Fuzzy Membership Function Evaluation by Non-Linear Regression: An Algorithmic Approach

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\textbf{ABSTRACT}
In most researches on fuzzy sets and its application, it is found that the consideration of membership function is predetermined and mostly linear in nature. Extraction and evaluation of non-linear fuzzy membership function that can update itself within different paradigms is still a matter of great concern to researchers. Here, we discuss 33 different membership function evaluation methodologies published between 1971 and 2016. In an approach to solve the problem, this paper presents a novel algorithm based non-linear fuzzy membership function evaluation scheme with the help of regression analysis and algebra. Three different case studies are done to check the applicability and tractability of the method. A comparative analysis with recent literature justifies the robustness of the proposed method.

\textbf{1. Introduction}

A fuzzy set is a collection of elements with a continuum of grades of membership. It is characterized by membership function, which assigns to each object a grade of membership in the real interval $[0, 1]$. The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework used in ordinary sets, and, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification, information processing and statistical process control.

Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership. For most control-oriented problems, it is assumed that the membership functions are linear and usually triangular. However, for other problems, these triangular membership functions are not appropriate, as they do not represent accurately the linguistic terms, which are being modeled. Therefore, it will have to elicit directly from the expert.

The number of possible membership functions should not be arbitrary. As the number of membership functions increases, the precision of the system may increase too; but its relevance will decrease. In the limit, when the number of membership functions approaches the number of data, a fuzzy system becomes a numeric system.
In science and technology, many problems related to reality are of great importance. To clear the vagueness in these problems, the membership grades are considered from different views (likelihood theory, random set theory, similarity theory, utility theory, measurement theory) by different authors [1]. Among these, a lot of work has been done in measurement theory. There are two important measurement problems in fuzzy set theory. The first kind deals with measuring the degree of membership of several subjects or objects in a single fuzzy set. Yager [2], Norwich and Turksen [3], Turksen [4], Bollman-Sdorra [5], Wong and Yao [6], Bilgic [7] and others have studied it. The second problem is to consider many fuzzy terms for a single agent. The justification of using different operators as connectives for fuzzy sets are considered by many authors like Bellman and Giertz [8], Kamp [9], Fung and Fu [10], Dubosis and Prade [11], Yager [12], Weber [13], Alsina [14], Bilgic and Turksen [1] and others. If we look back, we see that Zadeh [15] rightly enunciated the concept of ‘Principle of Incompatibility’. The conventional quantitative techniques are unsuitable, because as the complexity increases uncertainty increases until a threshold is reached beyond which precision and significant relevance becomes mutually exclusive characteristics. It has been established that Fuzzy Mathematics provides a very powerful tool for uncertainty management in science and society. However, membership evaluation continues to be an open problem. Numerous authors have studied the problem of assessment of membership function in different times. Many methods have been proposed and criticisms have been done, but reality is that this dilemma is not fully solved yet. However, the basic philosophical problem concerns the construction of a numerical scale for the membership values in such a way that the scale satisfies some conditions imposed on a rational measurement system. The main difficulty lies in constructing a homomorphism from a qualitative preference system to quantitative preference system and it is here that the ambiguity and idiosyncrasies are likely to be introduced. An approach free from this defeat has not yet been found, although modifications have been suggested to reduce it. In this relation, Kandel and Byatt [16] suggested to use statistical data whenever applicable.

In the literature, two kinds of measurement of membership are available. The first kind of elicitation, where we look for the degree of belongingness of a single agent (say $x$) to a fuzzy set (say $\tilde{A}$).

Since there lays the philosophical difference of interpreting the meaning of membership degrees, the elicitation process prescribed by different authors differ in conceptual trends. To be very explicit about the meaning of membership degrees, let us consider a simple example. Let $\tilde{A}$ be the fuzzy set ‘old’ and $x$ be any person. Also, let $\mu_{\tilde{A}}(x)$ be the degree of membership of $x$. Now if $\mu_{\tilde{A}}(x) = 0.35$, then it can be viewed from angles:

1. $35\%$ of the population of consideration declared that $x$ is old.
2. $35\%$ of the population of consideration thought ‘old’ as an interval containing the age of $x$.
3. If there exists a prototype which is completely ‘old’, then the age of $x$ is away from it to the degree $0.65 (= 1 - 0.35)$
4. On comparison with others, $x$ is older than some and on a scale representation, it can be symbolized as $0.35$. 

\[ \mu_{\tilde{A}}(x) = 0.35 \]
Therefore, these different views certainly produce different ways of elicitation techniques. On the context of the same example, some of them are briefly described.

1) Gathering opinion: In this method, opinions are sought from a given population about the belongingness of \( x \) in \( \tilde{A} \). In the said example the question will be ‘Is \( x \) old?’ The answer will be either ‘yes’ or ‘no’. The gathered answers are averaged to construct the membership grades.

2) Classifying the object: in this method, a given population is asked to classify the object \( x \) according to the fuzzy set \( \tilde{A} \). In the said example the question will be ‘How old is \( x \)?’ Many variations in this process have been made in the literature.

3) In the same category, in a reverse algorithm, the degree of membership is given to the population and they are asked to list the objects that are of that degree of their choice. In the said example, the question will be ‘who are old to the degree 0.35?’

4) Collecting Interval valued Sets: In this method, each member of the given population is asked to submit an interval (which must be a subset of \([0, 1]\)) to which the member thinks that the degree of membership \( \mu_{\tilde{A}}(x) \) lies. In the said example, the question will be, ‘which is the interval of membership grades of \( x \) in \( \tilde{A} \)?’

5) Comparing Pair wise: in this method, the objects are compared pair wise to construct the membership grades. In this example, the question may be, ‘which one is of greater membership, \( x \) or \( y \) in \( \tilde{A} \)?’

6) Exemplification: It is a direct method of obtaining membership degrees from the population. The question for the concerned example will be, ‘what is the degree of belonging of \( x \) in \( \tilde{A} \)?’

7) Neuro Fuzzy Techniques: Modern advancement in Neural Network models and machine learning methods has helped the study of constructing membership functions. Based on neural networks, many elicitation methods have been proposed.

In the membership function generation procedures, by analyzing the statistical/historical data, the experts prescribe to use different pre-defined forms (as if triangular, trapezoidal, bell shaped, S-shaped etc.) of membership functions to fit the data. As in most of the cases, the real data does not follow these forms strictly; it results in huge amount of information losses. So extraction of the actual form of membership function is still needed. To deal with this problem, in this paper, we attempt to develop a mathematically rigorous method based on non-linear regression analysis. Whenever statistical data is available, it constructs the membership function in the form of a polynomial within a bounded interval, the universe. As non-linear regression analysis is applied to generate the membership function, it replicates the actual path of the membership function with lesser amount of error.

In this paper, the proposed methodology and its significance are discussed in section 3. Prior to that, a review of the present status of membership evaluation methods is presented in section 2. Some case studies are used to illustrate the algorithm developed in section 4. In section 5, a comparative study is given. Finally, in section 6, some concluding remarks are specified.
2. Evaluations of Fuzzy Membership Function: A Brief Review

Numerous authors have studied the problem of assessment of membership functions in different times. However, the reality is that this dilemma is not fully solved yet. In this section, we discuss some important methods and their detailed/ brief descriptions.

(1) None other than Zadeh [15] does the first approach. This is a simple method with $U$ as a universe of objects and $\tilde{A}$ as a fuzzy set on $U$. The membership grades $\mu_{\tilde{A}}$ are estimated from partial information about $\tilde{A}$. The definition of $\tilde{A}$ by exemplification is an extension of the familiar linguistic notion of extensive definition. A discrete representation of the membership function is obtained by repeating the query for several heights. Suppose that we are trying to collect data by taking interview of persons. A person can answer in different linguistic ways that are finite in number. Let $(n + 1)$ be that number. The simplest method is then to translate these linguistic levels into numeric: $0, 1/n, 2/n, \ldots, n-1/n, 1$. The order will depend on the nature of $\tilde{A}$. This method suffers from proper solution of the problem of abstraction that plays a central role in pattern recognition.

(2) An automatic generation of membership functions has been accommodated by using the essential characteristic of inductive reasoning, which derives a consensus from the particular (derives the generic from the specific). The induction is performed by the entropy minimization principle, which clusters most optimally the parameters corresponding to the output classes [17]. This method is based on an ideal scheme that describes the input and output relationships for a well-established database, i.e. the method generate membership functions based solely on the data provided. The method can be quite useful for complex systems where the data are abundant and static. In situations where the data are dynamic, the method may not be useful, since the membership functions will continually change with time.

(3) Saaty [18] proposes his relative preference method in which he espouses a derived scale. He proceeds with assuming a finite set $A$ with $n$ elements $A_1, A_2, \ldots, A_n$, called the alternatives and a set of criteria $C$. He defines a binary relation on $A$ denoted by $\succ_C$ and read as ‘more preferred than with respect to criteria $C$’. Then he uses ‘values’ as a matrix $P_C$ of rational numbers taken from the set $\{1/9, 1/8, 1/7, \ldots, 1/2, 1, 2, \ldots, 9\}$, each level having a semantic interpretation. Each entry $t_{ij} = P_C(A_i, A_j)$ represents a pair wise judgment, i.e. the number that estimates the relative membership of $A_i$ when it is compared with $A_j$, the larger $t_{ij}$, the greater the membership of $x_i$ compared with that of $x_j$. Obviously $t_{ij} = 1/t_{ji}$ and $t_{ii} = 1$.

The membership values $w_i$ can be determined by finding Eigen vectors of $P_C$, which is assumed to be as consistent as possible. This method is limited in application because of Satty’s assumptions on fundamental scale as the inconsistency of $P_C$.

(4) Fung and Fu [10] proposes a method where for a fuzzy set $\tilde{A}$ in $U$ with membership $\mu_{\tilde{A}}$, another fuzzy set $\tilde{A}$ on $P(U)$ is induced, provided that $U$ is finite. Then, $\mu_{\tilde{A}}(\{x_1, x_2, \ldots, x_n \}) = (1/k) \sum_{i=1}^{k} \mu_{\tilde{A}}(x_i)$. This definition has the intuitive meaning of an ‘average membership’ of $\{x_1, x_2, \ldots, x_n\}$ in $\tilde{A}$. A preference relation, denoted by $\succeq$ is defined in $P(U)$ by for all $S_1, S_2 \in P(U)$, $S_1 \succeq S_2$ if and only if $\mu_{\tilde{A}}(S_1) \geq \mu_{\tilde{A}}(S_2)$. These inequalities determine more or less strongly the $\mu_{\tilde{A}}(x_i)$. 


(5) Bremermann [19] proposes a simple approach by giving the concept of deformable prototypes. Suppose we are given an object with prototype P. Also, suppose that P can be deformed by manipulating parameters \( p_1, p_2, \ldots, p_n \). Bremermann suggests that there is a combination of the matching of the object to the prototype and the distortion required to deform the prototype. Therefore, for identifying and deforming that object to match to P, distortions function and a matching function is required. The matching function is 

\[
\langle M(\Phi_i(p_1, p_2, \ldots, p_n)), \varphi \rangle
\]

where \( \varphi \) is the object to be identified, and \( \Phi_i \) is the prototype and the distortion function is \( D(\Phi_i(p_1, p_2, \ldots, p_n)) \). Then the cybernetic functional for the \( i \)th prototype can be defined by

\[
\langle (f_i, \varphi) \rangle = \min_{p_1, p_2, \ldots, p_n} < F_i(p_1, p_2, \ldots, p_n), \varphi >
\]

where \( < F_i(p_1, p_2, \ldots, p_n), \varphi > = M(\Phi_i(p_1, p_2, \ldots, p_n)) + cD(\Phi_i(p_1, p_2, \ldots, p_n)) \) and \( c \) is a constant.

The cybernetic functional is then used to generate a membership function \( \mu_P(x) = 1 - \langle f, \varphi > / \text{Max} \rangle \), where ‘Max’ is the least upper bound for \( f \).

Bremermann’s approach is successfully implemented on pattern recognition and in particular for fuzzy sets in ECG interpretation. However, the work still raises questions about arriving at a suitable prototype and choosing the various functions.

(6) Kochen and Badre [20] give another very interesting method. Assuming the membership function to be continuous and differentiable as an S shaped curve, they construct the differential equation 

\[
\left( \frac{d \mu_A}{dx} \right) = k \cdot \mu_A(x)[1 - \mu_A(x)]
\]

in which the marginal increase of a person’s strength of belief that ‘x is \( \tilde{A} \)’ (\( \tilde{A} \) being any adjective) is assumed proportional to the strength of his belief that ‘x is not \( \tilde{A} \)’. The solution of the differential equation is 

\[
\mu_A(x) = \frac{1}{1 + e^{(a-bx)}},\ a\ and\ b \ are\ parameters\ to\ be\ estimated\ from\ statistical\ data.
\]

Now this method is only made for S shaped membership curves only. The two assumptions they have taken, has made it more week. The quantitative estimation procedure of determining membership function generally is somehow missing.

(7) Besides these types of set-theoretic approaches, membership evaluation can be done by statistical polls. The implicit assumption is that the probabilities of a positive answer of the question ‘Does x belong to \( A \)?’ from a questioned person are proportional to \( \mu_A(x) \). This method is done by Hersh and Caramazza [21].

(8) MacVicar-Whelan [22] introduces filter functions in order to identify the membership functions of fuzzy sets modeling adjectives. A filter function \( F \) is characterized by two parameters, the location \( NP \) of the neutral point \( (F(NP) = \frac{1}{2}) \) and the width \( 2w \) of the transition between non membership and membership. More specifically

\[
F(x; NP, w) = \begin{cases} 
0, x \in (-\infty, a-w) \\
\frac{1}{2w}(x - NP + w), x \in [a - w, a + w] \\
1, x \in (a + w, \infty).
\end{cases}
\]

In a method MacVicar-Whelan points out that a sophistication of the shape of the transition is useless because of the imprecision. For more convenience, let \( x \) and \( \sigma \) be the parameters of the distribution ‘tallness’. Then person’s large height is modeled by the membership
function $\mu$ such that $\mu(x) = F(x; x + \alpha \sigma, \beta \sigma)$, $x$ being the height, where $\alpha$ and $\beta$ are to be determined from experimental data. 'Small height' is experienced in the same way by $1 - F(x; x + \alpha \sigma, \beta \sigma)$.

(9) According to Diskhant [23], in any calculation regarding membership function, if the final result depends on all the premises, then they must be connected with the strong conjunction ‘o’. However, if it depends only on one of the premises, then we must connect them with rigid conjunction $\land$. Formally in many valued logic $\mu_1 \rightarrow (\mu_2 \rightarrow \mu_1)$ is equivalent to $\mu_1 \circ \mu_2 \rightarrow \mu$, and $(\mu_1 \rightarrow \mu) \cup (\mu_2 \rightarrow \mu)$ is equivalent to $\mu_1 \land \mu_2 \rightarrow \mu$. Diskhant also gives a limit theorem, which consists of highly technical conditions. Roughly, they connote the membership functions of items decreasing from 1 at a maximum point to 0 at the ends of bearer uniformly, neither too rapidly nor too slowly. Practically this theorem determines, at the best, only a type of a membership function. There is a problem of two unknown parameters. In that case, we must pass to the problem of estimation of parameters. The main peculiarity of his consideration is the interpretation of unknown parameters as fuzzy variables.

(10) Earlier we have discussed about the preference relation. Let $\geq_F$ be such a binary relation on $\tilde{A}$ with the interpretation: $a \geq_F b \Leftrightarrow a$ is at least as $\tilde{F}$. Norwich and Turksen [24] consider this relation to present their method. The structure is assumed to be bounded. Then there are elements for which $\tilde{F}$ is completely true or false and it is clear that an ordinal scale representation exists for $< \tilde{A}, \geq_F >$ once the transitivity and connectedness of $\geq_F$ are accepted. The ordinal scale for measuring fuzziness allows only comparison of membership values and no other numerical manipulation. So there exists a function $\mu_F: \tilde{A} \rightarrow [0, 1]$ such that $a \geq_F b \Leftrightarrow \mu_F(a) \geq_F \mu_F(b)$.

(11) Making a little change in the concept in Nowakowska [25], Alsina [14] gives a measurement tool in which she assumed that the probability of a positive answer of that question is an increasing function of the value $\mu_{\tilde{F}}(x)$.

(12) Civanlar and Trussel [26] present a way to construct the membership functions for fuzzy sets whose elements have a defining feature with a known probability density function in the universe of discourse. The method finds the smallest fuzzy set which assigns high average membership values to those objects with the defining features distributed according to the given probability density function. It shows that for any probability density function, the method is capable to generate membership functions in accordance with the possibility – probability consistency principle.

(13) Takagi and Hayashi [27] propose a method by which fuzzy membership functions may be created for fuzzy classes of an input data set. First, a number of input data are selected and then divided into a training data set and a checking data set. The training data set is used to train the neural network. Once the neural network is ready, its final version can be used to determine the membership values (function) of any input data in the different regions.

(14) Kim and Russell [28] have portioned the set of data into classes based on minimization of the entropy $S$, the expected value of the information contained in the data set and $S$ is given by $S = k \sum_{i=1}^{N} [P_i \ln(P_i) + (1 - P_i) \ln(1 - P_i)]$, where the probability of the $i^{th}$ sample to be true is $P_i$ and $N$ is the number of samples. This entropy then leaves with points in the region that are used to determine triangular membership function.
Genetic algorithms are used to compute membership functions [29] in many cases. Given some functional mapping for a system, some membership functions and their shapes are assumed for the various fuzzy variables defined for a problem. These membership functions are then coded as bit strings that are then concatenated. An evaluation (fitness) function is used to evaluate the fitness of each set of membership functions (parameters that define the functional mapping).

Chen and Otto [30] have developed a constrained interpolation scheme for constructing a membership function. As we all know, membership functions are usually convex and bounded in \([0, 1]\). To preserve these constraints an advanced interpolation scheme using Bernstein polynomial is implemented by them. They assume that the slope (that requires some conditions to be constructed) of the end points of support of the membership functions ramps up slowly and the membership function is smooth about any peak points, i.e. \(\mu'(x) = 0\) for \(\mu(x) = 0\) or 1.

In Singh and Bailey’s approach [31], the authors have partially improved an evaluation procedure. In this method, the possibility probability consistency principle is applied to generate the optimal Membership Functions. The Membership Functions are generated for tracking data based on their probability density functions. The experimental results are compared according to the confidence levels of the probability density functions of the Membership Functions.

Royo and Verdegay [32] proposed another approach, which deals with those Membership Functions, whose measurement is somehow not obvious, but is an object to study itself. At first, the authors worked on the description of the possible cases where a Membership Function’s existence is not certain. Later, a new technique for the measurement of Linguistic Terms is proposed. The methodology is based on Norwich and Turksen’s Representation theorem and Uniqueness theorem, which established the connection between the latter structures and the Membership Functions.

Torra [33] revisits the last said method. Some errors were found; e.g. the assumption of null derivatives at the extreme points is not satisfied by their method. In addition, the conditions for constructing the slopes are not exhaustive.

Yam and Koczy [34] introduce a new approach for fuzzy interpolation and extrapolation of sparse rule base comprising of membership functions with finite number of characteristic points. The approach calls for representing membership functions as points in high-dimensional Cartesian spaces using the locations of their characteristic points as coordinates. Analysis of well-defined membership functions can be readily incorporated by it. Furthermore, the Cartesian representation enables separation between membership functions to be quantitatively measured by the Euclidean distance between their representing points, thereby allowing the interpolation and extrapolation problems to be treated using various scaling equations.

Wong et al. [35] generalize the geometric representation for membership function comprising of finite number of characteristic points. An extended class of membership functions satisfying certain monotonicity conditions is expressed as elements in the space of square, integrable functions. Specifically, bell-shaped membership functions, which are not possible before, are accommodated with this generalized representation. An example of interpolation problem treated using both the finite dimensional approach and generalized approach is included for illustration by them.
Zhao and Bose [36] proposed the evaluation of twelve regular shaped Membership Functions for fuzzy logic controlled inductive motor drive. Mainly these Membership Functions are triangular, trapezoidal, Gaussian, Bell and polynomial types. At first, the fuzzy controller sensitivity is analyzed and compared. The triangular membership function works here as the base. A load torque disturbance is considered to study the robustness of the drive system. The responses of the fuzzy controller are compared and examined.

Bağiş [37] proposes a new approach for the optimum determination of membership functions for a fuzzy logic controller based on the use of Tabu search algorithm. The simulation results show that the approach could be employed as a simple and effective optimization method for achieving optimal determination of membership functions.

Tan et al. [38] propose a dynamic input membership function scheme which combines the center width narrow and the center width constant scheme together with a new scheme where the center triangular input membership functions has a wider width than those that are located more towards the two ends of the input universal set (center width wide or CWW). The dynamic input membership scheme (DIMS) applies CWN, CWC and CWW at different stages of the transient states of the controller step response to achieve a superior step response for a DC motor controller as compared to applications of only one type of input membership functions throughout the controller operation.

In their method, Dombi and Gera [39] actually approximates the linear Membership Functions. In many applications, triangular and trapezoidal Membership Functions are employed for their linearity and computational simplicity. However, the nature of too much linearity sometimes causes errors. This work approximates piecewise linear functions with the help of sigmoid functions and some arithmetic operations. The obtained Membership Function magnifies the applicability of fuzzy methods to the operators and Memberships where the differentiability is worthy. By this methodology, the piecewise linear membership functions can be tuned by gradient based optimization in a fuzzy control system to obtain more effective outcomes. The cut function, basis of the Lukasiewicz operator class, is also used in the algorithm.

Wonget al. [40] generalize this approach for interpolating fuzzy rules comprised of membership functions with finite number of characteristic points. Instead of representing membership functions as points in Cartesian spaces, they now become elements in the space of square, integrable functions. Interpolation is thus conducted between the antecedent and consequent function spaces. The generalized representation allows an extended class of membership functions satisfying two monotonicity conditions to be accommodated in the interpolation process. The work also extends the similarity triangle-based interpolation technique from the previous Cartesian representation to the new representation.

Dutta Majumder et al. [41] propose a way of extraction of membership function. The approach is from numerical point of view with the help of interpolating statistical data. There are two methods, namely (i) Modified Newton’s divided difference Method and (ii) Modified Lagrange’s interpolation Method.

Chen et al. [42] propose a genetic algorithm (GA) based framework for finding membership functions suitable for fuzzy mining problems. Each individual represents a
possible set of membership functions for the items and is divided into two parts, control genes and parametric genes. Control genes are encoded into binary strings and used to determine whether membership functions are active or not. Each set of membership functions for an item is encoded as parametric genes with real-number schema. Seven fitness functions are proposed, each of which is used to evaluate the goodness of the obtained membership functions and used as the evolutionary criteria in GA.

Different studies have proposed methods for mining fuzzy association rules from quantitative data, where the membership functions are assumed to be known in advance though it is not a simple task to recognize a priori the most suitable fuzzy sets that cover the domains of quantitative attributes for mining fuzzy association rules. Alcalá-Fdez et al. [43] present a new fuzzy data-mining algorithm for extracting both fuzzy association rules and membership functions by means of a genetic learning of the membership functions. It is based on the 2-tuples linguistic representation model allowing us to adjust the context associated to the linguistic term membership functions.

Maniu [44] proposed yet another problem based method. The problem of multidimensional poverty is considered here. Only the monetary dimension of poverty is discussed and a new approach is prescribed to estimate it. The extension of the construction of the Membership Functions of fuzzy monetary measure is exhibited by considering the set of all sub populations of the main population. The mapping obtained is called the degree of poverty and is a monotone measure with respect to a specified set inclusion pre-order. The approach is established by a case study from Romania.

Zhu et al. [45] present a construction method of Membership Functions based on descriptive knowledge. It is specifically related to the predictive soil mapping under fuzzy logic. A set of Membership Functions are at first constructed by regression technique to represent the descriptive knowledge on soil-landscape relationships. This method is exemplified in a watershed located in Heshan farm of Nenjiang country. Comparison between results shows that the soil organic content map based on the fuzzy Membership Functions is better than the soil map based on the linear regression model.

Ouyang et al. [46] represents a methodology that is based on six regular shaped Membership Functions (Rectangular Distribution Function  / Gamma Distribution function, Normal Distribution Function, Cauchy Distribution Function, Trapezoidal Membership Function and Ridge Shaped distribution Function). A new Membership Function is constructed through weighting and combining these six Membership Functions by means of unbiased variance, which improves the stability and the accuracy of the absolute outcome. The formulated Membership Function is used in PQ (Power Quality) comprehensive evaluation. The experimental result is compared with the six distribution functions individually. At first, the six functions are separately applied to calculate the result of the membership. Then, a prescribed algorithm evaluates the weights of the Membership Functions. Then, the NFMF (Normalized Fuzzy Membership Function) is constructed by an aggregation technique, proposed in the methodology.
Sebastian and Ewa [47] present a study on data-driven diagnostic rules, which are easy to interpret by human experts. To this end, the Dempster-Shafer theory extended for fuzzy focal elements is used. Fuzzy focal elements are provided by membership functions having variable shapes. The aim of the study is to evaluate common membership function shapes and to introduce a rule elimination algorithm.

3. Proposed Algorithmic Approach Using Non-Linear Regression

Let us consider that the fuzzy linguistic term $\tilde{A}$ has a correspondence to the universe of real numbers. Let $E$ be the random experiment of selecting a source at random and collecting data from that source about this correspondence. For example, say $\tilde{A}$ is 'young men'. Then the outputs of $E$ can be obtained as interval of ages (in years) like (14, 30), (15, 36), (18, 32), (15, 40), (19, 35) etc.

Let $E$ be repeated $m$ times and let $a$ and $b$ be respectively the minimum and the maximum of the outputs obtained. Then $[a, b]$ can be considered as the universe of discourse of the fuzzy set $\tilde{A}$.

We consider a univariate existence population $S = [a, b]$ and let it be determined by the values of the random variable $X$. Let us chose and fix the $(n + 1)$ points $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ such that $(a =) x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n ( = b)$. Now we consider the $(n + 1)$ events $(X = x_0), (X = x_1), (X = x_2), \ldots, (X = x_n)$.

Let the number of occurrence of the event $(X = x_i)$ in the $m$ experiments be $m_i$. Let $y_i = m_i / m$ be the frequency ratio of the occurrence of the event $(X = x_i)$.

Thus for the points $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ we get $y_0, y_1, y_2, \ldots, y_{n-1}, y_n$ respectively where each $y_i \in [0, 1]$. Let $\mu(x)$ be the required membership function. Then $\mu(x_0) = y_0, \mu(x_1) = y_1, \mu(x_2) = y_2, \ldots, \mu(x_n) = y_n$, which is shown in Table 1.

Table 1. Membership values.

| $X$   | $x_0$ | $x_1$ | $x_2$ | $\ldots$ | $x_n$ |
|-------|-------|-------|-------|----------|-------|
| $Y = \mu(x)$ | $y_0$ | $y_1$ | $y_2$ | $\ldots$ | $y_n$ |

We consider the distribution of the 2-dimensional random variate $(X, Y)$ which is either discrete or continuous. To find the equation of the $k$th degree regression polynomial of $Y$ on $X$, we have to find the values of $c_0, c_1, c_2, \ldots, c_k$ for which $E[(Y - (c_0 + c_1X_1 + c_2X_2 + \ldots + c_kX_k))^2]$ is minimum. Let

\[ L = E[(Y - (c_0 + c_1X_1 + c_2X_2 + \ldots + c_kX_k))^2], \]

where we assume that $E(Y), E(Y^2), E(X_r), r = 1, 2, 3, \ldots, k$ and $E(X_rY), r = 1, 2, 3, \ldots, k$ exist. Then $L$ can be regarded as a function of $(k + 1)$ real variables $c_0, c_1, c_2, \ldots, c_k$ where $L$ and its partial derivatives of any order are continuous.

Now a necessary condition for $L$ to be minimum is that

\[ \frac{\partial L}{\partial c_0} = \frac{\partial L}{\partial c_1} = \ldots = \frac{\partial L}{\partial c_k} = 0 \]

which are the normal equations.
If \( c_0 = C_0, c_1 = C_1, c_2 = C_2, \ldots, c_k = C_k \) is the unique solution of the normal equations, then for \( i = 0, 1, 2, \ldots, k \), we have

\[
\begin{bmatrix}
\frac{\partial L}{\partial c_0} \\
\frac{\partial L}{\partial c_1} \\
\vdots \\
\frac{\partial L}{\partial c_k}
\end{bmatrix}(C_0, C_1, \ldots, C_k) = 0. \tag{3.3}
\]

i.e. \( E(X^iY) - C_0 E(X^i) - C_1 E(X^{i+1}) - C_2 E(X^{i+2}) - \ldots - C_k E(X^{i+k}) = 0 \).

i.e. \( \alpha_{i+1} - C_0 \alpha_{i,0} - C_1 \alpha_{i+1,0} - C_2 \alpha_{i+2,0} - \ldots - C_k \alpha_{i+k,0} = 0, \alpha_{i,j} = E(X^iY^j), j = 0, 1. \)

Thus we have the following system of normal equations:

\[
\begin{align*}
C_0 \alpha_{0,0} + C_1 \alpha_{1,0} + C_2 \alpha_{2,0} + \ldots & + C_k \alpha_{k,0} = \alpha_{0,1} \\
C_0 \alpha_{1,0} + C_1 \alpha_{2,0} + C_2 \alpha_{3,0} + \ldots & + C_k \alpha_{k+1,0} = \alpha_{1,1} \\
C_0 \alpha_{2,0} + C_1 \alpha_{3,0} + C_2 \alpha_{4,0} + \ldots & + C_k \alpha_{k+2,0} = \alpha_{2,1} \\
\vdots & \vdots \\
C_0 \alpha_{k,0} + C_1 \alpha_{k+1,0} + C_2 \alpha_{k+2,0} + \ldots & + C_k \alpha_{2k,0} = \alpha_{k,1}
\end{align*}
\tag{3.4}
\]

from which the values of \( C_0, C_1, C_2, \ldots, C_k \) can be obtained.

Then the equation of the \( k^{th} \) degree regression parabola of \( Y \) on \( X \) is

\[
y = C_0 + C_1 x + C_2 x^2 + \ldots + C_k x^k, \tag{3.5}
\]

which is the best fitting parabola of degree \( k \), to the distribution of \( (X, Y) \) from the family of parabolas given by

\[
y = c_0 + c_1 x + c_2 x^2 + \ldots + c_k x^k, \tag{3.6}
\]

where \( c_0, c_1, c_2, \ldots, c_k \) are parameters.

Hence the required membership function is obtained as

\[
\mu(x) = C_0 + C_1 x + C_2 x^2 + \ldots + C_k x^k. \tag{3.7}
\]

**Theorem 1:** If the membership function of degree \( n \) is normal, it will have at most \((n+1)/2\) points in its kernel.

**Proof:**

**Case 1.** \( \mu(x) \) is monotonic decreasing.

Then it is obvious that \( \mu(x) = 1 \) at \( x = x_0 \) and it follows that \( \ker(\mu(x)) = \{x_0\} \).

**Case 2.** \( \mu(x) \) is monotonic increasing.

Then it is obvious that \( \mu(x) = 1 \) at \( x = x_n \) and it follows that \( \ker(\mu(x)) = \{x_n\} \).

**Case 3.** \( \mu(x) \) is piecewise monotonic.

The possibility of a point to have the membership degree 1 (i.e. having maximum membership degree) comes with the first order derivative zero at that point. Since the membership function \( \mu(x) \) is a polynomial of degree \( n \), we must have its first order derivative as an \((n-1)^{th}\) degree polynomial and \( \frac{d\mu(x)}{dx} = 0 \) can give a maximum number of \((n-1)\) real roots in its domain. Again since the curve is continuous and piecewise monotonic, the critical points can give the value 1 at a maximum number of \( \frac{(n-1)}{2} + 1 = \frac{(n+1)}{2} \) points.
Therefore, the kernel of the membership function can have at most \((n+1)/2\) points.

**Theorem 2:** If \(\mu(x)\) is 0 or 1 at any point, \(\mu'(x) = 0\) at that point.

**Proof.** Let at the point \(x\), \(\mu(x)\) is either 0 or 1. Then, \(\mu(x)\) has either same value or there is a local maxima or minima for \(x\) at both sides of \(x\).

Now when \(\mu(x) = 0\), \(\mu(x)\) is differentiable (as it is a polynomial) and \(\mu'(x) = 0\) at any neighborhood of \(x\). Thus, \(\mu'(x) = 0\) at the point \(x\).

Again when \(\mu(x) = 1\), the membership function has a local maxima or minima at \(x\) and consequently \(\mu'(x) = 0\) at that point.

**Note 1:** The membership function \(\mu(x)\) may or may not be normal. Suppose we seek for the age intervals for which a person can be called ‘middle aged’. Then it is quite possible that the age of 40 comes for each interval. Then the point \(x = 40\) gets its maximum membership grade 1 while considered for constructing membership function of ‘middle aged’ persons. In this case, the membership function is normal. However, when a point is not common in each interval, we cannot get a normal membership function. For example, we refer examples 1 and 2 of the next section.

**Note 2:** According to Dombi [48], the membership functions should

(i) be continuous,
(ii) map an interval \([a, b]\) to \([0, 1]\)
(iii) be either monotonically increasing or monotonically decreasing or can be divided into monotonically increasing and decreasing parts,
(iv) maintain the linear form through its domain.

In our proposed method, all the above conditions of constructing membership functions are clearly satisfied except the last one where the linearization of the membership function is stated. Actually, it is quite unnatural that a membership function always behaves in a linear form. It depends on the nature of the fuzzy set. This prediction can lead to more errors and can mislead the way of formulation. Keeping this in mind the proposed method is more effective than the other attempts.

**Note 3:** The domain of the membership function obtained by this method is bounded by \(a\) and \(b\) by its definition. As \([a, b]\) is the universe of discourse, the points outside of \([a, b]\) are not taken into consideration.

The algorithm of the above method is as follows:

**Note 4:** In the above algorithm, in (44), the membership polynomial is set to be of 3rd order. It can be extended to any finite order. As the coefficients are becoming too small after \(C_3\), we have not included them.

### 4. Case Studies

The proposed method along with the algorithm can easily be applied to real life problems where statistical data are available. The perfection of this method certainly depends on the quality of the source of data as well as the cardinality of the data set. Like any other statistical rigorous methods, the result will be more accurate if the data set is large.
Read the table
(1) read the number of terms \( n \)
(2) for \( h = 1 \) to \( n \), do till (5)
(3) read \( x_h \)
(4) read \( M_h \)
(5) next \( h \)

Calculate the values of \( \alpha_{rs} \)
(6) for \( r = 0–6 \), do till (15)
(7) for \( s = 0–1 \), do till (14)
(8) \( S1 \leftarrow 0, a_{1rs} \leftarrow 0 \)
(9) for \( h = 1 \) to \( n \), do till (11)
(10) \( S1 \leftarrow S1 + x_h M_h^s \)
(11) next \( h \)
(12) \( a_{1rs} \leftarrow (1/k) * S1 \)
(13) write \( r, s \) and \( a_{1rs} \)
(14) next \( s \)
(15) next \( r \)

Read the augmented matrix \((a_{ij})^k \times (k+1)\)
(16) read the order of the matrix, \( k \)
(17) for \( i = 1 \) to \( k \), do till (24)
(18) for \( j = 1 \) to \( k+1 \), do till (23)
(19) if \( j \neq k+1 \), do step (20)
(20) \( a_{ij} \leftarrow a_{ij-2} - 0 \)
(21) If \( j = k+1 \), do step (22)
(22) \( a_{ij} \leftarrow a_{i-1,1} \)
(23) next \( j \)
(24) next \( i \)

Reduce into upper triangular
(25) for \( p = 0 \) to \( k-1 \), do till (32)
(26) for \( i = p+1 \) to \( k \), do till (31)
(27) ratio \( \leftarrow a_{ip} / a_{pp} \)
(28) for \( j = 1 \) to \( k+1 \), do till (30)
(29) \( a_{ij} \leftarrow a_{ij} - \text{ratio} * a_{pj} \)
(30) next \( j \)
(31) next \( i \)
(32) next \( p \)

Backward substitution
(33) \( C_k \leftarrow a_{kk} / a_{kk} \)
(34) for \( p = k-1 \) to \( 0 \), do till (40)
(35) \( C_p \leftarrow a_{pk} / a_{pp} \)
(36) for \( j = p+1 \) to \( k \), do till (38)
(37) \( C_p \leftarrow C_p - a_{pj} C_j \)
(38) next \( j \)
(39) \( C_p \leftarrow C_p / a_{pp} \)
(40) next \( p \)

Print answer
(41) for \( i = 1 \) to \( k \), do till (43)
(42) write \( i \) and \( C_i \)
(43) next \( i \)
(44) write the membership function \( M(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 \)
(45) end.

4.1. Case Study I

The authors have gone through a survey. We have collected data from 1000 different people of different age and social status. One thing is common among them. They all are regular
motorbike rider of a particular brand of motor cycle on National Highway 34, a busy road of West Bengal, India.

We ask them a simple question 'What interval of speed (in kilometer/hour) according to you is SAFE?'

We get the answer as different intervals; e.g. in [30, 50], in [25, 40], in [35, 50], in [40, 60], in [40, 70] etc.

Thus, we have a multi set of 1000 intervals. The minimum and the maximum speed we get there are 20 and 70 respectively.

Without losing generality, we choose the values 20, 30, 40, 50, 60 and 70 and find out the frequency of occurrences of these values among the elements of the multi-set we have. Then each output is divided by 1000, the total number of experiments, to obtain the fuzzy membership grades at speeds 20, 30, 40, 50, 60 and 70 kilometers per hour of the fuzzy set 'SAFE'. The final output we obtain is presented in Table 2.

Here the estimated membership function is obtained as:

$$
\mu(x) = 0.47142857143 + (0.00596825397)x + (0.00035000000)x^2 - (0.00000611111)x^3. \quad (4.1)
$$

The corresponding membership curve of (4.1) is drawn by MATLAB and is shown in Figure 1.

Clearly, it represents a \( \pi \)-shaped curve. It is continuous, convex and piecewise monotonic in [20, 70].

It definitely shows that

| x  | \( \mu(x) \) |
|----|---------------|
| 20 | 0.69          |
| 30 | 0.78          |
| 40 | 0.88          |
| 50 | 0.92          |
| 60 | 0.73          |
| 70 | 0.52          |

Table 2. Data set.

Figure 1. Membership function corresponding to (4.1).
(i) Speed between 40 and 50 kmph is most safe according to the riders.
(ii) In a highway, speeds 20 and 60 kmph are equally safe.
(iii) Speed more than 70 kmph is risky.

4.2. Case Study II

The methodology discussed is now applied to the measurement of the monetary dimension of poverty, a very significant socio-economic problem. The case study in this section is on the evaluation of the membership function for the poverty of monetary condition of a household. The measurement of poverty varies with respect to the location of the society. Here the real life example is based on the rural areas of India. The concept will be definitely different for the urban areas.

A group of 15 Experts \{E_1, E_2, \ldots, E_{15}\} is appointed to obtain the data for constructing the required membership function. Here the task is to evaluate the membership function of the fuzzy set ‘poverty of income per capita’ and let us denote this fuzzy set by \(\tilde{A}\). For this, at first the Experts are asked to submit the intervals of income per capita for a household, which can be considered as Poor condition of the household. The set \(\tilde{A}\) is defined in \([0, \infty]\) where the real numbers represent the income per capita of the household in US $. Now, according to the methodology, the Experts submit their decisions in terms of real intervals and those are displayed in Table 3.

The minimum and maximum of the outcomes are respectively 0 and 35. Now eight values are chosen from \([0, 35]\) and they are 0, 5, 10, 15, 20, 25, 30 and 35. The number of occurrence and the frequency ratios are displayed in Table 4. Although the methodology states that the universe of discourse should be restricted to \([0, 35]\), but keeping the concerned problem in mind, the universe of discourse is created as unbounded above. This is from the assumption that the poverty degree of the household whose monthly income per capita is greater than 35 US $ is 0 as it should not be considered as ‘Poor’ at all.

| Experts | Intervals |
|---------|-----------|
| E₁      | [0, 20]   |
| E₂      | [0, 18]   |
| E₃      | [0, 25]   |
| E₄      | [0, 22]   |
| E₅      | [0, 15]   |
| E₆      | [0, 10]   |
| E₇      | [0, 20]   |
| E₈      | [0, 22]   |
| E₉      | [0, 24]   |
| E₁₀     | [0, 17]   |
| E₁₁     | [0, 22]   |
| E₁₂     | [0, 30]   |
| E₁₃     | [0, 35]   |
| E₁₄     | [0, 27]   |
| E₁₅     | [0, 20]   |
Table 4. The number of occurrence and the frequency ratios.

| Specified values | Number of occurrence | Frequency ratio |
|------------------|----------------------|-----------------|
| 0                | 15                   | 1               |
| 5                | 15                   | 1               |
| 10               | 15                   | 1               |
| 15               | 14                   | 0.933           |
| 20               | 11                   | 0.733           |
| 25               | 4                    | 0.267           |
| 30               | 2                    | 0.133           |
| 35               | 1                    | 0.067           |

Figure 2. Membership function corresponding to (4.2).

The evaluated equation of the Membership Function is

\[(0.95356060606) + (0.04692222222)x - (0.00458948052)x^2 + (0.00007141414)x^3.\]  

(4.2)

The membership curve corresponding to (4.2) is shown in Figure 2.

This concept of evaluation definitely challenges the crisp methodology of determining people under poverty line, known popularly as BPL (Below Poverty Level) in India. The proposed algorithm executes the task in a much more logical way. In fact, identification of poor household in a region (rural or urban) is a multi attribute decision making problem, where the number and weights of the attributes are context dependent. One of the most important attributes is ‘income per capita’ of the individual household. Now the most important task is to map the income per capita of a household in the decision scale. The traditional method divides the ‘income per capita’ in some intervals and assigns crisp numeric value. The process is linear, easy to manipulate (at any level of the survey) and thus it is far away from reality. As a result of this, more often, it is observed that some people are struggling in poverty without being listed in BPL. The proposed algorithm may help to properly evaluate the households under the attribute ‘income per capita’. The process of decision determination will certainly be much more effective as the inherent uncertainty has been taken into consideration.

Now five households are selected from ‘Nutangram Gram Panchayet’ of West Bengal, India. Their house numbers are respectively HB/23, HB/49, HB/79, HB/135 and HB/511. For
past one year, their average monthly income per capita are respectively 18.75$, 20$, 23.55$, 16.20$ and 27.25$.

According to the present evaluation procedure, the scores (membership values) of poverty for these monthly incomes are respectively 5, 5, 3, 5 and 2 in the 10 point scale. Thus, the membership values are respectively 0.5, 0.5, 0.3, 0.5 and 0.2. However, according to the current approach, the membership values are respectively 0.690611, 0.627526, 0.445974, 0.812857 and 0.269268. Clearly, the evaluated outcome by the present approach leads to a better selection model for poor households. The problem of the identification of poor household is one of the important parts of the multi-dimensional poverty. Figure 3 clearly describes the differences of membership values between traditional crisp methodology and the proposed methodology based on fuzzy membership evaluation by non-linear regression.

4.3. Case Study III

We consider the data set used in the case study in Dutta Majumder et al. [41]. The authors collected some data to extract the membership function of the ‘risk factor’. The values $x$ corresponds to the speed of the motorbike in kmph and the values of $y$ are the average numeric of the risk assigned by the riders in [0, 1] Table 5.

The table is as follows:

Here the estimated membership function is obtained as

$$\mu(x) = 0.0380.0023030303(x) + 0.00021363636(x^2)0.00000106061(x^3).$$  (4.3)

The corresponding membership curve is drawn by MATLAB and is shown in Figure 4.
Table 5. Data set in Dutta Majumder et al. [41].

| x  | y = μ(x) |
|----|---------|
| 10 | 0.04    |
| 20 | 0.06    |
| 30 | 0.15    |
| 40 | 0.18    |
| 50 | 0.35    |
| 60 | 0.45    |
| 70 | 0.55    |
| 80 | 0.70    |
| 90 | 0.75    |
|100| 0.90    |

Clearly, it represents an S-shaped curve. In addition, it is monotonic and continuous in [10, 100]. If we compare the membership curve obtained in Figure 3 and the membership curve obtained in Dutta Majumder et al. [41], we see that the membership curve in Figure 3 is much smoother than the other one.

5. Comparative Study

In fuzzy mathematics, each set meets with a corresponding membership function. Some probabilistic distribution functions are also used to fit with the properties of membership functions as they range in [0, 1]. These functions are Rectangular Distribution Function, Γ Distribution functions, Normal Distribution Functions, Cauchy Distribution Functions and Ridge Shaped distribution Functions. The intermediate types of these membership functions are widely used.

In fuzzy mathematics, various types of membership functions are used to approximate or interpret the given conditions of the concerned problem. A brief discussion on these functions has already been provided in section 2. In this section, a comparative study is provided to illustrate the advantage of using the proposed algorithm. Theoretically, the approaches can be divided into two parts:
(1) Regular Shaped Membership Functions: For these membership functions the shapes are regular, i.e. smooth and predictable and the natures are known to at least some extent.

(2) Newly Constructed Membership Functions: These membership functions are neither predictable, nor can be expressed as the union of some regular shaped membership functions.

The advantage of using the second type membership functions is that they are best fitted with the given data. Therefore, the proposed algorithm is more effective when it is possible to achieve some significant data from the system. The more data is extracted, the more effective Membership Function is constructed.

In problem formulation, different types of application need different types of approaches, because the evaluation of the Membership Functions is context dependent. Hersh et al. [49] showed the effects of variable context in evaluating the Membership Functions. It is claimed there that the variable frequency of occurrence of the elements cannot influence the location and form of the Membership Functions. However, the unique number of elements does affect this.

Let us illustrate this by considering a simple example.

Consider the question: ‘Is 50 Km/hr is a risky speed?’ while determining the Membership Function of the risk factor of a motor bike in average crowd. Now, if the question is asked repeatedly to more than one person, the output membership is always of the same form. But if the question is asked to one more person, the output varies. That is, the context changes. Thus Bilgick and Turksen [1] claims that the Membership Function is not only a function of the object from the universe of discourse, but of the discourse as well. To be more specific, \( \mu_A(x) = f_A(x, X) \).

As a result of this, methodologies cannot be compared with each other with the numerical data sets or rule bases. In this part, the resulting Membership Function (in section (1)) is compared with the popular and effective state of the art Membership Functions. The comparison is demonstrated in Table 6.

Now some other drawbacks of these Membership Functions are pointed out.

Rectangular distribution is unable to directly describe the membership when the domain variable is out of the specified interval. The outcome is too much absolute as it only delivers 0 and 1. The parameters in \( \Gamma \) distribution, Normal distribution and Cauchy distributions are difficult to set or to be estimated. Triangular and Trapezoidal Membership

| Name                      | Whether applicable to the first type | Whether applicable to the second type | Whether having a specific shape |
|---------------------------|-------------------------------------|---------------------------------------|---------------------------------|
| Rectangular Distribution  | Yes (to some extent)                 | No                                    | Yes                             |
| \( \Gamma \) Distribution | Yes                                 | No                                    | Yes                             |
| Normal Distribution       | Yes                                 | No                                    | Yes                             |
| Cauchy Distribution       | Yes                                 | No                                    | Yes                             |
| Triangular Membership     | Yes                                 | No                                    | Yes                             |
| Trapezoidal Membership    | Yes                                 | No                                    | Yes                             |
| Gaussian Membership       | Yes                                 | No                                    | Yes                             |
| Ridge Shaped distribution | Yes                                 | No                                    | Yes                             |
| Proposed Membership       | Yes (to some extent)                 | Yes                                   | No                              |
Functions are limitedly defined in a certain interval. The nature of these Membership Functions is very much linear, whereas the problems in the real world are non-linear. The Gaussian Membership Function is non-linear in nature, but is unable to fit with any type of data set. The Ridge-shaped Membership Function is too much non-linear and thus the estimated computation error is very much high. Moreover, several parameters are to be decided for the evaluation procedure.

In accordance with the above discussion, it should be accepted that the arbitrary choice of the Membership Functions for a particular problem is not a satisfactory way to deal with. Sometimes, it is too much linear and sometimes, it is too much non-linear. Hence, the representation of the concerned fuzzy set does not generally reflect the real prototype, not even a closer of that. Therefore, the choice is upon the user. The user should select the right function according to the actual or real situation. In this respect, it can be claimed that the current approach is more effective than the existing ones.

In addition, for applicability, the proposed algorithm clearly can be used to any control-oriented problem. Again, for problems related to pattern recognition, after supplying numeric values to the distance function, it can be used [19]. Kochen and Badre’s [20] approach is a particular case of our approach. If the collected data is of a particular type, it will give us the said S shaped curve. All the parameters considered are clearly strongly related with the expression of the membership polynomial [23]. Again depending on the data, we may get the triangular membership function by piecewise linear regression [28].

### 6. Conclusions

In this paper, we have proposed a novice method to find out the membership function of a fuzzy set by applying non-linear regression analysis. The approach is expected to have a much wider scope of applicability. To any problem where the numeric values are available or can be made available, the polynomial we prescribe will produce us the required numeric value which is either accurate or near to accurate.

Since in our method the conditions strongly restrict $\mu$ in $[0, 1]$, and also since the population (domain) is predetermined, there will be no chance of getting $\mu < 0$ or $\mu > 1$ as in case of some previous methods.

Though this method is still far away from getting the generalized membership curve, it can be used to much more cases than others can. If we want to get the generalized one, it will be obtained from philosophical point of view, not so much of Mathematics.

There are lots of scopes of future study considering this way of constructing Fuzzy Membership Functions. As a theoretical development, new approaches of evaluation of the MFs can be constructed using other approximation tools. Secondly, some other algorithms, such as, genetic algorithm (GA), ant colony optimization (ACO), particle swarm optimization (PSO), Tabu search, simulated annealing, agent-based simulations, VEGA (vector evaluation genetic algorithm), NEGA (Non-dominated sorting genetic algorithm), NPGA (Nitched Pareto genetic algorithm) and PAES (Pareto archived evolution strategy) can be integrated with the proposed algorithm in order to determine the Membership Function, especially when the data set is significantly large. Finally, from the viewpoint of application, besides control system applications, it can be implemented in accessing the nature of the alternatives with respect to different attributes in uncertainty based Multi-Attribute Decision Making problems.
Disclosure statement

No potential conflict of interest was reported by the author(s).

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References

[1] Bilgiç T, Turksen IB. Measurement of membership functions: theoretical and empirical Work. Handbook of Fuzzy Sets and Systems, Vol. 1, Fundamental of Fuzzy Sets 1995.
[2] Yager RR. A measurement–informational discussion of fuzzy union and intersection. International Journal of Man–Machine Studies. 1979;11:189–200.
[3] Norwich AM, Turksen IB. The construction of membership functions. In: RR Yager, editor. Fuzzy Sets and possibility theory: recent developments. New York: Pergamon Press; 1982. p. 61–67.
[4] Turksen IB. Measurement of membership functions and their assessment. Fuzzy Sets Syst. 1991;40:5–38.
[5] Bollmann Sdorra P, Wong SKM, Yao Y. A measurement theoretic axiomatization of fuzzy sets. Fuzzy Sets Syst. 1993;60(3):295–307.
[6] Wong SKM, Yao Y. A probabilistic method for computing term-by-term relationships. JASIS. 1993;44(8):431–439.
[7] Bilgic T. Measurement theoretic frameworks for fuzzy set theory with applications to preference modelling, PhD thesis, University of Toronto, Dept. of Industrial Engineering Toronto Ontario M5S 1A4 Canada 1995.
[8] Bellman R, Giertz M. On analytic formalism of the theory of fuzzy sets. Inf Sci (Ny). 1973;5:149–156.
[9] Kamp JAW. Two theories about adjectives. In: EL Keenan, editor. Formal semantics of natural language. London: Cambridge University Press; 1975. p. 123–155.
[10] Fung LW, Fu SK. An axiomatic approach to rational decision making in a fuzzy environment. In: HJ Zimmermann, LA Zadeh, BR Gaines, editor. Fuzzy sets and their application to cognitive and decision processes. New York: Academic Press; 1975. p. 227–256.
[11] Dubosis D, Prade H. Fuzzy sets and systems: Theory and applications. New York: Academic Press 1980.
[12] Yager RR. On a general class of fuzzy connectives. Fuzzy Sets Syst. 1980;4:235–242.
[13] Weber S. A general concept of fuzzy connectives, negations and implications based on tnorms and tconorms. Fuzzy Sets Syst. 1983;11:115–134.
[14] Alsina C. On a family of connectives for fuzzy sets. Fuzzy Sets Syst. 1985;16:231–235.
[15] Zadeh LA. Similarity relations and fuzzy orderings. Inf Sci (Ny). 1971;3:177–200.
[16] Kandel A, Byatt JW. Fuzzy sets, fuzzy algebra and fuzzy statistics. Proc IEEE. 1978;66(12): 1619–1639.
[17] De Luca A, Termini S. A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory. Inf Control. 1972;20(4):301–312.
[18] Saaty TL. Measuring the fuzziness of sets. J. Cybern. 1974;4(4):53–61.
[19] Bremermann H. Pattern recognition. In: Hartmut Bossel, Salomon Klaczko, Norbert Müller, editors. Systems theory in the social sciences. Basel: Birkhäuser; 1976. p. 116–159.
[20] Kochen M, Badre AN. On the precision of adjectives which denote fuzzy sets. J. Cybern. 1976;4(1):49–59.
[21] Hersh H, Caramazza A. A fuzzy set approach to modifiers and vagueness in natural language. J Experimental Psychology: General. 1976;105(3):254–276.
[22] MacVicar-Whelan PJ. Fuzzy sets, the concept of height and the hedge very. IEEE Transactions Syst. Man Cybern. 1978;8:507–511.
[23] Diskant H. About membership function estimation. Fuzzy Sets Syst. 1981;5:141–147.
[24] Norwich AM, Turksen IB. A model for the measurement of membership and the consequences of its empirical implementation. Fuzzy Sets Syst. 1984;12:1–25.
[25] Nowakowska M. Methodological problems of measurement of fuzzy concepts in the social sciences. Behav Sci. 1977;22:107–115.
[26] Civanlar MR, Trussel HJ. Constructing membership functions using statistical data. Fuzzy Sets Syst. 1986;18:1–14.
[27] Takagi H, Hayashi I. Nn-driven fuzzy reasoning. Int J Approx Reason. 1991;5(3):191–213.
[28] Kim CJ, Russell BD. Automatic generation of membership function and fuzzy rule using inductive reasoning. Industrial Fuzzy Control and Intelligent Systems, 1993., IFIS’93., Third International Conference on, pp. 93–96. 1993.
[29] Karr CL, Gentry EJ. Fuzzy control of pH using genetic algorithms. IEEE Trans. Fuzzy Systems. 1993;1(1):46–53.
[30] Chen JE, Otto KN. Constructing membership functions using interpolation and measurement theory. Fuzzy Sets Syst. 1995;73:313–327.
[31] Singh RNP, Bailey WH. Fuzzy logic application to multisensor-multitarget correlation. IEEE Trans Aerosp Electron Syst. 1997;33:752–769.
[32] Royo AS, Verdegay JL. Methods for the construction of membership functions. Int J Intell Syst. 1999;14:1213–1230.
[33] Torra V. The WOWA operator and the interpolation function $W_i^*$: Chen and otto's interpolation method revisited. Fuzzy Sets Syst. 2000;113:389–396.
[34] Yam Y, Koczy LT. Representing membership functions as points in high-dimensional spaces for fuzzy interpolation and extrapolation. IEEE Trans Fuzzy Syst. 2000;8:761–772.
[35] Wong ML, Yam Y, Baranyi P. Representing membership functions as elements in function space. American Control Conference, Arlington, VA, USA, June 2001; Proceedings of the 2001. IEEE, 2001; pp. 1922–1927.
[36] Zhao J, Bose BK. Evaluation of membership functions for fuzzy logic controlled induction motor drive. IECON 02 [IEEE 2002 28th Annual Conference of the Industrial Electronics Society] 2002; 1: 229–234.
[37] Bağiş A. Determining fuzzy membership functions with tabu search- an application to control. Fuzzy Sets Syst. 2003;139:209–225.
[38] Tan HL, Rahim NA, Hew WP. A dynamic input membership scheme for a fuzzy logic DC motor controller. The IEEE Int Conf Fuzzy Systems. 2003: 426–429. https://keprofesianhmeitb.files.wordpress.com/2009/03/a-dynamic-input-membership-scheme-for-a-fuzzy-logic-dc-motor-controller.pdf
[39] Dombi J, Gera Z. The approximation of piecewise linear membership function and Lukasiewicz operators. Fuzzy Sets Syst. 2005;154:275–286.
[40] Wong ML, Yam Y, Baranyi P. Interpolation with function space representation of membership functions. IEEE Trans Fuzzy Syst. 2006;14(3):398–411.
[41] Dutta Majumder D, Bhattacharyya R, Mukherjee S. Methods of evaluation and extraction of membership functions—review with a new approach. In 2007 International Conference on Computing: Theory and Applications (ICCTA’07), Kolkata, India, 2007, pp. 277–281, doi:10.1109/ICCTA.2007.86.
[42] Chen CH, Hong TP, Tseng VS. A comparison of different fitness functions for extracting membership functions used in fuzzy data mining. In 2007 IEEE Symposium on Foundations of Computational Intelligence, Honolulu, HI, USA, 2007, pp. 550–555. doi:10.1109/FOCI.2007.371526.

[43] Alcalá-Fdez J, Alcalá R, Gacto MJ, et al. Learning the membership function contexts for mining fuzzy association rules by using genetic algorithms. Fuzzy Sets Syst. 2009;160(7):905–921.

[44] Maniu G. The construction of the membership functions in the fuzzy measuring of poverty. Buletinul. 2009;LXI:107–117.

[45] Zhu A, Yang L, Li B, et al. Construction of membership functions for predictive soil mapping under fuzzy logic. Geoderma. 2010;155:164–174.

[46] Ouyang S, Liao YJ, Liu Y, et al. The design of new fuzzy membership function and its application in power quality comprehensive evaluation. APPEEC ’11 [Proceedings of the 2011 Asia-Pacific Power and Energy Engineering Conference] 2011; 1-5.

[47] Sebastian P, Ewa S. Membership functions for fuzzy focal elements. Arch Control Sci. 2016;26(3):395–427.

[48] Dombi J. Membership function as an evaluation. Fuzzy Sets Syst. 1990;35:1–21.

[49] Hersh H, Caramazza A, Brownell H. Effects of context on fuzzy membership functions. In: Gupta M, Ragade R, Yager R, editor. Advances in fuzzy set theory and applications. Amserdam: Oxford; 1979. p. 389–408.