Dual algebraic structures for the two-level pairing model

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Abstract
Duality relations are explicitly established relating the Hamiltonians and basis classification schemes associated with the number-conserving unitary and number-nonconserving quasispin algebras for the two-level system with pairing interactions. These relations are obtained in a unified formulation for both bosonic and fermionic systems, with arbitrary and, in general, unequal degeneracies for the two levels. Illustrative calculations are carried out comparing the bosonic and fermionic quantum phase transitions.

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1. Introduction

The two-level pairing model describes a finite system which undergoes a second-order quantum phase transition between weak- and strong-coupling dynamical symmetry limits. This quantum phase transition is characterized by singularities in the evolution of various ground-state properties as the pairing interaction strength is varied: (1) a discontinuity in the second derivative of the ground-state eigenvalue, (2) a discontinuity in the first derivative of the quantum order parameter, which is defined by the relative population $\langle N_2 \rangle - \langle N_1 \rangle$ of the two levels and is analogous to the magnetization parameter in the Ising model, and (3) a vanishing energy gap $\Delta$ between the ground state and first excited state with the same conserved quantum numbers, and thus a singular-level density $\rho \sim \Delta^{-1}$. Although true singularities in these quantities only occur in the limit of infinite particle number, ‘precursors’ are found at finite $N$, which approach the singular limit according to definite power-law scalings [1–8]. The quantum phase transition in the two-level pairing model has long been of interest for applications to nuclei [9–11]. It has recently served as a testbed for considering phase transitional phenomena, including the finite-size scaling just described, excited-state

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quantum phase transitions [8, 12–15], thermodynamic properties [16], decoherence [17], and
quasidynamical symmetry [18], as well as for developing theoretical methods for treatment of
these phenomena, including continuous unitary transformation [5, 7] and Holstein–Primakoff
boson expansion [19].

Finite systems with pairing-type interactions, and consisting either of bosons or fermions,
occur in a broad variety of physical contexts. Fermionic examples include superconducting
grains (electrons) [20] and the atomic nucleus (nucleons) [21]. Bosonic examples include the
s- and d-wave nucleon pairs of the interacting boson model (IBM) [22], which themselves
undergo a bosonic pairing interaction in the description of nuclear quadrupole collectivity, and
condensates of trapped bosonic atoms [23, 24].

The Lie algebraic properties of the two-level bosonic and fermionic systems are closely
parallel, but the differences which do arise fundamentally affect the irreducible representations
(irreps) under which the eigenstates transform and are therefore essential to defining the
spectroscopy of the system. Two complementary algebraic formulations are relevant to the
description of finite pairing systems [25–30]: a unitary algebra is spanned by the bilinear
products of a creation and an annihilation operator [31, 32], and a quasispin algebra [25,
33, 34] is defined in terms of creation and annihilation operators for time-reversed pairs of
particles. These structures are intertwined by duality relations, in particular, relating irreps of
the quasispin algebra with those of an orthogonal (in the case of bosons) or symplectic (in the
case of fermions) subalgebra of the full unitary algebra. Such relations have often been used
[30, 35–40] to effect simplifications of the calculations for two- and multi-level systems.

In this paper, the duality relations between the unitary and quasispin algebraic structures
for the two-level system are systematically established. In particular, attempts to compare
results across two-level systems with different level degeneracies or between the bosonic
and fermionic cases (see [14]) raise the question as to which differences in spectroscopic
results are superficial, i.e. originating from an imperfect choice of correspondence between
the Hamiltonian parameters for the two cases, and which are due to more fundamental or
irreconcilable distinctions. Therefore, a main intent of this work is to resolve the relationships
between the disparate forms of the Hamiltonian which arise in the definitions of the dynamical
symmetries and in numerical studies of the transition between them. These Hamiltonians
include (1) the Casimir form defined in terms of the unitary algebra, (2) the pairing form
used in studies of the fermionic system, which is essentially defined in terms of quasispin
operators, and (3) the ‘multipole’ form traditionally considered for physical reasons in bosonic
studies. The relations are established in a fully general fashion which uniformly accommodates
arbitrary level degeneracies \( n_1 \) and \( n_2 \), for both the bosonic and fermionic cases.

Previous work on two-level systems has concentrated either on the so-called \( s-b \) boson
models, with level degeneracies \( n_1 = 1 \) and \( n_2 \geq 1 \) (i.e. for which one of the levels is a
singlet), or on fermionic models of equal degeneracies \( n_1 = n_2 \). The observations outlined
here are intended to provide a foundation for more detailed future work, allowing for the most
general choice of level degeneracies. The results are provided as a basis for algebraic studies
of the quantum phase transitions, excited-state spectroscopic structure, and classical geometry
[41, 42] of two- and multi-level pairing models. Although the discussion is presented for
two-level systems, for the sake of clarity, many of the results carry over to multi-level systems
essentially without modification.

After a brief summary of the dual algebraic structures for the many-body problem
in general (section 2), the unitary algebraic structure is presented in detail, including
the categorization of the subalgebra structure, classification of the irreps, construction
of the generators, and identification of the Casimir operators, all in a unified form for
bosonic and fermionic cases (section 3). The simpler quasispin structure is also reviewed
Duality relations are then established between the unitary (or Casimir) and quasispin (or pairing) formulations of the Hamiltonian (section 5). These are explicitly related to the spectral properties of the two-level system through numerical calculations across the quantum phase transition, illustrating basic distinctions between the bosonic and fermionic cases, when calculated for bosonic and fermionic systems with similar level degeneracies and/or similar particle number (section 6).

2. Bosonic and fermionic algebras

The fundamental Lie algebra describing transformations of a many-boson or many-fermion system is spanned by the bilinear products of creation and/or annihilation operators \( a_m^\dagger a_m \), \( a_m a_m^\dagger \), and \( a_m a_{m'} \) (e.g. [29, 30]). For bosons, the resulting algebra is \( \text{Sp}(2n, \mathbb{R}) \), and for fermions it is \( \text{SO}(2n) \), where \( m \) and \( m' = 1, \ldots, n \) range over the single-particle states of the system. Two important sets of subalgebras arise: number-conserving subalgebras and number-nonconserving (quasispin) subalgebras.

The restriction to number-conserving operators, spanned by the elementary one-body operators \( a_m^\dagger a_m \), constitutes a \( U(n) \) algebra. The \( U(n) \) algebra contains a subalgebra \( \text{SO}(n) \) for the bosonic case or \( \text{Sp}(n) \) for the fermionic case. If each single-particle creation operator \( a_m^\dagger \) is associated with a time-reversed partner \( a_m \), then these \( \text{SO}(n) \) or \( \text{Sp}(n) \) subalgebras are defined by the property that they leave invariant the ‘scalar’ pair state \( \sum_m a_m^\dagger a_m |0\rangle \) [30]. This special property underlies the duality relations with the quasispin pair algebra considered in this work. More specifically, we consider rotationally-invariant problems, for which the single-particle states may be identified as the \( 2j+1 \) substates of single-particle levels of various angular momenta \( j \) (i.e. \( j \)-shells, in the nomenclature of nuclear physics, which we adopt for either bosonic or fermionic levels). Then, the creation operators are of the form \( a_m \rightarrow a_m^\dagger \) and \( a_m^\dagger \rightarrow (-)^{h-m} a_m^\dagger \) for the \( k \)th level. For such rotationally-invariant systems, the \( \text{SO}(n) \) or \( \text{Sp}(n) \) subalgebras in turn contain the physical \( \text{SO}(3) \sim \text{SU}(2) \) angular momentum algebra. Although we follow the convention of denoting the angular momentum algebra \( \text{SO}(3) \) in the bosonic case and \( \text{SU}(2) \) in the fermionic case, there is no material distinction between the algebras. In general, there may also be other, intervening subalgebras in the chain.

Alternatively, the scalar pair creation operator \( S_+ = \frac{1}{2} \sum a_m^\dagger a_m \) and the number-conserving operator \( S_0 = \frac{1}{2} \sum (a_m a_m^\dagger + \theta a_m^\dagger a_m) \) close under commutation, where \( \theta = + \) for bosonic systems or \( \theta = - \) for fermionic systems. These operators define a number-nonconserving pair quasispin algebra, either \( \text{Sl}(2,1) \) for bosons or \( \text{Sl}(2) \) for fermions. The calligraphic notation for the quasispin algebras is adopted [30] to avoid ambiguity between the \( \text{Sl}(2) \) quasispin algebra and the \( \text{SU}(2) \) angular momentum algebra.

In summary, for a bosonic system, the subalgebras under consideration are

\[
\text{Sp}(2n, \mathbb{R}) \supset \left\{ \begin{array}{l}
U(n) \supset \text{SO}(n) \supset \cdots \supset \text{SO}(3) \\
\text{Sl}(1,1) \supset U(1),
\end{array} \right.
\]

and for a fermionic system, they are

\[
\text{SO}(2n) \supset \left\{ \begin{array}{l}
U(n) \supset \text{Sp}(n) \supset \cdots \supset \text{SU}(2) \\
\text{Sl}(2) \supset U(1).
\end{array} \right.
\]

The subalgebras of \( U(n) \) are useful in the classification of states not only for the pairing Hamiltonian (defined in section 5.2) but also for a much richer range of Hamiltonians [43].

A close relation between unitary chain and quasispin subalgebras arises since the quasispin and orthogonal or symplectic algebras may be embedded as mutually commuting ‘dual’
algebras within the larger $\text{Sp}(2n, \mathbb{R})$ or $\text{SO}(2n)$ algebra. The algebraic foundations are discussed in detail in [25–30]. Here, we simply note that duality denotes the situation in which the states within a space may be classified simultaneously in terms of two mutually commuting groups (or algebras) $G_1$ and $G_2$, such that the irrep labels of the two groups are in one-to-one correspondence. For the present problem, the embedding and associated labels are given, for the bosonic case, by
\[ \text{Sp}(2n, \mathbb{R}) \supset [\text{SO}(n) \supset \cdots \supset \text{SO}(3)] \otimes [\text{SU}(1, 1) \supset \text{U}(1)] \] (3)
and, for the fermionic case, by
\[ \text{SO}(2n) \supset [\text{Sp}(n) \supset \cdots \supset \text{SU}(2)] \otimes [\text{SU}(2) \supset \text{U}(1)]. \] (4)
The seniority label $v$ (section 3.2) and quasispin label $S$ (section 4) are in one-to-one correspondence, i.e. specifying the value of one uniquely determines the value of the other, and vice versa.

The duality relations hold equally well regardless of whether the $n$-dimensional single-particle space is construed to consist of a single $j$-shell ($n = 2j + 1$, odd for bosons or even fermions), two $j$-shells ($n = n_1 + n_2$), or, indeed, multiple $j$-shells. The one-level case has been considered in detail (e.g. [29]). However, we find that the detailed construction of operators for two-level systems within the context of this duality, as needed for spectroscopic studies of these systems, requires elaboration. Although, for simplicity, we consider only the case of two levels, the results may readily be generalized to additional levels.

3. Unitary algebra

3.1. Subalgebra chains

Consider the $\text{U}(n)$ subalgebra chains for the two-level system, consisting of either bosonic or fermionic levels, of possibly unequal degeneracies. If the levels are $j$-shells of angular momenta $j_1$ and $j_2$, the level degeneracies are $n_1 = 2j_1 + 1$ and $n_2 = 2j_2 + 1$, and the total degeneracy of the system is $n = n_1 + n_2$. For the bosonic case, we have
\[ \text{U}(n_1 + n_2) \supset \left\{ \begin{array}{c}
\text{SO}(n_1 + n_2) \\
\text{U}_1(n_1) \otimes \text{U}_2(n_2) \\
\text{SO}_1(n_1) \otimes \text{SO}_2(n_2) \\
\text{SO}_1(3) \otimes \text{SO}_2(3) \otimes \text{SO}_{12}(3)
\end{array} \right\} \] (5)
and for the fermionic case, we have
\[ \text{U}(n_1 + n_2) \supset \left\{ \begin{array}{c}
\text{Sp}(n_1 + n_2) \\
\text{Sp}_1(n_1) \otimes \text{Sp}_2(n_2) \\
\text{SU}_1(2) \otimes \text{SU}_2(2) \otimes \text{SU}_{12}(2),
\end{array} \right\} \] (6)
where $n_1 = 2j_1 + 1$ and $n_2 = 2j_2 + 1$. The irrep labels, indicated beneath the symbol for each algebra, are defined in section 3.2, and the algebras themselves are constructed explicitly in section 3.3. Throughout the following discussion, bosonic and fermionic cases will be considered in parallel.

The subalgebras summarized in (5) and (6) are generically present, regardless of the level degeneracies $n_1$ and $n_2$, for $n_1$ and $n_2 \geq 2$. However, several clarifying comments are in order:
(1) The important special case of a singlet bosonic level \( j_k = 0 \) leads to \( \nu_k = 1 \), and the corresponding orthogonal algebra \( \text{SO}(n_k) \) is undefined. The label \( \nu_k \) may still be defined, in a limited sense, through the quasispin, as noted in section 4. Two-level boson problems in which \( j_1 = 0 \) are termed  \( s-b \) boson models. These include the Schwinger boson realization \((j_1 = j_2 = 0)\) of the Lipkin model \([44]\). The subalgebra chains and labeling schemes for the \( s-b \) models were considered in \([14]\).

(2) For a fermionic level with \( j_k = \frac{1}{2} \), and therefore \( \nu_k = 2 \), the symplectic algebra \( \text{Sp}_4(n_k) \) in (6) is identical to the \( \text{SU}_4(2) \) angular momentum algebra.

(3) Additional subalgebras of \( U(n_1 + n_2) \) may also arise, parallel to chains indicated above and still containing the angular momentum algebra, e.g., for the IBM \((n_1 = 1 \text{ and } n_2 = 5)\), there is a physically relevant chain \( U(6) \supseteq \text{SU}(3) \supseteq \text{SO}_{12}(3) \) \([22]\). However, since these chains are not directly relevant to the pairing problem and cannot be treated in a uniform fashion for arbitrary \( n_1 \) and \( n_2 \), they are not considered further here.

(4) Further subalgebras may also intervene between \( \text{SO}(n) \) and \( \text{SO}(3) \), or between \( \text{Sp}(n) \) and \( \text{SU}(2) \), the classic example being the appearance of the exceptional algebra \( G_2 \) in the \( j \) labeling schemes for the angular momentum addition. For instance, addition of the angular momentum generators may also be made higher in the subalgebra chains (5) and (6), yielding \( U_1(n_1) \otimes U_2(n_2) \supseteq U_{12}(n_1) \supseteq \text{SO}_{12}(3) \), and \( \text{SO}_2(n_1) \otimes \text{SO}_2(n_2) \supseteq \text{SO}_{12}(n_1 \otimes n_2) \supseteq \text{SO}_{12}(3) \) for the bosonic case, or similarly \( U_1(n_1) \otimes U_2(n_2) \supseteq U_{12}(n_1) \supseteq \text{SU}_{12}(2) \) and \( \text{Sp}_1(n_1) \otimes \text{Sp}_2(n_2) \supseteq \text{Sp}_{12}(n_1) \otimes \text{SU}_{12}(2) \) for the fermionic case.

3.2. Branching

The branching rules for the irreps arising in the bosonic or fermionic realizations of the algebras in (5) and (6) provide the classification of states for the two-level pairing model. Some, but not all, of these branchings can be expressed in the closed form.

For the bosonic realization of \( U(n) \), the symmetric irreps \([N] \equiv [N0\ldots0] \) (with \( n \) labels) are obtained, where \( N \) is the occupation number. For \( \text{SO}(n) \), the irreps are \([v] \equiv [v0\ldots0] \) (with \([n/2]\) labels).

The \( U(n) \rightarrow \text{SO}(n) \) branching is of the type considered by Hammermesh \([45]\) and is given by

\[
v = (N \mod 2), \ldots , N - 2, N.
\]

This rule applies both to the branching \( U(n_1 + n_2) \rightarrow \text{SO}(n_1 + n_2) \) and to the branching associated with each of the two levels in \( U_1(n_1) \otimes U_2(n_2) \rightarrow \text{SO}_1(n_1) \otimes \text{SO}_2(n_2) \). Note that \( n \) is odd for a single bosonic level and is even for the two-level system. (The \( U(n) \rightarrow U_1(n_1) \otimes U_2(n_2) \) branching rule follows trivially from additivity of the number operators, \( N = N_1 + N_2 \).

For the branching \( \text{SO}(n_1 + n_2) \rightarrow \text{SO}_1(n_1) \otimes \text{SO}_2(n_2) \), the allowed \( v_1 \) and \( v_2 \) are obtained by considering all partitions of \( v \) as

\[
v = v_1 + v_2 + 2n_v 
\quad (n_v = 0, 1, \ldots, \lfloor v/2 \rfloor).
\]

This rule may be verified by dimension counting arguments, that is, \( \dim[v] = \sum_{v_1v_2} \dim[v_1] \dim[v_2] \), using the Weyl dimension formula \([45]\). Note that for the bosonic
system (in contrast to the fermionic case below) the branching rule for \( SO(n_1 + n_2) \rightarrow SO_1(n_1) \otimes SO_2(n_2) \) is independent of the level degeneracies \( n_1 \) and \( n_2 \), and the total occupation number \( N \) influences the allowed \( SO_1(n_1) \) and \( SO_2(n_2) \) irreps only through the constraint (7) on \( v \).

The allowed partitions \( v \rightarrow (v_1, v_2) \) are listed, for low \( v \), in table 1. As a concrete example, for \( N = 2 \), the allowed \( SO(n_1 + n_2) \) irreps have \( v = 0 \) and \( 2 \), with branchings to \( SO_1(n_1) \otimes SO_2(n_2) \) given by the corresponding rows of table 1. As a specific example of the equivalence of dimensions, consider the case of the two-level bosonic system with \( j_1 = j_2 = 1 \), thus described by \( SO(6) \supset SO_1(3) \otimes SO_2(3) \). The \( v = 2 \) irrep of \( SO(6) \) has dimension 20, while the corresponding \( SO_1(3) \otimes SO_2(3) \) irreps likewise have the total dimension \( \dim(20) = \dim(1, 1) + \dim(0, 2) + \dim(0, 0) = (5)(1)(1)(3)(3) + (1)(5) + (1)(1) = 20. \)

The branchings of the form \( SO(n) \rightarrow SO(3) \), needed for \( SO_1(n_1) \otimes SO_2(n_2) \rightarrow SO_1(3) \otimes SO_2(3) \), are more complicated and, in general, involve missing labels. However, such branchings occur widely in physical applications, and general methods exist for the solution based on weights or character theory [29, 46]. An explicit multiplicity formula is obtained in [47], applicable to the symmetric irreps arising in the present bosonic case (5).

Finally, the reduction \( SO_1(3) \otimes SO_2(3) \rightarrow SO_1(2) \) is governed by the usual triangle inequality for angular momentum addition.

The branching rules for the fermionic case are nearly identical, with a few modifications. For \( U(n) \), we obtain the \emph{antisymmetric} irreps \( \{N\} \equiv [1, 1, 0, 0, \ldots, 0], \) that is, with \( N \) unit entries (out of \( n \) labels total). Similarly, for \( Sp(n) \), we have \( \{v\} \equiv [1, 1, 0, 0, \ldots, 0], \) with \( v \) unit entries (out of \( n/2 \) labels total).

The branching rule for \( U(n) \rightarrow Sp(n) \) [45] requires the modification of (7) to

\[
v = (N' \mod 2), \ldots, N' - 2, N',
\]

where \( N' \equiv \min(N, n - N) \). Note, therefore, that \( v \leq \frac{1}{2}n \). This rule applies to \( U(n_1 + n_2) \rightarrow Sp(n_1 + n_2) \) and to \( U_1(n_1) \otimes U_2(n_2) \rightarrow Sp_1(n_1) \otimes Sp_2(n_2) \).

### Table 1.

| \( v \) | \( n \) | \( (v_1, v_2) \) |
|---|---|---|
| 0 | 0 | (0,0) |
| 1 | 0 | (1,0), (0,1) |
| 2 | 0 | (2,0), (1,1), (0,2) |
| 1 | | (0,0) |
| 3 | 0 | (3,0), (2,1), (1,2), (0,3) |
| 1 | | (1,0), (0,1) |
| 4 | 0 | (4,0), (3,1), (2,2), (1,3), (0,4) |
| 1 | | (2,0), (1,1), (0,2) |
| 2 | | (0,0) |
| 5 | 0 | (5,0), (4,1), (3,2), (2,3), (1,4), (0,5) |
| 1 | | (3,0), (2,1), (1,2), (0,3) |
| 2 | | (1,0), (0,1) |
| 6 | 0 | (6,0), (5,1), (4,2), (3,3), (2,4), (1,5), (0,6) |
| 1 | | (4,0), (3,1), (2,2), (1,3), (0,4) |
| 2 | | (2,0), (1,1), (0,2) |
| 3 | | (0,0) |
3.3. Generators

Of the generators for the algebras in chains (5) and (6), those involving a single level are well known [32]. Here, we must construct the generators for the two-level system. Since the

| N | v | \([\lambda_1, \lambda_2]_{\text{SO}(5)}\) | \((v_1, v_2)\) | \([\lambda_1, \lambda_2]_{\text{SO}(4)}\) |
|---|---|---|---|---|
| 0 | 0 | [0,0] | (0,0) | [0,0] |
| 1 | 1 | \([\frac{1}{2}, \frac{1}{2}]\) | (1,0), (0,1) | \([\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, -\frac{1}{2}]\) |
| 2 | 0 | [0,0] | (0,0) | [0,0] |
| 2 | 1 | [1,0] | (1,1), (0,0) | [1,0], [0,0] |
| 3 | 1 | \([\frac{1}{2}, \frac{1}{2}]\) | (1,0), (0,1) | \([\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, -\frac{1}{2}]\) |
| 4 | 0 | [0,0] | (0,0) | [0,0] |

The values of \(v_1\) and \(v_2\) arising in the branching \(\text{Sp}(n_1 + n_2) \to \text{Sp}(n_1) \otimes \text{Sp}(n_2)\) are again given by the partitioning condition (8), but now subject to an additional constraint, so

\[
v = v_1 + v_2 + 2n_v \quad (n_v = 0, 1, \ldots, [v/2])
\]

\[
|(v_1 - v_2) - \frac{1}{2}(n_1 - n_2)| \leq \frac{1}{2}(n_1 + n_2) - v,
\]

as can again be verified by dimensional counting. The branching rules (9) and (10) together automatically enforce \(v_1 \leq \frac{1}{2}n_1\) and \(v_2 \leq \frac{1}{2}n_2\). For two levels of equal degeneracy \((n_1 = n_2) \equiv n_{12}\), the constraint simplifies to

\[
|v_1 - v_2| \leq n_{12} - v
\]

and only serves to exclude \((v_1, v_2)\) values when \(v > \frac{1}{2}n_{12}\).

For illustration, branchings for the low-dimensional case \(\text{Sp}(4) \to \text{Sp}(2) \otimes \text{Sp}(2)\) (two \(j = \frac{1}{2}\) levels) are given in table 2. The chain \(\text{Sp}(4) \supset \text{Sp}(2) \otimes \text{Sp}(2)\) is isomorphic to the canonical chain \(\text{SO}(5) \supset \text{SO}(4)\) of orthogonal algebras, and the branchings given in table 2 therefore also follow from the \(\text{SO}(n)\) canonical branching rule [48–50]. The \(\text{SO}(5)\) Cartan labels are given by \([\lambda_1, \lambda_2] = [\frac{1}{2}(\lambda_1^1 + \lambda_2^1), \frac{1}{2}(\lambda_1^2 - \lambda_2^2)]\), where \([\lambda_1^1, \lambda_2^2]\) are the \(\text{Sp}(4)\) Cartan labels, and the \(\text{SO}(4)\) Cartan labels are given by \([\lambda_1, \lambda_2] = [\frac{1}{2}(v_1 + v_2), \frac{1}{2}(v_1 - v_2)]\) (see [51] for a summary of notation for this chain). Branchings for the higher-dimensional case \(\text{Sp}(20) \supset \text{Sp}(10) \otimes \text{Sp}(10)\) are given in table 3.

3.3. Generators

The values of \(v_1\) and \(v_2\) arising in the branching \(\text{Sp}(n_1 + n_2) \to \text{Sp}(n_1) \otimes \text{Sp}(n_2)\) are again given by the partitioning condition (8), but now subject to an additional constraint, so

\[
v = v_1 + v_2 + 2n_v \quad (n_v = 0, 1, \ldots, [v/2])
\]

\[
|(v_1 - v_2) - \frac{1}{2}(n_1 - n_2)| \leq \frac{1}{2}(n_1 + n_2) - v,
\]

as can again be verified by dimensional counting. The branching rules (9) and (10) together automatically enforce \(v_1 \leq \frac{1}{2}n_1\) and \(v_2 \leq \frac{1}{2}n_2\). For two levels of equal degeneracy \((n_1 = n_2) \equiv n_{12}\), the constraint simplifies to

\[
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3.3. Generators

Of the generators for the algebras in chains (5) and (6), those involving a single level are well known [32]. Here, we must construct the generators for the two-level system. Since the
subalgebra chains terminate in the two-level angular momentum algebra, SO_{12}(3) \sim SU_{12}(2), it is most natural to express the generators as spherical tensors with respect to this angular momentum algebra.

First, let us briefly review the results for the single \( j \)-shell, with the creation operators \( a_m^\dagger \) and annihilation operators \( a_m \) (\( m = -j, -j + 1, \ldots, j \)). In the case of a single bosonic level, with degeneracy \( n = 2j + 1 \) (\( j \) integer), the subalgebra chain for U(\( n \)) is U(\( n \)) \supset SO(n) \supset \cdots \supset SO(3) \) (see (1)). The generators of U(\( n \)), in the spherical tensor form, are the bilinears

\[
G^{(g)}_\gamma = (a_\gamma^\dagger \times \tilde{a})^{(g)}_\gamma \quad (g = 0, 1, \ldots, 2j),
\]

where \( \gamma = -g, -g + 1, \ldots, +g \). The product of two spherical tensor operators is defined by \((A^a \times B^b)_\gamma = \sum_{\alpha\beta} (a\alpha b\beta |c\gamma) A^{a}_\alpha B^{b}_\beta \). We follow the time-reversal phase convention \( \tilde{\lambda}_a^e = (-)^{e-a} \lambda^a_{-\alpha} \) [52]. Thus, e.g., \( \tilde{a}_m = (-)^{j+m} a_{-m} \), where the time-reversal factor is required for the annihilation operator to transform as a spherical tensor under rotation\(^4\).

The commutators of the generators are most conveniently expressed in the spherical tensor coupled form (see the appendix)

\[
[G^{(e)}, G^{(f)}]^{(g)} = (-)^g [1 - (-)^{ef+g\delta}] \hat{e} \begin{vmatrix} e & f & g \\ j & j & j \end{vmatrix} G^{(e)},
\]

where we adopt the shorthand \( j = (2j + 1)^{1/2} \). The coefficient on the right-hand side of (13) vanishes unless \( e + f + g \) is odd. Consequently, the generators \( G^{(g)}_\gamma \) with \( g \) odd \( (g = 1, 3, \ldots, 2j - 1) \) close under commutation, forming the basis for the subalgebra SO(\( n \)). Finally, the generators \( G^{(1)}_\gamma \), which span the SO(3) algebra, are proportional to the physical angular momentum generators \( L^{(1)}_k = [1/2(j + 1)(2j + 1)]^{1/2} (a_\gamma^\dagger \times \tilde{a})^{(1)}_\gamma \) for a single bosonic \( j \)-shell.

For a single fermionic level, with degeneracy \( n = 2j + 1 \) (\( j \) half-integer), we have instead the chain U(\( n \)) \supset Sp(n) \supset \cdots \supset SU(2) \) (see (2)). The generators \( G^{(g)}_\gamma \) of U(\( n \)) again obey the commutation relations (13), and the generators with \( g \) odd \( (g = 1, 3, \ldots, 2j) \) now span the Sp(\( n \)) algebra. The \( G^{(1)}_\gamma \) are proportional to the physical angular momentum operators, now given by \( L^{(1)}_k = [1/2(j + 1)(2j + 1)]^{1/2} (a_\gamma^\dagger \times \tilde{a})^{(1)}_\gamma \), closing as an SU(2) algebra.

Proceeding now to the algebras involving both levels of the two-level system, let us reduce the complexity of the subscripts, relative to the generic multi-level notation \( a_{\alpha m}^\dagger, \tilde{a}_m \), by denoting the creation operators for the two levels by \( a_{\alpha j}^\dagger = a_{j}^\dagger \), and \( b_{\beta j}^\dagger = \tilde{a}_{j} \), respectively, with the angular momenta \( j_a \equiv j_1 \) and \( j_b \equiv j_2 \), where \( \alpha = -j_a, -j_a + 1, \ldots, +j_a \), and \( \beta = -j_b, -j_b + 1, \ldots, +j_b \). The level degeneracies appearing in the algebra labels are \( n_1 = 2j_a + 1 \) and \( n_2 = 2j_b + 1 \). The algebra U(\( n_1 + n_2 \)) is spanned by

\[
\begin{align*}
G_{\alpha m}^{(g)} &= (a_\alpha^\dagger \times \tilde{a})^{(g)}_m \quad (g = 0, 1, \ldots, 2j_b) \\
G_{ab}^{(g)} &= (a_\alpha^\dagger \times b_\beta)^{(g)} \quad (g = |j_a - j_b|, \ldots, j_a + j_b) \\
G_{ba}^{(g)} &= (b_\beta^\dagger \times \tilde{a})^{(g)}_m \quad (g = |j_a - j_b|, \ldots, j_a + j_b) \\
G_{bb}^{(g)} &= (b_\beta^\dagger \times \tilde{b})^{(g)} \quad (g = 0, 1, \ldots, 2j_b). 
\end{align*}
\]

The commutation relations for these generators are listed in table 4. They may all be obtained from the general bilinear commutation relation (A.9). Note the nearly identical commutation relations for the bosonic and fermionic realizations of the U(\( n_1 + n_2 \)) algebra, with sign differences indicated by the presence of the symbol \( \theta \) in table 4 (recall that \( \theta = + \) for the

\(^4\) The convention \( \tilde{\lambda}_a^m = (-)^{m+\theta} \lambda^m_a \) also arises in the literature. The relative sign between these conventions implies straightforward modifications \( \tilde{\lambda}_a^m \rightarrow (-)^{2j} \tilde{\lambda}_a^{m} \) to signs throughout the following results.
The commutation relations for the SO(n) are given by: 

\[ F(g) = (-)^{\sigma + s} F(g) \]

for the fermionic case and \( \theta = - \) (for the fermionic case). Commutators not listed in Table 4, e.g., \([G^{(f)}_{ab}, G^{(f)}_{cd}]^{(g)}\), can be obtained from those given, by the coupled commutator symmetry relation (A.4). The subalgebra \( U_1(n_1) \otimes U_2(n_2) \) is obtained by simply omitting the 'mixed' generators \( G^{(f)}_{ab} \) and \( G^{(f)}_{ba} \).

The SO(n) subalgebra, for the bosonic case, or Sp(n) subalgebra, for the fermionic case, is then obtained by restricting (14) to the following generators: \( G^{(g)}_{ab} \) with \( g \) odd (\( g = 1, 3, \ldots, 2j_a - 1 \) or \( 2j_a \)), \( G^{(g)}_{ab} \) with \( g \) odd (\( g = 1, 3, \ldots, 2j_b - 1 \) or \( 2j_b \)), and certain linear combinations of the form \( F^{(g)} = \eta_g G^{(g)}_{ab} + \xi_g G^{(g)}_{ba} \), the coefficients of which are determined by the requirement of closure. Specifically, using the results of Table 4, it is found that closure is obtained if \( \xi_g/\eta_g = \sigma_0(-)^g \) for all \( g \). That is, the relative sign between the terms must alternate, between generators \( F^{(g)} \) with an even and an odd tensor rank \( g \), but an overall sign parameter \( \sigma_0 \) may be chosen as either \( \pm 1 \). Thus, we obtain

\[ F^{(g)} = \eta_g [(a^\dagger \times \hat{b})^{(g)} + \sigma_0(-)^g (\hat{b}^\dagger \times \hat{a})^{(g)}]. \]  

Note therefore that there are actually two distinct SO(n) subalgebras which may be included in (5), or two Sp(n) subalgebras in chain (6), distinguished by the relative sign \( \sigma_0 \) in the generators \( F^{(g)} \).

An overall arbitrary phase \( \eta_g \) remains in the definition of \( F^{(g)} \). If we choose \( F^{(g)} \) to be a 'self-adjoint' tensor in the sense that

\[ F^{(g)} = \eta_g [(a^\dagger \times \hat{b})^{(g)} + \sigma_0(-)^g (\hat{b}^\dagger \times \hat{a})^{(g)}]. \]  

i.e. \( F^{(g)} = (-)^{\eta_g} F^{(g)} \), the commutation relations among the generators take on a simple form and, moreover, involve only real coefficients. Let \( \sigma_0 = (-)^s \), with \( s = 0 \) or 1. Then, a self-adjoint tensor \( F^{(g)} \) is obtained for the choice

\[ \eta_g = \begin{cases} 1 & a + b + s \quad \text{even} \\ i & a + b + s \quad \text{odd.} \end{cases} \]

The commutation relations for the SO(n) or Sp(n) generators are listed in Table 5.

| \( E^{(g)} \) | \( F^{(g)} \) | \( [E^{(g)}, F^{(g)}] \) |
|---|---|---|
| \( G^{(g)}_{ab} \) | \( G^{(g)}_{cd} \) | \( (-)^s \eta_g G^{(g)}_{ab} + (-)^s \eta_g G^{(g)}_{cd} \) |
| \( G^{(g)}_{ab} \) | \( G^{(g)}_{cd} \) | \( (-)^s \eta_g G^{(g)}_{ab} + (-)^s \eta_g G^{(g)}_{cd} \) |
| \( G^{(g)}_{ab} \) | \( G^{(g)}_{cd} \) | \( (-)^s \eta_g G^{(g)}_{ab} + (-)^s \eta_g G^{(g)}_{cd} \) |
| \( G^{(g)}_{ab} \) | \( G^{(g)}_{cd} \) | \( (-)^s \eta_g G^{(g)}_{ab} + (-)^s \eta_g G^{(g)}_{cd} \) |

Table 4. Commutation relations for the generators of the \( U_1(n_1) \otimes U_2(n_2) \) algebra of the two-level bosonic system (\( \theta = + \)) or fermionic system (\( \theta = - \)), in coupled form.
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Table 5. Commutation relations for the generators of the SO($n_1 + n_2$) algebra of the two-level bosonic system ($\theta = +$) or the Sp($n_1 + n_2$) algebra of the fermionic system ($\theta = -$), in coupled form.

| $E^{(e)}$ | $F^{(f)}$ | $[E^{(e)}, F^{(f)}]$ |
|-----------|-----------|---------------------|
| $G^{(e)}_{aa}$ | $G^{(f)}_{aa}$ | $2(-\gamma^e_\mu \hat{e}^\mu)^i_j (\epsilon^{ef} f^g g^h j^i)$ |
| $G^{(e)}_{bb}$ | $G^{(f)}_{bb}$ | $2(-\gamma^e_\nu \hat{e}^\nu)^j_i (\epsilon^{ef} f^g g^h j^i)$ |
| $F^{(e)}$ | $F^{(f)}$ | $-2(-\gamma^e_\mu \hat{e}^\mu)^i_j (\epsilon^{ef} f^g g^h j^i)$ $G^{(e)}_{aa} - 2(-\gamma^e_\mu \hat{e}^\mu)^i_j (\epsilon^{ef} f^g g^h j^i)$ $G^{(e)}_{bb}$ |
| $G^{(e)}_{aa}$ | $G^{(f)}_{bb}$ | 0 |
| $G^{(e)}_{bb}$ | $G^{(f)}_{bb}$ | $\theta(-\gamma^e_\mu \hat{e}^\mu)^i_j (\epsilon^{ef} f^g g^h j^i)$ |
| $G^{(e)}_{bb}$ | $G^{(f)}_{bb}$ | $\theta(-\gamma^e_\mu \hat{e}^\mu)^i_j (\epsilon^{ef} f^g g^h j^i)$ |

The phase choice (17) for the two-level generator $F^{(e)}$ also offers consistency with the $s$–$b$ boson models, where $F^{(b)}$ plays an important role as a physical transition operator. For instance, in the IBM ($\hat{j}_a = 0$ and $\hat{j}_d = 2$), the choice $\sigma_0 = +$ (i.e. $\sigma = 0$) gives the SO(6) generator $F^{(2)} = (s^1 \times d^{(2)} + (d^1 \times \bar{s})^{(2)}$, which is the leading-order electric quadrupole operator [38]. The choice $\sigma_0 = -$ (i.e. $\sigma = 1$) instead yields the generator $F^{(2)} = i[(s^1 \times d)^{(2)} - (d^1 \times \bar{s})^{(2)}]$ of a distinct SO(6) subalgebra, denoted by SO(6) [53], which has been shown to be relevant to the decomposition of nuclear excitations into intrinsic and collective parts [54].

Finally, the construction of the remaining subalgebras in (5) and (6) follows by the application of the same principles. The SO$_1(n_1) \otimes$ SO$_2(n_2)$ or Sp$_1(n_1) \otimes$ Sp$_2(n_2)$ algebra is obtained by restriction to $G^{(e)}_{aa}$ and $G^{(f)}_{bb}$ with $g$ odd, and SO$_1(3) \otimes$ SO$_2(3)$ or SU$_1(2) \otimes$ SU$_2(2)$ is obtained by further restriction to $g = 1$. The combined angular momentum algebra, SO$_1(3)$ or SU$_1(2)$, then has the generators $L^{(1)}_1$, where

$$L^{(1)}_1 = \theta_k \left[ 2j_a(j_a + 1)(2j_a + 1) \right]^{1/2} G^{(1)}_{aa} + \theta_k \left[ 2j_b(j_b + 1)(2j_b + 1) \right]^{1/2} G^{(1)}_{bb}. \tag{18}$$

3.4. Casimir operators

To exploit the symmetry properties of the two-level pairing model with respect to the subalgebras of U($n_1 + n_2$), it will be necessary (section 5.1) to express the Hamiltonian in terms of the quadratic Casimir operators of the algebras in (5) and (6). Identification of the Casimir operator proceeds in two stages. First, a quadratic operator which commutes with the generators must be identified. This only defines the Casimir operator to within a normalization (and phase) factor. It is then desirable to choose the normalization such that the eigenvalues of the Casimir operator match the conventional eigenvalue formulas [43, 55], given in terms of the Cartan highest weight labels for the irrep in table 6. For the symmetric irreps of SO($n$) or antisymmetric irreps of Sp($n$) arising in the two-level pairing problem, the eigenvalues can be expressed in terms of the single unified formula

$$\left\{ \begin{array}{c} C_2[SO(n)] \\ C_2[Sp(n)] \end{array} \right\} = 2\nu(\theta v + n - 2\theta). \tag{19}$$

Thus, as the second stage of defining the Casimir operator, the normalization is evaluated by explicitly considering the action of the operator on the one-body states, for which the irrep labels are known.
For the single-level algebra, SO\((n)\) or Sp\((n)\), the operator \(G \circ G\), defined by\(^5\)
\[
G \circ G = - \sum_{g \text{ odd}} \hat{g}[G^{(g)} \times G^{(g)'})]_0^0,
\]
commutes with all the generators, i.e. \(G^{(g)}\) with \(g\) odd \([24, 31, 46]\). The result follows from the general theory of Casimir operators for an algebra, and it may be verified by explicitly evaluating the commutator \([G^{(g)}, G^{(g)'})]\). Using the product rule (A.6) and the product rule (A.6), in terms of the general theory of Casimir operators for an algebra, SO\((n)\) or Sp\((n)\), we start from the Casimir operators \(4\) and proceed to the product rule (A.6). In terms of the conventional spherical tensor scalar product, defined by \(A^{(g)} \cdot B^{(g')} = (-1)^g A^{(g)} \times B^{(g')}\), this operator is \(G \circ G = 2 \sum_{g \text{ odd}} G^{(g)} \cdot G^{(g)'})\). The eigenvalue of \(G \circ G\) acting on the one-body state \(|a\rangle\) may easily be evaluated by Wick’s theorem in coupled form \([57]\), using the commutator results described in the appendix. Comparison with the eigenvalue formula (19) gives the normalization
\[
\begin{align*}
\{ C_2[SO(n)] \} &= 4G \circ G, \\
\{ C_2[Sp(n)] \} &= 2G \circ G,
\end{align*}
\]
covering both the bosonic and fermionic cases.

Proceeding to the two-level problem, for the Casimir operator of SO\((n_1 + n_2)\) or Sp\((n_1 + n_2)\), we start from the Casimir operators \(4G_{aa} \circ G_{aa}\) and \(4G_{bb} \circ G_{bb}\) of each single-level subalgebra, as defined in (20). Although each of these operators commutes with each of the generators of SO\((n_1) \otimes SO(n_2)\) or Sp\((n_1) \otimes Sp(n_2)\), they do not commute with the two-level generators \(F^{(g)}\). We therefore introduce the operator
\[
F \circ F = \sum_{g} \hat{g}[F^{(g)} \times F^{(g)'})]_0^0,
\]
or, equivalently, \(F \circ F = \sum_{g} (-1)^g F^{(g)} \cdot F^{(g)'})\), with \(g = |j_a - j_b|, \ldots, j_a + j_b\). This quantity is invariant with respect to SO\((n_1) \otimes SO(n_2)\) or Sp\((n_1) \otimes Sp(n_2)\). Moreover, the combination
\[
\begin{align*}
\{ C_2[SO(n_1 + n_2)] \} &= 2\theta F \circ F + 4G_{aa} \circ G_{aa} + 4G_{bb} \circ G_{bb},
\{ C_2[Sp(n_1 + n_2)] \} &= 4\theta F \circ F + 4G_{aa} \circ G_{aa} + 4G_{bb} \circ G_{bb}
\end{align*}
\]
commutes with all the generators of SO\((n_1 + n_2)\) or Sp\((n_1 + n_2)\), as seen by the application of the commutators in table 5 and the product rule (A.6). That this combination of operators also has the correct normalization to match the eigenvalue formula for \(C_2[SO(n)]\) (n even) or \(C_2[Sp(n)]\) (Table 6) may be verified by explicitly calculating the one-body expectation value \(<a|C_2[a]⟩\) or \(<b|C_2[b]⟩\).

\(^{5}\) The generators of SO\((n)\) or Sp\((n)\) together transform as an SO\((n)\) or Sp\((n)\) tensor. Therefore, the circle in the notation \(G \circ G\) is meant to represent a scalar product with respect to SO\((n)\) or Sp\((n)\), following [56]. In (22), the notation is generalized to represent an SO\((n_1) \otimes SO(n_2)\) or Sp\((n_1) \otimes Sp(n_2)\) scalar.
Returning to the example of the IBM SO(6) ⊂ SO(5) chain, the Casimir operator (21) becomes
\[
C_2[\text{SO}(5)] = 4(d^1 \times \tilde{d}^{(1)}) \cdot (d^1 \times \tilde{d}^{(1)}) + 4(d^1 \times \tilde{d}^{(3)}) \cdot (d^1 \times \tilde{d}^{(3)}),
\]
for the single level consisting of the quadrupole boson \(d^{(2)}\). Then, for the SO(6) algebra of the two-level \(s-d\) system,
\[
C_2[\text{SO}(6)] = 2[(s^1 \times \tilde{s})^{(2)} + (d^1 \times \tilde{d})^{(2)}] \cdot [(s^1 \times \tilde{s})^{(2)} + (d^1 \times \tilde{d})^{(2)}] + C_2[\text{SO}(5)],
\]
consistent with the usual result [22].

Similar results apply to the quadratic Casimir operators of the unitary algebras in (5) and (6). For the single-level algebra, the linear invariant of \(U(n)\) is simply the occupation number operator \(N = \sum_n a_n^\dagger a_n\), or \(N = \theta J G_0^{(0)}\). The quadratic invariant is given by
\[
C_2[U(n)] = \sum_g \hat{g}(-)^g [G^{(e)} \times G^{(e)}]_0^{(0)},
\]
or \(C_2[U(n)] = \sum_g G^{(e)} \cdot G^{(e)}\). However, for the bosonic realization of \(U(n)\), only symmetric irreps arise, and, for the fermionic realization of \(U(n)\), only antisymmetric irreps arise, with eigenvalues as given in table 6. Therefore, in either situation, it is found that the quadratic invariant is simply a function of the linear invariant and can be expressed as
\[
C_2[U(n)] = N(\theta N + n - \theta). \tag{27}
\]
Likewise, the Casimir operator of the two-level system’s algebra \(U(n_1 + n_2)\) may be expressed as
\[
C_2[U(n_1 + n_2)] = 2\theta N_1 N_2 + n_2 N_1 + n_1 N_2 + C_2[U_1(n_1)] + C_2[U_2(n_2)]. \tag{28}
\]
This result is obtained by the comparison of the eigenvalues for \(C_2[U(n_1 + n_2)]\) with those for \(C_2[U(n_1)]\) and \(C_2[U(n_2)]\), together with the additivity of the number operators \(N = N_1 + N_2\).

### 4. Quasispin algebra

First, we note that a set of three operators \(S_+, S_-, \text{ and } S_z\) obeying the commutation relations
\[
[S_0, S_+]=S_+, \quad [S_0, S_-]=S-, \quad [S_+, S_-]=2\theta S_0, \tag{29}
\]
and obeying the unitarity conditions \(S_+^2 = S_-^2 = S_0^2 = S_0\), span a unitary realization either of the algebra SU(1, 1), for \(\theta = +\), or of the algebra SU(2), for \(\theta = -\). The SU(1, 1) or SU(2) invariant operator is given by
\[
S^2 = S_0^2 - \frac{1}{2} \theta (S_+ S_- + S_- S_+).
\]
\[
= S_0(S_0 - 1) - \theta S_+ S_- \tag{30}
\]
For an irrep of SU(1, 1), this operator takes on the eigenvalues \(S (S - 1)\), and the possible eigenvalues of \(S_0\) are given by \(M = S, S + 1, S + 2, \ldots\). For the ‘true’ group representations of SU(1, 1), \(S\) must be integer or half-integer, but the description of the bosonic system in the quasispin formalism as considered below requires the projective representations with \(S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\), for reasons described in [34]. As usual, for SU(2), \(S^2\) takes on the eigenvalues \(S (S + 1)\), with \(S = 0, 1, 2, \ldots\), and the eigenvalues of \(S_0\) are given by \(M = -S, \ldots, +S - 1, +S\).

Now, consider a system consisting of one or more \(j\)-shells of angular momentum \(j_k (k = 1, 2, \ldots)\). Regardless of whether the operators \(a_{km}^\dagger\) for the \(k\)th level are bosonic or fermionic, a quasispin algebra is defined following the prescription of section 2. The scalar
pair creation operator, scalar pair annihilation operator, and an operator simply related to the number operator for this level form a closed set under commutation. Specifically, let

\[ S_{k+} = \frac{1}{2} \sum_m a_{km}^\dagger a_{km}, \]
\[ S_{k-} = \frac{1}{2} \sum_m a_{km} a_{km}, \]
\[ S_{k0} = \frac{1}{4} \sum_m \left( a_{km}^\dagger a_{km} + \theta a_{km} a_{km}^\dagger \right). \]

These define either an SU(1, 1) quasispin algebra for bosons [34], which we denote by \( SU(1, 1) \), or an SU(2) quasispin algebra for fermions [25, 26], which we denote by \( SU(2) \). In terms of spherical tensor coupled products, these operators (31) may be represented as

\[ S_{k+} = \frac{1}{2} \theta \mathbf{j}_k \left( a_{k}^\dagger \times a_{k} \right)_0 \]
\[ S_{k-} = \frac{1}{2} \theta \mathbf{j}_k \left( a_{k}^\dagger \times \tilde{a}_{k} \right)_0 \]
\[ S_{k0} = \frac{1}{4} \theta \mathbf{j}_k \left[ \left( a_{k}^\dagger \times \tilde{a}_{k} \right)_0 + \left( \tilde{a}_{k}^\dagger \times a_{k} \right)_0 \right]. \]

The operator \( S_{k0} \) is related to the number operator \( N_k = \sum_m a_{km}^\dagger a_{km} \), which may be expressed in spherical tensor form as \( N_k = \mathbf{j}_k \left( a_{k}^\dagger \times \tilde{a}_{k} \right)_0 \), by

\[ S_{k0} = \frac{1}{4} (N_k + \theta \Omega_k), \] (33)

where the constant \( \Omega_k = \frac{1}{2} (2 j_k + 1) \) is the pair degeneracy of the level \( k \). The quasispin \( S_k \), moreover, is related to the seniority quantum number \( v_k \) by the duality relation for a single \( j \)-shell.

It is therefore possible to interconvert between quasispin quantum numbers \( S_k \) and \( M_k \), for each level, and occupation-seniority quantum numbers \( N_k \) and \( v_k \), according to

\[ S_k = \frac{1}{2} (N_k + \theta \Omega_k), \]
\[ M_k = \frac{1}{2} (N_k + \theta \Omega_k). \] (34)

Since the lowest weight state for a given quasispin (i.e. with \( M_k = \theta S_k \)) is destroyed by the pair annihilation operator, and since this state contains \( N_k = v_k \) particles, \( v_k \) may be interpreted as the number of unpaired particles, either bosons or fermions. The seniority \( v_k \) takes on values \( v_k = 0, 1, 2, \ldots \), subject to the constraint \( v_k \leq N_k \) for bosons or \( v_k \leq \min(N_k, 2\Omega_k - N_k) \) for fermions, by the \( \mathcal{M} \) contents of the irreps noted above. Note, therefore, in the fermionic case, that \( v_k \leq \Omega_k \), with the maximum value occurring for half-filling \( (N_k = \Omega_k) \).

A quasispin algebra for the two-level system—which we denote by \( SU_{12}(1, 1) \) for the bosonic case or \( SU_{12}(2) \) for the fermionic case—is spanned by the sum generators

\[ S_+ = S_{k+} + \sigma S_{k0}, \quad S_- = S_{k-} + \sigma S_{k0}, \quad S_0 = S_{k0} + S_{20}. \] (35)

A quasispin algebra is obtained with either choice of sign \( \sigma = \pm \) in the ladder operators. This algebra defines a total quasispin quantum number \( S \) which is dual to the two-level algebra seniority quantum number \( v \), by a relation of the same form as (34), namely

\[ S = \frac{1}{2} (\Omega + \theta v), \]
\[ M = \frac{1}{2} (N + \theta \Omega). \] (36)

As noted in section 3.1, the case of a \( j = 0 \) bosonic level is anomalous. There is no orthogonal algebra dual to the quasispin algebra, and thus no seniority quantum number, but the label \( v_k \) may still be defined from the quasispin via (34). The squared quasispin operator is identically \( S^2 = \frac{1}{\Theta} \) for such a level, as described in [34, 38]. Thus, \( S_k = \frac{1}{2} \) or \( \frac{1}{2} \), and hence \( v_k = 0 \) or 1. Since \( M = S \) must be integral, it follows that \( v_k = 0 \) for \( N_k \) even and \( v_k = 1 \) for \( N_k \) odd, i.e. \( v_k = N_k \mod 2 \). The natural interpretation of this value is that particles in a \( j = 0 \) level are automatically paired to zero angular momentum, except for the one unpartnered particle when the occupation is odd.
where \( N \) and \( \Omega \) are defined above as the sums of the single-level values. The allowed values for the total quasispin \( S \) are given for \( SU(1, 1) \) by \( S \geq S_1 + S_2 \) (i.e. \( S_1 \otimes S_2 \rightarrow S_1 + S_2, \ S_1 + S_2 + 1, \ldots \) ) and for \( SU(2) \) by the familiar triangle inequality \( |S_1 - S_2| \leq S \leq S_1 + S_2 \) (i.e. \( S_1 \otimes S_2 \rightarrow |S_1 - S_2|, \ldots, S_1 + S_2 - 1, S_1 + S_2 \)).

When re-expressed in terms of the seniority labels \( v_1, v_2, \) and \( v \), the \( SU(1, 1) \) coupling rule is equivalent to the \( SO(n_1 + n_2) \rightarrow SO(1) \otimes SO_2(n_2) \) branching rule (8), and the \( SU(2) \) coupling rule is equivalent to the \( Sp(n_1 + n_2) \rightarrow Sp(1) \otimes Sp_2(n_2) \) branching rule (9).

Similarly, the \( SU(n) \) branching rule (10).

When re-expressed in terms of the seniority labels \( v_1, v_2, \) and \( v \), the \( SU(1, 1) \) coupling rule is equivalent to the \( SO(n_1 + n_2) \rightarrow SO(1) \otimes SO_2(n_2) \) branching rule (8), and the \( SU(2) \) coupling rule is equivalent to the \( Sp(n_1 + n_2) \rightarrow Sp(1) \otimes Sp_2(n_2) \) branching rule (9).

Similarly, the \( SL(1, 1) \) Hamiltonian relations

5. Hamiltonian relations

5.1. Dynamical symmetries

Before considering the pairing Hamiltonian in particular, let us consider the Hamiltonian defined by the Casimir operators of the unitary subalgebra chains. A dynamical symmetry \([43]\) arises when the Hamiltonian is constructed in terms of the Casimir operators of a single chain of subalgebras. The eigenstates thus reduce the subalgebra chain, i.e. constitute irreps of the subalgebras. More generally, especially when considering phase transitions, it is useful to construct the Hamiltonian from terms consisting of Casimir operators from multiple, parallel chains, here the upper and lower chains of (5) or (6), as

\[
H = aN + b_1 N_1 + b_2 N_2 + b \left\{ \frac{C_2[SO(n_1 + n_2)]}{C_2[Sp(n_1 + n_2)]} \right\} + c_1 \left\{ \frac{C_2[SO_2(n_1)]}{C_2[Sp_1(n_1)]} \right\} + c_2 \left\{ \frac{C_2[SO_2(n_2)]}{C_2[Sp_2(n_2)]} \right\} + d_1 \mathbf{J}_1^2 + d_2 \mathbf{J}_2^2 + e \mathbf{J}^2,
\]

where higher-order invariants may also be included.

The upper chain in (5) or (6) defines an \( SO(n_1 + n_2) \) or \( Sp(n_1 + n_2) \) dynamical symmetry, and the lower chain defines a \( U_1(n_1) \otimes U_2(n_2) \) dynamical symmetry. The \( U_1(n_1) \otimes U_2(n_2) \) dynamical symmetry is obtained for the Hamiltonian (37) with \( b = 0 \), i.e.

\[
H = aN + b_1 N_1 + b_2 N_2,
\]

\[
+ c_1 \left\{ \frac{C_2[SO_1(n_1)]}{C_2[Sp_1(n_1)]} \right\} + c_2 \left\{ \frac{C_2[SO_2(n_2)]}{C_2[Sp_2(n_2)]} \right\} + d_1 \mathbf{J}_1^2 + d_2 \mathbf{J}_2^2 + e \mathbf{J}^2.
\]

The eigenstates \( |N N_1 N_2 v_1 v_2 \cdots J_1 J_2 J \rangle \) have definite occupation numbers for each of the levels and have energy eigenvalues

\[
E = aN + b_1 N_1 + b_2 N_2 + 2c_1 v_1 (\theta v_1 + n_1 - 2\theta) + 2c_2 v_2 (\theta v_2 + n_2 - 2\theta) + d_1 J_1 (J_1 + 1) + d_2 J_2 (J_2 + 1) + e J (J + 1).
\]

The \( SO(n_1 + n_2) \) or \( Sp(n_1 + n_2) \) dynamical symmetry is obtained for the Hamiltonian (37) with \( b_1 = b_2 = 0 \), i.e.

\[
H = aN + b \left\{ \frac{C_2[SO(n_1 + n_2)]}{C_2[Sp(n_1 + n_2)]} \right\} + c_1 \left\{ \frac{C_2[SO_1(n_1)]}{C_2[Sp_1(n_1)]} \right\} + c_2 \left\{ \frac{C_2[SO_2(n_2)]}{C_2[Sp_2(n_2)]} \right\} + d_1 \mathbf{J}_1^2 + d_2 \mathbf{J}_2^2 + e \mathbf{J}^2,
\]
Energy diagrams for the bosonic $SO(n_1 + n_2)$ and fermionic $Sp(n_1 + n_2)$ dynamical symmetries of the two-level pairing model. The degeneracy within irreps of the two-level algebra (brackets labeled by $v$) is split according to the single-level algebra irrep labels $(v_1 v_2)$. (a) Energy levels for the bosonic $SO(6) \supset SO(3) \otimes SO(3)$ dynamical symmetry Hamiltonian $(j_1 = j_2 = 1)$. (b) Energy levels for the fermionic $Sp(8) \supset Sp(4) \otimes Sp(4)$ dynamical symmetry Hamiltonian $(j_1 = j_2 = \frac{3}{2})$. The Hamiltonian in each case is chosen as $H = -\theta F \circ F$, i.e. $b = -\frac{1}{2}$ and $c_1 = c_2 = +\frac{1}{2}$ in (40). The total occupation in both cases is $N = 4$, which for the fermionic example gives half-filling.

with eigenstates $|N v_1 v_2 \cdots J_1 J_2 J \rangle$ and energy eigenvalues

$$E = aN + 2bv(\theta v + n_1 + n_2 - 2\theta) + 2c_1 v_1 (\theta v_1 + n_1 - 2\theta) + 2c_2 v_2 (\theta v_2 + n_2 - 2\theta) + d_1 J_1 (J_1 + 1) + d_2 J_2 (J_2 + 1) + eJ(J + 1).$$

The energy spectrum for the dynamical symmetry follows from the branching rules of section 3.2. Examples of energy level diagrams are shown for a bosonic system ($n_1 = n_2 = 3$) in figure 1(a) and for a fermionic system of similar degeneracies ($n_1 = n_2 = 4$) in figure 1(b).

5.2. Pairing Hamiltonian

The pairing Hamiltonian for a generic multi-level system, consisting of levels of angular momentum $j_1, j_2, \ldots$, is given by

$$H = \sum_{km} \varepsilon_{km} a_{km}^\dagger a_{km} + \frac{1}{4} \sum_{kk_1} \sum_{m'm'} G_{kk_{1}k_{1}} (\theta_{j_{1}'-m'} a_{k_{1}m'}^\dagger a_{k_{1}'-m}^\dagger - \theta_{j_{1}'-m} a_{k_{1}'-m'}^\dagger a_{k_{1}'-m})$$

where the summation indices $k$ and $k'$ run over the single-particle levels ($k = 1, 2, \ldots$), and $m$ and $m'$ run over their substates ($m = -j_k, -j_k + 1, \ldots, +j_k$). The creation operators $a_{km}^\dagger$ and annihilation operators $a_{km}$ obey either canonical commutation relations (bosons) or anticommutation relations (fermions). The first term represents the one-body energy contribution for each level, and the products in the second term are creation or annihilation operators for pairs involving time-reversed partner substates. Note that $G_{kk'} = G_{kk'}^*$ by Hermiticity of $H$, and that we take all coefficients to be real. That this Hamiltonian is
specifically constructed from angular momentum zero pair operators is seen by rewriting it in terms of angular-momentum coupled products, as

\[ H = \sum_k \varepsilon_k N_k + \frac{1}{4} \theta \sum_{k,k'} G_{kk'} J_k J_k' (a_k^* \times a_k')_0^0 \times (\tilde{a}_k \times \tilde{a}_k')_0^0 \tag{43} \]

The Hamiltonian (42) is integrable and may be solved using a generalized Gaudin algebra [59] or, under certain conditions, Bethe ansatz [60, 61] methods. Two limiting cases deserve special mention, since they are characterized by dynamical symmetries (section 5.1), as described below, and are solvable by more elementary methods. (1) Trivially, when all \( G_{kk} = 0 \), the problem reduces to that of a system of noninteracting particles (weak-coupling limit), with eigenstates characterized by occupations numbers \( N_k \). (2) When all \( \varepsilon_k = 0 \), physically corresponding to the situation in which the level energy difference is negligible relative to the pairing strength (strong-coupling limit), and if all \( G_{kk} \) are equal to within possible phase factors (uniform pairing), the problem is immediately solvable by the use of quasispin (section 5.3).

5.3. Quasispin Hamiltonian

The generic multi-level pairing Hamiltonian (42) can be expressed entirely in terms of the quasispin generators, by comparison with (32), as

\[ H = \sum_k \varepsilon_k (2S_{k0} - \theta \Omega_k) + \sum_{k,k'} G_{kk'} S_k S_k' \tag{44} \]

This Hamiltonian therefore conserves the quasispin \( S_q \) or, equivalently, the seniority \( v_k \), associated with each level. The Hamiltonian is also number conserving, so the total z-projection \( S_0 = \sum_k S_{0k} = \frac{1}{2}(N + \theta \Omega) \) is conserved, where \( N = \sum_k N_k \) and \( \Omega = \sum_k \Omega_k \). However, the individual \( S_{0k} \) are not in general conserved, unless the levels completely decouple, with \( G_{kk'} = 0 \) for all \( k' \neq k \).

Note that numerical diagonalization is straightforward in the weak-coupling basis, consisting of states \( |N_1 v_1 N_2 v_2 \cdots \rangle \) of good seniority and occupation for each level. The action of the Hamiltonian (44) on these states follows from the known action of the quasispin ladder operators, \( S_k |SM \rangle = \frac{-\theta (S \pm \theta M)(S \mp \theta M - \theta)}{\Omega (S \pm 1)} |SM \pm 1 \rangle \), once the quantum numbers are translated via (34). Specifically,

\[
\begin{align*}
S_{k+} |N_1 v_1 \cdots N_k v_k \cdots \rangle &= \frac{1}{2}[\theta (N_k - v_k + 2)(N_k + v_k + 2\theta \Omega_k)]^{1/2} |N_k + 2v_k \cdots \rangle \\
S_{k-} |N_1 v_1 \cdots N_k v_k \cdots \rangle &= \frac{1}{2}[\theta (N_k - v_k)(N_k + v_k + 2\theta \Omega_k - 2)]^{1/2} |N_k - 2v_k \cdots \rangle .
\end{align*}
\tag{45}
\]

Consequently, the matrix elements for the diagonal pairing terms are

\[
\langle \cdots |N_k v_k \cdots |S_k S_k^+ \cdots |N_k v_k \cdots \rangle = \frac{1}{2} \theta [N_k(N_k + 2\theta \Omega_k - 2) - v_k(v_k + 2\theta \Omega_k - 2)]
\tag{46}
\]

and for the off-diagonal terms are

\[
\langle \cdots (N_k' + 2)v_k' \cdots (N_k - 2)v_k \cdots |S_k S_k^+ \cdots N_k v_k^\prime \cdots N_k v_k \cdots \rangle = \frac{1}{4} [(N_k' - v_k' + 2)(N_k' + v_k' + 2\theta \Omega_k) \\
\times (N_k - v_k)(N_k + v_k + 2\theta \Omega_k - 2)]^{1/2},
\tag{47}
\]

as noted for the fermionic case in, e.g., [10, 40].

Returning to the two-level problem, the pairing Hamiltonian in quasispin notation is

\[
H = \varepsilon_1 (2S_{10} - \theta \Omega_1) + G_{11} S_{1+} S_{1-} + \varepsilon_2 (2S_{20} - \theta \Omega_2) + G_{22} S_{2+} S_{2-} \\
+ G_{12} (S_{1+} S_{2-} + S_{2+} S_{1-}).
\tag{48}
\]
The two-level pairing Hamiltonian has two dynamical symmetries [62] defined with respect to the quasispin algebras, corresponding to either the upper or lower subalgebra chains in

$$\mathcal{SU}_1(1, 1) \otimes \mathcal{SU}_2(1, 1) \supset \left\{ \mathcal{SU}_1(1, 1) \right\}_{v_1} \otimes \left\{ \mathcal{SU}_2(1, 1) \right\}_{v_2} \supset U_{12}(1)$$

(49)

for the bosonic case or

$$\mathcal{SU}_1(2) \otimes \mathcal{SU}_2(2) \supset \left\{ \mathcal{SU}_1(2) \right\}_{v_1} \otimes \left\{ \mathcal{SU}_2(2) \right\}_{v_2} \supset U_{12}(1)$$

(50)

for the fermionic case, with conserved quantum numbers as indicated. Here, the algebra $U_1(1)$ is the trivial Abelian algebra spanned by $S_{00}$, and $U_{12}(1)$ is spanned by their sum $S_{zz}$, as defined in (35). The occupation-seniority labels $(v$ and $N$) are indicated, rather than the quasispin labels $(S$ and $M)$, for a closer connection to the physical problem and easier comparison with the dual algebra’s dynamical symmetries.

The dynamical symmetry Hamiltonian for the upper subalgebra (i.e. $\epsilon_1 = \epsilon_2 = 0$) of the two-level pairing Hamiltonian, with uniform pairing strength, is defined in section 5.2. Specifically, let $G_{11} = \sigma G_{12} = \sigma G_{21} = G_{22} \equiv G$, for either sign $\sigma = \pm$. Then, the Hamiltonian is given by

$$H = GS_+S_-,$$

(51)

where $S_{\pm}$ are the sum-quasispin ladder operators of (35), defined in terms of the same sign $\sigma$. Since $S_{\pm} = \theta[S_0(S_0 - 1) - S_z^2]$, by (30), the strong-coupling Hamiltonian conserves the total quasispin $S$ (or seniority $v$), as well as the projection quantum number $M$ (or occupation $N$), and has the eigenvalues

$$\langle S_+ S_- \rangle = \frac{1}{2} \theta[N(N + 2\theta \Omega - 2)] - \nu(v + 2\theta \Omega - 2)],$$

(52)

as expressed in terms of the occupation-seniority labels. The eigenstates are identical to those of the $SO(n_1 + n_2)$ or $Sp(n_1 + n_2)$ dynamical symmetry Hamiltonian (40).7 The specific relationship between the Hamiltonians is determined below in section 5.4.

The dynamical symmetry Hamiltonian for the lower subalgebra chain is the weak-coupling limit of the pairing Hamiltonian ($\epsilon_1 = \epsilon_2 = 0$), as defined in section 5.2. The dynamical symmetry eigenstates are simply the level occupation eigenstates of good $N_1$ and $N_2$, as for the $U_1(n_1) \otimes U_2(n_2)$ dynamical symmetry Hamiltonian (38), i.e. the weak-coupling basis states considered above.

The full two-level pairing Hamiltonian (48) can be expressed entirely in terms of the invariant operators of algebras appearing in the upper and lower chains. Specifically,

$$H = \epsilon_1(2S_{10} - 3\theta S_1) + (G_{11} - \sigma G_{12})\theta[S_{10}(S_{10} - 1) - S_z^2]$$

$$+ \epsilon_2(2S_{20} - 3\theta S_2) + (G_{22} - \sigma G_{12})\theta[S_{20}(S_{20} - 1) - S_z^2]$$

$$+ G_{12}\sigma \theta[S_0(S_0 - 1) - S_z^2].$$

(53)

5.4. Duality relations for the Hamiltonian

The eigenstates for the dynamical symmetries of the two algebraic frameworks—one number-conserving unitary and number-nonconserving quasispin—are identical, that is, the irreps

7 More precisely, the quasispin Hamiltonian (51) has a higher degeneracy than the $SO(n_1 + n_2)$ or $Sp(n_1 + n_2)$ Hamiltonian (40), but the eigenstates can be chosen from within each degenerate subspace to match those of (40), i.e. of good $J_1$, $J_2$, and $J$.
which reduce the unitary algebra chains (5) and (6) reduce the quasispin algebra chains (49) and (50) as well, and the labels for the chains are connected through the duality relations. The pairing Hamiltonian is defined in section 5.2 (see (42)) in terms of certain combinations of operators which represent scalar pair creation, scalar pair annihilation, and number operators. These are noted in section 5.3 (see (44)) to be essentially the quasispin generators, and the Hamiltonian can also therefore be expressed directly in terms of the quasispin invariants (see (53)). However, the pairing Hamiltonian can just as well be expressed in terms of the Casimir operators of subalgebras of U(n1 + n2), as a special case of the Hamiltonian of section 5.1 (see (37)). That such a relation exists is implied by the duality of irreps, but it is explicitly obtained by appropriate recoupling and reordering of the bosonic or fermionic creation and annihilation operators in this section.

For a single j-shell, recoupling and commutation of creation operators yields

\[
(a^\dagger \times a^\dagger)_0 \otimes (\tilde{a} \times \tilde{a})_0 = \frac{\theta}{\gamma^2} \left[ -(a^\dagger \times \tilde{a})_0^0 + \sum_S g_S [a^\dagger \times \tilde{a}]_0^{(s)} \times (a^\dagger \times \tilde{a})_0^{(s)} \right].
\]

Thus, the relation between quasispin and Casimir Hamiltonians for a single level is

\[
4S_+S_- = -\theta N + C_2[U(n)] - \frac{1}{2} \left( C_2[SO(n)] \right); \tag{55}
\]

by comparison with the explicit realizations of the various operators, namely N from section 3.4, C_2[SO(n)] or C_2[Sp(n)] from (21), C_2[U(n)] from (26), and S_± from (32).

For the two-level system, which has quasispin generators given by (35), the product S_+,S_- involves both single-level terms (S_1+S_-, S_2+S_-, and S_1-S_-) and cross terms (\sigma S_1 S_2- and \sigma S_2 S_1-), which destroy a pair in one level and create a pair in the other. Recoupling and commutation of the mixed terms yield

\[
(a^\dagger \times a^\dagger)_0 \otimes (\tilde{b} \times \tilde{b})_0^0 + (b^\dagger \times b^\dagger)_0 \otimes (\tilde{a} \times \tilde{a})_0^0 = \frac{(-)^{j_a+j_b} \theta \sigma_0}{j_a j_b} \left[ F \circ F - 2N_a N_b - \theta (J_a^2 N_a + J_b^2 N_b)\right]. \tag{56}
\]

The one-level terms and mixed terms of S_+,S_- may thus be combined to give an expression involving the SO(n1 + n2) or Sp(n1 + n2) Casimir operator and U(n1 + n2) invariants, if and only if the sign \sigma arising in the definition of the sum quasispin algebra and the sign \sigma_0 entering into the definition of F^{(s)} are related by

\[
\frac{\sigma_0}{\sigma} = -\theta (-)^{j_a+j_b}. \tag{57}
\]

We again have an expression of the same form as (55):

\[
4S_+S_- = -\theta (N_1 + N_2) + C_2[U(n1 + n2)] - \frac{1}{2} \left( C_2[SO(n1 + n2)] \right). \tag{58}
\]

That the expression is of this form is to be expected from the general nature of the duality, which indeed makes no assumption (section 2) as to whether the single-particle states are considered to be arranged into a single j-shell, as for (55), or two j-shells, as here. From a practical standpoint, what is most useful is that the relation can now be expressed explicitly in terms of the two-level system operators given by (23), (28), and (35) as spherical-tensor products of creation and annihilation operators, with well-defined phases \sigma_0 and \sigma.

\[8\] The product of a pair creation operator and a pair annihilation operator is related to the product of spherical-tensor one-body operators by, e.g., identity (25a) of [57].
The two-level operator correspondence (58) relates the $SU_{12}(1, 1)$ or $SU_{12}(2)$ dynamical symmetry Hamiltonian in the quasispin scheme (51), i.e. strong coupling with uniform pairing, to the $SO(n_1 + n_2)$ or $Sp(n_1 + n_2)$ dynamical symmetry Hamiltonian in the unitary algebra scheme. Furthermore, taken in conjunction with the single-level operator correspondence (55), it allows the full two-level pairing Hamiltonian (not just at the dynamical symmetry limit) to be expressed in terms of Casimir operators of algebras appearing in the two parallel subalgebra chains of $U(n_1 + n_2)$. Starting from (48), one obtains

\[
H = \left( \epsilon_1 - \frac{1}{4} \delta G_{11} \right) N_1 + \frac{1}{4} \left( G_{11} - \sigma G_{12} \right) C_2[U_1(n_1)] - \frac{1}{8} (G_{11} - \sigma G_{12}) \begin{cases} C_2[SO(n_1)] \\ C_2[Sp(n_1)] \end{cases} + \left( \epsilon_2 - \frac{1}{4} \delta G_{22} \right) N_2 + \frac{1}{4} \left( G_{22} - \sigma G_{12} \right) C_2[U_2(n_2)] - \frac{1}{8} (G_{22} - \sigma G_{12}) \begin{cases} C_2[SO(n_2)] \\ C_2[Sp(n_2)] \end{cases} + \frac{1}{4} \sigma G_{12} C_2[U(n_1 + n_2)] - \frac{1}{8} \sigma G_{12} \begin{cases} C_2[SO(n_1 + n_2)] \\ C_2[Sp(n_1 + n_2)] \end{cases}.
\]  

(59)

For this relation to be valid, the phases $\sigma_0$ and $\sigma$, used in defining the generators for orthogonal or symplectic algebra and two-level quasispin algebra, respectively, must be related by (57).

5.5. Multipole Hamiltonian

The Hamiltonian for spectroscopic studies of the $s$-$b$ boson models is commonly expressed in terms of a ‘multipole’ term of the form $[(s^I \times \hat{b}^J)^{(L)} + (b^I \times \hat{s})^{(L)}] \cdot [(s^I \times \hat{b}^J)^{(L)} + (b^I \times \hat{s})^{(L)}]$, where $L = j_0$ [63]. For instance, the customary IBM $U(5)$–$SO(6)$ quadrupole Hamiltonian [64, 65] is

\[
H_{QQ} = \frac{(1 - \xi)}{N} N_a - \frac{\xi}{N^2} [(s^I \times \hat{a})^{(2)} + (d^I \times \hat{s})^{(2)}] \cdot [(s^I \times \hat{a})^{(2)} + (d^I \times \hat{s})^{(2)}].
\]

(60)

The $U(5)$ limit is obtained for $\xi = 0$ and the $SO(6)$ limit for $\xi = 1$. The operator $F \circ F$, appearing as the ‘cross term’ in $C_2[SO(n_1 + n_2)]$ or $C_2[Sp(n_1 + n_2)]$ (see (23)), generalizes the multipole term to the generic two-level model. In contrast, the Hamiltonian for spectroscopic studies of the fermionic system is commonly expressed in pairing or quasispin form [5, 20]. Thus, we seek to relate these distinct—pairing and multipole—forms of the Hamiltonian.

Recall that the operators considered thus far in connection with the strong-coupling limit—$S^2, S^z, S^x$, and $C_2[SO(n_1 + n_2)]$ or $C_2[Sp(n_1 + n_2)]$—differ only in normalization (or sign) and by addition of a function of $N$, the conserved total occupation number. Therefore, the eigenstates are identical and the eigenvalues differ only by a rescaling and a constant offset. However, the operator $F \circ F$ appearing in the multipole Hamiltonian differs from these again, see (23) by terms proportional to $C_2[SO(n_1)]$ and $C_2[SO(n_2)]$, in the bosonic case, or $C_2[Sp(n_1)]$ and $C_2[Sp(n_2)]$, in the fermionic case. The eigenstates are therefore again the same as for $S^2, S^z, S^x$, and $C_2[SO(n_1 + n_2)]$ or $C_2[Sp(n_1 + n_2)]$, but eigenvalues are no longer degenerate for states sharing the same value of $v$. They are rather now split by $v_1$ and $v_2$, as illustrated in figure 1.

9 The analog of (58) for the IBM was exploited in [38] to establish the properties of the IBM $SO(6)$ dynamical symmetry eigenstates. With $\theta = +$, the sign condition (57) gives $\sigma_0 = -\sigma$ (recall $j_a = 0$ and $j_b = 2$). Thus, the pairing operator $S^z, S^x$ for the quasispin defined with the negative sign ($\sigma = -$) relates to the Casimir operator of the ‘physical’ $SO(6)$ algebra ($\sigma_0 = +$), and the quasispin algebra defined with the positive relative sign is instead dual to the $SO(6)$ algebra (section 3.3).
For the explicit relationships among these operators, observe that, by (58),

\[ 4S_+S_- = C_2[U(n_1 + n_2)] - \theta N - \frac{1}{2} \left( \frac{C_2[SO(n_1 + n_2)]}{C_2[Sp(n_1 + n_2)]} \right), \tag{61} \]

and, in terms of (23),

\[ \begin{cases} C_2[SO(n_1 + n_2)] \\ C_2[Sp(n_1 + n_2)] \end{cases} = 2\theta F \circ F + \begin{cases} C_2[SO_1(n_1)] \\ C_2[Sp_1(n_1)] \end{cases} + \begin{cases} C_2[SO_2(n_2)] \\ C_2[Sp_2(n_2)] \end{cases}. \tag{62} \]

Therefore, the pairing \((S_+S_-)\) and multipole \((F \circ F)\) forms of the Hamiltonian, those most frequently encountered in applications, are related by

\[ -\theta F \circ F = 4S_+S_- - [C_2[U(n_1 + n_2)] - \theta N] + \frac{1}{2} \left( \frac{C_2[SO_1(n_1)]}{C_2[Sp_1(n_1)]} \right) + \frac{1}{2} \left( \frac{C_2[SO_2(n_2)]}{C_2[Sp_2(n_2)]} \right), \tag{63} \]

where

\[ C_2[U(n_1 + n_2)] - \theta N = \begin{cases} N(N + 2\Omega - 2) \\ N(2\Omega - N + 2) \end{cases} \tag{64} \]

simply contributes a \(c\)-number shift to the eigenvalue spectrum, without affecting the eigenfunctions.

A positive coefficient \((G > 0)\) for \(S_+S_-\) gives a positive pair energy, i.e. repulsive pairing, in both bosonic and fermionic cases, as may be seen from (52) with \(N = 1\) and \(v = 0\). The sign of the pairing interaction is of special interest in comparing the bosonic and fermionic two-level pairing models, since it should be noted (section 6) that the system undergoes a quantum phase transition for a repulsive \((G > 0)\) pairing interaction in the bosonic case and an attractive pairing interaction \((G < 0)\) in the fermionic case. Thus, it is essential to note that repulsive pairing is obtained for a negative coefficient on \(F \circ F\) in the bosonic case and a positive coefficient on \(F \circ F\) in the fermionic case, i.e. for \(H = -\theta F \circ F\), or vice versa for attractive pairing. Therefore, for repulsive pairing, a Hamiltonian

\[ H_{FF} = \left( \frac{1 - \xi}{N} \right) N_2 - \theta \frac{\xi}{N^2} F \circ F \tag{65} \]

is the natural generalization of the multipole form for the transitional Hamiltonian (60) to generic two-level pairing models, as considered in section 6.

The last two terms in (63), involving the Casimir operators of \(SO_1(n_1)\) and \(SO_2(n_2)\) or \(Sp_1(n_1)\) and \(Sp_2(n_2)\), contribute a common shift to the energy eigenvalues for each subspace of states characterized by a given pair of values of the conserved \((v_1v_2)\) quantum numbers, without affecting the eigenfunctions, i.e. these terms serve only to displace the different \((v_1v_2)\) subspaces relative to each other. If only \((v_1v_2) = (00)\) states are considered, the \(SO_1(n_1)\) and \(SO_2(n_2)\) or \(Sp_1(n_1)\) and \(Sp_2(n_2)\) terms have no effect at all. They will therefore not be considered further.

Now to consider the spectra, the strong-coupling Hamiltonian operator \(4S_+S_-\)—we include the factor of 4 arising in (63) for convenience—has eigenvalues given by (52), obtained with \(0 \leq v \leq N\) for bosonic pairing or with \(0 \leq v \leq \min(N, 2\Omega - N)\) for fermionic pairing, where only even values of \(v\) arise for \(N\) even, or odd values of \(v\) for \(N\) odd (see (7) and (9)). Thus, taking \(N\) even, the eigenvalues span the range

\[ \langle 4S_+S_- \rangle = 0, \ldots, N(N + 2\Omega - 2), \tag{66} \]

20
Figure 2. Eigenvalues of the pairing interaction term $4S_+S_-$, which determines the energy spectrum of the two-level pairing model in the strong-coupling limit, shown for the (a) bosonic and (b) fermionic systems, as a function of filling $N$. The eigenvalues are given by (66) and (67), respectively. The axes are labeled generically, to indicate the asymptotic (large-$\Omega$) dependences discussed in the text, but the specific points shown for illustration are calculated for $\Omega = 50$.

for bosonic pairing, or

$$\langle 4S_+S_- \rangle = \begin{cases} 0 & (N \leq \Omega) \\ \frac{4(N - \Omega)}{v = 2\Omega - N} & (\Omega \geq N) \\ 4(2\Omega - N + 2) & v = 0 \end{cases}$$

(67)

for fermionic pairing, in which case $0 \leq N \leq 2\Omega$. (If $N$ is odd, the sequences above would end instead with $v = 1$ rather than $v = 0$, but the large-$N$ dependence of the highest eigenvalue on $N^2$ is not changed.) The range of eigenvalues therefore depends upon the total occupation or ‘filling’ $N$ of the two-level system as sketched in figure 2(a) for the bosonic system and figure 2(b) for the fermionic system. The asymptotic dependences for large degeneracy ($\Omega \gg 1$) are indicated. Specifically, at a filling approximately equal to half the total degeneracy, i.e. $N \approx \Omega$, note that the bosonic eigenvalues span a range $\sim 3\Omega^2 (\approx 3N^2)$, while the fermionic eigenvalues for the same filling and degeneracy only span a range of $\sim \Omega^2 (\approx N^2)$. If, instead, the limit of large occupation is taken at fixed degeneracy ($\Omega \ll N$) in the bosonic case, the range of eigenvalues $\sim N^2$ is the same as for a fermionic pairing model of the same $N$ but at half-filling (which is obtained for a correspondingly larger degeneracy $\Omega = N$).

Note also that, for repulsive pairing, the bosonic ground state (for which $v = N$) has zero eigenvalue. In contrast, the fermionic ground state (for which $v = 0$ below half-filling and $v = 2\Omega - N$ past half-filling) has an eigenvalue which grows linearly with $N$ past half-filling. The nonzero ground-state pairing energy for the fermionic system may be understood since, past half-filling, Pauli exclusion enforces the existence of some particles in time-reversal conjugate orbits and hence some probability for pairs coupled to zero angular momentum.

For the multipole form $-\theta F \circ F$ of the pairing interaction operator, the spectrum is shifted downward by an $N$-dependent offset relative to that of $4S_+S_-$. (see (63)). The highest eigenvalue (obtained for $v = 0$) is always zero. The asymptotic form of the ground-state eigenvalue is, alternatively, $\sim -N^2$ for fermionic half-filling ($1 \ll \Omega \approx N$), $\sim -3N^2$ for bosonic ‘half-filling’ ($1 \ll \Omega = N$), and again $\sim -N^2$ for the bosonic system at larger boson number ($\Omega \ll N$).
6. Transitional Hamiltonian

A second-order quantum state phase transition occurs between the weak- and strong-coupling limits for the two-level pairing models. Specifically, for the bosonic system it occurs with the repulsive pairing interaction, and for the fermionic system it occurs with the attractive interaction. The quantum phase transition is apparent numerically from calculations for finite \( N \) and from semiclassical treatments of the large-\( N \) limit. The present duality relations (section 5) immediately help clarify the comparison of numerical eigenvalue spectra across the transition but are also intended to facilitate the construction of coherent states for the semiclassical treatment.

The simplest semiclassical 'geometry' for the two-level pairing model is obtained from the quasispin algebraic structure, most simply by replacing the quasispin operators with classical angular momentum vectors, which maps the pairing model onto an essentially one-dimensional coordinate space. This approach has been applied in both the bosonic \( s-b \) models and fermionic two-level pairing model with equal degeneracies \([5, 8, 42, 66, 67]\). In both these circumstances, the quantum phase transition is found to occur, in the large-\( N \) limit, at \( N|G|/\varepsilon = 1 \). For the \( s-b \) models, a higher-dimensional and richer classical geometry (see \([42]\)) has been established through the use of \( U(n_2 + 1)/U(n_2) \) coherent states \([42, 68–71]\). An extension of this treatment to \( U(n_1 + n_2)/[U(n_1) \otimes U(n_2)] \) for generic two-level pairing models might profitably be obtained using the explicit construction of generators for the \( SO(n_1 + n_2) \supset SO(n_1) \otimes SO(n_2) \) and \( Sp(n_1 + n_2) \supset Sp(n_1) \otimes Sp(n_2) \) chains considered in section 3.

However, at present, we confine ourselves to laying the groundwork for more detailed further work, allowing for the most general choice of level degeneracies and more uniformly treating the bosonic and fermionic cases. A pairing Hamiltonian

\[
H_{\text{pair}} = \frac{(1 - \xi)}{N} N_2 + \frac{4\xi}{N^2} S_+ S_-, \tag{68}
\]

may be defined with opposite signs \( \theta \) of the pairing term for the bosonic and fermionic cases, so that the quantum phase transition is obtained in either case. This Hamiltonian yields the weak-coupling limit for \( \xi = 0 \), the strong-coupling limit at \( \xi = 1 \), and the critical interaction strength \( N|G|/\varepsilon = 1 \) at \( \xi = 1/5 \). Scaling of the one-body term by \( N^{-1} \) and of the two-body term by \( N^{-2} \) ensures that the critical point remains fixed at the finite value \( \xi = 1/5 \) as \( N \to \infty \).

However, for this Hamiltonian, a grossly different 'envelope' to the eigenvalue spectrum (i.e. the range of eigenvalues, obtained as a function of the control parameter \( \xi \)) is found in the bosonic and fermionic cases (see figure 3).

To facilitate the direct comparison of the bosonic and fermionic quantum phase transitions, it is helpful to instead construct a Hamiltonian for which the ground-state energy follows the same trajectory as a function of \( \xi \) in the large \( N \) limit, and the eigenvalues span the same range at each of the limits, namely [0, 1] for \( \xi = 0 \) and [−1, 0] for \( \xi = 1 \). By the results of section 5, this is accomplished by choosing, for the repulsive pairing interaction, the Hamiltonian

\[
H_+ = \frac{(1 - \xi)}{N} N_2 + \frac{\xi}{N^2} \left[ 4S_+ S_- - \frac{N(N + 2\Omega - 2)}{N(2\Omega - N + 2)} \right], \tag{69}
\]

and, for the attractive pairing interaction, the usual Hamiltonian

\[
H_- = \frac{(1 - \xi)}{N} N_2 - \frac{4\xi}{N^2} S_+ S_- \tag{70}
\]

The \( c \)-number offset included in the definition of \( H_+ \), which arises as \( C_2 U(n_1 + n_2) - \theta N \), is included to achieve the same range of eigenvalues in the strong-coupling limit, as well
as a similar evolution of ground-state energy across the transition (figure 3), in the large-$N$
limit, thereby facilitating the comparison of the bosonic (repulsive pairing) and fermionic
(attractive pairing) quantum phase transition. With inclusion of this offset, \( H_s \) is equivalent
to the generalized multipole transitional Hamiltonian (65) when acting on the \((v_1 \, v_2) = (00)\)
subspace of any two-level pairing model, as may be seen from (63). In particular, for the \(s-b\)
models, inclusion of this offset makes \( H_s \) identical to the conventional multipole form (60) of
the transitional Hamiltonian, when acting on the \((v_1 \, v_2) = (00)\) subspace.

The evolution of the eigenvalue spectrum across the transition between weak coupling
and strong coupling is shown for representative bosonic and fermionic cases in figure 4.
Specifically, equal-degeneracy pairing models \((n_1 = n_2 \equiv \Omega)\) are considered, and the
\((v_1 \, v_2) = (00)\) states are shown. Here, a sufficiently large total occupancy \((N = 50)\) is
chosen such that the precursors of the phase transitional singularities are readily apparent.
Spectra are shown for both bosons (figure 4 (left)) and fermions (figure 4 (right)), with
repulsive (figure 4 (top)) and attractive (figure 4 (bottom)) interactions. Qualitatively similar
spectra in the bosonic and fermionic cases are obtained when level degeneracies for the bosonic
calculation \((\Omega = 5)\) are much less than the occupancy, while the degeneracies for the fermionic
calculation \((\Omega = 50)\) are such as to give half-filling. Then, the "envelope" of the spectrum
(the range of eigenvalues at a given value of the Hamiltonian parameter \(\xi\)) is essentially
identical for the bosonic case with repulsive pairing (figure 4(a)) and the fermionic case with
attractive pairing (figure 4(d)), i.e. the interactions signs which yield a ground-state quantum
phase transition. Features to observe include the essentially constant ground-state energy for
\(\xi < 1/5\) and downturn (from 0 to \(-1\)) for \(\xi > 1/5\), an approximately linear evolution of the
highest eigenvalue from +1 to 0, and a compression of the level density at \(E \approx 0\) for \(\xi > 1/5\),
a characteristic of the excited-state quantum phase transition [14].
Figure 4. Eigenvalues of the bosonic two-level pairing model, with level degeneracies \( n_1 = n_2 = 5 \) (at left), and fermionic two-level pairing model, with level degeneracies \( n_1 = n_2 = 50 \) (at right), for repulsive (at top) and attractive (at bottom) pairing interactions, shown for the \( (v_1,v_2) = (00) \) subspace, as the functions of the control parameter \( \xi \) between the weak- and strong-coupling limits. All calculations are for \( N = 50, \) thus for \( \Omega_1 \ll N \) in the bosonic case and \( \Omega_1 = N \) (half-filling) in the fermionic case. The alternative regime in which the bosonic system also has \( \Omega_1 \approx N \) is shown (specifically, for \( n_1 = n_2 = 51 \) and \( N = 50 \)) in the inset to panel (a). The Hamiltonians \( H_{\pm} \) of (69) and (70) are used in the calculations.

The structure of the eigenvalue spectrum is likewise similar when one compares the bosonic case with attractive pairing (figure 4(c)) and the fermionic case with repulsive pairing (figure 4(b)). This should hardly be surprising. Indeed, when \( n_1 = n_2 \), the eigenvalue spectra for Hamiltonians for opposite pairing signs (e.g. figures 4(a) and (c), or figures 4(b) and (d)) may be obtained from each other, by negation of the Hamiltonian and interchange of the level labels 1 and 2, to within addition of a \( c \)-number function of \( \xi \). Therefore, in the present example, the resemblance between figures 4(c) and (b) is a necessary consequence of the resemblance between figures 4(a) and (d).

The emergence of finite-size precursors to the infinite-\( N \) singularities associated with the quantum phase transition depends not only on \( N \) but also on the level degeneracies \( n_1 \) and \( n_2 \). An important distinction therefore arises between bosonic and fermionic models [14]. For fermionic systems, the total occupancy \( N \) is limited to \( n_1 + n_2 \). Therefore, the limit of large \( N \) can only be taken if the level degeneracies are simultaneously increased. Since at full filling \( (N = n_1 + n_2) \) the spectrum, like that for zero filling, is trivial, it is more informative
to take the limit $N \to \infty$ at or near half-filling ($N = \frac{1}{2}(n_1 + n_2) = \Omega$). However, no such restriction arises for bosonic systems, and $N \to \infty$ can be obtained even for fixed level degeneracies.

Indeed, for the bosonic two-level pairing models, we find numerically that the onset of critical phenomena requires $N \gg \Omega$, not $N \approx \Omega$. The evolution of eigenvalues for the bosonic system with the same occupancy ($N = 50$) as in figure 4(a), but with level degeneracies comparable to the occupation $n_1 = n_2 (= \Omega) = 49 \approx 50$, analogous to ‘half-filling’, is shown for comparison in figure 4 (inset). The eigenvalue spectrum is qualitatively different, as compared to figure 4(a) or 4(d), with respect to each of the properties noted above, e.g. the ground-state eigenvalue is not recognizably constant for $\xi < 1/5$, there is no apparent change in curvature at $\xi = 1/5$, and closer inspection reveals no level spacing compression of the type associated with the excited-state quantum phase transition. This is already anticipated from the different eigenvalue range ($\sim 3N^2$) in the strong-coupling limit, obtained in section 5.5.

Similar distinctions between the large-$N$ limit taken with $N \gg \Omega$ or $N \sim \Omega$ are obtained for the critical scaling properties, which we defer to a more comprehensive study. For now, we restrict attention to the basic energy spectra obtained with the present transitional Hamiltonian for the general two-level pairing model, and note that the spectrum for finite $N$ depends strongly not just on the total degeneracy $n = n_1 + n_2$ of the two levels but also on the equality or degree of inequality of the two level degeneracies $n_1$ and $n_2$.

The transitional spectra for two different bosonic models with total degeneracy $n = 6$, and taken with $N = 10$ (i.e. occupation substantially greater than the degeneracy), are compared in figure 5 (top): the $s$–$b$ model ($n_1 = 1$ and $n_2 = 5$) (figure 5(a)) and the choice of two levels with equal degeneracies ($n_1 = 3$ and $n_2 = 3$) (figure 5(b))10. The transitional spectra for the $(v_1v_2) = (00)$ subspaces in figures 5(a) and (b) are similar to each other. Although only irreps of type $(0v_2)$ or $(1v_2)$ are obtained in the former case, more general irreps $(v_1v_2)$ are possible in the latter case, naturally leading to a more complicated spectrum. In particular, it should be noted that the lowest state from each subspace of the form $(v_10)$ approximately tracks the lowest $(00)$ state in energy, and that these states are in fact lower in energy than the $(00)$ state everywhere between the dynamical symmetry limits (see the lowest curve for $(v_1v_2) = (20)$ in figure 5(b)). It is perhaps not surprising that, given a repulsive pairing interaction, the energy may be lowered by breaking pairs within the lower single-particle energy (i.e. increasing $v_1$). In contrast, increasing $v_2$ also enforces nonzero occupation $(N_2 \gtrsim v_2)$ of the higher single-particle energy level and is therefore not as energetically preferred.

For the fermionic system, the difference between the transitional spectra for near-equal versus highly imbalanced degeneracies for the two levels is marked. The transitional spectra for the different fermionic models with the total degeneracy $n = 20$, again taken with $N = 10$ (which now represents half-filling), are compared in figure 5 (bottom): for the most extremely imbalanced possible choice of degeneracies ($n_1 = 2$ and $n_2 = 18$) (figure 5(c)) and with equal degeneracies ($n_1 = 10$ and $n_2 = 10$) (figure 5(d)). The quantum phase transition which occurs for equal degeneracies is washed out in the limit of imbalanced degeneracies, as is evident in the simple, near-linear evolution of the ground-state energy across figure 5(c). Such an effect may be expected on the basis of the Pauli principle. The lower level, of degeneracy $n_1 = 2$, easily saturates at full occupancy, so that the dynamics are effectively those of a one-level system of degeneracy $n_2 = 18$, which does not support critical phenomena as a function of pairing interaction strength.

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10 Figure 5(b) also corrects a labeling error in the legend of figure 6(c) of [14].
Figure 5. Eigenvalues for the bosonic two-level pairing model, with level degeneracies (a) $n_1 = 1$ and $n_2 = 5$, or (b) $n_1 = 3$ and $n_2 = 3$, and for the fermionic two-level pairing model, with level degeneracies (c) $n_1 = 2$ and $n_2 = 18$, or (d) $n_1 = 10$ and $n_2 = 10$, as the functions of the control parameter $\xi$ between the weak- and strong-coupling limits. All calculations are for $N = 10$. Eigenvalues are shown only for the lowest seniority subspaces $(v_1 v_2)$, specifically, those with $v_1 + v_2 \leq 2$. The Hamiltonian $H_+ \text{ of (69)}$ is used for the bosonic calculations and $H_- \text{ of (70)}$ for the fermionic calculations.

7. Conclusion

Although the existence of duality relations between the number-conserving unitary and number-nonconserving quasispin algebras for the two-level system with pairing interactions is well known, and indeed these relations have proven useful in practical calculations for specific special cases of the two-level pairing model, here we have sought to establish a systematic treatment of the duality relations, both for bosonic and fermionic two-level pairing models and for an arbitrary choice of level degeneracies. A principal goal has been to clarify the relationships between the disparate forms of the Hamiltonian encountered in the study and application of these models. The results are intended to provide a foundation for a more comprehensive investigation of quantum phase transitions in two-level pairing models—including the dependence of scaling properties on the bosonic or fermionic nature of the system and on the level degeneracies—beyond the special cases conventionally considered, namely bosonic $s$–$b$ models and fermionic models with equal degeneracy. The duality between orthogonal or symplectic algebras and the quasispin algebras is also relevant to the analysis of the classical dynamics of the system, through the associated coset spaces \cite{41}. The dual algebras yield complementary descriptions involving classical coordinate spaces with different dimensionalities \cite{42}. Finally, although the present derivations were given for the case of two-level models, they may readily be extended to the $\text{SO}(n_1 + n_2 + \cdots)$ or $\text{Sp}(n_1 + n_2 + \cdots)$
algebras associated with multi-level systems, with generators directly generalizing those of (14) and (15), for which the quantum phase transitions have been much less completely studied. Physical realizations of interest in this more general case include the nuclear shell model and descriptions of superconductivity in metallic grains.

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Appendix. Spherical tensor commutation relations

When working with angular momentum coupled products of spherical tensor operators, it is convenient to consider the coupled commutator [57, 72], itself a spherical tensor operator, with components given by

\[ [A^{(a)}_{\alpha}, B^{(b)}_{\beta}]_{\gamma} = \sum_{\alpha\beta}^{\gamma\gamma} (\epsilon a b \beta | c \gamma) \left[ A^{(a)}_{\alpha}, B^{(b)}_{\beta} \right]. \]  

(A.1)

To clearly set out the identities used in establishing the commutators of the generators in tables 4 and 5, the basic definitions and properties are summarized in this appendix. The use of coupled commutation results bypasses the tedious process of uncoupling the operators, i.e. introducing multiple sums over products of the Clebsch–Gordan coefficients, taking commutators of the spherical tensor components, and then recoupling.

If both bosonic and fermionic operators are to be considered, a consistent set of definitions is obtained if the quantity in parentheses is taken to be the graded commutator, that is, either the commutator or the anticommutator according to the bosonic or fermionic nature of the operators. Specifically,

\[ [A^{(a)}_{\alpha}, B^{(b)}_{\beta}] = A^{(a)}_{\alpha} B^{(b)}_{\beta} - \theta_{ab} B^{(b)}_{\alpha} A^{(a)}_{\beta}, \]  

(A.2)

where \( \theta_{ab} = + \) if either \( A \) or \( B \) is a bosonic operator, and \( \theta_{ab} = - \) if both \( A \) and \( B \) are fermionic operators. (For the sake of these definitions, it is assumed that a bosonic operator has integer angular momentum and a fermionic operator has half-integer angular momentum.) The coupled commutator can be written directly in terms of coupled products as

\[ [A^{(a)}_{\alpha}, B^{(b)}_{\beta}] = (A^{(a)}_{\alpha} \times B^{(b)}_{\beta})_{\gamma} - \theta_{ab} (-)^{-a-b} (B^{(b)}_{\alpha} \times A^{(a)}_{\beta})_{\gamma}. \]  

(A.3)

and obeys the symmetry or antisymmetry relation

\[ [B^{(b)}_{\beta}, A^{(a)}_{\alpha}] = -\theta_{ab} (-)^{-a-b} [A^{(a)}_{\alpha}, B^{(b)}_{\beta}]_{\gamma}. \]  

(A.4)

The uncoupled commutators of the spherical tensor components may be recovered from the coupled commutators, if needed, by inverting (A.1) to give

\[ [A^{(a)}_{\alpha}, B^{(b)}_{\beta}] = \sum_{\gamma\gamma} \sum_{\gamma\gamma} (\epsilon a b \beta | c \gamma) [A^{(a)}_{\alpha}, B^{(b)}_{\beta}]_{\gamma}. \]  

(A.5)

The product rule for coupled commutators is [57, (6)]

\[ [(A \times B)^{c}, C]^{d} = \sum_{f} \sum_{f} (\epsilon a b c | d e) \left[ A^{(a)}_{\alpha} \times B^{(b)}_{\beta} \right] \left[ C^{(c)}_{\gamma} \times [A^{(a)}_{\alpha}, B^{(b)}_{\beta}]_{\gamma} \right]. \]  

(A.6)
A second application of this identity yields the double product rule needed for evaluating commutators of one-body or pair operators:

\[
[(A \times B)^{(g)}(C \times D)^{(g)}(f)]^{(e)} = \sum_{hk} \hat{e} \hat{f} \hat{h} \hat{k} \left[ (-)^{\epsilon_0 \epsilon_1 \epsilon_2 \epsilon_3} \right]_{abc \times d}^{\epsilon_0 \epsilon_1 \epsilon_2 \epsilon_3} \left[ A \times [B, C]^{(k)} \times D \right]^{(e)} \\
+ \theta_{ac}(\epsilon) \epsilon_0 \epsilon_1 \epsilon_2 \epsilon_3 \left[ A \times [B, D]^{(k)} \times C \right]^{(e)} \\
+ \theta_{bc}(\epsilon) \epsilon_0 \epsilon_1 \epsilon_2 \epsilon_3 \left[ C \times [A, D]^{(k)} \times B \right]^{(e)} \\
+ \theta_{cd}(\epsilon) \epsilon_0 \epsilon_1 \epsilon_2 \epsilon_3 \left[ C \times [A, B]^{(k)} \times D \right]^{(e)}.
\]

(A.7)

If operators $A^\dagger$ and $B^\dagger$ are creation operators, obeying canonical commutation or anticommutation relations, the canonical commutators are represented in coupled form by [57, (10)]

\[
[\hat{A}, \hat{B}]^{(e)} = \hat{a}\delta_{AB}\delta_{e0}
\]

(A.8)

and $[\hat{A}, \hat{B}]^{(e)} = [A^\dagger, B]^\dagger = 0$. Therefore, the coupled commutator of two-body operators is

\[
[(A^\dagger \times B)^{(e)}(C^\dagger \times D)^{(e)}(f)]^{(g)} = (-)^{2\theta}(\epsilon) \epsilon_0 \epsilon_1 \epsilon_2 \epsilon_3 \hat{e} \hat{f} \hat{g} \left[ A \times [B, C]^{(k)} \times D \right]^{(e)} \delta_{BC} \\
- \theta_{ad}(\epsilon) \left[ C \times [B, D]^{(k)} \times A \right]^{(e)} \delta_{AD},
\]

(A.9)

as needed, e.g., for the commutators of the generators of $U(n_1 + n_2)$.

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