Coefficient Bounds for Al-Oboudi Type Bi-univalent Functions based on a Modified Sigmoid Activation Function and Horadam Polynomials

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Abstract
Using the Al-Oboudi type operator, we present and investigate two special families of bi-univalent functions in $D$, an open unit disc, based on $\phi(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$, a modified sigmoid activation function (MSAF) and Horadam polynomials. We estimate the initial coefficients bounds for functions of the type $g_{\phi}(z) = z + \sum_{j=2}^{\infty} \phi(s)d_j z^j$ in these families. Continuing the study on the initial coefficients of these families, we obtain the functional of Fekete-Szeg"{o} for each of the two families. Furthermore, we present few interesting observations of the results investigated.

1 Preliminaries
Let the set of complex numbers be denoted by $\mathbb{C}$ and the set of normalized regular functions in $D = \{z \in \mathbb{C} : |z| < 1\}$ that have the power series of the form
\begin{equation}
    g(z) = z + d_2 z^2 + d_3 z^3 + \ldots = z + \sum_{j=2}^{\infty} d_j z^j,
\end{equation}
be indicated by $A$ and the set of all functions of $A$ that are univalent in $D$ is symbolized by $S$. The famous Koebe theorem (see [12]) ensures that any function
$g \in \mathcal{S}$ has an inverse $g^{-1}$ satisfying $z = g^{-1}(g(z))$, $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$ and $r_0(g) \geq 1/4$, $z, \omega \in \mathfrak{D}$, where

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \ldots \quad (1.2)$$

A function $g$ of $\mathcal{A}$ is said to be bi-univalent (or bi-schlicht) in $\mathfrak{D}$ if $g$ and its inverse $g^{-1}$ are both univalent (or schlicht) in $\mathfrak{D}$. The set of bi-univalent functions having the form (1.1) is indicated by $\Sigma$. Historically investigations of the family $\Sigma$ begun five decades ago by Lewin [23] and Brannan et al. [9]. After few years, Tan [40] found the initial coefficient bounds of bi-univalent functions. Later, Brannan and Taha [10] presented and investigated certain subsets of $\Sigma$ similar to convex and starlike functions of order $\sigma$ ($0 \leq \sigma < 1$) in $\mathfrak{D}$. Some interesting results concerning initial bounds for certain special sets of $\Sigma$ have been appeared in [11], [18] and [32].

Let the set of real numbers be $\mathbb{R} = (-\infty, \infty)$ and the set positive integers be $\mathbb{N} := \mathbb{N}_0 \setminus \{0\} = \{1, 2, 3, \ldots\}$.

Recently, Hörzum and Koçer [21] (see also [20]) examined the Horadam polynomials $H_j(x)$ (or $H_j(x, a, b; p, q)$ ). It is given by the recurrence relation

$$H_j(x) = pxH_{j-1}(x) + qH_{j-2}(x), \quad H_1(x) = a, \quad H_2(x) = bx, \quad (1.3)$$

where $j \in \mathbb{N} \setminus \{1, 2\}$, $x \in \mathbb{R}$, $p$, $q$, $a$ and $b$ are real constants. It is easy to see from (1.3) that $H_3(x) = pbx^2 + qa$. The generating function of the sequence $H_j(x)$, $j \in \mathbb{N}$, is as below (see [21]):

$$G(x, z) := \sum_{j=1}^{\infty} H_j(x)z^{j-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}, \quad (1.4)$$

where $z \in \mathbb{C}$ is independent of the argument $x \in \mathbb{R}$, that is $\Re(z) \neq x$.

Few particular cases of $H_j(x, a, b; p, q)$ are:

- $i)$ $H_j(x, 1, 1; 1, 1) = F_j(x)$,  
- $ii)$ $H_j(x, 1, 2; 2, -1) = U_j(x)$,
- $iii)$ $H_j(x, 1, 1; 2, -1) = T_j(x)$,  
- $iv)$ $H_j(x, 2, 1; 1, 1) = L_j(x)$,
- $v)$ $H_j(x, 2, 2; 2, 1) = Q_j(x)$ and $vi)$ $H_j(x, 1, 2; 2, 1) = P_j(x)$.

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They are named as Fibonacci polynomials, second type Chebyshev polynomials, first type Chebyshev polynomials, Lucas polynomials, Pell-Lucas polynomials and Pell polynomials, respectively.

In the literature, the estimates on $|d_2|, |d_3|$ and the famous inequality of Fekete-Szegő were determined for bi-univalent functions related to certain polynomials like Fibonacci polynomials, $(p,q)$-Lucas polynomials, second kind Chebyshev polynomials and Horadam polynomials. We also note that the above polynomials and other special polynomials are potentially important in statistical, physical, mathematical and engineering sciences. Additional information about these polynomials can be found in [7], [8], [16], [17], [24] and [42]. More details about the famous Fekete-Szegő problem connected with Haradam polynomials are available with the works of [1], [2], [3], [26], [31], [38] and [41].

The recent research trend is the study of bi-univalent functions linked with any one of the above mentioned polynomials using well-known operators, which can be seen in the research papers [4], [13], [25], [28], [34], [36], [37] and [39]. Generally interest was shown to estimate the initial Taylor-Maclaurin coefficients and the celebrated inequality of Fekete-Szegő for the special families of $\Sigma$ that are being introduced using known operators.

In this work, we present two special sets of $\Sigma$ using Al-Oboudi type operator which was precisely defined in the paper [19]. We determine the initial coefficient bounds and also obtain the relevant connection to the celebrated Fekete-Szegő functional for functions in the defined families.

Let $A_1$ denote the set of regular functions of the form

$$g_\phi(z) = z + \sum_{j=2}^{\infty} \phi(s) d_j z^j,$$

where $\phi(s) = \frac{2}{1+2^{-s}}, s \geq 0$, is a MSAF. Note that $\phi(0) = 1$ and hence $A_1 := A$ (see [14]).

**Definition 1.1.** For $g_\phi \in A_1$, $k \in \mathbb{N}_0, \beta \geq 0$, an Al-Oboudi type operator $D^k_{\beta}$ of
\[ A_φ \rightarrow A_φ, \text{ is defined by} \]
\[ D_0^{\beta} g_φ(z) = g_φ(z), \quad D_1^{\beta} g_φ(z) = (1 - \beta)g_φ(z) + \beta z g'_φ(z), \ldots, \quad D_k^{\beta} g_φ(z) = D_\beta(D_k^{\beta-1} g_φ(z)), \quad z \in \mathbb{D}. \]

**Remark 1.1.** If \( g_φ(z) = z + \sum_{j=2}^{\infty} \phi(s)d_j z^j \in A_φ, \ z \in \mathbb{D} \), then
\[ D_k^{\beta} g_φ(z) = z + \sum_{j=2}^{\infty} (1 + (j - 1)\beta)^k \phi(s)d_j z^j, \quad z \in \mathbb{D}. \]

When \( \phi(s) = 1 \), we get the Al-Oboudi operator [5], which reduces to the Sălăgean operator [29], if \( \beta = 1 \).

For regular functions \( g \) and \( f \) in \( \mathbb{D} \), \( g \) is said to subordinate to \( f \), if there is a Schwarz function \( \psi \) in \( \mathbb{D} \), such that \( \psi(0) = 0, \ |\psi(z)| < 1 \) and \( g(z) = f(\psi(z)), \ z \in \mathbb{D} \). This subordination is indicated as \( g < f \) or \( g(z) < f(z) \). Specifically, when \( f \in S \) in \( \mathbb{D} \), then \( g(z) < f(z) \) is equivalent to \( g(0) = f(0) \) and \( g(\mathbb{D}) \subset f(\mathbb{D}) \).

Inspired by the articles [6], [33] and the trends on functions \( \in \sum \), we present two special families of \( \sum \) by using Al-Oboudi type operator, which is as in Definition 1.1 and Horadam polynomials \( H_j(x) \) as in the relation (1.3) having the generating function (1.4).

Throughout this paper, \( f_\phi(\omega) = g_\phi^{-1}(\omega) \) is an extension of \( g^{-1} \) to \( \mathbb{D} \) given by (1.2), \( p, q, a \) and \( b \) are as in (1.3) and \( G \) is as in (1.4), unless and otherwise mentioned.

**Definition 1.2.** A function \( g \) in \( \sum \) having the power series (1.1) is said to be in the family \( S \sum_{\gamma, \mu, k, \beta, \phi(s)}(x, \gamma, \mu, k, \beta, \phi(s)), \ 0 \leq \gamma \leq 1, \mu \geq 0, \ k \in \mathbb{N}_0, \beta \geq 0 \) and \( \phi(s) \) the MSAF, if
\[
\frac{z(D_\beta^k g_\phi(z))' + \mu z^2(D_\beta^k g_\phi(z))''}{\gamma D_\beta^k g_\phi(z) + (1 - \gamma)z} < 1 - a + G(x, z), \quad z \in \mathbb{D}
\]
and
\[
\frac{\omega(D_\beta^k f_\phi(\omega))' + \mu \omega^2(D_\beta^k f_\phi(\omega))''}{\gamma D_\beta^k f_\phi(\omega) + (1 - \gamma)\omega} < 1 - a + G(x, \omega), \quad \omega \in \mathbb{D}.
\]
We note that i) $\mu = 0$, ii) $\gamma = 0$ and iii) $\gamma = 1$ lead the family $S\mathcal{E}_{\Sigma}(x, \gamma, \mu, k, \beta, \phi(s))$ to the below mentioned subfamilies:

1. $SK_{\Sigma}(x, \gamma, k, \beta, \phi(s)) \equiv S\mathcal{E}_{\Sigma}(x, \gamma, k, \beta, \phi(s))$ is the set of functions $g \in \sum$ satisfying

\[
\frac{z(D_k^2g_\phi(z))'}{\gamma D_\beta^k g_\phi(z) + (1 - \gamma)z} < 1 - a + G(x, z), \quad \text{and} \quad \frac{\omega(D_k^2f_\phi(\omega))'}{\gamma D_\beta^k f_\phi(\omega) + (1 - \gamma)\omega} < 1 - a + G(x, \omega),
\]

where $z, \omega \in \mathbb{D}$.

2. $SL_{\Sigma}(x, \mu, k, \beta, \phi(s)) \equiv S\mathcal{E}_{\Sigma}(x, 0, \mu, k, \beta, \phi(s))$ is the family of functions $g \in \sum$ satisfying

\[
(D_k^2g_\phi(z))' + \mu z(D_k^2g_\phi(z))'' < 1 - a + G(x, z)
\]

and

\[
(D_k^2f_\phi(\omega))' + \mu \omega(D_k^2f_\phi(\omega))'' < 1 - a + G(x, \omega),
\]

where $z, \omega \in \mathbb{D}$.

3. $SM_{\Sigma}(x, \mu, k, \beta, \phi(s)) \equiv S\mathcal{E}_{\Sigma}(x, 1, \mu, k, \beta, \phi(s))$ is the family of functions $g \in \sum$ satisfying

\[
\left(\frac{z(D_k^2g_\phi(z))'}{D_\beta^k g_\phi(z)}\right) + \mu \left(\frac{z(D_k^2g_\phi(z))''}{D_\beta^k g_\phi(z)}\right) < 1 - a + G(x, z)
\]

and

\[
\left(\frac{\omega(D_k^2f_\phi(\omega))'}{D_\beta^k f_\phi(\omega)}\right) + \mu \left(\frac{\omega(D_k^2f_\phi(\omega))''}{D_\beta^k f_\phi(\omega)}\right) < 1 - a + G(x, \omega),
\]

where $z, \omega \in \mathbb{D}$.

Letting $k = 0$ and $\phi(s) = 1$ in the Definition 1.2, we obtain the family $SN_{\Sigma}(x, \gamma, \mu) \equiv S\mathcal{E}_{\Sigma}(x, \gamma, \mu, 0, \beta, 1)$ of functions $g \in \sum$ satisfying

\[
\frac{zg'(z) + \mu z^2g''(z)}{\gamma g(z) + (1 - \gamma)z} < 1 - a + G(x, z) \quad \text{and} \quad \frac{\omega f'(\omega) + \mu \omega^2 f''(\omega)}{\gamma f(\omega) + (1 - \gamma)\omega} < 1 - a + G(x, \omega),
\]

where $z, \omega \in \mathbb{D}$, $f(\omega) = g^{-1}(\omega)$ is as given by (1.2), $a$ is as in (1.3) and $G$ is as in (1.4).
Definition 1.3. A function \( g \in \sum \) having the power series \((1.1)\) is said to be in the family \( S\mathfrak{B}_{\sum}(x, \gamma, \tau, k, \beta, \phi(s)) \), \( 0 \leq \gamma \leq 1, \tau \geq 1, k \in \mathbb{N}_0, \beta \geq 0 \) and \( \phi(s) \) the MSAF, if

\[
\frac{z\left[(D^k_\beta g_\phi(z))']\right]}{\gamma D^k_\beta g_\phi(z) + (1 - \gamma)z} < 1 - a + \mathcal{G}(x, z), \quad z \in \mathfrak{D}
\]

and

\[
\frac{\omega[(D^k_\beta f_\phi(\omega))']}{\gamma D^k_\beta f_\phi(\omega) + (1 - \gamma)\omega} < 1 - a + \mathcal{G}(x, \omega), \quad \omega \in \mathfrak{D}.
\]

Note that the certain choices of \( \gamma \) lead the family \( S\mathfrak{B}_{\sum}(x, \gamma, \tau, k, \beta, \phi(s)) \) to the following two subclasses:

1. \( SP_{\sum}(x, \tau, k, \beta, \phi(s)) \equiv \mathfrak{S}_{\sum}(x, 0, \tau, k, \beta, \phi(s)) \) is the set of functions \( g \in \sum \) satisfying

\[
[(D^k_\beta g_\phi(z))'] < 1 - a + \mathcal{G}(x, z), \quad z \in \mathfrak{D} \quad \text{and} \quad [(D^k_\beta f_\phi(\omega))'] < 1 - a + \mathcal{G}(x, \omega), \quad \omega \in \mathfrak{D},
\]

2. \( SM_{\sum}(x, \tau, k, \beta, \phi(s)) \equiv \mathfrak{S}_{\sum}(x, 1, \tau, k, \beta, \phi(s)) \) is the class of functions \( g \in \sum \) satisfying

\[
\frac{z[(D^k_\beta g_\phi(z))']}{D^k_\beta g_\phi(z)} < 1 - a + \mathcal{G}(x, z), \quad z \in \mathfrak{D} \quad \text{and} \quad \frac{\omega[(D^k_\beta f_\phi(\omega))']}{D^k_\beta f_\phi(\omega)} < 1 - a + \mathcal{G}(x, \omega), \quad \omega \in \mathfrak{D},
\]

\( SM_{\sum}(x, \tau, k, \beta, \phi(s)) \) is the family of Al-Oboudi type \( \tau \)-bi-pseudo-starlike functions associated with Horadam polynomials involving the MSAF.

On taking \( k = 0 \) and \( \phi(s) = 1 \) in Definition 1.3, we get the family \( SQ_{\sum}(x, \gamma, \tau) \equiv \mathfrak{S}_{\sum}(x, \gamma, \tau, 0, \beta, 1) \) of functions \( g(z) \in \sum \) satisfying

\[
\frac{z[g'(z)]}{g(z) + (1 - \gamma)z} < 1 - a + \mathcal{G}(x, z) \quad \text{and} \quad \frac{\omega[f'(\omega)]}{f(\omega) + (1 - \gamma)\omega} < 1 - a + \mathcal{G}(x, \omega),
\]

where \( z, \omega \in \mathfrak{D} \), \( f(\omega) = g^{-1}(\omega) \) is as given by \( (1.2) \), \( a \) is as in \( (1.3) \) and \( \mathcal{G} \) is as in \( (1.4) \).

Remark 1.2. We note that i) \( S\mathfrak{B}_{\sum}(x, \gamma, 1, k, \beta, \phi(s)) \equiv \mathfrak{S}_{K_{\sum}}(x, \gamma, k, \beta, \phi(s)) \), ii) \( SM_{\sum}(x, 1, k, \beta, \phi(s)) \equiv \mathfrak{S}_{K_{\sum}}(x, 1, k, \beta, \phi(s)) \equiv SM_{\sum}(x, 0, k, \beta, \phi(s)) \) and iii) \( SP_{\sum}(x, 1, k, \beta, \phi(s)) \equiv \mathfrak{S}_{K_{\sum}}(x, 0, k, \beta, \phi(s)) \equiv SL_{\sum}(x, 0, k, \beta, \phi(s)) \).
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Remark 1.3. i) For $\mu = \gamma = 0$, the class $SN_\Sigma(x, 0, 0) \equiv H_\Sigma(x)$ was studied by Alamoush [2] and ii) For $\mu = 0$ and $\gamma = 1$, the family $SN_\Sigma(x, 1, 0) \equiv S^*_\Sigma(x)$ was investigated by Srivastava et al. [32].

Remark 1.4. i) For $\gamma = 0$, the family $SQ_\Sigma(x, 0, \tau) \equiv S^*_\Sigma(x, \tau)$ was investigated by Abirami et al. [1] and ii) For $\beta = 1$, the family $S_{N\Sigma}(x, \tau, k, 1, \phi(s)) \equiv M_{\Sigma}(x, \tau, k, \phi(s))$ was considered in [25].

Remark 1.5. In a special situation, if we choose $a = 1$, $b = p = 2$, $q = -1$ and $x \to t$, the generating function (1.4) reduces to the second type Chebyshev polynomials $U_j(t)$, which is explicitly given by

$$U_j(t) = (j + 1) \, {}_2F_1\left(-j, j + 2; \frac{3}{2}, \frac{1 - t}{2}\right) = \frac{\sin(j + 1)\psi}{\sin\psi}, \quad (t = \sin\psi)$$

in terms of the Gauss hypergeometric function ${}_2F_1$. In this particular situation, the bi-univalent function families $S_{\Sigma\Sigma}(x, \gamma, \mu, k, \beta, \phi(s))$ and $S_{B\Sigma}(x, \tau, k, \beta, \phi(s))$ would become the families $S_{\Sigma\Sigma}(t, \gamma, \mu, k, \beta, \phi(s))$ and $S_{B\Sigma}(t, \gamma, \tau, k, \beta, \phi(s))$, respectively. The families $S_{B\Sigma}(t, 1, \tau, 0, \beta, \phi(s)) \equiv AO_{\Sigma}(t, \tau, \phi(s))$ and $S_{B\Sigma}(t, 1, \tau, 0, \beta, 1) \equiv AY_{\Sigma}(t, \tau)$ were studied earlier in [8] and [7], respectively.

In Section 2, we derive the estimates for $|d_2|$, $|d_3|$ and the inequality of Fekete-Szegő [15] for functions of the form (1.1) $\in S_{\Sigma\Sigma}(x, \gamma, \mu, k, \beta, \phi(s))$ and we also present some observations of our result. In Section 3, we derive the estimates for $|d_2|$, $|d_4|$ and the Fekete-Szegő inequality for functions of the form (1.1) $\in S_{B\Sigma}(x, \gamma, \tau, k, \beta, \phi(s))$. Few interesting consequences of the result are also considered.

2 Estimates for Function Family $S_{\Sigma\Sigma}(x, \gamma, \mu, k, \beta, \phi(s))$

In the following theorem, we determine the initial coefficients bounds and the inequality of Szegő for functions in $S_{\Sigma\Sigma}(x, \gamma, \mu, k, \beta, \phi(s))$. 

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Theorem 2.1. Let $0 \leq \gamma \leq 1$, $\mu \geq 0$, $k \in \mathbb{N}_0$, $\beta \geq 0$ and $\phi(s)$ be the MSAF. If the function $g \in S(\Sigma)(x, \gamma, \mu, k, \beta, \phi(s))$, then

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1 + \beta)^k \phi(s)\sqrt{|(\gamma^2 - (2\mu + 3)\gamma + 3(2\mu + 1))(bx)^2 - (2(\mu + 1) - \gamma)^2(pb x^2 + q \omega)|}},$$

where

$$|d_3| \leq \frac{1}{(1 + 2\beta)^k \phi(s)}\left[\frac{(bx)^2}{(2(\mu + 1) - \gamma)^2 + \frac{|bx|}{(2(\mu + 1) - 1)}}\right].$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2|$$

$$\leq \begin{cases} \frac{|bx|}{(1 + 2\beta)^k \phi(s)(3(2\mu + 1) - \gamma)} & \left[1 - \frac{(1 + 2\beta)^k \delta}{(1 + 2\beta)^k \phi(s)}\right] \leq J \\ \frac{|bx|}{(1 + 2\beta)^k \phi(s)(\gamma^2 - (2\mu + 3)\gamma + 3(2\mu + 1))(bx)^2 - (2(\mu + 1) - \gamma)^2(pb x^2 + q \omega)} & \left[1 - \frac{(1 + 2\beta)^k \delta}{(1 + 2\beta)^k \phi(s)}\right] \geq J \end{cases}$$

where

$$J = \frac{1}{(3(2\mu + 1) - \gamma)} \left|\gamma^2 - (2\mu + 3)\gamma + 3(2\mu + 1) - (2(\mu + 1) - \gamma)^2 \left(\frac{pb x^2 + q \omega}{b^2 x^2}\right)\right|.$$

Proof. Let $g \in S(\Sigma)(x, \gamma, \mu, k, \beta, \phi(s))$. Then, for two regular functions $\mathfrak{M}$, $\mathfrak{N}$ with $\mathfrak{M}(0) = 0$, $|\mathfrak{M}(z)| < 1$, $\mathfrak{N}(0) = 0$ and $|\mathfrak{N}(\omega)| < 1$, $z, \omega \in \mathfrak{D}$ and on account of Definition 1.2, we can write

$$\frac{z(D^k_{\beta}g_{\phi}(z))' + \frac{\mu z^2}{\gamma D^k_{\beta}g_{\phi}(z)}(D^k_{\beta}g_{\phi}(z))''}{\gamma D^k_{\beta}g_{\phi}(z) + (1 - \gamma)z} = 1 - a + \mathcal{G}(x, \mathfrak{M}(z))$$

and

$$\frac{\omega(D^k_{\beta}f_{\phi}(\omega))' + \frac{\mu \omega^2}{\gamma D^k_{\beta}f_{\phi}(\omega)}(D^k_{\beta}f_{\phi}(\omega))''}{\gamma D^k_{\beta}f_{\phi}(\omega) + (1 - \gamma)\omega} = 1 - a + \mathcal{G}(x, \mathfrak{N}(\omega)).$$

Or, equivalently

$$\frac{z(D^k_{\beta}g_{\phi}(z))' + \frac{\mu z^2}{\gamma D^k_{\beta}g_{\phi}(z)}(D^k_{\beta}g_{\phi}(z))''}{\gamma D^k_{\beta}g_{\phi}(z) + (1 - \gamma)z} = 1 - a + H_1(x) + H_2(x)z + H_3(x)(m(z))^2 + \ldots$$

(2.5)
and
\[
\frac{\omega(D_k^bf_\phi(\omega))' + \mu\omega^2(D_k^bf_\phi(\omega))''}{\gamma D_k^bf_\phi(\omega) + (1 - \gamma)\omega} = 1 - a + H_1(x) + H_2(x)n(\omega) + H_3(x)(n(\omega))^2 + \ldots
\] (2.6)

From (2.5) and (2.6), in view of (1.3), we find
\[
\frac{z(D_k^bg_\phi(z))' + \mu z^2(D_k^bg_\phi(z))''}{\gamma D_k^bg_\phi(z) + (1 - \gamma)z} = 1 + H_2(x)m_1z + [H_2(x)m_2 + H_3(x)m_3]z^2 + \ldots
\] (2.7)

and
\[
\frac{\omega(D_k^bf_\phi(\omega))' + \mu\omega^2(D_k^bf_\phi(\omega))''}{\gamma D_k^bf_\phi(\omega) + (1 - \gamma)\omega} = 1 + H_2(x)n_1\omega + [H_2(x)n_2 + H_3(x)n_3^2]\omega^2 + \ldots
\] (2.8)

It is well known that if \(|\Re(z)| = |m_1z + m_2z^2 + m_3z^3 + \ldots| < 1, z \in \mathcal{D}\) and \(|\Re(\omega)| = |n_1\omega + n_2\omega^2 + n_3\omega^3 + \ldots| < 1, \omega \in \mathcal{D}\), then
\[
|m_i| \leq 1 \text{ and } |n_i| \leq 1 \quad (i \in \mathbb{N}).
\] (2.9)

We easily get the following by equating the corresponding coefficients in (2.7) and (2.8):
\[
(1 + \beta)^k\phi(s)(2(\mu + 1) - \gamma)d_2 = H_2(x)m_1
\] (2.10)
\[
(1 + 2\beta)^k\phi(s)(3(2\mu + 1) - \gamma)d_3 - (1 + \beta)^2\phi^2(s)(2(\mu + 1) - \gamma)\gamma d_2^2 = H_2(x)m_2 + H_3(x)m_2^2
\] (2.11)
\[
- (1 + \beta)^k\phi(s)(2(\mu + 1) - \gamma) d_2 = H_2(x)n_1
\] (2.12)
\[
-(1 + 2\beta)^k\phi(s)(3(2\mu + 1) - \gamma)d_3 + (1 + \beta)^2\phi^2(s)(\gamma^2 - 2(\mu + 2)\gamma + 6(2\mu + 1))d_2^2
\]
\[
= H_2(x)n_2 + H_3(x)n_1^2.
\] (2.13)

From (2.10) and (2.12), we easily obtain
\[
m_1 = -n_1
\] (2.14)

and also
\[
2(1 + \beta)^2\phi^2(s)(2(\mu + 1) - \gamma)^2d_2^2 = (m_1^2 + n_1^2)(H_2(x))^2.
\] (2.15)
If we add \( (2.11) \) and \( (2.13) \), then we obtain
\[
2(1 + \beta)^{2k}\phi^2(s)(\gamma^2 - (2\mu + 3)\gamma + 3(2\mu + 1))d_2^2 = H_2(x)(m_2 + n_2) + H_3(x)(m_1^2 + n_1^2).
\]
(2.16)

Substituting the value of \( m_1^2 + n_1^2 \) from (2.15) in (2.16), we get
\[
d_2^2 = \frac{(H_2(x))^3(m_2 + n_2)}{2(1 + \beta)^{2k}\phi^2(s)[(\gamma^2 - (2\mu + 3)\gamma + 3(2\mu + 1))(h_2(x))^2 - (2(\mu + 1) - \gamma)^2h_3(x)]},
\]
(2.17)

which yields (2.1) on using (2.9).

After subtracting (2.13) from (2.11) and then using (2.14), we obtain
\[
d_3 = \frac{(1 + \beta)^{2k}\phi(s)}{(1 + 2\beta)^k}d_2^2 + \frac{H_2(x)(m_2 - n_2)}{2(1 + 2\beta)^k\phi(s)(3(2\mu + 1) - \gamma)}.
\]
(2.18)

Then in view of (2.15), (2.18) becomes
\[
d_3 = \frac{(H_2(x))^3(m_2 + n_2)}{2(1 + 2\beta)^k\phi(s)(2(\mu + 1) - \gamma)^2} + \frac{H_2(x)(m_2 - n_2)}{2(1 + 2\beta)^k\phi(s)(3(2\mu + 1) - \gamma)},
\]
which yields (2.2) on using (2.9).

From (2.17) and (2.18), for \( \delta \in \mathbb{R} \), we get
\[
|d_3 - \delta d_2^2| = |H_2(x)|\left|\left(T(\delta, x) + \frac{1}{2(1 + 2\beta)^k\phi(s)(3(2\mu + 1) - \gamma)}\right)m_2 + \left(T(\delta, x) - \frac{1}{2(1 + 2\beta)^k\phi(s)(3(2\mu + 1) - \gamma)}\right)n_2\right|,
\]
where
\[
T(\delta, x) = \frac{\left(\frac{(1+\beta)^{2k}\phi(s)}{(1+2\beta)^k} - \delta\right)(H_2(x))^2}{2(1 + \beta)^{2k}\phi^2(s)[(\gamma^2 - (2\mu + 3)\gamma + 3(2\mu + 1))(h_2(x))^2 - (2(\mu + 1) - \gamma)^2h_3(x)]}.
\]

In view of (1.3), we conclude that
\[
|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|H_2(x)|}{(1+2\beta)^k\phi(s)(3(2\mu+1) - \gamma)} : 0 \leq |T(\delta, x)| \leq \frac{1}{2(1+2\beta)^k\phi(s)(3(2\mu+1) - \gamma)} \\ 2|H_2(x)||T(\delta, x)| : |T(\delta, x)| \geq \frac{1}{2(1+2\beta)^k\phi(s)(3(2\mu+1) - \gamma)} \end{cases},
\]
which gets (2.3) with \( J \) as in (2.4). This evidently ends the proof of Theorem 2.1.
By setting i) \( \mu = 0 \), ii) \( \gamma = 0 \), iii) \( \gamma = 1 \) and iv) \( k = 0, \phi(s) = 1 \) in Theorem 2.1, we have the following four corollaries, respectively.

**Corollary 2.1.** If the function \( g \in SK\Sigma(x, \gamma, k, \beta, \phi(s)) \), then

\[
   |d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1 + \beta)^k\phi(s)\sqrt{(3(2\mu + 1)(bx)^2 - 4(\mu + 1)^2(pbxs^2 + qa))}},
\]

\[
   |d_3| \leq \frac{1}{(1 + 2\beta)^k\phi(s)} \left[ \frac{b^2x^2}{4(\mu + 1)^2} + \frac{|bx|}{3(2\mu + 1)} \right],
\]

and for some \( \delta \in \mathbb{R} \),

\[
   |d_3 - \delta d_2^2| \leq \begin{cases} 
   \frac{|bx|}{(1 + 2\beta)^k\phi(s)(3-\gamma)} & ; \left| 1 - \frac{(1 + 2\beta)^k\delta}{(1 + 2\beta)^k\phi(s)} \right| \leq J_1 \\
   \frac{|bx|^3}{(1 + 2\beta)^k\phi(s)(3\gamma - 3)(bx)^2 - (2 - \gamma)^2(pbxs^2 + qa)} & ; \left| 1 - \frac{(1 + 2\beta)^k\delta}{(1 + 2\beta)^k\phi(s)} \right| \geq J_1,
\end{cases}
\]

where \( J_1 = \frac{1}{3-\gamma} \left| \gamma^2 - 3\gamma + 3 - (2 - \gamma)^2 \left( \frac{pbxs^2 + qa}{b^2x^2} \right) \right| \).

**Remark 2.1.** For \( \gamma = \beta = 1 \), Corollary 2.1 reduce to Corollary 2.1 of Magesh et al. 25 and Corollary 2.1 further coincide with Corollary 2.1 of Abirami et al. 1, when \( k = 0 \) and \( \phi(s) = 1 \). Corollary 2.1 coincide with Theorem 2.2 of Alamoush 3, when \( \gamma = k = 0 \) and \( \phi(s) = 1 \) and also we obtain Corollary 1 and Corollary 3 of 31 for \( k = 0, \gamma = \phi(s) = 1 \).

**Corollary 2.2.** If the function \( g \in SL\Sigma(x, \gamma, k, \beta, \phi(s)) \), then

\[
   |d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1 + \beta)^k\phi(s)\sqrt{3(2\mu + 1)(bx)^2 - 4(\mu + 1)^2(pbxs^2 + qa)}},
\]

\[
   |d_3| \leq \frac{1}{(1 + 2\beta)^k\phi(s)} \left[ \frac{b^2x^2}{4(\mu + 1)^2} + \frac{|bx|}{3(2\mu + 1)} \right],
\]

and for \( \delta \in \mathbb{R} \),

\[
   |d_3 - \delta d_2^2| \leq \begin{cases} 
   \frac{|bx|}{3(2\mu + 1)(1 + 2\beta)^k\phi(s)} & ; \left| 1 - \frac{(1 + 2\beta)^k\delta}{(1 + 2\beta)^k\phi(s)} \right| \leq J_2 \\
   \frac{|bx|^3}{3(2\mu + 1)(bx)^2 - 4(\mu + 1)^2(pbxs^2 + qa)} & ; \left| 1 - \frac{(1 + 2\beta)^k\delta}{(1 + 2\beta)^k\phi(s)} \right| \geq J_2,
\end{cases}
\]

where \( J_2 = \left| 1 - \frac{4(\mu + 1)^2}{3(2\mu + 1)} \left( \frac{pbxs^2 + qa}{b^2x^2} \right) \right| \).
Remark 2.2. For $\mu = k = 0$ and $\phi(s) = 1$ Corollary 2.2 coincide with Theorem 2.2 of [3].

Corollary 2.3. If the function $g \in SM_\Sigma(x, \mu, k, \beta, \phi(s))$, then

$$|d_2| \leq \frac{|bx|}{(1 + \beta)^k \phi(s) \sqrt{[(4\mu + 1)(bx)^2 - (2\mu + 1)^2(pb^2x^2 + qa)]}},$$

$$|d_3| \leq \frac{1}{(1 + 2\beta)^k \phi(s)} \left[ \frac{b^2x^2}{(2\mu + 1)^2} + \frac{|bx|}{2(3\mu + 1)} \right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{2(3\mu + 1)} \left| (4\mu + 1) - (2\mu + 1)^2 \left( \frac{pb^2x^2 + qa}{b^2x^2} \right) \right| ; & |1 - \frac{1 + 2\beta \delta}{(1 + \beta)^{2k} \phi(s)}| \leq J_3, \\ \frac{|bx|^3}{(1 + 2\beta)^k \phi(s)(4\mu + 1)(bx)^2 - (2\mu + 1)^2(pb^2x^2 + qa))} \left| 1 - \frac{1 + 2\beta \delta}{(1 + \beta)^{2k} \phi(s)} \right| ; & |1 - \frac{1 + 2\beta \delta}{(1 + \beta)^{2k} \phi(s)}| \geq J_3, \end{cases}$$

where $J_3 = \frac{1}{2(3\mu + 1)} \left| (4\mu + 1) - (2\mu + 1)^2 \left( \frac{pb^2x^2 + qa}{b^2x^2} \right) \right|$.  

Remark 2.3. Corollary 2.3 coincide with Theorem 2.1 of Magesh et al. [26], when $k = 0$ and $\phi(s) = 1$. Also we obtain Corollary 2.1 of [25] from Corollary 2.3, when $\mu = 0$ and $\beta = 1$.

Corollary 2.4. If the function $g(z) \in SN_\Sigma(x, \gamma, \mu)$, then

$$|d_2| \leq \frac{|bx|}{\sqrt{[(\gamma^2 - (2\mu + 3)\gamma + 3(2\mu + 1))(bx)^2 - (2(\mu + 1) - \gamma)^2(pb^2x^2 + qa)]}},$$

$$|d_3| \leq \frac{(bx)^2}{(2(\mu + 1) - \gamma)^2 + (3(2\mu + 1) - \gamma)}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{3(2\mu + 1) - \gamma} ; & |1 - \frac{1}{3(2\mu + 1) - \gamma}| \leq J_4, \\ \frac{|bx|^3}{(\gamma^2 - (2\mu + 3)\gamma + 3(2\mu + 1))(bx)^2 - (2(\mu + 1) - \gamma)^2(pb^2x^2 + qa))} |1 - \frac{1}{3(2\mu + 1) - \gamma}| ; & |1 - \frac{1}{3(2\mu + 1) - \gamma}| \geq J_4, \end{cases}$$

where

$$J_4 = \frac{1}{(3(2\mu + 1) - \gamma)} \left| \gamma^2 - (2\mu + 3)\gamma + 3(2\mu + 1) - (2(\mu + 1) - \gamma)^2 \left( \frac{pb^2x^2 + qa}{b^2x^2} \right) \right|.$$
3 Estimates for the Function Family

Let $g \in S_{\mathcal{B}}(x, \gamma, \tau, k, \beta, \phi(s))$. Then, for some regular functions $M$ and $N$ such that $M(0) = 0, |M(z)| = |m_1z + m_2z^2 + m_3z^3 + ...| < 1, N(0) = 0$ and $|N(\omega)| = |n_1\omega + n_2\omega^2 + n_3\omega^3 + ...| < 1, z, \omega \in \mathcal{D}$ and on account of Definition 1.3, we can write

$$z[(D_{\beta}^{k}g_\phi(z))^\tau] = 1 - a + g(x, M(z)), \ z \in \mathcal{D}$$

and

$$\omega[(D_{\beta}^{k}f_\phi(\omega))^\tau] = 1 - a + g(x, N(\omega)), \ \omega \in \mathcal{D}.$$

Following the procedure similar to the proof of Theorem 2.1, one gets

$$(1 + \beta)^k(2\tau - \gamma)\phi(s)d_2 = H_2(x)m_1$$ (3.4)
(1 + \beta)^2 \phi^2(s)(\gamma^2 - 2\tau \gamma + 2\tau(\tau - 1))d_2^2 + (1 + 2\beta)^2 \phi(s)(3\tau - \gamma)d_3 = H_2(x)m_2 + H_3(x)m_3^2 (3.5)

- (1 + \beta)^2 (2\tau - \gamma)\phi(s)d_2 = H_2(x)n_1 (3.6)

(1 + \beta)^2 \phi^2(s)(\gamma^2 - 2(\tau + 1)\gamma + 2\tau(\tau + 2))d_2^2 - (1 + 2\beta)^2 \phi(s)(3\tau - \gamma)d_3 = H_2(x)n_2 + H_3(x)n_3^2. (3.7)

The results (3.1)-(3.3) now follow from (3.4)-(3.7) by adopting the procedure as in Theorem 2.1.

By setting i) \gamma = 0, ii) \gamma = 1 and iii) \kappa = 0, \phi(s) = 1 in Theorem 3.1, we have the following three corollaries.

Corollary 3.1. If the function \( g \in SP\sum_{x, \tau, \kappa, \beta, \phi(s)} \), then

\[ |d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1 + \beta)^k \phi(s)\sqrt{\tau(2\tau + 1)(bx)^2 - 4\tau^2(pb2x^2 + qa)}}; \]

\[ |d_3| \leq \frac{1}{(1 + 2\beta)^k \phi(s)} \left( \frac{(bx)^2}{4\tau^2} + \frac{|bx|}{3\tau} \right) \]

and for \( \delta \in \mathbb{R} \),

\[ |d_3 - \delta d_2| \leq \begin{cases} \frac{|bx|}{3\tau(1 + 2\beta)^k \phi(s)} & ; 1 - \frac{(1 + 2\beta)^k \delta}{(1 + \beta)^2 \phi(s)} \leq \Omega_1, \\ \frac{(1 + 2\beta)^k \phi(s)\tau(2\tau + 1)}{(1 + 2\beta)^k \phi(s)\tau(2\tau + 1)(bx)^2 - 4\tau^2(pb2x^2 + qa)} |bx|^3 & ; 1 - \frac{(1 + 2\beta)^k \delta}{(1 + \beta)^2 \phi(s)} \geq \Omega_1, \end{cases} \]

where \( \Omega_1 = \frac{1}{3} |(2\tau + 1) - 4\tau (\frac{pb2x^2 + qa}{b^2x^2})| \).

Remark 3.1. Corollary 3.1 coincides with Theorem 2.1 of [3], when \( k = 0 \) and \( \tau = \phi(s) = 1 \).

Corollary 3.2. If the function \( g \in S\Re\sum_{x, \tau, \kappa, \beta, \phi(s)} \), then

\[ |d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1 + \beta)^k \phi(s)\sqrt{((\tau(2\tau - 1))(bx)^2 - (2\tau - 1)^2(pb2x^2 + qa)}}; \]

\[ |d_3| \leq \frac{1}{(1 + 2\beta)^k \phi(s)} \left[ \frac{(bx)^2}{(2\tau - 1)^2} + \frac{|bx|}{(3\tau - 1)} \right]\]
and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{(1+2\beta)^h \phi(s)(3\tau-1)} \left| 1 - \frac{(1+2\beta)^h \delta}{(1+2\beta)^{2h} \phi(s)} \right| |bx|^3 \quad ; \quad 1 - \frac{(1+2\beta)^h \delta}{(1+2\beta)^{2h} \phi(s)} \leq \Omega_2 \\ \frac{|b(x)|}{(1+2\beta)^h \phi(s)(\tau(2\tau-1))(bx)^2 - (2\tau-1)^2(pb^2x^2 + qa)} \left| 1 - \frac{(1+2\beta)^h \delta}{(1+2\beta)^{2h} \phi(s)} \right| \geq \Omega_2, \end{cases}$$

where $\Omega_2 = \frac{1}{(3\tau-1)} \left| \tau(2\tau - 1) - (2\tau - 1)^2 \left( \frac{pb^2x^2 + qa}{b^2x} \right) \right|.$

**Remark 3.2.** Corollary 3.2 reduces to Theorem 2.1 of [25], when $\beta = 1$ and also the results of Corollary 3.2 coincide with Theorem 2.1 of Abirami et al. [1], when $k = 0$ and $\phi(s) = 1$.

**Corollary 3.3.** If the function $g \in SQ_{\sum}(x, \gamma, \tau)$, then

$$|d_2| \leq \frac{|bx|}{\sqrt{|bx|}} \left| \frac{\gamma^2}{\gamma^2 + (\tau - \gamma)(2\tau + 1)}(bx)^2 - (2\tau - \gamma)^2(pb^2x^2 + qa) \right|,$$

$$|d_3| \leq \frac{(bx)^2}{(2\tau - \gamma)^2} + \frac{|bx|}{(3\tau - \gamma)}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{3\tau - \gamma} \left( 1 - \delta \right) \left| \frac{|bx|^3}{(\gamma^2 + (\tau - \gamma)(2\tau + 1))(bx)^2 - (2\tau - \gamma)^2(pb^2x^2 + qa)} \right| \quad ; \quad \left| 1 - \delta \right| \leq \Omega_3 \\ \frac{|b(x)|}{(1+2\beta)^h \phi(s)(3\tau-1)} \left| 1 - \frac{(1+2\beta)^h \delta}{(1+2\beta)^{2h} \phi(s)} \right| |bx|^3 \quad ; \quad \left| 1 - \delta \right| \geq \Omega_3, \end{cases}$$

where $\Omega_3 = \frac{1}{(3\tau - \gamma)} \left| \frac{(\gamma^2 + (\tau - \gamma)(2\tau + 1)) - (2\tau - \gamma)^2}{b^2x^2} \left( \frac{pb^2x^2 + qa}{b^2x^2} \right) \right|.$

**Remark 3.3.** Corollary 3.3 reduces to Theorem 2.1 of [1], when $\gamma = 1$.

### 4 Conclusion

Two special families of holomorphic and bi-univalent (or bi-schlicht) functions are introduced by using Al-Oboudi type operator involving a modified sigmoid activation function associated with Horadam polynomials. Bounds of the first
two coefficients $|d_2|$, $|d_3|$ and the celebrated Fekete-Szegő functional have been fixed for each of the two families. Through corollaries of our main results, we have highlighted many interesting new consequences.

The special families examined in this research paper using Al-Oboudi type operator could inspire further research related to other aspects such as families using $q$-derivative operator [22], [35], meromorphic bi-univalent function families associated with Al-Oboudi differential operator [30] and families using integro-differential operators [27].

References

[1] C. Abirami, N. Magesh, J. Yamini and N. B. Gatti, Horadam polynomial coefficient estimates for the classes of $\lambda$-bi-pseudo-starlike and bi-Bazilevic functions, J. Anal. 28 (2020), 951-960. https://doi.org/10.1007/s41478-020-00224-2

[2] A. G. Alamouch, Certain subclasses of bi-univalent functions involving the Poisson distribution associated with Haradam polynomials, Malaya Journal of Matematik 7(4) (2019), 618-624. https://doi.org/10.26637/MJM0704/0003

[3] A. G. Alamouch, Coefficient estimates for certain subclass of bi-univalent functions associated the Horadam polynomials, arXiv:1812.10589 [math.CV] 22 Dec 2018, 7 pp.

[4] I. Aldawish, T. Al-Hawary and B. A. Frasin, Subclasses of bi-univalent functions defined by Frasin differential operator, Mathematics 8 (2020), 783. https://doi.org/10.3390/math8050783

[5] F. M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, Int. J. Math. Sci. 2004 (2004), Article ID 172525, 8 pp. https://doi.org/10.1155/S0161171204108090

[6] Ş. Altınkaya and S. R. Swamy, Fekete-Szegő functional for regular functions based on quasi-subordination, Inter. J. Nonlinear Anal. Appl. 13(1) (2022), 8 pp. (in press).

[7] Ş. Altınkaya and S. Yalçın, On the Chebyshev polynomial coefficient problem of some subclasses of bi-univalent functions, Gulf J. Math. 5(3) (2017), 34-40.

http://www.earthlinepublishers.com
[8] I. T. Awolere and A. T. Oladipo, Coefficients of bi-univalent functions involving pseudo-starlikeness associated with Chebyshev polynomials, *Khayyam J. Math.* 5(1) (2019), 140-149.

[9] D. A. Brannan and J. G. Clunie, Aspects of contemporary complex analysis, Proceedings of the NATO Advanced Study Institute held at University of Durhary, New York: Academic Press, 1979.

[10] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, *Studia Univ. Babeș-Bolyai Math.* 31(2) (1986), 70-77.

[11] S. Bulut, Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions, *C. R. Acad. Sci. Paris Sér I* 352 (2014), 479-484. [https://doi.org/10.1016/j.crma.2014.04.004](https://doi.org/10.1016/j.crma.2014.04.004)

[12] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, 1983.

[13] S. M. El-Deeb, T. Bulboaca and B. M. El-Matary, Maclaurin coefficient estimates of bi-univalent functions connected with the q-derivative, *Mathematics* 8 (2020), 418. [https://doi.org/10.3390/math8030418](https://doi.org/10.3390/math8030418)

[14] O. A. Fadipe-Joseph, B. B. Kadir, S. E. Akinwumi and E. O., Adeniran, Polynomial bounds for a class of univalent function involving Sigmoid function, *Khayyam J. Math.* 4(1) (2018), 88-101.

[15] M. Fekete and G. Szegő, Eine Bemerkung Über Ungerade Schlichte Funktionen, *J. Lond. Math. Soc.* s1-8 (1933), 85-89. [https://doi.org/10.1112/jlms/s1-8.2.85](https://doi.org/10.1112/jlms/s1-8.2.85)

[16] P. Filipponi and A. F. Horadam, Derivative sequences of Fibonacci and Lucas polynomials, in: G. E. Bergum, A. N. Philippou, A. F. Horadam, eds., Applications of Fibonacci Numbers, Springer, Dordrecht, 1991, pp. 99-108. [https://doi.org/10.1007/978-94-011-3586-3_12](https://doi.org/10.1007/978-94-011-3586-3_12)

[17] P. Filipponi and A. F. Horadam, Second derivative sequence of Fibonacci and Lucas polynomials, *Fibonacci Quart.* 31 (1993), 194-204.

[18] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.* 24 (2011), 1569-1573. [https://doi.org/10.1016/j.aml.2011.03.048](https://doi.org/10.1016/j.aml.2011.03.048)
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[19] B. A. Frasin, S. R. Swamy and J. Nirmala, Some special families of holomorphic and Al-Oboudi type bi-univalent functions related to $k$-Fibonacci numbers involving modified Sigmoid activation function, *Afr. Mat.* 32(3-4) (2021), 631-643. [https://doi.org/10.1007/s13370-020-00850-w](https://doi.org/10.1007/s13370-020-00850-w)

[20] A. F. Horadam and J. M. Mahon, Pell and Pell-Lucas polynomials, *Fibonacci Quart.* 23 (1985), 7-20.

[21] T. Hörçüm and E. Gökçen Koçer, On some properties of Horadam polynomials, *Int. Math. Forum.* 4 (2009), 1243-1252.

[22] B. Khan, H. M. Srivastava, M. Tahir, M. Darus, Q. Z. Ahmed and N. Khan, Applications of a certain $q$-integral operator to the subclasses of analytic and bi-univalent functions, *AIMS Mathematics* 6(1) (2020), 1024-1039.

[23] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.* 18 (1967), 63-68. [https://doi.org/10.1090/S0002-9939-1967-0206255-1](https://doi.org/10.1090/S0002-9939-1967-0206255-1)

[24] N. Magesh and S. Bulut, Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions, *Afr. Mat.* 29(1-2) (2018), 203-209. [https://doi.org/10.1007/s13370-017-0535-3](https://doi.org/10.1007/s13370-017-0535-3)

[25] N. Magesh, P. K. Mamatha, S. R. Swamy and J. Yamini, Horadam polynomial coefficient estimates for a class of $\lambda$-bi-pseudo-starlike functions, Presented in National Conference on Recent Trends in Mathematics and its Applications (NCRTMA-2019), GITAM (Deemed to be university), December 20-21, 2019, 7 pp.

[26] N. Magesh, J. Yamini and C. Abhirami, Initial bounds for certain classes of bi-univalent functions defined by Horadam polynomials, arXiv: 1812.04464vi [math.cv] 11 Dec 2018, 14 pp.

[27] A. O. Páll-Szabó and G. I. Oros, Coefficient related studies for new classes of bi-univalent functions, *Mathematics* 8 (2020), 1110. [https://doi.org/10.3390/math8071110](https://doi.org/10.3390/math8071110)

[28] F. M. Sakar and M. A. Aydoğan, Initial bounds for certain subclasses of generalized Sălăgean type bi-univalent functions associated with the Horadam polynomials, *Journal of Quality Measurement and Analysis* 15(1) (2019), 89-100.

http://www.earthlinepublishers.com
[29] G. S. Sălăgean, Subclasses of univalent functions, Lecture Notes in Mathematics, vol. 1013, Springer, Berlin, 1983, pp. 362-372. \[https://doi.org/10.1007/BFb0066543\]

[30] T. G. Shaba, M. G. Khan and B. Ahmed, Coefficient bounds for certain subclasses of meromorphic bi-univalent functions associated with Al-Oboudi differential operator, \textit{Palestine J. Math.} 9(2) (2020), 11 pp.

[31] H. M. Srivastava, Ş. Altunkaya and S. Yalçın, Certain subclasses of bi-univalent functions associated with the Horadam polynomials, \textit{Iran. J. Sci. Technol. Trans. Sci.} 43 (2019), 1873-1879. \[https://doi.org/10.1007/s40995-018-0647-0\]

[32] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, \textit{Appl. Math. Lett.} 23 (2010), 1188-1192. \[https://doi.org/10.1016/j.aml.2010.05.009\]

[33] S. R. Swamy, Ruscheweyh derivative and a new generalized multiplier differential operator, \textit{Annals of Pure and Applied Mathematics} 10(2) (2015), 229-238.

[34] S. R. Swamy, S. Bulut and Y. Sailaja, Some special families of holomorphic and Sălăgean type bi-univalent functions associated with Horadam polynomials involving a modified sigmoid activation function, \textit{Hacet. J. Math. Stat.} 50(3) (2021), 710-720. \[https://doi.org/10.15672/hujms.695858\]

[35] S. R. Swamy and P. K. Mamatha, Certain classes of bi-univalent functions associated with \(q\)-analogue of Bessel functions, \textit{South East Asian J. Math. Math. Sci.} 16(3) (2020), 61-82.

[36] S. R. Swamy, P. K. Mamatha, N. Magesh and J. Yamini, Certain subclasses of bi-univalent functions defined by Sălăgean operator associated with the \((p, q)\)-Lucas polynomials, \textit{Adv. Math. Sci. J.} 9(8) (2020), 6017-6025. \[https://doi.org/10.37418/amsj.9.8.70\]

[37] S. R. Swamy, J. Nirmala and Y. Sailaja, Some special families of holomorphic and Al-Oboudi type bi-univalent functions associated with \((m, n)\)-Lucas polynomials involving modified sigmoid activation function, \textit{South East Asian J. Math. Math. Sci.} 17(1) (2021), 1-16. \[https://doi.org/10.15672/hujms.695858\]

[38] S. R. Swamy and Y. Sailaja, Horadam polynomial coefficient estimates for two families of holomorphic and bi-univalent functions, \textit{Inter. J. Math. Trends and Tech.} 66(8) (2020), 131-138. \[https://doi.org/10.14445/22315373/IJMTT-V66I8P514\]
[39] S. R. Swamy, A. K. Wanas and Y. Sailaja, Some special families of holomorphic and Sălăgean type bi-univalent functions associated with \((m, n)\)-Lucas polynomials, *Communications in Mathematics and Applications* 11(4) (2020), 563-574. 
\[ http://dx.doi.org/10.26713/cma.v11i4.1411 \]

[40] D. L. Tan, Coefficient estimates for bi-univalent functions, *Chinese Ann. Math. Ser. A* 5 (1984), 559-568.

[41] A. K. Wanas and A. A. Lupas, Applications of Horadam polynomials on Bazilevic bi-univalent function satisfying subordinate conditions, *J. Phys.: Conf. Ser.* 1294 (2019) 032003. \[ https://doi.org/10.1088/1742-6596/1294/3/032003 \]

[42] T.-T. Wang and W.-P. Zhang, Some identities involving Fibonacci, Lucas polynomials and their applications, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* 55(103) (2012), 95-103.

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