A NOTE ON RIGIDITY FOR CROSSED PRODUCT VON NEUMANN ALGEBRAS

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Abstract. In this note, we will point out, as a corollary of Popa’s rigidity theory, that the crossed product von Neumann algebras for Bernoulli shifts cannot have relative property T. This is an operator algebra analogue of the theorem shown by Neuhauser and Cherix-Martín-Valette for discrete groups. Our proof is different from that for groups.

1. Introduction

Nowadays, the notion of relative property T plays one of the central roles for operator algebras (See [14] for information). The most cerebrated results in this area is the rigidity theory for malleable actions, established by Popa [12, 13]. For example, Popa solved Connes’ problem for a certain class of discrete groups by using his rigidity theory. In this note, we would like to point out that as a corollary of Popa’s rigidity theory, one can show the operator algebra analogue of the theorem shown by Neuhauser and Cherix-Martín-Valette for discrete groups.

Recently Neuhauser and Cherix-Martín-Valette independently showed the following theorem: (See [1, 11, 9] for the proof.)

Theorem 1.1 (Neuhauser, Cherix-Martín-Valette). Let $G$, $H$ be countable discrete groups with $H \neq \{e\}$. If $G$ is an infinite group, then the inclusion $\oplus_G H \subset (\oplus_G H) \rtimes G$ does not have relative property T.

Then it is natural to consider the operator algebra analogue of this result. The main result of this paper is:

Theorem 1.2. Let $A(\neq \mathbb{C})$ be a finite von Neumann algebra with a distinguished faithful normal tracial state $\tau$. Let $G$ be a countable discrete, infinite group. Consider the finite von Neumann algebra $Q = \otimes_G A$ with a trace $\otimes_G \tau$. Let $\sigma$ be the Bernoulli shift action of $G$ on $Q$. Then the inclusion $Q \subset P = Q \rtimes_\sigma G$ does not have relative property T in the sense of [14].

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Moreover, if $A$ is type I, for any diffuse von Neumann subalgebra $Q_0 \subset Q$, the inclusion $Q_0 \subset P = Q \rtimes \sigma G$ does not have relative property T.

Here we remark that if $A$ itself is a property T II$_1$ factor for example, then obviously the inclusion $A \subset P$ has relative property T. This means that in the above theorem, we cannot omit the condition that $A$ is type I.

The proof for the group case is elegant and understandable [1][4]. However it is difficult (at least for the author) to arrange it for the operator algebra setting. Thus we shall adopt a slightly different method. Our main tool is Popa’s rigidity theory, as noted above.

2. Proof

Before starting the proof, we shall fix some notations. For a finite von Neumann algebra $M$ with a faithful normal tracial state $\tau$, we denote its 2-norm by $||x||_2 = \tau(x^*x)^{1/2}$ for $x \in M$. The norm-unit ball of $M$ is denoted by $(M)_1 = \{x \in M; ||x|| \leq 1\}$. For a von Neumann subalgebra $N \subset M$, we denote by $E_N$ the trace-preserving conditional expectation onto $N$. The center of $M$ is denoted by $Z(M)$. For a discrete group $G$, we denote its group von Neumann algebra by $L(G)$. If $G$ acts on $M$, then $L(G)$ can be considered as a subalgebra of the crossed product von Neumann algebra $M \rtimes G$ in the natural way. We denote the canonical implementing unitary in $L(G)$ by $\lambda_g$ for any $g \in G$.

Let us now start the proof of theorem 1.2.

First we would like to give the proof that $Q \subset P$ does not have relative property T. This can be very much easily shown and the proof might be a folklore for specialists. I learned the following proof from the referee. I would like to thank him.

For any positive number $r > 0$, let $\phi_r(\cdot) = r\tau(\cdot) + (1-r)\text{id}$ and let $\Phi_r(\cdot) = \otimes_G \phi_r(\cdot)$. Then it is easy to see that this unital, normal, trace-preserving completely positive map on $Q$ can be extended to $P$ by $\Phi_r(x\lambda_g) = \phi_r(x)\lambda_g$ for any $x \in Q$ and $g \in G$. We use the same notation $\Phi_r$ for this extended map. If the inclusion $Q \subset P$ is rigid, then $\Phi_r$ must converge to identity uniformly on $(Q)_1$ with respect to the trace 2-norm $||\cdot||_2$ as $r \to 0$. But this is impossible. Indeed, Let $u \in A$ be some non-scalar unitary element so that $|\tau(u)| < 1$. (Since $A$ is not scalar, we can always find such a unitary.) Then $\|\phi_r(u)\|_2 = \|r\tau(u) + (1-r)u\|_2 \leq r|\tau(u)| + 1 - r$. For any finite subset $F \subset G$, define $u_F = \otimes_F u \in Q$. Then we have $\|\Phi_r(u_F)\|_2 \leq (r|\tau(u)| + 1 - r)|F| \leq 1$. Since $\lim_{r \to 0} \|\Phi_r(u_F)\|_2 = \|u_F\|_2 = 1$ uniformly for the choice of $F$, we see that $\lim_{r \to 0} (r|\tau(u)| + 1 - r)|F| = 1$ uniformly for the choice of $F$. This is obviously impossible.
Next we shall consider the case that $A$ is type I. Let $Q_0$ be a diffuse von Neumann subalgebra of $Q = \otimes G A$. We show that $Q_0 \subset P$ does not have relative property T. The following theorem plays a crucial role in our argument:

**Theorem 2.1** (Popa [12], special case (uniqueness of group algebra)). Let $C = A \otimes B$ where $A$ is a type I von Neumann algebra with a faithful normal tracial state $\tau_1$ and $B$ is diffuse abelian with faithful normal tracial state $\tau_2$. Let $G$ be a countable infinite group. ($G$ may be non-property T!) Consider the Bernoulli shift action $\sigma$ of $G$ on $N = \otimes G C$, where the von Neumann algebra $N = \otimes G C$ is a completion with respect to $\tau = \otimes G (\tau_1 \otimes \tau_2)$. Let $P \subset N \rtimes G$ be an irreducible $II_1$-subfactor. Under these assumptions, if there exists a diffuse von Neumann subalgebras $Q \subset P$ such that this inclusion is rigid, (for example, if $P$ has property $T$, we can take $Q = P$,) then there exists a non-zero partial isometry $v \in N \rtimes G$ such that $vQv^* \subset L(G)$.

We shall give the proof of theorem 1.2. in the case that $A$ is type I.

Let $A$, $G$, $Q$, $Q_0$, $P$, $\sigma$ be as in theorem 1.2 and we assume $A$ is type I. Let $B = L^\infty[0,1]$ and $C = A \otimes B$. We identify $P$ with $(\otimes G A \otimes 1) \rtimes G \subset M = \otimes G C \rtimes G$. Then by assumption, $P$ is an irreducible $II_1$-subfactor of $M$. (Irreducibility follows from $L(G)' \cap M = Z(L(G)) \subset P$.) Suppose that $Q_0 \subset P$ has relative property T. Since $Q_0$ is diffuse, by Popa’s theorem we can find a non-zero partial isometry $v \in M$ such that $vQ_0v^* \subset L(G)$. But this is obviously impossible. Indeed it is easily seen, for example, by using Popa’s orthogonal pair technique [11] as follows: Since $Q_0 \subset \otimes G A$ is diffuse, for any $\epsilon > 0$, we can take a family of orthogonal projections $\{e_i\}_{i=1}^n \subset Q_0$ such that $\sum_i e_i = 1$ and $\tau(e_i) < \epsilon$. Then, for any $x \in \otimes G C$ and $g \in G$ we see that

\[
|\tau(v^* x \lambda_g)|^2 = |\tau(\sum_i e_i v^* x \lambda_g e_i)|^2 \\
\leq \|\sum_i e_i v^* x \lambda_g e_i\|^2_2 \\
= \sum_i \|e_i v^* x \lambda_g e_i\|^2_2 \\
= \sum_i \tau(ve_i v^* x \sigma_g(e_i) x^*) \\
= \sum_i \tau(ve_i v^*) \tau(x \sigma_g(e_i) x^*) \\
\leq \epsilon \|x\|_2.
\]
Here we use the fact that $\nu e_i v^* \in L(G)$ and $x\sigma_g(e_i)x^* \in \otimes G C$. Since $\epsilon$ is arbitrary, $v$ must be 0. This is a contradiction and hence the inclusion $Q_0 \subset P$ does not have relative property T.

Remark 2.1. The proof in the group case (Theorem 1.1) is done by constructing the non-vanishing cocycle of $G$ to some Hilbert space explicitly. It is difficult (for the author) to modify this argument for the von Neumann algebra case.

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