Weighted Norm Inequalities for Fractional Bergman Operators

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Abstract
We prove in this paper one weight norm inequalities for some positive Bergman-type operators.

Keywords Békollè–Bonami weight · Bergman operator · Dyadic grid · Maximal function · Upper-half plane

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1 Introduction and Results
The set \( \mathbb{R}^2_+ := \{ z = x + iy \in \mathbb{C} : x \in \mathbb{R}, \text{ and } y > 0 \} \) is called the upper half-plane. By a weight, we will mean a nonnegative locally integrable function on \( \mathbb{R}^2_+ \). Let \( \alpha > -1 \) and \( 1 \leq p < \infty \). For \( \omega \) a weight, we denote by \( L^p(\mathbb{R}^2_+, \omega dV) \) the set of all functions \( f \) defined on \( \mathbb{R}^2_+ \) such that

\[
||f||_{p,\omega,\alpha}^p := \int_{\mathbb{R}^2_+} |f(z)|^p \omega(z) dV(z) < \infty,
\]

with \( dV(x+iy) = y^\alpha dx dy \). When \( \omega = 1 \), we simply write \( L^p(\mathbb{R}^2_+, dV) \) and \( ||\cdot||_{p,\alpha} \) for the corresponding norm.

For \( \alpha > -1 \) and \( 0 \leq \gamma < 2 + \alpha \), the positive fractional Bergman operator \( T_{\alpha,\gamma} \) is defined by
\[ T_{\alpha,\gamma} f(z) := \int_{\mathbb{R}^n_+} \frac{f(w)}{|z - w|^{2+\alpha-\gamma}} dV_\alpha(w). \]  

(1)

For \( \gamma = 0 \), the operator \( P^+_{\alpha} := T_{\alpha,0} \) is the positive Bergman projection.

The above operator can be seen as the upper half-plane analogue of the fractional integral operator (Riesz potential) defined by

\[ I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x - y|^{n-\alpha}}, \quad x \in \mathbb{R}^n \]

for \( 0 \leq \alpha < n \), \( n \in \mathbb{N} \). We recall that the weighted boundedness of the latter was obtained by Muckenhoupt and Wheeden [6]. More precisely, let \( Q \) be the set of all cubes in \( \mathbb{R}^n \). Let \( 1 \leq p, q < \infty \). We say a weight \( \omega \) belongs to the class \( A_{p,q} \) if and only if

\[ \sup_{Q \in Q} \left( \frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_Q \omega(x)^{-p'} dx \right)^{\frac{1}{p'}} < \infty. \]

When \( p = 1 \), we denote by \( A_{1,q} \), the class of all weights \( \omega \) such that

\[ [\omega]_{1,q} := \sup_{Q \in Q} \text{ess sup} \left( \frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{\frac{1}{q}} \omega(x)^{-1} < \infty. \]

The results obtained by Muckenhoupt and Wheeden [6] are summarized as follows.

**Theorem 1.1** Let \( 0 < \alpha < n \). Then the following are satisfied:

(a) Let \( q = \frac{n}{n-\alpha} \). Then \( \omega \in A_{1,q} \) if and only if there is a constant \( C > 0 \) such that

\[ \sup_{\lambda > 0} \lambda \omega^q \left( \left\{ x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda \right\} \right)^{1/q} \leq C \int_{\mathbb{R}^n} |f(x)| \omega(x) dx. \]

(2)

(b) Given \( 1 < p < \frac{n}{\alpha} \), let \( q \) be such that \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n} \). Then \( \omega \in A_{p,q} \) if and only if there is a constant \( C > 0 \) such that

\[ \left( \int_{\mathbb{R}^n} (\omega(x)|I_\alpha f(x)|)^q dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} (\omega(x)|f(x)|)^p dx \right)^{\frac{1}{p}}. \]

(3)

Our aim in this paper is to provide corresponding results for the operator \( T_{\alpha,\gamma} \). For this and to present our results, we need some other definitions.

For any interval \( I \subset \mathbb{R} \), we denote by \( Q_I \) its associated Carleson square, that is, the set

\[ Q_I := \{ z = x + iy \in \mathbb{C} : x \in I \text{ and } 0 < y < |I| \}. \]
Let $\alpha > -1$ and $1 < p < \infty$. Given a weight $\omega$, we say $\omega$ is in the Békollé–Bonami class $B_{p,\alpha}$ if the quantity

$$[\omega]_{B_{p,\alpha}} := \sup_{I \in \mathcal{I}} \left( \frac{1}{|Q_I|^\alpha} \int_{Q_I} \omega(z) dV(z) \right) \left( \frac{1}{|Q_I|^\alpha} \int_{Q_I} \omega(z)^{1-p'} dV(z) \right)^{p-1}$$

is finite. Here and throughout the paper, for a measurable set $E \subset \mathbb{R}^2_+$, 

$$|E|_{\omega,\alpha} := \int_E \omega(z) dV(z),$$

and we write $|E|_\alpha$ for $|E|_{1,\alpha}$. Also, we have used $\mathcal{I}$ to denote the set of all intervals of $\mathbb{R}$.

It is now well known that for $1 < p < \infty$, the operator $P_{\alpha}^+$ is bounded on $L^p(\mathbb{R}^2_+, \omega dV)$ if and only if $\omega \in B_{p,\alpha}$ (see [1, 2, 7]). For $p = 1$, we say $\omega \in B_{1,\alpha}$ if

$$[\omega]_{B_{1,\alpha}} := \sup_{I \in \mathcal{I}} \left( \frac{1}{|Q_I|^\alpha} \int_{Q_I} \omega(z) dV(z) \right) < \infty.$$

For $1 \leq p, q < \infty$, we introduce the two following classes of weights: we say a weight $\omega$ belongs to the set $B_{p,q,\alpha}$ with $p \neq 1$ if

$$[\omega]_{B_{p,q,\alpha}} := \sup_{I \in \mathcal{I}} \left( \frac{1}{|Q_I|^\alpha} \int_{Q_I} \omega^p dV(z) \right)^{\frac{1}{q}} \left( \frac{1}{|Q_I|^\alpha} \int_{Q_I} \omega^{-p'} dV(z) \right)^{\frac{1}{p'}} < \infty.$$

We say $\omega$ belongs to $B_{1,q,\alpha}$ if

$$[\omega]_{B_{1,q,\alpha}} := \sup_{I \in \mathcal{I}} \left( \frac{1}{|Q_I|^\alpha} \int_{Q_I} \omega^q dV(z) \right)^{\frac{1}{q}} \omega(z)^{-1} < \infty.$$

We observe that if $r = 1 + \frac{q}{p}$, then

$$[\omega^q]_{B_{r,\alpha}} = [\omega]_{B_{p,q,\alpha}}^{q}.$$ 

The above classes can be compared with the classes of weights $A_{p,q}$ introduced by Muckenhoupt and Wheeden in relation to the study of weighted norm inequality for the Riesz potential (see [6]).

For the strong inequality, we obtain the following.

**Theorem 1.2** Let $\alpha > -1$ and $0 \leq \gamma < 2 + \alpha$, and $1 < p < \frac{2+\alpha}{\gamma}$. Define $q$ by

$$\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{2+\alpha}.$$ 

Assume that the weight $\omega$ belongs to the class $B_{p,q,\alpha}$. Then $T_{\alpha,\gamma}$ is bounded from $L^p(\omega^p dV)$ to $L^q(\omega^q dV)$ if and only if $\omega \in B_{p,q,\alpha}$. Moreover,

$$\left( \int_{\mathbb{R}^2_+} (\omega(z)|T_{\alpha,\gamma} f(z)|)^q dV(z) \right)^{\frac{1}{q}} \leq C_{\alpha,p} ([\omega]_{B_{p,q,\alpha}})^{1+\frac{\gamma}{p} + \frac{q}{p'}} \left( \int_{\mathbb{R}^2_+} (\omega(z)|f(z)|)^p dV(z) \right)^{\frac{1}{p}}.$$  

(4)
We observe that if \( \gamma \geq 0 \), then for any positive function \( f \) and any \( z = x + iy \in \mathbb{R}^2_+ \), \( y^\gamma P^+_\alpha f(z) \leq T_{\alpha, \gamma} f(z) \). It follows that given two weights \( \sigma \) and \( \omega \) on \( \mathbb{R}^2_+ \), for \( 1 \leq p \leq q < \infty \), the (strong) boundedness of \( T_{\alpha, \gamma} \) from \( L^p(\mathbb{R}^2_+, \sigma \, dV_\alpha) \) to \( L^q(\mathbb{R}^2_+, \omega \, dV_\alpha) \) implies the boundedness of \( P^+_\alpha \) from \( L^p(\mathbb{R}^2_+, \sigma \, dV_\alpha) \) to \( L^q(\mathbb{R}^2_+, \omega dV_\eta) \), where \( \eta = \alpha + q \gamma \). In particular, taking \( \sigma = \omega \) and observing that when \( \gamma = (2 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right) \), \( \eta = (2 + \alpha)(\frac{4}{p} - 1) + \alpha \), we deduce the following from the above result.

**Corollary 1.3** Let \( \alpha > -1 \) and \( 0 \leq \gamma < 2 + \alpha \), and \( 1 < p < \frac{2+\alpha}{2+\alpha-\gamma} \). Define \( q \) by \( \frac{1}{p} - \frac{1}{q} = \frac{\gamma}{2+\alpha} \). Let \( \omega \) be a weight on \( \mathbb{R}^2_+ \). Assume that the weight \( \omega \in B_{p,q,\alpha} \). Then \( P^+_\alpha \) is bounded from \( L^p(\omega^p \, dV_\alpha) \) to \( L^q(\omega^q \, dV_\eta) \), with \( \eta = (2 + \alpha)(\frac{4}{p} - 1) + \alpha \). Moreover,

\[
\left( \int_{\mathbb{R}^2_+} (\omega(z) | P^+_\alpha f(z) |)^q \, dV_\eta \right)^{\frac{1}{q}} \leq C_{\alpha,p}(\omega) \left( \int_{\mathbb{R}^2_+} (\omega(z) | f(z) |)^p \, dV_\alpha \right)^{\frac{1}{p}}.
\]

(5)

For the limit case \( p = 1 \), we obtain the following weak boundedness of the operator \( T_{\alpha, \gamma} \).

**Theorem 1.4** Let \( \alpha > -1 \) and \( 0 < \gamma < 2 + \alpha \). Let \( q = \frac{2+\alpha}{2+\alpha-\gamma} \). Assume that the weight \( \omega \) belongs to the class \( B_{1,q,\alpha} \). Then \( T_{\alpha, \gamma} \) is bounded from \( L^1(\omega \, dV_\alpha) \) into \( L^{q,\infty}(\omega^q \, dV_\eta) \). Moreover,

\[
\sup_{\lambda > 0} \lambda \left| \left\{ z \in \mathbb{R}^2_+ : |T_{\alpha, \gamma} f(z)| > \lambda \right\} \right|^{\frac{1}{q}}_{\omega^q, \alpha} \leq C(\alpha, \gamma)(\omega)_{B_{1,q,\alpha}}^{\frac{q}{q-1}} \int_{\mathbb{R}^2_+} |f(z)| \, dV_\alpha(z).
\]

(6)

It is not clear how to deduce the weak boundedness of the positive Bergman operator \( P^+_\alpha \) from the one of \( T_{\alpha, \gamma} \). We will then also prove the following.

**Theorem 1.5** Let \( \alpha > -1 \) and \( 0 < \gamma < 2 + \alpha \). Let \( q = \frac{2+\alpha}{2+\alpha-\gamma} \). Assume that the weight \( \omega \) belongs to the class \( B_{1,q,\alpha} \). Then \( P^+_\alpha \) is bounded from \( L^1(\omega \, dV_\alpha) \) into \( L^{q,\infty}(\omega^q \, dV_\eta) \) with \( \eta = (2 + \alpha)(q - 1) + \alpha \). In this case,

\[
\sup_{\lambda > 0} \lambda \left| \left\{ z \in \mathbb{R}^2_+ : |P^+_\alpha f(z)| > \lambda \right\} \right|^{\frac{1}{q}}_{\omega^q, \eta} \leq C(\alpha, \gamma)(\omega)_{B_{1,q,\alpha}}^{q-1} \int_{\mathbb{R}^2_+} |f(z)| \, dV_\alpha(z).
\]

(7)

In comparison with the case of Riesz potentials (see [5]), our results are certainly not sharp in terms of the power of \( (\omega)_{B_{p,q,\alpha}} \) in the estimates (4), (5), (6), and (7). We also illustrate this fact with the following estimate for a concrete example of exponents that is inspired from [5].

Put

\[
p_0 = \frac{2 - \frac{\gamma}{2+\alpha}}{\frac{\gamma}{2+\alpha} - \left( \frac{\gamma}{2+\alpha} \right)^2 + 1}; \quad q_0 = \frac{2 - \frac{\gamma}{2+\alpha}}{1 - \frac{\gamma}{2+\alpha}}.
\]
Observe that

\[ \frac{1}{q_0} = \frac{1}{p_0} - \frac{\gamma}{2 + \alpha} \]

and

\[ \frac{q_0}{p'_0} = 1 - \frac{\gamma}{2 + \alpha}. \]

Note also that

\[ q_0 < 1 + \frac{p'_0}{p_0} + \frac{q_0}{p'_0}. \]

**Theorem 1.6** Let \( \omega \in B_{p_0, q_0, \alpha}, \alpha > -1. \) Then \( T_{\alpha, \gamma} : L^{p_0}(\omega^{p_0} dV_\alpha) \to L^{q_0}(\omega^{q_0} dV_\alpha) \) is bounded. Moreover,

\[ \|T_{\alpha, \gamma}\|_{L^{p_0}(\omega^{p_0} dV_\alpha) \to L^{q_0}(\omega^{q_0} dV_\alpha)} \leq C_{p_0, q_0, \alpha} \omega^{q_0}_{B_{p_0, q_0, \alpha}}. \]

For our proofs, we follow the now standard trend of techniques of sparse domination using dyadic grids. As observed above, our operators are clearly analogues of the Riesz potential. We note that a simplification of the proofs of the results in Theorem 1.1 was recently obtained by Cruz-Uribe [3]. We follow here the approach in the online version of [3] (the reader is advised to consult this online version and not the published one).

We note that there is a natural sparse family (see for example [3] for a definition of sparseness) on the upper half-plane made of Carleson squares.

The strong inequalities are easier to prove than the weak inequalities, and we only provide a proof here for the sake of the reader who is not familiar with these techniques. For the weak type estimates, we remark that one of the key arguments in the online version of [3] is the reverse Hölder’s inequality, a tool that is not available in our setting. To overcome this difficulty, we use a reverse doubling property satisfied by the Békolé–Bonami weights with a careful consideration of the involved constant. In the case of the weak type estimate for the positive Bergman operator, there is a further difficulty due to the change of weight (power of the distance to the boundary). There is another cost to pay to overcome this other difficulty which is illustrated by the change of the power in the constant in (7).

The question of sharp off-diagonal estimates for the Bergman projection has been partially answered in [8]. Its extension to the full upper-triangle and Sawyer-type characterizations are considered in a forthcoming paper. Note also that in [9], we obtained some bump-conditions for the two-weight boundedness of the above fractional Bergman operators.

In the next section, we recall some useful facts and results needed in our proofs. Here, we particularly point out the fact that our results essentially follow from their dyadic counterparts. In Sect. 3, we prove the weak type results. The strong inequalities are proved in Sect. 4.

Given two positive quantities \( A \) and \( B \), the notation \( A \lesssim B \) (resp. \( B \lesssim A \)) will mean that there is a universal constant \( C > 0 \) such that \( A \leq CB \) (resp. \( B \leq CA \)). When \( A \lesssim B \) and \( B \lesssim A \), we write \( A \simeq B \). Notation \( C_\alpha \) or \( C(\alpha) \) means that the constant \( C \) depends on the parameter \( \alpha \).
2 Preliminaries

2.1 Some Properties of Weights

The following is an easy consequence of Hölder’s inequality (see [4, Lemma 2.1] for the case $\alpha = 0$).

**Lemma 2.1** Let $1 < p < \infty$, and $\alpha > -1$. Let $I \subset \mathbb{R}$ be an interval, and denote by $T_I$ the upper half of the Carleson square $Q_I$. Assume that $\omega \in B_{p,\alpha}$. Then

$$|Q_I|_{\omega,\alpha} \leq C_{p,\alpha}[\omega]|T_I|_{\omega,\alpha},$$

(8)

where $C_{p,\alpha} := \max\left\{2, \left(\frac{2^{1+\alpha}}{2^{1+\alpha} - 1}\right)^p\right\}$.

As a consequence of the above lemma, we obtain the following reverse doubling property.

**Lemma 2.2** Let $1 < p < \infty$ and $\alpha > -1$. Let $I \subset \mathbb{R}$ be an interval, and denote by $B_I$ the lower half of the Carleson square $Q_I$. Assume that $\omega \in B_{p,\alpha}$. Then

$$\frac{|B_I|_{\omega,\alpha}}{|Q_I|_{\omega,\alpha}} \leq \theta,$$

where $\theta = 1 - \frac{1}{C_{p,\alpha}[\omega]|B_{p,\alpha}|}$, with $C_{p,\alpha}$ the constant in (8).

2.2 Maximal Functions and Their Boundedness

Let $\alpha > -1$, $0 \leq \gamma < 2 + \alpha$, and let $\sigma$ be a weight. The weighted fractional maximal function $M_{\sigma,\alpha,\gamma}$ is defined by

$$M_{\sigma,\alpha,\gamma}f(z) := \sup_{I \subset \mathbb{R}} \frac{1}{|Q_I|_{\sigma,\alpha,\gamma}} \int_{Q_I} |f(w)||\sigma(w)|dV_{\alpha}(w).$$

(9)

When $\gamma = 0$, the above operator is just the weighted Hardy–Littlewood maximal function denoted by $M_{\sigma,\alpha}$, and if, moreover, $\sigma = 1$, we simply write $M_{\alpha}$. The unweighted fractional maximal function is just the operator $M_{\alpha,\gamma} := M_{1,\alpha,\gamma}$.

We consider the following system of dyadic grids:

$$D^\beta := \left\{2^j \left([0, 1) + m + (-1)^j \beta\right) : m \in \mathbb{Z}, \ j \in \mathbb{Z}\right\}, \ \text{for} \ \beta \in \{0, 1/3\}.$$  

When $\beta = 0$, we observe that $D^0$ is the standard dyadic grid of $\mathbb{R}$, denoted by $D$.

For any $\beta \in \{0, 1/3\}$, we denote by $M_{\sigma,\alpha,\gamma}^{d,\beta}$ the dyadic analogue of $M_{\sigma,\alpha,\gamma}$, defined as in (9) but with the supremum taken over dyadic intervals in the grid $D^\beta$.

We have the following useful result.
Lemma 2.3 Let \( 1 \leq p \leq q < \infty, \alpha > -1 \) and \( 0 \leq \gamma < 2 + \alpha \). Let \( \omega \) be a weight on \( \mathbb{R}_+^2 \), and let \( \mu \) be a positive measure on \( \mathbb{R}_+^2 \). Then the following assertions are equivalent.

(a) There is a constant \( C_1 > 0 \) such that for any \( f \in L^p(\mathbb{R}_+^2, \omega dV_\alpha) \) and any \( \lambda > 0 \),

\[
\mu(\{ z \in \mathbb{R}_+^2 : M_{\alpha, \gamma} f(z) > \lambda \}) \leq \frac{C_1}{\lambda^q} \left( \int_{\mathbb{R}_+^2} |f(z)|^p \omega(z) dV_\alpha(z) \right)^{q/p}.
\]

(b) There is a constant \( C_2 > 0 \) such that for any \( f \in L^p(\mathbb{R}_+^2, \omega dV_\alpha) \), for any \( \beta \in \{0, \frac{1}{3}\} \), and any \( \lambda > 0 \),

\[
\mu(\{ z \in \mathbb{R}_+^2 : M_{\alpha, \gamma}^{d, \beta} f(z) > \lambda \}) \leq \frac{C_2}{\lambda^q} \left( \int_{\mathbb{R}_+^2} |f(z)|^p \omega(z) dV_\alpha(z) \right)^{q/p}.
\]

(c) There is a constant \( C_3 > 0 \) such that for any interval \( I \subset \mathbb{R} \),

\[
|Q_I|_{\alpha}^{\frac{\gamma}{2+\alpha} - \frac{1}{p}} \left( \frac{1}{|Q_I|_\alpha} \int_{Q_I} \omega^{1-p'}(z) dV_\alpha(z) \right)^{q/p'} \mu(Q_I) \leq C_3,
\]

where \( \left( \frac{1}{|Q_I|_\alpha} \int_{Q_I} \omega^{1-p'}(z) dV_\alpha(z) \right)^{1/p'} \) is understood as \( \left( \inf_{Q_I} \omega \right)^{-1} \) when \( p = 1 \).

Proof The equivalence (a) \( \iff \) (c) is from [9,Theorem 2.3]. Clearly, (a) \( \Rightarrow \) (b). That (b) \( \Rightarrow \) (a) follows from [9,Lemma 4.1] and is the main idea in the proof of [9,Theorem 2.3]. \qed

We refer to [9,Corollary 4.3] for the following.

Lemma 2.4 Let \( \alpha > -1 \), \( 0 \leq \gamma < 2 + \alpha \), and let \( \sigma \) be a weight. Let \( 1 < p < \frac{2+\alpha}{\gamma} \), and define \( q \) by \( \frac{1}{q} = \frac{1}{p} - \frac{\gamma}{2+\alpha} \). Then there exists a constant \( C = C(p, \alpha, \gamma) \) such that for any \( \beta \in \{0, \frac{1}{3}\} \),

\[
\left( \int_{\mathbb{R}_+^2} \left( M_{\sigma, \alpha, \gamma}^{d, \beta} f(z) \right)^q \sigma(z) dV_\alpha(z) \right)^{1/q} \leq C \left( \int_{\mathbb{R}_+^2} |f(z)|^p \sigma(z) dV_\alpha(z) \right)^{1/p}.
\]

2.3 Dyadic Analogue of Fractional Bergman Operators

Let \( \alpha > -1 \) and \( 0 \leq \gamma < 2 + \alpha \). For \( \beta \in \{0, 1/3\} \), we introduce the following dyadic operators:

\[
\mathcal{Q}_{\alpha, \gamma}^\beta f = \sum_{I \in D^\beta} |Q_I|_{\alpha}^{\frac{\gamma}{2+\alpha}} \left( f \cdot \frac{1}{|Q_I|_\alpha} \right)_{1/Q_I} 1_{Q_I}.
\]
Here, $\langle \cdot, \cdot \rangle_\alpha$ stands for the duality pairing

$$\langle f, g \rangle_\alpha = \int_{\mathbb{R}_+^2} f(z) g(z) dV_\alpha(z).$$

The operators $Q^\beta_{\alpha, \gamma}$ were introduced in [7] in the case $\gamma = 0$ in relation with the sharp estimate of the Bergman projection. The following result is obtained as in the case $\gamma = 0$ (see [7, Proposition 3.4]).

**Lemma 2.5** Let $\alpha > 0$ and $0 \leq \gamma < 2 + \alpha$. Then there exists a constant $C = C_{\alpha, \gamma} > 0$ such that for all $f \in L^1_{loc}(\mathbb{R}_+^2, dV_\alpha)$, $f \geq 0$, and $z \in \mathbb{R}_+^2$,

$$T_{\alpha, \gamma} f(z) \leq C \sum_{\beta \in \{0, \frac{1}{3}\}} Q^\beta_{\alpha, \gamma} f(z). \quad (13)$$

We also recall the following covering results. The first one is [7, Lemma 3.1] while the second one is [4, Lemma 2.3].

**Lemma 2.6** Let $I$ be any interval in $\mathbb{R}$. Then the following hold.

1. There exists a dyadic interval $J \in \mathcal{D}^\beta$ for some $\beta \in \{0, \frac{1}{3}\}$ such that $I \subseteq J$ and $|J| \leq 8|I|$.
2. For any $\beta \in \{0, \frac{1}{3}\}$, $I$ can be covered by two adjacent intervals $I_1$ and $I_2$ in $\mathcal{D}^\beta$ such that $|I| < |I_1| = |I_2| \leq 2|I|$.

**Remark 2.7** As the operators considered here are positive, we only need to consider positive functions in our proofs. Also from Lemma 2.5, it follows that to prove the norm inequalities for $T_{\alpha, \gamma}$, it suffices to prove them for the positive dyadic operators $Q^\beta_{\alpha, \gamma}$. Let us note that it is also enough to prove the norm inequalities for bounded and compactly supported functions as the general case will follow from Fatou’s lemma.

### 3 Proofs of Theorems 1.4 and 1.5

**Proof of Theorem 1.4** Assume that $\omega \in B_{1,q,\alpha}$. Following Remark 2.7, one only needs to show that the estimate (6) holds with $T_{\alpha, \gamma}$ replaced by $Q^\beta_{\alpha, \gamma}$. From the same remark, definition (12), and Lemma 2.6, we can assume that $f$ is supported on some dyadic square $Q_J$, $J \in \mathcal{D}^\beta$, as we also prove that the estimate obtained is independent of $J$.

Put for $\lambda > 0$,

$$\mathbb{E}_\lambda := \{z \in \mathbb{R}_+^2 : Q^\beta_{\alpha, \gamma} f(z) > \lambda\}.$$

Let $\Lambda > 0$ be fixed. We first check that

$$\sup_{0 < \lambda < \Lambda} \lambda^\theta |\mathbb{E}_\lambda|_{u, \alpha} < \infty, \quad u = \omega^\theta.$$
Indeed if \( z \notin Q_J \), then \( Q^\beta_{\alpha,\gamma} f(z) \neq 0 \) only if we can find \( I \in D^\beta \) such that \( J \subset I \) and \( z \in Q_I \). Let \( I_0 \) be the smallest \( I \) such that this holds. Then for any other \( I \in D^\beta \) such that \( J \subset I \) and \( z \in Q_I \), we have that \( Q_I \subseteq Q_J \), and so, \( |Q_I|_\alpha = 2^{(2+\alpha)k} |Q_{I_0}|_\alpha \) for some integer \( k > 0 \). Moreover,

\[
\int_{Q_I} f \, dV_\alpha = \int_{Q_{I_0}} f \, dV_\alpha.
\]

It follows that

\[
Q^\beta_{\alpha,\gamma} f(z) = \sum_{J \subset I} |Q_I|_{\frac{\gamma}{1+\alpha}}^{-1} \int_{Q_I} f \, dV_\alpha
\]

\[
= \left( \sum_{k=0}^{\infty} 2^{(1+\alpha)k \left( \frac{\gamma}{1+\alpha} - 1 \right)} \right) |Q_{I_0}|_{\frac{\gamma}{1+\alpha}}^{-1} \int_{Q_{I_0}} f \, dV_\alpha
\]

\[
\leq C \mathcal{M}^d_{\alpha,\beta} f(z).
\]

Hence putting

\[
\mathcal{F}_\lambda := \left\{ z \in \mathbb{R}^2_+ : \mathcal{M}^d_{\alpha,\beta} f(z) > \frac{\lambda}{C} \right\},
\]

we obtain

\[
\sup_{0 < \lambda < \Lambda} \lambda^q |\mathcal{E}_\lambda|_{u,\alpha} \leq \Lambda^q |Q_J|_{u,\alpha} + \sup_{0 < \lambda < \Lambda} \lambda^q \left\{ \left\{ z \in \mathbb{R}^2_+ \setminus Q_J : \mathcal{M}^d_{\alpha,\gamma} f(z) > \frac{\lambda}{C} \right\} \right\}_{u,\alpha}
\]

\[
\leq \Lambda^q |Q_J|_{u,\alpha} + \sup_{0 < \lambda < \Lambda} \lambda^q |\mathcal{F}_\lambda|_{u,\alpha} < \infty.
\]

Here we have used that \( u \) is locally integrable and that \( \mathcal{M}^d_{\alpha,\gamma} \) is bounded from \( L^1(u \, dV_\alpha) \) to \( L^{q,\infty}(u \, dV_\alpha) \) for \( u \) satisfying

\[
\supess \sup_{I \in Q_I} \left( \frac{1}{|Q_I|_\alpha} \int_{Q_I} u \, dV_\alpha \right)^{\frac{1}{q}} \omega^{-1}(z) < \infty
\]

(see Lemma 2.3).

Next, we observe that

\[
Q^\beta_{\alpha,\gamma} f(z) = Q^\beta_{\alpha,\gamma,\text{in}} f(z) + Q^\beta_{\alpha,\gamma,\text{out}} f(z),
\]

where

\[
Q^\beta_{\alpha,\gamma,\text{in}} f(z) = \sum_{I \in D^\beta, I \subseteq J} |Q_I|_{\frac{\gamma}{1+\alpha}} \left( \frac{1}{|Q_I|_\alpha} \int_{Q_I} f \, dV_\alpha \right) 1_{Q_I}(z)
\]
and

\[ Q_{\alpha,\gamma, J}^\beta \ f(z) = \sum_{I \in D^\beta \atop I \supseteq J} |Q_I|^{\frac{\gamma}{\alpha}} \left( \frac{1}{|Q_I|^{\alpha}} \int_{Q_I} f \, dV_\alpha \right) \chi_{Q_I}(z). \]

We recall that \( E_\lambda \) can be written as a union of maximal dyadic Carleson squares. Indeed, if \( z \in E_\lambda \), denote by \( Q(z) \) the smallest dyadic square containing \( z \). Let \( w \in Q(z) \). Then any dyadic square supported by an interval in \( D^\beta \) containing \( z \) contains \( Q(z) \) and hence contains \( w \). Thus

\[ \lambda < Q_{\alpha,\gamma, J}^\beta \ f(z) = \sum_{I \in D^\beta \atop z \in Q_I} |Q_I|^{\frac{\gamma}{\alpha}} \langle f, \frac{1}{|Q_I|^{\alpha}} \rangle \]
\[ \leq \sum_{I \in D^\beta \atop w \in Q_I} |Q_I|^{\frac{\gamma}{\alpha}} \langle f, \frac{1}{|Q_I|^{\alpha}} \rangle \]
\[ = Q_{\alpha,\gamma, J}^\beta \ f(w). \]

Hence \( Q(z) \subset E_\lambda \). That is, for any \( z \in E_\lambda \), there is a dyadic square containing \( z \) that is entirely contained in \( E_\lambda \). Moreover, as \( f \) is compactly supported, this dyadic square cannot be arbitrarily large. Thus \( E_\lambda \) is a union of maximal dyadic Carleson squares.

Let \( I \in D^\beta \) be such that \( Q_I \) is one of the maximal squares above. Let \( I \) be the dyadic parent of \( I \). Then there exists \( z_0 \in Q_j \setminus Q_I \) such that for any \( z \in Q_I \),

\[ \lambda \geq Q_{\alpha,\gamma, J}^\beta \ f(z_0) = Q_{\alpha,\gamma, J}^\beta \ f(z). \]

It follows that for any \( z \in Q_J \cap E_{2\lambda} \),

\[ Q_{\alpha,\gamma, J}^\beta \ f(z) > \lambda. \]

Next we fix \( \Lambda > 0 \). We also fix \( \lambda \) such that \( 0 < \lambda < \Lambda \). We recall that \( u = \omega^q \). Define \( L \) to be the family of maximal dyadic Carleson squares whose union is \( E_\lambda \). Put

\[ L_1 := \{ Q_I \in L : |Q_I|^{\lambda} \geq 2^{-q-1} |Q_I|_{u, \alpha} \} \]

and

\[ L_2 := L \setminus L_1. \]

Then

\[ (2\lambda)^q |E_{2\lambda}|_{u, \alpha} = (2\lambda)^q \sum_{Q_I \in L} |Q_I| \cap E_{2\lambda} |_{u, \alpha} \]
\[ = (2\lambda)^q \left( \sum_{Q_I \in L_1} |Q_I| \cap E_{2\lambda} |_{u, \alpha} + \sum_{Q_I \in L_2} |Q_I \cap E_{2\lambda} |_{u, \alpha} \right). \]
We have

\[
(2\lambda)^q \sum_{Q_I \in L_2} |Q_I \cap \mathbb{E}_{2\lambda}|_{u,\alpha} \leq (2\lambda)^q 2^{-q-1} \sum_{Q_I \in L_2} |Q_I|_{u,\alpha}
\]

\[
\leq \frac{1}{2} \lambda^q \sum_{Q_I \in L_2} |Q_I|_{u,\alpha}
\]

\[
\leq \frac{1}{2} \lambda^q |\mathbb{E}_{\lambda}|_{u,\alpha}
\]

\[
\leq \frac{1}{2} \sup_{0<\lambda<\Lambda} \lambda^q |\mathbb{E}_{\lambda}|_{u,\alpha} < \infty.
\]

Now

\[
L := (2\lambda)^q \sum_{Q_I \in L_1} |Q_I \cap \mathbb{E}_{2\lambda}|_{u,\alpha}
\]

\[
= (2\lambda)^q \sum_{Q_I \in L_1} |Q_I \cap \mathbb{E}_{2\lambda}|_{u,\alpha} |Q_I \cap \mathbb{E}_{2\lambda}|_{u,\alpha}^{1-q}
\]

\[
\leq 2^{q-1} (2\lambda)^q \sum_{Q_I \in L_1} |Q_I \cap \mathbb{E}_{2\lambda}|_{u,\alpha} |Q_I|_{u,\alpha}^{1-q}
\]

\[
\leq C(q) \sum_{Q_I \in L_1} (\lambda |Q_I \cap \mathbb{E}_{2\lambda}|_{u,\alpha})^q |Q_I|_{u,\alpha}^{1-q}
\]

\[
\leq C(q) \sum_{Q_I \in L_1} (\lambda |z \in Q_I : Q_{\alpha,\gamma,I}^{\beta,\text{in}} f(z) > \lambda|_{u,\alpha})^q |Q_I|_{u,\alpha}^{1-q}
\]

\[
\leq C(q) \sum_{Q_I \in L_1} \left( \int_{Q_I} \left( Q_{\alpha,\gamma,I}^{\beta,\text{in}} f(z) u(z) dV_{\alpha}(z) \right)^q |Q_I|_{u,\alpha}^{1-q} \right)
\]

\[
= C(q) \sum_{Q_I \in L_1} \left( \int_{Q_I} \left( Q_{\alpha,\gamma,I}^{\beta,\text{in}} u(z) f(z) dV_{\alpha}(z) \right)^q |Q_I|_{u,\alpha}^{1-q} \right).
\]

In the last inequality, we have used duality and the fact that \(Q_{\alpha,\gamma,I}^{\beta,\text{in}}\) is self-adjoint with respect to the pairing

\[
\langle f, g \rangle_{\alpha} = \int_{\mathbb{R}^2_+} f(z) \overline{g(z)} dV_{\alpha}(z).
\]

Let us estimate \(Q_{\alpha,\gamma,I}^{\beta,\text{in}} u\). We recall that

\[
q = \frac{2 + \alpha}{2 + \alpha - \gamma} \quad \text{and so} \quad \frac{1}{q'} = \frac{\gamma}{2 + \alpha}.
\]
We first write

\[
Q_{\beta, \alpha, I} u(z) = \sum_{K \subseteq I} |Q_K|^{\frac{\alpha}{\alpha}} \left( \frac{1}{|Q_K|^{\alpha}} \int_{Q_K} u dV_{\alpha} \right)^{\frac{1}{q}} 1_{Q_K}(z)
\]

\[
= \sum_{K \subseteq I} |Q_K|^{\frac{1}{\alpha}} \left( \frac{1}{|Q_K|^{\alpha}} \int_{Q_K} u dV_{\alpha} \right)^{\frac{1}{q} + \frac{1}{q'}} 1_{Q_K}(z).
\]

We observe that for any \(Q_K \subseteq Q_I, \ z \in Q_K\),

\[
\left( \frac{1}{|Q_K|^{\alpha}} \int_{Q_K} u dV_{\alpha} \right)^{\frac{1}{q}} \leq [\omega]_{B_{1,q}, \alpha} \omega(z).
\]

On the other hand, as \(u \in B_{1,\alpha} \subseteq B_{2,\alpha}\) with \([u]_{B_{2,\alpha}} \leq [u]_{B_{1,\alpha}}\), we have from Lemmas 2.1 and 2.2 that for any \(K \in \mathcal{D}^\beta\),

\[
|B_K|_{u,\alpha} \leq \delta,
\]

where \(0 < \delta < 1\), \(B_K\) is the lower half of \(Q_K\). It follows that if \(K_j\) is a descendant of \(K\) of the \(j\)-th generation, then

\[
|Q_{K_j}|_{u,\alpha} \leq \delta^j.
\]

Let us fix \(z \in Q_I\). Then each \(K\) in the sum \(Q_{\beta, \alpha, I} u(z)\) is a descendant of \(I\) of some generation, and it is the unique dyadic interval of this generation such that \(z \in Q_K\). It follows that

\[
\sum_{K \subseteq I} |Q_K|^{\frac{1}{\alpha}} \left( \frac{1}{|Q_K|^{\alpha}} \int_{Q_K} u dV_{\alpha} \right)^{\frac{1}{q'}} = \sum_{K \subseteq I} |Q_K|^{\frac{1}{\alpha}} \left( \frac{1}{|Q_K|^{\alpha}} \int_{Q_K} u dV_{\alpha} \right)^{\frac{1}{q}} \leq \sum_{k=0}^{\infty} \delta^k |Q_I|^{\frac{1}{q'}}_{u,\alpha} \leq \frac{1}{1 - \delta^q} |Q_I|^{\frac{1}{q'}}_{u,\alpha}.
\]

We observe that

\[
\frac{1}{1 - \delta^q} \leq \left( \frac{1}{1 - \delta} \right)^{\frac{1}{q'}} = C_{\alpha}^{\frac{q}{q'}} [\omega]_{B_{1,q}, \alpha}^{\frac{q}{q'}}.
\]
Hence
\[ Q_{\alpha,\beta} \leq C(q, \alpha)[\omega]_{B_{1,q,\alpha}}^{1+\frac{q}{q'}} \omega(z)|Q_I|_{u,\alpha}^{1/q}. \] (14)

It follows that
\[
L := (2\lambda)^q \sum_{Q_I \in L_1} |Q_I \cap E_{2\lambda}|_{u,\alpha} \\
\leq C(q, \alpha)[\omega]_{B_{1,q,\alpha}}^{q+\frac{q^2}{q'}} \sum_{Q_I \in L_1} \left( \int_{Q_I} f(z) \omega(z) dV_\alpha(z) \right)^{q} \left| Q_I \right|^{1-q+\frac{q}{q'}}_{u,\alpha} \\
= C(q, \alpha)[\omega]_{B_{1,q,\alpha}}^{q^2} \left( \sum_{Q_I \in L_1} \int_{Q_I} f \omega dV_\alpha(z) \right)^{q} \\
\leq C(\alpha)[\omega]_{B_{1,q,\alpha}}^{q^2} \left( \int_{\mathbb{R}^d_+} f \omega dV_\alpha(z) \right)^{q}.
\]

Putting the two estimates together, we obtain
\[
(2\lambda)^q |E_{2\lambda}|_{u,\alpha} \leq \frac{1}{2} \sup_{0<\lambda<\Lambda} \lambda^q |E_\alpha|_{u,\alpha} + C[\omega]_{B_{1,q,\alpha}}^{q^2} \left( \int_{\mathbb{R}^d_+} f \omega dV_\alpha(z) \right)^{q}. \tag{15}
\]

Recall that \( \sup_{0<\lambda<\Lambda} \lambda^q |E_\alpha|_{u,\alpha} < \infty \). Hence, taking the supremum on the left-hand side of the inequality (15), we get
\[
\sup_{0<\lambda<\Lambda} \lambda^q |E_\alpha|_{u,\alpha} \leq C[\omega]_{B_{1,q,\alpha}}^{q^2} \left( \int_{\mathbb{R}^d_+} f \omega dV_\alpha(z) \right)^{q}.
\]

Letting \( \Lambda \to \infty \), we obtain the estimate (6). \( \square \)

We next provide the modifications needed in the above proof to prove Theorem 1.5.

**Proof of Theorem 1.5** Assume that \( \omega \in B_{1,q,\alpha} \). We recall with Lemma 2.5 that for \( f \geq 0 \),
\[
P^+_{\alpha} f \leq C \sum_{\beta \in \{0,1\}} Q_{\alpha}^0 f,
\]
where
\[
Q_{\alpha}^0 f = \sum_{I \in D_{\beta}} \left( f \cdot \frac{1}{|Q_I|_{\alpha}} \right) \chi_{Q_I}.
\]

We also recall with Remark 2.7 that one only needs to show that the estimate (7) holds with \( P^+_{\alpha} \) replaced by \( Q_{\alpha}^0 \). We still assume that \( f \) is supported on some dyadic cube \( Q_J, J \in D_{\beta} \). We recall that \( \eta = (2+\alpha)(q-1) + \alpha \). Put for \( \lambda > 0 \),

\( \square \) Springer
\[ \mathbb{E}_\lambda := \{ z \in \mathbb{R}^2_+ : Q^\beta_\alpha f(z) > \lambda \} . \]

Let \( \Lambda > 0 \) be fixed. Let us check as above that
\[ \sup_{0 < \lambda < \Lambda} \lambda^q |\mathbb{E}_\lambda|_{u, \eta} < \infty, \quad u = \omega^q. \]

Still reasoning as in the previous proof, we obtain that there is a constant \( C > 0 \) such that for \( z \notin Q_I \), \( Q^\beta_\alpha f(z) \leq C \mathcal{M}^d,^\beta_\alpha f(z) \). Putting
\[ \mathbb{F}_\lambda := \left\{ z \in \mathbb{R}^2_+ | Q_I : \mathcal{M}^d,^\beta_\alpha f(z) > \frac{\lambda}{C} \right\}, \]
we obtain
\[ \sup_{0 < \lambda < \Lambda} \lambda^q |\mathbb{E}_\lambda|_{u, \eta} \leq \Lambda^q |Q_I|_{u, \eta} + \sup_{0 < \lambda < \Lambda} \lambda^q |\mathbb{F}_\lambda|_{u, \eta} < \infty, \]
where we have used the fact that \( u \) is locally integrable and that by Lemma 2.3, \( \mathcal{M}^d,^\beta_\alpha \) is bounded from \( L^1(udV_\alpha) \) to \( L^{q,\infty}(udV_\eta) \) since the measure \( d\mu(z) = u(z)dV_\eta(z) \) satisfies the estimate (10). Indeed, for any interval \( I \) and any \( z \in Q_I \),
\[
|Q_I|^{-q} \omega^{-q}(z) \mu(Q_I) = |Q_I|^{-q} \omega^{-q}(z) |Q_I|_{u, \eta} \\
\leq |Q_I|^{-q} \omega^{-q}(z) |Q_I|_{u, \eta}^{q-1} |Q_I|_{u, \alpha} \\
= \frac{|Q_I|_{u, \alpha}}{|Q_I|_{\alpha}^{q-1}} \omega^{-q}(z) \\
\leq [\omega]_{1, q, \alpha}^q.
\]

Now let us decompose \( Q^\beta_\alpha \) as follows:
\[ Q^\beta_\alpha f(z) = Q^\beta_\alpha,^\text{in} f(z) + Q^\beta_\alpha,^\text{out} f(z), \]
where
\[ Q^\beta_\alpha,^\text{in} f(z) = \sum_{\substack{I \in D^\beta_\alpha \cap I \subseteq J}} \left( \frac{1}{|Q_I|_{\alpha}} \int_{Q_I} f dV_\alpha \right) 1_{Q_I}(z) \]
and
\[ Q^\beta_\alpha,^\text{out} f(z) = \sum_{\substack{I \in D^\beta_\alpha \cap I \supseteq J}} \left( \frac{1}{|Q_I|_{\alpha}} \int_{Q_I} f dV_\alpha \right) 1_{Q_I}(z). \]
We also obtain as in the previous proof that for any $z \in Q_J \cap \mathbb{E}_{2\lambda}$,

$$Q_{\alpha,J}^\beta, \text{ in } f(z) > \lambda.$$  

Let us once more fix $\Lambda_1 > 0$. We then also fix $\lambda$ such that $0 < \lambda < \Lambda$. We still denote by $\mathcal{L}$ the family of maximal dyadic Carleson squares whose union is $\mathbb{E}_\lambda$. We also define

$$\mathcal{L}_1 := \{ Q_I \in \mathcal{L} : |Q_I \cap \mathbb{E}_{2\lambda}|_{u,\eta} \geq 2^{-q-1} |Q_I|_{u,\eta} \}$$

and

$$\mathcal{L}_2 := \mathcal{L} \setminus \mathcal{L}_1.$$  

Then

$$(2\lambda)^q |\mathbb{E}_{2\lambda}|_{u,\eta} = (2\lambda)^q \left( \sum_{Q_I \in \mathcal{L}_1} |Q_I \cap \mathbb{E}_{2\lambda}|_{u,\eta} + \sum_{Q_I \in \mathcal{L}_2} |Q_I \cap \mathbb{E}_{2\lambda}|_{u,\eta} \right).$$

We obtain once more that

$$(2\lambda)^q \sum_{Q_I \in \mathcal{L}_2} |Q_I \cap \mathbb{E}_{2\lambda}|_{u,\eta} \leq \frac{1}{2} \sup_{0 < \lambda < \Lambda} \lambda^q |\mathbb{E}_\lambda|_{u,\eta} < \infty.$$  

Now, still following the proof of Theorem 1.4, we obtain

$$L := (2\lambda)^q \sum_{Q_I \in \mathcal{L}_1} |Q_I \cap \mathbb{E}_{2\lambda}|_{u,\eta}$$

$$\leq C(q) \sum_{Q_I \in \mathcal{L}_1} \left( \int_{Q_I} \left( Q_{\alpha,I}^\beta, \text{ in } f(z) \right) u(z) dV_\eta(z) \right)^q |Q_I|_{u,\eta}^{1-q}$$

$$= C(q) \sum_{Q_I \in \mathcal{L}_1} \left( \sum_{K \subseteq I} \left( f, \frac{1}{|Q_K|_\alpha} \right)_{Q_K} \eta \right)^q |Q_I|_{u,\eta}^{1-q}.$$  

Now observe that as $1 - q < 0$ and $T_I \subset Q_I$, we have

$$|Q_I|_{u,\eta}^{1-q} \leq |T_I|_{u,\eta}^{1-q} \leq C_{\alpha,\gamma} |I|^{(2+\alpha)(1-q)(q-1)} |T_I|_{u,\alpha}^{1-q}.$$  

Hence for any $K \subset I$, we obtain

$$|Q_I|_{u,\eta}^{1-q} \leq C_{\alpha,\gamma} |Q_K|^{(1-q)(q-1)} |T_I|_{u,\alpha}^{1-q}.$$  

Note also that

$$|Q_K|_{u,\eta} \leq |Q_K|_{u,\alpha}^{q-1} |Q_K|_{u,\alpha}.$$
It follows from these observations that

\[
L \leq C(q) \sum_{Q_I \in L_1} \left( \sum_{K \subseteq I} |Q_K|^{\frac{1}{q'}} \left| \frac{1}{|Q_K|^{\alpha}} \langle f, \frac{1}{|Q_K|^{\alpha}} f \rangle \right| \right)^q |T_I|^{1-q} |u,\alpha|^{q}.
\]

Hence

\[
L \leq C(q) \sum_{Q_I \in L_1} \left( \int_{Q_I} \left( \mathcal{Q}_{\alpha,\gamma, I}^{\beta, \infty} f(z) \right) u(z) dV_\alpha(z) \right)^q |T_I|^{1-q} |u,\alpha|.
\]

Using (14) and Lemma 2.1, we obtain that

\[
\mathcal{Q}_{\alpha,\gamma, I}^{\beta, \infty} u(z) \leq C(q, \alpha)[\omega]_{B^{q+1} \alpha} \omega(z) |Q_I|^{\frac{1}{q'}} |u,\alpha|^{\frac{1}{q'}}.
\]

Taking this into (16), we conclude that

\[
L := (2\lambda)^q \sum_{Q_I \in L_1} |Q_I \cap \mathbb{R}^2 \alpha| |u,\eta|
\leq C(q, \alpha)[\omega]_{B^{q+1} \alpha} \sum_{Q_I \in L_1} \left( \int_{Q_I} f(z) \omega(z) dV_\alpha(z) \right)^q |T_I|^{\frac{1}{q'}} |u,\alpha|^{\frac{1}{q'}}.
\]

\[
\leq C(q, \alpha)[\omega]_{B^{2q} \alpha} \sum_{Q_I \in L_1} \left( \int_{\mathbb{R}^2} f(z) \omega(z) dV_\alpha(z) \right)^q |T_I|^{\frac{1}{q'}} |u,\alpha|^{\frac{1}{q'}}.
\]

The remainder of the proof then follows as in the last part of the proof of Theorem 1.4.

\[
\square
\]

4 Proofs of Theorems 1.2 and 1.6

We start this section with the proof of Theorem 1.2.

**Proof of Theorem 1.2** We start by considering the sufficiency. We recall that for any \( f \geq 0 \),

\[
T_{\alpha,\gamma} f \lesssim \sum_{\beta \in \{0, \frac{1}{2}\}} \mathcal{Q}_{\alpha,\gamma}^{\beta} f,
\]

\[\square\] Springer
where the dyadic operators $Q_{\alpha, \gamma}^\beta$ are given by (12). Thus the question reduces to proving that

$$Q_{\alpha, \gamma}^\beta : L^p(\omega^p dV_\alpha) \longrightarrow L^q(\omega^q dV_\alpha)$$

is bounded. Let us put $u = \omega^q$ and $\sigma = \omega^{-p'}$. For any $g \in L^q(\omega^{-q'} dV_\alpha)$, we would like to estimate

$$\langle T_{\alpha, \gamma} f, g \rangle_\alpha = \sum_{I \in D^\beta} |Q_I|_{\alpha, \alpha}^{\frac{\gamma}{\gamma + \alpha} - 1} \left( \int_{Q_I} f dV_\alpha \right) \left( \int_{Q_I} g dV_\alpha \right)$$

$$= \sum_{I \in D^\beta} |Q_I|_{\alpha, \alpha}^{\frac{\gamma}{\gamma + \alpha} - 1} \left( \int_{Q_I} (f \sigma^{-1}) dV_\alpha \right) \left( \int_{Q_I} (gu^{-1}) dV_\alpha \right).$$

Put

$$S_{\sigma, \alpha}(f, Q_I) = \frac{1}{|Q_I|_{\sigma, \alpha}} \int_{Q_I} f \sigma dV_\alpha$$

and

$$S_{u, \alpha, \gamma}(g, Q_I) = \frac{1}{|Q_I|_{u, \alpha}^{\frac{1}{\gamma + \alpha}}} \int_{Q_I} g u dV_\alpha.$$  

Then

$$L := \langle T_{\alpha, \gamma} f, g \rangle_\alpha$$

$$= \sum_{I \in D^\beta} |Q_I|_{\alpha, \alpha}^{\frac{\gamma}{\gamma + \alpha} - 1} |Q_I|_{\sigma, \alpha} |Q_I|_{u, \alpha}^{\frac{1}{\gamma + \alpha}} S_{\sigma, \alpha}(f \sigma^{-1}, Q_I) S_{u, \alpha}(gd^{-1}, Q_I).$$

Now observe that if $r = 1 + \frac{q}{p'}$, then $[u]_{B_r, \alpha} = [\omega]_{B_{p, q, \alpha}}^q$ and $[\sigma]_{B_{r, \alpha}} = [\omega]_{B_{p, q, \alpha}}^{p'}$. It follows using Lemma 2.1 that

$$|Q_I|_{\alpha, \alpha}^{\frac{\gamma}{\gamma + \alpha} - 1} |Q_I|_{\sigma, \alpha} |Q_I|_{u, \alpha}^{\frac{1}{\gamma + \alpha}} \leq [\omega]_{B_{p, q, \alpha}} |Q_I|_{\sigma, \alpha}^{\frac{1}{\gamma + \alpha}} |Q_I|_{u, \alpha}^{\frac{1}{\gamma + \alpha}}$$

$$\leq C_{p, \alpha, \gamma} [\omega]_{B_{p, q, \alpha}} [u]_{B_{r, \alpha}}^{\frac{1}{r}} [\sigma]_{B_{r, \alpha}}^{\frac{1}{r}} |T_I|_{\sigma, \alpha}^{\frac{1}{r}} |T_I|_{u, \alpha}^{\frac{1}{r}}$$

$$= C_{p, \alpha, \gamma} [\omega]_{B_{p, q, \alpha}}^{1 + \frac{p'}{p} + \frac{q}{p}} |T_I|_{\sigma, \alpha}^{\frac{1}{r}} |T_I|_{u, \alpha}^{\frac{1}{r}}.$$
Hence

\[ L := \langle T_{\alpha, \gamma} f, g \rangle_\alpha \]

\[ \leq C[\omega]^{1 + \frac{\beta}{p} + \frac{q}{p'}} \sum_{I \in D^\beta} S_{\sigma, \alpha}(f \sigma^{-1}, Q_I) |T_I|_{\sigma, \alpha}^{\frac{1}{p}} S_{u, \alpha}(g u^{-1}, Q_I) |T_I|_{u, \alpha}^{\frac{1}{p'}} \]

\[ \leq C[\omega]^{1 + \frac{\beta}{p} + \frac{q}{p'}} L_1 \times L_2, \]

where

\[ L_1 := \left( \sum_{I \in D^\beta} S_{\sigma, \alpha}(f \sigma^{-1}, Q_I)^p |T_I|_{\sigma, \alpha} \right)^{\frac{1}{p}} \]

and

\[ L_2 := \left( \sum_{I \in D^\beta} S_{u, \alpha}(f u^{-1}, Q_I)^{p'} |T_I|_{u, \alpha} \right)^{\frac{1}{p'}}. \]

We easily obtain with the help of Lemma 2.4 that

\[ L_1 := \left( \sum_{I \in D^\beta} S_{\sigma, \alpha}(f \sigma^{-1}, Q_I)^p |T_I|_{\sigma, \alpha} \right)^{\frac{1}{p}} = \left( \sum_{I \in D^\beta} \int_{T_I} S_{\sigma, \alpha}(f \sigma^{-1}, Q_I)^p \sigma dV_\alpha \right)^{\frac{1}{p}} \]

\[ \leq \left( \int_{R^2_+} (M_{\sigma, \alpha} f \sigma^{-1})(z)^p \sigma(z) dV_\alpha(z) \right)^{\frac{1}{p}} \]

\[ \leq C \left( \int_{R^2_+} (f \sigma^{-1})^p \sigma dV_\alpha \right)^{\frac{1}{p}} = C \left( \int_{R^2_+} (f \omega)^p dV_\alpha \right)^{\frac{1}{p}}. \]

Observing that \( \frac{1}{q'} - \frac{1}{p'} = \frac{\gamma}{\gamma + \alpha} \), we obtain with the help of Lemma 2.4 that

\[ L_2 := \left( \sum_{I \in D^\beta} S_{u, \alpha}(f u^{-1}, Q_I)^{p'} |T_I|_{u, \alpha} \right)^{\frac{1}{p'}} \]

\[ \leq C \left( \int_{R^2_+} (f \omega)^p dV_\alpha \right)^{\frac{1}{p'}}. \]
\[
\sum_{t \in D} \int_{T_I} S_{u,\alpha}(g u^{-1}, Q_I) \rho' u dV_\alpha \left( \int_{R^2_+} (M^d_{u,\alpha,\gamma}(g u^{-1}(z)) \rho' u(z) dV_\alpha(z)) \right)^{1/p'} \leq C \left( \int_{R^2_+} (g u^{-1})^{q'} u dV_\alpha \right)^{1/q'} = C \left( \int_{R^2_+} (g \omega^{-1})^{q'} dV_\alpha \right)^{1/q'}.
\]

Hence,
\[
L := \langle T_{\alpha,\gamma} f, g \rangle_\alpha \leq C \omega^{1+\frac{p'}{p} + \frac{q'}{p'}} \left( \int_{R^2_+} (f \omega)^p dV_\alpha \right)^{1/p} \left( \int_{R^2_+} (g \omega^{-1})^{q'} dV_\alpha \right)^{1/q'}. \]

That is,
\[
\langle T_{\alpha,\gamma} f, g \rangle_\alpha \leq C[\omega]_{B_{p,q,\alpha}}^{1+\frac{p'}{p} + \frac{q'}{p'}} \|f \omega\|_{p,\alpha} \|g \omega^{-1}\|_{q',\alpha}. \]

Hence taking the supremum over all \( g \in L^q'(\omega^{-q'}dV_\alpha) \) with \( \|g \omega^{-1}\|_{q',\alpha} = 1 \), we obtain
\[
\| (T_{\alpha,\gamma} f) \omega \|_{q,\alpha} \leq C[\omega]_{B_{p,q,\alpha}}^{1+\frac{p'}{p} + \frac{q'}{p'}} \|f \omega\|_{p,\alpha}. \]

The proof of the sufficiency is complete.

To prove that the condition \( \omega \in B_{p,q,\alpha} \) is necessary, recall that for any \( f > 0 \),
\[
M_{\alpha,\gamma} f(z) \leq T_{\alpha,\gamma} f(z), \quad \forall z \in R^2_+. \]

Hence the boundedness of \( T_{\alpha,\gamma} \) from \( L^p(\omega^p dV_\alpha) \) to \( L^q(\omega^q dV_\alpha) \) implies the boundedness of the maximal function \( M_{\alpha,\gamma} \) from \( L^p(\omega^p dV_\alpha) \) to \( L^q(\omega^q dV_\alpha) \). In particular, it implies that \( M_{\alpha,\gamma} \) is bounded from \( L^p(\omega^p dV_\alpha) \) to \( L^{q,\infty}(\omega^q dV_\alpha) \), which by Lemma 2.3 implies that \( \omega \in B_{p,q,\alpha} \). The proof is complete. \( \square \)

**Proof of Theorem 1.6** Recall that for any \( f > 0 \),
\[
T_{\alpha,\gamma} f \preceq \sum_{\beta \in [0, \frac{1}{2}]} Q_{\alpha,\gamma}^\beta f, \]

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where $Q_{\alpha, \gamma}^\beta$ is given by (12). The boundedness of $T_{\alpha, \gamma}$ then follows from the boundedness of

$$Q_{\alpha, \gamma}^\beta : L^{p_0}(\omega^{p_0} dV_\alpha) \longrightarrow L^{q_0}(\omega^{q_0} dV_\alpha).$$

We observe that the latter is equivalent to the boundedness of $Q_{\alpha, \gamma}^\beta (\sigma \cdot) : L^{p_0}(\sigma dV_\alpha) \longrightarrow L^{q_0}(\sigma dV_\alpha)$.

For any $0 < f \in L^{p_0}(\sigma dV_\alpha)$ and any $0 < g \in L^{q_0}(\sigma dV_\alpha)$, we only have to estimate the quantity $\langle Q_{\alpha, \gamma}^\beta (\sigma f), gu \rangle_\alpha$.

We have

$$L := \langle Q_{\alpha, \gamma}^\beta (\sigma f), gu \rangle_\alpha$$

$$= \sum_{I \in D^\beta} \left( \frac{1}{|Q_I|^{\frac{1}{2+\alpha-\gamma}}} \right)_{\alpha} \langle g, 1_{Q_I} \rangle_\alpha$$

$$= \sum_{I \in D^\beta} \frac{1}{|Q_I|^{\frac{1}{2+\alpha-\gamma}}} \langle \sigma f, 1_{Q_I} \rangle_\alpha \langle gu, 1_{Q_I} \rangle_\alpha$$

$$= \sum_{I \in D^\beta} \frac{|Q_I|_{u,\alpha}}{|Q_I|_\alpha} \left( \frac{|Q_I|_{\sigma,\alpha}}{|Q_I|_\alpha} \right)^{1-\frac{1}{2+\alpha-\gamma}} \langle \sigma f, \frac{1}{|Q_I|_{\sigma,\alpha}} \rangle_\alpha \langle gu, \frac{1}{|Q_I|_{u,\alpha}} \rangle_\alpha$$

$$\leq [\omega]_{B^{p_0,q_0,\alpha}}^{q_0} \sum_{I \in D^\beta} |Q_I|_\alpha \left( \frac{1}{|Q_I|_{\sigma,\alpha}} \int_{Q_I} f \sigma dV_\alpha \right) \left( \frac{1}{|Q_I|_{u,\alpha}} \int_{Q_I} gu dV_\alpha \right).$$

Observe that

$$u^{1-\frac{1}{2+\alpha-\gamma}} \sigma = 1 = u^{\frac{1}{q_0}} \frac{1}{1-\frac{1}{2+\alpha-\gamma}} \sigma^{\frac{1}{q_0}} = u^{\frac{1}{q_0}} \sigma^{\frac{1}{q_0}}$$

and so

$$|Q_I|_\alpha \sim |T_I|_\alpha = \int_{T_I} \frac{1}{q_0} \sigma^{\frac{1}{q_0}} dV_\alpha \leq |T_I|_{\sigma,\alpha}^{\frac{1}{q_0}} |T_I|_{u,\alpha}^{\frac{1}{q_0}}.$$

It follows using Hölder’s inequality and Lemma 2.4 that

$$L := \langle Q_{\alpha, \gamma}^\beta (\sigma f), gu \rangle_\alpha$$

$$\leq [\omega]_{B^{p_0,q_0,\alpha}}^{q_0} \sum_{I \in D^\beta} |T_I|_{\sigma,\alpha}^{\frac{1}{q_0}} \left( \frac{1}{|Q_I|_{\sigma,\alpha}} \int_{Q_I} f \sigma dV_\alpha \right)$$

$$\times |T_I|_{u,\alpha}^{\frac{1}{q_0}} \left( \frac{1}{|Q_I|_{u,\alpha}} \int_{Q_I} gu dV_\alpha \right).$$
≤ [\omega]_{B_{p_0,q_0,\alpha}}^{q_0} \left( \sum_{I \in \mathcal{D}^\beta} |T_I|_{\sigma,\alpha} \left( \frac{1}{|Q_I|_{\sigma,\alpha}} \int_{Q_I} f \sigma dV_\alpha \right) \right)^{q_0} \frac{1}{q_0} \\
\times \left( \sum_{I \in \mathcal{D}^\beta} |T_I|_{\sigma,\alpha} \left( \frac{1}{|Q_I|_{\sigma,\alpha}} \int_{Q_I} g \sigma dV_\alpha \right) \right)^{q_0} \frac{1}{q_0} \\
≤ [\omega]_{B_{p_0,q_0,\alpha}}^{q_0} \| \mathcal{M}_{d,\beta}^{\sigma,\alpha} f \|_{q_0,\sigma,\alpha} \| \mathcal{M}_{d,\beta}^{u,\alpha,\gamma} g \|_{q_0',u,\alpha} \\
≤ C_{\alpha,\gamma} [\omega]_{B_{p_0,q_0,\alpha}}^{q_0} \| f \|_{p_0,\sigma,\alpha} \| g \|_{q_0',u,\alpha} .

The proof is complete. □

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