On the Continual Theory of Flexoelectric Deformations

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Abstract

In many cases the correct theoretical description of flexoelectricity requires the consideration of the finite size of a body and is reduced to the solution of boundary problems for partial differential equations. Generally speaking, in this case one should solve jointly the equations of polarization equilibrium and equations of elastic equilibrium. However, due to the fact that typically flexoelectric moduli are very small, usually one can consider the solution of polarization equilibrium equations at a given elastic strain (direct flexoelectric effect) or the solution of elastic equilibrium equations at given polarization (converse flexoelectric effect). Derivation of the polarization equilibrium equations and boundary conditions for them can be made in the quite usual way. Solution of these equations usually is not too difficult problem. On the contrary description of converse flexoelectric effect is more complicated problem. Inter alia effective solution of corresponded boundary problems requires the development of special mathematical methods. The subject of this paper is a detailed discussion of the relevant theory focuses on the converse flexoelectric effect. It is also considered some particular examples illustrating the application of the general theory.

1 Introduction

A body in which flexoelectrical phenomena are observed always has a finite size. In some cases the finite sample size is not significant, and the sample
can be described as an infinite medium. Examples are the influence of the flexoelectric effect on the dispersion of acoustic phonons \[1\] or its influence on structure of ferroelectric domain wall \[2\]. In other cases, such as flexoelectric bending of a plate \[3\], finite size of a sample plays a fundamental role.

Obviously, the effects associated with the finite size of the sample caused by the presence of the sample boundary. In particular, this raises the surface piezoelectricity which, as it turns out, leads to the effects of the same order as the flexoelectric effect \[4\].

Less trivial fact is that even without surface piezoelectricity, in the presence of flexoelectricity classical theory of elasticity should be modified. In this case it turns out that, generally speaking, one should use elastic boundary conditions of non-classical form. These boundary conditions are consistent only if one considers the spatial dispersion of the elastic moduli (contribution to the thermodynamic potential which is a bilinear in elastic strain gradients) \[5\]. The latter leads to that not only the boundary conditions, but also differential equations of elastic equilibrium should be non-classical.

Equations of elastic equilibrium are needed to describe the converse flexoelectric effect (mechanical response to the polarization). As regards the direct flexoelectric effect (polarization response to an inhomogeneous elastic strain), to describe it the equations of polarization equilibrium are needed. It is essential that this equations and boundary conditions for them are modified by flexoelectricity not so radically as the equations and boundary conditions describing the elastic response (converse flexoelectric effect). This is related to the fact that the flexoelectric contributions to the thermodynamic potential contains only the first spatial derivatives of the polarization, while it contains the second spatial derivatives of the elastic displacements. Equations of polarization equilibrium and boundary conditions for them in the presence of flexoelectricity were derived in paper \[6\].

Thus, non-trivial features of flexoelectricity in finite samples reveal themselves mainly in the converse flexoelectric effect. This is why we do not discuss much equations of polarization equilibrium, focusing mainly on the converse flexoelectric effect at a given polarization. Moreover, in the particular examples (but not in the general theory) further we assume a homogeneous polarization at which non-trivial features of flexoelectric effect in finite samples reveal itself most dramatically.

The case of homogeneous polarization is interesting in the sense that a naive analysis based on constitutive equations leads to the (erroneous) conclusion that the homogeneous polarization does not cause a flexoelectric
deformation of a body. The consequence of this conclusion is the ability to create a sensor which not behave as an actuator [7, 8]. Such an ability contradicts the general principles of thermodynamics. But in reality violations of the principles of thermodynamics is not happening, homogeneous polarization does lead to a flexoelectric deformation of the body [4, 5].

In the paper [4] the conclusion that homogeneous polarization leads to a flexoelectric deformation of a thin plate was made on the basis of the particular ansatz for elastic displacement distribution over the sample and direct minimization of the thermodynamic potential. A more rigorous analysis of this problem requires the solution of differential equations of elastic equilibrium with appropriate boundary conditions. Such a solution for the case of homogeneously polarized ball was presented in the paper [9]. As was expected the solution confirmed the presence of flexoelectric strains caused by homogeneous polarization.

It is worth mentioning that in the presence of flexoelectricity the exact (within the framework of a continuum media theory) equations of elastic equilibrium are too complex to be used for a case differs from a simple case of a ball. Even for a ball the solution is very cumbersome and requires an introduction of non-standard functions presented by a series in powers of radial coordinate. So that the development of approximate method is worth. Such a method was developed in the paper [10].

Papers on the subject [9, 10] are too brief. More detailed description is presented here. General aspects of derivation of the equations of elastic equilibrium and the boundary conditions for them in the case when the thermodynamic potential depends on higher derivatives discussed in another paper [12]. Here we apply these results to the case of flexoelectricity and develop the theory further. Besides, it is presented a comparison of exact solution [9] and the solution of the same problem within the approximate theory [10]. Some additional examples of approximate theory application are presented also.
2 Thermodynamic potential, differential equations of elastic equilibrium, and boundary condition

It is well known [11] that within the theory of continuum flexoelectricity it is described by a contribution to thermodynamic potential density:

\[ H_{flx} = -f^{(1)}_{klij} P_k u_{i,j,l} - f^{(2)}_{klij} P_{k,l} u_{i,j}. \] (1)

Here \( P_k \) is polarization, \( u_i \) is elastic displacement, \( f^{(1)}_{klij} \) and \( f^{(2)}_{klij} \) are material tensors, \( \ldots, i = \partial (\ldots) / \partial x_i \), and Einstein summation rule is adopted. Note that usually strain tensor \( u_{ij} = (u_{i,j} + u_{j,i}) / 2 \) is used instead of displacement gradients \( u_{i,j} \) (distortion tensor). But since \( u_{i,j} \) is convolved with material tensors which are symmetrical in corresponding indices, it does not matter to write down \( u_{i,j} \) or \( u_{ij} \). Note also that here it is used definition of \( f^{(a)}_{klij} \) slightly different from that adopted in [11], pairs of indices \( ij \) and \( kl \) are swapped. This definition corresponds to [3, 9, 10] and more widely accepted. Certainly the opposite definition is also applicable.

Thermodynamic potential of a body \( H \) is integral over body volume \( V \) from thermodynamic potential density. Using integration by parts its flexoelectric contribution can be represented in the form:

\[ H_{flx} = \frac{f_{klij}}{2} \int (P_{k,l} u_{i,j} - P_k u_{i,j,l}) dV + \oint d_{ijk} u_{i,j} P_k dS, \] (2)

where \( f_{klij} = f^{(1)}_{klij} - f^{(2)}_{klij} \) is so-called flexocoupling tensor, \( d_{ijk} \) is surface piezoelectric tensor of special form:

\[ d_{ijk} = -\frac{1}{2} (f^{(1)}_{klij} + f^{(2)}_{klij}) n_l. \] (3)

Hereinafter \( n_l \) is unit vector normal to body surface \( S \). Note that \( d_{ijk} \) is different in different points of the surface and it is symmetric in indices \( i, j \).

Thus flexoelectric part of thermodynamic potential density [11] can be reduced to the Lifshitz-type form

\[ H_{flx} = \frac{1}{2} f_{klij} (P_{k,l} u_{i,j} - P_k u_{i,j,l}) \] (4)

together with additional surface piezoelectricity described by piezoelectric tensor [3]. Further effects of surface piezoelectricity are not described so that Lifshitz-type form [11] is used. 

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Certainly besides flexoelectric potential one should take into account elastic energy. It will be clear from what follows that besides conventional elastic energy \( c_{ijkl}u_{ij}u_{kl} / 2 \) higher elasticity term \( v_{ijklm}u_{ij,n}u_{k,l,m} / 2 \) should be added. So that total volume density of the thermodynamic potential \( \mathcal{H}_B \) is
\[
\mathcal{H}_B = \frac{1}{2} \left[ c_{ijkl}u_{ij}u_{kl} + v_{ijklm}u_{ij,n}u_{k,l,m} + f_{klj} \left( P_{k,l}u_{ij} - P_ku_{ij,l} \right) \right].
\] (5)
Possible surface contribution to the thermodynamic potential is not considered here. So that volume integral from (5) is total thermodynamic potential of a body.

Starting from the equation (5) it is possible to write down the equations of elastic equilibrium and the boundary conditions for them. In fact, one just need to calculate the tensors
\[
T_{ij} = \frac{\partial \mathcal{H}_B}{\partial u_{ij}} = c_{ijkl}u_{kj} + \frac{1}{2} f_{klj} P_{k,l},
\] (6)
\[
\Theta_{ijk} = \frac{\partial \mathcal{H}_B}{\partial u_{ijk}} = v_{ijnlm}u_{n,l,m} - \frac{1}{2} f_{nkij} P_n,
\] (7)
and use the results of [12]. In this way it turns out that “physical stress” \( \sigma_{ij} \) which obey the equations of equilibrium
\[
\sigma_{ij,j} = 0
\] (8)
are determined by equation:
\[
\sigma_{ij} = T_{ij} - \Theta_{ijk,k} = c_{ijkl}u_{kj} - v_{ijnlm}u_{n,l,m,k} + f_{klj} P_{k,l}.
\] (9)
Boundary conditions for (8) can be written in different equivalent forms, one should only substitute (6) and (7) to equations from [12]. Particularly they can be written in the form which coincides with that presented in [5]:
\[
\Theta_{ijk}n_jn_k |_{S} = 0,
\] (10)
\[
\sigma_{ij}n_j - \Theta_{ijk}n_k + \Theta_{ijk,m}n_jn_k - \Theta_{ijk}\gamma_{jk}|_{S} = 0.
\] (11)
Naturally terms contained \( \Sigma_{ij} \) and \( F_i \) do not appear because surface contributions to thermodynamic potential is not described.

It is obvious from (10) and (7) that if \( v_{ijklm} \to 0 \) and \( f_{ijkl}P_{i}|_{S} \neq 0 \) then generally \( u_{i,j,k} \to \infty \). So that the limit \( v_{ijklm} \to 0 \) is singular. This fact
explains the above statement that generally in the presence of flexoelectricity one should take into account the higher elasticity (spatial dispersion of elastic moduli). Higher elasticity term can be omitted only in very special (and physically doubtful) case when \(P_i = 0\) on the surface of a body. In this special case one can use classical boundary conditions, additional terms in them disappear.

Regarding the description of the direct flexoelectric effect, to obtain the corresponding differential equations and boundary conditions, one should add to (4) the following expression:

\[
\mathcal{H}_P = \frac{1}{2}g_{ijkl}P_{i,j}P_{k,l} - \frac{1}{2}a_{ij}P_iP_j - E_iP_i, \tag{12}
\]

where \(E_i\) is electric field. If surface contributions are not considered then it remains only to vary such thermodynamic potential in \(P_i\) and to obtain by standard way the following differential equations

\[
a_{lk}P_l - g_{ijkl}P_{i,j,l} - f_{klij}u_{i,j,l} - E_k = 0 \tag{13}
\]

and boundary conditions for them

\[
g_{ijkl}P_{i,j,n_l} + \frac{1}{2}f_{klij}u_{i,j,n_l} \bigg|_S = 0. \tag{14}
\]

It does not constitute a serious problem to take into account the surface contributions here. Usually such contributions contain only the polarization \(P_i\) but not polarization gradients \(P_{i,j}\). This is why to find the change of the equation written above is a simple exercise in this case, in which we are not resting here.

### 3 Exact solution for homogeneously polarized ball of isotropic dielectric

When solving a particular problem it is convenient to use covariant formalism (see [12]). To transform equations from Cartesian form to covariant form one should only change partial derivatives to covariant derivatives and provide an index balance (only superscript can be convolved with subscript). So that equations of mechanical equilibrium and boundary conditions for them take the form:

\[
\sigma^{\alpha_\beta} = 0, \tag{15}
\]
Here \( n_\alpha \) is covariant representation of unit vector normal to body surface \( S \), tensors \( \sigma^{\alpha\beta} \) and \( \Theta^{\alpha\beta\gamma} \) are determined as follows:

\[
\sigma^{\alpha\beta} = c^{\alpha\beta\gamma\delta} u_\gamma u_\delta - v^{\alpha\beta\gamma\delta\epsilon\zeta} u_\gamma u_\delta u_\epsilon u_\zeta + \frac{1}{2} f^{\delta\gamma\alpha\beta} P_\gamma P_\delta ,
\]

\[
\Theta^{\alpha\beta\gamma} = v^{\alpha\beta\epsilon\delta\gamma\zeta} u_\epsilon u_\delta u_\gamma - \frac{1}{2} f^{\delta\gamma\alpha\beta} P_\delta .
\]

It is also convenient to use curvilinear coordinate system of special class in which equation of body surface has the form \( x^3 = \text{const} \). In this coordinate system tensor \( \gamma^{\alpha\beta} \) can be presented in terms of Cristoffel symbols [12]:

\[
\gamma_{\beta\gamma} = \Gamma_{\alpha\beta\gamma}^\epsilon n_\beta n_\gamma \pm \Gamma_{\beta\gamma\alpha}^\epsilon n_\beta n_\gamma - \Gamma_{\gamma\beta\alpha}^\epsilon n_\delta .
\]

Besides \( n_\alpha \) has the only component:

\[
n_3 = \frac{1}{\sqrt{g^{33}}} .
\]

For a case of a ball it is naturally to use spherical coordinate system:

\[
\begin{align*}
x &= r \sin \theta \cos \psi , \\
y &= r \sin \theta \sin \psi , \\
z &= r \cos \theta .
\end{align*}
\]

Here \( x, y \) and \( z \) are Cartesian coordinates, \( \psi = x^1, \theta = x^2 \), and \( r = x^3 \) are curvilinear ones. Equation of the ball surface has the form \( r = R \), so that these curvilinear coordinates belong to the special class mentioned above.

Instead of curvilinear indices 1, 2, and 3 further it is used also the indices \( \psi, \theta \), and \( r \) respectively. So the letters \( r, \psi \) and \( \theta \) are excluded from notation of “running” indexes. Particular Cartesian indices are further denoted as \( x, y \) and \( z \).

Metric tensor components \( g_{\alpha\beta} \), its determinant \( g \), and Cristoffel symbols \( \Gamma^\alpha_{\beta\gamma} \) can be derived directly from \( \{ 22 \} \):

\[
\begin{align*}
g_{\psi\psi} &= r^2 \sin^2 \theta , & g_{\theta\theta} &= r^2 , & g_{rr} &= 1 , & g &= r^4 \sin^2 \theta , \\
\Gamma^\theta_{\psi\psi} &= \Gamma^\theta_{\theta\psi} = \cot \theta , & \Gamma^\psi_{\theta\theta} &= \Gamma^\psi_{r\psi} = r^{-1} , & \Gamma^\psi_{\psi\psi} &= - \sin \theta \cos \theta , \\
\Gamma^\theta_{\theta\theta} &= \Gamma^\theta_{r\theta} = r^{-1} , & \Gamma^\psi_{\theta\psi} &= - r \sin^2 \theta , & \Gamma^\theta_{\psi\psi} &= - r .
\end{align*}
\]
Other components are zero. Since tensor $g_{\alpha\beta}$ is diagonal, inverse tensor $g^{\alpha\beta}$ is diagonal also and corresponding components are simply equal to the inverse diagonal components of $g_{\alpha\beta}$. Unit vector normal to the surface has the only component $n_r = 1$.

Further one should specify the form of the material tensors for isotropic media. It is well-known that elastic tensor of such a media is defined by two parameters, say its components $c_{12}$ and $c_{44}$ (Voigt notation is used), and can be expressed as follows:

$$c_{ijkl} = c_{12}\delta_{ij}\delta_{kl} + c_{44}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

(24)

It is possible to show that flexocoupling tensor of isotropic media can be expressed analogously:

$$f_{ijkl} = f_{12}\delta_{ij}\delta_{kl} + f_{44}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

(25)

For six-rank tensor of higher elastic moduli of isotropic media further it is used a simplified expression

$$v_{ijklmn} = v_1(\delta_{ij}\delta_{lm}\delta_{nk} + \delta_{ij}\delta_{nl}\delta_{mk} + \delta_{ij}\delta_{nm}\delta_{lk} + \delta_{ik}\delta_{jm}\delta_{lm} + \delta_{il}\delta_{jn}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{im}\delta_{jl}\delta_{kn} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{in}\delta_{jm}\delta_{lk} + \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jk}\delta_{ln} + \delta_{im}\delta_{jl}\delta_{nk} + \delta_{in}\delta_{jl}\delta_{mk} + \delta_{in}\delta_{jm}\delta_{lk} + \delta_{in}\delta_{jm}\delta_{lk}) +$$

$$v_2(\delta_{ik}\delta_{jl}\delta_{nm} + \delta_{ik}\delta_{jm}\delta_{nl} + \delta_{il}\delta_{jk}\delta_{nm} + \delta_{il}\delta_{jm}\delta_{nk} + \delta_{im}\delta_{jk}\delta_{nl} + \delta_{im}\delta_{jl}\delta_{nk}),$$

(26)

where $v_1$ and $v_2$ are constant parameters.

Above the expressions of material tensors are presented in Cartesian components. To convert them to curvilinear components one should only change Kronecker delta to metric tensor with corresponding location (up or bottom) of indices.

If the ball is polarized along $z$-axis with polarization $P$ then polarization vector has two covariant components:

$$P_r = P\cos\theta,$$

(27)

$$P_\theta = -rP\sin\theta.$$  

(28)

The equations above mathematically completely determine the problem under study. To solve this problem one must first separate the variables. Generally this is done by decomposition of the vector field $u_\alpha$ in a series of spherical vectors. Note that since the covariant formalism is used here, spherical vectors do not coincide with their usual form [13]. However, the required
form of these vectors can easily be obtained from conventional. Covariant spherical vectors can be defined as follows

$$
\begin{align*}
Y^{(1)}_{lm|\theta} &= \frac{r}{\sqrt{l(l+1)}} (Y_{lm})_{,\theta}, \\
Y^{(1)}_{lm|\psi} &= \frac{r}{\sqrt{l(l+1)}} i m Y_{lm}, \\
Y^{(2)}_{lm|\theta} &= \frac{r}{\sqrt{l(l+1)}} \cdot \frac{-i m}{\sin \theta} Y_{lm}, \\
Y^{(2)}_{lm|\psi} &= \frac{r}{\sqrt{l(l+1)}} \sin \theta (Y_{lm})_{,\theta}, \\
Y^{(3)}_{lm|r} &= Y_{lm}, \\
Y^{(3)}_{lm|r} &= 0, \\
Y^{(3)}_{lm|\theta} &= 0, \\
Y^{(3)}_{lm|\psi} &= 0,
\end{align*}
$$

(29)

where $Y_{lm}$ is scalar spherical functions expressed in the usual manner in terms of associated Legendre polynomials.

In the case considered here, i.e. when the polarization is homogeneous, expansion in spherical vectors is simplified radically. Indeed, it is easy to see that in such expansion of the polarization field $P_{\alpha}$ there are only the terms with $l = 1$, $m = 0$ and top indices 1,3. Since the theory is linear, it follows that the expansion of the field $u_{\alpha}$ contains only the similar terms. As a result, this leads to the fact that $u_{r} \sim \cos \theta$, $u_{\theta} \sim \sin \theta$, $u_{\psi} = 0$. This is why one can use a substitution:

$$
\begin{align*}
&u_{\psi} = 0, \\
&u_{\theta} = -rf_{2}(r) \sin \theta \\
&u_{r} = f_{1}(r) \cos \theta.
\end{align*}
$$

(30) (31) (32)

Factor $r$ added into (31) to make radial function $f_{2}(r)$ analytical at $r = 0$. The consequence of using the covariant formalism is that $u_{\theta}$ is not a physical displacement in the meridional direction, such physical displacement is $u_{\theta} \sqrt{g^{\theta\theta}}$, while $g^{\theta\theta} = r^{-2}$. Certainly the physical component of the displacement should not have a singularity at the ball center. Note that covariant form of $Y^{(1)}_{lm|\theta}$ contains the same factor $r$.

Thus, the problem is reduced to finding the radial functions $f_{1}(r)$ and $f_{2}(r)$. To find the corresponding differential equations and boundary conditions one should substitute (30)–(32) in the above equations and make a very cumbersome but absolutely straightforward algebraic transformations.
As the result it yields the system of two differential equations

\[ v_3(\xi^4 f_1'''' + 4\xi^3 f_1''' - 8\xi^2 f_1'' + 16 f_1) + v_3(8\xi^2 f_2'' - 16 f_2) + \]

\[ v_4(\xi^4 f_1'''' + 4\xi^3 f_1''' - 6\xi^2 f_1'' + 12 f_1) - \]

\[ v_4(2\xi^3 f_2''' - 2\xi^2 f_2'' - 4\xi f_2' + 12 f_2) = \] (33)

\[ c_{44}^2(2\xi^2 f_1'' + 4\xi f_1' - 6 f_1) - c_{44}^2(2\xi f_1' - 6 f_2) + \]

\[ c_{12}^2(\xi^2 f_1'' + 2\xi f_1' - 2 f_1) - c_{12}^2(2\xi f_2' - 2 f_2), \]

\[ v_3(\xi^4 f_2'''' + 4\xi^3 f_2''' - 4\xi^2 f_2'' + 8 f_2) + v_3(4\xi^2 f_1'' - 8 f_1) - \]

\[ v_4(2\xi^2 f_2'' - 4 f_2) + v_4(\xi^3 f_1''' + 4\xi^2 f_1'' - 2\xi f_1' - 4 f_1) = \] (34)

\[ c_{44}^2(\xi^2 f_2'' + 2\xi f_2' - 4 f_2) + c_{44}^2(\xi f_1' + 4 f_1) - \]

\[ c_{12}^2(4 f_2 + c_{12}^2(\xi f_1' + 2 f_1), \]

and four boundary condition for it

\[ 2v_1(f_2''' + f_2'' - 18 f_2' + 34 f_2 + 8 f_1'' + 14 f_1' - 34 f_1) + \]

\[ + 2v_2(2f_2''' + 2f_2'' - 20 f_2' + 36 f_2 + 4 f_1'' + 16 f_1' - 36 f_1) - \] (35)

\[ -2c_{44}R^2(f_2' - f_2 + f_1)\Big|_{\xi=1} = f_{12}R^2 P, \]

\[ R^2[c_{12}(2f_1 - 2f_2) + (c_{12} + 2c_{44})f_1'] - \]

\[ v_1(9f_1''' + 18 f_1'' - 46 f_1' + 36 f_1 - 14 f_2'' + 32 f_2' - 36 f_2) - \] (36)

\[ 2v_2(3f_1''' + 6f_1'' - 22 f_1' + 28 f_1 - 2f_2'' + 16 f_2' - 28 f_2)|_{\xi=1} = R^2 f_{12} P, \]

\[ (18v_1 + 12v_2)f_1'' + 12v_1(3f_1' - 4 f_1 - 2 f_2' + 4 f_2)|_{\xi=1} = \]

\[ (f_{12} + 2 f_{44})R^2 P, \]
\begin{align*}
2v_1(f''_2 + 2f'_2 - 6f_2 + 2f'_1 + 6f_1) + \\
4v_2(f''_2 - 2f'_2 + 2f_2 + 2f'_1 - 2f_1)|_{\xi=1} = f_{44}R^2 P.
\end{align*}

(38)

Here \(\xi = r/R\) is dimensionless radial coordinate, prime denotes the derivative in \(\xi\), \(v_3 = (v_1 + 2v_2)R^{-2}\), \(v_4 = (8v_1 + 4v_2)R^{-2}\).

In general, the system of two fourth-order equations (33) — (34) requires eight boundary conditions, but here we have only four boundary conditions (35) — (38). However, as is usually in the case of such problems, the missing boundary conditions are replaced by the conditions that a solution should be analytic at zero. Therefore, the solution should be expressed in the form of series in non-negative powers of \(\xi\).

Practically it is more convenient to find the complete basis set of solutions expressed as such series, and to present a general solution as a linear combination of such basis functions. If the number of basis functions is equal to the number of boundary conditions then the coefficients in the linear combination can be easily founded by solving a system of linear algebraic equations.

In accordance with the above, we express a solution in the form:

\[ f_i(\xi) = \sum_{k=1}^{4} C_k \mathcal{B}_{ki}(\xi), \]

(39)

\[ \mathcal{B}_{ki}(\xi) = \sum_{n=0}^{\infty} a_{kin}\xi^n. \]

(40)

The constant coefficients \(a_{kin}\) are found from the condition that \(\mathcal{B}_{ki}(\xi)\) obeys the system of differential equations (33) - (34). This condition leads to the following linear algebraic equations for the coefficients (index \(k\) which is numbering the solutions is omitted):

\[
\begin{cases}
& v_3(n^4 - 2n^3 - 9n^2 + 10n + 16) + \\
& v_4(n^4 - 2n^3 - 7n^2 + 8n + 12)]a_{1n} + \\
& + [v_3(8n^2 - 8n - 16) - v_4(2n^3 - 8n^2 + 2n + 12)]a_{2n} = \\
& [c_{44}(2n^2 - 6n - 2) + c_{12}(n^2 - 3n)]a_{1n-2} - \\
& [c_{44}(2n - 10) + c_{12}(2n - 6)]a_{2n-2},
\end{cases}
\]

(41)

\[
\begin{cases}
& v_3(4n^2 - 4n - 8) + v_4(n^3 + n^2 - 4n - 4)]a_{1n} + \\
& [v_3(n^4 - 2n^3 - 5n^2 + 6n + 8) - v_4(2n^2 - 2n - 4)]a_{2n} = \\
& [c_{44}(n + 2) + c_{12n}]a_{1n-2} + [c_{44}(n^2 - 3n - 2) - c_{12}2]a_{2n-2}.
\end{cases}
\]
From (41) one can see that the system of linear algebraic equations, even though it is infinite, has the characteristic structure: the coefficients with greater \( n \) are expressed in terms of the coefficients with smaller \( n \). Therefore, one can find \( a_{in} \) for any finite \( n \) step by step, one needs only to analyse the case of small \( n \) and define some of the \( a_{in} \) for these small \( n \).

Naturally for small \( n \) system of equations (41) is degenerate; some \( a_{in} \) with small \( n \) can be defined arbitrarily. This arbitrariness leads to that it may be several functions \( B_{ki}(\xi) \) and their normalization may be arbitrary. Detailed analysis shows that the system of equations (41) is degenerate for \( n = 1, 2, 4 \) only. It turns out five arbitrary constants, and so it is the same number of basis functions. However, one of this functions is physically meaningless and should be omitted (more on this see below).

From an abstract point of view, arbitrary constants can be defined as anything, it is just needed that obtained basis functions be linearly independent. However, there is an important question: is it possible to choose these constants in such manner that the series representing at least some of the basic functions cut short, would constitute a finite sums? It turns out that this choice of constants is really possible.

From the system of equations (41) it is clear that in order to series cut short, it should be \( a_{3i} = a_{4i} = 0 \). For this to be possible, in any case, right-hand sides of (41) should be zero for \( n = 3, 4 \). Together with (41) for \( n = 0, 1, 2 \) these conditions lead to a system of linear algebraic equations which is quite amenable to analysis. The result is that there may be two basis functions which are finite sums.

One of them is determined by the conditions \( a_{10} = a_{20} = \text{const} \neq 0 \), \( a_{n>0} = 0 \). This function corresponds to the displacement of the ball as a whole. Such a displacement is physically meaningless and this basis function should be omitted. Certainly such a displacement does not change the thermodynamic potential, so that this function indeed is a solution of the equations derived from a stationary condition of this potential. But it is meaningless solution.

Another basis function, which is finite sum, is determined by condition \((c_{12} - c_{44})a_{22} = (3c_{44} + 2c_{12})a_{12} \), other coefficients are zero. So that first basis function is determined as follows:

\[
\begin{align*}
B_1 : \\
& a_{12} = 1, \\
& a_{22} = \frac{3c_{44} + 2c_{12}}{c_{12} - c_{44}}a_{12},
\end{align*}
\] (42)
other $a_{in}$ are zero.

It should be stressed that the function $B_1$ obeys not only the equations (33) — (34) but also the equations of the classical theory of elasticity, which can be obtained from (33) — (34) by setting $v_3 = v_4 = 0$. It can be verified by direct substitution.

Further analysis shows that there are three additional basic functions $B_2$ — $B_4$ which are expressed by infinite series and can be chosen, for example, as follows.

$$
\begin{align*}
\mathcal{B}_2 & : \\
& \begin{cases}
    a_{12} = 1, \\
    a_{22} = \frac{3c_{44} + 2c_{12}}{c_{44} - c_{12}}a_{12}, \\
    a_{24} = 0,
\end{cases} \\
& (40v_3 + 60v_4)a_{14} = \\
& = (6c_{44} + 4c_{12})a_{12} + (2c_{44} - 2c_{12})a_{22}, \\
& a_{i0} = a_{i1} = a_{i3} = 0 \\
\end{align*}
$$

coefficients with $n \geq 5$ are calculated by means of (41).

$$
\begin{align*}
\mathcal{B}_3 & : \\
& \begin{cases}
    a_{11} = 1, \\
    a_{21} = \frac{4v_3 + 3v_4}{4v_3 + 2v_4}a_{11}, \\
    a_{i0} = a_{i2} = a_{i4} = 0,
\end{cases} \\
& \quad \quad (44)
\end{align*}
$$

coefficients with $n = 3$ and $n \geq 5$ are calculated by means of (41).

$$
\begin{align*}
\mathcal{B}_4 & : \\
& \begin{cases}
    a_{14} = 1, \\
    a_{24} = \frac{2v_3 + 3v_4}{v_4 - 4v_3}a_{14}, \\
    a_{i0} = a_{i1} = a_{i2} = a_{i3} = 0,
\end{cases} \\
& \quad \quad (45)
\end{align*}
$$

coefficients with $n \geq 5$ are calculated by means of (41).

Graphs of the basis functions $\mathcal{B}_{ik}$ defined above for some set of parameters are shown in Fig[1] Turn our attention that in contrast with $B_1$ the functions $B_2$ — $B_4$ are concentrated near the surface and decay rapidly in the ball volume.

Above there is explicit representation of four basic functions $B_1$ — $B_4$. It only remains to substitute (39) in the boundary conditions (35) — (38) and from the resulting system of four linear algebraic equations to find numerically $C_k$. Thus, the problem is solved. Result of numerical calculations for some parameters is presented in the Fig[2].
Figure 1: Basis functions for \( R = 1 \cdot 10^{-5}, P = 1, c_{44} = 1.1 \cdot 10^{12}, c_{12} = 3.4 \cdot 10^{12}, f_{44} = f_{12} = 1 \cdot 10^{-3}, v_1 = 0.2, v_2 = 0.1 \).

Note that the graphs shows that, except for a thin layer near \( \xi = 1 \), the curves are parabolic, the more so the less \( v_1 \) and \( v_2 \) are. In the bulk of the ball the only function \( \mathbf{B}_{1i}(\xi) \) remains with a good approximation. Thus, for a small \( v_i \), except for a thin surface layer, solution approximately obeys the equations of classical elasticity theory (see discussion of function \( \mathbf{B}_1 \) properties above). This observation is used in the next section to construct an approximate method of flexoelectric deformations calculations. This method is needed to describe more complicated geometry then spherical geometry of a ball.
4 Approximate method to calculate flexoelectric deformations of finite size bodies

Exact (within the framework of a continuum media theory) equations of elastic equilibrium used in previous section are partial differential equations of the fourth order. They are too complex to be used in the cases interesting for applications. Even for a homogeneously polarized ball of isotropic media solution of them is cumbersome and requires a functions expressed by a series in powers of radial coordinate. So that to find flexoelectric deformation in more complicated cases one needs an approximate method. Such a method was proposed in [10] and it is described in this section.

When solving the problem for a ball (see previous section) we have seen that the solution is a sum of two parts. The first part corresponded basis function $B_1$ obeys the equations of the classical theory of elasticity, the second (contribution of remaining basis functions) is a non-classical part which, for small $\varepsilon$, is concentrated near the surface and rapidly decays inside the body. Hereafter the first part is called a volume or a classical part, and the
second one — a non-classical or a surface part.

It is naturally to assume that this property should be preserved in more general cases. The aim is to use this observation for construction of approximation that is useful for solution of more general problems than the case of a ball.

By means of (18) equations (15) can be represent in following form

\[ c^{\alpha\beta\gamma\delta} u_{\gamma;i;\beta} + f^{\gamma\delta\alpha\beta} P_{\gamma;\delta;\beta} - v^{\alpha\beta\gamma\delta\varepsilon\zeta} u_{\gamma;\delta;\varepsilon;\beta} = 0. \] (46)

Boundary conditions for this differential equations here are used in the form different from used in previous section. Particularly they are

\[ \Theta^{\alpha33} = 0, \] (47)
\[ \sigma^{\alpha3} - \Theta^{\alpha33}_{;\beta} + \Theta^{\alpha33}_{;3} + \Theta^{\alpha(\beta\gamma)} \Gamma^{3}_{(\beta\gamma)} = 0. \] (48)

Here the indices enclosed in parentheses runs only the values 1 and 2, if there are multiple indexes in one pair of parentheses then these indices are not equal to 3 simultaneously. Equivalence of this form of boundary conditions to (17) is proved in [12]. Certainly it is assumed coordinate system where equation of the body surface has a form \( x^3 = x^3_S, \) \( x^3_S \) is some constant.

According to the observation mentioned above the solution of the equation (46) is presented in the form \( u_\gamma = \tilde{u}_\gamma + \hat{u}_\gamma, \) where \( \tilde{u}_\gamma \) obey the classical equations:

\[ c^{\alpha\beta\gamma\delta} \tilde{u}_{\gamma;i;\beta} + f^{\gamma\delta\alpha\beta} P_{\gamma;\delta;\beta} = 0. \] (49)

From (49) and (46) it follows that \( \hat{u}_\gamma \) obey differential equations:

\[ c^{\alpha\beta\gamma\delta} \hat{u}_{\gamma;i;\beta} - v^{\alpha\beta\gamma\delta\varepsilon\zeta} \hat{u}_{\gamma;\delta;\varepsilon;i;\beta} = v^{\alpha\beta\gamma\delta\varepsilon\zeta} \tilde{u}_{\gamma;\delta;\varepsilon;i;\beta}. \] (50)

Substitution \( u_\gamma = \tilde{u}_\gamma + \hat{u}_\gamma \) and some transformations also yield that the boundary conditions (17) and (48) take the form:

\[ v^{\alpha3\varepsilon\delta\zeta} \tilde{u}_{\varepsilon;i;\delta;\zeta} = -\frac{1}{2} f^{\delta3\alpha3} P_{\delta} - v^{\alpha3\varepsilon\delta\zeta} \hat{u}_{\varepsilon;i;\delta;\zeta}, \] (51)
Equations above are exact. In order to turn into approximation one should note that real values of \( v_{\alpha\beta\gamma\delta\varepsilon\zeta} \) are small. It is convenient to assume that all of them are proportional to a scalar \( v \to 0 \). It is assumed also that all \( c_{\alpha\beta\gamma\delta} \) is proportional to a scalar \( c \). If \( v \to 0 \) then the right-hand side of (50) and the second term on the right hand side of (51) can be neglected. Indeed, since \( \hat{u}_\gamma \) obey the classical equation (49), in the case of sufficiently smooth surface and in the absence of polarization gradients tend to infinity in the limit \( v \to 0 \), the classical part can not have large gradients which can compensate smallness of \( v_{\alpha\beta\gamma\delta\varepsilon\zeta} \). As for the non-classical part \( \hat{u}_\gamma \), the situation is different: here such compensation is possible, but only if the derivatives are in \( x^3 \). Moreover, in a thin layer near the surface covariant derivatives in \( x^3 \) can be replaced by the usual derivatives, \( \Gamma \)-terms give only small corrections here. Thus, (50) and (51) can be approximately replaced by

\[
\begin{align*}
\hat{c}_{\alpha\beta\gamma\delta} \hat{u}_{\gamma,3,3} + \hat{f}_{\gamma\delta\alpha} P_{\gamma,\delta} - v^{\alpha\beta\gamma\delta\varepsilon\zeta} \hat{u}_{\gamma,\varepsilon,\delta,\zeta,\varepsilon} - v^{\alpha\beta(\varepsilon)} P_{\delta,\varepsilon,\delta,\zeta,\varepsilon} + \\
\frac{1}{2} f^{\delta\epsilon\alpha(\beta)} P_{\delta,\varepsilon,\gamma,\delta,\epsilon,\varepsilon} - v^{\alpha\beta\gamma\delta\varepsilon\zeta} \hat{u}_{\gamma,\varepsilon,\delta,\zeta,\varepsilon} - \\
\frac{1}{2} f^{\delta(\gamma)\alpha(\beta)} P_{\delta,\gamma,\delta,\gamma,\varepsilon,\varepsilon} - v^{\alpha\beta\gamma\delta\varepsilon\zeta} \hat{u}_{\gamma,\varepsilon,\delta,\zeta,\varepsilon} - \\
v^{\alpha(\beta)} e_{\delta\varepsilon\varepsilon,\delta,\gamma,\delta,\varepsilon,\varepsilon} + v^{\alpha(\beta)} \delta(\gamma) \hat{u}_{\gamma,\varepsilon,\delta,\zeta,\varepsilon} \Gamma_{(\beta)}^{3}(\gamma) - \hat{u}_{\gamma,\epsilon,\delta,\zeta,\varepsilon} \Gamma_{(\beta)}^{3}(\gamma) = 0.
\end{align*}
\]

Equations (55) are a system of three ordinal differential equations, the dependence on \( x^1 \) and \( x^2 \) is parametric here. Moreover, in a thin layer near
the body surface one can assume that the coefficients do not depend on $x^3$, they are approximately equal to the surface values. Solution of such a system is easy. In a standard way one should find a fundamental basis set of solutions in the form $\hat{u}_\gamma = \bar{u}_\gamma e^{\lambda (x^3 - x^3_S)}$. Obviously equations for amplitudes $\bar{u}_\gamma$ are

$$C^{\alpha \gamma 3} \bar{u}_\gamma = \lambda^2 v^{\alpha \gamma 333} \bar{u}_\gamma.$$  

(56)

Equation (56) is a standard generalized eigenvalue problem for symmetric positive definite $3 \times 3$ matrices. Therefore $\lambda_n$ and $\bar{u}_n^\gamma$ are calculable by standard way. If the body bulk corresponds to $x^3 \leq x^3_S$ for definiteness, then $\lambda_n$ equals the positive square root of the $n$-th eigenvalue, and general representation of non-classical part is

$$\hat{u}_\gamma = \sum_{n=1}^{3} a_n \bar{u}_n^\gamma e^{\lambda_n (x^3 - x^3_S)},$$  

(57)

where only the coefficients $a_n$ are unknown. The latter can be found easily by (54) which yields a simple system of linear algebraic equations:

$$\sum_{n=1}^{3} \lambda_n^2 v^{\alpha \gamma 333} \bar{u}_n^\gamma a_n = \frac{1}{2} f^{3 \alpha 3} P_\delta.$$  

(58)

Note that the smallness of $v^{\alpha \gamma 333}$ in (58) is compensated by $\lambda_n^2$, and using (56) this system of equations can be rewritten as follows:

$$\sum_{n=1}^{3} C^{\alpha \gamma 3} \bar{u}_n^\gamma a_n = \frac{1}{2} f^{3 \alpha 3} P_\delta.$$  

(59)

Thus, the non-classical part of elastic displacements is found completely in explicit form.

As it follows from above, to find completely the non-classical part of elastic displacements one needs only the boundary conditions (51). The remaining boundary conditions (52) yield the boundary conditions for the classical equations (49) in this way, one should only substitute known $\hat{u}_\gamma$ to (52) and select the terms of the appropriate order of accuracy.

However, within this derivation of the boundary conditions for the equations (49) a singularity arises requiring additional analysis. This singularity is related to the fact that it appears a subtraction of two close but large terms in which it is impossible to use the approximate expression for $\hat{u}_\gamma$, derived above. In these (and only these) terms it is necessary to consider a
corrections to such $\hat{u}_\gamma$. The appropriate analysis is made below and while $\hat{u}_\gamma$ means the exact non-classical part of elastic displacements rather then approximate one defined by (57).

According to the above, all the terms in (52), where gradients of $\tilde{u}_\gamma$ are convolved with $v^{\alpha\beta\gamma\delta\epsilon\zeta}$, should be omitted for the same reasons as in the case of the equation (50). Further it is convenient to introduce the notation:

$$s^\alpha = -c^{\alpha\beta\gamma\delta} \hat{u}_{\gamma;\delta} + v^{\alpha\beta\gamma\delta\epsilon\zeta} \hat{u}_{\gamma;\delta;\epsilon;\zeta} + v^{\alpha(\beta)\gamma\delta\epsilon\zeta} \hat{u}_{\gamma;\delta;\epsilon;\zeta}(\beta) - v^{\alpha(\beta)\gamma\delta(\epsilon)} \hat{u}_{\epsilon;\delta;\chi} \Gamma^3_{(\beta)(\gamma)}.$$

With this notation the boundary conditions to the classical equations can be written as follows:

$$c^{\alpha\beta\gamma\delta} \tilde{u}_{\gamma;\delta} = s^\alpha - \int^{\delta\alpha\beta} P_{\gamma;\delta} - \frac{1}{2} f^{\delta\alpha(\beta)} P_{\delta;\beta} + \frac{1}{2} f^{\delta(\gamma)\alpha(\beta)} P_{\delta} \Gamma^3_{(\beta)(\gamma)}.$$

Now all the terms to be further simplified are contained in $s^\alpha$. Carrying out simplification, first of all one should leave in the last term only the partial (not covariant) derivatives in $x^3$. The reason for this is the same as above. Next, one should express the remaining covariant derivatives in terms of partial derivatives and $\Gamma$-terms. The exact expression for the third covariant derivative of the vector is very cumbersome. But keeping in mind that these third covariant derivatives are convolved with $v^{\alpha\beta\gamma\delta\epsilon\zeta}$, one can omit all the terms where initial vector is not differentiated at least two times. So it turns out:

$$\hat{u}_{\alpha;\beta;\gamma;\zeta} \approx \hat{u}_{\alpha;\beta;\gamma;\zeta} - \hat{u}_{\delta;\beta;\zeta} \Gamma^\delta_{\alpha\gamma} - \hat{u}_{\delta;\gamma;\zeta} \Gamma^\delta_{\alpha\beta} - \hat{u}_{\delta;\beta;\gamma} \Gamma^\delta_{\alpha\zeta} - \hat{u}_{\delta;\gamma;\zeta} \Gamma^\delta_{\beta\gamma} - \hat{u}_{\alpha;\beta;\gamma} \Gamma^\delta_{\delta\zeta}.$$

(62)
With this equation and simplifications mentioned above we get:

\[ s^\alpha = -c^{\alpha \gamma \delta} \hat{u}_{\gamma \delta} + v^{\alpha \gamma \delta \epsilon \zeta} \hat{u}_{\gamma \delta, \epsilon \zeta} - v^{\alpha \gamma \delta \epsilon \zeta} \hat{u}_{\rho, \delta, \epsilon \zeta} \Gamma^\rho_{\gamma \zeta} - \\
v^{\alpha \gamma \delta \epsilon \zeta} \hat{u}_{\rho, \zeta, \epsilon} \Gamma^\rho_{\gamma \delta} - v^{\alpha \gamma \delta \epsilon \zeta} \hat{u}_{\rho, \delta, \epsilon} \Gamma^\rho_{\gamma \zeta} - \\
v^{\alpha \gamma \delta \epsilon \zeta} \hat{u}_{\gamma, \rho, \epsilon} \Gamma_{\delta \epsilon} - v^{\alpha \gamma \delta \epsilon \zeta} \hat{u}_{\gamma, \delta, \epsilon} \Gamma_{\rho \zeta} + v^{\alpha (\beta) \gamma \delta \epsilon \zeta} \hat{u}_{\gamma, \delta, \epsilon} \Gamma_{\beta (\gamma)} - \\
v^{\alpha (\beta) \gamma \delta \epsilon \zeta} \hat{u}_{\gamma, \rho, \epsilon} \Gamma_{\beta (\gamma)} - v^{\alpha (\beta) \gamma \delta \epsilon \zeta} \hat{u}_{\gamma, \delta, \epsilon} \Gamma_{\rho (\beta)} - \\
v^{\alpha (\beta) \epsilon 3 (\gamma) 3} \hat{u}_{\epsilon, 3, 3} \Gamma^3_{(\beta) (\gamma)} \].

Again, using the fact that to compensate smallness of \( v^{\alpha \beta \gamma \delta \epsilon \zeta} \) one needs at least two derivatives in \( x^3 \), this equation is further simplified:

\[ s^\alpha = -c^{\alpha \gamma \delta} \hat{u}_{\gamma \delta} + v^{\alpha \gamma \delta \epsilon \zeta} \hat{u}_{\gamma \delta, \epsilon \zeta} - v^{\alpha \gamma \delta \epsilon \zeta} \hat{u}_{\rho, 3, 3} \Gamma^\rho_{\gamma \zeta} - \\
v^{\alpha \gamma \delta 33} \hat{u}_{\rho, 3, 3} \Gamma^\rho_{\gamma \delta} - v^{\alpha \gamma \delta 33} \hat{u}_{\rho, 3, 3} \Gamma^\rho_{\gamma \zeta} - v^{\alpha \gamma \delta 33} \hat{u}_{\gamma, 3, 3} \Gamma^3_{\delta \epsilon} - \\
v^{\alpha \gamma \delta 33} \hat{u}_{\gamma, 3, 3} \Gamma^3_{\zeta \epsilon} - v^{\alpha \gamma \delta 33} \hat{u}_{\gamma, 3, 3} \Gamma^3_{\zeta \epsilon} + v^{\alpha (\beta) 333} \hat{u}_{\gamma, 3, 3} \Gamma^3_{\zeta (\beta)} - \\
v^{\alpha (\beta) 333} \hat{u}_{\gamma, 3, 3} \Gamma^3_{\zeta (\beta)} - v^{\alpha (\beta) \epsilon 3 (\gamma) 3} \hat{u}_{\epsilon, 3, 3} \Gamma^3_{(\beta) (\gamma)} \].

It is convenient to introduce yet another notation:

\[ t^{\alpha \rho} = v^{\alpha \gamma \delta 33} \Gamma^\rho_{\gamma \delta} + v^{\alpha \gamma \delta 33} \Gamma^\rho_{\gamma \delta} + v^{\alpha \gamma \delta 33} \Gamma^\rho_{\gamma \zeta} + v^{\alpha \gamma \delta 33} \Gamma^\rho_{\gamma \zeta} + \\
v^{\alpha \gamma \delta 33} \Gamma^3_{\delta \epsilon} + v^{\alpha \gamma \delta 33} \Gamma^3_{\delta \epsilon} + v^{\alpha (\beta) 333} \Gamma^3_{\zeta (\beta)} + \\
v^{\alpha (\beta) 333} \Gamma^3_{\zeta (\beta)} + v^{\alpha (\beta) \rho 333} \Gamma^3_{\zeta (\beta)} \].

Using the symmetry properties of \( v^{\alpha \beta \gamma \delta \epsilon \zeta} \) it can be rewritten as follows:

\[ t^{\alpha \beta} = (v^{\alpha \gamma \delta 33} + 2v^{\alpha \gamma \delta 33}) \Gamma^\beta_{\gamma \delta} + (2v^{\alpha \gamma \delta 33} + v^{\alpha \gamma \delta 34}) \Gamma^\gamma_{\delta \beta} + \\
2v^{\alpha \gamma \delta 33} \Gamma^3_{\gamma (\delta)} + v^{\alpha \gamma \delta 33} \Gamma^3_{\gamma (\delta)} + v^{\alpha (\gamma) 33 (\delta) 3} \Gamma^3_{(\gamma) (\delta)} \].
In this notation the expression for $s^\alpha$ can be written as follows:

$$s^\alpha = \alpha^\gamma_\delta \hat{u}_{\gamma,\delta} + v^{\alpha_\delta_3 \gamma_3 \epsilon} \hat{u}_{\gamma,\delta,\epsilon} + v^{\alpha_\delta_3 \gamma_3 \epsilon} \hat{u}_{\gamma,\delta,\epsilon} +$$

(67)

$$v^{\alpha(\beta)\gamma_3 \eta_3 \epsilon_3} \hat{u}_{\gamma,\eta_3,\epsilon_3,\beta} - h^{\alpha_\beta} \hat{u}_{\beta,\alpha,\gamma}.$$

Now one can proceed to the analysis of the singularity mentioned above. It appears in the first and third term of (67), when all the derivatives are in $x^3$. For all other terms there is no singularity and one can use the expression (57) for $\hat{u}_{\gamma}$. It is also clear that singular terms exactly cancel each other out because if approximate expression (57) is used. This is due to the explicit form of this expression. One might think that these terms should simply be excluded, but in reality the situation is more complicated.

The fact is that $-\alpha_\gamma_3 \hat{u}_{\gamma_3} + v^{\alpha_\gamma_3 \gamma_3 \epsilon} \hat{u}_{\gamma_3,\gamma_3,\epsilon}$ is equal to zero only for the main approximation for $\hat{u}_{\gamma}$. But the corrections to this approximation may result in the fact that this expression becomes finite. Henceforth we denote such corrections as $w_\gamma$ keeping notation $\hat{u}_{\gamma}$ only for the main approximation. In this notation it turns out:

$$s^\alpha = -\alpha^{\gamma_3} \hat{u}_{\gamma_3,\epsilon} + v^{\alpha_\gamma_3 \gamma_3 \epsilon} \hat{u}_{\gamma_3,\gamma_3,\epsilon} - c^{\alpha_\delta_3 \gamma_3 \epsilon} \hat{u}_{\gamma,\delta,\epsilon} +$$

(68)

$$v^{\alpha_\delta_3 \gamma_3 \epsilon} \hat{u}_{\gamma,\delta,\epsilon} + v^{\alpha(\beta)\gamma_3 \eta_3 \epsilon_3} \hat{u}_{\gamma,\eta_3,\epsilon_3,\beta} - h^{\alpha_\beta} \hat{u}_{\beta,\alpha,\gamma}.$$

To compensate the smallness of $v^{\alpha_\beta_3 \gamma_3 \epsilon}$ in the fifth term of this equation there should be at least two differentiations in $x^3$. Thus, after some reductions one can rewrite (68) as follows:

$$s^\alpha = -\alpha^{\gamma_3} \hat{u}_{\gamma_3,\epsilon} + v^{\alpha_\gamma_3 \gamma_3 \epsilon} \hat{u}_{\gamma_3,\gamma_3,\epsilon} - c^{\alpha_\delta_3 \gamma_3 \epsilon} \hat{u}_{\gamma,\delta,\epsilon} +$$

(69)

$$v^{\alpha_\delta_3 \gamma_3 \epsilon} \hat{u}_{\gamma,\delta,\epsilon} + 2(v^{\alpha_\gamma_3 \gamma_3 \epsilon}) \hat{u}_{\gamma_3,\gamma_3,\epsilon} - h^{\alpha_\beta} \hat{u}_{\beta,\alpha,\gamma}.$$

It is clear that in (69) one should use only corrections $w_\gamma$ of order $v^{1/2}$. Only such corrections yield a contribution of the same order as the other terms. To find the corresponding contributions, we should keep in the exact equation

$$c^{\alpha_\gamma_3 \gamma_3 \epsilon}(\hat{u}_{\gamma_3} + w_{\gamma_3})_{\beta,\delta,\epsilon} - v^{\alpha_\beta_3 \gamma_3 \epsilon}(\hat{u}_{\gamma_3} + w_{\gamma_3})_{\beta,\delta,\epsilon} = v^{\alpha_\beta_3 \gamma_3 \epsilon} \hat{u}_{\gamma,\delta,\epsilon}$$

(70)

only the terms of the order $v^{-1/2}$. Indeed if $w_{\gamma_3} \sim v^{1/2}$ then the main terms $w_{\gamma_3,\gamma_3}$ and $v^{\alpha_\gamma_3 \gamma_3 \epsilon} w_{\gamma_3,\gamma_3,\gamma_3}$ have just such an order. Note that although there
are terms $\hat{u}_{\gamma} \cdot 3$ and $v^{\alpha\gamma \cdot 33} \hat{u}_{\gamma} \cdot 3$ in the equation, which are of the order $\sim v^{-1}$, now they cancel each other completely due to the fact that now $\hat{u}_\gamma$ denotes the main approximation. Right hand side of (71) is $\sim v$, so it should be omitted. The terms $c^{\alpha\beta\gamma \delta} w_{\gamma(\delta;\beta)}$ and $v^{\alpha\beta\gamma \delta \varepsilon \zeta} w_{\gamma(\delta;\varepsilon;\gamma;\zeta;\beta)}$ should be omitted by the same reasons. Thus (71) is reduced to

$$-c^{\alpha\gamma \cdot 3} w_{\gamma} + v^{\alpha\gamma \cdot 33} w_{\gamma} = c^{\alpha\beta\gamma \delta} \hat{u}_{\gamma(\delta;\beta)} - v^{\alpha\beta\gamma \delta \varepsilon \zeta} \hat{u}_{\gamma(\delta;\varepsilon;\gamma;\zeta;\beta)}.$$  \hspace{1cm} (71)

Equation (71) requires further simplification: in the right-hand side we should keep only the terms $\sim v^{-1/2}$. For this in the first term there should be one partial derivative in $x^3$, and in the second term there should be three derivatives in $x^3$. This is why we simplify second and fourth covariant derivatives as follows:

$$\hat{u}_{\gamma,\delta;\varepsilon} \approx \hat{u}_{\gamma,\delta} \Gamma_{\varepsilon}^\gamma - \hat{u}_{\gamma,\varepsilon} \Gamma_{\delta}^\gamma - \hat{u}_{\gamma,\delta} \Gamma_{\varepsilon}^\gamma,$$ \hspace{1cm} (72)

$$\hat{u}_{\gamma,\delta;\varepsilon,\gamma} \approx \hat{u}_{\gamma,\delta,\varepsilon} \Gamma_{\gamma}^\gamma - \hat{u}_{\gamma,\gamma,\varepsilon} \Gamma_{\delta}^\gamma - \hat{u}_{\gamma,\gamma,\delta} \Gamma_{\varepsilon}^\gamma.$$ \hspace{1cm} (73)

Eventually the simplified equation (71) takes the form:

$$-c^{\alpha\gamma \cdot 3} w_{\gamma} + v^{\alpha\gamma \cdot 33} w_{\gamma} = (c^{\alpha\gamma \cdot (\beta)} + c^{\alpha \cdot (\beta) \cdot 3}) \hat{u}_{\gamma,(\beta)} - c^{\alpha\beta\gamma \delta} \Gamma_{\delta}^\gamma \hat{u}_{\gamma} - (c^{\alpha\gamma \cdot 3} + c^{\alpha \cdot 3 \cdot \gamma}) \Gamma_{\gamma}^{\varepsilon} \hat{u}_{\varepsilon} - 2(v^{\alpha\gamma \cdot (\beta) \cdot 33} + v^{\alpha \cdot (\beta) \cdot 33} \hat{u}_{\gamma}) + 2(v^{\alpha\gamma \cdot 33} \Gamma_{\gamma}^\gamma + v^{\alpha \cdot 33 \cdot \gamma} \Gamma_{\gamma}^\gamma) \hat{u}_{\gamma,\varepsilon,\gamma} + (4v^{\alpha\gamma \cdot 33} \Gamma_{\delta}^\gamma + v^{\alpha\gamma \cdot 33} \Gamma_{\delta}^\gamma + v^{\alpha \cdot 33 \cdot \delta} \Gamma_{\delta}^\gamma) \hat{u}_{\gamma}.$$ \hspace{1cm} (74)

Taking into account that near the body surface the Christoffel symbols and material tensors can be considered as constants, the equation (74) has a very specific form: all of its terms are differentiated with respect to $x^3$ at least once. Therefore, there exists a particular solution of this equation.
which obeys an equation with one less derivation:

\[-c^{\alpha\beta\gamma\delta}w_{\gamma,3} + v^{\alpha\beta\gamma\delta\delta}w_{\gamma,3,3,3} = (c^{\alpha\gamma\beta} + c^{\beta}(\gamma^3))\hat{u}_{\gamma,3} -
\]

\[c^{\alpha\beta\gamma\delta}\Gamma^{3}_{\delta\beta}\hat{u}_{\gamma} - (c^{\alpha\beta\gamma\delta} + c^{\alpha}(\beta^3))\Gamma^{\varepsilon}_{\gamma\beta}\hat{u}_{\varepsilon} - 2(v^{\alpha\beta\gamma}\beta^{33} +
\]

\[\]

\[v^{\alpha}(\beta^333)\hat{u}_{\gamma,3,3,3} + 2(v^{\alpha\beta\gamma}\beta^{33}\Gamma^{\varepsilon}_{\gamma\beta} + v^{\alpha\beta\gamma33}\Gamma^{\varepsilon}_{\gamma\delta})\hat{u}_{\rho,3,3} +
\]

\[\]

\[\]

\[\]

\[\text{(75)}
\]

Note that in the left-hand side of (75) there is exactly the expression that is needed in (69). The general solution of the inhomogeneous differential equation is the sum of a particular solution of the inhomogeneous equation and the general solution of the homogeneous equation. But the remarkable fact is that any solution of the homogeneous equation decaying away from the surface when substituted into \(-c^{\alpha\beta\gamma\delta}w_{\gamma,3} + v^{\alpha\beta\gamma\delta\delta}w_{\gamma,3,3,3}\) yields zero. So it is quite enough to consider only a particular solution, and one can simply replace \(-c^{\alpha\beta\gamma\delta}w_{\gamma,3} + v^{\alpha\beta\gamma33}w_{\gamma,3,3,3}\) in (69) by right-hand side of (75). Having made the replacement, we obtain \(s^\alpha\) in the form:

\[s^\alpha = -c^{\alpha\gamma\beta}(\delta)\hat{u}_{\gamma,(\delta)} + c^{\alpha\beta\gamma\delta}\hat{u}_{\varepsilon}\Gamma^{\varepsilon}_{\gamma\delta} + (c^{\alpha\gamma\beta}(\delta) + c^{\alpha\beta}(\gamma^3))\hat{u}_{\gamma,(\beta)} -
\]

\[c^{\alpha\beta\gamma\delta}\Gamma^{3}_{\delta\beta}\hat{u}_{\gamma} - (c^{\alpha\beta\gamma\delta} + c^{\alpha}(\beta^3))\Gamma^{\varepsilon}_{\gamma\beta}\hat{u}_{\varepsilon} - 2(v^{\alpha\beta\gamma}\beta^{33} +
\]

\[\]

\[\]

\[\text{(76)}
\]

Elementary transformations lead (76) to a rather simple form:

\[s^\alpha = c^{\alpha\beta}(\gamma^3)\hat{u}_{\gamma,(\beta)} - c^{\alpha\beta\gamma\delta}\Gamma^{3}_{\delta\beta}\hat{u}_{\gamma} - c^{\alpha\beta\gamma3}\Gamma^{\varepsilon}_{\gamma\beta}\hat{u}_{\varepsilon} - h^{\alpha\beta}\hat{u}_{\beta,3,3},
\]

\[\text{(77)}
\]

where \(h^{\alpha\beta}\) is redefined as

\[h^{\alpha\beta} = v^{\alpha\beta\gamma\delta}(\gamma^3)\Gamma^{3}_{\gamma\delta} - v^{\alpha\beta\gamma3}\Gamma^{3}_{\gamma3} - 2v^{\alpha\beta\gamma33}\Gamma^{3}_{\gamma3} - v^{\alpha\gamma\beta}(\varepsilon\delta\delta)\Gamma^{3}_{\delta\varepsilon}.
\]

\[\text{(78)}
\]

This equations together with (61) completely solves the problem considered in this section.

Remember that in used coordinate system unit vector normal to the body surface \(n_\alpha\) has the only component \(n_3 = 1/\sqrt{g_{33}}\). So that with respect to the
classical part of strain, \( \varepsilon^{\alpha\gamma\delta} u_{\gamma\delta} \big|_S \) actually defines the external force density \( \sigma^{\alpha\beta} n_\beta \) on the body surface, one should only multiply it by \( 1/\sqrt{g^{33}} \). This is why differential equations (49) appended by (61) correspond to a standard problem of classical theory of elasticity for a body under external forces on its surface. Methods of solution of such problems are well-known and do not require special consideration.

It should be emphasized that physically there are no external forces on the surface. Forces mentioned above are purely formal in nature and describe the interaction of non-classical and classical parts of deformation.

5 Comparison of exact and approximate solutions for a homogeneously polarized ball

It worth to find flexoelectric deformations of homogeneous polarized ball by approximate method described in previous section and compare this deformations with ones known from exact solution. This is a test of approximate method at least.

For isotropic material the matrices \( c^{\alpha\gamma\delta} \) and \( v^{\alpha\beta\gamma\delta\epsilon} \) are diagonal. So that in order to find \( \hat{u}_\gamma \) one does not even need to solve the generalized eigenvalue problem (56), the result is obtained at once:

\[
\begin{align*}
\hat{u}_r &= \frac{f_{12} + 2f_{44}}{2(c_{12} + 2c_{44})} P \cos \theta e^{\lambda_r R(\xi - 1)}, \\
\hat{u}_\theta &= -R \frac{f_{44}}{2c_{44}} P \sin \theta e^{\lambda_\theta R(\xi - 1)}, \\
\hat{u}_\psi &= 0,
\end{align*}
\tag{79}
\]

where

\[
\lambda_\theta = \sqrt{\frac{c_{44}}{v_1 + 2v_2}}, \tag{80}
\]

\[
\lambda_r = \sqrt{\frac{c_{12} + 2c_{44}}{9v_1 + 6v_2}}. \tag{81}
\]

For comparison with the exact solution it is useful to present \( \hat{u}_\gamma \) in terms of
the functions $\hat{f}_1$ and $\hat{f}_2$:

$$\hat{f}_1 = \frac{\hat{u}_r}{\cos \theta} = \frac{f_{12} + 2f_{44}}{2(c_{12} + 2c_{44})} Pe^{\lambda r R(\xi - 1)},$$  \hspace{1cm} (82)$$

$$\hat{f}_2 = -\frac{\hat{u}_\theta}{r \sin \theta} = \frac{1}{\xi} \cdot \frac{f_{44}}{2c_{44}} Pe^{\lambda \theta R(\xi - 1)}.$$  \hspace{1cm} (83)$$

Next step is to find $\bar{\sigma}^{\alpha_3 n_3} = c^{\alpha_3 \gamma \delta} \bar{u}_{\gamma \delta n_3}$ on the ball surface. To do this, one should just use the equations of the previous section. It turns out

$$\bar{\sigma}^{33 n_3} \big|_S = \frac{P \cos \theta}{R} \left[ \frac{2c_{44}(f_{12} + 2f_{44})}{c_{12} + 2c_{44}} - 2f_{44} \right],$$  \hspace{1cm} (84)$$

$$\bar{\sigma}^{23 n_3} \big|_S = \frac{P \sin \theta}{R^2} \left[ \frac{f_{12}}{2} - \frac{c_{12}(f_{12} + 2f_{44})}{2(c_{12} + 2c_{44})} \right].$$  \hspace{1cm} (85)$$

Certainly $\bar{\sigma}^{13} = 0$. This is obviously from symmetry but can be also obtained by direct calculations.

To solve the differential equations (49) actually is not necessary here. It is clear that in the terms of functions $\hat{f}_1 = \frac{\hat{u}_r}{\cos \theta}$ and $\hat{f}_2 = -\frac{\hat{u}_\theta}{r \sin \theta}$ the solution is proportional to the previously defined function $B_1$. Eventually the validity of this statement can be verified by direct substitution. So that one just needs to find the coefficient of proportionality, it is easily done by using (84) or (85). One can use any of these conditions, they both give the same result:

$$\hat{f}_1 = \frac{P(c_{12}f_{44} - c_{44}f_{12})(c_{12} - c_{44})}{c_{44}(c_{12} + 2c_{44})(3c_{12} + 2c_{44})} \xi^2,$$  \hspace{1cm} (86)$$

$$\hat{f}_2 = \frac{P(c_{12}f_{44} - c_{44}f_{12})(2c_{12} + 3c_{44})}{c_{44}(c_{12} + 2c_{44})(3c_{12} + 2c_{44})} \xi^2.$$  \hspace{1cm} (87)$$

It is also useful to derive the expression for the classical part of the elastic displacement in Cartesian components as a function of the Cartesian coordinates. Direct conversions yield

$$\begin{cases}
\bar{u}_z = a_1 z^2 + a_2 (x^2 + y^2), \\
\bar{u}_y = (a_1 - a_2) y z, \\
\bar{u}_x = (a_1 - a_2) x z,
\end{cases}$$  \hspace{1cm} (88)$$

25
where

\[ a_1 = \frac{P(c_{12}f_{44} - c_{44}f_{12})(c_{12} - c_{44})}{R^2 c_{44}(c_{12} + 2c_{44})(3c_{12} + 2c_{44})}, \]  

(89)

\[ a_2 = \frac{P(c_{12}f_{44} - c_{44}f_{12})(2c_{12} + 3c_{44})}{R^2 c_{44}(c_{12} + 2c_{44})(3c_{12} + 2c_{44})}. \]  

(90)

Now we can find \( f_i = \hat{f}_i + \tilde{f}_i \) and compare it with the results of exact calculations. This comparison is shown in Fig. This figure shows a good agreement between the approximate and the exact solution, the smaller higher elastic moduli match those better. It should also be emphasized that the latest version, when the difference is badly distinguishable, corresponds to most physically reasonable values of higher elastic moduli. Thus, the approximate method works very well at least for a homogeneously polarized ball with the given parameters.
Figure 3: Comparison of exact and approximate solutions for a ball. $R = 1 \cdot 10^{-5}$, $P = 1$, $c_{44} = 1.1 \cdot 10^{12}$, $c_{12} = 3.4 \cdot 10^{12}$, $f_{44} = f_{12} = 1 \cdot 10^{-3}$. $a$ – for $v_1 = 2.0 \cdot 10^{-1}$, $v_2 = 1.0 \cdot 10^{-1}$; $b$ – for $v_1 = 2.0 \cdot 10^{-2}$, $v_2 = 1.0 \cdot 10^{-2}$; $c$ – for $v_1 = 2.0 \cdot 10^{-3}$, $v_2 = 1.0 \cdot 10^{-3}$. Solid line is approximate solution, dashed line is exact solution.
6 Flexoelectric bending of homogeneously polarized circular rod

In this section we apply the above-described method to find the bending of a homogeneously polarized circular rod of an isotropic material [14]. Such a rod may be polarized in different directions. However, due to the fact that the flexoelectric effect is linear, we can restrict ourselves to the case of longitudinal and transverse polarization only. The response of the rod to the polarization of an arbitrary direction obviously is a superposition of responses to the longitudinal and transverse polarization. Besides, axial symmetry allows us not distinguish different directions of the transverse polarization.

As stated above, the classical part of strain is determined by the formal forces on the body surface. The surface of the rod consists of a cylindrical surface and end surfaces. First we consider the cylindrical surface of the rod. Here it is natural to use a cylindrical coordinate system. Let cylinder be located along the longitudinal axis corresponded to coordinate \( x \). This coordinate is also denoted as \( x^1 \). The other two curvilinear coordinates are the angular \( x^2 = \phi \) and radial \( x^3 = r \). Angle \( \phi \) is measured in respect to the direction of the Cartesian \( z \)-axis. Equation of the surface is \( x^3 = R \) where \( R \) is rod radius. So that this coordinate system belongs to the class that is needed in accordance with the method of calculations.

For such coordinates simple geometrical reasoning lead to the following equations defining the relationship between Cartesian and curvilinear coordinates:

\[
\begin{align*}
    x &= x^1, \\
    y &= r \sin \phi = x^3 \sin x^2, \\
    z &= r \cos \phi = x^3 \cos x^2.
\end{align*}
\]  

By means of simple differentiation it is easy to find the metric tensor, its determinants and the Christoffel symbols:

\[
\begin{align*}
    g_{11} &= g_{xx} = 1, & g_{22} &= g_{\phi\phi} = r^2, & g_{33} &= g_{rr} = 1, & g &= r^2, \\
    \Gamma^r_{\phi\phi} &= -r, & \Gamma^\phi_{r\phi} &= \Gamma^\phi_{\phi r} = \frac{1}{r}.
\end{align*}
\]  

The remaining components are zero. Since the metric tensor \( g_{\alpha\beta} \) is diagonal to find the inverse metric tensor \( g^{\alpha\beta} \) is easy: one just need to take the
reciprocals of the diagonal components. Note also that the vector normal to surface has the only component \( n_r = 1 \).

By making perfectly straightforward calculations similar to the case of a ball described above, the following expression can be obtained for the non-classical part of the elastic displacement \( \hat{u}_\alpha \) near the cylindrical surface. If the polarization \( P \) is longitudinal along the \( x \)-axis then the expression is as follows:

\[
\begin{align*}
\hat{u}_x &= \hat{u}_1 = P \frac{f_{44}}{2c_{44}} e^{\lambda_x R(\xi - 1)}, \\
\hat{u}_\phi &= \hat{u}_2 = 0, \\
\hat{u}_r &= \hat{u}_3 = 0.
\end{align*}
\] (93)

For transverse polarization along \( z \)-axis \( \hat{u}_\alpha \) is determined by the following equations:

\[
\begin{align*}
\hat{u}_x &= 0, \\
\hat{u}_\phi &= -PR \frac{f_{44}}{2c_{44}} \sin \phi e^{\lambda_\phi R(\xi - 1)}, \\
\hat{u}_r &= P \frac{f_{12} + 2f_{44}}{2(c_{12} + 2c_{44})} \cos \phi e^{\lambda_r R(\xi - 1)}.
\end{align*}
\] (94)

Here \( \xi = r/R \) is dimensionless radial coordinate,

\[
\lambda_x = \lambda_\phi = \sqrt{\frac{c_{44}}{v_1 + 2v_2}},
\] (95)

\[
\lambda_r = \sqrt{\frac{c_{12} + 2c_{44}}{9v_1 + 6v_2}}.
\] (96)

Further, direct application of the equations of section 4 gives the following boundary conditions for the classical part of the displacement. For transverse
polarization they are
\[
\begin{align*}
\tilde{\sigma}^{13} n_3 |_S &= 0, \\
\tilde{\sigma}^{23} n_3 |_S &= -\frac{P \sin \phi}{R^2} \cdot \frac{1}{2} \left[ \frac{c_{12} (f_{12} + 2f_{44})}{c_{12} + 2c_{44}} - f_{12} \right], \\
\tilde{\sigma}^{33} n_3 |_S &= \frac{P \cos \phi}{R} \left[ \frac{c_{44} (f_{12} + 2f_{44})}{c_{12} + 2c_{44}} - f_{44} \right].
\end{align*}
\]

If polarization is longitudinal then all \(\tilde{\sigma}^{\alpha 3}\) are zero.

Boundary conditions (97) can be also expressed in terms of the Cartesian tensor component. Elementary transformations yield:
\[
\begin{align*}
\tilde{\sigma}_{xi} n_i |_S &= 0, \\
\tilde{\sigma}_{yi} n_i |_S &= \frac{P (c_{44} f_{12} - c_{12} f_{44})}{R (c_{12} + 2c_{44})} \cdot \sin 2\phi, \\
\tilde{\sigma}_{zi} n_i |_S &= \frac{P (c_{44} f_{12} - c_{12} f_{44})}{R (c_{12} + 2c_{44})} \cdot \cos 2\phi.
\end{align*}
\]

In this form of boundary conditions it is immediately clear that the total force acting on the cylindrical surface is equal to zero, the integration over the angle turns these expressions to zero. It also can be easily calculated that the total bending moment of these forces vanishes while integrating over the angle \(\phi\).

Thus, not only the sum of formal surface forces, but also the sum of the bending moments of these forces, acting on a small part of the cylindrical surface length, are equal to zero. This means that the bending moment does not change along the homogeneously polarized rod. We emphasize that even so, the bending moment appears, but entirely by the boundary effects at the ends of the rod. So that in the case of homogeneously polarized rod the problem is reduced to the standard problem of the classical theory of elasticity, i.e. to the determination of rod bending under the bending moments applied to its ends.

Certainly the surface forces (97) slightly deform cross-section of the rod. Since the main effect is the bending of the rod, we do not discuss this deformation. Note only that qualitatively it is similar to the deformation of the ball in the meridian cross-section.
To calculate the bending moment appeared at the ends of the rod, we need to specify the shape of these ends. We emphasize that we can not limit the rod by planes because this gives sharp edges while the theory requires smooth surface. We should smooth out these sharp edges, say by a quarter of toroidal surface, or assume that the rod terminates by halves of the ball. Calculations were made for both these variants, and it was obtained the same result. Omitting the details of these rather simple calculations, we note only that in the case of spherical ends one can use the results of the previous section. For edges slightly smoothed by the toroidal surface one can approximately describe a small part of this toroidal surface as cylindrical one and use the equations given above in this section. Eventually it turns out that the bending moment is

\[
M_y = \frac{P\pi R^2(c_{44}f_{12} - c_{12}f_{44})}{c_{12} + 2c_{44}}.
\] (99)

The other components are equal to zero. It is implied here that the rod is transversely polarized along \(z\)-axis and the equation is written for the end in the positive direction of \(x\)-axis. For the other end of the rod \(M_y\) has the opposite sign.

As is already clear from symmetry, for the longitudinal polarized rod there is no bending moment at all, this can be also proved by direct calculations. Such calculations also show that the total force acting on each end of the rod is zero as for longitudinal and for transverse polarization.

Equation (99) actually solves the problem of flexoelectric bending of the homogeneously polarized rod. One should only substitute this bending moment into standard equations from textbooks (see [15] for instance).

7 Flexoelectric bending of a homogeneously polarized circular plate

In this section we apply the above-described method of approximate calculation to find the flexoelectric bending of a thin circular plate with a radius \(R\) and thickness \(h\), uniformly polarized normal to its plane [14]. Material of the plate is assumed isotropic. For definiteness, we assume that the average surface of the plate lies in the coordinate plane \(OXY\).

Calculations for the plate are generally similar to those made in the previous section for the rod. However, most part of the plate surface is flat, so
that one can immediately conclude, without any calculations, that the formal forces on this part of the surface are zero. As in the case of the rod, bending of the plate is determined by edge effects which are discussed below.

Thus, for most part of the plate, except for the edges, the classical part of the plate deformation is determined by standard equations without surface loading. It is well known [15] that under such conditions the displacements of the average surface of the plate $\zeta(x, y)$ obey two-dimensional differential equation

$$\Delta \Delta \zeta = 0,$$

where $\Delta$ is two-dimensional Laplace operator. The components of the strain tensor can be expressed in terms of $\zeta$ as follows:

$$u_{xx} = -z\zeta_{x,x}, \quad u_{yy} = -z\zeta_{y,y}, \quad u_{xy} = -z\zeta_{x,y}, \quad u_{xz} = u_{yz} = 0; \quad u_{zz} = z \frac{c_{12}}{c_{12} + 2c_{44}} (\zeta_{x,x} + \zeta_{y,y}).$$

(101)

Using polar coordinates it is easy to find an axially symmetric, regular at origin, solution of the equation (100):

$$\zeta(x, y) = -\frac{Gr^2}{2} = -\frac{G}{2} (x^2 + y^2).$$

(102)

This solution contains a single integration constant $G$ which is nothing but the curvature of the plate.

To find integration constant $G$ one needs first calculate the components of the strain tensor by means of (101) and second calculate the components of the strain tensor by means of standard equations $\sigma_{ij} = c_{ijkl} u_{ij}$. Thereafter it becomes obvious that there is the linear density of the bending moment $M$ on the boundary of the plate. For instance, at the point of intersection of the plate boundary with the coordinate axis $OX$ this bending moment density is

$$M = \int_{-h/2}^{+h/2} \sigma_{xx}(z) zdz = \frac{h^3}{6} G \frac{c_{44}(3c_{12} + 2c_{44})}{c_{12} + 2c_{44}}.$$

(103)

Due to axial symmetry $M$ is the same at all other points of boundary.

On the other hand $M$ can be calculated in terms of formal forces arising due to flexoelectricity on curved surfaces on the plate boundary. As in the case of the rod, it is necessary to smooth out sharp edges using one-quarter of toroidal surface (on each sharp edge) or one-half of the toroidal surface for
the whole boundary. A small part of a toroidal surface can be considered as a cylindrical one, so that we can apply the equations (98) (coordinate system should be rotated). For both variants of sharp edges smoothing it turns out

\[ M = \frac{Ph(c_{44}f_{12} - c_{12}f_{44})}{c_{12} + 2c_{44}}. \]

(104)

It only remains to equate the two expressions for \( M \) and express the curvature of the plate in terms of polarization \( P \): \n
\[ G = \frac{6P(c_{44}f_{12} - c_{12}f_{44})}{h^2c_{44}(3c_{12} + 2c_{44})}. \]

(105)

It is interesting to compare the equation (105) with the result obtained in [4] by direct minimization of the plate energy. Elementary transformations of the equations from [4] yield the equation different from (105) only in that there is 12 instead 6 in the numerator. However, it should be kept in mind that in [4] it is considered the case when in a thin layer near the plate surface polarization falls to zero. If, in accordance with calculations presented here, we modify the calculations [4] to the case, when the polarization is strictly homogeneous, then it appears exactly the equation (105).

A similar dependence on the boundary conditions imposed on the polarization also holds for direct flexoelectric effect, it is discussed in detail in another paper [16].

8 Conclusion

Above we have considered the continuum theory of the flexoelectric effect in the finite-size bodies. In the part of description of the converse flexoelectric effect, this theory turns out to be quite complicated. This is due to the fact that the independent variables are the elastic displacement. Flexoelectric part of the thermodynamic potential depends on the second spatial derivatives of the elastic displacement. Moreover in the case of flexoelectricity the elastic energy should be considered taking into account its dependence on the second spatial derivatives of elastic displacement (elastic spatial dispersion), or the theory is self-contradictory in the general case. All of this leads to several theoretical problems.

The first theoretical problem is that derivation of the boundary conditions for the differential equations of elastic equilibrium becomes non-trivial
in such a case. While the volume integration by parts, which is necessary to express the variation of the thermodynamic potential in terms of independent variates, its appear the surface integrals containing the gradients of these independent variates. Needed transformation of these surface integrals require special mathematical tools. For this purpose it was offered to use additional surface curvilinear coordinate system or to solve the problem in curvilinear coordinates of the general form from the beginning. From a practical point of view the second approach is more convenient although it has been proved that this two approaches are mathematically equivalent. It should also be noted that its appear that the curvature of the boundary surface plays an important role in the boundary conditions obtained in both approaches.

Solving the problem of elastic boundary conditions derivation, we get the opportunity to solve, in principle, any boundary problems needed to describe flexoelectricity in finite-size bodies. However, the corresponding boundary problems are extremely complex and difficult to solve. Such a boundary problem has been solved for a homogeneously polarized ball but for more complex geometries to make a similar is unrealistic. Even for the ball the solution is extremely cumbersome. Therefore the development of approximate methods for solving such boundary problems is desired. This is the second theoretical problem.

The problem of developing a method for the approximate solution of the corresponding boundary problems were also solved and the solution is described above. It turns out that in the framework of such problems elastic displacements can be approximated as the sum of two parts. The first part was called non-classical, it is concentrated near the surface of the body and decays exponentially inside the body. To find this part it is necessary to solve fairly simple one-dimensional equations. The second, classical part is determined by the equations of the classical theory of elasticity. It turns out that the boundary conditions for this equations have a standard classical form of the boundary conditions for the body under the external forces on the surface. Thus, the finding of the classical part is reduced to the standard classical problem which does not require a separate discussion. It should be noted that these forces on the surface are formal. Physically there is no forces on surface, formal forces describe the interaction between classical and non-classical part of the elastic displacement. So that these formal forces are expressed in terms of non-classical part of the displacement. The corresponding equations are given.

It is important to note that for homogeneous polarization the formal
forces, describing the influence of the non-classical part of displacement to
the classical one, appear only on the curved parts of the surface of the body.
Thus, the above statement that curvature of the surface plays an impor-
tant role gets a clear physical meaning. Note that the body can be, say, a
polyhedron. In this case the formal forces appear only on the edges which
can be treated as a limiting case of a curved surface. While calculating one
should slightly smooth these edges and tend to zero the radius of smoothing
at the end of the calculations. The similar approach is applied above for the
particular case of the rod and thin plate which also has the sharp edges.

Thus, in framework of the theory of continuum the description flexoelec-
tricity in finite-size bodies actually requires special theory. This theory is
described above in detail. The application of this theory is illustrated in par-
ticular problems of calculation of the bending of homogeneously polarized
rod and a thin plate.

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References

[1] J.D. Axe, J. Harada and G. Shirane, Anomalous acoustic dispersion in
centrosymmetric crystals with soft optic phonons, Phys. Rev. B 1, 1227
(1970).

[2] P.V. Yudin, A.K. Tagantsev, E.A. Eliseev, A.N. Morozovska and N. Set-
ter, Bichiral structure of ferroelectric domain walls driven by flexoelec-
tricity, Phys. Rev. B 86, 134102 (2012).

[3] E.V. Bursian, and O.I. Zaikovskii, Changes in the curvature of a fer-
roelectric film due to polarization, Sov. Phys.—Solid State 10, 1121
(1968).

[4] A.K. Tagantsev and A.S. Yurkov, Flexoelectric effect in finite samples,
J. Appl. Phys. 112(4), 044103 (2012).

[5] A. S. Yurkov, Elastic boundary conditions in the presence of the flexo-
electric effect, JETP Letters. 94(6), 455–458 (2011).
[6] E.A. Eliseev, A.N. Morozovska, M.D. Glinchuk and R. Blinc, Spontaneous flexoelectric/flexomagnetic effect in nanoferroics, *Phys. Rev. B* **79**, 165433 (2009).

[7] L.E. Cross, Flexoelectric effects: Charge separation in insulating solids subjected to elastic strain gradients, *J. Mater. Sci.* **41**(1), 53 (2006).

[8] B. Chu, W. Zhu, N. Li and L.E. Cross, Flexure mode flexoelectric piezoelectric composites, *J. Appl. Phys.* **106**(10), 104109 (2009).

[9] A.S. Yurkov, Flexoelectric deformation of a homogeneously polarized ball, [arXiv:1304.1868 [cond-mat.mtrl-sci]]. (2013).

[10] A. S. Yurkov, On the flexoelectric deformations of finite size bodies, *JETP Letters* **99**, 214 (2014).

[11] P.V. Yudin and A.K. Tagantsev, Fundamentals of flexoelectricity in solids, *Nanotechnology* **24**, 432001 (2013).

[12] A.S. Yurkov, Elastic boundary conditions in the theory with second gradients in the thermodynamic potential, [arXiv:1501.00822 [cond-mat.mtrl-sci]]. (2015).

[13] D.A. Varshalovich, A.N. Moskalev, V.K. Khersonskii, *Quantum theory of angular momentum*. World Scientific Pub. Co. (1987).

[14] A.S. Yurkov, *Fizika tverdogo tela* **57**, 450 (2015), (in press, in Russian). English translation to appear in *Phys. Solid State*.

[15] L.D. Landau, E.M. Lifshitz, *Theory of elasticity*. Pergamon, Oxford (1975).

[16] A.S. Yurkov, A.K. Tagantsev, Impact of surface phenomena on direct bulk flexoelectric effect in finite samples, [arXiv:1501.03365 [cond-mat.mtrl-sci]]. (2015).