LAGRANGIAN REDUCTION OF NONHOLONOMIC DISCRETE MECHANICAL SYSTEMS BY STAGES

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ABSTRACT. In this work we introduce a category \( \mathcal{LDP}_d \) of discrete-time dynamical systems, that we call discrete Lagrange–D’Alembert–Poincaré systems, and study some of its elementary properties. Examples of objects of \( \mathcal{LDP}_d \) are nonholonomic discrete mechanical systems as well as their lagrangian reductions and, also, discrete Lagrange-Poincaré systems. We also introduce a notion of symmetry group for objects of \( \mathcal{LDP}_d \) and a process of reduction when symmetries are present. This reduction process extends the reduction process of discrete Lagrange–Poincaré systems as well as the one defined for nonholonomic discrete mechanical systems. In addition, we prove that, under some conditions, the two-stage reduction process (first by a closed and normal subgroup of the symmetry group and, then, by the residual symmetry group) produces a system that is isomorphic in \( \mathcal{LDP}_d \) to the system obtained by a one-stage reduction by the full symmetry group.

1. Introduction. Mechanical systems are dynamical systems that are used to model a wide variety of aspects of the real world, from the falling apple to the movement of astronomical objects, including machinery and billiards (see, for instance, [17] and [1]). One of the flavors of Mechanics —Lagrangian or Variational Mechanics— describes the evolution of a mechanical system using a variational principle defined in terms of a function, the Lagrangian, \( L : TQ \to \mathbb{R} \), where \( Q \) is the configuration manifold of the system. Nonholonomic mechanical systems, which describe systems containing rolling or sliding contact (such as wheels or skates),

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add constraints —in the form of a non-integrable subbundle $\mathcal{D} \subset TQ$— to the variational principle (see, for instance, [3] and [6]).

**Numerical integrators and discrete mechanical systems.** As in many applications it is essential to predict the evolution of a mechanical system, the equations of motion that can be derived from the corresponding variational principle must be solved. Solving these ordinary differential equations can be quite difficult in practice, so numerical integrators are used to find approximate solutions to those equations. The standard methods for numerically approximating solutions of ODEs do not necessarily preserve the structural characteristics of the solutions of the equations of motion of mechanical systems (see [18]). Discrete mechanical systems were introduced as a way of modeling discrete-time analogues of mechanical systems; the evolution of a discrete mechanical system is also defined in terms of a variational principle for the discrete Lagrangian $L_d : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathbb{R}$; this formalism is extended to deal with more general systems, including forced discrete systems as well as discrete nonholonomic ones (see [24] and [9]). The equations of motion of discrete mechanical systems are algebraic equations whose solutions are numerical integrators for the corresponding continuous system. In many cases, these integrators have very good structural characteristics (especially when considering long-time evolution), that resemble those of the continuous system ([29] and [18]).

**Symmetries and symmetry reduction.** It is a natural idea to think that when a mechanical or, more generally, a dynamical system has some degree of symmetry, it should be possible to gain some insight into its dynamics by studying some other “simplified system” obtained by eliminating or locking the symmetry. This process is usually known as the reduction of the given system and the resulting system is known as the reduced system. In the case of Classical Mechanics, this idea seems to go back as far as the work of Lagrange. Over time, it has become a technique that has been applied in both the Lagrangian and Hamiltonian formalisms, for unconstrained systems as well as for holonomically and nonholonomically constrained ones (see among many other references, [3] [6], [2], [30, 31] [28], [25], [26], [4] and [23]). The reduction process has also been applied to discrete-time mechanical systems with and without constraints (see, for instance, [21], [27], [20] and [13]).

It is well known that, in most instances, the reduction of a mechanical system is not a mechanical system but, rather, a more general dynamical system: that is, while the dynamics of a mechanical system on $\mathcal{Q}$ is defined using a variational principle for the Lagrangian, defined on $T\mathcal{Q}$ in the continuous case or on $\mathcal{Q} \times \mathcal{Q}$ in the discrete case (and, maybe, other additional data), the dynamics of the reduced system is determined by a function that is usually not defined on a tangent bundle or a Cartesian product (of a manifold with itself). This can be problematic if one expects to analyze the reduced system with the same techniques as the original one. That issue has usually been solved by passing from the family of mechanical systems to a larger class of dynamical systems, where there is a reduction process that is closed within this larger class. Such is the case, for example, of mechanical systems on Lie algebroids and Lie groupoids (see [19] and [20]). In this paper we follow this guiding principle, but choose the larger class following ideas adapted from [7] and [14].

**Reduction by stages.** Sometimes, it may be convenient to eliminate part of the symmetric behavior of a mechanical system, while keeping some residual symmetry
that could be analyzed at a later point, if so desired. In this case a second reduction step to eliminate the residual symmetry is possible. A natural question is, in that case, whether the result of this two-stage reduction is equivalent to the full reduction of all the symmetries at one time. The equivalence of the two-stage and one-stage reduction processes has been established in several cases. For instance, for Lagrangian systems without constraints by H. Cendra, J. Marsden and T. Ratiu in [7], for Lagrangian systems with nonholonomic constraints by H. Cendra and V. Díaz in [5], for Hamiltonian systems by J. Marsden et al. in [26] and, for unconstrained discrete mechanical systems by the authors in [14].

**Aims.** The main purpose of the present work is to establish the same equivalence described in the previous paragraph for nonholonomically constrained discrete-time mechanical systems. In this respect, the paper is an extension of [14] to the constrained setting that parallels [5] in the discrete-time context. We stress that, for the present paper, whether a discrete-time system is related to a (continuous-time) mechanical system or not is irrelevant; in this respect, our analysis and results are completely independent of the discretization process chosen to produce the discrete dynamical system in question, if one was used at all. As an aside, we also mention that, at the moment, the very interesting subject of geometric discretization of nonholonomic mechanical systems should be regarded as “work in progress”, without conclusive hard results on the quality of the numerical integrators obtained.

**Constructions and results.** Except for a few special cases, the reduced system obtained from a nonholonomic discrete mechanical system via the general reduction process defined in [13] is not a discrete mechanical system, constrained or not. So, as we mentioned above, the first step is to construct a family of dynamical systems that contains all the systems of interest — nonholonomic discrete mechanical systems as well as their reductions. The discrete Lagrange–D’Alambert–Poincaré systems (DLDPSs) form such a family: one of these systems is determined by a fiber bundle $\phi : E \to M$, a function $L_d : E \times M \to \mathbb{R}$, the discrete Lagrangian, a nonholonomic infinitesimal variation chaining map $\mathcal{P}$ (see Definition 3.3) as well as a regular submanifold $D_d \subset E \times M$, the kinematic constraints, and a subbundle $D \subset p_1^*TE$ (where $p_1 : E \times M \to E$ is the projection), the variational constraints. All such systems are discrete-time dynamical systems whose trajectories are determined by a variational principle. Examples of DLDPSs include the nonholonomic discrete mechanical systems (Example 3.9) and, when they are symmetric, their reductions by the procedure defined in [13] (Section 3.2); also, the discrete Lagrange–Poincaré systems considered in [14] are DLDPSs. A convenient notion of morphism between DLDPSs is introduced and a category $\mathbb{LDP}_d$ is so defined. The category $\mathbb{LP}_d$ of discrete Lagrange–Poincaré systems defined in [14] is a full subcategory of $\mathbb{LDP}_d$.

Roughly speaking, a Lie group $G$ is a symmetry group of a DLDPS if it acts on the underlying fiber bundle in such a way that it preserves the different structures. When $G$ is a symmetry group of a DLDPS $\mathcal{M}$, we construct a new DLDPS $\mathcal{M}/G$ that we call the reduced system. In fact, the construction requires an additional piece of data: an affine discrete connection on a certain principal $G$-bundle; interestingly, we prove that the reduced systems obtained using different affine discrete connections are always isomorphic in $\mathbb{LDP}_d$ (Proposition 5.14). Also, the reduction mapping $\mathcal{M} \to \mathcal{M}/G$ is a morphism in $\mathbb{LDP}_d$; Corollary 5.16 and Theorem 5.17 prove that the reduction mapping determines a bijective correspondence between
the trajectories of $M$ and those of $M/G$. It is important to notice that both the notion of symmetry group and the reduction process extend the ones already in use for nonholonomic discrete mechanical systems as well as for discrete Lagrange–Poincaré systems.

When $G$ is a symmetry group of the DLDPS $M$ and $H \subset G$ is a closed and normal subgroup, $H$ is a symmetry group of $M$, so we can consider the reduced system $M^H := M/H$ using a discrete affine connection $A^H_d$. Then, under a condition on $A^H_d$, we prove that $G/H$ is a symmetry group of $M^H$, so that we can consider a new reduced system $M^{G/H} := M^H/(G/H)$. One of the main results of the paper, Theorem 6.6, is that $M^{G/H}$ is isomorphic in $LDP_d$ to $M/G$.

Plan for the paper. Section 2 reviews the notion of affine discrete connection as well as some basic results on principal bundles. Section 3 introduces the DLDPSs and their dynamics. Section 3.2 shows that both nonholonomic discrete mechanical systems as well as their reduction (in the sense of [13]) are examples of DLDPS and, also, that their dynamics as DLDPSs is the same as the “classical one”. Section 4 introduces the category $LDP_d$ whose objects are DLDPSs. Symmetries and a reduction process in $LDP_d$ are analyzed in Section 5; in particular, in Section 5.4, we illustrate how these ideas can be applied by studying the discrete LL systems on a Lie group $G$. Finally, Section 6 establishes the equivalence between the two-stage and the single-stage reduction process, under appropriate conditions.

Future work. It would be very interesting to connect the analysis of this paper with a discretization process for continuous mechanical systems. This would allow, for instance, the estimation of the error made when using a DLDPS as an approximation of a (continuous) mechanical system. Indeed, a first step would be to tackle this same problem with no constraints, that is, for discrete Lagrange–Poincaré systems ([14]). It should be noted that this error analysis is only known for unconstrained systems (see [29]) and forced mechanical systems (see [10] and [12]). Another avenue for exploration would be the study of possible Poisson structures in DLDPSs: even though DLDPSs do not have a canonical Poisson structure, some of them do (those coming from discrete mechanical systems, for instance) and it would be interesting to see how those structures behave under the reduction process.

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Notation. Throughout the paper many spaces are Cartesian products. In general we denote the corresponding projections by $p_k : \prod_{j=1}^N X_j \to X_k$ and the obvious adaptations. Also, $l^X$ and $r^X$ will denote left and right smooth actions of a Lie group on the manifold $X$. If $G$ acts on the left on $X$ we denote the corresponding quotient map by $\pi^{X,G} : X \to X/G$.

2. Revision of some discrete tools. In this section we review some basic notions and results about affine discrete connections and smooth fiber bundles.

2.1. Affine discrete connections. Let $l^Q : G \times Q \to Q$ be a smooth left action of the Lie group $G$ on the manifold $Q$. We consider several other actions of $G$; for example, we have the $G$ actions $l^{Q \times Q}$ and $l^{Q \times Q^2}$ on $Q \times Q$ defined by $l^{Q \times Q}_g(q_0, q_1) :=$
Given a discrete connection $A_d$ on $\pi^{Q,G} : Q \to Q/G$, the space $\mathcal{U} := l^Q_{g^{-1}}(\text{Hor}_{A_d}) = \{(q_0, l^Q_g(q_1)) \in Q \times Q : (q_0, q_1) \in \text{Hor}_{A_d}, g \in G\}$, is called the domain of $A_d$.

**Proposition 2.2.** The space $\mathcal{U}$ is an open set in $Q \times Q$.

**Proof.** See point 1 of Proposition 2.4 in [15]. □

**Proposition 2.3.** Let $A_d$ be a discrete connection with level $\gamma$ and domain $\mathcal{U}$ on the principal $G$-bundle $\pi^{Q,G} : Q \to Q/G$. For each $(q_0, q_1) \in \mathcal{U}$, there is a unique $g \in G$ such that $(q_0, l^Q_g((q_1))) \in \text{Hor}_{A_d}$.

**Proof.** See Proposition 2.5 in [15]. □

**Definition 2.4.** Given a discrete connection $A_d$ with domain $\mathcal{U}$ on the principal $G$-bundle $\pi^{Q,G} : Q \to Q/G$, we define its discrete connection form $A_d : \mathcal{U} \subset Q \times Q \to G$ by $A_d(q_0, q_1) := g$, where $g \in G$ is the element that appears in Proposition 2.3.

In what follows we consider the open set $\mathcal{U}' := (id \times \pi^{Q,G})(\text{Hor}_{A_d}) \subset Q \times (Q/G)$.

**Definition 2.5.** Let $A_d$ be a discrete connection on the principal $G$-bundle $\pi^{Q,G} : Q \to Q/G$. The discrete horizontal lift $h_d : \mathcal{U}' \to \text{Hor}_{A_d}$ is the inverse map of the injective local diffeomorphism $(id_Q \times \pi^{Q,G})|_{\text{Hor}_{A_d}} : \text{Hor}_{A_d} \to \mathcal{U}'$. That is $h^0_d(r_1) = h_d(q_0, r_1) := (q_0, q_1) \iff (q_0, q_1) \in \text{Hor}_{A_d}$ and $\pi^{Q,G}(q_1) = r_1$.

In addition we define $\overline{h^0_d} := p_2 \circ h^0_d$.

**Proposition 2.6.** Let $A_d$ be a discrete connection on the principal $G$-bundle $\pi^{Q,G} : Q \to Q/G$. Then,

1. the discrete connection form $A_d$ and the discrete horizontal lift $h_d$ are smooth maps and,
2. if we consider the left $G$-actions on $G$ and on $Q \times (Q/G)$ given by $l^G$ and $l^Q \times (Q/G)$ by $l^G$ and
   
   \[
   l^Q \times (Q/G)(g_0, r_1) := (l^Q_g(q_0), r_1),
   \]
   
   and the diagonal action $l^Q \times Q$ on $Q \times Q$ then $A_d$ and $h_d$ are $G$-equivariant.
3. In general, for any $g_0, q_1 \in G$,

\[
A_d(l^Q_{g_0}(q_0), l^Q_{g_1}(q_1)) = g_1 A_d(q_0, q_1) g_0^{-1} \quad \text{for all} \quad (q_0, q_1) \in \mathcal{U}. \quad (2.1)
\]
bundle maps (over the identity). In addition, if we consider the left $G$-action on $Q$ and $E$, with $A(d)$ instead of $A_{d}$, then $\mathcal{H}or := \{(q_0, q_1) \in Q \times Q: A(q_0, q_1) = e\}$ defines an affine discrete connection with level set $\gamma(q) := A(q, q)^{-1}$ and whose discrete connection 1-form is $A$.

Proof. This proof is analogue to the proof of Proposition 4.12 in [13].

2.2. Principal bundles. Here we review a few basic notions and results on principal bundles. We refer to Section 9 of [14] and its references for additional details.

Definition 2.8. Let $G$ be a Lie group and $(E, M, \phi, F)$ a fiber bundle. We say that $G$ acts on the fiber bundle $E$ if there are free left $G$-actions $t^E$ and $t^M$ on $E$ and $M$ respectively and a right $G$-action $r^F$ on $F$ such that

1. $t^M$ induces a principal $G$-bundle structure $\pi^{M, G}: M \to M/G$,
2. $\phi$ is a $G$-equivariant map for the given actions,
3. for every $m \in M$ there is a trivializing chart $(U, \Phi_U)$ of $E$ such that $U \subset M$ is $G$-invariant, $m \in U$ and, when considering the left $G$-action $t^U \times F$ on $U \times F$ given by $t^U \times F(m, f) := (t^M(m), r^F \circ (f))$, the map $\Phi_U$ is $G$-equivariant.

Remark 2.9. When a Lie group $G$ acts on the fiber bundle $(E, M, \phi, F)$ and on the manifold $F'$ by a right action, it is possible to construct an associated bundle on $M/G$ with total space $(E \times F')/G$ and fiber $F \times F'$. The special case when $F' = G$ acting on itself by $r_g(h) := g^{-1}hg$ is known as the conjugate bundle and is denoted by $\tilde{G}_E$.

Proposition 2.10. Let $G$ be a Lie group that acts on the fiber bundle $(E, M, \phi, F)$ and $A_{d}$ be a discrete connection on the principal $G$-bundle $\pi^{M, G}: M \to M/G$. We define $\bar{\Phi}_{A_{d}}: E \times M \to (E \times M) \times (M/G)$ and $\bar{\Psi}_{A_{d}}: E \times G \times (M/G) \to (E \times M)$ by

$$\bar{\Phi}_{A_{d}}(\epsilon, m) := (\epsilon, A_{d}(\phi(\epsilon), m), \pi^{M, G}(m)) \quad \text{and} \quad \bar{\Psi}_{A_{d}}(\epsilon, w, r) := (\epsilon, t^{M}_{w}(h^{M}_{d}(r))).$$

Then, $\bar{\Phi}_{A_{d}}$ and $\bar{\Psi}_{A_{d}}$ are smooth functions, inverses of each other. If we view $E \times M$ and $E \times G \times (M/G)$ as fiber bundles over $M$ via $\phi \circ p_1$, then $\bar{\Phi}_{A_{d}}$ and $\bar{\Psi}_{A_{d}}$ are bundle maps (over the identity). In addition, if we consider the left $G$-actions $t^{E \times M}$ and $t^{E \times G \times (M/G)}$ defined by

$$t^{E \times M}(\epsilon, m) := (t^{E}_{\epsilon}(\epsilon), t^{M}_{\epsilon}(m)) \quad \text{and} \quad t^{E \times G \times (M/G)}(\epsilon, w, r) := (t^{E}_{\epsilon}(\epsilon), t^{G}_{\epsilon}(w), r),$$

then $\bar{\Phi}_{A_{d}}$ and $\bar{\Psi}_{A_{d}}$ are $G$-equivariant and they induce diffeomorphisms $\Phi_{A_{d}}: (E \times M)/G \to \tilde{G}_E \times (M/G)$ and $\Psi_{A_{d}}: \tilde{G}_E \times (M/G) \to (E \times M)/G$.

Proof. This is Proposition 2.6 in [14] adapted to affine discrete connections.

Remark 2.11. The discrete connection $A_{d}$ need not be defined on $Q \times Q$ but, rather, on the open subset $U$. This restricts the domain of $\Psi_{A_{d}}$ and $\Phi_{A_{d}}$ to appropriate open sets, where the results of the Proposition 2.10 hold. We will ignore this point and keep working as if $A_{d}$ were globally defined in order to avoid a more involved notation.
Proposition 2.13. Let \( s \) be a discrete connection on the principal \( G \)-bundle \( \pi : M \to M/G \). Then, \( \Upsilon_{A_d} : E \times M \to \tilde{G}_E \times (M/G) \) is defined as
\[
\Upsilon_{A_d} := \Phi_{A_d} \circ \pi^E_{\times M,G} = (\pi^E_{\times G,G} \times \text{id}_{M/G}) \circ \tilde{\Phi}_{A_d}.
\]

Lemma 2.12. Let \( G \) be a Lie group that acts on the fiber bundle \((E, M, \phi, F)\) and \( A_d \) be a discrete connection on the principal \( G \)-bundle \( \pi^M,G : M \to M/G \). Then, \( \Upsilon_{A_d} : E \times M \to \tilde{G}_E \times (M/G) \) defined by (2.3) is a principal \( G \)-bundle.

Proof. This is Lemma 2.8 in [14] adapted to affine discrete connections.

All together, we have the following commutative diagram
\[
\begin{array}{ccc}
E \times M & \xrightarrow{\tilde{\Phi}_{A_d}} & (E \times G) \times (M/G) \\
\pi^E_{\times M,G} \downarrow & \nearrow \Upsilon_{A_d} & \pi^E_{\times G,G} \times \text{id}_{M/G} \\
(E \times M)/G & \sim & \tilde{G}_E \times (M/G)
\end{array}
\]

Proposition 2.13. Let \( \rho : X \to Y \) be a principal \( G \)-bundle, \( Z \subset X \) a \( G \)-invariant regular submanifold, and \( S := \rho(Z) \). Then \( S \) is a regular submanifold of \( Y \).

Proof. The statement can be proved locally, that is, it suffices to show that for each \( s \in S \) there is an open subset \( U \subset Y \) such that \( s \in U \) and \( (S \cap U) \subset U \) is a regular submanifold.

As \( \rho \) is a principal \( G \)-bundle, for each \( s \in S \), there are an open subset \( U \subset Y \) such that \( s \in U \) and a diffeomorphism \( \Phi_U : \rho^{-1}(U) \to U \times G \) that is \( G \)-equivariant (for \( p_g^U \times G(u, g') := (u, g'(u)) \)) and that \( p_1 \circ \Phi_U = \rho|_{\rho^{-1}(U)} \).

As \( Z \subset X \) is a regular submanifold and \( \rho^{-1}(U) \) is an open subset of \( X \), \( Z \cap \rho^{-1}(U) \) is a regular submanifold of \( \rho^{-1}(U) \). Then, as \( \Phi_U \) is a diffeomorphism, \( \tilde{Z} := \Phi_U(Z \cap \rho^{-1}(U)) \) is a regular submanifold of \( U \times G \). Furthermore, as \( Z \) is \( G \)-invariant and \( \Phi_U \) is \( G \)-equivariant, \( \tilde{Z} \) is \( G \)-invariant.

Let \( i : U \to U \times G \) be given by \( i(u) := (u, e) \), where \( e \) is the identity of \( G \); it is easy to check that \( S \cap U = i^{-1}(\tilde{Z}) \). Then \( i \) is smooth and, furthermore, for each \( s' \in S \cap U \), \( di(s')(T_u U) = T_u U \oplus \{0\} \subset T_{(s', e)}(U \times G) \). On the other hand, as \( (s', e) \in \tilde{Z} \) and \( \tilde{Z} \) is \( G \)-invariant, we have that \( \{0\} \oplus T_u G \subset T_{(s', e)}(U \times G) \). Then, \( T_{i(s')} \tilde{Z} \oplus di(s')(T_{s'} U) = T_{i(s')}(U \times G) \) and \( i \) is transversal to \( \tilde{Z} \), so that \( S \cap U = i^{-1}(\tilde{Z}) \) is a regular submanifold of \( U \) (see Theorem 6.30 in [22]).
3. Discrete Lagrange–D’Alembert–Poincaré systems. In this section we introduce a type of discrete-time dynamical system that contains, among other examples, all nonholonomic discrete mechanical systems as well as their reductions, as defined in [13].

3.1. Some definitions. Given a fiber bundle \( \phi : E \to M \) we denote \( C'(E) := E \times M \), seen as a fiber bundle over \( M \) by \( \phi \circ p_1 \). We define the discrete second order manifold \( C''(E) := (E \times M) \times_{p_2, \phi \circ p_1} (E \times M) \) considered as a fiber bundle over \( M \) by \( \tilde{p}_2 := p_2|_{C''(E)} \) for the projection \( p_2 : E \times M \times E \times M \to M \).

Remark 3.1. Given a fiber bundle \( \phi : E \to M \), the second order manifold \( \tilde{p}_2 : C''(E) \to M \) is isomorphic as a fiber bundle to the fiber bundle \( \phi \circ p_2 : E \times E \times M \to M \) with \( F_E((\epsilon_0, m_1), (\epsilon_1, m_2)) := (\epsilon_0, \epsilon_1, m_1) \).

Definition 3.2. Given a fiber bundle \( \phi : E \to M \), a discrete path in \( C'(E) \) is a set \((\epsilon, m.) = ((\epsilon_0, m_1), \ldots, (\epsilon_{N-1}, m_N)) \) where \(((\epsilon_k, m_{k+1}), (\epsilon_{k+1}, m_{k+2})) \in C''(E)\) for \( k = 0, \ldots, N - 2 \).

Definition 3.3. Let \( \phi : E \to M \) be a fiber bundle and \( D \) be a subbundle of the pullback bundle \( p_1^*(TE) \subset T(C'(E)) \). A nonholonomic infinitesimal variation chaining map (NIVCM) \( \mathcal{P} \) on \( (E, D) \) is a homomorphism of vector bundles over \( \tilde{p}_1 \), according to the following commutative diagram

\[
\begin{array}{cccc}
D & \leftarrow & \tilde{p}_{34}^*(D) & \xrightarrow{\mathcal{P}} & \ker(d\phi) \subset TE \\
\downarrow & & \downarrow & & \downarrow \\
E \times M & \leftarrow & C''(E) & \xrightarrow{\tilde{p}_1} & E
\end{array}
\]

where \( \tilde{p}_1((\epsilon_0, m_1), (\epsilon_1, m_2)) := \epsilon_0 \) and \( \tilde{p}_{34}((\epsilon_0, m_1), (\epsilon_1, m_2)) := (\epsilon_1, m_2) \).

Remark 3.4. The fiber of the bundle \( p_1^*(TE) \) on \( (\epsilon, m) \) consists of vectors of the form \((\delta \epsilon, 0) \in T(\epsilon, m)(C'(E))\).

Definition 3.5. Let \( \phi : E \to M \) be a fiber bundle, \( D \subset p_1^*TE \) be a subbundle, \( \mathcal{P} \) be a NIVCM on \((E, D)\) and \((\epsilon, m.) = ((\epsilon_0, m_1), \ldots, (\epsilon_{N-1}, m_N)) \) be a discrete path in \( C'(E) \). An infinitesimal variation over \((\epsilon, m.) \) is a tangent vector \((\delta \epsilon, \delta m.) = ((\delta \epsilon_0, \delta m_1), \ldots, (\delta \epsilon_{N-1}, \delta m_N)) \in T(\epsilon, m.)(C'(E)^N) \) such that

\[
\delta m_k = d\phi(\epsilon_k)(\delta \epsilon_k) \quad \text{with} \quad k = 1, \ldots, N - 1. \quad (3.1)
\]

A nonholonomic infinitesimal variation over \((\epsilon, m.) \) with fixed endpoints is an infinitesimal variation \((\delta \epsilon, \delta m.) \) over \((\epsilon, m.) \) such that

\[
\delta m_N = 0, \quad \delta \epsilon_{N-1} = \delta \epsilon_{N-1}, \quad (3.2)
\]

\[
\delta \epsilon_k = \delta \epsilon_k + \mathcal{P}((\epsilon_k, m_{k+1}), (\epsilon_{k+1}, m_{k+2}))(\delta \epsilon_{k+1}, 0), \quad \text{if} \quad k = 1, \ldots, N - 2,
\]

\[
\delta \epsilon_0 = \mathcal{P}((\epsilon_0, m_1), (\epsilon_1, m_2))(\delta \epsilon_1, 0),
\]

where \((\delta \epsilon_k, 0) \in D(\epsilon_k, m_{k+1})\) is arbitrary for \( k = 1, \ldots, N - 1 \).

Definition 3.6. Let \( \phi : E \to M \) be a fiber bundle. A discrete Lagrange–D’Alembert–Poincaré system (DLDPS) over \( E \) is a collection \( \mathcal{M} := (E, L_d, D_d, D, \mathcal{P}) \) where \( L_d : C'(E) \to \mathbb{R} \) is a smooth function, the discrete Lagrangian, \( D_d \subset C'(E) \) is a
regular submanifold, the kinematic constraints, \( D \) is a subbundle of \( p_1^*(TE) \), the variational constraints, and \( \mathcal{P} \) is a NIVCM over \((E,D)\).

**Definition 3.7.** Let \( \mathcal{M} = (E, L_d, D_d, \mathcal{D}, \mathcal{P}) \) be a DLDPS. The discrete action of \( \mathcal{M} \) is a function \( S_d : C'(E)^N \to \mathbb{R} \) defined by \( S_d(\epsilon, m) := \sum_{k=0}^{N-1} L_d(\epsilon_k, m_{k+1}) \). A trajectory of \( \mathcal{M} \) is a discrete path \((\epsilon, m) \in C'(E)^N\) such that \((\epsilon_k, m_{k+1}) \in D_d\) for all \(k = 0, \ldots, N - 1\) and

\[
dS_d(\epsilon, m)(\delta \epsilon, \delta m) = 0
\]

for all nonholonomic infinitesimal variations \((\delta \epsilon, \delta m)\) on \((\epsilon, m)\) with fixed endpoints.

Given a DLDPS \( \mathcal{M} = (E, L_d, D_d, \mathcal{D}, \mathcal{P}) \) we have the vector bundle \((\tilde{p}_34^*(D))^* \to C''(E)\). Let \( \nu_d \) be the smooth section of this bundle defined by

\[
\nu_d((\epsilon_0, m_1), (\epsilon_1, m_2)) := D_1 L_d(\epsilon_1, m_2) \circ d p_1(\epsilon_1, m_2) + D_2 L_d(\epsilon_0, m_1) \circ d(\phi \circ p_1)(\epsilon_1, m_2) + D_1 L_d(\epsilon_0, m_1) \circ \mathcal{P}(\epsilon_0, m_1, (\epsilon_1, m_2)).
\]

The next result characterizes the trajectories of a DLDPS in terms of its equations of motion.

**Proposition 3.8.** Let \( \mathcal{M} = (E, L_d, D_d, \mathcal{D}, \mathcal{P}) \) be a DLDPS and \((\epsilon, m)\) be a discrete path in \( C'(E) \). Then \((\epsilon, m)\) is a trajectory of \( \mathcal{M} \) if and only if

\[
(\epsilon_k, m_{k+1}) \in D_d \quad \text{for all} \quad k = 0, \ldots, N - 1 \quad \text{and} \quad \nu_d((\epsilon_{k-1}, m_k), (\epsilon_k, m_{k+1})) = 0 \quad \text{for all} \quad k = 1, \ldots, N - 1,
\]

where \( \nu_d \) is the section defined by \((3.3)\).

**Proof.** Let \((\delta \epsilon, \delta m)\) be a nonholonomic infinitesimal variation over \((\epsilon, m)\) with fixed endpoints. A straightforward but lengthy computation using Definition 3.5 shows that

\[
dS_d(\epsilon, m)(\delta \epsilon, \delta m) = \sum_{k=1}^{N-1} (D_1 L_d(\epsilon_k, m_{k+1}) \circ d p_1(\epsilon_k, m_{k+1})
+ D_2 L_d(\epsilon_{k-1}, m_k) \circ \mathcal{P}(\epsilon_{k-1}, m_k, (\epsilon_k, m_{k+1}))
+ D_1 L_d(\epsilon_{k-1}, m_k) \circ d(\phi \circ p_1)(\epsilon_k, m_{k+1}))(\delta \epsilon_k, 0).
\]

As the \( \tilde{\delta} \epsilon_k \in D_{(\epsilon_k, m_{k+1})} \) are arbitrary, the result then follows by Definition 3.7. \( \square \)

We refer to condition \((3.4)\) as the *equations of motion* of the system.

**Example 3.9.** We recall from [13] (Definition 3.1) that a discrete nonholonomic mechanical system is a collection \((Q, L_d, D_d, D^{nh})\) where \( Q \) is a differentiable manifold, \( L_d : Q \times Q \to \mathbb{R} \) is a smooth function, \( D^{nh} \) is a subbundle of \( TQ \) and \( D_d \) is a regular submanifold of \( Q \times Q \). In an analogous way to what happens with discrete mechanical systems and the discrete Lagrange–Poincaré systems in [14] (Example 3.12), a discrete nonholonomic mechanical system can be seen as a discrete Lagrange–D’Alembert–Poincaré system with \( \phi = \text{id}_Q \) (so that \( C'(E) = Q \times Q \)), \( \mathcal{P} = 0 \), the same \( D_d \) as kinematic constraints and \( D := p_1^*(D^{nh}) \subset p_1^*(TQ) \). In this
case, a discrete path in $C'(E)$ can be identified with path $q_\cdot = (q_0, \ldots, q_N) \in Q^{N+1}$ and the equations of motion (3.4) become

\[(q_k, q_{k+1}) \in D_d \quad \text{for all} \quad k = 0, \ldots, N - 1 \quad \text{and} \quad D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_k, q_0) \in (D^{nh})^o \quad \text{for all} \quad k = 1, \ldots, N - 1\]

for $k = 1, \ldots, N - 1$, which are the same equations of motion of the discrete nonholonomic mechanical system $(Q, L_d, D_d, D^{nh})$ given by (6) in [8] or (3) in [13].

**Remark 3.10.** Under appropriate regularity conditions on the discrete lagrangian $L_d$ and dimensional relation on the constraints spaces, the existence of trajectories of a DLDPS is guaranteed in a neighborhood of a given trajectory.

### 3.2. Nonholonomic discrete mechanical systems with symmetry.

Symmetries of a nonholonomic discrete mechanical system (Example 3.9) were considered in [13]. Even more, a reduction process was developed there so that a new discrete-time dynamical system—called the *reduced system*—was constructed starting from a symmetric nonholonomic discrete mechanical system and whose dynamics captured the essential features of that of the original system. Unfortunately, that reduced system is not usually a nonholonomic discrete mechanical system. The goal of this section is to recall those constructions and results from [13] and prove that, indeed, the reduced system can be interpreted as a DLDPS whose trajectories in the sense of Definition 3.7 are the same as those of the reduced system (in the sense of [13]).

Let $l^Q$ be a left $G$-action on $Q$ such that $\pi^{Q,G} : Q \to Q/G$ is a principal $G$-bundle and fix a discrete connection $\mathcal{A}_d$ on this bundle. In this case, the commutative diagram (2.2) turns into

\[
\begin{array}{ccc}
Q \times Q & \xrightarrow{\bar{\Phi}_{\mathcal{A}_d}} & (Q \times G) \times (Q/G) \\
\downarrow{\pi^{Q,Q,G}} & & \downarrow{\pi^{Q \times G,G \times id_{Q/G}}} \\
(Q \times Q)/G & \xrightarrow{\Upsilon_{\mathcal{A}_d}} & \bar{G} \times (Q/G)
\end{array}
\]

where $\bar{G} := (Q \times G)/G$ with $G$ acting on $Q$ by $l^Q$ and on $G$ by conjugation and

\[
\Upsilon_{\mathcal{A}_d}(q_0, q_1) := (\pi^{Q \times G,G}(q_0, \mathcal{A}_d(q_0, q_1)), \pi^{Q,G}(q_1)).
\]

A Lie group $G$ is a symmetry group of the discrete nonholonomic mechanical system $(Q, L_d, D_d, D^{nh})$ if $\pi^{Q,G} : Q \to Q/G$ is a principal bundle, $L_d$ and $D_d$ are invariant by the diagonal action $l^{Q \times Q}$ and $D^{nh}$ is invariant by the lifted action $l^{Q \times Q}$.

By the $G$-invariance of $L_d$, there is a well defined map $\bar{L}_d : \bar{G} \times (Q/G) \to \mathbb{R}$ such that $\bar{L}_d(v, r_1) = L_d(q_0, q_1)$ for any $(q_0, q_1) \in Q \times Q$ such that $\Upsilon_{\mathcal{A}_d}(q_0, q_1) = (v_0, r_1)$. The actions associated to $L_d$ and $\bar{L}_d$ are $S_d(q) := \sum_k L_d(q_k, q_{k+1})$ and $\bar{S}_d(v, r.) := \sum_k L_d(v_k, r_{r+1})$. Also by $G$-invariance, being $D_d \subset Q \times Q$ a regular submanifold, $D_d/G \subset (Q \times Q)/G$ is a regular submanifold by Proposition 2.13; as $\Phi_{\mathcal{A}_d}$ is a diffeomorphism, $D_d := \Phi_{\mathcal{A}_d}(D_d/G)$ is a regular submanifold of $\bar{G} \times (Q/G)$.

The next result of [13] relates the variational principle that describes the dynamics of $(Q, L_d, D_d, D^{nh})$ with a variational principle for its reduced system defined on $\bar{G} \times (Q/G)$.
Theorem 3.11. Let $G$ be a symmetry group of the discrete nonholonomic mechanical system $(Q, \mathcal{L}_d, \mathcal{D}_d, \mathcal{D}^{nh})$. Let $\mathcal{A}_d$ be a discrete connection on the principal $G$-bundle $\pi^{Q,G} : Q \to Q/G$. Let $q_k$ be a discrete path in $Q$, $r_k := \pi^{Q,G}(q_k)$, $w_k := \mathcal{A}_d(q_k, q_{k+1})$ and $v_k := \pi^{Q\times G,G}(q_k, w_k)$ be the corresponding discrete paths in $Q/G$, $G$ and $G$. Then, the following statements are equivalent.

1. $(q_k, q_{k+1}) \in \mathcal{D}_d$ for all $k$ and $q_k$ satisfies the criticality condition $dS_d(q_k)(\delta q_k) = 0$ for all fixed-endpoint variations $\delta q_k \in \mathcal{D}^{nh}_{q_k}$ for all $k$.
2. $(v_k, r_{k+1}) \in \mathcal{D}_d$ for all $k$ and $d\tilde{S}_d(v, r)(\delta v, \delta r) = 0$ for all $(\delta v, \delta r)$ such that $(\delta v_k, \delta r_{k+1}) := d\tilde{\mathcal{Y}}_{\mathcal{A}_d}(q_k, q_{k+1})(\delta q_k, \delta q_{k+1})$ (3.7) for $k = 0, \ldots, N - 1$ and where $\delta q_k$ is a fixed-endpoint variation on $q_k$ such that $\delta q_k \in \mathcal{D}^{nh}_{q_k}$ for all $k$.

Remark 3.12. Theorem 3.11 is part of Theorem 5.11 in [13]; this last result requires the additional data of a connection on the principal bundle $\pi^{Q,G} : Q \to Q/G$ to decompose the variations $\delta q_k$ in horizontal and vertical parts. We have omitted this requirement and adapted the result accordingly.

The reduced system associated to $(Q, \mathcal{L}_d, \mathcal{D}_d, \mathcal{D}^{nh})$ in Section 5 of [13] is the discrete-time dynamical system on $\tilde{G} \times (Q/G)$ whose trajectories are discrete paths that satisfy the variational principle of point 2 in Theorem 3.11. Next we construct a DLDPS that will, eventually, be equivalent to this reduced system. We define the fiber bundle $\phi : E \to M$ as $p^{Q/G} : \tilde{G} \to Q/G$, where $p^{Q/G}(\pi^{Q\times G,G}(q, w)) := \pi^{Q,G}(q)$. The reduced lagrangian $\tilde{L}_d : \tilde{G} \times (Q/G) \to \mathbb{R}$ is a smooth map on $C^\infty(\tilde{G}) = \tilde{G} \times (Q/G)$. We already saw that $\mathcal{D}_d$ is a regular submanifold of $C^\infty(\tilde{G}) = \tilde{G} \times (Q/G)$. The space $\mathcal{D} := d\tilde{\mathcal{Y}}_{\mathcal{A}_d}(p^*_d(\mathcal{D}^{nh}))$ denotes a subbundle of $T(C^\infty(\tilde{G}))$ (this is a special case of Lemma 5.8 in Section 5.2).

In order to define the NIVCM $\tilde{\mathcal{P}} \in \text{hom}(\tilde{p}_{34}^*(\mathcal{D}), \ker(dp^{Q/G}))$ we consider the application $\mathcal{Y}_{\mathcal{A}_d} : Q \times Q \to \tilde{G} \times (Q/G)$ given by (3.6) and define

$$\tilde{\mathcal{P}}((v_0, r_1), (v_1, r_2))(\delta v_{1,0}) := D_2(p_1 \circ \mathcal{Y}_{\mathcal{A}_d})(q_0, q_1)(\delta q_1) \in T_{v_0} \tilde{G}$$

(3.8) where $(q_0, q_1, q_2)$ are such that $(v_0, r_1) = \mathcal{Y}_{\mathcal{A}_d}(q_0, q_1)$ and $(v_1, r_2) = \mathcal{Y}_{\mathcal{A}_d}(q_1, q_2)$ and $\delta q_1 \in \mathcal{D}^{nh}_{q_1}$ satisfies that $\delta v_1 = D_1(p_1 \circ \mathcal{Y}_{\mathcal{A}_d})(q_0, q_1)(\delta q_1)$. That $\tilde{\mathcal{P}}$ is well defined follows from Lemma 3.13.

In this way, we associate a DLDPS $\mathcal{M} := (E, \tilde{L}_d, \mathcal{D}_d, \tilde{\mathcal{P}})$ to the reduced system and we will prove that the trajectories of both systems coincide.

Lemma 3.13. Let $Q$, $\mathcal{D}$, $\mathcal{A}_d$ and $\mathcal{Y}_{\mathcal{A}_d}$ be as before. Then, the following statements are true.

1. For $(q_0, q_1) \in Q \times Q$, $d\mathcal{Y}_{\mathcal{A}_d}|_{\mathcal{D}^{nh}_{q_0} \times \{0\}}(q_0, q_1) : (\mathcal{D}^{nh}_{q_0} \times \{0\}) \subset T_{(q_0, q_1)}(Q \times Q) \to \mathcal{D}_{\mathcal{Y}_{\mathcal{A}_d}(q_0, q_1)} \subset \tilde{p}^*_d(T(\tilde{G}))$ is an isomorphism of vector spaces.
2. For $((v_0, r_1), (v_1, r_2)) \in C^\infty(E)$ and $(\delta v_{1,0}) \in \mathcal{D}$ the map $\tilde{\mathcal{P}}$ given by (3.8) is well defined and it is linear in $\delta v_1$.
3. For $((v_0, r_1), (v_1, r_2)) \in C^\infty(E)$ and $(\delta v_{1,0}) \in \mathcal{D}$ we have $dp^{Q/G}(v_0)(\tilde{\mathcal{P}}((v_0, r_1), (v_1, r_2))(\delta v_{1,0})) = 0$.

Proof. See point 2 in Lemma 5.1 for point 1 and Lemma 5.10 for points 2 and 3. \qed
The following result of [14] proves that all discrete paths in $C'(E) = C'(\overline{G})$ arise from discrete paths in $C'(id_Q)$.

**Lemma 3.14.** Let $(v, r)$ be a discrete path in $C'(E)$ and $q_0 \in Q$ such that $p_{Q/G}(v_0) = \pi_{Q/G}(q_0)$. Then, there exists a unique discrete path in $C'(id_Q : Q \to Q)$ such that $\Upsilon_{A_d}(q_k, q_{k+1}) = (v_k, r_{k+1})$ for all $k = 0, \ldots, N - 1$.

The following result compares the nonholonomic infinitesimal variations with fixed endpoints on the discrete path $(v, r)$ in $C'(E) = C'(\overline{G})$ for the system $\mathcal{M}$, with the variations defined in point 2 of Theorem 3.11 on the same discrete path.

**Proposition 3.15.** Let $(v, r)$ be a be discrete path in $C'(E)$ such that $(v_k, r_{k+1}) = \Upsilon_{A_d}(q_k, q_{k+1})$ for all $k$, where $q$ is a discrete path in $C'(id_Q)$. Then, the following statements are true.

1. Given a fixed-endpoint variation $\delta q$ in $\mathcal{D}^{nh}$ on $q$, the infinitesimal variation $(\delta v, \delta r)$ defined in point 2 of Theorem 3.11 by (3.7) is a nonholonomic infinitesimal variation on $(v, r)$ with fixed endpoints (in the sense of Definition 3.5) for $\mathcal{M}$.

2. Given a nonholonomic infinitesimal variation $(\delta v, \delta r)$ on $(v, r)$ with fixed endpoints (for $\mathcal{M}$), there exists a fixed-endpoint variation $\delta q$ in $\mathcal{D}^{nh}$ on $q$ such that (3.7) is satisfied for all $k$.

**Proof.** Let $\delta q$ be a fixed-endpoint variation on $q$ in $Q$ such that $\delta q_k \in \mathcal{D}^{nh}$ and let $(\delta v, \delta r)$ be the variation defined by (3.7) in terms of $\delta q$. Let $(\delta v_k, 0) := d\Upsilon_{A_d}(q_k, q_{k+1})(\delta q_k, 0) \in d\Upsilon_{A_d}(p^*_G(Q)) = \mathcal{D}_{(v_k, r_{k+1})}$ for $k = 0, \ldots, N - 1$.

We want to see that $(\delta v, \delta r)$ is a nonholonomic infinitesimal variation on $(v, r)$ with fixed endpoints. Recall that

$$\Upsilon_{A_d}(q_k, q_{k+1}) = (\pi_{Q \times G}(q_k, A_d(q_k, q_{k+1})), \pi_{Q, G}(q_{k+1}))$$

$$= ((p_1 \circ \Upsilon_{A_d})(q_k, q_{k+1}), \pi_{Q, G}(q_{k+1})),$$

and given that $\delta q_k$ is a fixed-endpoint variation, we notice that

$$(\delta v_{N-1}, \delta r_{N}) = d\Upsilon_{A_d}(q_{N-1}, q_N)(\delta q_{N-1}, \delta q_N)$$

$$= d\Upsilon_{A_d}(q_{N-1}, q_N)(\delta q_{N-1}, 0) = (\tilde{\delta v}_{N-1}, 0),$$

and

$$\delta v_0 = d(p_1 \circ \Upsilon_{A_d})(q_0, q_1)(\delta q_0, \delta q_1) = d(p_1 \circ \Upsilon_{A_d})(q_0, q_1)(0, \delta q_1)$$

$$= D_2(p_1 \circ \Upsilon_{A_d})(q_0, q_1)(\delta q_1) = \tilde{\mathcal{P}}((v_0, r_1), (v_1, r_2))(\tilde{\delta v}_1, 0).$$

Also, taking into account that $\tilde{\delta v}_{k+1} = D_1(p_1 \circ \Upsilon_{A_d})(q_k, q_{k+1})(\delta q_{k+1})$, for $k = 1, \ldots, N - 2$ we have that

$$(\delta v_k, \delta r_{k+1}) = d\Upsilon_{A_d}(q_k, q_{k+1})(\delta q_k, \delta q_{k+1})$$

$$= d\Upsilon_{A_d}(q_k, q_{k+1})(\delta q_k, 0) + d\Upsilon_{A_d}(q_k, q_{k+1})(0, \delta q_{k+1})$$

$$= (\delta v_k, 0) + D_2\Upsilon_{A_d}(q_k, q_{k+1})(\delta q_{k+1})$$

$$= (\tilde{\delta v}_k, 0) + (D_2(p_1 \circ \Upsilon_{A_d})(q_k, q_{k+1})(\delta q_{k+1}),$$

$$D_2(p_2 \circ \Upsilon_{A_d})(q_k, q_{k+1})(\delta q_{k+1}))$$

$$= (\tilde{\delta v}_k, 0) + (\tilde{\mathcal{P}}((v_k, r_{k+1}), (v_{k+1}, r_{k+2}))(\tilde{\delta v}_{k+1}, 0), \delta r_{k+1}).$$
Thus, \((\delta v, \delta r)\) satisfies conditions (3.2). By construction of the discrete path \((v, r)\), \(r_k = \pi D/G(v_k)\) for all \(k\) and since \(\pi D/G \circ p_1 \circ \Upsilon_A = \pi Q/G \circ p_1\), then
\[
dp{(p_1 \circ \Upsilon_A)(q_k, q_{k+1})}(d(v_k, q_k, q_{k+1}) (\delta v_k, \delta q_k)) = \pi D/G(q_k)(\delta q_k)
\]
so that \(\delta r_k = dp D/G(v_k)(\delta v_k)\), where \(\delta v_k\) is given by the condition (3.7). Hence \((\delta v, \delta r)\) satisfies condition (3.1), hence part 1 is true.

2. We consider \((\delta v, \delta r)\) that satisfies (3.1) and (3.2) for some vectors \((\delta v_k, 0) \in \tilde{D}_{(v_k, r_{k+1})}\) with \(k = 1, \ldots, N - 1\). Let \(\delta q_0 := 0 \in D^{n^h}_{q_0}\) and \(\delta q_N := 0 \in D^{n_N}_q\) and, for each \(k = 1, \ldots, N - 1\), using point 1 of Lemma 3.13, let \(\delta q_k \in D^{n^h}_{q_k}\) such that \(d \Upsilon_A(q_k, q_{k+1})(\delta q_k, 0) = (\delta v_k, 0)\).

We have that \(r_k = \phi(v_k) = p D/G(v_k)\) for all \(k\). Also, using (3.9), we have
\[
\delta r_k = d\phi(v_k)(\delta v_k) = dp D/G(v_k)(\delta v_k)
\]
and
\[
\delta v_k = d\phi(v_k)(\delta v_k) = dp D/G(v_k)(\delta v_k)
\]
so that \(\delta r_k = dp D/G(v_k)(\delta v_k)\), where \(\delta v_k\) is given by the condition (3.7). Hence \((\delta v, \delta r)\) satisfies condition (3.1), hence part 1 is true.

\[\square\]

**Corollary 3.16.** A discrete path \((v, r)\) is a trajectory of \(\mathcal{M}\) if and only if it is a trajectory of the reduced system according to part 2 of Theorem 3.11.

**Proof.** The equivalence between the two descriptions of a trajectory follows immediately by the correspondence of the infinitesimal variations established in Proposition 3.15. \[\square\]

4. Categorical formulation.

**Definition 4.1.** We define the category of discrete Lagrange–D’Alembert–Poincaré systems \(\Sigma \mathcal{P}_d\) as the category whose objects are DLDPSs. Given \(\mathcal{M}, \mathcal{M}' \in \text{ob}_{\Sigma \mathcal{P}_d}\) with \(\mathcal{M} = (E', L_d', D_d', D', P)\) and \(\mathcal{M}' = (E', L_d', D_d', D', P')\), a map \(\Upsilon : C'(E') \rightarrow C'(E')\) is a morphism in \(\text{mor}_{\Sigma \mathcal{P}_d}(\mathcal{M}, \mathcal{M}')\) if

1. \(\Upsilon\) is a surjective submersion,
2. \(D_1(p_2 \circ \Upsilon) = 0\).
3. As maps from $C''(E)$ in $M'$

$$p_2^{C''(E'),M'} \circ \Upsilon \circ p_1^{C''(E),C'(E)} = \delta' \circ p_1^{C''(E'),E'} \circ \Upsilon \circ p_2^{C''(E),C'(E)} \tag{4.1}$$

where $p_{A,B}^d : A \to B$ are the maps induced by the canonical projections of the Cartesian product onto its factors,

4. $L_d = L_2^d \circ \Upsilon$,

5. $D_2' = \Upsilon(D_d)$,

6. $D' = d\Upsilon(D)$,

7. For all $(((\epsilon_0, m_1), (\epsilon_1, m_2)), (\delta\epsilon_1, 0)) \in \tilde{\mathcal{P}}^*(D)$,

$$\mathcal{P}'(\Upsilon(2)((\epsilon_0, m_1), (\epsilon_1, m_2)))(d\Upsilon(\epsilon_1, m_2)(\delta\epsilon_1, 0)) = d(p_1 \circ \Upsilon)((\epsilon_0, m_1), \mathcal{P}((\epsilon_0, m_1), (\epsilon_1, m_2))(\delta\epsilon_1, 0), d\phi(1)(\delta\epsilon_1)) \tag{4.2}$$

where $\Upsilon \times \Upsilon$ defines a map $\Upsilon(2) : C''(E) \to C''(E')$.

We recall the statement of Lemma 4.3 of [14].

**Lemma 4.4.** Let $\Upsilon \in \text{mor}_{\mathcal{LDP}_d} (\mathcal{M}, \mathcal{M}')$, $((\epsilon_0, m_1), (\epsilon_1, m_2)) \in C''(E)$ and also $(\epsilon'_0, m'_1) := \Upsilon(\epsilon_0, m_1)$. Then, if $\delta\epsilon_1 \in T_{\mathcal{E}} E$,

$$D_2(p_2 \circ \Upsilon)(\epsilon_0, m_1)(d\phi(1)(\delta\epsilon_1)) = d\phi(1)'(D_1(p_1 \circ \Upsilon)(\epsilon_1, m_2)(\delta\epsilon_1)).$$

**Proposition 4.3.** $\mathcal{LDP}_d$ is a category with the standard composition of functions and identity mappings.

**Proof.** This proof is analogous to the proof of Proposition 4.4 of [14]. \qed

**Remark 4.4.** Any discrete Lagrange–Poincaré system [14, Definition 3.4] can be seen as a Lagrange–D’Alembert–Poincaré system “without constraints”, that is, with $\mathcal{D}_d := C'(E)$ and $\mathcal{D} := p_1^d T E$. Also, with this interpretation any morphism of Lagrange–Poincaré systems is a morphism of Lagrange–D’Alembert–Poincaré systems. It is immediate to check that the category of Lagrange–Poincaré systems [14, Definition 4.1] is a full subcategory of $\mathcal{LDP}_d$.

**Lemma 4.5.** Let $\Upsilon' \in \text{mor}_{\mathcal{LDP}_d} (\mathcal{M}, \mathcal{M}')$ and $\Upsilon'' \in \text{mor}_{\mathcal{LDP}_d} (\mathcal{M}, \mathcal{M}'')$ where $\mathcal{M} = (E, L_d, D_d, \mathcal{D}, \mathcal{P})$, $\mathcal{M}' = (E', L_d', D_d', \mathcal{D}', \mathcal{P}')$ and $\mathcal{M}'' = (E'', L_d'', D_d'', \mathcal{D}'', \mathcal{P}'')$. If $F : C'(E') \to C'(E'')$ is a smooth map such that the diagram

$$\begin{array}{ccc}
C'(E) & \xrightarrow{F} & C'(E'') \\
\Upsilon' \downarrow & & \downarrow \Upsilon'' \\
C'(E') & \xrightarrow{F} & C'(E'')
\end{array}$$

is commutative, then $F \in \text{mor}_{\mathcal{LDP}_d} (\mathcal{M}', \mathcal{M}'')$. Also, if $F$ is a diffeomorphism, then $F$ is an isomorphism in $\mathcal{LDP}_d$.

**Proof.** The proof that $F$ satisfies the points 1 to 4 and 7 in Definition 4.1 and the last assertion of the statement is the same as in the proof of Lemma 4.5 in [14]. We want to prove that $F$ satisfies points 5 and 6 of Definition 4.1. Since $\Upsilon' \in \text{mor}_{\mathcal{LDP}_d} (\mathcal{M}, \mathcal{M}')$ and $\Upsilon'' \in \text{mor}_{\mathcal{LDP}_d} (\mathcal{M}, \mathcal{M}'')$ and the previous diagram is commutative, we have that

$$F(D'_d) = F(\Upsilon'(D_d)) = (F \circ \Upsilon')(D_d) = \Upsilon''(D_d) = D''_d,$$
and, then, point 5 in Definition 4.1 is satisfied. Similarly, as \( \Upsilon' \) and \( \Upsilon'' \) are morphisms in \( \mathcal{LDP}_d \), using the commutativity of the diagram, we have
\[
D'' = d\Upsilon''(D) = d(F \circ \Upsilon')(D) = dF(d\Upsilon'(D)) = dF(D').
\]
This proves that point 6 in Definition 4.1 is satisfied.

**Theorem 4.6.** Given \( \Upsilon \in \text{mor}_{\mathcal{LDP}_d}(\mathcal{M},\mathcal{M}') \) with \( \mathcal{M} = (E,L_d,D_d,D,P) \) and \( \mathcal{M}' = (E',L'_d,D'_d,D',P') \), let \( (\epsilon,m) = ((\epsilon_0,m_1),\ldots,(\epsilon_{N-1},m_N)) \) be a discrete path in \( C'(E) \) and define \( (\tilde{\epsilon}_k,m'_{k+1}) := \Upsilon(\epsilon_k,m_{k+1}) \) for \( k = 0,\ldots,N-1 \).

1. If \( (\epsilon,m) \) is a trajectory of \( \mathcal{M} \), then \( (\epsilon',m') \) is a trajectory of \( \mathcal{M}' \).
2. If \( D_d = \Upsilon^{-1}(D'_d) \) and \( (\epsilon',m') \) is a trajectory of \( \mathcal{M}' \), then \( (\epsilon,m) \) is a trajectory of \( \mathcal{M} \).

**Proof.** By hypothesis, \( (\epsilon,m) \) is a discrete path in \( C'(E) \). It follows from its definition, the fact that \( \Upsilon \) is a morphism and (4.1) that \( (\epsilon',m') \) is a discrete path in \( C'(E) \).

Assume that \( (\delta\epsilon,\delta m) \) is an infinitesimal variation in \( \mathcal{M} \) over \( (\epsilon,m) \) and that \( (\delta\epsilon',\delta m') \) is an infinitesimal variation in \( \mathcal{M}' \) over \( (\epsilon',m') \) satisfying
\[
d\Upsilon(\epsilon_k,m_{k+1})(\delta\epsilon_k,\delta m_{k+1}) = (\delta\epsilon'_k,\delta m'_{k+1}) \quad \text{for} \quad k = 0,\ldots,N-1. \tag{4.3}
\]
Then, using the chain rule and that \( L_d = L'_d \circ \Upsilon \), we see that
\[
dS_d(\epsilon,m)(\delta\epsilon,\delta m) = dS'_d(\epsilon',m')(\delta\epsilon',\delta m'). \tag{4.4}
\]
In order to prove point 1, we assume that \( (\epsilon,m) \) is a trajectory of \( \mathcal{M} \). Then \( (\epsilon_k,m_{k+1}) \in D_d \) for \( k = 0,\ldots,N-1 \). Then,
\[
(\epsilon'_{k+1},m'_k) = \Upsilon(\epsilon_k,m_{k+1}) \in \Upsilon(D_d) = D'_d \quad \text{for} \quad k = 0,\ldots,N-1. \tag{4.5}
\]

Let \( (\delta\epsilon',\delta m') \) be an infinitesimal variation with fixed endpoints in \( \mathcal{M}' \) over the discrete path \( (\epsilon',m') \). That is, there are \( (\delta\epsilon'_k,0) \in D'_d(\epsilon'_k,m'_{k+1}) \) for \( k = 1,\ldots,N-1 \) such that (3.1) and (3.2) hold with \( \delta\epsilon'_k \) and \( \tilde{\epsilon}_k \) instead of \( \delta\epsilon_k \) and \( \tilde{\epsilon}_k \).

By morphism’s property 6 applied to \( \Upsilon \), there exist \( (\tilde{\epsilon}_k,0) \in D(\epsilon_k,m_{k+1}) \) such that \( d\Upsilon(\epsilon_k,m_{k+1})(\tilde{\epsilon}_k,0) = (\delta\epsilon'_k,0) \) for \( k = 1,\ldots,N-1 \); we fix one such vector for each \( k \). Next apply (3.1) and (3.2) to define an infinitesimal variation \( (\delta\epsilon,\delta m) \) on \( (\epsilon,m) \) with fixed endpoints based on the \( \tilde{\epsilon}_k \) constructed above.

Direct computations using the morphism properties of \( \Upsilon \) show that condition (4.3) holds for the \( (\delta\epsilon,\delta m) \) and \( (\delta\epsilon',\delta m') \) variations. Then, using (4.4),
\[
dS'_d(\epsilon',m')(\delta\epsilon',\delta m') = dS_d(\epsilon,m)(\delta\epsilon,\delta m) = 0,
\]
where the last equality holds because \( (\delta\epsilon,\delta m) \) is an infinitesimal variation with fixed endpoints in \( \mathcal{M} \) over \( (\epsilon,m) \), that is a trajectory of \( \mathcal{M} \). Finally, as \( (\delta\epsilon,\delta m') \) was an arbitrary infinitesimal variation with fixed endpoints in \( \mathcal{M}' \) over the path \( (\epsilon',m') \), and we have (4.5), we conclude that \( (\epsilon',m') \) is a trajectory of \( \mathcal{M}' \). This proves point 1.

In order to prove point 2, assume that \( (\epsilon',m') \) is a trajectory of \( \mathcal{M}' \). Then, as \( D_d = \Upsilon^{-1}(D'_d) \) and \( (\epsilon_k,m'_{k+1}) \in D'_d \) for \( k = 0,\ldots,N-1 \), we have that \( \Upsilon(\epsilon_k,m_{k+1}) = (\epsilon'_k,m'_{k+1}) \in D'_d \), so that \( (\epsilon_k,m_{k+1}) \in \Upsilon^{-1}(D'_d) = D_d \), for \( k = 0,\ldots,N-1 \).
An argument similar to the one used in the proof of point 1 shows that \((\epsilon, m.\)) satisfies the criticality condition in \(\mathcal{M}\), so that it is a trajectory of \(\mathcal{M}\), thus proving point 2. \(\square\)

The following result, whose proof is immediate, is useful when working with concrete DLDPSs.

**Lemma 4.7.** Let \(\phi : E \to M\) and \(\phi' : E' \to M'\) be two fiber bundles and \((F, f)\) be a fiber bundle isomorphism from \(E\) to \(E'\). For any \(\mathcal{M} = (E, L_d, D_d, D, \mathcal{P}) \in \text{ob}_{\mathcal{DP}_d}\), let \(L'_d := L_d \circ (F \times f)^{-1}\), \(D'_d := (F \times f)(D_d), \mathcal{P}' := d(F \times f)(D)\) and \(\mathcal{P}'\) so that, for all \(((\epsilon_0, m_1), (\epsilon_1, m_2), (\delta\epsilon_1, 0)) \in \tilde{p}_{\mathcal{P}'d}(\mathcal{D}')\), we have

\[
\begin{align*}
P'(\epsilon_0, m_1, \epsilon_1, m_2)(\delta\epsilon_1, 0) &= dF(F^{-1}(\epsilon_0))((P(F^{-1}(\epsilon_0), f^{-1}(m_1)), (F^{-1}(\epsilon_1), f^{-1}(m_2))) \circ dF(\epsilon_1)(\delta\epsilon_1), 0). \\
\text{Then, } \mathcal{M}' := (E', L'_d, D'_d, \mathcal{P}', \mathcal{M}') \in \text{ob}_{\mathcal{DP}_d}\text{ and } F \times f \in \text{hom}_{\mathcal{DP}_d}(\mathcal{M}, \mathcal{M}')\text{ is an isomorphism in } \mathcal{DP}_d. \text{ In particular, the corresponding sections } \nu_d \text{ and } \nu'_d \text{ defined by } (3.3) \text{ satisfy } (F \times f)^{\ast}(\nu'_d) = \nu_d \text{ or, explicitly, } \\
\nu_d((\epsilon_0, m_1), (\epsilon_1, m_2))\delta\epsilon_1, 0 = \nu'_d((F(\epsilon_0), f(m_1)), (F(\epsilon_1), f(m_2)))(dF(\epsilon_1)(\delta\epsilon_1), 0).
\end{align*}
\]

5. **Reduction of discrete Lagrange–D’Alembert–Poincaré systems.** In this section we introduce the notion of symmetry group of a DLDPS and a reduction procedure to associate a “reduced” DLDPS system to a symmetric one. In addition, we prove that the reduction procedure is a morphism in \(\mathcal{DP}_d\) and compare the dynamics of the reduced system to that of the original one.

5.1. **Discrete Lagrange–D’Alembert–Poincaré systems with symmetry.**

Let \(G\) be a Lie group that acts on the fiber bundle \((E, M, \phi, F)\) as in Definition 2.8. We consider the \(G\)-actions on \(C'(E)\) and \(C''(E)\) given by

\[
\begin{align*}
I^C_g(E)(\epsilon_0, m_1) := (I^E_g(\epsilon_0), I^M_g(m_1)), \\
I^C_g(E)((\epsilon_0, m_1), (\epsilon_1, m_2)) := (I^{C'}_g(E)(\epsilon_0, m_1), I^{C'}_g(E)(\epsilon_1, m_2)).
\end{align*}
\]

Also, we consider the \(G\)-actions on \(\ker(d\phi) \subset TE\) and on \(\tilde{p}_{\mathcal{P}'d}T(C'(E)) \subset TC''(E)\) given by

\[
\begin{align*}
I^E_g(\epsilon_0, \delta\epsilon_0) &= dI^E_g(\epsilon_0)(\delta\epsilon_0), \\
I^E_g(C'(E))(\epsilon_0, m_1)(\delta\epsilon_1, \delta m_1) &= (dI^E_g(\epsilon_0)(\delta\epsilon_0), dI^M_g(m_1)(\delta m_1)).
\end{align*}
\]

\[
\begin{align*}
I^C_g(E)((\epsilon_0, m_1), (\epsilon_1, m_2), (\delta\epsilon_1, \delta m_2)) := (I^{C'}_g(E)((\epsilon_0, m_1), (\epsilon_1, m_2), (\delta\epsilon_1, \delta m_2)).
\end{align*}
\]

**Lemma 5.1.** Let \(G\) be a Lie group acting on the fiber bundle \(\phi : E \to M\) and \(A_d\) a discrete connection on the principal \(G\)-bundle \(\pi^{M, G} : M \to G\). We define \(\Upsilon_{A_d}^{(2)} : C''(E) \to C''(\tilde{G}_E)\) as the restriction of \(\Upsilon_{A_d} \times \Upsilon_{A_d} : C'(E) \times C'(E) \to C'(\tilde{G}_E) \times C'(\tilde{G}_E)\) to the corresponding spaces, where \(\Upsilon_{A_d}\) is defined by (2.3). Then,

1. \(\Upsilon_{A_d}^{(2)}\) is well defined.
2. \(d\Upsilon_{A_d}(\epsilon_0, m_1) : (p_1^* TE)_{(\epsilon_0, m_1)} \to (p_1^* T(\tilde{G}_E))_{\Upsilon_{A_d}(\epsilon_0, m_1)}\) is an isomorphism of vector spaces for every \((\epsilon_0, m_1) \in C'(E)\).
3. $\gamma_{\mathcal{A}_d}^{(2)}: C''(E) \to C''(G_E)$ is a principal $G$-bundle with structure group $G$. In particular, $C''(E)/G \simeq C''(G_E)$.

4. For $((v_0, r_1), (v_1, r_2)) \in C''(G_E)$ and $(\epsilon_0, m_1) \in C'(E)$ such that $\gamma_{\mathcal{A}_d}(\epsilon_0, m_1) = (v_0, r_1)$, there is a unique $(\epsilon_1, m_2) \in C'(E)$ such that $((\epsilon_0, m_1), (\epsilon_1, m_2)) \in C''(E)$ and $\gamma_{\mathcal{A}_d}^{(2)}((\epsilon_0, m_1), (\epsilon_1, m_2)) = ((v_0, r_1), (v_1, r_2))$.

**Proof.** This result is almost identical to Lemma 5.1 in [14], the only difference being that, here, we are using affine discrete connections instead of discrete connections. It is easy to see that the proof of Lemma 5.1 remains valid for affine discrete connections. □

**Proposition 5.2.** Let $G$ be a Lie group acting on the fiber bundle $\phi: E \to M$ and $\mathcal{A}_d$ be a discrete connection on the principal $G$-bundle $\pi_{M,G}: M \to M/G$. Given a discrete path $(v, r) = ((v_0, r_1), \ldots, (v_{N-1}, r_N))$ in $C'(G_E)$ and $(\epsilon_0, m_1) \in C'(E)$ such that $\gamma_{\mathcal{A}_d}(\epsilon_0, m_1) = (v_0, r_1)$, there is a unique discrete path $(\epsilon, m) \in C'(E)$ such that $(\epsilon_0, m_1) = (\epsilon_0, m_1)$ and $\gamma_{\mathcal{A}_d}(\epsilon_k, m_{k+1}) = (v_k, r_{k+1})$ for all $k$.

**Proof.** This is Proposition 5.2 in [14] except for using affine discrete connections instead of discrete connections, which doesn’t alter the proof. □

**Definition 5.3.** Let $\mathcal{M} = (E, L_d, D_d, \mathcal{D}, \mathcal{P}) \in \text{ob}_{\text{LDP}_d}$. A Lie group $G$ is a symmetry group of $\mathcal{M}$ if

1. $G$ acts on $\phi: E \to M$ (Definition 2.8),
2. $L_d$ is $G$-invariant by the action $l^C(E)$ (5.1),
3. $D_d$ is $G$-invariant by the action $l^C(E)$ (5.1),
4. $D$ is $G$-invariant by the lifted action $l^T E$ (5.3),
5. $\mathcal{P}$ is $G$-equivariant for the actions $l^{\mathcal{P}_E}(C(E))$ (5.5) and $l^T E$ (5.3).

**Remark 5.4.** In the context of Example 3.9, if $G$ is a symmetry group of the nonholonomic discrete mechanical system $(Q, L_d, D_d, D^n)$ in the sense of [13], then it is a symmetry group of $(Q, L_d, D_d, \mathcal{D}, 0) \in \text{ob}_{\text{LDP}_d}$ in the sense of Definition 5.3.

**Lemma 5.5.** Let $\mathcal{M} = (E, L_d, D_d, \mathcal{D}, \mathcal{P}) \in \text{ob}_{\text{LDP}_d}$ and $G$ be a Lie group. Then, for $g \in G$, if $\Upsilon := l^C_g(E)$ and $\mathcal{M}' = \mathcal{M}$,

1. $D_d$ is $G$-invariant by the diagonal action (5.1) if and only if point 5 in Definition 4.1 is satisfied for $\Upsilon$,
2. $D$ is $G$-invariant by the lifted action (5.4) if and only if point 6 in Definition 4.1 is satisfied for $\Upsilon$,
3. Point 5 in Definition 5.3 is equivalent to point 7 in Definition 4.1 for $\Upsilon$.

**Proof.** Points 1 and 2 are directly satisfied by the definitions of $l^C_g(E)$ and $l^T C(E)$.

To prove point 3 we start by noting that

\[ P(l^C_g(E)((\epsilon_0, m_1), (\epsilon_1, m_2))) = P(l^T \gamma_{\mathcal{P}_E}(C(E)))((\epsilon_0, m_1), (\epsilon_1, m_2), (\delta \epsilon_1, 0)) \]

and

\[ d(p_1 \circ l^C_g(E))(\epsilon_0, m_1) = P((\epsilon_0, m_1), (\epsilon_1, m_2), (\delta \epsilon_1, 0), d\phi(\epsilon_1) (\delta \epsilon_1)) \]

\[ = d l^E_g(\epsilon_0)(P((\epsilon_0, m_1), (\epsilon_1, m_2), (\delta \epsilon_1, 0)). \]
Then, by point 7 in Definition 4.1 for \( \Upsilon := i_g^{C(E)} \) and \( \mathcal{M}' = \mathcal{M} \) we have that the first members of the previous identities are the same, and by point 5 of Definition 5.3 the last members of the previous identities are the same, proving the equivalence of the conditions.

**Proposition 5.6.** Let \( \mathcal{M} = (E, L_d, D_d, D, \mathcal{P}) \in \text{ob}_{\mathcal{LDP}_d} \) and \( G \) be a Lie group. Then, \( G \) is a symmetry group of \( \mathcal{M} \) if and only if \( G \) acts on the fiber bundle \( \phi : E \to M \) and \( i_g^{C(E)} \in \text{mor}_{\mathcal{LDP}_d}(\mathcal{M}, \mathcal{M}) \) for all \( g \in G \).

**Proof.** Assume that \( G \) is a symmetry group of \( \mathcal{M} \). Then, by definition, \( G \) acts on the fiber bundle \( \phi : E \to M \). We have to prove that \( i_g^{C(E)} \in \text{mor}_{\mathcal{LDP}_d}(\mathcal{M}, \mathcal{M}) \) for all \( g \in G \).

Proving that \( i_g^{C(E)} \) satisfies conditions 1 to 4 of Definition 4.1 is analogous to what was done in the proof of the Proposition 5.6 in [14]. Lemma 5.5 proves that \( i_g^{C(E)} \) satisfies the remaining conditions of the Definition 4.1. Thus, \( \mathcal{M}' \in \text{mor}_{\mathcal{LDP}_d}(\mathcal{M}, \mathcal{M}) \).

Conversely, if \( G \) acts on the fiber bundle \( \phi : E \to M \) and \( i_g^{C(E)} \) \( \mathcal{M}', \mathcal{M} \), the first condition of Definition 5.3 is satisfied and the remaining conditions follow from morphism’s properties and Lemma 5.5.

### 5.2. Reduced discrete Lagrange–D’Alembert–Poincaré system

Let \( G \) be a symmetry group of \( \mathcal{M} = (E, L_d, D_d, D, \mathcal{P}) \in \text{ob}_{\mathcal{LDP}_d} \). Since \( G \) acts on \( (E, M, \phi, F) \) the conjugate bundle \( (G_E, M/G, p^{M/G}, F \times G) \) is a fiber bundle (see Section 9 in [14]).

Let \( \mathcal{A}_d \) be a discrete connection on the principal \( G \)-bundle \( \pi^{M,G} : M \to M/G \) and \( \Upsilon_{\mathcal{A}_d} : E \times M \to \bar{G}_E \times (M/G) \) be the map defined by (2.3) that is a principal bundle with structure group \( G \) by Lemma 2.12.

We define \( \bar{L}_d : \bar{G}_E \times (M/G) \to \mathbb{R} \) by \( \bar{L}_d(v_0, r_1) := L_d(\epsilon_0, m_1) \) for any \( (\epsilon_0, m_1) \in \Upsilon_{\mathcal{A}_d}^{-1}(v_0, r_1) \), that, by the \( G \)-invariance of \( L_d \), is well defined. Hence, \( \bar{L}_d \circ \Upsilon_{\mathcal{A}_d} = L_d \).

**Lemma 5.7.** The space \( \bar{D}_d := \Upsilon_{\mathcal{A}_d}(\mathcal{D}_d) \) is a regular submanifold of \( \bar{G}_E \times (M/G) \). Also, \( \mathcal{D}_d = \Upsilon_{\mathcal{A}_d}^{-1}(\mathcal{D}_d) \).

**Proof.** Since \( \Upsilon_{\mathcal{A}_d} : E \times M \to \bar{G}_E \times (M/G) \) is a principal \( G \)-bundle and \( \mathcal{D}_d \) is a \( G \)-invariant regular submanifold of \( E \times M \), by Proposition 2.13, \( \bar{D}_d \) is a regular submanifold of \( \bar{G}_E \times (M/G) \). Also, \( \mathcal{D}_d \) is \( G \)-invariant and \( \Upsilon_{\mathcal{A}_d} \) is a principal \( G \)-bundle, we have \( \mathcal{D}_d = \Upsilon_{\mathcal{A}_d}^{-1}(\Upsilon_{\mathcal{A}_d}(\mathcal{D}_d)) = \Upsilon_{\mathcal{A}_d}^{-1}(\bar{D}_d) \).

**Lemma 5.8.** The space \( \bar{D} := \text{d}\Upsilon_{\mathcal{A}_d}(\mathcal{D}) \) is a subbundle of \( p_1^*(T(\bar{G}_E)) \) where \( p_1 : C'(\bar{G}_E) \to \bar{G}_E \) is the projection onto the first factor. In addition, \( \text{rank}(\bar{D}) = \text{rank}(\mathcal{D}) \).

**Proof.** Given \( (v, r) \in C'(\bar{G}_E) \) and \( (\epsilon, m) \in C'(E) \) such that \( (v, r) = \Upsilon_{\mathcal{A}_d}(\epsilon, m) \), we want to prove that the subspace \( \text{d}\Upsilon_{\mathcal{A}_d}(v, r)(\mathcal{D}(\epsilon, m)) \subset T_{(v, r)}C'(\bar{G}_E) \) is independent of the particular choice of \( (\epsilon, m) \) in \( \Upsilon_{\mathcal{A}_d}^{-1}(v, r) \).
If \((v, r) = \Upsilon_{\mathcal{A}_d}(\bar{\epsilon}, \bar{m})\) then, since \(\Upsilon_{\mathcal{A}_d} : C'(E) \to C'(\tilde{G}_E)\) is a G-principal bundle, there is \(g \in G\) such that \((\bar{\epsilon}, \bar{m}) = l_g^{C'}(\epsilon, m)\). Since \(\mathcal{D}\) is \(G\)-invariant we have that
\[
\mathcal{D}(\epsilon, m) = \mathcal{D}(l_g^{C'}(\epsilon, m)) = dl_g^{C'}(\epsilon, m)(\mathcal{D}(\epsilon, m)),
\]
and, as \(\Upsilon_{\mathcal{A}_d}\) is \(G\)-invariant,
\[
d\Upsilon_{\mathcal{A}_d}(\epsilon, m)(\mathcal{D}(\epsilon, m)) = d\Upsilon_{\mathcal{A}_d}(l_g^{C'}(\epsilon, m))(dl_g^{C'}(\epsilon, m)(\mathcal{D}(\epsilon, m)))
\]
\[
= d(\Upsilon_{\mathcal{A}_d} \circ l_g^{C'})(\epsilon, m)(\mathcal{D}(\epsilon, m))
\]
\[
= d\Upsilon_{\mathcal{A}_d}(\epsilon, m)(\mathcal{D}(\epsilon, m)).
\]
Then \(d\Upsilon_{\mathcal{A}_d}(\epsilon, m)(\mathcal{D}(\epsilon, m))\) is a vector subspace of \(T(v, r)(C'(\tilde{G}_E))\) for each \((v, r) \in C'(\tilde{G}_E)\) and is independent of the particular \((\epsilon, m)\) chosen in \(\Upsilon_{\mathcal{A}_d}(v, r)\); we call it \(\mathcal{D}(v, r)\). This construction gives a fiberwise vector structure to \(\mathcal{D}\). That \(\text{rank}(\mathcal{D}) = \text{rank}(\mathcal{D})\) follows immediately from point 2 in Lemma 5.1.

Now we need to check that for every \((v, r) \in C'(\tilde{G}_E)\) there exist smooth sections defined in an open neighborhood of \((v, r)\) that generate \(\mathcal{D}(v', r')\) for all \((v', r')\) in that neighborhood. To do this, notice that given \((v, r) \in C'(\tilde{G}_E)\), for any \((\epsilon, m) \in \Upsilon^{-1}_{\mathcal{A}_d}(v, r)\), as \(\mathcal{D}\) is a subbundle of \(T(C'(E))\), there is an open neighborhood \(U \subset C'(E)\) of \((\epsilon, m)\) and \(d = \text{dim}(\mathcal{D}(\epsilon, m))\) smooth local sections \(\sigma_1, \ldots, \sigma_d : U \to T'(C'(E))\) such that \(\{\sigma_1(\epsilon', m'), \ldots, \sigma_d(\epsilon', m')\}\) is a basis of \(\mathcal{D}(\epsilon', m')\) for each \((\epsilon', m') \in U\) (Lemma 10.32 in [22]). Then, \(\Upsilon_{\mathcal{A}_d}(U)\) is an open neighborhood of \((v, r)\) (because \(\Upsilon_{\mathcal{A}_d}\), being a principal bundle map, is an open map; see Lemma 21.1 [22]). In addition, as \(\Upsilon_{\mathcal{A}_d}\) is a principal bundle, there is an open neighborhood \(V \subset \Upsilon_{\mathcal{A}_d}(U)\) of \((v, r)\) and a smooth section \(\Sigma : V \to U \subset \Upsilon_{\mathcal{A}_d}\). Define \(\eta_j := d\Upsilon_{\mathcal{A}_d} \circ \sigma_j \circ \Sigma, j = 1, \ldots, d\), which are smooth sections over \(V\) of \(TC'(\tilde{G}_E)\) such that, for each \((v', r') \in V\), \(\{\eta_1(v', r'), \ldots, \eta_d(v', r')\}\) generates \(\mathcal{D}(v', r')\).

**Remark 5.9.** As, by point 2 of Lemma 5.1, \(d\Upsilon_{\mathcal{A}_d}(\epsilon_0, m_1) : (p_1^*T\tilde{E}|_{(\epsilon_0, m_1)}) \to (p_1^*T(\Gamma G_E)|_{\Upsilon_{\mathcal{A}_d}(\epsilon_0, m_1)})\) is an isomorphism, \(d\Upsilon_{\mathcal{A}_d}(\epsilon_0, m_1)|_{\mathcal{D}(\epsilon_0, m_1)}\) is an isomorphism from \(\mathcal{D}(\epsilon_0, m_1)\) onto \(\mathcal{D}(\Upsilon_{\mathcal{A}_d}(\epsilon_0, m_1))\).

As, by point 3 of Lemma 5.1 \(\Upsilon_{\mathcal{A}_d}^{(2)}\) is a principal \(G\)-bundle, given \(((v_0, r_1), (v_1, r_2)) \in C''(\tilde{G}_E)\), there are \(((\epsilon_0, m_1), (\epsilon_1, m_2)) \in C''(E)\) such that \(\Upsilon_{\mathcal{A}_d}^{(2)}((\epsilon_0, m_1), (\epsilon_1, m_2)) = ((v_0, r_1), (v_1, r_2))\). We fix one element in the \(G\)-orbit formed by those elements. Using Remark 5.9, given \(((v_0, r_1), (v_1, r_2)), (\delta v_1, 0) \in \tilde{p}_3^*\mathcal{D}\) there is a unique \(((\epsilon_0, m_1), (\epsilon_1, m_2)), (\delta \epsilon_1, 0)\) \(\in \tilde{p}_3^*\mathcal{D}\) such that \(\delta v_1, 0 = d\Upsilon_{\mathcal{A}_d}(\epsilon_1, m_2)(\delta \epsilon_1, 0)\).

For the previous let,
\[
\tilde{\varphi}((v_0, r_1), (v_1, r_2))(\delta v_1, 0) := D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_0, m_1)(\mathcal{P}((\epsilon_0, m_1), (\epsilon_1, m_2))(\delta \epsilon_1, 0))
\]
\[
+ D_2(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_0, m_1)(d\phi(\epsilon_1)(\delta \epsilon_1))
\]
\[
= d(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_0, m_1)(\mathcal{P}((\epsilon_0, m_1), (\epsilon_1, m_2))(\delta \epsilon_1, 0), d\phi(\epsilon_1)(\delta \epsilon_1)).
\]
(5.6)

**Lemma 5.10.** Under the previous conditions, the map \(\tilde{\varphi}\) defined by (5.6) is a well defined element of \(\text{hom}(\tilde{p}_3^*\mathcal{D}, \text{ker}(dp^M/G))\).

**Proof.** The proof is similar to the proof of the Lemma 5.10 of [14] with \(\tilde{p}_3^*\mathcal{D}\) instead of \(\tilde{p}_3^*(T\tilde{G}_E)\) and taking into account the \(G\)-invariance of \(\mathcal{D}\).
Since the constraint spaces of the system $M$ maintained in Section 3.2 as associated to the reduction of $(Q, L)$ analogous to the proof of Proposition 5.13 in [14], with the proof that $\Upsilon_{A_d}$ is determined by $\tilde{\mathcal{L}}_{\theta}(\tilde{\mathcal{P}}, \delta q_k, 0) = D_2(p_1 \circ \Upsilon_{A_d})(q_{k+1}, q_k)$, where we have $\tilde{\mathcal{L}}_{\theta}(\tilde{\mathcal{P}}, \delta q_k, 0) = \tilde{\mathcal{L}}_{\theta}(\tilde{\mathcal{P}}, \delta q_k, 0)$ and $\delta q_k \in \mathcal{D}_{q_{k+1}}^h$. This DLDPS coincides with the one obtained in Section 3.2 as associated to the reduction of $(Q, L, D_d, D^n)$ modulo $G$ in the sense of [13]. Thus, the reduction process of DLDPSs extends the reduction construction of discrete nonholonomic systems introduced in [13].

**Proposition 5.13.** Let $G$ be a symmetry group of $M = (E, L_d, D_d, D, P) \in \text{ob}_\mathcal{D}_\mathcal{P}_d$ and $\mathcal{A}_d$ a discrete connection on the principal $G$-bundle $\pi_{M/G} : M \to M/G$. Then, $\Upsilon_{A_d} \in \text{mor}_\mathcal{D}_\mathcal{P}_d(M, M/(G, \mathcal{A}_d))$, where $\Upsilon_{A_d}$ is the map defined by (2.3).

**Proof.** The proof that $\Upsilon_{A_d}$ satisfies the conditions 1 to 4 and 7 of Definition 4.1 is analogous to the proof of Proposition 5.13 in [14], with $\tilde{\mathcal{L}}_{\theta}(\tilde{\mathcal{P}}, \delta q_k, 0) = D_2(p_1 \circ \Upsilon_{A_d})(q_{k+1}, q_k)$, where we have $\tilde{\mathcal{L}}_{\theta}(\tilde{\mathcal{P}}, \delta q_k, 0) = \tilde{\mathcal{L}}_{\theta}(\tilde{\mathcal{P}}, \delta q_k, 0)$ and $\delta q_k \in \mathcal{D}_{q_{k+1}}^h$. This DLDPS coincides with the one obtained in Section 3.2 as associated to the reduction of $(Q, L, D_d, D^n)$ modulo $G$ in the sense of [13]. Thus, the reduction process of DLDPSs extends the reduction construction of discrete nonholonomic systems introduced in [13].

**Proposition 5.14.** Let $G$ be a symmetry group of $M = (E, L_d, D_d, D, P) \in \text{ob}_\mathcal{D}_\mathcal{P}_d$ and $\mathcal{A}_d^1, \mathcal{A}_d^2$ be two discrete connections on the principal $G$-bundle $\pi_{M/G} : M \to M/G$. Then, the reduced systems $M/(G, \mathcal{A}_d^1)$ and $M/(G, \mathcal{A}_d^2)$ are isomorphic in $\mathcal{L}_\mathcal{D}_\mathcal{P}_d$.

**Proof.** The proof is analogous to the proof of Proposition 5.14 in [14], using Lemmas 2.12 and 4.5 and Proposition 5.13.

**5.3. Dynamics of the reduced discrete Lagrange–D’Alembert–Poincaré system.** In this section we consider the dynamics of the reduced system defined in Section 5.2.

**Theorem 5.15.** Let $G$ be a symmetry group of $M = (E, L_d, D_d, D, P)$, $\mathcal{A}_d$ a discrete connection on the principal $G$-bundle $\pi_{M/G} : M \to M/G$ and $M/(G, \mathcal{A}_d) = (\tilde{G}_E, \tilde{L}_d, \tilde{D}_d, \tilde{D}, \tilde{P}) \in \text{ob}_\mathcal{D}_\mathcal{P}_d$ be the corresponding reduced DLDPS. Assume that $(\epsilon, m.) = ((\epsilon_0, m_1), \ldots, (\epsilon_{N-1}, m_N))$ is a discrete path in $C'(E)$, and $(v, r.) = (v_{0}, r_1, \ldots, v_{N-1}, r_N)$, where $v_{0}, r_1, \ldots, v_{N-1}, r_N \in \mathcal{D}_{q_{k+1}}^h$. Then, the discrete Lagrange–D’Alembert–Poincaré system is given by

$$\mathcal{L}(\epsilon, m.) = \mathcal{L}_{\theta}(\mathcal{P}, \delta q_k, 0) = D_2(p_1 \circ \Upsilon_{A_d})(q_{k+1}, q_k),$$

$$\delta q_k \in \mathcal{D}_{q_{k+1}}^h.$$
be a discrete connection on the principal $G$-bundle $\pi^{M,G} : M \to M/G$. For the discrete path $(\epsilon, m)$ in $C'(E)$ we define a discrete path $(v, r)$ in $C'(\tilde{G}_E)$ as $(v_k, r_k) := \Upsilon_{A_d}(\epsilon_k, m_{k+1})$ for $k = 0, \ldots, N - 1$. Then, the following statements are equivalent.

1. $(\epsilon, m)$ is a trajectory of the system $\mathcal{M}_d$.
2. Condition (3.4) is satisfied for $\nu_d := \nu_d^{M}$ defined by (3.3) for $\mathcal{M}$.
3. $(v, r)$ is a trajectory of the system $\mathcal{M}/(G, A_d)$.
4. Condition (3.4) is satisfied for $\nu_d := \nu_d^{M/(G, A_d)}$ and $D_d := \hat{D}_d$ being those of $\mathcal{M}/(G, A_d)$.

Proof. The equivalence $1 \Leftrightarrow 2$ was demonstrated in Proposition 3.8. The equivalence $3 \Leftrightarrow 4$ follows from Proposition 3.8 applied to the system $\mathcal{M}/(G, A_d)$. The equivalence $1 \Leftrightarrow 3$ follows from Theorem 5.15.

Theorem 5.17. Let $G$ be a symmetry group of $\mathcal{M} = (E, L_d, D_d, D, P) \in \text{ob}_\Sigma^G$, and $A_d$ be a discrete connection on the principal $G$-bundle $\pi^{M,G} : M \to M/G$. Let $(v, r)$ be a trajectory of the system $\mathcal{M}/(G, A_d)$ and $(\tilde{\epsilon}_0, \tilde{m}_1) \in D_d$ such that $\Upsilon_{A_d}(\tilde{\epsilon}_0, \tilde{m}_1) = (v_0, r_1)$. Then, there exists a unique trajectory $(\epsilon, m)$ of $\mathcal{M}$ such that $(\epsilon_0, m_1) = (\tilde{\epsilon}_0, \tilde{m}_1)$ and $\Upsilon_{A_d}(\epsilon_k, m_{k+1}) = (v_k, r_{k+1})$ for all $k$.

Proof. By Proposition 5.2, the discrete path $(v, r)$ lifts to a unique discrete path $(\epsilon, m)$ in $C'(E)$ starting at $(\tilde{\epsilon}_0, \tilde{m}_1)$. Then, $(\epsilon_0, m_1) = (\tilde{\epsilon}_0, \tilde{m}_1)$ and $(v_k, r_{k+1}) = \Upsilon_{A_d}(\epsilon_k, m_{k+1})$ for all $k$. As $(v, r)$ is a trajectory of $\mathcal{M}/(G, A_d)$, by Theorem 5.15, $(\epsilon, m)$ is a trajectory of $\mathcal{M}$.

Remark 5.18. Theorem 5.17 states that all trajectories of a reduced discrete Lagrange–D’Alembert–Poincaré system $\mathcal{M}/(G, A_d)$ come from trajectories of the original system $\mathcal{M}$. A direct description of the reconstruction process in terms of the lifting of discrete paths is given in Remark 5.18 of [14]. In that respect, it should be kept in mind that, as $D_d = \Upsilon_{A_d}^{-1}(D_d)$ and the trajectories of $\mathcal{M}/(G, A_d)$ are in $D_d$, the lifted discrete paths are in $D_d$ automatically.

Next we study the relationship between the equation of motion of a symmetric DLPS and that of its reduction. Before we can state the result, we recall the pullback construction for sections of vector bundles. If $\rho_j : \mathcal{V}_j \to X_j$ for $j = 1, 2$ are smooth vector bundles and $F : \mathcal{V}_1 \to \mathcal{V}_2$ is a morphism of vector bundles over $f : X_1 \to X_2$, there is a pullback map $F^* : \Gamma(X_2, \mathcal{V}_2^* \to \Gamma(X_1, \mathcal{V}_1^*)$ determined by $F^*(\alpha_2)(x_1)(v_1) := \alpha_2(f(x_1))(F(v_1))$, for $\alpha_2 \in \Gamma(X_2, \mathcal{V}_2^*)$, $x_1 \in X_1$ and $v_1 \in (\mathcal{V}_1)_x$.

Let $G$ be a symmetry group of $\mathcal{M} = (E, L_d, D_d, D, P) \in \text{ob}_\Sigma^G$ and $A_d$ be a discrete connection on the principal $G$-bundle $\pi^{M,G} : M \to M/G$. Let $\mathcal{M}/(G, A_d)$ be the corresponding reduced system. We have the vector bundles
\[ \tilde{p}_{34}^*(D) \rightarrow C''(E) \] and \[ \tilde{p}_{34}^*(\tilde{D}) \rightarrow C''(\tilde{G}_E) \]. Then, as \( d\tilde{Y}^{(2)}_{A_d} : TC''(E) \rightarrow TC''(\tilde{G}_E) \) is a morphism of vector bundles (over \( T_{A_d}^{(2)} : C''(E) \rightarrow C''(\tilde{G}_E) \)) that restricts to a morphism \( d\tilde{Y}^{(2)}_{A_d}\bigl|_{\tilde{p}_{34}^*(D)} : \tilde{p}_{34}^*(D) \rightarrow \tilde{p}_{34}^*(\tilde{D}) \), we have the pullback map \( (d\tilde{Y}^{(2)}_{A_d}\bigl|_{\tilde{p}_{34}^*(D)})^* : \Gamma(C''(\tilde{G}_E), \tilde{p}_{34}^*(\tilde{D})^*) \rightarrow \Gamma(C''(E), \tilde{p}_{34}^*(D)^*) \). The following result relates the equation of motion of \( \mathcal{M} \), \( \nu_d^M \in \Gamma(C''(E), \tilde{p}_{34}^*(D)^*) \), to the one of \( \mathcal{M}/(G, A_d) \), \( \nu_d^{M/G} \in \Gamma(C''(\tilde{G}_E), \tilde{p}_{34}^*(\tilde{D})^*) \).

**Lemma 5.19.** With the previous notation, \( \nu_d^M = (d\tilde{Y}^{(2)}_{A_d}\bigl|_{\tilde{p}_{34}^*(D)})^*(\nu_d^{M/G}) \).

**Proof.** For any \( \mu := ((\epsilon_0, m_1), (\epsilon_1, m_2)) \in C''(E) \) and \( (\delta\epsilon_1, 0) \in (\tilde{p}_{34}^*(D))_\mu \), we have
\[
(d\tilde{Y}^{(2)}_{A_d}\bigl|_{\tilde{p}_{34}^*(D)})^*(\nu_d^{M/G})(\mu)(\delta\epsilon_1, 0) = \nu_d^{M/G}(\tilde{Y}^{(2)}_{A_d}(\mu))(d\tilde{Y}^{(2)}_{A_d}(\epsilon_1, m_2))(\delta\epsilon_1, 0).
\]
Then, a direct computation shows that
\[
\nu_d^{M/G}(\tilde{Y}^{(2)}_{A_d}(\mu))(d\tilde{Y}^{(2)}_{A_d}(\epsilon_1, m_2))(\delta\epsilon_1, 0) = \nu_d^M(\mu)(\delta\epsilon_1, 0),
\]
proving the statement. \( \Box \)

Let \( \mathcal{M} = (E, L_d, D_d, D, \mathcal{P}) \in \text{ob}_{\mathbb{P}_j} \) and \( G \) be a symmetry group of \( \mathcal{M} \). Consider the vertical bundle \( \ker(T\pi^{E,G}) \subset TE \) over \( E \) and let \( V := p^*_1 \ker(T\pi^{E,G}) \), where \( p_1 : C'(E) \rightarrow E \) is the projection map. Assume that \( S := D \cap V \) is a vector bundle over \( C'(E) \) and that there is another vector bundle \( \mathcal{H} \rightarrow C'(E) \) such that \( D = S \oplus \mathcal{H} \); the vector bundle \( \mathcal{H} \) may be constructed using a (continuous) connection on the principal bundle \( \pi^{E,G} : E \rightarrow E/G \). Then, as the section of motion \( \nu_d^M \), takes values in \( \tilde{p}_{34}^*(D)^* \simeq \tilde{p}_{34}^*(S^*) \oplus \tilde{p}_{34}^*(\mathcal{H}^*) \) we can decompose it as \( \nu_d^M = \left( (\nu_d^M)^S, (\nu_d^M)^H \right) \). Clearly, for any \( \mu \in C''(E) \), the condition \( \nu_d^M(\mu) = 0 \) is equivalent to
\[
(\nu_d^M)^S(\mu) = 0 \quad \text{and} \quad (\nu_d^M)^H(\mu) = 0.
\]
The last two conditions are usually called the vertical and horizontal equations of motion.

**Definition 5.20.** Let \( \mathcal{M} = (E, L_d, D_d, D, \mathcal{P}) \in \text{ob}_{\mathbb{P}_j} \) and \( G \) be a symmetry group of \( \mathcal{M} \). We define the discrete nonholonomic momentum map
\[
J_d : C'(E) \rightarrow (g^D)^* \quad \text{by} \quad J_d(\epsilon_0, m_1)(\xi) := -D_1L_d(\epsilon_0, m_1)(\xi_E(\epsilon_0))
\]
for \( (\epsilon_0, m_1) \in C'(E) \) and \( (\epsilon_0, m_1), \xi \in g^D := \{(\epsilon_0, m_1), \xi \in C'(E) \times g : (\xi_E(\epsilon_0), 0) \in D(\epsilon_0, m_1)\} \), where \( g := \text{Lie}(G) \); we assume that \( g^D \rightarrow C'(E) \) is a smooth vector bundle. We also define, for each \( \xi \in \Gamma(g^D) \), \( (J_d)_\xi : C'(E) \rightarrow \mathbb{R} \) by \( (J_d)_\xi(\epsilon_0, m_1) := J_d(\epsilon_0, m_1)(\xi(\epsilon_0, m_1)) \).

In the context of Definition 5.20 we have that \( L_{d}(l^E_\xi(\epsilon_0), l^M_\xi(m_1)) = L_{d}(\epsilon_0, m_1) \) for all \( g \in G \) and \( (\epsilon_0, m_1) \in C'(E) \). Then, for each \( ((\epsilon_0, m_1), \xi) \in C'(E) \times g \),
\[
0 = \left| \frac{d}{dt} \right|_{t=0} L_{d}(l^{E}_{\exp(\xi t)}(\epsilon_0), l^{M}_{\exp(\xi t)}(m_1))
\]
\[= D_1L_d(\epsilon_0, m_1)(\xi_E(\epsilon_0)) + D_2L_d(\epsilon_0, m_1)(\xi_M(m_1)), \]
thus, for \( ((\epsilon_0, m_1), \xi) \in C'(E) \times g^D \),
\[
J_d(\epsilon_0, m_1)(\xi) = -D_1L_d(\epsilon_0, m_1)(\xi_E(\epsilon_0)) = D_2L_d(\epsilon_0, m_1)(\xi_M(m_1)). \tag{5.7}
\]
By Proposition 3.8, if \((\epsilon, m.)\) is a trajectory of \(\mathcal{M}\) then (3.4) holds. Let \(\xi \in \Gamma(\mathfrak{g}^D)\), so that, for each \(k = 1, \ldots, N - 1\), \(\xi(\epsilon_k, m_{k+1}) \in \mathfrak{g}^D(\epsilon_k, m_{k+1})\); then, for each such \(k\), we can evaluate the second condition in (3.4) at \((\xi(\epsilon_k, m_{k+1}))_{E(\epsilon_k),0}) \in \mathcal{S}(\epsilon_k, m_{k+1}) \subset D(\epsilon_k, m_{k+1})\) to obtain

\[-D_1 L_d(\epsilon_k, m_{k+1})(\xi(\epsilon_k, m_{k+1}))_{E(\epsilon_k)})
\]

\[= D_2 L_d(\epsilon_{k-1}, m_k) \circ d\phi(\xi)(\xi(\epsilon_k, m_{k+1}))_{E(\epsilon_k)})
\]

\[= (\xi(\epsilon_k, m_{k+1}))_{E(\epsilon_k)}) \circ (\xi(\epsilon_k, m_{k+1}))_{E(\epsilon_k),0}).
\]

Plugging this last result into (5.7), we see that

\[\begin{align*}
(J_d)\xi(\epsilon_k, m_{k+1}) &= (J_d)\xi(\epsilon_{k-1}, m_k) \\
&+ D_1 L_d(\epsilon_{k-1}, m_k) \circ \mathcal{P}(\epsilon_{k-1}, m_k, \xi_{E(\epsilon_k)})_{E(\epsilon_k),0} (\xi(\epsilon_k, m_{k+1}))_{E(\epsilon_k),0}).
\end{align*}
\] (5.8)

for all \(k = 1, \ldots, N - 1\). The following result completes the previous discussion.

**Proposition 5.21.** Let \(\mathcal{M} = (E, L_d, D_d, D, \mathcal{P}) \in \text{ob}_\text{DLP}\) and \(G\) be a symmetry group of \(\mathcal{M}\). Also, let \((\epsilon, m.)\) be a discrete path in \(C'(E)\). The following statements are equivalent.

1. \((\epsilon, m.)\) satisfies \(\nu^S_d((\epsilon_{k-1}, m_k), (\epsilon_k, m_{k+1})) = 0\) for all \(k = 1, \ldots, N - 1\).
2. For all sections \(\xi \in \Gamma(\mathfrak{g}^D)\) and \(k = 1, \ldots, N - 1\), equation (5.8) is satisfied.

**Proof.** The previous argument leading to (5.8) shows that 1 \(\Rightarrow\) 2 is true. Conversely, assume that point 2 is valid. For \(k = 1, \ldots, N - 1\) and any \((\epsilon_k, m_{k+1}), s \in \mathcal{S}(\epsilon_k, m_{k+1})\), there is \(\xi \in \mathfrak{g}\) such that \(s = \xi_{E(\epsilon_k)}\). Then, as we are assuming that \(\mathfrak{g}^D\) is a smooth vector bundle, there is \(\xi \in \Gamma(\mathfrak{g}^D)\) such that \(\xi_{E(\epsilon_k)} = \xi\). Using this particular \(\xi\), equation (5.8) leads to \(\nu^S_d((\epsilon_{k-1}, m_k), (\epsilon_k, m_{k+1}))(s) = 0\) and, eventually, to the validity of point 1.

The equation (5.8) is known as the **discrete nonholonomic momentum evolution equation** and has been considered, for instance, in [8] and [13].

### 5.4. Discrete LL systems on Lie groups.

In this section we review the notions of discrete and continuous LL system on a Lie group \(G\) and then show how, in the discrete case, LL systems are examples of DLDPSs. We also show that their reduced and “momentum description” on \(Lie(G)^*\) are also examples of DLDPSs in a natural way and find their equations of motion, which agree with the ones that appear in the literature. As a concrete case, we explore the discrete Suslov system.

An LL system on the Lie group \(G\) is a nonholonomic mechanical system \((G, L, D)\) for whom \(G\), acting on itself by left multiplication, is a symmetry group. Such a system can be described alternatively as a reduced system on \(\mathfrak{g} := Lie(G)\) with reduced lagrangian \(\ell\) and constraint subspace \(\mathfrak{d} \subset \mathfrak{g}\). Yet another description, using a (reduced) Legendre transform, is as a dynamical system on \(\mathfrak{g}^*\) satisfying the Euler–Poincaré–Suslov equations (see, for instance [3]).

**Example 5.22.** A well known example of this type of system, due to G. Suslov [32], is a model for a rigid body, with a fixed point and constrained so that one of the components of its angular velocity relative to the body frame vanishes. Explicitly, the configuration space is the Lie group \(G := SO(3)\), with Lagrangian \(L(g, \dot{g}) :=\)
Example 5.23. The discrete version of the Suslov system has been extensively studied in \([11]\) and \([16]\) as a reduced discrete mechanical system on \(G\) as a discrete dynamical system obeying the discrete Euler–Lagrange–Suslov equations in \(\mathfrak{so}(3)^*\) that, in terms of the angular momentum \(M := \mathbb{I}\omega\) are

\[
\begin{aligned}
\dot{M} &= M \times (\mathbb{I}^{-1}M) + \lambda e_3, \\
M &\in \mathfrak{g}^* := \mathbb{I}(\mathfrak{d}).
\end{aligned}
\]

A discrete analogue of the LL systems has been considered by Yu. Fedorov and D. Zenkov in \([11]\) and, also, by R. McLachlan and M. Perlmutter in \([27]\); the purpose of this section is to show that all the discrete systems that have been considered (reduced, non-reduced and on \(\mathfrak{g}^*\)) can be seen as DLDPS. A discrete LL system on the Lie group \(G\) is a discrete nonholonomic system \((G, L_d, D_d, D_{nh})\) for whom \(G\), acting on itself by left multiplication, is a symmetry group. As seen in Example 3.9, such a discrete nonholonomic system can naturally be seen as a DLDPS \(M^{LL} = (E^{LL}, L_d^{LL}, D_d^{LL}, \mathcal{P}^{LL})\) where the fiber bundle \(E^{LL} \rightarrow M^{LL}\) is id\(_G\) : \(G \rightarrow G\) (with \(G\) acting by left multiplication on both \(G\)s), so that \(C'(id_G) = G \times G\) while \(L_d^{LL} = L_d, D_d^{LL} = D_d, \mathcal{P}^{LL} = 0\) and \(D^{LL} = p_1^*(\mathcal{P}^{nh})\) for \(p_1 : G \times G \rightarrow G\) the projection onto the first factor.

Example 5.23. The discrete version of the Suslov system has been extensively studied in \([11]\) and \([16]\) as a reduced discrete mechanical system on \(SO(3)\) and as a discrete dynamical system obeying the discrete Euler–Lagrange–Suslov equations. Here we follow the notation of \([16]\). This nonholonomic discrete mechanical system is defined on the space \(G := SO(3)\), with discrete Lagrangian \(L_d(y_0, g_1) := -\text{Tr}(g_1y_1g_0)\), where

\[
J := \begin{pmatrix}
\frac{1}{2}(I_{22} + I_{33} - I_{11}) & 0 & -I_{13} \\
0 & \frac{1}{2}(I_{11} + I_{33} - I_{22}) & -I_{23} \\
-I_{13} & -I_{23} & \frac{1}{2}(I_{11} + I_{22} - I_{33})
\end{pmatrix}
\]

is the mass tensor associated to the rigid body’s inertia tensor \(\mathbb{I}\) (which can be assumed to have component \(I_{12} = 0\)). The infinitesimal variations are determined by the subspace

\[
\mathfrak{d} := \left\{ \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \right\} \subset \mathfrak{so}(3)
\]

that corresponds to the subspace \(\mathfrak{d} \subset \mathbb{R}^3\) defined in Example 5.22 under the isomorphism \(\mathbb{R}^3 \simeq \mathfrak{so}(3)\) — through \(D_{nh}^g := dL_g(1)(\mathfrak{d}) = \{ g\omega \in \mathbb{R}^{3 \times 3} : \omega \in \mathfrak{d} \}\) for any \(g \in G\). In order to define the discrete dynamical constraints, we recall the Cayley transform \(\text{Cay} : \mathfrak{so}(3) \rightarrow G\) defined by \(\text{Cay}(\omega) := (1 + \frac{\omega}{2})(1 - \frac{\omega}{2})^{-1}\); then we define \(S_d := \text{Cay}(\mathfrak{d})\). An explicit parametrization of \(S_d\) is given in (25) of \([16]\), but it won’t be necessary for our purposes. It suffices to say that \(W \in G\) is in \(S_d\) if and only if its rotation axis is orthogonal to the \(e_3\) axis in \(\mathbb{R}^3\) and

\[1\]We consider here only the system whose discrete Lagrangian originates in \(\ell_d^{(1,\epsilon)}\) with \(\epsilon = 1\). Still, the analysis remains valid for arbitrary \(\epsilon\) and, also, for \(\ell_d^{(\infty,\epsilon)}\).
the angle of rotation is in \((-π, π)\). Then, the discrete dynamical constraint is
\[
D_d := \{(g_0, g_1) \in G \times G : g_0^{-1}g_1 \in S_d\}.
\]
All together, \((G, L_d, D_d, D^{nh})\) is a discrete nonholonomic mechanical system in the sense of Example 3.9. Hence, as seen in the same Example, \(\mathcal{M}^{LL} := (E^{LL}, L_d^{LL}, D_d^{LL}, D^{LL}, \mathcal{P}^{LL})\) is a DLDP, where the fiber bundle \(E^{LL} \to M^{LL}\) is \(id_G : G \to G\), \(L_d^{LL} := L_d\), \(D_d^{LL} := D_d\), \(D^{LL} := P_1(D^{nh})\) and \(\mathcal{P}^{LL} = 0\). We conclude that the discrete Suslov system can be seen as a discrete Lagrange–D’Alembert–Poincaré system in a natural way.

Back in the setting of an arbitrary discrete \(LL\) system, we are given that \(G\) is a symmetry group of the nonholonomic discrete mechanical system \((G, L_d, D_d, D^{nh})\) thus, as noted in Remark 5.4, \(G\) is also a symmetry group of \(\mathcal{M}^{LL}\) in \(\mathfrak{L}\mathfrak{D}\mathfrak{P}_d\).

In order to reduce \(\mathcal{M}^{LL}\) by \(G\) we need a discrete affine connection on the trivial principal bundle \(G \to G/G = \{[e]\}\). It is easy to check that \(A_d(g_0, g_1) := g_1g_0^{-1}\) is such a connection and, for completeness sake, we recall that, by Proposition 5.14, all the reduced systems \(\mathcal{M}^{LL}/G\) obtained using different discrete connections\(^2\) are isomorphic in \(\mathfrak{L}\mathfrak{D}\mathfrak{P}_d\). Thus, we have the reduced space \(\mathcal{M}^r := \mathcal{M}^{LL}/(G, A_d)\).

The total space of the fiber bundle underlying \(\mathcal{M}^r\) is \(\hat{G}_G = (G \times G)/G\) where \(G\)-action is \(\eta_g^G \times G\) \((g_0, g_1) := (gg_0, g_0g_1^{-1})\). The corresponding reduction morphism \(\Upsilon_{A_d} : G \times G \to \hat{G}_G \times \{[e]\}\) is given by \(\Upsilon_{A_d}(g_0, g_1) = (\pi^{G \times G} \cdot \pi_G)(g_0, g_0g_1^{-1}, [e])\).

It is easier to work with an isomorphic model of \(\mathcal{M}^r\). Let \(\tilde{\eta} : G \times G \to G\) be defined by \(\tilde{\eta}(g_0, g_1) := g_0^{-1}g_1g_0\); as \(\tilde{\eta}\) is \(G\)-invariant for the action \(\eta_g^G \times G\) defined above it induces a smooth map \(\eta : (G \times G)/G \to G\) that turns out to be a diffeomorphism. If we view \(\eta\) as an isomorphism of the fiber bundle \((G \times G)/G \to \{[e]\}\) onto \(G\) we can use Lemma 4.7 to obtain \(\mathcal{M}^\eta \in \mathfrak{L}\mathfrak{D}\mathfrak{P}_d\) that is isomorphic to \(\mathcal{M}^r\) in \(\mathfrak{L}\mathfrak{D}\mathfrak{P}_d\). Explicitly, \(\mathcal{M}^\eta = (E^\eta, L_d^\eta, D_d^\eta, D^\eta, \mathcal{P}^\eta)\), where \(E^\eta \to M^\eta\) is the fiber bundle \(G \to \{e\}\), \(L_d^\eta = \ell_d\) —where \(\ell_d(W) = L_d(e, W)\)—, \(D_d^\eta = S_d \times \{e\}\) —where \(W \in S_d\) if and only if \((e, W) \in D_d\) and \(D_d^\eta = dR_W(e)(\delta)\) and
\[
\mathcal{P}^\eta((W_0, e), (W_1, e)) \delta W_1, 0 = dL_W_0(\delta h)
\]
where \((\delta W_1, 0) = (-dR_W_1(e)(\delta h), 0) \in \tilde{\mathcal{P}}_34(D^\eta)\). With this information, the evolution in \(\mathcal{M}^\eta\) can be determined using the section \(\nu_d^\eta\) defined in (3.3).

**Proposition 5.24.** \(W_k \in G\) for \(k = 0, \ldots, N - 1\) is a trajectory of \(\mathcal{M}^\eta\) if and only if
\[
\begin{cases}
W_k \in S_d & \text{for} \quad k = 0, \ldots, N - 1, \\
R_{W_{k+1}}(T_{W_{k+1}}, \ell_d) - L_{W_k}(T_{W_{k-1}}, \ell_d) \in \mathfrak{d}^\circ & \text{for} \quad k = 0, \ldots, N - 2.
\end{cases}
\]

**Proof.** Indeed, by Proposition 3.8, \(W_k\) is a trajectory of \(\mathcal{M}^\eta\) if and only if
\[
\begin{cases}
W_k \in S_d & \text{for} \quad k = 0, \ldots, N - 1, \\
\nu_d^\eta((W_k, e), (W_{k+1}, e))(D_{W_{k+1}, e}) = 0 & \text{for} \quad k = 0, \ldots, N - 2.
\end{cases}
\]
As, in this case,
\[
\nu_d^\eta((W_k, e), (W_{k+1}, e))(D_{W_{k+1}, e}) = \nu_d^\eta((W_k, e), (W_{k+1}, e))(-dR_{W_{k+1}}(e)(\delta))
\]
\[
= (d\ell_d(W_{k+1}) \circ dR_{3k}(e) - d\ell_d(W_k) \circ dL_W(e))(\delta)
\]
\[
= (R_{W_{k+1}}^*(d\ell_d(W_{k+1})) - L_{W_k}^*(d\ell_d(W_k)))(\delta),
\]
the statement follows. \(\square\)

\(^2\)In this case it is easy to describe all possible affine discrete connections: their domain can be extended to \(G \times G\) and have discrete connection form \(A_d^h(g_0, g_1) = g_1h^{-1}g_0^{-1}\) for a fixed \(h \in G\).
Remark 5.25. When the discrete path $W$ is the reduction of the discrete path $g$, that is, when $W_k = p_1 \circ \eta \circ Y_{A_d}(g_k, g_{k+1}) = g_k^{-1}g_{k+1}$, by Corollary 5.16, $W$ is a trajectory of $\mathcal{M}^9$ if and only if $g$ is a trajectory of $\mathcal{M}^{LL}$, and by Example 3.9, if and only if $g$ is a trajectory of the discrete nonholonomic system $(G, L_d, D, D^{nh})$.

Remark 5.26. The result of Proposition 5.24 is part (iv) of Theorem 3.2 in [11]. The complete result can be read off Corollary 5.16 applied to $\mathcal{M}^{LL}$ and $\mathcal{M}^9$.

Example 5.27. It is immediate that the discrete Suslov system described in Example 5.23 is an LL-system, so that $\mathcal{M}^{LL} \in \text{ob}_L \mathfrak{M}_d$ can be reduced by $G = SO(3)$ and the connection $A_d(g_0, g_1) := g_1g_0^{-1}$, defining a reduced system $\mathcal{M}^* := \mathcal{M}^{LL}/(G, A_d)$. Just as it was described above, the diffeomorphism $\eta : (G \times G)/G \to G$ defined by $\eta((g, g_0)) := g_0^{-1}g_1$ can be used to define $\mathcal{M}^9 \in \text{ob}_L \mathfrak{M}_d$, with the property that $\eta$ turns out to be an isomorphism between $\mathcal{M}^*$ and $\mathcal{M}^9$.

Explicitly, the fiber bundle $E^9 \to \mathcal{M}^9$ is $G \to \{1\}$, $L^9_\eta(W) = \delta_W(W) = -\text{Tr}(\mathcal{J}W)$, $D^9_\eta = S_d \times \{1\} = \text{Cay}(\delta) \times \{1\}$, $D^0_{W_1} = dR_W(1)(\delta) = \{\delta hW \in \mathbb{R}^{3 \times 3} : \delta h \in \delta W_1\}$, and for any $\delta W_1 \in D^9_{W_1}$,

$$\mathcal{P}^9((W_0, 1), (W_0, 1))(\delta W_1, 0) = -W_0\delta W_1 W_1^{-1}.$$ This discrete system $\mathcal{M}^9$ provides the usual (reduced) description of the discrete Suslov system as a dynamical system on $SO(3)$. Specializing Proposition 5.24 to the current setting, we have that a discrete path $W$ in $G$ is a discrete trajectory of $\mathcal{M}^9$ if and only if

$$\begin{cases} W_k \in S_d = \text{Cay}(\delta) & \text{for all } k = 0, \ldots, \\
-(W_{k+1} - \mathcal{J}W_{k+1}^F) - (\mathcal{J}W_k - W_k^F) \in \delta W_{k+1} & \text{for all } k = 0, \ldots,
\end{cases}$$

where we identify $\mathfrak{so}(3)^*$ with $\mathfrak{so}(3)$ using the inner product $\langle A_1, A_2 \rangle := \frac{1}{2} \text{Tr}(A_1A_2)$; in particular, $\delta W_1$ corresponds to $\delta W_{k+1}$.

Just as in the continuous case, it is possible to give an alternative model for $\mathcal{M}^9$ as a dynamical system in (a submanifold of) $\mathcal{g}^*$. Recall that the (c) discrete Legendre transform of $L_d$ is the map $\mathbb{F}^*L_d : G \times G \to T^*G$ defined by $\mathbb{F}^*L_d(g_0, g_1) := -D_1L_d(g_0, g_1)$. Using the trivialization $\lambda : T^*G \to G \times \mathcal{g}^*$ defined by $\lambda(g) := (g, L^*_g(\alpha_g))$, we see that

$$\lambda(\mathbb{F}^*L_d)(g_0, g_1) = \lambda(-D_1L_d(g_0, g_1)) = (g_0, L^*_g(-D_1L_d(g_0, g_1))).$$

If we define $p : G \times G \to \mathcal{g}^*$ by $p(g_0, g_1) := L^*_g(-D_1L_d(g_0, g_1))$, for any $\xi \in \mathcal{g}$ we have

$$p(\xi) = L^*_g(-D_1L_d(g_0, g_1))(\xi) = -D_1L_d(g_0, g_1)(dL_{g_0}(e)(\xi)) = \frac{d}{ds}igg|_{s=0} L_d(g_0 \exp(s\xi), g_1) = \frac{d}{ds}igg|_{s=0} \ell_d(\exp(-s\xi)g_0^{-1}g_1) = dl_d(W_0)(\mathcal{J}^*R_{W_0}(\xi)) = R_{W_0}(dl_d(W_0)(\xi)),$$

for $W_0 := g_0^{-1}g_1$. Thus, we define the reduced Legendre transform $L : G \to \mathcal{g}^*$ by

$$L(W_0) := p = R_{W_0}(dl_d(W_0)).$$

In what follows we assume that $L$ is a diffeomorphism. Then, applying Lemma 4.7 to $L$ and $\mathcal{M}^9$, we see that there is $\mathcal{M}^S \in \text{ob}_L \mathfrak{M}_d$ such that $L$ is an isomorphism

\[\text{In fact, under the usual regularity conditions on } \ell_d, L \text{ is a local diffeomorphism and care must be taken, restricting the constructions to appropriate open subsets, where } L \text{ is diffeomorphism.}\]
from $\mathcal{M}^0$ into $\mathcal{M}^S$. Explicitly, $E^S \to M^S$ is $\mathfrak{g}^* \to \{0\}$, $L^S_d = \ell_d \circ \mathcal{L}^{-1}$, $\mathcal{D}^S_p = \mathcal{L}(\mathcal{S}_d)$, $\mathcal{D}^S_p = d(\mathcal{L}, 0)\mathcal{L}^{-1}(p) = (\mathcal{D}^\eta_{\mathcal{L}^{-1}(p)}(\mathcal{L}))|0\rangle$ and

$$\mathcal{P}^S((p_{k-1}, 0), (p_k, 0))(\delta p_k, 0) = R^\mathcal{L}_{-1}(p_{k-1}) (d\mathcal{L}(\mathcal{L}^{-1}(p_{k-1})))|0\rangle$$

if $\delta p_k = -R^\mathcal{L}_{-1}(p_k) (d\mathcal{L}(\mathcal{L}^{-1}(p_k)))|0\rangle$.

A direct application of Proposition 3.8 to $\mathcal{M}^S \in \text{ob}_{\mathcal{L}^\mathcal{D}^\mathcal{P}}$ leads to the following result.

**Proposition 5.28.** A discrete path $p$. in $\mathfrak{g}^*$ is a discrete trajectory of $\mathcal{M}^S$ if and only if

$$\begin{cases} p_k \in \mathcal{L}(\mathcal{S}_d), & \text{for } k = 0, \ldots, \\ (p_{k+1} - \text{Ad}^*_{\mathcal{L}^{-1}(p_k)}(p_k)) \in \mathfrak{h}^\mathcal{S} & \text{for } k = 0, \ldots, \end{cases}$$

(5.10)

where $\text{Ad}^*_g := L^\mathcal{S}_g \circ R^\mathcal{S}_{g^{-1}}$.

The system (5.10) is known as the discrete Euler–Poincaré–Suslov equations (see Theorem 3.3 in [11]).

**Proof.** As $\mathcal{M}^S$ is constructed out of $\mathcal{M}^0$ using the diffeomorphism $\mathcal{L}$ and Lemma 4.7, we know that

$$\begin{align*}
\nu^S_d((\mathcal{L}(W_k), 0), (\mathcal{L}(W_{k+1}), 0)) & (d\mathcal{L}(W_{k+1})|\delta W_{k+1}, 0) \\
& = \nu^S_d((W_k, e), (W_{k+1}, e))|\delta W_{k+1}k, 0).
\end{align*}$$

Using the computation of $\nu^S_d$ developed in the proof of Proposition 5.24,

$$\begin{align*}
\nu^S_d((p_k, 0), (p_{k+1}, 0)) & (\mathcal{D}^S_p(p_{k+1}, 0)) \\
& = \nu^S_d((\mathcal{L}(W_k), 0), (\mathcal{L}(W_{k+1}), 0)) (d\mathcal{L}(W_{k+1})|\mathcal{D}^\eta_{\mathcal{L}^{-1}(p_k)}(\mathcal{L})) \\
& = \nu^S_d((W_k, e), (W_{k+1}, e)) (\mathcal{D}^\eta_{\mathcal{L}^{-1}(p_k)}(\mathcal{L})) \\
& = (R^\mathcal{L}_{W_k}(d\mathcal{L}(W_{k+1})) - L^\mathcal{L}_{W_k}(d\mathcal{L}(W_k)))|0\rangle \\
& = (p_{k+1} - \text{Ad}_{\mathcal{L}^{-1}(p_k)}(p_k))|0\rangle = (p_{k+1} - \text{Ad}_{\mathcal{L}^{-1}(p_k)}(p_k))|0\rangle
\end{align*}$$

As $\mathcal{M}^S \in \text{ob}_{\mathcal{L}^\mathcal{D}^\mathcal{P}}$, the result follows from Proposition 3.8. \hfill $\Box$

**Example 5.29.** The discrete Legendre transform $\mathcal{L} : G \to \mathfrak{so}(3)^*$ associated to the (reduced) discrete Suslov system $\mathcal{M}^0$ described in Example 5.27 can be easily computed using (5.9): for $W \in G$ and $\xi \in \mathfrak{so}(3),$

$$\begin{align*}
\mathcal{L}(W)(\xi) = \frac{d}{ds} \bigg|_{s=0} \ell_d(\exp(s\xi)W) = \frac{d}{ds} \bigg|_{s=0} - \text{Tr}(\mathbb{J} \exp(s\xi)W) & = - \text{Tr}(\mathbb{J} \xi W) \\
= - \frac{1}{2}(\text{Tr}(\mathbb{J} \xi W) + \text{Tr}((\mathbb{J} \xi W)^t)) & = - \frac{1}{2}(\text{Tr}(W \mathbb{J} \xi) - \text{Tr}(\mathbb{J} W^t \xi)) \\
= \frac{1}{2} \text{Tr}((W \mathbb{J} - J W^t)\xi^t) & = (W \mathbb{J} - J W^t, \xi),
\end{align*}$$

and we conclude that $\mathcal{L}(W) = W \mathbb{J} - J W^t$. It is easy to check that $\mathcal{L}$ is a local diffeomorphism, but it is not globally injective (nor onto). Still, the previous arguments can be applied locally to a pair of domains where $\mathcal{L}$ restricts to a diffeomorphism. Hence, $\mathcal{L}$ can be used together with Lemma 4.7 to construct $\mathcal{M}^S \in \text{ob}_{\mathcal{L}^\mathcal{D}^\mathcal{P}}$ that is isomorphic to $\mathcal{M}^0$. Proposition 5.28 provides the equations of motion for $\mathcal{M}^S$. Let
p. be a discrete path in $\mathfrak{so}(3)^* \simeq \mathfrak{so}(3)$, and define $W_k := \mathcal{L}^{-1}(p_k) \in G$ for all $k$. Then, by Proposition 5.28, $p$ is a trajectory of $\mathcal{M}$ if and only if

\[
\begin{cases}
p_k \in \mathcal{L}(\mathcal{S}_d) & \text{for } k = 0, \ldots, \\
p_{k+1} - \text{Ad}_{W_k}(p_k) \in \mathfrak{d}^\perp & \text{for } k = 0, \ldots,
\end{cases}
\]

where, as before, $\mathfrak{d}^\circ$ is identified with $\mathfrak{d}^\perp$. As, for any $\xi \in \mathfrak{so}(3)$,

\[
\text{Ad}^*_W(p_k)(\xi) = p_k(\text{Ad}_W(\xi)) = p_k(W_k\xi W_k^{-1}) = \frac{1}{2} \text{Tr}(p_k(W_k\xi W_k^{-1})^t) = \frac{1}{2} \text{Tr}(W_k^tp_k W_k^t \xi^t) = (W_k^tp_k W_k)(\xi)
\]

we see that $p_{k+1} - \text{Ad}^*_W(p_k) = p_{k+1} - W_k^tp_k W_k$. Then, the discrete Euler–Poincaré–Suslov equations (5.10) for $\mathcal{M}$ are

\[
\begin{cases}
p_k \in \mathcal{L}(\mathcal{S}_d) & \text{for } k = 0, \ldots, \\
p_{k+1} - W_k^tp_k W_k \in \mathfrak{d}^\perp & \text{for } k = 0, \ldots.
\end{cases}
\]

This expression matches the ones given in (6.11) of [11] and (29) of [16].

6. Reduction by two stages. Having introduced a category of DLDPSs, a notion of symmetry group for a DLDPS and a process of reduction for such symmetric objects that is closed in the category, we study the problem of reduction by stages in this section. In other words, when $G$ is a symmetry group of $\mathcal{M}$ and $H$ is a subgroup of $G$ we want to compare the result of the reduction $\mathcal{M}/G$ with that of the iterated reduction $(\mathcal{M}/H)/(G/H)$ whenever possible.

6.1. Residual symmetry. Let $G$ be a symmetry group of $\mathcal{M} = (\mathcal{E}, \mathcal{L}_d, \mathcal{D}_d, \mathcal{D}, \mathcal{P}) \in \text{ob}_{\mathcal{LDP}_d}$ and $H \subset G$ a closed normal subgroup. The following result proves that $H$ is a symmetry group of $\mathcal{M}$.

**Proposition 6.1.** Let $G$ be a symmetry group of $\mathcal{M} \in \text{ob}_{\mathcal{LDP}_d}$. If $H \subset G$ is a closed Lie subgroup, then $H$ is a symmetry group of $\mathcal{M}$.

**Proof.** By Lemma 5.7 of [14] we have that if $G$ is a Lie group that acts on the fiber bundle $(\mathcal{E}, \mathcal{M}, \phi, F)$ and $H \subset G$ is a closed Lie subgroup, then $H$ acts on the fiber bundle $(\mathcal{E}, \mathcal{M}, \phi, F)$ so that condition 1 in Definition 5.3 is valid. The remaining conditions follow from the fact that $G$ satisfies them and that $H$ acts by the restriction of the corresponding actions of $G$. \qed

In what follows we consider the action of the group $G/H$ on the system obtained after having reduced the symmetry of $\mathcal{H}$. As a first step we recall the statement of Lemma 7.1 in [14], that establishes that $G/H$ acts on the fiber bundle obtained after the first reduction stage.

**Lemma 6.2.** Let $G$ be a Lie group that acts on the fiber bundle $(\mathcal{E}, \mathcal{M}, \phi, F)$ and $H \subset G$ be a closed normal subgroup. Define the maps

\[
\begin{align*}
\pi_{G/H}(\pi_E^G(\epsilon), \pi_M^G(m)) & := \pi_{G/H}(\pi_E^G(\epsilon), \pi_M^G(m)), \\
\pi_{G/H}(\pi_M^G(\epsilon), \pi_M^G(m)) & := \pi_{G/H}(\pi_M^G(\epsilon), \pi_M^G(m)).
\end{align*}
\]
Then \( l_{\mathcal{H}} \), \( l_{M/H} \) and the trivial right action on \( F \times H \) define an action of \( G/H \) on the fiber bundle \((\mathcal{H}, M/H, p_{M/H}, F \times H)\).

As in Section 5, these actions induce “diagonal” actions on \( C'(\mathcal{H}) \), \( C''(\mathcal{H}) \) through definitions (5.1) and (5.2) and “lifted” actions on the spaces \( T \mathcal{H} \) and \( \tilde{p}_{\mathcal{H}}(T') \) through definitions (5.3) and (5.5).

The reduction of \( \mathcal{M} \) by \( H \) requires the choice of a discrete connection \( \mathcal{A}^H_p \) on the principal \( H \)-bundle \( \pi^{M,H} : M \to M/H \). It turns out that, under some conditions on \( \mathcal{A}^H_p \) that we explore next, \( G/H \) is a symmetry group of \( \mathcal{M} : = \mathcal{M}/(H, \mathcal{A}^H_p) \).

**Lemma 6.3.** Let \( G \) be a Lie group that acts on \( \mathcal{M} \) by the action \( l^M \) in such a way that \( \pi^{M,G} : M \to M/G \) is a principal \( G \)-bundle. Assume that \( H \subset G \) is a closed normal subgroup and that \( \mathcal{A}^H_p \) is a discrete connection on the principal \( H \)-bundle \( \pi^{M,H} : M \to M/H \), whose domain \( \Omega \) is \( G \)-invariant by the diagonal action \( l^M \times M \).

Then, the following statements are equivalent.

1. For each \( g \in G \) and \((m_0, m_1) \in \Omega \),
   \[
   \mathcal{A}^H_p(l^M_g(m_0), l^M_g(m_1)) = g \mathcal{A}^H_p(m_0, m_1)g^{-1}.
   \]
2. The submanifold \( H \mathcal{A}^H_p \subset M \times M \) is \( G \)-invariant by the action \( l^M \times M \).

**Proof.** The proof is analogous to the proof of Lemma 7.2 in [14].

**Proposition 6.4.** Let \( G \) be a symmetry group of \( \mathcal{M} = (E, L_d, D_d, D, \mathcal{P}) \in \mathcal{Q}(E,D,\mathcal{P}) \) and \( H \subset G \) be a closed normal subgroup. Choose a discrete connection \( \mathcal{A}^H_p \) on the principal \( H \)-bundle \( \pi^{M,H} : M \to M/H \) so that one of the conditions of Lemma 6.3 holds. Then, \( G/H \) is a symmetry group of \( \mathcal{M} : = \mathcal{M}/(H, \mathcal{A}^H_p) = (\mathcal{H}, L_d, D_d, D, \mathcal{P}) \).

**Proof.** By Lemma 6.2, \( G/H \) acts on the fiber bundle \((\mathcal{H}, M/H, p_{M/H}, F \times H)\). Thus, we have the \( G/H \)-action \( l^{C'(\mathcal{H})}_{\pi^{G,H}(g)}(v_0, r_1) := (l^{\mathcal{H}}_{\pi^{G,H}(g)}(v_0), l^{M/H}_{\pi^{G,H}(g)}(r_1)) \). Unraveling the definitions, we have that, for \( g \in G \),

\[
\Upsilon_{\mathcal{A}^H_p} \circ l^{C'}_{\pi^{G,H}(g)} = l^{C'(\mathcal{H})}_{\pi^{G,H}(g)} \circ \Upsilon_{\mathcal{A}^H_p}. \tag{6.1}
\]

In addition, just as in the proof of Proposition 7.3 in [14], \( \hat{L}_d : \mathcal{H} \times (M/H) \to \mathbb{R} \) is \( l^{C'(\mathcal{H})} \)-invariant.

As \( \hat{D}_d := \Upsilon_{\mathcal{A}^H_p}(D_d) \), for any \( g \in G \), using (6.1) and the \( G \)-invariance of \( D_d \), we have that

\[
\Upsilon_{\mathcal{A}^H_p}(\hat{D}_d) = \Upsilon_{\pi^{G,H}(g)}(\Upsilon_{\mathcal{A}^H_p}(D_d)) = \Upsilon_{\mathcal{A}^H_p}(l^{C'}_{\pi^{G,H}(g)}(D_d)) = \Upsilon_{\mathcal{A}^H_p}(D_d) = \hat{D}_d,
\]

so that \( \hat{D}_d \) is \( G/H \)-invariant.

Also, as \( \hat{D} := d\Upsilon_{\mathcal{A}^H_p}(D) \), for any \( g \in G \), using (6.1) and the \( G \)-invariance of \( D \), we have

\[
\Upsilon_{\mathcal{A}^H_p}(\hat{D}) = d\Upsilon_{\pi^{G,H}(g)}(\Upsilon_{\mathcal{A}^H_p}(D)) = d(l^{C'}_{\pi^{G,H}(g)} \circ \Upsilon_{\mathcal{A}^H_p})(D)
\]

and

\[
\Upsilon_{\mathcal{A}^H_p}(l^{C'}_{\pi^{G,H}(g)}(D)) = d\Upsilon_{\mathcal{A}^H_p}(l^{C'}_{\pi^{G,H}(g)}(D)) = d\Upsilon_{\mathcal{A}^H_p}(D) = \hat{D},
\]

so that \( \hat{D} \) is \( G/H \)-invariant.
The proof of the $G/H$-invariance of $\tilde{\mathcal{P}}$ mimics the one given in Proposition 7.3 of [14], adapted to the current context.

6.2. Comparison with reduction by the full symmetry group. Let $G$ be a symmetry group of $\mathcal{M} = (E, L_d, D_d, D, \mathcal{D}) \in \text{ob}_{\mathcal{D}}$. Then, if we choose a discrete connection $\mathcal{A}_d^G$ on the principal $G$-bundle $\pi : M \to M/G$ we have the reduced system $\mathcal{M}^G := \mathcal{M}/(G, \mathcal{A}_d^G) \in \text{ob}_{\mathcal{D}}$. If $H \subset G$ is a closed normal Lie subgroup, then $H$ is a symmetry group of $\mathcal{M}$ ( Proposition 6.1) and choosing a discrete connection $\mathcal{A}_d^H$ on the principal $H$ bundle $\pi : M \to M/H$ we have the reduced system $\mathcal{M}^H := \mathcal{M}/(H, \mathcal{A}_d^H) \in \text{ob}_{\mathcal{D}}$. Last, if $\mathcal{A}_d^H$ satisfies either one of the conditions that appear in Lemma 6.3, then $G/H$ is a symmetry group of $\mathcal{M}^H$ (Proposition 6.4). Then, choosing a discrete connection $\mathcal{A}_d^{G/H}$ on the principal $G/H$-bundle $\pi : M \to (M/H)/(G/H)$, we have the discrete system $\mathcal{M}^{G/H} := \mathcal{M}/(G/H, \mathcal{A}_d^{G/H}) \in \text{ob}_{\mathcal{D}}$. The goal of this section is to prove that $\mathcal{M}^G$ and $\mathcal{M}^{G/H}$ are isomorphic in $\mathcal{D}$. The following diagram depicts the relation between the different discrete Lagrange–D’Alembert–Poincaré systems and morphisms.

At the “geometric level” the corresponding manifolds and smooth maps are

We can enlarge the previous diagram by adding the various diffeomorphisms associated to a discrete connection and by taking into account the diagram (2.2).
The following result introduces the new functions that appear in diagram (6.2) and explores their basic properties.

**Lemma 6.5.** Under the previous conditions,

1. \( \Phi_A^G : C'(E)^G \to C'(\tilde{H}_E) \) (see Proposition 2.10) is a \( G/H \)-equivariant diffeomorphism. Then, it induces a smooth diffeomorphism \( \overline{\Phi}_A^G : \frac{C'(E)^G}{\mathcal{G}^G} \to \frac{C'(\tilde{H}_E)}{\mathcal{G}^G} \).

2. \( \pi C'(E)^G : C'(E) \to \frac{C'(E)^G}{\mathcal{G}^G} \) is a smooth \( H \)-invariant map. Then, it induces a smooth map \( F_1 : \frac{C'(E)^G}{\mathcal{G}^G} \to \frac{C'(E)}{\mathcal{G}} \).

3. \( F_1 : \frac{C'(E)^G}{\mathcal{G}^G} \to \frac{C'(E)}{\mathcal{G}} \) is a smooth \( G/H \)-invariant map. Then, it induces a smooth map \( F_2 : \frac{C'(E)^G}{\mathcal{G}^G} \to \frac{C'(E)}{\mathcal{G}} \). Also, \( F_2 \) is a diffeomorphism.

4. Diagram (6.2) is commutative.

**Proof.** The proof is the same as that of Lemma 7.5 in [14].

**Theorem 6.6.** Consider the description given at the beginning of this section. Let \( F : C'(G/H_{H_E}) \to C'(G_E) \) be defined by \( F := \Phi_A^G \circ F_2 \circ (\overline{\Phi}_A^G)^{-1} \circ (\Phi_A^{G/H})^{-1} \) (see diagram (6.2)). Then, \( F \) is an isomorphism in \( \mathfrak{D} \mathfrak{P}_d \).

**Proof.** From Proposition 2.10 both \( \Phi_A^G \) and \( \Phi_A^{G/H} \) are diffeomorphisms and, by Lemma 6.5, both \( \overline{\Phi}_A^G \) and \( F_2 \) are diffeomorphisms. Thus, \( F \) is a diffeomorphism.

As \( \Upsilon_A^H \) and \( \Upsilon_A^{G/H} \) are morphisms in \( \mathfrak{D} \mathfrak{P}_d \), the same is true for \( \Upsilon_A^{G/H} \circ \Upsilon_A^H \).

Then, by Lemma 4.5 applied to \( \Upsilon_A^G, \Upsilon_A^{G/H} \circ \Upsilon_A^H \) and \( F \), we conclude that \( F \) is an isomorphism in \( \mathfrak{D} \mathfrak{P}_d \).

**Theorem 6.7.** Consider the description given at the beginning of this section.

1. Let \( (\epsilon, m.) = ((\epsilon_0, m_1), \ldots, (\epsilon_{N-1}, m_N)) \) be a discrete path in \( C'(E) \). For \( k = 0, \ldots, N-1 \) we define the discrete paths \( (v_k^H, r_{k+1}^H) := \Upsilon_A^H(\epsilon_k, m_{k+1}), (v_k^G, r_{k+1}^G) := \Upsilon_A^G(\epsilon_k, m_{k+1}) \) and \( (v_k^{G/H}, r_{k+1}^{G/H}) := \Upsilon_A^{G/H}(v_k^H, r_{k+1}^H) \) in \( C'(H_E), C'(G_E) \) and \( C'(G/H_{H_E}) \) respectively. Then, the following conditions are equivalent.
   (a) \( (\epsilon, m.) \) is a trajectory of \( \mathcal{M} \).
   (b) \( (v^G, r^G) \) is a trajectory of \( \mathcal{M}^G \).
   (c) \( (v^H, r^H) \) is a trajectory of \( \mathcal{M}^H \).
   (d) \( (v^{G/H}, r^{G/H}) \) is a trajectory of \( \mathcal{M}^{G/H} \).

2. Let \( F : C'(G/H_{H_E}) \to C'(G_E) \) be the diffeomorphism defined in Theorem 6.6.
   Then, \( F(v_k^{G/H}, r_{k+1}^{G/H}) = (v_k^G, r_{k+1}^G) \) for all \( k \).

3. The systems \( \mathcal{M}^G \) and \( \mathcal{M}^{G/H} \) are isomorphic in \( \mathfrak{D} \mathfrak{P}_d \).

**Proof.** Point 1 is verified by Theorem 5.15, while point 3 follows from Theorem 6.6.

The following computation proves point 2.

\[
(v_k^G, r_{k+1}^G) = \Upsilon_A^G(\epsilon_k, m_{k+1}) = (F \circ \Upsilon_A^{G/H} \circ \Upsilon_A^H)(\epsilon_k, m_{k+1})
\]

\[
= (F \circ \Upsilon_A^{G/H})(v_k^H, r_{k+1}^H) = F(v_k^{G/H}, r_{k+1}^{G/H}).
\]

\( \square \)
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