The Mixmaster Cosmological Model as a Pseudo-Euclidean Generalized Toda Chain

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The question of the integrability of the mixmaster model of the Universe, presented as a dynamical system with finite degrees of freedom, is investigated in present paper. As far as the model belongs to the class of pseudo-Euclidean generalized Toda chains [1], the method of getting Kovalevskaya exponents developed in [2] for chains of Euclidean type, is used. The generalized formula of Adler and van Moerbeke [3] for systems of an indefinite metric is obtained. There was shown that although by the formula we got integer values of Kovalevskaya exponents there were multivalued solutions, branched at particular points on a plane of complex time $t$. This class of solutions differs from ones considered in [2]. Apparently, the system does not possess additional algebraic [3],[4] and one-valued [5] first integrals because of complex and transcendental values of the exponents. In addition, in the section 4 a ten-dimensional mixmaster model [6] was studied. There were not integer exponents.

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1 The Pseudo-Euclidean Generalized Toda Chains

The mixmaster model belongs to pseudo-Euclidean generalized Toda chains [1] on a level $H = 0$. The hamiltonian has a form

$$H = \frac{1}{2}(-p^2_\alpha + p^2_\beta + p^2_\gamma) + \exp(4\alpha) V(\beta_+, \beta_-), \quad (1.1)$$

where the potential function $V(\beta_+, \beta_-)$ is an exponential polynomial:

$$V(\beta_+, \beta_-) = \exp(-8\beta_+) + \exp(4\beta_+ + 4\sqrt{3}\beta_-) + \exp(4\beta_+ - 4\sqrt{3}\beta_-)$$

$$-2\exp(4\beta_+) - 2\exp(-2\beta_+ + 2\sqrt{3}\beta_-) - 2\exp(-2\beta_+ - 2\sqrt{3}\beta_-). \quad (1.2)$$

The hamiltonian has a form:

$$H = \frac{1}{2} \langle p, p \rangle + \sum_{i=1}^{N} c_i v_i, \quad (1.3)$$

where $\langle , \rangle$ is a scalar product in a Minkowski space $R^{1,n-1}$, $c_i$ are some real coefficients, $v_i \equiv \exp(a_i, q)$, $( , )$ is a scalar product in a Euclid space $R^n$, $\vec{a}$ are real vectors. Pseudoeuclidity of a momentum space is a distinctive peculiarity of gravitational problems so they cannot be referred as analytical dynamics problems, where a quadratic by momenta form is a kinetic energy.

On the other hand, the cosmological models can be considered as dynamical systems. So it is possible to carry over strict methods of analysis, traditionally used in the analytical mechanics, and adapt them to the systems like (1.3). For analysis of integrability of the cosmological model which is a hamiltonian system with three degrees of freedom, let us apply the Painlevé test (see e.g. [7]) for calculation of Kovalevskaya exponents [8]. The term "Kovalevskaya exponents" was introduced in paper [4] thus it was marked an outstanding contribution of the Russian woman into solution of the important problem of dynamics-integration of a task of a rigid body rotation. It was found that a general solution is meromorphic if and only if the system possesses an additional one-valued [9] or algebraic [9] first integral. Further development of the theory of Kovalevskaya exponents is contained in [10] [11].

By expanding $2n$-dimensional phase space to $2N$-dimensional one by homeomorphism $(p, q) \mapsto (v, u)$:

$$v_i = \exp(a_i, q), \quad u_i = \langle a_i, p \rangle, \quad i = 1, ..., N,$$

(1.4)

generalizing the Flaska transformation [12], one gets a hamiltonian system whose movement equations are system of autonomic homogeneous differential equations with polynomial right side:

$$\dot{v}_i = u_i v_i,$$

$$\dot{u}_i = \sum_{j=1}^{N} M_{ij} v_j, \quad i = 1, ..., N. \quad (1.5)$$

The matrix $\hat{M}$ is constructed of scalar products of vectors $\vec{a}$ in a Minkowski space $R^{1,n-1}$:

$$M_{ij} \equiv -c_j <a_i, a_j> \quad . \quad (1.6)$$
The property of integrability of dynamical systems appears in a character of singularities of solutions what it is not possible to say about points of common position, so just singular points represent particular interest for investigation. The equations (1.5) have the following partial solutions:

\[ u_i = U_i/t, \quad v_i = V_i/t^2, \quad i = 1, 2, \ldots, N, \]  

(1.7)

coefficients \( U_i, V_i \) obey a system of algebraic equations

\[-2V_i = U_i V_i, \]

\[-U_i = \sum_{j=1}^{N} M_{ij} V_j. \]  

(1.8)

Now let us analyze all types of solutions of the system (1.8) in detail.

I. Let \( V_1 \neq 0 \), the rest \( V_2, V_3, \ldots, V_N = 0 \), then we get a solution: if \( M_{11} \neq 0 \), then \( V_1 = 2/M_{11}, U_1 = -2, U_2 = -2M_{21}/M_{11}, \ldots, U_N = -2M_{N1}/M_{11} \). Analogously the last solutions will be obtained. If for some \( i : V_i \neq 0 \), and \( V_j = 0 \) for all \( j \neq i \), then if \( M_{ii} \neq 0 \), and we get \( U_i = -2, U_j = -2M_{ji}/M_{ii} \) for all \( i \neq i \). It follows from the obtained solutions that the significant point of analysis is a nonequality of a corresponding diagonal element of the matrix \( \hat{M} \) to zero that is possible in case of an isotropy of a vector \( a_i \). It is a principal distinctive feature of pseudoeuclidean chains.

II. Let us pass to the next series of solutions. Let \( V_1 \neq 0 \) and \( V_2 \neq 0 \), the other \( V_i = 0 \), then \( U_1 = U_2 = -2 \), \( V_1 \) and \( V_2 \) are found from a system

\[ 2 = M_{11} V_1 + M_{12} V_2, \]

\[ 2 = M_{21} V_1 + M_{22} V_2. \]  

(1.9)

The condition of existence of solutions of the system (1.9) is a nondegeneracy of a matrix \( \hat{M}_{12} \). If \( \det \hat{M}_{12} \neq 0 \), then, obtaining from (1.9) values \( V_1 \) and \( V_2 \) and putting them in (1.8), one gets \( U_3, U_4, \ldots, U_N : \)

\[-U_3 = M_{31} V_1 + M_{32} V_2, \]

\[-U_4 = M_{41} V_1 + M_{42} V_2, \]  

(1.10)

\[ \ldots \]

\[-U_N = M_{N1} V_1 + M_{N2} V_2. \]

By the same way we obtain the rest of solutions of this series searching through all possible nonzero pairs \( V_i \neq 0, V_j \neq 0 \). The corresponding condition of existence of solutions is a nondegeneracy of a two-dimensional matrix: \( \det \hat{M}_{ij} \neq 0 \), constructed of elements \( M_{ii}, M_{ij}, M_{ji}, M_{jj} \).

III. The following series of solutions are obtained by considering all nonzero triple solutions \( V_i, V_j, V_k \) with a condition of nondegeneracy of corresponding three-dimensional matrices. The last possible solution is: \( U_i = -2, i = 1, 2, \ldots, N, V_i = -(M_{ij})^{-1} U_j, \det \hat{M} \neq 0. \)
2 A Calculation of Kovalevskaya Exponents

For investigation of a single-valuedness of obtained solutions we use Lyapunov method \[2\] based on studying of behaviour of solutions of equations in variations. First variations obey the following differential equations:

\[
\frac{d}{dt}(\delta u) = \sum_{j=1}^{N} M_{ij} \delta v_j,
\]

\[
\frac{d}{dt}(\delta v_i) = \frac{U_i \delta v_i}{t} + \frac{V_i \delta u_i}{t^2}, \quad i = 1, ..., N.
\] (2.1)

We seek their solutions in the form of

\[
\delta u_i = \xi_i t^{\rho - 1}, \quad \delta v_i = \eta_i t^{\rho - 2}, \quad i = 1, ..., N.
\] (2.2)

Then for searching of coefficients \(\xi_i, \eta_i\) one gets a linear homogeneous system of equations with a parameter \(\rho\):

\[
(\rho - 2 - U_i) \eta_i = V_i \xi_i,
\]

\[
(\rho - 1) \xi_i = \sum_{j=1}^{N} M_{ij} \eta_j, \quad i = 1, ..., N.
\] (2.3)

Values of a parameter \(\rho\) are called Kovalevskaya exponents.

I. Let us consider solutions of the first series when \(V_1 \neq 0\). Let \(\eta_1 \neq 0\) and the rest \(\eta_i = 0\), then from the first system of equations (2.3) one gets \(\xi_1 = M_{11} \rho \eta_1 / 2\), a substitution of it to the second system (2.3) gives a condition on values of a parameter \(\rho\): \(\rho(\rho - 1) - 2 = 0\), i.e. \(\rho_1 = -1, \rho_2 = 2\). The remaining equation (2.3) gives us solutions \(\xi_i = \xi_i(\eta_1, \rho)\).

Let then \(\eta_2 \neq 0\), then \(\eta_3, \eta_4, ..., \eta_N = 0, \rho = 2 - 2M_{21}/M_{11}, \rho\eta_1 = 2\xi_1/M_{11}\). The second system gets functions \(\xi_i = \xi_i(\eta_2), i = 1, 2, ..., N\).

Further, considering \(\eta_3 \neq 0\), we receive \(\rho = 2 - 2 < a_3, a_1 > < a_1, a_1 >\); \(\eta_2, \eta_4, ..., \eta_N = 0; \rho\eta_1 = 2\xi_1/M_{11}\); and from the second system: \(\xi_i = \xi_i(\eta_2), i = 1, 2, ..., N\).

As a result, having looked over all solutions of the first series for a case \(V_1 \neq 0\), we obtain a formula for a spectrum \(\rho: \rho = 2 - 2 < a_i, a_1 > / < a_1, a_1 >, i = 2, 3, ..., N\).

As a final result, having considered the rest solutions of the first series, we obtain a formula for Kovalevskaya exponents \(\rho\) that is the formula of Adler and van Moerbeke for the case of indefinite spaces:

\[
\rho = 2 - 2 \frac{< a_i, a_k >}{< a_k, a_k >}, \quad i \neq k, \quad < a_k, a_k > \neq 0.
\] (2.4)

The requirement \(\rho \in \mathbb{Z}\) is a necessary condition of a meromorphy of solutions on a complex plane of \(t\). It should be noticed that under derivation of the formula (2.4) no restriction on a metric signature was imposed. It is correct not only for spaces of the Minkowski signature.

II. The following series of solutions do not lead to the similar elegant result. But we may show that solutions for variations exist in principle. So, for example, let us take a solution of the second series when \(V_1 \neq 0, V_2 \neq 0\). Then we have:

\[
\rho \eta_1 = V_i \xi_i.
\]
\[ \rho \eta_2 = V_2 \xi_2, \]
\[ (\rho - 2 - U_3) \eta_3 = 0, \]
\[ \ldots \ldots \]
\[ (\rho - 2 - U_N) \eta_N = 0, \]
\[ (\rho - 1) \xi_i = \sum_{j=1}^{N} M_{ij} \eta_j, \quad i = 1, \ldots, N. \]

Let us assume, for instance, \( \eta_3 \neq 0 \), then \( \eta_4, \ldots, \eta_N = 0 \), and therefore \( \rho = 2 + U_3 \). Out of the first pair of equations, expressing \( \eta_1 = V_1 \xi_1 / \rho, \) \( \eta_2 = V_2 \xi_2 / \rho \), we substitute them to the second system. We obtain:

\[ (\rho(\rho - 1) - M_{11} V_1) \xi_1 = M_{12} V_2 \xi_2 + \rho M_{13} \eta_3, \]
\[ (\rho(\rho - 1) - M_{21} V_1) \xi_1 = M_{22} V_2 \xi_2 + \rho M_{23} \eta_3, \]
\[ \rho(\rho - 1) \xi_3 = M_{31} \xi_1 + M_{32} \xi_2 + M_{33} \rho \eta_3. \]
\[ \ldots \ldots \]
\[ \rho(\rho - 1) \xi_N = M_{N1} \xi_1 + M_{N2} \xi_2 + M_{N3} \rho \eta_3. \]

The first two equations represent a system of nonhomogeneous linear equations, whence we get solutions \( \xi_1 = \xi_1(\eta_3), \) \( \xi_2 = \xi_2(\eta_3) \). Substituting them to the rest equations, we get \( \xi_3 = \xi_3(\eta_3), \ldots, \xi_N = \xi_N(\eta_3) \). By the same way the other solutions are to be obtained.

III. For the last solution when all \( U_i = -2 \) for \( i = 1, 2, \ldots, N \), and \( V_i \) are solutions of a matrix equation

\[ 2 = \sum_{j=1}^{N} M_{ij} V_j \]

for the nondegenerate matrix \( \hat{M} \), we receive an eigenvalue problem:

\[ \sum_{j=1}^{N} (M_{ij} V_j - \delta_{ij} \rho(\rho - 1)) \xi_j = 0. \]

As it is seen from (2.7), a sum of elements of any line of a matrix \( M_{ij} V_j \) is two. Out of this property of the matrix we obtain that one eigenvector \( \xi_i \) is unit with an eigenvalue \( \rho(\rho - 1) = 2 \). From here we obtain already known roots: \( \rho = -1, 2 \).

### 3 The Mixmaster Model of the Universe

Now let us apply the method to analysis of integrability of the mixmaster model of the Universe, ”root vectors” of which have a form (see (1.2)):

\[ a_1(4, -8, 0), \quad a_4(4, 4, 0), \]
\[ a_2(4, 4, 4 \sqrt{3}), \quad a_5(4, -2, 2 \sqrt{3}), \]
\[ a_3(4, -8, 0), \quad a_6(4, 4, 0). \]
\[ a_3(4, 4, -4\sqrt{3}), \quad a_6(4, -2, -2\sqrt{3}). \]

"Cartan matrix", composed of scalar products of the vectors (3.1) in Minkowski space, then has a form:

\[
<a_i, a_j> = \begin{pmatrix}
1 & -1 & -1 & -1 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & -1 \\
-1 & -1 & 1 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & -1/2 & -1/2 \\
0 & 0 & -1 & -1/2 & 0 & -1/2 \\
0 & -1 & 0 & -1/2 & -1/2 & 0
\end{pmatrix} \cdot (3.2)
\]

One gets, three "root vectors" dispose out of a light cone (space-like vectors), the rest three are isotropic, are situated on the light cone. Using the generalised Adler-van Moerbeke formula (2.4), taking account of zero norm of three vectors, we get \(\rho_1 = 2, \rho_2 = 4\), i.e. they are integer.

Now we construct a matrix \(\hat{M}\) by the formula (1.6), using a vector \(c_i(1, 1, 1, -2, -2, -2)\):

\[
M_{ij} = \begin{pmatrix}
-1 & 1 & 1 & -2 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & -2 \\
1 & 1 & -1 & 0 & -2 & 0 \\
1 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 1 & -1 & 0 & -1 \\
0 & 1 & 0 & -1 & -1 & 0
\end{pmatrix} \cdot (3.3)
\]

By virtue of a specific form of the matrix (3.3), it is not difficult a consideration of all partial solutions of the second type, when a pair \(V_i, V_j\) under some \(i, j\) is not zero. One may notice that there is only one, by virtue of nondegeneracy of blocks, nontrivial solution: \(V_1 \neq 0, V_4 \neq 0\). It has a form: \(V_1 = 1/24, V_4 = -1/24, U_i = -2, i = 1, \ldots, 6\).

Equations on coefficients of variations of solutions are:

\[
(\rho(\rho - 1) + 2)\eta_1 + 4\eta_4 = 0,
\]

\[
2\eta_1 + \rho(\rho - 1)\eta_4 = 0,
\]

\[
(\rho - 1)\xi_2 = M_{21}\eta_1,
\]

\[
(\rho - 1)\xi_3 = M_{31}\eta_1,
\]

\[
(\rho - 1)\xi_5 = M_{54}\eta_4,
\]

\[
(\rho - 1)\xi_6 = M_{64}\eta_4.
\]

The condition of existence of solutions of the system of the first pair of equations is a factorized algebraic equation of a fourth order on to a spectral parameter \(\rho\):

\[
[\rho(\rho - 1) - 2][\rho(\rho - 1) + 4] = 0. \quad (3.5)
\]

The solutions of the first square equation are integer: \(\rho_1 = -1, \rho_2 = 2\), but roots of the second equation

\[
\rho_{1,2} = (1 \pm i\sqrt{15})/2 \quad (3.6)
\]
are complex and irrational. Thus, in a common case, a solution is not one-valued on the complex plane of time $t$. Under some additional conditions \[3,\overline{3},\overline{4},\overline{5}\], it leads to nonexistence additional algebraic or one-valued first integrals. Let us remark that meromorphy of a common solution is a criterion of existence of an algebraic first integral in dynamics of a rigid body (Gussont theorem) and in a problem of three bodies (Brunce theorem).

4 A Mixmaster Model with Geometry $R^1 \times S^3 \times S^3 \times S^3$

In paper \[6\] a model of a spatially homogeneous and isotropic vacuum universe with geometry $R^1 \times S^3 \times S^3 \times S^3$ was studied. A ten-dimensional model was chosen due to promising advances in a ten-dimensional superstring theory. Its hamiltonian in Misner coordinates $(\alpha, \beta, \theta, \phi, \eta, \psi)$ is:

$$H = -\frac{p_0^2}{24} + \sum_{j=1}^{8} \frac{p_j^2}{2} + \frac{e^{16\alpha}}{2} [e^{-4\beta/\sqrt{3}} g(\beta) + e^{2\beta/\sqrt{3}} (e^{-2\eta} g(\phi) + e^{2\eta} g(\psi))],$$

(4.1)

where $g(x)$ means a function

$$g(x) \equiv e^{4x_+ + 4\sqrt{3}x_-} + e^{4x_+ - 4\sqrt{3}x_-} + e^{-8x_+} - 2e^{4x_+} - 2e^{2x_+ + 2\sqrt{3}x_-} - 2e^{2x_+ - 2\sqrt{3}x_-}. \quad (4.2)$$

The problem also belongs to the generalized pseudo-Euclidean Toda chains. Its potential energy is a sum of 18 terms:

$$U = \sum_{l=1}^{18} c_l \exp(a_l, q).$$

(4.3)

So 18 "root vectors" $\vec{a}$ in 9-dimensional Euclidean space are:

$$a_1(16, 4, 4\sqrt{3}, -4/\sqrt{3}, 0, 0, 0, 0),$$
$$a_2(16, 4, -4\sqrt{3}, -4/\sqrt{3}, 0, 0, 0, 0),$$
$$a_3(16, -8, 0, -4/\sqrt{3}, 0, 0, 0, 0),$$
$$a_4(16, 4, 0, -4/\sqrt{3}, 0, 0, 0, 0),$$
$$a_5(16, 2, 2\sqrt{3}, -4/\sqrt{3}, 0, 0, 0, 0),$$
$$a_6(16, 2, -2\sqrt{3}, -4/\sqrt{3}, 0, 0, 0, 0),$$
$$a_7(16, 0, 0, 2/\sqrt{3}, 4, 4\sqrt{3}, -2, 0, 0),$$
$$a_8(16, 0, 0, 2/\sqrt{3}, 4, -4\sqrt{3}, -2, 0, 0),$$
$$a_9(16, 0, 0, 2/\sqrt{3}, -8, 0, -2, 0, 0),$$
$$a_{10}(16, 0, 0, 2/\sqrt{3}, 4, 0, -2, 0, 0),$$
$$a_{11}(16, 0, 0, 2/\sqrt{3}, 2, 2\sqrt{3}, -2, 0, 0),$$
$$a_{12}(16, 0, 0, 2/\sqrt{3}, 2, -2\sqrt{3}, -2, 0, 0),$$

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As before, we pass to redundant coordinates. Then get (1+8)-pseudo-Euclidean space with metric $\eta_{ik} = \text{diag}(-1/12, 1, ..., 1)$. By the same way as in section 3 we calculate Kovalevskaya exponents using the Adler-van Moerbeke formula (2.4). But besides integer numbers $\rho = 0, 2, 3, 4$ there are rational exponents $\rho_1 = 4/3$, $\rho_2 = 10/3$. Apparently the considered model is nonintegrable.

5 Discussion of Results

A search of first integrals in the form of algebraic and one-valued functions is too restrictive. A more interesting question is about existence of additional real-analytic integrals. Such a statement of the question goes back to Poincaré. For Euclidean Toda chains a question about existence of additional real-analytic polynomials by momenta first integrals was studied in paper [13] where a classification of integrable systems was brought.

Furthermore, in this case, we are not interested in integrability at the all phase space but at a concrete hypersurface "integral of energy" $H = 0$.

The next step of investigation of the presence of chaotic characteristics is a consideration of "truncated" model which to be obtain if one of the degrees of freedom of the mixmaster model is fixed. I.e. it is obtained if we rewrite a potential part of the hamiltonian (1.2), introducing coefficients in front of its terms

$$V(\beta_+, \beta_-) = C_1 \exp(-8\beta_+) + C_2 \exp(4\beta_+ + 4\sqrt{3}\beta_-) + C_3 \exp(4\beta_+ - 4\sqrt{3}\beta_-) - 2C_4 \exp(4\beta_+) - 2C_5 \exp(-2\beta_+ + 2\sqrt{3}\beta_-) - 2C_6 \exp(-2\beta_+ - 2\sqrt{3}\beta_-),$$

(5.1)

and put $C_2 = C_3 = C_5 = C_6 = 0$. Such a hamiltonian system in a four-dimensional phase system can be studied with the help of the numerical analysis of Poincaré mapping if its phase trajectories are situated in a compact region. An existence of chaos in the "truncated" model implies a stochasticity of the initial model. Let us remark that a calculation by the generalized formula of Adler and van Moerbeke (2.4) for this case yields integer $\rho$, as to additional exponents, they are also complex and irrational, coincide with (3.6).

Because the Killing metric in the generalized Adler-van Moerbeke formula is indefinite it should be pointed out at a classification scheme of noncompact Lie algebras by analogy with using the Cartan scheme of classification of compact simple Lie algebras for getting exact solutions of Toda lattices as it was done in [14].

It is worth to mention paper [15], where was made an attempt of proof of complicated behaviour of the mixmaster model trajectories out of an analysis of level lines of the potential energy function.
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