COMMON FIXED POINT THEOREMS FOR HYBRID GENERALIZED 
\((F, \varphi)\)-CONTRACTIONS UNDER COMMON LIMIT RANGE PROPERTY 
WITH APPLICATIONS

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Abstract. We consider a relatively new hybrid generalized \(F\)-contraction involving a pair of mappings and utilize the same to prove a common fixed point theorem for a hybrid pair of occasionally coincidentally idempotent mappings satisfying generalized \((F, \varphi)\)-contraction condition under common limit range property in complete metric spaces. A similar result involving a hybrid pair of mappings satisfying a Rational type Hardy-Rogers \((F, \varphi)\)-contractive condition is also proved. Our results generalize and improve several results of the existing literature. As applications of our results, we prove two theorems for the existence of solutions of certain system of functional equations arising in dynamic programming, and Volterra integral inclusion besides providing an illustrative example.

1. Introduction and preliminaries

Let \((X, d)\) be a metric space. Then, following the Nadler [28], we adopt the following notations:

- \(CL(X) = \{A : A \text{ is a non-empty closed subset of } X\}\).
- \(CB(X) = \{A : A \text{ is a non-empty closed and bounded subset of } X\}\).
- For non-empty closed and bounded subsets \(A, B\) of \(X\) and \(x \in X\),
  \[d(x, A) = \inf \{d(x, a) : a \in A\}\]
  and
  \[H(A, B) = \max \left\{ \sup \{d(a, B) : a \in A\}, \sup \{d(b, A) : b \in B\} \right\}.\]

Recall that \(CB(X)\) is a metric space with the metric \(H\) which is known as the Hausdorff-Pompeiu metric on \(CB(X)\).

In 1969, Nadler [28] proved that every multi-valued contraction mapping defined on a complete metric space has a fixed point. In proving this result, Nadler used the idea of Hausdorff metric to establish the multi-valued version of Banach Contraction Principle which runs as follows:

**Theorem 1.** Let \((X, d)\) be a complete metric space and \(T\) a mapping from \(X\) into \(CB(X)\) such that for all \(x, y \in X\),
\[H(Tx, Ty) \leq \lambda d(x, y),\]
where \(\lambda \in [0, 1)\). Then \(T\) has a fixed point, i.e., there exists a point \(x \in X\) such that \(x \in Tx\).

Hybrid fixed point theory involving pairs of single-valued and multi-valued mappings is a relatively new development in Nonlinear Analysis (e.g. [11,12,15,24,29,45] and references

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Definition 2. Let \((f, T)\) be a hybrid pair of mappings. Then a mapping \(f\) is\( T\) if for every \(x \in X\), the set of all fixed points of \(f\) (resp. \(T\)) is denoted by \(F(f)\) (resp. \(F(T)\)). If \(x \in X\) is a coincidence point of \(f, T\), then the set of all coincidence points of \(f, T\) is denoted by \(C(f, T)\). A point \(x \in X\) is a common fixed point of \(f, T\) if \(x \in \text{Fix}(f) \cap \text{Fix}(T)\). The set of all common fixed points of \(f, T\) is denoted by \(\text{Fix}(f, T)\).

Definition 1. Let \(f : X \to X\) and \(T : X \to CB(X)\) be a single-valued and multi-valued mapping respectively. Then

- A point \(x \in X\) is a fixed point of \(f\) (resp. \(T\)) if \(x = fx\) (resp. \(x \in Tx\)). The set of all fixed points of \(f\) (resp. \(T\)) is denoted by \(F(f)\) (resp. \(F(T)\)).
- A point \(x \in X\) is a coincidence point of \(f, T\) if \(fx \in Tx\). The set of all coincidence points of \(f, T\) is denoted by \(C(f, T)\).
- A point \(x \in X\) is a common fixed point of \(f, T\) if \(x = fx \in Tx\). The set of all common fixed points of \(f, T\) is denoted by \(\text{Fix}(f, T)\).
- \(T\) is a closed multi-valued mapping if the graph of \(T\) i.e., \(G(T) = \{(x, y) : x \in X, y \in Tx\}\) is a closed subset of \(X \times X\).

We also recall the following terminology often used in the considerations of a hybrid pairs of mappings.

Definition 2. Let \((X, d)\) be a metric space with \(f : X \to X\) and \(T : X \to CB(X)\). Then a hybrid pair of mappings \((f, T)\) is said to be:

- commuting on \(X\) if \(fTx \subseteq Tx \forall x \in X\).
- weakly commuting on \(X\) if \(h(fTx, Tf) \leq d(fx, Tx) \forall x \in X\).
- compatible if \(fTx \in CB(X) \forall x \in X\) and \(\lim_{n \to \infty} h(Tfx, Tx_n) = 0\), whenever \(\{x_n\}\) is a sequence in \(X\) such that

\[
\lim_{n \to \infty} Tx_n \to A \in CB(X) \text{ and } \lim_{n \to \infty} f x_n \to t \in A.
\]

- non-compatible if \(\exists\) at least one sequence \(\{x_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} Tx_n \to A \in CB(X) \text{ and } \lim_{n \to \infty} f x_n \to t \in A \text{ but } \lim_{n \to \infty} h(Tfx, Tx_n) \text{ is either non-zero or nonexistent.}
\]

- weakly compatible if \(Tfx = fTx\) for each \(x \in C(f, T)\).
- coincidently idempotent if for every \(v \in C(f, T)\), \(ffv = fv\) i.e., \(f\) is idempotent at the coincidence points of \(f\) and \(T\).
- occasionally coincidently idempotent if \(ffv = fv\) for some \(v \in C(f, T)\).
• enjoy the property (E.A) \[19\] if \(\exists\) a sequence \(\{x_n\}\) in \(X\) such that
  \[
  \lim_{n \to \infty} f x_n = t \in A = \lim_{n \to \infty} T x_n,
  \]
  for some \(t \in X\) and \(A \in CB(X)\).
• enjoy common limit range property with respect to the mapping \(f\) (in short \(CLR_f\) property) \[14\] if \(\exists\) a sequence \(\{x_n\}\) in \(X\) such that
  \[
  \lim_{n \to \infty} f x_n = f u \in A = \lim_{n \to \infty} T x_n,
  \]
  for some \(u \in X\) and \(A \in CB(X)\).

The following example demonstrates the interplay of the occasionally coincidentally idempotent property with other notions described in the preceding definition.

**Example 1.** [18, Example 1] Let \(X = \{1, 2, 3\}\) (with the standard metric),
\[
  f : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad T : \begin{pmatrix} 1 & 2 \\ \{1\} & \{1, 3\} \\ \{1, 3\} & \{1\} \end{pmatrix}.
\]

Then, it is straightforward to observe the following:
• \(C(f, T) = \{1, 2\}\), and \(F(f, T) = \{1\}\).
• \((f, T)\) is not commuting and not weakly commuting.
• \((f, T)\) is not compatible.
• \((f, T)\) is not weakly compatible.
• \((f, T)\) is not coincidentally idempotent since \(ff = f3 = 2 \neq 3 = f2\).
• \((f, T)\) is occasionally coincidentally idempotent since \(ff1 = 1 = f1\).

Obviously, in this case \((f, T)\) is also non-compatible, but simple modifications of this example can show that the occasionally coincidentally idempotent property is independent of this notion, too.

The following example (taken from [18]) demonstrates the relationship between (E.A) property and common limit range property.

**Example 2.** [18] Examples 2 and 3] Let \(X = [0, 2]\) be a metric space equipped with the usual metric \(d(x, y) = |x - y|\). Define \(f, g : X \to X\) and \(T : X \to CB(X)\) as follows:
\[
  f x = \begin{cases} 
  2 - x, & \text{if } 0 \leq x < 1, \\
  \frac{x}{2}, & \text{if } 1 \leq x \leq 2;
  \end{cases} \quad g x = \begin{cases} 
  2 - x, & \text{if } 0 \leq x \leq 1, \\
  \frac{x}{2}, & \text{if } 1 < x \leq 2;
  \end{cases} \quad T x = \begin{cases} 
  \left[\frac{1}{2}, \frac{3}{2}\right], & \text{if } 0 \leq x \leq 1, \\
  \left[\frac{1}{2}, \frac{1}{2}\right], & \text{if } 1 < x \leq 2.
  \end{cases}
\]

One can verify that the pair \((f, T)\) enjoys the property (E.A), but not the \(CLR_f\) property. On the other hand, the pair \((g, T)\) satisfies the \(CLR_g\) property.

**Remark 1.** If a pair \((f, T)\) satisfies the property (E.A) along with the closedness of \(f(X)\), then the pair also satisfies the \(CLR_f\) property.

Throughout this paper, we denote by \(\mathbb{R}\) the set of all real numbers, by \(\mathbb{R}^+\) the set of all positive real numbers and by \(\mathbb{N}\) the set of all positive integers. In what follows, \(\mathcal{F}\) denote the family of all functions \(F : \mathbb{R}^+ \to \mathbb{R}\) that satisfy the following conditions:

(F1) \(F\) is continuous and strictly increasing;
(F2) for each sequence \(\{\beta_n\}\) of positive numbers, \(\lim_{n \to \infty} \beta_n = 0 \iff \lim_{n \to \infty} F(\beta_n) = -\infty;\)
(F3) there exists \(k \in (0, 1)\) such that \(\lim_{\beta \to 0^+} \beta^k F(\beta) = 0.\)
Some examples of functions $F \in \mathcal{F}$ are $F(t) = \ln t$, $F(t) = t + \ln t$, $F(t) = -\frac{1}{\sqrt{t}}$, see [47].

**Definition 3.** [47] Let $(X, d)$ be a metric space. A self-mapping $T$ on $X$ is called an $F$-contraction if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that

$$
\tau + F(d(Tx,Ty)) \leq F(d(x,y)),
$$

(1.1)

for all $x, y \in X$ with $d(Tx,Ty) > 0$.

**Example 3.** [47] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping given by $F(x) = \ln x$. It is clear that $F$ satisfies $(F_1)$–$(F_3)$ for any $k \in (0,1)$. Under this setting, (1.1) reduces to

$$
d(Tx,Ty) \leq e^{-\tau}d(x,y), \text{ for all } x, y \in X, \ T x \neq T y.
$$

Notice that for $x, y \in X$ such that $Tx = Ty$, the previous inequality also holds and hence $T$ is a contraction.

In what follows, for a metric space $(X, d)$ and a multi-valued mapping $T : X \to CL(X)$, we denote

$$
M(x,y) = \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2} \left[ d(x,Ty) + d(y,Tx) \right] \right\}.
$$

**Definition 4.** [39] Let $(X, d)$ be a metric space. A multi-valued mapping $T : X \to CL(X)$ is called an $F$-contraction if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that for all $x, y \in X$ with $y \in Tx$, $\exists z \in Ty$,

$$
\tau + F(d(y,z)) \leq F \left( M(x,y) \right), \text{ whenever } d(y,z) > 0.
$$

(1.2)

**Example 4.** [39] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping given by $F(x) = \ln x$. Then for each multi-valued mapping $T : X \to CL(X)$ satisfying (1.2), we have

$$
d(y,z) \leq e^{-\tau}M(x,y), \text{ for all } x, y \in X, \ z \in Ty, \ y \neq z.
$$

It is clear that for $z, y \in X$ such that $y = z$ the previous inequality also holds.

Some fixed point results for single-valued (resp. multi-valued) $F$-contractions were obtained in [3, 23, 47] (resp. [39]).

Our aim in this paper is to prove a common fixed point theorem for a hybrid pair of occasionally coincidentally idempotent mappings satisfying generalized $(F, \varphi)$-contraction condition, under CLR property in complete metric spaces. Also, a similar result for a variant of Rational type Hardy-Rogers generalized $(F, \varphi)$-contractive condition is also derived. Here, it can be pointed out that Sgroi and Vetro [39] introduced and studied such conditions for multi-valued mappings while the similar conditions were earlier introduced and studied by Wardowski [47] for single-valued mappings. Our results generalize and improve several known results of the existing literature. Finally, we utilize our results to prove the existence of solutions of certain system of functional equations arising in dynamic programming, as well as Volterra integral inclusion besides providing an illustrative example.

2. The Main Results

This section is divided into two parts. In the first subsection, we prove a common fixed point theorem for a hybrid pair of occasionally coincidentally idempotent mappings satisfying a generalized $(F, \varphi)$-contractions condition via CLR property in complete metric spaces, while in the second one we obtain results for hybrid pairs which satisfy a Rational Hardy-Rogers type $(F, \varphi)$-contractive condition.
Definition 5. Let \((X, d)\) be a metric space, \(f : X \to X\) and \(T : X \to CB(X)\). Then hybrid pair \((f, T)\) is said to be a generalized \((F, \varphi)\)-contraction, if there exist an increasing, upper semicontinuous mapping from the right

\[
\Phi = \{ \varphi : [0, \infty) \to [0, \infty) \mid \limsup_{s \to t^+} \varphi(s) < \varphi(t), \ \varphi(t) < t, \forall t > 0 \},
\]

\(F \in F\) and \(\tau \in \mathbb{R}^+\) such that

\[
\tau + F(\mathcal{H}^p(Tx, Ty)) \leq F\left(\varphi\left(\max \left\{ \frac{d^p(fx, Tx)}{1+d^p(fx, fx)}, \frac{\beta}{1+d^p(fy, Ty)} \right\} \right) \right)
\]

for all \(x, y \in X\), \(p \geq 1\) with \(\mathcal{H}(Tx, Ty) > 0\).

Definition 6. Let \((X, d)\) be a metric space, \(f : X \to X\) and \(T : X \to CB(X)\). Then hybrid pair \((f, T)\) is said to be a Rational Hardy-Rogers \((F, \varphi)\)-contraction, if there exist an increasing, upper semicontinuous mapping from the right

\[
\Phi = \{ \varphi : [0, \infty) \to [0, \infty) \mid \limsup_{s \to t^+} \varphi(s) < \varphi(t), \ \varphi(t) < t, \forall t > 0 \},
\]

\(F \in F\) and \(\tau \in \mathbb{R}^+\) such that

\[
\tau + F(\mathcal{H}^p(Tx, Ty)) \leq F\left(\varphi\left(\max \left\{ \frac{d^p(fx, Tx)}{1+d^p(fx, fx)}, \frac{\beta}{1+d^p(fy, Ty)} \right\} \right) \right)
\]

for all \(x, y \in X\) with \(Tx \neq Ty\), where \(p \geq 1\), \(\alpha, \beta, \gamma, \delta \geq 0\), \(\alpha + \beta + 2\gamma + 2\delta \leq 1\).

Now we propose our first main result as follows:

**Theorem 2.** Let \((X, d)\) be a metric space, \(f : X \to X\) and \(T : X \to CB(X)\). If the hybrid pair \((f, T)\) satisfies generalized \((F, \varphi)\)-contraction condition \((2.1)\), and also enjoys the CLR\(_f\) property, then the mappings \(f\) and \(T\) have a coincidence point.

Moreover, if the hybrid pair \((f, T)\) is occasionally coincidentally idempotent, then the pair \((f, T)\) has a common fixed point.

**Proof.** Since the pair \((f, T)\) enjoys the CLR\(_f\) property, there exists a sequence \(\{x_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} fx_n = fu \in A = \lim_{n \to \infty} Tx_n,
\]

for some \(u \in X\) and \(A \in CB(X)\). We assert that \(fu \in Tu\). If not, then using condition \((2.1)\), we have

\[
\tau + F(\mathcal{H}^p(Tx_n, Tu)) \leq F\left(\varphi\left(\max \left\{ \frac{d^p(fx_n, Tx_n)}{1+d^p(fx_n, fx_n)}, \frac{\beta}{1+d^p(fy, Ty)} \right\} \right) \right).
\]

Passing to the limit as \(n \to \infty\), we have

\[
\tau + F(\mathcal{H}^p(A, Tu)) \leq F\left(\varphi\left(\max \left\{ \frac{d^p(fu, A)}{1+d^p(fu, fu)}, \frac{\beta}{1+d^p(fu, Tu)} \right\} \right) \right).
\]

By the CLR\(_f\) property, for all \(u \in X\),

\[
\tau + F(\mathcal{H}^p(A, Tu)) \leq F\left(\varphi\left(\max \left\{ \frac{d^p(fu, A)}{1+d^p(fu, fu)}, \frac{\beta}{1+d^p(fu, Tu)} \right\} \right) \right).
\]
Using \(fu \in A, \tau > 0\), (\(F_1\)) and property of \(\Phi\), we have
\[
\begin{align*}
\mathcal{H}(A,Tu) & \leq \varphi\left(\max\left\{0, d^p(fu,Tu), 0, \frac{1}{2}[d^p(fu,Tu) + 0], 0, 0\right\}\right) \\
& = \varphi(d^p(fu,Tu)) \\
& < d^p(fu,Tu).
\end{align*}
\]
Since \(fu \in A\) the above inequality implies
\[
d(fu,Tu) \leq \mathcal{H}(A,Tu) < d(fu,Tu),
\]
a contradiction. Hence \(fu \in T u\) which shows that the pair \((f,T)\) has a coincidence point (i.e., \(C(f,T) \neq \emptyset\)).

Now, assume that the hybrid pair \((f,T)\) is occasionally coincidentally idempotent. Then for some \(v \in C(f,T)\), we have \(fv = f v \in Tv\). Our claim is that \(Tv = Tf v\). If not, then using condition (2.1), we get
\[
\begin{align*}
\tau + F(\mathcal{H}(Tfv,Tv)) & \leq F\left(\varphi\left(\max\left\{d^p(ffv,Tfv), \frac{d^p(fv,f v)}{1+d^p(fv,f v)}, \frac{d^p(fv,f v)}{1+d^p(fv,f v)}\right\}\right)\right) \\
& = F\left(\varphi\left(\max\left\{d^p(fv,f v), 0, \frac{1}{2}\left[d^p(fv,f v) + d^p(fv,Tv)\right], d^p(fv,f v)\left[d^p(fv,Tv)d^p(fv,f v)/(1+d^p(fv,f v))\right]\right\}\right)\right).
\end{align*}
\]
Since \(fv \in Tv\), the above inequality implies
\[
\begin{align*}
\tau + F(\mathcal{H}(Tfv,Tv)) & \leq F\left(\varphi\left(\max\left\{d^p(fv,Tfv), 0, \frac{1}{2}d^p(fv,Tfv), 0, 0\right\}\right)\right) \\
& = F\left(\varphi(d^p(Tfv,fv))\right).
\end{align*}
\]
Using (\(F_1\)) and property of \(\Phi\), we get
\[
d^p(Tfv,fv) < d^p(Tfv,fv),
\]
which is a contradiction. Thus we have \(fv = ff v \in Tv = Tf v\) which shows that \(fv\) is a common fixed point of the mappings \(f\) and \(T\).

In view of Remark 1 we have the following natural result:

**Corollary 1.** Let \((X,d)\) be a metric space, \(f : X \to X\) and \(T : X \to CB(X)\). If the hybrid pair \((f,T)\) satisfies generalized \((F,\varphi)\)-contraction condition (2.1), and enjoys the (E.A) property along with the closedness of \(f(X)\), then the mappings \(f\) and \(T\) have a coincidence point.

Moreover, if the hybrid pair \((f,T)\) is occasionally coincidentally idempotent, then the pair \((f,T)\) has a common fixed point.

Notice that, a non-compatible hybrid pair always satisfies the property (E.A). Hence, we get the following corollary:

**Corollary 2.** Let \(f\) be a self mapping on a metric space \((X,d)\), \(T\) a mapping from \(X\) into \(CB(X)\) satisfying generalized \((F,\varphi)\)-contraction condition (2.1). If the hybrid pair \((f,T)\) is non-compatible and \(f(X)\) a closed subset of \(X\), then the mappings \(f\) and \(T\) have a coincidence point.

Moreover, if the pair \((f,T)\) is occasionally coincidentally idempotent, then the pair \((f,T)\) has a common fixed point.
If \( F : \mathbb{R}^+ \rightarrow \mathbb{R} \) is defined by \( F(t) = \ln t \) and denoting \( e^{-r} = k \), then we have the following corollary:

**Corollary 3.** Let \((X, d)\) be a metric space, \( f : X \rightarrow X \) and \( T : X \rightarrow CB(X) \). Assume that there exist \( k \in (0, 1) \), \( \varphi \in \Phi \) such that

\[
H^p(Tx, Ty) \leq k \varphi \left( \max \left\{ \frac{d^p(fx, Tx)}{1 + d^p(fx, fx)}, \frac{d^p(fy, Ty)}{1 + d^p(fy, fy)} \right\} \right)
\]

for all \( x, y \in X \) with \( H(Tx, Ty) > 0 \), \( p \geq 1 \), and the hybrid pair \((f, T)\) enjoys the CLR\(_f\). Then the mappings \( f \) and \( T \) have a coincidence point.

Moreover, if the hybrid pair \((f, T)\) is occasionally coincidentally idempotent, then the pair \((f, T)\) has a common fixed point.

Now, we present our second main result as follows:

**Theorem 3.** Let \((X, d)\) be a metric space, \( f : X \rightarrow X \) and \( T : X \rightarrow CB(X) \). If the hybrid pair \((f, T)\) satisfies a Rational Hardy-Rogers \((F, \varphi)\)-contraction condition \((2.2)\) and also enjoys the CLR\(_f\) property, then the mappings \( f \) and \( T \) have a coincidence point.

Moreover, if the hybrid pair \((f, T)\) is occasionally coincidentally idempotent, then the pair \((f, T)\) has a common fixed point.

**Proof.** As the pair \((f, T)\) shares the CLR\(_f\) property, there exists a sequence \( \{x_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} fx_n = fu \in A = \lim_{n \to \infty} Tx_n,
\]

for some \( u \in X \) and \( A \in CB(X) \). We assert that \( fu \in Tu \). If not, then using condition \((2.2)\), we have

\[
\tau + F(H^p(Tx_n, Tu)) \leq F\left( \varphi\left( \frac{\alpha d^p(fx_n, fu) + \beta[1 + d^p(fx_n, Tx_n)]d^p(fu, Tu)}{1 + d^p(fx_n, fu)} \right) + \gamma [d^p(fu, Tfu) + d^p(fu, Tu)] + \delta [d^p(fx_n, Tu) + d^p(fu, Tx_n)] \right).
\]

Passing to the limit as \( n \to \infty \) in the above inequality, we obtain

\[
\tau + F(H^p(A, Tu)) \leq F(\varphi(\beta + \gamma + \delta)d^p(fu, Tu)),
\]

Using \( \tau > 0 \) and \((F1)\) and property of \( \Phi \), it follows that

\[
d^p(fu, Tu) \leq d^p(A, Tu) < (\beta + \gamma + \delta)d^p(fu, Tu),
\]

a contradiction, as \( \beta + \gamma + \delta \leq 1 \). Hence, \( fu \in Tu \) which shows that the hybrid pair \((f, T)\) has a coincidence point (i.e., \( C(f, T) \neq \emptyset \)).

Now, if the mappings \( f \) and \( T \) are occasionally coincidentally idempotent, then there exists \( v \in C(f, T) \) such that \( ffv = fv \in Tv \). Our claim is that \( fu \) is the common fixed point of \( f \) and \( T \). It is sufficient to show that \( Tu = Tf \). If not, then using condition \((2.2)\), we have

\[
\tau + F(H^p(Tfu, Tv)) \leq F\left( \varphi\left( \frac{\alpha d^p(ffv, fv) + \beta[1 + d^p(fu, Tfu)]d^p(fv, Tv)}{1 + d^p(ffv, fv)} \right) + \gamma [d^p(ffv, Tfu) + d^p(fv, Tv)] + \delta [d^p(ffv, Tv) + d^p(fu, Tfu)] \right).
\]

Since \( fu \in Tv \), the above inequality implies

\[
\tau + F(d^p(Tfu, Tv)) \leq F(\varphi(\gamma + \delta)d^p(fv, Tv)).
\]

Using \((F1)\) and property of \( \Phi \), we can have

\[
d^p(Tfu, fv) < (\gamma + \delta)d^p(fv, Tv),
\]

and we conclude that \( Tu = Tf \).
a contradiction, as \( \gamma + \delta \leq 1 \). Thus, \( fv = ffv \in Tv = Tf \) which shows that \( fv \) is a common fixed point of the mappings \( f \) and \( T \).

If \( F : \mathbb{R}^+ \to \mathbb{R} \) is given by \( F(t) = \ln t \) and denoting \( e^{-\tau} = k \), then we have the following corollary:

**Corollary 4.** Let \((X,d)\) be a metric space, \( f : X \to X \) and \( T : X \to CB(X) \). Suppose that there exist \( k \in (0,1) \), \( \varphi \in \Phi \) such that

\[
\mathcal{H}(Tx,Ty) \leq k\varphi \left( \alpha d^p(fx,fy) + \frac{\beta [1 + d^p(fx,Tx)][d^p(fy,Ty)]}{1 + d^p(fx,fy)} + \gamma [d^p(fx,Tx) + d^p(fy,Ty)] \right)
\]

for all \( x, y \in X \) with \( Tx \neq Ty \), where \( p \geq 1 \), \( \alpha, \beta, \gamma, \delta \geq 0 \), \( \alpha + \beta + 2\gamma + 2\delta \leq 1 \), and the hybrid pair \((f,T)\) enjoys the CLR\(f\). Then the mappings \( f \) and \( T \) have a coincidence point.

Moreover, if the pair \((f,T)\) is occasionally coincidentally idempotent, then the pair \((f,T)\) has a common fixed point.

### 3. Illustrative Example

In this section, we provide an example to establish the genuineness of our extension.

**Example 5.** Let \( X = [0,3] \) be a metric space equipped with the metric \( d(x,y) = |x - y| \). Define \( f : X \to X \) and \( T : X \to CB(X) \) as follows:

\[
f(x) = \begin{cases} 
3 - x, & \text{if } x \in [0,2], \\
3, & \text{if } x \in (2,3].
\end{cases} \quad T(x) = \begin{cases} 
[1,2], & \text{if } x \in [0,2], \\
[0,\frac{1}{2}], & \text{if } x \in (2,3].
\end{cases}
\]

Let \( F : \mathbb{R}^+ \to \mathbb{R} \) such that \( F(t) = \ln(t) \), \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \varphi(t) = \frac{2}{10}t \) and \( \tau = \frac{1}{5} > 0 \) and \( p \geq 1 \). Then it is easy to verify that

- \( F \in \mathcal{F} \); \( \varphi \in \Phi \); \( f(X) = [1,3] \cup \{3\} \), a closed set in \( X \); \( \mathcal{C}(f,F) = [1,2] \);
- the hybrid pair \((f,T)\) satisfies CLR\(f\) property, as for the sequence \( \{1 + \frac{1}{n}\}_{n \in \mathbb{N}} \),

\[
\lim_{n \to \infty} f(x) = \lim_{n \to \infty} \left( 2 - \frac{1}{n} \right) = 2 = f1 \in [1,2] = \lim_{n \to \infty} T(1 + \frac{1}{n});
\]

- \((f,T)\) is not coincidentally idempotent because \( ff1 = f2 = 1 \neq 2 = f1 \);
- \((f,T)\) is occasionally coincidentally idempotent, because \( ff_{\frac{1}{2}} = f_{\frac{3}{2}} = \frac{5}{2} \).

Now, in order to verify condition \((2.1)\), we distinguish two cases:

Case I: If \( x \in [0,2] \) and \( y \in (2,3] \), then

\[
\mathcal{H}(Tx,Ty) = \mathcal{H}([1,2],[0,\frac{1}{2}]) = \max \left\{ d([1,2],[0,\frac{1}{2}]), d([0,\frac{1}{2}],[1,2]) \right\} = \max \left\{ \frac{3}{2}, 1 \right\} = \frac{3}{2},
\]

and \( d(fy,Ty) = d(3,[0,\frac{1}{2}]) = \frac{5}{2} \). Therefore, for all \( p \geq 1 \), we get

\[
\mathcal{H}^p(Tx,Ty) = \left( \frac{3}{2} \right)^p < e^{-\frac{2}{5}} \left( \frac{9}{10} \right)^{\frac{5}{2}} = e^{-\frac{1}{5}} \left( \frac{9}{10} d^p(fy,Ty) \right) = e^{-\tau} \varphi(d^p(fy,Ty))
\]

Taking the logarithms of the both side along with \( F(t) = \ln(t) \), we get

\[
\tau + F(\mathcal{H}^p(Tx,Ty)) < F(\varphi(d^p(fy,Ty))).
\]
Case II: If \( x \in (2, 3] \) and \( y \in [1, 2] \), then
\[
H(Tx, Ty) = H\left([0, \frac{1}{2}], [1, 2]\right) = \frac{3}{2} \quad \text{and} \quad d(fx, Tx) = d(3, [0, \frac{1}{2}]) = \frac{5}{2}.
\]
Therefore, for all \( p \geq 1 \), we get
\[
H^p(Tx, Ty) = \left(\frac{3}{2}\right)^p < e^{-\frac{1}{5} \left(\frac{9}{10} \left(\frac{5}{2}\right)^p\right)} = e^{-\frac{1}{5} \varphi\left(\frac{5}{2}\right)}.
\]
Taking the logarithm of both sides of the above inequality and using \( \ln(t) = \mathcal{F}(t) \), we get
\[
\tau + F\left(H^p(Tx, Ty)\right) < F\left(\varphi\left(d^p(fx, Tx)\right)\right).
\]
Notice that for \( x, y \in [1, 2] \) (or \( x, y \in (2, 3] \)) \( H(Tx, Ty) = 0 \).

Thus, all the hypotheses of Theorem 2 are satisfied and the hybrid pair \((f, T)\) has the common fixed point (namely \( \frac{3}{2} \)).

With a view to establish genuineness of our extension, notice that for \( x = 1, y = 3 \), we have
\[
H(Tx, Ty) = \frac{3}{2}; \quad d(fx, fy) = d(2, 3) = 1;
\]
\[
\frac{1}{2} [d(fx, Tx) + d(fy, Ty)] = \frac{1}{2} [d(2, [1, 2]) + d(3, [0, \frac{1}{2}])] = \frac{1}{2} (0 + \frac{5}{2}) = \frac{5}{4}
\]
and
\[
\frac{1}{2} [d(fx, Ty) + d(fy, Tx)] = \frac{1}{2} [d(2, [0, \frac{1}{2}]) + d(3, [1, 2])] = \frac{1}{2} (\frac{3}{2} + 1) = \frac{5}{4},
\]
which shows that the contractive condition of Theorem 11 (due to Kadelburg et al. [18]) is not satisfied. Thus, in all our Theorem 2 is applicable to the present example while Theorem 11 of Kadelburg et al. [18] is not which substantiates the utility of Theorem 2.

4. Applications

As applications of our main results, we prove an existence theorem on bounded solutions of a system of functional equations. Also, an existence theorem on the solution of integral inclusion is proved.

4.1. Application To Dynamic Programming. In 1978, Bellman and Lee [5] first studied the existence of solutions for functional equations wherein authors notice that the basic form of functional equations in dynamic programming can be described as follows:
\[
q(x) = \sup_{y \in D}\{G(x, y, q(\tau(x, y)))\}, \quad x \in W,
\]
where \( \tau : W \times D \to W \), \( G : W \times D \times \mathbb{R} \to \mathbb{R} \) are mappings, while \( W \subseteq U \) is a state space, \( D \subseteq V \) is a decision space, and \( U, V \) are Banach spaces.

In 1984, Bhakta and Mitra [6] obtained some existence theorems for the following functional equation which arises in multistage decision process related to dynamic programming
\[
q(x) = \sup_{y \in D}\{g(x, y) + G(x, y, q(\tau(x, y)))\}, \quad x \in W,
\]
where \( \tau : W \times D \to W \), \( g : W \times D \to \mathbb{R} \), \( G : W \times D \times \mathbb{R} \to \mathbb{R} \) are mappings, while \( W \subseteq U \) is a state space, \( D \subseteq V \) is a decision space, and \( U, V \) are Banach spaces.
In recent years, a lot of work have been done in this direction wherein a multitude of existence and uniqueness results have been obtained for solutions and common solutions of some functional equations, including systems of functional equations in dynamic programming using suitable fixed point results. For more details one can consults [26, 27, 31–33, 37] and the references therein.

Consider now a multistage process, reduced to the system of functional equations

\[ q_i(x) = \sup_{y \in D} \{ g(x, y) + G_i(x, y, q_i(\tau(x, y))) \}, \quad x \in W, \ i \in \{1, 2\}, \tag{4.1} \]

where \( \tau : W \times D \to W, \ g : W \times D \to \mathbb{R}, \ G_i : W \times D \times \mathbb{R} \to \mathbb{R} \) are given mappings, while \( W \subseteq U \) is a state space, \( D \subseteq V \) is a decision space, and \( U, V \) are Banach spaces. The purpose of this section is to prove the existence of solutions for a system of functional equations (4.1) using Theorem 2.

Let \( B(W) \) be the set of all bounded real-valued functions on \( W \). For an arbitrary \( h \in B(W) \) define \( \| h \| = \sup_{x \in W} |h(x)| \), with respective metric \( d \). Also, \( (B(W), \| \cdot \|) \) is a Banach space wherein convergence is uniform. Therefore, if we consider a Cauchy sequence \( \{ h_n \} \) in \( B(W) \), then the sequence \( \{ h_n \} \) converges uniformly to a function, say \( h^* \), so that \( h^* \in B(W) \).

We consider the operators \( T_i : B(W) \to B(W) \) given by

\[ T_i h_i(x) = \sup_{y \in D} \{ g(x, y) + G_i(x, y, h_i(\tau(x, y))) \}, \tag{4.2} \]

for \( h_i \in B(W), \ x \in W, \ for \ i = 1, 2; \) these mappings are well-defined if the functions \( g \) and \( G_i \) are bounded. Also, denote

\[ \Theta(h, k) = \max \left\{ \frac{d(T_2h, T_2k)}{1+d(T_2h, T_2k)}, \frac{d(T_1h, T_1k)}{1+d(T_1h, T_1k)}, \frac{d(T_1h, T_2k)+d(T_1k, T_2h)}{1+d(T_1h, T_1k)}, \frac{d(T_2h, T_1k)+d(T_1k, T_2h)}{1+d(T_1h, T_1k)} \right\}, \]

for \( h, k \in B(W) \).

**Theorem 4.** Let \( T_i : B(W) \to B(W) \) be given by (4.2), for \( i = 1, 2 \). Suppose that the following hypotheses hold:

1. There exist \( \tau \in \mathbb{R}^+ \) and \( \varphi \in \Phi \) such that
   \[ |G_1(x, y, h(x)) - G_2(x, y, k(x))| \leq e^{-\tau} \varphi(\Theta(h, k)) \]
   for all \( x \in W, \ y \in D; \)
2. \( g : W \times D \to \mathbb{R} \) and \( G_i : W \times D \times \mathbb{R} \to \mathbb{R} \) are bounded functions, for \( i = 1, 2; \)
3. There exists a sequence \( \{ h_n \} \) in \( B(W) \) and a function \( h^* \in B(W) \) such that
   \[ \lim_{n \to \infty} T_1h_n = \lim_{n \to \infty} T_2h_n = T_1h^*; \]
4. \( T_1T_1h = T_1h, \ whenever \ T_1T_1h = T_2h, \ for \ some \ h \in B(W). \)

Then the system of functional equations (4.1) has a bounded solution.

**Proof.** By hypothesis (3), the pair \( (T_1, T_2) \) shares the common limit range property with respect to \( T_1 \). Now, let \( \lambda \) be an arbitrary positive number, \( x \in W \) and \( h_1, h_2 \in B(W) \). Then

\[ q_i(x) = \sup_{y \in D} \{ g(x, y) + G_i(x, y, q_i(\tau(x, y))) \}, \quad x \in W, \ i \in \{1, 2\}, \]
there exist \( y_1, y_2 \in D \) such that
\[
T_1h_1(x) < g(x, y_1) + G_1(x, y_1, h_1(\tau(x, y_1))) + \lambda,  \\ T_2h_2(x) < g(x, y_2) + G_2(x, y_2, h_2(\tau(x, y_2))) + \lambda,  \\ T_1h_1(x) \geq g(x, y_2) + G_1(x, y_2, h_1(\tau(x, y_2))),  \\ T_2h_2(x) \geq g(x, y_1) + G_2(x, y_1, h_2(\tau(x, y_1))).
\] (4.3) (4.4) (4.5) (4.6)

Next, by using (4.3) and (4.6), we obtain
\[
T_1h_1(x) - T_2h_2(x) < G_1(x, y_1, h_1(\tau(x, y_1))) - G_2(x, y_1, h_2(\tau(x, y_1))) + \lambda
\]
\[
\leq |G_1(x, y_1, h_1(\tau(x, y_1))) - G_2(x, y_1, h_2(\tau(x, y_1)))| + \lambda
\]
\[
\leq e^{-\tau} \varphi(\Theta(h_1, h_2)) + \lambda
\] and so we have
\[
T_1h_1(x) - T_2h_2(x) < e^{-\tau} \varphi(\Theta(h_1, h_2)) + \lambda.  \\ (4.7)
\]

Analogously, by using (4.4) and (4.5), we get
\[
T_2h_2(x) - T_1h_1(x) < e^{-\tau} \varphi(\Theta(h_1, h_2)) + \lambda  \\ (4.8)
\]

Combining (4.7) and (4.8), we get
\[
|T_1h_1(x) - T_2h_2(x)| < e^{-\tau} \varphi(\Theta(h_1, h_2)) + \lambda,
\]

implying thereby
\[
d(T_1h_1, T_2h_2) \leq e^{-\tau} \varphi(\Theta(h_1, h_2)) + \lambda.
\]

Notice that, the last inequality does not depend on \( x \in W \) and \( \lambda > 0 \) is taken arbitrarily, therefore we obtain
\[
d(T_1h_1, T_2h_2) \leq e^{-\tau} \varphi(\Theta(h_1, h_2)).
\]

By passing to logarithms, we can write
\[
\tau + \ln(d(T_1h_1, T_2h_2)) \leq \ln(\varphi(\Theta(h_1, h_2))).
\]

If we consider \( F \in F \) defined by \( F(t) = \ln t \), for each \( t \in (0, +\infty) \), and put \( f = T_1, T = T_2 \), then all the hypotheses of Theorem \[2\] are satisfied for the pair \( (f, T) \) and \( p = 1 \). Moreover, in view of the hypotheses (4), the pair \( (T_1, T_2) \) is occasionally coincidentally idempotent, so by using Theorem \[2\], the mapping \( T_1 \) and \( T_2 \) have a common fixed point, that is, the system of functional equations (4.1) has a bounded solution. \( \square \)

### 4.2. Application to Volterra integral inclusions

Here, we present yet another application of Theorem \[3\] This application is essentially inspired by [10].

We establish new results on the existence of solutions of integral inclusion of the type
\[
x(t) \in q(t) + \int_0^{\sigma(t)} k(t, s)F(s, x(s))\, ds  \\ (4.9)
\]
for \( t \in J = [0, 1] \subset \mathbb{R} \), where \( \sigma : J \to J \), \( q : J \to E \), \( k : J \times J \to \mathbb{R} \) are continuous and \( F : J \times E \to C(E) \), where \( E \) is a Banach space with norm \( \| \cdot \|_E \) and \( C(E) \) denotes the class of all nonempty closed subsets of \( E \).

Let \( C(J, E) \) be the space of all continuous \( E \)-valued functions on \( J \). Define a norm \( \| \cdot \| \) on \( C(J, E) \) by
\[
\| x \| = \sup_{t \in J} \| x(t) \|_E.
\]
Definition 7. A continuous function \( a \in C(J, E) \) is called a lower solution of the integral inclusion \([4.9]\), if it satisfies
\[
a(t) \leq q(t) + \int_0^{\sigma(t)} k(t, s)v_1(s)ds,
\]
such that \( v_1(t) \in F(t, a(t)) \) almost everywhere (a.e.) for \( t \in J \), where \( B(J, E) \) is the space of all \( E \)-valued Bochner-integrable functions on \( J \). Similarly, a continuous function \( b \in C(J, E) \) is called an upper solution of the integral inclusion \([4.9]\), if it satisfies
\[
b(t) \geq q(t) + \int_0^{\sigma(t)} k(t, s)v_2(s)ds,
\]
such that \( v_2(t) \in F(t, b(t)) \) a.e. for \( t \in J \).

Notice that, all the solution lies between lower solution ‘\( a \)’ as well upper solution ‘\( b \)’. We can denote the solution set as an interval \([a, b]\).

Definition 8. A continuous function \( x : J \to E \) is said to be a solution of the integral inclusion \([4.9]\), if
\[
x(t) = q(t) + \int_0^{\sigma(t)} k(t, s)v(s)ds
\]
for some \( v \in B(J, E) \) satisfying \( v(t) \in F(t, x(t)) \), for all \( t \in J \).

In what follows, we also need the following definitions:

Definition 9. A multi-valued mapping \( F : J \to 2^E \) is said to be measurable if for any \( y \in E \), the function \( t \mapsto d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\} \) is measurable.

Definition 10. A multi-valued mapping \( \beta : J \times E \to 2^E \) is called Carathéodory if
(i) \( t \mapsto (t, x) \) is measurable for each \( x \in E \), and
(ii) \( x \mapsto (t, x) \) is upper semicontinuous almost everywhere for \( t \in J \).

Denote
\[
\|F(t, x)\| = \sup\{\|u\|_E : u \in F(t, x)\}.
\]

Definition 11. A Carathéodory multi-mapping \( F(t, x) \) is called \( L^1 \)-Carathéodory if for every real number \( r > 0 \), there exists a function \( h_r \in L^1(J, \mathbb{R}) \) such that
\[
\|F(t, x)\| \leq h_r(t) \text{ for almost every } t \in J
\]
and for all \( x \in E \) with \( \|x\|_E \leq r \).

Denote
\[
S_F^L(x) = \{v \in B(J, E) : v(t) \in F(t, x(t)) \text{ a.e. } t \in J\}.
\]

Lemma 1. \([23]\) If \( \text{diam}(E) < \infty \) and \( F : J \times E \to 2^E \) is \( L^1 \)-Carathéodory, then \( S_F^L(x) \neq \emptyset \) for each \( x \in C(J, E) \).

Lemma 2. \([26]\) Let \( E \) be a Banach space, \( F \) a Carathéodory multi-mapping with \( S_F^L \neq \emptyset \) and \( L : L^1(J, E) \to C(J, E) \) a continuous linear mapping. Then the operator
\[
L \circ S_F^L : C(J, E) \to 2^{C(J,E)}
\]
is a closed graph operator on \( C(J, E) \times C(J, E) \).
Let us list the following set of conditions:

\((H_0)\) The function \(k(t,s)\) is continuous and non-negative on \(J \times J\) with
\[
e^{-\tau} = \sup_{t,s \in J} k(t,s)
\]
for some \(\tau \in \mathbb{R}^+;\)

\((H_1)\) The multi-valued mapping \(F(t,x)\) is Carathéodory;

\((H_2)\) The multi-valued mapping \(F(t,x)\) is increasing in \(x\) almost everywhere for \(t \in J;\)

\((H_3)\) There exist \(\tau \in \mathbb{R}^+\) and \(\varphi \in \Phi\) such that
\[
|F(s,x(s)) - F(s,y(s))| \leq e^{-\tau} \varphi(\Delta(x,y))
\]
for all \(s \in J, x \in E,\) where
\[
\Delta(x,y) = \alpha |f_x - f_y| + \beta \left[\frac{1 + |f_x - T x| |f_y - T y|}{1 + |f_x - f_y|}\right] + \gamma \left[|f_x - T x| + |f_y - T y|\right]
\]
with \(\alpha, \beta, \gamma, \delta \geq 0, \alpha + \beta + 2\gamma + 2\delta \leq 1;\)

\((H_4)\) \(S_F(x) \neq \emptyset\) for each \(x \in C(J,E).\)

**Theorem 5.** Suppose that the conditions \((H_0)-(H_4)\) hold. Then the integral inclusion \([1.9]\) has a solution in \([a,b]\) defined on \(J.\)

**Proof.** Let \(X = C(J,E).\) Define a multi-valued mapping \(T : [a,b] \subset X \to 2^X\) given by
\[
T x = \left\{ u \in [a,b] : u(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v(s) ds; \ v \in S_F(x), \text{ for every } t \in [0,1] \right\}.
\]
Observe that \(T\) is well-defined, as owing to \((H_4),\) \(S_F(x) \neq \emptyset.\) To show that \(T\) satisfies all hypotheses of Theorem 3 defined on \([a,b].\)

For all \(\vartheta, \mu \in 2^X\) on \(t \in J\) and making use of \((H_0)\) and \((H_3),\) we have
\[
\|\vartheta(t) - \mu(t)\|_E = \left\| \int_0^{\sigma(t)} k(t,s)v_1(s) ds - \int_0^{\sigma(t)} k(t,s)v_2(s) ds \right\|_E
\leq \int_0^{\sigma(t)} k(t,s) ds \|v_1(s) - v_2(s)\|_E
\leq \sup_{t,s \in J} k(t,s) \varphi(\Delta(v_1,v_2)) \text{ for } v_1, v_2 \in S_F(x).
\]
This implies that
\[
\|\vartheta(t) - \mu(t)\|_E \leq e^{-\tau} \varphi(\Delta(v_1,v_2)).
\]
for each \(t \in J.\) Passing to logarithms, we can write this as
\[
\tau + \ln \|\vartheta(t) - \mu(t)\|_E \leq \ln(\varphi(\Delta(v_1,v_2))).
\]
If we consider \(F \in \mathcal{F}\) defined by \(F(z) = \ln z,\) for each \(z \in (0, +\infty),\) we deduce that the operator \(T\) satisfy condition \((2.2)\) where \(f\) is an identity mapping and \(p = 1.\) Also \(T\) is a closed mapping, using Theorem 3 we conclude that the given integral inclusion has a solution in \([a,b].\) \(\square\)
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