Putting Yukawa-like Modified Gravity (MOG) on the test in the Solar System

Lorenzo Iorio

INFN-Sezione di Pisa. Permanent address for correspondence: Viale Unità di Italia 68, 70125, Bari (BA), Italy. E-mail: lorenzo.iorio@libero.it

Received _______________; accepted _______________
ABSTRACT

We deal with a Yukawa-like long-range modified model of gravity (MOG) which recently allowed to successfully accommodate many astrophysical and cosmological features without resorting to dark matter. On Solar System scales MOG predicts retrograde secular precessions of the planetary longitudes of the perihelia $\varpi$ whose existence has been put on the test here by taking the ratios of the observationally estimated Pitjeva’s corrections to the standard Newtonian/Einsteinian perihelion precessions for different pairs of planets. It turns out that MOG, in the present form which turned out to be phenomenologically successful on astrophysical scales, is ruled out at more than $3\sigma$ level in the Solar System. If and when other teams of astronomers will independently estimate their own corrections to the usual precessions of the perihelia it will be possible to repeat such a test.

Subject headings: Experimental tests of gravitational theories; Modified theories of gravity; Celestial mechanics; Orbit determination and improvement; Ephemerides, almanacs, and calendars
1. Introduction

The modified gravity (MOG) theory put forth in (Moffat 2006) was used successfully to describe various observational phenomena on astrophysical and cosmological scales without resorting to dark matter (see Moffat and Toth (2008) and references therein). It is a fully covariant theory of gravity which is based on the existence of a massive vector field coupled universally to matter. The theory yields a Yukawa-like modification of gravity with three constants which, in the most general case, are running; they are present in the theory’s action as scalar fields which represent the gravitational constant, the vector field coupling constant and the vector field mass. Actually, the issue of the running of the parameters of modified models of gravity is an old one, known in somewhat similar contexts since the early 1990s (see, e.g., (Bertolami et al. 1993) and references therein). An approximate solution of the MOG field equations (Moffat and Toth 2007) allows to compute their values as functions of the source’s mass.

The resulting Yukawa-type modification of the inverse-square Newton’s law in the gravitational field of a central mass $M$ is (Moffat and Toth 2007, 2008)

$$A_{\text{MOG}} = -\frac{G_N M}{r^2} \{1 + \alpha [1 - (1 + \mu r) \exp(-\mu r)]\},$$

where $G_N$ is the Newtonian gravitational constant and

$$\alpha = \frac{M}{(\sqrt{M} + C'_1)^2} \left(\frac{G_\infty}{G_N} - 1\right), \quad G_\infty \approx 20 \ G_N, \ C'_1 \approx 25000 \ M_\odot^{1/2},$$

$$\mu = \frac{C'_2}{\sqrt{M}}, \ C'_2 \approx 6250 \ M_\odot^{1/2} \ \text{kpc}^{-1}.$$  

Such values have been obtained by (Moffat and Toth 2007) as a result of the fit of the velocity rotation curves of some galaxies in the framework of the searches for an explanation of the flat rotation curves of galaxies without resorting to dark matter.
In this paper we will put eq. (1) on the test in the Solar System in order to check if it is compatible with the latest determinations of the corrections \(\langle \Delta \dot{\omega} \rangle\) to the usual Newtonian/Einsteinian planetary perihelion precessions (Pitjeva 2005a,b). Such quantities are solved-for parameters phenomenologically estimated in a least-square sense by fitting about one century of observations of various kinds concerning planetary motions with their values theoretically predicted by simultaneously integrating the equations of motions of the major bodies of the Solar System written in terms of a complete suite of dynamical force models. They encompass all the known Newtonian effects along with those predicted by general relativity, except for the Lense-Thirring effect (Lense and Thirring 1918) which was left unmodelled (see below). Thus, \(\langle \Delta \dot{\omega} \rangle\) account for any unmodelled/mismodelled dynamical features of motion and are suitable, in principle, to put on the test MOG, provided that they are used in a suitable way.

Note that Moffat and Toth (2008) explicitly write that eq. (1), with eq. (2) and eq. (3), is not in contradiction with the present-day knowledge of Solar System dynamics. We will show that it is not so also for any other (non-zero) values of \(\alpha\) and \(\mu\), with the only quite general condition that \(\mu r \ll 1\) in Solar System, as it must be for any long-range modified model of gravity. It is interesting to point out that Yukawa-like modifications of Newton’s law might also be tested in the context of stellar dynamics (Bertolami and Páramos 2005). Here we outline the procedure that we will follow.

Generally speaking, let LRMOG (Long-Range Modified Model of Gravity) be a given exotic model of modified gravity parameterized in terms of, say, \(K\), in a such a way that \(K = 0\) would imply no modifications of gravity at all. Let \(\mathcal{P}(\text{LRMOG})\) be the prediction of a certain effect induced by such a model like, e.g., the secular precession of the perihelion
of a planet. For all the exotic models considered it turns out that

\[ \mathcal{P} \text{(LRMOG)} = K g(a, e), \]  

(4)

where \( g \) is a function of the system’s orbital parameters \( a \) (semimajor axis) and \( e \) (eccentricity); such \( g \) is a peculiar consequence of the model LRMOG (and of all other models of its class with the same spatial variability). Now, let us take the ratio of \( \mathcal{P} \) for two different systems A and B, e.g. two Solar System’s planets:

\[ \frac{\mathcal{P}_A \text{(LRMOG)}}{\mathcal{P}_B \text{(LRMOG)}} = \frac{g_A}{g_B}. \]

The model’s parameter \( K \) has now been canceled, but we still have a prediction that retains a peculiar signature of that model, i.e. \( g_A/g_B \). Of course, such a prediction is valid if we assume \( K \) is not zero, which is just the case both theoretically (LRMOG is such that should \( K \) be zero, no modifications of gravity at all occurred) and observationally because \( K \) is usually determined by other independent long-range astrophysical/cosmological observations. Otherwise, one would have the meaningless prediction \( 0/0 \). The case \( K = 0 \) (or \( K \leq \overline{K} \)) can be, instead, usually tested by taking one perihelion precession at a time. If we have observational determinations \( \mathcal{O} \) for A and B of the effect considered above such that they are affected also by LRMOG (it is just the case for the purely phenomenologically estimated corrections to the standard Newton-Einstein perihelion precessions, since LRMOG has not been included in the dynamical force models of the ephemerides adjusted to the planetary data in the least-square parameters’ estimation process by Pitjeva (Pitjeva 2005a,b)), we can construct \( \frac{\mathcal{O}_A}{\mathcal{O}_B} \) and compare it with the prediction for it by LRMOG, i.e. with \( g_A/g_B \). Note that \( \delta \mathcal{O}/\mathcal{O} > 1 \) only means that \( \mathcal{O} \) is compatible with zero, being possible a nonzero value smaller than \( \delta \mathcal{O} \). Thus, it is

\[ ^1 \text{In our case it will be } K = -\alpha \mu^2, \text{ as we will see in Section } \]

\[ ^2 \text{If they are differential quantities constructed by contrasting observations to predictions obtained by analytical force models of canonical Newtonian/Einsteinian effects, } \mathcal{O} \text{ are, in principle, affected also by the mismodelling in them.} \]
perfectly meaningful to construct $O_A/O_B$. Its uncertainty will be conservatively evaluated as $|1/O_B|\delta O_A + |O_A/O_B^2|\delta O_B$. As a result, $O_A/O_B$ will be compatible with zero. Now, the question is: Is it the same for $g_A/g_B$ as well? If yes, i.e. if

$$\frac{O_A}{O_B} = \frac{P_A(LRMOG)}{P_B(LRMOG)}$$

within the errors, or, equivalently, if

$$\left|\frac{O_A}{O_B} - \frac{P_A(LRMOG)}{P_B(LRMOG)}\right| = 0$$

within the errors, LRMOG survives (and the use of the single perihelion precessions can be used to put upper bounds on $K$). Otherwise, LRMOG is ruled out.

2. The predicted perihelion precessions and the confrontation with the measured non-standard rates

In the case of the Sun, eq. (2) and eq. (3) yield

$$\alpha_\odot \approx 3 \times 10^{-8}, \mu \approx 3 \times 10^{-5} \text{ AU}^{-1},$$

so that

$$\alpha_\odot \mu^2 = 3 \times 10^{-17} \text{ AU}^{-2}.$$  \hspace{1cm} (8)

Since in the Solar System $\mu r \approx 10^{-5} - 10^{-4}$, we can safely assume $\exp(-\mu r) \approx 1 - \mu r$, so that

$$A_{\text{MOG}} \approx -\frac{G_N M}{r^2} \left(1 + \alpha \mu^2 r^2 \right).$$

As a result, a radial, uniform perturbing acceleration

$$A = -G_N M \alpha \mu^2 \approx 10^{-19} \text{ m s}^{-2}$$

is induced.
The secular, i.e. averaged over one orbital revolution, effect of a small radial and uniform perturbing acceleration on the longitude of the perihelion of a planet \( \varpi \) has been worked out by, e.g., Sanders (2006); it amounts to

\[
\left\langle \frac{d\varpi}{dt} \right\rangle = A \sqrt{\frac{a(1 - e^2)}{G_NM}} = -\alpha \mu^2 \sqrt{G_NM} a(1 - e^2).
\]  

Clearly, using only one perihelion rate at a time would yield no useful information on MOG due to the extreme smallness of the perturbing acceleration, as told us by eq. (10). Thus, let us take the ratios of the perihelion precessions. It must be noted that the following analysis is, in fact, truly independent of the values of \( \alpha \) and \( \mu \), provided only that \( \alpha \mu^2 r^2 \ll 1 \) in the Solar System so as that the perturbative approach can be applied to eq. (9); the condition \( \mu r \ll 1 \) is the cornerstone of any long-range modified models of gravity, and should \( \alpha \approx 1 \) the planetary orbits would have been distorted in a so huge manner that it would have been detected since long time. Applying the scheme outlined in Section 1 to our case in which \( K = -\alpha \mu^2 \) and \( g(a, e) = \sqrt{G_NM} a(1 - e^2) \), one can construct

\[
\Pi \equiv \frac{\langle \Delta \dot{\varpi}_A \rangle}{\langle \Delta \dot{\varpi}_B \rangle}
\]

(12)

with the estimated corrections \( \langle \Delta \dot{\varpi} \rangle \) to the standard Newtonian/Einsteinian perihelion precessions of planets A and B, listed in Table 1 and compare them to the theoretical prediction

\[
\mathcal{A} \equiv \sqrt{\frac{a_A(1 - e_A^2)}{a_B(1 - e_B^2)}},
\]

(13)

obtained from eq. (11), for that pair of planets A and B. The results are in Table 2. \( |\Pi - \mathcal{A}| \) is different from zero at more than 3\( \sigma \) level for A = Venus, B = Mercury, A = Earth, B = Mercury and A = Mars, B = Mercury. It is important to note that the errors have been conservatively evaluated as

\[
\delta \Pi \leq |\Pi| \left( \frac{\delta \langle \Delta \dot{\varpi}_A \rangle}{|\langle \Delta \dot{\varpi}_A \rangle|} + \frac{\delta \langle \Delta \dot{\varpi}_B \rangle}{|\langle \Delta \dot{\varpi}_B \rangle|} \right)
\]

(14)
Table 1: Inner planets. First row: estimated corrections to the standard precessions of the
longitudes of the perihelia in $10^{-4}'' \text{cy}^{-1}$ ($'' \text{cy}^{-1} \rightarrow \text{arcseconds per century}$), from Table
3 of (Pitjeva 2005b) (apart from Venus). The quoted errors, in $10^{-4}'' \text{cy}^{-1}$, are not
the formal ones but are realistic. The formal errors are quoted in square brackets (E.V. Pitjeva,
personal communication to the author, November 2005). The units are $10^{-4}'' \text{cy}^{-1}$. Second
row: semimajor axes, in Astronomical Units (AU). Their formal errors are in Table IV of
(Pitjeva 2005a), in m. Third row: eccentricities. Fourth row: orbital periods in years. The
result for Venus have been recently obtained by including the Magellan radiometric data
(E.V. Pitjeva, personal communication to the author, June 2008).

|            | Mercury | Venus | Earth | Mars |
|------------|---------|-------|-------|------|
| $\langle \Delta \dot{\varpi} \rangle$ ($10^{-4}'' \text{cy}^{-1}$) | $-36 \pm 50[42]$ | $-4 \pm 5[1]$ | $-2 \pm 4[1]$ | $1 \pm 5[1]$ |
| $a$ (AU)   | 0.387   | 0.723 | 1.000 | 1.523 |
| $e$        | 0.2056  | 0.0067 | 0.0167 | 0.0934 |
| $P$ (yr)   | 0.24    | 0.61  | 1.00  | 1.88  |

Table 2: First column: pair of planets. Second column: $\Pi$ for that pair of planets. The
errors come from the realistic uncertainties in $\langle \Delta \dot{\varpi} \rangle$. Third column: $A$ for that pair of
planets. Fourth column: $\sigma$ level of discrepancy between $\Pi$ and $A$ for that pair of planets.

| A | B | $\Pi$ | $A$ | $\sigma$ |
|---|---|------|----|---------|
| Venus | Mercury | 0.1 ± 0.3 | 1.4 | 4 |
| Earth | Mercury | 0.05 ± 0.18 | 1.64 | 8 |
| Mars | Mercury | $-0.03 \pm 0.18$ | 2.02 | 11 |

because of the existing correlations among the estimated corrections to the precessions of

$^3$The maximum correlation, 26%, occurs for the Earth and Mercury (E.V. Pitjeva, per-
If we repeat our analysis by subtracting from $\langle \Delta \dot{\omega} \rangle$ the main canonical unmodelled effect, i.e. the general relativistic Lense-Thirring precessions (Lense and Thirring 1918) induced by the Sun’s angular momentum (Iorio 2007a) shown in Table 3, i.e. if we use

$$\Pi^* \equiv \frac{\langle \Delta \dot{\omega}_A \rangle^*}{\langle \Delta \dot{\omega}_B \rangle^*} = \frac{\Delta \dot{\omega}_A - \dot{\omega}^{(LT)}_A}{\Delta \dot{\omega}_B - \dot{\omega}^{(LT)}_B},$$

the situation does not substantially change, apart from the sigma level at which $|\Pi^* - A|$ is not compatible with zero, as shown in Table 4.

Table 3: General relativistic Lense-Thirring precessions of the longitudes of perihelia of the inner planets of the Solar System in $10^{-4}''$ cy$^{-1}$.

|       | Mercury | Venus | Earth | Mars |
|-------|---------|-------|-------|------|
| $\dot{\omega}^{(LT)}$ ($10^{-4}''$ cy$^{-1}$) | -20     | -3    | -1    | -0.3 |

Table 4: First column: pair of planets. Second column: $\Pi^*$ for that pair of planets including the unmodelled general relativistic Lense-Thirring effect. The errors come from the realistic uncertainties in $\langle \Delta \dot{\omega} \rangle$. Third column: $A$ for that pair of planets. Fourth column: $\sigma$ level of discrepancy between $\Pi$ and $A$ for that pair of planets.

| A     | B      | $\Pi^*$ | $A$ | $\sigma$ |
|-------|--------|---------|-----|---------|
| Venus | Mercury| 0.06 ± 0.51 | 1.4 | 2.7     |
| Earth | Mercury| 0.06 ± 0.44 | 1.64| 3.5     |
| Mars  | Mercury| -0.08 ± 0.56 | 2.02| 3.7     |

( Personal communication to the author, November 2005).
The availability of the corrections to the usual rates of perihelia of several planets allows us to put on the test MOG also in another way as well. The acceleration law of eq. (1) can also be recast in the commonly used Yukawa form (Moffat and Toth 2007)

\[ A_Y = -\frac{G_Y M}{r^2} \left[ 1 + \alpha_Y \left( 1 + \frac{r}{\lambda} \right) \exp \left( -\frac{r}{\lambda} \right) \right], \]  

where

\[ G_Y = \frac{G_N}{1 + \alpha_Y}, \]

\[ \alpha_Y = -\frac{(G_\infty - G_N)M}{(G_\infty - G_N)M + G_N(\sqrt{M} + C'_{\text{E}})^2}, \]

\[ \lambda = \frac{1}{\mu}. \]

In the case of the Sun

\[ \alpha_Y^\odot = -3.04 \times 10^{-8}, \ G_Y = 1.00000003040G_N, \ \lambda = 33000 \text{ AU}. \]  

A Yukawa-type acceleration of the form of eq. (16) has been tested by Iorio (2007b) in the Solar System without any a-priori assumption on the size of \( \alpha_Y \); concerning \( \lambda \), it was only assumed that \( \lambda \gtrsim a_e \). By using the corrections to the standard rates of the perihelia of \( A = \text{Earth} \) and \( B = \text{Mercury} \) quoted in Table 1 Iorio (2007b) found

\[ \lambda = \frac{a_B - a_A}{\ln \left( \frac{a_B}{a_A} \right)} = 0.182 \pm 0.183 \text{ AU}, \]  

which contradicts eq. (20). Using the data for Venus in the equation for \( \alpha_Y \) (Iorio 2007b)

\[ \alpha_Y = \frac{2\lambda^2 \langle \Delta \dot{\omega} \rangle}{\sqrt{G_Y M a}} \exp \left( \frac{a}{\lambda} \right) \]  

yields

\[ \alpha_Y = (-1 \pm 4) \times 10^{-11}, \]  

\footnote{The strength parameter \( \alpha \) used in (Iorio 2007b) can be identified with \( \alpha_Y \) here.}

\footnote{According to eq. (20), using \( G_N \) in eq. (22) instead of \( G_Y \), as done in (Iorio 2007b), does not produce appreciable modifications of the results.}
which is three orders of magnitude smaller than the result of eq. (20).

If we use Π* for the Earth and Mercury in eq. (21) and ⟨Δ\ddot{\omega}⟩* for Venus in eq. (22) the results does not change appreciably; indeed, we have

\[ \lambda = 0.2 \pm 0.4 \text{ AU}, \quad \alpha_Y = (-0.3 \pm 2.7) \times 10^{-11}. \]  

(24)

3. Conclusions

In the framework of the attempts of explaining certain astrophysical and cosmological features without invoking dark matter, MOG (Moffat 2006; Moffat and Toth 2007) is a long-range modified model of gravity, based on a vector field and three scalar fields representing running constants, which assumes a Yukawa-like form. Recent developments of this theory allowed their proponents to fix (Moffat and Toth 2007, 2008) the values of the constants entering it. We have shown that, on Solar System scales, MOG yields a uniform anomalous acceleration which would induce retrograde planetary perihelion precessions. We put on the test the possibility that such exotic precessions exist by comparing the ratio of them \( \mathcal{A} \) for different pairs of planets to the ratio \( \Pi \) of the corrections to the usual Newtonian/Einsteinian precessions estimated by E.V. Pitjeva which account for any unmodelled/mismodeleld dynamical effects. It turns out that \( \Pi \neq \mathcal{A} \) at more than 3σ level even by including in \( \Pi \) the main unmodelled canonical effect, i.e. the general relativistic Lense-Thirring precessions. Conversely, using the estimated corrections to the planetary perihelion rates to phenomenologically determine the strength parameter of the putative MOG Yukawa force and its range yields values which are neatly incompatible with those of MOG (Moffat and Toth 2007, 2008). In assessing the results presented here it must be considered that, at present, no other people have estimated the non-standard part of the planetary perihelion motions; it would certainly be useful to repeat the present analysis if and when other teams of astronomers will estimate their own set of corrections to the
standard perihelion precessions as well.
REFERENCES

Bertolami, O., Mourão, J.M., and Pérez-Mercader, J., *Phys. Lett. B*, 311, 27, 1993.

Bertolami, O., and Páramos, J., *Phys. Rev. D*, 71, 023521, 2005.

Iorio, L., *Planet. Space Sci.*, 55, 1290, 2007a.

Iorio, L., *J. High En. Phys.*, 10, 041, 2007b.

Lense J and Thirring H 1918 *Phys. Z*. 19 156 Translated and discussed in Mashhoon B, Hehl F W and Theiss D S 1984 *Gen. Relativ. Gravit.* 16 711. Reprinted in Ruffini R J and Sigismondi C (eds) 2003 *Nonlinear Gravitodynamics* (Singapore: World Scientific) pp 349–88

Moffat, J.W., *J. Cosmol. Astropart. Phys.*, 3, 004, 2006.

Moffat, J.W., and Toth, V.T., [arXiv:0712.1796v4 [gr-qc]], 2007.

Moffat, J.W., and Toth, V.T., *Astroph. J.*, 680, 1158, 2008.

Pitjeva, E.V., *Sol. Syst. Res.* 39, 176, 2005a.

Pitjeva, E.V., *Astron. Lett.*, 31, 340, 2005b.

Sanders, R.H., *Mon. Not. Roy. Astron. Soc.*, 370, 1519, 2006.

This manuscript was prepared with the AAS IEEE macros v5.2.