Abstract. We study intersection cohomology of moduli spaces of semistable vector bundles on a complex algebraic surface. Our main result relates intersection Poincaré polynomials of the moduli spaces to Donaldson-Thomas invariants of the surface. In support of this result, we compute explicitly intersection Poincaré polynomials for sheaves with rank two and three on ruled surfaces.

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1. Introduction

Let $S$ be a complex projective surface and $J$ be a polarization on $S$. For any $\gamma \in H^{\text{even}}(S)$, let $M_\gamma$ (resp. $\mathfrak{M}_\gamma$) denote the moduli space (resp. stack) of Gieseker $J$-semistable sheaves having Chern character $\gamma$. The moduli space $M_\gamma$ is a projective, possibly singular, variety. In this paper we will prove a relation between intersection cohomologies of $M_\gamma$ and invariants of $\mathfrak{M}_\gamma$ under certain technical conditions on the polarization.

To state our result more precisely, let $P(X) \in \mathbb{Z}[y]$ be the (motivic) Poincaré polynomial for an algebraic variety $X$ (see §2.6). It is defined for a smooth projective $X$ by the formula

$$P(X) = \sum_{n \geq 0} \dim H^n(X)(-y)^n,$$

and then is extended to arbitrary $X$ by additivity with respect to complements. We can represent the stack $\mathfrak{M}_\gamma$ as a global quotient $R_\gamma/G_\gamma$, where $G_\gamma$ is a general linear group [11, §4.3], and define

$$P(\mathfrak{M}_\gamma) = P(R_\gamma)/P(G_\gamma) \in \mathbb{Q}(y).$$
Let the polarization $J$ and surface $S$ be such that the following two conditions are satisfied:
(A) $J \cdot K_S < 0$, implying that the category of semistable sheaves with a fixed reduced Hilbert polynomial (see §2.2) has a vanishing second Ext.
(B) $J$ is generic (see Remark 2.1), implying that if $E$, $F$ have equal reduced Hilbert polynomials then $\chi(E, F) = \chi(F, E)$.

Under these conditions, generating functions of $P(\mathfrak{M}_\gamma)$ have been determined for sheaves with small rank and $S$ a rational or ruled surface [5, 16, 17, 18, 24, 31, 32]. Through the Hitchin-Kobayashi correspondence [15] and Donaldson-Uhlenbeck-Yau theorem [3, 29], these generating functions are of interest for the study of Yang-Mills theories. In particular, the partition function of topologically twisted $\mathcal{N} = 4$ supersymmetric Yang-Mills theory localizes on Hermitian-Yang-Mills connections [30], and it equals the generating function of Euler numbers $\chi(M_\gamma)$ of $M_\gamma$ if $\gamma$ is indivisible. In such cases, semi-stability implies stability and $P(\mathfrak{M}_\gamma)$ is related to $P(M_\gamma)$ by

$$P(\mathfrak{M}_\gamma) = \frac{P(M_\gamma)}{y^2 - 1}.$$  

In the following, we consider arbitrary $\gamma$. To state our main result, recall that for any algebraic variety $X$ the intersection Poincaré polynomial $IP(X)$ is defined by

$$IP(X) = \sum_n \dim \text{IH}^n(X)(-y)^n,$$

where $\text{IH}^n(X)$ are intersection cohomologies of $X$. Our main result relates the virtual Poincaré functions $P(\mathfrak{M}_\gamma)$ and $IP(M_{\gamma})$.

**Theorem 1.1.** Let $J$ satisfy the conditions (A) and (B) above. Then

$$1 + \sum_{pJ(\gamma) = p} (-y)^{-\dim \mathfrak{M}_\gamma} P(\mathfrak{M}_\gamma)z^\gamma = \text{Exp} \left( \frac{\sum_{pJ(\gamma) = p} (-y)^{-\dim M_\gamma} IP(M_\gamma)z^\gamma}{y - 1} \right)$$

where $\dim \mathfrak{M}_\gamma = \dim M_\gamma - 1 = -\chi(\gamma, \gamma)$, the sums run over all $\gamma$ with a fixed reduced Hilbert polynomial $p$ and Exp is a plethystic exponential (2.22) defined for $f(y, z) = \sum f_r(y)z^r$ as

$$\text{Exp}(f) = \exp \left( \sum_{n \geq 1} \frac{1}{n} f(y^n, z^n) \right).$$

The conditions (A) and (B) are in particular satisfied for the projective plane $\mathbb{P}^2$. For rank 2 sheaves on $\mathbb{P}^2$, $IP(M_{\gamma})$ were determined by Yoshioka [32, Remark 4.6] extending work of Kirwan on moduli spaces of rank 2 vector bundles on Riemann surfaces [14]. More recently, $IP(M_{\gamma})$ were determined for rank 3 and 4 sheaves on $\mathbb{P}^2$ [17, 18]. In further support of Theorem 1.1, Section 5 provides $IP(M_{\gamma})$ for rank 2 and 3 sheaves on ruled surfaces. For two ruled surfaces, we show that for a specific non-generic polarization, $J = -K_S$, Theorem 1.1 continues to hold. Note that the conditions (A) and (B) exclude the case of K3 surfaces which have a 2-Calabi-Yau category of coherent sheaves. Although our approach fails in this case, we expect that a similar relation between intersection cohomologies and stack invariants should be true.

Let us now reformulate the above result on the level of mixed Hodge structures and $E$-polynomials. For any algebraic variety $X$, we can consider the cohomology with compact support $H^*_c(X, \mathbb{Q})$ as an element in $K_0$(MHS) and define the $E$-polynomial of $X$ by taking the Hodge-Euler polynomial of $H^*_c(X, \mathbb{Q})$ (2.32)

$$E(X) = E(H^*_c(X, \mathbb{Q})) = \sum_{p+q} (-1)^q h^{p,q}(H^*_c(X, \mathbb{Q}))u^pv^q \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}].$$

Furthermore, define

$$E(\mathfrak{M}_\gamma) = E(R_\gamma)/E(G_\gamma),$$

$\mathbb{L} = E(A^1) = E(\mathbb{Q}(-1)[-2]) = uv.$

The (motivic) Poincaré polynomial is equal to $P(X, y) = E(X, y, y).$
On the other hand, for any quasi-projective variety \( X \) of dimension \( d \), let \( \text{IC}_X \) be its intersection complex \([28, \S 1.13]\) (considered as a pure Hodge module of weight \( d \)) and let

\[
\text{IH}^*(X) = H_c^*(X, \text{IC}_X)[-d]
\]

be its intersection cohomology (considered as an object in \( D^b \text{MHS} \) or as an element in \( K_0(\text{MHS}) \)). Define the intersection \( E \)-polynomial of \( X \) by taking the Hodge-Euler polynomial \((2.32)\)

\[
\text{IE}(X) = (\text{IH}^*(X)) = \sum_{p,q,n} (-1)^n h^{p,q}(\text{IH}^n(X))u^p v^q.
\]

Note that if \( X \) is projective, then \( \text{IH}^*(X) \) is pure of weight zero, and the intersection Poincaré polynomial is equal to \( \text{IP}(X, y) = \text{IE}(X, y, y) \).

**Theorem 1.2.** Let \( J \) satisfy the conditions (A) and (B) above. Then

\[
1 + \sum_{p, j(\gamma) = p} \mathbb{L}^{-\frac{1}{2}\dim \mathfrak{M}_\gamma} E(\mathfrak{M}_\gamma) z^\gamma = \text{Exp} \left( \sum_{p, j(\gamma) = p} \mathbb{L}^{-\frac{1}{2}\dim M_\gamma} \frac{\text{IE}(M_\gamma) z^\gamma}{\mathbb{L}^2 - \mathbb{L}^{-\frac{1}{2}}} \right),
\]

where \( \mathbb{L}^2 = -(uv)^{\frac{1}{2}} \).

Note that if \( \gamma \) is indivisible, then \( M_\gamma \) consists of stable sheaves and is smooth. In this case we obtain from the theorem

\[
E(\mathfrak{M}_\gamma) = \frac{\text{IE}(M_\gamma)}{\mathbb{L} - 1}
\]

which is straightforward as \( \text{IE}(M_\gamma) = E(M_\gamma) \) and all stabilizers of objects in \( \mathfrak{M}_\gamma \) are isomorphic to \( \mathbb{C}^* \).

The above results can be also formulated in terms of Donaldson-Thomas invariants \( \Omega_\gamma = \Omega_\gamma(u,v) \) defined by the formula (see \((4.4, 4.6)\) for an explicit expression)

\[
(1.1) \quad 1 + \sum_{p, j(\gamma) = p} \mathbb{L}^{-\dim \mathfrak{M}_\gamma} E(\mathfrak{M}_\gamma) z^\gamma = \text{Exp} \left( \sum_{p, j(\gamma) = p} \frac{\Omega_\gamma z^\gamma}{\mathbb{L}^2 - \mathbb{L}^{-\frac{1}{2}}} \right)
\]

Then the above theorem can be simply written in the form

\[
(1.2) \quad \Omega_\gamma = \mathbb{L}^{-\frac{1}{2}\dim M_\gamma} \text{IE}(M_\gamma).
\]

The idea of the proof of the above theorems goes back to \([22]\) (see also \([25, 21]\)). We introduce a smooth moduli space \( M^f_\gamma \) of framed vector bundles (see \(\S 3.2\)) which is equipped with a projective map \( \pi : M^f_\gamma \to M_\gamma \). Then we analyze the intersection complex of \( M_\gamma \) by studying the pushforward with respect to \( \pi \) of the intersection complex on \( M^f_\gamma \).

One may wonder if a similar result can be proved for the moduli spaces of Mumford (also called \( \mu \)-) semistable sheaves on a surface. On the one hand, there are no technical difficulties. If \( J \cdot K_S < 0 \) then the category of Mumford semistable sheaves with a fixed slope has a vanishing second Ext (see Lemma 3.1). And if a polarization is generic then sheaves having the same slope satisfy \( \chi(E, F) = \chi(F, E) \) (see Remark 2.1). On the other hand, one can show that although the moduli spaces (and stacks) of Gieseker and Mumford semistable sheaves are different in general, their Donaldson-Thomas invariants coincide (see Theorem 4.1). Therefore we can not expect to get any new invariants from the Mumford semistable sheaves. The reason for this phenomenon is that the moduli space \( M^{\mu, s}_\gamma \) of Mumford stable sheaves is not dense in the moduli space \( M^\mu_\gamma \) of Mumford semistable sheaves in general. Indeed, \( M^{\mu, s}_\gamma \) is contained in the moduli space \( M_\gamma \) of Gieseker semistable sheaves which is projective and therefore closed in \( M^\mu_\gamma \).

The paper is organized as follows. Section 2 reviews aspects of sheaves on surfaces, \( \lambda \)-rings and mixed Hodge structures. Section 3 proves the main results, Theorems 1.1 and 1.2. Section 4 discusses properties of Donaldson-Thomas invariants and their generating functions. This is applied in Section 5 to determine \( \text{IP}(M_\gamma) \) explicitly for sheaves with ranks 2 or 3 on a few ruled surfaces.
2. Preliminaries

Let $S$ be an algebraic surface with the canonical class $K_S$. Given a coherent sheaf $F$ on $S$, let $c_1$ and $c_2$ be its first and second Chern classes respectively and let $\gamma = (r, \gamma_1, \gamma_2) = \chi F$ be its Chern character (so that $\gamma_1 = c_1$ and $\gamma_2 = \frac{1}{2}c_1^2 - c_2$).

2.1. Hirzebruch-Riemann-Roch theorem. By the Hirzebruch-Riemann-Roch theorem, we have

$$
\chi(F) = \chi(S, F) = \int_S \gamma \cdot \text{td}(S) = \gamma_2 - \frac{1}{2}K_S \gamma_1 + \chi(O_S)r,
$$

where the Todd class $\text{td}(S)$ is defined by

$$
\text{td}(S) = 1 - \frac{1}{2}K_S + \chi(O_S) = 1 - \frac{1}{2}K_S + \frac{1}{12}(K_S^2 + e(S)).
$$

Applying this to $\text{Hom}(F, F') = F^* \otimes F'$ (assuming that $F$ is a vector bundle), we obtain

$$
\chi(\gamma, \gamma') := \chi(F, F') = (\gamma_2' r + \gamma_2 r' - \gamma_1 \gamma_1') + \frac{1}{2}(\gamma_1 r' - \gamma_1' r)K_S + \chi(O_S)r r'.
$$

This implies

$$
(\gamma, \gamma') := \chi(\gamma, \gamma') - \chi(\gamma', \gamma) = (\gamma_1 r' - \gamma_1' r)K_S,
$$

$$
\chi(\gamma, \gamma) = 2r \gamma_2 - \gamma_1^2 + \chi(O_S)r^2.
$$

Note that if $(r, \gamma_1)$ and $(r', \gamma_1')$ are proportional, then $\langle \gamma, \gamma' \rangle = 0$.

2.2. Semistability. Let $J$ be a polarizing line bundle on $S$. We denote its first Chern class also by $J$. We let furthermore

$$
\mu(F) = \frac{\gamma_1}{r}, \quad \mu_J(F) = \mu(F) \cdot J = \frac{\gamma_1 \cdot J}{r}
$$

and let the reduced Hilbert polynomial be $p_J(F, n) = \chi(F \otimes J^n)/r(F)$, or in terms of the Chern character

$$
p_J(\gamma, n) = \frac{J^2}{2} n^2 + \left(\frac{\gamma_1 \cdot J}{r} - \frac{1}{2}K_S \cdot J\right) n + \left(\frac{\gamma_2 - \frac{1}{2}K_S \cdot \gamma_1}{r} + \chi(O_S)\right).
$$

Remark 2.1. In the following, we let $J$ be any element of the ample cone $C(S) \subset H^2(S, \mathbb{R})$ rather than of $C(S) \cap H^2(S, \mathbb{Z})$. Note that if $\gamma$ and $\gamma'$ are proportional, then $p_J(\gamma) = p_J(\gamma')$. Conversely, assume that $J$ is generic in the sense that $J \cdot \gamma_1 = J \cdot \gamma_1'$ implies $\gamma_1 = \gamma_1'$ for elements in $H^2(S, \mathbb{Z})$. Equivalently, $J \cdot \gamma_1 = 0$ implies $\gamma_1 = 0$ for $\gamma_1 \in H^2(S, \mathbb{Z})$. If $\mu_J(\gamma) = \mu_J(\gamma')$, then $(r, \gamma_1)$ and $(r', \gamma_1')$ are proportional. We conclude from (2.3) that $\langle \gamma, \gamma' \rangle = 0$ in this case. If $p_J(\gamma) = p_J(\gamma')$, then $\gamma_1/r = \gamma_1'/r'$ and this implies that $\gamma_2/r = \gamma_2'/r'$. We conclude that $\gamma$ and $\gamma'$ are proportional. Note that for any $F \neq 0$, we have $\chi(F \otimes J^n) > 0$ for $n \gg 0$. Therefore, for $\chi(F) = (r, \gamma_1, \gamma_2)$, we have either $r > 0$ or $r = 0$ and $\gamma_1 \cdot J > 0$ or $r = \gamma_1 = 0$ and $\gamma_2 > 0$.

We recall that a sheaf $F$ is called Mumford (or $\mu$-) semi-stable with respect to the polarization $J$ if for each subsheaf $F' \subseteq F$,

$$
\mu_J(F') \leq \mu_J(F).
$$

Similarly, a sheaf $F$ is called Gieseker semi-stable with respect to the polarization $J$ if for each subsheaf $F' \subseteq F$,

$$
p_J(F', n) \leq p_J(F, n),
$$

where $\leq$ indicates the lexicographic ordering with respect to the monomials in $n$.

We recall that a Harder-Narasimhan filtration with respect to a stability condition $\varphi$ is a filtration $0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_l = F$ of a sheaf $F$, such that the quotients $E_i = F_i/F_{i-1}$ are semi-stable with respect to $\varphi$ and satisfy $\varphi(E_i) > \varphi(E_{i+1})$ for all $i$.

Note that for $J = \pm K_S$, the sheaves with equal slopes $\mu_J(\gamma) = \mu_J(\gamma')$ (or reduced Hilbert polynomials), have vanishing $\langle \gamma, \gamma' \rangle$. 

2.3. Discriminant. We define the discriminant (cf. [11, §3.4])

\[(2.9) \quad \Delta(\gamma) = \Delta(F) = \frac{1}{r} \left( c_2 - \frac{r-1}{2r} c_1^2 \right) = \frac{\gamma_1^2}{2r^2} - \frac{\gamma_2}{r}. \]

Then

\[(2.10) \quad \frac{\chi(\gamma, \gamma)}{r^2} = \frac{2\gamma_2}{r} - \frac{\gamma_1^2}{r^2} + \chi(O_S) = -2\Delta(\gamma) + \chi(O_S). \]

Note that

\[(2.11) \quad \log(\gamma/r) = \frac{\gamma_1 + \gamma_2}{r} - \frac{1}{2} \left( \frac{\gamma_1 + \gamma_2}{r} \right)^2 = \frac{\gamma_1}{r} + \left( \frac{\gamma_2}{r} - \frac{\gamma_1^2}{2r^2} \right) = \mu(\gamma) - \Delta(\gamma). \]

Therefore

\[(2.12) \quad \Delta(\gamma \cdot \gamma') = \Delta(\gamma) + \Delta(\gamma'). \]

Note that for a line bundle \(L\), we have \(r = 1\) and \(\gamma_2 = \gamma_1^2/2\). Therefore

\[(2.13) \quad \Delta(L) = 0, \quad \Delta(F \otimes L) = \Delta(F). \]

By the Bogomolov inequality [11, §3.4], if \(F\) is (Gieseker or Mumford) semistable, then

\[(2.14) \quad \Delta(F) \geq 0. \]

Consider a filtration \(0 \subset F_1 \subset \cdots \subset F_\ell = F\) of \(F\) whose quotients \(E_i = F_i/F_{i-1}\) have Chern character \(\gamma(E_i) = \gamma^{(i)}\). Then the discriminant \(\Delta(\gamma)\) is expressed in terms of \(\gamma^{(i)}\) as

\[(2.15) \quad r\Delta = \sum_{i=1}^\ell r^{(i)} \Delta^{(i)} = \sum_{i=2}^\ell \frac{1}{2r^{(i)}} \sum_{j=1}^{i-1} \frac{1}{r^{(j)}} \sum_{k=0}^{j-1} \frac{1}{r^{(k)}} \left( \sum_{l=1}^{i-1} r^{(i), \lambda_{j,k}} \right)^2. \]

2.4. \(\lambda\)-rings. For the definition and basic properties of \(\lambda\)-rings see e.g. [4, 23]. Assuming that \(R\) is a commutative algebra over \(\mathbb{Q}\), a \(\lambda\)-ring structure on \(R\) is given by a family of ring homomorphisms \((\psi_n : R \to R)_{n \geq 1}\), called Adams operations, satisfying

\[(1) \quad \psi_1 = \text{id}_R, \quad \psi_0 = 0, \quad \psi_m \circ \psi_n = \psi_{mn}. \]

Using the ring of symmetric functions

\[\Lambda = \lim_{\Psi} \mathbb{Z}[x_1, \ldots, x_n]^{S_n},\]

we can define a (unique) map \(\circ : \Lambda \times R \to R\), called a plethystic operation, such that

\[(1) \quad - \circ : \Lambda \to R \text{ is a ring homomorphism for all } r \in R, \]

\[(2) \quad p_n \circ r = \psi_n(r) \text{ for all } r \in R \text{ and } n \geq 1, \text{ where } p_n = \sum x_i^n \in \Lambda, \text{ called power sums.} \]

Assuming that the first of the above axioms is satisfied, the plethystic operation in its turn is uniquely determined by any of the following families of maps

\[(1) \quad \lambda_n(r) = e_n \circ r, \quad \text{where } e_n = \sum x_i \cdots x_i \in \Lambda, \text{ elementary symmetric functions.} \]

\[(2) \quad \sigma_n(r) = h_n \circ r, \quad \text{where } h_n = \sum x_i \cdots x_i \in \Lambda, \text{ complete symmetric functions.} \]

We will see later several examples of \(\lambda\)-rings. The key example arises from the Grothendieck ring \(K_0(\mathcal{A})\) of an abelian (or exact) symmetric monoidal category \(\mathcal{A}\). The \(\lambda\)- and \(\sigma\)-operations are defined in this case by taking exterior and symmetric powers, respectively

\[(2.16) \quad \lambda_n(V) = [\Lambda^n V], \quad \sigma_n(V) = [S^n V]. \]

For example, if \(\mathcal{A} = \text{Vect}^N\), the category of finite-dimensional \(N\)-graded vector spaces over a field \(k\), then

\[K_0(\mathcal{A}) \simeq \mathbb{Z}[x], \quad [k_i] \mapsto x^i, \quad i \geq 0, \]

where \(k_i\) is a one-dimensional vector space concentrated in degree \(i\). One can show that

\[\sigma_n(x^i) = x^{ni}, \quad \psi_n(x^i) = x^{ni} \]

and generally

\[\psi_n(f(x)) = f(x^n). \]
Generalizing this example further, we can equip the rings
\[ \mathbb{Q}[x_1, \ldots, x_k], \quad \mathbb{Q}(x_1, \ldots, x_k), \quad \mathbb{Q}[x_1, \ldots, x_k] \]
with a \( \lambda \)-ring structure
\[ \psi_n(f(x_1, \ldots, x_k)) = f(x_1^n, \ldots, x_k^n). \]
More generally, given a \( \lambda \)-ring \( R \) and a commutative monoid \( \Gamma \), we can equip the semigroup algebra
\[ R[\Gamma] = \bigoplus_{\gamma \in \Gamma} R \left\{ f = \sum_{\text{finite}} f_\gamma z^\gamma \mid f_\gamma \in R \forall \gamma \in \Gamma \right\}, \]
with a \( (\Gamma\text{-graded}) \lambda \)-ring structure
\[ \psi_n \left( \sum f_\gamma z^\gamma \right) = \sum \psi_n(f_\gamma) z^{n\gamma}. \]
Assuming that \( \Gamma \) is locally finite, that is,
\[ \# \left\{ (a, b) \in \Gamma \times \Gamma \mid a + b = c \right\} < \infty \quad \forall c \in \Gamma, \]
we can also equip the completion
\[ R[\Gamma] = \prod_{\gamma \in \Gamma} R \]
with a \( \lambda \)-ring structure. We define the plethystic exponential on \( R[\Gamma] \)
\[ \text{Exp}(f) = \sum_{n \geq 0} \sigma_n(f) = \exp \left( \sum_{n \geq 1} \frac{1}{n} \psi_n(f) \right), \quad f = \sum f_\gamma z^\gamma, \quad f_0 = 0. \]
Its inverse, the plethystic logarithm, is given by
\[ \text{Log}(f) = \sum_{n \geq 1} \frac{\mu(n)}{n} \psi_n(\log(f)), \quad f = \sum f_\gamma z^\gamma, \quad f_0 = 1, \]
where \( \mu \) is the classical Möbius function.

2.5. Some Grothendieck groups. Given a scheme (or an algebraic stack) \( S \) locally of finite type over \( \mathbb{C} \), let \( K_0(\text{Var}/S) \) denote the free abelian group generated by isomorphism classes of objects in \( \text{Var}/S \) (the category of finite type schemes over \( S \)), modulo relations
\[ [X \to S] = [Z \to S] + [U \to S], \]
where \( Z \subset X \) is a closed subvariety and \( U = X \setminus Z \) is its complement. Sometimes we denote \( K_0(\text{Var}/\mathbb{P}) \) (where \( \mathbb{P} = \text{Spec} \mathbb{C} \)) by \( K_0(\text{Var}/\mathbb{C}) \). It has a structure of a ring and the group \( K_0(\text{Var}/S) \) has a module structure over it. The element \( L = [A^1] \in K_0(\text{Var}/\mathbb{C}) \) is called the Lefschetz motive. One can show that by localizing \( K_0(\text{Var}/S) \) with respect to \( L \) and \( L^n - 1 \) for \( n \geq 1 \), one obtains the Grothendieck group \( K_0(\text{St}/S) \) of (finite type) stacks with affine stabilizers over \( S \) [2, §3.3]. Let \( K^{\text{mot}}(\mathbb{P}) \) be obtained from \( K_0(\text{Var}/\mathbb{P}) \otimes \mathbb{Q} \) by localizing with respect to the above elements and by adjoining the element \( L^{\frac{1}{2}} \). Generally, define
\[ K^{\text{mot}}(S) = K_0(\text{Var}/S) \otimes_{K_0(\text{Var}/\mathbb{P})} K^{\text{mot}}(\mathbb{P}). \]
If \( X \to S \) is a finite type stack with affine stabilizers over \( S \), we denote by \([X \to S]\) the corresponding element in \( K^{\text{mot}}(S) \). The rings \( K_0(\text{Var}/\mathbb{P}) \) and \( K^{\text{mot}}(\mathbb{P}) \) are equipped with (pre-)\( \lambda \)-ring structures
\[ \sigma^n(X) = [S^n X]. \]
The Adams operations act on \( \mathbb{L} \) as \( \psi_n(\mathbb{L}) = \mathbb{L}^n \). The action on \( \mathbb{L}^{\frac{1}{2}} \) is defined to be \( \psi_n(-\mathbb{L}^{\frac{1}{2}}) = (-\mathbb{L}^{\frac{1}{2}})^n \).
For any quasi-projective variety $S$, the Grothendieck group $K_0(MHM(S))$ of mixed Hodge modules over $S$ is a module over the ring $K_0(MHS) = K_0(MHM(pt))$. Similarly to the construction of $K^{mot}(S)$, we consider
\begin{equation}
(2.27) \quad L = H^*_c(A_1, \mathbb{Q}) = \mathbb{Q}(-1)[-2]
\end{equation}
as an element of $K_0(D^b MHS) = K_0(MHS)$ and define the ring $K^{mh}(pt)$ by localizing $K_0(MHS) \otimes \mathbb{Q}$ with respect to $L$ and $L^n - 1$, $n \geq 1$ and by adjoining the element $L^\frac{1}{2}$. Then we define
\begin{equation}
(2.28) \quad K^{mh}(S) = K_0(MHM(S)) \otimes_{K_0(MHM(pt))} K^{mh}(pt).
\end{equation}

There is a well-defined group homomorphism
\begin{equation}
(2.29) \quad \chi_c : K_0(Var/S) \to K_0(MHM(S)), \quad [f : X \to S] \mapsto [f_! Q_X]
\end{equation}
which extends to
\begin{equation}
(2.30) \quad \chi_c : K^{mot}(S) \to K^{mh}(S).
\end{equation}

**Remark 2.2.** To see that the map is indeed well-defined, consider $f : X \to S$, a closed embedding $i : Z \to X$ and its open complement $j : U \to X$. Then there is a distinguished triangle
\begin{equation}
j_!j^! F \to F \to i_* i^* F \to
\end{equation}
for any $F \in D^b(MHM(X))$. Considering $F = Q_X$ and applying $f_!$ for $f : X \to S$, we obtain a distinguished triangle
\begin{equation}
(fj)_! Q_U \to f_! Q_X \to (fi)_! Q_Z \to
\end{equation}
which implies
\begin{equation}
\chi_c(f) = \chi_c(f_i) + \chi_c(fj).
\end{equation}

In particular, we have a map
\begin{equation}
(2.31) \quad \chi_c : K_0(Var/\mathbb{C}) \to K_0(MHS), \quad [X] \mapsto H^*_c(X, \mathbb{Q})
\end{equation}
which was proved to be a homomorphism of (pre-)\(\lambda\)-rings in [20, §2.2].

2.6. **E-polynomial and Poincaré polynomial.** Given a mixed Hodge structure $V$, we define its Hodge-Euler polynomial [20, §3.1]
\begin{equation}
(2.32) \quad E(V, u, v) = \sum_{p,q} h^{p,q}(V) u^p v^q, \quad h^{p,q}(V) = \dim Gr_F^p Gr_{F+q}(V_{\mathbb{C}}).
\end{equation}
We can extend $E$ to a \(\lambda\)-ring homomorphism
\begin{equation}
E : K_0(MHM(pt)) = K_0(MHS) \to \mathbb{Q}[u^{\pm 1}, v^{\pm 1}],
\end{equation}
where the \(\lambda\)-ring structure on the right is given by $\psi_n(f(u, v)) = f(u^n, v^n)$. We can also extend $E$ to a \(\lambda\)-ring homomorphism
\begin{equation}
E : K^{mh}(pt) \to \mathbb{Q}(u^{\frac{1}{2}}, v^{\frac{1}{2}})
\end{equation}
with $E(L^\frac{1}{2}) = -(uv)^\frac{1}{2}$. We will also denote by $E$ the composition
\begin{equation}
(2.33) \quad K^{mot}(pt) \xrightarrow{\chi_c} K^{mh}(pt) \xrightarrow{E} \mathbb{Q}(u^{\frac{1}{2}}, v^{\frac{1}{2}})
\end{equation}
called the $E$-polynomial (or Hodge-Deligne polynomial), although it is a rational function in general. For an algebraic variety $X$, we have
\begin{equation}
(2.34) \quad E(X) = \sum_n (-1)^n \sum_{p,q} h^{p,q}(H^c(X, \mathbb{Q}))[u^p v^q].
\end{equation}
We define the (motivic) Poincaré polynomial
\begin{equation}
P : K^{mot}(pt) \to \mathbb{Q}(y), \quad P(X) = P(X, y) = E(X, y, y).
\end{equation}
If $X$ is a smooth projective variety then
\begin{equation}
(2.35) \quad P(X) = \sum_{n \geq 0} \dim H^n(X, \mathbb{C})(-y)^n.
\end{equation}
Moreover, \( P(L^{\frac{1}{2}}) = -y \).

2.7. **Virtual intersection complexes and motives.** Given an algebraic variety \( X \) of dimension \( d \), define the virtual intersection complex

\[
\text{IC}^\text{vir}_X = \text{IC}_X(d/2) = \mathbb{L}^{-\frac{d}{2}}\text{IC}_X[-d].
\]

This is a weight zero Hodge module. Given a map \( f : X \to S \), where \( X \) is an algebraic variety (or a finite type stack with affine stabilizers over \( S \)) of dimension \( d \), define

\[
[X \to S]_{\text{vir}} = \mathbb{L}^{-\frac{d}{2}}[X \to S].
\]

If \( X \) is smooth then

\[
\chi_c([X \to S]_{\text{vir}}) = \mathbb{L}^{-\frac{d}{2}}f_!(\mathbb{Q}_X) = f_!(\text{IC}^\text{vir}_X)
\]
as \( \text{IC}_X = \mathbb{Q}_X[d] \) and \( \text{IC}^\text{vir}_X = \mathbb{L}^{-\frac{d}{2}}\mathbb{Q}_X \). In particular, if \( X \) is an algebraic variety (or stack) of dimension \( d \), then

\[
[X]_{\text{vir}} = \mathbb{L}^{-\frac{d}{2}}[X] \in \mathbb{K}^{\text{mot}}(\text{pt})
\]
and if \( X \) is smooth, then

\[
\chi_c([X]_{\text{vir}}) = H^*(X, \text{IC}^\text{vir}_X).
\]

2.8. **Graded commutative monoids.** We will construct generalizations of the rings \( \mathbb{K}^{\text{mot}}(\text{pt}) \) and \( \mathbb{K}^{\text{mhm}}(\text{pt}) \). Let \( \Gamma \subset \mathbb{Z}^n \) be a monoid and let \( M = \bigsqcup_{\gamma \in \Gamma} M_\gamma \) be a \( \Gamma \)-graded commutative monoid in the category of complex algebraic varieties. This means that \( M_\gamma \) are complex algebraic varieties (we will assume that they are quasi-projective) equipped with an associative commutative multiplication

\[
\mu : M_\gamma \times M_{\gamma'} \to M_{\gamma + \gamma'}
\]
and with a unit \( \text{pt} \to M_0 \) satisfying the standard axioms. We will assume that the map \( \mu \) is finite. Define a \( \Gamma \)-graded group

\[
\mathbb{K}^{\text{mot}}(M) = \bigoplus_{\gamma \in \Gamma} \mathbb{K}^{\text{mot}}(M_\gamma)
\]
and equip it with a commutative ring structure

\[
[X \to M_\gamma] \cdot [Y \to M_{\gamma'}] = [X \times Y \to M_\gamma \times M_{\gamma'} \xrightarrow{\mu} M_{\gamma + \gamma'}].
\]

It has a (pre-)\( \lambda \)-ring structure defined by

\[
\sigma_n(X \to M_\gamma) = [S^n X \to S^n M_\gamma \xrightarrow{\mu} M_{n\gamma}].
\]

On the other hand, consider a \( \Gamma \)-graded category

\[
\mathcal{A} = \bigsqcup_{\gamma \in \Gamma} \mathcal{A}_\gamma, \quad \mathcal{A}_\gamma = \text{MHM}(M_\gamma)
\]
and equip it with the tensor product

\[
\odot : \mathcal{A}_\gamma \times \mathcal{A}_{\gamma'} \to \mathcal{A}_{\gamma + \gamma'}, \quad E \odot F = \mu_*(E \boxtimes F),
\]
where \( E \boxtimes F = p_1 E \otimes p_2 F \) with \( p_1 : M_\gamma \times M_{\gamma'} \to M_\gamma \), \( p_2 : M_\gamma \times M_{\gamma'} \to M_{\gamma'} \) being projections.

It is proved in [25] that \( \mathcal{A} \) equipped with this tensor product is a symmetric monoidal category.

The Grothendieck groups

\[
K_0(\mathcal{A}) = \bigoplus_{\gamma \in \Gamma} K_0(\mathcal{A}_\gamma), \quad \mathbb{K}^{\text{mhm}}(M) = \bigoplus_{\gamma \in \Gamma} \mathbb{K}^{\text{mhm}}(M_\gamma)
\]
are commutative \( \Gamma \)-graded rings with multiplication

\[
[E] \cdot [F] = [E \odot F].
\]

By [25, 1, 20, 4, 9] they are also \( \lambda \)-rings with \( \sigma \)-operations defined by

\[
\sigma_n(E) = S^nE = \text{im} \left( \frac{1}{n!} \sum_{\tau \in S_n} \tau \right) \subset E^\otimes n.
\]
One can prove that the map
\begin{equation}
\chi_e : K^{\text{mot}}(M) \to K^{\text{hm}}(M), \quad [f : X \to M_\gamma] \mapsto f_! \mathbb{Q}_X
\end{equation}
is a homomorphism of (pre-)\(\lambda\)-rings using results of [19]. If \(\Gamma\) is locally finite \((2.20)\), then we can equip
\begin{equation}
\hat{K}^{\text{mot}}(M) = \prod_\gamma K^{\text{mot}}(M_\gamma), \quad \hat{K}^{\text{hm}}(M) = \prod_\gamma K^{\text{hm}}(M_\gamma)
\end{equation}
with the structures of (pre-)\(\lambda\)-rings and extend \((2.48)\) to a homomorphism of (pre-)\(\lambda\)-rings
\begin{equation}
\chi_e : \hat{K}^{\text{mot}}(M) \to \hat{K}^{\text{hm}}(M).
\end{equation}

3. The main result

Let \(S\) be a projective surface over \(\mathbb{C}\) and \(J \in H^2(S, \mathbb{R})\) be a polarization on \(S\). We will assume that \(J\) is generic and \(J \cdot K_S < 0\). The latter requirement is needed because of the following result.

**Lemma 3.1.** Assume that \(J \cdot K_S < 0\). Then for any Gieseker (or Mumford) semistable sheaves \(E, F \in \text{Coh } S\) with \(p_J(E) = p_J(F)\) (or \(\mu_J(E) = \mu_J(F)\)), we have
\[\operatorname{Ext}^2(E, F) = 0.\]

**Proof.** Gieseker semistability implies Mumford semistability. Therefore we can assume that \(E, F\) are Mumford semistable and \(\mu_J(E) = \mu_J(F)\). Then the sheaf \(E \otimes K_S\) is also Mumford semistable and \(\mu_J(E \otimes K_S) < \mu_J(E) = \mu_J(F)\). This implies \(\operatorname{Hom}(F, E \otimes K_S) = 0\). By the Serre duality
\[\operatorname{Ext}^2(E, F) \simeq \operatorname{Hom}(F, E \otimes K_S)^* = 0.\]

Given a polynomial \(p\), let \(\Gamma^* \subset \frac{1}{2} H^{\text{even}}(S, \mathbb{Z})\) be a semigroup consisting of classes \(\gamma = (r, \gamma_1, \gamma_2)\) with \(p_J(\gamma) = p\) and \(r > 0\) and let \(\hat{\Gamma} = \Gamma^* \cup \{0\}\). If \(J\) is generic then \(\Gamma\) is isomorphic to \(\mathbb{N}\). For any \(\gamma \in \Gamma\), let \(M_\gamma\) (resp. \(\overline{M}_\gamma\)) denote the moduli space (resp. stack) of Gieseker semi-stable sheaves on \(S\) having Chern character \(\gamma\). We let \(M_0 = \mathfrak{pt}\). The schemes \(M_\gamma\) are projective, possibly singular [11, Theorem 4.3.4]. The goal of this section is to prove the following (cf. Theorem 1.2)

**Theorem 3.2.** If \(J\) is generic and \(J \cdot K_S < 0\) then
\[\sum_{\gamma \in \Gamma^*} \mathbb{L}^{-\frac{1}{2} \dim M_\gamma} \text{IE}(M_\gamma) z^\gamma = (\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}) \log \left( \sum_{\gamma \in \Gamma} \mathbb{L}^{-\frac{1}{2} \dim \overline{M}_\gamma} E(\overline{M}_\gamma) z^\gamma \right) \]
in \(\mathbb{Q}(u^\frac{1}{2}, v^\frac{1}{2})[\Gamma]\) with \(L = E(H^*_c(A^1, \mathbb{Q})) = uv\).

This theorem relates the \(E\)-polynomials of the intersection cohomologies \(IH^*(M_\gamma, \mathbb{Q})\) and the \(E\)-polynomials of the stacks \(\overline{M}_\gamma\) (or of the elements \(\chi_c([\overline{M}_\gamma]) \in K^{\text{hm}}(\mathfrak{pt})\)). We can formulate it on the level of mixed Hodge structures.

**Theorem 3.3.** If \(J\) is generic and \(J \cdot K_S < 0\) then
\[\sum_{\gamma \in \Gamma^*} \mathbb{L}^{-\frac{1}{2} \dim M_\gamma} IH^*(M_\gamma) z^\gamma = (\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}) \log \left( \sum_{\gamma \in \Gamma} \mathbb{L}^{-\frac{1}{2} \dim \overline{M}_\gamma} \chi_c([\overline{M}_\gamma]) z^\gamma \right) \]
in \(K^{\text{hm}}(\mathfrak{pt})[\Gamma]\) with \(L = [H^*_c(A^1, \mathbb{Q})] = [\mathbb{Q}(-1)[-2]]\).

This statement can be further generalized to an equation in the \(\lambda\)-ring \(\hat{K}^{\text{hm}}(M)\) \((2.49)\), where \(M = \bigsqcup_{\gamma \in \Gamma} M_\gamma\) is a \(\Gamma\)-graded commutative monoid with a finite multiplication
\begin{equation}
\mu : M_\gamma \times M_{\gamma'} \to M_{\gamma + \gamma'}, \quad (E, F) \mapsto E \oplus F.
\end{equation}
Remark 3.4. We can consider $\Gamma$ as a $\Gamma$-graded monoid in the category of algebraic varieties, which consists of a single point at every degree. Then $\mathbf{K}_{\text{hm}}^{\text{hm}}(\Gamma) \cong \mathbf{K}_{\text{hm}}^{\text{hm}}(\mathbf{pt})[\Gamma]$. The natural projection $a : M \to \Gamma$, $M_\gamma \to \{\gamma\}$ is a homomorphism of $\Gamma$-graded monoids and induces a $\lambda$-ring homomorphism

$$a_1 : \mathbf{K}_{\text{hm}}^{\text{hm}}(M) \to \mathbf{K}_{\text{hm}}^{\text{hm}}(\mathbf{pt})[\Gamma], \quad [F \in \text{HM}(M_\gamma)] \mapsto H^*_\gamma(M_\gamma, F)z^\gamma.$$ 

We note that

$$a_1(\text{IC}^\text{vir}_{M_\gamma}) = L^{-\frac{1}{2} \dim M_\gamma} \text{IH}^*(M_\gamma)z^\gamma, \quad a_1(\chi_e([\mathcal{M}_\gamma \to M_\gamma])) = \chi_e([\mathcal{M}_\gamma])z^\gamma.$$

\[ \Box \]

Theorem 3.5. If $J$ is generic and $J \cdot K_S < 0$ then

$$\sum_{\gamma \in \Gamma} \text{IC}^\text{vir}_{M_\gamma} = (L^\frac{1}{2} - L^{-\frac{1}{2}}) \log \left( \sum_{\gamma \in \Gamma} \chi_e([\mathcal{M}_\gamma \to M]\text{vir}) \right)$$

in $\mathbf{K}_{\text{hm}}^{\text{hm}}(M)$ with $L = [H^*_\gamma(A^1, \mathbb{Q})] = [\mathbb{Q}(-1)[-2]]$.

Our goal will be to prove this theorem. The strategy of the proof follows the ideas of [22, 25, 21]. Namely, we construct auxiliary smooth moduli spaces $M^\dagger_{\gamma}$ of framed sheaves together with projections $\pi : M^\dagger_{\gamma} \to M_\gamma$. Then we relate $\pi_!\text{IC}^\text{vir}_{M^\dagger_{\gamma}}$ to both sides of the above theorem. In contrast to [22, 25, 21], we will need to overcome a technical difficulty arising from the fact that the framing functors (see §3.2) needed in the construction of $M^\dagger_{\gamma}$ are not exact.

3.1. Motivic Hall algebra and DT invariants. Let

$$M = \bigsqcup_{\gamma \in \Gamma} M_\gamma, \quad \mathcal{M} = \bigsqcup_{\gamma \in \Gamma} \mathcal{M}_\gamma.$$ 

There are natural maps $p_\gamma : \mathcal{M}_\gamma \to M_\gamma$ and $p : \mathcal{M} \to M$. We have seen that $M$ is a commutative $\Gamma$-graded monoid. Therefore $\mathbf{K}_{\text{mot}}(M)$ is equipped with a (pre)-$\lambda$-ring structure. We define the motivic Hall algebra

$$H = \mathbf{K}_{\text{mot}}(\mathcal{M}) = \bigoplus_{\gamma \in \Gamma} H_\gamma, \quad H_\gamma = \mathbf{K}_{\text{mot}}(\mathcal{M}_\gamma)$$

with the Ringel-Hall multiplication as in [2, §4.2] (see also [13]). The following map, called an integration map,

$$I : H \to \mathbf{K}_{\text{mot}}(M), \quad [X \to \mathcal{M}_\gamma] \mapsto L^{-\frac{1}{2} \dim \mathcal{M}_\gamma} [X \to \mathcal{M}_\gamma \to M_\gamma]$$

is an algebra homomorphism by [13, 27] if the category of semistable sheaves with Chern characters $\gamma \in \Gamma$ has homological dimension one (this is the case under our assumptions by Lemma 3.1). Note that $\dim \mathcal{M}_\gamma = -\chi(\gamma, \gamma)$.

Remark 3.6. To make $I$ an algebra homomorphism, one actually defines multiplication in $\mathbf{K}_{\text{mot}}(M)$ as

$$[X \to M_\gamma] \cdot [Y \to M_{\gamma'}] = \mathbb{L}^\frac{1}{2}(\chi(\gamma, \gamma') - \chi(\gamma', \gamma))[X \times Y \to M_{\gamma + \gamma'}].$$

But $\chi(\gamma, \gamma') = \chi(\gamma', \gamma)$ if $p_J(\gamma) = p_J(\gamma')$ (or $\mu_J(\gamma) = \mu_J(\gamma')$) and $J$ is generic, hence we omit the twist. This is also true if $J = \pm K_S$ and all our arguments will also work in this situation. \[ \Box \]

Previously we defined ring homomorphisms

$$H \xrightarrow{I} \mathbf{K}_{\text{mot}}(M) \xrightarrow{\chi_{\text{mot}}} \mathbf{K}_{\text{hm}}^{\text{hm}}(M).$$

Similarly, we can define completions

$$\widehat{H} = \prod_{\gamma} H_\gamma, \quad \widehat{\mathbf{K}_{\text{mot}}(M)} = \prod_{\gamma} \mathbf{K}_{\text{mot}}(M_\gamma), \quad \widehat{\mathbf{K}_{\text{hm}}^{\text{hm}}(M)} = \prod_{\gamma} \mathbf{K}_{\text{hm}}^{\text{hm}}(M_\gamma),$$

and ring homomorphisms between them.

We define motivic Donaldson-Thomas invariants

$$\text{DT}_{\gamma}^{\text{mot}} \in \mathbf{K}_{\text{mot}}(M_\gamma)$$

and all our arguments will also work in this situation.
by the formula
\[(3.4) \quad I(1_M) = \sum_{\gamma \in \Gamma^*} [\mathcal{M}_\gamma \to M] = \text{Exp} \left( \sum_{\gamma \in \Gamma^*} \frac{DT_{\gamma}^{\text{mot}}}{L^+ - L^{-\frac{1}{2}}} \right), \]
where \(1_M : \mathcal{M} \to \mathcal{M}\) is an identity map. Equivalently,
\[(3.5) \quad \sum_{\gamma \in \Gamma^*} DT_{\gamma}^{\text{mot}} = \left( L^+ - L^{-\frac{1}{2}} \right) \log \left( \sum_{\gamma \in \Gamma^*} [\mathcal{M}_\gamma \to M]_{\text{vir}} \right).
\]
We define MHM Donaldson-Thomas invariants as
\[(3.6) \quad \chi_c I \left( \sum_{\gamma} 1_M \right) = \text{Exp} \left( \sum_{\gamma \in \Gamma^*} \frac{DT_{\gamma}^{\text{mhms}}}{L^+ - L^{-\frac{1}{2}}} \right). \]

3.2. Framed moduli spaces. Let \(A = \text{Coh} S\) and \(T \in A\) be some object. We consider a left exact functor
\[
\Phi : A \to \text{Vect}, \quad \Phi(E) = \text{Hom}(T, E)
\]
and define the category \(A_f\) of framed objects as follows. Its objects are triples \((E, V, s)\), where \(E \in A, V \in \text{Vect}\) and \(s : V \to \Phi(E)\) is a linear map. A morphism \(f : (E, V, s) \to (E', V', s')\) is a pair \((f_1, f_2)\), where \(f_1 : E \to E'\) and \(f_2 : V \to V'\) satisfy \(\Phi(f_1)s = s'f_2\). One can show that \(A_f\) is an abelian category. We will denote an object \((E, 0, 0)\) by \(E\) and call it an unframed object. We will denote an object \((E, C, s)\) by \((E, s)\) and consider \(s\) as an element of \(\Phi(E)\).

Given a coherent sheaf \(E\) with Chern character \(\gamma\), we define
\[(3.7) \quad \phi(\gamma) = \phi(E) := \sum_{i \geq 0} (-1)^i \dim R^i \Phi(E) = \chi(T, E).
\]
In order to construct moduli spaces of stable framed objects we will use results of [10] (or more precisely, the dual version of these results). First, we will reformulate the stability condition from [10]. Given a polynomial \(\delta \in \mathbb{Q}[n]\), define for any triple \(E = (E, V, s)\)
\[
ps(E) = \frac{P_J(E) + \dim V \cdot \delta}{\text{rk}(E)},
\]
where \(P_J(E)\) is the Hilbert polynomial of \(E\) (with respect to the polarization \(J\)). We will say that an object \(E\) is \(\delta\)-stable if for any proper \(G \subset E\) we have
\[
ps(G) < ps(E).
\]
In the case of a pair \(E = (E, s)\) this means
\[
(1) \quad \text{For any } G \subset E, \text{ we have } p_J(G) < ps(E).
\]
\[
(2) \quad \text{For any proper } G \subset E \text{ with } s \in \Phi(G), \text{ we have } ps(E) < p_J(E/G).
\]
Here \(p_J(E)\) is the reduced Hilbert polynomial of \(E\). In [10] the authors constructed the moduli spaces of \(\delta\)-stable objects. For our applications we will consider \(\delta\) to be a constant such that \(0 < \delta \ll 1\). Then \((E, s)\) is stable if and only if
\[
(1) \quad \text{For any } G \subset E, \text{ we have } p_J(G) \leq p_J(E). \text{ Equivalently, } E \text{ is Gieseker semistable.}
\]
\[
(2) \quad \text{For any proper } G \subset E \text{ with } s \in \Phi(G), \text{ we have } p_J(E) < p_J(E/G). \text{ Equivalently, } p_J(G) < p_J(E).
\]
Let \(M^f_\gamma\) denote the moduli space of stable pairs \((E, s)\) with \(c(E) = \gamma\). This is a projective variety and there is a projective map
\[
\pi : M^f_\gamma \to M_\gamma, \quad (E, s) \mapsto E,
\]
where \(M_\gamma\) denotes as before the moduli space of (Gieseker) semistable sheaves on \(S\) with Chern character \(\gamma\).
These moduli spaces are instrumental for a new description of the DT invariants. Let $\mathcal{B} \subset \mathcal{A} = \text{Coh} \, S$ be an abelian category of Gieseker semistable sheaves $E$ with the reduced Hilbert polynomial $p_J(E) = p$. Let $||-||$ be some norm on $H^{\text{even}}(S, \mathbb{R})$.

**Definition 3.7.** Given a constant $N > 0$ and a left exact functor $\Phi : \mathcal{A} \to \text{Vect}$, we will say that $\Phi$ is $N$-exact if $R^i\Phi(E) = 0$ for $i > 0$ and $\Phi(E) \neq 0$ for all semistable $E$ with $||\text{ch} \, E|| \leq N$.

**Remark 3.8.** For a fixed $N > 0$, the set of $\gamma \in \Gamma$ with $||\gamma|| \leq N$ is finite, hence the family of semistable sheaves of type $\gamma$ with $||\gamma|| \leq N$ is bounded [11, Theorem 3.3.7] in the sense of [11, §1.7]. This implies that we can choose $T = L^{-n}$, where $L$ is an ample line bundle and $n \gg 0$ such that

1. $\text{Ext}^i(T, E) = 0$, $i > 0$
2. $\text{Hom}(T, E) \neq 0$

for all semistable $E$ with Chern character $\gamma$ and $||\gamma|| \leq N$. Therefore $\Phi = \text{Hom}(T, -)$ is $N$-exact. This rather arbitrary choice of $T$ indicates that the moduli spaces $M_\gamma^f$ play a purely auxiliary role in our analysis of the moduli spaces $M_\gamma$.

**Remark 3.9.** Let us show that if $\Phi$ is $N$-exact then $M_\gamma^f$ are smooth for $||\gamma|| \leq N$ as we will need this fact when we will work with intersection complexes on $M_\gamma^f$. If $E$ is semistable and has Chern character $\gamma$, then $\dim \Phi(E) = \phi(\gamma)$ (under our assumptions on $\Phi$ and $\gamma$). The moduli stack $\mathcal{M}_\gamma$ of semistable framed objects is open in $\mathcal{M}_\gamma \times K^{\phi(\gamma)}$, where $\mathcal{M}_\gamma$ is a smooth moduli stack of semistable sheaves. As the automorphism groups of objects in $\mathcal{M}_\gamma$ are trivial, we conclude that $M_\gamma^f$ is smooth.

**Theorem 3.10.** We have

$$
\sum_{\rho_J(\gamma) = p} (-1)^{\phi(\gamma)} [M_\gamma^f \to M_\gamma]_{\text{vir}} = \text{Exp} \left( \sum_{\rho_J(\gamma) = p, \gamma \neq 0} (-1)^{\phi(\gamma)} [\mathcal{P}^{\phi(\gamma)-1}]_{\text{vir}} \text{DT}^\text{mot}_\gamma \right)
$$

in $K^\text{mot}(M)$, for summands with $||\gamma|| \leq N$.

**Proof.** Let $\mathcal{B} \subset \mathcal{A}$ be the subcategory of Gieseker semistable vector bundles $E$ with $p(E) = p$. One can show that for a pair $(E, s)$ with $E \in \mathcal{B}$ there exists a unique stable subobject $(E', s) \subset (E, s)$ with $E', E/E' \in \mathcal{B}$ (this is a Harder-Narasimhan filtration with respect to an appropriate stability condition on $B_f$). Let

$$
1^f_{\mathcal{B}} = \sum_{\gamma \in \Gamma} [\mathcal{M}_\gamma^f \to \mathcal{M}_\gamma] \in \hat{H}
$$

and similarly let $1^f_{\mathcal{B}} \in \hat{H}$ parametrize all pairs $(E, s)$ with $E \in \mathcal{B}$. Let

$$
1_{\mathcal{B}} = \sum_{\gamma \in \Gamma} [\mathcal{M}_\gamma \to \mathcal{M}_\gamma] \in \hat{H}.
$$

Then the above Harder-Narasimhan filtration translates to an equation

$$
1^f_{\mathcal{B}} = 1^f_{\mathcal{B}} \circ 1_{\mathcal{B}}
$$

in the Hall algebra $\hat{H}$. We should stress that this is a relation in the Hall algebra of $\mathcal{B}$, although we used the Harder-Narasimhan filtration in the category $B_f$. Applying the integration map $I : \hat{H} \to K^\text{mot}(M)$, we obtain the following relation

$$
\sum_{\gamma} L^{\phi(\gamma)} [\mathcal{M}_\gamma \to M_\gamma]_{\text{vir}} = \sum_{\gamma} L^{-\frac{1}{2} \dim \mathcal{M}_\gamma} [M_\gamma^f \to M_\gamma] \cdot \sum_{\gamma} [\mathcal{M}_\gamma \to M_\gamma]_{\text{vir}}
$$

for $||\gamma|| \leq N$. Using the formula

$$
\dim M_\gamma^f = -\chi(\gamma, \gamma) + \phi(\gamma) = \dim \mathcal{M}_\gamma + \phi(\gamma)
$$
we obtain
\[ \sum_{\gamma} \mathcal{L}^{\phi(\gamma)} [M_{\gamma}]_{\text{vir}} = \sum_{\gamma} \mathcal{L}^{\frac{1}{2} \phi(\gamma)} [M^f_{\gamma}]_{\text{vir}} \cdot \sum_{\gamma} [\mathcal{M}_{\gamma}]_{\text{vir}}. \]
This can be written in terms of DT invariants
\[ \sum_{\gamma} \mathcal{L}^{\frac{1}{2} \phi(\gamma)} [M^f_{\gamma}]_{\text{vir}} = \exp \left( \sum_{\gamma} \left( \mathcal{L}^{\phi(\gamma)} - 1 \right) \frac{\text{DT}_{\text{mot}}^\gamma}{\mathcal{L}^{\frac{1}{2}} - \mathcal{L}^{-\frac{1}{2}}} \right). \]
Applying the (plethystic) change of variables
\[ x \mapsto (-\mathcal{L}^{\frac{1}{2}})^{-\phi(\gamma)} x, \quad x \in \mathcal{M}_{\text{mot}}(M), \]
we obtain
\[ \sum_{\gamma} (-1)^{\phi(\gamma)} [M^f_{\gamma}]_{\text{vir}} = \exp \left( \sum_{\gamma} (-1)^{\phi(\gamma)} \frac{\mathcal{L}^{\phi(\gamma)} - \mathcal{L}^{-\frac{1}{2}\phi(\gamma)}}{\mathcal{L}^{\frac{1}{2}} - \mathcal{L}^{-\frac{1}{2}}} \text{DT}_{\text{mot}}^\gamma \right) = \exp \left( \sum_{\gamma} (-1)^{\phi(\gamma)} [\mathcal{M}_{\gamma}]_{\text{vir}} \text{DT}_{\text{mot}}^\gamma \right). \]

\[ \square \]

**Proof of Theorem 3.5.** By comparing the statement of the theorem and the definition of DT invariants (3.5) we have to prove
\[ \text{DT}^\gamma_{\text{mot}} = \text{IC}_{\mathcal{M}_{\gamma}}^{\text{vir}}, \]
We can assume by induction that \( \text{DT}^\gamma_{\text{mot}} = \text{IC}_{\mathcal{M}_{\gamma}} \) for \( ||\alpha|| < N := ||\gamma|| \). Let \( T = L^{-n} \), where \( L \) is an ample line bundle, and \( \Phi = \text{Hom}(T, -) \) be as in Remark 3.10. Then assumptions of Theorem 3.10 are satisfied and we can apply the map \( \chi_c : \mathcal{M}_{\text{mot}}(M) \to \mathcal{M}_{\text{mot}}(M) \) to its statement. As \( M^f_{\gamma} \) is smooth, we obtain from (2.38)
\[ \chi_c([M^f_{\gamma}]_{\text{vir}}) = \pi_! \text{IC}_{M^f_{\gamma}}^{\text{vir}}. \]
Therefore Theorem 3.10 implies
\[ (-1)^{\phi(\gamma)} \pi_! \text{IC}_{M^f_{\gamma}}^{\text{vir}} = \sum_{\sum m_\alpha a_\alpha = \gamma} \prod_{\alpha} S^{m_\alpha} \left( (-1)^{\phi(\alpha)} [\mathcal{M}_{\gamma}]_{\text{vir}} \text{DT}_{\alpha}^{\text{mot}} \right). \]
Now we literally repeat the arguments of [25, Theorem 5.4] to conclude that \( \text{DT}^\gamma_{\text{mot}} = \text{IC}_{\mathcal{M}_{\gamma}}^{\text{vir}} \).

For all these arguments to work it is enough to assume that \( \Phi \) is N-exact. \[ \square \]

**4. Some properties of DT invariants**

As before, we assume that \( J \) is a generic (ample) polarization on a surface \( S \) with \( J \cdot K_S < 0 \). Let \( \mathcal{M}_\gamma \) be the moduli space Gieseker semistable sheaves with Chern character \( \gamma \). Let \( \mathcal{M}_\gamma = \mathcal{M}_J^G(\gamma) \) be the moduli stack of Gieseker semistable sheaves and \( \mathcal{M}_\gamma^G = \mathcal{M}_J^G(\gamma) \) be the moduli stack of Mumford (or \( \mu \)-) semistable sheaves with Chern character \( \gamma \).

In the previous section we studied Donaldson-Thomas invariants (3.4, 3.6)
\[ \text{DT}^\gamma_{\text{mot}} \in \mathcal{M}_{\text{mot}}(\mathcal{M}_\gamma), \quad \text{DT}^\gamma_{\text{mot}} \in \mathcal{M}_{\text{mot}}(\mathcal{M}_\gamma). \]
For the actual computations it is more appropriate to study their images in \( \mathcal{M}_{\text{mot}}(pt) \) or merely their \( E \)-polynomials or (motivic) Poincaré polynomials. Thus, we define Donaldson-Thomas invariants
\[ \Omega_{\gamma} = P(a \text{ DT}^\gamma_{\text{mot}}) = P(a \chi_c \text{ DT}^\gamma_{\text{mot}}) \in \mathbb{Q}(y). \]
where \( a : M_{\gamma} \to pt \) is a projection. Applying (3.4), we can write equivalently
\[ 1 + \sum_{p, j(\gamma) = p} I_p z^p = \exp \left( \sum_{p, j(\gamma) = p} \frac{\Omega_{\gamma} z^p}{y^{1-p} - y} \right), \]
\[ (4.3) \quad \mathcal{I}_\gamma = \mathcal{I}(\gamma, y; J) = (-y)^{\chi(\gamma, -\gamma)} P(\mathcal{R}_\gamma). \]

Let us give an explicit formula. We define the rational invariant $\Omega_\gamma$ of $\mathcal{M}_\gamma$, which is given in terms of $\mathcal{I}_\gamma$ by [12, Definition 6.22]

\[ (4.4) \quad \bar{\Omega}_\gamma = \bar{\Omega}(\gamma, y; J) = \sum_{\gamma_1 + \cdots + \gamma_\ell = \gamma, \forall i, p_j(\gamma_i) = p_j(\gamma)} (-1)^{\ell-1} \prod_{i=1}^{\ell} \mathcal{I}(\gamma_i, y; J). \]

The inverse relation is given by

\[ (4.5) \quad \mathcal{I}(\gamma, y; J) = \sum_{\gamma_1 + \cdots + \gamma_\ell = \gamma, \forall i, p_j(\gamma_i) = p_j(\gamma)} \frac{1}{\ell!} \prod_{i=1}^{\ell} \bar{\Omega}(\gamma_i, y; J). \]

Finally, we define the Donaldson-Thomas invariant

\[ (4.6) \quad \Omega_\gamma = \Omega(\gamma, y; J) = (y^{-1} - y) \sum_{m | \gamma} \frac{\mu(m)}{m} \bar{\Omega}(\gamma/m, y^m; J), \]

with inverse relation

\[ (4.7) \quad \bar{\Omega}(\gamma, y; J) = \sum_{m | \gamma} \frac{1}{m} \Omega(\gamma/m, y^m; J) y^{m - y^m - y^m}. \]

The main result of the previous section implies

\[ (4.8) \quad \Omega_\gamma = (-y)^{-\dim M_\gamma} \sum_n \dim H^n(M_\gamma)(-y)^n, \]

therefore $\Omega_\gamma \in \mathbb{Q}[y^{\pm 1}]$.

Similarly, we define invariants $\bar{\mathcal{I}}_\gamma^M$, $\bar{\Omega}_\gamma^M$, and $\Omega_\gamma^M$ of the moduli stacks $\mathcal{M}_\gamma^M$. In particular, for any $\tau \in \mathbb{R}$,

\[ (4.9) \quad 1 + \sum_{\mu_\mathcal{J}(\gamma) = \tau} \bar{\mathcal{I}}_\gamma^M z^\gamma = \exp \left( \frac{\sum_{\mu_\mathcal{J}(\gamma) = \tau} \Omega_\gamma^M z^\gamma}{y - y^2} \right). \]

**Theorem 4.1.** If $J$ is generic and $J \cdot K_S < 0$, then

\[ \Omega_\gamma = \Omega_\gamma^M. \]

**Proof.** Every Mumford semistable sheaf $F$ has a unique Harder-Narasimhan filtration

\[ 0 = F_0 \subset F_1 \subset \ldots \subset F_n = F \]

with respect to the Gieseker stability. The factors of this filtration are Gieseker (hence Mumford) semistable with slope $\mu_\mathcal{J}(F)$. Let $\gamma = (r, \gamma_1, \gamma_2) = \chi F$ and $\gamma^{(i)} = (r^{(i)}, \gamma_1^{(i)}, \gamma_2^{(i)}) = \chi F_i/F_{i-1}$ for $i = 1, \ldots, n$. The sequence

\[ (\gamma^{(1)}, \ldots, \gamma^{(n)}) \]

is called the type of the HN filtration and we claim that there occurs a finite number of such types for the family of all Mumford semistable sheaves of fixed type $\gamma$. As $J$ is generic, we have $\gamma^{(i)}/r^{(i)} = \gamma_1/r$, hence the number of possible pairs $(r^{(i)}, \gamma^{(1)}/r^{(i)})$ is finite. We conclude by the Bogomolov inequality ($\Delta(\chi F) \geq 0$ for a Mumford semistable sheaf $F$) that $\gamma^{(2)}$ are bounded above and there is a finite number of possible classes $\gamma^{(1)}$ appearing in the HN filtrations.

For a fixed $\tau \in \mathbb{R}$, let $\Gamma^*$ be the set of all Chern characters $\gamma = (r, \gamma_1, \gamma_2)$ with $\mu_\mathcal{J}(\gamma) = \tau$ and $\Delta(\gamma) \geq 0$. Let $\Gamma = \Gamma^* \cup \{0\}$. Then $\gamma_1/r$ is independent of $\gamma \in \Gamma^*$ and therefore $\gamma_2/r \leq \gamma^{(2)}/2r^2 =: \nu_r$ is bounded above. This implies that $\Gamma$ is a locally finite monoid (2.20). Using the formula for the reduced Hilbert polynomial (2.6), we can write for any $\gamma \in \Gamma$

\[ (4.10) \quad p_\mathcal{J}(\gamma, n) = p_\mathcal{J}(\gamma, n) + \frac{\gamma_2}{r}, \]
where the polynomial $p_{J,r}$ is independent of $\gamma \in \Gamma$. This implies that for $\gamma, \gamma' \in \Gamma$

$$p_{J}(\gamma) \leq p_{J}(\gamma') \iff \gamma_2/r \leq \gamma'_2/r'.$$

For $\nu \in \mathbb{R}$, let

$$\Gamma^*_\nu = \{ \gamma \in \Gamma^\ast \mid p_{J}(\gamma, n) = p_{J,r} + \nu \}, \quad \Gamma^*_\nu = \Gamma^*_\nu \cup \{0\}.$$  

Uniqueness of the Harder-Narasimhan filtration implies a relation in the motivic Hall algebra (of the category of Mumford semistable sheaves) which translates into a relation in $K^{mot}(\mathfrak{pt})[\Gamma]$ (as well as in $Q(y)[\Gamma]$ after taking the Poincaré polynomials)

$$\sum_{\gamma \in \Gamma} T_{\gamma}^{M} z^{\gamma} = \prod_{\nu} \left( \sum_{\gamma \in \Gamma^*_\nu} I_{\gamma} z^{\gamma} \right).$$  

By the definition of DT invariants, we have

$$\sum_{\gamma \in \Gamma} T_{\gamma}^{M} z^{\gamma} = \exp \left( \frac{\sum_{\gamma \in \Gamma^\ast} \Omega_{\gamma}^{M} z^{\gamma}}{y^{1} - y} \right), \quad \sum_{\gamma \in \Gamma^*_\nu} I_{\gamma} z^{\gamma} = \exp \left( \frac{\sum_{\gamma \in \Gamma^*_\nu} \Omega_{\gamma} z^{\gamma}}{y^{1} - y} \right).$$

Therefore we obtain from (4.11)

$$\exp \left( \frac{\sum_{\gamma \in \Gamma^\ast} \Omega_{\gamma}^{M} z^{\gamma}}{y^{1} - y} \right) = \prod_{\nu} \exp \left( \frac{\sum_{\gamma \in \Gamma^*_\nu} \Omega_{\gamma} z^{\gamma}}{y^{1} - y} \right) = \exp \left( \sum_{\nu} \frac{\sum_{\gamma \in \Gamma^*_\nu} \Omega_{\gamma} z^{\gamma}}{y^{1} - y} \right)$$

and $\Omega^M_\gamma = \Omega_\gamma$. \hfill \qed

Let us give a slightly different formulation of the above theorem. Consider the generating functions in $Q[y, t]$ defined by

$$H_{r,\gamma_1}(y, t; J) = H_{r,\gamma_1} := \sum_{\gamma_2} T_{\gamma}^{M}(\gamma, y; J) t^{r, \Delta(\gamma)},$$

and

$$h_{r,\gamma_1}(y, t; J) = h_{r,\gamma_1} := \sum_{\gamma_2} \Omega(\gamma, y; J) t^{r, \Delta(\gamma)}.$$  

For rational and ruled surfaces, we can write explicit formulas for $H_{r,\gamma_1}(y, t; J)$. The following theorem relates these invariants to $h_{r,\gamma_1}(y, t; J)$.

**Theorem 4.2.** Assume that $J$ is generic and $J \cdot K_S < 0$. Then, for every $\tau \in \mathbb{R}$,

$$\sum_{\gamma \in \mathbb{J}_{J/r=\tau}} h_{r,\gamma_1} z^{\gamma_0} = \sum_{\gamma_0} L_{\gamma_0} \sum_{\gamma_1} H_{r,\gamma_1} z^{\gamma_1} = \log \left( 1 + \sum_{\gamma_1} H_{r,\gamma_1} z^{\gamma_1} \right),$$

which is equivalent to

$$h_{r,\gamma_1} = \sum_{\sum_{(\nu)}(\gamma_1) = (r, \gamma_1), \mu_{J}(\gamma_1) = \mu_{J}(\gamma)} (-1)^{1-2} \prod_{i=1}^{\ell} H_{r,\gamma_{1}^{(i)}},$$

**Proof.** We can write

$$\sum_{\gamma \in \mathbb{J}_{J/r=\tau}} H_{r,\gamma_1} z^{\gamma_0} = \sum_{\mu_{J}(\gamma) = \tau} T_{\gamma}^{M} z^{\gamma_0} t^{\gamma_1/2^{r} - \gamma_2} = \sum_{\mu_{J}(\gamma) = \tau} T_{\gamma}^{M} u^r t^{-\gamma_2},$$

$$\sum_{\gamma \in \mathbb{J}_{J/r=\tau}} h_{r,\gamma_1} z^{\gamma_0} = \sum_{\mu_{J}(\gamma) = \tau} \Omega_{\gamma} z^{\gamma_0} t^{\gamma_1/2^{r} - \gamma_2} = \sum_{\mu_{J}(\gamma) = \tau} \Omega_{\gamma} u^r t^{-\gamma_2},$$

where $u = z_0 z^{\gamma_1/2^{r}/2^{r^2}}$ is independent of $\gamma$ for fixed $\mu_{J}(\gamma) = \tau$. Therefore we have to prove

$$1 + \sum_{\mu_{J}(\gamma) = \tau} T_{\gamma}^{M} u^r t^{-\gamma_2} = \exp \left( \sum_{\mu_{J}(\gamma) = \tau} \Omega_{\gamma} u^r t^{-\gamma_2} \right).$$
Given $\nu \in \mathbb{R}$ and any class $\gamma$ with
\[(4.18)\]
$$\gamma_1 \cdot J/r = \tau, \quad \gamma_2/r = \nu,$$
we obtain from (2.6) and the assumption that $J$ is generic that
\[(4.19)\]$$p_J(\gamma, n) = p_J,\tau(n) + \nu,$$
where $p_{J,\tau}$ is a polynomial that depends only on $J$ and $\tau$. Moreover, if $\gamma$ satisfies (4.19), then it also satisfies (4.18). We can write equation (4.11) as
\[
1 + \sum_{\mu_J(\gamma) = \tau} I^M_{\gamma} z^\gamma = \prod_{\nu} \left(1 + \sum_{p_J(\gamma) = p_J,\tau + \nu} I_\gamma z^\gamma\right).
\]
where $z^\gamma = z_0^\gamma z_1^{\gamma_1} z_2^{\gamma_2}$. On the other hand equation (4.5) can be written as
\[
1 + \sum_{p_J(\gamma) = p} I_\gamma z^\gamma = \exp \left(\sum_{p_J(\gamma) = p} \bar{\Omega}_\gamma z^\gamma\right)
\]
for any polynomial $p$. Combining these two equations, we obtain
\[
1 + \sum_{\mu_J(\gamma) = \tau} I^M_{\gamma} z^\gamma = \prod_{\nu} \exp \left(\sum_{p_J(\gamma) = p_J,\tau + \nu} \bar{\Omega}_\gamma z^\gamma\right) = \exp \left(\sum_{\mu_J(\gamma) = \tau} \bar{\Omega}_\gamma z^\gamma\right).
\]
Using the substitution $u = z_0 z_1^{\gamma_1}/r$ and $t = z_2^{-1}$, we obtain (4.17).

**Remark 4.3.** Consider a possibly non-generic polarization $J = \pm K_S$. Then $(\gamma, \gamma') = 0$ whenever $\mu_J(\gamma) = \mu_J(\gamma')$ by (2.3). By formula (2.6), we can still write the reduced Hilbert polynomial in the form (4.10) $p_J(\gamma, n) = p_J,\tau(n) + \bar{\omega}$, where $\tau = \mu_J(\gamma)$. Assuming that $J \cdot K_S < 0$, we still obtain the relation (4.11) between Gieseker invariants $I_\gamma$ and Mumford invariants $I^M_{\gamma}$. This formula can be translated into a relation between invariants $H_{r,\gamma_1}$ and $h_{r,\gamma_1}$ similar to (4.16). More precisely, by equation (2.9), we can write $-\gamma_2 = r \Delta(\gamma) - \gamma_1^2/2r$ and consider the series
\[(4.20)\]
$$t^{-\gamma_1^2/2r} H_{r,\gamma_1}(y, t; J) = \sum_{\gamma_2} I^M(\gamma, y; J) t^{-\gamma_2}$$
which behaves better than $H_{r,\gamma_1}$ as the second Chern class respects short exact sequences. We have an analogue of (4.16)
\[(4.21)\]
$$t^{-\gamma_1^2/2r} h_{r,\gamma_1} = \sum_{\sum_{i = 1}^\ell (-1)^{i-1} \prod_{i=1}^\ell t^{-r^{(i)}(\gamma_1^{(i)})^2/2r} H_{r,\gamma_1}^{(i)}(\gamma_1^{(i)}).$$

\[\diamondsuit\]

5. **Explicit results for ruled surfaces**

In this section, we determine in a number of different cases the motivic DT-invariants giving the dimensions of intersection cohomology groups in cases where $M_\gamma$ is singular. For the projective plane, these invariants were computed earlier by Götsche [5] for $r = 1$, Yoshioka [32] for $r = 2$, and the first author [17, 18] for $r \geq 3$. This section gives explicit results for $\dim \HH^0(M_\gamma)$ for the ruled surfaces $\pi : \Sigma_{g,d} \to C$. 
5.1. Wall-crossing and suitable polarization. Let $\pi : \Sigma_{g,d} \to C$ be a ruled surface with fiber $f \simeq \mathbb{P}^1$ over a smooth projective curve $C$ of genus $g$. Here $\Sigma_{g,d} = \mathbb{P}(L \oplus O_C)$ with a line bundle $L$ of degree $d \geq 0$. The intersection numbers on $\Sigma_{g,d}$ are

$$C^2 = -d \leq 0, \quad C \cdot f = 1, \quad f^2 = 0,$$

and its canonical class is $K_{\Sigma_{g,d}} = -2C + (2g - 2 - d)f$. We parametrize a polarization $J$ by $J_{m,n} = m(C + df) + nf$ with

$$m = f \cdot J_{m,n} \in \mathbb{Q}_{\geq 0}, \quad n = C \cdot J_{m,n} \in \mathbb{Q}_{\geq 0}.$$

Then $\lambda J_{m,n}$ for appropriate $\lambda \in \mathbb{Z}$ is the first Chern class of a nef line bundle. Note that the requirement $J \cdot K_S < 0$ implies the further constraint $\frac{m}{f} \geq g - 1 - \frac{d}{2}$.

We recall the generating functions $H_{r,\gamma}(y,t;J_{0,1})$ for the boundary polarization $J_{0,1} = f$.

**Proposition 5.1** (cf. [17, 24]). For the “boundary” polarization $J_{0,1}$, $H_{r,\gamma}(y,t;J_{0,1})$ are given for all $r \geq 1$ by

$$H_{r,\gamma}(y,t;J_{0,1}) = \begin{cases} H_r(y,t), & \text{if } \gamma_1 \cdot f = 0 \pmod{r}, \\ 0, & \text{if } \gamma_1 \cdot f \neq 0 \pmod{r}. \end{cases}$$

with

$$H_r(y,t) = H_r := (-y)^{-r(1-g)} \prod_{n=1}^{\infty} \frac{(1 - y^{2r+1}t^n)^{2g}(1 - y^{2r-1}t^n)^{2g}}{(1 - y^{2r}t^n)(1 - y^{2r+2}t^n)(1 - t^n)^2} \times \prod_{k=1}^{r-1} \frac{(1 - y^{-2k+1}t^n)^{2g}(1 - y^{-2k-1}t^n)^{2g}}{(1 - y^{-2k}t^n)^2(1 - y^{-2k+2}t^n)^2}.$$

**Proof.** By [24, Corollary 5.2], if $r \mid \gamma_1 \cdot f$, then

$$\tilde{Z}_f(r,\gamma_1) := \sum_{\gamma_2} P(M_{r,\gamma_2}) t^{r \Delta} = P(\text{Bun}_{C,r}) \prod_{k \geq 1} \prod_{i=-r}^{r-1} Z_C(y^{2k+2i}t^k),$$

where $\text{Bun}_{C,r}$ is the stack of vector bundles of rank $r$ and degree zero over $C$ that has a Poincaré polynomial

$$P(\text{Bun}_{C,r}) = \frac{(1 - y)^{2g}}{y^2 - 1} \prod_{i=1}^{r-1} Z_C(y^{2i}).$$

and $Z_C(t)$ is the (Poincaré polynomial of the) zeta function of the curve $C$ which has the explicit form

$$Z_C(t) = \frac{(1 - yt)^{2g}}{(1-t)(1-y^2t)}.$$

The function of our interest is

$$H_r(y,t) = \sum_{\gamma_2} (-y)^{\chi(\gamma,\gamma)} P(M_{r,\gamma_2}) t^{r \Delta}.$$

Using that $\chi(\gamma, \gamma) = r^2(-2\Delta(\gamma) + 1 - g)$, we obtain

$$H_r(y,t) = (-y)^{r^2(1-g)} P(\text{Bun}_{C,r}) \prod_{k \geq 1} \prod_{i=-r}^{r-1} Z_C(y^{2i}t^k).$$

After substitution of (5.3) and (5.4), this expression is easily rewritten to Eq. (5.2). \qed

**Remark 5.2.**

1. For $r = 1$, (5.2) agrees with Götsche [5, Theorem 0.1], and for $r = 2$ with Yoshioka [32, Theorem 0.1].

2. The generalization of Proposition 5.1 to generating functions of $E(M_{r,\gamma})$ can be found in [17, Conjecture 4.3], where it is also shown that $t^{-\Delta(\gamma)} H_r(y,t)$ may be written in terms of Dedekind eta and Jacobi theta functions.
(3) Haghighat [8] provides a string theoretic explanation of the $H_r$ for the surfaces $\Sigma_{1,d}$.

To determine $H_{r,\gamma_1}(y, t; J_{\epsilon,1})$, we need to subtract from $H_{r,\gamma_1}(y, t; J_0,1)$ the contributions due to sheaves with HN-filtrations of length $> 1$ for $J_{\epsilon,1}$. A useful tool for this is the wall-crossing formula of Joyce for $\mathcal{I}^M(\gamma)$ [12], which we now recall. We will state this formula for $\mathcal{I}^M$, although it is more generally applicable.

**Definition 5.3.** Let $(\gamma^{(i)}) = (\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(\ell)})$ be a tuple of Chern characters with $\gamma^{(i)} = (r^{(i)}, \gamma_1^{(i)}, \gamma_2^{(i)})$ and $r^{(i)} \in \mathbb{N}_{>0} \forall i$. We define $S((\gamma^{(i)}), J, J')$ as follows. If for all $i = 1, \ldots, \ell - 1$, we have either

(a) $\mu_J(\gamma^{(i)}) \leq \mu_J(\gamma^{(i+1)})$ and $\mu_J(\sum_{j=1}^{\ell} \gamma^{(j)}) > \mu_J(\sum_{j=i+1}^{\ell} \gamma^{(j)})$, or

(b) $\mu_J(\gamma^{(i)}) > \mu_J(\gamma^{(i+1)})$ and $\mu_J(\sum_{j=1}^{\ell} \gamma^{(j)}) \leq \mu_J(\sum_{j=i+1}^{\ell} \gamma^{(j)})$,

then $S((\gamma^{(i)}), J, J') = (-1)^k$ where $k$ is the number of $i = 1, \ldots, \ell - 1$ such that (a) is correct. Otherwise, $S((\gamma^{(i)}), J, J') = 0$.

Then we have the following theorem of Joyce.

**Theorem 5.4** ([12, Theorem 6.21]). Under a change of polarization $J \to J'$ the invariants $\mathcal{I}^M(\gamma, y; J)$ are expressed in terms of $\mathcal{I}^M(\gamma, y; J)$ by

$$\mathcal{I}^M(\gamma, y; J') = \sum_{\sum_{i=1}^{\ell} \gamma^{(i)} = \gamma, \ r^{(i)} \geq 1 \forall i} S \left( (\gamma^{(i)}), J, J' \right) (-y)^{-\sum_{j<i} (r^{(i)} \gamma_1^{(i)} - r^{(j)} \gamma_1^{(j)}) \cdot K_2(\Pi_{i=1}^{\ell} \mathcal{I}^M(\gamma^{(i)}, y; J)).$$

Our first aim is to determine a generating function for the invariants $\mathcal{I}^M(\gamma, y; J)$ with the polarization $J$ sufficiently close to $J_{0,1} = f$, such that no walls exist between $J$ and $f$. Such a polarization clearly depends on $\gamma$ which is made precise in the following definition and proposition following [1, Remark 5.3.6].

**Definition 5.5.** A $\gamma$-suitable polarization $J$ is a polarization such that for any $\xi \in \text{Pic}(S)$ satisfying the following two conditions:

(1) $\xi^2$ is bounded by

$$-\frac{r^4}{2} \Delta(\gamma) \leq \xi^2 < 0,$$

(2) either $\xi \cdot f = 0$ or $(\xi \cdot f)(\xi \cdot J) > 0$.

**Proposition 5.6.** No walls exist between $f$ and a $\gamma$-suitable polarization $J$.

**Proof.** From Equations (2.7) and (2.8) we deduce that a wall for $\gamma$ exists between $f$ and $J$ if there exist $\gamma^{(1)}$ and $\gamma^{(2)}$ such that $(r^{(1)} \gamma_1^{(1)} - r^{(2)} \gamma_1^{(1)}) \cdot f$ and $(r^{(1)} \gamma_1^{(2)} - r^{(2)} \gamma_1^{(1)}) \cdot J$ have a different sign. We set $r^{(1)} \gamma_1^{(2)} - r^{(2)} \gamma_1^{(1)} = \xi \in H^2(S, \mathbb{Z})$. Then from (2.15) we find that

$$r \Delta(\gamma) = \sum_{i=1,2} r^{(i)} \Delta^{(i)} - \frac{1}{2r^{(1)}r^{(2)}r} \xi^2.$$

By the Bogomolov inequality $r \Delta \geq 0$, and therefore we arrive at

$$-2r^{(1)}r^{(2)} \Delta(\gamma) \leq \xi^2.$$

The left hand side is minimized by $r^{(1)} = r^{(2)} = r/2$. Moreover, $\xi$ is negative definite, $\xi^2 < 0$, since $\xi \cdot f$ and $\xi \cdot J$ have a different sign and the signature of $H^2(S, \mathbb{Z})$ is $(1, b_2(S) - 1) = (1, 1)$. Thus, the $\xi$ satisfy Condition (1) in Definition 5.5. However, the different sign violates Condition (2) and therefore no walls exist between $f$ and a suitable polarization $J$.

The next proposition gives a closed expression for the generating function for invariants $\mathcal{I}^M(\gamma, y; J)$ for a suitable polarization $J = J_{\epsilon,1}$. Since the generating function sums over all $\gamma_2$, the choice of suitable polarization $J_{\epsilon,1}$ is determined as follows. Truncate $H_{r,\gamma_1}(y, t; J_{\epsilon,1})$ at
some power $t^K$, with $K$ the largest value of $r\Delta$ of interest, and denote the corresponding $\gamma$ by $\gamma_{\text{max}}$. Then $J_{r,1}$ is chosen such that it is $\gamma_{\text{max}}$-suitable, which implies by Equation (5.6) that $J_{r,1}$ is $\gamma$-suitable for the terms of $H_{r,\gamma_1}$ with $r\Delta(\gamma) < K$.

We have the following proposition

**Proposition 5.7.** Assume $|y| < 1$ and $\gamma_1 = \beta C - \alpha f$, then $H_{r,\gamma_1}(J_{r,1})$ equals

\[
H_{r,\gamma_1}(J_{r,1}) = \begin{cases} 
\sum_{r^{(i)} + \ldots + r^{(\ell)} = r} y^{2\sum_{i=2}^r r^{(i)} + r^{(i-1)}} ((\frac{df}{d\gamma})^r + \sum_{k=1}^{r^{(k)}} r^{(k)})) \prod_{i=1}^\ell H_{r^{(i)}}, & \text{if } \gamma_1 \cdot f = 0 \pmod{r}, \\
0, & \text{if } \gamma_1 \cdot f \neq 0 \pmod{r},
\end{cases}
\]

where $\{ x \} = x - \lfloor x \rfloor$ is the fractional part of $x$.

**Proof.** The proof follows the proof of [18, Proposition 4.1]. We substitute the wall-crossing formula of Theorem 5.4 with $J = J_{0,1}$ and $J' = J_{r,1}$ in $H_{r,\gamma_1}(J_{r,1})$,

\[
H_{r,\gamma_1}(J_{r,1}) = \sum_{\gamma_1} \sum_{r^{(i)} \geq 1} S((\gamma^{(i)}), J_{0,1}, J_{r,1}) (-y)^{-\sum_{j<i}(r^{(j)} - \gamma_1^{(i)} - \gamma_1^{(j)})} K_S \prod_{i=1}^\ell r^{(i)}.
\]

To evaluate the sum we parametrize the first Chern classes $\gamma^{(i)}_1$ as $\gamma^{(i)}_1 = b^{(i)} C - a^{(i)} f$, such that $\sum_{i=1}^{\ell} a^{(i)} = \alpha$ and $\sum_{i=1}^{\ell} b^{(i)} = \beta$. Then we have from Theorem 5.4 that $S((\gamma^{(i)}), J_{0,1}, J_{r,1})$ is non-vanishing if for all $i = 1, \ldots, \ell$, we have

(a) either

\[
\frac{b^{(i)}}{r^{(i)}} \leq \frac{b^{(i+1)}}{r^{(i+1)}} \quad \text{and} \quad \frac{\sum_{j=1}^{i} b^{(j)} - a^{(j)}}{\sum_{j=1}^{i} r^{(j)}} > \frac{\sum_{j=i+1}^{\ell} b^{(j)} - a^{(j)}}{\sum_{j=i+1}^{\ell} r^{(j)}},
\]

(b) or

\[
\frac{b^{(i)}}{r^{(i)}} > \frac{b^{(i+1)}}{r^{(i+1)}} \quad \text{and} \quad \frac{\sum_{j=1}^{i} b^{(j)} - a^{(j)}}{\sum_{j=1}^{i} r^{(j)}} \leq \frac{\sum_{j=i+1}^{\ell} b^{(j)} - a^{(j)}}{\sum_{j=i+1}^{\ell} r^{(j)}}.
\]

Since $J_{r,1}$ is a suitable polarization we deduce that $S((\gamma^{(i)}), J_{0,1}, J_{r,1})$ can only be non-vanishing if $\frac{b^{(i)}}{r^{(i)}} = \frac{\beta}{\alpha}$ for all $i = 1, \ldots, \ell$. If in addition $\sum_{i=1}^{\ell} a^{(i)} < \sum_{j=1}^{\ell} r^{(j)}$ for all $i$, then $S((\gamma^{(i)}), J_{0,1}, J_{r,1}) = (-1)^{\ell-1}$. Thus we find in particular that for $\gamma_1 \cdot f \neq 0 \pmod{r}$, $H_{r,\gamma_1}(y, f, J_{r,1}) = 0$.

Next we make the change of variables from $a^{(i)}$ to $s^{(i)}$ defined by

\[
a^{(i)} = s^{(i)} - s^{(i+1)}, \quad i = 1, \ldots, \ell - 1, \quad a^{(\ell)} = s^{(\ell)},
\]

or inversely $s^{(i)} = \sum_{j=i}^{\ell} a^{(j)}$. Eliminating $a^{(1)}$ using $a^{(1)} = \alpha - s^{(2)}$, we find that the summation in Eq. (5.9) reduces to all $s^{(i)}$, $i \geq 2$ such that $s^{(i)} > \sum_{j<i} r^{(j)} \alpha$. The exponent of $y$ in (5.9) in terms of the new variables becomes

\[
- \sum_{j<i} (r^{(j)} \gamma^{(i)}_1 - r^{(i)} \gamma^{(j)}_1) \cdot K_S = -2 \sum_{j<i} (r^{(j)} a^{(i)} - r^{(i)} a^{(j)})
\]

(5.10)

\[
= 2\alpha (r - r^{(1)}) - 2 \sum_{j=2}^{\ell} (r^{(j)} + r^{(j-1)}) s^{(j)}.
\]

To evaluate the sum for $\gamma_1 \cdot f = 0 \pmod{r}$, we first note that (2.15) simplifies in the present situation to $r\Delta = \sum_{i=1}^{\ell} r^{(i)} \Delta^{(i)}$. Assuming that $|y| < 1$, the geometric sums over $s^{(i)}$ can be carried out, such that

\[
H_{r,\gamma_1}(J_{r,1}) = \sum_{r^{(i)} + \ldots + r^{(\ell)} = r} (-1)^{\ell-1} y^{2\alpha (r - r^{(1)}) - 2 \sum_{j=2}^{\ell} (r^{(j)} + r^{(j-1)}) (1 + |\sum_{i=1}^{\ell} r^{(i)} |)} \prod_{i=1}^{\ell} H_{r^{(i)}}.
\]
After multiplication of numerator and denominator by $\prod_{i=2}^{\ell} y - 2(r^{(i)} + r^{(i-1)})$ and using the identity $(r - r_1)r = \sum_{i=2}^{\ell} (r^{(i)} + r^{(i-1)}) \sum_{k=i}^{r^{(k)}} r^{(k)}$, we arrive at the desired result. \qed

**Remark 5.8.** $H_{r,\gamma}(y, t, J_{\varepsilon,1})$ can be analytically continued beyond $|y| = 1$. ◊

5.2. Rank 2. In this subsection, we apply the formulas discussed above to determine $\dim IH^r(M_\gamma)$ for rank 2 sheaves in a number of cases. Considering $(r, \gamma) = (2, 0)$, Proposition 5.7 gives for $H_{2,0}$

\begin{equation}
H_{2,0}(J_{\varepsilon,1}) = H_2 + \frac{1}{1 - y^4} H_1^2.
\end{equation}

Since the suitable polarization $J_{\varepsilon,1}$ is generic, we determine $h_{2,0}$ using (4.16),

\begin{equation}
h_{2,0}(J_{\varepsilon,1}) = H_{2,0}(J_{\varepsilon,1}) - \frac{1}{2} H_1^2.
\end{equation}

Following (4.6), the generating function of $\dim IH(M_\gamma)$, $\sum_{\gamma} \Omega_\gamma t^{r \Delta(\gamma)}$, is then given by

\begin{equation}
(y^{-1} - y) \left( h_{2,0}(y, t; J_{\varepsilon,1}) - \frac{1}{2} H_1(y^2, t^2) \right).
\end{equation}

We list $b_n := \dim IH^r(M_\gamma)$ and numerical DT invariants $\Omega_\gamma^{\text{num}} = \Omega(\gamma, -1; J)$ for $S = \Sigma_{0,d}$ and for small $\gamma$ in Table 1. Note that for a suitable polarization the $\Omega_\gamma$ are independent of $d$. The numbers are listed up to $\dim_{\mathbb{C}} M_\gamma$; those with $n > \dim_{\mathbb{C}} M_\gamma$ are determined by Poincaré duality $b_n = b_{2 \dim_{\mathbb{C}} M_\gamma - n}$.

| $\gamma_2$ | $b_0$ | $b_2$ | $b_4$ | $b_6$ | $b_8$ | $b_{10}$ | $b_{12}$ | $b_{14}$ | $b_{16}$ | $b_{18}$ | $b_{20}$ | $b_{22}$ | $b_{24}$ | $\Omega_\gamma^{\text{num}}$ |
|-----------|-------|-------|-------|-------|-------|----------|----------|----------|----------|----------|----------|----------|----------|----------------|
| 2         | 1     | 1     | 2     | 3     |       |          |          |          |          |          |          |          |          | -12          |
| 3         | 1     | 3     | 8     | 16    | 20    |          |          |          |          |          |          |          |          | -96          |
| 4         | 1     | 3     | 10    | 24    | 51    | 82       | 103      |          |          |          |          |          |          | -548         |
| 5         | 1     | 3     | 10    | 26    | 62    | 130      | 232      | 348      | 420      |          |          |          |          | -2464        |
| 6         | 1     | 3     | 10    | 26    | 65    | 144      | 301      | 555      | 913      | 1284     | 1518     |          |          | -9640        |
| 7         | 1     | 3     | 10    | 26    | 65    | 147      | 318      | 642      | 1203     | 2065     | 3172     | 4280     | 4964     | -33792       |

**Table 1.** Table with $b_n$ (with $n \leq \dim_{\mathbb{C}} M_\gamma$) and the numerical DT invariant $\Omega^{\text{num}}$ of $J_{\varepsilon,1}$-semi-stable sheaves on $\Sigma_{0,d}$ with $r = 2$, $\gamma_1 = 0$, and $2 \leq \gamma_2 \leq 7$.

Tensoring a sheaf $F$ on $\Sigma_{0,d}$ with a line bundle with $\gamma_1 = \gamma_2 = 0$ does not change $\gamma(F)$. As a result, the moduli space $M_\gamma$ is a fibration with fibre the moduli space of such line bundles, i.e. the Jacobian of $C$. This further implies that the intersection Poincaré polynomial involves a factor $(1 - y)^{2g}$. To concisely tabulate the motivic DT invariants, we define a new set of numbers $b'_n$ through

\begin{equation}
\text{IP}(M_\gamma) =: (1 - y)^{2g} \sum_{n=0}^{2 \dim_{\mathbb{C}} M_\gamma - 2g} b'_n (-y)^n.
\end{equation}

We list in the Tables 2 and 3 below the $b'_n$ for $g = 1$ and $g = 2$ for $n \leq \dim_{\mathbb{C}} M_\gamma - g$. The numbers $b'_n$ with $n > \dim_{\mathbb{C}} M_\gamma - g$ are again determined by Poincaré duality.

| $\gamma_2$ | $b'_0$ | $b'_1$ | $b'_2$ | $b'_3$ | $b'_4$ | $b'_5$ | $b'_6$ | $b'_7$ | $b'_8$ | $b'_9$ | $b'_{10}$ | $b'_{11}$ | $b'_{12}$ |
|-----------|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|-----------|-----------|-----------|
| 1         | 1     | 1      | 2      | 2      |        |        |        |        |        |        |           |           |           |
| 2         | 1     | 2      | 4      | 10     | 17     | 24     | 30     | 32     | 32     |        |           |           |           |
| 3         | 1     | 2      | 4      | 10     | 21     | 40     | 68     | 108    | 163    | 218    | 256       | 278       | 286       |

**Table 2.** Table with $b'_n$ (5.13) of $J_{\varepsilon,1}$-semi-stable sheaves on $\Sigma_{1,d}$ with $r = 2$, $\gamma_1 = 0$, and $1 \leq \gamma_2 \leq 3$.

For other polarizations, we can determine $H_{2,\gamma_1}(J_{m,n})$ using the wall-crossing formula of Theorem 5.4. Without loss of generality, we can set $\gamma_1 = \beta C - \alpha f$ with $\alpha$ and $\beta$ either 0 or 1. For
Table 3. Table with $b'_n$ (5.13) of $J_{r,1}$-semi-stable sheaves on $\Sigma_{2,d}$ with $r = 2$, $\gamma_1 = 0$, and $0 \leq \gamma_2 \leq 3$.

$$r = 2, \text{ we have either } \ell = 1 \text{ or } 2. \text{ Setting for } \ell = 2, a^{(1)} = -a \text{ and } a^{(2)} = a + \alpha, \text{ and similarly for } b^{(1)} \text{ and } b^{(2)}, \text{ we arrive at}$$

$$H_{2,\gamma_1}(J_{m,n}) = H_{2,\gamma_1}(J_{r,1})$$

$$+ \frac{1}{2} H_1^2 \sum_{a \in \mathbb{Z} + \frac{\alpha}{2}} \sum_{b \in \mathbb{Z} + \frac{\beta}{2}} (\sgn(2nb - 2ma + v) - \sgn(2b - 2a\varepsilon + v))$$

$$\times y^{2b(d-2+2g)-4a\varepsilon} d b^{2+2ab},$$

with $0 < v \ll \varepsilon$. Here we used the notation $\sgn(x) - \sgn(y)$, familiar from the theory of indefinite theta functions [6, 7, 33]. Note that for general $J_{m,n}$ the invariants do depend on $d$.

The infinite sum over $a$ is a geometric series and can be resummed. For example for $\gamma_1 = 0$, one has

$$H_{2,0}(J_{m,n}) = H_2 + H_1^2 \sum_{b \in \mathbb{Z} + \frac{\alpha}{2}} \sum_{a \in \mathbb{Z} + \frac{\beta}{2}} y^{2b(d-2+2g)-4a\varepsilon} d b^{2+2ab} \frac{1}{1 - y^2 t^{-2b}}.$$ 

Note that for given $b$, there is only one value of $a$ contributing to the sum on the right hand side.

As an example of a non-suitable polarization we take $J_{6,5}$, which is generic for small $r\Delta$. We list invariants $b'_n$ for $\Sigma_{1,0}$ and $\Sigma_{1,2}$ in the following tables

| $\gamma_2$ | $b'_0$ | $b'_1$ | $b'_2$ | $b'_3$ | $b'_4$ | $b'_5$ | $b'_6$ | $b'_7$ | $b'_8$ | $b'_9$ | $b'_{10}$ | $b'_{11}$ | $b'_{12}$ |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|----------|----------|----------|
| 0          | 1      | 0      | 1      |        |        |        |        |        |        |        |          |          |          |
| 1          | 1      | 4      | 3      | 12     | 21     | 20     | 23     | 24     |        |        |          |          |          |
| 2          | 1      | 4      | 9      | 20     | 48     | 80     | 139    | 224    | 304    | 364    | 387      | 408      |          |
| 3          | 1      | 4      | 9      | 24     | 60     | 124    | 234    | 432    | 762    | 1216   | 1820     | 2600     | 3359     | 3904     | 4251     | 4384    |

Table 4. Table with $b'_n$ (5.13) of $J_{6,5}$-semi-stable sheaves on $\Sigma_{1,0}$ with $r = 2$, $\gamma_1 = 0$, and $1 \leq \gamma_2 \leq 3$.

| $\gamma_2$ | $b'_0$ | $b'_1$ | $b'_2$ | $b'_3$ | $b'_4$ | $b'_5$ | $b'_6$ | $b'_7$ | $b'_8$ | $b'_9$ | $b'_{10}$ | $b'_{11}$ | $b'_{12}$ |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|----------|----------|----------|
| 1          | 1      | 2      | 2      | 2      |        |        |        |        |        |        |          |          |          |
| 2          | 1      | 2      | 4      | 10     | 18     | 26     | 32     | 34     | 34     |        |          |          |          |
| 3          | 1      | 2      | 4      | 10     | 21     | 21     | 20     | 24     | 30     | 32     | 32        |          |          |

Table 5. Table with $b'_n$ (5.13) of $J_{6,5}$-semi-stable sheaves on $\Sigma_{1,2}$ with $r = 2$, $\gamma_1 = 0$, and $1 \leq \gamma_2 \leq 3$.

Finally we consider a non-generic polarization, namely $J = -K_{\Sigma_{g,d}}$. For this choice of polarization the torus is commutative and the invariants $I(\gamma, y; J)$ can be related to $\dim \text{IH}(M_\gamma)$. The anti-canonical class $-K_{\Sigma_{g,d}}$ does only lie in the ample cone for $g = 0$ and $d = 0, 1$. For these cases we have $-K_{\Sigma_{0,d}} = J_{2,2-d}$. Since $J_{2,2-d}$ is non-generic, we need to consider in more detail partitions $\sum_{i=1}^{\ell} \gamma^{(i)} = \gamma$ of $\gamma$ such that $p_J(\gamma^{(i)}) = p_J(\gamma)$ for $i = 1, \ldots, \ell$. While this implies that $\gamma^{(i)}$ are proportional for generic $J$, for $J = -K_{\Sigma_{0,d}}$ this can also occur for $\gamma^{(i)}$ which are not proportional. Equation (4.4) shows that we need take these partitions in to account to determine
\[ \tilde{\Omega}_\gamma \text{ from } \mathcal{Z}_\gamma. \] We consider first the case \( \Sigma_{0,0} \) together with \( \gamma_1 = 0 \) such that the slope vanishes. Then the \( \gamma_1^{(i)} \) which satisfy \( -\gamma_1^{(i)} \cdot K_{\Sigma_{0,0}} = 0 \) are of the form \( a^{(i)}(C - f) \). Similarly for \( d = 1 \), the \( \gamma_1^{(i)} \) are of the form \( a^{(i)}(2C - f) \) lead to a vanishing slope. As prescribed by Equation (4.21), we sum over all \( a^{(1)} = -a^{(2)} = a \in \mathbb{Z} \) and find that for \( d = 0, 1 \) the function \( h_{2,0}(y, t; -K_{\Sigma_{0,0}}) \) is given by

\[ h_{2,0}(-K_{\Sigma_{0,0}}) = H_{2,0}(J_{2,2-d}) - \frac{1}{2} H_1^2 \sum_{a \in \mathbb{Z}} t^{2(1+3d)a^2}. \]

For this polarization the motivic DT invariants are listed in Tables 6 and 7.

| \( \gamma_2 \) | \( b_0 \) | \( b_2 \) | \( b_4 \) | \( b_6 \) | \( b_8 \) | \( b_{10} \) | \( b_{12} \) | \( b_{14} \) | \( b_{16} \) | \( b_{18} \) | \( b_{20} \) | \( b_{22} \) | \( b_{24} \) | \( \Omega_{\gamma_2}^{\text{num}} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | -12 |
| 3 | 1 | 3 | 8 | 16 | 20 | 83 | 104 | 10 | 16 | 20 | 354 | 428 | 1209 | 2091 | -96 |
| 4 | 1 | 3 | 10 | 24 | 51 | 83 | 104 | 130 | 234 | 354 | 428 | 1209 | 2091 | -552 |
| 5 | 1 | 3 | 10 | 26 | 62 | 130 | 234 | 354 | 428 | 1209 | 2091 | 3244 | 4416 | -2496 |
| 6 | 1 | 3 | 10 | 26 | 65 | 144 | 301 | 559 | 927 | 1316 | 1560 | 3244 | 4416 | -9824 |
| 7 | 1 | 3 | 10 | 26 | 65 | 147 | 318 | 642 | 1209 | 2091 | 3244 | 4416 | 5140 | -34624 |

Table 6. The motivic DT invariants \( b_\gamma \) and the numerical DT invariant \( \Omega_\gamma^{\text{num}} \) of \( J_{1,1,\text{-semi-stable sheaves on } \Sigma_{0,0} \) with \( r = 2, \gamma_1 = 0, \) and \( 2 \leq \gamma_2 \leq 7. \)

| \( \gamma_2 \) | \( b_0 \) | \( b_2 \) | \( b_4 \) | \( b_6 \) | \( b_8 \) | \( b_{10} \) | \( b_{12} \) | \( b_{14} \) | \( b_{16} \) | \( b_{18} \) | \( b_{20} \) | \( b_{22} \) | \( b_{24} \) | \( \Omega_{\gamma_2}^{\text{num}} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | -12 |
| 3 | 1 | 3 | 8 | 16 | 21 | 84 | 109 | 10 | 16 | 21 | 362 | 349 | 429 | 514 | -98 |
| 4 | 1 | 3 | 10 | 24 | 51 | 84 | 109 | 130 | 234 | 362 | 349 | 429 | 514 | -2558 |
| 5 | 1 | 3 | 10 | 26 | 62 | 130 | 234 | 362 | 349 | 429 | 514 | 4551 | 5379 | -564 |
| 6 | 1 | 3 | 10 | 26 | 65 | 144 | 301 | 561 | 939 | 1352 | 1634 | 3299 | 4551 | -10072 |
| 7 | 1 | 3 | 10 | 26 | 65 | 147 | 318 | 642 | 1212 | 2091 | 3299 | 4551 | 5379 | -35518 |

Table 7. The motivic DT invariants \( b_\gamma \) and numerical DT invariant \( \Omega_\gamma^{\text{num}} \) of \( J_{2,1,\text{-semi-stable sheaves on } \Sigma_{0,1} \) with \( r = 2, \gamma_1 = 0, \) and \( 2 \leq \gamma_2 \leq 7. \)

5.3. Rank 3. As example of higher rank sheaves, we tabulate in this section \( \dim \text{IH}(M_{\gamma}) \) with \( r = 3 \) in various cases. First we consider a suitable polarization for \( (r, \gamma_1) = (3, 0) \). Then Proposition 5.7 evaluates to

\[ H_{3,0}(J_{\epsilon,1}) = H_3 + \frac{2}{1 - y^6} H_1 H_2 + \frac{24}{1 - y^6} H_1 H_3, \]

Then the generating function \( h_{3,0}(y, t, J_{\epsilon,1}) \) (4.14) follows from Theorem 4.2 and is given by

\[ h_{3,0}(J_{\epsilon,1}) = H_{3,0}(J_{\epsilon,1}) - H_1 H_{2,0}(J_{\epsilon,1}) + \frac{1}{3} H_3. \]

The generating function of \( \Omega_\gamma \) is then given by \( h_{3,0}(y, t, J_{\epsilon,1}) - \frac{4}{3} H_1(y^3, t^3) \). We list in Tables 8, 9 and 10 the invariants \( b_\gamma \) defined as in Equation (5.13)

Invariants for other values of \( J \) can again be determined using the wall-crossing formula.

\[
S((\gamma^{(i)}); J_{0,1}, M_{n,m}) = \frac{(-1)^{t-1}}{2^{t-1}} \left( \prod_{i=2}^{t} \left( \text{sgn}(b^{(i)} - b^{(i-1)}) + v \right) - \text{sgn} \left( \sum_{j=1}^{t-1} \sum_{k=i}^{t} [r^{(j)} b^{(k)} n - a^{(k)} m] - \sum_{k=i}^{t} \sum_{j=1}^{t} [r^{(j)} b^{(k)} n - a^{(j)} m] + v \right) \right)
\]
To determine the contribution of partitions of $\gamma$ with $(r^{(1)}, r^{(2)}) = (2, 1)$, we set $\gamma_1^{(1)} = -2bC + af$ and $\gamma_1^{(2)} = 2bC - af$

\begin{equation}
H_2 H_1 \sum_{a, b \in \mathbb{Z}} (\text{sgn}(2bn - am + v) - \text{sgn}(b + v)) y^{-6b(2g - 2 + d) - 6a} \delta^{3db^2 + 3ab},
\end{equation}

with $0 < v \ll 1$. This can be resummed to

\begin{equation}
2 H_2 H_1 \sum_{2bn - am \in \{0, 1\}} \frac{y^{-6b(2g - 2 + d) - 6a} \delta^{3db^2 + 3ab}}{1 - y^6 e^{-3b}}.
\end{equation}

The contribution due to $(r^{(1)}, r^{(2)}) = (1, 2)$ is identical to the above.

For the contribution of partitions with $r^{(i)} = 1$ for $i = 1, 2, 3$, we set $\gamma^{(1)} = -\gamma^{(2)} - \gamma^{(3)}$ and $b^{(2)} = b_1, b^{(3)} = b_2, a^{(2)} = a_1, a^{(3)} = a_2$. Then we arrive at

\begin{equation}
\frac{1}{4} H_1^3 \sum_{b_1, a_1 \in \mathbb{Z}} (\text{sgn}(b_1 + b_2)n - (a_1 + a_2)m + v) - \text{sgn}(2b_1 + b_2 + v)) \times (\text{sgn}(b_2 n - a_2 m + v) - \text{sgn}(b_2 - b_1 + v))
\end{equation}

\begin{equation}
\times y^{-2(b_1 + 2b_2)(2g - 2 + d) - 4(a_1 + 2a_2)} \delta^{d(b_1^2 + b_2^2 + b_1 + b_2) + 2a_1 b_1 + 2a_2 b_2 + b_1 a_2 + b_2 a_1}.
\end{equation}

This can be resummed to

\begin{equation}
H_1^3 \sum_{m \in \{0, 1\}} \sum_{b_1, a_1 \in \mathbb{Z}} \sum_{m \in \{0, 1\}} y^{-2(b_1 + 2b_2)(2g - 2 + d) - 4(a_1 + 2a_2)} \delta^{d(b_1^2 + b_2^2 + b_1 + b_2) + 2a_1 b_1 + 2a_2 b_2 + b_1 a_2 + b_2 a_1}.
\end{equation}

We list in Tables 11 and 12 the invariants for the surfaces $\Sigma_{1,0}$ and $\Sigma_{1,2}$ with polarization $J_{0,5}$.

Finally, we consider the polarization $J = -K_{\Sigma_{0, d}}$ for $S = \Sigma_{0, d}$ with $d = 0, 1$. As in the case of $r = 2$, sheaves with equal slope do not necessarily have proportional Chern character for this non-generic polarization. Besides sheaves with Chern character proportional to $(3, 0, \gamma_2)$, all sheaves with $\gamma_1^{(i)} = a^{(i)}(C - f)$, have slope 0 for $d = 0$. For $r^{(i)} = 2$, we need to distinguish between
For the first few values of $a^{(i)}$ even and odd, since the invariants differ in these cases. For $d = 1$ the same comments apply except that in this case sheaves with $\gamma^{(i)} = a^{(i)}(2C - f)$ have a vanishing slope.

Subtracting the contributions of these sheaves from $H_{3,0}(J_{2,2,-d})$ as in Equation (4.21), we arrive at $h_{3,0}(J_{2,2,-d})$ for $d = 0, 1$

$$h_{3,0}(J_{2,2,-d}) = H_{3,0}(J_{2,2,-d}) - H_1 H_{2,0}(J_{2,2,-d}) \sum_{a \in \mathbb{Z}} 6(1 + 3d)a^2$$

$$- H_1 H_{2,(d-1)C+f}(J_{2,2,-d}) \sum_{a \in \mathbb{Z}} 6(1 + 3d)(a^2 + a + 4)$$

$$+ \frac{1}{3} H_1^3 \sum_{a_1, a_2 \in \mathbb{Z}} 2(1 + 3d)(a_1^2 + a_2^2 + a_1 a_2).$$

For the first few values of $\gamma_2$, the invariants are listed in Tables 13 and 14.

Table 11. Table with $b'_n$ of $J_{6,5}$-semi-stable sheaves on $\Sigma_{1,0}$ with $r = 3, \gamma_1 = 0, \gamma_2$ and $0 \leq \gamma_2 \leq 2$.

Table 12. Table with $b'_n$ (5.13) of $J_{6,5}$-semi-stable sheaves on $\Sigma_{1,2}$ with $r = 3, \gamma_1 = 0, \gamma_2 \leq 3$.

Table 13. The motivic DT invariants $b_n$ and numerical DT invariant $\Omega^{\text{num}}$ of $J_{1,1}$-semi-stable sheaves on $\Sigma_{0,0}$ with $r = 3, \gamma_1 = 0$, and $3 \leq \gamma_2 \leq 6$.

Table 14. The motivic DT invariants $b_n$ and numerical DT invariant $\Omega^{\text{num}}$ of $J_{2,1}$-semi-stable sheaves on $\Sigma_{0,1}$ with $r = 3, \gamma_1 = 0$, and $3 \leq \gamma_2 \leq 6$.

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