The study of dynamics in general relativity has been hampered by a lack of coordinate independent measures of chaos. Here we present a variety of invariant measures for quantifying chaotic dynamics in relativity by exploiting the coordinate independence of fractal dimensions. We discuss how preferred choices of time naturally arise in chaotic systems and how the existence of invariant signals of chaos allow us to reinstate standard coordinate dependent measures. As an application, we study the Mixmaster universes and find it to exhibit transient soft chaos.

The last century has seen the emergence of three branches of physics which have combined to overthrow the Newtonian ideal of a clockwork universe. Our new world view encompasses an uneasy alliance of quantum mechanics, relativity and, to a lesser extent, chaos theory. One of the most remarkable aspects of these theories, in their current form, is their marked mutual incompatibility. The existence of a minimum area in the phase space of quantum mechanics, $\hbar$, suppresses chaos, while the ubiquity of chaos in classical systems demands a new description of the semi-classical regime. The privileged role played by time in both quantum mechanics and chaos theory has led to a head-on collision with general relativity where the choice of a time coordinate is arbitrary.

In this letter we offer at least a partial solution to one of these problems, namely, the classification and quantification of chaos in general relativity. Extensive discussion of the problem can be found in Ref. [1], so we shall limit our discussion to a brief review. In general relativity both space and time are dynamical and intermixed. There is no such thing as the time direction. In contrast, chaos theory has been developed for Newtonian dynamics where time and space are absolute and the notion of a mechanical phase space is clear. It may appear that the standard tools of chaos theory can be directly applied to relativity in schemes such as the ADM formalism [2] where an explicit space-time split is made. This is not the case. The coordinate freedom remains in the choice of lapse and shift functions [2] which describe the 3 + 1 decomposition. The fundamentally different role played by time in relativity and Newtonian mechanics manifests itself in the coordinate, or gauge, dependence of chaotic measures such as Lyapunov exponents [3]. Lyapunov exponents are the standard method for quantifying chaotic behaviour as they directly measure sensitive dependence on initial conditions. If two initially close trajectories separate along a given eigendirection as

$$ \varepsilon(t) = \varepsilon_0 e^{\lambda t}, $$

then $\lambda$ represents the Lyapunov exponent along that direction. If $\lambda > 0$ the system is said to exhibit sensitive dependence on initial conditions with a characteristic chaotic, or Lyapunov, timescale $T_L = 1/\lambda$. Unfortunately, this nice picture breaks down when applied to general relativity. Consider the allowed coordinate transformation $t \rightarrow \ln \tau$. In terms of this time variable we find

$$ \varepsilon(\tau) = \varepsilon_0 \tau^\lambda, $$

which describes the standard power-law divergence of trajectories found in integrable system. In particular, the Lyapunov exponents in this coordinate system would all be zero. It should be mentioned that the Lyapunov exponents also depend on the choice of distance measure in phase space and are therefore variant under spatial coordinate transformations also. From the above discussion it is clear that standard coordinate dependent measures of chaos have to be either modified, abandoned or augmented in general relativity [4].

In order to find methods suitable for classifying and quantifying chaos in general relativity we shall exploit the remarkable connection between chaos and fractal curves. The utility of these methods comes from the coordinate or diffeomorphism invariance of fractal dimensions. Fractal structures can be found in the phase space of all chaotic system. They may be uncovered by taking Poincaré sections, plotting attractor basin boundaries or by finding the intersections of stable and unstable phase space manifolds. These methods reveal respectively, Cantor sets, fractal basin boundaries [6] and chaotic invariant tori [5]. Actually, fractal basin boundaries are a particular type of chaotic invariant set. The connection between chaos and fractals is deep. A non-chaotic, integrable system has sufficient isolating integrals (constants of the motion) to fully determine the dynamics. The trajectories of an integrable system are restricted by these isolating integrals to lie on smooth manifolds in phase space with the topology of $n$-dimensional tori. In chaotic systems there are insufficient isolating integrals and the smooth tori are replaced by fractal cantori - locally the product of a torus and a Cantor set. The fractal dimension of a cantorus captures topological information [7] about a trajectory while Lyapunov exponents measure metrical properties. The importance of such topological information as a qualitative sign of chaos in general relativity has been emphasised by Calzetta and El Hasi [8] and Dettmann et. al. [9]. In what follows, we discuss quantitative re-
relationships that relate fractal dimensions to important quantities such as final state sensitivity, Lyapunov exponents and chaotic entropy.

A unified description of chaotic dynamics is possible in terms of chaotic invariant sets \cite{1}. Chaotic invariant sets are formed by the intersection of stable and unstable manifolds in phase space. The complex, and often fractal, nature of these sets follows from the result that if a stable and unstable manifold intersect once, they intersect and infinite number of times. Familiar examples of chaotic invariant sets are strange attractors and fractal basin boundaries. A lesser known example is the strange repeller responsible for chaotic scattering \cite{12,13}.

For low-dimensional systems an intriguing result has been found relating the fractal dimension of chaotic invariant sets to the Kolmogorov-Sinai entropy \cite{14} and Lyapunov exponents. The importance of these relations in general relativity follows from the diffeomorphism invariance of the multifractal dimensions \( D_q \). The fractal dimensions are defined by

\[ D_q = \frac{1}{q - 1} \lim_{\epsilon \to 0} \frac{\ln \sum_{i=1}^{N(\epsilon)} p_i^q}{\ln \epsilon}, \]

where \( N(\epsilon) \) are the number of hypercubes of side length \( \epsilon \) needed to cover the fractal and \( p_i \) is the weight assigned to the \( i^{th} \) hypercube. The \( p_i \)'s satisfy \( \sum_{i=1}^{N(\epsilon)} p_i = 1 \). The standard capacity dimension is recovered when \( q = 0 \), the information dimension when \( q = 1 \), the correlation dimension when \( q = 2 \) etc. For homogeneous fractals all the various dimension yield the same result. The multifractal dimensions \( D_q \) are invariant under diffeomorphisms for all \( q \) and \( D_1 \) is additionally invariant under coordinate transformations that are not-invertible at a finite number of points \cite{15}. If a strange attractor or repeller has a multifractal spectrum \( D_q \) with continuous first derivative \( \partial_q D_q \) it is said to be uniformly hyperbolic. Physically this means that all trajectories on the attractor or repeller are unstable. In practice, most attractors and repellers are not uniformly hyperbolic and exhibit a phase transition at some \( q = q_r \) where \( D_q \) has a discontinuous derivative. The breakdown of hyperbolicity for \( q > q_r \) is due to the presence of stable orbits in an otherwise chaotic system. Such systems are said to exhibit soft chaos.

Remarkably, it has been shown that \( D_1 \) equals the Lyapunov dimension \( D_L \) for 2-dimensional systems. The Lyapunov dimension is defined by

\[ D_L = h(\mu) \left( \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right), \]

where \( \lambda_1 > 0 > \lambda_2 \) are the Lyapunov exponents in each eigendirection and \( h(\mu) \) is the metric or K-S entropy of the chaotic invariant set with respect to the set’s natural measure \( \mu \) (see Ref. \cite{11} or Ref. \cite{13} for definitions of \( h(\mu), \lambda \) and \( \mu \)). The metric entropy is bounded from above by the topological entropy \( H \) \cite{16}. As its name suggests, the topological entropy is diffeomorphism invariant while the metric entropy can be transformed to zero by a coordinate change. In practice, the topological entropy is much harder to calculate than the metric entropy, but when it can be calculated, it provides a gauge invariant signal of chaos in addition to the \( D_q \)'s.

The relation \( D_1 = D_L \) has been rigorously established for certain dynamical systems \cite{17} and has been numerically confirmed for many typical systems \cite{6}. Similar relations have been conjectured to hold for \( n \)-dimensional systems \cite{18}. In Hamiltonian systems conservation of phase space volume implies \( \lambda_1 = -\lambda_2 \). For typical strange attractors and repellers the metric entropy is given by

\[ h(\mu) = \lambda_1 - \frac{1}{\tau_d}, \]

where \( \tau_d \) is the decay time for trajectories leaving the repeller. Strange repellers are found in systems displaying chaotic scattering, examples of which have been explored in general relativity \cite{10,19,20}. For dissipative systems with strange attractors \( \tau_d = \infty \) and (6) reduces to the Kaplan-Yorke relation \cite{13}. The Lyapunov dimension of a strange repeller is a particularly useful measure of chaos as it does not require a compact phase space. Most dynamical systems in general relativity do not have compact phase spaces.

With these preliminaries out of the way, we can now focus on the important implications eqn. (4) has for chaos in general relativity. The result is direct: although neither Lyapunov exponents nor K-S entropies are coordinate invariant, their ratio is. This result allows us to reinstate both Lyapunov exponents and K-S entropies as useful chaotic measures in a fixed gauge, so long as we first verify that \( D_1 \) has a fractional value. In practice, it is easy to reconstruct chaotic invariant sets numerically \cite{21} and measure their fractal dimension. We use the fractal value of \( D_1 \) to prove the system is chaotic. Of course, even for chaotic systems there are an infinite number of gauge choices in which the Lyapunov exponents and K-S entropy vanish. Therefore, as a physical choice, we exclude all coordinate choices where chaotic systems have \( D_1 = D_L = 0/0 \). On reflection, it is clear that a preferred time choice is a natural consequence of chaos as the K-S entropy selects a chaotic arrow of time. In the confines of a good gauge choice, we can compare quantities such as Lyapunov times to the characteristic timescales of other physical processes. In practice these are the physically important, albeit gauge variant, questions we need to ask.

A related method of choosing non-pathological gauges employs Poincaré sections. Regardless of the choice of gauge, chaotic trajectories will appear as fractal curves (cantori and stochastic layers) while stable trajectories will appear as smooth curves (KAM tori). Pathological gauge choices can be defined as those which have vanishing Lyapunov exponents for cantori or non-vanishing
exponents for KAM tori. At present there appears to be no way to quantitatively relate the fractal dimension of cantori to their Lyapunov exponents. Some preliminary results have recently been found relating the gap structure of cantori to their Lyapunov exponents \[24\], and hopefully new results will soon emerge to bolster our qualitative picture.

For those uncomfortable with preferred time choices, there is another method for quantifying chaos that is entirely gauge independent. A defining feature of chaotic systems is their sensitive dependence on initial conditions. Generally, a range of possible outcomes can be assigned to a dynamical system, and each outcome has a basin of attraction in the space of initial conditions. For chaotic systems the basin boundaries are fractal, and the fractal dimension of the boundary provides a coordinate independent measure of the chaotic dynamics \[10,20\]. The quantitative importance of the fractal dimension is expressed in terms of the final state sensitivity \[f(\delta)\] \[29\]. This quantity describes how the unavoidable uncertainty in specifying initial conditions gets amplified in chaotic systems, leading to a large final state uncertainty. The function \[f(\delta)\] is the fraction of phase space volume which has an uncertain outcome due to the initial conditions being uncertain within a hypersphere of radius \(\delta\). It can be shown \[23\] that

\[
 f(\delta) \sim \delta^\alpha, \quad \alpha = N - D_0,
\]

where \(N\) is the phase space dimension and \(D_0\) is the capacity dimension of the basin boundary. For non-chaotic systems \(\alpha = 1\) and there is no amplification of initial uncertainties, while for chaotic systems \(0 < \alpha < 1\) and marked final state sensitivities can occur. For example, if \(\alpha = 0.1\) a 50% reduction in the initial uncertainty only results in a 7% reduction in the final state uncertainty.

By reversing the argument, eqn. (6) can be used to quickly determine the dimension of the basin boundary \[23\].

One application for the gauge invariant measures \(D_1\), \(H\), and \(f(\delta)\) advanced in this letter might be to settle the long running debate over the existence of chaos in the Mixmaster universe \[1,16\]. At most, the Mixmaster will only exhibit transient chaos as its trajectories are asymptotically regular \[24\]. To show the Mixmaster is chaotic, we need only consider the reduction of the full dynamics to the discrete \((u,v)\) map, as this map accurately describes the majority of Mixmaster trajectories \[23\]. The map is defined by

\[
 (u_{n+1}, v_{n+1}) = \begin{cases} 
 (u_n - 1, v_n + 1) & u_n > 2, \\
 \left(\frac{1}{u_n - 1}, v_n\right) & 1 < u_n < 2. 
\end{cases}
\]

Using the standard metrical measures of chaos it is unclear whether or not the \((u,v)\) map is chaotic. For a finite number of iterations, \(n\), the map has a short-time Lyapunov exponent given by \(\lambda_n = \pi^2/ (6 \ln 2 \ln u_{\text{max}})\), where \(u_{\text{max}}\) is the largest value of \(u\) visited during the \(n\) iterations \[23\]. However, as we take the limit \(n \to \infty\) to recover the true Lyapunov exponent we find for typical trajectories that \(u_{\text{max}} \to \infty\) and \(\lambda = 0\). This regular asymptotic behaviour, combined with positive short-time Lyapunov exponents is the hallmark of chaotic scattering.

For the \((u,v)\) map the chaotic scattering occurs when \(1 < u < 2\). In order to prove that the \((u,v)\) map harbours a strange repeller we must reconstruct its chaotic invariant set. For a map this corresponds to the set of fixed points \((u_{n+k}, v_{n+k}) = (u_n, v_n)\) for all integers \(k\). By numerically generating this set we can find its spectrum of multifractal dimensions \(D_q\). Since the \((u,v)\) map is invertible, we need only consider the chaotic future invariant set generated by the \(u\) map. The topological entropy of the set is given by

\[
 H = \lim_{k \to \infty} \frac{1}{k} \ln \text{[No. of fixed points at order } k\text{]} = \ln 2.
\]

The result \(H = \ln 2\) follows from a simple counting argument. Since \(H > 0\) we are assured that the \((u,v)\) map is chaotic and harbours a strange repeller. It is interesting to note that the Mixmaster \((u,v)\) map has the same topological entropy as the Smale Horseshoe. As we would expect, the compactified version of the \((u,v)\) map, the Gauss map, has a considerably larger topological entropy of \(H_G = \pi^2 / 6 / (\ln 2)^2\) \[16\]. Numerically the topological entropy was found to converge to \(\ln 2\) very quickly with \(k\), differing by less than 1 part in 1000 for \(k \geq 10\). To be on the safe side, we chose the finite approximation \(k = \{1,15\}\) when calculating the multifractal dimensions. The task of calculating \(D_q\) is made easier by the dense nature of the repeller for small \(u\). We find the fraction of all fixed points in a given integer interval \([1, u]\) to be

\[
 F(u) = 1 - 2^{-u+1},
\]

so the core of the strange repeller is strongly localised around \(u = 1\). For this reason, we chose to measure the

\[
 D_q = \frac{1}{q} \int_{0}^{\infty} (\ln 2)^q \left(\frac{F(u)}{q}\right) du.
\]

\[\text{FIG. 1. The multifractal dimension } D_q \text{ as a function of } q.\]
fractal dimension in the interval \( u = [1..10] \) as it contains 99.8% of the repeller. The above approximations are particularly good for evaluating multifractal dimensions with positive \( q \) as these focus on the dense regions of a fractal. Conversely, our truncation to finite \( k \) and \( u \) will lead to large errors as \( q \to -\infty \). The spectrum of dimensions \( D_q \) for the \( u \) map is displayed graphically in Fig. 1. For reference, \( D_0 = 0.91 \pm 0.01 \) and \( D_1 = 0.872 \pm 0.005 \). If the map is hyperbolic at \( q = 1 \) we would expect to find \( D_1 = 1 - 1/(\lambda \tau_d) \), in accordance with eqn.\((\ref{eq:1})\). However, the decay of orbits from the repeller proceeds as

\[
N(n) = N_1 \exp(-n/\tau_d) + N_2(n+1)^{-a}.
\] (10)

The power-law tail is typical for repellers punctured by stable periodic orbits, and leads to what is known as soft chaos. This indicates that the repeller is not hyperbolic stable periodic orbits, and leads to what is known as soft chaos. This distinction will not concern us as only invariant under diffeomorphisms, not homeomorphisms. This distinction will not concern us as only invariant under diffeomorphisms, not homeomorphisms.

On a grander scale, it may be that chaos theory has a role to play in reconciling quantum mechanics and general relativity in the context of quantum cosmology \cite{24}, where chaos related phenomena such as decoherence, Fokker-Planck diffusion and dynamical arrows of time are thought to be important.

This work builds on my collaborations with Carl Dettmann, Sam Drake, Norm Frankel and Janna Levin. I would like to thank Albert Fathi, Robert MacKay and Edward Ott for answering several questions concerning cantori and dimensions.

\[\begin{align*}
\text{[1]} & \text{ Deterministic chaos in general relativity eds. D. Hobill, A. Burd and A. Coley, (Plenum Press, New York, 1994).} \\
\text{[2]} & \text{ R. Arnowitt, S. Deser and C. W. Misner, in Gravitation: An introduction to current research ed. L. Witten, (Wiley, New York, 1962).} \\
\text{[3]} & \text{ V. I. Oseledec, Trans. Moscow Math. Soc. 19, 197, (1968).} \\
\text{[4]} & \text{ S. E. Rugh, Cand. Scient. Thesis, The Niels Bohr Institute, (1990); S. E. Rugh and B. J. T. Jones, Phys. Lett. A147, 353, (1990); J. Pullin, Talk given at VII Simposio Latinoamericano de Relatividad y Gravitacion, (1990).} \\
\text{[5]} & \text{ S. Aubry, in Solitons and condensed matter physics ed. A. R. Bishop and T. Schneider, (Springer, New York, 1978); I. C. Percival, in Nonlinear dynamics and the beam-beam interaction ed. M. Month and J. C. Herrera, 57, (AIP Conference Proceedings, 1979).} \\
\text{[6]} & \text{ C. Grebogi, E. Ott and J. A. Yorke, Phys. Rev. Lett. 50, 935 (1983).} \\
\text{[7]} & \text{ H. Kantz and P. Grassberger, Physica D 17, 75 (1985); T. Bohr and D. Rand, Physica D 25, 387 (1987); G. H. Hsu, E. Ott and C. Grebogi, Phys. Lett. A127, 199 (1988). Z. Kovacs and L. Wiesenfeld, Phys. Rev. E51, 5476 (1995).} \\
\text{[8]} & \text{ The fractal dimension is not a true topological invariant as it is only invariant under diffeomorphisms, not homeomorphisms. This distinction will not concern us as we only need to be concerned about diffeomorphisms in general relativity.} \\
\text{[9]} & \text{ E. Calzetta and C. El Hasi, Class. Quant. Grav. 10, 1825 (1993).} \\
\text{[10]} & \text{ C. P. Dettmann, N. E. Frankel and N. J. Cornish, Phys. Rev. D60, R618 (1994); Fractals, 3, 161 (1995).} \\
\text{[11]} & \text{ E. Ott, Chaos in dynamical systems, (Cambridge University Press, Cambridge, 1993).} \\
\text{[12]} & \text{ P. Gaspard and S. A. Rice, J. Chem. Phys. 90, 2225 (1989); S. Bleher, C. Grebogi and E. Ott, Physica D 46, 87 (1990); O. Biham and W. Wenzel, Phys. Rev. Lett. 63, 819 (1989).} \\
\text{[13]} & \text{ A. J. Lichtenberg and M. A. Lieberman, Regular and chaotic dynamics, (Springer-Verlag, New York, 1992).} \\
\text{[14]} & \text{ A. N. Kolmogorov, Dokl. Akad. Nauk SSSR 98, 527 (1954); Ya. G. Sinai, Dokl. Akad. Nauk SSSR, 124, 768 (1959).} \\
\text{[15]} & \text{ E. Ott, W. D. Withers and J. A. Yorke, J. Stat. Phys. 36, 687 (1984).} \\
\text{[16]} & \text{ J. D. Barrow, Phys. Rep. 85, 1 (1982); Phys. Rev. Lett. 46, 963 (1981); D. F. Chernoff and J. D. Barrow, Phys. Rev. Lett. 50, 134 (1983).} \\
\text{[17]} & \text{ L. S. Young, Ergod. Th. Dynam. Syst. 2, 109 (1982); A. Fathi, Commun. Math. Phys. 126, 240 (1989).} \\
\text{[18]} & \text{ J. L. Kaplan and J. A. Yorke, in Functional differential equations and approximations of fixed points eds. H. O. Peitgen and H. O. Walter, (Springer, Berlin, 1979).} \\
\text{[19]} & \text{ G. Contopoulos, Proc. R. Soc. Lond. A431, 183 (1990); A435, 551 (1991).} \\
\text{[20]} & \text{ N. J. Cornish and J. J. Levin, preprint CWRU-P12-95, (1995).} \\
\text{[21]} & \text{ H. E. Nusse and J. Yorke, Physica D 36, 137 (1989).} \\
\text{[22]} & \text{ R. Coutinho, R. Lima, R. Vilela Mendes and S. Vaiencti, Ann. Inst. Henri Poincaré, 56, 415 (1992).} \\
\text{[23]} & \text{ S. W. McDonald, C. Grebogi, E. Ott and J. A. Yorke, Physica D 17, 125 (1985); C. Grebogi, E. Kestelich, E. Ott and J. A. Yorke, in Dynamical systems ed. J. C. Alexander (Springer-Verlag, Berlin, 1988).} \\
\text{[24]} & \text{ P. K. -H. Ma and J. Wainwright in Ref. \cite{1}.} \\
\text{[25]} & \text{ B. K. Berger, Gen. Rel. Grav. 23, 1385 (1991); Phys. Rev. D47, 3222 (1993).} \\
\text{[26]} & \text{ L. Diosi, N. Gisin, J. Halliwell and I. C. Percival, Phys. Rev. Lett. 74, 203 (1995).} \]