Supersolid phase of extended Bose-Hubbard model with artificial gauge field

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(Dated: April 30, 2019)

We examine the zero and finite temperature phase diagram of the extended Bose-Hubbard model on a square optical lattice. To study various quantum phases and their transitions we employ single-site and cluster Gutzwiller mean-field theory. We have observed that the Mott insulator phase vanishes above a critical value of nearest-neighbour interaction and the supersolid phase occupies a larger region in the phase diagram. We show that the presence of artificial gauge field enlarges the domain of supersolid phase. The finite temperature destroys the crystalline structure of the supersolid phase and thereby favours normal fluid to superfluid phase transition. The presence of an envelope harmonic potential demonstrates coexistence of different phases and at $z k_B T \geq U$, the supersolidity of the system is destroyed.

I. INTRODUCTION

Ultracold atomic systems have played an important role in the study of quantum many-body systems. In particular, the novel experimental developments in manipulating ultracold atoms in optical lattices have lead to the realization of new quantum states and quantum phase transitions in strongly correlated systems [1, 2]. In recent years, there has been a surge of interest in understanding the supersolid (SS) phase which is characterized by the simultaneous appearance of a crystalline and an off-diagonal long-range order [3, 4]. This phase breaks two continuous symmetries: the phase invariance of the superfluid (SF) and translational invariance to form crystal. Although the SS phase was predicted in liquid 4He a long time ago [5, 6], the experimental observation of supersolidity in liquid 4He remains elusive [7–9]. However, the quest for SS phase has gained new impetus following the remarkable theoretical insights and experimental achievements in ultracold atoms in optical lattices, which are excellent quantum many-body systems to observe SS phase as these are clean and controllable. Recently, the characteristic signature of SS phase has been observed in ultracold atoms [10–14] and this phase may emerge by tuning the bosonic interactions of different length scales [15]. The existence and stability of SS phase have been confirmed in an optical cavity [16].

On theoretical front, the existence of SS phases has been studied using the extended Bose-Hubbard model (eBHM) with nearest-neighbour (NN) repulsions. However, the SS phase is fragile and previous studies have demonstrated that SS phase does not exist with NN interaction for square [17, 18] and honeycomb [19, 20] lattices. The checkerboard SS phase is unstable and the system undergoes phase separation into SF and solid phases [21]. However, dipolar interaction, which decays as the inverse cube of the distance, stabilize the SS phase [22, 23]. The SS phase can also be stabilized by tuning the anisotropy of the dipolar interaction [24, 25]. This phase has been explored in various lattice systems, such as one-dimensional (1D) chain [26, 27], two-dimensional (2D) square [21, 22, 28–31], triangular [32–37], honeycomb [19, 20], kagome [38], bilayer lattice of dipolar bosons [39], and three-dimensional (3D) cubic lattice [30, 40–42].

In this work we investigate theoretically the presence of SS phase in 2D square optical lattice with long-range interaction and artificial gauge field. The long-range interaction can be realized with the dipolar ultracold atoms [43, 44]. And, it is possible to introduce artificial gauge field with lasers [45–48]. We show that the combined effect of the long-range interaction and artificial gauge field increases the domain of the SS phase. In particular, we examine the effect of magnetic flux quanta on the SS-SF phase boundary in the presence of the NN interaction. Furthermore, to relate with experimental realizations, we incorporate the effects of thermal fluctuations arising from finite temperature. For our studies we use the single-site and cluster mean-field theories.

The paper is organized as follows. In Sec. II we introduce the model considered in our study and describe theoretical approach employed. Here we provide description of the single-site, cluster and finite temperature Gutzwiller (GW) mean-field theories. The ground-state phase diagrams and study of dipolar atoms in the confining potential are presented in Sec. III. Finally, we conclude with the key findings of the present work in Sec. IV.

II. THEORETICAL METHODS

A. Extended Bose-Hubbard model

Consider a system of bosonic atoms with long-range interactions in a 2D square optical lattice in the presence of synthetic magnetic field. The temperature of the system is low such that all the atoms occupy the lowest Bloch band. Such a system is well described by eBHM, and the Hamiltonian of
The model is

\[ \hat{H}_{\text{BHM}} = -\sum_{p,q} \left( J_x \hat{b}_{p+1,q}^\dagger \hat{b}_{p,q} + J_y \hat{b}_{p,q+1}^\dagger \hat{b}_{p,q} + \text{H.c.} \right) + \sum_{p,q} \left( \epsilon_{p,q} - \mu \right) + \frac{U}{2} \left( \hat{n}_{p,q} - 1 \right) + \sum_{\langle \xi' \xi \rangle} V_{\xi',\xi} \hat{n}_{\xi} \hat{n}_{\xi'}, \]

where \( p(q) \) is the lattice site index along \( x(y) \) direction, \( \hat{b}_{p,q}^\dagger \) is the bosonic operator which creates (annihilates) an atom at the lattice site \( (p,q) \), \( \hat{n}_{p,q} = \hat{b}_{p,q}^\dagger \hat{b}_{p,q} \) is the boson number operator, \( J_x \) and \( J_y \) are the tunneling or hopping strength between two NN sites along \( x \) and \( y \) directions, respectively, \( \epsilon_{p,q} \) is the offset energy arising due to the presence of external envelope potential, \( \mu \) is the chemical potential, and \( U > 0 \) is the on-site interatomic interaction. Here \( \xi \) is a combination of lattice indices in 2D, that is, \( \xi \equiv (p, q) \) and \( \xi' \equiv (p', q') \) are neighbouring sites to \( \xi \). The long-range interaction \( V_{\xi',\xi} \) for \( V_1 - V_2 \) model is given by

\[ V_{\xi',\xi} = \begin{cases} V_1 & \text{if } |r_{pq} - r_{p'q'}| = a, \\ V_2 & \text{if } |r_{pq} - r_{p'q'}| = \sqrt{2}a, \\ 0 & \text{otherwise,} \end{cases} \]

where \( a \) is the lattice spacing, \( r_{pq} = (pa, qa) \) is the lattice site coordinates. The parameters \( V_1 \geq 0 \) and \( V_2 \geq 0 \) are the NN and next NN (NNN) interactions, respectively. The other type of long-range interaction is the dipolar interaction. In the experiments of ultracold quantum gases, long-range interaction can arise from either electric or magnetic dipole moments. To examine the effect of the dipolar interaction, assume that the dipoles are polarized along \( z \)-axis. So that the atoms experience an isotropic repulsive dipole-dipole interaction in the \( xy \) plane. The long-range dipole interaction \( V_{\xi',\xi} \) in the 2D square lattice is

\[ V_{\xi',\xi} = \begin{cases} \frac{V_0 a^3}{|r_{pq} - r_{p'q'}|^3} & (p,q) \neq (p',q'), \\ 0 & (p,q) = (p',q'). \end{cases} \]

Since the dipolar interaction falls off as the inverse cube of the distance, the dipolar interaction is \( \{V, V/\sqrt{2}, V/\sqrt{3}, V/\sqrt{5}, \ldots\} \) for the NN, NNN, third, fourth neighbours, and so on. In theoretical studies the range of dipolar interaction is considered only up to certain neighbours. For the present work, we consider the first two terms, though simple this encapsulates all the observable effects of the dipole interactions in the system.

### B. Artificial gauge field

The long-range interaction in the above Hamiltonian, Eq. (1), is characteristic or inherent to the internal state of the atomic species. In terms of the many-body physics, the nature of the correlation can further be modified through the introduction of artificial gauge field. The presence of the artificial gauge field modifies the Hamiltonian to

\[ \hat{H}_{\text{BHM}} = -\sum_{p,q} \left( J_x e^{2\pi a q} \hat{b}_{p+1,q}^\dagger \hat{b}_{p,q} + J_y \hat{b}_{p,q+1}^\dagger \hat{b}_{p,q} + \text{H.c.} \right) + \sum_{p,q} \left( \epsilon_{p,q} - \mu \right) + \frac{U}{2} \left( \hat{n}_{p,q} - 1 \right) + \sum_{\langle \xi' \xi \rangle} V_{\xi',\xi} \hat{n}_{\xi} \hat{n}_{\xi'}, \]

where the strength of the magnetic field is reflected in the number of flux quanta per plaquette \( \alpha = (e/h) \int d^2r (A_x - A_y) \). Here, \( 0 \leq \alpha < 1 \), and \( A(r) \) is the vector potential which gives rise to synthetic magnetic field \( \mathbf{B} = \nabla \times \mathbf{A} \). In the presence of the synthetic magnetic field, atoms acquire a \( 2\pi \alpha \) phase when they hop around a plaquette. This results into a phase shift in the hopping strength of the model. Physically, the synthetic magnetic field introduces a force on the atoms which is equivalent of the Lorentz force on a charged particle in the presence of external magnetic field. The system then is a charge neutral analogue of the quantum Hall system in condensed matter systems. For the present study, we consider Landau gauge, where the vector potential \( A(r) = -By \hat{x} \). Hence, for the homogeneous system, at zero magnetic field the system possesses the translational invariance along both axes, whereas in the presence of magnetic field the system preserves the invariance only along the \( x \)-axis of the lattice.

### C. Gutzwiller mean-field theory

To study the ground-states of the systems described by the model Hamiltonians in Eq. (1) and (4) and their properties we use single-site Gutzwiller mean-field (SGMF) and cluster Gutzwiller mean-field (CGMF) theories. The later is the extension of SGMF which incorporates the correlation within a cluster of neighbouring sites exactly. In the SGMF theory [49–51], the bosonic operators are expanded about their expectation values as

\[ \hat{b}_{p,q} = \phi_{p,q} + \delta \hat{b}_{p,q}, \quad \hat{b}_{p,q}^\dagger = \phi_{p,q}^* + \delta \hat{b}_{p,q}^\dagger, \]

Therefore, the product of the creation and annihilation operators which occurs in the hopping term can be written as

\[ \hat{b}_{p,q}^\dagger \hat{b}_{p',q'} \sim \phi_{p,q}^* \phi_{p',q'} + \hat{b}_{p,q}^\dagger \phi_{p',q'} - \phi_{p,q} \phi_{p',q'}, \]

where second order terms in the fluctuation \( \delta \hat{b}_{p,q} \) are neglected. Here, \( \phi_{p,q} = (\hat{b}_{p,q}) \) is the SF order parameter of the system. Using above approximation in the Hamiltonian,
Eq. (4), the single-site mean-field Hamiltonian is
\[
\hat{H}_{\text{MF}}^{p,q} = -\left[ J_x e^{i2\pi \alpha q} \left( \phi_{p+1,q}^* \hat{b}_{p,q} - \phi_{p+1,q}^* \phi_{p,q} \right) + J_y \left( \phi_{p,q+1}^* \hat{b}_{p,q} - \phi_{p,q+1}^* \phi_{p,q} \right) \right] + \left( \epsilon_{p,q} - \mu \right) \hat{n}_{p,q} + \sum_{(\xi,\zeta)} V_{\xi,\zeta} \hat{n}_{\xi} \hat{n}_{\zeta},
\]
and the total Hamiltonian of the system is
\[
\hat{H}_{\text{MF}} = \sum_{p,q} \hat{H}_{\text{MF}}^{p,q}. \tag{8}
\]
Here, the neighbouring lattice sites are coupled through \(\phi_{p,q}\), the SF order parameter. And therefore the eigenstate of the entire lattice is the products of single-site states. Accordingly, the many-body wave function of the ground state of the system is given by the Gutzwiller ansatz
\[
|\Psi\rangle = \prod_{p,q} \langle \psi\rangle_{p,q} = \prod_{p,q} \left( \sum_{n=0}^{N_b} c_{n(p,q)}^{\dagger} |n\rangle_{p,q} \right), \tag{9}
\]
where \(|\psi\rangle_{p,q}\) is the single-site ground state, \(N_b\) is the number of occupation basis or maximum number of bosons at each lattice site, \(|n\rangle_{p,q}\) is the occupation or Fock state of \(n\) bosons occupying the site \((p, q)\) and \(c_{n(p,q)}^{\dagger}\) is the probability amplitude or coefficients of the occupation state. The normalization of the wave function leads to the normalization of \(c_{n(p,q)}\) at each lattice site as \(\sum_{n} |c_{n(p,q)}^{\dagger}|^2 = 1\). Using the above ansatz the SF order parameter \(\phi_{p,q} = \langle \Psi | \hat{b}_{p,q} | \Psi \rangle\) is obtained as
\[
\phi_{p,q} = \sum_{n=0}^{N_b} \sqrt{n} c_{n-1(p,q)}^{\dagger} c_{n(p,q)}. \tag{10}
\]
Similarly, the occupancy of each lattice site \(n_{p,q} = \langle \Psi | \hat{n}_{p,q} | \Psi \rangle\) is
\[
n_{p,q} = \sum_{n=0}^{N_b} n |c_{n(p,q)}|^2. \tag{11}
\]
The two parameters \(\phi_{p,q}\) and \(n_{p,q}\), together serve to define the quantum phases of the system. In the MI phase, \(\phi\) is zero and \(n_{p,q}\) is integer commensurate. The density-wave (DW) phase also has zero \(\phi\), and \(n_{p,q}\) integer but incommensurate with long-range crystalline order. In contrast, for the SF phase \(\phi\) is non-zero and \(n_{p,q}\) has real value, and both the parameters are commensurate across the lattice. The SS phase too has non-zero \(\phi\) and real \(n_{p,q}\), but both of these have long-range crystalline order.

Although, the quantum phases of the system can be defined based on \(\phi\) and \(n_{p,q}\), these do not serve as good markers of the phase boundaries. To identify the phase boundaries, the trends in the energy of the system is a reliable measure. From the mean-field Hamiltonian, Eq. (7), the total energy of the system \(E = \langle \Psi | \hat{H}_{\text{MF}} | \Psi \rangle\) is obtained as a sum of the single-site energies \(E_{p,q} = \langle \Psi | \hat{H}_{\text{MF}}^{p,q} | \Psi \rangle\). It is to be mentioned here that the computation of the total energy \(E = \sum_{p,q} E_{p,q}\) is also required to obtain the ground state in the SGMF theory. As in the SGMF theory for eBHM, \(E\) is minimized self consistently with the Eqs. (10) and (11).

In the CGMF theory, a lattice of dimension \(K \times L\) is partitioned into \(W\) clusters of size \(M \times N\), that is \(W = (K \times L)/(M \times N)\) [52–58]. Then, the hopping terms of the model are decomposed into two types. One is the exact term which corresponds to hopping within the cluster, and the other is the inter-cluster hopping between lattice sites which lie on the boundary of two neighbouring clusters. The latter is defined by coupling through the mean-field or the SF order parameter. The Hamiltonian of a cluster is
\[
\hat{H}_C = -\sum_{p,q \in C} \left( J_x e^{i2\pi \alpha q} \hat{b}_{p+1,q}^\dagger \hat{b}_{p,q} + J_y \hat{b}_{p,q+1}^\dagger \hat{b}_{p,q} + \text{H.c.} \right) + \sum_{p,q \in \partial C} \left( \epsilon_{p,q} - \mu \right) \hat{n}_{p,q} + U \hat{n}_{p,q} (\hat{n}_{p,q} - 1) + \sum_{(\xi,\zeta) \in \partial C} V_{\xi,\zeta} \hat{n}_{\xi} \hat{n}_{\zeta} \tag{12}
\]
where the model parameters \(J_x\), \(J_y\), \(U\) and \(V_{\xi,\zeta}\) are defined like in SGMF and prime in the first summation indicates that the \((p+1, q)\) and \((p, q+1)\) lattice sites are also inside the cluster. Here, \(\partial C\) in the second summation represents the lattice sites at the boundary of the clusters and \((\phi_{p,q}^C) = \sum_{p',q' \in \partial C} (\hat{b}_{p',q'}^\dagger)\) is the SF order parameter at the lattice site which lies at the boundary of neighbouring cluster. Like hopping, the long-range interaction term also has two contributions, one is within the cluster which is exact, and the other is inter-cluster interaction at the boundary which is defined through the mean occupancy \(\langle \hat{n}_{\xi} \rangle\). The matrix elements of \(\hat{H}_C\) are, then, calculated in terms of the cluster basis states
\[
|\Phi_{\ell}\rangle_\ell = \prod_{q=0}^{N_M-1} \prod_{p=0}^{M-1} |n_{p,q}^\ell\rangle, \tag{13}
\]
where \(|n_{p,q}^\ell\rangle\) is the occupation number basis at the \((p,q)\) lattice site, and \(\ell = \{n_0^0, n_1^0, \ldots, n_{M-1}^0, n_0^1, \ldots, n_{M-1}^1, \ldots, n_{M-1}^{N-1}\}\) is the index quantum number to identify the cluster state. After diagonalizing the Hamiltonian, we can get the ground state of the cluster as
\[
|\Psi_c\rangle = \sum_{\ell} C_{\ell} |\Phi_{\ell}\rangle_\ell, \tag{14}
\]
where \(C_{\ell}\) are components of the eigenvector, and naturally satisfy the normalization condition \(\sum_{\ell} |C_{\ell}|^2 = 1\). The ground state of the entire \(K \times L\) lattice, like in SGMF, is the direct
product of the cluster ground states
\[ |\Psi^c_{GW}\rangle = \prod_k |\Psi^c_k\rangle, \]  
where \( k \) is the cluster index and varies from 1 to \( W = (K \times L)/(M \times N) \). The SF order parameter \( \phi \), like in Eq. (10), can be computed in terms of the cluster states. The average occupancy of the \( k \)th cluster can also be computed similarly.

### D. Finite temperature Gutzwiller mean-field theory

At finite temperature, the thermal fluctuations modify the properties of the system, and observable properties are the thermal averages. To calculate the thermal averages we need the entire eigenspectrum. So, in the SGMF, we retain the entire energy spectrum \( E^l_{p,q} \) and the eigenstates \( |\psi^l_{p,q}\rangle \) obtained from the diagonalization of the single-site Hamiltonian \( \hat{H}^MF_{p,q} \) in Eq. (7). Then, we evaluate the single-site partition function of the system

\[ Z = \sum_{i=1}^{N_c} e^{-\beta E^i}, \]  
where \( \beta = 1/k_BT \) and \( T \) is the temperature of the system. At finite \( T \), the region in the phase diagram with \( \phi = 0 \) and the real commensurate occupancy \( \langle \hat{n}_{p,q} \rangle \) is identified as the normal fluid (NF) phase. Similarly, in the CGMF, the partition function is defined in terms of all the eigenvalues \( E^l_k \) and eigenfunctions \( |\Psi^l_k\rangle \) of each \( k \)th cluster from all the \( W \) clusters.

From the definition of the partition function, in the SGMF, the thermal average of \( \phi_{p,q} \) is

\[ \langle \phi_{p,q} \rangle = \frac{1}{Z} \sum_{i=0}^{N_c} \sum_{p,q} i \langle \hat{n}_{p,q} \rangle e^{-\beta E^i} |\psi^i_{p,q}\rangle^i, \]  
where \( \langle \ldots \rangle \) represents the thermal averaging. Similarly, the occupancy or the density at finite \( T \) is defined as

\[ \langle \hat{n}_{p,q} \rangle = \frac{1}{Z} \sum_{i=0}^{N_c} \sum_{p,q} i \langle \hat{n}_{p,q} \rangle e^{-\beta E^i} |\psi^i_{p,q}\rangle^i, \]  
The average occupancy is \( \langle n \rangle = \sum_{p,q} \langle \hat{n}_{p,q} \rangle / (K \times L) \). These definitions can be extended to the CGMF by replacing the single-site states and energies with that of the cluster.

### III. RESULTS AND DISCUSSIONS

The standard BHM shows two phases, the incompressible Mott insulator (MI) phase corresponding to commensurate integer filling, and the compressible SF phase which has finite \( \phi \). The SF-MI quantum phase transition was observed by tuning the depth of the optical lattice [59]. In eBHM, the introduction of the NN interaction changes the phase diagram through the emergence of two more phases. First is the DW, which sandwiches the MI lobes at low values of NN interaction, and second is the SS phase, it occurs as envelope around the DW lobes. In this work, we first examine the phase diagram for the homogeneous systems. We, then, study the impact of artificial gauge field on the phase diagram by considering \( \alpha = 1/2 \). For comparison with experimental realizations, we also study with envelope potential.

Besides \( \phi_{p,q} \) and \( \langle \hat{n}_{p,q} \rangle \), the relative average occupancy \( \Delta n \) is another order parameter which can distinguish the DW and SS phases from MI and SF phases. For a \( K \times L \) lattice and SGMF method it is defined as

\[ \langle \Delta n \rangle = \frac{1}{K \times L} \sum_{p,q} \langle \hat{n}_{p,q} \rangle - \langle \hat{n}_{p+1,q} \rangle, \]  
The similar expression can be defined for CGMF method. For the DW and SS phases \( \langle \Delta n \rangle \) is nonzero, and in particular, it is integer and real for the DW and SS phases, respectively. But, for MI and SF phases \( \langle \Delta n \rangle \) is zero as \( \langle \hat{n}_{p,q} \rangle \) is uniform.

![Phase Diagrams](image.png)

**FIG. 1.** The phase diagrams of the eBHM with uniform hopping ~\( J_x = J_y = J \)~ for \( V/U = 0.2, 0.27, 0.32, \) and 0.5 (a-d). The densities of the MI and DW phases are shown in the parenthesis. The plots show the phase boundaries for two cases \( \alpha = 0 \) and \( \alpha = 1/2 \). For the former case, the black line indicates the MI-SF or the DW-SS phase boundary and green line corresponds to the SS-SF phase boundary.

#### A. Homogeneous case

The phase diagram of the eBHM obtained from the SGMF is as shown in Fig. 1 for different values of \( V \). It is important to note that here \( V \) is NN interaction which is \( V_1 \) of model.
The nature of the DW-SS transition is better represented by $\langle \Delta n \rangle$ and values for $V = 0.22U$ and $0.32U$ corresponding to $V < V_c$ and $V > V_c$ are shown in Fig. 3. In the figure, the dark regions correspond to MI and SF phases and the region in other colors correspond to DW and SS phases. For $V = 0.22U$, the regions in yellow color are DW phases and regions in other shades correspond to SS. The gradient in the shades indicates that the transition from DW to SS in terms of $\langle \Delta n \rangle$ is smooth. For the case of $V = 0.32U$, there are no dark regions in the neighbourhood of $J/U \approx 0$. This is due to the absence of MI lobes, and is consistent with the phase diagram shown in Fig. 1 as all the MI lobes are transformed to DW lobes. The nature of the DW phases are apparent and visible from color gradient in Fig. 3 as the colors indicate the difference in
the occupancy of two neighbouring lattice sites. Like in the case of \( V = 0.22U \), regions with a color gradient indicate the SS phase and overall the relative average occupancy is in agreement with the phase diagram in Fig. 1. However, the phase diagram in terms of \( \langle \Delta n \rangle \) provides a richer descriptions of the two phases, DW and SS, unique to the eBHM vis-a-vis BHM. And, the appropriate order parameter to examine the regions of SS phase.

**B. Finite temperature effects**

FIG. 4. The finite temperature phase diagram of eBHM obtained from CGMF theory with \( 2 \times 2 \) clusters. The green striped region mark the DW and MI phases. (a) Phase diagram for \( \alpha = 0 \) at \( k_B T = 0.02U \). The black line marks MI-SF, NF-SF, DW-SS, and NF-SS phase boundaries. And, the red line marks the SS-SF phase boundary. (b) Phase diagram for \( \alpha = 1/2 \) at \( k_B T = 0.02U \). The blue line marks MI-SF, NF-SF, DW-SS, and NF-SS phase boundaries. And, the green line marks the SS-SF phase boundary. Phase diagrams for \( \alpha = 0 \) and \( 1/2 \) at (c) \( k_B T = 0.1U \) and (d) \( k_B T = 0.3U \) with the combined color scheme of (a) and (b).

Thermal fluctuations associated with finite temperatures are an essential feature of experimental observations. Although the zero temperature phase diagrams do provide key insights and qualitative understanding, to relate with the experimental results it is essential to incorporate thermal fluctuations. We do this through the approach outlined in Section II D. As mentioned earlier, the SS phase is yet to be observed in the eBHM and this could be due to the sensitivity of the phase to the thermal fluctuations. At zero temperature, SS phase appears in the system at a finite value of the NN interaction \( V \). In Fig. 4, we show the finite temperature phase diagram obtained using \( 2 \times 2 \) clusters in the CGMF method. As we have demonstrated and by others [52–58] that the results with CGMF are more reliable, hereafter we only consider the results from CGMF theory. From the plots in Fig. 4, a distinguishing feature of the thermal fluctuations is the emergence of the NF phase. The thermal fluctuation melts both the MI and DW phases and destroys the SF phase at the MI-DW or DW-DW boundaries.

To be more specific at \( k_B T = 0.02U \), as shown in Fig. 4(a-b), the green stripes mark the DW and MI phases and the NF phase exist outside of these. The MI and NF both have zero \( \phi \) and commensurate densities. The difference is that the MI has integer commensurate density, but NF has real commensurate density. The DW phase, on the other hand, has checkerboard density but with integer values. So, density can effectively be used to differentiate these three phases, namely MI, DW and NF. On comparing the plots in Fig. 4(a) and Fig. 4(b), it is clear that the larger MI and DW lobes with finite \( \alpha \) are retained at finite temperatures and hence the larger SS domain as well. At intermediate temperatures, both the MI and DW are entirely transformed into the NF phase but a portion of the SS lobes survives. This is visible in the phase diagram at \( k_B T = 0.1U \) shown in the Fig. 4(c). From the plots in the figure, the quantitative differences with and without the artificial gauge field is also visible. With artificial gauge field, the domain of the NF and SS phases are larger. For example, at \( \mu = 0 \) the NF extends upto \( J/U = 0.066 \) and \( J/U = 0.108 \) for zero and finite \( \alpha \), respectively. This trend of larger extent of NF phase with finite \( \alpha \) extends to higher values of \( \mu \). Upon further increase in temperature the crystalline order of the SS phase is destroyed and it vanishes from the phase diagram. At \( k_B T \approx 0.3U \), as shown in Fig. 4(d), only the NF and SF phases are present in the system. At \( \mu/U = 0 \) the NF-SF phase boundary is located at \( J/U = 0.132 \) and \( J/U = 0.220 \) for zero and finite \( \alpha \), respectively. The separation between the location of the phase boundaries is reduced as \( \mu/U \) increased. This is to be expected as the size of the DW lobes decrease with increasing \( \mu/U \).

FIG. 5. The zero temperature density distribution (upper panel) and SF order parameter (lower panel) of a \( 50 \times 50 \) square lattice for different values of dipolar interaction strengths \( V/U \) in \( V_{dip} \) model. The \( V/U \) value is shown at the top of the figures. Here \( x \) and \( y \) are in units of the lattice constant \( a \).
C. Inhomogeneous case

The system considered so far is uniform and we emulate it with a $12 \times 12$ lattice with periodic boundary conditions. However, in most of the quantum gas experiments the optical lattice has a confining envelope potential. Most often the external envelope potential is a harmonic oscillator. Hence, the inhomogeneity arising from this confining potential is another factor to be considered for comparison with the experimental observations. Therefore, we examine the ground-state of eBHM in a $50 \times 50$ square lattice with SGMF theory. In which the external harmonic potential is incorporated in the chemical potential through the offset energy $\epsilon_{p,q} = \Omega(p^2+q^2)$. Here $\Omega$ is the strength of the confining potential. The parameters of the system considered in the $V_{\text{dip}}$ model are $J/V = 0.1$, $\mu/V = 2.8$ and $\Omega/V = 0.01$ [61]. The value of $\Omega$ is such that the atomic density outside the lattice potential is zero.

To determine the changes in the competing phases we examine the ground state of the system at $V/U = 0.05$, 0.5, and 1.0. These values cover the weak and strong limits of the dipolar interaction. In the experiments these regimes are reachable using Feshbach resonance in the dipolar atoms like Cr [62], Er [63] and Dy [64, 65]. Like in the previous cases, to study the effects of the thermal fluctuations we consider three different values of $k_B T/U = 0.02$, and 0.3. At zero temperature profiles of $n_{p,q}$ and $\phi_{p,q}$ corresponding to $V/U = 0.05$, 0.5, and 1.0 are shown in Fig. 5. For weak dipolar interaction $V/U = 0.05$ the density is nearly uniform in the central region. The corresponding $\phi_{p,q}$ though nearly uniform shows a dip around the center. But, it is more uniform at the intermediate strength of the dipole interaction 0.5 $U$. In both of these cases, $V/U = 0.05$ and 0.5 $U$, the central SF region is surrounded by the $(1,0)$ DW phase and this is evident from the ring-shaped profile of the checkerboard density pattern in the figure. The other key feature is that the domain of the DW phase gets narrowed as $V/U$ is increased and above a critical value $V/U = 0.8$ there is a quantum phase transition from DW phase to SS phase. This happens when both the interaction strengths, on-site and dipole interactions are comparable. As shown in Fig. 5, for $V/U = 1$ there is a large region around the center where both the density and SF order parameter show checkerboard distributions. This is the signature of the SS phase.

Next, to relate to the experimental realizations we incorporate the finite temperature effects. For weak dipolar interaction the thermal fluctuations leads to the melting of the SF phase. This is evident from the density and SF order parameter corresponding to $V/U = 0.05$ at $k_B T/U = 0.2$ as shown in Fig. 6. A more detailed study, where $V/U$ is fixed and temperature is changed, shows that the SF phase at the central region does exist at lower temperatures. But, it melts to NF phase at the critical temperature of $k_B T/U = 0.16$. At the higher value of $V/U = 0.5$ the central SF region remerges and so does the SS phase at still higher value of $V/U = 1$. In short, with thermal fluctuations it is essential to have stronger dipolar interactions to observe SS phase. Considering the parameters of the experimental realization of dipolar condensates of $^{168}$Er in optical lattices [44], the corresponding temperature of $k_B T/U = 0.2$ is $\approx 40$ nK. This is within the experimental realm and hence, the combined effect of dipolar interaction and artificial gauge field can lead to the emergence of SS phase within experimentally achievable parameter domain.

At higher temperatures, $k_B T/U \approx 0.3$, the central region is in NF phase for weaker dipole interactions $V/U \leq 0.5$. Then, on increasing $V/U$ further the density assumes checkerboard pattern, but the SF order parameter is zero. That is the central region of the system is in the DW phase. This is to be compared and contrasted with the earlier result at $k_B T/U \approx 0.2$, where as shown in Fig. 6 the SS phase exists for $V/U = 1$. Thus, focusing on the strong interaction domain $V/U = 1$, our results show the existence of a SS-DW transition at $z k_B T = U$. In short, the SS phase exists in the system at lower temperatures, but the SS order melts into DW phase when $z k_B T \geq U$. On increasing the temperature further, the crystalline structure or diagonal long-range order of the DW phase starts to melt and at $z k_B T \approx 3 U$ system is fully in the NF phase. So, the melting of the SS phase occurs in two steps. First, the off-diagonal long-range SF order is destroyed. This transforms the SS phase into DW phase. And, second, the DW phase melts into NF phase. Like in the case of uniform system, there are no qualitative changes in the results with the introduction of artificial gauge field. However, the quantitative changes, which follows the same trends as in uniform system, can be important considerations in experimental realizations.

IV. CONCLUSIONS

We have examined the zero and finite temperature phase diagrams of eBHM in two dimensions using SGMF and CGMF.
theory. In the presence of artificial gauge field the domain of SS phase is enhanced and 2 × 2 CGMF theory provide better description of the system. At higher temperatures, the thermal fluctuations destroys the SS phase and the phase diagram exhibits NF-SF transition. Furthermore, we have studied the system of dipolar atoms in the presence of a harmonic confinement as these atoms are predicted to stabilize the SS phase. Our results show that beyond a critical threshold of confinement as these atoms are predicted to stabilize the SS phase and the phase diagram exhibits NF-SF transition. These suggest that the prospect of observing the SS phase is higher when the temperature $z k_B T < U$, this range of temperature is possible in the experiments of dipolar Bose gases loaded into optical lattices. This offers an opportunity to observe SS phase in quantum dipolar gas experiments.

ACKNOWLEDGMENTS

The results presented in the paper are based on the computations using Vikram-100, the 100TFLOP HPC Cluster at Physical Research Laboratory, Ahmedabad, India. We thank Arko Roy, S. Gautam and S. A. Silotri for valuable discussions.

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