A QUENCHED VARIATIONAL PRINCIPLE FOR DISCRETE RANDOM MAPS.

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ABSTRACT. We study the variational principle for discrete height functions (or equivalently domino tilings) where the underlying measure is perturbed by a random field. We show that the variational principle holds almost surely under the standard assumption that the random field is stationary and ergodic. The entropy functional in the variational principle homogenizes and is the same as for the uniform measure. Main ingredient in the argument is to show the existence, equivalence and characterization of the quenched and annealed surface tension. This is accomplished by a combination of the Kirszbraun theorem and two sub-additive ergodic theorems.

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1. Introduction

The broader scope of this article is the study of limit shapes as a limiting behavior of discrete systems. Limit shapes are a well-known and studied phenomenon in statistical physics and combinatorics (e.g. [Geo88]). Among others, models that exhibit limit shapes are domino tilings and dimer models (e.g. [Kas63, CEP96, CKP01]), polymer models, lozenge tilings (e.g. [Des98, LRS01, Wil04]), Ginzburg-Landau models (e.g. [DG100, FOC04]), Gibbs models (e.g. [She05]), the Ising model (e.g. [DKS92, Cer06]), asymmetric exclusion processes (e.g. [FS06]), sandpile models (e.g. [LP08]), the six vertex model (e.g. [BCG16, CS16, NR16]) and the Young tableaux (e.g. [LS77, VK77, PR07]).

Limit shapes appear whenever fixed boundary conditions force a certain response of the system. The main tool to explain those shapes is a variational principle. The variational principle asymptotically characterizes the number of microscopic states, i.e. the microscopic entropy $\text{Ent}_n$, via a variational problem. For large system sizes $n$, the entropy of the system is given by maximizing a macroscopic entropy $\text{Ent}(f)$ over all

![Figure 1. An Aztec diamond for domino tilings. The combinatorics of the model is similar to Lipschitz functions from $\mathbb{Z}^2$ to $\mathbb{Z}$. (see CKP01)](image-url)
admissible limiting profiles $f \in \mathcal{A}$. The boundary conditions are incorporated in the admissibility condition. In formulas, the variational principle can be expressed as (see for example Theorem 2.5 below)

$$\operatorname{Ent}_n \approx \inf_{f \in \mathcal{A}} \operatorname{Ent}(f),$$

where the macroscopic entropy

$$E(f) = \int \operatorname{ent}(\nabla f(x)) dx$$

can be calculated via a local quantity $\operatorname{ent}(\nabla f(x))$. This local quantity is called local surface tension.

Often, a consequence of a variational principle is large deviations principle which states that the uniform measure on the microscopic configurations concentrates around configurations that are close to the minimizer of the variational problem (see Theorem 2.7 below). This is related to the appearance of limit shapes on large scales.

In analogy to classical probability theory, one can see the variational principle as an elaborated version of the law of large numbers. On large scales the behavior of the system is determined by a deterministic quantity, namely the minimizer $f$ of the macroscopic entropy. Hence, deriving a variational principle is often the first step in analyzing discrete models, before one attempts to study other questions like the fluctuations of the model.

The numerous examples in the literature and many simulations (see for example [MT16]) show that limit shapes are a universal phenomenon. This article is part of a program that was started by the authors in [MT16]. The aim is to develop robust and universal methods to deduce variational principles. While in [MT16] the author chose to study how to generalize variational principles in target spaces where the usual cluster swapping methods do not work, in this article we explore a new direction and prove the robustness of the variational principle in a random environment.

In [MT16], two properties were identified that should be sufficient to derive a variational principle: The first one is a stability property. Perturbing the boundary condition on a microscopic scale does not change the macroscopic properties of the model. The second one is a concentration property. Else, one cannot hope that the model satisfies a
variational principle which is a type of law of large numbers.

While the first property can be deduced from previous works in [MT16], [She05] or [CKP01], finding robust methods to verify the second property is a lot harder. In this work we show that it is possible to use the subadditivity of the microscopic surface tension. This approach is inspired from the study of gradient-Gibbs models and classical statistical mechanics (see for example [FS97] and [Rue99]). Using sub-additivity makes the method to deduce variational principles very robust and one should be able to apply this approach in many different settings.

In this article, we work within the setting of the classical variational principle of height functions, which is combinatorially similar to the domino model considered in [CKP01] (see Figure 1). It is one of the fundamental results for studying domino tilings and the other integrable discrete models. A detailed analysis of the limit shapes for domino tilings was given in [KOS06]. It is a very active research area. Several new approaches were developed recently to make methods more robust (see for example [CJY15, BK16, CS16]).

Inspired from homogenization of random walks in random environment (see for example [Bis11]), we consider a perturbation of the uniform measure by a random field. On the random field we make standard assumptions that are well known from the study of random walks in random environments: We assume that the random field is stationary and ergodic. To make things easier, we also assume that the random field is bounded. One certainly can weaken the last assumption.

In our main results (see Theorem 2.13 and Theorem 2.15) we deduce a quenched variational principle: for almost every realization ω of the random field the variational principle holds. Moreover, we show that the quenched local surface tension exists almost surely, and is given by the local surface tension that is associated to the uniform measure. As a consequence, the model homogenizes on large scales and has the same limit shapes as the classical model.

This verifies another instance where the variational principle behaves like the law of large numbers: The mean behavior of the random walk in random environment is also the same as the random walk in the uniform environment. However, the random environment changes the fluctuations of the random walk. It would be very interesting to see if
a similar effect happens for height functions exposed to a random field.

The proof of the main result is based on the following observations and ingredients: The first main ingredient is the existence of the local surface tension \( \text{ent}(s, \omega) \) with fixed boundary data. In [MT16], one used a combination of the Kirszbraun theorem (or Lipschitz extendability property) and a concentration inequality. In this work, we show the existence of the quenched local surface tension \( \text{ent}(s, \omega) \) for a.e. \( \omega \) by only using sub-additivity in the form of a sub-additive ergodic theorem.

In [MT16], the second main ingredient is the equivalence of \( \text{ent}(s) \) and the local surface tension \( \text{ent}^{\text{free}}(s) \) with free boundary data i.e.

\[
\text{ent}(s) = \text{ent}^{\text{free}}(s). \tag{1}
\]

The identity (1) was deduced in [MT16] via a combination of the Kirszbraun theorem and the concentration inequality. Because it is very difficult to deduce the concentration inequality in the setting of this article we proceed a little bit different. We change the definition of the free local surface tension \( \text{ent}^{\text{free}}(s) \) to obtain an approximate version of (1) (see (2) and Lemma 3.8 below). A more detailed argument will show that the approximate version still is sufficient to deduce the variational principle. Let us fix \( \delta > 0 \). The local surface tension \( \text{ent}^{\text{free}(\delta)}(s, \omega) \) associated to \( \delta \) is defined as the local surface tension with boundary data that is within \( \delta \)-distance to the canonical linear boundary data. A more detailed analysis than in [MT16] shows that instead of (1), approximate equivalence is sufficient to deduce the variational principle i.e.

\[
|\text{ent}(s, \omega) - \text{ent}^{\text{free}(\delta)}(s, \omega)| \lesssim \delta. \tag{2}
\]

We show that the estimate (2) can be obtained as a consequence of the Kirszbraun theorem and does not require concentration.

The third ingredient in our argument is the characterization of the quenched local surface tension \( \text{ent}(s, \omega) \). Using sub-additivity, we first show that the quenched local surface tension \( \text{ent}(s, \omega) \) is the same as the annealed local surface tension \( \mathbb{E}[\text{ent}(s, \omega)] \) i.e.

\[
\text{ent}(s, \omega) = \mathbb{E}[\text{ent}(s, \omega)].
\]

Then, using a sub-additive ergodic theorem for height functions (see Theorem 4.4) we show that the annealed local surface tension coincides with the surface tension \( \text{ent}(s) \) of the classical model associated with
the uniform measure.

This article is organized in the following way. In Section 2 we state the main results. In Section 3 we study the local surface tension. In Section 4 we give proofs and in the Appendix A we proof the subadditive ergodic theorem.

2. Setting and main results

We start with describing the classical variational principle outlined in [CKP01].

We consider the following model. For \( n \in \mathbb{N} \), we consider a finite subset \( R_n \subset \mathbb{Z}^m \) of the \( m \)-dimensional lattice \( \mathbb{Z}^m \). We assume that for \( n \to \infty \) the scaled sublattice \( \frac{1}{n} R_n \) converges in the Gromov-Hausdorff sense to a compact and simply connected region \( R \subset \mathbb{R}^m \) with Lipschitz boundary \( \partial R \). The basic objective is to study graph homomorphisms \( h : R_n \to \mathbb{Z} \), also called height functions.

**Definition 2.1. (Graph homomorphism, height function)** Let \( \Lambda \subset \mathbb{Z}^m \) be a finite set. We denote with \( d_G \) the natural graph distance on a graph \( G \). A function \( h : \Lambda \to G \) is called graph-homomorphism, if

\[
|h(k) - h(l)| = 1
\]

for all \( k, l \in \Lambda \) with \( d_G(k, l) = 1 \). A graph homomorphism \( h : \Lambda \to G \) is called height function if \( G = \mathbb{Z} \). Let \( \partial \Lambda \) denote the inner boundary of \( \Lambda \subset \mathbb{Z}^m \) i.e.

\[
\partial \Lambda = \{ x \in \Lambda \mid \exists y \notin \Lambda : \text{dist}_{\mathbb{Z}^m}(x, y) = 1 \}.
\]

We call a homomorphism \( h : \partial \Lambda \to G \) boundary graph homomorphism or boundary height function if \( G = \mathbb{Z} \).
We want to study the question of how many height functions exist that extend a fixed prescribed boundary height function \( h_{\partial R_n} : \partial \Lambda_n \to \mathbb{Z} \). Hence, let us consider the set \( M(R_n, h_{\partial R_n}) \) that is defined as

\[
M(R_n, h_{\partial R_n}) = \{ h : R_n \to \mathbb{Z} \mid h \text{ is a height function} \\
\text{and } h(\sigma) = h_{\partial R_n}(\sigma) \quad \forall \sigma \in \partial R_n \}.
\] (3)

The goal of the variational principle is to derive an asymptotic formula as \( n \to \infty \) of the microscopic entropy

\[
\text{Ent} (R_n, h_{\partial R_n}) := -\frac{1}{|R_n|^2} \log M(R_n, h_{\partial R_n}).
\] (4)

For this purpose, let us identify the possible scaling limits of sequences of height functions \( h_{R_n} : R_n \to \mathbb{Z} \) and boundary height functions \( h_{\partial R_n} : \partial R_n \to \mathbb{Z} \). We call those objects asymptotic height profile and asymptotic boundary height profile.

**Definition 2.2 (Asymptotic height profile).** A function \( h_R : R \to \mathbb{R} \) is called asymptotic height profile if the map \( h_R \) is 1-Lipschitz with respect to the \( l_1 \)-norm, i.e. for all \( x, y \in R \)

\[
|h_R^1(x) - h_R^1(y)| \leq |x - y|_{l_1}.
\] (5)

**Definition 2.3 (Asymptotic boundary height profile).** We call a function \( h_{\partial R} : \partial R \to \mathbb{R} \) asymptotic boundary height profile if it satisfies the condition \( \text{(5)} \)

We want to note that every asymptotic boundary height profile can be extended to an asymptotic height profile by using the Kirszbraun theorem. This observation is important because otherwise the statement of the variational principle could be empty (see Theorem 2.5 and Theorem 2.13 below). For continuous metrics, Kirszbraun theorems state that under the right conditions a \( k \)-Lipschitz function defined on a subset of a metric space can be extended to the whole space (cf. [Kir34, Val43, Sch69]). A discrete Kirszbraun theorem was developed in the setting of tilings in [Tas14] and [PST16]. The Kirsbraun theorem was extended to trees in [MT16].

The next step toward the variational principle is to define in which sense a sequence of (boundary) graph homomorphisms \( h_{\partial R_n} : \partial R_n \to \mathbb{Z} \) converges to an asymptotic height profile \( h_R \).
Definition 2.4. Let $h_{\partial R_n} : \partial R_n \to \mathbb{R}$ be a sequence of boundary height functions and let $h_{\partial R}$ be an asymptotic boundary height profile (cf. Definition 2.2). For $z \in \partial R_n$ we define the set

$$S(z) := \partial R \cap \left\{ x \in \mathbb{R}^m : \left| x - \frac{z}{n} \right|_{\infty} \leq \frac{1}{2n} \right\}.$$ 

We say that the sequence $h_{\partial R_n}$ converges to $h_{\partial R}$ (i.e. $\lim_{n \to \infty} h_{\partial R_n} = h_{\partial R}$) if

$$\lim_{n \to \infty} \sup_{z \in \partial R_n : S(z) \neq \emptyset} \sup_{x \in S(z)} \frac{1}{n} \left| h_{\partial R_n}(z) - h_{\partial R}(x) \right| = 0.$$ 

Now, we are prepared to formulate the classical variational principle for height functions [CKP01]. As we outlined in the introduction, a variational principle contains two statements. The first statement, namely Theorem 2.5, gives a variational characterization of the entropy (cf. (4))

$$\text{Ent} (R, h_{\partial R}) = -\frac{1}{n^2} \ln |M(R_n, h_{\partial R_n})|.$$ 

It asymptotically characterizes the number of height functions $h_n \in M(R_n, h_{\partial R_n})$ with boundary data $h_{\partial R_n}$ (see also [MT16]).

Theorem 2.5 (Variational principle, Theorem 1.1 and Theorem 4.3 in [CKP01]). We assume that $R \subset \mathbb{R}^m$ is a compact, simply connected region with Lipschitz boundary $\partial R$. We consider a lattice discretization $R_n \subset \mathbb{Z}^m$ of $R$ such that the rescaled sublattice $\frac{1}{n} R_n$ converges to $R$ in the Gromov-Hausdorff sense. We assume that the boundary height functions $h_{\partial R_n}$ converge to an asymptotic boundary height profile $h_{\partial R}$ in the sense of Definition 2.4.

Let $AHP(h_{\partial R})$ denote the set of asymptotic height profiles that extend $h_{\partial R}$ from $\partial R$ to $R$. Given an element $h_R \in AHP(h_{\partial R})$, we define the macroscopic entropy via

$$\text{Ent} (R, h_{\partial R}) = -\frac{1}{n^2} \ln |M(R_n, h_{\partial R_n})|.$$ 

where the local surface tension $\text{ent}(s)$ is given by Lemma 3.6 from below. Then it holds that

$$\lim_{n \to \infty} \text{Ent} (\Lambda_n, h_{\partial R_n}) = \min_{h_R \in AHP(h_{\partial R})} \text{Ent} (R, h_R).$$

The local surface tension $\text{ent}(s)$ will be defined in Section 3 as a limit of carefully chosen entropies. It is striking that Cohn, Kenyon and Propp were able to deduce an explicit formula of $\text{ent}(s)$ in [CKP01] in the case of the two-dimensional lattice $\mathbb{Z}^2$ (see Remark 3.4 below). Using that formula they also show that on $\mathbb{Z}^2$, the local surface tension $\text{ent}(s)$ is
strictly convex. The strict convexity of $\text{ent}(s)$ for arbitrary lattices was deduced by Scott Sheffield in \cite{She05}.

It follows from the strict convexity of $\text{ent}(s)$ that the continuous entropy $\text{Ent}(R, h_R)$ is also strictly convex. Hence, there is a unique asymptotic height profile $h_{\min} \in AHP(h_{\partial R})$ that minimizes (7) i.e.

$$\lim_{n \to \infty} \text{Ent} (\Lambda_n, h_{\partial R_n}) = \min_{h_R \in AHP(h_{\partial R})} \text{Ent} (R, h_R) = \text{Ent} (R, h_{\min}).$$

Let us now turn to the second part of the variational principle, namely the profile theorem (see Theorem 2.7 from below). The profile theorem contains information about the profile of a graph homomorphisms $h_n$ that is chosen uniformly random from $M(R_n, h_{\partial R_n})$.

In a non-rigorous way, the statement of Theorem 2.7 is the following. Let us consider an asymptotic boundary height profile $h_R \in AHP(h_{\partial R})$. Then the continuous entropy $\text{Ent}(h_R)$ is given by the number of graph homomorphisms $h_n \in M(R_n, h_{\partial R_n})$ that are close to $h_R$. Applying this statement to the minimizer $h_{\min}$ of the continuous entropy $\text{Ent}(h)$ has the following consequence. The uniform measure on the set of graph homomorphisms $M(R_n, h_{\partial R_n})$ concentrates on graph homomorphisms $h_n$ that have a profile that is close to $h_{\min}$. As a consequence, a uniform sample of $M(R_n, h_{\partial R_n})$ will have a profile that is close to the minimizing profile $h_{\min}$ for large $n$.

Let us now make this discussion precise. For that purpose, we have to specify when the profile of a graph homomorphism $h_n$ is close to an asymptotic height profile $h$.

**Definition 2.6.** For fixed $\varepsilon > 0$, let us consider the grid $R_{\text{grid}, \varepsilon}$ with $\varepsilon$-spacing contained in $R$. More precisely, $R_{\text{grid}, \varepsilon}$ is given by (see Figure 2)

$$R_{\text{grid}, \varepsilon} := \{ x = (z_1, \ldots, x_m) \in R \mid \exists 1 \leq k \leq m : |x_k| \in \varepsilon \mathbb{N} \}.$$

For a given asymptotic height profile $h$, we define the ball $HP_n(h, \delta, \varepsilon)$ of size $\delta > 0$ on the scale $\varepsilon > 0$ by the formula

$$HP_n(h, \delta, \varepsilon) = \left\{ h_n \in M(R_n, h_{\partial R_n}) \mid \sup_{x \in R_n, z \in R_{\text{grid}, \varepsilon}} \left| \frac{1}{n} h_n(x) - h \left( \frac{x}{n} \right) \right| \leq \delta \right\},$$

where the set $M(R_n, h_{\partial R_n})$ of graph homomorphisms is given by \cite{3}.
Now, let us formulate the profile theorem.

**Theorem 2.7.** (*Profile theorem*) Let \( h_R \) be an extension of the asymptotic boundary height profile \( h_{\partial R} \). Then

\[
\text{Ent}(R, h_R) = -\frac{1}{|R_n|} \ln |HP_n(h_R, \delta, \varepsilon)| + \theta(\varepsilon) + \theta(\delta) + \theta\left(\frac{1}{\varepsilon n}\right),
\]

where \( \theta \) denotes a generic smooth function with \( \lim_{x \to 0} \theta(x) = 0 \).

A consequence of Theorem 2.7 is that the uniform measure on \( M(R_n, h_{\partial R_n}) \) exponentially concentrates around the minimizing profile \( h_{\text{min}} \). For a proof of Theorem 2.5 and Theorem 2.7 we refer to [CKP01], [She05] or [MT16]. The main difficulty in the argument is to show that the local surface tension \( \text{ent}(s) \) exists and that the local surface tension with free boundary condition is equivalent to the surface tension with fixed boundary condition. We will discuss the local surface tensions in more detail in Section 3.

Let us now turn to homogenization. The main change in the model is that instead of the uniform measure on \( M(R_m, h_{\partial R_m}) \) we consider a noisy perturbation \( \mu_\omega \) of the uniform measure. In homogenization one considers two different cases. In the quenched case one considers the measure \( \mu_\omega \) for a.e. \( \omega \). In the annealed case one takes the expectation wrt. the events \( \omega \). Our goal is to show deduce the quenched variational principle which means that the variational principle holds for the measure \( \mu_\omega \) and a.e. \( \omega \).

Before we turn to the definition of the random measure \( \mu_\omega \), let us describe the structure of \( \omega \).
Assumption 2.8 (Random field $\omega$). Let us consider a real valued random field $\omega = (\omega_e)_{e \in E(Z)} \in \mathbb{R}^{E(Z)}$ on the set of edges $E(Z)$ of $Z$. We assume that $\omega$ satisfies the following assumptions:

- We assume that the random field $\omega$ is uniformly bounded in the sense that there exists a constant $C_\omega < \infty$ such that almost surely
  \[ \sup_{e \in E(Z)} |\omega_e| \leq C_\omega. \]

Moreover, we assume bound $C_\omega$ is $L^1$, i.e. that $\mathbb{E}[|C_\omega|] < \infty$.

- We assume that the random field $\omega$ is shift invariant. More precisely, this means that for any finite number of edges $e_1, \ldots, e_k \in E(Z)$, any element $z \in Z$ and any bounded and measurable function $\xi : \mathbb{R}^k \to \mathbb{R}$
  \[ \mathbb{E}[\xi(\omega_{e_1}, \ldots, \omega_{e_k})] = \mathbb{E}[\xi(\omega_{\tau_z(e_1)}, \ldots, \omega_{\tau_z(e_k)})]. \]

- We assume that the random field $\omega$ is ergodic (see for example [Dur10]).

- We assume w.l.o.g. that
  \[ \mathbb{E}[\omega_{e_0,1}] = 0. \] \hfill (8)

Remark 2.9. We made the assumption that the random field $\omega$ is bounded out of convenience. It simplifies our calculations. The main result should also be true if one assumes that certain moments are bounded. The shift invariance and ergodicity are standard assumptions when studying homogenization of a random walk in random environment (see for example [Bis11]). The assumption (8) is just a normalization.

Let us now turn to the definition of the measure $\mu_\omega$.

Definition 2.10 (Quenched Gibbs measure $\mu_\omega$). Let $\omega$ be a random field satisfying the Assumption 2.8. We denote with $\mu_\omega$ the following probability measure on the state space $M(R_n, h_{\partial R_n})$. The probability to see a element $h \in M(R_n, h_{\partial R_n})$ is given by

\[
\mu_\omega(h) = \frac{1}{Z_{R_n,h_{\partial R_n},\omega}} \exp \left( \frac{1}{2} \sum_{x, y \in R \atop |x - y| = 1} \omega_{e_h(x), h(y)} \right),
\]
where $Z_{R_n, h_{\partial R_n}, \omega}$ denotes the normalization constant

$$Z_{R_n, h_{\partial R_n}, \omega} = \sum_{h \in M(R_n, h_{\partial R_n})} \exp \left( \frac{1}{2} \sum_{|x-y|=1} \omega_{\epsilon h(x), h(y)} \right).$$

We call the measure $\mu_\omega$ the quenched Gibbs measure.

Remark 2.11. If we choose the constant field $\omega = 0 = (0)_{E(\mathbb{Z})}$, then the associated quenched Gibbs measure $\mu_0$ is the uniform measure on $M(R_n, h_{\partial R_n})$. In this case we recover the original variational principle of [CKP01].

Let us now introduce the quenched and annealed microscopic entropies.

Definition 2.12 (The quenched and annealed microscopic entropy). The quenched microscopic entropy $\text{Ent}(R_n, h_{\partial R_n}, \omega)$ is given by

$$\text{Ent}(R_n, h_{\partial R_n}, \omega) := \frac{1}{|R_n|} \ln Z_{R_n, h_{\partial R_n}, \omega}. \tag{9}$$

The annealed microscopic entropy $\text{Ent}(R_n, h_{\partial R_n})$ is given by

$$\text{Ent}(R_n, h_{\partial R_n}) = \mathbb{E} \left[ \text{Ent}(R_n, h_{\partial R_n}, \omega) \right]. \tag{10}$$

The main result of this article is the following quenched variational principle.

Theorem 2.13 (Quenched variational principle). We assume the same hypothesis as in Theorem 2.5. Then for almost every $\omega$ it holds that

$$\lim_{n \to \infty} \text{Ent}(\Lambda_n, h_{\partial R_n}, \omega) = \min_{h_R \in \mathcal{AHP}(h_{\partial R})} \text{Ent}(R, h_R) = \text{Ent}(R, h_{\min}),$$

where $\text{Ent}(R, h_{\min})$ is the continuous entropy associated to the uniform measure given by (6).

Remark 2.14. Of course, the quenched variational principle of Theorem 2.13 immediately yields the annealed variational principle

$$\lim_{n \to \infty} \text{Ent}(\Lambda_n, h_{\partial R_n}) = \lim_{n \to \infty} \mathbb{E} \left[ \text{Ent}(\Lambda_n, h_{\partial R_n}, \omega) \right]$$

$$= \min_{h_R \in \mathcal{AHP}(h_{\partial R})} \text{Ent}(R, h_R) = \text{Ent}(R, h_{\min}).$$

The second main result is the following quenched version of the profile theorem.
Theorem 2.15. (Profile theorem) Let $h_R$ be an extension of the asymptotic boundary height profile $h_{\partial R}$. Then

$$\text{Ent}(R, h_R, \omega) = -\frac{1}{|R_n|} \ln \sum_{h \in HP_n(h_R, \delta, \varepsilon)} \exp \left( \frac{1}{2} \sum_{x,y \in R} \frac{\omega_{h(x), h(y)}}{|x-y|=1} \right) + \theta(\varepsilon) + \theta(\delta) + \theta \left( \frac{1}{\varepsilon n} \right),$$

where $\theta$ denotes a generic smooth function with $\lim_{x \downarrow 0} \theta(x) = 0$.

As in the uniform case, Theorem 2.15 yields exponential concentration of $\mu_\omega$ around the minimizing profile $h_{\text{min}}$.

For the proof of Theorem 2.13 and Theorem 2.15 we use the argument outlined in [MT16]. The main ingredient is to show that the quenched local surface tension exists (see Theorem 3.2) and that the microscopic quenched local surface tension with fixed boundary data is equivalent to the microscopic quenched local surface tension with free boundary data (see Lemma 3.5). We additionally show that the quenched local surface tension is the same as the surface tension for the uniform measure (see Theorem 3.6). After providing those ingredients, the proof of Theorem 2.13 and Theorem 2.15 is similar to the one in [MT16]. We omit the details.

3. The quenched and annealed local surface tension

In this section we show the existence and equivalence of various different local surface tensions. This is the main ingredient for the proof of the main results of Section 2. As outlined in the introduction our argument is based on a combination of the Kirszbraun theorem and subadditivity. The results of this section are not just a simple application of an sub-additive ergodic theorem. To show the existence, we need to derive a new, non-standard sub-additive ergodic theorem for height functions (see Theorem 4.4 below). To show the equivalence we take advantage of the Kirszbraun theorem in a different way than in [MT16]. We prove all statements in Section 4.

Let’s start with considering the quenched local surface tension $\text{ent}(s, \omega)$. The quenched local surface tension $\text{ent}(s, \omega)$ will be defined as the limit of the microscopic quenched local surface tension $\text{ent}_n(s, \omega)$. 
Definition 3.1 (Microscopic quenched local surface tension \( \text{ent}_n(s, \omega) \)). Let \( s \in \mathbb{R}^d \) be a vector that satisfies \(|s|_\infty \leq 1 \). Recall that \( S_n = \{0, \ldots, n - 1\}^m \) denotes the “square” (really, the hypercube) of size \( n \). Let \( h_{\partial S_n}^s \) denote a canonical choice of a boundary height function that minimizes the expression
\[
\sum_{x \in \partial S_n} |h_{\partial S_n}^s(x) - \lfloor x \cdot s \rfloor|.
\]
The microscopic quenched local surface tension \( \text{ent}_n(s, \omega) \) is given by
\[
\text{ent}_n(s, \omega) := \text{Ent}(S_n, h_{\partial S_n}^s, \omega),
\]
where the right hand side is the quenched microscopic entropy given by \([10]\). Because the boundary values are fixed by \( h_{\partial S_n}^s \) we also call \( \text{ent}_n(s, \omega) \) microscopic quenched local surface tension with fixed boundary data.

Theorem 3.2 (Existence of the quenched local surface tension). Let \( s \in \mathbb{R}^d \) be a vector that satisfies \(|s|_\infty \leq 1 \). Then for almost every realization \( \omega \) of the random field there exists the limit
\[
\text{ent}(s, \omega) := \lim_{n \to \infty} \text{ent}_n(s, \omega).
\]
We call \( \text{ent}(s, \omega) \) quenched local surface tension.

The proof of Theorem 3.2 is given in Section 4

Remark 3.3. Similarly to Remark 2.11 we obtain back the local surface tension \( \text{ent}(s) \) for the uniform measure if we consider a constant random field \( \omega = 0 = (0)_{E(\mathbb{Z})} \). More precisely, it holds
\[
\text{ent}(s) = \text{ent}(s, 0).
\]
For convenience, we call \( \text{ent}(s) \) the uniform local surface tension.

Remark 3.4. The uniform local surface tension \( \text{ent}(s, t) \) is characterized in \([\text{CKP01}]\) for the two-dimensional lattice in the following way. It holds that
\[
\text{ent}(s, t) = \frac{1}{\pi} \left( L(\pi p_a) + L(\pi p_b) + L(\pi p_c) + L(\pi p_d) \right),
\]
where \( L(\cdot) \) is the Lobachevsky function defined by
\[
L(z) = -\int_0^z \log |2 \sin(t)| \, dt.
\]
The quantities \( p_a, p_b, p_c, p_d \) are determined by the equations

\[
2(p_a - p_b) = s, \\
2(p_c - p_d) = t, \\
p_a + p_b + p_c + p_d = 1, \\
\sin(\pi p_a) \sin(\pi p_b) = \sin(\pi p_c) \sin(\pi p_d).
\]

Let us now consider the annealed local surface tension. Again, for a vector \( s \in \mathbb{R}^d \) that satisfies \( |s|_\infty \leq 1 \), we define the annealed local surface tension \( \text{ent}(s) \) by

\[
\text{ent}(s) = \mathbb{E} [\text{ent}(s, \omega)]
\]

**Theorem 3.5** (Equivalence of quenched and annealed local surface tension). Let \( s \in \mathbb{R}^m \) with \( |s|_\infty \leq 1 \). Then for almost every realization \( \omega \) it holds that

\[
\text{ent}(s, \omega) = \lim_{n \to \infty} \mathbb{E} [\text{ent}_n(s, \omega)] = \mathbb{E} [\text{ent}(s, \omega)].
\]

The proof of Theorem 3.5 is stated in Section 4.

The next statement shows that the quenched and annealed local surface tension coincides with the local surface tension \( \text{ent}(s) \) of the uniform measure.

**Theorem 3.6** (Characterization of the annealed local surface tension). It holds that

\[
\mathbb{E} [\text{ent}(s, \omega)] = \text{ent}(s, 0) = \text{ent}(s),
\]

where \( \text{ent}(s) \) denotes the local surface tension associated to the uniform measure.

The proof of Theorem 3.6 is given in Section 4.

Let us now turn to the second ingredient needed in the proof of the variational principle (see Theorem 2.13 and Theorem 2.15 and [MT16]). It is the equivalence of the local surface tension for fixed and free boundary data. Let us introduce the quenched local surface tension with free boundary data.

**Definition 3.7.** Let \( s \in \mathbb{R}^m \) be a vector that satisfies \( |s|_\infty \leq 1 \). For \( \delta > 0 \), we write \( M_n^{\text{free}(\delta)}(s) \) for the set of height functions

\[
M_n^{\text{free}(\delta)}(s) = \{ h_n : S_n \to \mathbb{Z} : h_n \text{ is a height function and} \ |h_n(x) - \lfloor s \cdot x \rfloor| \leq \delta n \text{ for } x \in \partial S_n \}.
\]
Then, we define the quenched microscopic entropy with free boundary data

\[ \text{ent}_{\text{free}}(s, \omega) = -\frac{1}{|S_n|} \ln \sum_{h \in M_n^{\text{free}}(s)} \exp \left( \frac{1}{2} \sum_{x,y \in S_n, |x-y|=1} \omega_{h(x), h(y)} \right) \].

Among the results that we will need is that the quenched microscopic entropy with free boundary data is approximately equivalent to the quenched microscopic entropy with fixed boundary data.

**Lemma 3.8.** Fix a slope \( s \in \mathbb{R}^m \) with \( |s|_{\infty} < 1 \). There exists a constant \( C = C(s) \) such that, for any \( \delta > 0 \), any sufficiently large \( n \in \mathbb{N} \), and almost any realization \( \omega \),

\[ \text{ent}_n(s, \omega) \geq \text{ent}_{\text{free}}^{\delta}(s, \omega) \geq (1 + C\delta)^{-m} \text{ent}_{n+[C\delta n]}(s, \omega) + \theta(\delta). \]

4. Proofs of results of Section 3

4.1. **Proof of Theorems 3.2 and 3.5.** First, we focus on the quenched local surface tension. Theorem 3.2 and Theorem 3.5 state respectively that the quenched local surface tension exists, and that it is almost surely equal to the annealed local surface tension. We shall prove both theorems at once, using an ergodic theorem for subadditive random processes.

To state the ergodic theorem, we will need some notation. Let \( B \) denote the set of all (non-empty) boxes in \( \mathbb{Z}^d_+ \). More precisely, define

\[ B = \left\{ ([a_1, b_1] \times \cdots \times [a_d, b_d]) \cap \mathbb{Z}^d_+ \mid a_1 < b_1, \ldots, a_d < b_d \in \mathbb{Z}^d_+ \right\} \]

Define a random process \( F = (F_B)_{B \in B} \) by

\[ F_B(\omega) = |B| \text{Ent}(B, h_{\partial B}, \omega) = \ln Z_{B, h_{\partial B}, \omega} \]

We will see that \( F \) satisfies the hypotheses of the following ergodic theorem, which is a slight modification of the theorem proven in [AK81]. The modifications are occasioned because our \( F \) is not superadditive, but rather almost superadditive in the sense of condition (11) below. For completeness, we give a proof of this ergodic theorem in Appendix A.

**Theorem 4.1** (Ergodic theorem for almost superadditive random processes). Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space and let \( \tau = (\tau_u)_{u \in \mathbb{Z}^d_+} \) be a semi-group of measure-preserving transformations on \( \Omega \). Let \( \tilde{F} = \)
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Let \((F_B)_{B \in \mathcal{B}}\) be an almost superadditive random process, i.e. a family of \(L^1\) random variables \(F_B\) indexed by \(B \in \mathcal{B}\) satisfying the following three conditions:

- For all \(u \in \mathbb{Z}^d_+\), \(F_{u+B} = F_B \circ \tau_u\), where \(u+B = \{u+x : x \in B\}\) is the translation of \(B\) by \(u\).
- (Almost superadditivity) \(F\) satisfies the almost superadditivity condition: for any disjoint boxes \(B_1, \ldots, B_n \in \mathcal{B}\) whose union \(B = B_1 \cup \cdots \cup B_n\) also lies in \(\mathcal{B}\),
  \[F_B \geq \sum_{i=1}^n F_{B_i} - A \sum_{i=1}^n |\partial B_i|\]  (11)
  where \(\partial B_i = \{x \in B_i | \exists y \notin B_i, x \sim y\}\) is the inner boundary of \(B_i\) and \(A = A(\omega) : \Omega \to [0, \infty)\) is a random variable.
- The quantity \(\tilde{\gamma}(F) = \limsup_{n \to \infty} \frac{1}{|S_n|} \mathbb{E}[F_{S_n}]\) is finite. (Recall that \(S_n = [0,n]^d \cap \mathbb{Z}^d_+ \in \mathcal{B}\) is the hypercube of side length \(n\) in the lattice.)

Then the limit \(\lim_{n \to \infty} \frac{1}{|S_n|} F_{S_n}\) exists almost surely. If moreover the semi-group of transformations \(\tau\) is ergodic, then
  \[\lim_{n \to \infty} \frac{1}{|S_n|} F_{S_n} = \lim_{n \to \infty} \frac{1}{|S_n|} \mathbb{E}[F_{S_n}] = \tilde{\gamma}(F).\]

**Proof of Theorem 3.2 and Theorem 3.5.** We fix a nonzero slope \(s \in \mathbb{R}^d\) with \(|s|_{\infty} \leq 1\).

We will apply Theorem 4.1. Recall that our \(\Omega = \{\omega = (\omega_e)_{e \in E(\mathbb{Z})}\}\) is a random field on the edge set of \(\mathbb{Z}\), not on \(\mathbb{Z}^d\). The semi-group of transformations \(\tau = (\tau_u)_{u \in \mathbb{Z}_+^d}\) will therefore be the translations
  \[(\tau_u \omega)_{(x,x+1)} = \omega_{(x+[u \cdot s],x+[u \cdot s]+1)} \quad \text{for } x \in \mathbb{Z} \text{ and } u \in \mathbb{Z}_+^d.\]

Our random process is \(F = (F_B)_{B \in \mathcal{B}}\) defined by
  \[F_B = -|B| \text{Ent}(B, h_{\partial B}, \omega) = \ln Z_{B, h_{\partial B}, \omega}.\]

First, one can easily derive an \(L^1\) bound on each \(F_B\) from the fact that \(|\omega_e| \leq C_\omega\) for all edges \(e \in E(\mathbb{Z})\), where \(C_\omega\) is in \(L^1\).

Next, we claim that \(F\) is compatible with translations, in the sense that \(F_{u+B} = F_B \circ \tau_u\) for \(u \in \mathbb{Z}_+^d\) and \(B \in \mathcal{B}\). This is clear from the definitions. Indeed, the innermost sum in \(F_{u+B}\), i.e. the sum inside the exponential, is
  \[\sum_{x,y \in u+B} \omega_{h(x),h(y)},\]
where \( h \) is one of the height functions in \( M(u + B, h_{\partial u + B}) \). On the other hand, the corresponding sum in \( F_B \circ \tau_u(e_{x,y}) \) is

\[
\sum_{x,y \in B, |x-y|=1} \tau_u(\omega_{h'(x),h'(y)}),
\]

where here \( h' \) is a height function defined on the untranslated box \( B \). Since there is an obvious correspondence \( h \leftrightarrow h' \) between \( M(u + B, h_{\partial u + B}) \) and \( M(B, h_{\partial B}) \) so that \( h(u + x) = \tau_u(h(x)) \) for all \( x \in B \), it follows that these two sums are equal.

The main condition needed to apply Theorem 4.1 is the almost superadditivity condition. To establish almost superadditivity, consider disjoint boxes \( B_1, \ldots, B_n \in B \) whose union \( B = B_1 \cup \cdots \cup B_n \) also lies in \( B \). We wish to show that

\[
F_B \geq \sum_{i=1}^n F_{B_i} - A \sum_{i=1}^n |\partial B_i|
\]

where \( \partial B_i = \{ x \in B_i | \exists y \not\in B_i, x \sim y \} \) is the inner boundary of \( B_i \) and \( A = A(\omega) : \Omega \to [0, \infty) \) is a random variable.

The key idea is that the evaluation of \( F_B \) involves summing over all the edges in the big rectangle \( B \), which is almost the same as summing over all the edges in all the smaller rectangles \( B_i \).

It will be useful to introduce the Hamiltonian

\[
H(h) = \frac{1}{2} \sum_{x \sim y \in B} \omega_{\epsilon h(x), h(y)}
\]

for \( h \in M(B, h_{\partial B}) \), where the sum runs over all adjacent pairs \( x, y \in B \).

We define \( H_i(h_i) \) analogously for \( h_i \in M(B_i, h_{\partial B_i}) \). The Hamiltonian is almost additive, in the following sense: for any height functions \( h_1 \in M(B_1, h_{\partial B_1}), \ldots, h_n \in M(B_n, h_{\partial B_n}) \), we may define a height function \( h \in M(B, h_{\partial B}) \) by \( h(x) = h_i(x) \) if \( x \in B_i \) (the boundary conditions ensure that \( h \) is still a graph homomorphism even at the
boundaries of the boxes \( B_i \). Then,

\[
\sum_{i=1}^{n} H_i(h_i) = \frac{1}{2} \sum_{i=1}^{n} \sum_{x \sim y \in B_i} \omega_{h_i(x), h_i(y)}
\]

\[
= \frac{1}{2} \sum_{x \sim y \in B} \omega_{h(x), h(y)} - \frac{1}{2} \sum_{x \sim y \in B} \omega_{h(x), h_j(y)}
\]

\[
\leq H(h) + \sum_{i=1}^{n} d|\partial B_i| C_\omega
\]

The factor \( d \) occurs in the last line because the error term is easily bounded in terms of the cardinality of the outer boundary of \( B_i \), but our symbol \( \partial B_i \) denotes the inner boundary.

Returning now to the almost superadditivity of \( F \), we can estimate

\[
\sum_{i=1}^{n} F_{B_i} = \sum_{i=1}^{n} \ln \sum_{h_i \in M(B_i, h_{\partial B_i})} \exp \left( H_i(h_i) \right)
\]

\[
= \ln \left[ \prod_{i=1}^{n} \sum_{h_i \in M(B_i, h_{\partial B_i})} \exp \left( H_i(h_i) \right) \right]
\]

\[
= \ln \left[ \sum_{h_1 \in M(B_1, h_{\partial B_1})} \cdots \prod_{i=1}^{n} \exp \left( H_i(h_i) \right) \right]
\]

\[
\leq \ln \left[ \sum_{h_1, \ldots, h_n} \exp \left( H(h) + dC_\omega \sum_{i=1}^{n} |\partial B_i| \right) \right]
\]

\[
= \ln \sum_{h_1, \ldots, h_n} \exp(H(h)) + dC_\omega \sum_{i=1}^{n} |\partial B_i|
\]

\[
\leq F_B + A \sum_{i=1}^{n} |\partial B_i|
\]

(In the last line, we define \( A = A(\omega) = dC_\omega \). This gives us the almost superadditivity relation that is a hypothesis of Theorem 4.1.)
The last remaining condition of Theorem 4.1 is finiteness of $\tilde{\gamma}(F) = \limsup_{n \to \infty} \frac{1}{|S_n|} \mathbb{E}[F_{S_n}]$. Here we again use the $L^1$ bound $C_\omega$, since

$$F_{S_n}(\omega) = \ln \sum_{h \in M(S_n, h_\partial S_n)} \exp(H(h)) \leq \ln \sum_{h \in M(S_n, h_\partial S_n)} \exp(C_\omega |E(S_n)|) = \ln |M(S_n, h_\partial S_n)| + C_\omega |E(S_n)|$$

Taking expectations and dividing by $|S_n|$, we have

$$\frac{1}{S_n} \mathbb{E}[F_{S_n}] \leq \frac{1}{S_n} \ln |M(S_n, h_\partial S_n)| + \mathbb{E}[C] \frac{|E(S_n)|}{|S_n|}.$$  

The first value on the right converges to the surface tension $\text{ent}(s, 0)$ arising from the uniform measure, which is finite. The second value also converges, since $\frac{|E(S_n)|}{|S_n|} \to d$ as $n \to \infty$.

At this point, we may apply Theorem 4.1 to conclude immediately that the pointwise limit

$$\text{ent}(s, \omega) = \lim_{n \to \infty} \text{ent}_n(s, \omega) = \lim_{n \to \infty} \frac{1}{|S_n|} F_{S_n}$$

exists almost surely. This proves Theorem 3.2.

Moreover, we assumed in Section 2 that $\omega$ is ergodic. Since $s \neq 0$, the semi-group $(\tau_u)_{u \in \mathbb{Z}_d^+}$ includes the translation by 1 on $E(\mathbb{Z})$, so $(\tau_u)_{u \in \mathbb{Z}_d^+}$ is an ergodic semi-group. By the last part of Theorem 4.1

$$\lim_{n \to \infty} \text{ent}_n(s, \omega) = \lim_{n \to \infty} \mathbb{E}[\text{ent}_n(s, \omega)].$$

This proves Theorem 3.3. \qed

4.2. **Proof of Theorem 3.6.** The statement of Theorem 3.6 follows directly from a combination of Lemma 4.2 and Lemma 4.3 from below.

**Lemma 4.2.** For every $n \in \mathbb{N}$ it holds that

$$\mathbb{E}[\text{ent}_n(s, \omega)] \geq \text{ent}_n(s, 0).$$

**Proof.** Fix $n \in \mathbb{N}$. Let us abbreviate $M_n = M(S_n, h_\partial S_n)$, and define $N = |M_n|$. Consider the “log-sum-exp” function $\Phi_N : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\Phi_N(x_1, \ldots, x_N) = \ln \sum_{i=1}^N e^{x_i}$$
This is a convex function, so by Jensen’s inequality,

\[
\mathbb{E}[\text{ent}_n(s, \omega)] = \mathbb{E} \left[ \frac{1}{n^d} \ln \sum_{h \in M_n} \exp \left( \frac{1}{2} \sum_{x,y \in S_n, |x-y|=1} \omega_{e(h(x),h(y))} \right) \right] \\
\geq \frac{1}{n^d} \ln \sum_{h \in M_n} \exp \left( \mathbb{E} \left[ \frac{1}{2} \sum_{x,y \in S_n, |x-y|=1} \omega_{e(h(x),h(y))} \right] \right) \\
= \frac{1}{n^d} \ln \sum_{h \in M_n} \exp(0) = \text{ent}_n(s, 0) 
\]

\[
\square
\]

**Lemma 4.3.** For almost every \( \omega \) it holds

\[
\mathbb{E}[\text{ent}_n(s, \omega)] \leq \text{ent}_n(s, 0).
\]

The main ingredient in the proof of Lemma 4.3 is a ergodic theorem (see Theorem 4.4 below).

**Proof of Lemma 4.3.** For \( n, k \in \mathbb{N} \), consider the random function

\[
\frac{1}{k^m} \sum_{i=1}^{k^m} \text{ent}_n(s, \tau_{u_i} \omega),
\]

where \( (u_i)_{i=1}^{k^m} \) enumerates the \( k^m \)-many lattice points in the set

\[
\{(a_1n, \ldots, a_mn) : 0 \leq a_1, \ldots, a_m \leq k-1, a_i \in \mathbb{Z}\}.
\]

On one hand, as \( k \to \infty \), this random function converges almost surely and in \( L^1 \) to \( \mathbb{E}[\text{ent}_n(s, \omega)] \) by the standard \( m \)-dimensional ergodic theorem.

On the other hand, we can expand the definition of \( \text{ent}_n \) as follows: First, recall that

\[
\text{ent}_n(s, \omega) = - \frac{1}{|S_n|} \ln \sum_{h \in M(S_n, h^*_\partial S_n)} \exp(H(h)),
\]

where \( H(h) \) is the Hamiltonian given by \( H(h) = \frac{1}{2} \sum_{x,y} \omega_{e(h(x),h(y))} \), with \( x \) and \( y \) in the sum running over all pairs \( (x, y) \) of adjacent vertices in the sub-graph \( S_n \subset \mathbb{Z}^m \). Below, we will write

\[
\text{ent}_n(s, \tau_{u_i} \omega) = - \frac{1}{|S_n|} \ln \sum_{h_i \in M_i} H_i(h_i),
\]
where $M_i = M(S_n + u_i, h_{\partial(S_n + u_i)})$ and $H_i$ is a similar Hamiltonian but with the sum running over edges in $S_n + u_i$. Armed with this notation, we use algebraic laws of the logarithm to write

$$
\frac{1}{k^m} \sum_{i=1}^{k^m} \text{ent}_n(s, \tau_{u_i} \omega)
= -\frac{1}{n} \ln \left\{ \prod_{i=1}^{k^m} \sum_{h_i \in M_i} \exp(H_i(h_i)) \right\}^{1/k^m}
= -\frac{1}{n} \ln \left\{ \sum_{h_1 \in M_1} \ldots \sum_{h_{k^m} \in M_{k^m}} \prod_{i=1}^{k^m} \exp(H_i(h_i)) \right\}^{1/k^m}, \quad (12)
$$

where in the last line we used the distributive rule to exchange the sum and product.
Now, we observe that by Hölder’s inequality (with conjugate exponents $k^m$ and $\frac{k^m}{k^m - 1}$)

$$
\left( \sum_{h_1 \in M_1} \ldots \sum_{h_{k^m} \in M_{k^m}} \prod_{i=1}^{k^m} \exp(H_i(h_i)) \right)^{1/k^m} \geq |M| \left( \frac{1}{|M|^{k^m}} \sum_{h_1 \in M_1} \ldots \sum_{h_{k^m} \in M_{k^m}} \exp \left( \frac{1}{k^m} \sum_{i=1}^{k^m} H_i(h_i) \right) \right).
$$
This yields that (12) can be estimated as

\[
\frac{1}{k^m} \sum_{i=1}^{k^m} \text{ent}_n(s, \tau_i \omega) \leq -\frac{1}{\eta^m} \ln |M_1| - \frac{1}{\eta^m} \ln \left( \frac{1}{|M|^{k^m}} \sum_{h_1 \in M_1 \ldots h_{k^m} \in M_{k^m}} \exp \left( \frac{1}{k^m} \sum_{i=1}^{k^m} H_i(h_i) \right) \right).
\]

\[
= \text{ent}_n(s, 0) - \frac{1}{\eta^m} \ln \left( \frac{1}{|M|^{k^m}} \sum_{h_1 \in M_1 \ldots h_{k^m} \in M_{k^m}} \exp \left( \frac{1}{k^m} \sum_{i=1}^{k^m} H_i(h_i) \right) \right). \tag{13}
\]

Now, consider the space \( M^\mathbb{N} = \{(h_i)_{i \in \mathbb{N}} : h_i \in M_i \text{ for each } i\} \) of infinite sequences of height functions, each belonging to the appropriate set \( M_i \). For each \((h_i) \in M^\mathbb{N}\), the ergodic theorem (see Theorem 4.4 below) implies that

\[
\frac{1}{k^m} \sum_{i=1}^{k^m} H_i(h_i) \to 0 \quad \text{as } m \to \infty.
\]

Furthermore, since the base square \( S_n \) is fixed and since the random field \( \omega \) is uniformly bounded by \( C_\omega \), the sequence of functions \( f_k : M^\mathbb{N} \to \mathbb{R} \) defined by

\[
f_k(h_i) = \frac{1}{k^m} \sum_{i=1}^{k^m} H_i(h_i)
\]

is uniformly bounded. We endow the space \( M^\mathbb{N} \) with the measure \( \mu \) whose marginals are the uniform probability measures on \( \prod_{i=1}^N M_i \subset M^\mathbb{N} \); such a measure exists by Kolmogorov extension. We observe now that

\[
\frac{1}{|M|^{k^m}} \sum_{h_1 \in M_1 \ldots h_{k^m} \in M_{k^m}} \exp \left( \frac{1}{k^m} \sum_{i=1}^{k^m} H_i(h_i) \right) = \int_{M^\mathbb{N}} \exp(f_k(h_i)) \, d\mu(h_i).
\]

Since the sequence \((\exp \circ f_k)_k\) is uniformly bounded and converges point-wise to 1 for each point \((h_i) \in M^\mathbb{N}\), we apply the dominated convergence


\[ \lim_{k \to \infty} \frac{1}{M^m} \sum_{h_i \in M_1} \cdots \sum_{h_{km} \in M_{km}} \exp \left( \frac{1}{k^m} \sum_{i=1}^{k^m} H_i(h_i) \right) = 1. \]

Returning to \([13]\), we see that

\[ \lim_{k \to \infty} \frac{1}{k^m} \sum_{i=1}^{k^m} \text{ent}_n(s, \tau_{n}, \omega) \leq \text{ent}_n(s, 0) - 0. \]

But we saw at the beginning that this (pointwise) limit in \(k\) is precisely \(\mathbb{E}[\text{ent}_n(s, \omega)]\), so this completes the proof.

\[ \square \]

**Theorem 4.4** (Ergodic theorem). Let \(s \in \mathbb{R}^d\) satisfying \(|s|_\infty \leq 1\). Let \(n \in \mathbb{N}\) fixed. For \(k \in \mathbb{Z}^d\) we choose independent elements \(h_k \in M(k + S_n, h^s_{\theta_h(k+S_n)})\) according to the uniform distribution. Let \(e_{xy} \in E(S_n)\) be a fixed edge of the box \(S_n\). Then it holds for almost every choice of \(\omega\) and \(h_k\) that

\[ \lim_{m \to \infty} \frac{1}{m^d} \sum_{k | \|k\| \leq m} \omega_{e_{h_k(k+x), h_k(k+y)}} = 0 \quad \text{a.s. and in } L^1. \]

For the proof of Theorem 4.4 we need the multidimensional ergodic theorem. For convenience of the reader, we now state the precise statement that is used in this article. The statement is standard and can be found for example in Lemma 3.1. in [GVQ16].

**Lemma 4.5** (Lemma 3.1. in [GVQ16]). Given a positive integer \(k\), let \(\theta_1, \theta_2, \ldots, \theta_k\) be the commutative measure-preserving transformations of a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Additionally, we assume that there is a \(s \in \{1, \ldots, k\}\) such that the transformation \(\theta_s\) is ergodic. Let \(m_n = (m_{1,n}, \ldots, m_{k,n})\) be a multiple sequence of integers such that

\[ \min \{m_{1,n}, \ldots, m_{k,n}\} \to \infty. \]

Then for any bounded function \(f : \mathbb{R} \to \mathbb{R}\) the multiple sequence converges almost surely and in \(L^1\) to the limit

\[ \lim_{n \to \infty} \frac{1}{m_{1,n} \cdots m_{k,n}} \sum_{i_1=0}^{m_{1,n}-1} \cdots \sum_{i_k=0}^{m_{k,n}-1} f(\theta_{i_1}^{i_1} \cdots \theta_{i_k}^{i_k} \omega) = \mathbb{E}[f(\omega)]. \]

The proof of Lemma 4.5 is a simple consequence of the standard Birkhoff ergodic theorem (see for example [Dur10]) and is left out in this article. Let’s turn to the proof of Theorem 4.4.
Proof of Theorem 4.4. The proof consists of an application of the multidimensional ergodic theorem of Lemma 4.5. For that purpose let us explain the commutative measure-preserving transformations \( \theta_1, \ldots, \theta_k \) that are used. We recall that \( e \in E(S(n)) \) denotes an edge in the box \( S(n) \). Let \( x, y \in S(n) \) such that \( e = e_{xy} \). Without loss of generality we may assume that for all \( i \in \{1, \ldots, d\} \)

\[
x_i \leq y_i.
\]

For \( i \in \{1, \ldots, d\} \) we define the index set \( I_i \subset \mathbb{Z} \) in the following way

\[
I_i = \{ \min \{ h(y), h(y) \} \in \mathbb{Z} \mid h \in M(S_n, h_{\partial S_n}) \}.
\]

Let \( l \in I_i \), then we define the transformations \( \theta_{\pm i,l} \) as the shift operator

\[
\theta_{\pm i,l}(\omega_{e_{z,l}}) = \omega_{e_{z\pm|z|^2, l\pm|z|^2}}.
\]

We observe that there are only finitely many such transformations exists and enumerate them for simplicity as \( \theta_1, \ldots, \theta_m \). It follows from the definition of the transformations and that the random field \( \omega \) is shift invariant that \( \theta_1, \ldots, \theta_m \) are measure preserving. The transformations \( \theta_1, \ldots, \theta_m \) are commutative because the addition on \( \mathbb{Z} \) is commutative.

Now, let \( (k_r)_{r \in \mathbb{N}} \) be a sequence of elements \( k_r \in \mathbb{Z}^d \) that exploits the whole lattice \( \mathbb{Z}^d \) such that for all \( r \in \mathbb{R} \)

\[
|k_r - k_{r+1}| = 1.
\]

The key observation of the argument is that for any independent sequence of height functions \( (h_{k_r})_{r \in \mathbb{N}} \) there exists a sequence \( (m_r)_{r \in \mathbb{N}} \) of numbers \( m_r \in \{1, \ldots, m\} \) such that for any \( r \in \mathbb{N} \)

\[
\omega_{e_{h_{k_r}(x), h_{k_r}(y)}} = \theta_{m_{r-1}} \cdots \theta_{m_1} \left( \omega_{e_{h_{k_1}(x), h_{k_1}(y)}} \right).
\]

It follows by the Borel-Cantelli Lemma that in the sequence \( (m_r)_{r \in \mathbb{N}} \) all numbers of the set \( \{1, \ldots, m\} \) appear infinitely often. Hence, we can apply directly the multidimensional ergodic theorem of Lemma 4.5 and get the desired statement of Theorem 4.4.

4.3. Proof of Lemma 3.8. Our proof of Lemma 3.8 is fundamentally a simple counting argument, although to complete the argument we will need the Kirszbraun theorem for graphs, as stated in [MT16].

Proof. We start with the straightforward observation that

\[
M(S_n, h_{\partial S_n}) \subset M_n^{\text{free}(\delta)}(s)
\]
because the fixed boundary data for the former set, namely the canonical boundary height profile $h_{\partial S_n}$ with slope $s$, satisfies the free boundary criterion for the latter set. It follows immediately that

$$\text{ent}_n(s, \omega) = -\frac{1}{|S_n|} \ln \sum_{h \in M(S_n, h_{\partial S_n})} \exp(H(h, \omega))$$

$$\geq -\frac{1}{|S_n|} \ln \sum_{h \in M_{\text{free}}(s)} \exp(H(h, \omega))$$

$$= \text{ent}_{n_{\text{free}}}(s, \omega),$$

where the symbol $H(h, \omega)$ denotes the sum of $\omega_{e(h(x), h(y))}$ over all pairs of adjacent vertices $x, y$ in $S_n$.

For the second inequality, we will use the Kirszbraun theorem. Let $C = C(s) = \frac{2}{1 - |s|}$. Let $n' = n + [C\delta n]$; then the desired inequality is

$$\text{ent}_{n_{\text{free}}}(s, \omega) \geq (1 + C\delta)^{-m} \text{ent}_{n'}(s, \omega) + \theta(\delta).$$

Of course, the set $M_{n_{\text{free}}}(s)$ is not (generally) a subset of $M(S_{n'}, h_{\partial S_{n'}})$. However, the Kirszbraun theorem allows us to construct an injection from the former set to the latter. Let $h_n \in M_{n_{\text{free}}}(s)$; we wish to extend $h_n$ to a height function $h_{n'} \in M(S_{n'}, h_{\partial S_{n'}})$. To this end, suppose $x \in \partial S_n$, $y \in \partial S_{n'}$, and $h_n \in M_{n_{\text{free}}}(s)$. The free boundary criterion implies that $|h_n(x) - \lfloor s \cdot x \rfloor| \leq \delta n$. On the other hand, by the properties of the canonical height function $h_{\partial S_{n'}}$, the desired value for $h_{n'}(y)$ satisfies $|h_{n'}(y) - \lfloor s \cdot y \rfloor| \leq 1$. Thus,

$$|h_n(x) - h_{n'}(y)| \leq \delta n + |s \cdot (x - y)| + 1$$

$$\leq \delta n + |s||x - y| + 1 \quad (14)$$

Now, since $|x - y| \geq n' - n \geq C\delta n$, we can say that $\delta n \leq \frac{1}{2} (1 - |s|)|x - y|$. Furthermore for $n$ large, we have $1 \leq \frac{1}{2} (1 - |s|)|x - y|$. Combining these inequalities with (14), we conclude that

$$|h_n(x) - h_{n'}(y)| \leq |x - y|$$
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Therefore an extension $h_{n'} \in M(S_{n'}, h_{\partial S_{n'}})$ exists by the Kirszbraun theorem. So, we can estimate

\[
\text{ent}_{n'}^{\text{free}(\delta)}(s, \omega) = -\frac{1}{|S_n'|} \ln \sum_{h_n \in M_{n'}^{\text{free}(\delta)}(s)} \exp(H_n(h_n, \omega)) \\
\geq -\frac{1}{|S_n|} \ln \sum_{h_{n'} \in M_{n'}(S_{n'}, h_{\partial S_{n'}})} \exp(H_{n'}(h_{n'}, \omega)) \\
\geq -\frac{1}{|S_n|} \ln \sum_{h_{n'} \in M_{n'}(S_{n'}, h_{\partial S_{n'}})} \exp(H_{n'}(h_{n'}, \omega)) \\
- \frac{1}{|S_n|} \max_{h_{n'} \in M_{n'}(S_{n'}, h_{\partial S_{n'}})} |H_{n'}(h_{n'}, \omega) - H_n(h_{n'}, \omega)|
\]

As before, $H_n(h_n, \omega)$ denotes the sum over all adjacent pairs $x, y$ in $S_n$ of the quantity $\omega_{e_{h_n(x), h_n(y)}}$; the symbol $H_{n'}(h_{n'}, \omega)$ used in the last line denotes a similar sum but with $x, y$ running through all pairs of adjacent vertices in $S_{n'}$. Because the lattice $\mathbb{Z}^m$ is $m$-regular, there are at most $m(|S_{n'}| - |S_n|)$ such pairs which contribute to the sum for $n'$ but not to the sum for $n$. Since

$$|S_{n'}| - |S_n| = (n')^m - n^m \leq C_1(n' - n)n^{m-1} = C_2 \delta n^m,$$

we can bound $|H_{n'}(h_{n'}, \omega) - H_n(h_{n'}, \omega)| \leq C_\omega C_3 \delta n^m$. So, we can continue the estimate (15) as follows:

\[
\text{ent}_{n'}^{\text{free}(\delta)}(s, \omega) \geq -\frac{1}{|S_n|} \ln \sum_{h_{n'} \in M_{n'}(S_{n'}, h_{\partial S_{n'}})} \exp(H_{n'}(h, \omega)) \\
- \frac{1}{|S_n|} C_\omega C_3 \delta n^m \\
= \frac{n^m}{(n')^m} \text{ent}_{n'}(s, \omega) - C_\omega C_3 \delta \\
= \left(\frac{1}{1 + C\delta}\right)^m \text{ent}_{n'}(s, \omega) - C_\omega C_3 \delta \\
= (1 + C\delta)^{-m} \text{ent}_{n'}(s, \omega) + \theta(\delta).
\]

\[\square\]

APPENDIX A. PROOF OF THE ERGODIC THEOREM FOR ALMOST SUPERADDITIVE MULTIVARIABLE RANDOM PROCESSES

For convenience, we first recall the relevant notation and the statement of the ergodic theorem, which is a slight modification of Theorem 2.4
from [AK81]. We write $B$ for the set of boxes:

$$B = \{(a_1, b_1) \times \cdots \times (a_m, b_m) \cap \mathbb{Z}^m : a_i < b_i \text{ for all } i, \text{ where } a_i, b_i \in \mathbb{Z}\}.$$  

We will also have reason to consider subsets:

$$B_k = \{(a_1, b_1) \times \cdots \times (a_m, b_m) \cap \mathbb{Z}^m \in B : \text{all } a_i \text{ and } b_i \text{ are divisible by } k\}.$$  

Fix a semi-group of measure-preserving transformations $\tau = (\tau_u)_{u \in \mathbb{Z}^d_+}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that a family of $L^1$ random variables $(F_B)_{B \in B}$, indexed by the boxes $B \in B$, is *almost superadditive* if:

- for all $u \in \mathbb{Z}^d_+$, $F_{u+B} = F_B \circ \tau_u$,
- (almost superadditivity) for any collection of disjoint boxes $B_1, \ldots, B_n \in B$
  
  whose union $B$ also lies in $B$, the inequality

$$F_B \geq \sum_{i=1}^n F_{B_i} - A \sum_{i=1}^n |\partial B_i|$$  

holds, where $\partial B_i = \{x \in B_i : \exists y \notin B_i, x \sim y\}$ is the inner boundary of $B_i$ and $A = A(\omega)$ is an arbitrary non-negative random variable, and

- the time constant $\tilde{\gamma} = \limsup_{n \to \infty} \frac{1}{|S_n|} \mathbb{E}[F_{S_n}]$ is finite, where

$$S_n = [0, n)^d \cap \mathbb{Z}^d \in B.$$  

The theorem is:

**Theorem A.1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\tau = (\tau_u)_{u \in \mathbb{Z}^d_+}$ be a semi-group of measure preserving transformations on $\Omega$, and let $F = (F_B)_{B \in B}$ be an almost superadditive random process. Then, the pointwise limit

$$\lim_{n \to \infty} \frac{1}{|S_n|} F_{S_n}$$  

exists almost surely. If moreover $\tau$ is ergodic, then

$$\lim_{n \to \infty} \frac{1}{|S_n|} F_{S_n} = \lim_{n \to \infty} \frac{1}{|S_n|} \mathbb{E}[F_{S_n}] = \tilde{\gamma}(F)$$  

almost surely.
Our proof of this theorem is take almost directly from [AK81], with minor modifications to replace their exact superadditivity by our almost superadditivity, as well as some minor notational changes. Following [AK81], we start with a few preliminary results:

**Lemma A.2** (A covering lemma; Lemma 3.1 of [AK81]). Let $W$ be a finite subset of $\mathbb{Z}^d_+$ (the letter $W$ is chosen because more obvious choices like $A$ or $F$ are already used for other purposes in the current article). For each $u \in W$ let $n(u) \geq 1$ be an integer. Then there is a set $W' \subseteq W$ such that $\{u + S_{n(u)} : u \in W'\}$ is a family of disjoint sets and such that

$$3^m \sum_{u \in W'} |S_{n(u)}| \geq |W|.$$ 

As mentioned in [AK81], this is a modification of a common covering lemma due to Wiener. The proof is standard.

**Theorem A.3** (A maximal inequality; Theorem 3.2 of [AK81]). Let $F$ be a non-negative almost superadditive process and let $\alpha > 0$. If

$$E = \{ \omega : \limsup_{n \geq 1} \frac{1}{|S_n|} F_{S_n}(\omega) > \alpha \},$$

then

$$\mathbb{P}(E) \leq \frac{3^m \gamma(F)}{\alpha}.$$ 

**Proof.** For $N, M \in \mathbb{N}$ with $N < M$ define

$$E_{N,M} = \left\{ \omega : \sup_{N \leq n \leq M} \frac{1}{|S_n|} F_{S_n}(\omega) > \alpha \right\}.$$ 

Fix for now a larger integer $K > M$; we will soon take $K \to \infty$. But first, consider a single $\omega \in \Omega$. Define the set $W = W(\omega) = \{u \in S_{K-M} : \tau_u \omega \in E_{N,M}\}$. For $u \in W$, there is an integer $n(u)$ (implicitly depending on $\omega$) such that $N \leq n(u) \leq M$ and $\frac{1}{|S_{n(u)}|} F_{S_{n(u)}}(\tau_u \omega) > \alpha$. Since $F_B \circ \tau_u = F_{u+B}$ for any $B \in \mathcal{B}$, we can rewrite this inequality as

$$F_{u+S_{n(u)}}(\omega) > \alpha |S_{n(u)}|. \quad (16)$$

Then, we can apply the covering lemma, Lemma A.2, to pick $u_1, \ldots, u_l \in W$ (again, implicitly depending on $\omega$) such that the boxes $u_i + S_{n(u_i)}$ are disjoint but $3^m \sum_{i=1}^l |S_{n(u_i)}| \geq |W|$. Combining this with (16), we
get

\[ |W| \leq 3^m \sum_{i=1}^{l} |S_{n(u_i)}| \leq \frac{3^m}{\alpha} \sum_{i=1}^{l} F_{u_i + S_{n(u_i)}} \]

\[ \leq \frac{3^m}{\alpha} F_{S_K} + A(\omega) \sum_{i=1}^{l} |\partial S_{n(u_i)}| \]

Now, let \( \epsilon(N) = \sup_{n \geq N} \frac{|\partial S_n|}{|S_n|} \) (since \( |\partial S_n| \rightarrow 0 \), this is actually a maximum; since \( n \mapsto \frac{|\partial S_n|}{|S_n|} \) is decreasing for \( n \) sufficiently large, eventually \( \epsilon(N) = \frac{|\partial S_n|}{|S_n|} \)). For each \( i \) we have \( |\partial S_{n(u_i)}| \leq \epsilon(N)|S_{n(u_i)}| \), and since the boxes \( u_i + S_{n(u_i)} \) are disjoint and contained in \( S_K \), we conclude that

\[ |W| \leq \frac{3^m}{\alpha} F_{S_K} + A(\omega) \epsilon(N) |S_K|. \]

Recall that the set \( W = W(\omega) \) depended on the choice of \( \omega \). Taking expectations and recalling that the shifts \( \tau \) are measure preserving, we have that

\[ \mathbb{E}[|W|] = \mathbb{E} \left[ \sum_{u \in S_{K-M}} 1_{\tau_u^{-1} E_{N,M}} \right] = (K - M)^m \mathbb{P}(E_{N,M}), \]

and thus

\[ (K - M)^m \mathbb{P}(E_{N,M}) \leq \frac{3^m}{\alpha} \mathbb{E}[F_{S_K}] + \mathbb{E}[A] \epsilon(N) |S_K|. \]

We can rearrange this inequality to

\[ \mathbb{P}(E_{N,M}) \leq \frac{3^m}{\alpha} \frac{K^m}{(K - M)^m} \frac{1}{K^m} \mathbb{E}[F_{S_K}] + \mathbb{E}[A] \epsilon(N) \frac{K^m}{(K - M)^m}. \]

Since \( \bar{\gamma}(F) = \limsup_{n \to \infty} \frac{1}{|S_n|} \mathbb{E}[F_{S_n}] \), we take \( K \to \infty \) to obtain

\[ \mathbb{P}(E_{N,M}) \leq \frac{3^m}{\alpha} \bar{\gamma}(F) + \mathbb{E}[A] \epsilon(N). \]

Next, let \( E_N = \{ \omega : \sup_{n \geq N} \frac{1}{|S_n|} F_{S_n}(\omega) > \alpha \} = \bigcup_{M \geq N} E_{N,M} \). Since this is an increasing union,

\[ \mathbb{P}(E_N) = \lim_{M \to \infty} \mathbb{P}(E_{N,M}) \leq \frac{3^m}{\alpha} \bar{\gamma}(F) + \mathbb{E}[A] \epsilon(N). \]

Finally, \( E = \{ \omega : \limsup_{n \to \infty} \frac{1}{|S_n|} F_{S_n}(\omega) > \alpha \} = \bigcap_{N \geq 1} E_N \) is a decreasing intersection of measurable sets, so

\[ \mathbb{P}(E) = \lim_{N \to \infty} \mathbb{P}(E_N) \leq \frac{3^m}{\alpha} \bar{\gamma}(F), \]

as desired.
Lemma A.4 (Convergence of expectations; Lemma 3.4 of [AK81]).

\[ \tilde{\gamma}(F) = \lim_{n \to \infty} \frac{1}{|S_n|} E[F_{S_n}]. \]

Moreover, if \( H = (H_B)_{B \in B_k} \) is almost superadditive but defined only on boxes in \( B_k \), the same equality holds (except that both in the definition of \( \tilde{\gamma}(H) \) and in the right-hand side above, we only consider values of \( n \) that are divisible by \( k \) as we take \( n \to \infty \)).

Proof. Recall that \( \tilde{\gamma} = \limsup_{n \to \infty} \frac{1}{|S_n|} E[F_{S_n}] \). We wish to show that \( \liminf_{n \to \infty} \frac{1}{|S_n|} E[F_{S_n}] \geq \tilde{\gamma} \). Let \( \epsilon > 0 \), and fix \( k \) large (just how large, we will see soon) and such that \( \frac{1}{|S_k|} E[F_{S_k}] > \tilde{\gamma} - \epsilon \). For \( n \) sufficiently large, we can subdivide the large box \( S_n \) into \( r \) translates of \( S_k \), say \( u_i + S_k \) for \( 1 \leq i \leq r \), and \( s \) translates of \( S_1 \), say \( v_j + S_1 \) for \( 1 \leq j \leq s \). By requiring that \( n \) be large enough, we can ensure that \( s \leq \epsilon |S_n| \). By almost superadditivity,

\[ F_{S_n} \geq \sum_{i=1}^{r} F_{S_k} \circ \tau_{u_i} + \sum_{j=1}^{s} F_{S_1} \circ \tau_{v_j} - A(r|\partial S_k| + s|\partial S_1|). \]

Taking expectations and dividing by \( |S_n| \), we have

\[ \frac{1}{|S_n|} E[F_{S_n}] \geq \frac{r}{|S_n|} E[F_{S_k}] + \frac{s}{|S_n|} E[F_{S_1}] - E[A]\left(\frac{r}{|S_n|} |\partial S_k| + \frac{s}{|S_n|} |\partial S_1|\right). \]

Now, since \( r|S_k| + s = |S_n| \) and \( 0 \leq s \leq \epsilon |S_n| \), it follows that \( r|S_k| \geq (1 - \epsilon)|S_n| \), or equivalently that \( \frac{r}{|S_n|} \geq \frac{1 - \epsilon}{|S_k|} \). Using this inequality together with the inequality \( \frac{s}{|S_n|} \leq \epsilon \), we obtain

\[ \frac{1}{|S_n|} E[F_{S_n}] \geq \frac{1 - \epsilon}{|S_k|} E[F_{S_k}] + \epsilon E[F_{S_1}] - E[A]\left(\frac{1 - \epsilon}{|S_k|} |\partial S_k| + \epsilon |\partial S_1|\right). \]

By choice of \( k \), we have that \( \frac{1}{|S_k|} E[F_{S_k}] \geq \tilde{\gamma} - \epsilon \); as promised, we see now that we want \( k \) large enough that \( \frac{|\partial S_k|}{|S_k|} < \epsilon \). Thus,

\[ \frac{1}{|S_n|} E[F_{S_n}] \geq (1 - \epsilon)(\tilde{\gamma} - \epsilon) + \epsilon E[F_{S_1}] - E[A]\left((1 - \epsilon)\epsilon + \epsilon |\partial S_1|\right). \]
Since this holds for all \( n \) large enough (depending on \( k \), which in turn depends on \( \epsilon \)), we conclude that
\[
\liminf_{n \to \infty} \frac{1}{|S_n|} \mathbb{E}[F_{S_n}] \geq \tilde{\gamma},
\]
as desired.

Let us deal quickly with the case where the almost superadditive process \( H = (H_B)_{B \in \mathcal{B}_k} \) is defined only on boxes in \( \mathcal{B}_k \), i.e. only on boxes whose vertices lie on points of \( \mathbb{Z}^n \) whose every coordinate is divisible by \( k \). We may define a process \( F_B = \frac{1}{|S_k|} H_{kB} \), where \( kB = \{ ku : u \in B \} \) is the \( k \)-fold rescaling of \( B \). Then
\[
\frac{1}{|S_n|} F_{S_n} = \frac{1}{|S_n||S_k|} H_{S_{kn}} = \frac{1}{|S_{kn}|} F_{S_{kn}}
\]
so that \( \tilde{\gamma}(F) = \tilde{\gamma}(H) \), and the result just proven for \( F \) also carries over (via linearity of the limit) to \( H \).

**Theorem A.5** (An ergodic theorem for almost subadditive processes; c.f. Theorem 2.4 of [AK81]). If \( F \) is an almost superadditive process, then \( \lim_{n \to \infty} \frac{1}{|S_n|} F_{S_n} \) exists almost surely. Moreover, if the measure preserving operators \( \tau \) are ergodic, then \( \lim_{n \to \infty} \frac{1}{|S_n|} F_{S_n}(\omega) = \tilde{\gamma}(F) \) independently of \( \omega \).

**Proof.** We reproduce the proof in [AK81] with only minor modifications.

Step 1 (Reduction to \( F \geq 0 \)). Here, our almost superadditivity assumption requires a slight modification of the proof from [AK81]. Rather than comparing the process \( F \) under consideration to the additive process
\[
G_B(\omega) = \sum_{u \in B} F_{S_1} \circ \tau_u(\omega)
\]
we shall instead use the additive process
\[
\tilde{G}_B(\omega) = \sum_{u \in B} F_{S_1} \circ \tau_u(\omega) - A(\omega)|B|
\]
With this modification, we define an almost superadditive process \( F' = F - \tilde{G} \). Almost superadditivity of \( F \) implies that \( F' \geq 0 \), since for any partition \( B = \bigcup B_i \) of a box \( B \) into disjoint sub-boxes \( B_i \), the size of the whole rectangle \( |B| \) is larger than the sum of the sizes of the inner boundaries \( |\partial B_i| \).

Now, the desired convergence result is known for additive processes, so if we can prove convergence for \( F' \) we may conclude convergence for \( F \) as well. So, from this point on we shall assume that the process \( F \) is non-negative.
Step 2 (Alternate rates of convergence). Let \( \bar{f} = \bar{f}(\omega) \) and \( f = f(\omega) \) denote respectively the pointwise lim sup and lim inf of \( \frac{1}{|S_n|} F_{S_n} \). We shall show that, for \( m \) fixed, these two functions are also the pointwise lim sup and lim inf of \( \frac{1}{|S_{km}|} F_{S_{km}} \) as \( k \to \infty \).

For convenience, we write \( f^{(m)} \) for the pointwise lim sup of the sequence \( \frac{1}{|S_{km}|} F_{S_{km}} \) as \( k \to \infty \). Clearly \( f^{(m)} \leq f \). We must prove the opposite inequality. Consider first any two boxes \( B \subseteq B' \). Since \( F \) is almost superadditive and non-negative, we have \( F_{B'} \geq F_B - o(|B'|) \).

For each \( n \), let \( k(n) \) be the smallest integer such that \( k(n)m \geq n \).

Then, \( \frac{1}{|S_n|} F_{S_k m} \geq \frac{1}{|S_n|} F_{S_n} - \frac{o(|S_k m|)}{|S_n|} \).

Taking the lim sup on both sides and using the fact that \( |S_k m| \sim |S_n| \), we have that \( f^{(m)} \geq f \). Thus \( f = \limsup_k \frac{1}{|S_{km}|} F_{S_{km}} \) as desired; the corresponding result for \( f \) is proved similarly.

Step 3 (Approximating \( F \)). Fix \( \alpha > 0 \), and let \( E = \{ \omega : f(\omega) - f(\omega) > \alpha \} \). In order to show that \( P(E) = 0 \), let \( \epsilon > 0 \). By Lemma A.4, there exist \( k \) arbitrarily large such that \( \frac{1}{|S_k|} E[F_{S_k}] > \tilde{\gamma} - \frac{\epsilon}{2} \) (we shall see soon why we need the extra \( \frac{\epsilon}{2} \)). Define an additive family \( H \) on \( B_k \) by

\[
H_B = \sum_{u \in B \cap k\mathbb{Z}^m} F_{S_k} \circ \tau_u - A|B \cap k\mathbb{Z}^m|\partial S_k,
\]

where \( k\mathbb{Z}^m = \{ku : u \in \mathbb{Z}^m\} \) is a sub-lattice of \( \mathbb{Z}^m \). Recall that \( B_k \) is the set of boxes whose vertices all lie in this same sub-lattice \( k\mathbb{Z}^m \).

We wish to show that \( F \geq H \), i.e. for every box \( B \in B_k \), \( F_B \geq H_B \).

So, fix a box \( B \in B_k \). Since \( B \) decomposes into the disjoint union of the sub-boxes \( \{u + S_k : u \in B \cap k\mathbb{Z}^m\} \), almost superadditivity of \( F \) gives us

\[
F_B \geq \sum_{u \in B \cap k\mathbb{Z}^m} F_{u+S_k} - |A| \sum_{u \in B \cap k\mathbb{Z}^m} |\partial(u + S_k)|.
\]

Of course, we have that \( F_{u+S_k} = F_{S_k} \circ \tau_u \), and each of the \( |B \cap k\mathbb{Z}^m| \)-many terms in the second sum above is equal to \( |\partial S_k| \). So, we see immediately that \( F \geq H \).
Next, we compute \( \tilde{\gamma}(H) \). Applying Lemma A.4:

\[
\tilde{\gamma}(H) = \lim_{n \to \infty} \left( \frac{1}{|S_{kn}|} \mathbb{E}[H_{S_{kn}}] \right)
= \lim_{n \to \infty} \left( \sum_{u \in S_{kn} \cap k\mathbb{Z}^m} \frac{\mathbb{E}[F_{S_k \circ \tau_u}]}{k^m n^m} - \frac{\mathbb{E}[A]|S_{kn} \cap k\mathbb{Z}^m|\partial S_k}{k^m n^m} \right)
= \lim_{n \to \infty} \left( \frac{1}{|S_k|} \mathbb{E}[F_{S_k}] - \frac{\mathbb{E}[A]|\partial S_k|}{|S_k|} \right)
\]

(In the last line, we used the fact that \( \tau \) is measure-preserving.) Note that \( n \) no longer appears in the final expression. We choose \( k \) large enough that \( \mathbb{E}[A]|\partial S_k| < \frac{\epsilon}{2} \), while still satisfying \( \frac{1}{|S_k|} \mathbb{E}[F_{S_k}] > \tilde{\gamma}(F) - \frac{\epsilon}{2} \).

This implies that \( \tilde{\gamma}(H) > \tilde{\gamma}(F) - \epsilon \).

Let \( F' = F - H \), so that \( F' \) is a non-negative random family defined on \( B_k \). By Lemma A.4, we can write \( \tilde{\gamma}(F') \) as a limit; importantly, \( \tilde{\gamma} \) is linear, so \( \tilde{\gamma}(F') = \tilde{\gamma}(F) - \tilde{\gamma}(H) < \epsilon \).

Since \( H \) is additive, it converges pointwise almost surely. Recalling step 2, we estimate

\[
\bar{f}(\omega) - \underline{f}(\omega) = \limsup_{n \to \infty} \frac{1}{|S_{kn}|} F_{S_{kn}} - \liminf_{n \to \infty} \frac{1}{|S_{kn}|} F_{S_{kn}}
= \limsup_{n \to \infty} \frac{1}{|S_{kn}|} F'_{S_{kn}} - \liminf_{n \to \infty} \frac{1}{|S_{kn}|} F'_{S_{kn}}
\leq \sup_{n \to \infty} \frac{1}{|S_{kn}|} F'_{S_{kn}}
\]

So, the event \( E = \{ \bar{f} - \underline{f} > \alpha \} \) is contained in \( \{ \sup_{n \geq 1} \frac{1}{|S_{kn}|} F'_{S_{kn}} > \alpha \} \).

By Lemma A.3

\[
\mathbb{P}(E) \leq \frac{3m \tilde{\gamma}(F')}{\alpha} \leq \frac{3m \epsilon}{\alpha}.
\]

Taking \( \epsilon \to 0 \), we see that \( \mathbb{P}(E) = 0 \).

Since \( \alpha > 0 \) was arbitrary, we conclude that \( \bar{f} = \underline{f} \) almost surely, and thus that \( \frac{1}{|S_n|} F_{S_n} \) converges pointwise almost surely.

Step 4 (Conclusions for ergodic \( \tau \)). We wish to show, assuming the measure preserving transformations \( \tau \) are ergodic, that \( \frac{1}{|S_n|} F_{S_n} \to \tilde{\gamma}(F) \) (for almost every \( \omega \)). To this end, we shall show that

\[
f(\omega) := \lim_{n \to \infty} \frac{1}{|S_n|} F_{S_n}(\omega)
\]

is shift-invariant, in the sense that \( f \circ \tau_u = f \) for each \( u \). Then, of course, it follows that \( f(\omega) \) is (almost surely) constant, and is equal to its expected value, which is \( \tilde{\gamma}(F) \).
Since $\tau = (\tau_u)_{u \in \mathbb{Z}^m}$ is a semi-group of measure-preserving transformations, it suffices to consider $u$ with $|u|_1 = 1$. Fix such a $u$. Then for every $n$, we have $S_n + u \subset S_{n+1}$, so by almost superadditivity (and positivity of $F$),

$$\frac{1}{|S_n|} F_{S_n+u}(\omega) \leq \frac{1}{|S_n|} F_{S_{n+1}}(\omega) + \frac{2A(\omega)|\partial S_n|}{|S_n|}$$

As $n \to \infty$, the term on the left side converges to $f \circ \tau_u(\omega)$, the first term on the right converges to $f(\omega)$, and the second term on the right vanishes. Thus we have $f \circ \tau_u \leq f$; by symmetry, $f \circ \tau = f$, which completes the proof. 

\[\square\]

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