Holographic Quantum Statistics
from Dual Thermodynamics

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Abstract:
We propose dual thermodynamics corresponding to black hole mechanics with the identifications \( E' \rightarrow A/4 \), \( S' \rightarrow M \), and \( T' \rightarrow T^{-1} \) in Planck units. Here \( A \), \( M \) and \( T \) are the horizon area, mass and Hawking temperature of a black hole and \( E' \), \( S' \) and \( T' \) are the energy, entropy and temperature of a corresponding dual quantum system. We show that, for a Schwarzschild black hole, the dual variables formally satisfy all three laws of thermodynamics, including the Planck-Nernst form of the third law requiring that the entropy tend to zero at low temperature. This is in contrast with traditional black hole thermodynamics, where the entropy is singular. Once the third law is satisfied, it is straightforward to construct simple (dual) quantum systems representing black hole mechanics. As an example, we construct toy models from one dimensional (Fermi or Bose) quantum gases with \( N \sim M \) in a Planck scale box. In addition to recovering black hole mechanics, we obtain quantum corrections to the entropy, including the logarithmic correction obtained by previous papers. The energy-entropy duality transforms a strongly interacting gravitational system (black hole) into a weakly interacting quantum system (quantum gas) and thus provides a natural framework for the quantum statistics underlying the holographic conjecture.

Keywords: Thermodynamics, Quantum Gravity, Holography.
1. Introduction

The quantum estimate of the cosmological vacuum energy density is about 123 orders of magnitude larger than its measured value [1, 2]. This discrepancy might be resolved by an upper limit on the entropy of the universe, since such a bound implies an upper limit on its total energy density. Along the lines of this argument, recently we conjectured that after imposing a holographic entropy limit, the resulting quantum theory leads exactly to the measured value of the total energy density [3]. In this work we construct an example, where imposing the holographic entropy limit on a quantum system indeed leads to an upper limit on its energy density. We choose a black hole as a representative holographic quantum system, since it is easy to relate the entropy and energy using black hole mechanics.

Black hole mechanics is one of the most tantalizing features of general relativity. Its formal analogy with thermodynamics was noticed early on [4, 5, 6]. The quantum nature of the underlying physics was clarified by the discovery of the Hawking radiation [7, 8]. While the quantum theory of black holes struggles with unsolved problems such as the information paradox [9], it successfully motivated the holographic conjecture [10, 11, 12]: the entropy in a spherical volume cannot exceed the quarter of its surface area in Planck units (see [13, 14] for alternative and generalized definitions). Despite the general belief that the explanation of the holographic conjecture must be of quantum nature, it is non-trivial to find simple quantum systems representing black hole mechanics. In this paper, we set out to construct a quantum statistical model displaying the holographic behavior of a Schwarzschild black hole.

Following [15], we summarize the laws of black hole mechanics for an isolated Schwarzschild black hole (i.e. one without electric charge or angular momentum) characterized with its mass $M$, surface gravity $\kappa = 1/(4M)$, and surface area $A$.

- Zeroth Law: $\kappa$ is the same everywhere on the horizon in a time independent black hole.
• First Law:
\[ \delta M = \frac{\kappa}{8\pi} \delta A \]  
(1.1)

• Second Law:
\[ \delta A \geq 0 \]  
(1.2)

Based on the analogy with thermodynamics one associates the temperature with the surface gravity \( T \rightarrow \kappa/2\pi \), the energy with the mass \( E \rightarrow M \), and the entropy with the horizon area \( S \rightarrow A/4 \). Holography in this context amounts to the conjecture that a black hole maximizes the entropy \[11\]. The traditional assignment of variables is further motivated by the Hawking effect: a black hole emits black body radiation with temperature \( \kappa/2\pi \). A black hole weakly interacting with its surrounding appears to obey the generalized version of the second law where the sum of the entropies of matter and the black hole always increases \[5\].

On the other hand, black hole evaporation due to Hawking radiation raises questions: is the unitary nature of quantum mechanics broken? It appears that an initial pure quantum state will evolve into a mixed thermal state through the process of black hole formation and evaporation; this amounts to information loss. In addition, it is not clear precisely what quantum degrees of freedom are responsible for the entropy of the black hole.

The resolution of the information paradox is still open. Either quantum gravity allows correlations to be restored through the horizon, in contrast with classical gravity, thus information could be restored during evaporation; or quantum mechanics needs to be modified to allow non-unitary evolution \[16, 17, 18\]. The degrees of freedom responsible for the entropy have stimulated vigorous research. Competing models involve weakly coupled string states \[19, 20\], conformal symmetry \[21, 22, 23\], string fuzzballs \[24\] spin network states at \[25\] or inside \[26\] the horizon, “heavy” degrees of freedom in induced gravity \[27\], or non-local topological properties of black hole spacetime \[28\]. In each of these models the entropy is proportional to the horizon area, yet none of them offer a clear and comprehensive picture associated with black hole thermodynamics.

Another peculiarity of black hole mechanics is rarely discussed: it violates the (strong version of the) third law of thermodynamics \[29\]. The entropy is in fact singular (tends to infinity rather than zero) as the temperature approaches zero. Since virtually every known quantum mechanical system in nature obeys the third law, we argue that this singularity expresses yet another way the difficulty of quantizing gravity.

In order to construct quantum models representing black hole mechanics, we propose dual thermodynamics with the following assignments \( E' \rightarrow A/4, S' \rightarrow M, \) and \( T' \rightarrow 2\pi/\kappa \) in Planck units. In the dual description, the entropy and energy exchange roles, and the inverse of the Hawking temperature plays the role of the temperature. By definition, the dual variables obey the first law while the validity of the second law follows from the fact that the mass of a Schwarzschild black hole can only increase if we neglect Hawking radiation. Note that at this point we do not aim to model the generalized second law which includes the sum of matter and black hole entropies. The advantage of our proposal is that
the dual entropy of a black hole is proportional to its dual temperature \( S' \propto T' \), i.e. the third law of dual thermodynamics is satisfied as well.

In the next section we formalize the duality transformation between two thermodynamical systems, and in Section 3 we construct a simple quantum model which reproduces the dual thermodynamics. In the last section we summarize our results after transforming back from the dual variables and compare the quantum corrections to the entropy with those available in the literature.

2. Energy-entropy duality

We define the energy-entropy duality by assuming that the first law in entropy representation corresponds to the first law in dual energy representation. We allow for pressure and chemical potential for full generality, although we exclude for the moment other entities, such as electric charge, magnetic susceptibility, etc. The first law in both representations reads

\[
dS = \frac{1}{T}dE + \frac{p}{T}dV - \frac{\mu}{T}dN, \tag{2.1}
\]

\[
dE' = T'dS' - p'dV' + \mu'dN', \tag{2.2}
\]

where \( S, E, V, N, T, p, \mu \) are the thermodynamic entropy, energy, volume, species number, temperature, pressure, and chemical potential, respectively, and primed symbols denote dual thermodynamical variables. Comparison of the two equations motivates the definition of the entropy-energy duality

\[
S \rightarrow E', \quad E \rightarrow S', \quad T \rightarrow \frac{1}{T'}, \quad \frac{\mu}{T} \rightarrow -\mu'. \tag{2.3}
\]

We assumed that the degrees of freedom \( N \) is the same in both spaces, which fixes the transformation of \( \mu \). From these transformation rules it follows that \( \frac{dV}{T} \rightarrow -p'dV' \) from the first law.

While we presented a definition which might be more generally applicable, we aim specifically at application to black hole mechanics. As shown at the end of the previous section, for vanishing chemical potential, and \( pdV = 0 \), all three dual thermodynamic laws will be satisfied as a consequence of black hole mechanics. For the more general case with non vanishing chemical potential, we can show that it is sufficient to have \( p'dV' = 0, \mu > 0 \) and \( N \sim M \) for the dual second law to hold. In this case

\[
dS' = \frac{1}{T'}(dE' + p'dV' - \mu'dN') \geq 0, \tag{2.4}
\]

since \( dE' = dS \geq 0 \) and \( -\mu'dN' \geq 0 \) separately. We will show that the specific quantum statistics we propose to represent the dual thermodynamics satisfies these sufficient conditions.

The above arguments show that it is reasonable to assume that the proposed entropy-energy duality is meaningful for the specific case of black hole mechanics, as all three laws of thermodynamics hold in the dual variables. The precise necessary conditions for the
applicability of the entropy-energy duality in a more general case are less clear, and are beyond the scope of the present paper.

Our duality proposal was motivated by the third law of thermodynamics. The behaviour $S' \propto T'$ and $E' \propto T'^2$ in dual variables is now easily reproducible by ordinary quantum statistics. In fact, probably multiple quantum systems could satisfy these sensible relations. We show in the next section, that one dimensional quantum gases (whether Fermi or Bose) provide a fully consistent dual quantum statistical model for the dual black hole thermodynamics.

3. Dual quantum statistics: one dimensional quantum gas

In this section, we summarize the main results for the statistical mechanics of one dimensional quantum gases. As we show in the next section, they provide a representation of dual black hole thermodynamics in the limit where $|\mu'| \ll T'$, with $\mu'$ corresponding to quantum corrections. All equations in this section are written in dual variables, therefore we omit the primes for convenience.

The energy, pressure and particle number of one dimensional quantum gases are calculated as

$$E = pL = \frac{gL}{2\pi} \int_0^\infty f(\epsilon) \epsilon \, d\epsilon, \quad (3.1)$$

$$N = \frac{gL}{2\pi} \int_0^\infty f(\epsilon) \, d\epsilon, \quad (3.2)$$

where $f(\epsilon) = 1/(e^{(\epsilon-\mu)/T} \pm 1)$. Hereafter the upper (lower) sign represents the Fermi-Dirac (Bose-Einstein) distribution. The coefficient $g$ gives the internal degrees of freedom of the gas particles, and $L$ is the size of the system. The entropy is obtained from the integral form of the first thermodynamic law

$$S = \frac{1}{T} (E + pL - \mu N). \quad (3.3)$$

The above integrals can be expressed in terms of the polylogarithm function $\text{Li}_n(z) \equiv \sum_{k=1}^\infty z^k/k^n$:

$$E = pL = \mp \frac{gL}{2\pi} \text{Li}_2 \left( \mp e^{\mp \frac{\mu}{T}} \right) T^2, \quad (3.4)$$

$$N = \pm \frac{gL}{2\pi} \log \left( 1 \pm e^{\mp \frac{\mu}{T}} \right) T. \quad (3.5)$$

We will use the expansions of these formulae in the limit of $|\mu| \ll T$. For a Fermi gas

$$E = pL = \frac{1}{24} gL \pi T^2 + \frac{gL}{2\pi} \log(2) \mu T + \frac{gL}{8\pi} \mu^2 + O(\mu^3), \quad (3.6)$$

$$N = \frac{gL}{2\pi} \log(2) \mu + \frac{gL}{4\pi} \mu + \frac{gL}{16\pi T} \mu^2 + O(\mu^3), \quad (3.7)$$

$$S = \frac{1}{12} gL \pi T + \frac{gL}{2\pi} \log(2) \mu + O(\mu^3). \quad (3.8)$$
and for a Bose gas

\[ E = pL = \frac{1}{12} gL \pi T^2 - \frac{gL}{2\pi} (\log (-\mu/T) - 1) T \mu - \frac{gL}{8\pi} \mu^2 + O(\mu^3), \]  \hspace{1cm} (3.9)

\[ N = -\frac{gL}{2\pi} \log (-\mu/T) T - \frac{gL}{4\pi} \mu - \frac{gL \mu^2}{48 \pi T} + O(\mu^3), \]  \hspace{1cm} (3.10)

\[ S = \frac{1}{6} gL \pi T - \frac{gL}{2\pi} (\log(-\mu/T) - 1) \mu + O(\mu^3). \]  \hspace{1cm} (3.11)

4. Quantum statistical model of a black hole

In this section we use the previous results to show that dual black hole thermodynamics is represented by one dimensional quantum gases. Note that we return to primed notation for dual variables. In addition, we derive quantum corrections to the black hole mechanics based on our toy model. When the dual temperature, which is proportional to the mass of the black hole, approaches the Planck scale, we expect quantum corrections. In the dual space quantum effects will manifest themselves when \( T' \) approaches \( \mu' \), therefore we can assume that \( \mu' \) is at most of the order of the Planck mass or smaller. In that case, for a large black hole, \( \mu N \) will be completely negligible.

We can set \( gL \) of our dual quantum theory to ensure consistency of the energy with that of the black hole:

\[ gL = \frac{3}{2\pi^2}, \frac{3}{4\pi^2}, \]  \hspace{1cm} (4.1)

for Fermi and Bose systems, respectively. This means that the one dimensional dual quantum system lives in a space of size order of Planck length.

Equation (4.1) fixes the parametric freedom in our quantum models. Transforming back from the dual variables, the leading terms of Eqs. (3.8) and (3.11) reproduce the black hole energy (assuming \( \mu \) is small). Meanwhile, the leading terms of Eqs. (3.6) and (3.9) reproduce the black hole entropy as demonstrated by the leading terms of Eqs. (4.4) and (4.6). This itself is a non-trivial result. After fixing \( gL \) to obtain the correct coefficient of the black hole energy, the dual system does not have any freely adjustable parameters. Yet, not only the temperature (or mass) dependences of the black hole energy and entropy are reproduced, but also the coefficient of the entropy term is given correctly. Finally, from \( dL = 0 \) follows \( pdV = 0 \), which means that black hole mechanics with its holographic aspects is perfectly represented by our dual model.

For the Fermi model, we also obtain that the number of degrees of freedom

\[ N = \frac{6 \log(2)}{\pi^2} M, \]  \hspace{1cm} (4.2)

is essentially given by the black hole mass in Planck mass units. This is an entirely sensible prediction: the number of degrees of freedom grow extensively with the black hole mass, while our model faithfully reproduces the holographic growth of entropy.

For the Bose model, \( N \) has a logarithmic sensitivity for \( \mu' \), but for a wide range of values \( N \simeq M \log M \) will hold, including the value we suggest from consistency with previous quantum gravity calculations for the entropy correction. Note that we do not discuss the possible effects Bose condensation on \( N \), as we assumed that \( \mu' \ll T' \).
Next, we calculate quantum corrections to the black hole entropy considering corrections to the energy of the dual quantum gas. The entropy according to our Fermi model is

\[ S = -\frac{48}{\pi} \text{Li}_2(-e^{-\mu})M^2 = 4\pi M^2 - \frac{48}{\pi} \log(2)M^2\mu + \frac{12}{\pi} M^2 \mu^2 - \frac{2}{\pi} M^2 \mu^3 + O(\mu^4), \tag{4.3} \]

while the Bose model gives

\[ S = \frac{24}{\pi} \text{Li}_2(e^{-\mu})M^2 = 4\pi M^2 + \frac{24}{\pi} \mu M^2 (\log(\mu) - 1) - \frac{6}{\pi} \mu^2 M^2 + O(\mu^4). \tag{4.4} \]

Quantum corrections to the entropy of various black holes were calculated using the Cardy formula [30, 31]. These corrections allow us to fix the exact value of \( \mu \). Comparing our expression with Refs. [32, 22, 33], we find that the Bose case easily lends itself for obtaining a logarithmic correction to the black hole entropy. Setting

\[ \mu = \frac{\pi}{16M^2}, \tag{4.5} \]

the black hole entropy derived from the dual Bose gas becomes

\[ S = 4\pi M^2 - \frac{3}{2} \log(4\pi M^2) - \frac{3}{2} \left( 1 + \log \left( \frac{4}{\pi^2} \right) \right) - \frac{3\pi}{128 M^2} + O(1/M^3). \tag{4.6} \]

We find it remarkable that our simplest toy model, namely a dual one dimensional Bose gas, is able to reproduce not only the correct semi classical black hole energy and entropy but also their highly non-trivial and minute logarithmic quantum corrections. It is possible to obtain a value of \( \mu \) for the Fermi gas as well which would be consistent with logarithmic correction to the entropy.

5. Conclusions

We proposed a dual thermodynamics for an isolated Schwarzschild black hole based on our conjectured energy-entropy duality. Our new formulation has the major advantage of obeying all laws of thermodynamics. This means that ordinary quantum statistics can represent the otherwise mysterious holography encoded in a black hole. While it is conceivable that that there are many dual quantum systems which would be equally suitable for this purpose, we present toy models based on one dimensional Fermi or Bose gases. We show that these exactly reproduce the holographic thermodynamics of black holes. In particular, since energy and entropy play dual roles, the holographic entropy cut off in real space corresponds to the energy cut off in dual space from the Fermi-Dirac or Bose-Einstein distribution and vice versa, as it was conjectured in Ref. [3].

The energy-entropy duality transforms a gravitating, strongly interacting, low temperature system into a weakly interacting, high temperature dual quantum system. This
weakly interacting system obeys the laws of ordinary quantum mechanics. The duality transformation allows the interpretation of results obtained in dual space where calculations are straightforward. We have demonstrated this by presenting quantum corrections to the black hole mechanics in the framework of our toy model. We find it remarkable that our simple model not only reproduces black hole mechanics, but it is also consistent with previously obtained logarithmic corrections to the entropy.

The most conservative interpretation of our result \( p dV = 0 \) would be that the pressure vanishes, \( p = 0 \). This is consistent with the integral form \( E = 2TS \), as can be easily checked from the explicit expressions of energy, entropy, and temperature. Our quantum holographic model reproduces all these quantities as well as their relationship. However, in extensive thermodynamics \( E = TS - pV \) should hold if the pressure is non-zero. A literal interpretation of our dual model suggests a non-zero negative pressure \( p = -TS/V \), yielding \( E = TS - pV = 2TS \), consistently with black hole mechanics. This pressure term, however, does not enter into the (differential) first law of black hole mechanics due to \( dL = 0 \) in our underlying quantum model. Thus, taken at face value, this picture predicts that a black hole, which classically has zero temperature and pressure, has both a non-zero temperature and pressure associated with it. Although this pressure does not manifest itself in the differential black hole mechanics due to a constraint (\( S \), and \( V \) are not independent variables), it appears in the integral of the first law. Just as the temperature associated with black hole mechanics, this pressure is entirely of quantum nature and has an equation of state \( w = -\frac{1}{2} \), the hallmark of dark energy.

The similarity of Friedmann universes to black holes raises the possibility of applying similar ideas toward the development of quantum cosmology. In particular, it is intriguing that negative pressure is entirely natural in this context. Further research into this area could shed light on the dark energy component of the universe, and thereby touch base with observations. In [3], based on simple thermodynamical considerations, we found that the present value of the cosmological constant is natural in the holographic context. We conjecture that, possibly when “bulk viscosity” effects corresponding to holographic entropy production are taken into account, this negative quantum gravitational pressure might account for the apparent acceleration of the universe [34].

The energy-entropy duality as defined in Eq. 2.3 establishes a transformation among partition functions as well. For example, the transformation of the grand canonical partition function should be

\[
q \equiv \frac{pV}{T} \rightarrow -q'T'.
\]

Exploring the partition functions should shed more light on the density of states for a black hole. Further investigation of this issue is left for future research.

It is clear that our considerations can be generalized for other types of black holes; e.g., angular momentum and charge add further terms into the dual thermodynamics. At the moment general relativity is taken into account only in a fairly implicit way, through the phenomenology of black hole mechanics. However, derivation of Einstein’s equation exists directly from on holographic thermodynamics [35]. This raises the possibility of covariant
formulation of these ideas, possibly based on the work in Ref. [13]. We speculate that this could lead to an energy-entropy dual of Einstein’s equation.

It is also clear that a range of dual models could reproduce the same large $M$ asymptotic thermodynamics, and possibly different models could produce different quantum effects. The present range of models can be characterized by $\mu$ and the choice of Fermi or Bose statistics, however, it is likely that other weakly interacting models could be producing similar results. Some of them, such as Ising type spin network models, could have more direct connection with previous work in this area, although one could hardly miss the striking similarity of one dimensional quantum gases with string excitations. Indeed, string theory appears to have a thermodynamic duality $T \rightarrow 1/T$ according to Ref. [36, 37] and our dual description is similar to the well known AdS/CFT duality [38]. We will explore the range possible models and connections with existing formulations in the future.

Ultimately, it would be desirable to extend these ideas to the interaction of a black hole with its surrounding. It is might be possible to relate the entropy current of the dual system to the energy current of the surrounding of the original system, and vice versa. Such calculations could shed more light on the information paradox associated with the Hawking radiation, and are left for future research.

Acknowledgments

We thank Nick Kaiser, Arjun Menon and Robert Wald for stimulating discussions. IS was supported by NASA through AISR NAG5-11996, and ATP NASA NAG5-12101 as well as by NSF grants AST02-06243, AST-0434413 and ITR 1120201-128440. Research at the HEP Division of ANL is supported in part by the US DOE, Division of HEP, Contract W-31-109-ENG-38. CB also thanks the Aspen Center for Physics for its hospitality and financial support.

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