Analytic Solutions of the Teukolsky Equation and their Properties

Shuhei Mano,* and Eiichi Takasugi†

Department of Physics, Osaka University
Toyonaka, Osaka 560, Japan

Abstract

The analytical solutions reported in our previous paper are given as series of hypergeometric or Coulomb wave functions. By using them, we can get the Teukolsky functions analytically in a desired accuracy. For the computation, the deep understanding of their properties is necessary. We summarize the main result: The relative normalization between the solutions with a spin weight \( s \) and \( -s \) is given analytically by using the Teukolsky-Starobinsky (T-S) identities. By examining the asymptotic behaviors of our solution and combined with the T-S identities and the Wronskian, we found nontrivial identities between the sums of coefficients of the series. These identities will serve to make various expression in simpler forms and also become a powerful tool to test the accuracy of the computation. As an application, we investigated the absorption rate and the evaporation rate of black hole and obtain interesting analytic results.

*e-mail address: mano@phys.wani.osaka-u.ac.jp
†e-mail address: takasugi@phys.wani.osaka-u.ac.jp
1 Introduction

In our previous paper[1], we reported the analytic solutions of Teukolsky equation[2] which consist of two type of series; one is given in the form of series of hypergeometric functions (hereafter we call it the hypergeometric type solution) and the other is given in the form of series of Coulomb wave functions (hereafter we call it the Coulomb type solution which was first given by Leaver[3]). The hypergeometric type solutions are shown to be convergent in the region except infinity with all finite $\epsilon = 2M\omega$, $M$ being the mass of black hole and $\omega$ an angular frequency. The Coulomb type solutions are convergent in the region far above the outer horizon for all finite $\epsilon$. We showed that the matching of these two types of solutions is perfect in the intermediate region where both solutions are convergent. In this way, we found the solutions which are convergent in the entire region of $r$ for all finite $\epsilon$. In addition, we showed that our solutions are suitable to get the Teukolsky functions in the $\epsilon$ expansion or the numerical computation. We also presented analytical solutions of Regge-Wheeler equation[4].

We showed[1] that the solution can be written in various forms so that we have to choose some special ones. Furthermore, it is necessary to investigate further the properties of solutions to calculate the Teukolsky functions, because the solutions are given as series. For this purpose, we examined the asymptotic behaviors of the solutions and compared with the general result derived before. We made a systematic study on the incoming solution on the outer horizon.

As for the incoming solution on the horizon, we take the same hypergeometric type one as in Ref.1, for which the boundary condition on the horizon is trivial. For the outgoing solution, we take a different form. As for the Coulomb type solutions, we take different forms from that given in Ref.1. In this paper, we choose a variable $z = \omega(r - r_\pm)$ which is different from the definition in Ref.1 where we took $z = \omega(r - r_-)$. Accordingly, the solutions adopted here are different. We took them because $z$ reduces to $\omega r$ in the Schwarzschild limit and thus the solutions for large $r$ can be compared with the previous analysis by Tagoshi and Nakamura, and Sasaki and companies[5] for Schwarzschild case.
and to our solutions of the Regge-Wheeler equation[4]. Therefore, some of solutions given here are new ones.

In order to investigate the properties of the solutions, we mainly examine the Teukolsky-Starobinsky (T-S) identities[6],[7] and the Wronskian[6]. Although the solutions of Teukolsky equation satisfies the T-S identities automatically, it is non trivial to see directly that our solutions satisfy them because our solutions are given as series. Fortunately, we were able to show analytically that the incoming solution satisfies one of T-S identities. By using this, we can fix analytically the relative normalization between solutions with a spin weight $s$ and $-s$. It was hard to prove that our solution satisfies another T-S identities directly. In order to investigate the T-S identities and the Wronskian, we considered the asymptotic behaviors. For this, we derived the expressions of asymptotic amplitudes on the horizon and at infinity from our solution. By examining the T-S identities and the Wronskian, we found two very nontrivial identities among sums of coefficients. Since the solutions are defined as series, various quantities expressed by Teukolsky functions necessary involve the sums of coefficients which are calculated by solving the three term recurrence relation. These identities will serve to make the expressions of them in compact forms form which we may be able to have a direct incite of the physical meaning. As an example, we considered the absorption rate and the evaporation rate of black hole and obtained compact forms of them. For computing Teukolsky functions, we will make either the $\epsilon$ expansion or the numerical computation of coefficients. In order to test the accuracy of the computation, we can use these identities. Since the sum of coefficients in these identities are proportional to the asymptotic amplitudes on horizon or at infinity, the check of these identities gives a direct check of accuracy of Teukolsky functions.

With these understandings of the solutions given in this paper, it is now in the position to make the numerical study of Teukolsky functions. By solving the three term recurrence relation, we can in principle obtain Teukolsky functions with a desired accuracy. The further numerical study is in pressing need.

In Sec.2, we give the outline of solutions of Teukolsky equation and new forms of them. In Sec.3, various properties of coefficients the proportionality factor between the
hypergeometric type and Coulomb type solutions are given. The asymptotic behaviors on the outer horizon and at infinity are expressed directly by using our solution in Sec.4. In Sec.5, we discuss properties of our solution by using the T-S identities. By using the Wronskian, we discuss the conserved quantity and the energy conservation in Sec.6. By using the T-S identities and the Wronskian, we derive two identities among the sums of coefficients in Sec.7. The absorption rate and the evaporation rate by black hole are discussed in Sec.8. Discussions are given in Sec.9. The outlines of proofs of various formulas in the text are given in Appendix A.

2 Analytic solutions

The hypergeometric type solution which we discuss here is the same as the one given in Ref.1. However, we take different solutions for the Coulomb type from the one in Ref.1. We do not present the derivations and the proofs of various equations and statements which one can find in Ref.1 or one can prove following the discussions given there.

(a) Notation

We start from the Teukolsky function[2],[6]

\[ \Upsilon_s \simeq e^{-i\omega t} e^{im\phi} s^m_l(\theta) s R_{\omega lm}(r), \]

where \( s^m_l(\theta) \) is a spin-weighted spheroidal function with a spin-weight parameter \( s \) and \( l \) is an angular momentum which satisfies \( l \geq \max(|m|, |s|) \), and \( s R_{\omega lm}(r) \) is a radial function which we write \( R_s \) for simplicity. Solutions of the radial equation are expressed as functions of variables \( x \) and \( z \) which are defined by

\[ x = -\frac{\omega}{\epsilon \kappa} (r - r_+), \quad z = \omega (r - r_-) = \epsilon \kappa (1 - x) = \epsilon \kappa \tilde{x}, \]

where

\[ \epsilon = 2M\omega, \quad \kappa = \sqrt{1 - q^2}, \quad q = \frac{a}{M}. \]

Here, \( M \) and \( a \) are the mass and the angular momentum of the Kerr black hole, respectively. The parameters \( r_\pm \) are poles of \( \Delta = r^2 + a^2 - 2Mr \) and is expressed by
\[ r_\pm = M \pm \sqrt{m^2 - a^2} = \epsilon(1 \pm \kappa)/2\omega \] where \( r_\pm \) give positions of the outer and the inner horizon. For later use, it is convenient to define

\[ \tau = \frac{\epsilon - mq}{\kappa}, \quad k = \omega - \frac{ma}{2Mr_+} = 2\frac{\kappa}{1 + \kappa} \left( \frac{\omega}{\epsilon} \right) \epsilon_+ \]

and

\[ \epsilon_\pm = \frac{(\epsilon \pm \tau)}{2}. \]

(b) Coefficients

Solutions of the radial part are expressed in the form of series of hypergeometric functions and also in the form of series of Coulomb wave functions. Coefficients of these series satisfy the same three term recurrence relation[1],[8]

\[ \alpha_n^{\nu}(s)a_{n+1}^{\nu}(s) + \beta_n^{\nu}(s)a_n^{\nu}(s) + \gamma_n^{\nu}(s)a_{n-1}^{\nu}(s) = 0, \]

where

\[ \alpha_n^{\nu}(s) = \frac{i\epsilon \kappa(n + \nu + 1 + s + i\epsilon)(n + \nu + 1 + s - i\epsilon)(n + \nu + 1 + i\tau)}{(n + \nu + 1)(2n + 2\nu + 3)}, \]
\[ \beta_n^{\nu}(s) = -\lambda_s - s(s + 1) + (n + \nu)(n + \nu + 1) + \epsilon^2 + \epsilon(\epsilon - mq) \]
\[ + \frac{\epsilon(\epsilon - mq)(s^2 + \epsilon^2)}{(n + \nu)(n + \nu + 1)}, \]
\[ \gamma_n^{\nu}(s) = -\frac{i\epsilon \kappa(n + \nu - s + i\epsilon)(n + \nu - s - i\epsilon)(n + \nu - i\tau)}{(n + \nu)(2n + 2\nu - 1)}, \]

where \( \lambda_s = E(s) - 2am\omega + a^2\omega^2 - s(s + 1) \) with \( E(s) \) being the eigenvalue of the spin-weighted function \( sS_i^n(\theta) \).

The series of coefficients converges if the transcendental equation for \( \nu \)

\[ R_n(\nu)L_{n-1}(\nu) = 1, \]

is satisfied, where \( R_n(\nu) \) and \( L_n(\nu) \) are the continued fractions defined by

\[ R_n(\nu) = \frac{a_n^{\nu}(s)/a_{n+1}^{\nu}(s)}{a_{n-1}^{\nu}(s)/a_n^{\nu}(s)} = \frac{\gamma_n^{\nu}(s)}{\beta_n^{\nu}(s) + \alpha_n^{\nu}(s)R_{n+1}(\nu)}, \]
\[ L_n(\nu) = \frac{a_n^{\nu}(s)/a_{n+1}^{\nu}(s)}{a_{n+1}^{\nu}(s)/a_n^{\nu}(s)} = \frac{\alpha_n^{\nu}(s)}{\beta_n^{\nu}(s) + \gamma_n^{\nu}(s)L_{n-1}(\nu)}. \]
The solutions are characterized \( \nu \) which we called the renormalized (shifted) angular momentum because \( \nu = l + O(\epsilon^2) \) as shown in Ref.1. We can prove that if \( \nu \) satisfies Eq.(8), then \(-\nu - 1\) also satisfies, i.e., \( R_n(-\nu - 1)L_{n-1}(-\nu - 1) = 1 \). In other words, if \( \nu \) is a solution, then \(-\nu - 1\) is also a solution. Note that \( E(s) \) is an even function of \( s \), \( E(-s) = E(s) \) as shown by Press and Teukolsky[9]. By using this, we can prove that \( \nu \) is an even function of \( s \)

\[
\nu(s) = \nu(-s),
\]

which is important and enables us to relate the solution of spin weight \( s \) to that \(-s\) by the Teukolsky-Starobinsky identities. This can be proved by noticing that the transcendental equation (8) contains \( \beta_\nu(s) \) and a combination \( \alpha_k^\nu(s) \gamma_{k+1}^\nu(s) \) which are even functions of \( s \). Thus, the equation (8) is invariant under the change of \( s \) to \(-s\) and its eigenvalue \( \nu(s) \) is an even function of \( s \).

For \( \nu \) which satisfies Eq.(10), we find

\[
\lim_{n \to \infty} n \frac{a_n^\nu(s)}{a_{n-1}^\nu(s)} = - \lim_{n \to -\infty} n \frac{a_n^\nu(s)}{a_{n+1}^\nu(s)} = \frac{i\epsilon \kappa}{2},
\]

which enables to establish the region of convergence of solutions.

(c) Analytic solutions

(c-1) Hypergeometric type solutions

The incoming solution is given by

\[
R_{\text{in}; s}^\nu = A_s e^{i\epsilon \kappa x} (-x)^{-s-i\epsilon}(1-x)^{i\epsilon} \sum_{n=-\infty}^{\infty} a_n^\nu(s) \times F(n + \nu + 1 - i\tau, -n - \nu - i\tau; 1 - s - 2i\epsilon_+; x),
\]

where \( a_n^\nu(s) \)'s are coefficients which satisfy the three term recurrence relation in Eq.(6) and \( A_s \) is the normalization constant which we take for \( s > 0 \) with the choice \( A_{-s} = 1 \)

\[
A_{-s} = 1, \quad A_s = C_s \left( \frac{\omega}{\epsilon \kappa} \right)^{2s} \frac{\Gamma(1 + s - 2i\epsilon_+)}{\Gamma(1 - s - 2i\epsilon_+)} \frac{\Gamma(\nu + 1 - s + i\epsilon)}{\Gamma(\nu + 1 + s + i\epsilon)},
\]

where \( C_s \)'s are Starobinsky constants (defined in Eqs.(49)-(52)). The relative normalization is determined so that the Teukolsky-Starobinsky identities are satisfied as we shall
see later. It may be worthwhile to mention that $R^{-\nu-1}_{\text{in};s} = R^\nu_{\text{in};s}$ so that the change $\nu$ into $-\nu-1$ does not lead to a new solution.

Another independent solution is the outgoing solution on the outer horizon and is given by

$$R^\nu_{\text{out};s} = \Delta^{-s}(R^\nu_{\text{in};s})^*$$

which is different from the one given in Ref.1. Solutions, $R^\nu_{\text{in};s}$ and $R^\nu_{\text{out};s}$ form a pair of independent solutions of Teukolsky equation.

By using large $|n|$ behaviors of coefficients in Eq.(11) and hypergeometric functions, we can prove (see Ref.1) that the solution in Eq.(12) is convergent for the entire complex plain except infinity for all finite $\epsilon$. If we restrict $x$ to the physical region, the convergence region is $0 \leq (-x) < \infty$, i.e., $r < \infty$.

The solution can be analytically continued by

$$R^\nu_{\text{in};s} = A_s e^{i\kappa} \left[ R^\nu_{0,s} + R^{-\nu-1}_{0,s} \right] ,$$

where

$$R^\nu_{0,s} = e^{-i\kappa x}(x)^{\nu+i\epsilon+1} - x - 1 - i\epsilon \sum_{n=-\infty}^{\infty} \frac{\Gamma(1-s-2i\epsilon+\nu)\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1-i\tau)\Gamma(n+\nu+1-s-i\epsilon)} a_n^\nu(s) \times x^n F\left( -n-\nu-i\tau, -n-\nu-s-i\epsilon; -2n-2\nu; \frac{1}{x} \right).$$

Then, $R^\nu_{0,s}$ and $R^{-\nu-1}_{0,s}$ form a pair of independent solutions.

(c-2) Coulomb type solutions

The solution is given by

$$R^\nu_{C;s} = z^{-1-s} \left( 1 - \frac{\epsilon\kappa}{z} \right)^{s-i\epsilon+} \sum_{n=-\infty}^{\infty} (-i)^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} a_n^\nu(s) F_{n+\nu,s}(z) ,$$

(17)
where \((a)_n = \Gamma(n + a)/\Gamma(a)\) and
\[
F_{n+\nu,s}(z) = e^{-iz}(2z)^{n+\nu}z^{\nu+1-s+i\epsilon} \frac{\Gamma(n+\nu+1-s+i\epsilon)}{\Gamma(2n+2\nu+2)} \times \Phi(n+\nu+1-s+i\epsilon, 2n+2\nu+2; 2iz).
\]  (18)

This solution and the definition of \(z\) are different from those given in Ref.1. It is amusing to observe that coefficients appearing in the hypergeometric type solution are the same as those in the Coulomb type solutions[1],[9]. Thus, the renormalized (shifted) angular momentum takes the same value both for the hypergeometric type and the Coulomb type solutions. This fact is quite important for matching of these two different types of solutions in the region where both solutions are convergent.

Another independent solution is obtained by changing \(\nu\) to \(-\nu-1\),
\[
R_{-\nu-1}^{\nu-1}\text{.}  \tag{19}
\]

Solutions \(R_{\nu}^{\nu}\) and \(R_{-\nu-1}^{\nu-1}\) form a pair of independent solutions of Teukolsky equation.

Coulomb type solutions given above contain both the incoming and the outgoing solutions at infinity. Thus, solutions are decomposed into another pair of solutions, the incoming solution at infinity \(R_{+i\nu}^{\nu}\) and the outgoing solution at infinity \(R_{-i\nu}^{\nu}\). Explicitly, we have
\[
R_{\nu}^{\nu} = R_{+i\nu}^{\nu} + R_{-i\nu}^{\nu}, \tag{20}
\]

where
\[
R_{+i\nu}^{\nu} = 2^\nu e^{-\pi e^{i\pi(\nu+1-s)}} \frac{\Gamma(\nu+1-s+i\epsilon)}{\Gamma(\nu+1-s-i\epsilon)} e^{-iz} z^{\nu+i\epsilon} (z - \epsilon\kappa)^{-s-i\epsilon} \times \sum_{n=-\infty}^{\infty} i^n a_n^{\nu}(s)(2z)^n \Psi(n+\nu+1-s+i\epsilon, 2n+2\nu+2; 2iz), \tag{21}
\]

\[
R_{-i\nu}^{\nu} = 2^\nu e^{-\pi e^{-i\pi(\nu+1-s)}} e^{iz} z^{\nu+i\epsilon} (z - \epsilon\kappa)^{-s-i\epsilon} \sum_{n=-\infty}^{\infty} i^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} a_n^{\nu}(s)(2z)^n \Psi(n+\nu+1+s-i\epsilon, 2n+2\nu+2:-2iz). \tag{22}
\]
By using the large \(| n |\) behaviors of coefficients in Eq.(11) and Coulomb wave functions, we can prove (see Ref.1) that solutions are convergent in the complex region of \(| z | > \epsilon \kappa\) for all finite \(\epsilon\). If we confine to the physical region, the convergence region is \(r > r_+\).

(c-3) Matching of two types of solutions

Let us compare \(R_0^\nu\) and \(R_C^\nu\). We first observe that they are solutions of Teukolsky equation and behaves like \(\tilde{x}^\nu\) multiplied by a single-valued function for large \(\tilde{x}\) (\(z = \epsilon \kappa \tilde{x}\)). Thus, \(R_0^\nu\) should be proportional to \(R_C^\nu\). The same is true for \(R_{-\nu-1}^\nu\) and \(R_{-\nu-1}^{C},\nu\). It should be noted that \(\nu - (-\nu - 1) = 2
\nu + 1\) is not an integer so that \(R_0^\nu\) and \(R_{-\nu-1}^{C},\nu\) are solutions with different characteristic exponents.

We define

\[
R_{0,s}^\nu = K_\nu(s)R_{C,s}^\nu.
\]

Then, by comparing each power of \(\tilde{x}\) in the region \(1 << \tilde{x} < \infty\) where both solutions converge, we find

\[
K_\nu(s) = \frac{(2\epsilon \kappa)^{-\nu + s - \tilde{r} - 2 - s \tilde{r}_\nu} \Gamma(1 - s - 2i\epsilon) \Gamma(\tilde{r} + 2\nu + 1) \Gamma(\tilde{r} + 2\nu + 2)}{\Gamma(\nu + 1 + i\tau) \Gamma(\nu + 1 - s - i\epsilon) \Gamma(\tilde{r} + \nu + 1 - s + i\epsilon)} \times \frac{\Gamma(\tilde{r} + \nu + 1 + i\tau) \Gamma(\tilde{r} + \nu + 1 + s + i\epsilon)}{\Gamma(\tilde{r} + \nu + 1 - s + i\epsilon)} \times \left( \sum_{n=\tilde{r}}^\infty \frac{(\nu + 1 + s - i\epsilon)_n a_n^\nu(s)}{(n - \tilde{r})!} \right)^* \times \left( \sum_{n=-\infty}^{\tilde{r}} \frac{(-1)^n (\nu + 1 - s - i\epsilon)_n a_n^\nu(s)}{(\tilde{r} - n)! (\nu + 1 + s + i\epsilon)_n} \right)^{-1},
\]

where \(\tilde{r}\) can be any integer and \(K_\nu(s)\) is independent of the choice of \(\tilde{r}\).

(c-4) Solutions valid in the entire plane of \(r\)

*Incoming solution on the horizon*

We take the hypergeometric type expression for \(R_{in,s}^\nu\) given in Eq.(12) for the incoming solution on the outer horizon which is convergent except infinity. By the matching
between the hypergeometric type solution and the Coulomb type solution in Eq.(23), we have the Coulomb type expression for $R^\nu_{in; s}$

$$R^\nu_{in; s} = A_s e^{i \epsilon \kappa} [K_\nu(s) R^\nu_{C; s} + K_{-\nu - 1}(s) R^{\nu - 1}_{C; s}],$$

(25)

which is convergent in the region $r > r_+$. The solution which is defined in this way is convergent in the entire region of $r$ for all finite $\epsilon$. That is, we use the hypergeometric type expression in Eq.(12) around the outer horizon and the Coulomb type expression in Eq.(25) around infinity. In the intermediate region, we use either the expression by using $R^\nu_{0; s}$ in Eq.(15) or the Coulomb type expression depending on the situations.

**Outgoing solution on the horizon**

By using $R^\nu_{in; -s}$, the outgoing solution on the outer horizon is given by $R^\nu_{out; -s} = \Delta_{-s} (R^\nu_{in; -s})^*$. Thus, this solution is valid in the entire region of $r$ for all finite $\epsilon$. The relative normalization between solutions with spin weight $s$ and $-s$ is fixed automatically following the normalization of $R^\nu_{in; s}$.

**Upgoing solution**

The upgoing solution is the one which satisfies the outgoing boundary condition at infinity so that we can write

$$R^\nu_{up; s} = B_s R^\nu_{-s;},$$

(26)

where $B_s$’s are normalization constants and the relative normalization between $B_s$ and $B_{-s}$ is fixed for $s > 0$ with the choice $B_{-s} = 1$

$$B_{-s} = 1, \quad B_s = C_s^* \omega^{2s},$$

(27)

where $C_s$’s are Starobinsky constants (defined in Eq.(49)-(53)). By using the relations

$$R^{\nu - 1}_{+; s} = -i e^{-i \pi \nu} \frac{\sin \pi (\nu - s + i \epsilon)}{\sin \pi (\nu + s - i \epsilon)} R^\nu_{+; s},$$

$$R^{\nu - 1}_{-; s} = i e^{i \pi \nu} R^\nu_{-; s},$$

(28)

we find

$$R^\nu_{up; s} = B_s \frac{e^{-\pi \epsilon}}{\sin 2\pi \nu} \left( -i \frac{\sin \pi (\nu + s - i \epsilon)}{K_{-\nu - 1}(s)} R^{\nu - 1}_{0; s} + e^{-i \pi \nu} \frac{\sin \pi (\nu - s + i \epsilon)}{K_{\nu}(s)} R^\nu_{0; s} \right).$$

(29)
Then, if needed, we can express $R^\nu_{0,s}$ in terms of $R^\nu_{\text{in};s}$ and $R^\nu_{\text{out};s}$ and then we obtain the hypergeometric type expression. Now, we obtained the upgoing solution which is written by the hypergeometric type expression which is convergent in the region $r < \infty$ and the Coulomb type expression which is convergent for $r > r_+$. Thus, we find the upgoing solution which is convergent in the entire region of $r$ for all finite $\epsilon$.

3 Some properties of coefficients and $K_\nu(s)$

In this section, we present various properties of coefficients and some relations between $K_\nu(s)$'s which become important in the following discussion. Hereafter, we consider $s = 0$ (massless scalar), $s = \pm 1/2$ (massless fermion, neutrino), $s = \pm 1$ (photon) and $s = \pm 3/2$ (massless spin 3/2 particle) and $s = \pm 2$ (graviton).

(a) Properties of coefficients

In below, we take initial values of coefficients for $s = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2$

\[ a^\nu_0(s) = a^{-\nu-1}_0(s) = 1. \]  

(30)

Then, from the three term recurrence relation, we have

\[ a^{-\nu-1}_n(s) = a^\nu_n(s). \]  

(31)

In addition, by using the explicit forms of $\alpha_n^\nu(s), \beta_n^\nu(s)$ and $\gamma_n^\nu(s)$ and the three term recurrence relation, we obtain the following relations:

\[ a^\nu_n(-s) = \frac{(\nu + 1 + s + i\epsilon)_n}{(\nu + 1 - s + i\epsilon)_n} a^\nu_n(s), \]  

(32)

\[ a^\nu_n(s)^* = (-1)^n \frac{(\nu + 1 + i\tau)_n}{(\nu + 1 - i\tau)_n} a^\nu_n(s), \]  

(33)

\[ a^\nu_n(-s)^* = (-1)^n \frac{(\nu + 1 + i\tau)_n}{(\nu + 1 - i\tau)_n} \frac{(\nu + 1 + s + i\epsilon)_n}{(\nu + 1 - s + i\epsilon)_n} a^\nu_n(s). \]  

(34)
To prove Eq. (32), we see that 
\[ \alpha_n^{-}(s) = \frac{(n + \nu + 1 - s + i\epsilon)}{(n + \nu + 1 + s + i\epsilon)} \alpha_n^{-}(s), \]
\[ \beta_n^{-}(s) = \beta_n^{-}(s) \text{ and } \gamma_n^{-}(s) = \frac{(n + \nu + s + i\epsilon)}{(n + \nu - s + i\epsilon)} \gamma_n^{-}(s) \]
and then, it is clear that \( a_n^{-}(s) \) defined in Eq. (32) satisfies the three term recurrence relation for \(-s\).
Similar considerations will lead to Eqs. (33) and (34).

(b) Useful relations for \( K_\nu(s) \)

The first one is

\[
\frac{K_\nu(-s)}{K_\nu(s)} = \frac{K_{-\nu-1}(-s)}{K_{-\nu-1}(s)} = \frac{1}{(\epsilon \kappa)^{2s}} \frac{\Gamma(1 + s - 2i\epsilon_+) \Gamma(\nu + 1 - s + i\epsilon)}{\Gamma(1 - s - 2i\epsilon_+) \Gamma(\nu + 1 + s + i\epsilon)}^{2s},
\]

which can be proved directly by using the relation between \( a_\nu^+(s) \) and \( a_\nu^{-}(s) \) in Eq. (32).

The second one is rather nontrivial, but can be proved directly for \( \bar{r} = 0 \),

\[
K_\nu(s)K_{-\nu-1}(-s)^* = \frac{2\epsilon \kappa \Gamma(1 + s + 2i\epsilon_+) \Gamma(1 - s - 2i\epsilon_+)}{\pi} \left| \frac{\sin \pi \nu + i\epsilon}{\sin 2\pi \nu} \right|^2,
\]

where we used Eqs. (31) and (32). Since \( K_\nu(s) \) is independent on the choice of \( \bar{r} \), the relation in Eq. (36) is valid for all integer value of \( \bar{r} \). These are useful relations which can be used to prove that the solution satisfies the T-S identity directly and to discuss the absorption rate and the evaporation rate of black hole.

4 Asymptotic behaviors

We consider asymptotic behaviors of our solution on the outer horizon and at infinity. Then, we write asymptotic amplitudes by using our solution explicitly following the definition given in the book of Chandrasekhar[10]. The general properties derived by using the asymptotic behaviors before are compared to the explicit forms derived here.
By doing this comparison, we can examine the structure of the solution and get a deep insight of our solution. The result presented here will be useful to derive various quantities by our solution.

(a) Definition of asymptotic amplitudes \( R_s^{(inc)}, R_s^{(ref)} \) and \( R_s^{(trans)} \)

The asymptotic behaviors are defined by[10]

\[
R_s \rightarrow R_s^{(inc)} \frac{1}{r} e^{-i\omega r^*} + \frac{1}{r^{1+2s}} e^{i\omega r^*} \quad (r \to \infty),
\]

\[
R_s^{(trans)} \Delta^{-s} e^{-ikr^*} \quad (r \to r_+),
\]

where \( k \) is defined in Eq.(4) and \( r_* \) is defined by \( dr_*/dr = (r^2 + a^2)/\Delta \). Then, we find

\[
\omega r_* \rightarrow z + \epsilon \ln z \quad (r \to \infty),
\]

\[
k r_* \rightarrow \epsilon_+ \ln(-x) \quad (r \to r_+).
\]

(b) Asymptotic amplitudes expressed by using our solution \( R_{\nu}^{\nu,s} \)

Here we express \( R_s^{(inc)}, R_s^{(ref)} \) and \( R_s^{(trans)} \) explicitly by using the analytic solution. In below, we consider the cases of \( s = 0, \pm 1/2 \) (neutrino), \( s = \pm 1 \) (photon), \( \pm 3/2 \) (massless spin 3/2 particle) and \( \pm 2 \) (graviton).

The asymptotic amplitude on the outer horizon \( R_s^{(trans)} \) is given from Eq.(12) by

\[
R_s^{(trans)} = A_s \left( \frac{\epsilon \kappa}{\omega} \right)^{2s} \sum_{n=-\infty}^{\infty} a_n^{\nu}(s). \quad (39)
\]

To derive asymptotic amplitudes at infinity \( R_s^{(inc)} \) and \( R_s^{(ref)} \), we need some nontrivial works. We define

\[
R_C^{\nu,s} \rightarrow A_{\nu,+;s}^{\nu} z^{-1} e^{-i(z+\epsilon \ln z)} + A_{\nu,-;s}^{\nu} z^{-1-2s} e^{i(z+\epsilon \ln z)}. \quad (40)
\]

Then, from Eq.(17), we find

\[
A_{\nu,+;s}^{\nu} = 2^{-1+s-i\epsilon} e^{i(\pi/2)(\nu+1-s)} e^{-\pi \epsilon/2} \frac{\Gamma(\nu + 1 - s + i\epsilon)}{\Gamma(\nu + 1 + s - i\epsilon)} \sum_{n=-\infty}^{\infty} (-1)^n \frac{(\nu + 1 + s - i\epsilon)_n}{(\nu + 1 - s + i\epsilon)_n} a_n^{\nu}(s),
\]

\[
A_{\nu,-;s}^{\nu} = 2^{-1-s+i\epsilon} e^{-i(\pi/2)(\nu+1+s)} e^{-\pi \epsilon/2} \sum_{n=-\infty}^{\infty} (-1)^n \frac{(\nu + 1 + s - i\epsilon)_n}{(\nu + 1 - s + i\epsilon)_n} a_n^{\nu}(s). \quad (41)
\]
Since the asymptotic behaviors of $R_{\nu,s}$ are expressed by the combination of $R_{C\nu,s}$ and $R_{C\nu-1,s}$, the asymptotic amplitudes look quite involved. It is quite important that these asymptotic amplitudes are expressed in simpler forms to examine the T-S identities by using the asymptotic behaviors, to derive the absorption rate and the numerical computations. We found various useful relations. Firstly, we relate $A_{\nu,s}^+;s$ to $A_{\nu-1,s}^-;s$. By using Eq.(31), we find

$$A_{\nu-1,s}^- = -i e^{-i \pi \nu} \frac{\sin \pi (\nu - s + i \epsilon)}{\sin \pi (\nu + s - i \epsilon)} A_{\nu,s}^+;s,$$

$$A_{\nu-1,s}^- = i e^{i \pi \nu} A_{\nu,s}^+;s.$$  (42)

By combining Eqs.(25), (40) and (42), we find

$$R_{s}^{(\text{inc})} = \frac{A_{\nu}}{\omega} \left[ K_{\nu}(s) - i e^{-i \pi \nu} \frac{\sin \pi (\nu - s + i \epsilon)}{\sin \pi (\nu + s - i \epsilon)} K_{\nu-1}(s) \right] A_{\nu,s}^+;s,$$

$$R_{s}^{(\text{ref})} = \frac{A_{\nu}}{\omega_{1+2s}} [K_{\nu}(s) + i e^{i \pi \nu} K_{\nu-1}(s)] A_{\nu,s}^-;s.$$  (43)

These are expressed in compact forms. In addition, we find the relation between $A_{\nu,s}^+;s$ to $A_{\nu,s}^-;s$ by using Eq.(32)

$$\frac{A_{\nu,s}^+}{A_{\nu,s}^-} = 2 e^{-i \pi s} \left[ \frac{\Gamma(\nu + 1 - s + i \epsilon)}{\Gamma(\nu + 1 + s + i \epsilon)} \right]^2 \frac{\sum_{n=-\infty}^{\infty} a_{\nu,n}(s)}{\sum_{n=-\infty}^{\infty} a_{\nu,n}(-s)},$$

$$\frac{A_{\nu,s}^-}{A_{\nu,s}^+} = 2^{-2s} e^{-i \pi s}.$$  (44)

These relations are useful to discuss the T-S identities.

5 The Teukolsky-Starobinsky identities

The Teukolsky-Starobinsky (T-S) identities[6],[7] for $s = 1/2, 1, 3/2$ and 2 can be expressed as

$$\Delta^s (D^\dagger)^2 s \Delta^s R_s = C^* R_{-s}, \quad (\text{T-S identity (A)})$$  (45)
\[(D)^{2s} R_{-s} = C_s R_s , \quad \text{\textup{(T – S identity (B))}} \quad (46)\]

where \(D\) and \(D^\dagger\) are differential operators

\[
D = \frac{\partial}{\partial r} - i \frac{K}{\Delta} = -\frac{\omega}{\epsilon \kappa} \left( \frac{d}{dx} + i \epsilon + i \frac{\epsilon_+}{x} + i \frac{\epsilon_-}{1-x} \right),
\]

\[
D^\dagger = \frac{\partial}{\partial r} + i \frac{K}{\Delta}, \quad (47)
\]

\[
K = (r^2 + a^2)\omega - am. \text{ Note that } K \text{ used here corresponds to } -K \text{ defined in the book of Chandrasekhar[6]. In the above expression, } (D^\dagger)^{2s} \text{ means that the differential operation } D^\dagger \text{ applies } 2s \text{ times and the same for } (D)^{2s}.
\]

The Starobinsky constants[7] \(C_s\)’ are given as follows:

\[
|C_2|^2 = (Q_2^2 + 4awm - 4a^2\omega^2)[(Q_2 - 2)^2 + 36awm - 36a^2\omega^2] + (2Q_2 - 1)(96a^2\omega^2 - 48awm) + 144\omega^2(M^2 - a^2) , \quad (49)
\]

\[
(C_3^2)^2 = (Q_3^2 - \frac{3}{4})^2(Q_3^2 + \frac{1}{4}) - 16a^2\omega^2(Q_3^2 - \frac{7}{4}) + 16am\omega(Q_3^2 - \frac{3}{4}) , \quad (50)
\]

\[
(C_1^2)^2 = Q_1^2 - 4a^2\omega^2 + 4am\omega , \quad (51)
\]

\[
(C_\frac{5}{2})^2 = Q_\frac{5}{2} + \frac{1}{4} , \quad (52)
\]

where

\[
Q_s = E(s) + a^2\omega^2 - 2awm .
\]

Trivially, \(C_0 = 1\). The coefficient \(C_2\) is a complex number, but others can be real. The coefficients \(C_2\) and \(C_1\) are first obtained by Teukolsky and Press[7] and \(C_\frac{5}{2}\) is given by Torres del Castillo[11].
Solutions of Teukolsky equation should satisfy the T-S identities. It is noted that if a function satisfies the T-S relation (A), it also satisfies the T-S relation (B), and vice versa. This is shown by identity

$$\Delta^s(D^\dagger)^{2s}\Delta^s(D)^{2s}R_{-s} = |C_s|^2 R_{-s},$$

for \( s = 0, 1/2, 1, 3/2, 2 \). The proof for \( s = 1 \) and 2 were given in the book of Chandrasekhar[10] and the other cases can be proved similarly. This fact is important for our solution because it may be hard to prove analytically that our solution satisfies both of the T-S identities, although it is a solution of the Teukolsky equation. This is because our solution is expressed as series.

Fortunately, we can show analytically that \( R^\nu_{i\mp s} \)'s for \( s = 0, 1/2, 1, 3/2, 2 \) satisfy the T-S identity (A). By using this, we can fix the relative normalization between solutions with spin weight \( s \) and \(-s\) analytically.

Firstly, we consider the hypergeometric type expression for \( R^\nu_{i\pm s} \) in Eq.(12). As we see in Appendix A, we find

$$\Delta^s(D^\dagger)^{2s}\Delta^s(D)^{2s}R^\nu_{i\mp s} = \frac{A^s}{A_{-s}} \left( \frac{\epsilon\kappa}{\omega} \right)^{2s} \frac{\Gamma(1 - s - 2i\epsilon_+)}{\Gamma(1 + s - 2i\epsilon_+)} \frac{|\Gamma(\nu + 1 + s + i\epsilon)|^2}{|\Gamma(\nu + 1 - s + i\epsilon)|^2} |C_s|^2 R_{i\mp s}.$$

Then, it is clear that by using the normalization factors in Eq.(13), the righthand side reduces to \( C^*_s R^\nu_{i\mp s} \). That is, by taking the relative normalization in Eq.(13), the hypergeometric type expression for \( R^\nu_{i\pm s} \) satisfies the T-S identity (A).

In addition, we can also show that the Coulomb type expression for \( R^\nu_{i\pm s} \) satisfies the T-S relation (A) as well. To prove this, we first observe that the following relation is satisfied:

$$\Delta^s(D^\dagger)^{2s}\Delta^sR^\nu_{\pm s} = \frac{1}{\omega^{2s}} R^\nu_{\pm s},$$

for \( s = 0, 1/2, 1, 3/2, 2 \). The proof of this relation is given in Appendix A. Then, we find from Eq.(20)

$$\Delta^s(D^\dagger)^{2s}\Delta^sR^\nu_{C; s} = \frac{1}{\omega^{2s}} R^\nu_{C; s}.$$
By using the Coulomb type expression in Eq.(25), we can immediately find
\[
\Delta^s(D^\dagger)^{2s}\Delta^s R_{\text{in};s}^\nu = \frac{A_s e^{i\kappa}}{\omega^{2s}} \left( (K_\nu(s) R_{C;-,s}^{\nu} + K_{-\nu-1}(s) R_{C;-,s}^{\nu-1}) \right) \\
= C_s R_{\text{in};-,s}^\nu,
\] (57)
where we used the relation among \(K_\nu(s)\) given in Eq.(35) and the relative normalization in Eq.(13) to derive the last equality. In summary, both expressions of \(R_{\text{in};s}^\nu\) satisfies the T-S identity (A).

It is not possible to prove analytically that our solution \(R_{\text{in};s}^\nu\) satisfies the T-S identity (B), although it should satisfy by the identity in Eq.(53). It is expected that the T-S identity (B) will give some nontrivial identities if it applies to our solution.

The analysis by using the asymptotic behaviors of the solution has been made by Teukolsky and Press[7]. By substituting the asymptotic behaviors in Eq.(37) on the outer horizon into the the T-S identity (B), one finds for \(s = 0, 1/2, 1, 3/2, 2\)
\[
R_{s}^{\text{(trans)}} = \frac{1}{C_s} \left( \frac{\epsilon \kappa}{\omega} \right)^{2s} \frac{\Gamma(1 + s - 2i\epsilon_\pm)}{\Gamma(1 - s - 2i\epsilon_\pm)} R_{s}^{\text{(trans)}}, 
\] (58)
Similarly, by considering the T-S identity (A) and (B) at infinity, one finds
\[
R_{s}^{\text{(inc)}} = e^{-i\pi s} \frac{(2\omega)^{2s}}{C_s} R_{s}^{\text{(inc)}}, \\
R_{s}^{\text{(ref)}} = e^{-i\pi s} \frac{C_s}{(2\omega)^{2s}} R_{s}^{\text{(ref)}},
\] (59)
The relations in Eq.(58) and (59) are essentially the same as those by Teukolsky and Press[7]. By requiring that the asymptotic amplitudes in Eqs.(39) and (43) satisfy the Teukolsky-Press relations, we can derive some identities between the sums of coefficients which we shall see in Sec.7.

Finally by using the relation in Eq.(55), we can show that the upgoing solution with the relative normalization in Eq.(27) satisfies the T-S identity (A). That is, we find
\[
\Delta^s(D^\dagger)^{2s}\Delta^s R_{\text{up};s}^{\nu} = B_s \Delta^s(D^\dagger)^{2s}\Delta^s R_{\text{up};-,s}^{\nu} \\
= C_s R_{\text{up};-,s}^{\nu},
\] (60)
Thus, the relative normalization between the solution with spin weight $s$ and $-s$ is determined analytically for the upgoing solution as given in Eq.(27).

6 The conserved current and the energy conservation

In below, we take $s$ to be 0, 1/2, 1, 3/2, 2. The Wronskian gives the conserved current[7]

$$[Y(-s)\frac{dY(s)}{dr_s} - Y(s)\frac{dY(-s)^*}{dr_s}]_{r=r_+} = [Y(-s)^*\frac{dY(s)}{dr_s} - Y(s)\frac{dY(-s)^*}{dr_s}]_{r=\infty},$$

(61)

where $Y(s) = \Delta s/2(r^2 + a^2)^{1/2}R_{in,s}$. By substituting the asymptotic behaviors in Eq.(37), one finds

$$(R_s^{(inc)}R_{-s}^{(inc)})^* = (R_s^{(ref)}R_{-s}^{(ref)})^* - i\frac{\epsilon K(s + 2i\epsilon_+)}{2\omega^2}(R_s^{(trans)}R_{-s}^{(trans)})^*.$$  

(62)

By using the Teukolsky-Press relations in Eqs.(58) and (59), we obtain

$$|R_s^{(inc)}|^2 = \frac{(2\omega)^4s}{C_s^2}|R_s^{(ref)}|^2 + \delta_s|R_s^{(trans)}|^2,$$

(63)

where

$$\delta_s = -ie^{i\pi s}\omega^{4s-2}(\frac{\epsilon K}{2})^{-2s+1}\frac{\Gamma(1-s+2i\epsilon_+)}{\Gamma(s+2i\epsilon_+)}. $$

(64)

Explicitly, we obtain

$$\delta_0 = \frac{\epsilon_+(\epsilon K)}{\omega^2},$$

$$\delta_1 = 1,$$

$$\delta_2 = \frac{\omega^2}{\epsilon_+(\epsilon K)},$$

$$\delta_3 = \frac{4\omega^4}{(\frac{1}{4} + 4\epsilon_+^2)(\epsilon K)^2},$$

$$\delta_2 = \frac{4\omega^6}{\epsilon_+(1 + 4\epsilon_+^2)(\epsilon K)^3},$$

(65)
The relation in Eq.(63) leads to the energy conservation:

\[
\frac{d^2 E_{\text{inc},s}}{dtd\Omega} = \frac{d^2 E_{\text{ref},s}}{dtd\Omega} + \frac{d^2 E_{\text{trans},s}}{dtd\Omega},
\]

(66)

where \(\frac{d^2 E_{\text{inc},s}}{dtd\Omega}\) is the incident energy going into the black hole, \(\frac{d^2 E_{\text{ref},s}}{dtd\Omega}\) is the energy reflected by the black hole and \(\frac{d^2 E_{\text{trans},s}}{dtd\Omega}\) is the energy absorbed by the black hole. It is clear that \(\delta_s\)'s are proportional to \(\epsilon_+\) for bosons and are positive definite for fermions. Since \(2\epsilon_+ = \epsilon + \tau = \epsilon(1 + \frac{1}{n}) - \frac{ma}{Mn}\), \(\epsilon_+\) can be negative for large angular momentum of black hole. That is, the super radiance occurs for bosons for \(a > \frac{c(1+\kappa)M}{m}\), but not fermions[12].

By substituting the explicit forms of asymptotic amplitudes in Eqs.(39) and (43), we have an identity involving the sums of coefficients which we shall see in the next section. In Sec.8, we shall derive a simple expression for the absorption rate.

### 7 Identities involving the sums of coefficients

The requirement that the asymptotic amplitude on the outer horizon in Eq.(39) satisfies the Teukolsky-Press relation in Eq.(58) leads to the identity for \(s = 0, 1/2, 1, 3/2, 2\)

\[
\sum_{n=-\infty}^{\infty} a_n^*(-s) = |C_s|^2 \left| \frac{\Gamma(\nu + 1 - s + i\epsilon)}{\Gamma(\nu + 1 + s + i\epsilon)} \right|^2 \sum_{n=-\infty}^{\infty} a_n^*(s). \quad (I - 1)
\]

(67)

The requirement that the asymptotic amplitudes at infinity in Eq.(43) satisfy the Teukolsky-Press relations in Eq.(59) is automatically satisfied. This can be seen by using the relation among \(K_\nu(s)\) in Eq.(35). No more identity arises from the T-S identities.

Next, we consider the constraint from the Wronskian. By substituting the asymptotic amplitudes in Eqs.(39) and (43) into Eq.(63) and by using the identity (I-1), we find

\[
\left\{ \sum_{n=-\infty}^{\infty} (-1)^n \frac{\Gamma(\nu + 1 + s + i\epsilon)}{\Gamma(\nu + 1 - s + i\epsilon)} a_n^*(s) \right\}^2 \left| \sum_{n=-\infty}^{\infty} a_n^*(s) \right|^2 = |C_s|^2 \left| \frac{\Gamma(\nu + 1 - s + i\epsilon)}{\Gamma(\nu + 1 + s + i\epsilon)} \right|^2. \quad (I - 2)
\]

(68)
Since the derivation is lengthy, we shall give a brief proof in Appendix A. The identity (I-2) can be written in a different form by using the identity (I-1) as

\[ \left| \sum_{n=-\infty}^{\infty} (-1)^n \frac{(\nu + 1 + s - i\epsilon)_n}{(\nu + 1 - s + i\epsilon)_n} a_n^\nu(s) \right|^2 = \left( \sum_{n=-\infty}^{\infty} a_n^\nu(s) \right) \left( \sum_{n=-\infty}^{\infty} a_n^\nu(-s) \right)^* \quad (I-2)' \]  

(69)

The identities (I-1) and (I-2) have simple structures and will serve to make various formulas into simpler ones. Since our solution is given by the sums of either hypergeometric or Coulomb wave functions, various formulas given by our solution are expressed in complicated forms involving the sums of coefficients. Therefore, these identities are powerful to compute various quantities analytically and also numerically.

Another important feature of these identities lies in the fact that they can be used as a good tool to test the accuracy of the computation. In order to obtain the Teukolsky function, we have to solve the three term recurrence relation by the \( \epsilon \) expansion or the numerical calculation. In the computation, the estimation of the accuracy becomes an important issue. The accuracy is directly tested by looking how accurately these identities are satisfied because the sums appearing in these identities correspond to the asymptotic amplitudes themselves. As for the accuracy test of the computation, to check the \( \tilde{r} \) independence of \( K_\nu(s) \) will be another good test. These identities and relation give nontrivial test of the accuracy of computation.

8 The absorption rate and the evaporation rate of black hole

In this section, we present the absorption rate formula expressed by using our solution. Since the calculation is lengthy, we give the brief derivation in Appendix A and only give the result

\[ \Gamma_s = \delta_s \left| \frac{R_s^{(\text{trans})}}{R_s^{(\text{inc})}} \right|^2 = 1 - \frac{(2\omega)^{4s}}{|C_s|^2} \left| \frac{R_s^{(\text{ref})}}{R_s^{(\text{inc})}} \right|^2 \]
\begin{equation}
= (2\epsilon\kappa)^{2\nu+1} \frac{e^{\pi\epsilon}}{\pi} \left\{ \sinh 2\pi\epsilon_+ \right\} \cosh 2\pi\epsilon_+ \n\times D'_s \left| 1 + i (2\epsilon\kappa)^{2\nu+1} e^{\pi\epsilon} (-1)^{2s} \sin \pi (\nu - i\tau) \left( \frac{\sin \pi (\nu - s - i\epsilon)}{\sin 2\pi\nu} \right)^2 D'_s \right| ^2 ,
\end{equation}

where the upper column (\(\sinh 2\pi\epsilon_+\)) is taken for bosons \((s = 0, 1, 2)\) and the lower column (\(\cosh 2\pi\epsilon_+\)) is taken for fermions \((s = 1/2, 3/2)\). Here,

\begin{equation}
D'_s = \left| \frac{\Gamma(\nu + 1 - i\tau) \Gamma(\nu + 1 - s + i\epsilon) \Gamma(\nu + 1 + s + i\epsilon)}{\Gamma(2\nu + 1) \Gamma(2\nu + 2)} \right|^2 d'_s
\end{equation}

To derive the above formula, we used \(\sin \pi (\nu + s - i\epsilon) = (-1)^{2s} \sin \pi (\nu - s - i\epsilon)\). It may be worthwhile to note that \(d'_s\) starts from 1 in the \(\epsilon\) expansion and is the only part which we have made a calculation by solving the three term recurrence relation. Since the mathematical structure is simple, it is easy to estimate the rate either in the expansion of \(\epsilon\) or the direct numerical calculation.

The evaporation rate from the black hole is given by[13]

\begin{equation}
<N> \equiv \frac{\Gamma_s}{e^{\frac{2\pi\epsilon_+}{\kappa'}(\omega - m\Omega_+)} + 1} = 2 \left\{ \left(\sinh 2\pi\epsilon_+\right)^{-1} \right\} \left(\cosh 2\pi\epsilon_+\right)^{-1} e^{-\frac{2\pi\epsilon_+}{\kappa'}(\omega - m\Omega_+)} \Gamma_s .
\end{equation}

Here, the upper column for bosons and the lower column for fermions, \(\kappa'\) is surface gravity, \(\Omega_+\) is angular velocity on horizon and we used

\begin{equation}
2\epsilon_+ = \epsilon + \tau \equiv 2M\omega + \frac{2M\omega - m\frac{a}{M}}{\sqrt{1 - (\frac{a}{M})^2}} = \frac{\omega - m\Omega_+}{\kappa'} .
\end{equation}
By substituting $\Gamma_s$ in Eq.(70) into the evaporation rate formula, we find

$$< N > = \frac{2}{\pi} e^{-\frac{i\pi}{\kappa}(1+m\Omega_+)}(2\epsilon\kappa)^{2\nu+1} D_s^{\nu} \times \frac{D_s^{\nu}}{1 + i(2\epsilon\kappa)^{2\nu+1} e^{2\nu}(-1)^{2s} \sin \pi \frac{(\nu-i\tau)}{\sin 2\pi\nu}} \left(\frac{\sin \pi (\nu-s-i\epsilon)}{\sin 2\pi \nu}\right)^2 D_s^{\nu} .$$

(75)

It is interesting to observe that the denominator in the evaporation $< N >$ which shows the difference between the boson and the fermion emissions is canceled by the sinh $2\epsilon$ or $\cosh 2\epsilon$ factor in $\Gamma_s$. As a result, the evaporation rate shows the Boltzman like behavior both for both boson and fermion emissions.

In below, we made the $\epsilon$ expansion for $\Gamma_s$. In Ref.1, we obtained the renormalized angular momentum $\nu$ and coefficients up to the order $\epsilon^2$. In the approximation neglecting the $O(\epsilon^2)$ terms, we can immediately derive the absorption rate because $\nu = l + O(\epsilon^2)$. We find

$$\Gamma_s = (2\epsilon\kappa)^{2l+1} e^{\pi\epsilon} \left\{ \frac{\tau \sinh \pi (\epsilon+i\tau)}{\sinh \pi \tau} \prod_{k=1}^l (k^2 + \tau^2) \right\} \left\{ \frac{\cosh \pi (\epsilon+i\tau)}{\cosh \pi \tau} \prod_{k=1/2} (k^2 + \tau^2) \right\} \times \left(\frac{(l-s)!(l+s)!}{2!(2l+1)!}\right)^2 d_s^{\nu} ,$$

(76)

where the upper cases for massless scalar, photons and gravitons, the lower case for neutrinos and massless spin 3/2 particles and

$$d_s^{\nu} = 1 + mqs^2 \left(\frac{1}{l^2} - \frac{1}{(l+1)^2}\right) \epsilon .$$

(77)

For massless scalar case ($s = 0$), it is understood that $\prod_{k=1}^l (k^2 + \tau^2) = 1$ for $l = 0$. In the zeroth order of $\epsilon$, it is easy to see that this expression reproduces the formulas first given by Page[14] for $s = 0, 1/2, 1$ and 2.

9 Summary

To obtain the Teukolsky functions in a desired accuracy is a quite important issue for the gravitational wave astrophysics. The analytic solution which we reported in our previous
paper is expected to serve for this purpose. Since the solution is given as series of special functions, the examination of the property of the solution is needed.

In this paper, we used new Coulomb type solutions and constructed the analytic solutions valid in the entire region of $r$ for all finite $\epsilon$. By using this solution, we examined various properties of solutions of Teukolsky equation. We found that the relative normalization between the solution with a spin weight $s$ and $-s$ is determined analytically by using the T-S identities. In addition, we found an identity (I-1) involving the sums of coefficients of series. By using the Wronskian, we found another identity (I-2). The sums of coefficients appeared in these identities are proportional to the asymptotic amplitudes on the horizon and at infinity and thus can be used to test the accuracy for the computation. The remarkable fact is that we can compute the Teukolsky functions systematically in a desired accuracy. By using these identities, we demonstrated that the asymptotic amplitudes can be expressed in compact forms. The fact that various quantities can be written simpler forms by using these identities is important to see the general properties of these quantities analytically without any approximation and also to make the numerical computation.

As an example, we derived the absorption rate and the evaporation rate of black hole. We found that they are expressed in compact forms from which we can extract the physics transparently.

Now, the properties of solutions are well understood and it is ready to go to derive the Teukolsky functions by either $\epsilon$ expansion or the numerical computation which we are now pursuing.

**Acknowledgment**

We would like to thank to H. Suzuki for discussions, M. Sasaki, M. Shibata, H. Tagoshi and T. Tanaka for comments and encouragements. This work is supported in part by the Japanese Grant-in-Aid for Scientific Research of Ministry of Education, Science, Sports and Culture, No. 06640396 and 08640374.
Appendix A: Proofs of various formulas

In this appendix, we give brief proofs of various formulas given in the text.

(a) The Teukolsky-Starobinsky identities

Proof of equation (54)

For \( s = 0, 1/2, 1, 3/2, 2 \), we rewrite \( R_{\text{in};s}^{\nu} \) as

\[
R_{\text{in};s}^{\nu} = A_s e^{i\kappa x} (-x)^{-s-i\epsilon_{+}} (1-x)^{-s-i\epsilon_{-}} \left\{ (1-x)^{s+2i\epsilon_{-}} \sum_{n=-\infty}^{\infty} a_n^{\nu}(s) \right. \\
	imes \left. F(n+\nu+1-i\tau,-n-\nu-i\tau;1-s-2i\epsilon_{+};x) \right\}.
\]

(A.1)

Then, one finds

\[
\Delta^{s}(D^{1})^{2s} \Delta^{s}R_{\text{in};s}^{\nu} = A_s \left( -\frac{\epsilon_{\kappa}}{\omega} \right)^{2s} e^{i\kappa x} (-x)^{-s-i\epsilon_{+}} (1-x)^{-s-i\epsilon_{-}} \frac{d^{2s}}{dx^{2s}} \left\{ (1-x)^{s+2i\epsilon_{-}} \sum_{n=-\infty}^{\infty} a_n^{\nu}(s) \right. \\
	imes \left. F(n+\nu+1-i\tau,-n-\nu-i\tau;1-s-2i\epsilon_{+};x) \right\}.
\]

(A.2)

Here we use the relation among hypergeometric function

\[
\frac{d^n}{dx^n}(1-x)^{a+b-c}F(a,b;c;x) = \frac{(c-a)_n(c-b)_n}{c^n} (1-x)^{a+b-c-n}F(a,b;c+n;x)
\]

(A.3)

and then the right hand side of Eq.(A.3) becomes

\[
A_s \left( -\frac{\epsilon_{\kappa}}{\omega} \right)^{2s} e^{i\kappa x} (-x)^{-s-i\epsilon_{+}} (1-x)^{i\epsilon_{-}} \times \sum_{n=-\infty}^{\infty} a_n^{\nu}(s) \frac{(-n-\nu-s-i\epsilon)_{2s}(n+\nu+1-s-i\epsilon)_{2s}}{(1-s-2i\epsilon_{-})_{2s}} \times F(n+\nu+1-i\tau,-n-\nu-i\tau;1+s-2i\epsilon_{+};x).
\]

(A.4)
By using

\[
(-n - \nu - s - i\epsilon)_{2s}(n + \nu + 1 - s - i\epsilon)_{2s} = (-1)^{2s}\left|\frac{\Gamma(n + 1 + s + i\epsilon)}{\Gamma(n + 1 - s + i\epsilon)}\right|^2 \frac{\left(n + 1 + s + i\epsilon\right)}{\left(n + 1 - s + i\epsilon\right)}
\]

(A.5)

and the relation between \(a^{\nu}_n(s)\) and \(a^{\nu}_n(-s)\) in Eq.(32), Eq.(54) is proven.

**Proof of Eq.(55)**

The outline of the proof is as follows:

\[
\omega_{4s} e^{-\pi \epsilon} e^{-i\pi(n+1-s)} \frac{\Gamma(n + 1 - s + i\epsilon)}{\Gamma(n + 1 + s - i\epsilon)} \Delta^s(D^\dagger)^{2s} \Delta^s R^{\nu}_{+;s}
\]

\[
= z^n (z - \epsilon \kappa)^n (D^\dagger)^{2s} e^{-iz} z^{-ie} (z - \epsilon \kappa)^{-ie} \sum_{n=-\infty}^{\infty} i^n a^{\nu}_n(s)
\]

\[
\times z^{\nu+s+ie} (2z)^n \Psi(n + \nu + 1 - s + i\epsilon, 2n + 2\nu + 2; 2iz)
\]

\[
= \omega_{4s} e^{-iz} z^{-ie} (z - \epsilon \kappa)^{-ie} \sum_{n=-\infty}^{\infty} i^n a^{\nu}_n(s)
\]

\[
\times \frac{d^{2s}}{dz^{2s}} z^{\nu+s+ie} (2z)^n \Psi(n + \nu + 1 - s + i\epsilon, 2n + 2\nu + 2; 2iz)
\]

\[
= \omega_{4s} e^{-iz} z^{\nu+ie} (z - \epsilon \kappa)^{-ie} \sum_{n=-\infty}^{\infty} i^n a^{\nu}_n(s)(n + \nu + 1 - s + i\epsilon)_{2s}
\]

\[
\times (-n - \nu - s + i\epsilon)_{2s} (2z)^n \Psi(n + \nu + 1 + s + i\epsilon, 2n + 2\nu + 2; 2iz)
\]

(A.6)

where we used the relation of confluent hypergeometric functions

\[
\frac{d^n}{dx^n} \left(x^{a+n-1}\Psi(a, c; x)\right) = (a)_n(a - c + 1)_n x^{a-1}\Psi(a + n, c; x)
\]

(A.7)

to derive the last line. By a careful calculation by using Eq.(A.6) and the relation between \(a^{\nu}_n(s)\) and \(a^{\nu}_n(-s)\) in Eq.(32), we can derive the relation for \(R^{\nu}_{+;s}\) in Eq.(55).
Similarly, we find
\[ \omega^{2s} \Gamma^{-\nu} e^{\pi \epsilon} \Gamma^{\nu+1+s} \Delta s (D^0)^2 e^{s \Delta} R^\nu_{-s} \]
\[ = e^{-iz} z^{s-i\epsilon} (z - \epsilon \kappa)^{s-i\epsilon} + \sum_{n=-\infty}^{\infty} i^n a_\nu^n(s) \left( \frac{\nu + 1 + s - i\epsilon}{\nu + 1 - s + i\epsilon} \right)_n \]
\[ \times \frac{d^n}{dz^n} e^{2iz} z^{s+i\epsilon} (2z)^n \Psi(n + \nu + 1 + s - i\epsilon, 2n + 2s + 2; -2iz) \]
\[ = (-1)^n e^{-iz} z^{\nu+i\epsilon} (z - \epsilon \kappa)^{s-i\epsilon} + \sum_{n=-\infty}^{\infty} i^n a_\nu^n(s) \left( \frac{\nu + 1 + s - i\epsilon}{\nu + 1 - s + i\epsilon} \right)_n \]
\[ \times (2z)^n \Psi(n + \nu + 1 - s - i\epsilon, 2n + 2s + 2; -2iz) , \]
(A.8)

where we used the relation
\[ \frac{d^n}{dx^n} (e^{-x} x^{-a-n-1} \Psi(a, c; x)) = (-1)^n e^{-x} x^{-a-1} \Psi(a - n, c; x). \]
(A.9)

By using the relation between \( a_\nu^n(s) \) and \( a_\nu^n(-s) \) in Eq.(32), we can derive the relation for \( R^\nu_{-s} \) in Eq.(55).

(b) The derivation of the absorption rate

We find from Eq.(24) with \( \tilde{r} = 0 \)
\[ |K_\nu(s)|^2 = \frac{\pi}{2^{2s} \sin \pi (s + 2i\epsilon_+)} \frac{\Gamma(1 - s + 2i\epsilon_+)}{\Gamma(1 - s + 2i\epsilon_+)} \left| \frac{\Gamma(\nu + 1 + s + i\epsilon)}{\Gamma(\nu + 1 - s + i\epsilon)} \right|^2 \frac{1}{D^\nu_s} , \]
(A.10)

where \( D^\nu_s \) is defined in Eq.(71). Then, by changing \( \nu \) to \( -\nu - 1 \) and using \( a_{-n}^\nu(s) = a_\nu^n(s) \), we find
\[ \frac{|K_{-\nu-1}(s)|^2}{|K_\nu(s)|^2} = \frac{(2e\kappa)^{4\nu+2}}{\pi^2} \sin \pi (\nu + i\tau) \left( \frac{\sin \pi (\nu + s + i\epsilon)}{\sin 2\pi \nu} \right)^2 (D^\nu_s)^2 . \]
(A.11)
We also find by using Eqs.(35) and (36)

$$\frac{K_{\nu}(s)K_{-\nu-1}(s)^*}{|K_{\nu}(s)|^2} = \frac{K_{\nu}(s)K_{-\nu-1}(-s)^*}{|K_{\nu}(s)|^2} \left( \frac{K_{-\nu-1}(s)}{K_{-\nu-1}(-s)} \right)^*$$

$$= \frac{(2\epsilon K)^{2\nu+1} \sin \pi (\nu - i\tau)}{\pi} \left| \frac{\sin \pi (\nu - s + i\epsilon)}{\sin 2\pi \nu} \right|^2 D_s^\nu .$$

(A.12)

By using Eqs.(A.11) and (A.12), we obtain

$$\left| K_{\nu}(s) - ie^{-i\pi \nu} \frac{\sin \pi (\nu - s + i\epsilon)}{\sin \pi (\nu + s - i\epsilon)} K_{-\nu-1}(s) \right|^2$$

$$= \frac{1}{D_s^\nu} \frac{\pi (2\epsilon K)^{2\nu+2s}}{2\pi \sin \pi (s + 2i\epsilon_+)} \frac{\Gamma(1 - s + 2i\epsilon_+)}{\Gamma(s + 2i\epsilon_+)} \frac{\Gamma(\nu + 1 + s + i\epsilon)}{\Gamma(\nu + 1 - s + i\epsilon)}$$

$$\times \left| 1 + \frac{i(-1)^{2s} e^{i\pi \nu} (2\epsilon K)^{2\nu+1} \sin \pi (\nu - i\tau)}{\pi} \left( \frac{\sin \pi (\nu - s + i\epsilon)}{\sin 2\pi \nu} \right)^2 D_s^\nu \right|^2 .$$

(A.13)

By using this formula, we can derive the formula for the absorption rate in Eq.(70).

(c) Identity (I-2) form the Wronskian

Here we show that the identity (I-2) is obtained from the Wronskian given in Eq.(63).

Let us first parameterize \( |R_s^{\text{inc}}|^2 \) as

$$|R_s^{\text{inc}}|^2 = \frac{|A_s|^2 2^{2s-2} e^{-\pi \epsilon}}{\omega^2} \left\{ \left| K_{\nu}(s) + ie^{i\pi \nu} K_{-\nu-1}(s) \right|^2 \right.$$

$$+ \left\{ \left| K_{\nu}(s) - ie^{-i\pi \nu} \frac{\sin \pi (\nu - s - i\epsilon)}{\sin \pi (\nu + s - i\epsilon)} K_{-\nu-1}(s) \right|^2 \right.$$

$$- \left| K_{\nu}(s) + ie^{i\pi \nu} K_{-\nu-1}(s) \right|^2 \right\} \frac{\Gamma(\nu + 1 - s + i\epsilon)}{\Gamma(\nu + 1 + s + i\epsilon)} \left( \sum_{n=-\infty}^{\infty} a_n^\nu(s) \right)^2 .$$

(A.14)

Then, by using the relation

$$\left| K_{\nu}(s) - ie^{-i\pi \nu} \frac{\sin \pi (\nu - s - i\epsilon)}{\sin \pi (\nu + s + i\epsilon)} K_{-\nu-1}(s) \right|^2 - \left| K_{\nu}(s) + ie^{i\pi \nu} K_{-\nu-1}(s) \right|^2$$

$$= -2ie^{i\pi \nu} (\epsilon K)^{2s+1} e^{i\pi \epsilon} \frac{\Gamma(1 - s + 2i\epsilon_+)}{\Gamma(s + 2i\epsilon_+)} \left( \frac{\Gamma(\nu + 1 + s + i\epsilon)}{\Gamma(\nu + 1 - s + i\epsilon)} \right)^2 ,$$

(A.15)
we can prove that the contribution from the second part in the parenthesis in Eq.(A.14), i.e., the part in Eq.(A.15) cancels the transition (absorption) part $\delta_s |R_{s}^{(\text{trans})}|^2$. Thus, the contribution from the first part in Eq.(A.14) should be equal the reflection part. By equating these two, we can derive the identity (I-2).
References

[1] S. Mano, H. Suzuki and E. Takasugi, Prog. Theor. Phys. 95, 1079 (1996).

[2] S.A. Teukolsky, Phys. Rev. Lett. 29, 1114 (1972).

[3] E.W. Leaver, J. Math. Phys. 27, 1238 (1986).

[4] S. Mano, H. Suzuki and E. Takasugi, Prog. Theor. Phys. 96, 549 (1996).

[5] H. Tagoshi and T. Nakamura, Phys. Rev. D 49, 4016 (1994).
   H. Tagoshi and M. Sasaki, Prog. Theor. Phys. 92, 745 (1994).
   M. Shibata, M. Sasaki, H. Tagoshi and T. Tanaka, Phys. Rev. D 51, 1646 (1995).

[6] S.A. Teukolsky and W.H. Press, Astorophys. J. 193, 443 (1974).

[7] A.A. Starobinsky and Churilov, Sov. Phys. JETP 38, 1 (1974).

[8] V.S. Otchik, in"Quantum Systems :New trends and Methods:",
edited by A.O. Barut et. al.,(World Scientific, 1995), 154.

[9] W.H. Press and S.A. Teukolsky, Astorophys. J. 185, 649 (1973).

[10] S. Chandrasekhar, The Mathematical Theory of Black Holes,
    (Oxford University Press, 1992).

[11] G.F. Torres del Castillo, J. Math. Phys. 30, 2114 (1989).

[12] C.W. Misner, Phys. Rev. Lett. 28, 993 (1972).
    W.H. Press and S.A. Teukolsky, Nature (London), 238, 211 (1972).
    Ya.B. Zel’dovich, JETP Lett. 14, 180 (1971); Sov. Phys. JETP 35, 1085 (1972).

[13] S.W. Hawking, Comm. Math. Phys. 43, 199 (1975).

[14] D.N. Page, Phys. Rev. D 13, 198 (1976).