SPECTRAL INVARIANCE OF PSEUDODIFFERENTIAL BOUNDARY VALUE PROBLEMS ON MANIFOLDS WITH CONICAL SINGULARITIES

PEDRO T. P. LOPES AND ELMAR SCHROHE

Abstract. We prove the spectral invariance of the algebra of classical pseudodifferential boundary value problems on manifolds with conical singularities in the \( \mathcal{L}_p \)-setting. As a consequence we also obtain the spectral invariance of the classical Boutet de Monvel algebra of zero order operators with parameters. In order to establish these results, we show the equivalence of Fredholm property and ellipticity for both cases.

1. Introduction

Elliptic boundary value problems on manifolds with conical singularities have been studied since the 60’s, where the work of V. A. Kondratiev [14] stands out, see also Kozlov, Maz’ya and Rossmann [15] for a detailed presentation. The pseudodifferential analysis started with the work of R. Melrose and G. Mendoza [19, 20], B. Plamenevsky [21], and B. -W. Schulze [29]. Algebras of pseudodifferential boundary value problems for conical singularities were constructed in the 90’s by A. O. Derviz [7] and E. Schrohe and B.-W. Schulze [25, 26]. The latter approach combines elements of the Boutet de Monvel calculus [3] with the pseudodifferential analysis developed by B. -W. Schulze [25, 26] and B. -W. Schulze [29, 29]. While initially only \( L_2 \)-based Sobolev spaces were used, S. Coriasco, E. Schrohe and J. Seiler established the continuity also on Bessel potential and Besov spaces [5], see also [4], relying on work of G. Grubb and N. Kokholm [9, 12].

Our main result is the spectral invariance of the algebra developed in [26] in the \( \mathcal{L}_p \)-setting, see Theorem [61]. This algebra contains, after the composition with order reducing operators, the classical differential boundary value problems studied by V. A. Kondratiev [14], hence also their inverses, whenever these exist. As a by-product we obtain the spectral invariance of the algebra of zero order classical Boutet de Monvel operators with parameters in the \( \mathcal{L}_p \)-setting, see Theorem [29]. This algebra includes, after composition with order reducing operators, the differential boundary value problems studied by M. S. Agranovich and M. I. Vishik in [11], which were an important ingredient for the work of Kondratiev.

It is an immediate consequence of Theorem [61] that the invertibility of a conically degenerate boundary value problem is to a large extent independent of the space it is considered on: It depends neither on the Sobolev regularity parameter \( s \) nor on \( 1 < p < \infty \). This is of great practical importance as it allows to check invertibility in the most convenient setting. A similar result holds for the Fredholm property, as we show in Corollary [61].
This article extends the results of [27] to conical manifolds with boundary. The need to work with Besov spaces led to interesting new features. In Theorem 29 for example, we consider a zero order parameter-dependent operator $A = \{ A(\lambda); \lambda \in \Lambda \}$ in Boutet de Monvel’s calculus. We show that the invertibility of $A(\lambda)$ for each $\lambda$ together with a norm estimate $\| A(\lambda)^{-1} \| \leq c(\lambda)^r$ for a constant $c \geq 0$ and sufficiently small $r > 0$ implies that the inverse also is parameter-dependent of order zero. In particular, the operator norm will then be uniformly bounded. Similar effects can be observed when showing the equivalence of parameter-ellipticity and the Fredholm property with parameters.

This paper is a step toward the analysis of nonlinear partial differential equations on manifolds with boundary and conical singularities, see e.g. [23] by Roidos and Schrohe, [30] by Shao and Simonett or [31] by Vertman for the case without boundary. A next step concerns the analysis of resolvents of closed extensions in the spirit of Gil, Krainer and Mendoza [8] or [24] in the case without boundary and Krainer [10] for conic manifolds with boundary.

2. Parameter-dependent Boutet de Monvel algebra

To make this article readable for non-experts, we briefly describe the parameter-dependent Boutet de Monvel algebra with classical symbols on compact manifolds with boundary in the $L_p$-setting. We first define several operator classes on the half-space $\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n; x_n > 0 \}$.

The set of parameters of the operators and symbols will always be a conical open set $\Lambda \subset \mathbb{R}^d$, that is, $p \in \Lambda$ implies that $tp \in \Lambda$ for $t > 0$. It can be the empty set, in which case we recover the usual symbols and operators. We write $\mathbb{N}_0 := \{0, 1, 2, \ldots \}$ and $\mathbb{R}^+ = \mathbb{R}_+ \times \mathbb{R}_+$. For a Fréchet space $W$, the Schwartz space $S(\mathbb{R}^n, W)$ consists of all $u \in C^\infty(\mathbb{R}^n, W)$ such that $\sup_{x \in \mathbb{R}^n} p \left( x^a \partial_x^b u(x) \right) < \infty$ for every continuous seminorm $p$ of $W$. We simply write $S(\mathbb{R}^n)$, if $W = \mathbb{C}$. If $\Omega \subset \mathbb{R}^n$ is an open set, $S(\Omega)$ denotes the restrictions to $\Omega$ of functions in $S(\mathbb{R}^n)$, and $C_c^\infty(\Omega)$ the space of smooth functions with compact support in $\Omega$. The operator of restriction of distributions defined in $\mathbb{R}^n$ to $\Omega$ is denoted by $r_\Omega : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\Omega)$. The extension by zero of a function $u$ defined in $\Omega$ to $\mathbb{R}^n$ will be denoted by $e_\Omega$:

$$e_\Omega(u)(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$$

If $\Omega = \mathbb{R}^n_+$, we denote $r_{\mathbb{R}^n_+}$ also by $r^+$ and $e_{\mathbb{R}^n_+}$ by $e^+$. The open ball in $\mathbb{R}^n$ with the Euclidean norm whose center is $x$ and radius is $r > 0$ will be denoted by $B_r(x)$. Our convention for the Fourier transform is $\mathcal{F}u(\xi) = \hat{u}(\xi) = \int e^{-ix\xi}u(x)dx$. We shall often use the function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\langle \xi \rangle := \sqrt{1 + |\xi|^2}$$

and sometimes we use $\langle \xi, \lambda \rangle := \sqrt{1 + |\langle \xi, \lambda \rangle|^2}$ and similar expressions, as well.

Finally, given two Banach spaces $E$ an $F$, we denote by $B(E, F)$ the bounded operators from $E$ to $F$ and use the notation $B(\mathcal{E}) := B(E, \mathcal{E})$.

**Definition 1.** The space $S^m(\mathbb{R}^n \times \mathbb{R}^n, \Lambda)$ of parameter-dependent symbols of order $m \in \mathbb{R}$ consists of all functions $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda)$ that satisfy

$$|\partial_x^a \partial_\xi^b p(x, \xi, \lambda)| \leq C_{a,b} \| \xi \|^m \lambda^{-|\lambda|}, \quad (x, \xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \Lambda. $$

A symbol $p$ defines a parameter-dependent pseudodifferential operator $Op(p)(\lambda) : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ by the formula:

$$Op(p)(\lambda) u(x) = (2\pi)^{-n} \int e^{ix\xi} p(x, \xi, \lambda) \hat{u}(\xi) d\xi.$$
We say that \( p \) is classical, if there are symbols \( p_{(m-j)} \in S^{m-j}(\mathbb{R}^n \times \mathbb{R}^n, \Lambda), j \in \mathbb{N}_0 \), such that

1. For all \( t \geq 1 \) and \(|(\xi', \lambda)| \geq 1 \), we have
   \[ p_{(m-j)}(x, t\xi', t\lambda) = t^{m-j}p_{(m-j)}(x, \xi', \lambda). \]

2. We have the asymptotic expansion
   \[ p \sim \sum_{j=0}^{\infty} p_{(m-j)} \text{ i.e. } p - \sum_{j=0}^{N-1} p_{(m-j)} \in S^{m-N}(\mathbb{R}^n \times \mathbb{R}^n, \Lambda), \text{ for all } N \in \mathbb{N}_0. \]

This subset is denoted by \( S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n, \Lambda) \). It is a Fréchet space with the natural seminorms.

**Definition 2.** Let \( p \in S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n, \Lambda), m \in \mathbb{Z} \), be written as a function of \((x', x_n, \xi', \xi_n, \lambda) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \Lambda\). We say that it satisfies the transmission condition, if
   \[ p \sim \sum_{j=0}^{\infty} p_{(m-j)} \text{ and if, for all } k \in \mathbb{N}_0 \text{ and for all } \alpha \in \mathbb{N}_0^{n+1}, \text{ we have} \]
   \[ D_{x_n}^k D_{\xi_n}^\alpha p_{(m-j)}(x', x_n, \xi', \xi_n, \lambda) = (-1)^{m-j-|\alpha|} D_{x_n}^k D_{\xi_n}^\alpha p_{(m-j)}(x', x_n, \xi', \xi_n, \lambda). \]

In this case, the operator \( P(\lambda)+ := r^+ op(p)(\lambda) e^{+} : S(\mathbb{R}^n_+) \to S(\mathbb{R}^n_+) \) is well defined.

Two more classes of functions are required. Our notation here follows G. Grubb [3].

**Definition 3.** We denote by \( S^m(\mathbb{R}^{n-1}, \mathbb{S}_+, \Lambda), m \in \mathbb{R} \), the space of all functions \( \hat{f} \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}^{n-1} \times \Lambda) \) that satisfy:

\[
\|x_n^k D_x^\alpha D_{\xi'}^\beta D_{\lambda}^\gamma \hat{f}(x', x_n, \xi', \lambda)\|_{L^\infty(\mathbb{R}_+ \times \xi_n)} \leq C_{k,k',\alpha',\beta',\gamma}(\xi', \lambda)^{m+k-k'-|\alpha'|-|\gamma|}.
\]

The subset \( S^m_{cl}(\mathbb{R}^{n-1}, \mathbb{S}_+, \Lambda) \) consists of all \( \hat{f} \) with an asymptotic expansion \( \hat{f} \sim \sum_{j=0}^{\infty} \hat{f}_{(m-j)} \), i.e. there are functions \( \hat{f}_{(m-j)} \in S^{m-j}(\mathbb{R}^{n-1}, \mathbb{S}_+, \Lambda) \), \( j \in \mathbb{N}_0 \), such that
   \[ \hat{f} - \sum_{j=0}^{N-1} \hat{f}_{(m-j)} \in S^{m-N}(\mathbb{R}^{n-1}, \mathbb{S}_+, \Lambda) \text{ for all } N \in \mathbb{N}_0, \]
   \[ \hat{f}_{(m-j)}(x', x_n, \xi', \lambda) = t^{m-j} t^{-|\xi'|} \hat{f}_{(m-j)}(x', x_n, \xi', \lambda) t \geq 1, |(\xi', \lambda)| \geq 1. \]

Similarly, \( S^m(\mathbb{R}^{n-1}, \mathbb{S}_+, \Lambda) \) with all \( \tilde{g} \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}^{n-1} \times \Lambda) \) with

\[
\|y_n^k D_y^\alpha D_{\xi'}^\beta D_{\lambda}^\gamma \tilde{g}(x', x_n, y_n, \xi', \lambda)\|_{L^\infty(\mathbb{R}_+ \times \xi_n)} \leq C_{k,k',\alpha',\beta',\gamma}(\xi', \lambda)^{m+2-k-k'-l+l'-|\alpha'|-|\gamma|}.
\]

Write \( \tilde{g} \in S^m_{cl}(\mathbb{R}^{n-1}, \mathbb{S}_+, \Lambda) \), if
   \[ \tilde{g} \sim \sum_{j=0}^{\infty} \tilde{g}_{(m-j)} \text{ with } \tilde{g}_{(m-j)} \in S^{m-j}(\mathbb{R}^{n-1}, \mathbb{S}_+, \Lambda) \]
   such that \( \tilde{g} - \sum_{j=0}^{N-1} \tilde{g}_{(m-j)} \in S^{m-N}(\mathbb{R}^{n-1}, \mathbb{S}_+, \Lambda) \) for all \( N \in \mathbb{N}_0, \)
   \[ \tilde{g}_{(m-j)}(x', x_n, y_n, \xi', \lambda) t \geq 1, |(\xi', \lambda)| \geq 1. \]

We may now define the operators that, together with the pseudodifferential ones, appear in the Boutet de Monvel calculus: the Poisson, trace and singular Green operators. We will always restrict ourselves to the classical elements. The notation \( \gamma_j : S(\mathbb{R}^{n-1}_+) \to S(\mathbb{R}^{n-1}_+) \), \( j \in \mathbb{N}_0 \), indicates the operator \( \gamma_j u(x') = \lim_{x_n \to 0} D_{x_n}^j u(x', x_n) \) as well as its extension to Sobolev, Bessel and Besov spaces.

**Definition 4.** Let \( \Lambda \in \mathbb{R}, m \in \mathbb{R} \) and \( d \in \mathbb{N}_0 \).

1. A classical parameter-dependent Poisson operator of order \( m \) is an operator family \( K(\lambda) : S(\mathbb{R}^{n-1}) \to S(\mathbb{R}^{n+1}_+) \) associated with \( k \in S^m_{cl}(\mathbb{R}^{n-1}, \mathbb{S}_+, \Lambda) \) of the form

   \[
   K(\lambda) u(x', x_n) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{ix\xi'} e^{ix\xi'} k(x', x_n, \xi', \lambda) \tilde{u}(\xi') d\xi',
   \]
For $\tilde{k} \sim \sum_{j=0}^{\infty} \tilde{k}_{(m-1)} \left( x', \xi', D_n, \lambda \right) : \mathbb{C} \rightarrow \mathcal{S} (\mathbb{R}_+)$ by
\[
\tilde{k}_{(m-1)} \left( x', \xi', D_n, \lambda \right) (v) = i \tilde{k}_{(m-1)} \left( x', x_n, \xi', \lambda \right).
\]

2) A classical parameter-dependent trace operator of order $m$ and class $d$ is an operator family $T (\lambda) : \mathcal{S}(\mathbb{R}^n_+) \rightarrow \mathcal{S}(\mathbb{R}^{n-1}_+)$ of the form
\[
T (\lambda) = \sum_{j=0}^{d-1} S_j (\lambda) \gamma_j + T' (\lambda),
\]

where $S_j (\lambda)$ is a parameter-dependent pseudodifferential operator of order $m-j$ on $\mathbb{R}^{n-1}_+$ and $T' (\lambda) : \mathcal{S}(\mathbb{R}^n_+) \rightarrow \mathcal{S}(\mathbb{R}^{n-1}_+)$ is of the form
\[
T' (\lambda) u (x') = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} e^{ix' \xi'} \int_{\mathbb{R}_+} \tilde{t} (x', x_n, \xi', \lambda) (\mathcal{F}_{x' \rightarrow \xi'} u) (\xi', x_n) dx_n d\xi',
\]

with $\tilde{t} \in \mathcal{S}_{cl}^m (\mathbb{R}^{n-1}_+, \mathbb{R}_+, \mathcal{L})$. For $\tilde{k} \sim \sum_{j=0}^{\infty} \tilde{t}_{(m-j)}$ we define $\tilde{t}_{(m)} (x', \xi', D_n, \lambda) : \mathcal{S} (\mathbb{R}_+) \rightarrow \mathbb{C}$ by
\[
\tilde{t}_{(m)} (x', \xi', D_n, \lambda) u = \int_{\mathbb{R}_+} \tilde{t}_{(m)} (x', x_n, \xi', \lambda) (u (x_n)) dx_n.
\]

3) A classical parameter-dependent singular Green operator of order $m$ and type $d$ is an operator family $G (\lambda) : \mathcal{S}(\mathbb{R}^n_+) \rightarrow \mathcal{S}(\mathbb{R}^n_+)$ of the form
\[
G (\lambda) = \sum_{j=0}^{d-1} K_j (\lambda) \gamma_j + G' (\lambda),
\]

where $K_j$ are Poisson operators of order $m-j$ and $G' (\lambda) : \mathcal{S}(\mathbb{R}^n_+) \rightarrow \mathcal{S}(\mathbb{R}^n_+)$ is an operator of the form
\[
G' (\lambda) u (x) = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} e^{ix' \xi'} \int_{\mathbb{R}_+} \tilde{g} (x', x_n, y_n, \xi', \lambda) (\mathcal{F}_{x' \rightarrow \xi'} u) (\xi', y_n) dy_n d\xi',
\]

where $\tilde{g} \in \mathcal{S}_{cl}^m (\mathbb{R}^{n-1}_+, \mathbb{R}_+, \mathcal{L})$. We define the operator $g_{(m-1)} (x', \xi', D_n, \lambda) : \mathcal{S} (\mathbb{R}_+) \rightarrow \mathcal{S} (\mathbb{R}_+)$ by
\[
g_{(m-1)} (x', \xi', D_n, \lambda) u (x_n) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix_n \xi_n} p_{(m)} (x', 0, \xi', D_n, \lambda) e^{-ix_n u (x_n)} dx_n d\xi_n.
\]

Remark 6. With a symbol $p \in \mathcal{S}_{cl}^m (\mathbb{R}^n \times \mathbb{R}_+, \mathcal{L})$ that satisfies the transmission condition, we associate the operator $p_{(m)} (x', 0, \xi', D_n, \lambda) : \mathcal{S} (\mathbb{R}_+) \rightarrow \mathcal{S} (\mathbb{R}_+)$ defined by:
\[
p_{(m)} (x', 0, \xi', D_n, \lambda) u (x_n) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix_n \xi_n} p_{(m)} (x', 0, \xi', x_n, \lambda) e^{-ix_n u (x_n)} dx_n d\xi_n.
\]

Definition 6. Let $n_1, n_2, n_3$ and $n_4 \in \mathbb{N}_0$. The set of classical parameter-dependent Boutet de Monvel operators on $\mathbb{R}^n_+$, denoted by $\mathcal{B}^{n,d}_{n_1} (\mathbb{R}^n_+, \mathcal{L})$ for $m \in \mathbb{Z}$ and $d \in \mathbb{N}_0$, or just by $\mathcal{B}^{n,d} (\mathbb{R}^n_+, \mathcal{L})$, consists of all operators $A$ given by
\[
A (\lambda) = \left( P_+ (\lambda) + G (\lambda) \right) \left( T (\lambda) \right) K (\lambda) : \mathcal{S} (\mathbb{R}^n_+)^{n_1} \oplus \mathcal{S} (\mathbb{R}^n_+)^{n_3} \rightarrow \mathcal{S} (\mathbb{R}^{n-1}_+)^{n_2} \oplus \mathcal{S} (\mathbb{R}^{n-1}_+)^{n_4},
\]

where $P_+$ is a pseudodifferential operator of order $m$ satisfying the transmission condition, $G$ is a singular Green operators of order $m$ and type $d$, $K$ is a Poisson
operator of order \( m \), \( T \) is a trace operator of order \( m \) and type \( d \) and \( S \) is a pseudodifferential operator of order \( m \). All are parameter-dependent in the respective classes.

The following algebra is also useful to prove spectral invariance:

**Definition 7.** Let \( n_1, n_2, n_3, n_4 \in \mathbb{N}_0 \) and \( 1 < p < \infty \). We define the set \( \mathcal{B}_{n_1,n_2,n_3,n_4}^p(\mathbb{R}^n, \Lambda) \), also denoted by \( \mathcal{B}^p(\mathbb{R}^n, \Lambda) \), as the set of all operators \( A \) of the form (2.4), where: \( P_+ \) is of order 0, \( G \) is of order 0 and type 0, \( K \) is of order \( \frac{1}{p} \), \( T \) is of order \( -\frac{1}{p} \) and type 0 and \( S \) is of order 0. All are parameter-dependent in the respective classes.

**Definition 8.** With \( A \in \mathcal{B}_{n_1,n_2,n_3,n_4}^{m,d} (\mathbb{R}^n, \Lambda) \), we associate the operator-valued principal boundary symbol \( \sigma_\partial (A) \), defined on \( \mathbb{R}^{n-1} \times ((\mathbb{R}^{n-1} \times \Lambda) \setminus \{0\}) \). The operator

\[
\sigma_\partial (A)(x', \xi', \lambda) : \mathcal{S}(\mathbb{R}^n_+ \oplus \mathcal{C}^{n_2}) \rightarrow \mathcal{S}(\mathbb{R}^n_+ \oplus \mathcal{C}^{n_4})
\]

is given by

\[
\begin{pmatrix}
(p_{(m)+} (x', 0, \xi', D_n, \lambda) + g_{(m-1)} (x', \xi', D_n, \lambda)) & k_{(m-1)} (x', \xi', D_n, \lambda) \\
\end{pmatrix}
\]

where the entries are the matrix version of the operators in Definition 4 and Remark 8.

Similarly, with \( A \in \mathcal{B}_{n_1,n_2,n_3,n_4}^{m,d} (\mathbb{R}^n, \Lambda) \), we associate an operator \( \sigma_\partial (A)(x', \xi', \lambda) \) acting as in (2.4), given as

\[
\begin{pmatrix}
(p_{(0)+} (x', 0, \xi', D_n, \lambda) + g_{(-1)} (x', \xi', D_n, \lambda)) & k_{(-1)} (x', \xi', D_n, \lambda) \\
\end{pmatrix}
\]

Let now \( M \) be a manifold with boundary, \( E_0 \) and \( E_1 \) two complex hermitian vector bundles over \( M \) and \( F_0 \) and \( F_1 \) two complex hermitian vector bundles over \( \partial M \). Let \( U_j \subset M, j = 1, \ldots, N \), be open cover of \( M \) consisting of trivializing sets for the vector bundles, \( \Phi_1, \ldots, \Phi_N \in C^{\infty} (M) \) be a partition of unity subordinate to \( U_1, \ldots, U_N \) and \( \Psi_1, \ldots, \Psi_N \in C^{\infty} (M) \) be supported in \( U_j \) such that \( \Psi_j \Phi_j = \Phi_j \).

A linear operator \( A(\lambda) : C^{\infty} (M, E_0) \oplus C^{\infty} (\partial M, F_0) \rightarrow C^{\infty} (M, E_1) \oplus C^{\infty} (\partial M, F_1) \), \( \lambda \in \Lambda \), can always be written as

\[
A(\lambda) = \sum_{j=1}^{N} \Phi_j A(\lambda) (1 - \Psi_j) + \sum_{j=1}^{N} \Phi_j A(\lambda) (1 - \Psi_j).
\]

Using the above definitions, we define the Boutet de Monvel algebra on \( M \):

**Definition 9.** A family \( A(\lambda) : C^{\infty} (M, E_0) \oplus C^{\infty} (\partial M, F_0) \rightarrow C^{\infty} (M, E_1) \oplus C^{\infty} (\partial M, F_1), \lambda \in \Lambda \), is called a parameter-dependent Boutet de Monvel operator of order \( m \in \mathbb{Z} \) and class \( d \in \mathbb{N}_0 \), if

1) The operators \( \Psi_j A(\lambda) \Phi : \mathcal{S}(\mathbb{R}^n_+ \oplus \mathcal{C}^{n_2}) \rightarrow \mathcal{S}(\mathbb{R}^n_+ \oplus \mathcal{C}^{n_4}) \) belong to \( \mathcal{B}_{n_1,n_2,n_3,n_4}^{m,d} (\mathbb{R}^n, \Lambda) \) after localization.

2) The Schwartz kernels of the operators \( \sum_{j=1}^{N} \Phi_j A(\lambda) (1 - \Psi_j) \) belong to

\[
\begin{pmatrix}
S(\Lambda, C^{\infty} (M \times M, \text{Hom}(\pi_2^* E_0, \pi_1^* E_1))) \\
S(\Lambda, C^{\infty} (\partial M \times M, \text{Hom}(\pi_2^* E_0, \pi_1^* E_1))) \\
S(\Lambda, C^{\infty} (\partial M \times \partial M, \text{Hom}(\pi_2^* F_0, \pi_1^* F_1))) \\
S(\Lambda, C^{\infty} (\partial M \times \partial M, \text{Hom}(\pi_2^* F_0, \pi_1^* F_1)))
\end{pmatrix},
\]

where \( \text{Hom} \) indicates the space of homomorphisms and \( \pi_i : M \times M \rightarrow M \) is given by \( \pi_i (x_1, x_2) = x_i \) for \( i = 1, 2 \).

If \( \partial M = \emptyset \), the algebra reduces to the classical parameter-dependent pseudodifferential operators. The above definition is independent of the partitions of unity and trivializing sets we choose.
A central notion is parameter-ellipticity:

**Definition 10.** Given a parameter-dependent Boutet de Monvel operator $A \in B^m_{E_0,F_0,E_1,F_1}(M,A)$ we define:

1) The interior principal symbol $\sigma_0(A) \in C^\infty((T^*M \times \Lambda) \setminus \{0\}, \text{Hom}(\pi_{T^*M \times \Lambda}^*E_0, \pi_{T^*M \times \Lambda}^*E_1))$, where $\pi_{T^*M \times \Lambda} : T^*M \times \Lambda \to M$ is the canonical projection. It is the principal symbol of the pseudodifferential operator part of the operator $A$.

2) The boundary principal symbol $\sigma_0(A)$. For $(z,\lambda) \in (T^*\partial M \times \Lambda) \setminus \{0\}$ we let

$$
\sigma_0(A)(z)(\lambda) : \pi_{T^*\partial M \times \lambda}(E_{0}|_{\partial M} \otimes S(\mathbb{R}_+)) \oplus F_0 \to \pi_{T^*\partial M \times \lambda}(E_{1}|_{\partial M} \otimes S(\mathbb{R}_+)) \oplus F_0,
$$

where $\pi_{T^*\partial M \times \Lambda} : (T^*\partial M \times \Lambda) \setminus \{0\} \to \partial M$ is the canonical projection. After localization, it corresponds to the symbol in Definition 8.

We say that $A(\lambda)$ is parameter-elliptic if both symbols are invertible. With obvious changes, we can also define parameter-ellipticity, interior and boundary principal symbols of operators $A \in B^m_{E_0,F_0,E_1,F_1}(M,A)$.

The above operators act continuously on Bessel and Besov spaces. First we fix a dyadic partition of unity $\{\varphi_j ; j \in \mathbb{N}_0\}$.

**Definition 11.** Let $\varphi_0 \in C^\infty_c(\mathbb{R}^n)$ be supported $\{\xi ; |\xi| < 2\}$, $0 \leq \varphi_0 \leq 1$ and $\varphi_0(\xi) = 1$ in a neighborhood of the closed unit ball. Define $\varphi_j \in C^\infty_c(\mathbb{R}^n)$, $j \geq 1$, by $\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi)$.

**Remark 12.** We use the following notation: $K_j := \{\xi \in \mathbb{R}^n ; 2^j-1 \leq |\xi| \leq 2^{j+1}\}$, for $j \geq 1$, and $K_0 := \{\xi \in \mathbb{R}^n ; |\xi| \leq 2\}$. The above definition implies that $\text{supp}(\varphi_j) \subset \text{interior}(K_j)$, for $j \geq 0$. Moreover, we see that $\varphi_j(\xi) = \varphi_1(2^{-j+1}\xi)$, for $j \geq 2$ and $\sum_{j=0}^{\infty} \varphi_j(\xi) = \chi_1(\xi)$, $\xi \in \mathbb{R}^n$.

**Definition 13.** For each $s \in \mathbb{R}$, we define the operator $\langle D \rangle^s : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ as the pseudodifferential operator with symbol $\xi \in \mathbb{R}^n \mapsto \xi^s$. Moreover, we write $\varphi_j(D)u = \text{op}(\varphi_j)u$.

1) The Bessel potential space $H^s_p(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) ; \langle D \rangle^s u \in L_p(\mathbb{R}^n)\}$, for $1 < p < \infty$ and $s \in \mathbb{R}$, is the Banach space with norm $\|u\|_{H^s_p(\mathbb{R}^n)} := \|\langle D \rangle^s u\|_{L_p(\mathbb{R}^n)}$.

2) The Besov space $B^s_p(\mathbb{R}^n)$, for $s \in \mathbb{R}$ and $1 < p < \infty$, is the Banach space of all tempered distributions $f \in S'(\mathbb{R}^n)$ that satisfy:

$$
\|f\|_{B^s_p(\mathbb{R}^n)} := \left( \sum_{j=0}^{\infty} 2^{jsp} \|\varphi_j(D)f\|_{L_p(\mathbb{R}^n)}^p \right)^{\frac{1}{p}} < \infty.
$$

For an open set $\Omega \subset \mathbb{R}^n$, we define the Bessel potential spaces $H^s_p(\Omega)$, as the set of restrictions of $H^s_p(\mathbb{R}^n)$ to $\Omega$ with norm

$$
\|u\|_{H^s_p(\Omega)} := \left\{ \inf \|v\|_{H^s_p(\mathbb{R}^n)} ; r_{\Omega}(v) = u \right\}.
$$

Similarly, we define the Besov spaces $B^s_p(\Omega)$. Together with partition of unity and local charts, this leads to the spaces $H^s_p(M)$, $H^s_p(M, E)$, $B^s_p(\partial M)$ and $B^s_p(\partial M, E)$, where $E$ is a vector bundle over $M$ or $\partial M$.

**Remark 14.** Let $s \in \mathbb{R}$, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

1) There are continuous inclusions $C^\infty_c(\mathbb{R}^n) \hookrightarrow S(\mathbb{R}^n) \hookrightarrow B^s_p(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n)$. Moreover the spaces $C^\infty_c(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ are dense in $B^s_p(\mathbb{R}^n)$. The same can be said of $H^s_p(\mathbb{R}^n)$.

2) The dual of $B^s_p(\mathbb{R}^n)$ is $B^{-s}_q(\mathbb{R}^n)$, where the identification is given by the $L^2$ scalar product. Again the same holds for $H^s_p(\mathbb{R}^n)$ and $H^{-s}_q(\mathbb{R}^n)$.
3) A pseudodifferential operator with symbol $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ extends to continuous operators $\text{op}(a) : H^s_p(\mathbb{R}^n) \to H^{s-m}_p(\mathbb{R}^n)$ and $\text{op}(a) : B^s_p(\mathbb{R}^n) \to B^{s-m}_p(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

4) The following interpolation holds: $(L^p(\mathbb{R}^n), H^1_p(\mathbb{R}^n))_{\theta,p} = B^\theta_p(\mathbb{R}^n)$, for all $0 < \theta < 1$, where $(X,Y)_{\theta,p}$ denotes the real interpolation space of the interpolation couple $(X,Y)$, as in A. Lunardi [13].

5) If $M$ is a compact manifold (with or without boundary) and $E$ is a vector bundle over $M$, then $H^s_p(M, E) \hookrightarrow H^s_p(M, E)$ and $B^s_p(M, E) \hookrightarrow B^s_p(M, E)$ are compact inclusions, whenever $s > \frac{1}{p}$.

6) The trace functional $\gamma_0 : S(\mathbb{R}^n) \to S(\mathbb{R}^{n-1})$ extends to a continuous and surjective map $\gamma_0 : H^s_p(\mathbb{R}^n) \to B^{s+\frac{1}{p}}(\mathbb{R}^{n-1})$ when $s > \frac{1}{p}$.

7) The Besov spaces do not depend on the choice of the dyadic partition of unity; different partitions yield equivalent norms.

Remark 15. Let $G$ be a UMD Banach space that satisfies the property $(\alpha)$. Using Bochner integrals, we can define $B^s_p(\mathbb{R}, G)$ and $H^s_p(\mathbb{R}, G)$ in the same way as before, see, for instance, [2, 6]. It is worth noting that $B^s_p(\mathbb{R}, G)$ and $B^s_p(\partial X, E)$ are UMD spaces with the property $(\alpha)$ for all $s \in \mathbb{R}$ and $1 < p < \infty$. Later, we also use that $B^s_p(\mathbb{R}, G) \subset H^s_p(\mathbb{R}, G) := \{ u \in L^p(\mathbb{R}, G) : \frac{\partial u}{\partial t} \in L^p(\mathbb{R}, G) \}$, for all $0 < s < 1$.

Let us now state the following properties of composition, adjoints and continuity of Boutet de Monvel operators [22, 10, 9, 12].

Theorem 16. 1) (Composition) Let $A \in B^{m,d}_{E_{1},F_{1},E_{2},F_{2}}(M, \Lambda)$, $B \in B^{m',d'}_{E_{0},F_{0},E_{1},F_{1}}(M, \Lambda)$. Then $AB \in B^{m+m',d+d'}_{E_{0},F_{0},E_{2},F_{2}}(M, \Lambda)$, where $d'' := \max\{m'+d', d'\}$. Similarly, if $A \in B^0_{E_{0},F_{0},E_{1},F_{1}}(M, \Lambda)$ and $B \in B^0_{E_{0},F_{0},E_{2},F_{2}}(M, \Lambda)$, then $AB \in B^0_{E_{0},F_{0},E_{2},F_{2}}(M, \Lambda)$.

2) (Adjoint) Let $A \in B^0_{E_{0},F_{0},E_{1},F_{1}}(M, \Lambda)$. Then $A^* \in B^0_{E_{1},F_{1},E_{0},F_{0}}(M, \Lambda)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $A^*$ is the only operator that satisfies, for every $u \in C^\infty(M, E_0) \oplus C^\infty(\partial M, F_0)$ and $v \in C^\infty(M, E_1) \oplus C^\infty(\partial M, F_1)$, the relation

\[
(A(v)u, v)_{L^2(M, E_1) \oplus L^2(M, F_1)} = (u, A^*(v)\lambda)_{L^2(M, E_0) \oplus L^2(M, F_0)}.
\]

3) (Continuity) An operator $A \in B^{m,d}_{E_{0},F_{0},E_{1},F_{1}}(M, \Lambda)$ induces bounded operators $A(\lambda) : H^s_p(M, E_0) \oplus B^{s-m}_{\partial M, F_0} \to H^{s+m}_{p}(M, E_1) \oplus B^{s-m}_{p}(\partial M, F_1)$ for all $s > d - 1 + \frac{1}{2}$. Similarly $A \in B^{m}_{E_{0},F_{0},E_{1},F_{1}}(M, \Lambda)$ induces bounded operators $A(\lambda) : H^s_p(M, E_0) \oplus B^{s}_{p}(\partial M, F_0) \to H^{s}_{p}(M, E_1) \oplus B^{s}_{p}(\partial M, F_1)$, for all $s > -1 + \frac{1}{2}$.

4) (Fredholm property) If $A \in B^{m,d}_{E_{0},F_{0},E_{1},F_{1}}(M, \Lambda)$, $\lambda = \max\{m, 0\}$, is parameter-elliptic, then there exists a $B \in B^{-m,d}_{E_{2},F_{2},E_{0},F_{0}}(M, \Lambda)$, such that

\[
AB - I \in B^{s-\frac{1}{p},d'}_{E_{0},F_{0},E_{1},F_{1}}(M, \Lambda) \quad \text{and} \quad BA - I \in B^{s-\frac{1}{p},d'}_{E_{0},F_{0},E_{1},F_{1}}(M, \Lambda).
\]

As a consequence, $A(\lambda)$ is a Fredholm operator of index $0$ for each $\lambda \in \Lambda$, and there exists a constant $\lambda_0 > 0$ such that $A(\lambda)$ is invertible, if $|\lambda| \geq \lambda_0$.

Similarly, if $A \in B^{m}_{E_{0},F_{0},E_{1},F_{1}}(M, \Lambda)$ is parameter-elliptic, then there exists a $B \in B^{m,d}_{E_{1},F_{1},E_{0},F_{0}}(M, \Lambda)$ such that Equation (2.6) holds for $d = d' = 0$.

2.1. The equivalence between ellipticity and Fredholm property. In this section, we prove that the Fredholm property together with some growth condition on $\lambda$ implies parameter-dependent ellipticity. The use of Besov spaces makes the proofs a little more elaborate than e.g. the proof in the parameter-independent $L^2$-case studied by S. Rempel and B.-W. Schulze [22]. To make it clearer, we first study the pseudodifferential term on Besov spaces and then the boundary terms.
2.1.1. Pseudodifferential operators with parameters on a manifold without boundary acting on Besov spaces. In this section, we prove the following theorem:

Theorem 17. Let $M$ be a compact manifold without boundary, $E$ and $F$ be vector bundles over $M$. Let $A(\lambda) : C^\infty(M,E) \to C^\infty(M,F)$, $\lambda \in \Lambda$, be a classical parameter-dependent pseudodifferential operator of order $0$. Then the following conditions are equivalent:

i) $A$ is parameter-elliptic.

ii) There exist uniformly bounded operators $B_j(\lambda) : B^0_p(M,F) \to B^0_p(M,E)$, $\lambda \in \Lambda$, $j = 1$ and 2, such that

$$B_1(\lambda) A(\lambda) = 1 + K_1(\lambda) \text{ and } A(\lambda) B_2(\lambda) = 1 + K_2(\lambda).$$

where $K_1(\lambda) : B^0_p(M,E) \to B^0_p(M,E)$ and $K_2(\lambda) : B^0_p(M,F) \to B^0_p(M,F)$ are compact operators for every $\lambda \in \Lambda$. Moreover, $\lim_{|\lambda| \to \infty} K_j(\lambda) = 0$.

iii) There exist bounded operators $B_j(\lambda) : B^0_p(M,F) \to B^0_p(M,E)$, $\lambda \in \Lambda$, $j = 1$ and 2, such that

$$B_1(\lambda) A(\lambda) = 1 + K_1(\lambda) \text{ and } A(\lambda) B_2(\lambda) = 1 + K_2(\lambda).$$

where $K_1(\lambda) : B^0_p(M,E) \to B^0_p(M,E)$ and $K_2(\lambda) : B^0_p(M,F) \to B^0_p(M,F)$ are compact operators for every $\lambda \in \Lambda$. Moreover, $\lim_{|\lambda| \to \infty} K_j(\lambda) = 0$ and there exist $M \in \mathbb{N}_0$ and $C > 0$ such that $\|B_j(\lambda)\|_{B^0_p(M,F), B^0_p(M,E)} \leq C (\ln(\lambda))^M$, for some sufficiently small $r$, as a careful study of our proof shows.

We note that $A(\lambda) B_2(\lambda) = 1 + K_2(\lambda)$, for every $\lambda \in \Lambda$, is equivalent to $B_2(\lambda)^* A(\lambda)^* = 1 + K_2(\lambda)^*$, where $^*$ indicates the adjoint. This is the condition that we shall need. Obviously condition i) implies that $\dim(E) = \dim(F)$.

If i) holds, then we can find a parametrix to $A(\lambda)$ by Theorem 16(4) so that ii) is true, and ii) trivially implies iii). So we only need to prove that iii) implies i).

Definition 18. Let $s > 0$, $0 < \tau < \frac{1}{4}$ and $(y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$. We define the operator $R_s(y, \eta) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, also denoted just by $R_s$, by

$$R_s u(x) = \int_{\mathbb{R}^n} e^{is\xi \eta} u(s^\tau (x - y)).$$

Below we collect some well-known facts about the operators $R_s$. The items 1, 2, 4, 5 and 6 can be found in [22, 27]. As we are dealing also with Besov spaces, some estimates must be done more carefully. The third item was not proven in the previous references. Statement 7 is stronger than usual. Both are necessary, as $R_s$ is not an isometry in the space $B^0_p(\mathbb{R}^n)$.

Lemma 19. The operator $R_s = R_s(y, \eta)$ has the following properties:

1) $\|R_s u\|_{L^p(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)}$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.

2) $\lim_{s \to \infty} R_s u = 0$ weakly in $L^p(\mathbb{R}^n)$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.

3) $R_s : B^0_p(\mathbb{R}^n) \to B^0_p(\mathbb{R}^n)$ is continuous for all $s > 0$ and $\|R_s u\|_{B^0_p(\mathbb{R}^n)} \leq C_0 (1 + s^\theta (\eta))^\theta \|u\|_{H^s(\mathbb{R}^n)}$, for every $\theta \in [0,1]$, $s \geq 1$ and $u \in \mathcal{S}(\mathbb{R}^n)$. The constant $C_0$ depends on $\theta$, but not on $y$, $\eta$ or $s$.

4) The operator $R_s$ is invertible. Its inverse is given by

$$R_s^{-1} u(x) = s^{-\frac{\tau}{\theta}} e^{-is(y + \tau x)} u(y + \tau x).$$

5) The Fourier transform of $R_s u$ is given by

$$F(R_s u)(\xi) = s^{\frac{\tau}{\theta} - \tau \eta} e^{-i\theta (\xi - \eta)} u(s^{-\tau}(\xi - \eta)).$$
6) Let \( a \in S^m(\mathbb{R}^n \times \mathbb{R}^n, \Lambda) \). Then
\[
R_x^{-1} \text{op}(a)(s\lambda) R_x u(x) = \text{op}(a_s)(\lambda) u(x),
\]
where \( a_s(x, \xi, \lambda) = a(y + s^{-\tau}x, s\eta + s^\tau \xi, s\lambda) \).

7) Let \( a \in S^0(\mathbb{R}^n \times \mathbb{R}^n, \Lambda) \) be classical, \( u \in \mathcal{S}(\mathbb{R}^n) \), \( \lambda \in \Lambda \) with \((\eta, \lambda) \neq (0, 0)\) and \( 0 < r < \tau \). Then
(2.7)
\[
\lim_{s \to 0} s^\tau \left\| \text{op}(a)(s\lambda) R_x u - a_{(0)}(y, \eta, \lambda) R_x u \right\|_{L_p(\mathbb{R}^n)} = 0.
\]

**Proof.** 1), 4) and 5) are just simple computations, and 6) follows from 4), 5) and the definition of pseudodifferential operators.

In order to prove 2), we just have to note that \( \lim_{s \to \infty} \int_{\mathbb{R}^n} R_x u(x) v(x) \, dx = 0 \) for all \( u \in \mathcal{S}(\mathbb{R}^n) \) and \( v \in \mathcal{S}(\mathbb{R}^n) \). The proof follows then from the fact that \( L_p(\mathbb{R}^n) = L_q(\mathbb{R}^n) \), for \( \frac{1}{2} + \frac{1}{q} = 1 \), and that \( R_x \) is an isometry.

3) The operator \( R_x : B_\rho^0(\mathbb{R}^n) \to B_\rho^0(\mathbb{R}^n) \) is continuous for all \( s \in \mathbb{R} \), as \( R_x \) is the composition of dilatation, translation and multiplication by \( e^{ix\xi} \). The estimate follows by interpolation. In fact, for \( s \geq 1 \), it is easy to see that \( \|R_x u\|_{H_s^1(\mathbb{R}^n)} \leq (1 + s \langle \eta \rangle) \|u\|_{H_s^1(\mathbb{R}^n)} \). As \( (L_p(\mathbb{R}^n), H_s^1(\mathbb{R}^n))_{\rho, p} = B_\rho^0(\mathbb{R}^n) \), we conclude (see \( \Lambda \). Lunardi [35, Corollary 1.1.7]) that there exists a constant \( C_\rho \) such that
\[
\|R_x u\|_{B_\rho^0(\mathbb{R}^n)} \leq C_\rho \|R_x u\|_{H_s^1(\mathbb{R}^n)} \|R_x u\|_{L_p(\mathbb{R}^n)}^{1-\theta} \leq C_\rho (1 + s \langle \eta \rangle)^\theta \|u\|_{H_s^1(\mathbb{R}^n)}.
\]

7) This is the longest statement we need to prove. We divide the proof into several steps. Our first goal is the \( L_p \)-convergence:
(2.8)
\[
\lim_{s \to \infty} s^\tau \left\| \text{op}(a_s)(\lambda) u - a_{(0)}(y, \eta, \lambda) u \right\|_{L_p(\mathbb{R}^n)} = 0, \text{ where } u \in \mathcal{S}(\mathbb{R}^n).
\]

In a **first step** let us show that, for every \((x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n\),
(2.9)
\[
|a \left( y + s^{-\tau}x, s\eta + s^\tau \xi, s\lambda \right) - a_{(0)}(y, \eta, \lambda)| \leq C_{\lambda, \eta} \langle x \rangle \langle \xi \rangle^2 s^{-\tau}.
\]

Let \( \chi : \mathbb{R}^n \times \Lambda \to \mathbb{C} \) be a smooth function that is equal to 0 in a neighborhood of the origin and equal to 1 outside a closed ball centered at the origin that does not contain \((\eta, \lambda)\). For \( s \geq 1 \), we have
\[
|a \left( y + s^{-\tau}x, s\eta + s^\tau \xi, s\lambda \right) - \chi(s\eta + s^\tau \xi, s\lambda) a_{(0)} \left( y + s^{-\tau}x, s\eta + s^\tau \xi, s\lambda \right)|
\leq \frac{C}{\langle \eta + s^\tau \xi, s\lambda \rangle} \leq C s^\tau (s\eta, s\lambda)^{-1} \langle \xi \rangle,
\]
where we have used Peetre’s inequality. Since \( a_{(0)}(y, \eta, \lambda) = a_{(0)}(y, \eta, \lambda) \),
\[
|\chi(s\eta + s^\tau \xi, s\lambda) a_{(0)} \left( y + s^{-\tau}x, s\eta + s^\tau \xi, s\lambda \right) - a_{(0)}(y, \eta, \lambda)|
\leq \sum_{j=1}^n \left( \int_0^1 s^{-\tau} |x_j| \left| \chi(s\eta + ts^\tau \xi, s\lambda) \partial_{x_j} a_{(0)} \right| \left( y + ts^{-\tau}x, s\eta + ts^\tau \xi, s\lambda \right) \, dt 
+ s^\tau \int_0^1 |\xi_j| \partial_{\xi_j} \left( \chi a_{(0)} \right) \left( y + ts^{-\tau}x, s\eta + ts^\tau \xi, s\lambda \right) \, dt \right)
\leq \sum_{j=1}^n \left( C_1 s^{-\tau} |x_j| + C_2 s^{2\tau} \langle \eta, s\lambda \rangle \langle \xi_j \rangle \langle \xi \rangle \right).
\]

The estimates \( \Box \) and \( \Box \) imply \( \Box \) for \( \tau < \frac{1}{4} \).

In a **second step** we are going to show the pointwise convergence of the integral of \( \Box \) for all \( u \in \mathcal{S}(\mathbb{R}^n) \) and all \( x \in \mathbb{R}^n \). We know that
\[
s^\tau \left( \text{op}(a_s)(\lambda) u(x) - a_{(0)}(y, \eta, \lambda) u(x) \right) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} s^\tau \left( a \left( y + s^{-\tau}x, s\eta + s^\tau \xi, s\lambda \right) - a_{(0)}(y, \eta, \lambda) \right) \hat{u}(\xi) \, d\xi.
\]
The integrand goes to zero, as we have seen in Equation (2.9). Moreover
\[ |s^r (a (y + s^{−r} x, s_\eta, s_\lambda) − a_0 (y, \eta, \lambda)) \hat{u} (\xi)| \leq C_{\lambda, n} \langle \xi \rangle^{2} |\hat{u} (\xi)|, \]
is integrable with respect to \( \xi \), so that the dominated convergence theorem applies.

In the third step we will finally prove (2.8). It is enough to show that the integrand is dominated. Indeed, integration by parts shows that
\[ s^r x^r (op (a_s (\lambda) u (x) − a_0 (y, \eta, \lambda) u (x))) \]
\[ (2.12) \quad = \quad (-1)^{\gamma} \sum_{\sigma \leq \gamma} \left( \frac{\gamma}{\sigma} \right) s^r (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} D_\xi^n (a(y + s^{−r} x, s_\eta, s_\lambda) \]
\[ − a_0 (y, \eta, \lambda)) D_\xi^{−\sigma} \hat{u} (\xi) d\xi. \]
For \( \sigma = 0 \), we recall (2.9); for \( \sigma \neq 0 \), we use that \( r + 2r |\sigma| − |\sigma| < 0 \) and obtain
\[ s^r |D_\xi^n (a(y + s^{−r} x, s_\eta, s_\lambda))| \leq C |(\eta, \lambda)|^{−|\sigma|} \langle \xi \rangle^{|\sigma|}. \]
As \( \xi \mapsto \langle \xi \rangle^M \hat{u} (\xi) \) is integrable for all \( M > 0 \), (2.12) can be estimated by \( \hat{C}_{\lambda, \eta, \gamma} \langle x \rangle \).
Hence, for arbitrary \( N \),
\[ s^r |op (a_s (\lambda) u (x) − a_0 (y, \eta, \lambda) u (x))| \leq C_{\lambda, \eta, N} \langle x \rangle^{−N}. \]
The dominated convergence then shows the desired \( L_p \)-convergence.

Our next goal is to show \( L_p \)-convergence of the derivative:
\[ (2.13) \quad \lim_{s \to \infty} \quad s^r \quad \|op (a_s (\lambda) u − a_0 (y, \eta, \lambda) u)\|_{H^1_p (\mathbb{R}^n)} = 0, \quad u \in \mathcal{S} (\mathbb{R}^n). \]
Let us first observe that
\[ \partial_x op (a_s (\lambda)) = op (a_s (\lambda) \partial_x u + s^{−r} op ((\partial_x a_s) (\lambda)) u). \]
Using Equation (2.8) and the fact that \( r < \tau, \) we conclude that
\[ \lim_{s \to \infty} \quad s^r \quad \|op (a_s (\lambda) \partial_x u − a_0 (y, \eta, \lambda) \partial_x u)\|_{L_p (\mathbb{R}^n)} = 0 \]
and
\[ \lim_{s \to \infty} \quad s^r \quad \|−s^{−r} op ((\partial_x a_s) (\lambda)) u\|_{L_p (\mathbb{R}^n)} \]
\[ \leq \quad \lim_{s \to \infty} \quad s^r \quad \|op ((\partial_x a_s) (\lambda) u − (\partial_x a_s (0)) (y, \eta, \lambda) u)\|_{L_p (\mathbb{R}^n)} \]
\[ + \quad \lim_{s \to \infty} \quad s^r \quad \|(\partial_x a_s (0)) (y, \eta, \lambda) u\|_{L_p (\mathbb{R}^n)} = 0 \]
for all \( u \in \mathcal{S} (\mathbb{R}^n). \) Hence
\[ (2.14) \quad \lim_{s \to \infty} \quad s^r \quad \|\partial_x op (a_s (\lambda) u − a_0 (y, \eta, \lambda) \partial_x u)\|_{L_p (\mathbb{R}^n)} = 0. \]
Equations (2.8) and (2.13) imply (2.14).

In order to finish the proof of item (7), choose \( \theta > 0 \) such that \( \theta + r < \tau \). Then item (3) implies that
\[ s^r \quad \|op (a_s (\lambda) R_s u − a_0 (y, \eta, \lambda) R_s u)\|_{B^\theta_p (\mathbb{R}^n)} \]
\[ \leq \quad s^r \quad \|R_s (R_s^{−1} op (a_s (\lambda) R_s u − a_0 (y, \eta, \lambda) u))\|_{B^\theta_p (\mathbb{R}^n)} \]
\[ \leq \quad C_{\theta} (1 + s (\eta))^{\theta} s^r \quad \|op (a_s (\lambda) u − a_0 (y, \eta, \lambda) u)\|_{H^1_p (\mathbb{R}^n)} . \]
As the last term goes to zero, we obtain (2.7). \( \square \)

**Corollary 20.** Let \( a \in S^0_0 (\mathbb{R}^n \times \mathbb{R}^n, \Lambda) \) satisfy the transmission condition and \( u \in \mathcal{S} (\mathbb{R}^n_+). \) Then
\[ \lim_{s \to \infty} \quad s^r \quad \|\partial^+ op (a_s (\lambda) R_s (\partial^+ u) − a_0 (y, \eta, \lambda) R_s \partial^+ u)\|_{L_p (\mathbb{R}^n_+)} = 0, \]
for \( (y, \eta, \lambda) \in \mathbb{R}^n_+ \times ((\mathbb{R}^n \times \Lambda) \setminus \{0\}) \) and \( 0 < r < \tau \), where \( R_s = R_s (y, \eta). \)
Proof. We use that \( r^+ : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n) \) is continuous, that \( R_s : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n) \) is an isometry mapping \( C_c^\infty(\mathbb{R}^n) \) to \( C_c^\infty(\mathbb{R}^n) \), and Equation (2.8). \( \square \)

In order to control the action of \( R_s \) on Besov spaces, we recall the equivalence of Besov norm and \( L_p \) norm on certain subsets of \( S(\mathbb{R}^n) \), see e.g. [17].

**Lemma 21.** (Besov space property) There is a constant \( C > 0 \) such that
\[
C^{-1} \| u \|_{B^s_p(\mathbb{R}^n)} \leq \| u \|_{L_p(\mathbb{R}^n)} \leq C \| u \|_{B^s_p(\mathbb{R}^n)},
\]
for all \( u \in S'(\mathbb{R}^n) \) with \( \text{supp} F(u) \subset \bigcup_{k=0}^{m+2} K_k \) for some \( m \geq 0 \). Here \( C \) does not depend on \( m \). In particular, under these circumstances, \( u \in L_p(\mathbb{R}^n) \) if and only if \( u \in B^s_p(\mathbb{R}^n) \).

The number 2 could be replaced by a different one. We recall that the sets \( K_j \) were defined in Remark [12].

**Proof.** As \( \varphi_j(\xi) = \varphi_1(2^{-j+1} \xi) \) for \( j \geq 1 \), and \( u = \sum_{j=m-1}^{m+3} \varphi_j(D) u \), the estimate
\[
\| \varphi_j(D) u \|_{L_p(\mathbb{R}^n)} = \| F^{-1}(\varphi_j) \ast u \|_{L_p(\mathbb{R}^n)} \leq \| F^{-1}(\varphi) \|_{L_2(\mathbb{R}^n)} \| u \|_{L_p(\mathbb{R}^n)}, j \geq 1,
\]
implies the result. \( \square \)

The operator \( R_s \) has important properties when acting on functions whose Fourier transform is supported in \( \tilde{K} := \{ \xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 1 \} \).

**Lemma 22.** There is a constant \( s_0 > 0 \), that depends only on \( \eta, \) for which the operator \( R_s = R_s(\eta, \eta) \) has the following properties:

1) If \( u \in S'(\mathbb{R}^n) \) and \( \text{supp}(F u) \subset \tilde{K} \), then, for every \( s \geq s_0 \), there is an \( m \in \mathbb{N} \) that depends on \( s \), such that \( \text{supp}(F (R_s u)) \subset \bigcup_{k=0}^{m+2} K_k \).

2) There exists a constant \( C > 0 \) such that \( C^{-1} \| u \|_{B^s_p(\mathbb{R}^n)} \leq \| R_s u \|_{B^s_p(\mathbb{R}^n)} \leq C \| u \|_{B^s_p(\mathbb{R}^n)} \) for all \( s > s_0 \) and all \( u \in B^s_p(\mathbb{R}^n) \) with \( \text{supp}(F u) \subset \tilde{K} \).

3) For \( u \in S(\mathbb{R}^n) \) with \( \text{supp}(F u) \subset \tilde{K} \), \( \lim_{s \to \infty} R_s u = 0 \) weakly in \( B^0_p(\mathbb{R}^n) \).

**Proof.** 1) By item 5) of Lemma [19] \( F(R_s u)(\xi) = 0 \), unless \( \frac{1}{2} < |\xi - s \eta| < 1 \). If \( \eta = 0 \), this means that \( \frac{1}{2} s^7 < |\xi| < s^7 \). If \( \eta \neq 0 \), choose \( s_0 > 0 \) such that \( 2s^7 < |\xi| < 2s^7 \eta \), for \( s > s_0 \). Then \( \text{supp}(F (R_s u)) \subset \{ \xi : \frac{1}{2} s \eta < |\xi| < 2s \eta \} \), for \( s > s_0 \).

The result now follows easily.

2) As \( \text{supp}(F(R_s u)) \subset \bigcup_{k=0}^{m+2} K_k \) and \( \text{supp}(F(u)) \subset \tilde{K} \), the result follows from Lemma [21] and the fact that \( R_s \) is an isometry in \( L_p(\mathbb{R}^n) \).

3) From item 2) of Lemma [19] we know that
\[
\lim_{s \to \infty} \int_{\mathbb{R}^n} R_s u(x) v(x) \, dx = 0, \quad v \in S(\mathbb{R}^n).
\]

However, \( B^s_p(\mathbb{R}^n) \cong B^s_p(\mathbb{R}^n)' \), for \( \frac{1}{p} + \frac{1}{q} = 1 \), and \( S(\mathbb{R}^n) \) is dense in \( B^0_p(\mathbb{R}^n) \). As, by item 2), \( \| R_s u \|_{B^s_p(\mathbb{R}^n)} \) is uniformly bounded in \( s \) for all fixed \( u \in S(\mathbb{R}^n) \) such that \( \text{supp}(F u) \subset \tilde{K} \), the result follows. \( \square \)

We now prove Theorem [17]. The next simple lemma will be useful:

**Lemma 23.** Let \( E \) and \( F \) be Banach spaces and \( E' \) and \( F' \) be their dual spaces. If \( A : E \to F \) is a bounded linear operator such that \( A \) is injective, has closed range and its adjoint \( A' : F' \to E' \) is also injective, then \( A \) is an isomorphism.

**Proof.** Suppose that the range \( R(A) \) of \( A \) is a proper subset of \( F \). By the Hahn-Banach Theorem, there is an \( f \in F^* \), \( f \neq 0 \), such that \( f |_{R(A)} = 0 \). This implies that \( A^*(f) = f \circ A = 0 \). As \( A^* : F' \to E' \) is injective, we conclude that \( f = 0 \), which is a contradiction. \( \square \)
Proof. (of Theorem 22)  
As it suffices to prove the implication \( iii \) \( \implies \) \( i \), consider \( A(\lambda), B(\lambda) \) and \( K(\lambda) \) as in \( iii \). Our aim is to prove that the principal symbol \( p_{(0)}(z, \lambda) \) of \( A \) is invertible for every \( (z, \lambda) \in (T^*M \times \Lambda) \setminus \{0\} \). We focus on a trivializing coordinate neighborhood \( U \) containing \( x = \pi(z) \). We choose smooth functions \( \Phi, \Psi \) and \( H \) supported in \( U \) such that \( \Phi \) equals 1 near \( x \) and \( \Phi = \Psi, H = \Psi \). Denote by \( \tilde{A}(\lambda) \in B(B^0_p(\mathbb{R}^n)^{N_1}, B^0_p(\mathbb{R}^n)^{N_2}) \) and \( B(\lambda) \in B(B^0_p(\mathbb{R}^n)^{N_2}, B^0_p(\mathbb{R}^n)^{N_1}) \) the operators \( HAx(\lambda)\Psi \) and \( \Phi B(\lambda)H \) in local coordinates. Then our assumptions imply that there are compact operators \( K(\lambda) \), tending to zero in \( B(B^0_p(\mathbb{R}^n)^{N_1}) \) as \( |\lambda| \to \infty \) such that

\[
(2.15) \quad \tilde{B}(\lambda)\tilde{A}(\lambda) = \tilde{\Phi} + \tilde{K}(\lambda),
\]

where \( \tilde{\Phi} \) is \( \Phi \) in local coordinates. Here we use the fact that \( \tilde{B}(\lambda) \) has logarithmic growth and that \( \Phi B(\lambda)H^2A(\lambda)\Psi \) differs from \( \Phi B(\lambda)A(\lambda)\Psi \) by a compact operator whose norm tends to zero as \( |\lambda| \to \infty \).

Denote by \( (y, \eta, \lambda) \in \mathbb{R}^n \times (\mathbb{R}^n \times \Lambda) \setminus \{0\} \) the point corresponding to \( (z, \lambda) \) and fix an element \( u = cv \in \mathbb{S}(\mathbb{R}^n)^{N_1} \), where \( c \in \mathbb{C}^{N_1} \) and \( 0 \neq v \in \mathbb{S}(\mathbb{R}^n) \) with \( \text{supp}(\mathbb{F}v) \subset \{1; \frac{1}{2} < |\xi| < 1\} \). Equation (2.15) together with item \( ii \) of Lemma 22 implies that

\[
\|u\|_{B^0_p(\mathbb{R}^n)^{N_1}} \leq C \left( \|\tilde{B}(\lambda)\|_{B(B^0_p(\mathbb{R}^n)^{N_2}, B^0_p(\mathbb{R}^n)^{N_1})} \|\tilde{A}(\lambda)\|_{B^0_p(\mathbb{R}^n)^{N_2}} \right)
+ \|\tilde{K}(\lambda)\|_{B^0_p(\mathbb{R}^n)^{N_1}} + \|(1 - \tilde{\Phi})R(0)\|_{B^0_p(\mathbb{R}^n)^{N_1}}.
\]

We claim that \( \lim_{s \to \infty} \|\tilde{K}(\lambda)\|_{B^0_p(\mathbb{R}^n)^{N_1}} = 0 \) for \( \lambda \neq 0 \), and \( \|R(0)\|_{B^0_p(\mathbb{R}^n)^{N_1}} \leq C \|u\|_{B^0_p(\mathbb{R}^n)^{N_1}} \). For \( \lambda = 0 \), we use that \( \tilde{K}(\lambda) \) is compact and the third item of Lemma 22 which implies that \( \lim_{s \to \infty} R(0)u = 0 \) weakly in \( B^0_p(\mathbb{R}^n)^{N_1} \).

Since \( \Phi \in \mathbb{C}^{\infty}_c(\mathbb{R}^n) \) is equal to 1 in a neighborhood of \( y \), \( \lim_{s \to \infty} (1 - \tilde{\Phi})R(0)u = 0 \) in the topology of \( \mathbb{S}(\mathbb{R}^n) \) and, therefore, also in the topology of \( B^0_p(\mathbb{R}^n)^{N_1} \). We moreover estimate

\[
\|\tilde{A}(\lambda)R(0)u\|_{B^0_p(\mathbb{R}^n)^{N_2}} \leq \|\tilde{A}(\lambda)\|_{B^0_p(\mathbb{R}^n)^{N_2}} \|R(0)u\|_{B^0_p(\mathbb{R}^n)^{N_2}}
+ C \|\tilde{p}(0)(y, \eta, \lambda)\|_{B_{(C^{N_1}, C^{N_2})}} \|v\|_{B^0_p(\mathbb{R}^n)^{N_2}}.
\]

Item \( 7 \) of Lemma 13 implies that \( \lim_{s \to \infty} s^r\|\tilde{A}(\lambda)\|_{B_{(B^0_p(\mathbb{R}^n)^{N_2}, B^0_p(\mathbb{R}^n)^{N_1})}} = 0 \) for \( r \) sufficiently small. By assumption, \( \|\tilde{B}(\lambda)\|_{B_{(B^0_p(\mathbb{R}^n)^{N_2}, B^0_p(\mathbb{R}^n)^{N_1})}} \leq \tilde{C}(\ln(\lambda))^{\tilde{M}} \). Taking \( s \) sufficiently large, we conclude that

\[
C \left( \|\tilde{B}(\lambda)\|_{B_{(B^0_p(\mathbb{R}^n)^{N_2}, B^0_p(\mathbb{R}^n)^{N_1})}} \|\tilde{A}(\lambda)\|_{B^0_p(\mathbb{R}^n)^{N_2}} \right)
+ \|\tilde{K}(\lambda)\|_{B^0_p(\mathbb{R}^n)^{N_1}} + \|(1 - \tilde{\Phi})R(0)\|_{B^0_p(\mathbb{R}^n)^{N_1}} \leq \frac{1}{2} \|u\|_{B^0_p(\mathbb{R}^n)^{N_1}}.
\]

Hence, for sufficiently large \( s \), we have

\[
\|v\|_{B^0_p(\mathbb{R}^n)^{N_1}} \leq \frac{1}{2} \|u\|_{B^0_p(\mathbb{R}^n)^{N_1}} \leq \tilde{C}(\ln(\lambda))^{\tilde{M}} \|\tilde{p}(0)(y, \eta, \lambda)\|_{B_{(C^{N_1}, C^{N_2})}} \|v\|_{B^0_p(\mathbb{R}^n)^{N_2}}.
\]

As \( v \neq 0 \), this clearly implies that \( \tilde{p}(0)(y, \eta, \lambda) \) is injective.

An analogous argument applies to the adjoint operator. We conclude that \( \tilde{p}(0)(y, \eta, \lambda) \), that is, the adjoint of \( \tilde{p}(0)(y, \eta, \lambda) \) and the principal symbol of \( A(\lambda) \), is also injective. Lemma 22 then tells us that \( \tilde{p}(0)(y, \eta, \lambda) \) is an isomorphism and, in particular, that \( N_2 = N_1 \). Therefore \( A(\lambda) \) is an elliptic operator.
2.1.2. Boutet de Monvel operators with parameters acting on $L_p$-spaces.

**Theorem 24.** Let $M$ be a compact manifold with boundary $\partial M$. Let $E_0$ and $E_1$ be vector bundles over $M$, $F_0$ and $F_1$ be vector bundles over $\partial M$ and $A \in \mathcal{B}_{E_0,F_0,E_1,F_1}(M,A)$. Then the following conditions are equivalent:

i) The operator $A(\lambda)$ is an elliptic parameter-dependent operator.

ii) We find bounded operators $B_1(\lambda) : L^p(M,E_0) \oplus B^0_\infty(M,F_0) \to L^p(M,E_1) \oplus B^0_\infty(M,F_1)$ and $B_2(\lambda) : L^p(M,E_1) \oplus B^0_\infty(M,F_1) \to L^p(M,E_0) \oplus B^0_\infty(M,F_0)$ such that

$$B_1(\lambda) A(\lambda) = 1 + K_1(\lambda) \quad \text{and} \quad A(\lambda) B_2(\lambda) = 1 + K_2(\lambda), \quad \lambda \in \Lambda,$$

where the $B_j(\lambda)$ are uniformly bounded in $\lambda$ and $K_1(\lambda) : L^p(M,E_0) \oplus B^0_\infty(M,F_0) \to L^p(M,E_0) \oplus B^0_\infty(M,F_0)$ and $K_2(\lambda) : L^p(M,E_1) \oplus B^0_\infty(M,F_1) \to L^p(M,E_1) \oplus B^0_\infty(M,F_1)$ are compact and $\lim_{|\lambda| \to \infty} K_j(\lambda) = 0$, $j = 1, 2$.

iii) Condition ii) holds with the uniform boundedness of the $B_j(\lambda)$ replaced by the condition that, for $j = 1, 2$ and some $M \in \mathbb{N}_0$,

$$||B_j(\lambda)||_{\mathcal{B}(L^p(M,E_1) \oplus B^0_\infty(M,F_1), L^p(M,E_0) \oplus B^0_\infty(M,F_0))} \leq C(\ln(\lambda))^M.$$

**Remark 25.** Let $A^*(\lambda)$ be the adjoint operator of $A(\lambda)$. Theorem 16 tells us that $A(\lambda) B_2(\lambda) = 1 + K_2(\lambda)$ is equivalent to

$$B^*_2(\lambda) A^*(\lambda) = 1 + K^*_2(\lambda),$$

which is the condition that we will need later.

Again a standard parametrix construction shows that i) implies ii). As ii) implies iii) trivially, we only have to prove that iii) implies i).

We fix a point $(y, \eta) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ and a constant $0 < \tau < \frac{1}{4}$. For every $s > 0$, we define the isometries $R_s = R_s(y, \eta) : L^p(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^{n-1})$, $S_s : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ and $R_s \otimes S_s : L^p(\mathbb{R}_+^2) \to L^p(\mathbb{R}_+^2)$ by

$$R_s u(x') = \frac{\tau(n-1)}{r^\tau} e^{ix'y/s\tau} u(s\tau (x') - y), \quad S_s w(x_n) = s^{\frac{1}{2}} w(sx_n),$$

$$R_s \otimes S_s u(x) = \frac{\tau(n-1)}{r^\tau} s^{\frac{1}{2}} e^{ix'y/s\tau} u(s\tau (x') - y), sx_n.$$

The following simple proposition will be useful. It is very similar to the results we have already seen.

**Proposition 26.** The operator $R_s \otimes S_s : L^p(\mathbb{R}_+^2) \to L^p(\mathbb{R}_+^2)$ satisfies:

1) $||R_s \otimes S_s u||_{L^p(\mathbb{R}_+^2)} = ||u||_{L^p(\mathbb{R}_+^2)}$, $u \in L^p(\mathbb{R}_+^2)$.

2) $\lim_{s \to \infty} R_s \otimes S_s u = 0$ in the weak topology of $L^p(\mathbb{R}_+^2)$.

**Proof.**

1) is easily verified.

2) Due to the first item and the fact that $L^q(\mathbb{R}_+^2) \cong L^p(\mathbb{R}_+^2)'$, it is enough to prove that if $u(x) = u_1(x') u_2(x_n)$ and $v(x) = v_1(x') v_2(x_n)$, where $u_1, v_1 \in C_c(\mathbb{R}^{n-1})$ and $u_2, v_2 \in C_c(\mathbb{R}_+)$, then

$$\lim_{s \to \infty} \int_{\mathbb{R}_+^2} R_s \otimes S_s u(x) v(x) \, dx = 0.$$

A simple computation shows that both terms on the right hand side go to zero as $s \to \infty$. \qed
Proposition 27. Let \( 0 < r < \tau \) and let \( v \in S(\mathbb{R}^{n-1}) \) be such that \( F(v) \) has compact support. Denote by \( C(s) \) a function such that \( \lim_{s \to \infty} C(s) = 0 \). Then

1) (Pseudodifferential operator in the interior) Let \( p \in S^0(\mathbb{R}^n \times \mathbb{R}^n, \Lambda) \) satisfy the transmission condition and \( p \sim \sum_{j \in \mathbb{N}_0} p(-j) \) be its asymptotic expansion. Then

\[
\| \text{op}(p)(s\lambda) (R_s v \otimes S_s w) \|_{L_p(\mathbb{R}^n)} \leq C(s) \|w\|_{L_p(\mathbb{R}^n)},
\]

where

\[
\lim_{s \to \infty} s^r \| R_s v \otimes S_s w \|_{L_p(\mathbb{R}^n)} = 0.
\]

2) (Singular Green operators) Let \( S^{-\frac{1}{2}}_{\text{cl}}(\mathbb{R}^{n-1}, S,+\Lambda) \ni \tilde{g} \sim \sum_{j \in \mathbb{N}_0} \tilde{g}(-1-j) \) and \( G(\lambda) : S(\mathbb{R}^n) \to S(\mathbb{R}^n) \) be defined by (2.22). Then, for \( w \in S(\mathbb{R}^n) \),

\[
\| G(s\lambda) (R_s v \otimes S_s w) \|_{L_p(\mathbb{R}^n)} \leq C(s) \|w\|_{L_p(\mathbb{R}^n)}.
\]

3) (Trace operators) Let \( S^{-\frac{1}{2}}_{\text{cl}}(\mathbb{R}^{n-1}, S,+\Lambda) \ni \tilde{t} \sim \sum_{j \in \mathbb{N}_0} \tilde{t}(-\frac{1}{2}-j) \) and \( T(\lambda) : S(\mathbb{R}^n) \to S(\mathbb{R}^{n-1}) \) be defined by (2.22). Then for \( w \in S(\mathbb{R}^n) \),

\[
\| T(s\lambda) (R_s v \otimes S_s w) - R_s \int_{\mathbb{R}^n} \tilde{t}(-\frac{1}{2})(y,x_n,\eta,\lambda) w(x_n) d\nu \|_{L_p(\mathbb{R}^{n-1})} \leq C(s) \|w\|_{L_p(\mathbb{R}^n)}.
\]

4) (Poisson operators) Let \( S^{-\frac{1}{2}}_{\text{cl}}(\mathbb{R}^{n-1}, S,+\Lambda) \ni \tilde{k} \sim \sum_{j \in \mathbb{N}_0} \tilde{k}(-1-j) \) and \( K(\lambda) : S(\mathbb{R}^{n-1}) \to S(\mathbb{R}^n) \) be defined by (2.22). Then for \( w \in S(\mathbb{R}^n) \),

\[
\lim_{s \to \infty} s^r \| K(s\lambda) (R_s v) - R_s \int_{\mathbb{R}^n} \tilde{k}(-\frac{1}{2})(y,x_n,\eta,\lambda) v(x') d\nu \|_{L_p(\mathbb{R}^n)} = 0.
\]

Proof. The items 1, 2) and 4) extend the results in [22, Section 2.3.4.2]. They can be obtained by replacing the operators \( R_s \) and \( S_s \) in [22] by the definitions given here and arguing similarly as for the third item.

The third item is more delicate, as the limit is taken in the Besov space: Let \( q \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Using item 4, 5 and 6 of Lemma [19] we find that

\[
R_s^{-1} T(s\lambda) (R_s v \otimes S_s w) (x') = \int_{\mathbb{R}^n} e^{ix \cdot \xi'} \left( \int_{\mathbb{R}^n} \tilde{t}(y,x_n,\eta,\lambda) w(x_n) d\nu \right) d\xi'.
\]

Fix \( (y,\eta,\lambda) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Lambda \) such that \( (\eta,\lambda) \neq (0,0) \). We will use the simple fact that if \( v \in S(\mathbb{R}^{n-1}) \) is such that \( \text{supp}(F(v)) \) is compact, then for all \( \theta \in [0,1] \) and for all \( \xi' \in \text{supp}(F(v)) \), there is a \( s_0 > 0 \) such that

\[
C^{-1} s^{M \cdot |(\eta,\lambda)|} \leq |s\eta + s^\theta \xi'|, s \geq s_0.
\]

The constant \( C \) does not depend on \( \theta, s \geq s_0 \) and \( \xi' \in \text{supp}(F(v)) \).

We start by establishing \( L_p \)-convergence: Let \( 0 < r < \tau \) and \( v \in S(\mathbb{R}^{n-1}) \) with \( \text{supp}(F(v)) \) compact. Then, for all \( w \in S(\mathbb{R}^n) \), we have

\[
s^r \| R_s^{-1} T(s\lambda) (R_s v \otimes S_s w) - v(x') \|_{L_p(\mathbb{R}^{n-1})} \leq C(s) \|w\|_{L_p(\mathbb{R}^n)},
\]

where \( C(s) \) is a constant that depends on \( s, (y,\eta,\lambda) \) and \( v \) but not on \( w \). Moreover, \( \lim_{s \to \infty} C(s) = 0 \).
We divide the proof into steps, always assuming that $s \geq s_0$. First we see that

$$s^r R_s^{-1}T(s \lambda) (R_s v \otimes S_w) - v(x') \int_{\mathbb{R}^+} \hat{I}(-\frac{r}{s}) (y, x_n, \eta, \lambda) w(x_n) \, dx_n$$

(2.17) \leq \|w\|_{L^p(\mathbb{R}^+)} \left( \int_{\mathbb{R}^+} \left| \int_{\mathbb{R}^{n-1}} e^{ix' \xi'} s^{-\frac{r}{s}} \hat{I}(y + s^{-r} x', \frac{x_n}{s}, \eta + s^r \xi', \lambda) 
- s^{\frac{r}{s}} \hat{I}(-\frac{r}{s}) (y, x_n, \eta, \lambda) \right| \hat{\nu}(\xi') \, d\xi' \right)^{\frac{1}{q}} \, dx_n.$$

In a first step we will prove that, for all $(x', x_n, \xi') \in \mathbb{R}^{n-1} \times \mathbb{R}_x \times \mathbb{R}^{n-1}$ and $M \in \mathbb{N}_0$, there is a constant that depends on $\eta, \lambda$ and $M$ such that

$$s^{-\frac{r}{s}} \hat{I}(-\frac{r}{s}) (y, x_n, \eta, \lambda) \hat{\nu}(\xi') \, d\xi' \leq C_{\eta, \lambda, M} (x_n)^{-M} s^{r-M} \hat{\nu}(\xi').$$

(2.18)

Let us fix a function $\chi \in C^\infty(\mathbb{R}^{n-1} \times \Lambda)$ that is zero near the origin and equal to 1 outside a closed ball that does not contain $(\eta, \lambda)$. We note that

$$\left| s^{-\frac{r}{s}} \hat{I}(-\frac{r}{s}) (y, x_n, \eta, \lambda) \hat{\nu}(\xi') \, d\xi' \right| \leq C_1 s^{-\frac{r}{s}+r+M} (\eta, \lambda)^{-\frac{1}{s}+M} \leq C_2 s^{-\frac{1}{s}+M} (\eta, \lambda)^{-\frac{1}{s}+M},$$

(2.19)

for $\xi' \in \text{supp} (\hat{\nu}(\eta))$. We now study the term

$$s^{-\frac{r}{s}+M} \left( \frac{x_n}{s} \right)^M \hat{I}(\chi (s \eta + s^r \xi')) \hat{\nu}(\xi') \left( y + s^{-r} x', \frac{x_n}{s}, \eta + s^r \xi', \lambda \right)$$

(2.20)

Using the fact that $\int_{\mathbb{R}^+} \hat{I}(-\frac{r}{s}) (y, x_n, \eta, \lambda) \hat{\nu}(\xi') \, d\xi' = \hat{I}(-\frac{r}{s}) (y, x_n, \eta, \lambda)$, and a Taylor expansion we conclude that the expression (2.20) is smaller or equal to

$$s^{-\frac{r}{s}+M} \left( \frac{x_n}{s} \right)^M \hat{I}(\chi (s \eta + s^r \xi')) \hat{\nu}(\xi') \left( y + s^{-r} x', \frac{x_n}{s}, \eta + s^r \xi', \lambda \right)$$

$$
+ s^{r+\frac{r}{s}+M} \sum_{|\beta|=1} |\xi| \hat{I}(\chi (s \eta + s^r \xi')) \hat{\nu}(\xi') \left( y + s^{-r} x', \frac{x_n}{s}, \eta + s^r \xi', \lambda \right)$$

$$|d| \hat{I}(\chi (s \eta + s^r \xi')) \hat{\nu}(\xi') \left( y + s^{-r} x', \frac{x_n}{s}, \eta + s^r \xi', \lambda \right)$$

(2.21)

As $0 < r < \tau < \frac{1}{2}$, we conclude that $-1 + r \tau - r + r - 1 < r - \tau$. Hence (2.18) follows from the estimates of (2.19) and (2.21).

In a second step we will next show that the limit of Equation (2.18) as $s \to \infty$ is zero. This is true, as it is smaller than or equal to

$$C_{\eta, \lambda} s^{r-M} \int_{\mathbb{R}^{n-1}} |\hat{\nu}(\xi')| \, d\xi', \quad M > 1.$$

In a third step we want to prove that, for all $M \in \mathbb{N}_0$, the expression (2.18) is bounded by $C_M (x')^{M}$, for a constant $C_M > 0$. Then Lebesgue’s dominated convergence theorem will imply that (2.17) holds. In order to do that, we note that

$$\int_{\mathbb{R}^+} \hat{I}(-\frac{r}{s}) (y, x_n, \eta, \lambda) \hat{\nu}(\xi') \, d\xi' \leq \int_{\mathbb{R}^+} \hat{I}(-\frac{r}{s}) (y, x_n, \eta, \lambda) \hat{\nu}(\xi') \, d\xi'.$$

is a linear combination of terms of the form

$$\int_{\mathbb{R}^{n-1}} e^{ix' \xi'} s^{-\frac{r}{s}} D_{\xi'} \left( \hat{I}(y + s^{-r} x', \frac{x_n}{s}, \eta + s^r \xi', \lambda) - s^{\frac{r}{s}} \hat{I}(-\frac{r}{s}) (y, x_n, \eta, \lambda) \right) D_{\xi'^{-\sigma}} \hat{\nu}(\xi') \, d\xi'.$$
If $\sigma = 0$, we have already proven that the above expression is smaller than $C_{\eta,\lambda, M} \langle x_n \rangle^{-M} s^{r-\tau}$. For $\sigma \neq 0$, we estimate
\[
\left| s^{r-\frac{2}{s}} x_n^{M} D^r \xi \left( \frac{y + s^{-r} x'}{s}, s\eta + s^r \xi', s\lambda \right) \right| 
\leq \left| s^{r-\frac{1}{s} + |\sigma| + M} \left( \frac{x_n}{s} \right)^{M} (D^r \xi) \left( \frac{y + s^{-r} x'}{s}, s\eta + s^r \xi', s\lambda \right) \right| 
\leq C_{1} s^{r-\frac{1}{s} + |\sigma| + M} \langle s\eta + s^r \xi', s\lambda \rangle^{\frac{1}{s} - 1 - |\sigma|} 
\leq C_{2} s^{r+|\sigma| - |\sigma| - \frac{1}{s} - 1 - |\sigma|}.
\]
Hence
\[
|s^{r-\frac{2}{s}} D^r \eta (y + s^{-r} x', \frac{x_n}{s}, s\eta + s^r \xi', s\lambda) | \leq C_{\eta,\lambda, M} \langle x_n \rangle^{-M} s^{r-\tau}.
\]
The result now follows easily.

We will next establish the $L_p$-convergence of the derivative. Let $0 < r < \tau$ and $v \in \mathcal{S}(\mathbb{R}^{n-1})$ with $\text{supp}(\mathcal{F}(v))$ compact. Then, for all $w \in \mathcal{S}(\mathbb{R}^{+})$, we have
\[
\begin{align*}
\int_{\mathbb{R}^{n-1}} & \eta \bigl( y, x_n, \eta, \lambda \bigr) w(x_n) \, dx_n \biggr|_{H^{r}_{p}(\mathbb{R}^{n-1})} 
\leq C(s) \|w\|_{L_p(\mathbb{R}^{+})},
\end{align*}
\]
where $C(s)$ is a constant that depends on $s$, $(\eta, \lambda)$ and $v$ but not on $w$. Moreover, $\lim_{r \to \tau} C(s) = 0$.

Let us first fix a notation. We denote by $(\partial_{x_j} T)(\lambda)$, $j = 1, ..., n-1$, the operator:
\[
(\partial_{x_j} T)(\lambda) (u)(x') = \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \int_{\mathbb{R}^{+}} \partial_{x_j} i \left( \frac{y}{s}, x_n, \eta, \lambda \right) (\mathcal{F} e^{ix' \cdot \xi} u)(\xi', x_n) \, dx_n \, d\xi'.
\]
Now, let us first observe that, for $j = 1, ..., n-1$,\n\[
\partial_{x_j} R^{-1}_{s} T (s\lambda) (R_{s} v \otimes S_{w})
= R^{-1}_{s} T (s\lambda) \left( R_{s} \right) (\partial_{x_j} v) \otimes S_{w} + s^{-\tau} R^{-1}_{s} (\partial_{x_j} T)(s\lambda) (R_{s} v \otimes S_{w}).
\]
Using Equation (2.17) and the fact that $r < \tau$, we conclude that
\[
\left| s^{r} \left| R^{-1}_{s} T (s\lambda) \left( R_{s} \right) (\partial_{x_j} v) \otimes S_{w} \right| \right|_{L_p(\mathbb{R}^{n-1})} \leq C(s) \|w\|_{L_p(\mathbb{R}^{+})}
\]
and
\[
\begin{align*}
\left| s^{r} \left| s^{-\tau} R^{-1}_{s} (\partial_{x_j} T)(s\lambda) (R_{s} v \otimes S_{w}) \right| \right|_{L_p(\mathbb{R}^{n-1})} 
\leq \left| s^{r-\tau} \left| R^{-1}_{s} (\partial_{x_j} T)(s\lambda) (R_{s} v \otimes S_{w}) \right| \right|_{L_p(\mathbb{R}^{n-1})} 
\leq C(s) \|w\|_{L_p(\mathbb{R}^{+})}.
\end{align*}
\]
The expressions (2.24), (2.25) and (2.26) imply that
\[
\begin{align*}
\left| s^{r} \left| \partial_{x_j} \left( R^{-1}_{s} T (s\lambda) (R_{s} v \otimes S_{w}) \right) \right| \right|_{L_p(\mathbb{R}^{n-1})} 
\leq C(s) \|w\|_{L_p(\mathbb{R}^{+})}.
\end{align*}
\]
Finally, (2.23) is a consequence of Equations (2.24) and (2.17).
We are now in the position to prove item 3. Choose $0 < \theta < \theta + r < \tau$. Then

\[
s^r \left\| T (s\lambda) (R_s v \otimes S_s w) - (R_s v) (x') \right\| = \int_{R_+} \left| \left( y, x_n, \eta, \lambda \right) w (x_n) \right| dx_n \leq \int_{R_+} \left| \left( y, x_n, \eta, \lambda \right) w (x_n) \right| dx_n \leq C_0 \left( 1 + s \langle \eta \rangle \right)^\theta s^r \left\| R_s^{-1} T (s\lambda) (R_s v \otimes S_s w) - v (x') \right\| \leq C_0 \left( 1 + s \langle \eta \rangle \right)^\theta \left\| R_s^{-1} T (s\lambda) (R_s v \otimes S_s w) \right\| \leq C_0 \left( 1 + s \langle \eta \rangle \right)^\theta \left\| R_s^{-1} T (s\lambda) (R_s v \otimes S_s w) \right\| \leq C_0 \left( 1 + s \langle \eta \rangle \right)^\theta \left\| R_s^{-1} T (s\lambda) (R_s v \otimes S_s w) \right\|^\theta \left\| \left( y, x_n, \eta, \lambda \right) w (x_n) \right\| \left. \right|_{H^1 (R^n)} \leq C \left( s \right) \left\| w \right\|_{L_p (R^n)}.
\]

We also need to understand the action of the singular Green and trace operators on the operators $R_s = R_s (y, \eta)$ for $(y, \eta) \in R^n \times R^n.$ Notice that $(y, \eta) \in R^n \times R^n$ instead of $R^{n-1} \times R^{n-1}$ as in the previous proposition.

**Proposition 28.** Let $R_s = R_s (y, \eta)$, where $\eta = (\eta', \eta_n) \in R^{n-1} \times R$ and $y = (y', 0) \in R^n \times R$. For $u \in C_c^\infty (R^n)$ the following properties hold:

1) (Green) For $\tilde{g} \in \mathcal{S}_d^+ (R^{n-1}, S_{1,+}, \Lambda)$ define $G (\lambda) : \mathcal{S} (R^n) \rightarrow \mathcal{S} (R^n)$ by Equation (2.24). Then $\lim_{s \rightarrow \infty} s^r \left\| G (s\lambda) (R_s) (e^s u) \right\|_{L_p (R^n)} = 0$ for all $r > 0$.

2) (Trace) For $\tilde{f} \in \mathcal{S}_d^+ (R^{n-1}, S_{1,+}, \Lambda)$ define $T (\lambda) : \mathcal{S} (R^n) \rightarrow \mathcal{S} (R^n)$ by Equation (2.22). Then $\lim_{s \rightarrow \infty} s^r \left\| T (s\lambda) (R_s) (e^s u) \right\|_{B_{p}^b (R^{n-1})} = 0$ for all $r > 0$.

**Proof.** The proof is analogous to that of Proposition 27. Let us sketch the proof of 2) as 1) is similar.

Let $\frac{1}{p} + \frac{1}{q} = 1, R_s^{-1} := R_s^{-1} (y', \eta') : \mathcal{S} (R^{n-1}) \rightarrow \mathcal{S} (R^n)$ and $R_s := R_s (y, \eta) : \mathcal{S} (R^n) \rightarrow \mathcal{S} (R^n)$. Using item 6 of Lemma 19 in $(x', \xi')$ and the definition of $R_s$, we obtain that

\[
R_s^{-1} T (s\lambda) (R_s u) (x') = \int_{R^n} e^{ix \cdot \xi'} \left( \int_{R^n} e^{ir s^{-1} x \cdot n_s \xi'} \frac{\xi}{s^N} \right) \mathcal{F} x \mapsto \xi, u (\xi', x_n) dx_n d\xi'.
\]

Now, we note that

\[
\left( \frac{x_n}{s^N} \right)^N \left| \int_{R^n} e^{i x \cdot \xi'} \left( \int_{R^n} e^{i r s^{-1} x \cdot n_s \xi'} \frac{\xi}{s^N} \right) \mathcal{F} x \mapsto \xi, u (\xi', x_n) dx_n d\xi' \right| \leq C_N \langle s \xi' \rangle \langle s \lambda \rangle^{-\frac{1}{p} + 1 - N},
\]

On the support of $u$, we have $x_n \geq R > 0$ for a certain constant $R > 0$. Hence

\[
\left| \int_{R^n} e^{i x \cdot \xi'} \left( \int_{R^n} e^{i r s^{-1} x \cdot n_s \xi'} \frac{\xi}{s^N} \right) \mathcal{F} x \mapsto \xi, u (\xi', x_n) dx_n d\xi' \right| \leq C_N \langle s \xi' \rangle \langle s \lambda \rangle^{-\frac{1}{p} + 1 - N} R^{-N}.
\]

As $2\tau - 1 < 0$, we can always choose $N \in N_0$ so large that, for all $(x', x_n, \xi', \lambda) \in R^{n-1} \times R_+ \times R^{n-1} \times \Lambda$ such that $x_n \geq R$ and for all $r > 0$, we have

\[
\lim_{s \rightarrow \infty} \left( s \right) \left( \int_{R^n} e^{i x \cdot \xi'} \left( \int_{R^n} e^{i r s^{-1} x \cdot n_s \xi'} \frac{\xi}{s^N} \right) \mathcal{F} x \mapsto \xi, u (\xi', x_n) dx_n d\xi' \right) = 0.
\]

For large $N \in N_0$, the dominated convergence theorem implies that

\[
\lim_{s \rightarrow \infty} \left( s \right) \left( \int_{R^n} e^{i x \cdot \xi'} \left( \int_{R^n} e^{i r s^{-1} x \cdot n_s \xi'} \frac{\xi}{s^N} \right) \mathcal{F} x \mapsto \xi, u (\xi', x_n) dx_n d\xi' \right) = 0, \quad r > 0.
\]

Now, to finish the proof, we just study $L_p$ and $H^1_p$ convergence. Using integration by parts in the expression $x'^\gamma R_s^{-1} T (s\lambda) (R_s u) (x')$, we see that we can dominate $R_s^{-1} T (s\lambda) (R_s u) (x')$ by $(x')^{-N}$ for every $N$. Hence

\[
\lim_{s \rightarrow \infty} \left( s \right) \left( \int_{R^n} e^{i x \cdot \xi'} \left( \int_{R^n} e^{i r s^{-1} x \cdot n_s \xi'} \frac{\xi}{s^N} \right) \mathcal{F} x \mapsto \xi, u (\xi', x_n) dx_n d\xi' \right) = 0.
\]
If we take derivatives of first order in \( x' \), we find that
\[
\lim_{s \to \infty} s^r \| R_x^{-1} T (s \lambda) (R_x u) \|_{H^s_1 (\mathbb{R}^{n-1})} = 0.
\]
The estimate of the norm of \( R_x \) on Besov space and the same argument with interpolation of Proposition 27 lead us to the conclusion that
\[
\lim_{s \to \infty} s^r \| T (s \lambda) (R_x u) \|_{B^s_1 (\mathbb{R}^{n-1})} = 0.
\]
\hfill \Box

Finally, we prove the main Theorem of this sub-section.

Proof. (of Theorem 24) Let \( A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} \in \mathcal{B}^p_{\mathcal{P}, \mathcal{E}, \mathcal{E}_1} (M, \Lambda) \) and \( B_1, B_2, K_1 \) and \( K_2 \) be as in Theorem 24 iii). Write \( B_3 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \) and decompose similarly \( K_1 \).

Next we choose smooth functions \( \Phi, \Psi \) and \( H \), supported in a trivializing neighborhood \( U \) of \( x = \pi(z) \), such that \( \Phi \) equals 1 near \( x \) and \( \Psi \Phi = \Phi, H \Psi = \Psi \). We denote by \( \tilde{P}_+ (\lambda), \tilde{G}(\lambda) \in \mathcal{B}(L^p(\mathbb{R}_+^n)^{n_1}, L^p(\mathbb{R}_+^n)^{n_1}) \), \( \tilde{T}(\lambda) \in \mathcal{B}(L^p(\mathbb{R}_+^n)^{n_1}, L^p(\mathbb{R}_+^n)^{n_1}) \) and \( \tilde{B}_{11}(\lambda) \in \mathcal{B}(L^p(\mathbb{R}_+^n)^{n_1}, L^p(\mathbb{R}_+^n)^{n_1}) \) and \( \tilde{B}_{12}(\lambda) \in \mathcal{B}(L^p(\mathbb{R}_+^{n-1})^{n_1}, L^p(\mathbb{R}_+^{n-1})^{n_1}) \) the operators \( H \tilde{P}_+(\lambda) \Psi, H \tilde{G}(\lambda) \Psi, H \tilde{B}_{11}(\lambda) \Lambda, H \tilde{T}(\lambda) \Psi \) and \( H \tilde{B}_{12}(\lambda) \Lambda \) in local coordinates.

The identity \( B_3 A = I + K_3 \) implies that
\[
\tilde{B}_{11}(\lambda) (\tilde{P}_+(\lambda) + \tilde{G}(\lambda)) + \tilde{B}_{12}(\lambda) \tilde{T}(\lambda) = \tilde{\Phi} + \tilde{K}(\lambda),
\]
where \( \tilde{\Phi} \) is the function \( \Phi \) in local coordinates and \( \tilde{K}(\lambda) \) is the operator which collects the terms arising from the localizations of \( \Phi K_{11}(\lambda) H, \Phi B_{11}(\lambda)(1 - H^2) (P_+(\lambda) + G(\lambda)) \Psi \) and \( \Phi B_{12}(\lambda)(1 - H^2) T(\lambda) \Psi \). As the latter two operators have smooth integral kernels, with seminorms rapidly decreasing with respect to \( \lambda, K(\lambda) \) is compact and its norm tends to zero as \( |\lambda| \to \infty \).

The interior principal symbol. In order to prove the invertibility of the interior principal symbol \( p_0 ([z, \lambda] : \pi_{T^* M \setminus \Lambda} (E_0) \to \pi_{T^* M \setminus \Lambda} (E_1)) \) for \( (z, \lambda) \in (T^* M \times \Lambda) \setminus \{ 0 \} \), fix \( u = ev \in C^\infty (\mathbb{R}_+^n, \mathbb{R}_+), \) where \( e \in C^\infty(\mathbb{R}_+^n) \) and \( 0 \neq v \in C^\infty (\mathbb{R}_+^n, \mathbb{R}_+^n) \) is the point corresponding to \( z \) in local coordinates. For \( R_{\lambda} = R_{\lambda} (e^+ u) \) we note that \( R_{\lambda} (e^+ u) \in C^\infty (\mathbb{R}_+^n) \), since \( \text{supp} R_{\lambda} (e^+ u) \subset \mathbb{R}_+^n \). In particular
\[
\| u \|_{L^p(\mathbb{R}_+^n)^{n_1}} \leq \| B_{11}(\lambda) \|_{L^p(\mathbb{R}_+^n)^{n_3}, L^p(\mathbb{R}_+^n)^{n_1}} \| \tilde{P}(\lambda) R_{\lambda} (e^+ u) \|_{L^p(\mathbb{R}_+^n)^{n_3}}
+ \| (\tilde{B}_{11} \tilde{G} + \tilde{B}_{12} \tilde{T})(\lambda) R_{\lambda} (e^+ u) \|_{L^p(\mathbb{R}_+^n)^{n_3}} + \| \tilde{K}(\lambda) R_{\lambda} (e^+ u) \|_{L^p(\mathbb{R}_+^n)^{n_3}};
\]
and hence we obtain from (228)
\[
(229)
\]
On the right hand side of Equation (229), we estimate
\[
\| \tilde{P}(\lambda) R_{\lambda} (e^+ u) \|_{L^p(\mathbb{R}_+^n)^{n_3}} \leq \| \tilde{P}(\lambda) R_{\lambda} (e^+ u) - p_0 (0, \eta, \lambda) R_{\lambda} (e^+ u) \|_{L^p(\mathbb{R}_+^n)^{n_3}}
+ C \| p_0 (0, \eta, \lambda) \|_{B^r (C^\infty \cap C^\infty)}, \| u \|_{L^p(\mathbb{R}_+^n)}
\]
and note that Corollary 20 implies that
\[
\lim_{s \to \infty} s^r \| \tilde{P}(\lambda) R_{\lambda} (e^+ u) - p_0 (0, \eta, \lambda) R_{\lambda} (e^+ u) \|_{L^p(\mathbb{R}_+^n)^{n_3}} = 0.
\]
We claim that also \( \tilde{K}(\lambda) R_{\lambda} (e^+ u) \) tends to zero: For \( \lambda = 0 \) we infer this from the fact that \( \tilde{K}(0) \) is compact, while \( R_{\lambda} (e^+ u) \) weakly tends to zero. For \( \lambda \neq 0 \) the norm of \( \tilde{K}(\lambda) \) tends to zero as \( s \to \infty \), whereas \( R_{\lambda} (e^+ u) \) is bounded. Finally, it
is easy to check that \( \lim_{s \to \infty} (1 - \hat{\Phi}) R_s(e^+ u) = 0 \) in \( \mathcal{S}(\mathbb{R}^n)^{\ast 1} \) and therefore also in \( L_p(\mathbb{R}^n_+)^{\ast 1} \).

If we assume, for instance, that the second summund on the right hand side of \((2.20)\) tends to zero as \( s \to \infty \), then, taking \( s \) sufficiently large, the boundedness of \( R_s \), Inequality \((2.30)\) and Equation \((2.29)\) imply together with the assumption that \( \|B_{11}(s\lambda)\|_{\mathcal{B}(L_p(\mathbb{R}^n_+)^{\ast 1}, L_p(\mathbb{R}^n_+)^{\ast 1})} \leq C(\ln(s\lambda))^{\frac{1}{M}} \), that

\[
\|c\|_{C^n_1} \|v\|_{L_p(\mathbb{R}^n_+)} = \|u\|_{L_p(\mathbb{R}^n_+)^{\ast 1}} \leq \tilde{C} \|p(0)(y, \eta, \lambda) c\|_{\mathcal{B}(C^n_1, C^n_1)} \|v\|_{L_p(\mathbb{R}^n_+)},
\]

Hence \( p(0)(y, \eta, \lambda) \) is injective. The same argument, applied to the adjoint operator, shows the injectivity of \( p(0)(y, \eta, \lambda)^* \) and thus the invertibility of \( p(0)(y, \eta, \lambda) \). In particular, \( n_1 = n_3 \). In order to establish the convergence to zero of the second summund in \((2.20)\), we distinguish two cases.

**Case 1:** \( x \notin \partial M \). Then \( U \) can be taken as a subset of the interior of \( M \). According to the rules of the calculus, \( \hat{T}(s\lambda) \) and \( \hat{G}(s\lambda) \) are regularizing elements in their respective classes; in particular, they are compact. For \( \lambda \neq 0 \), their operator norms are rapidly decreasing as \( s \to \infty \). Arguing as for \( \hat{K} \) above, we obtain the assertion from the assumptions on \( B \).

**Case 2:** \( x \in \partial M \). Here, statements 1) and 2) of Proposition \(25\) assert that, for every \( r > 0 \), the norms of \( s' \hat{G}(s\lambda) R_s(e^+ u) \) and \( s' \hat{T}(s\lambda) R_s(e^+ u) \) go to zero in the corresponding spaces as \( s \to \infty \). The assertion then follows from the fact that the norm of \( B(s\lambda) \) grows at most logarithmically in \( s \) by assumption.

The boundary principal symbol. We have to show that, for any given \( (z, \lambda) \in (T^* \partial M \times \Lambda) \setminus \{0\} \), \( \sigma_B(A)(z, \lambda) \) is invertible in

\[
\operatorname{Hom}(\pi_{\partial M}((E_0|_{\partial M} \otimes \mathcal{S}(\mathbb{R}^n_+)) \oplus F_0), \pi_{\partial M}((E_1|_{\partial M} \otimes \mathcal{S}(\mathbb{R}^n_+)) \oplus F_1)).
\]

Let \( \tilde{B} \) and \( \tilde{A} \) be the operators \( H \mathcal{A} \Psi \) and \( \Phi \mathcal{B} H \) in local coordinates, respectively. Write the principal boundary symbol of \( \tilde{A} \) in the form

\[
(2.31) \quad \begin{pmatrix}
\begin{array}{c}
p(0)(x', 0, \xi', D_n, \lambda) + g_{(-1)}(x', \xi', D_n, \lambda) \\
t(-\frac{1}{2})(x', \xi', D_n, \lambda)
\end{array}
\end{pmatrix}
\begin{pmatrix}
k_{(\frac{1}{2} - 1)}(x', \xi', D_n, \lambda) \\
s(0)(x', \xi', \lambda)
\end{pmatrix}
\]

and let \( (y, \eta) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \) be the point that corresponds to \( z \) in local coordinates.

Fix a function \( 0 \neq u' \in \mathcal{S}(\mathbb{R}^{n-1}_+) \) with \( \text{supp} \mathcal{F} u' \subset \{ \xi; \frac{1}{2} < |\xi| < 1 \} \). For \( u = (u_1, \ldots, u_n) \in \mathcal{S}(\mathbb{R}^n)^{\ast 1} \) and \( v = (v_1, \ldots, v_n) \in \mathbb{C}^{n_2} \), not both zero, denote by \( u' \otimes u \) and \( u' \otimes v \) the functions \( \mathbb{R}^{n-1}_+ \ni (x', x_n) \mapsto (u'(x') u_1(x_n), \ldots, u'(x') u_n(x_n)) \) and \( \mathbb{R}^{n-1}_+ \ni x' \mapsto (u'(x') v_1, \ldots, u'(x') v_{n_2}) \), respectively. According to Lemmas \(19 \) and \(21\) and \(22\) there are constants such that

\[
\left\| u' \right\|_{L_p(\mathbb{R}^{n-1})} = \left\| R_s u' \right\|_{L_p(\mathbb{R}^{n-1})} \leq C_1 \left\| u' \right\|_{B_p(\mathbb{R}^{n-1})} \leq C_2 \left\| R_s u' \right\|_{B_p(\mathbb{R}^{n-1})} \leq C_3 \left\| u' \right\|_{L_p(\mathbb{R}^{n-1})}, \quad s \geq 1.
\]
Writing \( \|u\|_{L^p(B^*_p)} \) for the norm in \( L^p(\mathbb{R}_+^{n_1}) \oplus B^0_p(\mathbb{R}^{n-1})^{n_2} \), in analogy with Equation 23, we conclude from the identity \( B_1 A = I + K_1 \) that

\[
\|u\|_{L^p(\mathbb{R}^{n_1})} \| (u') \|_{L^p(\mathbb{R}_+^{n_1})} \leq C \left( \| R_u u \otimes S_u u \|_{L^p(B^*_p)} + C \left( \| R_u u \otimes S_u u \|_{L^p(B^*_p)} + \| u' \|_{L^p(B^*_p)} \right) \right)
\]

\[
\leq C \left( \left\| \tilde{B} (s\lambda) \tilde{A} (s\lambda) \left( R_u \otimes S_u \right) \left( u' \otimes u' \right) \right\|_{L^p(B^*_p)} + \left\| \tilde{B} (s\lambda) \left( R_u \otimes S_u \right) \left( u' \otimes u' \right) \right\|_{L^p(B^*_p)} \right)
\]

\[
= C \left( \left\| \left( \langle x', 0, \xi', D_n, \lambda \rangle + g_{(-1)}(x', \xi', D_n, \lambda) \right) \left( \Phi_{(s\lambda)}(x', \xi', D_n, \lambda) \right) \left( u' \otimes u' \right) \right\|_{L^p(B^*_p)} \right)
\]

Let us first consider the case where \( \lambda \neq 0 \). We infer from Proposition 27 and the fact that the norm of \( \tilde{B}(s\lambda) \) is \( O((\ln(s\lambda))^M) \) that the first summand on the right hand side is \( o(||(u' \otimes u) + (u' \otimes v)||) \). The same is true for the third summand, since the norm of \( K(s\lambda) \) tends to zero as \( s \to \infty \). The fourth summand tends to zero in \( S(\mathbb{R}_+^{n_1}) \oplus S(\mathbb{R}^{n-1})^{n_2} \), a fortiori in the \( L^p(B^*_p) \)-norm. Taking \( s \) sufficiently large, we may achieve that the sum of the first, the third and the fourth summands is \( \leq \frac{1}{2}((u' \otimes u) + (u' \otimes v)) \). From the boundedness of \( \tilde{B}(s\lambda) \), \( R_u \) and \( S_u \) for this fixed value of \( s \), we conclude that, with norms taken in \( L^p(\mathbb{R}_+^{n_1}) \oplus C^{n_2} \) and \( L^p(\mathbb{R}_+^{n_1}) \oplus C^{n_4} \),

\[
\left\| \left( u' \right) \right\|_{L^p(B^*_p)} \leq C \left( \left\| \left( \langle x', 0, \xi', D_n, \lambda \rangle + g_{(-1)}(x', \xi', D_n, \lambda) \right) \left( \Phi_{(s\lambda)}(x', \xi', D_n, \lambda) \right) \left( u' \otimes u' \right) \right\|_{L^p(B^*_p)} \right)
\]

In case \( \lambda = 0 \), we obtain the same conclusion using the compactness of \( K(0) \).

Hence the operator from Equation 23 is injective and has closed range. As the same can be said of the adjoint, we conclude from Lemma 23 that the principal boundary symbol is an isomorphism.

### 2.2. The spectral invariance of the parameter-dependent Boutet de Monvel algebra.

**Theorem 29.** Let \( A \in \mathcal{B}_{E_0,F_0,E_1}^p (M, \Lambda) \) be a parameter-dependent operator. Suppose that, for each \( \lambda \in \Lambda \), the operator

\[
A(\lambda) : L^p(M, E_0) \oplus B^0_p(\partial M, F_0) \to L^p(M, E_1) \oplus B^0_p(\partial M, F_1)
\]

is invertible. If there are constants \( C > 0 \) and \( M \in N_0 \) such that

\[
\| A(\lambda)^{-1} \|_{\mathcal{B}(L^p(M,E_0) \oplus B^0_p(\partial M,F_0), L^p(M,E_0) \oplus B^0_p(\partial M,F_0))} \leq C (\ln(\lambda))^M, \lambda \in \Lambda,
\]

then \( A(\lambda)^{-1} \in \mathcal{B}_{E_1,F_1,E_0,F_0}^p (M, \Lambda) \).
Proof. By Theorem 24, $A$ is parameter-elliptic. Hence we find a parametrix $B \in B^p_{E_1,F_1,\Sigma,F_0}(M,\Lambda)$ and $K_1 \in B^{-\infty,0}_{E_1,F_1,\Sigma,F_0}(M,\Lambda)$ and $K_2 \in B^{-\infty,0}_{E_0,F_0,\Sigma,F_0}(M,\Lambda)$ such that $AB = I + K_1$ and $BA = I + K_2$. We conclude that
\[
A(\lambda)^{-1} = B(\lambda) - K_2(\lambda)A(\lambda)^{-1} = B(\lambda) - K_2(\lambda)\left( B(\lambda) - A(\lambda)^{-1}K_1(\lambda) \right).
\]
As $K_2B \in B^{-\infty,0}_{E_1,F_1,\Sigma,F_0}(M,\Lambda)$, $A(\lambda)^{-1}$ grows at most as $(\ln(\lambda))^M$ in $\lambda$ and $K_j(\lambda)$, $j = 1,2$, are integral operators with smooth kernels whose derivatives decay rapidly with respect to $\lambda$, we see that $K_2A^{-1}K_1 \in B^{-\infty,0}_{E_1,F_1,\Sigma,F_0}(M,\Lambda)$ and $A^{-1} \in B^p_{E_1,F_1,\Sigma,F_0}(M,\Lambda)$. □

3. Boundary value problems on manifolds with conical singularities

In this section, we provide the definitions and results concerning manifolds with boundary and conical singularities that we shall need. Details can be found in [25, 26].

Definition 30. A compact manifold with boundary and conical singularities of dimension $n$ is a triple $(D, \Sigma, F)$ formed by:

1) A compact Hausdorff topological space $D$.
2) A finite subset $\Sigma \subset D$, which we call conical points, such that $D \setminus \Sigma$ is an $n$-dimensional smooth manifold with boundary.
3) A set of functions $F_\Sigma = \{ \varphi : U_\sigma \to X_\sigma \times [0,1]/X_\sigma \times \{0\}, \sigma \in \Sigma \}$ such that:
   i) The sets $U_\sigma \subset D$ are open and disjoint sets. Moreover, each $U_\sigma$ is a neighborhood of $\sigma \in \Sigma$.
   ii) $X_\sigma$ is a compact smooth manifold with boundary for each $\sigma \in \Sigma$.
   iii) The function $\varphi_\sigma : U_\sigma \to X_\sigma \times [0,1]/X_\sigma \times \{0\}$ is a homeomorphism, $\varphi_\sigma(\sigma) = X_\sigma \times \{0\}/X_\sigma \times \{0\}$ and $\varphi_\sigma : U_\sigma \setminus \{\sigma\} \to X_\sigma \times [0,1]$ is a diffeomorphism.

Remark 31. For each $\sigma \in \Sigma$, we could use a different function $\tilde{\varphi}_\sigma : U_\sigma \to X_\sigma \times [0,1]/X_\sigma \times \{0\}$ with the same properties as in item iii), as long as, for each $\sigma$,
\[
\tilde{\varphi}_\sigma \circ \varphi^{-1}_\sigma : X_\sigma \times [0,1] \to X_\sigma \times [0,1]
\]
extends to a diffeomorphism $\tilde{\varphi}_\sigma \circ \varphi^{-1}_\sigma : X_\sigma \times [-1,1] \to X_\sigma \times [-1,1]$. These are the changes of variables that we allow to do near the singularities.

For the analysis of the typical (pseudo-) differential boundary value problems on these manifolds, we introduce the Fuchs type boundary value problems on a manifold with corners $\mathbb{B}$. It is obtained by gluing the sets $X_\sigma \times [0,1]$ in place of $U_\sigma$, using the functions $\varphi_\sigma$. In this way, the singularities are identified with the sets $X_\sigma \times \{0\}$. The above remark ensures that the use of different functions $\tilde{\varphi}_\sigma$ instead of $\varphi_\sigma$ leads to diffeomorphic manifolds with corners. In order to avoid unnecessary complications with the notation, we shall consider manifolds with just one point singularity. A neighborhood of the conical point will always be identified with $X \times [0,1]/X \times \{0\}$ and a neighborhood of the corner will always be identified with $X \times [0,1]$, where $X$ is a compact manifold with boundary. For a finite number of singularities the definitions and arguments are analogous.

We will denote by $\text{int}(\mathbb{B})$ the manifold with boundary $\mathbb{D} \setminus (X \times \{0\})$. By $\text{int}(\mathbb{B})$, we denote the boundary of $\text{int}(\mathbb{B})$. In a neighborhood of the singularity, it can be identified with $\partial X \times [0,1]$. Finally $\mathbb{B}$ is the manifold with boundary given by $\text{int}(\mathbb{B}) \cup (\partial X \times \{0\})$. In particular, in a neighborhood of the singularity, it can be identified with $\partial X \times [0,1]$. We will also use $2\mathbb{D}$ to denote a manifold with boundary in which $\mathbb{D}$ is embedded. The boundary of $2\mathbb{D}$ is $2\mathbb{B}$, a manifold without boundary.

We divide our presentation into two parts. First we define the classes of functions and distributions and then the operators. The operators acting on a neighborhood
of the singularity will be defined as operators on $X \times [0, 1]$. We denote by $E_0$ and $E_1$ two vector bundles over $\mathbb{D}$ and by $F_0$ and $F_1$ two vector bundles over $\mathbb{B}$. Let $\pi_X : X \times [0, 1] \to X$ be the projection operator, then there are vector bundles $E_0'$ and $E_1'$ over $X$ such that $E_0$ and $E_1$ can be identified with $\pi_X^{-1}(E_0')$ and $\pi_X^{-1}(E_1')$, respectively. Similarly, if $\pi_{\partial X} : \partial X \times [0, 1] \to \partial X$ is the projection operator, then there are vector bundles $F_0'$ and $F_1'$ over $\partial X$ such that $F_0$ and $F_1$ can be identified with $\pi_{\partial X}^{-1}(F_0')$ and $\pi_{\partial X}^{-1}(F_1')$, respectively. $E_0$ will denote $E_0$ and $E_1$ and the same will be done for $E_1$, $F_0$ and $F_1$. We also denote by $2E_0$, $2F_0$, ..., the vector bundles over $2\mathbb{D}$ and $2\mathbb{B}$, whose restriction to $\partial \mathbb{D}$ and $\partial \mathbb{B}$ are $E_0$ and $F_0$.

Finally, a cut-off function $\omega \in C_0^\infty(\mathbb{R}_+)$ is a smooth nonnegative function that is equal to 1 in a neighborhood of 0 and equal to 0, outside $[0, 1]$.

3.1. Classes of functions and distributions. In the following sections, $X$ is a manifold endowed with a Riemannian metric and with boundary $\partial X$. All vector bundles are assumed to be hermitian. We use the notation $X^\wedge := \mathbb{R}_+ \times X$ and $\partial X^\wedge := \mathbb{R}_+ \times X$ and we will denote by $E$, $E_0$ and $E_1$ vector bundles over $X$ or $\mathbb{D}$ and by $F$, $F_0$ and $F_1$ vector bundles over $\partial X$ or $\mathbb{B}$. The vector bundles $E$, $E_0$, $E_1$, $F$, $F_0$ and $F_1$ will also refer to the pullback bundles in $X \times \mathbb{R}$, $X^\wedge$, $\partial X \times \mathbb{R}$ and $\partial X^\wedge$. Finally we denote by $C^\infty(X, E)$ the set $C^\infty(X, E) \oplus C^\infty(\partial X, F)$.

**Definition 32.** Let $W$ be a Fréchet space and $\gamma \in \mathbb{R}$. We define the Fréchet space $\mathcal{T}_\gamma(\mathbb{R}_+, W)$ as the space of all functions $\varphi \in C^\infty(\mathbb{R}_+, W)$ that satisfy

$$\sup \left\{ (\ln(t))^k \left\| \frac{d^k}{dt^k} \varphi(t) \right\|, \quad t \in \mathbb{R}_+ \right\} < \infty,$$

for all $k, l \in \mathbb{N}$ and for all continuous seminorms $p$ of $W$. We write $\mathcal{T}_\gamma(\mathbb{R}_+)$ when $W = \mathbb{C}$.

**Definition 33.** Let $\omega \in C^\infty([0, 1])$ be a cut-off function. The space of functions $C^\infty_\omega(\mathbb{D})$, $\gamma \in \mathbb{R}$, consists of all functions $u \in C^\infty(\text{int}(\mathbb{D}))$ such that $\omega u \in \mathcal{T}_{\gamma - \frac{1}{\omega}}(X^\wedge)$. Similarly, $C^\infty_\omega(\mathbb{B})$ are all the functions $u \in C^\infty(\text{int}(\mathbb{B}))$ such that $\omega u \in \mathcal{T}_{\gamma - \frac{1}{\omega}}(\partial X^\wedge)$.

**Definition 34.** Let $X = \bigcup_{j=1}^{M} U_j$ be a cover of $X$ consisting of trivializing sets and $\varphi_j : U_j \subset X \to V_j \subset \mathbb{R}^+_{\mu}$ be coordinate charts and $(\psi_j)_{j=1}^{M}$ be a partition of unity subordinate to $U_j$, $j = 1, ..., M$. The space $H^p_\mu(X \times \mathbb{R}, E)$ is defined as the set of distributions $\mathcal{D}'(\mathbb{R} \times X, E)$ such that $(t, x) \in \mathbb{R} \times \mathbb{R}^n_+ \mapsto (\psi_j u)(t, \varphi^{-1}_j(x))$ belong to $H^p_\mu(\mathbb{R} \times \mathbb{R}^n_+ \times C^N)$, where $N$ is the dimension of $E$, with norm given by:

$$\|u\|_{H^p_\mu(X \times \mathbb{R}, E)} = \sum_{j=1}^{M} \|\psi_j u\|_{H^p_\mu(\mathbb{R} \times \mathbb{R}^n_+ \times C^N)}.$$

The space $H^p_\mu(X^\wedge, E)$ is the space of all distributions $u \in \mathcal{D}'(X^\wedge, E)$ such that $u(t, x) = t^{-\frac{N}{\mu}} \psi_j(t) \varphi_j^{-1}(x) v(\ln(t), x)$, where $v \in H^p_\mu(X \times \mathbb{R}, E)$. Its norm is given by:

$$\|u\|_{H^p_\mu(X^\wedge, E)} := \|v\|_{H^p_\mu(X \times \mathbb{R}, E)}.$$

Similarly, using the space $B^p_\mu(\mathbb{R}^n, C^N)$ instead of $H^p_\mu(\mathbb{R} \times \mathbb{R}^n_+, C^N)$, we define the space $B^p_\mu(\partial X \times \mathbb{R}, F)$, where $N$ is the dimension of $F$. Associated to it is the space $B^p_\mu(\partial X \times \mathbb{R}, F)$ of all distributions $u \in \mathcal{D}'(\partial X^\wedge, F)$ such that $u(t, x) = t^{-\frac{N}{\mu}} \psi_j(t) \varphi_j^{-1}(x) v(\ln(t), x)$, where $v \in B^p_\mu(\partial X \times \mathbb{R}, F)$. Its norm is given by:

$$\|u\|_{B^p_\mu(\partial X \times \mathbb{R}, F)} := \|v\|_{B^p_\mu(\partial X \times \mathbb{R}, F)}.$$

**Remark 35.** The above definition implies $u \mapsto \|\partial_t u\|_{L_\mu(X^\wedge, E, dx^{\mu, 1})} + \|u\|_{L_\mu(\mathbb{R}_+, H^p_\mu(X, E, \mu))}$ is an equivalent norm for $H^p_\mu(\mathbb{R}^n_+) \hat{\otimes} (X^\wedge, E)$. 
Finally, we need Bessel and Besov spaces with asymptotics. First let us define asymptotic types.

**Definition 36.** We say that \( P = \{ (p_j, m_j, L_j); j \in \{1, ..., M\} \} \) is an asymptotic type for \( C^\infty (X, E) \) with weight \((\gamma, k) \in \mathbb{R} \times \mathbb{N}_0 \) if \( p_j \in C, \frac{n+1}{2} - \gamma - k < \text{Re} (p_j) < \frac{n+1}{2} - \gamma \), are distinct numbers, \( m_j \in \mathbb{N}_0 \) and \( L_j \subset C^\infty (X, E) \) are finite dimensional spaces. The set of all asymptotic types is denoted by \( \text{As} (X, E, \gamma, k) \). Similarly, we say that \( Q = \{ (p_j, m_j, L_j); j \in \{1, ..., M\} \} \) is an asymptotic type for \( C^\infty (\partial X, F) \) with weight \((\gamma, k) \in \mathbb{R} \times \mathbb{N}_0 \) and write \( Q \in \text{As} (\partial X, F, \gamma, k) \), if \( p_j \in C, \frac{n}{2} - \gamma - k < \text{Re} (p_j) < \frac{n}{2} - \gamma \), are distinct numbers, \( m_j \in \mathbb{N}_0 \) and \( L_j \subset C^\infty (\partial X, F) \) are finite dimensional spaces.

**Definition 37.** The Bessel potential and Besov space with asymptotics, respectively, are defined as follows:

1) Let \( P = \{ (p_j, m_j, L_j); j \in \{1, ..., M\} \} \in \text{As} (X, E, \gamma, k) \). We define
\[
\mathcal{H}^{s, \gamma}_{p} (D, E) = \cap_{\epsilon > 0} \mathcal{H}^{s, \gamma + k - \epsilon} (D, E) \oplus \mathcal{E}_P (X),
\]
where \( \mathcal{E}_P := \left\{ X^{\gamma} \otimes (t, x) \mapsto \omega (t) \sum_{j=1}^{M} \sum_{k=0}^{m_j} t^{-p_j} \ln^k (t) v_{jk} (x), v_{jk} \in L_j \right\} \).

2) Let \( \tilde{P} = \{ (\tilde{p}_j, \tilde{m}_j, \tilde{L}_j); j \in \{ 1, ..., M \} \} \in \text{As} (\partial X, F, \gamma, k) \). We define
\[
B^{s, \gamma}_{p, \tilde{p}} (\mathbb{R}, F) = \cap_{\epsilon > 0} B^{s, \gamma + k - \epsilon} (\mathbb{R}, F) \oplus \mathcal{E}_{\tilde{P}} (\partial X),
\]
where \( \mathcal{E}_{\tilde{P}} := \{ \partial X^{\gamma} \otimes (t, x) \mapsto \omega (t) \sum_{j=1}^{M} \sum_{k=0}^{m_j} t^{-p_j} \ln^k (t) v_{jk} (x), v_{jk} \in \tilde{L}_j \} \) and \( \omega \) is a cut-off function.

**Remark 38.** 1) The scalar product of \( L^2 (\partial X^\gamma, F, dx \frac{dt}{t}) \) allows the identification \( B^{s, \gamma}_{p} (\partial X^\gamma, F) \cong B^{s, \gamma}_{p} (\partial X^\gamma, F) \), \( \frac{n+1}{2} - \frac{1}{2} = 1 \). As \( B^{s, \gamma}_{q} (\partial X^\gamma, F) = t^{-\gamma} B^{s, \gamma}_{q} (\partial X^\gamma, F) \) and \( B^{s, \gamma}_{q} (\partial X^\gamma, F) = t^{-\gamma} B^{s, \gamma}_{q} (\partial X^\gamma, F) \), we conclude that \( B^{s, \gamma}_{q} (\partial X^\gamma, F) \cong B^{s, \gamma}_{q} (\partial X^\gamma, F) \), if we use the scalar product of \( L^2 (\partial X^\gamma, F, t^{n-1} dt dx) \).

2) In the same way, \( \mathcal{H}^{s, \gamma}_{p} (X^\gamma, E) \cong \mathcal{H}^{s, \gamma}_{q} (X^\gamma, E) \), if we use the scalar product of \( L^2 (X^\gamma, E, dx \frac{dt}{t}) \), and \( \mathcal{H}^{s, \gamma}_{p} (X^\gamma, E) \cong \mathcal{H}^{s, \gamma}_{q} (X^\gamma, E) \), if we use that of \( L^2 (X^\gamma, E, t^{n} dt dx) \).

**Definition 39.** Let \( s, \gamma \in \mathbb{R} \) and \( 1 < p < \infty \).

1) We define \( \mathcal{H}^{s, \gamma}_{p} (D, E) \) as the space of all distributions \( u \in \mathcal{H}^{s, \gamma}_{loc} (\text{int} (D), E) \) such that, for any cut-off function \( \omega \), here considered as a function on \( D \), we have \( \omega u \in \mathcal{H}^{s, \gamma}_{p} (X^\gamma, E) \). Its norm is given by
\[
\| u \|_{\mathcal{H}^{s, \gamma}_{p} (D, E)} := \| \omega u \|_{\mathcal{H}^{s, \gamma}_{p} (X^\gamma, E)} + \| (1 - \omega) u \|_{\mathcal{H}^{s, \gamma}_{loc} (\text{int} (D), E)}.
\]

2) Similarly, we obtain \( B^{s, \gamma}_{p} (\mathbb{R}, F) \) from \( B^{s, \gamma}_{p} (\partial X^\gamma, F) \) and \( B^{s, \gamma}_{p} (\partial X^\gamma, F) \).

3.2. Classes of operators. We are going to use the natural identification
\[
\mathcal{T}, \mathbb{R} \subset, C^\infty (X, E, F) \cong \mathcal{T}, \mathbb{R} \subset, C^\infty (X, E, F) \]
and write \( \Gamma_{\sigma} := \{ z \in \mathbb{C}; \text{Re} (z) = \sigma \} \). The latter set will be obviously identified with \( \mathbb{R} \), when it is convenient to do so.

**Definition 40.** The weighted Mellin transform is the continuous linear operator
\[
\mathcal{M}_{\gamma} : \mathcal{T}, \mathbb{R} \subset, C^\infty (X, E, F) \rightarrow \mathcal{S} (\Gamma_{\frac{1}{2} - \gamma}, C^\infty (X, E, F))
\]
defined by
\[
\mathcal{M}_{\gamma} \varphi (z) = \int_{0}^{\infty} t^z \varphi (t) \left( \frac{dt}{t} \right), z \in \Gamma_{\frac{1}{2} - \gamma}.
\]
It is an invertible operator, whose inverse is given by
\[
\mathcal{M}_{-\gamma}^{-1} \varphi (t) = \frac{1}{2 \pi i} \int_{\Gamma_{\frac{1}{2} - \gamma}} t^{-z} \varphi (z) dz = \frac{1}{2 \pi} \int t^{-\left( \frac{1}{2} - \gamma + i \tau \right)} \varphi \left( \frac{1}{2} - \gamma + i \tau \right) d\tau.
\]
Definition 41. For \( m \in \mathbb{Z} \) and \( d \in \mathbb{N}_0 \), \( MB_{E_0,F_0,E_1,F_1}^{m,d}(X,\mathbb{R}_+;\Gamma_\gamma) \) is the space of all functions \( h \in C^\infty \left( \mathbb{R}_+, MB_{E_0,F_0,E_1,F_1}^{m,d}(X,\Gamma_\gamma) \right) \) that satisfy
\[
\sup \left\{ p \left( (t\partial_t)^{\gamma} h(t) \right), \, t \in \mathbb{R}_+ \right\} < \infty,
\]
for all continuous seminorms \( p \) of \( MB_{E_0,F_0,E_1,F_1}^{m,d}(X,\Gamma_\gamma) \). In a similar way we define \( MB_{E_0,F_0,E_1,F_1}^{p}(X,\mathbb{R}_+;\Gamma_\gamma) \).

To a function in \( MB_{E_0,F_0,E_1,F_1}^{m,d}(X,\mathbb{R}_+;\Gamma_\gamma) \) or \( MB_{E_0,F_0,E_1,F_1}^{p}(X,\mathbb{R}_+;\Gamma_\gamma) \) we associate the Mellin operator \( M \).

\[
\text{The asymptotic types are used to define the following meromorphic functions.}
\]

\[
\text{Definition 42. A discrete Mellin asymptotic type of order } d \in \mathbb{N}_0 \text{ is a set } \mathcal{P} = \{(p_j, m_j, L_j)\}_{j \in \mathbb{Z}},
\]
where \( p_j \in \mathbb{C} \) satisfy \( \text{Re}(p_j) \to \pm \infty \) as \( j \to \pm \infty \), \( m_j \in \mathbb{N}_0 \) and \( L_j \) are finite-dimensional subspaces of operators of finite rank in \( \mathcal{B}_{E_0,F_0,E_1,F_1}^{m,d}(X) \). The collection of all these asymptotic types is denoted by \( \text{As}\left( \mathcal{B}_{E_0,F_0,E_1,F_1}^{m,d}(X) \right) \). Moreover, we let \( \pi_{\mathbb{C}} \mathcal{P} := \{p_j : j \in \mathbb{Z}\} \subset \mathbb{C} \).

\[
\text{The asymptotic types are used to define the following meromorphic functions.}
\]

\[
\text{Definition 43. The space } \mathcal{M}_{E_0,F_0,E_1,F_1}(X), \mathcal{P} \in \text{As}\left( \mathcal{B}_{E_0,F_0,E_1,F_1}^{m,d}(X) \right), \text{ is the space of all meromorphic functions } a : \mathbb{C}\setminus \pi_{\mathbb{C}} \mathcal{P} \rightarrow \mathcal{B}_{E_0,F_0,E_1,F_1}^{m,d}(X) \text{ such that:}
\]
i) For every \( p_j \in \pi_{\mathbb{C}} \mathcal{P} \), there is a neighborhood of \( p_j \) where \( a \) can be written as
\[
a(z) = \sum_{k=0}^{m_j} \nu_{jk} (z - p_j)^{-k-1} + a_0(z).
\]

Above, \( a_0 \) is a holomorphic function near \( p_j \), with values in \( \mathcal{B}_{E_0,F_0,E_1,F_1}^{m,d}(X) \) and \( \nu_{jk} \in L_j \), for \( k = 0, \ldots, m_j \).

ii) For every \( N \in \mathbb{N}_0 \), the function \( \gamma \in [-N, N] \mapsto a_N(\gamma + i.) \in \mathcal{B}_{E_0,F_0,E_1,F_1}^{m,d}(X,\mathbb{R}) \) is continuous, where
\[
a_N(z) := a(z) - \sum_{|\text{Re}(p_j)| \leq N} \sum_{k=0}^{m_j} \nu_{jk} (z - p_j)^{-k-1}.
\]

For \( P \in \text{As}(\mathcal{B}_{E_0,F_0,E_1,F_1}^{m,d}(X)) \), we can also define \( \tilde{M}_{E_0,F_0,E_1,F_1}(X) \) replacing \( \mathcal{B}_{E_0,F_0,E_1,F_1}^{m,d}(X) \) by \( \tilde{\mathcal{B}}_{E_0,F_0,E_1,F_1}^{p}(X) \). When \( P = \emptyset \), we also use the notations \( \mathcal{M}_{E_0,F_0,E_1,F_1}(X) \) and \( \tilde{\mathcal{B}}_{E_0,F_0,E_1,F_1}^{p}(X) \).

The last operator that we need are the Green ones.

\[
\text{Definition 44. We define } \mathcal{C}_{\mathbb{D}}^{m,d}(E_0,F_0,E_1,F_1)(\mathbb{D}; \gamma, \gamma', k) \text{ as the space of operators of the form}
\]
\[
\mathcal{G} = \sum_{j=0}^{d} \mathcal{G}_j \begin{pmatrix} D_j & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{G}_0,
\]
where, for each $G_j$, there exist asymptotic types $P \in A_s(X, E_1, \gamma', k)$ and $P' \in A_s(X, E_0, -\gamma, k)$, $Q \in A_s(\partial X, F_1, \gamma' - \frac{1}{2}, k)$ and $Q' \in A_s(\partial X, F_0, -\gamma - \frac{1}{2}, k)$, such that $G_j$ and its formal adjoint with respect to $\mathcal{H}_0^\omega(D, E_j) \oplus B_{(\mathbb{B}, F_j)}^{s, \gamma - \frac{1}{2}}$, $\hat{G}_j^*: \mathcal{H}_{p, q}^\omega(D, E_1) \to \mathcal{H}_{p, q}^{\omega, \gamma}(D, E_0)$ for all $r \in \mathbb{R}, s > -1 + \frac{1}{p}$ on the left hand side and $s > -1 + \frac{1}{q}$ on the right hand side. Near the boundary $\mathbb{B}$ of $D$, the operators $D^j$ coincide with $(-i\partial_\nu)^j$ where $\partial_\nu$ is the normal derivative.

Similarly, $\tilde{C}_G^{p, \omega}(E_0, E_1, F_0, F_1) (\mathbb{D}; k)$ denotes the space of all operators $G$ for which there exist asymptotic types $P \in A_s(X, E_1, \frac{n+1}{2}, k)$, $P' \in A_s(X, E_0, \frac{n+1}{2}, k)$, $Q \in A_s(\partial X, F_1, \frac{n}{2}, k)$ and $Q' \in A_s(\partial X, F_0, \frac{n}{2}, k)$, such that $G$ and its formal adjoint $G^*$ with respect to $\mathcal{H}_2^\omega(D, E_0) \oplus B_{(\mathbb{B}, F_0)}^{s, \gamma - \frac{1}{2}}$, $\hat{G}_j: \mathcal{H}_{p, q}^\omega(D, E_1) \to \mathcal{H}_{p, q}^{\omega, \gamma}(D, E_0)$ for all $r \in \mathbb{R}, s > -1 + \frac{1}{p}$ on the left hand side and $s > -1 + \frac{1}{q}$ on the right hand side.

It is an immediate consequence of the embedding properties for cone Sobolev spaces that, for $r \in \mathbb{R}, s > d + 1/p - 1$ and arbitrary $r', s' \in \mathbb{R}$, an operator

$$C_{G, \omega}^d(E_0, E_1, F_0, F_1) (\mathbb{D}; \gamma, \gamma', k) \ni G: \mathcal{H}_{p, q}^\omega(D, E_0) \to \mathcal{H}_{p, q}^{\omega, \gamma}(D, E_1)$$

is compact. An analogous statement applies to operators in $\tilde{C}_G^{p, \omega}(E_0, E_1, F_0, F_1) (\mathbb{D}; k)$.

Finally, we can define the cone algebra for boundary value problems.

**Definition 45.** For $\gamma \in \mathbb{R}, m \in \mathbb{Z}, d \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ we define the space $C^{m, d}_{E_0, E_1, F_0, F_1} (\mathbb{D}, (\gamma, \gamma' - m, k))$ of all operators $A: C^\infty(\mathbb{D}, E_0) \oplus C^\infty(\mathbb{B}, F_0) \to C^\infty(\mathbb{D}, E_1) \oplus C^\infty(\mathbb{B}, F_1)$ of the form

$$(3.1) \quad A = \omega_0 A_M + (1 - \omega_2) A_{\psi} (1 - \omega_3) + M + G,$$

where $\omega_1, ..., \omega_4 \in C^\infty(0, 1]$ are cut-off functions. The operator $A_M$ is a Mellin operator: $A_M = t^{-m} op^{\gamma - \frac{d}{2}}_M (h)$, with $h \in C^\infty(\mathbb{R}_+^*, M^{\omega, d}_{E_0, E_0, F_0, F_0}(\mathbb{D})(\mathbb{X}))$. The operator $A_{\psi}$ is a Boutet de Monvel operator $A_{\psi} \in B^d_{2E_0, 2E_1, 2F_0, 2F_1}(\mathbb{D})$. The operator $M$ is a smoothing Mellin operator: $M = \omega_0 \left( \sum_{k=1}^{n-1} t^{-m+1} op^{\gamma' - \frac{d}{2}}_M (h_1) \right) \omega_1$ with $h_1 \in M_{F_1 E_0, E_0, E_1, F_1}(\mathbb{X}), \pi C P_1 \cap \Gamma^{\omega, \gamma_1} = \emptyset, \text{ and } \gamma - l \leq \gamma_1 \leq \gamma$. The operator $G$ is a Green operator: $G : C^d_{G, E_0, E_0, F_0, F_1} (\mathbb{D}; \gamma, \gamma - m, k)$.

Similarly, the algebra $\tilde{C}_G^{p, \omega}(E_0, E_1, F_0, F_1) (\mathbb{D}; k)$ is defined as the space of all continuous operators $A: C^{\omega, d}_{(\mathbb{D}, E_0)} \oplus C^\infty(\mathbb{B}, F_0) \to C^{\omega, \gamma}(\mathbb{D}, E_1) \oplus C^\infty(\mathbb{B}, F_1)$ of the form

$$(3.1), \text{ where } A_M = op^{\gamma - \frac{d}{2}}_M (h), \text{ with } h \in C^\infty \left( \mathbb{R}_+^*, M^{\omega, d}_{E_0, E_0, F_0, F_0}(\mathbb{X}) \right), A_{\psi} \in B^d_{2E_0, 2E_1, 2F_0, 2F_1}(\mathbb{D}), M = \omega_0 \left( \sum_{k=1}^{n-1} t^{m+1} op^{\gamma' - \frac{d}{2}}_M (h_1) \right) \omega_1 \text{ with } h_1 \in M_{F_1 E_0, E_0, E_1, F_1}(\mathbb{X}), \pi C P_1 \cap \Gamma^{\omega, \gamma_1} = \emptyset, \text{ and } \frac{n+1}{2} - l \leq \gamma_1 \leq \frac{n+1}{2}, \text{ and } G \in \tilde{C}_G^{p, \omega}(E_0, E_1, F_0, F_1) (\mathbb{D}; k).
Definition 46. (Ellipticity) Using the notation of Definition 45, we say that $A \in C^{m,d}_{E_0,F_0,E_1,F_1}(\mathbb{D},(\gamma,\gamma-m,k))$, $d \leq \max\{0,m\}$, is elliptic if:

1) Outside the singularity $X \times \{0\}$, $A$ is an elliptic Boutet de Monvel operator in $B^{m,d}_{E_0,F_0,E_1,F_1}(\text{int } \mathbb{D})$: Its interior symbol and boundary symbol are invertible at each point.

2) Its conormal symbol $\sigma_M(A)(z) := h(0,z) + h_0(z)$, is elliptic, so is its adjoint.

Similarly, we say that $A \in C^{m}_{E_0,F_0,E_1,F_1}(\mathbb{D},k)$ is an elliptic operator if, outside the singularity $X \times \{0\}$, $A$ is an elliptic operator in $B^{m}_{E_0,F_0,E_1,F_1}(\text{int } \mathbb{D})$ and its conormal symbol $\sigma_M(A)(z) := h(0,z) + h_0(z)$ is an elliptic Boutet de Monvel operator.

Remark 47. Definition 46 follows [26]. Instead one might ask that

1) The principal pseudodifferential symbol $\sigma_\psi(A)$ is invertible on $T^*(\text{int } \mathbb{D}) \setminus \{0\}$ and, in local coordinates $(t,x,\tau,\xi)$ for the cotangent space in a collar neighborhood of the conical point, $t^m\sigma_\psi(A)(t,x,\tau/t,\xi)$ is smoothly invertible up to $t = 0$.

2) The boundary principal symbol $\sigma_\partial(A)$ is invertible on $T^*(\text{int } \mathbb{B}) \setminus \{0\}$ and, in local coordinates $(t,y,\tau/t,\eta)$ for the cotangent space in a collar neighborhood of the conical point, $t^m\sigma_\partial(A)(t,y,\tau/t,\eta)$ is smoothly invertible up to $t = 0$.

3) The conormal symbol is pointwise invertible.

See [32] Section 6.2.1 for details.

Proposition 48. The operators in $\dot{C}^{m,d}_{E_0,F_0,E_1,F_1}(\mathbb{D},(\gamma,\gamma-m,k))$ and $\dot{C}^{m}_{E_0,F_0,E_1,F_1}(\mathbb{D},k)$ have the following properties:

1) If $B \in C^{m_1,d_1}_{E_0,F_0,E_2,F_2}(\mathbb{D},(\gamma-m_0,\gamma-m_2,k))$ and $A \in C^{m_2,d_2}_{E_1,F_1,E_2,F_2}(\mathbb{D},(\gamma-m_0,\gamma-m_2,k))$, then $AB \in C^{m_3 + d_3}_{E_0,F_0,E_2,F_2}(\mathbb{D},(\gamma,\gamma-m,k))$. If $B \in \dot{C}^{m}_{E_0,F_0,E_1,F_1}(\mathbb{D},k)$ and $A \in \dot{C}^{m}_{E_1,F_1,E_2,F_2}(\mathbb{D},k)$, then $AB \in \dot{C}^{m}_{E_0,F_0,E_2,F_2}(\mathbb{D},k)$.

2) If $A \in \dot{C}^{m,d}_{E_0,F_0,E_1,F_1}(\mathbb{D},(\gamma,\gamma-m,k))$, then $A$ extends to a continuous operator:

$$
\begin{align*}
H^p_{E_0,F_0}(\mathbb{D}) & \rightarrow H^p_{E_0,F_0}(\mathbb{D}) \\
\mathbb{B}_{E_0,F_0}^{s-m,\gamma-m}(\mathbb{B},F_0) & \rightarrow \mathbb{B}_{E_0,F_0}^{s-m,\gamma-m}(\mathbb{B},F_0),
\end{align*}
$$

if $A \in \dot{C}^{m}_{E_0,F_0,E_1,F_1}(\mathbb{D},k)$, then $A$ extends to a continuous operator:

$$
\begin{align*}
H^p_{E_0,F_0}(\mathbb{D}) & \rightarrow H^p_{E_0,F_0}(\mathbb{D}) \\
\mathbb{B}_{E_0,F_0}^{s-m,\gamma-m}(\mathbb{B},F_0) & \rightarrow \mathbb{B}_{E_0,F_0}^{s-m,\gamma-m}(\mathbb{B},F_0),
\end{align*}
$$

3) If $A \in \dot{C}^{m}_{E_0,F_0,E_1,F_1}(\mathbb{D},k)$, then its formal adjoint with respect to the inner product in $H^2_{E_0,F_0}(\mathbb{D}) \otimes \mathbb{B}_{E_0,F_0}^{0,\gamma}(\mathbb{B})$ belongs to $\dot{C}^{m}_{E_1,F_1,F_0}(\mathbb{D},k)$, for $\frac{1}{p} + \frac{1}{q} = 1$. If $A$ is elliptic, so is its adjoint.

4) If $A \in C^{m,d}_{E_0,F_0,E_1,F_1}(\mathbb{D},(\gamma,\gamma-m,k))$ is elliptic, $d := \max\{m,0\}$, then there is an operator $B \in C^{m,d}_{E_1,F_1,F_0}(\mathbb{D},(\gamma-m,k))$, $d' := \max\{m,0\}$, such that $BA - I \in C^{d'}_{E_0,F_0,E_0,F_0}(\mathbb{D},(\gamma,\gamma,k))$; $AB - I \in C^{d'}_{E_1,F_1,F_1,F_1}(\mathbb{D},(\gamma-m,\gamma-m,k))$. 


Similarly, if \( \tilde{A} \in \tilde{C}_{E_0,F_0,E_1,F_1}^p(\mathbb{D},k) \) is elliptic, then there is an operator \( \tilde{B} \in \tilde{C}_{E_1,F_0,E_0,F_0}^p(\mathbb{D},k) \), such that
\[
\tilde{B}\tilde{A} - I \in \tilde{C}_{G,E_0,F_0,E_0,F_0}^p(\mathbb{D},k) \quad \text{and} \quad \tilde{A}\tilde{B} - I \in \tilde{C}_{G,E_1,F_1,E_1,F_1}^p(\mathbb{D},k).
\]
In particular, \( A \) and \( \tilde{A} \) are then Fredholm operators.

5. (Existence of order reducing operators) For \( m \in \mathbb{Z}, m' \in \mathbb{R} \) and \( \gamma \in \mathbb{R} \), there are elliptic operators \( A \in t^mC_{E_0,E_0}(\mathbb{D},(\gamma,\gamma-m,k)) \) and \( B \in t^{m'}C_{F_0,F_0}(\mathbb{B},(\gamma,\gamma-m',k)) \), such that \( A : \mathcal{H}^{s,\gamma}_p(\mathbb{D},E_0) \to \mathcal{H}^{s-m,\gamma-m}_p(\mathbb{D},E_0) \) and \( B : B^s_{\frac{1}{p},\gamma-\frac{1}{p}}(\mathbb{B},F_0) \to B^s_{1-m',\frac{1}{p},\gamma-m'-\frac{1}{p}}(\mathbb{B},F_0) \) are invertible for all \( s > -1 + \frac{1}{p} \), see [13] Section 6.4.

3.3. The equivalence between the Fredholm property and the ellipticity.

**Theorem 49.** Let \( A \in \tilde{C}_{E_0,F_0,E_1,F_1}^p(\mathbb{D},k) \). Then the following conditions are equivalent:

i) \( A \) is elliptic.

\[
\mathcal{H}^{s,\gamma}_p(\mathbb{D},E_0) \quad \text{and} \quad \mathcal{H}^{s-m,\gamma-m}_p(\mathbb{D},E_1)
\]

ii) \( A : \bigoplus B^s_{\frac{1}{p},\gamma-\frac{1}{p}}(\mathbb{B},F_0) \to \bigoplus B^s_{1-m',\frac{1}{p},\gamma-m'-\frac{1}{p}}(\mathbb{B},F_1) \)

That i) implies ii) follows from the existence of a parametrix of an elliptic operator, as it is stated in item 4) of Proposition [13]. It remains to prove that ii) implies i). If \( A \) is Fredholm, then condition 1) of Definition [16] holds by Theorem [24]. In fact, the proof of Theorem [24] is local, so it applies in this context. In the next two subsections, we will show that condition 2) of Definition [16] holds. We rely on the arguments in [27] Section 3.1]; however, the Besov space estimates need more attention. Before, however, we note the following consequence.

**Corollary 50.** For \( A \in C_{E_0,F_0,E_1,F_1}^{m,d}(\mathbb{D},(\gamma,\gamma-m,k)) \), \( m \in \mathbb{Z}, d \leq \max\{m,0\} \), \( p \in [1,\infty[, s \in \mathbb{Z}, s \geq d \), the following conditions are equivalent:

i) \( A \) is elliptic.

ii)

\[
\mathcal{H}^{s,\gamma}_p(\mathbb{D},E_0) \quad \text{and} \quad \mathcal{H}^{s-m,\gamma-m}_p(\mathbb{D},E_1)
\]

\[ B^s_{\frac{1}{p},\gamma-\frac{1}{p}}(\mathbb{B},F_0) \quad \text{and} \quad B^s_{1-m',\frac{1}{p},\gamma-m'-\frac{1}{p}}(\mathbb{B},F_1) \]

In particular, the Fredholm property is independent of \( p \) and \( s \), subject to the condition \( s \in \mathbb{Z}, s \geq d \). The same is then true for the kernel and the index.

**Proof.** According to item 4) of Proposition [13], ellipticity implies the Fredholm property. In order to see the converse, we note that, by item 5 of Proposition [13], we find operators \( P^{-s} \in t^{-s}C_{E_0,E_0}(\mathbb{D},\frac{n+1}{2},\frac{n+1}{2}+s,k) \) and \( Q^{s-m} \in t^{-s-m}C_{E_1,E_1}(\mathbb{D},\frac{n+1}{2},\frac{n+1}{2}+s+m,k) \), defined on \( \mathbb{D} \), \( P^{-s+1/p} \in t^{-s+1/p}C^{-s+1/p}_{F_0,F_0}(\mathbb{B},\frac{n+1}{2},\frac{n+1}{2}+s-m,k) \), and \( Q^{s-m+1/p} \in t^{-s-m+1/p}C^{-s-m+1/p}_{F_1,F_1}(\mathbb{B},\frac{n+1}{2},\frac{n+1}{2}+s-m,k) \), defined on \( \mathbb{B} \), such that \( A : \mathcal{H}^{s,\gamma}_p(\mathbb{D},E_0) \to \mathcal{H}^{s-m,\gamma-m}_p(\mathbb{D},E_1) \), \( \tilde{A} : B^s_{\frac{1}{p},\gamma-\frac{1}{p}}(\mathbb{B},F_0) \to B^s_{1-m',\frac{1}{p},\gamma-m'-\frac{1}{p}}(\mathbb{B},F_1) \), and \( \tilde{A} : B^s_{\frac{1}{p},\gamma-\frac{1}{p}}(\mathbb{B},F_0) \to B^s_{1-m',\frac{1}{p},\gamma-m'-\frac{1}{p}}(\mathbb{B},F_1) \) are invertible. Here we use that \( s \in \mathbb{Z} \). Since \( A \) is a Fredholm operator, the operator \( \tilde{A} \in \tilde{C}_{E_0,F_0,E_1,F_1}^p(\mathbb{D},k) \), defined by

\[
\tilde{A} = \begin{pmatrix} P^{-s} & 0 \\ 0 & \tilde{Q}^{s-m+\frac{1}{p}} \end{pmatrix} t^{\frac{n+1}{2}-\gamma+m} A t^{-\frac{n+1}{2}+\gamma} \begin{pmatrix} P^{-s+\frac{1}{p}} & 0 \\ 0 & \tilde{P}^{-s+\frac{1}{p}} \end{pmatrix}
\]
is a Fredholm operator in $B(\mathcal{H}_p^{0,\frac{3}{2}}(\mathbb{D}, E_0) \oplus \mathcal{B}_p^{0,\frac{3}{2}}(\mathbb{B}, F_0), \mathcal{H}_p^{0,\frac{3}{2}}(\mathbb{D}, E_1) \oplus \mathcal{B}_p^{0,\frac{3}{2}}(\mathbb{B}, F_1))$. By Theorem 15, $\tilde{A}$ is elliptic, hence so is $A$. As a consequence, the Fredholm property is independent of $p$ and $s$.

Suppose $A$ is elliptic and $u \in \mathcal{H}_p^{t,\gamma}(\mathbb{D}, E_0) \oplus \mathcal{B}_p^{t,\frac{3}{2}-\gamma-\frac{\varepsilon}{2}}(\mathbb{B}, F_0)$ belongs to the kernel of $A$. Then the existence of a parametrix and the mapping properties of the Green operators imply that, for some $\varepsilon > 0$ and all $t \in \mathbb{R}$, $u \in \mathcal{H}_p^{t,\gamma+\varepsilon}(\mathbb{D}, E_0) \oplus \mathcal{B}_p^{t,\frac{3}{2}-\gamma-\frac{\varepsilon}{2}}(\mathbb{B}, F_0)$. Thus $u$ also is an element of $\mathcal{H}_p^{t,\gamma}(\mathbb{D}, E_0) \oplus \mathcal{B}_p^{t,\frac{3}{2}-\gamma-\frac{\varepsilon}{2}}(\mathbb{B}, F_0)$ for $1 < q < \infty$ and $t \in \mathbb{R}$ and belongs to the kernel of $\tilde{A}$ on that space. This shows the independence of the kernel on $s$ and $p$.

We next consider the formal adjoint $\tilde{A}'$ of $\tilde{A}$ in the sense of item 3) of Proposition 15, which is an elliptic element of $C^q_{E_1, F_1, E_0, F_0}(\mathbb{D}, k)$, where $1/p+1/q = 1$. Its extension to an operator in $B(\mathcal{H}_q^{0,\frac{3}{2}}(\mathbb{D}, E_1) \oplus \mathcal{B}_q^{0,\frac{3}{2}}(\mathbb{B}, F_1), \mathcal{H}_q^{0,\frac{3}{2}}(\mathbb{D}, E_0) \oplus \mathcal{B}_q^{0,\frac{3}{2}}(\mathbb{B}, F_0))$ furnishes the adjoint to the operator $\tilde{A}$ acting as in (3.3). The index of $\tilde{A}$ then is the difference of the kernel dimensions of $A$ and $\tilde{A}'$. By the same argument as above, these are independent of $p$ and $q$. Hence the index of $\tilde{A}$ is independent of $p$ and the index of $A$ is independent of $s$ and $p$. □

3.4. Besov-space preliminaries. Given dyadic partitions of unity $\{\varphi_j\}_{j \in \mathbb{N}_0} \subset C^\infty(\mathbb{R})$ and $\{\bar{T}_j\}_{j \in \mathbb{N}_0} \subset C^\infty(\mathbb{R}^{n-1})$ of $\mathbb{R}$ and $\mathbb{R}^{n-1}$, respectively, we define a dyadic partition of unity $\{\bar{\varphi}_j\}_{j \in \mathbb{N}_0} \subset C^\infty(\mathbb{R}^n)$ of $\mathbb{R}^n$ by

$$
\bar{\varphi}_j(t, x) := \varphi_j(t) \bar{T}_j(x),
$$

$$
\bar{\psi}_j(t, x) := \varphi_j(t) \left( \sum_{k=0}^{j-1} \bar{T}_k(x) \right) + \left( \sum_{k=0}^{j-1} \varphi_k(t) \right) \bar{T}_j(x)
$$

$$
= \psi_j \left( 2^{-j} t, 2^{-j} x \right) - \psi_0 \left( 2^{-(j+1)} t, 2^{-j-1} x \right), \ j \geq 1.
$$

Then supp $\psi_j \subset \{(t, x) \in \mathbb{R}^n; \|(t, x)\|_N < 2\}$ and supp $\bar{\psi}_j \subset \{(t, x) \in \mathbb{R}^n; 2^{j-1} < \|(t, x)\|_N < 2^{j+1}\}$, for $j \geq 1$. Here $\|(t, x)\|_N$ denotes the norm $\|(t, x)\|_N = \max \{|x|, |t|\}$, where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^{n-1}$ and $|t|$ denotes the modulus of $t \in \mathbb{R}$.

Item 8 of Remark 15 implies that we can choose the following norm for $B_p^q(\mathbb{R}^n)$:

$$
\|f\|_{B_p^q(\mathbb{R}^n)} := \left( \sum_{j=0}^{\infty} 2^{jqp} \|\psi_j(D) f\|^p_{L_p(\mathbb{R}^n)} \right)^{\frac{1}{p}}.
$$

The next spaces are very useful for computations.

Definition 51. Let $G$ be a Banach space that is a UMD Banach space with the property $(\alpha)$. We define $B_p^{\alpha,\frac{3}{2}}(\mathbb{R}, G)$ as the space of all $u \in D'(\mathbb{R}, G)$ such that $(\mathbb{R} \ni t \mapsto u(e^{-t})) \in B_p^{\alpha,0}(\mathbb{R}, G)$ and $H_p^{\alpha,0}(\mathbb{R}^+, G)$ as the set of all $u \in D'(\mathbb{R}^+, G)$ such that $(\mathbb{R}^+ \ni t \mapsto u(e^{-t})) \in H_p^{\alpha,0}(\mathbb{R}^+, G)$. In particular, $H_p^{\alpha,0}(\mathbb{R}^+, G) = L_p(\mathbb{R}^+, G, \frac{dt}{t})$ and $H_p^{\alpha,0}(\mathbb{R}^+, G) = \{u \in L_p(\mathbb{R}^+, G, \frac{dt}{t}); t \partial_t u \in L_p(\mathbb{R}^+, G, \frac{dt}{t})\}$.

Proposition 52. There is a constant $C > 0$ such that

$$
\|u\|_{B_p^{\alpha,0}(\partial \mathcal{X} \cap F)} \leq C \|u\|_{H_p^{\alpha,\frac{3}{2}}(\mathbb{R}^+, B_p^{\alpha}(\partial \mathcal{X} \cap F))},
$$

for all $u \in T_p^\perp(\mathbb{R}^+, C^\infty(\partial \mathcal{X}, F))$. 
Proof. In order to prove the proposition, we fix a constant \( \theta \in [0, 1] \) and a constant \( C_0 > 1 \) such that \( j + 1 \leq C_0 \theta^p \), for all \( j \in \mathbb{N}_0 \). The Hölder inequality implies that, for every non-negative real numbers \( a_0, \ldots, a_j \), we have
\[
\left( \sum_{k=0}^{j} a_k \right)^p \leq (j + 1)^{p-1} \sum_{k=0}^{j} a_k^p \leq C_0^{p-1} \sum_{k=0}^{j} a_k^p.
\]

Now, let us first prove that \( \|u\|_{L^p(B^p)} \leq C \|u\|_{H^1_p(B^p(B^{p}(\mathbb{R}^{n-1}))}: \)
\[
\|u\|_{B^p_p(\mathbb{R}^n)} = \left( \sum_{j=0}^{\infty} \left( \sum_{k=0}^{j} \|\varphi_j(D_t) \varphi_k(D_x) u\|_{L^p(\mathbb{R}^n)} \right)^p \right)^{\frac{1}{p}}
\]
\[
\begin{align*}
&\leq \left( \sum_{j=0}^{\infty} \left( \sum_{k=0}^{j-1} \|\varphi_j(D_t) \varphi_k(D_x) u\|_{L^p(\mathbb{R}^n)} \right)^p \right)^{\frac{1}{p}} \\
&\quad + \left( \sum_{j=0}^{\infty} \left( \sum_{k=0}^{j} \|\varphi_j(D_t) \varphi_k(D_x) u\|_{L^p(\mathbb{R}^n)} \right)^p \right)^{\frac{1}{p}} \\
&\leq 2C_0 \left( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{j\theta p} \|\varphi_j(D_t) \varphi_k(D_x) u\|_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{p}} \\
&= 2C_0 \left( \sum_{j=0}^{\infty} 2^{j\theta p} \int_{\mathbb{R}^n} \sum_{k=0}^{\infty} \|\varphi_j(D_t) \varphi_k(D_x) u\|_{L^p(\mathbb{R}^{n-1})} \, dt \right)^{\frac{1}{p}} \\
&= 2C_0 \left( \sum_{j=0}^{\infty} 2^{j\theta p} \|\varphi_j(D_t) u\|_{L^p_p(B(B^{p}(\mathbb{R}^{n-1}))} \right)^{\frac{1}{p}} = 2C_0 \|u\|_{B^p_p(B^{p}(\mathbb{R}^{n-1}))}.
\end{align*}
\]

Choosing \( \theta < 1 \), we conclude that
\[
\|u\|_{B^p_p(\mathbb{R}^n)} \leq C_0 \|u\|_{B^p_p(B^{p}(\mathbb{R}^{n-1}))} \leq C_0 \|u\|_{H^1_p(B^{p}(\mathbb{R}^{n-1}))}.
\]

Using a change of variable \( t \mapsto e^{-t} \), we obtain that
\[
\|u\|_{B^p_p(\mathbb{R}^n)} \leq C_0 \|u\|_{H^1_p(\mathbb{R}^{n-1})},
\]
where \( B^p_p(\mathbb{R}^{n-1}) = \{ v(\ln(t), x); v \in B^p_p(\mathbb{R}^n) \} \). Finally, using partition of unity and localization, we obtain the assertion.

For the following proposition we write \( HB_{pE}, F_j(X^\tau) := H_{0}^{p, \frac{\tau}{2}}(X^\tau, E_j) \oplus B^p_0(\partial X^\tau, F_j) \) and \( HB_{pE_j}, F_j(\mathbb{D}) := H_{0}^{p, \frac{\tau}{2}}(\mathbb{D}, E_j) \oplus B^p_0(\mathbb{R}, F_j) \) for \( j = 0, 1 \). We denote by \( K_j, j \in \mathbb{N}_0 \), the sets introduced in Remark 12 for \( n = 1 \).

Proposition 53. There exists a constant \( C \), independent of \( m \), such that for all \( u \in T_p(\mathbb{R}^n) \) and all \( v \in C^{\infty}(X, E, F) \) with \( \text{supp}(\tau \mapsto (\mathcal{M}_u(\tau)(i\tau)) \subset K_m \)
\[
\frac{1}{C} \frac{1}{(m + 1)} \|u\|_{L^p_p(R^{n})} \|v\|_{L^p_p(X, E) \oplus B^p_0(\partial X, F)} \leq \|u \otimes v\|_{\mathcal{H}B_{pE}, F_j(X^\tau)} \leq C \|u\|_{L^p_p(R^{n})} \|v\|_{L^p_p(X, E) \oplus B^p_0(\partial X, F)}.
\]

In order to make the proof more transparent, we first prove the following lemma.

Lemma 54. There exists a constant \( C > 0 \), independent of \( m \), such that for \( u \in \mathcal{S}(\mathbb{R}) \) and \( v \in \mathcal{S}(\mathbb{R}^{n-1}) \) with \( \text{supp}(F u) \subset K_m \)
\[
\frac{1}{C} \frac{1}{(m + 1)} \|u\|_{L^p_p(R^{n-1})} \|v\|_{B^p_0(\mathbb{R}^{n-1})} \leq \|u \otimes v\|_{B^p_p(\mathbb{R}^{n-1})} \leq C \|u\|_{L^p_p(\mathbb{R})} \|v\|_{B^p_0(\mathbb{R}^{n-1})}.
\]
Proof. Let \((t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}\) and \(C > 0\) such that \(\|\varphi_k (D_t) u\|_{L_p(\mathbb{R})} \leq C \|u\|_{L_p(\mathbb{R})}\) and \(\|\tilde{\varphi}_k (D_x) u\|_{L_p(\mathbb{R}^{n-1})} \leq C \|u\|_{L_p(\mathbb{R}^{n-1})}\) for all \(k \in \mathbb{N}_0\) and for all Schwartz functions \(u\). This constant exists, as we have seen in the proof of Lemma \[21\] In particular, \(\|\varphi_k (D_t) \tilde{\varphi}_j (D_x) (u)\|_{L_p(\mathbb{R} \times \mathbb{R}^{n-1})} \leq C^2 \|u\|_{L_p(\mathbb{R} \times \mathbb{R}^{n-1})}\), for all \(k, j \in \mathbb{N}_0\) and \(u \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^{n-1})\).

Using the conventions \(\varphi_k = 0\), \(\tilde{\varphi}_k = 0\), \(\psi_k = 0\) and \(K_k = 0\), whenever \(k \leq -1\), we see that

\[
\|u \otimes v\|_{B^p_{\mathbb{R}(\mathbb{R}^n)}} = \left( \sum_{j=0}^{\infty} \|\psi_j (D) (u \otimes v)\|_{L_p(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}
\]

\[
= \left( \sum_{j=0}^{\infty} \|\varphi_j (D_t) u \otimes \tilde{\varphi}_j (D_x) v + \sum_{k=0}^{j-1} \varphi_k (D_t) u \tilde{\varphi}_j (D_x) v\|_{L_p(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{j=0}^{m+1} \|\varphi_j (D_t) u\|_{L_p(\mathbb{R})}^p \left( \sum_{k=0}^{j} \varphi_k (D_t) u \tilde{\varphi}_j (D_x) v\right) \right)^{\frac{1}{p}}
\]

\[
+ \left( \sum_{j=m+1}^{m+1} \|\varphi_j (D_t) u\|_{L_p(\mathbb{R})}^p \left( \sum_{k=0}^{j} \varphi_k (D_t) u \tilde{\varphi}_j (D_x) v\right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{j=0}^{m+1} \|\varphi_j (D_t) u\|_{L_p(\mathbb{R})}^p \right) \left( \sum_{j=0}^{m+1} \|\tilde{\varphi}_j (D_x) v\|_{L_p(\mathbb{R}^{n-1})}^p \right)^{\frac{1}{p}}
\]

\[
+ \left( \sum_{j=m+1}^{m+1} \|\varphi_j (D_t) u\|_{L_p(\mathbb{R})}^p \right) \left( \sum_{j=m+1}^{m+1} \|\tilde{\varphi}_j (D_x) v\|_{L_p(\mathbb{R}^{n-1})}^p \right)^{\frac{1}{p}}
\]

\[
\leq (m + 2)^{\frac{1}{p} - \frac{1}{2}} 3^\frac{p}{2} C \|u\|_{L_p(\mathbb{R})} \left( \sum_{j=0}^{m+1} \|\tilde{\varphi}_j (D_x) v\|_{L_p(\mathbb{R}^{n-1})}^p \right)^{\frac{1}{p}}
\]

\[
+ 3C \|u\|_{L_p(\mathbb{R})} \left( \sum_{j=0}^{m+1} \|\tilde{\varphi}_j (D_x) v\|_{L_p(\mathbb{R}^{n-1})}^p \right)^{\frac{1}{p}}
\]

\[
\leq C (m + 1) \|u\|_{L_p(\mathbb{R})} \|v\|_{B^p_{\mathbb{R}(\mathbb{R}^{n-1})}}.
\]

On the other hand, with \(\|\cdot\|\) denoting the norm in \(L_p(\mathbb{R}^n)\),

\[
\|u\|_{L_p(\mathbb{R})}^p \|v\|_{B^p_{\mathbb{R}(\mathbb{R}^{n-1})}} = \sum_{k=0}^{\infty} \|u \otimes \tilde{\varphi}_k (D_x) v\|_{L_p(\mathbb{R}^n)}^p
\]

\[
= \sum_{j=0}^{\infty} \left\| \sum_{k=m+1}^{m+1} \varphi_k (D_t) u \tilde{\varphi}_j (D_x) v \right\|^p
\]

\[
\leq \sum_{j=0}^{m+1} \left\| \sum_{k=m+1}^{m+1} \varphi_k (D_t) u \tilde{\varphi}_j (D_x) v \right\|^p + \sum_{j=m+1}^{m+1} \left\| \sum_{k=m+1}^{m+1} \varphi_k (D_t) u \tilde{\varphi}_j (D_x) v \right\|^p
\]

\[
\leq \sum_{j=0}^{m+1} \sum_{k=m+1}^{m+1} \|\varphi_k (D_t) u \tilde{\varphi}_j (D_x) v\|^p + \|u \otimes v\|_{B^p_{\mathbb{R}(\mathbb{R}^n)}}^p
\]

\[
\leq \sum_{j=0}^{m+1} \sum_{k=m+1}^{m+1} \|\varphi_k (D_t) \psi_t (D) (u \otimes v)\|^p + \|u \otimes v\|_{B^p_{\mathbb{R}(\mathbb{R}^n)}}^p
\]

\[
\leq \sum_{j=0}^{m+1} \sum_{k=m+1}^{m+1} \|\varphi_k (D_t) \psi_l (D) (u \otimes v)\|^p + \|u \otimes v\|_{B^p_{\mathbb{R}(\mathbb{R}^n)}}^p
\]
Hence
\[
\begin{align*}
\sum_{j=m+2}^{\infty} \left( \sum_{k=m+1}^{m+1} \varphi_k(D_t) u \tilde{\varphi}_j(D_x) v \right)^p & \leq \sum_{j=m+2}^{\infty} \left( \sum_{k=m+1}^{m+1} \varphi_k(D_t) u \tilde{\varphi}_j(D_x) v \right)^p \\
+ \sum_{j=0}^{m+1} \left( \sum_{k=0}^{j} \varphi_k(D_t) u \tilde{\varphi}_j(D_x) v \right)^p & = \| u \otimes v \|_{B_p^p(\mathbb{R}^n)}^p.
\end{align*}
\]

We have used in (1) that \(\text{supp} (F u) \subset K_m\) and, therefore, we have
\[
\begin{align*}
\sum_{j=0}^{m+1} \left( \sum_{k=0}^{j} \varphi_k(D_t) u \tilde{\varphi}_j(D_x) v \right)^p & \leq \sum_{j=0}^{m+1} \left( \sum_{k=0}^{j} \varphi_k(D_t) u \tilde{\varphi}_j(D_x) v \right)^p \\
+ \sum_{j=0}^{m+1} \left( \sum_{k=0}^{j} \varphi_k(D_t) u \tilde{\varphi}_j(D_x) v \right)^p & = \| u \otimes v \|_{B_p^p(\mathbb{R}^n)}^p.
\end{align*}
\]

We have used in (2) that for \(j \in \{0, \ldots, m+1\}\) and \(k \in \{m-1, \ldots, m+1\}\)
\[
\sum_{l=m-2}^{m+2} \psi_l(D) (\varphi_k(D_t) \otimes \tilde{\varphi}_j(D_x)) = \varphi_k(D_t) \otimes \tilde{\varphi}_j(D_x).
\]

\[\square\]

Proof. (of Proposition 53) Let \(u \in \mathcal{T}_\mathcal{B}^p(\mathbb{R}^n, v \in \mathcal{S}(\mathbb{R}^n))\) and suppose that \(\text{supp}(\tau \mapsto \mathcal{M}_\lambda u(\tau \xi)) \subset K_m\), \(m \in \mathbb{N}_0\). Define \(\tilde{u} \in \mathcal{S}(\mathbb{R})\) by \(\tilde{u}(t) = u(e^{-t})\).

Hence \(\mathcal{F}_{\tau \to \xi} \tilde{u}(\xi) = \mathcal{F}_{\tau \to \xi} (u(e^{-t}))(\xi) = \mathcal{M}_\lambda u(i\xi)\). Therefore, there is a constant \(m \in \mathbb{N}_0\) such that \(\text{supp}(F \tilde{u}) \subset K_m\). Hence, Lemma 53 implies that
\[
\frac{1}{C(m+1)} \| u \|_{L_p(\mathbb{R}^n, \mathbb{R}^n)} \| v \|_{B_p^p(\mathbb{R}^n)} = \frac{1}{C(m+1)} \| \tilde{u} \|_{L_p(\mathbb{R}^n)} \| v \|_{B_p^p(\mathbb{R}^n)}
\]
\[
\leq \| \tilde{u} \otimes v \|_{B_p^p(\mathbb{R}^n)} \leq C(m+1) \| \tilde{u} \|_{L_p(\mathbb{R}^n)} \| v \|_{B_p^p(\mathbb{R}^n)}
\]
\[
\leq C(m+1) \| u \|_{L_p(\mathbb{R}^n, \mathbb{R}^n)} \| v \|_{B_p^p(\mathbb{R}^n)}.
\]

As \(\| \tilde{u} \otimes v \|_{B_p^p(\mathbb{R}^n)} = \| u \otimes v \|_{B_p^p(\mathbb{R}^n)}\), we conclude that
\[
\frac{1}{C(m+1)} \| u \|_{L_p(\mathbb{R}^n, \mathbb{R}^n)} \| v \|_{B_p^p(\mathbb{R}^n)} \leq C(m+1) \| u \|_{L_p(\mathbb{R}^n, \mathbb{R}^n)} \| v \|_{B_p^p(\mathbb{R}^n)}.
\]

The general result follows using a partition of unity. \[\square\]

3.5. Proof of the invertibility of the conormal symbol. We notice that \(\mathcal{T}_\mathcal{B}^p(\mathbb{R}^n, C^\infty(\mathcal{X}, E_j, F_j))\) is a dense space of \(\mathcal{H}B_pE_j, F_j(\mathcal{X}^\wedge)\).

Definition 55. Let \(W\) be a Fréchet space, \(\epsilon > 0, \tau_0 \in \mathbb{R}\). We define \(T_\epsilon : \mathcal{T}_\mathcal{B}^p(\mathbb{R}^n, W) \to \mathcal{T}_\mathcal{B}^p(\mathbb{R}^n, W)\) and \(R_\epsilon, \tau_0 : \mathcal{T}_\mathcal{B}^p(\mathbb{R}^n, W) \to \mathcal{T}_\mathcal{B}^p(\mathbb{R}^n, W)\) by \(T_\epsilon u(t) = u(\epsilon^{-1} t)\) and \(R_\epsilon, \tau_0 u(t) = e^{\frac{1}{\epsilon} t - \tau_0} u(\epsilon^{-1} t)\).

The above operators are invertible: \(T_\epsilon^{-1} = T_\epsilon\) and \(R_\epsilon, \tau_0^{-1} = R_\epsilon, \tau_0\). The next proposition is analogous to Lemma 14.

Proposition 56. For an UMD Banach space \(W\) with the property \((\alpha)\), the operators \(T_\epsilon, R_\epsilon, \tau_0 : \mathcal{T}_\mathcal{B}^p(\mathbb{R}^n, W) \to \mathcal{T}_\mathcal{B}^p(\mathbb{R}^n, W)\) have the following properties:

1. \(T_\epsilon\) extends to an isometry
\[
T_\epsilon : \mathcal{H}B_{1, \frac{1}{\epsilon}}^p(\mathbb{R}^n, W) \to \mathcal{H}B_{1, \frac{1}{\epsilon}}^p(\mathbb{R}^n, W)
\]

If \(W = C^\infty(\mathcal{X}, E_j, F_j), j = 0, 1\), then the operator \(T_\epsilon\) extends to an isometry \(T_\epsilon : \mathcal{H}B_{pE_j, F_j}(\mathcal{X}^\wedge) \to \mathcal{H}B_{pE_j, F_j}(\mathcal{X}^\wedge)\).
2) For all \( \varepsilon > 0 \), \( R_{\varepsilon, \tau_0} \) extends to a bijective continuous map

\[
R_{\varepsilon, \tau_0} : H^1_p \rightarrow H^1_p.
\]

There exists a \( C \geq 0 \) with \( \|R_{\varepsilon, \tau_0}\|_{B(H^1_p, \hat{\mathcal{H}})} \leq C (1 + |\tau_0|) \), \( \varepsilon < 1 \).

3) (i) Let \( h \in M \mathcal{B}_{E_0, F_0, E_1, F_1}(X, \mathbb{R}, \nu) \cap C(x^{\infty, (X, E_0, F_0)}) \) and \( h_0(z) := h(0, z) \). For any \( u \in T^{(\mathbb{R}^+)}(x^{\infty, (X, E_0, F_0)}) \) we then have

\[
\lim_{\varepsilon \to 0} \left\| \text{op}_p^\frac{1}{2} h_0 \right\|_{L_p(x^{\infty, (X, E_0, F_0)})} = 0.
\]

(ii) Let \( h \in \mathcal{B}^0_{E_0, F_0, E_1, F_1}(X, \Gamma_0) \) and \( u \in T^{(\mathbb{R}^+)}(x^{\infty, (X, E_0, F_0)}) \). Then

\[
\lim_{\varepsilon \to 0} \left\| \text{op}_p^\frac{1}{2} (h) R_{\varepsilon, \tau_0} u - R_{\varepsilon, \tau_0} (h (i\tau_0)) u \right\|_{L_p(x^{\infty, (X, E_0, F_0)})} = 0.
\]

Proof. 1) Since \( \|T_{\varepsilon} u\|_{L_p(x^{\infty, (X, E_0, F_0)})} = \|u\|_{L_p(x^{\infty, (X, E_0, F_0)})} \) and \( \|(t\partial_t) (T_{\varepsilon} u)\|_{L_p(x^{\infty, (X, E_0, F_0)})} = \|\partial_t u\|_{L_p(x^{\infty, (X, E_0, F_0)})} \), we conclude that \( \|T_{\varepsilon} u\|_{H^1_p(x^{\infty, (X, E_0, F_0)})} = \|u\|_{H^1_p(x^{\infty, (X, E_0, F_0)})} \).

In order to show that \( T_{\varepsilon} : H\mathcal{B}_{p E_1, F_1}(X^{\infty}) \to H\mathcal{B}_{p E_1, F_1}(X^{\infty}) \) is an isometry, it remains to prove that \( T_{\varepsilon} : \mathcal{B}^0_{E_0, F_0}(\partial X^\infty, F_j) \to \mathcal{B}^0_{E_1, F_1}(\partial X^\infty, F_j) \) is an isometry. This follows with a partition of unity and the fact that \( T_{\varepsilon} : \mathcal{B}^0_{E_0, F_0}(\partial X^\infty, F_j) \to \mathcal{B}^0_{E_1, F_1}(\partial X^\infty, F_j) \) is an isometry. In fact, if \( v(s, x) = u(e^{-s} x) \), then \( (T_{\varepsilon} u)(e^{-s} x) = v(s + \ln(e), x) \). Hence

\[
\|T_{\varepsilon} u\|_{B^0_{p, E_1}(\mathbb{R}^n)} = \|v(s + \ln(e), x)\|_{B^0_{p, E_1}(\mathbb{R}^n)} = \|v\|_{B^0_{p, E_1}(\mathbb{R}^n)} = \|u\|_{B^0_{p, E_1}(\mathbb{R}^n)}.
\]

2) It is easy to see that \( \|R_{\varepsilon, \tau_0} u\|_{L_p(x^{\infty, (X, E_0, F_0)})} = \|u\|_{L_p(x^{\infty, (X, E_0, F_0)})} \). As \( t\partial_t (R_{\varepsilon, \tau_0} u) = (-i\tau_0) R_{\varepsilon, \tau_0} u + \varepsilon R_{\varepsilon, \tau_0} (t\partial_t u) \), we conclude that

\[
\|R_{\varepsilon, \tau_0} u\|_{H^1_p(x^{\infty, (X, E_0, F_0)})} \leq (1 + |\tau_0|) \|u\|_{H^1_p(x^{\infty, (X, E_0, F_0)})}\]

3) We first show \( \mathbf{L}_{p}\text{-convergence} \): For \( u \in T^{(\mathbb{R}^+)}(x^{\infty, (X, E_0, F_0)}) \),

\[
\lim_{\varepsilon \to 0} \left\| T_{\varepsilon}^{-1} \text{op}_p^\frac{1}{2} (h) T_{\varepsilon} u - \text{op}_p^\frac{1}{2} (h_0) u \right\|_{L_p(x^{\infty, (X, E_0, F_0)})} = 0.
\]

The proof here is essentially the same as the proof of [27] Lemma 3.9. It relies on the fact that \( T_{\varepsilon}^{-1} \text{op}_p^\frac{1}{2} (h) = \text{op}_p^\frac{1}{2} (h) \), where \( h_\varepsilon(t, z) = h(\varepsilon t, z) \), and on Lebesgue’s dominated convergence theorem.

Next we establish the \( \mathbf{L}_{p}\text{-convergence of the derivative} \):

\[
\lim_{\varepsilon \to 0} \left\| T_{\varepsilon}^{-1} \text{op}_p^\frac{1}{2} (h) T_{\varepsilon} u - \text{op}_p^\frac{1}{2} (h_0) u \right\|_{H^1_p(x^{\infty, (X, E_0, F_0)})} = 0.
\]

This follows almost immediately from the fact that

\[
(-t\partial_t) \text{op}_p^\frac{1}{2} (h_\varepsilon) u = \text{op}_p^\frac{1}{2} ((-t\partial_t) h_\varepsilon) u + \text{op}_p^\frac{1}{2} (h_\varepsilon) ((-t\partial_t) u).
\]

Using \([39]\), the fact that \( T_{\varepsilon} \) are isometries and Proposition \([52]\), we conclude that, as \( \varepsilon \to 0 \),

\[
\left\| \text{op}_p^\frac{1}{2} (h) T_{\varepsilon} u - T_{\varepsilon} \text{op}_p^\frac{1}{2} (h_0) u \right\|_{H^1_p(x^{\infty, (X, E_0, F_0)})} \leq C \left\| T_{\varepsilon}^{-1} \text{op}_p^\frac{1}{2} (h) T_{\varepsilon} u - \text{op}_p^\frac{1}{2} (h_0) u \right\|_{H^1_p(x^{\infty, (X, E_0, F_0)})} \to 0.
\]
3. ii) It is straightforward to check that \( R_{e, \tau_0}^{-1} \cdot \Omega_M^j \cdot (h) \cdot R_{e, \tau_0} = \Omega_M^j \cdot (h) \cdot R_{e, \tau_0} \), where \( h_e(z) = h(\epsilon z + i \tau_0) \). Repeating the previous arguments, we conclude that
\[
\lim_{\epsilon \to 0} \| R_{e, \tau_0}^{-1} \cdot \Omega_M^j \cdot (h) \cdot R_{e, \tau_0} u - h(i\tau_0) u \|_{L^p(R_+, L_p(X,E_j) \oplus B^p_{H^1}(\partial X,F_j))} = 0.
\]
Moreover, \((-t\partial_t) \cdot \Omega_M^j \cdot (h(\epsilon z + i \tau_0)) = \Omega_M^j \cdot (h(\epsilon z + i \tau_0)) \cdot (-t\partial_t u)\). Hence
\[
\lim_{\epsilon \to 0} \| R_{e, \tau_0}^{-1} \cdot \Omega_M^j \cdot (h) \cdot R_{e, \tau_0} u - h(i\tau_0) u \|_{H^{1, \frac{1}{2}}(R_+, L_p(X,E_j) \oplus B^p_{H^1}(\partial X,F_j))} = 0.
\]
Finally, using Proposition \ref{prop:main_result} and item 2, we conclude that, as \( \epsilon \to 0 \),
\[
\| \Omega_M^j \cdot (h) \cdot R_{e, \tau_0} u - R_{e, \tau_0} h(i\tau_0) u \|_{H^{1, \frac{1}{2}}(R_+, E_j, E_j)} \\
\leq C \| R_{e, \tau_0} \left( R_{e, \tau_0}^{-1} \cdot \Omega_M^j \cdot (h) \cdot R_{e, \tau_0} u - h(i\tau_0) u \right) \|_{H^{1, \frac{1}{2}}(R_+, L_p(X,E_j) \oplus B^p_{H^1}(\partial X,F_j))} \\
\leq C \left(1 + |\tau_0|\right) \left\| R_{e, \tau_0}^{-1} \cdot \Omega_M^j \cdot (h) \cdot R_{e, \tau_0} u - h(i\tau_0) u \right\|_{H^{1, \frac{1}{2}}(R_+, L_p(X,E_j) \oplus B^p_{H^1}(\partial X,F_j))} \to 0.
\]

\[\square\]

The next lemma is analogous to Lemma \ref{lem:main_result}.

**Lemma 57.** The operators \( T_\epsilon \) and \( R_{e, \tau_0} \) satisfy the following properties:

1. If \( u \in \mathcal{T}_\omega^\pm(R_+) \) with \( \text{supp}(\mathcal{M}_j^\omega u) \subset \{ \xi \in \Gamma_0; |\xi| \leq \frac{1}{2} \} \), \( v \in C^\infty(X,E_j,F_j) \) and \( \epsilon < 1 \), then \( \text{supp}(\mathcal{M}_j^\omega (R_{e, \tau_0} u)) \subset K_m \), where \( K_0 := \{ \xi \in \Gamma_0; |\xi| \leq 2 \} \), \( K_j := \{ \xi \in \Gamma_0; 2^{j-1} \leq |\xi| \leq 2^{j+1} \}, j \in \mathbb{N}_0 \{0\} \). The number \( m \in \mathbb{N}_0 \) is equal to 0 if \( |\tau_0| + \frac{1}{2} < 2 \) and, for \( |\tau_0| + \frac{1}{2} > 2 \), \( m \) is the smallest number such that \( 2^{m-1} < |\tau_0| - \frac{1}{2} < |\tau_0| + \frac{1}{2} < 2^{m+1} \). Hence \( m \leq C (|\ln(\tau_0)|) \).

2. There is a constant \( C > 0 \) such that for all \( \epsilon < 1 \), \( v \in C^\infty(X,E_j,F_j) \) and \( u \in \mathcal{T}_\omega^\pm(R_+) \) with \( \text{supp}(\mathcal{M}_j^\omega u) \subset \{ \xi \in \Gamma_0; |\xi| \leq \frac{1}{2} \} \),
\[
\frac{1}{C (|\ln(\tau_0)|)} \| u \otimes v \|_{H^{1, \frac{1}{2}}(R_+, E_j, F_j)} \\
\leq \| R_{e, \tau_0} (u \otimes v) \|_{H^{1, \frac{1}{2}}(R_+, E_j, F_j)} \leq C (|\ln(\tau_0)|) \| u \otimes v \|_{H^{1, \frac{1}{2}}(R_+, E_j, F_j)}.
\]

3. For all \( u \in \mathcal{T}_\omega^\pm(R_+, C^\infty(X,E_j,F_j)) \), we have \( \lim_{\epsilon \to 0} T_\epsilon(u) = 0 \) weakly in \( H^{1, \frac{1}{2}}(R_+, E_j, F_j) \).

**Proof.** 1) An easy computation shows that \( \mathcal{M}_j^\omega(R_{e, \tau_0} u)(z) = e^{\frac{i}{\tau_0}} \mathcal{M}_j^\omega u \left( \frac{z - i\tau_0}{\epsilon} \right) \).

When \( \epsilon < 1 \), this means that, if \( x \in \mathbb{R} \) is such that \( \mathcal{M}_j^\omega(R_{e, \tau_0} u)(ix) \neq 0 \), then \( \tau_0 - \frac{1}{2} < x < \tau_0 + \frac{1}{2} \), which implies that \( \text{supp}(\mathcal{M}_j^\omega(R_{e, \tau_0} u)) \) is contained in some ball of radius \( \frac{1}{\epsilon} \).

2) As \( \text{supp}(\mathcal{M}_j^\omega u) \subset K_0 \), Proposition \ref{prop:main_result} implies that
\[
\| u \otimes v \|_{H^{1, \frac{1}{2}}(R_+, E_j, F_j)} \\
\leq C_1 \| u \otimes v \|_{L^p(R_+, L_p(X,E_j) \oplus B^p_{H^1}(\partial X,F_j))} \\
= C_1 \| (R_{e, \tau_0} u) \otimes v \|_{L^p(R_+, L_p(X,E_j) \oplus B^p_{H^1}(\partial X,F_j))} \\
\leq C_2 (|\ln(\tau_0)|) \| (R_{e, \tau_0} u) \otimes v \|_{H^{1, \frac{1}{2}}(R_+, E_j, F_j)}
\]
and
\[
\| (R_{e, \tau_0} u) \otimes v \|_{H^{1, \frac{1}{2}}(R_+, E_j, F_j)} \leq C_3 (|\ln(\tau_0)|) \| (R_{e, \tau_0} u) \otimes v \|_{L^p(R_+, L_p(X,E_j) \oplus B^p_{H^1}(\partial X,F_j))} \\
= C_3 (|\ln(\tau_0)|) \| u \otimes v \|_{L^p(R_+, L_p(X,E_j) \oplus B^p_{H^1}(\partial X,F_j))} \leq C_4 (|\ln(\tau_0)|) \| u \otimes v \|_{H^{1, \frac{1}{2}}(R_+, E_j, F_j)}.
\]
3) We identify the dual of $\mathcal{H}B_{pE_1,F_1}(X^\times)$ with $\mathcal{H}B_{qE_j,F_j}(X^\times)$, using the scalar product $L_2(R_+, L_2(X, E_j) \oplus L_2(\partial X, F_j), \frac{dt}{t})$. As $T_\epsilon$ is an isometry in $\mathcal{H}B_{pE_1,F_1}(X^\times)$, it is enough to prove that
\[
\lim_{\epsilon \to 0} \int_{R_+} \langle u(t/\epsilon), v(t) \rangle_{L_2(X,E_1)\oplus L_2(\partial X,F_1)} \frac{dt}{t} = 0
\]
for all $u, v \in C^\infty_c(R_+, C^\infty(X, E_j, F_j))$. But this is true. In fact, let $a, b, R > 0$ be such that supp$(u) \subset [0, R]$ and supp$(v) \subset [a, b]$, then, for $\epsilon < \frac{R}{2}$, we have supp$(T_\epsilon u) \cap$ supp$(v) = \emptyset$. Hence we obtain the result.

\textbf{Lemma 58.} Let $h \in B_{E_0,F_0,E_1,F_1}^p(X, \Gamma_0)$ and suppose that there is a constant $c > 0$ such that, for each $u \in T_{\frac{\tau}{2}}(R_+, C^\infty(X, E_0, F_0))$, we have
\[
\|u\|_{\mathcal{H}B_{pE_0,F_0}(X^\times)} \leq c \|\partial \tau M(h)(u)\|_{\mathcal{H}B_{pE_1,F_1}(X^\times)}.
\]
Then, for every $v \in C^\infty(X, E_0, F_0)$ and $\tau \in R$, we have
\[
\|v\|_{\mathcal{H}B_{pE_0,F_0}^p(\partial X, F_0)} \leq C (\ln(\tau))^2 \|h(i\tau) v\|_{\mathcal{H}B_{pE_0,F_0}^p(\partial X, F_1)},
\]
for some constant $C$ independent of $v$.

\textbf{Proof.} Let $0 \neq u \in T_{\frac{\tau}{2}}(R_+)$, be a function with supp$(\mathcal{M}^\frac{\tau}{2}(u)) \subset \{z \in \Gamma_0; |z| < \frac{1}{2}\}$ and $v \in C^\infty(X)$. Then item 2 of Lemma 57 implies that
\[
\|u \otimes v\|_{\mathcal{H}B_{pE_0,F_0}(X^\times)} \leq C_1 (\ln(\tau_0)) \|\partial \tau M(h)(R_{c,\tau_0}(u \otimes v) - R_{c,\tau_0} h(i\tau_0)(u \otimes v))\|_{\mathcal{H}B_{pE_1,F_1}(X^\times)} + C_2 (\ln(\tau_0))^2 \|R_{c,\tau_0} h(i\tau_0)(u \otimes v)\|_{\mathcal{H}B_{pE_1,F_1}(X^\times)}.
\]
As $\lim_{\tau \to 0} \|\partial \tau M(h)(R_{c,\tau_0}(u \otimes v) - h(i\tau_0)(u \otimes v))\|_{\mathcal{H}B_{pE_1,F_1}(X^\times)} = 0$, we see again from Lemma 57 that
\[
\|u \otimes v\|_{\mathcal{H}B_{pE_0,F_0}(X^\times)} \leq C_1 (\ln(\tau_0))^2 \|u \otimes h(i\tau_0) v\|_{\mathcal{H}B_{pE_1,F_1}(X^\times)} \leq C_2 (\ln(\tau_0))^2 \|u \otimes h(i\tau_0) v\|_{\mathcal{H}B_{pE_1,F_1}(X^\times)},
\]
where $(u \otimes h(i\tau_0) v)(t, x) := u(t)(h(i\tau_0) v)(x)$. Now, it is easy to conclude that
\[
\|v\|_{\mathcal{H}B_{pE_0,F_0}^p(\partial X, F_0)} \leq \frac{C}{\|L_{p}(\mathbb{R}_+, \frac{\tau}{2})\}} \|u \otimes v\|_{\mathcal{H}B_{pE_0,F_0}(X^\times)} \leq \frac{C}{\|L_{p}(\mathbb{R}_+, \frac{\tau}{2})\}} (\ln(\tau))^2 \|u \otimes h(i\tau_0) v\|_{\mathcal{H}B_{pE_1,F_1}(X^\times)} \leq C (\ln(\tau))^2 \|h(i\tau_0) v\|_{\mathcal{H}B_{pE_0,F_0}^p(\partial X, F_1)}.
\]

We finish with the following proposition that proves the invertibility of the conormal symbol.

\textbf{Proposition 59.} Let $A \in \mathcal{C}^p(D_k) be a Fredholm operator in the space $B(\mathcal{H}B_{pE_0,F_0}(X^\times), \mathcal{H}B_{pE_1,F_1}(X^\times))$. Then the conormal symbol is invertible on $\Gamma_0$, and its inverse is an element of $\mathcal{B}^p_{E_1,F_1,E_0,F_0}(X, \Gamma_0)$. 

\textbf{Proof.}
Proof. We are going to consider operators given as

\[ A = \omega P_{M}^{\frac{1}{2}}(h)\omega_{0} + (1 - \omega) P(1 - \omega_{1}) + G, \]

where \( P \in \mathcal{B}_{E_{0},E_{1}}^{p}(2\mathbb{D}) \), \( G \in \mathcal{C}^{p}_{C_{0},E_{0},E_{1},E_{1}}(\mathbb{D},k) \), and \( h(t,z) = a(t,z) + \tilde{a}(z) \) with functions \( a \in C^{\infty}(\mathbb{R}^{+},M_{p}^{\infty}(E_{0},E_{1},E_{1})(X)) \) and \( \tilde{a} \in M_{p}^{\infty}(E_{0},E_{1},E_{1})(X) \) for some asymptotic type \( P \) with \( \pi_{c}P \cap \Gamma_{0} = \emptyset \). In particular, \( h_{0}(z) := h(0,z) = \sigma_{M}^{0}(A)(z) \).

Let us first prove that for all \( u \in \mathcal{H}B_{p,E_{0},E_{0}}(X,\omega) \)

\[ \|u\|_{\mathcal{H}B_{p,E_{0},E_{0}}(X,\omega)} \leq c\|\text{op}_{M}^{\frac{1}{2}}(h_{0})(u)\|_{\mathcal{H}B_{p,E_{1},E_{1}}(X,\omega)}. \]  

It suffices to show this for \( u \in C_{c}^{\infty}(\mathbb{R}^{+},C_{c}^{\infty}(X,E_{0},F_{0})) \). We find operators \( B_{1} \in \mathcal{B}(\mathcal{H}B_{p,E_{1},E_{1}}(\mathbb{D}),\mathcal{H}B_{p,E_{0},E_{0}}(\mathbb{D}),\mathcal{H}B_{p,E_{0},E_{0}}(\mathbb{D})) \) and \( K_{1} \in \mathcal{B}(\mathcal{H}B_{p,E_{0},E_{0}}(\mathbb{D}),\mathcal{H}B_{p,E_{0},E_{0}}(\mathbb{D})) \), where \( K_{1} \) is compact, such that \( B_{1}A - 1 = K_{1} \). Let us choose \( \sigma \) and \( \sigma_{1} \) in \( C_{c}^{\infty}([0,1]) \) such that \( \sigma_{1} = \sigma, \sigma_{1}\omega_{1} = \sigma_{1} \) and \( \sigma_{1}\omega = \sigma_{1} \).

Thus, the supports of \( \sigma_{1} \) and \( 1 - \sigma_{1} \) are disjoint, the operator \((1 - \sigma_{1})A\sigma\) is a Green operator and therefore compact. Hence

\[ \sigma_{1}B_{1}\sigma_{1}A\sigma_{1} - \sigma = \sigma_{1}K_{1}\sigma_{1} - \sigma_{1}B_{1}(1 - \sigma_{1})A\sigma_{1} = \sigma_{1}K_{2}\sigma_{1}, \]

where \( K_{2} \) is a compact. Using Equation \( \text{[3.6]} \) for \( A\sigma \), we conclude that \( \sigma = B\sigma_{M}^{\frac{1}{2}}(h)\sigma - K \), where \( B = \sigma_{1}B_{1}\sigma_{1} \) and \( K = \sigma_{1}(K_{2} - B_{1}\sigma_{1}G)\sigma \) is compact.

Now let \( u \in C_{c}^{\infty}(\mathbb{R}^{+},C_{c}^{\infty}(X,E_{0},F_{0})) \). We know that \( T_{\epsilon}(u) = \sigma T_{\epsilon}(u) \), when \( \epsilon \) is small. As \( \sigma = B\sigma_{M}^{\frac{1}{2}}(h)\sigma - K \), we have that, for \( \epsilon \) sufficiently small,

\[ \|u\|_{\mathcal{H}B_{p,E_{0},E_{0}}(X,\omega)} = \|\sigma T_{\epsilon}(u)\|_{\mathcal{H}B_{p,E_{0},E_{0}}(X,\omega)}. \]

As \( T_{\epsilon}u \) weakly tends to zero and \( K \) is compact, \( \lim_{\epsilon \to 0} \|KT_{\epsilon}(u)\|_{\mathcal{H}B_{p,E_{0},E_{0}}(X,\omega)} = 0 \). Using that \( T_{\epsilon} \) is an isometry and item 3.iii) of Proposition \( \text{[3.6]} \) we conclude that Inequality \( \text{[3.7]} \) holds. This result together with Lemma \( \text{[3.8]} \) implies that

\[ \|v\|_{\mathcal{H}B_{p}(X,E_{0})\oplus B_{p}^{0}(\partial X,F_{0})} \leq C \langle \ln(\tau) \rangle^{2} \|h'(\tau)\|^2_{\mathcal{H}B_{p}(X,E_{1})\oplus B_{p}^{0}(\partial X,F_{1})}, \]

for some constant \( C \) independent of \( v \).

As \( A \in \mathcal{C}^{p}_{C_{0},E_{0},E_{1},E_{1}}(\mathbb{D},k) \) is Fredholm, so is \( A^{*} \in \mathcal{C}^{q}_{C_{0},E_{1},E_{0},E_{0}}(\mathbb{D},k) \). The above argument implies that \( \sigma_{M}^{0}(A^{*})(z) = h'(\tau)^{-1} \) also satisfies an estimate as \( \text{[3.8]} \), for \( q \) instead of \( p \), where \( \frac{1}{p} + \frac{1}{q} = 1 \). Hence, for all \( \tau \in \mathbb{R}, h'(\tau) \) is injective, has closed range and the same is true for its adjoint. Lemma \( \text{[3.9]} \) implies that \( h'(\tau) \) is bijective and

\[ \|h'(\tau)^{-1}\|_{\mathcal{B}(\mathcal{H}B_{p}(X,E_{1})\oplus B_{p}^{0}(\partial X,F_{1}),\mathcal{H}B_{p}(X,E_{0})\oplus B_{p}^{0}(\partial X,F_{0}))} \leq C \langle \ln(\tau) \rangle^{2}. \]

Theorem \( \text{[3.9]} \) implies that \( h_{0}^{-1} \in \mathcal{B}_{E_{1},E_{1}}^{p}(X,\Gamma_{0}). \) \( \square \)
3.6. Spectral invariance of boundary value problems with conical singularities. Once we know the equivalence of Fredholm property and ellipticity, we can establish the spectral invariance.

**Theorem 60.** Let \( A \in \mathcal{C}^p_{E_0,F_0,E_1,F_1}(\mathbb{D},k) \). Suppose that, for each \( \lambda \in \Lambda \), the operator

\[
A : \mathcal{H}_p^{0,m,\lambda} \oplus \mathcal{B}_p^{0,\lambda} \to \mathcal{H}_p^{0,m,\lambda} \oplus \mathcal{B}_p^{0,\lambda}
\]

is invertible. Then \( A^{-1} \in \mathcal{C}^p_{E_1,F_1,E_0,F_0}(\mathbb{D},k) \).

**Proof.** The operator \( A \) is invertible, hence it is Fredholm and there are operators \( B \in \mathcal{C}^p_{E_0,F_0,E_1,F_1}(\mathbb{D},k) \), \( K_1 \in \mathcal{C}^p_{G,E_1,F_1,E_0,F_0}(\mathbb{D},k) \) and \( K_2 \in \mathcal{C}^p_{G,E_0,F_0,E_0,F_0}(\mathbb{D},k) \) such that \( AB = I + K_1 \) and \( BA = I + K_2 \). These identities imply that

\[
A^{-1} = B - K_2B + K_2A^{-1}K_1.
\]

As \( B \in \mathcal{C}^p_{E_1,F_1,E_0,F_0}(\mathbb{D},k) \), \( K_2B \in \mathcal{C}^p_{G,E_1,F_1,E_0,F_0}(\mathbb{D},k) \) and \( K_2A^{-1}K_1 \) belongs to \( \mathcal{C}^p_{E_1,F_1,E_0,F_0}(\mathbb{D},k) \), we obtain the result. \( \square \)

**Theorem 61.** Let \( A \in \mathcal{C}^{m,d}_{E_0,F_0,E_1,F_1}(\mathbb{D},\gamma,\gamma-m,k) \), where \( m \in \mathbb{Z} \), \( d = \max\{m,0\} \). Suppose that there is an \( s \in \mathbb{Z} \), \( s \geq d \) such that

\[
A : \mathcal{H}_p^{s,\gamma} \oplus \mathcal{B}_p^{s,\gamma} \to \mathcal{H}_p^{s-m,\gamma-m} \oplus \mathcal{B}_p^{s-m,\gamma-m}
\]

is invertible. Then, \( A^{-1} \in \mathcal{C}^{-m,d'}_{E_1,F_1,E_0,F_0}(\mathbb{D},\gamma-m,\gamma,k) \), where \( d' := \max\{-m,0\} \). In particular, for all \( s > d - 1 + \frac{1}{q} \) and \( 1 < q < \infty \) the operator \( A \) is invertible in \( \mathcal{B}(\mathcal{H}_q^{s,\gamma} \oplus \mathcal{B}_q^{s,\gamma}) \).

**Proof.** As in the proof of Corollary 3.3, we consider the operator \( \tilde{A} \in \mathcal{C}^p_{E_0,F_0,E_1,F_1}(\mathbb{D},k) \) defined by (3.3). As \( A \) is invertible, so is \( \tilde{A} \). We infer from Theorem 60 that \( (\tilde{A})^{-1} \) belongs to \( \mathcal{C}^p_{E_1,F_1,E_0,F_0}(\mathbb{D},k) \) and hence \( A^{-1} \in \mathcal{C}^{-m,d'}_{E_1,F_1,E_0,F_0}(\mathbb{D},\gamma-m,\gamma,k) \). \( \square \)

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**Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508-090, São Paulo, SP, Brazil**

**E-mail address:** ppolpes@ime.usp.br

**Institut für Analysis, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany.**

**E-mail address:** schrohe@math.uni-hannover.de