graded set, and let \( A_m \) be the admissible sequences of excess \( < m \). In the proof of Proposition 5.2, it is shown that \( E'_\infty/d(H^*(Z)) \) is generated by the nonzero elements of the below list:

- \( (G1) \) \( S_{q_0} S_{q_1} t_s \) where \( S_{q_1} \in A_s \) has leading term \( S_{q_i} \) for \( i > n \).
- \( (G2) \) \( S_{q_1} t_s \) with leading term \( S_{q_i} \) for \( i < n \).
- \( (G3) \) \( S_{q_n} S_{q_1} t_s + S_{q_1} d S_{q_1} t_s \) for \( S_{q_n} S_{q_1} \in A_m \)

Moreover, the only relation among the generators is when \( |s| = n \), where the relation \( S_{q_0} t_s = (S_{q_1} t_s + t_s d t_s)^2 \) holds, and when \( |s| = 0 \), where the relation \( t_s^2 = t_s \) holds.

We can thus define an algebra map \( \zeta : E'_\infty(H^*(Z))/d \to \ell'_n(H^*(Z))/d \) via the following prescription:
• Send $Sq_0 S_{q_1 t}$ in (G1) to $\phi_0 (S_{q_1 t})$.
• Send $Sq_1 S_{q_1 t}$ in (G2) to $\phi_i (S_{q_1 t})$, where $I'$ is $I$ with $i$ removed.
• Send $Sq_n S_{q_1 t} + S_{q_1 t} d S_{q_1 t}$ in (G3) to $\phi_n (S_{q_1 t})$.

Relation (7) for $\ell_n / \delta$ shows that this does give an algebra homomorphism, and it is clear that that $\eta_n / d \circ \zeta = id$, implying that $\zeta$ is injective. Thus it remains to check that $\zeta$ is surjective. Due to the multiplicative and additive relations for $\phi_i$, it suffices to check that $\phi_i (g)$ is in the image for each multiplicative generator $g$ of $H^* (Z)$.

(1) $\phi_0 (S_{q_1 t})$ is automatically in the image for $Sq_i$ leading with $Sq_i$ for $i > n$.
For $i \leq n$, relation (7): $(\phi_i (a))^2 = \phi_0 (Sq_i a)$ shows that it is in the image.

(2) For $i > 0$, $\phi_i (S_{q_1 t})$ is automatically in the image for $Sq_i$ leading with $Sq_j$ for $j \geq i$. For $j < i$, the Adem relations (6) lets us express $\phi_i (S_{q_1 t})$ using terms involving
$$\sum \phi_\alpha (Sq_\beta t)$$
with $\alpha \leq \beta < j$ which is in the image.

Now we can complete the proof of Theorem C.

**Theorem C.** The natural transformation $\eta_n$ realizes $\ell_n$ as the zeroth derived functor of $Z \mapsto grH^* (F_n (Z))$.

**Proof.** By Lemma 2.1, it suffices to see that $\ell_n$ preserves reflexive coequalizers and that $\eta_n$ is an isomorphism for $Z \in EM_{F_2}$. The first is clear: any functor defined in terms of generators and relations preserves reflexive coequalizers. The second follows from Lemma 6.10 and Lemma 6.11.

7. Questions and Further Directions

In this section we offer some questions further directions to be explored.

In this paper, we compute the zeroth derived functor of the functor $grH^* (F_n (Z))$ as our functor $\ell_n$, but we do not compute the zeroth derived functor of $H^* (F_n (Z))$.

**Notation 7.1.** $\ell_n$ denotes the zeroth derived functor of $Z \mapsto H^* (F_n (Z))$.

We expect the following question to be answerable through it is not completely answered here.

**Question 7.2.** Can one present $\ell_n$, using $\phi_i, \delta, \omega_i$ as generators?

Theorem B implies that in order to answer the above question, one only needs to find lifts of the relations in Definition 2.7 to define $\ell_n$ that hold among the classes $\phi_i (a), \delta (a), \omega_i$, then $\ell_n$ would be presented using those relations$^2$. We do provide lifts$^2$.

$^2$In the case $n = 1$, all the relations are proven, recovering the presentation of the functor $\ell$ due to Ottosen and Bökstedt.
of some of the relations defining $\tilde{\ell}_n$ throughout this paper, which are amalgamated in Appendix B.

The zeroth derived functor $\ell_n$ is just the first in a family of (nonabelian) derived functors that approximate $H^*(F_n(Z))$. The higher derived functors are defined purely algebraically in relation to the zeroth one.

**Question 7.3.** What can be said about the higher derived functors of $\ell_n$?

For example, can they be nonzero for arbitrary $i$, and are they at all computable? There is a Bousfield homology spectral sequence whose $E_2$ term is the derived functors of $\tilde{\ell}_n(H^*(Z))$ and which (under favorable conditions) converges to $H^*(F_n(Z))$, which was worked out in the case $n = 1$ in [BO04]. Moreover for $n = 1$ and $Z = \mathbb{CP}^n$, Ottosen and Bökstedt find that the spectral sequence degenerates at $E_2$, converging to $H^*((\mathbb{CP}^n)^{S_1}_{hSO(2)})$.

Next, we speculate about the extent to which our results can be generalized.

**Question 7.4.** For which real $G$-representations $V$ can one compute $H^*(Z_{S(V)}^{gG})$, where $Z \in EM_{F_2}$?

In the above question, $S(V)$ are the unit vectors in the representation $G$. We suspect that for most of the classical groups with their standard representations, our results should generalize in a straightforward manner, namely for $U(n)$ and $Sp(n)$. Our techniques don’t immediately generalize to the groups $SO(2n)$ however (for example Construction 4.3 doesn’t immediately generalize), but it may nevertheless be possible to make the computation for this family as well.

**Question 7.5.** Are there analogs of our computations at odd primes?

In [Ott03], Ottosen constructs an analog of $\ell_1$ at odd primes, but it is unclear how a generalization of this would manifest for larger $n$. Namely, the construction of $\ell_1$ at odd primes fundamentally uses the subgroups $C_p \subset SO(2)$. At the prime 2, the analog of these for larger $n$ was $O(n) \subset SO(n + 1)$, but there isn’t an obvious odd primary analog.

Finally, it would be interesting to know exactly what dependence there is of the functor $\ell_n$ on the Steenrod operations. An interesting fact in the case $n = 1$ observed in [BO99] is that the underlying algebra of the functor $\ell_1(A)$ can be computed only using the algebra structure on $A$ and the operation $Sq_1$.

**Question 7.6.** Is there an $m$ such that the underlying algebra of the functor $\ell_n$ or $\tilde{\ell}_n$ can be computed using only $Sq_1, \ldots, Sq_m$?

In Appendix A, we examine the analog of the above question non-equivariantly (see Proposition A.6). It seems that the optimal $m$ in the above question must be at least $n$, and is seems likely that for $\tilde{\ell}_n$, the optimal answer is $m = n$. For $\ell_n$ and $n > 1$, we are not sure if there exist any $m$ that works for the above question.
Appendix A. The Top $k$ Squares and Computing Division

Here we explain how to compute Lannes’ division functors, as well investigate which Steenrod operations are necessary to compute the algebra underlying the division functor. This section generalizes to odd primes without significant change.

In [Ott03], the functor $(-): H^*(S^1)$ is constructed using only the operation $Sq_1$. In order to take this into account, Ottosen introduces the category $\mathcal{F}$ of algebras with only an action of $Sq_1$.

A similar category $\mathcal{U}_n$ for unstable modules over the Steenrod algebra has been studied in [Li20], where only the operations $Sq_i$ for $i \leq n$ are remembered. It is significantly simpler than modules over the Steenrod algebra: it is proven there for example that $\mathcal{U}_n$ has homological dimension $n$. Thus it would be useful to know if our approximation functors $L_n$ and $\ell_n$ can be constructed after forgetting from $\mathcal{U} \to \mathcal{U}_n$.

We want a version of $\mathcal{U}_n$ that remembers the algebra structure as well as the $Sq_i$ for $i \leq n$. We define an analogous category $\mathcal{K}_n$ which only encodes the top $n$ Steenrod squares, generalizing $\mathcal{F}$ which is the case $n = 1$.

**Definition A.1.** The objects of $\mathcal{K}_n$ are nonnegatively graded commutative $\mathbb{F}_2$-algebras $A_*$ together with for $0 \leq i \leq n$, linear maps $Sq_i : A_m \to A_{2m-i}$ satisfying the following requirements:

1. $Sq_0a = a^2$
2. $Sq_1a = a$
3. $Sq_i a = 0$ for $i > |a|$
4. $Sq_i(ab) = \sum_{j=0}^i Sq_j(a)Sq_{i-j}(b)$
5. $Sq_iSq_ja = \sum_k (k-j-1)_{2k-i-j} Sq_{i+2j-k} Sq_{k}a$ for $i > j$

The morphisms are those preserving all present structure. Except for (2), (3), the relations are essentially the relations of the Dyer-Lashof operations on $E_{n+1}$ $\mathbb{F}_2$-algebras with vanishing Browder bracket [Law20]. $\mathcal{K}_0$ is the category of graded $\mathbb{F}_2$-algebras with the degree 0 subalgebra a Boolean algebra, and $\mathcal{K}_1$ is the category $\mathcal{F}$ used in [Ott03; BO04].

There is a natural functor $\mathcal{K} \to \mathcal{K}_n$ forgetting some of the operations, as well as a functor $\mathcal{K}_n \to \mathcal{U}_n$ forgetting multiplicative structure. We investigate to what extent the Steenrod operations are necessary to compute the division functor. In otherwords, we would like to ask for which $m$ there is a factorization in the diagram:

$$
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{(-):N} & \mathcal{K} \\
\downarrow & & \downarrow \\
\mathcal{K}_m & \longrightarrow & \mathcal{K}_n.
\end{array}
$$
To do this, we give a description of the division functor on both $\mathcal{K}$ and $\mathcal{U}$. For another account of the division functor see [Sch94].

$(-) : \mathcal{U}$ $\mathcal{N}$ denotes the left adjoint of the functor $(-) \otimes \mathcal{N}$ from $\mathcal{U}$ to itself. This left adjoint exists when $\mathcal{N}$ is finite type, which is the condition for $(-) \otimes \mathcal{N}$ to preserve limits, at which point one can apply an adjoint functor theorem for existence. From now on, $\mathcal{N}$ is finite type.

As a left adjoint, $(-) : \mathcal{U}$ $\mathcal{N}$ preserves colimits. Since any module can be presented as a cokernel of free modules, in order to compute this functor, we can restrict our attention to free modules first. There is a forgetful functor from $\mathcal{U}$ to graded $\mathbb{F}_2$-vector spaces, let $\mathcal{F}$ be the left adjoint, taking a graded vector space $V_*$ to $\mathcal{F}(V_*)$, the free unstable module on $V_*$. An example of a graded vector space is $\Sigma^n \mathbb{F}_2$, which is $\mathbb{F}_2$ in degree $n$ and 0 elsewhere. Then $\mathcal{F}(\Sigma^n \mathbb{F}_2)$ is a vector space with basis given by $\text{Sq}_I^x$ where $x$ is in degree $n$ and $I$ is an admissible sequence of excess $\leq n$.

Lemma A.2. $\mathcal{F}(V_*) : \mathcal{U}$ $\mathcal{N}$ is canonically isomorphic to $\mathcal{F}(V_* \otimes \mathcal{N}^*)$. The adjunction isomorphism $\nu : \mathcal{U}$ $\text{Hom}(\mathcal{F}(V_* \otimes \mathcal{N}^*), M) \to \text{Hom}(\mathcal{F}(V_*), \mathcal{N} \otimes M)$ is given by the formula $f(\nu(h)(v)) = h(v \otimes f)$ where $f \in \mathcal{N}^*, v \in M$.

Proof. We can compute $\text{Hom}(\mathcal{F}(V_*), \mathcal{U}$ $\mathcal{N}, M) = \text{Hom}(\mathcal{F}(V_*), \mathcal{N} \otimes M) = \text{Hom}_{\mathbb{F}_2}(V_*, \mathcal{N} \otimes M) = \text{Hom}_{\mathbb{F}_2}(V_* \otimes \mathcal{N}^*, M) = \text{Hom}(\mathcal{F}(V_* \otimes \mathcal{N}^*), M)$, and use the Yoneda lemma to conclude. The formula is given by following the isomorphisms. \qed

We use the notation $d_f(v)$ to denote the element $v \otimes f$ in $\mathcal{F}(V_*) : \mathcal{U}$ $\mathcal{N}$. Note that if $v$ is an arbitrary element in $M$ of degree $m$, $d_f(v)$ makes sense: we can think of $v$ as a map from $\mathcal{F}(\Sigma^m \mathbb{F}_2)$ to $M$, and by applying $\mathcal{U}$ $\mathcal{N}$, $d_f(v)$ is the image of the corresponding class in $\mathcal{F}(\Sigma^m \mathbb{F}_2) : \mathcal{U}$ $\mathcal{N}$. The class $d_f(v)$ is clearly natural with respect to maps of modules: given a map $g : M \to M'$, we have $(g : \mathcal{U}$ $\mathcal{N})(d_f(v)) = d_f(g(v))$.

Recall that by convention $\mathcal{N}^*$ is nonzero in nonpositive degree. It has a right action of the Steenrod algebra given by precomposition with the action on $\mathcal{N}$.

Lemma A.3. $d_f(Sq^i(x)) = \sum_0^i Sq^kd_fSq^{i-k}(x)$.

Proof. Let $\nu$ be as in Lemma A.2, $f \in \mathcal{N}^*$ and $h : (M' : \mathcal{U}$ $\mathcal{N}) \to M$ a map. Then we have $h(d_f(Sq^i(a))) = f(\eta(h)(Sq^i(a))) = f(Sq^i\eta(h)(a))$. Choose a basis $x_\alpha$ of $\mathcal{N}$ with...
dual basis $x^*_a$. Then $\eta(h)(a)$ is $\sum_\alpha h(d_{x^*_a}(a)) \otimes x_\alpha$, so

$$f(Sq^i \eta(h)(a)) = f(\sum_i \sum_\alpha Sq^k h(d_{x^*_a}(a)) \otimes Sq^{i-k} x_\alpha)$$

$$= \sum_\alpha \sum_0^i Sq^k h(d_{x^*_a}(a)) f(Sq^{i-k} x_\alpha)$$

$$= \sum_0^i Sq^k h(d_{fSq^{i-k}}(a))$$

which completes the proof, since $h$ is an arbitrary map (it can be taken to be the identity, for example). □

**Proposition A.4.** $M : \mathcal{U} N$ is presented by the generators $df(v)$ for $f \in N^*, v \in M$ along with the following relations for $v, v' \in M$:

1. $df(v + v') = df(v) + df(v')$
2. $df + f(v) = df(v) + df(v')$
3. $Sq^i (df(v)) = df Sq^i (v) - \sum_1^i Sq^{i-j} df Sq^j v$
4. $Sq_i df(v) = 0$ for $i < 0$.

**Proof.** Any module $M$ in $\mathcal{U}$ can be presented via the exact sequence

$$F(\bigoplus_0^\infty \Sigma^i M) \to F(M) \to M \to 0$$

where the second map sends the generator $[v]$ to $v$, and for the first map, we send the generators in $F(\Sigma^i M)$ to $[Sq^i v] - Sq^i [v]$. Now apply the functor $\mathcal{U} N$ to this exact sequence and Lemma A.2 to obtain

$$F(\bigoplus_0^\infty \Sigma^i M \otimes N^*) \to F(M \otimes N^*) \to M : \mathcal{U} N \to 0.$$

Thus we have a presentation of $M : \mathcal{U} N$, and we just need to read off what it says. The surjectivity of second map shows that $df(v)$ generates $M : \mathcal{U} N$, and the relations (1), (2) are obvious. (4) comes from the instability condition which says that $Sq^I x = 0$ when $I$ is an admissible sequence of excess $> |x|$, but this is implied in general by $Sq^I x = 0$ for $i > |x|$ and repeatedly applying relation (3). We get the relation that $d_f([Sq^I x]) = df(Sq^I [x])$, and the right hand side is given by Lemma A.3, giving relation (3). □

This description of the division functor is quite useful. For example, given a module $N$, we can figure out how many operations $Sq_i$ we need to remember in order to produce the underlying $\mathbb{F}_2$-vector space of the functor $M : \mathcal{U} N$. When computing
the underlying $\mathbb{F}_2$-vector space, relation (3) should be thought of as giving a way of computing the Steenrod action, as opposed to imposing more relations. Rather, it is the instability condition (4) that imposes relations. Thus by putting together (4) and (3), we can deduce how much structure is needed to compute the underlying vector space. We sketch how to do this, but do not give all the details for the proofs.

Fix a finite type module $N$. For each $x$ in the Steenrod algebra, we define $\langle x, N \rangle \in \mathbb{N} \cup \infty$ to be the largest $k$ such that there exists a $y \in N$ of degree $k$ with $xy \neq 0$.

**Example A.4.1.** $(1, N)$ is infinite unless $N$ has its degree bounded.

Now define $\beta_N(x)$ inductively by:

- $\beta_N(x) = -\infty$ if $\langle x, N \rangle = 0$
- $\beta_N(x) = \sup_k(\langle x, N \rangle, 2k + \beta_N(xSq^k))$ otherwise.

**Example A.4.2.** Let $N = \Sigma^n\mathbb{F}_2$. Then $\beta_N(x) = -\infty$ if $x \neq 1$, and $\beta_N(1) = n$.

**Example A.4.3.** Let $N = H^*(\mathbb{R}P^4)$. Then $\beta_N(1) = 7, \beta_N(Sq^1) = 3, \beta_N(Sq^2) = 3, \beta_N(Sq^2Sq^1) = 1$, and $\beta_N(x) = -\infty$ for all other $x$.

We are interested in $\beta_N(1)$. In general, $\beta_N(1)$ falls in a predictable range.

**Lemma A.5.** Suppose that $0 < \langle 1, N \rangle = c_N < \infty$. Then $c_N \leq \beta_N(1) \leq 2c_N - 1$.

**Proposition A.6.** The composite $U \xrightarrow{(-)\otimes N} U \rightarrow U_n$ factors through $U_{n+\beta_N(1)}$. $U_{-1}$ can be taken to be graded $\mathbb{F}_2$-modules.

**Proof.** The complete proof is not given here, rather just a sketch. The point is that we can construct the factorization through $U_{n+\beta_N(1)-1}$ by using the same generators as in Proposition A.4, with relations (1), (2), and recursively expand out the relation (4) : $Sq_i d_f(x) = 0$ for $i < 0$ using relation (3).

Relation (3) says that in order to compute $Sq_i d_f(x)$, you need to know $d_f Sq_{i+|f|} (x)$ as well as $Sq_{i+2k} d_f Sq^k (x)$. This recursion relation is exactly complementary to the one defining $\beta_n(x)$; thus we learn that if you know the operations $Sq_i$ for $i \leq \beta_N(x) + k$, then you can compute $Sq_k d_f(x)$ for any $f$ and $a$. Thus to compute $Sq_i$ for $i \leq n$ (i.e factor the functor through $U_n$) we need to know $Sq_i$ for $i \leq n + \beta_N(1)$. □

We believe that the value $n + \beta_N(1)$ in the above proposition is the optimal one making it true, but do not attempt to prove this.

Next we run the the same analysis for the category $\mathcal{K}$, $(-) : \mathcal{K} N$ is the left adjoint of the functor $A \rightarrow A \otimes N$ from $\mathcal{K}$ to itself.

Let $UF$ denote the free unstable algebra generated by a graded vector space, analogous to the functor $F$, except for $\mathcal{K}$. For example $UF(\Sigma^n\mathbb{F}_2)$ is a polynomial algebra on $Sq^I x$, where $x$ is the generator in degree $n$ and $I$ runs over all admissible sequences of excess $< n$ (i.e the cohomology of $K(\mathbb{Z}/2, n)$). We can replace $F$ with $UF$ in the proof of Lemma A.2 to obtain the next lemma.
Lemma A.7. \( UF(V) : N \) is canonically isomorphic to \( UF(V \otimes N^*) \). The adjunction isomorphism \( \nu : \text{Hom}(UF(V \otimes N^*), M) \to \text{Hom}(UF(V), N \otimes M) \) is given by the formula \( f(\nu(h)(v)) = h(v \otimes f) \) where \( f \in N^*, v \in M \).

Once again, we have classes \( d_f(v) \) that satisfy all the relations satisfied in case of modules. However now there are more relations coming from the multiplicative structure. Since \( N \) has a multiplication, \( N^* \) has a comultiplication, which is denoted \( \Delta \).

Lemma A.8. Suppose that \( \Delta(f) = \sum_i (\alpha_i \otimes \beta_i) \). Then \( d_f(ab) = \sum_i d_{\alpha_i}(a)d_{\beta_i}(b) \).

Proof. The proof is similar to Lemma A.3. Let \( \nu \) be as in Lemma A.2, \( f \in N^* \) and \( h : (M' : N) \to M \) a map. Then we have
\[
\begin{align*}
\nu(h(ab)) & = f(\eta(h)(ab)) = f(\eta(h)(a)\eta(h)(b)) = \\
& = \sum_i \alpha_i(\eta(h)(a))\beta_i(\eta(h)(b)) = h(\sum_i d_{\alpha_i}(a)d_{\beta_i}(b)).
\end{align*}
\]
We can take \( h \) to be the identity, to obtain the result.

Note that since \( d_f \) is linear in \( f \), the above lemma does not depend on how \( \Delta(f) \) is presented as a sum of simple tensors. Thus we denote \( \sum_i d_{\alpha_i}(a)d_{\beta_i}(b) \) by \( \Delta d_f(a \otimes b) \), so that the above Lemma can concisely be written as \( d_f(ab) = (\Delta d_f)(a \otimes b) \).

The next Proposition is the analog of Proposition A.4 for \( \mathcal{K} \).

Proposition A.9. \( M :_{\mathcal{K}} N \) is presented by \( d_f(v) \) for \( f \in N^*, v \in M \) along with the following relations:

1. \( d_f(v + v') = d_f(v) + d_f(v') \)
2. \( d_{f + f'}(v) = d_f(v) + d_{f'}(v) \)
3. \( Sq^i(d_f(v)) = d_f Sq^i(v) - \sum_i Sq^{i-j}d_f Sq^j v \)
4. \( Sq^i d_f(v) = 0 \) for \( i < 0 \)
5. \( Sq^0 d_f(v) = (d_f(v))^2 \)
6. \( d_f(vv') = (\Delta d_f)(v \otimes v') \).

Proof. \( M \) can be presented as the free algebra generated by \( M \) as a vector space, along with relations that enforce the algebra structure as well as the Steenrod action. We can view this as the following pushout diagram in \( \mathcal{K} \):

\[
\begin{array}{ccc}
UF(M \otimes M) \otimes UF(\bigoplus_{0}^{\infty} \Sigma^i M) & \longrightarrow & UF(M) \\
& \downarrow & \downarrow \\
0 & \longrightarrow & M
\end{array}
\]

The vertical nonzero map is given by \( [m] \mapsto m \) and the horizontal map sends \( [m \otimes m'] \) to \( [m][m'] - [mm'] \), and \( m \in \Sigma^i M \) to \( Sq^i[m] - [Sq^i m] \).

Applying \(- : N \) since \(- : N \) preserves pushouts, we get the pushout square
\[
UF(M \otimes M \otimes N^*) \otimes UF(\bigoplus_0^\infty \Sigma^i M \otimes N^*) \rightarrow UF(M \otimes N^*)
\]

As in Proposition A.4, relations (1) – (4) follow as in the case for \(U\). The relation (5) is enforced by being an object in \(K\), and (6) is the relation enforced by the \(UF(M \otimes M \otimes N^*)\) according to Lemma A.9.

We can repeat our analysis of the structure needed to compute division for algebras.

**Proposition A.10.** The composite \(K \rightarrow \mathcal{K} \rightarrow \mathcal{K}_n \rightarrow \mathcal{K}_{n+\beta_N(1)}\).

**Appendix B. Relations**

Here we collect some relations that hold among the classes in the image of the natural transformations \(\phi_i, \delta\) defined in Construction 4.1. The relations below numbered so as to agree with the numbering in Definition 2.7.

**Proposition B.1.** The following relations hold (\(\delta_{i,j}\) is the Kronecker delta).

1. \(\phi_i(a+b) = \phi_i(a) + \phi_i(b) + w_{n-i}\delta(ab)\) (\(w_1 = 0, w_0 = 1\))
2. \(\delta(a+b) = \delta(a) + \delta(b)\)
3. \(w_{n+1}\delta(a) = 0\)
4. \(\phi_k(ab) = \sum_{i+j=k} \phi_i(a)\phi_j(b) + \sum_{\ell=n+1}^{2n} \sum_{i+j=\ell} \phi_i(a)\phi_j(b)\)
5. \(\sum_{2 \leq \alpha_1 \ldots \alpha_m \leq n+1} \prod_{f=1}^{\rho_0} w_{\alpha_f}\)
6. \(Sq\phi_{n-k}(a) = \sum_{j \geq 0} (k+j-2j)\phi_{n-k+2j}(Sq^j a) + \delta_{k0} \sum_{2j<\ell} \delta(Sq^j a \times Sq^{j-1} a)\)
7. \(Sq\delta(a) = \delta(Sq^j a)\)
8. \(S_{\delta} = 0\) for \(i < 0\)
9. \(S_{\delta} = 0\) for \(\delta(a) = (a)^2\)

**Proof.** Relations (9), (10) hold for any object of \(\mathcal{K}\). Relation (4) was proven in Lemma 4.6 and Relations (7), (8) were proven in Lemma 4.8.

We now prove the remaining relations, using the notation of Section 4. First observe that in \(H^*(\mathbb{Z}_{SO(n)}^2)\), \(\tau^{SO(n)}_{SO(n)}(a \otimes b) = P(a+b) - P(a) - P(b)\). Thus multiplying by \(t^i\) and looking at the image in \(H^*(F_n(Z))\), we can use the definition of the \(\phi_i\) to get relation (1). Since \(ev_0^*\) and \(\tau_{SO(n+1)}^{SO(n+1)}\) are additive, relation (2) holds. To see relation (3), note that the image of \(P(a)\) in \(Z_{SO(n)}^n\) is \(\sum_i t^i \phi_i(a)\). Since \(P(ab) = P(a)P(b)\),
and \( t^{n+1} = t^{n-1}q^*w_2 + t^{n-2}q^*w_3 + \cdots + q^*w_{n+1} \), isolating the \( t^k \) coefficient gives

\[
\phi_k(ab) = \sum_{i+j=k} \phi_i(a)\phi_j(b) + \sum_{\ell=n+1}^{2n} \sum_{i+j=\ell} \phi_i(a)\phi_j(b) \left( \sum_{2\leq \alpha_1\ldots\alpha_m\leq n+1} \sum_{\alpha_1+\ldots+\alpha_m=\ell-k} \prod_{f=1}^m w_{\alpha_f} \right).
\]

\[\square\]

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