Algebraic characterizations of some relative notions of size

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Abstract

We obtain algebraic characterizations of relative notions of size in a discrete semigroup that generalize the usual combinatorial notions of syndetic, thick, and piecewise syndetic sets. “Filtered” syndetic and piecewise syndetic sets were defined and applied earlier by Shuungula, Zelenyuk, and Zelenyuk [25]. Other instances of these relative notions of size have appeared explicitly (and more often implicitly) in the literature related to the algebraic structure of the Stone–Čech compactification. Building on this prior work, we observe a natural duality and demonstrate how these notions of size may be composed to characterize previous notions of size (like piecewise syndetic sets) and serve as a convenient description for new notions of size.

Keywords Stone–Čech compactification, syndetic sets, thick sets, piecewise syndetic sets, ultrafilters

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1 Introduction

In this paper, we study certain “relative notions of size” that can be succinctly characterized by the algebraic structure of the Stone–Čech compactification of a discrete semigroup. All notions of size we consider are ultimately motivated by van der Waerden’s theorem on arithmetic progressions [29].
This well known classical result in Ramsey theory has many equivalent formulations. But, one early observation (see [26, Chapter 33, p. 319]) of Kakeya and Morimoto [17, Sections 1 and 2] shows that van der Waerden’s theorem can be reformulated to assert that certain “large” subsets of positive integers contain arbitrarily long arithmetic progressions:

**Theorem 1.1** (Reformulation of van der Waerden’s theorem). Let \( A \subseteq \mathbb{N} \) have **bounded gaps**, that is, there exists a positive integer \( b \) such that for all positive integers \( x \) we have

\[
(\{1, 2, \ldots, b\} + x) \cap A \neq \emptyset.
\]

Then for all positive integers \( k \in \mathbb{N} \) there exist \( a, d \in \mathbb{N} \) with

\[
\{a, a + d, a + 2d, \ldots, a + (k - 1)d\} \subseteq A.
\]

This reformulation is important since it’s the first **documented** example (known to the authors) of a fundamental heuristic that underlies a significant proportion of current research in Ramsey theory. Roughly stated, this heuristic asserts that underlying many Ramsey theoretic phenomena is at least one **notion of size** (for instance, bounded gaps) which contains enough “**structure**” to imply an interesting (**combinatorial**) **pattern** (for instance, arbitrarily long arithmetic progressions). In **Theorem 1.1** one can use the structure of minimal left ideals in the Stone–ˇCech compactification, as first shown by Bergelson, Furstenberg, Hindman, and Katznelson [3], to obtain an algebraic proof of van der Waerden’s theorem. Often the most difficult part of this trio is to identify the right structures to leverage to deduce the combinatorial consequence.

**Remark 1.2.** Historically, however, the dissemination of the “Ramsey theory heuristic” was more directly influenced from developments surrounding the Erdős and Turán Conjecture [12] leading up to Szemerédi’s Theorem [27] and beyond. Tao’s essay [28] provides a nice brief nontechnical overview on how Szemerédi’s Theorem, and especially its proofs, have influenced and motivated a lot of work in Ramsey theory and its applications. Additionally, Bergelson’s survey article [2] provides several examples and theorems demonstrating the effectiveness of approaching Ramsey theory from this heuristic.

**Three notions of size in semigroups**

In general, given a subset \( A \) of a (for us, infinite) semigroup \( S \), we can classify \( A \) as “large” in several different ways. A recent paper of Hindman [15] surveys 52 notions of size which have interesting connections to either Ramsey theory, dynamics, or the algebraic structure of the Stone–Čech compactification of a discrete semigroup. However in this paper we’ll consider far fewer notions — essentially **only three**.
For an arbitrary semigroup a set with bounded gaps is not guaranteed to make sense because most semigroups don’t have a natural ordering. But, we can capture the essential properties of bounded gaps via the notion of “syndetic”. In the following definition, and in the rest of this paper, given a set $X$ we let $\mathcal{P}_f(X)$ denote the collection of all nonempty finite subsets of $X$; and, if $(S, \cdot)$ is a semigroup, $A \subseteq S$, and $x \in S$ we define $x^{-1}A = \{y \in S : x \cdot y \in A\}$. Beside defining syndetic, we also introduce two more notions closely related to syndetic sets:

**Definition 1.3.** Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

(a) We call $A$ **syndetic** if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{h \in H} h^{-1}A = S$. We let $\text{Syn}$ denote the collection of all syndetic subsets of $S$.

(b) We call $A$ **thick** if and only if for all $H \in \mathcal{P}_f(S)$ we have $\bigcap_{h \in H} h^{-1}A \neq \emptyset$. We let $\text{Thick}$ denote the collection of all thick subsets of $S$.

(c) We call $A$ **piecewise syndetic** if and only if there exist $B \in \text{Syn}$ and $C \in \text{Thick}$ such that $A = B \cap C$. We let $\text{PS}$ denote the collection of all piecewise syndetic subsets of $S$.

We leave it as an exercise to verify, in $(\mathbb{N}, +)$ syndetic sets are precisely those sets with bounded gaps. It’s also helpful to keep in mind that, again in $(\mathbb{N}, +)$, a set is thick if and only if it contains arbitrarily long blocks of consecutive positive integers, and a set is piecewise syndetic if and only if there exists a fixed bound such that the set contains arbitrarily long subsets of positive integers whose gaps are no bigger than the fixed bound. Furstenberg in [13, Definition 1.11] defined piecewise syndetic sets, in $\mathbb{N}$ or $\mathbb{Z}$, as the intersection of a syndetic and thick set (and also noted the equivalence we just stated above).

In Section 2, we’ll give a brief review of the algebraic structure of the Stone–Čech compactification $(\beta S, \cdot)$ of a discrete semigroup $(S, \cdot)$, but, for now, we’ll simply note that all three notions have succinct characterizations in terms of this algebraic structure [16, Theorems 4.48 and 4.40]:

**Theorem 1.4.** Let $(S, \cdot)$ be a semigroup.

(a) $\text{Syn} = \{A \subseteq S : \text{for every minimal left ideal } L \text{ of } \beta S \text{ we have } L \cap c\ell_{\beta S}(A) \neq \emptyset\}$.

(b) $\text{Thick} = \{A \subseteq S : \text{there exists a minimal left ideal } L \text{ of } \beta S \text{ such that } L \subseteq c\ell_{\beta S}(A)\}$.

(c) $\text{PS} = \{A \subseteq S : \text{there exists a minimal left ideal } L \text{ of } \beta S \text{ such that } L \cap c\ell_{\beta S}(A) \neq \emptyset\}$.

**Organization of article**

The goal of this paper is to define and study “relative” notions of syndetic, thick, and piecewise syndetic sets by investigating how these notions “compose” with each other and prove algebraic characterizations of these relative notions that generalize Theorem 1.4.
**Relative notions in the literature**  “Filtered” notions of syndetic and piecewise syndetic sets were previously defined and considered by Shuungula, Zelenyuk, and Zelenyuk [25]. Their paper, and a related older paper of Davenport [10], both form the starting part for our own investigations. (In a sense, this paper can be partially viewed as a synthesis of their work.) “Filtered” notions of thick sets have also appeared implicitly in much of the literature related to the algebraic structure of the Stone–Čech compactification. And, in special cases, appeared more or less explicitly in the context of ‘finite embeddability’ by Blass and Di Nasso and Baglini [7, 19, 18] and in a note of Protasov and Slobodianiuk [22]. We also note that Zucker [30] considers some related ideas in the context of a different generalization of syndetic, thick, and piecewise syndetic sets.

In Section 2, we state standard and known results in Proposition 2.5 and fix some notation and terminology. (As we note in Remark 2.2 and Remark 2.4 some of the notation and terminology we introduce, while previously appearing in the literature, is not standardized. But, our choices are suitable and flexible for our purpose.) We also end this section with a brief review of the algebraic structure of the Stone–Čech compactification.

In Section 3, we give the definitions for relative syndetic and thick sets, observe a duality between them, illustrate how these notions can be “composed”, and prove, in Lemma 3.9, a result that algebraically characterizes a large number of these notions. As one (almost immediate) consequence of this lemma, we obtain combinatorial characterizations of closed subsemigroups, closed left ideals, and closed right ideals of $\beta S$. These latter combinatorial results were previously obtained by Davenport [10] and, independently, by Papazyan [21].

In Section 4, we define relative piecewise syndetic sets and immediately observe this notion is ‘partition regular’ in Theorem 4.2. Our definition of relative piecewise syndetic is different from the one defined earlier by Shuungula, Zelenyuk, and Zelenyuk [25]. We show that our definition satisfies their notion of relative piecewise syndetic. However, since we’ve been unable to verify the converse, we end this section with an open question asking if these two definitions, in fact, coincide.

We also note that in several of our proofs below given three (or more) statements $P$, $Q$, and $R$, following the usual mathematical practice, when we write “$P \iff Q \iff R$” we mean “($P \iff Q$) and ($Q \iff R$)”.

## 2 Preliminaries: Notions of size and the Stone–Čech compactification

Instead of considering a single “large” set, it’s usually convenient to consider a collection of all large subsets, as we’ve done in Definition 1.3 via $\text{Syn}$, $\text{Thick}$, and $\text{PS}$. Moreover, it’s reasonable to assume that such a collection satisfies some minimal requirements. (For example, any such collection should be nonempty and not contain the empty set.) To this end, we introduce some terminology defined using four conditions, that, in a sense, axiomatizes “notions of size” in set-theoretic terms:
Definition 2.1. Let $X$ be a nonempty set and let $\mathcal{F} \subseteq \mathcal{P}(X)$.

(a) We call $\mathcal{F}$ a **stack on** $X$ if and only if $\mathcal{F}$ satisfies two conditions:

1. $\emptyset \notin \mathcal{F}$ and $\emptyset \notin \mathcal{F}$ and
2. $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$ implies $B \in \mathcal{F}$.

(b) We call $\mathcal{F}$ a **filter on** $X$ if and only if $\mathcal{F}$ is a stack and $\mathcal{F}$ satisfies

3. $A \in \mathcal{F}$ and $B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

(c) We call $\mathcal{F}$ a **grill on** $X$ if and only if $\mathcal{F}$ is a stack and $\mathcal{F}$ satisfies

4. $A \cup B \in \mathcal{F}$ implies $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

(d) We call $\mathcal{F}$ an **ultrafilter on** $X$ if and only if $\mathcal{F}$ is both a filter and a grill.

Remark 2.2 (Origin of some of the nonstandard terminology). We take “stack” from “Stapel” of [14, p. 321] since it is relatively short and somewhat descriptive. Stacks are also called, especially in the dynamics literature, “(Furstenberg) families”. (For example, see [1, Introduction and Chapter 2].)

The terms “filter” and “ultrafilter” are both well known and completely standard, but the term “grill” seems like a reasonable standard but is not, perhaps, well known. This latter term we take from “grille” in [9].

From Definition 1.3 it’s easy to verify $\text{Syn}$, $\text{Thick}$, and $\text{PS}$ are all stacks on $S$. Simple examples in $(\mathbb{N}, +)$ show that neither the collections $\text{Syn}$ or $\text{Thick}$ are guaranteed to be either a filter or grill, however $\text{PS}$ is a grill.

Showing that $\text{PS}$ is a grill is not trivial (but not too hard either). In fact, the assertion that for the semigroup $(\mathbb{N}, +)$ the collection $\text{PS}$ is a grill is (commonly referred to as) Brown’s lemma [8, Lemma 1]. We refer you to [15, Section 2] for more details on the historical appearance of this notion. In contrast to the algebraic [16, a direct consequence of Theorem 4.40], the combinatorial [4, Theorem 2.5], and the dynamical [13, Theorem 1.24] proofs that $\text{PS}$ is a grill, we’ll provide another combinatorial proof of this fact in Corollary 2.6.

The reason we introduce this additional terminology is that our point-of-view will be to think of stacks, filters, grills, and ultrafilters as each describing a different aspect of a notion of size. Moreover, all four notions are connected by a certain operator on $\mathcal{P}(\mathcal{P}(X))$:

**Definition 2.3.** Let $X$ be a nonempty set and let $\mathcal{F} \subseteq \mathcal{P}(X)$. The **mesh of** $\mathcal{F}$ is $\mathcal{F}^* = \{A \subseteq X : X \setminus A \notin \mathcal{F}\}$.

**Remark 2.4 (No standard term for mesh operator).** There is no standard nor well known terminology for what we call the “mesh operator”. Perhaps the closest attempt, from which we derive
our terminology, is “Verzahnung” in [24, Kapitel II]. (Actually Schmidt [24, Kapitel II] defines his mesh operator as $F^* = \{A \subseteq X : (\forall B \in F) A \cap B \neq \emptyset\}$. When $F$ is a stack, both these definitions coincide (see Proposition 2.5(d).) In the dynamics literature, $F^*$ is called a “dual family”, and if $F$ is a filter, $F^*$ is called a filterdual (see [1, Chapter 2]).

As an example of using the mesh operator, first observe that there is a duality between syndetic and thick: $A \subseteq S$ is syndetic if and only if $S \setminus A$ is not thick. This duality, using the mesh operator, can also be written as $\text{Syn} = \text{Thick}^*$. While this may seem like a triviality, we’ll soon see this duality is a fundamental fact that helps show $PS$ is a grill (see Corollary 2.6 and its use of Proposition 2.5(h)).

The following proposition states some of the fundamental properties of the mesh operator. A significant subset of these statements was noted at least as early as Choquet [9]. Schmidt [24, Kapitel II] also proves a significant subset of these statements. A more contemporary reference for all of these statements is Akins [1, Propositions 2.1, 2.2, and 2.3]. The reader can easily verify the following proposition, but, for completeness and convenience, we’ll include the proof.

**Proposition 2.5.** Let $F, F_1,$ and $F_2$ be stacks on $X$.

(a) $F^*$ is a stack on $X$.

(b) $F = (F^*)^*$.

(c) $F_1 \subseteq F_2$ if and only if $F_2^* \subseteq F_1^*$.

(d) $F = \{A \subseteq X : (\forall B \in F^*) A \cap B \neq \emptyset\}$.

(e) $F$ is a filter if and only if $F^*$ is a grill.

(f) If $F$ a filter, then $F$ is an ultrafilter if and only if $F = F^*$.

(g) If $F$ a filter and $p$ is an ultrafilter, then $F \subseteq p$ if and only if $p \subseteq F^*$.

(h) The collection $\{B \cap C : B \in F \text{ and } C \in F^*\}$ is a grill on $X$ that contains $F$ and $F^*$.

**Proof.**

(a) Since $X \setminus X = \emptyset \notin F$ and $X \setminus \emptyset = X \in F$ we have $X \in F^*$ and $\emptyset \notin F^*$. Now let $A \in F^*$ and $A \subseteq B \subseteq X$. Since $X \setminus B \subseteq X \setminus A$ and $X \setminus A \notin F$ we have $X \setminus B \notin F$ (since $F$ is a stack). Hence $B \in F^*$. This shows $F^*$ is a stack on $X$.

(b) This follows from definition: $A \in (F^*)^* \iff X \setminus A \notin F^* \iff A \in F$.

(c) By statement (b), it suffices to show $F_1 \subseteq F_2$ implies $F_2^* \subseteq F_1^*$. Suppose $F_1 \subseteq F_2$ then $A \in F_2^*$, that is, $X \setminus A \notin F_2$ and its follows that $X \setminus A \notin F_1$. Hence $A \in F_1^*$.

(d) We have $A \in F \iff X \setminus A \notin F^* \iff \neg(\exists B \in F^*) B \subseteq X \setminus A \iff (\forall B \in F^*) B \cap A \neq \emptyset$, where the middle equivalence follows from statement (a).
(e) By assumption and statement (a) we automatically know that $\mathcal{F}$ and $\mathcal{F}^*$ are both stacks on $X$.

First, suppose that $\mathcal{F}$ is a filter and let $A \cup B \in \mathcal{F}^*$, that is, $(X \setminus A) \cap (X \setminus B) = X \setminus (A \cup B) \not\in \mathcal{F}$. It follows, since $\mathcal{F}$ is a filter, that either $X \setminus A \not\in \mathcal{F}$ or $X \setminus B \not\in \mathcal{F}$, that is, either $A \in \mathcal{F}^*$ or $B \in \mathcal{F}^*$. Therefore $\mathcal{F}^*$ is a grill on $X$.

Now, suppose $\mathcal{F}^*$ is a grill and let $A \in \mathcal{F}$ and $B \in \mathcal{F}$, that is, $X \setminus A \not\in \mathcal{F}^*$ and $X \setminus B \not\in \mathcal{F}^*$. It follows, since $\mathcal{F}^*$ is a grill, that $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B) \not\in \mathcal{F}^*$, that is, $A \cap B \in \mathcal{F}$. Therefore $\mathcal{F}$ is a filter on $X$.

(f) Observe that when $\mathcal{F}$ is a filter we have $A \in \mathcal{F} \implies X \setminus A \not\in \mathcal{F}$, and so $\mathcal{F} \subseteq \mathcal{F}^*$.

If $\mathcal{F}$ is an ultrafilter, then $\mathcal{F}$ is a grill and by statement (e) we have $\mathcal{F}^*$ is a filter. Hence by our observation we have $\mathcal{F}^* \subseteq \mathcal{F}$. If $\mathcal{F}^* = \mathcal{F}$, then, since $\mathcal{F}$ is a filter, by statement (e) we have $\mathcal{F}^*$ is a grill and so $\mathcal{F}$ is an ultrafilter.

(Note that we need the assumption that $\mathcal{F}$ is a filter. For if $X = \{1, 2, 3\}$ and $\mathcal{F} = \{A \subseteq X : |A| \geq 2\}$, then $\mathcal{F} = \mathcal{F}^*$ but $\mathcal{F}$ is not an ultrafilter, since it is not a grill.)

(g) This follows directly from statements (c) and (f).

(h) Put $\mathcal{G} = \{B \cap C : B \in \mathcal{F}$ and $C \in \mathcal{F}^*\}$ and observe that $\mathcal{F} \subseteq \mathcal{G}$ implies $\emptyset \not\in \mathcal{G}$ and statement (d) implies $\emptyset \not\in \mathcal{G}$. If $A \in \mathcal{G}$, so $A = B \cap C$ for some $B \in \mathcal{F}$ and $C \in \mathcal{F}^*$, and $A \subseteq D \subseteq X$, then $B \cup D \in \mathcal{F}$ (since $\mathcal{F}$ is a stack) and $C \cup D \in \mathcal{F}^*$ (since, by statement (a), $\mathcal{F}^*$ is a stack) implies $D = A \cup D = (B \cap C) \cup D = (B \cup D) \cap (C \cup D) \in \mathcal{G}$. This shows that $\mathcal{G}$ is a stack.

To see $\mathcal{G}$ is a grill let $A_1 \cup A_2 \in \mathcal{G}$ and pick $B \in \mathcal{F}$ and $C \in \mathcal{F}^*$ with $A_1 \cup A_2 = B \cap C$. Then $A_2 \setminus A_1 = (X \setminus A_1) \cap (A_1 \cup A_2) = (B \setminus A_1) \cap C$. If $B \setminus A_1 \in \mathcal{F}$, then $A_2 \setminus A_1 \in \mathcal{G}$ and so $A_2 \in \mathcal{G}$. If $B \setminus A_1 \not\in \mathcal{F}$, then $(X \setminus B) \cup A_1 \in \mathcal{F}^*$ and so $B \cap A_1 = B \cap ((X \setminus B) \cup A_1) \in \mathcal{G}$ and hence $A_1 \in \mathcal{G}$. □

For us Proposition 2.5(f) and (g) are of fundamental importance to the ultrafilter description of the Stone–Čech compactification and its connections to Ramsey Theory. For instance see [16, Theorems 3.11 and 5.7]. The earliest reference we’ve found for Proposition 2.5(h) is [1, Proposition 2.1(e)] (in Akin’s formulation it asserts that the set $\{B \cap C : B \in \mathcal{F}$ and $C \in \mathcal{F}^*\}$ is a filter). Our proof follows a blogpost of Moreira [20, Proposition 3].

One application of Proposition 2.5(h) is that it implies $\text{PS}$ is a grill (this implication is also noted by Moreira [20, Corollary 4]):

**Corollary 2.6.** Let $(S, \cdot)$ be a semigroup. Then the collection of all piecewise syndetic sets $\text{PS}$ is a grill on $S$.

**Proof.** By definition $\text{PS} = \{B \cap C : B \in \text{Syn}$ and $C \in \text{Thick}\}$, and so by Proposition 2.5(h) and the duality $\text{Syn} = \text{Thick}^*$ we have $\text{PS}$ is a grill. □
Brief review of algebraic structure of the Stone–Čech compactification

We’ll end this section by fixing some additional notation and giving a brief review of the algebraic structure of the Stone–Čech compactification of a discrete semigroup.

A standing convention throughout this paper is $\beta S$ will denote an infinite discrete semigroup with $\cdot$ as its binary operation. We take $\beta S$ to be the collection of all ultrafilters on $S$. Given $A \subseteq S$ we put $\overline{A} = \{p \in \beta S : A \in p\}$. Then the collection $\{\overline{A} : A \subseteq S\}$ is a basis for a compact Hausdorff topology on $\beta S$ and $\text{cl}_{\beta S}(A) = \overline{A}$. This topology is the Stone–Čech compactification of $S$.

If $F$ is a filter on $S$, we put $F = \{p \in \beta S : F \subseteq p\}$. Then $F$ is a nonempty closed subset of $\beta S$; conversely, any nonempty closed subset of $\beta S$ is uniquely generated by some filter. The empty subset of $\beta S$ is generated by the so-called “improper filter” $\mathcal{P}(S)$. (More precisely, if $C \subseteq \beta S$ is a closed subset, then $\bigcap \{p \in \beta S : p \in C\}$ is the filter that generates it.)

The proofs of these assertions can be found in [16, Sections 3.2 and 3.3].

The semigroup operation on $S$ can be extended to a semigroup operation on $\beta S$ [16, Theorem 4.1] such that for every $p, q \in \beta S$ we have $A \in p \cdot q$ if and only if $\{x \in S : x^{-1} A \in q\} \in p$ [16, Theorem 4.12]. This extension makes $(\beta S, \cdot)$ a compact right topological semigroup. Right topological means that for every $q \in \beta S$ the map $p \mapsto p \cdot q$ for all $p \in \beta S$ is continuous.

**Important Note:** Several of our references, [6, 11, 10], take $\beta S$ to be left topological. We take our algebraic structure on $\beta S$ to be right topological. In all the cases we cite, the appropriate left-right switches of their proofs and statements give the corresponding right topological version.

Compact Hausdorff right topological semigroups contain significant algebraic structure. For instance, $(\beta S, \cdot)$ has idempotent elements (usually many of them) [16, Theorem 2.5] and a smallest ideal $K(\beta S)$ which is the union of all minimal left ideals of $\beta S$ and also the union of all minimal right ideals of $\beta S$ [16, Theorem 2.8].

The characterization $A \in p \cdot q \iff \{x \in S : x^{-1} A \in q\} \in p$ can be taken as the definition for $F \cdot G$ when both $F$ and $G$ are stacks. That is, we can define $F \cdot G = \{A \subseteq S : \{x \in S : x^{-1} A \in G\} \in F\}$. Berglund and Hindman proved [6, Lemma 5.15] that when both $F$ and $G$ are filters, this product is also a filter and associative. We simply note that their proof (with appropriate left-right switches) also shows, if both $F$ and $G$ are stacks, then this product is also a stack and is associative.

3 Relative notions of syndetic and thick sets

The goal of this section is to demonstrate how the algebraic structure of $\beta S$ can be used to concisely characterize many instances of relative notions of syndetic and thick sets.

**Definition 3.1.** Let $A \subseteq S$ and let $F$ and $G$ both be stacks on $S$. 

8
(a) $A$ is $(\mathcal{F}, \mathcal{G})$-syndetic if and only if for every $B \in \mathcal{F}$ there exists $H \in \mathcal{P}(B)$ such that $\bigcup_{h \in H} h^{-1}A \in \mathcal{G}$.

(b) $A$ is $(\mathcal{F}, \mathcal{G})$-thick if and only if there exists $B \in \mathcal{F}$ such that for every $H \in \mathcal{P}(B)$ we have $\bigcap_{h \in H} h^{-1}A \in \mathcal{G}^*$.

(c) We also define the following two collections:

(i) $\text{Syn}(\mathcal{F}, \mathcal{G}) = \{ A \subseteq S : A$ is $(\mathcal{F}, \mathcal{G})$-syndetic $\}$.

(ii) $\text{Thick}(\mathcal{F}, \mathcal{G}) = \{ A \subseteq S : A$ is $(\mathcal{F}, \mathcal{G})$-thick $\}$.

The notation and name of “$(\mathcal{F}, \mathcal{G})$-syndetic” is due to Shuungula, Zelenyuk, and Zelenyuk [25, third paragraph of Section 2], and our approach to these notions are influenced by the results and methods in their paper (for instance, see Remark 3.3). They used this notion, when $\mathcal{F}$ and $\mathcal{G}$ are both filters, to prove a simple characterization of the smallest ideal in a closed subsemigroup of $\beta S$ [25, Theorem 2.2] analogous to a characterization of the smallest ideal of $\beta S$ [16, Theorem 4.39]. Davenport obtained a similar characterization (after a bit of rewriting) earlier [10, Theorem 3.4].

Observe $\text{Syn} = \text{Syn}(\{S\}, \{S\})$ and $\text{Thick} = \text{Thick}(\{S\}, \{S\})$.

The notions of $(\mathcal{F}, \mathcal{F})$-syndetic and $(\mathcal{F}, \mathcal{F})$-thick, again when $\mathcal{F}$ is a filter, also appears in a note of Protasov and Sloboadianiuk [22] as “$\tau$-large” and “$\tau$-thick”, respectively. Among other things, they also characterize the smallest ideal of a closed subsemigroup of $\beta S$ [22, Theorem 3.1].

Additionally, Blass and Di Nasso [7] also studied the notion of “finite embeddability” and its connection to the algebraic structure of the Stone–Čech compactification. In our terminology and notation, given two subsets $A$ and $B$ of the nonnegative integers, $\mathbb{Z}_{\geq 0}$, they defined $A$ is finitely embeddable in $B$ if and only if $B$ is $(\mathcal{F}, \{\mathbb{Z}_{\geq 0}\})$-thick where the first component is the principal filter $\mathcal{F} = \{ C \subseteq \mathbb{Z}_{\geq 0} : A \subseteq C \}$ generated by $A$. Baglini further studied [19] and extended [18] the concept of finite embeddability.

Similar to syndetic and thick, it’s easy to observe these relative notions share a duality:

**Proposition 3.2.** Let $\mathcal{F}$ and $\mathcal{G}$ both be stacks on a semigroup $S$.

(a) $\text{Syn}(\mathcal{F}, \mathcal{G}) = \text{Thick}(\mathcal{F}, \mathcal{G})^*$.

(b) $\text{Syn}(\mathcal{F}, \mathcal{G}) = \{ A \subseteq S : (\forall B \in \text{Thick}(\mathcal{F}, \mathcal{G})) A \cap B \neq \emptyset \}$.

**Proof.** (a) The justification for statement (a) is ultimately an easy application of De Morgan’s
laws:

\[ A \in \text{Syn}(\mathcal{F}, \mathcal{G}) \iff (\forall B \in \mathcal{F})(\exists H \in \mathcal{P}_f(B)) \bigcup_{h \in H} h^{-1} A \in \mathcal{G} \]

\[ \iff \neg(\exists B \in \mathcal{F})(\forall H \in \mathcal{P}_f(B)) \bigcap_{h \in H} h^{-1}(S \setminus A) \in \mathcal{G}^* \]

\[ \iff S \setminus A \notin \text{Thick}(\mathcal{F}, \mathcal{G}) \iff A \in \text{Thick}(\mathcal{F}, \mathcal{G})^*. \]

(b) Statement (b) then follows immediately from Proposition 2.5(d). \[\square\]

**Remark 3.3.** The reader may wonder why we refer to the mesh of a collection \( \mathcal{G}^* \) in the definition of \((\mathcal{F}, \mathcal{G})\)-thick and not indicate this directly in the notation, perhaps by calling such sets “\((\mathcal{F}, \mathcal{G})^*\)thick” instead. Our choice was made so we can follow the notation as used by Shuungula, Zelenyuk, and Zelenyuk [25] and maintain the duality principle between the notions of \((\mathcal{F}, \mathcal{G})\)-thick and \((\mathcal{F}, \mathcal{G})\)-syndetic (see Proposition 3.2 above).

**Example 3.4.** As the reader probably suspects, our relative notion of thickness does not imply the usual notion of thickness even in \((\mathbb{N}, +)\). Consider the principal filter generated by the even positive integers \( \mathcal{F} = \{ A \subseteq \mathbb{N} : 2\mathbb{N} \subseteq A \} \). Observe that \( A \in \text{Thick}(\mathcal{F}, \{\mathbb{N}\}) \) if and only if for all \( F \in \mathcal{P}_f(\mathbb{N}) \) there exists \( x \in \mathbb{N} \) such that \( 2F + x \subseteq A \). In particular, the set of even positive integers is \((\mathcal{F}, \{\mathbb{N}\})\)-thick but is not thick.

Hence \((\mathcal{F}, \{\mathbb{N}\})\)-thick is a weaker notion than thick (since it allows more sets to be “thick”), that is, \( \text{Thick} \subseteq \text{Thick}(\mathcal{F}, \{\mathbb{N}\}) \). Also, by Proposition 3.2, this is equivalent to writing \( \text{Syn}(\mathcal{F}, \{\mathbb{N}\}) \subseteq \text{Syn} \), that is, \((\mathcal{F}, \{\mathbb{N}\})\)-syndetic is a stronger notion than syndetic (since it allows fewer sets to be “syndetic”).

As Example 3.4 indicates, we also have the following order-reversing [order-preserving] implications for relative syndetic and thick sets:

**Proposition 3.5.** Let \( S \) be a semigroup and let \( \mathcal{F}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{G}, \mathcal{G}_1, \) and \( \mathcal{G}_2 \) all be stacks on \( S \).

(a) If \( \mathcal{F}_1 \subseteq \mathcal{F}_2 \), then \( \text{Syn}(\mathcal{F}_2, \mathcal{G}) \subseteq \text{Syn}(\mathcal{F}_1, \mathcal{G}) \).

(b) If \( \mathcal{G}_1 \subseteq \mathcal{G}_2 \), then \( \text{Syn}(\mathcal{F}, \mathcal{G}_1) \subseteq \text{Syn}(\mathcal{F}, \mathcal{G}_2) \).

(a’) If \( \mathcal{F}_1 \subseteq \mathcal{F}_2 \), then \( \text{Thick}(\mathcal{F}_1, \mathcal{G}) \subseteq \text{Thick}(\mathcal{F}_2, \mathcal{G}) \).

(b’) If \( \mathcal{G}_1 \subseteq \mathcal{G}_2 \), then \( \text{Thick}(\mathcal{F}, \mathcal{G}_2) \subseteq \text{Thick}(\mathcal{F}, \mathcal{G}_1) \).

**Proof.** By Proposition 3.2(c), statement (a) is equivalent to statement (a’) and statement (b) is equivalent to statement (b’). The proofs of statements (a) and (b) are each straightforward one-line verifications.
Proposition 3.7. \textit{PS} of all piecewise syndetic sets of relative thick are known. For instance, we have the following characterization for the collection of nine different notions of being (relatively) thick. In this case we have at most $K_\text{Thick}$.

(a) Let $A \in \text{Syn}(\mathcal{F}_2, \mathcal{G})$ and let $B \in \mathcal{F}_1 \subseteq \mathcal{F}_2$. Pick $H \in \mathcal{P}_f(B)$ such that $\bigcup_{h \in H} h^{-1}A \in \mathcal{G}$ as guaranteed by $A$.

(b) Let $A \in \text{Syn}(\mathcal{F}, \mathcal{G}_1)$ and let $B \in \mathcal{F}$. Pick $H \in \mathcal{P}_f(B)$ such that $\bigcup_{h \in H} h^{-1}A \in \mathcal{G}_1 \subseteq \mathcal{G}_2$.

We also note that that relative syndetic and thick collections do not necessarily preserve strict inclusion among either the left or right components:

Example 3.6. In $(\mathbb{N}, +)$ we have $\text{Thick}(\{\mathbb{N}\}, \{\mathbb{N}\}) = \text{Thick}(\mathcal{C}, \{\mathbb{N}\}) = \text{Thick}(\{\mathbb{N}\}, \mathcal{C}) = \text{Thick}(\mathcal{C}, \mathcal{C})$, where $\mathcal{C} = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite}\}$ is the cofinite filter on $\mathbb{N}$. (This can either be checked directly from the definitions or noting that $C$ is a closed ideal of $\beta\mathbb{N}$ and applying Lemma 3.9(c) and Theorem 1.4.)

One advantage in following Shuungula, Zelenyuk, and Zelenyuk’s notation is that it allows us to “compose” these notions to produce (possibly new) relative notions of size. To get a sense of what we mean by composing these notions consider $\text{Thick}(\mathcal{F}, \mathcal{G})$ where $\mathcal{F}$ and $\mathcal{G}$ are in $\{\{S\}, \text{Syn, Thick}\}$. In this case we have at most nine different notions of being (relatively) thick. Some of these notions of relative thick are known. For instance, we have the following characterization for the collection of all piecewise syndetic sets PS as a composition (starting with a well known characterization PS):

Proposition 3.7. Let $S$ be a semigroup.

(a) $\text{PS} = \text{Syn}(\{S\}, \text{Thick})$.

(b) $\text{PS} = \text{Thick}(\text{Syn}, \text{PS}^*)$.

(c) $\text{PS}^* = \text{Thick}(\{S\}, \text{Thick})$.

(d) $\text{PS} = \text{Thick}(\text{Syn}, \text{Thick}(\{S\}, \text{Thick}))$.

\textbf{Proof.} (a) It is a result of Bergelson, Hindman, and McCutcheon [4, Theorem 2.4(d)] (or also see [16, Theorem 4.49]) that $A \in \text{PS}$ if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{h \in H} h^{-1}A$ is thick.

(b) $(\subseteq)$ Let $A \in \text{PS}$. By [16, Theorem 4.40], we have that $A$ is piecewise syndetic if and only if $K(\beta S) \cap \overline{A} \neq \emptyset$. Pick $q \in K(\beta S) \cap \overline{A}$ as guaranteed, and by [16, Theorem 4.39], we have $\{x \in S : x^{-1}A \in q\}$ is syndetic. Putting $B = \{x \in S : x^{-1}A \in q\}$ we see that for every $H \in \mathcal{P}_f(B)$ we have that $\bigcap_{x \in H} x^{-1}A \in q \subseteq \text{PS}$, since $q$ is an ultrafilter in $K(\beta S)$. Hence $A \in \text{Thick}(\text{Syn}, \text{PS}^*)$.

$(\supseteq)$ We’ll provide a combinatorial proof of this direction. Let $A \in \text{Thick}(\text{Syn}, \text{PS}^*)$ and pick $B \in \text{Syn}$ as guaranteed for $A$. Pick $H \in \mathcal{P}_f(S)$ such that $\bigcup_{h \in H} h^{-1}B = S$. We claim that $\bigcup_{h \in H} h^{-1}A$ is thick.

Let $F \in \mathcal{P}_f(S)$ and for each $f \in F$ pick $h_f \in H$ such that $h_f \cdot f \in B$. Then $\{h_f \cdot f : f \in F\}$ is a finite nonempty subset of $B$. We can pick $X \in \text{PS}$ such that $\{h_f \cdot f : f \in F\} \cdot X \subseteq A$. It follows that $F \cdot X \subseteq \bigcup_{h \in H} h^{-1}A$ and so $A \in \text{PS}$.

11
This completes the proof of statement (b).

(c) Applying Proposition 3.2 to statement (a) yields \( PS^* = \text{Thick}(\{S\}, \text{Thick}) \).

(d) Combining statements (b) and (c) immediately proves this equivalence.

A combinatorial proof of a special case (in our notation, \( PS = \text{Thick}(\text{Syn}, \{S\}) \)) of statement (a), in the context of \((\mathbb{N}, +)\), is given in [20, Lemma 5]. The characterization for \( PS^* \), at least in the context of a group, appears at least as early as [5, Definition 1.4] under the name “permanently syndetic”. It appears that statement (d) is a new characterization of piecewise syndetic sets. (We wouldn’t be surprised if this latter characterization or its dual has appeared previously in the literature or folklore.)

As this result illustrates, it is not immediately clear how all of these notions are related. Moreover, developing an algebraic theory of their classification or characterization seems it could be a challenging but important project. (Important because the algebraic characterizations of notions of size indicate the underlying structure of “large” sets.) Therefore we propose the following characterization problem:

**Problem 3.8** (Characterization problem on composed notions of size). Develop an algebraic characterization of composed notions of size. Given stacks \( F \) and \( G \) is it possible to develop a systemic characterization of \( \text{Syn}(F, G) \) and \( \text{Thick}(F, G) \) using the algebraic structure of \( \beta S \)?

In Lemma 3.9 we solve special instances of the above “characterization problem” in terms of closed subsets of \( \beta S \), but, as Proposition 3.7 indicates, we believe more can be said on this point.

**Lemma 3.9.** Let \( F \) and \( G \) both be filters on a semigroup \( S \).

(a) \( \text{Thick}(F, G) = \{ A \subseteq S : \text{there exists} \ q \in G \text{ such that } F \cdot q \subseteq A \} \)

(b) \( \text{Thick}(F, G^*) = \{ A \subseteq S : F \cdot G \subseteq A \} \)

(c) \( \text{Thick}(F^*, G) = \{ A \subseteq S : F \cdot G \cap A \neq \emptyset \} \)

(d) \( \text{Thick}(F^*, G^*) = \{ A \subseteq S : \text{there exists} \ p \in F \text{ such that } p \cdot G \cap A \neq \emptyset \} \)

(a') \( \text{Syn}(F, G) = \{ A \subseteq S : \text{there exists} \ p \in F \text{ such that } p \cdot G \subseteq A \} \)

(b') \( \text{Syn}(F, G^*) = \{ A \subseteq S : F \cdot G \cap A \neq \emptyset \} \)

(c') \( \text{Syn}(F^*, G) = \{ A \subseteq S : F \cdot G \subseteq A \} \).

(d') \( \text{Syn}(F^*, G^*) = \{ A \subseteq S : \text{for every} \ p \in F \text{ we have } p \cdot G \cap A \neq \emptyset \} \).

**Proof.** By Proposition 3.2 statements (a), (b), (c), and (d) are equivalent to the corresponding statements (a'), (b'), (c'), and (d'), respectively.
(a) \((\subseteq)\) Let \(A \in \text{Thick}(\mathcal{F}, \mathcal{G})\) and pick \(B \in \mathcal{F}\) as guaranteed for \(A\). By [16, Theorem 3.11], we can pick \(q \in \mathcal{G}\) with \(\{x^{-1}A : x \in B\} \subseteq q\). If \(p \in \mathcal{F}\), then \(B \in p\) (since \(\mathcal{F} \subseteq p\)). Now \(B \subseteq \{x \in S : x^{-1}A \in q\}\) implies \(\{x \in S : x^{-1}A \in q\} \in p\), that is, \(A \in p \cdot q\). Hence \(\mathcal{F} \cdot q \subseteq \overline{A}\).

(\(\supseteq\)) Now assume we have \(A \subseteq S\) such that there exists \(q \in \overline{\mathcal{G}}\) with \(\overline{\mathcal{F}} \cdot q \subseteq \overline{A}\). Then \(A \in p \cdot q\) for every \(p \in \mathcal{F}\), that is, \(\{x \in S : x^{-1}A \in q\} \in p\) for every \(p \in \mathcal{F}\). Therefore \(\{x \in S : x^{-1}A \in q\} \in \mathcal{F}\), and if we put \(B = \{x \in S : x^{-1}A \in q\}\) and let \(H \in \mathcal{P}_f(B)\), we have \(\bigcap_{h \in H} h^{-1}A \in q \subseteq \mathcal{G}^*\). Hence \(A \in \text{Thick}(\mathcal{F}, \mathcal{G})\).

This completes the proof of statement (a).

(b) Recall, by [6, Lemma 5.15], we have \(\mathcal{F} \cdot \mathcal{G}\) is a filter on \(S\).

(\(\subseteq\)) Let \(A \in \text{Thick}(\mathcal{F}, \mathcal{G}^*)\) and pick \(B \in \mathcal{F}\) as guaranteed for \(A\), that is, for all \(H \in \mathcal{P}_f(B)\) we have \(\bigcap_{h \in H} h^{-1}A \in \mathcal{G}\). Then \(B \subseteq \{x \in S : x^{-1}A \in \mathcal{G}\}\) and \(B \in \mathcal{F}\) implies \(\{x \in S : x^{-1}A \in \mathcal{G}\} \in \mathcal{F}\), that is, \(A \in \mathcal{F} \cdot \mathcal{G}\). Hence \(\overline{\mathcal{F} \cdot \mathcal{G}} \subseteq \overline{A}\).

(\(\supseteq\)) Now assume we have \(A \subseteq S\) such that \(\mathcal{F} \cdot \mathcal{G} \subseteq \overline{A}\). Then \(A \in \mathcal{F} \cdot \mathcal{G}\), that is, \(\{x \in S : x^{-1}A \in \mathcal{G}\} \in \mathcal{F}\). Put \(B = \{x \in S : x^{-1}A \in \mathcal{G}\}\) and let \(H \in \mathcal{P}_f(B)\). We have \(\bigcap_{h \in H} h^{-1}A \in \mathcal{G}\) and hence \(A \in \text{Thick}(\mathcal{F}, \mathcal{G}^*)\).

This completes the proof of statement (b).

(c) \((\subseteq)\) Let \(A \in \text{Thick}(\mathcal{F}^*, \mathcal{G})\) and pick \(B \in \mathcal{F}^*\) as guaranteed for \(A\). Similar to our proof for statement (a), by [16, Theorem 3.11], we can pick \(q \in \mathcal{G}\) such that \(\{x^{-1}A : x \in B\} \subseteq q\). Pick \(p \in \mathcal{F}\) with \(B \in p\). Then \(B \subseteq \{x \in S : x^{-1}A \in q\}\) and \(B \in p\) implies \(\{x \in S : x^{-1}A \in q\} \in p\), that is, \(A \in p \cdot q\).

(\(\supseteq\)) Now assume we have \(A \subseteq S\) such that \(\mathcal{F} \cdot \mathcal{G} \cap \overline{A} \neq \emptyset\). Pick \(p \in \mathcal{F}\) and \(q \in \overline{\mathcal{G}}\) with \(p \cdot q \in \overline{A}\), that is, \(\{x \in S : x^{-1}A \in q\} \in p\). Put \(B = \{x \in S : x^{-1}A \in q\}\) and let \(H \in \mathcal{P}_f(B)\). Then \(\bigcap_{h \in H} h^{-1}A \in q \subseteq \mathcal{G}^*\). Hence \(A\) is \((\mathcal{F}^*, \mathcal{G})\)-thick.

This completes the proof of statement (c).

(d) \((\subseteq)\) Let \(A \in \text{Thick}(\mathcal{F}^*, \mathcal{G}^*)\) and pick \(B \in \mathcal{F}^*\) as guaranteed for \(A\). Then it follows that \(B \subseteq \{x \in S : x^{-1}A \in \mathcal{G}\}\), since \(\mathcal{G}\) is a filter. Pick \(p \in \mathcal{F}\) with \(B \in p\). Then \(A \in p \cdot \mathcal{G}\), that is, \(\overline{p \cdot \mathcal{G}} \subseteq \overline{A}\).

(\(\supseteq\)) Now assume we have \(A \subseteq S\) such that \(\overline{p \cdot \mathcal{G}} \subseteq \overline{A}\) for some \(p \in \mathcal{F}\). Then \(\{x \in S : x^{-1}A \in \mathcal{G}\} \in p\). Put \(B = \{x \in S : x^{-1}A \in \mathcal{G}\}\) (so \(B \in p \subseteq \mathcal{F}^*\)) and let \(H \in \mathcal{P}_f(B)\). Then \(\bigcap_{h \in H} h^{-1}A \in \mathcal{G}\) and hence \(A \in \text{Thick}(\mathcal{F}^*, \mathcal{G}^*)\).

This completes the proof of statement (d).

When \(\mathcal{F} = \mathcal{G} = \{S\}\), Lemma 3.9(a) and (a’) imply the characterizations for thick and syndetic given earlier in Theorem 1.4. Lemma 3.9 also generalizes a characterization for \(\text{Syn}(\mathcal{F}, \mathcal{G})\), when both \(\mathcal{F}\) and \(\mathcal{G}\) are filters and \(\mathcal{F} \subseteq \mathcal{G}\), proved by Shuungula, Zelenyuk, and Zelenyuk [25, Lemma 2.1]. Additionally, it generalizes the characterizations for \(\tau\)-large and \(\tau\)-thick in [22, Theorems 2.1 and 2.2], and generalizes Blass and Di Nasso’s formulations of finite embeddability [7, Theorem 4].
The remaining characterizations appear to be mostly new, but we suspect that they appear both explicitly and (in a sense, necessarily) implicitly in much of the literature on algebra in $\beta S$.

If $p, q \in \beta S$, then $A$ is $(p, q)$-thick if and only if $A$ is $(p, q)$-syndetic if and only if $A \in p \cdot q$, that is, all of these notions collapse to a product of two ultrafilters when we substitute $p, q$ for $F, G$. Hence Lemma 3.9 can be thought of as a generalization of [16, Theorem 4.12], but, of course, in the proof of our lemma we used this latter result implicitly throughout.

One convenient aspect of Lemma 3.9 is that we can easily create new filters and hence describe the corresponding closed subsets of $\beta S$ in a “combinatorial way”. We note two important examples of this in the following theorem:

**Theorem 3.10.** Let $F$ and $G$ both be filters on a semigroup $S$.

(a) $\text{Syn}(F^*, G)$ is a filter on $S$ and $\overline{\text{Syn}(F^*, G)} = \text{cl}(F \cdot G)$.

(b) $\text{Thick}(F, G^*)$ is a filter on $S$ and $\overline{\text{Thick}(F, G^*)} = F \cdot \overline{G}^*$.

*Proof.* (a) From Lemma 3.9(c’) we have $\text{Syn}(F^*, G) = \{ A \subseteq S : F \cdot G \subseteq A \}$. With this characterization we can easily verify that $\text{Syn}(F^*, G)$ is a filter on $S$. We have $S \in \text{Syn}(F^*, G)$ (since $F \cdot G \subseteq \beta S = S$) and $\emptyset \notin \text{Syn}(F^*, G)$ (since $\overline{\emptyset} = \emptyset$). If $A \in \text{Syn}(F^*, G)$ and $A \subseteq B \subseteq S$, then $F \cdot G \subseteq A \subseteq B$ and hence $B \in \text{Syn}(F^*, G)$. Finally, if $A, B \in \text{Syn}(F^*, G)$, then $F \cdot G \subseteq A \cap B = A \cap B$ and hence $A \cap B \in \text{Syn}(F^*, G)$.

Also from Lemma 3.9(c’) we have $F \cdot \overline{G} \subseteq \overline{\text{Syn}(F^*, G)}$, and since $\overline{\text{Syn}(F^*, G)}$ is closed we have $\text{cl}(F \cdot G) \subseteq \overline{\text{Syn}(F^*, G)}$. To see the reverse inclusion $\text{Syn}(F^*, G) \subseteq \text{cl}(F \cdot G)$ let $p \in \text{Syn}(F^*, G)$ and $A \in p$. By Proposition 2.5(g), Proposition 3.2, and Lemma 3.9(c) we have $(F \cdot G) \cap \overline{A} \neq \emptyset$. Hence $p \in \text{cl}(F \cdot G)$.

(b) The proof of this statement is similar to (a).

First, recall again from [6, Lemma 5.15], we have $F \cdot G$ is a filter on $S$ and so $F \cdot \overline{G}$ is nonempty closed subset of $\beta S$. From Lemma 3.9(b) we have $\text{Thick}(F, G^*) = \{ A \subseteq S : F \cdot \overline{G} \subseteq A \}$ and we can use this characterization to verify $\text{Thick}(F, G^*)$ is a filter on $S$. We have $S \in \text{Thick}(F, G^*)$ (since $F \cdot \overline{G} \subseteq \beta S = S$) and $\emptyset \notin \text{Thick}(F, G^*)$ (since $\overline{\emptyset} = \emptyset$). If $A \in \text{Thick}(F, G^*)$ and $A \subseteq B \subseteq S$, then $F \cdot \overline{G} \subseteq A \subseteq B$ and hence $B \in \text{Thick}(F, G^*)$. If $A, B \in \text{Thick}(F, G^*)$, then $F \cdot \overline{G} \subseteq A \cap B = A \cap B$ and hence $A \cap B \in \text{Thick}(F, G^*)$.

Therefore we have $F \cdot \overline{G} \subseteq \text{Thick}(F, G^*)$. To see the reverse inclusion $\overline{\text{Thick}(F, G^*)} \subseteq F \cdot \overline{G}$ let $p \in \overline{\text{Thick}(F, G)}$ and let $A \in p$. Then by Proposition 2.5(g), Proposition 3.2, and Lemma 3.9(b’) we have $F \cdot \overline{G} \cap \overline{A} \neq \emptyset$. Hence $p \in F \cdot \overline{G}$. \qed

In [11, Theorem 2.3], Davenport and Hindman obtained a special case of Theorem 3.10(a). In our notation, they proved that $\text{Syn}(p, C)$ is a filter, where $C$ is the cofinite filter on $\mathbb{N}$. Protasov implicitly uses $\text{Syn}(C^*, C)$, where $C$ is the cofinite filter on an infinite group $G$ to characterize $\text{cl}_{\beta G}(G^* \cdot G^*)$.
[23, Theorem 3.20], where $G^*$ is the collection of all non-principal ultrafilters on $G$. Finally, we also note that the inclusion (again, in our notation) $\text{Thick}(\mathcal{F}, G^*) \subseteq \text{Syn}(\mathcal{F}^*, \mathcal{G})$ was first proved by Berglund and Hindman [6, Lemma 5.15].

As a consequence of Theorem 3.10(a) is we can easily characterize closed subsemigroups, left ideals, and right ideals of $\beta S$. These results were previously proved by Davenport in [10] and, independently, by Papazyan in [21]. We’ll derive this characterizations from a more general result:

**Theorem 3.11.** Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be filters on a semigroup $S$. Then $\mathcal{F} \cdot \mathcal{G} \subseteq \mathcal{H}$ if and only if $\mathcal{H} \subseteq \text{Syn}(\mathcal{F}^*, \mathcal{G})$.

**Proof.** Since $\mathcal{H}$ is closed, we have $\mathcal{F} \cdot \mathcal{G} \subseteq \mathcal{H} \iff \text{cl}(\mathcal{F} \cdot \mathcal{G}) \subseteq \mathcal{H} \iff \text{Syn}(\mathcal{F}^*, \mathcal{G}) \subseteq \mathcal{H} \iff \mathcal{H} \subseteq \text{Syn}(\mathcal{F}^*, \mathcal{G})$, where the middle and last equivalences use Theorem 3.10(a).

**Corollary 3.12.** Let $\mathcal{F}$ be a filter on a semigroup $S$.

(a) $\overline{\mathcal{F}}$ is a closed subsemigroup of $\beta S$ if and only if $\mathcal{F} \subseteq \text{Syn}(\mathcal{F}^*, \mathcal{F})$.

(b) $\mathcal{F}$ is a closed left ideal of $\beta S$ if and only if $\mathcal{F} \subseteq \text{Syn}({S}^*, \mathcal{F})$.

(c) $\mathcal{F}$ is a closed right ideal of $\beta S$ if and only if $\mathcal{F} \subseteq \text{Syn}(\mathcal{F}^*, \{S\})$.

(d) $\mathcal{F}$ is a closed (two-sided) ideal of $\beta S$ if and only if $\mathcal{F} \subseteq \text{Syn}({S}^*, \mathcal{F}) \cap \text{Syn}(\mathcal{F}^*, \{S\})$.

**Proof.** In the justifications of (a), (b), and (c) below each of the second equivalences follows since $\mathcal{F}$ is closed and each of the third equivalences follows from Theorem 3.11:

(a) $\mathcal{F}$ is a subsemigroup $\iff \mathcal{F} \cdot \mathcal{F} \subseteq \mathcal{F} \iff \text{cl}(\mathcal{F} \cdot \mathcal{F}) \subseteq \mathcal{F} \iff \mathcal{F} \subseteq \text{Syn}(\mathcal{F}^*, \mathcal{F})$.

(b) $\mathcal{F}$ is a left ideal $\iff \beta S \cdot \mathcal{F} \subseteq \mathcal{F} \iff \text{cl}(\beta S \cdot \mathcal{F}) \subseteq \mathcal{F} \iff \mathcal{F} \subseteq \text{Syn}({S}^*, \mathcal{F})$.

(c) $\mathcal{F}$ is a right ideal $\iff \mathcal{F} \cdot \beta S \subseteq \mathcal{F} \iff \text{cl}(\mathcal{F} \cdot \beta S) \subseteq \mathcal{F} \iff \mathcal{F} \subseteq \text{Syn}(\mathcal{F}^*, \{S\})$.

(d) This follows directly from statements (b) and (c).

4 Relative notions of piecewise syndetic sets

In this section, inspired by Corollary 2.6, we define relative piecewise syndetic sets in such a way that any such collection forms a grill on $S$. We show, under the conditions considered in [25], that our definition of relative piecewise syndetic satisfies Shuungula, Zelenyuk, and Zelenyuk’s earlier previously defined notion of relative piecewise syndetic. Whether the converse implication is true is stated as an open question.
**Definition 4.1.** Let $A \subseteq S$ and let $\mathcal{F}$ and $\mathcal{G}$ both be stacks on $S$.

(a) $A$ is **piecewise $(\mathcal{F}, \mathcal{G})$-syndetic** if and only if there exist $B \in \text{Syn}(\mathcal{F}, \mathcal{G})$ and $C \in \text{Thick}(\mathcal{F}, \mathcal{G})$ such that $A = B \cap C$.

(b) We also define the collection $\text{PS}(\mathcal{F}, \mathcal{G}) = \{ A \subseteq S : A \text{ is piecewise } (\mathcal{F}, \mathcal{G})\text{-syndetic} \}$.

Note that we have $\text{PS} = \text{PS}(\{S\}, \{S\})$. Of course, our main motivation in defining piecewise $(\mathcal{F}, \mathcal{G})$-syndetic sets as an intersection of $(\mathcal{F}, \mathcal{G})$-syndetic and $(\mathcal{F}, \mathcal{G})$-thick sets is we can apply Proposition 2.5(h) to obtain a generalization of Corollary 2.6:

**Theorem 4.2.** Let $\mathcal{F}$ and $\mathcal{G}$ both be stacks on $S$. Then $\text{PS}(\mathcal{F}, \mathcal{G})$ is a grill on $S$ with $\text{Syn}(\mathcal{F}, \mathcal{G}) \subseteq \text{PS}(\mathcal{F}, \mathcal{G})$ and $\text{Thick}(\mathcal{F}, \mathcal{G}) \subseteq \text{PS}(\mathcal{F}, \mathcal{G})$.

**Proof.** By definition $\text{PS}(\mathcal{F}, \mathcal{G}) = \{ B \cap C : B \in \text{Syn}(\mathcal{F}, \mathcal{G}) \text{ and } C \in \text{Thick}(\mathcal{F}, \mathcal{G}) \}$, and by Proposition 3.2 we have $\text{Syn}(\mathcal{F}, \mathcal{G}) = \text{Thick}(\mathcal{F}, \mathcal{G})^*$. Hence from Proposition 2.5(h) it follows that $\text{PS}(\mathcal{F}, \mathcal{G})$ is a grill on $S$ that contains both $\text{Syn}(\mathcal{F}, \mathcal{G})$ and $\text{Thick}(\mathcal{F}, \mathcal{G})$.

Similar to Lemma 3.9, if we assume $\mathcal{F}$ and $\mathcal{G}$ are both filters on $S$, then we can (partially) solve a few special instances of the classification problem for relative piecewise syndetic sets:

**Proposition 4.3.** Let $\mathcal{F}$ and $\mathcal{G}$ both be filters on $S$.

(a) $\text{PS}(\mathcal{F}, \mathcal{G}) \subseteq \text{Thick}(\mathcal{F}^*, \mathcal{G})$ and the inclusion can be strict.

(b) $\text{PS}(\mathcal{F}, \mathcal{G}^*) = \text{Syn}(\mathcal{F}, \mathcal{G}^*)$.

(c) $\text{PS}(\mathcal{F}^*, \mathcal{G}) = \text{Thick}(\mathcal{F}^*, \mathcal{G})$.

(d) $\text{PS}(\mathcal{F}^*, \mathcal{G}^*) \subseteq \{ A \subseteq S : \text{ there exists } p \in \overline{\mathcal{F}} \text{ such that } \overline{p \cdot \mathcal{G} } \cap \overline{A} \neq \emptyset \}$.

**Proof.** (a) Observe, via Lemma 3.9(c), that $\{ A \subseteq S : (\exists q \in \mathcal{G}) (\mathcal{F} \cdot q \cap \overline{A} \neq \emptyset) \} = \text{Thick}(\mathcal{F}^*, \mathcal{G})$.

Now let $A \in \text{PS}(\mathcal{F}, \mathcal{G})$ and pick $B \in \text{Syn}(\mathcal{F}, \mathcal{G})$ and $C \in \text{Thick}(\mathcal{F}, \mathcal{G})$ as guaranteed for $A$. By Lemma 3.9(a), pick $q \in \mathcal{G}$ with $\mathcal{F} \cdot q \subseteq \overline{C}$. Then

$$(\mathcal{F} \cdot q) \cap \overline{A} = (\mathcal{F} \cdot q) \cap \overline{B} \cap \overline{C} = (\mathcal{F} \cdot q) \cap \overline{B} \cap \overline{C} = (\mathcal{F} \cdot q) \cap \overline{B} \neq \emptyset,$$

where the last relation follows from Lemma 3.9(a’).

To see that the inclusion can be strict, in $(\mathbb{N}, +)$ we note $\text{PS} \subsetneq \text{Thick}(\{\mathbb{N}\}^*, \{\mathbb{N}\})$ since, for example, $\{2\} \in \text{Thick}(\{\mathbb{N}\}^*, \{\mathbb{N}\})$ but $\{2\} \notin \text{PS}$.

(b) By Theorem 4.2 it suffices to verify $\text{PS}(\mathcal{F}, \mathcal{G}^*) \subseteq \text{Syn}(\mathcal{F}, \mathcal{G}^*)$.\[16]
Let \( A \in \text{PS}(\mathcal{F}, \mathcal{G}^*) \) and pick \( B \in \text{Syn}(\mathcal{F}, \mathcal{G}^*) \) and \( C \in \text{Thick}(\mathcal{F}, \mathcal{G}^*) \) as guaranteed for \( A \). Then
\[
\mathcal{F} \cdot \mathcal{G} \cap A = \mathcal{F} \cdot \mathcal{G} \cap B \cap C = \mathcal{F} \cdot \mathcal{G} \cap B \cap C = \mathcal{F} \cdot \mathcal{G} \cap B \neq \emptyset,
\]
where the third equality follows from Lemma 3.9(b) and last relation follows from Lemma 3.9(b'). Hence by Lemma 3.9(b') we have \( A \in \text{Syn}(\mathcal{F}, \mathcal{G}^*) \).

(c) By Theorem 4.2 it suffices to verify \( \text{PS}(\mathcal{F}^*, \mathcal{G}) \subseteq \text{Thick}(\mathcal{F}^*, \mathcal{G}) \).

Let \( A \in \text{PS}(\mathcal{F}^*, \mathcal{G}) \) and pick \( B \in \text{Syn}(\mathcal{F}^*, \mathcal{G}) \) and \( C \in \text{Thick}(\mathcal{F}^*, \mathcal{G}) \) as guaranteed for \( A \). Then
\[
\mathcal{F} \cdot \mathcal{G} \cap A = \mathcal{F} \cdot \mathcal{G} \cap B \cap C = \mathcal{F} \cdot \mathcal{G} \cap B \cap C = \mathcal{F} \cdot \mathcal{G} \cap B \neq \emptyset,
\]
where the third equality follows from Lemma 3.9(c') and last relation follows from Lemma 3.9(c). Hence by Lemma 3.9(c) we have \( A \in \text{Thick}(\mathcal{F}^*, \mathcal{G}) \).

(d) Let \( A \in \text{PS}(\mathcal{F}^*, \mathcal{G}^*) \) and pick \( B \in \text{Syn}(\mathcal{F}^*, \mathcal{G}^*) \) and \( C \in \text{Thick}(\mathcal{F}^*, \mathcal{G}^*) \) as guaranteed for \( A \). By Lemma 3.9(d) pick \( p \in \mathcal{F} \) such that \( \overline{p \cdot \mathcal{G}} \subseteq \mathcal{C} \). Then
\[
\overline{p \cdot \mathcal{G}} \cap A = \overline{p \cdot \mathcal{G}} \cap B \cap C = \overline{p \cdot \mathcal{G}} \cap B \cap C = \overline{p \cdot \mathcal{G}} \cap B \neq \emptyset,
\]
where the last relation follows Lemma 3.9(d').

The point of statements (b) and (c) is that for filters \( \mathcal{F} \) and \( \mathcal{G} \), neither \( \text{PS}(\mathcal{F}, \mathcal{G}^*) \) nor \( \text{PS}(\mathcal{F}^*, \mathcal{G}) \) produce any new notion of size beyond relative syndetic and thick sets, respectively. We suspect that the inclusion in statement (d) is strict even in \((\mathbb{N},+)\), but we don’t know of an example to prove it. Hence from our point-of-view, statement (a) represents the main interesting new notion of size and for the rest of this section we’ll restrict our attention to \( \text{PS}(\mathcal{F}, \mathcal{G}) \) for filters \( \mathcal{F} \) and \( \mathcal{G} \).

Observe from Lemma 3.9(a) we have \( \text{Thick} = \bigcup_{q \in \beta S} \text{Thick}({\{S\}}, q) \), and so from Proposition 3.7(a) we can conclude \( A \in \text{PS} \) if there exists \( q \in \beta S \) with \( A \in \text{Syn}({\{S\}}, \text{Thick}({\{S\}}, q)) \).

For a filter \( \mathcal{F} \) such that \( \mathcal{F} \) is a closed subsemigroup of \( \beta S \), the notion of a piecewise \( \mathcal{F} \)-syndetic set was defined earlier by Shuungula, Zelenyuk, and Zelenyuk [25, p. 534, second paragraph] as \( A \subseteq S \) is \textbf{piecewise} \( \mathcal{F} \)-\textbf{syndetic} if and only if there exists \( q \in \mathcal{F} \) such that \( A \in \text{Syn}(\mathcal{F}, \text{Thick}(\mathcal{F}, q)) \). It’s not immediately clear, in this case, that the two definitions of relative piecewise syndetic sets are equivalent. We prove, under the conditions considered in their paper, that \( A \in \text{PS}(\mathcal{F}, \mathcal{F}) \) implies \( A \) is piecewise \( \mathcal{F} \)-syndetic (the converse implication is open). We’ll derive this from a more general result:

**Theorem 4.4.** Let \( \mathcal{F} \) and \( \mathcal{G} \) both be filters on \( S \) with \( \mathcal{F} \) is a closed subsemigroup of \( \beta S \) and \( \mathcal{F} \cdot \mathcal{G} \subseteq \mathcal{G} \). If \( A \in \text{PS}(\mathcal{F}, \mathcal{G}) \), then there exists \( q \in \mathcal{G} \) with \( A \in \text{Syn}(\mathcal{F}, \text{Thick}(\mathcal{F}, q)) \).

**Proof.** Pick \( B \in \text{Syn}(\mathcal{F}, \mathcal{G}) \) and \( C \in \text{Thick}(\mathcal{F}, \mathcal{G}) \) such that \( A = B \cap C \). By Lemma 3.9(a) pick
$q \in \mathcal{G}$ such that $\mathcal{F} \cdot q \subseteq \mathcal{C}$. Since $\mathcal{F}$ is a subsemigroup we have $\mathcal{F} \cdot \mathcal{F} \cdot q \subseteq \mathcal{F} \cdot q \subseteq \mathcal{C}$. Since $\mathcal{F} \cdot \mathcal{G} \subseteq \mathcal{G}$ and $B \in \text{Syn}(\mathcal{F}, \mathcal{G})$, by Lemma 3.9(a') it follows that for all $p \in \mathcal{F}$ we have $\mathcal{F} \cdot p \cdot q \cap B \neq \emptyset$. Hence for all $p \in \mathcal{F}$ we have

$$(\mathcal{F} \cdot p \cdot q) \cap A = (\mathcal{F} \cdot p \cdot q) \cap B \cap \mathcal{C} = (\mathcal{F} \cdot p \cdot q) \cap B \cap \mathcal{C} = (\mathcal{F} \cdot p \cdot q) \cap B \neq \emptyset.$$ 

\[ \square \]

**Corollary 4.5.** Let $\mathcal{F}$ be a filter on $S$ with $\mathcal{F}$ a closed subsemigroup of $\beta S$. If $A \in \text{PS}(\mathcal{F}, \mathcal{F})$, then there exists $q \in \mathcal{F}$ with $A \in \text{Syn}(\mathcal{F}, \text{Thick}(\mathcal{F}, q))$.

**Question 4.6.** Let $\mathcal{F}$ be a filter on $S$ with $\mathcal{F}$ a closed subsemigroup of $\beta S$. If $A \subseteq S$ such that there exists $q \in \mathcal{F}$ with $A \in \text{Syn}(\mathcal{F}, \text{Thick}(\mathcal{F}, q))$, must $A$ be a member of $\text{PS}(\mathcal{F}, \mathcal{F})$?

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