Analytical model of high-frequency energy flow response for a beam with free layer damping

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Abstract
Energy flow analysis (EFA) is developed to predict the vibrational energy density of beam structures with both full free layer damping (FFLD) and partial free layer damping (PFLD) treatments in the high frequency range. Both equivalent flexural stiffness and structural damping loss factor of a beam with free layer damping are obtained using the equivalent complex stiffness method. Then the energy density governing equation considering high damping effect is derived for a beam with FFLD treatment. Following obtainment of the energy transfer coefficients at both ends of free damping layer, the energy density within a beam with PFLD treatment is evaluated by solving the presented energy governing equation. To verify the proposed formulation, numerical simulations are performed for the pinned-pinned beams with FFLD and PFLD treatments. The EFA results are compared with the exact solutions from wave analysis at various frequencies, and good correlations are observed between the developed EFA results and the exact solutions.

Keywords
Beam with free layer damping treatment, energy flow analysis, equivalent complex stiffness model, energy density, high-frequency vibration

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Introduction
As one of typical structural components, lightweight beams are extensively used in mechanical and aerospace engineering in recent years. However, these lightweight beams are always sensitive to high frequency vibration excited by noise, turbulent flow, engine vibrating, and so on. Particularly, the undesirable vibration possibly leads to failure and degraded performance of structures or machines. To prohibit the unfavorable vibration, free layer damping treatment is widely adopted in beam-type structures. Free layer damping treatment uses high damping viscoelastic materials firmly attached to the surface of the lightweight beams.¹,² Therefore, the high frequency vibration analysis of lightweight beams with free layer damping (FLD) is significant for the design of beam-like structures under high frequency excitation.

Over the last few decades, advances in technology have drawn attention to high-frequency noise and vibration. Statistical energy analysis (SEA), a representative stochastic energy-based method, is widely employed to predict time and space averaged response of structures at high frequencies.³–⁵ However, SEA has some difficulties in offering detailed information, such as the distribution of energy density at the local domain.

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of interest. As an alternative method for high frequency vibration analysis, energy flow analysis (EFA) is a recently developed effective tool for the dynamic response of built-up structures at high frequency. Since the energy governing equation for EFA is a parabolic partial differential equation, the solution can present the spatial variation of energy densities.6 The governing equations of EFA were developed for structural components such as rods,7–9 beams,7,8 membranes,10 and plates.11,12 To predict the vibrational response of the structures governing equation of beams with thermal effects was derived by Zhang et al.,23 whose work was afterwards extended by Wang et al.24 to plates in non-uniform thermal environment. Recently, Yeo et al.25 develop an energy flow model for a dilatational wave in 3D elastic solids vibrating at medium- to high-frequency ranges. Thus, EFA has been an efficient method for high-frequency structural vibration analysis.

However, few publications are available on the high-frequency vibration analysis of a structure with viscoelastic damping treatment. The previous energy flow models are mostly developed for low damping and homogeneous structures. Since a free layer damping structure uses high damping viscoelastic materials firmly attached to the surface of structural elements, the traditional energy flow models can’t accurately predict the dynamic behavior of the viscoelastic laminated structures. Thus, this paper motivates to develop a new energy flow model to predict the high-frequency response of beam structures with viscoelastic damping treatment.

The paper is organized as follows. Firstly, the energy density governing equation of a beam with full free layer damping (FFLD) treatment is derived from the energy balance equation, the energy transmission equation and the energy loss equation. Secondly, the energy transfer coefficients are deduced from wave theory to indicate how wave energy is distributed when the flexural wave propagates through a beam with partially free layer damping (PFLD) treatment. Then, in order to verify the developed energy flow model, various numerical analyses are performed for pinned-pinned beam structures with FFLD and PFLD treatments at several excitation frequencies. The developed EFA results are compared with exact wave solutions for different cases. Finally, some conclusions on the proposed EFA model are presented.

**Energy flow model of a beam with FFLD treatment**

**Equivalent complex bending stiffness and structural damping loss factor**

Figure 1(a) depicts a free layer damping beam structure, where \( h_b \) and \( h_l \) denote the thickness of the base beam and viscoelastic layer, respectively. The elastic material properties for the base beam and viscoelastic layer are as follows: Young’s moduli \( E_b \) and \( E_d \), and density \( \rho_b \) and \( \rho_d \). When the structural damping is considered, the complex Young’s moduli of base beam and free layer damping can be represented by \( E_b' = E_b(1 + j \eta_b) \) and \( E_d' = E_d(1 + j \eta_d) \), where \( \eta_b \) and \( \eta_d \) are structural damping loss factors for the base beam and viscoelastic layer, respectively.

For a differential element of a laminated beam, as shown in Figure 1(b), the complex longitudinal strain \( \varepsilon^* \) and stress \( \sigma^*_i \) of the arbitrary axial fiber \( ab \) can be expressed as

\[
\varepsilon^* = y \frac{\partial^2 w(x,t)}{\partial x^2}, \sigma^*_i = E_i' \varepsilon^*(i = d, b) \tag{1}
\]

where \( w(x,t) \) is the transverse displacement of the laminated beam in the \( y \) direction, \( \sigma^*_d \) and \( \sigma^*_b \) are the normal stresses within the viscoelastic layer and base beam.

When the beam vibrates transversely under an external loading, the longitudinal force equilibrium is obtained. We have

\[
\int_{h_d + h_l}^{h_b + h_l} \sigma^*_i dy = 0 \tag{2}
\]

By substitution of equation (1) into equation (2), the location of neutral axis of the laminated beam is written as

\[
h_N = \frac{E_d' h_b^2 + E_b' h_l^2}{2(E_d' h_b + E_b' h_l)} \tag{3}
\]
where \( h_N \) is complex-valued, whose real part represents the physical location of the neutral axis.

For a beam with FFDL treatment, the complex bending moment resulting from the complex longitudinal stress is given by

\[
M^* = \int_{(h_n-h_b)}^{h_b} y\sigma_t^* dy \quad (4)
\]

\[
M^* = D_{eq} \frac{\partial^2 w(x, t)}{\partial x^2} \quad (5)
\]

where \( M^* \) is the complex bending moment of the laminated beam and \( D_{eq} \) is the equivalent complex flexural rigidity of the laminated beam.

By substituting equations (1) and (5) into equation (4), the equivalent complex flexural rigidity of the laminated beam is written by

\[
D_{eq} = \int_{(h_n-h_b)}^{h_b} y^2 E_t^* dy \quad (6)
\]

By substituting equation (3) into equation (6), the equivalent complex flexural stiffness of the laminated beam is obtained

\[
D_{eq} = D_b \left( 1 + 2\epsilon(2h + 3h^2 + 2h^3) + \epsilon^2 h^4 \right) \quad (7)
\]

where \( h \) is the thickness ratio of the free damping layer \( h_d \) to the base beam \( h_b \), \( \epsilon \) is the complex elastic modulus ratio of the free damping layer \( E_t^* \) to the base beam \( E_b^* \), and \( D_b \) is the complex flexural rigidity of the base beam.

According to the energy theory,\(^2\) the equivalent complex flexural stiffness can be given by

\[
D_{eq} = \text{Re}(D_{eq}) \left( 1 + j\eta_{eq} \right),
\]

where the real and imaginary components represent the storage and dissipation of the mechanical energy, respectively. The dissipative property of the laminated beam may be characterized by the equivalent structural damping loss factor \( \eta_{eq} \)

\[
\eta_{eq} = \frac{\text{Im}(D_{eq})}{\text{Re}(D_{eq})} \quad (8)
\]

where functions \( \text{Re}(\cdot) \) and \( \text{Im}(\cdot) \) are employed to extract the complex real and imaginary parts of \( D_{eq} \), respectively.

**Energy density governing equation**

To perform the energy flow analysis of the laminated beam, the energy density governing equation is deduced from the energy transfer equation, the energy loss equation and the energy balance equation.

For a beam with FFDL treatment, excited by a transverse harmonic load \( F(x_0)e^{j\omega t} \) at point \( x = x_0 \), as depicted in Figure 2, the equation of motion for the transversely vibration beam is expressed as

\[
D_{eq} \frac{\partial^4 w(x, t)}{\partial x^4} + m_{eq} \frac{\partial^2 w(x, t)}{\partial t^2} = F(x_0)e^{j\omega t} \quad (9)
\]

where \( m_{eq} = \rho_d S_d + \rho_b S_b \) is the equivalent mass of per unit length for the laminated beam, \( S_d \) and \( S_b \) are the cross sectional area of the viscoelastic layer and the base beam, respectively.

Hence, the general solution of equation (9) in arbitrary domain is written as

\[
w_t(x, t) = [Ae^{-jk_{10}x} + Be^{jk_{10}x} + Ce^{-jk_{20}x} + De^{jk_{20}x}]e^{j\omega t} \quad (10)
\]

where \( w_t(x, t) \) is the transverse displacement of a beam with FFDL treatment, \( A, B, C, \) and \( D \) are the constants depending on boundary conditions. The complex flexural wavenumber and the phrase velocity are presented as

\[
k_t = \sqrt{\frac{m_{eq}\omega^2}{D_{eq}}} = k_{1t} + jk_{2t} \quad (11)
\]

\[
c_t = \sqrt{\frac{D_{eq}\omega^2}{m_{eq}}} \quad (12)
\]

where \( k_{1t} \) and \( k_{2t} \) are the real and imaginary parts of the complex flexural wavenumber \( k_t \), and \( c_t \) is the phrase velocity for the laminated beam. In a light damping homogeneous beam, the terms with high-order structural damping can be ignored, and the flexural wavenumber can be linearly approximated as

\[
k_t' = \sqrt{\frac{m'\omega^2}{\text{Re}(D')}} \left( 1 - \frac{\eta}{4} \right) = k_{1t}' + jk_{2t}' \quad (13)
\]

where \( k_{1t}' \) and \( k_{2t}' \) are the real and imaginary terms of wavenumber \( k_t' \), \( \eta' \) is the structural damping loss factor, \( m' \) is the mass of per unit length and \( D' \) is the complex flexural rigidity for the homogeneous beam.

The first two terms of equation (10) are far-field solutions describing the propagating waves in the positive and negative \( x \) directions, respectively. The last two terms of equation (10) are near-field solutions
The time averaged energy density can be represented as  

\[
\langle q_{i} \rangle = \frac{1}{2} \text{Re} \left[ D_{eq} \left\{ \frac{\partial^2 w_{f}}{\partial x^2} \left( \frac{\partial w_{f}}{\partial t} \right)^* - \frac{\partial^2 w_{f}}{\partial x^2} \left( \frac{\partial^2 w_{f}}{\partial x \partial t} \right)^* \right\} \right]
\]  

(18)

To eliminate spatially harmonic terms in equations (17) and (18), a spatially averaged operation over a wavelength is applied to the time averaged energy density and intensity, as indicated by Wohlever.\textsuperscript{8,26} By substitution of equation (16) into equations (17) and (18), and following locally space averaging operation, the time and space averaged energy density and intensity are obtained as

\[
\langle \tilde{e}_{i} \rangle = \frac{1}{4} m_{eq} \omega^2 \left( Re(D_{eq}) + 1 \right) \left( \alpha_{1} |A|^2 e^{2k_{x}x} + \alpha_{2} |B|^2 e^{-2k_{x}x} \right)
\]  

(19)

\[
\langle \tilde{q}_{i} \rangle = \omega Re(D_{eq}) \left( k_{T1}^2 + k_{T2}^2 - 2 \eta_{eq} k_{T1} k_{T2} \right) \left( \alpha_{1} |A|^2 e^{2k_{x}x} - \alpha_{2} |B|^2 e^{-2k_{x}x} \right)
\]  

(20)

where \( \langle \cdot \rangle \) denotes the time and space averaged operation, \( \alpha_{1} \) and \( \alpha_{2} \) are the space averaging values of \( e^{2k_{x}x} \) and \( e^{-2k_{x}x} \), respectively. The smoothed energy density and intensity are still capable of predicting spatial variations of the energy density and intensity.

From equations (19) and (20), the energy transfer equation, representing the relationship between the time and space averaged energy density and intensity, can be expressed as

\[
\langle \tilde{q}_{i} \rangle = \frac{2 \omega Re(D_{eq}) \left( 1 - \tan^2 \frac{\varphi}{2} + 2 \eta_{eq} \tan \frac{\varphi}{2} \right) \frac{\partial \langle \tilde{e}_{i} \rangle}{\partial x}}{\tan \frac{\varphi}{2} m_{eq} \left( \frac{Re(D_{eq})}{|D_{eq}|} + 1 \right) \left( Re(\langle \tilde{e}_{i} \rangle) \right)^2 \frac{\partial}{\partial x}}
\]  

(21)

where \( \varphi = \arctan(\eta_{eq}) \) is the equivalent damping angle of the laminated beam.

In addition, from equation (19), the relationship between the time-space averaged potential energy density and the total energy density can be derived as

\[
\langle U_{i} \rangle = \frac{Re(D_{eq})}{Re(D_{eq}) + |D_{eq}|} \langle \tilde{e}_{i} \rangle
\]  

(22)

where \( \langle U_{i} \rangle \) are the time-space averaged potential energy density.

From the work of Lase et al.,\textsuperscript{7} the energy loss equation for a laminated beam with high structural damping loss factor is

\[
\langle U_{i} \rangle = \frac{Re(D_{eq})}{Re(D_{eq}) + |D_{eq}|} \langle \tilde{e}_{i} \rangle
\]  

(22)
\[ \langle \pi_{\text{diss}} \rangle = 2 \eta_{\text{eq}} \omega \langle U_f \rangle \]  

(23)

In a light damping homogeneous beam, the time-space averaged potential energy density is the same as time-space averaged kinetic energy density and hence the time-space averaged dissipated power is represented as \( \dot{q}_{\text{diss}} \)

\[ \langle \pi'_{\text{diss}} \rangle = \eta \omega \langle v_f^2 \rangle \]  

(24)

However in a high damping beam, the time-space averaged potential energy density is different from time-space averaged kinetic energy according to equations (19) and (22). It implies that equation (24) derived from low damping assumption is unsuitable for high damping systems. From this, substituting equation (22) into equation (23), the energy loss equation for a laminated beam is rewritten as

\[ \langle \pi_{\text{diss}} \rangle = 2 \eta_{\text{eq}} \omega \frac{\text{Re}(D_{\text{eq}})}{\text{Re}(D_{\text{eq}}) + |D_{\text{eq}}|} \langle \dot{e}_f \rangle \]  

(25)

For an elastic vibration system, the energy balance equation at the steady state can be described as

\[ \nabla \cdot \langle \dot{q}_f \rangle + \langle \pi_{\text{diss}} \rangle = \langle \pi_{\text{in}} \rangle \delta(x - x_0) \]  

(26)

Where \( \langle \pi_{\text{in}} \rangle \) is the time averaged input power injected into the laminated beam due to the harmonic external force.

Substituting equations (21) and (25) into equation (26), the energy density governing equation of the laminated beam can be derived as

\[ -\frac{2\omega \text{Re}(D_{\text{eq}})\left(1 - \tan^2 \frac{\theta}{4} + 2\eta_{\text{eq}} \tan \frac{\theta}{4}\right)}{\tan \frac{\theta}{4} \eta_{\text{eq}} \frac{\text{Re}(D_{\text{eq}})}{|D_{\text{eq}}|} + 1} \frac{\partial^2 \langle \dot{e}_f \rangle}{\partial x^2} + 2\eta_{\text{eq}} \omega \frac{\text{Re}(D_{\text{eq}})}{\text{Re}(D_{\text{eq}}) + |D_{\text{eq}}|} \langle \dot{e}_f \rangle = \langle \pi_{\text{in}} \rangle \delta(x - x_0) \]  

(27)

The conventional energy density governing equation is developed with low damping assumption by using linearly approximated wavenumber to evaluate the time and space averaged energy density and intensity. Unlike the traditional energy flow model which is only suitable for lightly damped systems, the developed energy density equation is deduced by using the exact wavenumber to evaluate the time and space averaged energy density and intensity due to high damping effect of the laminated beam.

**Energy flow analysis**

As shown in Figure 2, the laminated beam is divided into two subdomains \( \Phi \) and \( \Psi \) at the point \( x_0 \). By convention, the subscripts \( i \) (\( i = 1, 2 \)) represents the physical quantities attached to the \( i \)th subdomain. According to the equation (27), the general energy density solution of each subdomain can be expressed as

\[ \langle \dot{e}_f \rangle = F_{f1} e^{-\lambda_f x} + F_{f2} e^{\lambda_f x} \]  

(28)

where \( F_{f1} \) and \( F_{f2} \) are the coefficients determined from boundary conditions, and \( \lambda_f \) is the eigenvalue of equation (27).

Substituting equation (28) into equation (27), there is

\[ \lambda_f = \sqrt{\frac{\eta_{\text{eq}} |\text{Re}(c_f)|^2 \tan(\theta/4) \eta_{\text{eq}}}{\text{Re}(D_{\text{eq}}) \left(1 - \tan^2 \frac{\theta}{4} + 2\eta_{\text{eq}} \tan \frac{\theta}{4}\right)}} \]  

(29)

Substituting equation (28) into equation (21), the energy intensity of the \( i \)th subdomain of the laminated beam is written by

\[ \langle \dot{q}_{fi} \rangle = P_f (F_{fi} e^{-\lambda_f x} - F_{f2} e^{\lambda_f x}) \]  

(30)

where

\[ P_f = \frac{2\omega \text{Re}(D_{\text{eq}})\left(1 - \tan^2 \frac{\theta}{4} + 2\eta_{\text{eq}} \tan \frac{\theta}{4}\right)}{\tan \frac{\theta}{4} \eta_{\text{eq}} \frac{\text{Re}(D_{\text{eq}})}{|D_{\text{eq}}|} + 1} \text{Re}^2(c_f) \lambda_f \]  

(31)

Because the energy intensity across the boundary is zero for a pinned-pinned beam, the following intensity boundary conditions can be written as

\[ \langle \dot{q}_{f1} \rangle (0) = 0, \langle \dot{q}_{f2} \rangle (L) = 0 \]  

(32)

At the driving point \( x = x_0 \), the energy density is continuous while the energy intensity is discontinuous owing to the time averaged input power \( \langle \pi_{\text{in}} \rangle \) caused by the transverse concentrated load

\[ \langle \dot{e}_{f1} \rangle (x_0) = \langle \dot{e}_{f2} \rangle (x_0) \]  

(33)

\[ \langle \dot{q}_{f1} \rangle (x_0) = \langle \dot{q}_{f2} \rangle (x_0) + \langle \pi_{\text{in}} \rangle \]  

(34)

For the solutions of equation (27), it is necessary to convert the external load to the input power. By impedance method, time averaged input power can be obtained as

\[ \langle \pi_{\text{in}} \rangle = |F(x_0)|^2 / [8 \eta_{\text{eq}} \text{Re}(c_f)] \]  

(35)

Substituting equations (32)–(34) into equations (28) and (30), the flexural energy density and intensity for the vibrating laminated beam can be calculated.

**Energy flow model of a beam with PFLD treatment**

**Energy transfer coefficients**

For the coupled structures, the key step of EFA is to the energy transmission and reflection coefficients in...
the joint. Figure 4 depicts a beam with PFLD treatment, which can be treated as coupled collinear beams. The coupled beam is divided into four parts, by the driving force and both ends of the free damping layer, respectively. The lengths of the base beam and the free damping layer are specified to be $L$ and $|x_2 - x_1|$, respectively.

In the wave transmission approach, the energy transfer coefficients are evaluated from the displacement solutions of infinite members. Figure 5 shows a semi-infinite bold beam jointed to a semi-infinite beam with free damped treatment at $x = 0$. When a right travelling flexural wave of amplitude, $A_1$, is incident upon the joint, the incident flexural wave is partially transmitted and reflected. The transverse displacement in the incident beam $\oplus$ can be represented as

$$\bar{w}_{p1}(x, t) = [A_1e^{-jk_{lb}x} + B_1e^{jk_{lb}x} + D_1e^{jk_{lb}x}]e^{j\omega t}$$ (36)

where $k_{lb}$ is the flexural wavenumber in beam $\oplus$. The transverse displacement in the receiving beam $\ominus$ can be represented as

$$\bar{w}_{p2}(x, t) = [A_2e^{-jk_{lb}x} + C_2e^{-jk_{lb}x}]e^{j\omega t}$$ (37)

where $k_{lb}$ is the flexural wavenumber in beam $\ominus$. The complex amplitudes $A_1$, $B_1$, $D_1$, $A_2$, and $C_2$ are determined from the equilibrium and continuum conditions at the joint.

At the joint, the moment and shear force satisfy the equilibrium equation as below

$$D_b \frac{d^2 \bar{w}_{p1}(0, t)}{dx^2} = D_{eq} \frac{d^2 \bar{w}_{p2}(0, t)}{dx^2}$$ (38)

$$D_b \frac{d^3 \bar{w}_{p1}(0, t)}{dx^3} = D_{eq} \frac{d^3 \bar{w}_{p2}(0, t)}{dx^3}$$ (39)

The other two conditions impose continuity of transverse displacement and slopes. The continuity of transverse displacements and slopes lead to the following relationship

$$\bar{w}_{p1}(0, t) = \bar{w}_{p2}(0, t)$$ (40)

$$\frac{d\bar{w}_{p1}(0, t)}{dx} = \frac{d\bar{w}_{p2}(0, t)}{dx}$$ (41)

Equations (38)--(41) are solved simultaneously for the complex amplitudes, $B_1$, $D_1$, $A_2$, and $C_2$. From equation (15), the time averaged energy intensity of incident, reflected and transmitted flexural waves at the joint can be calculated

$$\frac{\langle q_{p1inc} \rangle}{\langle q_{p1inc} \rangle} = \frac{Re(D_b k_{lb}^3) |A_1|^2}{|A_1|^2}$$ (42)

According to equation (42), the energy reflection and transmission coefficients can be expressed as

$$\gamma_{f11} = \frac{\langle q_{p1ref} \rangle}{\langle q_{p1inc} \rangle} = \frac{|B_1|^2}{|A_1|^2}$$ (43)

$$\tau_{f12} = \frac{\langle q_{p2trans} \rangle}{\langle q_{p1inc} \rangle} = \frac{Re(D_{eq} k_{lb}^3) |A_2|^2}{Re(D_b k_{lb}^3) |A_1|^2}$$ (44)

The principle of conversation of energy applied to the semi-infinite beam joint model is

$$\frac{\langle q_{p1inc} \rangle}{\langle q_{p1inc} \rangle} = \frac{\langle q_{p1ref} \rangle}{\langle q_{p1inc} \rangle} + \frac{\langle q_{p2trans} \rangle}{\langle q_{p1inc} \rangle}$$ (45)

From equation (45), the relationship between the energy reflection and transmission coefficients is subjected to $\gamma_{f11} + \tau_{f12} = 1$. Considering the incident flexural wave which comes from the beam $\ominus$, the energy transfer coefficients $\gamma_{f12}$ and $\tau_{f11}$ can be evaluated in a similar way. Unlike the traditional energy model for the coupled beam, the energy transfer coefficients is derived from exact wavenumber without linear approximation because the influence of high damping is not neglected for a beam with PFLD treatment.

**Energy flow analysis**

Similar to the fully laminated beam, the flexural energy density and intensity of the partially laminated beam, as shown in Figure 6, is expressed as

$$\langle \varepsilon_p \rangle = F_{p1}e^{-\lambda_1 x} + F_{p2}e^{\lambda_2 x}$$ (46)
in a similar way, as can be expressed as

tive and negative traveling waves in each subdomain partially laminated beam. The energy intensity of posi-
g where \( \gamma \) and \( P \) can be obtained from equations (29) and (31).

For a pinned-pinned beam, the energy boundary conditions at both ends are expressed as

\[
\begin{align*}
\mathbb{q}_p^1(0) &= 0, \quad \mathbb{q}_p^4(L) = 0
\end{align*}
\]  

in a similar way, \( \lambda \) and \( P \) can be obtained from equations (29) and (31).

At the driving point between subdomain \( \Theta \) and \( \Omega \), the continuities of energy density and the equilibriums of energy flow are satisfied

\[
\begin{align*}
\mathbb{q}_p^1(x_0) &= \mathbb{q}_p^2(x_0) + \mathbb{q}_p^3(x_0) + \mathbb{q}_p^4(x_0)
\end{align*}
\]

From the conversation of energy, the net energy flow away from the joints in each subdomain, can be written as

\[
\begin{align*}
\mathbb{q}_p^1(x_1) &= \gamma t_{111} \mathbb{q}_p^1(x_1) + \tau_{112} \mathbb{q}_p^2(x_1) + \tau_{121} \mathbb{q}_p^3(x_1) + \tau_{111} \mathbb{q}_p^4(x_1) \\
\mathbb{q}_p^2(x_1) &= \gamma t_{222} \mathbb{q}_p^2(x_1) + \tau_{212} \mathbb{q}_p^1(x_1) + \tau_{221} \mathbb{q}_p^3(x_1) + \tau_{222} \mathbb{q}_p^4(x_1) \\
\mathbb{q}_p^3(x_2) &= \gamma t_{333} \mathbb{q}_p^3(x_2) + \tau_{312} \mathbb{q}_p^1(x_2) + \tau_{321} \mathbb{q}_p^2(x_2) + \tau_{332} \mathbb{q}_p^4(x_2) \\
\mathbb{q}_p^4(x_2) &= \gamma t_{444} \mathbb{q}_p^4(x_2) + \tau_{412} \mathbb{q}_p^1(x_2) + \tau_{421} \mathbb{q}_p^2(x_2) + \tau_{442} \mathbb{q}_p^3(x_2)
\end{align*}
\]

where \( \gamma_i \) is the energy reflection coefficient in the \( i \)th subdomain of the coupled beam due to the incident wave in the \( i \)th subdomain of the coupled beam, and \( \tau_i \) is the energy transmission coefficient in the \( j \)th subdomain of the coupled beam due to the incident wave in the \( j \)th subdomain of the coupled beam.

From reciprocity and conversation of energy principles, we have

\[
\tau_{ij} = \tau_{ji}, \quad \gamma_{ij} = \gamma_{ji} \quad (i, j = 1 \sim 4)
\]

Similar to the previous case, the time averaged input power of partially laminated beam can be obtained using the impedance of finite beam. Substituting equations (49)–(56) to equations (46)–(48), the energy densities for the coupled beam can be predicted.

**Verification and discussion**

In order to verify the developed energy flow analysis method for laminated beams with FFLD and PFLD treatments, some numerical simulations are performed on beams with both pinned end conditions, as depicted in Figures 2 and 4. The energy density distribution obtained by the proposed EFA model are compared with the exact time and time-space averaged solutions obtained by wave analysis method (refer to Appendix A and B).

**A beam with FFLD treatment**

The base beam is made of aluminum alloy, whose material properties and dimensions are as follows: \( \rho_b = 2700 \text{kg/m}^3 \), \( E_b = 71 \text{GPa} \), \( \eta_b = 0.01 \), \( L = 1 \text{m} \), \( b = 0.008 \text{m} \), \( h_b = 0.002 \text{m} \). The free damping layer is made of rubber with the viscouselastic free damping layer, whose material properties are as follows: \( \rho_d = 980 \text{kg/m}^3 \), \( E_d = 0.33 \text{GPa} \), and \( \eta_d = 1.3 \), and whose dimensions are the same as those of the base beam except \( h_d = 0.004 \text{m} \). The laminated beam is excited by a transverse harmonic point force with amplitude of 1 N at the center of the laminated beam.

Figures 7 to 10 illustrates the energy density distribution by EFA when the analysis frequencies are 800 Hz, 2 kHz, 4 kHz, and 6 kHz, respectively. As in all the following figures, the reference energy density is \( 10^{-12} \text{J/m} \). The exact time averaged energy density fluctuates spatially. Because the spatially harmonic terms of energy density are excluded in EFA, the EFA predictions are smooth representing the trend of the exact time averaged energy density. It is also found that the spatially harmonic terms lessen with increasing frequency, which implies that the EFA predictions are more accurate at higher frequencies. However, it is observed that the developed EFA results obviously differ from the exact time averaged wave solution in the vicinity of the driving point and the boundaries due to negligence of evanescent wave. Fortunately, this difference only exists within one or more wavelengths away from the driving point and the boundaries. As shown in Figure 11, the far-field solutions representing travelling waves are far greater than the near-field solutions representing evanescent waves except for neighborhood of the driving point and the boundaries, as a result of the rapid
dissipation of the evanescent wave. Particularly with increasing frequency, the far-field solutions overwhelm the near-field solutions. It indicates that the developed EFA solutions with negligence of near-field solutions are more consistent with the exact time averaged wave solution at higher analysis frequencies, as depicted in Figures 7 to 10. In addition, it is obviously observed that the energy density abruptly decays near the driving point owing to the dissipation of energy caused by high damping. The attenuation of energy density is larger at higher frequency. Moreover, the developed EFA results are consistent with the exact time and space averaged solutions obtained by wave analysis.

A beam with PFLD treatment

As shown in Figure 4, the free damping layer is partially laid above the central region of the base beam. The geometrical dimensions except the free damping layer length, the material properties as well as boundary conditions of the partially laminated beam are the same as those of fully laminated beam as depicted in Figure 2.

Figures 12 to 15 show the energy density distribution by EFA when the analysis frequencies are 800 Hz, 2 kHz, 4 kHz, and 6 kHz, respectively. As in all the following figures, the reference energy density is $10^{-12}$ J/m. The developed EFA result and exact time averaged
solutions differ near the point force and the boundaries because the developed energy flow model is derived by exclusion of the evanescent wave. Despite this, the developed EFA results are in good agreement with the global variation of the exact time averaged solutions at each frequency. As expected, there is a jump in the developed energy density by EFA at the two joints between bold beam and laminated beam. Nevertheless, the developed EFA results present a smooth response away from the discontinuities resulting from dissimilar cross-sectional area and material properties. It is also noted that the energy density severely attenuates near the driving point due to the dissipation of energy caused by high damping. The attenuation of energy density is larger at higher frequency. Additionally, the developed EFA results concur with the exact time and space averaged solutions obtained by wave analysis.

**Conclusion**

The energy density governing equations for beam structures with FFLD and PFLD treatments are developed to predict their high-frequency dynamic response by EFA method. Using equivalent complex stiffness method, the equivalent complex bending stiffness and structural damping loss factor are obtained to constitute the motion equation of a beam with FFLD. Due to high structural damping loss factor, the energy density governing equation of a beam with FFLD is derived by using the exact wavenumber obtained without linear
approximation to evaluate the time and space averaged energy density and intensity. For a beam with PFLD, the transmission and reflection coefficients are derived to determine how wave energy is distributed when the flexural wave propagates through both ends of the free damping layer.

To verify the developed energy flow model, the pinned-pinned beams with FFLD and PFLD treatments are adopted in the numerical simulations. The developed EFA results are compared with the exact solutions from wave analysis. The developed EFA results smoothly represent the trend of time averaged wave solutions at various analysis frequencies, and are in good agreement with time and space averaged wave solution. Particularly, to obtain the energy density distribution within a beam with PFLD treatment, the energy transfer coefficients is derived from exact wave-number without linear approximation because the influence of high damping is not simplified. Furthermore, it is found that the energy density within the laminated beam is more significantly attenuated by high damping effect at higher frequencies. The proposed method is expected to be useful for the prediction of the high frequency vibration of structures with FFLD and PFLD treatments.

Figure 14. Energy density distribution of a beam with PFLD treatment under excited frequency of 4 kHz.

—: EFA solution; —: time averaged wave solution; —: time and space averaged wave solution.

Figure 15. Energy density distribution of a beam with PFLD treatment under excited frequency of 6 kHz.

—: EFA solution; —: time averaged wave solution; —: time and space averaged wave solution.

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**Appendix A**

**Exact solution from wave analysis for a beam with FFLD treatment**

As shown in Figure 2, a beam with FFLD treatment is split into two components at $x_0$, where a transverse harmonic force is imposed. The governing equation of motion for each component can be expressed as

$$D_{eq} \frac{\partial^4 w_i(x, t)}{\partial x^4} + m_{eq} \frac{\partial^2 w_i(x, t)}{\partial t^2} = 0 \quad (A.1)$$

The exact general solutions of equation (A.1) is illustrated in equation (10), so that the displacement of the $i$th laminated beam can be obtained as

$$w_i(x, t) = [A_{fi} e^{-k_x x} + B_{fi} e^{k_x x} + C_{fi} e^{-k_y x} + D_{fi} e^{k_y x}] e^{iwt} \quad (A.2)$$

where $A_{fi}$, $B_{fi}$, $C_{fi}$, and $D_{fi}$ are the constants depending on the following boundary conditions.

The pinned-pinned boundary conditions at both ends of the laminated beam can be written as

$$w_{f1}(0, t) = 0, w_{f2}(L, t) = 0 \quad (A.3)$$

$$D_{eq} \frac{\partial^2 w_{f1}(0, t)}{\partial x^2} = 0, D_{eq} \frac{\partial^2 w_{f2}(L, t)}{\partial x^2} = 0 \quad (A.4)$$

At the interface between subdomain $\square$ and subdomain $\uncover$, the displacements, and slopes remain continuous, and the bending moment and the shear force remain equilibrium.

$$w_{f1}(x_0, t) = w_{f2}(x_0, t) \quad (A.5)$$

$$\frac{\partial w_{f1}(x_0, t)}{\partial x} = \frac{\partial w_{f2}(x_0, t)}{\partial x} \quad (A.6)$$

$$D_{eq} \frac{\partial^2 w_{f1}(x_0, t)}{\partial x^2} = D_{eq} \frac{\partial^2 w_{f2}(x_0, t)}{\partial x^2} \quad (A.7)$$

$$D_{eq} \frac{\partial^3 w_{f1}(x_0, t)}{\partial x^3} - D_{eq} \frac{\partial^3 w_{f2}(x_0, t)}{\partial x^3} = F(x_0) e^{iwt} \quad (A.8)$$

Substituting equation (A.2) into equations (A.3)–(A.8), the coefficients $A_{fi}$, $B_{fi}$, $C_{fi}$, and $D_{fi}$ can be determined and the exact displacement solution of the whole laminated beam can be evaluated. Then substituting the exact displacement solution into equation (17), the exact time averaged energy density is obtained. Following locally special averaging, the exact time and space averaged energy density is also obtained.
Appendix B

Exact solution from wave analysis for a beam with PFLD treatment

As shown in Figure 4, a beam with PFLD treatment is split into four components at driving point and coupled joints. The exact displacement solution of the $i$th beam can be written as

$$w_{pi}(x, t) = \left[A_{pi}e^{-jk_i x} + B_{pi}e^{jk_i x} + C_{pi}e^{-k_i x} + D_{pi}e^{k_i x}\right]e^{jwt}$$  \hspace{1cm} (B.1)

where $A_{pi}$, $B_{pi}$, $C_{pi}$, and $D_{pi}$ are the constants determined from the following boundary conditions.

The boundary conditions for a pinned-pinned beam are given by

$$w_{pi}(0,t) = 0, w_{pi}(L, t) = 0$$  \hspace{1cm} (B.2)

$$D_b \frac{\partial^2 w_{pi}(0,t)}{\partial x^2} = 0, D_b \frac{\partial^2 w_{pi}(L, t)}{\partial x^2} = 0$$  \hspace{1cm} (B.3)

At the joints ($x = x_1$ and $x = x_2$) between bold and laminated beams, the displacement, and slope are continuous, and the bending moment and the shear force satisfy the equilibrium condition.

$$w_{p1}(x_1, t) = w_{p2}(x_1, t), w_{p3}(x_2, t) = w_{p4}(x_2, t)$$  \hspace{1cm} (B.4)

$$\frac{\partial w_{p1}(x_1, t)}{\partial x} = \frac{\partial w_{p2}(x_1, t)}{\partial x}, \frac{\partial w_{p3}(x_2, t)}{\partial x} = \frac{\partial w_{p4}(x_2, t)}{\partial x}$$  \hspace{1cm} (B.5)

At the interface between subdomain $\odot$ and subdomain $\odot$, the continuity of displacements and slopes, as well as the equilibrium of bending moment and shear force are satisfied.

$$w_{p2}(x_0, t) = w_{p3}(x_0, t)$$  \hspace{1cm} (B.8)

$$\frac{\partial w_{p2}(x_0, t)}{\partial x} = \frac{\partial w_{p3}(x_0, t)}{\partial x}$$  \hspace{1cm} (B.9)

$$D_e \frac{\partial^2 w_{p2}(x_0, t)}{\partial x^2} = D_e \frac{\partial^2 w_{p3}(x_0, t)}{\partial x^2}$$  \hspace{1cm} (B.10)

$$D_e \frac{\partial^3 w_{p2}(x_0, t)}{\partial x^3} - D_e \frac{\partial^3 w_{p3}(x_0, t)}{\partial x^3} = F(x_0)e^{jwt}$$  \hspace{1cm} (B.11)

Similar to the previous case, the exact displacement solution of the whole beam can be evaluated from the coefficients by solving equations (B.2)–(B.11). Then the exact time and time-space averaged energy density can be obtained.