EXCHANGE LEAVITT PATH ALGEBRAS AND STABLE RANK

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Abstract. We characterize Leavitt path algebras which are exchange rings in terms of intrinsic properties of the graph and show that the values of the stable rank for these algebras are 1, 2 or $\infty$. Concrete criteria in terms of properties of the underlying graph are given for each case.

Introduction

For a row-finite graph $E$, the Leavitt path algebra $L(E)$ is the algebraic analogue of the Cuntz-Krieger algebra $C^*(E)$ described in [16]. The pioneer papers where $L(E)$ is introduced and studied are [1, 2]. In [3], Ara, Moreno and Pardo carry out a study of the monoid $V(L(E))$. Concretely they show that there is a natural isomorphism between the lattices of graded ideals of $L(E)$ and that of order ideals of $V(L(E))$. In [1] and [2] Abrams and Aranda Pino provide characterizations of the simplicity and purely infinite simplicity, respectively, of the Leavitt path algebra $L(E)$ in terms of properties involving the graph $E$ only.

An associative unital ring $R$ is said to be an exchange ring if $R_R$ has the exchange property introduced by Crawley and Jónsson. The structure of exchange rings has been intensively investigated by several authors; in the non necessarily unital case, their study was initiated by Ara in [3]. On the other hand, the concept of stable rank, introduced by Bass for unital rings (see e.g. [8]), is very useful in treating the stabilization problem in K-theory. In [19], Vasershtein opens with the definition of stable rank for a non necessarily unital ring. For the more specific case of $C^*$-algebras, the exchange property is closely related with the real rank: The $C^*$-algebras having real rank zero are precisely those which are exchange rings.

Following the philosophy of [1, 2], the aim of this paper is to study the exchange property for Leavitt path algebras and, focusing on this type of algebras, their stable rank.

Some of our sources of inspiration for the characterization of the exchange property are the works of Jeong and Park [13] and Bates, Hong, Raeburn and Szymański [9], while for the stable rank it is the paper by Deicke, Hong and Szymański [10]. The proofs presented here significantly differ to those of the analytic setting of $C^*$-algebras and the arguments are necessarily different in the purely algebraic context since many of the tools used there are not available in our case.

Date: February 13, 2022.

2000 Mathematics Subject Classification. Primary 16D70.

Key words and phrases. graph, Leavitt path algebra, exchange ring, stable rank.

The first author was partially supported by a FPU grant AP2001-1368 by the MEC. The first and third authors were partially supported by the MCYT and Fondos FEDER, BFM2001-1938-C02-01, MTM2004-06580-C02-02 and the “Plan Andaluz de Investigación y Desarrollo Tecnológico”, FQM 336. The second author was partially supported by the DGI and European Regional Development Fund, jointly, through Project MTM2004-00149, by PAI III grant FQM-298 of the Junta de Andalucía, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.
The paper is divided into seven sections. After some preliminaries, we begin by stating basic properties concerning special subsets of graphs. In particular, we study the ideals generated by hereditary and saturated subsets of vertices and cofinality of the graph.

Condition (K), studied in the third section, plays a central role in the paper. On the one hand, it is precisely the condition we need to impose on \( E \) so that \( L(E) \) is exchange; on the other hand, the development of results concerning stable rank occur under this hypothesis.

The main result characterizing exchange Leavitt path algebras appears in Section 4:

**Theorem 4.5.** For a graph \( E \), the following conditions are equivalent:

1. \( L(E) \) is an exchange ring.
2. \( E/H \) satisfies condition (L) for every hereditary saturated subset \( H \) of \( E^0 \).
3. \( E \) satisfies condition (K).
4. \( \mathcal{L}_N(L(E)) = \mathcal{L}(L(E)) \).
5. \( E_H \) and \( E/H \) satisfy condition (K) for every hereditary saturated subset \( H \) of \( E^0 \).
6. \( E_H \) and \( E/H \) satisfy condition (K) for some hereditary saturated subset \( H \) of \( E^0 \).

The rest of the sections are devoted to compute the stable rank in Leavitt path algebras satisfying condition (K). The first step towards this aim is done in Section 5: First, by investigating the absence of unital purely infinite simple quotients of \( L(E) \) (Proposition 5.4). Secondly, by relating prime graded ideals with maximal tails (Proposition 5.6). Then, in Section 6, we calculate the stable rank for Leavitt path algebras which do not have nonzero bounded graph traces and for which every vertex lying on a closed simple path is left infinite (Corollary 6.8). The paper finishes in Section 7 with a criterion to compute the stable rank for exchange Leavitt path algebras:

**Theorem 7.6.** Let \( E \) be a graph satisfying condition (K). Then, the values of the stable rank of \( L(E) \) are:

1. \( \text{sr}(L(E)) = 1 \) if \( E \) is acyclic.
2. \( \text{sr}(L(E)) = \infty \) if there exists \( H \in \mathcal{H}_E \) such that the quotient graph \( E/H \) is nonempty, finite, cofinal and contains no sinks.
3. \( \text{sr}(L(E)) = 2 \) otherwise.

1. Preliminaries

Along this paper, we describe Leavitt path algebras following the presentation of [6, Sections 2 and 4] but using the notation of [1] for the elements.

A (directed) graph \( E = (E^0, E^1, r, s) \) consists of two countable sets \( E^0, E^1 \) and maps \( r, s : E^1 \to E^0 \). The elements of \( E^0 \) are called vertices and the elements of \( E^1 \) edges.

A vertex which emits no edges is called a sink. A graph \( E \) is finite if \( E^0 \) is a finite set. If \( s^{-1}(v) \) is a finite set for every \( v \in E^0 \), then the graph is called row-finite. A path \( \mu \) in a graph \( E \) is a sequence of edges \( \mu = (\mu_1, \ldots, \mu_n) \) such that \( r(\mu_i) = s(\mu_{i+1}) \) for \( i = 1, \ldots, n - 1 \). In such a case, \( s(\mu) := s(\mu_1) \) is the source of \( \mu \) and \( r(\mu) := r(\mu_n) \) is the range of \( \mu \). An edge \( e \) is an exit for a path \( \mu \) if there exists \( i \) such that \( s(e) = s(\mu_i) \) and \( e \neq \mu_i \). If \( s(\mu) = r(\mu) \) and \( s(\mu_i) \neq s(\mu_j) \) for every \( i \neq j \), then \( \mu \) is called a cycle. If \( v = s(\mu) = r(\mu) \) and \( s(\mu_i) \neq v \) for every \( i > 1 \), then \( \mu \) is called a closed simple path based at \( v \). We denote by \( \text{CSP}_E(v) \) the set of closed simple paths in \( E \) based at \( v \). For a path \( \mu \) we denote by \( \mu^0 \) the set of its vertices, i.e., \( \{ s(\mu_i), r(\mu_i) \mid i = 1, \ldots, n \} \). For \( n \geq 2 \) we define \( E^n \) to be the set of paths of length \( n \),
and $E^* = \bigcup_{n \geq 0} E^n$ the set of all paths. We define a relation $\geq$ on $E^0$ by setting $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. A subset $H$ of $E^0$ is called hereditary if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set is saturated if every vertex which feeds into $H$ and only into $H$ is again in $H$, that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. The set $T(v) = \{ w \in E^0 \mid v \geq w \}$ is the tree of $v$, and it is the smallest hereditary subset of $E^0$ containing $v$. We extend this definition for an arbitrary set $X \subseteq E^0$ by $T(X) = \bigcup_{x \in X} T(x)$. Denote by $\mathcal{H}$ (or by $\mathcal{H}_E$ when it is necessary to emphasize the dependence on $E$) the set of hereditary saturated subsets of $E^0$. The hereditary saturated closure of a set $X$ is defined as the smallest hereditary and saturated subset of $E^0$ containing $X$. It is shown in [3] that the hereditary saturated closure of a set $X$ is $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

1. $\Lambda_0(X) = T(X)$,
2. $\Lambda_n(X) = \{ y \in E^0 \mid s^{-1}(y) \neq \emptyset \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X) \} \cup \Lambda_{n-1}(X)$, for $n \geq 1$.

Let $E = (E^0, E^1, r, s)$ be a graph, and let $K$ be a field. We define the Leavitt path $K$-algebra $L_K(E)$ associated with $E$ ($L(E)$ when the based field is understood) as the $K$-algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

1. $s(e)e = er(e) = e$ for all $e \in E^1$.
2. $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
3. $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
4. $v = \sum_{\{e \in E^1 \mid s(e) = v\}} ee^*$ for every $v \in E^0$ that emits edges.

Note that the relations above imply that $\{ee^* \mid e \in E^1\}$ is a set of pairwise orthogonal idempotents in $L(E)$. Note also that if $E$ is a finite graph then we have $\sum_{v \in E^0} v = 1$. In general the algebra $L(E)$ is not unital, but it can be written as a direct limit of unital Leavitt path algebras (with non-unital transition maps), so that it is an algebra with local units. Along this paper, we will be concerned only with row-finite graphs.

2. Basic properties of graphs

Let $E$ be a graph. For any subset $H$ of $E^0$, we will denote by $I(H)$ the ideal of $L(E)$ generated by $H$.

**Lemma 2.1.** If $H$ is a subset of $E^0$, then $I(H) = I(H)$, and $I(H) = I(H) \cap E^0$.

**Proof.** Take $G = I(H) \cap E^0$. By [1] Lemma 3.9, $G \in \mathcal{H}$. Thus, by minimality, we get $H \subseteq \overline{H} \subseteq G$, whence $I(H) \subseteq I(\overline{H}) \subseteq I(G)$. Since $G \subseteq I(H)$, we have $I(G) \subseteq I(H)$, so we get the desired equality. The second statement holds by [6] Proposition 4.2 and Theorem 4.3, as desired. \qed

For a graph $E$ and a hereditary subset $H$ of $E^0$, we denote by $E/H$ the quotient graph

$$(E^0 \setminus H, \{ e \in E^1 \mid r(e) \not\in H \}, r|_{(E/H)^1}, s|_{(E/H)^1}),$$

and by $E_H$ the restriction graph

$$(H, \{ e \in E^1 \mid s(e) \in H \}, r|_{(E_H)^1}, s|_{(E_H)^1}).$$

Observe that while $L(E_H)$ can be seen as a subalgebra of $L(E)$, the same cannot be said about $L(E/H)$. 
Now, we recall that $L(E)$ has a $\mathbb{Z}$-grading. For every $e \in E^1$, set the degree of $e$ as 1, the degree of $e^*$ as -1, and the degree of every element in $E^0$ as 0. Then we obtain a well-defined degree on the Leavitt path $K$-algebra $L(E)$, thus, $L(E)$ is a $\mathbb{Z}$-graded algebra:

$$L(E) = \bigoplus_{n \in \mathbb{Z}} L(E)_n, \quad L(E)_n L(E)_m \subseteq L(E)_{n+m}, \text{ for all } n, m \in \mathbb{Z}.$$

An ideal $I$ of a $\mathbb{Z}$-graded algebra $A = \bigoplus_{n \in \mathbb{Z}} A_n$ is a graded ideal in case $I = \bigoplus_{n \in \mathbb{Z}} (I \cap A_n)$.

**Remark 2.2.** An ideal $J$ of $L(E)$ is graded if and only if it is generated by idempotents; in fact, $J = I(H)$, where $H = J \cap E^0 \in \mathcal{H}_E$. (See the proofs of [6] Proposition 4.2 and Theorem 4.3.)

**Lemma 2.3.** Let $E$ be a graph and consider a proper $H \in \mathcal{H}_E$. Define $\Psi : L(E) \to L(E/H)$ by setting $\Psi(v) = \chi_{(E/H)_0}(v)v$, $\Psi(e) = \chi_{(E/H)_1}(e)e$ and $\Psi(e^*) = \chi_{((E/H)_1)^*}(e^*)e^*$ for every vertex $v$ and every edge $e$, where $\chi_{(E/H)_0} : E^0 \to K$ and $\chi_{(E/H)_1} : E^1 \to K$ denote the characteristic functions. Then:

1. The map $\Psi$ extends to a $K$-algebra epimorphism of $\mathbb{Z}$-graded algebras with $\text{Ker}(\Psi) = I(H)$ and therefore $L(E)/I(H) \cong L(E/H)$.
2. If $X$ is hereditary in $E$, then $\Psi(X) \cap (E/H)^0$ is hereditary in $E/H$.
3. For $X \supseteq H$, $X \in \mathcal{H}_E$ if and only if $\Psi(X) \cap (E/H)^0 \in \mathcal{H}_{E/H}$.
4. For every $X \supseteq H$, $\Psi(X) \cap (E/H)^0 = \Psi(X) \cap (E/H)^0$.

**Proof.**

1. It was shown in [11] Proof of Theorem 3.11] that $\Psi$ extends to a $K$-algebra morphism. Since $H \in \mathcal{H}_E$, $\Psi$ extends to a well-defined morphism. By definition, $\Psi$ is $\mathbb{Z}$-graded and onto. Moreover, $I(H) \subseteq \text{Ker}(\Psi)$.

Since $\Psi$ is a graded morphism, $\text{Ker}(\Psi) \in \mathcal{L}(L(E))$. By [6] Theorem 4.3], there exists $X \in \mathcal{H}_E$ such that $\text{Ker}(\Psi) = I(X)$. By Lemma 2.1, $H = I(H) \cap E^0 \subseteq I(X) \cap E^0 = X$. Hence, $I(H) \neq \text{Ker}(\Psi)$ if and only if there exists $v \in X \setminus H$. But then $\Psi(v) = (v \neq 0$ and $v \in \text{Ker}(\Psi)$, which is impossible.

2. It is clear by the definition of $\Psi$.

3. Since $\Psi$ is a graded epimorphism, there is a bijection between graded ideals of $L(E/H)$ and graded ideals of $L(E)$ containing $I(H)$. Thus, the result holds by [6] Theorem 4.3).

4. It is immediate by part (3).

Recall that a ring $R$ is said to be an idempotent ring if $R = R^2$. For an idempotent ring $R$ we denote by $R-\text{Mod}$ the full subcategory of the category of all left $R$-modules whose objects are the “unital” nondegenerate modules. Here a left $R$-module $M$ is said to be unital if $M = RM$, and $M$ is said to be nondegenerate if, for $m \in M$, $Rm \setminus 0$ implies $m = 0$. Note that if $R$ has an identity then $R-\text{Mod}$ is the usual category of left $R$-modules.

We will use the well-known definition of a Morita context in the case where the rings $R$ and $S$ have not necessarily an identity. Let $R$ and $S$ be idempotent rings. We say that $(R, S, M, N, \varphi, \psi)$ is a (surjective) Morita context if $RM_S$ and $SN_R$ are unital bimodules and $\varphi : N \otimes_R M \to S$, $\psi : M \otimes_S N \to R$ are surjective $S$-bimodule and $R$-bimodule maps, respectively, satisfying the compatibility relations: $\varphi(n \otimes m)n' = n\psi(m \otimes n')$, $m'\varphi(n \otimes m) = \psi(m' \otimes n)m$ for every $m, m' \in M, n, n' \in N$.

In [11] (see Proposition 2.5 and Theorem 2.7) it is proved that if $R$ and $S$ are two idempotent rings, then $R-\text{Mod}$ and $S-\text{Mod}$ are equivalent categories if and only if there exists a
(surjective) Morita context \((R, S, M, N, \phi, \psi)\). In this case, we will say that the rings \(R\) and \(S\) are Morita equivalent and we will refer to as the (surjective) Morita context \((R, S, M, N)\).

**Lemma 2.4.** Let \(E\) be a graph and \(H \subseteq E^0\) a proper hereditary subset. Then \(L(E_H)\) is Morita equivalent to \(I(H)\).

*Proof.* Let \(H = \{v_i \mid i \geq 1\}\), and consider the ascending family of idempotents \(e_n = \sum_{i=1}^n v_i\), \((n \geq 1)\). By Lemma 1.6, \(\{e_n \mid n \geq 1\}\) is a set of local units for \(L(E_H)\), so that \(L(E_H) = \bigcup_{i \geq 1} e_i L(E) e_i\). Since \(I(H)\) is generated by the idempotents \(v_i \in H\), it is a non-degenerated idempotent ring. Moreover, \(I(H) = \bigcup_{i \geq 1} L(E) e_i L(E)\). It is not difficult to see that \((\sum_{i \geq 1} e_i L(E) e_i, \sum_{i \geq 1} L(E) e_i L(E), \sum_{i \geq 1} L(E) e_i, \sum_{i \geq 1} e_i L(E))\) is a (surjective) Morita context for the idempotent rings \(\sum_{i \geq 1} e_i L(E) e_i = L(E_H)\) and \(\sum_{i \geq 1} L(E) e_i L(E) = I(H)\), hence \(I(H)\) is Morita equivalent to \(L(E_H)\). \(\square\)

Under certain conditions we will see in Section 5 that \(I(H)\) is not only Morita equivalent to a Leavitt path algebra; in fact it is isomorphic to a Leavitt path algebra.

**Lemma 2.5.** Let \(H \in \mathcal{H}_E\), and let \(X \subseteq H\) be any subset. Then, \(X \in \mathcal{H}_E\) if and only if \(X \in \mathcal{H}_{E_H}\).

*Proof.* First, suppose that \(X \in \mathcal{H}_{E_H}\). Since \(X \subseteq H\), if \(e \in E^1\) and \(s(e) \in X \subseteq H\), we have \(e \in E_H\). Hence, \(r(e) \in X\), so that \(X\) is hereditary into \(E\). Now, let \(v \in E^0\) such that \(s^{-1}_E(v) \neq \emptyset\) and \(r_E(s^{-1}_E(v)) \subseteq X \subseteq H\). Then, \(v \in H\), so that \(s^{-1}_{E_H}(v) = s^{-1}_E(v)\) (in particular, it is nonempty), and \(r_{E_H}(s^{-1}_{E_H}(v)) = r_E(s^{-1}_E(v)) \subseteq X\), whence \(v \in X\). Thus, \(X \in \mathcal{H}_E\).

Suppose that \(X \in \mathcal{H}_E\). Clearly, \(X\) is hereditary in \(E_H\) (because \(X \subseteq H\)). Now, let \(v \in H\) such that \(s^{-1}_{E_H}(v) \neq \emptyset\) and \(r_{E_H}(s^{-1}_{E_H}(v)) \subseteq X \subseteq H\). Since \(v \in H\), \(s^{-1}_E(v) = s^{-1}_{E_H}(v)\) (in particular, it is nonempty), and \(r_E(s^{-1}_E(v)) = r_{E_H}(s^{-1}_{E_H}(v)) \subseteq X\), whence \(v \in X\). Thus, \(X \in \mathcal{H}_{E_H}\). \(\square\)

**Lemma 2.6.** Let \(E\) be a graph and \(H \in \mathcal{H}_E\). Then, the canonical map

\[K_0(L(E)) \rightarrow K_0(L(E)/I(H))\]

is an epimorphism.

*Proof.* If \(H = E^0\) or \(H = \emptyset\), the result follows trivially. Now, suppose \(H\) to be a proper subset of \(E^0\). By Lemma 2.3 (1) we have \(L(E)/I(H) \cong L(E/H)\). By Lemma 5.6,

\[V(L(E))/V(I(H)) \cong V(L(E/H)) \cong V(L(E)/I(H)).\]

Since \(L(E)\) and \(L(E/H)\) have a countable unit, we have that \(K_0(L(E)) = \text{Grot}(V(L(E)))\) and \(K_0(L(E/H)) = \text{Grot}(V(L(E/H)))\). Hence, the canonical map \(K_0(L(E)) \rightarrow K_0(L(E)/I(H))\) is clearly an epimorphism, as desired. \(\square\)

We denote by \(E^\infty\) the set of infinite paths \(\gamma = (\gamma_n)_{n=1}^\infty\) of the graph \(E\) and by \(E^{\leq \infty}\) the set \(E^\infty\) together with the set of finite paths in \(E\) whose end vertex is a sink. We say that a vertex \(v\) in a graph \(E\) is cofinal if for every \(\gamma \in E^{\leq \infty}\) there is a vertex \(w\) in the path \(\gamma\) such that \(v \geq w\). We say that a graph \(E\) is cofinal if so are all the vertices of \(E\).

Observe that if a graph \(E\) has cycles, then \(E\) cofinal implies that every vertex connects to a cycle.

**Lemma 2.7.** If \(E\) is cofinal, and \(v \in E^0\) is a sink, then:
Remark 3.1.

1. The only sink of $E$ is $v$.
2. For every $w \in E^0$, $v \in T(w)$.
3. $E$ contains no infinite paths. In particular, $E$ is acyclic.

Proof.

1. It is obvious from the definition.
2. Since $T(v) = \{v\}$, the result follows from the definition of $T(v)$ by considering the path $\gamma = v \in E^{\leq \infty}$.
3. If $\alpha \in E^{\infty}$, then there exists $w \in \alpha^0$ such that $v \geq w$, which is impossible. Thus, in particular, $E$ contains no closed simple paths, and therefore no cycles.

Next result is known in the case of graphs without sinks. Since we have no knowledge of the existence of a (published) version of the result in the general case, we give a proof for the sake of completeness.

Lemma 2.8. A graph $E$ is cofinal if and only if $\mathcal{H} = \{\emptyset, E^0\}$.

Proof. Suppose $E$ to be cofinal. Let $H \in \mathcal{H}$ with $\emptyset \neq H \neq E^0$. Fix $v \in E^0 \setminus H$ and build a path $\gamma \in E^{\leq \infty}$ such that $\gamma^0 \cap H = \emptyset$: If $v$ is a sink, take $\gamma = v$. If not, then $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \notin H$; otherwise, $H$ saturated implies $v \in H$, which is impossible. Hence, there exists $e_1 \in s^{-1}(v)$ such that $r(e_1) \notin H$. Let $\gamma_1 = e_1$ and repeat this process with $r(e_1) \notin H$. By recurrence either we reach a sink or we have an infinite path $\gamma$ whose vertices are not in $H$, as desired. Now consider $w \in H$. By the hypothesis, there exists $z \in \gamma$ such that $w \geq z$, and by hereditariness of $H$ we get $z \in H$, contradicting the definition of $\gamma$.

Conversely, suppose that $\mathcal{H} = \{\emptyset, E^0\}$. Take $v \in E^0$ and $\gamma \in E^{\leq \infty}$, with $v \notin \gamma^0$ (the case $v \in \gamma^0$ is obvious). By hypothesis the hereditary saturated subset generated by $v$ is $E^0$, i.e., $E^0 = \bigcup_{n \geq 0} \Lambda_n(v)$. Consider $m$, the minimum $n$ such that $\Lambda_n(v) \cap \gamma^0 \neq \emptyset$, and let $w \in \Lambda_m(v) \cap \gamma^0$. If $m > 0$, then by minimality of $m$ it must be $s^{-1}(w) \neq \emptyset$ and $r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$. The first condition implies that $w$ is not a sink and since $\gamma = (\gamma_i) \in E^{\leq \infty}$, there exists $i \geq 1$ such that $s(\gamma_i) = w$ and $r(\gamma_i) = w' \in \gamma^0$, the latter meaning that $w' \in r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$, contradicting the minimality of $m$. Therefore $m = 0$ and then $w \in \Lambda_0(v) = T(v)$, as we needed.

3. Condition (K)

We begin this section by recalling the two following well-known notions which will play a central role in the sequel.

1. A graph $E$ satisfies condition (L) if every closed simple path has an exit, equivalently [11 Lemma 2.5], if every cycle has an exit.
2. A graph $E$ satisfies condition (K) if for each vertex $v$ on a closed simple path there exists at least two distinct closed simple paths $\alpha$, $\beta$ based at $v$, or, following [2], $V_1 = \emptyset$.

Remark 3.1.

1. Notice that if $E$ satisfies condition (K) then it satisfies condition (L).
2. According to [2 Lemma 7], if $L(E)$ is simple then it satisfies condition (K).
It is not difficult to see that if $E$ satisfies condition (L) then so does $E_H$, whereas $E/H$ need not. Condition (K) has a better behaviour as it is shown in the following result.

**Lemma 3.2.** Let $E$ be a graph and $H$ a hereditary subset of $E^0$. If $E$ satisfies condition (K), so do $E_H$ and $E/H$.

**Proof.** We will see $CSP_E(v) = CSP_{E_H}(v)$ and $CSP_E(v) = CSP_{E/H}(v)$ for every $v \in H$ and $w \in E^0 \setminus H$. Clearly, $CSP_{E_H}(v) \subseteq CSP_E(v)$; conversely, let $\alpha \in CSP_E(v)$, and suppose $\alpha = (\alpha_1, \ldots, \alpha_n)$. Since $H$ is hereditary and $s(\alpha_1) = v \in H$, we get $r(\alpha_1) = s(\alpha_2) \in H$. Thus, by recurrence, $\alpha \in CSP_{E_H}(v)$ and the result holds.

Now, let $v \in E^0 \setminus H$ and consider $\alpha = (\alpha_1, \ldots, \alpha_n) \in CSP_E(v)$. Since $r(\alpha_n) = v \not\in H$ we get $n \in (E/H)^1$. If $r(\alpha_n) = v \not\in H$ then $r(\alpha_{n-1}) = s(\alpha_n) \in H$ and $H$ hereditary implies $v = r(\alpha_n) \in H$, a contradiction. By recurrence, $\alpha \in CSP_{E/H}(v)$; since the converse is immediate, the result follows. $\square$

For an algebra $A$, denote by $\mathcal{L}(A)$ and $\mathcal{L}_{gr}(A)$ the lattices of ideals and graded ideals, respectively, of $A$. The following proposition provides a description of the ideals of $L(E)$ for $E$ a graph satisfying condition (K).

**Proposition 3.3.** If a graph $E$ satisfies condition (K) then, for every ideal $J$ of $L(E)$, $J = I(H)$, where $H = J \cap E^0$ is a hereditary saturated subset of $E^0$. In particular, $\mathcal{L}_{gr}(L(E)) = \mathcal{L}(L(E))$.

**Proof.** Let $J$ be a nonzero ideal of $L(E)$. By [1] Lemma 3.9 (which can be applied because $E$ satisfies condition (L) by Remark 3.1(1)) and [2] Proposition 6, $H = J \cap E^0 \neq \emptyset$ is a hereditary saturated subset of $E^0$. Therefore, and taking into account Remark 2.2, $I(H)$ is a graded ideal of $L(E)$ contained in $J$.

Suppose $I(H) \neq J$. Then, by Lemma 2.3(1),

$$0 \neq J/I(H) \not\trianglelefteq L(E)/I(H) \cong L(E/H).$$

Thus, $E/H$ satisfies condition (L) by Lemma 3.2 and Remark 3.1(1). Now, consider the isomorphism (of $K$-algebras) $\overline{\Psi} : L(E)/I(H) \to L(E/H)$ given by $\overline{\Psi}(x + I(H)) = \Psi(x)$ (for $\overline{\Psi}$ as in Lemma 2.3). By [2] Proposition 6, $\emptyset \neq \overline{\Psi}(J/I(H)) \subseteq (E^0 \setminus H) \subseteq \overline{\Psi}(J) \subseteq (E^0 \setminus H)$, so there exists $v \in J$ with $\overline{\Psi}(v) \in \overline{\Psi}(J)$. But $v \in E^0 \cap J = H$ and, on the other hand, $v = \overline{\Psi}(v) \in E^0 \setminus H$, which is impossible.

To finish, take into account that $J$ is an ideal generated by idempotents and apply Remark 2.2. $\square$

In the following section the converse of the previous result is proved.

**Corollary 3.4.** If $E$ satisfies condition (K), then for every ideal $I$ of $L(E)$, the canonical map

$$K_0(L(E)) \to K_0(L(E)/I)$$

is an epimorphism.

**Proof.** By Proposition 3.3, $I = I(H)$ for the hereditary saturated subset $H = I \cap E^0$ of $E^0$. Then, the result holds by Lemma 2.6. $\square$
Recall that a matricial algebra is a finite direct product of full matrix algebras over $K$, while a locally matricial algebra is a direct limit of matricial algebras.

The following result can be obtained as a corollary of Proposition 3.3. However we do not include its proof because it can be reached by doing slight changes on that of Corollary 2.3.

**Corollary 3.5.** If $E$ is a finite acyclic graph, then $L(E)$ is a $K$-matricial algebra.

**Corollary 3.6.** If $E$ is an acyclic graph, then $L(E)$ is a locally matricial $K$-algebra.

**Proof.** By Lemma 2.2, $L(E) \cong \lim_{\to} L(X_n)$, where $X_n$ is a finite subgraph of $E$ for all $n \geq 1$. Hence, $X_n$ is a finite acyclic graph for every $n \geq 1$, whence the result holds by Corollary 3.5. □

Recall that a graph homomorphism $f : E = (E^0, E^1) \to F = (F^0, F^1)$ is given by two maps $f^0 : E^0 \to F^0$ and $f^1 : E^1 \to F^1$ such that $r_F(f^1(e)) = f^0(r_E(e))$ and $s_F(f^1(e)) = f^0(s_E(e))$ for every $e \in E^1$. We say that a graph homomorphism $f$ is complete in case $f^0$ is injective and $f^1$ restricts to a bijection from $s_E^1(v)$ onto $s_F^1(f^0(v))$ for every $v \in E^0$ that emits edges. Note that under the assumptions above, the map $f^1$ must also be injective.

**Lemma 3.7.** If $E$ is a graph satisfying condition (K) then there exists an ascending family $\{X_n\}_{n \geq 0}$ of finite subgraphs such that:

1. For every $n \geq 0$, $X_n$ satisfies condition (K).
2. For every $n \geq 0$, the inclusion map $X_n \subseteq E$ is a complete graph homomorphism.
3. $E = \bigcup_{n \geq 0} X_n$.

**Proof.** We will construct $X_n$ by recurrence on $n$. First, we enumerate $E^0 = \{v_n \mid n \geq 0\}$. Then, we define $X_0 = \{v_0\}$. Clearly, $X_0$ satisfies condition (K) and also $X_0 \subseteq E$ is a complete graph homomorphism.

Now, suppose we have constructed $X_0, X_1, \ldots, X_n$ satisfying (1) and (2). Consider the graph $\tilde{X}_{n+1}$ with: (a) $\tilde{X}_{n+1}^1 = X_n^1 \cup \{e \in E^1 \mid s(e) \in X_n^0\}$; (b) $\tilde{X}_{n+1}^0 = X_n^0 \cup \{v_{n+1}\} \cup \{r(e) \mid e \in \tilde{X}_{n+1}^1\}$. Clearly, $\tilde{X}_{n+1}$ is finite and satisfies (2). If it also satisfies (1), we define $X_{n+1} = \tilde{X}_{n+1}$.

Suppose that $\tilde{X}_{n+1}$ does not satisfy condition (K). Consider the set of all cycles based at vertices in $\tilde{X}_{n+1}$, $\mu_1, \ldots, \mu_k \subseteq \tilde{X}_{n+1}$ such that: (i) $\mu_i \nsubseteq X_n$ for any $1 \leq i \leq k$; (ii) for every $1 \leq i \leq k$ and every $v \in \mu_i$, $\text{card}(\text{CSP}_{\tilde{X}_{n+1}}(v)) = 1$. Since $\tilde{X}_{n+1} \subseteq E$ and $E$ satisfies condition (K), there exist closed simple paths $\mu_1^0, \ldots, \mu_k^0 \subseteq E$ such that, for each $1 \leq i \leq k$, $\mu_i^0 \neq \mu_i^0$ and $\mu_i^0 \cap \mu_i^0 \neq \emptyset$. For each $1 \leq i \leq k$, let $\mu_i^0 = (e_{i,1}, \ldots, e_{i,j_i})$.

We consider the finite subgraph $\tilde{Y}_{n+1}$ of $E$ such that: (a) $\tilde{Y}_{n+1}^1 = \tilde{X}_{n+1}^1 \cup \{e_{i,l}^0 \mid 1 \leq i \leq k, 1 \leq l \leq j_i\}$; (b) $\tilde{Y}_{n+1}^0 = \tilde{X}_{n+1}^0 \cup \{s(e_{i,l}^0), r(e_{i,l}^0) \mid 1 \leq i \leq k, 1 \leq l \leq j_i\}$. Clearly, $\tilde{Y}_{n+1}$ satisfies (1).

Now, let $X_{n+1}$ be the finite subgraph of $E$ such that: (a) $X_{n+1}^1 = \tilde{Y}_{n+1}^1 \cup \{f \in E^1 \mid s(f) \in (\mu_i^0)^0 \text{ for some } 1 \leq i \leq k\}$; (b) $X_{n+1}^0 = \tilde{Y}_{n+1}^0 \cup \{r(e) \mid e \in X_{n+1}^1\}$. If $\mu \nsubseteq X_{n+1}$ is a closed simple path such that $\mu \nsubseteq \tilde{Y}_{n+1}$, then either it appears because one of the $e \in X_{n+1}^1 \setminus \tilde{Y}_{n+1}^1$ is a single loop based at some vertex in one $\mu_i^0$, or $s(e) \in (\mu_i^0)^0$ and $r(e)$ connects to a path that comes back to $s(e)$. In any case, the (potential) new closed simple paths are based at vertices of $\mu_i^0$ for some $i$, whence $X_{n+1}$ satisfies (1). Also, since the step from $\tilde{Y}_{n+1}$ to $X_{n+1}$ adds all
the exits of all the vertices in the cycles $\mu_i^1$, we conclude that for any vertex $v \in X_{n+1}^0$, $v$ is either a sink, or every $e \in E^1$ with $s(e) \in X_{n+1}^0$ belongs to $X_{n+1}^1$. Hence, $X_{n+1} \subseteq E$ is a complete graph homomorphism. This completes the recurrence argument.

Finally, since $v_n \in X_n$ for every $n \geq 0$, we conclude that $E^0 = \bigcup_{n \geq 0} X_n^0$ and by (2), $E^1 = \bigcup_{n \geq 0} X_n^1$.

The following definitions can be found in [13] Definition 3.2. Let $F$ be a subgraph of a graph $E$. Then:

1. The loop completion $\ell_E(F)$ of $F$ in $E$ is the subgraph of $E$ obtained as the union of $F$ with every closed path based at an element of $F^0$.
2. The exit completion $F_e$ of $F$ is a subgraph obtained by adding to $F$ the edges $V = \{ e \in E^1 \mid s(e) = s(f) \text{ for some } f \in F^1 \}$, and the vertices $\{ r(e) \mid e \in V \}$. We say that $F$ is exit complete if $F = F_e$.

**Lemma 3.8.** If $F$ is an exit complete subgraph of a graph $E$, then $L(F)$ is isomorphic to a subalgebra of $L(E)$.

**Proof.** Denote by $e'$ the edges of $F$ seen inside $E$, and by $v'$ the vertices of $F$ seen inside $E$. Since $F = F_e$, for every vertex $v \in F^0$ we have that either $v$ is a sink or $s^{-1}_F(v) = s^{-1}_E(v')$. Thus, the relations defining $L(F)$ and $L(F') \subseteq L(E)$ are exactly the same, so that there is a natural injective morphism form $L(F)$ to $L(E)$, as desired.

**Lemma 3.9.** If $F$ is a subgraph of a graph $E$ then:

1. $F_e$ is exit complete.
2. If $F$ is finite then so is $F_e$ whereas $l_E(F)$ need not be.

**Proof.**

1. Clearly $F^1_e \subseteq (F_e)^1$. Let us see the other inclusion. Take $g \in (F_e)^1$. If $g \in F^1_e$ we have finished. If not, there exists $f \in F^0_e$ with $s(g) = s(f)$. We have two possibilities: If $f \in F^1_e$, then, by definition, $g \in F^1_e$. If $f \notin F^1_e$ we can find $h \in F^1$ for which $s(f) = s(h)$. Therefore $s(g) = s(h)$ and again $g \in F^1_e$. Now it easily follows $F^0_e = (F_e)^0$.

2. Since $F$ is finite (and row-finite) then $F^1$ is finite. Now, for each $f \in F^1$ there are finitely many edges $e \in E^1$ with $s(e) = s(f)$ (because $E$ is row-finite), and therefore we are adding a finite number of edges and consequently of vertices. Thus, $F_e$ is finite. To show that $l_E(F)$ can be infinite, consider the infinite graph $E$

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\[ \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \]
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Then $F = (\{ v \}, \emptyset)$ is finite while $l_E(F) = E$ is not.

**Lemma 3.10.** Let $E$ be a graph and $T$ be any subgraph of $E$. Define $F = l_E(T)$, $G = l_E(T)_e$, $S$ the set of sinks of $G$ and $J = G/\overline{S}$. Then:

1. $\text{CSP}_F(v) = \text{CSP}_E(v)$ for every $v \in F^0$.
2. $\text{CSP}_G(v) = \text{CSP}_E(v)$ for every $v \in G^0$ such that $\text{CSP}_G(v) \neq \emptyset$.
3. $\text{CSP}_J(v) = \text{CSP}_E(v)$ for every $v \in J^0$ such that $\text{CSP}_J(v) \neq \emptyset$.
4. If $E$ satisfies condition (K) then so do $F$, $G$ and $J$.

**Proof.** (1) is evident from the definition of the loop completion.
Consider $v \in G^0$ such that $\text{CSP}_G(v) \neq \emptyset$ and take $p = (p_1, \ldots, p_n) \in \text{CSP}_G(v)$. Suppose $p^0 \cap F^0 = \emptyset$; then for an arbitrary edge $p_i$ we have $r(p_i) \notin F^0$. The construction of the exit completion yields that $p_i$ is a new added edge and consequently it exists $f \in F^1$ with $s(f) = s(p_i)$ and hence $s(p_i) \in p^0 \cap F^0$, a contradiction. Therefore $p^0 \cap F^0 \neq \emptyset$. Take $w$ in the previous intersection. Then $\text{CSP}_G(w) = (v$ and $w$ are in the same closed path) $\text{CSP}_G(w) \supseteq (\because F$ is a subgraph of $G) \text{CSP}_F(w) = (\text{by (i)}) \text{CSP}_E(w) = (v$ and $w$ are in the same closed path) $\text{CSP}_E(v) \supseteq \text{CSP}_G(v)$. Hence, (2) holds. Note that the result may fail for $\text{CSP}_G(v) = \emptyset$.

Let $v \in J^0$ such that $\text{CSP}_J(v) \neq \emptyset$. Obviously $\text{CSP}_J(v) \subseteq \text{CSP}_G(v)$. Now consider $p = (p_1, \ldots, p_k) \in \text{CSP}_G(v)$. We claim that $p^0 \cap S = \emptyset$. If not, there exists $m = \min\{n \in \mathbb{N} \mid p^0 \cap \Lambda_n(S) \neq \emptyset\}$. Take $v \in p^0 \cap \Lambda_m(S)$. If $m > 0$ then by minimality we have that $r(s^{-1}(v)) \subseteq \Lambda_{m-1}(S)$. In particular, if $v = s(p_i)$ then $r(p_i) \in p^0 \cap \Lambda_{m-1}(S)$, which contradicts the minimality of $m$. If $m = 0$ then $v \in p^0 \cap \Lambda_0(S) = p^0 \cap S = (S$ is the set of sinks) $p^0 \cap S$. This is absurd since $p$ has no sinks. Any possibility leads to a contradiction so $p^0 \subseteq J^0$ and, consequently, $p_1, \ldots, p_k \in J^1$. Thus, $p \in \text{CSP}_J(v)$. Now (2) gives the result.

(3) follows directly from (1), (2) and (3).

4. Exchange Leavitt path algebras

We will say that a (not-necessarily unital) ring $R$ is an exchange ring (see (3)) if for every element $x \in R$ the equivalent conditions in the next lemma are satisfied.

Lemma 4.1. ([3, Lemma 1.1]) Let $R$ be a ring and let $R'$ be a unital ring containing $R$ as a two-sided ideal. Then the following conditions are equivalent for an element $x \in R$:

1. There exists $e^2 = e \in R$ with $e - x \in R'(x - x^2)$,
2. there exist $e^2 = e \in Rx$ and $c \in R'$ such that $(1 - e) - c(1 - x) \in J(R')$,
3. there exists $e^2 = e \in Rx$ such that $R' = Re + R'(1 - x)$,
4. there exists $e^2 = e \in Rx$ such that $1 - e \in R'(1 - x)$,
5. there exist $r, s \in R$, $e^2 = e \in R$ such that $e = rx = s + x - sx$.

(Here $J(R')$ denotes the Jacobson radical of $R'$.)

Observe that $R$ being an exchange ring does not depend on the particular unital ring where $R$ is embedded as an ideal (look at condition (v) in the previous Lemma). Other characterizations of the exchange property for not necessarily unital rings can be found in [3].

Remark 4.2. Since any $K$-matricial algebra is an exchange ring, then so is any $K$-locally matricial algebra (apply [3, Theorem 3.2].)

Theorem 4.3. Let $E$ be a graph. If $L(E)$ is an exchange ring, then $E$ satisfies condition (K).

Proof. The first step will be to show that $E$ satisfies condition (L). Suppose that there exist a vertex $v$ and a cycle $\alpha$ with $s(\alpha) = v$ such that $\alpha$ has no exits. Denote by $H$ the hereditary saturated subset of $E^0$ generated by $\alpha^0$. By Lemma 2.4 $I(H)$ is Morita equivalent to $L(E_H)$. If $M$ is the graph having only a vertex $w$ and an edge $e$ such that $r(e) = s(e) = w$, then $L(M) \cong K[x, x^{-1}]$ by [11, Example 1.4 (ii)]). Consider the map $f : L(M) \rightarrow L(E_H)$ given by $f(w) = v$, $f(e) = \alpha$, $f(e^*) = \alpha^*$. It is well defined because the relations in $M$ are consistent with those in $L(E_H)$ (the only non trivial one being $\alpha\alpha^* = v$, which holds due to
the absence of exits for $\alpha$, as in [11 p. 12]). It is a (non-unital) monomorphism of $K$-algebras; clearly, $\text{Im} f \subseteq vL(E_H)v$. Now, we prove $vL(E_H)v \subseteq \text{Im} f$. To this end, it is enough to see $vpq^*v \in \text{Im} f$ for every $p = e'_1 \ldots e'_r, q = e_1 \ldots e_s$, with $e'_1, \ldots, e'_r, e_1, \ldots, e_s \in E^1_H$. Reasoning as in [11 Proof of Theorem 3.11] we get that $vpq^*v$ has the form: $v, v\alpha^n v$ or $v(\alpha^*)^m v$, with $m, n \in N$. Hence our claim follows.

Apply [3] Theorem 2.3] to have that $I(H)$ is an exchange ring; moreover, $L(E_H)$ is an exchange ring by Lemma 2.4 and [5, Theorem 2.3], and the same can be said about the corner $vL(E_H)v$ by [5, Corollary 1.5]. But $vL(E_H)v \cong L(H) \cong K[x, x^{-1}]$ is not an exchange ring, what leads to a contradiction.

Now, we will prove that $E$ satisfies condition (K). Suppose on the contrary that there exists a vertex $v$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \text{CSP}(v)$, with $\text{card}(\text{CSP}(v)) = 1$ (in fact, $\alpha$ must be a cycle). Consider $A = \{e \in E^1 \mid e \text{ exit of } \alpha\}$, $B = \{r(e) \mid e \in A\}$, and let $H$ be the hereditary saturated closure of $B$. With a similar argument to that used in [2 p. 6] we get that $H \cap \alpha^0 = \emptyset$, so that, $H$ is a proper subset of $E^0$. Then, $\alpha^0 \subseteq (E/H)^0$ and $\{\alpha_1, \ldots, \alpha_n\} \subseteq (E/H)^1$, whence $\alpha$ is a cycle in $E/H$ with no exits.

Since $L(E/H) \cong L(E)/I(H)$ (Lemma 2.3 (1)), $L(E/H)$ is an exchange ring [3, Theorem 2.2] and, by the previous step, $E/H$ satisfies condition (L), a contradiction. 

Recall that an idempotent $e$ in a ring $R$ is called infinite if $eR$ is isomorphic as a right $R$-module to a proper direct summand of itself. The ring $R$ is called purely infinite in case every right ideal of $R$ contains an infinite idempotent.

**Proposition 4.4.** If $E$ is a graph satisfying condition (K) and $\mathcal{L}(L(E))$ is finite, then $L(E)$ is an exchange ring.

**Proof.** Since $\mathcal{L}(L(E))$ is finite, we can construct an ascending chain of ideals

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = L(E)$$

such that, for every $0 \leq i \leq n - 1$, $I_i$ is maximal among the ideals of $L(E)$ contained in $I_{i+1}$. Now, let us prove the result by induction on $n$.

If $n = 1$, then $L(E)$ is a simple ring and then $E$ is cofinal by Lemma 2.8 [11 Theorem 3.11]. Since $E$ satisfies condition (K), it can occur exactly two possibilities:

1. $E$ has no closed simple paths, whence it is acyclic and thus, by Corollary 3.6 $L(E)$ is a locally matricial algebra, and so an exchange ring by Remark 4.2.

2. $E$ has at least one closed simple path, whence $L(E)$ is a purely infinite simple ring by cofinality, [11 Theorem 3.11] and [2 Theorem 11]. Thus, $L(E)$ is an exchange ring by [3 Corollary 1.2].

In any case, $L(E)$ turns out to be an exchange ring.

Now, suppose that the result holds for $k < n$. By Proposition 3.3 and [6 Theorem 4.3], there exist hereditary saturated sets $H_i$ ($1 \leq i \leq n$) such that:

1. $I_i = I(H_i)$ for every $0 \leq i \leq n$; in particular, $H_i \subseteq H_{i+1}$ for every $0 \leq i \leq n - 1$.
2. For any $0 \leq i \leq n - 1$, it does not exist an hereditary saturated set $T$ such that $H_i \not\subseteq T \not\subseteq H_{i+1}$.

Consider the restriction graph $E_{H_{n-1}}$. By Lemma 3.2 $E_{H_{n-1}}$ satisfies condition (K), so that $\mathcal{L}_{\text{gr}}(L(E_{H_{n-1}})) = \mathcal{L}(L(E_{H_{n-1}}))$ by Proposition 3.3. If for each $0 \leq i \leq n - 1$, $J_i \prec L(E_{H_{n-1}})$
is an exchange ring, then the previous remarks imply that
\[ 0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{n-1} = L(E_{H_{n-1}}) \]
where, for every \( 0 \leq i \leq n-2 \), \( J_i \) is maximal among the ideals of \( L(E) \) contained in \( J_{i+1} \); otherwise, since \( L_{gr}(L(E_{H_{n-1}})) = L(L(E_{H_{n-1}})) \), Lemma 2.6 would contradict property (ii) satisfied by the set \( H_i \). Thus, by induction hypothesis, \( L(E_{H_{n-1}}) \) is an exchange ring. Since \( I(H_{n-1}) \) is Morita equivalent to \( L(E_{H_{n-1}}) \) by Lemma 2.3, \( I(H_{n-1}) \) is an exchange ideal by [8, Theorem 2.3]. Now, by Lemma 2.3 (1), \( L(E)/I(H_{n-1}) \cong L(E/H_{n-1}) \). Hence, \( E/H_{n-1} \) is a graph satisfying condition (K) by Lemma 3.2 and \( L(E/H_{n-1}) \) is simple by construction. Following the same dichotomy for \( E/H_{n-1} \) as in (1) and (2) above, we get that \( L(E/H_{n-1}) \) is an exchange ring. Then, by using Lemma 2.6 and [8, Theorem 3.5], we conclude that \( L(E) \) is an exchange ring, as desired. \( \square \)

We would like to thank Gene Abrams for showing that (4) \( \Rightarrow \) (3) in the following theorem is true.

**Theorem 4.5.** For a graph \( E \), the following conditions are equivalent:

1. \( L(E) \) is an exchange ring.
2. \( E/H \) satisfies condition (L) for every hereditary saturated subset \( H \) of \( E^0 \).
3. \( E \) satisfies condition (K).
4. \( L_{gr}(L(E)) = L(L(E)) \).
5. \( E_H \) and \( E/H \) satisfy condition (K) for every hereditary saturated subset \( H \) of \( E^0 \).
6. \( E_H \) and \( E/H \) satisfy condition (K) for some hereditary saturated subset \( H \) of \( E^0 \).

**Proof.**

1. \( \Rightarrow \) (2). By Lemma 2.3 (1), \( L(E)/I(H) \cong L(E/H) \). Then, by [8, Theorem 2.2], \( L(E/H) \) is an exchange ring. Apply Theorem 4.3 and Remark 3.1 (1) to obtain (2).

2. \( \Rightarrow \) (3) is just the first paragraph in the proof of Theorem 4.3.

3. \( \Rightarrow \) (4) is Proposition 3.3.

4. \( \Rightarrow \) (3). Suppose on the contrary that \( E \) does not satisfy condition (K). Apply (2) \( \Rightarrow \) (3) to find a hereditary saturated subset \( H \) of \( E^0 \) such that \( E/H \) does not verify condition (L), that is, there exists a cycle \( p \) in \( E/H \) based at \( v \) without an exit. Now [11, Theorem 3.11, pp. 12, 13] shows that in this situation we have \( v \not\in J := I(v+p) \), meaning in particular that the ideal \( J \) is not graded. Now if \( H \neq \emptyset \), Lemma 2.3 shows that there exists a graded isomorphism \( \Phi : L(E)/I(H) \to L(E/H) \) so that we can lift \( \Phi^{-1}(J) \) to an ideal \( J \) of \( L(E) \), which cannot be graded (a quotient of graded ideals is again graded). If \( H = \emptyset \) then clearly \( J \) is an ideal of \( L(E/H) = L(E) \) which is not graded. In any case we get a contradiction.

3. \( \Rightarrow \) (1). We have two different proofs of this fact. The first one is inspired in the results of [8], while the second one follows the style of [13, Proof of Theorem 4.1].

   i. By Lemma 3.7, there exists a family \( \{ X_n \}_{n \geq 0} \) of finite subgraphs such that, for every \( n \geq 0 \), \( X_n \) satisfies condition (K), \( E = \bigcup_{n \geq 0} X_n \) and the natural inclusion maps \( f_n : X_n \hookrightarrow E \) are complete graph homomorphisms (therefore so are the inclusions \( f_{n,n+1} : X_n \hookrightarrow X_{n+1} \)). By [8, Lemma 2.2], we have induced maps \( L(f_{n,n+1}) : L(X_n) \to L(X_{n+1}) \) and \( L(f_n) : L(X_n) \to L(E) \) such that \( L(E) \cong \liminf(L(X_n), L(f_{n,n+1})). \)

   Fix \( n \geq 0 \). Since \( X_n \) satisfies condition (K), by Proposition 3.3 and [8, Theorem 4.3], \( L(L(X_n)) \) is isomorphic to the lattice of hereditary saturated subsets of \( X_n^0 \).
Hence, $\mathcal{L}(L(X_n))$ is finite. Thus, $L(X_n)$ is an exchange ring by Proposition 4.3. Since $L(E)$ is a direct limit of exchange rings, it is itself an exchange ring, as desired.

(ii) Take an element $x \in L(E)$. By [11, Lemma 1.5], there exist a finite family of vertices $V = \{v_1, \ldots, v_m\}$, and a finite family of edges $W = \{e_1, \ldots, e_n\}$, such that $x$ is in the linear span of $V$ and the set consisting of expressions $e_{i_1} \ldots e_{i_k} s_{i_k}^{l_k} \ldots s_{i_1}^{l_1}$ with $e_{i_l}$ edges in $W$. For $T^1 = W$ and $T^0 = V \cup \{r(e), s(e), i = 1, \ldots, n\}$, let $T$ be the graph $T = (T^0, T^1, r|_{T^1}, s|_{T^1})$ and consider $G$, $J$ and $S$ as in Lemma 3.10. It can be proved, as in [13, p. 224], that the number of hereditary subsets of $J^0$ is finite. Apply Lemma 3.10 (4), Proposition 3.3, and [6, Theorem 4.3] to obtain that $\mathcal{L}(L(J))$ is finite. Now, Proposition 4.4 shows that $L(J)$ is an exchange ring. By Lemma 2.3 (1), $L(J) \cong L(G)/I(S)$, and by Lemma 2.6 and [3, Theorem 3.5], $L(G)$ is an exchange ring. This means (use condition (5) in Lemma 4.1) that given $x$ there exist $e^2 = e, r, s \in L(G) \subseteq L(E)$ (Lemmas 3.9 (1) and 3.8) such that $e = r x = s + x - s x$. Whence, $L(E)$ is an exchange ring.

(3) $\Rightarrow$ (5) is Lemma 3.2.

(5) $\Rightarrow$ (6) is a tautology.

(6) $\Rightarrow$ (1). By (3) $\Rightarrow$ (1), $L(E_H)$ and $L(E/H)$ are exchange rings. Since $L(E_H)$ is Morita equivalent to $I(H)$ by Lemma 2.4 then $I(H)$ is an exchange ring because both are idempotent rings and we may apply [3, Theorem 2.3]. By Lemma 2.3 (1), $L(E/H) \cong L(E)/I(H)$. Now, $L(E)/I(H)$ and $I(H)$ exchange rings, Lemma 2.6 and [3, Theorem 3.5] imply that $L(E)$ is an exchange ring.

5. SOME SPECIAL FACTS

The following definitions are particular cases of those appearing in [10, Definition 1.3]:

Let $E$ be a graph, and let $\emptyset \neq H \in \mathcal{H}_E$. Define

$$F_E(H) = \{\alpha = (\alpha_1, \ldots, \alpha_n) | \alpha_i \in E^1, s(\alpha_i) \in E^0 \setminus H, r(\alpha_i) \in E^0 \setminus H \text{ for } i < n, r(\alpha_n) \in H\}.$$  

Denote by $\overline{F}_E(H)$ another copy of $F_E(H)$. For $\alpha \in F_E(H)$, we write $\overline{\alpha}$ to denote a copy of $\alpha$ in $\overline{F}_E(H)$. Then, we define the graph $H E = (H E^0, H E^1, s', r')$ as follows:

1. $H E^0 = (H E)^0 = H \cup F_E(H)$.
2. $H E^1 = (H E)^1 = \{e \in E^1 | s(e) \in H\} \cup \overline{F}_E(H)$.
3. For every $e \in E^1$ with $s(e) \in H$, $s'(e) = s(e)$ and $r'(e) = r(e)$.
4. For every $\overline{\alpha} \in \overline{F}_E(H)$, $s'(\overline{\alpha}) = \alpha$ and $r'(\overline{\alpha}) = r(\alpha)$.

Lemma 5.1. Let $E$ be a graph, and let $\emptyset \neq H \in \mathcal{H}_E$. Then:

1. If $E_H$ satisfies condition (L), then so does $H E$.
2. If $E_H$ satisfies condition (K), then so does $H E$.

Proof. Notice that each vertex $\alpha \in F_E(H)$ is a source emitting exactly one edge $\overline{\alpha} \in \overline{F}_E(H)$ which ends in $H$. Thus, every closed simple path in the graph $H E$ comes from a closed simple path in $E_H$, hence, the result follows.

The class of Leavitt path algebras is closed under quotients (Lemma 2.3 (1)). A direct consequence of the next result is that under condition (L), this class is also closed for ideals.

Lemma 5.2. (c.f. [10, Lemma 1.5]) Let $E$ be a graph, and let $\emptyset \neq H \in \mathcal{H}_E$. If $E_H$ satisfies condition (L), then $I(H) \cong L(E_H)$ (as nonunital rings).
Proof. We define a map \( \phi : L(\mu E) \to I(H) \) as follows: (i) For every \( v \in H \), \( \phi(v) = v \); (ii) for every \( \alpha \in F_E(H) \), \( \phi(\alpha) = \alpha \alpha^* \); (iii) for every \( e \in E^1 \) with \( s(e) \in H \), \( \phi(e) = e \) and \( \phi(e^*) = e^* \); (iv) for every \( \overline{\alpha} \in \overline{F_E(H)} \), \( \phi(\overline{\alpha}) = \alpha \) and \( \phi(\overline{\alpha}^*) = \overline{\alpha}^* \).

By definition, it is tedious but straightforward to check that the images of the relations in \( L(\mu E) \) satisfy the relations defining \( L(E) \). Thus, \( \phi \) is a well-defined \( K \)-algebra morphism.

Since for any \( v \in H \), \( \phi(v) = v \), to see that \( \phi \) is surjective, by [1, Lemma 1.5], it is enough to show that every finite path \( \alpha \) of \( E \) with \( r(\alpha) \) or \( s(\alpha) \) in \( H \) is in the image of \( \phi \). So let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be with \( \alpha_i \in E^1 \). If \( s(\alpha) \in H \), then \( s(\alpha_i) \in H \) for every \( i \) because \( H \) is hereditary and thus \( \alpha = \phi(\alpha_1) \cdots \phi(\alpha_n) = \phi(\alpha) \).

Suppose that \( s(\alpha_1) \in E^0 \setminus H \) and \( r(\alpha_n) \in H \). Then, there exists \( 1 \leq j \leq n - 1 \) such that \( r(\alpha_j) \in E^0 \setminus H \) and \( r(\alpha_{j+1}) \in H \). Thus, \( \alpha = (\alpha_1, \ldots, \alpha_{j+1})(\alpha_{j+2}, \ldots, \alpha_n) \), where \( \beta = (\alpha_1, \ldots, \alpha_{j+1}) \in F_E(H) \). Hence, \( \alpha = \phi(\beta)\phi(\alpha_{j+2}) \cdots \phi(\alpha_n) \).

Analogously it can be proved that \( \alpha^* \in \text{Im} \phi \).

Finally, if \( 0 \neq \text{Ker}(\phi) \), then \( \text{Ker}(\phi) \cap (HE)^0 \neq \emptyset \) by [2, Proposition 6] and Lemma 5.1(2), contradicting the definition of \( \phi \). \( \square \)

Note that the isomorphism above is not \( \mathbb{Z} \)-graded because while \( \overline{\alpha} \) has degree 1 in \( HE \) for every \( \alpha \in F_E(H) \), \( \phi(\overline{\alpha}) = \alpha \) has not necessarily degree 1.

Lemma 5.3. Let \( E \) be a graph satisfying condition (K). Then:

1. If \( J \triangleleft I \triangleleft L(E) \), then \( J \triangleleft L(E) \).
2. In particular, if \( H \in \mathcal{H}_E \) and \( J \triangleleft I(H) \), then there exists \( X \in \mathcal{H}_E \) such that \( X \subset H \) and \( J = I(X) \).

Proof. (1). By Proposition 3.3 \( I = I(H) \) for \( H = I \cap E^0 \in \mathcal{H}_E \), and by Proposition 5.2 (and Remark 3.1(1)) \( I(H) \) is isomorphic to the Leavitt path algebra \( L(\mu E) \) and therefore it has a set of local units. Take \( x \in J \) and \( z \in L(E) \), then there exits \( y \in I \) such that \( x = xy = yx \). Now \( xz = (zy)x \in IJ \subseteq J \) and analogously \( xz \in J \).

(2). Again Proposition 3.3 gives that \( J = I(X) \) for \( X = J \cap E^0 \), and therefore \( X = J \cap E^0 \subseteq I \cap E^0 = H \). \( \square \)

Proposition 5.4. Let \( E \) be a graph satisfying condition (K), let

\[ X_0 = \{ v \in E^0 \mid \forall e \neq f \in E^1 \text{ with } s(e) = s(f) = v, r(e) \geq v, r(f) \geq v \}, \]

and let \( X \) be the hereditary saturated closure of \( X_0 \). If \( L(E) \) has no unital purely infinite simple quotients, then neither has \( I(X) \).

Proof. We will suppose that \( X_0 \neq \emptyset \), because otherwise there is nothing to prove.

Case 1. We will begin by proving that if \( L(E) \) has no unital purely infinite simple quotients, then \( I(X) \) cannot be a unital purely infinite simple ring. Suppose that this statement is false. By Lemma 5.2 and Remark 3.1(1), \( I(X) \cong L(XE) \), thus, since \( I(X) \) is unital, \( XE \) is a finite graph; in particular, both \( X \) and \( FE(X) \) are finite, and so are

\[ X_1 = \{ v \in E^0 \mid v = s(\alpha_i) \text{ for some } \alpha = (\alpha_1, \ldots, \alpha_n) \in F_E(X) \} \]

and \( Y = X \cup X_1 \). We claim that \( K = E^0 \setminus Y \) belongs to \( \mathcal{H}_E \). Let \( v \in K \), \( w \in E^0 \), \( e \in E^1 \) be such that \( s(e) = v \) and \( r(e) = w \). We want to prove \( w \in K \). Suppose on the contrary that \( w \in Y \). If \( w \in X \), then \( e \in F_E(X) \) and so \( v = s(e) \in X_1 \subseteq Y \), a contradiction, hence \( w \in X_1 \setminus X \). In this case there exists a path \( \alpha = (\alpha_1, \ldots, \alpha_n) \in F_E(X) \) such that \( w = s(\alpha_i) \),
for some \( i \in \{1, \ldots, n - 1\} \). Then \( \beta = (e, \alpha_i, \ldots, \alpha_n) \in F_E(X) \) and \( v = s(\beta) \in X_1 \subseteq Y \), a contradiction. This shows that \( K \) is hereditary. Now we prove that it is saturated. Consider \( v \in E^0 \) and \( \emptyset \neq r(s^{-1}(v)) \subseteq K \). Suppose \( v \notin K \). Then \( v \in X \) or \( v \in X_1 \setminus X \). In the first case, since \( X \) is hereditary, \( \emptyset \neq r(s^{-1}(v)) \subseteq X \), a contradiction. In the second one, there exists \( \alpha = (\alpha_1, \ldots, \alpha_n) \in F_E(X) \) such that \( v = s(\alpha_i) \) for some \( i \in \{1, \ldots, n - 1\} \). Then \( r(\alpha_i) \in r(s^{-1}(v)) \subseteq K \), a contradiction because \( r(\alpha_i) \in Y \), by the definition of \( Y \).

The following step consists of showing that \( L(E/K) \), which is isomorphic to \( L(E)/I(K) \) by Lemma 2.3, is a unital purely infinite simple ring. First note that \( (E/K)^0 = Y \) is finite and therefore \( L(E/H) \) is a unital ring.

Now, since \( X \) is finite, \( L(E_X) \) is unital. As \( L(E_X) \) is Morita equivalent to the unital purely infinite simple ring \( I(X) \) by Lemma 2.4, \( L(E_X) \) is purely infinite simple. By [2] Proposition 10, \( E_X \) is cofinal, satisfies condition (L), and every vertex in \( E_X^0 \) connects to a cycle. As \( E \) satisfies condition (K), so does \( E/K \) by Lemma 3.2, whence \( E/K \) satisfies condition (L). Observe that \( E/K \) contains at least a cycle; moreover, since every vertex in \( F_E(X) \) connects to a vertex in \( X \), then every vertex in \( E/K \) connects to a cycle. Finally, notice that \( E/K \) has no sinks, as otherwise, since any sink would be in \( X \) because \( (E/K)^0 = X \cup X_1 \) and \( X_1 \setminus X \) clearly does not have sinks, \( xE \) would have a sink, which is not possible because \( L(xE) \cong I(X) \) is a unital purely infinite simple ring. Then, \( (E/K)^{\leq \infty} = (E/K)^{\infty} \). Hence, if \( v \in (E/K)^0 \) and \( \alpha \) is in \( (E/K)^{\infty} \), then \( \text{card}(\alpha^0) < \infty \) because \( Y \) is finite, so \( \alpha \) contains a cycle \( \beta \) (note that for a cycle contained in \( E/K \), \( \beta^0 \cap X_0 \neq \emptyset \) because \( E \) satisfies condition (K), and then by hereditariness \( \beta^0 \subseteq X \), thus \( E/K \) is cofinal. By [2] Theorem 11 and Lemma 2.8 \( L(E/K) \) is a unital purely infinite simple ring, a contradiction.

**Case 2.** \( I(X) \) has no unital purely infinite simple quotients. Suppose that \( I(X)/J \) is a unital purely infinite simple ring for some ideal \( J \) of \( I(X) \). By Lemma 3.2 \( E_X \) satisfies condition (K), whence so does \( xE \) by Lemma 3.1 (2). Lemma 3.3 implies that there exists \( H \in \mathcal{H}_E \) such that \( H \subseteq X \) and \( J = I(H) \). By Lemma 2.5 \( L(E)/I(H) \cong L(E/H) \), and by Lemma 3.2 \( E/H \) satisfies condition (K). This isomorphism shows that \( L(E/H) \) has no unital purely infinite simple quotients because neither has \( L(E) \). If \( \Psi \) is the isomorphism in Lemma 2.5 and \( Z_0 = \Psi(X_0) \), then \( Z = \overline{Z_0} = \Psi(X) \) by Lemma 2.3 (4), and in particular \( I(Z) = \Psi(I(X)) \). Thus, by case 1, applied to \( E/H, Z_0 \) and \( Z \), we get a contradiction.

The rest of this section is devoted to characterize the primeness of an ideal of the form \( I(H) \), for \( H \) hereditary and saturated, in terms of the so-called maximal tails.

The following definition is a particular case of that of [9]: Let \( E \) be a graph. A nonempty subset \( M \subseteq E^0 \) is a maximal tail if it satisfies the following properties:

- (MT1) If \( v \in E^0, w \in M \) and \( v \geq w \), then \( v \in M \).
- (MT2) If \( v \in M \) with \( s^{-1}(v) \neq \emptyset \), then there exists \( e \in E^1 \) with \( s(e) = v \) and \( r(e) \in M \).
- (MT3) For every \( v, w \in M \) there exists \( y \in M \) such that \( v \geq y \) and \( w \geq y \).

**Remark 5.5.** Let \( E \) be a graph. If \( J, K \in \mathcal{H}_E \), then \( I(J)I(K) = I(J \cap K) \). To see this, notice that by Remark 2.2 \( I(J) \cap I(K) = I(J \cap K) \). It is clear that \( I(J)I(K) \subseteq I(J \cap K) \). Since every vertex is an idempotent, the reverse inclusion is clear.

Recall that a graded ideal \( I \) of a graded ring \( R \) is said to be graded prime if for every pair of graded ideals \( J, K \) of \( R \) such that \( JK \subseteq I \), it is satisfied that either \( J \subseteq I \) or \( K \subseteq I \). The definition of prime ideal is analogue to the previous one by eliminating the condition of being
graded. It follows by [13] Proposition II.1.4] that for an ordered group (as it is our case), a graded ideal is graded prime if and only if it is prime.

**Proposition 5.6.** Let $E$ be a graph, and let $H \in \mathcal{H}_E$. Then, the following are equivalent:

1. The ideal $I(H)$ is prime.
2. $M = E^0 \setminus H$ is a maximal tail.

**Proof.** (1) ⇒ (2). It is not difficult to see that $M$ satisfies (MT1) and (MT2). Suppose that there exist $v, w \in M$ such that no $y \in M$ satisfies:

$$(*) v \geq y \text{ and } w \geq y.$$  
Fix such $v, w$. We will prove that $\{v\} \cap \{w\} \cap M = \emptyset$. Suppose that this is false. Let $m$ be the smallest number such that $\Lambda_m(T(v)) \cap \{w\} \cap M \neq \emptyset$ and take $y \in \Lambda_m(T(v)) \cap \{w\} \cap M$. If $m > 0$, then $s^{-1}(y) \neq \emptyset$ and $\emptyset \neq r(s^{-1}(y)) \subseteq \Lambda_{m-1}(T(v)) \cap \{w\}$ because $\{w\}$ is hereditary. By the minimality of $m$, $\Lambda_{m-1}(T(v)) \cap \{w\} \cap M = \emptyset$, hence $r(s^{-1}(y)) \subseteq M$. Since $M$ is saturated, this implies $y \in M$, a contradiction. Analogously it can be proved that 0 is the smallest number $n$ such that $T(v) \cap \Lambda_n(T(w)) \cap M \neq \emptyset$, that is, $T(v) \cap T(w) \cap M \neq \emptyset$, but this is a contradiction by $(*)$. Now, $I(v)I(w) = (\text{by Lemma 2.1})I(\{v\})I(\{w\}) = (\text{by Remark 5.5}) I(\{v\} \cap \{w\}) \subseteq (\text{as we have just proved}) I(H)$. By (1), and taking into account Lemma 2.1, this implies $v \in I(H)$ or $w \in I(H)$, a contradiction.

(2) ⇒ (1). Consider two ideals $J_1$ and $J_2$ in $L(E)$ such that $J_1J_2 \subseteq I(H)$. By Remark 2.2 there exist $H_1, H_2 \in \mathcal{H}_E$ such that $J_1 = I(H_1)$ and $J_2 = I(H_2)$. By Remark 5.5 $H_1 \cap H_2 \subseteq H$. If $H_i \not\subseteq H$ for $i = 1, 2$, then there exist $v_i \in H_i \setminus H$ $(i = 1, 2)$. In particular, $v_1, v_2 \in M$, so that there exists $x \in M$ such that $v_i \geq x$ $(i = 1, 2)$. Hence, $x \in H_1 \cap H_2 \subseteq H$, which contradicts $x \in M$. Thus, either $H_1 \subseteq H$ or $H_2 \subseteq H$, and thus either $I(H_1) \subseteq I(H)$ or $I(H_2) \subseteq I(H)$, as desired. □

**Corollary 5.7.** If $E$ is a graph satisfying condition (K), then there is a bijection between maximal tails and prime ideals. In particular, if $E$ has no proper maximal tails, then $L(E)$ is simple.

**Proof.** The first statement is a consequence of Proposition 5.6 and Proposition 3.3. This implies the second statement because the absence of proper maximal tails is equivalent to the absence of nonzero prime ideals. □

6. Stable rank for quasi stable rings

Let $S$ be any unital ring containing an associative ring $R$ as a two-sided ideal. The following definitions can be found in [19]. A column vector $b = (b_i)_{i=1}^n$ is called $R$-unimodular if $b_1 - 1, b_i \in R$ for $i > 1$ and there exist $a_1 - 1, a_i \in R$ $(i > 1)$ such that $\sum_{i=1}^n a_i b_i = 1$. The *stable rank* of $R$ (denoted by $sr(R)$) is the least natural number $m$ for which for any $R$-unimodular vector $b = (b_i)_{i=1}^{m+1}$ there exist $v_i \in R$ such that the vector $(b_i + v_i b_{m+1})_{i=1}^m$ is $R$-unimodular. If such a natural $m$ does not exist we say that the stable rank of $R$ is infinite.

Recall that a ring $R$ is said to be *stable* if $R \cong M_\infty(R)$. In this section, we cover the final step of the proof of Lemma 1.2. To this end, we need to compute the stable rank of some rings with local units whose behaviour is similar to that of stable rings with local units. It is not known if the property we consider should be equivalent to stability of the ring.
Lemma 6.1. Let $R$ be a ring with ascending local unit $\{p_n\}_{n \geq 1}$. If for every $n \geq 1$ there exists $m > n$ such that $p_n \lessdot p_m - p_n$, then $sr(R) \leq 2$.

Proof. Fix $S$ a unital ring containing $R$ as two-sided ideal. Let $a_1, a_2, a_3, b_1, b_2, b_3 \in S$ such that $a_1 - 1, a_2, a_3, b_1 - 1, b_2, b_3 \in R$, while $a_1 b_1 + a_2 b_2 + a_3 b_3 = 1$. By hypothesis, there exists $n \in \mathbb{N}$ such that $a_1 - 1, a_2, a_3, b_1 - 1, b_2, b_3 \in p_n R p_n$. Let $m > n$ such that $p_n \lessdot p_m - p_n$. Then, there exists $g_n \sim p_n, q_n \leq p_m - p_n$. In particular, $q_n p_n = p_n q_n = 0$. Now, there exist $u \in p_n R q_n, v \in q_n R p_n$ such that $uv = p_n, vu = q_n, u = p_n u = u q_n$ and $v = q_n v = v p_n$.

Fix $v_1 = 0, v_2 = u, c_1 = b_1$, and $c_2 = b_2 + v b_3$. Notice that $(a_1 + a_3 v_1) - 1, c_1 - 1, (a_2 + a_3 v_2), c_2 \in R$. Also, $a_3 u v b_3 = a_3 p_n b_3 = a_3 b_3, a_3 u b_2 = a_3 u q_n p_n b_2 = 0$, and $a_2 v b_3 = a_2 p_n q_n v b_3 = 0$. Hence,

$$(a_1 + a_3 v_1) c_1 + (a_2 + a_3 v_2) c_2 = a_1 b_1 + a_2 b_2 + a_3 b_3 = 1.$$ 

Thus, any unimodular 3-row is reducible, whence the result holds.

A monoid $M$ is cancellative if whenever $x + z = y + z$, for $x, y, z \in M$, then $x = y$. And $M$ is said to be unperforated in case for all elements $x, y \in M$ and all positive integers $n$, we have $nx \leq ny$ implies $x \leq y$.

Given an abelian monoid $M$, and an element $x \in M$, we define

$$S(M, x) = \{ f : M \to [0, \infty] \mid \text{additive map such that } f(x) = 1 \}.$$ 

Standard arguments show that, when $M$ is a cancellative monoid, then $S(M, x)$ is nonempty for every nonzero element $x \in M$.

Lemma 6.2. Let $R$ be a nonunital ring with ascending local unit $\{p_n\}_{n \geq 1}$ such that $V(R)$ is cancellative and unperforated, and let $S_R = \{ s : V(R) \to \mathbb{R}^+ \mid \text{morphisms of monoids} \}$. If for every $s \in S_R$, $\sup_{n \geq 1} \{s([p_n])\} = \infty$, then for every $n \geq 1$ there exists $m > n$ such that $p_n \lessdot p_m - p_n$.

Proof. Fix $n \in \mathbb{N}$, and consider $S_n = S(V(R), [p_n])$. For every $t \in S_n$, $\sup_{m \geq 1} t([p_m]) = \infty$. Otherwise, there exists $t \in S_n$ such that $\sup_{n \geq 1} t([p_m]) = \alpha \in \mathbb{R}^+$. Since $\{p_n\}_{n \geq 1}$ is a local unit, we conclude that $\alpha < \infty$ for every $x \in V(R)$, so that $t \in S_R$, contradicting the hypothesis. Thus, the supremum $\sup_{k \geq 1} \tilde{p}_k = \infty$. Since $S_n$ is compact, there exists $m > n$ such that $1 < \tilde{p}_m$, i.e. for every $s \in S_n^\ast$, $s[2[p_n]] < s([p_m])$.

Now, take $t \in S(V(R), [p_m])$. Since $p_n < p_m$, $0 \leq t(2[p_n]) = a \leq 2$. If $a = 0$, then clearly $0 = t(2[p_n]) < t([p_m]) = 1$. If $a \neq 0$, then $t'(-) := a^{-1} \cdot t(-)$ belongs to $S_n$, whence $1 = t'(2[p_n]) < t'([p_m])$ by the argument above. So, $t([p_m]) < t([p_m]) = 1$. Thus, for every $t \in S(V(R), [p_m])$, we have $t(2[p_n]) < t([p_m]) = 1$. By [17, Proposition 3.2], $2p_n \lessdot p_m = p_n + (p_m - p_n)$. Then, since $V(R)$ is cancellative, we get $p_n \lessdot p_m - p_n$, as desired.

Definition 6.3. Let $E$ be a graph. For every $v \in E^0$, we define $L(v) = \{ w \in E^0 \mid w \geq v \}$. We say that $v \in E^0$ is left infinite if $\text{card}(L(v)) = \infty$.

Definitions 6.4. Let $E$ be a graph. A graph trace on $E$ is a function $g : E^0 \to \mathbb{R}^+$ such that, for every $v \in E^0$ with $s^{-1}(v) \neq \emptyset$, $g(v) = \sum_{s(e) = v} g(r(e))$. We define the norm of $g$ to be the (possibly infinite) value $\|g\| = \sum_{v \in E^0} g(v)$. We say that $g$ is bounded if $\|g\| < \infty$. 


Remark 6.5. Let $E$ be a graph, let $E^0 = \{v_i \mid i \geq 1\}$, let $p_n = \sum_{i=1}^{n} v_i$, and let

$$S_E = \{s : V(L(E)) \rightarrow \mathbb{R}^+ \mid \text{morphisms of monoids}\}.$$  

By [6, Theorem 2.5], any element $s \in S_E$ induces a graph trace by the rule $g_s(v) = s([v])$. Moreover, $g_s$ is bounded if and only if $\sup_{n \in \mathbb{N}} \{s(p_n)\} < \infty$.

Conversely, by [6, Theorem 2.5] and [6, Lemma 3.3], if $g$ is a graph trace on $E$, and $v, w \in E^0$ with $[v] = [w] \in V(L(E))$, then $g(v) = g(w)$. So, the rule $s_g([v]) = g(v)$ is well-defined and extends by additivity to an element $s_g \in S_E$. Certainly, $g$ is bounded if and only if $\sup_{n \in \mathbb{N}} \{s_g([p_n])\} < \infty$.

Next result in the context of $C^*$-algebras is [18] Lemma 3.8. Here, we follow a different approach to prove it.

Lemma 6.6. Let $E$ be a graph, let $H \in \mathcal{H}_E$, and let $\pi : L(E) \rightarrow L(E)/I(H)$ be the natural projection map. If $e \in L(E)$ is an idempotent, $W \subseteq E^0 \setminus H$ is a finite set, and $\ell(e) \leq \sum_{w \in W} \pi(w)$ in $L(E)/I(H)$, then there exists a finite set $X \subseteq H$ such that $e \lesssim \sum_{w \in W} w + \sum_{x \in X} x$.

Proof. By [6, Theorem 2.5 and Lemma 5.6], $V(L(E))/V(I(H)) \cong V(L(E/H))$. Thus, $[\pi(e)] \leq \sum_{w \in W} [\pi(w)] \in V(L(E/H))$ implies that there exist $a, b \in V(I(H))$ such that $[e] + a \leq \sum_{w \in W} [w] + b \in V(L(E))$. Since $V(I(H)) = \{[v] \mid v \in H\}$, there exists a finite set $X \subseteq H$ such that $b \lesssim \sum_{x \in X} x$. Then, $[e] \leq \sum_{w \in W} [w] + \sum_{x \in X} [x]$, as desired.

Proposition 6.7. (c.f. [18] Theorem 3.2] Let $E$ be a graph. If every vertex of $E$ lying on a closed simple path is left infinite and $E$ has no nonzero bounded graph traces, then for every finite set $V \subseteq E^0$ there exists a finite set $W \subseteq E^0$ with $V \cap W = \emptyset$ and $\sum_{v \in V} v \lesssim \sum_{w \in W} w$.

Proof. The proof of this result corresponds to $(d) \Rightarrow (e) \Rightarrow (f)$ of [18] Theorem 3.2, with suitable adaptation of the arguments except for the Case II in $(d) \Rightarrow (e)$, in which the way to prove the following statement is different: If $F \subseteq E^0$ is a finite set, and $n = \max \{i \in \mathbb{N} \mid w_i \in F\}$, there exists $m > n$ such that $p_n \lesssim p_m - p_n$.

Suppose then $v \notin \overline{P}$. List the vertices of $E/\overline{P} = \{w_i \mid i \geq 1\}$, in such a way that $w_1 = v$. Let $\pi : L(E) \rightarrow L(E)/I(H)$ be the natural projection map. For every $n \geq 1$, set $p_n = \sum_{i=1}^{n} \pi(w_i)$. Clearly, $\{p_n\}_{n \geq 1}$ is an ascending local unit for $L(E/\overline{P})$. Since every vertex on a closed simple path is left infinite, no vertex on $E/\overline{P}$ lies on a closed simple path. Thus, $E/\overline{P}$ is acyclic, whence $L(E/\overline{P})$ is locally matricial by Corollary 3.6. In particular, $V(E/\overline{P})$ is cancellative and unperforated. Moreover, since $E$ has no nonzero bounded graph traces, neither has $E/\overline{P}$. Otherwise, by Remark 6.5 there exists a monoid morphism $s : V(L(E/\overline{P})) \rightarrow \mathbb{R}^+$ with $\sup_{n \in \mathbb{N}} \{s([p_n])\} < \infty$. Hence, $s$ induces a monoid morphism $s \circ \pi : V(L(E)) \rightarrow \mathbb{R}^+$ such that $\sum_{v \in E^0} (s \circ \pi)([v]) = \sum_{v \in E^0} \pi(v) s([v]) < \infty$, consequently there exists a bounded graph trace on $E$, contradicting the assumption. By Remark 6.5 and Lemma 6.2, for every $n \geq 1$ there exists $m > n$ such that $p_n \lesssim p_m - p_n$.

Corollary 6.8. Let $E$ be a graph. If every vertex of $E$ lying on a closed simple path is left infinite and $E$ has no nonzero bounded graph traces, then $sr(L(E)) \leq 2$. 


Proof. Let $E^0 = \{v_i \mid i \geq 1\}$, and for each $n \in \mathbb{N}$ consider $p_n = \sum_{i=1}^{n} v_i$. Then, $\{p_n\}_{n \geq 1}$ is an ascending local unit for $L(E)$. Fix $n \geq 1$ and set $V = \{v_1, \ldots, v_n\}$. By Proposition 6.7, there exists a finite subset $W \subseteq E^0$ such that $V \cap W = \emptyset$ and $p_n = \sum_{v \in V} v \lesssim \sum_{w \in W} w$. If $m$ is the largest subindex of $w \in W$, notice that $m > n$ and that $\sum_{w \in W} w \leq p_m - p_n$. Hence, the result holds because $L(E)$ satisfies the hypotheses of Lemma 6.1. □

7. Stable rank for exchange Leavitt path algebras

In this section, we characterize the stable rank of exchange Leavitt path algebras in terms of intrinsic properties of the graph.

Lemma 7.1. Let $E$ be an acyclic graph. Then, the stable rank of $L(E)$ is 1.

Proof. If $E$ is finite, then $L(E)$ is a $K$-matricial algebra by Corollary 3.1 whence $sr(L(E)) = 1$. Now suppose that $E$ is infinite. By Corollary 3.1, there exists a family $\{X_n\}_{n \geq 0}$ of finite subgraphs of $E$ such that $L(E) \cong \varprojlim L(X_n)$. By the definitions of direct limit and stable rank,

$$(*) \quad sr(L(E)) \leq \liminf_{n \to \infty} sr(L(X_n)).$$

If $E$ is acyclic, then so are the $X_n$’s, whence $sr(L(E)) = 1$ by the result above and $(*)$. □

Lemma 7.2. Let $E$ be a graph satisfying condition (K). Then, $L(E)$ has a unital purely infinite simple quotient if and only if there exists $H \in \mathcal{H}_E$ such that the quotient graph $E/H$ is nonempty, finite, cofinal and contains no sinks.

Proof. First, suppose that $J$ is an ideal of $L(E)$ such that $L(E)/J$ is a unital purely infinite simple ring. By Proposition 3.3, there exists $H \in \mathcal{H}_E$ such that $J = I(H)$. By Lemma 2.3 (1), $L(E)/J \cong L(E/H)$. Moreover, $E/H$ satisfies condition (K) by Lemma 3.2. Hence, since $L(E/H)$ is unital, $E/H$ is finite. Since $L(E)/J$ is purely infinite simple, $E/H$ is cofinal and every vertex connects to a closed simple path by [2, Theorem 11], whence $E/H$ has no sinks.

Conversely, suppose that there exists $H \in \mathcal{H}_E$ such that the quotient graph $E/H$ is nonempty, finite, cofinal and contains no sinks. Thus, it contains a closed simple path and every vertex connects to a closed simple path. Then, since $E/H$ satisfies condition (K) by Lemma 3.2, $L(E/H)$ is unital, purely infinite and simple by [2, Theorem 11] and 2.8. By Lemma 2.3 (1), $L(E)/M \cong L(E/H)$ and the proof is complete. □

Corollary 7.3. Let $E$ be a graph satisfying condition (K). If there exists $H \in \mathcal{H}_E$ such that the quotient graph $E/H$ is nonempty, finite, cofinal and contains no sinks, then the stable rank of $L(E)$ is $\infty$.

Proof. By Lemma 7.2, there exists a maximal ideal $M \triangleleft L(E)$ such that $L(E)/M$ is a unital purely infinite simple ring. Thus, $sr(L(E)/M) = \infty$ (see [7]). Since $sr(L(E)/M) \leq sr(L(E))$ (see [19, Theorem 4]), we conclude that $sr(L(E)) = \infty$. □

The proof of the following result closely follows that of [10, Lemma 3.2].

Lemma 7.4. Let $E$ be a non acyclic graph satisfying condition (K). If $L(E)$ does not have any unital purely infinite simple quotient, then there exists a graded ideal $J \triangleleft L(E)$ with $sr(J) = 2$ such that $L(E)/J$ is a locally matricial $K$-algebra.
Proof. Let 

$$X_0 = \{v \in E^0 \mid \exists e \neq f \in E^1 \text{ with } s(e) = s(f) = v, r(e) \geq v, r(f) \geq v\},$$

and let $X$ be the hereditary saturated closure of $X_0$. Consider $J = I(X)$, and notice that $L(E)/J \cong L(E/X)$ by Lemma 2.3 (1). Moreover, since $E$ satisfies condition (K), then so does $E/X$ by Lemma 3.2. If there is a closed simple path $\alpha$ in $E/X$, then every $v \in \alpha^0$ satisfy $\text{card}(\text{CSP}_{E/X}(v)) \geq 2$, therefore, there exists a vertex $v_0 \in \alpha^0 \cap X_0 \subseteq X$, contradicting the assumption. So, $E/X$ contains no closed simple paths, whence it is an acyclic graph, and thus $L(E)/J$ is locally matricial by Corollary 3.6.

Now, by Remark 3.1 (1), Lemma 3.2 and Lemma 5.2, $J \cong L(\chi E)$. We will show that every vertex lying in a closed simple path of $\chi E$ is left infinite, and that $\chi E$ has no nonzero bounded graph traces, as a way of contradiction.

Suppose that there exists a closed simple path $\alpha$ in $\chi E$ such that the set $Y$ of vertices of $\chi E$ connecting to the vertices of $\alpha^0$ is finite. It is not difficult to see that $\alpha^0 \cup Y$ is a maximal tail in $\chi E$. Let $M$ be a maximal tail of the smallest cardinal contained in $\alpha^0 \cup Y$. Observe that $M \cap X_0 \neq \emptyset$; otherwise $X \setminus M$, which is a hereditary saturated proper subset of $X$, would contain $X_0$, which is impossible. Denote by $\tilde{M}$ the quotient graph of $\chi E$ by the hereditary saturated set $H = \chi E^0 \setminus M$, i.e. $\tilde{M} = \chi E/H$. Then, since $M$ is finite, $L(\tilde{M})$ is a unital ring. As $E$ satisfies condition (K), so does $\chi E$ (by Lemma 3.2 and Lemma 5.1 (2)) and thus $\tilde{M}$ (by Lemma 3.2 again). Then, since $M$ does not contain smaller maximal tails, $L(\tilde{M})$ is simple by Corollary 5.7. As $M \cap X_0 \neq \emptyset$, $\tilde{M}$ is non cyclic. Thus, $L(\tilde{M}) \cong L(\chi E)/I(H)$ (by Lemma 2.3 (1)) is a unital purely infinite simple ring by [1, Theorem 3.11], Lemma 2.8 and [2, Theorem 11]. By Proposition 5.4 $L(E)$ has a unital purely infinite simple quotient, contradicting the hypothesis. Hence, every vertex lying in a closed simple path in $\chi E$ is left infinite.

Now, suppose that there exists a nonzero bounded graph trace $g$ on $\chi E$. By Remark 6.5 $s_g : V(L(\chi E)) \to \mathbb{R}^+$ is a nonzero morphism such that $\sum_{v \in \chi E^0} s_g([v]) < \infty$. But for any $v \in X_0$ we have $2s_g([v]) \leq s_g([v])$, so that $g(v) = 0$. Hence, $X_0 \subseteq \{w \in \chi E^0 \mid g(w) = 0\}$, which is a hereditary saturated subset of $\chi E$ by [18, Lemma 3.7]. Thus, since $\chi E = \chi E/X_0$ we conclude that $g \equiv 0$, contradicting the assumption. Hence, there exist no nonzero bounded graph traces on $\chi E$.

Thus, $\text{sr}(J) = \text{sr}(L(\chi E)) \leq 2$ by Corollary 6.8. Since every vertex in $X_0$ is properly infinite as an idempotent of $L(\chi E)$, $\text{sr}(L(\chi E)) \neq 1$, so that $\text{sr}(J) = 2$, as desired. \qed

Corollary 7.5. Let $E$ be a non acyclic graph satisfying condition (K). If $L(E)$ does not have any unital purely infinite simple quotient, then $\text{sr}(L(E)) = 2$.

Proof. Consider $J$ the graded ideal obtained in the previous Lemma. By [19, Theorem 4],

$$2 = \max\{\text{sr}(J), \text{sr}(L(E)/J)\} \leq \text{sr}(L(E)) \leq \max\{\text{sr}(J), \text{sr}(L(E)/J) + 1\} = 2.$$ 

Then, $\text{sr}(L(E)) = 2$, as desired. \qed

Theorem 7.6. Let $E$ be a graph satisfying condition (K). Then, the values of the stable rank of $L(E)$ are:

(1) $\text{sr}(L(E)) = 1$ if $E$ is acyclic.
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(2) \(sr(L(E)) = \infty\) if there exists \(H \in \mathcal{H}_E\) such that the quotient graph \(E/H\) is nonempty, finite, cofinal and contains no sinks.

(3) \(sr(L(E)) = 2\) otherwise.

Proof. Statement (1) holds by Lemma 7.1, and statement (2) by Corollary 7.3. If \(E\) is non-acyclic and it does not exist \(H \in \mathcal{H}_E\) such that the quotient graph \(E/H\) is nonempty, finite, cofinal and contains no sinks, then \(L(E)\) does not have any unital purely infinite simple quotient by Lemma 7.2. Hence, statement (3) holds by Corollary 7.5. \(\square\)

ACKNOWLEDGMENTS

Part of this work was done during a visit of the second author to the Department of Mathematics of the University of Colorado at Colorado Springs (U.S.A.) and to the Centre de Recerca Matemàtica (U.A.B., Spain). The second author want to thanks both host centers for their warm hospitality, and in particular to Gene Abrams and Pere Ara for valuable discussions on the topics of this paper.

REFERENCES

[1] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, J. Algebra (to appear).
[2] G. Abrams, G. Aranda Pino, Purely infinite simple Leavitt path algebras, J. Pure Appl. Algebra (to appear).
[3] P. Ara, Extensions of Exchange Rings, J. Algebra 197 (1997), 409–423.
[4] P. Ara, The exchange property for purely infinite simple rings, Proc. Amer. Math. Soc. 132(9) (2004), 2543–2547.
[5] P. Ara, M. Gómez Lozano, M. Siles Molina, Local rings of exchange rings, Comm. Algebra 26 (1998), 4191–4205.
[6] P. Ara, M.A. Moreno, E. Pardo, Nonstable K-Theory for graph algebras, Algebra Rep. Th. (to appear).
[7] P. Ara, G. K. Pedersen, F. Perera, An Infinite Analog of Rings with Stable Rank One, J. Algebra 230 (2000), 608–655.
[8] H. Bass, Introduction to some methods of the algebraic K-theory, Regional Conference Series in Mathematics 21, Amer. Math. Soc., Providence, (1974).
[9] T. Bates, J. H. Hong, I. Raeburn, W. Szymański, The ideal structure of the \(C^*\)-algebras of infinite graphs, Illinois J. Math 46 (2002), 1159–1176.
[10] K. Deicke, J.H. Hong, W. Szymański, Stable rank of graph algebras. Type I graph algebras and their limits, Indiana Univ. Math. J. 52(4) (2003), 963–979.
[11] J. L. García, J. J. Simón, Morita equivalence for idempotent rings, J. Pure Appl. Algebra 76 (1991), 39–56.
[12] J.H. Hong, W. Szymański, The primitive ideal space of the \(C^*\)-algebra of infinite graphs, arXiv:math.OA/0211162.
[13] J.A. Jeong, G.H. Park, Graph \(C^*\)-algebras with real rank zero, J. Funct. Anal. 188 (2002), 216–226.
[14] A. Kumjian, D. Pask, I. Raeburn, Cuntz-Krieger algebras of directed graphs, Pacific J. Math. 184 (1998), 161–174.
[15] C. Năstăsescu, F. van Oystaeyen, Graded ring theory, North-Holland, Amsterdam (1982).
[16] I. Raeburn, Graph algebras. CBMS Regional Conference Series in Mathematics, 103, Amer. Math. Soc., Providence, (2005).
[17] M. Rørdam, The stable and the real rank of \(\mathcal{Z}\)-absorbing \(C^*\)-algebras, International J. Math. 15(10) (2004), 1065–1084.
[18] M. Tomforde, Stability of \(C^*\)-algebras associated to graphs, Proc. Amer. Math. Soc. 132(6) (2004), 1787–1795.
[19] L.N. Vasershtein, Stable rank of rings and dimensionality of topological spaces, *Func. Anal. Applic.* 5 (1971), 102–110.

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