Multiple Nash-equilibrium in Quantum Game

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Abstract

Methods of exploring Nash equilibrium in quantum games are studied. Analytical conditions of the existence, the uniqueness or the multiplicity of the equilibria are found.

Several aspects of an antagonistic game, with one of the parties demonstrating opportunism were studied in [1, 2, 3]. Since the opportunistic behavior is enabled by quantum logic [4], a quantum version of the game was considered. It was found that using quantum strategies rather than usual mixed ones can augment the medium payoff of one of the players. However, in contrast with [5], where this phenomenon is caused by entangled states, in [3] this effect is due to the breach of distributivity.

In the first paper [1] quantum equilibrium was found approximately, using methods of numeric simulation, which was an obstacle to study qualitative effects, and only in [3] some analytic approaches were put forward. The present paper analysis the existence, the uniqueness and the multiplicity of Nash equilibria.

Quantum game. The analytic game considered in [1] reduces to the of the pay-operator of the form

\[ H = c_3 A_1 \otimes B_3 + c_1 A_3 \otimes B_1 + c_4 A_2 \otimes B_4 + c_2 A_4 \otimes B_2 \]

where \( c_j \) are non-negative numbers and \( A_j, B_k \) are self-adjoint operators in Hilbert spaces \( \mathcal{H}_A, \mathcal{H}_B \). These projectors correspond to pure strategies of the players and satisfy the commutation relations

\[
\begin{align*}
A_1 + A_3 &= I = A_2 + A_4, & A_1 A_3 &= A_2 A_4 = 0, & [A_1 A_2] &\neq 0 \\
B_1 + B_3 &= I = B_2 + B_4, & B_1 B_3 &= B_2 B_4 = 0, & [B_1 B_2] &\neq 0 
\end{align*}
\]

If the players use the quantum strategies \( \varphi \in \mathcal{H}_A, \psi \in \mathcal{H}_B \), then the average payoff of the first player is

\[
\langle H \rangle = \langle \varphi \otimes \psi | H | \varphi \otimes \psi \rangle = c_3 p_1 q_3 + c_1 p_3 q_1 + c_4 p_2 q_4 + c_2 p_4 q_2
\]

where \( p_j = \langle \varphi | A_j | \varphi \rangle, \ q_k = \langle \psi | B_k | \psi \rangle \) are the probability for the players to use the pure strategies \( j, k \).

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The reduction of quantum game. In the model [1, 3], two-dimensional real spaces $\mathcal{H}_A, \mathcal{H}_B$ were used. In this case $\text{rk} A_j = \text{rk} B_j = 1$, therefore for some rotations $U(\theta), U(\tau) \in SO(2)$ the following relations hold

$$A_2 = U^\dagger(\theta) A_1 U(\theta), \quad B_2 = U^\dagger(\tau) B_1 U(\tau), \quad (0 < \theta, \tau < \frac{\pi}{2}) \quad (3)$$

The quantum strategies of the players were represented by the vectors on the plane:

$$\varphi = (\cos \alpha, \sin \alpha), \quad \psi = (\cos \beta, \sin \beta)$$

In this case the probabilities of pure strategies satisfy the equations

$$p_1 = \cos^2 \alpha, \quad p_2 = \cos^2(\alpha - \theta), \quad q_1 = \cos^2 \beta, \quad q_2 = \cos^2(\beta - \tau) \quad (4)$$

where $\theta, \tau$ are angular parameters, related to the entangling relations (3).

Using the linear exchange of the variable

$$2p = M_\theta x + e, \quad 2q = M_\tau y + e$$

where

$$M_\gamma = \begin{bmatrix} \cos \gamma & -\sin \gamma \\ \cos \gamma & \sin \gamma \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the equations (4) read

$$x_1^2 + x_2^2 = 1, \quad y_1^2 + y_2^2 = 1$$

So, each player chooses a point on the unit circle and the quantum game is reduced to a classical one on a torus.

Denote, following [3]

$$a = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad b = \begin{bmatrix} c_3 \\ c_4 \end{bmatrix}, \quad \omega = b - a, \quad n = c_1 + c_3, \quad m = c_2 + c_4, \quad C = \begin{bmatrix} n & 0 \\ 0 & m \end{bmatrix}$$

$$A = M_\theta^\dagger CM_\tau, \quad u = M_\theta^\dagger \omega, \quad v = M_\tau^\dagger \omega \quad (5)$$

Then the average payoff (2) takes the form

$$g(x, y) + \text{tr} C$$

where

$$g(x, y) = -\langle x, Ay \rangle + \langle x, u \rangle - \langle v, y \rangle$$

Proposition 1. (The equilibrium criterion, see [3]) A pair of unit vectors $(x, y)$ forms a Nash equilibrium if and only if when nonnegative numbers $\lambda, \mu$ exist, for which the following holds

$$-Ay + u = \lambda x, \quad A^\dagger x + v = \mu y \quad (6)$$

Proposition 2. (see [3]) If the equilibrium exists, then $\omega \neq 0$. 

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Eigenequilibria. An equilibrium \((x, y)\) is called \textit{eigenequilibrium}, if it is an eigenvector of the matrix 
\[
\begin{bmatrix}
0 & A \\
A^\dagger & 0
\end{bmatrix}
\]

\textbf{Proposition 3.} (see [3]) \textit{If the eigenequilibrium exists, then \(\omega\) is a common eigenvector of the matrices \(CM_0M_0^\dagger, CM_1M_1^\dagger\).}

A game is said to be \textit{non-degenerate}, if
\[
\Delta = \left| \begin{array}{cc} n & m \\ \omega_1^2 & \omega_2^2 \end{array} \right| \neq 0
\]

\textbf{Proposition 4.} (see [3]) \textit{If the game is non-degenerate, then the necessary condition for the eigenequilibrium to exist is the coincidence of the angular parameters \(\theta = \tau\). In this case their values are completely determined by the payoff coefficients of the game \(\{c_j\}\):

\[
\cos 2\theta = \cos 2\tau = \frac{(m - n)\omega_1\omega_2}{\Delta}
\]

Further finding \textit{eigenequilbria} of \textit{non-degenerate} games, calculate \(\theta\) using (8) and put \(M = M_0\), \(z = M^1\omega\). In this case \(A = A^\dagger = M^1CM\) and the matrix \(A\) nonnegatively defined. The equilibrium criterion (6) becomes simpler:

\[
\begin{align*}
z - Ay &= \lambda x, \\
z + Ax &= \mu y
\end{align*}
\]

\textbf{Theorem 5.} (First existence theorem) \textit{Let a vector \(\omega\) be an eigenvector of the matrix \(CMM_0^\dagger\) and \(\langle Az, z \rangle \leq |z|^3\). Then the strategies \(x = y = z/|z|\) form an eigenequilibrium.}

\textbf{Proof.} If \(CMM_0^\dagger = \alpha \omega\), then \(M^1CM_1^\dagger = \alpha M_1^\dagger\omega\), that is, \(Az = \alpha z\). Since the matrix \(A\) is symmetric, the vector \((x, y)\) is an eigenvector of \(A\). It remains to check that it forms an equilibrium.

\[
\begin{align*}
z - Ay &= z - \alpha y = z - \frac{\langle Az, z \rangle}{|z|^2} \cdot \frac{z}{|z|} = (1 - \frac{\langle Az, z \rangle}{|z|^3})|z| \cdot x \\
z + Ax &= z + \alpha x = z + \frac{\langle Az, z \rangle}{|z|^2} \cdot \frac{z}{|z|} = (1 + \frac{\langle Az, z \rangle}{|z|^3})|z| \cdot y
\end{align*}
\]

Since
\[
(1 - \frac{\langle Az, z \rangle}{|z|^3})|z| \geq 0, \quad (1 + \frac{\langle Az, z \rangle}{|z|^3})|z| \geq 0
\]

the sufficient conditions of the equilibrium are satisfied.

\textbf{Theorem 6.} (Second existence theorem) \textit{Let a vector \(\omega\) be an eigenvector of the matrix \(CMM_0^\dagger\) and \(\langle Az, z \rangle = |z|^3\). Then there are two eigenequilbria \(x = y = z/|z|\), \(x = -z/|z|\), \(y = z/|z|\).}
Proof. The first equilibrium was already obtained in the first theorem, and it remains to prove that the second vector is also an equilibrium.

\[ z - Ay = z - \frac{\langle Az, z \rangle}{|z|^2} \cdot \frac{z}{|z|} = \left( 1 - \frac{\langle Az, z \rangle}{|z|^3} \right) |z| \cdot x = 0 \cdot x \]

\[ z + Ax = z + \alpha x = z - \frac{\langle Az, z \rangle}{|z|^2} \cdot \frac{z}{|z|} = \left( 1 - \frac{\langle Az, z \rangle}{|z|^3} \right) |z| \cdot y = 0 \cdot y \]

So, the sufficient conditions of the theorem are satisfied. □

Theorem 7. (Uniqueness theorem) Let there is a game with a non-degenerate equilibrium \( \langle Az, z \rangle \neq |z|^3 \). Then all possible equilibria are exhausted by it.

Proof. Let \((a, b)\) be an arbitrary equilibrium, and \((x, y)\) be an eigenequilibrium. According to the well-known ‘rectangular’ property of antagonistic games, \((a, y)\) and \((x, b)\) are also equilibria, furthermore, the value of the game is the same in all these points. Since \((x, y)\) is an eigenequilibrium, the vectors \(x, y\) are proportional to \(z\). It is trivially checked that there are only two possibilities for an eigenequilibrium:

\[ x = y = \frac{z}{|z|} \quad \text{or} \quad x = -\frac{z}{|z|}, \quad y = \frac{z}{|z|}. \]

When \(x = y = \frac{z}{|z|}\) we have

\[ g(a, y) = -\langle a, Ay \rangle + \langle a, z \rangle - \langle z, y \rangle = (|z| - \alpha)\langle a, x \rangle - |z| \]

Comparing it with

\[ g(x, y) = -\langle x, Ay \rangle + \langle x, z \rangle - \langle z, y \rangle = -\langle x, \alpha x \rangle = -\alpha \]

we obtain \((|z| - \alpha)\langle a, x \rangle - |z| = -\alpha\), hence \((|z| - \alpha)\langle a, x \rangle = |z| - \alpha\). The condition

\[ \alpha = \frac{\langle Az, z \rangle}{|z|^2} \neq |z| \quad (10) \]

implies \((a, x) = 1\). Since \(a\) and \(x\) are unit vectors, \(a = x\).

Consider further

\[ g(x, b) = -\langle x, Ab \rangle + \langle x, z \rangle - \langle z, b \rangle = -(|z| + \alpha)\langle y, b \rangle + |z| \]

and comparing it with \(g(x, y) = -\alpha\), we get \(-(|z| + \alpha)\langle a, x \rangle + |z| = -\alpha\), hence \((|z| + \alpha)\langle y, b \rangle = |z| + \alpha\) so \(y = b\).

For the second candidate for the equilibrium \(x = -z/|z|, \quad y = z/|z|\) We have

\[ g(a, y) = -\langle a, Ay \rangle + \langle a, z \rangle - \langle z, y \rangle = (\alpha - |z|)\langle a, x \rangle - |z| \]

Comparing it with

\[ g(x, y) = -\langle x, Ay \rangle + \langle x, z \rangle - \langle z, y \rangle = \alpha - 2|z| \]

we get \((\alpha - |z|)\langle a, x \rangle - |z| = \alpha - 2|z|\), hence \((\alpha - |z|)\langle a, x \rangle = \alpha - |z|\). From \([10]\) we have \(\langle a, x \rangle = 1\) therefore \(a = x\). In a similar way we obtain \(b = y\). □
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