Non-extremal Intersecting p-branes in Various Dimensions

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Abstract

Non-extremal intersecting p-brane solutions of gravity coupled with several antisymmetric fields and dilatons in various space-time dimensions are constructed. The construction uses the same algebraic method of finding solutions as in the extremal case and a modified “no-force” conditions. We justify the “deformation” prescription. It is shown that the non-extremal intersecting p-brane solutions satisfy harmonic superposition rule and the intersections of non-extremal p-branes are specified by the same characteristic equations for the incidence matrices as for the extremal p-branes. We show that \( S \)-duality holds for non-extremal p-brane solutions. Generalized \( T \)-duality takes place under additional restrictions to the parameters of the theory, which are the same as in the extremal case.

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A recent study of duality [1–5] and the microscopic interpretation of the Bekenstein-Hawking entropy within string theory [6, 7] has stimulated investigations of the intersecting (composite) p-brane solutions. An extension of the D-brane entropy counting to near-extremal black p-branes is a problem of the valuable interest. There has been recently considerable progress in the study of classical extremal [8–28] and non-extremal p-brane solutions [30–34] in higher dimensional gravity coupled with matter.

Heuristic scheme of construction of p-brane intersections was based on string theory representation of the branes, duality and supersymmetry. This construction involves the harmonic function superposition rule for the intersecting p-branes. The rule was formulated in [8] based on [10] for extremal solutions in $D = 11$ and $D = 10$ space-time dimensions. This rule was proved in [25] in arbitrary dimensions by using an algebraic method [18, 19, 22] of solutions of the Einstein equations.

It has been shown that there is a prescription for “deformation” of a certain class of extremal p-branes to give non-extremal ones [30, 31]. The harmonic function superposition rule for the intersecting p-branes has also been extended to non-extremal solutions with a single “non-extremality” parameter specifying a deviation from the BPS-limit [33]. Non-extremal black hole solutions from intersecting M-branes which are characterized by two non-extremal deformation parameters have found in [34].

Another approach to the construction of p-brane solutions was elaborated in the papers [18, 19, 22, 24, 25, 26, 28]. In these papers a general class of p-brane intersection solutions was found. One starts from the equations of motion and by using a special ansatz for the metric and the matter fields specified by the incidence matrix one reduces them to the Laplace equation and to a system of algebraic equations.

The aim of this letter is to extend this approach to the non-extremal case and to derive an explicit formula (40), see below, for solutions. We justify the harmonic function superposition rule and show that S- and T-dualities for intersecting non-extremal p-branes hold.

Let us consider the theory with the following action

$$I = \frac{1}{2\kappa^2} \int d^D X \sqrt{-g} \left( R - \frac{1}{2} (\nabla \phi)^2 - \sum_{j=1}^k \frac{e^{-\tilde{\alpha}^{(j)} \tilde{\phi}}}{(d_j + 1)!} F_{d_j+1}^{(j)} \right).$$

(1)

where $F_{d_j+1}^{(j)}$ is a $d_j + 1$ differential form, $F_{d_j+1}^{(j)} = dA_{d_j}$, $\phi$ is a set of dilaton fields.

Extremal solutions have been found starting from the following ansatz for the metric

$$ds^2 = e^{2F_i(x)} \eta_{i k} dx^i dx^k + e^{2B(x)} \sum_{\gamma=D-s-2}^{D-1} dx^\gamma dx^\gamma,$$

(2)

where $\eta_{\mu\nu}$ is a flat Minkowski metric.

In the non-extremal case we shall start from the following ansatz for metric. We choose a direction $i_0$ belonging to $0 < i < D - s - 3$ and consider the following ansatz

$$ds^2 = \sum_{i, k=0}^{D-s-3} e^{2F_i(r)} \eta_{i k} f^{d_{i_0}}(r) dx^i dx^k + e^{2B(r)} \left( r^2 d\Omega_{s+1}^2 + f^{-1}(r) dr^2 \right),$$

(3)

Here $r = \sqrt{x^i x^i}$ and $f(r)$ is an arbitrary function of $r$. We shall work in the following gauge

$$\sum_{L=0}^{D-3} F_L = 0,$$

(4)
where we assume $F_L = B$ for every $L = D - s - 2, \ldots, D - 1$. In the extremal case such gauge is Fock–De Donder one, but in non-extremal case it is not true.

To specify the ansatz for the antisymmetric fields we will use incidence matrices [25, 29]

$$\Delta^{(I)} = (\Delta_{aL}^{(I)}), \quad a = 1, \ldots, E, \quad L = 0, \ldots, D - 1,$$

$$\Lambda^{(I)} = (\Lambda_{bL}^{(I)}), \quad b = 1, \ldots, M, \quad L = 0, \ldots, D - 1. \quad (5)$$

The entries of the incidence matrix are equal to 1 or 0. Their rows correspond to independent branches of the electric (magnetic) gauge field and columns refer to the space-time indices of the metric (2). We assume $\Delta_{a0} = 1$, $\Delta_{a\alpha} = 0$, $\Lambda_{b0} = 0$, $\Lambda_{b\alpha} = 1$ and

$$F^{(I)} = \sum_{a=1}^E dA_{a}^{(I)} + \sum_{b=1}^M F_{b}^{(I)}, \quad (7)$$

where the coefficients of the differential forms $A_{a}^{(I)}$ and $F_{b}^{(I)}$ are

$$A_{a}^{(I)}_{M_1 \cdots M_{dI}} = \epsilon_{M_1 \cdots M_{dI} a(\alpha)} h_{\alpha} e_{D a} \prod_{i=1}^{dI} \Delta_{aM_i}, \quad (8)$$

$$F_{b}^{(I)}_{M_0 \cdots M_{dI}} = \epsilon_{M_0 \cdots M_{dI} \alpha} h_{\alpha} e_{D b} \prod_{i=0}^{dI} \Lambda_{bM_i}. \quad (9)$$

Here $\epsilon^{01 \cdots} = \epsilon_{01 \cdots} = 1$ are totally antisymmetric symbols, $D_a$ and $D_b$ are functions of $X^\alpha$. The product $\prod_{i=1}^{dI} \Delta_{aM_i}$ selects the non-zero components of electric part the form $A$ and the product $\prod_{i=0}^{dI} \Lambda_{bM_i}$ selects the non-zero components of magnetic components of its strength.

We will use Einstein equations in the form $R_{KL} = G_{KL}$, there $G_{KL}$ is related to the stress-energy tensor $T_{KL}$ as

$$G_{KL} = T_{KL} - \frac{g_{KL}}{D-2} T_P P. \quad (10)$$

For the above ansatz the tensor $G_{KL}$ is

$$G_{KL} = \frac{1}{2} \partial_K \phi \partial_L \phi + \sum_R \frac{h_R^2}{2} e^{2\tilde{F}_L - 2B + \tilde{F}_R} \left( -\partial_K D_R \partial_L D_R - \varsigma_R \eta_{KL} \left\{ \Delta_{RL} - \frac{d_R}{D-2} \right\} (\partial D_R)^2 \right), \quad (11)$$

where $R = a$ or $b$, $\Delta_{bL} = \Lambda_{bL},$

$$\tilde{F}_R = 2D_R - 2 \sum_{N=0}^{D-3} \Delta_{RN} F_N - \alpha \phi, \quad (12)$$

$$\varsigma_a = -1, \quad \varsigma_b = +1, \quad (13)$$

and

$$\tilde{F}_L \text{ is } F_L \text{ if } L \neq i_0, r, \quad \tilde{F}_{i_0} = F_{i_0} + 1/2 \ln f, \quad \tilde{F}_r = F_r - 1/2 \ln f. \quad (14)$$
Now let us suppose the following “no-force” condition

\[ \mathcal{F}_R \equiv 2C_R - 2 \sum_{N=0}^{D-3} \Delta_{RN} F_N - \alpha \phi = 0 \] (15)

where \( C_R \) is an one-center function

\[ e^{-C_R} \equiv H_R = 1 + \frac{Q_R}{r^s}, \] (16)

and \( Q_R \) is a constant.

Under “no-force” conditions (15) the field equations for the electric components and the Bianchi identities for magnetic components are reduced to

\[ \Box D_R + (\partial D_R, \partial(D_R - 2C_R)) = 0. \] (17)

This equation for the one-center functions \( C_R \) has the following solution

\[ e^{-D_R} \equiv \tilde{H}_R = 1 + \frac{Q_R + \mathcal{F}_R}{r^s - \mathcal{F}_R}, \] (18)

where \( \mathcal{F}_R \) is a constant.

Under the “no-force” conditions the \( G \)-tensor can be written in abridged notations as

\[ G_{KL} = g_{KL} \frac{s_R h_R^2}{2\kappa^2} \left[ \hat{\Delta}_{RK} - \frac{d_R}{D - 2} \right] e^{2(D_R + \mathcal{F}_R - C_R - B)(\partial D_R)^2}, \] (19)

\[ \hat{\Delta}_{RM} = \begin{cases} \Delta_{RK}, & K \neq D - 1 \\ \Delta_{RK} - s_R, & K = D - 1 \end{cases}. \] (20)

Field equation for dilaton reads

\[ f \Box \phi + (\partial \phi, \partial f) = - \sum_R s_R \alpha_R \frac{h_R^2}{2} e^{2(D_R - C_R)(\partial D_R)^2}. \] (21)

To solve the Einstein equations let us write the Ricci tensor for the metric (3) explicitly. For simplicity we calculate it in the stereographic parametrization of sphere,

\[ d\Omega_{s+1}^2 = \frac{4dz^{\hat{\alpha}}dz^{\hat{\beta}}}{(1 + z^{\hat{\gamma}}z^{\hat{\gamma}})^2}, \] (22)

at the point \( z^{\hat{\alpha}} = 0 \). We get

\[ R_{ij} = -g_{ij} e^{-2B} [f \Box F_i + (\partial F_i, \partial f)], \] (23)

\[ R_{i_0i_0} = -g_{i_0i_0} e^{-2B} \left[ f \Box F_{i_0} + (\partial F_{i_0}, \partial f) + \frac{\Delta f}{2f} \right], \] (24)

\[ R_{\dot{\alpha}\dot{\beta}} = -g_{\dot{\alpha}\dot{\beta}} e^{-2B} [f \Box F_{\dot{\alpha}} + (\partial F_{\dot{\alpha}}, \partial f)] + 4\eta_{\dot{\alpha}\dot{\beta}} [s(1 - f) - r \partial_r f], \] (25)

\[ R_{rr} = - \sum_{N=0}^{D-3} \partial_r A_N^2 - \eta_{rr} f^{-1} [f \Box B + (\partial F_{i_0}, \partial f)] - \frac{\Delta f}{2f}. \] (26)
To kill the last terms in (24)–(26) we suppose that
\[ s(1 - f) - r \partial_r f = 0. \]  
(27)

Then
\[ f = 1 - \frac{2\mu}{r^s}, \]  
(28)
where \( \mu \) is a constant.

Under condition (27) the Ricci tensor takes the form
\[ R_{\lambda\mu} = -g_{\lambda\nu}e^{-2B}[f \Box F_\lambda + (\partial F_\lambda, \partial f)], \]  
(29)
\[ R_{rr} = -\sum_{N=0}^{D-3} \partial_r A_N^2 - \eta_{rr}f^{-1}[f \Box B + (\partial F_{i0}, \partial f)]. \]  
(30)

Combining the Einstein equations and the field equation for dilaton we see that it is natural to set
\[ f \Box C_R + (\partial C_R, \partial f) - e^{2(D_R - C_R)}(\partial D_R)^2 = 0. \]  
(31)

Using the “no-force” conditions we derive equations for the constants \( h_R \) and the characteristic equations, which are the same as in the extremal case:
\[ (1 - \delta_{RR'}) \left\{ \frac{\tilde{\alpha}_R \tilde{\alpha}'_R}{2} - \frac{d_R d_{R'}}{D - 2} + \sum_{L=0}^{D-3} \Delta_R \Delta_{R'L} \right\} = 0, \]  
(32)

Substituting \( C_R \) and \( D_R \) into equation (31) we get
\[ \mathcal{F}_R = -Q_R \pm \sqrt{Q_R^2 + 2\mu Q_R}. \]  
(33)

As in the case non-extremal M-branes \[33\] it is convenient to parametrize the deformations of the harmonic functions \( H_R \) in the following way
\[ H_R \rightarrow \tilde{H}_R, \]  
(34)
\[ \tilde{H}_R = 1 + \frac{Q_R + \mathcal{F}_R}{r^s - \mathcal{F}_R}, \]  
(35)
\[ Q_R = 2\mu \sinh \gamma_R \cosh \gamma_R, \]  
(36)
\[ \mathcal{F}_R = 2\mu \sinh^2 \gamma_R, \]  
(37)

From Einstein equation for (rr)-component one can find the following restriction for \( i_0 \)
\[ \Delta_{ai0} = 1, \quad \Lambda_{bi0} = 0. \]  
(39)

Let us write the final expression for the metric
\[ ds^2 = \prod_{l=1}^{k} \left( H_1^{(l)} H_2^{(l)} \cdots H_{E_l}^{(l)} \right)^{2\alpha_{i(l)}^{(l)} r^{(l)}} \left( H_1^{(l)} H_2^{(l)} \cdots H_{M_l}^{(l)} \right)^{2\beta_{i(l)}^{(l)}} \]  
\[ \left\{ \sum_{L=0}^{D-3} \prod_{l=1}^{k} H_{a_l}^{(l)} \Delta_{ai_l}^{(l)} \prod_{b_l} H_{b_l}^{(l)} \right\}^{-\alpha_{i(l)}^{(l)}} f^{k-2} \eta_{KL} dy_K dy_L + d\Omega_{s+1}^2 + f^{-1} dr^2 \right\}, \]  
(40)
where
\[ t^{(I)} = \frac{D - 2 - d_I}{2(D - 2)}, \quad u^{(I)} = \frac{d_I}{2(D - 2)}. \] (41)

Constants in the ansatz for the antisymmetric fields have the form
\[ h_a^{(I)} = h_b^{(I)} = \sigma^{(I)}, \quad \text{where} \quad \sigma^{(I)} = \frac{1}{(t^{(I)} + u^{(I)})^2/4}. \] (42)

The incident matrices satisfy to the characteristic equations (32).

Let us note that the harmonic superposition rule is obvious from the expression (40).
Since the characteristic equations for the non-extremal case are the same as for the extremal case the duality properties are the same in both cases. In particular, \( S \)-duality takes place for all values of parameters, as to \( T \)-duality it takes place under the same restrictions on the parameters of the theory. Note that one can perform the \( T \)-duality transformation only along \( i \)-directions such that \( i \neq i_0 \).

All above results may be generalized for the space-time with an arbitrary signature. Kaluza-Klein theory with extra time-like dimensions has been considered in \([35]\). One deals with a modification of the action (1) in which the standard \( - \) signs before the \( F^2 \) terms are changed by \( -s_I \), where \( s_I = \pm 1 \), and \( \sqrt{-g} \) is changed by \( \sqrt{|g|} \).

In this case one has to replace in the metric (40) the term \( f^{-1}dr^2 \) by the \( \eta_{rr}f^{-1}dr^2 \), and set the following new equations
\[ r = \sqrt{|\eta_{\alpha\beta}x^\alpha x^\beta|} = \sqrt{\eta_{rr}\eta_{\alpha\beta}x^\alpha x^\beta}, \] (43)
\[ d\Omega_{s+1}^2 = \frac{4\eta_{\alpha\beta}dz^\alpha dz^\beta}{(1 + \eta_{rr}\eta_{\hat{\gamma}\hat{\gamma}}z^\gamma z^\gamma)^2}, \] (44)
\[ \eta_{\hat{\alpha}\hat{\beta}} = \text{diag}(\pm 1, \ldots, \pm 1), \] (45)
where the matrix
\[ \begin{pmatrix} \eta_{rr} & 0 \\ 0 & \eta_{\hat{\alpha}\hat{\beta}} \end{pmatrix}, \]
has the same signature as \( \eta_{\alpha\beta} \).

For the incidence matrix instead of two old restrictions
\[ \Delta_{a0} = 1, \quad \Lambda_{b0} = 0, \] (46)
one has
\[ s_I \prod_{L=0}^{D-1} (\eta_{LL})^{\Delta_{RL}} = s_R. \] (47)
So one can see that in the case of standard signature conditions (46) always guarantee the existence of \( i_0 \) (\( i_0 = 0 \) always exist), but in the general case the conditions (39) are not trivial.

To summarize, using an algebraic method of solution of Einstein equations in various dimensions we have constructed the non-extremal intersecting p-brane solutions (40) which satisfy the harmonic function superposition rule and possess \( S \)- and \( T \)-dualities. The intersections of non-extremal p-branes are controlled by the same characteristic equations as for the extremal cases.
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