Hyperbolic complex contact structures on $\mathbb{C}^{2n+1}$

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Abstract In this paper we construct complex contact structures on $\mathbb{C}^{2n+1}$ for any $n \geq 1$ with the property that every holomorphic Legendrian map $\mathbb{C} \rightarrow \mathbb{C}^{2n+1}$ is constant. In particular, these contact structures are not globally contactomorphic to the standard complex contact structure on $\mathbb{C}^{2n+1}$.

Keywords complex contact structures, hyperbolicity, Fatou-Bieberbach domains.

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1. Introduction and main results

Let $M$ be a complex manifold of odd dimension $2n + 1 \geq 3$, where $n \in \mathbb{N} = \{1, 2, \ldots\}$. A holomorphic vector subbundle $\xi \subset TM$ of complex codimension one in the tangent bundle $TM$ is a holomorphic contact structure on $M$ if every point $p \in M$ admits an open neighborhood $U \subset M$ such that $\xi|_U = \ker \alpha$ for a holomorphic 1-form $\alpha$ on $U$ satisfying $\alpha \wedge (d\alpha)^n \neq 0$.

A 1-form $\alpha$ satisfying this nondegeneracy condition is called a holomorphic contact form, and $(M, \xi)$ is a complex contact manifold. We shall also write $(M, \alpha)$ when $\xi = \ker \alpha$ holds on all of $M$. The model is the complex Euclidean space $(\mathbb{C}^{2n+1}, \xi_0 = \ker \alpha_0)$ where $\alpha_0$ is the the standard complex contact form

$$\alpha_0 = dz + \sum_{j=1}^{n} x_j dy_j.$$  

By Darboux’s theorem, every holomorphic contact form equals $\alpha_0$ in suitably chosen local holomorphic coordinates at any given point (see e.g. Geiges [11, Theorem 2.5.1, p. 67] for the smooth case and [1] Theorem A.2 for the holomorphic one). This standard case has recently been considered by Alarcón, López and the author in [1]. They proved in particular that every open Riemann surface $R$ admits a proper holomorphic embedding $f: R \hookrightarrow (\mathbb{C}^{2n+1}, \alpha_0)$ as a Legendrian curve, meaning that $f^{*}\alpha_0 = 0$ holds on $R$. In the same paper, the authors asked whether there exists a holomorphic contact form $\alpha$ on $\mathbb{C}^3$ which is not globally equivalent to the standard form $\alpha_0$ (cf. [1] Problem 1.5, p. 4). In this paper we provide such examples in every dimension.

**Theorem 1.1.** For every $n \in \mathbb{N}$ there exists a holomorphic contact form $\alpha$ on $\mathbb{C}^{2n+1}$ such that any holomorphic map $f: \mathbb{C} \rightarrow \mathbb{C}^{2n+1}$ satisfying $f^{*}\alpha = 0$ is constant. In particular, the complex contact manifold $(\mathbb{C}^{2n+1}, \alpha)$ is not contactomorphic to $(\mathbb{C}^{2n+1}, \alpha_0)$.

Indeed, a contactomorphism sends Legendrian curves to Legendrian curves, and $(\mathbb{C}^{2n+1}, \xi_0)$ admits plenty of embedded Legendrian complex lines $\mathbb{C} \hookrightarrow \mathbb{C}^{2n+1}$. Indeed,
given a point \( p = (x_0, y_0, z_0) \in \mathbb{C}^3 \) and a vector \( \nu = (\nu_1, \nu_2, \nu_3) \in \ker \alpha_0|_p \), the quadratic map \( f: \mathbb{C} \to \mathbb{C}^3 \) given by

\[
f(\xi) = (x_0 + \nu_1 \xi, y_0 + \nu_2 \xi, z_0 + \nu_3 \xi - \nu_1 \nu_2 \xi^2 / 2)
\]

is a holomorphic Legendrian embedding satisfying \( f(0) = p \) and \( f'(0) = \nu \).

We expect that our construction actually gives many nonequivalent holomorphic contact structures on \( \mathbb{C}^{2n+1} \); however, at this time we do not know how to distinguish them. Eliashberg showed that on \( \mathbb{R}^3 \) there exist countably many isotopy classes of smooth contact structures [8, 9]. His classification is based on the study of overtwisted disks in contact 3-manifolds; it is not clear whether a similar invariant could be used in the complex case.

In order to prove Theorem 1.1 we consider the directed Kobayashi metric associated to a contact complex manifold \((M, \xi)\). Let \( \mathbb{D} = \{ \xi \in \mathbb{C} : |\xi| < 1 \} \) denote the open unit disk. Given a holomorphic subbundle \( \xi \subset TM \), we say that a holomorphic disk \( f: \mathbb{D} \to M \) is tangental to \( \xi \) or horizontal if

\[
f'(\xi) \in \xi | f(\xi) \quad \text{holds for all } \xi \in \mathbb{D}.
\]

Consider the function \( \xi \to \mathbb{R}_+ \) given for any point \( p \in M \) and vector \( v \in \xi_p \) by

\[
|v|_\xi = \inf \left\{ \frac{1}{|\lambda|} : \exists f: \mathbb{D} \to M \text{ horizontal, } f(0) = p, f'(0) = \lambda v \right\}.
\]

When \( \xi = TM \), this is the Kobayashi length of the tangent vector \( v \in T_p M \), and its integrated version is the Kobayashi metric on \( M \) (cf. Kobayashi [14, 15]). The directed version of the Kobayashi metric was studied by Demailly [5] and several other authors, mainly on complex projective manifolds. More general metrics, obtained by integrating a Riemannian metric along horizontal curves in a smooth directed manifold \((M, \xi)\), have been studied by Gromov [13] under the name Carnot-Carathéodory metrics. (See also Bellaïche [2].) For this reason, we propose the name Carnot-Carathéodory-Kobayashi metric, or CCK metric, for the pseudodistance function \( d_\xi: M \times M \to \mathbb{R}_+ \) defined by

\[
d_\xi(p, q) = \inf_\gamma \int_0^1 |\gamma'(t)|_\xi dt, \quad p, q \in M,
\]

where the infimum is over all piecewise smooth paths \( \gamma: [0, 1] \to M \) satisfying \( \gamma(0) = p \), \( \gamma(1) = q \) and \( \gamma'(t) \in \xi_{\gamma(t)} \) for all \( t \in [0, 1] \). (By Chow’s theorem [4], a horizontal path connecting any given pair of points in \( M \) exists when the repeated commutators of vector fields tangential to \( \xi \) span the tangent space of \( M \) at every point. A discussion and proof of Chow’s theorem can also be found in Gromov’s paper [13] p. 86 and p. 113. Another source is Sussman [17, 18].)

The directed complex manifold \((M, \xi)\) is said to be (Kobayashi) hyperbolic if \( d_\xi \) given by (1.2) is a distance function on \( M \) (i.e., if \( d_\xi(p, q) > 0 \) holds for all pairs of distinct points \( p, q \in M \)), and is complete hyperbolic if \( d_\xi \) is a complete metric on \( M \). Clearly, the directed Kobayashi metric on \((M, \xi)\) dominates the standard Kobayashi metric on \( M \).

Now, Theorem 1.1 is an obvious corollary to the following result.

**Theorem 1.2.** For every \( n \in \mathbb{N} \) there exists a holomorphic contact form \( \alpha \) on \( \mathbb{C}^{2n+1} \) such that the complex contact manifold \((\mathbb{C}^{2n+1}, \xi = \ker \alpha)\) is Kobayashi hyperbolic.

The contact 1-forms that we shall construct in the proof of Theorem 1.2 are of the form

\[
\alpha = \Phi^* \alpha_0
\]
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where $\alpha_0$ is the standard contact form (1.1) and $\Phi: \mathbb{C}^{2n+1} \to \mathbb{C}^{2n+1}$ is a Fatou-Bieberbach map, i.e., an injective holomorphic map from $\mathbb{C}^{2n+1}$ onto a proper subdomain $\Omega = \Phi(\mathbb{C}^{2n+1}) \subset \mathbb{C}^{2n+1}$ such that $(\Omega, \alpha_0|_\Omega)$ is a hyperbolic contact manifold. Let us describe this construction. Let $C_N > 0$ for $N \in \mathbb{N}$ be a sequence diverging to $+\infty$ and

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b_{(x,y)}D^n \times C_N \mathbb{D}_z.$$

Here, $b_{(x,y)}D^n \subset \mathbb{C}^n$ denotes the boundary of the unit polydisk in the $(x, y)$-space and $\mathbb{D}_z$ is the closed unit disk in the $z$ direction. Thus, $K$ is the union of a sequence of compact cylinders $K_N = 2^{N-1} b_{(x,y)}D^n \times C_N \mathbb{D}_z$ tending to infinity in all directions. Theorem 1.2 follows immediately from the following two results of possible independent interest. In both results, $K$ is the set given by (1.3).

**Proposition 1.3.** If $C_N \geq 2^{3N+1}$ holds for all $N \in \mathbb{N}$ then the domain $\Omega_0 = \mathbb{C}^{2n+1} \setminus K$ is $\alpha_0$-hyperbolic. (Here, $\alpha_0$ is the contact form (1.1).)

**Proposition 1.4.** For every choice of constants $C_N > 0$ there exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^{2n+1} \setminus K$.

Indeed, if a domain $\Omega_0 \subset \mathbb{C}^{2n+1}$ is $\alpha_0$-hyperbolic then so is any subdomain $\Omega \subset \Omega_0$. Furthermore, a biholomorphic map $\Phi: \mathbb{C}^{2n+1} \to \Omega$ is an isometry in the directed Kobayashi metric from the contact manifold $(\mathbb{C}^{2n+1}, \alpha)$ with $\alpha = \Phi^* \alpha_0$ onto the contact manifold $(\Omega, \alpha_0)$. Since $(\Omega, \alpha_0)$ is hyperbolic by Proposition 1.3, Theorem 1.2 follows.

Proposition 1.3 is proved in Section 2, the proof uses Cauchy estimates and the explicit expression (1.1) for the standard contact form $\alpha_0$. The set $K$ given by (1.3) presents obstacles which impose a limitation on the size of holomorphic $\alpha_0$-Legendrian disks.

Proposition 1.4 is a special case of Theorem 5.1 which provides a more general result concerning the possibility of avoiding certain unions of cylinders in $\mathbb{C}^n$ by Fatou-Bieberbach domains. Its proof is inspired by a result of Globevnik [12, Theorem 1.1] who constructed Fatou-Bieberbach domains in $\mathbb{C}^n$ whose intersection with a ball $R \mathbb{B}^n$ for a given $R > 0$ is approximately equal to the intersection of the cylinder $\mathbb{D}^{n-1} \times \mathbb{C}$ with the same ball. His result implies that one can avoid any cylinder $K_N$ in the set $K$ (1.3) by a Fatou-Bieberbach domain $\Omega$. We shall improve the construction so that $\Omega$ avoids all cylinders $K_N$ at the same time. For this purpose we will use a sequence of holomorphic automorphisms $\Theta_k \in \text{Aut}(\mathbb{C}^n)$ such that the sequence of their compositions $\Theta_k = \theta_k \circ \cdots \circ \theta_1$ converges on a certain domain $\Omega$ and diverges to infinity on the set $K$; hence $K \cap \Omega = \emptyset$. We ensure in addition that each $\theta_k$ approximates the identity map on the polydisk $k \mathbb{D}^{n}$, and hence the limit $\Theta = \lim_{k \to \infty} \Theta_k: \Omega \to \mathbb{C}^{2n+1}$ is a biholomorphic map of $\Omega$ onto $\mathbb{C}^{2n+1}$.

Several interesting questions remain open. One is whether there exists a complete hyperbolic complex contact structure on $\mathbb{C}^{2n+1}$. Another is whether there exist algebraic contact forms $\alpha$ on $\mathbb{C}^{2n+1}$ (i.e., with polynomial coefficients) such that $(\mathbb{C}^{2n+1}, \alpha)$ is hyperbolic. (Our construction only furnishes transcendental examples.) If so, what is the minimal degree of such examples, and for which degrees is a generic (or very generic) contact form hyperbolic? In the integrable case, for affine algebraic and projective manifolds, this is the famous Kobayashi Conjecture; see Demailly [6], Brotbek [3] and Deng [7] for recent results on this subject.
Perhaps the most ambitious question is to classify complex contact structures on Euclidean spaces up to isotopy, in the spirit of Elishashberg’s classification \[8, 9\] of smooth contact structures on \(\mathbb{R}^3\).

Holomorphic contact structures on compact complex manifolds \(M = M^{2n+1}\) seem much better understood than those on open manifolds; see for example the paper by LeBrun \[16\] and the references therein. In particular, the space of all holomorphic contact subbundles of \(TM\), if nonempty, is a connected complex manifold \[16\] p. 422]. Furthermore, if \(M\) is simply connected then any two holomorphic contact structures on \(M\) are equivalent via some holomorphic automorphism of \(M\) \[16, Proposition 2.3\]. In particular, the only complex contact structure on the projective space \(\mathbb{C}P^{2n+1}\) (up to projective linear automorphisms) is the standard one, given in homogeneous coordinates by the 1-form \(\theta = \sum_{j=0}^{n}(z_j d\bar{z}_{n+j+1} - z_{n+j+1}dz_j)\). This structure is obtained by contracting the holomorphic symplectic form \(\omega = \sum_{j=0}^{n} dz_j \wedge d\bar{z}_{n+j+1}\) on \(\mathbb{C}P^{2n+2}\) with the radial vector field \(\sum_{k=0}^{2n+1} z_k \frac{\partial}{\partial z_k}\). Its restriction to any affine chart \(\mathbb{C}P^{2n+1} \subset \mathbb{C}P^{2n+1}\) is equivalent to the standard contact structure given by (1.1). It follows that the projective space \(\mathbb{C}P^{2n+1}\) does not carry any hyperbolic complex contact structures.

### 2. Hyperbolic contact structures on domains in \(\mathbb{C}^{2n+1}\)

In this section we prove Proposition \[1.3\]. For simplicity of notation we consider the case \(n = 1\); the same proof applies in every dimension.

Thus, let \((x, y, z)\) be complex coordinates on \(\mathbb{C}^3\) and \(\alpha_0 = dz + xdy\) be the standard contact form \[1.1\] on \(\mathbb{C}^3\). Recall that \(\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}\) and \(\overline{\mathbb{D}} = \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}\). The definition of the directed Kobayashi metric shows that Proposition \[1.3\] is an immediate corollary to the following lemma.

**Lemma 2.1.** Assume that \(C_N \geq 2^{3N+1}\) for every \(N \in \mathbb{N}\) and let

\[
K = \bigcup_{N=1}^{\infty} 2^{N-1}b\mathbb{D}^2(x,0) \times C_N \overline{\mathbb{D}}.
\]

For every holomorphic \(\alpha_0\)-horizontal disk \(f(\zeta) = (x(\zeta), y(\zeta), z(\zeta)) \in \mathbb{C}^3 \setminus K (\zeta \in \mathbb{D})\) with \(f(0) \in 2N_0 \mathbb{D}^3\) for some \(N_0 \in \mathbb{N}\) we have the estimates

\[
|\z'(0)| < 2^{N_0+1}, \quad |y'(0)| < 2^{N_0+1}, \quad |z'(0)| < 2^{2N_0+1}.
\]

**Proof.** Replacing \(f\) by the disk \(g(\zeta) \rightarrow f(r\zeta)\) for some \(r < 1\) close to 1 we may assume that \(f\) is holomorphic on \(\mathbb{D}\). Pick a number \(N \in \mathbb{N}\) with \(N > N_0\) such that \(|x(\zeta)| < 2^N\) and \(|y(\zeta)| < 2^N\) for all \(\zeta \in \mathbb{D}\). By the Cauchy estimates applied with \(\delta = 2^{-N}\) we then have

\[
|\z'(\zeta)| < 2^{2N} \quad \text{and} \quad |x(\zeta)y'(\zeta)| < 2^{3N} \quad \text{for} \quad |\zeta| \leq 1 - 2^{-N}.
\]

Since \(f\) is a horizontal disk, we have \(z'(\zeta) = -x(\zeta)y'(\zeta)\) for \(\zeta \in \mathbb{D}\) and hence

\[
|z(\zeta)| \leq |z(0)| + \int_0^\zeta xdy < 2^{N_0} + 2^{3N} < 2^{3N+1} \leq C_N \quad \text{for} \quad |\zeta| \leq 1 - 2^{-N}.
\]

From this estimate, the definition of the set \(K\) and the fact that \(f(\mathbb{D}) \cap K = \emptyset\) it follows that

\[
(x(\zeta), y(\zeta)) \notin 2^{N-1}b\mathbb{D}^2 \quad \text{for} \quad |\zeta| \leq 1 - 2^{-N}.
\]
Since $2^{-N+1}b\mathbb{D}^2$ disconnects the bisk $2^N\mathbb{D}^2$ and we have $(x(0), y(0)) \in 2^{N_0}\mathbb{D}^2 \subset 2^{-N+1}\mathbb{D}^2$, we conclude that

$$(x(\zeta), y(\zeta)) \in 2^{N-1}\mathbb{D}^2 \quad \text{for } |\zeta| \leq 1 - 2^{-N}.$$  

If $N - 1 > N_0$, we can repeat the same argument with the restricted horizontal disk $f: (1 - 2^{-N})\overline{\mathbb{D}} \to \mathbb{C}^3$ to obtain

$$(x(\zeta), y(\zeta)) \in 2^{N_0}\mathbb{D}^2 \quad \text{for } |\zeta| \leq 1 - 2^{-N} - 2^{-(N_0+1)}.$$  

After finitely steps of the same kind we get that

$$(x(\zeta), y(\zeta)) \in 2^{N_0}\mathbb{D}^2 \quad \text{for } |\zeta| \leq 1 - 2^{-N} - \ldots - 2^{-(N_0+1)}.$$  

Since $2^{-N} + \ldots + 2^{-(N_0+1)} < 1/2$, we see that $(x(\zeta), y(\zeta)) \in 2^{N_0}\mathbb{D}^2$ for $|\zeta| \leq 1/2$. Applying once again the Cauchy estimates gives $|x'(0)|, |y'(0)| \leq 2^{N_0+1}$ and hence $|z'(0)| = |x(0)y'(0)| \leq 2^{N_0+1}$; these are precisely the conditions in (2.1). □

### 3. Fatou-Bieberbach domains avoiding a union of cylinders

In this section we prove the following result on avoiding certain closed cylindrical sets in $\mathbb{C}^n$ by Fatou-Bieberbach domains. This includes Proposition 1.4 as a special case.

**Theorem 3.1.** Let $0 < a_1 < b_1 < a_2 < b_2 < \ldots$ and $c_1 > 0$ be sequences of real numbers such that $\lim_{i \to \infty} a_i = \lim_{i \to \infty} b_i = +\infty$. Let $n > 1$ be an integer and

$$K = \bigcup_{i=1}^{\infty} \left( b_i \overline{\mathbb{D}}^{n-1} \setminus a_i \mathbb{D}^{n-1} \right) \times c_i \overline{\mathbb{D}} \subset \mathbb{C}^n.$$  

Then there exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^n \setminus K$.

As said in the Introduction, the proof is inspired by [12] proof of Theorem 1.2] to a certain point and is based on the so called push-out method. Since the set $K$ (3.1) is noncompact, the construction of automorphisms used in the proof is somewhat more involved in our case. On the other hand, since our goal is merely to avoid $K$ by a Fatou-Bieberbach domain, and not to approximate a given cylinder as Globevnik did in [12], the construction is less precise in certain other aspects.

**Proof.** We denote by $\text{Aut}(\mathbb{C}^n)$ the group of all holomorphic automorphisms of $\mathbb{C}^n$. We first give the proof for $n = 2$ and explain in the end how to treat the general case.

Let $(z_1, z_2)$ be complex coordinates on $\mathbb{C}^2$, and let $K = K_1$ be the set (3.1). Up to a dilation of coordinates, we may assume without loss of generality that $a_1 > 1$.

Pick sequence $\epsilon_k \in (0, 1)$ satisfying $\sum_{k=1}^{\infty} \epsilon_k < +\infty$. We shall construct sequences of automorphisms $\phi_k, \psi_k \in \text{Aut}(\mathbb{C}^2)$ ($k \in \mathbb{N}$) of the following form:

$$\phi_k(z_1, z_2) = (z_1, z_2 + f_k(z_1)), \quad \psi_k(z_1, z_2) = (z_1 + g_k(z_2), z_2),$$

where $f_k$ and $g_k$ are suitably chosen entire functions on $\mathbb{C}$ to be specified. Set

$$\theta_k = \psi_k \circ \phi_k, \quad \Theta_k = \theta_k \circ \cdots \circ \theta_1, \quad k \in \mathbb{N}.$$  

We will also ensure that for every $k \in \mathbb{N}$ we have

$$|\theta_k(z) - z| < \epsilon_k \quad \text{for } z \in k\overline{\mathbb{D}}^2.$$  

Granted the last condition, it follows (cf. [10 Proposition 4.4.1 and Corollary 4.4.2]) that the sequence \( \Theta_k \in \text{Aut}(\mathbb{C}^2) \) converges uniformly on compacts in the open set

\[
\Omega = \bigcup_{k=1}^{\infty} \Theta_k^{-1}(k \mathbb{D}^2) = \{ z \in \mathbb{C}^2 : (\Theta_k(z))_{k \in \mathbb{N}} \text{ is a bounded sequence} \}
\]
to a biholomorphic map \( \Theta = \lim_{k \to \infty} \Theta_k : \Omega \to \mathbb{C}^2 \) of \( \Omega \) onto \( \mathbb{C}^2 \). We will also ensure that

\[
|\Theta_k(z)| \to +\infty \quad \text{for all points } z \in K,
\]
and hence \( K \cap \Omega = \emptyset \). This will prove the theorem when \( n = 2 \).

We begin by explaining how to choose the first two maps \( \phi_1 \) and \( \psi_1 \); all subsequent steps will be analogous. Set \( b_0 = 1 \). Pick a sequence \( r_j \) satisfying \( b_{j-1} < r_j < a_j \) for all \( j = 1, 2, \ldots \). Let \( N_j \in \mathbb{N} \) be a sequence of integers to be specified later. Set

\[
f(\zeta) = \sum_{j=1}^{\infty} \left( \frac{\zeta}{r_j} \right)^{N_j}.
\]

This function will define the first automorphism \( \phi_1 \) (cf. (3.2)). Let \( f_i(\zeta) = \sum_{j=1}^{i} \left( \frac{\zeta}{r_j} \right)^{N_j} \) denote the \( i \)-th partial sum of the series defining \( f(\zeta) \), where we set \( f_0 = 0 \). By choosing the exponent \( N_i \) big enough, we can ensure that the summand \( (\zeta/r_j)^{N_i} \) is arbitrarily small on the disk \( b_{i-1} \mathbb{D} \) and is arbitrarily big on the annulus

\[
A_i := b_i \mathbb{D} \setminus a_i \mathbb{D} = \{ \zeta : a_i \leq |\zeta| \leq b_i \}.
\]

In particular, we may ensure that for every \( i \in \mathbb{N} \) we have

\[
\sup_{|\zeta| \leq b_{i-1}} \left| \frac{\zeta}{r_1} \right|^{N_i} < 2^{-i-1} \epsilon_1.
\]

It follows that the power series defining \( f(\zeta) \) converges on all of \( \mathbb{C} \) and satisfies

\[
\sup_{|\zeta| \leq b_{i-1}} |f(\zeta) - f_{i-1}(\zeta)| < 2^{-i} \epsilon_1, \quad i \in \mathbb{N}.
\]

Note that the inequalities (3.6) and (3.7) persist if we increase the exponents \( N_i \). We can inductively choose the sequence \( N_i \in \mathbb{N} \) to grow fast enough such that the following inequalities hold for every \( i \in \mathbb{N} \) with an increasing sequence of numbers \( M_i \geq i + 1 \):

\[
\sup_{|\zeta| \leq b_{i-1}} |f_{i-1}(\zeta)| + c_{i-1} + \epsilon_1 < M_i < \inf_{\zeta \in A_i} \left( \left| \frac{\zeta}{r_1} \right|^{N_i} - |f_{i-1}(\zeta)| \right) - c_i - \epsilon_1.
\]

(Recall that \( A_i \) is the annulus (3.5). Here, \( c_0 \geq 0 \) is arbitrary while \( c_i > 0 \) for \( i \in \mathbb{N} \) are the constants in the definition (3.3) of the set \( K \).) In view of the inequalities (3.6), (3.7) and (3.8) there exist numbers \( \beta_{i-1} < \alpha_i \) such that for all \( i \in \mathbb{N} \) we have

\[
\sup_{|\zeta| \leq b_{i-1}} |f(\zeta)| + c_{i-1} < \beta_{i-1} < M_i < \alpha_i < \inf_{\zeta \in A_i} |f(\zeta)| - c_i.
\]

This gives increasing sequences \( 0 < \beta_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots \) diverging to \( \infty \). Set

\[
\phi_1(z_1, z_2) = (z_1, z_2 + f(z_1)).
\]

The right hand side of (3.9) shows that for every point \( z = (z_1, z_2) \in A_i \times c_i \mathbb{D} \) we have

\[
|z_2 + f(z_1)| \geq |f(z_1)| - c_i > \alpha_i,
\]

while the left hand side of (3.9) gives

\[
|z_2 + f(z_1)| \leq c_i + |f(z_1)| < \beta_i.
\]
Since these inequalities hold for every $i \in \mathbb{N}$, it follows that
\[ \phi_1(K) \subset L := \bigcup_{i=1}^{\infty} b_i D \times (\beta_i D \setminus \alpha_i D) \subset \mathbb{C}^2. \]

Note that the set $L$ is of the same kind as $K$ (3.11) with the reversed roles of the variables, i.e., the cylinders in $L$ are horizontal instead of vertical. Furthermore, since the sequence $\alpha_i$ is increasing and $\alpha_1 > M_1 \geq 2$ by (3.9), we also see that
\[ L \cap (\mathbb{C} \times 2 \overline{D}) = \emptyset. \]

The same argument as above with the set $L$ furnishes a shear automorphism
\[ \psi_1(z_1, z_2) = (z_1 + g(z_2), z_2) \]
for some $g \in \mathcal{O}(\mathbb{C})$ (cf. (3.2)) and a set $K_2$ of the same kind as $K = K_1$ (3.11) (this time again with vertical cylinders) such that, setting $\theta_1 := \psi_1 \circ \phi_1 \in \text{Aut}(\mathbb{C}^2)$, we have
\[ \theta_1(K_1) \subset K_2, \quad K_2 \cap 2 \overline{D}^2 = \emptyset, \quad \sup_{z \in 2 \overline{D}^2} |\theta_1(z) - z| < \epsilon_1. \]

Continuing inductively, we find a sequence of automorphisms $\theta_k \in \text{Aut}(\mathbb{C}^2)$ and of closed sets $K_k \subset \mathbb{C}^2$ of the form (3.1) such that for every $k \in \mathbb{N}$ we have
\[ \theta_k(K_k) \subset K_{k+1}, \quad K_k \cap k \overline{D}^2 = \emptyset, \quad \sup_{z \in k \overline{D}^2} |\theta_k(z) - z| < \epsilon_k. \]

Each step of the recursion is of exactly the same kind as the initial one. This implies that
\[ \Theta_k(K) \subset K_{k+1} \subset \mathbb{C}^2 \setminus (k+1)\overline{D}^2, \quad k \in \mathbb{N} \]
and hence (3.4) also holds. This completes the proof when $n = 2$.

Suppose now that $n > 2$. In this case, each automorphism $\theta_k = \psi_k \circ \phi_k \in \text{Aut}(\mathbb{C}^n)$ in the sequence (3.3) is a composition of two shear-like maps of the form
\[ \phi_k(z_1, z_2, \ldots, z_n) = \left( z_1 + f_k(z_1), z_2 + f_k(z_2), \ldots, z_n + f_k(z_{n-1}) \right), \]
\[ \psi_k(z_1, z_2, \ldots, z_n) = \left( z_1 + g_k(z_2), z_2 + g_k(z_3), \ldots, z_{n-1} + g_k(z_n), z_n \right). \]

A suitable choice of entire functions $f_k, g_k \in \mathcal{O}(\mathbb{C})$ ensures as before that condition (3.11) holds for each $k$ (with $\mathbb{D}^2$ replaced by $\overline{D}^n$). We leave the details to an interested reader. Further details in the case $n > 2$ are also available in [12, proof of Theorem 1.2]. □

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