DECOMPOSITIONS INVOLVING ANICK’S SPACES

BRAYTON GRAY

The goal of this work is to continue the investigation of the Anick fibration and the associated spaces. Recall that this is a $p$-local fibration sequence:

$$\Omega^2 S^{2n+1} \to S^{2n} \to T \to \Omega S^{2n+1}$$

where $\pi_n$ is a compression of the $p^n$th power map on $\Omega^2 S^{2n+1}$. This fibration was first described for $p \geq 5$ as the culmination of a 270 page book [A]. In [AG], the authors described an $H$ space structure for the fibration sequence. Its relationship to EHP spectra was discussed [G3] as well as first steps to developing a universal property.

Much work has been done since then to find a simpler construction, and this was obtained for $p \geq 3$ in [GT]. This new construction also reproduces the results of [AG]. It is in the context of these new methods that this work is developed and we assume a familiarity with [GT].

One of the main features of the construction is a certain fibration sequence:

$$\Omega G \xrightarrow{h} T \xrightarrow{i} R \xrightarrow{\rho} G$$

where $h$ has a right homotopy inverse $g : T \to \Omega G$ and the adjoint of $g$:

$$\tilde{g} : \Sigma T \to G$$

also has a right homotopy inverse $f : G \to \Sigma T$. Together these maps define an $H$ space structure on $T$ and a co-$H$ space structure on $G$, and both $G$ and $T$ are atomic. Furthermore $R \in W^\infty$, the class of spaces that are the one point union of mod $p^r$ Moore spaces for $r \leq s$.

For some applications it would be helpful to have a better understanding of the map $\rho$. In order to accomplish this, we reconstruct a space $D$ from [A]. $D$ is closely related to $G$. Although the formal properties of $D$ are not as simple as $G$ (it is not a co-$H$ space) other properties are simpler (for example Theorem A part (b) below). Define

$$C = \bigvee_{i=1}^{\infty} P^{2np^i+1} (p^r i - 1).$$

Theorem A. There is a cofibration sequence:

$$C \to G \to D.$$
and a fibration sequence:

\[ F \to D \to S^{2n+1} \]

such that

(a) \( H^j(D) = \begin{cases} 
Z/p^r & \text{if } j = 2n \\
Z/p & \text{if } j = 2np^s \quad s > 0 \\
0 & \text{otherwise} 
\end{cases} \)

(b) \( H^j(F) = \begin{cases} 
Z(p) & \text{if } j = 2ni \\
0 & \text{otherwise} 
\end{cases} \)

(c) There is a diagram of fibration sequences:

\[
\begin{array}{ccc}
T & \longrightarrow & T \\
\downarrow & & \downarrow \\
R & \longrightarrow & W \\
\rho & & \downarrow \\
G & \longrightarrow & D
\end{array}
\]

where \( i \) and \( i' \) are null homotopic.

(d) \( H^j(W) = \begin{cases} 
Z(p)/ip^{r-1} & \text{if } j = 2ni = 2np^s \\
Z(p)/ip^r & \text{if } j = 2ni, \text{otherwise} \\
0 & \text{otherwise} 
\end{cases} \)

(e) Furthermore there is a homotopy commutative diagram of fibration sequences:

\[
\begin{array}{ccc}
S^{2n-1} & \longrightarrow & T \\
\downarrow & & \downarrow \\
W & \longrightarrow & W \\
\downarrow & & \downarrow \\
F & \longrightarrow & D \\
\downarrow & & \downarrow \\
& \longrightarrow & S^{2n+1}
\end{array}
\]

with \( \Omega F \simeq S^{2n-1} \times \Omega W \).

**Theorem B.** If \( n > 1 \), \( R \simeq (C \rtimes T) \vee W \) and the composition:

\[ C \rtimes T \to R \overset{\rho}{\to} G \]

is homotopic to the composition:

\[ C \rtimes T \overset{1 \ltimes G}{\longrightarrow} C \rtimes \Omega G \overset{\omega}{\longrightarrow} C \vee G \overset{c \vee 1}{\longrightarrow} G \]

*Note that \( Z(p)/m \) is isomorphic to \( Z/p\nu(m) \) where \( \nu(m) \) is the number of powers of \( p \) in \( m \).
where $\omega$ is the Whitehead product map which lies in the fibration sequence:

$$C \times \Omega G \xrightarrow{\omega} C \vee G \xrightarrow{\pi_2} G$$

Furthermore, $\Omega G \simeq \Omega G \times \Omega(C \times \Omega D)$.

In addition, some partial results are obtained for $\rho|W$, but much is still unknown.

This paper is organized as follows. In section 1 we revisit some constructions in [GT] and sharpen some of the results. In section 2 we embark on a multifaceted induction, constructing the space $W$ via a sequence of approximations and prove Theorem A. Section 3 is devoted to proving Theorem B.

1.

In the course of the constructions in [GT], $G$ was constructed inductively as the union of spaces $G_k$ where

$$G_k = G_{k-1} \cup CP^{2np^k} (p^{r+k})$$

$G_k$ was constructed as a retract of $\Sigma T^{2np^k}$, the suspension of the $2np^k$ skeleton of $T$. We need to make a refinement of this construction. In the proof of 4.3(d) a map

$$e: P^{2np^k} (p^{r+k-1}) \vee P^{2np^k+1} (p^{r+k-1}) \to \Sigma T^{2np^k}$$

was constructed with the sole property that it induced an epimorphism in mod $p$ homology in dimensions $2np^k$ and $2np^k + 1$. The components of $e$ were given as compositions:

$$P^{2np^k} (p^{r+k-1}) \to \Sigma \left( T^{2np_k-1} \times T^{2np_k-1} \times \cdots \times T^{2np_k-1} \right) \xrightarrow{\Sigma \tilde{\mu}} \Sigma T^{2np^k}$$

$$P^{2np^k+1} (p^{r+k-1}) \to \Sigma \left( T^{2np_k-1} \times \cdots \times T^{2np_k-1} \right) \xrightarrow{\Sigma \tilde{\mu}'} \Sigma T^{2np^k}$$

where the middle space in each case is the suspension of a product of $p$ factors and lies in $\mathcal{W}^{r+k-1}$ and the maps $\tilde{\mu}$ and $\tilde{\mu}'$ are obtained from the action given in 4.3(n) for the case $k = 1$.

**Proposition 1.1.** There is a choice of a map $e$ which is a mod $p$ homology epimorphism in dimensions $2np^k$ and $2np^k + 1$ and such that the diagram:

$$
\begin{array}{ccc}
P^{2np^k} (p^{r+k-1}) \vee P^{2np^k+1} (p^{r+k-1}) & \xrightarrow{e} & \Sigma T^{2np^k} \\
\downarrow & & \downarrow \\
\Sigma(T \wedge T) & \xrightarrow{H(\mu)} & \Sigma T
\end{array}
$$


homotopy commutes where $H(\mu)$ is the Hopf construction on the multiplication $\mu: T \times T \to T$.

Proof. Since $\tilde{\mu}$ and $\tilde{\mu}'$ are obtained by iteration of the restriction of $\mu_{k-1}$ in 4.3(n):

$$T^{2np^{k-1}} \times T^{2nmp^{k-1}} \to T^{2n(m+1)p^{k-1}}$$

for $1 \leq m < p$, it follows that $e$ factors through

$$\Sigma(\mu): \Sigma \left( T^{2np^{k-1}} \times T^{2n(p-1)p^{k-1}} \right) \to \Sigma T^{2np^k}.$$ 

Now the standard splitting

$$\Sigma T^{2np^{k-1}} \lor \Sigma T^{2n(p-1)p^{k-1}} \lor \Sigma \left( T^{2np^{k-1}} \land T^{2n(p-1)p^{k-1}} \right)$$

$$\to \Sigma \left( T^{2np^{k-1}} \times T^{2n(p-1)p^{k-1}} \right)$$

is induced by the inclusions of the axes and the Hopf construction on the identity map of the product. Let $e_1, e_2, e_3$ be the idempotent self maps of $\Sigma \left( T^{2np^{k-1}} \times T^{2n(p-1)p^{k-1}} \right)$ corresponding to these three retracts. Then in homology we have

$$1 = (e_1)_* + (e_2)_* + (e_3)_*.$$ 

However, in dimensions $2np^k$ and $2np^k + 1$, $(e_1)_* = (e_2)_* = 0$. Consequently the composition:

$$P^{2np^k} (p^{r+k-1}) \lor P^{2np^k+1} (p^{r+k-1}) \to \Sigma \left( T^{2np^{k-1}} \times T^{2n(p-1)p^{k-1}} \right)$$

$$\xrightarrow{e_3} \Sigma \left( T^{2np^{k-1}} \times T^{2n(p-1)p^{k-1}} \right)$$

$$\to \Sigma T^{2np^k}$$

is also an epimorphism in mod $p$ homology in dimensions $2np^k$ and $2np^k + 1$. However $e_3$ is the composition:

$$\Sigma \left( T^{2np^{k-1}} \times T^{2n(p-1)p^{k-1}} \right) \to \Sigma \left( T^{2np^{k-1}} \land T^{2n(p-1)p^{k-1}} \right)$$

$$\xrightarrow{H} \Sigma \left( T^{2np^{k-1}} \times T^{2n(p-1)p^{k-1}} \right)$$

where $H$ is the Hopf construction on the identity. Thus $\Sigma\tilde{\mu} \circ H$ is the Hopf construction on $\tilde{\mu}$:

$$\Sigma \left( T^{2np^{k-1}} \land T^{2n(p-1)p^{k-1}} \right) \xrightarrow{H} \Sigma \left( T^{2np^{k-1}} \times T^{2n(p-1)p^{k-1}} \right) \xrightarrow{\Sigma\tilde{\mu}} \Sigma T^{2np^k}$$
We now apply these considerations to the induced fibration determined by $\bar{g}$:

\[
\begin{array}{c}
T \longrightarrow T \\
\downarrow \downarrow \downarrow \\
Q \longrightarrow R \\
\downarrow \downarrow \downarrow \\
\Sigma T \longrightarrow G
\end{array}
\]

The structure of the fibration $\pi'$: $Q \to \Sigma T$ is completely determined by the action map:

\[
\Omega \Sigma T \times T \to T
\]

which is given by the composition

\[
\Omega \Sigma T \times T \xrightarrow{\Omega \bar{g} \times 1} \Omega G \times T \to T.
\]

This is the action described in the proof of 4.3(n), so $Q \simeq \Sigma T \wedge T$ and $\pi' \sim H(\mu)$. We conclude

**Proposition 1.2.** There is a lifting $\tilde{e}$ of $\bar{g}e$ to $R$

\[
P^{2np^k}(p^{r+k-1}) \vee P^{2np^{k+1}}(p^{r+k-1}) \longrightarrow \Sigma T \wedge T \longrightarrow R
\]

\[
\Sigma T \xrightarrow{\bar{g}} G
\]

We will designate the components of $\bar{g}e$ as

\[
a_k : P^{2np^k}(p^{n+k-1}) \to G_k
\]

\[
c_k : P^{2np^{k+1}}(p^{r+k-1}) \to G_k
\]

and their lifts to $R_k$ as $\tilde{a}_k$ and $\tilde{c}_k$ respectively.

**Proposition 1.3.** There is a homotopy commutative diagram of cofibrations sequences:

\[
P^{2np^k}(p^{r+k}) \xrightarrow{\beta_k} P^{2np^k}(p^{r+k}) \xrightarrow{d} P^{2np^k}(p^{r+k-1}) \vee P^{2np^{k+1}}(p^{r+k-1})
\]

\[
P^{2np^k}(p^{r+k}) \xrightarrow{a_k \vee c_k} G_{k-1} \longrightarrow G_k
\]

**Proof.** The homotopy fiber of the projection $G_k \to P^{2np^{k+1}}(p^{r+k})$ is the relative James construction $(G_{k-1}, P^{2np^k}(p^{r+k}))_\infty$ (see [GI]) which is $G_{k-1} +$ cells of dimension $\geq 2np^k + 2n - 1$. Thus the map $\beta_k$ exists.
The next term is the James construction \((P^{2np^k+1}(p^{r+k}))_{\infty}\). The upper cofibration sequence is completely determined by the fact that the composition

\[
P^{2np^k}(p^{r+k}) \vee P^{2np^k+1}(p^{r+k-1}) \xrightarrow{a_k \vee c_k} G_k \to P^{2np^k+1}(p^{r+k})
\]

is an epimorphism in homology.

**Proposition 1.4.** There is a unique lifting \(\tilde{\beta}_k\) of \(\beta_k\) to \(R_{k-1}\):

\[
\begin{array}{ccc}
R_{k-1} & \to & R_k \\
\downarrow & & \downarrow \\
G_{k-1} & \to & G_k
\end{array}
\]

\(\beta_k\)

\(\tilde{\beta}_k\)

\(P^{2np^k}(p^{r+k})\)

Proof. The existance follows since \(R_{k-1}\) is a pullback:

\[
\begin{array}{ccc}
R_{k-1} & \to & R_k \\
\downarrow & & \downarrow \\
G_{k-1} & \to & G_k
\end{array}
\]

and \((\tilde{a}_k \vee \tilde{c}_k)d\): \(P^{2np^k}(p^{r+k}) \to R_k\) is a lifting of the composition

\[
P^{2np^k}(p^{r+k}) \xrightarrow{\beta_k} G_{k-1} \to G_k
\]

by 1.3. To prove uniqueness, suppose we have two liftings \(\tilde{\beta}_k\) and \(\tilde{\beta}'_k\). Their difference consequently factors through \(T\). But any map

\[
P^{2np^k}(p^{r+k}) \to T
\]

is necessarily trivial in mod \(p\) cohomology, for \(H^{2np^k-1}(T; \mathbb{Z}/p)\) is decomposable and the mod \(p^{r+k}\) Bockstein is nontrivial in \(H^{2np^k-1}(T; \mathbb{Z}/p)\). It follows that the difference \(\tilde{\beta}_k - \tilde{\beta}'_k\) factors through \(T^{2np^k-2}\) and hence though \(\Omega G_{k-1}\) by [GT, 4.3(b)]. Consequently the difference is trivial in \(R_{k-1}\).

The following result is a special case of [GT, 2.3]
Theorem 1.5. Suppose all spaces are localized at a prime \( p > 2 \), and in the diagram:

\[
\begin{array}{ccc}
F & \longrightarrow & F \\
\downarrow & & \downarrow \\
S^{m-1} & \longrightarrow & E_0 \\
\uparrow p & & \downarrow \\
S^{m-1} & \longrightarrow & B \\
& & \downarrow \\
& & B \cup e^m
\end{array}
\]

the middle column is a pullback and the bottom row is a cofibration. Then

\[
[E, BW_n] \rightarrow [E_0, BW_n]
\]

is onto.

2.

In this section we will construct the fibration

\[
T \rightarrow W_k \rightarrow D_k
\]

and prove Theorem A. The spaces \( D_k \) were first considered in [A] and their relationship to \( G_k \) was discussed in [AG]. We will construct them directly from the ideas of [GT]. We begin with

\[
C_k = \bigvee_{i=1}^{k} p^{2np^i+1} (p^{r+i-1})
\]

and define \( c: C_k \rightarrow G_k \) by \( c | P^{2np^i+1}(p^{r+i-1}) = c_i \). Now define \( D_k \) as the cofiber:

\[
C_k \xrightarrow{c} G_k \rightarrow D_k.
\]

Since \( c_i: P^{2np^i+1}(p^{r+i-1}) \rightarrow G_i \rightarrow G_k \) is an integral homology monomorphism, we immediately have

**Proposition 2.1.** \( H_i(D_k) = \begin{cases} 
Z/p^r & \text{if } i = 2n \\
Z/p & \text{if } i = 2np^j, 1 \leq j \leq k \\
0 & \text{otherwise.}
\end{cases} \)

By construction, the map \( c \) lifts to a map

\[
C_k \xrightarrow{\tilde{c}} R_k \rightarrow E_k
\]
where $E_k$ is described in [GT, 4.3(h)] and we have

\[
\begin{array}{cccccc}
\hat{c} & E_k & \rightarrow & J_k & \rightarrow & F_k \\
\downarrow & \downarrow \xi_k & & \downarrow & \downarrow & \\
C_k & G_k & \rightarrow & D_k & \rightarrow & D_k \\
\downarrow & \downarrow & & \downarrow & \downarrow & \\
S^{2n+1} \{p^r\} & S^{2n+1} \{p^r\} & \rightarrow & S^{2n+1}
\end{array}
\]

where the diagram of vertical fibrations defines the spaces $J_k$ and $F_k$. We are about to embark on a multipart induction and wish to make one observation first. Consider the Serre spectral sequence for the homology of the fibration:

\[
\Omega S^{2n+1} \rightarrow F_k \rightarrow D_k
\]

where

\[
E^2_{p,q} = H_p(D_k; H_q(\Omega S^{2n+1}))
\]

This is only nonzero when both $p$ and $q$ are divisible by $2n$. Hence $E^2_{p,q} \cong E^\infty_{p,q}$ and $H_i(F_k) = 0$ unless $i$ is divisible by $2n$. In particular

\[
(**) \quad H_{2np^k}(F_k) \rightarrow H_{2np^k}(D_k)
\]

is an epimorphism.

**Theorem 2.2.** Let $k \geq 0$. Then

(a) $H_r(F_k) = \begin{cases} 
Z(p) & \text{if } r = 2ni \\
0 & \text{otherwise} 
\end{cases}$

and the homomorphism $H_{2ni}(F_{k-1}) \rightarrow H_{2ni}(F_k)$ has degree $p$ if $i \geq p^k$ and degree 1 if $i < p^k$.

(b) There is a map $\theta_i: P^{2ni}(p^r) \rightarrow J_k$ for each $i \geq p^k$ such that the composition:

\[
P^{2ni}(p^r) \xrightarrow{\theta_i} J_k \rightarrow F_k
\]

induces an isomorphism in $H^{2ni}(\; Z/p)$.

(c) There is a map $\tilde{\gamma}_k: J_k \rightarrow BW_n$ such that the composition:

\[
\Omega^2 S^{2n+1} \rightarrow J_k \xrightarrow{\tilde{\gamma}_k} BW_n
\]

is homotopic to the map $\nu: \Omega^2 S^{2n+1} \rightarrow BW_n$ (see [G2]).
(d) Let $W_k$ be the homotopy fiber of $\tilde{\gamma}_k$. Then we have a homotopy commutative diagram of vertical fibration sequences:

\[
\begin{array}{cccc}
T & \longrightarrow & T & \longrightarrow & \Omega S^{2n+1} \\
\downarrow & & \downarrow & & \downarrow \\
R_k & \longrightarrow & W_k & \longrightarrow & F_k \\
\downarrow & & \downarrow & & \downarrow \\
G_k & \longrightarrow & D_k & \longrightarrow & D_k
\end{array}
\]

and two diagrams of fibration sequences:

\[
\begin{array}{cccc}
S^{2n-1} & \longrightarrow & \Omega^2 S^{2n+1} & \longrightarrow & BW_n \\
\downarrow & & \downarrow & & \downarrow \\
W_k & \longrightarrow & J_k & \longrightarrow & BW_n \\
\downarrow & & \downarrow & & \downarrow \\
F_k & \longrightarrow & F_k & \longrightarrow & S^{2n+1}
\end{array}
\]

\[
\begin{array}{cccc}
S^{2n-1} & \longrightarrow & \Omega S^{2n+1} & \longrightarrow & T & \longrightarrow & \Omega S^{2n+1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
W_k & \longrightarrow & J_k & \longrightarrow & BW_n & \longrightarrow & W_k \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_k & \longrightarrow & F_k & \longrightarrow & D_k & \longrightarrow & S^{2n+1}
\end{array}
\]

(e) $\Omega F_k \simeq S^{2n-1} \times \Omega W_k$

(f) the homomorphism $H^r(F_k) \to H^r(W_k)$ is an epimorphism and

\[
H^m(W_k) = \begin{cases} 
\mathbb{Z}_{(p)}/ip^{r-1} & \text{if } m = 2ni = 2np^s \ 0 < s \leq k \\
\mathbb{Z}_{(p)}/ip^r & \text{if } m = 2ni > 2n \text{ otherwise} \\
0 & \text{otherwise}
\end{cases}
\]

(g) The image of the homomorphism:

\[H^{2np^k+1}(W_k) \to H^{2np^{k+1}}(T)\]

has order $p$.

(h) The map $\alpha_{k+1}$ lifts to a map $\tilde{\alpha}_{k+1}: P^{2np^{k+1}}(p^{r+k+1}) \to R_k$ such that the composition:

\[P^{2np^{k+1}}(p^{r+k+1}) \xrightarrow{\tilde{\alpha}_{k+1}} R_k \to W_k\]

is nonzero in integral cohomology.

Proof. We prove these results inductively on $k$ using earlier results for a given value of $k$ and all results for lower values of $k$. In case $k = 0$, (a) is well known (see [CMN]).

Proof of (b). In case $k = 0$, this is [GT, 3.1]. Suppose $k > 0$. Since the homomorphism

\[H_{2np^k}(F_k) \to H_{2np^k}(D_k)\]

is onto by (**) and $H_{2np^k}(F_k)$ is free on one generator, the homorphism:

$$H_{2np^k}(F_k; \mathbb{Z}/p) \to H_{2np^k}(D_k; \mathbb{Z}/p)$$

is an isomorphism. Now consider the diagram:

$$E_k \to J_k \to F_k$$

Since the lower composition also induces an isomorphism in $H_{2np^k} (\; ; \mathbb{Z}/p)$, we conclude that the upper composition does as well. Let $\theta_{p^k}$ be the composition:

$$P_{2np^k} (p^r) \to P_{2np^k} (p^{r+k-1}) \to E_k \to J_k.$$  

This satisfies (b) in case $i = p^k$. We now construct $\theta_{m+1}$ for $m \geq p^k$ by induction. Having constructed $\theta_m$, consider the diagram of vertical fibration sequences:

$$P^{2(m+1)n} (p^r) \to P^{2mn} (p^r) \times_{\Omega P^{2n+1} (p^r)} J_k \to F_k$$

$$\downarrow \quad \xi_k \quad \downarrow$$

$$P^{2n+1} (p^r) \vee P^{2mn} (p^r) \overset{\epsilon \vee \xi_k \theta_m}{\to} D_k \quad \cong \quad D_k$$

$$\downarrow \quad \downarrow$$

$$P^{2n+1} (p^r) \quad \to \quad S^{2n+1} (p^r) \to S^{2n+1}$$

We will compare the mod $p$ cohomology spectral sequences for the first and last fibration, and in particular, the differential:

$$d_{2n+1}: E_{0,2(m+1)n}^{2n+1} \to E_{2n+1,2mn}^{2n+1}.$$  

In the righthand spectral sequence, the differential is an isomorphism as both groups are $\mathbb{Z}/p$ and the dimension of $D_k$ is less than $2(m+1)n$. The map of fibrations induces the following homomorphism on $E_{2n+1,2mn}^{2n+1}$ (where the coefficients are $\mathbb{Z}/p$):

$$H^{2mn} (F_k) \otimes H^{2n+1} (S^{2n+1}) \to$$

$$H^{2mn} (P^{2mn} (p^r) \times_{\Omega P^{2n+1} (p^r)} H^{2n+1} (P^{2n+1} (p^r)).$$

Since the composition $\pi_k \theta_m$ induces an isomorphism in $H^{2mn} (\; ; \mathbb{Z}/p)$, this homomorphism is an isomorphism as well. It follows that the homomorphism induced on $E_{0,2(m+1)n}$ is nonzero and hence an isomorphism. Thus $\pi_k \theta_{m+1}$ induces an isomorphism in $H^{2(m+1)n} (\; ; \mathbb{Z}/p)$.
Proof of (c). In case \( k = 0 \), this is \([\text{GT}, 3.5]\). Suppose that \( k > 0 \).

Write \( F_k(m) \) for the \( 2mn \) skeleton of \( F_k \) and \( J_k(m) \) for the total space of the induced fibration over \( F_k(m) \):

\[
\begin{array}{ccc}
\Omega^2 S^{2n+1} & \longrightarrow & \Omega^2 S^{2n+1} \\
^i_k \downarrow & & \downarrow \\
J_k(m) & \longrightarrow & J_k \\
\downarrow & & \downarrow \\
F_k(m) & \longrightarrow & F_k \\
\end{array}
\]

We will construct a compatible sequence of maps:

\( \tilde{\gamma}_k(m) : J_k(m) \to BW_n \)

with \( \tilde{\gamma}_k(m)i_k \sim \nu : \Omega^2 S^{2n+1} \to BW_n \) for a fixed \( k \) and \( m \geq 1 \). Since \( D_k = D_{k-1} \cup CP^{2np^k} \), the pair \( (F_k, F_{k-1}) \) is \( 2np^k - 1 \) connected. Consequently if \( m < p^k \), \( F_{k-1}(m) = F_k(m) \) and \( J_{k-1}(m) = J_k(m) \). We begin the induction on \( m \) by defining \( \tilde{\gamma}_k(m) = \tilde{\gamma}_{k-1}(m) \) when \( m < p^k \). Now \( F_k(m) = F_k(m-1) \cup \gamma_m e^{2mn} \). We wish to apply Theorem 1.5 to the diagram:

\[
\begin{array}{ccc}
\Omega^2 S^{2n+1} & \longrightarrow & \Omega^2 S^{2n+1} \\
\downarrow & & \downarrow \\
J_k(m-1) & \longrightarrow & J_k(m) \\
\downarrow & & \downarrow \\
F_k(m-1) & \longrightarrow & F_k(m) \\
\end{array}
\]

It suffices to show that there is a lifting \( \gamma'_m \) of \( \gamma_m \) which is divisible by \( p \).

\[
\begin{array}{ccc}
J_k(m-1) & \newline \gamma'_m \\
\downarrow & \newline \gamma_m \\
S^{2mn-1} & \longrightarrow & F_k(m-1) \\
\end{array}
\]

In fact, we will construct a lifting \( \gamma'_m \) of \( \gamma_m \) which is divisible by \( p^r \).

The composition:

\[
S^{2mn-1} \to P^{2mn}(p^r) \xrightarrow{\theta_m} J_k(m) \xrightarrow{\pi_k} F_k(m)
\]
factors through $F_k(m - 1)$ for dimensional reasons:

$$
\begin{align*}
S^{2mn-1} & \longrightarrow P^{2mn}(p^r) \\
x' & \downarrow \pi_k \theta_m \\
F_k(m - 1) & \longrightarrow F_k(m)
\end{align*}
$$

with $p^r x' \sim \gamma_m$, since $\pi_k \theta_m$ induces an isomorphism in $H^{2mn}(\mathbb{Z}/\rho)$. (For complete details, apply [GT, 3.2] with $M = S^{2mn-2}$, $X = F_k(m - 1)$, $f = \gamma_m$, $x = \pi_k \theta_m$, and $s = r$). Since $J_k(m - 1)$ is a pullback, $x'$ factors through $J_k(m - 1)$:

$$
\begin{align*}
S^{2mn-1} & \longrightarrow P^{2mn}(p^r) \\
\gamma_m & \downarrow \theta_m \\
J_k(m - 1) & \longrightarrow J_k(m) \\
\pi_k & \downarrow \pi_k \\
F_k(m - 1) & \longrightarrow F_k(m)
\end{align*}
$$

and $\pi_{k-1} \gamma'_m = x'$ so $p^r \pi_{k-1} \gamma'_m \sim \gamma_m$. Thus we have constructed $\tilde{\gamma}_k: J_k \to BW_n$ for each $k \geq 1$. $\tilde{\gamma}_k|_{J_{k-1}}$ may not be homotopic to $\tilde{\gamma}_{k-1}$, but they are homotopic on $J_{k-1}(m)$ for $m < p^k$. Thus we may define

$$
\tilde{\gamma}_\infty: J_\infty \to BW_n
$$

by taking the direct limit of the $\tilde{\gamma}_k$ and then redefine $\tilde{\gamma}_k$ as the restriction of $\tilde{\gamma}_\infty$.

**Proof of (d).** The map $\nu_k: E_k \to BW_n$ defined in [GT, 4.3(h)] was an arbitrary map such that the composition:

$$
\begin{align*}
\Omega^2 S^{2n+1} & \to \Omega S^{2n+1} \{p^r\} \to E_0 \to E_k \xrightarrow{\nu_k} BW_n
\end{align*}
$$

is homotopic to $\nu$. Since $J_0 = E_0$, $\tilde{\gamma}_0 = \nu_0$ and $\tilde{\gamma}_k$ is an arbitrary extension of $\tilde{\gamma}_{k-1}$, we can redefine $\nu_k$ as the composition

$$
E_k \to J_k \xrightarrow{\tilde{\gamma}_k} BW_n
$$

from which it follows that we have a commutative diagram of fibration sequences

$$
\begin{align*}
R_k & \longrightarrow E_k \xrightarrow{\nu_k} BW_n \\
\downarrow & \downarrow \| \\
W_k & \longrightarrow J_k \xrightarrow{\tilde{\gamma}_k} BW_n
\end{align*}
$$
where $W_k$ is the fiber of $\tilde{\gamma}_k$. Consequently the square:

$$
\begin{array}{ccc}
R_k & \longrightarrow & W_k \\
\downarrow & & \downarrow \\
G_k & \longrightarrow & D_k 
\end{array}
$$

is the composition of two pullback squares:

$$
\begin{array}{ccc}
R_k & \longrightarrow & E_k & \longrightarrow & G_k \\
\downarrow & & \downarrow & & \downarrow \\
W_k & \longrightarrow & J_k & \longrightarrow & D_k 
\end{array}
$$

so it is a pullback square and first diagram in (d) is a diagram of vertical fibration sequences. The second diagram follows from the definition of $W_k$ and the third is a combination of the first two.

Proof of (e). Extending the third diagram of (d) to the left yields a diagram:

$$
\begin{array}{ccc}
\Omega^2 S^{2n+1} & \longrightarrow & S^{2n-1} \\
\downarrow & & \downarrow \\
* & \longrightarrow & W_k \\
\downarrow & & \downarrow \\
\Omega^2 S^{2n+1} & \longrightarrow & F_k
\end{array}
$$

Both horizontal maps have degree $p^r$ in the lowest dimension, so $W_k$ is $4n - 2$ connected and the map $S^{2n-1} \to W_k$ is null homotopic. From this it follows that

$$\Omega F_k \simeq S^{2n-1} \times \Omega W_k$$

Proof of (f). Let $\phi: \Omega S^{2n+1} \to F_k$ be the connecting map in the fibration that defines $F_k$. Let $u_i \in H^{2ni}(\Omega S^{2n+1})$ be the generator dual to the $i^{th}$ power of a generator in $H_{2n}(\Omega S^{2n+1})$. Then

$$u_i u_j = \binom{i+j}{i} u_{i+j}.$$ 

Choose generators $e_i \in H^{2ni}(F_k)$ so that

$$\phi^* (e_i) = \begin{cases} 
p^{r+d} u_i & \text{if } p^d \leq i < p^{d+1}, \ d \leq k \\
p^{r+k} u_i & \text{if } i \geq p^k. \end{cases}$$
Since $\phi^*$ is a monomorphism, it is easy to check that

$$e_1 e_{i-1} = \begin{cases} ipr^{-1} e_i & \text{if } i = p^s \quad 0 < s \leq k \\ ipr^s e_i & \text{otherwise.} \end{cases}$$

It now follows from the integral cohomology spectral sequence for the fibration:

$$S^{2n-1} \to W_k \to F_k$$

that

$$d_{2n} (e_{i-1} \otimes u) = \begin{cases} ipr^{-1} e_i & \text{if } i = p^s \quad 0 < s \leq k \\ ipr^s e_i & \text{otherwise.} \end{cases}$$

From this one can read off the cohomology groups of $W_k$ since $H^j(F_k) = Z_{(p)}$ or 0 according as to whether $j$ is a multiple of $2n$.

**Proof of (g).** From (d) we have a homotopy commutative square:

$$
\begin{array}{ccc}
T & \longrightarrow & \Omega S^{2n+1} \\
\downarrow & & \downarrow \phi \\
W_k & \longrightarrow & F_k
\end{array}
$$

Applying cohomology we get:

$$
\begin{array}{ccc}
H^{2np^{k+1}}(T) & \longrightarrow & H^{2np^{k+1}}(\Omega S^{2n+1}) \\
\downarrow & & \downarrow \\
H^{2np^{k+1}}(W_k) & \longleftarrow & H^{2np^{k+1}}(F_k)
\end{array}
$$

which evaluates as:

$$
\begin{array}{ccc}
Z/p^{r+k+1} & \longrightarrow & Z_{(p)} \\
\uparrow & & \uparrow \ p^{r+k} \\
Z/p^{r+k+1} & \longleftarrow & Z_{(p)}
\end{array}
$$

where the two horizontal arrows are epimorphisms. It follows that the homorphism $H^{2np^{k+1}}(W_k) \to H^{2np^{k+1}}(T)$ has image of order $p$.

**Proof of (h).** Recall from the proof of [GT] 4.3(c) that the composition:

$$p^{2np^{k+1}} \left( p^{r+k+1} \right) \to T/T^{2np^{k+1} - 2} \to R_k \to G_k$$


is homotopic to $\alpha_{k+1}$. Write $\tilde{\alpha}_{k+1}$ for the composition of the first two maps, so we get a homotopy commutative diagram:

\[
\begin{array}{cccccc}
T^{2np^{k+1}} & \longrightarrow & P^{2np^{k+1}} (p^{r+k+1}) & \longrightarrow & R_k & \\
\downarrow & & \downarrow & & \downarrow & \\
T^{2np^{k+1}}/T^{2np^{k+1}-2} & \longrightarrow & T/T^{2np^{k+1}-2} & \longrightarrow & W_k & \\
\end{array}
\]

By (g), the composition on the left and bottom is nonzero in integral cohomology, so the composition on the top and right is nonzero in integral cohomology also.

**Proof of (a) in the case $k+1$.** We consider the diagram

\[
\begin{array}{cccccccc}
P^{2np^{k+1}} (p^{r+k+1}) & \longrightarrow & R_k & \longrightarrow & W_k & \longrightarrow & F_k & \longrightarrow & F_{k+1} \\
\alpha_{k+1} \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_k & \longrightarrow & D_k & \longrightarrow & D_k & \longrightarrow & D_{k+1} & \\
\end{array}
\]

where the map $\Gamma$ exists since the lower composite factors as

\[
P^{2np^{k+1}} (p^{r+k+1}) \xrightarrow{\alpha_{k+1}} G_k \rightarrow G_{k+1} \rightarrow D_{k+1}
\]

We now show that the homorphism:

\[
H^{2np^{k+1}} (F_{k+1}) \rightarrow H^{2np^{k+1}} (F_k)
\]

is not an epimorphism. If it were, the entire composition:

\[
H^{2np^{k+1}} (F_{k+1}) \rightarrow H^{2np^{k+1}} (P^{2np^{k+1}} (p^{r+k+1}))
\]

would be nonzero by (f) and (h). But $\delta_{k+1}$ factors as

\[
\Omega S^{2n+1} \xrightarrow{\delta_{k+1}} F_k \rightarrow F_{k+1}
\]

so the image of $\delta_{k+1}^* : H^{2np^{k+1}} (F_{k+1}) \rightarrow H^{2np^{k+1}} (\Omega S^{2n+1})$ is divisible by $p^{r+k}$ by (a). This implies that

\[
\Gamma^* : H^{2np^{k+1}} (\Omega S^{2n+1}) \rightarrow H^{2np^{k+1}} (P^{2np^{k+1}} (p^{r+k+1}))
\]

is an epimorphism. This contradicts Hopf invariant one as follows. The composition:

\[
P^{2np^{k+1}} (p^{r+k+1}) \rightarrow \Omega S^{2n+1} \xrightarrow{H_{p^{k}}} \Omega S^{2np^{k+1}}
\]
would also be nonzero in cohomology so the Whitehead element

$$\omega_{np^k} \in \pi_{2np^{k+1} - 3}(S^{2np^k - 1})$$

would be divisible by $p^{r+k+1}$. We have thus shown that

$$H^{2np^{k+1}}(F_{k+1}) \rightarrow H^{2np^{k+1}}(F_k)$$

is not onto. From the Serre spectral sequence for the cohomology of the fibration

$$\Omega S^{2n+1} \rightarrow F_{k+1} \rightarrow D_{k+1}$$

we obtain the following exact sequence:

$$0 \leftarrow H^{2np^{k+1}+1}(F_{k+1}) \leftarrow \mathbb{Z}/p \leftarrow H^{2np^{k+1}}(F_{k+1}) \leftarrow H^{2np^{k+1}}(F_k) \leftarrow 0$$

where $H^*(F_k)$ is obtained from the restriction of the fibration to $D_k$.

It follows that

$$H^{2np^{k+1}+1}(F_{k+1}) = 0$$

$$H^{2np^{k+1}}(F_{k+1}) \simeq \mathbb{Z}/p$$

and

$$H^{2np^{k+1}}(F_{k+1}) \rightarrow H^{2np^{k+1}}(F_k)$$

has degree $p$, and $(\delta_{k+1})^*$ has degree $p^{r+k+1}$ in dimension $2np^{k+1}$. We now switch to integral homology and use the principal action

$$\Omega S^{2n+1} \times \Omega S^{2n+1} \longrightarrow \Omega S^{2n+1}$$

$$\downarrow$$

$$\Omega S^{2n+1} \times F_{k+1} \longrightarrow F_{k+1}$$

to study $H_{2ni}(F_{k+1})$ when $i > p^{k+1}$. Observe that the new generator $e_i$ comes from the term

$$E^{2np^{k+1}, 2ni-2np^{k+1}}_{\infty}$$

and consequently $e_i = u_{i-p^{k+1}} \cdot e_{p^{k+1}}$, so

$$p^{r+k+1}e_i = u_{i-p^{k+1}} \cdot (p^{r+k+1}e_{p^{k+1}})$$

$$= u_{i-p^{k+1}} \cdot u_{p^{k+1}}$$

$$= u_i$$

so $(\delta_{k+1})^*$ has degree $p^{k+1}$ in $H_{2ni}$ for $i \geq p^{k+1}$. This completes the proof of 2.2.

Theorem A follows by taking limits. The cofibration $C \xrightarrow{\sim} G \rightarrow D$ is the limit of $C_k \rightarrow G_k \rightarrow D_k$ and the fibrations

$$T \rightarrow W \rightarrow D$$

$$F \rightarrow D \rightarrow S^{2n+1}$$
are the respective limits of
\[ T \to W_k \to D_k \]
and \[ F_k \to D_k \to S^{2n+1}. \]

3.

In this section we will prove Theorem B. To this end, consider the limiting diagram of the fibrations in 2.2(d) over \( k \):

\[
\begin{array}{ccc}
T & \longrightarrow & T \\
\downarrow & & \downarrow \\
R & \longrightarrow & W \\
\downarrow & & \downarrow \\
G & \longrightarrow & D
\end{array}
\]

Since \( D \) is the mapping cone of the map \( c: C \to D \), the induced fibration over \( C \) is trivial and we have a map:
\[ C \times T \to R. \]

Since the inclusion of \( T \) into \( R \) is null homotopic, this extends to a map
\[ C \times T \to R. \]

According to [AG, Lemma A6] we get a cofibration sequence:
\[ C \times T \to R \to W \]

**Theorem 3.1.** If \( n > 1 \) there is a split short exact sequence
\[ 0 \to \tilde{H}_*(C \times T) \to \tilde{H}_*(R) \to \tilde{H}_*(W) \to 0. \]

**Proof.** We first check that the connecting homomorphism:
\[ H_j(W) \to H_{j-1}(C \times T) \]
is trivial. By 2.2(f), \( H_j(W) \neq 0 \) only when \( j = 2ni - 1 \). However,
\[ C \times T \simeq \bigvee_{i=1}^{\infty} P^{2np^i+1} (p^{r+i-1}) \times T \]
so \( H_{j-1}(C \times T) \) is only nonzero when \( j - 1 = 2n\ell \) or \( 2n\ell - 1 \). So if \( n > 1 \), one or the other of these groups is trivial. To see that the sequence splits, note that by 2.2(f)
\[ H_{2nk-1}(W) = \begin{cases} 
\mathbb{Z}_{(p)}/kp^{r-1} & \text{if } k = p^s \\
\mathbb{Z}_{(p)}/kp^r & \text{otherwise.}
\end{cases} \]
It suffices to show that
\[ kp^r \tilde{H}_{2nk-1}(R) = 0 \]
\[ kp^{-1} H_{2nk-1}(R) = 0 \text{ if } k = p^s \]

Now according to [G4, Theorem C], \( R \) is a retract of \( \Sigma T \wedge T \), so it suffices to prove

**Lemma 3.2.** \( kp^r \tilde{H}_{2nk-1}(\Sigma T \wedge T) = 0 \) and if \( k = p^s \)
\[ kp^{-1} \tilde{H}_{2nk-1}(\Sigma T \wedge T) = 0. \]

**Proof.** \( \Sigma T \wedge T \) is a wedge of Moore spaces by [GT, 4.3(m)] and the dimensions and orders can be read off from the homology groups. Let \( \nu(x) \) be the number of powers of \( p \) in \( x \). Then
\[ \Sigma T \wedge T = \bigvee_{j \geq 1} \bigvee_{i \geq 1} P^{2ni} (p^r + \nu(i)) \wedge P^{2nj} (p^r + \nu(j)). \]

This is a wedge of Moore spaces of dimension \( 2nk \) and \( 2nk + 1 \). An element of \( H_{2nk-1}(\Sigma T \wedge T) \) of order \( p^m \) must lie in
\[ \Sigma P^{2ni} (p^r + \nu(i)) \wedge P^{2nj} (p^r + \nu(j)) \]
where \( k = i + j, m \leq r + \nu(i) \) and \( m \leq r + \nu(j) \). Consequently \( p^{m-r} \) must divide both \( i \) and \( j \). Since \( k = i + j \), \( p^{m-r} \) divides \( k \) and the element has order dividing \( kp^r \). But if \( k = p^s \) we must have \( \nu(i) < s \) and \( \nu(j) < s \) so \( p^{m-r} \) divides \( p^{s-1} \); i.e., \( p^m \) divides \( kp^{-1} \).

Since \( R \) is a wedge of Moore spaces and the homomorphism
\[ \tilde{H}_*(R) \rightarrow \tilde{H}_*(W) \]
has a right inverse, we can use [AG, Lemma A3] to construct a right homotopy inverse for the map \( R \rightarrow W \). We obtain

**Proposition 3.3.** If \( n > 1 \), \( R \simeq (C \times T) \vee W \).

**Proposition 3.4.** Suppose \( n > 1 \). Then there is a homotopy fibration sequence:
\[ C \times \Omega D \rightarrow G \rightarrow D \]
and \( \Omega G \simeq \Omega D \times \Omega(C \times \Omega D) \).

**Proof.** Since the map \( R \rightarrow W \) has a right homotopy inverse, so does
\[ \Omega G \simeq T \times \Omega R \rightarrow T \times \Omega W \simeq \Omega D. \]

We use this together with [AG, Corollary A7] to prove that the fiber of the map \( G \rightarrow D \) is \( C \times \Omega D \).
Proposition 3.5. The composition

\[ C \times T \to R \to G \]

factors as

\[ C \times T \xrightarrow{1 \times g} C \times \Omega G \xrightarrow{\omega} C \vee G \xrightarrow{\omega^{\vee 1}} G \]

where \( \omega \) is the “Whitehead product map” which is the fiber of the projection \( C \vee G \xrightarrow{\pi_2} G \).

Proof. \( C \times T \simeq C \times T \cup * \times C^*(T) \) where \( C^*(T) \) is the reduced cone on \( T \). The composition:

\[ C \times T \cup * \times C^*(T) \simeq C \times T \to R \to G \]

is given by:

\[ C \times T \xrightarrow{\pi_1} C \xrightarrow{c} G \]

\[ * \times C^*T \to \Sigma T \xrightarrow{\tilde{\omega}} G \]

On the other hand, the inclusion of the fiber of the projection \( C \vee G \xrightarrow{\pi_2} G \) is given by

\[ C \times \Omega G \cup * \times PG \simeq C \times \Omega G \to C \vee G \]

\[ C \times \Omega G \xrightarrow{\pi_1} C \to C \vee G \]

\[ * \times PG \xrightarrow{e \nu} G \to C \vee G \]

where \( PG = \{ \omega : I \to G \mid \omega(0) = * \} \) and \( e \nu(\omega) = \omega(1) \).

Theorem B follows from 3.3, 3.4, and 3.5.

It would be desirable to have a better understanding of the restriction:

\[ W \to R \xrightarrow{\rho} G \]

What is clear is that the composition

\[ P^{2np^k} \left( p^{r+k-1} \right) \xrightarrow{a_k} W \to D \]

is nonzero in mod \( p \) homology, so \( a_k \) induces an isomorphism in integral cohomology in dimension \( snp^k \). It seems difficult to identify the map

\[ P^{4np^k} \left( p^{r+k} \right) \to W \]

as it is the first class of that order.
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Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 S. Morgan Street, Chicago, IL, 60607-7045

E-mail address: brayton@uic.edu