BOSON-FERMION CORRESPONDENCE FROM FACTORIZATION SPACES

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Abstract. We give a proof of the boson-fermion correspondence (an isomorphism of lattice and fermion vertex algebras) in terms of isomorphism of factorization spaces.

0. Introduction

0.1. The notion of factorization space is a non-linear analog of the factorization algebra, which was introduced by Beilinson and Drinfeld [BD04] in their theory of chiral algebras developing a geometric framework of vertex algebras and the study of geometric Langlands correspondence [BD]. A factorization space over a scheme $X$ is a family of ind-schemes $G_I$ over $X^I$ for each finite set $I$ with “factorization structure” given by isomorphisms between these $G_I$’s. Given a factorization space on $X$, one can apply a linearization procedure, and if $X$ is a smooth curve then one obtains a factorization algebra, an equivalent notion of chiral algebra.

Although the formulation of factorization space looks highly complicated at first glance, it fits moduli problems over algebraic curves very well. It captures the intimate connection between two-dimensional quantum conformal field theories and moduli problems on curves.

Along this line, one can ask a question: are there enhanced versions of properties of vertex algebras in the level of factorization spaces? In this note, we consider this kind of problem for the boson-fermion correspondence.

Recall that the two-dimensional boson-fermion correspondence is stated as an isomorphism

$$V_Z \cong \bigwedge$$

between the lattice vertex algebra $V_Z$ attached to lattice $Z$ and the free fermion vertex super algebra $\bigwedge$. Here we borrowed the notations in [FB04]. The essence of the correspondence is that the vertex operators

$$V_\varphi(z) := e^{\varphi(z)}, \quad \varphi(z) := q + a_0 \log(z) - \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{a_n}{n} z^{-n}$$

acting on $V_Z$ obey the same commutation relation with the fermionic operators $\psi(z), \psi^*(z)$ acting on $\bigwedge$. Here $a_n$’s and $q$ denote the Heisenberg generators with the commutation relation $[a_m, a_n] = m \delta_{m+n, 0}$ and $[q, a_n] = \delta[n, 0]$. Thus schematically we have

$$V_+(z) \leftrightarrow \psi(z), \quad V_-(z) \leftrightarrow \psi^*(z)$$

of vertex operators. See [FB04, §5.3] for the detail.

Our main statement in this note is Theorem 5.1 giving an isomorphism

$$G_r(X, Z) \cong G_r(X, SC(2))$$

between factorization spaces over a smooth curve $X$. Here $G(X, Z)$ is the factorization space arising from the Picard scheme (moduli space of line bundles) on $X$. $G_r(X, SC(2))$ is the Beilinson-Drinfeld Grassmannian for the Clifford group $SC(2)$.

These factorization spaces are equipped with factorization super line bundles respectively, and the twisted linearization procedure give the lattice and Clifford chiral algebras. These chiral algebras correspond to the vertex algebras $V_Z$ and $\bigwedge$ respectively. Thus the isomorphism (0.3) can be considered as an enhanced version of the boson-fermion correspondence (0.1).

0.2. Organization. Let us explain the organization of this article.

In §1 we recall the definition of factorization spaces, and factorization line bundles. We also recall the Beilinson-Drinfeld Grassmann $G_r(X, G)$ for a smooth algebraic curve $X$ and a reductive group $G$, the standard example of factorization space,

§2 reviews the linearization procedure which produces a chiral algebra from factorization space.

In §3 we recall the factorization space $G(X, Z)$ arises from the Picard functor. By the linearization procedure it gives the lattice vertex algebra $V_Z$.

In §4 we consider the factorization space $G_r(X, SC(Q))$ for the Clifford group $SC(Q)$ attached to the non-degenerate quadratic form $Q$. We explain that by the linearization procedure it yields the Clifford chiral algebra.

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The final §5 gives the proof of the main theorem. The proof is not difficult, and the key observation is that we have a natural identification of moduli stacks

\[ M(X, SC(Q)) \cong \text{Pic}(X) \times \text{Pic}(X) \]

of the moduli space of SC(Q)-bundles on X with two pieces of Picard varieties. It reflects the correspondence (0.2) of vertex operators \((V_+, V_-) \leftrightarrow (\psi, \psi^*)\).

The ingredients in §§1–3 are basically known facts, and the main sources of our presentation are [BD04, FB04]. The content in §4 is also a variant of [FB04, Chap. 20].

0.3. Notation. We will work over the field \(\mathbb{C}\) of complex numbers unless otherwise stated. The symbol \(\otimes\) denotes the tensor product of \(\mathbb{C}\)-linear spaces.

For a scheme or an algebraic stack \(Z, \mathcal{O}_Z, \Theta_Z, \Omega_Z\) and \(D_Z\) denote the structure sheaf, the tangent sheaf, the sheaf of 1-forms and the sheaf of differential operators on \(Z\) respectively (if they are defined). By “an \(\mathcal{O}\)-module on \(Z\)” we mean a quasi-coherent sheaf on \(Z\). By “a \(D\)-module on \(Z\)” we mean a sheaf of right \(D_Z\)-modules quasi-coherent as \(\mathcal{O}_Z\)-modules.

For a morphism \(f : Z_1 \to Z_2\), the symbols \(f^!\) and \(f\) denote the inverse and direct image functors of \(\mathcal{O}\)-modules respectively.

Finally we will use the symbols \(\mathcal{O} := k[[t]]\) for the algebra of formal series and \(\mathcal{K} := k((t))\) for the field of formal Laurent series.

1. Factorization space

We follow [BD04, §3.10.16], [FB04, §20.4.1], [G99, Chap. 5] and [KV04].

1.1. The category \(\mathcal{S}\) of finite sets and surjections. We will repeatedly use the following category of sets.

**Definition 1.1.** Let \(\mathcal{S}\) be the category of finite non-empty sets and surjections. For \(\pi : J \twoheadrightarrow I\) in \(\mathcal{S}\) and \(i \in I\) we set \(J_i := \pi^{-1}(i) \subset J\).

Let \(X\) be a smooth complex curve as before. For \(\pi : J \twoheadrightarrow I\) in \(\mathcal{S}\), denote by

\[ \Delta^{(\pi)} \equiv \Delta^{(J/I)} : X^I \hookrightarrow X^J \]

an embedding of the locus such that \(x_j = x_j'\) if \(\pi(j) = \pi(j')\) for \(j, j' \in J\). Also set

\[ U^{(\pi)} \equiv U^{(J/I)} := \{(x_j)_{j \in J} \in X^J | x_j \neq x_j' \text{ if } \pi(j) \neq \pi(j')\}. \]

One can also see that \(U^{(\pi)}\) is the complement of the diagonals that are transversal to \(\Delta^{(\pi)} : X^I \hookrightarrow X^J\). Denote the open embedding \(U^{(\pi)} \subset X^J\) by

\[ j^{(\pi)} \equiv j^{(J/I)} : U^{(\pi)} \hookrightarrow X^J. \]

1.2. Recollection on ind-schemes. Before introducing the factorization spaces, we need to recall the notion of ind-schemes, since otherwise we have no good examples. We follow [KV04, §1.4] for the presentation.

For any category \(\mathcal{C}\), an ind-objects of \(\mathcal{C}\) is a filtering inductive system over \(\mathcal{C}\). Ind-objects form a category where a morphism is a collection of morphisms between the objects in the inductive systems satisfying some compatibility conditions. An ind-object will be represented by the symbol “\(\varinjlim_n C_n\)” with \(C_n \in \mathcal{C}\).

Denote by \(\mathcal{S}ch\) the category of separated schemes over \(\mathcal{C}\). An ind-scheme is an ind-object of \(\mathcal{S}ch\) represented by an inductive system of schemes. Formal schemes are ind-schemes, like

\[ \text{Spf } k[[t]] = \text{“} \varprojlim_n \text{Spec } k[[t]]/(t^{n+1}). \]

A strict ind-scheme is an ind-scheme with an inductive system given by closed embeddings of quasi-compact schemes.

Recall that a scheme \(S\) is equivalent to the functor of points \(\text{Spec } k : T \mapsto \text{Hom}_{\mathcal{S}ch}(T, S)\). This functor is a sheaf of sets on \(\mathcal{S}ch\) which is seen as the Zariski site. Calling such a sheaf a \(\mathcal{C}\)-space. An ind-scheme is a \(\mathcal{C}\)-spaces represented by an inductive system of schemes.

One can define an ind-scheme over a scheme \(Z\) similarly by replacing the category \(\mathcal{S}ch\) by the category \(\mathcal{S}ch_Z\) of schemes over \(Z\). For a morphism \(f : Z_1 \to Z_2\) of schemes, denote by \(f_*\) and \(f^*\) the push-forward and the pull-back of ind-schemes over \(Z_1/\text{s}’s\).

1.3. Definition of factorization space. Now we turn to our main object, the factorization space. It is a non-linear analog of factorization algebras, and may be considered as sheaves on the Ran space [R00] (see also [BD04, Chap. 4]). Recall once again the category \(\mathcal{S}\) in Definition 1.1.

**Definition.** Let \(X\) be a scheme. A factorization space \(\mathcal{S}\) on \(X\) consists of the following data.

- A formally smooth ind-scheme \(\mathcal{S}_I\) over \(X^I\) for each \(I \in \mathcal{S}\).
- An isomorphism

\[ \nu^{(\pi)} \equiv \nu^{(J/I)} : \Delta^{(\pi)*}\mathcal{S}_J \xrightarrow{\sim} \mathcal{S}_I \]

of ind-schemes over \(X^I\) for each \(\pi : J \twoheadrightarrow I\) in \(\mathcal{S}\).
• An isomorphism
\[ \kappa(\pi) \equiv \kappa^{(J/I)} : \left( \prod_{i \in I} G_{J_i} \right) \overset{\sim}{\rightarrow} j^{(\pi)} \ast \mathcal{O} \]
of ind-schemes over \( U(\pi) \) for each \( \pi : J \rightarrow I \) in \( \mathcal{S} \), called the factorization isomorphism.

These should satisfy the following compatibility conditions.
1. \( \nu(\pi) \)'s are compatible with compositions of surjections \( \pi \).
2. For any \( \pi : J \rightarrow I \) and \( \rho : K \rightarrow J \), we should have
\[ \kappa^{(K/J)} = \kappa^{(K/I)} \circ (\mathcal{O}_{K/I} \mathcal{O}) \]
3. For any \( J \rightarrow I \) and \( K \rightarrow J \), we should have
\[ \nu^{(K/J)}(\Delta^{(K/J)}(\kappa^{(K/I)})) = \kappa^{(J/I)}(\mathcal{O}_{K/I} \mathcal{O}) \]

A factorization space \( \mathfrak{S} \) is attached with the structure morphism \( \mathfrak{r} : \mathfrak{S} \rightarrow X \) of ind-schemes.

Let us also introduce the object corresponding to the unit of factorization algebra.

**Definition.** A **unit** of a factorization space \( \mathfrak{S} \) on a scheme \( X \) is the data of morphisms
\[ u^{(I)} : X^I \rightarrow \mathfrak{S}_I \]
of ind-schemes for each \( I \in \mathcal{S} \) such that for any morphism \( f : U \rightarrow \mathfrak{S}_{\{1\}} \) with open \( U \subset X \), \( u^{(1)} \mathcal{O} f \), which can be seen as a morphism \( U^2 \setminus \Delta \rightarrow \mathfrak{S}_{\{1,2\}} \) by \( \kappa^{(1,2)} = u^{(1)} \mathcal{O} \), extends to a morphism \( U^2 \rightarrow \mathfrak{S}_{\{1,2\}} \), and \( \Delta^{(\pi)}(u^{(1)} \mathcal{O} f) = f \).

**Fact 1.2.** A factorization space \( \mathfrak{S} \) on \( X \) together with a unit has a connection along \( X \).

Let us explain precisely what a connection on \( \mathfrak{S} \) along \( X \) is. Assume that we are given
- \( I \) an Artin scheme \( I \) of length 1
- a morphism \( f : I \times S \rightarrow X \) of schemes with \( S \) a scheme,
- \( g_0 : I_0 \times S \rightarrow \mathfrak{S} \) of ind-schemes with \( I_0 := I_{\text{red}} \simeq \text{Spec}(k) \) the reduced scheme of \( I \)

such that \( r^{(I)} \circ g_0 : I_0 \times S \rightarrow \mathfrak{S} \rightarrow X \) coincides with the composition \( I_0 \times S \rightarrow I \times S \rightarrow X \). A connection on \( \mathfrak{S} \) along \( X \) is equivalent to the property that for given \( (I, f, g_0) \) there is a map \( g : I \times S \rightarrow \mathfrak{S} \) such that \( r \circ g = f \) extending \( g_0 \).

**Definition.** A factorization space with a unit will be called a factorization monoid.

We followed [KV04] for the terminology. In [BD04, §3.10.16] factorization monoid is called chiral monoid. We also need a line bundle living over factorization space.

**Definition 1.3.** Let \( \mathfrak{S} \) be a factorization space over \( X \). A factorization line bundle \( L \) over \( \mathfrak{S}_I \) is a collection of line bundles \( L_I \) on \( \mathfrak{S}_I \) together with isomorphisms
\[ j^{(J/I)} L_I \overset{\sim}{\rightarrow} j^{(J/I)} \mathcal{O} \]
of ind-schemes over \( U^{(J/I)} \) which should satisfy the factorization property.

For later use we will name a standard example of factorization space, namely the Beilinson-Drinfeld Grassmannian \( \mathfrak{B}(X, G) \). Here \( G \) is a reductive group and \( X \) is a smooth algebraic curve. As a preparation we recall some facts on the moduli space of \( G \)-bundles on an algebraic curve.

1.4. **Moduli space of \( G \)-bundles on curve.** Let \( G \) be a reductive algebraic group over \( \mathbb{C} \). Recall that the **affine Grassmannian**
\[ G(\mathbb{C})/G(\mathbb{O}) = G(k[[z]])/G(k[[z]]) \]
can be considered as the moduli space of \( G \)-bundles on the disc \( D = \text{Spec}(\mathbb{O}) \) together with a trivialization on \( D^\times = \text{Spec}(\mathbb{C}) \). Strictly speaking, \( G(\mathbb{C})/G(\mathbb{O}) \) is an ind-scheme, and \( G(\mathbb{C})/G(\mathbb{O}) \) is a formally smooth strict ind-scheme.

Denote by \( \mathcal{M}(X, G) \) the category of \( G \)-torsors on \( X \). By the result of [DS95], we have

**Fact 1.4.** Let \( X \) be a smooth algebraic curve and \( x \in X \) be a point.
1. A choice of local coordinate \( z \) at \( x \) gives an identification
\[ \mathfrak{B}(X, G)_z := \{ (P, \varphi) \mid P \in \mathcal{M}(X, G), \varphi : \text{trivialization of } P|_{X \setminus \{x\}} \rightarrow G(\mathbb{C})/G(\mathbb{O}) \} \]
2. If \( G \) is semi-simple, then any \( G \)-bundle on \( X \setminus \{x\} \) is trivial.

Let \( \mathfrak{M}(X, G) \) be the moduli stack of \( G \)-torsors on a smooth projective curve \( X \). We have a natural morphism \( \mathfrak{B}(X, G)_x \rightarrow \mathfrak{M}(X, G) \) by forgetting the trivialization \( \varphi \). If \( G \) is semi-simple, then Fact 1.4 implies the following adelic description of \( \mathfrak{M}(X, G) \).
\[ \mathfrak{M}(X, G) \simeq G(\mathbb{Q}_x)^{\text{out}} \times G(\mathbb{Q}_x)/G(\mathbb{O}_x). \]
Here \( G(\mathbb{Q}_x)^{\text{out}} \) denotes the space of regular functions \( X \setminus \{x\} \rightarrow G \), which is naturally a subgroup of \( G(\mathbb{Q}_x) \).
1.5. **Beilinson-Drinfeld Grassmannian.** Let \( X \) be a smooth algebraic curve as before. Denote by \( \text{Sch} \) the category of schemes over \( \mathbb{C} \).

For \( I \in \mathbb{S} \), consider the functor which maps \( S \in \text{Sch} \) to the data \((f^I, \mathcal{P}, \varphi)\) consisting of:

- a morphism \( f^I : S \to X^I \) of schemes
- a \( G \)-torsor \( \mathcal{P} \) on \( S \times X \)
- a trivialization \( \varphi \) of \( \mathcal{P} \) on \( S \times X \setminus \{ \Gamma(f^I_i) \}_{i \in I} \), where \( f^I_i : S \to X \) is the composition of \( f^I \) with the \( i \)-th factor projection \( X^I \to X \), and \( \Gamma(s) \subset S \times X \) is the graph scheme of morphism \( s : S \to X \).

We have a natural morphism

\[
\mathcal{P} : \text{Sch} \to \text{Sch}
\]

The collection \( \{ \mathcal{P}, \{ x_i \}_{i \in I}, \varphi \} \) is the fiber of \( \mathcal{P} \setminus \{ \Gamma(f^I_i) \}_{i \in I} \).

By Fact 1.4, this functor can be represented by an ind-scheme \( \mathcal{G}(X, G) \). For \( S = \text{Spec} \, k \), \( \mathcal{G}(X, G) \) is identified with

\[
\{ (\mathcal{P}, \{ x_i \}_{i \in I}, \varphi) \mid \mathcal{P} \in \mathcal{M}_G(X), x_i \in X, \varphi : \text{trivialization of } \mathcal{P} \}_{\{ x_i \}_{i \in I}}.
\]

We have a natural morphism

\[
r^{(I)} : \mathcal{G}(X, G) \to X^I,
\]

and the space \( \mathcal{G}(X, G) \) in (1.1) is the fiber of \( r^{(I)} \) with \( I = \{ 1 \} \).

**Fact 1.5.** The collection

\[
\mathcal{G}(X, G) := \{ \mathcal{G}(X, G)r \}_{I \in \mathbb{S}}
\]

has a structure of factorization monoid on \( X \), called the **Beilinson-Drinfeld Grassmannian**.

Here we explain the factorization isomorphism only for \( I = \{ 1, 2 \} \). Denote by \( \mathcal{G}(x_i) \) the fiber of \( r^{(I)} \) over \( \{ x_i \}_{i \in I} \subset X^I \). By Fact 1.4, in the case \( |I| = 1 \), we have

\[
\mathcal{G}(x) \simeq G(\mathbb{K})/G(0).
\]

If \( x_1 \neq x_2 \), then \( \{ x_1, x_2 \} \subset X^2 \setminus \Delta \), and we have a morphism \( \mathcal{G}(x_1, x_2) \to \mathcal{G}(x_1) \times \mathcal{G}(x_2) \) by restricting the data to \( X \setminus \{ x_i \} \). We also have the other direction map \( \mathcal{G}(x_1) \times \mathcal{G}(x_2) \to \mathcal{G}(x_1, x_2) \) by gluing the \( G \)-bundles over \( X \setminus \{ x_1, x_2 \} \).

The unit on \( \mathcal{G}(X, G) \) is given by the trivial \( G \)-bundle. Namely, define \( u^{(I)} : X^I \to \mathcal{G}(X, G) \) by setting \( u^{(I)}(\{ x_i \}_{i \in I}) \) to be the trivial \( G \)-bundle with the obvious trivialization away from \( x_i \)’s.

\( \mathcal{G}(X, G) \) has a natural factorization line bundle. Assume the Lie algebra \( \text{Lie}(G) \) is equipped with an invariant inner product. It induces a central extension

\[
1 \to \mathbb{G}_m \to \mathcal{G} \to G(\mathbb{K}) \to 1
\]

of algebraic group ind-schemes, and hence a \( \mathbb{G}_m \)-torsor

\[
\mathcal{G}(\mathbb{K})/G(0) \to G(\mathbb{K})/G(0)
\]

over the affine Grassmannian. It defines a factorization line bundle \( \mathcal{L}(G) \) over \( \mathcal{G}(X, G) \).

2. **Linearization**

2.1. **Recollection on \( \mathcal{D} \)-modules.** Let us collect some notations on \( \mathcal{D} \)-modules.

Let \( Z \) be a smooth algebraic variety over \( \mathbb{C} \). Recall that the canonical sheaf

\[
\omega_Z := \wedge^{\dim_Z} \Omega_Z
\]

has a natural right \( \mathcal{D} \)-module structure determined by

\[
\nu \cdot \tau := -\mathcal{L}\text{ie}_\tau(\nu), \quad \nu \in \omega_Z, \quad \tau \in \Theta_Z \subset \mathcal{D}_Z
\]

where \( \mathcal{L}\text{ie}_\tau \) is the Lie derivative with respect to \( \tau \). Then the categories of left and right \( \mathcal{D} \)-modules are equivalent under the functor

\[
\mathcal{L} \mapsto \mathcal{L}^r := \omega_Z \otimes_{\mathcal{O}_Z} \mathcal{L},
\]

where the structure of right \( \mathcal{D} \)-module on \( \mathcal{L}^r \) is determined by

\[
(\nu \otimes l) \cdot \tau := (\nu \cdot \tau) \otimes l - \nu \otimes (\tau \cdot l).
\]

The inverse functor is given by

\[
\mathcal{M} \mapsto \mathcal{M}^l := \omega_Z^{-1} \otimes_{\mathcal{O}_Z} \mathcal{M}.
\]  

(2.1)

Hereafter the term a “\( \mathcal{D} \)-module” means a right \( \mathcal{D} \)-module.
2.2. Linearization of factorization spaces. Let $\mathcal{G}$ be a factorization monoid over a smooth variety $X$. Recall the morphisms $r^{(I)} : \mathcal{G}_I \to X$ and $u^{(I)} : X \to \mathcal{G}_I$. Now consider the $O$-module

$$A_{\mathcal{G},I} := r^{(I)}_*, u^{(I)}_! \omega_{X,I}.$$  

This sheaf can be considered as the space of delta functions on $\mathcal{G}_I$ along the section $u^{(I)}(X^I)$.

The connection on $\mathcal{G}$ along $X$ given in Fact 1.2 defines a right $D$-module structure on $A_{\mathcal{G},I}$ and the section $u^{(I)}$ defines an embedding $\omega_{X,I} : \to A_{\mathcal{G},I}$. Then the axiom of factorization space implies

**Fact 2.1.** If $X$ is a smooth curve, then the collection $\{A_{\mathcal{G},I}^a\}_{I \in S}$ has a structure of factorization algebra on $X$, and hence $A_{\mathcal{G},\{1\}}$ has a structure of chiral algebra on $X$.

Here we used the notation (2.1). We also omit the definition of chiral and factorization algebras. See [BD04, §§3.1–3.4]. We call the obtained chiral algebra $A_{\mathcal{G},\{1\}}$ the *chiral algebra associated to $\mathcal{G}$.*

Let us explain the twisted version of this construction. Assume $X$ is a curve and a factorization space $\mathcal{G}$ over $X$ is equipped with a factorization line bundle $L = \{L_I\}$. Then consider the collection of $D$-modules

$$A_{\mathcal{G},I}^\ell := r^{(I)}_* (L_I \otimes_{\mathcal{G}_I} u^{(I)}_! (\omega_{X,I})).$$

The construction in Fact 2.1 can be applied to this family of $L$-twisted sheaves. The result is

**Fact 2.2.** Assume that $\mathcal{G}$ is a monoid over a smooth curve $X$, and that a factorization linear bundle $L$ over $\mathcal{G}$ is given. Then $A_{\mathcal{G},\{1\}}^\ell$ has a structure of chiral algebra on $X$.

We call the resulting chiral algebra the *$L$-twisted chiral algebra associated to $\mathcal{G}$.*

3. Lattice chiral algebra

Now we recall the factorization monoid arising from the Picard functor whose associated chiral algebra is the lattice chiral algebra [BD04, §3.10], [G99, Chap. 6], [FB04, §§20.4.4–9].

Following the symbols in [FB04, Chap. 5], let $V_{\mathbb{Z}} = \oplus_{n \in \mathbb{Z}} \pi_n$ be the lattice vertex algebra associated to the lattice $\mathbb{Z}$. Each component $\pi_n$ is the Fock representation of the Heisenberg algebra generated by $\{a_n \mid n \in \mathbb{Z}\}$ and 1 with the defining relation $[a_m, a_n] = m \delta_{m+n,0}$. Recall that $V_{\mathbb{Z}}$ is actually a vertex super algebra.

There is a universal chiral algebra $\mathcal{A}_{\mathbb{Z}}$ in the meaning of [BD04, §3.3.14] corresponding to the lattice vertex algebra $V_{\mathbb{Z}}$. We call it the *lattice chiral algebra.*

3.1. Some super language. Below we will use some super language, so let us collect the notation.

For a commutative ring $R$, a super $R$-module means a $\mathbb{Z}/2\mathbb{Z}$-graded $R$-module. Denote by $p$ the parity of homogeneous elements of $M$. Consider the tensor product of super $R$-modules with the commutativity constraint $a \otimes b = (-1)^{p(a)p(b)} b \otimes a$. A super $R$-line is an invertible object of the monoidal category formed by super $R$-modules with the tensor product described as above. Super $R$-lines constitute a Picard groupoid.

One can consider a similar setting for any $C$-linear category $M$. Namely a $\mathbb{Z}/2\mathbb{Z}$-graded object form a $C$-category $M^*$, and if $M$ is a monoidal category, then $M^*$ is also a super monoidal category with Koszul rule of sags as described above.

Below we will use super line bundles on a scheme (or stack) $X$. It is nothing but a super object in the monoidal category $\operatorname{Pic}(X)$. In particular, replacing line bundles by super line bundles in Definition 1.3, we have the notion of *factorization super line bundle* on a factorization space.

3.2. Abelian Beilinson-Drinfeld Grassmannian and lattice chiral algebras. Consider the Beilinson-Drinfeld Grassmannian $\mathfrak{G}(X, G)$ with $G = \operatorname{GL}(1)$. Since $\operatorname{GL}(1)$-torsor is nothing but a line bundle, we have $\mathfrak{G}(X, \operatorname{GL}(1)) = \operatorname{Pic}(X)$. For $I \in S$, we also have

$$\mathfrak{G}(X, \operatorname{GL}(1))_I = \{ (\mathcal{L}, S, \varphi) \mid \mathcal{L} \in \operatorname{Pic}(X), S \in X^I, \varphi : \text{trivialization of } \mathcal{L}|_{X \setminus S} \}. $$

One can naturally construct a factorization super line bundle on $\mathfrak{G}(X, \operatorname{GL}(1))$ by the theta line bundle $\Theta$ on $\operatorname{Pic}(X)$. Choosing a theta characteristic $\Omega_X^{1/2}$, we can describe the fiber of $\Theta$ on $\mathcal{L} \in \operatorname{Pic}(X)$ as

$$\Theta|_\mathcal{L} \simeq \det H^*(\mathcal{L} \otimes_{\operatorname{O}_X} \Omega_X^{1/2}).$$  

(3.1)

Considered as an odd line bundle on $\operatorname{Pic}(X)$, it induces a factorization super line bundle on $\mathfrak{G}(X, \operatorname{GL}(1))$. Denote it by the same letter $\Theta = \{ \Theta_I \}$.

Recall that the chiral algebra associated to a factorization monoid is constructed along the unit $u^{(1)} : X \to \mathcal{G}_{\{1\}}$. In the present situation $\mathcal{G} = \mathfrak{G}(X, \operatorname{GL}(1))$, the unit is given by $x \mapsto (\mathcal{O}_X, \varphi)$ with $\varphi$ the obvious trivialization away from $x$. Now we modify this construction and consider

$$u(n)_x : X \longrightarrow \mathfrak{G}(X, \operatorname{GL}(1))_x, \quad x \mapsto (\mathcal{O}_X(nx), \varphi)$$
for $n \in \mathbb{Z}$. It induces $u(n)^{(1)} : X \to \mathcal{O}(X, GL(1))^{(1)}$ and $u(n)^{(I)} : X \to \mathcal{O}(X, GL(1))^{I}$ for any $I \in \mathbb{N}$. Now define the $\mathcal{O}$-module

$$A_{Z,I} := \bigoplus_{n \in \mathbb{Z}} r_n^{(I)} \left( \Theta_I \otimes u(n)^{(I)}(\omega_X) \right).$$

The factorization structure makes $A_{Z,(1)}$ into a chiral algebra. Denote it by $A_{Z}$.

**Fact 3.1.** The chiral algebra $A_Z$ is isomorphic to the lattice chiral algebra.

For later use let us restate this result. Set

$$\mathcal{O}(X,Z) := \{ (L \otimes \mathcal{O}_X(S), S, \varphi) \mid L \in \text{Pic}(X), S \in X^I, \varphi : \text{trivialization of } L|_{X \setminus S} \}$$

Here $\mathcal{O}_X(S)$ means $\mathcal{O}_X(nS)$ for some $n \in \mathbb{Z}$. Then $\mathcal{O}(X,Z)$ is a factorization monoid with the unit $u : X \to \mathcal{O}(X,Z)$ given by the trivial line bundle. $\Theta$ is a factorization super line bundle on $\mathcal{O}(X,Z)$. Now $A_Z$ is isomorphic to the $\Theta$-twisted chiral algebra associated to $\mathcal{O}(X,Z)$. Thus Fact 3.1 can be restated as

**Proposition 3.2.** The $\Theta$-twisted chiral algebra associated to $\mathcal{O}(X,Z)$ is isomorphic to the lattice chiral algebra.

**Remark.** The $\Theta$-twisted chiral algebra associated to $\mathcal{O}(X, GL(1))$ is nothing but the Heisenberg chiral algebra $\pi_0^{ch}$, and the corresponding vertex algebra $\pi_0$ is the 0-th part of the lattice vertex algebra $V_Z = \oplus_{n \in \mathbb{Z}} V_n$.

4. **Clifford chiral algebras**

4.1. **Clifford bundles.** Here we collect some notations on the Clifford algebra. See [Bo, §IX.9] for the detail.

Let us denote by $\mathfrak{c}(Q)$ the Clifford algebra associated to non-degenerate quadratic form $Q$ on a linear space $V$ of finite dimension. Denote by $\mathfrak{c}^+(Q)$ the even part and $\mathfrak{c}^-(Q)$ the odd part of $\mathfrak{c}(Q)$. The involution $u \mapsto \pm u$ for $u \in \mathfrak{c}^+(Q)$ is denoted by $\alpha$. The Clifford group $C(Q)$ and the special Clifford group $SC(Q)$ are defined to be

$$C(Q) := \{ u \in \mathfrak{c}^+(Q) \mid \alpha(u)u^{-1} \subset V \}, \quad SC(Q) := C(Q) \cap \mathfrak{c}^+(Q),$$

where $\mathfrak{c}^+(Q)$ is the group of units in $\mathfrak{c}(Q)$. SC($Q$) is a connected reductive group. The spinor norm is denoted by $N : SC(Q) \to \mathbb{C}^\times$.

Using the notations in §1.4, we have a moduli stack $\mathfrak{M}(SC(Q), X)$ of $SC(Q)$-torsors on a smooth algebraic curve $X$. Applying Fact 1.5 to $G = SC(Q)$, we have a factorization monoid $\mathcal{O}(X, SC(Q))$. The spinor norm $N$ on $SC(Q)$ induces a factorization super line bundle $\mathcal{L}(SC(Q))$ by a similar argument as in the latter part of §1.5.

4.2. **Clifford chiral algebras.** Let us introduce the Clifford chiral algebra following [BD04, §3.8.6] with minor modifications.

Let $V, V'$ be vector $D_X$-bundles (locally projective right $D_X$-module of finite rank) and assume that we have a skew-symmetric pairing $(\langle , \rangle) \in P_2^+ \{ (V, V'), \omega_X \}$ in the $*$-pseudo-tensor category [BD04, Chap. 2] such that it vanishes on $V$ and $V'$ respectively. Then $\nabla \otimes V' \oplus \omega_X$ is a Lie algebra with the commutator $[\langle , \rangle]$. The resulting chiral algebra is denoted by $W^{ch}(V, V', \langle , \rangle)$.

For a positive integer $n$, let $V_n := \mathbb{Q}[\mathbb{Z}]$ considered as $D$-module, and $V := V_0 \otimes_D \mathbb{A}^X$ considered as $D$-module. Set $V^0 := \mathcal{H}om^0((V, \mathcal{O}_X))$ with $*$ denoting the linear dual. As in [BD04, §2.2.16], the linear pairing induces a non-degenerate pairing $(\langle , \rangle) \in P_2^+ \{ (\mathbb{C}[1], V^0 [-1]), \omega_X \}$ with $[\pm 1]$ the shift as complexes. Now we set

$$\mathfrak{c}^{ch}(n) := W^{ch}(V^0[1], V^0 [-1], \langle , \rangle)$$

and call it the fermionic chiral algebra. Thus, strictly speaking, it is a chiral super algebra and generators behave as odd elements. It is a universal chiral algebra associated to the free fermionic vertex algebra $\Lambda$.

4.3. Let us set $SC(m) := SC(Q)$ with $Q$ the standard quadratic form $z_1^2 + \cdots + z_m^2$ on $\mathbb{C}^m$. Then we have the factorization monoid $\mathcal{O}(X, SC(m))$ with the factorization super line bundle $\mathcal{L}(SC(m))$. Let us denote by $A(SC(2m))$ the $\mathcal{L}(SC(m))$-twisted chiral algebra associated to $\mathcal{O}(X, SC(m))$.

Now we have

**Proposition 4.1.** $A(SC(2))$ is isomorphic to $\mathfrak{c}^{ch}(1)$.

The proof is a variant of the argument in [BF04, Chap. 20]. It says that for a semi-simple $G$, the $L(G)^{\otimes k}$-twisted chiral algebra associated to the Beilinson-Drinfeld Grassmannian $\mathcal{O}(X,G)$ is the affine chiral algebra $V_k^Q(g)$ of level $k$.

**Proof.** Denote by $\Lambda$ the free fermionic vertex super algebra (following the symbol in [BF04, §5.3]). Thus as a linear space $\Lambda$ is the fermionic Fock space of the infinite-dimensional Clifford algebra $\mathfrak{c}^{ch}(\infty)$. The algebra $\mathfrak{c}^{ch}(\infty)$ is generated by \{ $\psi_n, \psi^*_n \mid n \in \mathbb{Z}$ \} with the relation

$$[\psi_m, \psi_n]^+ = [\psi_m^*, \psi_n^*]^+ = 0, \quad [\psi_m, \psi_n^*]^+ = \delta_{m+n,0}. $$
The $\mathfrak{e}_l^{\infty}$-module $\wedge$ is generated by the element $|0\rangle$ such that

$$\psi_n|0\rangle = 0 \quad (n \geq 0), \quad \psi_n^*|0\rangle = 0 \quad (n > 0).$$

By the description of linearization procedure, we have an identification $A(SC(2))_x \simeq \mathfrak{e}_l^{\infty} \otimes \mathfrak{e}_l^{\infty}_x$ for any $x \in X$. Here $\mathbb{C}$ is the one-dimensional representation of $\mathfrak{e}_l^{\infty}$ corresponding to the fiber of $L(SC(2))_{(1)}$. Thus $A(SC(2))_x$ is identified with the free fermion vertex algebra $\wedge$ as linear space.

The factorization structure induces a connection on $A^f(SC(2))$ giving a left $\mathcal{D}$-module structure. Denote by $\psi(z)z^{1/2}dz$ and $\psi^*(z)z^{-1/2}dz$ the generating local sections of $A^f(SC(2))$. Here $z^{\pm 1/2}$ corresponds to the theta characteristic appearing in (3.1). $\psi(z)$ and $\psi^*(z)$ are considered to be odd fields. Then as in the affine chiral algebra case, we have $(z - w)[\psi_i(z), \psi^*_j(w)]_+ = 0$, which is nothing but the Clifford chiral super algebra relation. \hfill $\Box$

5. BOSON-FERMION CORRESPONDENCE VIA FACTORIZATION SPACES

**Theorem 5.1.** We have an isomorphism of factorization monoids

$$\mathfrak{g}(X, \mathbb{Z}) \simeq \mathfrak{g}_r(X, SC(2)).$$

Under this isomorphism we have an identification of factorization super line bundles

$$\Theta \simeq L(SC(2)).$$

Then Propositions 3.2 and 4.1 yield $A_{\mathbb{Z}} \simeq \mathfrak{e}_l^{ch}(1)$ as chiral algebras. In terms of vertex algebra, it means the isomorphism $V_{\mathbb{Z}} \simeq \wedge$ which is nothing but the boson-fermion correspondence.

**Proof of Theorem 5.1.** Recall the notations of $G$-bundles in §1.4. Since we have $SC(2) \simeq GL(1)^{\times 2}$, it holds that

$$\mathfrak{m}(SC(2), X) \simeq \mathfrak{m}(GL(1), X)^{\times 2} \simeq Pic(X)^{\times 2}.$$ 

Then by Fact 1.4 on the trivialization we have

$$\mathfrak{g}_r(X, SC(2))_x \simeq (\mathfrak{g}_r(X, GL(1))_x)^{\times 2} \simeq \mathfrak{g}(X, \mathbb{Z})_x$$

for each $x \in X$. It yields the isomorphism $\mathfrak{g}_r(X, SC(2))_I \simeq \mathfrak{g}(X, \mathbb{Z})_I$ of ind-schemes for any $I \in \mathcal{I}$.

The factorization structures on $\mathfrak{g}_r(X, SC(2))$ and $\mathfrak{g}(X, \mathbb{Z})$ are the same essentially because they are both Beilinson-Drinfeld Grassmannians. The units coincide because they are given by trivial bundles. Thus we have $\mathfrak{g}(X, \mathbb{Z}) \simeq \mathfrak{g}_r(X, SC(2))$.

Recall that the factorization super line bundle $L(SC(2))$ comes from the spinor norm $N$ on $SC(2)$. Under the isomorphism $SC(2) \simeq GL(1)^{\times 2}$, $N$ is given by $N(x, y) = xy$, and it can be identified with the linear pairing. Now the isomorphism $\Theta \simeq L(SC(2))$ is an obvious one. \hfill $\Box$

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