P-FINITÉ RECURRENCES FROM GENERATING FUNCTIONS
WITH ROOTS OF POLYNOMIALS

RICHARD J. MATHAR

Abstract. We derive the P-finite recurrences for classes of sequences with
ordinary generating function containing roots of polynomials. The focus is on
establishing the D-finite differential equations such that the familiar steps of
reducing their power series expansions apply.

1. Aim

The manuscript derives P-finite recurrences for some classes of generating func-
tions $g(x)$ which have a self-replicating property under differentiation, mainly
involving roots and exponentials of polynomials of $x$. The standard application of
the recurrences are

- numerical generation of a deep list of the expansion coefficients with con-
  stant time and with constant requirements on memory for each additional
term. [George Fischer’s work on efficiently implementing a large set of holon-
nomic sequences of the Online Encyclopedia of Integer Sequences (OEIS)
was in fact the main inspiration of this work, see https://github.com/archmageirvine/joeis.
- asymptotic estimates for large $n$ [31][13, Thm. VII.10], where the generat-
ing functions cover statistics of walks, biological cycles and similar growths
to higher generations.
- reverse engineering of generating functions if the P-finite recurrences match
  one of the emerging types.

2. Inverse Root

This section is a tutorial which demonstrates the mechanism of coefficient shifts
for differentiation of power series. Let $g(x)$ be the (ordinary) generating function
of a sequence which is an inverse $r$-th root of a polynomial $p(x)$,

$$g(x) \equiv \sum_{n \geq 0} g_n x^n = \frac{1}{\sqrt[r]{p(x)}}, \quad r \neq 0,$$

$$p(x) \equiv \sum_{n=0}^{\text{deg } p} p_n x^n.$$

Then the chain rule of differentiation yields a first derivative of (1),

$$g'(x) = -\frac{1}{r} \frac{p'(x)}{p^{1+1/r}(x)} = -\frac{1}{r} \frac{p'(x)}{p(x)} g(x).$$

Date: September 7, 2021.

2020 Mathematics Subject Classification. Primary 11Y40; Secondary 05A15, 11B37, 11Y55.
Key words and phrases. P-finite, D-finite, holonomic recurrence, generating function.
Obviously \( g(x) \) is a D-finite function \([27, 17]\):

\[
(4) \quad rp(x)g'(x) + p'(x)g(x) = 0.
\]

The P-finite recurrence is derived by insertion of the two power series:

\[
(5) \quad r \sum_{i=0}^{\deg p} p_i x^i \sum_{j \geq 1} j g_j x^{j-1} + \sum_{i=1}^{\deg p} i p_i x^i \sum_{j \geq 0} g_j x^j = 0.
\]

Resummation with \( k \equiv i + j - 1 \) yields

\[
(6) \quad r \sum_{k=0}^{\deg p-1} k+1 \sum_{j=1}^{k+1} p_{k+1-j} j g_j x^k + r \sum_{k=\deg p}^{k+1} \sum_{j=\deg p}^{k+1} p_{k+1-j} j g_j x^k + \sum_{k=0}^{\deg p-2} \sum_{j=0}^{k} (k+1-j) p_{k+1-j} g_j x^k = 0.
\]

Comparison of coefficients for \( x^k \), \( k \geq \deg p \), on both sides gives

\[
(7) \quad r \sum_{j=\deg p}^{k+1} p_{k+1-j} j g_j + \sum_{j=\deg p}^{k} (k+1-j) p_{k+1-j} g_j = 0, \quad k \geq \deg p.
\]

Setting \( n \equiv k + 1 \) gives

\[
(8) \quad r \sum_{j=\deg p}^{n} j p_{n-j} g_j + \sum_{j=\deg p}^{n-1} (n-j) p_{n-j} g_j = 0, \quad n \geq \deg p + 1;
\]

\[
(9) \quad r np_0 g_n + \sum_{j=\deg p}^{n-1} [r j + n - j] p_{n-j} g_j = 0, \quad n \geq \deg p + 1.
\]

Setting \( j = n - t \) gives

\[
(10) \quad r np_0 g_n + \sum_{t=1}^{\deg p} (rn - rt + t) p_t g_{n-t} = 0, \quad n \geq \deg p + 1.
\]

As an extension of Noe’s \([20]\) recurrences we find:

**Theorem 1.** The coefficients of the generating function \((1)\) obey the P-finite recurrence

\[
(11) \quad \sum_{t=0}^{\deg p} (rn - rt + t) p_t g_{n-t} = 0, \quad n \geq \deg p + 1.
\]

**Remark 1.** For \( r = 1 \) this equation may be divided through a common factor \( n \),

\[
(12) \quad \sum_{t=0}^{\deg p} p_t g_{n-t} = 0, \quad n \geq \deg p + 1, \quad r = 1,
\]

which is a recurrence with constant coefficients \( p_t \). We recover the well known result: The sequences with a rational generating function \( p(x)/q(x) \) have recurrences where the D-finite equation does not contain derivatives of \( g(x) \), so the \( g(n) \) obey C-finite recurrences.
Remark 2. Our formulas for recurrences normalize the representation by using indices $g_{n-t}$, $t \geq 0$, because that is the most readable convention while implementing computer programs that derive sequences from lower-order terms [19, 23].

Example 1. Examples of (1), square roots \( r = 2 \), degree \( \deg p = 1, 2 \), including the OEIS numbers of the coefficient sequences [14]

\[
\begin{align*}
p(x) & \quad \text{OEIS number} \\
1 - 2x - 3x^2 & \quad A002426 \\
1 - 6x - 3x^2 & \quad A122868 \\
1 - 6x + x^2 & \quad A001850 \\
1 - 6x + 5x^2 & \quad A026375 \\
1 - 4x - 4x^2 & \quad A006139 \\
1 - 4x & \quad A000984 \\
1 - 2x + 5x^2 & \quad A098331 \\
1 - 4x^2 & \quad A126869
\end{align*}
\]

Example 2. Examples of (1), square roots \( r = 2 \), degree \( \deg p = 3 \)

\[
\begin{align*}
p(x) & \quad \text{OEIS number} \\
1 - 4x^2 - 4x^3 & \quad A115962 \\
1 - 2x - 7x^2 + 8x^3 & \quad A098477 \\
1 - 2x - 3x^2 + 4x^3 & \quad A026569 \\
1 - 2x - 3x^2 - 4x^3 & \quad A191354 \\
1 - 2x + x^2 - 4x^3 & \quad A098479 \\
1 - 2x + x^2 - 8x^3 & \quad A098480 \\
1 - 4x - 8x^2 - 4x^3 & \quad A137635 \\
1 - 4x + 8x^3 & \quad A165431 \\
1 - 4x + 4x^3 & \quad A157004
\end{align*}
\]

Example 3. Examples of (1), roots \( r \neq 2 \)

\[
\begin{align*}
p(x) & \quad r & \quad \text{OEIS number} \\
1 - 4x & \quad \frac{2}{3} & \quad A002457 \\
1 - 8x & \quad \frac{2}{3} & \quad A115902 \\
1 - 36x & \quad \frac{6}{11} & \quad A004998 \\
1 + 9x + 9x^3 & \quad -3 & \quad A298308 \\
1 - 9x - 27x^3 & \quad 3 & \quad A095776
\end{align*}
\]

3. Generalized Inverse Root

The case with a non-trivial numerator polynomial \( q(x) \) and denominator polynomial \( v(x) \) generalizes the content of Chapter 2:

\[
g(x) = \frac{q(x)}{v(x) \sqrt[p]{p(x)}}
\]

where expansion coefficients \( q_n \) and \( v_n \) are defined via

\[
q(x) = \sum_{n=0}^{\deg q} q_n x^n; \quad v(x) = \sum_{n=0}^{\deg v} v_n x^n.
\]

Multiply (13) by \( v(x) \), omitting the argument \( x \) for brevity:

\[
v g = q p^{-1/r}
\]
The derivative of this equation with respect to \( x \) is

\[
vg' + v'g = q'p^{-1/r} - \frac{1}{r}qq'p^{-1-1/r}
\]

\[
= q'[qp^{-1/r}/v] - p'^vq^{-1/r} - \frac{1}{r}p'v^{1/r}.
\]

Multiplied by \( rqp \) this is a first-order D-finite differential equation with polynomial coefficients:

\[
rqpvg' + (rqp'v - rpq'v + qp'v)g = 0.
\]

Assemble two auxiliary polynomials with coefficients \( R_n \) and coefficients \( Q_n \):

\[
R(x) \equiv rqpv = \sum_{n=0}^{\deg R} R_n x^n; \quad Q(x) \equiv rqp'v - rpq'v + qp'v = \sum_{n=0}^{\deg Q} Q_n x^n,
\]

such that (17) reads

\[
Rg' + Qg = 0.
\]

**Remark 3.** (13) is a convolution of the sequence with generating function \( q/v \) by the sequence with generating function (1). Compatible with Stoll’s remark [28], the product \( R = rqpv \) indicates that the product of the degrees of the recurrences of the convoluted sequences is a bound for the degree of the recurrence.

**Remark 4.** If \( p(x) \) is not a polynomial but a rational polynomial, the structure is preserved if \( R(x) \) and \( Q(x) \) are multiplied by the denominators that appear through the evaluation of \( p \) and \( p' \), so the format (19) stays. The generic background is: replacing a generating function \( g(x) \rightarrow q(x) v(x)p(x) \) with polynomials \( q, v, w \) and \( p \) preserves holonomicity (which is obvious because the chain rule applied to the complicated generating function merely emits additional rational polynomial factors in comparison to the differential equation of the simple \( g(x) \)). The binomial transformation [4] of a P-finite sequence is again P-finite, for example.

\[
deg R_i x^j \sum_{j=1}^{\deg R} jg_j x^{i-1} + \sum_{i=0}^{\deg Q} Q_i x^j \sum_{j=0}^{\deg Q} g_j x^j = 0.
\]

\[
\sum_{k=0}^{\deg Q-1} k Q_k g_j x^k = 0.
\]

Comparison of coefficients \([x^k]\) for sufficiently large \( k \) on both sides yields the P-finite recurrence

\[
\sum_{j=k+1-\deg R}^{k+1} jR_{k+1-j}g_j + \sum_{j=k-\deg Q}^{k} Q_{k-j}g_j = 0, \quad k \geq \max(\deg R, \deg Q).
\]
Flip the direction in both $j$-sums:

\[(23) \quad \sum_{j=0}^{\deg R} (k + 1 - j) R_j g_{k+1-j} + \sum_{j=0}^{\deg Q} Q_j g_{k-j} = 0, \quad k \geq \max(\deg R, \deg Q).\]

Substitute $k + 1 = n$:

\[(24) \quad \sum_{j=0}^{\deg R} (n-j) R_j g_{n-j} + \sum_{j=0}^{\deg Q} Q_j g_{n-j-1} = 0, \quad n > \max(\deg R, \deg Q).\]

Replace $j \to j - 1$ in the second term:

\[(25) \quad \sum_{j=0}^{\deg R} (n-j) R_j g_{n-j} + \sum_{j=1}^{\deg Q + 1} Q_{j-1} g_{n-j} = 0, \quad n > \max(\deg R, \deg Q).\]

In the case of (18) the degree of $Q$ is one less than the degree of $R$ because it contains one more derivative. If we define coefficients $Q_n$ or $R_n$ to be zero if $n < 0$ or $n$ larger than the degrees, this may be condensed as:

**Theorem 2.** The $P$-finite recurrence of a sequence with the generating function (13) is

\[(26) \quad \sum_{j=0}^{\deg R} [(n-j) R_j + Q_{j-1}] g_{n-j} = 0, \quad n > \deg R,\]

where $R(x)$ and $Q(x)$ are the polynomials defined by the sums and derivatives (18) of the three polynomials $p(x)$, $q(x)$ and $v(x)$.

**Remark 5.** To keep the recurrences simple, common polynomial factors of $x$ in homogeneous differential equations like (19) should be eliminated (for example by finding the greatest common divisor with the Euclidean algorithm) before defining $R$ and $Q$.

**Example 4.** Examples of (13) with square roots $r = 2$:

| $q(x)$ | $p(x)$ | $v$ | A-number |
|--------|--------|-----|---------|
| $1 - x$ | $1 - 6x + x^2$ | $1$ | A110170 |
| $1 + x$ | $1 - 6x + x^2$ | $1$ | A241023 |
| $1 - x$ | $1 - 6x + 5x^2$ | $1$ | A085362 |
| $1 - x$ | $1 - 2x - 3x^2$ | $1$ | A025178 |
| $x(1 + x)$ | $1 - 2x - 3x^2$ | $1$ | A025565 |
| $1 + 2x$ | $1 - 4x^2$ | $1$ | A063886 |
| $1 + x$ | $1 + 4x^2$ | $1$ | A128057 |
| $1 - 4x$ | $1 - x^2$ | $1$ | A106188 |
| $1$ | $1 + 4x$ | $1 - 4x$ | A091520 |

We did not require that $r$ is an integer or positive. So formats like $g(x) = \sqrt{q(x)/p(x)}$, $r = -2$ or $g(x) = \sqrt{q(x)}/p^{\frac{r}{2}}(x)$, $r = 2/3$ are also covered.

The format

\[(27) \quad g = \sqrt{q(x)/p(x)}\]

is reduced to the format (13) by multiplying numerator and denominator by $\sqrt{q(x)}$ such that the numerator is root-free. The shortcut is:
Theorem 3. The generating function (27) obeys the differential equation

\[ Rg' + Qg = 0, \]

with polynomials \( R \equiv 2qp \) and \( Q = qp' - q'p \), such that (26) applies.

If an additive polynomial \( w(x) \) appears on the right hand side like

\[ g(x) = w(x) + \frac{q(x)}{v(x) \sqrt{p(x)}}, \]

this modifies the coefficients \( g_n \) for \( n \) up to the degree of the polynomial \( w(x) \). It delays the validity of (26) to the point that all indices \( j \) of the coefficients \( g_j \) must be larger than the degree of \( w(x) \), so \( n \) in Theorem 2 must be larger than the sum of \( d \) and the polynomial degree of \( w \).

4. Generalized Inverse Root II

4.1. Rooted Denominator. The case with a non-trivial numerator polynomial \( q(x) \) and denominator polynomials \( v(x) \) and \( w(x) \) generalizes the content further:

\[ g(x) = \frac{q(x)}{w(x) + v(x) \sqrt{p(x)}}, \]

where expansion coefficients \( q_n, v_n \) and \( w_n \) are defined via

\[ q(x) \equiv \sum_{n \geq 0} q_n x^n; \quad v(x) \equiv \sum_{n \geq 0} v_n x^n; \quad w(x) \equiv \sum_{n \geq 0} w_n x^n. \]

With the strategy of Section 3 one ends up with a differential equation which contains terms proportional to \( g'^2 \), which (apparently) does not lead to recurrences with a finite number of terms. If \( r \) is a positive integer, \( g \) is algebraic with \( g^r v^p = (q - gw)^r \). So \( g(x) \) is also holonomic [27, Thm. 2.1][3]. Comtet’s long division algorithm then yields the polynomial coefficients of the associated linear differential equation of \( g(x) \) [12, 11, 5]. Final reshuffling of the coefficients of these polynomials by lowering the exponents gives the P-finite recurrence.

Some progress can be made for square roots, \( r = 2 \), multiplying numerator and denominator of the fraction by \( w - v \sqrt{p} \):

\[ (w^2 - v^2 p)g = wq - vq \sqrt{p}. \]

The first derivative is

\[ (w^2 - v^2 p)g' + (w^2 - v^2 p)g = (wq)' - (vq)' \sqrt{p} - vq p' \frac{1}{2 \sqrt{p}} \]

\[ = (wq)' - \left( (vq)' + \frac{vq p'}{2p} \right) \sqrt{p}. \]

Multiply by \( 2p \) to eliminate all denominators,

\[ 2p(w^2 - v^2 p)g' + 2p(w^2 - v^2 p)g = 2p(wq)' - [2(vq)'p + vqp'] \sqrt{p}. \]

To eliminate the square root, multiply this equation by \( vq \), multiply (32) by \( 2(vq)'p + vqp' \), and subtract both equations

\[ 2pvq(w^2 - v^2 p)'g + 2pvq(w^2 - v^2 p)'g + [2(vq)'p + vqp'](w^2 - v^2 p)g \]
\[ = -2pvq(wq)' + [2(vq)'p + vqp']wq. \]
This is a first order differential equation with polynomial coefficients $R(x)$, $Q(x)$ and $H(x)$,
\begin{equation}
R(x)g' + Q(x)g = H(x),
\end{equation}
where
\begin{equation}
R(x) \equiv 2pvq(v^2 p - w^2) \equiv \sum_{j=0}^{\deg R} R_j x^j;
\end{equation}
\begin{equation}
Q(x) \equiv -4pvqw' + 2pq(pv^2 + w^2) + vq(pv^2 + w^2) p' - 2pv(pv^2 - w^2) q' \equiv \sum_{j=0}^{\deg Q} Q_j x^j;
\end{equation}
\begin{equation}
H(x) \equiv -q^2(2pv^2 - 2wpv - wp^2).
\end{equation}

**Remark 6.** For $w = 0$ the polynomials reduce to $H = 0$, $R = 2p^2 v^3 q$, $Q = pv(2pqv + vq' - 2pq')$. The differential equation can be divided by the common factor $pv^2$ of $R$ and $Q$ because $H$ is zero, and (18)–(19) emerge as a special case.

**Remark 7.** Unlike (36), holonomic functions are defined to obey a differential equation where no term such as $H(x)$ exists, which is not $g$ or a derivative of $g$. That format is obtained by differentiating (36) $d/dx 1 + \deg H$ times, such that $H$ disappears in the final higher-order differential equation [27]. The philosophy in this paper is to keep the order of the differential equation as low as possible, even if that means that the recurrence steps in belatedly due to the influence of the $H(x)$ on the low powers of $x$.

The further reduction follows exactly the path of Section 3, paying attention to eliminate the early $a$-coefficients where the low, non-vanishing orders of $H(x)$ may interfere:

**Theorem 4.** The P-finite recurrence of a sequence with the generating function (30) at $r = 2$ is
\begin{equation}
\sum_{j=0}^{\deg R} (n - j)R_j + Q_{j-1} g_{n-j} = 0, \quad n - \deg R > \deg H,
\end{equation}
where $R(x)$ and $Q(x)$ are the polynomials defined by the sums and derivatives (37)–(38) of the four polynomials $p(x)$, $q(x)$, $v(x)$ and $w(x)$.

**Example 5.** Examples of (30):
\[
p \quad q \quad v \quad w \quad r \quad 1 - 2x - 3x^2 \quad 1 + x \quad -x \quad 2 \quad A116394
\]

4.2. **Rooted Numerator.** If the generating function has the shape
\begin{equation}
g(x) = \frac{w(x) + v(x) \sqrt{p(x)}}{q(x)}
\end{equation}
multiply this equation by $q$, derive it, and eliminate $\sqrt{p}$ with the aid of the previous equation:
\begin{equation}
qg' + q' g = w' + v' \sqrt{p} + \frac{1}{r} v' \frac{p'}{p} \sqrt{p} = w' + \left[ v' + \frac{vp'}{rp} \right] \frac{qg - w}{v};
\end{equation}
Multiply by \( rpv \) to obtain an equation which fits (36), this time with
\[
R(x) \equiv rpv = \sum_{n=0}^{\text{deg } R} R_n x^n;
\]
(43)
\[
Q(x) \equiv rp(q'v - qv') - vp'q = \sum_{n=0}^{\text{deg } Q} Q_n x^n;
\]
(44)
\[
H(x) \equiv rp(w'v - wv') - vp'w;
\]
(45)

Theorem 5. The P-finite recurrence of a sequence with the generating function (41) is given by (40) where \( R(x) \) and \( Q(x) \) are the polynomials defined by the sums and derivatives (43)-(44) of the four polynomials \( p(x) \), \( q(x) \), \( v(x) \) and \( w(x) \).

This formula and Section 2 cover Callan’s generating functions [9].

The generating functions of the form
\[
g = \frac{u(x)}{w(x)} + \frac{q(x)}{v(x) \sqrt{p(x)}}
\]
with polynomials \( p(x) \), \( v(x) \), \( u(x) \) and \( w(x) \) are also covered by the form (41) because they can be rewritten as
\[
g = \frac{u(x)v(x)p(x) + w(x)q(x)p^{1-1/c}(x)}{w(x)v(x)p(x)}
\]
(47)

This allows to find P-recurrences of sequence which are sums of C-finite sequences represented by \( u(x)/w(x) \) and sequences represented by \( q(x)/[v(x) \sqrt{p(x)}] \), such as transiting from [14, A026375] to [14, A242586].

The form with a common square root in numerator and denominator is also in this class:
\[
g = \frac{u(x)v(x)p(x) - q(x)w(x) + |w(x)v(x) - q(x)u(x)| \sqrt{p(x)}}{v^2(x)p(x) - q^2(x)}
\]
(48)

4.3. Orthogonal Polynomials. Some orthogonal polynomials have generating functions which are in our category of rational square root expressions [1, §22.9]. If the argument \( x \) of these orthogonal polynomials is kept fixed and their degree \( n \) used as the index of the \( g(n) \), their well-known 3-term recurrences appear [1, 22.7].

5. Generalized Hypergeometric Function

5.1. Power Series. The Generalized Hypergeometric Functions \( _pF_q(x) \) with a set of constant “numerator” \( \{ \alpha \}_p \) and “denominator” \( \{ \beta \}_q \) are another special case with simple P-finite recurrences [26]:
\[
g(x) = \sum_{n \geq 0} \prod_{i=1}^{p} (\alpha_i)_n \prod_{j=1}^{q} (\beta_j)_n \frac{x^{t+rn}}{n!c^n},
\]
where \( (\cdot)_n \) are Pochhammer symbols [1, (13.1.2)]
\[
(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + n - 1).
\]
(50)
The non-vanishing coefficients of the power series are
\[(51)\]
\[g_{r_{n+t}} = \prod_i (\alpha_i)_n \prod_j (\beta_j)_n n! e^{\alpha_i}.\]

The associated P-finite 2-term recurrence is
\[(52)\]
\[c(n+1) \prod_j (\beta_j + n)g_{r_{n+r+t}} = \prod_i (\alpha_i + n)g_{r_{n+t}}.\]

Example 6. The generating function
\[(53)\]
\[(1 - x)^\alpha = F_0(-\alpha; x)\]
is a borderline case between (1) and (49). Also
\[(54)\]
\[\arcsin x = x_2 F_1(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x^2),\]
\[(55)\]
\[J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1)} F_1(\nu + 1; -z^2/4),\]
and
\[(56)\]
\[\ln(1 + x) = x_2 F_1(1, 1; 2; -x)\]
fit in here [7, §0.7.2][16].

5.2. Sequence Index in Parameters. There is another family of recurrences associated with hypergeometric functions. If the sequence entries \(g(n)\) are hypergeometric functions of some constant argument \(x\) where the index \(n\) appears in the parameters such that \(\alpha_p\) or \(\beta_p\) are polynomials of \(n\), the 3-term contiguous relations of the Gaussian Hypergeometric Functions [1, §15.2], the 3-term recurrences of the Confluent Hypergeometric Functions [1, §13.4][22] and the contiguous relations for other pairs of \((p, q)\) [21, 2, 29] provide the P-finite recurrences for \(g(n)\).

A side aspect is that for the terminating hypergeometric functions in that case, i.e., polynomials of \(n\), the P-finite recurrences are even C-finite.

6. Nested Roots

6.1. Elliptic Integrals. Let
\[(57)\]
\[g(x) = \sqrt[1]{f(x)}\]
be a generating function with a discriminant function \(f(x)\). Then
\[(58)\]
\[g' = \frac{1}{r} \frac{f'}{f} g;\]

Suppose also that \(f(x)\) obeys a D-finite first-order differential equation of the form
\[(59)\]
\[L(x)f' + H(x)f = 0,\]
with polynomials \(L(x)\) and \(H(x)\), which implies that \(\log f(x) = -\int \frac{H(x)}{L(x)} dx\) are Elliptic Integrals. Then (58) becomes
\[(60)\]
\[r L(x)g' + H(x)g = 0.\]

This means if we have obtained a D-finite first-order differential equation for a generating function, the D-finite differential equation equation for a generating function that is some (fractional) power of the original generating function is inherited (apart from the factor \(r\)) by the derived generating function [16]. For squares of hypergeometric functions see e.g. Chaundy and Vidunas [10, 30].
6.2. Polynomial Discriminants. Let

\[ g(x) = \sqrt{w(x) + \sqrt{p(x)}} \]

be a generating function with polynomials \( w(x) \) and \( p(x) \).

The first and second derivatives of (61) are

\[ g' = \frac{1}{r}(w' + \frac{1}{2} \frac{p'}{\sqrt{p}})(w + \sqrt{p})^{1/r - 1}; \]

\[ g'' = \frac{1}{r}[\frac{1}{r - 1}(w' + \frac{1}{2} \frac{p'}{\sqrt{p}})^2(w + \sqrt{p})^{3/2 - 2} + (w'' + \frac{1}{2} \frac{p''}{\sqrt{p}} - \frac{1}{4} \frac{p'^2}{p^{3/2}})(w + \sqrt{p})^{1/r - 1}] \]

With the ansatz

\[ T(x)g'' + R(x)g' + Q(x)g = 0 \]

we assume that this generating function is D-finite with three polynomials \( T(x) \), \( R(x) \) and \( Q(x) \). This requires

\[ T(x)\frac{1}{r}[\frac{1}{r - 1}(w' + \frac{1}{2} \frac{p'}{\sqrt{p}})^2(w + \sqrt{p})^{3/2 - 2} + (w'' + \frac{1}{2} \frac{p''}{\sqrt{p}} - \frac{1}{4} \frac{p'^2}{p^{3/2}})(w + \sqrt{p})^{1/r - 1}] \]

\[ + R(x)\frac{1}{r}(w' + \frac{1}{2} \frac{p'}{\sqrt{p}})(w + \sqrt{p})^{1/r - 1} + Q(x)(w + \sqrt{p})^{1/r} = 0, \]

and multiplied by \( r^2(w + \sqrt{p})^{2 - \frac{3}{r}} \)

\[ T(x)\left[(1 - r)(w' + \frac{1}{2} \frac{p'}{\sqrt{p}})^2 + r(w'' + \frac{1}{2} \frac{p''}{\sqrt{p}} - \frac{1}{4} \frac{p'^2}{p^{3/2}})(w + \sqrt{p})\right] \]

\[ + R(x)r(w' + \frac{1}{2} \frac{p'}{\sqrt{p}})(w + \sqrt{p}) + Q(x)r^2(w + \sqrt{p})^2 = 0. \]

Expanding squares and products this reads

\[ T(x)[(1 - r)w'' + (1 - r)w' \frac{p'}{\sqrt{p}} + (\frac{1}{4} - \frac{r}{2}) \frac{p'^2}{p}] \]

\[ + rw'' + \frac{1}{2} rw' \frac{p'}{\sqrt{p}} - \frac{1}{4} rw \frac{p'^2}{p^{3/2}} + rw' \sqrt{p} + \frac{1}{2} rp'' \]

\[ + R(x)(rw' w + \frac{1}{2} rw w' \frac{p'}{\sqrt{p}} + rw' \sqrt{p} + \frac{1}{2} rp') + Q(x)(r^2 w^2 + 2r^2 w \sqrt{p} + r^2 p) = 0. \]

This has the structure

\[ T(x)[\alpha_1(x) + \frac{1}{p^{3/2}} \alpha_4(x)] + R(x)[\alpha_2(x) + \frac{1}{p^{3/2}} \alpha_5(x)] + Q(x)[\alpha_3(x) + \frac{1}{p^{3/2}} \alpha_6(x)] = 0, \]
where 6 α-coefficients are rational polynomials in x defined as

\[\alpha_1(x) = (1 - r)w'^2 + \frac{1}{4} - \frac{r}{2}p'^2 + rw''w + \frac{1}{2}rp'';\]

\[\alpha_2(x) = rw'w + \frac{1}{2}rp';\]

\[\alpha_3(x) = r^2w^2 + r^2p;\]

\[\alpha_4(x) = (1 - r)wp''p + \frac{1}{2}rwp(p - \frac{1}{4}rwp'^2 + rw''p^2;\]

\[\alpha_5(x) = \frac{1}{2}rwp'p + rw'p^2;\]

\[\alpha_6(x) = 2r^2wp^2.\]

Instead of solving (68) in general we continue with a separation ansatz, where the components which depend on \(1/p^{3/2}\) and do not depend on it are individually zero:

\[(75) \quad T(x)\alpha_1(x) + R(x)\alpha_2(x) + Q(x)\alpha_3(x) = 0;\]

\[(76) \quad \land \quad T(x)\alpha_4(x) + R(x)\alpha_5(x) + Q(x)\alpha_6(x) = 0.\]

In the language of 3-dimensional vector algebra, the vector \((T, R, Q)\) is orthogonal to the vector \((\alpha_1, \alpha_2, \alpha_3)\) as well as orthogonal to the vector \((\alpha_4, \alpha_5, \alpha_6)\), so it is the vector cross product of the two α-vectors:

\[(77) \quad T(x) = \alpha_2\alpha_6 - \alpha_3\alpha_5;\]

\[(78) \quad R(x) = \alpha_3\alpha_4 - \alpha_1\alpha_6;\]

\[(79) \quad Q(x) = \alpha_1\alpha_5 - \alpha_2\alpha_4.\]

**Remark 8. This construction of a d-dimensional vector \(V\) that is orthogonal to \(d - 1\) vectors \(v^{(1)}, v^{(2)}, \ldots, v^{(d-1)}\) such that \(\sum_{i=1}^{d} V_i v_i^{(1)} = \sum_{i=1}^{d} V_i v_i^{(2)} = \cdots = 0\) carries over to more than 3 dimensions [25]: the i-th component of \(V\) is the tensor sum-product (determinantal mix) of the \(\epsilon\)-Tensor (parity of the permutation of its indices) with the product of the components of the \(v\)-vectors:

\[(80) \quad V_i = \sum_{j=1}^{d} \sum_{k=1}^{d} \sum_{m=1}^{d} \epsilon_{ijk\ldots m} v_j^{(1)} v_k^{(2)} \cdots v_m^{(d-1)}.\]

Insertion of the α-terms and multiplying \(T, R\) and \(Q\) with a common factor \(8/r\) yields:

\[(81) \quad T = 4r^2(\omega^2 + p)p(-2w'p + wp');\]

\[(82) \quad R = -2r(-4w^2wp'p + 4w^2w'p^2p - 4w^3wp''p + rw^3p'^2 + 4p^2w''p^2 - 4p^2w'p' + 4p^3w'p'\]

\[+ 2r^2wp'' + 5rwp^2 - 4rwp^2p'' - 8wp^2w'^2 - 8rwp^2w'^2 - 2wp^2);\]

\[(83) \quad Q = -4w^2wp'p + 8w^2p^2 + 4w^2w'p^2 - 8w^3p^2 - p^3w + 6p^2wp' + 3p^3w +
\[8p^2wp' + 4p^3w^2p'p + 4p^2w'p^2 + 2w'w^2p''p + 2w'w^2p^2 - 4p^2w'^2.\]

Working backwards through the logic shows that the ansatz (64) is indeed satisfied.
Theorem 6. The coefficients of the generating function (61) obey the P-finite recurrence

\[ \sum_{j \geq 0} [(n-j)(n-j-1)T_j + (n-j)R_{j-1} + Q_{j-2}] g_{n-j} = 0, \]

where \( T = \sum_{n \geq 0} T_n x^n \), \( R = \sum_{n \geq 0} R_n x^n \) and \( Q = \sum_{n \geq 0} Q_n x^n \) are the polynomials (81)–(83).

If the polynomials of \( n \) in front of the \( g_{n-j} \) are given in the standard basis of powers of \( n \), the coefficients \( T_j \), \( R_j \) and \( Q_j \) are easily recovered by accumulating [1, 24.1.4] [24]

\[ n^t = \sum_{m=0}^{t} S_t^{(m)} n(n-1)(n-2) \cdots (n-m+1), \]

where \( S \) are the Stirling Numbers of the Second Kind.

6.3. Degenerate cases. \( T(x) \) of (81) is zero if \( w^2 = p \) or \( wp' = 2w'p \), and the simpler (19) applies; the P-recurrence only involves first-degree polynomials. The case \( w^2 = p \) is not interesting: then \( g = \sqrt{2w} \) has the format (1) and would be treated accordingly. The case \( wp' = 2w'p \) means \( 2w'/w = p'/p \), therefore \( 2 \ln w = \ln p + C \), therefore \( \ln w^2 = \ln p + C \), therefore \( w^2 = Cp \), and again (1) is the underlying format.

6.4. Deeper Nests. The roots may be nested deeper where

\[ g(x) = \sqrt[r_k]{w_k(x)} \sqrt[r_{k-1}]{w_{k-1}(x)} \cdots \sqrt[r_1]{w_1(x)}, \]

were the \( w_i(x) \) are polynomials and where the \( r_i \) are integers. [A first approach to obtain the coefficients \( g_n \) numerically is to regard this as the composition of roots [6].]

\( g(x) \) is an algebraic function: take the \( r_k \)-th power of both sides, move \( w_k \) to the left side, take the \( r_{k-1} \)-st power, move \( w_{k-1}(x) \) to the left side and so on to establish its algebraic equation. The highest power is \( g^{r_k r_{k-1} \cdots r_1}(x) \). The generic strategy of Section 4.2 establishes a recurrence.

7. Exponentials

7.1. Exponential of Root. The class of generating functions

\[ g(x) = \exp[w(x) \pm \sqrt[p]{p(x)}] \]

with polynomials \( w(x) \) and \( p(x) \) has similar regenerative properties as the nested roots:

\[ g' = \left( w' \pm \frac{1}{2} p' p^{-1/2} \right) \exp[w \pm \sqrt[p]{p}]; \]

\[ g'' = \left( w'' \pm \frac{1}{2} p'' p^{-1/2} + \frac{1}{4} p'^2 p^{-3/2} \right) \exp[w \pm \sqrt[p]{p}] + \left( w' \pm \frac{1}{2} p' p^{-1/2} \right)^2 \exp[w \pm \sqrt[p]{p}]. \]

Remark 9. If \( p(x) = 0 \), the subsequent sections are superfluous because then \( g' - w'g = 0 \), which is a special case of (19), so the recurrence (26) applies.
The same procedure as in Section 6 unfolds:

\[(90)\]

\[
T \left[ \left( w'' \pm \frac{1}{2} p'' p^{-1/2} \mp \frac{1}{4} p'^2 p^{-3/2} \right) \exp[w \pm \sqrt{p}] + \left( w' \pm \frac{1}{2} p' p^{-1/2} \right)^2 \exp[w \pm \sqrt{p}] 
\right.
\]

\[
+ R \left( w' \pm \frac{1}{2} p' p^{-1/2} \right) \exp[w \pm \sqrt{p}] + Q \exp[w \pm \sqrt{p}] = 0.
\]

\[(91)\]

\[
T \left[ w'' \pm \frac{1}{2} p'' p^{-1/2} \mp \frac{1}{4} p'^2 p^{-3/2} \pm \left( w' \pm \frac{1}{2} p' p^{-1/2} \right)^2 \exp[w \pm \sqrt{p}] \right] + R \left( w' \pm \frac{1}{2} p' p^{-1/2} \right) + Q = 0.
\]

\[(92)\]

\[
T \left[ w'' \pm \frac{1}{2} p'' p^{-1/2} \mp \frac{1}{4} p'^2 p^{-3/2} \pm w'^2 + w' p' p^{-1/2} + \frac{1}{4} p'^2 + R \left( w' \pm \frac{1}{2} p' p^{-1/2} \right) \right] + Q = 0.
\]

Remark 10. If $\sqrt{p(x)}$ is replaced by $\sqrt[p]{p}$ in the generating function (87), terms proportional to $p^{1/r}$, proportional to $p^{2/r}$ and not related to $p^{1/r}$ appear, and a higher-dimensional ansatz following Remark 8 is needed to disentangle the three algebraic branches.

The coefficients matching \((68)\) are

\[(93)\]

\[
\alpha_1 = w'' + w'^2 + \frac{1}{4} p'^2 \frac{1}{p};
\]

\[(94)\]

\[
\alpha_2 = w';
\]

\[(95)\]

\[
\alpha_3 = 1;
\]

\[(96)\]

\[
\alpha_4 = \pm \frac{1}{2} p'' p \pm \frac{1}{4} p'^2 \pm w' p';
\]

\[(97)\]

\[
\alpha_5 = \pm \frac{1}{2} p' p;
\]

\[(98)\]

\[
\alpha_6 = 0.
\]

With these we gather three polynomials \((77)\)–\((79)\) and multiply $T$, $R$ and $Q$ by the common factor 8 to maintain integer coefficients:

\[(99)\]

\[
T(x) = \pm 4 p' p = \sum_{n \geq 0} T_n x^n;
\]

\[(100)\]

\[
R(x) = \pm 2(2 p'' p - p'^2 + 4 w' p p) = \sum_{n \geq 0} R_n x^n;
\]

\[(101)\]

\[
Q(x) = \pm (4 p'' p w'' - 4 p' p w'^2 + p^3 - 4 w' p'' p + 2 w' p') = \sum_{n \geq 0} Q_n x^n.
\]

Theorem 7. The coefficients of the generating function (87) obey the P-finite recurrence (84) where $T(x)$, $R(x)$ and $Q(x)$ are the polynomials \((99)\)–\((101)\).

7.2. Exponential Times Arithmetic. If the generating function is of the kind

\[(102)\]

\[
g(x) = \exp[q(x)/v(x)] \frac{1}{\sqrt[p]{p(x)}}
\]
with polynomials \( p(x), q(x) \) and \( v(x) \), then by the product and chain rules

\[
g'(x) = \left( \frac{q'}{v} - \frac{qv'}{v^2} \right) \exp(q/v) \frac{1}{\sqrt{p}} - \frac{1}{r} \exp(q/v) \frac{p'}{p^{1+1/r}} = \left( \frac{q'}{v} - \frac{qv'}{v^2} \right) g - \frac{1}{r} \frac{p'}{p} g.
\]

Multiplication with \( r p v^2 \) yields the first order differential equation

\[
rpv^2 g' = [rp(q' v - q v') - v^2 p'] g.
\]

So we face (19) but this time with the polynomials

\[
R(x) \equiv r p v^2, \quad Q(x) \equiv v^2 p' - rp(q' v - q v').
\]

and apply the recurrence (26).

8. Logarithm of Rational

If the generating function is of the kind

\[
g(x) = \log[q(x)/v(x)]
\]

with polynomials \( p(x) \) and \( q(x) \), then

\[
g'(x) = \frac{v}{q} \left( \frac{q'}{v} - \frac{qv'}{v^2} \right).
\]

This is a special case of (36) substituting

\[
R(x) = q v; \quad Q(x) = 0; \quad H(x) = v q' - q v'
\]

and obeys the recurrence (40). If in addition the Wronskian \( H \) is zero (i.e., if the two polynomials have a common factor), this simplifies furthermore to (19).

Remark 11. The logarithm of the ratio is the difference between the logarithms of numerator and denominator. These logarithms are separately holonomic, and by the closure property one may also generate the P-finite recurrence from the recurrences of the two terms. The technique to combine the two recurrences for a Hadamard sum has been demonstrated by Mallinger [18, Thm. 1.4.3] and Kauers [15].

9. Summary

We validated a set of P-recurrences of sequences which involve generating function with roots.

Appendix A. Inhomogeneous P-finite

If the sequence \( a \) obeys a P-finite recurrence with a polynomial \( I(n) \),

\[
\sum_{j \geq 0} P_j(n) g_{n-j} + I(n) = 0,
\]

it can be rewritten as a homogeneous P-finite recurrence by shifting the index by 1:

\[
\sum_{j \geq 0} P_j(n-1) g_{n-j-1} + I(n-1) = 0,
\]

multiplying (109) by \( I(n-1) \) and multiplying (110) by \( I(n) \) and subtracting both equations. This results in a recurrence which is one longer than (109) and has polynomial coefficients of degrees which are the sum of the degrees in (109) and the degree of \( I(n) \).
Appendix B. Exponential Generating Functions

A generating function \( g \) may be interpreted as an ordinary generating function for a sequence \( g(n) \) and at the same time as an exponential generating function for a sequence \( b(n) \):

\[
\sum_{n \geq 0} g_n x^n = \sum_{n \geq 0} b_n \frac{x^n}{n!}.
\]

If the \( g_n \) obey a P-finite recurrence with polynomials \( P \) and \( J + 1 \) terms,

\[
\sum_{j=0}^{J} P_j(n) g_{n-j} = 0,
\]

substituting \( g_n = b_n/n! \) and multiplication of the recurrence with \( n! \) yields an associated P-finite recurrence for the \( b_n \):

\[
\sum_{j=0}^{J} (n-j+1) P_j(n) b_{n-j} = 0.
\]

So the polynomials in the P-finite recurrence of the \( b \)-terms of the exponential generating function are the polynomials of the \( a \)-terms of the ordinary generating function multiplied by first-order polynomials with can be represented as Pochhammer symbols.

Remark 12. An equivalent match applies to logarithmic generating functions \( g(x) = \sum_{n \geq 0} c_n x^n/n \).

Example 7. If we reinterpret (13) as an exponential generating function and multiply the coefficients of (26) with the Pochhammer symbols, some OEIS sequences are covered:

\[
\begin{array}{cccc}
p(x) & q(x) & v(x) & r \\
1 - 4x + x^2 & 1 & 1 & 2 & A285199 \\
1 - 8x + x^2 & 1 & 1 & 2 & A006438 \\
1 + 2x + 4x^2 & 1 & 1 & 2 & A182827 \\
1 - 2x - 2x^2 & 1 & 1 & 2 & A098460 \\
1 - 2x - 3x^2 & 1 & 1 & 2 & A098461 \\
1 - 10x & 1 & 1 & 10 & A144773
\end{array}
\]

Example 8. If we reinterpret (102) as an exponential generating function and multiply the coefficients of (26) with the Pochhammer symbols, additional OEIS sequences are covered:
Appendix C. Reduction of the Number of Terms

In [14, A122877] the generating function

\[
g = \frac{1 - 2x - 3x^2 - (1 - x)\sqrt{1 - 2x - 7x^2}}{8x^3}
\]

matches (41) with polynomials \( q = 8x^3, w = 1 - 2x - 3x^2, v = -(1 - x), p = 1 - 2x - 7x^2 \) and \( r = 2 \), such that the differential equation (36) is set up with \( R = -16x^3(1 - x)(1 - 2x - 7x^2), Q = -16x^2(3 - 7x - 11x^2 + 7x^3) \) and \( H = -64x^3 \) defined in (43)–(45). The greatest common factor \(-16x^2\) of \( R, Q \) and \( H \) can be dropped in the differential equation:

\[
x(1 - x)(1 - 2x - 7x^2)g' + (3 - 7x - 11x^2 + 7x^3)g = 4x.
\]

Only the indices \( j = 1 \)–4 contribute to the recurrence (40), so the generating function supports a 4-term recurrence:

\[
(n + 3)g_n - (3n + 4)g_{n-1} - (5n + 1)g_{n-2} + 7(n - 2)g_{n-3} = 0
\]

with first degree polynomials. Differentiating of (36) yields a second order differential equation

\[
Rg'' + (R' + Q)g' + Q'g = H',
\]

here

\[
x(1 - x)(1 - 2x - 7x^2)g'' + (1 + x)(4 - 9x - 35x^2)g' + (-7 - 22x + 21x^2)g = 4.
\]

The number of the terms in the recurrence derived from the first-order differential equation is based on:

- The factor \( R \) contributes powers \( x^{0} \)– \( x^{\deg R} \); \( q' \) represents \( \sum ng_nx^{n-1} \), so the product has powers \( x^{n-1} \) up to \( x^{n-1 + \deg R} \).
- The factor \( Q \) contributes powers \( x^{0} \)– \( x^{\deg Q} \); \( g \) represents \( \sum g_nx^n \), so the product has powers \( x^n \) up to \( x^{n + \deg Q} \).

The range of powers is \( x^{n-1} \) up to the larger of \( x^{n-1 + \deg R} \) or \( x^{n + \deg Q} \), and the spread of exponents determines the number of coupled \( a \)-coefficients. The equivalent analysis of the second-order differential equation (using \( g'' = \sum n(n - 1)g_nx^{n-2} \))
shows that the exponents have been decremented by one, but the spread of exponents remains the same. [This preservation remains valid, even if some lower coefficients $R_n$ vanish, like in our example where $R_0 = 0$.] The numbers of terms in the P-recurrences derived from (36) and (117) are the same.

The penalty in (117), induced by $g'' \sim \sum n(n - 1)g_n$, is that the polynomials in the P-recurrences are of degree 2, not 1. However, if $\sum R_n = 0$ [equivalent: a factor $1 - x$ in the factorization of $R(x)$], the contribution of the $n^2$ terms in the recurrence derived from (117) vanishes. In that circumstance the P-recurrence from (117) also has coefficients which are polynomials of first degree. In the synoptical view on both recurrences of the same number of terms and the same polynomial degrees, one may multiply each recurrence with the polynomial in front of $g_n$ of the other recurrence, subtract both, to obtain a recurrence with one term less and with polynomial coefficients with a degree which is the sum of the individual degrees.

In the example considered here, the requirement on $R(x)$ is fulfilled, and besides (116) there is a 3-term recurrence

$$-(n + 3)(n - 1)g_n + n(2n + 1)g_{n-1} + 7n(n - 1)g_{n-2} = 0$$

with quadratic polynomials.

**Remark 13.** The derivative of D-finite differential equations with polynomial coefficients yields differential equations of higher order, equivalent to P-recurrences with polynomials of higher degrees, and potentially of smaller length. We take the stand that recurrences derived from differential equations of lower order are preferable, even if the number of terms in the P-recurrences (the length of the recurrences) is larger, because the step from the P-recurrences to the D-equation plus differentiation is straightforward, whereas the opposite direction (one integration of the D-equation) may be difficult and introduces further constants.

**References**

1. Milton Abramowitz and Irene A. Stegun (eds.), *Handbook of mathematical functions*, 9th ed., Dover Publications, New York, 1972. MR 0167642
2. W. N. Bailey, *Contiguous hypergeometric functions of the type $3f_2$*, Proc. Glasg. Math. Ass. 2 (1954), no. 2, 62–65. MR 0064918
3. Cyril Banderier and Michael Drmota, *Formulae and asymptotics for coefficients of algebraic functions*, Combin. Probab. Comput. 24 (2015), no. 1, 1–53. MR 3318039
4. Mira Bernstein and Neil J. A. Sloane, *Some canonical sequences of integers*, Lin. Alg. Applic. 226–228 (1995), 57–72, (E: [s]). MR 1344554
5. Alin Bostan, Frédéric Chyzak, Bruno Salvy, Grégoire Lecerf, and Éric Schost, *Differential equations for algebraic functions*, Proceedings of the 2007 Intl. Symp. on Symbolic and algebraic computations, 2007, pp. 25–32. MR 2396180
6. Richard P. Brent and H. T. Kung, *Fast algorithms for manipulating formal power series*, J. ACM 25 (1978), 581–595. MR 0520733
7. I. N. Bronstein and K. A. Semendjajew, *Teubner’s taschenbuch der mathematik*, Teubner, 1996.
8. Richard A. Brualdi, *From the editor-in-chief*, Lin. Alg. Applic. 320 (2000), no. 1–3, 209–216. MR 1796542
9. David Callan, *On generating functions involving the square root of a quadratic polynomial*, J. Integer Seq. 10 (2007), # 07.5.2. MR 2304410
10. T. W. Chaundy, *An extension of hypergeometric functions (i)*, Q. J. Math. 14 (1943), 55–78. MR 0010749
11. Jamens Cockle, *On transcendental and algebraic solution*, Phil. Mag. Series 4 21 (1861), no. 141, 379–383.
12. Louis Comtet, *Calcul pratique des coefficients de Taylor d’une fonction algébrique*, L’Enseignement Mathématique 10 (1964), 267–270. MR 0164441
13. Philippe Flajolet and Robert Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2009. MR 2483235
14. O. E. I. S. Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2021), https://oeis.org/. MR 3822822
15. Manuel Kauers, *The holonomic toolkit*, Texts and Monographs in Symbolic Computation, pp. 119–144, Springer, Wien, Heidelberg, NY, 2013. MR 3616749
16. Wilfrid Koeppf, *Power series in computer algebra*, J. Symb. Comput. 13 (1992), no. 6, 581–604. MR 1177710
17. L. Lipshitz, *D-finite power series*, J. Algebra 122 (1989), no. 2, 353–373. MR 0999079
18. Christian Mallinger, *Algorithmic manipulations and transformations of univariate holonomic functions and sequences*, Master’s thesis, Johannes Kepler Univ. Linz, 1996.
19. István Nemes and Marko Petkovšek, *RCComp: A Mathematica package for computing with recursive sequences*, J. Symb. Comput. 20 (1995), no. 5-6, 745–753. MR 1395425
20. Tony D. Noe, *On the divisibility of generalized trinomial coefficients*, J. Int. Seq. 9 (2006), #06.2.7. MR 2247938
21. Earl D. Rainville, *The contiguous function relations for \( pfq \) with application to Bateman’s \( j_{\mu}^{n} \) and Rice’s \( h_{0}(\zeta, p, \nu) \),* Bull. Amer. Math. Soc. 51 (1945), no. 10, 714–723. MR 0012726
22. Medhat A. Rakha, Arjun K. Rathie, and Purnima Chopra, *On some new contiguous relations for the gauss hypergeometric function with applications*, Comput. Math. Appl. 61 (2011), 620–629. MR 2764057
23. Christophe Reutenauer, *On a matrix representation for polynomially recursive sequences*, El. J. Combinat. 19 (2012), no. 2, #P36. MR 2988858
24. Bruno Salvy and Paul Zimmermann, *Gfun: a maple package for the manipulation of generating and holonomic functions in one variable*, ACM Trans. Math. Softw. 20 (1994), no. 2, 163–177.
25. Ronald Shaw, *Vector cross products in n dimensions*, Int. J. Math. Educ. Sci. Technol. 18 (1987), no. 6, 803–816.
26. Lucy Joan Slater, *Generalized hypergeometric functions*, Cambridge University Press, 1966. MR 0201688
27. Richard P. Stanley, *Differently finitely power series*, Eur. J. Combin. 1 (1980), no. 2, 175–188. MR 0585753
28. Michael Stoll, *Bounds for the length of recurrence relations for convolutions of p-recursive sequences*, Eur. J. Comb. 18 (1997), no. 6, 707. MR 1468339
29. Raimundas Vidūnas, *Contiguous relations of hypergeometric series*, J. Comput. Appl. Math. 153 (2003), no. 1–2, 507–519. MR 1985719
30. ———, *A generalization of Clausen’s identity*, Raman. J. 26 (2011), 133–146. MR 2837722
31. Jet Wimp and Doron Zeilberger, *Resurrecting the asymptotics of linear recurrences*, J. Math. Anal. Appl. 111 (1985), 162–176. MR 0808671

URL: https://www.mpia-hd.mpg.de/~mathar

MAX-PLANCK INSTITUTE OF ASTRONOMY, KÖNIGSTUHL 17, 69117 HEIDELBERG, GERMANY

Email address: mathar@mpia-hd.mpg.de