INARIANT MEANS OF THE WOBBLING GROUP

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Abstract. Given a metric space \((X, d)\), the wobbling group of \(X\) is the group of bijections \(g : X \to X\) satisfying \(\sup_{x \in X} d(g(x), x) < \infty\). Consider the set of all finite subsets of \(X\), \(\mathcal{P}_f(X)\), as a group with multiplication given by symmetric difference. We study algebraic and analytic properties of \(W(X)\) in relation with the metric space structure of \(X\), such as amenability of the action of the lamplighter group \(W(X) \rtimes \bigoplus \chi Z_2\) on \(\bigoplus \chi Z_2\) and property (T).

1. Introduction

In this paper we deal with amenable actions of discrete groups. In our setting an action of a group \(G\) on a set \(X\) is called amenable if there is a \(G\)-invariant mean on \(X\). A linear map \(\mu\) on \(\ell_\infty(X)\) is a mean on \(X\) if it is unital and \(\|\mu\| = 1\). A group \(G\) is amenable if and only if its action on itself by left translation is amenable, in this case all actions of \(G\) are amenable. Thus the question of determining whether an action is amenable is interesting only in the case when \(G\) is not amenable.

Let \(G\) be a discrete group acting transitively on a set \(X\). Let \(\{0, 1\}^X\) be the set of all subsets of \(X\) considered as abelian group with multiplication given by the symmetric difference of sets. Denote by \(\mathcal{P}_f(X)\) the subgroup of \(\{0, 1\}^X\) which consists of all finite subsets of \(X\). The group \(G\) acts in a natural way on \(\mathcal{P}_f(X)\) by \(g(\{x_1, \ldots, x_n\}) = \{g(x_1), \ldots, g(x_n)\}\) for a finite set \(\{x_1, \ldots, x_n\}\) of \(X\). This action induces an action of the semidirect product \(G \rtimes \mathcal{P}_f(X)\) (also called wreath product of the action) on \(\mathcal{P}_f(X)\) by the formula

\[(g, E)(F) = g(F) \Delta E\]

for \(E, F \in \mathcal{P}_f(X)\) and \(g \in W(X)\).

In [11], Nekrashevych and authors showed that the action of \(G \rtimes \mathcal{P}_f(X)\) on \(\mathcal{P}_f(X)\) is amenable if the Schreier graph of the action of \(G\) on \(X\) is recurrent. However the character-ization of metric spaces \(X\) for which the answer to the following question is positive is still open:

Question 1.1. Is the action of \(G \rtimes \mathcal{P}_f(X)\) on \(\mathcal{P}_f(X)\) amenable?

An easy necessary condition for [11] is that the action of \(G\) on \(X\) is amenable. A consequence of our results is that this is not sufficient, see Proposition 2.1.

A metric space \(X\) has bounded geometry if for every \(R > 0\), the balls of radius \(R\) have bounded cardinality. We will mainly be interested in a special case of Question 1.1 when \((X, d)\) is a metric space with bounded geometry and \(G\) is a group of bijections \(g\) of \(X\) with bounded displacement, i.e. with the property that \(|g|_w := \sup\{d(x, g(x)) : x \in X\} < \infty\). Following [6] (see also [3]) we will call the group of all such bijections of \(X\) the wobbling group of \(X\) and denote it by \(W(X)\). In [12], [9, Remark 0.5.C′′1] and [6] the wobleings were introduced as tools to prove non-amenability results. In [10], they were used to prove amenability results

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(see below for details). When $X$ is a Cayley graph of a finitely generated group $\Gamma$ with word metric we will denote the wobbling group of $X$ shortly by $W(\Gamma)$. The group $W(\Gamma)$ does not depend on a finite generating set of $\Gamma$ and it coincides with the group of piecewisewise translations of $\Gamma$. In parallel to the Question 1.1 we can ask:

**Question 1.2.** Is the action of $W(X) \ltimes \mathcal{P}_f(X)$ on $\mathcal{P}_f(X)$ amenable?

The motivation for the question above is based on the recent result of the first named author and N. Monod, [10], where the authors show that the full topological group of Cantor minimal system is amenable, which was previously conjectured by Grigorchuk and Medynets in [8]. The combination of this result with the result of H. Matui, [14], produces the first examples of infinite simple finitely generated amenable groups. The technical core of [10] is to show that the Question 1.2 has positive answer for the particular case $X = \mathbb{Z}$.

Our goal would be to give a necessary and sufficient condition on $(X, d)$ for Question 1.2 to have a positive answer. Theorem 1.4 summarizes our results in this direction.

**Definition 1.3.** Let $(X, d)$ be a metric space with bounded geometry and fix $x_0 \in X$. $(X, d)$ is called transient if there is $R > 0$ such that the random walk starting at $x_0$ and jumping from a point $x$ uniformly to $B(x, R)$ is transient. Otherwise it is called recurrent.

This notion does not depend on $x_0$, and when $(X, d)$ is a connected graph with graph distance this notion is equivalent to the transience of the usual random walk on this graph (Proposition 3.2).

**Theorem 1.4.** Let $(X, d)$ be a metric space with bounded geometry.

- If $(X, d)$ is recurrent, then the action of $W(X) \ltimes \mathcal{P}_f(X)$ on $\mathcal{P}_f(X)$ is amenable. This includes $X = \mathbb{Z}, \mathbb{Z}^2$ or more generally a metric space $(X, d)$ with bounded geometry that embeds coarsely in $\mathbb{Z}^2$.
- If $X$ contains a Lipschitz and injective image of the infinite binary tree, then the action of $W(X) \ltimes \mathcal{P}_f(X)$ on $\mathcal{P}_f(X)$ is not amenable.

By Remark 4.4 Question 1.2 has a negative answer for many Cayley graphs of groups with exponential growth. By [15, Theorem 3.24], the first criterion applies to a finitely generated group $X = \Gamma$ if and only if $\Gamma$ is virtually $\{0\}, \mathbb{Z}, \mathbb{Z}^2$. The case when $X = \mathbb{Z}^d, d \geq 3$ remains an intriguing open question.

The sufficient condition in terms of the random walk is in fact a necessary and sufficient condition for a stronger property to hold, which we now describe.

The Pontryagin dual of $\mathcal{P}_f(X)$ is the compact group $\{0, 1\}^X$, for the pairing $\phi(E, \omega) = \exp(i \pi \sum_{j \in E} \omega_j), E \in \mathcal{P}_f(X), \omega \in \{0, 1\}^X$. By Fourier transform Questions 1.1 is equivalent to Question 1.5 below (see Lemma 3.1 of [10] for details). Fix a point $x_0 \in X$ and denote by $L_2(\{0, 1\}^X, \mu)$ the Hilbert space of functions on $\{0, 1\}^X$ with the Haar probability measure $\mu$. Let $A(x_0) = \{\omega_x \in X \in \{0, 1\}^X : \omega_{x_0} = 0\}$ be the cylinder set which fixes $\omega_{x_0}$ as zero.

**Question 1.5** (Dual form of the Question 1.1). Does there exist a net of unit vectors $\{f_n\} \in L_2(\{0, 1\}^X, \mu)$ such that

1. $\|g f_n - f_n\|_2 \to 0$ for every $g \in G$,
2. $\|f_n \cdot \chi_{A(x_0)}\|_2 \to 1$?
It was shown in [10] that for $X = \mathbb{Z}$ and $G = W(\mathbb{Z})$ the sequence 

$$f_n(\omega) = \exp(-n \sum_{j \in \mathbb{Z}} \omega_j \exp(-|j|/n))$$

satisfies the conditions (i) and (iii). Observe that the functions $f_n$ satisfy the following property :

(iii) $f \in L_2(\{0,1\}^X, \mu)$ can be represented as a product of independent random variables, i.e., there are functions $f_x \in L_2(\{0,1\}, m)$ such that $f(\omega) = \prod_{x \in X} f_x(\omega_x)$.

If we see $L_2(\{0,1\}^X, \mu)$ as the infinite tensor product of the Hilbert space $L_2(\{0,1\}, m)$ with unit vector 1, where $m(\{0\}) = m(\{1\}) = 1/2$, the condition (iii) above means that $f$ is an elementary tensor in $L_2(\{0,1\}^X, \mu)$.

The Schreier graph of the action of $G$ on $X$ with respect to a finite generating set $S$ is the graph with vertices $X$ and with an edge between $x$ and $y$ for each $g \in S$ with $gx = y$. It was shown in [11], that there exists a sequence of unit vectors $\{f_n\}$ in $L_2(\{0,1\}^X, \mu)$ that satisfy (i), (ii) and (iii) if and only if the random walk on the Schreier graph of the action of $G$ on $X$ with respect to $S$ is recurrent. We prove analog of this result in terms of the wobbling groups.

**Theorem 1.6.** Let $(X,d)$ be a metric space with bounded geometry and let $x_0 \in X$, and take $G = W(X)$. There exists a sequence of unit vectors $\{f_n\}$ in $L_2(\{0,1\}^X, \mu)$ that satisfy (i), (ii) and (iii) if and only if $X$ is recurrent.

As Narutaka Ozawa pointed out to us (personal communication), one can give a more direct proof (without Pontryagin duality) that this implies a positive answer to Question 1.1. Indeed, if and only if the random walk on the Schreier graph of the action of $G$ on $X$ with respect to $S$ is recurrent. We prove analog of this result in terms of the wobbling groups.

The Schreier graph of the action of $G$ on $X$ with respect to $S$ carries a recurrent random walk if and only if for every finite subset $F \subset G$ and $\varepsilon > 0$ there exists a finitely supported function $a : X \rightarrow [0,1]$ that satisfies:

1. $a(x_0) = 1$,
2. $\|g.a - a\|_{\ell_2(X)} < \varepsilon$ for every $g \in F$.

As Narutaka Ozawa pointed out to us (personal communication), one can give a more direct proof (without Pontryagin duality) that this implies a positive answer to Question 1.1. Indeed, for $a$ as above, let $\xi(B) = \prod_{x \in B} a(x)$ for $B \in \mathcal{P}_f(X)$ and $\xi(\emptyset) = 1$. Then $\xi \in \ell^2(X)$ is $\{x_0\}$-invariant and

$$\log \frac{\langle \xi, \xi \rangle}{\langle g \xi, \xi \rangle} = \log \prod_{x \in B} \frac{1 + a(x)^2}{1 + a(x)a(gx)} \leq \sum (a(x)^2 - a(x)a(gx)) = \frac{1}{2} \|a - g.a\|^2.$$

Thus $\xi$ is almost $G$-invariant. Any weak-* cluster point in $\ell_\infty(\mathcal{P}_f(X))^*$ of the net $|\xi|^2/\|\xi\|^2$ will therefore be a $G \rtimes \mathcal{P}_f(X)$-invariant mean.

As one may expect there is a strong relation between group structure of $W(X)$ and metric space structure of $X$. We show that if $X$ is of uniform subexponential growth, then $W(X)$ does not contain infinite property $(T)$ subgroups, see Theorem 4.3. On the other hand, an example of R. Tessera, see Theorem 4.7 shows that there exists a solvable group $\Gamma$ such that $W(\Gamma)$ contains $SL_3(\mathbb{Z})$.

The paper is organized as follows. Section 2 deals with Question 1.1. In Section 3 we start the study Question 1.2: we study the notion transience for metric spaces with bounded geometry and prove Theorem 1.6. In the last section we prove the second half of Theorem 1.4 (Proposition 1.3), and we study when $W(X)$ contains property $(T)$ groups.
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2. Schreier graphs of the actions and almost invariant vectors in the standard Bernoulli space.

2.1. A necessary condition. Let $G$ be a group acting on $X$. We start by recording the following result. The second assertion follows from results of Section 4 but is not used in the rest of the paper.

Proposition 2.1. If the action of $G \rtimes \mathcal{P}_f(X)$ on $\mathcal{P}_f(X)$ amenable, then so is the action of $G$ on $X$. The converse is not true.

Proof. Assume that the action of $G \rtimes \mathcal{P}_f(X)$ on $\mathcal{P}_f(X)$ amenable. Consider the unital positive $G$-equivariant map $T : \ell_\infty(X) \to \ell_\infty(\mathcal{P}_f(X) \setminus \{\emptyset\})$ given by $Tf(A) = \frac{1}{m}(f)$, for all $A$ nonempty finite subset of $X$. By (ii) implies (iv) in [10, Lemma 3.1], the set of non-empty finite subsets of $\Gamma$ carries a $W(\Gamma)$-invariant mean $m$. The composition $m \circ T$ is a $G$-invariant mean on $X$.

To see that the converse is not true, take for $X$ the Cayley graph of a finitely generated amenable group $\Gamma$ that contains an infinite binary tree (such a group exists by Remark 4.4). By Theorem 1.4, the action of $W(\Gamma) \rtimes \mathcal{P}_f(X)$ on $\mathcal{P}_f(X)$ is not amenable. On the other hand the action of $W(\Gamma)$ on $X$ is amenable; more precisely any $\Gamma$-invariant mean $m$ on $X$ is also $W(\Gamma)$-invariant mean $m$. Indeed, for any $g \in W(\Gamma)$ there is a finite partition $A_1, \ldots, A_n$ of $X$ and elements $\gamma_1, \ldots, \gamma_n$ such that $g$ acts as the translation by $\gamma_k$ on $A_k$. Then for every $f \in \ell_\infty(X)$, using that $(\gamma_i(A_i))_{i=1}^n$ forms a partition of $X$ we get

$$m(g \cdot f) = \sum_i m(\gamma_i \cdot (f1_{\gamma_i(A_i)})) = \sum_i m(f1_{A_i}) = m(f). \tag*{$\square$}$$

3. Almost invariant vectors for the action of $W(X)$

We will use a characterization of transience of a random walk on a locally finite connected graph $(V, E)$ in terms of electrical network. The capacity of a point $x_0 \in V$ is the quantity defined by

$$cap(x_0) = \inf \left\{ \left( \sum_{(x,x') \in E} |a(x) - a(x')|^2 \right)^{1/2} \right\}$$

where the infimum is taken over all finitely supported functions $a : V \to \mathbb{C}$ with $a(x_0) = 1$. We will use the following

Theorem 3.1 ([15, Theorem 2.12]). The random walk on a locally finite connected graph $(V, E)$ is transient if and only if $cap(x_0) > 0$ for some (equivalently every) $x_0 \in V$. 

Before we prove Theorem 1.6, we state some properties of transience for metric spaces.

**Proposition 3.2.** Let \((X, d)\) be a metric space with bounded geometry and \(x_0 \in X\). For \(R > 0\), the random walk starting at \(x_0\) and jumping from a point \(x\) uniformly to \(B(x, R)\) is transient if and only if there exists \(C > 0\) such that for every \(f : X \to \mathbb{C}\) with finite support

\[
|f(x_0)| \leq C \left( \sum_{x, x' \in X, d(x, x') \leq R} |f(x') - f(x)|^2 \right)^{1/2}.
\]

The notion of transience given by Definition 1.3 is independent of \(x_0\).

In the case \((X, d)\) is a connected graph with bounded geometry, the transience in the sense of Definition 1.3 is equivalent to the transience of the usual random walk on the graph.

**Proof.** The first point is Theorem 3.1 applied to the (connected component of \(x_0\) in the) graph structure on \(X\) where there is an edge between two points of \(X\) at distance at most \(R\). The independence from \(x_0\) in Definition 1.3 follows because the existence of \(R > 0\) such that (1) holds is easily seen to be independent from \(x_0\).

Assume that \((X, E)\) is a connected graph with bounded geometry, take \(x_0 \in X\) and \(R \geq 1\). Let \((X, E')\) be the graph structure on \(X\) in which there is an edge between two points of \(X\) at distance at most \(R\). The formal identity between \((X, d)\) and \((X, d')\) is a bilipschitz bijection, so that by [15] Theorem 3.10] the random walk on \((X, E)\) is transient if and only if the random walk on \((X, E')\) is transient. \(\square\)

**Lemma 3.3.** Let \(q : (X, d_X) \to (Y, d_Y)\) be a Lipschitz map between two metric spaces with bounded geometry. Assume that the preimage \(q^{-1}(y)\) of every \(y \in Y\) has cardinality less than some constant \(K\). If \((X, d_X)\) is transient then so is \((Y, d_Y)\).

**Proof.** Let \(C, R\) as in (1). There exists \(R'\) such that \(d_Y(q(x), q(x')) \leq R'\) whenever \(d_X(x, x') \leq R\). Then for every \(f \in c_0(Y), f \circ q \in c_0(X)\) so that

\[
|f(q(x_0))| \leq C \left( \sum_{x, x' \in X, d(x, x') \leq R} |f(q(x')) - f(q(x))|^2 \right)^{1/2} \leq CK \left( \sum_{y, y' \in Y, d(y, y') \leq R'} |f(y) - f(y')|^2 \right)^{1/2}.
\]

This proves that \(Y\) satisfies (1), and hence is transient by Proposition 3.2. \(\square\)

**Proof of Theorem 1.6.** The action of \(W(X)\) on \(X\) satisfies (i), (ii) and (iii) if and only if the action on \(X\) of every finitely generated subgroup of \(W(X)\) satisfies (i), (ii) and (iii).

Let us assume that \(X\) is recurrent. Let \(G\) be a subgroup of \(W(X)\) generated by a finite set \(S\). Let \(X_0\) be the orbit of \(x_0\) under the action of \(G\), and consider the Schreier graph \((X_0, E)\) of the action of \(G\) on \(X_0\) relatively to \(S\). Then the formal identity map from \(X_0\) to \((X, d_X)\) is an injective Lipschitz map. Lemma 3.3 implies that \((X_0, E)\) is recurrent. By Theorem 2.8, [11], the action of \(G\) on \(X\) satisfies (i), (ii) and (iii).

Reciprocally assume that \(X\) is transient, and take \(R\) as in Definition 1.3. Define a graph structure on \(X\) by putting an edge between \(x\) and \(x'\) if \(d(x, x') \leq R\). We obtain a (not necessarily connected) graph \((X, E)\) with bounded geometry on which the random walk starting from \(x_0\) is transient. Denote by \(d_E\) the associated graph distance. We now construct a finite subset \(S\) of \(W(X)\) such that the associated Schreier graph contains \((X, E)\). By Lemma 3.3
applied to the identity from the connected component of \( x_0 \) in \((X, E)\) to the Schreier graph of the action of \( S \) on \( X \), this Schreier graph is transient so that by Theorem 2.8 of [11] the action of the group generated by \( S \) does not satisfy (i), (ii) and (iii). Here is the construction of \( S \). Take a finite collection \((X_i)_{i \leq l}\) of subsets of \( X \) such that \( \cup_i X_i = X \) and \( d_E(x, y) \geq 3 \) for all \( x, y \in X_i \) and all \( i \). Take \( k \in \mathbb{N} \), and for every \( x \in X \) take a sequence \( y_1(x), \ldots, y_k(x) \) that covers all neighbours of \( x \) in \((X, E)\). The existence of such collection \((X_i)\) and such \( k \) follows from the bounded geometry assumption. Then for every \( i \leq l \) and every \( j \leq k \), consider the element \( s_{i,j} \) of \( W(X) \) that permutes \( x \) and \( y_j(x) \) for every \( x \in X_i \) and acts as the identity on the rest of \( X \). Then \( S = \{ s_{i,j} : i \leq l, j \leq k \} \) works. Indeed by construction for every neighbours \((x, x') \in (X, E)\) there is at least one (in fact two) element of \( S \) that permutes \( x \) and \( x' \).

**4. Properties of the wobbling groups**

4.1. **Negative answers to the Question [1.2]**. When \( X \) is the Cayley graph of a finitely generated group, the first assertion in Proposition 2.1 implies

**Lemma 4.1.** Let \( \Gamma \) be a finitely generated group. If there exists a \( W(\Gamma) \ltimes \mathcal{P}_f(\Gamma) \)-invariant mean on \( \mathcal{P}_f(\Gamma) \) then \( \Gamma \) is amenable.

We can also give a negative answer to Question [1.2] for some amenable groups. One ingredient for this is the following monotonicity property.

**Lemma 4.2.** Let \( i : X \to Y \) an injective map such that \( \sup_{d(x, x') \leq R} d(i(x), i(x')) < \infty \) for every \( R > 0 \). If Question [1.2] has a positive answer for \( Y \), then is also has positive answer for \( X \).

**Proof.** The map \( i \) allows to define an embedding \( W(X) \subset W(Y) \) by defining, for \( g \in W(X) \), \( g \cdot i(x) = i(g \cdot x) \) and \( g \cdot y = y \) if \( y \notin i(X) \).

Assume that Question [1.2] has a positive answer for \( Y \), and take \( x_0 \in X \). By [10, Lemma 3.1] there is a mean \( m \) on \( \mathcal{P}_f(Y) \) that is \( W(Y) \)-invariant and that gives full weight to the collection of sets containing \( i(x_0) \). Then the push-forward mean on \( \mathcal{P}_f(X) \) (given by \( \varphi \in \ell_\infty(\mathcal{P}_f(X)) \mapsto m(A \mapsto \varphi(i^{-1}(A))) \) is \( W(X) \)-invariant and gives full weight to the collection of sets containing \( x_0 \). By [10, Lemma 3.1] again, Question [1.2] has a positive answer for \( X \).

**Proposition 4.3.** Let \((X, d)\) be a metric space with bounded geometry with an injective and Lipschitz map from the infinite binary tree \( T \) to \( X \). Then there is no \( W(X) \ltimes \mathcal{P}_f(X) \)-invariant mean on \( \mathcal{P}_f(X) \).

**Proof.** There is a Lipschitz injective map from the free group with two generators in \( T \), and hence in \( X \) if \( X \) contains an injective and Lipschitz image of \( T \). The Proposition therefore follows from Lemma 4.1 and Lemma 4.2.

**Remark 4.4.** The class of groups for which this proposition applies, \( i.e. \) for which there is a Cayley graph that contains a copy of the infinite binary tree as a subgraph, contains in particular all non-amenable groups ([2, Theorem 1.5]), as well as all elementary amenable groups with exponential growth (by [3] such groups contain a free subsemigroup). In [4], R. Grigorchuk disproveing a conjecture of Rosenblatt proved that the lamplighter group \( \mathbb{Z}_2 \wr \mathbb{Z} \) contains an infinite binary tree, here \( G \) is Grigorchuk’s 2-group of intermediate growth. We do not know whether all groups with exponential growth contain such a tree.
4.2. Property (T) subgroups. It is an interesting question to extract properties of the group \( W(X) \) using the properties of the underlying metric space. Below we prove that \( W(X) \) cannot contain property (T) groups when \( X \) is of subexponential growth. Alain Valette (personal communication) pointed out to us that a very similar observation (attributed to Kazhdan) was made by Gromov in [9].

**Theorem 4.5.** Let \( X \) be a metric space with uniform subexponential growth:

\[
\lim \log \sup_{n} |B(x, n)|/n = 0.
\]

Then \( W(X) \) does not contain an infinite countable property (T) group.

**Proof.** Assume \( G < W(X) \) is a finitely generated property (T) group, with finite symmetric generating set \( S \). We will prove that \( G \) is finite. To do so we prove that the \( G \)-orbits on \( X \) are finite, with a uniform bound. Assume that \( 1 \in S \). If \( m = \max \{|g|_w : g \in S\} \), then \( S^n x \subset B(x, mn) \) for every \( x \in X \), so that by assumption, the growth of \( S^n x \) is subexponential (uniformly in \( x \in X \)). The classical expanding properties for actions of (T) groups will imply that the orbit of \( x \) is finite (uniformly in \( x \)).

Indeed, by (T), there exists \( \varepsilon > 0 \) such that for every unitary action of \( G \) on a Hilbert space \( H \) without invariant vectors, the inequality \( \sum_{\xi \in S} \| g \cdot \xi - \xi \|^2 \geq \varepsilon \| \xi \|^2 \) holds for every \( \xi \in H \). As a consequence, for every transitive action of \( G \) on a set \( Y \), we have \( \sum_{g \in S} |gF \Delta F| \geq \varepsilon |F| \) for every finite subset \( F \) of \( Y \) satisfying \( |2F| \leq |Y| \) (take \( H = \ell_2(Y) \) if \( Y \) is infinite and \( H = \text{the subspace of } \ell_2(Y) \text{ orthogonal to the vector with all coordinates equal otherwise} \), and apply the preceding equality with \( \xi = \chi_F - |F|/|Y| \chi_Y \setminus F \)). Here \( \chi_F \) is the indicator function of \( F \), and \( |F|/|Y \setminus F| \) is by convention 0 if \( Y \) is infinite). By induction, we therefore have that for \( x \in Y \) and \( n \in \mathbb{N} \), \( |S^n x| \geq (1 + \varepsilon/4)^n \) unless \( |Y| \leq 2(1 + \varepsilon/4)^n \). Applying it to the orbit of some \( x \in X \), we get

\[
|S^n x| < (1 + \varepsilon/4)^n \implies |\text{Orb}_G(x)| < 2(1 + \varepsilon/4)^n.
\]

Hence, subexponential growth gives an \( n \in \mathbb{N} \) such that \( |\text{Orb}_G(x)| < 2(1 + \varepsilon/4)^n \). QED.

To construct spaces such that \( W(X) \) contains property (T) groups, we first remark that the groups \( W(X) \) behave well with respect to coarse embeddings. A map \( q : (X, d_X) \to (Y, d_Y) \) between metric spaces is a coarse embedding if there exists nondecreasing functions \( \varphi_+, \varphi_- : [0, \infty[ \to \mathbb{R} \) such that \( \lim_{t \to \infty} \varphi_+(t) = \infty \) and

\[
\varphi_-(d_X(x, x')) \leq d_Y(q(x), q(x')) \leq \varphi_+(d_X(x, x'))
\]

for every \( x, x' \in X \).

**Lemma 4.6.** Let \( q : (X, d_X) \to (Y, d_Y) \) be a map such that there is an increasing function \( \varphi_+ : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( d_Y(qx, qy) \leq \varphi_+(d_X(x, y)) \), and such that the preimage \( q^{-1}(y) \) of every \( y \in Y \) has cardinality less than some constant \( K \) (e.g. \( q \) is a coarse embedding and \( X \) has bounded geometry). Let \( F \) be a finite metric space of cardinality \( K \). Then \( W(X) \) is isomorphic to a subgroup of \( W(Y \times F) \).

**Proof.** In this statement \( Y \times F \) is equipped with the distance \( d((y, f), (y', f')) = d_Y(y, y') + d_F(f, f') \). Since \( F \) is bigger than \( q^{-1}(y) \) for all \( y \), there is a map \( f : X \to F \) such that the map \( \tilde{q} : x \in X \mapsto (q(x), f(x)) \in Y \times F \) is injective. We can therefore define an action of \( W(X) \) on \( Y \times F \) by setting \( g(\tilde{q}(x)) = \tilde{q}(gx) \) and \( g(y, f) = (y, f) \) if \( (y, f) \notin \tilde{q}(X) \). The assumption on
φ+ guarantees that this action is by wobbings, ie that it defines an embedding of \(W(X)\) in \(W(Y × F)\).

In a contrast to Theorem 4.5 we have the following result by Romain Tessera. With his kind permission we include a proof.

**Theorem 4.7.** There is a solvable group \(\Gamma\) such that \(W(\Gamma)\) contains the property \((T)\) group \(SL(3,\mathbb{Z})\).

**Proof.** The proof uses the notion of asymptotic dimension (see [1]). By [1] Corollary 94, \(SL(3,\mathbb{Z})\) has finite asymptotic dimension. By [1] Theorem 44 this implies that \(SL(3,\mathbb{Z})\) embeds coarsely into a finite product of binary trees. Take \(\Gamma_0\) a solvable group with a free semigroup. In particular it coarsely contains a binary tree, so \(SL(3,\mathbb{Z})\) embeds coarsely in \(\Gamma_0^n\) for some \(n\). By Lemma 4.6 there is a finite group \(F\) such that \(W(SL(3,\mathbb{Z}))\) embeds as a subgroup in \(W(F × \Gamma_0^n)\). But \(W(SL(3,\mathbb{Z}))\) contains \(SL(3,\mathbb{Z})\) (action by translation).

**Remark 4.8.** The proof actually shows that for every group \(\Lambda\) with finite asymptotic dimension, there is an integer \(n\) such that \(\Lambda\) is isomorphic to a subgroup of \(W(\Gamma^n)\) whenever there is a Cayley graph of \(\Gamma\) that contains an infinite binary tree as a subgraph. By Remark 4.4 this includes lots of groups \(\Gamma\) with exponential growth. In some sense this says that the assumptions of Theorem 4.5 are not so restrictive.

**References**

[1] G. Bell, A. Dranishnikov, *Asymptotic Dimension* Topology Appl., 12 (2008) 1265–1296.

[2] I. Benjamini and O. Schramm, *Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant* Geometric and Functional Analysis 7 (1997), 3, 403–419

[3] T. Ceccherini-Silberstein, R. Grigorchuk, P. de la Harpe, *Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces*, Proc. Steklov Inst. Math. 1999, no. 1 (224), 57–97.

[4] C. Chou, *Elementary amenable groups* Illinois J. Math. 24 (1980), 3, 396–407.

[5] Y. de Cornulier, *Groupes pleins-topologiques, d’après Matui, Juschenko-Monod..., Séminaire Bourbaki, Janvier 2013.

[6] W. A. Deuber, M. Simonovits, V. T. Sós, *A note on paradoxical metric spaces*, Studia Sci. Math. Hungar. 30 (1995), no. 1-2, 17–23.

[7] R. Grigorchuk, *Superamenability and the occurrence problem of free semigroups*, (Russian) Funktsional. Anal. i Prilozhen. 21 (1987), no. 1, 74–75.

[8] R. Grigorchuk, K. Medynets, *Topological full groups are locally embeddable into finite groups*, Preprint, [http://arxiv.org/abs/math/1105.0719v3](http://arxiv.org/abs/math/1105.0719v3).

[9] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2, London Math. Soc. Lecture Note Ser. 182 (1993).

[10] K. Juschenko, N. Monod, *Cantor systems, piecewise translations and simple amenable groups*, arXiv:1204.2132

[11] Extensions of amenable groups by recurrent groupoids/Extensions of amenable groups by recurrent groupoids, arXiv:1305.2637

[12] M. Laczkovich, *Equidecomposability and discrepancy: a solution of Tarski’s circle-squaring problem*, J. Reine Angew. Math. 404 (1990) 77–117.

[13] T. Lyons, *A simple criterion for transience of a reversible Markov chain*, Ann. Probah. 11 (1983), no. 2, 393–402.

[14] H. Matui, *Some remarks on topological full groups of Cantor minimal systems*, Internat. J. Math. 17 (2006), no. 2, 231–251.

[15] W. Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics (2000).

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