Field Equations for the Simplest Multi-Particle Higher-Spin Systems

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Abstract

We derive the product law for the simplest multi-particle higher-spin algebra $M_2(A)$ and its factor-algebra $M^2(A)$ with rank-one fields factored out. Equations of motion for the systems resulting from these algebras are analysed. We conclude that the equations for $M_2(A)$ describe the conformal off-shell higher-spin system with the rank-two fields representing the off-shell degrees of freedom of the originally massless system. On the other hand the equations resulting from $M^2(A)$ describe the infinite system of conserved currents unrelated to massless fields.
1 Introduction

One of the major problems of the theory of fundamental interactions is to understand relation between String Theory \cite{1}, that contains infinite towers of massive higher-spin (HS) fields, and HS gauge theories in which all HS fields are massless and which exhibit infinite-dimensional HS symmetries (for review see, e.g., \cite{2}). Arguments that String Theory possesses higher symmetries in the high-energy limit were given long ago in \cite{3,4}. This suggests that String Theory should be related to HS theory. For related discussion see, e.g., \cite{5,6}.

More recently, further proposals on the relation between HS gauge theories and String Theory were put forward in \cite{7,8}. In \cite{9} it was conjectured that there exists a broad class of HS gauge theories based on the HS algebras and their further multi-particle extensions. In \cite{9} it was argued that dynamics based on certain algebras has no room for usual massless fields while other models properly describe HS gauge fields. In this paper, we consider in some detail how this can be seen at the level of field equations associated with one or another algebra in the framework of the simplest multi-particle extension proposed originally in \cite{10}.

Specifically, we consider two versions of the two-particle algebra and discuss the form of the related field equations. The corresponding algebras denoted $M_2$ and $M^2$, respectively, are factor-algebras of the simplest multi-particle algebra of \cite{10}. To these algebras we associate two HS systems and derive their equations of motions. It will be shown that the $M_2$ system properly describes massless HS fields as well as two-particle fields that can be interpreted as conserved currents. In particular, it will be explained how the $M_2$ system is related to the system of HS currents built from products of HS fields according to \cite{11,12}. On the other hand, it will be explicitly shown that, in agreement with the general group-theoretical argument of \cite{9}, the $M^2$ system, that can be understood as resulting from the oscillator (Weyl) algebra with two sets of oscillators, cannot describe massless fields.

The rest of the paper is organized as follows. In Section 2 the construction of the multi-particle algebras is recalled. In Sections 3 and 4 we analyse field equations of the $M_2$ and $M^2$ systems, respectively. Brief conclusions are in Section 5. Appendix contains some details of the derivation of the relevant multi-particle algebras.

2 Multi-particle algebras

First, we recall relevant elements of the algebraic construction of \cite{10}. Let $A$ be some associative algebra $A$ with basis elements $t_i$, the product law $\ast$ and structure coefficients

\[ t_i \ast t_j = f_{ij}^{\ k} t_k \]  

(2.1)

obeying associativity condition

\[ (t_i \ast t_j) \ast t_k = t_i \ast (t_j \ast t_k) \in A, \quad t_i, t_j, t_k \in A. \]  

(2.2)

$A$ is also assumed to be unital with the unit element $e_\ast$ obeying

\[ e_\ast \ast t = t \ast e_\ast = t, \quad \forall t \in A. \]  

(2.3)
For any such $A$ it is possible to build an associative multi-particle algebra $M(A)$ having the meaning of the universal enveloping of $A$ (more precisely, of the Lie algebra associated with $A$). As a linear space

$$M(A) = \bigoplus_{n=0}^{\infty} \text{Sym} \ A \otimes \cdots \otimes A,$$

i.e., the basis for this algebra consists of symmetric monomials.

The monomial of degree zero in $t_i$ is identified with the unity $\text{Id}$ of $M(A)$.

Due to symmetrization, elements of $M(A)$ can be represented as functions of the commuting variables $\alpha_i$

$$F = \sum_{n=0}^{\infty} F^{i_1,\ldots,i_n} T_{i_1,\ldots,i_n} \sim F(\alpha) = \sum_{n=0}^{\infty} F^{i_1,\ldots,i_n} \alpha_{i_1} \cdots \alpha_{i_n}. \quad (2.6)$$

Associative product $\circ$ in $M(A)$ is generated by the $\star$ product in $A$ as follows [10]

$$F(\alpha) \circ G(\alpha) = F(\alpha) \exp \left( \frac{\partial}{\partial \alpha_{i_j}} f_{ij} \frac{\partial}{\partial \alpha_{i_k}} \right) G(\alpha), \quad (2.7)$$

where $f_{ij}$ are the structure constants of $A$ and derivatives $\frac{\partial}{\partial \alpha_i}$ and $\frac{\partial}{\partial \alpha_j}$ act on $F(\alpha)$ and $G(\alpha)$, respectively.

One can easily check that associativity of the product $\circ$ in $M(A)$ follows from that of $A$

$$(F_1(\alpha) \circ F_2(\alpha)) \circ F_3(\alpha) = F_1(\alpha) \circ (F_2(\alpha) \circ F_3(\alpha)). \quad (2.8)$$

$M(A)$ has a family of two-sided ideals $\mathcal{I}^M$:

$$\{ F \in \mathcal{I}^M : F(\alpha) = \sum_{n=M}^{\infty} F^{i_1,\ldots,i_n} \alpha_{i_1} \cdots \alpha_{i_n} \}. \quad (2.9)$$

Specifically, in this paper we are interested in the ideal $\mathcal{I}^3$ spanned by polynomials of degree three or higher.

The factor-algebra $M_2(A) := M(A)/\mathcal{I}^3$ consists of polynomials of degrees $p \leq 2$ with the basis monomials $1, \alpha_i, \alpha_i \alpha_j$. Discarding polynomials of degree three or higher, the product law (2.7) induces the following product law $M_2 \circ$ in $M_2$

$$\alpha_{i_1} \overset{M_2}{\circ} 1 = \alpha_{i_1} \exp \left( \frac{\partial}{\partial \alpha_{i_1}} f^{i_k}_{j} \frac{\partial}{\partial \alpha_{i_k}} \right) 1 = \alpha_{i_1}, \quad (2.10)$$

1For simplicity, in this paper we consider purely bosonic models. To include fermions one has to consider an appropriate graded extension of the multi-particle algebra introduced in [1].
The consideration of the previous section was applicable to any associative algebra

\[ M \]

3.1 Higher-spin algebra

For example, the coefficient 2 is prescribed because, as shown in [10], otherwise it would not be an ideal.

\[ e^\circ M \]

The algebra \( e^\circ M \) with the simplest HS algebra (see e.g. [2] and references therein), which is the unit in the algebra \( e^\circ M \), contains unit element \( e^\circ 1 \), which was unit in the algebra \( e^\circ f \) of the oscillator (\( i.e., \) Weyl) algebra with the doubled set of oscillators.

Thus, \( e^\circ f^2 \) gives rise to another factor-algebra \( M^2(A) := M_2(A)/\mathcal{I}_{e^\circ -2Id} \). We can chose the following basis in the ideal:

\[ \alpha' \equiv e^\circ -2Id \in \mathcal{I}_{e^\circ -2Id} \]  

\[ \alpha_i' \equiv (e^\circ -2Id)^M_2 \alpha_i = e^\circ \alpha_i - \alpha_i \in \mathcal{I}_{e^\circ -2Id} \]  

\[ (e^\circ -2Id)^M_2 \alpha_i \alpha_j = 0 \in \mathcal{I}_{e^\circ -2Id} \]

The algebra \( M_2(A) \) spanned by 1, \( \alpha_i, \alpha_i \alpha_j \) has dimension \( \frac{1}{2} n(n+1) + n + 1 \) for \( i, j = 1, \ldots, n \). The ideal has dimension \( n + 1 \). Its basis can be completed to that of \( M_2(A) \) using elements \( \alpha_i \alpha_j \). Thus, \( M^2(A) = M_2(A)/\mathcal{I}_{e^\circ -2Id} \) can be represented by the elements \( \alpha_i \alpha_j \) with the product law

\[ (\alpha_i \alpha_j) \circ (\alpha_i \alpha_j) = (\alpha_i \alpha_j) (\alpha_i \alpha_j) + (\alpha_i \alpha_j) (\alpha_i \alpha_j) \]

which is the product law of the oscillator (\( i.e., \) Weyl) algebra with the doubled set of oscillators.

Now we are in a position to write down the field equations for the HS systems associated with \( M_2(A) \) and \( M^2(A) \).

3 \( M_2 \) system

3.1 Higher-spin algebra

The consideration of the previous section was applicable to any associative algebra \( A \). Now we identify \( A \) with the simplest HS algebra (see e.g. [2] and references therein), which is the...
associative algebra of polynomials $F(Y; K) \in A$

$$F(Y; K) = \sum_{n=0}^{\infty} \sum_{p,q=0,1} F_{A_1}^{A_0} Y_{A_1} \cdots Y_{A_n} k^p \bar{k}^q, \quad A_1, \ldots, A_n = 1, \ldots, 4$$  (3.1)

with the star product of $K = (k, \bar{k})$–independent functions

$$(F_1 \ast F_2)(Y) = F_1(Y) \exp \[ i \bar{A} C_{AB} \partial^B ] F_2(Y),$$  (3.2)

where $\partial^A := \frac{\partial}{\partial Y^A}$ and

$$C_{AB} = -C_{BA}$$  (3.3)

is a non-degenerate symplectic form. In other words, the algebra of $K$–independent functions is the Weyl algebra $A_2$. The presence of the Klein operators $K = (k, \bar{k})$, which extends $A_2$ to its semidirect product with the group algebra of $\mathbb{Z}_2 \times \mathbb{Z}_2$, is important in many respects and, in particular, for the formulation of nonlinear HS equations of [13, 14]. The product law in the resulting algebra $A$ supplements (3.2) by the relations

$$k \ast F(y, \bar{y}) = F(-y, \bar{y}) \ast k, \quad \bar{k} \ast F(y, \bar{y}) = F(y, -\bar{y}) \ast \bar{k}, \quad k \ast k = \bar{k} \ast \bar{k} = 1, \quad k \ast \bar{k} = \bar{k} \ast k,$$

(3.4)

where $y_\alpha$ and $\bar{y}_\dot{\alpha}$ are the left and right spinor components of $Y_A = (y_\alpha, \bar{y}_\dot{\alpha})$, $\alpha, \dot{\alpha} = 1, 2$. In other words $k$ and $\bar{k}$ generate automorphisms of the Weyl algebra $A_2$, that change signs of the left and right spinors, respectively.

Denoting $Y_A, K$ by $\mathcal{Y} = (Y_A, K)$, algebra $A$ is the algebra of functions $f(\mathcal{Y})$. As a linear space $A$ can be decomposed into direct sum of its even and odd parts

$$A = A^0 \oplus A^1$$  (3.5)

formed, respectively, by the elements $f(\mathcal{Y})$ even and odd under the automorphism

$$\tau f(Y; k, \bar{k}) = f(Y; -k, -\bar{k}),$$  (3.6)

i.e.,

$$f_0(Y; k, \bar{k}) \in A^0 : \quad f_0(Y; k, \bar{k}) = f_0(Y; -k, -\bar{k}),$$  (3.7)

$$f_1(Y; k, \bar{k}) \in A^1 : \quad f_1(Y; k, \bar{k}) = -f_1(Y; -k, -\bar{k}).$$  (3.8)

Clearly, $A^0$ forms a subalgebra of $A$ which we call proper HS algebra. The one-form gauge fields of this algebra

$$\omega_h(Y; K|x) = \omega_h(Y; -K|x)$$  (3.9)

describe the genuine massless HS fields of all spins $s \geq 1$ including the spin-two field describing the graviton [13]. The gauge fields

$$\omega_{top}(Y; K|x) = -\omega_{top}(Y; -K|x)$$  (3.10)
associated with $A^1$ describe an infinite set of topological gauge fields, each describing at most a finite number of degrees of freedom. (For details see [13].)

Another fundamental object of the HS theory is a zero-form $C(Y; K|x)$ related to $\omega(Y; K|x)$ by virtue of the field equations of [13]. In this case however, massless (topological) fields are associated with odd (even) zero-forms $C(Y; K|x)$

$$C_{hs}(Y; K|x) = -C_{hs}(Y; -K|x), \quad (3.11)$$

$$C_{top}(Y; K|x) = C_{top}(Y; -K|x). \quad (3.12)$$

Note that zero-forms $C_{hs}(Y; K|x)$ belong to so-called twisted adjoint module over the proper HS algebra. In this paper we focus on the equations on the HS fields neglecting possible contributions to the sector of topological fields.

### 3.2 $M_2$

Then elements of $M_2(A)$ can be represented by the unit element $Id$ and symmetric polynomials of two types

$$F(Y) \quad (3.13)$$

and

$$F(Y^1; Y^2) = F(Y^2; Y^1). \quad (3.14)$$

The product law is

$$F(Y) \circ G(Y) = F(Y^1)G(Y^2) + F(Y^2)G(Y^1) + F(Y) \star G(Y), \quad (3.15)$$

$$F(Y) \circ G(Y^1; Y^2) = F(Y^1) \star_{1,1} G(Y^1; Y^2) + F(Y^2) \star_{2,2} G(Y^1; Y^2), \quad (3.16)$$

$$F(Y^1; Y^2) \circ G(Y^1; Y^2) = F(Y^1; Y^2) \star_{1,1} \star_{2,2} G(Y^1; Y^2), \quad (3.17)$$

where star product for the $Y$ variables is defined as

\[ \star_{i,i} = \exp \left[ i \left( \frac{\partial}{\partial Y_A^i} C_{AB} \frac{\partial}{\partial Y_B^i} \right) \right], \quad i = 1, 2. \quad (3.18) \]

(No summation over $i$. For more detail on the derivation of this product law see Appendix.) Analogously,

$$K_i \circ Y^j = Y^j \circ K_i, \quad i \neq j, \quad (3.19)$$

while the product law of $K_i$ with $Y_i$ at the same $i$ keeps the form (3.4) unchanged.

With this associative product the Lie bracket is defined as usual

$$[F, G] = F \circ G - G \circ F. \quad (3.20)$$
3.3 Higher-spin equations

Usual HS algebra is the Lie algebra associated with associative algebra $A^0$ defined in Section 3.1. Lie algebra $M_2$ contains usual HS algebra as a subalgebra since, as is easy to see from (3.15),

$$[F(Y), G(Y)]_0 = [F(Y), G(Y)]_* .$$ (3.21)

(To simplify notations we will use the same letter for an associative algebra and associated Lie algebra.) Let us write the unfolded equations (for more detail on the unfolded formulation see, e.g., [2]) for the Lie algebra $M_2$ to see the difference with the usual HS theory.

We will work with the system in $AdS$ background described in the two-component spinor notations by the connection

$$\omega_0 = -\frac{i}{4}(\omega_0(x)_{\alpha\beta}y_{\alpha}y_{\beta} + \overline{\omega}_0(x)_{\dot{\alpha}\dot{\beta}}\overline{y}_{\dot{\alpha}}\overline{y}_{\dot{\beta}} + 2\lambda h_{\delta\dot{\delta}}^\alpha(x)y_{\alpha}\overline{y}_{\dot{\beta}})$$ (3.22)

obeying the $sp(4)$ flatness conditions

$$d\omega_o + \omega_o \star \omega_o = 0 , \quad d := dx^n \frac{\partial}{\partial x^n} .$$ (3.23)

The latter can be solved, for instance, in Poincaré coordinates

$$h_{\delta\dot{\delta}}^\alpha(x) = z^{-1} g_{\delta\dot{\delta}}^\alpha , \quad \omega_{\dot{\alpha} \dot{\alpha}}^\alpha = -\lambda^2 z^{-1} g_{\beta\dot{\alpha}}^\alpha x^\alpha$$ (3.24)

where we use notation assuming symmetrization over upper indices denoted by the same letter $\alpha$.

The dynamical fields we are interested in are described by the zero-form

$$C_{hs} = (C_{hs}(Y|x), C_{hs}(Y^1; Y^2|x))$$ (3.25)

and one-form

$$\omega_{hs} = (\omega_{hs}(Y|x), \omega_{hs}(Y^1; Y^2|x))$$ (3.26)

where the rank-two fields $\omega_{hs}(Y^1; Y^2|x)$ and $C_{hs}(Y^1; Y^2|x)$ obey conditions (3.9) and (3.11) with respect to each argument

$$\omega_{hs}(Y_1; K_1; Y_2; K_2|x) = \omega_{hs}(Y_1; -K_1; Y_2; K_2|x) = \omega_{hs}(Y_1; K_1; Y_2; -K_2|x) ,$$ (3.27)

$$C_{hs}(Y_1; K_1; Y_2; K_2|x) = -C_{hs}(Y_1; -K_1; Y_2; K_2|x) = -C_{hs}(Y_1; K_1; Y_2; -K_2|x).$$ (3.28)

Let us start with the equations on zero-forms in the twisted adjoint HS module setting

$$C_{hs}(Y; K|x) := \sum_{p+q=1} C_{pq}(Y|x)k^p\overline{k}^q ,$$ (3.29)

$$C_{hs}(Y^1, Y^2; K^1; K^2|x) := \sum_{p_1+q_1=1, p_2+q_2=1} C_{p_1p_2q_1q_2}(Y^1, Y^2|x)k^1 p_1 k^2 p_2 \overline{k}^1 q_1 \overline{k}^2 q_2 .$$ (3.30)
The field equations on the zero-forms \( C_{hs} \) read as

\[
dC_{hs} + \omega_0 \circ C_{hs} - C_{hs} \circ \omega_0 = 0, \tag{3.31}
\]

where

\[
C_{hs} = C_{hs}(Y; K|x) + C_{hs}(Y^1; Y^2; K^1; K^2|x). \tag{3.32}
\]

From (3.31) we obtain that the equation in the \( p + q = 1 \) sector of \( C_{hs}(Y; K|x) \) is

\[
D_{tu} C_{pq}(Y|x) = 0, \tag{3.33}
\]

where

\[
D_{tu} C_{pq}(Y|x) := D^L C_{pq}(Y|x) - i \lambda h_0^{\alpha \beta} (y_\alpha \overline{y}_\beta - \partial_\alpha \overline{\partial}_\beta) C_{pq}(Y|x), \tag{3.34}
\]

\[
D^L C_{pq}(Y|x) := dC_{pq}(Y|x) + (\omega_0^{\alpha \beta} y_\alpha \partial_\beta + \overline{\omega}_0^{\alpha \beta} \overline{y}_\alpha \overline{\partial}_\beta) C_{pq}(Y|x). \tag{3.35}
\]

Equation (3.33) is the usual equation for massless fields in the twisted adjoint representation obtained originally in [13] (see also [2]).

In sector \( p_1 + q_1 = 1, p_2 + q_2 = 1 \) we get

\[
(D^L - i \lambda h_0^{\alpha \beta} (y_\alpha \overline{y}_\beta + y_\beta \overline{y}_\alpha - \partial_\alpha \overline{\partial}_\beta - \partial_\alpha \overline{y}_\beta) + \eta \Gamma_{loc}(C(Y^1; Y^2); Y) + \bar{\eta} \Gamma_{loc}(C(Y^1; Y^2); Y) - \mathcal{H}_{loc}(C(Y^1; Y^2); Y)) = 0. \tag{3.36}
\]

This equation exactly matches the rank-two equation on currents of [11]. Thus the part of zero-forms with two arguments can be interpreted as describing symmetrized conserved currents.

Since field equations (3.33) and (3.36) are insensitive to the choice of \( p, p_1, p_2, q, q_1, q_2 \) obeying \( p + q = 1, p_1 + q_1 = 1, p_2 + q_2 = 1 \) in the sequel these labels will be discarded.

The current deformation of the HS equations found in [12], suggests the deformed form of HS equations relating zero-forms to one-forms

\[
\begin{cases}
D_{ad} \omega(Y) = L(C(Y)) + Q(C(Y), \omega(Y)) + \Gamma_{loc}(C(Y^1; Y^2); Y), \\
D_{tu} C(Y) + [\omega, C(Y)]_\ast = -\mathcal{H}_{loc}(C(Y^1; Y^2); Y) - \mathcal{H}_{loc}(C(Y^1; Y^2); Y),
\end{cases} \tag{3.37}
\]

where

\[
D_{ad} \omega = D^L \omega + \lambda h_0^{\alpha \beta} (y_\alpha \overline{y}_\beta + \partial_\alpha \overline{\partial}_\beta) \omega, \tag{3.38}
\]

\[
L(C(Y)) = \frac{i}{4} \left( \eta \Gamma^{\alpha \beta \gamma} \overline{\partial}_\alpha \overline{\partial}_\beta C(0, \overline{y}|x) + \bar{\eta} \Gamma^{\alpha \beta \gamma} \partial_\alpha \partial_\beta C(y, 0|x) \right), \tag{3.39}
\]

\[
Q(C, \omega) = \eta \int dSdT \exp(iS_A T^A) \int_0^1 d\tau \left( h(t, \tau \overline{t} - \overline{y}) (1 - \tau) y + s, \overline{y} + \overline{\tau} \right) C(\tau t, \overline{y} + \overline{\tau}|x) + \\
+ h(s, \overline{\tau} - \overline{t}) C(-\tau s, \overline{y} + \overline{\tau}|x) \omega(-(1 - \tau) y - t, \overline{y} + \overline{\tau}) + c.c., \tag{3.40}
\]

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\[ \Gamma_{\mu\nu}^i(C(Y^1; Y^2); Y) = \frac{i}{8} \eta \int \frac{d^4 \tau}{\tau^4} \delta(1 - \tau_3 - \tau_4) \delta'(1 - \tau_1 - \tau_2) \theta(\tau_1) \theta(\tau_2) \theta(\tau_3) \theta(\tau_4) \]

\[ \left( \overline{H}^{\alpha\beta} \partial_\alpha \partial_\beta \exp(i \tau_3 \overline{\partial}_1 \partial_2 \overline{\partial}_2) C(\tau_1 y, \tau_4 \tau_2 \overline{y}, -\tau_2 y, -\tau_4 \tau_1 \overline{y}) + \right. \\

\left. + H^{\alpha\beta} \partial_\alpha \partial_\beta \exp(i \tau_3 \partial_1 \partial_2 \overline{\partial}_2) C(\tau_4 \tau_1 y, \tau_2 \overline{y}, -\tau_4 \tau_2 y, -\tau_1 \overline{y}) \right), \quad (3.41) \]

\[ \mathcal{H}_{\text{loc}}^\text{cur} (C(Y^1; Y^2); Y) = \frac{1}{2} \eta \exp(i \overline{\partial}_1 \partial_2 \overline{\partial}_2) \int_0^1 d\tau h(y, (1 - \tau) \overline{\partial}_1 - \tau \overline{\partial}_2) C(\tau y, \overline{y}, -(1 - \tau) y, \overline{y}), \quad (3.42) \]

\[ \mathcal{H}_{\overline{\text{loc}}}^\text{cur} (C(Y^1; Y^2); Y) = \frac{1}{2} \overline{\eta} \exp(i \partial_1 \overline{\partial}_2 \partial_2) \int_0^1 d\tau h((1 - \tau) \partial_1 - \tau \partial_2, \overline{y}) C(y, \tau \overline{y}, y, -(1 - \tau) \overline{y}) \quad (3.43) \]

where \( \eta \) is a complex deformation parameter, \( \partial_\alpha \) and \( \overline{\partial}_\alpha \) are derivatives over the \( i \)th undotted and dotted spinor arguments, respectively, with upper indices and

\[ h(a, b) := h_0^{\alpha\beta} a_\alpha b_\beta, \quad H^{\alpha\beta} := h_0^{\alpha\bar{\alpha}} h_0^{\beta\bar{\beta}}, \quad \overline{H}^{\bar{\alpha}\bar{\beta}} := h_0^{\bar{\alpha}\bar{\alpha}} h_0^{\beta\bar{\beta}}. \quad (3.44) \]

The difference between the system (3.37) and that of [12] is that in our case the rank-two fields \( C(Y^1; Y^2) \) are algebraically independent from the rank-one fields \( C(Y) \) while in [12] the former were realized as bilinear combinations of the latter. Hence, in [12] the rank-two fields were indeed describing the conformal currents built from the dynamical rank-one fields \( C(Y) \). In the system described in this section the fields \( C(Y^1; Y^2) \) that appear exactly on the same place as usual currents are independent fields that are expressed via derivatives of \( C(Y) \) by virtue of equations (3.37). In fact, this means that the resulting system can be interpreted as a conformal off-shell deformation of the usual massless equations. As such it is anticipated to be related to the off-shell HS equations proposed recently in [15].

Indeed, for instance in the spin-one sector the deformation of Maxwell equations is

\[ \partial_\nu F^{\mu\nu} = 4\pi J^\mu, \quad F = dA. \quad (3.45) \]

In the system considered in this paper \( J^\mu \) is a component of the rank-two field \( C(Y^1; Y^2) \) independent of the rank-one fields \( C(Y) \) where the two-form field strength \( F \) lives as well as all other massless fields of the system. Hence, equation (3.45) simply expresses the rank-two field \( J^\mu \) via the rank-one field \( F(Y) \). In [12] the currents like \( J^\mu \) were not independent, being expressed via bilinears of the rank-one fields \( C(Y) \) thus describing a nonlinear deformation of the equations of motion for massless fields.

Note that formal consistency of Maxwell equations demands the current \( J \) be conserved

\[ \partial_\nu J^\nu = 0. \quad (3.46) \]

The situation with higher spins is analogous. The respective HS conservation conditions are the only differential conditions obeyed by the primary current fields obeying the rank-two equations [11]. More precisely, the deformation (3.37) is conformal invariant which
means that the physical HS currents are traceless while the modules of the Poincare algebra considered in [15] include the traceful components of the conserved currents as well. As shown in [15], the traceful components of the currents do not contribute to the equations for the traceless ones. This means that the system considered in this paper remains off-shell in the conformal zero-form sector. It would be interesting to reach more precise interpretation of the results of [15] in terms of conformal symmetry.

Group theoretically equations (3.37) describe a nontrivial deformation of the direct sum of the massless modules realized by the rank-one fields \( C(Y) \) and the rank-two field \( C(Y_1; Y^2) \) with the deformation parameter \( \eta \). At \( \eta = 0 \) the system decomposes into two independent subsystems of rank-one and rank-two fields while at \( \eta \neq 0 \) the system describes the semi-direct indecomposable sum of the two models, becoming off-shell.

Let us note that the full set of field variables also includes fields \( \omega(Y^1; Y^2) \). The undeformed equation for the one-forms starts with the flatness condition

\[
\text{d}\omega(Y^1; Y^2) + \omega(Y^1; Y^2) \circ \wedge \omega(Y^1; Y^2) = 0. \tag{3.47}
\]

Its linearized version resulting from (3.15)-(3.17) reads as

\[
D^\ell \omega(y^1, \bar{y}^1; y^2, \bar{y}^2) + \lambda h_{\alpha\beta} (y^1_{\alpha} \bar{\partial}_{\beta} + \bar{y}^1_{\alpha} \partial_{\beta} + y^2_{\alpha} \bar{\partial}_{\beta} + \bar{y}^2_{\alpha} \partial_{\beta}) \omega(y^1, \bar{y}^1; y^2, \bar{y}^2) = 0. \tag{3.48}
\]

We anticipate that the nontrivial deformation of these equations including equations (3.37) will result from the nonlinear equations of [9] very much as the nonlinear deformation of usual HS equations was derived in [12] from the nonlinear HS equations of [14].

4 \( M^2(A) \) system

In this section we analyse the dynamical system associated with the algebra \( M^2(A) \). As explained in Section 2, algebra \( M^2(A) \) acts on fields depending on two spinor variables \( F(Y^1; Y^2) \). This raises the question whether it is possible to identify some of these fields with the rank-one fields describing massless fields such as background \( AdS \) connection \( \omega_0 \).

First we observe that Lie algebra \( M^2(A) \) contains a subalgebra of polynomials of the form \( F(Y^1; Y^2) = f(Y^1) + f(Y^2) \). Indeed,

\[
[f(Y^1) + f(Y^2), g(Y^1) + g(Y^2)]_0 = \\
= f(Y^1) \ast_{11} g(Y^1) + f(Y^1)g(Y^2) + f(Y^2)g(Y^1) + f(Y^2) \ast_{22} g(Y^2) - \\
- g(Y^1) \ast_{11} f(Y^1) - g(Y^1)f(Y^2) - g(Y^2)f(Y^1) - g(Y^2) \ast_{22} f(Y^2) = \\
= h(Y^1) + h(Y^2), \tag{4.1}
\]

\[
h(Y) = [f(Y), g(Y)]_*. \tag{4.2}
\]

This suggests to try to identify the rank-one fields as

\[
\omega_0(Y^1; Y^2) = \omega_0(Y^1) + \omega_0(Y^2). \tag{4.3}
\]
However, this does not work in the zero-form sector. Indeed, let us decompose the zero-form \( C \) into two parts: one containing only polynomials of \( Y^1 \) or \( Y^2 \) separately and the other one containing elements that depend nontrivially on both \( Y^1 \) and \( Y^2 \),

\[
C(Y^1; Y^2) = C^1(Y^1) + C^1(Y^2) + C^2(Y^1; Y^2), \quad C^2(Y^1; 0) = C^2(0; Y^2) = 0, \quad (4.4)
\]

i.e.,

\[
C^1(Y^i) = \frac{1}{2i} \sum_{m,n \geq 0} \frac{1}{m!n!} C_{\alpha_1...\alpha_n,\beta_1...\beta_m}^1 \dot{y}^{\alpha_1} \cdot ... \cdot \dot{y}^{\alpha_n} \beta_1 \cdot ... \cdot \beta_m, \quad i = 1, 2, \quad (4.5)
\]

\[
C^2(Y^1; Y^2) = \frac{1}{2i} \sum_{m+n>0 \atop l+p>0} \frac{1}{m!n!l!p!} C_{\alpha_1...\alpha_n,\beta_1...\beta_m,\gamma_1...\gamma_l,\delta_1...\delta_p}^2 \dot{y}^{\alpha_1} \cdot ... \cdot \dot{y}^{\alpha_n} \beta_1 \cdot ... \cdot \beta_m \gamma_1 \cdot ... \cdot \gamma_l \delta_1 \cdot ... \cdot \delta_p. \quad (4.6)
\]

Then the equation in the zero-form sector is

\[
dC + \omega_0 \circ C - C \circ \omega_0 = 0, \quad (4.7)
\]

where

\[
\tilde{f}(y, \overline{y}) := f(y, -\overline{y}). \quad (4.8)
\]

This gives three different equations on \( C^1(Y^1) \), \( C^1(Y^2) \) and \( C^2(Y^1; Y^2) \):

\[
0 = D_{tu}^1 C^1(Y^1) - \frac{i\lambda \hbar_0^{\alpha\beta}}{m!} \sum_{m+n>0} C_{\alpha_1...\alpha_n,\beta_1...\beta_m,\alpha,\beta}^2 \dot{y}^{\alpha_1} \cdot ... \cdot \dot{y}^{\alpha_n} \beta_1 \cdot ... \cdot \beta_m, \quad (4.9)
\]

\[
0 = D_{tu}^2 C^1(Y^2) - \frac{i\lambda \hbar_0^{\alpha\beta}}{m!} \sum_{m+n>0} C_{\alpha_1...\alpha_n,\beta_1...\beta_m,\alpha,\beta}^2 \dot{y}^{\alpha_1} \cdot ... \cdot \dot{y}^{\alpha_n} \beta_1 \cdot ... \cdot \beta_m, \quad (4.10)
\]

\[
0 = (D^L - i\lambda \hbar_0^{\alpha\beta} (y^{1}_{\alpha} \overline{y}^{\beta}_{\beta} + y^{2}_{\alpha} \overline{y}^{\beta}_{\beta} - \partial^{1}_{\alpha} \overline{y}^{\beta}_{\beta} - \partial^{2}_{\alpha} \overline{y}^{\beta}_{\beta})) C^2(y^1, \overline{y}^1; y^2, \overline{y}^2) - i\lambda \hbar_0^{\alpha\beta} \left( y^{1}_{\alpha} \overline{y}^{\beta}_{\beta} C^1(Y^1) + y^{2}_{\alpha} \overline{y}^{\beta}_{\beta} C^1(Y^1) \right), \quad (4.11)
\]

where

\[
C^2(y^1, \overline{y}^1; y^2, \overline{y}^2) =
\]

\[
= \sum_{m+n>0 \atop l+p>0} \frac{1}{m!n!l!p!} C_{\alpha_1...\alpha_n,\beta_1...\beta_m,\gamma_1...\gamma_l,\delta_1...\delta_p}^2 \dot{y}^{\alpha_1} \cdot ... \cdot \dot{y}^{\alpha_n} \beta_1 \cdot ... \cdot \beta_m \gamma_1 \cdot ... \cdot \gamma_l \delta_1 \cdot ... \cdot \delta_p. \quad (4.12)
\]

\[12\]
The first and second equations look analogously to the equations on zero-forms with nontrivial right hand side. These equations are formally consistent as follows from their derivation and can also be checked directly. Naively, one might think that they also describe an off-shell version of the massless system. This is however not the case because of the last term in the third equation which means that the fields \( C^1 \) source \( C^2 \). In turn, this means that the rank-two fields \( C^2(Y^1; Y^2) \) form an irreducible module that admits no limiting procedure allowing to put the massless system on the mass shell. This fact is in agreement with the analysis of [10] where it was explained that rank-two fields carry lowest energies inappropriate for the description of massless fields. The field \( C^2(Y^1; Y^2) \) describes the module of conserved currents which are not linked to the massless fields as a result of factorization of the ideal in \( M_2 \). Note that, as mentioned in Section 2, \( M^2 \) is isomorphic to the Weyl algebra with the doubled set of generators (oscillators).

5 Conclusion

In this paper we have derived the product law for the simplest multi-particle HS algebra \( M_2(A) \) and its factor-algebra \( M^2(A) \) with the rank-one fields factored out. We analysed the equations of the system resulting from these algebras. We conclude that the system for \( M_2(A) \) describes the off-shell HS systems with the rank-two fields describing conformal off-shell degrees of freedom of the originally massless system. On the other hand, the system resulting from \( M^2(A) \) describes the infinite system of conserved currents unrelated to massless fields. Note that the off-shell completion of massless field equations was recently proposed in [15]. It would be interesting to compare the construction of [15] with the one found in this paper and, especially, to clarify the role of the traceful components of the currents considered in [15].

Analogously, one can consider multi-particle algebras of higher ranks \( k \). We expect that unfactorised algebras also describe off-shell HS systems. One of the features illustrated by the analysis of this paper is that all multi-particle algebras of finite ranks have the property that higher-rank fields do not contribute to the field equations of lower-rank fields. However, as shown in the same papers, this is not true for the full multi-particle algebra of infinite rank. This property is crucial for the spontaneous breaking of HS symmetries.

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Appendix. Derivation of $M_2(A)$ higher-spin product law

Let us explicitly write down the realization of the elements of the algebra $M_2(A)$ in terms of elements of the HS algebra $A$.

The basis of $M_2(A)$ consists of the elements

$$1, \quad \alpha_i, \quad \alpha_i \cdot \alpha_j = \alpha_j \cdot \alpha_i, \quad i, j = 1, \ldots, n.$$  \hfill (5.1)

Here the dot product replaces indices 1 and 2 of the $y$-arguments and is introduced to distinguish it from the product of elements of the original HS star-product algebra realized by the polynomials of one spinor argument $F(Y)$.

Other way around the dot product can be replaced by rewriting formulas using two arguments $y^1$ and $y^2$. For instance, for the element $\alpha_i \cdot \alpha_j$ with $\alpha_i = F(y)$ and $\alpha_j = G(y)$

$$\alpha_i \cdot \alpha_j \sim F(y^1)G(y^2) + F(y^2)G(y^1).$$  \hfill (5.2)

Then the product law can be checked easily for any monomial

$$(y_{i_1} \ldots y_{i_n} \cdot y_{j_1} \ldots y_{j_m} \circ y_{k_1} \ldots y_{k_p} \cdot y_{l_1} \ldots y_{l_o}) =$$

$$(y_{i_1} \ldots y_{i_n} \circ y_{j_1} \ldots y_{j_m} \circ y_{k_1} \ldots y_{k_p} \cdot y_{l_1} \ldots y_{l_o}) + (y_{i_1} \ldots y_{i_n} \circ y_{j_1} \ldots y_{j_m} \cdot y_{k_1} \ldots y_{k_p} \circ y_{l_1} \ldots y_{l_o}).$$  \hfill (5.3)

The same can be done in terms of $y^1$ and $y^2$ using (3.17)

$$\left((y^1_{i_1} \ldots y^1_{i_n})(y^2_{j_1} \ldots y^2_{j_m}) + (y^2_{i_1} \ldots y^2_{i_n})(y^1_{j_1} \ldots y^1_{j_m})\right) \circ \left((y^1_{k_1} \ldots y^1_{k_p})(y^2_{l_1} \ldots y^2_{l_o}) + (y^2_{k_1} \ldots y^2_{k_p})(y^1_{l_1} \ldots y^1_{l_o})\right) =$$

$$= (y^1_{i_1} \ldots y^1_{i_n} \circ y^2_{j_1} \ldots y^2_{j_m}) + (y^1_{i_1} \ldots y^1_{i_n} \circ y^2_{j_1} \ldots y^2_{j_m} \circ y^2_{k_1} \ldots y^2_{k_p}) +$$

$$+(y^1_{i_1} \ldots y^1_{i_n} \circ y^1_{j_1} \ldots y^1_{j_m} \circ y^2_{k_1} \ldots y^2_{k_p}) + (y^1_{i_1} \ldots y^1_{i_n} \circ y^1_{j_1} \ldots y^1_{j_m} \circ y^2_{k_1} \ldots y^2_{k_p} \circ y^2_{l_1} \ldots y^2_{l_o}) +$$

$$+(y^1_{i_1} \ldots y^1_{i_n} \circ y^2_{j_1} \ldots y^2_{j_m} \circ y^2_{k_1} \ldots y^2_{k_p} \circ y^2_{l_1} \ldots y^2_{l_o}) + (y^2_{j_1} \ldots y^2_{j_m} \circ y^1_{k_1} \ldots y^1_{k_p} \circ y^2_{l_1} \ldots y^2_{l_o}).$$  \hfill (5.4)

So the right hand sides of (5.3) and (5.4) coincide up to equivalence (5.2). This proves (3.17).

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