Soft Temporal Switching of Transmission Line Parameters: Wave-Field, Energy Balance, and Applications

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(Invited Paper)

Abstract—Time-varying guiding structures introduce an additional degree of freedom, besides spatial variation, that enables better control over the guided wave in a device. Periodically, time-modulated structures which are usually considered enable wave control over narrowband signals. However, for ultrawideband short-pulse signals, time variation in the form of temporal discontinuities is required. Such a setup has recently been proposed as a means to overcome the Bode–Fano bound on impedance matching. While hard (abrupt) temporal discontinuities are relatively simple to analyze by employing continuity of magnetic flux and electric charge, soft (gradual) temporal switching of the guiding structure parameters is more challenging. This article explores the case of a short-pulse dynamics in a 1-D, metamaterial transmission line (TL), medium with general smooth time variation of its parameters. In this time-varying TL, wave-field solutions are obtained by a Wentzel-Kramers-Brillouin (WKB) approach which is more common in the context of gradual spatial variations. Using this methodology, a leading order transmitted and reflected waves due to time variation are derived, followed by a discussion of the energy balance in such switched media. A canonical example of capacitor discharge into a long time-varied TL is given. These results may be used as analysis/synthesis tools for time-varying wave devices in electromagnetics and acoustics.

Index Terms—Bode–Fano bound, temporal-discontinuity, time-varying medium, transmission-lines (TLs), Wentzel-Kramers-Brillouin (WKB) approach.

I. INTRODUCTION

Wave engineering is the art of tailoring wave phenomena to achieve various functionalities, such as information transmission, signal processing, power absorption, radiation, imaging, and detection. Conventional wave engineering is based on wave manipulation via spatial means [1], [2]. Namely, the desired wave phenomena are obtained and tuned using structural properties of the guiding media to enforce the required bulk wave equations and boundary conditions. Two typical building blocks are spatial discontinuities and spatial periodicity or modulation of the guiding configuration. Discontinuities, such as an iris or a step in a waveguide create reflection, affect transmission, and introduce a region of stored reactive energy, thus emulating the presence of a reactive lumped component. Periodicity, on the other hand, gives rise to another set of effects such as bandgaps, passbands, distributed resonances, and grating lobes. These two building blocks are commonly used in wave devices.

While the use of spatial means for wave control is at the core of wave engineering, already a long time ago an additional venue for wave manipulation has been proposed. That is, the use of time variation of the guiding media. This approach has been suggested for various purposes such as for signal processing, parametric amplification, and delaylins [3]–[15], energy amplification [15]–[18] and accumulation [19] by a continuous nonperiodic time variation of a lumped load, nonreciprocity in non-Hermitian time-Floquet systems [20], inverse prism functionality [21], spatial and temporal control of light spectra [22], temporal photonic crystal [23] with its real-space moving analog [24], unusual electromagnetic modes [25], mixer-duplexer antenna system [26], for wave pattern engineering [27], [28], as a means to implement synthetic magnetic field [29], as well as to break time-reversal symmetry and achieve magnet-less nonreciprocity [30]–[40]. The latter is of a particular interest since it hints on the plausibility of utilizing time variation to overcome fundamental bounds that are based on the assumption of time invariance.

The examples above expose some of the benefits that are attainable by time variation. However, despite the vast research, up to now the spotlight has been primarily aimed at time variation in the form of periodic time-modulation, and consequently, the goals have been typically centered on narrowband, continuous-wave, applications. In contrast, one may wonder about possibilities to use time-varying media for ultrawideband and short-pulse applications by exploiting wave dynamics with temporal hard and soft switching of the guiding media. In a previous publication [41], we have suggested that an immediate benefit of this approach is the ability to overcome fundamental limitations that are associated with the Bode–Fano bound [42]–[48] on the bandwidth of effective impedance matching, with possible further extensions to the Chu–Harrington limit [49]–[59] on small antennas [60]–[62] and constraints on metastructures [63], [64]. Motivated by [41], we believe that temporal discontinuities have a
great potential to yield wave functionalities, well beyond the state-of-the-art, thus expanding the frontiers of wave engineering in electromagnetics and acoustics.

Nonetheless, temporal discontinuities are not realistic, and moreover, occasionally, in order to have more degrees of freedom, we would like to have a sequence of temporal discontinuities rather than a single one. This boils down to the problem of wave dynamics in media with softly varying parameters which is the focus of this article. See Fig. 1 for a visualization of the various possible switching schemes, hard (abrupt), soft (smooth), and staircase. A more precise distinction will be provided in Sections II and III. Specifically, we consider a 1-D guiding medium with simultaneous, albeit generally independent, variation in the wave velocity and characteristic impedance, or equivalently, permeability and permittivity. To achieve that goal, this article is organized as follows. Section II presents the layout of the problem, concise review on abrupt switching, and a discussion on forward and backward wave propagations in an infinite soft switching transmission line (TL). Section III discusses the heart of the work which is the application of the Wentzel-Kramers-Brillouin (WKB) formalism for pulsed wave in smoothly time-varying, infinite, TL. Then, Section IV presents an excitation problem, namely, the case of wave propagation due to a terminal voltage at a finite/semi-infinite TL undergoing temporal switching as a generalization of the infinite case. An example of capacitor discharging into a softly switched TL is given in Section V with some additional details in the Appendix. Discussion and conclusions are given in Section VI.

II. Layout and Preliminaries

We explore wave dynamics in meta-material TL with time-varying characteristics, impedance, and wave velocity, or equivalently, per-unit-length inductance and capacitance. Such a TL may be emulated as a circuit ladder composed of an infinite arrangement of periodic unit cells each consisting a series inductor \( L(t) \) and parallel capacitor \( C(t) \). The TL is assumed to be aligned along the \( z \)-axis.

![Diagram](Image)

**Fig. 1.** Illustration of hard, soft, and staircase switching schemes between the initial and final states. The hard switching takes place instantaneously at \( t = t_i \), as opposed to the extended but finite switching duration for the staircase and soft switching schemes, it starts at \( t = t_i \) and ends at \( t = t_f \). In each of the cases, during the switching time, the line state \((Z(t), v(t))\) experiences a transition between the initial and final states: \((Z_i, v_i) \rightarrow (Z_f, v_f)\). Note that the time-variation trajectory of the impedance and/or velocity may not be a monotonic function of time. The analysis described below is valid in all such cases.

We denote by \((L_i, C_i)\) \( [(L_f, C_f)]\) the per-unit-length TL inductance and capacitance at some initial [final] time \( t_i \) \( [t_f] \), where \( t_f - t_i \geq T_s \). Thus, \( T_s \) is the switching time between the initial and final times. Equivalently, during the switching time, the characteristic properties of the TL impedance \( Z(t) = \sqrt{L(t)/C(t)} \) and wave velocity \( v(t) = 1/\sqrt{L(t)C(t)} \) change between \((Z_i, v_i) \rightarrow (Z_f, v_f)\).

In a previous publication [41], we showed that temporally switched TLs may be used to overcome the Bode–Fano criterion for impedance matching. In that work, we have developed an analytical model for the wave dynamics in the idealized case of abrupt switching with \( T_s \rightarrow 0 \) at time \( t_i \), namely, with \((Z(t_i^-), v(t_i^-)) = (Z_i, v_i)\) and \((Z(t_i^+), v(t_i^+)) = (Z_f, v_f)\).

Assuming that initially at time \( t < t_i \), there is a pulsed voltage wave with waveform \( V_i^+(t) \) propagating in the forward \( z \)-direction. Continuity of the magnetic flux and electric charge in the TL structure renders that upon switching at \( t_i \), \( V_f^+(t) \) is split into two waveforms, one corresponding to a forward propagation wave \( V_i^+(t) \) and the other to a backward propagation wave-field, \( V_f^-(t) \).

\[
V_f^+(z, t) = TV_i^+\left(\frac{v_f}{v_i} \left(t - t_i - \frac{z - z_0}{v_i}\right)\right)
\]

\[
V_f^-(z, t) = \Gamma V_i^+\left(-\frac{v_f}{v_i} \left(t - t_i - \frac{z - z_0}{v_i}\right)\right)
\]

(1a)

where \( z_0 = v_i t_i \), and \( T \) and \( \Gamma \) are the transmission and reflection coefficients, respectively. These are given by

\[
T = \frac{1}{2} \left(\frac{v_f}{v_i}\right) \left[\frac{Z_f}{Z_i} + 1\right], \quad \Gamma = \frac{1}{2} \left(\frac{v_f}{v_i}\right) \left[\frac{Z_f}{Z_i} - 1\right].
\]

(1b)

Under this temporal switching, the energy change in the wave system is given by [41]

\[
\Delta E = \left\{\frac{1}{2} \left(\frac{v_f}{v_i}\right) \left[\frac{Z_f}{Z_i} + 1\right] - 1\right\} \mathcal{E}_i
\]

(1c)

where \( \mathcal{E}_i = \|V_i^+(t)\|^2 \) is the energy of the initial pulse. The transmission and reflection coefficients in (1b) are due to temporal discontinuity, i.e., for abrupt switching. However, for soft (smooth) switching extending over a period of time \( T_s > 0 \), these expressions cannot be used as is but instead will be utilized for the construction of the forward and backward waveforms within a consecutive sequence of small abrupt switching framework. This process yields to a WKB-type solutions as discussed next.

III. WKB-Type Derivation and the Bremmer Series

In the following section, we derive the expressions for the forward- and backward-propagating waveforms upon propagation in a temporally softly switched (time-varying) TL. Since the TL characteristics smoothly change with time over an extended duration, \( T_s > 0 \), the derivation can use WKB-type arguments by, basically, following two main approaches: 1) a Bremmer-type analysis, see [5], [8], [65], or the more rigorous and 2) “generalized” WKB solution for first-order linear differential equation of [66]. Both formulations give similar results at least for the first-order terms. The Bremmer series approach is more appealing due to its transparent connection to...
the branches in Fig. 2 are either the back reflected wave diagram for this switching process. The weights on each of Fig. 2 shows the corresponding binary-tree-like bouncing in the staircase approximation, wave-bouncing process occurs. Thus, it is possible to aggregate wave contributions according to their order of magnitude. See the red line for the transmitted wave, and the green line for the leading order of the reflected waves components.

the physical properties of wave propagation, i.e., transmission and reflection, hence we shall follow it for the evaluation of both waveforms and the energy change during switching [for the extension of (1)].

A. Forward Wave-Field Derivation

We assume that the characteristic properties of the TL \((Z(t), v(t))\) softly vary with time. Consequently, a WKB-Bremer-derivation type that is based on “staircase” approximation of the TL’s characteristic can be applied. In this approximation, it is required that the change in the impedance and wave velocity between adjacent steps is small. To that end, we divide the switching interval \((t_i, t_f)\) into \(N\) sub intervals \((t_0, t_1), (t_1, t_2), \ldots, (t_{N-1}, t_N)\) with corresponding TL’s characteristics \((Z_0, v_0), (Z_1, v_1), (Z_2, v_2), \ldots (Z_N, v_N)\) [i.e., \(t_0 = t_i, t_N = t_f\)]. This staircase approximation suggests that at each switch time \(t_n, n = 1, \ldots, N\), an abrupt switching takes place with forward and backward wave-fields. Thus, in the staircase approximation, wave-bouncing process occurs. Fig. 2 shows the corresponding binary-tree-like bouncing diagram for this switching process. The weights on each of the branches in Fig. 2 are either the back reflected wave reflection coefficient \(\Gamma_{n,n+1}\) or the forward-propagating wave transmission coefficient \(T_{n,n+1}\), as in (1b)

\[
T_{n,n+1} = \frac{V_{n+1}}{V_n} \left[ \frac{Z_{n+1}}{Z_n} + 1 \right]
\]

\[
\Gamma_{n,n+1} = \frac{V_{n+1}}{V_n} \left[ \frac{Z_{n+1}}{Z_n} - 1 \right].
\]

It is easily noted from the bouncing diagram in Fig. 2 that at time \(t > t_0\), both the forward and backward wave-fields are composed of many partial wave contributions that are weighted according to the amount of transmission or reflection it encountered during switching. Since soft switching is considered, we may approximate \(Z_{n+1} = Z_n + \delta Z_n\) with \(\delta Z_n \ll Z_n\). By inserting this approximation into (2) and keeping the lead order terms in \(\delta Z_n\), we obtain

\[
T_{n,n+1} = \frac{V_{n+1}}{V_n} \sqrt{\frac{Z_{n+1}}{Z_n}} + O\left( \frac{\delta Z_n}{Z_n} \right)^2 \tag{3a}\]

\[
\Gamma_{n,n+1} = \frac{1}{2} \frac{V_{n+1}}{V_n} \delta Z_n + O\left( \frac{\delta Z_n}{Z_n} \right)^2 \tag{3b}\]

Clearly, while the transmission coefficient through a single infinitesimal temporal discontinuity is \(O(1)\), the reflection coefficient is much smaller and behaves as \(O(\delta Z_n/Z_n)\). Now, consider a scattering process that involves \(n\) temporal discontinuities in the time interval \([t_0, t_N]\). In this case, the overall waveform at time \(t_n\) can be decomposed into \(2^n\) waveform components as shown in the bouncing diagram in Fig. 2. Each of these components can be associated with a certain order of magnitude that depends on the number of transmissions and reflections that it experienced during propagation. For example, after \(m\) transmissions and \(n-m\) reflections, the wave component will be \(O(T^m \Gamma^{n-m})\). In light of that, since the impedance and wave velocity profiles are assumed smooth, the leading order wave component is \(O(T^n)\) (no reflections), and thus, a predominantly forward propagation process takes place. This dominant contribution follows the red trajectory marked in Fig. 2. This contribution is identified as the WKB approximation of the wave-field and reads

\[
V^+_j(Z, t_n) = \left[ \prod_{m=0}^{n-1} T_{m,m+1} \right] \times V^+_j \left( \frac{Z_{n+1}}{Z_n} - 1, \ldots, \frac{Z_m}{Z_{m+1}} - 1 \right) \tag{4}\]

where in view of (3a)

\[
\prod_{m=0}^{n} T_{m,m+1} \approx \prod_{m=0}^{n} \frac{V_{m+1}}{V_m} \sqrt{\frac{Z_{m+1}}{Z_m}} \frac{Z_{n+1}}{Z_n} \tag{5}\]

Approaching the continuum limit, where \(t_{n+1} - t_n \to 0\), rendering the summation in (4) as integration, the final forward wave-field contribution is given by

\[
V^+_j(Z, t) = \int_{t_0}^{t_f} dr \frac{V(t')}{Z_0} \sqrt{\frac{Z(t)}{Z_0}} \tag{6}\]

This forward term is the WKB approximation of the solution.

B. Derivation of the Reflected Wave

The first-order correction term to the WKB forward wave is a backward-propagating wave-field that consists of all wave components of order \(O(T^{n-1})\). The wave-bouncing trajectories that correspond to these components are marked by the green lines in Fig. 2. Notably, during \(n\) discontinuities, there
are \( n \) wave components of this type that should be summed up. The derivation of this term follows closely the discussion in (4)–(6) albeit somewhat more involved. Therefore, for the sake of clarity, before presenting the general expression, we shall discuss in detail, the derivation of the reflection term for the special case of \( t_3 < t < t_4 \) that is depicted in Fig. 2. From Fig. 2, it follows that at this temporal range, there are three backward-propagating contributions that in view of the discussion in (1a) yield

\[
V_f^r (z, t) = \Gamma_0 T_{12} T_{23} V_i^+ \left[ -\frac{v_3}{v_0} (t - t_3) - \frac{Z - z}{v_0} \right] + T_{01} \Gamma_{12} T_{23} V_i^+ \left[ -\frac{v_5}{v_0} (t - t_3) - \frac{Z - z_2}{v_0} \right] + T_{01} \Gamma_{12} T_{23} V_i^+ \left[ -\frac{v_3}{v_0} (t - t_3) - \frac{Z - z_3}{v_0} \right]
\]

(7a)

with \( z_1 = v_0 (t_1 - t_0) - v_1 (t_2 - t_1) - v_2 (t_3 - t_2) = v_0 (t_1 - t_0) - v_0 (t_1 - t_0) - v_1 (t_2 - t_1) - v_2 (t_3 - t_2) \), similarly \( z_2 = 2 v_0 (t_1 - t_0) + 2 v_1 (t_2 - t_1) - v_0 (t_1 - t_0) - v_1 (t_2 - t_1) - v_2 (t_3 - t_2) \) and \( z_3 = 2 v_0 (t_1 - t_0) + 2 v_1 (t_2 - t_1) + 2 v_2 (t_3 - t_2) - v_0 (t_1 - t_0) - v_1 (t_2 - t_1) - v_2 (t_3 - t_2) \). Inserting into (7a) with the approximation in (3b) gives

\[
V_f^r (z, t) = \frac{1}{2} T_{01} T_{12} T_{23} \times \left[ \frac{\delta Z_0}{Z_0} V_i^+ \left[ -\tau (t) + 2 \frac{v_0}{v_0} (t_1 - t_0) - \frac{Z - z}{v_0} \right] + \frac{\delta Z_1}{Z_1} V_i^+ \left[ -\tau (t) + 2 \frac{v_0}{v_0} (t_1 - t_0) + 2 \frac{v_1}{v_0} (t_2 - t_1) - \frac{Z - z}{v_0} \right] + \frac{\delta Z_2}{Z_2} V_i^+ \left[ -\tau (t) + 2 \frac{v_0}{v_0} (t_1 - t_0) + 2 \frac{v_1}{v_0} (t_2 - t_1) + 2 v_2 (t_3 - t_2) - \frac{Z - z}{v_0} \right] + 2 \frac{v_2}{v_0} (t_3 - t_2) - \frac{Z - z}{v_0} \left] \right]
\]

(7b)

with \( \tau (t) = (v_3 / v_0) (t - t_3) + (v_2 / v_0) (t_3 - t_2) + (v_1 / v_0) (t_2 - t_1) + (v_0 / v_0) (t_1 - t_0) \). Similar structure follows for later times, thus inductively it yields for \( t_n < t < t_{n+1} \):

\[
V_f^r (z, t) = \frac{1}{2} \left[ \prod_{m=0}^{n} T_{n,m+1} \right] \times \sum_{m=0}^{n} \frac{\delta Z_m}{Z_m} V_i^+ \left[ -\frac{v_0}{v_0} (t - t_n) - \frac{v_0}{v_0} (t_{i+1} - t_i) + 2 \sum_{k=0}^{m} \frac{v_k}{v_0} (t_{k+1} - t_k) - \frac{z}{v_0} \right]
\]

(8)

Approaching the continuum limit, we note that \( (\delta Z_m / Z_m) \rightarrow (d \ln Z(t) / dt) dt \), and the final expression for the first-order correction term is obtained

\[
V_f^r (z, t) = \frac{1}{2} \left[ \frac{v_0}{v_0} \sqrt{\frac{Z(t)}{Z_0}} \right] \times \int_{t_0}^{t} dt' \frac{d \ln Z(t')}{dt'} V_i^+ \left[ -\frac{v_0}{v_0} (t - t_n) + 2 \sum_{k=0}^{m} \frac{v_k}{v_0} (t_{k+1} - t_k) - \frac{z}{v_0} \right].
\]

(9)

As a correction term to the forward-propagating WKB wave-field in (6), this term is identified as the leading order term of the reflected wave-field.

The next correction term involves all wave components of order \( O(T^{-2}) \). This is a forward-propagating wave-field that compensates the WKB contribution for the changes in time of \( (Z(t), v(t)) \). However, since it consists of two reflections (at two different switches), it is negligibly small and will not be considered further. Nevertheless, its derivation follows the same lines as above.

Lastly, we emphasize that while the discussion so far did not make any explicit reference to the field’s waveform, implicitly this formalism assumes that the duration of the switching, \( T_s \), is much larger than the pulsewidth. Moreover, in the aforementioned derivation of the forward wave and the first order reflection, we have followed Bremmer-type arguments that are intimately connected with the wave-field dynamics. An alternative approach that gives similar results is based on an asymptotic solution of linear differential equation to give a generalized type of WKB solution was treated in [66]. The latter, while easily used for the two first-order terms in the approximation of the wave-field, yields less transparent higher order terms.

C. Energy Balance Derivation

The discussion in the previous section concerned the forward and backward wave dynamic. However, the discussion is incomplete without the assessment of the energy balance during switching. The energy balance in a single abrupt switching was discussed in [41] and outlined above in (1c). In this section, we extend the discussion to the soft switching following arguments akin to that used in Sections III-A and III-B for the wave-fields.

The energy balance at time \( t_n < t < t_{n+1} \) is obtained by summing up all the partial energy balances during each of the step-like abrupt switchings

\[
\Delta E = \sum_{m=0}^{n-1} \Delta E_m
\]

(10)

where \( \Delta E_m = \Delta e_{m,m+1} \| V^+_m (t) \|^2 / Z_m \) denotes the energy change during switching at \( t = t_{m+1} \). We use the energy balance expression in (1c) for a single abrupt switching in order to write

\[
\Delta e_{n,n+1} = \frac{1}{2} \left( \frac{v_{n+1}}{v_n} \right) \left[ \frac{Z_{n+1}}{Z_n} \frac{Z_n}{Z_n + 1} - 1 \right] \approx \frac{\delta_{n}}{v_n}
\]

(11)

where the first-order approximation on the right-hand side applies for soft switching with \( Z_{n+1} = Z_n + \delta Z_n \) and \( v_{n+1} = v_n + \delta v_n \) where \( \delta Z_n \ll Z_n \) and \( \delta v_n \ll v_n \). Interestingly, with regard to the characteristic impedance, this expression is variational which means that small variation in the impedance during switching appears as second-order corrections. Looking at the limited time scenario depicted in Fig. 2, it can be seen that at the first switching, the balance is given by (1c) where with it yields

\[
\Delta E_0 = \Delta e_{01} \| V^+_0 (t) \|^2 / Z_0 \approx \frac{\delta v_0}{v_0} \| V^+_0 (t) \|^2 / Z_0.
\]

(12a)
For the second switching at \( t = t_2 \), there are two contributions that gives the first-order approximation

\[
\Delta \mathcal{E}_1 = \frac{\left| T_{01} V_i^+ (\frac{v_{10}}{v_0} t) \right|^2}{Z_1} + \Delta e_{12} \frac{\left| T_{01} V_i^+ (\frac{v_{10}}{v_0} t) \right|^2}{Z_1} = \frac{\left| T_{01} \right|^2 + \Delta e_{12} \left| T_{01} \right|^2}{Z_1} \frac{\left| V_i^+ (t) \right|^2}{Z_0}\]

\[
= \frac{1}{2} \Delta e_{12} \frac{1}{\frac{v_1}{v_0}} \left[ \frac{Z_1}{Z_0} + \frac{Z_0}{Z_1} \right] \frac{\left| V_i^+ (t) \right|^2}{Z_0} = \frac{\delta v_1 v_1}{v_0} \frac{\left| V_i^+ (t) \right|^2}{Z_0} (12b) \]

where we used

\[
T_{n,n+1}^2 + \Gamma_{n,n+1}^2 = \frac{1}{2} \left( \frac{v_{n+1}}{v_n} \right)^2 \left[ \frac{Z_{n+1}}{Z_n} \right] \frac{Z_{n+1} + Z_n}{Z_{n+1}} \right]. (12c) \]

In a similar manner, for the \( n \)th switching at \( t = t_n \), it follows that:

\[
\Delta \mathcal{E}_n \approx \frac{\delta v_1 v_1}{v_0} \frac{\left| V_i^+ (t) \right|^2}{Z_0} (12d) \]

Then, by summing over all the partial contributions in (10) and approaching the continuum limit, we obtain the following remarkably simple result for the energy change in a smooth (soft) variation:

\[
\Delta \mathcal{E} = \left( \frac{v(t)}{v_0} - 1 \right) \frac{\left| V_i^+ (t) \right|^2}{Z_0} (13) \]

which is similar to (1c) for a single abrupt switching of the velocity only, albeit here we have switched both the velocity and the impedance. The omission of the “impedance” term comes due to the first-order nature of the approximation and since any small changes in the impedance appear as a second-order correction. Equation (13) implies an important consequence and introduces an interesting degree of freedom. Specifically, by choosing different soft switching trajectories \((Z(t), v(t))\) between the identical initial and final states, the propagating signal in the line will be manipulated differently, but nevertheless there will be no net energy change in the wave system due to the time variation of the TL parameters if the initial and final switching states have identical velocities.

IV. TERMINAL VOLTAGE WKB-TYPE DERIVATION

The discussion so far assumed that the TL is infinite, and moreover that, initially, prior to the switching time at \( t = t_i = t_0 \), there is a voltage waveform \( V_i^+ (t) \) propagating in it. However, it would be useful to consider the more realistic scenario in which the TL is finite or semi-infinite and it is excited by a source at its terminal. Let us set the terminal location at \( z = 0 \), there, a voltage \( V_0(t) \) and current \( I_0(t) \) are imposed by an external source. At the TL terminal, the relation \( V_0(t) = Z(t) I_0(t) \) is satisfied where, as before, \( Z(t) \) is the time-dependent TL’s characteristic impedance. This section is devoted for the derivation of the forward-propagating WKB-type voltage wave. This derivation uses the expression derived in Section III and in particular of (6). Given a terminal voltage waveform \( V_0(t) \) that initiates at \( t = 0 \), it may be expressed as

\[
V_0(t) = \int_0^\infty V_0(\tau) \delta(t - \tau) d\tau \]

where \( \delta(t) \) is dirac’s delta. This representation implies that the input signal can be expressed as a superposition of weighted impulses, each excited at time \( t = \tau \). However, since the TL system is linear (albeit time variant), we may write the excited wave as a superposition integral of weighted impulse responses \( h_j^+ (z, t; \tau) \). For an impulse signal at time \( t = \tau \), \( V_i^+ (t) = \delta(t - \tau) \), using (6), the forward-propagating wave at time \( t \) and location \( z \) on the line, reads

\[
h_j^+ (z, t; \tau) = \left[ \frac{v(t)}{v(\tau)} \sqrt{\frac{Z(t)}{Z(\tau)}} \right] \delta \left( \int_{\tau}^t dt' \frac{v(t') - z}{v_0} \right). (15) \]

It is important to emphasize that this solution that was obtained for the infinite line case is valid, also, here only because the excited pulsewidth is zero. This (15) represents the impulse response of the time-variant line. Once we have it, we can calculate the response for any excitation using superposition

\[
V_j^+ (z, t) = \int_0^\infty d\tau V_0(\tau) h_j^+ (z, t; \tau) \]

\[
= \frac{v(t)}{v(\tau_0)} \sqrt{\frac{Z(t)}{Z(\tau_0)}} V_0(\tau_0) (16) \]

where \( \tau_0 \) satisfies the integral equation

\[
\int_{\tau_0}^t dt' v(t') - z = 0, \quad \tau_0 > 0. (17) \]

The solution of this equation gives the time \( \tau_0 = \tau_0(z, t) \) at which an infinitesimal wave contribution appears at the terminal of the TL in order to reach \( z \) at time \( t \) (for the special case of \( v(t) = v_0 \), it follows that \( \tau_0 = t - z/v_0 \)). It is immediately noted that the expression in (16) is different than the one in (6). However, this is obvious upon noting the different scenarios (as mentioned in the beginning of this section): in (6), it was assumed that the wave-field is wholly contained in the TL before switching and during switching, while here the wave-field is imposed at the boundary (at \( z = 0 \)) during switching. Moreover, in the present case, different sections of the waveform (along the time) undergo different switching, i.e., while the leading edge “senses” the entire switching period, the trailing edge “senses” only parts of it. This has further implications as, for example, nonuniform different temporal compression/expansion and amplification of the waveform. Note that this result generalizes previous results in, for example, [8] for the normal incidence case that assumed switching of only the electric permittivity. Here, on the other hand, it is assumed the both the impedance and velocity, i.e., generally, the permeability and permittivity, can independently change.

The backward-propagating (reflected) wave-field \( V_j^- (z, t) \) due to the forward-propagating wave \( V_j^+ (z, t) \) can be calculated in much the same way by starting with (9). However, special attention should be given to the fact that this wave propagates toward the terminal at \( z = 0 \). Consequently, due
to impedance mismatch at this point, parts of the impinging wave are reflected-back as delayed forward waves. These may overlap in time and space in the vicinity of the terminal. Assuming gradual impedance and velocity changes during switching renders weak (negligible) $V_f^-$ such that $V_f^+$ gives an accurate approximation of the propagating wave.

While the discussion in this and previous sections was focused on finding the wave-field solutions in a heuristic and intuitive way using the Bremmer series and tracing wavefronts as representing fundamental solutions of the wave-equation, one may solve the TL's set of differential equations: 

$$
\partial_t Q(z, t) = -C(t) \partial_t \Phi(z, t) \quad \text{and} \quad \partial_t \Phi(z, t) = -L(t) \partial_t Q(z, t),
$$

where $Q(z, t) = C(t) V(z, t)$ and $\Phi(z, t) = L(t) I(z, t)$ and with an appropriate boundary conditions, either for the infinite TL case or for the finite case (see Sections III and IV, respectively). A general solution of these equations can follow a spectral transformation of the $z$ coordinate, rendering them a standard set of first-order linear differential equations admitting an exponential-type integral solution. Applying successive approximation yields the corresponding WKB-type solution in the spectral domain [66]. Transforming back the spectral solution to the spatial, $z$, coordinate gives similar results for the wave-fields to those obtained in Sections III and IV. In these special cases, where $C(t)$ and $L(t)$ are “convenient” functions of time, an exact solution of the wave equation may be formulated (see [14]).

V. Example and Design Considerations

In the above discussion, we have derived analytical solutions for ultrawideband short-pulse propagation in, and excitation of, a softly switched time-varying TL. Here, we show how this formalism can be used in order to solve a simple canonical problem of lumped capacitor discharge into a smoothly time-varying TL. The analyzed configuration is shown in Fig. 3(a).

The TL characteristics vary with time between two states, initial and final. The specific variation trajectory $(Z(t), \nu(t))$ is optimized to yield a certain design goal. Here, for instance, we seek to maximize the power delivered from the charged capacitor to the TL. In another work [41], the optimization was performed to maximize the energy delivery between a pulsed source and a load. The idea to optimize the time-variation trajectory of a particular component in a system was also recently discussed in [19]. There, for time-harmonic signals, a lumped reactive load was dynamically tuned to prevent any reflections into the feed TL and, hence, to maximize the stored energy on the reactive load. We stress that the example given here is merely a single demonstration for the use of the wave-dynamic formulas developed in previous sections. It is by no means the only possible application of the theory above. In the optimization process considered here, both the impedance and the wave velocity of the line may be changed with time; however, in the WKB approximation, since reflection is negligibly small, the discharged capacitor effectively sees only the time-varying TL characteristic impedance as a load, with no additional correction. Once the pulse is on the line, the velocity variation in time may also affect the propagation dynamics. This highlights an advantage of the use of softly switched TL systems as they raise a potential to decouple between distinct mechanisms and get separate control on the excitation and the propagation processes. Being said, we also note that the fact that the source sees a time-varying load that depends only on the characteristic line impedance but not on the velocity (as opposed to abrupt temporal switching) reduces the available degrees of freedom of the system. Thus, as common in engineering, from different standpoints, this feature can be considered as an advantage or disadvantage of soft switching when compared to abrupt switching.

Getting back to our capacitor discharge problem. Assume that for $t < 0$, the capacitor, $C_a$, is charged with $Q_0$. At $t = t_0 = 0$, a switch, $s_{sw}$, is closed and the capacitor starts to discharge through a fixed resistor $R_a$ and a softly varying TL with characteristic impedance $Z(t)$. The initial TL impedance at $t = 0$ is $Z_t = Z_0$. In the discharge process, some of the capacitor’s initial energy is dissipated on the series resistor $R_a$, while the rest is transmitted in the line. Our goal is to optimize the energy delivery into the line by changing the load $Z(t)$ (i.e., changing the TL’s characteristic impedance). For that purpose, we solve analytically the variational problem that leads to the best characteristic impedance trajectory in time $Z(t)$ (see the Appendix for the optimization details) to give

$$Z(t) = R_a \zeta(t), \quad \zeta(t) = \zeta_0 - \frac{t}{\tau_a} \quad (18)$$

where $\zeta_0 = Z_0/R_a$ and $\tau_a = R_a C_a$. This time-varying impedance renders constant discharge current $I_d = Q_0/C_a (R_a + Z_0)$. As can be seen in (18), during discharge $Z(t)$ is a strictly monotonic decreasing function of $t$ (see the dash-dotted brown line in Fig. 3(b) for illustration). Since $Z(t)$ is restricted to be positive for all $t$, the minimal

![Figure 3](image-url)
value of the TL characteristic impedance is set as a design parameter to \( Z_m = R_a\zeta_m < Z_0 \), and this takes place at \( t_m = (\zeta_0 - \zeta_m)\tau_a \) (see the continuous light-blue line in Fig. 3(b) for illustration). Exactly at this instant, \( t = t_m \), the switch, \( S_{sw} \), is opened again, thus ending the discharge process. The remaining charge on the capacitor \( C_a \) is given by \( Q_m = -Q_0(1 + \zeta_m)/(1 + \zeta_0) \ll Q_0 \). Since the current \( I_d \) flows into the time-varying TL with characteristic impedance \( Z(t) \), it produces a terminal voltage at \( z = 0 \) of

\[
V_0(t) = \frac{Q_0}{C_a}\left[1 + \zeta_0\right] \frac{H(t) - H(t - t_m)}{1 + \zeta_0}.
\]

which, as illustrated in Fig. 3(c), has a trapezoidal-like waveform, where \( H(t) \) denotes the Heaviside step function.

The discussion above was about the capacitor’s discharge that creates a voltage \( V_0(t) \) at the TL’s terminal at \( z = 0 \). This voltage, in turn, generates a forward-propagating voltage wave in the TL (which may be either semi-infinite or with a matched load) according to (16). The following discussion aims to explore and demonstrate some of the propagation characteristics.

Let us assume that the time-varying TL is implemented as a periodic circuit of series inductors \( L(t) \) and shunt capacitors \( C(t) \) per-unit length, such that \( Z(t) = \sqrt{L(t)/C(t)} \) and \( v(t) = 1/\sqrt{L(t)C(t)} \). Following (18) and the discussion above, the normalized TL’s characteristic impedance, \( \zeta(t) = Z(t)/R_a \), can be extended for times \( t > t_m \) as

\[
\zeta(t) = \zeta_0 - \frac{t}{\tau_a}H(t) - \left(\zeta_0 - \zeta_m - \frac{t}{\tau_a}\right)H(t - t_m).
\]

See an illustration of the characteristic impedance profile in Fig. 3(b). As discussed above, the choice of \( v(t) \) is detached from that of \( Z(t) \) since it affects only the wave propagation and can be done arbitrarily. Nevertheless, there are some few favorable such choices, as follows.

1) **Constant Velocity Scheme:** The illustration in Fig. 3(c) shows that the pulsedwidth of the terminal voltage, \( V_0(t) \), is \( t_m \), as evident by (19). However, in view of (16) and (17), while propagating in a TL with time-dependent velocity, \( v(t) \), the pulsedwidth generally scales up/down (pulse compression/expansion). In order to maintain the same pulsedwidth, it follows that the velocity should be time independent, \( v(t) = v_0 \). Thus, following (17), leading to \( t_0 = t - z/v_0 > 0 \) while propagating. However, this requirement is redundant since \( V_0(t_0) = 0 \) for \( t_0 < 0 \). Moreover, since \( Z(t) \) and \( v_0 \) are set, (16) renders

\[
V_j^0(z, t) = \sqrt{\frac{Z(t)}{Z(t - z/v_0)}}V_0(t - z/v_0)
\]

\[
= \frac{Q_0}{C_a}\left[1 + \zeta_0\right] \times\left[H(t - z/v_0) - H(t - z/v_0 - t_m)\right].
\]

Though this wave velocity setup prevents pulse compression, yet there is an unavoidable multiplicative time-dependent amplitude scaling of the propagating contribution. In order to maintain this “constant velocity” setting, a synchronous time variation of both the inductance and capacitance during switching is required (see a more general case below).

Fig. 4 depicts space–time snapshots of the wave propagation for the capacitor’s discharging circuit (see Fig. 3) in a constant velocity TL with \( Q_0/C_a = 1 \), \( \zeta_m = 1 \) at \( z = z/L_0 = 0, 5, 10, 20, 30, 40, 50, 75, 100 \) (with \( L_a = \tau_a/v_0 \)), \( \tilde{t} = t/\tau_a \) and for several cases of the initial characteristic impedances \( \zeta_0 = 2, 5, 10, 20 \) in Fig. 4(a)–(d), respectively. Viewing these figures: 1) the effect of the impedance switching on the waveform is readily noted by comparing the terminal voltage at \( z = 0 \) and the voltages at points \( z \) deep inside the TL; 2) for early arrival times of the leading edge of the waveform that are prior to the cessation of the switching (the latter takes place at \( t_m = t_m/\tau_a = 1, 4, 9, 19 \), in Fig. 4(a)–(d), respectively), the behavior of the waveform is different than for later times (\( \tilde{t} > \tilde{t}_0 \)) due to the interpolation between the \( \zeta(t) \) and \( \zeta(t - z/v_0) \) terms in (21) that change with relative spatial delay; and 3) moreover, for \( t > t_m \), where \( \zeta(t) = \zeta_m \), the wave propagation becomes a shift invariant process.

2) **Capacitor-Only Switching Scheme:** Though favorable in keeping the same pulsedwidth during propagation, the “constant velocity” scheme requires the synchronous change of both TL per-unit-length capacitance and inductance. However, while time-varying capacitance is relatively simple to implement (e.g., using varactor diodes), a continuous time variation of inductance is somewhat more challenging. Therefore, it is interesting to analyze the more practical case of a metamaterial TL with only time-varying capacitance.

To that end, assume that the per-unit-length inductance is maintained constant \( L(t) = L_0 \). In that case, the per-unit-length capacitance is given by \( C(t) = L_0/[Z(t)]^2 \). Yielding

\[
v(t) = \frac{Z(t)}{L_0} = \frac{R_a}{L_0}\zeta(t).
\]

It is interesting to note that following (18), \( v(t) \) is also a monotonic decreasing function of \( t \). Consequently, during propagation, the wave-field slows down from the initial velocity \( \zeta_0R_a/L_0 \) to the final velocity \( \zeta_mR_a/L_0 \) and, thus, energy is absorbed from the wave system during switching.
that \( \bar{\tau} = z / l_a \) with \( l_a = t_a v_0 \) and \( \tau = t / t_a \), and in all cases, we fix \( Q_0 / C_a = 1, \) \( \zeta_m = 1 \). 

For \( \zeta_0 = 2 \) and \( \zeta_0 = 5 \), we explore several cases: 1) different interplay between the \( \zeta(t) \) and \( \zeta(t_0 z, t) \) terms [compare (24) and (21)]; 2) nonlinearity of the arrival time \( t_0(z, t) \) following (17) and due to the velocity variation during switching; and 3) the increase in the waveform width, i.e., “pulse up scaling,” i.e., expansion due to the change in the TL wave-propagation velocity. Noting that the velocity changes linearly (smoothly) between \( v_0 = v(t_0) = (R_a / L_0) \zeta_0 \) and \( v_m = v(t_m) \), it can be shown that pulsewidth scaling ratio, i.e., the ratio of the pulsewidth after and before to switching is given by \( (\zeta_0 + \zeta_m) / 2 \zeta_m \), suggesting an increase of \( 1.5, 3, 5.5, 10, 5 \) for \( \zeta_0 = 2, 5, 10, 20 \) in Fig. 5(a)–(d), respectively.\(^3\)

3) Inductor-Only Switching Scheme: The dual, more difficult to implement, time-varying inductor case with constant capacitance \( C(t) = C_0 \) gives inductance \( L(t) = C_0 Z(t)\) and \( v(t) = (C_0 R_a \zeta_m)^{-1} \). Note that \( v(t) \) is a monotonic increasing function of the time from an initial value \( [C_0 R_a \zeta_m]^{-1} \) to its final value \( [C_0 R_a \zeta_m]^{-1} \). Similar to the discussion in the previous case, the corresponding wave-field is, thus, given by

\[
V_f^+(z, t) = \sqrt{Z(t_0)} \frac{Z(t)}{Z(t_0)} V_0(t_0)
\]

\[
= \frac{Q_0}{C_0(1 + \zeta_0)} \frac{[\zeta(t_0 z, t)]^{3/2}}{\zeta(t)} \times [H(t_0(z, t)) - H(t_0(z, t) - t_m)].
\]

\( \zeta(z, t) \) is the switching function and \( \zeta(t_0 z, t) \) is the spatial derivative of \( \zeta(t) \). The scaling ratio follows by observing that: 1) modeling the soft switching as a series of abrupt switchings where at each such switches, the pulsewidth changes by the ratio \( \zeta(t_0 z, t) / \zeta(t_s) \), with \( \zeta(t_s) \) and \( \zeta(t_s) \) the velocities before and after the switching time \( t_s \), [see (1a)]; 2) recalling that such series of velocity changes in addition to a continuous (incremental) “injection” of signal at the TL’s terminal due to the external source renders an infinitesimal temporal recursive-like increase in the propagating pulsewidth; and 3) approaching the continuity limit as was carried out in previous sections for the analysis of waveforms.

Fig. 4. Normalized space–time snapshots of the pulse propagation in the softly time-varying TL with constant velocity \( v_0 \). The axes are normalized such that \( \bar{z} = z / l_a \) with \( l_a = t_a v_0 \) and \( \bar{t} = t / t_a \), and in all cases, we fix \( Q_0 / C_a = 1, \zeta_m = 1 \). (a) \( \zeta_0 = 2 \). (b) \( \zeta_0 = 5 \). (c) \( \zeta_0 = 10 \). (d) \( \zeta_0 = 20 \).
Fig. 5. Normalized space–time snapshots of the pulse propagation in the softly time-varying TL. The axes are normalized such that $\bar{z} = z/l_a$ with $l_a = v_a t_m$ and $\bar{t} = t/t_a$, and in all cases, we fix $Q_0/C_a = 1$. (a) $\zeta_0 = 2$. (b) $\zeta_0 = 5$. (c) $\zeta_0 = 10$. (d) $\zeta_0 = 20$.

Fig. 6. Normalized characteristic inductance $L(t)$ and capacitance $C(t)$ per-unit length of the switched TL for $\zeta_0 = 5$, $\zeta_m = 1$ with $t_m = (\zeta_0 - \zeta_m)\tau_a = 4\tau_a$ and $v_0/v_f = 5$ with $t_f = 16\tau_a$.

Though the behavior of the impedance/velocity here is different than that treated in the previous case, corresponding switching effects, though different in detail, can be discern, i.e., pulse compression and waveform deformation. Thus, for brevity, they will not be further discussed here.

4) Impedance and Velocity Change: In this example, we explore a case of noncoordinated temporal variation of the TL’s velocity and characteristic impedance. The impedance changes softly from the initial, $Z_0$, to final, $Z_m$, values over a duration $t_m$ as described in (20). The velocity, on the other hand, changes from the initial value $v_0$ at time $t = 0$ to the final value $v_f$ at time $t_f$ (which is generally different than $t_m$), along a linear trajectory

$$v(t) = v_0 - (v_0 - v_f)\frac{t-t_f}{t_f} - (v_0 - v_f)\frac{H(t) - H(t-t_f)}{t_f}. \quad (26)$$

The velocity and impedance change is achieved by temporal gradual switching of the TL’s capacitance and inductance per-unit length as depicted in Fig. 6 for the case of $\zeta_0 = 5$, $\zeta_m = 1$ with $t_m = (\zeta_0 - \zeta_m)\tau_a = 4\tau_a$ and $v_0/v_f = 5$ with $t_f = 16\tau_a$. The nonlinear trajectory of $L(t)$ and $C(t)$ is due to the different rates and durations of changes of the impedance and velocity. Fig. 7(a) depicts waveforms of the forward-propagating wave for different ratios of the initial to the final velocities, $\tilde{v} = v_0/v_f$, for $\zeta_0 = 5$, $\zeta_m = 1$ with $t_m = 4\tau_a$ and $t_f = 16\tau_a$ and at observation point $z$ such that the arrival of the wavefront takes place after the switching ends. Fig. 7(b) depicts the ratio of the energy of the forward-propagating wave $E_f = \|V_f(z,t)\|^2/Z_m$ to the energy injected into the TL by the discharging capacitor $E_i = \|l_d\sqrt{Z(t)}\|^2$, where $\| \|$ is the energy norm as in (1c). Following the discussion in the previous examples, it is interesting to note in Fig. 7 that whenever $v_0 > v_f$, energy is absorbed from the wave system and the pulsewidth widened, whereas for $v_0 < v_f$, energy is injected into the wave system by the switching of the TL’s properties and the pulsewidth become narrower. Whenever no
change in the velocity occurs during switching, i.e., $v_f = v_0$, no energy is injected nor absorbed by the switching system ($\mathcal{E}_F/\mathcal{E}_i = 1$). This behavior of no energy exchange with the switching system is in accordance with (13). Moreover, this case corresponds also to the discussion above in the first example of “Constant velocity scheme.”

To conclude this example section, it should be emphasized that the analysis here assumes that any back-reflected wave due to the forward propagation wave in this time-varying media is negligibly small and does not cause major deformations to the forward, first-order, WKB contributions.

VI. CONCLUSION AND DISCUSSION

In this article, we have developed analytical formulas describing the transmitted and reflected waves due to a propagating short pulse that experiences gradual temporal switching of the guiding medium. The analysis is based on the WKB approach, originally developed as a general tool for differential equations with softly varying coefficients. While this approach has been extensively used in quantum mechanics and for electromagnetic and acoustic wave propagation in spatially varying media, its application to time-varying wave systems was limited so far. We have developed the WKB formalism for medium that experiences simultaneous and independent variation of both impedance and wave velocity (capacitance and inductance per-unit-length or, synonymously, permittivity, and permeability). We have moreover extended our formalism to be applicable also for excitation problems by a lumped source, using the concept of impulse response and superposition in linear time-varying systems. Finally, we concluded by demonstration of the analytical formalism on a canonical problem of capacitor discharge into a semi-infinite TL. These examples demonstrate that gradual change in the velocity enables energy absorption (or harvesting) from the wave system to the switching system (with negligible reflected wave-field) and can form the basis for an energy-based electromagnetic absorber/shield of pulsed energy. The opposite case where energy is invested into the wave system by the switching system implies a type of parametric amplification or an energy-based wave accelerator. Both cases have numerous application in electromagnetics and in acoustics.

APPENDIX

This appendix describes the derivation of the discharge current $I_d$ and the TL’s characteristic impedance $Z(t)$ in (18). In the present discussion, the interest is only on the capacitor’s discharge, thus the TL can be modeled by its characteristic impedance and to appear as a lumped impedance element $Z(t)$. The design goal in the discharge process is to obtain an optimal energy delivery from an initially charged capacitor to the TL, i.e., $Z(t)$.

The capacitor’s charge $Q(t)$ during discharge is given by

$$Q(t) = -Q_0 e^{-\int_0^t dt / \tau(t)}, \quad \tau = C_a (R_a + Z(t)) \quad (27)$$

where we assumed that initially at $t = 0$, the capacitor is charged with a negative charge $-Q_0$. The discharge current is $i(t) = dQ(t)/dt = -Q(t)/\tau(t)$, the instantaneous power delivered to the TL is $p(t) = i^2(t) Z(t) = Q^2(t) Z(t)/\tau^2(t)$, and the energy delivered during discharge is

$$\mathcal{E} = \int_0^\infty dt \left( \frac{Q(t)}{\tau(t)} \right)^2 Z(t)$$

$$= \frac{R_a Q_0^2}{\tau_a} \int_0^\infty dt \left[ y(t) - \tau_a (y'(t))^2 \right] e^{-2y(t)} \quad (28)$$

where $y(t) = \int_0^t dt / \tau(t)$, and $Q(t) = -Q_0 e^{-y(t)}$.

To obtain an optimal solution to the capacitor discharge and energy delivery to $Z(t)$, the corresponding Euler–Lagrange equation [67] is formulated

$$\frac{d}{dy} \mathcal{L}(y, y', t) - \frac{d}{dt} \frac{d}{dy} \mathcal{L}(y, y', t) = 0$$

$$\mathcal{L}(y, y', t) = [y'(t) - \tau_a (y'(t))^2] e^{-2y(t)} \quad (29)$$
with \( y'(t) = (d y(t)/d t) \). Using \( L(y, y', t) \) in Euler–Lagrange equation gives the differential equation

\[
y''(t) - (y'(t))^2 = 0.
\]

Noting that \( r(t) = r_0(1 + \zeta(t)) = (y'(t))^{-1} \) with \( \zeta(t) = Z(t)/R_0 \), see (18) gives

\[
\zeta(t) = \frac{1}{\tau_d} y'(t) - 1.
\]

Assuming that initially at \( t = 0 \) the TL characteristic impedance is \( Z_0 = R_0 \), it gives the initial condition for the solution of the differential equation in (30): \( y'(0) = [\tau_d(1 + \zeta_0)]^{-1} \). It further follows that:

\[
y'(t) = \frac{1}{\tau_d(1 + \zeta_0) - t}
\]

\[
y(t) = -\ln\left[1 - \frac{t}{\tau_d(1 + \zeta_0)}\right]
\]

and in view of (31), it gives \( \zeta(t) \) of (18). The capacitor charge is, thus, given by

\[
Q(t) = -Q_0 \left[1 - \frac{t}{\tau_d(1 + \zeta_0)}\right]
\]

leading to the constant discharge current \( I_d = (Q_0/\tau_d(1 + \zeta_0)) \). It follows that a complete discharge of the capacitor is achieved at \( I_d = \tau_d(1 + \zeta_0) \). However, at such \( \zeta_d(1 + \zeta_d) > 0 \) \((Z_d < 0)\), which is not physical. Furthermore, achieving \( \zeta \to 0 \) is impractical, hence we set the lowest possible characteristic impedance as \( Z_m = R_0 > 0 \), which is reached at time \( t_m = (\zeta_0 - \zeta_m) \tau_d \).

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