GAUGE INVARIANT FORMULATION OF THE SELF-INTERACTING DUFFIN-KEMMER-PETIAU EQUATIONS.

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Abstract. We show that the Duffin-Kemmer-Petiau equation, minimally coupled to an Abelian gauge field, can be regarded as a matrix equation for the gauge potential produced internally from the matter fields. This can be solved as a rational expression in terms of currents bilinear in the matter wavefunction, together with a similar expression for the field strength tensor, thus providing a gauge invariant formulation of the self-interacting DKP equations. We give the derivation of this result for the 5 component DKP system, by analogy with the Dirac equation case. To this end, we establish the algebraic structure of the set of bilinear currents, and the properties of the minimal generating set, which consists of two scalars and two four-vectors, together with a single quadratic constraint.

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1. Introduction

The Dirac equation, minimally coupled with an electromagnetic field, can be regarded as a set of algebraic equations for the gauge potential, whose solution is a rational expression in terms of currents bilinear in the Dirac wavefunction, and their derivatives. This result was obtained by Radford [1], and subsequently developed in higher dimensional [2] and non-Abelian cases [3]. The resulting Maxwell-Dirac equations have been shown to admit monopole-like solutions [1, 4, 5, 6, 7].

In this paper we provide the corresponding inversion construction for the self-interacting Duffin-Kemmer-Petiau (DKP) equation [8, 9, 10]. We concentrate here on the 5 component representation, with the 10 component system to be treated in a separate work.

In section 2 below, we briefly review the DKP equation and the DKP algebra, and study the algebra of linearly independent bilinear currents, and that of their algebraically independent generating set, together with the Fierz-DKP [11] rearrangement identities appropriate to the 5 component system. In section 3 these results are used to obtain the expressions for the gauge potential and the field strength tensor, and hence arrive at a gauge-invariant formulation of the self-interacting DKP equations.

2. DKP Equation and DKP Algebra

The Dirac equation together with the DKP equation are the unique instances of relativistic first order equations based on wave functions belonging to representations of a five-dimensional orthogonal group [12, 13, 14] which describe single-mass systems. When interactions are introduced through minimal coupling to an Abelian gauge potential,

\[
(i(\partial^\mu + ieA^\mu)\Gamma_\mu + m)\Phi = 0, \tag{2.1}
\]

it is notable that there is a simple rearrangement whereby the system can be viewed as a linear matrix equation for the potential, \(R^\mu A_\mu = \Psi\), for which a matrix inversion, if it exists, would yield an algebraic expression \(A_\mu = (R^{-1})^\mu_\mu \Psi\) for the gauge potential itself. Here \(R^\mu \equiv \Gamma^\mu \Phi\) is the rectangular matrix of coefficients of the potential, and \(\Psi\) represents the terms independent of the potential, occurring in the equation. As mentioned above, this procedure can indeed be implemented in the case of the Dirac equation, and the solution for the gauge potential is a rational expression in terms of a set of real tensor quantities, or ‘current bilinears’, which are quadratic in the Dirac wavefunction and its gradient. These currents are central to classical interpretations of the Dirac equation in ‘relativistic fluid’ formulations [15, 16], and have been analyzed in this context by Crawford [17].

In practice, the inversion of the coefficient matrix in the Dirac case (\(\Gamma_\mu \equiv \gamma_\mu\)) proceeds indirectly, by using properties of the Dirac algebra or Clifford algebra of \(\gamma_\mu\) matrices,

\[
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_\mu\nu \mathbb{I}, \tag{2.2}
\]

where \(\eta_{\mu\nu} := \text{diag}(1, -1, -1, -1)\) is the flat spacetime Minkowski metric. In this paper we show that in the DKP case (\(\Gamma_\mu \equiv \beta_\mu\)) analogous manipulations are also possible, starting with the defining relations of the Kemmer \(\beta_\mu\) matrices, namely

\[
\beta_\mu \beta_\rho \beta_\nu + \beta_\nu \beta_\rho \beta_\mu = \eta_{\mu\rho} \beta_\nu + \eta_{\nu\rho} \beta_\mu. \tag{2.3}
\]

The algebraic structure of the \(\beta_\mu\) was analyzed by Kemmer [9], and in particular in great detail by Harish-Chandra [18]. An immediate effect of the fact that the equations are not inhomogeneous, is the existence of a 1-dimensional representation with \(\beta_\mu = 0\), and indeed [9, 18] the 126-dimensional enveloping algebra splits into 1−, 25− and 100− dimensional sectors spanned by the 1−, 5− and 10− component irreducible DKP
representations. Below, we proceed with an investigation of the interacting DKP equation
for the 5 component system, with the 10 component system to be treated in a later work.

Analyzing the Casimir operator eigenvalues of the Lorentz symmetry algebra generators
\(\frac{1}{4}[^{\mu}_1, ^{\nu}_1]\), or adopting a concrete matrix basis, reveals in particular that, for the 5
component case, the combination \(\mathbb{1} - \beta^\mu_\mu \equiv \mathbb{1} - \beta^2\) is a projector [9]; in consequence,
young element in the DKP enveloping algebra can be resolved covariantly into block form,
corresponding to mappings between its eigenspaces. Carrying this out for the generators
\(\beta^\mu_\mu\) leads to the definition of the companion generators
\[
\hat{\beta}^\mu_\mu := \frac{1}{3}(\beta^\mu_\mu \beta^2 - \beta^2 \beta^\mu_\mu).
\]  
(2.4)

The set \(\{1, \beta^\mu_\mu, \beta^\nu_\nu, \hat{\beta}^\mu_\mu, \beta^2\} \) (where \(\beta^2 = \beta^\nu_\nu\beta^\nu_\nu\)) generates a basis (of 25 linearly independent
elements) of the algebra. These elements are not trace-orthogonal, as a consequence
of the reducibility of \(\beta^\mu_\mu\) elements. These elements are not trace-orthogonal, as a consequence
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\[
\text{Tr}(\beta^\mu_\mu \beta^\nu_\nu) = 2\eta^\mu_\nu = -\text{Tr}(\hat{\beta}^\mu_\mu \hat{\beta}^\nu_\nu);
\]
(2.5)

and others zero to this degree. Using these trace identities, and the DKP algebra defining
relations, allows elements of the DKP algebra, of any degree in the \(\beta^\mu_\mu\) and \(\hat{\beta}^\mu_\mu\), to be re-
written in terms of the basic set. A compilation of such identities is given in the Appendix.

Finally, following Kemmer [9], we introduce the real, symmetric, involutive matrix \(\eta\)
which implements the equivalence of \(\beta^\mu_\mu\) with its transpose, for which

\[
\beta^\mu_\mu = \eta \beta^{\mu\top}_\mu; \quad \hat{\beta}^\mu_\mu = -\eta \hat{\beta}^{\mu\top}_\mu,
\]
(2.7)

we define the charge conjugate wavefunction \(\Phi := \Phi^\dagger \eta\), and introduce the set of real
bilinear DKP currents: scalars \(S, S^\dagger\); charge vector current \(J^\mu\) and companion vector
current \(H^\mu\); and tensor current \(K_{\mu\nu}\) (with \(H^\mu = -H^\mu\) and \(K^\top_{\mu\nu} = K_{\nu\mu}\)), defined as follows:

\[
S := \Phi \Phi; \quad S^\dagger := \Phi \beta^2 \Phi; \quad J^\mu := \Phi \beta^\mu \Phi; \quad H^\mu := \Phi \hat{\beta}^\mu \Phi; \quad K_{\mu\nu} := \Phi \beta^\mu \beta^\nu \Phi,
\]
(2.8)

(with \(\eta^{\mu\nu} K_{\mu\nu} \equiv S^\dagger\)). Correspondingly, from the trace properties, we extract the Fierz-
DKP rearrangement identity

\[
\Phi \Phi = \left(\frac{2}{3} S - \frac{2}{3} S^\dagger\right) \mathbb{1} + \frac{1}{4} J^\mu \beta^\mu + K^{\mu\nu} \beta^\mu \beta^\nu - \frac{1}{2} H^\mu \beta^\mu - \left(\frac{2}{3} S + \frac{2}{3} S^\dagger\right) \beta^2.
\]
(2.9)

(see Appendix for details). Using this identity, the expansion of products of the form
\((\Phi \Delta \Phi) \cdot (\Phi \Delta' \Phi)\), where \(\Delta, \Delta'\) are DKP matrices, generates a system of homogeneous
quadratic relations, or Fierz-DKP identities, expressing the algebraic dependence amongst
the current bilinears. For example, if \(\Delta = \Delta' = \mathbb{1}\), we have immediately

\[
\frac{1}{9}(2S + S^\dagger)^2 = \frac{1}{2}(J \cdot J - H \cdot H) + K : K^\top
\]
(2.10)

with \(J \cdot J = \eta^{\mu\nu} J^\mu J^\nu\), \(H \cdot H = \eta^{\mu\nu} H^\mu H^\nu\), and \(K : K^\top = \eta^{\mu\nu} \eta^{\rho\sigma} K_{\mu\rho} K_{\sigma\nu}\). From these
and similar identities, as discussed in the Appendix, it is possible to eliminate the tensor
current \(K_{\mu\nu}\), namely

\[
K_{\mu\nu} = -\frac{1}{3}(S - S^\dagger) \eta_{\mu\nu} - \frac{3}{4} \frac{(J^\mu + H^\mu)(J^\nu - H^\nu)}{(S - S^\dagger)}.
\]
(2.11)

The algebraically independent currents are thus \(S, S^\dagger, J^\mu\), and \(H^\mu\), subject to the single
constraint (either from the trace of \(K_{\mu\nu}\), or by substitution for \(K : K^\top\) in the above scalar equation)

\[
\frac{1}{4}(J \cdot J - H \cdot H) + \frac{1}{9}(S - S^\dagger)(4S - S^\dagger) = 0,
\]
(2.12)

which can itself be regarded as a condition to eliminate the scalar combination \((4S - S^\dagger)\)
in terms of \((S - S^\dagger)\), for example.
3. Inversion of the DKP Equation for $A_\mu$ and $F_{\mu\nu}$.

As mentioned in the introduction, the algebraic inversion of the DKP equation proceeds by indirect algebraic manipulation rather than direct matrix inversion. By pre-multiplying the DKP equation with chosen elements $\mathbf{\bar{F}} \Delta \times \cdots$ and combining these with the corresponding complex conjugate forms (given that $A_\mu$ is real), and the algebraic identities established above, the form of $A_\mu$ itself, and hence of the field strength $F_{\mu\nu}$, can be derived, as we now show.

Starting with the DKP equation and its complex conjugate,

\begin{equation}
(i \beta^\mu \partial_\mu - e \beta^\mu A_\mu - m) \Phi = 0,
\end{equation}

\begin{equation}
\mathbf{\bar{F}}(i \beta^\mu \bar{\partial}_\mu + e \beta^\mu A_\mu + m) = 0,
\end{equation}

and pre-and post-multiplying by $\mathbf{\bar{F}}$, $\Phi$ and $\mathbf{\bar{F}} \beta^2$, $\beta^2 \Phi$, we obtain the two pairs of relations,

\begin{equation}
\partial_\mu J^\mu = 0,
\end{equation}

\begin{equation}
\partial_\mu H^\mu = \frac{1}{2} i (4 S^8 - 10 S^8),
\end{equation}

which entail the standard DKP current conservation condition, and also a companion current non-conservation condition, as well as additional vector-gauge potential and companion vector-gauge potential quadratic constraints. Here the product relations in the DKP algebra

\begin{equation}
\beta^2 \beta_\mu = \frac{2}{3} \beta_\mu - \frac{1}{2} \beta^\mu, \quad \beta_\mu \beta^2 = \frac{2}{3} \beta_\mu + \frac{1}{2} \beta^\mu,
\end{equation}

have been used (see Appendix).

Repeating this procedure, in this case by pre-and post-multiplication with $\mathbf{\bar{F}} \beta^\nu$, $\beta^\nu \Phi$ and $\mathbf{\bar{F}} \beta^\nu \Phi$, leads similarly to two pairs of relations, expressing quadratic tensor current-gauge potential constraints on $e(K^{\mu\nu} \pm K^{\nu\mu}) A_\nu$. In the second pair, however, the additional inhomogeneous term in the relevant product relations,

\begin{equation}
\beta_\mu \beta_\nu = - \beta_\mu \beta_\nu - \frac{1}{2} \eta_\mu \eta_\nu (1 - \beta^2) = - \beta_\mu \beta_\nu,
\end{equation}

throws up a contribution proportional to $e A_\mu (S - S^8)$. Elimination of the $e(K^{\mu\nu} - K^{\nu\mu}) A_\nu$ tensor current contraction terms yields an equation for the companion vector as a gradient of the scalar current,

\begin{equation}
H_\mu = \frac{i}{3 m} \partial_\mu (S - S^8),
\end{equation}

while elimination of the $e(K^{\mu\nu} + K^{\nu\mu}) A_\nu$ tensor current contraction terms allows the gauge potential to be written as

\begin{equation}
A_\mu = \frac{3 m}{2 e} \frac{J_\mu}{(S - S^8)} + \frac{1}{2 e} \frac{i (\mathbf{\bar{F}} (\partial_\mu \Phi) - (\partial_\mu \mathbf{\bar{F}}) \Phi) - i (\mathbf{\bar{F}} \beta^2 (\partial_\mu \Phi) - (\partial_\mu \mathbf{\bar{F}}) \beta^2 \Phi)}{(S - S^8)}.
\end{equation}

In this expression, the first term contains the gauge invariant, conserved current four-vector, whereas the second, gauge-dependent, term contains derivatives acting ‘internally’ on the DKP wavefunction itself, and so is not in bilinear form.

The gauge dependence can still be accommodated in bilinear form, by introducing a further 15 complex bilinear currents associated with the corresponding symmetric DKP generators $\eta_{\mu \nu}$, $\eta_{\mu \nu}$, and $\eta_{\mu \nu}$. Defining $\Phi := \Phi^\dagger \eta$, these are $\bar{S} := \Phi \Phi$, $\bar{J}_\mu := \Phi \beta_\mu \Phi$, and $\bar{K}_{\mu \nu} := \Phi \beta_{\mu \nu} \Phi \equiv \bar{K}_{\mu \nu}$ (with $\bar{H}_\mu := \Phi \beta_\mu \Phi \equiv 0$).

A Fierz-DKP rearrangement identity for $\Phi \Phi$ in terms of these complex currents, equivalent to that given above for $\Phi \Phi$ in terms of hermitian currents, is derived in the Appendix from the general identity for $\Phi \Phi$ given there, by taking $\Psi = \Phi^\dagger$, so that $\Phi = \Psi \eta = \Phi^\dagger \eta \equiv \Phi$. The coefficients are in fact identical in form (see equation (3.15)), with the omission of the $\beta_\mu$ term. In view of the special form of the gauge dependent part of the
expression for $A_\mu$, we require only a single special case, however: defining $\zeta := I - \beta^2$ and $\tilde{Z} := \tilde{S} - \tilde{S}^\circ$, $Z := S - S^\circ$, we find using the $\beta$ identities given in the Appendix,

$$\zeta \Phi \tilde{\Phi} \zeta = \tilde{Z} \zeta,$$

(3.9)

which can be used to transcribe the expression into complex bilinear currents, as follows:

$$\frac{\Phi \zeta \partial_\mu \Phi}{Z} = \frac{\tilde{\Phi} \zeta \partial_\mu \tilde{\Phi} \cdot Z}{Z^2},$$

(3.10)

wherein (inserting the transpose of the second factor and rearranging using equation (2.7))

$$Z^2 = (\Phi \zeta \Phi)(\tilde{\Phi} \zeta \tilde{\Phi}) = \Phi \zeta (\tilde{\Phi} \zeta \Phi)^* \tilde{Z} \equiv \tilde{Z}^* \tilde{Z}.$$  

(3.11)

Similarly

$$\frac{i(\Phi \zeta \partial_\mu \Phi - \partial_\mu \Phi \zeta)}{Z} = \frac{1}{2} i \left( \frac{\partial_\mu \tilde{Z}}{Z} - \frac{\partial_\mu \tilde{Z}^*}{Z^*} \right).$$

(3.13)

Thus the additional gauge-dependent part in the expression for $A_\mu$ above can be written formally in terms of the imaginary part of $\partial_\mu (\ln \tilde{Z})$, and so is indeed a pure gauge which will not contribute to the field strength. Making the choice $\tilde{Z} = \tilde{Z}^* \equiv Z$, we have therefore in this gauge

$$A_\mu = \frac{3m}{2e} \frac{J_\mu}{(S - S^\circ)}.$$  

(3.14)

Using the equation (3.7) above for the companion vector current, the field strength $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ becomes

$$F_{\mu\nu} = \frac{3m}{2e} \frac{D_{\mu} J_{\nu}}{(S - S^\circ)}, \quad \text{with} \quad D_\mu := \partial_\mu + 3m \frac{H_\mu}{(S - S^\circ)}.$$  

(3.15)

Equations (3.14) and (3.15) complete the task of algebraic inversion of the self-interacting DKP equation, expressing the field quantities (the gauge potential (for a specific gauge choice) and its field strength) in terms of gauge invariant bilinear DKP currents and their derivatives. Given that the source term $eJ$ for the equation of motion for the Abelian gauge field is as usual [9] given by the coupling to the DKP vector charge current, the field equations thereby also attain a gauge invariant form in the DKP bilinears. Using again equation (3.7), the system reduces to a set of nonlinear differential equations in the vector current $J$ itself, together with the scalar density $Z := (S - S^\circ)$. Introducing the locally scaled vector current $J_\mu := J_\mu Z^{-1}$, so that $A_\mu = (3m/2e) J_\mu$, we have finally

$$(\partial^2 \delta_\mu \nu - \partial_\mu \partial_\nu) J_\nu = \frac{2e^2}{m} Z J_\mu.$$  

(3.16)

Furthermore, from the LHS of (3.5) and (3.14), in addition with (2.12) and (5.7), $Z$ and $J^\mu$ must also satisfy the constraint equations:

$$Z \partial_\mu J^\mu + J^\mu \partial_\mu Z = 0,$$

(3.17)

$$J_\mu J^\mu = \frac{2}{9m^2} \left[ \frac{\partial_\mu \partial^\mu Z}{Z} - \frac{(\partial_\mu Z)(\partial^\mu Z)}{2Z^2} \right] + \frac{4}{9}.$$  

(3.18)
4. Conclusions

Since their original discovery, the DKP equations have remained candidate relativistic particle equations, and appear in traditional texts on quantum field theory [19] along with the Dirac equation, and the corresponding complex scalar Klein-Gordon, and massive vector Proca equations, with which they are usually regarded as equivalent (see for example [20] for a joint analysis of both cases). While this is accepted at least in the free field case (for an historical review and further analysis see [21] and references therein), it is an open question as to whether the interacting, second-quantized DKP theory, including the 5 component case, remains equivalent to standard field theories (in curved spacetime for example [22]).

In this paper we have given a gauge invariant reformulation of the self-coupled DKP equations, in terms of the set of real bilinear DKP currents. Our work provides the basis for a systematic examination of classical solutions of the self-interacting DKP equations under different spacetime symmetry group reductions [23], and for the development of the bilinear method in an Einstein-Cartan setting [24]. We expect analogous methods to be applicable also to the 10 component DKP system. More generally, a functional change of variables would allow progress towards reformulation of the interacting DKP system as a nonlinear field theory. These topics will form the subject of future investigations.

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**Appendix A. Algebraic structure in the 5 component DKP system**

From the defining relations of the Kemmer algebra

\[
\beta_{\mu} \beta_{\rho} \beta_{\nu} + \beta_{\nu} \beta_{\rho} \beta_{\mu} = \eta_{\mu\rho} \beta_{\nu} + \eta_{\nu\rho} \beta_{\mu},
\]

and the trace identities, the following degree three and four relations follow by covariance:

\[
\beta_{\lambda\mu
u} = \frac{1}{4} (\eta_{\lambda\mu} \beta_{\nu} + \eta_{\mu \nu} \beta_{\lambda}) + \frac{1}{4} (\eta_{\nu \mu} \beta_{\lambda} - \eta_{\lambda \mu} \beta_{\nu}),
\]

\[
\beta_{\kappa\lambda\mu\nu} = \eta_{\lambda \mu} \beta_{\kappa \nu} + \frac{1}{2} (\eta_{\kappa \lambda} \eta_{\mu \nu} - \eta_{\mu \kappa} \eta_{\lambda \nu}) \beta^{2} - \frac{1}{4} (\eta_{\kappa \lambda} \eta_{\mu \nu} - \eta_{\mu \lambda} \eta_{\kappa \nu}) \mathbb{I},
\]

where \(\eta^{\mu\nu} \beta_{\mu} \beta_{\nu} := \beta^{2} \equiv \beta_{0}^{2} - \beta_{1}^{2} - \beta_{2}^{2} - \beta_{3}^{2}\), and \(\hat{\beta}_{\mu} := \frac{1}{2} (\beta_{\mu} \beta^{2} - \beta^{2} \beta_{\mu})\). Specific cases following from these basic identities are as follows:

\[
\hat{\beta}_{\mu} \hat{\beta}_{\nu} = - \beta_{\mu} \beta_{\nu};
\]

\[
\hat{\beta}_{\mu} \beta_{\nu} = - \beta_{\mu} \hat{\beta}_{\nu} = \beta_{\mu} \beta_{\nu} - \frac{2}{3} \eta_{\mu \nu} (\beta^{2} - \mathbb{I});
\]

\[
\beta^{2} \beta_{\mu} = \frac{1}{2} \hat{\beta}_{\mu} + \frac{1}{2} \hat{\beta}_{\mu},
\]

\[
\beta^{2} \beta_{\mu} = \frac{1}{2} \hat{\beta}_{\mu} - \frac{1}{2} \hat{\beta}_{\mu};
\]

together with the contraction identities

\[
\beta^{\mu} \beta_{\rho} \beta_{\mu} = \beta_{\rho},
\]

\[
\beta^{\mu} \beta_{\rho} \beta_{\sigma} \beta_{\mu} = \eta_{\rho\sigma} \mathbb{I}.
\]

Further \(\beta\) matrix identities at degree 5 and higher can be derived from (A.2) by associativity. In particular, defining the special combination (scaled projection) \(\zeta := \mathbb{I} - \beta^{2}\), we have

\[
\zeta^{2} = -3 \zeta, \quad \zeta \beta^{2} \zeta = -12 \zeta, \quad \zeta \beta_{\mu} \zeta = 0, \quad \zeta \beta_{\mu} \beta_{\nu} \zeta = -3 \eta_{\mu \nu} \zeta.
\]

The basic Fierz-DKP rearrangement identity, equation (2.9), is derived as follows. Any Hermitian combination \(\Phi \bar{\Psi}\) of DKP wavefunctions \(\Phi, \Psi\) may be expanded in terms of the basic set with arbitrary (real) coefficients

\[
\Phi \bar{\Psi} = a \mathbb{I} + j^{\mu} \beta_{\mu} + \frac{1}{2} k^{\mu
u} \beta_{\mu} \beta_{\nu} + h^{\mu} \hat{\beta}_{\mu},
\]

and the coefficients determined by evaluating traces of the form \(Tr(\Delta \Phi \bar{\Psi}) = \bar{\Psi} \Delta \Phi\) for and DKP matrix \(\Delta\). For example, if \(\Psi = \Phi\) and \(\Delta = \mathbb{I}\), we have trivially \(\bar{\Phi} \Phi = 5a + k_{\mu \rho} \beta_{\mu} \beta_{\rho}\), while with \(\Delta = \beta_{\rho} \beta_{\sigma}\) we have \(\bar{\Phi} \beta_{\rho} \beta_{\sigma} \Phi = 2a \eta_{\rho\sigma} + \frac{1}{2} (k_{\rho \sigma} + \eta_{\rho \sigma} k_{\mu \mu})\). Solving these equations and the further relations following from tracing with \(\beta_{\mu}\) and \(\hat{\beta}_{\sigma}\) establishes the solution
(2.9) (for the case $\Psi = \Phi$):
\[
a = \frac{2}{9} \Phi \Phi - \frac{2}{9} \Phi \beta^2 \Phi; \quad (A.12)
\]
\[
j_{\mu} = \frac{1}{2} \Phi \beta_{\mu} \Phi; \quad (A.13)
\]
\[
h_{\mu} = -\frac{1}{2} \Phi \beta_{\mu} \Phi; \quad (A.14)
\]
\[
\frac{1}{2} k_{\mu\nu} = \Phi \beta_{\nu\mu} \Phi - \eta_{\mu\nu} \left( \frac{2}{9} \Phi \Phi + \frac{1}{9} \Phi \beta^2 \Phi \right). \quad (A.15)
\]

Homogeneous quadratic identities amongst the bilinear currents are derived in turn by expanding the $(\Phi \Phi)$ matrix in products of the form $(\Phi \Delta \Phi) \cdot (\Phi \Delta' \Phi)$; for example
\[
(\Phi \Phi)^2 = \Phi \left( \left( \frac{5}{9} S - \frac{2}{9} S^\delta \right) I + \frac{1}{2} J^\mu \beta_{\mu} + K^\nu \beta_{\mu} \beta_{\nu} - \frac{1}{2} H^\mu \beta_{\mu} - \left( \frac{2}{9} S + \frac{1}{9} S^\delta \right) \beta^2 \right) \Phi, \quad (A.16)
\]
recovering the scalar identity equation (2.10) above.

For the Fierz-DKP rearrangement identity for complex bilinear currents required in the final reduced form of the inversion, the expansion basis is restricted to the symmetric elements of the DKP algebra, $\eta, \eta \beta_{\mu}$, and $\eta \{ \beta_{\mu}, \beta_{\nu} \}$. Taking the case $\Psi = \Phi^*$, in the expansion of $\Phi \Phi$ (so that $\Phi = \Psi^\dagger \eta = \Phi^\top \eta \equiv \Phi$) the corresponding expression for $\Phi \Phi$ in terms of these complex currents, equivalent to that given above for $\Phi \Phi$ in terms of Hermitian currents, has coefficients identical in form, with the omission of the $\beta_{\mu}$ term ($\bar{h}_{\mu} = 0$), and such that $\bar{k}_{\mu\nu} = \bar{k}^{\nu\mu}$:
\[
\Phi \bar{\Phi} = \left( \frac{5}{9} \bar{S} - \frac{2}{9} \bar{S}^\delta \right) I + \frac{1}{2} \bar{J}^\mu \beta_{\mu} + \bar{K}^\nu \beta_{\mu} \beta_{\nu} - \left( \frac{2}{9} \bar{S} + \frac{1}{9} \bar{S}^\delta \right) \beta^2. \quad (A.17)
\]