Instabilities of (near) extremal rotating black holes in higher dimensions

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Recently, Durkee and Reall have conjectured a criterion for linear instability of rotating, extremal, asymptotically Minkowskian black holes in $d \geq 4$ dimensions, such as the Myers-Perry black holes. They considered a certain elliptic operator, $\mathcal{A}$, acting on symmetric traceless tensors intrinsic to the horizon. Based in part on numerical evidence, they suggested that if the lowest eigenvalue, $\lambda$, of this operator is less than the critical value $-1/4$ (called “effective BF-bound”), then the black hole is linearly unstable. In this paper, we prove their conjecture. Our proof uses a combination of methods such as (i) the “canonical energy method” of Hollands-Wald, (ii) algebraically special properties of the near horizon geometries associated with the black hole, and (iii) the structure of the (linearized) constraint equations. Our method of proof is also applicable to rotating, extremal asymptotically Anti-deSitter black holes. In that case, our methods show that all such black holes are unstable. Although we explicitly discuss in this paper only extremal black holes, we argue that our methods can be generalized straightforwardly to obtain the same results for near extremal black holes.

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1. Introduction

Whether one believes that extra dimensions must play a role in fundamental theories of Nature, or whether one just employs them as a tool in holographic approaches to strongly correlated real-life systems [1, 2], one needs to understand the nature of black holes in higher dimensional gravity theories. Apart from the obvious interest in finding new, in particular stationary, black hole solutions, it is also very important to understand the stability properties of known solutions, see [3] for a review. Stable black holes are of obvious relevance. But also unstable ones are interesting, because instabilities can evolve to new, as yet unknown, black holes, or they can correspond to new, stationary black holes branching off a known solution.

To analyze the (in)stability of a background, the first step is to study linear perturbations, i.e. solutions to the linearized Einstein equations (in this paper we consider the vacuum Einstein equations with $\Lambda$). If these settle down in a sufficiently strong sense, then one can hope that small non-linear perturbations will do the same. On the other hand, if there are linear perturbations which do not settle down, then the background is clearly unstable, although a linear analysis cannot be used to predict what might be the endpoint of the non-linear evolution. In this paper, we want to identify criteria for linear instabilities of (rotating) black hole backgrounds in $d \geq 4$ dimensions.

Unfortunately, the problem of understanding the long-time behavior of solutions to the linearized Einstein equations on black hole backgrounds is a highly non-trivial problem. It has been solved in generality only for the – already far from trivial – case of Schwarzschild spacetime [4, 5, 6], and its higher-dimensional cousins [7], where no unstable modes$^1$ were found. For the Kerr-metric, one can cast the perturbation equations in Teukolsky form [8], and thereby analyze stability. Again, no unstable modes were found [9]. This success suggests to find an analogous ‘Teukolsky’ form for the perturbation equations of rotating black holes also in higher dimensions, e.g. for the Myers-Perry solutions [10, 11], which can be viewed as generalizations of Kerr/Kerr-AdS, or the black rings [12, 13]. Since the existence of the Teukolsky form appears to be related to the profound algebraically special properties of the Kerr metric, one is naturally led to generalize such notions to higher dimensions, as was in fact done in a series of papers by [14, 15, 16, 17, 18]. Unfortunately, the bottom line of these investigations is that the known rotating black holes are not of sufficiently algebraically special nature in order to cast the perturbation equations in Teukolsky form. It appears that, mainly for this reason, there has been limited success in the analytical understanding of the (even linear) stability of generic rotating black holes in higher dimensions, although there are by now several very interesting partial numerical results [19, 20].

In [17] Durkee and Reall observed that, while the perturbation equations on the known asymptotically flat rotating backgrounds in $d > 4$ cannot be put in Teukolsky form, this is possible for their near horizon (NH) limits [21, 22, 23, 24]. In fact, [17] showed that the Teukolsky equations on the NH geometry separate into an “$(R, T)$”-part obeying a charged Klein-Gordon equation in an auxiliary$^2$ $AdS_2$-space, and an “angular part”. The modes of the angular part are eigenfunctions of an elliptic operator, $\mathcal{A}$, acting on symmetric traceless tensors intrinsic to the

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$^1$ Mode stability does not imply uniform boundedness in time of generic perturbations with bounded initial data. This problem has been studied e.g. in [25, 26, 27].

$^2$ Note that the original black hole background has vanishing cosmological constant, and is asymptotically Minkowskian, rather than asymptotically anti-deSitter.
(d − 2)-dimensional horizon cross section, \( \mathcal{B} \). Its eigenvalues effectively become a mass term in the AdS\(_2\)-Klein-Gordon equation for the \( (R, T) \)-part. By looking at the properties of that equation, [17] made a conjecture about the stability properties of the corresponding extremal black hole (assumed to be asymptotically flat, \( \Lambda = 0 \)):

**Conjecture 1:** Assuming generic\(^4\) values of the angular velocities of the black hole, if the lowest eigenvalue \( \lambda \) of the operator \( \mathcal{A}(a = 0) \) (see eq. (73)) is below the critical value of \(-\frac{1}{4}\) (called the “effective BF-bound”), then the original extremal black hole is unstable.

In order to support their conjecture, [17] worked out explicitly the spectrum of \( \mathcal{A} \) in the case of the cohomogeneity-1 Myers-Perry black holes, and compared the implications of their conjecture to the numerical results of [19, 20]. The conjecture was thereby found to hold up to dimension \( d = 15 \). In this paper, we prove conjecture 1. The precise statements are given below in thm. 2, which also includes a statement concerning “nongeneric” values of the spin parameter. In order to show conjecture 1, the first idea might be to look at the explicit form of the \( (R, T) \)-part of the perturbations in the NH geometry corresponding to a mass below the effective BF-bound. Unfortunately, while these modes can be given in closed form (see e.g. [31, 32] and also appendix C), it is hard to see what one learns from them directly about the behavior of perturbations on the original black hole background. The point is that the modes fail to be \( L^2 \)-normalizable at the “infinity”, \( R \to \infty \), of the NH geometry. But the NH geometry is supposed to be a reasonable description (“blow up”) of the black hole only for finite \( R \), so it is rather unclear how one could use those modes directly to prove or disprove the above conjecture. It is also unclear how to implement the dynamical evolution of compactly supported initial data for our AdS\(_2\)-Klein-Gordon equation, because the case \( \lambda < -\frac{1}{4} \) corresponds precisely to the situation where it is essentially impossible to have a well-defined dynamics [33].

For these reasons, we will use a different approach which is based on the “canonical energy method” introduced by [28], which is a sort of variational principle\(^5\). The method proceeds by defining the “canonical energy”, \( \mathcal{E} \), of the perturbation, \( \gamma_{ab} \). This is a quadratic expression depending on up to two derivatives of the perturbation, and depending on a Cauchy surface \( \Sigma \) outside the black hole. The concrete form of \( \mathcal{E} \) and its key properties are recalled below in sec. 2.2. These are: 1) \( \mathcal{E} \) is gauge invariant, 2) \( \mathcal{E} \) is monotonically decreasing for any axi-symmetric perturbation, in the sense that \( \mathcal{E}(\Sigma_2) \leq \mathcal{E}(\Sigma_1) \) as long as \( \Sigma_2 \) is later than \( \Sigma_1 \) [see fig. 1]. 3) \( \mathcal{E} \) vanishes if and only if \( \gamma_{ab} \) represents a perturbation towards another black hole in the family up to a gauge transformation. Properties 1), 2), and 3) together imply that if we can

\(^3\) The conjecture stated in [17] is stronger than that given here, in that an eigenvalue \( \lambda < -\frac{1}{4} \) was also suggested to be necessary for an instability of the black hole, under certain conditions on the lapse function of the near horizon geometry. Our version of the conjecture only gives a sufficient condition for an instability of the black hole, and is somewhat weaker in this sense. On the other hand, the condition on the lapse functions has been replaced here by the condition that \( \Omega \) be generic, see footnote 4.

\(^4\) We call the angular velocities \( \Omega = (\Omega_1, \ldots, \Omega_n) \) generic if the components are linearly independent over \( \mathbb{Q} \), which is the same as saying that there is no non-trivial vector of integers \( m \) such that \( m \cdot \Omega = 0 \). The generic values form a dense set of full Lebesgue measure.

\(^5\) In [28] this approach was introduced in the context of general non-extremal black holes with \( \Lambda = 0 \), i.e. asymptotically flat boundary conditions. This method suitably generalizes to the extreme case, and it also generalizes to \( \Lambda < 0 \), i.e. asymptotically AdS-boundary conditions. Theories with various additional types of matter fields were considered in [29, 54].
find a perturbation with $\mathcal{E} < 0$, then such a perturbation cannot settle down to a perturbation to another stationary black hole in the family. Hence, such a perturbation must correspond to a linear instability.

Thus, to establish an instability, we must find a perturbation $\gamma_{ab}$ for which $\mathcal{E} < 0$. Since $\mathcal{E}$ can be expressed in terms of the initial data for the perturbation, we basically have a variational problem involving initial data. However, a major complication arises from the fact that the initial data must satisfy the linearized constraint equations. Since these have a rather complicated structure, this may at first sight appear to render our method rather impractical. Fortunately, it turns out that, in order to construct the desired perturbation with $\mathcal{E} < 0$, we can proceed by a roundabout route which effectively avoids having to solve explicitly the constraints. There are basically three steps:

1. We pass to the NH limit of the black hole. In the NH limit, linearized perturbations can be constructed via a higher dimensional generalization [34] of the “Hertz-potential” ansatz [35, 36]. Using the Hertz-potential ansatz, and the separability property of the linearized Einstein equations on the NH geometry background established by [17], we reduce the canonical energy $\mathcal{E}$ in the NH geometry to an “energy-like” expression involving only the complex scalar Klein-Gordon-type field on the auxiliary $AdS_2$-space. It involves the lowest eigenvalue, $\lambda$ of the elliptic, second order, hermitian operator $\mathcal{A}$ (see eq. (73)) on the $(d - 2)$-dimensional horizon cross section. It is shown that the $AdS_2$-energy can become negative for compactly supported data outside the horizon if $\lambda$ is below the critical value $-\frac{1}{4}$. From these data, we get a gravitational perturbation with compactly supported initial data in the NH geometry, having $\mathcal{E} < 0$.

2. The perturbation of the NH geometry obtained in step 1) is next scaled, using the isometries of the NH geometry, to a perturbation having support in a neighborhood of “size” $\varepsilon$ near the horizon. In such a neighborhood, the NH geometry is by construction approximately equal to the original black hole geometry. It is therefore plausible – and will be shown – that the initial data of the scaled perturbation satisfy, to within a small error of order $\varepsilon$, the linearized constraint equations of the original geometry.

3. We show by the powerful methods of Corvino-Schoen [37] (see also [38]) that for sufficiently small $\varepsilon$, the initial data of the scaled perturbation on the NH geometry can be modified to give a perturbation on the original black hole geometry still having $\mathcal{E} < 0$.

The above mentioned properties of $\mathcal{E}$ then imply that conjecture 1 is true for any of the known extremal, asymptotically flat black holes, i.e. the Myers-Perry black holes and the black rings. Finding the lowest eigenvalue $\lambda$ of $\mathcal{A}$ in those concrete geometries is a much simpler problem than that of analyzing the perturbed Einstein equations (3), although even this problem probably has to be solved on a computer for generic values of the spin-parameters.

The techniques of this paper also apply to the case of rotating, extremal, asymptotically AdS black holes ($\Lambda < 0$) of the MP-type. In this case, we simply find that all such black holes are unstable (!), see thm. 3. The dS-case is briefly discussed in the conclusions section, where

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6 The $AdS_2$ Klein-Gordon field involves a complex charge parameter. This implies, among other things, that the energy like expression has an unusual form, containing up to 3 derivatives.
we also formulate a conjecture (conjecture 2) generalizing conjecture 1 to near extremal black holes. The same methods as for the gravitational perturbations also work for a test Maxwell field, and analogues of all of the above results are shown to be true for that case too.

**Conventions:** Our conventions for the signature and definition of the Riemann tensor are identical with those used in Wald’s text [39].

### 2. Stationary black holes and canonical energy

#### 2.1. Stationary black holes and their perturbations

In this paper, we consider \( d \)-dimensional stationary black hole spacetimes \((\mathcal{M}, g)\) with Killing horizons satisfying the Einstein equations

\[
R_{ab} - \frac{1}{2}Rg_{ab} = -\Lambda g_{ab}
\]

with cosmological constant \( \Lambda \).

A stationary spacetime with Killing horizon by definition has a Killing vector field (KVF) \( K \) that is tangent to the generators of the horizon \( \mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^- \), where \( \pm \) means the future/past horizon. This implies that, on \( \mathcal{H} \), we have

\[
K^b \nabla_b K^a = \kappa K^a .
\]

(1)

The quantity \( \kappa \geq 0 \) is the surface gravity and necessarily constant over \( \mathcal{H} \). A black hole is called extremal if \( \kappa = 0 \), in which case \( K \) is tangent to affinely parameterized null-geodesic generators of \( \mathcal{H} \).

For the remainder of this paper, we restrict attention to extremal black holes (BH’s). If the black hole is rotating, i.e. if \( K \) does not coincide with the asymptotic time-like KVF, then one can show in a very general setting [40] that there must exist rotational Killing vector fields, written

\[
\partial / \partial \phi_I , I = 1, \ldots, n > 0
\]

in suitable coordinates, such that

\[
K = \frac{\partial}{\partial t} + \Omega_I \frac{\partial}{\partial \phi_I} .
\]

(2)

The constants \( \Omega_I \in \mathbb{R}, I = 1, \ldots, n > 0 \) are called “angular velocities” of the horizon, and “rotational” means that \( \partial / \partial \phi_I \) generate an isometric action of \( \text{U}(1)^n \cong \text{SO}(n) \) on the spacetime corresponding to shifts in the angular coordinates \( \phi_I \). Concrete examples of such black holes are the Myers-Perry solutions [10, 11] (briefly reviewed in sec. 2), or the black rings [12, 13]. For these examples, \( n = \lfloor (d - 1)/2 \rfloor \).

When \( \Lambda = 0 \), one is dealing with asymptotically flat spacetimes, see [41, 42] and [43] for a precise definition of this concept in higher dimensions. For even \( d \), this notion can be formulated within the formalism of conformal infinity, used throughout this paper. In this framework, one considers a conformal compactification \((\tilde{\mathcal{M}}, \tilde{g} = \Omega^2 g)\) of \((\mathcal{M}, g)\). Future/past infinity correspond to the conformal boundary \( \mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^- \), which is a (conformal) null surface defined by \( \Omega = 0 \). When \( \Lambda < 0 \), the spacetime is asymptotically \( \text{AdS} \). In this case, the conformal boundary \( \mathcal{I} \) is timelike, see e.g. [45] for further explanation.

A metric perturbation is a linearized solution to the Einstein equations, i.e. a symmetric tensor field \( \gamma_{ab} \) for which the following equation holds:

\[
0 = (\mathcal{L}_\gamma)_{ab} \equiv -\frac{1}{2} \nabla_a \nabla_b \gamma - \frac{1}{2} \nabla^c \nabla_c \gamma_{ab} + \nabla^c (\nabla_c (a \gamma)_{bc})
- \frac{1}{2} g_{ab} \left( \nabla^c \nabla^d \gamma_{cd} - \nabla^c \nabla_c \gamma - \frac{2\Lambda}{d-2} \gamma \right) - \frac{2\Lambda}{d-2} \gamma_{ab} ,
\]

(3)
This equation has the gauge-invariance under $\gamma_{ab} \mapsto \gamma_{ab} + \varepsilon X g_{ab}$ for any smooth vector field $X$. In this paper, we will consider only perturbations having initial data of compact support on some Cauchy surface of the exterior region, $\Sigma$, i.e. the support is bounded away from the black hole $B = \mathcal{H} \cap \Sigma$ and infinity $\mathcal{C} = \mathcal{I} \cap \Sigma$. See fig. 1 for an illustration of this situation with $\Sigma = \Sigma_1$.

The linearized Einstein equation is not hyperbolic in nature due to its gauge invariance. But, as is well known, if one fixes the gauge (e.g. the transverse-traceless gauge), then the system becomes hyperbolic, and possesses a well-posed initial value formulation. This means that, if we prescribe compactly supported initial data on $\Sigma$ (satisfying the linearized constraints, see below), then the solution $\gamma_{ab}$ exists, is smooth, and unique inside the domain of dependence $D(\Sigma)$. In the asymptotically flat ($\Lambda = 0$) spacetimes considered in this paper, i.e. the extremal Myers-Perry black holes, if we take $\Sigma$ to be a slice as shown in fig. 3, then $D(\Sigma)$ comprises an entire exterior region. In the asymptotically $AdS$ black hole spacetimes ($\Lambda < 0$) considered in this paper, if we take $\Sigma$ to be a slice as shown in fig. 4, then $D(\Sigma)$ is only a subset of an exterior region. This corresponds to the well-known fact that these regions are not globally hyperbolic. In order to get a solution in an entire exterior region, we must specify what happens at the $AdS$-conformal boundary $\mathcal{I}$. For this, one has to specify (conformal) boundary conditions on $\gamma_{ab}$, which for an exact $AdS$-background were motivated thoroughly in [33], and correspond to keeping the conformal metric fixed to first order. The boundary-initial value problem for asymptotically $AdS$-spacetimes in the fully non-linear regime has been covered by [44]. His results also imply that the initial-boundary value problem has a globally regular solution in any exterior region, with the standard asymptotic expansions near $\mathcal{I}$ as given e.g. in [45].

### 2.2. Canonical energy of gravitational perturbations

We next recall the definition of the canonical energy of a perturbation of a stationary asymptotically $AdS$ ($\Lambda < 0$) or flat ($\Lambda = 0$) black hole with Killing horizon, and its key properties, referring to [28] for details. The main ingredient is the "symplectic current" of two solutions to the linearized Einstein equations, given by

$$w^a = \frac{1}{16\pi} p^{abcdef} (\gamma_{2bc} \nabla_d \gamma_{1ef} - \gamma_{1bc} \nabla_d \gamma_{2ef}) \tag{4}$$

where

$$p^{abcde} = \varepsilon^{[a} e_{bc]d} g^{ef} - \frac{1}{2} e^{[a} g^{bc]d} g_{ef} - \frac{1}{2} e^{[a} g^{bd} g_{ef} - \frac{1}{2} e^{[a} g^{cd} g_{ef} + \frac{1}{2} e^{[a} g^{cd} g_{bf]}. \tag{5}$$

This current is shown to be conserved, $\nabla^a w_a = 0$. The symplectic form $W(\Sigma; \gamma_1, \gamma_2)$ is defined by integrating the dual $\ast w$ over a $(d-1)$-dimensional submanifold $\Sigma$,

$$W(\Sigma; \gamma_1, \gamma_2) \equiv \int_\Sigma \ast w(g; \gamma_1, \gamma_2). \tag{6}$$

We typically take $\Sigma$ to run between a cut $\mathcal{C}$ of infinity $\mathcal{I}$, and a section $\mathcal{R}$ of the future horizon $\mathcal{H}^+$, or a slice "running down the throat", see $\Sigma = \Sigma_1$ or $= \Sigma_2$ in figs. 1, 2 for examples of such slices.
In order to define the canonical energy associated with such a slice, we need to introduce two boundary terms, and we need to fix the gauge (near) \( \mathcal{H}^\pm \) and \( \mathcal{I} \). The gauge conditions and boundary terms are needed, as in \([28]\), so that (i) \( \mathcal{E} \) has appropriate gauge invariance properties, and such that (ii) \( \mathcal{E} \) has suitable monotonicity properties. We begin by stating our gauge conditions. In the asymptotically flat case, we impose, near \( \mathcal{I}^\pm \), that the perturbation is in transverse-traceless gauge. The decay near the null-infinities \( \mathcal{I}^\pm \) of solutions with compactly supported data on a Cauchy surface \( \Sigma = \Sigma_1 \) (see fig. 1) in this gauge has been analyzed in sec. 2 of \([41]\). The analysis shows in particular that the integral (6) converges also for a Cauchy surface of the type \( \Sigma = \Sigma_2 \), see fig. 1. In the asymptotically \( AdS \)-case, we impose on \( \gamma_{ab} \) the linearized version of the Graham-Fefferman type gauge, implying again convergence of (6) (see e.g. \([45]\)). Near \( \mathcal{H}^\pm \), we can first impose the linearized “Gaussian normal null form” gauge conditions described in \([28]\). As in that reference, we would additionally like to impose as a gauge condition that the perturbed expansion, \( \delta \vartheta \), of \( \gamma_{ab} \), vanishes on \( \mathcal{H}^\pm \). In \([28]\) a proof was given that such a gauge always exists, but this proof does not appear to generalize to extremal black holes. We circumvent this problem in the present paper by only considering perturbations \( \gamma_{ab} \) having compact support on a slice “going down the throat”, as shown by \( \Sigma = \Sigma_1 \) in fig. 1. In this situation \( \delta \vartheta = 0 \) on \( \mathcal{H}^\pm \) for a suitable gauge can be established via the linearized Raychaudhuri equation,

\[
\frac{d}{du} \delta \vartheta = -\frac{2}{d-2} \vartheta \delta \vartheta - 2\sigma_{ab} \delta \sigma^{ab} - \delta R_{ab} K^a K^b = 0 ,
\]

where \( \sigma_{ab} \) and \( \vartheta \) are the (vanishing) shear and expansion of the background and \( \delta \sigma_{ab} \) and \( \delta \vartheta \) their first order variation under \( \gamma_{ab} \). The point is that, for example in the Lorentz gauge, \( \gamma_{ab} \) must be supported in the region shaded in red in fig. 1 by the usual rules for the propagation of disturbances for hyperbolic PDE’s. Thus, \( \delta \vartheta \) must clearly vanish for sufficiently negative values of the affine parameter \( u \) on \( \mathcal{H}^+ \), and therefore, by (7), for all \( u \) (and similarly for \( \mathcal{H}^- \)). It then also follows that the perturbed area, \( \delta A \), of a horizon cross section, must vanish on \( \mathcal{H}^\pm \) in our gauge, so

\[
\delta A|_B = 0 = \delta \vartheta|_B ,
\]

for any cross section \( B \subset \mathcal{H} \). It is not hard to see that the vector fields \( X^a \) preserving this gauge under \( \gamma_{ab} \to \gamma_{ab} + L_x g_{ab} \) must be tangent to \( \mathcal{H} \). We next define the boundary terms. The first boundary term, \( B(\mathcal{B}, \gamma) \) is associated with the section \( \mathcal{B} \) of the future horizon, and is defined as

\[
B(\mathcal{B}, \gamma) = \frac{1}{32\pi} \int_{\mathcal{B}} \gamma^{ab} \delta \sigma_{ab} ,
\]

where the natural integration element \( d\Vol_{\mathcal{B}} \) of \( \mathcal{B} \) is understood. The definition of the boundary term from infinity, \( B(\mathcal{I}, \gamma) \), depends on the asymptotic structure. In the asymptotically \( AdS \)-case, it is simply zero. In the asymptotically flat case, the boundary term at infinity is given by replacing, roughly speaking, \( \delta \sigma_{ab} \) with the perturbed news tensor\(^7\),

\[
\delta N_{ab} = \tilde{q}_a \sigma^d_b \Omega^{\frac{d+4}{d-2}} \left( \frac{2}{d-2} \tilde{R}_{cd} - \frac{1}{(d-1)(d-2)} \tilde{g}_{cd} \tilde{R} \right) - \tilde{q}_{ab} \text{trace} ,
\]

\(^7\) It has been shown in \([41]\) that the decay of \( \gamma_{ab} \) in the transverse-traceless gauge is sufficiently strong that the (linearized) Bondi news tensor at \( \mathcal{I}^\pm \) is finite.
Figure 1: Conformal diagram of the exterior of the black hole. To obtain the balance equation, we integrate $\nabla^a w_a = 0$ over rectangle shaded in blue. The region shaded in red indicates the support of a perturbation $\gamma_{ab}$ whose initial data are compactly supported.

where $\tilde{q}^b_a$ is the projector onto a cross section $\mathcal{C}$ of $\mathcal{I}$, and where tildas refer to the conformal metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$. We also replace $\gamma_{ab}$ with $\Omega^{- (d-6)/2} \gamma_{ab}$, and we replace the volume element with the natural integration element on $\mathcal{I}$ coming from $\tilde{g}^{ab}$, i.e. we define

\[ C(\mathcal{C}, \gamma) = -\frac{1}{32\pi} \int_{\mathcal{C}} \tilde{\gamma}^{ab} \delta \tilde{N}_{ab}. \] (11)

For details see [28]. With these notions at hand, we can make the following

Definition 2.1: The canonical energy of a perturbation is defined as the quadratic form

\[ E(\Sigma, \gamma) \equiv W(\Sigma, \gamma, \psi K_{\gamma}) - B(\mathcal{B}, \gamma) - C(\mathcal{C}, \gamma). \] (12)

The boundary terms are added in the definition of $E$ in order for $E$ to have a very important monotonicity property under ‘time evolution’. This property comes about as follows. Since the symplectic current is conserved, it follows that $d(\ast w) = 0$. We can integrate this equation over a ‘quadrangle-shaped’ domain of $\mathcal{M}$ as shown in fig. 1.

By Stokes’ theorem, the result is a contribution from the boundaries. The contributions from $\Sigma_1$ resp. $\Sigma_2$ give $W(\Sigma_1, \gamma, \psi K_{\gamma})$ resp. $-W(\Sigma_2, \gamma, \psi K_{\gamma})$ whereas the contributions from $\mathcal{H}_{12}$ resp. $\mathcal{I}_{12}$ represent ‘fluxes’. One can compute these fluxes using the consequences of the linearized Raychaudhuri equation on $\mathcal{H}$, and the asymptotic expansion of the metric and perturbation near $\mathcal{I}$. Combining these with the boundary terms in the definition of $E$, one reaches the following important conclusion:

Lemma 2.1: Let $\gamma$ be a perturbation having smooth compactly supported initial data on $\Sigma_1$ (i.e. with support intersecting neither $\mathcal{H}$ nor $\mathcal{I}$). Let $\Sigma_2 \subset J^+(\Sigma_1)$, as in figs. 1, 2.

1. In the asymptotically flat case, assume that the perturbation is axisymmetric in the sense that

\[ \mathcal{L}_\psi \gamma_{ab} = 0, \quad \psi = \Omega t \frac{\partial}{\partial \phi_t}. \] (13)

\[ \text{It is understood here also that } \Omega \text{ has been chosen such that } \tilde{n}^a = \tilde{\psi}^a \Omega = (\partial/\partial t)^a \] is an affinely parameterized null field tangent to scri.
Figure 2: Conformal diagram of the exterior of the AdS black hole. To obtain the balance equation, we integrate $\nabla^a w_a = 0$ over the shaded rectangle. In this case, there is no flux across $I_{12}$ due to the AdS-boundary conditions.

Then it follows that\footnote{Here natural integration elements on $H$ and $I$ are understood.}

$$\mathcal{E}(\Sigma_1) - \mathcal{E}(\Sigma_2) = \frac{1}{4\pi} \int_{H_{12}} \delta \sigma_{ab} \delta \sigma^{ab} + \frac{1}{16\pi} \int_{I_{12}} \delta \tilde{N}_{ab} \delta \tilde{N}^{ab} \geq 0,$$

meaning that $\mathcal{E}(\Sigma_2) \leq \mathcal{E}(\Sigma_1)$.

2. In the asymptotically AdS-case, we have $\mathcal{E}(\Sigma_2) \leq \mathcal{E}(\Sigma_1)$ also for non-axisymmetric perturbations.

A proof is given for the case of asymptotically flat non-extremal black holes in thm. 1 of [28], and the proof for extremal black holes generalizes straightforwardly (modulo the change regarding how to show $\delta \vartheta = 0$ mentioned earlier). The axisymmetry restriction in the asymptotically flat case is imposed, as in [28], to eliminate any indefinite flux terms at infinity, corresponding physically to the radiation of angular momentum. The same proof also works for the asymptotically AdS-case, illustrated in fig. 2. The key difference in the AdS-case is due to the fact that there simply is no flux at infinity, due to the “reflecting nature” of the AdS boundary conditions, \cite{45}, so no “axisymmetry” restriction needs to be imposed.

A key property of $\mathcal{E}$ is its gauge invariance under $\gamma_{ab} \rightarrow \gamma_{ab} + \xi_{X} g_{ab}$, see lemma 2 of [28]. Although that lemma was formulated for stationary, non-extremal black holes, inspection of the proof shows that the lemma also applies to a slice $\Sigma$ “going down the throat” in an extremal black hole such as $\Sigma = \Sigma_1$ in figs. 1, 2), if the perturbation $\gamma_{ab}$ has compact support on $\Sigma$, as will be the case in our applications. Gauge invariance also holds for a slice $\Sigma = \Sigma_2$ as drawn in figs. 1, 2 intersecting the future horizon if $X^a$ becomes tangent to the generators on $H$ and approaches a BMS-transformation at $I^\pm$. This follows again by inspecting the proof of lemma 2 of [28], noting that the perturbed area and expansion of $H$ must vanish in our case. By arguing as in prop. 4 of [28], it then follows also that $\mathcal{E}(\gamma, \Sigma)$ is a perturbation towards another stationary black hole (where $\Sigma = \Sigma_1$ or $\Sigma_2$).

With these properties of $\mathcal{E}$ at hand, we may now explain how we will exploit $\mathcal{E}$ in order to show that a stationary black hole spacetime is linearly unstable, in exact parallel with the
discussion in [28]. Suppose that, on a slice \( \Sigma = \Sigma_1 \) as in figs. 1, 2 (for asymptotically flat resp. AdS black holes), we can find a compactly supported perturbation – axisymmetric in the asymptotically flat case – for which \( \mathcal{E}(\Sigma_1, \gamma) < 0 \). By lemma (2.1), \( \mathcal{E}(\Sigma_2, \gamma) \) must then be less than or equal to \( \mathcal{E}(\Sigma_1, \gamma) \) for any later slice as drawn in figs. 1 resp. 2. On the other hand, if \( \gamma_{ab} \) is to approach a pure gauge transformation (compatible with our gauge conditions on \( \mathcal{H} \) and \( \mathcal{I} \)), then \( \mathcal{E}(\Sigma_2, \gamma) \) must go to zero on a sufficiently late slice. This cannot be the case, and so \( \gamma_{ab} \) cannot go to a pure gauge transformation at late times. Likewise, \( \gamma_{ab} \) cannot converge to a perturbation to another stationary black hole. Thus, the black holes is linearly unstable.

Below, it is useful to work also with a formulation of \( \mathcal{E} \) in terms of initial data of the background and perturbations. Let \( \Sigma \) be a spatial slice, with unit normal \( v^a \). We denote the induced metric by \( h_{ij} \) and an extrinsic curvature by \( K_{ij} \). Recall that the canonical momentum \( p^{ij} \) is defined in terms of the extrinsic curvature of \( \Sigma \) as

\[
p^{ij} = h^2(K^{ij} - Kh^{ij}).
\]

Here and in the following we introduce a fixed (e.g. coordinate-) \((d - 1)\)-form field \( d^{-1}x \) on \( \mathcal{M} \) related to the volume form on \( \Sigma \) by \( dvol_h = h^2 d^{-1}x \). With these definitions, \( h_{ij} \) and \( p^{ij} \) are canonically conjugate pairs. The lapse and shift of \( K^a \) are denoted by \( N \) resp. \( N^i \). The initial data of a perturbation \( \gamma_{ab} \) are written \( (\delta h_{ij}, \delta p^{ij}) \) and are, throughout this article, assumed to be of compact support on \( \Sigma \). In terms of these, we have \( \mathcal{E} = (1/16\pi) \int_\Sigma \rho dvol_h \), where \( \rho \) is given by:

\[
\rho = N \left[ \frac{1}{2} \text{Ric}(h)_{ij} \delta h^k_i \delta h^{kj} - 2 \text{Ric}(h)_{ik} \delta h^{ij} \delta h^k_j + \frac{1}{2} (D_i \delta h^k)D_k \delta h^i - \left( \frac{1}{2} (D^i \delta h^k)D_j \delta h_{ik} - (D^i \delta h^k)D_k \delta h_{jk} \right) + \right.
\]

\[
N \left[ 2 \delta p_{ij} \delta p^{ij} + \frac{1}{2} \rho \delta p^{ij} (\delta h^k)^2 - \rho \delta p^{ij} \delta h^k - 3 \rho \delta p^{ij} \delta h^i \delta h^j - \frac{2}{d-2} (\delta p_i)^2 + \frac{3}{d-2} \rho \delta p_i^j \delta h^k - \frac{3}{d-2} \rho p^k_i \rho p^j_i \delta h_{ij} + 8 \rho^j \delta h^j \delta p^{ij} + \rho^{jk} \delta h_{ij} \delta h^{ij} + 2 \rho^{ij} \rho^{kl} \delta h_{ik} \delta h^i_j - \frac{1}{d-2} (p^i)^2 (\delta h_{ij})^2 - \frac{1}{(d-2)^2} (\delta h_{ij})^2 \right] - \frac{4}{d-2} \rho \delta p^i \delta h^i - \frac{2}{d-2} (\delta p^{ij} \delta h^i)^2 - \frac{4}{d} \rho \delta p^{ij} \delta h^i \delta h^i \right] h^{-1} - N^i [ -2 \delta p^j D_i \delta h_{ik} + 4 \delta p^j D_i \delta h_{ik} + 2 \delta h_{ik} D_j \delta p^{ik} - 2 \rho^{ik} \delta h_{ii} D_j \delta h^k_+ + \rho^{ik} \delta h_{ii} D_j \delta h^k_+ ] h^{-1},
\]

see [28] for a derivation.\(^{10}\)

### 2.3. Canonical energy of electromagnetic perturbations

One may also study a test electromagnetic field, \( A_a \), propagating on the background black hole spacetime \( (\mathcal{M}, g_{ab}) \). We will call these “electromagnetic perturbations”. The field equation is the Maxwell equation \( 0 = \nabla^a \nabla_a A_b \), and the field strength is as usual \( F_{ab} = 2 \nabla_a A_b \). The symplectic \((d - 1)\)-form for two perturbations \( A_1, A_2 \) can be derived from the Lagrangian formulation as described in [47], with the result

\[
w_a = \frac{1}{2\pi} (A_1^b \nabla_a A^2_b - A_2^b \nabla_a A_1^b). \quad (17)
\]

\(^{10}\)Note that the boundary terms in \( \mathcal{E} \) in [28] can be omitted for perturbations having compact support on \( \Sigma \).
As always, $\nabla^a w_a = 0$. The symplectic form $W(\Sigma; A_1, A_2)$ is obtained, just as in the gravitational case, by integrating $\star w$ over a $(d - 1)$-dimensional submanifold $\Sigma$.

$$W(\Sigma; A_1, A_2) \equiv \int_{\Sigma} \star w(g; A_1, A_2).$$  \hfill (18)

As in the gravitational case, we impose gauge conditions on $A_a$ near infinity and near the horizon. Near the horizon, our gauge condition on $A_a$ analogous to (8) is that perturbed electrostatic potential vanishes, $-K^a A_a|_{\mathcal{H}} = 0$. Similar to the case of gravitational perturbations, if this condition is satisfied on one cross section $\mathcal{B}$ of $\mathcal{H}^+$, then it is automatically satisfied everywhere on $\mathcal{H}^+$, and similarly “−” [47]. Near infinity, we impose the Lorentz gauge condition $\nabla^a A_a = 0$. It is shown in appendix E that this condition implies the following behavior of $A_a$ near infinity in the asymptotically flat case ($\Lambda = 0$): In terms of the unphysical metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$, we have that $\tilde{A}_a = \Omega^{(d-4)/2} A_a$ and $\Omega^{-1} n^a \tilde{A}_a$ are finite and smooth at $\mathcal{I}$, where $\tilde{n}^a = g^{ab} \tilde{\nabla}_b \Omega$.

In order to define the canonical energy for an electromagnetic perturbation associated with such a slice, we must, as in the gravitational case, introduce certain boundary terms. The boundary term on the horizon is

$$B(\mathcal{B}, A) = \frac{1}{2\pi} \int_{\mathcal{B}} A^a \xi_K A_a,$$  \hfill (19)

whereas the boundary term at infinity is

$$C(\mathcal{\mathcal{I}}, A) = \frac{1}{2\pi} \int_{\mathcal{I}} \tilde{A}^a \xi_{\tilde{n}} \tilde{A}_a,$$  \hfill (20)

where, as in the gravitational case, natural integration elements are understood. For asymptotically AdS-spacetimes ($\Lambda < 0$), the boundary term from infinity is again simply set to zero. The canonical energy in the electromagnetic case is then defined in precise analogy to the gravitational case:

**Definition 2.2:** The canonical energy of an electromagnetic perturbation is defined as the quadratic form

$$\mathcal{E}^r(\Sigma, A) \equiv W(\Sigma; A, \xi_K A) - B(\mathcal{B}, A) - C(\mathcal{I}, A).$$  \hfill (21)

Proceeding as in the case of gravitational perturbation, one can derive a monotonicity property analogous to that described in lemma (2.1). In the case $\Lambda = 0$, we assume, as in the case of gravitational perturbations, that $A_a$ is axisymmetric, $\xi_\psi A_a = 0$, compare eq. (13). One obtains the balance equation ($n^a = K^a$ on $\mathcal{H}$)

$$\mathcal{E}^r(\Sigma_1) - \mathcal{E}^r(\Sigma_2) = \frac{1}{2\pi} \int_{\mathcal{H}_{12}} (\xi_n A^a) \xi_n A_a + \frac{1}{2\pi} \int_{\mathcal{I}_{12}} (\xi_{\tilde{n}} \tilde{A}^a) \xi_{\tilde{n}} \tilde{A}_a \geq 0,$$  \hfill (22)

meaning that $\mathcal{E}^r(\Sigma_2) \leq \mathcal{E}^r(\Sigma_1)$ as in the gravitational case. For $\Lambda < 0$, one obtains the same balance equation without the second term on the right side even for non-axisymmetric perturbations. More details on how to derive (22) are given in appendix E.
Again, one can also write $E$ in terms of the initial data of the perturbations. In the case of electromagnetic perturbations these are given by $(A_i, E^i)$, where $E^i$ is the densitized\(^{11}\) electric field. In terms of these, we have $E = (1/4\pi) \int_E \rho dvol_h$, where $\rho$ is given by:

$$\rho = N \left( \frac{1}{2} h^{-1} E_i E^i + \frac{1}{4} F_{ij} F^{ij} \right) + N^i E^j F_{ij} h^{-\frac{1}{2}}. \quad (23)$$

### 3. Extremal black holes and their near horizon limit

A key role is played in conjecture 1 by the notion of near horizon (NH) limit of an extremal black hole. We now recall this construction, referring to the review [24] for more details, and establish some notation used in the subsequent sections. In an open neighborhood of the horizon $H^+$ we may introduce Gaussian normal coordinates. These depend on the choice of a section $B \subset H^+$ with coordinates $x^A, A = 1, \ldots, d-2$, which are supplemented by coordinates $\rho, u$. The coordinate $u$ parameterizes affine null geodesics ruling $H^+$, whereas $\rho$ parameterizes null geodesics transversal to $H^+$ and orthogonal to $B$. The metric takes the form

$$ds^2 = 2du(d\rho - \frac{1}{2} \rho^2 \alpha du - \rho \beta_A dx^A) + \mu_{AB} dx^A dx^B. \quad (24)$$

The tensor fields $\alpha, \beta_A dx^A, \mu_{AB} dx^A dx^B$ are defined independently of the choice of coordinates $x^A$. The Gaussian null coordinates may be chosen, by adjusting $B$ if necessary, so that the Killing field is $K = \partial/\partial u$. A key role is played by the 1-parameter group of diffeomorphisms defined in a neighborhood of $H^+$ by

$$\phi_\varepsilon : (u, \rho, x^A) \mapsto \left( \varepsilon u, \frac{1}{\varepsilon} \rho, x^A \right). \quad (25)$$

The form for the metric (24) and the fact that $K = \partial/\partial u$ is a Killing field implies that the limit

$$g^{NH} = \lim_{\varepsilon \to 0} \phi_\varepsilon^* g \quad (26)$$

defines a new metric, called “NH limit”. In the following, we will omit the superscript “NH” to avoid clutter. The near horizon metric can again be represented in the form (24). In these coordinates, $\beta_A, \mu_{AB}$ are independent of the coordinates $u, \rho$ and are obtained from the counterparts of the original BH metric simply by setting $\rho = 0$. The diffeomorphisms $\phi_\varepsilon$ by construction form a 1-parameter group of isometries of the NH geometry which together with the group generated by $K$ forms generates an action of the 2-dimensional group $\mathbb{R} \times \mathbb{R}_+$. These general constructions can be applied, in particular, to the known extremal, vacuum stationary black holes, i.e. the Myers-Perry (MP) black holes and black rings [22]. The former are known in any dimension $d \geq 5$, whereas the latter only in $d = 5$. For definiteness, we will focus on the MP black holes. We recall the results in the case $\Lambda = 0$ following Ref. [22], and refer to [46] for the case $\Lambda \neq 0$. First we describe the MP black holes themselves. These solutions are parameterized by their mass parameter $\mu > 0$ and rotation parameters $a_I \in \mathbb{R}$ with $I = 1, \ldots, n$. Their properties differ somewhat in even and odd dimension, so for definiteness

\(^{11}\)We choose $E^i$ to be a density so that it is canonically conjugate to $A_i$, but of course we could also work with the undensitized electric field.
and simplicity we focus on the odd dimensional case \( d = 2n + 1 \). The horizon topology is \( \mathcal{B} \cong S^{2n-1} \), and the topology of a Cauchy surface for the exterior region is \( \mathbb{R}^{d-1} \) minus a ball. The exterior region is parameterized by coordinates \( t, r > r_+ \), \( n \) azimuthal coordinates \( \phi_I \in [0, 2\pi] \) and \( n \) latitudinal coordinates \( \mu_I \in [0, 1] \) subject to \( \sum \mu_I^2 = 1 \). In terms of these, the MP metric is

\[
g = -dt^2 + \frac{\mu r^2}{\Pi F} \left( dt + \sum_{I=1}^n a_I \mu_I^2 d\phi_I \right)^2 + \frac{\Pi F}{\Pi - \mu r^2} dr^2 + \sum_{I=1}^n (r^2 + a_I^2) (d\mu_I^2 + \mu_I^2 d\phi_I^2) .
\]  

(27)

Here,

\[
\Pi = \prod_{I=1}^n (r^2 - a_I^2) , \quad F = 1 - \sum_{I=1}^n \frac{a_I^2 \mu_I^2}{r^2 + a_I^2} .
\]

(28)

The location of the event horizon \( \mathcal{B} \) in these coordinates is at \( r = r_+ > 0 \), where \( \Pi(r_+) = \mu r_+^2 \), and the angular velocities of the horizon are given by

\[
\Omega_I = -\frac{a_I}{r_+^2 + a_I^2} .
\]

(29)

The isometry group of the MP black holes is \( \mathbb{R} \times U(1)^n \) corresponding to shifts in \( t \) respectively the \( \phi_I \).

As for the case of the Kerr metric, there are extremal limits. In odd \( d = 2n + 1 \geq 5 \) these are characterized by the condition\(^{12}\)

\[
1 = \sum_{I=1}^n \frac{r_+^2}{a_I^2 + r_+^2} .
\]

(30)

A Penrose diagram for the extreme MP spacetime is given in figure 3.

The NH-limit for the extremal MP solutions was given in [22]. It has the general form

\[
d\hat{s}^2 = L^2 \, d\hat{s}^2 + g_{IJ}(d\hat{\phi}^I + k^I \hat{\mathcal{A}})(d\hat{\phi}^J + k^J \hat{\mathcal{A}}) + d\hat{\sigma}_{d-n-2}^2 .
\]

(31)

The geometry can be thought of as a fibration \( \mathcal{B} \to \mathcal{M} \to \hat{\mathcal{M}} \) with leaves \( \mathcal{B} \cong S^{2n-1} \), total space \( \mathcal{M} \), and orbit space \( \hat{\mathcal{M}} \). The quantities \( L > 0, d\hat{\sigma}_{d-n-2}^2 \) are intrinsic to \( \mathcal{B} \), whereas hatted quantities refer to the base space \( \hat{\mathcal{M}} \). The base space is geometrically \( \hat{\mathcal{M}} = AdS_2 \) with a uniform electric field \( d\hat{\mathcal{A}} = \text{vol}_2 \hat{\mathcal{A}} \):

\[
d\hat{s}^2 = -R^2 d\hat{T}^2 + \frac{dR^2}{R^2} , \quad \hat{\mathcal{A}} = -Rd\hat{T} .
\]

(32)

\(^{12}\)Note that this requires that all \( a_I \neq 0 \).
Figure 3: Conformal diagram of the extremal $\Lambda = 0$ MP spacetime [10]. The Cauchy surface for the exterior is a complete manifold, i.e. the proper distance of $\mathcal{B}$ from any point on $\Sigma$ is infinite. The near horizon region is shaded, and the upward curvy lines are the orbits of the Killing field $K$. 

Figure 4: Conformal diagram of the extremal $\Lambda < 0$ MP spacetime.

The other quantities appearing in the NH-metric of the extremal MP solutions are explicitly:

\[
L^2 = \frac{F(r_+)}{C^2}, \quad (33)
\]

\[
k^I = \frac{2r_+a_I}{C^2(r_+^2 + a_I^2)^2}, \quad (34)
\]

\[
d\sigma_{d-n-2}^2 = \sum_{l=1}^{n} (r_+^2 + a_l^2) d\mu_l^2, \quad (35)
\]

\[
g_{IJ} = (r_+^2 + a_l^2) \mu_l^2 \delta_{IJ} + \frac{a_Ia_J\mu_l^2\mu_j^2}{L^2}, \quad (36)
\]

where $C^2 = \Pi''(r_+) / 2\Pi(r_+) > 0$. If the coordinates on $\mathcal{B} = S^{2n-1}$ are denoted collectively by $\mathbf{x}^A = (\phi_I, \mu_J)$, the relationship to the Gaussian null form (24) is:

\[
\rho = L^2 \cdot R, \quad u = T - 1/R \quad (37)
\]

\[
\beta_A dx^A = \frac{1}{L^2} \left( g_{IJ} k^I d\phi^J + dL^2 \right) \quad (38)
\]

\[
\mu_{AB} dx^A dx^B = g_{IJ} d\phi^I d\phi^J + d\sigma_{d-n-2}^2 \quad (39)
\]

\[
\alpha = \frac{1}{L^2} \left( 1 - \frac{1}{L^2} g_{IJ} k^I k^J \right) \quad (40)
\]

implying also that the horizon Killing field in the coordinates (31) is $K = \partial / \partial T$. The relationship between the coordinates $(\rho, u, x^A)$ and those used to represent the MP metric (27) can
Figure 5: Conformal diagram of the NH limit of the extremal MP spacetime [10], i.e. AdS$_2$. This should be thought of as corresponding to the shaded region in the diagrams 3 or 4 of the extreme MP black hole, to be taken “infinitely thin”. The Cauchy surface $\Sigma$ in that conformal diagram corresponds to the surface $\Sigma$ drawn here. The curvy upward lines show the orbits of $K = \partial / \partial T$, whereas the curvy horizontal lines the surfaces of constant $T$.

be found in [23]. A Penrose diagram of the NH geometry illustrating the relationship to the extremal MP spacetime is given in figure 5.

The presence of the AdS$_2$ factor is crucial for the considerations of this paper. It implies for example that the NH geometry has a larger isometry group than what can be inferred straight-forwardly from the general construction leading to eq. (26) [21]. This enhanced isometry group is $\text{SL}(2, \mathbb{R}) \times U(1)^n$, with the $\text{SL}(2, \mathbb{R})$ factor corresponding to AdS$_2$. The metric $d\sigma_{d-2-\nu}^2$ may be thought as that inherited on the orbit space $\mathcal{M} / [\text{SL}(2, \mathbb{R}) \times U(1)^n]$.

For later purposes, it is useful to know the (asymptotic) form of the induced metric and extrinsic curvature on the slice $\Sigma = \{ T = 0 \}$ “going down the throat” of the extreme MP metric resp. the NH geometry. These can straightforwardly be calculated e.g. noting that, by (33), $R \propto r$, and recalling that the components $\alpha, \beta_A, \mu_{AB}$ of the BH and NH metrics (see (24)) differ by terms of order $r$ from the BH metric. With $(x^i) = (y = \log R, x^A)$, and $h = h_{ij}d\chi^i d\chi^j, K = K_{ij}d\chi^i d\chi^j$, one finds

$$h = L^2 dy \otimes dy + g_{IJ}d\phi^I \otimes d\phi^J + d\sigma_{d-2-\nu}^2 + O(e^y), \quad (41)$$

$$K = \frac{g_{IJ}k^I}{\sqrt{L^2 + g_{MN}k^M k^N}} (d\phi^J \otimes dy + dy \otimes d\phi^J) + O(e^y). \quad (42)$$
To obtain the formula for $K_{ij}$, one can make use e.g. of the well-known formula (see e.g. appendix E of [39]) $K_{ij} = (2N)^{-1}[\partial_T h_{ij} - 2D_iN_j]$ in terms of the lapse $N$ and shift $N^j$ of $K = \partial / \partial T$, together with $\partial_T h_{ij} = 0$. The expression $O(\epsilon^0)$ represents terms whose coordinate components w.r.t. $(y,x^A)$, including their $y$-derivatives, decay as $e^y$ for $y \to -\infty$, i.e. in the throat. The form of $h_{ij}$ shows explicitly that the slice $\Sigma$ is a complete manifold, which is a characteristic feature of extremal black holes.

4. Hertz potentials

4.1. Gravitational perturbations

The NH limits arising from the known black hole solutions in various dimensions (and in particular of the Myers-Perry family) have further special properties that allow one, to a certain extent, to decouple and separate the linearized Einstein equations (3). These properties have to do with the presence of certain null vector fields with special optical properties, and with the fact that the Weyl-tensor of the NH geometries is algebraically special in a sufficiently strong sense.

The properties are formulated in terms of a special pair of null vector fields $l^a, n^a$. They are normalized so that

$$n^a l_a = 1, \quad n^a n_a = l^a l_a = 0, \quad g_{ab} = 2n(a l_b) + q_{ab},$$

so $q^a_b$ projects onto the subspace of $\mathcal{M}$ orthogonal to $n^a, l^a$. The ‘algebraically special property’ of the Weyl tensor, $C_{abcd}$ which we referred to is:

$$C_{abcd}q^a e^f l^b l^d = 0 = C_{abcd}q^a f^b e^c q^d, \quad \text{and same for } n^a \leftrightarrow l^a.$$

(44)

The ‘special optical properties’ for $n^a, l^a$ are that they should be geodesic, shear free, expansion free, and twist free; in formulas:

$$q_{cb} l^a \nabla_a l^b = 0, \quad q_{ac} q_{bd} \nabla^c l^d = 0, \quad \text{and same for } n^a \leftrightarrow l^a.$$

(45)

In the terminology of [17], spacetimes satisfying (44) and (45) are “doubly Kundt”. The NH geometries (31) studied in this paper all fall into this class, with $l^a, n^a$ concretely given by (69).

For the considerations of this section, the explicit forms (31) and (69) are not needed. It is enough to know their general properties.

These properties can be exploited as follows. From the background Einstein equations and Bianchi-identity, we always have

$$0 = \nabla_{[a} C_{bc]de}$$

(46)

and taking $\nabla^a$ of this equation and using again the background Einstein equations, we get the wave equation

$$0 = \nabla^a \nabla_a C_{bcde} + 2C_{bc}^{\quad \ell} C_{\ell ade} + 2C_{[b|d|}^{\quad \ell} C_{c]a\ell e} + 2C_{[b|e|}^{\quad \ell} C_{c]\ell ad} - \frac{4\Lambda}{d-1} C_{bcde}.$$

(47)

First order perturbed versions of these equations are derived by considering 1-parameter families of background metrics satisfying the Einstein equations. By dotting equations (46), (47)
and their perturbed counterparts in all possible ways into \( l^a, n^a, q^a_b \), using the optical properties (45) and the algebraically special properties of the Weyl tensor (44), [17] were able to find a decoupled tensorial wave equation for the quantity

\[
\Omega_{ab} = \delta C_{cdef} q^c_l q^d_l l^f,
\]

(48)
which is a traceless symmetric tensor field whose indices are projected by \( q^a_b \). This wave equation can be written as

\[
(\partial \Omega)_{ab} = 0,
\]

(49)
where \( \partial \) is the differential operator\(^{13} \)

\[
(\partial \Omega)_{ab} \equiv \left\{ 2\hat{\beta}' \beta + \delta^c \delta_c - 6 \tau^c \delta_c + 4 C_{cdef} l^e q^d q^f - \frac{4d \Lambda}{(d - 1)(d - 2)} \right\} \Omega_{ab} + \frac{4 \tau^c (q^e_c \delta_{(a} - q^e_{(a} \delta^f)) + 2 l^e n^c C_{cdeg} (5 q^f g q^d_{(a} - 3 q^d_{(a} q^f g))}{(d - 1)(d - 2)} \Omega_{b)f}.
\]

(50)
The operators \( \delta_{ab} \), \( \beta \), \( \hat{\beta}' \) depend on a real number \( b \in \mathbb{R} \) (“boost weight”) and act on tensor fields \( t_{a_1...a_s} \) whose indices are projected by \( q^{a}_{b} \). They are defined by

\[
\delta_{b} t_{a_1...a_s} = q_{a_1}^{b_1} \cdots q_{a_s}^{b_s} \left[ q_{b} c \nabla_c - b \cdot q_{b} n^d (\nabla_{c} l_d) \right] t_{b_1...b_s},
\]

(51)
\[
\hat{\beta} t_{a_1...a_s} = q_{a_1}^{b_1} \cdots q_{a_s}^{b_s} \left[ \nabla c - b \cdot l^e n^d (\nabla_{c} l_d) \right] t_{b_1...b_s}.
\]
The boost weight of a quantity is defined to be its scaling power under \( l^a \rightarrow \lambda l^a, n^a \rightarrow \lambda^{-1} n^a \), so for example \( \Omega_{ab} \) has \( b = 2 \), \( \beta \) raises the boost weight by one unit, and \( \hat{\beta}' \) decreases the boost weight by one unit. We also use the ‘prime convention’, which means that a ‘’ on any object means that \( n^a \) and \( l^a \) are to be exchanged in its definition. Since \( n^a \) and \( l^a \) are on the same footing, the ‘primed’ version of equation (49) also holds for \( \Omega'_{ab} \). We use the shorthands:

\[
\tau_a = q^b_{a} n^c \nabla_c l_b, \quad \tau'_a = q^b_{a} l^c \nabla_c n_b,
\]

(52)
where the second expression is an example of the priming operation.

In \( d = 4 \), a traceless symmetric tensor field \( \Omega_{ab} \) that is projected by \( q^a_b \) can be identified with a complex scalar via a choice of complex 2-bein for \( q_{ab} \). Namely, if \( q_{ab} = m(a \bar{m}b) \), then we can write \( \Omega_{ab} = \Phi_0 m_a m_b + \bar{\Phi}_0 \bar{m}_a \bar{m}_b \) for some complex scalar function \( \Phi_0 \). The equation \( \partial \Omega_{ab} = 0 \) is then equivalent to the Teukolsky equation [8] for \( \Phi_0 \). For this reason, we will refer to eq. (49) as the “Teukolsky equation” also in \( d > 4 \).

From the quantity \( \Omega_{ab} \) in (48) one can in principle reconstruct the perturbation \( \gamma_{ab} \) itself up to gauge transformations, and up to a finite dimensional space of special perturbations. (In our case the latter would be perturbations to other NH-geometries.) But since \( \Omega_{ab} \) involves derivatives of \( \gamma_{ab} \), this relationship would necessarily be non-local, depend on awkward choices of

\(^{13}\) Note that the highest derivative terms in \( 2\beta' \beta + \delta^c \delta_c \) coincide with those of \( \nabla \nabla_a \).
boundary conditions, etc. For our purpose, it is much better to construct perturbations $\gamma_{ab}$ satisfying the linearized Einstein equations directly. We will do this by introducing a certain “potential” for gravitational perturbations, whose existence, like that of the Teukolsky equation (49), is closely related to the optical and algebraically special properties of the background. The desired potential $U_{ab}$, called “Hertz-potential”, satisfies an equation that is closely related to the operator $\mathcal{O}$ defined above in eq. (50). If this equation holds, then one can define a corresponding gravitational perturbation $\gamma_{ab}$ by acting on $U_{ab}$ with a certain second-order partial differential operator. This gravitational perturbation then satisfies (3).

To set things up, it is necessary to recall the standard notion of “adjoint” of a linear partial differential operator $P$ from smooth sections of a real vector bundle $E$ to smooth sections of a real vector bundle $F$ over $\mathcal{M}$. Let $E'$ and $F'$ be the dual bundles whose fibres are given by the vector space dual of the fibres of $E$ and $F$, and let $\wedge^d T^* \mathcal{M}$ be the line bundle of densities of weight one half over $\mathcal{M}$.

If $f \in C^\infty(\mathcal{M}, E), h \in C^\infty(\mathcal{M}, F'),$ then the “transpose” $P^*$ is the differential operator from sections of $F' \otimes \wedge^d T^* \mathcal{M}$ to sections of $E' \otimes \wedge^d T^* \mathcal{M}$, defined uniquely by

$$
\langle h, Pf \rangle = \langle P^* h, f \rangle + dX,
$$

where $X$ is a $(d - 1)$-form on $\mathcal{M}$ and angles denote the dual pairing. (Another way of saying the same thing is that $P^*$ is obtained by the usual rules of partial integration under an integral.) For example, in the case of the operator $P = \mathcal{O}$, $E = F$ is the bundle of contravariant, symmetric, traceless tensors that are projected by $q_{ab}$, or in the case of the linearized Einstein operator, $\mathcal{L}$ (cf. (3)), $E = F$ is the space of contravariant, symmetric tensors. In these cases, the bundles possess a natural inner product induced by $q_{ab}$ respectively $g_{ab}$, and the manifold has a natural volume element induced from the metric. These may be used to identify $E$ with $E'$, $F$ with $F'$, to identify densities with scalars, and hence to identify the transpose of these operators with operators from $F$ to $E$, as we will do in the following. With this convention, the adjoints of the operators $\mathfrak{p}, \mathfrak{d}_a$ as defined in (51) are found to be

$$
\mathfrak{p}^* = -\mathfrak{p}, \quad \mathfrak{d}_a^* = -\mathfrak{d}_a + \tau_a + \tau_a'.
$$

After these preliminaries, we can construct the Hertz-potentials, which were first found in 4 dimensions in [35, 36]. The derivation is in fact a straightforward adaptation of the elegant proof of Wald [36] from four to higher dimensions. Consider an arbitrary smooth symmetric tensor field $\gamma_{ab}$. If this tensor field happens to satisfy the linearized Einstein equations (3), then (49) holds. Therefore, for an arbitrary $\gamma_{ab}$, the right side of (49) is non-zero but must have the form of a linear partial differential operator applied to (3), which we abbreviate as $(\mathcal{L} \gamma)_{ab} \equiv T_{ab}$. Concretely, one finds after a lengthy calculation that

$$
(\mathcal{O} \Omega)_{ab} = +2\{\mathfrak{p} \mathfrak{d}_a - (2\tau_a + \tau_a')\mathfrak{p} - (\mathfrak{p} \tau_a)(q_{ab}^d T_{cd}) - \mathfrak{p}^2 (q_{ab} q_{cd}^d T_{cd}) + \frac{1}{d-2} g_{ab} \mathfrak{p}^2 (g^{cd} T_{cd})
- \frac{1}{d-2} q_{ab} \mathfrak{d} C_{cde}^f l^e n^f T_{mn}^m n^p \}
+ \frac{1}{d-2} g_{ab} \left\{ 2\mathfrak{p} \mathfrak{p} + \mathfrak{d}^c \mathfrak{d}_c - 6\tau_a \mathfrak{d}_a + 4 C_{cde}^f n^e l^d l^f - \frac{2d \Lambda}{d-1} \right\} T_{mn}^m n^p,
$$

\[14\] After we completed our calculation, we have learned that an almost identical analysis had been carried out previously by [34].
Following [36], the relation (55) can be succinctly written in terms of operators. Let $\mathcal{F}$ be the linear second order differential operator which associates with a symmetric tensor field $\gamma_{ab} = \delta g_{ab}$ the quantity $(\mathcal{F} \gamma)_{ab} = \Omega_{ab} = q_{ab} q^{cd} \delta C_{cedf} l^e l^f$. $\mathcal{F}$ is given concretely by the second line of eq. (48). Let $\mathcal{F}$ be the linear second order differential operator representing the particular linear combinations of derivatives of $T_{ab}$ on the right side of (55). $\mathcal{F}$ is a linear operator from symmetric tensors to traceless second rank tensors projected by $\gamma^L_{ab}$.

The key point is now that the linearized Einstein operator is symmetric $\mathcal{L} = \mathcal{L}^*$, which is a direct consequence of the fact that it arises from an action principle. Whence, if $(\mathcal{F}^* U)_{ab} = 0$, then $\gamma_{ab} := (\mathcal{F}^* U)_{ab}$ is a symmetric tensor satisfying the linearized Einstein equations $(\mathcal{L} \gamma)_{ab} = 0$. Working out explicitly the operator $\mathcal{F}^*$ gives:

$$
(\mathcal{F}^* U)_{ab} = -l_a l_b C_{cedf} l^e n^f U^{cd} + 2l_{(ab} \delta^c \delta_{bc) \tau} + 2l_{(a} (\tau^c + \tau^c)_{b)c} - \Lambda U_{ab} ,
$$

and we conclude (see also [34]):

**Lemma 4.1:** (Hertz potentials for gravitational perturbations) Consider a background solution to the vacuum Einstein equations with $\Lambda$ having null vector fields $l^a, n^a$ with the optical properties (45) and an algebraically special property (44). Let $U_{ab}$ be a smooth symmetric, trace free tensor field satisfying $q^a c q^b d U_{ab} = U_{bd}$. together with

$$
(\mathcal{F}^* U)_{ab} = 0 .
$$

Here, $\mathcal{F}^*$ is the transpose of the operator $\mathcal{F}$ defined above in eq. (50) in terms of the operators $\bar{p}, \delta_a$ given in eq. (51) with $b = 2$. Then

$$
\gamma_{ab} = -l_a l_b (C_{cedf} l^e n^f U^{cd}) + 2l_{(ab} \delta^c \delta_{bc) \tau} + 2l_{(a} (\tau^c + \tau^c)_{b)c} - \Lambda U_{ab}
$$

is a solution to the linearized Einstein equation (3). We call $U_{ab}$ the Hertz-potential for $\gamma_{ab}$. Note that by definition $\gamma_{ab} l_b = 0 = q^{ab} \gamma_{ab}$.

### 4.2. Electromagnetic perturbations

Hertz potentials in higher dimensions can also be introduced in the case of electromagnetic perturbations. The Maxwell equations are

$$
\nabla^a F_{ab} = 0 , \quad \nabla_{[a} F_{bc]} = 0 .
$$

(60)

Taking derivatives of these equations, there follows the equation

$$
\nabla^c \nabla_c F_{ab} + R_{abcd} F^{cd} + R_{ad} F_{bc} + R_{bd} F_{ca} = 0 .
$$

(61)

We now assume again the background Einstein equation $R_{ab} - \frac{1}{2} g_{ab} R = -\Lambda g_{ab}$ and that the background has the optical and algebraically special properties as in (45), (44). We define
Ω_a = F_{cb}q^c_a l^b. By contracting eqs. (61), (60) in all possible ways into n^a, l^a, q_{ab}, one finds again that Ω_a satisfies a decoupled equation analogous to (50). It is [17]:

\[
(\mathcal{O}\Omega)_a \equiv \left\{ 2\mathcal{O}_a + \partial_c \partial^c - 4\mathcal{O}_a + q_{ec}l^d C_{cdef} - \frac{2\Lambda(2d-3)}{(d-1)(d-2)} \right\} \Omega_a + \left\{ -4\mathcal{O}_a q^d_{[a} \partial_{b]} + l^e C_{cdef} (3q^d_{[a} q^f_{b]} - q^d_{[a} q^f_{b]} \right\} \Omega_b = 0
\]  

(62)

The operators \( \mathcal{O}, \partial_a \) are defined as in eq. (51) with \( b = 1 \) in the present case. \( \mathcal{O} \) again has the character of a wave operator. In order to derive a Hertz potential for \( A_a \), we proceed just as in the case of gravitational perturbations. Let \( A_a \) be an arbitrary 1-form, not necessarily satisfying the Maxwell equations. One derives

\[
(\mathcal{O}\Omega)_a = \mathcal{O}(q^b_a j_b) - (\partial_a - 2\tau_a - \tau'_a)(l^b j_b),
\]  

(63)

where \( \Omega_a = l^b q^c_a F_{cb} \) is as above, and where \( J_a = \nabla^b F_{ba} \). As an equation for \( A_a \), we write this again in the form \( (\mathcal{O}\mathcal{T}A)_a = (\mathcal{T}\mathcal{L}A)_a \), where \( \mathcal{L} \) is now the Maxwell operator defined by

\[
(\mathcal{L}A)_a = 2\nabla^c \nabla_{[c} A_{a]},
\]  

(64)

where \( \mathcal{T} \) is defined by

\[
(\mathcal{T}A)_a = 2l^b q^c_a \nabla_{[c} A_{b]} ,
\]  

(65)

and where \( \mathcal{T} \) is defined by the right hand side of eq. (63). The transpose of that operator is

\[
(\mathcal{T}^*U)_a = -\mathcal{O}_a + l_a (\partial_b + \tau_b) U^b.
\]  

(66)

Taking the transpose of the operator equation \( \mathcal{O} \mathcal{T} = \mathcal{T} \mathcal{L} \) and applying both sides of the resulting equation to \( U_a \) now gives the desired Hertz potential (see also [34]):

**Lemma 4.2:** (Hertz potentials for electromagnetic perturbations) Consider a background solution to the vacuum Einstein equations with \( \Lambda \) having null vector fields \( l^a, n^a \) with the optical properties (45) and an algebraically special property (44). Let \( U_a \) be a smooth tensor field satisfying \( q_{bc} U_c = U_b \), together with

\[
(\mathcal{O}^*U)_a = 0.
\]  

(67)

Here, \( \mathcal{O}^* \) is the transpose of the operator \( \mathcal{O} \) defined above in eq. (62) in terms of the operators \( \mathcal{O}_a, \partial_a \) given in eq. (51) with \( b = 1 \). Then the field strength \( F_{ab} = 2\nabla_{[a} A_{b]} \) of

\[
A_a = -\mathcal{O}_a + l_a (\partial_b + \tau_b) U^b
\]  

(68)

is a solution to the Maxwell equations (60). We call \( U_a \) the Hertz-potential for \( A_a \). Note that by definition \( \nabla^a A_a = 0 = A_a l^a \).
5. Construction of a perturbation with \( \mathcal{E} < 0 \) in the NH geometry

5.1. Gravitational sector

We will now employ the Hertz potentials to construct a gravitational perturbation with \( \mathcal{E} < 0 \) in the NH geometry in the case when the operator (73) has an eigenvalue \( \lambda < -\frac{1}{4} \).

To do this, we must use the algebraically special properties of the near horizon geometries. Define null vector fields \( n, l \) by

\[
\begin{align*}
l &= \frac{1}{L \sqrt{2}} \left( R \frac{\partial}{\partial R} - \frac{1}{R} \frac{\partial}{\partial T} - k_I \frac{\partial}{\partial \phi_I} \right), \\
n &= \frac{1}{L \sqrt{2}} \left( R \frac{\partial}{\partial R} + \frac{1}{R} \frac{\partial}{\partial T} + k_I \frac{\partial}{\partial \phi_I} \right),
\end{align*}
\]

where the coordinates \( R, T, \phi_I \) refer to the form (31) of the NH geometry. These vector fields can both be shown to satisfy the optical properties (45), and the algebraically special properties (44) [17, 18]. In particular, since both \( n^a, l^a \) are twist-free we get

\[
\left[ n^a, l^b \right] \nabla^c l^d = \left[ n^a, l^b \right] n^c d^d,
\]

the subspaces perpendicular to \( n^a, l^a \) are integrable (by Frobenius’ theorem). The corresponding family of \((d - 2)\)-dimensional submanifolds (all diffeomorphic to \( B \)) establish an isomorphism \( M \cong \hat{M} \times B \), where \( \hat{M} \) is the base space of the foliation. Also, since the properties of being geodesic, null, shear, expansion, and twist-free are geometrical features that are invariant under any isometry, it follows at once that \( n^a, l^a \) are, up to a factor, Lie-derived by any Killing field, and it then also follows that the foliation is invariant under all the isometries. The leaves of this foliation in fact correspond precisely to the surfaces of constant \( T, R \) in eq. (31), which partially explains the geometrical significance of these coordinates from the point of view of algebraically special geometry.

Since the Hertz-potential \( U^{ab} \) has only components tangent to the foliation (i.e. because it is projected by \( q^{ab} \)), we may write

\[
U^{ab} = U^{AB} \left( \frac{\partial}{\partial x^A} \right)^a \left( \frac{\partial}{\partial x^B} \right)^b,
\]

where \( x^A \) are coordinates of \( B \). Tensor fields such as the Hertz-potentials \( U = U^{AB} \partial_A \otimes \partial_B \) that are tangent to the \((d - 2)\)-dimensional submanifolds \( B \) of \( M \cong \hat{M} \times B \) may be decomposed in a Fourier series with respect to the isometries \( U(1)^n \cong T^n \) acting on \( B \). We focus on Hertz potentials with a specific “mode number” \( m \in \mathbb{Z}^n \), and ones where the \( R, T \) dependence is separated from the \( x^A \) dependence. The ansatz is therefore

\[
U^{AB} = \psi \cdot Y^{AB} \cdot \exp(im \cdot \phi),
\]

where \( Y = Y^{AB} \partial_A \otimes \partial_B \) is a symmetric traceless tensor intrinsic to \( B \) that Lie derived by \( \partial / \partial \phi_I \) (i.e. dependent only on \( x^A \) and invariant under \( U(1)^n \)), and where \( \psi \) is a complex valued smooth function of \( R, T \). Inserting these definitions and using also (70), one finds, just as in [17], that for such \( U \), the action of \( \Theta^* \) (cf. condition (67)) is written as

\[
(\hat{D}^2 - q^2 - \Theta)U = \Theta^* U
\]
where $\hat{D} = \hat{V} - iq\hat{A}$ (see eq.(32)) and where $\mathcal{A}$ is the second order elliptic operator acting on symmetric traceless tensor fields on $\mathcal{B}$ given in coordinate components by

$$(\mathcal{A} U)^C_B = \left(- L^2 \mathcal{D} A - (\mathcal{D} A L^2) \mathcal{D} A + L^2 (\mathcal{D} A L^2)(\mathcal{D} A L^2) + \mathcal{D} A L^2 \right) U^C_B - 4 \left( k_A - \mathcal{D} A L^2 \right) U_{AC} + 6 - 2a - 4L^2 k_A - 2a(2-d) - 4L^2 \right) U^C_B +$$

$$L^2 \left( \mathcal{D} A B + \mathcal{D} \mu_{AB} \right) U^C_B + L^2 \left( \mathcal{D} C A + \mathcal{D} \mu_C \right) U^C_B - 2 \mathcal{D} C A B D U^{AD} \tag{73}$$

Capital Roman indices are always raised and lowered with $\mu^{AB}$, $\mu_{AB}$, and $\mathcal{D} A B C D \omega_D = 2 \mathcal{D} A B \omega_{C D}$ is the Riemann tensor of the Levi-Civita connection $\mathcal{D} A$ associated with $\mu_{AB}$. The quantities $\mu_{AB}, k^A, L$ are given concretely by equations (33) for the MP solutions. The “charge” $q \in \mathbb{C}$ is given by

$$q = -ib + a \quad \text{where} \quad \begin{cases} b = 2 \\ a = k \cdot m. \end{cases} \tag{74}$$

Since $\mathcal{A}$ is elliptic and $\mathcal{B}$ is compact, $\mathcal{A}$ has a basis of smooth, square integrable eigentensors. As noted [17], $\mathcal{A}$ is actually symmetric with respect to the inner product on $(\mathcal{B}, \mu)$ given by

$$(Y, Y)_{\mathcal{B}} = \int_{\mathcal{B}} |Y|^2 L^2 \sqrt{\mu} d^{d-2}x. \tag{75}$$

Therefore, the eigenvalues are all real. We may thus solve the condition $(\mathcal{D}^* U)_{ab} = 0$ required from a Hertz-potential by a separation of variables ansatz: Let $\lambda$ be an eigenvalue of $\mathcal{A}$, and $Y$ be a corresponding eigentensor that is invariant under $U(1)^n$, i.e.

$$\mathcal{A} Y = \lambda Y, \quad \ell_{\partial / \partial \psi} Y = 0 \quad I = 1, ..., n. \tag{76}$$

Then, if $\psi$ satisfies the “charged $AdS_2$ Klein-Gordon equation”

$$(\hat{D}^2 - q^2 - \lambda) \psi = - \frac{1}{R^2} \frac{\partial^2 \psi}{\partial T^2} + \frac{\partial}{\partial R} \left( R^2 \frac{\partial \psi}{\partial R} \right) - \frac{2iq}{R} \frac{\partial \psi}{\partial T} - \lambda \psi = 0, \tag{77}$$

it follows that $U_{ab} = U^{AB}(\partial_A)^a(\partial_B)^b$ as in eq. (71) satisfies $(\mathcal{D}^* U)_{ab} = 0$. Consequently, for such a $U_{ab}$, the perturbation $\gamma_{ab}$ given by eq. (59) satisfies the linearized Einstein equations on the NH geometry. By first solving the eigenvalue equation (76) to get $\lambda$, and then the Klein-Gordon equation (77), one can thus get solutions to the linearized Einstein equations.

We wish to evaluate the canonical energy of such a perturbation. To do this, we evaluate, at first, the symplectic form $W(\gamma_1, \gamma_2)$ of two perturbations, each given by eq. (59) in terms of two Hertz potentials as in (71). We record the lengthy expression in the next lemma:

**Lemma 5.1:** Let $\mathcal{A} Y = \lambda Y$, $\|Y\|_{\mathcal{B}} = 1$, assume $Y$ is Lie-derived by $\partial / \partial \phi_I, I = 1, ..., n$ (i.e. a tensor field on $\mathcal{B}$ that is invariant under $U(1)^n$), let $\psi_1, \psi_2$ be two (complex) solutions to the equation (77). Let $U_1, U_2$ be the corresponding (complex) Hertz-potential as in eq. (71), and

\[\text{Solutions to (77) are straightforward to obtain as we recall in appendix C. Solutions to the eigenvalue problem (76) must in general be found numerically in concrete examples.}\]
let \( \gamma_1, \gamma_2 \) be the corresponding (complex) perturbations as in eq. (59). Then the symplectic form is\(^{16}\)

\[
W(\Sigma, \gamma_1, \gamma_2) = -2 \int_\Sigma \hat{\ast} \hat{w}(\psi_1, \psi_2)
\]

(78)

where the conserved current \( \hat{w} \) on AdS\(_2\) is given up to a total divergence (i.e. up to changing \( \hat{\ast} \hat{w} \) by an exact 1-form) by

\[
16\pi \hat{w} = (R^{-1} \partial_T + R \partial_R + ia)^2 \hat{\psi}_1 (d + ia RdT) (R^{-1} \partial_T + R \partial_R - ia)^2 \psi_2 +
\]

\[
5(\partial_T + R \partial_R + ia) \hat{\psi}_1 (d + ia RdT) (R^{-1} \partial_T + R \partial_R - ia) \psi_2 +
\]

\[
4 \hat{\psi}_1 (d + ia RdT) \psi_2 +
\]

\[
8(\partial_T + R \partial_R + ia) \hat{\psi}_1 (R^{-1} \partial_T + R \partial_R + ia) \psi_2 (RdT + R^{-1} dR) -
\]

\[
\{3(\lambda + a^2) + 13i\alpha\} (\partial_T + R \partial_R + ia) \hat{\psi}_1 \psi_2 (RdT + R^{-1} dR) -
\]

\[
4(\partial_T + R \partial_R + ia) \hat{\psi}_1 \psi_2 (RdT + R^{-1} dR) +
\]

\[
3i a (\partial_T + R \partial_R + ia) \hat{\psi}_1 (R^{-1} \partial_T + R \partial_R - ia) \psi_2 (RdT + R^{-1} dR) +
\]

\[
4i a \hat{\psi}_1 \psi_2 (RdT + R^{-1} dR) - (\psi_1 \leftrightarrow \psi_2)
\]

(79)

As before, \( a = k \cdot m \).

Proof: The formula for \( \hat{w} \) can in principle be obtained by inserting the ansatz (71) into (59), then substituting that into the symplectic form \( W \), see eq. (4) and (6), then carrying out the integration of \( \mathcal{B} \) and then the integration over AdS\(_2\) afterwards, taking advantage of \( \mathcal{A} \psi = \lambda \psi \). However, looking at these expressions, it is clear that one will easily get several 100’s of terms, even before splitting the various covariant derivatives on \( \mathcal{M} \) into components along AdS\(_2\) resp. \( \mathcal{B} \). Another way of proceeding is to note that \( \hat{\ast} \hat{w} \), whatever its formula may be, must automatically be conserved because the symplectic form \( \ast w \) is closed. Furthermore, \( \hat{w}(\psi_1, \psi_2) \) must be anti-linear in \( \psi_1 \) and linear in \( \psi_2 \), and \( \hat{w}(\psi_2, \psi_1) = -\hat{w}(\psi_1, \psi_2)^* \) from the anti-symmetry of the symplectic form \( W \). There is essentially only one such current that can be constructed (i.e. any other differs at most by a total derivative or a multiplicative constant). Therefore, trying to construct an expression with these properties using only the charged AdS\(_2\)-Klein-Gordon equation (77) for \( \psi_1, \psi_2 \) is an alternative way to proceed. At first sight, one might think that such a current \( \hat{w} \) could only depend on at most terms with 1 derivative. Indeed, if the charge (74) were \( q = a \) (i.e. real) instead of \( q = -ib + a \) (i.e. complex), then the unique conserved current would be given by the standard expression

\[
\hat{J} = (d - iqRdT) \hat{\psi}_1 \psi_2 - \hat{\psi}_1 (d + iqRdT) \psi_2
\]

(80)

for a charged Klein-Gordon field in the vector potential \( \hat{A} = -RdT \). However, that expression is not conserved in our case, since the charge \( q = -ib + a \) is complex. Formula (80) for \( \hat{J} \) can also not be correct in our case since \( W \) clearly depends on up to 5 derivatives [note that the symplectic form \( W(\gamma_1, \gamma_2) \) depends on 1 derivative, and each perturbation \( \gamma_1, \gamma_2 \) is given by up to 2 derivatives of the corresponding Hertz-potential as in eq. (59)].

It turns out, as it must be the case, that a conserved current \( \hat{J} \) containing 5 derivatives can be constructed using only (77). We found it after quite a bit of trial and error. To check that \( \hat{J} \) is

\(^{16}\)For complex perturbations, we continue \( W \) anti- linearly in the first entry.
indeed conserved, we found it useful to work with the combination of derivatives as given in the definition of \( l, n \). The normalization factor is found comparing the highest derivative terms.

We can now construct the canonical energy as in eq. (12). As our Cauchy surface, we take, for simplicity \( \Sigma = \{ T = 0 \} \), and we take a perturbation \( \gamma_{ab} \) given in terms of a Hertz-potential \( U_{ab} \) as in eq. (59). For \( U_{ab} \), we make the separation of variables ansatz (71) in terms of some \( \psi \) with compact support on \( \hat{\Sigma} = \{ T = 0, R > 0 \} = \mathbb{R}_+ \). Recalling that \( K = \partial / \partial T \) in Poincaré coordinates, we get from the previous lemma

\[
\mathcal{E} = W(\Sigma; \gamma, \xi_K \gamma) = -2 \int_{\hat{\Sigma}} \hat{\omega}(\psi, \frac{\partial}{\partial T} \psi),
\]

where \( \hat{\omega} \) is the Hodge operator on \( AdS_2 \). When we evaluate \( \mathcal{E} \), we may use eq. (77) in order to write it in terms of \( \psi|_{\xi}, \partial_T \psi|_{\xi} \), because any \( T \)-derivative of order \( > 1 \) may be eliminated in favor of terms containing only up to one \( T \)-derivative. Thus, \( \mathcal{E} \) becomes a quadratic form (i.e. quadratic functional) \( \mathcal{E}(f_0, f_1) \) of the initial data

\[
(f_0, f_1) \equiv \left( \psi_{|T=0}, \frac{\partial}{\partial T} \psi \right|_{T=0} \in C_0^\infty(\mathbb{R}_+; \mathbb{C}) \times C_0^\infty(\mathbb{R}_+; \mathbb{C}),
\]

on \( \hat{\Sigma} \), i.e. at \( T = 0 \). The resulting formula is rather long and given in the appendix A. It simplifies somewhat for initial data having \( f_1 = 0 \) and \( f_0 \) real valued. In this case

\[
\mathcal{E} = \frac{1}{8\pi} \int_{-\infty}^{\infty} \left\{ \left( \frac{2d^3f_0}{dy^3} + \frac{d^2f_0}{dy^2} - (2\lambda + 1)\frac{df_0}{dy} + \lambda f_0 \right)^2 \right. \\
+ \left( \frac{2d^3f_0}{dy^3} + \frac{d^2f_0}{dy^2} - (\lambda + a^2)\frac{df_0}{dy} \right)^2 + 5 \left( \frac{d^2f_0}{dy^2} + \frac{df_0}{dy} - \lambda f_0 \right)^2 \\
+ (5 + 4a^2) \left( \frac{d^2f_0}{dy^2} \right)^2 + (\lambda - 3) \left( \frac{2d^2f_0}{dy} + \frac{df_0}{dy} - (\lambda + a^2)f_0 \right)^2 \\
+ (2\lambda - 4a^2 + 4a^2\lambda) \left( \frac{df_0}{dy} \right)^2 + (2\lambda + 3a^2)f_0^2 \\
- 2 \left( \frac{d^2f_0}{dy^2} + \frac{df_0}{dy} - \lambda f_0 \right) \frac{df_0}{dy} - 2(4 - 3\lambda - 3a^2) \left( \frac{d^2f_0}{dy^2} + \frac{df_0}{dy} - \lambda f_0 \right) f_0 \right\} e^y dy.
\]

Here, \( a = m \cdot k \) as before and \( y = \log R \). This expression is not manifestly positive definite, so there is a possibility of having \( \mathcal{E} < 0 \) for a suitable \( f_0 \) and \( \lambda \). To this end, we make the variational ansatz

\[
f_0(R) = \frac{R^N}{(R + \epsilon)^{N+1/2}(1 + R^N e^{1/(1-R)})}, \quad f_1(R) = 0,
\]

for \( 0 < R < 1 \) and \( f_0(R) = 0 \) for \( R \geq 1 \), where \( N \) is any number \( \geq 3 \) in the case of gravitational perturbations, so that the derivatives up to order 3 vanish at \( R = 0 \). Inserting the ansatz into our expression for \( \mathcal{E} \) gives, after a lengthy calculation:

\[
\mathcal{E} = \frac{1}{8\pi} (\lambda + \frac{1}{4})(\lambda^2 + 2a^2\lambda + a^4 - 9a^2 + \frac{2}{\epsilon}) \log \epsilon^{-1} + O(1),
\]

25
where \( O(1) \) stands for terms having a finite limit as \( \epsilon \to 0^+ \). From this expression, we are able to draw the following conclusions about the positivity properties of \( \mathcal{E} \) on the NH geometry:

(i) If \( a = 0 \), then the right side of eq. (84) is dominated by \( 2(\lambda + \frac{1}{4})(\lambda^2 + \frac{7}{4}) \log \epsilon^{-1} \), which becomes negative for \( \lambda < -\frac{1}{4} \) and sufficiently small \( \epsilon > 0 \). Whence, there are initial data of the form (102) with \( \mathcal{E} < 0 \).

(ii) Let \( \underline{m} \in \mathbb{Z}^n \) be such that \( a = \underline{m} \cdot \underline{k} \neq 0 \) (w.l.o.g. take \( a > 0 \)). In that case, we note that the operator \( \mathcal{A} \) in (73) can be written as \( \mathcal{A}_0 - a^2 \), where \( \mathcal{A}_0 \) means that we set \( a = 0 \) in eq. (73). Hence, the eigenvalues of \( \mathcal{A} \) are \( \lambda - a^2 \), where \( \lambda \) is some eigenvalue of \( \mathcal{A}_0 \). Inserting this eigenvalue into the right side of eq. (84), we trivially find that \( \mathcal{E} = 2(\lambda - a^2 + \frac{1}{4})(\lambda^2 - 9a^2 + \frac{7}{4}) \log \epsilon^{-1} \) up to \( O(1) \)-terms. This expression is negative e.g. if \( a^2 - \frac{1}{4} > \lambda > 3a \). We now rescale \( \underline{m} \rightarrow N\underline{m} \) by a natural number \( N \), so that \( a \rightarrow Na \). Then, in order to get a negative \( \mathcal{E} \), we need \( a^2 - 1/(4N^2) > \lambda/N^2 > 3a/N \). It is easy to satisfy these inequalities by choosing e.g. \( N = \lceil \sqrt{\lambda + 1/a} \rceil \) and \( \lambda \) sufficiently large. Therefore, there are again initial data of the form (102) with \( \mathcal{E} < 0 \) if we rescale \( \underline{m} \) by a sufficiently large natural number.

Our ansatz (102) for the initial data does not have compact support on \( R > 0 \), since the support clearly includes \( R = 0 \). However, because \( \mathcal{E} \) only depends on up to 3 derivatives with respect to \( R \), and because \( f_0(R) \) is a three times differentiable function whose derivatives vanish up to third order at \( R = 0 \), it is possible to slightly translate \( f_0(R) \) to the right and modify so that the new \( f_0(R) \) is compactly supported away from \( R = 0 \), smooth, and still has \( \mathcal{E} < 0 \). Whence, we have shown the following theorem:

**Theorem 1:** (i) Let \( \underline{m} \) be such that \( a = 0 \), and let \( \lambda \) be the smallest eigenvalue of the operator \( \mathcal{A} \) (see eq. (73)). If \( \lambda < -\frac{1}{4} \), then there exists a perturbation \( \gamma_{ab} \) of the form (59), with Hertz potential \( U^{ab} \) of the form (71), such that \( \mathcal{E} < 0 \) on the NH geometry and such that the initial data for \( \gamma_{ab} \) are compactly supported on \( \Sigma = \{ T = 0 \} \).

(ii) Let \( \underline{m} \) be such that \( a \neq 0 \). Then we can rescale \( \underline{m} \rightarrow N\underline{m} \) such that there exists of perturbation as described in (i), with rescaled \( \underline{m} \) such that \( \mathcal{E} < 0 \) on the NH geometry.

### 5.2. Electromagnetic sector

A similar analysis is possible in the case of electromagnetic perturbations. In that case, the Hertz potential is

\[
U^a = U^A \left( \frac{\partial}{\partial x^A} \right)^a \tag{85}
\]

and it must satisfy \( (\partial^* U)^a = 0 \), where \( \partial^* \) is now as in eq. (62). The separation ansatz is now

\[
U^A = \psi \cdot Y^A \cdot \exp(i\underline{m} \cdot \phi) \, , \tag{86}
\]

where \( Y = Y^A \partial_A \) is a tensor intrinsic to \( \mathcal{B} \) that is Lie derived by \( \partial/\partial \phi_i \) (i.e. dependent only on \( x^A \) and invariant under \( U(1)^n \)), and where \( \psi \) is a complex valued smooth function of \( R, T \). Inserting these definitions and using also (70), one finds, just as in [17], that for such \( U \), the action of \( \partial^* \) (cf. condition (67)) is as written in (72) with complex charge \( q = -ib + a, a = \)
\( k \cdot m, b = 1 \) in the electromagnetic case. \( \mathcal{A} \) is now the second order elliptic operator acting on vector fields on \( \mathcal{B} \), given in coordinate components by

\[
(\mathcal{A} U)_A = -L^{-2} D^B (L^A D_B U^A) + (2 - a^2 - \frac{s}{4L^2} k b k - \frac{d}{2} \Lambda L^2) U_A \\
+ L^2 (\mathcal{R}_{AB} - \frac{1}{2} \mathcal{H}_{AB}) U_B + \left( - D_A k_B + 2(k_A - 2L^A L) D_B - 2L^{-1} D_A k_B \right) U_B.
\]

(87)

Since \( \mathcal{A} \) is elliptic and \( \mathcal{B} \) is compact, \( \mathcal{A} \) has a basis of smooth, square integrable eigenvector fields. As noted by Ref., \( \mathcal{A} \) is actually symmetric with respect to the inner product on \( (\mathcal{B}, \mu) \) given by (75). Therefore, the eigenvalues are again real. The symplectic form is now given by the following lemma, which is demonstrated just as in the gravitational case:

**Lemma 5.2:** Let \( \mathcal{A} Y = \lambda Y, \| Y \|_{\mathcal{B}} = 1 \), assume \( Y \) is Lie-derived by \( \partial / \partial \phi, I = 1, ..., n \) (i.e. a vector field on \( \mathcal{B} \) that is invariant under \( U(1)^n \)), let \( \psi_1, \psi_2 \) be two (complex) solutions to the equation (77). Let \( U_1, U_2 \) be the corresponding (complex) Hertz-potential as in eq. (86), and let \( A_1, A_2 \) be the corresponding (complex) perturbations as in eq. (68). Then the symplectic form

\[
W(\Sigma, A_1, A_2) = -2 \int_\Sigma \hat{\mathcal{W}}(\psi_1, \psi_2),
\]

where the conserved current \( \hat{\mathcal{W}} \) on \( \text{AdS}_2 \) is given up to a total divergence (i.e. up to changing \( \hat{\mathcal{W}} \) by an exact 1-form) by

\[
4\pi \hat{\mathcal{W}} = (-R^{-1} \partial_T + R \partial_R + ia) \psi_1 (d + iaRdT) (-R^{-1} \partial_T + R \partial_R - ia) \psi_2 + \bar{\psi}_1 (d + iaRdT) \psi_2 - [\psi_1 (R^{-1} \partial_T + R \partial_R + ia) \psi_2 - ia \psi_1 \psi_2] (RdT + R^{-1} dR) - (\psi_1 \leftrightarrow \psi_2),
\]

(89)

where as before, \( a = m \cdot k \).

We can now construct the canonical energy as in eq. (21). We take a perturbation \( A_\alpha \) given in terms of a Hertz-potential \( U^\alpha \) as in eq. (68). For \( U^\alpha \), we make the separation of variables ansatz (86) in terms of some \( \psi \) with compact support on \( \Sigma = \{ T = 0, R > 0 \} = \mathbb{R}_+ \). Recalling that \( K = \partial / \partial T \) in Poincaré coordinates, we get from the previous lemma

\[
\mathcal{E} = W(\Sigma; A, LkA) = -2 \int_\Sigma \hat{\mathcal{W}}(\psi, \frac{\partial}{\partial T} \psi).
\]

(90)

When we evaluate \( \mathcal{E} \), we may again use eq. (77) (this time with \( q = -i + a \)) in order to eliminate terms containing more than one \( T \)-derivative. Thus, \( \mathcal{E} \) becomes a quadratic form (i.e. quadratic functional) \( \mathcal{E}(f_0, f_1) \) of the initial data (82) on the \( \Sigma \), i.e. at \( T = 0 \). The resulting formula is rather long and given in the appendix B. It simplifies somewhat for initial data having \( f_1 = 0 \) and \( f_0 \) real valued. A calculation reveals that, in this case

\[
\mathcal{E} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left( \frac{d^2 f_0}{dy^2} + \frac{df_0}{dy} - \lambda f_0 \right)^2 + \left( \frac{d^2 f_0}{dy^2} \right)^2 + (\lambda + a^2) \left( \frac{df_0}{dy} \right)^2 + \lambda a^2 f_0^2 \right\} e^y dy.
\]

\(^{17}\)For complex perturbations, we continue \( W \) anti-linearly in the first entry.
Here, \( a = m \cdot k \) as before and \( y = \log R \). We substitute the variational ansatz (102) for \( f_0, f_1 \), taking any \( N \geq 2 \). A lengthy calculation shows that, for this choice

\[
\mathcal{E} = \frac{1}{2\pi} (\lambda + \frac{1}{4}) (\lambda + \frac{1}{2} + a^2) \log \varepsilon^{-1} + O(1),
\]

(92)

where \( O(1) \) stands for terms that do not diverge as \( \varepsilon \to 0^+ \). A relatively simple form of \( \mathcal{E} \) is also found for initial data having \( f_0 = 0 \) and \( f_1 \) real valued. In this case:

\[
\mathcal{E} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \left( \frac{d f_1}{dy} - 2 f_1 \right)^2 + \left( \frac{d f_1}{dy} - f_1 \right)^2 + (\lambda + 4a^2 - 2) f_1^2 \right\} e^{-y} dy.
\]

(93)

We substitute the variational ansatz (102) with \( f_1 \leftrightarrow f_0 \) and \( R \leftrightarrow 1/R \). A calculation shows that, for this choice

\[
\mathcal{E} = \frac{1}{2\pi} (\lambda + \frac{1}{4} + a^2) \log \varepsilon^{-1} + O(1).
\]

(94)

From the expressions (92), (94), we are able to draw the following conclusions about the positivity properties of \( \mathcal{E} \) in the NH geometry. (i) If \( a = 0 \) and if \( -\frac{1}{2} < \lambda < -\frac{1}{4} \), then for \( \varepsilon > 0 \) and sufficiently small, we get \( \mathcal{E} < 0 \) from (92), whereas for \( \lambda < -\frac{1}{2} \), we get \( \mathcal{E} < 0 \) from (94). (ii) If \( a \neq 0 \), then the same statements as in the gravitational case are true. The trial functions \((f_0, f_1)\) employed to get (92), (94) are not of compact support on \( \mathbb{R}_+ \), but we may argue as in the case of gravitational perturbations that we can slightly modify it to such a function keeping \( \mathcal{E} < 0 \).

In summary, our discussion shows that thm. 1 also holds in the electromagnetic case, except possibly in the case when \( \lambda = -\frac{1}{2} \) and \( a = 0 \).

6. Construction of a perturbation with \( \mathcal{E} < 0 \) in the extremal BH geometry

6.1. Outline of the construction

In theorem 1 have identified cases (depending generically on spectrum the operator \( \mathcal{A} \)) in which there is a gravitational perturbation \( \gamma_{ab} \) of the form (59), with Hertz potential \( U_{ab} \) as in (71), which: (i) is of compact support on the Cauchy surface \( \Sigma = \{ T = 0 \} \) of the NH geometry, (ii) satisfies the linearized Einstein equations (3), and (iii) has \( \mathcal{E} < 0 \) in the NH spacetime. Starting from such a perturbation, we will construct in this section a perturbation of the corresponding BH spacetime which is of compact support on \( \Sigma \), which satisfies the perturbed Einstein equations and still has \( \mathcal{E} < 0 \) in the BH spacetime. This will lead to the main results of this paper given in theorem 2 for \( \Lambda = 0 \), assumed from now on. A simple extension to asymptotically AdS solutions (\( \Lambda < 0 \)) will give theorem 3.

We repeat that \( \gamma_{ab} \) as given by thm. 1 is, by construction, a solution to the linearized Einstein equations (3) on the NH background, but of course not on the BH background, \( (L\gamma)_{ab} \neq 0 \), where from now on and in the following \( g_{ab}, \nabla_a \) refer to the BH background, and \( L \) in this equation is the linearized Einstein operator of the BH background, see eq. (3). We will construct the desired perturbation of the BH background in two steps:
1. We identify tensor fields on the NH geometry (in particular $\gamma_{ab}$) with tensor fields in the BH geometry by identifying points in both spacetimes (near $\mathcal{H}^\pm$) if they carry the same Gaussian null coordinates. Under this identification the slice $\Sigma = \{ T = 0 \}$ in the NH geometry corresponds via eqs. (37) to a slice $\Sigma$ in the BH spacetime “running down the throat”.

2. We then apply the scaling isometry $\phi_\varepsilon$ to $\gamma_{ab}$, see (25), and define, for small $\varepsilon > 0$

$$\gamma_{ab}(\varepsilon) \equiv \frac{1}{\sqrt{\varepsilon}} \phi_\varepsilon^* \gamma_{ab}.$$  

(95)

Since $\phi_\varepsilon$ is an isometry of the NH geometry, $\gamma_{ab}(\varepsilon)$ is a new solution to the linearized Einstein equations on the NH geometry, having compactly supported initial data on $\Sigma$. The support “moves down the throat” as $\varepsilon \to 0$. Moreover, since $\phi_\varepsilon^* K = \varepsilon K$, it follows that the canonical energy of the NH-spacetime remains unchanged,

$$\mathcal{E}(\Sigma, \gamma(\varepsilon)) = \mathcal{E}(\Sigma, \gamma) < 0.$$  

(96)

3. Let

$$(\delta h_{ij}(\varepsilon), \delta p^{ij}(\varepsilon))$$  

$\equiv$ initial data of $\gamma_{ab}(\varepsilon)$ on $\Sigma$.  

(97)

By construction, these $\varepsilon$-dependent initial data (we omit the reference to $\varepsilon$ in the following) satisfy the constraints of the NH-spacetime, but not the BH-spacetime. We then add to these initial data a small perturbation (for small $\varepsilon$), such that the modified initial data are still of compact support, satisfy the constraints of the BH-spacetime [under the identification in 1)], and still have negative canonical energy in the BH-spacetime. As described in the introduction, the time-evolution of these modified initial data in the BH spacetime cannot settle down to a perturbation that is pure gauge or represents a perturbation to another stationary black hole in the family. Thus, such a black hole is linearly unstable.

6.2. Correcting the variational ansatz for initial data

We now turn to a more precise explanation of the strategy we have just outlined. Steps 1) and 2) do not require further explanation, but point 3) is of a rather technical nature and needs to be discussed. Generally speaking, we have the following problem. We have an ansatz $(\delta h_{ij}, \delta p^{ij})$ – in our case given by eq. (97) – for the initial data having support in a bounded set $A_0 \subset \subset \Sigma$ (in the BH spacetime). The linearized constraints are not satisfied. We would like to modify our ansatz by adding a correction so that the new initial data are compactly supported in a some (possibly slightly larger) bounded set $A \supset \supset A_0$, and solve the linearized constraints. There is a well-known general method for achieving just this, developed in [37], and also in [38]. We now describe this method, following with minor modifications the original references and paying attention in particular to the key question for us by how much the original ansatz has to be modified depending on how much it violated the linearized constraints.

The constraints are (assuming $\Lambda = 0$ from now on):

$$C = h^{\frac{1}{2}} \begin{pmatrix} -\text{Scal}_h + h^{-1} p_{ij} p^{ij} - \frac{1}{2} h^{-1} p^2 & -2D_j (h^{-\frac{1}{2}} p^{ij}) \end{pmatrix} = 0.$$  

(98)
The linearized constraints will generally use boldface letters for a tuple consisting of a scalar (or density) on \( \Sigma \), and a vector (or density) on \( \Sigma \). The linearized constraints \( \delta C \) may be viewed as the result of acting on \( (\delta h_{ij}, \delta p^{ij}) \) by a linear operator, \( \mathcal{C} \). It is explicitly given by

\[
\mathcal{C} \left( \frac{\delta h_{ij}}{\delta p^{ij}} \right) = \left( h^{\frac{1}{2}} (D^i D_j \delta h_{ij} - D^i D_j \delta h_{ij}) + h^{\frac{1}{2}} (\delta h_{ij} p^i p^j) u + 2p_i \delta p^{ij} + 2p^i p_j \delta h_{ij} - \frac{1}{2} \delta h_{ij} p^k p^j p^k) \right). \tag{99}
\]

Since \( \mathcal{C} \) is a differential operator that maps the pair \((\delta h_{ij}, \delta p^{ij})\) consisting of a symmetric tensor, \( \delta h_{ij} \), and a symmetric tensor density, \( \delta p^{ij} \) on \( \Sigma \) into a pair \((u, X^i)\) consisting of a scalar density and dual vector density on \( \Sigma \), its adjoint differential operator, \( \mathcal{C}^* \), maps a pair \( X = (u, X^i) \) consisting of a scalar and vector field on \( \Sigma \) into a pair \((\delta h_{ij}, \delta p^{ij})\) consisting of a symmetric tensor density and symmetric tensor on \( \Sigma \). One can straightforwardly calculate that \( \mathcal{C}^* \) is given by

\[
\mathcal{C}^* \left( \frac{u}{X^i} \right) = \left( h^{\frac{1}{2}} (\delta h_{ij} h^{ij} + D^i D^j u + Ric(h)_{ij} u) + h^{\frac{1}{2}} (\delta h_{ij} p^i p^j) u + 2p_i \delta p^{ij} + 2p^i p_j \delta h_{ij} - \frac{1}{2} \delta h_{ij} p^k p^j p^k) \right). \tag{100}
\]

The idea is to make particular ansatz for the correction to \((\delta h_{ij}, \delta p^{ij})\) in order to satisfy the linearized constraints. Let \( s : A \rightarrow \mathbb{R} \) be a function \( 1 \geq s > 0 \) such that near the boundary \( \partial A \), we have

\[
s(x) = \text{dist}_h(x, \partial A), \tag{101}
\]

where we mean the geodesic distance relative to the metric \( h \) on \( \Sigma \). We also ask that \( s(x) = 1 \) in \( A_0 \subset \subset A \). The ansatz is:

\[
\left( \frac{\delta h_{ij}}{\delta p^{ij}} \right) \equiv e^{-2/s^a} \left( \begin{array}{cc} s^4 \alpha + 4 & 0 \\ 0 & s^2 \alpha + 2 \end{array} \right) \mathcal{C}^* \left( \frac{u}{X^i} \right). \tag{102}
\]

The ‘cutoff functions’ involving \( s \) are being put in since we anticipate extending the solution by \( 0 \) across the boundary \( \partial A \) in a smooth way. The tensors \( X = (u, X^i) \) are to be determined. The matrix of cutoff functions appearing on the right side will appear often, so we introduce the shorthand:

\[
\Phi \equiv e^{-1/s^a} \left( \begin{array}{cc} s^{2\alpha + 2} & 0 \\ 0 & s^{\alpha + 1} \end{array} \right). \tag{103}
\]

Our ansatz can then be written more compactly as

\[
\left( \frac{\delta \bar{h}}{\delta \bar{p}} \right) \equiv \left( \frac{\delta h}{\delta p} \right) - \Phi^2 \mathcal{C}^* X. \tag{104}
\]

\[18\]In terms of the Einstein tensor \( G_{ab} \) and unit normal \( \nu^a \), the Hamiltonian constraint is given by \( = G_{ab} \nu^a \nu^b + \Lambda \), whereas the vector constraint is given as \( = G_{cb} \nu^b h^c_a \).
We want \( (\delta h_{ij}, \delta p^{ij}) \) to satisfy the linearized constraints of the BH background. Acting with \( \mathcal{C} \) shows that \( \mathbf{X} \) must satisfy the equation:

\[
\mathcal{C} \Phi^2 \mathcal{C}^* \mathbf{X} = \mathbf{f}, \tag{105}
\]

where \( \mathbf{f} \equiv \delta \mathcal{C} = \mathcal{C} (\delta h, \delta p) \) is the violation of the linearized constraints of the ansatz \( (\delta h_{ij}, \delta p^{ij}) \).

The question is of course whether (105) has a suitable solution at all, which is far from obvious. In order to construct such a solution, and to control its properties, one uses the technology of weighted Sobolev spaces. We begin by defining the weighted Sobolev norms

\[
\|u\|_{W^{p,k},\alpha} = \left( \sum_{n=0}^{k} \int_{A} |D^n u|^p s^n(\alpha+1) e^{-2/s^\alpha} \text{dvol}_{A} \right)^{1/p}.
\tag{106}
\]

on \( C^\infty_0 (A) \) tensor fields \( u \) on \( A \subset \Sigma \). We let \( W^{p,k},\alpha} (A) \) be the completion of the space of such tensor fields under this norm. Since we will mostly consider \( p = 2 \), and sometimes \( \alpha = 0 \), we introduce the notations \( H^{k,\alpha} = W^{2,k,\alpha}, L^{2,\alpha} = H^{0,\alpha}, L^2 = L^{2,0} \). We also use the notation \( H^k \) for the ordinary Sobolev spaces without any weights. Our weights differ slightly from those used by [37]. The following lemma is the key to prove the existence of a weak solution to (105):

**Lemma 6.1:** (Generalized weighted Friedrichs-Poincaré inequality) For sufficiently large \( \alpha \), there is a constant \( c = c(\alpha, A) \) such that

\[
c \| \Phi \mathcal{C} \mathbf{X} \|_{L^2} \geq \| \mathbf{X} - P_A \mathbf{X} \|_{H^{2,\alpha} \oplus H^{1,\alpha}}, \tag{107}
\]

for any tensor field \( \mathbf{X} \in H^{2,\alpha} (A) \oplus H^{1,\alpha} (A) \). Here \( P_A \) is the orthogonal projector (in \( L^{2,\alpha} (A) \)) onto the subspace \( \mathfrak{k} \) spanned by the KVF’s, i.e. if \( \mathcal{Y}_i \) is a basis of Killing vector fields on \( M \) that has been orthonormalized (in \( L^{2,\alpha} \)) with the Gram-Schmidt process, we have

\[
P_A \mathbf{X} = \sum_i \mathcal{Y}_i (\mathcal{Y}_i, X)_{L^2} \mathcal{Y}_i. \tag{108}
\]

The proof of this lemma is given in appendix D below using a method which is somewhat different from [37, 38]. Using this key lemma, one can show existence:

**Lemma 6.2:** Let \( \mathbf{f} \in C^\infty_0 (A) \) with support in \( A_0 \subset A \). Then there exists a solution \( \mathbf{X} \) to (105) which is in \( H^{2,\alpha} (A) \oplus H^{1,\alpha} (A) \) and which in fact additionally satisfies for all \( k = 0, 1, 2, \ldots \)

\[
\int_A s^{2k\beta} |D^k (\Phi \mathcal{C} \mathbf{X})|^2 \text{dvol}_A \leq c \| \mathbf{f} \|_{H^2 \oplus H^k}^2 \tag{109}
\]

for a sufficiently large \( \beta > 0 \), and a constant \( c = c(A, \alpha, k) \).

**Remarks:** a) Note that in our definition of the corrected initial data (104), we have on the right side the expression \( \Phi^2 \mathcal{C}^* \mathbf{X} \), i.e. we have the square of \( \Phi \). Since \( \Phi \) is a multiplication operator involving the exponential cutoff factor \( e^{-1/s^\alpha} \) [cf. (103)], it follows from the estimate in the previous lemma (because \( s^{-N} e^{-1/s^\alpha} \rightarrow 0 \) for any \( N \) when \( s \rightarrow 0 \)) that \( s^{-N} \Phi^2 \mathcal{C}^* \mathbf{X} \) is in each (unweighted) Sobolev space of arbitrary order \( k \) for any \( N \). Thus, by the usual Sobolev
embedding theorem, \( C^\omega(\tilde{A}) \subset \cap_k H^k(A) \), it follows that \( \Phi^2 \mathcal{E}^* \mathbf{X} \) is smooth up to and including the boundary \( \partial A \), and that it can in fact be smoothly extended by 0 across \( \partial A \). Thus, the corrected initial data (104) are smooth up to and including the boundary \( \partial A \) and can be extended by 0 across \( \partial A \).

b) Below, we will consider applying this result to an annular domain of the form \( A = \{ x \in \Sigma \mid y_0 - \log \varepsilon < y(x) < y_1 - \log \varepsilon \} \). We claim that for \( \varepsilon \to 0 \) (i.e., for the annular domain going down the throat), the constant \( c = c(\alpha, k, A) \) may be chosen to be independent of \( \varepsilon \). This is in essence a direct consequence of the fact that the background \( h_{ij} \) and \( K_{ij} \) (hence also \( p^{ij} \)) are nearly translation invariant under shifts of \( y \) in the throat \((y \to -\infty)\), see eq. (41), and follows by inspecting the constants in (127) and (109).

\textbf{Proof of Lemma 6.2:} The proof of this lemma consists of an application of standard tools from PDE-theory for elliptic operators. The only differences are the weight factors, and the fact that the operator in question \( \mathcal{E} \Phi^2 \mathcal{E}^* \) is a system of mixed order (up to order 4), see [49, 50] for the corresponding classical results. Existence is proved with the help of the weighted Poincaré-Friedrichs inequality. One considers the weak formulation of the PDE problem (105) which consists in finding an element \( \mathbf{X} \in H^2_0(A) \oplus H^1_0(A) \) such that

\[
B[\mathbf{X}, \mathbf{Y}] = F[\mathbf{Y}], \quad \text{for all } \mathbf{Y} \in H^2_0(A) \oplus H^1_0(A),
\]

(110)

where the bilinear form \( B \) is \( B[\mathbf{X}, \mathbf{Y}] = (\Phi \mathcal{E}^* \mathbf{X}, \Phi \mathcal{E}^* \mathbf{X})_{L^2} \) and where the functional \( F \) is \( F[\mathbf{X}] = (\mathbf{f}, \mathbf{X})_{L^2} \). This weak formulation is obtained as usual by formally multiplying the PDE with \( \mathbf{Y} \), integrating over \( A \), and performing (formally) partial integrations to bring the operator \( \mathcal{E} \) to the other factor as \( \mathcal{E}^* \). The subscript “0” in our choice of Sobolev space anticipates/reflects a choice of “boundary conditions”, and the weight \( \alpha \) in the Sobolev space corresponds that in \( \Phi \), see (103). Note that if \( \mathbf{Y} \) corresponds to a KVF, then, since \( \mathcal{E}^* \mathbf{Y} = 0 \), it also follows that \( F[\mathbf{Y}] = 0 \), and similarly that \( B[\mathbf{X}, \mathbf{Y}] = 0 \). Thus, it is sufficient to satisfy the above identity for all \( \mathbf{Y} \in H^2_0(A) \oplus H^1_0(A) \) that are orthogonal to the span \( \mathfrak{t} \) of KVF’s. On that subspace the quadratic form is bounded from below by a positive multiple of the norm (coercive) by the Poincaré-Friedrichs inequality, whereas \( F \) is bounded in the \( H^2_0(A) \oplus H^1_0(A) \)-norm\(^{19}\). Existence of a weak solution then follows from the standard Lax-Milgram theorem (see e.g. [48]), and one has, in fact,

\[
\| \mathbf{X} \|_{H^2_0(A) \oplus H^1_0(A)} \leq c_0 \| \mathbf{f} \|_{L^2(A) \oplus L^2} ,
\]

(111)

for some constant \( c_0 = c_0(\alpha, A) \).

It remains to show the regularity statement (109). Here, we proceed by the standard method of finite difference quotients, combined with the Poincaré-Friedrichs inequality. First, we slice \( A \) near the boundary like an onion into the ‘skins’ \( O_n = A_{2^{n+1}} \setminus A_{2^{n-1}} \), where each set \( A_\delta \) is characterized by the condition that \( s(x) < \delta \). Next, we choose test functions \( \zeta_n \geq 0 \) having support in \( O_n \), and such that \( \Sigma \zeta_n = 1 \). We may assume that \( |D^k \zeta_n| \leq c_1 2^{nk} \) for some constants \( c_1 = c_1(k, A) \), and we shall pretend, in order avoid a more cumbersome notation, that the support of each \( \zeta_n \) is contained in a single coordinate chart. This could always be achieved by subdividing \( O_n \) further into a fixed (independent of \( n \)) number of subregions. Points in \( O_n \) are then identified with their coordinate vectors in \( \mathbb{R}^{d-1} \). The finite difference operator in the \( j \)-th

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\(^{19}\)Here it is used that \( \mathfrak{t} \) is supported away from the boundary \( \partial A \), so that the weight factors do not play a role.
We also have factors of 2, which, for sufficiently small \( \delta \), the r.h.s. is now bounded from below as explained e.g. in sec. 6.3.2 of [48].

The factor of 2 arises from the fact that a 'partial integration' results in a factor \( \alpha \). Using similar arguments, the l.h.s. of eq. (110) with the test-function

\[
Y = \Delta_j^{-\delta} (\zeta_n^2 \Delta_\delta X), \quad 0 < \delta \ll 1
\]

so that

\[
F[\Delta_j^{-\delta} (\zeta_n^2 \Delta_\delta X)] = B[\Delta_j^{-\delta} (\zeta_n^2 \Delta_\delta X), X].
\]

The r.h.s. is now bounded from below as explained e.g. in sec. 6.3.2 of [48], where the only differences in our case are the presence of weights in \( B \), and the fact that \( B \) contains higher derivatives. As in the standard case, the basic idea is simply to 'move \( \Delta_\delta \) to the other factor' using the 'partial integration' and 'Leibniz' rules for finite difference operators. One finds, for sufficiently small \( \delta > 0 \):

\[
\text{r.h.s.} \geq B[\zeta_n \Delta_j^\delta X, \zeta_n \Delta_j^\delta X] - c_3 \{ \delta \| \zeta_n \Delta_j^\delta X \|_{H^2,a \oplus H^1,a}^2 + 2^{n(1+\alpha)} \| X \|_{H^2,a \oplus H^1,a} \| \zeta_n \Delta_j^\delta X \|_{H^2,a \oplus H^1,a} \}
\]

The factor of \( 2^{1+\alpha} \) arises from the fact that a 'partial integration' results in a factor \( \Delta_\delta^\alpha (s^{p(1+\alpha)} e^{-2/s^a}) \), which, for sufficiently small \( \delta > 0 \) is bounded on \( O_n \) by

\[
|\Delta_j^{-\delta} (s^{p(1+\alpha)} e^{-2/s^a})| \leq c_{12} 2^{n(1+\alpha)} s^{p(1+\alpha)} e^{-2/s^a}.
\]

We also have factors of \( 2^n \) arising from the fact that pulling \( \zeta_n \) through various derivative operators will result in \( D\zeta_n \), \( 2^2 \zeta_n \), which are bounded by a constant times \( 2^n \) resp. \( 2^{2n} \leq 2^{n(1+\alpha)} \), choosing \( \alpha > 1 \). Employing the 'Peter-Paul' trick \( 2|ab| \leq a^2/\epsilon + \epsilon b^2 \) on the last term (giving small weight to the norm \( b = \| \Delta_j^\delta X \|_{H^2,a \oplus H^1,a} \) results altogether in

\[
\begin{align*}
\geq & B[\zeta_n \Delta_j^\delta X, \zeta_n \Delta_j^\delta X] - c_3 (\delta + \epsilon) \| \zeta_n \Delta_j^\delta X \|_{H^2,a \oplus H^1,a}^2 + c_3 2^{n(1+\alpha)} \| X \|_{H^2,a \oplus H^1,a}^2 \\
\geq & [c_4 - c_3 (\delta + \epsilon)] \| \zeta_n \Delta_j^\delta X \|_{H^2,a \oplus H^1,a}^2 - c_5 2^{n(1+\alpha)} \| X \|_{H^2,a \oplus H^1,a}^2 \\
\geq & [c_4 - c_3 (\delta + \epsilon)] \| \zeta_n \Delta_j^\delta X \|_{H^2,a \oplus H^1,a}^2 - c_5 2^{n(1+\alpha)} \| X \|_{H^2,a \oplus H^1,a}^2 \quad \text{(116)}
\end{align*}
\]

applying in the second line the Friedrichs-Poincaré inequality (giving rise to the constant \( c_4 \)), and in the third line the inequality (111), combining the constants into \( c_5 \). We choose \( \epsilon, \delta \) so small that \( c_4 > c_3 (\delta + \epsilon) \). Then the coefficient in front of the first term on the right side is positive. Using similar arguments, the l.h.s. of eq. (114) is bounded by

\[
\text{l.h.s.} \leq c_6 \left\{ 2^{n(1+\alpha)} \| f \|_{H^{1,a}}^2 + \| X \|_{H^{2,a \oplus H^1,a}}^2 \right\} \leq c_7 2^{n(1+\alpha)} \| f \|_{H^{1,a}}^2 \quad \text{(117)}
\]
using again (111). Combining the bounds for the left and right hand sides, we find for some constant \( c_8(\alpha, A) \) and sufficiently small \( \delta > 0 \) that
\[
\| \zeta_n \Delta_j X \|_{H^2, a \oplus H^1, a} \leq c_8 2^{n(1+\alpha)} \| f \|_{H^1 \oplus H^1},
\]
and the same bound in fact then also holds for \( \zeta_n D_j X \) by the properties of the finite difference quotients and (111). Therefore
\[
\| s^\beta D X \|_{H^2, a \oplus H^1, a} = \| s^\beta (\sum_n \zeta_n) D X \|_{H^2, a \oplus H^1, a}
\leq \sum_n \| s^\beta \zeta_n D X \|_{H^2, a \oplus H^1, a}
\leq c_9 \sum_n 2^{-\beta n} \| \zeta_n D X \|_{H^2, a \oplus H^1, a}
\leq c_{10} \sum_n 2^{-\beta n} 2^{n(1+\alpha)} \| f \|_{H^1 \oplus H^1}
\leq c_{11} \| f \|_{H^1 \oplus H^1}
\]
assuming \( \beta > 1 + \alpha \) in the last step. This proves the statement of the theorem for \( k = 1 \), because the \( L^2 \)-norm of \( s^\beta D (\Phi \xi^* X) \) is bounded by a constant times the \( H^2, a \oplus H^1, a \)-norm of \( s^\beta D X \). The case of general \( k \) is treated with an induction in \( k \), considering in the \( k \)-th step the testfunction
\[
Y = \prod_l \Delta_{j_l}^{-\delta} \left( \zeta_2^2 \prod_{m} \Delta_{j_m}^\delta \right) X.
\]
Since there are no new ideas need in that step, and since the details closely resemble standard constructions as given e.g. in sec. 6.3 of [48], we do not elaborate on these constructions.

6.3. Construction of a gravitational perturbation with \( \xi < 0 \) in the extremal BH geometry

After these preliminaries, we turn back to the construction of the modified gravitational perturbation from step 3) in the outline section 6.1, using the general construction from the previous subsection. Let \( \delta C_\xi \) be the constraints of the perturbation (97) in the BH background. They are given by
\[
\delta C_\xi = \xi \begin{pmatrix} \delta h_{ij}(\xi) \\ \delta p^{ij}(\xi) \end{pmatrix} = \xi \begin{pmatrix} \mathcal{L} \gamma_a b(\xi) v^a v^b \\ \mathcal{L} \gamma_a b(\xi) v^{b} h^{a} \end{pmatrix}
\]
where \( \mathcal{L} \) is the linearized Einstein operator (3) for the BH background, and where \( \xi \) is the linearized constraint operator for the BH background (from now on we drop the reference to \( \xi \) in the initial data). The next lemma tells us that \( \delta C_\xi \) is small:

Lemma 6.3: We have \( |\partial_\xi^a \delta C_\xi| \leq c \sqrt{\xi} \) on \( \Sigma \), where \( c = c(n) \) and \( \text{supp } \delta C_\xi \subset A_0 \). Here, \( A_0 = A_0(\xi) \) is an ‘annular’ domain of the form
\[
A_0 = \{ x \in \Sigma \mid y_0 - \log \xi < y(x) < y_1 - \log \xi \}
\]
for some \( y_0, y_1 > 0 \) independent of \( \xi \).
Proof: This lemma relies on the following simple facts. First, by construction the perturbation $\gamma_{ab}$ is of the form eq. (59) for a suitable Hertz potential $U^{ab}$. As a consequence, the perturbation $\gamma_{ab}$ has the schematic form $\gamma_{ab} = x_l u_{ab} + y_i (a b) + z_{ab}$, where $x_l, y_i, z_{ab}$ are projected by $q_{ab}$. Furthermore, it follows from eqs. (25) and (37) that the diffeo $\phi_\varepsilon$ acts as $(T, R) \mapsto (\varepsilon T, R/\varepsilon)$ or equivalently as $(T, y) \mapsto (\varepsilon T, y - \log \varepsilon)$. Thus, it is just a shift in $y$, from which the support property is immediately obvious. Then, e.g. from the explicit expressions of the dual 1-forms of $n^a, l^a$ [cf. eq. (69)]

$$n = e^y dT + dy, \quad l = -e^y dT + dy,$$

it follows that $\phi_\varepsilon^* l = l, \phi_\varepsilon^* n = n$. It is not difficult to see from these facts that $(\delta h_{ij}, \delta p^{ij})$, defined as in eq. (97), must have coordinate expressions in $(y, x^A)$ that are of order $O(1/\sqrt{\varepsilon})$ together with all their $(y, x^A)$-derivatives.

It also follows by construction that the background $(h_{ij}, p^{ij})$ of the BH and NH-backgrounds as in eq. (41) agree up to terms of $O(e^y)$ in the coordinates $(y, x^A)$. For $y \to -\infty$. Substituting this information into the definition of the linearized constraint operator on the BH background, and using that $(\delta h_{ij}, \delta p^{ij})$ is annihilated by the linearized constraint operator on the NH background, gives the statement of the lemma.

With this in mind, we are now ready to state and prove the main two theorems of this paper concerning instability criteria of extremal BH’s in the asymptotically flat resp. asymptotically AdS case.

Theorem 2: Let $(\mathcal{M}, g)$ be an extremal MP black hole with $\Lambda = 0$. Let $\mathcal{A}$ be the associated elliptic operator (73) on the horizon cross section $\mathcal{B}$ for $a = m \cdot k$ (and notation referring to (31)).

(i) Assume that the angular velocities of the horizon are generic, in the sense that the equation

$$0 = m \cdot \Omega \equiv -\sum_{l=1}^{n} \frac{m_l a_l}{r^2 + a_l^2} \text{ for MP solutions},$$

has no solution $m \in \mathbb{Z}^n$ with $a \neq 0$. Let $\lambda$ be the smallest eigenvalue of $\mathcal{A}(a = 0)$. Then, if $\lambda < -\frac{1}{4}$, there is a perturbation that is of compact support on the Cauchy surface $\Sigma$ which cannot settle down to perturbation to another stationary black hole (or a pure gauge transformation). In other words, the black hole is linearly unstable.

(ii) If (124) has a solution $m \in \mathbb{Z}^n$ such that $a \neq 0$, then the black hole is unstable (in the same sense).

Remark: Case (i) may be called the “generic case”, because it holds for all but a measure zero set of spin-parameters. Case (ii) may be called the “resonant case”. When all the spin parameters $a_l$ are equal then it follows from the explicit form of the quantities $k^l$ in the MP solution, eq. (33) that $k < \Omega$, hence we are always in case (i). This case corresponds to the cohomogeneity-1 MP-black holes investigated in ref. [17], where the lowest eigenvalue $\lambda$ of $\mathcal{A}$ has been calculated analytically. These authors also identified the cases where $\lambda < -\frac{1}{4}$ and found agreement with the conclusions of the numerical investigations of linear perturbations in cohomogeneity-1 black holes by [19].

Proof: (i) Since $\lambda < -\frac{1}{4}$, (i) of theorem 1 applies and can construct a smooth $\varepsilon$-dependent perturbation $\gamma_{ab}(\varepsilon)$ having $\varepsilon < 0$ in the NH-geometry, as described in 1) and 2) in the outline.
subsection 6.1. Let \((\delta h_{ij}(\epsilon), \delta p^{ij}(\epsilon))\) be the initial data of this perturbation as in (97), which are compactly supported in an annular domain of the form \(A_0\) (122) in the slice \(\Sigma\) of the NH geometry (from now on we drop the reference to \(\epsilon\) in the initial data). By lemma 6.2 and the following remark, there exists a solution \(X\) to (105) with \(f := \delta C_e\) such that \((\tilde{\delta} h_{ij}, \tilde{\delta} p^{ij})\) as in (104) are \(C_0^\infty\) tensor fields on \(\Sigma\) supported in an annular domain, called \(A\), slightly larger than (122). By the estimate in lemma 6.2, the \(L^2\)-norms of \(k\)-th derivatives of \(\tilde{\delta} h_{ij} - \delta h_{ij}\) respectively of \(\tilde{\delta} p^{ij} - \delta p^{ij}\) given by those of \(\Phi^2\epsilon^+X\) are bounded by the \(H^k \oplus H^k\) Sobolev norms of \(f\), which are of order \(O(\sqrt{\epsilon})\) by the previous lemma 6.3. Thus, in this sense, our correction to the original variational ansatz given by eq. (121) are small. The background initial data \((h_{ij}, p^{ij})\) of the NH-geometry and the BH-geometry \((\tilde{h}_{ij}, \tilde{p}^{ij})\) are both given by (41) and hence differ by terms of order \(O(\epsilon^3)\) for \(y \to -\infty\), i.e. by terms of order \(O(\epsilon)\) within \(A\). Let \(\mathcal{E}\) be the canonical energy of \((\delta h_{ij}, \delta p^{ij})\) in the NH-geometry, and let \(\tilde{\mathcal{E}}\) be the canonical energy of \((\tilde{\delta} h_{ij}, \tilde{\delta} p^{ij})\) in the BH-geometry.

Using now the concrete form of \(\mathcal{E}\) resp. \(\tilde{\mathcal{E}}\) in the NH- respectively BH-geometry given by eq. (16), using that \(N, N^j\) are of order \(O(\epsilon^3)\) (hence of order \(O(\epsilon)\) within \(A\), using that the \(H^1\) norm of \(\tilde{\delta} h_{ij} - \delta h_{ij}\) and the \(L^2\)-norm of \(\tilde{\delta} p^{ij} - \delta p^{ij}\) is of order \(O(\sqrt{\epsilon})\), it follows that \(\tilde{\mathcal{E}} - \mathcal{E} = O(\epsilon^2)\). Since it is already known that \(\delta \epsilon < 0\) (independently of \(\epsilon\)), we conclude that \(\tilde{E} < 0\) for sufficiently small \(\epsilon\).

We may now appeal to the general arguments of [28]. Pick an \(\epsilon\) so that \(\tilde{E} = \tilde{E}(\Sigma) < 0\) for the compactly supported perturbation \((\delta h_{ij}, \delta p^{ij})\). Let \(\gamma_{ab}\) be the spacetime perturbation on the BH-background defined by these initial data obeying the transverse-traceless gauge condition. By eq. (124), the Hertz-potential \(U^{ab}\) as defined through (71) is Lie-derived by \(\psi = \Omega \partial/\partial \phi^I\). Therefore, the perturbation \(\gamma_{ab}\) of the NH-geometry as in (59) also is Lie-derived by \(\psi\). Then, since \(\psi\) is tangent to \(\Sigma\), also its initial data \((\delta h_{ij}, \delta p^{ij})\) are Lie-derived by \(\psi\). Furthermore, since the construction of the “corrected” initial data \((\tilde{\delta} h_{ij}, \tilde{\delta} p^{ij})\) is unique and only involves auxiliary data that are Lie-derived by \(\psi\), the corrected initial data as in (104) are also Lie-derived by \(\psi\), and whence also \(\gamma_{ab}\). Thus, the flux lemma 2.1 applies, and \(\tilde{E}(\Sigma') \leq \tilde{E}(\Sigma) < 0\) for any later slice \(\Sigma'\) (see fig. 1 with \(\Sigma_1 \to \Sigma, \Sigma_2 \to \Sigma'\). Whence, \(\tilde{E}\) cannot go to zero on an asymptotically late slice, and, therefore, as argued in [28], the corresponding perturbation cannot settle down to a perturbation towards another stationary black hole (modulo gauge).

(ii) In this case, (ii) of theorem 1 applies. The rest of the argument is as in (i).

We have a similar, but stronger, version of the theorem in the asymptotically AdS-case:

**Theorem 3:** Let \((\mathcal{M}, g)\) be an extremal MP black hole with \(\Lambda < 0\). If the black hole is rotating (i.e. \(\Omega \neq 0\)), then it is unstable in the same sense as thm. 2.

**Remarks:** 1) It is possible that the instability identified in this theorem is related to the so called “superradiant instability” which has been discussed in [51, 52, 53]. This point deserves further study. At any rate, there should be an independent proof of the superradiant instability by the canonical energy method which is not restricted to extremal black holes and does not depend on the use of NH geometries. We have learnt from S. Green that he is working on such a proof [54].

2) At present, it is not known whether extremal non-rotating (static) AdS black holes are linearly unstable. For a review of perturbations of static black holes, see [55].
**Proof:** The condition $m \cdot \Omega = 0$ is needed in the asymptotically flat case in order to get a perturbation $\gamma_{ab}$ that is Lie-derived by $\psi^a$ (cf. previous proof). This is necessary in turn to apply the flux lemma 2.1 in the asymptotically flat case, which only holds for perturbations that are Lie-derived by $\psi^a$. In the asymptotically AdS-case, the flux formula applies also for perturbations that are not Lie-derived by $\psi^a$, whence the condition $m \cdot \Omega = 0$ is unnecessary. If $\Omega \neq 0$, then there is clearly an $m$ such that $a \neq 0$. The same reasoning as in (ii) of the previous theorem then gives the result.

6.4. Construction of an electromagnetic perturbation with $\mathcal{E} < 0$ in the extremal BH geometry

The strategy of the previous sections can also be applied to electromagnetic perturbations and directly leads to the exact analogs of thms. 2 and 3 for electromagnetic fields (except possibly for the case $\lambda = -\frac{1}{2}$): Simply replace ‘gravitational perturbation’ by ‘electromagnetic perturbation’ in these statements [and the operator $\mathcal{A}$ now refers to (87)]. Because the strategy is so similar, we only outline the changes required for electromagnetic fields. As in the outline subsec. 6.1 for gravitational perturbations, there are three steps. The first step 1) is again to take an electromagnetic perturbation $A_a$ on the NH geometry having $\mathcal{E} < 0$ and compact support on $\Sigma$, as guaranteed by thm. 1, which was shown to hold in the electromagnetic case in sec. 5.2 (except possibly for the case $\lambda = -\frac{1}{2}$). Step 2) proceeds as in the gravitational case, leading to initial data $(E^i(\epsilon), A_i(\epsilon))$ that have been scaled “down the throat”. Step 3), i.e. correcting these initial data to give a perturbation satisfying the constraints in the BH spacetime, is actually easier in the electromagnetic case. Here, the constraint is simply Gauss’ law, $D_i(h^{-\frac{1}{2}}E^i) = 0$, which for abelian gauge fields as considered here does not involve $A_i$ and is linear. The ansatz for the corrected initial data on the BH spacetime is now

$$\tilde{E}^i = E^i - s^{2\alpha+2} e^{-2/s^a} h^\frac{1}{2} D^i u, \quad (125)$$

instead of (102) where we are using the same notations, and where $u$ is to be determined. Thus, letting $f = D_i(h^{-\frac{1}{2}}E^i)$ be the violation of the Gauss law constraint the original initial data $E^i = E^i(\epsilon)$, in order for $\tilde{E}^i$ to satisfy Gauss’ law, we now need to solve

$$D_i(s^{2\alpha+2} e^{-2/s^a} D^i u) = f \quad (126)$$

instead of eq. (105). The main tool is again a weighted Friedrichs-Poincaré inequality, which in this case is:

**Lemma 6.4:** (Weighted Friedrichs-Poincaré inequality) For each $\alpha > 0$, there is a constant $c = c(\alpha, A)$ such that

$$c \int_A |Du|^2 s^{2\alpha+2} e^{-2/s^a} dvol_A \geq \|u - \langle u \rangle_A\|_{H^{1,\alpha}}^2, \quad (127)$$

for any $u \in H^{1,\alpha}(A)$. Here the weighted mean value is defined as

$$\langle u \rangle_A = \int_A u s^{2\alpha+2} e^{-2/s^a} dvol_A / \int_A s^{2\alpha+2} e^{-2/s^a} dvol_A. \quad (128)$$
The proof of this lemma is analogous, but simpler than, that given for the generalized weighted Poincaré-Friedrichs inequality given in D. (Note that \( \langle u \rangle_A \) plays a similar role as \( P_A X \) in the gravitational case, because it may be viewed as the projection of \( u \) onto the ‘constant mode’.) With this inequality at hand, one again proves the existence of a suitably regular solution to (126), with bounds on the norms of \( u \) in terms of those of \( f \) of the type \( \int s^{2\beta k} |D^k (s^{\alpha+1} e^{-1/s^\alpha} u)|^2 \text{dvol}_A \leq c \| f \|_{H^{k}}^2 \) for sufficiently large \( \beta \) and all \( k \). In a similar way as in the gravitational case, it follows that \( \tilde{E}_i \) is smooth and of compact support on \( A \). The rest of the proof is also similar to the gravitational case, so we omit the details to avoid repetition.

7. Conclusions and outlook

In this paper, we have proved conjecture 1 for all known extremal stationary asymptotically flat vacuum black holes. A corresponding, but much stronger, result was also obtained for asymptotically anti-deSitter black holes. Our proof combined a number of different, and powerful, methods. Due to its conceptual nature, it should be possible to apply our strategy, in suitably modified form, to a variety of other interesting situations in which some interesting tractable limiting spacetime (analogous to the NH-geometry) can be identified. For example, it seems very likely that an analogue of conjecture 1 should also be true for near extremal black holes, i.e. one is tempted to conjecture:

**Conjecture 2:** Consider an extremal, stationary, asymptotically flat black hole spacetime, and assume that the angular velocities are generic. If the lowest eigenvalue \( \lambda \) of the operator \( \mathcal{A} (a = 0) \) (see eq. (73)) is below the critical value of \( -\frac{1}{2} \), then there exists a neighborhood of the black hole parameters near extremality for which the corresponding regular, non-extremal BHs are linearly unstable.

With regard to asymptotically \( AdS \) black holes, we conjecture similarly that all known near extremal, rotating asymptotically AdS-black holes are unstable. In order to prove these conjectures, one would show again the existence of suitable initial data in the near extremal spacetime with \( \varepsilon < 0 \). Since, by the results of the present paper, such data are already available on the extremal limit, one can modify these by exactly the same procedure as described in sec. 6.2 for black hole parameters sufficiently close to the extremal limit. The only difference would be that \( \varepsilon \) would now have the interpretation of a measure of “how close”. For example, for a given mass we could take \( \varepsilon \) simply to be the norm of the difference between the values \( a^I \) of the near extremal parameters to the extramal value \( a^I_{\text{extr}} \). We hope to come back to this issue in a future publication.

Another interesting question is whether conjecture 1 (and 2) is true also in the deSitter case. A conformal diagram for a deSitter-MP black hole is given in fig. 6. There are actually two ways in which to take the extremal limit, described in figs. 7 and 8. To have a reasonable notion of stability, one should look at the “stationary regions” in these extremal limits. It is clear that this makes sense only in the case described in the case depicted in fig. 7, wherein one would look at the region whose boundaries are the event- and cosmological horizons. Since the cosmological horizon has geometrical properties that are very similar to those of an event horizon, it seems

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20This was suggested to us by R. M. Wald.

21See thm. 2.
plausible that an analogue of the monotonicity result for $\mathcal{E}$, expressed in lemma 2.1, should also hold in the asymptotically deSitter case (the region $\mathcal{J}^+$ in the asymptotically flat case would now effectively be replaced by a portion of the cosmological horizon.). Whence, it seems likely that conjecture 1 (and 2) continue to be true also in the deSitter case.

We should finally comment on the relationship between our results and those of Aretakis\textsuperscript{22} [56], and as well as those of Dain et al. [57]. Dain et al. consider, for the extremal 4-dimensional Kerr spacetime, an “energy” for axisymmetric linear perturbations, which they show to be positive definite. We have every reason to believe that their quantity is actually identical to our canonical energy $\mathcal{E}$, when expressed in terms of the variables and gauges chosen by Dain et al. If so, this would preclude the possibility of finding compactly supported, axisymmetric initial

\textsuperscript{22} Aretakis considered a test scalar field in extreme Kerr. The generalization to linear gravitational perturbations was given in [58], and to non-linear gravitational perturbations in [59].
data having $\mathcal{E} < 0$ for extreme Kerr, which would be consistent with conjecture 1 because (in the case of extreme Kerr, the smallest eigenvalue $\lambda$ is above the effective BF bound). As argued by Dain et al., a positive definite $\mathcal{E}$ should translate into pointwise bounds on the perturbation outside the black hole, indicating that in this sense, the extreme Kerr black hole should be regarded as stable.

On the other hand, Aretakis has argued that sufficiently high transverse derivative (in fact, second derivatives) of linear perturbation with smooth initial data blows up on the horizon, so that, in this sense, an extremal Kerr black hole is in fact unstable. Aretakis’ result is not in contradiction with that of Dain et al., because their canonical energy (likely to be equal to our $\mathcal{E}$) contains first derivatives of the perturbation only, and hence should be insensitive to the phenomenon discovered by Aretakis. The kind of generic instability discovered by Aretakis has not yet been shown to exist in higher dimensions, although it very likely does. At any rate, the Aretakis-type instability should be very different in nature from the instabilities identified by conjecture 1 of this paper. Maybe the best way to see this is that the unstable modes implied by conjecture 1 have compactly supported initial data strictly outside the horizon, and that they should, as explained above in the context of conjecture 2, continue to give rise to instabilities even for near extremal black holes. The analysis of Aretakis does not apply to these.

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A. Formula for $\mathcal{E}$ in gravitational case

Here we give the formula for $\mathcal{E}$ for a perturbation $\gamma_{ab}$ of the form (59), where $U^{ab}$ is given by eq. (71), and where $Y^{AB}$ is an eigenfunction of the operator $\mathcal{A}$ (cf. eq. (73)) with eigenvalue $\lambda$. As above, we let $a = k \cdot m$. Let $f_0, f_1$ be the initial data of $\psi$ as in eq. (82). Both of them are compactly supported, complex valued, smooth functions of the variable $R \in \mathbb{R}_+$, and $\mathcal{E}$ is a quadratic functional of these under those conditions. As above, we find it convenient to express it in terms of the variable $y = \log R \in \mathbb{R}$. To write down $\mathcal{E}$, define

$$X_4 = 2e^{-y} \frac{d^2 f_1}{dy^2} + (1 - 2i) e^{-y} \frac{df_1}{dy} + (4 - 2ia - \lambda - \alpha^2) e^{-y} f_1 - \lambda f_0 + e^{-y} f_0$$

$$X_5 = -\frac{d^2 f_0}{dy^2} - \frac{df_0}{dy} + (ia - 4) e^{-y} f_1 + \lambda f_0 + e^{-y} \frac{df_1}{dy}, \quad X_6 = e^{-y} f_1$$

$$X_3 = f_0, \quad X_1 = 2 \frac{d^2 f_0}{dy^2} + (1 - 2i) \frac{df_0}{dy} + 5 e^{-y} f_1 - (\lambda + a^2) f_0 - 2 \frac{df_1}{dy} e^{-y}$$

$$X_2 = -e^{-y} f_1 - iaf_0 + \frac{df_0}{dy}$$

$$X_7 = +e^{-y} f_1 + iaf_0 + \frac{df_0}{dy}$$

Then $\mathcal{E}$ is given by $\mathcal{E} = (1/8\pi) \int_{-\infty}^{\infty} \rho(y) e^y dy$, where

$$\rho(f_0, f_1) = \left| \frac{d}{dy} X_1 \right|^2 + 5 \left| \frac{d}{dy} X_2 \right|^2 + 4 \left| \frac{d}{dy} X_3 \right|^2 + (2\lambda + 3a^2 - 4) \left| X_2 \right|^2 + (\lambda - 3) \left| X_1 \right|^2 + 4a^2 \left| X_3 \right|^2 + \left| X_4 \right|^2 + 5 \left| X_5 \right|^2 + 4 \left| X_6 \right|^2 + 2 \text{Re}\{-7ia\tilde{X}_5 X_3 + 8ia\tilde{X}_6 X_3\} + 2 \text{Re}\{8\tilde{X}_5 X_7 - 3(\lambda + a^2)\tilde{X}_5 X_3 + 4\tilde{X}_5 X_3\} + 2 \text{Re}\{3iaX_3 \frac{d}{dy} X_2 + 4iaX_3 \frac{d}{dy} X_3\}.$$  \hspace{1cm} (129)

B. Formula for $\mathcal{E}$ in electromagnetic case

Here we give the formula for $\mathcal{E}$ for a perturbation $A_a$ of the form (68), where $U^a$ is given by eq. (86), and where $Y^A$ is an eigenfunction of the operator $\mathcal{A}$ (cf. eq. (87)) with eigenvalue $\lambda$. As above, we let $a = k \cdot m$. Let $f_0, f_1$ be the initial data of $\psi$ as in eq. (82). Both of them are compactly supported, complex valued, smooth functions of the variable $R \in \mathbb{R}_+$, and $\mathcal{E}$ is a quadratic functional of these under those conditions. To write down $\mathcal{E}$, define

$$X_1 = -\frac{d^2 f_0}{dy^2} - \frac{df_0}{dy} + \lambda f_0 - e^{-y} (2 - ia) f_1 + e^{-y} \frac{df_1}{dy}, \quad X_2 = e^{-y} f_1$$

$$X_3 = -e^{-y} f_1 - iaf_0 + \frac{df_0}{dy}, \quad X_4 = f_0$$

$$X_5 = +e^{-y} f_1 + iaf_0 + \frac{df_0}{dy}$$

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Then $B$ is given by $B = (1/4\pi) \int_{-\infty}^{\infty} \rho(y)e^y dy$, where

$$
\rho(f_0, f_1) = |X_1|^2 + |X_2|^2 + \left| \frac{d}{dy}X_3 \right|^2 + \left( \frac{d}{dy}X_4 \right)^2 + (\lambda - 1)|X_5|^2 + a^2|X_6|^2 - 2 \text{Re} \left\{ 2iaX_4X_2 - ia\tilde{X}_4 \frac{d}{dy}X_4 + \tilde{X}_5X_2 \right\} .
$$

(130)

C. Mode-type solutions to eq. (77)

Here we study the solutions of eq. (77) of the form $\psi(T, R) = f(R)e^{i\omega T}$ with $\omega \in \mathbb{C}$. The resulting equation for $f(R)$ is

$$
0 = \frac{d}{dR} \left( R^2 \frac{df}{dR} \right) - (\lambda + q^2)f + \left( q + \frac{\omega}{R} \right)^2 f .
$$

(131)

Solutions of the equation can be given in terms of the hypergeometric function ${}_1F_1(a, b; z)$ or Whittaker functions [60, 31, 32]. Except for degenerate cases which are not relevant for this paper, the general solution is a linear combination of

$$
f_{\pm}(R) = e^{i\omega/R} \left( \frac{-i\omega}{R} \right)^{1/2 \pm i\delta} {}_1F_1 \left( \pm \delta + \frac{1}{2} - iq, 1 \pm 2\delta; \frac{-2i\omega}{R} \right) ,
$$

(132)

where

$$
\delta = \sqrt{\frac{1}{4} + \lambda} .
$$

(133)

In the case of most interest for this paper, $\lambda < -\frac{1}{4}$, so $\delta$ is imaginary. The solutions to (131) behave generically as linear combinations of $e^{\pm i\omega/R}$ for $R \to 0$ and as $R^{\frac{1}{2} \pm \delta}$ for $R \to \infty$. In order to get a solution whose derivatives vanish at $R = 0$ (i.e. the horizon), we need to take a particular linear combination of $f_{\pm}$, and we need to take $\omega$ to have a non-vanishing imaginary part. If, for example, we let $\text{Im}(\omega) > 0$, then the desired linear combination having vanishing derivatives at $R = 0$ to all orders is

$$
f(R) = A_+ f_+(R) + A_- f_-(R) ,
$$

(134)

where $A_{\pm} = 2^{\delta}\Gamma(-2\delta)/\Gamma(\frac{1}{2} \pm \delta - iq)$. With this choice, $f(R)$ behaves as $\sim e^{i\omega/R}$ near $R = 0$, and as $\sim A_+ R^{-\frac{1}{2} + \delta} + A_- R^{-\frac{1}{2} - \delta}$ as $R \to \infty$. For example, take $\omega = i$. The corresponding solution $\psi(T, R)$ is

$$
\psi(T, R) = e^{T \frac{-1}{R}} \left[ A_+ R^{-1/2 - i\delta} {}_1F_1 \left( +\delta + \frac{1}{2} - iq, 1 + 2\delta; \frac{2}{R} \right)
+ A_- R^{-1/2 + i\delta} {}_1F_1 \left( -\delta + \frac{1}{2} - iq, 1 - 2\delta; \frac{2}{R} \right) \right] .
$$

(135)

This solution is exponentially growing in $T$, and regular on the future horizon (as is seen e.g. by the fact that $R, u = T - 1/R$ provide regular coordinates on the future horizon), but not $L^2$-normalizable near infinity.
D. Proof of generalized Friedrichs-Poincaré inequality, lemma 6.4

We repeat the statement of lemma 6.4:

**Lemma D.1:** *(Generalized Friedrichs-Poincaré inequality)* For sufficiently large $\alpha$, there is a constant $c = c(\alpha, A)$ such that

$$ c \| \Phi^\alpha X \|_{L^2} \geq \| X - P_A X \|_{H^2(\alpha) \oplus H^1(\alpha)} , \quad (136) $$

for any tensor field $X \in H^2_0(A) \oplus H^1_0(A)$. Here $P_A$ is the orthogonal projector (in $L^2(\alpha)$) onto the subspace $\mathfrak{k}$ spanned by the KVF’s, i.e. if $Y_i$ is a basis of Killing vector fields on $\mathcal{M}$ that has been orthonormalized with the Gram-Schmidt process, we have

$$ P_A X = \sum_i Y_i(\dot{Y}_i, X)_{L^2(\alpha)} . \quad (137) $$

The proof is divided into several steps.

**Step 1)** We recall the 1-dimensional weighted Hardy inequality by Kufner, see sec. 5 of [61]:

Let $\sigma$ be a non-negative, smooth function on the interval $[0, 1]$ such that $\sigma(0) = 0$ whereas $\sigma(s) > 0$ for $s > 0$. For $1 < p < \infty$, define

$$ \phi(s) = (p - 1) \frac{\int_0^s \sigma(t) dt}{\sigma(s)} . \quad (138) $$

Then there holds the inequality

$$ \int_0^1 |u(s)|^p \sigma(s) ds \leq \left( \frac{p}{p-1} \right)^p \int_0^1 |\phi u'(s)|^p \sigma(s) ds , \quad (139) $$

for all $u \in C^1_0(0,1)$ i.e. with vanishing boundary values at the ends of the interval. We apply this inequality to the case when $\sigma(s) = e^{-2/s^\alpha}$ for some $\alpha > 0$. One finds that $|\phi(s)| \leq (p - 1)s^{(\alpha+1)/\alpha}$. This gives rise to the inequality

$$ \int_0^1 |u(s)|^p e^{-2/s^\alpha} ds \leq \left( \frac{p}{\alpha} \right)^p \int_0^1 |u'(s)|^p s^{p(\alpha+1)} e^{-2/s^\alpha} ds , \quad (140) $$

for the same class of functions $u$.

**Step 2)** We now generalize this inequality to tensors $u \in C^1_0(A)$ on some open, domain $A$ with smooth boundary in a Riemannian manifold $(\Sigma, h)$. We let $0 \leq s \leq 1$ be a function which is equal to $s(x) = \text{dist}_h(x, \partial A)$ in a neighborhood of the boundary, and which is positive and smooth in the interior. We denote $A_\varepsilon = \{ x \in A \mid s(x) < \varepsilon \}$ and first consider only tensors $u$ compactly supported in $A_\varepsilon$. Using parallell transport, we may identify such a tensor field with an $s$-dependent tensor field that is defined on $T_0AA$. Then for each fixed $y \in \partial A$, we can apply
the 1-dimensional Hardy inequality from step 1) to the function \( u(y, s) \) of \( s \), and afterwards integrate with respect to \( y \) using the integration element on \( \partial A \). This gives

\[
\int_A |u(x)|^p e^{-2/s^\alpha} \, dvol_A = \int_{\partial A} \left( \int_0^\varepsilon |u(s, y)|^p J(s) e^{-2/s^\alpha} \, ds \right) \, dvol_{\partial A}
\]

\[
\leq \left( \frac{p}{\alpha} \right)^p \int_{\partial A} \left( \int_0^\varepsilon \left| \frac{\partial}{\partial s} [u^{1/p}(s, y)] \right|^p \right) \, dvol_{\partial A}
\]

where

\[
J(s, y) = \frac{dvol_{\partial A(s)}}{dvol_{\partial A}}, \quad (141)
\]

and with \( \partial A(s) = \{ x \in A \mid \text{dist}(x, \partial A) = s \} \). We distribute the \( s \)-derivatives using the Leibniz rule, we use \( \partial \log J/\partial s = \theta \) (i.e. equal to the expansion of the generators of geodesics orthogonal to \( \partial A \)), and we use the Minkowski inequality for \( L^p \)-norms. We find:

\[
\int_A |u|^p e^{-2/s^\alpha} \, dvol_A \leq \frac{1}{\alpha^p} \left\{ e^{p(1 + \varepsilon)} (\sup_{s \leq \varepsilon} \theta \|\|)^p \int_A |u|^p e^{-2/s^\alpha} \, dvol_A + \int_A |D_{\partial/\partial s} u|^{p\varepsilon} e^{-2/s^\alpha} \, dvol_A \right\}.
\]

(142)

We clearly have \( |D_{\partial/\partial s} u| \leq |Du| \), and if we furthermore choose \( \varepsilon \) so small that

\[
(e^{\alpha + 1} \sup \theta / \alpha)^p < 1/2
\]

(143)

then we get

\[
\int_A |u|^p e^{-2/s^\alpha} \, dvol_A \leq \frac{2}{\alpha^p} \int_A |Du|^{p\varepsilon} e^{-2/s^\alpha} \, dvol_A,
\]

(144)

holding for all \( u \in C^\infty_0(A) \) whose support is contained in \( A_\varepsilon \). One can apply the same kind of estimate again to the right side (noting that it holds for tensors). For example, for \( p = 2 \), we get in this way

\[
\int_A |Du|^2 s^{2(\alpha + 1) e^{-2/s^\alpha}} \, dvol_A
\]

\[
\leq \frac{2}{\alpha^2} \int_A |D(s^{\alpha} Du)|^2 s^{2(\alpha + 1) e^{-2/s^\alpha}} \, dvol_A
\]

\[
\leq \frac{2}{\alpha^2} \int_A |D^2 u|^2 s^{4(\alpha + 1) e^{-2/s^\alpha}} \, dvol_A + \frac{2(\alpha + 1)^2 \varepsilon^{2\alpha}}{\alpha^2} \int_A |Du|^2 s^{2(\alpha + 1) e^{-2/s^\alpha}} \, dvol_A.
\]

(145)

We now let \( \varepsilon \) be so small that \( \frac{2(\alpha + 1)^2 \varepsilon^{2\alpha}}{\alpha^2} < 1/2 \). Then the second term on the r.h.s. can be absorbed by the left hand side, resulting in

\[
\int_A |Du|^2 s^{2(\alpha + 1) e^{-2/s^\alpha}} \, dvol_A \leq \frac{4}{\alpha^2} \int_A |D^2 u|^2 s^{4(\alpha + 1) e^{-2/s^\alpha}} \, dvol_A,
\]

(146)

for smooth tensor fields \( u \) supported in \( A_\varepsilon \). Combining this with eq. (144), we may write, for some \( c_1 \).

\[
\|u\|_{H^{2, \alpha}}^2 \leq c_1 \int_A |D^2 u|^2 s^{4(\alpha + 1) e^{-2/s^\alpha}} \, dvol_A
\]

(147)
for smooth $u$ compactly supported in $A_\varepsilon$. (Here we recall the notations $H^{k,\alpha} = W^{2,k,\alpha}$ and $L^{2,\alpha} = H^{0,\alpha}$.)

Step 3) We now wish to obtain, in the case $p = 2$, an inequality similar to step 2) for vector fields $u = X$ but with $DX$ replaced by $\ell_X h$. Note that such an inequality does not follow directly from (144), because $(\ell_X h)_{ij} = D_i X_j + D_j X_i$, whereas $Du = DX$ corresponds to $D_i X_j$ with tensor indices written out, i.e. we have an additional symmetrization of the tensor indices. In fact, we have instead

$$|DX|^2 = \frac{1}{2} |\ell_X h|^2 - (\text{div}_h X)^2 + \text{Ric}_h(X, X) - \text{div}_h(D_X X - X\text{div}_h X),$$  

(148)

by an elementary computation. Using this identity on the r.h.s. of eq. (144) gives us

$$\int_A |X|^2 e^{-2/s^\alpha}dvol_A \leq \frac{2}{\alpha^2} \int_A |DX|^2 s^{2(\alpha+1)} e^{-2/s^\alpha} dvol_A$$  

(149)

The total divergence terms are treated with a partial integration, which yields

$$- \int_A \text{div}_h(D_X X - X\text{div}_h X)s^{2(\alpha+1)} e^{-2/s^\alpha} dvol_A$$

$$= \int_A \langle ds, D_X X - X\text{div}_h X \rangle \frac{d}{ds} \{ s^{2(\alpha+1)} e^{-2/s^\alpha} \} dvol_A$$  

(150)

$$= \int_A \left\{ \langle ds, D_X X \rangle - \langle ds, X \rangle \text{div}_h X \right\} (2\alpha + (2\alpha + 2)s^\alpha)s^{\alpha+1} e^{-2/s^\alpha} dvol_A$$

To estimate the second term under the integral, we now use the Cauchy-Schwarz-inequality, together with the elementary inequality $(\text{div}_h X)^2 \leq \frac{d-1}{4} |\ell_X h|^2$. To estimate the first term, we use that

$$\langle ds, D_X X \rangle = \langle ds, X \rangle \{ (\ell_X h)(ds, ds) - \text{div}_h X \} - \text{H}_{\partial A}(X, X) - \partial \langle ds, X \rangle^2 + \text{div}_{\partial A}(X_{\partial A} \langle ds, X \rangle)$$  

(151)

where in the last term, $X_{\partial A}$ denotes the projection of $X$ along the surfaces of constant $s$, and $\text{div}_{\partial A}$ is the intrinsic divergence on these surfaces (so that this term does not contribute under an integral). $H_{\partial A}$ is the extrinsic curvature of these surfaces, and $\partial$ half its trace (expansion). Using also the Cauchy-Schwarz inequality gives the bound

$$- \int_A \text{div}_h(D_X X - X\text{div}_h X)s^{2(\alpha+1)} e^{-2/s^\alpha} dvol_A$$

$$\leq \frac{(4\alpha + 2)(d + 2)}{4} \int_A |\ell_X h|^2 s^{2(\alpha+1)} e^{-2/s^\alpha} dvol_A +$$

$$+ (4\alpha + 2) \left\{ 2 + e^{1+\alpha} \sup_{s < \varepsilon} (|H| + |\partial|) \right\} \int_A |X|^2 e^{-2/s^\alpha} dvol_A. $$  

(152)
We choose $\epsilon$ so small that $\epsilon^{1+\alpha} \text{sup}(|H| + |\vartheta|) < 1$. We also have

$$\int_A \text{Ric}(X,X)s^{2(\alpha+1)}e^{-2/s^a} \text{dvol}_A \leq \epsilon^{2\alpha+2} \sup_{s < \epsilon} |\text{Ric}_h| \int_A |X|^2s^{2(\alpha+1)}e^{-2/s^a} \text{dvol}_A,$$

and we additionally choose $\epsilon$ so small that $\epsilon^{2+2\alpha} \text{sup} |\text{Ric}_h| < 1$. Combining these inequalities with eq. (149), we get

$$\int_A |X|^2s^{2(\alpha+1)}e^{-2/s^a} \text{dvol}_A \leq \frac{1 + (2\alpha + 1)(d + 2)}{\alpha^2} \int_A |\mathcal{E}_Xh|^2s^{2(\alpha+1)}e^{-2/s^a} \text{dvol}_A + \frac{24\alpha + 14}{\alpha^2} \int_A |X|^2s^{2(\alpha+1)}e^{-2/s^a} \text{dvol}_A.$$

We now choose $\alpha$ so large that $(24\alpha + 14)/\alpha^2 < 1/2$. In that case, we get from (154) the relation

$$\int_A |X|^2s^{2(\alpha+1)}e^{-2/s^a} \text{dvol}_A \leq c_0 \int_A |\mathcal{E}_Xh|^2s^{2(\alpha+1)}e^{-2/s^a} \text{dvol}_A.$$

In fact, because we have really estimated $\int_A |\mathcal{D}X|^2s^{2(\alpha+1)}e^{-2/s^a} \text{dvol}_A$ by the above argument as well, we get the same type of upper bound for that quantity, too. Thus, we can write

$$\|X\|^2_{H^1,a} \leq c_2 \int_A |\mathcal{E}_Xh|^2s^{2(\alpha+1)}e^{-2/s^a} \text{dvol}_A,$$

which holds for all $X \in C_0^\infty(A)$, and a sufficiently large $\alpha > 0$.

Step 4) We shall now combine eqs. (147) and (156) to obtain, for $X = (u,X)$ supported in $A_\epsilon$, the inequality

$$\|u\|^2_{H^2,a} + \|X\|^2_{H^1,a} = \|X\|^2_{H^2,a \oplus H^1,a} \leq c_3 \|\Phi \mathcal{E}^*X\|^2_{L^2 \oplus L^2},$$

where $\mathcal{E}^*$ is the adjoint of the constraint operator defined above in eq. (100), and where we recall that $\Phi$ is the matrix multiplication operator (103). Since the highest derivative parts of $\mathcal{E}^*$ on $u$ resp. $X$ are $D^2u$ resp. $\mathcal{E}_Xh$ it follows immediately from the definition of $\mathcal{E}^*$ and the definition of the norms $\| \cdot \|_{H^k,a}$ that inequality (157) holds true in the special cases when $X$ is either $(u,0)$ or $(0,X)$. To deal with the general case, one only has to estimate the ‘cross terms’ between $X$ and $u$ in a fairly straightforward way. One can write, using the Cauchy-Schwarz inequality and that $X$ has its support for $s < \epsilon$:

$$\|\Phi \mathcal{E}^*X\|^2_{L^2 \oplus L^2} \geq \int_A |\mathcal{E}_Xh|^2s^{2(\alpha+1)}e^{-2/s^a} \text{dvol}_A + \int_A |D^2u|^2s^{4(\alpha+1)}e^{-2/s^a} \text{dvol}_A - c_4\epsilon^{1+\alpha} \left\{ \begin{array}{l}
\sup_{s < \epsilon} (|\text{Ric}_h| + |p|^2) \cdot \|u\|_{H^2,a} \|u\|_{L^2,a} + \\
\sup_{s < \epsilon} (|p| \cdot \|u\|_{H^2,a} \|X\|_{H^1,a} + \\
\sup_{s < \epsilon} (|\text{Ric}_h|^2 + |p|^2 + |\text{Ric}_h||p|^2) \cdot \|u\|_{L^2,a}^2 + \\
\sup_{s < \epsilon} (|\text{Ric}_h| \cdot \|X\|_{H^1,a} \|u\|_{L^2,a} + \\
\sup_{s < \epsilon} (|p|^2 \cdot \|X\|_{H^1,a}^2) \right\}.$$
The first two terms on the r.h.s. represent the ‘diagonal terms’ and are bounded from below respectively by (147) and (156). The terms in curly brackets represent the ‘cross terms’ and are bounded from below by the trivial inequality $-2ab \geq -a^2 - b^2$. Then the desired in equality (157) immediately follows for sufficiently small $\varepsilon > 0$.

Step 5) The next step is to establish an inequality of the form (157) on the ‘complement’ of the set $A_\varepsilon$ (inside $A$). More precisely, let $O_\varepsilon = A \setminus A_{\varepsilon/2}$, so that $A = A_\varepsilon \cup O_\varepsilon$, and so that $O_\varepsilon$ and $A_\varepsilon$ have an open overlap. On $O_\varepsilon$, we clearly have $s > \varepsilon/2 > 0$, so the weight functions involving $s$ appearing in the various integrals are bounded away from zero and basically have no influence. Alternatively speaking, for tensors supported in $O_\varepsilon$, the $H^{k,\alpha}$ norms are all equivalent to the norms with $\alpha = 0$, i.e. ordinary Sobolev space $H^k$-norms with no weight factors. In this setting, inequalities (147) and (156) are standard consequences of the ellipticity of the operators $D^i D^j D_i u$ and $D^i D_j X_i$ together with the fact that none of these operators has a kernel in $H^2_0(O_\varepsilon)$ respectively $H^1_0(O_\varepsilon)$, as such objects would correspond to KVFs $D^i u$ resp. $X^i$ with vanishing boundary values on $\partial O_\varepsilon$, which do not exist. To derive from this a result of the type (157) for $X$ supported in $O_\varepsilon$, one can proceed in a similar way as in step 4) and estimate ‘diagonal terms’ and ‘cross terms’, the details of which are given in lemma 2.8

One obtains:

$$\|X\|_{H^2, a \otimes H^{1, a}} \leq c_3 \left\{ \|\Phi \mathcal{E}^a X\|_{L^2, a \otimes L^2} + \|X\|_{H^1, a \otimes H^{0, a}} \right\}$$

holding for $X \in C^\infty(A)$ supported in $O_\varepsilon$.

Step 6) One next combines (157) (for $X$ supported in $A_\varepsilon$) and (159) (for $X$ supported in $O_\varepsilon$). Suppose that the Poincaré-Friedrichs inequality does not hold. Then there is a sequence $X_n \in H^2_0, a(A) \oplus H^1_0, a(A)$, such that $\|X_n - P_A X_n\|_{H^2, a \otimes H^{1, a}} = 1$, but $\|\Phi \mathcal{E}^a X_n\|_{L^2, a \otimes L^2} \to 0$. Let $\chi_1 + \chi_2 = 1$ be a partition of unity such that $\text{supp}\chi_1 \subset O_\varepsilon, \text{supp}\chi_2 \subset A_\varepsilon$. Then we estimate

$$\|X_n\|_{H^2, a \otimes H^{1, a}} \leq \|\chi_1 X_n\|_{H^2, a \otimes H^{1, a}} + \|\chi_2 X_n\|_{H^2, a \otimes H^{1, a}}$$

with possibly new constants in each line. In the last step we used that the commutator $[\mathcal{E}^a, \psi]$ with a smooth compactly supported function $\psi$ decreases the order of each entry of the matrix operator $\mathcal{E}^a$ by one unit (unless the order of the entry is already $0$), so that $[\mathcal{E}^a, \chi] : H^2 \oplus H^1 \to H^1 \oplus H^0$ is bounded. Now, by assumption, $\|\Phi \mathcal{E}^a X_n\|_{L^2 \otimes L^2} \to 0$ for the first term on the right side. On the other hand, since $\{X_n\}$ is by assumption bounded in $H^2(O_\varepsilon) \oplus H^1(O_\varepsilon)$, it follows from the Rellich-Kondrachov compactness theorem (see e.g. [48]) that $\{X_n\}$ (or a subsequence thereof) is Cauchy in $H^1(O_\varepsilon) \oplus H^0(O_\varepsilon)$. Hence, the above inequality shows that a subsequence of $\{X_n\}$ is Cauchy in $H^2, a(A) \oplus H^1, a(A)$, hence convergent with limit $X$ in this space. By the continuity of $\Phi \mathcal{E}^a : H^2, a(A) \oplus H^1, a(A) \to L^2(A) \oplus L^2(A)$, we learn that $\Phi \mathcal{E}^a X = 0$. So $X$ must be equal, almost everywhere, to a non-trivial Killing vector field. But then, clearly $\|X - P_A X\|_{H^2, a \otimes H^{1, a}} = 0$ which is in contradiction with our assumption $\|X_n - P_A X_n\|_{H^2, a \otimes H^{1, a}} = 1$ and the convergence of the sequence.

\(^{23}\)Our situation corresponds to $\psi = e^{-1/s^a}, \phi = s^{1+a}$ in the notation of that lemma, which satisfies assumption A.2 of [38].
E. Behavior of electromagnetic perturbations near $\mathcal{I}$ in higher dimensions

Here we study the behavior of electromagnetic perturbations near null infinity in asymptotically flat backgrounds solving the vacuum Einstein equations following the method described in [41] for gravitational perturbations. We impose the Lorentz gauge $\nabla^a A_a = 0$, and define the unphysical metric as $\bar{g}_{ab} = \Omega^2 g_{ab}$. We also set

$$\bar{A}_a = \Omega^{-(d-4)/2} A_a, \quad \bar{\phi} = \Omega^{-1} \bar{n}^a \bar{A}_a, \quad \bar{n}^a = \bar{g}^{ab} \bar{\nabla}_b \Omega,$$  \hfill (161)

and we adopt the convention that indices on tensors with a tilde are raised and lowered using $\bar{g}^{ab}$. The smoothness of $\bar{g}_{ab}$ at $\mathcal{I}$ evidently implies the smoothness of the corresponding Ricci tensor, $\bar{R}_{ab}$, at $\mathcal{I}$. It follows from the background Einstein equations that $\bar{\phi} = \Omega^{-1} \bar{n}^a \bar{A}_a$ is smooth at $\mathcal{I}$, and whence that $\bar{n}^a$ is null there. After a lengthy calculation using the background Einstein equations, one finds that the Maxwell equation $\bar{\nabla}^a \bar{\nabla}_{[a} A_{b]} = 0$ in Lorentz gauge is equivalent to the coupled system of equations

$$\bar{\nabla}^b \bar{\nabla}_b \bar{A}_a = 2 \bar{\nabla}_a \bar{\phi} + \frac{2}{d-2} \bar{R}_a \bar{A}_b + \frac{d-4}{2(d-1)(d-2)} \bar{R} \bar{A}_a$$  \hfill (162)

$$\bar{\nabla}^b \bar{\nabla}_b \bar{\phi} = -\frac{2}{d-2} \bar{R}^{ab} \bar{\nabla}_a \bar{\phi} + \frac{1}{2(d-1)} \bar{A}^a \bar{\nabla}_a \bar{R} + \frac{d^2}{2(d-1)(d-2)} \bar{R} \bar{\phi}$$  \hfill (163)

for $(\bar{A}_a, \bar{\phi})$ in the unphysical spacetime metric $\bar{g}_{ab}$. The first key point is that these equations have the character of wave equations, i.e. the highest derivative part is $\bar{\nabla}^b \bar{\nabla}_b$. Therefore, the initial value problem is well posed in the unphysical spacetime $\mathcal{M}$. The second key point is that all inverse powers of $\Omega$ have cancelled out (!), meaning that, on the right side, all coefficient tensors are manifestly smooth at $\mathcal{I}$. Whence, if $A_a$, and hence $(\bar{A}_a, \bar{\phi})$, have initial data of compact support on some Cauchy surface as drawn in fig. 1, then the solution $(\bar{A}_a, \bar{\phi})$ to the above system of equations will be smooth at $\mathcal{I}$. In particular, it follows that $A_a$ decays as $\Omega^{(d-4)/2}$ near $\mathcal{I}$, and it follows from the definition of $\bar{\phi}$ that $\bar{n}^a \bar{A}_a$ has to vanish on $\mathcal{I}$.

Consider now a quadrangle shaped domain as in fig. 1. The fall-off behavior at $\mathcal{I}$ of $A_a$ implied by the smoothness of $\bar{A}_a$ and the vanishing of $\bar{n}^a \bar{A}_a$ on $\mathcal{I}$ allow one to write

$$\int_{\mathcal{I}_1} \star w = \frac{1}{2\pi} \int_{\mathcal{I}_1} (\mathcal{E}_n A^a) \mathcal{E}_n A_a - C(\mathcal{E}_1, A) + C(\mathcal{E}_2, A),$$  \hfill (164)

where the boundary terms are as in eq. (20), and where the symplectic current $w_a$ is as in eq. (17). Similarly, using the gauge condition $A_a K^a |_{\mathcal{I}} = 0$, one can write, with $\bar{n}^a = K^a$,

$$\int_{\mathcal{I}_2} \star w = \frac{1}{2\pi} \int_{\mathcal{I}_2} (\mathcal{E}_n A^a) \mathcal{E}_n A_a - B(\mathcal{B}_1, A) + B(\mathcal{B}_2, A),$$  \hfill (165)

where the boundary terms are as in (19). Now integrate $d \star w = 0$ over the quadrangle, and use Gauss’ theorem to write the integral as a sum of boundary integrals over $\Sigma_1, \Sigma_2, \mathcal{I}_1, \mathcal{I}_2$. The last two integrals were just evaluated, whereas the first two give the canonical energy associated with $\Sigma_1, \Sigma_2$, respectively. Equation (22) follows.
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