Analytic results for Gaussian wave packets in four model systems:

II. Autocorrelation functions

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Abstract

The autocorrelation function, $A(t)$, measures the overlap (in Hilbert space) of a time-dependent quantum mechanical wave function, $\psi(x,t)$, with its initial value, $\psi(x,0)$. It finds extensive use in the theoretical analysis and experimental measurement of such phenomena as quantum wave packet revivals. We evaluate explicit expressions for the autocorrelation function for time-dependent Gaussian solutions of the Schrödinger equation corresponding to the cases of a free particle, a particle undergoing uniform acceleration, a particle in a harmonic oscillator potential, and a system corresponding to an unstable equilibrium (the so-called ‘inverted’ oscillator.) We emphasize the importance of momentum-space methods where such calculations are often more straightforwardly realized, as well as stressing their role in providing complementary information to results obtained using position-space wavefunctions.
I. INTRODUCTION

Despite widely varying approaches to the subject, every introductory quantum mechanics text of which we are aware discusses the subject of wave function normalization and conservation of probability, proving that

\[ \langle \psi_t | \psi_t \rangle = \langle \psi_0 | \psi_0 \rangle = 1 \]  

provided the Hamiltonian describing the system is Hermitian. This important relationship is perhaps most often discussed in position-space using

\[ \langle \psi_t | \psi_t \rangle = \int_{-\infty}^{+\infty} \psi^*(x,t) \psi(x,t) \, dx = 1 \]  

but, using the Fourier transform,

\[ \psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{ipx/h} \phi(p,t) \, dp, \]  

it is equally well written in the form

\[ \langle \psi_t | \psi_t \rangle = \int_{-\infty}^{+\infty} \phi^*(p,t) \phi(p,t) \, dp = 1. \]  

For bound state problems, where any time-dependent state can be expanded in terms of energy eigenstates, \( u_n(x) \), with quantized energies, \( E_n \), we can also write

\[ \psi(x,t) = \sum_n a_n u_n(x) e^{-iE_nt/h} \quad \text{where} \quad a_n = \int_{-\infty}^{+\infty} [u_n(x)]^* \psi(x,0) \, dx \]  

so that the normalization condition is given by

\[ \langle \psi_t | \psi_t \rangle = \sum_{n=1}^{\infty} |a_n|^2 = 1. \]

An important related concept, namely the autocorrelation function \([1], A(t)\), is defined as the overlap of a time-dependent state, \( \psi_t \), with its initial value, \( \psi_0 \), namely

\[ A(t) \equiv \langle \psi_t | \psi_0 \rangle = \int_{-\infty}^{+\infty} \psi^*(x,t) \psi(x,0) \, dx = \int_{-\infty}^{+\infty} \phi^*(p,t) \phi(p,0) \, dp \]  

and is extensively used in the study of the time-development of quantum wave packets in bound state systems. One clearly has \( A(0) = 1 \), by definition, if the initial wave function is itself properly normalized, but generally \( |A(t)| < 1 \) for later times as the wave packet develops in time and the different energy/momentum components evolve differently. Besides
being of obvious theoretical value in the analysis of time-dependent systems, the autocorrelation function is physically important because it is very directly related to the observable ionization signal in the pump-probe type experiments where the time-development of atomic wave packets is studied experimentally \[2], \[3].

For a single (bound) energy eigenstate or stationary state, where \(\psi(x, t) = u_n(x)e^{-iE_n t/\hbar}\), one clearly has

\[
A(t) = e^{+iE_n t/\hbar}
\]
giving

\[
|A(t)|^2 = 1.
\]

For more general bound state systems as in Eqn. (5), however, the most appropriate form is

\[
A(t) = \langle \psi_t | \psi_0 \rangle = \sum_{n=1}^{\infty} |a_n|^2 e^{+iE_n t/\hbar}
\]

so that information on the non-trivial time-development of the wave packet is encoded in the energy eigenvalue spectrum. For wave packets which are constructed from energy eigenstates centered around some large value of \(n = n_0\), one can write

\[
E(n) \approx E(n_0) + E'(n_0)(n - n_0) + \frac{E''(n_0)}{2}(n - n_0)^2 + \cdots
\]

which gives the time-dependence of each individual quantum eigenstate as

\[
e^{-iE_n t/\hbar} = \exp \left( -i/\hbar \left[ E(n_0)t + (n - n_0)E'(n_0)t + \frac{1}{2}(n - n_0)^2E''(n_0)t + \cdots \right] \right)
\]

\[
= \exp \left( -i\Omega_0 t - 2\pi i(n - n_0)t/T_{cl} - 2\pi i(n - n_0)^2t/T_{rev} + \cdots \right)
\]

in terms of which the classical period and quantum mechanical revival time (discussed below) are given respectively by

\[
T_{cl} = \frac{2\pi\hbar}{|E'(n_0)|} \quad \text{and} \quad T_{rev} = \frac{2\pi\hbar}{|E''(n_0)|/2}
\]

and the common first term, \(\exp(-i\Omega_0 t) = \exp(-iE(n_0)t/\hbar)\), is an unimportant overall phase. The most familiar example is the harmonic oscillator, with \(E_n = (n + 1/2)\hbar\omega\), and any wavepacket in this system is periodic with period \(T_{cl} = 2\pi/\omega\) and \(T_{rev} \to \infty\) since \(E''(n_0) = 0\).

This form of the autocorrelation function is especially useful in the context of quantum wave packet revivals \[1\], that is, systems where initially localized states which have a short-term, quasi-classical time evolution, can spread significantly over several orbits, only to reform later in the form of a quantum revival in which the spreading reverses itself, the
wave packet relocalizes, and the semi-classical periodicity is once again evident. The presence of an approximate quantum revival at a later time, \( t = T_{\text{rev}} \), is indicated by \( |A(T_{\text{rev}})| \approx 1 \), accompanied by evidence of the return of the short-term periodicity in \( A(t) \) with the classical period, \( T_{cl} \). The phase-structure of \( A(t) \) can also yield useful information, as parametric plots of \( Re[A(t)] \) versus \( Im[A(t)] \) (Argand diagrams) can provide striking visualizations of the highly correlated Schrödinger cat-like states which evolve at fractional multiples of the revival time, the mathematics of which was first worked out in detail by Averbukh and Perelman.

Such revival phenomena have been observed in a wide variety of physical systems, especially in Rydberg atoms. In many cases, the autocorrelation function is closely related to experimentally observable quantities, and its use has become very familiar for analyzing model systems exhibiting exact or approximate wave packet revivals.

We note that early investigators commented on the existence of a ‘quantum recurrence theorem’ which utilized the notion of ‘distance’ (in Hilbert space) between a time-dependent quantum state and its initial value as

\[
||\psi_t - \psi_0||^2 = \int_{-\infty}^{+\infty} |\psi(x, t) - \psi(x, 0)|^2 dx
\]

\[
\begin{align*}
\quad & = \int_{-\infty}^{+\infty} |\psi(x, t)|^2 dx + \int_{-\infty}^{+\infty} |\psi(x, 0)|^2 dx \\
& \quad - \int_{-\infty}^{+\infty} \psi^*(x, t)\psi(x, 0) dx - \int_{-\infty}^{+\infty} \psi^*(x, 0)\psi(x, t) dx \\
& \quad = 2 \left( 1 - \Re \left[ \int_{-\infty}^{+\infty} \psi^*(x, t)\psi(x, 0) dx \right] \right) = 2(1 - \Re[A(t)])
\end{align*}
\]

which is also related to \( A(t) \); we also note that Baltz [10] has considered similar ideas in a more pedagogical context.

In contrast to the research literature, where \( A(t) \) is now a standard tool, there are few, if any, examples of the evaluation of the autocorrelation function for the many familiar and frequently studied model systems of introductory quantum mechanics. Besides being useful as a diagnostic for the rate of time-evolution of a quantum state, such calculations of \( A(t) \) can also help answer questions such as How similar are two quantum states?, both in magnitude and phase, and we will stress the complementary roles that position-space and momentum-space approaches can have in addressing such issues in what follows.

In this note, we will use the special properties of Gaussian wave packet solutions to
explicitly evaluate $A(t)$, obtaining closed form expressions, for four familiar and accessible model systems. We focus on the cases of a free particle (Sec. II), a particle undergoing uniform acceleration (Sec. III), a particle in a harmonic oscillator potential (Sec. IV), and a system corresponding to an unstable equilibrium (the ‘inverted’ oscillator, in Sec. V.) The first two cases do not correspond to bound state systems, but do provide useful results for comparison to the short-term time-dependence of wave packets in systems such as the infinite well \[1] - [7] and the so-called ‘quantum bouncer’ [18] - [21] where, for small times at least, the wave packet propagation is similar to the corresponding unbound case. For the case of the harmonic oscillator, any wave packet solution (Gaussian or not) is explicitly periodic with $T_{cl} = 2\pi/\omega$ and the wavepacket never enters a truly ‘collapsed’ phase; the oscillator does, however, provide a useful explicit example illustrating the exactly periodic behavior of $A(t)$, seen more approximately in many bound state systems. For example, Nauenberg [1] has provided an elegant general description of the medium-term time-development of $A(t)$ for a general, one-dimensional, bound state system and the exact results presented here for the harmonic oscillator can be used as an efficient ‘benchmark’ for comparison to that more general analysis. While we will focus on closed-form results for oscillator wave packets (obtained using propagator methods), we can also make contact with the expansion in eigenstates in Eqn. (5) as well.

Such studies of the general behavior of $A(t)$ for many standard examples are also useful as they complement existing work on the rate of wave packet spreading [22] - [24] and especially on the time evolution of quantum states [25], [26]. In this last context, the example provided here for the free particle (in Sec. II) can be used as a specific case to confirm a (hopefully) well-known result for isolated quantum systems [27] - [29], namely

$$|\langle \psi_t | \psi_0 \rangle|^2 \geq \cos^2 \left( \frac{\Delta H t}{\hbar} \right) \quad \text{for} \quad 0 \leq t \leq \frac{\pi \hbar}{2\Delta H}$$

(14)

where $\Delta H = \sqrt{\langle H^2 \rangle - \langle H \rangle^2}$ is the uncertainty in the free-particle energy of the wave packet.

Because our presentation here will use the same notation and many of the same methods as the companion paper [30], we will refer extensively to results from that paper, especially for properties of the standard Gaussian wave packets we utilize.
II. FREE-PARTICLE GAUSSIAN WAVE PACKETS

The most general free-particle, momentum-space and position-space Gaussian wave packets, with arbitrary initial values of \( \langle x \rangle_0 = x_0 \) and \( \langle p \rangle_0 = p_0 \), can be written in the form

\[
\phi(p, t) = \phi_0(p) e^{-i p^2 t / 2m \hbar} = \sqrt{\frac{\alpha}{\sqrt{\pi}}} e^{-\alpha^2 (p-p_0)^2 / 2} e^{-i p x_0 / \hbar} e^{-i p^2 t / 2m \hbar} \tag{15}
\]

and

\[
\phi(p, 0) = \sqrt{\frac{\alpha}{\sqrt{\pi}}} e^{-\alpha^2 (p-p_0)^2 / 2} e^{-i p x_0 / \hbar} \tag{16}
\]

and

\[
\psi(x, t) = \frac{1}{\sqrt{\sqrt{\pi} \alpha \hbar (1 + it/\hbar t_0)}} e^{ip_0(x-x_0)/\hbar} e^{-i p^2 t / 2m \hbar} e^{-(x-x_0-\hbar \alpha t / m)^2 / 2(\hbar \alpha)^2 (1 + it/\hbar t_0)} \tag{17}
\]

\[
\psi(x, 0) = \frac{1}{\sqrt{\sqrt{\pi} \alpha \hbar}} e^{ip_0 (x-x_0)/\hbar} e^{-(x-x_0)^2 / 2(\hbar \alpha)^2} \tag{18}
\]

where \( t_0 \equiv m \hbar \alpha^2 \) defines the spreading time. The calculation of \( A(t) \) is done most straightforwardly in momentum-space where

\[
A(t) = \int_{-\infty}^{+\infty} \phi^*(p, t) \phi(p, 0) \, dp = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\alpha^2 (p-p_0)^2} e^{i p^2 t / 2m \hbar} \tag{19}
\]

where

\[
|A(t)|^2 = \frac{1}{\sqrt{1 + (t/2t_0)^2}} \exp \left[ -2 \alpha^2 \frac{p_0^2}{\hbar} \frac{(t/2t_0)^2}{1 + (t/2t_0)^2} \right]. \tag{20}
\]

The same result can, of course, be obtained in position-space, with

\[
A(t) = \int_{-\infty}^{+\infty} \psi^*(x, t) \psi(x, 0) \, dx = \frac{1}{\sqrt{1 - it / 2t_0}} e^{i p_0 x / \hbar} e^{i p^2 t / 2m \hbar} \exp \left[ -\frac{(p_0 t / m)^2}{4(\hbar \alpha)^2 (1 - it / 2t_0)} \right] \tag{21}
\]

which is easily seen to be identical to Eqn. (17) with a minimum of manipulation.

We note that for times satisfying \( t << t_0 \) there is an increasing exponential suppression of the overlap between \( \psi_t \) and \( \psi_0 \), but the exponential factor does ‘saturate’ for long times, giving

\[
|A(t >> 2t_0)|^2 \rightarrow \frac{2t_0}{t} \exp \left[ -\frac{p_0^2}{\Delta p_0^2} \right] \text{ since } \Delta p_0 = \frac{1}{\alpha \sqrt{2}}. \tag{22}
\]

The asymptotic form of the exponential factor can be understood by noting that the ‘distance in position space’ between the initial ‘peak’ at \( \langle x \rangle_0 = x_0 \), and that at later times when \( \langle x \rangle_t = x_0 + \hbar \alpha t / m \), grows linearly with \( t \), while for long times the position spread,

\[
\Delta x_t = \Delta x_0 \sqrt{1 + (t/t_0)^2} \quad \rightarrow \quad \Delta x_0 \frac{t}{t_0}, \tag{23}
\]
grows in the same way. This leads to factors in the exponent of the form
\[
\frac{(x(t) - x(0))^2}{(\Delta x_t)^2} \rightarrow \frac{(p_0 t/m)^2}{(\Delta x_0(t/t_0))^2} \approx \left( \frac{p_0 t_0}{m \Delta x_0} \right)^2 \approx (p_0)^2 \text{ since } \Delta x_0 = \frac{\Delta \hbar}{\sqrt{2}}.
\] (24)

This argument is most transparent using the position-space wave functions where the exponentially small overlap of magnitudes is clear, but the same suppression arises in the momentum-space formulation, this time due to cancellations arising from the rapidly oscillating phase factor in the integrand in Eqn. (19). We will occasionally distinguish the exponential suppression factors (which we can describe as ‘dynamical’ as they depend on the initial wave packet parameter \(p_0\)) from the more intrinsic pre-factor term (containing only the spreading time) which is due to the natural dispersion of the wave packet (which we can therefore describe as ‘dispersive’.)

For this case of an isolated quantum system, the specific result for the autocorrelation function in Eqn. (20) must also satisfy the general theorem in Eqn. (14). To lowest non-trivial order in \(t\) (\(\mathcal{O}(t^2)\)), the modulus of \(A(t)\) in this case for short times is
\[
|A(t)|^2 = \frac{1}{\sqrt{1 + (t/t_0)^2}} \exp \left[-2\alpha^2 p_0^2 \frac{(t/2t_0)^2}{(1 + (t/2t_0)^2)}\right] 
\rightarrow \left(1 - \frac{2\alpha^2 p_0^2 t^2}{4t_0^2} + \cdots \right) \left(1 - \frac{t^2}{8t_0^2} + \cdots \right)
\approx 1 - \frac{\alpha^2 t^2}{2t_0^2} \left(p_0^2 + \frac{1}{4\alpha^2}\right) + \cdots.
\] (25)

We note that this result arises from both the exponential (‘dynamical’) suppression, as well as the (‘dispersive’) prefactor term. The general result of Eqn. (14) requires the calculation of
\[
\langle H \rangle = \frac{1}{2m} \left(p_0^2 + \frac{1}{2\alpha^2}\right) \quad \text{and} \quad \langle H^2 \rangle = \left(\frac{1}{2m}\right)^2 \left(p_0^4 + \frac{3p_0^2}{\alpha^2} + \frac{3}{4\alpha^4}\right)
\] (26)

which give
\[
(\Delta H)^2 = \left(\frac{1}{2m}\right)^2 \frac{2}{\alpha^2} \left(p_0^2 + \frac{1}{4\alpha^2}\right).
\] (27)

When the right-hand-side of Eqn. (14) is expanded to lowest order in \(t\) (again, \(\mathcal{O}(t^2)\)), using this result, it is found that
\[
\cos^2 \left(\frac{\Delta H t}{\hbar}\right) = \left(1 - \frac{1}{2} \left(\frac{\Delta H t}{\hbar}\right)^2 + \cdots \right)^2
= 1 - \frac{1}{2m^2\alpha^2} \left(p_0^2 + \frac{1}{4\alpha^2}\right) \left(\frac{t^2}{\hbar^2}\right) + \cdots
= 1 - \frac{\alpha^2 t^2}{2t_0^2} \left(p_0^2 + \frac{1}{4\alpha^2}\right) + \cdots
\] (28)
since \( t_0 \equiv m\alpha^2 \) and the result in Eqn. (25) is found to satisfy (in fact, to saturate, at this order) the inequality in Eqn. (14) for short times, confirming the general result. We note that it is easy to show that the inequality is also satisfied to \( O(t^4) \) where the left-hand-side is indeed larger than the right at this order.

III. UNIFORM ACCELERATION

The explicit form for Gaussian solutions, in both momentum-space and position-space, for the problem of a particle undergoing uniform acceleration (constant force \( F \) or linear potential given by \( V(x) = -Fx \)) are given in Ref. [30]. For the momentum-space form we have

\[
\phi(p, t) = \Phi(p - Ft)e^{-ip^2 t/6mF\hbar} = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha^2((p-Ft)-p_0)^2/2}e^{-i(p-Ft)x_0/\hbar}e^{i((p-Ft)^3-p^3)/6mF\hbar} \quad (29)
\]

\[
\phi(p, 0) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha^2(p-p_0)^2/2}e^{-ipx_0/\hbar} \quad (30)
\]

with arbitrary initial position \( (x_0) \) and momentum \( (p_0) \). The corresponding position-space wavefunction is

\[
\psi(x, t) = \left[ e^{iFt(x-x_0-Ft^2/6m)/\hbar} e^{i(p_0+Ft)(x-x_0-p_0t/2m)/\hbar} \right] \left( \frac{1}{\sqrt{\sqrt{\pi}\alpha\hbar(1+it/t_0)}} \right) \times e^{-(x-(x_0+pt/m+Ft^2/2m))^2/2(\alpha\hbar^2)^2(1+it/t_0)}. \quad (31)
\]

with \( \psi(x, 0) \) given in Eqn. (18).

The calculation of \( A(t) \) can be done using either form to obtain

\[
A(t) = \frac{1}{\sqrt{1-it/2t_0}} \exp \left[ \frac{(2ip_0^2 t/m\hbar - (\alpha Ft)^2(1+(t/2t_0)^2))}{4(1-it/2t_0)} \right] e^{-iFt(x_0-Ft^2/6m)/\hbar}, \quad (32)
\]

and the same factors of \( 1-it/2t_0 \) as in Eqn. (19) are obtained; this expression also reduces to that case in the free particle limit when \( F \to 0 \), as it must. The modulus-squared is given by

\[
|A(t)|^2 = \frac{1}{\sqrt{1+(t/2t_0)^2}} \exp \left[ -2\alpha^2(p_0^2+(Ft_0)^2(1+(t/2t_0)^2)) \left( \frac{(t/2t_0)^2}{1+(t/2t_0)^2} \right) \right] \quad (33)
\]

and we note that this result can be obtained from Eqn. (20) by the simple substitution

\[
p_0^2 \to p_0^2 + (Ft_0)^2(1+(t/2t_0)^2). \quad (34)
\]
For this case of uniform acceleration, the wave packet spreading is identical (same $\Delta x$) as in the free-particle case, which can be understood by noting that the distance between two classical particles starting at the same initial location, undergoing the same force, but with slightly different initial velocities (or momenta, $p^{(A)}_0 - p^{(B)}_0 = \Delta p_0$) would be

$$x_A(t) - x_B(t) = (x_0 + p^{(A)}_0 t/m + Ft^2/2m) - (x_0 + p^{(B)}_0 t/m + Ft^2/2m) = \frac{\Delta p_0 t}{m} \quad (35)$$

which increases linearly with time, in exactly the same way as for the free-particle solutions (when $F = 0$). The ‘distance’ between the peaks in $\psi_0$ and $\psi_t$, however, eventually grows as $t^2$ so that the exponential (‘dynamical’) suppression in $A(t)$ does not saturate, while the ‘dispersive’ pre-factor is the same as for the free-particle case.

We also note that the factors of $p_0$ and $F$ in $|A(t)|^2$ appear in quadrature, and not in a combination such as $p_0 + Ft$. One might naively expect that in cases where $p_0$ and $F$ have opposite signs, so that at a time given by $t_{ret} = 2p_0/F$ when the classical particle (and central value of the quantum wave packet) has returned to $x_0$, the magnitudes of $\psi(x,t)$ and $\psi(x,0)$ would be similar (only differing in possibly small spreading effects due to the $(1+it/t_0)$ factors) and so would give rise to a relatively large value of $|A(t)|$. While this does indeed occur for the magnitudes, at that time the classical momentum is of the opposite sign ($p(0) = p_0 \to p(t_{ret}) = -p_0$), giving rise to rapidly oscillating phase factors in the quantum wave function, which still gives the expected exponential suppression.

IV. SIMPLE HARMONIC OSCILLATOR WAVE PACKETS

The free particle and case of uniform acceleration, corresponding to unbound motions, do not provide examples for direct comparison to the (quasi) periodic behavior seen in bound state systems. The harmonic oscillator admits Gaussian wave packet solutions which can be written (using propagator techniques) in closed form for arbitrary initial values of position and momentum $(x_0, p_0)$ and for which the evaluation of $A(t)$ is therefore possible. In this case, we can also write any general wave packet as an expansion in eigenstates as in Eqn. [5], using $E_n = (n + 1/2)\hbar \omega$, as

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n u_n(x)e^{-i(n+1/2)\omega t} \quad (36)$$

from which it is clear that the observable probability density, $|\psi(x,t)|^2$, is periodic with the classical period, $T_{cl} = 2\pi/\omega$. 


Using the propagator techniques outlined in [30], and the initial position-space wave function
\[
\psi(x, 0) = \frac{1}{\sqrt{\beta \sqrt{\pi}}} e^{i p_0 x / \hbar} e^{-(x-x_0)^2 / 2 \beta^2},
\] (37)
where \( \beta \equiv \alpha \hbar \), one can evaluate the time-development in closed form as
\[
\psi(x, t) = \frac{1}{\sqrt{L(t) \sqrt{\pi}}} \exp \left[ \frac{S[x, t]}{2 \beta L(t)} \right]
\] (38)
where
\[
L(t) \equiv \beta \cos(\omega t) + \frac{i \hbar}{m \omega \beta} \sin(\omega t)
\] (39)
and
\[
S[x, t] \equiv -x_0^2 \cos(\omega t) + 2x x_0 - x^2 \left[ \cos(\omega t) + \frac{i m \omega \beta^2 \sin(\omega t)}{\hbar} \right] - \frac{2x_0 p_0 \sin(\omega t)}{m \omega} + \frac{2i \beta^2 p_0 x}{\hbar} - \frac{i \beta^2 p_0^2 \sin(\omega t)}{m \omega \hbar}.
\] (40)
The corresponding position-space probability density can be written as
\[
|\psi(x, t)|^2 = \frac{1}{\sqrt{\pi} |L(t)|} \exp \left[ - \frac{(x - x_0 \cos(\omega t) - p_0 \sin(\omega t) / m \omega)^2}{|L(t)|^2} \right]
\] (41)
with
\[
\langle x \rangle_t = x_0 \cos(\omega t) + \frac{p_0 \sin(\omega t)}{m \omega} \quad \text{and} \quad \Delta x_t = \frac{|L(t)|}{\sqrt{2}}.
\] (42)
Thus, the expectation value moves in accordance with classical expectations [31], while the width oscillates (from wide to narrow, or vice versa.) For the special case of the ‘minimum uncertainty’ wave packet where
\[
\beta^2 = \frac{\hbar}{m \omega} \equiv \beta_0^2,
\] (43)
the width of the packet is fixed as
\[
\Delta x_t = \Delta x_0 = \frac{\beta_0}{\sqrt{2}}
\] (44)
which is the same as the ground state oscillator energy eigenvalue state, but simply oscillates at the classical frequency. We note that the momentum-space wavefunctions can also be written using a propagator formalism [32] and used to evaluate \( A(t) \) in a parallel fashion, obtaining the same result.

The evaluation of \(|A(t)|^2\) for general values of \( \beta, \ x_0, \) and \( p_0 \) is straightforward enough, but the resulting expressions are somewhat cumbersome, so we will focus on several special cases as illustrative.
**Case I:** Minimum uncertainty wave packets, $\beta = \beta_0$.

In this case the evaluation of $A(t)$ gives

$$A(t) = \sqrt{\cos(\omega t) + i \sin(\omega t)} \exp \left[ - \left( \frac{x_0^2}{2\beta_0^2} + \frac{\beta_0^4 p_0^2}{2\hbar^2} \right) \left( (1 - \cos(\omega t)) - i \sin(\omega t) \right) \right]$$  \hspace{1cm} (45)

where great simplifications have been made by noting that

$$\frac{1}{\cos(\omega t) - i \sin(\omega t)} = \cos(\omega t) + i \sin(\omega t).$$  \hspace{1cm} (46)

We note that a very similar expression arises in analyses of the macroscopic wavefunction for Bose-Einstein condensates [33] and the collapse and revival of the matter wave field for such systems has been observed experimentally [34].

Once again, the two important parameters appear together in quadrature, as in the uniform acceleration case. This gives

$$|A(t)|^2 = \exp \left[ - \left( \frac{x_0^2}{2\beta_0^2} + \frac{\beta_0^4 p_0^2}{2\hbar^2} \right) \left( 1 - \cos(\omega t) \right) \right]$$  \hspace{1cm} (47)

which clearly exhibits the expected periodicity. All of the suppression can be attributed to the ‘dynamical’ factors (those in the exponential, containing $x_0$ and $p_0$) as there is no ‘dispersive’ pre-factor component for this constant width packet.

For this case, the minimum degree of overlap at any point during a single classical period is

$$|A(T_{cl}/2)|^2 = \exp \left[ -2 \left( \frac{x_0^2}{\beta_0^2} + \frac{\beta_0^4 p_0^2}{\hbar^2} \right) \right]$$  \hspace{1cm} (48)

so there is no time at which the wave packet is ever truly orthogonal to its initial state.

**Case II:** Arbitrary $\beta$, but $x_0, p_0 = 0$. For this case, the wave packet does not oscillate, but only ‘pulsates’, and the time-dependent wave function simplifies to

$$\psi(x, t) = \frac{1}{\sqrt{\pi} \sqrt{(\beta \cos(\omega t) + (i\hbar/m\omega\beta) \sin(\omega t))}} \exp \left[ -x^2 \left( \cos(\omega t) + (i\omega^2/\hbar) \sin(\omega t) \right) \right]$$  \hspace{1cm} (49)

It is convenient to define the parameters

$$r \equiv \frac{\hbar}{m\omega\beta^2} = \frac{\beta_0^2}{\beta^2}$$

so that $\frac{1}{r} = \frac{\beta^2}{\beta_0^2}$  \hspace{1cm} (50)

in terms of which the resulting autocorrelation function in this case has the very simple form

$$A(t) = \sqrt{\frac{2}{2 \cos(\omega t) - i(r + 1/r) \sin(\omega t)}}$$  \hspace{1cm} (51)
or

\[ |A(t)|^2 = \frac{1}{\sqrt{\cos^2(\omega t) + (r + 1/r)^2 \sin^2(\omega t)/4}} \] (52)

all of which can be attributed to a ‘dispersive’ (but in this case periodic) pre-factor.

We first note that in this case \( A(t) \) is invariant under the transformation \( r \to 1/r \), in other words, the time-dependence is the same for both initially wide (\( \beta > \beta_0 \)) or narrow (\( \beta < \beta_0 \)) packets. It is clear that the larger the deviation from the ‘minimum uncertainty’ wavepacket, the faster the wavepacket ‘pulsates’ away from its initial shape. It is also noteworthy that in this case \( |A(T_{cl}/2)| = 1 \) so that the wave packet returns to its initial form (up to a constant complex phase) twice each classical period. This can be understood from the expansion of this wave form in terms of energy eigenstates. In this case, where the parameters \( x_0, p_0 \) both vanish, one is expanding an even-parity function in Eqn. (36), so that only the even \( (a_{2n}) \) terms are nonvanishing and the \( n \)-dependent exponential factors in Eqn. (5) oscillate twice as rapidly as in the general case.

Finally, for the very special case where \( \beta = \beta_0 \) \((r = 1)\) as well, we recover the ground state energy eigenstate of the oscillator, with its trivial stationary-state time-dependence \((\psi_0(x, t) = u_0(x) \exp(-i E_0 t/\hbar))\) and Eqn. (51) indeed reduces to

\[ A(t) \xrightarrow{r \to 1} \sqrt{\frac{2}{2 \cos(\omega t) - 2i \sin(\omega t)}} = \sqrt{\exp(i \omega t)} = e^{+i \omega t/2} \] (53)

as expected.

Wave packet solutions (Gaussian or not) of the harmonic oscillator can be shown (using the expansion in Eqn. (36), for example) to satisfy

\[ \psi(x, t + mT_{cl}) = (-1)^m \psi(x, t) \] (54)

with a similar result for the momentum-space version as well. Because of the specially symmetric nature of the potential, we also have

\[ \psi(-x, t + T_{cl}/2) = (-i) \psi(x, t) \quad \text{and} \quad \phi(-p, t + T_{cl}/2) = (-i) \phi(p, t) \] (55)

so that half a period later, the wave-packet is reproduced, but at the opposite ‘corner’ of phase space, namely with \( x \leftrightarrow -x \) and \( p \leftrightarrow -p \): we note that two applications of Eqn. (55) reproduce Eqn. (54). One can also show these connections using the propagator techniques in Ref. 30, provided one properly identifies the complex pre-factors as described in detail in Ref. 35.
This type of behavior can be diagnosed using a variation on the standard autocorrelation function, namely
\[
\overline{A}(t) \equiv \int_{-\infty}^{+\infty} \psi^*(-x, t) \psi(x, 0) \, dx = \int_{-\infty}^{+\infty} \phi^*(-p, t) \phi(p, 0) \, dp \quad (56)
\]
which measures the overlap of the initial state with the ‘out-of-phase’ version of itself at later times. Given the simple connections in Eqn. (55), we can immediately write, for Case I considered above,
\[
|\overline{A}(t)|^2 = \exp \left[ -\left( \frac{x_0^2}{\beta_0^2} + \frac{\beta_0^2 p_0^2}{\hbar^2} \right) (1 + \cos(\omega t)) \right] \quad (57)
\]
which is exponentially suppressed at integral multiples of \(T_d\), but unity at \(t = (2k + 1)T_d/2\).

This type of anti-correlation function finds use in the study of wave packet revivals [4] where quantum wave packets may reform near \(t = T_{rev}\), as in Eqn. (12), but out of phase with the original packet.

V. ‘INVERTED’ OSCILLATOR WAVE PACKETS FOR UNSTABLE EQUILIBRIUM

As described in detail in Ref. [30], the case of the ‘inverted’ oscillator, defined by
\[
\tilde{V}(x) \equiv -\frac{1}{2} m\tilde{\omega}^2 x^2 \quad (58)
\]
can be studied using wave packet results for the standard harmonic oscillator by making the substitutions
\[
\omega \to i\tilde{\omega}, \quad \sin(\omega t) \to i \sinh(\tilde{\omega} t), \quad \text{and} \quad \cos(\omega t) \to \cosh(\tilde{\omega} t). \quad (59)
\]
The general wave packet solution in Eqn. (38), for example, can be carried over in this way to obtain the ‘runaway’ wavepacket, with probability density given by
\[
|\psi(x, t)|^2 = \frac{1}{\sqrt{\pi}|B(t)|} \exp \left[ - \left( \frac{x - x_0 \cosh(\tilde{\omega} t) - p_0 \sinh(\tilde{\omega} t)/m\tilde{\omega}}{|B(t)|^2} \right)^2 \right] \quad (60)
\]
with
\[
\langle x \rangle_t = x_0 \cosh(\tilde{\omega} t) + \frac{p_0 \sinh(\tilde{\omega} t)}{m\tilde{\omega}} \quad \text{and} \quad \Delta x_t = \frac{|B(t)|}{\sqrt{2}} \quad (61)
\]
where
\[
|B(t)| = \sqrt{\beta^2 \cosh^2(\tilde{\omega} t) + (\hbar/m\tilde{\omega}\beta)^2 \sinh^2(\tilde{\omega} t)}. \quad (62)
\]
As above, the expression for $A(t)$ for the general case is cumbersome, so we only examine it for one specific case as an example, namely the case where $\beta = \beta_0 = \sqrt{\hbar/m\tilde{\omega}}$. This situation no longer corresponds to a constant width wave packet, since

$$\Delta x_t \longrightarrow \frac{\beta_0}{\sqrt{2}} \sqrt{\cosh^2(\tilde{\omega}t) + \sinh^2(\tilde{\omega}t)}$$

increases exponentially, as the individual momentum components comprising the wave packet quickly diverge in $p$-space. For the case of $x_0 = 0$, we have the general expression

$$A(t) = \frac{1}{\sqrt{\cosh(\tilde{\omega}t)}} \exp \left[ \left( \frac{p_0^2}{2m\tilde{\omega}\hbar} \right) \left\{ \frac{\cosh(\tilde{\omega}t) - 1 + i \sinh(\tilde{\omega}t)(2 \cosh(\tilde{\omega}t) - 1)}{\cosh(\tilde{\omega}t)(\cosh(\tilde{\omega}t) - i \sinh(\tilde{\omega}t))} \right\} \right]$$

In the limit when $t >> 1/\tilde{\omega}$, the hyperbolic functions both approach $\exp(\tilde{\omega}t)/2$ and we have the limiting case

$$A(t >> 1/\tilde{\omega}) \longrightarrow \frac{1}{\sqrt{\exp(\tilde{\omega}t)/2}} \exp \left[ -\frac{p_0^2}{2m\tilde{\omega}\hbar}(1 - i) \right]$$

The exponential (‘dynamical’) suppression once again is seen to ‘saturate’, as in the free-particle case, and for the same reason, namely that both $x(t) - x_0$ and $\Delta x_t$ have the same large $t$ (here exponential) behavior. The resulting modulus is given by

$$|A(t)|^2 \longrightarrow 2e^{-\tilde{\omega}t} \exp \left[ -\frac{p_0^2}{m\tilde{\omega}\hbar} \right]$$

which still becomes exponentially small, but now due to the (‘dispersive’) prefactor. If one also has $x_0 \neq 0$, the expression above includes an additional factor of $\exp(-x_0^2/\beta_0^2)$ (similar to that in Eqn. (47), with no cross-term involving $x_0$ times $p_0$.

### VI. CONCLUSIONS AND DISCUSSION

We have evaluated the autocorrelation function for time-dependent Gaussian wave packet solutions for four quite different quantum mechanical systems, examining the behavior of $A(t)$ in terms of both classical analogs and the quantum mechanical time-evolution of the wave function magnitude and phases in both position- and momentum-space. We have focused attention on the different contributions to the suppression in $A(t)$ arising from ‘dispersive’ (prefactor) and more ‘dynamical’ (typically exponential) effects (which depend on the initial conditions) which may or may not saturate to a small but finite value, depending
on the relationship between the classical dynamical behavior \((x(t)\) or \(\langle x \rangle_t\)) and the quantum mechanical spreading \((\Delta x_t)\). We have extended earlier results of Baltz \[10\] in terms of a now standard analysis tool, namely the autocorrelation function, directly comparing the behavior of \(A(t)\) for several distinct classes of classical behavior. Using the free-particle solution, we have also been able to exhibit a useful test case for quite general theorems on the time-development of isolated quantum systems, while providing other non-trivial examples of closed-form results for \(A(t)\) for Gaussian solutions. Additional examples which are simple extensions of the results presented here are multi-dimensional free-particle, uniformly accelerated particle, or harmonic oscillator solutions, where the autocorrelation function factorizes as \(A(t) = A_x(t) \cdot A_y(t)\), or the related problem of a Gaussian wave packet in a uniform magnetic field which admits Gaussian solutions corresponding to classical circular orbits.
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