Canonical approach to 2D supersymmetric WZNW model
coupled to supergravity

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Abstract

Starting from the known representation of the Kac-Moody algebra in terms of the coordinates and momenta, we extend it to the representation of the super Kac-Moody and super Virasoro algebras. Then we use general canonical method to construct an action invariant under local gauge symmetries, where components of the super energy-momentum tensor \( L_\pm \) and \( G_\pm \) play the role of the diffeomorphisms and supersymmetries generators respectively.

We obtain covariant extension of WZNW theory with respect to local supersymmetry as well as explicit expressions for gauge transformations.

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1 Introduction

The two-dimensional Wess-Zumino-Novikov-Witten (WZNW) model has been discussed from various viewpoints. For example, at the infrared-stable fixed point it provides non-Abelian bosonization rules describing non-interacting massless fermions. It is conformally invariant and exactly soluble theory.

The Kac-Moody (KM) and Virasoro algebras play a vital role in understanding the model and for constructing its Lagrangian, particularly if it is coupled to the gauge fields. Coupling to 2D gravity is straight because the components of the energy-momentum tensors \( L_\pm \), as reparametrization generators, satisfy anomaly free Virasoro algebra. On the other hand, the currents \( j_{\pm a} \) satisfy the KM algebra with central charge, so it is not possible to gauge the full non-Abelian symmetry group, but only the “anomaly free” subgroup. Both features are consequences of the fact that WZNW model describes axial anomaly while the reparametrization symmetry is not anomalous.
Supersymmetric extension of the WZNW model has also been studied using superfield formalism as well as the component notation. It was shown that it was possible to choose the fermionic field such that fermions were completely decoupled.

In this paper we are going to construct supersymmetric WZNW model coupled to 2D supergravity. In our approach the super Kac-Moody (SKM) and super Virasoro algebras will play important role, because the super WZNW (SWZNW) model will appear as a Lagrangian realization of the super Virasoro algebra. We will use the Hamiltonian formalism where the generators, satisfying the super Virasoro Poisson brackets (PB) algebra, are the constraints so that the components of the metric tensor and Rarita-Schwinger field will appear naturally as Lagrange multipliers.

In Sec. II we introduce general canonical formalism in order to construct a gauge invariant effective action from the known representation of the constraints in terms of coordinates and momenta. Then we extend the known representation of the KM algebra to the representation of the SKM algebra. With the help of these expressions we construct the representation of the energy-momentum super tensor, firstly in terms of superfields and then in component notation. The components of super energy-momentum tensor are ordinary energy-momentum tensor and supersymmetric generator and they satisfy the supersymmetric extension of Virasoro algebra which is a starting point for our construction.

In Sec. III we construct the effective Lagrangian for SWZNW model in external supergravity field. We treat the components of energy-momentum super tensor as the constraints and use the general canonical formalism to obtain the corresponding Lagrangian. These components are the first class constraints enabling us to find the gauge symmetry transformations (reparametrizations and supersymmetry transformations) of the matter and the gauge multiplets.

The Lagrangian constructed in Sec. III is not manifestly Lorentz invariant because, in the Hamiltonian formalism, only some components of the gauge fields appear. In Sec. IV we introduce additional components to complete the gauge multiplet, so that we obtain the full covariant description of the fields and the full local gauge invariance. These new components are pure gauge, because they can be gauge away by local Weyl, local Lorentz and local super Weyl symmetries.

Sec. V is devoted to conclusion. In the Appendix we introduce the notation and establish connection between our Hamiltonian components and corresponding Lagrangian fields.

2 Super Virasoro Generators

We are going to use the canonical formalism to construct an action invariant under the gauge transformations for a given algebra of the group \( G \). Firstly, we need a representation of this algebra as the Poisson brackets algebra

\[
\{G_m, G_n\} = U_{mn} \gamma^r G_r, \quad \{G_m, H_0\} = V_m \gamma^r G_r, \tag{1}
\]

which elements \( G_m \) and the Hamiltonian \( H_0 \) are functions of the coordinates \( q^i \) and canonically conjugate momenta \( p_i \). Note that, by definition, \( G_m \) are the first class constrains. Then the canonical action, defined in usual way as

\[
I[q, p, u] = \int dx \left( p_i \dot{q}^i - H_0 - u^m G_m \right), \tag{2}
\]
is invariant under the gauge transformations of any quantity on phase space \( F(q^i, p_i) \),
\[
\delta F = \{ F, \varepsilon^m G_m \},
\]
and of the Lagrange multipliers \( u^m \),
\[
\delta u^m = \dot{\varepsilon}^m + u^s \varepsilon^s U_{sr}^m + \varepsilon^r V_r^m.
\]
The multipliers will be identified as gauge fields, later.

Similar approach has been used for construction of the action for W-strings propagating on group manifold and on curved backgrounds [4] and for 2D induced gravity [5]. Covariant extension of the WZNW model has been obtained using the first class constrains as generators of \textit{diffeomorphisms}. In the present paper we are going to supersymmetrize above approach.

Following the same steps, let us firstly construct the representation of the SKM algebra. We shall start from the known representation of KM algebra and then introduce a fermionic field in such a way that extended algebra closes and forms SKM.

Let us introduce notation. The field \( g \) is a mapping from a two-dimensional Riemannian spacetime \( \Sigma \) to a semi-simple Lie group \( G \), parametrized by local coordinates \( q^i \), \( g = g(q^i) \). The generators of the group \( G \), \( t_a \), satisfy the Lie algebra \([t_a, t_b] = f_{abc} t_c \). The expansions of one-forms \( g^{-1} dq \equiv dq^i E_{\pm i}^a t_a \) and \( gdg^{-1} \equiv dq^i E_{\pm i}^a t_a \) define vielbeins on the group manifold \( E_{\pm i}^a \). The Cartan metric in the tangent space is \( \gamma_{ab} = \frac{1}{2} \text{Tr}(t_at_b) \) (the trace is taken in the adjoint representation of \( G \)), and in the coordinate basis \( \gamma_{ij} \equiv E_{\pm i}^a E_{\pm j}^b \gamma_{ab} = E_{\pm i}^a E_{\pm j}^b \). Variables \( E_{\pm i}^a \) and \( \gamma^{ij} \) are inverses of the \( E_{\pm i}^a \) and \( \gamma_{ij} \) respectively. On the basis of the theorem that any closed form is locally exact, the equation \( d (gdg^{-1}gdg^{-1}gdg^{-1}) = 0 \) can be written in the form \( \frac{1}{4} \text{Tr}(gdg^{-1})^3 = -6d\tau \), where \( \tau \equiv \frac{1}{2} dq^i dq^j \tau_{ij} \) is a two-form. Rewriting in components, it becomes \( \partial_i \tau_{jk} + \partial_j \tau_{ki} + \partial_k \tau_{ij} = \pm \frac{1}{2} f_{abc} E_{\pm i}^a E_{\pm j}^b E_{\pm k}^c \).

Now, we are ready to introduce the phase space representation of bosonic part of KM currents in 2D Minkowski space-time \((\tau, \sigma)\) (well known from [8], [9]) as a
\[
\dot{j}_{\pm a} = -E_{\pm a} \dot{j}_{\mp i}, \quad j_{\pm i} = p_i + kP_{\pm ij} q^j,
\]
where the prime denotes the space derivative, and momentum independent part is
\[
P_{\pm ij} = \tau_{ij} \pm \frac{1}{2} \gamma_{ij}.
\]
PB of currents [3] defines two independent KM algebras of the group \( G \), with the central charges \( \pm k \):
\[
\{ j_{\pm a}(x), j_{\pm b}(y) \} = f_{abc} j_{\pm c}(x) \delta(\sigma_x - \sigma_y) \pm k \gamma_{ab} \delta'(\sigma_x - \sigma_y).
\]
In order to supersymmetrize the above algebra, let us introduce a Lie algebra valued fermionic fields \( \dot{\chi}_{\pm a} \). Because the fermionic part of the Lagrangian should be linear in time derivative,
there always exist the second class constraints $S_{±a} ≡ π_{±a} - ik \hat{χ}_{±a}$ linear in coordinate $\hat{χ}_{±a}$ and in corresponding canonical momenta $π_{±a}$. Dirac brackets for the fermionic fields are $\{\hat{χ}_{±a}, \hat{χ}_{±b}\}^* = -\frac{i}{2k} γ_{ab} δ$, while for bosonic currents $j_{±a}$ they remain the same as the PB. So, we can start from the relation (7) and

$$\{\hat{χ}_{±a}(x), \hat{χ}_{±b}(y)\} = -\frac{i}{2k} γ_{ab} δ(σ_x - σ_y) ,$$  

where, from now, we omit star because of simplicity. Note that in both bosonic and fermionic cases all quantities of the opposite chirality commute.

It is easy to check that bilinears in the fermionic fields $\tilde{J}_{±a} ≡ -ik f_{abc} \hat{χ}^b_{±} \hat{χ}^c_{±}$ satisfy the KM algebra without central charges

$$\{\tilde{J}_{±a}(x), \tilde{J}_{±b}(y)\} = f_{abc} \tilde{J}_{±c}(x) δ(σ_x - σ_y) ,$$

and have nontrivial brackets with $\hat{χ}_{±b}$,

$$\{\tilde{J}_{±a}(x), \hat{χ}_{±b}(y)\} = f_{abc} \hat{χ}_{±c}(x) δ(σ_x - σ_y) .$$

We can introduce new currents $J_{±a} ≡ j_{±a} + \tilde{J}_{±a}$ (such that KM algebra remains unchanged) which with its supersymmetric partners $\hat{χ}_{±a}$ satisfy two independent SKM algebras:

$$\{J_{±a}(x), J_{±b}(y)\} = f_{abc} J_{±c}(x) δ(σ_x - σ_y) ± k γ_{ab} δ(σ_x - σ_y) ,$$
$$\{J_{±a}(x), \hat{χ}_{±b}(y)\} = f_{abc} \hat{χ}_{±c}(x) δ(σ_x - σ_y) ,$$
$$\{\hat{χ}_{±a}(x), \hat{χ}_{±b}(y)\} = -\frac{i}{2k} γ_{ab} δ(σ_x - σ_y) .$$

Next step is the construction of the components of energy-momentum super tensors as functions of the SKM currents, keeping in mind that they have to be group invariants. The easiest way to do this is to introduce the superfields

$$I_{±a}(z) ≡ \sqrt{2} k \hat{χ}_{±a}(x) + θ_{±a}(x) ,$$

and rewrite the algebra (11) in the form

$$\{I_{±a}(z_1), I_{±b}(z_2)\} = \delta_{±12} f_{abc} I_{±c}(z_1) - ik γ_{ab} D^± δ_{±12} ,$$

where $D^± = \frac{∂}{∂θ_{±}} ± i θ_{±} \frac{∂}{∂σ}$ is the super covariant derivative, while $δ_{±12} = (θ_{±1} - θ_{±2}) δ(σ_1 - σ_2)$ is a generalization of the Dirac $δ$-function to the super $δ$-function. Derivative is always taken over the first argument of $δ$-function. Notation is given in the Appendix.

Up to the third power of $I_{±a}$, there are only two invariants $γ_{ab} D^± I^a_{±} I^b_{±}$ and $f_{abc} I^a_{±} I^b_{±} I^c_{±}$. Note that $γ_{ab} I^a_{±} I^b_{±}$ is identically equal to zero because super currents are odd variables. A requirement for the closed algebra determines the ratio of coefficients multiplying two invariants, so we take for super energy-momentum tensor

$$T_{±} ≡ \frac{1}{2k} \left( γ_{ab} D^± I^a_{±} I^b_{±} + \frac{i}{3k} f_{abc} I^a_{±} I^b_{±} I^c_{±} \right) .$$

In components notation, we have

$$T_{±} = θ_{±} G_{±} + θ_{±} L_{±} ,$$
where the bosonic part is
\[ L_{\pm} = \mp \frac{1}{2k} \left( J_{\pm a} J_{\pm a} + 2ik^2 \hat{\chi}_{\pm a} \hat{\chi}_{\pm a} + 2ikf_{abc} \hat{\chi}_{\pm a} \hat{\chi}_{\pm b} J_{\pm c}^c \right), \] (16)

while its supersymmetric partner is
\[ G_{\pm} = \frac{1}{\sqrt{2}} J_{\pm a} \hat{\chi}_{\pm a} + \frac{i \sqrt{2k}}{3} f_{abc} \hat{\chi}_{\pm a} \hat{\chi}_{\pm b} \hat{\chi}_{\pm c}. \] (17)

It is useful to express (16) and (17) in terms of quantities \( j_{\pm a} \) and \( \hat{\chi}_{\pm a} \):
\[ L_{\pm} = \mp \frac{1}{2k} j_{\pm a} j_{\pm a} + i k \hat{\chi}_{\pm a} \hat{\chi}_{\pm a}, \]
\[ G_{\pm} = \frac{1}{\sqrt{2}} j_{\pm a} \hat{\chi}_{\pm a} - \frac{i k}{3 \sqrt{2}} f_{abc} \hat{\chi}_{\pm a} \hat{\chi}_{\pm b} \hat{\chi}_{\pm c}. \] (18)

Using the SKM algebra (11), we can obtain the following brackets between the components of energy-momentum tensor and currents
\[ \{ L_{\pm}, J_{\pm a} \} = -J_{\pm a} \delta', \quad \{ G_{\pm}, J_{\pm a} \} = \frac{1}{\sqrt{2}} \hat{\chi}_{\pm a} \delta', \]
\[ \{ L_{\pm}, \hat{\chi}_{\pm a} \} = \frac{1}{2} (\hat{\chi}_{\pm a} \delta - \hat{\chi}_{\pm a} \delta'), \quad \{ G_{\pm}, \hat{\chi}_{\pm a} \} = -\frac{i}{2 \sqrt{2k}} J_{\pm a} \delta, \] (19)
as well as the brackets between the components of energy-momentum tensor themselves
\[ \{ L_{\pm}, L_{\pm} \} = -(L_{\pm} \delta + 2L_{\pm} \delta') = -[L_{\pm}(x) + L_{\pm}(y)] \delta', \]
\[ \{ G_{\pm}, G_{\pm} \} = \pm \frac{i}{2} L_{\pm} \delta, \]
\[ \{ L_{\pm}, G_{\pm} \} = -\frac{1}{2} (G_{\pm} \delta + 3G_{\pm} \delta'), \]
\[ \{ G_{\pm}, L_{\pm} \} = -\frac{1}{2} \left( 2G_{\pm} \delta + 3G_{\pm} \delta' \right). \] (20)

In terms of the super fields, we have instead of (19)
\[ \{ T_{\pm}(z_1), I_{\pm a}(z_2) \} = \pm \frac{i}{2} \left( D^\pm I_{\pm a} D^\pm \delta_{\pm 12} + I_{\pm a} D^{\pm 2} \delta_{\pm 12} \right), \] (21)

and instead of (20)
\[ \{ T_{\pm}(z_1), T_{\pm}(z_2) \} = \pm \frac{i}{2} \left( 2D^\pm 2 T_{\pm} \delta_{\pm 12} + D^\pm T_{\pm} D^\pm \delta_{\pm 12} + 3T_{\pm} D^{\pm 2} \delta_{\pm 12} \right). \] (22)

This is a supersymmetric extension of the Virasoro algebra \textit{without central charge}. Since \( L_{\pm} \) and \( G_{\pm} \) are the first class constraints, we shall apply the general canonical method to construct a theory invariant under \textit{diffeomorphisms} generated by \( L_{\pm} \) and under local \textit{supersymmetry} generated by \( G_{\pm} \). Because we know from (7) that, in the bosonic case with the similar approach, we have got covariant extension of the WZNW theory with respect to diffeomorphisms, here we expect to obtain covariant extension of WZNW theory with respect to local supersymmetry.
3 Effective Lagrangian and Gauge Transformations

In order to construct a covariant theory we start with the generators

\[ H_0 = 0, \quad G_m = (L_-, L_+, G_-, G_+) \tag{23} \]

with the explicit expressions given in equations (18), and the PB algebra (20) instead of the first equation (1). According to (2), we introduce the canonical Lagrangian

\[ \hat{\mathcal{L}} = \dot{q}_i p_i + ik \dot{\hat{\chi}}^a + ik \dot{\hat{\chi}}^a - h^- L_+ - h^+ L_- - i\psi^- G_- - i\psi^+ G_+ \tag{24} \]

with multipliers \( u^m = (h^-, h^+, \psi^-, \psi^+) \). Note that, on Dirac brackets, the second class constraints are equal to zero \( (S_{\pm a} = 0) \), so we have \( \pi_{\pm a} = i\hat{\chi}_{\pm a} \). We can eliminate remain momentum variables with the help of their equations of motion

\[ p^i = -\frac{k}{h^- - h^+} \left[ \dot{q}^i + (h^+ P^i_j - h^- P^-_j) q^j + i \sqrt{2} \left( \dot{\psi}^- \hat{\chi}^i + \dot{\psi}^+ \hat{\chi}^i \right) \right], \tag{25} \]

where \( \hat{\chi}^i = E_{\pm a} \hat{\chi}_{\pm i} \). On the equations of motion (25), the currents (5) become

\[ j^i_{\pm} = \frac{k}{2} \left[ \hat{\partial}_{\pm} \dot{q}^i + i \frac{\sqrt{2}}{\sqrt{-\hat{g}}} \left( \dot{\psi}^- \hat{\chi}^i_{\pm} + \dot{\psi}^+ \hat{\chi}^i_{\pm} \right) \right], \tag{26} \]

so that the Lagrangian (24) can be written in the form:

\[ \hat{\mathcal{L}} = \hat{\mathcal{L}}_{WZ} + \hat{\mathcal{L}}_f + \hat{\mathcal{L}}_{int} \]

\[ \hat{\mathcal{L}}_{WZ} = -\frac{k}{2} \sqrt{-\hat{g}} P_{ij} \hat{\partial}_- q^i \hat{\partial}_+ q^j \]

\[ \hat{\mathcal{L}}_f = -ik \sqrt{-\hat{g}} \left( \hat{\chi}^a_- \hat{D}_- \hat{\chi}^a_+ + \hat{\chi}^a_+ \hat{D}_+ \hat{\chi}^a_- \right) \]

\[ \hat{\mathcal{L}}_{int} = \frac{ik}{2\sqrt{2}} \left( \psi^+ \hat{\chi}_+ \hat{\partial}_+ q^i + \psi^- \hat{\chi}_- \hat{\partial}_- q^i \right). \tag{27} \]

Here, \( \hat{\mathcal{L}}_{WZ} \) and \( \hat{\mathcal{L}}_f \) are WZNW and fermion Lagrangians respectively, covariantized in external super gravitational fields \( \hat{g}_{\mu\nu} \) and \( \hat{\psi}^\pm \), while \( \hat{\mathcal{L}}_{int} \) describes interaction between bosonic fields \( q^i \) and fermionic fields \( \hat{\chi}_{\pm i} \). The tensor \( \hat{g}_{\mu\nu} \) is introduced instead of variables \( (h^+, h^-) \) (see the Appendix)

\[ \hat{g}_{\mu\nu} \equiv -\frac{1}{2} \begin{pmatrix} -2h^+ h^- & h^+ + h^- \\ h^+ + h^- & -2 \end{pmatrix}. \tag{28} \]

Covariant derivatives, acting on fermionic fields \( \hat{\chi}_{\pm}^a \), are defined by

\[ \hat{D}_{\pm} \hat{\chi}_+^a \equiv \hat{\partial}_{\pm} \hat{\chi}_+^a + \frac{i}{3\sqrt{2}\sqrt{-\hat{g}}} f^a_{bc} \psi_b^\pm \hat{\chi}_\pm^c \pm \frac{i}{8\hat{g}} \psi^- \hat{\chi}_+^a, \tag{29} \]

where \( \hat{\partial}_\pm = \epsilon^\mu_{\pm} \partial_\mu \), and \( \epsilon^\mu_{\pm} \) are also given in the Appendix.
The general canonical method provides a mechanism to write out gauge symmetries of the Lagrangian (27). Instead of relations (3), with the help of (18), we find the following gauge transformations of the fields,

\[ \delta q^i = \frac{1}{k} \left( \varepsilon^- j^i_+ - \varepsilon^+ j^i_- \right) - \frac{i}{\sqrt{2}} \left( \eta^+ \hat{\chi}^i_+ + \eta^- \hat{\chi}^i_- \right), \]

\[ \delta \hat{\chi}^a_\pm = -\varepsilon^\pm \partial_1 \hat{\chi}^a_\pm - \frac{1}{2} \left( \partial_1 \varepsilon^\pm \right) \hat{\chi}^a_\pm - \frac{1}{2k\sqrt{2}} \eta^\pm J^a_\pm, \] (30)

and instead of (4) using (20), we obtain the gauge transformations of the multipliers,

\[ \delta h^\pm = \partial_0 \varepsilon^\pm + h^\pm \partial_1 \varepsilon^\pm - \varepsilon^\pm \partial_1 h^\pm \pm \frac{i}{2} \psi^\pm \eta^\pm, \]

\[ \delta \psi^\pm = \frac{1}{2} \psi^\pm \partial_1 \varepsilon^\pm - \left( \partial_1 \psi^\pm \right) \varepsilon^\pm + \partial_0 \eta^\pm + h^\pm \partial_1 \eta^\pm - \frac{1}{2} \left( \partial_1 h^\pm \right) \eta^\pm. \] (31)

Bosonic fields \( \varepsilon^\pm \) and fermionic fields \( \eta^\pm \) are parameters of diffeomorphisms and local supersymmetry transformations respectively.

4 Lagrangian Formulation

It turns out that the Lagrangian (27) is invariant under the following rescaling of fields by two arbitrary parameters \( F(x) \) and \( f(x) \):

\[ \hat{e}^\pm _\mu \rightarrow e^\pm _\mu \equiv e^{F\pm f} \hat{e}^\pm _\mu, \]

\[ \psi^\pm \rightarrow \psi^\pm(\mp) \equiv \frac{1}{2\sqrt{-g}} e^{-\frac{1}{2}(F\mp 3f)} \psi^\pm, \]

\[ \hat{\chi}^a_\pm \rightarrow \chi^a_\mp \equiv e^{-\frac{1}{2}(F\mp f)} \hat{\chi}^a_\pm. \] (32)

As a consequence, we have:

\[ \sqrt{-g} \rightarrow \sqrt{-g} = e^{2F} \sqrt{-g}, \]

\[ \partial_\pm \rightarrow \partial_\pm \equiv e^{-\left(F\mp f\right)} \partial_\pm. \] (33)

In terms of rescaled fields, the rescaled Lagrangian has the same form as the original one (27),

\[ \mathcal{L} = \mathcal{L}_{WZ} + \mathcal{L}_f + \mathcal{L}_{int}, \]

\[ \mathcal{L}_{WZ} = -\frac{k}{2} \sqrt{-g} P_{ij} \partial_- q^i \partial_+ q^j, \]

\[ \mathcal{L}_f = -ik \sqrt{-g} \left( \chi^a_+ D_- \chi^a + \chi^a_- D_+ \chi^a \right), \]

\[ \mathcal{L}_{int} = \frac{ik}{\sqrt{2}} \sqrt{-g} \left[ \psi_{(-)} \chi_{+} \partial_+ q^i + \psi_{(+)} \chi_{-} \partial_- q^i \right]. \] (34)

where \( D_{\mp} \chi^a_\pm \equiv \partial_{\mp} \chi^a_\pm + \frac{1}{\sqrt{2}} f^{a}{}_{bc} \psi_{\mp}(\mp) \chi^b_+ \chi^c_\pm + \frac{1}{4} \psi_{\mp}(\mp) \psi_{\mp}(\pm) \chi^a_\pm. \) Note that the term with derivatives over \( F \) and \( f \) vanishes because of nilpotency of the field \( \hat{\chi}_i^a \).
Introduction of the new fields \( F \) and \( f \) gives additional gauge freedom to the Lagrangian (34). It becomes invariant under the local Weyl transformations
\[
\delta_\sigma e^\pm_\mu = \sigma e^\pm_\mu, \\
\delta_\sigma \psi_{\pm}(\pm) = -\frac{1}{2} \sigma \psi_{\pm}(\pm), \\
\delta_\sigma \chi_\pm^a = -\frac{1}{2} \sigma \chi_\pm^a, \\
\]
as a consequence of the transformation \( \delta_\sigma F = \sigma \), while all \( F \) independent fields remain Weyl invariant. Furthermore, the Lagrangian (34) does not change under the local Lorentz transformations
\[
\delta_\ell e^\pm_\mu = \mp \ell e^\pm_\mu, \\
\delta_\ell \psi_{\pm}(\pm) = \pm \frac{3}{2} \ell \psi_{\pm}(\pm), \\
\delta_\ell \chi_\pm^a = \pm \frac{1}{2} \ell \chi_\pm^a, \\
\]
(35)
generated by the transformation \( \delta_\ell f = -\ell \). The vielbein \( e^\pm_\mu \) is a Lorentz vector, the fields \( \psi_{\pm}(\pm) \) and \( \chi_\pm^a \) transform like components of a spinor field with the spin \( \frac{3}{2} \) and \( \frac{1}{2} \) respectively, while all \( f \) independent fields are Lorentz scalars.

The Lagrangian (34) depends only on two components \( \psi_{\pm}(\pm) \) of the Rarita-Schwinger spinor field \( \psi_{\mu\alpha} = e^a_\mu \psi_{a(\alpha)} \) (\( \mu = 0, 1, \alpha = +, - \)). It means that we also have the additional local super Weyl symmetry
\[
\delta_\lambda \psi_{\pm}(\mp) = \pm \lambda_\pm, \\
\]
while all other fields are super Weyl invariant. Transformations (35) – (37) can be written in a covariant form,
\[
\text{Lorentz:} \quad \delta_\ell e^\pm_\mu = -\ell e^\pm_\mu, \quad \delta_\ell \psi_{\pm}(\pm) = \pm \frac{3}{2} \ell \psi_{\pm}(\pm), \quad \delta_\ell \chi_\pm^a = \pm \frac{1}{2} \ell \chi_\pm^a, \\
\text{Weyl:} \quad \delta_\sigma e^\pm_\mu = \sigma e^\pm_\mu, \quad \delta_\sigma \psi_{\pm}(\pm) = \frac{1}{2} \sigma \psi_{\pm}(\pm), \quad \delta_\sigma \chi_\pm^a = \frac{1}{2} \sigma \chi_\pm^a, \\
\text{super Weyl:} \quad \delta_\lambda e^\pm_\mu = 0, \quad \delta_\lambda \psi_{\pm}(\pm) = \gamma_{\mu} \lambda, \quad \delta_\lambda \chi_\pm^a = 0, \\
\]
(38)
where \( \gamma_{\mu} \equiv e^a_\mu \gamma_a \). The representation of the \( \gamma \)-matrices is given in the Appendix. Note that the fields \( \hat{e}^\pm_\mu \), \( \hat{\psi}^\pm \) and \( \hat{\chi}^a_\pm \), introduced in the Hamiltonian approach are Lorentz, Weyl and super Weyl invariants.

The fields \( F \) and \( f \) do not enter the original Lagrangian (27) but are introduced by rescaling the fields (32). Thus we cannot find their change under diffeomorphisms and SUSY transformations just applying the general Hamiltonian rules (3), (4). We have introduced them in such a way that the new fields \( e^\pm_\mu \) and \( \psi_{\mu\alpha} \) have proper Lorentz, Weyl and super Weyl transformations. Now, we demand that \( e^\pm_\mu \) transforms like a vector and \( \psi_{\mu\alpha} \) transforms like a Rarita-Schwinger field under general coordinate transformations with a local parameter \( \varepsilon^\mu(x) \) and under \( N = 1 \) supersymmetric transformations with a local spinor parameter \( \zeta_\alpha(x) \). It means that (see for example (4))
\[
\delta e^\pm_\mu = -\varepsilon^\nu \partial_\nu e^\pm_\mu - e^a_\nu \partial_\mu e^\nu_{\nu} - i \frac{1}{2} \bar{\zeta} \gamma^a \psi_\mu, \\
\delta \psi_\mu = -\varepsilon^\nu \partial_\nu \psi_\mu - \psi_\nu \partial_\mu \varepsilon^\nu + i \frac{1}{2} \gamma_\mu \zeta, \\
\]
(39)
where $\nabla_\mu \zeta = e^a_\mu \nabla_a \zeta$, $\nabla_a \zeta = (\partial_a + \frac{1}{2} \gamma_5 \omega_a) \zeta$. Writing out in components, it gives

$$\delta (F \pm f) = \chi^2_0 e^0 - \partial_1 e^1 - e^\mu \partial_\mu (F \pm f) \pm ie^{-(F \pm f)} \zeta_{\mp} \psi_{1 \mp},$$

$$\delta h^\pm = \partial_0 e^1 - h^\pm (\partial_0 e^0 - \partial_1 e^1) - (h^\pm)^2 \partial_1 e^0 - e^\mu \partial_\mu h^\pm \mp i e^{-\frac{1}{2} (F \pm f)} \zeta_{\mp} \psi^\pm,$$

$$\delta \psi^\pm = \left[ \frac{1}{2} \partial_1 (e^1 - h^\pm e^0) + \frac{1}{2} (\partial_1 h^\pm) e^0 - \partial_0 e^0 - h^\pm \partial_1 e^0 \right] \psi^\pm - e^\mu \partial_\mu \psi^\pm + \left( \partial_0 + h^\pm \partial_1 - \frac{1}{2} \partial_1 h^\pm \right) \left( \zeta_{\mp} e^{-\frac{1}{2} (F \pm f)} \right) + \frac{i}{4} \zeta_{\mp} \psi_{\mp(\mp)} \psi^\pm,$$  \hspace{1cm} (40)

where $\psi^\mp \equiv 2 e^{-\frac{1}{4} (F \pm f)} (\psi_0 \pm h^\mp \psi_1 \pm)$ in according with (32).

In order to establish relation between Hamiltonian and Lagrangian transformations we should compare the Hamiltonian transformations (31) with Lagrangian one, the last two equations of (10). We find that we can identify them choosing the following relation between the gauge parameters

$$\varepsilon^\pm \equiv e^1 - h^\pm e^0, \hspace{1cm} \eta^\pm \equiv 2 \zeta_{\mp} e^{-\frac{1}{2} (F \pm f)} - e^0 \psi^\pm,$$  \hspace{1cm} (41)

and imposing gauge fixing $\psi_{\mp(\mp)} = 0$, because in the Hamiltonian approach all quantities are super Weyl invariant. Substituting equations of motion (26) in (30) we can obtain the momentum independent formulation of transformation law of matter variables $q^i$, $\hat{\chi}^a_{\pm}$. In terms of the Lagrangian variables $e^\mu$ and $\zeta_{\pm}$, we have

$$\delta q^i = - e^\mu \partial_\mu q^i - i \sqrt{2} \left( \zeta_{\mp} \hat{\chi}^a_+ + \zeta_+ \hat{\chi}^a_- \right),$$

$$\delta \hat{\chi}^a_\pm = - e^\mu \partial_\mu \hat{\chi}^a_{\pm} + e^0 \sqrt{-g} \nabla_{\mp} \hat{\chi}^a_{\pm} + \frac{1}{2} \left( h^\pm \partial_1 e^0 - \partial_1 e^1 \right) \hat{\chi}^a_{\pm} +$$

$$+ \frac{1}{2k \sqrt{2}} \left( e^0 \psi^\pm - 2 \zeta_{\mp} e^{-\frac{1}{2} (F \pm f)} \right) J^a_{\pm}. \hspace{1cm} (42)$$

In the flat space limit $e^a_\mu \rightarrow \delta^a_\mu$ and $\psi_\mu \rightarrow 0$, the Lagrangian (34) becomes

$$\mathcal{L}_0 = - \frac{k}{2} P_{ij} \partial_- q^i \partial_+ q^j - ik \left( \hat{\chi}^a_+ \partial_+ \hat{\chi}^a_- + \hat{\chi}^a_- \partial_\alpha \hat{\chi}^a_+ + \right)$$

$$\hspace{2cm} \left( \chi^a_+ \partial_+ \chi^a_- + \chi^a_- \partial_+ \chi^a_+ \right), \hspace{1cm} (43)$$

and bosonic and fermionic parts are decoupled. The first term is the bosonic Wess-Zumino action, while the second one is the Lagrangian of free spinor fields $\chi^a_{\pm}$. The Lagrangian (34) describes the $N = 1$ supersymmetric WZNW theory [2].

5 Conclusion

Using the general canonical method we construct the Lagrangian for the SWZNW model coupled to 2D supergravity. The basic ingredients of our approach are the symmetry generators, which are functions of the coordinates and momenta and satisfy the SKM and super Virasoro PB
algebras. Application of the Hamiltonian method naturally incorporates gauge fields (components of the metric tensor and Rarita-Schwinger fields) as Lagrange multipliers of the symmetry generators. This method also gives a prescription for finding gauge transformations for both the matter and gauge fields.

The result of Sec. II is the representation of the super Virasoro algebra elements (the components of energy-momentum tensor $L_{\pm}$ and supersymmetric generators $G_{\pm}$) as functions of the coordinates and momenta, eq. (18). In this approach, firstly we had to construct the KM and SKM algebras, where we introduced superfields as a useful method for going from the SKM algebra to the super Virasoro one.

In Sec. III the effective Lagrangian for the theory which symmetry is defined by super Virasoro algebra (20) has been constructed. Treating generators of the algebra as the first class constraints, the general canonical method (introduced in Sec. II) has been applied. After elimination of the bosonic momentum variables on the basis of their equations of motion, we obtain the effective action containing WZNW theory, fermionic Lagrangian and the part describing their interaction in external supergravity field. The Hamiltonian version of reparametrization and supersymmetry transformations has been found for the matter fields $q^i$ and $\hat{\chi}^a_{\pm}$ (30), and for the Lagrange multipliers $h_{\pm}$ and $\psi_{\pm}$ (31).

The Hamiltonian formalism deals with the Lagrangian multipliers $h_{\pm}$ and $\psi_{\pm}$, which are just the part of the gauge fields necessary to represent the symmetry of the algebra. In Lagrangian formulation we need the covariant description of the fields. In order to complete vielbeins $e^a_{\mu}$, it was necessary to introduce the new bosonic components $F$ and $f$, while for completing the Rarita-Schwinger fields $\psi_{\mu a}$ we need the new fermionic fields $\psi_{\pm(\mp)}$. The new components are not physical because they do not appear in the Lagrangian, but they give additional gauge freedom corresponding to the additional gauge symmetries. These are local Weyl and local Lorentz symmetries for the bosonic fields $F$ and $f$ respectively, and local super Weyl symmetry for the fermionic field $\psi_{\mp(\pm)}$. The fields $F$ and $f$ are not parts of the Hamiltonian formalism, so we find their transformation laws under reparametrizations and supersymmetry requiring that vielbeins $e^a_{\mu}$ transform as vectors and $\psi_{\mu a}$ transform as a Rarita-Schwinger field. Consequently, in Sec. IV we establish the complete relation between Hamiltonian and Lagrangian formulations. We find the connection between corresponding fields, gauge parameters and gauge transformations.

We can conclude that our main result is derivation of the Lagrangian for SWZNW model coupled to supergravity, eq. (34). By construction, this Lagrangian is invariant under local Lorentz, local Weyl and local super Weyl transformations (38), and under reparametrizations and N=1 supersymmetry transformations, (39) and (42).

Thanks to using the canonical approach, we can interpret the results in a different way, as the complete canonical analysis of our Lagrangian. It means that momenta are defined by eq. (25), the canonical Hamiltonian is zero (see (23)) and $L_{\pm}$ and $G_{\pm}$ defined in (18) are the first class constraints satisfying super Virasoro PB algebra (20).

In the flat space limit we reproduce the result of ref. [2].

A Noteation

At each point of the curved two-dimensional space-time $\Sigma$ with signature (−+) and coordinates $x^\mu$ ($\mu = 0, 1$), there is a light cone basis of 1-forms $dx^a \equiv e^a_\mu dx^\mu$. Vielbeins $e^a_\mu$ are expressed...
in terms of variables \((h^-, h^+, F, f)\) as

\[
e^{\pm}_\mu = e^{F \pm f} \delta^\pm_\mu, \quad \hat{e}\mu^a = \frac{1}{2} \begin{pmatrix} -h^+ & 1 \\ h^- & -1 \end{pmatrix} (a = +, -; \mu = 0, 1). \tag{A.1}
\]

Inverse vielbeins \(e^{\mu}_a (e_a^\mu e_\mu^b = \delta^b_a \text{ and } e_\mu^a e^\nu_a = \delta^\nu_\mu)\) are

\[
e_{\pm}^\mu = e^{-(F \pm f)} \hat{e}\mu^a, \quad \hat{e}_a^\mu = \frac{2}{h^- - h^+} \begin{pmatrix} 1 & 1 \\ h^- & h^+ \end{pmatrix}. \tag{A.2}
\]

Related basis of tangent vectors \(\partial_a \equiv e^\mu_a \partial_\mu\) can be written as

\[
\partial_\pm = e^{-(F \pm f)} \hat{\partial}_\pm, \quad \hat{\partial}_\pm = \frac{2}{h^- - h^+} (\partial_0 + h^\mp \partial_1). \tag{A.3}
\]

It follows from (A.1) that the components of the metric tensor \(g_{\mu \nu} = \eta_{ab} e^a_\mu e^b_\nu\) are

\[
g_{\mu \nu} = e^{2F} g_{\mu \nu}, \quad \hat{g}_{\mu \nu} \equiv -\frac{1}{2} \begin{pmatrix} -2h^+ h^- & h^+ + h^- \\ h^+ + h^- & -2 \end{pmatrix}, \tag{A.4}
\]

while the inverse metric \(\hat{g}^{\mu \nu}\) is

\[
g^{\mu \nu} = e^{-2F} \hat{g}^{\mu \nu}, \quad \hat{g}^{\mu \nu} \equiv -\frac{2}{(h^- - h^+)^2} \begin{pmatrix} 2 & h^+ + h^- \\ h^+ + h^- & 2h^- h^+ \end{pmatrix}. \tag{A.5}
\]

Here \(\sqrt{-g} = e^{2F} \sqrt{-g} \equiv e^{2F} (h^- - h^+)\).

In tangent Minkowski space we also introduce light cone coordinates \(x^a (a = +, -)\), with \(x^{\pm} \equiv \frac{1}{2} (x^0 \pm x^1)\), so that raising and lowering of the tangent space indices are performed as \(A_\pm = -2A^\mp\).

The Riemannian connection on \(\Sigma\) is defined by

\[
\omega_a = \varepsilon^{bc} e_\mu^b \partial_c e_{a \mu}, \tag{A.6}
\]

where \(\varepsilon^{mn} (\varepsilon^{01} = 1)\) is the constant totally antisymmetric tensor in the Minkowski space or, in light cone basis, \(\varepsilon^{-+} = -\varepsilon^{+-} = \frac{1}{2}\). Written in terms of variables (A.2) and (A.3), the connection becomes

\[
\omega_\pm = e^{-(F \pm f)} \left[ \hat{\omega}_\pm \mp \hat{\partial}_\pm (F \mp f) \right], \quad \hat{\omega}_\pm \equiv \mp \frac{2 \partial_1 h^\mp}{h^- - h^+}. \tag{A.7}
\]

Covariant derivative acting on a field with \(n\) spinor indices has to be \(\nabla_a = \partial_a + \frac{n}{2} \omega_a\), or

\[
\nabla_\pm = e^{(\pm n - 1)F - (n \pm 1) f} \hat{\nabla}_\pm \varepsilon^{\mp nF + nf}, \tag{A.8}
\]

where \(\hat{\nabla}_a = \hat{\partial}_a + \frac{n}{2} \hat{\omega}_a\).

Dirac matrices, defined in tangent Minkowski space, satisfy the Clifford algebra

\[
\{\gamma^m, \gamma^n\} = 2\eta^{mn}. \tag{A.9}
\]

We use a representation

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 \equiv \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.10}
\]
where we have the identity $\text{Tr} (\gamma^m \gamma^n \gamma_5) = 2 \varepsilon^{mn}$. Also for projective $\gamma$-matrices $\gamma^\pm = \frac{1}{2} (\gamma^0 \pm \gamma^1)$ we have:
\[
\gamma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.
\] (A.11)

Majorana spinor is a Dirac spinor $\theta_\alpha \equiv \begin{pmatrix} \theta_+ \\ \theta_- \end{pmatrix}$ satisfying Majorana condition $\theta = C \bar{\theta}^T$, where $\bar{\theta} \equiv \theta^T \gamma^0$, and $C$ is the charge conjugation matrix ($C^{-1} \gamma_\mu C = -\gamma^T_\mu$). In the representation (A.10) and with $C = \gamma^0$, the Majorana spinors are real, $\theta^*_\alpha = \theta_\alpha$. Tensor $C^{\alpha \beta}$ performs the raising of spinor indices ($\theta^\alpha = C^{\alpha \beta} \theta_\beta$), while $C^{\alpha \beta}$ performs their lowering ($\theta_\alpha = C^{\alpha \beta} \theta_\beta$). In the
components, it gives $\theta_{\pm} = \pm \theta^\mp$. Spinor contraction is denoted by $\theta \xi \equiv \theta^\alpha \xi_\alpha = -\theta_\alpha \xi^\alpha$.

The spinor covariant derivative is
\[
D_\alpha = \bar{\partial}_\alpha + i (\gamma^m \theta)_\alpha \partial_m,
\] (A.12)
where $\partial_m \equiv \frac{\partial}{\partial x^m}$ and $\bar{\partial}_\alpha \equiv \frac{\partial}{\partial \bar{\theta}_\alpha}$. More explicitly, the derivative (A.12) in the representation (A.10) is
\[
D^\pm \equiv \bar{\partial}_\mp - i \theta_+ \partial_\pm,
\] (A.13)
where $D_\alpha \equiv \begin{pmatrix} -D^+ \\ D^- \end{pmatrix}$.

A generalization of the $\delta$-function to the super $\delta$-function is
\[
\delta_{\pm 12} \equiv \theta_{\mp 12} \delta(x_1^\pm - x_2^\pm),
\] (A.14)
where $\theta_{12} = \theta_1 - \theta_2$. Its properties are
\[
\int d^4 z_1 \delta_{\pm 12} = 1
\]
\[
F(z_1) \delta_{\pm 12} = F(z_2) \delta_{\pm 12}
\]
\[
\delta_{\pm 21} = -\delta_{\pm 12}, \quad D^\pm_2 \delta_{\pm 12} = -D^\mp_2 \delta_{\pm 12},
\] (A.15)
where $d^4 z \equiv d^2 x d^2 \theta$ and basic integrals for Grassman odd numbers are $\int d\theta = 0$ and $\int d\theta \theta = 1$. Four real coordinates $z^A = (x^\pm, \theta_\pm)$, with $x^\pm$ light cone coordinates and $\theta_\alpha$ a Majorana spinor, parametrize N=1 superspace.

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