EXPONENTIAL STABILITY TO THE BRESSE SYSTEM WITH BOUNDARY DISSIPATION CONDITIONS

M. S. ALVES, OCTAVIO VERA, JAIME MUÑOZ RIVERA, AND AMELIE RABBAUD

Abstract. We consider the Bresse model with three control boundary conditions. We prove the exponential stability of the system using the semigroup theory of linear operators and a result obtained by Prüss [15].

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1. Introduction

In this work, we study the stabilization of a problem arising from engineering motivation, the so-called circular arch problem also known as the Bresse system (see [8]) which is given by

\[ \begin{align*}
\rho_1 \varphi_{tt} - \kappa (\varphi_x + \psi + \ell w)_x - k_0 \ell (w_x - \ell \varphi) &= 0, & \text{in} & (0, L) \times (0, +\infty), \\
\rho_2 \psi_{tt} - b \psi_{xx} + \kappa (\varphi_x + \psi + \ell w) &= 0, & \text{in} & (0, L) \times (0, +\infty), \\
\rho_1 w_{tt} - k_0 (w_x - \ell \varphi)_x + \kappa \ell (\varphi_x + \psi + \ell w) &= 0, & \text{in} & (0, L) \times (0, +\infty),
\end{align*} \]

(1.1)

where \( L \) is the length of the beam, \( \rho_1 = \rho A, \rho_2 = \rho I, \kappa = \kappa' G A, \kappa_0 = E A, b = E T, \ell = R^{-1}, \)
\( \rho \) is the density of the material, \( E \) is the modulus of elasticity, \( G \) is the shear modulus, \( \kappa' \) is the shear factor, \( A \) is the cross-sectional area, \( I \) is the second moment of area of the cross-section and \( R \) is the radius of curvature. The functions \( w, \varphi \) and \( \psi \) are the longitudinal, vertical and shear angle displacements, respectively.

One of the main issues, both from a mathematical and physical point of view is the question of stability in long time (\( t \to \infty \)), in order to prevent the problem from infinite vibrations. This question has been studied by many authors. We refer to the book of Liu and Zheng [11] for a general survey on this topic.

Concerning the Bresse system above, few results about the asymptotic behavior exist. Let us briefly review the different kinds of stabilization that have been conducted. An important problem in the Bresse system is to find a minimum dissipation by which the solution decays uniformly to zero in time. In this direction we have the paper of Fatori and Rivera [5], which improved the paper by Liu and Rao [10], and more recently the article [12], where the polynomial decay rate of the energy is improved. In these papers, the authors show that, in general, the Bresse system is not exponentially stable but that there exists polynomial stability with rates that depend on the wave propagations and the regularity of the initial data. Moreover, they introduced a necessary condition for the dissipative semigroup to decay polynomially. This result allowed them to show some optimality to the polynomial rate of decay. The Bresse system with frictional damping was considered by Alabau-Boussouira et al. [1]. In that paper the authors showed that the Bresse system is exponentially stable if and only if the velocities of waves propagations are the same. Also, they showed that when the velocities are not the same, the system is not exponentially stable, and they proved that the solution in this case goes to zero polynomially, with rates that can be improved by taking more regular initial data. This rate of polynomial decay was improved by Fatori and Monteiro [4]. The indefinite damping acting on the shear angle displacement was considered by Palomino et al. [16]. In [13] Noun and Wehbe...
extended the results of Alabau-Boussouira et al. and considered the important case when the dissipation law is locally distributed. Finally, Lima et al. considered the Bresse system with past history acting in the shear angle displacement. They show the exponential decay of the solution if and only if the wave speeds are the same. If not, they show that the Bresse system is polynomial stable with optimal decay rate.

In the present work, we attack the delicate problem where the dissipative effect (some how the control we may have on the system) takes place at the boundary. However, the consideration of only one dissipative effect, or only two, seems to be difficult to treat. Let us mention some known results related to the boundary stabilization of the Timoshenko beam. Kim and Renardy in proved the exponential stability of the system under two boundary controls. In, Ammar-Khodja and his co-authors studied the decay rate of the energy of the nonuniform Timoshenko beam with only one boundary controls acting in the rotation-angle equation. In, Bassam and his co-authors studied the indirect boundary stabilization of the Timoshenko system with only one dissipation law.

As a first step towards the stability of such systems with one control at the boundary, we consider in the present article the following boundary conditions that complement system (1.1):

\[
\begin{align*}
\varphi(L, t) &= 0, \quad \psi(L, t) = 0, \quad w(L, t) = 0 \quad \text{in} \ (0, +\infty), \\
\kappa (\varphi_x + \psi + \ell w)(0, t) &= \gamma_1 \varphi_t(0, t), \quad \text{in} \ (0, +\infty), \\
b \psi_x(0, t) &= \gamma_2 \psi_t(0, t), \quad \text{in} \ (0, +\infty), \\
k_0 (w_x - \ell \varphi)(0, t) &= \gamma_3 w_t(0, t), \quad \text{in} \ (0, +\infty),
\end{align*}
\]  

where \(\gamma_j > 0, \ j = 1, 2, 3\). In other words, we investigate three dissipative effects at the boundary. The system is finally completed with initial conditions

\[
\begin{align*}
\varphi(x, 0) &= \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad \text{in} \ (0, L), \\
\psi(x, 0) &= \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \quad \text{in} \ (0, L), \\
w(x, 0) &= w_0(x), \quad w_t(x, 0) = w_1(x), \quad \text{in} \ (0, L).
\end{align*}
\]

Let us define the energy functional associated to the system: for \((\varphi, \psi, w)\) a regular solution to (1.1)-(1.3), its associated total energy is defined by

\[
\mathcal{E}(t) = \frac{1}{2} \int_0^L \left( \rho_1 |\varphi_t|^2 + \rho_2 |\psi_t|^2 + \rho_1 |w_t|^2 + \kappa |\varphi_x + \psi + \ell w|^2 + b |\psi_x|^2 + k_0 |w_x - \ell \varphi|^2 \right) dx.
\]

Then a straightforward computation gives

\[
\frac{d}{dt} \mathcal{E}(t) = -\gamma_1 |\varphi_t(0)|^2 - \gamma_2 |\psi_t(0)|^2 - \gamma_3 |w_t(0)|^2 \leq 0,
\]

consequently the system (1.1)-(1.3) is dissipative in the sense that the energy is non-increasing.

**Remark 1.1.** We observe that if \(R \to \infty\) then \(\ell \to 0\) and this model reduces to the well-know Timoshenko beam equations (see [6] and [8] for details).

The main result of this paper is to prove that the exponential stability of the system (1.1)-(1.3) holds. As far as the authors know, there have been no contributions made in this sense. Our main tools are semigroup techniques, a result by Prüss as well as spectral arguments.

The remaining part of this paper is organized as follows. Section 2 outlines briefly the notations and well-posedness of the system. In section 3, we show the exponential stability of the corresponding semigroup. Through this paper, \(C\) is a generic constant, not necessarily the same at each occasion (it will change line to line), which depends in an increasing way on the indicated quantities.
2. Existence and uniqueness

The aim of this section is to prove the existence and uniqueness of solutions for the problem (1.1)-(1.3). Given a Banach space $X$, let $\| \cdot \|_X$ be the usual norm defined on $X$. In particular, we denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and the norm defined on $L^2(0, L)$, respectively. Before stating the existence and the uniqueness result of problem (1.1)-(1.3), we first set-up the following short-hand notation for the function space

$$H^1_L(0, L) = \{ \phi \in H^1(0, L) : \phi(0) = 0 \}.$$ 

Putting $\Phi = \varphi_t$ and $\Psi = \psi_t$, the phase space of our problem is

$$\mathcal{H} = [H^1_L(0, L)]^3 \times [L^2(0, L)]^3,$$ 

normed by

$$\| (\varphi, \psi, w, \Phi, \Psi, W) \|_{\mathcal{H}}^2 = \kappa \| \varphi_x + \psi + \ell w \|^2 + \rho_1 \| \Phi \|^2 + b \| \psi_x \|^2 + \rho_2 \| \Psi \|^2 + \rho_1 \| W \|^2 + k_0 \| w_x - \ell \varphi \|^2.$$ 

We denote by $C^T$ the transpose of a matrix $C$ and introducing the state vector

$$U(t) = (\varphi(t), \psi(t), w(t), \Phi(t), \Psi(t), W(t))^T,$$ 

system (1.1)-(1.2) can be written as a linear ordinary differential equation in $\mathcal{H}$ of the form

$$\frac{d}{dt} U(t) = \mathcal{A} U(t),$$ 

where the domain $\mathcal{D}(\mathcal{A})$ of the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ is given by

$$\mathcal{D}(\mathcal{A}) = \{ U \in \mathcal{H} : \varphi, \psi, w \in H^2(0, L), \Phi, \Psi, W \in H^1_L(0, L),$$

$$\kappa (\varphi_x + \psi + \ell w)(0) = \gamma_1 \Phi(0), \quad b \psi_x(0) = \gamma_2 \Psi(0),$$

$$\kappa_0 (w_x - \ell \varphi)(0) = \gamma_3 W(0) \}$$

and

$$\mathcal{A} U = \begin{pmatrix} \Phi \\ \Psi \\ W \end{pmatrix} = \begin{pmatrix} \kappa (\varphi_x + \psi + \ell w)_x + \frac{k_0 \ell}{\rho_1} (w_x - \ell w) \\ \frac{b}{\rho_2} \psi_{xx} - \frac{\kappa}{\rho_2} (\varphi_x + \psi + \ell w) \\ \frac{k_0}{\rho_1} (w_x - \ell \varphi)_x - \frac{\kappa \ell}{\rho_1} (\varphi_x + \psi + \ell w) \end{pmatrix}.$$

**Proposition 2.1.** The operator $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup $\{ S_A(t) \}_{t \geq 0}$.

**Proof.** The operator $\mathcal{A}$ is dissipative. Indeed, for every $U \in \mathcal{D}(\mathcal{A})$, it is not difficult to see that

$$\text{Re} \langle \mathcal{A} U, U \rangle_{\mathcal{H}} = - \gamma_1 |\Phi(0)|^2 - \gamma_2 |\Psi(0)|^2 - \gamma_3 |W(0)|^2 \leq 0.$$ 

Moreover, the domain $\mathcal{D}$ of $\mathcal{A}$ is clearly dense in the Hilbert $\mathcal{H}$ and the operator is closed. Finally, for all $F = (f_1, f_2, f_3, f_4, f_5, f_6)$ there exists a unique $U = (\varphi, \psi, w, \Phi, \Psi, W) \in \mathcal{D}(\mathcal{A})$. 

such that $AU = F$ (that is to say, that is solution to the resolvent system of the operator). Indeed, the system reads, in terms of components:

\begin{align}
\Phi &= f_1,
(2.3) \\
\Psi &= f_2,
(2.4) \\
W &= f_3,
(2.5) \\
\kappa (\varphi_x + \psi + \ell w)_x + \kappa_0 \ell (w_x - \ell \varphi) &= \rho_1 f_4,
(2.6) \\
b \psi_{xx} - \kappa (\varphi_x + \psi + \ell w) &= \rho_2 f_5,
(2.7) \\
\kappa_0 (w_x - \ell \varphi)_x - \kappa \ell (\varphi_x + \psi + \ell w) &= \rho_1 f_6.
(2.8)
\end{align}

From (2.3)-(2.5) we know $\Phi$, $\Psi$ and $W$. To show the existence and uniqueness of $(\varphi, \psi, w)$ satisfying (2.6)-(2.8) we consider the continuous and coercive sesquilinear form

\[ B((\varphi, \psi, w), (u, v, p)) = \kappa \int_0^L (\varphi_x + \psi + \ell w) (u_x + v + \ell p) \, dx + b \int_0^L \psi_x \varphi_x \, dx + \kappa_0 \int_0^L (w_x - \ell \varphi) (p_x - \ell u) \, dx, \]

for $(\varphi, \psi, w), (u, v, p)$ belong to $[H^1_L(0, L)]^3$ and the continuous sesquilinear function

\[ F(u, v, p) = \rho_1 \int_0^L f_4 \varphi_x \, dx + \rho_2 \int_0^L f_5 \psi_x \, dx + \rho_1 \int_0^L f_6 p_x \, dx + \gamma_1 f_1(0) \varphi(0) + \gamma_2 f_2(0) \psi(0) + \gamma_3 f_3(0) p(0). \]

By the Lax-Milgram Theorem there exists a unique $(\varphi, \psi, w)$ in $[H^1_L(0, L)]^3$ such that

\[ B((\varphi, \psi, w), (u, v, p)) = F(u, v, p), \quad \forall (u, v, p) \in [H^1_L(0, L)]^3 \]

Hence $0 \in \rho(A)$, and the conclusion of Propostion 2.1 follows from the Lumer-Phillips Theorem (see for example [14]).

As a direct consequence of Proposition 2.1 we claim:

**Theorem 2.2.** Given $U_0 = (\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1) \in \mathcal{H}$ there exists a unique solution $U(t) = S(t)U_0 = (\varphi(t), \psi(t), w(t), \varphi_1(t), \psi_1(t), w_1(t))$ to (2.1) such that

\[ U \in C(0, \infty; \mathcal{H}). \]

If moreover, $U_0 \in \mathcal{D}(A)$, then

\[ U \in C^1([0, \infty[: \mathcal{H}) \cap C([0, \infty[: \mathcal{D}(A)). \]

3. **Exponential stability**

The main goal of this section is to prove the exponential decay of solutions. Our main tool is the well known result (see [15]):

**Theorem 3.1.** Let $S(t) = e^{At}$ be a $C_0$-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if $i \mathbb{R} \subset \rho(A)$ and

\[ \lim_{|\lambda| \to \infty} \| (i \lambda I - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} < \infty. \]
Therefore we will need to study the resolvent equation \((i \lambda I - A)U = F\), for \(\lambda \in \mathbb{R}\), namely
\begin{align*}
(3.2) & \quad i \lambda \varphi - \Phi = f_1, \\
(3.3) & \quad i \lambda \psi - \Psi = f_2, \\
(3.4) & \quad i \lambda w - W = f_3, \\
(3.5) & \quad i \lambda \rho_1 \Phi - \kappa (\varphi_x + \psi + \ell w)_x - \kappa_0 \ell (w_x - \ell \varphi) = \rho_1 f_4, \\
(3.6) & \quad i \lambda \rho_2 \Psi - b \psi_{xx} + \kappa (\varphi_x + \psi + \ell w) = \rho_2 f_5, \\
(3.7) & \quad i \lambda \rho_1 W - \kappa_0 (w_x - \ell \varphi)_x + \kappa \ell (\varphi_x + \psi + \ell w) = \rho_1 f_6,
\end{align*}
where \(F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}\). Taking inner product in \(\mathcal{H}\) with \(U\) and using (2.2) we get
\begin{equation}
(3.8) \quad |\text{Re} \langle AU, U \rangle_{\mathcal{H}}| \leq \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}.
\end{equation}
This implies that
\begin{equation}
(3.9) \quad |\Phi(0)|^2 + |\Psi(0)|^2 + |W(0)|^2 \leq C \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}},
\end{equation}
and, applying (3.2)-(3.4), we obtain
\begin{equation}
(3.10) \quad |\varphi(0)|^2 + |\psi(0)|^2 + |w(0)|^2 \leq \frac{C}{|\lambda|^2} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2} \|F\|^2_{\mathcal{H}}.
\end{equation}
Moreover, since
\[|\varphi_x(0) + \psi(0) + \ell w(0)|^2 + |\psi_x(0)|^2 + |w_x(0) - \ell \varphi(0)|^2 \leq C \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}},\]
it follows that
\begin{equation}
(3.11) \quad |\varphi_x(0)|^2 + |\psi_x(0)|^2 + |w_x(0)|^2 \leq C \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2} \|F\|^2_{\mathcal{H}}.
\end{equation}
We will now establish a couple of lemmas in order to prove our stability result.

**Lemma 3.2.** The imaginary axis \(i\mathbb{R}\) is contained in the resolvent set \(\rho(A)\).

*Proof.* Because the domain of \(A\) has compact immersion over the phase space \(\mathcal{H}\), we only need to prove that there is no imaginary eigenvalues. We will argue by contradiction. Let us suppose that there is \(\lambda \in \mathbb{R}, \lambda \neq 0\), and \(U \in \mathcal{D}(A), U \neq 0\), such that \(AU = i \lambda U\). Then, from (2.2) we have
\begin{equation}
(3.12) \quad \Phi(0) = 0, \quad \Psi(0) = 0, \quad W(0) = 0.
\end{equation}
Hence, from (3.2) and (1.3) we obtain
\begin{equation}
(3.13) \quad \varphi(0) = 0, \quad \psi(0) = 0, \quad w(0) = 0 \quad \text{and} \quad \varphi_x(0) = 0, \quad \psi_x(0) = 0, \quad w_x(0) = 0.
\end{equation}
From (3.2)- (3.6) we have
\begin{align*}
- \lambda^2 \rho_1 \phi - \kappa (\varphi_x + \psi + \ell w)_x - \kappa_0 \ell (w_x - \ell \varphi) &= 0, \\
- \lambda^2 \rho_2 \psi - b \psi_{xx} + \kappa (\varphi_x + \psi + \ell w) &= 0, \\
- \lambda^2 \rho_1 w - \kappa_0 (w_x - \ell \varphi)_x + \kappa \ell (\varphi_x + \psi + \ell w) &= 0.
\end{align*}
Consider \(X = (\varphi, \psi, \omega, \varphi_x, \psi_x, \omega_x)\). Then we can rewrite (3.13) and (3.14) as the initial value problem
\begin{align*}
(3.15) & \quad \frac{d}{dx} X = AX, \\
& \quad X(0) = 0,
\end{align*}
Proof. To get (3.16), let us multiply the equation (3.3) by \( C \) for a positive constant where
\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\frac{k_0 \ell^2}{\kappa} & -\frac{1}{\kappa} & 0 & -\frac{\rho_1 \chi_1}{\kappa} & 0 & -\frac{(k_0 + \kappa) \ell}{\kappa} \\
0 & \frac{k_0 \ell}{\kappa} & -\frac{\rho_1 \chi_1 \ell + \kappa \ell^2}{\kappa} & \frac{k_0}{\kappa} & 0 & 0 \\
0 & \frac{k_0}{\kappa} & -\frac{\rho_1 \chi_1}{\kappa} & \frac{\rho_1 \chi_1 \ell + \kappa \ell^2}{\kappa} & 0 & 0
\end{pmatrix}
\]

By the Picard Theorem for ordinary differential equations the system (3.15) has a unique solution \( X = 0 \). Therefore \( \varphi = 0, \psi = 0, w = 0 \). It follows from (3.2)-(3.4), for \( f_1 = f_2 = f_3 = 0 \), that \( \Phi = 0, \Psi = 0, W = 0 \), i.e., \( U = 0 \).

\[ \square \]

Let us introduce the following notation
\[
\begin{align*}
I_{\varphi}(\alpha) &= \rho_1 |\Phi(\alpha)|^2 + \kappa |\varphi_x(\alpha)|^2, \\
I_{\psi}(\alpha) &= \rho_2 |\Psi(\alpha)|^2 + b |\psi_x(\alpha)|^2, \\
I_w(\alpha) &= \rho_1 |W(\alpha)|^2 + k_0 |w_x(\alpha)|^2, \\
I(\alpha) &= I_{\varphi}(\alpha) + I_{\psi}(\alpha) + I_w(\alpha)
\end{align*}
\]
\[
E_\psi(L) = \int_0^L I_\psi(s) \, ds, \quad E_\varphi(L) = \int_0^L I_\varphi(s) \, ds, \quad E_w^n(L) = \int_0^L I_w(s) \, ds.
\]

Lemma 3.3. Let \( q \in H^1(0, L) \). We have that
\[
E_{\psi}(L) = q I_{\psi}|_{I_0}^L - \kappa_0 \ell^2 q |\varphi|^2|_{I_0}^L + 2 \kappa \Re \int_0^L q \varphi_x \overline{\varphi_x} \, dx + \kappa_0 \ell^2 \int_0^L q(\varphi) |\varphi|^2 \, dx
\]
\[+ 2(\kappa + k_0) \ell \Re \int_0^L q w_x \overline{\varphi_x} \, dx + R_1\]
\[
E_{\psi}(L) = q I_{\psi}|_{I_0}^L - \kappa q |\psi|^2|_{I_0}^L - 2 \kappa \Re \int_0^L q \varphi_x \overline{\psi_x} \, dx
\]
\[+ \kappa \int_0^L q(s) |\psi|^2 \, dx - 2 \kappa \ell \Re \int_0^L q w \overline{\psi_x} \, dx + R_2,\]

(3.17) and
\[
E_{w}(L) = q I_w|_{I_0}^L - \kappa \ell^2 q |w|^2|_{I_0}^L - 2 \kappa \ell \Re \int_0^L q \varphi_x \overline{\varphi}_x \, dx
\]
\[+ 2(k + k_0) \ell \Re \int_0^L q \varphi_x \overline{\varphi}_x \, dx + R_3,
\]
where \( R_i \) satisfies
\[
|R_i| \leq C \|U\| \|F\|, \quad i = 1, 2, 3,
\]
for a positive constant \( C \).

Proof. To get (3.16), let us multiply the equation (3.3) by \( q \varphi_x \). Integrating on \((0, L)\) we obtain
\[
i \lambda \rho_1 \int_0^L \Phi q \varphi_x \, dx - \kappa \int_0^L (\varphi_x + \psi + \ell w)_x q \varphi_x \, dx
\]
\[= -\kappa_0 \ell \int_0^L (w_x - \ell \varphi) q \varphi_x \, dx = \rho_1 \int_0^L f_4 q \varphi_x \, dx
\]
or

\[- \rho_1 \int_0^L \Phi q(\lambda \phi_x) \, dx - \kappa \int_0^L q \phi_{xx} \overline{\phi}_x \, dx - \kappa \int_0^L q \psi_x \overline{\psi}_x \, dx\]

\[- (\kappa + \kappa_0) \ell \int_0^L q w_x \overline{\phi}_x \, dx + \kappa_0 \ell^2 \int_0^L q \phi_x \overline{\phi}_x \, dx = \rho_1 \int_0^L f_4 q \overline{\phi}_x \, dx.\]

Since \(i \lambda \phi_x = \Phi_x + f_{1x}\) taking the real part in the above equality results in

\[- \rho_1 \int_0^L \frac{d}{dx} |\phi_x|^2 \, dx - \frac{\kappa}{2} \int_0^L \frac{d}{dx} |\phi_x|^2 \, dx = \rho_1 \int_0^L \Re f_4 q \overline{\phi}_x \, dx\]

\[+ \rho_1 \Re \int_0^L \Phi q \overline{f}_{1x} \, dx + \kappa \Re \int_0^L q \phi_x \overline{\phi}_x \, dx + (\kappa + \kappa_0) \ell \Re \int_0^L q w_x \overline{\phi}_x \, dx + R_1\]

where

\[R_1 = 2 \rho_1 \Re \int_0^L \Phi q \overline{f}_{1x} \, dx + 2 \rho_1 \Re \int_0^L f_4 q \overline{\phi}_x \, dx\]

Similarly, multiplying equation (3.5) by \(q \overline{\psi}_x\), integrating on \((0, L)\) and taking the real part we obtain

\[- \frac{\rho_2}{2} \int_0^L \frac{d}{dx} |\psi|^2 \, dx - \frac{b}{2} \int_0^L \frac{d}{dx} |\psi_x|^2 \, dx = \rho_2 \Re \int_0^L f_5 q \overline{\psi}_x \, dx\]

\[+ \rho_2 \Re \int_0^L \Psi q \overline{f}_{2x} \, dx - \kappa \ell \Re \int_0^L q \psi_x \overline{\phi}_x \, dx - \kappa \ell \Re \int_0^L q w \overline{\phi}_x \, dx\]

\[- \frac{\kappa_0}{2} \int_0^L \frac{d}{dx} |\psi|^2.\]

Performing an integration by parts we get

\[\int_0^L q'(s) [\rho_2 |\Psi(s)|^2 + b |\psi_x(s)|^2] \, ds\]

\[= q \mathcal{I}_\psi |^L_0 - \kappa q |\psi|^2 |^L_0 - 2 \kappa \Re \int_0^L q \phi_x \overline{\psi}_x \, dx\]

\[+ \kappa \int_0^L q'(s) |\psi|^2 \, dx - 2 \kappa \ell \Re \int_0^L q w \overline{\phi}_x \, dx + R_2\]

where

\[R_2 = 2 \rho_2 \Re \int_0^L \Psi q \overline{f}_{2x} \, dx + 2 \rho_2 \Re \int_0^L f_5 q \overline{\psi}_x \, dx.\]
Finally, multiplying equation (3.6) by $q \bar{w}_x$, integrating on $(0, L)$ and taking the real part, after some algebraic manipulations we obtain (3.18) for

$$R_3 = 2 \rho_1 \text{Re} \int_0^L W q \overline{f}_3 x \, dx + 2 \rho_1 \text{Re} \int_0^L f_6 q \bar{w}_x \, dx.$$ 

Our conclusion follows. \hfill \Box

We are now ready to state our main stability result.

**Theorem 3.4.** The semigroup $\{S_A(t)\}_{t \geq 0}$ is exponentially stable, that is, there exist positive constants $M$ and $\mu$ such that

$$\|S_A(t)\|_{L(H)} \leq M \exp(-\mu t), \quad \forall t \geq 0.$$ 

**Proof.** By Lemma 3.2 we know that $i \mathbb{R} \subset \rho(A)$. Therefore, by Theorem 3.1 it suffices to show that the estimate (3.1) holds. Given $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in H$ and $\lambda \in \mathbb{R}$ let be $U$ the unique function satisfying

$$(i \lambda I - A)U = F.$$ 

If we take $q(x) = x - \ell$ in Lemma 3.3 and if we add (3.16)-(3.18) we arrive at

$$E_\varphi(L) + E_\psi(L) + E_w(L)$$

$$= L \mathcal{I}_\varphi(0) - \kappa_0 \ell^2 L |\varphi(0)|^2 + \kappa_0 \ell^2 \int_0^L |\varphi|^2 \, dx + L \mathcal{I}_\psi(0) - L \kappa |\psi(0)|^2 + \kappa \int_0^L |\psi|^2 \, dx + L \mathcal{I}_w(0) - \kappa \ell^2 L |w(0)|^2 + \kappa \ell \int_0^L |w|^2 \, dx + R_1 + R_2 + R_3$$

$$- 2 \kappa \ell \text{Re} \int_0^L (x - L) w \overline{\psi}_x \, dx - 2 \kappa \ell \text{Re} \int_0^L (x - L) \psi \bar{w}_x \, dx.$$ 

Since

$$- 2 \kappa \ell \text{Re} \int_0^L q w \overline{\psi}_x \, dx - 2 \kappa \ell \text{Re} \int_0^L q \psi \bar{w}_x \, dx$$

$$= - 2 \kappa \ell L \text{Re} w(0) \overline{\psi}(0) + 2 \kappa \ell \text{Re} \int_0^L \psi \bar{w} \, dx$$

using Lemma 3.3 and the Young inequality we get

$$E_\varphi(L) + E_\psi(L) + E_w(L)$$

$$\leq L \mathcal{I}_\varphi(0) + \kappa_0 \ell^2 \int_0^L |\varphi|^2 \, dx + L \mathcal{I}_\psi(0) + \kappa \ell |\psi(0)|^2 + \kappa (1 + \ell) \int_0^L |\psi|^2 \, dx + L \mathcal{I}_w(0) + \kappa \ell |w(0)|^2 + 2 \kappa \ell \int_0^L |w|^2 \, dx + C \|U\|_H \|F\|_H$$

$$= L \mathcal{I}_\varphi(0) + \kappa_0 \ell^2 \int_0^L |\varphi|^2 \, dx + L \mathcal{I}_\psi(0) + \kappa \ell |\psi(0)|^2 + \kappa (1 + \ell) \int_0^L |\psi|^2 \, dx + L \mathcal{I}_w(0) + \kappa \ell |w(0)|^2 + 2 \kappa \ell \int_0^L |w|^2 \, dx + C \|U\|_H \|F\|_H$$

Finally, multiplying equation (3.6) by $q \bar{w}_x$, integrating on $(0, L)$ and taking the real part, after some algebraic manipulations we obtain (3.18) for

$$R_3 = 2 \rho_1 \text{Re} \int_0^L W q \overline{f}_3 x \, dx + 2 \rho_1 \text{Re} \int_0^L f_6 q \bar{w}_x \, dx.$$ 

Our conclusion follows. \hfill \Box

We are now ready to state our main stability result.

**Theorem 3.4.** The semigroup $\{S_A(t)\}_{t \geq 0}$ is exponentially stable, that is, there exist positive constants $M$ and $\mu$ such that

$$\|S_A(t)\|_{L(H)} \leq M \exp(-\mu t), \quad \forall t \geq 0.$$ 

**Proof.** By Lemma 3.2 we know that $i \mathbb{R} \subset \rho(A)$. Therefore, by Theorem 3.1 it suffices to show that the estimate (3.1) holds. Given $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in H$ and $\lambda \in \mathbb{R}$ let be $U$ the unique function satisfying

$$(i \lambda I - A)U = F.$$ 

If we take $q(x) = x - \ell$ in Lemma 3.3 and if we add (3.16)-(3.18) we arrive at

$$E_\varphi(L) + E_\psi(L) + E_w(L)$$

$$= L \mathcal{I}_\varphi(0) - \kappa_0 \ell^2 L |\varphi(0)|^2 + \kappa_0 \ell^2 \int_0^L |\varphi|^2 \, dx + L \mathcal{I}_\psi(0) - L \kappa |\psi(0)|^2 + \kappa \int_0^L |\psi|^2 \, dx + L \mathcal{I}_w(0) - \kappa \ell^2 L |w(0)|^2 + \kappa \ell \int_0^L |w|^2 \, dx + R_1 + R_2 + R_3$$

$$- 2 \kappa \ell \text{Re} \int_0^L (x - L) w \overline{\psi}_x \, dx - 2 \kappa \ell \text{Re} \int_0^L (x - L) \psi \bar{w}_x \, dx.$$ 

Since

$$- 2 \kappa \ell \text{Re} \int_0^L q w \overline{\psi}_x \, dx - 2 \kappa \ell \text{Re} \int_0^L q \psi \bar{w}_x \, dx$$

$$= - 2 \kappa \ell L \text{Re} w(0) \overline{\psi}(0) + 2 \kappa \ell \text{Re} \int_0^L \psi \bar{w} \, dx$$

using Lemma 3.3 and the Young inequality we get

$$E_\varphi(L) + E_\psi(L) + E_w(L)$$

$$\leq L \mathcal{I}_\varphi(0) + \kappa_0 \ell^2 \int_0^L |\varphi|^2 \, dx + L \mathcal{I}_\psi(0) + \kappa \ell |\psi(0)|^2 + \kappa (1 + \ell) \int_0^L |\psi|^2 \, dx + L \mathcal{I}_w(0) + \kappa \ell |w(0)|^2 + 2 \kappa \ell \int_0^L |w|^2 \, dx + C \|U\|_H \|F\|_H$$
for a positive constant $C$. It results by (3.9), (3.10) and (3.11) that we can find a positive constant $C$ such that

$$
\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) + \mathcal{E}_w(L) \\
\leq \kappa_0 \ell^2 \int_0^L |\varphi|^2 \, dx + \kappa (1 + \ell) \int_0^L |\psi|^2 \, dx + 2 \kappa \ell \int_0^L |w|^2 \, dx \\
+ \frac{C}{|\lambda|^2} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + C \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2} \|F\|_{\mathcal{H}}^2,
$$

for $\lambda \neq 0$. Since that $\varphi = \frac{\Phi + f_1}{i\lambda}$, $\psi = \frac{\Psi + f_2}{i\lambda}$ and $w = \frac{W + f_3}{i\lambda}$ we obtain

$$
\|U\|_{\mathcal{H}}^2 \leq \frac{C}{|\lambda|^2} \|U\|_{\mathcal{H}}^2 + \frac{C}{|\lambda|^2} \|F\|_{\mathcal{H}}^2 + \frac{C}{|\lambda|^2} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + C \|F\|_{\mathcal{H}}^2
$$

for $\lambda \neq 0$. If $|\lambda| > 1$ we get

$$
\left(1 - \frac{C}{|\lambda|}\right) \|U\|_{\mathcal{H}}^2 \leq C \|F\|_{\mathcal{H}}^2.
$$

Consequently, since $\lambda \mapsto (i\lambda I - \mathcal{A})$ is continuous it follows that

$$
\|(i\lambda I - \mathcal{A})\|_{\mathcal{L}(\mathcal{H})} \leq C, \ \forall \lambda \in \mathbb{R},
$$

for a positive constant $C$. The conclusion then follows by applying the Theorem 3.1.

4. Conclusion

In this paper, we provide a result of exponential stability for the Bresse system when three dissipative effects are concentrated at the boundary. It is a step towards complete understanding of boundary stabilization of such system. Indeed, we expect to be able to obtain similar results as the ones existing for Timoshenko type models [2, 3, 7], but it seems for now, that there are more mathematical difficulties for the Bresse model.

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Department of Mathematics, Viçosa University. CEP 36570-000. Viçosa, MG-Brazil.

E-mail address: malves@ufv.br

Department of Mathematics, Universidad del Bío-Bío, Av. Collao 1202, Casilla 5-C, Concepción, Chile.

E-mail address: overa@ubiobio.cl, octaviovera49@gmail.com

LNCC, Av. Getulio Vargas 333. Quintadinha. CEP 25651-075. Petrópolis.RJ.Brasil. Inst. de Matemática, UFRJ, Av. da Silva Ramos. CEP 21945-970.RJ. Brasil.

E-mail address: rivera@lncc.br, rivera@im.ufrj.br

Department of Mathematics, grupo de investigación GIMNAP 151408/VC, Universidad del Bío-Bío, Av. Collao 1202, Casilla 5-C, Concepción, Chile.

E-mail address: arambaud@ubiobio.cl