On the asymptotic dimension of the curve complex

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Abstract

We give a bound, linear in the complexity of the surface, to the asymptotic dimension of the curve complex as well as the capacity dimension of the ending lamination space.

1 Introduction

Let \( \Sigma \) be a closed orientable surface, possibly with punctures. The curve complex \( \mathcal{C}(\Sigma) \) of \( \Sigma \) has played the fundamental role in recent work on the geometry of mapping class groups. Its hyperbolicity was established by Masur and Minsky [MM99], who also introduced many tools used to study its geometry. In [BF08] Bell and Fujiwara used the notion of tight geodesics of [MM99] and a finiteness theorem of Bowditch [Bow08] to prove that \( \mathcal{C}(\Sigma) \) has finite asymptotic dimension. This fact was then used in [BBFT14] to show that mapping class groups have finite asymptotic dimension.

Recall that a metric space \( X \) has asymptotic dimension \( \leq n \) provided for every \( R > 0 \) there exists a cover of \( X \) by uniformly bounded sets so that every metric \( R \)-ball in \( X \) intersects at most \( n + 1 \) elements of the cover.

Bowditch’s finiteness theorem was nonconstructive and as a result Bell and Fujiwara were not able write any explicit upper bounds on the asymptotic dimension of \( \mathcal{C}(\Sigma) \). More recently, Richard Webb [Web] gave a constructive proof of Bowditch’s theorem and gave an explicit upper bound, exponential in the complexity of the surface, to the asymptotic dimension of \( \mathcal{C}(\Sigma) \).

Asymptotic dimension of any visual \( \delta \)-hyperbolic space \( X \) is closely related to the topology of its Gromov boundary \( \partial X \). Buyalo [Buy05a] introduced the notion of the capacity dimension of a metric space and showed that \( \text{asdim} \ X \leq \text{capdim} \ \partial X + 1 \), where \( \partial X \) is equipped with a visual metric. (In the context of this paper, capacity dimension is the same as the Assouad-Nagata dimension).

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Subsequently, Buyalo-Lebedeva [BL07] showed that when $X$ is a hyperbolic group, then equality holds above, and moreover, $\text{capdim} \partial X = \dim \partial X$.

Klarreich [Kla99] identified the boundary of the curve complex with the space $\mathcal{EL}$ of ending laminations, which is a subquotient of the space $\mathcal{PML}$ of projective measured laminations.

In his work on the topology of the ending lamination space, Gabai [Gab] produced upper bounds to the covering dimension of $\mathcal{EL}$: $\dim \mathcal{EL} \leq 4g + p - 4$ if $\Sigma$ has genus $g$ and $p > 0$ punctures, and $\dim \mathcal{EL} \leq 4g - 5$ if $\Sigma$ is closed of genus $g$. We also note that the case of the 5 times punctured sphere was worked out earlier by Hensel and Przytycki [HP11].

**Main Theorem.** $\text{capdim} \mathcal{EL} \leq 4g + p - 4$ if $p > 0$ and $\text{capdim} \mathcal{EL} \leq 4g - 5$ if $p = 0$.

**Corollary 1.1.** $\text{asdim} \mathcal{C}(\Sigma) \leq 4g + p - 3$ if $p > 0$ and $\text{asdim} \mathcal{C}(\Sigma) \leq 4g - 4$ if $p = 0$.

We note that these numbers are very close to the virtual cohomological dimension $\text{vcd} \text{MCG}(\Sigma)$ of the mapping class group, established by Harer [Har86]: if $p = 0$ then $\text{vcd} = 4g - 5$, if $p > 0, g > 0$ then $\text{vcd} = 4g + p - 4$ and if $g = 0, p \geq 3$ then $\text{vcd} = p - 3$.

Our method is to directly construct required covers of $\mathcal{EL}$ via train track neighborhoods in $\mathcal{PML}$. Exactly such a strategy was employed by Gabai in proving his upper bounds on covering dimension but we will need to do extra work to gain more metric control of the covers. Roughly speaking, train tracks give a cell structure on $\mathcal{PML}$ and a cell structure has a natural dual “handle decomposition” which gives an open cover of the space of multiplicity bounded by the dimension of the cell structure. By making the cell structure finer and showing that the multiplicity of the the cover does not increase in $\mathcal{EL}$ Gabai obtains his upper bound. Note that cells of small dimension will not contain ending laminations which is why in both Gabai’s work and ours the dimension bound is smaller than the dimension of $\mathcal{PML}$.

To bound the capacity dimension one needs to find for any sufficiently small $\epsilon > 0$ covers that have bounded multiplicity and where all elements have diameter bounded above by $\epsilon$ while the Lebesgue number is bounded below by a fixed fraction of $\epsilon$. This last property will not be satisfied by family of covers constructed by Gabai.

For readers familiar with train tracks we give a brief sketch of the construction of the cover which will highlight the difficulties in our approach. The set of laminations carried by a train track $\sigma$ is naturally parameterized by a polyhedron $P(\sigma)$ in $\mathbb{R}^n$. (In what follows we will blur the distinction between a measured lamination and a projective measured lamination.) Note that $\sigma$ carries both ending laminations and simple closed curves. We denote the former as $P_\infty(\sigma)$ and the latter as $S(\sigma)$. A basepoint $*$ in $\mathcal{C}(\Sigma)$ determines visual metric $\rho$ on $\mathcal{EL}$. To estimate the visual diameter of $P_\infty(\sigma)$ we take the
curve $a \in S(\sigma)$ that is closest to $*$ in $C(\Sigma)$ and then the diameter of $P_\infty(\sigma)$ is coarsely $A^{-d(a,*)}$ for some fixed constant $A$.

To construct our cover we will repeatedly split train tracks along large branches. The process of splitting $\sigma$ gives two train tracks $\sigma_+$ and $\sigma_-$ such that $P(\sigma_+) \cup P(\sigma_-) = P(\sigma)$ and $P(\sigma_+) \cap P(\sigma_-) = P(\tau)$ where $\tau = \sigma_+ \cap \sigma_-$ is a train track with $P(\tau)$ a codimension one face of both $P(\sigma_+)$ and $P(\sigma_-)$. To start the construction we take a cell structure on $PML$ determined by a finite collection of train tracks. If the visual diameter of any of the top dimensional cells is larger than a fixed $\epsilon > 0$ then we split. We continue this process and stop splitting a top dimensional cell only when its diameter is $\leq \epsilon$.

At any finite stage of this construction we will obtain a cell structure on all of $PML$. In particular every simple closed curve will be carried some train track. For example one of the cells must contain the basepoint $*$ and therefore will have large visual diameter. It immediately follows that we will need to split infinitely many times to get a collection of cells that have small visual size.

At the end of the construction we will have a countable collection of train tracks $\sigma_1, \sigma_2, \ldots$ each determining a top dimensional cell. The collection of these cells is locally finite and covers all filling laminations. To complete the proof we will need to establish the following facts:

- **(Lemma 5.6)** All cells $P_\infty(\sigma_i)$ have visual diameter bounded above by $\epsilon$ and bounded below by a fixed fraction of $\epsilon$.

- **(Proposition 3.23)** The cells of dimension less than $\dim PML$ obtained by intersecting $P(\sigma_i)$ also have the form $P(\sigma)$ and if $P_\infty(\sigma)$ is nonempty its visual diameter is also bounded below by a fraction of $\epsilon$.

- **(Proposition 4.4)** If $a \in S(\sigma_i)$ and $b \in S(\sigma_j)$ are curves that are close in $C(\Sigma)$ then either
  
  (i) both $a$ and $b$ are close to a curve in $S(\tau) = S(\sigma_i) \cap S(\sigma_j)$ where $\tau$ is a subtrack of both $\sigma_i$ and $\sigma_j$ or
  (ii) both $a$ and $b$ are close to the basepoint $*$ (when compared to $\max\{d(*, S(\sigma_i)), d(*, S(\sigma_j))\}$).

The key to proving the first bullet is the work of Masur-Minsky on splitting sequences (see Theorem 3.12). The second bullet follows from an adaptation of the work of Hamenstädt [Ham09, Lemma 5.4] (see Propositions 3.21 and 3.22). The third bullet is the key technical advance of the paper and is proved using a version of Sela’s shortening argument. (See Lemma 3.25.)

**Conjecture.** $\text{asdim} C = \dim E\mathcal{L} + 1$ and $\text{asdim} MCG(\Sigma) = \text{vcd} MCG(\Sigma)$.

Except in genus 0, $\dim E\mathcal{L}$ is not known. The known upper and lower bounds differ by the genus [Gab].

**Plan of the paper.** In Section 2 we consider subdivision process on polyhedral cell structures in abstract. In Section 3 we review train track theory, and prove our main technical result, Lemma 3.25. In Section 4 we
apply this analysis and show that the visual size of the cover of \(\mathcal{FPML}\) we produce is controlled. In Section 5 we finish the argument by producing the required “handle decomposition” from our cover and checking that it satisfies the definition of capacity dimension. Finally, in the appendix we prove a technical result (Corollary A.6) about train tracks that is presumably known to the experts. It was a surprise to us that there are nonorientable train tracks that carry only orientable laminations, and large birecurrent train tracks that do not carry filling laminations. These phenomena are discussed in the appendix.

2 Good cell structures

In this section we consider abstract cell structures obtained by successively subdividing cells in an initial cell structure.

2.1 Polytopes

A polytope in a finite dimensional vector space \(V \cong \mathbb{R}^n\) is a finite intersection of closed half-spaces. The dimension of a polytope \(U \subset V\) is the dimension of its affine span. A face of \(U\) is the intersection \(U \cap H\) for a hyperplane \(H \subset V\) such that \(U\) is contained in one of the two closed half-spaces of \(H\). Faces of a polytope are also polytopes, a polytope has finitely many faces, and a face of a face is a face. The union of proper faces of a polytope is its boundary, and the complement of the boundary is the (relative) interior. See [Gru03] or [Zie95]. Our main example of a polytope is the set (a cone) \(V(\sigma)\) of measured laminations carried by a train track \(\sigma\) on a surface \(\Sigma\).

2.2 Cell structures

Definition 2.1. Let \(U \subset V\) be a polytope. A finite collection \(C\) of subsets of \(U\) which are also polytopes of various dimensions, called cells, is a cell structure on \(U\) if:

(C1) \(\bigcup_{C \in C} C = U\),
(C2) when two cells intersect, their intersection is a union of cells,
(C3) distinct cells have disjoint relative interiors,
(C4) every face of every cell in \(C\) is a union of cells.

Remark 2.2. We are really thinking about the filtration (into skeleta) \(U^0 \subset U^1 \subset \cdots \subset U^n = U\) so that the components of \(U^i - U^{i-1}\) are open \(i\)-dimensional convex polytopes whose faces are subcomplexes.

We need the following fact.

1Some authors require polytopes to be compact. Our polytopes will be cones on compact spaces.
Lemma 2.3. Let $U$ be an $n$-polytope and $\mathcal{C}$ a cell structure on $U$. If $E \in \mathcal{C}$ is a cell of dimension $i < n$ then there is a cell $F \in \mathcal{C}$ of dimension $i + 1$ such that $E$ is contained in a proper face of $F$.

Proof. By considering a plane complementary to (the affine span of) $E$ at an interior point, the statement is reduced to the case $i = 0$. There must be 1-cells attached to $E$ as long as $U$ is not a point. \qed

Definition 2.4. A cell structure $\mathcal{C}$ on an $n$-dimensional polytope $U$ is good if (C5) for every $i < n$, every $i$-dimensional cell $C \in \mathcal{C}$ is the intersection of $i$-dimensional faces of $> i$-dimensional cells in $\mathcal{C}$ that contain $C$.

For example, a convex polygon with subdivided edges is not a good cell structure since (C5) fails. However, starting with a convex polygon and subdividing by line segments results in a good cell structure. See Figure 2.1.

Figure 2.1: A hexagon subdivided 3 times results in a good cell structure with 12 0-cells, 15 1-cells and 4 2-cells.

2.3 Subdivision

Let $\mathcal{C}$ be a good cell structure on a polytope $U$ of dimension $n$ and let $W$ be the intersection of a codimension 0 cell $\Omega \in \mathcal{C}$ with a hyperplane (thus we are assuming $\dim \Omega = \dim U = n$). We will assume that the hyperplane intersects the relative interior of $\Omega$. Construct a new collection $\mathcal{C}'$ by “cutting by $W$”. More precisely, replace each cell $E \in \mathcal{C}$ which is contained in $\Omega$ and with the property that $E - W$ is disconnected by the following three cells: $E \cap W$ and the closures $E_1, E_2$ of the two complementary components of $E - W$. Thus $W$ is a codimension 1 cell of $\mathcal{C}'$. The cells $E_i$ have the same dimension as $E$, while $\dim(E \cap W) = \dim E - 1$. Figure 2.1 represents 3 consecutive subdivisions of a good cell structure consisting of a hexagon and its faces.

Lemma 2.5. The collection $\mathcal{C}'$ obtained from a good cell structure $\mathcal{C}$ by subdividing is a good cell structure.

Proof. As in the notation of the definition of subdivision we subdivide a codimension 0 cell $\Omega \in \mathcal{C}$ by a codimension 1 cell $W$. We leave it as an exercise to
prove that $C'$ is a cell structure and argue only that it is good. We show that an $i$-cell $C'$ of $C'$ ($i < n$) is the intersection of $i$-faces of $> i$-cells containing $C'$. Let $D$ be this intersection. Note that $D \supset C'$ so we only need to show that $C'$ is not a proper subset of $D$.

Let $C \in C$ be the smallest cell containing $C'$. Note that either dim $C = \dim C'$ (and possibly $C = C'$) or $\dim C = \dim C' + 1$. Let $E \in C$ be a cell that has a face $F$ that contains $C$. Then there will be a cell $E' \subset E$ (possibly equal to $E$) in $C'$ with a face $F' \subset F$ and $F' \supset C'$. By letting $E$ vary over all cells that have faces containing $C$ we see that $D \subset C$. If $C = C'$ we are now done. If not then $C$ is disconnected by $W$ and in $C$ becomes the 3 cells $C_1, C_2$ and $C \cap W$ with $C'$ is one of these three cells. Similarly, after subdivision $\Omega$ becomes three cells $\Omega_1, \Omega_2$ and $\Omega \cap W = W$ with $C_1$ and $C_2$ contained in a faces of $\Omega_1$ and $\Omega_2$ respectively. In particular if $C' = C_1$ (or $C' = C_2$) then $C$ is contained in a face of $\Omega_1$ ($\Omega_2$) but that face doesn’t contain any points in $C \setminus C'$ so we must have that $C = D$. If $C' = C \cap W$ then $C'$ is a face of $W$ but since $W$ doesn’t contain any points in $C \setminus W$ we have that $C = D$ in this case also.

Remark 2.6. When $E$ has codimension 1 and $U$ is a manifold (e.g. when $U$ is a polytope), the intersection in $(C5)$ consists of (at most) two elements. But when codimension is $> 1$ the argument does not produce a uniform bound on the number of faces required.

**Corollary 2.7.** Suppose $C$ is a good cell structure. If a cell $E \in C$ of dimension $i < n$ has $m$ codimension 1 faces, then $E$ can be written as the intersection of $\leq m$ $i$-dimensional faces of cells in $C$ of dimension $> i$.

**Definition 2.8.** A (finite or infinite) sequence $C_0, C_1, \cdots$ of good cell structures on $U$ is excellent if:

- $(C6)$ $C_0$ consists of $U_p$’s and their faces,
- $(C7)$ for $i \geq 1$, $C_i$ is obtained from $C_{i-1}$ by the subdivision process along codimension 0 cells described above, or else $C_i = C_{i-1}$.

**Remark 2.9.** Easy examples in $\mathbb{R}^3$ show that it is not true in general that an $i$-cell is the intersection of $i$-faces of codimension 0 cells. E.g. consider the plane $x = 0$ and half-planes $z = 0, x \geq 0$ and $y = 0, x \leq 0$.

**Remark 2.10.** This lemma is where our cell structure differs from Gabai’s. For our cell structure we only subdivide cells of positive co-dimension if they are induced by subdivisions of top dimensional cells. The proposition insures that when doing this all cells are defined via train tracks (i.e. they are of the form $V(\theta)$ where $\theta$ is a train track, see Proposition 3.21). Gabai also needs this property but he achieves it by subdividing cells of positive codimension. We do not want to do this as the visual diameter of these cells may become arbitrarily small. See Figure 2.2.
3 Train tracks

3.1 Notation and background

We will assume the reader is familiar with the theory of train tracks. The standard reference is [PH92]. See also [MM99] and [Ham09] for introductions to the theory. A quick definition is that a train track in a surface $\Sigma$ is a smooth graph with a well-defined tangent line at every point, including at the vertices, so that no complementary component is a (smooth) disk, a monogon, a bigon, or a punctured disk, and so that every edge can be extended in both directions to a smoothly immersed path (these are called legal paths or train paths). All our train tracks will be generic (i.e. all vertices have valence 3), recurrent and transversely recurrent (birecurrent). A train track $\sigma \subset \Sigma$ is large if each complementary component is homeomorphic to a disk or a once punctured disk.

3.1.1 Transverse measures

The edges of the train track are branches and the vertices are switches. At each switch of a generic train track $\sigma \subset \Sigma$ there are three incident half-branches. Two of these are tangent (i.e. determine the same unit tangent vector) and are called small, while the third is a large half-branch. A branch whose both half-branches are large is called large. If both half-branches are small then the branch is small. Otherwise the branch is mixed.

A transverse measure on a (generic) train track is an assignment of non-negative weights to each branch that satisfy the switch equations. That is, at each switch the sum of the weights of the two small half branches should be equal to the weight of the large half branch. A transverse measure determines a unique measured lamination on $\Sigma$. These are the laminations carried by $\tau$.

The set of all measured laminations on $\Sigma$ is denoted $\mathcal{ML}$ and the set of measured laminations carried by $\sigma$ is denoted $V(\sigma)$. Thus $V(\sigma)$ is the closed positive cone in the vector space of real weights on the branches of $\sigma$ satisfying...
the (linear) switch equations; in particular, $V(\sigma)$ is a polytope. We denote by $\mathcal{PML}$ the projective space of measured laminations and for a train track $\sigma$ we let $P(\sigma) \subset \mathcal{PML}$ be the set of projective measured laminations carried by $\sigma$. Then $P(\sigma)$ can be identified with the projectivization of $V(\sigma) - \{0\}$. We will often blur the distinction between a measured lamination and its projective class.

We also denote by $\mathcal{FPML} \subset \mathcal{PML}$ the subset of those laminations that are filling, i.e. whose complementary components are disks or punctured disks. Given a measured lamination $\lambda \in \mathcal{ML}$ (or $\mathcal{PML}$) we let $[\lambda]$ be the underlying geodesic lamination.

We have a quotient map $\mathcal{FPML} \to \mathcal{EL}$ to the space of ending laminations defined by $\lambda \mapsto [\lambda]$. Recall that Klarreich [Kla99] showed that $\mathcal{EL}$ is the Gromov boundary of the curve complex $\mathcal{C}(\Sigma)$. Note that in general if $\lambda_i \in \mathcal{FPML}$ is a sequence with limit $\lambda$ then $[\lambda]$ may be a proper subset of the Hausdorff limit of $[\lambda_i]$.

For a train track $\sigma$ let $P_{\infty}(\sigma) = P(\sigma) \cap \mathcal{FPML}$.

At each switch the tangent direction gives a way to compare the orientation of each branch adjacent to the switch. A train track is orientable if each branch can be given an orientation that is consistent at each switch.

When $\sigma$ is a generic birecurrent train track we have $|b|/3 = |v|/2 = -\chi(\sigma)$, where $|b|, |v|$ denote the numbers of branches and switches respectively.

**Lemma 3.1** ([PH92 Lemma 2.1.1]). Let $\sigma$ be a connected recurrent train track. Then the dimension of $V(\sigma)$ is $|b|/3$ if $\sigma$ is non-orientable and $|b|/3 + 1$ if $\sigma$ is orientable.

**Sketch of the proof.** Suppose first that $\sigma$ is nonorientable. Given a switch $v$, there is a train path that starts and ends at $v$, and the initial and terminal half branches are the two small half branches at $v$. This path assigns weights to the branches of $\sigma$ that satisfy all switch equations except at $v$. This shows that the switch equations are linearly independent, proving the assertion.

Now suppose $\sigma$ is orientable. Choose an orientation and write each switch equation as the sum of incoming branch(es) equals the sum of outgoing branch(es). Then summing all switch equations yields an identity, with each branch occurring once on both sides. Thus one switch equation is redundant, and we need to argue that the others are independent. Let $v, w$ be two distinct switches. Choose a train path that connects $v$ to $w$. This path assigns weights to all edges, and the switch equations are satisfied except at $v$ and $w$. This proves the claim.

### 3.1.2 Faces of $V(\sigma)$

There is a bijection between faces of $V(\sigma)$ and recurrent subtracks of $\sigma$. (Here we allow train tracks to be disconnected and to contain components that are simple closed curves.) A subtrack of $\sigma$ may not be recurrent but any track has a unique maximal recurrent subtrack.
3.1.3 Splitting

If $b$ is a large branch of $\sigma$, one can produce two new train tracks $\sigma_1, \sigma_2$ by splitting $b$. See Figure 3.1. We say that $\sigma_1$ is obtained by the left split and $\sigma_2$ by the right split.

![Figure 3.1: A large branch $e$ in the middle is split in two ways to give train tracks $\sigma_1$ and $\sigma_2$.](image)

Every lamination that is carried by $\sigma$ will be carried by either $\sigma_1$ or $\sigma_2$. If a lamination is carried by both $\sigma_1$ and $\sigma_2$ then it will be carried by the central split $\tau = \sigma_1 \cap \sigma_2$, obtained from either $\sigma_1$ or $\sigma_2$ by removing the diagonally drawn branch.

We have the following facts:

- ([PH92] Lemma 1.3.3(b)) If $\sigma$ is transversely recurrent, so are $\sigma_1, \sigma_2$ and $\tau$.
- ([PH92] Lemma 2.1.3) If $\sigma$ is recurrent, then either all three of $\sigma_1, \sigma_2, \tau$ are recurrent, or exactly one is recurrent.

It is also easy to see that $\sigma_1, \sigma_2$ are orientable if and only if $\sigma$ is.

3.2 Subdivision

Now suppose $\sigma$ is a birecurrent train track and $b$ a large branch of $\sigma$. We describe a process that subdivides $V(\sigma)$. There are several cases. Denote by $\sigma_1, \sigma_2, \tau$ the left, right, and central splits of $\sigma$ along $b$.

(S1) If all three of $\sigma_1, \sigma_2, \tau$ are recurrent, the cell $V(\tau)$ is a codimension 1 hyperplane in $V(\sigma)$ and cuts it into $V(\sigma_1)$ and $V(\sigma_2)$. Thus $\dim V(\sigma) = \dim V(\sigma_1) = \dim V(\tau) + 1$.

(S2) If $\sigma_1$ is recurrent but $\sigma_2$ and $\tau$ are not recurrent, we replace $\sigma$ by $\sigma_1$ and we have $V(\sigma) = V(\sigma_1)$. We proceed analogously if $\sigma_2$ is the only recurrent track.
Suppose $\tau$ is recurrent, but $\sigma_1, \sigma_2$ are not. Then $\tau$ is the maximal recurrent subtrack of both $\sigma_i$ and $V(\sigma) = V(\sigma_i) = V(\tau)$. Here we replace $\sigma$ by $\tau$. Since $\dim V(\sigma) = \dim V(\tau)$ Lemma 3.1 implies that $\sigma$ is nonorientable while $\tau$ is orientable. Note that if this case occurs every lamination carried by $\sigma$ is orientable. It may also happen that $\sigma$ is large while $\tau$ is not, so we have a situation that a large birecurrent train track does not carry any filling laminations.

There is also a converse to the last possibility.

**Lemma 3.2.** Suppose $\tau$ is obtained from $\sigma$ by a central split. If $\sigma$ is nonorientable and $\tau$ is orientable then $V(\sigma) = V(\tau)$.

**Proof.** By Lemma 3.1, we have $\dim V(\sigma) = \dim V(\tau)$. If $\lambda$ is carried in $\partial V(\sigma)$ then the realization of $\lambda$ in $\tau$ will have some branch of zero weight and hence the realization of $\lambda$ in $\sigma$ will also have a weight zero branch and $\lambda \in \partial V(\sigma)$. This implies that the inclusion of $V(\tau)$ in $V(\sigma)$ is a proper map and thus $V(\tau) = V(\sigma)$.

### 3.2.1 Carrying maps

A map $\sigma \to \tau$ between two train tracks is a carrying map if it is locally injective on each edge and takes legal train paths to legal train paths. We say that a carrying map $\sigma \to \tau$ is fully carrying if it is a homotopy equivalence, and we then write $\sigma \to \tau$. If $\lambda$ is a filling lamination carried by $\tau$, we write $\lambda \to \tau$ for the carrying map. If moreover this map induces a bijection between complementary components that preserves numbers of sides and punctures, we say that $\tau$ fully carries $\lambda$ and we write $\lambda \to \tau$. Thus in this case splitting $\tau$ according to $\lambda$ always produces train tracks that fully carry $\lambda$.

Our definition of a track fully carrying a lamination is stronger than what is used in [Gab] where it is only assumed that any realization of $\lambda$ as a measured lamination will be in the relative interior of $V(\tau)$. If $\lambda \to \tau$ in our sense and $\sigma$ is another track with $\lambda \to \sigma \to \tau$ than $\lambda \to \sigma$. This is not always true for the definition of fully carried in [Gab].

If $\sigma_1$ is a splitting of $\sigma$ there is a unique (up to isotopy of $\sigma_1$) full carrying map $\sigma_1 \to \sigma$ that is a bijection on vertices. If $\sigma_1$ is obtained from a finite sequence of splittings of $\sigma$ we will always assume that the carrying map $\sigma_1 \to \sigma$ is a composition of such maps.

If $\tau$ is obtained from $\sigma$ by some finite combination of splits and central splits we write $\tau \to^s \sigma$. If $\tau$ is obtained by a finite sequences of splits only then $\tau$ is fully carried by $\sigma$ and we write $\tau \to^s \sigma$.

We also use the notation $\sigma \to^s \tau$ to mean that $\sigma$ is obtained from $\tau$ by a sequence of splits, central splits, and passing to birecurrent subtracks whose corresponding cells have co-dimension one. The number of moves in $\sigma \to^s \tau$ is the number of splits, central splits and subtracks in the sequence. Note that if $\sigma$ is a subtrack of $\tau$ then the number of moves in $\sigma \to^s \tau$ is at least the
codimension of $V(\sigma)$ in $V(\tau)$. When we write $\sigma \xrightarrow{s^s} \tau$ we will be implicitly assuming that some sequence of moves has been chosen. However, the choice of a sequence is not unique and different choices of sequences may have a different number of moves.

Given a sequence of $\sigma \xrightarrow{s^s} \tau$ we now want to define the set of active and stationary branches. To do so we first make some general comments about sets of branches and half-branches and their complements. If $S$ is a collection of branches and half-branches of a train track $\tau$ then the complementary branch set $A$ contains a branch $b$ if neither $b$ nor any of its half-branches are in $S$ and contains a half-branch $h$ if neither $h$ nor the full branch containing $h$ is in $S$. We also emphasize that we distinguish between $S$ containing a branch $b$ and $S$ containing both half branches in $b$ but not $b$ itself. Let $|S|$ be the union of branches and half-branches in $S$. We think of half branches as really germs, so $|S| \cup |A|$ is not quite all of $\tau$ because each full branch $b$ that is not contained in either $S$ nor $A$ the union $|S| \cup |A|$ will be missing an interval in the interior of $b$.

If $\sigma \xrightarrow{s^s} \tau$ is a single move corresponding to a split or central split then the active branches are the large branch being split along with the adjacent half branches. If the move is passing to a subtrack then the active branches are the one or two small branches being removed along with all adjacent half branches. The branches and half branches that aren’t active are stationary.

**Lemma 3.3.** If $\sigma \xrightarrow{s^s} \tau$ is a single move than there is a carrying map such that the pre-image of each stationary (half-)branch is a (half-)branch.

Using the lemma we can then define a branch or half branch of $\sigma$ to be stationary if it is a pre-image of a stationary branch by the carrying map $\sigma \xrightarrow{s^s} \tau$. If $\sigma \xrightarrow{s^s} \tau$ is more than one move we can then inductively define a branch or half branch to be stationary if it is stationary at each move. The stationary branches are defined for each track in the sequence so if $\sigma'$ appears in $\sigma \xrightarrow{s^s} \tau$ we let $S(\sigma \xrightarrow{s^s} \tau; \sigma')$ be the stationary branches at $\sigma'$. The active branches at $\sigma'$ are denoted $A(\sigma \xrightarrow{s^s} \tau; \sigma')$ and are the complement branch set of $S(\sigma \xrightarrow{s^s} \tau; \sigma')$. Given a sequence $\sigma \xrightarrow{s^s} \tau$ there will be a carrying map that is a homeomorphism on the stationary branches and maps active branches to active branches. We record this as a lemma.

**Lemma 3.4.** If $\sigma \xrightarrow{s^s} \tau$ and $\sigma_1$ and $\sigma_2$ are train tracks in the sequence with $\sigma_1 \xrightarrow{s^s} \sigma_2$ then there is a carrying map that is a switch-preserving homeomorphism from $|S(\sigma \xrightarrow{s^s} \tau; \sigma_1)|$ to $|S(\sigma \xrightarrow{s^s} \tau; \sigma_2)|$ and maps $|A(\sigma \xrightarrow{s^s} \tau; \sigma_1)|$ to $|A(\sigma \xrightarrow{s^s} \tau; \sigma_2)|$.

The effect of a sequence of moves is completely determined by what happens on the active branches. The following lemma formalizes this.

**Lemma 3.5.** Let $\tau$, $\sigma$ and $\tau'$ be birecurrent train tracks with $\sigma \xrightarrow{s^s} \tau$. Let $B_\sigma \supset A(\sigma \xrightarrow{s^s} \tau; \sigma)$ be a collection of branches and half-branches of $\sigma$ and $B_\tau$
its image in $\tau$ under the carrying map $\sigma \xrightarrow{ss} \tau$. Assume that $\mathcal{B}_\sigma'$ is a collection of branches and half-branches of $\tau'$ such that there is a switch preserving homeomorphism from $|\mathcal{B}_\sigma| \to |\mathcal{B}_\sigma'|$. Then there exists a train track $\sigma'$ and a collection $\mathcal{B}_\sigma'$ of branches and half-branches in $\sigma'$ such that $\sigma'' \xrightarrow{ss} \tau'$ with the same number of moves as $\sigma \xrightarrow{ss} \tau$ and there is a switch preserving homeomorphism $|\mathcal{B}_\sigma| \to |\mathcal{B}_\sigma'|$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\sigma & \xrightarrow{ss} & \mathcal{B}_\sigma \\
\downarrow & & \downarrow \\
\tau & \xrightarrow{ss} & \mathcal{B}_\tau \\
\end{array}
\begin{array}{ccc}
|\mathcal{B}_\sigma| & \xrightarrow{ss} & |\mathcal{B}_\sigma'| \\
\downarrow & & \downarrow \\
|\mathcal{B}_\tau| & \xrightarrow{ss} & |\mathcal{B}_\tau'| \\
\end{array}
\begin{array}{ccc}
\sigma' & \xrightarrow{ss} & \tau' \\
\end{array}
\]

This lemma has several important applications which we record without proof. Formal proofs are straightforward inductions on the number of moves.

First we observe that if we have two sequences $\sigma_1 \xrightarrow{ss} \tau$ and $\sigma_2 \xrightarrow{ss} \tau$ whose active branches are disjoint then there is a minimal sequence $\tilde{\sigma} \xrightarrow{ss} \tau$ that realizes both sequences.

**Lemma 3.6.** Let $\tau$, $\sigma_1$ and $\sigma_2$ be birecurrent train tracks such that $\sigma_i \xrightarrow{ss} \tau$ with $m_i$ moves. Assume that $A(\sigma_1 \xrightarrow{ss} \tau; \tau) \cap A(\sigma_2 \xrightarrow{ss} \tau; \tau) = \emptyset$. Then there exists a birecurrent train track $\sigma$ with $\sigma \xrightarrow{ss} \sigma_i$ for $i = 1, 2$ with $m_{3-i}$ moves and $V(\sigma) = V(\sigma_1) \cap V(\sigma_2)$. Furthermore the carrying map $\sigma_1 \xrightarrow{ss} \tau$ restricted to $A(\sigma_1 \xrightarrow{ss} \sigma_i; \sigma_i)$ is a switch preserving homeomorphism to $A(\sigma_1 \xrightarrow{ss} \tau; \tau)$.

Our next observation is that if we have sequences $\sigma \xrightarrow{ss} \sigma'$ and $\sigma' \xrightarrow{ss} \tau$ where the active moves of the first sequence are contained in the stationary moves of the second sequence then we can switch the order of the moves. Here is the formal statement.

**Corollary 3.7.** Let $\sigma$, $\sigma'$ and $\tau$ be birecurrent train tracks with $\sigma \xrightarrow{ss} \sigma'$ and $\sigma' \xrightarrow{ss} \tau$. If $A(\sigma \xrightarrow{ss} \sigma'; \sigma') \subset S(\sigma' \xrightarrow{ss} \tau; \sigma')$ then there exists a birecurrent train track $\sigma''$ with $\sigma \xrightarrow{ss} \sigma''$ and $\sigma'' \xrightarrow{ss} \tau$ and $A(\sigma \xrightarrow{ss} \sigma'; \sigma') = A(\sigma'' \xrightarrow{ss} \tau; \tau)$. Furthermore $\sigma \xrightarrow{ss} \sigma'$ and $\sigma'' \xrightarrow{ss} \tau$ have the same number of moves as do $\sigma \xrightarrow{ss} \sigma''$ and $\sigma' \xrightarrow{ss} \tau$.

The main application of Corollary 3.7 is to show that the if a large branch is active then the first move of the sequence can be chosen to be a split or central split on that branch. As there is a small subtlety in proving this we provide a detailed proof.

**Corollary 3.8.** Let $b \in A(\sigma \xrightarrow{ss} \tau; \tau)$ be a large branch in $\tau$. Then there exists a birecurrent train track $\sigma'$ with $\sigma' \xrightarrow{ss} \tau$ a single move on $b$ and $\sigma \xrightarrow{ss} \sigma'$ with the sequence having at most the same number of moves as $\sigma \xrightarrow{ss} \tau$.
Figure 3.2: The subtrack obtained by removing a is isotopic to a subtrack of the track obtained by splitting.

Proof. Let \( \tau_1 \) and \( \tau_2 \) be birecurrent train tracks with \( \sigma \xrightarrow{s} \tau_2 \xrightarrow{s} \tau_1 \xrightarrow{s} \tau \), \( \tau_2 \xrightarrow{s} \tau_1 \) a single move, \( b \in S(\tau_1 \xrightarrow{s} \tau; \tau) \) but \( b \notin S(\tau_2 \xrightarrow{s} \tau; \tau) \). If \( \tau_2 \xrightarrow{s} \tau_1 \) is a split or central split on \( b \) then we apply Corollary 3.7 to \( \tau_2, \tau_1 \) and \( \tau \) to make the split on \( b \) the first move in \( \tau_2 \xrightarrow{s} \tau \).

If \( \tau_2 \xrightarrow{s} \tau_1 \) is not a split or central split on \( b \) then it must be a subtrack move on a branch adjacent to \( b \). But a subtrack move on a small branch adjacent to a large branch is equivalent to first splitting the large branch and then performing the subtrack move on the small branch. See Figure 3.2. Therefore we can assume that \( \tau_2 \xrightarrow{s} \tau_1 \) is a splitting on \( b \).

Lemma 3.9. If \( \sigma \xrightarrow{s} \tau \) and \( V(\sigma) \) intersects the relative interior of \( V(\tau) \) then \( \sigma \xrightarrow{s} \tau \).

Proof. If \( V(\sigma) \) intersects the relative interior of \( V(\tau) \) then the carrying map \( \sigma \xrightarrow{s} \tau \) must be surjective. Let \( \tau' \xrightarrow{s} \tau \) be the first move. This map must be surjective and for a single move this can only happen for a split or central split. If \( \sigma \xrightarrow{} \tau' \) is not surjective, then \( \tau' \xrightarrow{s} \tau \) is a split and the image of \( \sigma \) in \( \tau' \) includes all edges except the diagonal. Thus we can replace the first split with the central split and proceed by induction.

Lemma 3.10. Let \( \tau \) be a birecurrent train track and \( b \) a branch. Then there exists a collection of large branches \( B \) such that if \( \sigma \xrightarrow{s} \tau \) with \( b \in A(\sigma \xrightarrow{s} \tau; \tau) \) then every branch in \( B \) is in \( A(\sigma \xrightarrow{s} \tau; \tau) \).

Proof. If \( b \) is large then \( B = \{ b \} \). If not, any branch that contains a large half-branch adjacent to \( b \) must be active at some move before \( b \) can become active. For each large half-branch adjacent to \( b \) this will inductively will define a path from \( b \) to a large branch that must be active for \( b \) to be active.
3.3 Splitting sequences and excellent cell structures

We now fix a birecurrent train track $\sigma$. We will define a sequence of excellent cell structures $C_j, j = 0, 1, \cdots$ on the polytope $V(\sigma)$.

We start by defining $C_0$ to consist of $V(\sigma)$ and its faces. Inductively, each top dimensional cell $E$ of $C_j$ will correspond to a birecurrent track $\theta_E$ such that $E = V(\theta_E)$.

To define $C_{j+1}$, choose a top dimensional cell $E$ of $C_j$ and a large branch $b$ of $\theta_E$. Let $\theta_1, \theta_2, \tau$ be the left, right and central splits of $\theta_E$ along $b$. We now consider the three cases (S1)-(S3).

If all three $\theta_1, \theta_2, \tau$ are recurrent we split $E = V(\theta_E)$ along the hyperplane $V(\tau)$ yielding new top dimensional cells $V(\theta_1)$ and $V(\theta_2)$, and we subdivide all cells that are cut by this hyperplane as described in Section 2.3.

If $\theta_1$ is recurrent, but $\theta_2$ and $\tau$ are not, we define $C_{j+1} = C_j$ and $\theta_E = \theta_1$. We proceed similarly if $\theta_2$ is recurrent, but $\theta_1$ and $\tau$ are not.

If $\tau$ is recurrent, but $\theta_1, \theta_2$ are not, we set $C_{j+1} = C_j$ and $\theta_E = \tau$.

A sequence $C_j$ obtained in this way is said to be obtained by a splitting process from $\sigma$. Note that if $E = V(\theta_E)$ is a top dimensional cell in $C_i$ and if $E' = V(\theta_{E'})$ is a top dimensional cell in $C_j$ such that $j > i$ and $E \subset E'$, then $\theta_{E'} \to \theta_E$ and the sequence of splits and central splits contains at most one central split.

We will see later in Proposition 3.23 that every cell of $C_j$ has the form $V(\theta)$ for a suitable birecurrent train track $\theta$.

3.3.1 The curve graph and vertex cycles

By $C(\Sigma)$ we denote the curve graph of $\Sigma$. Its vertices are isotopy classes of essential simple closed curves on $\Sigma$, and two vertices are connected by an edge if the corresponding classes have disjoint representatives. When $\Sigma$ has low complexity $C(\Sigma)$ can be empty or discrete, and in the sequel we will always assume that $C(\Sigma)$ contains edges. In that case $C(\Sigma)$ is connected and the edge-path metric is $\delta$-hyperbolic [MM99].

The train track $\sigma$ carries a curve that crosses each branch at most twice, and if it crosses a branch twice it does so with opposite orientations. Such curves are the vertex cycles of $\sigma$. To a train track $\sigma \subset \Sigma$ we associate the sets $B(\sigma) \subset C(\Sigma)$ consisting of all vertex cycles for $\sigma$, and the set $S(\sigma) \subset C(\Sigma)$ of all curves carried by $\sigma$. We think of $B(\sigma) \subset S(\sigma)$ as a thick basepoint of $S(\sigma)$. It is a nonempty uniformly bounded subset of $S(\sigma)$.

3.3.2 Splitting sequences and the geometry of the curve graph

We begin with elementary lemma relating a single splitting to the geometry of the curve graph.

Lemma 3.11. Suppose $\sigma \leftrightarrow \tau$ is a single move. Then

$$d(B(\sigma), B(\tau))$$
is uniformly bounded.

Proof. Vertex cycles in subtracks are also vertex cycles in the track. In the case of splittings, the intersection number between a vertex cycle of $\sigma$ and a vertex cycle of $\sigma_i$ is uniformly bounded, and so is the distance in $C(\Sigma)$. 

Given a sequence $\sigma_s s \to \tau$, the previous lemma implies that the corresponding sequence of vertex cycles is a coarse path in $C(\Sigma)$. The following theorem is central to what follows.

**Theorem 3.12** ([MM04 Theorem 1.3],[Aou14 Theorem 1.1]). Let $\sigma_i$ be a sequence of birecurrent train tracks such that $\sigma_i+1 s s \to \sigma_i$ is a single move. Then the sequence $B(\sigma_i)$ is a reparametrized quasi-geodesic in $C(\Sigma)$ with uniform constants.

**Lemma 3.13.** Let $\tau$ be a train track. Then $S(\tau)$ is quasi-convex, with uniform constants.

**Proof.** Let $a \in S(\tau)$. Split $\tau$ towards $a$. This gives a nested sequence of tracks and thus a quasi-geodesic $g_a$ from $B(\tau)$ to $a$ that remains in $S(\tau)$.

If $a, b \in S(\tau)$ then by hyperbolicity $[a, b]$ is coarsely contained in $g_a \cup g_b \subset S(\tau)$.

The proof of the following lemma uses a technical result (Corollary [A.6]) whose proof is deferred to the appendix.

**Lemma 3.14.** Assume that $P_\infty(\tau) \neq \emptyset$. Then $S(\tau)$ is the coarse convex hull of the set of ending laminations carried by $\tau$.

**Proof.** As $C(\Sigma)$ is hyperbolic any quasi-convex subset contains the coarse convex hull of its Gromov boundary. By Klarreich’s Theorem the Gromov boundary of $C(\Sigma)$ is the space of ending laminations. If $\gamma_i \in S(\tau)$ converge to the boundary then there exists $\lambda_i \in P(\tau)$ with $\lambda_i \to \lambda \in P(\tau)$ such that $\gamma_i$ is a component of $[\lambda_i]$ and the Hausdorff limit of $\gamma_i$ contains the ending lamination $[\lambda]$. In particular the Gromov boundary of $S(\tau)$ is exactly the ending laminations in $P_\infty(\tau)$ so $S(\tau)$ coarsely contains its convex hull.

By Corollary [A.6] for any $a \in S(\tau)$ either $a$ is uniformly close to $B(\tau)$ or there exists a sequence of ending laminations $\lambda_i \in P_\infty(\tau)$ such that the Hausdorff limit of $[\lambda_i]$ contains $a$. Then the projections of $\lambda_i$ to the curve complex of the annulus around $a$ go to infinity and so, by the Bounded Geodesic Image Theorem ([MM00]), when $j >> i$ the geodesic between $[\lambda_i]$ and $[\lambda_j]$ passes within distance one of $a$. Therefore either $a$ is distance at most one from the convex hull of $S(\tau)$ or it is a bounded distance from $B(\tau)$. However, as $S(\tau)$ is quasi-convex, it is coarsely connected. Therefore $S(\tau)$ is the coarse convex hull of the ending laminations carried by $\tau$.

**Lemma 3.15.** Let $\tau$ and $\sigma$ be birecurrent train tracks with $\sigma s s \to \tau$. Then $B(\sigma)$ is coarsely the closest point within $S(\sigma)$ to $B(\tau)$.
Proof. Consider a splitting sequence from \( \tau \) to \( \sigma \). It determines a quasi-geodesic from \( B(\tau) \) to \( B(\sigma) \). Now if \( a \in S(\sigma) \) is any curve, the splitting sequence and the quasi-geodesic can be continued until \( a \) crosses every branch at most once. This extended quasi-geodesic ends at \( a \) and this proves the claim. \( \square \)

Lemma 3.16. Let \( \tau \) and \( \sigma \) be birecurrent train tracks with \( \sigma \xrightarrow{s} \tau \). There exists a constant \( C = C(\Sigma) \) such that if \( \sigma \xrightarrow{s} \tau \) has \( C \) or more moves then \( A(\sigma \xrightarrow{s} \tau; \tau) \) contains a vertex cycle.

Proof. There is a bound, depending only on \( \Sigma \), on the number of moves that are central splits and passing to subtracks. For each regular split there will be two half branches that are mapped to a one and a half branches so by increasing the number of moves we can guarantee that there is a branch that is mapped to a legal path of arbitrary length. Any legal path that is sufficiently long will contain a vertex cycle as it must cross some branch twice in the same direction. As all constants will only depend on \( \Sigma \) this implies the lemma. \( \square \)

Lemma 3.17. Let \( \tau \), \( \sigma_1 \) and \( \sigma_2 \) be birecurrent train tracks with \( A(\sigma_1 \xrightarrow{s} \tau; \tau) \cap A(\sigma_2 \xrightarrow{s} \tau; \tau) = \emptyset \). Then there exists a birecurrent train track \( \sigma \xrightarrow{s} \tau \) such that if \( \sigma \xrightarrow{s} \tau \) has \( i \), \( i=1,2 \) with \( V(\sigma) = V(\sigma_1) \cap V(\sigma_2) \) and \( \min\{d(B(\sigma), B(\sigma_1)), d(B(\sigma), B(\sigma_2))\} \) uniformly bounded.

Proof. Let \( C \) be the constant from Lemma 3.16. If either \( \sigma \xrightarrow{s} \sigma_1 \) or \( \sigma \xrightarrow{s} \sigma_2 \) has less than \( C \) moves then let \( \sigma \) be the track supplied by Lemma 3.6 and the distance bound follows from Lemma 3.11. Now we assume that \( \sigma \xrightarrow{s} \sigma_1 \) has at least \( C \) moves and let \( \tau_1 \) be the track in the sequence such that \( \sigma \xrightarrow{s} \tau_1 \) has exactly \( C \) moves. Since \( A(\tau_1 \xrightarrow{s} \tau; \tau) \subset A(\sigma_1 \xrightarrow{s} \tau; \tau) \) we still have \( A(\tau_1 \xrightarrow{s} \tau; \tau) \cap A(\sigma_2 \xrightarrow{s} \tau; \tau) = \emptyset \) and we can apply Lemma 3.6 to get a track \( \tau_2 \) with \( \tau_2 \xrightarrow{s} \tau_1 \), \( \tau_2 \xrightarrow{s} \tau \) and \( V(\tau_2) = V(\tau_1) \cap V(\tau_2) \).

We also observe that since \( A(\tau_2 \xrightarrow{s} \tau_1; \tau_1) \subset S(\sigma_1 \xrightarrow{s} \tau; \tau_1) \) we have \( A(\sigma_1 \xrightarrow{s} \tau_1; \tau_1) \cap A(\tau_2 \xrightarrow{s} \tau_1; \tau_1) = \emptyset \) and we can again apply Lemma 3.6 to obtain a track \( \sigma \) with \( \sigma \xrightarrow{s} \sigma_1 \), \( \sigma \xrightarrow{s} \tau_2 \), \( V(\sigma) = V(\sigma_1) \cap V(\tau_2) \) and both \( \sigma \xrightarrow{s} \tau_1 \) and \( \sigma \xrightarrow{s} \tau_2 \) having \( C \) moves. In particular both \( d(B(\sigma_1), B(\tau_1)) \) and \( d(B(\sigma), B(\tau_2)) \) are uniformly bounded. By Lemma 3.16 there is a vertex cycle \( c \) in \( A(\sigma_1 \xrightarrow{s} \tau_1; \tau_1) \subset S(\tau_2 \xrightarrow{s} \tau_1; \tau_1) \). Since the carrying map \( \tau_2 \xrightarrow{s} \tau_1 \) is a homeomorphism on the stationary branches, \( c \) is a vertex cycle in \( \tau_2 \) also and therefore \( d(B(\tau_1), B(\tau_2)) \) and, by the triangle inequality, \( d(B(\sigma), B(\tau_1)) \) are also uniformly bounded. Finally we note that \( V(\sigma) = V(\sigma_1) \cap V(\tau_2) = V(\sigma_1) \cap V(\tau_2) \) since \( V(\tau_2) = V(\tau_1) \cap V(\tau_2) \) and \( V(\sigma_1) \subset V(\tau_1) \). \( \square \)

When \( A \) is a geodesic lamination on \( \Sigma \), we denote by \( M(A) \) the lamination obtained from \( A \) by removing all isolated non-closed leaves. Thus \( M(A) \) consists of closed leaves and of minimal components and it is the maximal sublamination of \( A \) that supports a transverse measure. We call \( M(A) \) the measurable part of \( A \).
Lemma 3.18. Suppose sequences $a_i, b_i$ of closed geodesics converge to geodesic laminations $A, B$ respectively in Hausdorff topology. Assume

(i) both sequences go to infinity in the curve complex $C(\Sigma)$, and
(ii) $d(a_i, b_i)$ is uniformly bounded.

Then $A$ and $B$ have equal measurable parts, i.e. $M(A) = M(B)$.

Proof. It suffices to prove the claim when $d(a_i, b_i) \leq 1$ for all $i$. Then $A$ and $B$ have no transverse intersections. If $C$ is a minimal component of $M(A)$ that does not belong to $M(B)$, then it does not belong to $B$ either, and so for large $i$ the curve $b_i$ is disjoint from the subsurface supporting $C$ (which may be an annulus), contradicting (i). 

3.4 Train tracks for cells

Given birecurrent train tracks $\tau, \sigma_1$ and $\sigma_2$ with $\sigma_i \xrightarrow{ss} \tau$ we would like to find a fourth track $\sigma$ with $V(\sigma) = V(\sigma_1) \cap V(\sigma_2)$. If all three tracks are maximal and the relative interior of $V(\sigma_1) \cap V(\sigma_2)$ is open in $V(\tau)$ then this is due to Hamenstädt [Ham09]. We begin with two preliminary results.

Lemma 3.19. Suppose $\sigma \xrightarrow{ss} \tau$ and $\tau' \subset \tau$ is a birecurrent subtrack. Then there exists a birecurrent subtrack $\sigma'$ of $\sigma$ with $V(\sigma') = V(\sigma) \cap V(\tau')$, $\sigma' \xrightarrow{ss} \tau'$ and the number of moves not exceeding the sum of the number of moves in $\sigma \xrightarrow{ss} \tau$ and $\dim V(\tau') - \dim V(\sigma')$.

Proof. We first assume that $\sigma \xrightarrow{ss} \tau$ is a single move. The general case will follow by induction.

The intersection $V(\sigma) \cap V(\tau')$ will be a face of $V(\tau)$ and hence there will be a subtrack $\sigma' \subset \sigma$ with $V(\sigma') = V(\sigma) \cap V(\tau')$. To show that $\sigma' \xrightarrow{ss} \tau'$ there are several cases for each of the type of move in $\sigma \xrightarrow{ss} \tau$.

(1) $\sigma \xrightarrow{ss} \tau$ is a subtrack move. Then $V(\sigma)$ is a face of $V(\tau)$ and it follows that $V(\sigma')$ is a face of $V(\tau')$ so $\sigma' \xrightarrow{ss} \tau'$ with the number of moves $\dim V(\tau') - \dim V(\sigma')$. Here we are using Lemma 2.3 and the fact that faces of $V(\sigma)$ correspond to subtracks of $\sigma$.

(2) $\sigma \xrightarrow{s} \tau$ is a split or central split along a large branch $b$ and $\tau'$ contains $b$ and all its adjacent branches. Then $\sigma' \xrightarrow{s} \tau'$ is a single move on the same branch $b$.

(3) $\sigma \xrightarrow{c} \tau$ is a split along $b$ and one or more of the two large half-branches adjacent to $b$ in $\sigma$ is not in $\tau'$. Then the restriction of the carrying map $\sigma \xrightarrow{s} \tau$ to $\sigma'$ will be a switch preserving homeomorphism so $\sigma'$ is a subtrack of $\tau$ and we proceed as in (1). See Figure 3.3.

(4) $\sigma \xrightarrow{s} \tau$ is a central split and one or more of the half-branches adjacent to $b$ in $\tau$ is not in $\tau'$. Then as in (3) $\sigma'$ is a subtrack of $\tau$. 

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Figure 3.3: If the branch $a$ is removed in $\tau'$ then it also must be removed in $\sigma'$. However, then both small branches adjacent to $a$ must be removed and $\sigma'$ will be a subtrack of $\tau$.

(5) $\sigma \xrightarrow{\Delta} \tau$ is a split and $\tau'$ contains both of the large half-branches adjacent to $b$ and does not contain one or more of the two adjacent small half branches. In this case $\tau'$ is isotopic to a subtrack of $\sigma$ and $\sigma' = \tau'$.

Lemma 3.20. Let $\tau$, $\sigma_1$ and $\sigma_2$ be birecurrent train tracks such that $\sigma_i \xrightarrow{\Delta} \tau$ for $i = 1, 2$ and $A(\sigma_1 \xrightarrow{\Delta} \tau ; \tau) \cap A(\sigma_2 \xrightarrow{\Delta} \tau ; \tau) \neq \emptyset$. Then there exist birecurrent train tracks $\tau'$, $\sigma'_1 \subset \sigma_1$ and $\sigma'_2 \subset \sigma_2$ such that $\tau' \xrightarrow{\Delta} \tau$ and $\sigma'_i \xrightarrow{\Delta} \tau'$ with each sequence $\sigma'_i \xrightarrow{\Delta} \tau'$ having less moves than $\sigma_i \xrightarrow{\Delta} \tau$. Furthermore $V(\sigma'_1) \cap V(\sigma'_2) = V(\sigma_1) \cap V(\sigma_2)$.

Proof. If there is branch that is active in both sequences then by Lemma 3.10 there must be a large branch $b$ that is active in both sequences. By Lemma 3.8 we can assume that the first move in both sequences is along $b$. If it is the same move then $\tau'$ is the track obtained from this first move and $\sigma'_i = \sigma_i$. If not than we let $\tau'$ be the central split on $b$. In at least one of the sequences,
say $\sigma_1 \xrightarrow{\delta} \tau$, the first move $\tau_1 \xrightarrow{\delta} \tau$ will be a right or left split on $b$ and $\tau'$ will be obtained from $\tau_1$ by removing the diagonal. We then use Lemma 3.19 to find a subtrack $\sigma'_1 \subset \sigma_1$ with $\sigma'_1 \xrightarrow{\delta} \tau'$ having at most the same number of moves as $\sigma \xrightarrow{\delta} \tau_1$ which is one less move than $\sigma_1 \xrightarrow{\delta} \tau$.

If the first move of $\sigma_2 \xrightarrow{\delta} \tau$ is a right of left split we similarly replace $\sigma_2$ with a subtrack $\sigma'_2$. If the first move is a central split then $\sigma'_2 = \sigma_2$.

Note that in all cases $V(\tau') \supset V(\sigma_1) \cap V(\sigma_2)$ and $V(\sigma'_2) = V(\sigma_1) \cap V(\sigma_2)$. It follows that $V(\sigma'_1) \cap V(\sigma'_2) = V(\sigma_1) \cap V(\sigma_2)$.

**Proposition 3.21.** Let $\tau$, $\sigma_1$ and $\sigma_2$ be birecurrent train tracks such that $\sigma_i \xrightarrow{ss} \tau$ for $i = 1, 2$. Assume that $V(\sigma_1) \cap V(\sigma_2) \neq \emptyset$. Then there exist birecurrent train tracks $\sigma^\pm$ and subtracks $\sigma'_i \subset \sigma_i$ such that

1. $\sigma^- \xrightarrow{ss} \sigma'_1 \xrightarrow{ss} \sigma^+ \xrightarrow{ss} \tau$ for $i = 1, 2$;
2. $V(\sigma^-) = V(\sigma_1) \cap V(\sigma_2)$;
3. $\min\{d(B(\sigma_1), B(\sigma^-)), d(B(\sigma_2), B(\sigma^-))\}$ is uniformly bounded.

**Proof.** If $A(\sigma_1 \xrightarrow{ss} \tau; \tau) \cap A(\sigma_2 \xrightarrow{ss} \tau; \tau) = \emptyset$ then the proposition follows from Lemma 3.17 with $\sigma^+ = \tau$ and $\sigma^-$ the track given by Lemma 3.17. If not we repeatedly apply the following two steps:

1. Let $\tau' \subset \tau$ be the smallest birecurrent subtrack such that $V(\sigma_1) \cap V(\sigma_2) \subset V(\tau')$ and let $\tau_1 \subset \tau_i$ be the subtracks given by Lemma 3.19. Then $V(\tau_1)$ will intersect the relative interior of $V(\tau')$ so $\tau_1 \xrightarrow{\delta} \tau'$. If $A(\tau_1 \xrightarrow{ss} \tau'; \tau') \cap A(\tau_2 \xrightarrow{ss} \tau'; \tau') = \emptyset$ then the proposition follows from Lemma 3.17 with $\sigma^+ = \tau'$ and $\sigma^-$ the track given by Lemma 3.17. If not we go to step (2):
2. Apply Lemma 3.20 to $\tau_1 \xrightarrow{ss} \tau'$ reduce the number of moves in $\sigma'_i \xrightarrow{ss} \tau'$ and then check if we can apply Lemma 3.17 to the new sequences. If not we go back to step (1).

Each time will apply step (1) we reduce the dimension of the cell so there is a bound on the number of times it will apply with the bound only depending on the complexity of $\Sigma$. Step (2) will decrease the length of the sequences so
eventually \( \mathcal{A}(\tau_1 \overset{s}{\to} \tau'; \tau') \cap \mathcal{A}(\tau_2 \overset{s}{\to} \tau'; \tau') = \emptyset \) and the process will terminate.

\[
\square
\]

Proposition 3.22. Assume that \( \sigma_i \overset{s}{\to} \tau, i = 1, 2 \) with the \( V(\sigma_i) \) proper co-dimension zero cells in \( V(\tau) \) meeting along a common co-dimension 1 face and \( V(\tau) = V(\sigma_1) \cup V(\sigma_2) \). Then there exist \( \tau', \sigma'_i \) such that \( \tau' \overset{s}{\to} \tau \) with \( V(\tau') = V(\tau), V(\sigma'_i) = V(\sigma_i) \) and the \( \sigma_i \) are the right and left split of \( \tau' \) along a large branch.

Proof. Let \( \tau_i \) be the first track in the sequence \( \sigma_i \overset{s}{\to} \tau \) such that \( V(\tau_i) \) is a proper subset of \( V(\tau) \). If \( V(\tau_i) = V(\sigma_1) \) we then let \( \tau' \) be the track that appears in the sequence just before \( \tau_i \). Then \( V(\tau') = V(\tau) \) and \( \tau_i \overset{s}{\to} \tau' \) must be a single split along a large branch \( b \). We then let \( \sigma'_1 = \tau_1 \) and \( \sigma'_2 \) the other split along \( b \). We argue similarly if \( V(\tau_2) = V(\sigma_2) \).

If \( V(\sigma_i) \) is a proper subset of \( V(\tau_i) \) for both \( i = 1 \) and \( i = 2 \) then we apply Proposition 3.23 to \( \tau_i \). Since \( \dim V(\tau_i) = \dim V(\tau) \) we do not need to pass to subtracks and have a track \( \sigma^+ \overset{s}{\to} \tau \) with \( \tau_i \overset{s}{\to} \sigma^+ \) and \( \mathcal{A}(\tau_1 \overset{s}{\to} \sigma^+, \sigma^+) \cap \mathcal{A}(\tau_2 \overset{s}{\to} \sigma^+, \sigma^+) = \emptyset \). By Corollary 3.1 all the moves in \( \tau_1 \overset{s}{\to} \sigma^+ \) commute with the moves in \( \tau_2 \overset{s}{\to} \sigma^+ \). Let \( \tau' \overset{s}{\to} \sigma^+ \) be all the moves in \( \tau_1 \overset{s}{\to} \sigma^+ \) that don’t decrease the size of \( V(\sigma^+) \) followed by the same sequence for \( \tau_2 \overset{s}{\to} \sigma^+ \). In particular \( V(\tau') = V(\sigma^+) = V(\tau) \) and there are large branches \( b_1 \) and \( b_2 \) and single splits \( \sigma'_i \overset{s}{\to} \tau' \) on the \( b_i \) with \( V(\sigma'_i) = V(\tau_i) = V(\sigma_i) \). However, if \( b_1 \neq b_2 \) then \( V(\sigma'_1) \cup V(\sigma'_2) \) will be a proper subset of \( V(\tau') \), a contradiction. Therefore \( b_1 = b_2 \) and the \( \sigma'_i \overset{s}{\to} \tau' \) are the two distinct splits on a single large branch.

\[
\square
\]

Proposition 3.23. Let \( \mathcal{C}_j \) be an excellent sequence of cell structures obtained by splitting a train track \( \tau \). To every cell \( E \in \mathcal{C}_j \) one can assign a recurrent and transversely recurrent track \( \theta_E \) satisfying the following:

1. \( E = V(\theta_E) \).

2. If \( E \) is a top dimensional cell, then \( \theta_E \) is the track associated to \( E \) in the definition of the splitting sequence.

3. If \( F \subset E \) are cells then \( \theta_F \overset{s}{\to} \theta_E \).

4. There is a constant \( C = C(\Sigma) \) such that for each cell \( F \in \mathcal{C}_k \) there is a top dimensional cell \( E \in \mathcal{C}_k \) with \( F \subset E \) and \( d(B(\theta_E), B(\theta_F)) \leq C \).

In particular, if all top dimensional cells in \( \mathcal{C}_j \) have vertex cycles distance at most \( B \) from \( B(\tau) \) then \( d(B(\tau), B(\theta_E)) \leq B + C \) while if \( E \subset F \) for a cell \( F \in \mathcal{C}_j \) with \( d(B(\tau), B(\theta_F)) \geq A \) then \( d(B(\tau), B(\theta_F)) \geq A - C \).

Proof. We define \( \theta_E \) for \( E \in \mathcal{C}_j \) by induction on \( j \). When \( j = 0 \) each cell \( E \) is naturally associated to a subtrack of \( \tau \) and we define \( \theta_E \) to be this subtrack. Now suppose that \( \theta_E \) has been defined for all cells in \( \mathcal{C}_k \) of dimension \( > i \) for a certain \( i < n \). Let \( F \in \mathcal{C}_k \) with \( \dim F = i \). By property (C5) of an excellent sequence if \( E_1, \ldots, E_\ell \) are all \( i \)-dimensional cells in \( \mathcal{C}_k \) with \( F \subset E_i \)
then $F = \cap E_s$. Let $F_j = E_1 \cap \cdots \cap E_j$. Via induction we have tracks $\theta_{F_j}$ with $V(\theta_{F_j}) = F_j$ and if $E \in C_k$ with $E_s \subset E$ for some $s = 1, \ldots, j$ then $\theta_{F_j} \overset{\delta}{\rightarrow} \theta_E$. The track $\theta_{F_j}$ is defined by applying Proposition \ref{3.21} to $\theta_{F_{j-1}}$ and $\theta_{E_j}$. We then set $\theta_{F} = \theta_{F_j}$ and this track will satisfy properties (1)-(3).

To get the distance bound in (4) we observe that (c) of Proposition \ref{3.21} gives a linear bound to this distance in $\ell$. While we cannot control the size of $\ell$, once we know that $F = V(\theta_F)$ for the train track $\theta_F$ we observe that the number of co-dimension one faces of $F$ is bounded by the number of small branches of $\theta_F$ and hence a constant only depending on $\Sigma$. In particular there is a subcollection of the $E_1, \ldots, E_\ell$ of uniformly bounded size whose intersection gives $F$ by Corollary \ref{2.7}. Applying the argument of the previous paragraph to this subcollection we get a track $\theta'_{J'}$ with $V(\theta'_{J'}) = F$ and the distance bound in (4).

Finally we note that while $\theta_F$ and $\theta'_{J'}$ may not be the same track (and $\theta'_{J'}$ may not satisfy (3)) since $V(\theta_F) = V(\theta'_{J'})$ the two tracks have the same vertex cycles and therefore (4) holds for $\theta_F$ also.

Given a lamination $\lambda \in P_\infty(\tau)$ let $\tau_i$ be a sequence of tracks such that $\tau_0 = \tau$, $\tau_{i+1} \overset{ss}{\rightarrow} \tau_i$ is a single move and $\lambda \in P_\infty(\tau_i)$ for all $i$. We say that the sequence is a full splitting sequence if for every $i$ and every large branch $b$ in $\tau_i$ there exists an $i_n$ such that $b \in A(\tau_{i_n} \overset{ss}{\rightarrow} \tau_i; \tau_i)$.

**Proposition 3.24.** Assume that $\lambda$ is fully carried by $\tau$. Then there exists a full splitting sequence $\tau = \tau_1, \tau_2, \cdots$ such that $\lambda$ is fully carried by every $\tau_i$. Moreover, any infinite splitting sequence starting at $\tau$ and carrying $\lambda$ is a full splitting sequence. Furthermore if $\lambda'$ is carried by every $\tau_i$ then $[\lambda] = [\lambda']$.

**Proof.** The first statement follows from \cite{Ago11} Lemma 2.1. In fact, the proof of \cite{Ago11} Lemma 2.1 proves the stronger second statement. The third statement is probably well known but as we could not find a proof we provide one here. Assume that $\lambda'$ is carried by all $\tau_i$ but $[\lambda] \neq [\lambda']$. By \cite{PH92} Corollary 1.7.13 we can find a birecurrent train track $\tau'$ that carries $\lambda$, does not carry $\lambda'$ and is carried by $\tau$. Hence it will fully carry $\lambda$, but it may not come from a sequence of splits and central splits of $\tau$. Instead we use \cite{PH92} Theorem 2.3.1 to find a track $\sigma$ with $\sigma \overset{ss}{\rightarrow} \tau'$, $\sigma \overset{ss}{\rightarrow} \tau$ and $\lambda$ carried by $\sigma$. As all three tracks fully carry $\lambda$ we in fact have $\sigma \overset{ss}{\rightarrow} \tau'$ and $\sigma \overset{ss}{\rightarrow} \tau$.

We will show that for sufficiently large $i$ we have $\tau_i \rightarrow \sigma$. As $\tau_i$ carries $\lambda'$ but $\sigma$ does not this will be a contradiction. We repeatedly apply Proposition \ref{3.21} Let $\sigma^1 = \tau_i$ and assume that we have constructed tracks $\sigma^1, \ldots, \sigma^j$ with $\sigma^j \overset{ss}{\rightarrow} \sigma_{j-1}^+, \sigma^j \overset{ss}{\rightarrow} \sigma_{j-1}$ and $A(\sigma \overset{ss}{\rightarrow} \sigma^1; \sigma^j) \cap A(\sigma^1 \overset{ss}{\rightarrow} \sigma^j; \sigma^1) = \emptyset$. We then construct $\sigma^j$ by applying Proposition \ref{3.21} to $\sigma_{j-1}$, $\sigma$ and $\tau_i$. Note that since $\lambda$ is fully carried by all of the tracks all the carrying maps given by Proposition are fully carrying. This also implies that $\lambda$ is in the relative interior of the associated cells so we also never need to pass to subtracks.
Note that the sequence $\sigma_i^+; \sigma_i^3; \ldots$ must eventually be constant for if not the number of moves in $\sigma \xrightarrow{s} \sigma_i^+ \rightarrow \tau$ would go to infinity as $i \to \infty$ but the number of moves will be bounded by the number of moves in the original sequence $\sigma \xrightarrow{s} \tau$. Therefore there exists an $N > 0$ such that if $i > N$ then $\sigma_i^+ = \sigma_N^+$. The sequence is $\sigma_N^+, \tau_N, \tau_{N+1}, \ldots$ is a full splitting sequence so for $i$ sufficiently large $A(\tau_i \rightarrow \sigma_N; \sigma_N)$ is all of $\sigma_N$. The active branches for $\sigma \rightarrow \sigma_N$ must be disjoint from $A(\tau_i \rightarrow \sigma_N; \sigma_N)$ so we must have $\sigma = \sigma_N$ and $\tau_N \rightarrow \sigma$ as desired.

3.5 A shortening argument

In this section we assume that $\sigma, \tau, \rho$ are partial train tracks, i.e. each is a subgraph of a train track. We allow valence two vertices with the turn illegal, or even valence 1 vertices. Even though the main result is used only when $\tau, \rho$ are train tracks, the extra flexibility of passing to subgraphs will make the proof easier. More precisely, we assume:

- $\tau, \rho$ are two partial train tracks on $\Sigma$,
- $\sigma$ is the graph that consists of edges that $\tau$ and $\rho$ have in common,
- branches of $\tau - \sigma$ and $\rho - \sigma$ intersect transversally and any vertex in common to $\tau$ and $\rho$ is also a vertex of $\sigma$,
- any lamination carried by both $\tau, \rho$ is carried by $\sigma$, i.e.

$$V(\tau) \cap V(\rho) = V(\sigma) \quad (*)$$

Given a triple $\sigma = (\sigma; \tau, \rho)$ as above define the complexity $\chi(\sigma)$ to be the pair $(e(\sigma), I(\tau, \rho))$, ordered lexicographically, where $e(\sigma)$ is the number of edges of $\sigma$ and $I(\tau, \rho)$ is the number of transversal intersections between the branches of $\tau$ and $\rho$. Note that for a given complexity there are only finitely many $\sigma$ up to the action of the mapping class group.

The number of branches of $\sigma$ is uniformly bounded depending only on the surface $\Sigma$, so the bound on $\chi(\sigma)$ really only amounts to the bound on the intersection between the branches of $\tau$ and $\rho$.

As an example of the extra flexibility, note that if we remove an edge of $\sigma$ from all three graphs $\sigma, \tau, \rho$ the listed conditions continue to hold, but the new triple has smaller complexity.

Denote by $Supp(\sigma)$ the support of $\sigma$, i.e. the smallest subsurface that contains $\sigma$ (possibly $\emptyset$, or disconnected, or all of $\Sigma$). Thus $Supp(\sigma) = \emptyset$ if and only if $\sigma$ is contained in a disk.

**Lemma 3.25.** For every $C$ and every $\chi$ there is $C' = C'(\Sigma, C, \chi)$ such that if $\chi(\sigma) \leq \chi$, $a \in S(\tau)$, $b \in S(\rho)$, $d(a, b) \leq C$, then

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(i) If \( \text{Supp}(\sigma) = \emptyset \) then
\[
d(a, B(\tau)) \leq C', d(b, B(\rho)) \leq C'
\]

(ii) If \( \text{Supp}(\sigma) \neq \emptyset, \Sigma \) then
\[
d(a, \mathcal{C}(\text{Supp}(\sigma))) \leq C', d(b, \mathcal{C}(\text{Supp}(\sigma))) \leq C'
\]

(iii) If \( \text{Supp}(\sigma) = \Sigma \) then
\[
d(a, S(\sigma)) \leq C', d(b, S(\sigma)) \leq C'
\]

In (ii) by \( \mathcal{C}(\text{Supp}(\sigma)) \) we mean the set of curves carried by \( \text{Supp}(\sigma) \), even when \( \text{Supp}(\sigma) \) is disconnected.

Most of the time when we apply Lemma 3.25, we will have that \( \tau, \rho \) are subtracks of some large track \( \omega \) and \( \sigma = \tau \cap \rho \), and then the condition (*) is standard and quickly follows from the fact that legal paths in the universal cover are quasi-geodesics and that they are uniquely determined by their endpoints on the circle at infinity. The proof of Lemma 3.25 is by modifying the tracks and then \( \tau, \rho \) may develop intersecting branches.

If \( \sigma \) is a train track or a partial train track, and \( a \) is carried by \( \sigma \) then the combinatorial length \( \ell_{\tau}(a) \) is the sum of the weights of \( a \).

**Proof of Lemma 3.25.** We will suppose such \( C' \) does not exist and obtain a contradiction.

**Proof of (i).** If the lemma fails for a particular \( \sigma \), there are sequences of curves \( a_n \in S(\tau), b_n \in S(\rho) \) such that \( d(a_n, b_n) \leq C, d(a_n, B(\tau)) > n, d(b_n, B(\rho)) > n \). After passing to a subsequence \( n_j \), we may assume that \( a_n \rightarrow A, b_n \rightarrow B \) in Hausdorff topology, where \( A, B \) are geodesic laminations.

By Lemma 3.18 \( A \) and \( B \) have the same (nonempty) measurable part \( \Lambda \), which must be carried by \( \sigma \) by assumption (*). This contradicts the assumption that \( \sigma \) is contained in a disk.

**Proof of (ii).** We induct on the complexity.

For each \( \sigma \) with \( \chi(\sigma) \leq \chi \) where the lemma fails, for every \( n \) there are curves \( a_n^\sigma \in S(\tau), b_n^\sigma \in S(\rho) \) with \( d(a_n^\sigma, b_n^\sigma) \leq C, d(a_n^\sigma, \mathcal{C}(\text{Supp}(\sigma))) > n, d(b_n^\sigma, \mathcal{C}(\text{Supp}(\sigma))) > n \). We will assume that subject to these conditions
\[
\ell_{\tau}(a_n^\sigma) + \ell_{\rho}(b_n^\sigma)
\]
is minimal possible.

To obtain a contradiction we will find a sequence of triples \( \sigma_i = (\sigma_i; \tau_i, \rho_i) \) where the lemma fails with \( \sigma = \sigma_1 \) and for each \( \sigma_i \) an infinite sequence \( \{n_j^i\} \) such that

1. \( n_j^1 = j \) and \( \{n_j^i\} \) is a subsequence of \( \{n_j^{i-1}\} \) for \( i > 1 \);
2. \( a_{n_j^i}^\sigma \in S(\tau_i), b_{n_j^i}^\sigma \in S(\rho_i) \);

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(3) \( d(a_{n_j}^{\sigma_i}, b_{n_j}^{\sigma_i}) \leq C; \)

(4) \( d(a_{n_j}^{\sigma_i}, C(\text{Supp}(\sigma_i))) > n_j^i, \) \( d(b_{n_j}^{\sigma_i}, C(\text{Supp}(\sigma_i))) > n_j^i; \)

(5) \( \ell_{\tau_i}(a_{n_j}^{\sigma_i}) + \ell_{\rho_i}(b_{n_j}^{\sigma_i}) < \ell_{\tau_i}(a_{n_j}^{\sigma_i-1}) + \ell_{\rho_i}(b_{n_j}^{\sigma_i-1}); \)

(6) for every \( i, j, \) \( \ell_{\tau_i}(a_{n_j}^{\sigma_i}) + \ell_{\rho_i}(b_{n_j}^{\sigma_i}) \) is minimal possible subject to (2)-(4);

(7) \( \chi(\sigma_i) \leq \chi(\sigma_{i-1}); \)

(8) \( \sigma_i \) satisfies (*) i.e. \( V(\tau_i) \cap V(\rho_i) = V(\sigma_i). \)

By (7) our sequence \( \sigma_i \) will eventually repeat (up to \( \text{Mod}(\Sigma) \)) so there are \( k < l \) with \( \sigma_k = \phi(\sigma_l) \) for some mapping class \( \phi. \) By repeated applications of (5), we have

\[ \ell_{\tau_i}(a_{n_j}^{\sigma_i}) + \ell_{\rho_i}(b_{n_j}^{\sigma_i}) < \ell_{\tau_i}(a_{n_j}^{\sigma_k}) + \ell_{\rho_i}(b_{n_j}^{\sigma_k}) \]

obtaining our contradiction to (6), since for large \( j \) the curves \( \phi(a_{n_j}^{\sigma_i}), \phi(b_{n_j}^{\sigma_i}) \) satisfy (2)-(4) (for \( i = k \)) and have smaller total combinatorial length than \( a_{n_j}^{\sigma_k} \) and \( b_{n_j}^{\sigma_k}. \)

We will construct the sequence \( \sigma_i \) inductively. Assume that \( \sigma_i \) and the sequence \( \{n^i_j\} \) have been defined satisfying the above conditions. We then define a subsequence \( \{n_{j+1}^i\} \) of \( \{n_j^i\} \) and show that there exists a \( \sigma_{i+1} \) so that (1)-(8) hold with suitable choices of curves. We first choose the subsequence \( \{n_{j+1}^i\} \) such that \( a_{n_j}^{\sigma_{i+1}} \rightarrow A \) and \( b_{n_j}^{\sigma_{i+1}} \rightarrow B \) where \( A, B \) are two geodesic laminations and convergence is with respect to Hausdorff topology. The construction of \( \sigma_{i+1} \) is more involved.

Lemma 3.18 implies that \( A \) and \( B \) have the same measurable part \( \Lambda \) and differ only in isolated non-closed leaves. By assumption (8), \( \Lambda \) is carried by \( \sigma_i. \)

Let \( \bar{\sigma} \subset \sigma_i \) be the union of the branches crossed by \( \Lambda. \) Thus \( \bar{\sigma} \) is a train-track.

Case 1. \( \bar{\sigma} \) has at least one illegal turn. Note that \( \Lambda \) supports a transverse measure of full support and in particular \( \bar{\sigma} \) has a large branch (one with maximal transverse measure). Split along this branch so that \( \Lambda \) is still carried to obtain a new track \( \sigma_{i+1}. \)

Case 1a. The non-degenerate case that such a split is unique (i.e. the pairs of weights at the two ends are distinct) is pictured in Figure 3.4. The vertical segment represents a large branch of \( \bar{\sigma} \) and the two branches at the top and at the bottom are also in \( \bar{\sigma}. \) The branches pictured on the sides are branches of \( \sigma_i - \bar{\sigma}, \tau_i - \bar{\sigma}, \) or \( \rho_i - \sigma_i. \) The splitting operation consists of cutting along the large branch thus producing two vertical branches of the split \( \bar{\sigma}, \) adding the suitable diagonal branch so that \( \Lambda \) is carried, and attaching the side branches at exactly the same point, to either the left or the right vertical branch. We define \( \sigma_{i+1} \) to be the split version of \( \sigma_i. \) Thus \( \sigma_{i+1} \) includes the two vertical branches, the two branches at the top, the two branches at the bottom, the diagonal branch, and any side branches that came from \( \sigma_i - \bar{\sigma}. \) The track \( \tau_{i+1} \)
contains $\sigma_{i+1}$ and includes side branches that came from $\tau_i - \tau$, and similarly for $\rho_{i+1}$. Observe that $\chi(\sigma_{i+1}) = \chi(\sigma_i)$, so (7) holds.

**Claim.** For large $j$, $a_{n_{i+1}^j} \in S(\tau_{i+1})$ and $b_{n_{i+1}^j} \in S(\rho_{i+1})$.

Indeed, there are leaves of $\Lambda$ that cross from the upper left [right] to the lower left [right] branch on the left diagram in Figure 3.4, and likewise from upper left to lower right. The same is therefore true for segments of $a_{n_{i+1}^j}$ for large $j$. This prevents $a_{n_{i+1}^j}$ from entering the vertical segment say from a side branch on the left and exiting through a side branch on the right, or the top or bottom right branch. Since such configurations do not occur, the Claim holds.

Thus after discarding an initial portion of each sequence, properties (2)-(5) hold (for (4) note that $\text{Supp}(\sigma_{i+1}) = \text{Supp}(\sigma_i)$ and for (5) note that since $a_{n_{i+1}^j}$ contains segments that cross from upper left to lower left, from upper right to lower right, and from upper left to lower right, the combinatorial length strictly decreases after the split). Now define $a_{n_{i+1}^j}$ and $b_{n_{i+1}^j}$ to be a pair of curves that minimize the sum of the combinatorial lengths, subject to (2)-(5).

It remains to prove (8). Let $\Omega$ be a lamination carried by $\tau_{i+1}$ and by $\rho_{i+1}$. It is therefore carried by $\tau_i$ and $\rho_i$, so by (8) for $\sigma_i$ it is carried by $\sigma_i$. Now we again have to argue that certain configurations do not occur, e.g. that leaves of $\Omega$ do not enter on a left side branch and exit on a right side branch. If this occurs then $\Omega$ would not be carried by $\tau_{i+1}$ or $\rho_{i+1}$.

**Case 1b.** In the degenerate case when both splits carry $\Lambda$ (i.e. when $\Lambda$ does not cross the diagonally drawn branches in Figure 3.5), we define $\sigma_{i+1}$
to be the track obtained from $\sigma_i$ by cutting open along the vertical segment. Thus $\sigma_{i+1}$ does not include either of the diagonal branches. See Figure 3.5.

Figure 3.5:

Next, we observe that for large $j$ the curves $a^{\sigma_{i+1}}_{n_j}$ cannot cross both from top left to bottom right and from top right to bottom left, and the same is true for $b^{\sigma_{i+1}}_{n_j}$. Thus after passing to a further subsequence we can add one of the two diagonal branches to $\tau_{i+1}$ and ensure that $a^{\sigma_{i+1}}_{n_j} \in S(\tau_{i+1})$, and likewise $b^{\sigma_{i+1}}_{n_j} \in S(\rho_{i+1})$ after including one of the two diagonals. It is possible that one diagonal is added to $\tau_{i+1}$ and the other to $\rho_{i+1}$ and then the intersection number increases by 1. But the number of branches of $\sigma_{i+1}$ decreased, so we still have $\chi(\sigma_{i+1}) < \chi(\sigma_i)$ and we are done by induction. If the same diagonal is added to both $\tau_{i+1}$ and to $\rho_{i+1}$ we will also add it to $\sigma_{i+1}$. The rest of the argument is similar to the non-degenerate case.

Case 2. $\bar{\sigma}$ does not have any illegal turns. Thus $\bar{\sigma}$ is a collection of legal simple closed curves and so is $\Lambda$. In $A$ and $B$ there must be isolated leaves spiraling towards each component of $\Lambda$, in opposite directions on the two sides. The spiraling directions are the same for both $A$ and $B$, since otherwise the projection distance on the curve complex of the annulus would be large. In other words, both $a^{\sigma_i}_{n_j}$ and $b^{\sigma_i}_{n_j}$ wind around the same annulus and in the same direction a large number of times. Applying the Dehn twist (left, or right, as appropriate) shortens both curves as they wind around the annulus one less time. At the same time this operation does not change the distance to $C(Supp(\sigma))$. This contradicts the minimality and we are done.

Proof of (iii). Again the proof is by induction on the complexity. We will inductively assume (1)-(8) except that (4) is replaced with
(4’) \( d(a^{\sigma_i}_{n^i_j}, S(\sigma_i)) > n^i_j, \quad d(b^{\sigma_i}_{n^i_j}, S(\sigma_i)) > n^i_j. \)

The proof follows closely our proof of (ii). As in that proof, we pass to a further subsequence and construct limiting laminations \( A \) and \( B \) that have a common measurable part \( \Lambda \) which is carried by \( \sigma_i \), and \( \bar{\sigma} \) is the union of the edges of \( \sigma \) crossed by \( \Lambda \).

There are two cases as in (ii).

Case 1. \( \bar{\sigma} \) contains an illegal turn. We split along a large branch of \( \bar{\sigma} \) as before and define \( \sigma_{i+1} \) in the same way (in both subcases, whether the split is degenerate or non-degenerate). The only change is that now we have to argue that (4’) holds, instead of (4). The reason now is that \( S(\sigma_{i+1}) \subset S(\sigma_i) \).

Case 2. \( \bar{\sigma} \) is a collection of legal loops. Now we cannot simply apply a Dehn twist since this does not necessarily preserve \( S(\sigma_i) \). Note that there must be branches of \( \sigma_i \) attached to both sides of \( \bar{\sigma} \) for otherwise we would be in situation (ii).

Case 2a. All branches of \( \sigma_i \) attached to a component of \( \bar{\sigma} \) are attached in the same direction. See Figure 3.6.

![Figure 3.6](image)

Thus there is a unique curve in \( S(\sigma_i) \) that crosses an edge of this component of \( \bar{\sigma} \) (and it equals the component). Therefore the Dehn twist preserves \( S(\sigma_i) \) and we proceed as before.

Case 2b. There are branches of \( \sigma_i \) attached to a component of \( \bar{\sigma} \) in opposite directions. We will assume here that every branch of \( \sigma_i \) is crossed by either \( a^{\sigma_i}_{n^i_j} \) or by \( b^{\sigma_i}_{n^i_j} \) (or both) for every \( j \), for otherwise we can remove this edge from all three \( \sigma_i \), \( \tau_i \), \( \rho_i \) and use induction.

Then we can find two branches of \( \sigma_i \) attached in opposite directions and on opposite sides of this component of \( \bar{\sigma} \) (the curves \( a^{\sigma_i}_{n^i_j} \) or \( b^{\sigma_i}_{n^i_j} \) spiral and cannot
escape on the same side). In other words, we have a picture as in Figure 3.4 where the vertical segment as well as top left and lower right branches [or top right and lower left branches] belong to $\sigma$, and the top right and the lower left branches [or top left and lower right branches] belong to $\sigma_i - \bar{\sigma}$. Perform the split as in Figure 3.4 so that $\Lambda$ is carried. If there are any side branches attached to the vertical segment, then after the split the number of side branches attached to $\bar{\sigma}$ is strictly smaller and we may induct on this number. If there are no such side branches, then the combinatorial lengths of $a_{n_j^i}$ and $b_{n_j^i}$ strictly decrease after the split (e.g. consider a piece of $a_{n_j^i}$ that enters $\bar{\sigma}$ through the top branch which is not part of $\bar{\sigma}$). Then proceed as before, by defining $a_{n_j^{i+1}}$ and $b_{n_j^{i+1}}$ to be curves that minimize combinatorial length subject to (2)-(4').

We will only use two special cases of Lemma 3.25, and we state them below.

Corollary 3.26. For every $C > 0$ there is $C' > 0$ depending only on the surface $\Sigma$ so that the following holds. Let $\tau$ be a large track on $\Sigma$. Assume one of the following.

(I) $\tau_1, \tau_2$ are large subtracks of $\tau$. Let $\sigma = \tau_1 \cap \tau_2$. After pruning dead ends, $\sigma$ becomes a track (possibly empty) and $V(\sigma) = V(\tau_1) \cap V(\tau_2)$.

(II) $\tau_1, \tau_2$ are the two tracks obtained from $\tau$ by splitting along a large branch $e$, and $\sigma$ the track obtained by a central split at $e$. Thus $P(\tau) = P(\tau_1) \cup P(\tau_2)$ and $P(\sigma) = P(\tau_1) \cap P(\tau_2)$.

Then one of the following holds.

(1) $\sigma$ is not large (possibly it is empty), and for any two curves $a_i \in P(\tau_i)$ with $d(a_1, a_2) \leq C$ it follows that $d(a_i, B(\tau_i)) \leq C'$, or

(2) $\sigma$ is large and for any two curves $a_i \in S(\tau_i)$ with $d(a_1, a_2) \leq C$ there is a curve $c \in S(\sigma)$ such that $d(a_i, c) \leq C'$.

4 Cell structures via splittings

Now we take $U = V(\tau)$ for a recurrent, transversely recurrent, maximal train track $\tau$.

Let $C_j$ be the excellent sequence obtained by repeating the subdivision process, at every step choosing one of the codimension 0 cells $V(\sigma)$, $\sigma$ a recurrent, transversely recurrent, maximal train track, and splitting in a selected large branch. Thus inductively, codimension 0 cells in $C_j$ are in 1-1 correspondence with a set of train tracks, each obtained from $\tau$ by a splitting sequence, and all tracks in the splitting sequence correspond to codimension 0 cells in $C_k$ for $k \leq j$.  

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4.1 Interpolating curves process

In this section we set the groundwork for proving that the distance between disjoint cells is not too small. This follows easily from Lemma 3.25 when the associated train tracks have bounded intersection number. To handle the general case we define a certain iterative procedure that constructs sequences of curves relating different cells in \( C_j \).

We start by defining a sequence \( C_0, C_1, \ldots \) inductively. Here \( C_0 > 0 \) is a fixed constant, and \( C_{i+1} \) is defined as \( C' \) in Corollary 3.26 for the constant \( C = 2^n C_i \), where \( n = \dim \mathcal{ML} \).

When \( a \) is a simple closed curve we denote by \( \text{Carr}_j(a) \) the carrier of \( a \) in \( C_j \), i.e. the smallest cell of \( C_j \) that contains \( a \).

**Definition 4.1.** A sequence of curves \( a = a_0, a_2, \ldots, a_m \) in \( \Sigma \) is good with respect to the cell structure \( C_j \) (or \( C_j \)-good) if for any two adjacent curves \( a_i, a_{i+1} \) in the sequence the carriers \( \text{Carr}_j(a_i) \) and \( \text{Carr}_j(a_{i+1}) \) are nested (or possibly equal), i.e. \( \text{Carr}_j(a_i) \subseteq \text{Carr}_j(a_{i+1}) \) or \( \text{Carr}_j(a_{i+1}) \subseteq \text{Carr}_j(a_i) \).

A sequence which is \( C_j \)-good may not be \( C_{j+1} \)-good. We now describe an inductive procedure that consists of inserting curves to produce \( C_k \)-good sequences with \( k \) large.

We start with a \( C_0 \)-good sequence of bounded length. For example, we might start with a sequence \( a_0, a_1 \) of length 2 consisting of two curves in the interior of the same cell in \( C_0 \). Inductively assume that we inserted some curves in the sequence and obtained a \( C_j \)-good sequence \( a \).

Suppose \( a, b \) are two consecutive curves in \( a \) that fail to satisfy the definition of \( C_{j+1} \)-good, that is, the carriers \( A = \text{Carr}_{j+1}(a) \) and \( B = \text{Carr}_{j+1}(b) \) are not nested. There are several cases.

![Figure 4.1: Interpolating points to achieve goodness at the next stage.](image)

(i) \( \text{Carr}_j(a) = \text{Carr}_j(b) \), we call this cell \( C \). Thus the subdivision operation splits \( C \) into \( A \) and \( B \) (if the cut contains either point, \( A \) and \( B \) would be nested) and \( A \cap B = W \) is the codimension 1 cut. See the left diagram in Figure 4.1.

We now apply Corollary 3.26(II) to the train tracks \( \tau_1, \tau_2, \sigma \) such that \( V(\tau_1) \) and \( V(\tau_2) \) are the two splits of \( V(\tau) = C \) (so \( \tau_1, \tau_2 \) have branches \( e_1, e_2 \) that intersect) and \( \sigma \) is a common subtrack of \( \tau_1, \tau_2 \).
obtained by deleting $e_1$ from $\tau_1$ or $e_2$ from $\tau_2$ (these tracks exist by Proposition 3.22). Therefore we obtain a curve $w \in P(\sigma) = W$, and if $d(a, b) \leq C_i$ we have in addition that $d(a, w) \leq C_{i+1}$, $d(b, w) \leq C_{i+1}$. We insert $w$ in the sequence between $a$ and $b$. The consecutive curves in $a, w, b$ satisfy the $C_{j+1}$-goodness condition.

(ii) $\text{Carr}_j(a) \subsetneq \text{Carr}_j(b)$. This is depicted in the other two diagrams in Figure 4.1. Notice that the cut $W$ cannot contain $a$, or else the goodness condition would hold in $C_{j+1}$. There are two further subcases. If $b$ does not belong to $W$ either, we are in the situation of the middle diagram. First apply Corollary 3.26(II) as in (i) above to find $w \in W$. Then apply Corollary 3.26(I) to curves $a$ and $w$ to find a curve $v$ carried by the intersection of the $C_{j+1}$-carriers of $a$ and $w$. Finally, interpolate to get the sequence $a, v, b$. The other subcase is that $b \in W$, depicted in the right diagram in Figure 4.1. We again interpolate $v$ in the intersection of $C_{j+1}$-carriers of $a$ and $b$.

Whenever we apply Corollary 3.26 it may happen that conclusion (1) occurs. In that case we stop the process and do not attempt to define a $C_{j+1}$-good sequence.

To the $C_j$-good sequence $a_j = a_0, a_1, \ldots, a_m$ constructed in this way we will associate a dimension sequence $D(a_j)$ inductively. This is a sequence of nonnegative integers $d_0, d_1, \ldots, d_m$ with the requirement that the dimension of $\text{Carr}_j(a_i)$ is $\leq d_i$. It is also constructed inductively. For the initial sequence we take the dimensions of the $C_0$-carriers. Inductively, we extend the dimension sequence. For each curve $x$ that is inserted when extending the sequence from $a_j$ to $a_{j+1}$ define the corresponding integer as the dimension of $\text{Carr}_{j+1}(x)$. For curves that were part of the sequence $a_j$ leave the value unchanged. Thus the number associated to a curve in the sequence is the dimension of its carrier when the curve first appeared. The dimension of the carrier of a curve may decrease, but the value in the dimension sequence is unchanged.

The following proposition summarizes the essential features of the construction.

**Proposition 4.2.** Suppose that a curve $x$ got inserted between the curves $a, b$ in a $C_j$-good sequence $a_j$.

(i) The value of the dimension sequence at $x$ is strictly less than at both $a$ and $b$.

(ii) If $d(a, b) \leq C_i$ for some $i$ then $d(a, x) \leq C_{i+2}$ and $d(b, x) \leq C_{i+2}$.

In (ii) we may be applying Corollary 3.26 twice, and this is why the conclusion involves $C_{i+2}$.

The following lemma can be proved by a straightforward induction on $n$. We will apply it to dimension sequences.
Lemma 4.3. Let $D_i = (x_{i0}, x_{i1}, \ldots, x_{ij})$, $i = 0, 1, 2, \ldots, N$ be a sequence of finite sequences of nonnegative integers. Assume the following:

(a) $D_0 = (x_{00}, x_{01})$ has length 2 and $x_{00}, x_{01} \leq n$,

(b) for $i \geq 0$ the sequence $D_{i+1}$ is obtained from $D_i$ by inserting between some consecutive terms a nonnegative integer strictly smaller than each of the two terms.

Then $j_N \leq 2^n$.

For example, $33, 323, 31213, 301020103$ is such a sequence with $n = 3$, $N = 3, j_3 = 8$.

Proposition 4.4. For every $C > 0$ there is $C' = C'(C, \Sigma)$ so that the following holds. Let $C_j$ be an excellent sequence of cell structures with all cells (i.e. their vertex cycles) at distance $\leq K$ from $\ast$. Suppose $A, B$ are two cells in $C_j$. If $a \in \text{int}(A)$ and $b \in \text{int}(B)$ and $d(a, b) \leq C$ then either

- $d(\ast, a), d(\ast, b) \leq K + C'$, or

- there is a curve $c$ contained in a cell of $C_j$ which is contained in a face of each $A, B$ such that $d(a, c), d(b, c) \leq C'$.

Proof. First assume that $a, b$ is a $C_0$-good sequence. We set $C_0 = C$ and define $C_j$ inductively as above. Run the process starting with $a, b$. There are now two possibilities.

Case 1. The process produces a $C_j$-good sequence $a = a_0, a_1, \ldots, a_N$. From Lemma 4.3 we see that $N \leq 2^n$ where $n = \dim ML$. Thus there were at most $2^n - 1$ insertions and this implies that $d(a_i, a_{i+1}) \leq C_2(2^n - 1)$ for any two consecutive curves $a_i, a_{i+1}$.

The sequence of $C_j$-carriers $\text{Carr}_j(a_i), i = 0, 1, \ldots$ either increases or decreases (or stays the same) at every step. We now modify the sequence, by “pushing the peaks down” so that an initial part of the sequence of carriers is nonincreasing, and the rest is nondecreasing. Let $a_i, a_{i+1}, \ldots, a_k$ be a subsequence of consecutive curves such that

$\text{Carr}_j(a_i) \subset \text{Carr}_j(a_{i+1}) = \text{Carr}_j(a_{i+2}) = \cdots = \text{Carr}_j(a_{k-1}) \supset \text{Carr}_j(a_k)$

First we pass to the length 3 subsequence $a_i, a_{i+1}, a_k$. The distance $d(a_{i+1}, a_k) \leq 2^n C_2(2^n - 1)$ (we are happy with very crude estimates), so applying Corollary 3.26(i) we find a curve $x$ with $\text{Supp}_j(x) \subset \text{Carr}_j(a_i) \cap \text{Carr}_j(a_k)$ and with $d(a_i, x), d(a_k, x) \leq C_2(2^n - 1) + 2$. (If conclusion (1) occurs see Case 2.) Continuing in this way produces the desired sequence. The number of steps that consist of pushing the peaks down is bounded, e.g. by $n2^n$, so at the end the distance between any two consecutive curves is bounded by $C_2(2^n - 1) + 2n2^n$.

Finally, pass to a length 3 sequence $a, c, b$ where $c$ has the minimal carrier, and set $C' = 2^n C_2(2^n - 1) + 2n2^n$.

Case 2. At some stage in the process, when applying Corollary 3.26 conclusion (1) occurs. This applies also to the part of the procedure when
we push the peaks down. Thus we have a sequence \( a = a_0, a_1, \ldots, a_N = b \) with \( N \leq 2^n \), \( d(a_i, a_{i+1}) \leq C_2(2^n - 1) + 2n2^n \), and for some \( i \) we have \( d(\ast, a_i) \leq C_2(2^n - 1) + 2n2^n + 1 \). This implies by the triangle inequality that
\[
d(\ast, a), d(\ast, b) \leq C_2(2^n - 1) + 2n2^n + 1 + 2nC_2(2^n - 1) + 2n2^n
\]
and we may take \( C' \) to be this bound.

Finally, consider the general case when \( a, b \) is not a \( C_0 \)-good sequence. Let \( A = \text{Car}_0(a), B = \text{Car}_0(b) \). Thus \( A = V(\alpha), B = V(\beta) \) for certain tracks \( \alpha, \beta \). Lemma 3.25 gives that either \( d(\ast, a), d(\ast, b) \) are uniformly bounded, as functions of \( \Sigma \) and \( C \), or there is a curve \( c \in A \cap B \) within uniform distance, call it \( C_0 \), from \( a, b \). (Note here that since \( C_0 \) is a fixed cell structure, the intersection number between any two tracks defining it is uniformly bounded, so Lemma 3.25 applies uniformly.) Thus \( a, c, b \) is a \( C_0 \)-good sequence and the procedure above proves the statement. \( \square \)

## 5 Capacity dimension of \( \mathcal{E}L \)

### 5.1 Capacity dimension

Let \( (Z, \rho) \) be a metric space. The notion of capacity dimension of \( Z \) was introduced by Buyalo in [Bu05a]. One of several possible equivalent definitions is the following, see [Bu05a] Proposition 3.2. We also note that for bounded metric spaces, such as the boundary of a hyperbolic space with visual metric, capacity dimension agrees with the Assouad-Nagata dimension. See [LS05].

**Definition 5.1.** The capacity dimension of a metric space \( Z \) is the infimum of all integers \( m \) with the following property. There exists a constant \( c > 0 \) such that for all sufficiently small \( s > 0 \), \( Z \) has a \( cs \)-bounded covering with \( s \)-multiplicity at most \( m + 1 \).

The covering \( L \) is \( cs \)-bounded if all elements have diameter \( < cs \) and the \( s \)-multiplicity of \( L \) is \( \leq m + 1 \) if every \( z \in Z \) is at distance \( < s \) from at most \( m + 1 \) elements of \( L \).

We will produce covers that resemble cell structures and whose thickenings resemble handle decompositions. It is more convenient here to index the handles starting with 1, rather than with 0. We will use following form of the definition of capacity dimension.

**Proposition 5.2.** Suppose that there is a constant \( c > 10 \) such that for all sufficiently small \( s > 0 \) there is a cover \( K \) of \( Z \) with the following properties:

- The collection \( K \) is the disjoint union of subcollections \( K_1, K_2, \ldots, K_{m+1} \).
- The diameter of any set in \( K \) is \( \leq s \).
- If \( A, B \in K_i \) are distinct elements in the same subcollection and if \( a \in A, b \in B, \rho(a, b) < s/c^{3i-1} \) then there is some \( e \in E \in K_k \) for some \( k < i \) so that \( \rho(a, e) < s/c^{3i-2} \) and \( \rho(b, e) < s/c^{3i-2} \).
Then the capacity dimension of $Z$ is at most $m$.

**Proof.** Inductively on $i$, for each $K \in K_i$ define the associated “handle”

$$H(K) = N_{s/c^i}(K) - \cup_{K' \in K_{k, k < i}} H(K')$$

It is clear that the collection of all handles forms a cover of $Z$ and that the diameter of each element is bounded by $s + 2s/c^3$. We will argue that the $s/c^m$-multiplicity of the cover is $\leq m + 1$. Suppose $z \in Z$ is at distance $< s/c^m$ from $m + 2$ handles. Then two of the handles have the same index, say $H(A), H(B)$ with $A, B \in K_i$. Thus we have $a_0 \in H(A)$ and $b_0 \in H(B)$ with $\rho(a_0, b_0) < 2s/c^{m+4}$. Choose $a \in A$ and $b \in B$ with $\rho(a, a_0) < s/c^i$ and $\rho(b, b_0) < s/c^i$. Then $\rho(a, b) < 2s/c^{m+4} + 2s/c^i < s/c^{i-1}$. By assumption there is $e \in K \in K_k$ with $k < i$ and with $\rho(a, e) < s/c^{i-2}$, $\rho(b, e) < s/c^{i-2}$. Thus $\rho(a_0, e) < s/c^{i-2} + s/c^i < s/c^{i-3} \leq s/c^k$, so $a_0 \in H(K)$ or it belongs to a lower index handle, and similarly for $b_0$. This is a contradiction since $H(A)$ and $H(B)$ are disjoint from lower index handles. 

**Remark 5.3.** In section 5.3 we will show that the cover of $\mathcal{EL}$ induced by $P_\infty(\sigma)$ as $V(\sigma)$ range over cells in $C_j$ with $\sigma$ large satisfy the assumptions of Proposition 5.2. For example, shapes like in Figure 5.1 are ruled out by Proposition 4.3.

We also have the following general fact. We thank Vera Tonić for pointing it out to us.

**Proposition 5.4 (LS05).** Suppose $Z$ is written as a finite union of (closed) subsets $Z_i$. If $\text{capdim } Z_i \leq n$ for all $i$, then $\text{capdim } Z \leq n$. 

Figure 5.1: The handle decomposition associated to a triangulation. Cells shaped like the one pictured on the right will result in handle decompositions with bad Lebesgue number.
5.2 Visual size and distance

Recall that a metric $\rho$ on the boundary $Z$ of a $\delta$-hyperbolic space $X$ is said to be visual if there is a basepoint $\ast \in X$ and constants $a > 1$ and $c_1, c_2 > 0$ such that

$$c_1 a^{-\frac{d(\ast, x) + d(\ast, x') - d(x, x')}{2}} \leq \rho(z, z') \leq c_2 a^{-\frac{d(\ast, x) + d(\ast, x') - d(x, x')}{2}}$$

for all $z, z' \in Z$, where $(\cdot|\cdot)$ denotes the Gromov product

$$(x|x') = \frac{1}{2}(d(\ast, x) + d(\ast, x') - d(x, x'))$$

on $X$, extended naturally to $Z$. See [GdlH90, Ch. 7] for more details and for the construction of visual metrics.

Also recall that, in a $\delta$-hyperbolic space, the Gromov product $(a|b)$ is, to within a uniform bound that depends on $\delta$, the distance between the basepoint $\ast$ and any geodesic $[a, b]$. The same is true when $a, b$ are distinct points at infinity. We may also replace geodesics $[a, b]$ with quasi-geodesics, but then the uniform bound depends also on quasi-geodesic constants.

A $\delta$-hyperbolic space $X$ is visual [BS00] for some (every) basepoint $x_0$ there exists $C > 0$ such that for every $x \in X$ there is a $(C, C)$-quasi-geodesic ray in $X$ based at $x_0$ and passing through $x$. Equivalently, $X$ is the coarse convex hull of the boundary $\partial X$. Any $\delta$-hyperbolic space whose isometry group acts coboundedly and that contains a biinfinite quasi-geodesic line is visual. Thus a curve complex is visual.

**Theorem 5.5.** [Buy05a] Let $X$ be a visual $\delta$-hyperbolic metric space and $Z$ its Gromov boundary endowed with a visual metric. Then

$$\operatorname{asdim}(X) \leq \operatorname{capdim}(Z) + 1$$

**Lemma 5.6.** Assume that $\sigma$ is a large track obtained from $\tau$ by a sequence of splittings. Using $B(\tau)$ as a basepoint, the visual diameter of $P_\infty(\sigma)$ is $n^{-d(\ast, B(\sigma))}$, to within a bounded factor.

**Proof.** This follows from Lemmas 3.14 and 3.15. \hfill \Box

**Proposition 5.7.** Let $C_j$ be an excellent sequence of cell structures obtained by splitting tracks. For all sufficiently large constants $c$ (depending only on $\Sigma$) the following holds for all sufficiently small $s > 0$. Suppose for a certain $j$ the visual diameter of each $P_\infty(\sigma)$ is $> s$, where $\sigma$ ranges over all maximal train tracks such that $V(\sigma) \in C_j$. Then

- The visual diameter of each $P_\infty(\sigma)$ is $> s/c$ for every large track $\sigma$ determining a cell $V(\sigma)$ in $C_j$.
- Suppose cells $V(\sigma)$ and $V(\sigma')$ in $C_j$ are distinct with both $\sigma, \sigma'$ large. Suppose $a \in P_\infty(\sigma), b \in P_\infty(\sigma')$, and $\rho(a, b) < s/c^2$. Then there is a cell $V(\mu) \subset V(\sigma) \cap V(\sigma')$ and there is an $e \in P_\infty(\mu)$ so that $\rho(a, e) < s/c, \rho(b, e) < s/c$. 

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Proof. The first bullet follows from Lemma 5.6 and Proposition 3.14. Consider a quasi-geodesic ray from \( * \) to \( a \) that passes through \( B(\sigma) \) and between \( B(\sigma) \) and \( a \) stays in \( S(\sigma) \). By Lemmas 3.13 and 3.14 we may assume that these rays are uniformly quasi-geodesic. Likewise, construct such a ray from \( * \) to \( b \). Also choose a uniform quasi-geodesic from \( a \) to \( b \). We now have a uniformly thin triangle with two vertices at infinity. Choose \( a' \) on the first ray and \( b' \) on the second ray, just past the thick part viewed from \( * \). Thus \( d(a', b') \) is uniformly bounded, say by \( C \). Also note that \( d(\cdot, a') \), \( d(\cdot, b') \) is a definite amount larger than \( d(\cdot, B(\sigma)) \), \( d(\cdot, B(\sigma')) \) by the assumption that \( \rho(a, b) < s/c \) with \( c \) sufficiently large. In particular, \( a' \in S(\sigma), b' \in S(\sigma') \). This satisfies the conclusions.

5.3 The cover

It is well known that \( \mathcal{PML} \) can be covered by finitely many sets of the form \( P(\tau) \) for \( \tau \) a large train track (for a concrete cover see [PH92]). Thus \( \mathcal{EL} \) is finitely covered by sets of the form \( P(\sigma) \). In view of Proposition 5.4 we need to find an upper bound to \( \text{capdim} P(\sigma) \).

Here we fix a large birecurrent train track \( \tau \) and describe a cover of \( Z = P(\sigma) \) that satisfies Proposition 5.2 for a certain \( m \) depending on \( \Sigma \) and for small \( s > 0 \). The dimensions of cones \( V(\sigma) \) for large train tracks \( \sigma \subset \Sigma \) belong to a certain interval \([A, A + K]\) that depends on \( \Sigma \). We put \( m = K \). We also fix a large constant \( c \).

Now fix a small \( s > 0 \) and start with the standard cell structure on \( V(\tau) \). This is \( C_0 \). Now suppose \( C_j \) has been constructed and the visual size of each \( P(\sigma) \) for a top dimensional cell \( V(\sigma) \in C_j \) is \( > s/c \). Enumerate all top dimensional cells \( V(\sigma) \in C_j \) such that the visual size of \( P(\sigma) \) is \( \geq s \) and also enumerate all large branches in the corresponding tracks \( \sigma \). Then construct \( C_{j+1}, \ldots, C_k \) by splitting along these branches, in any order. We call this collection of splits a multisplit.

This gives an infinite excellent sequence. Note that once some \( P(\sigma) \) (with \( V(\sigma) \) maximal) reaches visual size \( < s \) at the end of a multisplit, it never gets subdivided again (see Lemma 5.11). Coarsely, reaching a certain visual size is equivalent to \( B(\sigma) \) reaching a certain distance from the basepoint \( * \) (see Lemma 5.6).

**Lemma 5.8.** Let \( \lambda \) be a filling lamination, and for every \( j \) let \( E_j = V(\sigma_j) \) be the cell in \( C_j \) that contains \( \lambda \) in its interior. Then the sequence \( E_j \) eventually
Proof. We argue by contradiction. From Proposition 3.23 we have that $\sigma_{j+1} \xrightarrow{\mathcal{A}} \sigma_j$. Let $a_j$ be a vertex cycle of $\sigma_j$. Then we may assume, perhaps after a subsequence, that $a_j \to \lambda'$ in $\mathcal{PML}$ and the lamination $\lambda'$ is necessarily carried by all $\sigma_j$. By Lemma A.4 for large $j$, $\sigma_j$ will fully carry $\lambda$, so by Lemma 3.24 we have $[\lambda'] = [\lambda]$. By an argument of Kobyashi (see p. 124 of [MM99]) the sequence $a_j$ goes to infinity in the curve complex, so the visual size of $E_j$ goes to 0 by Lemma 5.6. But in the construction of $\mathcal{C}_j$ the visual diameter of all top dimensional cells is bounded below and by Proposition 3.23 this bounds below the visual diameter of all cells, giving a contradiction.

We let the cover $\mathcal{K}$ consist of the sets of the form $P_{\infty}(\sigma)$ such that $V(\sigma)$ is a stable cell. We partition the sets in $\mathcal{K}$ according to the dimension of the cell.

**Theorem 5.9.**

$$\text{capdim}(EL) \leq K(\Sigma)$$

where $K = K(\Sigma)$ is the smallest integer such that every recurrent, transversely recurrent, large track $\sigma$ on $\Sigma$ has dim $P(\sigma) \in [A, A + K]$ for some $A = A(\Sigma)$.

**Proof.** We only need to argue that the cover $\mathcal{K}$ satisfies the conditions of Proposition 5.2. This is clear from the construction and Propositions 3.23 and 4.4 applied to $\mathcal{C}_j$ for large $j$.

**Corollary 5.10.**

$$\text{asdim}(\mathcal{C}(\Sigma)) \leq K(\Sigma) + 1$$

**Example 5.11.** One can see easily what happens in the case of the punctured torus. Then $\mathcal{FPM}L = \mathcal{EL}$ is homeomorphic to the set of irrational numbers, or equivalently to $\mathbb{Z}^\infty$. The visual metric is complete, and the cover $\mathcal{K}$ constructed above will be infinite and will consist of pairwise disjoint sets, all of the same index, and all of comparable sizes. For example, consider a standard track that supports laminations whose slope is in the interval $[1, \infty]$. Splitting produces two tracks, one carrying laminations in the interval $[1, 2]$ and the other in the interval $[2, \infty]$. We can take the curve with slope $\infty$ as the basepoint and agree to stop subdividing when the distance from $\infty$ to $B(\sigma)$ is $> 0$, i.e. when $\infty$ is no longer carried by $\sigma$. Thus we stop splitting the track carrying $[1, 2]$ and we split the other track. We get tracks carrying $[2, 3]$ and $[3, \infty]$. Continuing in this way we get an infinite cover $P_{\infty}(\sigma_n)$ of $\mathcal{EL}$ where $\sigma_n$ carries ending laminations with slope in $[n, n + 1]$.

**Remark 5.12.** There are two other closely related notions to asymptotic dimension. In the linearly controlled asymptotic dimension, or the asymptotic Assouad-Nagata dimension $\ell$ – asdim one insists on the linear control on the size of the cover. Also, say that $\text{eco} – \text{dim}(X) = n$ if $X$ quasi-isometrically
embeds in a product of $n$ trees and $n$ is smallest possible. Then there is a chain of inequalities

$$\text{asdim} \leq \ell - \text{asdim} \leq \text{ecodim} - \dim$$

for any metric space, and Buyalo shows in [Buy05b] that when $X$ is $\delta$-hyperbolic then

$$\text{ecodim}(X) \leq \text{capdim} \partial X + 1$$

See also the discussion in [MS13]. Therefore our arguments also give the same bound on $l - \text{asdim}$ and $\text{ecodim}$ for $\mathcal{C}(\Sigma)$. Previously, Hume observed that $\text{ecodim}(\mathcal{C}(\Sigma)) < \infty$ [Hum].

### Appendix A  Ending laminations

A **splitting path** is a legal embedded path in a thickened train track that begins and ends at a switch. See Figure A.1.

![Figure A.1](image)

Figure A.1: The dashed line is the splitting path. It must start and end at a switch in the thickened track.

If $\tau$ is a train track and $\theta$ is the central split along the splitting path then $\theta$ will have one or two connected components and a total of three less branches and two less switches then $\tau$. By Lemma 3.1 we then have:

**Lemma A.1.** Either $\dim V(\theta) + 1 = \dim V(\tau)$ or $\dim V(\theta) = \dim V(\tau)$. If $\dim V(\theta) = \dim V(\tau)$ then one of the following holds.

1. $\tau$ is non-orientable and $\theta$ is connected and orientable,
2. $\tau$ is orientable and $\theta$ has two components (both necessarily orientable)
   or
3. $\tau$ is non-orientable and $\theta$ has one orientable component and one non-orientable component.

If $\dim V(\theta) = \dim V(\tau)$ we say that the splitting path is a **full dimension splitting path** or fd-path. While a splitting path will be embedded in the thickened train track, in the actual train track it may cross a single branch
multiple times. However, an fd-path can cross any branch at most twice and this strong restriction implies that there is a uniform bound on the number of fd-paths in a given track.

**Lemma A.2.** An fd-path of types (1) or (2) crosses each branch of $\tau$ at most once and an fd-path of type (3) crosses each branch at most twice.

**Proof.** Orient the splitting path. If the path crosses a branch more than once, we examine two consecutive strands of the path in the branch.

- If the two strands have the same orientation then $\theta$ will be connected and we must be in case (1). Then $\tau$ will be non-orientable and $\theta$ will be orientable so we can choose an orientation for $\theta$. On opposite sides of the splitting path the orientation of $\theta$ cannot agree or $\tau$ would be orientable. Using the orientation of the splitting path (and the surface) we can assume that to the right of the splitting path the orientation of $\theta$ and the splitting path agree while to the left they are opposite of each other. However, this is not possible if there are two consecutive strands in the same branch with the same orientation. Therefore we can never have consecutive branches with the same orientation.

- If consecutive branches in the same branch have opposite orientation then the component of $\theta$ between the two strands will be non-orientable and we must be in case (3). If the splitting path crosses the same branch three or more times then orientation of consecutive branches will always be opposite so both components of $\theta$ will non-orientable, a contradiction.

**Lemma A.3.** Let $\tau_1$ and $\tau_2$ be birecurrent train tracks with $\tau_2 \xrightarrow{ss} \tau_1$ and $\dim V(\tau_1) = \dim V(\tau_2)$. Let $\theta_2$ be the central split of $\tau_2$ along a splitting path. Then there is a splitting path in $\tau_1$ with central split $\theta_1$ such that $V(\theta_2) = V(\theta_1) \cap V(\tau_2)$.

**Proof.** As usual via induction we can reduce this to the case when $\tau_2 \xrightarrow{ss} \tau_1$ is a single move. In $\tau_2$ the splitting path starts and ends at large half branches. If these large half branches and their adjacent half branches are not active in the move then the composition of the splitting path with the carrying map is a splitting path. If not the carrying map is a “fold” along the switch and, in $\tau_1$, we extend the path along the fold.

If the move is a right or left split along a large branch $b$ in $\tau_1$ then $\theta_1 = \theta_2$ if the path crosses $b$ in $\tau_2$. If not that $\theta_2$ is a split of $\theta_1$ along $b$. If the move is a central split then $\theta_2$ is a central split of $\theta_1$.

One consequence of the existence of fd-paths is that there are train tracks that do not fully carry any lamination. The next lemma shows that this is the only obstruction.
Lemma A.4. Let \( \tau \) be a birecurrent train track and \( \lambda \in V(\tau) \) such that for any track \( \sigma \) with \( \sigma \overset{ss}{\rightarrow} \tau \) and \( \lambda \in V(\sigma) \) it follows that \( \dim V(\sigma) = \dim V(\tau) \) and \( \lambda \) is contained in the relative interior of \( V(\sigma) \). Then for any non-constant sequence \( \tau = \tau_1, \tau_2, \ldots \) with \( \tau_{i+1} \overset{ss}{\rightarrow} \tau_i \) and \( \lambda \in V(\tau_i) \) we have that \( \lambda \) is fully carried by \( \tau_i \) for \( i \) sufficiently large.

Proof. By [Thu79, Proposition 8.9.2] there exists birecurrent train track \( \tau' \) that fully carries \( \lambda \). Note that while fully carrying is not discussed in this proposition, one sees that if the \( \epsilon \) in the construction is chosen to be sufficiently small then the track will be fully carrying. Now the rest of the proof follows exactly as in Proposition 3.24 with one exception: To see that we don’t need to pass to subtracks in the application of Proposition 3.21 we use the assumption that \( \lambda \) is in the relative interior of \( V(\tau_i) \) for all \( i \). With this one change we see that there exists a birecurrent train track \( \sigma \) that fully carries \( \lambda \) and an \( N \) such that \( \tau_N \overset{ss}{\rightarrow} \sigma \). Since \( \sigma \) fully carries \( \lambda \), we have that \( \tau_N \) and \( \tau_i \) for \( i \geq N \) will also fully carry \( \lambda \).

There is a simple condition that guarantees that a lamination satisfies the assumptions of Lemma A.4. Namely, if \( \sigma \overset{ss}{\rightarrow} \tau \) and \( \lambda \in \partial V(\sigma) \) then there is hyperplane \( P \) defined by equations that have rational coefficients and such that \( P \) has positive co-dimension in \( V(\tau) \) with \( \lambda \in V(\tau) \cap P \) so if \( \lambda \) does not lie in any proper, rational hyperplane than we can apply Lemma A.4.

Proposition A.5. Let \( \tau \) be a birecurrent large train track and assume that \( \lambda \in V(\tau) \) is not contained in a rational hyperplane of positive co-dimension. Then either \( \tau \) contains an fd-path with central split \( \theta \) and \( \lambda \in V(\theta) \) or \( \lambda \) is an ending lamination.

Proof. By Lemma A.4 there exists a track \( \sigma \) such that \( \sigma \overset{ss}{\rightarrow} \tau \) and \( \sigma \) fully carries \( \lambda \). If \( \tau \) fully carries \( \sigma \) then it also fully carries \( \lambda \) and \( \lambda \) must be an ending lamination. If not a central split must occur in the sequence \( \sigma \overset{ss}{\rightarrow} \tau \). Let \( \tau_1 \) be the track in the sequence that occurs just before the central split and \( \theta_1 \) the central split of \( \tau_1 \) along the fd-path. By Lemma A.3 there exists an fd-path in \( \tau \) with central split \( \theta \) such that \( V(\theta_1) = V(\tau) \cap V(\theta) \) so \( \lambda \in V(\theta) \).

Corollary A.6. Let \( \tau \) be a birecurrent train track and \( a \in S(\tau) \) a simple closed curve. There exists a \( C = C(\Sigma) \) such that either \( d(B(\tau), a) \leq C \) or there exists a sequence \( \lambda_i \in P_\infty(\tau) \) such that \( a \) is contained in the Hausdorff limit of the \( [\lambda_i] \).

Proof. By Lemma A.2 \( \tau \) contains finitely many fd-paths. Assume there are \( k \geq 0 \) such paths. We begin by splitting on each of these paths to obtain \( k \) new tracks which we label \( \theta_1, \ldots, \theta_k \). If \( a \) is in the complement of \( \cup V(\theta_i) \) then the corollary follows from Proposition A.5 applied to a sequence of laminations in \( V(\tau) \) converging to \( a \) and not contained in proper rational planes. If not \( a \in S(\theta_i) \) for some \( i \). If \( \theta_i \) is small then \( d(B(\theta_i), a) \leq 2 \) and \( d(B(\theta_i), B(\tau)) \)
is uniformly bounded since an fd-path is at worst two-to-one by Lemma A.2. Therefore \( d(B(\tau), a) \) is uniformly bounded.

If \( \theta_i \) is large then it is connected and, by Lemma A.1, orientable. As in the previous paragraph we split along all fd-paths to get a collection of tracks \( \theta^1_i, \ldots, \theta^k_i \). Since \( \theta_i \) is orientable, Lemma A.1 implies that the \( \theta^j_i \) are disconnected and hence small. If \( a \in \theta^j_i \) for some \( j \) then \( a \) is uniformly close to \( B(\theta^j_i) \), and therefore to \( B(\tau) \), as above. If not, we again apply Proposition A.5.

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