Exceptional Times for the Dynamical Discrete Web

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Abstract

The dynamical discrete web (DyDW), introduced in recent work of Howitt and Warren, is a system of coalescing simple symmetric one-dimensional random walks which evolve in an extra continuous dynamical time parameter $\tau$. The evolution is by independent updating of the underlying Bernoulli variables indexed by discrete space-time that define the discrete web at any fixed $\tau$. In this paper, we study the existence of exceptional (random) values of $\tau$ where the paths of the web do not behave like usual random walks and the Hausdorff dimension of the set of exceptional such $\tau$. Our results are motivated by those about exceptional times for dynamical percolation in high dimension by Häggstrom, Peres and Steif, and in dimension two by Schramm and Steif. The exceptional behavior of the walks in the DyDW is rather different from the situation for the dynamical random walks of Benjamini, Häggstrom, Peres and Steif. For example, we prove that the walk from the origin $S_0^\tau$ violates the law of the iterated logarithm (LIL) on a set of $\tau$ of Hausdorff dimension one. We also discuss how these and other results extend to the dynamical Brownian web, the natural scaling limit of the DyDW.
1 Introduction

In this paper, we present a number of results concerning a dynamical version of coalescing random walks, which was recently introduced in [HoW07]. Our results concern sets of dynamical times of Hausdorff dimension less than or equal to one (and of zero Lebesgue measure) where the system of coalescing walks behaves exceptionally. The results are analogous to and were motivated by the model of dynamical percolation and its exceptional times [HPS97, SS05]. In this section, we define the basic model treated in this paper, which we call the dynamical discrete web (DyDW), recall some facts about dynamical percolation, and then briefly describe our main results. The justification for calling this model a discrete web is that there is a natural scaling limit, the dynamical Brownian web (DyBW), which was proposed by Howitt and Warren in [HoW07] and completely constructed in [NRS08]. As we shall explain (see Section 6), the exceptional times results for the DyDW extend to the continuum DyBW.

We note that exceptional times for other dynamical versions of random walks in various spatial dimensions have been studied in [BHPS03, Ho06, AH06] and elsewhere, but, as we shall see, these are quite different from the dynamical random walks of the DyDW.

The Discrete Web

The discrete web is a collection of coalescing one-dimensional simple random walks starting from every point in the discrete space-time domain $\mathbb{Z}^2_{\text{even}} = \{(x,t) \in \mathbb{Z}^2 : x + t \text{ is even}\}$. The Bernoulli percolation-like structure is highlighted by defining $\xi_{x,t}$ for $(x,t) \in \mathbb{Z}^2_{\text{even}}$ to be the increment of the random walk at location $x$ at time $t$. These Bernoulli variables are symmetric and independent and the paths of all the coalescing random walks can be reconstructed by assigning to each point $(x,t)$ an arrow from $(x,t)$ to $\{x + \xi_{x,t}, t + 1\}$ and considering all the paths starting from arbitrary points in $\mathbb{Z}^2_{\text{even}}$ that follow the arrow configuration $\mathcal{N}$.

The Dynamical Discrete Web

In the DyDW, there is, in addition to the random walk discrete time parameter, an additional (continuous) dynamical time parameter $\tau$. The system starts at $\tau = 0$ as an ordinary DW and then evolves in $\tau$ by randomly switching the direction of each arrow at a fixed rate independently of all other arrows. We will generally do the switching by having at each $(x,t) \in \mathbb{Z}^2_{\text{even}}$ a Poisson clock ring at rate one and then reset the direction of the arrow at random; thus the rate of switching will be $1/2$. This amounts to extending the percolation substructure $\xi^0$ to time varying functions $\xi^\tau$ defining a (right continuous) dynamical arrow configuration $\tau \mapsto \mathcal{N}(\tau)$ and $W(\tau)$, the dynamical discrete web at time $\tau$, is defined as the web constructed from $\mathcal{N}(\tau)$. 

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If one follows the arrows starting from the (space-time) origin $(0,0)$, the dynamical path $S^\tau_0$ begins at $\tau = 0$ as a simple symmetric random walk and then evolves dynamically in $\tau$. At any fixed time $\tau$, $S^\tau_0$ has the same law as at time $\tau = 0$. As a consequence, if $\mu$ is the probability distribution of a simple symmetric random walk starting from the origin and $A$ is any event with $\mu(A) = 1$, we have for any deterministic $\tau$ that $\mathbb{P}(S^\tau_0 \in A) = 1$. By a straightforward application of Fubini’s Theorem this implies that
\[ \mathbb{P}(S^\tau_0 \in A \text{ for Lebesgue a.e. } \tau) = 1. \] (1.1)
Following [BHPS03], for any event such that (1.1) holds, a natural question is whether (1.1) can be strengthened to
\[ \mathbb{P}(S^\tau \in A \text{ for all } \tau \geq 0) = 1, \] (1.2)
i.e., do there exist some exceptional times $\tau$ at which $S^\tau_0$ violates some almost sure properties of the standard random walk? or stated differently, is the random walk sensitive to the dynamics introduced on the DW?

**Analogies With Dynamical Percolation**

Similar questions have been investigated in percolation. Static (site) percolation models are defined also in terms of independent Bernoulli variables $\xi^0_z$, indexed by points $z$ in some $d$-dimensional lattice, which in general are asymmetric with parameter $p$. There is a critical value $p_c$ when the system has a transition from having an infinite cluster (connected component) with probability zero to having one with probability one. It is expected that at $p = p_c$ there are no infinite clusters and this is proved for $d = 2$ and for high $d$ (see, e.g., [Gr99]). In dynamical percolation, one extends $\xi^0_z$ to time varying functions $\xi^\tau_z$, as in the case of coalescing walks, except that the transition rates for the jump processes $\xi^\tau_z$ are chosen to have the critical asymmetric $(p_c, 1-p_c)$ distribution to be invariant.

The question raised in [HPS97] was whether there were exceptional times when an infinite cluster (say, one containing the origin) occurs, even though this does not occur at deterministic times. This was answered negatively in [HPS97] for large $d$ and, more remarkably, was answered positively by Schramm and Steiff for $d = 2$ in [SS05], where they further obtained upper and lower bounds on the Hausdorff dimension (as a subset of the dynamical time axis) of these exceptional times. In [GPS08], the exact Hausdorff dimension was obtained.

**Main Results**

We apply in this paper the approaches used for dynamical percolation to the dynamical discrete web. Although we restrict attention to one-dimensional random walks whose paths are in two-dimensional space-time and hence analogous to $d = 2$ dynamical percolation, we use both the high $d$ and $d = 2$ methods of [HPS97] [SS05].
- Tameness. A natural initial question is whether there might be exceptional dynamical times $\tau$ for which the walk from the origin $S_0^\tau(t)$ is transient (say to $+\infty$). Our first main result (see Theorem 2.1), modeled after the high-$d$ dynamical percolation results of [HPS97], is that there are no such exceptional times.

- Existence of Exceptional Times. For a simple symmetric random walk $S$, it is well known that $\liminf_{t \to \infty} S(t)/\sqrt{t} = -\infty$ a.s. (and, of course, $\limsup_{t \to \infty} S(t)/\sqrt{t} = +\infty$ a.s.). In the following, we will say that a path is subdiffusive if it violates this a.s. property of the standard random walk.

**Definition 1.1.** [K$^+$-subdiffusivity] Let $K \in (0, \infty)$. A path $\pi$ starting at $x = 0$ at time $t = 0$ is said to be $K^+$-subdiffusive iff there exists $j \geq 0$ such that

$$\forall t > 0, \quad \pi(t) \geq -j - K \sqrt{t}. \quad (1.3)$$

We say that $\pi$ is subdiffusive iff there exists $K \in (0, \infty)$ such that $\liminf_{t \to \infty} \pi(t)/\sqrt{t} \geq -K$ or $\limsup_{t \to \infty} \pi(t)/\sqrt{t} \leq K$.

In Proposition 4.1 we prove that for $K$ large enough, there is a strictly positive probability for having a dynamical time $\tau \in [0, 1]$ at which $S_0^\tau$ is $K^+$-subdiffusive. Propositions 5.2 and 5.3 gives lower and upper bounds on the (deterministic) Hausdorff dimension of these exceptional times in $[0, \infty)$. Interestingly, the Hausdorff dimensions depend nontrivially on the constant $K$ so that the dimension tends to zero (respectively, one) as $K \to 0$ (respectively, $K \to \infty$). In particular, as a direct consequence of Proposition 5.2, we obtain the following result.

**Theorem 1.1.** The set of times $\tau \in [0, \infty)$ at which $S_0^\tau$ is subdiffusive has Hausdorff dimension one. Hence, the set of exceptional times for the law of the iterated logarithm (LIL) also has Hausdorff dimension one.

Since a set of exceptional times has zero Lebesgue measure (see (1.1)), we see that the set of exceptional times for the LIL (or for subdiffusivity) is in a sense as large as it can be. This is strikingly in contrast with the dynamical one-dimensional random walk of [BHPS03] where there are no exceptional times for which the LIL fails (in [BHPS03], we recall that the analogue of $S_0^\tau$ is simply defined as

$$\bar{S}_0^\tau(n) = \sum_{i=1}^n X_i^\tau; \quad (1.4)$$

where $\{X_i^\tau\}_i$ are independent $\{-1, +1\}$-valued Markov jump processes with rate 1 and uniform initial distribution). To explain why the walks of [BHPS03] can behave so differently from those of the discrete web, we note that a single switch in the dynamical random walk of (1.4) affects only one increment of the walk while single switches in the discrete web can change the path of the walker by
a “macroscopic” amount. Indeed, the difference between the path $S^\tau_0$ before and after a single switch is given by the difference between two independent simple random walks starting two spatial units apart. This corresponds to the excursion of a (non-simple) random walk from zero whose mean duration is infinite. It follows that a simple random walk is more sensitive to the extra noise induced by the dynamics on the discrete web than to the one induced by the dynamics considered in [BHPS03]. Rephrasing [SS05] in our context, since our dynamical random walk “changes faster” than the one in [BHPS03], it has “more chances” to exhibit exceptional behavior. Mathematically, “changing fast” corresponds to having small correlations over short time intervals and the main ingredient for proving our exceptional times results will be the correlation estimate (3.15) of Proposition 3.1.

By an obvious symmetry argument, there are also exceptional dynamical times $\tau$ for which $S^\tau(t) \leq j + K \sqrt{t}$ for all $t$. One may ask whether there are exceptional $\tau$ for which $|S^\tau(t)| \leq j + K \sqrt{t}$ for all $t$. Proposition 5.5 below shows, at least for small $K$, that there are no such exceptional times. The case of large $K$ is unresolved.

Scaling Limits

In Section 6, we discuss the continuum analogue of the dynamical random walk, the dynamical Brownian motion constructed in [NRS08]. We briefly recall there the main ideas of the construction along with some elementary properties of that object. Then, we outline the main ideas that are needed to extend the results for exceptional times from the discrete level to the continuum.

2 Tameness

In this section, we prove the following theorem.

**Theorem 2.1.** $\mathbb{P}(S^\tau_0 \text{ is recurrent for all } \tau \geq 0) = 1$.

Recall the definition of the dynamical percolation model given in the introduction. For Bernoulli percolation on a homogeneous graph with critical probability $p_c$, let $\theta(p)$ be the probability that the origin belongs to an infinite cluster. In Section 3 of [HPS97], it is proved that if for some $C < \infty$

$$\theta(p) \leq C(p - p_c) \quad \text{for } p \geq p_c,$$

(2.5)

then in the corresponding dynamical percolation model, there is almost surely no dynamical time $\tau$ at which percolation occurs. In our setting, an entirely parallel argument can be used to show tameness of the dynamical discrete web with respect to recurrence.

Following [HPS97], we start by giving a very general tameness criterion. Let $\mathbb{P}_p$ be the probability measure for the static web when the probability for having a right arrow at a given site of $\mathbb{Z}_{even}^2$ is $p$. Let $S_0$ be the simple random walk starting from the origin and let $A$ be a measurable set of
paths such that $P_{1/2}(S_0 \in A) = 0$. In the following, we denote $P_p(A)$ by $\theta_A(p)$. Our first lemma is the analogue to Lemma 3.1 in [HPS97].

**Lemma 2.1.** Let $A$ be such that $\{S_0 \in A\}$ is an increasing event w.r.t. the basic Bernoulli $\{\xi_z\}$ variables and such that $P_{1/2}(S_0 \in A) = 0$. Let $N_A$ be the cardinality of the set $\{\tau \in [0, 1] : S_0^\tau \in A\}$. Suppose there exists $c < \infty$ such that

$$\theta_A(p) \leq c(p - \frac{1}{2}) \quad \text{for all } p \geq \frac{1}{2},$$

(2.6)

Then $E(N_A) < \infty$.

**Proof.** Let $m > 1$. We first estimate $E(N_m)$ where $N_m$ is the number of $i \in \{1, 2, ..., m\}$ such that there exists $\tau \in \left[\frac{i-1}{m}, \frac{i}{m}\right]$ for which $S_0^\tau \in A$. For a given $i \leq m$, define

$$\bar{\xi} = \left\{ \sup_{\tau \in \left[\frac{i-1}{m}, \frac{i}{m}\right]} \xi_z(\tau) \right\}_{z \in \mathbb{Z}^2_{even}}.$$

(2.7)

This naturally induces a new arrow configuration $\bar{\mathcal{A}}$ for which the probability to find a right arrow at any given site is given by

$$\bar{p} = 1 - \frac{1}{2} \exp(-\frac{1}{2m}) \leq \frac{1}{2} + \frac{1}{4m}.$$  

(2.8)

For such a configuration, the path $\bar{S}_0$ starting from the origin is a drifting random walk coupled with $S_0^\tau$ in such a way that

$$\forall \tau \in \left[\frac{i-1}{m}, \frac{i}{m}\right], \quad S_0^\tau \leq \bar{S}_0,$$

(2.9)

which implies

$$P(\exists \tau \in \left[\frac{i-1}{m}, \frac{i}{m}\right] \text{ with } S_0^\tau \in A) \leq P(\bar{S}_0 \in A) \leq c(\bar{p} - \frac{1}{2}) \leq \frac{c}{4m}.$$  

(2.10)

Hence, $E(N_m) \leq m \frac{c}{4m} = c/4$ for all $m > 1$. Since $N_A = \lim \inf_{m \to \infty} N_m$, Fatou’s lemma completes the proof.

We now turn to the proof of Theorem 2.1. For any $n \geq 0$, let $A_n$ be the set of (piecewise linear) simple random walks $\pi$ starting from the origin and such that for all $t \geq 0$, $\pi(t) > -n$. It is well known that

$$\theta_{A_n}(p) = 1 - \left(1 - \frac{2p - 1}{p}\right)^n, \quad \text{for } p \in \left[\frac{1}{2}, 1\right].$$

(2.11)

Clearly, $A_n$ satisfies the hypotheses of Lemma 2.1 implying that $E(N_{A_n}) < \infty$.

In Lemma 3.2 of [HPS97], it is proved that for any homogeneous graph with critical probability $p_c$, the number $N$ of times $\tau \in [0, 1]$ such that in dynamical percolation the origin belongs to an infinite
cluster is a.s. either 0 or \( \infty \). By exactly the same reasoning, one can show that \( N_{A_n} \) is either 0 or \( \infty \). Since \( \mathbb{E}(N_{A_n}) < \infty \) for every \( n \), we have that \( N_{A_n} = 0 \) for every \( n \) and this together with the corresponding result for transience to \(-\infty\) completes the proof of Theorem 2.1.

**Remark 2.2.** Another property of the static discrete web with respect to which the dynamical one is tame is the almost sure coalescence of all of its paths. Indeed, the difference between two independent random walks is again a (non-simple) random walk and the proof of Theorem 2.1 can easily be adapted to show that at every dynamical time \( \tau \) two walkers always meet and coalesce after some finite time \( t \).

### 3 Sensitivity to the Dynamics

In the following, \((C([0, 1]), |.|_{\infty})\) denotes the space of continuous functions on \([0, 1]\) equipped with the sup norm. In order to prepare for our results about exceptional times, we need to prove that the arrow configuration in the DyDW decorrelates fast enough to allow exceptional behavior for the dynamical random walk. This will be done by proving that on a large (diffusive) scale and for \( \tau \neq \tau' \), the paths \( S_0^\tau \) and \( S_0^{\tau'} \) evolve almost independently. More precisely, if for any (small) \( \delta > 0 \) and any \( \pi \in C([0, 1]) \) we set \( \tilde{\pi}(t) \equiv \pi(t/\delta^2) \delta \), we will prove that for a certain open set \( O \subset C([0, 1]) \), we have the following decorrelation inequality:

\[
P(\tilde{S}_0^\tau \in O, \tilde{S}_0^{\tau'} \in O) \leq \mathbb{P}(\tilde{S} \in O)^2 + K(\frac{\delta}{|\tau - \tau'|})^a, \tag{3.13}
\]

where \( S \) is a simple symmetric random walk and \( K \) does not depend on \( \delta, \tau \) and \( \tau' \). In other words, the inequality (3.13) estimates the sensitivity of the event \( O \) to the dynamics.

We now turn to our specific choice for \( O \). Recall that we aim to prove that at some exceptional \( \tau \)'s the path \( S_0^\tau \) is \( K^{+}\text{-subdiffusive} \), which requires that the walk starting from the origin is abnormally tilted to the right. Hence, it is natural to study the noise sensitivity of the event

\[
O = \{ \forall t \in [0, 1], \, \pi(t) > -1 \text{ and } \pi(1) > 1 \} \tag{3.14}
\]

which occurs for paths slightly tilted to the right. Studying noise sensitivity for this event is analogous to the corresponding question concerning left-right crossing of a square in dynamical percolation as studied in [SS05]. The previous discussion motivates the following proposition.

**Proposition 3.1.** For \( O = \{ \forall t \in [0, 1], \, \pi(t) > -1 \text{ and } \pi(1) > 1 \} \), there exist \( K, a \in (0, \infty) \) (independent of \( \delta, \tau \) and \( \tau' \)) such that

\[
P(\tilde{S}_0^\tau \in O, \tilde{S}_0^{\tau'} \in O) \leq \mathbb{P}(\tilde{S} \in O)^2 + K(\frac{\delta}{|\tau - \tau'|})^a, \tag{3.15}
\]

where \( S \) is a simple symmetric random walk.
In order to prove the proposition, we start by highlighting the fact that along the $t$-axis, the pair $(S_0^*, S_0^{r*})$ alternates between times at which the two paths are equal (they “stick together”) and times at which they move independently. Recall that if $S_0^*$ and $S_0^{r*}$ coincide at time $t$ and if the clock at $(S_0^*(t), t)$ does not ring on $[\tau, \tau')$, the increments of $S_0^*$ and $S_0^{r*}$ at time $t$ are equal (i.e., $S_0^*$ and $S_0^{r*}$ stick together). Otherwise, the two increments are independent. This suggests the following time decomposition of the pair $(S_0^*, S_0^{r*})$. Define inductively $\{T_k\}_{k \geq 0}$ with $T_0 = 0$ and for any $k \geq 0$,

$$T_{2k+1} = \inf\{n \in \mathbb{N}, n \geq T_{2k} : \text{the clock at } (S_0^*(n), n) \text{ rings in } [\tau, \tau')\},$$

$$T_{2k+2} = \inf\{n \in \mathbb{N}, n > T_{2k+1} : S_0^*(n) = S_0^{r*}(n)\},$$

$$\Delta T_k = T_{2k+1} - T_{2k} \quad \text{with } \mathbb{P}(\Delta T_k \geq j) = e^{-|\tau-\tau'|j}.$$  

On the interval of integer time $[T_{2k}, T_{2k+1}]$, the paths $S_0^*, S_0^{r*}$ coincide and at time $T_{2k+1}$ they move independently until meeting at time $T_{2k+2}$. Hence, if we skip the intervals $\{[T_{2k}, T_{2k+1}]\}_{k \geq 0}$, $(S_0^*, S_0^{r*})$ behave as two independent random walks $(S_d^*, S_d^{r*})$, while if we skip $\{[T_{2k+1}, T_{2k+2}]\}_{k \geq 0}$, the two walks coincide with a single random walk $S_x$. Furthermore, since $S_x$ is constructed from the arrow configuration at different sites than the ones used to construct $(S_d^*, S_d^{r*})$, it is independent of $(S_d^*, S_d^{r*})$.

Now, skipping the intervals $\{[T_{2k}, T_{2k+1}]\}_{k \geq 0}$ corresponds to making the random time change $t \rightarrow C(t)$ where $C$ is the right continuous inverse of

$$t + \sum_{k \leq (t)} \Delta T_k$$

with $l(t) = \#\{i \in \mathbb{N}, i \leq t : S_d(i) = S_d^{r*}(i)\}$.  

(3.16)

Skipping $\{[T_{2k+1}, T_{2k+2}]\}_{k \geq 0}$ corresponds to making the time change $t \rightarrow t - C(t)$. This analysis yields the following lemma.

**Lemma 3.1.** There exist three independent simple symmetric random walks $(S_x, S_d^*, S_d^{r*})$ and an independent sequence of independent non-negative integer valued random variables $\{\Delta T_k\}_{k \geq 0}$ with $\mathbb{P}(\Delta T_k \geq j) = e^{-|\tau-\tau'|j}$ such that

$$S_0^*(t) = S_d^*(C(t)) + S_x(t - C(t)), \quad (3.17)$$

$$S_0^{r*}(t) = S_d^{r*}(C(t)) + S_x(t - C(t)), \quad (3.18)$$

where $C$ is the right continuous inverse of (3.16).

In the following, the pair $(S_0^*, S_0^{r*})$ will be referred to as a sticky pair of random walks. We note that the previous lemma has a continuous analogue called a sticky pair of Brownian motions—see Section for more details.
Heuristically, in order to prove Proposition 3.1, we need to show that at large (diffusive) scales Equations (3.17)-(3.18) become

\[ S_0(t) \approx S_0(\tau), \quad (3.19) \]
\[ S_0'(t) \approx S_0'(\tau), \quad (3.20) \]
or equivalently that \( C(t) \approx t \) (see Lemma 3.3 below). The following three lemmas prepare the justification of this informal approximation. Let \( \delta > 0 \). We recall that for a path \( S, \tilde{S}(\cdot) \equiv S(\cdot/\delta^2) \).

In the following, we set \( \Delta \equiv \delta/|\tau - \tau'| \) and for \( O \subset C([0,1]) \) and any \( r \geq 0 \), we define \( O + r \equiv \{ \pi \in C([0,1]) \text{ s.t. } \exists \, \bar{\pi} \in O \text{ s.t. } |\pi - \bar{\pi}|_{\infty} \leq r \} \).

**Lemma 3.2.** Let \( S \) be a simple symmetric random walk. For the \( O \) defined in (3.14) and any \( \alpha < \frac{1}{2} \),

\[ \mathbb{P}(\tilde{S} \in [O + \Delta^\alpha] \setminus O) \leq c' \Delta^\alpha \quad (3.21) \]
where \( c' \in (0,\infty) \) is independent of \( \Delta \) and \( \delta \).

**Proof.**

\[ \mathbb{P}(\tilde{S} \in [O + \Delta^\alpha] \setminus O) \leq \mathbb{P}(\inf_{t \in [0,1]} \tilde{S}(t) \in (-1 - \Delta^\alpha, -1]) + \mathbb{P}(\tilde{S}(1) \in (1 - \Delta^\alpha, 1]). \]

We will prove that the second term on the right hand side of of the inequality is of order \( \Delta^\alpha \). The first term can be handled similarly.

In [Fr73], it is proved that a sequence of rescaled standard random walks \( \{S(\cdot/\delta^2)\}_{\delta > 0} \) and a Brownian motion \( B \) can be constructed on the same probability space in such way that for any \( \alpha < \frac{1}{2} \), the quantity \( \mathbb{P}(|B - S(\cdot/\delta^2)|_{\infty} > \delta^\alpha) \) goes to 0 faster than any power of \( \delta \). On this probability space,

\[ \mathbb{P}(\tilde{S}(1) \in (1 - \Delta^\alpha, 1]) \]
\[ \leq \mathbb{P}(B(1) \in [1 - 2\Delta^\alpha, 1 + \Delta^\alpha]) + \mathbb{P}(|\tilde{S} - B|_{\infty} \geq \Delta^\alpha), \quad (3.22) \]

Let \( \alpha < \frac{1}{2} \). Because \( \Delta^\alpha > \delta^\alpha \) (since \( |\tau - \tau'| \leq 1 \)), the last term on the right-hand side of (3.22) is bounded by \( O(\delta) \), and consequently by \( O(\Delta) \). By a density argument, the first term on the right hand side of the inequality is clearly bounded by \( c\Delta^\alpha \) and this completes the proof of the lemma. \( \square \)

**Lemma 3.3.** Define \( \tilde{C}(t) = C(t/\delta^2)\delta^2 \) where \( C \) is defined in (3.10) (note that the random clock \( C \) and the paths are rescaled in a different manner).

For any \( 1 > \beta > 0 \)

\[ \mathbb{P}(\sup_{t \in [0,1]} (t - \tilde{C}(t)) \geq \Delta^\beta) \leq \tilde{c} \Delta^{1-\beta}, \quad (3.23) \]
where \( \tilde{c} \in (0,\infty) \) is independent of \( \Delta \) and \( \delta \).
Proof. Recall from Lemma 3.1 that,

\[ C^{-1}(t) = t + L(t), \]

where \[ L(t) = \sum_{k \leq t} \Delta T_k, \] (3.24)

\( \{\Delta T_k\} \) are independent geometric random variables, \( l \) is the discrete local time at the origin of \( S^*_d - S^*_d' \) and \( C^{-1} \) is the right continuous inverse of \( C \). In the following, we set \( \bar{L}(t) \equiv L(t/\delta^2) \delta^2 \).

We first prove that

\[ \mathbb{P}(\bar{L}(1) \geq \Delta^{\beta}) \leq \tilde{c} \Delta^{1-\beta}. \] (3.25)

By the Markov inequality,

\[ \mathbb{P}(\bar{L}(1) \geq \Delta^{\beta}) \leq \mathbb{E} \left( \left( l(1/\delta^2) \delta \right) \left( \mathbb{E}(\Delta T_1) \delta \right) \frac{1}{\Delta^2} \right) \] (3.26)

with \( \mathbb{E}(\Delta T_1) = \sum_{k=1}^{\infty} e^{-\left|\tau - \tau'\right|} k = \frac{\exp\left(-\left|\tau - \tau'\right|\right)}{1 - \exp\left(-\left|\tau - \tau'\right|\right)} \). (3.27)

Now

\[ \mathbb{E}(l(1/\delta^2)) = \sum_{k \leq 1/\delta^2} \mathbb{P}(S^*_d(k) - S^*_d'(k) = 0) \]

and it is a standard fact that the probability in the summation is \( O(1/\sqrt{k}) \) as \( k \to \infty \); thus \( \mathbb{E}(l(1/\delta^2)) \) is uniformly bounded in \( \delta \) as \( \delta \to 0 \). Furthermore, since \( \mathbb{E}(\Delta T_1) = 0(\left|\tau - \tau'\right|)^{-1} \), we have \( \delta \mathbb{E}(\Delta T_1) = O(\Delta) \) and thus (3.25) follows.

Next, on the event \( \{\bar{L}(1) \leq \Delta^{\beta}\} \), (3.24) implies that for any \( t \in [0,1] \):

\[ (\bar{C})^{-1}(t) \leq t + \Delta^{\beta}. \] (3.28)

Since \( \bar{C}(t) \leq t \) and \( \bar{C} \) is an increasing function of \( t \), it follows that on \( \{\bar{L}(1) \leq \Delta^{\beta}\} \), for all \( t \in [0,1] \), we have

\[ t - \bar{C}(t) \leq \Delta^{\beta}. \] (3.29)

The lemma thus follows from (3.25).

\[ \square \]

Lemma 3.4. For any continuous function \( f \), define \( \omega_f(\epsilon) = \sup_{s,t \in [0,1],|s-t|<\epsilon}|f(t) - f(s)| \) to be the modulus of continuity of \( f \) on \([0,1]\).

Let \( \alpha, \beta \in (0,\infty) \) be such that \( \beta/2 > \alpha \). For any \( r \geq 0 \), there exists \( c \) (independent of \( \Delta \) and \( \delta \)) such that

\[ \mathbb{P}(\omega_f(\Delta^{\beta}) \geq \frac{\Delta^{\beta}}{2}) \leq c \Delta^r. \] (3.30)
Proof. Let \( m, n \geq 0 \) and define
\[
\tilde{M} = \int_0^1 \int_0^1 \frac{|\tilde{S}(t) - \tilde{S}(s)|^n}{|t-s|^m} \; dt \; ds. 
\] (3.31)

By the Garsia, Rodemich and Rumsey inequality [GRR70], we have for \( m > 2 \) and all \( s, t \in [0, 1] \)
\[
|\tilde{S}(t) - \tilde{S}(s)| \leq \frac{8m}{m-2} (4\tilde{M})^{\frac{1}{m-2}} |t-s|^{\frac{m-2}{m}}. 
\] (3.32)

It is well known that \( \mathbb{E}(|\tilde{S}(t) - \tilde{S}(s)|^n) \leq c'|t-s|^n \), where \( c' \) is uniform in \( \delta \). Hence, (3.31) implies that if \( \frac{m}{2} - m > -1 \), then \( \mathbb{E}(\tilde{M}) \leq c < \infty \) so that for every \( r \geq 0 \),
\[
\mathbb{P}(\tilde{M} > \Delta^{-r}) \leq c \Delta^r. 
\] (3.33)

On the other hand, on \{ \( \tilde{M} \leq \Delta^{-r} \)\}, (3.32) yields
\[
\omega_{\tilde{S}}(\Delta^\beta) \leq \frac{8m}{m-2} (4\Delta^{-r})^{\frac{1}{m-2}} |\Delta^\beta|^{\frac{m-2}{m}} 
\] (3.34)
\[
\leq c(n, m) \Delta^{\frac{1}{2}(\beta(m-2)-r)}. 
\] (3.35)

Since \( \beta/2 > \alpha \), one can always take (for fixed \( \alpha, \beta, r \)) \( n, m \) large enough such that both \( \frac{m}{2} - m > -1 \) and \( \frac{1}{2}(\beta(m-2)-r) > \alpha \). For such a choice, and taking \( \Delta \) small enough so that \( c(n, m)\Delta^{\frac{1}{2}(\beta(m-2)-r)} \leq \Delta^\alpha/2 \), we obtain than on \{ \( \tilde{M} \leq \Delta^{-r} \)\}
\[
\omega_{\tilde{S}}(\Delta^\beta) \leq \Delta^\alpha/2. 
\] (3.36)

Hence, for small enough values of \( \Delta \), the claim of the lemma follows from (3.33). The claim is obviously satisfied for larger values of \( \Delta \). \( \Box \)

We are now ready to prove Proposition 3.1. Recall the definition of \( S_d^\tau \) and \( S_d^{\tau'} \) in Lemma 3.1
For any \( \alpha > 0 \), we have
\[
\mathbb{P}(\tilde{S}_0^\tau \in O, \tilde{S}_0^{\tau'} \in O) \leq \mathbb{P}(\tilde{S}_d^\tau \in O + \Delta^\alpha, \tilde{S}_d^{\tau'} \in O + \Delta^\alpha) + 2\mathbb{P}(\tilde{S}_0^\tau \in O, \tilde{S}_d^\tau \in (O + \Delta^\alpha)^c). 
\] (3.37)

where \((O + \Delta^\alpha)^c\) is the complementary set of \( O + \Delta^\alpha \). Note that we used the equidistribution of \((\tilde{S}_d^\tau, \tilde{S}^\tau)\) and \((\tilde{S}_d^{\tau'}, \tilde{S}^{\tau'})\). We start by dealing with the first term on the right-hand side of the inequality. Since \( \tilde{S}_d^\tau, \tilde{S}_d^{\tau'} \) are independent and distributed like a rescaled simple symmetric random walk \( \tilde{S} \), we have
\[
\mathbb{P}(\tilde{S}_d^\tau \in O + \Delta^\alpha, \tilde{S}_d^{\tau'} \in O + \Delta^\alpha) = \mathbb{P}(\tilde{S}_d^\tau \in O + \Delta^\alpha) \; \mathbb{P}(\tilde{S}_d^{\tau'} \in O + \Delta^\alpha) 
\leq \mathbb{P}(\tilde{S} \in O)^2 + 2\mathbb{P}(\tilde{S} \in [O + \Delta^\alpha] \setminus O). 
\] (3.38)
The latter inequality and Lemma 3.2 above imply that
\[ \Pr(\tilde{S}_d^\tau \in O + \Delta^\alpha, \tilde{S}_d^\prime \in O + \Delta^\alpha) \leq \Pr(\tilde{S} \in O)^2 + 2c\Delta^\alpha, \]  
(3.39)
for any \( \alpha < 1/2 \). By (3.37) and (3.39), Proposition 3.1 follows if there are \( c'', a' \in (0, \infty) \) such that
\[ \Pr[\tilde{S}_0^\tau \in O, \tilde{S}_d^{\tau'} \in (O + \Delta^\alpha)^c] \leq c'' \Delta^{a'}. \]  
(3.40)
This inequality can be justified as follows. Let \( 0 < \beta < 1 \). By Lemma 3.1
\[ \tilde{S}_0^\tau(t) = \tilde{S}_d(t) + \tilde{S}_s(t - \bar{C}(t)), \]  
(3.41)
\[ = \tilde{S}_d(t) + [\tilde{S}_d(\bar{C}(t)) - \tilde{S}_d(t)] + \tilde{S}_s(t - \bar{C}(t)). \]  
(3.42)
The last equality implies that for any \( 0 < \beta < 1 \) with \( \alpha < \beta/2 \),
\[ \Pr[\tilde{S}_0^\tau \in O, \tilde{S}_d^{\tau'} \in (O + \Delta^\alpha)^c] \leq \Pr(|\tilde{S}_0^\tau - \tilde{S}_d^\tau|_\infty \geq \Delta^\alpha) \]
\[ \leq \Pr(|\tilde{S}_s(t - \bar{C}(t))|_\infty \geq \frac{\Delta^\alpha}{2}) + \Pr(|\tilde{S}_d(t) - \tilde{S}_d(\bar{C}(t))|_\infty \geq \frac{\Delta^\alpha}{2}) \]
\[ \leq 2\Pr(\omega_{\tilde{S}}(\Delta^\beta) \geq \frac{\Delta^\alpha}{2}) + 2\Pr(|t - \bar{C}(t)|_\infty \geq \Delta^\beta) \]
\[ \leq 2c\Delta^\gamma + 2\tilde{c}\Delta^{1-\beta}. \]
where \( r > 0 \) and the last inequality is given by Lemmas 3.3 and 3.4 above. So far we have only needed \( \alpha \in (0,1/2) \) and thus we can indeed choose \( \beta \in (0,1) \) and then \( \alpha < \beta/2 \) so that Proposition 3.1 follows.

4 Existence of Exceptional Times

In this section we prove the following result.

**Proposition 4.1.** For \( K \) large enough
\[ \Pr(\exists \tau \in [0,1], \ s.t. \ S_0^\tau \text{ is } K^+ - \text{subdiffusive}) > 0. \]  
(4.43)
(For a definition of \( K^+ - \text{subdiffusivity}, \text{ see Definition 1.1.} \)

Let \( \gamma > 2 \) and \( d_k = 2(\lfloor \frac{k}{\gamma} \rfloor + 1) \), where \( \lfloor x \rfloor \) is the integer part of \( x \). We construct inductively a sequence of “diffusive” boxes \( R_k \) in the following manner (see Figure 1).

- \( R_0 \) is the rectangle with vertices \((-d_0,0), (+d_0,0), (-d_0,d_0^2) \) and \((+d_0,d_0^2)\).
Figure 1: Construction of the first three boxes \((R_0, R_1, R_2)\) with \(t\) the vertical and \(x\) the horizontal coordinate. The thin curves represent segments of the paths starting from \(\bar{z}_i\), for \(i = 1, 2, 3\), for which the events \(A_\tau^0, A_\tau^1\) and \(A_\tau^2\) occur.

- Let \(\bar{z}_k = (x_k, t_k)\) be the middle point of the lower edge of \(R_k\) (e.g., \(\bar{z}_0 = (0, 0)\)). \(R_{k+1}\) is the rectangle of height \(d_{2k+1}^2\) and width \(2d_{k+1}\) such that \(\bar{z}_{k+1}\) coincides with the upper right vertex of \(R_k\) (see Figure 1).

Note that for our particular choice of \(d_k\), \(\bar{z}_k\) always belongs to \(\mathbb{Z}_{\text{even}}^2\) for \(k \geq 0\) and a simple computation leads to the following lemma.

**Lemma 4.1.** Let \(\partial^\gamma = \partial^\gamma(t)\) denote the right-continuous function obtained by joining together the left boundaries \(\partial_k\) of \(R_k\). For any \(K > 0\), let \(\gamma(K)\) be the solution in \((2, \infty)\) of \(K = (\gamma - 2)\sqrt{\gamma + \frac{3}{4}}\). Then,

\[
\forall t \geq 0, \quad \partial^\gamma(K)(t) \geq -3 - K\sqrt{t}. \quad (4.44)
\]

**Proof.** On \([t_n, t_{n+1})\), we have \(\partial^\gamma(t) = \partial^\gamma(t_n) = x_n - d_n = (d_0 + d_1 + \cdots + d_{n-1} - d_n)\). If \(\gamma\) is such that

\[
\partial^\gamma(t_n) \geq -(3 + K\sqrt{t_n}) \quad \text{for } n = 0, 1, 2, \ldots ,
\]

then we will have \(\partial^\gamma(t) \geq -(3 + K\sqrt{t})\) for all \(t \geq 0\) as desired.

The inequality \((4.45)\) can be rewritten as

\[
d_n \leq 3 + d_0 + \cdots + d_{n-1} + K[d_0^2 + \cdots + d_{n-1}^2]^{1/2}. \quad (4.46)
\]
Lemma 4.2. There exists for \( n \) \( \gamma^n \leq 1 + \frac{\gamma^n - 1}{\gamma - 1} + K \sqrt[\gamma^n - 1]{\gamma^2 - 1}. \) (4.47)

Using the elementary bound \( \sqrt[\gamma^n - 1]{\gamma^2 - 1} \geq \gamma^n(1 - \gamma^{-2n}) \) (for \( \gamma \geq 1 \)), we see that in order to verify (4.47), it suffices to have, for \( n = 0, 1, 2, \ldots, \)

\[ \gamma^n \leq 1 + \frac{1}{\gamma - 1} + K \sqrt[\gamma - 1]{\gamma^2 - 1}. \] (4.48)

Choosing \( \gamma \) such that \( K = (\gamma - 2)\sqrt[\gamma^2 - 1]{\gamma + 1} \) yields (4.44). The lemma follows from the fact that \( \gamma \to (\gamma - 2)\sqrt[\gamma^2 - 1]{\gamma + 1} \) is a continuous increasing function mapping \( (2, \infty) \) onto \( (0, \infty) \).

By Lemma 4.1, \( S_0^\tau \) is \( K^\tau \)-subdiffusive if \( S_0^\tau(t) \geq \partial(\gamma(K))(t) \). Let \( S_{\bar{z}_k}^\tau \) be the path in \( W(\tau) \) starting from \( \bar{z}_k = (x_k, t_k) \) and define the event

\[ A_k^\tau = A_k^\tau(K) = \{ \forall t \in [t_k, t_{k+1}] \ S_{\bar{z}_k}^\tau(t) > \partial_k(t), S_{\bar{z}_k}^\tau(t_{k+1}) > x_{k+1} \}. \] (4.49)

(Here \( \partial_k \) depends implicitly on \( \gamma(K) \).) Since paths in \( W(\tau) \) do not cross, if \( \cap_{k \leq n} A_k^\tau \) occurs, \( S_0^\tau \) is forced to remain to the right of \( \partial_k \) on \( [t_k, t_{k+1}] \) for every \( k \leq n \) (see Figure 1). This implies that if we have

\[ \mathbb{P}(\exists \tau \in [0, 1], \ \cap_{k \geq 0} A_k^\tau(K) \text{ occurs}) > 0, \] (4.50)

then

\[ \mathbb{P}(\exists \tau \in [0, 1], \ S_0^\tau \text{ is } K^\tau \text{-subdiffusive}) > 0. \] (4.51)

In the rest of the section we proceed to verify (4.50).

In the following, \( K \) is temporarily fixed and to ease the notation we write \( A_k^\tau \) for \( A_k^\tau(K) \) and \( \gamma \) for \( \gamma(K) \). In order to verify (4.50), we start by proving the following lemma using Proposition 3.1.

**Lemma 4.2.** There exists \( c \in (0, \infty) \) such that for \( \tau, \tau' \in [0, 1] \)

\[ \forall n \geq 0, \ \prod_{k=0}^{n} \frac{\mathbb{P}(A_k^\tau \cap A_k^{\tau'})}{\mathbb{P}(A_k^\tau)} \leq c \frac{1}{|\tau - \tau'|^b}, \] (4.52)

where \( A_k \equiv A_k^0 \) and \( b = \log(\sup_k [\mathbb{P}(A_k)^{-1}]) / \log \gamma > 0. \)

**Proof.** Let \( (S^\tau, S'^{\tau'}) \) be the paths starting at \( (0, 0) \), defined as the translated version of the pair \( (S_{\bar{z}_k}^\tau, S_{\bar{z}_k}'^{\tau'}) \in (W(\tau), W(\tau')) \) starting at \( \bar{z}_k \). By translation invariance, \( (S^\tau, S'^{\tau'}) \) is a sticky pair of random walks starting at \( (0, 0) \) whose distribution is described in Lemma 3.1 and by definition

\[ A_k^\tau = \{ S^\tau(d_k^\tau) > d_k, \ \inf_{[0, d_k^\tau]} S^\tau(t) > -d_k \}. \] (4.53)
By Proposition 3.1 for $\delta = d_k^{-1}$, there exists $c, a \in (0, \infty)$ such that
\[
\mathbb{P}(A_k^c, A_k^c) \leq \mathbb{P}(A_k)^2 + c \left( \frac{1}{\gamma^k|\tau - \tau'|} \right)^a.
\] (4.54)

Defining $N_0 = \lceil -\log(\log(\gamma)) \rceil + 1$ so that $(\gamma_N|\tau - \tau'|) \geq 1$, we have for $n > N_0$
\[
\prod_{k=N_0+1}^n \left( \frac{\mathbb{P}(A_k^c \cap A_k^c')}{\mathbb{P}(A_k)^2} \right) \leq \prod_{k=N_0+1}^\infty \left( 1 + \frac{c}{\mathbb{P}(A_k)^2} \frac{1}{\gamma^a(n-k)} \right)
\leq \prod_{k=1}^\infty \left( 1 + \frac{c}{\inf_{n} \mathbb{P}(A_k)^2} \frac{1}{\gamma^a(n-k)} \right).
\] (4.55)

The right-hand side of (4.55) is independent of $|\tau - \tau'|$ and is finite. Indeed, we have $0 < \inf_n \mathbb{P}(A_n)$ since the boxes $R_k$ have diffusively scaled sizes and therefore $\mathbb{P}(A_k) \to \mathbb{P}(A)$ as $k \to \infty$, where $A$ is the event that a Brownian motion $B(t)$ starting at 0 at time 0 has $B(1) > 1$ and $\inf_{t \in [0,1]} B(t) > -1$.

On the other hand, for $n \leq N_0$
\[
\prod_{k=0}^n \frac{\mathbb{P}(A_k^c \cap A_k^c')}{\mathbb{P}(A_k)^2} \leq (\sup_k \frac{1}{\mathbb{P}(A_k)^2})^{N_0+1},
\leq c'' \exp \left( \frac{\log[\sup_k(\mathbb{P}(A_k)^{-1})]}{\log(\gamma)} \right) \log \left( \frac{1}{|\tau - \tau'|} \right),
= c''/|\tau - \tau'|^b,
\] (4.56)

where $c'' = \sup_k (\mathbb{P}(A_k)^{-1})$ and $b = \log[\sup_k(\mathbb{P}(A_k)^{-1})] / \log \gamma$ are in $(0, \infty)$. This and (4.55) imply (4.52).

Following [SS05], the Cauchy-Schwarz inequality and the previous lemma imply that for every $n \geq 0$, we have
\[
\mathbb{P} \left( \int_0^1 \prod_{k=0}^n 1_{A_k^c} d\tau > 0 \right) \geq \frac{\left( \mathbb{E} \left[ \int_0^1 \prod_{k=0}^n 1_{A_k^c} d\tau \right] \right)^2}{\mathbb{E} \left[ \int_0^1 \prod_{k=0}^n 1_{A_k^c} d\tau \right]^2},
\] (4.57)
\[
= \left( \left[ \int_0^1 \left( \prod_{k=0}^n \frac{\mathbb{P}(A_k^c \cap A_k^c')}{\mathbb{P}(A_k)^2} d\tau d\tau' \right) \right] \right)^{-1}
\] (4.58)
\[
\geq c^{-1} \left( \left[ \int_0^1 \int_0^1 \frac{1}{|\tau - \tau'|^b} d\tau d\tau' \right] \right)^{-1},
\] (4.59)

where the equality is a consequence of the stationarity of $\tau \to W(\tau)$ and the independence between the arrow configurations in different boxes $R_k$. Recall that $\gamma$ has an implicit dependence on $K$ and that $\gamma$ increases from 0 to $\infty$ as $K$ increases on $(0, \infty)$ (see Lemma 4.1). Hence, for $K$ large enough such that $\gamma = \gamma(K) > \sup_k \mathbb{P}(A_k)^{-1}$, we have
\[
b = \log(\sup_k [\mathbb{P}(A_k)^{-1}]) / \log(\gamma(K)) < 1
\]
\[(\tau, \tau') \rightarrow |\tau - \tau'|^{-b} \in L^1([0,1] \times [0,1], d\tau \; d\tau')\]. Then \((4.59)\) then implies that
\[
\inf_n P\left( \int_0^1 \prod_{k=0}^n 1_{A^\tau_k} \; d\tau > 0 \right) \geq p > 0.
\]

Let \(E_n\) be the set of times \(\tau\) in \([0,1]\) such that \(\bigcap_{k=0}^n A^\tau_k\) occurs. \((4.60)\) implies that \(P(\bigcap_{n=0}^\infty \{E_n \neq \emptyset\}) \geq p > 0\). Since \(\{E_n\}\) is obviously decreasing in \(n\), if the \(E_n\) were closed subsets of \([0,1]\) it would follow that \(P(\bigcap_{n=0}^\infty E_n) \neq \emptyset \) \(\geq p > 0\).

Unfortunately, the set of times at which one arrow is (or any finitely many are) oriented to the right (resp., to the left) is not in general a closed subset of \([0,1]\) since we have a right continuous process, and thus \(E_n\) is not in general a closed set. This extra technicality is handled like in Lemma 5.1 in [SS05], as follows. On the one hand, there are only countably many switching times for all \(\xi^\tau_z\)'s (recall that \(\xi^\tau_z\) represents the arrow direction at location \(z\)). On the other hand, at any switching time \(\tau\), \(\bigcap_{n=0}^\infty A^\tau_n\) does not occur by independence of the \(\xi^\tau_z\)'s. Since there are countably many switching times, this implies that almost surely, the closures \(\bar{E}_n\) of \(E_n\) satisfy
\[
\bigcap_{n=1}^\infty \bar{E}_n = \bigcap_{n=1}^\infty E_n.
\]

This completes the verification of \((4.50)\) and thus the proof of Proposition 4.1.

5 Hausdorff Dimension Of Exceptional Times

In this section, we derive some lower and upper bounds for the Hausdorff dimension of the set of exceptional dynamical times \(\tau \in [0, \infty)\) at which \(S^\tau_0\) becomes subdiffusive.

**Definition 5.1.** We say that \(\tau\) is a \(K\)-exceptional time if the path \(S^\tau_0\) in \(W(\tau)\) does not cross the moving boundary \(t \mapsto -K\sqrt{t}\). \(T(K)\) is then defined as the set of all \(K\)-exceptional times \(\tau \in [0, \infty)\).

Clearly, the set consisting of all the \(K\)-exceptional times in \([0, \infty)\) is a non-decreasing function of \(K\). The next proposition asserts that for fixed \(K\), the Hausdorff dimension \(\dim_H\) of the set of exceptional times is unchanged if \(-K\sqrt{t}\) is replaced by \(-j - K\sqrt{t}\) for any \(j \geq 0\). We note that as in dynamical percolation (see Sec. 6 of [HPS97]), \(\dim_H(T(K))\) is a.s. a constant by the ergodicity in \(\tau\) of the dynamical discrete web.

**Proposition 5.1.** The Hausdorff dimension \(\dim_H\) of the set \(T_j = T_j(K)\) of exceptional times \(\tau \in (0, \infty)\) such that \(S^\tau_0\) does not cross the moving boundary \(t \mapsto -j - K\sqrt{t}\) does not depend on \(j \geq 0\) (for fixed \(K\)).

**Proof.** By monotonicity in \(j\), it is enough to prove that \(\dim_H(T_j) \leq \dim_H(T_0)\) for \(j\) any positive integer.
First, \( T_0 \supset T_j \bigcap \{ \tau \in [0, 1] : \xi_{m,m}(\tau) = +1 \} \) for \( m < j \) where \( T_j \) is the set of \( \tau \in [0, \infty) \) such that \( S_{(j,j)}(n) \geq -K \sqrt{n} \) for \( n \geq j \). Furthermore, \( T_j \supset T_T \), where \( T_T \) is the set of \( \tau \in [0, \infty) \) such that \( S_{(j,j)}(n) - j \geq -j - K \sqrt{n} - j \) for \( n \geq j \). Note that \( T_T \) is just the translation (from \((0, 0)\) to \((j, j)\)) of \( T_j \). Hence, \( \dim_H(T_0) \) is at least the dimension of \( \{ \tau \in [0, 1] : \forall m < j, \xi_{(m,m)} = +1 \} \bigcap T_T \).

By ergodicity in \( \tau \), the a.s. constant \( \dim_H(T_T) \) is the essential supremum of the random variable \( \dim_H(T_T \bigcap [0, 1]) \). On the other hand, since \( T_T \bigcap [0, 1] \) and \( \{ \tau \in [0, 1] : \forall m < j, \xi_{(m,m)} = +1 \} \) are independent and the probability to have \( \{ \forall \tau \in [0, 1], \forall m < j, \xi_{(m,m)} = +1 \} \) is strictly positive, it follows that \( \dim_H(\{ \tau \in [0, 1] : \forall m < j, \xi_{(m,m)} = +1 \} \bigcap T_T) \) has the same essential sup as \( \dim_H(T_T \bigcap [0, 1]) \). Hence \( \dim_H(T_j) = \dim_H(T_T) \leq \dim_H(T_0) \) and the conclusion follows. \( \square \)

### 5.1 Lower Bound

Set \( \gamma_0 = \sup_{k,K} 1/\mathbb{P}(A_k(K)) \), where \( A_k(K) \) is defined by (4.30) with \( \tau = 0 \). (Note that \( \gamma_0 > 2 \).) We recall that \( \gamma(K) \) is the solution in \((2, \infty)\) of \( K = K(\gamma) = (\gamma - 2)\sqrt{\frac{2}{\gamma - 1}} \) for \( K > 0 \). In this section, we prove the following proposition using Lemma 4.2 and then arguments identical to ones in [SS05].

**Proposition 5.2.**

\[
\dim_H(T(K)) \geq 1 - \frac{\log \gamma_0}{\log \gamma(K)} \quad \text{for} \quad K > K(\gamma_0).
\]

Thus, \( \lim_{K \to \infty} \dim_H(T(K)) = 1 \).

Let \( K > K(\gamma_0) \). Note that since \( K \to \gamma(K) \) is increasing, \( \gamma(K) > \gamma_0 \). In the following and as in Section 4, we drop the dependence on \( K \) in the notation. Consider the random measure \( \sigma_n \), such that for any Borel set \( E \) in \([0, 1]\)

\[
\sigma_n(E) = \int_E \prod_{k=0}^{n} \frac{1}{\mathbb{P}(A_k)} \, d\tau.
\]

We note that \( \sigma_n \) is supported by \( \bar{E} \), the closure of \( E_n \) with

\[
E_n = \{ \tau \in [0, 1] : \cap_{k \leq n} A_k \text{ occurs} \}.
\]

For any positive measure \( \sigma \), define the \( \alpha \)-energy of \( \sigma \) as

\[
\mathcal{E}_\alpha(\sigma) = \int_0^1 \int_0^1 \frac{1}{|\tau - \tau'|^\alpha} \, d\sigma(\tau) \, d\sigma(\tau') \, d\tau.
\]

Following [SS05], we will need the following extension of Frostman’s Lemma.

**Lemma 5.1.** [SS05] Let \( D_1 \supset D_2 \supset \ldots \) be a decreasing sequence of compact subsets of \([0, 1]\), and let \( \mu_1, \mu_2, \ldots \) be a sequence of positive measures with \( \mu_n \) supported on \( D_n \). Suppose that there exists
a constant $C \in (0, \infty)$ and $\alpha \in (0, 1)$ such that for infinitely many values of $n$,
\[
\mu_n([0, 1]) \geq 1/C, \quad \mathcal{E}_\alpha(\mu_n) \leq C.
\] (5.65)

Then the Hausdorff dimension of $\cap D_n$ is at least $\alpha$.

Using the ergodicity of the dynamical web in the variable $\tau$, we will prove Proposition 5.2 by showing that for $\alpha < 1 - \frac{\log(\gamma_0)}{\log(\gamma(\mathbb{A}))}$, $\{\sigma_n\}$ satisfies the hypotheses of this lemma with strictly positive probability. By Lemma 4.2, we have for all $n$ that
\[
E[\sigma_n([0, 1])^2] = \int_0^1 \int_0^1 \prod_{k=0}^n \frac{P(A_k \cap A_k')}{P(A_k)} d\tau d\tau' \\
\leq c \left( \int_0^1 \int_0^1 \frac{1}{|\tau - \tau'|} d\tau d\tau' \right),
\]
where
\[
b = \log [\sup_k (P(A_k)^{-1})] / \log \gamma \leq \frac{\log(\gamma_0)}{\log(\gamma)} < 1.
\] (5.66)

By the Cauchy-Schwarz inequality
\[
E [\sigma_n([0, 1])]^2 \geq P \left[ \sigma_n([0, 1]) > \frac{1}{2} \right] \geq E [\sigma_n([0, 1]) \cdot 1_{\sigma_n([0, 1]) > 1/2}] \\
\geq E [\sigma_n([0, 1])] - \frac{1}{2} = \frac{1}{2},
\]
which implies that $P(\sigma_n([0, 1] > \frac{1}{2})) > c_1$ for some $c_1 > 0$ not depending on $n$.

By Fubini’s Theorem and Lemma 4.2,
\[
E(\mathcal{E}_\alpha(\sigma_n)) = \int_0^1 \int_0^1 |\tau - \tau'|^{-\alpha} \prod_{k=0}^n \frac{P(A_k \cap A_k')}{P(A_k)} d\tau d\tau'.
\]
\[
\leq c \int_0^1 \int_0^1 \frac{1}{|\tau - \tau'|^{\alpha + 1}} d\tau d\tau'.
\] (5.67)

Taking $\alpha$ such that
\[
\alpha < 1 - \frac{\log(\gamma_0)}{\log(\gamma)},
\] (5.68)
we have from (5.66) that $b + \alpha < 1$ and therefore
\[
\sup_{n \geq 0} E(\mathcal{E}_\alpha(\sigma_n)) \leq c_2 < \infty.
\] (5.69)

By Markov’s inequality, for all $n$ and all $T$,
\[
P(\mathcal{E}_\alpha(\sigma_n) \geq c_2 T) \leq 1/T.
\] (5.70)

Choose $T$ such that $1/T < c_1/2$. Letting
\[
U_n^\alpha = \{ \sigma_n([0, 1]) > \frac{1}{2} \} \cap \{ \mathcal{E}_\alpha(\sigma_n) \leq c_2 T \},
\] (5.71)
by the choice of \( T \), we have that
\[
P(U^\alpha_n) \geq c_1/2. \tag{5.72}
\]
By Fatou’s lemma,
\[
P(\limsup_{n \to \infty} U^\alpha_n) \geq c_1/2. \tag{5.73}
\]
By Lemma 5.1 it follows that for \( \alpha \) satisfying (5.68), \( \bigcap_{n \geq 0} \bar{E}_n \) has Hausdorff dimension at least \( \alpha \) with positive probability. Since \( \bigcap_{n \geq 0} \bar{E}_n = \bigcap_{n \geq 0} E_n \) (see (4.61)), the same statement holds for \( \bigcap_{n \geq 0} E_n \) and we are done.

## 5.2 Upper Bound

We will prove the following proposition.

**Proposition 5.3.** \( \dim_H(T(K)) \leq 1 - p(K) \) where \( p(K) \in (0, 1) \) is the solution of the equation
\[
f(p, K) \equiv \frac{\sin(\pi p/2) \Gamma(1 + p/2)}{\pi} \sum_{n=1}^{\infty} \frac{(\sqrt{2}K)^n}{n! \Gamma((n - p)/2)} = 1. \tag{5.74}
\]
Furthermore, \( K \to p(K) \) is a continuous decreasing function on \((0, \infty)\) with
\[
\lim_{K \to 0} p(K) = 0 \quad \text{and more significantly} \quad \lim_{K \to 1} p(K) = 1. \tag{5.75}
\]

To prove Proposition 5.3 we need the following lemma proved in the Appendix.

**Lemma 5.2.** Let \( 0 < l < 1 \). Let \( S_e \) be the simple asymmetric random walk with
\[
P(S_e(n + 1) - S_e(n) = +1) = \frac{1}{2} + \frac{1}{2}(1 - e^{-t}). \tag{5.76}
\]
Then there exists \( c(l) \) such that
\[
P(\forall n, \ S_e(n) \geq -1 - K\sqrt{n}) \leq c(l) e^{p(K/l)} \tag{5.77}
\]
where \( p(K) \) is the real solution in \((0, 1)\) of (5.74) (which satisfies (5.75)).

Let us partition \([0, 1]\) into intervals of equal length \( 2\epsilon \), and select the intervals containing a \( K \)-exceptional time. The union of those is a cover of \( T(K) \) and we now estimate the number \( n(\epsilon) \) of intervals in the cover.

Let \( U_\epsilon \) be the event that there is a time \( \tau \) in \([0, 2\epsilon]\) such that \( \tau \in T(K) \). From the full dynamical arrow configuration for all \( \tau \in [0, 2\epsilon] \), we construct a static arrow configuration as follows. We declare the static arrow at \((i, j)\) to be right oriented if and only if the dynamical arrow is right oriented (i.e., \( \xi_{i,j}(\tau) = +1 \)) at some \( \tau \in [0, 2\epsilon] \) (a similar construction was used in Section 2). In this
configuration, the path $S_\varepsilon$ starting from the origin and following the arrows is a slightly right-drifting random walk with $\mathbb{P}(S_\varepsilon(n + 1) - S_\varepsilon(n) = +1) = \frac{1}{2} + \frac{1}{2}(1 - e^{-\varepsilon})$. Clearly,

$$\mathbb{P}(U_\varepsilon) \leq \mathbb{P}(\forall n, S_\varepsilon(n) \geq -1 - K\sqrt{n}).$$

(5.78)

Lemma 5.2 implies that for any $l < 1$

$$\mathbb{P}(U_\varepsilon) \leq c(l)\varepsilon^{p(\frac{K}{l})}.$$  

(5.79)

Hence

$$\mathbb{E}(n(\varepsilon)) = O(\varepsilon^{p(\frac{K}{l})-1})$$  

(5.80)

so that

$$\limsup_{\varepsilon \to 0} \mathbb{E}\left(\frac{n(\varepsilon)}{\varepsilon^{p(\frac{K}{l})-1}}\right) < \infty.$$  

(5.81)

By Fatou’s Lemma, $\liminf_{\varepsilon \downarrow 0} n(\varepsilon) e^{1 - p(K/l)}$ is almost surely bounded, which implies that $dim_H T(K)$ (which is equal to $dim_H(T_1(K))$ by Proposition 5.1) is bounded above by $1 - p(K/l)$ for any $l < 1$. Since $p(K)$ is continuous in $K$, Proposition 5.3 follows.

**Remark 5.4.** We conjecture that $1 - p(K)$ is the exact Hausdorff dimension of $T(K)$.

Finally, Lemma 5.2 also yields the following tameness result.

**Proposition 5.5.** Let $K_1, K_2 > 0$ be small enough so that $p(K_1) + p(K_2) > 1$, where $p(K)$ is defined in Proposition 5.3. For any $j \geq 0$,

$$\mathbb{P}(\exists \tau \in [0, 1] \ s.t. \ \forall t \geq 0, \ - j - K_1\sqrt{t} \leq S_\varepsilon^*(t) \leq +j + K_2\sqrt{t}) = 0.$$  

(5.82)

**Proof.** Define $U^+_\varepsilon$ (resp., $U^-\varepsilon$) to be the event that for some $\tau \in [0, 2\varepsilon]$ and all $t \geq 0$, $S^*(t) \leq +k + K_2\sqrt{t}$ (resp., $S^*(t) \geq -k - K_1\sqrt{t}$). $U^+_\varepsilon$ (resp., $U^-\varepsilon$) is a decreasing (resp., increasing) event with respect to the basic $\xi^{(i,j)}$ processes. Hence, using the FKG inequality, we have

$$\mathbb{P}(U^+_\varepsilon \cap U^-\varepsilon) \leq \mathbb{P}(U^+_\varepsilon) \cdot \mathbb{P}(U^-\varepsilon).$$

Reasoning as in Proposition 5.3 for any $l < 1$, we have

$$\mathbb{P}(U^-\varepsilon) \leq \mathbb{P}(\forall t \geq 0, \ S_\varepsilon(t) \geq -j - K_1\sqrt{t})$$  

(5.83)

$$\leq c_1\varepsilon^{p(\frac{K_1}{l})},$$  

(5.84)

where $S_\varepsilon$ is defined as in the proof of Proposition 5.3. The second inequality is given by (5.79) immediately for $j \leq 1$ and with a little bit of extra effort for all $j$. Symmetrically,

$$\mathbb{P}(U^+_\varepsilon) \leq c_2\varepsilon^{p(\frac{K_2}{l})},$$  

(5.85)
which implies that
\[ \mathbb{P}(U^+_\varepsilon \cap U^-_\varepsilon) \leq c_1 c_2 \, \varepsilon^{p(K_1/l) + p(K_2/l)}. \] (5.86)

Take \( l \) close enough to 1 so that \( p(K_1/l) + p(K_2/l) > 1 \) and define \( N \) as the cardinality of \( \{ \exists \tau \in [0,1] \text{ s.t. } \forall t \geq 0, -j - K_1 \sqrt{t} \leq S_{\tau}(t) \leq +j + K_2 \sqrt{t} \} \). Reasoning as in Lemma 2.1, we have
\[ \mathbb{E}(N) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \mathbb{P}(U^+_\varepsilon \cap U^-_\varepsilon) = 0, \] (5.87)
which completes the proof of the proposition.

6 Scaling Limit

In this section, we discuss the existence of a dynamical Brownian motion (constructed, using the Brownian web, in \cite{NRS08}) and the occurrence of exceptional times for this object.

6.1 Brownian Web and \((1, 2)\) points

Under diffusive scaling, individual random walk paths converge to Brownian motions. In \cite{FINR04}, it was proved (extending the results of \cite{Ar81, TW98}) that the entire collection of discrete paths in the DW converges (in an appropriate sense) to the continuum Brownian web (BW), which can be loosely described as the collection of graphs of coalescing one-dimensional Brownian motions starting from every possible location in \( \mathbb{R}^2 \) (space-time).

Formally, the Brownian web (BW) is a random collection of paths with specified starting points in space-time. The paths are continuous graphs in a space-time metric space \( (\mathbb{R}^2, \rho) \) which is a compactification of \( \mathbb{R}^2 \). \((\Pi, d)\) denotes the space whose elements are paths with specific starting points. The metric \( d \) is defined as the maximum of the sup norm of the distance between two paths and the distance between their respective starting points. (Roughly, the distance between two paths is small when they start from close (space-time) points and remain close afterwards). The Brownian web takes values in a metric space \( (\mathcal{H}, d_{\mathcal{H}}) \), whose elements are compact collections of paths in \((\Pi, d)\) with \( d_{\mathcal{H}} \) the induced Hausdorff metric. Thus the Brownian web is an \((\mathcal{H}, \mathcal{F}_{\mathcal{H}})\)-valued random variable, where \( \mathcal{F}_{\mathcal{H}} \) is the Borel \( \sigma \)-field associated to the metric \( d_{\mathcal{H}} \). The next theorem, taken from \cite{FINR04}, gives some of the key properties of the BW.

**Theorem 6.1.** There is an \((\mathcal{H}, \mathcal{F}_{\mathcal{H}})\)-valued random variable \( W \) whose distribution is uniquely determined by the following three properties.

(o) from any deterministic point \((x, t)\) in \( \mathbb{R}^2 \), there is almost surely a unique path \( B_{(x, t)} \) starting from \((x, t)\).
(i) for any deterministic, dense countable subset $D$ of $\mathbb{R}^2$, almost surely, $W$ is the closure in $(\mathcal{H}, d_H)$ of $\{B(x,t) : (x,t) \in D\}$.

(ii) for any deterministic $n$ and $(x_1,t_1), \ldots, (x_n,t_n)$, the joint distribution of $B(x_1,t_1), \ldots, B(x_n,t_n)$ is that of coalescing Brownian motions (with zero drift and unit diffusion constant).

This characterization provides a practical construction of the Brownian web. For $D$ as above, construct coalescing Brownian motion paths starting from $D$. This defines a skeleton for the Brownian web. $W$ is simply defined as the closure of this precompact set of paths.

We note that generic (e.g., deterministic) space-time points have almost surely only $m_{\text{out}} = 1$ outgoing (to later times) paths from that point and $m_{\text{in}} = 0$ incoming paths passing through that point (from earlier times). An interesting property of the BW is related to the existence of special points with other values of $(m_{\text{in}}, m_{\text{out}})$. In the following, a dominant role is played by the $(1, 2)$ points as we shall explain. Back in the lattice, $(1, 2)$ points correspond to locations where a path starts at a “microscopic” distance from an old path (that started from earlier time; we note that in the count of paths, incoming paths that coalesce at some earlier time are identified) and coalesces with it only after some “macroscopic” amount of time. For such a point, the single incident path continues along exactly one of the two outward paths. The $(1, 2)$ point is either left-handed or right-handed according to whether the incoming path connects to the left or right outgoing path. See Figure 2 for a schematic diagram of the “left-handed” case. Both varieties occur and it is known [FINR04] that each of the two varieties, as a subset of $\mathbb{R}^2$, has Hausdorff dimension 1.

Figure 2: A schematic diagram of a left $(m_{\text{in}}, m_{\text{out}}) = (1, 2)$ point. In this example the incoming path connects to the leftmost outgoing path, the right outgoing path is a newly born path.
6.2 The Dynamical Brownian Web and Exceptional Times

It is natural that there should also exist scaling limits of the DyDW (including of the random walk from the origin evolving in $\tau$, i.e., a dynamical Brownian motion). Indeed, this was proposed by Howitt and Warren [HoW07] who also studied (two dynamical time distributional) properties of any such limit. In [NRS08], we provided a complete construction that we now briefly describe.

A priori, a direct construction in the continuum appears difficult since the DyDW is entirely based on a modification of the discrete arrow structure of the DW, while in the BW it was unclear a priori whether there even is any arrow structure to modify. Two of the main themes of [NRS08] are thus: (i) “Where is the arrow structure of the BW?” and (ii) “How is it modified to yield the DyBW (including a dynamical Brownian motion from the origin)?”. The answer to the first question is that the arrow structure of the BW comes from the $(1,2)$ points. Indeed, one can change the direction of the “continuum” arrow at a given $(1,2)$ point $z$ by simply connecting the incoming path to the newly born path starting from $z$ rather than to the original continuing path. (Back in the lattice, this amounts to changing the direction of an arrow whose switch induces a “macroscopic” effect in the web.) The answer to question (ii) is based on the construction of a Poissonian marking of the $(1,2)$ points (see [NRS08] for details) that indicates which $(1,2)$ points get switched and at what value of $\tau$ does the switch occur. We note that the main difficulty in the construction of the DyBW lies in the fact that between two dynamical times $\tau < \tau'$, one needs to switch the direction of a set of $(1,2)$ points dense in $\mathbb{R}^2$ in order to deduce the web at time $\tau'$ from the one at time $\tau$.

We proceed to discuss the existence of exceptional times for $B_0^\tau$, the dynamical Brownian motion starting from the origin at dynamical time $\tau$. (We remark that our tameness results, Theorem 2.1 and Remark 2.2 can be extended to the continuum DyBW, but the arguments involve some extra Brownian web technology.) Recall that the key ingredient for proving our existence results for the dynamical discrete web is contained in Proposition 3.1 where we estimate how fast the dynamical discrete web decorrelates. The proof of that proposition mostly relies on the observation that $(S_0^\tau, S_0^\tau')$ form a sticky pair of random walks. More precisely, we showed in Lemma 3.1 that along the $t$-axis the pair alternates between periods during which the two paths evolve as a single path (they stick) and periods during which they move independently.

In [NRS08], we proved that $\tau \rightsquigarrow B_0^\tau$ has a similar structure (as suggested in [HoW07]), in that for two distinct dynamical times $\tau, \tau'$, the paths $B_0^\tau, B_0^\tau'$ form a $1/(2|\tau - \tau'|)$-sticky pair of Brownian motion. Such a pair can be simply expressed in terms of three independent standard Brownian
motions \((B^\tau_d, B^\tau_0', B_s)\) in the following way.

\[
B^\tau_0(t) = B^\tau_d(C(t)) + B_s(t - C(t)),
\]
\[
B^\tau_0'(t) = B^\tau_0'(C(t)) + B_s(t - C(t)),
\]
(6.88)

where \(C\) is the continuous inverse of the function

\[
C^{-1}(s) = s + \frac{1}{\sqrt{2}|\tau - \tau'|} l_0(s)
\]
(6.89)

and \(l_0\) is the local time at the origin of the process \((B^\tau_d - B^\tau_0')/\sqrt{2}\). We note that the paths \(B^\tau_0, B^\tau_0'\) always spend a strictly positive Lebesgue measure of time together, hence the name sticky Brownian motions. Finally, the time the two paths spend together is directly related to the parameter \(1/(2|\tau - \tau'|)\) commonly referred to as the “amount of stick” of the pair.

If we denote by \(\tilde{\pi}(\cdot) = \pi(\cdot/\delta^2)\delta\), the scaling invariance for the Brownian motion combined with (6.88) implies that \((\tilde{B}^\tau_0, \tilde{B}^\tau_0')\) is identical in law to a \(\delta/(\sqrt{2}|\tau - \tau'|)\)-sticky pair of Brownian motions. In other words, the amount of stick of the pair \((\tilde{B}^\tau_0, \tilde{B}^\tau_0')\) vanishes as \(\delta \to 0\) and from (6.88) and (6.89) we see that for small \(\delta\),

\[
B^\tau_0(t) \approx B^\tau_d(t) \quad \text{and} \quad B^\tau_0'(t) \approx B^\tau_0'(t),
\]
(6.90)
i.e., the two paths become “almost independent”. This can be made more precise by establishing (along the same lines as the proof of Proposition 3.1) that for

\[
O = \{\forall t \in [0,1], \pi(t) > -1 \text{ and } \pi(1) > 1\}
\]
and \(\delta > 0\), there exist \(K, a \in (0, \infty)\) (independent of \(\delta, \tau\) and \(\tau'\)) such that

\[
P(\tilde{B}^\tau_0 \in O, \tilde{B}^\tau_0' \in O) \leq P(\tilde{B} \in O)^2 + K\left(\frac{\delta}{|\tau - \tau'|}\right)^a
\]
(6.91)
\[
= P(B \in O)^2 + K\left(\frac{\delta}{|\tau - \tau'|}\right)^a,
\]
(6.92)

where \(B\) is a standard Brownian motion.

Since all the results of Section 4 and Subsection 5.1 for our dynamical random walk are based on the discrete analog of this result, Propositions 4.1 and 5.2 can easily be extended to the continuum in the following manner. If we define

\[
A = \{ \inf_{t \in [0,1]} B(t) > -1, B(1) > 1\}
\]
and let \(\gamma_0 = 1/P(A)\), we have:
Theorem 6.2. Let $\bar{T}(K)$ be the set of $\tau$’s belonging to $[0, \infty)$ such that

$$\forall t \geq 0, \ B_\tau(t) \geq -1 - K \sqrt{t}. \quad (6.93)$$

Then $\bar{T}(K)$ is non-empty and

$$\dim_H(\bar{T}(K)) \geq 1 - \frac{\log \gamma_0}{\log \gamma(K)} \text{ for } K > K(\gamma_0). \quad (6.94)$$

Thus, $\lim_{K \to \infty} \dim_H(\bar{T}(K)) = 1$.

We conclude by noting that also our upper bound results on the Hausdorff dimension, Propositions 5.3 and 5.5, can be extended to the continuum DyBW, but, like the tameness results, that extension requires some extra Brownian web technology beyond what is described in this paper.

A Some Estimates On Random Walks (Proof of Lemma 5.2)

We start with the two following lemmas.

Lemma A.1. [S77] Let $j, K \in (0, \infty)$ and let $B$ be a standard Brownian motion. Then there exists $q \in (0, \infty)$ such that

$$\lim_{t \to \infty} t^{p(K)} P(\forall s \in [0, t], B(s) \geq -j - K \sqrt{s}) = q, \quad (A.95)$$

where $p(K)$ is the solution in $(0, 1)$ of the equation

$$f(p, K) \equiv \frac{\sin(\pi p/2) \Gamma(1 + p/2)}{\pi} \sum_{n=1}^{\infty} \frac{(\sqrt{2}K)^n}{n!} \Gamma((n - p)/2) = 1. \quad (A.96)$$

Furthermore, $p(K)$ is a continuous decreasing function on $(0, \infty)$ with

$$\lim_{K \to \infty} p(K) = 0 \quad \text{and} \quad \lim_{K \downarrow 0} p(K) = 1. \quad (A.97)$$

Lemma A.2. Let $K \in (0, \infty), l \in (0, 1)$ and let $S$ be a simple symmetric random walk. Then there exists $c(K, l) \in (0, \infty)$ such that for every $n$

$$n^{p(K/l)^{2/3}} P(\forall k \leq n, \ S(k) \geq -1 - K \sqrt{k}) \leq c(K, l). \quad (A.98)$$

Proof. By Lemma A.1, it suffices to prove that for every $l < 1$, there exists $c(K, l)$ such that

$$P(\forall k \leq n, \ S(k) \geq -1 - K \sqrt{k}) \leq c(K, l) \quad (A.99)$$

We now prove the latter inequality. Consider $S$ the discrete time random walk embedded in the Brownian motion $B$. Namely, we define inductively a sequence of stopping times $t_i$ with $t_0 = 0$ and

$$t_{i+1} = \inf\{t > t_i : |B(t) - B(t_i)| \geq 1\} \quad (A.100)$$
and then we define $S(i) = B(t_i)$. Note that $S$ and $\{t_i\}$ are independent and therefore

$$
\mathbb{P}(\forall k \leq n, S(k) \geq -1 - K\sqrt{k})
$$

$$
= \mathbb{P}(\forall k \leq n, S(k) \geq -1 - K\sqrt{k}, t^2k \leq t_k) / \mathbb{P}(\forall k \leq n, t^2k \leq t_k \leq k),
$$

$$
= \mathbb{P}(\forall k \leq n, B(t_k) \geq -1 - K\sqrt{k}, t^2k \leq t_k) / \mathbb{P}(\forall k \leq n, t^2k \leq t_k \leq k),
$$

$$
\leq \mathbb{P}(\forall k \leq n, B(t_k) \geq -1 - K\sqrt{t_k}/l, t^2k \leq t_k) / \mathbb{P}(\forall k \leq n, t^2k \leq t_k \leq k).
$$

If for every $k \leq n$, we have $B(t_k) \geq -1 - K\sqrt{t_k}/l$ and moreover $t^2k \leq t_k$, then on $[0, l^2n]$, every time $B$ takes an integer value, $B$ is to the right of $t \sim -1 - K\sqrt{t}/l$. Hence, $B$ remains to the right of $t \sim -2 - K\sqrt{t}/l$ on $[0, l^2n]$ which implies that

$$
\mathbb{P}(k \leq n, S(k) \geq -1 - K\sqrt{k})
$$

$$
\leq \mathbb{P}(\forall t \leq l^2n, B(t) \geq -2 - K\sqrt{t}/l) / \mathbb{P}(\forall k \leq n, t^2k \leq t_k \leq k).
$$

Finally, $t_k$ is a sum of $k$ i.i.d. random variables with mean 1 (whose common distribution includes 1 in its support). Therefore, $\mathbb{P}(\forall k \in \mathbb{N}, t^2k \leq t_k \leq k) > 0$ and (A.99) follows. This completes the proof of the lemma.

We are now ready to prove Lemma 5.2. Recall that $S_\epsilon$ is a simple random walk with

$$
\mathbb{P}(S_\epsilon(k + 1) - S_\epsilon(k) = 1) = \frac{1}{2} + \frac{1}{2}(1 - e^{-\epsilon}).
$$

Since $\frac{1}{2} + \frac{1}{2}(1 - e^{-\epsilon}) \leq \frac{1}{2}(1 + \epsilon)$, it is enough to show the conclusions of the lemma for the simple walk $\tilde{S}_\epsilon$ where $p^\pm_\epsilon = \mathbb{P}(\tilde{S}_\epsilon(k + 1) - \tilde{S}_\epsilon(k) = \pm 1) = \frac{1}{2}(1 \pm \epsilon)$.

Let $T_\epsilon = \inf\{n > 0 : \tilde{S}_\epsilon(n) < -1 - K\sqrt{n}\}$. We have

$$
\mathbb{P}(T_\epsilon = n) = \mathbb{P}(T_0 = n) \ f_\epsilon(n) \quad \text{(A.101)}
$$

with

$$
\begin{align*}
\ f_\epsilon(n) &= (2p^-_\epsilon)^{\left(\lfloor n - \lceil 1 + K\sqrt{\epsilon}\rceil \right)} (2p^+_\epsilon)^{\left(\lfloor n - \lceil 1 + K\sqrt{\epsilon}\rceil - 1 \right)},
\end{align*}
$$

(A.102)

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Since a simple symmetric random walk $S$ eventually hits the moving boundary $t \sim -1 - K\sqrt{t}$ we have

$$
\mathbb{P}(T_\epsilon = \infty) = 1 - \sum_{n \geq 1} f_\epsilon(n)\mathbb{P}(T_0 = n) = \sum_{n \geq 1} (1 - f_\epsilon(n))\mathbb{P}(T_0 = n).
$$

(A.103)
Then, proceeding to a summation by parts we have

\[
P(T_\epsilon = \infty) = \sum_{n \geq 1} (P(T_0 \geq n) - P(T_0 \geq n + 1)) [1 - f_\epsilon(n)],
\]

(A.104)

\[
= \sum_{n \geq 1} P(T_0 \geq n + 1)(f_\epsilon(n) - f_\epsilon(n + 1)) + (1 - f_\epsilon(1)),
\]

(A.105)

\[
= S_\epsilon + (1 - f_\epsilon(1)).
\]

(A.106)

with \( S_\epsilon \equiv \sum_{n \geq 1} P(T_0 \geq n + 1)f_\epsilon(n)(1 - \frac{f_\epsilon(n + 1)}{f_\epsilon(n)}) \)

(A.107)

We proceed to estimate \( S_\epsilon \). First, for \( \epsilon \in (0, 1) \), we have

\[
\ln(1 + \epsilon) \leq \epsilon, \quad \ln(1 - \epsilon) \leq -\epsilon,
\]

implying that

\[
f_\epsilon(n) = \exp\left(\frac{1}{2} \ln(1 - \epsilon)(n + [1 + K\sqrt{n}] + 1) + \frac{1}{2} \ln(1 + \epsilon)(n - [1 + K\sqrt{n}] - 1)\right)
\]

\[
\leq \exp\left\{-\frac{\epsilon}{2}(n + [1 + K\sqrt{n}] + 1) + \frac{\epsilon}{2}(n - [1 + K\sqrt{n}] - 1)\right\}
\]

\[
\leq \exp\{-\epsilon([1 + K\sqrt{n}] + 1)\}
\]

\[
\leq \exp\{-\epsilon K\sqrt{n}\}.
\]

Next, if we set \( \Delta_n \equiv [1 + K\sqrt{n + 1}] - [1 + K\sqrt{n}] = [K\sqrt{n + 1}] - [K\sqrt{n}] \), we have

\[
1 - \frac{f_\epsilon(n + 1)}{f_\epsilon(n)} = 1 - \exp\{-\epsilon\Delta_n - \frac{\epsilon^2}{2} + o(\epsilon)\Delta_n + o(\epsilon^2)\} = \epsilon\Delta_n + \frac{\epsilon^2}{2} + o(\epsilon)\Delta_n + o(\epsilon^2).
\]

By Lemma \ref{lem:A2} for every \( l < 1 \) there exists \( c(K, l) \) such that

\[
S_\epsilon \leq \sum_{n \geq 1} P(T_0 \geq n + 1) \exp\{-\epsilon K\sqrt{n}\} (\epsilon\Delta_n + \frac{\epsilon^2}{2} + o(\epsilon)\Delta_n + o(\epsilon^2))
\]

(A.108)

\[
\leq c(K, l) \sum_{n \geq 1} \exp\{-\epsilon K\sqrt{n}\} \left(\epsilon\Delta_n + \frac{\epsilon^2}{2n} + o(\epsilon)\Delta_n + \frac{o(\epsilon^2)}{n}\right).
\]

(A.109)

where \( p \equiv p(K/l) \) is as in Lemma \ref{lem:A2}. Since \( \Delta_n = [K\sqrt{n + 1}] - [K\sqrt{n}] \), and \( (\sqrt{n + 1} - \sqrt{n})\sqrt{n} \to 1/2 \) as \( n \to \infty \), it is natural to expect that

\[
\lim_{\epsilon \to 0} \epsilon^2 \sum_{n \geq 1} \exp\{-\epsilon K\sqrt{n}\} \frac{\Delta_n}{\epsilon} \frac{1}{(\sqrt{n}^2)^p}
\]

\[
= \lim_{\epsilon \to 0} \epsilon^2 \sum_{n \geq 1} \exp\{-\epsilon K\sqrt{n}\} \frac{K}{2\sqrt{ne^2}} \frac{1}{(\sqrt{n}^2)^p}
\]

\[
= \frac{1}{2} \int_0^{\infty} \exp\{-Kt\} \frac{K}{t^{(p + 1)/2}} \, dt
\]

(A.110)
where the second equality is due to the Riemann sum on the right hand side of the first equality. To justify the first equality one may note that $\Delta_n$ is (for large $n$) either 0 or 1 and then define $N_\ell(n)$ (resp., $N_u(n)$) to be the largest $m \leq n$ (resp., smallest $m > n$) such that $\Delta_m \neq 0$. It is straightforward to show first that that $N_u(n) - N_\ell(n)/\sqrt{n} \to 2/K$ as $n \to \infty$ and then to obtain the first equality of (A.110) as a consequence. It is also the case that

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n \geq 0} \exp\{-K\sqrt{\epsilon^2 n}\} \frac{1}{(\epsilon^2 n)^{p/2}} = \int_0^\infty \exp\{-K\sqrt{t}\} \frac{1}{t^{p/2}} \, dt.$$  (A.111)

Since $0 < p < 1$ both integrals in (A.110) and (A.111) are finite. Thus, (A.109) yields $S_\epsilon = O(\epsilon^p) = O(\epsilon^{p(K/l)})$.

Finally, it is easy to prove that $f_\epsilon(1) - 1 = O(\epsilon)$. Since $p(K/l) < 1$, Lemma 5.2 follows from (A.106).

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