CONVERGENCE OF KRASULINA ESTIMATOR

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Abstract. Principal component analysis (PCA) is one of the most commonly used statistical procedures with a wide range of applications. Consider the points $X_1, X_2, ..., X_n$ are vectors drawn i.i.d. from a distribution with mean zero and covariance $\Sigma$, where $\Sigma$ is unknown. Let $A_n = X_nX_n^T$, then $E[A_n] = \Sigma$. This paper consider the problem of finding the least eigenvalue and eigenvector of matrix $\Sigma$. A classical such estimator are due to Krasulina\cite{9}. We are going to state the convergence proof of Krasulina for the least eigenvalue and corresponding eigenvector, and then find their convergence rate.

1. Introduction

Principal component analysis (PCA) is one of the most widely used dimension reduction techniques in data analysis. Suppose $X_1, X_2, ..., X_n$ are vectors drawn i.i.d. from a distribution with mean zero and covariance $\Sigma$, where $\Sigma \in \mathbb{R}^{d \times d}$ is unknown. Let $A_n = X_nX_n^T$, then $E[A_n] = \Sigma$. We are interested in finding eigenvalues of matrix $\Sigma$ and the corresponding eigenvectors if identifiable.

This problem has been studied for many aspects, especially in the offline setting when all the observations are available at once, see \cite{2, 4, 5, 12, 13, 14, 16}. For instance, \cite{4} find sharp bound of optimal rates of convergence on the loss function $E[\|\Theta^T - \hat{\Theta}^T\|_F^2]$ for all offline estimation, where $\Theta = [\theta_1, \theta_2, ..., \theta_r]$ is a matrix for eigenvector and $\hat{\Theta}$ is the corresponding estimator. Also the paper \cite{8} states the bound of standard PCA of estimated covariance matrix and true covariance matrix.

However, in high dimensional data, when one performs PCA in a large data set, one may need to consider computational complexity constantly. Indeed, for data in $\mathbb{R}^d$, the default method need storage space in $O(d^2)$. Therefore, it is interesting to find online incremental schemes that only take one data point at a time, updating with each new point. Some of these methods only need $O(d)$ space in computing one eigenvector. Assume matrix $\Sigma$ has the standard decomposition:

\begin{equation}
\Sigma = \sum_{j=1}^{d} \lambda_j \theta_j \otimes \theta_j,
\end{equation}

where $\lambda_j$ is the $j$th eigenvalue satisfy: $\lambda_1 < \lambda_2 \leq \lambda_3 \leq ... < \lambda_d$, $\theta_j$ is the corresponding eigenvector, identifiable up to sign. To compute the least eigenvalue and corresponding eigenvector, Krasulina\cite{9} suggested a very elegant scheme. At time $n + 1$, estimation of the least eigenvector $V_{n+1}$ is updated by

\begin{equation}
V_{n+1} = V_n - \gamma_n + 1 \xi_{n+1},
\end{equation}

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where \( \{\gamma_n\} \) is the learning rate, typically, \( \{\gamma_n\} \) is chosen such that

\[
\sum \gamma_n = \infty, \quad \sum \gamma_n^2 < \infty.
\]

For example, \( \gamma_n = \frac{c}{n} \) where \( c \) is an absolute constant. And

\[
\xi_{n+1} = <X_{n+1}, V_n> \cdot X_{n+1} - \frac{<X_{n+1}, V_n>^2}{\|V_n\|^2} \cdot V_n = A_{n+1} \cdot V_n - \frac{<A_{n+1}V_n, V_n>}{\|V_n\|^2} \cdot V_n.
\]

There has been a lot of effort to compute the spectrum decomposition. Oja and Karhunen [11] suggested a method which is closely related to Krasulin, they use the update for the leading eigenvector as follows:

\[
V_{n+1} = \frac{V_n + \gamma_{n+1} <X_{n+1}, V_n>}{\|V_n + \gamma_{n+1} <X_{n+1}, V_n>\|} \cdot X_{n+1}
\]

[9, 11] proved that these estimators converge almost surely under the assumption (1.1), (1.3) and \( E[\|X_n\|^k] < \infty \) for some suitable \( k \).

There are many other incremental estimators which convergence has not been established yet. [15] introduce a candid covariance-free incremental PCA algorithm with assumption (1.1), they suggest the estimator:

\[
V_{n+1} = \frac{n - 1 - l}{n} V_{n-1} + \frac{1 + l}{n} X_n X_n^T V_{n-1} / \|V_{n-1}\|
\]

where \( l \) is called the amnesic parameter. With the presence of \( l \), larger weight is given to new samples and the effect of old samples will fade out gradually, typically, \( l \) ranges from 2 to 4. They also addressed the estimation of additional eigenvectors by first subtracting from the data its projection on the estimated eigenvectors, then apply (1.5). [1] consider PCA problem as stochastic optimization problem, they consider an unknown source distribution over \( \mathbb{R}^d \), and would like to find the \( k \)-dimensional subspace maximizing the variance of the distribution inside the subspace. They solve the problem by stochastic gradient descent, and suggest the updates:

\[
V_{n+1} = P_{orth}(V_n + \eta_n X_n X_n^T V_n),
\]

where \( P_{orth}(V) \) performs a projection with respect to the spectral norm of \( VV^T \) onto the set of \( d \times d \) matrices with \( k \) eigenvalues equal to 1 and the rest 0, \( \eta_n \) is the step size.

There also exists many results which analyze incremental PCA from the statistical perspective. They mainly obtain the asymptotic consistency of estimators under certain conditions. For example, [10] suggest a Block-Stochastic Power Method with assumption:

\[
X_n = AZ_n + E_n,
\]

where \( A \) is a fixed matrix, \( Z_n \) is a multivariate normal random variable, i.e. \( Z_n \sim N(0, I) \), and \( E_n \) is the noise vector, also sampled from multivariate normal random variable, i.e. \( E_n \sim N(0, \sigma^2 I) \). For a fixed block size \( B \), they update the estimator as:

\[
V_{n+1} = \frac{1}{B} \sum_{t \in [n-B,n]} <V_n, X_t > X_t \cdot V_n / \|V_n\|^2
\]

They prove that under (1.6) for any \( \epsilon > 0 \), estimator (1.7) satisfies

\[
\mathbb{P}(\|V_n - \theta_1\| \leq \epsilon) = 0.99,
\]
given \( n = O\left(\frac{\log(d/\epsilon)}{\log(\alpha^2 + 0.75\alpha^2)}\right) \) and block size \( B = O\left(\frac{(1+3(\sigma^2+\tau^2)^2)\sqrt{eta}}{\epsilon^3}\right) \). \([7]\) finds an upper bound in probability \( 1 - \delta \) of alignment loss function \( 1 - \frac{\langle V_n, \theta_1 \rangle^2}{\|V_n\|^2} \) for Oja’s estimator \([1,4]\) with assumption: \( \|A_i - \Sigma\| \leq A \) and max\( \{\|E[(A_i - \Sigma)(A_i - \Sigma)^T]\|,\|E[(A_i - \Sigma)^T(A_i - \Sigma)]\|\} \leq B \). For choice of step size \( \gamma_n = O\left(\frac{\alpha}{g_1(\lambda_n + \beta)}\right) \), where \( \alpha > \frac{1}{2} \), \( g_1 = \lambda_1 - \lambda_2 \), \( \beta = O\left(\frac{4\lambda_1^2\sqrt{\delta}}{g_1^2 + \delta}\right) \), and \( n > \beta \).

As for nonasymptotic result, \([8]\) derives suboptimal bound on the alignment loss \( E[1 - \frac{\langle V_n, \theta_1 \rangle^2}{\|V_n\|^2}] \), for the following choice of the learning rate: \( \gamma_n = \frac{1}{g_1 n} \), where \( g_1 = \lambda_1 - \lambda_2 \), provided that \( n > \tilde{p} \).

\([6]\) introduce Mini-batch Power Method, for batch size \( B \):

\[
V_{n+1} = \frac{1}{B} \sum_{t \in (n-B,n]} V_n, X_t > X_t - \beta V_{n-1} \quad \text{where } \beta \text{ is the Momentum parameter.}
\]

When \( \beta \) is chosen between \( \left[\frac{\lambda_2^2}{4}, \frac{\lambda_2^2}{4}\right] \), and initial vector satisfy \( \|V_0\| = 1 \) and \( \|V_0, \theta_1 \| \geq \frac{1}{2} \), for any \( \delta \in (0,1) \) and \( \epsilon \in (0,1) \), if we assume: \( n = \frac{\sqrt{\sigma}}{\sqrt{\lambda_2^2 - 4\beta}} \log\left(\frac{32\lambda_2}{\delta}\right) \) and max\( \{\|E[(A_i - \Sigma)(A_i - \Sigma)^T]\|,\|E[(A_i - \Sigma)^T(A_i - \Sigma)]\|\} \leq \frac{(\lambda_2^2 - 4\beta)^2}{2\delta^2 \lambda_2} \), then

\[
1 - \frac{\langle V_n, \theta_1 \rangle^2}{\|V_n\|^2} \leq \epsilon
\]

with probability at least \( 1 - 2\delta \).

Krasulina states the convergence of the least eigenvalue and eigenvector estimators, but did not provide convergence rate. In this paper, we find the rate of convergence for both eigenvalue and eigenvector estimator of Krasulina \([1,2]\) under a relatively mild assumption. Our analysis reveals a slower rate of convergence of eigenvalue estimator \( \tilde{\lambda}_1 = \frac{A_n V_n, V_n}{\|V_n\|^2} \) and corresponding eigenvector estimator \( \tilde{\theta}_1 = \frac{V_n}{\|V_n\|} \) as compared to the offline setting for Krasulina scheme.

**Notations:** for any vector \( x \in \mathbb{R}^d \), we denote by \( \|x\| \) the \( l^2 \)-norm of \( x \). For the sake of simplicity, for any matrix \( A \), \( \|A\| \) will refer to the operator norm of \( A \), specifically, \( \|A\| = \sup_{u,v} \frac{\langle Au, v \rangle}{\|v\|} \). For series \( \{x\}_n, \{y\}_n \), \( x_n \sim_p y_n \) is defined as: \( \forall \epsilon > 0 \), there exists a finite \( M > 0 \) and a finite \( N > 0 \), such that \( P\left(\|x_n - y_n\| < \epsilon M\right) < 1 - \epsilon \), \( \forall n > N \). \( y_n \leq_p x_n \) is defined as: \( \forall \epsilon > 0 \), there exists a finite \( M > 0 \) and a finite \( N > 0 \), such that \( P\left(\|y_n\| < \epsilon M\right) < 1 - \epsilon \).

\[
2. \text{Main Results}
\]

We now state our main result:

**Theorem 2.1.** Assume \( \lambda_1 < \lambda_2 \), \([1,3]\) and \( E\|A_n\|^2 < \infty \), let \( g = \lambda_2 - \lambda_1 \), by Krasulina scheme \([1,2]\), we have

\[
|\tilde{\lambda}_1 - \lambda_1| \leq_p \frac{\|\Sigma\|}{\sqrt{n}} \cdot \sqrt{E[\|A_n\|^2]} \sqrt{\|\Sigma\|}
\]

and

\[
1 - \frac{\langle V_n, \theta_1 \rangle^2}{\|V_n\|^2} \geq_p \frac{\|\Sigma\|}{g^2 \sqrt{n}} \cdot \sqrt{E[\|A_n\|^2]} \sqrt{\|\Sigma\|}
\]
Lemma 3.1. Let \( \{Y_n\}_n \) be a sequence of real-valued independent random variables. We assume that for all \( n \geq 1 \), \( Y_n \) is zero mean and square integrable. Define \( S_n = \sum_{i=1}^{n} Y_i \). If \( \sum_{n=1}^{\infty} E[Y^2_n] < \infty \), then \( \{S_n\}_n \) converges to a real-valued random variable in probability.

Proof. Since \( Y_n \) is centered, \( E[Y^2_n] = \text{Var}(Y_n) \), thus:

\[
E[|S_{n+r} - S_n|^2] = \text{Var}(\sum_{i=n+1}^{n+r} Y_i) = \sum_{i=n+1}^{n+r} \text{Var}(Y_i) = \sum_{i=n+1}^{n+r} E[Y^2_i] \leq \sum_{i=n}^{\infty} E[Y^2_i],
\]

this is the remainder term of a convergence series, thus \( \{S_n\}_n \) is Cauchy, so \( \{S_n\}_n \) converges to a real-valued random variable in \( L^2 \). By Kolmogorov inequality, Lemma 3.1 follows.

Now, we start by bound the asymptotic expectation of \( ||V_n||^2 \):

Lemma 3.2. \( \lim_{n \to \infty} E[||V_n||^2] = \infty \).

Proof. First, we prove that \( V_n \) and \( \xi_{n+1} \) are orthogonal for any \( n \geq 1 \). Let \( X_{n+1} - \frac{\langle X_{n+1}, V_n \rangle}{||V_n||^2} \cdot V_n = W_n \), we have:

\[
\xi_{n+1} = \langle X_{n+1}, V_n \rangle \cdot X_{n+1} - \frac{\langle X_{n+1}, V_n \rangle^2}{||V_n||^2} \cdot V_n
\]

\[
= \langle X_{n+1}, V_n \rangle (X_{n+1} - \frac{\langle X_{n+1}, V_n \rangle}{||V_n||^2} \cdot V_n)
\]

\[
= \langle X_{n+1}, V_n \rangle \cdot W_n.
\]

We note that \( \langle W_n, V_n \rangle = 0 \), so \( ||\xi_{n+1}|| = \langle X_{n+1}, V_n \rangle \cdot ||W_n|| \leq \langle X_{n+1}, V_n \rangle \cdot ||X_{n+1}|| = ||A_{n+1}V_n|| \), thus:

\[
E[||\xi_{n+1}||^2] \leq E[||A_{n+1}||^2] \cdot ||V_n||^2 = ||\Sigma||^2 ||V_n||^2.
\]

Now since \( \xi_n \perp V_{n-1} \), we have \( ||V_n||^2 = ||V_{n-1} - \gamma_n \xi_n||^2 = \langle V_{n-1}, V_{n-1} \rangle > + \gamma_n^2 ||\xi_n||^2 \), thus:
Lemma 3.3. \( \mu \) to \( \lambda \)gence in probability of the sequence of \( V_n \).

Proof. Since:

\[ \sum_{i=1}^{\infty} \gamma_i^2 \| \Sigma \|^2 < \infty, \]  

thus \( \prod_{i=1}^{n-1} (1 + \gamma_i^2 \| \Sigma \|^2) < \infty, \)  

thus \( \lim_{n \to \infty} E[\| V_n \|^2] < \infty. \)

Next, let \( \mu(V_n) = \frac{< \Sigma V_n, V_n >}{\| V_n \|^2}, \) and \( a_1^{(n)} = < V_n, \theta_1 >. \) We first prove the convergence in probability of the sequence of \( V_n \) and \( a_1^{(n)}. \) Specifically, \( \mu(V_n) \) converges to \( \lambda_1, \) and \( V_n \) converges to a vector which is aligned with \( \theta_1. \) To prove that, we can recursively properly apply the inequality, to show the Cauchy property of sequence \( \mu(V_n) \) and \( a_1^{(n)}. \)

Lemma 3.3. \( \mu(V_n) = \frac{< \Sigma V_n, V_n >}{\| V_n \|^2} \) converges a.s. to \( \mu \) as \( n \to \infty. \)

Proof.

\[
\mu(V_{n+1}) = \frac{< \Sigma V_n - \gamma_{n+1} \cdot \Sigma \xi_{n+1}, V_n - \gamma_{n+1} \xi_{n+1} >}{\| V_n - \gamma_{n+1} \xi_{n+1} \|^2} \\
= \frac{< \Sigma V_n, V_n > + \gamma_{n+1}^2 < \Sigma \xi_{n+1}, \xi_{n+1} > - 2 \gamma_{n+1} < \xi_{n+1}, \Sigma V_n >}{\| V_n \|^2 + \gamma_{n+1}^2 \| \xi_{n+1} \|^2} \\
= \frac{1}{1 + \gamma_{n+1}^2 \| V_n \|^2} (\mu(V_n) - 2 \gamma_{n+1} \frac{< \xi_{n+1}, \Sigma V_n >}{\| V_n \|^2} + \gamma_{n+1}^2 \frac{< \Sigma \xi_{n+1}, \xi_{n+1} >}{\| V_n \|^2})
\]

Since:

\[
< \xi_{n+1}, \Sigma V_n > = < A_{n+1} V_n, \Sigma V_n > - \frac{< A_{n+1} V_n, V_n > < \Sigma V_n, V_n >}{\| V_n \|^2} \\
= \| \Sigma V_n \|^2 - \frac{< \Sigma V_n, V_n >^2}{\| V_n \|^2} + < A_{n+1} V_n, \Sigma V_n > - \| \Sigma V_n \|^2 - \frac{< A_{n+1} V_n, V_n > < \Sigma V_n, V_n >}{\| V_n \|^2} + \frac{< \Sigma V_n, V_n >^2}{\| V_n \|^2} \\
= ( < A_{n+1} - \Sigma V_n, \Sigma V_n > - \frac{< (A_{n+1} - \Sigma) V_n, V_n >}{\| V_n \|^2} \cdot < \Sigma V_n, V_n > ) + ( \| \Sigma V_n \|^2 - \frac{< \Sigma V_n, V_n >^2}{\| V_n \|^2} )
\]

Let

\[
f(V_n) = \frac{\| \Sigma V_n \|^2}{\| V_n \|^2} - \frac{< \Sigma V_n, V_n >^2}{\| V_n \|^4},
\]

(3.1)
(3.2) \[ Z_n = \frac{<(A_{n+1} - \Sigma)V_n, \Sigma V_n>}{\|V_n\|^2} - \frac{<(A_{n+1} - \Sigma)V_n, V_n>}{\|V_n\|^4} \cdot <\Sigma V_n, V_n>, \]

thus: \[ \frac{\xi_{n+1} \Sigma V_n}{\|V_n\|^2} = f(V_n) + Z_n. \]

so \( \mu(V_{n+1}) = \frac{1}{1 + \frac{\xi_{n+1}}{\|V_n\|^2} \gamma_{n+1}} \left( \mu(V_n) - 2\gamma_{n+1} f(V_n) - 2\gamma_{n+1} Z_n + \frac{\xi_{n+1} \xi_{n+1}}{\|V_n\|^2} \right). \]

Let \( a_n = \gamma_{n+1} Z_n, b_n = \frac{\xi_{n+1} \xi_{n+1}}{\|V_n\|^2}, \]
and \( c_n = \frac{1}{1 + \gamma_{n+1} \frac{\xi_{n+1}}{\|V_n\|^2}}, \]
thus:

\[ \mu(V_{n+1}) = c_n \cdot (\mu(V_n) - 2\gamma_{n+1} f(V_n) - 2a_n + b_n). \]

Now we have:

(3.4) \[ \mu(V_{n+1}) - c_n \cdot \mu(V_n) = -2\gamma_{n+1} c_n f(V_n) - 2a_n c_n + b_n c_n. \]

For series \( \{a_n\}, \) since \( Z_n \) is centered and bounded, by lemma 3.1:

\[ \sum_{i>k} \gamma_i^2 < \infty, \]

thus \( \sum_{n=1}^{\infty} a_n < \infty. \)

For series \( \{b_n\}, \) by lemma 3.2:

\[ E[\|\xi_n\| F_{n-1}] \leq \|\Sigma\|^2 \|V_n\|^2, \]

thus \( E[\frac{\xi_{n+1}}{\|V_n\|^2} < \infty. \) Since \( \gamma \approx \frac{1}{\pi}, \) we have \( \sum_{n=1}^{\infty} b_n < \infty. \)

For series \( \{c_n\}, \prod c_n = \prod \frac{1}{1 + \gamma_{n+1} \frac{\xi_{n+1}}{\|V_n\|^2}}, \)
which is equivalent to

\[ \prod \frac{1}{1 + \gamma_{n+1} \frac{\xi_{n+1}}{\|V_n\|^2}}, \]

which has the same convergence properties as \( \sum \gamma_{n+1} \frac{\xi_{n+1}}{\|V_n\|^2}, \) since \( \frac{\xi_{n+1}}{\|V_n\|^2} \) is bounded. \( \prod_{n=1}^{\infty} c_n < \infty. \)

And by Cauchy-Schwartz inequality:

(3.5) \[ f(V_n) = \frac{\|\Sigma V_n\|^2}{\|V_n\|^2} - \frac{<\Sigma V_n, V_n>^2}{\|V_n\|^4} \geq 0. \]

Now, if \( \liminf \mu(V_n) < \limsup \mu(V_n), \) choose \( a, b \) such that \( \liminf \mu(V_n) < a < b < \limsup \mu(V_n), \) find \( m_1, n_1 \) large enough, such that \( \mu(V_{n_1}) < a, \mu(V_{m_1}) > b, \) and for all \( n_1 < j < m_1, \) we have \( a \leq \mu(V_j) \leq b. \) Thus:

\[ \mu(V_{m_1}) - \mu(V_n) \prod_{i=n_1}^{m_1-1} c_i > b - a. \]

On the other hand:

(3.6) \[ \mu(V_{m_1}) - \mu(V_n) \prod_{i=n_1}^{m_1-1} c_i = \sum_{j=n_1}^{m_1-1} \left( (-2\gamma_{j+1} \cdot f(V_j) - 2a_j + b_j) \cdot \prod_{i=j}^{m_1-1} c_i \right) \]

\[ \leq \sum_{j=n_1}^{m_1-1} \left( (-2a_j + b_j) \cdot \prod_{i=j}^{m_1-1} c_i \right) \]

\[ \rightarrow 0 \text{ as } n_1, m_1 \rightarrow \infty, \]
which is a contradiction, thus $\mu(V_n) \to \mu$ with probability 1.

\text{Lemma 3.4.} \; a_1^{(n)} = \langle V_n, \theta_1 \rangle, \text{ where } \theta_1 \text{ is the eigenvector of } \lambda_1, \; a_1^{(n)} \text{ converges to some value } a_1 \text{ as } n \to \infty.

\text{Proof.} \; \text{Since } V_{n+1} = V_n - \gamma_{n+1} \xi_{n+1}, \; \xi_{n+1} = A_{n+1} V_n - \frac{\langle A_{n+1} V_n, V_n \rangle}{\|V_n\|^2} V_n, \text{ by definition of } a_1^{(n)} = \langle V_n, \theta_1 \rangle \text{ and } \mu(V_n) = \frac{\langle \Sigma V_n, V_n \rangle}{\|V_n\|^2}, \text{ also by the nature: } \langle \Sigma V_n, \theta_1 \rangle = \langle V_n, \Sigma \theta_1 \rangle = \langle V_n, \lambda_1 \theta_1 \rangle = \lambda_1 a_1^{(n)}, \text{ we have:}

\begin{align*}
a_1^{(n+1)} &= \langle V_{n+1}, \theta_1 \rangle = \langle V_n - \gamma_{n+1} \xi_{n+1}, \theta_1 \rangle \\
&= \langle V_n, \theta_1 \rangle - \gamma_{n+1} \langle A_{n+1} V_n, V_n \rangle - \frac{\langle A_{n+1} V_n, V_n \rangle}{\|V_n\|^2} \langle V_n, \theta_1 \rangle \\
&= a_1^{(n)} + \gamma_{n+1} < \frac{\Sigma V_n, \gamma V_n}{\|V_n\|^2} V_n + \frac{(\Sigma - A_{n+1}) V_n, \theta_1 \rangle \\
&= a_1^{(n)} + \gamma_{n+1}(\mu(V_n) - \lambda_1) a_1^{(n)} + \gamma_{n+1} Z'_n \\
&= a_1^{(n)} (1 + \gamma_{n+1}(\mu(V_n) - \lambda_1)) + \gamma_{n+1} Z'_n,
\end{align*}

where $Z'_n = (\Sigma - A_{n+1}) V_n, \theta_1 > + \frac{\langle A_{n+1} - \Sigma \rangle V_n, V_n \rangle}{\|V_n\|^2} a_1^{(n)}$.

Since $E[\|V_n\|^2] = E[\|V_{n-1}\|^2] + \gamma_n^2 E[\|\xi_n\|^2] \leq E[\|V_{n-1}\|^2] + \gamma_n^2 E[\|\Sigma\|^2 E[\|V_{n-1}\|^2] \leq \prod_{i=1}^\infty (1 + \gamma_n^2 \|\Sigma\|^2) \leq \infty, Z'_n \text{ is centered and bounded, by lemma 3.1, } \sum_{n=1}^\infty \gamma_n Z'_n < \infty.$

Now, if $\liminf a_1^{(n)} < \limsup a_1^{(n)}$, choose $a, b$ such that $\liminf a_1^{(n)} < a < b < \limsup a_1^{(n)}$, find $m_1, n_1$, such that: $m_1 \geq n_1 \geq N, \; a_1^{(m_1)} < a, \; a_1^{(n_1)} > b$, for $j \in (n_1, m_1), a \leq a_j^{(j)} \leq b$. Since $\lambda_1$ is the least eigenvalue, $\mu(V_k) \geq \lambda_1$.

Thus:

\begin{align*}
a_1^{(m_1)} - a_1^{(n_1)} \prod_{k=n_1}^{m_1} (1 + \gamma_{k+1}(\mu(V_k) - \lambda_1)) \leq a_1^{(m_1)} - a_1^{(n_1)} < a - b \leq 0.
\end{align*}

On the other hand:

\begin{align*}
a_1^{(m_1)} - a_1^{(n_1)} \prod_{k=n_1}^{m_1} (1 + \gamma_{k+1}(\mu(V_k) - \lambda_1)) \\
&= \sum_{j=n_1}^{m_1-1} \gamma_j Z'_j \prod_{i=j}^{m_1-1} (1 + \gamma_{j+1}(\mu(V_j) - \lambda_1)) \\
&\geq \sum_{j=n_1}^{m_1-1} \gamma_j Z'_j
\end{align*}

Since $\sum_{j=1}^\infty \gamma_j Z'_j < \infty$, let $n_1 \to \infty$, we can let $\sum_{j=n_1}^{m_1-1} \gamma_j Z'_j$ as closed to 0 as we want, which is a contradiction.

Thus $a_1^{(n)} \to a_1$ with probability 1.

Now we get the idea that $\mu(V_n)$ and $a_1^{(n)}$ are both convergence with probability 1, and by the proof above, all coefficients in (3.4) are convergence with probability 1.
1, so does the part $\gamma_{n+1}c_{n}f(V_{n})$. By find the convergence rate for each of these parts, we can find the convergence rate for $\mu(V_{n})$.

**Lemma 3.5.** (1) $\mu(V_{n}) \to \lambda_{1}$ as $n \to \infty$ with probability 1, and (2) the convergence rate of $\frac{\mathbb{E}[\sum_{i=1}^{n}V_{n}]}{\sqrt{n}}$ to $\lambda_{1}$ is in the order of $O\left(\frac{1}{\sqrt{n}} \cdot (\sqrt{\mathbb{E}[A_{n}]} \sqrt{\Sigma})\right)$.

**Proof.** (1) By assumption 2, $|a_{1}^{(n)}| > 0$, thus:

$$a_{1}^{(n+1)} = <V_{n+1}, \theta_{1}> = <V_{n+1}, \theta_{1}> = <V_{n} - \gamma_{n+1}x_{n+1}, \theta_{1}>$$

$$= <V_{n}, \theta_{1}> - \gamma_{n+1} <A_{n+1}V_{n}, \theta_{1}>$$

$$= a_{1}^{(n)} + \gamma_{n+1} \frac{<A_{n+1}V_{n}, \theta_{1}>}{\|V_{n}\|^2} a_{1}^{(n)} - \gamma_{n+1} <A_{n+1}V_{n}, \theta_{1}>$$

$$= a_{1}^{(n)} + \gamma_{n+1} \frac{<A_{n+1}V_{n}, \theta_{1}>}{\|V_{n}\|^2} a_{1}^{(n)} - \gamma_{n+1} <A_{n+1}V_{n}, \theta_{1}>$$

$$+ \gamma_{n+1} \frac{<A_{n+1}V_{n}, \theta_{1}>}{\|V_{n}\|^2} a_{1}^{(n)} + \gamma_{n+1} <A_{n+1}V_{n}, \theta_{1}>$$

$$= a_{1}^{(n)} \left(1 + \gamma_{n+1}(\mu(V_{n}) - \lambda_{1})\right) + \gamma_{n+1}Z'_{n},$$

where $Z'_{n} = \frac{\sqrt{\sum_{i=1}^{n}V_{n}}}{\sqrt{n}}\frac{1}{\sqrt{n}}(\frac{E[\mu(V_{n})]}{E[a_{1}^{(n)}]} - \lambda_{1})$, which is centered and bounded, then by Jensen’s inequality:

$$E[a_{1}^{(n+1)}] \geq E\left[a_{1}^{(n)}(1 + \gamma_{n+1}(\frac{E[\mu(V_{n})]}{E[a_{1}^{(n)}]} - \lambda_{1}))\right]$$

$$\geq \prod_{k=1}^{n} \left(1 + \gamma_{k+1} \frac{E[\mu(V_{k})]}{E[a_{1}^{(k)}]} - \lambda_{1}\right) E[a_{1}^{(1)}]$$

By Lemma 3.4, $\{a_{1}^{(n)}\}$ convergence, thus $\prod_{k=1}^{\infty} \left(1 + \gamma_{k+1} \frac{E[\mu(V_{k})]}{E[a_{1}^{(k)}]} - \lambda_{1}\right) < \infty$, and $\sum_{k=1}^{\infty} \gamma_{k+1}(\frac{E[\mu(V_{k})]}{E[a_{1}^{(k)}]} - \lambda_{1}) < \infty$. Since $\gamma_{k+1} \approx \frac{1}{n}$, $\lim_{k \to \infty} \frac{E[\mu(V_{k})]}{E[a_{1}^{(k)}]} = \lambda_{1} = 1$.

By dominant convergence theorem: $\lim_{k \to \infty} a_{1}^{(k)} = a_{1}$, $\lim_{k \to \infty} \mu(V_{k}) = \mu$. Thus:

$$\mu = \lambda_{1}$$

(2)

$$\lambda_{1} = \frac{<A_{n}V_{n}, V_{n}>}{\|V_{n}\|^2} = (\lambda_{1} - \mu(V_{n})) + (\mu(V_{n}) - \frac{<A_{n}V_{n}, V_{n}>}{\|V_{n}\|^2})$$

$$= (\lambda_{1} - \mu(V_{n})) + \frac{<\sum_{i=1}^{n}A_{n}V_{n}, V_{n}>}{\|V_{n}\|^2}$$

Since $E[\frac{<\sum_{i=1}^{n}A_{n}V_{n}, V_{n}>}{\|V_{n}\|^2}] = 0$, we only need to consider $|\lambda_{1} - \mu(V_{n})|$. From (3.3) we have: $\mu(V_{n+1}) - c_{n}, \mu(V_{n}) = -2\gamma_{n+1}c_{n}f(V_{n}) - 2a_{n}c_{n} + b_{n}c_{n} = (-2\gamma_{n+1}f(V_{n}) - 2a_{n} + b_{n})c_{n}, a_{j}, b_{j}$ and $c_{j}$ are defined the same as (3.3). The same way as we get (3.6), keep increase $V_{n}$ to $V_{m}$ recursively, we have:

$$\mu(V_{m}) - \mu(V_{n}) \prod_{i=n}^{m-1} c_{i} = \sum_{j=n}^{m-1} (b_{j} - 2\gamma_{j+1}f(V_{j}) - 2a_{j}) \prod_{i=j}^{m-1} c_{i}.$$
For $b_j$ part,

$$\sum_{j=n}^{\infty} b_j = \sum_{j=n}^{\infty} \gamma_{j+1}^2 \frac{\langle \Sigma \xi_{j+1}, \xi_{j+1} \rangle}{\|V_j\|^2} \leq \sum_{j=n}^{\infty} \gamma_{j+1}^2 \frac{\|\Sigma\| \|\xi_{j+1}\|^2}{\|V_j\|^2} = \sum_{j=n}^{\infty} \gamma_{j+1}^2 \cdot c,$$

thus its rate of convergence is $O\left(\frac{1}{n}\right)$

For $a_j$ part, $\sum_{j=n}^{\infty} a_j = \sum_{j=n}^{\infty} \gamma_{j+1} Z_j$, $Z_j$ is centered and bounded, by lemma 3.1, $E[|S - S_n|^2] \leq \sum_{i>n} E[a_i^2]$, whose rate of convergence is $O\left(\frac{1}{n}\right)$, thus $\sum_{j=n}^{\infty} a_j$ has the rate of convergence $O\left(\frac{1}{\sqrt{n}}\right)$.

For $c_j$ part, by proof of the lemma 3.3, $\prod_{i=n}^{\infty} c_i$ has the same convergence properties as $\sum_{i=n}^{\infty} \gamma_{i+1}^2 \|\xi_{i+1}\|^2$. Since $\|\xi_{i+1}\|^2$ is bounded, $\prod_{i=n}^{\infty} c_i$ has the rate of convergence $O\left(\frac{1}{n}\right)$.

For $f(V_j)$ part, by assumption 2, rewrite $V_n = \sum_{i=1}^{d} a_{i}^{(n)} \theta_i$, where $d$ is the dimension. From (3.6), we have: $\sum_{n=1}^{\infty} \gamma_{n+1} f(V_n) \prod_{k=1}^{n-1} \left(1 + \frac{\gamma_{k+1}^2 \|\xi_{k+1}\|^2}{\|V_k\|^2}\right)^{-1} < \infty$ with probability 1. Since we have $\gamma_n \approx \frac{1}{n}$ and $f(V_n) \geq 0 \forall n$, if $\liminf_{n \to \infty} f(V_n) = c$, then $\sum_{n=1}^{\infty} \gamma_{n+1} f(V_n) \prod_{k=1}^{n-1} \left(1 + \frac{\gamma_{k+1}^2 \|\xi_{k+1}\|^2}{\|V_k\|^2}\right)^{-1} = \infty$, thus $c = 0$.

Now, by nature of eigenvector and eigenvalue, as well as assumption 2: $\theta_i^2 = 1$, $\theta_i \theta_j = 0$ for $i \neq j$, and $\|V_n\|^2 = \sum_{i=1}^{d} (a_{i}^{(n)})^2$. Thus:

$$f(V_n) = \frac{\|\Sigma V_n\|^2}{\|V_n\|^2} - \frac{\langle \Sigma V_n, V_n \rangle^2}{\|V_n\|^4} = \frac{\sum_{i=1}^{d} (a_{i}^{(n)})^2 \lambda_i \theta_i^2}{\|V_n\|^2} - \mu(V_n)^2$$

$$= \frac{\sum_{i=1}^{d} (a_{i}^{(n)})^2 (\lambda_i^2 - \mu(V_n)^2)}{\|V_n\|^2},$$

which leads to the result: $f(V_j) \to 0$ with the same rate of $\mu(V_n) \to \lambda_1$.

Thus, $\frac{\langle A_n V_n, V_n \rangle}{\|V_n\|^2}$ converges to $\lambda_1$ the same rate as $a_j$ part, has the rate of convergence $O\left(\frac{1}{\sqrt{n}}\right)$. More precisely, by proof of the Lemma 3.1, $E[|S_{n+r} - S_n|^2] \leq \sum_{i>n} E|X_i^2|$ if $\{X_n\}$ is 0 mean. Then for $a_j = \gamma_{j+1} Z_j$, we have

$$E[|S - S_n|^2] \leq \sum_{i>n} E[a_i^2] \leq \sum_{i>n} \frac{1}{\gamma_{i+1}^2} E[Z_i^2].$$
Now for $Z_n$, by (3.2), we have:

$$
\|Z_n\| = \left\| \frac{(A_{n+1} - \Sigma)V_n + \Sigma V_n}{\|V_n\|^2} - \frac{(A_{n+1} - \Sigma)V_n + \Sigma V_n}{\|V_n\|^4} \right\|
$$

$$
\leq \frac{\| (A_{n+1} - \Sigma)V_n + \Sigma V_n \|}{\|V_n\|^2} + \frac{\| (A_{n+1} - \Sigma)V_n + \Sigma V_n \|}{\|V_n\|^4} \cdot \|\Sigma\| + \frac{\| (A_{n+1} - \Sigma)V_n + \Sigma V_n \|}{\|V_n\|^2} \cdot \|\Sigma\|
$$

$$
\lesssim_p \frac{\| (A_{n+1} - \Sigma)V_n + \Sigma V_n \|}{\|V_n\|^2} \cdot \|\Sigma\|
$$

$$
\leq \| A_{n+1} - \Sigma \| \cdot \|\Sigma\|
$$

$$
\leq (\|A_{n+1}\| + \|\Sigma\|) \cdot \|\Sigma\|.
$$

Thus:

$$
E[Z_n^2] \leq \|\Sigma\|^2 E[\|A_{n+1}\|^2 + \|\Sigma\|^2 + 2\|A_{n+1}\| \cdot \|\Sigma\|] \lesssim_p \|\Sigma\|^2 E[\|A_{n+1}\|^2 + \|\Sigma\|^2] \asymp_p \|\Sigma\|^2 \cdot \left( E[\|A_{n}\|^2] \sqrt{\|\Sigma\|^2} \right).
$$

So $E[\|S - S_n\|^2]$ has rate of convergence $O(\frac{1}{n^2} \cdot \|\Sigma\|^2 \cdot \left( E[\|A_n\|^2] \sqrt{\|\Sigma\|} \right))$, thus $\sum_{j=0}^{\infty} a_j$ has rate of convergence $O(\frac{1}{\sqrt{n}} \cdot \left( E[\|A_n\|^2] \sqrt{\|\Sigma\|} \right))$.

\[\Box\]

**Lemma 3.6.** (1) $V_n \to a_1^{(n)} \theta_1$ with probability 1 and (2) $\frac{V_n}{\|V_n\|^2}$ approach to 1 in the order of $\frac{\|\Sigma\|}{\sqrt{n}}$ with probability 1.

**Proof.** (1) We already proved that $f(V_n) \to 0$ and $\mu(V_n) \to \lambda_1$ in lemma 3.5, thus $\lambda_i - \mu(V_n) > 0$ for $i \neq 1$ when $n$ is large enough. By (3.7), $0 = \lim_{n \to \infty} f(V_n) = \lim_{n \to \infty} \sum_{i=1}^{d}(a_i^{(n)})^2(\lambda_i^2 - \mu(V_n)^2)$, $a_i^{(n)} = 0$ when $i \neq 1$, thus $V_n \to a_1^{(n)} \theta_1$ with probability 1.

(2) By previous argument, we have:

$$
f(V_n) = \frac{\sum_{i=1}^{d}(a_i^{(n)})^2(\lambda_i^2 - \mu(V_n)^2)}{\|V_n\|^2}
$$

$$
= \frac{(a_1^{(n)})^2(\lambda_1^2 - \mu(V_n)^2)}{\|V_n\|^2} + \sum_{i=2}^{d}(a_i^{(n)})^2(\lambda_i^2 - \mu(V_n)^2),
$$

correspondence with the same rate of $\mu(V_n) \to \lambda_1$, we have $\sum_{i=2}^{d}(a_i^{(n)})^2(\lambda_i^2 - \mu(V_n)^2) \to 0$ at least the same rate as $\frac{(a_1^{(n)})^2(\lambda_1^2 - \mu(V_n)^2)}{\|V_n\|^2} \to 0$.

By first part, $\mu(V_n)$ has rate of convergence $O(\frac{\|\Sigma\|}{\sqrt{n}} \cdot \left( E[\|A_n\|^2] \sqrt{\|\Sigma\|} \right))$, we have $\sum_{i=1}^{d}(a_i^{(n)})^2(\lambda_i^2 - \lambda_1^2) \lesssim_p \frac{\|\Sigma\|}{\sqrt{n}} \cdot \left( E[\|A_n\|^2] \sqrt{\|\Sigma\|} \right) \cdot \left( \frac{(a_1^{(n)})^2\lambda_1}{\|V_n\|^2} \right)$, let $g = |\lambda_1 - \lambda_2|$, 


thus:
\[
\sum_{i=2}^{\infty} (a_i^{(n)})^2 \leq_P \frac{\|\Sigma\|}{\sqrt{n}} \cdot \left(\sqrt{E[\|A_n\|^2]} \vee \|\Sigma\|\right) \cdot \frac{(a_1^{(n)})^2 \lambda_1}{(\lambda_i - \lambda_1)(\lambda_i + \lambda_1)}
\]
\[
\lesssim_P \frac{\|\Sigma\|}{g\sqrt{n}} \cdot \left(\sqrt{E[\|A_n\|^2]} \vee \|\Sigma\|\right)
\]

Now by assumption 2, \(\|V_n\|^2 = \sum_{i=1}^{d} (a_i^{(n)})^2\), thus:
\[
\|V_n\|^2 - (a_1^{(n)})^2 = \sum_{i=2}^{\infty} (a_i^{(n)})^2 \lesssim_P \frac{\|\Sigma\|}{g\sqrt{n}} \cdot \left(\sqrt{E[\|A_n\|^2]} \vee \|\Sigma\|\right).
\]

Above all:
\[
1 - \frac{\langle V_n, \theta_1 \rangle^2}{\|V_n\|^2} \lesssim_P \frac{\|\Sigma\|}{g\sqrt{n}} \cdot \left(\sqrt{E[\|A_n\|^2]} \vee \|\Sigma\|\right).
\]

\[\Box\]

4. Experiment

The dataset \(X \in \mathbb{R}^{10^6 \times 10}\) was just generated through its singular value decomposition. Specifically, we fix a \(10 \times 10\) diagonal matrix \(\Sigma = \text{diag}\{1, 0.9, \cdots, 0.9\}\) and generate random orthogonal projection matrix \(U \in \mathbb{R}^{10^6 \times 10}\) and random orthogonal matrix \(V \in \mathbb{R}^{10 \times 10}\). And the dataset \(X = \sqrt{n}U\Sigma V^T\), which guarantees that the matrix \(A = \frac{1}{n}X^TX\) has eigen-gap 0.1.

5. Open problems

In this paper, we find the rate of convergence of a famous online scheme, Krassulina scheme. However, we can see that the rate of convergence of online scheme is much slower than offline setting, which can achieve the rate of convergence at \(O(\frac{1}{n})\). An open question is: whether we can achieve offline rate of convergence with a online setting. On the other hand, we prove the rate of convergence in Krassulina scheme, and it seems like we cannot improve this result anymore. its interesting to prove the same type result in other schemes, such as Oja scheme and naive PCA.

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