On the Integrability of the Abel and of the Extended Liénard Equations

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Abstract We present some exact integrability cases of the extended Liénard equation
\[ y'' + f(y)(y')^n + k(y)(y')^m + g(y)y' + h(y) = 0, \]
with $n > 0$ and $m > 0$ arbitrary constants, while $f(y)$, $k(y)$, $g(y)$, and $h(y)$ are arbitrary functions. The solutions are obtained by transforming the equation Liénard equation to an equivalent first kind first order Abel type equation given by
\[ dv/dy = f(y)v^3 - n + k(y)v^3 - m + g(y)v^2 + h(y)v^4, \]
with $v = 1/y'$. As a first step in our study we obtain three integrability cases of the extended quadratic-cubic Liénard equation, corresponding to $n = 2$ and $m = 3$, by assuming that particular solutions of the associated Abel equation are known. Under this assumption the general solutions of the Abel and Liénard equations with coefficients satisfying some differential conditions can be obtained in an exact closed form. With the use of the Chiellini integrability condition, we show that if a particular solution of the Abel equation is known, the general solution of the extended quadratic cubic Liénard equation can be obtained by quadratures. The Chiellini integrability condition is extended to generalized Abel equations with $g(y) \equiv 0$ and $h(y) \equiv 0$, and arbitrary $n$ and $m$, thus allowing to obtain the general solution of the corresponding Liénard equation. The application of the generalized Chiellini condition to the case of the reduced Riccati equation is also considered.

Keywords Abel equation; Liénard equation; general solutions; integrability conditions

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1 Introduction

The Liénard type ordinary second order nonlinear differential equation of the form$^{[24,25]}$,
\[ \ddot{x}(t) + f(x)\dot{x}(t) + g(x) = 0, \] (1)

as well as its generalization, the Levinson-Smith type equation$^{[23]}$,
\[ \ddot{x}(t) + f(x, \dot{x})\dot{x}(t) + g(x) = 0, \] (2)

where a dot represents the derivative with respect to the independent variable $t$, $g$ is an arbitrary $C^1$ functions of $x$, while $f$ is an arbitrary function of $x$ and $\dot{x}$, play an important role in many areas of physics, biology and engineering$^{[1]}$. These types of equations have been intensively studied from both mathematical and physical point of view, and these investigation still remain an active and quickly developing field of research in pure mathematics and mathematical physics$^{[2,4,27,34,39,40,45]}$.

One of the important applications of the generalized Liénard type equations is represented by the mathematical analysis of the non-linear oscillations, which in many situations can be
described by the equation\cite{35,38}

\[ \ddot{x} + f(\dot{x})g(x) + h(x) = 0, \tag{3} \]

which is a particular case of (2), where \( f, g \) and \( h \) are arbitrary \( C^1 \) functions. From a physical point of view the Liénard equation can also be interpreted as the generalization of the equation of motion of the damped oscillator,

\[ \ddot{x} + \gamma \dot{x} + \omega^2 x = 0, \tag{4} \]

with \( \gamma = \text{constant} \) and \( \omega^2 = \text{constant} \), respectively\cite{7}, describing, from a physical point of view, nonlinear or anharmonic oscillations\cite{8,12}. A new criterion for integrability of the Liénard equation using an approach based on nonlocal transformations, was obtained in \cite{20}. Some previously known criteria for integrability were also reobtained, and several new examples of integrable Liénard equations were given. A new family of the Liénard-type equations which admits a non-standard autonomous Lagrangian was found in \cite{22}, and autonomous first integrals for each member were obtained. Four new integrability criteria of a particular type of Liénard type equations were obtained in \cite{21} by studying the connections between this family of Liénard–type equations and type III Painlevé–Gambier equations. The results were illustrated by providing examples of some integrable Liénard-type equations.

The connection between the linear harmonic oscillator equation and some classes of second order nonlinear ordinary differential equations of Liénard and generalized Liénard type, which physically describe important oscillator systems, was investigated in \cite{18}. By using a method inspired by quantum mechanics, and which consist on the deformation of the phase space coordinates of the harmonic oscillator, one can generalize the equation of motion of the classical linear harmonic oscillator to several classes of strongly non-linear differential equations. The first integrals, and a number of exact solutions of the corresponding equations can then be explicitly obtained. The procedure can be further generalized to derive explicit general solutions of nonlinear second order differential equations unrelated to the harmonic oscillator.

An important relation can be established between the Liénard type equations, and the first kind first order Abel differential equation\cite{19,37,41},

\[ \frac{dy}{dx} = p(x)y^3 + q(x)y^2. \tag{5} \]

This relation allows to find some exact general solutions of the Liénard type equations by using the integrability conditions of the Abel equation. Some general integrability conditions for the Abel equation were obtained in \cite{11,28,29}.

By using the Liénard equation-Abel equation equivalence, a class of exact solutions of the Liénard differential equation was obtained in \cite{14}. With the use of an exact integrability condition for the Abel equation (the Chiellini lemma)\cite{6,13}, as a first step one can obtain the exact general solution of the Abel equation. Once the solution of the Abel equation is known, a class of exact solutions of the Liénard equation, expressed in a parametric form can be found. The Chiellini integrability condition of the Abel equation was extended to the case of the generalized Abel equation in \cite{13}. The exact solutions of some particular Liénard type equations, including a generalized van der Pol type equation, were also explicitly obtained. The interesting connection between dissipative nonlinear second order differential equations and the Abel equations with cubic and quadratic terms was considered in \cite{32}. The general solution of the corresponding integrable dissipative equations was derived by using the Chiellini integrability condition. In \cite{42} nonsingular dissipative oscillators, Darboux related to the classical harmonic oscillator, and having periodic dissipative/gain properties, have been found through a modified factorization method. The same method can be also applied to the so-called upside-down (hyperbolic)
“oscillator”. The barotropic Friedmann-Robertson-Walker cosmologies in the comoving time lead in the radiation-dominated case to scale factors of identical form as for the Chielini dissipative scale factors in conformal time$^{[43]}$. This is due to the Ermakov equation, which is obtained in this case. Some results in the integrability of the Abel equations were discussed and reformulated in $^{[33]}$. Analytic techniques for the solutions of nonlinear oscillators with damping using the Abel equation, the use of the Chielini integrability condition for the study of planar isochronous systems and of the Hamiltonian structures of the Liénard equation, as well as the relation between the generalized damped Milne-Pinney equation and the Chielini method were investigated in $^{[9,10,36]}$. Travelling waves solutions in reaction-diffusion-convection systems were obtained, with the use of the Chielini and Lenke integrability conditions, in $^{[16,17]}$.

There are several methods that can be used for obtaining an exact solution of the general Abel equation

$$\frac{dy}{dx} = p(x)y^3 + q(x)y^2 + r(x)y + s(x).$$  \(6\)

An important property of the Abel differential equations is that they can be systematized into equivalence classes$^{[3,5,26]}$. Two Abel ordinary differential equations are defined to be equivalent if they can be obtained one from the other by means of the transformation$^{[3]}$

$$(y, x) \rightarrow (Y, X) : x = F(X), \quad y = P(X)Y(X) + Q(X), \quad F'P \neq 0,$$  \(7\)

where $X$ and $Y(X)$ are the new independent and dependent variables, while $F$, $P$ and $Q$ are arbitrary functions of $X$. Then the equivalence class for an ordinary differential equation is the set of all ordinary differential equations equivalent to the given one. As shown in $^{[3,5,26]}$, to each equivalence class we can associate an infinite sequence of absolute invariants (see $^{[5]}$ for the definition of absolute and relative invariants). As shown initially in $^{[26]}$ (see also $^{[3]}$ and $^{[5]}$ for detailed discussions), one can associate to the Abel equation the relative invariant $S_3$ of weight 3,

$$S_3(x) \equiv s(x)p^3(x) + \frac{1}{3} \left[ \frac{2q^2(x)}{9} - r(x)q(x)p(x) + p(x)q'(x) - q(x)p'(x) \right].$$  \(8\)

$S_3$ can be used to recursively generate an infinite sequence of relative invariants $S_{2m+1}$ of weights $2m + 1$ through the relations

$$S_{2m+1}(x) = p(x)S_{2m-1}(x) - (2m - 1)S_{2m-1}(x) \left[ p'(x) + r(x)p(x) - \frac{q^2(x)}{3} \right].$$  \(9\)

With the help of the relative invariants $S_{2m+1}$ one can construct the absolute invariants $I_n$, given by $^{[3,5]}$,

$$I_1(x) = \frac{S_3(x)}{S_3(x)}, \quad I_2(x) = \frac{S_3(x)S_7(x)}{S_5(x)}, \quad I_3(x) = \frac{S_9(x)}{S_3(x)},$$  \(10\)

If $I_1(x)$ is a constant, then all the other invariants are also constant. The early results on the invariants of the Abel equation were used in $^{[5]}$ to obtain a set of new integrable Abel ordinary differential equation classes, with some depending on arbitrary parameters, and to introduce an explicit method for verifying or refuting the equivalence between two given Abel ordinary differential equations. The same approach was also used in $^{[3]}$ for the analytic study of the buoyancy-drag equation with a time-dependent acceleration $\gamma(t)$, by determining its equivalence class under the point transformations, which allowed to define for some values of $\gamma(t)$ a time-dependent Hamiltonian from which the buoyancy-drag equation can be derived.

There are also several other methods for obtaining solutions of the Abel equation. If we introduce the transformations$^{[19,37,41]}$

$$y(x) = \omega(x)\eta(\xi(x)) - \frac{q(x)}{3p(x)},$$  \(11\)
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with

\[ \omega(x) = e^{\int [r(x) - \frac{q^2(x)}{3p(x)}]dx}, \quad \xi(x) = \int p(x)\omega^2(x)dx, \]  

then the Abel equation can be reduced to the normal form

\[ \frac{dy}{dx} = y^3 + I(x), \]  

where

\[ p(x)\omega^3(x)I(x) = s(x) + \frac{1}{3} \frac{d}{dx} p(x) - \frac{r(x)q(x)}{3p(x)} + \frac{2q^3(x)}{27p^2(x)}. \]  

Then, if \( I(x) \) is constant, the Abel equation can be integrated.

On the other hand, if \( y = y_1(x) \) is a particular solution Eq.(6), then by means of the transformations

\[ u(x) = \frac{E(x)}{y(x) - y_1(x)}, \]  

where

\[ E(x) = \exp \left\{ \int \left[ 3p(x)y_1^2 + 2q(x)y_1 + r(x) \right]dx \right\}, \]

then the Abel equation can be transformed into the form

\[ \frac{du}{dx} + \Phi_1 u + \Phi_2 = 0, \]  

where

\[ \Phi_1(x) = p(x)E^2(x), \quad \Phi_2(x) = \left[ 3p(x)y_1 + q(x) \right]E(x). \]

Therefore, if the particular solution is given by

\[ y_1 = -\frac{q(x)}{3p(x)}, \]

then \( \Phi_2 = 0 \), and the general solution of the Abel equation can be found by integrating an ordinary differential with separable variables.

It is the goal of the present paper to present some exact integrability cases of the extended Liénard equation

\[ y'' + f(y)(y')^n + k(y)(y')^m + g(y)y' + h(y) = 0, \]  

where \( f(y), k(y), g(y), \) and \( h(y) \) are arbitrary functions. This second order highly nonlinear differential equation can be transformed to an equivalent generalized first kind first order Abel type equation.

As a first step in our analysis we study in detail the quadratic-cubic extended Liénard equations, with \( n = 2 \) and \( m = 3 \), respectively. For this case, we first obtain the general solutions of the Liénard equation under the assumption that one particular solution of the associated Abel equation is known. Under this assumption, we obtain three integrability cases of the extended quadratic-cubic Liénard equation, corresponding to \( n = 2 \) and \( m = 3 \).

With the use of the Chiellini integrability condition, the general solutions of the Abel and quadratic-cubic Liénard equations with coefficients satisfying some differential conditions can be obtained in an exact closed form. We also show that if a particular solution of the Abel equation is known, the general solution of the extended quadratic cubic Liénard equation can be obtained by quadratures, if the coefficients of the equation and the particular solution satisfy a given differential condition.
As a next step in our study, we extend the Chiellini integrability condition to generalized Abel equations with \( g(y) \equiv 0 \) and \( h(y) \equiv 0 \), and arbitrary \( n \) and \( m \). Thus, with the help of the generalized Chiellini Lemma, we obtain the general solution of the corresponding extended Liénard equation for arbitrary \( m \) and \( n \). The application of the generalized Chiellini condition to the case of the reduced Riccati equation is also considered, thus leading to a new integrability case of this equation (see [15,30,31]) for alternative integrability conditions of the Riccati equation.

In the present paper we use the term integrable or exactly integrable differential equation as corresponding to a differential equation that can be solved by quadratures, or by integrals, definite or indefinite, respectively.

The present paper is organized as follows. In Section II we define the quadratic-cubic Liénard equation, and we obtain its associated Abel equation. Then, by assuming that a particular solution of the associated Abel equation is known, we present two cases of exact integrability of the Abel and Liénard equations, respectively. In Section III we obtain the integrability condition of the extended Liénard equation under the assumption that the coefficients \( g(y) \) and \( h(y) \) identically vanish, so that \( g(y) = h(y) \equiv 0 \). We discuss and conclude our results in Section IV.

2 Exact Solutions of the Extended Liénard Equation

In the following we denote \( y' = \frac{du}{dy} \) and \( y'' = \frac{d^2u}{dy^2} \), respectively. We introduce the extended Liénard equation by means of the following definition.

**Definition 2.1.** The second order non-linear ordinary differential equation

\[
y'' + f(y)(y')^n + k(y)(y')^m + g(y)y' + h(y) = 0,
\]

where \( f(y), k(y), g(y), h(y) \in C^\infty(I) \) are arbitrary functions defined on a real interval \( I \subseteq \mathbb{R} \), \( f(y), k(y), g(y), h(y) \neq 0 \), \( \forall y \in I \), and \( m, n \in \mathbb{R} \) are constants satisfying the conditions \( m, n > 0 \), and \( n \neq m \), respectively, is called the extended Liénard differential equation.

By introducing a new dependent variable \( u = y' \), Eq.(21) becomes

\[
\frac{du}{dy} + f(y)u^{n-1} + k(y)u^{m-1} + g(y) + \frac{h(y)}{u} = 0.
\]

Let \( u = \frac{1}{v} \), then we rewrite Eq.(22) as

\[
\frac{dv}{dy} = f(y)v^{\alpha} + k(y)v^\beta + g(y)v^2 + h(y)v^3, \quad \alpha = 3 - n, \quad \beta = 3 - m.
\]

**Definition 2.2.** We call Eq.(23) the generalized Abel equation associated to the extended Liénard equation.

Hence once the solutions of the first order differential Eq.(23) are known, the solutions of the corresponding extended Liénard equation can also be found. In the following we will obtain three classes of exact solutions of Eq.(23) for \( n = 2 \) and \( m = 3 \), which will generate three classes of exact solutions of the extended Liénard equation.

2.1 The Quadratic-cubic Extended Liénard Equation

Assume that \( n = 2 \) and \( m = 3 \), then the extended Liénard equation takes the form

\[
y'' + f(y)(y')^2 + k(y)(y')^3 + g(y)y' + h(y) = 0.
\]
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We call Eq.(24) the quadratic-cubic extended Liénard equation. For this choice of $n$ and $m$ the generalized associated Abel Eq.(23) becomes a standard first kind first order Abel type differential equation, given by

$$\frac{dv}{dy} = k(y) + f(y)v + g(y)v^2 + h(y)v^3. \quad (25)$$

We assume that a particular solution $v_p$ that satisfies Eq.(25) is known, that is, $v_p$ satisfies the equation

$$\frac{dv_p}{dy} = k(y) + f(y)v_p + g(y)v_p^2 + h(y)v_p^3. \quad (26)$$

Subtracting Eq.(25) and Eq.(24) and introducing the new function $F$ defined as

$$F(y) = v - v_p$$

then we obtain the Abel equation for $F(y)$

$$\frac{dF}{dy} = [f(y) + 2g(y)v_p + 3h(y)v_p^2]F + [g(y) + 3h(y)v_p]F^2 + h(y)F^3. \quad (27)$$

By introducing a new function $w$, defined as

$$U(y) = E(y)w(y), \quad (28)$$

where

$$E(y) = e\int[f(y) + 2g(y)v_p + 3h(y)v_p^2]dy, \quad (29)$$

it follows that Eq.(27) takes the form

$$\frac{dw}{dy} = [g(y) + 3h(y)v_p]Ew^2 + h(y)E^2w^3, \quad (30)$$

or, equivalently,

$$\frac{dw}{dy} = [g(y) + 3h(y)v_p]e^2\int[f(y) + 2g(y)v_p + 3h(y)v_p^2]dy w^2 + h(y)e^2\int[f(y) + 2g(y)v_p + 3h(y)v_p^2]dy w^3. \quad (31)$$

2.1.1 General Solution of the Abel and Extended Liénard Equation for

$v_p = v_{p0} = -g(y)/3h(y)$

If the particular solution of the Abel Eq.(25) is given by

$$v_{p0}(y) = -\frac{g(y)}{3h(y)}, \quad (32)$$

then Eq.(30) becomes

$$\frac{dw(y)}{dy} = h(y)E^2(y)w^3(y), E(y) = e^{\int[f(y)+g^2(y)/3h(y)]dy}. \quad (33)$$

Eq.(33) can be straightforwardly integrated, giving

$$w(y) = \pm \frac{1}{\sqrt{2}C - \int h(y)e^2\int[f(y)+g^2(y)/3h(y)]dy dy}, \quad (34)$$

where $C$ is an arbitrary constant of integration.
Then we immediately obtain

\[
U(y) = \pm \frac{e^{\int [f(y) - g^2(y)/3h(y)]dy}}{\sqrt{2} \sqrt{C - \int h(y)e^2 \int [f(y) - g^2(y)/3h(y)]dy} dy}
\] (35)

Hence

\[
v = \frac{1}{y'} = U(y) + v_{p0}(y) = \pm \frac{e^{\int [f(y) - g^2(y)/3h(y)]dy}}{\sqrt{2} \sqrt{C - \int h(y)e^2 \int [f(y) - g^2(y)/3h(y)]dy} dy} - \frac{g(y)}{3h(y)}. \] (36)

Therefore we have obtained the following

**Theorem 1.** If the generalized Abel equation associated to the extended quadratic-cubic Liénard equation with \( n = 2 \) and \( m = 3 \) has the particular solution given by Eq.(32), and the coefficients of the Liénard equation satisfy the condition

\[
\frac{d}{dy} \left[ \frac{g(y)}{h(y)} \right] = -3k(y) + \frac{f(y)g(y)}{h(y)} - \frac{2 g^3(y)}{9 h^2(y)}, \] (37)

then the general solution of the quadratic-cubic Liénard equation is given by

\[
x = x_0 = \int \left\{ \pm \frac{e^{\int [f(y) - g^2(y)/3h(y)]dy}}{\sqrt{2} \sqrt{C - \int h(y)e^2 \int [f(y) - g^2(y)/3h(y)]dy} dy} - \frac{g(y)}{3h(y)} \right\} dy, \] (38)

where \( x_0 \) is an arbitrary integration constant. Eq.(37) is an Abel equation for \( g(y) \), and a Riccati equation for \( 1/h(y) \).

### 2.1.2 General Solutions of Abel and Extended Liénard Equation for \( v_p \neq -g(y)/3h(y) \) via the Chiellini Integrability Condition

If the particular solution of the generalized Abel equation associated to the extended Liénard equation is not given by Eq.(32), \( v_p \neq -g(y)/3h(y) \), a solution of Eq.(31) can be found with the use of the Chiellini Lemma, which can be formulated as follows[6,19].

**Lemma 1[6].** A first kind Abel type differential equation of the form \( y' + p(x)y^3 + q(x)y^2 = 0 \) can be exactly integrated if the functions \( q(x) \) and \( p(x) \) satisfy the condition

\[
\frac{d}{dx} \left[ \frac{p(x)}{q(x)} \right] = Sq(x), \quad S = \text{constant}, \quad S \neq 0. \] (39)

In order to solve the Abel equation with the help of the Chiellini Lemma we introduce a new function \( \theta(x) \), defined as

\[
y(x) = \frac{q(x)}{p(x)} \theta(x). \] (40)

Then, with the use of the Chiellini integrability condition the Abel equation can be transformed into a first order separable differential equation,

\[
\frac{d\theta}{dx} = \frac{q^2(x)}{p(x)} \theta^2 + \theta + S, \] (41)
with the general solution given by
\[
p(x) = K_0^{-1} e^{G(\theta, S)},
\]
where \(K_0^{-1}\) is an arbitrary constant of integration, and
\[
e^{G(\theta, S)} = \begin{cases} 
\frac{\theta}{\sqrt{\theta^2 + \theta + S}} \exp \left[ -\frac{1}{\sqrt{4S - 1}} \arctan \left( \frac{1 + 2\theta}{\sqrt{4S - 1}} \right) \right], & S > \frac{1}{4}, \\
\frac{\theta}{1 + 2\theta} e^{1/(1+2\theta)}, & S = \frac{1}{4}, \\
\frac{\theta}{\sqrt{\theta^2 + \theta + S}} \exp \left[ \frac{1}{\sqrt{1 - 4S}} \arctanh \left( \frac{1 + 2\theta}{\sqrt{1 - 4S}} \right) \right], & S < \frac{1}{4},
\end{cases}
\]
respectively. Eq.(42) determines \(\theta\) as a function of \(x\) and of the constant \(S\), respectively.

For the Abel Eq.(31) the Chiellini integrability condition is given by
\[
\frac{d}{dy} \left\{ \frac{h(y) e^{\int[f(y) + 2g(y)v_p + 3h(y)v_p^2]dy}}{[g(y) + 3h(y)v_p]} \right\} = S[g(y) + 3h(y)v_p] e^{\int[f(y) + 2g(y)v_p + 3h(y)v_p^2]dy},
\]
or, equivalently
\[
\frac{d}{dy} \left[ \frac{E(y) - v_p(y) - v_{po}(y)}{v_p(y) - v_{po}(y)} \right] = 9Sh(y)[v_p(y) - v_{po}(y)]E(y).
\]

2.1.3 The Case \(E(y) \equiv 1\)

If the particular solution of the Abel Eq.(25) satisfies the condition
\[
3h(y)v_p^2 + 2g(y)v_p + f(y) = 0,
\]
or
\[
v_p(y) = -\frac{g(y)}{3h(y)} \pm \sqrt{\frac{g^2(y)}{9h^2(y)} - \frac{f(y)}{3h(y)}},
\]
then, without any loss of generality, one can choose \(E(y) \equiv 1\). For this case the Chiellini integrability condition becomes
\[
\frac{d}{dy} \left[ \frac{1}{v_p(y) - v_{po}(y)} \right] = 9Sh(y)[v_p(y) - v_{po}(y)],
\]
or
\[
\frac{d}{dy} \left\{ \frac{g^2(y)}{9h^2(y)} - \frac{f(y)}{3h(y)} \right\}^{-1/2} = 9Sh(y) \frac{g^2(y)}{9h^2(y)} - \frac{f(y)}{3h(y)},
\]
By taking into account the identity
\[
\frac{1}{\sqrt{b(y)}} \frac{d}{dy} \frac{1}{\sqrt{b(y)}} = -\frac{1}{2b^2(y)} \frac{d}{dy} b(y),
\]
the Condition (49) can be first reformulated as
\[
\frac{d}{dy} \left[ \frac{g^2(y)}{9h^2(y)} - \frac{f(y)}{3h(y)} \right] = -18Sh(y) \left[ \frac{g^2(y)}{9h^2(y)} - \frac{f(y)}{3h(y)} \right]^2.
\]
Hence we have obtained the following

**Theorem 2.** a) If the coefficients of the general Abel Eq.(25) satisfy the condition

\[
\frac{g^2(y)}{3h^2(y)} - \frac{f(y)}{h(y)} \frac{1}{6S \int h(y)dy + C_0/3},
\]

where \(C_0\) is an arbitrary constant, then the Abel equation is exactly integrable.

b) The general solution of the Abel Eq.(52) with coefficients satisfying Condition (52) is given by

\[
v(y, S) = \pm \sqrt{\frac{g^2(y) - 3f(y)h(y)}{h(y)}} \theta(y, S),
\]

where \(\theta(y, S)\) is a solution of the algebraic equation

\[
K_0^{-1}e^{G[y(y), S]} = \pm \frac{h(y)}{\sqrt{g^2(y) - 3f(y)h(y)}}.
\]

c) The extended quadratic-cubic Liénard Eq.(24) with coefficients satisfying Condition (52) is exactly integrable, and its general solution is given by

\[
x - x_0 = \pm \frac{1}{3} \int \frac{1}{h(y)} \{ \sqrt{g^2(y) - 3f(y)h(y)}[3\theta(y, S) + 1] - g(y) \} dy,
\]

where \(x_0\) is an arbitrary constant of integration.

### 2.2 The General Solution of the Extended Quadratic-cubic Liénard for Arbitrary Particular Solutions of the Associated Abel Equation

We consider now the case of an arbitrary particular solution of the associated Abel equation. From the Chiellini integrability Condition (44), it follows that the arbitrary coefficient \(f(y)\) of the quadratic-cubic Liénard equation must satisfy the following differential condition

\[
f(y) = \frac{d^2}{dy^2} \left[ \ln \left( \frac{g(y)}{h(y)} + 3v_p \right) \right] + S \frac{d}{dy} \left\{ h(y) \left( \frac{g(y)}{h(y)} + 3v_p \right)^2 \right\} - 2g(y)v_p - 3h(y)v_p^2.
\]

With the help of the transformation given by

\[
w = \frac{[g(y) + 3h(y)v_p]}{h(y)e^{\int[f(y)+2g(y)v_p+3h(y)v_p^2]dy}} \theta,
\]

immediately inserting Eq.(57) into Eq.(31), the latter becomes a separate variable differential equation

\[
\frac{d\theta}{dy} = \frac{[g(y) + 3h(y)v_p]^2}{h(y)} \theta^2 + \theta + S.
\]

Or equivalently,

\[
\int \frac{1}{\theta^2 + \theta + S} d\theta = \int \frac{[g(y) + 3h(y)v_p]^2}{h(y)} dy.
\]

Now in view of the relation

\[
w = \frac{[g(y) + 3h(y)v_p]}{h(y)e^{\int[f(y)+2g(y)v_p+3h(y)v_p^2]dy}} \theta = (v - v_p)e^{-\int[f(y)+2g(y)v_p+3h(y)v_p^2]dy},
\]
we obtain
\[ v(y) = \frac{g(y) + 3h(y)v_p}{h(y)} \theta(y) + v_p. \] (61)

Next substituting Eq.(25) into Eq.(25), the latter takes the form
\[
\frac{dv}{dy} = k(y) + \left\{ \frac{d^2}{dy^2} \ln \left[ \frac{g(y)}{h(y)} + 3v_p \right] \right\} + \frac{d}{dy} \left\{ \ln \left[ \frac{g(y)}{h(y)} + 3v_p \right] \right\} v + g(y)v^2 + h(y)v^3. \] (62)

Therefore we have obtained the following

**Theorem 3.** a) If a particular solution \( v_p \) of the Abel Eq.(25) is known, then the general solution of the Abel Eq. (62) is given by Eq.(61).

b) The general solution of the quadratic-cubic extended Liénard equation with coefficient \( f(y) \) satisfying the condition (56) is given by
\[
x - x_0 = \int \left[ \frac{g(y) + 3h(y)v_p}{h(y)} \theta(y) + v_p \right] dy, \] (63)

where \( x_0 \) is the arbitrary integration constant.

The function \( \theta(y) \), given by Eq.(61), and which determines the solution of the Abel equation, can be determined from Eq.(59). Subsequently, we have obtained the general solution of the extended quadratic-cubic Liénard Eq.(24) under the assumption that a particular solution of the associated Abel equation is known.

### 3 General Solution of the Extended Liénard Equation for

**Arbitrary \( n, m \) and \( g(y) = h(y) \equiv 0 \)**

We assume now that the coefficients \( g(y) \) and \( h(y) \) of the extended Liénard equation identically vanish, so that \( g(y) = h(y) \equiv 0 \). In this case the extended Liénard equation and its associated Abel equation take the form
\[
g'' + f(y)(y')^n + k(y)(y')^m = 0, \] (64)

and
\[
\frac{dv}{dy} = f(y)v^\alpha + k(y)v^\beta, \quad \alpha = 3 - n, \quad \beta = 3 - m, \] (65)

respectively, where \( v = 1/y' \). Then we can formulate the following **Generalized Chiellini Lemma**

**Lemma 2** (Generalized Chiellini Lemma). If the coefficients \( f(y) \) and \( k(y) \) of the generalized Abel equation satisfy the differential conditions
\[
\frac{d}{dy} \left[ \left( \frac{f(y)}{k(y)} \right)^{\frac{n}{\alpha}} \right] = SP^{\beta-1}f(y)\left[ \frac{f(y)}{k(y)} \right]^{\frac{n}{\alpha}}, \quad \alpha \neq \beta \] (66)
or
\[
\frac{d}{dy} \left[ \left( \frac{f(y)}{k(y)} \right)^{\frac{\beta}{\alpha}} \right] = SP^{\alpha-1}k(y)\left[ \frac{f(y)}{k(y)} \right]^{\frac{\beta}{\alpha}}, \quad \alpha \neq \beta, \] (67)

where \( \alpha, \beta, S \) and \( P \) are arbitrary constants, then the generalized Abel equation is exactly integrable.
Proof. By means of a simple transformation the generalized Abel Eq. (65) can be transformed into the equivalent form
\[ \frac{d \ln v}{dy} = f(y)v^{\alpha-1} + k(y)v^{\beta-1}. \]  
(68)

Let’s assume now the function \( v \) can be represented as
\[ v(y) = F[f(y), k(y)]\theta(y), \]  
(69)
where \( F \) and \( \theta \) are two arbitrary functions. Then we have \( \ln v = \ln F + \ln \theta \), and
\[ \frac{d \ln v}{dy} = \frac{1}{F} \frac{dF}{dy} + \frac{1}{\theta} \frac{d\theta}{dy} = fF^{\alpha-1}\theta^{\alpha-1} + kF^{\beta-1}\theta^{\beta-1}. \]  
(70)

Let’s assume that the condition
\[ \frac{1}{S} \frac{dF}{dy} = P^{\beta-\alpha}fF^{\alpha-1} = kF^{\beta-1}, \]  
(71)
where \( S \) and \( P \) are arbitrary constants, holds for all \( F, f \) and \( k \). Then the above condition determines first the function \( F \) as
\[ F = P \left( \frac{f}{k} \right)^{\frac{\alpha}{\beta - \alpha}}. \]  
(72)

Then it follows that the function \( F \) must satisfy the conditions
\[ \frac{d}{dy} \left[ \left( \frac{f}{k} \right)^{\frac{\alpha}{\beta - \alpha}} \right] = SP^{\beta-1}f \left( \frac{f}{k} \right)^{\frac{\alpha}{\beta - \alpha}}, \]  
(73)
or, equivalently,
\[ \frac{d}{dy} \left[ \left( \frac{f}{k} \right)^{\frac{\alpha}{\beta - \alpha}} \right] = SP^{\beta-1}k \left( \frac{f}{k} \right)^{\frac{\beta}{\beta - \alpha}}. \]  
(74)

Hence Eq. (69) becomes
\[ v = F[f, k]\theta = P \left( \frac{f}{k} \right)^{\frac{\alpha}{\beta - \alpha}} \theta. \]  
(75)

Eq. (70) immediately gives
\[ Sk^{\beta-1} + \frac{1}{\theta} \frac{d\theta}{dy} = P^{\alpha-\beta}kF^{\beta-1}\theta^{\alpha-1} + kF^{\beta-1}\theta^{\beta-1}, \]  
(76)
which leads to the following equation with separable variables for \( \theta \),
\[ \frac{d\theta(y)}{dy} = P^{\beta-1}k(y) \left[ \frac{f(y)}{k(y)} \right]^{\frac{\alpha}{\beta - \alpha}} \left[ P^{\alpha-\beta}\theta^{\alpha}(y) + \theta^{\beta}(y) - S\theta(y) \right]. \]  
(77)

This ends the proof of the Generalized Chiellini Lemma.

By using Lemma 2 we can obtain the general solution of the extended Liénard Eq. (64) by means of the following

**Theorem 4.** If the coefficients \( f(y) \) and \( k(y) \) of the extended Liénard Eq. (64) satisfy the condition
\[ \frac{d}{dy} \left[ \left( \frac{f(y)}{k(y)} \right)^{\frac{n-m}{m}} \right] = SP^{2-m}f(y) \left[ \frac{f(y)}{k(y)} \right]^{\frac{n-m}{m}}, \quad n \neq m \]  
(78)
or
\[ \frac{d}{dy} \left[ \left( \frac{f(y)}{k(y)} \right)^{\frac{n-m}{m}} \right] = SP^{2-m}k(y) \left[ \frac{f(y)}{k(y)} \right]^{\frac{n-m}{m}}, \quad n \neq m, \]  
(79)
where $S$ and $P$ are arbitrary constants, then the general solution of the extended Liénard equation can be obtained as

$$x - x_0 = P \int \left[ \frac{f(y)}{k(y)} \right]^{\frac{n-m}{m}} \theta(y) dy,$$  \hspace{1cm} (80)

where $x_0$ is the arbitrary integration constant and $\theta(y)$ is the solution of the equation

$$\theta(y) = H^{-1} \left\{ P^{2-m} \int k(y) \left[ \frac{f(y)}{k(y)} \right]^{\frac{n-m}{m}} dy \right\}, \hspace{1cm} n \neq m. \hspace{1cm} (81)$$

and we have denoted

$$H(\theta) = \int \frac{d\theta}{P^{m-n} \theta^3 - n + \theta^3 - m - S \theta} = P^{2-m} \int k(y) \left[ \frac{f(y)}{k(y)} \right]^{\frac{n-m}{m}} dy, \hspace{1cm} n \neq m. \hspace{1cm} (82)$$

The proof of Theorem 4 is immediate. \hspace{1cm} $\Box$

### 3.1 The Chiellini Integrability Condition for the Riccati Equation

In the case $\alpha = 0$, $\beta = 2$, the generalized Abel Eq.(65) becomes a reduced Riccati type equation, given by

$$\frac{dv}{dy} = f(y) + k(y)v^2.$$  \hspace{1cm} (83)

Then the generalized Chiellini Lemma allows us to obtain an integrability condition for the Riccati equation, which is given by the following

**Theorem 5.** If the coefficients $f(y)$ and $g(y)$ of the reduced Riccati Eq.(83) satisfy the condition

$$\frac{d}{dy} \left[ \sqrt{\frac{f(y)}{k(y)}} \right] = K f(y),$$  \hspace{1cm} (84)

where $K$ is an arbitrary constant, then the Riccati equation is exactly integrable, and its solution is given by

$$v(y) = \sqrt{\frac{f(y)}{k(y)}} \left\{ \sqrt{1 - \frac{K^2}{4}} \tan \left\{ \sqrt{1 - \frac{K^2}{4}} \left[ \int \sqrt{f(y)k(y)dy + C} \right] \right\} + \frac{K}{2} \right\}, \hspace{1cm} K \neq 2, \hspace{1cm} (85)$$

$$v(y) = \sqrt{\frac{f(y)}{k(y)}} \left[ - \frac{1}{\sqrt{f(y)k(y)dy + C}} \mp 1 \right], \hspace{1cm} K = \mp 2. \hspace{1cm} (86)$$

**Proof.** We look for a solution of the Riccati Eq.(83) of the form

$$v(y) = \sqrt{\frac{f(y)}{k(y)}} \theta(y).$$  \hspace{1cm} (87)

By substituting into the Riccati equation we obtain

$$\frac{d}{dy} \left[ \sqrt{\frac{f(y)}{k(y)}} \right] \theta(y) + \sqrt{\frac{f(y)}{k(y)}} \frac{d\theta(y)}{dy} = f(y) [1 + \theta^2(y)].$$  \hspace{1cm} (88)
With the use of the Condition (84), Eq.(88) becomes a first order separable differential equation, which can be integrated as

\[
\int \frac{d\theta}{\theta^2 - K\theta + 1} = \int \sqrt{f(y)k(y)}dy.
\]  

(89)

Hence we obtain

\[
\theta(y) = \sqrt{1 - \frac{K^2}{4}} \tan \left\{ \sqrt{1 - \frac{K^2}{4}} \left[ \int \sqrt{f(y)k(y)}dy + C \right] \right\} + \frac{K}{2}, \quad K \neq 2, \quad (90)
\]

\[
\theta(y) = -\frac{1}{\sqrt{f(y)k(y)}dy + C} \mp 1, \quad K = \mp 2,
\]  

(91)

where \(C\) is an arbitrary integration constant. Therefore the general solution of the Riccati Eq.(83) with coefficients satisfying a generalized Chiellini type integrability immediately follows from Eq.(87), and this completes the Proof of the Theorem.

\[\Box\]

4 Concluding Remarks

In the present paper, by using the Chiellini integrability condition of the Abel equation we have obtained three exact solutions of the extended quadratic-cubic Liénard equation, and of the associated Abel equation. The solutions have been obtained under the assumption that a particular solution of the Abel equation is known. We have considered the cases in which these particular solutions have a specific form, as well as the general case in which the functional form of \(v_p\) was not imposed in advance. Once the particular solutions of the associated Abel equation are known, the general solution of the nonlinear quadratic-cubic second order extended Liénard differential equation can be obtained by quadratures, if the four coefficients of the equation satisfy some consistency conditions.

We have also presented a generalization of the Chiellini integrability condition for the case of generalized Abel equations of the type (65), which gives the possibility of obtaining some exact solutions of the extended Liénard equation for arbitrary \(n\) and \(m\), and for \(g(y) = h(y) \equiv 0\). As a particular application of the generalized Chiellini integrability condition we have also obtained an integrability case of the reduced Riccati equation.

All the solutions obtained in the present paper require the existence of some differential relations between the coefficients of the differential equations. These constraints impose strong restrictions on the functional form of the solutions, thus limiting the number of possible solutions of the Abel and extended Liénard equations that can be obtained in this way. Some applications of the present results to some extended Liénard equations that describe some physically interesting mathematical models will be presented in a future paper.

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