ON THE COHOMOLOGICAL DIMENSION
OF THE MODULI SPACE OF RIEMANN SURFACES

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Abstract. The moduli space of Riemann surfaces of genus \( g \geq 2 \) is (up to a finite étale cover) a complex manifold and so it makes sense to speak of its Dolbeault cohomological dimension. The conjecturally optimal bound is \( g - 2 \). This expectation is verified in low genus and supported by Harer’s computation of its de Rham cohomological dimension and by vanishing results in the tautological intersection ring. In this paper we prove that such dimension is at most \( 2g - 2 \). We also prove an analogous bound for the moduli space of Riemann surfaces with marked points. The key step is to show that the Dolbeault cohomological dimension of each stratum of translation surfaces is at most \( g \). In order to do that, we produce an exhaustion function whose complex Hessian has controlled index: in the construction of such a function some basic geometric properties of flat surfaces come into play.

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1. Introduction

1.1. Cohomological dimension of $\mathcal{M}_{g,n}$. The moduli space $\mathcal{M}_{g,n}$ of compact connected Riemann surfaces of genus $g \geq 2$ with $n \geq 0$ distinct marked points is a non-compact orbifold and so it makes sense to speak of its de Rham cohomological dimension $\text{coh-dim}_{dR}(\mathcal{M}_{g,n})$, that is the greatest degree for which its de Rham cohomology with coefficients in some flat vector bundle does not vanish. Using topological methods, Harer [12] proved that $\text{coh-dim}_{dR}(\mathcal{M}_{g,n}) = 4g - 5 + n + \varsigma_n$, where $\varsigma_0 = 0$ and $\varsigma_n = 1$ for $n > 0$.

In a similar fashion, as $\mathcal{M}_{g,n}$ can also be given the structure of a complex-analytic orbifold, it makes sense to speak of Dolbeault cohomology with coefficients in a holomorphic vector bundle and of Dolbeault cohomological dimension (see Sections A.1 and A.2).

**Question** (Looijenga). Is $\text{coh-dim}_{Dol}(\mathcal{M}_{g,n}) = g - 2 + \varsigma_n$?

A few pieces of evidence support the above conjecture:

(1) using the (twisted) holomorphic de Rham complex, it would imply Harer’s result;
(2) it would imply Diaz’s result [6] that a compact holomorphic subvariety of $\mathcal{M}_g$ has dimension at most $g - 2$;
(3) it would imply Looijenga’s vanishing [15] (proven to be sharp by Faber in [7]) of the tautological classes of degree $q > g - 2$, which live in $H^{2,q}(\mathcal{M}_g, \Omega^1_{\mathcal{M}_g})$;
(4) it is verified in the unmarked case for $g \leq 5$ in [10] (see also [9]).
Conversely, Lemma A.2 together with Harer’s theorem gives \( \text{coh-dim}_{Dol}(M_{g,n}) \geq \text{coh-dim}_{dR}(M_{g,n}) - \text{dim}_C(M_{g,n}) = (4g - 5 + n + \varsigma_n) - (3g - 3 + n) = g - 2 + \varsigma_n \). The same conclusion can be drawn from Faber’s non-vanishing result of \( \kappa_1^{g-2+\varsigma_n} \in H^{0, g-2+\varsigma_n}_{\overline{M}_{g,n}; \Omega_{\overline{M}_{g,n}}^{g-2+\varsigma_n}} \), where \( \kappa_1 \) is the class of a certain Kähler form.

**Remark 1.1.** Because \( M_{g,n} \) is also a Deligne-Mumford stack, one could also consider cohomology with coefficients in an algebraic coherent sheaf. The same conjecture on such an “algebraic cohomological dimension”, namely whether \( \text{coh-dim}_{alg}(M_{g,n}) \leq g - 2 + \varsigma_n \), supported by the same pieces of evidence mentioned above, is open too. However, in this paper we will only deal with Dolbeault cohomological dimensions.

**Remark 1.2.** From this point of view, the situation for the moduli spaces \( M_{0,n} \) of Riemann surfaces of genus 0 with \( n \geq 3 \) distinct marked points and for \( M_{1,n} \) with \( n \geq 1 \) is completely understood. Indeed, \( M_{0,n} \) is an affine algebraic variety and \( M_{1,n} \) has an affine finite étale cover. Hence, they have Dolbeault and algebraic cohomological dimension 0.

**Remark 1.3.** These conjectures have also been formulated for suitable partial compactifications of \( M_{g,n} \), such as the moduli space of surface of compact type, of surfaces with rational tails, of surfaces with at most \( k \) rational components, of irreducible surfaces. Ionel [13] and Graber-Vakil [11] proved the analogous of Looijenga’s vanishing theorem; the parallel topological vanishing results for de Rham cohomology follow from Harer’s work and were analyzed in [17].

At present there seem to be three basic ideas to prove similar vanishing theorems for the (Dolbeault or algebraic) cohomology of \( X \) in degrees greater or equal than \( q \), namely:

- (a) exhibiting a cover of \( X \) made of at most \( q + 1 \) open affine (or just Stein) subsets;
- (b) exhibiting a stratification of \( X \) made of at most \( q + 1 \) layers, such that each locally closed stratum is affine;
- (c) exhibiting an exhaustion function on \( X \) whose complex Hessian has index at most \( q \).

The classical example of a stratification of \( M_g \) with \( g - 1 \) layers (and of \( M_{g,1} \) with \( g \) layers) is due to Arbarello [2]. However, though the smallest stratum is affine because it coincides with the hyperelliptic locus and the top-dimensional stratum is also affine (as remarked in [3]), in most cases the other strata are not (see [3]).

### 1.2. Main results

As \( M_{g,n} \) is irreducible and not compact, clearly \( \text{coh-dim}_{Dol}(M_{g,n}) < \text{dim}_C(M_{g,n}) = 3g - 3 + n \). Though we are not able to answer Looijenga’s question, we can improve the above bound. The following is the first main result of the paper.

**Theorem A.** \( \text{Coh-dim}_{Dol}(M_{g,n}) \leq 2g - 2 + \varsigma_n \) for all \( g \geq 2 \) and \( n \geq 0 \).
The idea of the proof is to bound the cohomological dimension of the projectivized
Hodge bundle $\mathbb{P}H_{g,n}$, consisting of triples $(C, P, [\varphi])$, where $(C, P) \in \mathcal{M}_{g,n}$ is a
Riemann surface $C$ endowed with distinct marked points $P = \{p_1, \ldots, p_n\} \to C$
and $\varphi$ is a nonvanishing holomorphic $(1,0)$-form on $C$. As $\mathbb{P}H_{g,n} \to \mathcal{M}_{g,n}$ is a
holomorphic $\mathbb{P}^{g-1}$-fibration and as the cohomological dimension behaves well under
such fibrations, the above theorem is equivalent to the following.

**Theorem B.** $\text{Coh-dim}_{Dol}(\mathbb{P}H_{g,n}) \leq 3g - 3 + \varsigma_n$ for all $g \geq 2$ and $n \geq 0$.

Now we exploit a stratification of $\mathbb{P}H_{g,n}$ well-known to people in translation surfaces
(and described in Section 2), whose locally closed strata are the loci $\mathbb{P}H_n(m_1, \ldots, m_{n+k})$
of triples $(C, P, [\varphi])$ such that

1. $P \cup Z(\varphi)$ consists of $n + k$ distinct points $\{p_1, \ldots, p_n, q_1, \ldots, q_k\}$, which will
be called “special”;
2. $\text{ord}_{p_i} \varphi = m_i \geq 0$ for all $i = 1, \ldots, n$ and $\text{ord}_{q_j} \varphi = m_{n+j} > 0$ for all $j = 1, \ldots, k$.

The hierarchy of such stratification is as expected: going to a deeper stratum corre-
sponds to coalescing some special points. It seems to be known by folklore that
each locally closed stratum is a regular submanifold of $\mathbb{P}H_{g,n}$ and that the relative
periods of $\varphi$ give local holomorphic coordinates, but in Section 3 we give a proof of
these facts and of their extension to closed strata for completeness.

The proof of Theorem B is in Section 4. The first step consists in boun-
ding the cohomological dimension of such strata, thus obtaining a result which could be of
its own interest.

**Theorem C.** $\text{Coh-dim}_{Dol}(\mathbb{P}H_n(m_1, \ldots, m_{n+k})) \leq g$ for all $g \geq 2$ and $n \geq 0$ and for
all integers $m_1, \ldots, m_n \geq 0$ and $m_{n+1}, \ldots, m_{n+k} > 0$ such that $m_1 + \cdots + m_{n+k} = 2g - 2$.

We stress that the above result is already non-optimal in genus 2 (in which case all
strata are affine) and in genus 3 (in which case, $\mathbb{P}H_1(4)$ and $\mathbb{P}H_2(3, 1)$ are affine: see [16]).

The proof of Theorem C exploits a result by Andreotti-Grauert [1] and reduces to
producing a real-valued exhaustion function $\xi$ on $\mathbb{P}H_n(m_1, \ldots, m_{n+k})$ whose complex
Hessian has controlled index (see Subsection A.4). The point is that $\varphi$ induces a flat
metric $|\varphi|^2$ on $C$ with conical singularities of angle $2\pi(m_i+1)$ at the $i$-th special point
and so we can exploit the geometry of such a flat metric. In particular, $\xi(C, P, [\varphi])$
will involve the total area of $|\varphi|^2$ and the lengths of certain short segments joining
two special points.

The second step is to build a Mayer-Vietoris spectral sequence that assembles the
information relative to each stratum into a statement about $\mathbb{P}H_{g,n}$. This would
be easier in the algebraic context; but in order to apply Corollary A.11 in the
analytic setting, we need to prove the analogue of Theorem C for certain open
subsets of $\mathbb{P}H_{g,n}$ obtained by suitably thickening the above strata and to check that
the produced exhaustion functions combine well, so that every intersection of such 
open subsets has Dolbeault cohomological dimension at most $g$. As the stratification 
has depth $2g - 3 + \varsigma_n$, we obtain an upper bound of $3g - 3 + \varsigma_n = g + (2g - 3 + \varsigma_n)$ 
for $\text{coh-dim}_{\text{Dol}}(\mathbb{P}H_{g,n})$.

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topic.

2. Moduli spaces of Abelian differentials

2.1. Moduli of Riemann surfaces. Let $g \geq 2$ and $n \geq 0$ and denote by $M_{g,n}$ 
the moduli space of Riemann surfaces of genus $g$ with $n$ marked points and by 
$\pi : C_{g,n} \rightarrow M_{g,n}$ be the universal family. A point in $M_{g,n}$ will thus represent a 
couple $(C, P)$, where $C$ is the isomorphism class of a compact connected Riemann 
surface of genus $g$ and the injective map $P : \{1, \ldots, n\} \hookrightarrow C$ is the marking.

Remark 2.1. $M_{g,n}$ has a natural structure of complex-analytic orbifold, and even of 
Deligne-Mumford stack, which is a global quotient of a smooth variety $\tilde{M}_{g,n}$ by a 
finite group $G$. All constructions can be intended to be performed $G$-equivariantly 
on $\tilde{M}_{g,n}$. Indeed, if $\rho : \tilde{M}_{g,n} \rightarrow M_{g,n}$ is the covering map and $E \rightarrow M_{g,n}$, is a 
holomorphic vector bundle, then $\rho^*E \rightarrow \tilde{M}_{g,n}$ is a $G$-equivariant holomorphic vector 
bundle and $H^{0,q}_\partial(M_{g,n}; E) := H^{0,q}_\partial(\tilde{M}_{g,n}; \rho^*E)^G$ is well-defined and independent of 
the choice of the finite Galois cover. Similar considerations hold for $C_{g,n}$ and all the 
other moduli spaces that appear in this paper.

A variant of the previous construction with is useful for technical purposes is to 
allow certain points to coalesce. For every $k > 0$ we will denote by $M^k_{g,n}$ the moduli 
space of triples $(C, P)$, where $C$ is a compact Riemann surfaces of genus $g$ and the 
restriction of $P : n + k \rightarrow C$ to $n \subset n + k$ is injective. Forgetting the last $k$ marked 
points defines a map $f_k : M^k_{g,n} \rightarrow M_{g,n}$, so that $M^k_{g,n}$ can be identified to the $k$-th fiber product $C_{g,n} \times \cdots \times C_{g,n} \times M_{g,n}$ and the map $M^k_{g,n+1} \rightarrow M^k_{g,n}$ that forgets 
the last marked point can be identified to the universal family $\pi_k : C^k_{g,n} \rightarrow M^k_{g,n}$. 
So, analogous considerations as in Remark 2.1 apply in this case.

Let $S_n(k, l)$ the set of surjections $\sigma : n + k \twoheadrightarrow n + l$ which restrict to the identity 
on $n$. For each $\sigma \in S_n(k, l)$, we obtain a map $b_\sigma : M^l_{g,n} \rightarrow M^k_{g,n}$ by letting 
b_\sigma(C, Q) = (C, Q \circ \sigma), where $Q : n + l \rightarrow C$. We denote by $\delta_\sigma \subset M^k_{g,n}$ the image of 
b_\sigma.

If $\sigma' = \tau \circ \sigma$ for some $\tau \in S_n(l, h)$, then we will write $\sigma' \preceq \sigma$. 

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Notation. As $\delta_\sigma = \delta_{\sigma \alpha}$ for every permutation $\tau \in \mathfrak{S}_n(l)$ of $n + l$ that restricts to the identity on $n$, we will select a preferred surjection in each $\mathfrak{S}_n(l)$-orbit: we call $\sigma$ a lexicographic surjection if for every $i = 1, \ldots, k$ there exists a $j = 1, \ldots, l$ such that $\sigma\{(n + 1, \ldots, n + i)\} = \{n + 1, \ldots, n + j\}$. We will denote by $\mathfrak{S}^\text{lex}_n(k, l) \subseteq \mathfrak{S}_n(k, l)$ the subset of lexicographic surjections.

For $i < j$ and $j > n$, it will be occasionally useful to denote by $\delta_{i,j}$ the Cartier divisor of $\mathcal{M}_{g,n}^k$ on which $p_i = p_j$ and by $\delta$ the union of all such $\delta_{i,j}$.

2.2. The Hodge bundle. Consider $\omega_\pi \to \mathcal{C}_{g,n}$ the $\pi$-vertical holomorphic cotangent bundle. The so-called Hodge bundle

$$\mathcal{H}_{g,n} := \pi_\ast \omega_\pi \to \mathcal{M}_{g,n}$$

is a holomorphic vector bundle of rank $g$: a point of $\mathcal{H}_{g,n}$ represents a triple $(C, P, \varphi)$ up to isomorphism, where $\varphi$ is a holomorphic $(1, 0)$-form on the $P$-marked compact Riemann surface $C$ of genus $g$. For $k > 0$ we similarly define $\mathcal{H}^k_{g,n} := \pi_{k,\ast} \omega_{g_k}$ to be the Hodge bundle over $\mathcal{M}_{g,n}^k$, so that $\mathcal{H}^k_{g,n} = f^\ast \mathcal{H}_{g,n}$. We will use the symbols $\mathcal{H}_{g,n}$ and $(\mathcal{H}_{g,n}^k)^\ast$ to denote the complement of the zero sections.

2.3. Stratification of the Hodge bundle. Inside $\mathcal{H}_{g,n}^{2g-2}$ define the locus

$$(\mathcal{H}_{g,n}^k)^\ast := \{(C, P, \varphi) \in \mathcal{H}_{g,n}^{2g-2} \mid \varphi \neq 0, \quad \varphi(p_i) = 0 \quad \forall i > n\}$$

Moreover, given $m : \mathbb{N} \to \mathbb{N}$ such that $|m| := \sum_{i=1}^k m_i = 2g - 2$ and $m_i > 0$ for $i > n$, we define

$$\mathcal{H}_{g,n}^k(m)^\ast := \{(C, P, \varphi) \in \mathcal{H}_{g,n}^k \mid C \in \mathcal{M}_{g,n}^k, \quad \text{ord}_{p_i}(\varphi) = m_i \quad \forall i\}$$

inside $\mathcal{H}_{g,n}^k$. Clearly, $(\mathcal{H}_{g,n}^k)^\ast$ is the closure of $\mathcal{H}_n^k(0^n, 1^{2g-2})^\ast$ inside $(\mathcal{H}_{g,n}^{2g-2})^\ast$, and it maps finitely over $\mathcal{H}_{g,n}^k$ as a $\mathfrak{S}_{2g-2}$-branched cover.

Notice that rescaling the differential $\varphi$ induces a $\mathbb{C}^\ast$-action on all these spaces. We will denote by $\mathbb{P}\mathcal{H}_n^k(m)$ the quotient of $\mathcal{H}_n^k(m)^\ast$ by $\mathbb{C}^\ast$ and by

$$\mathcal{H}_n^k(m) := \{(C, P, \varphi) \in \mathcal{H}_{g,n}^k \mid C \in \mathcal{M}_{g,n}^k, \quad mP \in |K_C|, \quad \varphi \in H^0(C, K_C(-mP))\}$$

where $mP$ is the effective divisor $m_1P_1 + \cdots + m_{n+k}P_{n+k}$ on $C$. Clearly, $\mathcal{H}_n^k(m)^\ast \subseteq \mathcal{H}_n^k(m)$ coincides with the locus where $\varphi \neq 0$; indeed, $\mathcal{H}_n^k(m)$ can be identified with the total space of the tautological line bundle $\mathcal{O}_{\mathbb{P}\mathcal{H}_n^k(m)}(-1)$ over $\mathbb{P}\mathcal{H}_n^k(m)$. Similar notations will be used for $\mathcal{H}_{g,n}^k, (\mathcal{H}_{g,n}^k)^\ast, (\mathcal{H}_{g,n}^k)^\ast$.

Notice that the natural projection identifies $\mathbb{P}\mathcal{H}_n^k(m)$ with a subset of $\mathcal{M}_{g,n}^k$. The image of $\mathcal{H}_n^k(m)$ through the forgetful map $\mathcal{H}_{g,n}^k \to \mathcal{H}_{g,n}$ will be denoted by $\mathcal{H}_n(m)$, and similarly $\mathbb{P}\mathcal{H}_n^k(m)$ will be the corresponding locus inside $\mathbb{P}\mathcal{H}_{g,n}$. Thus, $\mathcal{H}_n(m) = \mathcal{H}_n(m \circ \sigma)$ for every $\sigma \in \mathfrak{S}_n(k)$. It is clear that the induced map $\mathbb{P}\mathcal{H}_n^k(m) \to \mathbb{P}\mathcal{H}_n(m)$ is an unramified Galois cover with group $\text{Aut}_{n}(m) = \{\lambda \in \mathfrak{S}_n(k) \mid m_{\lambda(i)} = m_i\}$.

The loci $\mathbb{P}\mathcal{H}_n^k(m)$ and $\mathbb{P}\mathcal{H}_n(m)$ are part of an algebraic stratification of $\mathbb{P}\mathcal{H}_n^k$ and of $\mathbb{P}\mathcal{H}_{g,n}$ respectively that we want to describe.
By abuse of notation, we will denote by \( \omega_i \) both the vertical cotangent line bundle \( \omega_i : \mathcal{M}_{g,n}^k \to \mathcal{M}_{g,n}^k \) associated to the map \( f_i \) the forgets the \( i \)-th marking and its pull-back to \( \mathbb{P}\mathcal{H}^k_{g,n} \) or to any locus \( \mathbb{P}\mathcal{H}'^k_{n}(m) \). We will write \( J^r \omega_i \) for the \( r \)-th jet bundle (relative to the family \( f_i \)) associated to \( \omega_i \). As usual \( J^0 \omega_i = \omega_i \) and there is a natural exact sequence

\[
0 \to \omega_i^{(r+1)} \otimes \omega_i \to J^{r+1} \omega_i \to J^r \omega_i \to 0
\]

Here we want to show the following.

**Lemma 2.2.** \( \overline{\mathbb{P}\mathcal{H}'^k_{n}(m)}^{sm} \) is a local complete intersection inside \( \mathbb{P}\mathcal{H}^k_{g,n} \), and its class in the Chow ring \( A^*(\mathbb{P}\mathcal{H}^k_{g,n}) \) is

\[
\left[ \overline{\mathbb{P}\mathcal{H}'^k_{n}(m)}^{sm} \right] = \prod_{i=1}^{k} \prod_{r=0}^{m_i-1} \left( h + \frac{(r+1)(r+2)}{2} c_1(\omega_i) \right)
\]

where \( h = c_1(\mathcal{O}_{\mathbb{P}\mathcal{H}^k_{g,n}}(1)) \). The complement of \( \mathbb{P}\mathcal{H}'^k_{n}(m) \) inside its closure is the Cartier divisor \( \mathbb{P}\mathcal{H}'^k_{n}(m) \cap \delta \), and so \( \mathbb{P}\mathcal{H}'^k_{n}(m) \subset \mathbb{P}\mathcal{H}^k_{g,n} \) is locally closed.

**Proof.** The closure inside \( \mathbb{P}\mathcal{H}^k_{g,n} \) of each \( \mathbb{P}\mathcal{H}'^k_{n}(m) \) is exactly

\[
\overline{\mathbb{P}\mathcal{H}'^k_{n}(m)}^{sm} = \bigcap_{m_i > 0} Z(\text{Ev}_{i}^{m_i-1}) =: Z(\text{Ev}^m)
\]

where \( Z \) denotes the zero locus and \( \text{Ev}_{i}^{m_i-1} \) and \( \text{Ev}^m \) are the evaluation maps

\[
\text{Ev}_{i}^{m_i-1} : \mathcal{O}_{\mathbb{P}\mathcal{H}^k_{g,n}}(-1) \longrightarrow J^{m_i-1} \omega_i
\]

\[
\text{Ev}^m : \mathcal{O}_{\mathbb{P}\mathcal{H}^k_{g,n}}(-1) \longrightarrow \bigoplus_{m_i > 0} J^{m_i-1} \omega_i
\]

Thus we have described \( \overline{\mathbb{P}\mathcal{H}'^k_{n}(m)}^{sm} \) as the zero locus of a section of the holomorphic vector bundle \( \bigoplus_{m_i > 0} J^{m_i-1} \omega_i \otimes \mathcal{O}_{\mathbb{P}\mathcal{H}^k_{g,n}}(1) \) on \( \mathcal{H}^k_{g,n} \) of rank \( \sum m_i = 2g - 2 \). As \( \dim(\mathbb{P}\mathcal{H}^k_{g,n}) = 4g - 4 + n + k \), we expect \( \overline{\mathbb{P}\mathcal{H}'^k_{n}(m)}^{sm} \) to have pure dimension \( 2g - 2 + n + k \). We will see in Subsection 3.3 that this is indeed the case, and so we are done. \( \square \)

**2.4. Hierarchy of strata.** Let \( k \) and \( m \) be as above and let \( \sigma \in \mathfrak{S}_{n}(k,l) \). There is a map

\[
\overline{\mathbb{P}\mathcal{H}'^k_{n}(\sigma \ast m)}^{sm} \sim \overline{\mathbb{P}\mathcal{H}'^k_{n}(m)}^{sm} \cap \delta_{\sigma}
\]

given by \((C, Q, [\varphi]) \mapsto (C, Q \circ \sigma, [\varphi])\), where \( \sigma : m : n + l \to \mathbb{N} \) is defined as \( (\sigma \ast m)_j := \sum_{\sigma(i) = j} m_i \).

**Notation.** We will use the simplified symbol \( \mathbb{P}\mathcal{H}'^k_{n}(\sigma \ast m) \) for \( \mathbb{P}\mathcal{H}'^k_{n}(m) \), if \( m = (0^n, 1^{2g-2}) \) corresponds to the open stratum \( \mathbb{P}\mathcal{H}'^k_{n}(0^n, 1^{2g-2}) \) inside \( \mathbb{P}\mathcal{H}'^k_{n} \).
So we inductively obtain the stratification
\[
\mathbb{P}H'_n(m)^{sm} = \bigsqcup_{l=0}^{k-1+c_n} \mathbb{P}H'_n(m)^l, \quad \text{where } \varsigma_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n > 0 \end{cases}
\]
and
\[
\mathbb{P}H'_n(m)^l \cong \bigsqcup_{\sigma \in \mathcal{E}^{l \times (k,k-l)}} \mathbb{P}H'_n(\sigma \ast m)
\]
and so
\[
\mathbb{P}H'_{g,n} \cong \bigsqcup_{l=0}^{2g-3+c_n} \bigsqcup_{\sigma \in \mathcal{E}^{l \times (2g-2,2g-2-l)}} \mathbb{P}H'_n(\sigma).
\]
As a consequence of the above description, we have the following.

**Lemma 2.3.** The closure of \( \mathbb{P}H'_n(\sigma \ast m) \) is a complete intersection inside \( \mathbb{P}H'_n(m)^{sm} \). In particular, the complement of \( \mathbb{P}H'_n(\sigma \ast m) \) inside its closure is a Cartier divisor and so the complement of \( \mathbb{P}H_n(m) \) inside \( \mathbb{P}H_n(m)^{sm} \) is \( \mathbb{Q} \)-Cartier.

**Proof.** The first claim follows from the fact that \( \delta_{i,j} \) is Cartier and so \( \delta_{\sigma} \) is a complete intersection. The complement of \( \mathbb{P}H'_n(\sigma \ast m) \) in its closure is a union of smaller strata and so it is a union of Cartier divisors, and so Cartier. Finally, the forgetful map \( \mathbb{P}H'_n(m)^{sm} \to \mathbb{P}H_n(m)^{sm} \) is a finite (branched) Galois cover that preserves the locus where some zeroes coalesce: this implies the third claim. \( \square \)

In terms of evaluation maps and of the classes \( h \) and \( c_1(\omega_i) \), we have the following.

**Lemma 2.4.** The locus \( \delta \cap \mathbb{P}H'_n(m)^{sm} \) is the support of the effective divisor
\[
(n + k)h + \sum_{i=1}^{n+k} (m_i + 1)c_1(\omega_i) = \sum_{i<j, n<j} (m_i + m_j)\delta_{i,j} \in A^1\left(\mathbb{P}H'_n(m)^{sm}\right)
\]

**Proof.** Consider the evaluation map
\[
ev_i : \mathcal{O}_{\mathbb{P}H'_n(m)}(-1) \to \omega_1^{\otimes (m_1+1)}
\]
defined on \( \mathbb{P}H'_n(m)^{sm} \), which vanishes exactly \( m_j \) times on \( \delta_{i,j} \) for all \( j \neq i \). Thus
\[
h + (m_i + 1)c_1(\omega_i) = [Z(ev_i)] = \sum_{j \neq i} m_j[\delta_{i,j}] \in A^1\left(\mathbb{P}H'_n(m)^{sm}\right)
\]
and we conclude summing up over \( i = 1, \ldots, n + k \) and remembering that \( \delta_{i,j} = \emptyset \) if \( i < j \leq n \). \( \square \)

As \( \mathbb{P}H_{g,n} = \mathbb{P}H_n(0^n, 1^{2g-2})^{sm} \), we also obtain the stratification \( \mathbb{P}H_{g,n} = \bigsqcup_{l=0}^{2g-3+c_n} (\mathbb{P}H_{g,n})^l \)
where
\[
(\mathbb{P}H_{g,n})^l = \bigsqcup_{m_1, \ldots, m_n, m_{n+1} \geq \cdots \geq m_{n+2g-2-l}} \mathbb{P}H_n(m_1, \ldots, m_n, 2g-2-l).
\]
2.5. A naive compactification of strata. It is well-known that $\mathcal{M}_{g,n}$ can be compactified as the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable Riemann surfaces of genus $g$ with $n$ distinct marked points. The boundary $\partial \mathcal{M}_{g,n} := \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is a normal crossing divisor and it can be written as the union of the Cartier divisor of surfaces with at least one non-disconnecting node, and the (reducible) Cartier divisor $\partial_{red}\mathcal{M}_{g,n}$ of surfaces with at least one disconnecting node.

The universal family extends as a flat holomorphic map $\overline{\pi} : \overline{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ between complex orbifolds. Similarly to the case of smooth surfaces, $\overline{\mathcal{M}}^k_{g,n}$ is the moduli of space of stable genus $g$ Riemann surfaces with $n$ distinct marked points, endowed with a choice of $k$ further (not necessarily distinct) marked points, so that $\overline{\mathcal{M}}^k_{g,n}$ can be identified to the $k$-th fiber product $\overline{C}_{g,n} \times \overline{C}_{g,n} \cdots \times \overline{C}_{g,n}$. We will denote by $\partial_{irr}\mathcal{M}^k_{g,n}$ and $\partial_{red}\mathcal{M}^k_{g,n}$ the loci inside $\overline{\mathcal{M}}^k_{g,n}$ that project to $\partial_{irr}\mathcal{M}_{g,n}$ and to $\partial_{red}\mathcal{M}_{g,n}$ respectively. Moreover, $\overline{\delta}$ will denote the closure of $\delta$ inside $\overline{\mathcal{M}}^k_{g,n}$.

The relative dualizing line bundle $\omega_{\overline{\pi}} \to \overline{\mathcal{C}}_{g,n}$ and the Hodge bundles $\overline{\mathcal{H}}_{g,n} := \overline{\pi}_*\omega_{\overline{\pi}} \to \overline{\mathcal{M}}^k_{g,n}$ and $\overline{\mathcal{H}}^k_{g,n} \to \overline{\mathcal{M}}^k_{g,n}$ extend the ones defined above over the locus of smooth surfaces.

Taking the closure of each stratum $\mathbb{P}\mathcal{H}'_{n}(m)$ inside $\mathbb{P}\overline{\mathcal{H}}^k_{g,n}$ provides a compactification $\overline{\mathbb{P}\mathcal{H}}_{n}(m) = \mathbb{P}\mathcal{H}'_{n}(m) \cup \partial \mathbb{P}\mathcal{H}'_{n}(m)$, obtained by adding three Cartier divisors:

- the divisor $\partial_{irr}\mathbb{P}\mathcal{H}'_{n}(m)$ of triples $(C, P, [\varphi])$ that sit over $\partial_{irr}\mathcal{M}^k_{g,n}$;
- the divisor $\partial_{red}\mathbb{P}\mathcal{H}'_{n}(m)$ of triples $(C, P, [\varphi])$ that sit over $\partial_{red}\mathcal{M}^k_{g,n}$;
- the divisor $\overline{\partial}\mathbb{P}\mathcal{H}'_{n}(m)$ where any two marked points coincide.

Similar considerations hold for the closure of $\mathbb{P}\mathcal{H}_{n}(m)$ inside $\mathbb{P}\mathcal{H}_{g,n}$. Notice that the surjective forgetful map $\mathbb{P}\overline{\mathcal{H}}_{n}(m) \to \mathbb{P}\mathcal{H}_{n}(m)$ is not necessarily finite at the reducible boundary. Also, $\mathbb{P}\mathcal{H}'_{g,n}(m)$ and $\mathbb{P}\mathcal{H}_{n}(m)$ need not be a smooth fibration.

3. Infinitesimal study of strata of smooth surfaces

3.1. Smoothness of locally closed strata. Let $g \geq 2$ and $n, k \geq 0$ and $m : n + k \to \mathbb{N}$ such that $|m| = 2g - 2$ and $m_i > 0$ for $i > n$. The following statement is well-known to the experts and commonly accepted. Here we give a detailed proof for completeness.

**Proposition 3.1.** $\mathcal{H}'_{n}(m)$ and $\mathcal{H}_{n}(m)$ are smooth of pure dimension $2g - 1 + n + k$.

Because $\mathcal{H}'_{n}(m) \to \mathcal{H}_{n}(m)$ is finite étale and surjective, and because $\mathcal{H}'_{n}(m)^* \hookrightarrow \mathcal{H}'_{n}(m)$ is the complement of the zero section of the total space of $\mathcal{O}_{\mathbb{P}\mathcal{H}'_{n}(m)}(-1) \to \mathbb{P}\mathcal{H}'_{n}(m)$, it is enough to prove the statement for $\mathcal{H}'_{n}(m)^*$. Fix $(C, P, \varphi) \in \mathcal{H}'_{n}(m)^*$, where $P : n + k \to C$ is injective and $Z(\varphi) = \sum_i m_i p_i$. 

Notation. As the Riemann surface $C$ is compact, we will often use the isomorphisms between the cohomology $H^q(C, E)$ of an algebraic coherent locally free sheaf, Dolbeault cohomology $H^{p,q}_{\overline{\partial}}(C, E)$ and Čech cohomology $\check{H}^q(\mathfrak{U}, E)$ of the holomorphic vector bundle $E$ associated to an open cover $\mathfrak{U}$ of $C$ made of Stein open subsets.

A deformation of complex structure on $(C, P)$ can be encoded in a Čech cocycle $v = \{v_{hj}\} \in \check{Z}^1(\mathfrak{U}, T_C(-P))$, where

- $\mathfrak{U} = \{V_j\}$ is an open cover of $C$ by small disks;
- $z_j : V_j \to \mathbb{C}$ is a holomorphic coordinate;
- $v_{hj} = v_{hj}(z_j)_{\overline{\partial}z_j}$ is a holomorphic vector field on $V_{hj} = V_h \cap V_j$ that vanishes on $P$.

Denote by $\Phi_{hj}^\varepsilon(z_j) = z_j + \varepsilon v_{hj}(z_j) + o(\varepsilon)$ is the infinitesimal flow on $V_j \cap V_h$ determined by $v_{hj}$.

Now, call $\varphi_j$ the restriction of $\varphi$ to $V_j$ and let $\varphi_j^\varepsilon$ be an infinitesimal deformation of $\varphi_j$ on $V_j$. In the local coordinate $z_j$ we can write $\varphi_j^\varepsilon = \varphi_j + \varepsilon \psi_j + o(\varepsilon)$, for deformation data $(\{v_{hj}\}, \{\psi_j\})$ to glue into a first-order deformation of $(C, P, \varphi)$, we need $\Phi_{hj}^\varepsilon = d(\varphi_j \cdot v_{hj}) + o(\varepsilon)$ to vanish at first order on $V_j \cap V_h$.

Hence, a deformation of the triple $(C, P, \varphi)$ is determined by a couple $(v, \psi)$, where $\psi = \{\psi_j\} \in \check{C}^0(\mathfrak{U}, K_C(-mP))$, that satisfies

$$\delta \psi + d(\varphi \cdot v) = 0$$

that is, $\psi_j - \psi_h + d(\varphi \cdot v_{hj}) = 0$ on $V_j \cap V_h$ for every $j, h$. Also, $(v, \psi)$ corresponds to a trivial deformation if and only if there exist $w = \{w_j\} \in \check{C}^0(\mathfrak{U}, T_C(-P))$ such that $v = \delta w$ and $\psi + d(\varphi \cdot w) = 0$, i.e. $v_{hj} = v_j - v_h$ and $\psi_j + d(\varphi \cdot w_j) = 0$.

In view of the above considerations, it is natural to define the map of holomorphic line bundles $\beta : T_C(-P) \to K_C(-mP)$ over $C$ as $\beta(w) := d(\varphi \cdot w)$.

Summarizing what we have done, a nonzero section $\varphi \in H^0(C, K_C(-mP))$ can be extended along the first-order deformation of complex structure $(v) \in H^1(C, T_C(-P))$ if and only if $(v) \in \ker(H^1(\beta))$, where $H^1(\beta) : H^1(C, T_C(-P)) \to H^1(C, K_C(-mP))$, namely

$$T_{(C, P, \varphi)} \mathcal{P}H_n^m(m)^* \cong \ker(H^1(\beta)).$$

By Serre duality, we recognize that

$$H^1(\beta)^* = H^0(\beta^* : H^0(C, \mathcal{O}_C(mP)) \to H^0(C, K_C^{\otimes 2}(P))$$
where $\beta' : \mathcal{O}_C(mP) \to K_C^{\leq 2}(P)$ is the non-$\mathcal{O}_C$-linear map defined as $\beta'(f) := -\varphi \otimes df$. Indeed, $f d(v \cdot \varphi) = d(f v \cdot \varphi) - (\varphi \otimes df)v$ and so

$$
\langle f, \beta(v) \rangle = \int_C f d(v \cdot \varphi) = \int_C (-\varphi \otimes df)v = \langle -\varphi \otimes df, v \rangle.
$$

Clearly, $\ker(H^0(\beta'))$ is given by the constant functions and so $\text{coker}(H^1(\beta))$ is given by $H^1(K_C(-mP)) \to H^1(K_C)$.

We can summarize our considerations in the following lemma.

**Lemma 3.2.** The tangent space to $\mathcal{H}_n'(m)^*$ at $(C, P, \varphi)$ can be identified to the hypercohomology group $\mathbb{H}^0$ of the complex $C(\beta) := [T_C(-P) \xrightarrow{\beta} K_C(-mP)]$ of holomorphic vector bundles on $C$ in degrees $[-1, 0]$. In particular, it fits in the commutative diagram

$$
\begin{array}{cccc}
\mathbb{C} \cdot \varphi & \xrightarrow{\cong} & T_{(C,P,\varphi)} \mathcal{H}_n'(m)^* & \xrightarrow{\cong} & T_{(C,P)} \mathcal{M}_{g,n}^k \\
\mathbb{H}^0(C(\beta)) & \xrightarrow{\cong} & H^1(T_C(-P)) & \xrightarrow{H^1(\beta)} & H^1(K_C(-mP)) & \xrightarrow{\cong} & H^1(C(\beta)) & \xrightarrow{0}
\end{array}
$$

where the right part of the sequence can be interpreted as

$$
0 \to N_{(C,P)} \mathbb{P} \mathcal{H}_n'(m)^*/\mathcal{M}_{g,n}^k \to H^1(K_C(-mP)) \to H^1(C(\beta)) \cong \mathbb{C} \to 0
$$

**Proof of Proposition 3.1.** By the above lemma, $\dim T_{(C,P,\varphi)} \mathcal{H}_n'(m)^* = \chi(K_C(-mP)) + h^1(T_C(-P)) + 1 = (1 - g) + (3g - 3 + n + k) + 1 = 2g - 1 + n + k$. \hfill $\square$

The above result also shows that each $\mathbb{P} \mathcal{H}_n(m)$ is equidimensional, which is not obvious, considering that some strata $\mathbb{P} \mathcal{H}_n(m)$ are not connected (see [14]).

**Remark 3.3.** We underline that $\beta$ is not a morphism of $\mathcal{O}_C$-modules and so $C(\beta)$ must be considered as the mapping cone of $\beta$ in the Abelian category of sheaves of $\mathbb{C}$-vector spaces over $C$.

### 3.2. The period map for locally closed strata

Given a point $(C, P, \varphi) \in \mathcal{H}_n'(m)^*$, consider an open contractible neighbourhood $U \subset \mathcal{H}_n'(m)^*$. By abuse of notation, we will denote the subset $\{p_1, \ldots, p_{n+k}\} \subset C$ of distinct marked points still by $P$.

The universal family $\mathcal{C}_U \to U$ has a $C^\infty$ trivialization $\mathcal{C}_U \cong (C, P) \times U$. So, fixed the trivialization, each point of $U$ corresponds to a couple $(J_u, \varphi_u)$, where $J_u$ is a complex structure on $C$ and $\varphi_u$ is a $J_u$-holomorphic $(1,0)$-form on $C$.

Hence, we can define the (local) period map

$$
P_m : U \to H^1(C, P; \mathbb{C})
$$

$$
u \mapsto \begin{cases} 0 & \text{if } \nu = 0 \\ P_m(\varphi_u) & \text{otherwise} \end{cases}
$$

where $P_m(\varphi_u) : H_1(C, P; \mathbb{Z}) \to \mathbb{C}$ associates to $\gamma$ the period $\int_\gamma \varphi_u$. Clearly, $P_m$ is not affected by changing the trivialization by an isotopy that fixes the marked
points. When no confusion can occur, we will denote the period map $\mathcal{P}_m$ simply by $\mathcal{P}$.

Remark 3.4. If all $m_i > 0$, then $\mathcal{P}(\varphi_u) \in H^1(C, P; \mathbb{C})$ indeed coincides with the cohomology class $(\varphi_u)$ associated to the closed 1-form $\varphi_u$ that vanishes on $P$.

Notice that $\mathcal{P}$ globalizes to a $\tilde{\mathcal{P}} : \mathcal{H}_n'(m)^* \to H^1(C, P; \mathbb{C})$, where $\mathcal{H}_n'(m)^*$ is the universal cover of $\mathcal{H}_n'(m)$. Equivalently, one can look at $\mathcal{P}$ as of a section of the $\mathbb{C}$-local system $H^1(C, P; \mathbb{C})$ of rank $2g + n + k - 1$ over $\mathcal{H}_n'(m)$.

The following result is well-known, but we provide a proof that we could not find in the literature.

**Proposition 3.5.** The map $\mathcal{P} : U \to H^1(C, P; \mathbb{C})$ is a local biholomorphism.

The main point is to express the differential of $\mathcal{P}$ at $(J_u, \varphi_u)$ as a map in cohomology. Given the transcendental nature of $\mathcal{P}$, Dolbeault cohomology (relative to the complex structure $J_u$) is more suited for this purpose: in particular, elements of $\mathbb{H}^0_\partial(C, C(\beta))$ are couples $(\dot{\varphi}^{1,0}, \dot{\mu})$, where $\dot{\varphi}^{1,0} \in \mathcal{A}^{0,0}(C, K_C(-mP))$, $\dot{\mu} \in \mathcal{A}^{0,1}(C, T_C(-P))$ and $\overline{\partial}\dot{\varphi}^{1,0} = \partial(\dot{\mu} \cdot \varphi_u)$.

Remark 3.6. By suitably chosing the trivialization, it is always possible to represent a class in $\mathbb{H}^0_\partial(C, C(\beta))$ by a couple $(\dot{\varphi}^{1,0}, \dot{\mu})$ as above, where $\dot{\varphi}^{1,0}$ vanishes in a neighbourhood of $P$.

**Lemma 3.7.** The map $\mathcal{P}$ is holomorphic and its differential at $(J_u, \varphi_u)$ is given by

$$
\mathbb{H}^0_\partial(C, C(\beta)) \xrightarrow{(\dot{\varphi}^{1,0}, \dot{\mu})} H^1(C, P; \mathbb{C}) \xrightarrow{(\dot{\varphi}^{1,0} + \dot{\mu} \cdot \varphi_u)}
$$

where $\dot{\varphi}^{1,0}$ is chosen as in the above remark.

**Proof.** The map $\mathcal{P}$ is clearly smooth.

As the zeroes of $\varphi_u$ have constant multiplicities, suitably choosing a trivialization we can assume that $u \mapsto \varphi_u$ is constant in a neighbourhood of $P$. Thus, a tangent vector to $\mathcal{H}_n'(m)$ at $(C, P, \varphi_u)$ can be always represented by a first-order deformation $\dot{\varphi}$ of $\varphi_u$ that vanishes in a neighbourhood of $P$. Moreover, $\dot{\varphi}$ can be split as $\dot{\varphi} = \dot{\varphi}^{1,0} + \dot{\varphi}^{0,1}$ according to $J_u$. Because

$$
\varphi_u + \varepsilon \dot{\varphi} = \varphi_u \left(1 + \varepsilon \frac{\dot{\varphi}^{1,0}}{\varphi_u} \right) \left[1 + \varepsilon \frac{\dot{\varphi}^{0,1}}{\varphi_u} \right] + o(\varepsilon^2)
$$

we conclude that $\dot{\varphi}^{0,1} = \dot{\mu} \cdot \varphi_u$ and so $\dot{\varphi} = \dot{\varphi}^{1,0} + \dot{\varphi}^{0,1} = \dot{\varphi}^{1,0} + \dot{\mu} \cdot \varphi_u$. Hence, $d\mathcal{P}$ has the above expression and $\mathcal{P}$ is holomorphic. $\square$
Proof of Proposition 3.5. Multiplication by \( \varphi \) induces an isomorphism between \( T_C(-P) \) and \( \mathcal{O}_C(-mP - P) \), so that the following diagram

\[
\begin{array}{ccc}
T_C(-P) & \xrightarrow{\beta} & K_C(-mP) \\
\downarrow{\varphi} & & \downarrow{d} \\
\mathcal{O}_C(-mP - P) & \xrightarrow{d} & \mathcal{O}_C(-mP - P)
\end{array}
\]

is commutative. Notice that \( d : \mathcal{O}_C(-mP - P) \rightarrow K_C(-mP) \) is surjective and its kernel is given by \( j_! \mathcal{O} \hookrightarrow \mathcal{O}_C(-mP - P) \), where \( j : C \setminus P \hookrightarrow C \) and \( j_! \mathcal{O} \) is the sheaf of locally constant functions that vanish at \( P \). As a consequence, \( C(\beta) \cong j_! \mathcal{O}[-1] \) and in particular

\[
H^0(C(\beta)) \cong H^1(j_! \mathcal{O})
\]

We conclude by noticing that the above map in Dolbeault cohomology incarnates in the formula for \( d\mathcal{P} \) exhibited in Lemma 3.7. \( \square \)

3.3. Smoothness of closed strata. In this subsection we want to show the following.

Proposition 3.8. For every \( m : n + k \rightarrow \mathbb{N} \) with \( m_i > 0 \) for all \( i > n \) and such that \( |m| = 2g - 2 \), the moduli space \( \mathcal{H}'_n(m) \) is smooth of pure dimension \( 2g - 1 + n + k \).

Again, it is enough to prove the statement for \( \mathcal{H}'_n(m)^{sm} \). Consider \( (C, P, \varphi) \in \mathcal{H}'_n(m)^{sm} \) with \( \varphi \neq 0 \). It is well-known that the complex

\[
T_{C,P} := \left[ T_C \xrightarrow{ev} \bigoplus_{i=1}^{n+k} T_C|_{p_i} \right]
\]

in degrees \([0, 1]\) controls the first-order deformations of \( (C, P) \in \mathcal{M}_{g,n}^k \). Indeed, let us work in Cech cohomology with an open cover \( \mathfrak{U} \) of \( C \) by small disks.

An element of \( \check{H}^1(\mathfrak{U}, T_{C,P}) \) is represented by a \((n + k + 1)\)-uple \((v, v^1, \ldots, v^{n+k})\) with \( v = \{v_{i,j}\} \in \check{Z}^1(\mathfrak{U}, T_C) \) and \( v^i \in T_C|_{p_i} \), up to the equivalence relation that declares it trivial if there exists a \( w = \{w_j\} \in \check{Z}^0(\mathfrak{U}, T_C) \) such that \( \delta w = v \) and \( w_j(p_i) = v^i \) whenever \( p_i \in V_j \). Thus,

\[
\check{H}^1(C, T_{C,P}) \cong T_{(C,P)} \mathcal{M}_{g,n}^k \quad \text{and} \quad \check{H}^0(C, T_{C,P}) = \check{H}^2(C, T_{C,P}) = 0
\]

Notice that, if the marked points are distinct, then \( T_{C,P} \) is quasi-isomorphic to \( T_C(-P) \) and so \( \check{H}^1(C, T_{C,P}) \cong H^1(C, T_C(-P)) \) as expected.

Now consider the complex in degrees \([0, 1]\)

\[
k_{C,P,m} := \left[ K_C \xrightarrow{ev} \bigoplus_{i=1}^{n+k} J_{p_i}^{m_i} K_C \right]
\]
where \( J_{p_i}^0 K_C = K_C \mid_{p_i} \) and \( J_{p_i}^{m_i - 1} K_C \) is the \((m_i - 1)\)-th jet of \( K_C \) at \( p_i \), and where we have set \( J_{p_i}^{-1} K_C = 0 \).

We can define a (non-\( O_C \)-linear) map \( \beta : T_{C,P} \to K_{C,P,m}^\bullet \) as

\[
\begin{align*}
T_{C,P} & := \left[ T_C \longrightarrow \bigoplus_{i=1}^{n+k} T_C \mid_{p_i} \right] \\
\beta & \downarrow \quad \beta_0 \downarrow \quad \beta_1 \\
K_{C,P,m}^\bullet & := \left[ K_C \longrightarrow \bigoplus_{i=1}^{n+k} J_{p_i}^{m_i - 1} K_C \right]
\end{align*}
\]

where \( \beta_0(v) = d(v \cdot \varphi) \) and \( \beta_1(v^i) = (d(v^i \cdot \varphi)_{p_i}) \). To be precise, in order to define \( \beta_1 \) we should pick a local extension \( \tilde{\varphi} \) of \( \varphi \), compute \( d(\tilde{\varphi}) \) and then evaluate at \( p_i \). However, this turns out to be independent of the chosen extension and indeed it is linear in \( v^i \), so that in coordinates it even makes sense to write \( \beta_1(v^i) = v^i \cdot (d\varphi)_{p_i} \).

As before, the main point is the computation of the tangent space to \( \mathcal{H}_n^\beta(m)^{sm,*} \). We notice that the smoothness of \( \mathcal{H}_n^\beta(m)^{sm,*} \) is an immediate corollary.

**Lemma 3.9.** The tangent space to \( \mathcal{H}_n^\beta(m)^{sm,*} \) at \((C,P,\varphi)\) fits into the following commutative diagram

\[
\begin{array}{c}
\mathbb{C} \cdot \varphi \ar[r] & T_{(C,P,\varphi)} \mathcal{H}_n^\beta(m)^{sm,*} \ar[r] & T_{(C,P)} \mathcal{M}_{g,n} \\
0 \ar[r] & \mathbb{H}^0(K_{C,P,m}^\bullet) \ar[r] \ar[d]^{\cong} & \mathbb{H}^0(C(\beta)) \ar[r] \ar[d]^{\cong} & \mathbb{H}^1(T_{C,P}^\bullet) \ar[r] \ar[d]^{\cong} & \mathbb{H}^1(K_{C,P,m}^\bullet) \ar[r] & \mathbb{H}^1(C(\beta)) \ar[r] & 0
\end{array}
\]

As before, the right portion of the exact sequence can be reinterpreted as

\[
0 \longrightarrow N_{(C,P)} \mathcal{P} \mathcal{H}_n^\beta(m)^{sm} \mathcal{M}_{g,n} \longrightarrow \mathbb{H}^1(K_{C,P,m}^\bullet) \longrightarrow \mathbb{H}^1(C(\beta)) \cong \mathbb{C} \longrightarrow 0.
\]

Before proving the lemma, we notice that the smoothness of \( \mathcal{H}_n^\beta(m)^{sm,*} \) is an immediate corollary.

**Proof of Proposition 3.8.** From the above lemma, it follows that

\[
\dim T_{(C,P,\varphi)} \mathcal{H}_n^\beta(m)^{sm,*} = \chi(K_{C,P,m}^\bullet) + h^1(T_{C,P}^\bullet) + 1 = (1-g) + (3g-3+n+k) + 1 = 2g-1+n+k
\]

and so \( \mathcal{H}_n^\beta(m)^{sm,*} \) is smooth and equidimensional.

In order to prove Lemma 3.9 we mimick the argument in the previous subsection and we construct a \( \beta^\vee \). As \( T^\bullet \) is quasi-isomorphic to \[ T_C \oplus (\bigoplus_i T_C(-p_i)) \to \bigoplus_i T_C \] where the map is given by the diagonal on the first summand \( T_C \) and the inclusion on the second summand, then

\[
\mathcal{H}om(T^\bullet, K_C) \cong q_i \left[ \bigoplus_i K_C \otimes^2 \to K_C \otimes^2 \oplus \bigoplus_i K_C \otimes^2(p_i) \right]
\]
in degrees $[-1, 0]$, where the map to the first $K_C^{\otimes 2}$ is given by the sum and the map to the remaining summands is the inclusion. Similarly,

$$\mathcal{H}om(K^\bullet, K_C) \simeq_q \left[ \bigoplus_i \mathcal{O}_C \rightarrow \mathcal{O}_C \oplus \bigoplus_i \mathcal{O}_C(m_i p_i) \right]$$

in degrees $[-1, 0]$. Moreover, $\beta_0'(f_i) = -\varphi \otimes df_i$ and $\beta_i'(f, g_i) = (-\varphi \otimes df, (-\varphi \otimes dg_i)_{p_i})$. It is easy to see that the kernel of $\mathbb{H}^0(\beta')$ is given the constant functions and so $\text{coker}(\mathbb{H}^1(\beta)) \cong H^1(K_C)$ is also one-dimensional, which concludes the proof.

3.4. Local structure of substrata. We want to study the local structure of $\overline{\mathbb{P}H}^n_n(\sigma, m)^{sm} \subset \overline{\mathbb{P}H}^n_n(m)^{sm}$, where $m : n + k \rightarrow \mathbb{N}$ and $\sigma, m : n + l \rightarrow \mathbb{N}$ with $\sigma \in \mathfrak{S}_n(k, l)$.

**Proposition 3.10.** $\overline{\mathbb{P}H}^n_n(\sigma, m)^{sm}$ is a regular submanifold of $\overline{\mathbb{P}H}^n_n(m)^{sm}$.

**Proof.** Let $(C, Q, \varphi) \in \overline{\mathbb{P}H}^n_n(\sigma, m)^{sm}$ with $\varphi \neq 0$ and let $(C, P, \varphi) \in \overline{\mathbb{P}H}^n_n(m)^{sm}$ be the corresponding point, where $P = Q \circ \sigma : n + k \rightarrow C$.

The differential at $(C, Q)$ of the inclusion $\mathcal{M}_{g,n} \hookrightarrow \mathcal{M}_{g,n}^k$ is encoded by $\mathbb{H}^1$ of the natural morphism $\mathcal{T}_{C,Q}^* \rightarrow \mathcal{T}_{C,P}^*$, induced by the diagonal map $\mathcal{T}_C|_{q_i} \rightarrow \bigoplus_{\sigma(i) = j} \mathcal{T}_C|_{p_i}$.

Similarly, we have a morphism $K^\bullet_{C,Q,\sigma, m} \rightarrow K^\bullet_{C,P, m}$, where $J_{q_i(\sigma, m)}^{-1} K_C \rightarrow \bigoplus_{\sigma(i) = j} J_{p_i}^{-1} K_C$ is the composition of diagonal and restriction maps. Clearly, these morphisms are compatible with the $\beta$ maps and we obtain the following commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & T_{(C, Q, \varphi)} \overline{\mathbb{P}H}^n_n(\sigma, m)^{sm} & \rightarrow & \mathbb{H}^1(\mathcal{T}_{C,Q}^*) & \rightarrow & \mathbb{H}^1(K^\bullet_{C,Q, m}) & \rightarrow & H^1(K_C) \cong \mathbb{C} & \rightarrow & 0 \\
\bigg\downarrow & & \bigg\downarrow & & \bigg\downarrow & & \bigg\downarrow & & \cong & & \bigg\downarrow \\
0 & \rightarrow & T_{(C, P, \varphi)} \overline{\mathbb{P}H}^n_n(m)^{sm} & \rightarrow & \mathbb{H}^1(\mathcal{T}_{C,P}^*) & \rightarrow & \mathbb{H}^1(K^\bullet_{C,P, m}) & \rightarrow & H^1(K_C) \cong \mathbb{C} & \rightarrow & 0
\end{array}
$$

where the second vertical map is injective and so the first one is. The third vertical map is an isomorphism because $\mathbb{H}^1(C, K^\bullet_{C,P, m})$ and $\mathbb{H}^1(C, K^\bullet_{C,Q,\sigma, m})$ have the same dimension. $\square$

3.5. The period map for closed strata. Let $m : n + k \rightarrow \mathbb{N}$ with $|m| = 2g - 2$ and $m_i > 0$ for $i > n$. We want to study the local period map for $\overline{\mathbb{P}H}^n_n(m)^{sm,*}$.

First of all, we need to determine the correct period domain. Viewing the period map as a section of a $\mathbb{C}$-local system, the problem consists in finding the right extension of $H^1(C, P; \mathbb{C}) \rightarrow H^1(m)^{sm,*}$ over $\overline{\mathbb{P}H}^n_n(m)^{sm}$. 

**Lemma 3.11.** The local system $H^1(C, P; \mathbb{C})$ on $\mathcal{H}^n_n(m)$ extends as a $\mathbb{C}$-local system $H^1(\mathbb{C}, P)$ of rank $2g + n + k - 1$ on $\overline{\mathbb{P}H}^n_n(m)^{sm}$, where $\mathbb{C}_{\mathbb{C}, P} := \left[ \mathbb{C}_C \xrightarrow{\text{ev}} \bigoplus_{i=1}^{n+k} \mathbb{C}_{p_i} \right]$.
is a complex of constructible sheaves of \( \mathbb{C} \)-vector spaces on \( C \) in degrees \([0, 1] \).

**Proof.** First of all, it is easy to check that \( H^d(\mathbb{C}_C; P) \) is indeed a local system of constant rank over \( \mathcal{M}_g^n \) for \( d = 0, 1, 2 \), which can be pulled back to \( \mathcal{H}_{g,n} \) and restricted to \( \overline{\mathcal{H}_n}(m)^{sm,*} \). Moreover, if the points \( \{p_i\} \) are distinct, then \( \mathbb{H}^d(C, \mathbb{C}_C; P) = H^d(C, P; \mathbb{C}) \) for \( d = 0, 1, 2 \). So indeed \( H^1(\mathbb{C}_C; P) \) is an extension of \( H^1(C, P; \mathbb{C}) \).

The next step is to actually define the period map. In order to study its local properties, we pick a contractible open subset \( U \) of \( \overline{\mathcal{H}_n}(m)^{sm,*} \). This way we can topologically trivialize the unmarked universal family \( \mathcal{C}_U \cong C \times U \to U \), which we can thus think of as a map \( u \mapsto (J_u, P_u, \varphi_u) \), where \( P_u \) denotes the subset of marked points on the fixed surface \( C \) at \( u \in U \).

**Proposition 3.12.** The period map \( \mathcal{H}_n(m)^* \to H^1(C, P; \mathbb{C}) \) extends as a holomorphic section

\[
P : \overline{\mathcal{H}_n}(m)^{sm,*} \to H^1(\mathbb{C}_C; P)
\]

by defining \( \mathcal{P}(C, P, \varphi) \) as

\[
\mathcal{P}(C, P, \varphi) :\ H_1(C, P; \mathbb{Z}) \to \mathbb{C}
\]

and observing that \( \text{Hom}(H_1(C, P; \mathbb{Z}), \mathbb{C}) \cong H^1(C, P; \mathbb{C}) \cong H^1(\mathbb{C}_C; P, \varphi) \).

**Proof.** Let \( j : C \setminus P \hookrightarrow C \) be the inclusion and \( j_! \mathcal{C} \) the sheaf of locally constant \( \mathbb{C} \)-valued functions that vanish at \( P \), considered as a complex in degree 0. The natural morphism \( j_! \mathcal{C} \to \mathbb{C}_C; P \) induces a map \( H^d(C, P; \mathbb{C}) \to H^d(\mathbb{C}_C; P) \) in cohomology, which is injective for \( d = 1 \) and an isomorphism for \( d = 0, 2 \).

If the marked points are distinct, then the map \( H^1(C, P; \mathbb{C}) \to H^1(\mathbb{C}_C; P) \) is an isomorphism and so \( \mathcal{P} \) restricts to the previously defined period map on \( \overline{\mathcal{H}_n}(m)^* \).

To show that \( \mathcal{P} \) is holomorphic at \( o \in U \), pick \( D', D \subset C \) open disks that contain \( P_o \) and such that \( D' \subset D \). Up to restricting \( U \) to a smaller neighbourhood of \( o \), we can assume that \( P_u \subset D' \) for all \( u \in U \). By Mayer-Vietoris sequence,

\[
H^1(\mathbb{C}_C; P) \hookrightarrow H^1(C, \mathbb{C}) \oplus H^1(\mathbb{C}_D; P)
\]

where \( H^1(C, \mathbb{C}) \) is a trivial local system and \( \mathcal{C}_{D, P} := [\mathcal{C}_D \to \mathbb{C}_D] \). Thus, it is enough to show that the composition of \( \mathcal{P} \) with either of the two projections to \( H^1(C, \mathbb{C}) \) or \( H^1(\mathbb{C}_D; P) \) is holomorphic.

In the former case, we already know that \( u \mapsto (\varphi_u) \in H^1(C, \mathbb{C}) \) is holomorphic. The latter case is a consequence of the following lemma.

**Lemma 3.13.** Let \( D \subset \mathbb{C} \) be an open disc and let \( U \) be a contractible complex manifold. Given integers \( m_1, \ldots, m_{n+k} \geq 0 \) and holomorphic maps \( p_1, \ldots, p_{n+k} : U \to D \) and \( a : U \times D \to \mathbb{C}^* \), consider the family of Abelian differentials \( u \mapsto \alpha_u := a(u, z)(z - p_1(u))^{m_1} \cdots (z - p_{n+k}(u))^{m_{n+k}}dz \) on \( D \) parametrized by \( U \). Then
(a) For every $i,j$, the function
\[
\int_{\gamma_{ij}} \alpha : U \to \mathbb{C} \\
\mu \to \int_{\gamma_{ij}} \alpha_{\mu}
\]
is holomorphic. Hence, for a fixed $i$ this proves (a). As for (b), by dimensional reasons it is enough to show that, for all $\alpha$.

Proof. As $D$ and $U$ are contractible, the local system $H^1(C_D, p)$ reduces to $H^0(C_p \oplus \cdots \oplus C_{p_{n+k}})/H^0(C_D)$ and $\alpha$ can be written as $dg$, where $g : U \times D \to \mathbb{C}$ is a holomorphic function. Hence, $\int_{\gamma_{ij}(u)} \alpha_u = g(p_j(u)) - g(p_i(u))$ is holomorphic and this proves (a). As for (b), by dimensional reasons it is enough to show that, for a fixed $i$, the elements $\int_{\gamma_{ij}}$ for $j \neq i$ are linearly independent. Indeed, if $\int_{\gamma_{ij}} \alpha_u = 0$ for all $j$, this means that $g(p_j(u)) = c$ for all $j$ and so the class of $g_u$ in $H^0(C_{p_1} \oplus \cdots \oplus C_{p_{n+k}})/H^0(C_D)$ reduces to $H^0(C_D)$. Finally, (c) is a consequence of (b) and (a).

Over the contractible open set $U$ it is possible to trivialize the local system $H^1(C_D, p)$ as $H^1(C_D, p) \times U$, and so it is possible to view $P$ as a holomorphic map.

Lemma 3.14. The differential of the local period map $P : U \to H^1(C_D, p)$ at $u$ can be written as
\[
\begin{array}{ccc}
\mathbb{H}^1(C, C(\beta)) & \to & H^1(C_D, p) \\
(\phi_{1,0}, \mu, v^1, \ldots, v^{n+k}) & \mapsto & (\phi_{1,0} + \mu \cdot \varphi_u, \varphi_u(v^1), \ldots, \varphi_u(v^{n+k}))
\end{array}
\]

Proof. As in Lemma 3.7 we have that $\dot{\phi} = \phi_{1,0} + \mu \cdot \varphi_u$, where $\phi_{1,0} \in A^{0,1}(C, C_{-mP})$ and $\dot{\mu} \in A^{0,1}(C, C_{-P})$. On the other hand, we can represent the 1-cocycle $dP_u(\dot{\phi}_{1,0} + \mu \cdot \varphi_u, v^1, \ldots, v^{n+k})$ of $C_D, p$ as a couple $(\alpha, t_1, \ldots, t_{n+k})$, where $\alpha$ is a closed 1-form on $C$ and $t_i \in \mathbb{C}$.

If $\gamma$ is an oriented arc on $C$ that ends at $p_j$ and starts at $p_i$, then it defines a functional on $H^1(C_D, p)$ and
\[
\left( \int_\gamma \varphi \right)' = \int_\gamma \dot{\phi} + (\varphi_u(v^j) - \varphi_u(v^i))
\]
which shows that $\alpha = \phi = \phi_{1,0} + \mu \cdot \varphi_u$ and $t_i = \varphi_u(v^i)$.
Notice moreover that we needed not require that \( \varphi \) vanishes near the marked points. \( \square \)

Now we are ready to prove the following.

**Proposition 3.15.** The local period map \( \mathcal{P} : U \to \mathbb{H}^{1}(\mathbb{C}_{C,P}) \) is a local biholomorphism.

**Proof.** Let \( \mathcal{D}^{\bullet}_{C,P,m} \) be the complex (of coherent sheaves on \( C \)) in degrees \([0,1]\) defined as

\[
\mathcal{D}^{\bullet}_{C,P,m} := \big[ \mathcal{O}_{C} \xrightarrow{ev} \bigoplus_{i=1}^{n+k} J_{p_{i}}^{m} \mathcal{O}_{C} \big]
\]

and notice that the multiplication by \( \varphi \) induces a quasi-isomorphism \( \mathcal{T}_{C,P}^{\bullet} \cong \mathcal{D}^{\bullet}_{C,P,m} \). From the exact sequence

\[
0 \to \underline{\mathbb{C}}_{C,P} \to \mathcal{D}^{\bullet}_{C,P,m} \xrightarrow{d} K^{\bullet}_{C,P,m} \to 0
\]

and \( \mathbb{C} \cong \mathbb{H}^{0}(\underline{\mathbb{C}}_{C,P}) \cong \mathbb{H}^{0}(\mathcal{D}^{\bullet}_{C,P,m}) \), we obtain

\[
0 \to \mathbb{H}^{0}(K^{\bullet}_{C,P,m}) \to \mathbb{H}^{1}(\underline{\mathbb{C}}_{C,P}) \to \mathbb{H}^{1}(\mathcal{D}^{\bullet}_{C,P,m}) \to \mathbb{H}^{1}(K^{\bullet}_{C,P,m}) \to \mathbb{H}^{2}(\underline{\mathbb{C}}_{C,P}) \to 0
\]

The explicit expression of \( d\mathcal{P} \) given in Lemma 3.14 shows that the following diagram commutes.

\[
\begin{array}{ccc}
0 & \to & \mathbb{H}^{0}(K^{\bullet}_{C,P,m}) \\
\downarrow & & \downarrow \cong \\
0 & \to & \mathbb{H}^{1}(\underline{\mathbb{C}}_{C,P}) \\
\end{array}
\begin{array}{ccc}
& & \\
\downarrow d\mathcal{P} & & \downarrow \cong \\
& & \\
0 & \to & \mathbb{H}^{1}(\mathcal{D}^{\bullet}_{C,P,m}) \\
\end{array}
\begin{array}{ccc}
\to & & \\
\to & & \\
\mathbb{H}^{1}(K^{\bullet}_{C,P,m}) & \to & 0
\end{array}
\]

Hence, \( d\mathcal{P} \) is an isomorphism. \( \square \)

Finally, we discuss the compatibility between the stratification and the various period maps.

Assume that there exist an integer \( l \in [1,k] \), a surjection \( \sigma \in \mathfrak{S}_{n}(k,l) \) and a \( (C, Q, \varphi) \in \mathcal{H}_{n}^{l}(\sigma_{*}m) \) with \( \sigma_{*}m : n + l \to \mathbb{N} \) such that \( P = Q \circ \sigma : n + k \to C \). In other words, assume that \( (C, Q, \varphi) \) corresponds to \( (C, P, \varphi) \) under the identification of \( \mathcal{H}_{n}^{l}(\sigma_{*}m) \) with \( \mathcal{H}_{n}^{l}(m) \cap \delta_{g} \).

**Proposition 3.16.** The period maps \( \mathcal{P}_{m} \) and \( \mathcal{P}_{\sigma_{*}m} \) induce an isomorphism of the normal bundle of \( \mathcal{H}_{n}^{l}(\sigma_{*}m) \) inside \( \mathcal{H}_{n}^{l}(m) \) with \( H^{1}(\mathbb{C}_{Q,P}) \), where \( \mathbb{C}_{Q,P} = \big[ \bigoplus_{j} \mathbb{C}_{q_{j}} \xrightarrow{ev} \bigoplus_{i} \mathbb{C}_{p_{i}} \big] \).

**Proof.** For every \( j = 1, \ldots, n + l \), let \( \Delta_{j} \subset C \) be a small open disk that contains \( q_{j} \).

If \( q_{j} = q_{j'} \), then take \( \Delta_{j} = \Delta_{j'} \).

Up to restricting to smaller \( U \), we can assume that \( (p_{i})_{i} \in \Delta_{j} \) for every \( \sigma(i) = j \).

Thus we can decompose \( \mathcal{C}_{U} \to U \) into the union of an unmarked family \( \mathcal{C}_{U} - \bigcup_{j} \Delta_{j} \times \)
U \to U$ and a family of disks $\Delta_j \times U \to U$ with marking $P|_{\sigma^{-1}(j)} : \sigma^{-1}(j) \times U \to \Delta_j \times U$ over $U$. By abuse of notation, we will denote by $P_j$ the set of marked points $P(\sigma^{-1}(j))$, which are contained inside the disk $\Delta_j$.

Similarly to what we have seen above for $(C, P)$, the vector space $\bigoplus_j H^1(\Delta_j, P_j; \mathbb{C})$ includes inside $H^1(C \Delta, P)$, where $C \Delta, P = \bigoplus_j C \Delta_j \overset{ev}{\to} \bigoplus_i C P_i$, and so the local system $\bigoplus_j H^1(\Delta_j, P_j; \mathbb{C})$ also extends over $U$ as $H^1(C \Delta, P)$. Thus, we have the following commutative diagram

$$
\begin{array}{c}
U \cap \mathcal{H}'_n(\sigma \cdot m) & \overset{\mathcal{P}_{\sigma \cdot m}}{\twoheadrightarrow} & U \\
\mathcal{P}_m \downarrow & & \mathcal{P}_m \downarrow \\
H^1(C \Delta) & \overset{\mathcal{P}_{C \Delta}}{\longrightarrow} & H^1(C \Delta, P)
\end{array}
$$

and the exact sequence of the triple $(C, \Delta, P)$ induces

$$
0 \longrightarrow T_{(C, Q, \varphi)} \mathcal{H}'_n(\sigma \cdot m) \overset{sm \cdot \mathcal{P}_{\sigma \cdot m}}{\longrightarrow} T_{(C, P, \varphi)} \mathcal{H}'_n(m) \overset{sm \cdot \mathcal{P}_m}{\longrightarrow} N_{(C, P, \varphi)} \mathcal{H}'_n(\sigma \cdot m) \overset{sm \cdot \mathcal{P}_m}{\longrightarrow} 0
$$

The conclusion follows by observing that the obvious map $C \Delta, P \to C Q, P$ is a quasi-isomorphism. □

4. LENGTH AND AREA FUNCTIONALS

Let $g \geq 2$ and $n, k \geq 0$ and let $m : n + k \to \mathbb{N}$ with $|m| = 2g - 2$ and such that $m_i > 0$ for all $i > n$.

The purpose of this section is to study certain real-valued homogeneous functions on $\mathcal{H}'_n(m)^*$ defined in terms of the flat metric induced by the Abelian differential. Then we will combine them to obtain one that descends to $P \mathcal{H}'_n(m)$ and which is $\text{Aut}_n(m)$-invariant.

In taking products and sums of such functions, we need to keep control of the directions along which their complex Hessian is positive-definite. For this reason, we will often refer to the technical condition described in the following subsection.

4.1. Technical interlude. Consider a function $\eta : \Omega \to \mathbb{R}$ defined on an open cone $\Omega \subseteq \mathcal{H}'_n(m)^*$, that is $\mathbb{C}^* \cdot \Omega = \Omega$, and let $(C, P, \varphi) \in \Omega$.

**Definition 4.1.** The real-valued function $\eta$ is **homogeneous of degree** $d$ if $\eta(C, P, \lambda \varphi) = |\lambda|^d \eta(C, P, \varphi)$ for all $\lambda \in \mathbb{C}^*$.

Because of Proposition 3.5 there exists a contractible open neighbourhood $U \subset \Omega$ of $(C, P, \varphi)$ such that the period map $\mathcal{P} : U \to H^1(C, P; \mathbb{C})$ is biholomorphic onto an open subset $\mathcal{P}(U) \subset H^1(C, P; \mathbb{C})$. 

Notation. The Hodge decomposition of $H^1(C; \mathbb{C})$ with respect to the complex structure of $(C, P, \varphi)$ will be denoted by $H^{1,0}_\varphi(C) \oplus H^0_{\varphi,1}(C)$.

Denote by $\Pi_\varphi \subset H^1(C, P; \mathbb{C})$ the inverse image of $H^{1,0}_\varphi(C)$, so that

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^0(P; \mathbb{C}) & \longrightarrow & H^1(C, P; \mathbb{C}) & \longrightarrow & H^1(C; \mathbb{C}) & \longrightarrow & 0 \\
0 & \longrightarrow & \widetilde{H}^0(P; \mathbb{C}) & \longrightarrow & \widetilde{H}^1(P; \mathbb{C}) & \longrightarrow & \Pi_\varphi & \longrightarrow & H^0_{\varphi,1}(C) & \longrightarrow & 0
\end{array}
$$

commutes, where $\widetilde{H}^0(P; \mathbb{C}) = H^0(P; \mathbb{C})/\mathbb{C}$.

Let $\Pi'_\varphi$ be a complement of the line $\mathbb{C}(\varphi)$ inside $\Pi_\varphi$.

**Definition 4.2.** A function $\eta$ weakly satisfies condition $(\star)$ at $(C, P, \varphi)$ if, up to taking a smaller $U$, the restriction of $\eta$ to $U \cap \mathcal{P}^{-1}((\varphi) + \Pi_\varphi)$ is plurisubharmonic. It satisfies condition $(\star)$ if it satisfies weakly $(\star)$ and moreover the restriction to $U \cap \mathcal{P}^{-1}((\varphi) + \Pi'_\varphi)$ is strictly plurisubharmonic. We will say that $\eta$ (weakly) satisfies condition $(\star)$ in $\Omega$ if it does at every point of $\Omega$.

**Remark 4.3.** If $\eta$ is $C^2$, then it weakly satisfies $(\star)$ at $(C, P, \varphi)$ if the restriction of $(i\partial \bar{\partial} \eta)_{(C, P, \varphi)}$ to the subspace $d\mathcal{P}^{-1}_{(C, P, \varphi)}(\Pi_\varphi)$ is semipositive-definite; and it satisfies $(\star)$ at $(C, P, \varphi)$ if moreover the restriction of $(i\partial \bar{\partial} \eta)_{(C, P, \varphi)}$ to the subspace $d\mathcal{P}^{-1}_{(C, P, \varphi)}(\Pi'_\varphi)$ is positive-definite.

We collect some obvious properties in the following statement.

**Lemma 4.4.** The following properties hold.

(a) If $\eta$ satisfies weak $(\star)$, then $\lambda \eta$ satisfies weak $(\star)$ for all $\lambda \geq 0$.

(b) If $\zeta$ satisfies $(\star)$, then $\nu \zeta$ satisfies $(\star)$ for all $\nu > 0$.

(c) If $\eta$ satisfies $(\star)$ at $(C, P, \varphi)$, then $\eta$ satisfies weak $(\star)$ at $(C, P, \varphi)$.

(d) If $\eta_1, \eta_2$ satisfy weak $(\star)$ at $(C, P, \varphi)$, then $\eta_1 + \eta_2$ satisfies weak $(\star)$ at $(C, P, \varphi)$.

(e) If $\zeta$ satisfies $(\star)$ and $\eta$ satisfies weak $(\star)$ in $\Omega$, then $\zeta + \eta$ satisfies $(\star)$ in $\Omega$.

(f) If $\eta_1$ and $\eta_2$ are positive functions such that $\log(\eta_1)$ and $\log(\eta_2)$ satisfy weak $(\star)$, then $\log(\eta_1 + \eta_2)$ satisfies weak $(\star)$. Suppose moreover that $\eta_1, \eta_2 \in C^2$ and, for every $0 \neq \hat{u} \in T_uU$ such that $d\mathcal{P}_u(\hat{u}) \in \Pi'_\varphi$, either $i\partial \bar{\partial} \log(\eta_1)_u(\hat{u}, \hat{u}) > 0$ or $i\partial \bar{\partial} \log(\eta_2)_u(\hat{u}, \hat{u}) > 0$. Then $\log(\eta_1 + \eta_2)$ satisfies $(\star)$.

**Proof.** Properties (a), (b), (c) and (d) are obvious.

As for (e), let $U \subset \Omega$ be an open neighbourhood of $(C, P, \varphi)$ as above and let $W'_\varphi := U \cap \mathcal{P}^{-1}((\varphi) + \Pi'_\varphi)$.

Notice that, along the segment $U \cap \mathcal{P}^{-1}(C(\varphi))$ both $\zeta$ and $\eta$ are plurisubharmonic and so $\zeta + \eta$ is.

Because the restriction of $\zeta$ to $W'_\varphi$ is strictly plurisubharmonic, there exists a smooth $h : W' \rightarrow \mathbb{R}$ such that $|\zeta|_{W'} - h$ is plurisubharmonic and $(i\partial \bar{\partial} h) > 0$ on $W'$. Hence, $h + |\eta|_{W'}$ is strictly plurisubharmonic and so $(\zeta + \eta)|_{W'}$ is too.
As for (f), restrict to a germ of complex curve through \( u \in U \) in direction \( \dot{u} \). Then by direct computation

\[
i \partial \bar{\partial} \log(\eta_1 + \eta_2) = \frac{i \eta_1 \eta_2 (\partial \log(\eta_1/\eta_2) \wedge \bar{\partial} \log(\eta_1/\eta_2)) + (\eta_1 + \eta_2) (\eta_1 \partial \bar{\partial} \log(\eta_1) + \eta_2 \partial \bar{\partial} \log(\eta_2))}{(\eta_1 + \eta_2)^2}
\]

and the conclusion follows. \( \square \)

**Notation.** A function \( \mathbb{P} \mathcal{H}'_n(m) \to \mathbb{R} \) can be seen as a \( \mathbb{C}^* \)-invariant function \( \mathcal{H}'_n(m)^* \to \mathbb{R} \). We will say that the former (weakly) satisfies condition \( \ast \) if the latter does. Similarly, we say that \( \mathcal{H}_n(m)^* \to \mathbb{R} \) (weakly) satisfies condition \( \ast \) if the \( \text{Aut}_n(m) \)-invariant induced \( \mathcal{H}'_n(m)^* \to \mathbb{R} \) does.

The same definitions can be given for real-valued functions defined on an open subset of \( \mathbb{P} \mathcal{H}'_n(m)^m \). Indeed, in this case the period map lands in \( H^1(\mathbb{C}_C, \mathbb{P}) \); so we still have an exact sequence

\[
0 \to \mathbb{C}^k / \mathbb{C} \to H^1(\mathbb{C}_C, \mathbb{P}) \to H^1(\mathbb{C}; \mathbb{C}) \to 0
\]

and we will denote by \( \Pi_{\mathcal{P}} \) the subspace of \( H^1(\mathbb{C}_C, \mathbb{P}) \) that maps to \( H^{1,0}(\mathbb{C}) \).

**4.2. The area functional.** Let \( \pi : C_g \to \mathcal{M}_g \) be the universal family and let \( H^1 := R^1 \pi_* \mathcal{P} \) be the flat vector bundle over \( \mathcal{M}_g \), whose fiber over \( C \in \mathcal{M}_g \) is \( H^1(\mathbb{C}; \mathbb{C}) \). We define a Hermitian pairing \( h \) on \( H^1 \) as

\[
h_C(\varphi_1, \varphi_2) := \frac{i}{2} \int_C \varphi_1 \wedge \bar{\varphi}_2.
\]

**Lemma 4.5.** The Hermitian form \( h \) has the following properties.

(a) For every \( C \in \mathcal{M}_g \), the pairing \( h_C \) has signature \((n_+, n_-, n_0) = (g, g, 0)\).

(b) The restriction of \( h_C \) to \( H^{1,0}(\mathbb{C}) \) is positive-definite and its restriction to \( H^0,1(\mathbb{C}) \) is negative-definite, where \( H^1_C = H^{1,0}(\mathbb{C}) \oplus H^{0,1}(\mathbb{C}) \) is Hodge decomposition with respect to \( C \in \mathcal{M}_g \).

(c) \( h \) is smooth and it restricts to a smooth Hermitian metric on the holomorphic vector bundle \( \mathcal{H}_g \to \mathcal{M}_g \).

**Proof.** About (b), a holomorphic form \( \varphi \) can be locally written as \( f(z)dz \) with respect to a holomorphic coordinate \( z = x + iy \). So \( \frac{i}{2} \varphi \wedge \bar{\varphi} = |f(z)|^2 dx \wedge dy \), which implies that \( h|_{H^{1,0}} \) is positive-definite. An analogous argument shows that \( h|_{H^{0,1}} \) is negative-definite. Clearly, (a) follows from (b).

Finally, \( h \) is smooth because it is locally constant and (c) then follows from (b), because \( \mathcal{H}_g \) is the subbundle of \( H^1 \) consisting of \( (1, 0) \)-holomorphic forms. \( \square \)

**Definition 4.6.** The Area functional \( A : \mathcal{H}_g \to \mathbb{R} \) is

\[
A(C, \varphi) := h_C(\varphi, \varphi) = \frac{i}{2} \int_C \varphi \wedge \bar{\varphi}.
\]
Notice that indeed $A$ computes the area of the surface $C$ endowed with the flat metric $|\varphi|^2$ with conical singularities.

**Notation.** By a little abuse, we will use the same symbol $A$ for the pull-back of the area functional to $\mathcal{H}^k_{g,n}$ and for its restrictions to $\mathcal{H}'_n(m)$ and $\mathcal{H}_n(m)$. Similarly, the restriction of the pull-back of $h$ provides a smooth Hermitian metric on $\mathcal{O}_\mathcal{H}_n(m)(-1)$ and $\mathcal{O}_\mathcal{H}_n(m)(-1)$.

**Lemma 4.7.** When regarded as $A : \mathcal{H}_n(m)^* \to \mathbb{R}_+$, the function $A$ has the following properties:

(a) $A$ and $\log(A)$ are smooth;

(b) $A$ is homogeneous of degree 2;

(c) the signature of $i\partial\bar{\partial}A$ is $(g, g, k - 1)$ and the signature of $i\partial\bar{\partial}\log(A)$ is $(g - 1, g, k)$;

(d) both $A$ and $\log(A)$ satisfy condition weak $(*);$ 

(e) $A$ has logarithmic divergence at points of $\partial_{\text{ir}} \mathcal{H}_n(m)^*$ representing surfaces with only one irreducible component.

The same statements hold when regarding $A$ as a function on $\mathcal{H}'_n(m)^*$.

**Proof.** Because of Lemma 4.5(c), the functional $A$ is smooth and so is $\log(A)$ too. Moreover, $A$ is manifestly homogeneous of degree 2. Hence, the same properties hold on $\mathcal{H}'_n(m)^*$.

Claims (c) and (d) are local, so we can directly work on $\mathcal{H}'_n(m)^*$, because $\mathcal{H}'_n(m)^*$ and $\mathcal{H}_n(m)^*$ are locally biholomorphic.

Let $(C, P, \varphi) \in \mathcal{H}'_n(m)^*$. Because of Proposition 3.5 there exists a small open neighbourhood $U \subset \mathcal{H}'_n(m)$ of $(C, P, \varphi)$ such that the period map realizes a biholomorphism between $U$ and an open subset $\mathcal{P}(U) \subset H^1(C, P; \mathbb{C})$. Being the statement local, we can work in period coordinates. As for (c), consider the exact sequence

$$0 \to \tilde{H}^0(P; \mathbb{C}) \to H^1(C, P; \mathbb{C}) \to H^1(C; \mathbb{C}) \to 0$$

and notice that $A$ factorizes through $H^1(C; \mathbb{C})$, so that $\tilde{H}^0(P; \mathbb{C})$, which has dimension $k - 1$, certainly belongs to the radical of $i\partial\bar{\partial}A$. More explicitly, if $\varphi \in H^1(C, P; \mathbb{C}) \cong T_{(C, P, \varphi)} \mathcal{H}_n(m)$, then

$$A(\varphi + \varepsilon \dot{\varphi}) = A(\varphi) + \varepsilon b(\dot{\varphi}, \varphi) + \varepsilon b(\varphi, \dot{\varphi}) + |\varepsilon|^2 b(\dot{\varphi}, \dot{\varphi})$$

Claims (c) and (d) for $A$ then follow from Lemma 4.5.

As for $\log(A)$, we compute $i\partial\bar{\partial} \log(A) = iA^{-2}(\partial\bar{\partial}A - \partial A \wedge \bar{\partial}A)$ and so

$$i\partial\bar{\partial} \log(A)(\dot{\varphi}, \dot{\varphi}) = \frac{A(\varphi)A(\dot{\varphi}) - |b(\varphi, \dot{\varphi})|^2}{A(\varphi)^2}$$
from which we conclude that, as a Hermitian form on $H^1(C; \mathbb{C})$, the Hessian $i\bar{\partial} \log(A)$ is negative-definite on $H^{0,1}(C)$, vanishes along the line $\mathbb{C} \cdot (\varphi)$ and it is positive-definite on $H^{1,0}(C) \cap (\varphi)^{\perp}$, which shows that $\log(A)$ satisfies condition weak ($\ast$). As $i\bar{\partial} \log(A)$ has signature $(g - 1, g, 1)$ as a Hermitian form on $H^1(C; \mathbb{C})$, it follows that it has signature $(g - 1, g, k)$ as a Hermitian form on $H^1(C, P; \mathbb{C})$. Hence, (c) and (d) are proven.

Finally, let $U = \Delta$ be the open unit disk and let $U \to \overline{H_n(m)^*}$ be a holomorphic map to the locus of nonvanishing Abelian differentials such that the image of $U \setminus \{0\}$ sits in $H_n(m)^*$. Up to a (polynomial) base change, we can assume that this map lifts to $U \to \overline{H'_n(m)}$. Thus claim (e) for $A$ defined on $H_n(m)^*$ is equivalent to that for $A$ defined on $H'_n(m)^*$.

Let $\mathcal{C}_U \to U$ be the induced flat family of stable Riemann surfaces endowed with a nonzero Abelian differential and call $(C_u, P_u, \varphi_u)$ the fiber over $u \in U$.

If the differential $\varphi_0$ has no poles, then $A(C_u, P_u, \varphi_u)$ is bounded for all $u \in U$ in a compact neighbourhood of 0.

Thus, let us assume that $C_0$ has a disconnecting node $\nu$ and $\varphi_0$ acquires a simple pole at $\nu$.

Up to shrinking $U$, in local analytic coordinates $(z, w)$ around $\nu \in C_0$, the family looks like $\mathcal{C}_U = \{(z, w, u) \in \Delta^2 \times U \mid zw = u^d\} \to U$ with $z(\nu) = w(\nu) = 0$, for a certain $d \geq 1$.

We recall that the local generator of the relative dualizing line bundle near $\nu$ can be written as $\frac{dz}{z}$ and that, along each $C_u$, the relation $\frac{dz}{z} + \frac{dw}{w} = 0$ holds. Thus, there exists a holomorphic function $f(z, w)$ such that $\varphi_u = f(z, w)\frac{dz}{z}$, with $f(0, 0) = \rho \neq 0$, and there is a polydisk $B = \{|z| < R, |w| < R\}$ centered at $\nu$ such that no zero of $\varphi_u$ ever enters $B$. As $z$ is a local coordinate on $C_u$ for $u \neq 0$, we have $B \cap C_u = \{w = u^{d}/z, |u^{d}/R| < |z| < R\}$ and so the $|\varphi_u|^2$-area of $B \cap C_u$ is

\[
A(B \cap C_u, \varphi_u) = \frac{i}{2} \int_{B \cap C_u} \varphi_u \wedge \overline{\varphi_u} = \int_{B \cap C_u} \frac{|f(z, w)|^2}{|z|^2} i \frac{dz}{2} \wedge d\overline{z} = \\
= \int_0^{2\pi} d\theta \int_{|u|^{d}/R}^R |\rho|^2 \frac{dr}{r} + O(1) = 2\pi d|\rho|^2 \log \left(\frac{1}{|u|}\right) + O(1)
\]

Notice that $\varphi_0$ is regular outside a neighbourhood of the poles of $\varphi_0$. Thus, if $\nu_1, \ldots, \nu_s$ are the nodes of $C_0$ of multiplicities $d_1, \ldots, d_s \geq 1$ at which $\varphi_0$ has simple pole, and if $\pm \rho_j \neq 0$ are the residues of $\varphi_0$ at the two branches of $\nu_j$, then

\[
A(C_u, P_u, \varphi_u) = \left(2\pi \sum_{j=1}^{2} d_j |\rho_j|^2 \right) |\log |u|| + O(1)
\]

which proves (e).
4.3. **Length functions.** As degenerations on flat surfaces of bounded area occur as conical singularities clash together, it makes sense to study distance functions between couples of marked points along prescribed paths.

**Definition 4.8.** An arc on the surface $C$ with marked points $P$ is a nontrivial homotopy class (with fixed endpoints) of a simple oriented path $\gamma$ on $C$ that intersects $P$ at its endpoints. A set of arcs $B = \{\gamma_j\}$ is a basis if the set of homology classes $\{(\gamma_j)\}$ is a basis of the real vector space $H_1(C, P; \mathbb{R})$. We denote by $\mathcal{B}$ the set of all bases of arcs.

Let $\varphi$ be a nonzero Abelian differential on $(C, P)$ and assume that the set of its zeroes $Z(\varphi)$ is contained inside $P$. The associated metric $|\varphi|^2$ on $C$ is flat outside $Z(\varphi)$ and it has a conical singularity of angle $2\pi(d + 1)$ at a zero of order $d$.

**Definition 4.9.** A segment for $(C, P, \varphi) \in \mathcal{H}_n(m)^*$ is a smooth $|\varphi|^2$-geodesic path between two (not necessarily distinct) points in $P$. We will say that a segment is proper if it intersects $P$ at its endpoints only.

**Remark 4.10.** People in translation surfaces often call saddle connection a segment joining two zeroes of $\varphi$.

As $|\varphi|^2$ is nonpositively curved, there always exists a unique geodesic representing a given arc $\gamma$. Such a geodesic is a concatenation of proper segments $\gamma = \gamma^1 \cup \cdots \cup \gamma^s$. We denote by $\ell_\gamma(\varphi)$ the length of such a geodesic with respect to the metric $|\varphi|^2$. Clearly, $\ell_\gamma(\varphi) = \ell_{\gamma^1}(\varphi) + \cdots + \ell_{\gamma^s}(\varphi)$.

**Definition 4.11.** The systole of $(C, P, \varphi) \in \mathcal{H}_n(m)^*$ is the length $\ell_{\text{sys}}(\varphi)$ of the shortest (nontrivial) arc on $C$ with respect to the metric $|\varphi|^2$.

We recall some preliminary and well-known results on the geometry of such flat surfaces with conical singularities.

**Lemma 4.12.** For every $(C, P, \varphi) \in \mathcal{H}_n(m)^*$, the following hold:

(a) for every $L > 0$ there are finitely many $(C, P, \varphi)$-segments of length less than $L$;
(b) the systole $\ell_{\text{sys}}(\varphi)$ is always realized at a proper segment.

**Proof.** Let $\tilde{C} \to C$ be the universal cover and let $\tilde{\varphi}$ be the pullback of $\varphi$ to $\tilde{C}$. Denote by $\tilde{P} \subset \tilde{C}$ the subset of points that project to $P \subset C$. Let $D \subset \tilde{C}$ be a fundamental domain and $R$ its diameter (with respect to the $|\tilde{\varphi}|^2$ metric). Fix a point $\tilde{x} \in D$. Then each segment of length at most $L$ is contained in the ball $B = B(\tilde{x}, L + R) \subset \tilde{C}$. Because $\tilde{C}$ is compact and so its systole is positive, $B$ intersects only finitely many translates of $D$ and so $B \cap \tilde{P}$ is finite. Thus, there are only finitely many segments completely contained in $B$ and this proves (a). As a consequence, the systole of $(C, P, \varphi)$ is certainly attained at some arc $\gamma$. Write $\gamma$ as a concatenation of proper segments $\gamma^1 \cup \cdots \cup \gamma^s$. If $s > 1$, then $\ell_{\gamma^1}(\varphi) <
\[ \ell_s(\varphi) = \ell_{sys}(\varphi) \] would give a contradiction. Hence, \( s = 1 \) and \( \gamma \) is a proper segment for \((C, P, \varphi)\), which proves (b).

\[
\text{Proof.}
\]

Let \((C, P, \varphi) \in \mathcal{H}_n(m)^*\) and consider \( U \subset \mathcal{H}_n'(m)^*\) a small contractible open neighbourhood of \((C, P, \varphi)\), so that the universal family of marked Riemann surfaces topologically trivializes over \( U \) and \( \mathcal{P} \) maps \( U \) biholomorphically onto an open subset of \( H^1(C, P; \mathbb{C}) \). We can think of points of \( u \in U \) as of closed differential 1-forms \( \varphi_u \) on the fixed marked surface \((C, P)\).

We recall that, for every \((\gamma) \in H_1(C, P; \mathbb{Z})\), the local period

\[ \mathcal{P}_\gamma : U \rightarrow \mathbb{C} \]

defined as \( \mathcal{P}_\gamma(u) := \int_\gamma \varphi_u \) is holomorphic by Proposition 3.5. Moreover, if \( \gamma = \gamma^1 \cup \cdots \cup \gamma^s \) is a decomposition of \( \gamma \) in a union of segments, then \( |\mathcal{P}_\gamma_i(u)| = \ell_i(u) \) and \( \ell_\gamma(u) = \sum_i \ell_i(u) = \sum_i |\mathcal{P}_\gamma_i(u)| \geq |\mathcal{P}_\gamma(u)|. \)

**Notation.** Let \( U_\gamma \subset U \) the locus of differentials for which \( \gamma \) is a proper segment and \( U_B := \bigcap_{\gamma \in B} U_\gamma \) the locus of differentials for which \( B \) is a basis made of proper segments.

**Lemma 4.13.** The following properties hold.

(a) For every arc \( \gamma \) and basis \( B \), the loci \( U_\gamma \subset U \) and \( U_B \subset U \) are open.

(b) Let \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_{\geq 0} \) be a strictly decreasing continuous function. Then

\[
f_B := \sup_{B \in B} f_B : U \rightarrow \mathbb{R}_{\geq 0}
\]

is well-defined, where \( f_B := \sum_{\gamma \in B} (f \circ \ell_\gamma) : U \rightarrow \mathbb{R}_{\geq 0} \).

Moreover, \( f_B \) is locally the maximum of finitely many \( f_B|_{U_B} \).

**Proof.** To prove (a), it is clearly enough to prove that \( U_\gamma \) is a neighbourhood of each \( u \in U_\gamma \).

Let \( \tilde{C} \rightarrow C \) be the universal cover and denote by \( \tilde{P} \subset \tilde{C} \) the points that map to \( P \).

The arc \( \gamma \) lifts to an arc \( \tilde{\gamma} \) with distinct endpoints and every \( \varphi_u \) on \( C \) pulls back to a \( \tilde{\varphi}_u \) on \( \tilde{C} \). Call \( \tilde{\gamma}_u \) the \( \tilde{\varphi}_u \)-geodesic on \( \tilde{C} \) homotopic to \( \tilde{\gamma} \).

Because \( \tilde{\gamma}_u \) is a proper segment, there exists an \( \varepsilon > 0 \) such that the \( \varepsilon \)-neighbourhood \( N_\varepsilon \) of \( \tilde{\gamma}_u \) with respect to \( |\tilde{\varphi}_u|^2 \) does not contain any other point of \( \tilde{P} \).

Thus, there exists a neighbourhood \( U' \subset U \) of \( u \) such that \( \ell_\gamma(\varphi_u') < \ell_\gamma(\varphi_u) + \varepsilon/4 \) and \( \text{dist}_{|\varphi_u'|^2}(\tilde{\gamma}_u, \partial N_\varepsilon) > \varepsilon/2 \) for all \( u' \in U \). Hence, \( \gamma_u \) is realized by a \( |\varphi_u'|^2 \)-geodesic contained inside \( N_\varepsilon \), which is thus a proper segment, for all \( u' \in U' \). This shows that \( U' \subset U_\gamma \) and so \( U_\gamma \) is open.

In order to prove (b), we want to show that, for every \( u \in U \) there exist finitely many bases \( B_1, \ldots, B_r \) and a neighbourhood \( V \subset U \) of \( u \) such that \( V \subset U_{B_1} \cap \cdots \cap U_{B_r} \) and \( f_B|_V = \max \{ f_{B_j}|_V \mid j = 1, \ldots, r \} \).
Observe that a $B$ such that $f_B$ realizes the sup at $u$ always exists, because of Lemma 4.12(a). Indeed, for every relatively compact neighbourhood $K \subset U$ of $u$ and for every $\varepsilon > 0$ the set

$$\{ B \in \mathcal{B} \mid K \cap U_B \neq \emptyset, \text{ and } f_B > f_B - \varepsilon \text{ on } U_B \cap K \}$$

is finite.

To complete the proof of (b), we need to show that a $B$ such that $f_B(u) = f_B(u)$ is made of proper segments. By contradiction, let $B = \{ \gamma_1, \ldots, \gamma_{2g+n+k-1} \}$ and suppose that $\gamma_i$ is not a proper segment. Then $\gamma_i$ is the concatenation of two paths $\gamma_i'$ and $\gamma_i''$, where $\gamma_i'$ and $\gamma_i''$ are union of proper segments. Because $B$ is a basis of $H(C, P; \mathbb{R})$, then we can write $(\gamma_i') = \sum_h a_h'(\gamma_h)$ and $(\gamma_i'') = \sum_h a_h''(\gamma_h)$. As $a_i' + a_i'' = 1$, at least one of the two must be nonzero: without loss of generality, we can assume that $a_i' \neq 0$. Then $B' := (B \setminus \{ \gamma_i \}) \cup \{ \gamma_i \}$ is again a basis of (possibly not proper) segments for $\varphi_u$, but $f_{B'}(u) > f_B(u) = f_B(u)$. \hfill \Box

As the $f_B$ above defined in Lemma 4.13(b) is the sup of all bases, it is clearly independent of the local trivialization of the family and so it globalizes over $\mathcal{H}_n(m)^*$. Moreover, it is $\text{Aut}_n(m)$-invariant and so we obtain the following.

**Corollary 4.14.** For every continuous strictly decreasing $f : \mathbb{R}_+ \to \mathbb{R}_{\geq 0}$, there is a global $f : \mathcal{H}_n(m)^* \to \mathbb{R}_{\geq 0}$, which is locally the maximum of finitely many functions which are as regular as $f$. Moreover, if $f$ is homogeneous of weight $d$, then $f_B$ is homogeneous of weight $d$.

The decreasing function we will be interested in is the following.

**Lemma 4.15.** The function $\ell_B^{-2} : \mathcal{H}_n(m)^* \to \mathbb{R}_+$ associated to $f(x) := x^{-2}$ has the following properties:

(a) $\ell_B^{-2}$ is homogeneous of degree $-2$;

(b) $\ell_B^{-2}$ is strictly plurisubharmonic on $\mathcal{H}_n(m)^*$; $\log(\ell_B^{-2})$ is plurisubharmonic on $\mathcal{H}_n(m)^*$ and it is strictly plurisubharmonic when restricted to a codimension 1 submanifold which is transverse to the fibers of $\mathcal{H}_n(m)^* \to \mathbb{P}\mathcal{H}_n(m)$;

(c) $\ell_B^{-2}$ and $\log(\ell_B^{-2})$ satisfy condition $(\ast)$;

(d) $\ell_B^{-2}$ has polynomial divergence at points in $\partial_{\text{red}} \mathcal{H}_n(m)^*$ and $\overline{\mathcal{H}_n(m)^*}$ and it remains bounded and positive at the remaining points of $\partial_{\text{irr}} \mathcal{H}_n(m)^*$.

**Proof.** Part (a) is obvious. About (b), we can work on $\mathcal{H}_n(m)^*$, because $\mathcal{H}_n(m)^* \to \mathcal{H}_n(m)^*$ is a finite étale cover. By Lemma 4.13(b), every $(C, P, \varphi) \in \mathcal{H}_n(m)^*$ has a neighbourhood on which $\ell_B^{-2}$ can be written as the maximum of $\ell_{B_1}^{-2}, \ldots, \ell_{B_n}^{-2}$ and the arcs in $B_1, \ldots, B_n$ are realized by proper segments with respect to any Abelian differential in such a neighbourhood. Hence, $\ell_{B_j}^{-2} = \sum_{\gamma \in B_j} |\mathcal{P}_\gamma|^{-2}$, which is strongly plurisubharmonic, because $B_j$ is a basis. Hence, $\ell_B^{-2}$ is strongly plurisubharmonic by Remark A.8. It is straightforward to check that $i\partial\overline{\partial} \log(\ell_B^{-2})$ is semipositive-definite and its radical coincides.
with the direction given by rescaling. This proves (b). Moreover, (c) is an immediate consequence.

As for part (d), notice that polynomial divergence is invariant under base change. Thus, we can suppose that the zeroes of the differential are marked and so we can work on $\overline{\mathcal{H}_n'(m)}$ rather than $\overline{\mathcal{H}_n(m)}$.

Let $U = \Delta$ be the open unit disk and let $U \to \overline{\mathcal{H}_n'(m)}$ be a holomorphic map such that the image of $U \setminus \{0\}$ sits in $\mathcal{H}_n'(m)^*$, and let $\mathcal{C}_U \to U$ be the induced flat family of stable Riemann surfaces endowed with a nonzero Abelian differential and call $(C_u, P_u, \varphi_u)$ the fiber over $u \in U$. Again, by invariance of polynomial divergence under base change, we can assume that $\mathcal{C}_U$ is smooth.

Notice that a length function associated to some arc goes to zero as $u \to 0$ if and only if at least one of the following conditions hold:

(i) (at least) two zeroes of $\varphi_u$ clash together at a smooth point of $C_0$;

(ii) there is an arc $\gamma$ between two (not necessarily distinct) $P$-marked points which is entirely contained in the union $C_0^0$ of the components of $C_0$ on which $\varphi$ vanishes.

If $(C_0, P_0, \varphi_0) \in \overline{\mathcal{H}_n'(m)^*}$ then case (i) occurs; if $(C_0, P_0, \varphi_0) \in \partial_{\text{red}} \mathcal{H}_n'(m)^*$, then case (ii) occurs. Thus, $\ell_{\mathcal{B}^2}$ does not diverge at points of $\partial_{\text{red}} \mathcal{H}_n'(m)^*$ that do not also lie on the other above mentioned boundary loci.

Now suppose that (i) occurs and that, up to rearranging the label, $p_1(u), \ldots, p_s(u)$ are clashing together at a smooth point $q = p_1(0) = \cdots = p_s(0) \in C_0$. Up to shrinking the base of the family, a neighbourhood of $q$ inside $\mathcal{C}_U$ looks like $\Delta_z \times U$ and $\varphi_u = (z-p_1(u))^{m_1} \cdots (z-p_s(u))^{m_s}(1+O(u))dz$ with $c \neq 0$, where $p_1, \ldots, p_s : U \to \Delta_z$ are holomorphic. If $\gamma_{ij}(u)$ is a path from $(p_i(u), u)$ to $(p_j(u), u)$ entirely contained inside $\Delta_z \times \{u\}$, then it is easy to check that

$$\int_{\gamma_{ij}(u)} \varphi_u = \int_{p_i(u)}^{p_j(u)} (z-p_1(u))^{m_1} \cdots (z-p_s(u))^{m_s}(1+O(u))dz \asymp u^{d_{ij}(m_1+\cdots+m_s+1)}$$

where $d_{ij} = \text{ord}_{u=0}(p_i(u) - p_j(u)) > 0$. Hence, if such a $\gamma_{ij}(u)$ is homotopic to a segment with respect to $\varphi_u$, then

$$\ell_{\gamma_{ij}(u)}(\varphi_u)^{-2} = O(|u|^{-2d_{ij}(m_1+\cdots+m_s+1)})$$

and so it diverges polynomially in $|u|$. As the shortest $\gamma_{ij}(u)$ contained in $\Delta_z \times \{u\}$ is certainly a proper segment (at least up to shrinking $U$ further), we conclude that clashing of zeroes contributes with a polynomial divergence to $\ell_{\mathcal{B}^2}$.

Suppose finally that (ii) occurs, and let $C_0'$ be the union of components of $C_0$ touched by the path $\gamma$. The differential $\varphi$ is a section of $\omega_{\mathcal{C}_U/U}$ over $\mathcal{C}_U$ and there exists a maximum $d \geq 0$ such that $\varphi$ is also a section of $\omega_{\mathcal{C}_U/U}(-d C_0') \subset \omega_{\mathcal{C}_U/U}$. So $\varphi$ can be locally written as $u^d \eta$, where $\eta$ is a holomorphic section of $\omega_{\mathcal{C}_U/U}$ on a neighbourhood of $C_0'$ that does not completely vanish on $C_0''$. Clearly, $d > 0$ because $C_0' \subset C_0^0$ by assumption.
Now, let \([0, 1) \to U\) be the radial path \(r \mapsto r\zeta\) for a fixed complex number \(\zeta\) with \(|\zeta| = 1\). The pull-back family over \([0, 1)\) can be topologically trivialized over \((0, 1)\) and \(C_0\) is obtained from \(C_{\zeta/2}\) by shrinking disjoint disconnecting loops to nodes. So the total space is homeomorphic to \(C \times [0, 1) / \sim\), where \(C = C_{\zeta/2}\) and \((x, 0) \sim (x', 0)\) if and only if \(c(x) = c(x')\), where \(c : C_{\zeta/2} \to C_0\) is the shrinking map. Let \(\hat{\gamma}\) be a fixed path inside \(C\) such that \(c(\hat{\gamma}) = \gamma\) and call \(\gamma(r\zeta) \subset C_{r\zeta}\) the path obtained transporting \(\hat{\gamma}\) through the above trivialization.

The differential \(\varphi_{r\zeta} = r^d\zeta^d \eta_{r\zeta}\) in a neighbourhood of \(\gamma(r\zeta)\) and \(\eta_0\) does not vanish on \(C'_0\).

If the geodesic in the class of \(\gamma(r\zeta)\) breaks into a union \(\gamma^1(r\zeta) \cup \cdots \cup \gamma^n(r\zeta)\) of segments with respect to \(\varphi_{r\zeta}\), then

\[
\ell_{\gamma(r\zeta)}(\varphi_{r\zeta}) = \sum_{j=1}^s \left| \int_{\gamma^j(r\zeta)} \varphi_{r\zeta} \right| = r^d \sum_{j=1}^s \left| \int_{\gamma^j(0)} \eta_0 \right| \approx r^d
\]

Thus,

\[
\ell_{\gamma(u)}(\varphi_u)^{-2} = O(|u|^{-2d})
\]

and again it diverges polynomially in \(|u|\). Hence, case (ii) contributes with a polynomial divergence to \(\ell^{-2}_{B}\), and so (d) is proven. \(\square\)

5. Exhaustion functions and strata

5.1. Cohomological dimension of strata. Let \(g \geq 2\) and \(n, k \geq 0\) and let \(m : n + k \to \mathbb{N}\) such that \(|m| = 2g - 2\) and \(m_i > 0\) for all \(i > n\).

Consider the \(\text{Aut}(m)\)-equivariant function \(\Xi_m : \mathcal{H}'_n(m)^* \to \mathbb{R}_+\) defined as \(\Xi_m(C, P, \varphi) := A(C, P, \varphi)\ell^{-2}_B(C, P, \varphi)\) and define

\[
\xi_m := \log(\Xi_m) : \mathcal{H}'_n(m)^* \to \mathbb{R}
\]

The first purpose of this section is to prove the following.

**Proposition 5.1.** \(\xi_m\) descends to an \(\text{Aut}_n(m)\)-invariant exhaustion function on \(\mathbb{P}\mathcal{H}'_n(m)\) that satisfies condition \((\ast)\), and so is strongly \((g + 1)\)-convex.

**Proof.** As seen in Lemma 4.7(b) and Lemma 4.15(a), the functions \(A\) and \(\ell^{-2}_B\) are homogenous of degrees 2 and \(-2\) respectively. So \(\Xi_m\) and \(\xi_m\) descend to \(\mathbb{P}\mathcal{H}'_n(m)\).

Moreover, by Lemma 4.15(d) the function \(\ell^{-2}_B\) blows up along \(\overline{\delta}\) and \(\partial_{\text{red}}\). This shows that \(\Xi\) diverges along \(\overline{\delta}\) and \(\partial_{\text{red}}\). On the other hand, if \((C, P, \varphi) \in \partial_{\text{irr}} \mathcal{H}'_n(m)^*\) acquires a simple pole at a nondisconnecting node with residue \(\rho \neq 0\), then \(A(C, P, \varphi)\) blows up by Lemma 4.7(c). Because the smooth closed \(|\varphi|^2\)-geodesic curve that winds parallel to the node has length \(\rho\), we can conclude that \(\ell^{-2}_B(C, P, \varphi) \geq \frac{1}{|\varphi|^2} > 0\) and so \(\Xi_m\) blows up at \((C, P, \varphi)\). Hence, \(\Xi_m\) and \(\xi_m\) are exhaustion functions on \(\mathbb{P}\mathcal{H}'_n(m)\).

Considering \(\xi_m\) as a function on \(\mathcal{H}'_n(m)^*\), clearly \(i\overline{\partial} \overline{\partial} \xi_m = i\overline{\partial} \overline{\partial} \log(A) + i\overline{\partial} \overline{\partial} \log(\ell^{-2}_B)\). We recall that, by Lemma 4.7(d) and Lemma 4.15(c), the functions \(\log(A)\) and
log(\ell_B^{-2}) satisfy condition weak (\ast) and condition (\ast) respectively. Hence, \xi_m satisfies condition (\ast) by Lemma [4.4(e)] and so \xi_m : \mathbb{P}H'_n(m) \to \mathbb{R} is strictly (g + 1)-convex. Finally, \xi_m is \text{Aut}_n(m)-invariant because both \mathcal{A} and \ell_B^{-2} are.

As a consequence, we obtain our first main result.

**Proof of Theorem C.** By Proposition [5.1] and Theorem [A.6], \mathbb{P}H'_n(m) has Dolbeault cohomological dimension at most \( g \). As the cover \mathbb{P}H'_n(m) \to \mathbb{P}H_n(m) is finite étale and surjective, the same holds for \mathbb{P}H_n(m) by Lemma [A.3].

### 5.2. Good covers of \( \mathbb{P}H'_{g,n} \)

In order to invoke Corollary [A.11] we need to produce an open cover of \( \mathbb{P}H'_{g,n} \), which are subsets of \( \mathcal{H} \). We recall that \( \mathbb{P}H'_{g,n} = \mathbb{P}H'_n(0^n, 1^{2g-2})^{sm} \) is stratified by locally closed submanifolds \( \mathbb{P}H'_n(\sigma) \) of codimension \( d(\sigma) \), where \( \sigma \in \mathfrak{S} \). The first property we require from our cover \( \mathfrak{V} = \{ V_\sigma \mid \sigma \in \mathfrak{S} \} \) is to be adapted to the stratification \( \{ \mathbb{P}H'_n(\sigma) \mid \sigma \in \mathfrak{S} \} \) in the sense of Definition [A.9].

The second property we ask our cover is that for all \( (C, P, [\varphi]) \in \mathcal{V}_\sigma \) all marked points \( P(\sigma^{-1}(j)) \) are “about to clash”.

**Definition 5.2.** Let \( \mathfrak{V} \) be an adapted cover of \( \mathbb{P}H'_{g,n} \) and let \( \mathcal{V}_\sigma \to \mathcal{V}_\sigma \) the induced tautological family of Riemann surfaces. A **clashing datum** for \( \mathcal{V}_\sigma \) is a collection \( \{ \Delta_{j,V_\sigma} \mid 1 \leq j \leq 2g - 2 + n - d(\sigma) \} \) of open subsets of \( \mathcal{C}_{V_\sigma} \) that satisfies the following properties:

1. \( \Delta_{j,V_\sigma} \to \mathcal{V}_\sigma \) is a disk bundle for all \( j = 1, \ldots, 2g - 2 + n - d(\sigma) \);
2. \( \Delta_{j,V_\sigma} \cap \Delta_{h,V_\sigma} = \emptyset \) if \( j \neq h \);
3. \( P \) restricts to \( P_j : V_\sigma \times \sigma^{-1}(j) \to \Delta_{j,V_\sigma} \subset \mathcal{C}_{V_\sigma} \) for every \( j = 1, \ldots, 2g - 2 + n - d(\sigma) \);
4. at each \( (C, P, [\varphi]) \in \mathcal{V}_\sigma \), proper \( \varphi \)-segments contained in

\[
\Delta_{\sigma,[\varphi]} = \bigcup_{j=1}^{2g-2+n-d(\sigma)} \Delta_{j,V_\sigma} \big|_{(C, P, [\varphi])}
\]

are shorter than \( \frac{1}{2g+n} \) all the others and their classes generate \( H_1(\Delta_{\sigma,[\varphi]}, P; \mathbb{R}) \).

A **good cover** is an adapted cover \( \mathfrak{V} \) endowed with a clashing datum for each \( \mathcal{V}_\sigma \in \mathfrak{V} \).

Our purpose is to show the following.

**Lemma 5.3.** There exists a good cover \( \mathfrak{V} \) of \( \mathbb{P}H'_{g,n} \). Moreover, every refinement of a good cover that satisfies property (AS1) in Definition [A.9] is also good.
Clearly, it is enough to work with $\mathbb{C}^*$-invariant subsets of $(\mathcal{H}'_{g,n})^*$ and to produce a good cover $\tilde{\mathfrak{V}} = \left\{ \tilde{V}_{\sigma} \right\}$ of $(\mathcal{H}'_{g,n})^*$ that descends to the wished $\mathfrak{V}$.

**Definition 5.4.** The injectivity radius of $(C, P, \varphi)$ relative to $\sigma$ is the function $R_{\sigma} : \mathcal{H}'_{g,n} \rightarrow \mathbb{R}_{\geq 0}$ defined as

$$R_{\sigma}(C, P, \varphi) := \frac{1}{2} \min \left\{ \text{dist}_{|\varphi|}\{p_i, p_k\} \left| \sigma(i) \neq \sigma(k) \right\} \cup \left\{ \ell_\gamma(\varphi) \left| \gamma \text{ nontrivial loop} \right\} \right\}$$

Notice that $R_{\sigma}$ is 1-homogeneous, it vanishes on $(\partial_{red} \mathcal{H}'_{g,n})^*$ and on those $\mathcal{H}'(\tau)^*$ such that $\sigma \not\equiv \tau$, and it is strictly positive elsewhere.

**Proof of Lemma 5.3.** For the second claim, let $\mathfrak{U} = \{U_{\sigma}\}$ be a refinement of a good cover $\mathfrak{V}$ and assume that $\mathfrak{U}$ satisfies property (AS1). If $\sigma \leq \tau$, then $U_{\sigma} \cap \mathbb{P} \mathcal{H}'_{g,n}(\tau) \neq \emptyset$ and so $U_{\sigma} \cap U_{\tau} \neq \emptyset$. On the other hand, if $\sigma \not\equiv \tau$ and $\tau \not\equiv \sigma$, then $U_{\sigma} \cap U_{\tau} \subset V_{\sigma} \cap V_{\tau} = \emptyset$. Hence, $\mathfrak{U}$ satisfies property (AS2) and so is adapted. As the restriction of a clashing datum to a smaller open subset of $\mathbb{P} \mathcal{H}'_{g,n}$ is still a clashing datum, the result follows.

Now we concentrate on the first claim. Fix $\sigma \in \mathfrak{S}$. For every $j = 1, \ldots, 2g - 2 + n - d(\sigma)$ denote by $i_j \in [1, 2g - 2 + n]$ the smallest integer such that $\sigma(i_j) = j$.

For every $(C, P, \varphi) \in (\mathcal{H}'_{g,n})^*$ at which $R_{\sigma}(\varphi) > 0$, denote by $\Delta_{j,\sigma,\varphi} \subset C$ the ball centered at $p_{i_j}$ of radius $\frac{1}{2g + n(j)} R_{\sigma}(\varphi)$, which is a topological 2-disk by the choice of $R_{\sigma}$. Moreover, $\Delta_{j,\sigma,\varphi} \subset C$ is invariant under rescaling of $\varphi$ because $R_{\sigma}(\varphi)$ is 1-homogeneous.

The disks are disjoint. In fact, if $\exists q \in \Delta_{j,\sigma,\varphi} \cap \Delta_{h,\sigma,\varphi}$ with $j \neq h$, then $R_{\sigma}(\varphi) \leq \frac{1}{2} \text{dist}_{|\varphi|}\{p_{i_j}, p_{i_h}\} \leq \frac{1}{2} \text{dist}_{|\varphi|}\{p_{i_j}, q\} + \frac{1}{2} \text{dist}_{|\varphi|}\{q, p_{i_h}\} < \frac{1}{2g - 2 + n(j)} R_{\sigma}(\varphi)$, which is a contradiction.

Now define

$$\tilde{V}_{\sigma} := \left\{ (C, P, \varphi) \in (\mathcal{H}'_{g,n})^* \mid p_i \in \Delta_{j,\sigma,\varphi} \text{ whenever } \sigma(i) = j \right\}$$

which is clearly an open $\mathbb{C}^*$-invariant neighbourhood of the locally closed stratum $\mathcal{H}'_{\sigma}(\sigma)^*$, and so it descends to an open neighbourhood $V_{\sigma}$ of the locally closed stratum $\mathbb{P} \mathcal{H}'_{\sigma}(\sigma)$.

**Claim.** The cover $\mathfrak{V} = \{V_{\sigma}\}$ is adapted to the stratification, namely $V_{\sigma} \cap V_{\tau} \neq \emptyset$ if and only if $\sigma \preceq \tau$ or $\tau \preceq \sigma$.

Clearly, it is enough to work with $\tilde{\mathfrak{V}} = \left\{ \tilde{V}_{\sigma} \right\}$. Notice that, because $\mathcal{H}'_{\sigma}(\sigma)^* \subset \tilde{V}_{\sigma}$ for all $\sigma \in \mathfrak{S}$, we clearly have $\tilde{V}_{\sigma} \cap \tilde{V}_{\tau} \neq \emptyset$ if $\sigma \preceq \tau$ or $\tau \preceq \sigma$.

Conversely, let’s proceed by contradiction and assume there exists $(C, P, \varphi) \in \tilde{V}_{\sigma} \cap \tilde{V}_{\tau}$ with $\sigma \not\equiv \tau$, and $\tau \not\equiv \sigma$.

As $\sigma \not\equiv \tau$, there exist $i, k \in \{1, \ldots, 2g - 2 + n\}$ such that $\sigma(i) \neq \sigma(k)$ but $\tau(i) = \tau(k)$. The first condition implies that $R_{\sigma}(\varphi) \leq \frac{1}{2} \text{dist}_{|\varphi|}\{p_i, p_k\}$ and the second
condition implies that $\operatorname{dist}_{\varphi}^2(p_i, p_k) < \frac{2}{(2g+n)} R_\tau(\varphi)$, so that $2(2g+n) \cdot R_\tau(C, P, \varphi) < R_\tau(\varphi)$. Vice versa, $\tau \not\subseteq \sigma$ implies $2(2g + n) \cdot R_\tau(\varphi) < R_\sigma(\varphi)$, giving the required contradiction.

Finally, we want to show that $\mathcal{Y}$ is a good cover. As $R_\sigma$ is a continuous function and the restriction of $R_\sigma$ to $\tilde{V}_\sigma$ is positive, the family $\Delta_{j,\sigma,\varphi}$ defines a disk bundle $\Delta_{j,V_\sigma} \to V_\sigma$. Thus, we are left to prove the following.

Claim. The disk bundles $\{\Delta_{j,V_\sigma}\}$ endow $V_\sigma$ with a clashing datum.

The only nontrivial property still to check is the one in Definition 5.2(iv). Let $(C, P, \varphi) \in \tilde{V}_\sigma$ and let $\sigma(k) = j$. Because $p_k \in \Delta_{j,\sigma,\varphi}$, the shortest $\varphi$-geodesic between $p_k$ and $p_i$ is contained inside $\Delta_{j,\sigma,\varphi}$. Hence, $\varphi$-segments contained in $\Delta_{j,\sigma,\varphi}$ generate $H_1(\Delta_{j,\sigma,\varphi}, P; \mathbb{R})$.

Finally, fix $(C, P, \varphi) \in \tilde{V}_\sigma$. As $\Delta_{j,\sigma,\varphi}$ is convex, for every pair of points $q, q' \in \Delta_{j,\sigma,\varphi}$, there always exists a geodesic that joins $q$ and $q'$ and that is completely contained inside $\Delta_{j,\sigma,\varphi}$. Moreover, the shortest such geodesic has length less than $\frac{1}{2g+n} R_\sigma(\varphi)$.

In fact, segments completely contained inside $\Delta_{j,\sigma,\varphi}$ have length at most $c(\varphi) = \frac{1}{2g+n} R_\sigma(\varphi)$.

Consider then a segment $\gamma$ on $(C, P, \varphi) \in \tilde{V}_\sigma$ joining $p_i$ and $p_k$. Assume first that $\sigma(i) = \sigma(k) = j$ and let $\beta$ the shortest arc from $p_k$ to $p_i$ contained inside $\Delta_{j,\sigma,\varphi}$. If $\gamma$ is not contained inside $\Delta_{j,\sigma,\varphi}$, then $\gamma \cup \beta$ is a nontrivial closed loop and so $\ell_\gamma(\varphi) + \ell_\beta(\varphi) \geq 2R_\sigma(\varphi)$, which implies $\frac{1}{2g+n} \ell_\gamma(\varphi) \geq \frac{1}{2g+n}(2 - \frac{1}{2g+n}) R_\sigma(\varphi) > c(\varphi)$. On the contrary, suppose that $\sigma(i) = j \neq h = \sigma(k)$, so that $\operatorname{dist}_{\varphi}^2(p_i, p_k) \geq \operatorname{dist}_{\varphi}^2(p_i, p_i) - \operatorname{dist}_{\varphi}^2(p_i, p_k) \geq \left(2 - \frac{1}{2g+n}\right) R_\sigma(\varphi) > R_\sigma(\varphi)$. Hence, $\frac{1}{2g+n} \ell_\gamma(\varphi) \geq \frac{1}{2g+n} \operatorname{dist}_{\varphi}^2(p_i, p_k) > c(\varphi)$.

5.3. Cohomological dimension of $\mathcal{M}_{g,n}$. The main purpose of this subsection is to exhibit a suitable refinement of the good cover found above, whose open sets carry exhaustion functions which are somehow compatible with each other.

Proposition 5.5. There exists a good open cover $\mathcal{U} = \{U_\sigma\}$ of $\mathcal{P} H'_{g,n}$ and exhaustion functions $\xi_\sigma : U_\sigma \to \mathbb{R}$ that satisfy condition $(\ast)$ for all $\sigma \in \mathcal{S}$.

Before proving the proposition, we show how this leads to the wished conclusion.

Proof of Theorems A and B Let $\sigma_* = (\sigma_0 \preceq \sigma_1 \preceq \cdots \preceq \sigma_d)$ and call $U_{\sigma_*}$ the intersection $U_{\sigma_0} \cap \cdots \cap U_{\sigma_d}$. Thus, if $U_{\sigma_*} \neq \emptyset$, then necessarily $d \leq 2g - 3 + \varsigma_n$.

Define

$$\xi_{\sigma_*} : U_{\sigma_*} \to \mathbb{R}$$

as $\xi_{\sigma_*} := \xi_{\sigma_0} + \cdots + \xi_{\sigma_d}$, which is an exhaustion function. Because of Proposition 5.3, all functions $\xi_{\sigma_i}$ satisfy condition $(\ast)$. By Lemma 4.4(e) so does their sum $\xi_{\sigma_*}$, which is thus $(g + 1)$-convex. Hence, Corollary A.11 applied to $\mathcal{U}$ implies that
coh-dim_{Dol}(\mathbb{P}H'_{g,n}) \leq (2g - 3 + \varsigma_n) + g = 3g - 3 + \varsigma_n. Applying Lemma \text{A.3}(\text{a}) to the map \mathbb{P}H'_{g,n} \to \mathbb{P}H_{g,n}, we also obtain coh-dim_{Dol}(\mathbb{P}H_{g,n}) \leq 3g - 3 + \varsigma_n and so Theorem \text{B} follows.

Finally, Theorem \text{A} is a consequence of Theorem \text{B} and Corollary \text{A.4} applied to the projective bundle \mathbb{P}H_{g,n} \to M_{g,n} with (g - 1)-dimensional fibers. \hfill \square

Let \((C, P, [\varphi])\) be a point of \(V_\sigma\). We recall that \(\Delta_{\sigma,\varphi}\) denotes the union of the disks \(\Delta_{j,\sigma,\varphi}\), defined in the proof of Lemma \text{5.3}.

Consider the arcs in \((C, P)\) entirely contained inside \(\Delta_{\sigma,\varphi}\). A collection of such arcs will be called a \(\Delta\)-basis if their classes form a basis of \(H_1(\Delta_{\sigma,\varphi}, P; \mathbb{R})\). We will denote by \(\mathcal{B}_{\sigma,\varphi}\) the set of \(\Delta\)-bases.

On the other hand, consider the arcs in \((C, P)\) that cannot be realized inside \(\Delta_{\sigma,\varphi}\). A collection of such arcs will be called a \(\sigma\)-relative basis if their classes form a basis of \(H_1(C, \Delta_{\sigma,\varphi}; \mathbb{R})\). We will denote by \(\mathcal{B}'_{\sigma,\varphi}\) the set of \(\sigma\)-relative bases.

Remark 5.6. For a point \((C, P, [\varphi]) \in V_\sigma\), segments contained inside each \(\Delta_{\sigma,\varphi}\) disk are shorter than \(\frac{1}{2g+n}\) any other segment not contained in \(\Delta_{\sigma,\varphi}\) essentially by definition. Thus, a basis \(B\) of \(H_1(C, P; \mathbb{Z})\) of segments that maximizes \(\sum_{\gamma \in \mathcal{B}} \frac{4}{\ell^2_{\gamma}}\) at \([\varphi]\) must be obtained as \(B = B^\Delta \cup B',\) with \(B^\Delta \in \mathcal{B}^\Delta_{\sigma,\varphi}\) and \(B' \in \mathcal{B}'_{\sigma,\varphi}\).

Consider the two functions \(\eta_\sigma, \zeta_\sigma : V_\sigma \to \mathbb{R}\) defined as

\[
\eta_\sigma := \sup_{B' \in \mathcal{B}'} \eta_{\sigma, B'}, \quad \eta_{\sigma, B'} := \sum_{\gamma \in B'} \frac{A}{\ell^2_{\gamma}} \\
\zeta_\sigma := \sup_{B = B^\Delta \cup B'} \zeta_{\sigma, B}, \quad \zeta_{\sigma, B} := \left(\sum_{\beta \in B^\Delta} |P_{\beta}|^2\right) \left(\sum_{\gamma \in B'} \frac{1}{\ell^2_{\gamma}}\right)
\]

meaning that \(\mathcal{B}' = \mathcal{B}'_{\sigma,\varphi}\) and \(B^\Delta = \mathcal{B}^\Delta_{\sigma,\varphi}\), whenever we are evaluating the above functions at \((C, P, [\varphi])\). Notice that, as usual, the sup over \(\mathcal{B}'\) is always attained at bases of \(H_1(C, \Delta_{\sigma,\varphi}; \mathbb{R})\) made of segments.

Lemma 5.7. Let \(c > 0\) and let \(\chi : [0, c) \to \mathbb{R}_+\) be smooth function with \(\chi(0) = 1\) such that \(\chi' > 0\) and \((\log \chi)'' > 0\). Then the function \(\xi_\sigma : U_\sigma \to \mathbb{R}\) defined as

\[\xi_\sigma := \log(\eta_\sigma + \chi \circ \zeta_\sigma)\]

satisfies condition (\(*)\), where \(U_\sigma := \{\zeta_\sigma < c\} \subseteq V_\sigma\).

Clearly, for every \(B = B' \cup B^\Delta\) with \(B' \in \mathcal{B}'\) and \(B^\Delta \in \mathcal{B}^\Delta\), we can locally define \(\xi_{\sigma, B} := \log(\eta_{\sigma, B'} + \chi \circ \zeta_{\sigma, B})\) so that \(\xi_\sigma = \sup_B \xi_{\sigma, B}\).

Proof. Let \((C, P, \varphi) \in \tilde{V}_\sigma\). As taking the sup can only improve subharmonicity, in order to check condition (\*) at \((C, P, \varphi)\) it is enough to work with a fixed basis \(B = B' \cup B^\Delta\) such that \(\eta_\sigma(\varphi) = \eta_{\sigma, B'}(\varphi)\) and \(\zeta_\sigma(\varphi) = \zeta_{\sigma, B}(\varphi)\). Thus, arcs in \(B'\)
are realized by proper \( \varphi \)-segments and so \( \eta_{\sigma,B'} \) and \( \zeta_{\sigma,B} \) are smooth near \((C,P,\varphi)\). Hence, it is sufficient to compute the complex Hessian.

As usual, we can work in period coordinates in a small open neighbourhood of \((C,P,\varphi)\) by Proposition \ref{prop:4.15}. Let \( \Pi_\varphi \subset H^1(C,P) \) be the kernel of the projection onto \( H^{0,1}_\varphi(C) \) and let \( \Pi'_\varphi \subset \Pi_\varphi \) be a complement of the line \( \mathbb{C}(\varphi) \). The conclusion will follow if we prove that the restriction of \( i\partial\bar{\partial}\log(\xi_{\sigma,B})_\varphi \) to \( \Pi'_\varphi \) is positive-definite. Consider a deformation \( (\varphi_\varepsilon) = (\varphi + \varepsilon \varphi) \) for a small \( \varepsilon \in \mathbb{C} \), with \( 0 \neq (\varphi) \in \Pi'_\varphi \). We want to show that \( i\partial\bar{\partial}\log(\eta_{\sigma,B'})_\varphi(\varphi_\varepsilon,\varphi) \) and \( i\partial\bar{\partial}\log(\chi \circ \xi_{\sigma,B})_\varphi(\varphi_\varepsilon,\varphi) \) are \( \geq 0 \) and in fact at least one of the two is strictly positive. By Lemma \ref{lemma:4.4}(f), we can then conclude that \( i\partial\bar{\partial}\log(\xi_{\sigma,B})_\varphi(\varphi,\varphi) > 0 \).

A computation analogous to the ones in Lemma \ref{lemma:4.7}(c) and in Lemma \ref{lemma:4.15}(c) shows that \( i\partial\bar{\partial}\log(\eta_{\sigma,B'})_\varphi(\varphi,\varphi) \geq 0 \). On the other hand,

\[
i\partial\bar{\partial}\log(\chi \circ \xi_{\sigma,B})_\varphi = i\left( \chi' \circ \xi_{\sigma,B} \right) \partial\bar{\partial}\xi_{\sigma,B} + \left[ \left( \chi' \circ \xi_{\sigma,B} \right) \chi'' \circ \xi_{\sigma,B} - \left( \chi' \circ \xi_{\sigma,B} \right)^2 \right] \partial\xi_{\sigma,B} \wedge \partial\xi_{\sigma,B} \frac{\chi \circ \xi_{\sigma,B}}{(\chi \circ \xi_{\sigma,B})^2}
\]

As \( \chi, \chi' > 0 \) and \( \chi \chi'' - (\chi')^2 = (\log \chi)' > 0 \), we certainly have \( i\partial\bar{\partial}\log(\chi \circ \xi_{\sigma,B})_\varphi(\varphi_\varepsilon,\varphi) \geq 0 \). Suppose now by contradiction that \( i\partial\bar{\partial}\log(\eta_{\sigma,B'})_\varphi(\varphi,\varphi) = 0 \) and \( i\partial\bar{\partial}\log(\chi \circ \xi_{\sigma,B})_\varphi(\varphi,\varphi) = 0 \).

The first condition can be satisfied only if there is a fixed constant \( c \in \mathbb{C} \) such that \( \int_\gamma \varphi = c \int_{\gamma'} \varphi \) for all \( \gamma \in B' \). But then \( \xi_{\sigma,B}(|\varphi|) \) reduces to

\[
\frac{a}{|1 + c\varepsilon|^2} \sum_{\beta \in B^A} |P_\beta(\varphi_\varepsilon)|^2
\]

where \( a = \sum_{\gamma \in B'} \ell_\gamma(\varphi)^{-2} \) is a constant and we deduce that \( (i\partial\bar{\partial}\xi_{\sigma,B})_\varphi(\varphi_\varepsilon,\varphi) = 0 \) only if \( \partial_\varepsilon [(1 + c\varepsilon)^{-1}P_\beta(\varphi_\varepsilon)] = 0 \) for all \( \beta \in B^A \). This is equivalent to asking that \( \int_\beta \varphi = c \int_\beta \varphi \) for all \( \beta \in B^A \).

Because \( B = B^A \cup B' \) is a basis of \( H_1(C,P;\mathbb{R}) \), this implies that \( (\varphi) = c(\varphi) \) and so we have reached a contradiction. \( \square \)

**Lemma 5.8.** View \( V_\sigma \subset \mathbb{P}H^\prime_{g,n} \) and let \( \{(C_{\sigma},P_{\sigma},[\varphi_0])\}_{s \in \mathbb{N}} \subset V_\sigma \) be a sequence that converges to a point \((C,P,[\varphi])\) at the boundary of \( V_\sigma \).

(a) If \((C,P,[\varphi])\) lies in a deeper stratum in \( \partial \mathbb{P}H^\prime_{g,n} \), then \( \xi_\sigma(C_{\sigma},P_{\sigma},[\varphi_0]) \to +\infty \) as \( s \to \infty \).

(b) If \((C,P,[\varphi]) \in \mathbb{P}H^\prime_{g,n} \) and it does not lie in a deeper stratum, then \( \lim \inf \xi_\sigma(C_{\sigma},P_{\sigma},[\varphi_0]) \geq c \), where \( c > 0 \) is a constant that depends only on \( g \) and \( n \).

**Proof.** The same argument as in the proof of Proposition \ref{prop:5.1} shows that, in case (a), \( \eta_\sigma(C_{\sigma},P_{\sigma},[\varphi_0]) \to +\infty \), and so does \( \xi_\sigma(C_{\sigma},P_{\sigma},[\varphi_0]) \).

In case (b), \( \xi_\sigma \) is continuous at \((C,P,[\varphi])\) and so it is enough to prove that \( \xi_\sigma(C,P,[\varphi]) \geq c \) for a suitable \( c = c(g,n) > 0 \).

Because \((C,P,[\varphi]) \) belongs to the boundary of \( V_\sigma \) but not to a deeper stratum, there exist two distinct \( i,k \in \{1,\ldots,2g-2+n\} \) such that \( \sigma(i) = \sigma(k) = j \) and
Thus, one can speak of de Rham cohomology $\tilde{X}$ carries on, and affine connections are defined in the usual way.

The correspondence between locally free $E$-sheaves $A$-flat vector bundle and $H^\beta$ looks like $\{\cdots\}$ we have that $U$ is an exhaustion function because of Lemma 5.8 and of the above choice of $\chi$. Moreover, $\xi_\sigma: U_\sigma \to \mathbb{R}$ is an exhaustion function because of Lemma 5.8 and of the above choice of $\chi$. Finally, $\xi_\sigma$ satisfies condition $(\ast)$ because of Lemma 5.7.

Appendix A. Cohomological dimension

A.1. Cohomology of orbifolds. Let $X$ be a smooth orbifold. Thus $X$ locally looks like $[\tilde{U}/G]$, where $\tilde{U}$ is an open subset of a Euclidean space and $G$ is a finite group acting on $\tilde{U}$. In particular, change of charts $\tilde{U}_i \leftarrow \tilde{U}_{ij} \to \tilde{U}_j$ are étale and so local diffeomorphisms.

By definition, smooth functions on $[\tilde{U}/G]$ are $G$-invariant smooth functions on $\tilde{U}$: this defines a sheaf $\mathcal{E}_X$ of smooth functions on $X$. Analogously, one can define the sheaves $\mathcal{A}_X^q$ of smooth differential $q$-form on $X$.

The correspondence between locally free $\mathcal{E}_X$-modules and smooth vector bundles on $X$ carries on, and affine connections are defined in the usual way.

Thus, one can speak of de Rham cohomology $H^\ast_{dR}(X; \mathbb{L})$ of the orbifold $X$ with coefficients in a flat vector bundle $\mathbb{L}$ as the cohomology of the complex

$$0 \to (\mathcal{A}_X^0 \otimes \mathcal{E}_X \mathbb{L})(X) \xrightarrow{d} (\mathcal{A}_X^1 \otimes \mathcal{E}_X \mathbb{L})(X) \xrightarrow{d} (\mathcal{A}_X^2 \otimes \mathcal{E}_X \mathbb{L})(X) \xrightarrow{d} \ldots$$

The case of an orbifold $X = [\tilde{X}/G]$ which is a global quotient of a manifold $\tilde{X}$ by a finite group $G$ is rather special and easier to deal with. Indeed, if $\rho: \tilde{X} \to X$ is the quotient map and $\mathbb{L} \to X$ is a flat vector bundle, then $\rho^\ast \mathbb{L} \to X$ is a $G$-equivariant flat vector bundle and $H^\ast_{dR}(X; \mathbb{L}) = H^\ast_{dR}(\tilde{X}; \rho^\ast \mathbb{L})^G$. Moreover, for any flat vector bundle $\tilde{\mathbb{L}} \to \tilde{X}$, the push-forward $\rho_\ast \tilde{\mathbb{L}} \to X$ is also a flat vector bundle because $\rho$.
is finite étale (in the orbifold sense) and surjective, and so $\rho_*\tilde{L} \to X$ is a flat vector bundle and $H^q_{dR}(\tilde{X}; \tilde{L}) = H^q_{dR}(X; \rho_*\tilde{L})$. Hence, de Rham cohomology theories of $X$ and $\tilde{X}$ with coefficients in flat vector bundles are somehow equivalent.

All considerations can be transported to the complex-analytic world, when $X$ is a complex-analytic orbifold. Indeed, in this case it will make sense to speak of the sheaf $\mathcal{O}_X$ of holomorphic functions, of the sheaf $\mathcal{A}^{p,q}_X$ of smooth differential $(p,q)$-forms, holomorphic vector bundles and of Dolbeault cohomology groups $H^0_{\bar{\partial}}(X; E)$ with coefficients in the holomorphic vector bundle $E$ over $X$ as the cohomology of the complex

$$0 \to (\mathcal{A}^{0,0}_X \otimes \mathcal{E}_X)(X) \xrightarrow{\bar{\partial}} (\mathcal{A}^{0,1}_X \otimes \mathcal{E}_X)(X) \xrightarrow{\bar{\partial}} (\mathcal{A}^{0,2}_X \otimes \mathcal{E}_X)(X) \to \ldots$$

Again, if $X = [\tilde{X}/G]$ is a global quotient, then $H^0_{\bar{\partial}}(X; E) = H^0_{\bar{\partial}}(\tilde{X}; \rho^*E)^G$, where $\rho : \tilde{X} \to X$. Moreover, if $\tilde{E} \to \tilde{X}$ is a holomorphic vector bundle, then $\rho_*\tilde{E} \to X$ is too and $H^0_{\bar{\partial}}(\tilde{X}; \tilde{E}) = H^0_{\bar{\partial}}(X; \rho_*\tilde{E})$.

A.2. Cohomological dimensions. Let $X$ be a complex manifold of (complex) dimension $N$.

**Definition A.1.** The Dolbeault cohomological dimension of $X$ is

$$\text{coh-dim}_{Dol}(X) := \max\{q \in \mathbb{N} \mid H^0_{\bar{\partial}}(X; E) \neq 0 \text{ for some hol. vector bundle } E\}.$$ 

Clearly, $\text{coh-dim}_{Dol}(X) \leq N$ and the equality is attained if and only if $X$ has a compact component of top dimension. On the opposite extreme, if $X$ is Stein, then $\text{coh-dim}_{Dol}(X) = 0$. It is also easy to see that, if $Z \subset X$ is a closed complex submanifold, then $\text{coh-dim}_{Dol}(Z) \leq \text{coh-dim}_{Dol}(X)$.

A relation between the Dolbeault cohomological dimension and the de Rham cohomological dimension

$$\text{coh-dim}_{dR}(X) := \max\{s \in \mathbb{N} \mid H^s_{dR}(X; \mathbb{L}) \neq 0 \text{ for some } \mathbb{C}\text{-local system } \mathbb{L}\}$$

is the following.

**Lemma A.2.** Let $X$ be a complex-analytic orbifold. Then

$$\text{coh-dim}_{dR}(X) \leq \text{coh-dim}_{Dol}(X) + \dim_X(X).$$

**Proof.** The holomorphic de Rham complex on $X$

$$0 \to \mathcal{O}_X = \Omega^0_X \xrightarrow{\partial} \Omega^1_X \xrightarrow{\partial} \ldots \xrightarrow{\partial} \Omega^N_X \to 0$$

is a resolution by holomorphic vector bundles of the locally constant sheaf $\mathbb{C}_X$. So, tensoring by a $\mathbb{C}$-local system $\mathbb{L}$ on $X$, we obtain the spectral sequence

$$E^q_2 = H^{0,q}_{\bar{\partial}}(X; \mathbb{L} \otimes \Omega^p_X) \Longrightarrow H^{p+q}_{dR}(X; \mathbb{L})$$
and so the conclusion follows as \( E_2^{2,p} = 0 \) unless \( p \leq \dim_C(X) \) and \( q \leq \text{coh-dim}_{\text{Dol}}(X) \). \( \square \)

It is well-known that \( \text{coh-dim}_{dR}(X) \leq 2N \) and the equality holds if and only if \( X \) has a compact top-dimensional component.

A.3. **Fibrations.** Let \( \pi : Y \to X \) be a holomorphic map of complex manifolds whose fibers have dimension \( \leq r \). Then Leray spectral sequence \( H_d^{a,q}(X; R^p \pi_* E) \to H_d^{0,p+q}(Y; E) \) immediately shows that \( \text{coh-dim}_{\text{Dol}}(Y) \leq \text{coh-dim}_{\text{Dol}}(X) + r \).

With extra assumptions on \( \pi \), we can give an exact estimate. (Actually, the hypotheses can be weakened but the following is enough for our purposes.)

**Lemma A.3.** Assume \( X \) and \( Y \) connected.

(a) If \( \pi \) is finite and surjective, then \( \text{coh-dim}_{\text{Dol}}(Y) = \text{coh-dim}_{\text{Dol}}(X) \).

(b) If \( \pi \) is submersive and with compact connected fibers, then \( \text{coh-dim}_{\text{Dol}}(Y) = \text{coh-dim}_{\text{Dol}}(X) + r \).

**Proof.** In case (a), the inequality \( \text{coh-dim}_{\text{Dol}}(Y) \leq \text{coh-dim}_{\text{Dol}}(X) \) follows from the above discussion. For the reverse inequality notice that, if \( \pi \) is finite and surjective of degree \( d \geq 1 \), then the trace \( \text{tr}_{Y/X} : \pi_* \mathcal{O}_Y \to \mathcal{O}_X \) is a map of \( \mathcal{O}_X \)-modules and the composition \( \mathcal{O}_X \to \pi_* \mathcal{O}_Y \xrightarrow{\text{tr}} \mathcal{O}_X \) is the multiplication by \( d \) (and so an isomorphism, as we are in characteristic 0). Tensoring by a coherent sheaf \( F \) on \( X \), we obtain \( F \to \pi_* \pi^* F \to F \), which is again the multiplication by \( d \). Hence, \( H_d^{a,q}(X, F) \to H_d^{a,q}(X, \pi_* \pi^* F) \to H_d^{a,q}(X, F) \) is also an isomorphism. Thus, if \( H_d^{0,q}(X, F) \neq 0 \), then \( H_d^{0,q}(Y, \pi^* F) = H_d^{0,q}(X, \pi_* \pi^* F) \neq 0 \), which shows that \( \text{coh-dim}_{\text{Dol}}(Y) \geq \text{coh-dim}_{\text{Dol}}(X) \).

In case (b), let \( \omega_\pi \) be the line bundle of vertical \((r,0)\)-holomorphic forms, so that \( R^p \pi_* \omega_\pi = 0 \) for \( p \neq r \) and \( R^p \pi_* \omega_\pi \cong \mathcal{O}_X \) by (fiberwise) Serre duality. Applying Leray spectral sequence to \( E = \omega_\pi \otimes \pi^* F \), and using the projection formula \( R^p \pi_* E \cong F \otimes R^p \pi_* \omega_\pi \), we obtain \( H^q(X; F) \cong H^{r+q}(Y; E) \), and so \( \text{coh-dim}_{\text{Dol}}(Y) \geq r + \text{coh-dim}_{\text{Dol}}(X) \).

**Lemma A.3(b) has the following immediate consequence.**

**Corollary A.4.** Let \( E \to X \) be a holomorphic vector bundle of rank \( r + 1 \) and let \( \mathbb{P}E \) be its projectivization. Then \( \text{coh-dim}_{\text{Dol}}(\mathbb{P}E) = r + \text{coh-dim}_{\text{Dol}}(X) \).

A.4. **\( q \)-convex exhaustions.** Let \( X \) be a complex manifold and \( \xi : X \to \mathbb{R} \) be a continuous function. We recall that \( \xi \) is an **exhaustion function** if it is proper and bounded from below.

**Definition A.5.** We say that a smooth \( \xi \) is strongly \( q \)-convex if its complex Hessian \( \sqrt{\bar{\partial} \partial \bar{\partial} \xi} \) has at most \( q - 1 \) nonpositive eigenvalues at each point of \( X \). We say that \( X \) is strongly \( q \)-complete if it admits a smooth \( q \)-convex exhaustion function.
We would like to elaborate upon the following result (see also Chapter IX, Corollary 4.11 [5]).

**Theorem A.6** (II). Let $X$ be a strongly $(q+1)$-complete complex manifold. Then

$$\operatorname{coh-dim}_{\text{Dol}}(X) \leq q.$$ 

We will need to be able to work with nonsmooth functions; this is relatively minor and standard.

**Lemma A.7.** Let $X$ be a complex manifold of dimension $N$ and let $\xi : X \to \mathbb{R}$ be a continuous function such that for every $x \in X$ there exists a locally closed complex submanifold $L_x \subset X$ through $x$ of dimension $N - q + 1$ such that $\xi|_{L_x}$ is strictly plurisubharmonic. Then, for every $\delta > 0$ there exists a smooth $\tilde{\xi} : X \to \mathbb{R}$ such that $|\tilde{\xi}(x) - \xi(x)| < \delta$ and $\tilde{\xi}|_{L_x}$ is still strictly plurisubharmonic for every $x \in X$. Thus, $\tilde{\xi}$ is $q$-convex.

In light of the above Lemma A.7, we will call strongly $q$-convex such a (possibly) nonsmooth $\xi$ too.

**Remark A.8.** We will be interested in a function $\xi : X \to \mathbb{R}$ that satisfies:

- for every $x \in X$ there exist an open neighbourhood $U$ of $x$ and finitely many smooth functions $\xi_i : U \to \mathbb{R}$ such that $\xi|_U = \max\{\xi_i\}$;
- there exists a complex submanifold $L_x \subset U$ through $x$ of dimension $N - q + 1$ such that $\xi_i|_{L_x}$ is strongly plurisubharmonic for all $i$.

For every $x \in X$, such a $\xi|_{L_x}$ is strongly plurisubharmonic (see [3], Chapter I, Lemma 5.18(e)). So, Lemma A.7 applies to $\xi$ and the smooth $\tilde{\xi}$ thus obtained is an exhaustion function whenever $\xi$ is.

**A.5. Stratifications and coverings.** Let $X$ be a complex manifold and suppose that $X$ is stratified through locally closed strata $X_\sigma$, where $\sigma \in \mathcal{S}$ and $\mathcal{S}$ is a partially ordered set of indices, so that $X_\tau \subset X_\sigma$ if and only if $\tau \preceq \sigma$.

**Definition A.9.** An open cover $\mathcal{U} = \{U_\sigma \mid \sigma \in \mathcal{S}\}$ of $X$ is adapted to the stratification $\{X_\sigma \mid \sigma \in \mathcal{S}\}$ if it satisfies the following two properties:

- (AS1) $U_\sigma$ is an open neighbourhood of $X_\sigma$ for all $\sigma \in \mathcal{S}$;
- (AS2) $U_\sigma \cap U_\tau \neq \emptyset$ if and only if $\sigma \preceq \tau$ or $\tau \preceq \sigma$.

We state the following for Dolbeault cohomology even though it holds in greater generality.

**Lemma A.10.** The Dolbeault cohomological dimension of $X$ can be estimated as follows

$$\operatorname{coh-dim}_{\text{Dol}}(X) \leq \max\left\{\operatorname{coh-dim}_{\text{Dol}}(U_{\sigma_0} \cap \cdots \cap U_{\sigma_d}) + d \mid \sigma_0 \preceq \sigma_1 \preceq \cdots \preceq \sigma_d\right\}$$
Proof. By Mayer-Vietoris spectral sequence applied to the cover \( \mathcal{U} \) for \( E \)-valued \((0,q)\)-differential forms

\[
E_{2}^{q,d} = \bigoplus_{\sigma_{0} \preceq \cdots \preceq \sigma_{d}} H_{\mathcal{U}}^{0,q}(U_{\sigma_{0}} \cap \cdots \cap U_{\sigma_{d}}, E) \implies H_{\mathcal{U}}^{0,q+d}(X; E)
\]

the result immediately follows.

We will apply the above estimate to the following particular case.

Corollary A.11. If each nonempty \( U_{\sigma_{0}} \cap \cdots \cap U_{\sigma_{d}} \) is \((q + 1)\)-convex, then

\[
\text{coh-dim}_{\text{Dol}}(X) \leq q + d_{\text{max}}
\]

where \( d_{\text{max}} \) is the maximum depth of the stratification, i.e. \( d_{\text{max}} = \max \{ d \mid \sigma_{0} \preceq \cdots \preceq \sigma_{d} \} \).

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