EFFECT OF WARRANTY AND QUANTITY DISCOUNTS ON A
DETERIORATING PRODUCTION SYSTEM WITH A
MARKOVIAN PRODUCTION PROCESS AND
ALLOWABLE SHORTAGES

Tien-Yu Lin*
Economics and Management College
Zhaoqing University
Zhaoqing City 526061, Guangdong Province, China

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Abstract. This paper explores the retailer’s optimal lot sizing and quantity
backordering for a deteriorating production system with a two-state Markov
production process in which quantity discounts are provided by the supplier.
The products are sold with the policy of free reasonable repair warranty em-
ploying the fraction of nonconforming items in a lot size. Unlike the traditional
economic production quantity (EPQ) model with warranty policy based on the
elapsed time of the system in the control state follows an exponential distri-
bution, this paper not only constructs an alternative mathematical model for
EPQ model based on the fraction of nonconforming items in a lot size for an
imperfect production system but also extends the topics of optimal quantity
and shortage to a wider scope of academic research and further finds that some
results are different from the traditional EPQ models. We seek to minimize
the expected total relevant cost through optimal lot sizing and quantity back-
ordering. We also demonstrate that the optimal lot size is bounded in a finite
interval. An efficient algorithm is developed to determine the optimal solution.
Moreover, a numerical example is given and sensitivity analysis is conducted
to highlight management insights.

1. Introduction. How to maintain appropriate inventory level is always an im-
portant economic issue for many industries [37]. A lot of research showed that
good inventory management could lead to increased revenue, lower handling and
holding costs, and improved cash flow. In his famous economic production quantity
(EPQ) inventory model, Taft [39] assumed that production process and products
were failure-free. However, in practice product quality is not perfect and usually
depends on the state of the production process [11]. To reflect the reality, many
researchers extend the EPQ model to consider deteriorating production processes
and imperfect product quality.

In their pioneer work, Porteus [32] and Rosenblatt and Lee [33] employed the
classic economic order quantity (EOQ) model to study the effect of process dete-
rioration on the relationship between lot size and quality imperfection. Since then
many researchers extended the work of Proteus [32] and Rosenblatt and Lee [33],

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* Corresponding author: Tien-Yu Lin.
including Van Beek and Van Putten [45], Tapiero et al. [44], Chand [4], and Zhang and Gerchak [54]. Yano and Lee [50] and Wright and Mehrez [49] conducted detailed surveys of inventory models with random yields and imperfect product quality. Recently, Hou et al. [11] considered a deteriorating production process with maintenance cost incurred while in uncontrolled state. Maddah et al. [22] proposed an EOQ-type model for a two-state Markovian production process with imperfect quality items. Sana [34] and Yoo et al. [53] developed an EPQ model for a deteriorating production process in which the fraction of items in a lot size is defective under the uncontrolled state of the production system. For related researches of production processes with imperfect product quality, see for examples Lin [19]; Pal et al. [29–31]; Sarkar [36]; Jaber, et al. [14]; Zhou et al. [55]; and Moussawi-Haidar et al. [24]. Refer to Khan et al. [15] for a recent review. The aforementioned models with imperfect production systems assume that shortages are not allowed. However, in practice, stock out is inevitable because of various uncertainties. Therefore, several researchers (e.g. [7–10]) have extended the traditional deteriorating EPQ model with the assumption of allowable shortages. Recently, Tai [40] developed two EPQ models for imperfect items with a rework process in which he assumed that shortages are allowed and backlogged. Ouyang and Chang [28] explored the effects of imperfect quality items with rework procedures and the permissible delay in payment on an imperfect production system for the economic production quantity model with complete backordering. Taleizadeh et al. [41] proposed an EPQ model with an interruption in the process, scrap, and rework, wherein the inventory model is for a multiple-product single-machine system considering backorders. San-José et al. [35] presented an inventory model assuming that the unit holding cost has significant components: a fixed cost and a variable, where shortages are allowed and a mixture of backorders and lost sales is assumed during the stock-out period. Taleizadeh et al. [42], developed an imperfect EPQ model with defective items and a reworking process by considering two levels for the credit period in the supply chain from the manufacturer’s side in which shortages are allowed. Huang and Wu [12] developed a continuous-time Markov decision process for analyzing the optimal control for a production-inventory system with a shortage cost function involving a nonzero fixed term. For further review of the EPQ model with imperfect quality and permissible shortages, the reader can refer to the works of Sarkar et al. [38]; Kumar and Goswami [16]; Barron and Hermel [2]; and Taleizadeh et al. [43].

For an imperfect production, several researchers (e.g., Lee and Park [18]; Makis [23]; Wang and Sheu [48]) have employed maintenance and inspection to minimize the imperfect quality items in a deteriorating production process. For those imperfect quality items, one may either reprocess the imperfect items or sell these items with warranty, which incurs extra warranty costs. Several researchers have dedicated their efforts in the issue of products sold with warranty, especially for the deteriorating production system. A detailed survey and review of various issues related to warranty policies can be found in Blischke and Murthy [3], Murthy and Blischke [25, 26], and Murthy and Djamaludin [27]. Recently, Yeh et al. [52] and Wang [47] developed the EPQ models with a two-state Markovian deteriorating production process where the items are sold with the policy of free reasonable repair warranty employing the fraction of nonconforming items in a lot size. Chen and Lo [5] explored an imperfect production system with allowable shortages for products sold with free minimal repair warranty; in this system, they assumed that
the elapsed time is an exponential distribution under the controlled production stage. Yeh et al. [51] explored the optimal replacement strategy minimizing the total cost for the repairable items under the free reasonable repair warranty. Van der Heijden and Iskandar [46] considered the combined optimization of repair and replacement decisions as well as the items sold with minimal repair warranty. Most recently, some researchers (e.g., Liu et al. [20], Lee et al. [17], Chen et al. [21], Luo and Wu [1], Alqahtani et al. [6], Chien et al. [13]) explored the effects of warranty policy with free-repair or/and free-replacement for the items sold in a deteriorating production system. All of the aforementioned models assume that the product direct cost is irrelevant to the quantity purchased and that the elapsed time of the system in the control state follows an exponential distribution. The assumption, the product direct cost is irrelevant to the quantity purchased, for the above literature sources may not be true in some cases. Recent researchers have employed quantity discounts to obtain economic advantages, including lower ordering costs, lower per-unit purchasing costs, and the less probability of shortages [19]. They also recognized that quantity discounts provided by the supplier may significantly influence the buyer’s decision making. Employing quantity discounts in the EPQ model with free minimal repair warranty and shortages, which had not been discussed in previous studies, is therefore worth further study.

Although Yeh et al. [52] incorporated the warranty cost in the EPQ model is intuitively meaningful for mathematical easiness, they employed a McClaurin series to approximate optimal production run length and thus over evaluate the total cost. Three special cases with bounds for solving the optimal production period are further discussed. Previous research assumed that in a production system the time elapsed from a controlled to an uncontrolled state is an exponential distribution. Alternatively, this paper is based on the fraction of nonconforming items in a lot size and investigates an imperfect production system with allowable shortages and quantity discounts. Thus, we deal with the deterioration process system with a two-state Markovian production process (as in Porteus [32]) when the products are sold with a free minimal repair warranty, employing the fraction of nonconforming items in a lot size. The best example to use for this study is AC/DC (Alternating Current / Direct Current) cooling fan manufacture in China. The contributions of this paper are: (1) This work not only constructs an alternative mathematical model for the EPQ model with a free-repair warranty policy but also extends the topics of optimal quantity and shortage to a wider scope of academic research and further finds that some results are different from the traditional EPQ models. (2) This paper presents unique optimal lot sizing and backordering quantity that minimize the expected total relevant cost, regardless of whether the probability that the system will shift from the controlled to the uncontrolled state is low. Hence, this contribution made by this paper is different from Porteus’s [32] work. (3) The study provides new insights into production management strategies and inventory control. (4) An efficient solution algorithm that considerably reduces the procedures for finding the optimal lot size and backordering quantity was developed.

The remainder of this paper is organized as follows. Section 2 describes the notations and model environment including assumptions. Section 3 formulates the proposed problem as a minimization model where the lot size and backordering quantity are decision variables. Section 4 derives some theorems and an algorithm to determine the optimal lot size and backordering quantity. A numerical example and sensitivity analysis are provided in section 5. Section 6 makes conclusions.
2. Model environment and notations. In reality, production processes continuously deteriorate because of operating issues such as erosion, fatigue, and destruction. Due to continuous deterioration, the operating state of a manufacturing process may be in either of the two possible states: controlled (in control) or uncontrolled (out of control). Production process is always in a controlled state at the beginning of a production run, and it may transfer to an uncontrolled state with probability $p$ (or stay in controlled state with probability $1-p$).

Due to manufacturing variability, an item is nonconforming with probability $\lambda_1$ (or $\lambda_2$) defined in the notations section. The production process is controlled (or out of control) when $\lambda_1 < \lambda_2$. Because a nonconforming item can only be detected after the production process has been employed for a specific time, all items produced are released for sale with a free minimal repair warranty (Yeh et al. [45]). Under the policy of reasonable free-repair warranty, the defective items identified within the warranty period lead to effective warranty assertions and are repaired without imposing any cost on the buyer. After an overhaul, the hazard ratio of a product remains the same as before the breakdown. Each minimal repair incurs a warranty cost to the manufacturer. Additionally, assumed shortages are allowable in a deteriorating production system. This paper employs an all-unit quantity discount policy to handle the cost structure.

The following notations, adapted from Porteus [32] and Yeh et al. [52], are used in the development of our mathematical model for a deteriorating production system with Markovian production process and allowable shortages:

- $i$: the number of goods in a lot sizing
- $n$: index of all-unit quantity discount category, $n = 1, 2, \ldots, r$
- $D$: demand rate
- $M$: production rate, $M > D$
- $K$: setup cost for each production cycle
- $c_n$: purchasing cost of material per unit for $n$th level
- $I$: unit-holding cost per unit, represented as a fraction of dollar value
- $L$: lot sizing (decision variable) per cycle, $L \in [L_{n-1}, L_n]$ in which $[L_{n-1}, L_n)$ is the internal of lot sizing corresponding to the purchasing cost per unit $c_n$.
- $S$: the maximum backordering quantity (decision variable), $S$:
- $R$: restoring cost for the production system from uncontrolled state back to controlled state
- $b$: the backordering cost per unit
- $p$: the transaction probability that the production system from controlled state transfers to uncontrolled state
- $\bar{p}$: the probability that the production system remains in controlled state and $\bar{p} = 1 - p$
- $\lambda_1$: the percentage of nonconforming items when the production system stays in controlled state
- $\lambda_2$: the percentage of nonconforming items when the production system stays in uncontrolled state, $0 < \lambda_1 < \lambda_2 < 1$
- $\alpha(L)$: the fraction of the nonconforming items
- $V$: the maximum inventory level per cycle
- $N$: number of nonconforming items manufactured per cycle
- $T$: cycle time for each manufacturing run
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$ATC_n(L,S)$: the expected total relevant cost per unit time for lot sizing $L$ and maximum backordering quantity $S$, which corresponds to the purchasing cost per unit $c_p$

$c_w$: unit-warranty cost incurred at each minimal repair

$h_1(\tau)$: failure rate function for a conforming product

$h_2(\tau)$: failure rate function for a nonconforming product

$\omega$: warranty period

$T_1$: production time when backorder is replenished

$T_2$: production time when inventory builds up

$T_3$: time period when no production and inventory deplete

$T_4$: time period when no production but shortage occurs

3. Mathematical formulation. The operating stage of the production system can be classified into a controlled (in-control) or an uncontrolled (out-of-control) status. The system is not self-recovered and the deteriorating process can be formulated by a two-state discrete-time Markov chain with the one-step transition matrix $\begin{bmatrix} p & p \\ 0 & 1 \end{bmatrix}$. Let $\alpha(L)$ be the fraction of the nonconforming items and $\bar{p}$ be the probability that the system remains controlled state within the production of an item. We therefore need the following lemma to help model development.

**Lemma 1.** The fraction of nonconforming quantities in a lot size $L$ is

$$\alpha(L) = \lambda_2 + (\lambda_1 - \lambda_2) \cdot \frac{1}{L} \sum_{k=1}^{L} \bar{q}^k$$  \hspace{1cm} (1)

**Proof.** See Appendix A.

The total relevant cost per unit is composed of the purchasing cost, setup cost, holding cost, backorder cost, restoration cost and warranty cost. Based on the four inventory stages shown in Fig. 1, these costs are derived as follows.

**Figure 1.** The inventory level for imperfect manufacturing system with allowable shortages
The total relevant cost per unit is composed of the purchasing cost, setup cost, holding cost, backorder cost, restoration cost and warranty cost. Based on the four inventory stages shown in Fig. 1, these costs are derived as follows.

The purchasing cost \((PC)\) per unit time corresponding to the quantity discount scheme is given by

\[
PC = c_n D, \quad n = 1, 2, \ldots, r
\]  

(2)

The setup cost \((SC)\) per unit time is given by

\[
SC = \frac{KD}{L}
\]  

(3)

Because the inventory holding cost \((HC)\) is complex, we employ Fig. 1 to illustrate the inventory behavior, which indicates the relationship between \(T_2\), \(T_3\), and \(V\). From Fig. 1, we know the inventory level per cycle is the upper triangular area above the x-axis and thus the maximum inventory level \(V\) is

\[
V = L(1 - D/M) - S
\]

\(T_2\), the time of manufacturing production when the inventory is accumulated, is

\[
T_2 = V/(M - D)
\]

and \(T_3\), the time period when no production and inventory deplete, is

\[
T_3 = V/D
\]

Therefore, the average over the cycle is the area under the inventory triangle, divided by \(T\). Hence, we have average inventory \((AI)\) per cycle as

\[
AI = \left[ L \left( 1 - \frac{D}{M} \right) - S \right] / 2L \left( 1 - \frac{D}{M} \right)
\]  

(4)

According to Eq. (4), the holding cost per unit time corresponding to the unit-purchasing \(c_n\) is evaluated as

\[
HC = c_n I \left[ L \cdot \left( 1 - \frac{D}{M} \right) - S \right] / 2L \left( 1 - \frac{D}{M} \right)
\]  

(5)

Similarly, we employ Fig. 1 to illustrate the backordering cost occurs during \(T_1\) and \(T_4\), in which \(T_1 = S/(M - D)\) represents the manufacturing time when backorder is refilled and \(T_4 = S/D\) represents the time interval when no manufacturing item but shortage occurs. The average backordering level \((ABL)\) is

\[
ABL = \left\{ S^2 / \left[ 2L \left( 1 - \frac{D}{M} \right) \right] \right\}
\]  

(6)

According to Eq. (6), the backordering cost \((BC)\) per unit time is

\[
BC = bS^2 / \left[ 2L \left( 1 - \frac{D}{M} \right) \right]
\]  

(7)

For the formulation of the restoration cost \((RC)\) per unit time, we note it only occurs when the manufacturing process is uncontrolled at the end of a manufacturing cycle for a lot size \(L\). Hence, the restoration cost per unit time is

\[
RC = \frac{DR (1 - \varphi L)}{L}
\]  

(8)

As to the warranty cost \((WC)\) per unit time, we note that the failure process for a conforming (or a nonconforming) item is a non-homogeneous process with intensity \(h_1(\tau)\) (or \(h_2(\tau)\)) (Yeh et al. [45]) under the free minimal repair warranty. Therefore,
given a compliant product, the probability that this compliant product fails during the warranty time interval $\omega$ is $\int_0^{\omega} h_1(\tau) d\tau$. Alternatively, given a nonconforming item, the probability that this nonconforming item fails within the warranty period $\omega$ is $\int_0^{\omega} h_2(\tau) d\tau$. According to Lemma 1, the fraction of nonconforming products in a lot size is $\alpha(L)$. Therefore, for a given item, the probability for this item to fail during the warranty time interval (Prob($w$)) is

$$\text{Prob}(w) = [1 - \alpha(L)] \int_0^{\omega} h_1(\tau) d\tau + \alpha(Q) \int_0^{\omega} h_2(\tau) d\tau$$

(9)

Therefore, we have the expected warranty cost for the sold items per unit time as follows:

$$WC = c_n D \left\{ \frac{KD}{L} + c_n I \left[ L \left( 1 - \frac{D}{M} \right) - S \right] \right\}^{2} / \left[ 2L \left( 1 - \frac{D}{M} \right) \right]$$

$$+ \left[ \lambda_2 + (\lambda_1 - \lambda_2) \frac{1}{L} \sum_{i=1}^{L} p_i \right] \left[ \int_0^{\omega} h_2(\tau) d\tau \right]$$

$$= c_n D \left\{ \frac{SD}{L} + \frac{D}{L} (1 - p^L) + c_w D \left\{ \left[ \int_0^{\omega} h_1(\tau) d\tau \right] + \alpha(Q) \frac{1}{L} \sum_{i=1}^{L} p_i \right\} \left[ \int_0^{\omega} h_2(\tau) d\tau \right] \right\}$$

$$, n = 1, 2, \ldots, r$$

(10)

Therefore, the expected total relevant cost per unit time $ATC_n(L, S)$, is given by

$$ATC_n(L, S) = c_n D + \frac{KD}{L} + c_n I \left[ L \left( 1 - \frac{D}{M} \right) - S \right] / \left[ 2L \left( 1 - \frac{D}{M} \right) \right]$$

$$+ \frac{bS^2}{2L \left( 1 - \frac{D}{M} \right)} + \frac{DR}{L} (1 - p^L)$$

$$+ c_w D \left\{ \frac{1 - \lambda_2 - (\lambda_1 - \lambda_2) \frac{1}{L} \sum_{i=1}^{L} p_i \right\} \left[ \int_0^{\omega} h_1(\tau) d\tau \right]$$

$$+ \left[ \lambda_2 + (\lambda_1 - \lambda_2) \frac{1}{L} \sum_{i=1}^{L} p_i \right] \left[ \int_0^{\omega} h_2(\tau) d\tau \right] \right\}$$

(11)

In fact, we cannot easily prove Eq. (11) is jointly concave in $L$ and $S$. The Hessian Matrix was too complex. Therefore, we develop an alternative method in the next section to find the optimal lot size and backordering quantity.

4. Solution methodology and algorithm. The following proposition is needed for optimal solution derivation in the expected total relevant cost $ATC_n(L, S)$.

**Proposition 1.** For fixed $Q$, $ATC_n(L, S)$ is convex in $S$, for $n = 1, 2, \ldots, r$

**Proof.** See Appendix B.

Proposition 1 shows that there is a unique solution that minimizes Eq. (11) when the lot size is given. Thus, setting $\partial ATC_n(L, S)/\partial S = 0$, we have

$$S = c_n IL \left( 1 - \frac{D}{M} \right) / (c_n I + b)$$

(12)
Substituting Eq. (12) into Eq. (11) the expected total cost can be expressed as a single variable function as follows

\[ \text{ATC}_n(L) = c_n D + \frac{KD}{L} + \frac{c_n I L (1 - D/M)}{2} \left( \frac{b}{c_n I + b} \right) + \frac{DR (1 - p^L)}{L} + c_w D \left\{ \frac{1}{L} \sum_{i=1}^L \tilde{p}_i \left[ \int_0^\infty h_1(\tau) d\tau \right] \right\} + \lambda_2 (\lambda_1 - \lambda_2) \frac{1}{L} \left[ \int_0^\infty h_2(\tau) d\tau \right] , n = 1, 2, \ldots, r \]  

(13)

We then take the first derivative for Eq. (13) with respect to L and have the following formulation:

\[ \frac{\partial \text{ATC}_n(L)}{\partial L} = \begin{cases} 
\frac{-D(K + R)}{L^2} + \frac{c_n I}{2} (1 - D/M) \left( \frac{b}{c_n I + b} \right) , p = 1; n = 1, 2, \ldots, r & (14a) \\
\frac{-DK}{L^2} + \frac{c_n I}{2} (1 - D/M) \left( \frac{b}{c_n I + b} \right) - \psi \frac{D}{L^2} (1 + L\tilde{p} \ln p - \tilde{p}^L) & (14b) \\
\frac{-DK}{L^2} + \frac{c_n I}{2} (1 - D/M) \left( \frac{b}{c_n I + b} \right) , p = 0; n = 1, 2, \ldots, r & (14c) 
\end{cases} \]

where \( \psi = \left( R - c_w (\lambda_2 - \lambda_1) \left( \frac{\tilde{p}}{1 - \tilde{p}} \right) \right) \left[ \int_0^\infty h_2(\tau) d\tau - \int_0^\infty h_1(\tau) d\tau \right] \)

Setting \( \text{ATC}_n'(L) = 0 \), we have the following results from Eqs. (14a) and (14c)

\[ L_{1n} = \sqrt{\frac{2D(K + R)}{c_n I (1 - D/M)} \cdot \left( \frac{c_n I + b}{b} \right)} , n = 1, 2, \ldots, r \]  

(15)

\[ L_{2n} = \sqrt{\frac{2DK}{c_n I (1 - D/M)} \cdot \left( \frac{c_n I + b}{b} \right)} , n = 1, 2, \ldots, r \]  

(16)

We know the candidate optimal \( L^* \) exists at \( \text{ATC}_n'(L) = 0 \). In addition, Theorem 1 illustrates that a unique \( L \) exists satisfying \( \text{ATC}_n'(L) = 0 \). Theorem 2 provides the bounds for the candidate lot size.

**Theorem 1.** The candidate optimal lot size \( L^*_n \) exists and is unique for \( n = 1, 2, \ldots, r \)

**Proof.** See Appendix C

**Theorem 2.** Given \( L_{1n} = \sqrt{\frac{2D(K + R)}{c_n I (1 - D/M)} \cdot \left( \frac{c_n I + b}{b} \right)} \) from Eq. (15) and \( L_{2n} = \sqrt{\frac{2DK}{c_n I (1 - D/M)} \cdot \left( \frac{c_n I + b}{b} \right)} \) from Eq. (16)

(a) If \( \psi \leq 0 \), then \( 0 < L^*_n = L_{2n} \leq L_{1n} , n = 1, 2, \ldots, r \).

(b) If \( \psi > 0 \), then \( 0 < L^*_n = L_{2n} < L_{1n} , n = 1, 2, \ldots, r \).

**Proof.** See Appendix D

Although the candidate optimal lot size \( L^*_n \) cannot be expressed in closed form, it can be obtained through employing any nonlinear search method (e.g., Fibonacci). Due to purchasing cost per unit depends on the ordering lot size, which corresponds to different cost curves, one cannot directly access the optimal lot size from the
candidate optimal lot size $L_n^\Delta$. We then know that $L_n^\Delta$ is valid when $L_{n-1} \leq L_n^\Delta \leq L_n$ holds given the all-unit quantity discount scheme. However, two cases occur when $L_n^\Delta$ is invalid under the ordering lot size $L_n^\Delta$ out of its range corresponding to the unit purchasing $c_n$:

**Case A.** $L_n^\Delta > L_n$ where $L_n^\Delta$ is the upper bound of ordering lot size corresponding to the purchasing cost per unit $c_n$.

In general, the producer may adopt the lower unit-purchasing cost (say $c_a$ for $c_a < c_n$) to meet his ordering lot size $L_n^\Delta$. From Eq. (13) we know that $ATC_a(L_a^\Delta) < ATC_n(L_n^\Delta)$. It indicates there are no additional computation works are required because the possible candidates under $c_n$ could not occur in this situation.

**Case B.** $L_n^\Delta < L_{n-1}$, where $L_{n-1}$ is the lower bound of ordering lot size corresponding to the purchasing cost per unit $c_n$.

Because $L_{n-1}$ is the lower bound of ordering lot size, this implies the candidate optimal lot sizing may occur at the breakpoint $L_{n-1}$ with its corresponding $c_n$. Therefore, the close form used to obtain the candidate backordering quantity for $S_n^\Delta$ is shown as follows:

$$S_n^\Delta = \frac{L_{n-1} c_n I(1 - D/M)}{c_n I + b}, \forall n = 1, 2, \ldots, r$$

(17)

One now has determined the candidate optimal ordering lot size and backordering quantity corresponding to the purchasing cost unit $c_n$ but has not obtained the unique solution. We then develop an algorithm in which the complexity of the proposed algorithm is $O(n)$ to obtain the benefits of overall optimal solution. The motivation for our proposed algorithm is to make logic clear and easy to understandable by user for solving the proposed model. We note that the complexity of the proposed algorithm $O(n)$ means the runtime increase directly in proportion to $n$. This implies our proposed algorithm is an efficient method. Furthermore, the proposed algorithm indicated one may employ any nonlinear search method (e.g., Fibonacci), or Secant search methods to obtain the candidate optimal solution and then obtain optimal solution. The superiority of the proposed algorithm is: (1) It is a step-wise expression of a solution to a given problem, which makes it easy to understand. (2) The problem is broken down into sub-steps. Hence, it is easier for programmer to convert it into an actual program. (3) Every step in the algorithm has its own logical sequence, and hence, it is easy to debug.

**Algorithm:** Finding the overall optimal ordering lot size and allowable shortages.

**Step 1:** Compute $\psi$, Where

$$\psi = \left\{ R - c_w(\lambda_2 - \lambda_1) \left( \frac{p}{1 - p} \right) \left[ \int_0^\omega h_2(\tau) d\tau - \int_0^\omega h_1(\tau) d\tau \right] \right\}$$

**Step 2:** Set $n = 0$.

**Step 3:** Set $n = n + 1$.

**Step 4:** If $n > r$, go to Step 9.

**Step 5:** Compute $L_{1n}$ and $L_{2n}$, where

$$L_{1n} = \sqrt{\frac{2D(K + R)}{c_n I(1 - D/M)} \cdot \left( \frac{c_n I + b}{b} \right)}$$

$$L_{2n} = \sqrt{\frac{2DK}{c_n I(1 - D/M)} \cdot \left( \frac{c_n I + b}{b} \right)}$$
Step 6: If \( \psi \leq 0 \), set \( L_{tn} = 0 \) and \( L_{un} = L_{2n} \); Otherwise, set \( L_{tn} = L_{2n} \) and \( L_{un} = L_{tn} \).

Step 7: Find \( L_n^\Delta \in [L_{tn}, L_{un}] \) such that \( ATC_n'(L_n^\Delta) = 0 \) employing any nonlinear search method (e.g. Fibonacci) or Secant search methods and record \( L_n^\Delta \).

Step 8: Go to Step 3.

Step 9: Set \( n = 0 \).

Step 10: Set \( n = n + 1 \).

Step 11: If \( n > r \), go to Step 15.

Step 12: If \( L_n^\Delta > L_n \), go to Step 10.

Step 13: If \( L_{n-1} \leq L_n^\Delta < L_n \), Do{
\[
\begin{align*}
\{ \text{Compute } S_n^\Delta, \text{ where } S_n^\Delta &= \frac{(1 - D/M)c_n IL_n^\Delta}{c_n I + b} \} \\
\{ \text{Compute } ATC_n(L_n^\Delta, S_n^\Delta) \text{ from Eq. (11) and record them } \} \\
\{ \text{Go to Step 10} \}
\end{align*}
\]

Step 14: If \( L_n^\Delta < L_{n-1} \), Do{
\[
\begin{align*}
\{ \text{let } L_n^\Delta = L_{n-1} \} \\
\{ \text{Compute } S_n^\Delta, \text{ where } S_n^\Delta &= \frac{(1 - D/M)c_n IL_{n-1}}{c_n I + b} \} \\
\{ \text{Compute } ATC_n(L_n^\Delta, S_n^\Delta) \text{ from Eq. (11) and record them } \} \\
\{ \text{Go to Step 10} \}
\end{align*}
\]

Step 15: Compare \( ATC_n(L_n^\Delta, S_n^\Delta) \) recorded in steps 13 and 14. The solution corresponding to the lowest expected total cost provides the optimal lot size, the backordering quantity, and the unit-purchasing cost.

Step 16: Stop.

5. Numerical example and sensitivity analysis. To verify the developed model and algorithm, a numerical example is provided. We further conduct a sensitivity analysis illustrate the effects of parameters on the deteriorating production system. As shown in the Introduction section, a manufacture of \( AC/DC \) cooling fan in China is employed in this example. Due to the trade secret, all necessary costs are estimated from the relative data and listed as Table 1.

Following Yeh et al.'s [45] example, we suppose that the life time distribution of both conforming and nonconforming items are Weibull with hazard function \( h_1(\tau) = \theta_1^\beta_1 \tau^{\beta_1 - 1} \) and \( h_2(\tau) = \theta_2^\beta_2 \tau^{\beta_2 - 1} \), respectively. We further let the scale parameters are \( \theta_1 = 1/36 \) and \( \theta_2 = 12 \). The shape parameters are assumed \( \beta_1 = \beta_2 = 2 \). Therefore, the mean time to failure is \( 12 \gamma(3/2) \) for a nonconforming item and \( 36 \gamma(3/2) \) for a conforming item. In addition, a price discount schedule is offered by the supplier with the following intervals: \( [1, 150] \) corresponding to \( c_1 = 40.04 \), \( [150, 350] \) corresponding to \( c_2 = 40.03 \), \( [350, 650] \) corresponding to \( c_3 = 40.02 \), \( [650, 950] \) corresponding to \( c_4 = 40.01 \), \( [950, \infty) \) corresponding to \( c_5 = 40 \). Applying the above parameters, we know \( \psi \) is positive (\( \psi = 193.5 \)). Using the algorithm developed in section 4, we have the optimal solution for the given parameter set is \( L^* = 883.1 \) units and \( S^* = 448 \) units, which corresponds to the unit-purchasing cost \( c_4 = 40.01 \). The expected total relevant cost is $84652.73.

Figure 2 illustrates the expected total relevant cost as a function of \( L \) and \( S \) given unit-purchase cost known as \( c_4 = 40.01 \). The three-dimension graph indicates the expected total cost is a convex and unique solution for \( L \) and \( S \), which minimizes...
the expected annual cost. The optimal solution for the given parameter set is same as the results obtained in our developed Algorithm. Note that the optional lot size $L^*$ is larger than that in the traditional EPQ model (that is, $L^* > L_{\text{trad}} = 558.9$) for the case $\psi > 0$. This illustrates that the optimal lot size is affected by the policies for shortages, warranty, and quantity discounts. Note also that we found the optimal solution which minimize the whole system may not occur at the lowest unit-purchasing cost of quantity discount system.

| Description and parameters | Value | Unit |
|-----------------------------|-------|------|
| Production rate ($M$)       | 10,000| units/year |
| Demand rate ($D$)           | 2,000 | units/year |
| Setup cost ($K$)            | 500   | $/cycle |
| Holding cost rate for a unit (a fraction of dollar value) ($I$) | 0.26 | $/unit/year |
| Backordering cost ($b$)     | 6     | $/unit/year |
| Repair cost/warranty cost ($c_w$) | 5   | $/unit |
| Restoration cost ($R$)      | 100   | $/cycle |
| Probability that the system from controlled state shifts to uncontrolled state ($p$) | 0.1 | N/A |
| Percentage of nonconforming items when the process is controlled state ($\lambda_1$) | 0.1 | N/A |
| Percentage of nonconforming items when the process is in uncontrolled ($\lambda_2$) | 0.75 | N/A |

Realizing the model parameters affect the optimal solution in Example, all parameters should theoretically be studied. However, investigating the effect of these parameters on the optimal lot sizing and backordering quantity is a hard and laborious work. According to Taguchi experimental design, five important parameters ($p, K, c_w, I, \text{ and } R$) are set at two levels (low and high) and illustrated as below: $p = (0.1, 0.13); K = (500, 650); c_w = (6, 7.8); I = (0.2, 0.26); R = (0.2, 0.26)$. The other parameters remain unchanged. Table 2 shows the optimal policy under 32 combinations of $p, K, c_w, I, \text{ and } R$. This paper findings are described as follows:

**Figure 2.** The three-dimension graph of the expected total cost
Table 2 The values of $L^*$, $S^*$, and $ATC^*$ corresponding to 32 combinations of $p, K, c_w, I, R$

| $p$ | $K$ | $c_w$ | $I$ | $R$ | $L^*$ | $S^*$ | $ATC^*$ |
|-----|-----|-------|-----|-----|-------|-------|---------|
| 6   | 500 | 0.2   | 100 | 1050| 480   |       | 84514.92|
|     |     |       | 130 | 1050| 480   |       | 84572.06|
|     |     | 0.26  | 100 | 883.1| 448   |       | 84652.73|
|     |     |       | 130 | 905.1| 459.2 |       | 84719.84|
| 7.8 | 0.2 | 100   | 1050| 480  |       |       | 85348.04|
|     |     |       | 130 | 1050| 480   |       | 85905.19|
|     |     | 0.26  | 100 | 879.8| 446.4 |       | 85984.22|
|     |     |       | 130 | 901.9| 457.6 |       | 86051.58|
| 0.1 | 6   | 0.2   | 100 | 1050| 480   |       | 84800.63|
|     |     |       | 130 | 1062.1| 485.5 |       | 84857.59|
|     |     | 0.26  | 100 | 1050| 532.7 |       | 84958.68|
|     |     |       | 130 | 1050| 532.7 |       | 85015.83|
| 7.8 | 0.2 | 100   | 1050| 480  |       |       | 86133.76|
|     |     |       | 130 | 1059| 484.1 |       | 86190.8 |
|     |     | 0.26  | 100 | 1050| 532.7 |       | 86291.81|
|     |     |       | 130 | 1050| 532.7 |       | 86348.95|
| 6   | 500 | 0.2   | 100 | 1050| 480   |       | 84518.1 |
|     |     |       | 130 | 1050| 480   |       | 84575.24|
|     |     | 0.26  | 100 | 884.3| 448.7 |       | 84656.5 |
|     |     |       | 130 | 906.3| 459.8 |       | 84723.52|
| 7.8 | 0.2 | 100   | 1050| 480  |       |       | 85853.41|
|     |     |       | 130 | 1050| 480   |       | 85910.55|
|     |     | 0.26  | 100 | 881.9| 447.4 |       | 85990.62|
|     |     |       | 130 | 903.9| 458.6 |       | 86057.8 |
| 0.13| 6   | 0.2   | 100 | 1050| 480   |       | 84803.81|
|     |     |       | 130 | 1050| 480   |       | 84860.95|
|     |     | 0.26  | 100 | 1050| 532.7 |       | 84961.86|
|     |     |       | 130 | 1050| 532.7 |       | 85019|
| 7.8 | 0.2 | 100   | 1050| 480  |       |       | 86139.12|
|     |     |       | 130 | 1060.9| 485  |       | 86196.11|
|     |     | 0.26  | 100 | 1050| 532.7 |       | 86297.17|
|     |     |       | 130 | 1050| 532.7 |       | 86354.32|

(1) Because $p$ is defined as the transaction probability and indicates the system will shift from the controlled state into the uncontrolled state, we therefore know that $p$ could be recognized as the system reliability. This implies that the more unreliable the system is, the higher the lot size, the backordering quantity, and the expected total cost are. That is, $L^*$, $S^*$, and $ATC^*$ all increase with $p$. As the unreliable probability increases the manufacturer produces greater lot sizes to satisfy the customer's needs, which matches Jaber et al.'s [56] result. Furthermore, the allowable shortages are also increased to reduce the shock from a variant production system. We note that the larger the unreliable system the more frequently the production process will shift into the “uncontrolled” state and the more restoration cost will occur. This indicates the total expended cost rises as $p$ rises. However, as the quantity discount effect occurs, $L^*$ and $S^*$ remain unchanged with $p$, which is another scenario for this case. This effect may stimulate the retailer to order greater quantity, which usually occurs at the ordering quantity break point, to enjoy the
quantity discount benefit. Thus, when the optimal ordering lot size occurs at the break point of ordering quantity scheme, the backordering quantity remains at the same value while the expected total cost increases as q increases.

(2) Increasing setup cost implies that the fixed production cost increases. This leads the retailer to order greater quantity \((L)\) to apportion the fixed cost. In the meantime, the backordering quantity \((S)\) is also increasing with the setup cost \((K)\). Thus, as expected, \(L^*, S^*,\) and \(ATC^*\) all increase with \(K\). We note that as the setup cost increases, the unit-purchasing cost may decrease in some situations. This is because the retailer enjoys quantity discount benefits and thus employs the lower unit-purchasing cost.

(3) In general, \(L^*\) and \(S^*\) decrease in \(c_w\); while \(ATC^*\) increases with \(c_w\). To our knowledge a higher warranty cost leads to extra repair cost and thus increases the expected total relevant cost. Alternatively, as \(L^*\) and \(S^*\) decrease with \(c_w\), we know that decreasing the number of failures occurring in the warranty duration could reduce greater warranty cost (or minimal repair cost). By reducing the lot size, one can reduce the number of nonconforming items to decrease the failures occurring in the warranty duration and reduce additional warranty cost. Thus, the lot size \(L^*\) decreases. Because the backordering quantity will be replenished in the next cycle, we know, similar to the lot size analysis, the allowable shortages \(S^*\) decrease when \(w\) increases. Alternatively, another situation for this case is that \(L^*\) and \(S^*\) remain unchanged with \(c_w\). This scenario usually occurs when the retailer employs the quantity discount benefit and the ordering lot size at the break point corresponds to its unit-purchasing cost. In this scenario, \(ATC^*\) also increases with \(c_w\).

(4) A higher holding cost usually increases the expected total cost. Thus, to avoid the extra holding cost, the retailer in general will order less quantity. This leads to \(L^*\) decreasing with \(I\); while \(ATC^*\) is increasing with \(I\). As to the backordering quantity, two scenarios occur: (i) \(S^*\) is increasing with \(I\) if the unit-purchasing cost remains unchanged. It is the often case that the retailer employs the less ordering quantity policy with more backordering quantity to reduce the additional cost under the insignificant quantity discount effect. (ii) \(S^*\) decreases with \(I\) if the purchasing cost per unit increases. In this scenario, the quantity discount effect is significant. Thus, the retailer may allow less backordering quantity to decrease the extra cost caused by the increasing purchasing cost per unit. This indicates the retailer employs the less ordering quantity policy and backordering quantity to reduce the additional cost.

(5) As expected, \(L^*, S^*,\) and \(ATC^*\) all increase with \(R\). This implies that the more the restoration cost for renewing the system from uncontrolled state back to controlled state is, the higher the lot size, the backordering quantity, and the expected total cost are. If the maintenance cost increases, the manufacturer produces greater lot sizes and permits the more shortages to reduce the cycle runs. It is intuitively the total expended cost increases as \(R\) increases. Note that, as the quantity discount effect occurs, \(L^*\) and \(S^*\) remain unchanged with \(R\), which is another scenario for this case. This effect may stimulate the retailer ordering greater lot size, which similar to Case (1) in this paper. Thus, when the optimal ordering quantity occurs at the break point of ordering quantity scheme, the backordering quantity remains at the same value while the expected total cost increases with \(R\).

(6) From Table 2 it is also observed that none of the second order interactions of \(p, K, w, I,\) and \(R\) are significant. We therefore conclude that an additive
model could appropriately explain the relationship between the five parameters \((p, K, w, I, \text{and} R)\) and \(L^\ast (S^\ast \text{and} ATC^\ast)\).

6. Conclusions. In this paper, we considered a deteriorating two-state Markovian production process with allowable shortages and quantity discounts. We derived the optimal backordering quantity and unique optimal lot size, which were bounded within a finite interval. An efficient algorithm was developed to determine the optimal solution. Sensitivity analysis on the optimal ordering lot size and backordering quantity was conducted through numerical examples to evaluate their performance levels. Numerical results display that warranty and quantity discount policies could have a significant impact on decision-making.

Further, the numerical results are summarized as follows: (1) The optimal lot size identified in this paper is larger than that in the traditional EPQ model. (2) The lowest unit-purchasing cost may not minimize the expected total cost of the system. (3) In general, the higher the transaction probability (i.e., \(p\)), the higher the optimal ordering lot size, backordering quantity, and expected total cost (with some exceptions, alongside the occurrence of quantity discount effect). (4) The optimal lot size, backordering quantity, and expected total cost all increase with the setup cost. (5) If the warranty period extends, the expected total cost increases, while the lot size and backordering quantity decrease (If the quantity discount effect occurs, the lot size and backordering quantity remain unchanged.) (6) In general, as the unit-holding cost increases, the optimal lot size decreases, while the backordering quantity and the expected total cost increase, if the unit-purchasing cost remains unchanged. However, if the quantity discount effect occurs, the optimal ordering lot size and backordering quantity decrease, while the expected total cost increases.

Our findings in this paper provide new insights into production management strategies, scheduling, and inventory control. Extensions of the proposed model can be used to investigate more practical issues such as incremental quantity discounts, different warranty policies (e.g., pro-rata warranty), preventive maintenance, and probability demand.

Appendix A.

Proof for Lemma 1. Let \(\overline{p} = 1 - p\) and in a lot of size \(L\), the probability density function for the number of goods in the controlled state, \(Y\), is

\[
P_r\{Y = i\} = \begin{cases} \overline{p}^i p, & 0 \leq i < L \\ \overline{p}^L, & i = L \end{cases}
\]

Then, the expected value of \(Y\) is expresses as follows:

\[
E[Y] = p \sum_{i=1}^{L-1} i\overline{p}^i + L\overline{p}^L
\]

\[
= \sum_{i=1}^{L} i\overline{p}^i
\]

Moreover, in a lot of size \(L\), the number of nonconforming goods is given by \(N = \lambda_1 Y + \lambda_2 (L - Y)\). This indicates the expected value of \(N\) becomes

\[
E[N] = \lambda_2 L + (\lambda_1 - \lambda_2) \sum_{i=1}^{L} i\overline{p}^i
\]
Furthermore, the fraction of nonconforming items, denoted by \( \alpha(L) \), in a lot size \( L \) is given by

\[
\alpha(L) = \frac{E[N]}{L} = \lambda_2 + (\lambda_1 - \lambda_2) \cdot \frac{1}{L} \sum_{i=1}^{L} \bar{p}^j
\]

The proof is completed.

**Appendix B.**

**Proof for Proposition 1.** For fixed \( L \), \( ATC_n(L, S) \) in Eq. (14) is then reduced as a single variable function. Thus, the sign of \( \partial^2 ATC_x(L, S) / \partial S^2 \) shows its convexity. One therefore has

\[
\frac{\partial^2 ATC_n(L, S)}{\partial S^2} = \frac{(c_n I + b)}{2} \left[ L \left( 1 - \frac{D}{M} \right) \right] > 0, \text{ for } n = 1, 2, \ldots, r
\]

It implies \( ATC_n(L, S) \) given a fixed value of \( L \) is convex in \( S \) for all \( n = 1, 2, \ldots, r \). This completes the proof.

**Appendix C.**

**Proof for Theorem 1.** Form Eq. (14a), Theorem 1 holds in which \( p = 1 \); Alternatively, if \( p = 0 \), Theorem 1 holds from Eq. (14c). For the case that \( 0 < p < 1 \), we let (14b) as

\[
g_n(L) = L^2 \cdot ATC'(L)
\]

\[
= -DK + c_n I \cdot \frac{b}{2} \left( 1 - \frac{D}{M} \right) \cdot L^2 - \psi D \left( 1 + L \bar{p}^L \ln \bar{p} - \bar{p}^L \right)
\]

for \( n = 1, 2, \ldots, r \) (A1)

Because \( g_n(L) \) is a continuous function with \( \lim_{L \to 0^+} g_n(L) = -DK < 0 \) and \( \lim_{L \to \infty} g_n(L) = \infty > 0 \), we learn there is a sign change of \( g_n(L) \) from negative to positive. The first derivative of \( g_n(L) \) can further be obtained as follows:

\[
g_n'(L) = L \left[ c_n I - \psi D \ln \left( \bar{p}^L \bar{p}^L \right) \right] \quad (A2)
\]

The unique characteristic of the candidate optimal ordering quantity is shown as the following two scenarios:

(i) \( \psi \leq 0 \)

In this scenario, we know \( g_n'(L) > 0 \) for all \( L > 0 \). This indicates \( g_n(L) \) is strictly increasing on \( L \). Furthermore, we learn the equation \( g_n(L) = 0 \) has a unique positive solution and so does \( ATC_n'(L) = 0 \).

(ii) \( \psi > 0 \)

In this scenario, one has

\[
g_n'(L) \leq 0 \cdot \text{ for } L \leq L_\ell = \frac{1}{\ln \bar{p}} \ln \left[ c_n I \left( 1 - \frac{D}{M} \right) \left\{ \frac{b}{c_n I + b} \left[ D \psi \left( \ln \bar{p} \right)^2 \right] \right\} \right]
\]

and

\[
g_n'(L) > 0 \cdot \text{ for } L > L_\ell = \frac{1}{\ln \bar{p}} \ln \left[ c_n I \left( 1 - \frac{D}{M} \right) \left\{ \frac{b}{c_n I + b} \right\} \left[ D \psi \left( \ln \bar{p} \right)^2 \right] \right]
\]

This implies \( g_n(L) \) first decreases and then increases to infinity when \( L \) increases. Therefore, \( g_n(L) \) is strictly increasing on \( (L_\ell, \infty) \) and is a convex function respecting
L > L_L. We further learn there is a sign change of \(g_n(L)\) from negative to positive and so does \(ATC'_n(L) = 0\).

Because there is a same sign for \(ATC'_n(L) = 0\) and \(g_n(L)\) given \(L > 0\), one knows \(ATC'_n(L) = 0\) alters its sign just once from negative to positive. Thus, the candidate optimal ordering quantity \(L_n^\Delta\) is unique and exists. \(\square\)

Appendix D.

Proof for Theorem 2. Let \(g(L_n) = L_n^2 ATC'(L_n)\). Since we cannot determine the sign of the value \(\psi\), two case occurs

Case I. \(\psi \leq 0\)

In this case, we have
\[
\begin{align*}
g(L_{1n}) &= DR - D\psi (1 + Lp^L (\ln p - p^L)) \\
g(L_{2n}) &=-D\psi (1 + Lp^L (\ln p - p^L))
\end{align*}
\]

Let \(Y(L) = 1 + Lp(\ln p) - p^L\), we can easily obtain \(Y(L) = 1 + Lp(\ln p) - p^L\) for \(L > 0\). Therefore, given \(\psi \leq 0\), in this case, we have \(g(L_{1n}) > 0\) and \(g(L_{2n}) > 0\) obtained from Eqs. (A3) and (A4), respectively. In addition, we know \(g(L)\) is strictly increasing from Theorem 1. This illustrates \(L_n^\Delta \leq L_{1n}\) and \(L_n^\Delta = L_{2n}\) for \(\psi \leq 0\). Furthermore, from Eqs. (15) and (16), we know \(0 < L_{2n} \leq L_{1n}\). Therefore, we have \(0 < L_n^\Delta = L_{2n} \leq L_{1n}, n = 1, 2, \ldots, r\).

Case II. \(\psi > 0\)

If \(L_n^\Delta\) is the candidate optimal ordering quantity obtained from Eqs. (A1) and (15), we know
\[
g(L_{1n}) = D \left\{ R [1 - Y(L_{1n})] + c_w (\lambda_2 - \lambda_1) \left( \frac{p}{1 - p} \right) \left[ \int_0^\omega h_2(\tau) d\tau \right. \right.
\]
\[
\left. \left. - \int_0^\omega h_1(\tau) d\tau \right]\right \} Y(L_{1n})
\]

Because \(0 < Y(L) < 1\), one has \(g(L_{1n}) > 0 = g(L_n^\Delta)\), which indicates \(L_n^\Delta < L_{1n}\). From Eqs. (A1) and (19), one further has \(g(L_{2n}) = -D\psi Y(L_{2n})\). We know, in this case, \(\psi > 0\). This implies \(g(L_{2n}) < 0 = g(L_n^\Delta)\), which implies \(L_{2n} = L_n^\Delta\). Therefore, if \(\psi > 0\), then \(L_{2n} < L_n^\Delta < L_{1n}\), for \(n = 1, 2, \ldots, r\).

This completes the proof by combining Case I and Case II. \(\square\)

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REFERENCES

[1] A. Y. Alqahtani, S. M. Gupta and K. Nakashima, Warranty and maintenance analysis of sensor embedded products using internet of things in industry 4.0, Int. J. Prod. Econ., 208 (2019), 483–499.
[2] Y. Barron and D. Hermel, Shortage decision policies for a fluid production model with MAP arrivals, Int. J. Prod. Res., 55 (2017), 3946–3969.
[3] W. R. Blischke and D. N. P. Murthy, Product warranty management III: A review and mathematical models, Eur. Oper. Res., 62 (1992), 1–34.
[4] S. Chand, Lot sizes and setup frequency with learning in setups and process quality, Eur. J. Oper. Res., 42 (1989), 190–202.
[5] C.-K. Chen and C.-C. Lo, Optimal production run length for products sold with warranty in an imperfect production system with allowable shortages, Math. Comput. Model., 44 (2006), 319–331.
[6] Y.-H. Chien, Z. G. Zhang and X. L. Yin, On optimal preventive-maintenance policy for generalized Polya process repairable products under free-repair warranty, *Eur. J. Oper. Res.*, **279** (2019), 68–78.

[7] K.-J. Chung and K.-L. Hou, An optimal production run time with imperfect production processes and allowable shortages, *Comput. Oper. Res.*, **30** (2003), 483–490.

[8] A. Eroglu and G. Ozdemir, An economic order quantity model with defective items and shortages, *Int. J. Prod. Econ.*, **106** (2007), 544–549.

[9] P. A. Hayek and M. K. Salameh, Production lot sizing with the reworking of imperfect quality items produced, *Prod. Plan. Control*, **12** (2001), 584–590.

[10] K.-L. Hou, Optimal production run length for deteriorating production system with a two-state continuous-time Markovian processes under allowable shortages, *J. Oper. Res. Soc.*, **56** (2005), 346–350.

[11] K.-L. Hou, L.-C. Lin and T.-Y. Lin, Optimal lot sizing with maintenance actions and imperfect production processes, *Int. J. Syst. Sci.*, **46** (2015), 2749–2755.

[12] B. Huang and A. Wu, Reduce shortage with self-reservation policy for a manufacturer paying both fixed and variable stockout expenditure, *Eur. J. Oper. Res.*, **262** (2017), 944–953.

[13] M. Y. Jaber, M. Bonney and I. Moualek, An economic order quantity model for an imperfect production process with entropy cost, *Int. J. Prod. Econ.*, **118** (2009), 26–33.

[14] M. Y. Jaber, S. Zanoni and L. E. Zavanella, Economic order quantity models for imperfect items with buy and repair options, *Int. J. Prod. Econ.*, **155** (2014), 126–131.

[15] M. Khan, M. Y. Jaber, A. L. Guiffrida and S. Zolfaghari, A review of the extensions of a modified EOQ model for imperfect quality items, *Int. J. Prod. Econ.*, **132** (2011), 1–12.

[16] R. S. Kumar and A. Goswami, EPQ model with learning consideration, imperfect production and partial backlogging in fuzzy random environment, *Int. J. Syst. Sci.*, **46** (2015), 1486–1497.

[17] H. Lee, J. H. Cha and M. Finkelstein, On information-based warranty policy for repairable products from heterogeneous population, *Eur. J. Oper. Res.*, **253** (2016), 204–215.

[18] J. S. Lee and K. S. Park, Joint determination of production cycle and inspection intervals in a deteriorating production system, *J. Oper. Res. Soc.*, **42** (1991), 775–783.

[19] T.-Y. Lin, Coordination policy for a two-stage supply chain considering quantity discounts and overlapped delivery with imperfect quality, *Comput. Ind. Eng.*, **66** (2013), 53–62.

[20] B. Liu, J. Wu and M. Xie, Cost analysis for multi-component system with failure interaction under renewing free-replacement warranty, *Eur. J. Oper. Res.*, **243** (2015), 874–882.

[21] M. Luo and S. M. Wu, A comprehensive analysis of warranty claims and optimal policies, *Eur. Oper. Res.*, **276** (2019), 144–159.

[22] B. Maddah, L. Moussawi and M. Y. Jaber, Lot sizing with a Markov production process and imperfect items scrapped, *Int. J. Prod. Econ.*, **124** (2010), 340–347.

[23] V. Makis, Optimal lot sizing and inspection policy for an EMQ model with imperfect inspections, *Nav. Res. Log.*, **45** (1998), 165–186.

[24] L. Moussawi-Haidar, M. Salameh and W. Nasr, Production lot sizing with quality screening and rework, *Appl. Math. Model.*, **40** (2016), 3242–3256.

[25] D. N. P. Murthy and W. R. Blischke, Product warranty management:II: An integrated framework for study, *Eur. J. Oper. Res.*, **62** (1992), 261–281.

[26] D. N. P. Murthy and W. R. Blischke, Product warranty management:III: A review of mathematical models, *Eur. J. Oper. Res.*, **63** (1992), 1–34.

[27] D. N. P. Murthy and I. Djimaludin, New product warranty: A literature review, *Int. J. Prod. Econ.*, **79** (2002), 231–260.

[28] L.-Y. Ouyang and C.-T. Chang, Optimal production lot with imperfect production process under permissible delay in payments and complete backlogging, *Int. J. Prod. Econ.*, **144** (2013), 610–617.

[29] B. Pal, S. S. Sana and K. Chaudhuri, Three-layer Supply Chain- a Production- inventory model for reworkable items, *Appl. Math. Comput.*, **219** (2012), 530–543.

[30] B. Pal, S. S. Sana and K. Chaudhuri, Maximizing profits for an EPQ model with unreliable machine and rework of random defective items, *Int. J. Syst. Sci.*, **44** (2013), 582–594.

[31] B. Pal, S. S. Sana and K. Chaudhuri, A mathematical model on EPQ for stochastic demand in an imperfect production system, *J. Manuf. Sys.*, **32** (2013), 260–270.

[32] E. L. Porteus, Optimal lot sizing, process quality improvement and setup cost reduction, *Oper. Res.*, **34** (1986), 137–144.

[33] M. J. Rosenblatt and H. L. Lee, Economic production cycle with imperfect production processes, *IIE Trans.*, **18** (1986), 48–55.
[34] S. S. Sana, An economic production lot size model in an imperfect production system, Eur. J. Oper. Res., 201 (2010), 158–170.

[35] L. A. San-José, J. Sicilia and J. García-Laguna, Analysis of an EOQ inventory model with partial backordering and non-linear unit holding cost, Omega, 54 (2015), 147–157.

[36] B. Sarkar, An inventory model with reliability in an imperfect production process, Appl. Math. Comput., 218 (2012), 4881–4891.

[37] B. Sarkar, L. E. Cárdenas-Barrón, M. Sarkar and M. L. Singgih, An economic production quantity model with random defective rate, rework process and backorders for a single stage production system, J. Manuf. Syst., 33 (2014), 423–435.

[38] B. Sarkar, S. Saren and L. E. Cárdenas-Barrón, An inventory model with trade-credit policy and variable deterioration for fixed lifetime products, Ann. Oper. Res., 229 (2015), 677–702.

[39] E. W. Taft, The most economical production lot, The Iron Age, 101 (1918), 1410–1412.

[40] A. H. Tai, Economic production quantity models for deteriorating/imperfect products and service with rework, Comput. Ind. Eng., 66 (2013), 879–888.

[41] A. A. Taleizadeh, L. E. Cárdenas-Barrón and B. Mohammadi, A deterministic multi product single machine EPQ model with backordering, scraped products, rework and interruption in manufacturing process, Int. J. Prod. Econ., 150 (2014), 9–27.

[42] A. A. Taleizadeh, S. S. Kalantari and L. E. Cárdenas-Barrón, Pricing and lot sizing for an EPQ inventory model with rework and multiple shipments, Top, 24 (2016), 143–155.

[43] A. A. Taleizadeh, H. R. Zarei and B. R. Sarker, An optimal control of inventory under probabilistic replenishment intervals and known price increase, Eur. Oper. Res., 257 (2017), 777–791.

[44] C. S. Tapiero, P. H. Ritchken and A. Reisman, Reliability, pricing and quality control, Eur. J. Oper. Res., 31 (1987), 37–45.

[45] B. Van Beek and C. Van Putten, OR contributions to flexibility improvement in production/inventory systems, Eur. J. Oper. Res., 31 (1987), 52–60.

[46] M. van der Heijden and B. P. Iskandar, Last time buy decisions for products sold under warranty, Eur. J. Oper. Res., 224 (2013), 302–312.

[47] C.-H. Wang, The impact of free-repair warranty policy on EMQ model for imperfect production systems, Comput. Oper. Res., 31 (2004), 2021–2035.

[48] C.-H. Wang and S.-H. Sheu, Optimal lot sizing for products sold under free-repair warranty, Eur. J. Oper. Res., 149 (2003), 131–141.

[49] C. M. Wright and A. Mehrez, An overview of representative research of the relationships between quality and inventory, Omega, 26 (1998), 29–47.

[50] C. A. Yano and H. L. Lee, Lot sizing with random yields: A review, Oper. Res., 43 (1995), 311–334.

[51] R. H. Yeh, M. Y. Chen and C. Y. Lin, Optimal periodic replacement policy for repairable products under free-repair warranty, Eur. J. Oper. Res., 176 (2007), 1678–1686.

[52] R. H. Yeh, W. T. Ho and S. T. Tseng, Optimal production run length for products sold with warranty, Eur. J. Oper. Res., 120 (2000), 575–582.

[53] S. H. Yoo, D. S. Kim and M. S. Park, Lot sizing and quality investment with quality cost analyses for imperfect production and inspection processes with commercial return, Int. J. Prod. Econ., 140 (2012), 922–933.

[54] X. Zhang and Y. Gerchak, Joint lot sizing and inspection policy in an EOQ model with random yield, IIE Trans., 22 (1990), 41–47.

[55] Y.-W. Zhou, J. Y. Chen, Y. Z. Wu and W. H. Zhou, EPQ models for items with imperfect quality and one-time-only discount, Appl. Math. Model., 39 (2015), 1000–1018.

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E-mail address: 3146375768@qq.com