Small–Angle Electron–Positron Scattering with a Per Mille Accuracy

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Abstract

The elastic and inelastic high–energy small–angle electron–positron scattering is considered. All radiative corrections to the cross–section with the relative accuracy $\delta\sigma/\sigma = 0.1\%$ are explicitly taken into account. According to the generalized eikonal representation for the elastic amplitude, in higher orders only diagrams with one exchanged photon may be considered. Single photon emission with radiative corrections and next–to–leading two–photon and pair production diagrams are evaluated, together with leading three–loop corrections. All contributions have been calculated analytically. We integrate the calculated distributions over typical for LEP 1 experiments intervals of angles and energies. To the leading approximation, the results are shown to be described in terms of kernels of electron structure functions. Some numerical results are presented.

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1 Introduction

An accurate verification of the Standard Model is one of the primary aims of LEP [1]. While electroweak radiative corrections to the $s$-channel annihilation process and to large–angle Bhabha scattering allow a direct extraction of the Standard Model parameters, small angle Bhabha cross–section affects, as an overall normalization condition, all observable cross–sections and represents an equally unavoidable condition toward a precise determination of the Standard Model parameters. The small–angle Bhabha scattering process is used to measure the luminosity of electron–positron colliders. At LEP an experimental accuracy on the luminosity of

$$|\frac{\delta \sigma}{\sigma}| < 0.001$$

has been reached [2]. However, to obtain the total accuracy, a systematic theoretical error must also be added. This precision calls for an equally accurate theoretical expression for the Bhabha scattering cross–section in order to extract the Standard Model parameters from the observed distributions. An accurate determination of the small–angle Bhabha cross–section and of the luminosity directly affects the determination of absolute cross–sections such as, for example, the determination of the invisible width and of the number of massless neutrino species $N_\nu$ [3].

In recent years a considerable attention has been devoted to the Bhabha process [4, 5, 6]. The reached accuracy is however still inadequate [2]. According to these evaluations the theoretical estimates are still incomplete; moreover, they are somewhat larger ($\sim$ a factor 2) than the projected theoretical and experimental precision [2] and are comparable to the currently published experimental precision.

The process that will be considered in this work is that of Bhabha scattering when electrons and positrons are emitted at small angles with respect to the initial electron and positron directions. We have examined the radiative processes inclusively accompanying the main $e^+e^-\rightarrow e^+e^-$ reaction at high energies, when both the scattered electron and positron are tagged within the counter aperture.

We assume that the center-of-mass energies are within the range of the LEP collider $2\epsilon = \sqrt{s} = 90 – 200$ GeV and the scattering angles are within the range $\theta \simeq 10 – 150$ mrad. We assume that the charged-particle detectors have the following polar angle cuts:

$$\theta_1 < \theta_- = p_1q_1 \equiv \theta < \theta_3, \quad \theta_2 < \theta_+ = p_2q_2 < \theta_4, \quad 0.01 \lesssim \theta_i \lesssim 0.1 \text{ rad},$$

where $p_1, q_1$ ($p_2, q_2$) are the momenta of the initial and of the scattered electron (positron) in the center-of-mass frame.

In this paper we present the results of our calculations of the electron–positron scattering cross–section with an accuracy of $\mathcal{O}(0.1\%)$. The squared matrix elements of the various exclusive processes inclusively contributing to the $e^+e^-\rightarrow e^+e^-$ reaction are integrated in order to define an experimentally measurable cross–section according to suitable restrictions on the angles and energies of the detected particles. The different contributions to the electron and positron distributions, needed for the required accuracy, are presented using analytical expressions.

In order to define the angular range of interest and the implications on the required accuracy, let us first briefly discuss, in a general way, the angle-dependent corrections to the cross–section.
We consider $e^+e^-$ scattering at angles as defined in Eq. (2). Within this region, if one expresses the cross-section by means of a series expansion in terms of angles, the main contribution to the cross-section $d\sigma/d\theta^2$ comes from the diagrams for the scattering amplitudes containing one exchanged photon in the $t$-channel. These diagrams, as well known, show a singularity of the type $\theta^{-4}$ for $\theta \to 0$, e.g.

$$\frac{d\sigma}{d\theta^2} \sim \theta^{-4}. $$

Let us now estimate the correction of order $\theta^2$ to this contribution. If

$$\frac{d\sigma}{d\theta^2} \sim \theta^{-4}(1 + c_1\theta^2),$$

then, after integration over $\theta^2$ in the angular range of Eq. (2), we obtain:

$$\int_{\theta^2_{\text{min}}}^{\theta^2_{\text{max}}} \frac{d\sigma}{d\theta^2} d\theta^2 \sim \theta^{-2}(1 + c_1\theta^2_{\text{min}} \ln \frac{\theta^2_{\text{max}}}{\theta^2_{\text{min}}}).$$

We see that, for $\theta^2_{\text{min}} = 50$ mrad and $\theta^2_{\text{max}} = 150$ mrad (we have taken the case where the $\theta^2$ corrections are maximal), the relative contribution of the $\theta^2$ terms is about $5 \times 10^{-3}c_1$. Therefore, the terms of relative order $\theta^2$ must be kept only in the Born cross-section where the coefficient $c_1$ is not small. In higher orders of the perturbative expansion the coefficient $c_1$ contains at least one factor $\alpha/\pi$ and therefore these terms can be safely omitted. This implies that, within our accuracy, only radiative corrections from the scattering-type diagrams contribute $[7]$. Furthermore only diagrams with one photon exchanged in the $t$-channel should be taken into account according to the generalized eikonal representation see Eq.(16) below.

Having as a final goal for the experimental cross-section the relative accuracy of Eq. (1), and taking into account that the minimal value of the squared momentum transfer $Q^2 = 2\epsilon^2(1 - \cos \theta)$ in the region defined in Eq. (2) is of the order of 1 GeV$^2$, we may omit in the following also the terms appearing in the radiative corrections of the type $m^2/Q^2$, with $m$ equal to the electron mass.

The contents of this paper can be outlined as follows. In Section 2 we discuss the Born cross-section $d\sigma^B$ by taking the $Z^0$-boson exchange into account and we compute the corrections to it due to the virtual and real soft-photon emission. We define also an experimentally measurable cross-section $\sigma_\text{exp}$ with the experimental cuts on angles and energies taken into account and we discuss how to obtain it from the differential distributions. We present the results, as discussed above, in the form of an expansion in terms of the scattering angle $\theta$. The ratio $\Sigma = \sigma_\text{exp}/\sigma_0$ is introduced by normalizing $\sigma_\text{exp}$ with respect to the cross-section $\sigma_0 = 4\pi\alpha^2/\epsilon^2\theta^2$. In Section 3, by using a simplified, but still suitable to obtain the required accuracy, version of the differential cross-section for the small-angle scattering, we discuss the contribution to $\sigma_\text{exp}$ from the single bremsstrahlung process. The details of the Sudakov
technique which we use to calculate the hard-photon emission are given in Appendix A. In Section 4 we find all corrections of \( \mathcal{O}(\alpha^2) \) to \( \sigma_{\text{exp}} \) caused by virtual and real photons emission as well as pair production. In Section 5 we consider the virtual and soft-photon emission accompanying the single photon bremsstrahlung process. The details of this derivation are given in Appendices B and C. In Section 6 we consider the double hard-photon emission process in both the same-side and opposite-side cases. Details are given in Appendix D. In Section 7 we consider the hard pair production process in both the collinear and semi-collinear kinematical region. The details of this derivation are given in Appendix F. In Appendix D the expressions for the leading logarithmic approximation in terms of structure functions factorization are given. The details of the cancellation of the \( \Delta \)-dependence are presented in Appendix E, \( \Delta \) is a small auxiliary parameter \((\Delta \ll 1)\) which separate the processes of soft and hard real photon emission (see eq. (15)). In Section 8 the expressions to leading logarithmic \( \mathcal{O}(\alpha^3) \) for the \( e^+e^- \) and \( e^+e^-\gamma \) radiative processes are obtained. In Section 9, finally, estimates of the neglected terms together with numerical results are presented.

A less detailed derivation of these results has been reported elsewhere [8].

2 Born cross-section and one-loop virtual and soft corrections

The Born cross-section for Bhabha scattering within the Standard Model is well known [6]:

\[
\frac{d\sigma^B}{d\Omega} = \frac{\alpha^2}{8s} \{ 4B_1 + (1-c)^2B_2 + (1+c)^2B_3 \},
\]

where

\[
B_1 = \left( \frac{s}{t} \right)^2 \left[ 1 + (g_v - g_a)^2 \right], \quad B_2 = \left[ 1 + (g_v - g_a)^2 \right],
\]

\[
B_3 = \frac{1}{2} \left[ 1 + \frac{s}{t} + (g_v + g_a)^2 \left( \frac{s}{t} \xi + \chi \right) \right]^2 + \frac{1}{2} \left[ 1 + \frac{s}{t} + (g_v - g_a)^2 \left( \frac{s}{t} \xi + \chi \right) \right]^2,
\]

\[
\chi = \frac{\Lambda s}{s - m_Z^2 + i M_Z \Gamma_Z}, \quad \xi = \frac{\Lambda t}{t - M_Z^2}, \quad \Lambda = \frac{G_F M_Z^2}{2\sqrt{2}\pi\alpha}, \quad g_v = -\frac{1}{2}, \quad g_a = -\frac{1}{2}(1 - 4\sin^2 \theta_w),
\]

\[
s = (p_1 + p_2)^2 = 4\varepsilon^2, \quad t = -Q^2 = (p_1 - q_1)^2 = -\frac{1}{2} \left[ 1 + \right] s (1-c), \quad c = \cos \theta, \quad \theta = \hat{p}_1 \cdot \hat{q}_1.
\]

Here \( \theta_w \) is the Weinberg angle. In the small–angle limit \((c = 1 - \theta^2/2 + \theta^4/24 + \ldots)\), expanding formula (5) leads to

\[
\frac{d\sigma^B}{\theta d\theta} \approx \frac{8\pi\alpha^2}{\varepsilon^2\theta^4} \left( 1 - \frac{\theta^2}{2} + \frac{9\theta^4}{40} + \delta_{\text{weak}} \right),
\]

3
where $\varepsilon = \sqrt{s}/2$ is the electron or positron initial energy and the weak correction term $\delta_{\text{weak}}$, connected with diagrams with $Z^0$-boson exchange, is given by the expression:

$$
\delta_{\text{weak}} = 2g_v^2\xi - \frac{\theta^2}{4}(g_v^2 + g_a^2)\text{Re}\chi + \frac{\theta^4}{32}(g_v^4 + g_a^4 + 6g_v^2g_a^2)|\chi|^2.
$$

(7)

One can see from Eq. (7) that the contribution $c_w$ of the weak correction $\delta_{\text{weak}}$ into the coefficient $c_1$ introduced in Eq. (3)

$$
c_w \simeq 2g_v^2 + \left(\frac{g_v^2 + g_a^2}{4}\Gamma_Z\right) + \theta^2 {\text{max}} \frac{(g_v^4 + g_a^4 + 6g_v^2g_a^2)}{32} \Gamma_Z^2 \simeq 1.
$$

(8)

The contribution connected with $Z^0$-boson exchange diagrams does not exceed 0.3% for typical energy and angle ranges. Radiative corrections to $Z^0 - \gamma$ interference contributions were considered in detail in papers by W. Beenakker and B. Pietrzyk [4]. They are not small and should be taken into account in an analysis of experimental data. We will not touch the subject in this publication.

In the pure QED case one-loop radiative corrections to Bhabha cross-section were calculated a long time ago [3]. Taking into account a contribution from soft-photon emission with energy less than a given finite threshold $\Delta\varepsilon$, we have here for the cross-section $d\sigma^{(1)}_{\text{QED}}$, in the one-loop approximation:

$$
\frac{d\sigma^{(1)}_{\text{QED}}}{dc} = \frac{d\sigma^B_{\text{QED}}}{dc} (1 + \delta_{\text{virt}} + \delta_{\text{soft}}),
$$

(9)

where $d\sigma^B_{\text{QED}}$ is the Born cross-section in the pure QED case (it is equal to $d\sigma^B$ with $g_a = g_v = 0$) and

$$
\delta_{\text{virt}} + \delta_{\text{soft}} = 2\frac{\alpha}{\pi} \left[ 2 \left( 1 - \ln \frac{4\varepsilon^2}{m^2} + 2 \ln(\cot \frac{\theta}{2}) \right) \ln \frac{\varepsilon}{\Delta\varepsilon} + \frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} \int \frac{dx}{x} \ln(1 - x)
\right]
$$

$$
- \frac{23}{9} + \frac{11}{6} \ln \frac{4\varepsilon^2}{m^2}
+ \frac{\alpha}{\pi} \frac{1}{(3 + c^2)^2} \left[ \frac{\pi^2}{3} (2c^4 - 3c^3 - 15c)
\right]
$$

$$
+ 2 (2c^4 - 3c^3 + 9c^2 + 3c + 21) \ln^2(\sin \frac{\theta}{2})
$$

$$
- 4 (c^4 + c^2 - 2c) \ln^2 \cos \frac{\theta}{2} - 4 (c^3 + 4c^2 + 5c + 6) \ln(\tan \frac{\theta}{2})
$$

$$
+ \frac{2}{3} (11c^3 + 33c^2 + 21c + 111) \ln(\sin \frac{\theta}{2}) + 2 (c^3 - 3c^2 + 7c - 5) \ln(\cos \frac{\theta}{2})
$$

$$
+ 2 (c^3 + 3c^2 + 3c + 9) \delta_t - 2 (c^3 + 3c)(1 - c) \delta_s
\right].
$$

The value $\delta_t (\delta_s)$ is defined by contributions to the photon vacuum polarization function $\Pi(t)$ ($\Pi(s)$) as follows:

$$
\Pi(t) = \frac{\alpha}{\pi} \left( \delta_t + \frac{1}{3}L - \frac{5}{9} \right) + \frac{1}{4} \left( \frac{\alpha}{\pi} \right)^2 L,
$$

(10)
where
\[ L = \ln \frac{Q^2}{m^2}, \quad Q^2 = -t = 2\varepsilon^2(1 - c), \] (11)
and we took into account the leading part of the two-loop contribution in the polarization operator. In the Standard Model, \(\delta_t\) contains contributions of muons, tau-leptons, W-bosons and hadrons:
\[ \delta_t = \delta_t^\mu + \delta_t^\tau + \delta_t^W + \delta_t^H, \quad \delta_s = \delta_t (Q^2 \to -s), \] (12)
the first three contributions are theoretically calculable and can be given as:
\[ \delta_t^\mu = \frac{1}{3} \ln \frac{Q^2}{m^2} - \frac{5}{9}, \]
\[ \delta_t^\tau = \frac{1}{2} v_\tau \left(1 - \frac{1}{3} v_\tau^2\right) \ln \frac{v_\tau + 1}{v_\tau - 1} + \frac{1}{3} v_\tau^2 - \frac{8}{9}, \quad v_\tau = \sqrt{1 + \frac{4m_\tau^2}{Q^2}}, \] (13)
\[ \delta_t^W = \frac{1}{4} v_W \left(v_W^2 - 4\right) \ln \frac{v_W + 1}{v_W - 1} - \frac{1}{2} v_W^3 + \frac{11}{6}, \quad v_W = \sqrt{1 + \frac{4M_W^2}{Q^2}}. \]
The contribution of hadrons cannot be calculated theoretically; instead, it can be given as integration of the experimentally measurable cross-section:
\[ \delta_t^H = \frac{Q^2}{4\pi\alpha^2} \int_{4m^2_s}^{+\infty} \frac{\sigma_{e^+e^\to h}(x)}{x + Q^2} \, dx. \] (14)

For numerical calculations we will use for \(\Pi(t)\) the results of Ref. [10].

In the small scattering angle limit we can present (1) in the following form:
\[ \frac{d\sigma^{(1)}_{QED}}{dc} = \frac{d\sigma^{B}_{QED}}{dc} (1 - \Pi(t))^{-2} (1 + \delta), \] (15)
\[ \delta = 2\frac{\alpha}{\pi} \left[2(1 - L) \ln \frac{1}{\Delta} + \frac{3}{2} L - 2\right] + \frac{\alpha}{\pi} \theta^2 \Delta_\theta + \frac{\alpha}{\pi} \theta^2 \ln \Delta, \]
\[ \Delta_\theta = \frac{3}{16} l^2 + \frac{7}{12} l - \frac{19}{18} + \frac{1}{4} (\delta_t - \delta_s), \]
\[ \Delta = \frac{\Delta_\varepsilon}{\varepsilon}, \quad l = \ln \frac{Q^2}{s} \simeq \ln \frac{\theta^2}{4}. \]

This representation gives us a possibility to verify explicitly that the terms of relative order \(\theta^2\) in the radiative corrections are small. Taking into account that the large contribution proportional to \(\ln \Delta\) disappears when we add the cross-section for the hard emission, we can verify again that such terms can be neglected. Therefore we will omit in higher orders
the annihilation diagrams and multiple-photon exchange diagrams in the scattering channel. The second simplification is justified by the generalized eikonal representation for small–angle scattering amplitudes. In particular, for the case of elastic processes we have [11]:

\[ A(s, t) = A_0(s, t) F_1^2(t) \left(1 - \Pi(t)\right)^{-1} e^{i\varphi(t)} \left[1 + \mathcal{O}\left(\frac{Q^2}{s}\right)\right], \quad s \gg Q^2 \gg m^2, \quad (16) \]

where \( A_0(s, t) \) is the Born amplitude, \( F_1(t) \) is the Dirac form factor and \( \varphi(t) = -\alpha \ln(Q^2/\lambda^2) \) is the Coulomb phase, \( \lambda \) is the photon mass auxiliary parameter. The eikonal representation is violated at a three–loop level, but, fortunately, the corresponding contribution to the Bhabha cross–section is small enough (\( \sim \alpha^5 \)) and can be neglected for our purposes. We may consider the eikonal representation as correct within the required accuracy.

Let us now introduce the dimensionless quantity \( \Sigma = Q_1^2 \sigma_{\text{exp}}/(4\pi\alpha^2) \), with \( Q_1^2 = \varepsilon^2 \theta_1^2 \), where \( \sigma_{\text{exp}} \) is the Bhabha–process cross–section integrated over the typical experimental energy and angular range:

\[ \Sigma = \frac{Q_1^2}{4\pi\alpha^2} \int dx_1 \int dx_2 \Theta(x_1 x_2 - x_c) \int d^2 q_1^+ \Theta^c \int d^2 q_2^+ \Theta^e \frac{d\sigma_{e^+e^-\rightarrow e^+e^-}(q_1^+, x_2) e^-(q_2^+, x_1) + X}{dx_1 d^2 q_1^+ dx_2 d^2 q_2^+}, \quad (17) \]

where \( x_{1,2}, q_{1,2} \) are the energy fractions and the transverse components of the momenta of the electron and positron in the final state, \( s x_c \) is the experimental cut–off on their invariant mass squared and the functions \( \Theta^c \) do take into account the angular cuts (2):

\[ \Theta_i^c = \Theta(\theta_3 - \frac{q_1^+}{x_1 \varepsilon}) \Theta(\frac{q_1^+}{x_1 \varepsilon} - \theta_1), \quad \Theta_2^c = \Theta(\theta_4 - \frac{q_2^+}{x_2 \varepsilon}) \Theta(\frac{q_2^+}{x_2 \varepsilon} - \theta_2). \quad (18) \]

In the case of a symmetrical angular acceptance (we restrict ourselves further to this case only) we have:

\[ \theta_2 = \theta_1, \quad \theta_4 = \theta_3, \quad \rho = \frac{\theta_3}{\theta_1} > 1. \quad (19) \]

We will present \( \Sigma \) as the sum of various contributions:

\[ \Sigma = \Sigma_0 + \Sigma^\gamma + \Sigma^{2\gamma} + \Sigma^{e^+e^-} + \Sigma^{3\gamma} + \Sigma^{e^+e^-\gamma} = \Sigma_{00}(1 + \delta_0 + \delta^\gamma + \delta^{2\gamma} + \delta^{e^+e^-} + \delta^{3\gamma} + \delta^{e^+e^-\gamma}), \quad \Sigma_{00} = 1 - \rho^{-2}, \quad (20) \]

4 Result obtained in paper [12], we believe, is incorrect. It contradicts to the well established result of D. Yennie et al. [13] about cancelation of infrared singularities.

5 Really this quantity corresponds to some ideal detectors. It is intended for comparisons with the results of Monte Carlo event generators.
where $\Sigma_0$ stands for a modified Born contribution, $\Sigma^\gamma$ for a contribution of one-photon emission (real and virtual) and so on. The values of the $\delta^i$ as function of $x_c$ are given in Table 1 (see in Section 9). Being stimulated by the representation in Eq. (16), we shall slightly modify the perturbation theory, using the full propagator for the $t$-channel photon, which takes into account the growth of the electric charge at small distances. By integrating Eq. (16) with this convention, we obtain:

$$\Sigma_0 = \theta_1^2 \int \frac{d\theta^2}{\theta^4} (1 - \Pi(t))^{-2} + \Sigma_W + \Sigma_\theta,$$

(21)

where $\Sigma_W$ is the correction due to the weak interaction:

$$\Sigma_W = \theta_1^2 \int \frac{d\theta^2}{\theta^4} \delta_{\text{weak}},$$

(22)

and the term $\Sigma_\theta$ comes from the expansion of the Born cross-section in powers of $\theta^2$,

$$\Sigma_\theta = \theta_1^2 \int \frac{d^2z}{z} (1 - \Pi(-zQ_1^2))^{-2} \left( -\frac{1}{2} + z\theta_1^2 \frac{9}{40} \right).$$

(23)

The remaining contributions to $\Sigma$ in (21) are considered below.

### 3 Single hard-photon emission

In order to calculate the contribution to $\Sigma$ due to the hard-photon emission we start from the corresponding differential cross-section written in terms of energy fractions $x_{1,2}$ and transverse components $q_{1,2}^\perp$ of the final particle momenta [14]:

$$\frac{d\sigma_B^{e^+e^-\rightarrow e^+e^-\gamma}}{dx_1 d^2q_1^\perp dx_2 d^2q_2^\perp} = \frac{2\alpha^3}{\pi^2} \left\{ \frac{R(x_1; q_1^\perp, q_2^\perp)}{(q_1^\perp)^4} \frac{\delta(1 - x_2)}{(1 - \Pi(-(q_2^\perp)^2))^2} \right\} (1 + O(\theta^2)),$$

(24)

where

$$R(x; q_1^\perp, q_2^\perp) = 1 + \frac{x^2}{1 - x} \left[ \frac{(q_2^\perp)^2(1 - x)^2}{d_1 d_2} - \frac{2m^2(1 - x)^2 x}{1 + x^2} \frac{(d_1 - d_2)^2}{d_1^2 d_2^2} \right],$$

(25)

$$d_1 = m^2(1 - x)^2 + (q_1^\perp - q_2^\perp)^2, \quad d_2 = m^2(1 - x)^2 + (q_1^\perp - xq_2^\perp)^2.$$
and we use the full photon propagator for the $t$-channel photon. Performing a simple azimuthal angle integration of Eq. (24) we obtain for the hard-photon emission the contribution $\Sigma^H$:

$$\Sigma^H = \frac{\alpha}{\pi} \int_{x_c}^{1-\Delta} dx \frac{1 + x^2}{1 - x} F(x, D_1, D_3; D_2, D_4),$$

(26)

with

$$F = \int_{D_1}^{D_3} dz_1 \int_{D_2}^{D_3} dz_2 (1 - \Pi(-zQ_1^2))^{-2}\left\{\frac{1 - x}{z_1 - z_2} \left(a_1^{-\frac{1}{2}} - xa_2^{-\frac{1}{2}}\right) - \frac{4x\sigma^2}{1 + x^2}[a_1^{-\frac{3}{2}} + x^2a_2^{-\frac{3}{2}}]\right\},$$

(27)

where

$$a_1 = (z_1 - z_2)^2 + 4z_2\sigma^2, \quad a_2 = (z_1 - x^2z_2)^2 + 4x^2z_2\sigma^2, \quad \sigma^2 = \frac{m^2}{Q_1^2}(1 - x)^2,$$

(28)

and the integration limits in (27) in the symmetrical case are:

$$D_1 = x^2, \quad D_2 = 1, \quad D_3 = x^2\rho^2, \quad D_4 = \rho^2.$$  

(29)

From Eqs. (26)–(29) we have that:

$$\Sigma^H = \frac{\alpha}{\pi} \int_{x_c}^{1-\Delta} dx \frac{1 + x^2}{1 - x} \int_{1}^{\rho^2} \frac{dz}{z^2} (1 - \Pi(-zQ_1^2))^{-2}
\times \left\{[1 + \Theta(x^2\rho^2 - z)](L - 1) + k(x, z)\right\},$$

(30)

$$k(x, z) = \frac{(1 - x)^2}{1 + x^2} \left[1 + \Theta(x^2\rho^2 - z)\right] + L_1 + \Theta(x^2\rho^2 - z) L_2 + \Theta(z - x^2\rho^2) L_3,$$

where $L = \ln(zQ_1^2/m^2)$ and

$$L_1 = \ln \left|\frac{x^2(z - 1)(\rho^2 - z)}{(x - z)(x^2\rho^2 - z)}\right|, \quad L_2 = \ln \left|\frac{(z - x^2)(x^2\rho^2 - z)}{x^2(x - z)(x^2\rho^2 - z)}\right|,$$

(31)

$$L_3 = \ln \left|\frac{(z - x^2)(x^2\rho^2 - z)}{(x - z)(x^2\rho^2 - z)}\right|.$$  

It is seen from Eq. (30) that $\Sigma^H$ contains the auxiliary parameter $\Delta$. This parameter disappears, as it should, in the sum $\Sigma^\gamma = \Sigma^H + \Sigma^{V+S}$, where $\Sigma^{V+S}$ is the contribution of virtual and soft real photons which can be obtained using Eq. (15):

$$\Sigma^\gamma = \frac{\alpha}{\pi} \int_{\rho^2}^{1} \frac{dz}{z^2} \int_{x_c}^{1} dx (1 - \Pi(-zQ_1^2))^{-2}
\times \left\{[1 + \Theta(x^2\rho^2 - z)] + \frac{1 + x^2}{1 - x}k(x, z) - \delta(1 - x)\right\},$$

(32)
where
\[
P(x) = \left(\frac{1 + x^2}{1 - x}\right)_+ = \lim_{\Delta \to 0} \left\{ \frac{1 + x^2}{1 - x} \theta(1 - x - \Delta) + \left(\frac{3}{2} + 2 \ln \Delta\right) \delta(1 - x) \right\}
\]
(33)
is the non–singlet splitting kernel (see Appendix A for details).

4 Radiative corrections to \(\mathcal{O}(\alpha^2)\)

A systematic treatment of all \(\mathcal{O}(\alpha^2)\) contributions is absent up to now. This is mainly due to the extreme complexity of the analysis (more than 100 Feynman diagrams are to be taken into account considering elastic and inelastic processes). Nevertheless in the case of small scattering angles we may restrict ourselves by considering only diagrams of the scattering type. It is enough to make some rough estimates of other contributions. Contributions of pure annihilation–type diagrams, describing some \(\mathcal{O}(\alpha^2)\) RC, have so-called double–logarithmical enhancement \[15\] but, fortunately, it is proportional to the 4th power of the small scattering angle:

\[
(\Sigma_{\gamma\gamma})_{\text{annih}} \sim (\Sigma_{e^+e^-})_{\text{annih}} \sim \theta^4 \left(\frac{\alpha}{\pi}\right)^2 L^4.
\]
(34)

The contribution of interference of the scattering–type and the annihilation–type amplitudes can be estimated as

\[
(\Sigma_{\gamma\gamma})_{\text{interf}} \sim (\Sigma_{e^+e^-})_{\text{interf}} \sim \theta^2 \left(\frac{\alpha}{\pi}\right)^2 \ln^4 \left(\frac{Q^2}{s}\right).
\]
(35)

The absence of large logarithms here has a similar origin as in (15). The uncertainties comes from the discussed contributions are considered in sect. 9.

We consider first virtual two–loop corrections \(d\sigma_{VV}^{(2)}\) to the elastic scattering cross–section. Using the representation \[16\] and the loop expansion for the Dirac form factor \(F_1\)

\[
F_1 = 1 + \frac{\alpha}{\pi} F_1^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 F_1^{(2)}
\]
(36)

one obtains

\[
\frac{d\sigma_{VV}^{(2)}}{dc} = \frac{d\sigma_0}{dc} \left(\frac{\alpha}{\pi}\right)^2 (1 - \Pi(t))^{-2} \left[ 6(F_1^{(1)})^2 + 4F_1^{(2)} \right].
\]
(37)

The one-loop contribution to the form factor is well known:

\[
F_1^{(1)} = (L - 1) \ln \frac{\lambda}{m} + \frac{3}{4} L - \frac{1}{4} L^2 - 1 + \frac{1}{2} \zeta_2.
\]
(38)
The two–loop correction can be obtained from the results of Ref. [16]. Let us present it in the form

$$F_1^{(2)} = F_1^{\gamma\gamma} + F_1^{e^+e^-},$$

(39)

where the contribution $F_1^{e^+e^-}$ is related to the vacuum polarization by $e^+e^-$ pairs:

$$F_1^{e^+e^-} = -\frac{1}{36}L^3 + \frac{19}{72}L^2 - \left(\frac{265}{216} + \frac{1}{6}\z_2\right)L + O(1),$$

(40)

$$F_1^{\gamma\gamma} = \frac{1}{32}L^4 - \frac{3}{16}L^3 + \left(\frac{17}{32} - \frac{1}{8}\z_2\right)L^2 + \left(-\frac{21}{32} - \frac{3}{8}\z_2 + \frac{3}{2}\z_3\right)L,$$

(41)

$$+ \frac{1}{2}(L - 1)^2 \ln^2 \frac{m}{\lambda} + (L - 1)\left[-\frac{1}{4}L^2 + \frac{3}{4}L - 1 + \frac{1}{2}\z_2\right] \ln \frac{\lambda}{m} + O(1),$$

$$\z_2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \z_3 = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202.$$  

The photon mass $\lambda$ entering Eqs. (38)–(41) is cancelled in the expression $d\sigma_2^{(2)}/d\epsilon$ for the sum of the virtual and soft-photon corrections of the second order $d\sigma^{(2)}_{VV}/d\epsilon$ (see Eq. (37)), $d\sigma^{(2)}_{SS}/d\epsilon$ and $d\sigma^{(2)}_{SV}/d\epsilon$.

The cross–section $d\sigma^{(2)}/d\epsilon$ for the emission of two soft photons, each of energy smaller than $\Delta\epsilon = \epsilon\Delta$, is ($\Delta \ll 1$):

$$d\sigma^{(2)}_{SS} = d\sigma_0 \left(\frac{\alpha}{\pi}\right)^2 \left(1 - \Pi(t)\right)^{-2} 8 \left[(L - 1) \ln \frac{m\Delta}{\lambda} + \frac{1}{4}L^2 - \frac{1}{2}\z_2\right]^2,$$

(42)

and the virtual correction $d\sigma^{(2)}_{SV}/d\epsilon$ to the cross–section of the single soft-photon emission is:

$$d\sigma^{(2)}_{SV} = d\sigma_0 \left(\frac{\alpha}{\pi}\right)^2 \left(1 - \Pi(t)\right)^{-2} 16F_1^{(1)} \left[(L - 1) \ln \frac{m\Delta}{\lambda} + \frac{1}{4}L^2 - \frac{1}{2}\z_2\right].$$

(43)

The contribution to $\Sigma$ of this sum, except the part coming from $F_1^{e^+e^-}$ connected with the vacuum polarization, contains no more than a second power of $L$. It has the following form:

$$\Sigma_{SV} = \Sigma_{VV} + \Sigma_{VS} + \Sigma_{SS} = \left(\frac{\alpha}{\pi}\right)^2 \int \frac{dz}{z^2} (1 - \Pi(-zQ^2))^{-2} R_{SV}^{\gamma\gamma}. $$

(44)

It is convenient to separate the $R_{SV}^{\gamma\gamma}$ in the following way:

$$R_{SV}^{\gamma\gamma} = R_{SV}^{\gamma\gamma} + r_{SV\gamma\gamma} + r_{SV\gamma\gamma},$$

(45)

$$r_{SV}^{\gamma\gamma} = r_{SV\gamma\gamma} = L^2 \left(2 \ln^2 \Delta + 3 \ln \Delta + \frac{9}{8}\right),$$

$$+ L \left(-4 \ln^2 \Delta - 7 \ln \Delta + 3\z_3 - \frac{3}{2}\z_2 - \frac{45}{16}\right),$$

$$r_{SV\gamma\gamma} = 4[(L - 1) \ln \Delta + \frac{3}{4}L - 1]^2.$$
The contribution to $\Sigma$ coming from $F_1^{e^+e^-}$ contains an $L^3$ term, which is also cancelled when we take into account the soft pair production contribution

$$
d\sigma_{S}^{e^+e^-} = \left(\frac{\alpha}{\pi}\right)^2 d\sigma_0 (1 - \Pi(t))^{-2} R_s^{e^+e^-} = \left(\frac{\alpha}{\pi}\right)^2 d\sigma_0 (1 - \Pi(t))^{-2} \left[\frac{1}{9}(L + 2 \ln \Delta)^3 \right.
$$

$$
- \frac{5}{9}(L + 2 \ln \Delta)^2 + \left(\frac{56}{27} - \frac{2}{3} \zeta_2\right)(L + 2 \ln \Delta) + O(1)\right].
$$

Thus for the contribution of the virtual and soft $e^+e^-$ pairs to $\Sigma$ we have

$$
\Sigma_{S+V}^{e^+e^-} = \left(\frac{\alpha}{\pi}\right)^2 \int \frac{dz}{z^2} (1 - \Pi(-zQ_1^2))^{-2} R_{S+V}^{e^+e^-}, \tag{47}
$$

$$
R_{S+V}^{e^+e^-} = R_S^{e^+e^-} + 4F_1^{e^+e^-} = L^2 \left(\frac{2}{3} \ln \Delta + \frac{1}{2}\right) + L \left(-\frac{17}{6} + \frac{4}{3} \ln^2 \Delta\right) 
$$

$$
- \frac{20}{9} \ln \Delta - \frac{4}{3} \zeta_2 + O(1).
$$

In expressions (45)–(47), $\Delta = \delta\varepsilon/\varepsilon$ is the energy fraction carried by the soft pair, and it is assumed that $2m \ll \delta\varepsilon \ll \varepsilon$. Here we have taken into account only $e^+e^-$ pair production. The order of magnitude of the radiative correction due to pair production is less than 0.5%. A rough estimate of the muon pair contribution gives less than 0.05% since $\ln(Q^2/m^2) \sim 3 \ln(Q^2/m^2)$.

Contributions of pion and tau lepton pairs give corrections that are still smaller. Therefore, within the 0.1% accuracy, we may omit any pair production contribution except the $e^+e^-$ one.

5 Virtual and soft corrections to the hard-photon emission

By evaluating the corrections arising from the emission of virtual and real soft photons which accompany a single hard-photon we will consider two cases. The first case corresponds to the emission of the photons by the same fermion. The second one occurs when the hard-photon is emitted by another fermion:

$$
d\sigma \bigg|_{H(S+V)} = d\sigma^H(S+V) + d\sigma_{H(S+V)} + d\sigma_{H(S+V)} + d\sigma_{H(S+V)}^H. \tag{48}
$$

In the case when both fermions emit, one finds that:

$$
\Sigma_{H(S+V)}^H + \Sigma_{H(S+V)}^H = 2\Sigma^H_0 \left(\frac{\alpha}{\pi}\right) \left[(L - 1) \ln \Delta + \frac{3}{4}L - 1\right], \tag{49}
$$
where $\Sigma_H$ is given in Eq. (30). A more complex expression arises when the radiative corrections are applied to the same fermion line. Here the cross-section may be expressed in terms of the Compton tensor with an off-shell photon [17], which describes the process

$$\gamma^*(q) + e^-(p_1) \rightarrow e^-(q_1) + \gamma(k) + (\gamma_{\text{soft}}).$$

(50)

In the limit of small-angle photon emission we have:

$$d\sigma^H_{S+V} = \frac{\alpha^4}{4\pi(1-x)(q^2_{\gamma})^4\pi^3}[B_{11}(s_1, t_1) + x^2 B_{11}(t_1, s_1)]h + T,$$

(51)

$$T = T_{11}(s_1, t_1) + x^2 T_{11}(t_1, s_1) + x(T_{12}(s_1, t_1) + T_{12}(t_1, s_1)),$$

$$h = 2\left(L - \ln \left(\frac{q^2_{\gamma}}{u_1}\right) - 1\right)(2 \ln \Delta - \ln x) + 3L - \ln^2 x - \frac{9}{2},$$

where $\Delta = (\Delta\varepsilon/\varepsilon) \ll 1$, $\Delta\varepsilon$ is the maximal energy of the soft photon, escaping the detectors, $B_{11}(s_1, t_1) = (-4(q^2_{\gamma})^2)/(s_1 t_1) - 8m^2/s_1^2$ is the Born Compton tensor component, and the invariants are:

$$s_1 = 2q_1 k, \quad t_1 = -2p_1 k, \quad u_1 = (p_1 - q_1)^2, \quad s_1 + t_1 + u_1 = q^2.$$

The final result (see Appendix C for details) has the form:

$$\Sigma^H_{S+V} = \Sigma^H_{H(S+V)} = \frac{1}{2}(\frac{\alpha}{\pi})^2 \int_0^1 \int_{x_e}^1 \frac{dz}{z^2} \frac{dx(1+x^2)}{1-x} \frac{1}{L} \left\{ \left(2 \ln \Delta - \ln x + \frac{3}{2}\right) \right. \times \left. [(L - 1)(1 + \Theta) + k(x, z)] + \frac{1}{2} \ln^2 x + (1 + \Theta)[-2 + \ln x - 2\ln \Delta] + (1 - \Theta)\left[\frac{1}{2}L \ln x + 2 \ln \Delta \ln x - \ln x \ln(1-x) \right. \right. \left. - \ln^2 x - \text{Li}_2(1-x) - \frac{x(1-x) + 4x \ln x}{2(1+x^2)} \right] - \left. \frac{(1-x)^2}{2(1+x^2)} \right\},$$

(52)

$$\text{Li}_2(x) \equiv -\int_0^x \frac{dt}{t} \ln(1-t),$$

where $k(x, z)$ is given in Eq. (30) and $\Theta \equiv \Theta(x^2 \rho^2 - z)$.

### 6 Double hard-photon bremsstrahlung

We now consider the contribution given by the process of emission of two hard photons. We will distinguish two cases: a) the double simultaneous bremsstrahlung in opposite directions along electron and positron momenta, and b) the double bremsstrahlung in the same direction along electron or positron momentum. The differential cross-section in the first case can be obtained by using the factorization property of cross-sections within the impact parameter.
representation \[18\]. It takes the following form \[14\] (see Appendix A):

\[
\frac{d\sigma^{e^+e^→(e^+\gamma)(e^-\gamma)}}{dx_1dx_2^2q_1^2dx_2q_2^2} = \frac{\alpha^4}{\pi^3} \int \frac{d^2k^\perp}{\pi(k^\perp)^4} (1 - \Pi(-k^\perp)^2)^{-2} R(x_1; q_1^\perp, k^\perp) R(x_2; q_2^\perp, -k^\perp),
\]

where \(R(x; q^\perp, k^\perp)\) is given by Eq. (25). The calculation of the corresponding contribution \(\Sigma^H_H\) to \(\Sigma\) is analogous to the case of the single hard-photon emission and the result has the form:

\[
\Sigma^H_H = \frac{1}{4} \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{dz}{z^2} (1 - \Pi(-zQ_1^2))^{-2} \int_{x_c}^{1-\Delta} dx_1 \int_{1-x_1}^{1-\Delta} dx_2 \frac{1 + x_1^2}{1 - x_1} \frac{1 + x_2^2}{1 - x_2} \Phi(x_1, z) \Phi(x_2, z),
\]

where (see Eq. (31)):

\[
\Phi(x, z) = (L - 1)[\Theta(z - 1)\Theta(\rho^2 - z) + \Theta(z - x^2)\Theta(x^2\rho^2 - z)]
+ L_3[-\Theta(x^2 - z) + \Theta(z - x^2\rho^2)] + \left(L_2 + \frac{(1 - x^2)}{1 + x^2}\right)\Theta(z - x^2)\Theta(x^2\rho^2 - z)
+ \left(L_1 + \frac{(1 - x^2)}{1 + x^2}\right)\Theta(z - 1)\Theta(\rho^2 - z)
+ (\Theta(1 - z) - \Theta(z - \rho^2)) \ln \left|\frac{(z - x)(\rho^2 - z)}{(x\rho^2 - z)(z - 1)}\right|.
\]

Let us now turn to the double hard-photon emission in the same direction and the hard \(e^+ e^-\) pair production. Here we use the method developed by one of us \[19, 20\]. We will distinguish the collinear and semi–collinear kinematics of final particles. In the first case all produced particles move in the cones within the polar angles \(\theta_i < \theta_0 \ll 1\) centered along the charged-particle momenta (final or initial). In the semi–collinear region only one produced particle moves inside those cones, while the other moves outside them. For the totally inclusive cross–section, such a distinction no longer has physical meaning and the dependence on the auxiliary parameter \(\theta_0\) disappears. We underline that in this way all double and single–logarithmical contributions may be extracted rigorously. The contribution of the region when both the photons move outside the small cones does not contain any large logarithm \(L\). The systematic omission of those contributions in the double bremsstrahlung and pair production processes is the source of uncertainties of order \((\alpha/\pi)^2 \leq 0.6 \cdot 10^{-5}\).

The contribution of both collinear and semi–collinear regions (we consider for definiteness the emission of both hard photons along the electron, since the contribution of the emission along the positron is the same) has the form (see Appendix B for details):

\[
\Sigma^{HH}_H = \Sigma^H_H = \frac{1}{4} \left(\frac{\alpha}{\pi}\right)^2 \int_{\rho^2}^{1} \frac{dz}{z^2} (1 - \Pi(-zQ_1^2))^{-2}
\]

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One may see that the combinations
\[ r^{\gamma\gamma} + \Sigma^{H(S+V)} + \Sigma^{HH}, \quad r^2 + \Sigma^{H(S+V)} + \Sigma^{S+V} + \Sigma^{H} \] (57)
with \( r^{\gamma\gamma} \) and \( r^2 \) normalized (see Eqs. (42,43)) to

\[ r^{\gamma\gamma} \rightarrow \left( \frac{\alpha}{\pi} \right)^2 \int_0^\rho \frac{dz}{z^2} (1 - \Pi(-zQ_1^2))^{-2} r_{S+V}^{\gamma\gamma}, \]

and

\[ r^2 \rightarrow \left( \frac{\alpha}{\pi} \right)^2 \int_0^\rho \frac{dz}{z^2} (1 - \Pi(-zQ_1^2))^{-2} r_{S+V}^{\gamma}, \]

respectively, do not depend on \( \Delta \) for \( \Delta \to 0 \) (see Appendix E).

The total expression \( \Sigma^{2\gamma} \), which describes the contribution to (20) from the two-photon (real and virtual) emission processes is determined by expressions (43), (47), (49), (51), (53)
and (55). Furthermore it does not depend on the auxiliary parameter \( \Delta \) and has the form:

\[
\Sigma^{2\gamma} = \Sigma_{S+S} + 2\Sigma_{H(V+S)} = \Sigma_{S} + 2\Sigma_{H} + 2\Sigma_{HH}
\]

(58)

\[
= \Sigma^{\gamma\gamma} + \Sigma_{\gamma} + \left( \frac{\alpha}{\pi} \right)^2 \mathcal{L}(\phi^{\gamma\gamma} + \phi_{\gamma}), \quad \mathcal{L} = \ln \frac{\varepsilon^2 \theta^2}{m^2}.
\]

The leading contributions \( \Sigma^{\gamma\gamma}, \Sigma_{\gamma} \) have the following forms (see Appendix D):

\[
\Sigma^{\gamma\gamma} = \frac{1}{2} \left( \frac{\alpha}{\pi} \right)^2 \int \frac{dz}{z^2} L^2 (1 - \Pi (-Q_1^2 z))^{-2} \int \frac{dx}{x_c} \left\{ \frac{1}{2} P^{(2)}(x) \left[ \Theta (x^2 \rho^2 - z) + 1 \right] \right. \right.
\]

\[
+ \left. \int_{x} \frac{dt}{t} P(t) \left( \frac{x}{t} \right) \Theta (t^2 \rho^2 - z) \right\},
\]

(59)

\[
P^{(2)}(x) = \int_{x} \frac{dt}{t} P(t) \left( \frac{x}{t} \right) = \lim_{\Delta \to 0} \left\{ \left[ \left( 2 \ln \Delta + \frac{3}{2} \right)^2 - 4\zeta_2 \right] \delta (1 - x) \right. \right.
\]

\[
+ \left. 2 \left[ \frac{1 + x^2}{1 - x} \left( 2 \ln (1 - x) - \ln x + \frac{3}{2} \right) + \frac{1}{2} (1 + x) \ln x - 1 + x \right] \Theta (1 - x - \Delta) \right\},
\]

(60)

\[
\Sigma_{\gamma} = \frac{1}{4} \left( \frac{\alpha}{\pi} \right)^2 \int_{0}^{\infty} \frac{dz}{z^2} L^2 (1 - \Pi (-Q_1^2 z))^{-2} \int_{x_c}^{1} \frac{dx_1}{x_c} \int_{x_c/x_1}^{1} \frac{dx_2}{x_2} P(x_1) P(x_2)
\]

\[
\times \left[ \Theta (z - 1) \Theta (\rho^2 - z) + \Theta (z - x_1^2) \Theta (x_1^2 \rho^2 - z) \right]
\]

\[
\times \left[ \Theta (z - 1) \Theta (\rho^2 - z) + \Theta (z - x_2^2) \Theta (x_2^2 \rho^2 - z) \right].
\]

We see that the leading contributions to \( \Sigma^{2\gamma} \) may be expressed in terms of kernels for the evolution equation for structure functions.

The functions \( \phi^{\gamma\gamma} \) and \( \phi_{\gamma} \) in expression Eq. \((58)\) collect the next-to-leading contributions which cannot be obtained by the structure functions method \([21]\). They have a form that can be obtained by comparing the results in the leading logarithmic approximation with the logarithmic ones given above.

7 **Pair production**

Pair production process in high-energy e\(^+\) e\(^-\) collisions was considered about 60 years ago (see \([14]\) and references therein). In particular it was found that the total cross-section contains cubic terms in large logarithm \( L \). These terms come from the kinematics when the scattered electron and positron move in narrow (with opening angles \( \sim m/\varepsilon \)) cones and the created pair have the invariant mass of the order of \( m \) and moves preferably along either the electron beam direction or the positron one. According to the conditions of the LEP detectors, such a kinematics can be excluded. In the relevant kinematical region a parton-like description could be used giving \( L^2 \) and \( L\)-enhanced terms.
We accept the LEP 1 conventions whereby an event of the Bhabha process is defined as one in which the angles of the simultaneously registered particles hitting opposite detectors (see Eq. (64)).

The method, developed by one of us (N.P.M.) [19, 20], of calculating the real hard pair production cross-section within logarithmic accuracy (see the discussion in sect. 6) consists in separating the contributions of the collinear and semi-collinear kinematical regions. In the first one (CK) we suggest that both electron and positron from the created pair go in the narrow cone around the direction of one charged particle [the projectile (scattered) electron \( p_1 (q_1) \) or the projectile (scattered) positron \( p_2 (q_2) \)]:

\[
\hat{p}_+ p_\mp p_i < \theta_0 \ll 1, \quad \varepsilon \theta_0/m \gg 1, \quad p_i = p_1, p_2, q_1, q_2. \quad (62)
\]

The contribution of the CK contains terms of order \( (\alpha L/\pi)^2, (\alpha/\pi)^2 L \ln(\theta_0/\theta) \) and \( (\alpha/\pi)^2 L \), where \( \theta = \hat{p}-q_1 \) is the scattering angle. In the semi-collinear region only one of conditions (62) on the angles is fulfilled:

\[
\hat{p}_+ p_- < \theta_0, \quad p_\mp p_i > \theta_0; \quad \text{or} \quad p_- p_i < \theta_0, \quad p_\mp p_i > \theta_0; \quad (63)
\]

or \( \hat{p}_- p_+ > \theta_0, \quad p_\mp p_i < \theta_0 \).

The contribution of the SCK contains terms of the form:

\[
\left( \frac{\alpha}{\pi} \right)^2 L \ln \frac{\theta_0}{\theta}, \quad \left( \frac{\alpha}{\pi} \right)^2 L. \quad (64)
\]

The auxiliary parameter \( \theta_0 \) drops out in the total sum of the CK and SCK contributions.

All possible mechanisms for pair creation (singlet and non-singlet) and the identity of the particles in the final state are taken into account [24]. In the case of small-angle Bhabha scattering only a part of the total 36 tree-type Feynman diagrams are relevant, i.e. the scattering diagrams.

The sum of the contributions due to virtual pair emission (due to the vacuum polarization insertions in the virtual photon Green’s function) and of those due to the real soft pair emission does not contain cubic \( \sim L^3 \) terms but depends on the auxiliary parameter \( \Delta = \delta \varepsilon / \varepsilon \) \( (m_e \ll \delta \varepsilon \ll \varepsilon, \text{where} \delta \varepsilon \text{is the sum of the energies of the soft pair components}) \). The \( \Delta \)-dependence disappears in the total sum after the contributions due to real hard pair production are added. Before summing one has to integrate the hard pair contributions over the energy fractions of the pair components, as well as over those of the scattered electron and positron:

\[
\Delta = \frac{\Delta \varepsilon}{\varepsilon} < x_1 + x_2, \quad x_c < x = 1 - x_1 - x_2 < 1 - \Delta, \quad (65)
\]

\[
x_1 = \frac{\varepsilon_+}{\varepsilon}, \quad x_2 = \frac{\varepsilon_-}{\varepsilon}, \quad x = \frac{q_0}{\varepsilon},
\]

\( ^6 \)We have verified that the interference between the amplitudes describing the production of pairs moving in the electron direction and the positron one cancels. This is known as up-down (interference) cancellation [24].

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where $\varepsilon_{\pm}$ are the energies of the positron and electron from the created pair. We consider for definiteness the case when the created hard pair moves close to the direction of the initial (or scattered) electron.

Consider first the collinear kinematics. There are four different CK regions, when the created pair goes in the direction of the incident (scattered) electron or positron. We will consider only two of them, corresponding to the initial and the final electron directions. For the case of pair emission parallel to the initial electron, it is useful to decompose the particle momenta into longitudinal and transverse components:

\begin{align}
 p_+ &= x_1 p_1 + p^\perp_+,
 p_- &= x_2 p_1 + p^\perp_-,
 q_1 &= x p_1 + q^\perp_1,
 x &= 1 - x_1 - x_2,
 q_2 &\approx p_2,
 p^\perp_+ + q^\perp_+ = 0,
 \end{align}

where $p^\perp_i$ are the two–dimensional momenta of the final particles, which are transverse with respect to the initial electron beam direction. It is convenient to introduce dimensionless quantities for the relevant kinematical invariants:

\begin{align}
 z_i &= \left(\frac{\varepsilon \theta_i}{m}\right)^2, \quad 0 < z_i < \left(\frac{\varepsilon \theta_0}{m}\right)^2 \gg 1, \\
 A &= \frac{(p_+ + p_-)^2}{m^2} = (x_1 x_2)^{-1}[(1 - x)^2 + x_1^2 x_2^2(z_1 + z_2 - 2 \sqrt{z_1 z_2} \cos \phi)], \\
 A_1 &= \frac{2p_1 p_-}{m^2} = x_2^{-1}[1 + x_2^2 + x_2^2 z_2], \quad A_2 = \frac{2p_1 p_+}{m^2} = x_1^{-1}[1 + x_1^2 + x_1^2 z_1], \\
 C &= \frac{(p_1 - p_-)^2}{m^2} = 2 - A_1, \quad D = \frac{(p_1 - q_1)^2}{m^2} - 1 = A - A_1 - A_2,
\end{align}

where $\phi$ is the azimuthal angle between the $(p_1 p^\perp_1)$ and $(p_1 p^\perp)$ planes.

Keeping only the terms from the sum over spin states of the square of the absolute value of the matrix element, which give non–zero contributions to the cross–section in the limit $\theta_0 \to 0$, we find that only 8 from the total 36 Feynman diagrams are essential [24].

The result has the factorized form:

\begin{align}
 \sum_{\text{spins}} \left| M \right|^2 \big|_{p_+ p_- \parallel p_1} = \sum_{\text{spins}} \left| M_0 \right|^2 2^7 \pi^2 \alpha^2 \frac{I}{m^4},
\end{align}

where one of the multipliers corresponds to the matrix element in the Born approximation (without pair production):

\begin{align}
 \sum_{\text{spins}} \left| M_0 \right|^2 = 2^7 \pi^2 \alpha^2 \left( \frac{s^4 + t^4 + u^4}{s^2 t^2} \right),
 s = 2p_1 p_2, \quad t = -Q^2 x, \quad u = -s - t,
\end{align}

and the quantity $I$, which stands for the collinear factor, coincides with the expression for $I_a$ obtained in [20]. We write it here in terms of our kinematical variables:

\begin{align}
 I &= (1 - x_2)^{-2} \left( \frac{A(1 - x_2) + D x_2}{DC} \right)^2 + (1 - x)^{-2} \left( \frac{C(1 - x) - D x_2}{AD} \right)^2.
\end{align}
We rewrite the phase volume of the final particles as

\[
+ \frac{1}{2x AD} \left[ \frac{2(1 - x_2)^2 - (1 - x)^2}{1 - x} + \frac{x_1 x - x_2}{1 - x} + 3(x_2 - x) \right] \\
+ \frac{1}{2x CD} \left[ \frac{(1 - x_2)^2 - 2(1 - x)^2}{1 - x} + \frac{x - x_1 x_2}{1 - x} + 3(x_2 - x) \right] \\
+ \frac{x_2(x^2 + x_2^2)}{2x(1 - x_2)(1 - x) AC} + \frac{3x}{D^2} + \frac{2C}{AD^2} + \frac{2A}{CD^2} + \frac{2(1 - x_2)}{x A^2 D} \\
- \frac{4C}{x A^2 D^2} - \frac{4A}{D^2 C^2} + \frac{1}{D C^2} \left[ \frac{(x_1 - x)(1 + x_2)}{x(1 - x_2)} - 2 \frac{1 - x}{x} \right].
\]

We rewrite the phase volume of the final particles as

\[
d\Gamma = \frac{d^3 q_1 d^3 q_2}{(2\pi)^6 2q_1^2 q_2^2} (2\pi)^4 \delta^{(4)}(p_1 x + p_2 - q_1 - q_2) (71)
\]

\[
\times m^4 2^{-8} \pi^{-4} x_1 x_2 d x_1 d x_2 d z_1 d z_2 d\phi \frac{d\phi}{2\pi}.
\]

Using the table of integrals given in Appendix F we further integrate over the variables of the created pair. Following a similar procedure in the case when the pair moves in the direction of the scattered electron, integrating the resulting sum over the energy fractions of the pair components, and finally adding the contribution of the two remaining CK regions (when the pair goes in the positron directions), we obtain\(^7\):

\[
d\sigma_{coll} = \frac{\alpha^4 d x}{\pi Q_1^2} \int_1^{\rho^2} \frac{d z}{z^2} L \left\{ R_0(x) \left( L + 2 \ln \frac{\lambda^2}{z} \right) (1 + \Theta) \right\} (72)
\]

\[
+ \frac{4 R_0(x) \ln x + 2\Theta f(x) + 2 f_1(x)}{\Theta}, \quad \lambda = \frac{\theta_0}{\theta_{\text{min}}}, \\
\Theta \equiv \Theta(x^2 \rho^2 - z) = \begin{cases} 
1, & x^2 \rho^2 > z, \\
0, & x^2 \rho^2 \leq z,
\end{cases}
\]

\[
R_0(x) = \frac{2}{3} \frac{1 + x^2}{1 - x} + \frac{(1 - x)}{3 x} (4 + 7 x + 4 x^2) + 2(1 + x) \ln x,
\]

\[
f(x) = -\frac{107}{9} + \frac{136}{9} x - \frac{2}{3} x^2 - \frac{4}{3 x} - \frac{20}{9(1 - x)} + \frac{2}{3} \left[ -\frac{4}{3} x^2 - 5 x + 1 \right]
\]

\[
+ \frac{4}{x(1 - x)} \ln(1 - x) + \frac{1}{3} \left[ 8 x^2 + 5 x - 7 - \frac{13}{1 - x} \right] \ln x - \frac{2}{1 - x} \ln^2 x
\]

\[
+ 4(1 + x) \ln x \ln(1 - x) - \frac{2(3 x^2 - 1)}{1 - x} \operatorname{Li}_2(1 - x),
\]

\[
f_1(x) = -x \Re f\left(\frac{1}{x}\right) = -\frac{116}{9} + \frac{127}{9} x + \frac{4}{3} x^2 + \frac{2}{3 x} - \frac{20}{9(1 - x)} + \frac{2}{3} \left[ -4 x^2 
\]

\(^7\) Some misprints, which occur in the expressions for \(f(x)\) and \(f_1(x)\) in [23, 24], are corrected here.
\[-5x + 1 + \frac{4}{x(1 - x)} \ln(1 - x) + \frac{1}{3}[8x^2 - 10x - 10 + \frac{5}{1 - x}] \ln x\]
\[-(1 + x) \ln^2 x + 4(1 + x) \ln x \ln(1 - x) - \frac{2(x^2 - 3)}{1 - x} \text{Li}_2(1 - x),\]

\[Q_1 = \varepsilon \theta_{\min}, \quad L = \ln \frac{2Q_1^2}{m^2}.\]

Consider now semi–collinear kinematical regions. We will restrict ourselves again to the case in which the created pair goes close to the electron momentum (initial or final). A similar treatment applies in the CM system in the case in which the pair follows the positron momentum. There are three different semi–collinear regions, which contribute to the cross–section within the required accuracy. The first region includes the events for which the created pair has very small invariant mass:

\[4m^2 \ll (p_+ + p_-)^2 \ll |q^2|,\]

and the pair escapes the narrow cones (defined by \(\theta_0\)) in both the incident and scattered electron momentum directions. We will refer to this SCK region as \(p_+ \parallel p_-\). The reason is the smallness (in comparison with \(s\)) of the square of the four–momentum of the virtual photon converting to the pair \[24\].

The second SCK region includes the events for which the invariant mass of the created positron and the scattered electron is small, \(4m^2 \ll (p_+ + q_1)^2 \ll |q^2|\), with the restriction that the positron should escape the narrow cone in the initial electron momentum direction. We refer to it as \(p_+ \parallel q_1\) \[24\].

The third SCK region includes the events in which the created electron goes inside the narrow cone in the initial electron momentum direction, but the created positron does not. We refer to it as \(p_- \parallel p_1\) \[24\].

The differential cross–section takes the following form:

\[d\sigma = \frac{\alpha^4}{8\pi^4s^2} \frac{|M|^2}{q^4} \frac{d^2x_1d^2x_2}{x_1x_2x} \frac{d^2p_+^\perp d^2p_-^\perp d^2q_1^\perp d^2q_2^\perp \delta(1 - x_1 - x_2 - x)}{\delta(2)(p_+^\perp + p_-^\perp + q_1^\perp + q_2^\perp)},\]

where \(x_1 (x_2), x\) and \(p_+^\perp (p_-^\perp), q_1^\perp\) are the energy fractions and the perpendicular momenta of the created positron (electron) and the scattered electron (positron) respectively; \(s = (p_1 + p_2)^2\) and \(q^2 = -Q^2 = (p_2 - q_2)^2 = -\varepsilon^2\theta^2\) are the center-of-mass energy squared and the momentum transferred squared; the matrix element squared \(|M|^2\) takes different forms according to the different SCK regions.

For the differential cross–section in the \(p_+ \parallel p_-\) region we have (see, for details, \[23\]):

\[d\sigma_{p_+\parallel p_-} = \frac{\alpha^4}{\pi} L \frac{dx_1d^2x_2}{x_1x_2x} \frac{d(q_1^\perp)^2}{(q_2^\perp)^2} \frac{d(q_1^\perp)}{q_1^\perp + q_2^\perp} \delta(1 - x_1 - x_2) \delta(1 - x_1 - x_2 - x) \]
\[-\frac{d\phi}{2\pi}(q_1^\perp + q_2^\perp)^2 \left[ (1 - x_1)^2 + (1 - x_2)^2 - \frac{4x_1x_2}{(1 - x)^2} \right],\]
where $\phi$ is the angle between the two–dimensional vectors $q_1^\perp$ and $q_2^\perp$. $q_{1,2}^\perp$ are the transverse momentum components of the final electrons, $x_{1,2}$ are their energy fractions ($x = 1 - x_1 - x_2$). At this stage it is necessary to use the restrictions on the two–dimensional momenta $q_1^\perp$ and $q_2^\perp$. They appear when the contribution of the CK region (which here represents the narrow cones with opening angle $\theta_0$ in the momentum directions of both incident and scattered electrons) is excluded:

$$\frac{|p_+^\perp|}{\varepsilon_+} > \theta_0, \quad \frac{|q_1^\perp - q_2^\perp|}{\varepsilon_2} > \theta_0,$$  \tag{75}

where $\varepsilon_+$ and $\varepsilon_2$ are the energies of the created positron and the scattered electron respectively.

In order to exclude $p_+^\perp$ from the above equation we use the conservation of the perpendicular momentum, in this case:

$$q_1^\perp + q_2^\perp + \frac{1-x}{x_1} p_+^\perp = 0.$$  

In the semi–collinear region $p_+ \parallel q_1$, we obtain:

$$d\sigma_{p_+\parallel q_1} = \frac{\alpha^4}{\pi} L \frac{d(q_2^\perp)^2}{(|q_2^\perp|)^2} \frac{d(q_1^\perp)^2}{(|q_1^\perp|)^2} \frac{1}{2\pi} \frac{x^2}{(q_1^\perp + xq_2^\perp)^2 (1 - x_2)^2} \left[ (1 - x)^2 + (1 - x_1)^2 - x \frac{4x_1 x_2}{(1 - x_2)^2} \right],$$  

with the restrictions

$$\frac{|p_+^\perp - q_1^\perp|}{\varepsilon_+} > \theta_0, \quad p_+^\perp + q_2^\perp + \frac{1-x_2}{x} q_1^\perp = 0.$$  \tag{77}

Finally for the $p_- \parallel p_1$ semi–collinear region we get:

$$d\sigma_{p_-\parallel p_1} = \frac{\alpha^4}{\pi} L \frac{d(q_2^\perp)^2}{(|q_2^\perp|)^2} \frac{d(q_1^\perp)^2}{(|q_1^\perp|)^2} \frac{1}{2\pi} \frac{(1-x)^2 + (1-x_1)^2}{(1-x_2)^2 - \frac{4x_1 x_2}{(1 - x_2)^4}}.$$  \tag{78}

The restriction due to the exclusion of the collinear region when the created pair moves inside a narrow cone in the direction of the initial electron has the form

$$\frac{|p_+^\perp|}{\varepsilon_1} > \theta_0, \quad p_+^\perp + q_1^\perp + q_2^\perp = 0.$$  \tag{79}

In order to obtain the finite expression for the cross–section we have to add $d\sigma_{p_+\parallel q_+} + d\sigma_{p_+\parallel q_1} + d\sigma_{p_-\parallel p_}$ to the contribution of the collinear kinematics region (72) and those due to the production of virtual and soft pairs. Taking into account the leading and next-to-leading
terms we can write the full hard pair contribution including also the pair emission along the positron direction, after the integration over $x_2$ as

$$\sigma_{\text{hard}} = 2 \frac{\alpha^4}{\pi Q_1^2} \int_1^{\rho^2} \frac{dz}{z^2} \int \frac{dx}{x} \left\{ L^2 (1 + \Theta) R(x) + \mathcal{L}[\Theta F_1(x) + F_2(x)] \right\}. \tag{80}$$

$$F_1(x) = d(x) + C_1(x), \quad F_2(x) = d(x) + C_2(x),$$

$$d(x) = \frac{1}{1-x} \left( \frac{8}{3} \ln(1-x) - \frac{20}{9} \right),$$

$$C_1(x) = \frac{113}{9} - \frac{142}{9} x - \frac{2}{3} x^2 - \frac{4}{3} (1+x) \ln(1-x) + \frac{2}{3} \frac{1+x^2}{1-x} \left[ \ln \left( \frac{x^2 \rho^2 - z^2}{(x \rho^2 - z)^2} \right) - 3 \text{Li}_2(1-x) \right] + \frac{8}{3} x^2 - 3 x - 9 - \frac{8}{x} - \frac{7}{1-x} \ln x + \frac{2}{1-x} (5x^2 - 6) \ln^2 x + \beta(x) \ln \left( \frac{x^2 \rho^2 - z^2}{\rho^4} \right),$$

$$C_2(x) = \frac{122}{9} + \frac{133}{9} x + \frac{4}{3} x^2 + \frac{2}{3} (1+x) \ln(1-x) + \frac{2}{3} \frac{1+x^2}{1-x} \left[ \ln \left( \frac{(z-x^2)(\rho^2 - z)(z-1)}{(x \rho^2 - z)(z-x^2)} \right) + 3 \text{Li}_2(1-x) \right] + \frac{1}{3} (-8x^2 - 32x - 20 + \frac{13}{1-x} + \frac{8}{x}) \ln x + \frac{1}{3} (1+x) \ln^2 x + \beta(x) \ln \left( \frac{(z-x^2)(\rho^2 - z)(z-1)}{x^2 \rho^2 - z} \right), \quad \beta = 2R(x) - \frac{21+4x^2}{3} \frac{1-x}{1-x},$$

$$R(x) = \frac{11+4x^2}{3} + \frac{1}{6} \frac{1-x}{x} \left( 4 + 7x + 4x^2 \right) + (1+x) \ln x. \tag{81}$$

Eq. (80) describes the small–angle high–energy cross–section for the pair production process, provided that the created hard pair can move along both electron and positron beam directions.

The contribution to the cross–section of the small–angle Bhabha scattering connected with the real soft (with energy lower than $\Delta \epsilon$) and virtual pair production can be defined by the formula:

$$\sigma_{\text{soft+virt}} = 4 \frac{\alpha^4}{\pi Q_1^2} \int_1^{\rho^2} \frac{dz}{z^2} \left\{ L^2 \left( \frac{2}{3} \ln \Delta + \frac{1}{2} \right) + \mathcal{L} \left( \frac{17}{6} + \frac{4}{3} \ln^2 \Delta - \frac{20}{9} \ln \Delta - \frac{4}{3} \zeta_2 \right) \right\}. \tag{82}$$

Using Eqs. (80) and (82) it is easy to verify that the auxiliary parameter $\Delta$ is cancelled in the
sum $\sigma_{\text{pair}} = \sigma_{\text{hard}} + \sigma_{\text{soft+virt}}$. We can, therefore, write the total contribution $\sigma_{\text{pair}}$ as

$$
\sigma_{\text{pair}} = 2\frac{\alpha^4}{\pi Q_1^2} \int \frac{dz}{z^2} \left( L^2(1 + \frac{4}{3} \ln(1-x_c)) - \frac{2}{3} \int \frac{dx}{1-x} \bar{\Theta} \right) + \mathcal{L} \left[ -\frac{17}{3} \right]
$$

(83)

$$
-\frac{8}{3} \zeta_2 - \frac{40}{9} \ln(1-x_c) + \frac{8}{3} \ln^2(1-x_c) + \int \frac{dx}{1-x} \bar{\Theta} \cdot \frac{20}{9} - \frac{8}{3} \ln(1-x)
$$

$$
+ \int \frac{dx}{x_c} \left[ L^2(1+\Theta) \bar{R}(x) + \mathcal{L}(\Theta C_1(x) + C_2(x)) \right]
$$

$$
\bar{R}(x) = R(x) - \frac{2}{3(1-x)},
$$

$\bar{\Theta} = 1 - \Theta$.

The right-hand side of Eq. (83) gives the contribution to the small–angle Bhabha scattering cross-section for pair production. It is finite and can be used for numerical estimations. The leading term can be described by the electron structure function $D_e^e(x)$ [22].

### 8 Terms of $\mathcal{O}(\alpha \mathcal{L})^3$

In order to evaluate the leading logarithmic contribution represented by terms of the type $(\alpha \mathcal{L})^3$, we use the iteration up to $\beta^3$ of the master equation [21] obtained in Ref. [22]. To simplify the analytical expressions we adopt here a realistic assumption about the smallness of the threshold for the detection of the hard subprocess energy and neglect terms of the order of:

$$
x_c^n(\frac{\alpha}{\pi} \mathcal{L})^3 \leq 3 \cdot 10^{-5}, \quad n = 1, 2, 3.
$$

(84)

We may, therefore, limit ourselves to consider the emission by the initial electron and positron. Three photons (virtual and real) contribution to $\Sigma$ have the form:

$$
\Sigma^{3\gamma} = \frac{1}{4} (\frac{\alpha}{\pi} \mathcal{L})^3 \int \frac{dz}{z^2} \int \frac{dx_1}{x_c} \int \frac{dx_2}{x_c} \Theta(x_1 x_2 - x_c) \left[ \frac{1}{6} \delta(1-x_2) P^{(3)}(x_1) \right]
$$

(85)

$$
\times \Theta(x_1^2 \rho^2 - z) + \frac{1}{2x_1^2} \delta(1-x_1) \Theta(1-x_1 - \frac{2}{x_1^2}) P^{(2)}(x_1) P(x_2) \theta_1 \theta_2 \right] (1 + \mathcal{O}(x_1^3)),
$$

where $P(x)$ and $P^{(2)}(x)$ are given by Eqs. (33) and (59) correspondingly:

$$
\theta_1 \theta_2 = \delta(z - \frac{x_2}{x_1}) \Theta\left(\rho^2 \frac{x_2^2}{x_1^2} - z\right),
$$

$$
P^{(3)}(x) = \delta(1-x) \Delta_t + \Theta(1-x) \bar{\Theta} t,
$$

22
\[ \Delta_t = 48 \left[ \frac{1}{3} \zeta_3 - \frac{1}{2} \zeta_2 \left( \ln \Delta + \frac{3}{4} \right) \right] + \frac{1}{6} \left( \ln \Delta + \frac{3}{4} \right)^3, \quad (86) \]

\[ \Theta_t = 48 \left\{ \frac{1}{2} \ln z + \frac{1}{2} \ln \left( \frac{1}{1 - x} \right) + \frac{1}{2} \ln \left( \frac{1}{1 - x} \right) \right\} + \frac{1}{8} (1 + x) \ln \left( \frac{1}{1 - x} \right) + \frac{1}{16} (1 + x) \ln \left( \frac{1}{1 - x} \right) \}

The contribution to \( \Sigma \) of the process of pair production accompanied by photon emission when both, pair and photons, may be real and virtual has the form (with respect to paper by M. Skrzypek [22] we include also the non–singlet mechanism of pair production):

\[ \Sigma_{e^+e^-\gamma} = \frac{1}{4} (\frac{\alpha}{\pi})^3 \int \frac{dz}{z^2} \int \frac{dx_1}{x_c} \int \frac{dx_2}{x_c} \Theta(x_1 x_2 - x_c) \]

\[ \times \left\{ \frac{1}{3} [R^P(x_1) - \frac{1}{3} R^s(x_1)] \delta(1 - x_2) \Theta(x_1^2 \rho^2 - z) + \frac{1}{2} P(x_2) R(x_1) \Theta_1 \Theta_2 \right\}, \]

where

\[ R(x) = R^s(x) + \frac{2}{3} P(x), \quad R^s(x) = \frac{1 - x}{3x} (4 + 7x + 4x^2) + 2(1 + x) \ln x, \quad (87) \]

\[ R^P(x) = R^s(x) (\frac{3}{2} + 2 \ln(1 - x)) + (1 + x) (- \ln^2 x + 4 \text{Li}_2(1 - x) \]

\[ + \frac{1}{3} (-9 - 3x + 8x^2) \ln x + \frac{2}{3} (-\frac{3}{x} - 8 + 8x + 3x^2) + \frac{2}{3} P^{(2)}(x). \]

The total expression for \( \Sigma \) in Eq. (20) is the sum of the contributions in Eqs. (21), (32), (56), (60), (66) and (68). The quantity \( \Sigma \) depends on the parameters \( x_c, \rho \) and \( Q_2^2 \).

### 9 Estimates of neglected terms and numerical results

The uncertainty of our calculations is defined by neglected terms. Let us list them.

a) Terms of the first order RC coming from annihilation–type diagrams (15):

\[ \frac{\alpha}{\pi} \theta^2 \int \frac{d\theta}{\theta^2} \Delta_\theta \leq 0.10 \cdot 10^{-4}. \quad (88) \]

b) Similar terms in the second order do not exceed (see sect. 4)

\[ (\frac{\alpha}{\pi})^2 \int \frac{d\theta}{\theta^2} \leq 0.23 \cdot 10^{-4}, \quad (89) \]

\[ (\frac{\alpha}{\pi})^2 (\theta^2 - \theta^4) \leq 0.5 \cdot 10^{-5}. \]
c) We neglect terms which violate the eikonal approximation:

\[ \frac{\alpha}{\pi} \frac{Q^2}{s} \leq 0.3 \cdot 10^{-6}. \]  
(90)

d) We omit term of the second order which are not enhanced by large logarithms:

\[ (\frac{\alpha}{\pi})^2 = 0.5 \cdot 10^{-5}. \]  
(91)

e) Creation of heavy pairs (\(\mu\mu, \tau\tau, \pi\pi, \ldots\)) gives in sum at least one order of magnitude smaller than the corresponding contribution due to light particle production [25]:

\[ \Sigma_{\pi\pi} + \Sigma_{\mu\mu} + \Sigma_{\tau\tau} \leq 0.1 \Sigma^{ee+e-} \leq 0.5 \cdot 10^{-4}. \]  
(92)

f) Higher–order corrections, including soft and collinear multi-photon contributions, can be neglected since they only give contributions of the type \((\alpha L/\pi)^4 \leq 0.2 \cdot 10^{-5}\) or less.

g) The terms in the third order associated with the emission off the final particles:

\[ x_c (\frac{\alpha L}{\pi})^3 \leq 0.3 \cdot 10^{-4} \quad (\text{for } x_c = 0.5). \]  
(93)

Regarding all the uncertainties a)–g) and (82) as independent ones we conclude the total theoretical uncertainty of our results to be \(\pm 0.006\%\). ”

Let us define \(\Sigma^0\) to be equal to \(\Sigma_0|_{\Pi=0}\) (see Eq. (21)), which corresponds to the Born cross–section obtained by switching off the vacuum polarization contribution \(\Pi(t)\). For the experimentally observable cross–section we obtain:

\[ \sigma = \frac{4\pi \alpha^2}{Q^2} \Sigma^0_0 (1 + \delta_0 + \delta^\gamma + \delta^{2\gamma} + \delta^{ee+e-} + \delta^{3\gamma} + \delta^{ee+e-\gamma}), \]  
(94)

where

\[ \Sigma^0_0 = \Sigma_0|_{\Pi=0} = 1 - \rho^{-2} + \Sigma_W + \Sigma_{\theta}|_{\Pi=0} \]  
(95)

and

\[ \delta_0 = \frac{\Sigma_0 - \Sigma^0_0}{\Sigma^0_0}; \delta^\gamma = \frac{\Sigma^\gamma}{\Sigma^0_0}; \delta^{2\gamma} = \frac{\Sigma^{2\gamma}}{\Sigma^0_0}; \cdots . \]  
(96)

The numerical results are presented below in Table 1.

**Table 1:** The values of \(\delta^i\) in per cent for \(\sqrt{s} = 91.161\) GeV, \(\theta_1 = 1.61^\circ, \theta_2 = 2.8^\circ,\)
\(\sin^2 \theta_W = 0.2283, \Gamma_Z = 2.4857\) GeV.

---

8 Usually, in a calorimetric experimental set–up such terms do not contribute.
Each of these contributions to $\sigma$ has a sign that can change because of the interplay between real and virtual corrections. The cross-section corresponding to the Born diagrams for producing a real particle is always positive, whereas the sign of the radiative corrections depends on the order of perturbation theory. For the virtual corrections at odd orders it is negative, and at even orders it is positive. When the aperture of the counters is small the compensation between real and virtual corrections is not complete. In the limiting case of small aperture ($\rho \to 1$, $x_c \to 1$) the virtual contributions dominate.

The numerical results were obtained by using the NLLBHA fortran code [26].

The analytical and the numerical calculations for the cross-section in the non symmetrical Narrow-Wide configuration are in progress and will be presented elsewhere.

| $x_c$ | $\delta_0$ | $\delta^\gamma$ | $\delta_2^{\text{leading}}$ | $\delta_2^{\text{nonleading}}$ | $\delta^{e^+e^-}$ | $\delta^{e^+e^-\gamma}$ | $\delta^{3\gamma}$ | $\sum \delta^i$ |
|-------|-------------|-----------------|-----------------------------|-----------------------------|-------------------|----------------------|-----------------|-----------------|
| 0.1   | 4.120       | -8.918          | 0.657                       | 0.162                       | -0.016            | -0.017               | -0.019          | -4.031 ± 0.006  |
| 0.2   | 4.120       | -9.226          | 0.636                       | 0.156                       | -0.027            | -0.011               | -0.016          | -4.368 ± 0.006  |
| 0.3   | 4.120       | -9.626          | 0.615                       | 0.148                       | -0.033            | -0.008               | -0.013          | -4.797 ± 0.006  |
| 0.4   | 4.120       | -10.147         | 0.586                       | 0.139                       | -0.039            | -0.005               | -0.010          | -5.356 ± 0.006  |
| 0.5   | 4.120       | -10.850         | 0.539                       | 0.129                       | -0.044            | -0.003               | -0.006          | -6.115 ± 0.006  |
| 0.6   | 4.120       | -11.866         | 0.437                       | 0.132                       | -0.049            | -0.002               | -0.001          | -7.229 ± 0.006  |
| 0.7   | 4.120       | -13.770         | 0.379                       | 0.130                       | -0.057            | -0.001               | 0.005           | -9.194 ± 0.006  |
| 0.8   | 4.120       | -17.423         | 0.608                       | 0.089                       | -0.069            | 0.001                | 0.013           | -12.661 ± 0.006 |
| 0.9   | 4.120       | -25.269         | 1.952                       | -0.085                      | -0.085            | 0.005                | 0.017           | -19.379 ± 0.006 |
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References

[1] G. Altarelli, Lectures given at the E. Majorana Summer School, Erice, Italy, July 1993, CERN-TH.7072/93.

[2] LEP Electroweak Working Group, A Combination of Preliminary LEP Electroweak Results from the 1995 Summer Conferences, 1995, CERN report LEPEWWG/95–02.
B. Pietrzyk, High Precision Measurements of the Luminosity at LEP, Proceedings of the "Tennessee International Symposium on Radiative Corrections: Status and Outlook", Gatlinburg, Tennessee, 27 June– 1 July, 1994, Ed. B.F.L. Ward (World Scientific, Singapore 1995).

[3] Neutrino Counting in Z Physics at LEP, G. Barbiellini et al., L. Trentadue (conv.), G. Altarelli, R. Kleiss and C. Verzegnassi eds., CERN Report 89-08 (1989).

[4] W. Beenakker, F.A. Berends and S.C. van der Marck, Nucl. Phys. B355 (1991) 281;
M. Cacciari, A. Deandrea, G. Montagna, O. Nicrosini and L. Trentadue, Phys. Lett. B271 (1991) 431;
M. Caffo, H. Czyz and E. Remiddi, Nuovo Cim. 105A (1992) 277;
K.S. Bjorken, G. Fälldt and P. Osland, Nucl. Phys. B386 (1992) 280, 303;
W. Beenakker and B. Pietrzyk, Phys. Lett. B296 (1992) 241; and B304 (1993) 366.
M. Cacciari, G. Montagna, O. Nicrosini and F. Piccinini, Comput. Phys. Commun. 90 (1995) 301.

[5] S. Jadach et al., Phys. Rev. D47 (1993) 3733;
S. Jadach, E. Richter-Was, B.F.L. Ward and Z. Was, Phys. Lett. B260 (1991) 438, and 268 (1991) 253;
S. Jadach, M. Skrzypek and B.F.L. Ward, Phys. Lett. B257 (1991) 173.
S. Jadach et al., Phys. Lett. B353 (1995) 362.

[6] R. Budny, Phys. Lett. 55B (1975) 227;
D. Bardin, W. Hollik and T. Riemann, Z. Phys. C49 (1991) 485;
M. Boehm, A. Denner and W. Hollik, Nucl. Phys. B304 (1988) 687.

[7] V.S. Fadin, E.A. Kuraev, L.N. Lipatov, N.P. Merenkov and L. Trentadue, Yad. Fiz. 56 (1993) 145;
S. Jadach et al., Phys. Lett. B253 (1991) 469.

[8] V.S. Fadin, E.A. Kuraev, L.N. Lipatov, N.P. Merenkov and L. Trentadue, Yad. Fiz. 56 (1993) 145.
V.S. Fadin, E.A. Kuraev, L.N. Lipatov, N.P. Merenkov and L. Trentadue, Proceedings of the XXIX Rencontres de Moriond, Méribel, March 1994.
V.S. Fadin, E.A. Kuraev, L.N. Lipatov, N.P. Merenkov and L. Trentadue, same Proceedings as in Ref. 2 p.168.

[9] R.V. Polovin, JEPT 31 (1956) 449;
F.A. Redhead, Proc. Roy. Soc. 220 (1953) 219;
F.A. Berends et al., Nucl. Phys. B68 (1974) 541.

[10] S. Eidelman, F. Egerlehner, Z. Phys. C67 (1995) 585.

[11] E.A. Kuraev, L.N. Lipatov and N.P. Merenkov, Phys Lett. 47B (1973) 33; preprint 46
LNPI, 1973;
H. Cheng, T. T. Wu, Phys. Rev. 187 (1969) 1868;
V.S. Fadin, E.A. Kuraev, L.N. Lipatov, N.P. Merenkov and L. Trentadue, Yad. Fiz. 56
(1993) 145.

[12] G. Fäl dt and P. Osland, Nucl. Phys. B413 (1994) 16; Erratum ibidem B419 (1994) 404.

[13] D.R. Yennie, S.C. Frautchi, H. Suura, Ann. Phys. 13 (1961) 379.

[14] V.N. Baier, V.S. Fadin, V. Khoze and E.A. Kuraev, Phys. Rep. 78 (1981) 294;
V.M. Budnev, I.F. Ginzburg, G.V. Meledin and V.G. Serbo, Phys. Rep. C15 (1975) 183.

[15] V.G. Gorshkov, Uspekhi Fiz. Nauk 110 (1973) 45.

[16] R. Barbieri, J.A. Mignaco and E. Remiddi, Il Nuovo Cimento 11A (1972) 824.

[17] E.A. Kuraev, N.P. Merenkov, V.S. Fadin, Sov. J. Nucl. Phys. 45 (1987) 486.

[18] H. Cheng and T.T. Wu, Expanding Protons: Scattering at High Energies, London, Eng-
land, 1986.

[19] N.P. Merenkov, Sov. J. Nucl. Phys. 48 (1988) 1073.

[20] N.P. Merenkov, Sov. J. Nucl. Phys. 50 (1989) 469.

[21] L.N. Lipatov, Sov. J. Nucl. Phys. 20 (1974) 94;
G. Altarelli and G. Parisi, Nucl. Phys. B126 (1977) 298;
E. A. Kuraev and V. S. Fadin, Sov. J. of Nucl. Phys. 41 (1985) 466; Preprint INP 84-44,
Novosibirsk, 1984;
O. Nichro sini and L. Trentadue, Phys. Lett. 196B (1987) 551.

[22] M. Skrzypek, Acta Phys. Pol. B23 (1993), 135;
E.A. Kuraev, N.P. Merenkov and V.S. Fadin, Sov. J. Nucl. Phys. 47 (1988) 1009.
S. Jadach, M. Skrzypek, B.F.L. Ward, Phys. Lett. B257 (1991) p.173.
[23] A.B. Arbuzov, E.A. Kuraev, N.P. Merenkov and L. Trentadue, JETP 108 (1995) 1164; preprint CERN–TH/95–241, JINR E2–95–110, Dubna, 1995.

[24] A.B. Arbuzov, V.S. Fadin, E.A. Kuraev, L.N. Lipatov, N.P. Merenkov and L. Trentadue, Report CERN 95-03 (1995) p. 369.

[25] V.N. Baier, V.S. Fadin, V.M. Katkov, *Emission of relativistic electrons*, Moscow, Atomizdat, 1973.

[26] A short write-up of the NLLBHA code can be found in the CERN Yellow Report CERN-96-01, vol.2. A copy of the program is available, upon request, from the authors.
Appendix A

Infinite momentum frame kinematics

In this Appendix we will consider the kinematics we use to obtain the electron–positron
and photon distributions. Due to the peculiar range of momenta and angles of the reaction,
it is particularly convenient to use the Sudakov parametrization or infinite momentum frame
kinematics. For the reaction

\[ e^+(p_2) + e^-(p_1) \rightarrow e^+(q_2) + e^-(q_1) + \gamma(k) \]  

(A.1)

let us introduce the Sudakov decomposition:

\[
q_1 = \alpha_1 \tilde{p}_2 + \beta_1 \tilde{p}_1 + q_1^\perp, \quad q_2 = \alpha_2 \tilde{p}_2 + \beta_2 \tilde{p}_1 + q_2^\perp,
\]

\[
k = \alpha \tilde{p}_2 + \beta \tilde{p}_1 + k^\perp, \quad (A.2)
\]

where \( \tilde{p}_{1,2} \) are almost light-like four–vectors:

\[
q_1^\perp p_1 = q_2^\perp p_2 = 0, \quad (q_1^\perp)^2 = -(q_2^\perp)^2 < 0, \quad (A.3)
\]

\[
\tilde{p}_1 = p_1 - m^2 s p_2, \quad \tilde{p}_2 = p_2 - m^2 s p_1,
\]

\[
p_1^2 = q_1^2 = q_2^2 = m^2, \quad k^2 = 0, \quad \tilde{p}_1^2 = \tilde{p}_2^2 = m^2 s^2,
\]

\[
s = 2p_1 p_2 = 2\tilde{p}_1 \tilde{p}_2 = 2\tilde{p}_1 p_2 = 2\tilde{p}_2 p_1 \gg m^2, \quad (A.4)
\]

where \( q_i^\perp \) are Euclidean two–dimensional vectors in the center-of-mass reference frame.

We consider the kinematical configuration when the photon is emitted in the direction
close to the initial electron. We have the mass-shell conditions:

\[
q_1^2 = s \alpha_1 \beta_1 - (q_1^\perp)^2 = m^2, \quad \alpha_1 = \frac{(q_1^\perp)^2 + m^2}{s \beta_1}, \quad (A.5)
\]

\[
(q_2')^2 = s \alpha_2 \beta_2 - (q_2^\perp)^2 = m^2, \quad \beta_2 = \frac{(q_2^\perp)^2 + m^2}{s \alpha_2},
\]

\[
k^2 = s \alpha \beta - (k^\perp)^2 = 0, \quad s \alpha = \frac{(k^\perp)^2}{\beta},
\]

\[
\alpha_2 = 1, \quad |\beta_2| \sim |\alpha_1| \sim |\alpha| \ll 1, \quad \beta_1 \sim \beta \sim 1.
\]

The components along \( \tilde{p}_1 \) of the jets containing \( e^-(q_1) \) and \( \gamma(k) \) have a value of \( \mathcal{O}(1) \). The
phase volume decomposition with \( d^4q_1 = \frac{s}{2} d\alpha_1 d\beta_1 d^2q_1^\perp \) is:

\[
d\phi = \frac{d^3q_1 d^3q_2 d^3k}{2q_1^0 q_2^0 k^0 2\omega} \delta^{(4)}(p_1 + p_2 - q_1 - q_2 - k) \]

\[
= \frac{1}{4s \beta \beta_1} d\beta d\beta_1 \delta(1 - \beta - \beta_1) d^2k^\perp d^2q_1^\perp d^2q_2^\perp \delta^{(2)}(q_1^+ + q_2^+ + k^+). \]
The conservation law reads (we introduce a new four-momentum $q$ of the exchanged photon):

$$p_1 + q = q_1 + k, \quad p_2 = q_2 + q. \quad (A.7)$$

The inverse propagators are (here and further we use $\beta_1 = x$):

$$\begin{align*}
(p_1 - k)^2 - m^2 &= -\frac{1}{1 - x}d_1, \quad (p_1 + q)^2 - m^2 = \frac{1}{x(1 - x)}d, \\
q^2 &= -(q_2^\perp)^2, \quad d = m^2(1 - x)^2 + (q_1^\perp + q^\perp x)^2, \quad d_1 = m^2(1 - x)^2 + (q_1^\perp + q^\perp)^2.
\end{align*} \quad (A.8)$$

The matrix element reads

$$M = \frac{g^{\mu\nu}}{q^2}v(p_2)\gamma_\mu v(q_2)\bar{u}(q_1)O_\nu u(p_1)$$

$$O_\nu = \gamma_\nu \frac{\hat{p}_1 - \hat{k} + m}{(p_1 - k)^2 - m^2} \hat{e} + \hat{e} \frac{\hat{p}_1 + \hat{q} + m}{(p_1 + q)^2 - m^2} \gamma_\nu. \quad (A.9)$$

The following decomposition of the metric tensor $g_{\mu\nu}$ is used:

$$g_{\mu\nu} = g_{\mu\nu}^\perp + \frac{p_1^\perp p_2^\perp + p_1^\perp p_2^\perp}{p_1 p_2} \simeq \frac{2p_1^\perp p_2^\perp}{s} \left( 1 + \mathcal{O}(\frac{q^\perp}{s}) \right). \quad (A.10)$$

We use also the identity

$$p_2^\perp \bar{u}(q_1)O_\nu u(p_1) \equiv \bar{u}(q_1)\hat{v}_\rho u(p_1)e_\rho(k). \quad (A.11)$$

The generalized vertex $v_\rho$ has the form [14]:

$$v_\rho = s\gamma_\rho x(1 - x) \left( \frac{1}{d} - \frac{1}{d_1} \right) - \gamma_\rho \hat{k} \hat{p}_2 \frac{d}{d_1} x(1 - x) - \frac{\hat{p}_2 \hat{k} \gamma_\rho}{d_1}(1 - x). \quad (A.12)$$

The evaluation of the spin sum of the squared matrix element gives

$$\sum_{\text{spin}} |\bar{v}(q_2)\hat{p}_1 v(p_2)|^2 = \text{Tr} \hat{p}_2 \hat{p}_1 \hat{p}_2 \hat{p}_1 = 2s^2, \quad (A.13)$$

The squared matrix element for the single photon radiation is given by

$$R = -\frac{1}{4s^2} \text{Tr} (\hat{p}_1 + m) \hat{v}_\mu (\hat{p}_1 + \hat{k} - \hat{q} + m) \hat{v}_\mu$$

$$= x[-2xm^2(d - d_1)^2 + (q_2^\perp)^2(1 + x^2)dd_1] \frac{1}{d^2d_1^2}. \quad (A.14)$$

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Finally we obtain that
\[
\begin{align*}
\frac{d\sigma^{e^+e^-\to e^+(e^-\gamma)}}{d^2q_1^\perp d^2q_2^\perp dx_1 dx_2} &= 2\alpha^3 \frac{d^2q_1^\perp dq_2^\perp dx(1-x)}{\pi^2((a_2^2)^2)(dd_1)^2[-2xm^2(d-d_1)^2 + (q_2^\perp)^2(1+x^2)dd_1]}. \\
&= \frac{\alpha^4(1+x_1^2)(1+x_2^2)}{\pi^2(1-x_1)(1-x_2)} \int \frac{d^2q^\perp}{((q^\perp)^2)^2} \\
&\times \left[\frac{(q^\perp)^2(1-x_1)^2}{d_1d_2} - \frac{2x_1 m^2(1-x_1)^2(d_1-d_2)^2}{1+x_1^2} \right] \\
&\times \left[\frac{(q^\perp)^2(1-x_2)^2}{d_1d_2} - \frac{2x_2 m^2(1-x_2)^2(d_2-\tilde{d}_1)^2}{1+x_2^2} \right],
\end{align*}
\]  
(A.16)

where \(x_1, q_1^\perp\) and \(x_2, q_2^\perp\) are the energy fractions and the components transverse to the beam axis of the scattered electron and positron, respectively; \(q^\perp\) is the transverse two-dimensional momentum of the exchanged photon;

\[
\begin{align*}
\begin{array}{l}
d_1 = (1-x_1)^2m^2 + (q_1^\perp - q^\perp x_1)^2, \\
\tilde{d}_1 = (1-x_1)^2m^2 + (q_1^\perp + q^\perp x_1)^2,
\end{array}
\end{align*}
\]  
(A.17)

Let us now discuss the restrictions on the \(d^2q_1^\perp, d^2q_2^\perp\) integration imposed by experimental conditions of the electron and positron tagging. We consider the emission of a hard photon along the electron direction. We will consider the symmetric case:

\[
\begin{align*}
\theta_1 < \theta_e = \frac{|q_1^\perp|}{xe} < \theta_2, \\
\theta_1 < \theta_e = \frac{|q_2^\perp|}{e} < \theta_2,
\end{align*}
\]  
(A.18)

Here \(\theta_1\) and \(\theta_2\) are the minimal and maximal angles of aperture for the counters. It is convenient to introduce dimensionless quantities \(\rho = \theta_2/\theta_1, z_{1,2} = (q_{1,2})^2/Q_1^2\) \((Q_1 = e\theta_1)\).

The region in the \(z_1, z_2\) plane that gives the largest contribution to \(\Sigma\) is made by two narrow strips along the lines \(z_1 = z_2\) and \(z_1 = x^2z_2\). Therefore the leading logarithmic contribution will appear only in the cases where these lines cross the rectangle defined by \(x^2 < z_1 < \rho^2 x^2, 1 < z_2 < \rho^2\). Note that the line \(z_1 = x^2z_2\), which corresponds to the emission of one hard photon along the momentum of the scattered electron, is the diagonal of the rectangle defined above. As for the line \(z_1 = z_2\), which corresponds to the emission along the initial electron momentum, it crosses the rectangle only if \(x^2\rho^2 > z_2, x\rho > 1\). This last condition defines the appearance of leading contributions to \(\Sigma^H\).
For the contribution from the photon emission by the initial electron we have:

\[
F_1 = \Theta(1 - \rho x) \int_1^{\rho^2} \frac{dz_2}{z_2^2} \frac{x^2 \rho^2}{z_2^2} \int_x^{1 - \rho x} \frac{dz_1}{(z_1 - x z_2)(z_2 - z_1)} (1 - x)
\]

\[
+ \Theta(x \rho - 1) \int_1^{x^2 \rho^2} \frac{dz_2}{z_2^2} \frac{x^2 \rho^2}{z_2^2} \int_x^{1} \frac{dz_1}{(z_1 - x z_2)(z_2 - z_1)} (1 - x)
\]

\[
+ \Theta(x \rho - 1) \int_1^{x^2 \rho^2} \frac{dz_2}{z_2^2} \frac{x^2 \rho^2}{z_2^2} \int_x^{z_2 + \eta} \frac{dz_1}{(z_1 - x z_2)(z_2 - z_1)} + \int_1^{z_2 + \eta} \frac{dz_1}{(z_1 - x z_2)(z_2 - z_1)}
\]

\[
+ \int_{z_2 - \eta}^{1} \frac{dz_1}{\sqrt{R}} - \frac{2 x \sigma^2}{1 + x^2} \int_{z_2 - \eta}^{z_2 + \eta} \frac{dz_1}{\sqrt{R^3}}, \quad R = (z_2 - z_1)^2 + 4 \sigma^2 z_2,
\] (A.19)

where we introduced the auxiliary parameter \( \eta, \, \sigma^2 \ll \eta \ll 1 \). Neglecting the terms of order \( \eta \) we obtain:

\[
F_1 = \int_1^{\rho^2} \frac{dz}{z^2} \left\{ \Theta(x \rho - 1) \Theta(x^2 \rho^2 - z) \left( L - \frac{2x}{1 + x^2} \right) \right\}
\]

\[
+ \Theta(x^2 \rho^2 - z) L_2 + \Theta(z - x^2 \rho^2) L_3 \},
\] (A.20)

where \( L_i \) are given in eq. (B1) and we used the identity \( \Theta(1 - \rho x) + \Theta(\rho x - 1) \Theta(z - x^2 \rho^2) = \Theta(z - x^2 \rho^2) \).

In the same way we obtain for the final electron emission:

\[
F_2 = \int_1^{\rho^2} \frac{dz}{z^2} \left\{ L - \frac{2x}{1 + x^2} + L_1 \right\}. \] (A.21)

The total contribution due to one hard photon emission in small–angle Bhabha scattering therefore reads:

\[
\Sigma^H = \frac{\alpha}{\pi} \int_{x_c}^{1 - \Delta} dx \frac{1 + x^2}{1 - x} (F_1 + F_2). \] (A.22)
Appendix B

The contribution to $\Sigma$ from the semi-collinear region of emission of two hard photons in the same direction

An alternative way to use the quasi-real electron approximation is to compute the cross-section directly. To logarithmic accuracy we may restrict ourselves to considering only two regions i) the one with photon with momentum $q_1$ emitted along the momentum direction of the initial electron inside a narrow cone with opening angle $\theta_0 \ll 1$, and ii) the region with the photon emitted along the scattered electron. Taking into account the identity of photons with the statistical factor $\frac{1}{2}$ we obtain the cross-section:

$$d\sigma_{\Sigma SC} = \frac{\alpha^4}{2\pi} \int \frac{d^2 q_1^\perp}{\pi((q_1^\perp)^2)^2} \int \frac{d^2 q_2^\perp}{\pi} \frac{1-2\Delta}{x_c} \int dx \quad \text{(B.1)}$$

where

$$\int R \frac{d^2 k_1^\perp}{\pi} = 2(q_1^\perp)^2 Q_1^4 \int \frac{d^2 k_1^\perp}{\pi} \left\{ \frac{[1 + (1 - x_1)^2][x^2 + (1 - x_1)^2]}{x_1(1 - x_1)^2(2p_1 k_1)(2p_2 k_2)(2q_1 k_2)(k_1 \parallel p_1)} \right\} \quad \text{(B.2)}$$

It is convenient to specify the kinematics: in the case of the emission of the collinear photon with momentum $k_1$ parallel to $p_1$ we have

$$2p_1 k_1 = \frac{Q_1^2}{x_1}[(k_1^\perp)^2 + \sigma^2 x_1^2], \quad 2p_2 k_2 = \frac{Q_1^2}{x_2} (k_2^\perp)^2, \quad (B.3)$$

$$2q_1 k_2 = \frac{Q_1^2}{x_2 x} [x q_2^\perp - (1 - x_1) q_1^\perp]^2, \quad k_2^\perp = -q_2^\perp - q_1^\perp;$$

in the case when the photon is emitted along $q_1$ we have

$$2k_1 q_1 = \frac{Q_1^2}{x_1 x} [\sigma^2 x_1^2 + (x k_1^\perp - q_1^\perp)^2], \quad 2p_1 k_2 = \frac{Q_1^2}{x_2} (k_2^\perp)^2, \quad (B.4)$$

$$2q_1 k_2 = \frac{Q_1^2}{x_2 x} (q_1^\perp - x q_2^\perp)^2, \quad k_2^\perp = -q_2^\perp - q_1^\perp \frac{1 - x_2}{x},$$

where $Q_1^2 = \epsilon^2 \theta_1^2$, $\sigma^2 = m^2/Q_1^2$, and we introduced two-dimensional vectors $k_2^\perp$, $q_1^\perp$ and $q_2^\perp$ so that $(q_1^\perp)^2 = z_1$, $(q_2^\perp)^2 = z_2$ and $q_1^\perp q_2^\perp = \phi$. 

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The integration over $d^2k^\perp_1$ can be done with single logarithmic accuracy:

$$Q_1^2 \int \frac{d^2k^\perp_1}{\pi(2p_1k_1)} |_{k_1||p_i} = x_1 L, \quad Q_1^2 \int \frac{d^2k^\perp_1}{\pi(2q_1k_1)} |_{k_1||q_1} = \frac{x_1}{x} L. \quad (B.5)$$

It is also necessary, here, to consider the kinematical restrictions on the integration variables $\phi$ and $z_1$. When the photon is emitted within an angle $\theta_0$ along the direction of the momentum of the initial electron, $\theta_0$ represents the angular range to be filled by collinear kinematics events. We assign to the semi–collinear kinematics the events characterized by

\[ i) \quad \left| \frac{k^\perp_2}{x_2} \right| > \theta_0, \quad ii) \quad \left| \frac{q^\perp}{x} - \frac{k^\perp_2}{x_2} \right| > \theta_0, \quad (B.6) \]

where the region $i)$ the photon with four–momentum $k_2$ escapes the narrow cone with opening angle $\theta_0$ along the momentum direction of the initial electron. In the region $ii)$ the same happens for the final electron.

We can rewrite the conditions above in terms of the variables $z_1$ and $\phi$ as follows:

\[ i) \quad 1 > \cos \phi > -1 + \frac{\lambda^2 - (\sqrt{z_1} - \sqrt{z_2})^2}{2\sqrt{z_1}z_2}, \quad |\sqrt{z_1} - \sqrt{z_2}| < \lambda, \]

\[ ii) \quad 1 > \cos \phi > -1, \quad |z_1 - z_2| > 2\sqrt{z_2}\lambda, \]

\[ iii) \quad 1 > \cos \phi > -1 + \frac{x^2}{(1-x_1)}\lambda^2 - (\sqrt{z_1} - \frac{x}{1-x_1}\sqrt{z_2})^2}{2\sqrt{z_1}z_2\frac{x}{1-x_1}}, \quad |\sqrt{z_1} - \frac{x\sqrt{z_2}}{1-x_1}| < \lambda \frac{x}{1-x_1}, \]

\[ iv) \quad 1 > \cos \phi > -1, \quad |z_1 - \frac{x^2}{(1-x_1)^2z_2^2}| > 2\lambda\sqrt{z_2}(\frac{x^2}{(1-x_1)^2}), \]

where $\lambda = x_2\theta_0/\theta_1$. In our calculation we take the parameter $\lambda \ll 1$. Indeed, the restrictions on $\theta_0$ for collinear kinematics calculations are $\varepsilon\theta_0 \gg m$ or $\theta_0 \gg 10^{-5}$ at LEP energies. On the other hand the experimental conditions on $\theta_1$ are $\theta_1 > 10^{-2}$. Therefore we can take $\lambda \ll 1$ within our accuracy.

Analogous considerations can be made for the case when a photon with momentum $k_1$ is emitted along the direction of the final electron. In regions $ii)$ and $iv)$ we may do the integration over the azimuthal angle:

\[ \int_0^{2\pi} \frac{d\phi}{2\pi(2p_1k_2)(2q_1k_2)} |_{k_1||p_i} = \frac{x_2 x Q_1^{-4}}{(1-x_1)z_1 - x_2} \left[ \frac{1}{z_1 - z_1} - \frac{x(1-x_1)}{x^2 z_2 - (1-x_1)^2 z_1} \right]. \quad (B.7) \]

\[ \int_0^{2\pi} \frac{d\phi}{2\pi(2p_1k_2)(2q_1k_2)} |_{k_1||q_1} = \frac{x_2 x^3(1-x_2)^{-2}Q_1^{-4}}{z_1 - x^2 z_2/(1-x_2)} \left[ \frac{1}{z_1 - z_2} - \frac{1-x_2}{(1-x_2)^2 z_2} - \frac{1-x_2}{z_1 - x^2 z_2} \right]. \quad (B.8) \]
The integration of regions $i), iii)$ has the form

$$\mathcal{I} = 2 \int \frac{d\phi}{|z_1 - z_2|} \arctan\left\{ \left( \frac{\sqrt{|z_1| - |z_2|}}{|z_1 - z_2|} \tan \frac{\phi_0}{2} \right) \right\},$$

where

$$\phi_0 = \arccos\left(-1 + \frac{\lambda^2 - (\sqrt{|z_1| - |z_2|})^2}{2\sqrt{|z_1| z_2}}\right).$$

The result reads

$$\mathcal{I} = 2 \ln 2.$$  \hspace{1cm} (B.10)

We give here the complete contribution of the semi–collinear region:

$$d\sigma_{HH}^{\mu-coll} = \frac{\alpha^2 L}{4\pi^2} \int_{x_c}^{1-2\Delta} dx \int_{-\Delta}^{1-x-\Delta} dx_1 dx_2 \delta(1-x-x_1-x_2) \left[ 1 + (1-x_1)^2 \right] \left[ x^2 + (1-x_1)^2 \right]$$

$$\times \int \frac{dz}{z^2} \left[ \ln \frac{z\theta^2}{\lambda_0^2} \left[ 1 + \Theta(\rho^2 x^2 - z) + 2\Theta(\rho^2 (1-x_1)^2 - z) \right] \right]$$

$$+ \Theta(\rho^2 x^2 - z) \ln \frac{(z-x^2)(\rho^2 x^2 - z)}{x^2(1-x_1)^2(z-x(1-x_1))}$$

$$+ \Theta(z - \rho^2 (1-x_1)^2) \left[ \ln \frac{(z - \rho^2 (1-x_1)^2)(z - (1-x_1)^2)}{(\rho^2 (1-x_1)^2 - z)(z-x(1-x_1))} \right]$$

$$+ \frac{\ln \frac{\rho^2 (1-x_1)^2 - z}{(z - (1-x_1)^2)(\rho^2 (1-x_1)^2 - z)}}{\ln \frac{(z - (1-x_1)^2)(\rho^2 (1-x_1)^2 - z)}{(z-x(1-x_1))}}.$$
We see from the above expression that the dependence on $\theta_0 / \theta_1$ disappears in the sum of the contributions for the collinear and semi-collinear regions. The total sum is given by Eq. (56).
Appendix C

Virtual corrections to single photon emission cross-section

The cross-section for single hard photon bremsstrahlung containing virtual and real soft photon corrections may be written as follows:

$$d\sigma^{H(S+V)} = \frac{\alpha^3 dx d^2 q_2^\perp d^2 q_1^\perp}{2\pi^2 x(1-x)(q_2^\perp)^4} R, \quad R = \lim_{(2p_1p_2) \to \infty} \frac{4p_2\rho p_2\sigma K_{\rho\sigma}}{(2p_1p_2)^2}. \quad (C.1)$$

We define the Compton tensor with a heavy photon as [17]:

$$K_{\rho\sigma} = \frac{1}{8\pi\alpha} \sum_{spins} M_{\rho} M_{\sigma}^*, \quad (C.2)$$

where $M_{\rho}$ is the matrix elements of the Compton scattering process

$$\gamma^\nu(q) + e(p_1) \longrightarrow \gamma(k) + e(p_2), \quad (C.3)$$

$$k^2 = 0, \quad p_1^2 = p_2^2 = m^2,$$

$ho$ is the polarization index of the heavy photon ($q^2 \neq 0$). The Bose symmetry and the gauge-invariance requirements provide the following general form of $K_{\rho\sigma}$:

$$K_{\rho\sigma} = \frac{1}{2} (P_{\rho\sigma} + P_{\sigma\rho}^*), \quad (C.4)$$

$$P_{\rho\sigma} = \tilde{g}_{\rho\sigma} (B_g + \frac{\alpha}{2\pi} T_g) + \tilde{p}_{1\rho} \tilde{p}_{1\sigma} (B_{11} + \frac{\alpha}{2\pi} T_{11}) + \tilde{p}_{2\rho} \tilde{p}_{2\sigma} (\tilde{B}_{11} + \frac{\alpha}{2\pi} \tilde{T}_{11}) + \tilde{p}_{1\rho} \tilde{p}_{2\sigma} (B_{12} + \frac{\alpha}{2\pi} T_{12}) + \tilde{p}_{2\rho} \tilde{p}_{1\sigma} (\tilde{B}_{12} + \frac{\alpha}{2\pi} \tilde{T}_{12}),$$

$$q_{\rho} P_{\rho\sigma} = 0, \quad q_{\sigma} P_{\rho\sigma} = 0,$$

where

$$\tilde{g}_{\rho\sigma} = g_{\rho\sigma} - \frac{q_{\rho} q_{\sigma}}{q^2}, \quad \tilde{p}_{1\rho} = p_{1\rho} - q_{\rho} \frac{p_1 q}{q^2}, \quad \tilde{p}_{2\rho} = p_{2\rho} - q_{\rho} \frac{p_2 q}{q^2}. \quad (C.5)$$

Coefficients $B_i$ and $T_i$ before different tensor structures depend on the kinematical invariants

$$s = (p_2 + k)^2 - m^2, \quad t^2 = (p_1 - k)^2 - m^2, \quad u = (p_1 - p_2)^2, \quad s + t + u = q^2. \quad (C.6)$$

In the Born approximation we have

$$B_g = \frac{1}{st} [(s + u)^2 + (t + u)^2] - 2m^2 q^2 \left( \frac{1}{s^2} + \frac{1}{t^2} \right), \quad B_{12} = 0,$$

$$B_{11} = \frac{4q^2}{st} - \frac{8m^2}{s^2} \quad B_i = B_i(s, t, u), \quad \tilde{B}_i = B_i(t, s, u), \quad i = g, 12, 11. \quad (C.7)$$
Contributions in the one–loop approximation are:

\[ T_g = r B_g + \left\{ -\frac{1}{s} \left( \frac{u}{s} q^2 + \frac{1}{t} (2u b + s^2) \right) G + u \left( \frac{q^2}{st} - \frac{a}{c} \right) (l_q - l_u) \right\} \]

\[ T_{12} = \frac{2}{st} \left\{ \frac{q^2}{s^2} c(u - s) G + \frac{q^2}{t^2} (uq^2 - st) \tilde{G} - 2q^2 \left( \frac{uq^2}{st} + \frac{2u - s + t}{a} \right) (l_q - l_u) \right\} \]

\[ T_{11} = r B_{11} + \frac{2}{st} \left\{ -q^2 \left( 1 + \frac{u^2}{s^2} \right) G - q^2 \left( 2 + \frac{b^2}{t^2} \right) \tilde{G} + 2q^2 \left( \frac{b^2}{st} + \frac{2u}{a} \right) (l_q - l_u) \right\} \]

where the following notation is used:

\begin{align*}
\bar{t} &= t + m^2, \quad \bar{s} = s + m^2, \quad a = s + t, \quad b = s + u, \\
c &= t + u, \quad s_a = s - \frac{m^2}{u} a, \quad t_a = t - \frac{m^2}{u} a, \\
G &= (l_q - l_u)(l_q + l_u - 2l_t) - 2 \text{Li}_2(1) - 2 \text{Li}_2 \left( 1 - \frac{q^2}{u} \right) + 2 \text{Li}_2 \left( 1 - \frac{t}{q^2} \right), \\
F &= -\text{Li}_2 \left( 1 + \frac{t}{m^2} \right) + \text{Li}_2(1), \quad N = 2l_t(l_q - l_u) + 2 \text{Li}_2 \left( 1 - \frac{q^2}{u} \right) - 2F,
\end{align*}

\[ r = 4(l_u - 1) \ln \frac{\lambda}{m} - l_a^2 + 3l_q + \frac{\pi^2}{3} - \frac{9}{2}, \]
\[ Q = \frac{4c}{s} \left( 1 - \frac{m^2}{s} \right) \tilde{F} - 4b \frac{m^2}{t^2} F, \quad n = \frac{m^2}{t} \left( \frac{t}{l} - 1 \right), \]

\[ R = \frac{u}{l_\alpha} \left( 1 + \frac{m^2}{t_\alpha} \right) N - \frac{2u^2}{l_\alpha} \left[ \frac{1}{\alpha} (l_q - l_u) - \frac{1}{\alpha} \frac{q_\alpha l_t}{s} \right], \]

\[ l_t = \ln \left( -\frac{t}{m^2} \right), \quad l_q = \ln \left( -\frac{q^2}{m^2} \right), \quad l_u = \ln \left( -\frac{u}{m^2} \right), \quad l_s = \ln \left( -\frac{s - i0}{m^2} \right). \]

In order to avoid the dependence of the Compton tensor on the photon mass we need to take into account the emission of an additional soft photon with energy \( \omega \leq \Delta \varepsilon \). The corresponding contribution, which has to be added to the quantity \( r \), in the case of small angle Bhabha scattering reads [15]:

\[ \rho = 2 (l_u - 1) (2 \ln \frac{m}{\lambda} + 2 \ln \Delta - \ln x) + l_u^2 - \ln^2 x - \frac{\pi^2}{3}, \quad (C.10) \]

Note that in the case of small–angle Bhabha scattering the contributions of the \( \tilde{g}_{\rho\sigma} \) tensor structure is suppressed by factor \( \theta^2 \) relatively to terms containing \( \tilde{p}_\rho \tilde{p}_\sigma \).

We need only limited values of \( T_{ik} \) in the cases of \( s_1 \ll |t_1| \) and \( |t_1| \ll s_1 \) at fixed \( q^2 \) and \( u_1 = -2p_1q_1 \). In the case of small \( s_1 \) we have \( s_1 \equiv s = (m^2(1 - x)^2 + (q_x^\perp x + q_t^\perp)^2)/[x(1 - x)] \) (we omit in the remaining part of this Appendix the subscript 1 in the notation of invariants of the Compton subprocess).

Taking into account that, at small \( s \), \( q^2 = -(q_x^\perp)^2 \), \( t = -(1 - x)(q_x^\perp)^2 \) and \( u = -(q_t^\perp)^2 x \), we derive the following expressions for \( h = \rho + r \) and \( T \) in this limit:

\[ h_{s \ll |t|} = 2(L - 1 + \ln x) (2 \ln \Delta - \ln x) + 3L - \ln^2 x - \frac{9}{2}, \quad (C.11) \]

\[ T_{s \ll |t|} = \frac{2}{s(1 - x)} \left\{ 4(1 + x^2) \left[ \ln x \ \ln \frac{(q_x^\perp)^2}{s} - \text{Li}_2(1 - x) \right] \right. \]

\[ - \left. 1 + 2x + x^2 \right\} - \frac{16m^2}{s^2} \ln xL. \]

In the case of small \( |t| \) we have:

\[ h_{|t| \ll s} = 2(L - 1 - \ln x) (2 \ln \Delta - \ln x) + 3L - \ln^2 x - \frac{9}{2}, \quad (C.12) \]

\[ T_{|t| \ll s} = \frac{2x}{t(1 - x)} \left\{ 4(1 + x^2) \left[ \ln x \ \ln \frac{(q_x^\perp)^2}{-t} - \frac{1}{2} \ln^2 x - \text{Li}_2(1 - x) \right] \right. \]

\[ - \left. 1 - 2x + x^2 \right\} + \frac{16m^2x^2}{t^2} \ln xL. \]

One can be convinced that all terms reinforced by large logarithms \( (L \) or \( L^2 \)) arise only from these regions. We omit here only terms of order \( (\alpha/\pi)^2 \).
The further integration is straightforward. We show here the most important moments. The contribution of the $h$ containing terms gives (in close analogy with the Born contribution):

\[
\Sigma_h^{(S+S+)} = \frac{1}{2} \frac{\alpha}{\pi} \int \frac{d^2 \rho}{z^2} L \int \frac{dz}{x_c} \int \frac{dx}{1-x} \left\{ (1 + \Theta(\rho^2 x^2 - z)) \right\} \tag{C.13}
\]

\[
\times \left[ L(2 \ln \Delta - \ln x + \frac{3}{2}) + (2 \ln \Delta - \ln x)(\ln x - 2) - \frac{1}{2} \ln^2 x - \frac{15}{4} \right] \nonumber
\]

\[
+ (2 \ln \Delta - \ln x + \frac{3}{2}) h(x, z) - 2 \ln x(2 \ln \Delta - \ln x) \Theta(\rho^2 x^2 - z) \right\}
\]

To obtain the contributions from $T$ we consider at first new types of integrals:

\[
I_{s(t)} = \frac{Q_1^2}{\pi^2} \int \frac{d^2 q_2}{q_2^2} \int \frac{d^2 q_1}{q_1^2} \ln \left( \frac{q_2^2}{s}\right)^2, \tag{C.14}
\]

\[
i_{s(t)} = \frac{Q_1^2}{\pi^2} \int \frac{d^2 q_2}{q_2^2} \int \frac{d^2 q_1}{q_1^2} \pi s \{t\}, \quad m_{s(t)} = \frac{Q_1^2}{\pi q_2^2} \int \frac{d^2 q_2}{q_2^2} \int \frac{d^2 q_1}{q_1^2} m^2 \}
\]

Denoting $\sigma^2(1 - x)^2 + (q_2^2 x - q_1^2)^2/Q_1^2$ as $a + b \cos \phi$ and using the expressions

\[
\frac{1}{2\pi^2} \int_0^{2\pi} \frac{d\phi}{a + b \cos \phi} = \frac{1}{\sqrt{a^2 - b^2}}, \tag{C.15}
\]

\[
\frac{1}{2\pi^2} \int_0^{2\pi} \frac{d\phi}{a + b \cos \phi} = \ln \left( \frac{2(a^2 - b^2)}{a + \sqrt{a^2 - b^2}} \right)
\]

with $a^2 - b^2 = (z_1 - x^2 z_2)^2 + 4\sigma^2(1 - x)^2 x^2 z_2$, $\sigma^2 = m^2/Q_1^2$ and $z_{1,2} = (q_{1,2}^2)^2/Q_1^2$ we derive that

\[
I_s = (1 - x) \int \frac{d^2 z_2}{z_2^2} \int \frac{d^2 z_1}{z_1^2} \int \frac{d^2 \rho}{z_2^2} \int \frac{d^2 \rho}{x^2} \left[ \ln \left( \frac{z_2^2(1 - x) x^3}{z_1 - x^2 z_2} \right) - \ln \left( \frac{z_1 - z_2 x^2}{\sqrt{z_1 - x^2 z_2}^2 + 4\sigma^2 x^2(1 - x)^2 z_2} \right) \right] \nonumber
\]

Since we are evaluating with logarithmic accuracy, we may consider the contribution of the region $|z_1 - x^2 z_2| < \eta$, $\sigma^2 \ll \eta \ll 1$ when integrating over $z_1$. The result reads

\[
I_s = x(1 - x) \int \frac{d^2 z_2}{z_2^2} L \left[ \frac{1}{2} L + \ln \frac{x}{1 - x} \right]. \tag{C.17}
\]
The rest of the integrals can be calculated in the same way, and we have:

\[ I_t = -(1 - x) \int_1^{z_2^2} \frac{dz_2}{z_2^2} L \left[ \frac{1}{2} L + \ln \frac{1}{1 - x} \right], \quad i_t = -(1 - x) \int_1^{z_2^2} \frac{dz_2}{z_2^2} L, \]  
\[ i_s = x(1 - x) \int_1^{z_2^2} \frac{dz_2}{z_2^2} L, \quad m_t = \int_1^{z_2^2} \frac{dz_2}{z_2^2}, \quad m_s = x^2 \int_1^{z_2^2} \frac{dz_2}{z_2^2}. \]  

Using (C.17) and (C.18) we may represent the final result for the contribution to \( \Sigma^{H(S+V)} \) due to the \( T \) term as

\begin{align*}
\Sigma_T^{H(S+V)} &= \frac{1}{2} \alpha_0^2 \int_1^{z_2^2} \frac{dz}{z^2} L \int_1^{1 - \Delta} dx \frac{1 + x^2}{1 - x} \left\{ \left( \frac{1}{2} L + \ln \frac{1}{1 - x} \right) \ln x ight. \\
&\quad + \zeta_2 - \text{Li}_2(x) + \frac{x^2 + 2x - 1}{4(1 + x^2)} - \frac{2x \ln x}{1 + x^2} - \Theta(x^2 - z) \left( \frac{1}{2} L + \ln \frac{1}{1 - x} \right) \ln x \\
&\quad - \left. \frac{1}{2} \ln^2 x - \text{Li}_2(1 - x) - \frac{1 + 2x - x^2}{4(1 + x^2)} - \frac{2x \ln x}{1 + x^2} \right\}. \\
\end{align*}

The total contribution to \( \Sigma^{H(S+V)} \) (one-side hard photon emission with virtual and soft photon corrections) is the sum of (C.13) and (C.19):

\[ \Sigma^{H(S+V)} = \Sigma_\rho^{H(S+V)} + \Sigma_T^{H(S+V)}. \]  

This quantity is given in Eq. (52).
Appendix D

Leading logarithmic contribution to $\Sigma^{\gamma\gamma}$

Here we show that the main logarithmic terms can be summed according to the renormalization group. The sum, as given in the text, may be written as:

\[
S = 2 \left( \frac{\alpha}{\pi} \right)^2 \int_1^{\rho^2} \frac{dz}{z^2} L^2 \left\{ \int_{x_c}^1 dx \delta(1-x) \left( \ln^2 \Delta + \frac{3}{2} \ln \Delta + \frac{9}{16} \right) \right. \\
\times \left(1 + \Theta(x^2 \rho^2 - z)\right) + \frac{1}{2} \int_{x_c}^{1-\Delta} \frac{1+x^2}{1-x} \left(2 \ln \Delta - \ln x + \frac{3}{2} \right) (1 + \Theta(x^2 \rho^2 - z)) \\
+ \frac{1}{4} \int_{x_c}^{1-2\Delta} dx \left[ \frac{2(1+x^2)}{1-x} \ln \frac{1-x-\Delta}{\Delta} + \frac{1}{2} (1+x) \ln x - 1 + x \right] [1 + 3 \Theta(x^2 \rho^2 - z)] \\
+ \frac{1}{4} \int_{x_c}^1 dx \left[ \frac{2(1+x^2)}{1-x} \ln \left( \frac{(1-x-\Delta)(\rho - \sqrt{z})}{\Delta(\sqrt{z} - \rho x)} \right) \right] + x - 1 \\
- \frac{1}{2} (1+x) \ln \frac{\rho^2}{z} + \frac{\sqrt{z}}{\rho} - \frac{x \rho}{\sqrt{z}} \right \} \Theta(z - x^2 \rho^2) \right\}.
\]

One can see that the dependence on the parameter $\Delta$ disappears in the expression above.

We will now show that eq. (D.1) is equivalent to eq. (59). Let us transform eq. (59) using the substitution

\[
\Theta(t^2 \rho^2 - z) = \frac{1}{2} (1 + \Theta(x^2 \rho^2 - z)) + \frac{1}{2} \Theta(z - x^2 \rho^2) - 3 \Theta(z - t^2 \rho^2),
\]

and changing the order of integration in the last term:

\[
\int_1^{\rho^2} \frac{dt}{x} \int_{\rho^2 x^2}^{\rho^2} \frac{dz}{z^2} = \int_{\rho^2 x^2}^{\rho^2} \frac{dz}{z^2} \int_1^{\rho^2} \frac{dt}{x} = \int_1^{\rho^2} dz \Theta(z - \rho^2 x^2) \int x \, dt.
\]

By evaluating the integral over $t$, and using the explicit expressions for the splitting functions one can verify the coincidence of Eqs. (D.1) and (59). In an analogous way one can prove the validity of the representation (61) for $\Sigma^\gamma_\gamma$.

Using the representation in Eq. (60) for the function $P^{(2)}$ one can see that the above expression is equivalent to Eq. (D.1). In an analogous way one can prove the validity of representation in Eq. (61) for $\Sigma^\gamma_\gamma$.
Appendix E

Cancellation of the $\Delta$ dependence in the nonleading contributions to $\Sigma^{2\gamma}$

Let us consider the singular nonleading terms in $\Sigma^{2\gamma}$ in the limiting case $\Delta \to 0$. Dropping the common factor $(\alpha/\pi)^2 L \int dz/z^2$, we give below the various contributions separately.

Let us consider first $\Sigma^{\gamma\gamma}$. The contributions from the soft photon radiation and virtual corrections are:

$$ (\Sigma^{VV+VS+SS})_\Delta = \ln \Delta(-7 - 4 \ln \Delta)(1, \rho^2), \quad (E.1) $$

where we denote by $(a, b)$ the limits of the integration over $z$: $(a, b) = \Theta(z - a)\Theta(b - z)$.

The contribution due to the virtual corrections to the single hard photon emission gives:

$$ (\Sigma^{H(S+V)})_\Delta = \frac{1}{2} \ln \Delta \left\{ (1, \rho^2)(16 \ln \Delta + 14) + 4 \int \limits_{\tilde{x}_c}^1 dx \, \frac{1 + x^2}{1 - x} \left[ (1, \rho^2) - (1, \rho^2 x^2) \right] \right\}, \quad (E.2) $$

where $\tilde{x}_c = \max(x_c, 1/\rho)$ and the quantity $k(x, z)$ is defined in Eq. (30). The singular part in the contribution in Eq. (56) due to double hard photon bremsstrahlung reads:

$$ (\Sigma^{HH})_\Delta = \ln \Delta \left\{ \int \limits_{x_c}^1 dx \, \frac{1 + x^2}{1 - x} \left[ -(1, \rho^2) L_1 - (1, \rho^2 x^2) L_2 - ((1, \rho^2) - (1, \rho^2 x^2)) L_3 \right] \right\} (E.3) $$

It is possible to verify the cancellation:

$$ (\Sigma^{VV+VS+SS})_\Delta + (\Sigma^{H(S+V)})_\Delta + (\Sigma^{HH})_\Delta = 0. \quad (E.4) $$

In our contribution to the Tennessee–94 workshop (see [2]) there are some misprints, which are corrected here.
The corresponding contributions to $\Sigma_\gamma$ are:

\[
\begin{align*}
(\Sigma_{S+V})_\Delta &= \ln \Delta (-14 - 8 \ln \Delta)(1, \rho^2), \\
(\Sigma_{S+V} + \Sigma_{H}^{S+V})_\Delta &= \ln \Delta \left\{ 2 \int_{x_c}^1 dx \frac{1 + x^2}{1 - x} k(x, z) + (1, \rho^2) \right\} \left[ 16 \ln \Delta + 14 - 8 \ln(1 - x_c) \\
&\quad - 8 \ln(1 - \tilde{x}_c) + 4 \int_{x_c}^1 dx (1 + x) + 4 \int_{\tilde{x}_c}^1 dx (1 + x) \right] \\
&\quad + 4 \int_{\tilde{x}_c}^1 dx \frac{1 + x^2}{1 - x} [(1, \rho^2) - (1, \rho^2 x^2)], \\
(\Sigma_{H})_\Delta &= \ln \Delta \left\{ -8(1, \rho^2) \right\} \left[ \ln \Delta - \ln(1 - x_c) - \ln(1 - \tilde{x}_c) \right] \\
&\quad - 8 \int_{x_c}^1 dx \frac{(1, \rho^2) - (1, \rho^2 x^2)}{1 - x} - 2 \int_{x_c}^1 dx k(x, z) \frac{1 + x^2}{1 - x} \\
&\quad - 4 \int_{x_c}^1 dx (1 + x) [(1, \rho^2) + (1, \rho^2 x^2)] \right\}.
\end{align*}
\]

Rearranging the last term in $(\Sigma_{H})_\Delta$ as

\[
-4 \int_{x_c}^1 dx (1 + x) [(1, \rho^2) + (1, \rho^2 x^2)] = -4 \int_{x_c}^1 dx (1 + x)(1, \rho^2)
\]

\[= -4 \int_{\tilde{x}_c}^1 dx (1 + x)(1, \rho^2) + 4 \int_{\tilde{x}_c}^1 dx (1 + x) [(1, \rho^2) - (1, \rho^2 x^2)],
\]

we can see again the cancellation of the $\Delta$-dependence in the sum:

\[
(\Sigma_{S+V})_\Delta + (\Sigma_{S+V} + \Sigma_{H}^{S+V})_\Delta + (\Sigma_{H})_\Delta = 0.
\]
Appendix F

Relevant integrals for collinear pair production

We give here a list of the relevant integrals, calculated within the logarithmic accuracy, for
the collinear kinematical region of hard pair production.

We use the definitions in Eq. (67) and we imply, in the left-hand side of the expressions
below, the general operation:

$$\langle \ldots \rangle \equiv \int_0^{z_0} \int_0^{z_0} \frac{2\pi d\phi}{2\pi} (\ldots),$$

with the conditions $z_0 = (\varepsilon \theta_0/m)^2 \gg 1$, $L_0 = \ln z_0 \gg 1$. The details of the calculations can
be found in the Appendix of Ref. [19]. The results are:

$$\langle \frac{x_2 D + (1 - x_2) A}{DC} \rangle^2 = \frac{L_0}{(1 - x_2)^2} \left\{ L_0 + 2 \ln \frac{x_1 x_2}{x} - 8 \right\},$$

(F.2)

$$\langle \frac{(x_2 A_1 - x_1 A_2)^2}{AD} \rangle = \frac{L_0}{x_1 x_2 (1 - x_2)} \left\{ L_0 + 2 \ln \frac{x_1 x_2}{x} - 8 + \frac{(1 - x)^2}{x x_1 x_2} - \frac{4(1 - x)}{x} \right\},$$

$$\langle \frac{x_1 A_2 - x_2 A_1}{AD^2} \rangle = \frac{(x_1 - x_2) L_0}{x_1 x_2 (1 - x)}, \quad \langle \frac{1}{AD} \rangle = \frac{L_0}{x_1 x_2 (1 - x)}, \quad \langle \frac{1}{A^2 D} \rangle = \frac{L_0}{(1 - x)^3},$$

$$\langle \frac{1}{A} \rangle = \frac{L_0}{x_1 x_2 (1 - x)}, \quad \langle \frac{1}{C^2 D} \rangle = \frac{L_0}{x_1 (1 - x)^3},$$

$$\langle \frac{A}{C^2 D^2} \rangle = \frac{x_2 L_0}{x_1 (1 - x)^4}, \quad \langle \frac{C}{A^2 D^2} \rangle = \frac{x_2 L_0}{(1 - x)^4},$$

$$\langle \frac{A}{CD^2} \rangle = \frac{-L_0}{x_1 (1 - x)^2} \left\{ \frac{1}{2} L_0 + \ln \frac{x_1 x_2}{x} \right\} + \frac{x_2 x - x}{x x_1 x_2 (1 - x)^2},$$

$$\langle \frac{C}{AD^2} \rangle = \frac{-L_0}{x_1 (1 - x)^2} \left\{ \frac{1}{2} L_0 + \ln \frac{x_1 x_2}{x} \right\} - L_0 \left( \frac{x_1 - x_2}{x x_2 (1 - x)} + \frac{1}{xx_2 (1 - x)} \right).$$