ONE NONCOMMUTATIVE DIFFERENTIAL CALCULUS
COMING FROM THE INNER DERIVATION

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ABSTRACT. We define a noncommutative differential calculus constructed from the inner derivation, then several relevant examples are showed. It is of interest to note that for certain $C^*$-algebra, this calculus is closely related to the classical one when the algebra associates a deformation parameter.

INTRODUCTION

Noncommutative geometry is developed by Alain Connes. In his work [8], the cyclic cohomology, which is considered to be the classical differential calculus counterpart, is introduced. In this paper, we are dealing with one new noncommutative differential calculus. When considering some $C^*$-algebra $A_\theta$ which is parametrized by a number $\theta$, ({$A_\theta$} is a family of continuous field of $C^*$-algebra, when $\theta = 0$, $A_0$ is commutative, for instance, quantum tori, quantum plane etc), this differential calculus will be deforming to the classical one when $\theta$ goes to 0. The important point to note here is that the definition is trivial if the underlying algebra is commutative.

When combining the condition (2) of the following definition from Rieffel [12], it is very easy to see that applying this noncommutative differential calculus for the strict deformation quantization algebra, it will be deforming to the classical differential calculus.

Definition [12] 0.1. A strict deformation quantization of $A$ in the direction of $\Lambda$ means an open interval $I$ of real numbers containing 0, together with, for each $h \in I$, an associative product $\ast_h$, an involution $\ast_h$, and a $C^*$-norm $\| \cdot \|_h$ on $A$, which for $h = 0$ are the original pointwise product, complex conjugation involution, and supremum norm, such that

1. for every $f \in A$, the function $h \mapsto \|f\|_h$ is continuous
2. for every $f, g \in A$, $\|(f *_h g - g *_h f)/i\hbar - \{f, g\}\|_h$ converges to 0 as $h$ goes to 0.

The notation $\Lambda$ means a skew 2-vector field and $\{,\}$ is a Poisson bracket. For a full treatment of the strict deformation quantization theory, we refer the reader to [12].

This paper is organized as follows. In the first section, we define carefully this new noncommutative differential calculus and its basic properties. In the second section, we proceed to the study of the Dirichlet form constructed from the inner derivation. We will show this Dirichlet form is symmetric, Markov, conservative, completely positive and strongly local. In section 3, 4, 5, 6, 7, we apply this calculus for

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\[ M_n(\mathbb{C}), \text{graph } C^* \text{-algebra, quantum tori, quantum plane, and quantum Heisenberg manifolds. The last three are the examples of the deformation quantization } C^* \text{-algebra. In the case of quantum tori, the exterior derivative is realized as a finite difference operator. It is also worth pointing out that in section 7, we will look more closely at the structure of the quantum Heisenberg manifolds, it turns out the von Neumann algebra of the quantum Heisenberg manifolds is nothing but a 3-dimensional quantum tori.} \]

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1. Noncommutative differential calculus

We are dealing with a differential calculus on some non-commutative algebras, for instance, \( C^* \)-algebra, Banach algebra, von Neumann algebra. In order to generalize this differential calculus idea, we put the definition in the context of \( C^* \)-algebra. The reason to choose \( C^* \)-algebra is because it always has the adjoint, and consequently we define in the same way as from the complex differential calculus.

Let’s review first about the complex calculus. Suppose \( \Omega \subset \mathbb{C}^n \) is an open set and \( f \in \mathcal{C}^\infty(\Omega) \), we have the first order calculus form

\[
\delta f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j,
\]

here \( z_j, \bar{z}_j \) are the coordinate basis. Having this complex calculus in mind, we come up the definition below.

**Definition 1.1.** Given a noncommutative \( C^* \)-algebra \( \mathcal{A} \), the elements \( U_1, \ldots, U_n \in \mathcal{A} \) is called a differential basis if the following two conditions are satisfied:

(I) Each \( U_j \) is non self-adjoint;

(II) The elements \( U_1, U_1^*, \ldots, U_n, U_n^* \) mutually commute.

We emphasize that the condition (I) above is not necessary, at least for certain algebra which has no adjoin, even in the \( C^* \) algebra situation. The reason we put here is to make it similar as \( z, \bar{z} \) from the complex calculus. The condition (II) is crucial in order to prove \( \delta^2 = 0 \) in Proposition 1.4. The differential basis can always be achieved from \( \mathcal{A} \), one good method is to choose non self-adjoint elements from any maximal abelian subalgebra of \( \mathcal{A} \). Besides the differential basis defined above, we also need to have a derivation operator associated with the differential calculus.

**Definition 1.2.** A derivation of a \( C^* \)-algebra \( \mathcal{A} \) is a linear map \( \delta : \mathcal{A} \to \mathcal{A} \) such that it satisfies the Leibniz’s rule, i.e. \( \delta(xy) = \delta(x)y + x\delta(y) \). The derivation is called inner, if \( \exists U \in \mathcal{A} \), such that \( \delta a = [U, a] = Ua - aU, \forall a \in \mathcal{A} \).

We find inner derivation is a good choice. The advantage of using it lies in the fact that in certain algebra, for instance, the von Neumann algebra, the derivation is always inner\[11\]. Equipped with the differential basis and the inner derivation, we define the first order calculus form \( \delta_{U_1, \ldots, U_n} \) (we omit the subscript \( U_1, \ldots, U_n \) if it’s clear in the context) on \( \mathcal{A} \) as below

\[
\delta a = \sum_{j=1}^{n} [U_j, a]dU_j + \sum_{j=1}^{n} [U_j^*, a]dU_j^*.
\]
The symbol \( dU_j, dU^*_k \) above is merely for the notation convenience, it can be viewed as the basis of \( \delta a \) which belongs to a direct sum of 2n copies of \( \mathcal{A} \). For the high order, assume \( e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(r)} = \text{sgn}(\sigma)e_1 \wedge \ldots \wedge e_r \), where \( \sigma \) is any permutation of \( r \) numbers and \( e_1, \ldots, e_r \in \{dU_1, \ldots, dU_n, dU^*_1, \ldots, dU^*_n\} \). Then we put

\[
\Omega^r \mathcal{A} = \bigoplus_{p+q=r} \Omega^{p,q} \mathcal{A},
\]

where \( \Omega^{p,q} \mathcal{A} \) = \{space spanned by \( dU_1 \wedge \ldots \wedge dU_p \wedge dU^*_j \wedge \ldots \wedge dU^*_q \} \), \( \Omega^{p,q} \mathcal{A} \) is called type \( (p,q) \)-form and its dimension is

\[
dim \Omega^{p,q} \mathcal{A} = \binom{2n}{p+q} = \frac{(2n)!}{(p+q)!(2n-p-q)!}, 0 \leq p + q \leq 2n.
\]

Having disposed of the above information, the definition for the high order differential calculus is immediately obtained, namely if

\[
\alpha = \sum a_{I,J} dU_I \wedge dU^*_j \in \Omega^{p,q} \mathcal{A},
\]

where \( I = (i_1, \ldots, i_p) \), \( J = (j_1, \ldots, j_q) \) and \( dU_I = dU_{i_1} \wedge \ldots \wedge dU_{i_p}, dU^*_j = dU^*_{j_1} \wedge \ldots \wedge dU^*_{j_q} \), then

\[
\delta \alpha = \sum_{I,J} \delta a_{I,J} \wedge dU_I \wedge dU^*_j = \sum_{I,J} (\sum_{j=1}^n [U_I, a_{I,J}]dU_J + \sum_{j=1}^n [U^*_j, a_{I,J}]dU_I) \wedge dU_I \wedge dU^*_j.
\]

Definition 1.3. The exterior product, or wedge product, of a type \( (p,q) \)-form \( \alpha = \sum a_{I,J} dU_I \wedge dU^*_j \) and a type \( (r,s) \)-form \( \beta = \sum b_{M,N} dU_M \wedge dU^*_N \) is a type \( (p+s,q+t) \)-form defined by

\[
\alpha \wedge \beta = \sum a_{I,J} b_{M,N} dU_I \wedge dU^*_j \wedge dU_M \wedge dU^*_N.
\]

The preceding definition for wedge product shows the relation \( \alpha \wedge \beta = (-1)^{(p+q)(t+s)} \beta \wedge \alpha \) need not hold in general if the algebra \( \mathcal{A} \) is not commutative.

Proposition 1.4. \( \delta^2 = 0 \)

Proof. It is based on the condition (II) from definition (1.1), and we have

\[
[U_I, [U_K, a]] = [U_K, [U_I, a]], [U^*_J, [U_K, a]] = [U_K, [U^*_J, a]], [U^*_J, [U^*_K, a]] = [U^*_K, [U^*_J, a]].
\]

\( \square \)

Proposition 1.5. Let \( \alpha \) be a \( (p,q) \)-form and \( \beta \) be a \( (s,t) \)-form, then \( \delta(\alpha \wedge \beta) = \delta \alpha \wedge \beta + (-1)^{p+q} \alpha \wedge \delta \beta \).

Proof. This follows from the derivation property \( [U, ab] = [U, a]b + a[U, b] \). \( \square \)

For each \( (p,q) \)-form \( \alpha = \sum a_{I,J} dU_I \wedge dU^*_j \), we have a \( \ast \) operation defined by \( \alpha^* = \sum a^*_{I,J} dU_I \wedge dU^*_j \).

Proposition 1.6. The set \( \Omega^{p,q} \mathcal{A} \) of type \( (p,q) \)-forms on \( \mathcal{A} \) is a \( \mathcal{A} \)-module, it has the following properties:

1. If \( \alpha \in \Omega^{p,q} \mathcal{A} \), then \( \alpha^* \in \Omega^{p,q} \mathcal{A} \).
2. If \( \alpha \in \Omega^{p,q} \mathcal{A} \) and \( \beta \in \Omega^{r,s} \mathcal{A} \), then \( \alpha \wedge \beta \in \Omega^{p+r,q+s} \mathcal{A} \).
3. If \( \alpha \in \Omega^{p,q} \mathcal{A} \), then \( \delta \alpha \in \Omega^{p+1,q} \mathcal{A} \bigoplus \Omega^{p,q+1} \mathcal{A} \).
4. \( \Omega^{p,q} \mathcal{A} = 0 \), if \( \min\{p,q\} > n \).
Proof. The proof is straightforward by the easy computation. □

The point of the proceeding proposition (3) is that it allows one to define the operators
\[ \partial : \Omega^{p,q} A \rightarrow \Omega^{p+1,q} A \]
and
\[ \partial^* : \Omega^{p,q} A \rightarrow \Omega^{p,q+1} A \]
through the direct sum projection, namely,
\[ \partial \alpha = \sum_{I,J} [U_j, a_{I,J}] dU_j \land dU_I \land dU_J, \]
\[ \partial^* \alpha = \sum_{I,J} [U_j^*, a_{I,J}] dU_j^* \land dU_I \land dU_J. \]
From the relation
\[ \delta = \partial + \partial^*, \]
and the property \( \delta^2 = 0 \), we have \( 0 = (\partial + \partial^*)^2 = \partial^2 + \partial\partial^* + \partial^*\partial + \partial^* \), hence \( \partial^2 = \partial^* = \partial\partial^* + \partial^*\partial = 0 \).

Definition 1.7. A form \( \alpha \) is called closed if \( \delta \alpha = 0 \) and is called \( \partial^* \) closed if \( \partial^* \alpha = 0 \).

Let \( \mathbb{C}_{p,q}(A) = \{ \alpha \in \Omega^{p,q} A : \delta \alpha = 0 \} \), \( \mathbb{H}\mathbb{C}_{p,q}(A) = \{ \alpha \in \Omega^{p,q} A : \partial^* \alpha = 0 \} \), then \( \mathbb{C}_{p,q}(A) \) is the set of closed \( (p, q) \)-forms and \( \mathbb{H}\mathbb{C}_{p,q}(A) \) is the set of \( \partial^* \) closed \( (p, q) \)-forms. The property \( \delta^2 = 0 \) gives us a deRham complex,
\[ 0 \rightarrow \Omega^0 A \xrightarrow{\delta_0} \Omega^1 A \xrightarrow{\delta_1} \Omega^2 A \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{2n-1}} \Omega^{2n} A \xrightarrow{\delta_{2n}} 0. \]
And the \( k \)-th deRham cohomology is defined by
\[ H^k_{dR}(A) = \ker \delta_k / \text{img } \delta_{k-1}. \]

The property \( \partial^* \delta^2 = 0 \) gives us a Dolbeault complex,
\[ 0 \rightarrow \Omega^{0,0} A \xrightarrow{\delta^*_0} \Omega^{1,0} A \xrightarrow{\delta^*_1} \Omega^{2,0} A \xrightarrow{\delta^*_2} \cdots \xrightarrow{\delta^*_{n-1}} \Omega^{2n,0} A \xrightarrow{\delta^*_n} 0. \]
And the \( (p, q) \)-th Dolbeault cohomology is defined by
\[ H^p,q_{D}(A) = \ker \delta^*_q / \text{img } \delta^*_{q-1}. \]

Remark 1.8. The cohomology groups depend on the choice of the differential basis. In general, \( H^0_{dR}(A) \) contains the algebra generated by the basis. If the basis generates a maximal abelian subalgebra, then \( H^1_{dR}(A) \) is the same algebra generated by this basis. If \( A \) is a chosen differential basis and \( B \subseteq A \), then the cohomology groups computed by \( A \) and the cohomology groups computed by \( B \) need not have the subgroup relation. This can be shown by computing \( H^1_{dR}(A) \).

Theorem 1.9. \( H^0_{dR}(A) = H^0_{D}(A) = \mathbb{C}_{0,0}(A) \).

Proof. The proof is based on Fuglede-Putnam theorem which says that \( aU_i = U_i a \Leftrightarrow aU_i^* = U_i a \) if \( U \) is normal. Then it is a simple matter to check that \( H^0_{dR}(A) = \mathbb{C}_{0,0}(A), H^0_{D}(A) = \mathbb{H}\mathbb{C}_{0,0}(A) = \mathbb{C}_{0,0}(A) \) and \( H^0_{D}(A) = \mathbb{H}\mathbb{C}_{0,0}(A) \), and the proof is complete. □
The interest of the above theorem is that \( \mathbb{C}_{0,0} = \mathbb{H} \mathbb{C}_{0,0} \). Considering the classical complex differential calculus, if \( f \in \mathbb{H} \mathbb{C}_{0,0} \Rightarrow \frac{df}{dq} = 0 \Rightarrow f \) is holomorphic. So \( \mathbb{H} \mathbb{C}_{0,0} \) is the set of holomorphic functions, and \( \mathbb{C}_{0,0} \) is the set of constant functions, therefore in the classical case \( \mathbb{C}_{0,0} \neq \mathbb{H} \mathbb{C}_{0,0} \). The reason for this difference lies in the fact that in the Fuglede-Putnam theorem above, \( U \) need to be bounded. On the other hand, if we require \( f \) to be bounded, then by the Liouville’s theorem, \( f \) is constant.

2. Dirichlet forms

Dirichlet form, in \( C^* \)-algebras setting, is introduced by Albeverio and Høegh-Krohn[2], it shares a flavor of geometry in the sense of Connes’ noncommutative geometry[9]. For a recent account of the theory, we refer the reader to[6][7]. From the classical complex case, we have the Laplace operator defined by \( \sum \frac{\partial^2}{\partial x_i \partial x_i} \). When transformed to the inner derivation \([U, \cdot] \), the Laplace operator of the counterpart is

\[
\Delta = \sum_{j=1}^{n} [U^*_j, [U_j, \cdot]].
\]

If on this \( C^* \)-algebra \( \mathcal{A} \), it has the following additional assumption:

(III) It has a lower semicontinuous faithful trace \( \tau \).

Then we denote by \( L^2(\mathcal{A}, \tau) \) the Hilbert space of the GNS representation \( \pi_\tau \) associated to \( \tau \), and by \( L^\infty(\mathcal{A}, \tau) \) or \( \mathcal{M} \) the von Neumann algebra \( \pi_\tau(\mathcal{A})'' \) in \( \mathcal{B}(L^2(\mathcal{A}, \tau)) \) generated by \( \mathcal{A} \) in the GNS representation. \( 1_\mathcal{M} \) stands for the unit of \( \mathcal{M} \).

**Definition 2.1.** Given a strongly continuous semigroup \( \Phi_t (t \in \mathbb{R}^+) \) of operators defined on \( L^\infty(\mathcal{A}, \tau) \),

1. it is symmetric, if \( \tau(\Phi_t(x)y) = \tau(xy\Phi_t(y)) \).
2. it is Markov, if \( 0 \leq x \leq 1_\mathcal{M} \) implies that \( 0 \leq \Phi_t(x) \leq 1_\mathcal{M} \).
3. it is conservative, if \( \Phi_t(1_\mathcal{M}) = 1_\mathcal{M} \).
4. it is completely positive, if for any \( n \) we have \( \sum_{i,j=1}^{n} b^*_i \Phi_t(a^*_i a_j) b_j \geq 0 \) where \( a_i, b_i \in \mathcal{M}, i = 1, \ldots, n \).

**Proposition 2.2.** \( \Delta \) is the generator of a norm continuous, completely positive, symmetric, conservative Markov semigroup.

**Proof.** Let \( \Phi_t = e^{-t \Delta} \), then \( \Phi_t \) is norm continuous because \( \Delta \) is bounded. \( \Phi_t \) is symmetric and conservative:

\[
\tau(\Delta(a)b) = \sum_{j=1}^{n} \tau([U^*_j, [U_j, a]]b) = \sum_{j=1}^{n} \tau(a[U^*_j, [U_j, b]]) = \tau(a \Delta(b)) = \tau(a \Delta^n(b)) \Rightarrow \tau(\Phi_t(a)b) = \tau(a \Phi_t(b)), \text{ also } \Phi_t(1_\mathcal{M}) = 1_\mathcal{M}.
\]

\( \Phi_t \) is completely positive: 

\[
-\Delta a = \sum_{j=1}^{n} (U^*_j a U_j + U_j a^* U^*_j - U^*_j U_j a - a U_j U^*_j), \text{ let } K_1(a) = \sum_{j=1}^{n} (U^*_j a U_j + U_j a^* U^*_j) \text{ and } K_2(a) = Aa + aA \text{ where } A = -\sum_{j=1}^{n} U^*_j U_j = A^*. \text{ As } K_1 \text{ is completely positive, so is } K_1^n, \text{ hence } e^{t K_1} \text{ is completely positive.} (1 + \frac{m}{1+m})^n \rightarrow e^{tx}, \text{ for } x \in \mathbb{R}. \text{ From the functional calculus theorem, } A_m = (1 + \frac{m}{1+m})^n \rightarrow e^{t K_2} \text{ in norm. Since } \sum_{j,i=1}^{n} b^*_i K_2(a^*_i a_j) b_j \text{ is self-adjoint, } 1 + \frac{m}{1+m} \text{ is completely positive when } m \text{ large enough} \Rightarrow A_m \text{ is completely positive} \Rightarrow e^{t K_2} \text{ is completely positive. Then from the Trotter product formula, } (e^{\frac{m}{1+m} K_1} e^{\frac{m}{1+m} K_2})^n \rightarrow e^{-t \Delta} \text{ in the strong operator topology.}
Hence $\Phi_t$ is completely positive.

$\Phi_t$ is Markov:
From above, in particular, $\Phi_t$ is positive, then for $0 \leq a \leq 1_M$, $e^{-t\Delta}(a - 1_M) \leq 0 \Rightarrow 0 \leq e^{-t\Delta}a \leq 1_M$. $\square$

From [2], $E(a,b) = \langle \Delta a, \Delta b \rangle_{L^2(A,\tau)}$ is the Dirichlet form with the corresponding generator $\Delta$. With this Dirichlet form, we can construct a $C^*$-Hilbert $A \otimes A$-bimodule $A \otimes A$ [13] and a sesquilinear form with values in $A$ by the formula

\begin{equation}
(a \otimes b, c \otimes d)_A = b^* (a^* \Delta c + \Delta(a^*)c - \Delta(a^*)c) d.
\end{equation}

Definition [13] 2.3. A completely positive Markov semigroup $(\Phi_t)_{t \geq 0}$ and its infinitesimal generator $\Delta$ are strongly local if there exists a Hilbert space $H$ and an isometry $W$ in $L_A(A \otimes A, H \otimes A)$.

Replacing $\Delta$ defined in (2.1) to (2.2), we have

\begin{equation}
(a \otimes b, c \otimes d)_A = \sum_{j=1}^{n} b^* ([U_j, a]^*[U_j, c] + [U_j^*, a]^*[U_j^*, c]) d.
\end{equation}

Take $H$ to be $C^{2n}$, and define the mapping $W : A \otimes A \rightarrow C^{2n} \otimes A$ by

$$a \otimes b \mapsto [U_1, a]b \oplus \ldots \oplus [U_n^*, a]b,$$

and the $A$-bimodule structure by

$$\langle [U_1, a]b \oplus \ldots \oplus [U_n^*, a]b, [U_1, c]d \oplus \ldots \oplus [U_n^*, c]d \rangle_A = b^* \sum_{j=1}^{n} ([U_j, a]^*[U_j, c] + [U_j^*, a]^*[U_j^*, c]) d.$$

It’s not hard to see $W$ turns out to be an isometry, and this is precisely the assertion of the following theorem.

Theorem 2.4. $\Delta$ defined in (2.1) is strongly local.

Since $\Phi_t$ is conservative, by theorem (4.7)[7], this Dirichlet form has a representation

\begin{equation}
E(a,a) = \tau(\langle \delta a, \delta a \rangle_A).
\end{equation}

3. Matrix algebra

For the $n \times n$ matrix algebra $M_n(\mathbb{C})$, the diagonal matrices is its maximal abelian subalgebra. The elements $p_1, \ldots, p_n$ where $p_j$ is the $n \times n$ matrix with 1 in entry $(j,j)$ and 0 elsewhere are the generators of the diagonal matrices. Therefore $p_1, \ldots, p_n$ can be chosen as the differential basis. Thus we actually have constructed the noncommutative differential calculus on $M_n(\mathbb{C})$. Its first order differential calculus becomes

$$\delta_n a = [p_1, a] dp_1 + \ldots + [p_n, a] dp_n.$$
Theorem 3.2. The Laplace operator is
\[ E_\delta \]
we see that \( M \) therefore, \( a \) the diagonal entries of \( \operatorname{Tr} \) is
\[ n \]
\[ \text{Proposition 3.1. The Dirichlet form on } M_n(\mathbb{C}) \text{ constructed from } \text{Tr} \text{ and the Laplace operator } \sum_{j=1}^n [p_j, [p_j, \cdot]] \text{ is} \]
\[ E_n(a, b) = 2 \text{Tr}(\hat{a}^* \hat{b}). \]

Proof. Given the Laplace operator and \( \text{Tr} \), from (2.3), this Dirichlet form defined on \( M_n(\mathbb{C}) \) is
\[ E_n(a, b) = \text{Tr}(\sum_{j=1}^n [p_j, a]^* [p_j, b]). \]
As
\[ \text{Tr}([p_j, a]^* [p_j, b]) = \sum_{i=1, i \neq j}^n (a_{i,j}^* b_{i,j} + a_{j,i}^* b_{j,i}) = \sum_{i=1}^n (\hat{a}_{i,j}^* \hat{b}_{i,j} + \hat{a}_{j,i}^* \hat{b}_{j,i}), \]
therefore, \( E_n(a, b) = \sum_{i,j} (\hat{a}_{i,j}^* \hat{b}_{i,j} + \hat{a}_{j,i}^* \hat{b}_{j,i}) = 2 \text{Tr}(\hat{a}^* \hat{b}). \)

So far, we have defined the Dirichlet form \( E_n \) on \( M_n(\mathbb{C}) \). Regard \( M_n(\mathbb{C}) \) as a subalgebra of \( M_{n+1}(\mathbb{C}) \) via the embedding
\[ a \rightarrow \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \]
we see that \( \delta_{n+1} \) is compatible with \( \delta_n \), i.e., \( \delta_{n+1}(a) = \delta_n(a), \forall a \in M_n(\mathbb{C}) \), and \( E_{n+1} \) is compatible with \( E_n \), i.e., \( E_{n+1}(a, b) = E_n(a, b), \forall a, b \in M_n(\mathbb{C}) \). We are thus led to define the differential calculus, Dirichlet form on \( \cup_n M_n(\mathbb{C}) \). As the norm closure of \( \cup_n M_n(\mathbb{C}) \) is \( \mathcal{K}(l^2) \), i.e., the compact operators on \( l^2 \), we have the following theorem after this extension. In the theorem below, we use the notation \( \mathcal{HS}(l^2) \) to represent the Hilbert-Schmidt operators on \( l^2 \), i.e., \( x \in \mathcal{HS}(l^2) \iff \text{Tr}(x^* x) < \infty \).

Theorem 3.2. \( E(a, b) = 2 \text{Tr}(\hat{a}^* \hat{b}) \) is a Dirichlet form on \( \mathcal{HS}(l^2) \), the corresponding Laplace operator is \( \sum_{j=1}^\infty [p_j, [p_j, \cdot]] \). The restriction of \( E \) on \( M_n(\mathbb{C}) \) is \( E_n \).

Proof. We see at once that \( L^\infty(\mathcal{K}(l^2), \text{Tr}) = \mathcal{B}(l^2), L^2(\mathcal{K}(l^2), \text{Tr}) = \mathcal{HS}(l^2), \) and \( \text{Dom}(E) = \mathcal{HS}(l^2) \). If we let \( \Delta = \sum_{j=1}^\infty [p_j, [p_j, \cdot]] \), then \( \Delta(a) = \hat{a} \) for \( a \in \mathcal{B}(l^2) \).
In this way, the semigroup $\Phi_t(a) = e^{-t\Delta}a = a - a + e^{-t}a$. An easy computation shows $Tr(\bar{a}^* \bar{b}) = Tr(\Delta(a)^*b)$ and

$$-\Delta(a) = 2 \sum_{i=1}^{\infty} p_j a p_j \equiv 2a.$$ 

By using the same argument as from proposition (2.2), $\Delta$ is seen to be the generator of a symmetric, conservative, completely positive Markov semigroup, which completes the proof. \hfill $\square$

If we regard $M_{2n}(\mathbb{C})$ as a subalgebra of $M_{2n+1}(\mathbb{C})$ via the embedding

$$a \to \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

and consider the fact there exists a unique tracial state $tr$, we can perform the same construction as above and define a Dirichlet form on hyperfinite $II_1$ factor. \cite{16} provides a detailed study for this construction.

The following is the deRham complex of $M_n(\mathbb{C})$ with the differential basis $p_1, \ldots, p_n$.

$$0 \rightarrow M_n(\mathbb{C}) \xrightarrow{\delta} \Omega^1 M_n(\mathbb{C}) \xrightarrow{\delta} \Omega^2 M_n(\mathbb{C}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Omega^n M_n(\mathbb{C}) \xrightarrow{\delta} 0.$$ 

The 0-th deRham cohomology group of $M_n(\mathbb{C})$, which we will show from section 4, is

$$H^0_{dR}(M_n(\mathbb{C})) = \mathbb{C}^n.$$

4. Graph $C^*$-algebras

For a more detailed introduction to graph $C^*$-algebras we refer to \cite{3},\cite{10} and the reference therein. A directed graph $E = (E^0, E^1, r, s)$ consists of countable sets $E^0$ of vertices and $E^1$ of edges, and maps $r, s : E^1 \rightarrow E^0$ identifying the range and source of each edge. The graph is row-finite if each vertex emits at most finitely many edges. We write $E^\infty$ for the set of paths $\mu = \mu_1 \mu_2 \ldots \mu_n$ of length $|\mu| := n$; that is, sequences of edges $\mu_i$ such that $r(\mu_i) = s(\mu_{i+1})$ for $1 \leq i < n$. The map $r, s$ extend to $E^* := \cup_{n \geq 0} E^n$ in an obvious way, and $s$ extends to the set $E^\infty$ of infinite paths $\mu = \mu_1 \mu_2 \ldots$. A sink is a vertex $v \in E^0$ with $s^{-1}(v) = \emptyset$, a source is a vertex $w \in E^0$ with $r^{-1}(w) = \emptyset$.

A Cuntz-Krieger $E$-family in a $C^*$-algebra $B$ consists mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ satisfying the Cuntz-Krieger relations

$$s_e^* s_e = p_{r(e)} \text{ for } e \in E^1 \text{ and } p_v = \sum_{\{e : s(e) = v\}} s_e s_e^* \text{ whenever } v \text{ is not a sink}.$$ 

It is proved in \cite{10} that there is a universal $C^*$-algebra $C^*(E)$ generated by a non-zero Cuntz-Krieger $E$-family $\{s_e, p_v\}$. A product $s_\mu := s_{\mu_1} s_{\mu_2} \ldots s_{\mu_n}$ is non-zero precisely when $\mu = \mu_1 \mu_2 \ldots \mu_n$ is a path in $E^n$. Since the Cuntz-Krieger relations imply that the projections $s_e s_e^*$ are also mutually orthogonal, we have $s_e^* s_f = 0$ unless $e = f$, and words in $\{s_e, s_f^*\}$ collapse to products of the form $s_\mu s_\nu^*$ for $\mu, \nu \in E^*$ satisfying $r(\mu) = r(\nu)$. Indeed, because the family $\{s_\mu s_\nu^*\}$ is closed under multiplication and involution, we have

\begin{equation}
C^*(E) = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}.
\end{equation}
We adopt the conventions that vertices are paths of length 0, that \( s_v := p_v \) for \( v \in E^0 \), and all the paths \( \mu, \nu \) appearing in (4.1) are non-empty; we recover \( s_\mu \), for example, by taking \( \nu = r(\mu) \), so that \( s_\mu^* s_\mu = s_\mu p_{r(\mu)} = s_\mu \).

For simplicity, we will only consider the graph which has the finite vertices. It is then easily seen that \( \{p_v : v \in E^0\} \) satisfies the condition (I)(II) from definition 1.1, in this way, we have defined the noncommutative differential calculus on \( \mathcal{C}^*(E) \).

**Lemma 4.1.** Fix a vertex \( v_0 \in E^0 \) and a path \( \mu \). If \( |\mu| = n(n \neq 0) \), let \( \mu = \mu_1 \mu_2 \ldots \mu_n \), otherwise \( \mu \) is a vertex.

1. if \( s(\mu) \neq r(\mu) \), then
   \[
   [p_{v_0}, s_\mu] = \begin{cases} 
   s_\mu & \text{if } v_0 = s(\mu), \\
   -s_\mu & \text{if } v_0 = r(\mu), \\
   0 & \text{otherwise}.
   \end{cases}
   \]

2. if \( s(\mu) = r(\mu) \), i.e., path \( \mu \) is a loop, then \( [p_{v_0}, s_\mu] = 0 \).

3. \( \delta(s_\mu) = s_\mu dp_{s(\mu)} - s_\mu dp_{r(\mu)} \).

**Proof.** The case \( |\mu| = 0 \) is easy. So we assume \( |\mu| = n \), and \( \mu = \mu_1 \mu_2 \ldots \mu_n \). From the identification that \( p_{v_0} = s_{v_0} \), we know \( s_{v_0} s_\mu = s_\mu \) iff \( s(\mu) = v_0 \) and \( s_\mu s_{v_0} = s_\mu \) iff \( r(\mu) = v_0 \), which proves the lemma. \( \square \)

The remainder of this section will be devoted to discuss the 0-th deRham cohomology of \( \mathcal{C}^*(E) \).

**Proposition 4.2.** \( C_{0,0} = \{ s_\mu s_\nu^*, \mu, \nu \in E^*, r(\mu) = r(\nu), s(\mu) = s(\nu) \} \)

**Proof.** The above lemma and relation \( [p_{v_0}, s_\mu]^* = -[p_{v_0}, s_\mu] \) shows
\[
\delta(s_\mu s_\nu^*) = \delta(s_\mu) s_\nu^* + s_\mu \delta(s_\nu^*) = s_\mu s_\nu^* dp_{s(\mu)} - s_\mu s_\nu^* dp_{r(\mu)} - s_\mu (s_\nu^* dp_{s(\nu)} - s_\nu^* dp_{r(\nu)}) = s_\mu s_\nu^* dp_{s(\mu)} - s_\mu s_\nu^* dp_{s(\nu)}
\]
\( \square \)

**Corollary 4.3.** If \( E \) has no loops, then \( C_{0,0} = \{ s_\mu s_\nu^*: \mu \in E^* \} \).

Notice, \( s_\mu \) is a nonzero partial isometry with \( s_\mu s_\mu^* \leq p_{s(\mu)} \). Given a path \( \mu \), the next proposition gives a criteria for when \( s_\mu s_\mu^* = p_{s(\mu)} \) is true.

**Proposition 4.4.** Suppose \( \mu \) is a path and \( \mu = \mu_1 \mu_2 \ldots \mu_n \), then \( s_\mu s_\mu^* = p_{s(\mu)} \) iff \( \{ e \in E^1 : s(e) = s(\mu_j) \} = \{ \mu_j \} \) for \( j = 1, \ldots, n - 1 \). Intuitively, it is the kind of path, on which all the nodes, except the range node, have only one exit.

**Proof.** The proof follows from the Cuntz-Krieger relations
\[
s_e^* s_e = p_{r(e)} \text{ for } e \in E^1 \text{ and } p_v = \sum_{\{e : s(e) = v\}} s_e s_e^*
\]
whenever \( v \) is not a sink. \( \square \)

The above two propositions provide a way to get the deRham cohomology \( H^0_{dR}(\mathcal{C}^*(E)) \), which has special meaning behind the graph itself. If the directed graph \( E \) has no loops, then \( H^0_{dR}(\mathcal{C}^*(E)) \cong \mathbb{C}^m \), where \( m \geq \# \) vertices. In particular, if the graph \( E \) is a tree with only one sink, then \( m = \# \) vertices. The matrix algebra \( M_n(C) \) discussed in section 3 is a tree with one root and \( n - 1 \)
nodes connecting to the root. And we have $H^0_{dR}(M_a(\mathbb{C})) = n$. When $E$ has loops, $H^0_{dR}(C^*(E))$ is a bit more complicated, say if $E$ contains a simple loop with the property that this loop doesn’t have exit, then $H^0_{dR}(C^*(E))$ contains $C(\mathbb{T})$. (the loop is simple when its vertices are distinct)

5. 2N-DIMENSIONAL QUANTUM TORI

Let’s look at the 2n-dimensional quantum tori, $U_1, \ldots , U_{2n}$ are its unitary generators which subject to the relation $U_kU_j = e^{i\theta_{jk}}U_jU_k$. $\Theta = (\theta_{j,k})$ is a skew symmetric matrix which by the general spectral theorem, it can be orthogonal transformed to a block matrix, with the entry (we use the same letter) $\theta_{j,k} = 0$ except possibly these $\theta_{2j-1,2j}$ and $\theta_{2j,2j-1}$ ($j = 1, \ldots , n$). Then the unitary operators $U_{2j-1}, j = 1, \ldots , n$ satisfy the conditions (I) (II) from definition (1.1). So we have defined the noncommutative differential calculus on this quantum tori. Its first order differential form is

$$\delta A = \sum_{j=1}^{n} [U_{2j-1}, A]dU_{2j-1} + \sum_{j=1}^{n} [U_{2j-1}^*, A]dU_{2j-1}^* .$$

Take for simplicity, let’s assume $n=1$, and the unitary generators $U, V$ satisfies the relation $UV = e^{i\theta}VU$. In order to compute its cohomology groups, we need first to understand the set $\mathbb{C}_{0,0} = \{ A \in \mathcal{A} : [U, A] = 0 \}$. Define $\hat{\theta}_{s,t}(U^kV^l) = e^{i(sk+tl)}U^kV^l ((s, t) \in \mathbb{R}^2)$, then [15] shows $\hat{\theta}$ defines an action of $\mathbb{R}^2$ by automorphism of $\mathcal{A}$.

**Theorem 5.1.** If $\theta/\pi$ is irrational, then $\mathbb{C}_{0,0} = < U >$(the algebra generated by $U$). If $\theta/\pi = \frac{2}{q}(p \neq 2)$, then $\mathbb{C}_{0,0} = < U, V^{2q} > = \{ A \in \mathcal{A} : \hat{\theta}_{0,\frac{2}{q}}(A) = A \}$. If $\theta/\pi = \frac{2}{q}$, then $\mathbb{C}_{0,0} = < U, V^q > = \{ A \in \mathcal{A} : \hat{\theta}_{0,\frac{2}{q}}(A) = A \}$.

**Proof.** The prove is based on the concept of Fourier analysis method, which is already shown from [15]. The Fourier coefficient of $A$ is $a_{k,l}(A) = < Ae_{0,0}, e_{k,l} >$, thereafter

$$a_{k,l}(UA) = < UAe_{0,0}, e_{k,l} > = < Ae_{0,0}, e^{-i\theta/2}e_{k-1,l} > = e^{i\theta/2}a_{k-1,l}(A),$$

$$a_{k,l}(AU) = < AUe_{0,0}, e_{k,l} > = < Ae_{1,0}, e_{k,l} > = e^{i\theta/2}a_{k-1,l}(A),$$

hence if $A \in \mathbb{C}_{0,0}$, then $UA = AU$, 0 = $e^{i\theta/2}(1 - e^{-i\theta})a_{k-1,l}$, and the proof is complete. \[\square\]

The following is the deRham complex of $\mathcal{A}$ from this differential operator $\delta$,

$$0 \xrightarrow{\delta} \mathcal{A} \xrightarrow{\delta} \Omega^1\mathcal{A} \xrightarrow{\delta} \Omega^2\mathcal{A} \xrightarrow{\delta} 0 .$$

From above theorem, when $\theta$ is irrational,

$$H^0_{dR}(\mathcal{A}) = < U > \cong C(\mathbb{T}) .$$

The principal significance of the above introduced differential calculus is that it allows to deform to the classical one. The remainder of this section will be devoted to show it.

First let’s investigate the geometric meaning behind the derivation $\delta$. Suppose an element $a \in \mathcal{A}$ for the moment has the form $a = f(U, V) = \sum a_{m,n}U^mV^n$. In
fact, it is true that the above elements are dense in $A$. Take $U$ as the differential basis first, then the first order form is

$$
\delta_U f(U, V) = U(f(U, V) - f(U, e^{-i\theta}V))dU + U^*(f(U, V) - f(U, e^{i\theta}V))dU^*
$$

$$
= U(a - \hat{\theta}_{-\theta,0}(a))dU + U^*(a - \hat{\theta}_{\theta,0}(a))dU^*
$$

As $\hat{\theta}_{-\theta,0}$ is continuous in norm, for any $a \in A$, we actually have

$$
\delta_U a = U(a - \hat{\theta}_{-\theta,0}(a))dU + U^*(a - \hat{\theta}_{\theta,0}(a))dU^*.
$$

More general, we could take $U^{k_1}V^{k_2}$ as the differential basis. A slight change in the above observation actually leads to the following theorem.

**Theorem 5.2.** Given $a \in A$, the first order form by taking $U^{k_1}V^{k_2}$ as the differential basis is

$$
\delta_{U^{k_1}V^{k_2}} a = U^{k_1}V^{k_2}(a - \hat{\theta}_{-k_1\theta,k_2\theta}(a))dU^{k_1}V^{k_2} + (U^{k_1}V^{k_2})^*(a - \hat{\theta}_{k_1\theta,-k_2\theta}(a))d(U^{k_1}V^{k_2})^*.
$$

Interestingly, this derivation involves finite difference rather than derivations. Namely, $[U^{k_1}V^{k_2}, a]$ is the changing in $a$ along the $V$ direction by $-k_1\theta$ difference and the $U$ direction by $k_2\theta$ difference. This is a purely noncommutative phenomenon because difference quotients can’t be used as the basis of a differential calculus in the commutative case.

Next, coming back to the $2n$-dimensional quantum tori, based on the above theorem, we will show that the above first order form can be deformed to the classical one on $T^n$. In the sequel, $\theta_j$ denotes $\theta_{2j-1,2j} \neq 0, j = 1, \ldots, n$. We consider the derivative operator $\partial$ and $C^*$ subalgebra $B$ of $A$ generated by $U_2, U_4, \ldots, U_{2n}$. The relation matrix $\Theta$ tells us $B$ is commutative, and hence it is $*$ isomorphic to $C(T^n)$.

For our purpose, instead of taking the differential basis from the beginning of the section, we slightly change to $\theta_j^{-1}U_1, \ldots, \theta_j^{-1}U_{2j-1}, \ldots, \theta_j^{-1}U_{2n-1}$. Indeed, this again satisfies condition (I) and (II). For $a = f(U_2, U_4, \ldots, U_{2n}) \in B$,

$$
\partial a = \sum_{j=1}^{n} [\theta_j^{-1}U_{2j-1}, a]dU_{2j-1} = \sum_{j=1}^{n} \frac{1}{\theta_j} [U_{2j-1}, a]dU_{2j-1}.
$$

Then,

$$
\partial f = \sum_{j=1}^{n} \frac{1}{\theta_j} (f(U_2, \ldots, U_{2j}e^{i\theta}, \ldots, U_{2n}) - f(U_2, \ldots, U_{2j}, \ldots, U_{2n}))U_{2j-1}dU_{2j-1}.
$$

Since

$$
f(U_2, U_4, \ldots, U_{2n}) \cong f(e^{ix_2}, e^{ix_4}, \ldots, e^{ix_{2n}}),
$$
\[ \partial f \approx \sum_{j=1}^{n} f(e^{ix_2}, \ldots, e^{i(x_j+\theta_j)}, \ldots, e^{i2n}) - f(e^{ix_2}, \ldots, e^{ix_2}, \ldots, e^{i2n}) U_{2j-1} dU_{2j-1} \]

\[ \rightarrow \sum_{j=1}^{n} \frac{\partial f(e^{ix_2}, \ldots, e^{ix_2}, \ldots, e^{ix_2})}{\partial e^{ix_2}} i e^{ix_2} U_{2j-1} dU_{2j-1} \]

\[ = \sum_{j=1}^{n} \frac{\partial f(U_2, \ldots, U_2, \ldots, U_2)}{\partial U_{2j}} i U_{2j} U_{2j-1} dU_{2j-1} \]

\[ \approx \sum_{j=1}^{n} \frac{\partial f(U_2, \ldots, U_2, \ldots, U_2)}{\partial U_{2j}} dU_{2j} \]

as \( \theta_j \to 0, j = 1, \ldots, n \), which turns out to be quite similar as the classical first order differential calculus on \( T^n \). The same argument works for any high order. The example from quantum tori demonstrates rather strikingly that this noncommutative differential calculus is closely related to the classical one.

6. Quantum Plane

In physics, it has the “quantization procedure” which from a classical Hamiltonian

\[ H(p_1, q_1, \ldots, p_n, q_n) \]

on the 2n-dimensional phase space \( \mathbb{R}^{2n} \) to the quantum mechanical Hamiltonian version

\[ H(P_1, Q_1, \ldots, P_n, Q_n). \]

Here \( Q_j \) denotes the multiplication operator on \( L^2(\mathbb{R}^n) \) corresponding to the coordinate mapping \( x_j \), and \( P_j = -i\hbar \frac{\partial}{\partial x_j} \) is a partial derivative operator. The procedure for getting the differential calculus on quantum plane is quite similar as from the quantum tori, we continue in the fashion by first investigating the simple case \( n = 1 \).

**Definition 6.1.** A canonical pair is informally described as a pair of self-adjoint operators \( P, Q \) on a Hilbert space \( \mathcal{H} \), satisfying (Heisenberg’s) canonical commutation relation (CCR)

\[ [P, Q] = -i\hbar I \]

The momentum operator defined on \( L^2(\mathbb{R}^2) \) by \( Pf(x, y) = yf(x, y) - i\frac{\hbar}{2} \frac{\partial f(x, y)}{\partial x} \)

and the position operator defined by \( Qf(x, y) = xf(x, y) + i\frac{\hbar}{2} \frac{\partial f(x, y)}{\partial y} \) satisfy the CCR relation.

By using the Fourier transform, we obtain the quantum observable

\[ f(P, Q) = \frac{1}{2\pi} \int \hat{f}(t_1, t_2) e^{i(t_1 P + t_2 Q)} dt_1 dt_2, \]

where \( f \) is in the Schwartz class \( S(\mathbb{R}^2) \).

Fix \( k_1, k_2 \in \mathbb{R} \), we do the differential calculus with the basis \( \frac{1}{\hbar} e^{i k_1 P + i k_2 Q} \). Then

\[ \delta f(P, Q) = \frac{e^{i k_1 P + i k_2 Q}}{\hbar} f(P, Q) d\epsilon_{i k_1 P + i k_2 Q} + \frac{e^{-i k_1 P - i k_2 Q}}{\hbar} f(P, Q) d\epsilon_{-i k_1 P - i k_2 Q} \]
In conclusion, when \( \hbar \) linear sum in two directions.

The above deformed differential calculus, comparing to the classical one, is the one noncommutative differential calculus coming from the inner derivation 13.

where \( f \in \mathcal{S}(\mathbb{R}^2) \).

\[
\lim_{\hbar \to 0} \frac{e^{ik_1P + ik_2Q}}{\hbar} \left[ f(P, Q) \right] = \lim_{\hbar \to 0} \frac{1}{2\pi} \int \hat{f}(t_1, t_2) \frac{1}{\hbar} \left[ e^{ik_1P + ik_2Q}, e^{i(t_1P + t_2Q)} \right] dt_1 dt_2
\]

\[
= \lim_{\hbar \to 0} \frac{1}{2\pi} \int \hat{f}(t_1, t_2) e^{i(t_1P + t_2Q)} e^{ik_1P + ik_2Q} (e^{ik_1t_2h - ik_2t_1h} - 1) dt_1 dt_2
\]

\[
= \frac{1}{2\pi} \int \hat{f}(t_1, t_2) e^{i(t_1P + t_2Q)} i(k_1t_2 - k_2t_1) dt_1 dt_2 e^{ik_1P + ik_2Q}
\]

\[
= (k_1 \frac{\partial f(P, Q)}{\partial Q} - k_2 \frac{\partial f(P, Q)}{\partial P}) e^{ik_2P + ik_2Q}.
\]

In conclusion, when \( \hbar \to 0 \), \( \delta f(P, Q) \) deforms to

\[
(k_1 \frac{\partial f(P, Q)}{\partial Q} - k_2 \frac{\partial f(P, Q)}{\partial P}) e^{ik_2P + ik_2Q} - (k_1 \frac{\partial f(P, Q)}{\partial Q} - k_2 \frac{\partial f(P, Q)}{\partial P}) e^{-ik_1P - ik_2Q} d e^{-ik_1P - ik_2Q}.
\]

The above deformed differential calculus, comparing to the classical one, is the linear sum in two directions.

Having disposed of the simple case \( n = 1 \), we now return to the general \( 2n \)-dimensional quantum plane. Quite the same way as from the quantum tori, we take the differential basis to be \( \frac{1}{\hbar} e^{iP_1}, \ldots, \frac{1}{\hbar} e^{iP_n} \), it satisfies the condition (I) and (II) from definition (1.1). We consider only the derivative operator \( \partial \) and subalgebra \( \mathcal{B} \) generated by \( Q_1, Q_2, \ldots, Q_n \). By the commutative relations of \( Q_j, \mathcal{B} \) is isomorphic to \( C_0(\mathbb{R}^n) \). We start doing the first order calculus of the element \( f(Q_1, \ldots, Q_n) \) where \( f \in \mathcal{S}(\mathbb{R}^n) \).

\[
\lim_{\hbar \to 0} \frac{\partial f(e^{iP_1}, \ldots, e^{iP_n})}{\partial Q_j} f(Q_1, \ldots, Q_n)
\]

\[
= \lim_{\hbar \to 0} \sum_{j=1}^{n} \frac{1}{\hbar} \left[ e^{iP_j}, f(Q_1, \ldots, Q_n) \right] e^{iP_j}
\]

\[
= \lim_{\hbar \to 0} \sum_{j=1}^{n} \frac{1}{2\pi} \int \hat{f}(t_1, \ldots, t_n) e^{i(t_1Q_1 + \ldots + t_nQ_n)} \frac{e^{it_j} - 1}{\hbar} dt_1 \ldots dt_n e^{iP_j}
\]

\[
= \sum_{j=1}^{n} \frac{1}{2\pi} \int \hat{f}(t_1, \ldots, t_n) e^{i(t_1Q_1 + \ldots + t_nQ_n)} it_j dt_1 \ldots dt_n e^{iP_j}
\]

\[
= \sum_{j=1}^{n} \frac{\partial f(Q_1, \ldots, Q_n)}{\partial Q_j} e^{iP_j} e^{iP_j}
\]

\[
= \sum_{j=1}^{n} \frac{\partial f(Q_1, \ldots, Q_n)}{\partial Q_j} dQ_j
\]

which deforms to the classical first order differential calculus on \( \mathcal{S}(\mathbb{R}^n) \). We have thus showed the above result for Schwartz class functions, combining the fact that \( \mathcal{S}(\mathbb{R}^n) \) is dense in \( C_0(\mathbb{R}^n) \), we actually have showed its relation with the classical calculus for each \( C_0(\mathbb{R}^n) \) functions.
7. Quantum Heisenberg Algebra

The quantum Heisenberg manifolds $D^c_{\mu,\nu}$, a continuous field of $C^*$-algebra first introduced by Rieffel [12], recently, [1] uses the Fell bundles method to get the same quantum Heisenberg algebra as defined in [12]. The construction here to get the 3-dimensional quantum Heisenberg manifolds follows from [1]. Let

$$G = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

be the Heisenberg group of $3 \times 3$ matrices. The Heisenberg manifold $M_\epsilon$ is the quotient $G/H_\epsilon$ where $H_\epsilon$ is the discrete subgroup of $G$ when $x, y, cZ \in \mathbb{Z}$. To facilitate the notation, we identify the above matrix as the vector $(x, y, z)$ and the deformation actions $\theta_c$ be the Heisenberg group of $3$ matrices. The Heisenberg manifold $M_c$ is the quotient $G/H_c$ where $H_c$ is the discrete subgroup of $G$ when $x, y, cz \in \mathbb{Z}$. To facilitate the notation, we identify the above matrix as the vector $(x, y, z)$ and the multiplication rule is simply

$$(x, y, z)(m, n, p) = (x + m, y + n, z + p + ym)$$

So $M_c$ can be described as the quotient of the Euclidean space $\mathbb{R}^3$ by the right action of $H_c$ given by above multiplication. Denote $[x, y, z]$ as the quotient class of $(x, y, z)$ in $M_c$.

$$\phi_{(a,b)}[x, y, z] = [x + 2b\mu, y + 2b\nu, z + 2b\mu x + 2b^2\mu\nu + a/c]$$

defines an action of $\mathbb{R}^2$ on $M_c$. This action generates the deformation data $(\mathbb{Z}, \gamma, \theta^h)(\text{see definition 4.2 [1]})$, the gauge action $\gamma$ is an action on the circle group defined by

$$\gamma_c([x, y, z]) = [x, y, z + t/c]$$

and the deformation actions $\theta_h$ of $\mathbb{Z}$ is defined by

$$\theta_h([x, y, z]) = [x + 2h\mu, y + 2h\nu, z + 2h\mu x + 2h^2\mu\nu].$$

For each $k \in \mathbb{Z}$, let

$$B_k = \{ f \in C(M_c) : \gamma_{c^{2\pi i k}}(f) = e^{2\pi i \theta k}f, \forall \theta \in \mathbb{R} \} = \{ e^{2\pi i k c}f(x, y) : f(x + 1, y) = e^{-2\pi i k c}f(x, y), f(x, y) \in C(\mathbb{R} \times \mathbb{T}) \}.$$

Take $f_k \in B_k$, $g_j \in B_j$, define the product and involution by

$$f_k \times g_j = f_k \theta_{kh}(g_j)$$

$$f_k^* = \theta_{kh}^{-1}(f_k)$$

Then $C(M_c)_{\gamma}^h = (B_k, k \in \mathbb{Z}, \times, *)$ becomes a Fell bundle by keeping the linear, topological and norm structure from $C(M_c)$. Theorem (7.1) [1] shows $C(M_c)_{\gamma}^h \cong D^c_{\mu,\nu}$. We denote $N_h$ as the von Neumann algebra of the weak operator closure of $C(M_c)_{\gamma}^h$ on $L^2(\mathbb{R} \times \mathbb{T}^2)$. For the time being, without further notice, let us assume $h = 1$ or, what amounts to the same, that $\mu$ and $\nu$ are replaced, respectively, by $\hbar \mu$ and $\hbar \nu$.

In order to study the structure of Heisenberg manifolds, one possible way, as from the quantum tori, is to represent this $C^*$ algebra by generators. From (8) [1], we know that $C(M_c)_{\gamma}^h$ is generated by $B_0, B_1$. As $B_0$ is isomorphic to $C(\mathbb{T}^2)$, we get two generators $e^{2\pi i x}$, $e^{2\pi i y}$ if having a third generator, the generator $f_0(x, y)$ should not vanish at any point.

**Proposition 7.1.** Elements in $B_k(k \neq 0)$ always vanish somewhere.
Proposition 7.2. Each element in \( \bar{f} \) is invariant from the Heisenberg group action.

Proof: We give the proof only for the case \( k = 1 \), the other cases are the same.

Otherwise, assume \( 0 \neq f_0 \in B_1 \) and is also smooth, then \( e^{2\pi i k c z} \frac{df_0}{dx} \in C(\mathbb{T}^2) \), hence we can find a function \( g \in C(\mathbb{T}^2) \) such that \( \frac{df_0}{dx} = g f_0 \).

By solving this ODE, we have \( f_0(x, y) = C e^{\int_0^x g(s, y) ds} \) where \( C \) is some constant.

As \( f_0 \in B_1 \), \( f_0(x + 1, y) = e^{-2\pi i k c y} f_0(x, y) \), this gives \( e^{-2\pi i k c y} = e^{\int_0^1 g(x, y) dx} \), but we don’t have such a function \( g \in C(\mathbb{T}^2) \).

For an arbitrary \( f \in B_1 \), we find a smooth family \( \{ f_n \}_{n \geq 0} \in B_1 \) which uniformly converges to \( f \). For each \( f_n \), it has a vanishing point \( x_n \). Since \( f_n(x + 1, y) = e^{-2\pi i k c y} f_n(x, y) \), there is no loss of generality in assuming \( x_n \in [0, 1] \). Further, we can find a subsequence(we use the same \( n \)), such that \( x_n \to x^* \). Notice \( f \) is uniformly continuous on \([0, 1]\). Given \( \epsilon > 0 \), \( \exists N, \delta \), such that when \( n > N \) and \( |s - t| < \delta \),

\[
|f_n(s) - f_n(t)| \leq |f_n(s) - f(s)| + |f(s) - f(t)| + |f(t) - f_n(t)| < \epsilon.
\]

In particular, \( |f_n(x_m) - f_n(x^*)| < \epsilon \) when \( |x_m - x^*| < \delta \) and \( n > N \).

Let \( n = m \) be large enough, we conclude \( |f_n(x^*)| < \epsilon \). Let \( n \to \infty \), we have \( |f(x^*)| < \epsilon \) which shows \( x^* \) is a vanishing point of \( f \).

Above proposition tells us elements in \( B_1 \) always vanish somewhere, we can’t represent \( C(M_c)^0 \) simply by three generators. We have to find some other way to represent \( C(M_c)^0 \). Notice \( C(M_c)^0 \) is the closure of the direct sum \( \bigoplus_{k \in \mathbb{Z}} B_k \), any \( f(x, y, z) \in C(M_c)^0 \) can be decomposed uniquely by the Fourier transform on the \( z \)-coordinate, namely,

\[
f(x, y, z) = \sum_{k \in \mathbb{Z}} e^{2\pi i k c z} f_k(x, y)
\]

where \( e^{2\pi i k c z} f_k(x, y) \in B_k \), and

\[
f_k(x, y) = \int_0^1 f(x, y, z) e^{-2\pi i k c z} dz
\]

If we denote \( \Phi(x, y, k) = f_k(x, y) \), it turns out \( \Phi \) is a function on \( \mathbb{R} \times \mathbb{T} \), which is the original definition of \( L^c_{\mu, \nu} \) by Rieffel [12].

From [12] and above correspondence, \( \tau(f) = \int_0^1 \int_0^1 \int_0^1 f(x, y, z) dxdydz \) is a faithful normal finite trace state and is invariant from the Heisenberg group action. As \( C(M_c)^0 \) is the closure of the direct sum \( \bigoplus_{k \in \mathbb{Z}} B_k \), we hence get \( L^2(C(M_c)^0, \tau) = \bigoplus_{k \in \mathbb{Z}} \tilde{B}_k \), where \( \tilde{B}_k \) denotes the completion of \( B_k \) under \( \tau \).

Proposition 7.2. \( \tilde{B}_k \) and \( L^2(\mathbb{T}^2) \) is isomorphic.

Proof. Each element in \( \tilde{B}_k \) has the form \( e^{2\pi i k c x} f(x, y) \), where \( f(x + 1, y) = e^{-2\pi i k c y} f(x, y) \) and \( f(x, y + 1) = f(x, y) \), which is easy to infer from the element from \( B_k \) and completion under \( \tau \). So we define the linear map \( \tilde{U} : \tilde{B}_k \to L^2(\mathbb{T}^2) \) by

\[
\tilde{U} e^{2\pi i k c x} f(x, y) = e^{2\pi i k c x} e^{2\pi i k c y} f(x, y).
\]
Proposition 7.5

Proof. This is because \( \langle \phi \rangle \) is generated by \( U,V,W \).

\[ \tau(e^{2\pi i k z} f(x,y)) = \tau(f(x-2k\mu, y-2k\nu) f(x-2k\mu, y-2k\nu)) \]
\[ = \int_0^1 \int_0^1 f(x-2k\mu, y-2k\nu) f(x-2k\mu, y-2k\nu) dx dy \]
\[ = \int_0^1 \int_0^1 \overline{f(x,y)} f(x,y) dx dy. \]

Hence \( \langle e^{2\pi i k z} f(x,y), e^{2\pi i k z} f(x,y) \rangle_{\tilde{B}_k} = \langle \tilde{U} e^{2\pi i k z} f(x,y), e^{2\pi i k z} f(x,y) \rangle_{L^2(\mathbb{T}^2)} \)
and \( \tilde{U} \) is an isometry. As \( \varphi_{m,n} = e^{2\pi i (m\mu + n\nu)} e^{2\pi i k z} e^{-2\pi i c x} \in \tilde{B}_k \)(see [5]),
and \( \{ \tilde{U} \varphi_{m,n} \}_{m,n} \in \mathbb{Z} \) is an orthonormal basis \( L^2(\mathbb{T}^2) \), which implies \( \tilde{U} \) is also surjective.

From above, we know \( \{ \varphi_{m,n} \}_{m,n} \) is an orthonormal basis of \( \tilde{B}_k \), but the basis doesn’t derive the multiplication structure from \( \tilde{B}_1 \). Now let \( U = e^{2\pi i x}, V = e^{2\pi i y}, W = e^{2\pi i c} e^{-2\pi i c z} \), where the notation \( \{ x \} \) means the fraction part of \( x \).
Next proposition will show \( \{ U^n V^m W^k \}_{m,n} \in \mathbb{Z} \) is an orthonormal basis of \( \tilde{B}_k \).

Proposition 7.3. \( \{ U^n V^m W^k \}_{m,n} \in \mathbb{Z} \) is an orthonormal basis of \( \tilde{B}_k \).

Proof. Still denote \( \varphi_{m,n} \) as in Proposition 7.2, and let
\[ \psi_{m,n} = U^n V^m W^k \]
\[ = e^{2\pi i m x} e^{2\pi i n y} \prod_{j=0}^{k-1} e^{2\pi i c (z+2j\mu x+2j^2 \nu \mu)} \]
then \( \varphi_{m,n} = \frac{\psi_{m,n}}{\psi_{m,n}} \), and \( \frac{\psi_{m,n}}{\psi_{m,n}} = 1 \). \( \psi_{m,n} \) is also the orthonormal basis by the unitary transformation.

Corollary 7.4. \( \{ U^n V^m W^k \}_{m,n,k} \in \mathbb{Z} \) is an orthonormal basis for \( L^2(C(M_{c})^\theta, \tau) \).

Proposition 7.5. \( N_h \subset L^2(C(M_{c})^\theta, \tau) \).

Proof. This is because \( \tau \) is a finite normal tracial state.

As \( U,V,W \in N_h \subset L^2(C(M_{c})^\theta, \tau) \), we conclude \( N_h \) is the weak operator closure generated by \( U,V,W \).
The relation between \( U,V,W \) as is follows:
\[ U V = V U, U W = e^{-4\pi i \mu} W U, V W = e^{-4\pi i \nu} W V, \]
\[ U U^* = U^* U = V V^* = V^* V = W W^* = W^* W = 1. \]
The above relation gives more, namely, \( N_h \) is a 3 dimensional quantum tori. So we actually have proved the following theorem.

Theorem 7.6. \( C(M_{c})^\theta \) (or \( D_{\mu,\nu} \)) is not a quantum tori, while the von Neumann algebra \( N_h \) is a 3-dimensional quantum tori.

In the language of the generators, another way to state the faithful normal finite tracial state \( \tau \) defined above is
\[ \tau(\sum_{m,n,k} a_{m,n,k} U^n V^m W^k) = \delta_{0,0,0}. \]
In $D^\mu_{\nu}$, it has three canonical unbounded derivation operators $D_1, D_2, D_3$ defined by

$$D_1 f = \frac{\partial f}{\partial x} = \sum_{k \in \mathbb{Z}} e^{2\pi i k c z} \frac{\partial f_k(x, y)}{\partial x},$$

$$D_2 f = \frac{\partial f}{\partial y} + x \frac{\partial f}{\partial z} = \sum_{k \in \mathbb{Z}} e^{2\pi i k c z} \left( \frac{\partial f_k(x, y)}{\partial y} + 2\pi i k c f_k(x, y) \right),$$

$$D_3 f = \frac{\partial f}{\partial z} = \sum_{k \in \mathbb{Z}} e^{2\pi i k c z} 2\pi i c f_k(x, y).$$

We can now rephrase above unbounded derivation operators on $N_\hbar$ in terms of generators as follows:

$$D_1 U = 2\pi i U, D_1 V = 0, D_1 W = \sum_{k \neq 0} c_k^{-1} V^k W,$$

$$D_2 U = 0, D_2 V = 2\pi i V, D_2 W = 0,$$

$$D_3 U = 0, D_3 V = 0, D_3 W = 2\pi i c W.$$

The summation above is in the weak operator convergence sense. The elementary relation properties of these derivative operators are listed below:

$$[D_1, D_2] = D_3, [D_1, D_3] = [D_2, D_3] = 0.$$

So far, we studied the structure of the quantum Heisenberg manifolds, instead of using the quantum Heisenberg $C^*$-algebra to apply the noncommutative differential calculus, we find it much easier to use it in the von Neumann algebra case, as it is nothing but a 3 dimensional quantum tori. The remainder of this section is quite the same as from section 5.

Suppose an element $a \in N_\hbar$ has the form

$$a = f(U, V, W) = \sum_{m, n, k} a_{m, n, k} U^m V^n W^k,$$

then

$$[U^k_1 V^k_2 W^k_3, f(U, V, W)]$$

$$= \sum_{m, n, k} a_{m, n, k} (e^{4\pi i k_3 (m + n) - 4\pi i (k_1 + k_2 \nu)} - 1) U^m V^n W^k U^{k_1} V^{k_2} W^{k_3}$$

$$= (f(e^{4\pi i k_3 \mu} U, e^{4\pi i k_3 \nu} V, e^{-4\pi i (k_1 + k_2 \nu)} W) - f(U, V, W)) U^{k_1} V^{k_2} W^{k_3}.$$

The viewpoint of above derivation sheds some light on the changing of the phase space in three directions, namely, with $4\pi i k_3 \mu$ in $x$-direction, $4\pi i k_3 \nu$ in $y$-direction, and $4\pi i (k_1 \mu + k_2 \nu)$ in $z$-direction. When $\hbar \to 0$, the deformed derivation is as follows:

$$\lim_{\hbar \to 0} \frac{1}{\hbar} [W, f(U, V, W)] = (4\pi i \mu \frac{\partial f(U, V, W)}{\partial U} U + 4\pi i \nu \frac{\partial f(U, V, W)}{\partial V} V) W,$$

$$\lim_{\hbar \to 0} \frac{1}{\hbar} [U, f(U, V, W)] = -4\pi i \mu \frac{\partial f(U, V, W)}{\partial W} WU,$$

$$\lim_{\hbar \to 0} \frac{1}{\hbar} [V, f(U, V, W)] = -4\pi i \nu \frac{\partial f(U, V, W)}{\partial W} WV.$$
References

1. Beatriz Abadie, Ruy Exel, Deformation quantization via Fell bundles, Math. Scand. 89 (2001), no. 1, 135-160.
2. Sergio Albeverio, Raphael Hoegh-Krohn, Dirichlet Forms and Markov Semigroups on C*-algebras, Comm. Math. Phys. 56 (1977), no. 2, 173-187.
3. Teresa Bates, David Pask, Iain Raeburn, Wojciech Szymański, The C*-algebras of row-finite graphs, New York J. Math. 6 (2000), 307-324.
4. Ola Bratteli, Derek Robinson, Operator algebras and quantum statistical mechanics. 2., Springer-Verlag, Berlin, 1997.
5. Partha Sarathi Chakraborty, Kalyan B. Sinha, Geometry on the quantum Heisenberg manifold, J. Funct. Anal. 203 (2003), no. 2, 425-452.
6. Fabio Ciprini, Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras, J. Funct. Anal. 147 (1997), no. 2, 259-300.
7. Fabio Ciprini, Jean-Luc Sauvageot, Derivation as square roots of Dirichlet forms, J. Funct. Anal. 201 (2003), no. 1, 78-120.
8. Alain Connes, Noncommutative differential geometry, Inst. Hautes tudes Sci. Publ. Math. No. 62, (1985), 257-360.
9. Alain Connes, Noncommutative geometry, Academic Press, Inc., San Diego, CA, 1994.
10. Alex Kumjian, David Pask, Iain Raeburn, Cuntz-Krieger algebras of directed graphs, Pacific J. Math. 184 (1998), no. 1, 161-174.
11. Gert K. Pedersen, C*-algebras and their automorphism groups, Academic Press, Inc. London-New York, 1979.
12. Marc A. Rieffel, Deformation quantization of Heisenberg manifolds, Comm. Math. Phys. 122 (1989), no. 4, 531-562.
13. J.-L. Sauvageot, Tangent bimodule and locality for dissipative operators on c* algebras, Quantum probability and applications, IV (Rome, 1987), 322-338.
14. Nik Weaver, Sub-Riemannian metrics for quantum Heisenberg manifolds, J. Operator Theory 43 (2000), no. 2, 223-242.
15. Nik Weaver, Mathematical Quantization, Chapman & Hall/CRC, Boca Raton, FL, 2001.
16. Bo Zhao, Dirichlet Forms on hyperfinite II1 factor, preprint.

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