Period Doubling in Four-Dimensional Volume-Preserving Maps

Sang-Yoon Kim

Department of Physics
Kangwon National University
Chunchon, Kangwon-Do 200-701, Korea

Abstract

We numerically study the scaling behavior of period doublings at the zero-coupling critical point in a four-dimensional volume-preserving map consisting of two coupled area-preserving maps. In order to see the fine structure of period doublings, we extend the simple one-term scaling law to a two-term scaling law. Thus we find a new scaling factor $\delta_3 (= 1.8505 \ldots)$ associated with scaling of the coupling parameter, in addition to the previously known scaling factors $\delta_1 (= -8.7210 \ldots)$ and $\delta_2 (= -4.4038 \ldots)$. These numerical results confirm the renormalization results reported by Mao and Greene [Phys. Rev. A 35, 3911 (1987)].
PACS numbers: 05.45.+b, 03.20.+i, 05.70.Jk
Universal scaling behavior of period doubling has been found in area-preserving maps [1–7]. As a nonlinearity parameter is varied, an initially stable periodic orbit may lose its stability and give rise to the birth of a stable period-doubled orbit. An infinite sequence of such bifurcations accumulates at a finite parameter value and exhibits a universal limiting behavior. However these limiting scaling behaviors are different from those for the one-dimensional dissipative case [8].

An interesting question is whether the scaling results of area-preserving maps carry over higher-dimensional volume-preserving maps. Thus period doubling in four-dimensional (4D) volume-preserving maps has been much studied in recent years [7,9–13]. It has been found in Refs. [11–13] that the critical scaling behaviors of period doublings for two symmetrically coupled area-preserving maps are much richer than those for the uncoupled area-preserving case. There exist an infinite number of critical points in the space of the nonlinearity and coupling parameters. It has been numerically found in [11,12] that the critical behaviors at those critical points are characterized by two scaling factors, \( \delta_1 \) and \( \delta_2 \). The value of \( \delta_1 \) associated with scaling of the nonlinearity parameter is always the same as that of the scaling factor \( \delta \) (\( = 8.721 \ldots \)) for the area-preserving maps. However the values of \( \delta_2 \) associated with scaling of the coupling parameter vary depending on the type of bifurcation routes to the critical points.

The numerical results [11,12] agree well with an approximate analytic renormalization results obtained by Mao and Greene [13], except for the zero-coupling case in which the two area-preserving maps become uncoupled. Using an approximate renormalization method including truncation, they found three relevant eigenvalues, \( \delta_1 = 8.9474 \), \( \delta_2 = -4.4510 \) and \( \delta_3 = 1.8762 \) for the zero-coupling case [14]. However they believed that the third one \( \delta_3 \) is an artifact of the truncation, because only two relevant eigenvalues \( \delta_1 \) and \( \delta_2 \) could be indentified with the scaling factors numerically found.

In this paper we numerically study the critical behavior at the zero-coupling point in two symmetrically coupled area-preserving maps and resolve the inconsistency between the numerical results on the scaling of the coupling parameter and the approximate renormaliza-
tion results for the zero-coupling case. In order to see the fine structure of period doublings, we extend the simple one-term scaling law to a two-term scaling law. Thus we find a new scaling factor \( \delta_3 = 1.8505 \ldots \) associated with coupling, in addition to the previously known coupling scaling factor \( \delta_2 = -4.4038 \ldots \). The numerical values of \( \delta_2 \) and \( \delta_3 \) are close to the renormalization results of the relevant coupling eigenvalues \( \delta_2 \) and \( \delta_3 \). Consequently the fixed map governing the critical behavior at the zero-coupling point has two relevant coupling eigenvalues \( \delta_2 \) and \( \delta_3 \) associated with coupling perturbations, unlike the cases of other critical points.

Consider a 4D volume-preserving map consisting of two symmetrically coupled area-preserving Hénon maps with a periodic boundary condition,

\[
T : \begin{aligned}
\vec{x}(t + 1) &= -\vec{y}(t) + \vec{F}(\vec{x}(t)), \\
\vec{y}(t + 1) &= \vec{x}(t),
\end{aligned}
\]  

(1)

where \( \vec{x} = (x_1, x_2) \), \( \vec{y} = (y_1, y_2) \), \( \vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x})) \). Here \( z_m(t) \equiv (x_m(t), y_m(t)) \) \((m = 1, 2)\) is the state vector of the \( m \)th element at a discrete time \( t \), and the periodic boundary condition imposes \( z_m = z_{m+2} \) for all \( m \). The \( m \)th component \( F_m(\vec{x}) \) of the vector-valued function \( \vec{F}(\vec{x}) \) is given by

\[
F_m(\vec{x}) = F(x_m, x_{m+1}) = f(x_m) + g(x_m, x_{m+1}),
\]  

(2)

where \( f \) is the nonlinear function of the uncoupled Hénon’s quadratic map \([13] \), i.e.,

\[
f(x) = 1 - ax^2,
\]  

(3)

and \( g(x_1, x_2) \) is a coupling function obeying a condition

\[
g(x, x) = 0 \text{ for any } x.
\]  

(4)

The two-coupled map (1) is called a symmetric map \([11,12] \) because it has an exchange symmetry such that

\[
\sigma^{-1}T\sigma(\vec{z}) = T(\vec{z}) \text{ for all } \vec{z},
\]  

(5)
where \( \mathbf{z} = (z_1, z_2) \), \( \sigma \) is an exchange operator acting on \( \mathbf{z} \) such that \( \sigma \mathbf{z} = (z_2, z_1) \), and \( \sigma^{-1} \) is its inverse. The set of all fixed points of \( \sigma \) is a two-dimensional (2D) subspace of the 4D state space, on which \( x_1 = x_2 \) and \( y_1 = y_2 \). It follows from Eq. (5) that the exchange operator \( \sigma \) commutes with the symmetric map \( T \), i.e., \( \sigma T = T \sigma \). Hence the 2D subspace becomes invariant under \( T \). An orbit is called an in-phase orbit if it lies on the 2D invariant subspace, i.e., it satisfies
\[
  x_1(t) = x_2(t) \equiv x(t), \quad y_1(t) = y_2(t) \equiv y(t) \quad \text{for all } t.
\]
Otherwise it is called an out-of-phase orbit. Here we study only in-phase orbits. They can be easily found from the uncoupled Hénon map since the coupling function \( g \) obeys the condition (4).

The Jacobian matrix \( DT \) of the two-coupled map \( T \) is:
\[
  DT = \begin{pmatrix}
    D\mathbf{F} & -I \\
    I & 0
  \end{pmatrix},
\]
where \( D\mathbf{F} \) is the Jacobian matrix of the function \( \mathbf{F}(\mathbf{x}) \), \( I \) is the \( 2 \times 2 \) identity matrix, and \( 0 \) is the \( 2 \times 2 \) null matrix. Since \( \text{Det}(DT) = 1 \), the map \( T \) is a 4D volume-preserving map. Furthermore, if \( D\mathbf{F} \) is a symmetric matrix, i.e., \( D\mathbf{F}^t = D\mathbf{F} \) (\( t \) denotes transpose), then the map \( T \) is a symplectic map because its Jacobian matrix satisfies the relation \( DT^t J DT = J \) [16], where
\[
  J = \begin{pmatrix}
    0 & I \\
    -I & 0
  \end{pmatrix}.
\]

Stability analysis of an orbit in coupled maps is conveniently carried out by Fourier-transforming with respect to the discrete space \( \{m\} \) [17]. Consider an orbit \( \{\mathbf{x}(t)\} \equiv \{x_m(t) ; m = 1, 2\} \) of the two-coupled map \( T \) expressed in the form of second-order difference equations,
\[
  T : x_m(t + 1) + x_m(t - 1) = F_m(\mathbf{x}(t))
  = f(x_m(t)) + g(x_m(t), x_{m+1}(t)), \quad m = 1, 2.
\]
The discrete spatial Fourier transform of the orbit is:

\[ \mathcal{F}[x_m(t)] \equiv \frac{1}{2} \sum_{m=1}^{2} e^{-\pi imj} x_m(t) = \xi_j(t), \quad j = 0, 1. \]  

(10)

The wavelength of a mode with index \( j \) is \( \frac{2}{j} \).

To determine the stability of an in-phase orbit \( [x_1(t) = x_2(t) \equiv x(t) \text{ for all } t] \), we consider an infinitesimal perturbation \( \{\delta x_m(t)\} \) to the in-phase orbit, i.e., \( x_m(t) = x(t) + \delta x_m(t) \) for \( m = 1, 2 \). Linearizing the two-coupled map (9) at the in-phase orbit, we have

\[ \delta x_m(t + 1) + \delta x_m(t - 1) = f'(x(t)) \delta x_m(t) + \sum_{l=1}^{N} G_l(x(t)) \delta x_{l+m-1}(t), \]

(11)

where

\[ f'(x) = \frac{df}{dx}, \quad G_l(x) = \frac{\partial g(\bar{x})}{\partial x_l} \bigg|_{x_1=x_2=x} . \]  

(12)

Hereafter we will call the functions \( G_l \)’s “reduced” coupling functions of \( g(\bar{x}) \).

Let \( \delta \xi_j(t) \) be the Fourier transform of \( \delta x_m(t) \), i.e.,

\[ \delta \xi_j(t) = \mathcal{F}[\delta x_m(t)] = \frac{1}{2} \sum_{m=1}^{2} e^{-\pi imj} \delta x_m(t), \quad j = 0, 1. \]  

(13)

Then the Fourier transform of Eq. (11) becomes:

\[ \delta \xi_j(t + 1) + \delta \xi_j(t - 1) = [f'(x(t)) + \sum_{l=1}^{N} G_l(x(t)) e^{\pi i(l-1)j}] \delta \xi_j(t), \quad j = 0, 1. \]  

(14)

This equation can be also put into the following form:

\[
\begin{pmatrix}
\delta \xi_j(t + 1) \\
\delta \xi_j(t)
\end{pmatrix} = L_j(t)
\begin{pmatrix}
\delta \xi_j(t) \\
\delta \xi_j(t - 1)
\end{pmatrix}, \quad j = 0, 1,
\]

(15)

where

\[ L_j(t) = \begin{pmatrix}
  f'(x(t)) + \sum_{l=1}^{N} G_l(x(t)) e^{\pi i(l-1)j} & -1 \\
  1 & 0
\end{pmatrix}. \]  

(16)

Note that the determinant of \( L_j \) is one, i.e., \( Det(L_j) = 1 \).

It follows from the condition (4) that the reduced coupling functions satisfy
\[
\sum_{l=1}^{2} G_l(x) = 0.
\]

(17)

Hence there exists only one independent reduced coupling function \( G(x) \) such that

\[
G_2(x) = G(x), \quad G_1(x) = -G(x).
\]

(18)

Substituting \( G_l \)'s into the \((1, 1)\) entry of the matrix \( L_j(t) \), we have:

\[
\sum_{l=1}^{2} G_l(x(t)) e^{\pi i (l-1)j} = \begin{cases} 
0 & \text{for } j = 0, \\
-2G(x(t)) & \text{for } j = 1
\end{cases}
\]

(19)

Hence the matrices \( L_j \)'s of Eq. (16) are real matrices.

Stability of an in-phase orbit of period \( q \) is determined by iterating Eq. (15) \( q \) times:

\[
\begin{pmatrix}
\delta \xi_j(t+q) \\
\delta \xi_j(t+q-1)
\end{pmatrix} = M_j \begin{pmatrix}
\delta \xi_j(t) \\
\delta \xi_j(t-1)
\end{pmatrix}, \quad j = 0, 1,
\]

(20)

where

\[
M_j = \prod_{k=t}^{t+q-1} L_j(k).
\]

(21)

That is, the stability of each mode with index \( j \) is determined by the \( 2 \times 2 \) matrix \( M_j \). Since \( Det(M_j) = 1 \), each matrix \( M_j \) has a reciprocal pair of eigenvalues, \( \lambda_j \) and \( \lambda_j^{-1} \). These eigenvalues are called the stability multipliers of the mode with index \( j \). Associate with a pair of multipliers \( (\lambda_j, \lambda_j^{-1}) \) a stability index,

\[
\rho_j = \lambda_j + \lambda_j^{-1}, \quad j = 0, 1,
\]

(22)

which is just the trace of \( M_j \), i.e., \( \rho_j = Tr(M_j) \). Since \( M_j \) is a real matrix, \( \rho_j \) is always real. Note that the matrix \( M_0 \) for the case of the \( j = 0 \) mode is just the Jacobian matrix of the uncoupled Hénon map. Hence the coupling affects only the stability index \( \rho_1 \) of the \( j = 1 \) mode.

It follows from the reality of \( \rho_j \) that the reciprocal pair of eigenvalues of \( M_j \) lies either on the unit circle, or on the real line in the complex plane, i.e., they are a complex conjugate
pair on the unit circle, or a reciprocal pair of reals. Each mode with index \( j \) is stable if and only if its stability index \( \rho_j \) is real with \( |\rho_j| \leq 2 \), i.e., its stability multipliers are a pair of complex conjugate numbers of modulus unity. A period-doubling (tangent) bifurcation occurs when the stability index \( \rho_j \) decreases (increases) through \(-2 (2)\), i.e., two eigenvalues coalesce at \( \lambda_j = -1 (1) \) and split along the negative (positive) real axis.

An in-phase orbit is stable only when all its modes are stable. Hence its stable region in the space of the nonlinearity and coupling parameters is bounded by four bifurcation lines associated with tangent and period-doubling bifurcations of both modes (i.e., those curves determined by the equations \( \rho_j = \pm 2 \) for \( j = 0, 1 \)). When the stability index \( \rho_0 \) decreases through \(-2\), the in-phase orbit loses its stability via in-phase period-doubling bifurcation and gives rise to the birth of the period-doubled in-phase orbit. Here we are interested in scaling behaviors of such in-phase period-doubling bifurcations.

As an example we consider a linearly-coupled case in which the coupling function is

\[
g(x_1, x_2) = \frac{c}{2}(x_2 - x_1).
\]

Here \( c \) is a coupling parameter. Figure [1] shows the stability diagram of in-phase orbits with period \( q = 1, 2, \) and \( 4 \) [13]. As previously observed in [11,12], each “mother” stability region bifurcates into two “daughter” stability regions successively in the parameter plane; hereafter we call the direction of the left (right) branch of the two daughter stability regions \( L \) (\( R \)) direction. Consequently the stability region of the in-phase orbit with period \( q = 2^n \) \( (n = 0, 1, 2, \ldots) \) consists of \( 2^n \) branches. Each branch can be represented by its address \([a_0, \ldots, a_n]\), which is a sequence of symbols \( L \) and \( R \) such that \( a_0 = L \) and \( a_i = L \) or \( R \) for \( i \geq 1 \).

An infinite sequence of connected stability branches (with increasing period) is called a bifurcation “route” [11,12]. Each bifurcation route is also represented by an infinite sequence of symbols \( L \) and \( R \). A “self-similar” bifurcation “path” in a bifurcation route is formed by following a sequence of parameters \((a_n, c_n)\), at which the in-phase orbit of level \( n \) (period \( 2^n \)) has some given stability indices \((\rho_0, \rho_1)\) (e.g., \( \rho_0 = -2 \) and \( \rho_1 = 2 \)) [11,12]. All bifurcation
paths within a bifurcation route converge to an accumulation point \((a^*, c^*)\), where the value of \(a^*\) is always the same as that of the accumulation point for the area-preserving case (i.e., \(a^* = 4.136\,166\,803\,904 \ldots\)), but the value of \(c^*\) varies depending on the bifurcation routes. Thus each bifurcation route ends at a critical point \((a^*, c^*)\) in the parameter plane.

It has been numerically found that scaling behaviors near a critical point are characterized by two scaling factors, \(\delta_1\) and \(\delta_2\) \([11,12]\). The value of \(\delta_1\) associated with scaling of the nonlinearity parameter is always the same as that of the scaling factor \(\delta \approx 8.721 \ldots\) for the area-preserving case. However the values of \(\delta_2\) associated with scaling of the coupling parameter vary depending on the type of bifurcation routes. These numerical results agree well with analytic renormalization results \([13]\), except for the case of one specific bifurcation route, called the E route. The address of the E route is \([L, R, \ldots] \equiv [L, R, L, R, \ldots]\) and it ends at the zero-coupling critical point \((a^*, 0)\).

Using an approximate renormalization method including truncation, Mao and Greene \([14]\) obtained three relevant eigenvalues, \(\delta_1 = 8.9474\), \(\delta_2 = -4.4510\), and \(\delta_3 = 1.8762\) for the zero-coupling case; hereafter the two eigenvalues \(\delta_2\) and \(\delta_3\) associated with coupling will be called the coupling eigenvalues (CE’s). The two eigenvalues \(\delta_1\) and \(\delta_2\) are close to the numerical results of the nonlinearity-parameter scaling factor \(\delta_1(\approx 8.721 \ldots)\) and the coupling-parameter scaling factor \(\delta_2(\approx -4.403 \ldots)\) for the E route. However they believed that the second relevant CE \(\delta_3\) is an artifact of the truncation, because it could not be identified with anything obtained by a direct numerical method.

In order to resolve the inconsistency between the numerical results and the renormalization results for the zero-coupling case, we numerically reexamine the scaling behavior associated with coupling. Extending the simple one-term scaling law to a two-term scaling law, we find a new scaling factor \(\delta_3 = 1.8505 \ldots\) associated with coupling in addition to the previously found coupling scaling factor \(\delta_2 = -4.4038 \ldots\), as will be seen below. The values of these two coupling scaling factors are close to the renormalization results of the relevant CE’s \(\delta_2\) and \(\delta_3\).

We follow the in-phase orbits of period \(2^n\) up to level \(n = 14\) in the E route and
obtain a self-similar sequence of parameters \((a_n, c_n)\), at which the pair of stability indices, \((\rho_{0,n}, \rho_{1,n})\), of the orbit of level \(n\) is \((-2, 2)\). The scalar sequences \(\{a_n\}\) and \(\{c_n\}\) converge geometrically to their limit values, \(a^*\) and 0, respectively. In order to see their convergence, define \(\delta_n \equiv \Delta a_{n+1}/\Delta a_n\) and \(\mu_n \equiv \Delta c_{n+1}/\Delta c_n\), where \(\Delta a_n = a_n - a_{n-1}\) and \(\Delta c_n = c_n - c_{n-1}\). Then they converge to their limit values \(\delta\) and \(\mu\) as \(n \to \infty\), respectively. Hence the two sequences \(\{\Delta a_n\}\) and \(\{\Delta c_n\}\) obey one-term scaling laws asymptotically:

\[
\Delta a_n = C(a)\delta^{-n}, \quad \Delta c_n = C(c)\mu^{-n} \quad \text{for large } n,
\]

where \(C(a)\) and \(C(c)\) are some constants.

The two sequences \(\{\delta_n\}\) and \(\{\mu_n\}\) are shown in Table I. The second column shows rapid convergence of \(\delta_n\) to its limit value \(\delta (= 8.721 \ldots)\), which is close to the renormalization result of the first relevant eigenvalue (i.e., \(\delta_1 = 8.9474\)). The sequence \(\{\mu_n\}\) also seems to converge to \(\mu = -4.403 \ldots\) (see the third column). However its convergence is not as fast as that for the case of the sequence \(\{\delta_n\}\). This is because two relevant CE’s \(\delta_2\) and \(\delta_3\) are involved in the scaling of the sequence \(\{\mu_n\}\).

Taking into account the effect of the second relevant CE \(\delta_3\) on the scaling of the sequence \(\{\Delta c_n\}\), we extend the simple one-term scaling law (24) to a two-term scaling law:

\[
\Delta c_n = C_1\mu_1^{-n} + C_2\mu_2^{-n} \quad \text{for large } n,
\]

where \(|\mu_1| > |\mu_2|\). This is a kind of multiple scaling law \[19,20\]. Eq. (24) gives

\[
\Delta c_n = t_1\Delta c_{n+1} - t_2\Delta c_{n+2},
\]

where \(t_1 = \mu_1 + \mu_2\) and \(t_2 = \mu_1\mu_2\). Then \(\mu_1\) and \(\mu_2\) are solutions of the following quadratic equation,

\[
\mu^2 - t_1\mu + t_2 = 0.
\]

To evaluate \(\mu_1\) and \(\mu_2\), we first obtain \(t_1\) and \(t_2\) from \(\Delta a_n\)'s using Eq. (26):

\[
t_1 = \frac{\Delta c_n\Delta c_{n+1} - \Delta c_{n-1}\Delta c_{n+2}}{\Delta c_{n+1}\Delta c_{n+2} - \Delta c_n\Delta c_{n+2}}, \quad t_2 = \frac{\Delta c_{n+1}^2 - \Delta c_{n+1}\Delta c_{n-1}}{\Delta c_{n+1}\Delta c_{n+2} - \Delta c_n\Delta c_{n+2}}.
\]
Note that Eqs. (25)-(28) hold only for large \( n \). In fact the values of \( t_i \)'s and \( \mu_i \)'s (\( i = 1, 2 \)) depend on the level \( n \). Therefore we explicitly denote \( t_i \)'s and \( \mu_i \)'s by \( t_{i,n} \)'s and \( \mu_{i,n} \)'s, respectively. Then each of them converges to a constant as \( n \to \infty \):

\[
\lim_{n \to \infty} t_{i,n} = t_i, \quad \lim_{n \to \infty} \mu_{i,n} = \mu_i, \quad i = 1, 2.
\] (29)

Three sequences \( \{\mu_{1,n}\}, \{\mu_{2,n}\}, \) and \( \{\mu_{1,n}^2/\mu_{2,n}\} \) are shown in Table II. The second column shows rapid convergence of \( \mu_{1,n} \) to its limit values \( \mu_1 = -4.403897805 \). Comparing this sequence with the sequence \( \{\mu_n\} \) in Table I, we find that the finite sequence \( \{\Delta c_n\} \) obeys the two-term scaling law better than the one-term scaling law. That is, the accuracy for the scaling factor \( \mu_1 \) obtained from the two-term scaling law is much higher than that obtained from the one-term scaling law. The value of this scaling factor \( \mu_1 \) is close to the renormalization result of the first relevant CE (i.e., \( \delta_2 = -4.4510 \)). From the third and fourth columns, we also find that the second scaling factor \( \mu_2 \) is given by a product of two relevant CE’s \( \delta_2 \) and \( \delta_3 \),

\[
\mu_2 = \frac{\delta_2^2}{\delta_3},
\] (30)

where \( \delta_2 = \mu_1 \) and \( \delta_3 = 1.85065 \). It has been known that every scaling factor in the multiple-scaling expansion of a parameter is expressed by a product of the eigenvalues of a linearized renormalization operator \[19,20\]. Note that the value of \( \delta_3 \) is close to the renormalization result of the second relevant CE (i.e., \( \delta_3 = 1.8762 \)).

We now study the coupling effect on the stability index \( \rho_{1,n} \) of the \( j = 1 \) mode of the in-phase orbit of period \( 2^n \) near the zero-coupling critical point \( (a^*, 0) \). Figure 2 shows three plots of \( \rho_{1,n}(a^*, c) \) versus \( c \) for \( n = 4, 5, \) and 6. For \( c = 0 \), \( \rho_{1,n} \) converges to a constant \( \rho_1^* \) \( (= -2.54351020\ldots) \), called the critical stability index \[12\], as \( n \to \infty \). However, when \( c \) is non-zero \( \rho_{1,n} \) diverges as \( n \to \infty \), i.e., its slope \( S_n \) \( (\equiv \frac{\partial \rho_{1,n}}{\partial c}_{(a^*,0)} \) at the zero-coupling critical point diverges as \( n \to \infty \).

The sequence \( \{S_n\} \) obeys a two-term scaling law,

\[
S_n = D_1 \nu_1^n + D_2 \nu_2^n \quad \text{for large } n,
\] (31)
where $|\nu_1| > |\nu_2|$. This equation gives

$$S_{n+2} = r_1 S_{n+1} - r_2 S_n, \quad (32)$$

where $r_1 = \nu_1 + \nu_2$ and $r_2 = \nu_1 \nu_2$. As in the scaling for the coupling parameter, we first obtain $r_1$ and $r_2$ of level $n$ from $S_n$’s:

$$r_{1,n} = \frac{S_{n+1}S_n - S_{n+2}S_{n-1}}{S_n^2 - S_{n+1}S_{n-1}}, \quad r_{2,n} = \frac{S_{n+1}^2 - S_n S_{n+2}}{S_n^2 - S_{n+1}S_{n-1}}, \quad (33)$$

Then the scaling factors $\nu_{1,n}$ and $\nu_{2,n}$ of level $n$ are given by the roots of the quadratic equation, $\nu_n^2 - r_{1,n}\nu_n + r_{2,n} = 0$. They are listed in Table III and converge to constants $\nu_1$ ($= -4.403 897 805 09$) and $\nu_2$ ($= 1.850 535$) as $n \to \infty$, whose accuracies are higher than those of the coupling-parameter scaling factors. Note that the values of $\nu_1$ and $\nu_2$ are also close to the renormalization results of the two relevant CE’s $\delta_2$ and $\delta_3$.

We have also studied several other coupling cases with the coupling function, $g(x_1, x_2) = \frac{c}{2} (x_n^2 - x_1^n)$ ($n$ is a positive integer). In all cases studied ($n = 2, 3, 4, 5$), the scaling factors of both the coupling parameter $c$ and the slope of the stability index $\rho_1$ are found to be the same as those for the above linearly-coupled case ($n = 1$) within numerical accuracy. Hence universality also seems to be well obeyed.

**ACKNOWLEDGMENTS**

I thank Professor S. Y. Lee for a critical reading of the manuscript.
REFERENCES

* Electronic Address: sykim@cc.kangwon.ac.kr (Internet)

[1] G. Benettin, C. Cercignani, L. Galgani, and A. Giorgilli, Lett. Nuovo Cimento 28, 1 (1980); 29, 163 (1980).

[2] P. Collet, J. P. Eckmann, and H. Koch, Physica D 3, 457 (1981).

[3] J. M. Greene, R. S. MacKay, F. Vivaldi, and M. J. Feigenbaum, Physica D 3, 468 (1981).

[4] T. C. Bountis, Physica D 3, 577 (1981).

[5] R. H. G. Helleman, in Long-Time Prediction in Dynamics, edited by W. Horton, L. Reichl, and V. Szebehely (Wiley, New York, 1982), pp. 95-126.

[6] M. Widom and L. P. Kadanoff, Physica D 5, 287 (1982).

[7] R. S. MacKay, Ph.D. Thesis, Princeton University, 1982.

[8] M. J. Feigenbaum, J. Stat. Phys. 19, 25 (1978); 21, 669 (1979).

[9] T. Janssen and J. A. Tjon, J. Phys. A 16, 673 (1983); 16, 697 (1983).

[10] J.-m. Mao, I. Satija, and B. Hu, Phys. Rev. A 32, 1927 (1985); 34, 4325 (1986).

[11] J.-m. Mao and R. H. G. Helleman, Phys. Rev. A 35, 1847 (1987).

[12] S.-Y. Kim and B. Hu, Phys. Rev. A 41, 5431 (1990).

[13] J.-m. Mao and J. M. Greene, Phys. Rev. A 35, 3911 (1987).

[14] See the table I in Ref. [13]. The $\delta_3$ in the text corresponds to $\delta'_2$ in the table.

[15] M. Hénon, Quart. Appl. Math. 27, 291 (1969).

[16] J. E. Howard and R. S. MacKay, J. Math. Phys. 28, 1036 (1987).

[17] I. Waller and R. Kapral, Phys. Rev. A 30, 2047 (1984).
[18] Only some upper part of its whole stable region is shown for the case of the period-1 orbit in order to make the stable regions for the cases of higher periods ($q = 2, 4$) clearer.

[19] J.-m Mao and B. Hu, J. Stat. Phys. 46, 111 (1987); Int. J. Mod. Phys. B 2, 65 (1988).

[20] C. Reick, Phys. Rev. A 45, 777 (1992).
TABLES

TABLE I. Scaling factors $\delta_n$ and $\mu_n$ in the one-term scaling for the nonlinearity and coupling parameters are listed.

| $n$ | $\delta_n$          | $\mu_n$  |
|-----|---------------------|----------|
| 6   | 8.721 086 300 39    | -4.399 51|
| 7   | 8.721 096 265 95    | -4.405 75|
| 8   | 8.721 097 056 68    | -4.403 12|
| 9   | 8.721 097 188 39    | -4.404 22|
| 10  | 8.721 097 198 70    | -4.403 76|
| 11  | 8.721 097 200 44    | -4.403 96|
| 12  | 8.721 097 200 58    | -4.403 87|
| 13  | 8.721 097 200 60    | -4.403 91|

TABLE II. Scaling factors $\mu_{1,n}$ and $\mu_{2,n}$ in the two-term scaling for the coupling parameter are shown in the second and third columns, respectively. A product of them, $\frac{\mu_{1,n}^2}{\mu_{2,n}}$, is shown in the fourth column.

| $n$ | $\mu_{1,n}$          | $\mu_{2,n}$ | $\frac{\mu_{1,n}^2}{\mu_{2,n}}$ |
|-----|----------------------|-------------|----------------------------------|
| 5   | -4.403 908 128       | 10.437 4    | 1.858 17                         |
| 6   | -4.403 899 694       | 10.465 9    | 1.853 09                         |
| 7   | -4.403 898 736       | 10.458 2    | 1.854 46                         |
| 8   | -4.403 897 867       | 10.474 8    | 1.851 52                         |
| 9   | -4.403 897 847       | 10.473 9    | 1.851 68                         |
| 10  | -4.403 897 806       | 10.478 4    | 1.850 89                         |
| 11  | -4.403 897 807       | 10.478 6    | 1.850 85                         |
| 12  | -4.403 897 805       | 10.479 7    | 1.850 65                         |
TABLE III. Scaling factors $\nu_{1,n}$ and $\nu_{2,n}$ in the two-term scaling for the slope of the stability index of the $j = 1$ mode are shown.

| $n$ | $\nu_{1,n}$       | $\nu_{2,n}$       |
|-----|--------------------|--------------------|
| 5   | -4.403 898 453 59  | 1.851 433 5        |
| 6   | -4.403 897 730 29  | 1.850 782 6        |
| 7   | -4.403 897 813 85  | 1.850 603 6        |
| 8   | -4.403 897 804 07  | 1.850 553 8        |
| 9   | -4.403 897 805 21  | 1.850 540 0        |
| 10  | -4.403 897 805 07  | 1.850 536 1        |
| 11  | -4.403 897 805 09  | 1.850 535 0        |
| 12  | -4.403 897 805 09  | 1.850 534 9        |
FIGURES

FIG. 1. Stability diagram of in-phase orbits for the linearly-coupled case. The horizontal (non-horizontal) solid and dashed lines denote the period-doubling and tangent bifurcation lines of the $j = 0$ (1) mode, respectively. For other details see the text.

FIG. 2. Plots of the stability index $\rho_{1,n}(a^*, c)$ versus $c$ for $n = 4, 5, 6$. 