Theoretical Analysis of Primal-Dual Algorithm for Non-Convex Stochastic Decentralized Optimization

Yuki Takezawa  
Kyoto University and RIKEN AIP  
yuki-takezawa@ml.ist.i.kyoto-u.ac.jp

Kenta Niwa  
NTT Communication Science Laboratories  
kenta.niwa.bk@hco.ntt.co.jp

Makoto Yamada  
Kyoto University and RIKEN AIP  
myamada@i.kyoto-u.ac.jp

Abstract

In recent years, decentralized learning has emerged as a powerful tool not only for large-scale machine learning, but also for preserving privacy. One of the key challenges in decentralized learning is that the data distribution held by each node is statistically heterogeneous. To address this challenge, the primal-dual algorithm called the Edge-Consensus Learning (ECL) was proposed and was experimentally shown to be robust to the heterogeneity of data distributions. However, the convergence rate of the ECL is provided only when the objective function is convex, and has not been shown in a standard machine learning setting where the objective function is non-convex. Furthermore, the intuitive reason why the ECL is robust to the heterogeneity of data distributions has not been investigated. In this work, we first investigate the relationship between the ECL and Gossip algorithm and show that the update formulas of the ECL can be regarded as correcting the local stochastic gradient in the Gossip algorithm. Then, we propose the Generalized ECL (G-ECL), which contains the ECL as a special case, and provide the convergence rates of the G-ECL in both (strongly) convex and non-convex settings, which do not depend on the heterogeneity of data distributions. Through synthetic experiments, we demonstrate that the numerical results of both the G-ECL and ECL coincide with the convergence rate of the G-ECL.

1 Introduction

Neural networks have achieved promising results in many tasks such as natural language processing [6, 4] and image processing [25, 17]. To train a large-scale neural network efficiently, decentralized learning is a powerful tool. Decentralized learning allocates data into multiple nodes (e.g., servers) and trains a neural network in parallel. Because decentralized learning allows us to train a neural network without aggregating all the data in one server, it has also attracted considerable attention from the perspective of privacy preservation.

One of the most widely used algorithms for decentralized learning is the decentralized parallel SGD (D-PSGD) [14] (a.k.a. the Gossip algorithm). Recently, the effect of various variables on the convergence rate of the Gossip algorithm has been well investigated. (for example, noise of stochastic gradient, the structure of the network, and the heterogeneity of data distributions held by each node) [14, 15, 11, 12]. These theoretical results show that the convergence rate of the Gossip algorithm slows down when the data distribution held by each node is statistically heterogeneous.

Preprint. Under review.
To address the heterogeneity of data distributions, the primal-dual algorithm using Douglas-Rachford splitting [7] called the Edge-Consensus Learning (ECL) [20] has been proposed. In image classification tasks, it was shown that the ECL outperforms the Gossip algorithm when the data distributions held by each node are statistically heterogeneous, and the ECL has been experimentally shown to be robust to the heterogeneity of data distributions. Recently, Rajawat and Kumar [23] provided the convergence rate of the ECL when the objective function is convex and showed that it does not depend on the heterogeneity of data distributions. However, in a standard machine learning setting where the objective function is non-convex, the convergence rate of the ECL has not been shown. Furthermore, the relationship between the ECL and Gossip algorithm has not been investigated, including the differences between the ECL and Gossip algorithm and an intuitive reason why the ECL is robust to the heterogeneity of data distributions.

In this work, we propose the Generalized ECL (G-ECL), which contains the ECL as a special case, and provide the convergence rates of the G-ECL in both (strongly) convex and non-convex settings. More specifically, we investigate the relationship between the ECL and Gossip algorithm and show that the update formulas of the ECL can be regarded as correcting the stochastic gradient of each node in the Gossip algorithm. Then, to make the convergence analysis tractable, we increase the degrees of freedom of the hyperparameters of the ECL and propose the G-ECL, which contains the ECL as a special case. By using the proof techniques of the Gossip algorithm [11], we provide the convergence rates of the G-ECL in (strongly) convex and non-convex settings and show that they do not depend on the heterogeneity of data distributions. Table 1 summarizes the convergence rates of the Gossip algorithm, ECL and G-ECL. Through synthetic experiments, we demonstrate that the numerical results of both the ECL and G-ECL coincide with the convergence rates of the G-ECL.

Our contributions are summarized as follows:

- In this work, we investigate the relationship between the Gossip algorithm and ECL and show that the update formulas of the ECL can be regarded as correcting the stochastic gradient of each node in the Gossip algorithm (Sec. 3).
- We propose the G-ECL, which contains the ECL as a special case, and provide the convergence rates of the G-ECL in (strongly) convex and non-convex settings (Sec. 4 and 6). Then, we show that the convergence rates of the G-ECL do not depend on the heterogeneity of data distributions.
- Through synthetic experiments, we demonstrate that the numerical results of both the ECL and G-ECL coincide with the convergence rates of the G-ECL (Sec. 7).

**Notation:** In this work, we denote $[n] = \{1, 2, \ldots, n\}$ for any $n \in \mathbb{N}$. We write $I$ for the identity matrix, $\mathbf{1}$ for the vector with all ones, $\|\cdot\|$ for L2 norm, and $\|\cdot\|_F$ for the Frobenius norm.

## 2 Preliminary

In this section, we briefly introduce the problem setting of decentralized learning, Gossip algorithm, and ECL. A more detailed discussion of related works is presented in Sec. A.

### 2.1 Problem Setting

We introduce a problem setting for decentralized learning. Let $G = (\mathcal{V}, \mathcal{E})$ be an undirected graph representing the network topology of nodes where $\mathcal{V}$ denotes a set of nodes and $\mathcal{E}$ denotes a set of edges. In the following, we denote $\mathcal{V}$ as a set of integers $\{1, 2, \ldots, |\mathcal{V}|\}$ for simplicity. We denote the set of neighbors of node $i$ as $\mathcal{N}_i := \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$ and we denote $\mathcal{N}_i^+ := \mathcal{N}_i \cup \{i\}$. In decentralized learning, nodes $i$ and $j$ are allowed to communicate the parameters only if $(i, j) \in \mathcal{E}$.

The decentralized learning problem is formulated as follows:

$$
\min_{x \in \mathbb{R}^d} \left[ f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right], \quad f_i(x) := \mathbb{E}_{\xi_i \sim D_i} [F_i(x; \xi_i)],
$$

(1)

---

1The algorithm called the decentralized primal-dual algorithm in [23] is equivalent to the ECL when $\theta = 1$.
2The previous work [21] attempted to analyze the convergence rates of the ECL on both (strongly) convex and non-convex settings. However, strong approximations were used in the proofs, which do not hold in practice. This is discussed in detail in Sec. C.
Table 1: Convergence rates of the D-PSGD, ECL, and G-ECL. We define \( \tilde{\zeta}^2 := \frac{1}{p} \zeta^2 \), and the definition of other parameters is shown in Sec. 2, 3, and 5. Note that the D-PSGD requires the assumption about the heterogeneity of data distributions, while the G-ECL and ECL do not require.

|                | Non-Convex | Strongly Convex | General Convex |
|----------------|------------|-----------------|----------------|
| D-PSGD [11]    | \( \tilde{O} \left( \sqrt{\frac{n^2 \sigma^2 \eta}{n^2 \eta}} + \left( \frac{\eta (\sigma^2 + \zeta^2) (1 - \mu)}{pn^2} \right)^\frac{1}{2} + \frac{\zeta}{n} \right) \) | \( \tilde{O} \left( \frac{n^2 \eta}{n^2 \eta} \right) \) | \( \tilde{O} \left( \frac{\eta (\sigma^2 + \zeta^2) (1 - \mu)}{pn^2} \right)^\frac{1}{2} + \frac{\zeta}{n} \) |
| ECL [23]       | N/A        | N/A             | N/A            |
| G-ECL (this work) | \( \tilde{O} \left( \sqrt{\frac{n^2 \sigma^2 \eta}{n^2 \eta}} + \left( \frac{\eta (\sigma^2 + \zeta^2) (1 - \mu)}{pn^2} \right)^\frac{1}{2} + \frac{\zeta}{n} \right) \) | \( \tilde{O} \left( \frac{n^2 \eta}{n^2 \eta} \right) \) | \( \tilde{O} \left( \frac{\eta (\sigma^2 + \zeta^2) (1 - \mu)}{pn^2} \right)^\frac{1}{2} + \frac{\zeta}{n} \) |


*Rajawat and Kumar [23] evaluated \( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[D_{f_i}(x_i^{(r)}, x^*)] \) where \( D_{f_i} \) is Bregman divergence associated with \( f_i \). Note that if \( f_i \) is strictly convex, \( D_{f_i}(x_i^{(r)}, x^*) = 0 \) is equivalent to \( x_i^{(r)} = x^* \).

where \( x \in \mathbb{R}^d \) denotes the model parameter, \( f_i : \mathbb{R}^d \rightarrow \mathbb{R} \) denotes the objective function of node \( i \), \( n := |\mathcal{V}| \) is the number of nodes, and \( D_{f_i} \) is the data distribution held by node \( i \). In this work, we assume that only the stochastic gradient \( \nabla F_i(x; \xi_i) \) is accessible and the full gradient \( \nabla f_i(x) \) is inaccessible and analyze the convergence rate in both cases when \( f_i \) is (strongly) convex and when \( f_i \) is non-convex.

### 2.2 Gossip Algorithm

One of the most popular algorithms for decentralized learning is the D-PSGD [14] (a.k.a. the Gossip algorithm). In the Gossip algorithm, each node updates its model parameters as follows:

\[
x_i^{(r+1)} = \sum_{j \in \mathcal{N}_i} W_{ij} \left( x_j^{(r)} - \eta \nabla F_j(x_j^{(r)}; \xi_j^{(r)}) \right),
\]

where \( x_i \in \mathbb{R}^d \) is the model parameter of node \( i \), \( \eta > 0 \) is the step size, and \( W_{ij} \in [0, 1] \) is the weight of the edge \((i, j) \in \mathcal{E}\). Let \( \mathbf{W} \) be an \( n \times n \) matrix whose \((i, j)\)-element is \( W_{ij} \) if \( j \in \mathcal{N}_i^+ \) and 0 otherwise. In the Gossip algorithm, \( \mathbf{W} \) is assumed to be a mixing matrix defined as follows.

**Definition 1 (Mixing Matrix).** If \( \mathbf{W} \in [0, 1]^{n \times n} \) is symmetric (\( \mathbf{W} = \mathbf{W}^T \)) and doubly stochastic (\( \mathbf{W} \mathbf{1} = \mathbf{1}, \mathbf{W}^T \mathbf{1} = \mathbf{1}^T \)), then \( \mathbf{W} \) is called a mixing matrix.

### 2.3 Edge-Consensus Learning

In this section, we briefly introduce the primal-dual algorithm using Douglas-Rachford splitting called the Edge-Consensus Learning (ECL) [20]. Reformulating Eq. (1), we can define the primal problem as follows:

\[
\min_{x_i, \ldots, x_n} \frac{1}{n} \sum_{i=1}^{n} f_i(x_i) \quad \text{s.t.} \quad A_{ij} x_i + A_{ji} x_j = 0, \quad (\forall (i, j) \in \mathcal{E}),
\]

where \( A_{ij} = \mathbf{I} \) when \( i > j \) and \( A_{ij} = -\mathbf{I} \) when \( i < j \) for any \( i, j \in [n] \). Then, by solving the dual problem of Eq. (3) using Douglas-Rachford splitting [7], the update formulas can be derived as follows [26, 20]:

\[
x_i^{(r+1)} = \arg\min_{x_i} \left\{ f_i(x_i) + \sum_{j \in \mathcal{N}_i} \frac{\alpha_{ij}}{2} \| A_{ij} x_i - z_{ij}^{(r)} \|^2 \right\},
\]

\[
y_{ij}^{(r+1)} = y_{ij}^{(r)} - 2 A_{ij} x_i^{(r+1)},
\]

\[
z_{ij}^{(r+1)} = (1 - \theta) z_{ij}^{(r)} + \theta y_{ij}^{(r+1)},
\]

...
Algorithm 1: Update procedure at node $i$ in the ECL.

1: **Input:** Set $\{\alpha_{ij}\}_{ij}$ such that $\alpha_{ij} = \alpha_{ji} \geq 0$ for all $(i, j) \in \mathcal{E}$. Initialize $z_i^{(0)}$ to zero and $x_i^{(0)}$ with the same parameters for all $i \in [n]$. 
2: **for** $r = 0, 1, \ldots, R$ **do** 
3: Sample $\xi_i^{(r)}$ and compute $g_i^{(r)} := \nabla F_i(x_i^{(r)}; \xi_i^{(r)})$.
4: $x_i^{(r+1)} \leftarrow (1 + \eta \sum_{j \in N_i} \alpha_{ij})^{-1} \{x_i^{(r)} - \eta(g_i^{(r)} - \sum_{j \in N_i} \alpha_{ij} A_{ij} z_j^{(r)})\}$.
5: **for** $j \in N_i$ **do**
6: $y_{ij}^{(r+1)} \leftarrow z_{ij}^{(r)} - 2 A_{ij} x_i^{(r+1)}$.
7: Transmit$_{i \rightarrow j}(y_{ij}^{(r+1)})$.
8: Receive$_{i \rightarrow j}(y_{ij}^{(r+1)})$.
9: $z_{ij}^{(r+1)} \leftarrow (1 - \theta) z_{ij}^{(r)} + \theta y_{ij}^{(r+1)}$.
10: **end for**
11: **end for**

where $y_{ij} \in \mathbb{R}^d$ and $z_{ij} \in \mathbb{R}^d$ are dual variables, and $\theta \in (0, 1]$ and $\alpha_{ij} \geq 0$ are hyperparameters. In previous works \cite{20, 21}, hyperparameter $\{\alpha_{ij}\}_{ij}$ is set such that $\alpha_{ij} = \alpha_i$ for all $(i, j) \in \mathcal{E}$. However, in this work, we increase the degrees of freedom of the hyperparameter $\{\alpha_{ij}\}_{ij}$, which plays an important role in discussing the relationship between the ECL and Gossip algorithm in Theorem 2. Douglas-Rachford splitting has been well studied in the convex optimization literature and converges linearly to the optimal solution \cite{24, 2, 8}. Therefore, $\{x_i^{(r)}\}_i$ generated by Eqs. (4-6) converges linearly to the optimal solution when $f_i$ is convex.

However, when $f_i$ is non-convex (e.g., a loss function of a neural network), Eq. (4) can not be solved in general. Subsequently, Niwa et al. \cite{20} proposed solving Eq. (4) approximately as follows:

$$x_i^{(r+1)} = \arg \min_{x_i} \{\langle x_i, \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) \rangle + \frac{1}{2\eta} \| x_i - x_i^{(r)} \|^2 + \sum_{j \in N_i} \alpha_{ij} \frac{1}{2} \| A_{ij} x_i - z_{ij}^{(r)} \|^2 \}, \quad (7)$$

where $\eta > 0$ corresponds to the step size. The update formulas Eqs. (5-7) are called the Edge-Consensus Learning (ECL). The pseudo-code of the ECL is presented in Alg. 1 where Transmit$_{i \rightarrow j}(\cdot)$ denotes the operator that transmits parameters from node $i$ to node $j$ and Receive$_{i \rightarrow j}(\cdot)$ denotes the operator for node $i$ to receive parameters from node $j$. Then, Niwa et al. \cite{20, 21} experimentally showed that the ECL is robust to the heterogeneity of data distributions. Recently, Rajawat and Kumar \cite{12} provided the convergence rate of the ECL when $f_i$ is convex and showed that it does not depend on the heterogeneity of data distributions. However, when $f_i$ is non-convex, the convergence rate of the ECL has not been provided. Furthermore, the relationship between the Gossip algorithm and ECL has not yet been investigated, including the differences between the ECL and Gossip algorithm and an intuitive reason why the ECL is robust to the heterogeneity of data distributions.

3 Relationship between ECL and Gossip Algorithm

It can be observed that the ECL is different from the Gossip algorithm. However, it is difficult to discuss what is different between the ECL and Gossip algorithm using Eq. (2) and Eqs. (5-7). In this section, we discuss the relationship between the ECL and Gossip algorithm. All proofs are provided in Appendix.

**Organization:** The remainder of this section is organized as follows. In Sec. 3.1, we show that each node implicitly computes the weighted sum with its neighbors in the ECL as well as in the Gossip algorithm. In Sec. 3.2, we show that these weights in the ECL become a mixing matrix when the hyperparameter is set appropriately as well as in the Gossip algorithm. In Sec. 3.3, we discuss the property of the sequence of the average $\frac{1}{n} \sum_{i=1}^{n} x_i^{(r)}$ in the ECL.
3.1 Reformulation

To discuss the relationship between the ECL and Gossip algorithm, we reformulate the update formulas of the ECL as follows.

**Theorem 1.** Suppose that the hyperparameter $\theta$ is $\frac{1}{2}$, the dual variable $z_{ij}^{(0)}$ is initialized to $A_{ij}x_j^{(0)}$, and the hyperparameter $\{\alpha_{ij}\}_{ij}$ is set such that $\alpha_{ij} = \alpha_{ji} \geq 0$ for all $(i, j) \in \mathcal{E}$. Then, the update formulas Eq. (7) and Eqs. (5-6) are equivalent to the following:

$$
\hat{x}_i^{(r)} = \sum_{j \in \mathcal{N}^+} W_{ij} x_j^{(r)}, \quad (8)
$$

$$
x_i^{(r+1)} = \hat{x}_i^{(r)} - \eta_i \left( \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) - c_i^{(r)} \right), \quad (9)
$$

$$
c_i^{(r+1)} = \sum_{j \in \mathcal{N}^+} W_{ij} \left( c_j^{(r)} - \nabla F_j(x_j^{(r)}; \xi_j^{(r)}) \right) + \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) + \sum_{j \in \mathcal{N}_i} \alpha_{ij} (\hat{x}_j^{(r)} - x_j^{(r)}), \quad (10)
$$

where $c_i^{(0)} := \frac{1}{2} \sum_{j \in \mathcal{N}_i} \alpha_{ij} (x_j^{(0)} - x_i^{(0)})$, and $W_{ij}$ and $\eta_i$ are defined as follows:

$$
\eta_i := \frac{\eta}{1 + \eta \sum_{j \in \mathcal{N}_i} \alpha_{ij}}, \quad W_{ij} := \begin{cases} 
\frac{2 + \eta \sum_{k \in \mathcal{N}_i} \alpha_{ik}}{2(1 + \eta \sum_{k \in \mathcal{N}_i} \alpha_{ik})} & \text{if } i = j \\
\frac{\eta \alpha_{ij}}{2(1 + \eta \sum_{k \in \mathcal{N}_i} \alpha_{ik})} & \text{if } (i, j) \in \mathcal{E} \\
0 & \text{otherwise}
\end{cases}, \quad (11)
$$

At first glance, the update formulas of the ECL Eqs. (7) do not explicitly compute the weighted average, in contrast to that of the Gossip algorithm. However, Theorem 1 shows that as well as the update formulas of the Gossip algorithm Eq. (2), the update formula Eq. (8) computes the weighted sum whose weights are determined by $\eta$ and $\{\alpha_{ij}\}_{ij}$. Subsequently, the update formula Eq. (9) can be regarded as the one where $c_i$ modifies the local stochastic gradient $\nabla F_i(x_i; \xi_i)$ in the update formulas of the Gossip algorithm Eq. (2). To investigate how the term $c_i$ modifies the local stochastic gradient $\nabla F_i(x_i; \xi_i)$, we discuss the relationship between the ECL and gradient tracking methods in Sec. 3.2

3.2 Assumption of Hyperparameters

Let $W$ be an $n \times n$ matrix whose $(i, j)$-element is $W_{ij}$. In general, $W$ defined by Eq. (11) is not a mixing matrix because $W$ is not symmetric. In this section, to further discuss the relationship between the ECL and Gossip algorithm, we discuss the conditions of hyperparameters for $W$ to be a mixing matrix. The following assumption and theorem show that if we set the hyperparameter $\{\alpha_{ij}\}_{ij}$ appropriately, $W$ is a mixing matrix.

**Assumption 1.** The hyperparameter $\{\alpha_{ij}\}_{ij}$ is set such that $\alpha_{ij} = \alpha_{ji} \geq 0$ for all $(i, j) \in \mathcal{E}$, and there exists $\alpha > 0$ that satisfies $\sum_{k \in \mathcal{N}_i} \alpha_{ik} = \alpha$ for all $i \in [n]$.

**Theorem 2.** Suppose that Assumption 1 holds, then $W$ defined by Eq. (11) is a mixing matrix.

Moreover, as a by-product, when Assumption 1 holds, $\eta_i'$ defined by Eq. (11) are same for all $i \in [n]$.

**Remark 1.** Suppose that the hyperparameter $\{\alpha_{ij}\}_{ij}$ is set such that Assumption 1 holds, there exists $\eta' > 0$ that satisfies for all $i \in [n]$,

$$
\eta' = \eta_i' = \frac{\eta}{1 + \eta \alpha}. \quad (12)
$$

Then, when $G$ is a regular graph, we can set the hyperparameter $\{\alpha_{ij}\}_{ij}$ that satisfies Assumption 1 as follows.

\[\text{The Gossip algorithm and ECL also differ in the order of calculation of weighted average and the stochastic gradient descent. We discuss the effect of this difference in Sec. 6.2.}\]
Example 1. Suppose that $G$ is a $k$-regular graph with $k > 0$. If we set $\alpha_{ij} = \frac{c}{2}$ for all $(i, j) \in E$, then the hyperparameter $\{\alpha_{ij}\}_{ij}$ satisfies Assumption 1 and $W$ is defined as follows:

$$W_{ij} := \begin{cases} 
\frac{2 + \eta \alpha}{2(1 + \eta \alpha)} & \text{if } i = j \\
\frac{\eta \alpha}{2k(1 + \eta \alpha)} & \text{if } (i, j) \in E \\
0 & \text{otherwise}
\end{cases}$$

3.3 Property of Average Sequence

In Sec. 3.2, we show that if the hyperparameter $\{\alpha_{ij}\}_{ij}$ is set appropriately, $W$ defined by Eq. (11) is a mixing matrix as well as in the Gossip algorithm. In this section, we discuss the relationship between the Gossip algorithm and ECL from the property of the sequence of the average $\frac{1}{n} \sum_{i=1}^{n} x_i^{(r)}$.

In the Gossip algorithm, when $W$ is a mixing matrix, the average $\bar{x}^{(r)} := \frac{1}{n} \sum_{i=1}^{n} x_i^{(r)}$ generated by Eq. (2) satisfies the following:

$$\bar{x}^{(r+1)} = \bar{x}^{(r)} - \frac{\eta}{n} \sum_{i=1}^{n} \nabla F_i(x_i^{(r)}, \xi_i^{(r)}). \quad (13)$$

That is, the update formula of the Gossip algorithm is almost equivalent to that of the SGD, which plays an important role in the convergence analysis of the Gossip algorithm. Similarly, the property of Eq. (13) is satisfied in the ECL, as the following lemma indicates.

**Lemma 1** (Average Sequence). Suppose that the hyperparameter $\{\alpha_{ij}\}_{ij}$ is set such that Assumption 1 holds. Then, under the same assumptions as those in Theorem 1, it holds that $\sum_{i=1}^{n} x_i^{(r)} = 0$ for any round $r$, and the average $\bar{x}^{(r)} := \frac{1}{n} \sum_{i=1}^{n} x_i^{(r)}$ generated by Eqs. (8-10) satisfies the following:

$$\bar{x}^{(r+1)} = \bar{x}^{(r)} - \frac{\eta}{n} \sum_{i=1}^{n} \nabla F_i(x_i^{(r)}, \xi_i^{(r)}). \quad (14)$$

4 Generalized Edge-Consensus Learning

In Sec. 3, we show that a node computes the average with its neighbors using the mixing matrix $W$ in the ECL as well as in the Gossip algorithm and then updates the model parameter by the stochastic gradient descent modified by the term $\xi_i$. However, in contrast with the Gossip algorithm, $W$, $\eta'$ and $\{\alpha_{ij}\}_{ij}$ depend on each other in the ECL, which makes the convergence analysis difficult. Then, we refer to the update formulas Eqs. (8-10) as the Generalized ECL (G-ECL) when $W$, $\eta'$, and $\{\alpha_{ij}\}_{ij}$ are set independently as hyperparameters and provide the convergence rate of the G-ECL in Sec. 6.

Then, we experimentally demonstrate that the ECL converges at the same rate as the G-ECL in Sec. 7. The pseudo-code of the G-ECL is illustrated in Sec. 11. Note that because the G-ECL is equivalent to the ECL when $W$ and $\eta'$ are set as in Eq. (11), the ECL is a special case of the G-ECL.

5 Setup

In this section, we introduce the assumptions and notations used in the convergence analysis in the next section. We define $b' := \|W - I\|_F^2$ and $\bar{x}^{(r)} := \frac{1}{n} \sum_{i=1}^{n} x_i^{(r)}$ and denote $f^*$ as the optimal value of Eq. (11). When $f_i$ is convex for all $i \in [n]$, we denote $x^* \in \mathbb{R}^d$ as the optimal solution of Eq. (11). Next, we introduce the assumptions used for the convergence analysis of the G-ECL.

**Assumption 2** (Mixing Matrix). There exists $p \in (0, 1]$ such that for any $x_1, \ldots, x_n \in \mathbb{R}^d$,

$$\|XW - \bar{X}\|_F^2 \leq (1 - p) \|X - \bar{X}\|_F^2,$$  

where $X = (x_1, \ldots, x_n) \in \mathbb{R}^{d \times n}$ and $\bar{X} := \frac{1}{n} X 1 1^\top$.

**Assumption 3** ($L$-smoothness). For any $i \in [n]$, there exists $L > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq L \|x - y\|.$$  

(16)
Assumption 4 (Bounded Gradient Noise). For any \( i \in [n] \), there exists \( \sigma \geq 0 \) such that for all \( x_i \in \mathbb{R}^d \),
\[
\mathbb{E}_{\xi_i \sim \mathcal{D}_i} \| \nabla F_i(x_i; \xi_i) - \nabla f_i(x_i) \|^2 \leq \sigma^2.
\] (17)

Assumption 5 (\( \mu \)-convexity). For any \( i \in [n] \), there exists \( \mu \geq 0 \) such that for all \( x, y \in \mathbb{R}^d \),
\[
f_i(x) \geq f_i(y) + \langle \nabla f_i(y), x - y \rangle + \frac{\mu}{2} \| x - y \|^2.
\] (18)

Assumptions 2, 3, 4, and 5 are commonly used in convergence analyses of decentralized learning algorithms \([10, 15, 29]\). In addition, the following assumption, which represents the heterogeneity of data distributions, is commonly used \([15, 29]\). However, this assumption is not necessary for the convergence analysis of the G-ECL shown in Theorem 3.

Assumption 6 (Bounded Heterogeneity). There exists \( \zeta \geq 0 \) such that for all \( x \in \mathbb{R}^d \),
\[
\frac{1}{n} \sum_{i=1}^{n} \| \nabla f_i(x) - \nabla f(x) \|^2 \leq \zeta^2.
\]

6 Convergence Results

In this section, we present the convergence results of the G-ECL. Our convergence analysis is based on the analysis of the Gossip algorithm \([11]\), and all the proofs are presented in Sec. 4.

6.1 Main Theorem

Theorem 3. Suppose that Assumptions 2, 3, 4 hold, \( \{\alpha_{ij}\} \) is set such that \( \alpha_{ij} = \alpha_{ji} \geq 0 \) for all \( (i, j) \in E \), \( W \) is set to be a mixing matrix, and \( x_i^{(0)} \) is initialized with the same parameters for all \( i \in [n] \).

Non-convex: In addition, suppose that \( c_i \) is initialized to \( \nabla f_i(x_i^{(0)}) - \nabla f(x_i^{(0)}) \). Then, there exists a step size \( \eta' < \frac{1}{\mu} \) such that the average \( \bar{x}^{(r)} \) generated by the G-ECL satisfies
\[
\frac{1}{R} \sum_{r=0}^{R-1} \mathbb{E} \| \nabla f(\bar{x}^{(r)}) \|^2 \leq \mathcal{O} \left( \sqrt{\frac{r_0 \sigma^2 L}{nR}} + \left( \frac{r_0^2 L^2 \bar{\sigma}^2}{pR^2} \right)^{\frac{1}{2}} + \frac{sr_0}{R} \right),
\]
where \( r_0 := f(\bar{x}^{(0)}) - f^* \) and \( \bar{\sigma}^2 := (1 + \frac{\mu'}{\mu})\sigma^2 \).

General Convex: In addition, suppose that \( f_i \) is convex for all \( i \in [n] \). Assumption 5 holds with \( \mu = 0 \), and \( c_i \) is initialized to \( \nabla f_i(x^*) \). Then, there exists a step size \( \eta' < \frac{1}{\mu} \) such that the average \( \bar{x}^{(r)} \) generated by the G-ECL satisfies
\[
\frac{1}{R} \sum_{r=0}^{R-1} \mathbb{E} \| f(\bar{x}^{(r)}) \| = f^* \leq \mathcal{O} \left( \sqrt{\frac{r_0 \sigma^2}{nR}} + \left( \frac{r_0^2 L \bar{\sigma}^2}{pR^2} \right)^{\frac{1}{2}} + \frac{sr_0}{R} \right),
\]
where \( r_0 := \| \bar{x}^{(0)} - x^* \|^2 \) and \( \bar{\sigma}^2 := (1 + \frac{\mu}{\mu'})\sigma^2 \).

Strongly Convex: In addition, suppose that \( f_i \) is convex for all \( i \in [n] \). Assumption 5 holds with \( \mu > 0 \), and \( c_i \) is initialized to \( \nabla f_i(x^*) \). Let \( w^{(r)} := (1 - \frac{\mu}{2})^{-(r+1)} \) and \( W_R := \sum_{r=0}^{R} w^{(r)} \). Then, there exists a step size \( \eta' < \frac{1}{\mu} \) such that the average \( \bar{x}^{(r)} \) generated by the G-ECL satisfies
\[
\sum_{r=0}^{R} \frac{w^{(r)}}{W_R} \mathbb{E} \| f(\bar{x}^{(r)}) \| - f^* + \mu \mathbb{E} \| \bar{x}^{(R+1)} - x^* \|^2 \leq \mathcal{O} \left( r_0 \exp \left[ \frac{-\mu (R+1)}{s} \right] + \frac{\sigma^2}{\mu nR} + \frac{L \bar{\sigma}^2}{\mu^2 nR^2} \right),
\]
where \( r_0 := \| \bar{x}^{(0)} - x^* \|^2 \), \( \bar{\sigma}^2 := (1 + \frac{\mu}{\mu'})\sigma^2 \), and \( \mathcal{O}(\cdot) \) hides polylogarithmic factors.

Limitation: Theorem 3 shows the convergence rates of the G-ECL, but does not show that of the ECL because \( W \), \( \eta' \), and \( \{\alpha_{ij}\} \) depend on each other in the ECL. Specifically, our analysis does not prove that there exists a step size \( \eta' \) for the ECL to achieve the convergence rate shown in Theorem 3. In this work, we only provide the convergence rate of the G-ECL and experimentally demonstrate that the ECL also converges at the same rate as the G-ECL in Sec. 4. A more detailed discussion is provided in Sec. 4.
6.2 Discussion

In this section, we discuss the convergence rate of the G-ECL compared with that of the D-PSGD [14]. Table 1 lists the convergence rates of the G-ECL and D-PSGD. Here, we discuss only the strongly convex case, but this discussion holds for the convex and non-convex cases.

First, we discuss the effect of the heterogeneity of data distributions $\zeta$ in Assumption 6 on the convergence rate. Table 1 shows that the convergence rate of the D-PSGD depends on the heterogeneity of data distributions $\zeta$, while the convergence rate of the G-ECL does not depend on $\zeta$. Therefore, Theorem 3 indicates that the G-ECL is robust to the heterogeneity of data distributions, which is consistent with previous works [20,21] that experimentally demonstrated that the ECL is robust to the heterogeneity of data distributions.

Next, we discuss the factor $(1 - p)$ contained in the convergence rate of the D-PSGD. In the D-PSGD, the third term in the convergence rate is multiplied by $(1 - p)$, but in the G-ECL, the third term in the convergence rate is not multiplied by $(1 - p)$. That is, in the D-PSGD, the third term is 0 when $p = 1$ (i.e., $G$ is fully connected graph and $W = \frac{1}{n}11^\top$), but in the G-ECL, the third term is not 0 for any $p$. This is because the orders of calculation of the weighted average and stochastic gradient descent are different. The analysis of the D-PSGD evaluates the average $x^{(r)}$ after computing the weighted average of Eq. (8), whereas the analysis of the G-ECL evaluates the average $\bar{x}^{(r)}$ before computing the weighted average of Eq. (8). Thus, the third term is not multiplied by $(1 - p)$ in the G-ECL.

7 Experiments

In this section, using the synthetic dataset, we experimentally demonstrate that the numerical results of the G-ECL and ECL are consistent with the convergence rate of the G-ECL shown in Theorem 3. Following the previous work [11], we focus only on the strongly convex case.

Comparison Methods: We compare the D-PSGD [14], ECL [20], and G-ECL. In the D-PSGD, we use Metropolis-Hasting weights (i.e., $W_{ij} = W_{ji} = 1/(|\mathcal{N}_i|+1)$) and set the step size $\eta = 10^{-3}$. In the ECL, we set $\{\alpha_{ij}\}_{ij}$ as in Example 1. Then, we set $\eta = 0.5$, $\alpha = 10^{3}$ (i.e., $\eta' \approx 10^{-3}$). In the G-ECL, we set $W_{ij} = W_{ji} = 1/(|\mathcal{N}_i|+1)$, $\eta' = 10^{-3}$, and $\alpha_{ij} = 0$. Note that the ECL can be regarded as a special case of the G-ECL.

Synthetic Dataset and Network Topology: We set the dimension of the parameter $d = 50$ and the number of nodes $n = 25$. We set the objective function as $f_i(x) := \frac{1}{2}\|x - b_i\|^2$ and $b_i$ is drawn from $\mathcal{N}(0, \frac{\sigma_i^2}{d} I)$ for each $i \in [n]$. The stochastic gradient is defined as $\nabla f_i(x; \xi_i) := \nabla f_i(x) + \epsilon$ where $\epsilon$ is drawn from $\mathcal{N}(0, \frac{\sigma_i^2}{d} I)$ at each time. Note that the parameters $\zeta$ and $\sigma$ correspond to Assumptions 6 and 4. We evaluate the D-PSGD, ECL, and G-ECL on three network topologies consisting $n$ nodes: ring, torus, and fully connected graph. We implement all comparison methods with PyTorch [22], and all the experiments are executed on a machine with Intel Xeon CPU E7-8890 v4.

7.1 Numerical Results

In this section, we demonstrate that the convergence rate of the G-ECL shown in Theorem 3 coincides with the numerical results of both the G-ECL and ECL. Fig. 1 shows the error $\frac{1}{n}\sum_{i=1}^{n}\|x_i^{(r)} - x^*\|^2$ at each round $r$ when varying the heterogeneity of data distributions $\zeta$ and the noise of stochastic gradient $\sigma$.

Effect of Heterogeneity of Data Distributions: First, we discuss the effect of the heterogeneity of data distributions $\zeta$ on the convergence rate. When $\sigma_2 = 0$, the results show that the G-ECL and ECL converge with $\mathcal{O}(\exp(-R))$ in both cases when $\zeta_2 = 0$ and $\zeta_2^2 = 10$ for all network topologies. By contrast, when $\sigma_2 = 0$ and $G$ is ring or torus (i.e., $p < 1$), the convergence of the D-PSGD slows down when $\zeta_2^2 = 10$ compared to when $\zeta_2 = 0$. When $\sigma_2 = 0$ and $G$ is fully connected graph (i.e., $W = \frac{1}{n}11^\top$ and $p = 1$), the D-PSGD converges with $\mathcal{O}(\exp(-R))$ in both cases when $\zeta_2^2 = 0$ and when $\zeta_2^2 = 10$, as the convergence rate of the D-PSGD shown in Table 1 indicates. Therefore, these numerical results show that the convergence rates of both the G-ECL and ECL do not depend on the heterogeneity of data distributions $\zeta$. This is consistent with Theorem 3.
Figure 1: Convergence of the error $\frac{1}{n} \sum_{i=1}^{n} \| x_i^{(r)} - x^* \|^2$ when varying the heterogeneity of data distributions $\zeta$ and the noise of the stochastic gradient $\sigma$.

Effect of Noise of Stochastic Gradient: Next, we discuss the effect of the noise of the stochastic gradient $\sigma$ on the convergence rate. When $\zeta^2 = 0$, the results show that the convergence rates of the D-PSGD, ECL, and G-ECL slow down when $\sigma^2 = 10$ compared to when $\sigma^2 = 0$ on all network topologies, which is consistent with Theorem 3 and the convergence rate of the D-PSGD.

Comparison with ECL and G-ECL: Next, we compare the results of the ECL and G-ECL. As we discuss in Sec. 6 and 7, Theorem 3 provides only the convergence rates of the G-ECL and does not show that there exists a step size $\eta'$ for the ECL to achieve the convergence rates shown in Theorem 3. However, Fig. 1 shows that the results of the ECL and G-ECL are almost equivalent for all settings, and, as discussed above, the numerical results of both the ECL and G-ECL coincide with the convergence rate of the G-ECL. Thus, experimentally, the ECL also converges with the convergence rates provided in Theorem 3.

8 Conclusion

In this work, we first investigate the relationship between the Gossip algorithm and ECL. Specifically, we show that if the hyperparameter of the ECL is set such that Assumption 1 holds, a node computes the average with its neighbors in the ECL as well as in the Gossip algorithm, and the update formulas of the ECL can be regarded as correcting the local stochastic gradient $\nabla f_i(x_i; \xi_i)$ in the Gossip algorithm. Subsequently, to make the convergence analysis tractable, we increase the degrees of freedom of hyperparameters of the ECL and propose the G-ECL, which contains the ECL as a special case. By using the proof techniques of the Gossip algorithm [11], we provide the convergence rate of the G-ECL in (strongly) convex and non-convex settings and show that they do not depend on the heterogeneity of data distributions. Through the synthetic experiments, we demonstrate that the numerical results of both the ECL and G-ECL coincide with the convergence rate of the G-ECL.
References

[1] Allen-Zhu, Z. and Yuan, Y. (2016). Improved SVRG for non-strongly-convex or sum-of-non-convex objectives. In International Conference on Machine Learning.

[2] Bauschke, H. H. and Combettes, P. L. (2017). Convex analysis and monotone operator theory in hilbert spaces. Springer, 2nd edition.

[3] Boyd, S., Parikh, N., Chu, E., Peleato, B., and Eckstein, J. (2011). Distributed optimization and statistical learning via the alternating direction method of multipliers. In Foundations and Trends in Machine Learning.

[4] Brown, T., Mann, B., Ryder, N., Subbiah, M., Kaplan, J. D., Dhariwal, P., Neelakantan, A., Shyam, P., Sastry, G., Askell, A., Agarwal, S., Herbert-Voss, A., Krueger, G., Henighan, T., Child, R., Ramesh, A., Ziegler, D., Wu, J., Winter, C., Hesse, C., Chen, M., Sigler, E., Litwin, M., Gray, S., Chess, B., Clark, J., Berner, C., McCandlish, S., Radford, A., Sutskever, I., and Amodei, D. (2020). Language models are few-shot learners. In Advances in Neural Information Processing Systems.

[5] Defazio, A., Bach, F., and Lacoste-Julien, S. (2014). SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives. In Advances in Neural Information Processing Systems.

[6] Devlin, J., Chang, M.-W., Lee, K., and Toutanova, K. (2019). BERT: Pre-training of deep bidirectional transformers for language understanding. In Conference of the North American Chapter of the Association for Computational Linguistics.

[7] Douglas, J. and Rachford, H. H. (1956). On the numerical solution of heat conduction problems in two and three space variables. In Transactions of the American mathematical Society.

[8] Giselsson, P. and Boyd, S. (2017). Linear convergence and metric selection for douglas-rachford splitting and admm. In IEEE Transactions on Automatic Control.

[9] Hong, M., Hajinezhad, D., and Zhao, M.-M. (2017). Prox-PDA: The proximal primal-dual algorithm for fast distributed nonconvex optimization and learning over networks. In International Conference on Machine Learning.

[10] Koloskova, A., Lin, T., and Stich, S. U. (2021). An improved analysis of gradient tracking for decentralized machine learning. In Advances in Neural Information Processing Systems.

[11] Koloskova, A., Loizou, N., Boreiri, S., Jaggi, M., and Stich, S. (2020). A unified theory of decentralized SGD with changing topology and local updates. In International Conference on Machine Learning.

[12] Kong, L., Lin, T., Koloskova, A., Jaggi, M., and Stich, S. (2021). Consensus control for decentralized deep learning. In International Conference on Machine Learning.

[13] Kovalev, D., Koloskova, A., Jaggi, M., Richtarik, P., and Stich, S. (2021). A linearly convergent algorithm for decentralized optimization: Sending less bits for free! In International Conference on Artificial Intelligence and Statistics.

[14] Lian, X., Zhang, C., Zhang, H., Hsieh, C.-J., Zhang, W., and Liu, J. (2017). Can decentralized algorithms outperform centralized algorithms? a case study for decentralized parallel stochastic gradient descent. In Advances in Neural Information Processing Systems.

[15] Lian, X., Zhang, W., Zhang, C., and Liu, J. (2018). Asynchronous decentralized parallel stochastic gradient descent. In International Conference on Machine Learning.

[16] Liu, X., Li, Y., Wang, R., Tang, J., and Yan, M. (2021a). Linear convergent decentralized optimization with compression. In International Conference on Learning Representations.

[17] Liu, Z., Lin, Y., Cao, Y., Hu, H., Wei, Y., Zhang, Z., Lin, S., and Guo, B. (2021b). Swin transformer: Hierarchical vision transformer using shifted windows. In International Conference on Computer Vision.
[18] Lorenzo, P. D. and Scutari, G. (2016). NEXT: in-network nonconvex optimization. In *IEEE Transactions on Signal and Information Processing over Networks*.

[19] Nedić, A., Olshevsky, A., and Shi, W. (2017). Achieving geometric convergence for distributed optimization over time-varying graphs. In *SIAM Journal on Optimization*.

[20] Niwa, K., Harada, N., Zhang, G., and Kleijn, W. B. (2020). Edge-consensus learning: Deep learning on p2p networks with nonhomogeneous data. In *International Conference on Knowledge Discovery and Data Mining*.

[21] Niwa, K., Zhang, G., Kleijn, W. B., Harada, N., Sawada, H., and Fujino, A. (2021). Asynchronous decentralized optimization with implicit stochastic variance reduction. In *International Conference on Machine Learning*.

[22] Paszke, A., Gross, S., Massa, F., Lerer, A., Bradbury, J., Chanan, G., Killeen, T., Lin, Z., Gimelshein, N., Antiga, L., Desmaison, A., Kopf, A., Yang, E., DeVito, Z., Raison, M., Tejani, A., Chilamkurthy, S., Steiner, B., Fang, L., Bai, J., and Chintala, S. (2019). Pytorch: An imperative style, high-performance deep learning library. In *Advances in Neural Information Processing Systems*.

[23] Rajawat, K. and Kumar, C. (2020). A primal-dual framework for decentralized stochastic optimization. In *arXiv*.

[24] Ryu, E. K. and Boyd, S. P. (2015). A primer on monotone operator methods. In *Applied and Computational Mathematics*.

[25] Sarlin, P.-E., DeTone, D., Malisiewicz, T., and Rabinovich, A. (2020). SuperGlue: Learning feature matching with graph neural networks. In *IEEE Conference on Computer Vision and Pattern Recognition*.

[26] Sherson, T. W., Heusdens, R., and Kleijn, W. B. (2019). Derivation and analysis of the primal-dual method of multipliers based on monotone operator theory. In *IEEE Transactions on Signal and Information Processing over Networks*.

[27] Stich, S. (2019). Unified optimal analysis of the (stochastic) gradient method. In *arXiv*.

[28] Stich, S. U. and Karimireddy, S. P. (2020). The error-feedback framework: Better rates for SGD with delayed gradients and compressed communication. In *Journal of Machine Learning Research*.

[29] Vogels, T., Karimireddy, S. P., and Jaggi, M. (2020). Practical low-rank communication compression in decentralized deep learning. In *Advances in Neural Information Processing Systems*.

[30] Yuan, K., Chen, Y., Huang, X., Zhang, Y., Pan, P., Xu, Y., and Yin, W. (2021). DecentLaM: Decentralized momentum SGD for large-batch deep training. In *International Conference on Computer Vision*.

[31] Zhang, G. and Heusdens, R. (2018). Distributed optimization using the primal-dual method of multipliers. In *IEEE Transactions on Signal and Information Processing over Networks*.

[32] Zhang, R. and Kwok, J. (2014). Asynchronous distributed ADMM for consensus optimization. In *International Conference on Machine Learning*.
A Related Work

A.1 Gossip Algorithm

One of the most widely used algorithms for decentralized learning is the D-PSGD [14] also known as the Gossip algorithm. Recently, the convergence rate of the Gossip algorithm has been well investigated. Lian et al. [15] extended the Gossip algorithm to the asynchronous setting and analyzed the convergence rate. Koloskova et al. [11] provided the convergence rate of the Gossip algorithm when the network topology $G$ changes over time or when using the local steps. Yuan et al. [30] analyzed the convergence rate of the Gossip algorithm when using the momentum SGD instead of the SGD. These theoretical analyses indicate that the convergence rate of the Gossip algorithm slows down when the data distribution held by each node is statistically heterogeneous.

A.2 Primal-Dual Algorithm

In addition to the Gossip algorithm, primal-dual algorithms are applicable to decentralized learning [9,16,13]. As shown in Eq. (3), the decentralized learning problem can be formulated as a linearly constrained problem. One of the most famous algorithms for solving a linearly constrained problem is the ADMM, which has been applied to decentralized learning [3,32]. Zhang and Heusdens [31] proposed the PDMM and showed the PDMM converges faster than the ADMM. Recently, Sherson et al. [26] showed that the PDMM can be naturally derived by using Douglas-Rachford splitting [7], and Niwa et al. [20] applied it to a neural network, which is called the ECL. Recently, Rajawat and Kumar [23] provided the convergence rate of the ECL in the convex case and proposed to apply stochastic variance reduction methods [1,5] to the ECL.

A.3 Gradient Tracking Method

One of the most popular algorithms whose convergence rate does not depend on the heterogeneity of the data distributions is the gradient tracking method [18,19,10]. In addition, in Sec. B we discuss the relationship between the ECL and gradient tracking method.
B Relationship between Gradient Tracking Method and ECL

In this section, we discuss the relationship between the gradient tracking methods \cite{18,19,10} and ECL.

B.1 Gradient Tracking Method

In the gradient tracking method, the model parameter $x_i$ is updated as follows:

$$x_i^{(r+1)} = \sum_{j \in \mathcal{N}_i^+} W_{ij} \left( x_j^{(r)} - \eta p_j^{(r)} \right),$$  \hspace{1cm} (19)

$$p_i^{(r+1)} = \sum_{j \in \mathcal{N}_i^+} W_{ij} p_j^{(r)} + \left( \nabla F_i(x_i^{(r+1)}; \xi_i^{(r+1)}) - \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) \right),$$  \hspace{1cm} (20)

where $W$ is assumed to be a mixing matrix, as in the Gossip algorithm.

B.2 Discussion

To discuss the relationship between the gradient tracking method and ECL, we further reformulate the update formulas of the ECL.

Theorem 4. Suppose that the hyperparameter $\theta = \frac{1}{2}$, the dual variable $z_{ij}^{(0)}$ is initialized to $A_{ij} x_j^{(0)}$, and the hyperparameter $\{\alpha_{ij}\}_{ij}$ is set such that $\alpha_{ij} = \alpha_{ji} \geq 0$ for all $(i, j) \in \mathcal{E}$. Then, the update formulas Eq. (7) and Eqs. (5-6) are equivalent to the following:

$$\tilde{x}_i^{(r)} = \sum_{j \in \mathcal{N}_i^+} W_{ij} x_j^{(r)},$$  \hspace{1cm} (21)

$$x_i^{(r+1)} = \tilde{x}_i^{(r)} - \eta' p_i^{(r)},$$  \hspace{1cm} (22)

$$p_i^{(r+1)} = \sum_{j \in \mathcal{N}_i^+} W_{ij} p_j^{(r)} + \left( \nabla F_i(x_i^{(r+1)}; \xi_i^{(r+1)}) - \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) \right)$$

$$- \sum_{j \in \mathcal{N}_i} \alpha_{ij} \frac{\tilde{x}_j^{(r)} - \tilde{x}_i^{(r)}}{2},$$  \hspace{1cm} (23)

where $W_{ij}$ and $\eta'$ are defined by Eq. (11).

Proof. Defining $p_i^{(r)} := \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) - c_i^{(r)}$, the statement follows from Theorem 1. \hfill \square

If we omit the term $T$ in Eq. (23), the update formulas of the gradient tracking method Eqs. (19-20) and that of the ECL Eqs. (21-23) are almost equivalent. The only difference is the order of the calculation of the weighted average and the parameter update. Moreover, since Theorem 3 indicates that the G-ECL converges when $\alpha_{ij} = 0$ for all $(i, j) \in \mathcal{E}$, the term $T$ in Eq. (23) does not play an important role in the convergence of the G-ECL. Therefore, the ECL modifies the local stochastic gradient $\nabla F_i(x_i; \xi_i)$ in the update formulas of the Gossip algorithm as well as the gradient tracking methods, which makes the ECL robust to the heterogeneity of data distributions.
C Issues of Existing Convergence Analysis of Edge-Consensus Learning

In this section, we point out the issues in the proofs of the previous study [21] that attempted to analyze the convergence rate of the ECL.

The previous work [21] analyzed the ECL and proposed setting \{\alpha_{ij}\}_{ij} as follows:

\[
\alpha_{ij} = \frac{1}{\eta|\mathcal{V}_i|(K - 1)},
\]

where \(K\) denotes the number of local steps. Note that in this work, we provide the convergence rate of the G-ECL without local steps (i.e., we provide the convergence rate when each node communicates with its neighbors at each update). Then, when \{\alpha_{ij}\}_{ij} is set as in Eq. (24), the ECL is named the ECL-ISVR, and the previous work [21] attempted to analyze the convergence rate of the ECL-ISVR in both (strongly) convex and non-convex cases.

However, there are some errors in the proofs. In the strongly convex and convex cases, strong approximations were used in the first and third equations in [21, Sec. C.1], and these equations do not hold for either the ECL or ECL-ISVR in practice. Similarly, in the non-convex case, strong approximations were used in the first and third equations in [21, Sec. C.2], and these equations do not hold. Therefore, the convergence rates shown in this previous work can not be regarded as those of the ECL and ECL-ISVR.
D Additional Experiments

In this section, we present a more detailed analysis of the effect of the heterogeneity of data distributions $\zeta$ and noise of the stochastic gradient $\sigma$ on the convergence rate.

**Effect of Heterogeneity of Data Distributions ($\sigma^2 = 0$):** We first discuss the effect of $\zeta$ on the convergence rate when $\sigma^2 = 0$. Fig. 2 shows the error $\frac{1}{n} \sum_{i=1}^{n} \|x_i^{(r)} - x^*\|^2$ after $10^4$ rounds when varying $\zeta$ and setting $\sigma^2 = 0$. The results show that when $G$ is a ring or torus (i.e., $p < 1$), the error of the D-PSGD increases linearly with respect to $\zeta^2$, and when $G$ is a fully connected graph (i.e., $p = 1$), the error of the D-PSGD is almost the same even if $\zeta^2$ is increased. In contrast, the errors of the ECL and G-ECL are almost the same, even if $\zeta^2$ is increased for all network topologies. Therefore, the numerical results are consistent with the theoretical results in Theorem 3.

**Effect of Noise of Stochastic Gradient ($\zeta^2 = 0$):** Next, we discuss the effect of $\sigma$ on the convergence rate when $\zeta^2 = 0$. Fig. 3 shows the error $\frac{1}{n} \sum_{i=1}^{n} \|x_i^{(r)} - x^*\|^2$ after $10^4$ rounds when varying $\sigma$ and setting $\zeta^2 = 0$. The results show that the errors of all comparison methods increase linearly with respect to $\sigma^2$ for all network topologies. The theoretical results shown in Table 1 indicate that the convergence rates of the D-PSGD and G-ECL are $O(\sigma^2)$. Thus, the theoretical results are consistent with the numerical results. Moreover, Fig. 3 shows that in all comparison methods, the effect of $\sigma$ on the convergence is almost the same for all network topologies. In the convergence rate of both the D-PSGD and G-ECL, the second term $O(\frac{\sigma^2}{\mu n R})$, which does not depend on the network topology, is more dominant than the third term when the number of round $R$ is sufficiently large. Therefore, the numerical results are consistent with our theoretical results.
Effect of Heterogeneity of Data Distributions ($\sigma^2 = 10$): Next, we discuss the effect of $\zeta$ on the convergence rate when $\sigma^2 = 10$. Fig. 4 shows the error $\frac{1}{n} \sum_{i=1}^{n} \| x^{(r)}_i - x^* \|^2$ after $10^4$ rounds when varying $\zeta$ and setting $\sigma^2 = 10$. The results show that when $G$ is a ring, the error of the D-PSGD increases linearly with respect to $\zeta^2$. When $G$ is a torus or fully connected graph, the error of the D-PSGD is almost the same even if $\zeta^2$ is increased. This is because the effect of $\sigma^2$ is more dominant than the one of $\zeta^2$. Figs. 2 and 3 show that when $G$ is a torus, the error of the D-PSGD is approximately $1.0 \times 10^{-5}$ when $\zeta^2 > 0$ and $\sigma^2 = 0$ and is approximately $1.0 \times 10^{-4}$ when $\zeta^2 = 0$ and $\sigma^2 > 0$. Therefore, Fig. 4 indicates that when $G$ is a torus or fully connected graph, the error of the D-PSGD is almost the same even if $\zeta^2$ is increased. In contrast, the errors of the ECL and G-ECL are almost the same, even if $\zeta^2$ is increased for all network topologies. Therefore, the numerical results are consistent with the theoretical results in Theorem 3.

Effect of Noise of Stochastic Gradient ($\zeta^2 = 10$): Next, we discuss the effect of $\sigma$ on the convergence rate when $\zeta^2 = 10$. Fig. 5 shows the error $\frac{1}{n} \sum_{i=1}^{n} \| x^{(r)}_i - x^* \|^2$ after $10^4$ rounds when varying $\sigma$ and setting $\zeta^2 = 10$. The results show that the errors of all comparison methods increase linearly with respect to $\sigma^2$ for all network topologies. When $G$ is a ring, the error of the D-PSGD is consistently larger than those of the G-ECL and ECL. This is because the error of the D-PSGD is larger than those of the G-ECL and ECL when $\zeta^2 > 0$, as Figs. 2 and 4 indicate. When $G$ is a torus or fully connected graph, the errors of all comparison methods are almost the same. This is because the effect of $\sigma^2$ is more dominant than that of $\zeta^2$, as Fig. 4 indicates. Therefore, the numerical results are consistent with convergence rates of the G-ECL.
E. Proof of Theorem 1

Lemma 2. Suppose that the hyperparameter \( \theta = \frac{1}{2} \), the dual variable \( z^{(0)}_{ij} \) is initialized to \( A_{ij}x^{(0)}_j \), and the hyperparameter \( \{\alpha_{ij}\}_{ij} \) is set such that \( \alpha_{ij} = \alpha_{ji} \geq 0 \) for all \( (i, j) \in E \). Then, the update formulas Eq. (7) and Eqs. (5-6) are equivalent to the following:

\[
x_i^{(r+1)} = \sum_{j \in N_i^r} W_{ij} x_j^{(r)} - \frac{\eta}{1 + \eta \sum_{j \in N_i} \alpha_{ij}} \left( \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) - c_i^{(r)} \right), \tag{25}
\]

\[
c_i^{(r+1)} = c_i^{(r)} + \frac{1}{2} \sum_{j \in N_i} \alpha_{ij} (x_i^{(r+1)} - x_i^{(r+1)}), \tag{26}
\]

where \( c_i^{(0)} := \frac{1}{2} \sum_{j \in N_i} \alpha_{ij} (x_j^{(0)} - x_i^{(0)}) \), and \( W_{ij} \) is defined as follows:

\[
W_{ij} := \begin{cases} 
2 + \eta \sum_{k \in N_i^r} \alpha_{ik} & \text{if } i = j \\
\frac{\eta \alpha_{ij}}{2(1 + \eta \sum_{k \in N_i^r} \alpha_{ik})} & \text{if } (i, j) \in E \\
0 & \text{otherwise}
\end{cases} \tag{27}
\]

Proof. The update formulas of the ECL can be written as follows:

\[
x_i^{(r+1)} = \sum_{j \in N_i^r} \frac{1}{1 + \eta \sum_{j \in N_i} \alpha_{ij}} \left( x_i^{(r)} - \eta \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) + \eta \sum_{j \in N_i} \alpha_{ij} x_j^{(r)} \right), \\
z_{ij}^{(r+1)} = \frac{1}{2} (z_{ij}^{(r)} + z_{ji}^{(r)}) - A_{ij}x_i^{(r+1)}.
\]

Defining \( u_{ij}^{(r)} := z_{ij}^{(r)} + A_{ij}x_j^{(r)} \), the update formulas of the ECL can be rewritten as follows:

\[
x_i^{(r+1)} = \frac{1}{1 + \eta \sum_{j \in N_i} \alpha_{ij}} \left( x_i^{(r)} + \eta \left( \sum_{j \in N_i} \alpha_{ij} x_j^{(r)} \right) - \eta \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) + \eta \sum_{j \in N_i} \alpha_{ij} x_j^{(r)} \right), \\
u_{ij}^{(r+1)} = \frac{1}{2} (u_{ij}^{(r)} - A_{ij}x_j^{(r)}) + \frac{1}{2} (u_{ji}^{(r)} - A_{ji}x_i^{(r)}).
\]

From the above update formula for \( u_{ij} \), the update formulas for \( u_{ij} \) and \( u_{ji} \) are equivalent. Then, it holds that for any round \( r > 0 \),

\[
u_{ij}^{(r)} = u_{ji}^{(r)}. \tag{28}
\]

Moreover, because \( u_{ij}^{(0)} = 0 \), Eq. (28) holds for any round \( r \geq 0 \).

We define \( b_i^{(r)} := \sum_{j \in N_i^r} \alpha_{ij} A_{ij} u_{ij}^{(r)} \). From Eq. (28), \( b_i^{(r)} = \sum_{j \in N_i^r} \alpha_{ij} A_{ij} x_j^{(r)} \) holds for any round \( r \). The update formulas of the ECL are then rewritten as follows:

\[
x_i^{(r+1)} = \frac{1}{1 + \eta \sum_{j \in N_i} \alpha_{ij}} \left( x_i^{(r)} + \eta \left( \sum_{j \in N_i} \alpha_{ij} x_j^{(r)} \right) - \eta \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) + \eta b_i^{(r)} \right), \\
b_i^{(r+1)} = b_i^{(r)} + \frac{1}{2} \sum_{j \in N_i} \alpha_{ij} (x_j^{(r)} - x_i^{(r)}).
\]
Defining $c_i^{(r)} := b_i^{(r)} + \frac{1}{2} \sum_{j \in N_i} \alpha_{ij} (x_j^{(r)} - x_i^{(r)})$, the update formulas of the ECL are rewritten as follows:

$$
\begin{align*}
    x_i^{(r+1)} &= \frac{1}{2(1 + \eta \sum_{j \in N_i} \alpha_{ij})} \left( (2 + \eta \sum_{j \in N_i} \alpha_{ij}) x_i^{(r)} + \eta \sum_{j \in N_i} \alpha_{ij} x_j^{(r)} \right) \\
    &\quad - \frac{\eta}{1 + \eta \sum_{j \in N_i} \alpha_{ij}} \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) - c_i^{(r)}, \\
    c_i^{(r+1)} &= c_i^{(r)} + \frac{1}{2} \sum_{j \in N_i} \alpha_{ij} (x_j^{(r+1)} - x_i^{(r+1)}).
\end{align*}
$$

This concludes the proof.

**E.1 Proof of Theorem 1**

*Proof.* We define $\tilde{x}_i^{(r)} \in \mathbb{R}^d$ as follows:

$$
\tilde{x}_i^{(r)} := \sum_{j \in N_i^+} W_{ij} x_j^{(r)}.
$$

Then, from Lemma 2 we have

$$
\begin{align*}
    c_i^{(r+1)} &= c_i^{(r)} + \frac{1}{2} \sum_{j \in N_i} \alpha_{ij} (x_j^{(r+1)} - x_i^{(r+1)}) \\
    &= c_i^{(r)} + \frac{1}{2} \sum_{j \in N_i} \alpha_{ij} (\tilde{x}_j^{(r)} - x_i^{(r)}) - \frac{1}{2} \left( \sum_{j \in N_i} \alpha_{ij} \right) x_i^{(r+1)} \\
    &= c_i^{(r)} + \frac{1}{2} \sum_{j \in N_i} \alpha_{ij} \left( \tilde{x}_j^{(r)} - \eta' \left( \nabla F_j(x_j^{(r)}; \xi_j^{(r)}) - c_j^{(r)} \right) \right) \\
    &\quad - \frac{1}{2} \left( \sum_{j \in N_i} \alpha_{ij} \right) \left( \tilde{x}_i^{(r)} - \eta' \left( \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) - c_i^{(r)} \right) \right) \\
    &= \left( 1 - \frac{\eta' \sum_{j \in N_i} \alpha_{ij}}{2} \right) c_i^{(r)} + \frac{1}{2} \sum_{j \in N_i} \alpha_{ij} \eta' c_j^{(r)} \\
    &\quad - \left( 1 - \frac{\eta' \sum_{j \in N_i} \alpha_{ij}}{2} \right) \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) + \frac{1}{2} \sum_{j \in N_i} \alpha_{ij} \eta' \nabla F_j(x_j^{(r)}; \xi_j^{(r)}) \\
    &\quad + \frac{1}{2} \left( \sum_{j \in N_i} \alpha_{ij} \left( \tilde{x}_j^{(r)} - x_i^{(r)} \right) \right) + \nabla F_i(x_i^{(r)}; \xi_i^{(r)}).
\end{align*}
$$

Using Eq. (11), we get

$$
\begin{align*}
    c_i^{(r+1)} &= \sum_{j \in N_i^+} W_{ij} \left( c_j^{(r)} - \nabla F_j(x_j^{(r)}; \xi_j^{(r)}) \right) + \frac{1}{2} \left( \sum_{j \in N_i} \alpha_{ij} \left( \tilde{x}_j^{(r)} - x_i^{(r)} \right) \right) + \nabla F_i(x_i^{(r)}; \xi_i^{(r)}).
\end{align*}
$$

This concludes the proof.

**F Proof of Theorem 2**

*Proof.* When $(i, j) \in \mathcal{E}$, we have

$$
W_{ij} = \frac{\eta \alpha_{ij}}{2(1 + \eta \alpha)} = \frac{\eta \alpha_{ji}}{2(1 + \eta \alpha)} = W_{ji}.
$$
Therefore, $W$ is symmetric. Next, we prove that $W$ is doubly stochastic. We have
\[
\sum_{j=1}^{n} W_{ij} = \frac{2 + \eta}{2(1 + \eta \sum_{k\in\mathcal{N}_i} \alpha_{ik})} + \sum_{j\in\mathcal{N}_i} \frac{\eta \alpha_{ij}}{2(1 + \eta \sum_{k\in\mathcal{N}_i} \alpha_{ik})} = 1,
\]
\[
\sum_{i=1}^{n} W_{ij} = \sum_{i=1}^{n} W_{ji} = 1.
\]
This concludes the proof. \hfill \qed

**G Proof of Lemma 1**

**Proof.** From Lemma 2 we have
\[
\sum_{i=1}^{n} c_i^{(r+1)} = \sum_{i=1}^{n} c_i^{(r)} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j\in\mathcal{N}_i} \alpha_{ij} x_j^{(r+1)} - \frac{1}{2} \sum_{i=1}^{n} \sum_{j\in\mathcal{N}_i} \alpha_{ij} x_i^{(r+1)}.
\]
We define $\mathcal{E}^> := \{(i, j) \in \mathcal{E} | i > j\}$ and $\mathcal{E}^< := \{(i, j) \in \mathcal{E} | i < j\}$. Then, we get
\[
\sum_{i=1}^{n} \sum_{j\in\mathcal{N}_i} \alpha_{ij} x_j^{(r+1)} = \sum_{(i,j)\in\mathcal{E}^>} \alpha_{ij} x_j^{(r+1)} + \sum_{(i,j)\in\mathcal{E}^<} \alpha_{ij} x_j^{(r+1)}
\]
\[
= \sum_{(j,i)\in\mathcal{E}^<} \alpha_{ij} x_j^{(r+1)} + \sum_{(j,i)\in\mathcal{E}^>} \alpha_{ij} x_j^{(r+1)}
\]
\[
= \sum_{i=1}^{n} \sum_{j\in\mathcal{N}_j} \alpha_{ij} x_j^{(r+1)}
\]
\[
= \sum_{i=1}^{n} \sum_{j\in\mathcal{N}_i} \alpha_{ij} x_i^{(r+1)}
\]
where we use $\alpha_{ij} = \alpha_{ji}$ in the last equation. Then, we get
\[
\sum_{i=1}^{n} c_i^{(r+1)} = \sum_{i=1}^{n} c_i^{(r)}.
\]
From the initial value of $c_i$, we have $\sum_{i=1}^{n} c_i^{(0)} = 0$. Therefore, $\sum_{i=1}^{n} c_i^{(r)} = 0$ for any round $r$.

From Theorems 1 and 2 we have
\[
\frac{1}{n} \sum_{i=1}^{n} x_j^{(r+1)} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j\in\mathcal{N}_i} W_{ij} x_j^{(r)} - \eta \frac{1}{n} \sum_{i=1}^{n} \left( \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) - c_i^{(r)} \right)
\]
\[
= \frac{1}{n} \sum_{j=1}^{n} W_{ij} x_j^{(r)} - \eta \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(x_i^{(r)}; \xi_i^{(r)}) + \eta \frac{1}{n} \sum_{i=1}^{n} c_i^{(r)}
\]
\[
= \frac{1}{n} \sum_{j=1}^{n} x_j^{(r)} \sum_{i=1}^{n} W_{ij} - \eta \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(x_i^{(r)}; \xi_i^{(r)})
\]
\[
= \frac{1}{n} \sum_{j=1}^{n} x_j^{(r)} - \eta \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(x_i^{(r)}; \xi_i^{(r)}).
\]
This concludes the proof. \hfill \qed
Algorithm 2: Update procedure at node $i$ in the G-ECL.

1: **Input:** Set the step size $\eta' > 0$, the mixing matrix $W \in [0, 1]^{n \times n}$, and $\{\alpha_{ij}\}_{ij}$ that satisfies $\alpha_{ij} = \alpha_{ji} \geq 0$ for all $(i, j) \in E$ and $i \in [n]$. Initialize $x_i^{(0)}$ with the same parameter for all $i \in [n]$. Note that $\eta'$, $W$, and $\{\alpha_{ij}\}_{ij}$ can be set independently as hyperparameters in the G-ECL.

2: for $r = 0, 1, \ldots, R$ do

3: for $j \in N_i$ do

4: Transmit $i \rightarrow j(x_i^{(r)})$.

5: Receive $i \leftarrow j(x_j^{(r)})$.

6: end for

7: $\tilde{x}_i^{(r)} \leftarrow \sum_{j \in N_i^+} W_{ij} x_j^{(r)}$.

8: Sample $\xi_i^{(r)}$ and compute $g_i^{(r)} := \nabla F_i(x_i^{(r)}; \xi_i^{(r)})$.

9: $x_i^{(r+1)} \leftarrow \tilde{x}_i^{(r)} - \eta' \left(g_i^{(r)} - c_i^{(r)}\right)$.

10: for $j \in N_i$ do

11: Transmit $i \rightarrow j(\tilde{x}_j^{(r)}, c_j^{(r)} - g_j^{(r)})$.

12: Receive $i \leftarrow j(\tilde{x}_j^{(r)}, c_j^{(r)} - g_j^{(r)})$.

13: end for

14: $c_i^{(r+1)} \leftarrow \sum_{j \in N_i^+} W_{ij} \left(c_j^{(r)} - g_j^{(r)}\right) + g_i^{(r)} + \sum_{j \in N_i} \alpha_{ij} (\tilde{x}_j^{(r)} - \tilde{x}_i^{(r)})$.

15: end for

Alg. 2 shows the pseudo-code of the G-ECL. Note that from Theorem 1, the update formulas of Alg. 1 and Alg. 2 are equivalent when the hyperparameter $\theta$ is $\frac{1}{2}$, and $\eta'$, $W$, and $\{\alpha_{ij}\}_{ij}$ are set as in Eq. (11).
I Proof of Theorem [3]

I.1 G-ECL in Matrix Notation

We define \( X^{(r)}, C^{(r)}, \bar{X}^{(r)} \in \mathbb{R}^{d \times n}, \nabla F(X^{(r)}; \xi^{(r)}), \) and \( \nabla f(X^{(r)}) \) as follows:

\[
X^{(r)} := \begin{bmatrix} x_1^{(r)}, \ldots, x_n^{(r)} \end{bmatrix}, \quad C^{(r)} := \begin{bmatrix} c_1^{(r)}, \ldots, c_n^{(r)} \end{bmatrix}, \quad \bar{X}^{(r)} := \begin{bmatrix} \bar{x}^{(r)}, \ldots, \bar{x}^{(r)} \end{bmatrix},
\]

\[
\nabla F(X^{(r)}; \xi^{(r)}) := \begin{bmatrix} \nabla F_1(x_1^{(r)}; \xi_1^{(r)}), \ldots, \nabla F_n(x_n^{(r)}; \xi_n^{(r)}) \end{bmatrix},
\]

\[
\nabla f(X^{(r)}) := \begin{bmatrix} \nabla f_1(x_1^{(r)}), \ldots, \nabla f_n(x_n^{(r)}) \end{bmatrix},
\]

where \( \bar{x}^{(r)} := \frac{1}{n} \sum_{i=1}^{n} x_i^{(r)} \). We define \( E := \text{diag}(\sum_{k \in N_1} \alpha_{k1}, \ldots, \sum_{k \in N_n} \alpha_{kn}) \in \mathbb{R}^{n \times n} \) and \( D \in \mathbb{R}^{n \times n} \) whose \((i, j)\)-element is

\[
D_{ij} = \begin{cases} \alpha_{ij} & \text{if } (i, j) \in \mathcal{E} \\ 0 & \text{otherwise}. \end{cases}
\]

Note that because we assume that \( \alpha_{ij} = \alpha_{ji} \) for all \((i, j) \in \mathcal{E}, D \) is symmetric. Then, the update formulas Eqs. (8, 10) can be rewritten as follows:

\[
X^{(r+1)} = X^{(r)} W - \eta' (\nabla F(X^{(r)}; \xi^{(r)}) - C^{(r)}), \tag{30}
\]

\[
C^{(r+1)} = (C^{(r)} - \nabla F(X^{(r)}; \xi^{(r)})) W + \frac{1}{2} X^{(r)} W (D - E) + \nabla F(X^{(r)}; \xi^{(r)}), \tag{31}
\]

I.2 Preliminary and Technical Lemma

Definition 2 (\( \tau \)-Slow Increasing [28]). The sequence \( \{a_r\}_{r \geq 0} \) of a positive value is called \( \tau \)-slow increasing if it holds that for any \( r \geq 0 \),

\[
a_r \leq a_{r+1} \leq \left(1 + \frac{1}{2\tau}\right) a_r.
\]

Lemma 3. For any \( x, y \in \mathbb{R}^d, \gamma > 0 \), it holds that

\[
\|x + y\|^2 \leq (1 + \gamma)\|x\|^2 + (1 + \gamma^{-1})\|y\|^2. \tag{32}
\]

Lemma 4. For any \( x_1, \ldots, x_n \in \mathbb{R}^d \), it holds that

\[
\left\| \sum_{i=1}^{n} a_i \right\| \leq n \sum_{i=1}^{n} \|a_i\|. \tag{33}
\]

Lemma 5. For any \( x, y \in \mathbb{R}^d \) and \( \gamma > 0 \), it holds that

\[
2\langle x, y \rangle \leq \gamma \|x\|^2 + \gamma^{-1} \|y\|^2. \tag{34}
\]

Lemma 6. Suppose that Assumption [3] holds and \( f_i \) is convex. Then, it holds that for any \( x, y \in \mathbb{R}^d \),

\[
\|\nabla f_i(x) - \nabla f_i(y)\|^2 \leq 2L(f_i(x) - f_i(y) - \langle x - y, \nabla f_i(y) \rangle). \tag{35}
\]

I.3 Convergence Analysis for Convex Cases

I.3.1 Additional Notation

In Sec. I.3, we define \( \Xi^{(r)}, C^{(r)}, b, \) and \( b' \) as follows to simplify the notation:

\[
\Xi^{(r)} := \frac{1}{n} E \sum_{i=1}^{n} \|x_i^{(r)} - \bar{x}^{(r)}\|^2, \quad C^{(r)} := \frac{1}{n} E \|\nabla f(X^*) - C^{(r)}\|^2, \tag{36}
\]

\[
b := \left\| \frac{1}{2} (D - E) \right\|^2, \quad b' := \|I - W\|^2. \tag{37}
\]
1.3.2 Convergence Analysis

**Lemma 7** (Descent Lemma for Convex Cases). Suppose that Assumptions 2, 3, 4 and 5 hold. If \( \eta' \leq \frac{1}{2T} \), we have:

\[
\mathbb{E}_{r+1} \left\| \bar{x}^{(r+1)} - x^* \right\|^2 \\
\leq \left( 1 - \frac{\eta' \mu}{2} \right) \left\| \bar{x}^{(r)} - x^* \right\|^2 + \frac{\eta'^2 \sigma^2}{n} \mathbb{E}_{r+1} \left( f(\bar{x}^{(r)}) - f(x^*) \right) + \frac{3L\eta' n}{n} \sum_{i=1}^{n} \left\| x_i^{(r)} - x^* \right\|^2.
\]

**Proof.** The statement follows from Lemma 8 in [11].

**Lemma 8** (Recursion for Consensus Distance). Suppose that Assumptions 2, 3, 4 and 5 hold, and \( \{\alpha_{ij}\}_{ij} \) is set such that \( \alpha_{ij} = \alpha_{ji} \geq 0 \) for all \((i, j) \in \mathcal{E}\). Then, it holds that

\[
\Xi^{(r)} \leq \left( 1 - \frac{\eta' \mu}{2} \right) \Xi^{(r-1)} + \frac{9L^2\eta^2}{p} \Xi^{(r-1)} + \frac{18L^2\eta^2}{p} \Xi^{(r-1)} + \frac{9}{p} \eta'^2 \sigma^2.
\]

**Proof.** By using \( \sum_{i=1}^{n} \| a_i - \bar{a} \|^2 \leq \sum_{i=1}^{n} \| a_i \|^2 \) for any \( a_1, \ldots, a_n \in \mathbb{R}^d \), we have

\[
n\Xi^{(r)} = \mathbb{E} \left\| \bar{x}^{(r)} - \bar{x}^{(r-1)} \right\|^2 \\
= \mathbb{E} \left\| \bar{x}^{(r)} - \bar{x}^{(r-1)} - (\bar{x}^{(r)} - \bar{x}^{(r-1)}) \right\|^2 \\
\leq \mathbb{E} \left\| \bar{x}^{(r)} - \bar{x}^{(r-1)} \right\|^2.
\]

Then, we get

\[
\mathbb{E}_{r+1} \left\| \bar{x}^{(r)} - \bar{x}^{(r-1)} \right\|^2 \\
\leq \mathbb{E}_{r+1} \left\| \bar{x}^{(r-1)} - \bar{x}^{(r-1)} \right\|^2 \\
\leq \left( 1 - \frac{\eta' \mu}{2} \right) \mathbb{E}_{r+1} \left\| \bar{x}^{(r-1)} - \bar{x}^{(r-1)} \right\|^2 + \frac{\eta'^2 \sigma^2}{n} + \frac{9L\eta' n}{n} \sum_{i=1}^{n} \left\| x_i^{(r)} - x^* \right\|^2.
\]

By substituting \( \gamma = \frac{\eta'}{2} \), we get

\[
\mathbb{E}_{r+1} \left\| \bar{x}^{(r)} - \bar{x}^{(r-1)} \right\|^2 \\
\leq \left( 1 - \frac{\eta' \mu}{2} \right) \mathbb{E}_{r+1} \left\| \bar{x}^{(r)} - \bar{x}^{(r-1)} \right\|^2 + \frac{3 \eta'^2 \sigma^2}{n} + \frac{6Ln(\bar{x}^{(r-1)} - f(x^*))}{n} + 3 \mathbb{E}_{r+1} \left\| \bar{x}^{(r)} - \bar{x}^{(r-1)} \right\|^2.
\]

This concludes the proof.
Lemma 9. Suppose that Assumptions 2, 3, 4 and 5 hold, and \( \{\alpha_{ij}\}_{ij} \) is set such that \( \alpha_{ij} = \alpha_{ji} \geq 0 \) for all \( (i, j) \in \mathcal{E} \). Then, it holds that

\[
\mathcal{E}^{(r+1)} \leq (1 - \frac{p}{2}) \mathcal{E}^{(r)} + \left( \frac{9(1 - p)b}{p} + \frac{9L^2b'}{p} \right) \Xi^{(r)} + \frac{18Lb'}{p} (\mathbb{E} f(\bar{x}^{(r)}) - f(x^*)) + b'\sigma^2.
\]

Proof. We have

\[
\begin{align*}
\mathbb{E}_{r+1} \left\| C^{(r+1)} - \nabla f(X^*) \right\|^2_F \\
&\leq (1 - \frac{p}{2}) \left\| C^{(r)} - \nabla f(X^*) \right\|^2_F \\
&\quad + \frac{3}{p} \left\| \frac{1}{2} X^{(r)} W(D - E) + (\nabla f(X^*) - \nabla f(\bar{x}^{(r)}))(W - I) \right\|^2_F \\
&\quad + n b'\sigma^2 \\
&= (1 - \frac{p}{2}) \left\| C^{(r)} - \nabla f(X^*) \right\|^2_F \\
&\quad + \frac{3}{p} \left\| \frac{1}{2} X^{(r)} W(D - E) + (\nabla f(X^*) - \nabla f(\bar{x}^{(r)}))(W - I) \right\|^2_F \\
&\quad + n b'\sigma^2 \\
&\leq (1 - \frac{p}{2}) \left\| C^{(r)} - \nabla f(X^*) \right\|^2_F \\
&\quad + \frac{9}{p} \left\| \frac{1}{2} X^{(r)} W(D - E) \right\|^2_F \\
&\quad + n b'\sigma^2 \\
&\leq (1 - \frac{p}{2}) \left\| C^{(r)} - \nabla f(X^*) \right\|^2_F + \frac{9}{p} \left\| \frac{1}{2} X^{(r)} W(D - E) \right\|^2_F + 18Lb'/p (f(\bar{x}^{(r)}) - f(x^*)) \\
&\quad +\frac{18Lb'/p}{p} (f(\bar{x}^{(r)}) - f(x^*)) + b'\sigma^2.
\end{align*}
\]

From Lemma 4 we have \( \frac{1}{n} C^{(r)} 11^T = 0 \) and \( \frac{1}{n} \nabla f(X^*) 11^T = 0 \). Then, by substituting \( \gamma = \frac{p}{2} \), we get
Using the definitions of $D$ and $E$, we have $\bar{X}^{(r)}(D - E) = 0$. Then, we get

$$
\mathbb{E}_{r+1} \left\| C^{(r+1)} - \nabla f(X^*) \right\|_F^2
\leq (1 - \frac{p}{2}) \left\| C^{(r)} - \nabla f(X^*) \right\|_F^2 + \frac{9p}{2} \left\| \frac{1}{2} (X^{(r)} W - \bar{X}^{(r)})(D - E) \right\|_F^2
+ \frac{18Lb'n}{p} (f(\bar{x}^{(r)}) - f(x^*)) + \frac{9L^2 b'}{p} \left\| X^{(r)} - \bar{X}^{(r)} \right\|_F^2 + nb'\sigma^2
\leq (1 - \frac{p}{2}) \left\| C^{(r)} - \nabla f(X^*) \right\|_F^2 + \left( \frac{9(1 - p)b}{p} + \frac{9L^2 b'}{p} \right) \left\| X^{(r)} - \bar{X}^{(r)} \right\|_F^2
+ \frac{18Lb'n}{p} (f(\bar{x}^{(r)}) - f(x^*)) + nb'\sigma^2.
$$

This concludes the proof. \( \square \)

**Lemma 10.** Suppose that Assumptions 2, 4, and 5 hold and $\{\alpha_{ij}\}_{ij}$ is set such that $\alpha_{ij} = \alpha_{ji} \geq 0$ for all $(i, j) \in \mathcal{E}$. If $\eta'$ satisfies

$$
\eta' \leq \frac{p}{6\sqrt{L^2 + \frac{36(1-p)b + L^2b'}{p^2}}} \tag{36}
$$

then it holds that

$$
\Xi^{(r+1)} + \frac{36}{p^2} \eta'^2 \mathcal{E}^{(r+1)} \leq \left( 1 - \frac{p}{4} \right) \left( \Xi^{(r)} + \frac{36}{p^2} \eta'^2 \mathcal{E}^{(r)} \right)
+ \left( \frac{18L}{p} + \frac{648Lb'}{p^3} \right) \eta'^2 (\mathbb{E} f(\bar{x}^{(r)}) - f(x^*)) + \left( 1 + \frac{36b'}{p^2} \right) \eta'^2 \sigma^2,
$$

**Proof.** From Lemmas 8 and 9 we have

$$
\Xi^{(r+1)} \leq (1 - \frac{p}{2}) \Xi^{(r)} + \frac{9L^2 \eta'^2}{p} \Xi^{(r)} + \frac{18L}{p} \eta'^2 (\mathbb{E} f(\bar{x}^{(r)}) - f(x^*)) + \frac{9}{p} \eta'^2 \mathcal{E}^{(r)} + \eta'^2 \sigma^2,
$$

$$
\frac{36}{p^2} \mathcal{E}^{(r+1)} \leq (1 - \frac{p}{2}) \frac{36}{p^2} \mathcal{E}^{(r)} + \frac{36}{p^2} \left( \frac{9(1-p)b}{p} + \frac{9L^2 b'}{p} \right) \Xi^{(r)}
+ \frac{648Lb'}{p^3} (\mathbb{E} f(\bar{x}^{(r)}) - f(x^*)) + \frac{36b'}{p^2} \sigma^2.
$$

Then, we have

$$
\Xi^{(r+1)} + \frac{36}{p^2} \eta'^2 \mathcal{E}^{(r+1)} \leq \left( 1 - \frac{p}{4} \right) \left( \Xi^{(r)} + \frac{9L^2 \eta'^2}{p} \Xi^{(r)} + \frac{9L^2 \eta'^2}{p^3} \right) \Xi^{(r)}
+ \left( \frac{18L}{p} \eta'^2 + \left( 1 - \frac{p}{2} \right) \frac{36}{p^2} \eta'^2 \right) \mathcal{E}^{(r)}
+ \left( \frac{18L}{p} \eta'^2 + \frac{648Lb'}{p^3} \eta'^2 \right) (\mathbb{E} f(\bar{x}^{(r)}) - f(x^*))
+ \left( \eta'^2 + \frac{36b'}{p^2} \eta'^2 \right) \sigma^2.
$$

Using $\eta'^2 \leq \frac{p^2}{(36(L^2 + \frac{36(1-p)b + L^2 b'}{p^2}))}$, we get

$$
1 - \frac{p}{2} + \frac{9L^2 \eta'^2}{p} + \frac{324(1-p)b + 324L^2 b'}{p^3} \eta'^2 \leq 1 - \frac{p}{4}.
$$

This concludes the proof. \( \square \)
Lemma 11. Suppose that Assumptions 2, 3, 4, and 5 hold, and \( \{\alpha_{ij}\}_{ij} \) is set such that \( \alpha_{ij} = \alpha_{ji} \geq 0 \) for all \((i, j) \in \mathcal{E}\). If \( \{w(r)\}_r \) is \( \frac{4}{p} \) slow increasing positive sequence of weights and the step size \( \eta' \) satisfies

\[
\eta' \leq \min \left\{ \frac{p}{6 \sqrt{L^2 + \frac{36 \epsilon (1 - p) \eta + L \eta')}{p^2}}, \frac{p}{24L \sqrt{2 + \frac{72 \epsilon}{p^2}}} \right\},
\]

then it holds that

\[
3L \sum_{r=0}^{R} w(r) \Xi(r) \leq \frac{1}{2} \sum_{j=0}^{R} w(j) (\mathbb{E} f(\check{x}(j)) - f(x^*)) + \eta'^2 \left( 12 + \frac{432 b'}{p^2} \right) L \sigma^2 \sum_{r=0}^{R} w(r).
\]

Proof. We define \( \Theta(r) := \Xi(r) + \frac{36 \epsilon}{p^2} \eta'^2 \mathcal{E}(r) \). From Lemma 10 we get

\[
\Theta(r) \leq \eta'^2 \left( \frac{18L}{p} + \frac{648 L b'}{p^3} \right) \sum_{j=0}^{r-1} \left( 1 - \frac{p}{4} \right)^{r-j-1} (\mathbb{E} f(\check{x}(j)) - f(x^*))
\]

\[+ \eta'^2 \left( 1 + \frac{36 L b'}{p^2} \right) \sigma^2 \sum_{j=0}^{r-1} \left( 1 - \frac{p}{4} \right)^{r-j-1} \]

\[\frac{4}{p} \leq \eta'^2 \left( \frac{24L}{p} + \frac{864 L b'}{p^3} \right) \sum_{j=0}^{r-1} \left( 1 - \frac{p}{4} \right)^{r-j} (\mathbb{E} f(\check{x}(j)) - f(x^*)) + \eta'^2 \left( 4 + \frac{144 b'}{p^2} \right) \sigma^2 \frac{R}{p} \sum_{r=1}^{R} w(r).
\]

for any round \( r \geq 1 \). Then, we get

\[
\sum_{r=1}^{R} w(r) \Theta(r) \leq \eta'^2 \left( \frac{24L}{p} + \frac{864 L b'}{p^3} \right) \sum_{r=1}^{R} w(r) \sum_{j=0}^{r-1} \left( 1 - \frac{p}{8} \right)^{r-j} (\mathbb{E} f(\check{x}(j)) - f(x^*))
\]

\[+ \eta'^2 \left( 4 + \frac{144 b'}{p^2} \right) \sigma^2 \frac{R}{p} \sum_{r=1}^{R} w(r).
\]

By using that \( \{w(r)\}_r \) is \( \frac{4}{p} \) slow increasing (i.e., \( w(r) \leq (1 + \frac{p}{8})^{r-j} w(j) \)), we get

\[
\sum_{r=1}^{R} w(r) \Theta(r) \leq \eta'^2 \left( \frac{24L}{p} + \frac{864 L b'}{p^3} \right) \sum_{r=1}^{R} \sum_{j=0}^{r-1} \left( 1 - \frac{p}{8} \right)^{r-j} w(j) (\mathbb{E} f(\check{x}(j)) - f(x^*))
\]

\[+ \eta'^2 \left( 4 + \frac{144 b'}{p^2} \right) \sigma^2 \frac{R}{p} \sum_{r=1}^{R} w(r)
\]

\[= \eta'^2 \left( \frac{24L}{p} + \frac{864 L b'}{p^3} \right) \sum_{j=0}^{R-1} w(j) (\mathbb{E} f(\check{x}(j)) - f(x^*)) \sum_{r=j+1}^{R} \left( 1 - \frac{p}{8} \right)^{r-j}
\]

\[+ \eta'^2 \left( 4 + \frac{144 b'}{p^2} \right) \sigma^2 \frac{R}{p} \sum_{r=1}^{R} w(r)
\]

\[\leq \eta'^2 \left( \frac{192 L}{p^2} + \frac{6912 L b'}{p^3} \right) \sum_{j=0}^{R-1} w(j) (\mathbb{E} f(\check{x}(j)) - f(x^*))
\]

\[+ \eta'^2 \left( 4 + \frac{144 b'}{p^2} \right) \sigma^2 \frac{R}{p} \sum_{r=1}^{R} w(r).
\]
By using that $\Theta^{(r)} \geq \Xi^{(r)}$, we get
\[
\sum_{r=1}^{R} w^{(r)} \Xi^{(r)} 
\leq \eta^2 \left( \frac{192L}{p^2} + \frac{6912b'}{p^2} \right) \sum_{j=0}^{R-1} w^{(j)} (E f(\bar{x}^{(j)}) - f(x^*)) + \eta^2 \left( 4 + \frac{144b'}{p^2} \right) \frac{\sigma^2}{p} \sum_{r=1}^{R} w^{(r)}.
\]

By using that $\Xi^{(0)} = 0$, $w^{(R)} (E f(\bar{x}^{(R)}) - f(x^*)) \geq 0$, and $w^{(0)} \geq 0$, we get
\[
\sum_{r=0}^{R} w^{(r)} \Xi^{(r)} 
\leq \eta^2 \left( \frac{192L}{p^2} + \frac{6912b'}{p^2} \right) \sum_{j=0}^{R} w^{(j)} (E f(\bar{x}^{(j)}) - f(x^*)) + \eta^2 \left( 4 + \frac{144b'}{p^2} \right) \frac{\sigma^2}{p} \sum_{r=0}^{R} w^{(r)}.
\]

By multiplying the above equation by $3L$, we get
\[
3L \sum_{r=0}^{R} w^{(r)} \Xi^{(r)} 
\leq \eta^2 \left( 576 + \frac{20736b'}{p^2} \right) \frac{L^2}{p^2} \sum_{j=0}^{R} w^{(j)} (E f(\bar{x}^{(j)}) - f(x^*)) + \eta^2 \left( 12 + \frac{432b'}{p^2} \right) L \sigma^2 \sum_{r=0}^{R} w^{(r)}.
\]

Using that $\eta^2 \leq p^2/(1152L^2(1 + \frac{36b'}{p}))$, we get the statement.  

**Lemma 12.** Suppose that Assumptions \ref{assum:lip} and \ref{assum:zero} hold, and $\{\alpha_{ij}\}_{i,j}$ is set such that $\alpha_{ij} = \alpha_{ji} \geq 0$ for all $(i,j) \in E$. If $\{w^{(r)}\}_r$ is a slow increasing positive sequence of weights and the step size $\eta'$ satisfies
\[
\eta' \leq \min \left\{ \frac{1}{12L}, \frac{p}{6 \sqrt{L^2 + \frac{36b\sigma}{p}}} \cdot \frac{p}{24L \sqrt{2 + \frac{72\sigma}{p}}} \right\},
\]
then it holds that
\[
\frac{1}{2W_R} \sum_{r=0}^{R} w^{(r)} (E f(\bar{x}^{(r)}) - f(x^*)) 
\leq \frac{1}{\eta' W_R} \sum_{r=0}^{R} w^{(r)} \left( \left( 1 - \frac{\eta' \mu}{2} \right) E \|\bar{x}^{(r)} - x^*\|^2 - E \|\bar{x}^{(r+1)} - x^*\|^2 \right)
+ \eta' \frac{\sigma^2}{n} + \eta^2 \left( 12 + \frac{432b'}{p^2} \right) \frac{L \sigma^2}{p}.
\]

where $W_R := \sum_{r=0}^{R} w^{(r)}$.

**Proof.** From Lemma \ref{lem:main} we have
\[
\sum_{r=0}^{R} w^{(r)} (E f(\bar{x}^{(r)}) - f(x^*)) \leq \frac{1}{\eta' \eta} \sum_{r=0}^{R} w^{(r)} \left( \left( 1 - \frac{\eta' \mu}{2} \right) E \|\bar{x}^{(r)} - x^*\|^2 - E \|\bar{x}^{(r+1)} - x^*\|^2 \right)
+ \frac{\eta' \sigma^2}{n} \sum_{r=0}^{R} w^{(r)} + 3L \sum_{r=0}^{R} w^{(r)} \Xi^{(r)}.
\]
Using Lemma 1, we get
\[ \frac{1}{2} \sum_{r=0}^{R} w^{(r)} (E f(x^{(r)}) - f(x^*)) \leq \frac{1}{\eta} \sum_{r=0}^{R} w^{(r)} \left( \left( 1 - \frac{\eta \mu}{2} \right) E \| x^{(r)} - x^* \|^2 - E \| x^{(r+1)} - x^* \|^2 \right) + \frac{\eta' \sigma^2}{n} \sum_{r=0}^{R} w^{(r)} + \eta^2 \left( 12 + \frac{432 b'}{p^2} \right) \frac{L \sigma^2}{p} \sum_{r=0}^{R} w^{(r)}. \]

This concludes the proof.

**Lemma 13** (Convergence Rate for Strongly Convex Cases). Suppose that Assumptions 2, 3, and 5 hold with \( \mu > 0 \), and \( \{\alpha_{ij}\}_{ij} \) is set such that \( \alpha_{ij} = \alpha_{ji} \geq 0 \) for all \( (i,j) \in \mathcal{E} \). For weights \( w^{(r)} := (1 - \frac{\mu \eta'}{2})^{-(r+1)} \) and \( W_R := \sum_{r=0}^{R} w^{(r)} \), there exists a step size \( \eta' < \frac{1}{\mu} \) such that it holds that
\[ \frac{1}{2W_R} \sum_{r=0}^{R} w^{(r)} (E f(x^{(r)}) - f(x^*)) \leq \tilde{O} \left( r_0 d \exp \left[ -\frac{\mu (R+1)}{d} \right] + \frac{\sigma^2}{n \mu R} + \frac{(1 + \frac{\eta' p}{\mu^2 R^2 p}) L \sigma^2}{p} \right), \]
where \( r_0 := \| x^{(0)} - x^* \|^2 \).

**Proof.** We define \( w^{(r)} := (1 - \frac{\mu \eta'}{2})^{-(r+1)} \). Suppose \( \eta' \leq \min \{ \frac{2p}{\mu(R+1)^2}, \frac{2}{\mu} \} \). Then, we have
\[ \frac{1}{1 - \frac{\mu \eta'}{2}} \leq 1 + \frac{p}{8}, \]
\[ 1 - \frac{\mu \eta'}{2} > 0. \]

Therefore, \( \{w^{(r)}\}_r \) is \( \frac{4}{p} \)-slow increasing. From Lemma 12, substituting \( w^{(r)} := (1 - \frac{\mu \eta'}{2})^{-(r+1)} \), we get
\[ \frac{1}{2W_R} \sum_{r=0}^{R} w^{(r)} (E f(x^{(r)}) - f(x^*)) \leq \frac{1}{\eta W_R} \left( \| x^{(0)} - x^* \|^2 - w^{(R)} \| \tilde{x}^{(R+1)} - x^* \|^2 \right) + \eta' \frac{\sigma^2}{n} + \eta^2 \left( 12 + \frac{432 b'}{p^2} \right) \frac{L \sigma^2}{p}. \]

Unrolling the above equation, we get
\[ \frac{1}{2W_R} \sum_{r=0}^{R} w^{(r)} (E f(x^{(r)}) - f(x^*)) + \frac{w^{(R)}}{\eta' W_R} \| \tilde{x}^{(R+1)} - x^* \|^2 \leq \frac{\| x^{(0)} - x^* \|^2}{\eta W_R} + \eta' \frac{\sigma^2}{n} + \eta^2 \left( 12 + \frac{432 b'}{p^2} \right) \frac{L \sigma^2}{p}. \]

Using the followings:
\[ \frac{1}{W_R} \leq (1 - \frac{\mu \eta'}{2}) \leq \exp \left[ -\frac{\mu \eta'}{2} (R+1) \right], \]
\[ W_R = (1 - \frac{\mu \eta'}{2})^{-(R+1)} \sum_{r=0}^{R} (1 - \frac{\mu \eta'}{2})^r \leq \frac{2w^{(R)}}{\mu \eta'}, \]
we get
\[ \frac{1}{2W_R} \sum_{r=0}^{R} w^{(r)} (E f(x^{(r)}) - f(x^*)) + \frac{p}{2} E \| x^{(R+1)} - x^* \|^2 \leq \frac{1}{\eta} \| x^{(0)} - x^* \| \exp \left[ -\frac{\mu \eta'}{2} (R+1) \right] + \eta' \frac{\sigma^2}{n} + \eta^2 \left( 12 + \frac{432 b'}{p^2} \right) \frac{L \sigma^2}{p}. \]

Then, by tuning \( \eta' \) as in Lemma 15 in [11] and Lemma 2 in [27], we can get the statement. \( \square \)
Lemma 14 (Convergence Rate for General Convex Cases). Suppose that Assumptions 2, 3, 4 and 5 hold with \( \mu = 0 \), and \( \{\alpha_{ij}\}_{ij} \) is set such that \( \alpha_{ij} = \alpha_{ji} \geq 0 \) for all \( (i, j) \in \mathcal{E} \). There exists a step size \( \eta' < \frac{1}{L} \) such that it holds that

\[
\frac{1}{2(R+1)} \sum_{r=0}^{R} (E_f(\bar{x}^{(r)}) - f(x^*)) \\
\leq \mathcal{O} \left( \left( \frac{\sigma^2 \|\bar{x}^{(0)} - x^*\|^2}{n(R+1)} \right)^{\frac{1}{2}} + \left( \frac{1 + \frac{\eta'}{p} L \sigma^2}{p} \right)^{\frac{1}{4}} \left( \frac{\|\bar{x}^{(0)} - x^*\|^2}{R+1} \right)^{\frac{3}{4}} + \frac{d\|\bar{x}^{(0)} - x^*\|^2}{R+1} \right).
\]

Proof. From Lemma 12 by substituting \( w^{(r)} := 1 \), we get

\[
\frac{1}{2(R+1)} \sum_{r=0}^{R} (E_f(\bar{x}^{(r)}) - f(x^*)) \leq \frac{1}{\eta'(R+1)} \|\bar{x}^{(0)} - x^*\|^2 + \eta' \frac{\sigma^2}{n} + \eta'^2 \left( 12 + \frac{432b'}{p^2} \right) L \sigma^2.
\]

Then, using Lemma 17 in [11], we can get the statement. \( \square \)

I.4 Convergence Analysis for Non-convex Case

I.4.1 Additional Notation

In Sec. I.4, we define \( \Xi^{(r)} \), \( \mathcal{E}^{(r)} \), \( b \), and \( b' \) as follows to simplify the notation:

\[
\Xi^{(r)} := \frac{1}{n} E \sum_{i=1}^{n} \|X_i^{(r)} - \bar{x}^{(r)}\|^2, \quad \mathcal{E}^{(r)} := \frac{1}{n} E \left\| \nabla f(\bar{X}^{(r)}) - C^{(r)} - \frac{1}{n} \nabla f(\bar{X}^{(r)}) 11^\top \right\|^2, \\
b := \left\| \frac{1}{2} (D - E) \right\|^2_F, \quad b' := \|W - I\|_F^2.
\]

I.4.2 Convergence Analysis

Lemma 15 (Descent Lemma for Non-convex Case). Suppose that Assumptions 2, 3, 4 hold, and \( \{\alpha_{ij}\}_{ij} \) is set such that \( \alpha_{ij} = \alpha_{ji} \geq 0 \) for all \( (i, j) \in \mathcal{E} \). If \( \eta' \leq \frac{1}{2L} \), it holds that

\[
E_{r+1} f(\bar{x}^{(r+1)}) \leq f(\bar{x}^{(r)}) - \frac{\eta'}{4} \left\| \nabla f(\bar{x}^{(r)}) \right\|^2 + \frac{L^2 \eta'}{n} \sum_{i=1}^{n} \|X_i^{(r)} - \bar{x}^{(r)}\|^2 + \frac{L \sigma^2 \eta'^2}{2n}.
\]

Proof. We have

\[
E_{r+1} f(\bar{x}^{(r+1)}) \leq E_{r+1} f(\bar{x}^{(r)}) - \frac{\eta'}{n} \sum_{i=1}^{n} \nabla F_i(X_i^{(r)}, \xi_i^{(r)}) + \frac{L \eta'^2}{2} E_{r+1} \left\| \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(X_i^{(r)}, \xi_i^{(r)}) \right\|^2.
\]

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Then, we can estimate $T_1$ as follows:

\[
T_1 = -\eta' \frac{1}{n} \sum_{i=1}^{n} \left\langle \nabla f(\bar{x}^{(r)}), \nabla f_i(\bar{x}^{(r)}) \right\rangle \\
= -\eta' \frac{1}{n} \sum_{i=1}^{n} \left\langle \nabla f(\bar{x}^{(r)}), \nabla f_i(\bar{x}^{(r)}) - \nabla f_i(\bar{x}^{(r)}) + \nabla f_i(\bar{x}^{(r)}) \right\rangle \\
= -\eta' \left\| \nabla f(\bar{x}^{(r)}) \right\|^2 + \eta' \frac{1}{2n} \sum_{i=1}^{n} \left\| \nabla f_i(\bar{x}^{(r)}) - \nabla f_i(\bar{x}^{(r)}) \right\|^2 \\
\leq -\eta' \frac{1}{2} \left\| \nabla f(\bar{x}^{(r)}) \right\|^2 + \frac{\eta' L^2}{2n} \sum_{i=1}^{n} \left\| \bar{x}_i^{(r)} - \bar{x}^{(r)} \right\|^2.
\]

Then, we can estimate $T_2$ as follows:

\[
T_2 = \mathbb{E}_{r+1} \left\| \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(\bar{x}^{(r)}; \xi_i^{(r)}) \right\|^2 \\
\leq \left\| \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\bar{x}^{(r)}) \right\|^2 + \mathbb{E}_{r+1} \left\| \frac{1}{n} \sum_{i=1}^{n} (\nabla F_i(\bar{x}^{(r)}; \xi_i^{(r)}) - \nabla f_i(\bar{x}^{(r)})) \right\|^2 \\
\leq \left\| \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\bar{x}^{(r)}) \right\|^2 + \frac{\sigma^2}{n} \\
= \left\| \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\bar{x}^{(r)}) - \nabla f_i(\bar{x}^{(r)}) + \nabla f_i(\bar{x}^{(r)}) \right\|^2 + \frac{\sigma^2}{n} \\
\leq \frac{2L^2}{n} \sum_{i=1}^{n} \left\| \bar{x}_i^{(r)} - \bar{x}^{(r)} \right\|^2 + 2 \left\| \nabla f(\bar{x}^{(r)}) \right\|^2 + \frac{\sigma^2}{n}.
\]

Combining the above equations, we get

\[
\mathbb{E}_{r+1} f(\bar{x}^{(r+1)}) \\
\leq f(\bar{x}^{(r)}) - \left( \frac{\eta'}{2} - L\eta'^2 \right) \left\| \nabla f(\bar{x}^{(r)}) \right\|^2 + \left( \frac{L^2 \eta'}{2n} + \frac{L^3 \eta'^2}{n} \right) \sum_{i=1}^{n} \left\| \bar{x}_i^{(r)} - \bar{x}^{(r)} \right\|^2 + \frac{L\sigma^2}{2n} \eta'^2.
\]

Using that $\eta' \leq \frac{1}{4L}$, we get the statement. \qed

**Lemma 16** (Recursion for Consensus Distance). Suppose that Assumptions 2, 3, and 4 hold, and \{\alpha_{ij}\}_{ij} is set such that \alpha_{ij} = \alpha_{ji} \geq 0 for all (i, j) \in E. Then, it holds that

\[
\Xi^{(r+1)} \leq (1 - \frac{p}{2}) \Xi^{(r)} + \frac{6L^2}{p} \eta'^2 \Xi^{(r)} + \frac{6}{p} \eta'^2 \mathbb{E} \left\| \nabla f(\bar{x}^{(r)}) \right\|^2 + \frac{6}{p} \eta'^2 \xi^{(r)} + \eta'^2 \sigma^2.
\]

**Proof.** As in Lemma 8, we get

\[
n\Xi^{(r+1)} \leq \mathbb{E} \left\| \bar{X}^{(r+1)} - \bar{X}^{(r)} \right\|_F^2.
\]
Then, we can estimate as follows:

\[
\mathbb{E}_{r+1} \left\| X^{(r+1)} - \bar{X}^{(r)} \right\|_F^2 \\
= \mathbb{E}_{r+1} \left\| X^{(r)} W - \eta' \left( \nabla f(X^{(r)}; \xi^{(r)}) - C^{(r)} \right) - \bar{X}^{(r)} \right\|_F^2 \\
\leq (1 - \frac{p}{2}) \left\| X^{(r)} - \bar{X}^{(r)} \right\|_F^2 + 3 \eta^2 \left\| \nabla f(X^{(r)}) - C^{(r)} \right\|_F^2 + \eta^2 n\sigma^2
\]

By substituting \( \gamma = \frac{p}{2} \), we get

\[
\mathbb{E}_{r+1} \left\| X^{(r+1)} - X^{(r)} \right\|_F^2 \\
\leq (1 - \frac{p}{2}) \left\| X^{(r)} - \bar{X}^{(r)} \right\|_F^2 + 3 \eta^2 \left\| \nabla f(X^{(r)}) - C^{(r)} \right\|_F^2 + \eta^2 n\sigma^2
\]

From Lemma 17 we have \( \sum_{i=1}^n c_i^{(r)} = 0 \). Using \( \sum_{i=1}^n \| a_i - \bar{a} \|^2 = \sum_{i=1}^n \| a_i \|^2 - n\|\bar{a}\|^2 \) for any \( a_1, \ldots, a_n \in \mathbb{R}^d \), we get

\[
T = \sum_{i=1}^n \left\| \nabla f_i(\bar{x}^{(r)}) - c_i^{(r)} \right\|^2 \\
= \sum_{i=1}^n \left\| \nabla f_i(\bar{x}^{(r)}) - \bar{c}_i^{(r)} - \nabla f(\bar{x}^{(r)}) \right\|^2 + n \| \nabla f(\bar{x}^{(r)}) \|^2 \\
= \left\| \nabla f(\bar{x}^{(r)}) - C^{(r)} - \frac{1}{n} \nabla f(X^{(r)}) 11^T \right\|^2 + n \| \nabla f(\bar{x}^{(r)}) \|^2.
\]

Then, we can get the statement. \(\square\)

**Lemma 17.** Suppose that Assumptions 2, 3, and 4 hold, and \{\( c_{ij} \)\}_{ij} is set such that \( c_{ij} = \alpha_{j|i} \geq 0 \) for all \( (i, j) \in \mathcal{E} \). Then, it holds that

\[
\mathcal{E}^{(r+1)} \leq (1 - \frac{p}{2}) \mathcal{E}^{(r)} + \left( \frac{12L^2b'}{p} + \frac{12(1-p)b}{p} \varepsilon^{(r)} + \frac{48L^4}{p} \eta^2 \right) \varepsilon^{(r)} \\
+ \frac{48L^2}{p} \eta^2 \left\| \nabla f(\bar{x}^{(r)}) \right\|^2 + \left( \frac{24L^2\sigma^2}{np} + \frac{12b'}{p} \right) \sigma^2.
\]
Proof. We have

\[
E_{r+1} \left\| \nabla f(\bar{x}^{(r+1)}) - C^{(r+1)} - \frac{1}{n} \nabla f(\bar{x}^{(r+1)}) 11^\top \right\|_F^2 \\
\leq \frac{1}{2} x^{(r)} W (D - E) - \nabla f(X^{(r)}; \xi^{(r)}) - \frac{1}{n} \nabla f(X^{(r+1)}) 11^\top \\
+ \nabla f(\bar{x}^{(r)}) (W - I) - \nabla f(X^{(r)}) (W - I) + \frac{1}{n} \nabla f(X^{(r)}) 11^\top - \frac{1}{n} \nabla f(X^{(r)}) 11^\top \right\|_F^2. 
\]

Substituting $\gamma = \frac{p}{2}$, we get

\[
E_{r+1} \left\| \nabla f(\bar{x}^{(r+1)}) - C^{(r+1)} - \frac{1}{n} \nabla f(\bar{x}^{(r+1)}) 11^\top \right\|_F^2 \\
\leq \frac{1}{2} x^{(r)} W (D - E) - \nabla f(X^{(r)}; \xi^{(r)}) - \frac{1}{n} \nabla f(X^{(r+1)}) 11^\top \\
+ \nabla f(\bar{x}^{(r)}) (W - I) - \nabla f(X^{(r)}) (W - I) + \frac{1}{n} \nabla f(X^{(r)}) 11^\top - \frac{1}{n} \nabla f(X^{(r)}) 11^\top \right\|_F^2. 
\]

Substituting $\gamma = \frac{p}{2}$, we get

\[
E_{r+1} \left\| \nabla f(\bar{x}^{(r+1)}) - C^{(r+1)} - \frac{1}{n} \nabla f(\bar{x}^{(r+1)}) 11^\top \right\|_F^2 \\
\leq \frac{1}{2} x^{(r)} W (D - E) - \nabla f(X^{(r)}; \xi^{(r)}) - \frac{1}{n} \nabla f(X^{(r+1)}) 11^\top \\
+ \nabla f(\bar{x}^{(r)}) (W - I) - \nabla f(X^{(r)}) (W - I) + \frac{1}{n} \nabla f(X^{(r)}) 11^\top - \frac{1}{n} \nabla f(X^{(r)}) 11^\top \right\|_F^2. 
\]
Then, using Assumption 3, we get
\[
\begin{align*}
\mathbb{E}_{r+1} \left\| \nabla f(\mathbf{X}^{(r+1)}) - C^{(r+1)} - \frac{1}{n} \nabla f(\mathbf{X}^{(r+1)}) \mathbf{1} \right\|^2_F \\
\leq (1 - \frac{p}{2}) \left\| \nabla f(\mathbf{X}^{(r)}) - C^{(r)} - \frac{1}{n} \nabla f(\mathbf{X}^{(r)}) \mathbf{1} \right\|^2_F + \frac{24L^2b}{p} \left\| \mathbf{X}^{(r)} - \mathbf{X}^{(r)} \right\|^2_F \\
+ \frac{24L^2}{p} \mathbb{E}_{r+1} \left\| \mathbf{X}^{(r+1)} - \mathbf{X}^{(r)} \right\|^2_F + \frac{12}{p} \left\| \frac{1}{2} \mathbf{X}^{(r)} \mathbf{W} \mathbf{(D - E)} \right\|^2_F + \frac{12nb\sigma^2}{p}.
\end{align*}
\]

From Lemma 1 we get
\[
\begin{align*}
n T_1 &= \mathbb{E}_{r+1} \left\| \frac{\eta'}{n} \sum_{i=1}^n \nabla f_i(\mathbf{x}_i^{(r)}; \xi_i^{(r)}) \right\|^2 \\
&\leq \left\| \frac{\eta'}{n} \sum_{i=1}^n \nabla f_i(\mathbf{x}_i^{(r)}) \right\|^2 + \mathbb{E}_{r+1} \left\| \frac{\eta'}{n} \sum_{i=1}^n (\nabla f_i(\mathbf{x}_i^{(r)}; \xi_i^{(r)}) - \nabla f_i(\mathbf{x}_i^{(r)})) \right\|^2 \\
&\leq \eta'^2 \left\| \frac{1}{n} \sum_{i=1}^n \nabla f_i(\mathbf{x}_i^{(r)}) \right\|^2 + \frac{\eta'^2 \sigma^2}{n} \\
&= \eta'^2 \left\| \frac{1}{n} \sum_{i=1}^n (\nabla f_i(\mathbf{x}_i^{(r)}) - \nabla f_i(\bar{\mathbf{x}}^{(r)}) + \nabla f_i(\bar{\mathbf{x}}^{(r)})) \right\|^2 + \frac{\eta'^2 \sigma^2}{n} \\
&\leq 2\eta'^2 \left\| \frac{1}{n} \sum_{i=1}^n (\nabla f_i(\mathbf{x}_i^{(r)}) - \nabla f_i(\bar{\mathbf{x}}^{(r)})) \right\|^2 + 2\eta'^2 \left\| \nabla f(\bar{\mathbf{x}}^{(r)}) \right\|^2 + \frac{\eta'^2 \sigma^2}{n}.
\end{align*}
\]

From the definitions of $E$ and $D$, we have
\[
T_2 = \left\| \frac{1}{2} \mathbf{X}^{(r)} \mathbf{W} - \bar{\mathbf{X}}^{(r)} \right\|^2_F \leq (1 - p)b \left\| \mathbf{X}^{(r)} - \bar{\mathbf{X}}^{(r)} \right\|^2_F.
\]

Then, we get the statement.

\textbf{Lemma 18.} Suppose that Assumptions 2, 3, and 4 hold, and \( \{\alpha_{i,j}\}_{i,j} \) is set such that \( \alpha_{i,j} = \alpha_{j,i} \geq 0 \) for all \( (i, j) \in \mathcal{E} \). Then, if \( \eta' \) satisfies
\[
\eta' \leq \min \left\{ \frac{p}{2\sqrt{6(L^2 + \frac{48L^2b + 12L^2 + 48(1-p)b}{p^2})}}, \frac{\sqrt{\eta'}^2}{24L^2}, \frac{1}{4L} \right\}.
\]

\begin{align*}
\Xi^{(r+1)} + \frac{24}{p^2} \eta'^2 \mathcal{E}^{(r+1)} &
\leq \left(1 - \frac{p}{4}\right) \left( \Xi^{(r)} + \frac{24}{p^2} \eta'^2 \mathcal{E}^{(r)} \right) + \frac{54}{p} \eta'^2 \mathbb{E} \left\| \nabla f(\bar{\mathbf{x}}^{(r)}) \right\|^2 + \left( 2 + \frac{288b}{p^3} \right) \eta'^2 \sigma^2.
\end{align*}

\textbf{Proof.} From Lemma 1 we have
\[
\begin{align*}
\frac{24}{p^2} \mathcal{E}^{(r+1)} &\leq \left(1 - \frac{p}{2}\right) \frac{24}{p^2} \mathcal{E}^{(r)} + \frac{24}{p^2} \left( \frac{12L^2b'}{p} + \frac{12(1-p)b}{p} + \frac{48L^4}{p} \eta'^2 \right) \Xi^{(r)} \\
&+ \frac{1152L^2}{p^3} \eta'^2 \left\| \nabla f(\bar{\mathbf{x}}^{(r)}) \right\|^2 + \frac{24}{p^2} \left( \frac{24L^2 \eta'^2}{np} + \frac{12b'}{p} \right) \sigma^2.
\end{align*}
\]
By using that \( \eta'^2 \leq \frac{1}{16L^2} \) and \( \eta'^2 \leq \frac{np^3}{576L^2} \), we get
\[
\frac{24}{p^2} \xi^{(r+1)} \leq (1 - \frac{p}{2}) \frac{24}{p^2} \xi^{(r)} + \frac{24}{p^2} \left( \frac{12L^2b'}{p} + \frac{12(1-p)b'}{p} + \frac{3L^2}{p} \right) \xi^{(r)} + \frac{1152L^2}{p^3} \eta'^2 \left\| \nabla f(\hat{x}^{(r)}) \right\|^2 + \left( 1 + \frac{288b'}{p^3} \right) \sigma^2.
\]
Because \( \eta'^2 \leq p^2/(24(L^2 + 48L^2b' + 12L^2 + 48(1-p)b')) \) implies that \( \eta'^2 \leq \frac{p^2}{24L^2} \), we get
\[
\frac{24}{p^2} \xi^{(r+1)} \leq (1 - \frac{p}{2}) \frac{24}{p^2} \xi^{(r)} + \frac{24}{p^2} \left( \frac{12L^2b'}{p} + \frac{12(1-p)b'}{p} + \frac{3L^2}{p} \right) \xi^{(r)} + \frac{48}{p} \left\| \nabla f(\hat{x}^{(r)}) \right\|^2 + \left( 1 + \frac{288b'}{p^3} \right) \sigma^2.
\]
Combining this with Lemma 16, we get
\[
\xi^{(r+1)} + \frac{24}{p^2} \eta'^2 \xi^{(r+1)} \leq \left( 1 - \frac{p}{2} \right) \left( \xi^{(r)} + \frac{24}{p^2} \eta'^2 \xi^{(r)} \right) + \frac{54}{p} \eta'^2 \xi^{(r)} + \left( 2 + \frac{288b'}{p^3} \right) \eta'^2 \sigma^2.
\]
Using that \( \eta'^2 \leq p^2/(24(L^2 + 48L^2b' + 12L^2 + 48(1-p)b')) \), we get
\[
\xi^{(r+1)} + \frac{24}{p^2} \eta'^2 \xi^{(r+1)} \leq \left( 1 - \frac{p}{4} \right) \left( \xi^{(r)} + \frac{24}{p^2} \eta'^2 \xi^{(r)} \right) + \frac{54}{p} \eta'^2 \xi^{(r)} + \left( 2 + \frac{288b'}{p^3} \right) \eta'^2 \sigma^2.\]
This concludes the proof. \( \square \)

**Lemma 19.** Suppose that Assumptions 2, 3, and 4 hold, and \( \{\alpha_{i,j}\}_{ij} \) is set such that \( \alpha_{i,j} = \alpha_{j|i} \geq 0 \) for all \( (i, j) \in \mathcal{E} \). Then, if \( \eta' \) satisfies
\[
\eta' \leq \min \left\{ \frac{p}{24L\sqrt{3}}, \frac{p}{2\sqrt{6}(L^2 + 48L^2b' + 12L^2 + 48(1-p)b')}, \frac{\sqrt{np^3}}{24L}, \frac{1}{4L} \right\}, \tag{39}
\]
we have
\[
\frac{L^2}{R+1} \sum_{r=0}^{R} \xi^{(r)} \leq \frac{1}{8(R+1)} \sum_{r=0}^{R} \mathbb{E}\|\nabla f(\hat{x}^{(r)})\|^2 + \left( 8 + \frac{1152b'}{p^3} \right) \frac{L^2\sigma^2}{p} \eta'^2.
\]

**Proof.** We define \( \Theta^{(r)} := \xi^{(r)} + \frac{24}{p^2} \eta'^2 \xi^{(r)} \). From Lemma 18, we get
\[
\Theta^{(r)} \leq \left( 1 - \frac{p}{4} \right) \Theta^{(r-1)} + \frac{54}{p} \eta'^2 \mathbb{E}\|\nabla f(\hat{x}^{(r-1)})\|^2 + \left( 2 + \frac{288b'}{p^3} \right) \eta'^2 \sigma^2
\]
\[
= \frac{54}{p} \eta'^2 \sum_{j=0}^{r-1} \left( 1 - \frac{p}{4} \right)^{r-j-1} \mathbb{E}\|\nabla f(\hat{x}^{(j)})\|^2 + \left( 2 + \frac{288b'}{p^3} \right) \eta'^2 \sigma^2 \sum_{j=0}^{r-1} \left( 1 - \frac{p}{4} \right)^{r-j-1}
\]
\[
\leq \frac{54}{p} \eta'^2 \sum_{j=0}^{r-1} \left( 1 - \frac{p}{4} \right)^{r-j-1} \mathbb{E}\|\nabla f(\hat{x}^{(j)})\|^2 + \left( 8 + \frac{1152b'}{p^3} \right) \frac{\sigma^2}{p} \eta'^2,
\]

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for any round $r > 0$. By recursively adding both sides, we get
\[
\sum_{r=1}^{R} \Theta^{(r)} \leq \frac{54}{p} \eta'^2 \sum_{r=1}^{R} \sum_{j=0}^{r-1} \left(1 - \frac{p}{4}\right)^{r-j-1} \mathbb{E} \|\nabla f(\bar{x}^{(j)})\|^2 + \left(8 + \frac{1152 b'}{p^3}\right) \frac{\sigma^2}{p} \eta'^2 R
\]
\[
= \frac{54}{p} \eta'^2 \sum_{j=0}^{R-1} \mathbb{E} \|\nabla f(\bar{x}^{(j)})\|^2 \sum_{r=j+1}^{R} \left(1 - \frac{p}{4}\right)^{r-j-1} + \left(8 + \frac{1152 b'}{p^3}\right) \frac{\sigma^2}{p} \eta'^2 R
\]
\[
\leq \frac{216}{p^2} \eta'^2 \sum_{j=0}^{R-1} \mathbb{E} \|\nabla f(\bar{x}^{(j)})\|^2 \left(8 + \frac{1152 b'}{p^3}\right) \frac{\sigma^2}{p} \eta'^2 R.
\]

By using $\Xi^{(r)} \leq \Theta^{(r)}$, we get
\[
\sum_{r=1}^{R} \Xi^{(r)} \leq \frac{216}{p^2} \eta'^2 \sum_{j=0}^{R-1} \mathbb{E} \|\nabla f(\bar{x}^{(j)})\|^2 + \left(8 + \frac{1152 b'}{p^3}\right) \frac{\sigma^2}{p} \eta'^2 R.
\]

Using $\Xi^{(0)} = 0$, we get
\[
\frac{1}{R+1} \sum_{r=0}^{R} \Xi^{(r)} \leq \frac{216}{p^2(R+1)} \eta'^2 \sum_{r=0}^{R} \mathbb{E} \|\nabla f(\bar{x}^{(r)})\|^2 + \left(8 + \frac{1152 b'}{p^3}\right) \frac{\sigma^2}{p} \eta'^2 R.
\]

By multiplying the above equation by $L^2$, we get
\[
\frac{L^2}{R+1} \sum_{r=0}^{R} \Xi^{(r)} \leq \frac{216L^2}{p^2(R+1)} \eta'^2 \sum_{r=0}^{R} \mathbb{E} \|\nabla f(\bar{x}^{(r)})\|^2 + \left(8 + \frac{1152 b'}{p^3}\right) \frac{L^2 \sigma^2}{p} \eta'^2 R.
\]

By using $\eta'^2 \leq \frac{p^2}{1152 L^2}$, we get
\[
\frac{L^2}{R+1} \sum_{r=0}^{R} \Xi^{(r)} \leq \frac{1}{8(R+1)} \sum_{r=0}^{R} \mathbb{E} \|\nabla f(\bar{x}^{(r)})\|^2 + \left(8 + \frac{1152 b'}{p^3}\right) \frac{L^2 \sigma^2}{p} \eta'^2 R.
\]

This concludes the proof. \(\square\)

**Lemma 20 (Convergence Rate for Non-convex Case).** Suppose that Assumptions 2, 3, and 4 hold, and \(\alpha_{ij} = 0\) for all \((i, j) \in \mathcal{E}\). Then, there exists \(\eta' < \frac{p}{8}\) such that it holds that
\[
\frac{1}{R+1} \sum_{r=0}^{R} \mathbb{E} \|\nabla f(\bar{x}^{(r)})\|^2 \leq O \left( \left( \frac{L^2 \sigma^2 r_0}{n(R+1)} \right)^{\frac{1}{2}} + \left( \frac{(1 + \frac{b'}{p^2}) L^2 \sigma^2}{p} \right) \left( \frac{r_0}{R+1} \right)^{\frac{1}{2}} + \frac{d r_0}{R+1} \right),
\]
where \(r_0 := f(\bar{x}^{(0)}) - f^*\).

**Proof.** From Lemma 15, we have
\[
\frac{1}{4(R+1)} \sum_{r=0}^{R} \mathbb{E} \left\| \nabla f(\bar{x}^{(r)}) \right\|^2 \leq \frac{1}{\eta' (R+1)} \sum_{r=0}^{R} \left( \mathbb{E} f(\bar{x}^{(r)}) - \mathbb{E} f(\bar{x}^{(r+1)}) \right) + \frac{L^2}{R+1} \sum_{r=0}^{R} \Xi^{(r)} + \frac{L \sigma^2 \eta'}{2n}.
\]

Using Lemma 19 we get
\[
\frac{1}{8(R+1)} \sum_{r=0}^{R} \mathbb{E} \left\| \nabla f(\bar{x}^{(r)}) \right\|^2 \leq \frac{f(\bar{x}^{(0)}) - f^*}{\eta'(R+1)} + \frac{L \sigma^2 \eta'}{2n} + \left(8 + \frac{1152 b'}{p^3}\right) \frac{L^2 \sigma^2}{p} \eta'^2 R.
\]

Using Lemma 16 in [11], we get the statement. \(\square\)
J  Limitation of Theorem 3

In this section, we discuss the limitations of Theorem 3 and describe why the convergence rates shown in Theorem 3 cannot be regarded as that of the ECL.

In Lemmas 10, 11, 12, 18, and 19, we assume that the step size $\eta'$ is upper bounded. In the G-ECL and Gossip algorithm, there exists a step size $\eta'$ that satisfies the assumptions of Lemmas 10, 11, 12, 18, and 19 because the mixing matrix $W$ and step size $\eta'$ can be set independently as hyperparameters. However, in the ECL, $W$ and $\eta'$ are determined by $\eta$ and $\{\alpha_{ij}\}_{ij}$ as in Eq. (11) and depend on one another. That is, $\eta'$, $p$ in Assumption 2, and $b' := ||W - I||^2$ depend on each other. Therefore, to prove that the convergence rates of the ECL are that shown in Theorem 3, we need to prove that there exists a step size $\eta'$ that satisfies the assumptions of Lemmas 10, 11, 12, 18, and 19. In this work, it is left to future work to prove whether there exists a step size $\eta'$ that satisfies the assumptions of Lemmas 10, 11, 12, 18, and 19 and we experimentally demonstrate that the ECL converges at the same convergence rate as the G-ECL in Sec. 7.