Constraining the initial entropy of directly-detected exoplanets

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15 January 2014

ABSTRACT

The post-formation, initial entropy $S_i$ of a gas giant planet is a key witness to its mass-assembly history and a crucial quantity for its early evolution. However, formation models are not yet able to predict reliably $S_i$, making unjustified the use solely of traditional, ‘hot-start’ cooling tracks to interpret direct imaging results and calling for an observational determination of initial entropies to guide formation scenarios. Using a grid of models in mass and entropy, we show how to place joint constraints on the mass and initial entropy of an object from its observed luminosity and age. This generalises the usual estimate of only a lower bound on the real mass through hot-start tracks. Moreover, we demonstrate that with mass information, e.g. from dynamical stability analyses or radial velocity, tighter bounds can be set on the initial entropy. We apply this procedure to 2M1207 b and find that its initial entropy is at least 9.2 $k_B$/baryon, assuming that it does not burn deuterium. For the planets of the HR 8799 system, we infer that they must have formed with $S_i > 9.2$ $k_B$/baryon, independent of uncertainties about the age of the star. Finally, a similar analysis for $\beta$ Pic b reveals that it must have formed with $S_i > 10.5$ $k_B$/baryon, using the radial-velocity mass upper limit. These initial entropy values are respectively ca. 0.7, 0.5, and 1.5 $k_B$/baryon higher than the ones obtained from core accretion models by Marley et al., thereby quantitatively ruling out the coldest starts for these objects and constraining warm starts, especially for $\beta$ Pic b.

Key words: planets and satellites: gaseous planets – planets and satellites: fundamental parameters – techniques: imaging spectroscopy – stars: individual: 2MASSJ1207334–393254 – stars: individual: HR 8799 – stars: individual: $\beta$ Pictoris.

1 INTRODUCTION

While only a handful of directly-detected exoplanets is currently known, the near future should bring a statistically significant sample of directly-imaged exoplanets, thanks to a number of surveys underway or coming online soon. Examples include VLT/SPHERE, Gemini/GPI, Subaru/HiCIAO, Hale/P1640 (Vigan et al. 2010; Chauvin et al. 2010; McBride et al. 2011; McElwain et al. 2012; Yamamoto et al. 2013; Hinkley et al. 2011b). An important input for predicting the yields of such surveys and for interpreting their results is the cooling history of gas giant planets. In the traditional approach (Stevenson 1983; Burrows et al. 1997; Baraffe et al. 2003), objects begin their cooling, fully formed, with an arbitrarily high specific entropy $S_i$ and hence radius and luminosity. In the past, the precise choice of initial conditions for the cooling has been of no practical consequence because only the 4.5-Gyr-old Solar System’s gas giants were known, while high initial entropies are forgotten on the short Kelvin–Helmholtz time-scale $GM^2/RL$ (Stevenson 1983).

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\textsuperscript{1} The ‘initial’ entropy refers to the entropy at the beginning of the pure cooling phase, once the planet’s mass is assembled.

\textsuperscript{1} Using the standard core-accretion formation model (Pollack et al. 1996; Boss 1997; Bodenheimer et al. 2000; Hubickyj et al. 2005), Marley et al. (2007, hereafter M07) found that newly-formed gas giants produced by core accretion should be substantially colder than what the usual cooling tracks that begin with arbitrarily hot initial conditions assume. These outcomes are known as ‘cold start’ and ‘hot start’, respectively. M07 however noted that there are a number of assumptions and approximations that go into the core accretion models that make the predicted entropy uncertain. In particular, the accretion shock at the surface of the planet is suspected to play a key role as most of the mass is processed through it (M07; Baraffe, Chabrier, & Barman 2010) but there does not yet exist a satisfactory treatment of it. Furthermore, there may also be an accretion shock in the other main formation scenario, gravitational instability, such that it too could yield planets cooler than usually expected (see section 8 of Mordasini et al. 2012b). Conversely, Mordasini (2013) recently found that the initial entropy varies strongly with the mass of the

Marley et al. (2007). However, direct-detection surveys are aiming specifically at young ($\leq 500$ Myr) systems, and the traditional models, as their authors explicitly recognised, are not reliable at early ages.

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\textsuperscript{1} Mon. Not. R. Astron. Soc. 000, 000–000 (0000) Printed 15 January 2014 (MN \LaTeX{} style file v2.2)
core, leading nearly to warm starts for reasonable values of core masses in the framework of core accretion.

The most reasonable viewpoint for now is therefore to consider that the initial entropy is highly uncertain and may lie almost anywhere between the cold values of $M_{\odot}$ and the hot starts. In fact, $M_{\odot}$ calculated ‘warm start’ models that were intermediate between the cold and hot starts, and recently, Spiegel & Burrows (2012) calculated cooling tracks and spectra of giant planets beginning with a range of initial entropies. This uncertainty in the initial entropy means that observations are in a privileged position to inform models of planet formation, for which the entropy (or luminosity) at the end of the accretion phase is an easily accessible output. Its determination both on an individual basis as well as for a statistically meaningful sample of planets would be very valuable, with the latter allowing quantitative comparisons to planet population synthesis (Ida & Lithium, 2004; Mordasini et al., 2012). Moreover, upcoming and ongoing surveys should bring, in a near future, both qualitative and quantitative changes to the collection of observed planets to reveal a number of light objects at moderate separations from their parent star (i.e. Jupiter-like planets), which contrasts with the few currently known direct detections. With this in mind, we stress the importance of interpreting direct observations in a model-independent fashion and of thinking about the information that these yield about the initial (i.e. post-formation) conditions of gas giants.

In this paper, we investigate the constraints on the mass $M$ and initial, post-formation entropy $S_i$ that come from a luminosity and age determination for a directly-detected object, focusing on exoplanets. Since the current luminosity increases monotonically both with mass and initial entropy, these constraints take the form of a ‘trade-off curve’ between $M$ and $S_i$. We show that, in the planetary region, the allowed values of $M$ and $S_i$ can be divided into two regions. The first is a hot-start region where the initial entropy can be arbitrarily high but where the mass is essentially unique. This corresponds to the usual mass determination by fitting hot-start cooling curves. The second corresponds to solutions with a lower entropy in a narrow range and for which the planet mass has to be larger than the hot-start mass. (A priori, this may even reach into the mass regime where deuterium burning is important, which is discussed in a forthcoming paper.) The degeneracy between mass and initial entropy means that in general the mass and entropy cannot be constrained independently from a measurement of luminosity alone. When additional mass information is available, for example for multiple-planet systems or systems with radial velocity, it is possible to constrain the formation entropy more tightly.

We start in Section 2 by describing our gas-giant cooling models, discussing the luminosity scalings with mass and entropy, and comparing to previous calculations in the literature. In Section 3 we show in general how to derive constraints on the mass and initial entropy of a directly-detected exoplanet by comparing its measured luminosity and age to cooling curves with a range of initial entropies. We briefly consider in Section 4 similar constraints from their parent star (i.e. Jupiter-like planets), which contrasts with the few currently known direct detections. With this in mind, we stress the importance of interpreting direct observations in a model-independent fashion and of thinking about the information that these yield about the initial (i.e. post-formation) conditions of gas giants.

2 COOLING MODELS WITH ARBITRARY INITIAL ENTROPY

Given its importance in the initial stages of a planet’s cooling, we focus in this paper on the internal entropy of a gas giant as a fundamental parameter which, along with its mass, determines its structure and controls its evolution. Very few authors have explicitly considered entropy as the second fundamental quantity even though this approach is more transparent than the usual, more intuitive use of time as the second independent variable. In this section, before comparing our models to standard ones, we discuss interior temperature–pressure profiles and semi-analytically explain the different scalings of luminosity on mass and entropy which appear in the models.

2.1 Calculation of time evolution

We calculate the evolution of cooling gas giants by a ‘following the adiabats’ approach (Hubbard, 1977; Fortney & Hubbard, 2004; Arras & Bildsten, 2006). We generate a large grid of planet models with ranges of interior specific entropy $S$ and mass $M$ and determine for each model the luminosity $L = L(M,S)$ at the top of the convection zone. We then use it to calculate the rate of change of the convective zone’s entropy $dS/dt$, which is defined by writing the entropy equation $\partial L/\partial S = -T \partial S/\partial t$ as

$$L = \int dS/t = \int T \, dm + L_0,$$

with $L_0$ the luminosity due to deuterium burning in the convection zone. With the current $S$ and $dS/dt$ in hand, calculating a cooling curve becomes a simple matter of stepping through the grid of models at a fixed mass. This is computationally expedient and gives results nearly identical to the usual procedure based on the energy equation (Henyey et al., 1964; Kippenhahn & Weigert, 1990), as discussed in Section 2.6.

The assumptions in the ‘following the adiabats’ approach are that $dS/dt$ is constant throughout the convective zone and that no luminosity is generated in the atmosphere. For these to hold, we require both the thermal time-scale $t_{\text{therm}} = P_r / g F$ (Arras & Bildsten, 2006) – where $P_r$ is the specific heat capacity, $g$ the gravity, and $F$ the local flux – in the radiative zone overlying the convective core and the convective turnover time-scale to be much shorter than the time-scale on which the entropy is changing, $t_s = -S/(dS/dt) = M T S/L$, with $T$ the mass-weighted mean temperature in the convection zone. A similar approach was used to study the evolution of ohmically-heated irradiated gas giants by Huang & Cumming (2012).

We do not include deuterium burning directly in the planet models since it always occurs inside the convection zone. Instead, we calculate the deuterium burning luminosity per unit deuterium mass fraction for each model in the grid and use it in equation (1) to follow the cooling of the planet and the time evolution of the deuterium mass fraction averaged over the convective region. However, we focus in this work on masses below the (parameter-dependent) deuterium-burning limit near 12–14 $M_J$ (Spiegel, Burrows, & Milson, 2011; Molière & Mordasini, 2012; Bodenheimer et al., 2013), such that deuterium burning does not appear in the models.

2.2 Comparison to previous calculations

We now compare our cooling models to previous calculations in the literature. The most reasonable viewpoint for now is therefore to consider that the initial entropy is highly uncertain and may lie almost anywhere between the cold values of $M_{\odot}$ and the hot starts. In fact, $M_{\odot}$ calculated ‘warm start’ models that were intermediate between the cold and hot starts, and recently, Spiegel & Burrows (2012) calculated cooling tracks and spectra of giant planets beginning with a range of initial entropies. This uncertainty in the initial entropy means that observations are in a privileged position to inform models of planet formation, for which the entropy (or luminosity) at the end of the accretion phase is an easily accessible output. Its determination both on an individual basis as well as for a statistically meaningful sample of planets would be very valuable, with the latter allowing quantitative comparisons to planet population synthesis (Iida & Lithium, 2004; Mordasini et al., 2012). Moreover, upcoming and ongoing surveys should bring, in a near future, both qualitative and quantitative changes to the collection of observed planets to reveal a number of light objects at moderate separations from their parent star (i.e. Jupiter-like planets), which contrasts with the few currently known direct detections. With this in mind, we stress the importance of interpreting direct observations in a model-independent fashion and of thinking about the information that these yield about the initial (i.e. post-formation) conditions of gas giants.

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play a role in the objects’ evolution. We defer a detailed exploration of deuterium burning in our models to an ultrior publication but already mention the interesting result that cooling curves starting at low entropy can exhibit an initial increase in their luminosity due to deuterium burning. We subsequently noticed that the colder starts in fig. 8 of Mollère & Mordasini (2012) show a similar behaviour, and that Bodenheimer et al. (2013) also find in their formation simulations the possibility of deuterium ‘flashes’.

2.2 Calculation of gas giant models

To construct a model with a given mass \( M \) and internal entropy \( S \), we integrate inwards from the photosphere and outwards from the centre, adjusting the central pressure, the cooling time \( \tau_c \), and the luminosity \( L \) at the top of the inner zone until the two integrations match at a pressure of 30 kbar. We use the Eddington approximation at the photosphere, setting \( T = T_{\text{eff}} \) at \( P_{\text{phot}} = 2g/3\kappa \), and take the solar-metallicity (based on Lodders 2003) radiative opacity from Friedman, Marley, & Lodders (2008). In the deep interior, the contribution from the electron conductive opacity of Cassisi et al. (2007) is also included. The equation of state for the hydrogen–helium mixture is that of Saumon, Chabrier, & van Horn (1995, hereafter SCvH) with a helium mass fraction \( Y \approx 0.25 \). Since we focus on gas giants, we do not include a rocky core, which was found in a test grid to increase the luminosities by at most a few per cent at the lower masses, as in Saumon et al. (1996).

The grid has a lower entropy limit of \( S \approx 7–7.5 \) k_{\text{B}}/baryon\(^{3}\). The upper \( S \) limit in the grid is set by the requirement that \( R \lesssim 7 \) R\(_{\odot} \), a value found to make numerical convergence straightforward. The upper limits are \( S_{\text{max}} \approx 12 \) near 1 M\(_{\odot} \) and \( S_{\text{max}} \approx 14 \) for the higher masses.

Fig. 1 shows interior profiles in the \( T–P \) plane for a range of entropies and masses. Schematically, since hydrostatic balance dictates \( P_{c} \sim M^{2}/R^{4} \), increasing the mass at a fixed entropy extends the centre to a higher pressure along the adiabat. This is exacerbated at high entropies, where the planet substantially shrinks with increasing mass, while low-entropy objects have a roughly constant radius. (Radii as a function of \( M \) and \( S \) are presented in Appendix A.) At fixed mass, increasing the entropy mostly shifts the centre to a higher \( T \) or to lower \( P \). The first case obtains for low-entropy objects, which are essentially at zero temperature in the sense that \( k_{\text{B}}T_{c} \ll E_{F} \), where \( T_{c} \) is the central temperature and \( E_{F} \) is the Fermi energy level at the centre, taken to approximate the electron chemical potential. Increasing the entropy partially lifts the degeneracy since the degeneracy parameter \( Y \approx k_{\text{B}}T_{c}/E_{F} \approx T_{c}/G_{c}^{2} \) is a monotonic function of \( S \), and \( P_{c} \) remains constant because of the constant radius. At entropies higher than a turn-over value of \( \approx 10.4 \) k_{\text{B}}/baryon, the central temperature does not increase (and even decreases) with entropy. As pointed out in Paxton et al. (2013), this entropy value is given by \( k_{\text{B}}T_{c} \sim E_{F} \) – we find that \( Y \approx 0.15 \) for \( S = 10.4 \) – and is thus independent of mass. As for \( P_{c} \), it decreases because the radius increases. These behaviours also hold at higher masses not shown in the figure.

As for the 1-M\(_{\odot} \) planet with \( S = 9 \) shown in Fig. 1, some models with entropy \( S \approx 8–9.5 \) show a second, ‘detached’ convective zone at lower pressures, which follows from a re-increase of the Rosseland mean opacity (see section 3.1 of Burrows et al. 1997). This second convection zone, which is at most at an entropy 0.2 k_{\text{B}}/baryon higher than the convective core, will not affect the evolution of the object since the radiative thermal time-scale is much shorter than the cooling time \( \tau_{c} \) throughout the atmosphere. This holds in particular at the inner radiative-convective boundary (RCB); for example, the 1-M\(_{\odot} \), \( S = 10 \) model has \( \tau_{c} = 14 \) Myr and \( t_{\text{therm}} = 0.2 \) Myr at its RCB. Planets with a second convective zone can equivalently be thought of as having a radiative shell interrupting their convective zone, as originally predicted for Jupiter’s

\( \text{Figure 1. Top panel: Temperature–pressure profiles of core-less planets with entropy } S = 7.5–12 \text{ } k_{\text{B}}/\text{baryon (bottom to top), in steps of } 0.5 \text{ } k_{\text{B}}/\text{baryon}, \text{and } M = 1, 3, \text{ and } 10 \text{ } M_{\odot} \text{ (full, dashed, and dotted lines, respectively). Squares indicate the centre and triangles mark where the enclosed mass is 99 per cent of the total mass for each model. Curves end at the photosphere in the Eddington approximation, and the thick part(s) of each curve show the radiative zone(s), while the thin part(s) of the curve are convective; the radiative-convective boundary relevant for the thermal evolution is highlighted by a ring. Thin grey lines at low } P \text{ and } T \text{ indicate contours of constant Rosseland mean opacity (Freedman et al. 2008), with } \log k = -3.5 \text{ to } -1.0 \text{ (cm}^{2} \text{ g}^{-1}) \text{ in steps of 0.5 dex, increasing with } P. \text{ See text for a mention of the dominant sources of opacity in some parts of the diagram. Bottom panel: Same as top panel but focussing on the atmospheres. The three diagonal dashed lines mark log } R = \log g/T_{c}^{3} = 2, 4, \text{ and } 6 \text{ (cm}^{3} \text{ K}^{-1}), \text{ from top to bottom.} \)
adiabat by Guillot et al. (1994) based on too low opacities (see e.g. the brief review in Freedman et al. 2008).

Finally, we note that the higher-entropy objects ($S \gtrsim 11.5$, with some dependence on mass) are convective from the centre all the way to the photosphere.

2.3 Luminosity as a function of mass and entropy

To provide a model-independent way of thinking about an object’s brightness and thus to facilitate comparison with other models, we show in Fig. 2 the luminosity of the planet as a function of its internal temperature. We focus on objects without significant deuterium burning. Two main regimes are apparent: at lower entropies, the scaling with mass is roughly $L \sim M$, while at high entropies, the luminosity becomes almost insensitive to mass. Looking more closely, the high-luminosity regime is described by $L \propto M^3$ (for $M \gtrsim 1 M_\odot$), and the brief steepening of the luminosity slope with respect to entropy between $L \approx 10^{-6}$ and $10^{-2} L_\odot$ (at $S \sim 8.2–9$, depending on mass) marks a transition from $L \propto M^1$ to $L \propto M^{0.7}$ at fixed intermediate and low entropy respectively.

To try to understand these luminosity scalings, we firstly note that the radiative opacity at the radiative-convective boundary, which is equal to the total luminosity, can always be written as (Arras & Bildsten 2006)

$$L = \left(4 \pi T^4 \frac{\kappa_{\text{RCB}}}{3P} \frac{dT}{dr}\right)_{\text{RCB}} = \frac{16 \pi G c_s M T_{\text{RCB}}^4 \nabla_{\text{adiRCB}}}{3 \kappa_{\text{RCB}} P_{\text{RCB}}},$$

approximating the convection zone to contain the whole mass (cf. the 99-per-cent mass labels in Fig. 1) and radius. Thus, one way of obtaining $L(M, S)$ is to express the four quantities $P_{\text{RCB}}$, $T_{\text{RCB}}$, $\nabla_{\text{adiRCB}}$, and $\kappa_{\text{RCB}}$ in terms of $M$ and $S$. This will now be done for the three regimes in turn, starting with low entropies.

2.3.1 Low-entropy regime: $L \propto M^{0.7} f_{\text{low}}(S)$

To begin, consider the entropy dependence of luminosity at a fixed mass when $S \lesssim 9$–8 (at 1–10 $M_\odot$ respectively). Fig. 1 reveals that the opacity at the radiative-convective boundary $\kappa_{\text{RCB}}$ remains constant at a given mass, which provides a first relation. Secondly, over the temperature and pressure range of interest in this regime, the hydrogen and helium remain respectively molecular and neutral. This implies that $\nabla_{\text{adiRCB}}$ is constant and that the entropy has a simple functional form in the ideal-gas approximation, given by the Sackur–Tetrode expression (e.g. Callen 1985). For an H$_2$–He mixture with $Y = \frac{1}{4}$, this is

$$S = 9.6 + \frac{45}{32} \ln(T/1600 \text{ K}) - \frac{7}{16} \ln(P/3 \text{ bar})$$

(hence $\nabla_{\text{adi}} = 0.31$), where $S$ is the entropy per baryon in multiples of $k_B$. In $10 = 2.3$, and the reference $T$ and $P$ values were chosen for Section 4.3.2. Finally, $T_{\text{RCB}}$ is approximately fitted by $T_{\text{RCB}} \propto 10^{0.195}$. Combining the four relations (constant $\nabla_{\text{adiRCB}}$ and the three non-trivial ones) with equation (2) yields $L \propto 10^{1.545}$. This is quite close to a power-law fit of Fig. 2 which gives $L \propto 10^{3.35}$.

Next, consider the position of the RCB at fixed low entropy for different masses. Fixed entropy immediately implies $T \propto P^{0.2}$, and the constant $\nabla_{\text{adi}} = 0.31$ gives the second relation. The profiles of Fig. 1 indicate that in this low-entropy regime (for $T \approx 200–400$ K, $P \approx 0.01–10$ bar), the opacity depends approximately only on the pressure, with $\kappa \propto P^{0.3}$. Finally, we find that $P_{\text{RCB}} \propto M^{0.5}$, thus providing the third relation. Combining these and using equation (2) gives $L \propto M^{0.8}$, in good agreement with a direct fit, which gives $L \propto M^{0.7}$.

Thus, a constant adiabatic gradient, an RCB opacity dependent only on the mass, and a few power laws suffice to show that in the low-entropy regime

$$L_{\text{low}} = 10^{-7.7} L_\odot \left(\frac{M}{M_\odot}\right)^{0.7} 10^{1.545(S-7.5)},$$

which defines $f_{\text{low}}(S)$ up to a constant. For Jupiter’s adiabat with $T = 165–170$ K at 1 bar (Saumon & Guillot 2004), and thus an entropy of 6.71–6.75 $k_B$/baryon, this predicts $\approx 1.9 \times 10^{-0.5} L_\odot$, in good agreement with the current value $L_J = 8.7 \times 10^{-10} L_\odot$.

2.3.2 Intermediate-entropy regime: $L \propto M f_{\text{int}}(S)$

We now look at intermediate entropies, which are in the range 9–10.2 at 1 $M_\odot$ to 8.2–9.6 $k_B$/baryon at 10 $M_\odot$. (More concisely, this corresponds to planets with $L \sim 10^{-6}–10^{-4} L_\odot$.) As explained by Arras & Bildsten (2006), equation (2) is of the form $L = M f(S)$, where $f(S)$ is a function of the entropy in the convection zone, only if the quantity $T^2 \nabla_{\text{adi}}/\kappa P$ at the RCB is a function of a unique variable, $S$. The intermediate-$S$ behaviour can then be understood by noticing in Fig. 1 that objects at those entropies have a $T_{\text{RCB}}$ independent of internal entropy and an extended atmosphere (interrupted or not by a second convection zone, i.e. including the deeper radiative window when present). For these planets, the photosphere is sufficiently far from the RCB ($T_{\text{RCB}} \gtrsim 10^3 P_{\text{phot}}$, $T_{\text{RCB}} \gtrsim 5 T_{\text{adi}}$) that the atmosphere merges on to the radiative-zero profile, which solves (Cox & Giuli 1968)

$$\frac{dT}{dP} = \frac{3}{16 \pi c_s M T^3},$$

where the boundary condition is by definition of no consequence. The solution is a $T(P)$ relation, which yields the radiative gradient $\nabla_{\text{rad}} = d \ln T/d \ln P$ along it. The important point is that both the atmosphere profile and its gradient depend only on the quantity $L/M$. Now, choosing an internal entropy fixes the adiabat, i.e. sets a second $T(P)$ relation. We require that at the intersection of the
atmosphere and the adiabat $\nabla_{ad}$ be equal to $\nabla_{rad}$, which is the slope of the chosen adiabat. This thus pins down $\nabla_{rad}$ in the atmosphere and hence $\nabla_{rad}$ in the atmosphere profile itself. Therefore, there is a unique $L/M$ associated with an $S$, which means that $L$ must be of the form $L/M = f(S)$.

Since we determine convective instability through the Schwarzschild criterion $\nabla_{ad} < \nabla_{rad}$, where $\nabla_{ad}$ is relatively constant and the radiative gradient is given by

$$\nabla_{rad} = \frac{3L}{16 \pi acGM T^4},$$ (6)

near the surface, a slow inward increase of $\kappa$ will ensure a deep RCB. Consequently, the $L \propto M f(S)$ scaling will hold for those $M$ and $S$ such that, starting at the photosphere with $P_{\text{phot}} \approx f(M)$ and $T_{\text{eff}} \approx f(P_{\text{phot}}, S)$, the opacity increases only slowly along the adiabats. (For a power-law opacity $\kappa \propto P^n T^m$, this means $n/(n + s) \leq \nabla_{ad}$.) This is indeed the case for intermediate-$S$ models in Fig. [4], where the $T(P)$ profiles are nearly aligned with contour lines of constant opacity. For this argument to hold, the radius must be rather independent of the mass (cf. Zapolsky & Salpeter 1969) and of the entropy. Also, one needs to approximate the opacity at the photosphere to be that of the interior, which is reasonable: even in the extended atmospheres, the entropy increases by at most $\approx 0.5 \kappa_0$/baryon over our grid of models.

The functional form of $f(S)$ can be obtained by fixing $T_{\text{RCB}}$ and using equation (5) for the entropy of an ideal gas. The reference temperature of 1600 K was chosen based on the radiative-convective boundaries of Fig. [1]. For convenience, $S$ (3 bar, 1600 K) was computed using the interpolated Sun [CVH] tables. These include the contribution from the ideal entropy of mixing, a remarkably constant $S_{\text{mix}} = 0.18 \kappa_0$/baryon in a large region away from $\mathrm{H}_2$ dissociation. Combining this $S(P,T)$ with equation (2), fixing $T = 1600$ K, and taking $\kappa(P, 1600 \text{ K}) \approx 0.0104(3 \text{ bar})^2 \text{ cm}^2 \text{ g}^{-1}$ from the interpolated table yields

$$\log_{10} \frac{L_{\text{rad}}/L_\odot}{M/M_\odot} = -5.05 + 1.51(S - 9.6),$$ (7)

i.e. $L = M f_0(S)$ with $f_0(S) = 3.4 \times 10^{28} \text{ erg s}^{-1} \text{ M}_\odot^{-1} 10^{5.5(S-9.6)}$. (Thus, if $\kappa \propto P^n$ and $dS/d\ln P = -b$, $\log_{10} f_0(S) \propto (n + 1)b/ln b$.) The subscript ‘zc’ highlights that the solution applies when the radiative-zero solution is reached. This fits excellently (being mainly only a 0.1 dex too high in $\log L$) the $L/M$–$S$ relation found in our grid of models at intermediate entropies.

2.3.3 High-entropy regime: $L \propto M^{0.3} f_\lambda(S)$

For $S \gtrsim 10$–$9$ (at 3–10 M$_\odot$, respectively), the luminosity becomes almost independent of mass at a given entropy. This indicates that the radiative solution does not hold anymore, and indeed Fig. [1] shows that planets with high entropy have atmospheres extending only over a small pressure range, with the more massive objects fully convective from the centre to the photosphere. The shortness of the atmosphere is due to the opacity’s rapid increase inward, as constant-$\kappa$ contours are almost perpendicular to $T(P)$ profiles in that region. As in Section 2.3.2 we look at the behaviour of $T$, $P$, $\kappa$, and $\nabla_{ad}$ at the RCB as a function of $M$ and $S$.

Fig. [1] shows that at a fixed mass, $T_{\text{RCB}}$ is almost independent of the (high) entropy, with the actual scaling closer to $T_{\text{RCB}} \propto M^{0.3}$, with $M_{\text{ad}}$ again mostly independent of $S$, as for the low entropies. At high entropies, $\nabla_{rad}$ drops continuously with increasing $S$ (decreasing $P_{\text{RCB}}$), with a very rough $\nabla_{ad} \propto 10^{-0.135}$. For the fourth relation, we can fit $P_{\text{RCB}} \propto 10^{-0.65}$. Combining all this with equation (3) then gives $L \propto 10^{1.75}$, which is quite close to a fit $L \propto 10^{0.65}$.

At fixed high entropy, $\nabla_{ad}$ is somewhat constant at 0.25–0.13 for $S = 10$–$12$. Also, above $\approx 1500$ K and at $P < 1$ bar, the opacity is approximately independent of pressure, scaling only with temperature as $\kappa \propto T^4$. Finally, as in the low-entropy regime, $P_{\text{RCB}} \propto M^{0.5}$; combining the four relations, we should have $L \propto M^4$ with $\alpha = 0.5$–$0.4$. This is not far from the direct fit $L \propto M^{0.7}$.

Therefore, the luminosity in the high-entropy regime is

$$L_{\text{high}} = 10^{-3.88} \left( \frac{M}{M_\odot} \right)^{0.3} 10^{(5.3 - 10.5)},$$ (8)

which defines $f_{\text{high}}(S)$ up to a multiplicative constant. Fig. [2] indicates that the luminosity at the lower masses ($M \lesssim 2$ M$_\odot$) depends more steeply on $S$ ($\approx 10^{0.85}$) at 1 M$_\odot$, but this was ignored when obtaining equation (3).

Before summarising, let us briefly digest about the $P_{\text{RCB}} \propto M^{0.5}$ scaling seen both at fixed low entropy and at fixed high entropy. In both cases, the model grid shows that, as a reasonable approximation, $P_{\text{RCB}} \propto P_{\text{phot}}$. At low entropies, $R$ is constant, such that $P_{\text{RCB}} \propto P_{\text{phot}} = 2g/3k \propto M/k$. Then, the opacity’s scaling of $\kappa \propto P$ (see Section 2.3.2), the exponent is actually closer to $\approx 0.95$) immediately implies roughly $P_{\text{RCB}} \propto M^{0.5}$ At high entropies, planetary radii are significantly larger and vary, such that the radius dependence of the photospheric pressure should not be neglected: thus $P_{\text{phot}} \propto M^{2/3}$. Fitting the $R(M$, fixed high $S$) relation in our models, we find $R \propto M^{1.1}$. The same is not far from $M^{1.2}$, depending on the entropy. Then, $\nabla_{ad} \approx 0.2$ and $\kappa \propto T^4$ yield $\kappa \propto P^{0.8}$ at fixed $S$, and thus, isolating, $P_{\text{RCB}} \propto M^{0.6}$ or $P_{\text{RCB}} \propto M^{0.7}$. This is a slightly stronger dependence on mass than what is found in the grid, but the argument shows how the rough scaling can be derived.

2.3.4 Summary of luminosity scalings

In summary, we found from fitting the $L(M,S)$ relation across our planet models that $L \propto M^{0.35}$, with $a = (0.7, 1.0, 0.3)$ and $a = (1.3, 1.5, 0.6)$ at low ($S \lesssim 9$–$8$, for 1–10 M$_\odot$, respectively), intermediate ($S \lesssim 10$–$9$), or high entropy, respectively, from considering the behaviour of the different factors in equation (2). Fitting directly the relations in the grid gives very similarly

$$L_{\text{low}} = 1.5 \times 10^{-7} L_\odot M^{0.72} 10^{3.1(S-8.2)};$$ (9a)

$$L_{\text{inter}} = 7.2 \times 10^{-6} L_\odot M^{0.98} 10^{5.8(S-5.9)};$$ (9b)

$$L_{\text{high}} = 8.7 \times 10^{-5} L_\odot M^{0.29} 10^{5.8(S-10.2)};$$ (9c)

with $M \equiv M/M_\odot$, to $\approx 0.01$ dex in $\log L$ except for $M \lesssim 2$ M$_\odot$ at high entropies.

Our approximate understanding of the different regimes is the following. For the conditions found in the atmospheres of intermediate-entropy planets, contours of constant opacity are almost parallel to adiabats, which is equivalent to saying that $\kappa$ is almost constant along lines of constant $R$ (see Fig. [1]), where $\log R \equiv \log e/T_e^3$ and $T_e = T/10^6$ K. Since $\nabla_{ad} \propto \kappa$ by equation
and, for a constant \( \nabla_{\text{ad}} \), only \( \nabla_{\text{ad}} \) determines when the atmosphere becomes convective, a slow inward increase of the opacity causes the radiative zone to extend over a large pressure range. This in turn means that the atmosphere can reach the radiative-zero solution, which we have shown necessarily implies \( L = Mf(S) \). For the conditions found in the atmospheres of planets with high and low entropy, however, opacity increases relatively quickly along an adiabat. Since the \( T(P) \) slope in the atmosphere is not too different from that of the convective zone’s adiabat, this means that opacity increases rapidly in the atmosphere, which therefore cannot join on to the radiative-zero solution before becoming convective. It is interesting to note that the transition from low to intermediate entropies is accompanied by a ‘second-order’ (i.e. relatively small) change in \( a \) and \( \lambda \), while the physical explanation changes to ‘zeroth-order’.

The different \( L(M, S) \) scalings then reflect in part the approximate temperature- and pressure-independence of the opacity at low and high \( S \), respectively, and the fact that opacity increases relatively little along adiabats at intermediate \( S \). While developing even a rough analytical understanding of the various opacity scalings would be interesting but outside the scope of this work, we briefly indicate the major contributors in the high- and intermediate-\( S \) regimes. (The following temperatures should all be understood as somewhat approximate; cf. Fig. 1). Moving from 2800 K (or 3000 K at higher \( P \)), above which continuum sources dominate, down to 1300 K, the decrease in opacity is due to the settling of the rovibrational levels of \( \text{H}_2 \). Similarly, the settling of the rovibrational levels of \( \text{CH}_4 \) dominates from 800 K down to 480 K, with \( \text{H}_2 \text{O} \) also contributing. Around 2000 K, where opacity is nearly independent of pressure across a wide pressure range, \( \text{H}_2 \) dominates the opacity, with its abundance remaining rather constant. In the intermediate-entropy range, from 1300 or 1600 K to 800 K, \( \text{H}_2 \) dominates the composition but it is the appearance of \( \text{CH}_4 \), which is crucial for the opacity. The same qualitative behaviours can be found in the data of Ferguson et al. (2005) (J. Ferguson 2013, priv. comm., who also provided the information just presented). Ferguson et al. (2005) do not include the powerful alkali Na, K, Cs, Rd, and Li as Freedman et al. (2008) do, which can raise the opacity by some 0.2 dex near the RCB of intermediate-entropy planets.

### 2.4 Luminosity as a function of helium fraction

The standard grid used for analyses in this work uses a helium mass fraction \( Y = 0.25 \) but results can be easily scaled to a different \( Y \). Following a suggestion by D. Saumon (2013, priv. comm.), one can write

\[
\left( \frac{dL}{dY} \right)_{M,S} \approx \left( \frac{\partial L}{\partial \ln S} \right)_{M,T} \times \frac{d\ln S}{dY}(r_0,t_0),
\]

where \( (r_0,t_0) \) is some appropriate location, and compute \( \gamma S \equiv dS/dY \approx S_{\text{ff}} S_{\text{He}} + dS_{\text{max}}/dY \), where \( S = -(1-Y)S_{\text{H}} + Y S_{\text{He}} + S_{\text{max}} \), \( S_{\text{H}}, S_{\text{He}}, S_{\text{max}} \) are respectively the total, hydrogen, helium, and mixing entropies per baryon \( (\text{SCVs}) \). Given equation (2), one might heuristically expect \( (r_0,t_0) \) to be near the radiative–convective boundary (RCB), at least when determining \( dS/dY \) at constant \( L \). Indeed, we find this to be the case, with \( \gamma \approx -0.63 \) at the RCB of the \( Y = 0.25 \) or 0.30 models for intermediate to high entropies. Thus, for \( Y \) sufficiently close to 0.25,

\[
X(M, S', Y') = X(M, S, Y)
\]

if \( S' = S \times [1 - 0.63(Y' - 0.25)] \),

where \( X \) is a planet property such as \( L \) or \( \tau_\lambda \). For \( Y > 0.27 \), this is at worse accurate only to 0.05 dex in luminosity towards high entropies for masses below \( \approx 12 \text{M}_\odot \) when \( S \lesssim 8 \). This rescaling is also adequate (in the same domain) for \( X = R \), to 15 per cent towards high entropies, but a more accurate fit can be obtained with \( \gamma \approx -0.3 \pm 0.1 \). Interestingly, nowhere within a planet structure in the grid does \( \gamma \) drop below 0.55; the explanation in this case (in analogy to \( \gamma \) being evaluated at the RCB for \( L \)) is not clear.

Since at the RCBS in the grid the hydrogen is molecular, one might try to obtain \( \gamma \) analytically with the Sackur–Tetrode formula. Neglecting the subdominant contribution from the rotational degrees of freedom of \( \text{H}_2 \) yields \( S_{\text{ff}} S_{\text{He}} \approx \mu_{\text{He}}/\mu_\text{H} = 2 \), where \( \mu_x \) is the molecular mass of \( X \). This leads to \( \gamma = -0.57 \), which is not far from the accurate result \( \gamma \approx 0.63 \) yet shows that the \( dS_{\text{max}}/dY \) term cannot be neglected.

Equation (12) makes it simple to convert e.g. constraints on the initial entropy based on cooling curves with a particular \( Y \) to another set with a different \( Y \), regardless of the approach used for the cooling. The typically small change in the entropy \( (\approx 3 \text{per cent}) \) is nevertheless significant because of the strong dependence of \( L \) on \( S \).

### 2.5 Cooling

One can also derive the functional form of the cooling tracks with a few simple arguments. With \( L \propto M^{\gamma} \), equation (1) implies \( dS/dt = -M^{\gamma - 1} f(S)/T \). Since hydrostatic balance yields \( P_c \propto GM^2/R^4 \) and the convective core is adiabatic, we expect \( T \approx T_c \propto (M^2/R^7)^{1/3} \). (Across the whole grid, \( T/T_c \approx 0.55-0.63 \).) Therefore, ignoring the radius dependence (since \( 4\pi a_{\text{ad}} \) is not large and \( R \) is rather constant at lower entropies), \(-dS/f(S) \propto M^{2\gamma - 1 - \alpha} \text{d}t \).

This means that the entropy of a cooling planet should be approximately a function of \( t/M^2a_{\text{ad}}^{-1-\alpha} \), and that more massive planets cool more slowly since \( a \) is always smaller than \( a_{\text{ad}} + 1 \approx 5/3 \). In fact, using that \( f(S) \approx 10^5 \), one can compute the integral to obtain an analytic expression for the cooling tracks:

\[
\left( \frac{1}{L(t)} \right)^{1/4} \approx \frac{1}{L_i} + \frac{1}{L_{\text{bc}}(t)}, \quad L_{\text{bc}}(t) = \frac{M^2a_{\text{ad}}^{-1-\alpha}}{\beta t}, \tag{13}
\]

where \( L_i \) and \( L_{\text{bc}} \) are the initial and hot-start luminosities, respectively, and \( \beta = C \gamma \ln 10 \) with \( C \) a dimensional constant governing prefactors. We have assumed that \( a, \lambda \), and \( \nabla_{\text{ad}} \) do not change as the planet cools. Thus, \( L \propto M^{\gamma} \) at fixed entropy, with \( a = 0.3-1 \), but at a fixed time the luminosity has a steeper dependence on \( M \), \( L \propto M^{2\gamma - 1 - \alpha} \approx L \propto M^2 \) for intermediate entropies. A more detailed analytic understanding of the cooling curves for irradiated planets along these lines was developed by Arras & Bildsten (2004), and a careful, approximate but surprisingly accurate analysis for brown dwarfs may be found in Burrows & Liebert (1993).

Equation (13) approximately describes cooling tracks with arbitrary initial entropy, with our cooling curves well fitted (to roughly 0.1 dex) below \( 10^{6.5} \text{L}_\odot \) by \( C = 3 \times 10^{-2} \text{cgs} \) when \( \lambda = 1.5 \) and \( \nabla_{\text{ad}} = 1/3 \). At higher luminosities, in particular for hot starts, a better fit (with the restriction of \( L \gtrsim 10^{6.5} \text{L}_\odot \) if \( M \lesssim 2.5 \text{M}_\odot \)) which captures the average shape and spacing of the cooling curves is provided directly by the classical result of Burrows & Liebert (1993), who find

\[
L_{\text{bc}} = 7.85 \times 10^{-6} \text{L}_\odot \left( \frac{M/3 \text{M}_\odot}{(t/10 \text{Myr})^{6/7}} \right)_{2.97} \tag{14}
\]
Figure 3. Cooling curves for (bottom to top) $M = 1, 2, 3, 5, 8, 10, 15$, and $20 \, M_J$ (solid curves) compared with the Marley et al. (2007) hot starts (for $M = 1, 2, 3, 5, 8, 10, 15$ only; dash-dotted curves), the COND03 tracks (dots) and the Burrows et al. (1997) models (dashed curves). The COND03 data is shown as dots because of insufficient sampling of the cooling tracks at higher masses.

(for $\kappa = 0.01 \, \text{cm}^2 \, \text{g}^{-1}$), i.e. a somewhat different mass and time dependence.

2.6 Comparison with classical hot starts and other work

We now compare our cooling curves to classical hot starts. Fig. 3 shows cooling curves for large initial entropy compared to the hot-start models of Marley et al. (2007), the COND03 models (Baraffe et al. 2003) and those of Burrows et al. (1997), all of which use non-grey atmospheres with detailed opacities. The agreement is excellent, with our luminosities within the first 3 Gyr approximately within $-30$ and $20$ per cent and $-30$ and $60$ per cent above those of Baraffe et al. and Burrows et al. Of interest for the example of Section 4, our radii along the cooling sequence are at most approximately two to five per cent greater at a given time.

This difference is comparable to the effect of neglecting heavy elements in the equation of state (EOS) or not including a solid core (Saumon et al. 1996) and not significant for our purposes. See also Appendix A). Our deuterium-burning phase at $20 \, M_J$ ends slightly earlier than in Burrows et al. (1997) but this might be due to our simplified treatment of the screening factor.

Fig. 3 shows cooling curves for lower initial entropies than in Fig. 2. The cooling curves show the behaviour found by Marley et al. (2007) and Spiegel & Burrows (2012) in which the luminosity initially varies very slowly, with the cooling time at the initial entropy much larger than the age of the planet. Eventually, the cooling curve joins the hot start cooling curve once the cooling time becomes comparable to the age.

Comparing to the cold starts in fig. 4 of Marley et al. (2007), our models are a factor of $\approx 3.5$--$3.9$ lower in luminosity for the same initial entropies, as given in their fig. 2. Increasing our initial entropies by $0.38 \, k_b/\text{baryon}$ brings our cooling curves into agreement with theirs when the planet has not yet started cooling. (For this comparison, we do not correct the time offset for the higher masses (see figs. 2 and 4 of Marley et al., which are already on hot-start cooling curves at the earliest times shown.) As mentioned in Appendix B this implies a real difference between our $L(S)$ of only $0.14 \, k_b/\text{baryon}$. Spiegel & Burrows (2012) computed the evolution of gas giants starting with a wide range of initial entropies. To compute the bolometric luminosity of their models, we take the published spectra and integrate the flux in the wavelength range given, $0.8$--$15 \, \mu \text{m}$. The bottom panel of Fig. 4 shows the comparison to all four model types, with or without clouds and at solar or three times solar metallicity. Increasing our entropy by $0.45 \, k_b/\text{baryon}$ – e.g. comparing the Spiegel & Burrows (2012) model with $S = 9.0$ to ours with $S = 9.45$ – yields very good agreement, with our luminosities overlaying their curves or within the spread due to the different atmospheres. As discussed in Appendix B this is mainly due to a constant entropy offset of $0.52 \, k_b/\text{baryon}$ between the tables used.
by the Burrows et al. group and the published SCvH tables used in the present work, leaving a net offset of merely 0.07 $k_b$/baryon.

The apparent disagreement with Spiegel & Burrows (2012) at late times comes from the increasing fraction of the flux in the Rayleigh-Jeans tail of the spectrum beyond 15 $\mu$m. For comparison, the implied required bolometric correction is equal to 10 to 50 per cent of the flux in 0.8–15 $\mu$m for a blackbody with $T_{\text{eff}} = 700–300$ K. From this and the hot-start $T_{\text{eff}}$ tracks shown in Spiegel & Burrows (2012), we estimate that integrating the spectrum should give a reasonable estimate (to ca. 30 per cent) of the bolometric luminosity only up to $\approx$ 50, 200, and 1000 Myr for objects with $M = 1, 3$, and 10 $M_\odot$ respectively. This is indeed seen in Fig. 4. Bolometric luminosities kindly provided by the authors (D. Spiegel 2013, priv. comm.), which are rather insensitive to the atmosphere type, are also shown for a more direct comparison and confirm the reasonable match of our cooling curves with those of Spiegel & Burrows (2012).

We have also computed cooling curves with the MEsa stellar evolution code (Paxton et al. 2011; 2013, revision 4723), and they are in excellent agreement with our results. We compared our $L(S)$ relation to ones obtained from MEsa, (also with $Y = 0.25$) at different masses and found that they are very nearly the same, with an entropy offset $\Delta S \lesssim 0.1$. Moreover, Fig. 4 shows that the agreement of the time evolution is quite good, with in particular the late-time ‘bumps’ due to opacity when the cooling curves enter the intermediate-entropy regime (cf. Figs. 12 and 16). We also produced grids with other opacities, using the MEsa tables with the default $Y = 0.28$ and $Z = 0.02$ (for the opacity calculation only), and the Freedman et al. (2008) tables with $[\text{M/H}] = \pm 0.3$ dex, and found that this changed the luminosity by at most $\approx$ 10 per cent at a given mass and internal entropy. Similarly, small differences were found to result from a changed helium mass fraction in the bulk of the planet at a given mass and entropy per nucleus.

The upshot of these comparisons to classical, non-grey-atmosphere hot and cold starts is that cooling tracks computed with the simple and numerically swift cooling approach described above can reproduce models which explicitly calculate the time dependence of the luminosity. When comparing models from different groups, one should keep in mind that there can be a systematic offset in the entropy values of $\Delta S_{\text{sys}} = 0.52$ (at $Y = 0.25$) due to different versions of the SCvH EOS, which however has no physical consequence for the cooling. Moreover, the remaining difference $|\Delta S|$ $\lesssim$ 0.15 is small compared to the entropy range between hot and cold starts.

3 GENERAL CONSTRAINTS FROM LUMINOSITY MEASUREMENTS

Masses of directly-detected exoplanets are usually inferred by fitting hot-start cooling curves (e.g. Burrows et al. 1997; Baraffe et al. 2003) to the measured luminosity of the planet, using the stellar age as the cooling time. Since the hot-start luminosity at a given age is a function only of the planet mass, the measured luminosity determines the planet mass. Equation (14) provides a quick estimate of this ‘hot-start mass’:

$$M_{\text{hot}} = 3 M_\odot \left( \frac{L}{7.85 \times 10^{-6} L_\odot} \right)^{0.379} \left( \frac{t}{10 \text{ Myr}} \right)^{0.491}. \quad (15)$$

Moreover, a planet’s luminosity, at a given time and for a given mass, can never exceed that of the hot starts, since a larger initial entropy would have merely cooled on to the hot-start cooling track at an earlier age.

However, we have seen above that the luminosity at a given mass can be lowered by considering a sufficiently smaller initial entropy, which might be the outcome of more realistic formation scenarios (Marley et al. 2007; Spiegel & Burrows 2012). With the fact that luminosity increases with planet mass at a given entropy, this simple statement has important consequences for the interpretation of direct-detection measurements, namely that there is not a unique mass which has a given luminosity at a given age. Cold-start solutions correspond to planets not having forgotten their initial conditions, specifically their initial entropy $S_i$, and every different initial entropy is associated to a different mass. In other words, a point in $(t, L)$ space – a single brightness measurement – is mapped to a curve in $(M, S_i)$ space. Since Marley et al. (2007) (but see also Baraffe et al. 2002; Fortney et al. 2008), it is generally recognised that direct detections should not be interpreted to yield a unique mass solution, but, with the exception of Bonnefoy et al. (2013), who used infrared photometry, this is the first time that this degeneracy is calculated explicitly.

3.1 Shape of the $M$–$S_i$ constraints

The top panel of Fig. 5 shows the allowed masses and initial entropies for different values of luminosity $L_{\text{bol}}/L_\odot = 10^{-7}, 10^{-6}$ and $10^{-5}$, at ages of 10, 30, and 100 Myr. Below the deuterium-burning mass, constant-luminosity curves in the $M$–$S_i$ plane have two regimes. At high initial entropies, the derived mass is the hot-start mass independent of $S_i$ since all $S_i$ greater than a certain value have cooling times shorter than the age of the system. There, uncertainty in the stellar age translates directly into uncertainty in the planet mass; since $L \propto 1/t$ and $L \sim M^2$, the mass uncertainty is $\Delta M / M \approx \Delta \log_{10} L_{\text{bol}}$. At lower entropies, the luminosity measurement occurs during the early, almost constant-luminosity evolution phase. Given a luminosity match in this region, one can obtain another by assuming a lower (higher) initial entropy and compensating by increasing (decreasing) the mass. As seen in Section 2, $L$ is a very sensitive function of $S_i$ at low and intermediate entropies, so that a small decrease in initial entropy must be compensated by a large increase in mass to yield the same luminosity at a given time; this yields the approximately flat portion of the curves. As long as the cooling time for a range of masses and entropies remains shorter than the age, the entropy constraints do not significantly depend on the age. The uncertainty in the initial entropy is $\Delta S_i \approx 1/2 \Delta \log_{10} L_{\text{bol}}$, where $1/2 \approx 0.7$ or 1.7 at low or high entropy (see Section 2.4).

The circles in the top panel of Fig. 5 show the initial entropies for cold- and hot-start models from M07, the ‘tuning-fork diagram of exoplanets’. The entropies were increased by 0.38 $k_b$/baryon as in the top panel of Fig. 4 to match our models. Since the cooling time increases with mass (see Section 2.3), heavier planets of the hot-start (upper) branch, i.e. with arbitrarily high initial luminosity, have cooled less and are therefore at higher $S_i$. For their part, the cold-start entropies, which are still the post-formation ones, lie close to a curve of constant luminosity $L = 2 \times 10^{-6} L_\odot$. This reflects the fact that the post-formation luminosities in M07, as seen in their fig. 3, have similar values for all masses. The cause of this ‘coincidence’ is presently not clear; it might be a physical process or an artefact of the procedure used to form planets of different masses. Putting uncertainties in the precise values aside, the two prongs of the tuning fork in Fig. 5 give an approximate bracket within which just-formed planets might be found.
Constraining initial entropies of exoplanets

![Image of two graphs]

**Figure 5.** Top panel: Allowed values of planet mass $M$ and initial entropy $S_i$ corresponding to $L_{\text{bol}} = 10^{-7}$, $10^{-6}$, and $10^{-5} L_\odot$ (bottom to top sets of curves) and ages $t = 10, 30,$ and 100 Myr (left to right or dotted, full and dashed lines). The circles show the results from Marley et al. (2007), increased by 0.38 $k_B$/baryon (cf. Fig. 4), for cold starts (filled circles) and hot starts (open circles), where the entropy is 1 Myr after the onset of cooling. Red and blue squares indicate the illustrative $M$ and $S_i$ used for the lightcurves in the bottom panel, with errorbars corresponding to the hatched regions there. Bottom panel: Two examples of cooling curves that have $L = 10^{-6} L_\odot$ at 30 Myr. The dot-dashed (red) line has $M = 1.85 M_\oplus$ and $S_i = 11.9$ (hot start), while the full (blue) one has $M = 12 M_\oplus$ and $S_i = 8.33$ (cold start). Hashed regions indicate a spread of $0.5 M_\oplus$ ($0.1 \, k_B$/baryon) for the hot- (cold-)start curve. The respective relative uncertainties are $\Delta M/M = \frac{\Delta t}{t}$ and $\Delta S_i \approx 0.7 \Delta \log_{10} L_{\text{bol}}$ for the cold- and hot-start cases, respectively (see text).

Mass information for a directly-detected planet can put useful constraints on its initial entropy and also potentially on its age and luminosity simultaneously. For instance, dynamical-stability analyses and radial-velocity observations (see Section 5 and Appendix E) typically provide upper bounds on $S_i$. Since $S_i$ decreases monotonically with $M$ at a fixed luminosity, this translates into a lower limit on the initial entropy. This has the potential of excluding the coldest-start formation scenarios. Conversely, if a lower limit on the planet’s mass implies an upper bound on $S_i$, if it is greater than the hot-start mass, this lower limit on the mass of a planet would be very powerful, due to the verticalness of the hot-star branch. Combined with the flatness of the ‘cold branch’ of the $M(S_i)$ curve, this could easily restrict the initial entropy to a dramatically small $\Delta S_i \approx 0.5 \sim \Delta S_{\text{sys}}$. Also, the top panel of Fig. 5 shows that not all age and luminosity combinations are consistent with a given mass upper limit. Given the often important uncertainties in the age and the bolometric luminosity, this may represent a valuable input.

3.2 Solutions on the hot- vs. cold-start branch

The bottom panel of Fig. 5 shows lightcurves illustrating the two regimes of the $M(S_i)$ curves discussed above. The hot-start mass is $1.85 M_\oplus$, whereas the selected cold-start case ($S_i = 8.33$) has $M = 12 M_\oplus$ – i.e. a six times larger mass – and both reach log $L_{\text{bol}}/L_\odot = -6$ at $t = 30$ Myr, with the cold-start values essentially independent of age. The hatched region around the $1.85$-$M_\oplus$ curve comes from hot-start solutions between 1.35 and 2.35 $M_\oplus$ and is within a factor of two of the target luminosity, showing the moderate sensitivity of the cooling curves to the mass. However, in the cold-start phase, a variation by a factor of two can also be obtained by varying at a fixed mass the initial entropy from 8.23 to 8.43 $k_B$/baryon. This great sensitivity implies that combining a luminosity measurement with information on the mass would yield, if some of the hot-start masses can be excluded, tight constraints on the initial entropy.

3.3 Definition of ‘hot-start mass’

By showing the entropy of hot starts as a function of time, Fig. 6 provides another way of looking at Fig. 5. Given a mass obtained from hot-start cooling tracks, the entropy value at that time read off from the curves indicates what ‘hot’ is, i.e. provides a lower bound on the initial entropy if the hot-start mass is the true mass. If however the planet is more massive, this entropy value is instead an upper bound on the post-formation entropy. As a rule of thumb, the entropy slope is $\approx -2$ or $-1 \, k_B$/baryon per time decade at early or late times, approximately, with the break coming from the change in the entropy regime (cf. Fig. 2).
4 GENERAL CONSTRAINTS FROM GRAVITY AND EFFECTIVE-TEMPERATURE MEASUREMENTS

Before applying the analysis described in the previous section to observed systems, it is worth discussing a second way by which constraints on the mass and initial entropy can be obtained. The idea is to firstly derive an object’s effective temperature and surface gravity by fitting its photometry and spectra with atmosphere models. Integrating the best-fitting model spectrum gives the luminosity, and this can be combined with \( T_{\text{eff}} \) and \( \log g \) to yield the radius and the mass. This procedure was described and carefully applied by Mohanty et al. (2004) and Mohanty, Jayawardhana, & Basri (2004). However, one can go further: considering models coupling detailed atmospheres with interiors at an arbitrary entropy, the mass and radius translate into a mass and current entropy. Then, using cooling tracks beginning with a range of initial conditions, the initial state of the object can be deduced given the age. Thus, in contrast to the case when only luminosity is used, \( M \) and \( S_1 \) can both be determined without any degeneracy between the two.

In practice, however, there seems to be too much uncertainty in atmosphere models for this method to currently be reliable, as the work of Mohanty et al. shows. Their sample comprised a dozen young (\( \sim 5 \) Myr) objects with \( T_{\text{eff}} \approx 2600–2900 \) K with high-resolution optical spectra. The combined presence of a gravity-sensitive Na I doublet and effective-temperature-sensitive TiO band near 0.8 \( \mu \)m allowed a relatively precise determination of \( \log g \) and \( T_{\text{eff}} \) for most objects, with uncertainties of 0.25 dex and 150 K (Mohanty et al. 2004). However, there were significant offsets in the gravity (\( \sim 0.5 \) dex) of the two coldest objects with respect to model predictions by Baraffe et al. (1998) and Chabrier et al. (2000, Mohanty et al. 2004) came to the conclusion that the models’ treatment of deuterium burning, convection or accretion – i.e. the assumed initial conditions – are most likely responsible for this disagreement at lower \( T_{\text{eff}} \). Moreover, the more recent work of Barman et al. (2011b) (see Section C1.1 below) indicates that ‘unexpected’ cloud thickness and non-equilibrium chemistry may compromise a straightforward interpretation of spectra in terms of gravity and temperature for young, low-mass objects. (See also Moses (2013) for a review of photochemistry and transport-induced quenching in cool exoplanet atmospheres.) Nevertheless, with the hope that future observations will allow a reliable calibration of atmosphere models, we illustrate with an example how \( M \) and \( S_1 \) can be determined for an object from its measured \( T_{\text{eff}} \) and \( g \).

Fig. 7 shows the constraints on the current mass and entropy of a planet with \( \log g = 4.00 \pm 0.25 \) (\( \text{cm s}^{-2} \)) and \( T_{\text{eff}} = 900 \pm 50 \) K, where the uncertainties in the gravity and effective temperature are the possible accuracy reported by Mohanty et al. (2004) and thus correspond to an optimistic scenario. The constraints are obtained by simultaneously solving for \( M, S, R, \) and \( L \) given \( T_{\text{eff}} \) and \( \log g \), with the \( L(M, S) \) and \( R(M, S) \) relations given by our model grid. The required mass and current entropy are \( M = 6.5 \, M_\odot \) and \( S = 9.2 \), with the 1-\( \sigma \) ellipse within 4.0–10.1 \( M_\odot \) and 9.0–9.3 \( k_B/\text{baryon} \). The large uncertainty in the mass (the width of the ellipse) is dominated by the uncertainty in the gravity, since radius is roughly constant at these entropies. We note that non-Gaussian errorbars are trivial to propagate through when determining the mass and entropy in this way since it is only a matter of mapping each \( (\log g, T_{\text{eff}}) \) pair to an \( (M, S) \) point.

Since this \( M-S \) determination concerns only the current state, it is independent of the cooling sequence, in particular of the ‘hot vs. cold start’ issue. Nevertheless, with this approach, it is immediately apparent what constraints the age imposes on the initial conditions. Indeed, not all \( (M, S) \) are consistent with an age since no planet of a given mass can be at a higher entropy than the hot-start model at that time (see Fig. 5). The entropy of hot starts is shown in Fig. 7 after 10, 30, and 50 Myr. The 30-Myr age excludes objects with \( M \leq 6 \, M_\odot \), which is the hot-start mass of this example. Considering only hot-start evolution sequences would have been equivalent to requiring the solution to be on one of the hot-start (grey) curves. This is however a restriction which currently could not be justified, given our ignorance about the outcome of the formation processes.

The solid line in Fig. 7 indicates the derived constraints on the initial entropy, assuming an age of 30 Myr. These constraints are similar to ones based on luminosity (see Fig. 5) but are somewhat tighter since an upper mass limit is provided by the measurement of \( \log g \). Even within a set of models, i.e. putting aside possible systematic issues with the atmospheres of young objects, it is however often the case that the surface gravity is rather ill determined (as for the objects discussed below in Section 5). In this case, provided a sufficiently large portion of the spectrum is covered, we expect the approach based on the bolometric luminosity presented in this work to be more robust than the derivation of constraints from effectively only the surface temperature. Indeed, the former avoids compoundung uncertainties in \( T_{\text{eff}} \) with those in the radius in an evolutionary sequence, which can be further affected by the presence of a core (of unknown mass). With both sustained modelling efforts and the detailed characterisation of an increasing number of detections,
one may hope that reliable atmosphere models for young objects will become available in a near future, allowing accurate determinations of the mass, radius, and initial entropy of directly-detected exoplanets.

5 COMPARISON WITH OBSERVED OBJECTS

5.1 Directly-detected objects

Neuhäuser & Schmidt (2012) recently compiled and homogeneously analysed photometric and spectral data for directly-imaged objects and candidates, selecting only those for which a mass below \( \simeq 25 \, M_\odot \) is possible.\footnote{This value was chosen by Schneider et al. (2011) as an approximation to the ‘brown-dwarf desert’, which is a gap in the mass spectrum between \( \simeq 25–90 \, M_\odot \) (Marcy & Butler 2000; Grether & Lineweaver 2004; Luhman et al. 2007; Dieterich et al. 2012).} They report luminosities and effective temperatures, which they either take from the discovery articles or calculate, usually from bolometric corrections when a spectral type or colour index is available or brightness differences between different masses. This is an update of the analogues figures of Marley et al. (2007) and Janson et al. (2011), which had only a handful of data points. Given doubts about its nature (e.g. Janson et al. 2012; Currie et al. 2012; Kalas et al. 2012; Kenworthy et al. 2013; Galicher et al. 2013; Currie et al. 2013b), we do not include the reported upper limit for Fomalhaut b in this plot. Since their uncertainties are large, the central age values are taken as the geometric mean of the upper and lower bounds reported if no value is given. For RXJ1609 B/b, we use instead \( 11 \pm 2 \, \text{Myr} \) (Pecaut, Mamajek, & Bubar 2012). The errorbars for the HR 8799 planets go up to 160 Myr and do not include the controversial Moya et al. (2010b) asteroseismology measurement of 1.1–1.6 Gyr since it is not used in our analysis (see discussion in Section C2). Finally, since no luminosity was given for WD 0806-661 B/b, we crudely estimate from the hot-start mass of 6–9 \( M_\odot \) from Luhman et al. (2012) and the Spiegel & Burrows (2012) models a luminosity of \( \log L/L_\odot = -7.0 \pm 0.3 \) at 1.5–2.5 Gyr.

Fig. 8 also includes four recent objects discovered since the analysis of Neuhäuser & Schmidt (2012): 2M0122 b (Bowler et al. 2013), GI 504 b (Kuzuhara et al. 2013), 2M0103 ABB (Delorme et al. 2013), and \( \kappa \) And b (Carson et al. 2013). For 2M0103 ABB, we estimate a bolometric luminosity of \( \log L/L_\odot = -4.87 \pm 0.12 \) as done in Appendix C5 for \( \beta \) Pic b. The same approach with \( \kappa \) And b yields \( \log L/L_\odot = -3.83 \pm 0.15 \), which is entirely consistent with the published value of \( -3.76 \pm 0.06 \) dex (Bonnefoy et al. 2013b). We show the conservative age range of \( 30^{+10}_{-12} \) Myr for \( \kappa \) And b.

A recent detection which is not included in Fig. 8 is a candidate companion to HD 95086 with a hot-start mass near 4 \( M_\odot \) (Rameau et al. 2013; Meshkat et al. 2013), since only an \( L^* \)-band measurement is available. Nevertheless, we report a prediction for its luminosity of \( \log L_{\text{bol}}/L_\odot \simeq -4.8 \pm 0.4 \) from the estimated \( T_{\text{eff}} = 1000 \pm 200 \, \text{K} \) and \( \log g = 3.9 \pm 0.5 \) (cm s\(^{-2}\)), with the \( L_{\text{bol}} \) errorbars entirely dominated by those on \( T_{\text{eff}} \) and ignoring that the atmospheric parameters were in fact estimated from hot-start models.

This value was chosen by Schneider et al. (2011) as an approximation to the ‘brown-dwarf desert’, which is a gap in the mass spectrum between \( \simeq 25–90 \, M_\odot \) (Marcy & Butler 2000; Grether & Lineweaver 2004; Luhman et al. 2007; Dieterich et al. 2012). The interpretation of this fact is not obvious given that the data points forming the lower envelope come from multiple surveys and that different observational biases apply at different ages (e.g. due to the relatively low number of young objects in the solar neighbourhood).

The second feature is the absence of detections between the hot-start 10- and 15-\( M_\odot \) cooling curves, roughly between 20 and 100 Myr. More accurately, there is, in a given age bin in that range, a lower density of data points with luminosities around \( 10^{3.4–3.8} \) than at higher or lower luminosities. A proper assessment of the statistical significance of this ‘gap’ in the data points would require taking both the smallness of the number of detections around 40 Myr and the biases and non-detections of the various surveys into account. However, it would not be surprising if the underdensity in the luminosity function \( dN/dL \) of the data points, at a fixed age, proved to be real, since there is also a suggestive underdensity in the cooling curves. Indeed, the onset of deuterium burning near 13.6 \( M_\odot \) slows down the cooling, which breaks the hot-start scaling \( L \propto t^{-1} \) (see Section 2.5) and leads to a greater distance between the hot-start cooling curves for 10 and 15 \( M_\odot \) than for 5 and 10 \( M_\odot \). (This is clearly visible in fig. 1 of Burrows et al. 2001), which also
Cold-start curves too show this gap, to the extent that the luminosity rise due to deuterium burning is very sensitive to the initial entropy (see Fig. 8), which would need to be set accordingly finely to have the lightcurves pass. This consideration is thus particularly relevant for GG Tau Bb, DH Tau B/B, GQ Lup b, and CT Cha b (data points 1, 5, 6, and 12), which are all possibly younger than 1 Myr, and would require a closer investigation.

We now provide detailed constraints for three planetary systems, chosen for the low hot-start mass of the companion (2M1207) or because additional mass information is available (HR 8799 and β Pic).

5.2 2M1207

The companion to the brown dwarf 2MASSWJ 1207334–393254 (2M1207 A, also known as TWA 27 A; Gizis 2002) is the first directly-imaged object with a hot-start planetary mass (Chauvin et al. 2004, 2005). Since the age and luminosity of 2M1207 b are the inputs for our analysis, they are discussed in some detail in Section 4, along with tentative information on the mass. We adopt an age of 8.2 ± 0.6 Myr (Chauvin et al. 2004; Song et al. 2006) and a luminosity of log $L/L_\odot = -4.68 ± 0.05$ (Barman et al. 2011b), and assume that deuterium-burning masses above $\sim 14 M_\odot$ (Spiegel et al. 2011) are excluded.

Fig. 8 shows the joint constraints on the mass and initial entropy of 2M1207 b based on its luminosity and age. We recover the hot-start mass of 3–5 $M_\odot$ (Barman et al. 2011b), with equation (15) predicting $\approx 3.9 M_\odot$, but also find solutions at higher masses. If deuterium-burning masses can be excluded, the formation of 2M1207 b must have led to an initial entropy of $S_i > 9.2$, with an approximate formal uncertainty on this lower bound of 0.04 $k_B$/baryon (see Section 5.1) due solely to the luminosity’s statistical error, independent of the age’s. This initial entropy implies that the cold starts are too cold by 0.7 $k_B$/baryon, roughly independently of the mass, to explain the formation of this planet. This is consistent with the time-scale-based conclusion of Lodato, Delgado-Donate, & Clarke (2005) that core accretion cannot be responsible for the formation of this system if one also accepts the received wisdom that core accretion necessarily leads to the coldest starts. However, our robust quantitative finding is more general, in that it provides constraints on the result of the formation process which are model-independent.

To show how these $M(S_i)$ constraints can easily be made even more quantitative and thus suitable for statistical analyses, we ran a Metropolis–Hastings Markov-chain Monte Carlo (MCMC; e.g. Gregory 2005) in mass and entropy with constant priors on these quantities. Uncertainties in the luminosity $L$ and age $t$ were included in the calculation of the $\chi^2$ by randomly choosing an ‘observed’ $L_{\text{obs}}$ and a stopping time for the cooling curve $t_{\text{stop}}$ at every step in the chain. The quantity $\log L_{\text{obs}}$ was drawn from a Gaussian defined by the reported best value and its errorbars, and $t_{\text{stop}}$ from significant the ‘gap’ is and how it evolves as data points are added to this diagram.

In Fig. 8 the best-fitting age is calculated as the geometric mean of the reported upper and lower bounds since, in most cases, no best-fitting value is provided, and the bounds are typically estimates from different methods, which cannot be easily combined. In fact, the ages of young ($< 500$ Myr) stars are in general a challenge to determine, as Soderblom (2010) reviews, and represent the main uncertainty in direct observations. Moreover, as Fortney et al. (2005) point out, assuming co-evatility of the companion and its primary may be problematic for the youngest objects. Indeed, a formation time-scale of $\approx 1–10$ Myr in the core-accretion scenario would mean that some data points of Fig. 8 below $\sim 10$ Myr may need to be shifted to significantly lower ages, by an unknown amount. This consideration is thus particularly relevant for GG Tau Bb, DH Tau B/B, GQ Lup b, and CT Cha b (data points 1, 5, 6, and 12), which are all possibly younger than 1 Myr, and would require a closer investigation.

shows that there is a similar gap for low-mass stars at $10^4$ L$_\odot$ and 1–10 Gyr, due to the hydrogen main sequence.) In particular, cooling tracks for objects of 15, 20, and 25 $M_\odot$ nearly overlap at $\sim 100$ Myr around $10^4$ L$_\odot$, where data points and their errorbars collect too. This tentative indication of an agreement between the detections and the cooling tracks suggests that the latter might be consistent with the data. It will therefore be interesting to see how
a distribution which is either constant in $t$ between the adopted upper and lower limits $t_1 = 5$ Myr and $t_2 = 12$ Myr and zero otherwise, or lognormal in $t$, centred at $t_0 = \sqrt{5} \times t_2 = 7.7$ Myr and with $\sigma_{\log t} = \log t_2/t_1 = 0.19$ dex. The results are shown along the vertical axis of Fig. 9 for four different assumptions. To obtain the two solid lines, we applied an upper mass cut at 13 $M_\odot$ and took a lognormal (less peaked curve) or a top-hat (more peaked) distribution in time. The dashed curves come from the same MCMC chains but with a mass cut-off of 20 $M_\odot$, i.e. including deuterium-burning objects. Because of the delayed cooling due to deuterium burning, the required initial entropy drops down faster with mass than in the cold-start branch, such that lower $S_i$ values are possible. At 20 $M_\odot$, the required $S_i$ is 8.2, but it is still only 9 $K_b$/baryon at 15 $M_\odot$. However, the phase space for high masses is very small since the initial entropy needs to be extremely finely tuned; hence the smallness of the effect on the posteriors. As Fig. 9 shows, different assumptions on the luminosity, age, and mass priors all lead to similar results for the initial entropy, namely that $S_i \geq 9.2$ and that there are more solutions near this lower limit.

5.3 HR 8799

We now turn to the only directly-imaged system with multiple objects for which planetary masses are possible, HR 8799. The age of the system and the luminosities of the companions are discussed in Section 4.2 along with information on the mass. We consider ages of 20 to 160 Myr, close to the values of Marois et al. (2008), and use the standard luminosities of $\log L_{bol}/L_\odot = -5.1 \pm 0.1$ (HR 8799 b), $-4.7 \pm 0.1$ (cd) and $-4.7 \pm 0.2$ (e) from Marois et al. (2008, 2010).

The entropy of the HR 8799 system must have formed with an initial entropy $S_i = 8.2$, but it is still only 9 $K_b$/baryon at 15 $M_\odot$. This is indicated by the vertical line segments with arrows in the direction of the excluded masses. The circles show the results from Marois et al. (2007), increased by 0.38 $K_b$/baryon (cf. Fig. 4), for cold starts (filled circles) and hot starts (open circles), where the entropy is 1 Myr after the onset of cooling.

It also seems reasonable to assume that deuterium-burning masses can be excluded for all objects, thanks to the (preliminary) results from simulations of the system’s dynamical stability.

Fig. 10 shows the joint constraints on the masses and initial entropies of HR 8799 b, d, and e. Uncertainties in the age are taken into account by considering the two extremes of 20 Myr and 160 Myr separately, while the 1-$\sigma$ errors in the luminosities are reflected by the width of the hatched regions. We find hot-start masses for 20 Myr of 4.4 $M_\odot$ (b), 6.3 $\pm$ 0.6 (cd), and 6 $\pm$ 1 (e), where the errors here come only from those on the luminosity, fully consistent with the prediction by equation (15) of $\approx 4.2$ or 6 $M_\odot$. These values are in good agreement with Neuhauser & Schmidt (2012) and are similar to the usually-cited 30-Myr values of (5, 7, 7) $M_\odot$ (Marois et al. 2010). The hot-start masses for 160 Myr are above 12 ($HR 8799$ b and 13 $M_\odot$ (cde), and the $M(S_i)$ constraints for the latter three are not shown since they are within the deuterium-burning regime.

Excluding deuterium-burning masses for all objects and using only the luminosity measurements, Fig. 10 shows that all planets of the HR 8799 system must have formed with an initial entropy greater than 9 $K_b$/baryon, with $S_i \geq 8.9$ for b, $S_i \geq 9.2$ for c and d, and $S_i \geq 9.1$ for e (using as throughout this work the published SC14 entropy table; see Appendix B). Using tentative upper mass limits of 7 and 10 $M_\odot$, respectively, the lower bounds on the initial entropies can be raised to 9.2 (b) and 9.3 $K_b$/baryon (cde) if one takes the conservative scenario of the 1-$\sigma$ lower luminosities value at 20 Myr. These lower bounds on the entropy are however...
mostly independent of the age because they are set by cold-start solutions, where the age is much smaller than the cooling time at that entropy. Formal uncertainties on the lower bounds due to those in the luminosities are (see Section S4) approximately 0.07 (bcd) or 0.14 kg/baryon (c) and thus negligible.

Here too we ran an MCMC to derive quantitative constraints on the initial entropy of each planet. The quantity log $L_{bol}$ was drawn from a Gaussian defined by the reported best value and its errorbars, and $t_{bol}$ from a distribution which is either constant in $t$ between the adopted upper and lower limits $t_1 = 20$ Myr and $t_2 = 160$ Myr and zero otherwise, or lognormal in $t$, centred at $t_0 = \sqrt{t_1 t_2} = 57$ Myr and with $\sigma_{\log t} = \log t_2/t_1 = 0.9$ dex. Posteriors on the initial entropy for each of the HR 8799 planets are shown on the vertical axis in Fig. 10 using a flat prior in $S_i$ and a mass prior constant up to an $M_{max}$ and zero afterward. The cases ‘without mass constraints’ ($M_{max} = 13$ M$_J$) are nearly constant in $S_i$, especially for planet b, but show a peak near cold-branch values of 9 and 9.5 kg/baryon for HR 8799 b and cde. Adding mass information from dynamical-stability simulations by taking $M_{max} = (7, 10, 10, 10)$ M$_J$ flattens the $S_i$ posterior and shifts the minimum bounds at half-maximum from (8.9, 9.2, 9.2, 9.2) to (9.3, 9.5, 9.5, 9.4) kg/baryon, respectively. We note that these results are insensitive to both the form of the uncertainty in $t$ and the use of a non-flat prior in mass (as shown below for $\beta$ Pic b in Section S4).

Comparing to the ‘tuning fork’ entropy values reproduced in Fig. 10 we find that the coldest-start models of M07 cannot explain the luminosity measurements for the HR 8799 planets. Spiegel & Burrows (2012) and Marley et al. (2012) also came to this conclusion, with the latter noting that ‘warm starts’ match the luminosity constraints. It is now possible to say specifically that the cold starts would need to be made $\Delta S \approx 0.5$ kg/baryon hotter to explain the formation of the HR 8799 planets. Given that the precise outcomes of the core accretion and gravitational instability scenarios are uncertain and that this system represents a challenge for both (e.g. Marois et al. 2010 and Currie et al. 2011 review), quantitative comparisons such as our procedure allows should be welcome to help evaluate the plausibility of the one or the other.

We note in passing that one needs to take care also when interpreting the measurements of Hinkley et al. (2011) and Close & Males (2010). These authors measured upper limits on the brightness of companions within 10 AU and between 200 and 600 AU from the star, respectively. However, both groups then used the hot-start models of Baraffe et al. (2003) to translate the brightness limits into masses (11M$_J$ at 3–10 AU and 3M$_J$ within 600 AU, respectively). Therefore, since colder-start companions would need to be more massive to have the same luminosity, what they provide are really “lower upper limits” on the mass of possible companions. How much higher the masses could realistically be in this case is difficult to estimate without a bolometric luminosity, but there is an important general point: without the restriction of considering only hot-start evolutionary tracks, luminosity upper limits do not provide unambiguous mass constraints. Incidentally, this more general view of the results of Hinkley et al. (2011) means that the unseen companion evoked by Su et al. (2006) as the possible cause of the inner hole (at $\approx 6$ AU) does not have to be of small mass. However, this inner object would nevertheless have to be consistent with the results of dynamical stability simulations, with those of Goździewski & Migaszewski (2013) indicating a mass less than $1–8$ M$_J$.

5.4 $\beta$ Pic

A companion to the well-studied star $\beta$ Pic was first observed in 2009 (Lagrange et al. 2009; Bonnefoy et al. 2011) and, very recently, it became the first directly-detected object with a planetary mass for which radial-velocity data are also available. The age of the system is taken as $12^{+4}_{-4}$ Myr (Zuckerman et al. 2000), and we discuss in detail in Section C how we derive a bolometric luminosity of log $L_{bol}$/$L_\odot = -3.90^{+0.05}_{-0.12}$.

Fig. 11 shows the $M(S_i)$ constraints available for $\beta$ Pic b from our luminosity estimate and the radial-velocity (RV) constraint. We recover a hot-start mass $\approx 9.5 \pm 2.5$ M$_J$ (cf. $\approx 9.4$ M$_J$ from equation 15), in agreement with Quanz et al. (2010) and Neuhauser & Schmid (2012), but additionally find that higher masses are consistent with the luminosity measurement. Excluding solutions where deuterium burning plays an important role in the evolution of the object (recognisable by the extreme thinness of the constant-luminosity $M(S_i)$ curve) implies that $S_i \gtrsim 9.8$. Using the RV mass upper limit, these constraints on the initial entropy can be made tighter: with an age of 12 Myr, it must be that $S_i \gtrsim 10.5$. Since this corresponds to a warm start, both uncertainties on the age and on the luminosity contribute to that on the minimum $S_i$, of order 0.5 kg/baryon.

This lower limit on $S_i$ implies that closest-start objects of any mass are too cold by a significant 1.5–2.0 kg/baryon. Various authors (e.g. Quanz et al. 2010; Bonnefoy et al. 2011) recognised that the classical cold starts (M07; Fortney et al. 2008) cannot explain the observations, but it is now possible to quantify this. These results are mostly insensitive to the uncertainty on the age range; using instead 12–22 Myr as summarised by Fernández et al. (2008) would not change the conclusions.

As for the objects in the 2M1207 and HR 8799 systems, we ran an MCMC to obtain a posterior distribution on the initial entropy. This is shown along the vertical axis of Fig. 11 for four different assumptions. In all cases, we assumed lognormal uncertainties on the age and the luminosity, with asymmetric upper and lower errorbars for the latter. For the full curve closer to the vertical axis (in blue), we applied an upper mass cut at 12 M$_J$ and took into account that the underlying (real) mass distribution is possibly biased towards lower masses, as radial velocity measurements indicate (Cumming et al. 2008; Nielsen & Close 2010). Out of simplicity, this was done by taking the $(M, S_i)$ distribution obtained with a flat prior in mass and using importance sampling to put in a posteriori a $dn/dM \propto M^\alpha$ prior, with $\alpha = -1.3$, thus weighing lower masses more. Of course, the value of $\alpha$ might depend

\begin{itemize}
  \item[11] As this manuscript was being prepared, we became aware of the first robust estimate of the bolometric luminosity, by Bonnefoy et al. (2013a). They find log $L_{bol}$/$L_\odot = -3.87 \pm 0.08$, which excellently agrees with our value and thus does not change our conclusions. In particular, they find similar constraints on the initial entropy of $\beta$ Pic b, although this depends on which band they use (cf. their fig. 11 with our Fig. 11). We also note the more recent estimate by Currie et al. (2013), of log $L_{bol}$/$L_\odot = -3.80 \pm 0.02$ (very near our approximate 1-σ upper limit), from which they estimate a (hot-start) mass in the 3–11 M$_J$ range. Importantly, they obtain the lower masses by considering an age of 7 Myr for $\beta$ Pic b, i.e. by relaxing the assumption that it is co-eval with its star (see Section S4).
  \item[12] Wahhaj et al. (2013) derive in a recent analysis of the NICI campaign results for debris-disk stars a similar slope: if the linear semi-major axis power-law index $\beta_a = -0.61$ as Cumming et al. (2008) found for radial-velocity planets within 3 AU, the 66 non-detections combined with the Vigan et al. (2012) survey imply $\alpha < -1.7$ to 2, with the most likely values at $0.3 \omega + \beta_a < -1$. When however $\beta$ Pic b and HR 8799 bcd are included
\end{itemize}
Recently, Bodenheimer et al. (2013) and Mordasini (2013) showed that in the framework of formation models (which seek to pre-
form initial entropies of exoplanets

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\beta \text{ Pic b}
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\[
\text{excluded by radial velocity}
\]

\[
8 \text{ Myr}
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\[
1111\ 12 \text{ Myr}
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\[
\\//\\ 20 \text{ Myr}
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\text{Figure 11. Allowed values of planet mass } M \text{ and initial entropy } S_i \text{ for } \beta \text{ Pic b using our estimated } \log L_{\text{bol}}/L_\odot = -3.90^{+0.10}_{-0.05} \text{ and an age of } 12.5 \text{ Myr. The hatched regions correspond to luminosity values within the (asymmetrical) 1-\sigma interval, with the central values marked by thick lines. The curves along the vertical show the marginalised posterior distribution on } S_i \text{ from an MCMC simulation using the luminosity and age values and their uncertainties. These were taken to be lognormal but asymmetric for the luminosity, and lognormal in time, centred at } t_0 = \sqrt{8 \times 20} = 12.7 \text{ Myr with a standard deviation of } 0.35 \text{ dex. The full blue line (closer to the vertical axis) results from the } (M, S_i) \text{ distribution being multiplied by a } dN/dM \propto M^\alpha \text{ prior, with } \alpha = -1.3 \text{ (Cumming et al. 2008; Nielsen & Close 2014), and zero for } M > 12 \text{ M}_\odot \text{ while the full grey posterior has only a mass cut at } 12 \text{ M}_\odot \text{, from the Lagrange et al. (2012b) radial-velocity constraint. The dashed and dotted curves both have an upper mass cut of } 20 \text{ M}_\odot \text{, with and without the } dN/dM \text{ prior applied, respectively. The circles show the results from Marley et al. (2007), increased by } 0.38 \text{ km/s/baryon (cf. Fig. 2), for cold starts (filled circles) and hot starts (open circles), where the entropy is 1 Myr after the onset of cooling.}

\text{on the formation mechanism relevant to the object but this serves to illustrate the effects of a non-constant prior on mass. The other three posterior distributions on } S_i \text{ of Fig. 11 (in grey) correspond to the remaining combinations of 'with mass cut or not' and 'with power-law mass prior or not'. These curves are all similar, with the radial-velocity measurement increasing the minimum bound at half maximum from 10.2 to 10.5 km/s/baryon, quite insensitively to the use of the } dN/dM \text{ prior.}

\text{At the distance from its primary where } \beta \text{ Pic b is currently located (} \simeq 9 \text{ AU), core accretion is expected to be efficient and thus a likely mechanism for its formation Lagrange et al. (2013); Bonnefoy et al. (2013). Thus, the question posed to formation models is whether core accretion can be made hotter (by 1.5–2 km/s/baryon) than what traditional cold starts predict. Very recently, Bodenheimer et al. (2013) and Mordasini (2013) showed that in the framework of formation models (which seek to predict } S_i \text{), different rocky core masses are associated with a significantly different initial entropies at a fixed total mass; for instance, Bodenheimer et al. (2013) found } S_i = 7.5 \text{ for a } 12 \text{ M}_\odot \text{ object with}

\text{in the analysis, } \alpha > 2.2 \text{ for } \beta_0 = -0.61, \text{ with more solutions at } \beta_0 = -2.1, \alpha > 2.2. \text{(Too few detected objects prevent Biller et al. (2013) from inferring constraints on } \alpha \text{ and } \beta \text{ in a similar analysis of young moving-group stars.) Note finally that hot-start models were used to convert magnitudes to masses and that most targets are less than 100 Myr old, with a significant fraction near 10 Myr; ignoring cold starts at these ages can skew the inferred (limits on the) mass distribution. Also considering colder starts should yield more negative constraints on } \alpha \text{ given the same luminosity constraints.}

\text{a core of } 5 \text{ M}_\odot \text{ but } S_i = 9.1 \text{ when a different choice of parameters lead to a core mass of } 31 \text{ M}_\odot. \text{ Since these coldest starts assume that all the accretion energy is radiated away at the shock, the constraints on the initial entropy stress the need to investigate the physics of the shock (and its dependence on physical quantities such as the accretion rate), when the initial energy content of the planet is claimed to be set.}

\text{6 SUMMARY}

\text{The entropy of a gas giant planet immediately following its formation is a key parameter that can be used to help distinguish planet formation models (Marley et al. 2007). In this paper, we have explored the constraints on the initial entropy that can be obtained for directly-detected exoplanets with a measured bolometric luminosity and age. When the initial entropy is assumed to be very large, a 'hot start' evolution, the measured luminosity and age translate into a constraint on the planet mass. In contrast, when a range of initial entropies are considered ('cold starts' or 'warm starts'), the hot-start mass is in fact only a lower limit on the planet mass: larger-mass planets with lower initial entropies can also reproduce a given observed luminosity and age. Fig. 5 shows the allowed values of mass and entropy for different ages and luminosities, and can be used to quickly obtain estimates of mass and initial entropy for any given system.}

\text{To derive these constraints, we constructed a grid of gas giant models as a function of mass and internal entropy which can then be stepped through to calculate the time evolution of a given planet. In a hot-start evolution, the structure and luminosity of cooling gas giant planets are usually thought of as being a function of mass and time only; this leads to a 'hot-start mass' as given by equation (15). Once the assumption of hot initial conditions is removed, however, a more convenient variable is the entropy of the planet. One way to think of this is that gas giants obey a Vogt–Russell theorem in which the internal structure, luminosity, and radius of a planet depend only on its mass and entropy (as well as its composition, as for stars). Fig. 4 shows the luminosity as a function of entropy for different masses, and a general fitting formula for } L(M, S) \text{ is given by equation (9). (Similarly, cooling tracks with arbitrary initial entropy are well described analytically by equations (13–14).) A noteworthy result is that in the intermediate-entropy regime (} S \simeq 8.5–10 \text{ km/s/baryon}), where the outer radiative zone is thick and follows a radiative-zero solution, the luminosity obeys } L = M f(S) \text{ as found for irradiated gas giants by Arras & Bildsten (2006), with } f(S) \text{ a steeply increasing function of the entropy. We also note that constraints obtained for models with a particular helium mass fraction } Y \text{ can be easily translated to another } Y \text{ with equation (15) as it provides an approximate value for } dS/dL \text{ at constant } L. \text{ This is general and independent of the approach used to compute the cooling.}

\text{We find that our models are in good agreement (within tens of per cent) with the hot-start models of Burrows et al. (1997) and Baraffe et al. (2003), as well as the cold-start models of Marley et al. (2007), and cooling models calculated with the MBSA stellar evolution code (Paxton et al. 2011, 2013). We caution that the Spiegel & Burrows (2012) and Molière & Mordasini (2012) models, for example, use a version of the Saumon et al. (1995) equation of state whose entropy is offset by a constant 0.52 km/s/baryon from the published tables (which the present work uses); this difference is not significant physically but needs to be taken into account when comparing results of various groups. Details and (} Y, P, T, Q, S) \text{ points for a quick comparison are provided in the appendix.}
Appendix B. The remaining intrinsic difference in entropy between our models and those just cited is then approximately \( |\Delta S| \approx 0.15 \) at worse. This is more important than differences in opacities or composition; for example, we estimate from Saumon et al. (1996) that the uncertainty in the helium \((Y)\) and metal \((Z)\) mass fractions introduces variations of at most 10 per cent in the luminosity at a given age.

We stress again that when the initial entropy is allowed to take a range of values, the hot-start mass (equation (15)) is only a lower limit on the companion mass. The larger range of allowed masses means that the hot-start mass could actually lead to the mischaracterization of an object, with a hot-start mass in the planetary regime actually corresponding to an object with a mass above the deuterium-burning limit for low enough entropies. One way to break the degeneracy between mass and entropy is an accurate determination of the radius from spectral fitting (or actually a determination of \( \log g \) and \( T_{\text{eff}} \) from the spectrum), which would yield the mass and (current) entropy of the object without any degeneracy (see e.g. Fig. 1). As discussed in Section 4 however, current atmosphere models have significant uncertainties that make this approach difficult. Another possibility is to obtain independent constraints on the mass of a companion, for example from dynamical considerations.

In Section 5 we applied our models to three directly-imaged objects which have hot-start masses in the planetary-mass regime and for some of which there are additional constraints on the planet mass. We find that the initial entropy of 2M1207 b is at least 9.2 \( k_b/\text{baryon} \), assuming that it does not burn deuterium. For the planets of the HR 8799 system, we infer that they must have formed with \( S_i > 9.2 \ k_b/\text{baryon} \), independent of the age uncertainties for the star. Finally, a similar analysis for \( \beta \) Pic b reveals that it must have formed with \( S_i > 10.5 \ k_b/\text{baryon} \), using the radial-velocity mass upper limit of 12 \( M_J \). These initial entropy values are respectively ca. 0.7, 0.5, and 1.5 \( k_b/\text{baryon} \) higher than the ones obtained from core accretion models by Marley et al. (2007). This quantitatively rules out the coldest starts for these objects and constrains warm starts, especially for \( \beta \) Pic b.

An important point is that the uncertainties in age and luminosity impact the derived hot-start mass and the lower bound on initial entropy in different ways. The major uncertainty in direct detections is the age of the star and, relevant for very young systems, all the more that of the planet. This age uncertainty translates into errorbars for the hot-start mass which are \( \Delta M/M \approx \frac{1}{\Delta t} \). However, the uncertainty on the initial entropy on the mass-independence branch of the \( M(S_i) \) curve is due only to that in the bolometric luminosity, with \( \Delta S_i \approx 1/\Delta \log L_{\text{bol}} \), where \( \lambda \approx 0.7 \) when \( S_i \lesssim 9.6 \) (or, less accurately, when \( L \sim 10^{-6} - 10^{-4} \ L_\odot \)). This uncertainty \( \Delta S_i \) is typically very small, which means that, up to systematic errors, the initial entropy can be determined quite accurately.

It has been pointed out before that current direct-imaging detections are all inconsistent with the cold-start predictions from core-accretion models. It is important to stress however that the cold starts are in some sense an extreme case, as they assume complete radiation efficiency at the shock during runaway accretion, which is argued to set the low initial entropy of planet. By varying the nebula temperature or the accretion rate, Marley et al. (2007) were able to change the entropy by barely \( \Delta S \approx 0.1 \); however, Mordasini (2013) and Bodenheimer et al. (2013) report that it is possible within the core-accretion scenario to obtain considerably higher entropies, increased by as much as 1–2 \( k_b/\text{baryon} \), when considering different masses for the solid core (through a self-amplifying process explained in Mordasini (2013)). With \( \beta \) Pic b a likely candidate for formation by core accretion, this indicates that it is essential to gain a deeper understanding about what sets the initial entropy, for instance by looking in more detail at the properties of the shock during runaway accretion.

The derived bounds on \( S_i \) for the HR 8799 objects and \( \beta \) Pic b made use of information on the mass, which comes from dynamical stability analyses and radial velocity, respectively. Radial-velocity data of directly-detected planets are currently available only for \( \beta \) Pic b, but this should change in a near future as close-in planets start being detected directly. In the absence of dynamical information, an upper limit to the mass (and thus a lower limit on the initial entropy) should be obtainable from \( \log g \), even if its errorbars are large; thus, in practice, the \( M(S_i) \) constraint curve does not extend to arbitrarily high masses as a pure luminosity measurement would imply.

Finally, we ran Markov-chain Monte Carlo simulations to derive more detailed quantitative constraints on the mass and entropy of directly-detected objects. When taking the uncertainties in the age and luminosity into account, we considered normal, lognormal, and flat distributions and found the chosen form to make little difference. The advantage of this approach is that it allows one to derive posterior distributions on the initial entropy, which are suitable for statistical comparisons. Given the small semi-major axis of \( \beta \) Pic b (9 AU), we also tried a prior \( dN/dM \propto M^{-1.3} \) (which describes the population of radial-velocity planets) in addition to a flat prior on \( M \). The latter case yielded posterior distributions on \( S_i \) with a more pronounced peak.

The benefits that the expected large increase in the number of directly-detected exoplanets in the near future should bring are at least twofold. Firstly, each new detection will yield a new constraint on the initial entropy and therefore formation mechanism. Particularly with the ability to detect lower mass gas giants at small semi-major axes (\( \lesssim 20 \) AU), which should be possible with instruments such as GPI or SPHERE, there will be an opportunity to constrain the state of the gas giant immediately following core accretion. Secondly, with a larger sample of objects comes a chance for a statistical comparison with formation models. For example, as noted in Section 5.1 the onset of deuterium burning leads to a relative underdensity of planets with luminosities \( \sim 10^{-4} \ L_\odot \) and ages of tens of Myr (assuming that the masses of substellar objects are smoothly distributed near the deuterium-burning limit). Indeed, it is interesting that there appears to be such an underdensity in the current data sample (see Fig. 8), although the small number of detections so far means that this could be due to a statistical fluctuation. The best constraints on initial conditions for planet cooling will come from improved spectral models that can give reliable determinations of \( \log g \) and \( T_{\text{eff}} \).

ACKNOWLEDGMENTS

We thank P. Bodenheimer, D. Saumon, J. Ferguson, K. Goździewski, A. Burrows, J. Fortney, A. Showman, M. Marley, X. Huang, C. Mordasini, D. Spiegel, T. Guillot, T. Schmidt, M. Bonnefoy, B. Biller, and J. Carson for useful and often detailed discussions, helpful and rapid answers to inquiries, and generosity with data. This work was supported by the National Sciences and Engineering Research Council of Canada (NSERC) and the Canadian Institute for Advanced Research (CIFAR), and by a scholarship from the Fonds de recherche du Québec – Nature et technologies (FRQNT). GDM warmly thanks the MPIA for support during the last stages of this work.
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APPENDIX A: RADI

For completeness, we present and compare the radii $R$ in our models as a function of mass $M$ and entropy $S$. This comparison thus separates out possible differences in the treatment of deuterium burning since the planetary structure at a given $(M, S)$ is independent of the nuclear energy generation, which only affects the time evolution.

Fig. A1 shows $R(S)$ for different masses, using the standard grid with a helium mass fraction $Y = 0.25$ and without a solid core. At low entropies, objects of a given mass have a roughly constant radius asymptotically tending to the zero-temperature value $Zapolsky & Salpeter (1969)$. $Hubbard (1973) & Arras & Bildsten (2006)$, while the radius increases at high entropies. This increase is less pronounced for higher masses, with more massive objects being smaller at any given entropy. Finally, there is at a given mass a maximum entropy for which a finite radius is possible, as the upturn of the curves suggests. In fact, each curve turns over on to a hot branch (not calculated) where the ion thermal pressure dominates, leading to distorted objects; see for instance the analytic one-zone model of $Delove & Bildsten (2003)$.

The inset of Fig. A2 shows the relative difference $\Delta R/R_{\text{ref}} = (R - R_{\text{ref}})/R_{\text{ref}}$ between our radius at a given mass and entropy and that from more detailed calculations, either $Burrows et al. (1997)$ or $mesa (revised 4723) & Paxton et al. (2011, 2013)$. Over the range where data are available, there are larger deviation at higher entropies but the overall agreement for masses between 1 and 20 $M_J$ is excellent, with our models systematically a few per cent smaller.

Fig. A2 also shows $R(M)$ curves at fixed $S$ in a grid with a 20-$M_J$ core at a constant density of 8 g cm$^{-3}$ and in a grid with a helium mass fraction $Y = 0.30$ (and no core). Differences are small when including a solid core but larger when varying the helium fraction, with differences of the order of tens of per cent at smaller entropies. Note that a realistic equation of state for the core, such as a half-half rock–ice mixture from $ANEOS (Marleav et al. 2007)$, as formation by core accretion may produce, would yield average core densities closer to $10–100$ g cm$^{-3}$. The effect on the radius should however still be within tens of per cent.

$^{13}$ The presence of a turnover in the mass–radius relationship is due to the balance between the attractive Coulomb forces in the ion-electron plasma and the repulsive Fermi forces between the degenerate electrons.
Constraining initial entropies of exoplanets

APPENDIX C: AGE, LUMINOSITY, AND MASS CONSTRAINTS

C1 2M1207

C1.1 Age and luminosity

The first directly-imaged object with a hot-start planetary mass (Chauvin et al. 2004, 2005) is located 0.8" from its M8 brown-dwarf primary, a well-studied (see Gizis 2002 for the report of the discovery and Skemer et al. 2011 for a summary) member of the young (8−15 Myr) TW Hydrae association (Chauvin et al. 2004; Song et al. 2006) at a distance of 53 ± 1 pc (as averaged by Skemer et al. 2011 from Gizis 2002; Gizis et al. 2003; Biller & Close 2007; Mamajek & Meyer 2003; Margot et al. 2006), implying a projected orbital separation of 41 ± 1 AU.

Determining the luminosity of 2M1207 b is not straightforward. Photometry is available in the J (Mohanty et al. 2007), H, K, and L’ bands from NaCo observations at the VLT (Chauvin et al. 2004; Mohanty et al. 2007), at 0.8−1.0, 1.0−1.2, 1.35−1.55, and 1.4−1.8 μm (similar to H) from the Hubble Space Telescope (Song et al. 2006), and in the Herschel SPIRE bands of 250 and 350 μm (Riaz et al. 2012; Skemer et al. 2011) also obtained an upper limit at 8.3−9.1 μm. Spectroscopy is available at 1.1−1.35 μm, 1.4−1.8 μm (Chauvin et al. 2004; Mohanty et al. 2007; Patience et al. 2010), and 1.95−2.5 μm (Patience et al. 2011) and 2.0−2.4 μm (Mohanty et al. 2007), i.e. in JHK, and slightly redwards. Summing up the fluxes in J, H, K, and L’ (listed in Mohanty et al. 2007) and using the known distance gives a luminosity of 2.1 × 10^{27} erg s^{-1}, or 27−30 per cent of a bolometric luminosity log L_{bol}/L_{⊙} = −6.86 to −4.74 (see below).

A ‘direct luminosity determination’ is thus not possible.

As summarised by Barman et al. 2011b, there are inconsistencies between the luminosity of 2M1207 b and its spectral type, determined by Chauvin et al. 2004 to be L5−L9.5. With the corresponding bolometric correction BC_K of Golimowski et al. 2004, the K-band magnitude implies a luminosity of log L/L_{⊙} = −4.7 ± 0.1 Barman et al. 2011b. The hot-start, equilibrium models of Barber et al. 2003 then yield from the age and luminosity an effective temperature of 1010 ± 80 K Barman et al. 2011b in disagreement with T_eff = 1600 K implied by the spectral type.
C2 HR 8799

C2.1 Age and luminosities

Several properties of HR 8799 let its age be estimated: variability from non-radial oscillations, low abundance of iron-peak elements, and far-IR excess due to circumstellar dust by Marois et al. (2008). Along with its Galactic space motion and position in a Hertzsprung–Russell diagram, these lead Marois et al. (2008) to estimate an age range of 30–160 Myr with a preferred value of 60 Myr, consistent with the 20–150 Myr range of Moor et al. (2006) based on membership in the Local Association. Recently, Baines et al. (2013) used interferometric measurements of HR 8799’s radius to derive a stellar mass and age. They found best-fitting ages of 33 or 90 Myr, depending on whether the star is approaching or moving away from the main sequence. However, the statistical errorbars, which do not take uncertainties in the stellar models into account, are considerable in the second case (the 1-σ ranges are 20–40 and 40–471 Myr, respectively). Nevertheless, if Baines et al.’s measurement of the stellar radius and the deduced metallicity are correct, the age range of 1.1–1.6 Gyr from the asteroseismological analysis of Moya et al. (2010a) would be compromised, as Baines et al. (2013) point out. Indeed, they estimate a near-solar metallicity, which contrasts with the [M/H] = −0.3 or −0.1 result of Moya et al. (2010b), while metallicity is an important input of asteroseismological analyses. Moreover, there is the statistical argument put forth by Marois et al. (2008) that massive discs (such as HR 8799’s of 0.1 M*) are unlikely to be found around older stars. In our analysis, we shall therefore ignore the 1.1 Gyr result and instead use 20 and 160 Myr as lower and upper limits, which brackets the ranges reviewed in Moya et al. (2010a) and Baines et al. (2013).

Marois et al. (2008, 2010) estimated luminosities of log \( \log(L/L_\odot) = -5.1 \pm 0.1 \) (HR 8799 b), \(-4.7 \pm 0.1 \) (cd) and \(-4.7 \pm 0.2 \) (e) from the known distance of 39.4 ± 0.1 pc and six infrared magnitudes, covering a total of the bolometric luminosity, and also from bolometric corrections for brown dwarfs. For their part, Marley et al. (2012) recently derived luminosities of log \( \log(L/L_\odot) = -4.95 \pm 0.06, -4.90 \pm 0.10, \) and \(-4.80 \pm 0.09 \) for planets b, c, and d, respectively, by self-consistently obtaining the radius from evolutionary models. This contrasts with the usual procedure of optimizing (\( R/\sigma \)) along with \( T_{\text{eff}} \) and \( g \) to fit the photometry, which yields unphysically small radii of \( 0.8 R_\odot \) Baines et al. (2014). Marley et al. (2013) similarly, in a recent study using “atmospheric retrieval” (non-parametric determination of the \( P-T \) and composition structure) including a simple cloud model, Lee, Heng, & Irish (2013) also find for HR 8799 b a small radius of 0.66±0.04 R\(_\odot\), which implies with their \( T_{\text{eff}} = 900$±$90 \) K and \( g = 5.0$±$0.1 \) (cm s\(^{-2}\)) a bolometric luminosity \( \log(L/L_\odot) = -5.57 \). Since Marley et al. considered only hot stars, i.e. fixed \( \tau_1 \) high, the cooling tracks are \( \lambda_{\text{bol}}(M, t) \) and \( \pi(M, t) \) relations, which let \( R \) and \( t \) be uniquely determined from \( \log g \) and \( T_{\text{eff}} \). This gives an age of 360 Myr for HR 8799 b and age ranges of 40–100 Myr and 30–100 Myr for c and d, consistent with other literature estimates. Given the difficulties in obtaining a reasonable fit, Marley et al. (2012) warn that the first result should not be taken seriously, and note that the tension in the age would be reduced by considering colder initial conditions. In our analysis, the luminosity values of Marois et al. (2008, 2010) will be used since they are standard and almost or marginally consistent with those of Marley et al. (2012).

Finally, an age of 30 or 60 Myr for the system leads Marois et al. (2010) to derive from the luminosity and the cooling tracks of Baraffe et al. (2003) masses of (5, 7, 7, 7) or (7, 10, 10, 10) M\(_J\). The uncertainty in the age implies (see Section [3]) AM \( \approx 1–1.3 \) M\(_J\) for the hot-start values.

16 See comment in section 1.2.2 of Marley et al. (2012).

17 In fact, the errorbars on the luminosity of HR 8799 are c are clearly non-Gaussian, but this will not be taken into account out of simplicity.

18 Note also their careful and detailed review of atmospheric modelling efforts for HR 8799 b and their problems.
C2.2 Dynamical stability

Since it is to date the only directly-imaged multiple-planet system, HR 8799 has received a considerable amount of attention with regard to its dynamical stability (e.g. Marois et al. 2008; Reidemeister et al. 2009; Fabrycky & Murray-Clay 2010; Moro-Martin et al. 2011; Marois et al. 2010; Benno et al. 2011; Currie et al. 2013; Sudol & Haghighipour 2012; Esposito et al. 2013; Currie et al. 2013b; see reviews in Sudol & Haghighipour 2012 and Goździewski & Migaszewski 2013). However, only the most recent studies were able to consider all four planets. Crucial questions include whether there are two- or three-planet mean-motion resonances (MMRs; as Goździewski & Migaszewski 2009 and Fabrycky & Murray-Clay 2010 suggest), what the inclination and eccentricities of the orbits are (for instance, Lafrenière et al. 2009 estimated and Fabrycky & Murray-Clay 2010 suggest), and whether there are two- or three-planet co-planar (against which Currie et al. 2012b recently point out that if it is young with respect to its main-sequence lifetime, the planets should have as low masses as allowed, with how- ever somewhat higher masses permitted if some orbits are resonant. For this reason, we shall consider as approximate upper limits from stability analysis masses of 7, 10, 10, and 10 M\(_{\oplus}\) (bcde). In particular, as Goździewski & Migaszewski 2013 also find, it seems very likely that none is a deuterium-burning object.

C3 β Pic

C3.1 Age and luminosity

The namesake A5 dwarf of the nearby (9–73 pc; Malo et al. 2013) β Pictoris moving group has an age of 12 ± 6 Myr (Zuckerman et al. 2001) and asymmetric outer and warped inner discs, which have been observed for more than two decades (Lagrange et al. 2009, 2012a, see review in Lagrange et al. 2011). A companion was first detected in \(L^\prime\) (Lagrange et al. 2009) and subsequently confirmed at 4 μm (Quanz et al. 2010) and in \(K_s\) (Bonnefoy et al. 2011). Very recently, Bonnefoy et al. (2013a) added to these observations photometry in \(J\), \(H\), and \(M^\prime\). The distance of 19.44 ± 0.05 pc to \(β\) Pic b implies an orbital separation of 8–9 AU (Chauvin et al. 2013; Bonnefoy et al. 2013b), which is the smallest of all low-mass directly-detected objects. An object at this position had been predicted from the disc morphology by Freistetter, Kivio, & Löhne (2007).

19 Note that in a very recent study, Goździewski & Migaszewski 2013, finding, using a novel approach which assumes multiple MMRs but yields masses independently, that broad mass rangeloses which include the hot-start values (though not perfectly for HR 8799 c and d) are 4–8, 8–12, 8–12, and 7–10.5 M\(_{\oplus}\) (bcde). Using these values instead would barely change our derived minimum bounds on \(S_1\), lowering some by \(0.1\) kJ/baryon.

20 This is the value obtained from a re-reduction of Hipparcos data by van Leeuwen (2007). However, a number of recent studies still use the value of 19.3 ± 0.2 pc (Crislo et al. 1997).

C3.2 Mass information

The object \(β\) Pic b is particularly interesting because it is the first directly-imaged companion for which radial-velocity data are also available (Lagrange et al. 2012). Using new and archival data spanning eight years and thanks to the high inclination of the system (88 ± 2°; Chauvin et al. 2013; Lagrange et al. 2012a), were able to place tentative lower mass limits of 1–2 M\(_{\oplus}\), which is fully consistent with all reasonable age and luminosity combinations, even allowing for very large lower errorbars on the latter. However, Lagrange et al. place upper limit of 10–25 M\(_{\oplus}\) for the most probable orbit \(β\) Pic A (Lagrange et al. 2009; Chauvin et al. 2013) is an important result which excludes high-mass solutions and puts the object quite likely in the planetary (non-deuterium burning) range.

We conclude with a brief digression. To the rarity of objects observable simultaneously in radial velocity (RV) and in direct imaging contribute both intrinsic detection biases – direct imaging favours planets further out from their star, resulting in a small RV signal – as well as selection biases – target stars are usually chosen based on the presence of a disc, which implies that the systems are preferentially seen face-on. A further hindrance is that young stars – young systems being of greater interest because of a smaller brightness contrast – are usually active and thus less amenable to radial-velocity measurements. See also Lagrange et al. 2013, for a discussion of radial-velocity searches around young nearby stars and example prospects of coupling to direct imaging.

Constraining initial entropies of exoplanets

Until recently (Bonnefoy et al. 2013a; Currie et al. 2013a), the only bolometric luminosity estimate for \(β\) Pic b was due to Neuhäusler & Schmidt (2012), who report \(\log L_{\text{bol}} / L_\odot = -3.90^{+0.07}_{-0.08}\). They firstly derived, from the data T\(_{\text{eff}} = 1700 ± 300\) K (Bonnefoy et al. 2013a), a spectral type SpT = L–T4 using the T\(_{\text{eff}}\)–SpT relation of (Golimowski et al. 2004, hereafter G04). They then estimated from their SpT–BC\(_K\) curve a bolometric correction BC\(_K\) = 3.3 ± 0.3 mJy (T. Schmidt 2012, priv. comm.). Thus, the large, asymmetric lower errorbar on the bolometric luminosity comes from the large, asymmetric lower errorbar on the bolometric correction, which itself is due to the flat T\(_{\text{eff}}\)–SpT relation between L7 and T4 in G04. A more direct approach to the bolometric luminosity consists of converting the colour to a spectral type and obtaining from this a bolometric correction. Also using the fit\(_K\) of G04, this gives the same luminosity as found by Neuhäusler & Schmidt (2012) but with a smaller lower errorbar of 0.12 dex. One should however note that BC\(_K\) (SpT) is not a monotonic function (see fig. 6a of G04), so that the errorbars are strongly non-Gaussian. With a maximum BC\(_K\) of 3.3 mJy near L3.5, BC\(_K\) cannot formally be above ±3.9 dex. However, the BC\(_K\)–SpT relation of G04 was derived for field dwarfs (see also Stephens et al. 2009), and the spectral classification of young objects is not yet well understood, nor in fact, well defined (see e.g. Liu et al. 2011; Faherty et al. 2012).

Keeping in mind these uncertainties in interpreting the photometry, we shall use for the analysis \(\log L_{\text{bol}} / L_\odot = -3.90^{+0.07}_{-0.08}\), where the upper errorbar reflects the residuals of the G04 fit. This is at some variance with the value of \(\log L_{\text{bol}} / L_\odot = -3.80 ± 0.02\) of Currie et al. (2013a) but compares favourably with \(\log L_{\text{bol}} / L_\odot = -3.87 ± 0.08\) from Bonnefoy et al. (2013a).

21 It was brought to our attention that there is an extension of G04 by Liu et al. (2010), who use updated spectral types and removed binary systems from the sample. However, differences in BC\(_K\) only begin appearing later than \(L_1\), whereas the SpT we consider for \(β\) Pic b is \(L_1–T0\) (Bonnefoy et al. 2011), cf. the constraints of L0–L4 by Bonnefoy et al. (2013a) or L2–L5 by Currie et al. (2013a). According to the fit of Liu et al. (2010), BC\(_K\) = 3.05 mJy, while G04 gives 2.99 mJy; the difference (0.06 mJy) is less than the root-mean-square fit residuals (0.08 mJy and 0.13 mJy, respectively).