ORTHOGONALITY PRESERVING PROPERTY FOR PAIRS OF OPERATORS ON HILBERT $C^\ast$-MODULES

MICHAEL FRANK$^1$, MOHAMMAD SAL MOSLEHIAN$^2$ and ALI ZAMANI$^3$

Abstract. We investigate the orthogonality preserving property for pairs of operators on inner product $C^\ast$-modules. Employing the fact that the $C^\ast$-valued inner product structure of a Hilbert $C^\ast$-module is determined essentially by the module structure and by the orthogonality structure, pairs of linear and local orthogonality-preserving operators are investigated, not a priori bounded. We obtain that if $\mathcal{A}$ is a $C^\ast$-algebra and $T, S : \mathcal{E} \to \mathcal{F}$ are two bounded $\mathcal{A}$-linear operators between full Hilbert $\mathcal{A}$-modules, then $\langle x, y \rangle = 0$ implies $\langle T(x), S(y) \rangle = 0$ for all $x, y \in \mathcal{E}$ if and only if there exists an element $\gamma$ of the center $Z(M(\mathcal{A}))$ of the multiplier algebra $M(\mathcal{A})$ of $\mathcal{A}$ such that $\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle$ for all $x, y \in \mathcal{E}$. Varying the conditions on the operators $T$ and $S$ we obtain further affirmative results for local operators and for pairs of a bounded and an unbounded $\mathcal{A}$-linear operator with bounded inverse.

1. Introduction

The starting point of considerations about orthogonality-preserving operators on Hilbert spaces was Wigner’s theorem [37] with its first complete proof by Uhlhorn [36, Lemma 3.4, Theorems 4.1 and 4.2]: For two complex Hilbert spaces $(\mathcal{H}, [\cdot, \cdot])$ and $(\mathcal{K}, [\cdot, \cdot])$ with dim$(\mathcal{H}) \geq 3$ and for a bijective operator $T : \mathcal{H} \to \mathcal{H}$ with the property that $[T(x), T(y)] = 0$ if and only if $[x, y] = 0$, there exist a bijective isometry $U : \mathcal{H} \to \mathcal{H}$ and a scalar-valued function $\phi : \mathcal{H} \to \mathbb{C}$ of modulus one such that $T(x) = \phi(x)U(x)$ for each $x \in \mathcal{H}$; see [20, 26, 32] and the references therein. Uhlhorn gave a counterexample for dimension 2, and he found a similar statement for two-dimensional Hilbert spaces under additional assumptions; see [36, Theorem 5.1]. For a historical account on further variations and extensions we refer the readers to the survey by Chevalier [8]. The situation of two unknown bijective operators $T, S : \mathcal{H} \to \mathcal{H}$ on a given Hilbert space $\mathcal{H}$ of dimension at least 3 was treated by Molnár [27, Theorem 1]: In the case when $[T(x), S(y)] = [x, y]$ for each $x, y \in \mathcal{H}$ there are bounded invertible either both

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linear or both conjugate-linear operators $U, V : \mathcal{H} \to \mathcal{H}$ such that $V = U^{*-1}$, $T = U$, and $S = V$. Varying the conditions on $S$ and $T$, Chmieliński [10] obtained a number of exceptional vs. further affirmative results.

In [22, Theorem 4] Lukasik and Wójcik were able to classify the Hilbert space situations in which $[T(x), S(y)] = [x, y]$ for every $x, y \in \mathcal{H}$ and some functions $T, S : \mathcal{H} \to \mathcal{H}$. This takes place if and only if there exist three suitable closed subspaces $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \subseteq \mathcal{K}$ such that $\mathcal{K} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$ with $\mathcal{M}_j \perp \mathcal{M}_k$ for $j \neq k$, and $T, S$ can be written as the following decompositions

$$T = A + \phi, \quad S = (A^*)^{-1} + \psi$$

for an invertible operator $A : \mathcal{H} \to \mathcal{M}_1$ and for some operators $\phi : \mathcal{H} \to \mathcal{M}_2$ and $\psi : \mathcal{H} \to \mathcal{M}_3$. See [10, 21, 23] for more results.

A first generalization of Wigner’s theorem to Hilbert $C^*$-modules over standard $C^*$-algebras was found by Ilišević and Turnšek [15, Theorem 3.1]. They also considered the case of approximate orthogonality-preservation. In parallel, a generalization of Wigner’s theorem to (full) Hilbert $\mathcal{A}$-modules $(\mathcal{E}, \langle \cdot, \cdot \rangle)$ was found by Frank et al. [14, Theorem 4] and by Leung et al. [17, Theorem 3.2, Corollary 3.3, and Theorem 3.4] characterizing bounded/unbounded $\mathcal{A}$-linear operators $T : \mathcal{E} \to \mathcal{E}$ with the property that $(T(x), T(y)) = 0$ whenever $\langle x, y \rangle = 0$ for $x, y \in \mathcal{E}$, by the equality $\langle T(x), T(y) \rangle = u\langle x, y \rangle$ for a so-called $T$-specific positive central element $u$ of the multiplier algebra of the $C^*$-algebra of coefficients and for all $x, y \in \mathcal{E}$. The operator $T$ turns out to be bounded, and hence continuous. Further, Leung et al. [17] gave a characterization of such operators $T$ as $T(x) = W(wx) = wW(x)$, $(x \in \mathcal{E})$, for a $T$-specific positive central element $w$ of the multiplier algebra of the $C^*$-algebra of coefficients and for a (not necessarily bijective) Hilbert $C^*$-module isomorphism $W : \mathcal{E} \to \mathcal{T}(\mathcal{E})$. So, the orthogonality structure and the $C^*$-module structure of a Hilbert $C^*$-module determine the Hilbert $C^*$-module, without further topological characterizations beyond the definition. Also, non-trivial orthogonality-preserving $\mathcal{A}$-linear operators on Hilbert $\mathcal{A}$-modules with an injective $T$-specific positive central element $w$ (considered as a multiplication on the multiplier algebra) are always injective, and hence, strongly orthogonality-preserving. Further conditions equivalent to the orthogonality-preserving property of (bounded) $\mathcal{A}$-linear operators on Hilbert $\mathcal{A}$-modules have been investigated by several authors, cf. [2, 3, 16, 18, 28] and others.

In the present paper, we investigate several conditions on (not necessarily bijective) $\mathcal{A}$-linear operators $T$ and $S$ acting on Hilbert $\mathcal{A}$-modules and preserving
the orthogonality of elements as a pair in one direction, that is, we do not require the bijectivity of the operators, in general. Also, we change orthogonality-preservation to approximate orthogonality preservation in some situations, or we consider merely local operators. The exceptional cases for Hilbert space situations indicate more complicated situations to appear for the more general Hilbert $C^*$-module settings. We give some examples. Our focus is on affirmative results of wide generality which can be obtained and on some proving techniques to get more information on the background of the phenomena.

2. Preliminaries

Let $(\mathcal{H}, [\cdot, \cdot])$ be an inner product space. Recall that vectors $\eta, \zeta \in \mathcal{H}$ are said to be orthogonal, written as $\eta \perp \zeta$, if $[\eta, \zeta] = 0$. For inner product spaces $\mathcal{H}, \mathcal{K}$ and two functions $T, S : \mathcal{H} \to \mathcal{K}$, the orthogonality preserving property

$$\eta \perp \zeta \implies T(\eta) \perp S(\zeta) \quad (\eta, \zeta \in \mathcal{H})$$

was introduced in [10]. The following characterization was proved.

**Theorem 2.1.** [10, Theorem 3.9] Let $\mathcal{H}$ and $\mathcal{K}$ be inner product spaces, and let $T, S : \mathcal{H} \to \mathcal{K}$ be linear operators. The following conditions are equivalent:

(i) $\eta \perp \zeta \implies T(\eta) \perp S(\zeta)$ for all $\eta, \zeta \in \mathcal{H}$.

(ii) There exists $\gamma \in \mathbb{C}$ such that $[T(\eta), S(\zeta)] = \gamma [\eta, \zeta]$ for all $\eta, \zeta \in \mathcal{H}$.

Some results in [10] have been generalized in various ways by Lukasik and Wójcik in [22, 23]. Other related topics can be found in [21, 33].

Notice that orthogonality preserving functions may be nonlinear and discontinuous, i.e. far from linear; see [9, Example 2]. The theorem describes e.g. two situations: Either the operators $S$ and $T$ may have orthogonal ranges and so $\gamma = 0$ (and they may have non-trivial kernels), or in other situations $\ker(S) = \ker(T) = \{0\}$ for both kernels and $\text{ran}(S)^{\perp \perp} = \text{ran}(T)^{\perp \perp}$ for the ranges of both $S$ and $T$. We will see later that in fact $\text{ran}(S) = \text{ran}(T)$, by Corollary 3.9. However, for Hilbert $C^*$-modules over $C^*$-algebras of coefficients with non-trivial centers the latter assumption may fail, cf. Example 4.2 below.

For a given $\theta \in [0, 1)$ two vectors $\eta, \zeta \in \mathcal{H}$ are approximately orthogonal or $\theta$-orthogonal, denoted by $\eta \perp^\theta \zeta$, if $|[\eta, \zeta]| \leq \theta \|\eta\| \|\zeta\|$. Two operators $S, T : \mathcal{H} \to \mathcal{H}$ are approximately orthogonality preserving operators if for given $\delta, \varepsilon \in [0, 1)$ one has

$$\eta \perp^\delta \zeta \implies T(\eta) \perp^\varepsilon S(\zeta) \quad (\eta, \zeta \in \mathcal{H}).$$
Often $\delta = 0$ has been considered. The approximate orthogonality preserving operators and the orthogonality equations have been investigated recently in [24, 29, 38, 39, 40]. Chmielinski [9] and Turnšek [35] studied the approximate orthogonality preserving property for one linear operator with $\delta = 0$. In addition, Chmielinski et al. [11] verified the approximate orthogonality preserving property for two linear operators.

An inner product module over a $\mathcal{A}$-module $E$ equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$, which is linear and $\mathcal{A}$-linear in the first variable and has the properties $\langle x, y \rangle^* = \langle y, x \rangle$ as well as $\langle x, x \rangle \geq 0$ with equality if and only if $x = 0$. The space $E$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\| x \|_E^2 = \| \langle x, x \rangle \|^{1/2}$. An inner product $\mathcal{A}$-module $E$ has an "$\mathcal{A}$-valued norm" $| \cdot |$, defined by $| x | = \langle x, x \rangle^{1/2}$. By $\langle E, E \rangle$ we denote the closure of the span of $\{ \langle x, y \rangle : x, y \in E \}$. We say that a Hilbert $\mathcal{A}$-module $E$ is full if $\langle E, E \rangle = \mathcal{A}$. An isometry between inner product $\mathcal{A}$-modules $E$ and $F$ is an operator $U : E \rightarrow F$ which preserves inner products, i.e. $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x, y \in E$. An operator $T : E \rightarrow F$ called $\mathcal{A}$-linear if it is linear and $T(ax) = aT(x)$ for all $x \in E$, $a \in \mathcal{A}$. Further, $T$ is called local if it is linear and

$$ax = 0 \implies aT(x) = 0 \quad (a \in \mathcal{A}, x \in E).$$

Examples of local operators include multiplication and differential operators. Note, that every $\mathcal{A}$-linear operator is local, but the converse is not true, in general (take linear differential operators into account). However, every bounded local operator between inner product modules is $\mathcal{A}$-linear; see [18].

An operator $T : E \rightarrow F$ between Hilbert $\mathcal{A}$-modules $E$ and $F$ is called adjointable if there exists an operator $T^* : F \rightarrow E$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in E$ and $y \in F$. It is easy to see that every adjointable operator $T$ is a bounded $\mathcal{A}$-linear operator; see [25].

Although inner product $C^*$-modules generalize inner product spaces by allowing inner products to take values in a certain $C^*$-algebra instead of the $C^*$-algebra of complex numbers, some fundamental properties of inner product spaces are no longer valid in inner product $C^*$-modules in their full generality. For instance, they may not possess orthonormal bases or even (normalized tight) frames, cf. [19], and norm-closed or even orthogonally closed submodules may not be orthogonal summands, cf. [25]. Therefore, when we are studying inner product $C^*$-modules, it is always of interest under which conditions the results analogous to those for inner product spaces can be reobtained, as well as which more general
situations might appear. We refer the reader to [25] for more information on the basic theory of Hilbert $C^*$-modules.

It is natural to explore the (approximate) orthogonality preserving property between inner product $C^*$-modules. Elements $x$ and $y$ in an inner product $C^*$-module $\mathcal{E}$ are said to be orthogonal, written as $x \perp y$, if $\langle x, y \rangle = 0$. Analogously to the Hilbert space situation, for a given $\theta \in [0, 1)$ two elements $x, y \in \mathcal{E}$ are approximately orthogonal or $\theta$-orthogonal, denoted by $x \perp_{\theta} y$, if $\|\langle x, y \rangle\| \leq \theta \|x\| \|y\|$.

An operator $T : \mathcal{E} \to \mathcal{F}$ between inner product $C^*$-modules is approximately orthogonality preserving if for given $\delta, \varepsilon \in [0, 1)$ one has

$$x \perp_{\delta} y = \implies T(x) \perp_{\varepsilon} T(y) \quad (x, y \in \mathcal{E}).$$

This definition was introduced and investigated in [15, 28].

Two natural problems are to describe such a class of approximately orthogonality preserving operators and to determine the stability of the orthogonality preserving property. Let $\mathbb{K}(\mathcal{H})$ and $\mathbb{B}(\mathcal{H})$ be the $C^*$-algebras of all compact linear operators and of all bounded linear operators on a Hilbert space $\mathcal{H}$, respectively. Recall that $\mathcal{A}$ is a standard $C^*$-algebra on a Hilbert space $\mathcal{H}$ if $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$.

In the case when $\mathcal{A}$ is a standard $C^*$-algebra and $\delta = 0$, Iliševič and Turnšek [15] studied the approximate orthogonality preserving property on $\mathcal{A}$-modules. In [28], the authors gave some sufficient conditions for a linear operator between Hilbert $C^*$-modules to be approximately orthogonality preserving. Moreover, it was obtained in [28, Theorem 3.9], that if $\mathcal{A}$ is a standard $C^*$-algebra and $T : \mathcal{E} \to \mathcal{F}$ is a nonzero $\mathcal{A}$-linear $(\delta, \varepsilon)$-orthogonality preserving operator between $\mathcal{A}$-modules, then

$$\|\langle T(x), T(y) \rangle - \|T\|^2 \langle x, y \rangle\| \leq \frac{4(\varepsilon - \delta)}{(1 - \delta)(1 + \varepsilon)} \|T(x)\| \|T(y)\| \quad (x, y \in \mathcal{E}).$$

Now, we will concentrate our investigations on the following condition,

$$x \perp y = \implies T(x) \perp S(y) \quad (x, y \in \mathcal{E}),$$

which we call the orthogonality preserving property for two linear operators $T, S : \mathcal{E} \to \mathcal{F}$. In the case when $S = T$, the orthogonality preserving property has been treated by Frank et al. [14], by Leung et al. [17], and others.

In the present paper, we show (Theorem 3.8) that if $\mathcal{A}$ is a standard $C^*$-algebra and $T, S : \mathcal{E} \to \mathcal{F}$ are two nonzero local operators between inner product $\mathcal{A}$-modules, then $x \perp y = \implies T(x) \perp S(y)$ for all $x, y \in \mathcal{E}$ if and only if there
exists $\gamma \in \mathbb{C}$ such that $\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle$ for all $x, y \in \mathcal{E}$. In particular, $T$ and $S$ are $\mathcal{A}$-linear. In fact, this result can be considered as a generalization of Theorem 2.1. We then apply it in Theorem 4.1 to prove that if $\mathcal{A}$ is a $C^*$-algebra and $T, S : \mathcal{E} \to \mathcal{F}$ are two nonzero bounded $\mathcal{A}$-linear operators between full Hilbert $\mathcal{A}$-modules such that $x \perp y \implies T(x) \perp S(y)$ for all $x, y \in \mathcal{E}$, then there exists an element $\gamma$ of the center $Z(M(\mathcal{A}))$ of the multiplier algebra $M(\mathcal{A})$ of $\mathcal{A}$ such that $\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle$ for all $x, y \in \mathcal{E}$. In the case of pairs of merely bounded linear operators $S$ and $T$, the invertibility of $S$ implies the $\mathcal{A}$-linearity and adjointability of these operators, and $T = (S^*)^{-1}$.

3. Linear and local orthogonality-preserving operators

The aim of this section is to prove an analogue of Theorem 2.1 for two unknown linear operators in inner product $C^*$-modules, and subsequently, a generalization of Theorem 2.1 for $\mathcal{A}$-linear operators between Hilbert $\mathcal{A}$-modules. Let us start with some observations. The following result is a consequence of [2, Theorem 3.1] and [40, Lemma 4.1].

**Proposition 3.1.** Let $\mathcal{E}$ and $\mathcal{F}$ be two inner product $\mathcal{A}$-modules and $x, y \in \mathcal{E}$. Let $T, S : \mathcal{E} \to \mathcal{F}$ be two nonzero operators. The following statements are mutually equivalent:

(i) $x \perp y \implies T(x) \perp S(y)$.
(ii) $|x - \lambda y| = |x + \lambda y| \implies |T(x) - \lambda S(y)| = |T(x) + \lambda S(y)|$ for all $\lambda \in \mathbb{C}$.
(iii) $|x - ay| = |x + ay| \implies |T(x) - a S(y)| = |T(x) + a S(y)|$ for all $a \in \mathcal{A}$.
(iv) $|x|^2 \leq |x + \lambda y|^2 \implies |T(x)|^2 \leq |T(x) + \lambda S(y)|^2$ for all $\lambda \in \mathbb{C}$.
(v) $|x|^2 \leq |x + ay|^2 \implies |T(x)|^2 \leq |T(x) + a S(y)|^2$ for all $a \in \mathcal{A}$.
(vi) $|x| \leq |x + ay| \implies |T(x)| \leq |T(x) + a S(y)|$ for all $a \in \mathcal{A}$.

**Remark 3.2.** For inner product $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ and nonzero operators $T, S : \mathcal{E} \to \mathcal{F}$, we do not know whether the following statements for $x, y \in \mathcal{E}$ are mutually equivalent:

(i) $x \perp y \implies T(x) \perp S(y)$.
(ii) $|x| \leq |x + \lambda y| \implies |T(x)| \leq |T(x) + \lambda S(y)|$ for all $\lambda \in \mathbb{C}$.

Now, we consider the $C^*$-algebra $\mathbb{M}_2(\mathbb{C})$ of all complex $2 \times 2$ matrices, as an inner product $C^*$-module over itself. Let $A, B \in \mathbb{M}_2(\mathbb{C})$ and let $T, S : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})$ be two nonzero operators. Then, by [2, Proposition 3.6], the following statements are mutually equivalent:

(i) $A \perp B \implies T(A) \perp S(B)$. 
(ii) $|A| \leq |A + \lambda B| \implies |T(A)| \leq |T(A) + \lambda S(B)|$ for all $\lambda \in \mathbb{C}$.

Employing the polarization identity, we obtain the next result.

**Proposition 3.3.** Let $\mathcal{E}$ and $\mathcal{F}$ be two inner product $\mathcal{A}$-modules. Let $T, S : \mathcal{E} \to \mathcal{F}$ be two nonzero linear operators such that $\langle T(x), S(x) \rangle = \gamma |x|^2$ for all $x \in \mathcal{E}$ and for some $\gamma \in \mathbb{Z}(\mathcal{M}(\mathcal{A}))$. Then

$$x \perp y \implies T(x) \perp S(y) \quad (x, y \in \mathcal{E}).$$

Notice that the converse of the above proposition is not true, even in the case $T = S$; see [28, Example 3.14]. In the next theorem, we prove that the converse of the above proposition is true if $\mathcal{A}$ is a standard $C^*$-algebra, in particular, whenever $\mathbb{Z}(\mathcal{M}(\mathcal{A})) = \mathbb{C}$.

To achieve the next theorem we state some prerequisites. Given two vectors $\eta$ and $\zeta$ in a Hilbert space $\mathcal{H}$, we shall denote the one-rank operator defined by $(\eta \otimes \zeta)(\xi) = [\xi, \zeta]\eta$ by $\eta \otimes \zeta \in \mathcal{K}(\mathcal{H})$. Observe that $\eta \otimes \eta$ is a minimal projection. Recall that a projection $e$ in a $C^*$-algebra $\mathcal{A}$ is called minimal if $e\mathcal{A}e = C\mathcal{e}$.

Now let $\mathcal{A}$ be a standard $C^*$-algebra on a Hilbert space $\mathcal{H}$ and let $\mathcal{E}$ be an inner product (respectively, Hilbert) $\mathcal{A}$-module. Let $e = \eta \otimes \eta$ for some unit vector $\eta \in \mathcal{H}$ be a minimal projection. Then $\mathcal{E}_e = \{ex : x \in \mathcal{E}\}$ is a complex inner product (respectively, Hilbert) space contained in $\mathcal{E}$ with respect to the inner product $[x, y] = \text{tr}(\langle x, y \rangle), x, y \in \mathcal{E}_e$; see [5]. Note that if $x, y \in \mathcal{E}_e$, then $\langle x, y \rangle = [x, y]e$ and $\|x\|_{\mathcal{E}_e} = \|x\|_\mathcal{E}$, where the norm $\|\cdot\|_{\mathcal{E}_e}$ comes from the inner product $[\cdot, \cdot]$. This enables us to apply Hilbert space theory by lifting results from the Hilbert space $\mathcal{E}_e$ to the whole $\mathcal{A}$-module $\mathcal{E}$.

**Theorem 3.4.** Let $\mathcal{A}$ be a standard $C^*$-algebra on a Hilbert space $\mathcal{H}$ and let $\mathcal{E}$ and $\mathcal{F}$ be two inner product $\mathcal{A}$-modules. Suppose, $T, S : \mathcal{E} \to \mathcal{F}$ are two nonzero $\mathcal{A}$-linear operators such that

$$x \perp y \implies T(x) \perp S(y) \quad (x, y \in \mathcal{E}).$$

Then there exists $\gamma \in \mathbb{C}$ such that

$$\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle \quad (x, y \in \mathcal{E}).$$

**Proof.** The following proof is a modification of the one given by Ilišević and Turnšek [15, Theorem 3.1]. Let $e = \zeta \otimes \zeta$ and $f = \eta \otimes \eta$ be minimal projections in $\mathcal{A}$ and let $u = \zeta \otimes \eta$. Also, let $T_e = T|_{\mathcal{E}_e}$ and $S_e = S|_{\mathcal{E}_e}$. For linear operators $T_e, S_e : \mathcal{E}_e \to \mathcal{F}_e$ we have $[x, y] = 0 \implies [T_e(x), S_e(y)] = 0$ for all $x, y \in \mathcal{E}_e$. 

Hence, by Theorem 2.1, there exists $\gamma_e \in \mathbb{C}$ such that

$$[T(ex), S(ex)] = \gamma_e \|ex\|^2 = \gamma_e [ex, ex] \quad (x \in \mathcal{E}).$$

This yields $[T(ex), S(ex)]e = \gamma_e [ex, ex]e$, thus

$$\langle T(ex), S(ex) \rangle = \gamma_e \langle ex, ex \rangle = \gamma_e |ex|^2,$$

or equivalently,

$$e \langle T(x), S(x) \rangle e = \gamma_e |x|^2 e \quad (x \in \mathcal{E}). \quad (3.1)$$

Similarly, there exists $\gamma_f \in \mathbb{C}$ such that

$$f \langle T(x), S(x) \rangle f = \gamma_f |x|^2 f \quad (x \in \mathcal{E}). \quad (3.2)$$

Since $ufu^* = e$, it follows from (3.1) and (3.2) that

$$\gamma_e [ex, ex] e = \gamma_f [ex, ex] e = \gamma_e \langle x, x \rangle e$$

$$= e \langle T(x), S(x) \rangle e = uf u^* \langle T(x), S(x) \rangle u f^*$$

$$= uf \langle T(u^* x), S(u^* x) \rangle f u^* = u \gamma_f |u^* x|^2 u f^*$$

$$= \gamma_f uf u^* |x|^2 uf u^* = \gamma_f \langle x, x \rangle e$$

$$= \gamma_f \langle ex, ex \rangle e.$$

Thus

$$\gamma_e [ex, ex] = \gamma_f [ex, ex] \quad (x \in \mathcal{E}).$$

Replacing $x$ with $\frac{x}{|x|}$, we conclude $\gamma_e = \gamma_f = \gamma$. Hence, by (3.1), we get

$$e \langle T(x), S(x) \rangle e = e \gamma |x|^2 e \quad (x \in \mathcal{E})$$

for all minimal projections $e \in \mathcal{A}$. Having in mind that $\mathcal{A}$ is a standard $C^*$-algebra, we deduce that

$$\langle T(x), S(x) \rangle = \gamma |x|^2 \quad (x \in \mathcal{E}).$$

Now, by the polarization identity, we arrive at

$$\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle \quad (x, y \in \mathcal{E}).$$

□

**Corollary 3.5.** Let $\mathcal{A}$ be a standard $C^*$-algebra and let $\{\mathcal{E}, \langle \cdot, \cdot \rangle_1 \}$ be an inner product $\mathcal{A}$-module. Suppose that $\langle \cdot, \cdot \rangle_2$ is a second $\mathcal{A}$-valued inner product on $\mathcal{E}$. If $\langle x, y \rangle_1 = 0$ implies $\langle x, y \rangle_2 = 0$ for every $x, y \in \mathcal{E}$, then there exists a positive constant $\gamma \in \mathbb{C}$ such that $\langle x, y \rangle_2 = \gamma \langle x, y \rangle_1$ for each $x, y \in \mathcal{E}$. 
Proof. Take $\mathcal{E} = \mathcal{F}$ as $\mathcal{A}$-modules and set $T = S = I : (\mathcal{E}, \langle \cdot, \cdot \rangle_1) \to (\mathcal{E}, \langle \cdot, \cdot \rangle_2)$, where $I$ denotes the identity operator. Applying Theorem 3.4 the assertion follows. \qed

Remark 3.6. According to [28, Example 3.15], the assumption of $\mathcal{A}$-linearity, even in the case $T = S$, is necessary in Theorem 3.4. If either $S$ or $T$ is adjointable and $\gamma \neq 0$, then $\text{ran}(S) = \text{ran}(T) = \mathcal{E}$, because e.g. $S^* T = \gamma I$ for a real number $\gamma > 0$.

In the following result, we employ some ideas of [17] to consider local operators between inner product $\mathcal{A}$-modules, i.e. linear operators $T : \mathcal{E} \to \mathcal{F}$ such that $ax = 0$ for $x \in \mathcal{E}$ and $a \in \mathcal{A}$ forces $aT(x) = 0$ in $\mathcal{F}$. We are interested in operators which preserve orthogonality. Since we do not suppose that the local operators under consideration are bounded, the $\mathcal{A}$-linearity should be obtained separately.

**Theorem 3.7.** Let $\mathcal{A}$ be a standard C*-algebra on a Hilbert space $\mathcal{H}$ and let $\mathcal{E}$ and $\mathcal{F}$ be two inner product $\mathcal{A}$-modules. Suppose that $T, S : \mathcal{E} \to \mathcal{F}$ are two nonzero local operators such that
\[
x \perp y \implies T(x) \perp S(y) \quad (x, y \in \mathcal{E}).
\]
Then there exists $\gamma \in \mathbb{C} \setminus \{0\}$ such that
\[
\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle \quad (x, y \in \mathcal{E}).
\]
Moreover, the operators $T$ and $S$ are $\mathcal{A}$-linear.

Proof. Let $(f_j)_{j \in J}$ be an approximate unit in $\mathbb{K}(\mathcal{H})$ consisting of finite rank positive operators. Suppose that $p \in \mathbb{K}(\mathcal{H})$ is a projection. Since $T$ is local, the known equality $p(1-p)x = 0$ ensures $pT((1-p)x) = 0$, and analogously, we get $(1-p)T(px) = 0$. From the complex linearity of the operator $T$ we derive $pT(x) = T(px)$ from these two equalities. Consequently, $T(ax) = aT(x)$ for all finite rank operators $a \in \mathcal{A}$ and all $x \in \mathcal{E}$. Now, for each $x \in \mathbb{K}(\mathcal{H}) \cdot \mathcal{E}$, there exist $c \in \mathbb{K}(\mathcal{H})$ and $z \in \mathcal{E}$ such that $x = cz$. Hence
\[
\lim_j \|f_j T(x) - cT(z)\| = \lim_j \|f_j cT(z) - cT(z)\| = 0.
\]
Define $\widetilde{T} : \mathbb{K}(\mathcal{H}) \cdot \mathcal{E} \to \mathbb{K}(\mathcal{H}) \cdot \mathcal{F}$ by setting $\widetilde{T}(x)$ to be the norm limit of $f_j T(x)$. Notice that $\lim_j \|f_j T(x) - cT(z)\| = 0$ shows that $\widetilde{T}(x)$ depend on neither the choice of $(f_j)_{j \in J}$ nor the decomposition $x = cz$. Also, $\widetilde{T}$ is $\mathbb{K}(\mathcal{H})$-linear since,
\[
\widetilde{T}(ax) = \widetilde{T}(acz) = acT(z) = a\widetilde{T}(x).
\]
for all $x \in \mathcal{E}$ and all $a \in \mathbb{K}(\mathcal{H})$. Similarly, define $\tilde{S}: \mathbb{K}(\mathcal{H}) \cdot \mathcal{E} \to \mathbb{K}(\mathcal{H}) \cdot \mathcal{F}$ by setting $\tilde{S}(y)$ to be the norm limit of $f_jS(y)$. Now, if $x, y \in \mathbb{K}(\mathcal{H}) \cdot \mathcal{E}$ with $\langle x, y \rangle = 0$, then $\langle T(x), S(y) \rangle = 0$, which secures $\langle f_jT(x), f_kS(y) \rangle = 0$ for all $j, k \in J$. Thus $\langle \tilde{T}(x), \tilde{S}(y) \rangle = 0$. Hence for $\mathbb{K}(\mathcal{H})$-linear operators $\tilde{T}$ and $\tilde{S}$ we have

$$x \perp y \implies \tilde{T}(x) \perp \tilde{S}(y) \quad (x, y \in \mathbb{K}(\mathcal{H}) \cdot \mathcal{E}).$$

So, by Theorem 3.4, there exists $\gamma \in \mathbb{C}$ such that $\langle \tilde{T}(x), \tilde{S}(x) \rangle = \gamma|x|^2$ for all $x \in \mathbb{K}(\mathcal{H}) \cdot \mathcal{E}$. Thus

$$f_j\langle T(x), S(x) \rangle f_j = \langle \tilde{T}(f_jx), \tilde{S}(f_jx) \rangle = \gamma|f_jx|^2 = f_j\gamma|x|^2 f_j$$

for all $x \in \mathcal{E}$ and all $j \in J$. Consequently, if $\langle T(x), S(x) \rangle - \gamma|x|^2 \in \mathcal{A}$, then always $f_j((\langle T(x), S(x) \rangle - \gamma|x|^2)f_j = 0$, which yields $\langle T(x), S(x) \rangle - \gamma|x|^2 = 0$. Hence, $\langle T(x), S(x) \rangle = \gamma|x|^2$ for all $x \in \mathcal{E}$. Utilizing the polarization identity, we obtain

$$\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle \quad (x, y \in \mathcal{E}).$$

\[\square\]

Combining Proposition 3.3 and Theorem 3.7, we reach the next result. In fact, this result is a generalization of [15, Theorem 3.1] and [40, Theorem 4.10]. The result also generalizes [17, Corollary 3.2].

**Theorem 3.8.** Let $\mathcal{A}$ be a standard $C^*$-algebra and let $\mathcal{E}$ and $\mathcal{F}$ be two inner product $\mathcal{A}$-modules. Suppose that $T, S: \mathcal{E} \to \mathcal{F}$ are two nonzero local operators. The following statements are mutually equivalent:

(i) $x \perp y \implies T(x) \perp S(y)$ for all $x, y \in \mathcal{E}$.

(ii) There exists $\gamma \in \mathbb{C} \setminus \{0\}$ such that $\langle T(x), S(x) \rangle = \gamma|x|^2$ for all $x \in \mathcal{E}$.

(iii) There exists $\gamma \in \mathbb{C} \setminus \{0\}$ such that $\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle$ for all $x, y \in \mathcal{E}$.

Under these conditions the operators $T$ and $S$ are $\mathcal{A}$-linear.

**Corollary 3.9.** Let $\mathcal{A}$ be a standard $C^*$-algebra and let $\mathcal{E}$ be an inner product $\mathcal{A}$-module. Suppose that $T: \mathcal{E} \to \mathcal{E}$ is a nonzero local operator such that

$$x \perp y \implies T(x) \perp y \quad (x, y \in \mathcal{E}).$$

Then there exists $\gamma \in \mathbb{C} \setminus \{0\}$ such that $T(x) = \gamma x$ for all $x \in \mathcal{E}$.

**Proof.** Applying Theorem 3.8 to $T$ and $S = I$ we obtain, with some $\gamma \in \mathbb{C}$,

$$\langle T(x), y \rangle = \gamma \langle x, y \rangle$$

for all $x, y \in \mathcal{E}$. Hence, $\langle T(x) - \gamma x, y \rangle = 0$ for all $x, y \in \mathcal{E}$. Thus $T(x) = \gamma x$ for all $x \in \mathcal{E}$. \[\square\]
Corollary 3.10. Let $\mathcal{A}$ be a standard $C^*$-algebra and let $\mathcal{E}$ and $\mathcal{F}$ be two inner product $\mathcal{A}$-modules. Let $T_0, S_0 : \mathcal{E} \to \mathcal{F}$ be two nonzero local operators such that

\[ x \perp y \implies T_0(x) \perp S_0(y) \quad (x, y \in \mathcal{E}). \]

Suppose, the linear operators $T, S : \mathcal{E} \to \mathcal{F}$ are sufficiently close to $T_0$ and $S_0$, respectively, namely that for $\theta_1, \theta_2 \in [0, 1)$ and for all $x, y \in \mathcal{E}$

\[ \|T(x) - T_0(x)\| \leq \theta_1\|T(x)\| \quad \text{and} \quad \|S(y) - S_0(y)\| \leq \theta_2\|S(y)\|. \]

Then

\[ x \perp y \implies T(x) \perp^\varepsilon S(y) \quad (x, y \in \mathcal{E}), \]

where $\varepsilon = \theta_1\varepsilon_2 + \theta_1(\varepsilon_2 + 1) + (\varepsilon_1 + 1)\varepsilon_2$.

Proof. From the assumption, we obtain

\[ \|T_0(x)\| \leq (\theta_1 + 1)\|T(x)\| \quad \text{and} \quad \|S_0(y)\| \leq (\theta_2 + 1)\|S(y)\| \quad (x, y \in \mathcal{E}). \] (3.3)

Also, by Theorem 3.8, there exists $\gamma_0 \in \mathbb{C}$ such that

\[ \langle T_0(x), S_0(y) \rangle = \gamma_0 \langle x, y \rangle \quad (x, y \in \mathcal{E}). \] (3.4)

Now let $x, y \in \mathcal{E}$ and $x \perp y$. From (3.3) and (3.4) we get

\[
\|\langle T(x), S(y) \rangle\| = \|\langle T(x), S(y) \rangle - \langle T_0(x), S_0(y) \rangle\|
= \|\langle T(x) - T_0(x), S(y) - S_0(y) \rangle + \langle T(x) - T_0(x), S_0(y) \rangle + \langle T_0(x), S(y) - S_0(y) \rangle\|
\leq \|T(x) - T_0(x)\| \|S(y) - S_0(y)\| + \|T(x) - T_0(x)\| \|S_0(y)\|
+ \|T_0(x)\| \|S(y) - S_0(y)\|
\leq \left(\theta_1\theta_2 + \theta_1(\varepsilon_2 + 1) + (\varepsilon_1 + 1)\varepsilon_2\right) \|T(x)\| \|S(y)\|
= \varepsilon \|T(x)\| \|S(y)\|.
\]

Thus $\|\langle T(x), S(y) \rangle\| \leq \varepsilon \|T(x)\| \|S(y)\|$, and hence $T(x) \perp^\varepsilon S(y)$. \qed

4. $C^*$-linear orthogonality preserving operators

In this section, we intend to show the properties of pairs of bounded $C^*$-linear operators $\{T, S\}$ for which the orthogonality of two elements $x$ and $y$ of the domain ensures the orthogonality of their respective images $T(x)$ and $S(y)$. To get reasonable results we have either to suppose or to derive the $C^*$-linearity of the operators. The proof of the key equality relies, e.g., on the theory of the universal $\ast$-representation of the $C^*$-algebra of coefficients, in which the bicommutant of
the faithfully represented \( C^* \)-algebra is *-isomorphic to its bidual Banach space. Also, we make use of the existence of predual spaces of von Neumann algebras and for self-dual Hilbert \( C^* \)-modules over them. For a concise explanation we refer the reader to [34, Ch. II, 2+3; Ch. V, 1] and to [30, §§ 3+4].

We need some prerequisites for the next theorem; see [12, 14]. For a Hilbert \( \mathcal{A} \)-module \( \mathcal{E} \) over a \( C^* \)-algebra \( \mathcal{A} \), one can extend \( \mathcal{E} \) canonically to a Hilbert \( \mathcal{A}^{**} \)-module \( \mathcal{E}^# \) over the bidual Banach space and von Neumann algebra \( \mathcal{A}^{**} \) of \( \mathcal{A} \) [30, Theorem 3.2, Proposition 3.8, and §4]. For this the \( \mathcal{A}^{**} \)-valued pre-inner product can be defined by the formula

\[
[a \otimes x, b \otimes y] = a^* \langle x, y \rangle b,
\]

for elementary tensors of \( \mathcal{A}^{**} \otimes \mathcal{E} \), where \( a, b \in \mathcal{A}^{**}, \ x, y \in \mathcal{E} \). The quotient module of \( \mathcal{A}^{**} \otimes \mathcal{E} \) by the set of all isotropic vectors is denoted by \( \mathcal{E}^# \). Denote the canonical isometric module embedding of \( \mathcal{E} \) into \( \mathcal{E}^# \) described in [30] by \( \pi' \). The quotient module can be canonically completed to a self-dual Hilbert \( \mathcal{A}^{**} \)-module \( \mathcal{G} \) which is isometrically algebraically isomorphic to the \( \mathcal{A}^{**} \)-module of \( \mathcal{E}^# \). In addition, \( \mathcal{G} \) is a dual Banach space itself; cf. [30, Theorem 3.2, Proposition 3.8, and §4]. Every \( \mathcal{A} \)-linear bounded operator \( T : \mathcal{E} \rightarrow \mathcal{E} \) can be continued to a unique \( \mathcal{A}^{**} \)-linear operator \( T : \mathcal{E}^# \rightarrow \mathcal{E}^# \) preserving the operator norm and obeying the canonical embedding \( \pi'(\mathcal{E}) \) of \( \mathcal{E} \) into \( \mathcal{E}^# \). Similarly, \( T \) can be further extended to the self-dual Hilbert \( \mathcal{A}^{**} \)-module \( \mathcal{G} \). The extension is such that the isometrically algebraically embedded copy \( \pi'(\mathcal{E}) \) of \( \mathcal{E} \) in \( \mathcal{G} \) is a \( w^* \)-dense \( \mathcal{A} \)-submodule of \( \mathcal{G} \), and that \( \mathcal{A} \)-valued inner product values of elements of \( \mathcal{E} \) embedded in \( \mathcal{G} \) are preserved with respect to the \( \mathcal{A}^{**} \)-valued inner product on \( \mathcal{G} \) and to the canonical isometric embedding \( \pi \) of \( \mathcal{A} \) into its bidual Banach space \( \mathcal{A}^{**} \). Every bounded \( \mathcal{A} \)-linear operator \( T \) on \( \mathcal{E} \) can be extended to a unique bounded \( \mathcal{A}^{**} \)-linear operator on \( \mathcal{G} \) preserving the operator norm, cf. [30, Proposition 3.6, Corollary 3.7, and §4]. The extension of bounded \( \mathcal{A} \)-linear operators from \( \mathcal{E} \) to \( \mathcal{G} \) is continuous with respect to the \( w^* \)-topology on \( \mathcal{G} \). A Hilbert \( C^* \)-module \( \mathcal{K} \) over a \( W^* \)-algebra \( \mathcal{B} \) is self-dual, if and only if its unit ball is complete with respect to the topology induced by the semi-norms \( \{ |f(\langle x, \cdot \rangle)| : x \in \mathcal{K}, \ f \in \mathcal{B}^*, \ |x| \leq 1, \ |f| \leq 1 \} \), if and only if its unit ball is complete with respect to the topology induced by the semi-norms \( \{ f(\langle \cdot, \cdot \rangle)^{1/2} : f \in \mathcal{B}^*, \ |f| \leq 1 \} \). The first topology coincides with the \( w^* \)-topology on \( \mathcal{K} \) in that case, see [12, § 4].
Note, that in the construction above \( E \) is always \( w^\ast \)-dense in \( G \), as well as for each subset of \( E \) the respective construction is \( w^\ast \)-dense in its biorthogonal complement with respect to \( G \). However, starting with a subset of \( G \), its biorthogonal complement with respect to \( G \) might not have a \( w^\ast \)-dense intersection with the embedding of \( E \) into \( G \); cf. [31, Proposition 3.11.9].

Further, we want to consider only discrete \( W^* \)-algebras, i.e. \( W^* \)-algebras for which the supremum of all minimal projections contained in them equals their identity. (We prefer to use the word discrete instead of atomic.) To connect to the general \( C^* \)-case we make use of a theorem of Akemann [1, p. 278] stating that the \( * \)-homomorphism of a \( C^* \)-algebra \( A \) into the discrete part of its bidual von Neumann algebra \( A^{**} \) which arises as the composition of the canonical embedding \( \pi \) of \( A \) into \( A^{**} \) followed by the projection to the discrete part of \( A^{**} \) is an injective \( * \)-homomorphism \( \rho \). This injective \( * \)-homomorphism \( \rho \) is implemented by a central projection \( p \in Z(A^{**}) \) in such a way that \( A^{**} \) multiplied by \( p \) gives the discrete part of \( A^{**} \).

For topological characterizations of self-duality of Hilbert \( C^* \)-modules over \( W^* \)-algebras and the properties of the modules and operators we refer the reader to [30].

Now, we are in a position to state one of the main results of this section that generalizes [10, Theorem 3.2].

**Theorem 4.1.** Let \( \mathcal{A} \) be a \( C^* \)-algebra and let \( \mathcal{E} \) and \( \mathcal{F} \) be two full Hilbert \( \mathcal{A} \)-modules. Suppose that \( T, S : \mathcal{E} \to \mathcal{F} \) are two nonzero bounded \( \mathcal{A} \)-linear operators such that
\[
x \perp y \implies T(x) \perp S(y) \quad (x, y \in \mathcal{E}).
\]
Then there exists an element \( \gamma \) of the center \( Z(M(\mathcal{A})) \) of the multiplier algebra \( M(\mathcal{A}) \) of \( \mathcal{A} \) such that
\[
\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle \quad (x, y \in \mathcal{E}).
\]
In the complementary case, if \( \gamma = 0 \), the ranges of the operators \( T \) and \( S \) are orthogonal to each other, and no further information on them can be derived.

**Proof.** First, we make use of the existing canonical non-degenerate isometric \( * \)-representation \( \pi \) of a \( C^* \)-algebra \( \mathcal{A} \) in its bidual Banach space and von Neumann algebra \( \mathcal{A}^{**} \) of \( \mathcal{A} \), as well as of its extension \( \pi' : \mathcal{E} \to \mathcal{E}^\# \to \mathcal{G} \), \( \pi' : \mathcal{F} \to \mathcal{F}^\# \to \mathcal{H} \) and of the unique \( w^* \)-continuous \( \mathcal{A}^{**} \)-linear bounded operator extensions \( T, S : \mathcal{G} \to \mathcal{H} \). In other words, we extend the set \( \{ \mathcal{A}, \mathcal{E}, \mathcal{F}, T, S \} \) to the set \( \{ \mathcal{A}^{**}, \mathcal{G}, \mathcal{H}, T, S \} \). The Hilbert \( \mathcal{A}^{**} \)-modules \( \mathcal{G} \) and \( \mathcal{H} \) are self-dual and admit
a predual Banach space, hence, a \( w^* \)-topology. The extended operators \( T \) and \( S \) are \( w^* \)-continuous, \( \mathcal{A}^{**} \)-linear and bounded by the same constants as the original operators \( T \) and \( S \).

Secondly, we have to demonstrate that for each pair of elements \( x \perp y \) with \( x, y \in \mathcal{G} \) the property \( T(x) \perp S(y) \) still holds for the extended operators. In the sequel, we identify \( \mathcal{E} \) with its image \( \pi'(\mathcal{E}) \subseteq \mathcal{G} \). Note, that \( \pi'(\mathcal{E}) \) is \( w^* \)-dense in \( \mathcal{G} \). Applying techniques developed in [30], let us fix a non-trivial normal state \( f \in \mathcal{A}^* \) and form the inner product spaces \( \mathcal{E}_f' = \{ \mathcal{E}, f((\cdot, \cdot)) \}/\ker(f((\cdot, \cdot))) \) and \( \mathcal{G}_f = \text{cl}(\{ \mathcal{G}, f((\cdot, \cdot)) \}/\ker(f((\cdot, \cdot))) \) where \( \mathcal{E}_f \) is norm-dense in \( \mathcal{G}_f \). The operators \( T \) and \( S \) are restricted to this pre-Hilbert space and are still linear and continuous. Now, form the biorthogonal complements of the two remainder classes \( [x]_f \) and \( [y]_f \) of \( x, y \in \mathcal{G}_f \). Their intersection with \( \mathcal{E}_f \) is not empty, and these intersections form norm-dense subsets in them, respectively. For a pair of elements \( t \in [x]_f \cap \mathcal{E}_f \) and \( s \in [y]_f \cap \mathcal{E}_f \) we always have \( T(t) \perp_f S(s) \). Since inner products on pre-Hilbert spaces are separately weakly continuous in each of their arguments the domain of \( T \), for instance, can be extended to \( \mathcal{G}_f \) preserving the orthogonality relation to \( S(\mathcal{E}_f) \). By symmetry, the same is true for \( S \). Therefore, \( T([x]_f) \perp_f S([y]_f) \) for all \( x, y \in \mathcal{G}_f \) because of the norm-density of the respective subsets and of the continuity of the operators. Since the normal state \( f \) on \( \mathcal{A} \) was selected arbitrarily the relation \( T(x) \perp S(y) \) follows.

Since the von Neumann algebra \( p\mathcal{A}^{**} \) is discrete, its identity \( p \) can be represented as the \( w^* \)-sum of a maximal set of pairwise orthogonal atomic projections \( \{ q_\alpha : \alpha \in J \} \) of the center \( Z(p\mathcal{A}^{**}) \) of \( p\mathcal{A}^{**} \). Note, that \( w^* \sum_{\alpha \in J} q_\alpha = p \). Take a single atomic projection \( q_\alpha \in Z(p\mathcal{A}^{**}) \) of this collection and consider the part \( \{ q_\alpha p\mathcal{A}^{**}, q_\alpha p\mathcal{E}, q_\alpha p\mathcal{F}, q_\alpha pT, q_\alpha pS \} \) of the problem.

Since \( q_\alpha p\mathcal{A}^{**} \) is a discrete (type I) \( W^* \)-factor (finite- or infinite-dimensional), the equality \( \langle T(x), S(y) \rangle = \lambda_{q_\alpha} \langle x, y \rangle = \langle x, y \rangle \lambda_{q_\alpha} \) holds for specific \( T \) and \( S \) and complex number \( \lambda_{q_\alpha} \), cf. Theorem 3.4.

Now, we can follow the decomposition process in reverse direction. Note, that the multiplier algebra of \( p\mathcal{A} \) is *-isometrically embedded in \( p\mathcal{A}^{**} \). Since \( |\lambda_{q_\alpha}| \leq ||S||\ ||T|| \) for all \( \alpha \in J \), the sum \( \sum_{\alpha \in J} \lambda_{q_\alpha} q_\alpha \) is \( w^* \)-convergent in \( Z(p\mathcal{A}^{**}) \). Moreover, since \( \lambda_{q_\alpha} q_\alpha \) commutes with \( p(x, y) \) for each \( \alpha \), the sum \( \sum_{\alpha \in J} \lambda_{q_\alpha} q_\alpha \) commutes with \( p(x, y) \). What is more, since \( p(\mathcal{G}, \mathcal{F}) \) is dense in \( p\mathcal{A}^{**} \) and the \( \mathcal{C}^* \)-valued inner product \( \sum_{\alpha \in J} \lambda_{q_\alpha} q_\alpha \langle x, y \rangle = \sum_{\alpha \in J} \langle x, y \rangle \lambda_{q_\alpha} q_\alpha \) belongs to \( p\mathcal{A}^{**} \) for all \( x, y \in \mathcal{G} \), the element \( \sum_{\alpha \in J} \lambda_{q_\alpha} q_\alpha \) is in \( pZ(M(\mathcal{A})) \). Since \( \mathcal{E} \) and \( \mathcal{F} \) are full Hilbert \( \mathcal{A} \)-modules, we arrive at the equality \( p(T(x), S(y)) = \sum_{\alpha \in J} \lambda_{q_\alpha} q_\alpha \langle x, y \rangle \) for all \( x, y \in \mathcal{G} \).
The remaining step is to pull back this equality to the initial context along the two injective $\ast$-homomorphisms used.

Note, that $\gamma = 0$ forces only $T(E) \perp S(E)$ without further conditions on $T$ and $S$. □

**Example 4.2.** Let $\mathcal{A} = C_0((0,1])$ be the $C^*$-algebra of all continuous functions vanishing at zero, where $(0,1]$ is the unit interval with the usual metric topology. Set $E = F = \mathcal{A}$ with the usual $\mathcal{A}$-valued inner product derived from the multiplication and the involution on $\mathcal{A}$. Set

$$S(f(t)) = \sin \left( \frac{1}{t} \right) \cdot f(t), \quad T(f(t)) = \cos \left( \frac{1}{t} \right) \cdot f(t)$$

for all $f \in \mathcal{A}$ and $t \in (0,1]$. Then $\text{ran}(S) \neq \text{ran}(T)$ and they are neither equal to $E$, but $\text{ran}(S)^\perp = \text{ran}(T)^\perp = E$. Moreover, $\gamma = \sin \left( \frac{1}{t} \right) \cos \left( \frac{1}{t} \right) \in M(\mathcal{A})$ is not invertible. Both the operators are bounded, adjointable (and hence $\mathcal{A}$-linear), injective and orthogonality-preserving, but they are not invertible.

**Corollary 4.3.** Let $\mathcal{A}$ be a $C^*$-algebra and $\mathcal{E}$ be a full Hilbert $\mathcal{A}$-module. Let $T : \mathcal{E} \to \mathcal{E}$ be a bounded $\mathcal{A}$-linear operator such that $x \perp y$ implies $T(x) \perp y$ for every suitable elements $x, y \in \mathcal{E}$. Then there exists an element $\gamma \in Z(M(\mathcal{A}))$ such that $T(x) = \gamma x$ for each $x \in \mathcal{E}$.

The proof is the same as for Corollary 3.9 changing the origin of $\gamma$ and the theorem referred to.

**Corollary 4.4.** Let $\mathcal{A}$ be a $C^*$-algebra and $\mathcal{E}$ be a full Hilbert $\mathcal{A}$-module. Let $U, V : \mathcal{E} \to \mathcal{E}$ be isometries with $UU^* = VV^* = I$. Then $x \perp y$ forces $U(x) \perp V(y)$ for every suitable pair $x, y \in \mathcal{E}$ if and only if either $U(\mathcal{E}) \perp V(\mathcal{E})$ or $U(x) = \gamma V(x)$ for a fixed unitary element $\gamma \in Z(M(\mathcal{A}))$ and every $x \in \mathcal{E}$.

Proof. Let $x, y \in \mathcal{E}$ be a pair of orthogonal elements. Then $\langle V^*U(x), y \rangle = \langle U(x), V(y) \rangle = 0$ by the assumption. Applying Corollary 4.3 we obtain either $V^*U = 0$ or $V^*U(x) = \gamma x$ for a certain non-zero $\gamma \in Z(M(\mathcal{A}))$, i.e. $U = \gamma V$, and $\gamma$ has to be unitary since we are treating co-isometries. □

Geometrically this means that two isometric copies of a Hilbert $\mathcal{A}$-module $\mathcal{E}$ embedded into $\mathcal{E}$ as orthogonal direct summands additionally preserve orthogonality of the different images of each pair of initially orthogonal elements of $\mathcal{E}$, if and only if either these two images of $\mathcal{E}$ are orthogonal to each other as $\mathcal{A}$-submodules or these two isometric embeddings as orthogonal direct summands.
only differ by multiplication by a unitary from $Z(M(\mathcal{A}))$, i.e. coincide as $\mathcal{A}$-modules.

The next theorem generalizes [14, Theorem 3] and [17, Theorem 2.3], and gives a partial solution of [14, Problem 1]. It extends results of [10].

**Theorem 4.5.** Let $\mathcal{A}$ be a C*-algebra and $\mathcal{E}$ be a full Hilbert $\mathcal{A}$-module. Let $T : \mathcal{E} \rightarrow \mathcal{E}$ be a linear operator and let $S : \mathcal{E} \rightarrow \mathcal{E}$ be an invertible linear operator with bounded inverse operator. Suppose, $\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle$ for some invertible element $\gamma \in Z(M(\mathcal{A}))$. Then $T$ and $S$ are bounded, $\mathcal{A}$-linear, adjointable, invertible, and $ST^* = \gamma x$ for all $x \in \mathcal{E}$, i.e. the pairs of operators $\{S, T^*\}$ and $\{S^*, T\}$ commute, and also $S = \gamma^{-1}(T^*)^{-1}$ and $T = \gamma^*(S^*)^{-1}$.

In the special situation of $T = S$ the element $\gamma$ is positive and $T = \sqrt{\gamma} U$ for some unitary $\mathcal{A}$-linear operator $U$ on $\mathcal{E}$.

**Remark 4.6.** In the case where the Hilbert $\mathcal{A}$-module $\mathcal{E}$ admits some invertible bounded $\mathcal{A}$-linear operator $S$ that is not adjointable, it can not fulfill the equality $\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle$ for any bounded $\mathcal{A}$-linear operator $T$ on $\mathcal{E}$ and any $\gamma \in Z(M(\mathcal{A}))$; see [7, Example 6.2] and [13, Example 7.3].

**Proof.** Since $S$ is boundedly invertible, we derive the equality $\gamma \langle x, S^{-1}(z) \rangle = \langle T(x), z \rangle$ for every $x, z \in \mathcal{E}$. By the boundedness of $S^{-1}$, the operator $T$ is adjointable, bounded and $T^* = \gamma S^{-1}$. Since adjointable linear operators on Hilbert $\mathcal{A}$-modules are $\mathcal{A}$-linear and bounded, both $T$ and $S$ have to be $\mathcal{A}$-linear, bounded, invertible with bounded inverses and adjointable. This proves the last assertion. So $TT^{-1} = \gamma^{-1} TS^* = I$ and $T^{-1} T = \gamma^{-1} S^* T = I$. We arrive at $TS^* = \gamma^* I$ and $S^* T = \gamma^* I$. We obtain the commutation result.

If, in particular, $T = S$ in our initial equality we derive $TT^* = \gamma I$ and $T^* T = \gamma^* I$. Consequently, $\gamma$ is positive and $T = \sqrt{\gamma} U$ for some $\mathcal{A}$-linear unitary operator $U$ on $\mathcal{E}$. This shows the last assertion. \[\square\]

**Remark 4.7.** Checking the conditions on suitable bijective operators $T$ on Hilbert C*-modules fulfilling the conditions of Theorem 4.5 one recognizes that every bounded adjointable invertible operator $T$ together with the operator $S = \gamma^{-1}(T^*)^{-1}$ satisfies the equality $\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle$ for all elements $x$ and $y$. The adjointability of $T$ is necessary. In the particular case of $T$ being unitary and $\gamma = 1$, the operator $S$ has to be unitary, too, and $T = S$. Moreover, for a given operator $T$, the operator $S$ is unique. Comparing these observations with the results in [10] in the setting of Hilbert spaces, the case of non-surjective operators $T$ and
of non-injective elements $\gamma$ have to be investigated in more details. The (norm-closure of the) range of $T$ might not be an orthogonal summand, in particular. So more various situations will appear, see Section 6.

**Corollary 4.8.** Let $\mathcal{A}$ be a $C^*$-algebra and let $\langle \cdot, \cdot \rangle_2$ be another $\mathcal{A}$-valued inner product on a full Hilbert $\mathcal{A}$-module $\{E, \langle \cdot, \cdot \rangle_1\}$ inducing an equivalent norm to the given one. Suppose, $\langle x, y \rangle_1 = 0$ implies $\langle x, y \rangle_2 = 0$ for every suitable $x, y \in E$. Then there exists an invertible positive element $\gamma \in Z(M(\mathcal{A}))$ such that $\langle x, y \rangle_1 = \gamma \langle x, y \rangle_2$ for all $x, y \in E$.

**Proof.** Set $F = E$ as an $\mathcal{A}$-module, and add the alternative $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle_2$. Set $T = S = I$ and note, that both of these operators are bounded if considered as operators on $E$. Then Theorem 4.1 and Corollary 4.3 yield $\langle x, y \rangle_1 = \gamma \langle x, y \rangle_2$ for all $x, y \in E$. \qed

5. **Results in $C^*$-algebra of real rank zero**

Recall that a $C^*$-algebra $\mathcal{A}$ has real rank zero if every self-adjoint element in $\mathcal{A}$ can be approximated in norm by invertible self-adjoint elements. Note that if $\mathcal{A}$ has real rank zero, then the *-algebra generated by all the idempotents in $\mathcal{A}$ is dense in $\mathcal{A}$; see, for example, [6]. The result extends [17, Theorem 2.3].

**Theorem 5.1.** Let $\mathcal{A}$ be a $C^*$-algebra of real rank zero and with identity $e$, and let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert $\mathcal{A}$-modules. Suppose that $\mathcal{A}$-linear operators $T, S : \mathcal{E} \to \mathcal{F}$ satisfy the condition

$$x \perp y \implies T(x) \perp S(y) \quad (x, y \in \mathcal{E}).$$

Suppose that there is $z \in \mathcal{E}$ with $\langle z, z \rangle$ being invertible and $\langle T(z), S(z) \rangle$ is self-adjoint. Then, there exits a self-adjoint element $\gamma$ in the center $Z(\mathcal{A})$ of $\mathcal{A}$ such that

$$\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle \quad (x, y \in \mathcal{E}).$$

**Proof.** By replacing $z$ with $|z|^{-1}z$, we assume $\langle z, z \rangle = e$. For every symmetry (i.e. a self-adjoint unitary) $u \in \mathcal{A}$, we have

$$\langle z + uz, z - uz \rangle = |z|^2 + u|z|^2 - u|z|^2 - u|z|^2u = 0$$

whence, $z + uz \perp z - uz$. Hence our assumption yields $T(z + uz) \perp S(z - uz)$, or equivalently

$$\langle T(z), S(z) \rangle + u(T(z), S(z)) - \langle T(z), S(z) \rangle u - u(T(z), S(z))u = 0.$$
Now, let $\gamma := \langle T(z), S(z) \rangle$. So, the above equality becomes $\gamma + u\gamma - \gamma u - u\gamma u = 0$. Since $\gamma$ is self-adjoint, by taking adjoint $\gamma + \gamma u - \gamma u - u\gamma u = 0$. Thus $u\gamma = \gamma u$. As $\mathcal{A}$ is generated by projections, and thus also by symmetries, we get $\gamma \in Z(\mathcal{A})$. On the other hand, for each $x \in \mathcal{A}$ we have $\langle z, x - \langle x, z \rangle z \rangle = \langle z, x \rangle - |z|^2 \langle z, x \rangle = 0$. Hence

$$z \perp x - \langle x, z \rangle z \quad \text{and} \quad x - \langle x, z \rangle z \perp z. \quad (5.1)$$

So, our assumption yields

$$T(z) \perp S(x - \langle x, z \rangle z) \quad \text{and} \quad T(x - \langle x, z \rangle z) \perp S(z). \quad (5.2)$$

Furthermore, from (5.1) we infer that

$$\begin{align*}
\langle x - \langle x, z \rangle z + |x - \langle x, z \rangle z|^2, x - \langle x, z \rangle z - |x - \langle x, z \rangle z|z \rangle \\
= |x - \langle x, z \rangle z|^2 - \langle x - \langle x, z \rangle z, z \rangle |x - \langle x, z \rangle z| \\
+ |x - \langle x, z \rangle z|\langle z, x - \langle x, z \rangle z \rangle - |x - \langle x, z \rangle z|\langle z, z \rangle |x - \langle x, z \rangle z| \\
= |x - \langle x, z \rangle z|^2 - |x - \langle x, z \rangle z|^2 = 0.
\end{align*}$$

Then $x - \langle x, z \rangle z + |x - \langle x, z \rangle z|z$ is orthogonal to $x - \langle x, z \rangle z - |x - \langle x, z \rangle z|z$ and hence $T\left(x - \langle x, z \rangle z + |x - \langle x, z \rangle z|z \right)$ is orthogonal to $S\left(x - \langle x, z \rangle z - |x - \langle x, z \rangle z|z \right)$. Thus, it follows from (5.2) that

$$0 = \langle T\left(x - \langle x, z \rangle z + |x - \langle x, z \rangle z|z \right), S\left(x - \langle x, z \rangle z - |x - \langle x, z \rangle z|z \right) \rangle$$

$$= \langle T(x - \langle x, z \rangle z), S(x - \langle x, z \rangle z) \rangle - \langle T(x - \langle x, z \rangle z), S(z) \rangle |x - \langle x, z \rangle z|$$

$$+ |x - \langle x, z \rangle z|\langle T(z), S(x - \langle x, z \rangle z) \rangle - |x - \langle x, z \rangle z|\langle T(z), S(z) \rangle |x - \langle x, z \rangle z|$$

$$= \langle T(x - \langle x, z \rangle z), S(x - \langle x, z \rangle z) \rangle - |x - \langle x, z \rangle z|\gamma |x - \langle x, z \rangle z|. \quad (5.3)$$

Then $\langle T(x - \langle x, z \rangle z), S(x - \langle x, z \rangle z) \rangle = |x - \langle x, z \rangle z|\gamma |x - \langle x, z \rangle z|$. Since $\gamma \in Z(\mathcal{A})$, by (5.1), we obtain

$$\langle T(x - \langle x, z \rangle z), S(x - \langle x, z \rangle z) \rangle = \gamma |x - \langle x, z \rangle z|^2$$

$$= \gamma \langle x - \langle x, z \rangle z, x \rangle - \gamma \langle x - \langle x, z \rangle z, \langle x, z \rangle z \rangle$$

$$= \gamma |x|^2 - \gamma |\langle x, z \rangle|^2. \quad (5.3)$$
From (5.2) and (5.3) we infer that
\[
\langle T(x), S(x) \rangle = \left\langle T(x - \langle x, z \rangle z) + \langle x, z \rangle T(z), S(x - \langle x, z \rangle z) + \langle x, z \rangle S(z) \right\rangle \\
= \left\langle T(x - \langle x, z \rangle z), S(x - \langle x, z \rangle z) \right\rangle + \langle x, z \rangle \left\langle T(z), S(x - \langle x, z \rangle z) \right\rangle \\
+ \left\langle T(x - \langle x, z \rangle z), S(z) \right\rangle \langle z, x \rangle + \langle x, z \rangle \left\langle T(z), S(z) \right\rangle \langle z, x \rangle \\
= \gamma |x|^2 - \gamma |\langle x, z \rangle|^2 + \gamma |\langle x, z \rangle|^2 = \gamma |x|^2.
\]
Hence
\[
\langle T(x), S(x) \rangle = \gamma |x|^2 \quad (x \in \mathcal{E}). \tag{5.4}
\]
Now, by (5.4) and the polarization identity, we obtain
\[
\langle T(x), S(y) \rangle = \gamma \langle x, y \rangle \quad (x, y \in \mathcal{E}).
\]

\[\square\]

Remark 5.2. Notice that orthogonality preserving functions may be very non-linear and discontinuous; see [9, Example 2] and [10]. Now, let \( \mathcal{E} \) be a Hilbert \( \mathbb{K}(\mathcal{H}) \)-module and let \( \mathcal{F} \) be a Hilbert \( \mathcal{A} \)-module. Let \( g, h : \mathcal{E} \to \mathcal{F} \) be additive functions such that
\[
x \perp y \implies g(x) \perp h(y) \quad (x, y \in \mathcal{E}).
\]
Suppose that function \( f : \mathcal{E} \to \mathcal{A} \) defined by \( f(x) := \langle g(x), h(x) \rangle \) is continuous. Fix \( x, y \in \mathcal{E} \) such that \( x \perp y \). Hence \( \langle x, y \rangle = \langle y, x \rangle = 0 \). Therefore \( \langle g(x), h(y) \rangle = \langle g(y), h(x) \rangle = 0 \). Then
\[
f(x + y) = \langle g(x + y), h(x + y) \rangle \\
= \langle g(x), h(x) \rangle + \langle g(x), h(y) \rangle + \langle g(y), h(x) \rangle + \langle g(y), h(y) \rangle \\
= \langle g(x), h(x) \rangle + \langle g(y), h(y) \rangle = f(x) + f(y).
\]
Thus \( f \) is orthogonally additive. It follows from [16, Theorem 5.4 (ii)] that there are a unique continuous additive function \( A : \mathcal{E} \to \mathcal{A} \) and a unique operator \( \Phi : \langle \mathcal{E}, \mathcal{E} \rangle \to \mathcal{A} \) such that
\[
f(x) = A(x) + \Phi(\langle x, x \rangle) \quad (x \in \mathcal{E}).
\]

6. Additional comments

Let us briefly discuss some obstacles in the theory of Hilbert \( C^* \)-modules which prevent a straightforward generalization of Hilbert space results on the subject of the present paper, cf. [10], [21, Theorem 11], [22, Theorem 4]. First of all,
biorthogonally closed Hilbert $C^*$-submodules very often cannot divided out as (any kind of) direct summand of the hosting Hilbert $C^*$-module.

**Example 6.1.** Let us take $\mathcal{A} = C([0,2\pi])$, regarded as a Hilbert $C^*$-module over itself in the natural way. Consider two multiplication operators

\[
T(x)(t) = \cos(t) \cdot x(t) \quad \text{for} \quad t \in [0,\pi/2] \quad \text{and} \quad T(x)(t) = 0 \quad \text{for} \quad t \in [\pi/2,2\pi]
\]

\[
S(x)(t) = 0 \quad \text{for} \quad t \in [0,3\pi/2] \quad \text{and} \quad S(x)(t) = \cos(t) \cdot x(t) \quad \text{for} \quad t \in [3\pi/2,2\pi].
\]

Then $\langle T(x), S(y) \rangle = \langle x, y \rangle = 0$ for every pairwise orthogonal elements $x, y \in \mathcal{A}$. However, the intersection of the kernels of $T$ and $S$ is neither a direct orthogonal nor a direct topological summand of $\mathcal{A}$, beside it is both norm-closed and biorthogonally complemented. And the operators $T$ and $S$ are bounded and self-adjoint, and hence $\mathcal{A}$-linear and even positive. In fact, both they lack polar decomposition in $\mathcal{A}$. Similarly, both the biorthogonally complemented images of $T$ and $S$ have analogous properties like the intersection of the two kernels. Finally, there does not exist any bounded invertible module operator on the intersection of the kernels of $T$ and $S$ such that it could be continuously extended to $\mathcal{A}$ in such a way that the orthogonal complement of the intersection of the two kernels is contained in its kernel. This situation is different from the Hilbert space situation. For an example involving unbounded module operators we refer to [4, Example 2.1].

A second obstacle to be considered is the existence of direct topological summands in certain Hilbert $C^*$-modules which are not direct orthogonal summands. Here non-adjointable module operators come into play.

**Example 6.2.** Let $\mathcal{A}$ be a unital $C^*$-algebra with a non-trivial norm-closed two-sided ideal $\mathcal{I}$. Consider the Hilbert $\mathcal{A}$-module $\mathcal{E} = \mathcal{A} \oplus \mathcal{I}$ and its Hilbert $\mathcal{A}$-submodules $\mathcal{E}_1 = \{ (i, i) : i \in \mathcal{I} \}$ and $\mathcal{E}_2 = \mathcal{A} \oplus \{0\}$. Their intersection is the set $\{0\}$, $\mathcal{E}_1$ is not a direct orthogonal summand, however $\mathcal{E}_1$ coincides with its biorthogonal complement with respect to $\mathcal{E}$. But, $\mathcal{E}_1$ is a direct topological summand of $\mathcal{E}$, since $(a, i) = (i, i) + (a - i, 0)$ is the unique decomposition of any element of $\mathcal{E}$ into an element of $\mathcal{E}_1$ and an element of $\mathcal{E}_2$. So, there exists a bounded $\mathcal{A}$-linear idempotent operator $P : \mathcal{E} \to \mathcal{E}_1$ such that $P$ is non-adjointable.

Similarly, $\mathcal{E}_3 = \{ (i, -i) : i \in \mathcal{I} \} = \mathcal{E}_1$ and $\mathcal{E}_2$ are such a pair according to the decomposition $(a, i) = (-i, i) + (a + i, 0)$ for arbitrary elements of $\mathcal{E}$. So, there is a bounded, $\mathcal{A}$-linear, non-adjointable, idempotent operator $Q : \mathcal{E} \to \mathcal{E}_3$. 

Consequently, $\langle P(x), Q(y) \rangle = \langle x, y \rangle = 0$ for every pairwise orthogonal elements $x, y \in \mathcal{E}$. And by definition, the pair of operators $P$ and $Q$ is orthogonality-preserving, but both these operators are non-adjointable. And their ranges are norm-closed, biorthogonally complemented Hilbert $\mathcal{A}$-submodules that are not orthogonal summands, but they are topological direct summands. Moreover, $P$ and $Q$ are idempotents. Also, $PQ \neq 0$ and $QP \neq 0$. Such situations cannot appear for Hilbert spaces, cf. [21, Theorem 11], [22, Theorem 4].

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1 Hochschule f"ur Technik, Wirtschaft und Kultur (HTWK) Leipzig, Fakult"at IM, PF 301166, 04251 Leipzig, Germany

   Email address: michael.frank@htwk-leipzig.de

2 Department of Pure Mathematics, Ferdowsi University of Mashhad, Center of Excellence in Analysis on Algebraic Structures (CEAAS), P.O. Box 1159, Mashhad 91775, Iran

   Email address: moslehian@um.ac.ir, moslehian@yahoo.com

3 Department of Mathematics, Farhangian University, Tehran, Iran

   Email address: zamani.ali85@yahoo.com