Parabolic equations with the second-order Cauchy conditions on the boundary

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Received 5 July 2007, in final form 6 August 2007
Published 25 September 2007
Online at stacks.iop.org/JPhysA/40/12409

Abstract
The paper studies some ill-posed boundary value problems on semi-plane for parabolic equations with the homogenous Cauchy condition at initial time and with the second-order Cauchy condition on the boundary of the semi-plane. A class of inputs that allows some regularity is suggested and described explicitly in the frequency domain. This class is everywhere dense in the space of square integrable functions.

PACS numbers: 02.60.Lj, 02.30.Jr, 02.30.Fn, 02.30.Tb
Mathematics Subject Classification: 35K20, 35Q99, 32A35

Parabolic equations such as heat equations have fundamental significance for natural sciences, and various boundary value problems for them were widely studied including well-posed problems as well as the so-called ill-posed problems that are often significant for applications. The present paper introduces and investigates a special boundary value problem on semi-plane for parabolic equations with the homogenous Cauchy condition at initial time and with the second-order Cauchy condition on the boundary of the semi-plane. The problem is ill posed. A set of solvability or a class of inputs that allows some regularity in the form of prior energy-type estimates is suggested and described explicitly in the frequency domain. This class is everywhere dense in the class of $L^2$-integrable functions. This result looks counterintuitive, since these boundary conditions are unusual; solvability of this boundary value problem for a wider class of inputs is inconsistent with basic theory.

1. The problem setting
Let us consider the following boundary value problem:

\[
\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + b \frac{\partial u}{\partial x}(x, t) + cu(x, t) + f(x, t),
\]

\[
u(x, 0) \equiv 0,
\]

\[
u(0, t) \equiv g_0(t), \quad \frac{\partial u}{\partial x}(0, t) \equiv g_1(t).
\]

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Here, \( x > 0, t > 0 \) and \( a > 0; b, c \in \mathbb{R} \) are constants; \( g_k \in L_2(0, +\infty) \), \( k = 1, 2 \); and \( f \) is a measurable function such that \( \int_0^\infty \int_0^\infty |f(x, t)|^2 \, dt < +\infty \) for all \( y > 0 \).

This problem is ill posed (see Tikhonov and Arsenin (1977)).

Let \( \mu \doteq b^2/4 - c \). We assume that \( \mu > 0 \). Note that this assumption does not reduce generality for the cases when we are interested in a solution on a finite time interval, since we can rewrite the parabolic equation as that with \( c \) replaced by \( c - M \) for any \( M > 0 \) and \( g_k(t) \) replaced by \( e^{-Mt}g_k(t) \); the solution \( u_M \) of the new equation is related to the solution \( u \) of the old one as \( u_M(x, t) = e^{-Mt}u(x, t) \).

Definitions and special functions

Let \( R^* \doteq [0, +\infty) \), \( C^* \doteq \{ z \in C : \text{Re} z > 0 \} \). For \( v \in L_2(\mathbb{R}) \), we denote by \( \mathcal{F}v \) and \( \mathcal{L}v \) the Fourier and the Laplace transforms, respectively:

\[
V(i\omega) = (\mathcal{F}v)(i\omega) \doteq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega t} v(t) \, dt, \quad \omega \in \mathbb{R},
\]

\[
V(p) = (\mathcal{L}v)(p) \doteq \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-pt} v(t) \, dt, \quad p \in C^*.
\]

Let \( H^r \) be the Hardy space of holomorphic on \( C^* \) functions \( h(p) \) with finite norm \( \|h\|_{H^r} = \sup_{s>0} \|h(k+i\omega)\|_{L_2(\mathbb{R})} \), \( r \in [1, +\infty] \) (see, e.g., Duren (1970)).

For \( y > 0 \), let \( \mathcal{W}(y) \) be the Banach space of the functions \( u : (0, y) \times \mathbb{R}^* \to \mathbb{R} \) with the finite norm

\[
\|u\|_{\mathcal{W}(y)} \doteq \sup_{x \in (0, y)} \left( \|u(x, \cdot)\|_{L_2(\mathbb{R}^*)} + \left\| \frac{\partial u}{\partial x} (x, \cdot) \right\|_{L_2(\mathbb{R}^*)} + \left\| \frac{\partial^2 u}{\partial x^2} (x, \cdot) \right\|_{L_2(\mathbb{R}^*)} + \left\| \frac{\partial u}{\partial t} (x, \cdot) \right\|_{L_2(\mathbb{R}^*)} \right).
\]

The class \( \mathcal{W}(y) \) is such that all the equations presented in problem (1) are well defined for any \( u \in \mathcal{W}(y) \) and in the domain \( (0, y) \times \mathbb{R}^* \). For instance, if \( v \in \mathcal{W}(y) \), then, for any \( t_* > 0 \), we have that \( v|_{(0, y)\times[0,t_*]} \in C([0, t_*], L_2(0, y)) \) as a function of \( t \in [0, t_*] \). Hence, the initial condition at time \( t = 0 \) is well defined as an equality in \( L_2([0, y]) \). Further, we have that \( v|_{(0, y)\times\mathbb{R}^*} \in C([0, y], L_2(\mathbb{R}^*)) \) and \( \frac{\partial u}{\partial t} |_{(0, y)\times\mathbb{R}^*} \in C([0, y], L_2(\mathbb{R}^*)) \) as functions of \( x \in [0, y] \). Hence the functions \( v(0, t), \frac{\partial u}{\partial t}(x, t)|_{t=0} \) are well defined as elements of \( L_2(\mathbb{R}^*) \), and the boundary value conditions at \( x = 0 \) are well defined as equalities in \( L_2(\mathbb{R}^*) \).

Special smoothing kernel

Let us introduce the set of the following special function:

\[
K(p) = K_{\alpha, \beta, q}(p) \doteq e^{-\alpha(p+\beta)^q}, \quad p \in C^*.
\]

Here \( \alpha > 0, \beta > 0 \) are reals and \( q \in \left( \frac{1}{2}, 1 \right) \) is a rational number. We mean the branch of \( (p+\beta)^q \) such that its argument is \( q \text{ Arg}(p+\beta) \), where \( \text{Arg} z \in (-\pi, \pi] \) denotes the principal value of the argument of \( z \in \mathbb{C} \).

The functions \( K_{\alpha, \beta, q}(p) \) are holomorphic in \( C^* \), and

\[
\ln |K(p)| = -\text{Re} \left( \alpha(p+\beta)^q \right) = -\alpha |p + \beta|^q \cos[q \text{ Arg}(p+\beta)].
\]

In addition, there exists \( M = M(\beta, q) > 0 \) such that \( \cos[q \text{ Arg}(p+\beta)] > M \) for all \( p \in C^* \).

It follows that

\[
|K(p)| \leq e^{-aM|p+\beta|^q} < 1, \quad p \in C^*.
\]

Hence, \( K \in H^r \) for all \( r \in [1, +\infty] \).
Parabolic equations with the second-order Cauchy conditions on the boundary

Proposition 1. Let \( \beta > 0 \) and a rational number \( q \in \left( \frac{1}{2}, 1 \right) \) be given. Let \( v \in L_2(\mathbb{R}^*) \), \( V = \mathcal{L} v \in H^2 \). For \( \alpha > 0 \), set \( v_a = K_{\alpha, \beta, q} V \), \( v_a = \mathcal{F}^{-1} V_a(\omega) \). Then \( V_a \in H^2 \) and \( v_a \to v \) in \( L_2(\mathbb{R}^*) \) as \( \alpha \to 0, \alpha > 0 \).

Proof. Clearly, \( V_a(\omega) \to V(\omega) \) as \( \alpha \to 0 \) for a.e. \( \omega \in \mathbb{R} \). By (4), \( V_a \in H^2 \). In addition, \( |K_{\alpha, \beta, q}(\omega)| \leq 1 \). Hence, \( |V_a(\omega) - V(\omega)| \leq 2|V(\omega)| \). We have that \( \|V(\omega)\|_{L_2(\mathbb{R})} = \|v\|_{L_2(\mathbb{R}^*)} < +\infty \). By the Lebesgue dominance theorem, it follows that

\[
\|V_a(\omega) - V(\omega)\|_{L_2(\mathbb{R})} \to 0 \quad \text{as} \quad \alpha \to 0.
\]

Hence, \( v_a \to v \) in \( L_2(\mathbb{R}^*) \) as \( \alpha \to 0 \). Then the proof follows.

The inverse Fourier transform \( k(t) = \mathcal{F}^{-1}K_{\alpha, \beta, q}(\omega) \) can be viewed as a smoothing kernel; \( k(t) = 0 \) for \( t < 0 \). It can be seen that \( k \) has derivatives of any order.

Denote by \( C \) the set of functions \( v : \mathbb{R}^* \to \mathbb{R} \) such that there exist \( \alpha > 0, \beta > 0 \), and a rational number \( q \in \left( \frac{1}{2}, 1 \right) \), such that \( V \in H^2 \), where \( V(p) = K_{\alpha, \beta, q}(p)^{-1}V(p), \mathcal{L}v \).

The set \( C \) includes outputs of the convolution integral operators with the kernels \( k(t) \). By proposition 1, it follows that the set \( C \) is everywhere dense in \( L_2(\mathbb{R}^*) \).

2. The main result

Set \( F(x, \cdot) \equiv \mathcal{L} f(x, \cdot), \) where \( x > 0 \) is given, and \( G_k \equiv \mathcal{L} g_k, k = 0, 1 \).

Theorem 1. Let the functions \( f \) and \( g_k \) are such that there exist \( y > 0, \alpha > 0, \beta > 0, \) a rational number \( q \in \left( \frac{1}{2}, 1 \right) \), such that \( G_k \in H^2, \hat{F}(x, \cdot) \in H^2 \) for a.e. \( x > 0 \) and

\[
\int_0^\infty \|\hat{F}(s, \cdot)\|_{H^2} ds < +\infty, \text{where}
\]

\[
\hat{F}(x, p) = \frac{F(x, p)}{K(p)}, \quad \hat{G}_k(p) = \frac{G_k(p)}{K(p)},
\]

and where the function \( K = K_{\alpha, \beta, q} \) is defined by (4) (in particular, this means that \( g_k \in C \) and \( f(x, \cdot) \in C \) for a.e. \( x \in [0, y] \)). Then there exists an unique solution \( u(x, t) \) of problem (1) in the domain \((0, y) \times \mathbb{R}^* \) in the class \( \mathcal{W}(y) \). Moreover, there exists a constant \( C(y) = C(a, b, c, \alpha, \beta, q, \gamma) \) such that

\[
\|u\|_{\mathcal{W}(y)} \leq C(y) \left( \|\hat{G}_1\|_{H^2} + \|\hat{G}_2\|_{H^2} + \int_0^y \|\hat{F}(s, \cdot)\|_{H^2} ds \right).
\]

Remark 1. Theorem 1 requires that functions \( f \) and \( g_k \) are smooth in \( t \); in particular, they belong to \( C^\infty \) in \( t \). However, it is not required that \( f(x, t) \) is smooth in \( x \).

Proof of theorem 1. Instead of (1), consider the following problems for \( p \in \mathbb{C}^+ \):

\[
ap U(x, p) = \frac{\partial^2 U}{\partial x^2}(x, p) + b \frac{\partial U}{\partial x}(x, p) + c U(x, p) + F(x, p), \quad x > 0,
\]

\[
U(0, p) = G_0(p), \quad \frac{\partial U}{\partial x}(0, p) = G_1(p).
\]

Let \( \lambda_k = \lambda_k(p) \) be the roots of the equation \( \lambda^2 + b \lambda + (c - ap) = 0 \). Clearly, \( \lambda_{1,2} \equiv \frac{-b/2 \pm \sqrt{b^2 - 4ac}}{2} \). Recall that \( \mu > 0 \). It follows that the functions \( (\lambda_1(p) - \lambda_2(p))^{-1} \) and \( \lambda_k(p)(\lambda_1(p) - \lambda_2(p))^{-1}, k = 1, 2 \), belong to \( H^\infty \).
For $x \in (0, y]$, the solution of (7) is

$$U(x, p) = \frac{1}{\lambda_1 - \lambda_2} \left( (G_1(p) - \lambda_2 G_0(p)) e^{\lambda_1 x} - (G_1(p) - \lambda_1 G_0(p)) e^{\lambda_2 x} - \int_0^x e^{\lambda_1 (x-s)} F(s, p) \, ds + \int_0^x e^{\lambda_2 (x-s)} F(s, p) \, ds \right).$$

This can be derived, for instance, using the Laplace transform method applied to the linear ordinary differential equation (7), and having in mind that

$$\frac{1}{\lambda^2 + b \lambda + c - ap} = \frac{1}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = \frac{1}{\lambda_1 - \lambda_2} \frac{1}{\lambda - \lambda_1} - \frac{1}{\lambda - \lambda_2}.$$

Let $x \in (0, y), s \in [0, x]$. The functions $e^{(x-s)\lambda_k(p)}, k = 1, 2$, are holomorphic in $\mathbb{C}^*$. We have

$$\ln |e^{(x-s)\lambda_k(p)}| = \Re ((x-s)\lambda_k(p)) = (x-s) \left( -\frac{b}{2} \pm |ap + \mu|^{|1/2|} \cos \frac{\text{Arg}(ap + \mu)}{2} \right),$$

where $k = 1, 2, p \in \mathbb{C}^*$. It follows that

$$|K(p)e^{(x-s)\lambda_k(p)}| \leq e^{(x-s)|-b/2+|ap+\mu|^{1/2}|} - a M |p + \beta|^{|1/2|},$$

$k = 1, 2, p \in \mathbb{C}^*$. Similarly,

$$|K(p)e^{\lambda x}| \leq e^{(x-s)|-b/2+|ap+\mu|^{1/2}|} - a M |p + \beta|^{|1/2|}.$$
It follows that the corresponding inverse Fourier transforms \( u(x, \cdot) = F^{-1}[U(x, i\omega)]_{\omega \in \mathbb{R}} \) are well defined and are vanishing for \( t < 0 \). In addition, we have that \( \tilde{U}(x, i\omega) = U(x, -i\omega) \) (for instance, \( \tilde{K}(i\omega) = K(-i\omega), e^{(x-x')i\lambda_k(\omega)} = e^{(x-x')i\lambda_k(-i\omega)} \), etc). It follows that the inverse of the Fourier transform \( u(x, \cdot) = F^{-1}U(x, \cdot) \) is real.

Further, we have that

\[
\frac{\partial U}{\partial x}(x, p) = \frac{1}{\lambda_1 - \lambda_2} \left( (G_1(p) - \lambda_2 G_0(p))\lambda_1 e^{\lambda_1 x} - (G_1(p) - \lambda_1 G_0(p))\lambda_2 e^{\lambda_2 x} - \lambda_1 \int_0^x e^{\lambda_1 (x-s)} F(s, p) \mathrm{d}s + \lambda_2 \int_0^x e^{\lambda_2 (x-s)} F(s, p) \mathrm{d}s \right).
\]

Since \( \lambda_1(p)\lambda_2(p) = c - ap \), we again obtain that

\[
\left\| \frac{\partial U}{\partial x}(x, p) \right\|_{H^2} \leq C_3(y) \left( \sum_{k=1,2} \left\| \hat{G}_k \right\|_{H^2} + \int_0^x \left\| \hat{F}(s, p) \right\|_{H^2} \mathrm{d}s \right). \tag{11}
\]

By (7), \( \partial^2 U/\partial x^2 \) can be expressed as a linear combination of \( F, G_k, U, pU, \partial U/\partial x \). By (9)–(11),

\[
\left\| \frac{\partial^2 U}{\partial x^2}(x, p) \right\|_{H^2} \leq C_4(y) \left( \left\| \frac{\partial U}{\partial x}(x, p) \right\|_{H^2} + \sum_{m=0,1} \left\| p^m U(x, p) \right\|_{H^2} + \left\| F(x, p) \right\|_{H^2} \right). \tag{12}
\]

We have that \( |K(p)| < 1 \) on \( C^+ \) and \( \|F(s, p)\|_{H^2} \leq \|\hat{F}(s, p)\|_{H^2} \). It follows that

\[
\left\| \frac{\partial^2 U}{\partial x^2}(x, p) \right\|_{H^2} \leq C_5(y) \left( \sum_{k=1,2} \left\| \hat{G}_k \right\|_{H^2} + \int_0^x \left\| \hat{F}(s, p) \right\|_{H^2} \mathrm{d}s \right). \tag{12}
\]

Here \( C_5(y) \) are constants that depend on \( a, b, c, \alpha, \beta, q, y \). By (9)–(12), estimate (6) holds.

Therefore, \( u(x, \cdot) = F^{-1}[U(x, i\omega)]_{\omega \in \mathbb{R}} \) is the solution of (1) in \( \mathbb{V}(y) \). The uniqueness is ensured by the linearity of the problem, by estimate (6), and by the fact that \( \mathcal{L}u(x, \cdot), \mathcal{L}(\partial^k u(x, \cdot)/\partial x^k) \) and \( \mathcal{L}(\partial u(x, \cdot)/\partial t) \) are well defined on \( C^+ \) for any \( u \in \mathbb{V}(y) \). This completes the proof of theorem 1.

\[\square\]

Remark 2. It can be seen from the proof that it is crucial that \( u(x, 0) \equiv 0 \). Non-zero initial conditions cannot be included.

References

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