The Distribution of the Sum of Independent Generalized Gaussian Signals

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Abstract. The distribution of the sum of independent generalized Gaussian signals is studied in the paper. The properties of the sum of signals in detail are firstly analyzed. The conclusion that the distribution of the sum of generalized Gaussian signals is not generalized Gaussian distribution was obtained; the reasoning for the special case is given to support the proposed conclusion. The simulation results show that applying generalized Gaussian distribution to model the distribution of the sum based on high order statistics, they coincide with each other well except for the vicinity of mean.

Introduction

Generalized Gaussian Distribution (GGD) has a symmetric unimodal density characteristic with variable tail length and contains the Laplacian, Gaussian and uniform distributions as special cases. Therefore, GGD can be used to describe many kinds of noise [1-3], and to model the Probability Density Function (PDF) of transform and subband coefficients such as DCT [4] and DWT [5-6] coefficients. So, the signals with generalized Gaussian distribution (GGD) became the research hotspot in the field of signal processing and information security in recent years. Sometimes the sum of independent generalized Gaussian signals need be studied, or the linear system driven by generalized Gaussian white noise, whose PDF is approximated by PDF of generalized Gaussian [7]. It is well known the GGD has analytical expression for its probability density function. However, our research shows such is not the case for the sum of independent GG signals. The paper will show that the sum of independent GG signals is no longer GGD, and a new distribution similar to GGD.

Generalized Gaussian Distribution

A detailed treatment of GGD can be found in [8]. The generalized Gaussian PDF [9] is given by

$$f(x; \mu, \beta, \alpha) = \frac{\alpha}{2\beta^{\alpha}} \exp \left\{ -\frac{|x-\mu|^\alpha}{\beta} \right\}, \quad x \in \mathbb{R}$$

(1)

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$, $\beta = \frac{\sigma^{\alpha} \Gamma(1/\alpha)}{\Gamma(3/\alpha)}$, and $\mu$, $\alpha$, $\sigma^2$ are mean, shape parameter and variance respectively. The shape parameter $\alpha > 0$ determines the rate of exponential decay of the PDF. Note that for $\alpha = 2$, the density reduces to the Gaussian density, whereas for $\alpha = 1$ it becomes the Laplacian density. Some members of the class are illustrated in Figure 1 for $\mu = 0$, $\sigma = 10$, $\alpha = 0.7, 1, 2$ and 100.

![PDF of Generalized Gaussian family.](image-url)
Based on the PDF of GGD in (1), we can characterize some of the properties as follows.

(1.a) Symmetry $f(\mu - x) = f(\mu + x)$;
(2.a) Monotonicity The PDF is strictly monotone increasing for $x \leq \mu$, strictly monotone decreasing for $x \geq \mu$, and reaches the maximum for $x = \mu$.

(3.a) Convexity Since $f^\prime\prime(x) = \frac{\alpha^2}{2\beta^3} \Gamma(\frac{1}{\alpha}) \exp\left(-\left(x - \frac{\mu}{\beta}\right)^\gamma\right) [x - \mu]^{\gamma-2} \left[\frac{\alpha}{\beta^n} |x - \mu|^{-\gamma} (\gamma - 1)\right]$, it is obvious that

if $f^\prime(x) > 0$ for any $x$. If $\alpha > 1$, $f^\prime(x) > 0$, for $|x - \mu| > \left(1 - \frac{1}{\alpha}\right)^{\gamma} \beta$ and $f^\prime(x) < 0$ for $|x - \mu| < \left(1 - \frac{1}{\alpha}\right)^{\gamma} \beta$.

(4.a) $f(\mu) = \frac{\alpha}{2\beta^3(\frac{1}{\alpha})}$.

(5.a) Due to the symmetry of the PDF in (1), the odd moments $EX^{2k+1}$ vanish,

where $X$ is with zero mean. The even moments are given as follows

$$EX^{2k} = \beta^{2k} \frac{\Gamma(2k+1/\alpha)}{\Gamma(1/\alpha)}$$

(6.a) Eq. (2) and the cumulant properties give the $n$th-order cumulant $C_x^n$

$$C_x^n = \sigma^2 \cdot C_x = \sigma^2 \left(\frac{\Gamma(5/\alpha) \Gamma(1/\alpha)}{\Gamma^2(3/\alpha)} - 3\right), C_x^{2k+1} = 0$$

Sum of Independent Signals with GGD

The sum of independent signals with GGD in this Section is considered. Suppose $Z = X + Y$ and let the density functions of $X$ and $Y$ be $f_1(x)$ and $f_2(x)$ respectively, where $X$ and $Y$ are independent GGD random variables with respective parameters $(\mu_1, \beta_1, \alpha_1)$ and $(\mu_2, \beta_2, \alpha_2)$. Then the PDF of $Z$ is

$$f_z = f_1 * f_2$$

Thus, we can get some properties of $Z$ as follows

(1.b) Symmetry $f_z(\mu_1 + \mu_2 - z) = f_z(\mu_1 + \mu_2 + z)$
(2.b) Monotonicity

**Lemma 1**: $f_z(z)$ is strictly monotone increasing for $z \leq \mu_1 + \mu_2$, strictly monotone decreasing for $z \geq \mu_1 + \mu_2$, and reaches the maximum at $z = \mu_1 + \mu_2$.

(3.b) Convexity

$$f_z^\prime\prime(z) = \int_{-\infty}^{\infty} f_z(x)f_z^\prime(z-x)dx = \frac{\alpha_1 \alpha_2}{4\beta_1 \beta_2 \Gamma(\alpha_1/\alpha) \Gamma(\alpha_2/\alpha)} \left[\frac{\alpha_1}{\beta_1^n} \right] \left[\frac{\alpha_2}{\beta_2^n} \right] \left[\frac{\alpha_1}{\beta_1^n} \right] \left[\frac{\alpha_2}{\beta_2^n} \right] |x - \mu_1|^{\gamma-2} \left[\frac{\alpha_1}{\beta_1^n} |x - \mu_1|^{-\gamma} (\gamma - 1)\right] dx$$

It implies that

if $\alpha_i \leq 1$, $f_z^\prime(z) > 0$ for any $x$. Furthermore, we note that $\alpha_i$ is arbitrary. That is to say, if anyone of $\alpha_i$ and $\alpha_j$ is

less than or equal to 1, then $f_z^\prime(z) > 0$.

**Lemma 2**: $f_z^\prime(z) > 0$ if at least one of $\alpha_i$ and $\alpha_j$ is least than or equal to 1.

(4.b) $f_z(\mu_1 + \mu_2) = \frac{\alpha_1 \alpha_2}{4\beta_1 \beta_2 \Gamma(\alpha_1/\alpha) \Gamma(\alpha_2/\alpha)} \left[\frac{\alpha_1}{\beta_1^n} \right] \left[\frac{\alpha_2}{\beta_2^n} \right] \left[\frac{\alpha_1}{\beta_1^n} \right] \left[\frac{\alpha_2}{\beta_2^n} \right] |x - \mu_1|^{\gamma-2} \left[\frac{\alpha_1}{\beta_1^n} |x - \mu_1|^{-\gamma} (\gamma - 1)\right] dx$

(5.b) Consider $X$ and $Y$ with zero means. Then the moments of $Z$ follow that $EZ^{2n+1} = EX^{2n+1} Y^{2n+1} = 0$.

$$EZ^{2n} = \sum_{i=0}^{2n} C_{2n-i} C_x E X^{2n-i} Y^{2n-k} = \sum_{i=0}^{2n} C_{2n-i} C_x E X^{2n-i} Y^{2n-k}, EX^{2n}$ and $EY^{2n}$ are showed in Eq. (2).

(6.b) Using the property of cumulant, the nth-order cumulant $C_z^n$ is $C_z^{2n+1} = 0$, $C_z^{2n+2} = C_x^{2n+2} + C_y^{2n+2}$

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If we consider the special model $Z = X + cY$, where $c$ is a constant and $X$, $Y$ are iid with the parameters $(\mu, \beta, \alpha)$, thus we can also get the same properties of $Z$ from (1.b) to (3.b) motioned above. Similarly, considering the GG signals $X$ and $Y$ with zero means, properties from (4.c) to (6.c) are as follows

(4.c) $f_z(0) = \frac{\alpha}{(1 + |z|^\alpha)^{1/\alpha}}$. 

(5.c) $EZ^{2\alpha+1} = 0$, $EZ^{2\alpha} = \sum_{k=0}^{\infty} C_x^{2\alpha} 2^{2\alpha} \frac{\Gamma(2k+1/\alpha)\Gamma(n-2k+1/\alpha)}{\Gamma(2\alpha)\Gamma(n/\alpha)}$

(6.c) $C_x^{2\alpha+1} = 0$, $C_x^{2\alpha} = (1 + c^{2\alpha}) C_x^{2\alpha}$

From (6.c) and (7.c), we notice that if we denote $Z_1 = X + cY$ and $Z_2 = X - cY$, then the nth-order moments and cumulants of $Z_1$ and $Z_2$ are the same for any $n$. For the relationship between moment and eigenfunction, $Z_1$ and $Z_2$ have the same distribution.

**Lemma 3:** Let $Z_1 = X + cY$ and $Z_2 = X - cY$. Then $Z_1$ and $Z_2$ have the same distribution if GG random variables $X$ and $Y$ are iid with zero means, where $c$ is a constant. Furthermore, if let $c = 1$, properties from (4.c) to (6.c) can be simplified as follows

(4.d) $f_z(0) = 2^{\alpha/2} \frac{\alpha}{2\beta^{(1/\alpha)}}$,

(5.d) $EZ^{2\alpha+1} = E(X+Y)^{2\alpha+1} = 0$

$EZ^{2\alpha} = \sum_{k=0}^{\infty} C_x^{2\alpha} 2^{2\alpha} \frac{\Gamma(2k+1/\alpha)\Gamma(n-2k+1/\alpha)}{\Gamma(2\alpha)\Gamma(n/\alpha)}$, 

(6.d) $C_z^{2\alpha+1} = 0$, $C_z^{2\alpha} = 2C_x^{2\alpha+2}$

use the Cumulant-Moment formula to obtain the cumulants of $Z$. But it is more convenient to get $C_z^{\alpha}$ with (6.d).

**Distribution of the Sum of GGD Signals**

Comparing the properties of $Z$ mentioned in Section III with those of GGD in Section II, it is obvious that there are many similarities between them. Therefore, the question of interest is whether $Z$ has generalized Gaussian distribution. We will show that the distribution of $Z$ cannot be GGD, but a new distribution similar to GGD.

According to the PDF definition in (3), it is impossible to express the density of $Z$ for any $\alpha$ with an analytically exact expression. So we consider the special case that $X$ and $Y$ are independent identical distribution (i.i.d) GGD with $\alpha = 1$ and $\mu = 0$ now. Since $f_z(z)$ is symmetric, we have the analytical expression of the density

$$f_z(z) = \left( \frac{|z|}{4\beta^2} + \frac{1}{4\beta} \right) \exp \left( -\frac{|z|}{\beta} \right)$$

Therefore $f_z(0) = \frac{1}{4\beta^2}$, which satisfies (4.d) for $\alpha = 1$. Comparing Eq.(4) with Eq.(1), it is clear that the distribution of the sum of two Laplacian distributions is not GGD. In order to show that the conclusion is also true for the other values of $\alpha$, we give a proof of the case for $\alpha = \frac{1}{2}$

**Lemma 4:** Let $Z = X + Y$. Then $Z$ cannot be a GG random variable if GG random variables $X$ and $Y$ are i.i.d with mean zero, shape parameter $\alpha = \frac{1}{2}$ and variance $\sigma^2 = 1$.

**Proof** Prove it with reduction to absurdity, assume $Z$ to be a GG random variable. Hence, the PDF of $Z$ satisfies Eq. (1) under this hypothesis. Let $t$ denote the shape parameter of $Z$. Using the properties of GGD (4.a)-(6.a), we obtain
\[ f_z(0) = \frac{t}{2^\beta \Gamma(\frac{1}{\beta})}, \quad C_i^2 = \sigma_i^4 \left( \frac{\Gamma(\frac{5}{i}) \Gamma(\frac{1}{i})}{\Gamma^2(\frac{3}{i})} - 3 \right) \] where \( \rho_2 = \sigma_2 \sqrt{\frac{\Gamma(\frac{1}{\beta})}{\Gamma^2(\frac{3}{i})}} \). On the other hand, for the properties 4.d)-6.d), we have

\[ f_z(0) = \frac{2^{-\frac{\alpha}{\beta}}}{\sqrt{2\pi}} \left( \frac{\Gamma(\frac{5}{i}) \Gamma(\frac{1}{\alpha})}{\Gamma^2(\frac{3}{\alpha})} \right), \sigma_z^2 = 2 \sigma^2; \quad C^2 = 2 \sigma^4 \left( \frac{\Gamma(\frac{5}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma^2(\frac{3}{\alpha})} - 3 \right) \]. So, we get two equations

\[ \frac{r^2 \Gamma(\frac{3}{i})}{2 \Gamma(\frac{1}{i})} = \frac{2^{\frac{\alpha}{\beta}} \Gamma(\frac{3}{\alpha})}{2^{\frac{\alpha}{\beta}} \Gamma(\frac{1}{\alpha})} \quad (5) \]

\[ 2 \left( \frac{\Gamma(\frac{5}{i}) \Gamma(\frac{1}{i})}{\Gamma^2(\frac{3}{i})} \right) = \frac{\Gamma(\frac{5}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma^2(\frac{3}{\alpha})} + 3 \quad (6) \]

When \( \alpha = \frac{1}{2} \), Eqs. (5)-(6) become

\[ \frac{r^2 \Gamma(\frac{3}{i})}{\Gamma(\frac{1}{i})} = \frac{15}{4} \quad (7) \]

\[ \frac{\Gamma(\frac{5}{i}) \Gamma(\frac{1}{i})}{\Gamma^2(\frac{3}{i})} = \frac{141}{10} \quad (8) \]

It shows that \( p(\alpha) = \frac{\Gamma(\frac{5}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma^2(\frac{3}{\alpha})} \) is strictly monotone decreasing, and \( q(\alpha) = \frac{\alpha^{2} \Gamma(\frac{3}{\alpha})}{\Gamma(\frac{1}{\alpha})} \) is also strictly monotone decreasing, which is plotted in Figure 2. We have \( p(0.75) < p(t) \), \( q(0.75) > q(t) \). Hence, Eq.(7) implies \( t > 0.75 \), while Eq.(8) implies \( t < 0.75 \). There is a contradiction. Therefore, \( Z \) cannot be a GG random variable.

It is clear that if \( Z \) is GG, the equations (7) and (8) must have the same solution. Solving equations (7) and (8) We can get the solutions \( t \approx 0.8033 \) and \( t \approx 0.6260 \) respectively, which indicates that \( Z \) is not a GG random variable. When the shape parameter \( \alpha \) takes other values, we can also prove the conclusion with the method employed in the proof in Lemma 4.

![Figure 2. Curves for p(\alpha) and q(\alpha).](image)

The conclusion to the model \( Z = X + cY \) is extended, where \( X, Y \) are i.i.d with zero means. Since the distribution of \( Z \) has nothing with the sign of \( c \) as mentioned in Lemma 3, we only consider the case that \( 0 < c \leq 1 \). When \( c = 1 \), the model turns to \( Z = X + Y \). Assuming \( Z \) to be a GG random variable with shape parameter \( t \), we can get two equations in the same way employed in the proof Lemma 4.

\[ \frac{r^2 \Gamma(\frac{3}{i})}{(1 + c^2)^2 \Gamma(\frac{1}{\beta})} = \frac{\alpha^{2} \Gamma(\frac{3}{\alpha})}{(1 + c^2)^2 \Gamma(\frac{1}{\alpha})} \quad (9) \]

\[ (1 + c^2)^2 \left( \frac{\Gamma(\frac{5}{i}) \Gamma(\frac{1}{i})}{\Gamma^2(\frac{3}{i})} - 3 \right) = (1 + c^2) \left( \frac{\Gamma(\frac{5}{\alpha}) \Gamma(\frac{1}{\alpha})}{\Gamma^2(\frac{3}{\alpha})} - 3 \right) \quad (10) \]

In order to show the relationship between \( \alpha \) and \( t \), we define the \( \alpha - t \) curves satisfying equations (9) and (10), which are presented in Figure 3 for several values of \( c \). In each figure, the curves corresponding to equations (9) and (10) are curve \( c_1 \) and curve \( c_2 \) respectively.
From Figure 3, it is easy to see that if $\alpha \neq 2$, it can't satisfy equations (9) and (10) at the same time, so $Z$ is no longer GGD. In the other words, $Z$ is a GG random variable if $X$ and $Y$ are Gaussian random variables, and here $Z$ is also a Gaussian random variable. To sum up, the distribution of $Z$ is a kind of new distribution similar to GGD.

**Simulation**

From the model analysis above, we notice that the distribution of $Z$ is different from GGD, though there are many similarities between $Z$ and GGD. If we fit $Z$ with GGD model $Z_G$ based on HOS. In this Section, we will show the relation between $Z$ and $Z_G$ with simulation.

Let $f(z)$ and $g(z)$ be the density functions of $Z=X+Y$ and $Z_G$, where $X, Y$ are i.i.d with shape parameter $\alpha$ and $Z_G$ with shape parameter $t$ satisfies

$$EZ_G = EZ = 0, \quad E(Z_G)^i = EZ^i, \quad E(Z_G)^i = EZ^i$$

Then $t$ and $\alpha$ satisfy Eq.(6), from which we can get the solution of $t$. The corresponding shape parameters of $Z_G$ are shown in table 1 for $\mu =0$ and $\alpha = 0.7, 1, 2$ and 3, which are corresponding to the cases that $X$ are super-Gaussian, Gaussian and sub-Gaussian distributions respectively. $f(z)$ and $g(z)$ are illustrated in Figure 4.

| $\alpha$ | 0.7 | 1   | 2   | 3    |
|---------|-----|-----|-----|------|
| $t$     | 0.8996 | 1.2568 | 2   | 2.3663 |
| $f(0)$  | 0.0459 | 0.0354 | 0.0282 | 0.0272 |
| $g(0)$  | 0.0576 | 0.0390 | 0.0282 | 0.0262 |

Table 1 shows that if $X$ has super-Gaussian distribution such as $\alpha =0.7$ and 1, so does $Z_G$. Similarly, if $X$ is Gaussian or sub-Gaussian distribution, it is the same with $Z_G$.

Note that in Figure 4, the two densities coincide with each other well except for the vicinity of zero which is the mean of $Z$ and $Z_G$. In addition, $g(0) > f(0)$ when $Z_G$ is super- Gaussian distribution in (a); $g(0) < f(0)$ when $Z_G$ is sub- Gaussian in (b).

A comparison between the histograms (computed on 65 536 samples of $Z$ each) and the proposed HOS-based GGD models can be deduced from Figure 5: there is some difference especially in the
vicinity of zero among the histogram and g(x) as shown in Figure 5. All of these implies that the distribution of Z=X+Y cannot be GGD except that X has Gaussian distribution.

Figure 5. Fitting of histogram with GG density g(z) for several values of $\alpha$ (a) $\alpha =0.7$ (b) $\alpha =3$.

Conclusion
The properties of Z=X+Y and GGD were derived. From these properties, we obtained the conclusion that the distribution of the sum of independent GG signals cannot be GGD. As special cases, the PDF of Z for GG signals with $\alpha =1$ and the proof for GG signals with $\alpha =\frac{1}{2}$ were given. The simulation results also lend support to the proposed conclusion and show that there is some difference especially in the vicinity of mean among the densities if applying GGD to model the sum of independent GG signals based on HOS. These properties and results can be useful in analyzing the sum of independent signals with GGD, and the linear system driven by of generalized Gaussian signals.

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