ISOPERIMETRIC REGIONS IN NONPOSITIVELY CURVED MANIFOLDS

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ABSTRACT. Isoperimetric regions minimize the size of their boundaries among all regions with the same volume. In Euclidean and Hyperbolic space, isoperimetric regions are round balls. We show that isoperimetric regions in two and three-dimensional nonpositively curved manifolds are not necessarily convex balls, and need not even be connected.

1. Introduction

A round ball in Euclidean space $\mathbb{E}^n$ minimizes the size of its boundary among all regions with the same volume. The same is true in hyperbolic space $\mathbb{H}^n$. In this paper we examine the extent to which this isoperimetric property carries over to spaces of nonpositive curvature.

Let $|W|$ denote the volume of a Riemannian manifold. An isoperimetric region in an $n$-manifold is a region $W$ ($n$-dimensional submanifold with $(n-1)$-dimensional boundary) such that $\partial W$ minimizes $(n-1)$-volume among all regions enclosing the same volume. So an isoperimetric region satisfies

$$|\partial W| = \inf \{|\partial U| : U \subset M, |U| = |W|\}.$$

A Cartan–Hadamard manifold is a complete, simply-connected Riemannian manifold $(M,g)$ with nonpositive sectional curvature. The standard examples are $\mathbb{E}^n$, with curvature zero, and $\mathbb{H}^n$, with curvature -1, but more general examples have variable nonpositive curvature. Cartan–Hadamard manifolds share many geometric properties with $\mathbb{E}^n$ and $\mathbb{H}^n$. For example, any Cartan–Hadamard manifold is diffeomorphic to $\mathbb{E}^n$ and any two points in such a manifold are connected by a unique geodesic.

The classical isoperimetric inequality for $\mathbb{E}^n$ states that for any region $W \subset \mathbb{E}^n$, we have $|\partial W| \geq |\partial B^n|$ where $B^n \subset \mathbb{E}^n$ is a round ball with $|B^n| = |W|$ [19]. A similar result holds for $\mathbb{H}^n$, for which the isoperimetric inequality states that $|\partial W| \geq |\partial B^\kappa_{n-1}|$, where $B^\kappa_{n-1}$ is the the ball in hyperbolic space satisfying $|B^\kappa_{n-1}| = |W|$.

The Generalized Cartan–Hadamard Conjecture was posed in various forms by Aubin, Burago-Zalgaller and by Gromov [1, 4, 11]. It states that if $M$ is a Cartan–Hadamard manifold with sectional curvatures bounded above by $\kappa \leq 0$, and if $W$ is any region in $M$, then $|\partial W| \geq |\partial B^\kappa_n|$, where $B^\kappa_n$ is the ball in the space form of constant curvature $\kappa$ with $|B^\kappa_n| = |W|$. A summary of known results concerning this
conjecture is found in the recent paper of Kloeckner and Kuperberg [13]. Various cases were proved by Weil \((n = 2, \kappa = 0)\) [25], Bol \((n = 2, \kappa < 0)\) [2], Croke \((n = 4, \kappa = 0)\) [7], Kleiner \((n = 3, \kappa \leq 0)\) [12], Morgan and Johnson (small domains) [18], Druet (under a scalar curvature hypothesis) [9], and Kloeckner and Kuperberg (under a variety of geometric conditions) [13].

In Euclidean and hyperbolic space, isoperimetric regions that realize equality in the isoperimetric inequality are uniquely realized by round balls. Isoperimetric regions were classified for special classes of Riemannian surfaces by Benjamini and Cao [3] and by Howards, Hutchings and Morgan [17]. Some cases in flat 3-manifolds were derived by in Hauswirth, Perez, Romon and Ros [16]. The nature of general isoperimetric regions remains mysterious, even in well understood manifolds such as tori and complex hyperbolic space [20].

Kloeckner and Kuperberg posed the following question [13, Question 4.1]:

**Question.** If \(M\) is a Cartan-Hadamard manifold and \(W\) minimizes \(|\partial W|\) for some fixed value of \(|W|\), then is it convex? Is it a topological ball?

We show that the answer to both parts of this question is no in dimensions two and three. In fact the isoperimetric region need not even be connected.

**Theorem 1.1.** There are Cartan–Hadamard manifolds in dimensions two and three that contain disconnected isoperimetric regions.

We establish this by constructing explicit Cartan–Hadamard manifolds for which the isoperimetric regions appear as islands of zero curvature in a large hyperbolic ocean. We give an explicit construction of such manifolds for dimension two in Theorem 2.1 of Section 2 and for dimension three in Theorem 3.1 of Section 3. It is straightforward to generalize the construction to arrange for any isoperimetric region to have at least \(k\) components, for \(k\) any positive integer.

The construction in Theorem 1.1 appears to have been overlooked. One possible reason is that the derivation of the associated isoperimetric inequalities can often be reduced to the study of connected regions. Consider for example a region \(W \subset \mathbb{E}^2\), for which the isoperimetric inequality states

\[
|W| \leq \frac{|\partial W|^2}{4\pi}.
\]

Suppose that a region \(W\) has two components \(W_1\) and \(W_2\) whose areas sum to those of \(W\), so that \(|W_1| + |W_2| = |W|\). If Equation 1 holds for each of \(W_1, W_2\), then it also holds for \(W\), since

\[
|W| = |W_1| + |W_2| \leq \frac{|\partial W_1|^2}{4\pi} + \frac{|\partial W_2|^2}{4\pi} \leq \frac{(|\partial W_1| + |\partial W_2|)^2}{4\pi} = \frac{|\partial W|^2}{4\pi}.
\]

The convexity property used to obtain the inequality in Equation 2 extends to the isoperimetric inequalities conjectured to hold in all Cartan–Hadamard spaces [13], so it suffices to establish the Cartan–Hadamard Conjecture for connected regions. Furthermore, it was known that the conjecture holds for the regions consisting of geodesic balls of fixed radius in a Cartan–Hadamard manifold. Finally, connectedness of isoperimetric regions can be established in many cases. For example in a homogeneous space, a component of a disconnected isoperimetric region can be translated by an isometry until it touches a second component without overlapping, contradicting the known boundary regularity properties of isoperimetric regions. Nevertheless connectivity does not hold in general Cartan–Hadamard manifolds.
The existence of isoperimetric regions in Cartan–Hadamard manifolds is not automatic, since a sequence of increasingly efficient regions enclosing a given volume may drift off to infinity, yielding no limiting region. The manifolds that we construct are isometric with hyperbolic space away from a compact set, and existence holds in this setting. A general existence result that covers our examples is given by Ritore and Sinestrari [21, Theorem 1.21].

In Section 2 we establish Theorem 1.1 in dimension two. In Section 3 we present additional arguments needed to extend the construction to three dimensions. We end with some concluding remarks in Section 4.

2. Dimension Two

In this section we construct an isoperimetric region in a two-dimensional Cartan–Hadamard manifold that consists of two components. A similar construction gives an isoperimetric region possessing any finite number of components.

Theorem 2.1. There are two-dimensional Cartan–Hadamard manifolds that contain disconnected isoperimetric regions.

2.1. Curvature in Normal Coordinates. A metric described in geodesic normal (polar) coordinates \((r,\theta)\) on \(\mathbb{R}^2\) by

\[
g = dr \otimes dr + h(r)^2 d\theta \otimes d\theta,
\]

has Gauss curvature given by

\[
K(r,\theta) = -\frac{h''(r)}{h(r)}.
\]

A derivation can be found in Spivak [24, p. 136].

The function \(h(r) = r\) results in the flat Euclidean \(\mathbb{E}^2\), while choosing \(h(r) = \sinh(r)\) gives the hyperbolic metric on \(\mathbb{H}^2\), with curvature \(K(r,\theta) = -1\). In these coordinates the length of the circle of radius \(r\) is \(2\pi h(r)\). The first variation of arc length formula then shows that the normal curvature of the radius \(r\) circle is \(k(r) = h'(r)/h(r)\). Fix a positive constant \(R > 1\) and let

\[
C = R - \sinh^{-1} R \quad \text{and} \quad \delta = \frac{1}{4\pi R \sinh(2R)}.
\]

Lemma 2.1. There is a smooth function \(h : \mathbb{R}^+ \to \mathbb{R}\) with the following properties:

1. \(h(r) = r\) for \(0 \leq r \leq R - \delta\),
2. \(h(r) = \sinh(r - C)\) for \(r > R + \delta\),
3. \(h(r)\) is convex for all \(r > 0\).

Proof. The graphs of \(y = x\) and \(y = \sinh(r - C)\) cross at \((R, R)\). For \(R > 1\) we have \(\delta < R\), and the two graphs can be smoothly interpolated by an increasing convex function over the interval \((R - \delta, R + \delta)\). The graph of the resulting function \(h\) is shown in Figure 2.

We use Lemma 2.1 to explicitly construct a 2-dimensional Cartan-Hadamard manifold \(P(R)\). We describe the Riemannian metric on \(P(R)\) using normal coordinates around the origin. The manifold \(P(R)\) is the simply connected surface with smooth metric

\[
g = dr \otimes dr + h(r)^2 d\theta \otimes d\theta.
\]
Lemma 2.2. The 2-dimensional manifold $P(R)$ has nonpositive curvature. It is flat on the disk of radius $R - \delta$ about the origin and the exterior of the disk of radius $R + \delta$ is isometric to the exterior of a disk of radius $C + \delta$ in the hyperbolic plane $H^2$. On the annulus with radii $R - \delta$ and $R + \delta$, $P(R)$ has negative curvature, and the area of this annulus is less than $1/R$.

Proof. The curvature properties follow from Equation (3). Note that $\delta < R$ and therefore Fubini’s Theorem gives an area bound for the annulus with radii $R - \delta$ and $R + \delta$ of

$$2\pi \sinh(R + \delta - C)(2\delta) = 4\pi \sinh(\delta + \sinh^{-1} R)(\delta) < 4\pi \sinh(2R)(\delta) = 1/R.$$ 

The metrics on the surface $P(R)$ outside a disk of radius $R + \delta$ and on the hyperbolic plane outside a hyperbolic disk of radius $C + \delta$ agree. as indicated in Figure 2. Thus $P(R)$ can be obtained from the hyperbolic plane by changing the hyperbolic metric on a disk of radius $C + \delta$. Given a constant $d > 0$, we can change the metric on the hyperbolic plane in the same way on two subdisks $D_1$ and $D_2$ of $H^2$ whose distance from one another is $d$. We call the resulting surface $Q(R,d)$. 

Figure 2. The Cartan-Hadamard surface $P(R)$ at left has a rotationally symmetric nonpositively curved Riemannian metric. It is isometric to hyperbolic 2-space in the complement of a disk $D$. The Cartan-Hadamard surface $Q(R,d)$ at right is isometric to $H^2$ in the complement of two disks $D_1$ and $D_2$. Each disk is isometric to $D$ and they are separated by distance $d$. 

Figure 1. Graph of the radial function $h$. 

\[ y = \sinh(x-C) \]
2.2. Isoperimetric Regions in $P$ and $Q$. We now show that for suitable choices of $R$ and $d$, there is an area $A$ such that the isoperimetric region for area $A$ in $Q(R, d)$ is disconnected. For simplicity we use explicit values for $R$ and $d$, taking the values $R = 100$ and $d = 1000$ to construct $Q = Q(100, 1000)$, and investigate the isoperimetric region in $Q$ with area $A = 62830$.

For these values we have

$$C = R - \sinh^{-1}(R) \approx 94.70.$$  

and

$$\delta = \frac{1}{4\pi R \sinh(200)} \approx (2.2) \times 10^{-90}.$$

By Lemma 2.2, the area of each of the two annuli in $Q$ that transition from flat to hyperbolic metrics is less than .01. This is sufficiently small to ensure that these annuli will not play a significant role in the area comparisons we will make below.

A Euclidean disk of radius 100 has area $A_0 \approx 31416$ and circumference $\approx 628$, and a hyperbolic disk of radius $100 - C \approx 5.3$ has area $A_1 \approx 629$ and circumference $\approx 628$. Thus $A$ is slightly smaller than the area enclosed in two Euclidean disks of radius 100. For comparison, a hyperbolic disk with area $A_0$ has radius $\approx 9.9$, and circumference $\approx 62838$.

Now consider an isoperimetric region of area $A = 62830$ in $Q(100, 1000)$. One possibility for such a region consists of a disconnected region $W_0$ containing two disjoint disks, with one contained in each of the two flat subdisks $D_1$ and $D_2$ of $Q$. We will show that this $W_0$ has shorter boundary than any connected region with the same area.

**Lemma 2.3.** A connected isoperimetric region $W$ in $Q$ has $|\partial W| > 1257$.

**Proof.** We consider first the case that $W$ is disjoint from $(D_1 \cup A_1) \cup (D_2 \cup A_2)$. Then $W$ is contained in a subsurface of $Q$ that is isometric to $H^2$. The length of the boundary of $W$ can then be bounded below by applying the isoperimetric inequality in $H^2$ which states that a region enclosing an area of $A_0$ has boundary length greater than $62837 > 1257$.

Suppose now that $W$ intersects $D_1 \cup A_1$ and is disjoint from $D_2 \cup A_2$. The area of $D_1$ is approximately $A/2$, so that $|W \cap (D_1 \cup A_1)| < 2A/3$ and $W$ intersects the complement of $D_1 \cup A_1$ in a region of area at least $A/3 = 20610$.

The question of how short can one make the boundary curve of a region in hyperbolic space that lies in the complement of a convex set (the complement of $D_1 \cup A_1$) was studied by Choe and Ritore [6]. They show that in two and three-dimensional hyperbolic space, a region in the complement of a convex domain has longer boundary length than a half-disk of the same area. A half-disk in a half-space of $H^3$ with area 20610 has boundary length greater than 20606. As a result, a connected region $W$ that meets $D_1$ must have boundary length no less than 20606 in the complement of $D_1$. Since 20606 $> 1257$, the Lemma holds for such regions.

Finally assume that a region $W$ meets both $D_1 \cup A_1$ and $D_2 \cup A_2$. Since the two disks are distance 1000 apart, we must have $|\partial W| > 2000$ for $W$ connected. Again 2000 $> 1257$ and a connected region that intersects both $D_1$ and $D_2$ has $|\partial W| > 1257$. $\square$

**Proof of Theorem 2.1.** The surface $Q$ contains two disjoint flat disks $D_1$, $D_2$ with combined area 62830. Each disk is surrounded by a radius 26 annulus of area less than .01, which we call $A_1$ and $A_2$ respectively. Outside these annuli, $Q$ is isometric
to the complement in hyperbolic space of two round disks. The union of these two
disks forms a surface $W_0$ with $|\partial W_0| < 1256$

Now let $W$ be an isoperimetric region having area $A$ and assume for contradiction
that $W$ is connected. By Lemma 2.3 we have $|\partial W| > 1257$. Since $|\partial W_0| < 1256$,
we conclude that an isoperimetric region of area $A$ cannot be connected. □

3. Dimension Three

In this section we extend the two-dimensional arguments of Section 2 to construct
a disconnected isoperimetric region in a three-dimensional Cartan–Hadamard man-
ifold.

**Theorem 3.1.** There are three-dimensional Cartan–Hadamard manifolds that con-
tain disconnected isoperimetric regions.

We start by constructing for each $R > 0$ a Cartan-Hadamard 3-manifold $P_3(R)$
with a spherically symmetric Riemannian metric. The metric in geodesic normal
coordinates is given by

$$g = dr \otimes dr + h(r)^2 \omega$$

where $\omega$ is the Riemannian metric on the unit 2-sphere and $h(r)$ was given in Sec-

tion 2. This gives a Riemannian manifold $(M, g)$ that is symmetric under rotations
around the origin. At each point $p \in M$ we can construct an orthonormal frame
with one vector in the radial direction $\partial/\partial r$ and two additional vectors tangent
to the sphere $S(r)$ with $r = r(p)$. The sectional curvatures in the tangent planes
containing the vector $\partial/\partial r$ are determined by the totally geodesic plane through $p$
and the origin, and coincide with those computed for $P(R)$ in Section 2. Thus these
sectional curvatures are nonpositive. It remains to compute the sectional curvature
$K_S$ corresponding to a 2-plane tangent to the sphere $S(r)$.

We determine this with the Gauss Equation, which relates the ambient sectional
curvature $K_r$ to the intrinsic curvature $K_S$ of the 2-sphere $S(r)$ and the determinant
of the second fundamental form of $S(r)$:

$$K_r = K_S + det(II)$$

The constant curvature sphere $S(r)$ has area $4\pi h(r)^2$, so the Gauss-Bonnet Theorem
implies that $K_r = 1/h(r)^2$. The principle curvature $k_n(r)$ at a point on $S(r)$ can
then be determined from the first variation formula for the length $L(r) = 2\pi h(r)$
of a great circle, which states that $L'(r) = L(r) k_n(r)$. This implies that $k_n(r) = L'(r)/L(r) = h'(r)/h(r)$. So $det(II) = h'(r)^2/h(r)^2$, and

$$K_S = K_r - det(II) = \frac{1 - h'(r)^2}{h(r)^2}.$$}

Since $h'(r) \geq 1$ we see that $K_S$ is nonpositive. Note that $K_S = 0$ when $h(r) = r$,
and that $K_S = -1$ when $h(r) = \sinh(r - C)$, as expected.

Having shown that $(M, g)$ is a Cartan–Hadamard manifold, we construct $P_3(100)$
and $Q_3(100, \delta)$ as before, but with a different value fo $\delta$. The manifold $Q_3(100, \delta)$
contains two flat balls $B_1, B_2$ of radius 100, surrounded by thin collars $C_1$ and $C_2$
of radius $2\delta$. The exterior $(B_1 \cup C_1) \cup (B_2 \cup C_2)$ is hyperbolic, with the two balls
separated by a distance $d$. We take $d$ to be a large distance to be fixed later. For the
value $R = 100$, the volume enclosed by a Euclidean sphere of radius $R$ is $\approx 4188790$
and the area of the enclosing sphere is $A \approx 125664$. We choose $\delta$ sufficiently small
so that collars $C_1$ and $C_2$ around $B_1, B_2$ each have volume less than .01.
We now look for an isoperimetric region $W$ in $Q^3(100, d)$ that encloses volume $V = 8377580$, approximately the volume of $B_1 \cup B_2$. We first note that an isoperimetric region exists, since $Q_3(100, d)$ is isometric to $\mathbb{H}^3$ outside a compact set, [21 Theorem 1.21]. Again we consider three cases, according to the three possible ways that a region $W$ can intersect $B_1 \cup B_2$.

One candidate for an isoperimetric region enclosing volume $8377580$ is $W_0$, which consists of two flat balls in each of $B_1, B_2$. Its boundary area satisfies $|\partial W_0| < 251229$. We will show that this is less than the boundary area of any competing connected region.

We first consider a region $W \subset Q_3(100, d)$ enclosing volume $V$ and disjoint from both $B_1 \cup C_1$ and $B_2 \cup C_2$. Thus $W$ is isometric to a subset of hyperbolic space and we can bound its boundary area using the isoperimetric inequality for $\mathbb{H}^3$. In hyperbolic 3-space, the surface area of a sphere of radius $r$ is $4\pi \sinh^2(r) = 2\pi (\cosh(2r) - 1)$ and the volume of a radius $r$ ball is $\pi \sinh(2r) - 2\pi r$. Using this, we see that the volume of a ball of radius $7.74475$ is approximately $8377580$, and the boundary area of this ball is $|\partial E| > 1.67 \times 10^7$ which is larger than $|\partial W_0| \approx 251327$. So an isoperimetric region cannot be disjoint from both $B_1 \cup C_1$ and $B_2 \cup C_2$.

Now suppose that a connected isoperimetric region $W$ meets just one of $B_1 \cup C_1$ and $B_2 \cup C_2$, say $B_1 \cup C_1$, and is disjoint from the second. Since the volume of $B_1 \cup C_1$ is roughly half the volume enclosed by $W$, we must have that $W$ intersects the complement of $B_1 \cup C_1$ in a region whose volume is greater than $V/3 > 2792526$. We now apply a result of Choe and Ritore, who established isoperimetric inequalities for regions in the complement of a convex region in hyperbolic space. These give a lower bound on the area $|\partial W|$ in the complement of $B_1 \cup C_1$. Choe and Ritore show that any region in the complement of a convex region in $\mathbb{H}^3$ that encloses volume $V/3$ has boundary area no less than a hemisphere enclosing the same volume of $V/3$ [3 Theorem 3.2]. In particular, a computation shows that the area of such a boundary surface is at least $5.585 \times 10^6$ which again is larger than $|\partial W_0| \approx 251327$. Thus a connected isoperimetric region cannot be disjoint from $B_2 \cup C_2$.

In the remaining case a connected isoperimetric region $W$ meets both $B_1 \cup C_1$ and $B_2 \cup C_2$. We show that for $d$ sufficiently large, the boundary of such an isoperimetric region must have area $|\partial W| > 2.6 \times 10^5$, eliminating it as a possible isoperimetric region. Note that it is not true that any connected region meeting both $B_1 \cup C_1$ and $B_2 \cup C_2$ has very large area, since it is possible to span the region between these balls by a thin tube of arbitrarily low area. Thus we cannot say that $|\partial W|$ is large simply because $d$ is large, as in dimension two. However we know that the boundary surface $\partial W$ of an isoperimetric region is a stable constant mean curvature, so that there is a tubular neighborhood of $\partial W$ that contains no nearby surface of less area that encloses the same volume. We now show that the area of any connected stable constant mean curvature surface that meets both $B_1$ and $B_2$ goes to infinity with $d$, and in particular that for such $W$, $|\partial W| > 251328$ for large $d$.

Suppose that $W$ is an isoperimetric region that meets both $(B_1 \cup C_1)$ and $(B_2 \cup C_2)$. Then $\partial W$ is a stable constant mean curvature surface in $\mathbb{H}^3$. Let $\gamma \subset Q$ be a geodesic connecting $\partial C_1$ and $\partial C_2$. Then there is a family of totally geodesic planes $H_t$, $0 \leq t \leq d$ in $Q$ that are perpendicular to $\gamma$ and separate $B_1$ from $B_2$. For constants $0 \leq a < b \leq d$ define the region $I_{[a,b]}$ to be the submanifold of $Q$ between $H_a$ and $H_b$, isometric to a submanifold of hyperbolic space bounded by
two hyperbolic planes at distance $b - a$. We consider the intersection of the stable constant mean curvature surface $\partial W$ with $I_{[a,b]}$.

We claim that there is a constant $C$ such that the second fundamental form of $\partial W \cap I_{[d/4,3d/4]}$ is bounded above by $C$ for all $d \geq 4$. Our argument follows an argument given by Meeks, Perez and Ros [15, Theorem 2.16]. If there is a sequence of values $d_i$ and a sequence of isoperimetric regions $W_i$ with no upper bound for the second fundamental form on the surfaces $\partial W_i \cap I_{[d_i/4,3d_i/4]}$, then a sequence of blow ups, or rescalings, at a point of increasing second fundamental form yields a sequence of stable constant mean curvature surfaces with second fundamental form having norm one at that point. The limit of such a sequence of rescalings gives a complete stable minimal surface in $\mathbb{R}^3$ that is not flat. However a theorem of do Carmo and Peng [8], and independently Fischer-Colbrie and Schoen [10], shows that the plane is the only stable minimal surface in $\mathbb{R}^3$. Thus the second fundamental form of $\partial W \cap I_{[d/4,3d/4]}$ is uniformly bounded above by some constant $C$, independent of $d$.

For $w \in H^3$, let $B(w, \epsilon)$ denote the ball about $w$ of radius $\epsilon$. An upper bound on the second fundamental form of $\partial W$ implies a uniform lower bound on $\text{Area}(\partial W \cap B(w, \epsilon))$ for $w \in \partial W$ and for some fixed small $\epsilon > 0$. Indeed, for small enough $\epsilon$, $B(w, \epsilon)$ meets $\partial W$ in a topological disk containing $w$, and the area of such a disk is at least that of a radius $\epsilon$ hyperbolic disk, or $2\pi(cosh(\epsilon) - 1)$.

If we take

$$d > \frac{251328(4\epsilon)}{\pi(cosh(\epsilon) - 1)},$$

then there is enough room in $I_{[d/4,3d/4]}$ for $251328/(4\epsilon)$ disjoint balls centered on points of $\partial W$, each intersecting $\partial W$ in an area of no less than $2\pi(cosh(\epsilon) - 1)$. We conclude that the area of $\partial W$ is at least 251328 for such $d$ and that $W$ cannot be an isoperimetric region enclosing volume $V$. Thus we have eliminated the possibility of a connected isoperimetric region in $Q(d)$ for $d$ large. We conclude that the isoperimetric region in $Q(d)$ for volume $V = 8377580$ is not connected for sufficiently large values of $d$. \hfill \Box

4. Remarks

The constructions we gave for dimension two and three can be extended to all higher dimensions, and it is likely that the resulting Cartan–Hadamard manifolds have similar disconnected isoperimetric regions. However the arguments we gave rely on results known only in two and three dimensions. In particular, we do not know the analogs of the results of Choe and Ritore [6], do Carmo and Peng [8] or Fischer-Colbrie and Schoen in higher dimensions.

While disconnected regions are certainly not convex, it is natural to ask whether each connected component of an isoperimetric regions is convex. In a two-dimensional Cartan–Hadamard manifold, this is indeed the case, and each component of an isoperimetric region is convex.

**Lemma 4.1.** Each component of an isoperimetric region $W$ in a two-dimensional Cartan–Hadamard manifold is a convex disk.

**Proof.** Suppose not. Then there is a geodesic arc $\gamma$ from $a \in \partial W$ to $b \in \partial W$ whose interior lies outside of $W$. Deforming $\partial W$ inwards slightly at a point disjoint from the arc between $a, b$ results in a shortening of length that decreases the enclosed area. Replacing the adjacent arc of $\partial W$ between $a, b$ with $\gamma$ gives a shorter curve
enclosing more area. By sliding the point \( b \) closer to \( a \) we can arrange that the increase in area matches the decrease from the first deformation, resulting in a shorter curve enclosing an equal area. This contradicts our assumption that \( W \) is an isoperimetric region.

\[\square\]

Nonconvex isoperimetric regions appear to exist in three dimensions, though we have not constructed an explicit example. One plausible approach to construction of a nonconvex isoperimetric region follows the methods used to prove Theorem 3.1. Namely, we move \( B_1 \) and \( B_2 \) closer together, decreasing \( d \), until a connected region becomes more efficient at enclosing volume \( V \) than disconnected regions with components in each of \( B_1 \) and \( B_2 \). Note that this happens for some positive \( d \). As in Theorem 3.1 an isoperimetric region \( W \) will necessarily contain large subregions in each of \( B_1 \) and \( B_2 \). These will have a boundary surface with mean curvature close to zero. Convexity implies that these regions are connected by a surface that spans the hyperbolic region between \( B_1 \) and \( B_2 \). If we look at the intersection of this surface with the hyperbolic plane separating \( B_1 \) and \( B_2 \) then we see that its intersection with this plane is a convex curve whose mean curvature is greater than \( 1/2 \). Since \( \partial W \) has constant mean curvature, this gives a contradiction to the assumption of convexity. This suggests that convexity is unlikely to hold for connected isoperimetric regions.

The examples we constructed were flat on two balls and non-positively curved everywhere. It is straightforward to make a small perturbation of the constructed metrics to give examples of disconnected isoperimetric regions on manifolds with strictly negative curvature. One simply perturbs the function \( h(r) \) of Section 2 slightly so that it is everywhere strictly convex.

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