Multicomponent Painlevé ODEs and related nonautonomous KdV stationary hierarchies

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Abstract
First, starting from two hierarchies of autonomous Stäckel ordinary differential equations (ODEs), we reconstruct the hierarchy of Korteweg de Vries (KdV) stationary systems. Next, we deform considered autonomous Stäckel systems to nonautonomous Painlevé hierarchies of ODEs. Finally, we reconstruct the related nonautonomous KdV stationary hierarchies from respective Painlevé systems.

KEYWORDS
Painlevé equations, soliton equations, Stäckel equations

1 | INTRODUCTION

Two particular classes of second-order nonlinear ordinary differential equations (ODEs) playing important role in a variety of branches of modern mathematics and physics. The first class is represented by separable (Stäckel) equations with autonomous Hamiltonian representations. The second class is represented by Painlevé equations with nonautonomous Hamiltonian representations. Thus, both types of ODE's can be alternatively considered as respective autonomous and nonautonomous Hamiltonian dynamical systems. The Stäckel equations can be written in the so-called Lax representation in the form of isospectral deformation equations, whereas the Painlevé equations can be written in the Lax representation in the form of isomonodromic deformation equations. Both, separable and Painlevé equations, appear in a wide range of applications in physics and mathematics, so are definitely worth of investigation.

A significant progress in construction of new Stäckel and Painlevé equations took place because the modern theory of nonlinear integrable partial differential equations (PDEs) has been born (the so-called soliton theory). It was found that both types of equations are inseparably connected with the soliton systems with whom they share many properties. Actually, they have been constructed under particular reductions of soliton PDEs.
The systematic construction of Stäckel systems of arbitrary degrees of freedom from stationary flows and restricted flows of soliton hierarchies as well as constrained flows of respective Lax hierarchies is nowadays well developed (see also review of these methods in Ref. 5 and references therein). A bit less is known about similar constructions of Painlevé systems with arbitrary number of degrees of freedom. Nevertheless, many interesting results the reader can find in Refs. 6–16. In this article, we present an inverse approach to the relation between Painlevé ODEs and soliton PDEs on the example of the KdV family. Actually, from particular hierarchies of Painlevé systems, we construct the related hierarchies of nonautonomous and nonhomogeneous deformations of the KdV PDEs.

Let me briefly sketch this idea. Recently, we have developed a deformation theory of autonomous Stäckel equations to nonautonomous Painlevé equations. To be more precise, the literature was presented so far several constructions of Painlevé hierarchies with increasing number of degrees of freedom. What important, each number of degrees of freedom was related with a single equation. In that sense, our deformation approach contains more. We start from a hierarchy of autonomous separable systems with increasing number of degrees of freedom, where a system of \( n \) degrees of freedom consists \( n \) commuting (i.e., Frobenious integrable) evolution equations. After deformation, we obtain a hierarchy of nonautonomous Painlevé systems with increasing number of degrees of freedom, where system of \( n \) degrees of freedom consists of \( n \), Frobenious integrable, nonautonomous Painlevé evolution equations. So, from now on, I will use the phrase a hierarchy of Painlevé systems rather than a Painlevé hierarchy known from the literature.

On the other hand, in articles, we have made an interesting observation that the complete soliton hierarchies can be reconstructed from particular finite-dimensional Stäckel systems, representing their stationary flows. In the following article, we take the advantages from that observation and taking the Painlevé deformations of Stäckel systems related to KdV stationary flows, and we construct related hierarchies of nonautonomous and nonhomogeneous deformations of the KdV hierarchy. Such new hierarchy seems to be interesting objects for further investigation, as their stationary flows reconstruct a particular hierarchies of Painlevé systems.

This paper is organized as follows. In Section 2, we briefly collect the necessary information on the KdV hierarchy. In Section 3, we reconstruct the KdV hierarchy from the hierarchy of Stäckel systems, which are representation of the KdV stationary systems related to the first KdV Hamiltonian structure. In Section 4, we do the same from the hierarchy of Stäckel systems that are representation of the KdV stationary systems related to the second KdV Hamiltonian structure. In Section 5, applying recently developed theory, we deform autonomous Stäckel hierarchies from Sections 3 and 4 to nonautonomous Painlevé hierarchies. Finally, in Section 6, we reconstruct, like in the autonomous case, two nonautonomous KdV hierarchies of stationary systems, related to particular nonhomogenous KdV hierarchies.

2 | KdV HIERARCHY

Let us remind some elementary facts about the KdV hierarchy, important for our further considerations. The KdV equation

\[
\frac{u_t}{u} = \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x
\] (1)
is a member of the following bi-Hamiltonian chain of nonlinear PDEs

\[ u_n = \mathcal{K}_n = \pi_0 dH_n = \pi_1 dH_{n-1}, \quad n = 1, 2, \ldots, \]

(2)

where two Poisson operators are

\[ \pi_0 = \partial_x, \quad \pi_1 = \frac{1}{4} \partial_x^3 + \frac{1}{2} u \partial_x + \frac{1}{2} \partial_x u. \]

(3)

The hierarchy (2) can be generated by a recursion operator and its adjoint in a following way:

\[ N = \pi_1 \pi_0^{-1} = \frac{1}{4} \partial_x^2 + u + \frac{1}{2} u \partial_x^{-1}, \quad N^\dagger = \frac{1}{4} \partial_x^2 + u - \frac{1}{2} \partial_x^{-1} u_x, \]

(4)

\[ \mathcal{K}_{n+1} = N^n \mathcal{K}_1, \quad \gamma_n = dH_n = (N^\dagger)^{n} \gamma_0, \quad n = 1, 2, \ldots \]

(5)

In particular, conserved one forms are

\[ \gamma_0 = 2, \]
\[ \gamma_1 = u, \]
\[ \gamma_2 = \frac{1}{4} u_{xx} + \frac{3}{4} u^2, \]
\[ \gamma_3 = \frac{1}{16} u_{4x} + \frac{5}{8} u u_{xx} + \frac{5}{16} u_x^2 + \frac{5}{8} u^3, \]
\[ \gamma_4 = \frac{1}{64} u_{6x} + \frac{7}{32} u u_{4x} + \frac{7}{16} u_x u_{3x} + \frac{21}{64} u_{2x}^2 + \frac{35}{32} u^2 u_{xx} + \frac{35}{32} u u_x^2 + \frac{35}{64} u^4, \]

\[ \vdots \]

and related symmetries are

\[ \mathcal{K}_1 = u_x, \]
\[ \mathcal{K}_2 = \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x, \]
\[ \mathcal{K}_3 = \frac{1}{16} u_{5x} + \frac{5}{8} u u_{3x} + \frac{5}{4} u_x u_{xx} + \frac{15}{8} u_x^2 u_x, \]
\[ \mathcal{K}_4 = \frac{1}{64} u_{7x} + \frac{7}{32} u u_{5x} + \frac{21}{32} u_x u_{4x} + \frac{35}{32} u_{xx} u_{3x} + \frac{35}{32} u_{x} u_{xx}^2 + \frac{35}{8} u u_{x} u_{xx} + \frac{35}{32} u^2 u_{3x} + \frac{35}{16} u^3 u_x, \]

\[ \vdots \]

As \( u \) belongs to the whole hierarchy (2), it depends on infinitely many evolution parameters: \( u = u(t_1, t_2, t_3, \ldots) \).

In addition, with the KdV, hierarchy of symmetries is related a hierarchy of master symmetries

\[ \sigma_m = N^{m+1} \sigma_{-1}, \quad \tau_{-1} = 1, \]

(8)
nonlocal in general
\[ \sigma_{-1} = 1, \]
\[ \sigma_0 = u + \frac{1}{2}xu_x, \]  
\[ \sigma_1 = \frac{1}{2}u_{xx} + \frac{1}{8}xu_{3x} + u^2 + \frac{1}{2}xu_xu_x + \frac{1}{4}u_x\delta_x^{-1}u, \]  
\[ \vdots \]
Both, symmetries \( \mathcal{K}_n (5) \) and master symmetries \( \sigma_m (8) \) constitute so-called Virasoro algebra (hereditary algebra)
\[ [\mathcal{K}_m, \mathcal{K}_n] = 0, \quad [\mathcal{K}_m, \mathcal{K}_n] = \begin{pmatrix} n - \frac{1}{2} \end{pmatrix} \mathcal{K}_{n+m}, \quad [\sigma_m, \sigma_n] = (n-m)\sigma_{n+m}. \]  
(10)
Alternatively, the hierarchy (2) can be reconstructed from the isospectral Lax representation. Actually, consider some eigenvalue problem together with time evolutions of its eigenfunctions
\[ L(u)\psi = \lambda \psi, \lambda_{i_n} = 0, \]
\[ \psi_{t_n} = B_n(u)\psi, n = 1, 2, \ldots, \]  
(11)
where \( L \) and \( B_r \) are some differential operators. The compatibility conditions (Frobenius integrability conditions) for (11) take the form
\[ L_{t_n} = [B_n, L], \quad n = 1, 2, \ldots, \]  
(12)
known as isospectral deformation equations, as the eigenvalues of the operator \( L \) are independent of all times \( t_r \), and are equivalent with evolutionary hierarchy of PDEs. For the KdV hierarchy,
\[ L = \partial_x^2 + u, \quad B_n = (L^{n-\frac{1}{2}})_{\geq 0} = \sum_{i=0}^{n-1} \left( -\frac{1}{4} \mathcal{K}_i + \frac{1}{2} \gamma_i \partial_x \right) L^{n-i-1}, \]  
(13)
where explicitly
\[ B_1 = \partial_x, \]
\[ B_2 = \partial_x^3 + \frac{3}{2}u \partial_x + \frac{3}{4}u_x, \]
\[ B_3 = \partial_x^5 + \frac{5}{2}u \partial_x^3 + \frac{15}{4}u_x^2 \partial_x^2 + \frac{5}{8}(3u^2 + 5u_{xx}) \partial_x + \frac{15}{16}(u_{3x} + 2u_x), \]
\[ \vdots \]
Isospectral deformation equations (12) and (13) can be presented in the equivalent form of so-called zero curvature equations, more suitable for our further considerations. Rewriting Equations (11) for the KdV hierarchy in the form
\[ \Psi_x = U(\lambda; u)\Psi, \quad \Psi = (\psi, \psi_x)^T \]  
(15)
the compatibility conditions for (15) and (16) take the form
\[
\frac{d}{dt_n} U - \frac{d}{dx} V_n + [U, V_n] = 0 \iff u_{t_n} = \mathcal{K}_n, n = 1, 2, \ldots, \tag{17}
\]
known as zero curvature conditions and reconstruct the KdV hierarchy. In (17), \(\frac{d}{dx}\) means the total \(x\)-derivative and \(\frac{d}{dt_n}\) means the \(t_n\)-evolutionary derivative. Actually, we have
\[
V_k = \begin{pmatrix}
-\frac{1}{2} P_x & P \\
P(\lambda - u) - \frac{1}{2} P_{xx} & 1 - \frac{1}{2} P_x
\end{pmatrix}, \quad P_k = \frac{1}{2} \sum_{i=0}^{k-1} \gamma_i \lambda^{k-i-1}. \tag{18}
\]
In consequence,
\[
V_1 = U = \begin{pmatrix}
0 & 1 \\
\lambda - u & 0
\end{pmatrix}, \quad V_2 = \begin{pmatrix}
-\frac{1}{4} u_x & \lambda + \frac{1}{2} u \\
\lambda^2 - \frac{1}{2} u \lambda - \frac{1}{2} u^2 - \frac{1}{4} u_{xx} & \frac{1}{4} u_x
\end{pmatrix}, \tag{19}
\]
\[
V_3 = \begin{pmatrix}
-\frac{1}{4} u_x \lambda - \frac{1}{16} (u_{3x} + 6u u_x) & \lambda^2 + \frac{1}{2} u \lambda + \frac{1}{8} (u_{xx} + 3u^2) \\
\lambda^3 - \frac{1}{2} u \lambda^2 - \frac{1}{8} (u_{xx} + u^2 ) \lambda - \left( \frac{1}{16} u_{4x} + \frac{1}{2} u u_{xx} + \frac{3}{8} u^2 + \frac{3}{8} u^3 \right) & \frac{1}{4} u_x \lambda + \frac{1}{16} (u_{3x} + 6u u_x)
\end{pmatrix}, \tag{20}
\]
Finally, the compatibility conditions between equations from (16) take the form of zero curvature conditions
\[
\frac{d}{dt_r} V_s - \frac{d}{dt_s} V_r + [V_s, V_r] = 0, \quad r, s = 1, 2, \ldots \tag{21}
\]
and are valid as differential consequences of commutativity of \(r\)th and \(s\)th KdV vector fields.
Let us define the \(n\)th KdV stationary systems as\(^{22}\)
\[
u_{t_r} = \mathcal{K}_r, \quad 0 = \mathcal{K}_{n+1}, \quad r = 1, \ldots, n. \tag{22}
\]
The stationary restriction \(0 = \mathcal{K}_{n+1}\) provides constraint on the infinite-dimensional (functional) manifold, on which the KdV hierarchy is defined, reducing it to the finite-dimensional (stationary) submanifold \(M_n\) of dimension \((2n + 1)\), parameterized by jet coordinates \(u, u_x, \ldots, u_{(2n)x}\). The system (22) has two different \(2n\)-dimensional representations, related to two different foliations of \(M_n\). The first one is of the form
\[
u_{t_r} = \mathcal{K}_r, \quad 0 = \gamma_{n+1} + c, \quad r = 1, \ldots, n, \quad c \in \mathbb{R}, \tag{23}
\]
where last equation in (23) is integrated an \((n + 1)\) stationary flow of the KdV hierarchy (2) in the first Hamiltonian representation
\[
0 = \mathcal{K}_{n+1} = \partial_x \gamma_{n+1} \implies 0 = \gamma_{n+1} + c. \tag{24}
\]
The second one is

\[ u_r = K_r, \ 0 = \frac{1}{2} \gamma_n' \gamma_n'' - \frac{1}{4} \left( \gamma_n' \right)^2 + u \gamma_n^2 + c, \ r = 1, \ldots, n, \]  

(25)

where the last equation in (25) is integrated an \((n + 1)\) stationary flow of the hierarchy (2) in the second Hamiltonian representation. Indeed, differentiating it by \(x\) and dividing by \(2 \gamma_n\), we get

\[ 0 = \left( \frac{1}{4} \partial_x^3 + \frac{1}{2} u \partial_x + \frac{1}{2} \partial_x u \right) \gamma_n = K_{n+1}. \]  

(26)

Both representations constitute systems of ODEs with \(n\) degrees of freedom, on \(2n\)-dimensional phase space with jet coordinates \(u, u_x, \ldots, u_{(2n-1)x}\), as higher derivatives of \(u\) with respect to \(x\) are eliminated by the constraint (24) and (25), respectively.

Moreover, Lax representation of the system (23) is given by

\[ \frac{d}{dt} V_{n+1} = [V_r, V_{n+1}], \ r = 2, \ldots, n, \quad \frac{d}{dx} V_{n+1} = [V_1, V_{n+1}], \]  

(27)

following from (21), with constraint \(0 = \gamma_{n+1} + c\), imposed on \(V_{n+1}\), whereas Lax representation of the system (25) is given by (27) with constraint \(0 = \frac{1}{2} \gamma_n' \gamma_n'' - \frac{1}{4} \left( \gamma_n' \right)^2 + u \gamma_n^2 + c\), imposed on \(V_{n+1}\).

Stäckel representations of stationary systems (23) and (25) were derived in Ref. 22 and are presented in the next two sections.

3 HAMILTONIAN REPRESENTATION OF THE FIRST KdV HIERARCHY OF STATIONARY SYSTEMS

Let us briefly systematize known facts about the hierarchy of Stäckel systems, being the Hamiltonian representation of the KdV hierarchy of stationary systems (23).\(^2\)\(^5\)\(^20\)\(^22\)

Consider finite-dimensional separable Hamiltonian systems, so-called Stäckel systems, generated by the following hyperelliptic separation (spectral) curves on \((\lambda, \mu)\)-plane

\[ \lambda^{2n+1} + c \lambda^n + \sum_{r=1}^{n} h_r \lambda^{n-r} = \mu^2, \ n \in \mathbb{N}, \ c = \text{const.} \]  

(28)

For fix \(n \in \mathbb{N}\), consider a \(2n\)-dimensional Poisson manifold \((M, \pi)\) and a particular set \(\xi = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) \in M\) of Darboux (canonical) coordinates on \(M\), that is, \(\{\lambda_i, \lambda_j\}_\pi = \{\mu_i, \mu_j\}_\pi = 0, \{\lambda_i, \mu_j\}_\pi = \delta_{ij}\). By taking \(n\) copies of (28) at points \((\lambda, \mu) = (\lambda_i, \mu_i), i = 1, \ldots, n\), we obtain a system of \(n\) linear equations (separation relations) for \(h_r\)

\[ \lambda_i^{2n+1} + c \lambda_i^n + \sum_{r=1}^{n} h_r \lambda_i^{n-r} = \mu_i^2, \ i = 1, \ldots, n, \ n \in \mathbb{N}. \]  

(29)

Solving (29) with respect to \(h_r\) yields \(n\) functions (Hamiltonians) \(h_r\) on \((M, \pi)\)

\[ h_r = \sum_{i=1}^{n} (-1)^r \partial x_{s_i} \frac{\partial \mu_i^2}{\partial \lambda_i} \Delta_i + \sum_{i=1}^{n} (-1)^r \frac{\partial s_i}{\partial \lambda_i} \frac{\lambda_i^{2n+1} + c \lambda_i^n}{\Delta_i}. \]
\[ = \mu^T K_r G_0 \mu + V_r^{(2n+1)} + cV_r^{(n)}, \quad r = 1, \ldots, n, \] (30)

where

\[ G_0 = \text{diag}\left( \frac{1}{\Delta_1}, \ldots, \frac{1}{\Delta_n} \right), \quad \Delta_j = \prod_{k \neq j} (\lambda_j - \lambda_k), \] (31)

\[ K_r = (-1)^{r+1} \text{diag}\left( \frac{\partial s_r}{\partial \lambda_1}, \ldots, \frac{\partial s_r}{\partial \lambda_n} \right), \quad r = 1, \ldots, n, \] (32)

\( s_r \) are elementary symmetric polynomials in \( \lambda_i \) and \( V_r^{(k)}, \ k \in \mathbb{Z} \) are basic separable potentials. Matrix \( G_0 \) can be interpreted as a contravariant metric tensor on an \( n \)-dimensional manifold \( Q \) such that \( M = T^* Q \). It can be shown that the metric \( G_0 \) is flat. Matrices \( K_r \) are (1,1)-Killing tensors for the metric \( G_0 \).

Thus, we have constructed from scratch infinite hierarchy of separable autonomous Hamiltonian systems

\[ \xi_{t_r} = \pi dh_r = X_r, \quad r = 1, \ldots, n, \quad n \in \mathbb{N}, \] (33)

where each system consists of \( n \) evolution ODEs. The first Hamiltonian \( h_1 \) of each system can be interpreted as the Hamiltonian of a particle in the pseudo-Riemannian \( n \)-dimensional configuration space \( (Q, g_0 = G_0^{-1}) \), moving under particular potential force and the remaining Hamiltonians are its constants of motion.

By their very construction from separation relations, the Hamiltonians \( h_r \) Poisson commute for all \( r, k = 1, \ldots, n \)

\[ \{h_r, h_k\}_\pi = \pi(dh_r, dh_k) = 0, \] (34)

and so that \([X_r, X_k] = 0\). It means that each system is Liouville integrable and hence is Frobenius integrable, that is, the system (33) has a common, unique solution \( \xi(t_1, \ldots, t_n, \xi_0) \) through each point \( \xi_0 \in M \), depending in general on all the evolution parameters \( t_s \), constructed from separation relations (29) by quadratures.

In what follows we will work in so called Viéte canonical coordinates \((q, p) \in T^*Q\)

\[ q_i = (-1)^i s_i(\lambda), \quad p_i = -\sum_{k=1}^{n} \frac{\lambda_k^{n-i} \mu_k}{\Delta_k}, \quad i = 1, \ldots, n \] (35)

in which all functions \( h_r \) are polynomial functions of their arguments. Explicitly

\[ G_0^{ij} = q_{i+j-n-1}, \quad (K_r)_j^i = \begin{cases} q_{i+j-r-1}, & i \leq j \text{ and } r \leq j \\ -q_{i+j-r-1}, & i > j \text{ and } r > j \\ 0, & \text{otherwise,} \end{cases} \] (36)
where we set \( q_0 = 1, q_k = 0 \) for \( k < 0 \) or \( k > n \). Elementary separable potentials \( V^{(\alpha)}_r \) can be explicitly constructed by the recursion formula\(^{33}\)

\[
V^{(\alpha)}_r = R^\alpha V^{(0)}_r, \quad V^{(\alpha)} = (V^{(\alpha)}_1, \ldots, V^{(\alpha)}_n)^T, \quad R = \begin{pmatrix}
-q_1 & 1 & 0 & 0 \\
\vdots & 0 & \ddots & 0 \\
\vdots & 0 & 0 & 1 \\
-q_n & 0 & 0 & 0
\end{pmatrix}, \quad (37)
\]

with \( V^{(0)} = (0, \ldots, 0, -1)^T \). The first \( n \) basic separable potentials are trivial

\[
V^{(\alpha)}_k = -\delta_{k,n-\alpha}, \quad \alpha = 0, \ldots, n-1.
\]

The first nontrivial positive and negative potentials are

\[
V^{(n)} = (q_1, \ldots, q_n)^T, \quad V^{(-1)} = \begin{pmatrix} 1 \\ q_n \\ \vdots \\ q_n \end{pmatrix}^T,
\]

and higher positive and negative potentials are more complicated polynomials and rational functions in all \( q_i \).

Besides, the autonomous Hamiltonian equations (33) are represented by (i.e., are differential consequences of) Lax isospectral deformation equations

\[
\frac{d}{dr} L(\lambda; \xi) = [U_r(\lambda; \xi), L(\lambda; \xi)], \quad r = 1, \ldots, n, \quad (38)
\]

where \( \frac{d}{dr} \) is the evolutionary derivative along the \( r \)th flow from (33)

\[
\frac{d}{dr} A = \frac{\partial A}{\partial t_r} + \{A, h_r\} = \frac{\partial A}{\partial t_r} + A'[X_r], \quad (39)
\]

with \( L(\lambda; \xi) \) and \( U_r(\lambda; \xi) \) being matrices belonging to some Lie algebra and depending rationally on the spectral parameter \( \lambda \).

Actually, for the hierarchy generated by Hamiltonians (30), there is an infinitely many nonequivalent Lax representations.\(^{24}\) Here we chose the one compatible with (17)–(20).

**Theorem 1** (24). The \( L(\lambda; \xi) \) matrix for the system (33) takes a form

\[
L(\lambda; \xi) = \begin{pmatrix}
v(\lambda; \xi) & u(\lambda; \xi) \\
w(\lambda; \xi) & -v(\lambda; \xi)
\end{pmatrix}, \quad (40)
\]

where, in Viète coordinates \( \xi = (q, p) \), we have

\[
u(\lambda; q) = \lambda^n + \sum_{k=1}^{n} q_k \lambda^{n-k}, \quad v(\lambda; q, p) = -\sum_{k=1}^{n} M_k(q, p) \lambda^{n-k}, \quad M_k = \sum_{j=1}^{n} G^{kj}_q p_j \quad (41)
\]

and

\[
w(\lambda; q, p, t) = -\left[ \frac{v(\lambda)^2 - \lambda^{2n+1} - c \lambda^n}{u(\lambda)} \right]_+. \quad (42)
\]
Here, the operation \([\cdot]_+\) means the projection on the uniquely defined quotient of the division of an analytic function \(A\) over a (pure) polynomial \(u(\lambda)\) such that the following decomposition holds:

\[
A = \left[ \frac{A}{u} \right]_+ u + r,
\]

where the (unique) remainder \(r\) is a lower degree polynomial than the polynomial \(u\), see for details. In particular, when \(A\) is a Laurent polynomial, we have

\[
\left[ \frac{A}{u} \right]_+ \equiv \left[ \frac{[A]_{\geq 0}}{u} \right]_{\geq 0} + \left[ \frac{[A]_{< 0}}{u} \right]_{< 0},
\]

where \([\cdot]_{\geq 0}\) is the projection on the part consisting of nonnegative degree terms in the expansion into Laurent series at \(\infty\) and \([\cdot]_{< 0}\) is the projection on the part consisting of negative degree terms in the expansion into Laurent series at \(0\). Moreover,

\[
U_r(\lambda; \xi) = \left[ \frac{u_r(\lambda)L(\lambda)}{u(\lambda)} \right]_+, \quad r = 1, \ldots, n,
\]

where

\[
u_r(\lambda) = \lambda^{r-1} + \sum_{k=1}^{r-2} q_k \lambda^{r-k-1}.
\]

What is interesting, the separation curve (28) is reconstructed from Lax matrix \(L(\lambda; \xi)\) through

\[
\det[L(\lambda; \xi) - \mu I] = 0.
\]

The relation between the Stäckel hierarchy (33) and the KdV stationary hierarchy (23) is as follows.

**Theorem 2** (22). For fixed \(n \in \mathbb{N}\) and identification of \(t_1\) with \(x\), we get the following equivalence between the Stäckel system (33) and the \(n\)th KdV stationary system (23)

\[
\xi_{t_r} = X_r, \quad r = 1, \ldots, n
\]

\[
\forall\quad u_{t_r} = K_r, \quad 0 = \gamma_{n+1} + c, \quad r = 1, \ldots, n,
\]

where the transformation between jet and Viéte coordinates is as follows:

\[
q_k = \frac{1}{2} \gamma_k, \quad p_k = \frac{1}{2} \sum_{j=1}^{n} (G_0^{-1})_{kj}(q_j)_x, \quad k = 1, \ldots, n.
\]

In particular, the first flow \(\xi_x = X_1\) reconstructs (46) and the constraint

\[
0 = (p_1)_x - (X_1)^{n+1} \Leftrightarrow 0 = \gamma_{n+1} + c.
\]
Besides, the first component of the rth flow (44) reconstructs the rth KdV equation
\[(q_1)_r = (X_r)^1 \Leftrightarrow u_{t_r} = K_r, r = 1, \ldots, n,\] (48)
while the remaining components of systems (44) for \(r = 2, \ldots, n\) are differential consequences of (47) and (48). On the level of Lax representation of the Stäckel hierarchy (38) and the Lax representation (27) of the KdV stationary system (23), fixing \(n\), we have the following identities:
\[U_i = V_i, \quad i = 1, \ldots, n,\] (49)
and additionally,
\[L = V_{n+1} \quad \text{under constraint} \quad 0 = \gamma_{n+1} + c.\] (50)
Both systems (44) and (45) share the same solutions, which are the so-called finite gap solutions and rational solutions of related KdV hierarchy. \(^{20}\)

**Example 1.** Consider the case \(n = 3\). In \((q, p)\) coordinates, three Hamiltonians \(h_r = p^T K_r G_0 p + V_r^{(7)} + c V_r^{(3)}\), metric tensor \(G_0\), and its inverse are
\[
h_1 = (2p_1 p_3 + \frac{p_2^2}{2} + 2q_1 p_2 p_3 + q_2 p_3^2) + (q_1^5 - 4q_1^3 q_2 + 3q_1^2 q_3 + 3q_1 q_2^2 - 2q_2 q_3) + c q_1,\]
\[
h_2 = [2q_1 p_1 p_3 + 2q_1 p_2^2 + 2p_1 p_2 + (q_1 q_2 - q_3) p_3^2 + 2q_1^2 p_3 p_2]
+ (q_1^4 q_2 - q_1^3 q_3 - 3q_1^2 q_2^2 + 4q_1 q_2 q_3 + q_3^2 - q_2^2) + c q_2,\]
\[
h_3 = p_1^2 + 2q_1 p_1 p_2 + 2q_2 p_1 p_3 + q_1^2 p_2^2 + (q_2^2 - q_1 q_3) p_3^2 + 2(q_1 q_2 - q_3) p_2 p_3
+ (q_1^4 q_3 - 3q_1^2 q_2 q_3 + 2q_1 q_2^2 + q_2^2 q_3) + c q_3,\]
\[
G_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & q_1 \\ 1 & q_1 & q_2 \end{pmatrix}, \quad G_0^{-1} = \begin{pmatrix} q_1^2 - q_2 & -q_1 & 1 \\ -q_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

The related autonomous evolution equations are
\[
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = X_1 = \begin{pmatrix} 2p_3 \\ 2q_1 p_3 + 2p_2 \\ 2q_1 p_2 + 2q_2 p_3 + 2p_1 \\ -2p_2 p_3 - 5q_1^4 + 12q_1^2 q_2 - 6q_1 q_3 - 3q_2^2 - c \\ -p_2^2 + 4q_1^3 - 6q_1 q_2 + 2q_3 \\ -3q_1^2 + 2q_2 \end{pmatrix}, \quad (51)
\]
\[
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = X_2 = \begin{pmatrix} 2p_2 + 2q_1 p_3 \\ 2p_1 + 2q_1^2 p_3 + 4q_1 p_2 \\ 2q_1 p_1 + 2(q_1 q_2 - q_3) p_3 + 2q_1^2 p_2 \\ -2p_1 p_3 - 2p_2^2 - 4q_1 p_2 p_3 - 2q_2 p_3^2 + 6q_1 q_2^2 - 4q_1^3 q_2 + 3q_1^2 q_3 - 4q_2 q_3 \\ -q_1 p_3^2 - q_1^4 + 6q_1^2 q_2 - 4q_1 q_3 - 3q_2^2 - c \\ p_2^2 + q_1^3 - 4q_1 q_2 + 2q_3 \end{pmatrix}, \quad (52)
\]
\[
\begin{pmatrix}
q_1 \\
q_2 \\
q_3 \\
p_1 \\
p_2 \\
p_3
\end{pmatrix}
= X_3 = \begin{pmatrix}
2q_1p_2 + 2q_2p_3 + 2p_1 \\
2q_1^2p_2 + 2q_1p_1 + 2(q_1q_2 - q_3)p_3 \\
2q_2p_1 + 2(q_2^2 - q_1q_3)p_3 + 2(q_1q_2 - q_3)p_2 \\
-2q_1p_2^2 - 2q_2p_2p_3 + q_3p_3^2 - 4q_1^3q_3 + 6q_1q_2q_3 - 2p_1p_2 - 2q_3^2 \\
-2p_1p_2 - 2q_1p_2p_3 - 2q_2p_3^2 + 3q_3^2q_2 - 2q_2q_3 \\
q_1p_3^2 + 3q_1^2q_2 + 2p_2p_3 - q_4^4 - 4q_1q_3 - q_2^2 - c
\end{pmatrix}.
\]

Lax representations (38) of considered equations are as follows:

\[
L = \begin{pmatrix}
-p_3\lambda^2 - (q_1p_3 + p_2)\lambda - p_1 - q_1p_2 - q_2p_3 & \lambda^3 + q_1\lambda^2 + q_2\lambda + q_3 \\
\lambda^4 - q_1\lambda^3 - (q_2 - q_1q_3)\lambda^2 - (p_3^2 + q_1^2 - 2q_1q_2 + q_3)\lambda & p_3\lambda^2 + (q_1p_3 + p_2)\lambda + p_1 + q_1p_2 + q_2p_3 \\
-q_1p_2^2 - 2p_2p_3 + q_1^2 - 3q_3^2q_2 + 2q_1q_3 + q_2^2 + c & \lambda^3 - q_1\lambda^2 + (q_2^2 - q_1q_3)\lambda - p_2^2 - q_3^2 + 2q_1q_2 - 2q_3 - p_3\lambda + p_2 + q_1p_3
\end{pmatrix},
\]

\[
U_1 = \begin{pmatrix}
0 & 1 \\
\lambda - 2q_1 & 0
\end{pmatrix}, \quad U_2 = \begin{pmatrix}
-p_3 & \lambda + q_1 \\
\lambda^2 - q_1\lambda + q_1^2 & -2q_2
\end{pmatrix},
\]

\[
U_3 = \begin{pmatrix}
-p_3\lambda - p_2 - q_1p_3 & \lambda^2 + q_1\lambda + q_2 \\
\lambda^3 - q_1\lambda^2 + (q_2^2 - q_1q_3)\lambda - p_2^2 - q_3^2 + 2q_1q_2 - 2q_3 & p_3\lambda + p_2 + q_1p_3
\end{pmatrix}.
\]

Substituting \(q_1 = \frac{1}{2} \varphi_1 = \frac{1}{2} u\), we obtain recursively from (51)

\[
(q_1)_x = (X_1)^1 \iff p_3 = \frac{1}{4} u_x,
\]

\[
(p_3)_x = (X_1)^6 \iff q_2 = \frac{1}{8} u_{xxx} + \frac{3}{8} u^2 = \frac{1}{2} \varphi_2,
\]

\[
(q_2)_x = (X_1)^2 \iff p_2 = \frac{1}{16} u_{3x} + \frac{1}{4} u u_x,
\]

\[
(p_2)_x = (X_1)^5 \iff q_3 = \frac{1}{32} u_{4x} + \frac{5}{16} u u_{xx} + \frac{5}{32} u_x^2 + \frac{5}{16} u^3 = \frac{1}{2} \varphi_3,
\]

\[
(q_3)_x = (X_1)^3 \iff p_1 = \frac{1}{64} u_{5x} + \frac{9}{32} u_x u_{xx} + \frac{1}{8} u u_{3x} + \frac{1}{4} u^2 u_x,
\]

\[
(p_1)_x = (X_1)^4 \iff u_{6x} + \frac{7}{32} u u_{4x} + \frac{7}{16} u_x u_{3x} + \frac{21}{64} u_x^2 + \frac{35}{32} u_x^2 u_{xx} + \frac{35}{32} u_x u_{2x}^2 + \frac{35}{64} u_x^4 + c = \varphi_4 + c = 0,
\]

which coincide with (46) and the last equation of (45). From evolution equation (52), we get

\[
(q_1)_t = (X_2)^1 \iff u_{t_2} = \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x = \kappa_2.
\]
and the remaining equations are differential consequences of \( u_{t_2} = \mathcal{K}_2 \) and \( \gamma_4 + c = 0 \). For example,

\[
(q_3)_{t_2} = (X_2)^3 \iff (\gamma_4)_x = 0, (p_1)_{t_2} = (X_2)^4 \iff (\gamma_4)_{xx} = 0,
\]

and so on. Finally, from evolution equation (53), we get

\[
(q_1)_{t_3} = (X_3)^1 \iff u_{t_3} = \frac{1}{16} u_{5x} + \frac{5}{8} u_{3x} + \frac{5}{4} u_x u_{xx} + \frac{15}{8} u^2 u_x = \mathcal{K}_3,
\]

and again, the remaining equations are differential consequences of \( u_{t_3} = \mathcal{K}_3 \) and \( \gamma_4 + c = 0 \). Thus, indeed, we have the equivalence between both representations (44) and (45). Moreover, under above substitution, Stäckel Lax matrices \( U_r \) (55) turn into KdV matrices \( V_r \) (18)–(20), while \( L \) matrix (54) is related to the KdV matrix \( V_4 \) under constraint \( 0 = \gamma_4 + c \).

Summarizing results of this section, the \( n \)th KdV stationary system (23) has the \( n \)-dimensional Stäckel representation (33), generated by separation curve (28).

## 4 Hamiltonian Representation of the Second KdV Hierarchy of Stationary Systems

Let us briefly collect known facts about the hierarchy of Stäckel systems being the Hamiltonian representation of the KdV hierarchy of stationary systems (25).5,20,22

Consider Stäckel systems generated by the following separation curves on \((\lambda, \mu)\)-plane:

\[
\lambda^{2n} + c\lambda^{-1} + \sum_{r=1}^{n} h_r \lambda^{n-r} = \lambda \mu^2, \quad n \in \mathbb{N}.
\] (56)

Following the procedure from the previous section, we construct \( n \) Hamiltonian functions in involution of the form

\[
h_r = \sum_{i=1}^{n} (-1)^{r+1} \frac{\partial s_r}{\partial \lambda_i} \frac{\lambda_i \mu^2}{\Delta_i} + \sum_{i=1}^{n} (-1)^{r} \frac{\partial s_r}{\partial \lambda_i} \frac{\lambda_i^{2n} + c\lambda_i^{-1}}{\Delta_i}
\]

\[
= \mu^r K_r G_1 \mu + V_r^{(2n)} + cV_r^{(-1)}, \quad r = 1, \ldots, n,
\] (57)

where

\[
G_1 = \text{diag} \left( \frac{\lambda_1}{\Delta_1}, \ldots, \frac{\lambda_n}{\Delta_n} \right),
\] (58)

and Killing tensors \( K_r \) of \( G \) takes again the form (32). In Viéte coordinates (35),

\[
G_1^{ij} = \begin{cases} q_{i+j-n}, & i, j = 1, \ldots, n-1 \\ -q_n, & i = j = n. \end{cases}
\] (59)

Consider the hierarchy of Hamiltonian autonomous dynamical systems, where each system consists of \( n \) evolution equations

\[
\xi_{t_r} = X_r = \pi dh_r, \quad r = 1, \ldots, n, \quad n \in \mathbb{N}
\] (60)
generated by \( n \) Hamiltonian functions \( h_r \) (57). The related Lax isospectral equations are of the form (38), where \( L(\lambda; \xi) \) matrix takes the form (40), where now, in Viéte coordinates \( \xi = (q, p) \), we have\(^{24}\)

\[
  u(\lambda; q) = \lambda^n + \sum_{k=1}^{n} q_k \lambda^{n-k}, \quad v(\lambda; q, p) = -\sum_{k=1}^{n} M_k(q, p) \lambda^{n-k}, \quad M_k = \sum_{j=1}^{n} G_1^{kj} p_j
\]  

(61)

and

\[
  w(\lambda; q, p) = -\lambda \left[ \frac{v(\lambda)^2 \lambda - 1 - \lambda^2 n - c \lambda^{-1}}{u(\lambda)} \right].
\]  

(62)

Moreover, matrices \( U_r(\lambda; \xi) \) are given by the same formula (43).

Again, the separation curve (56) is reconstructed from Lax matrix \( L(\lambda; \xi) \) through

\[
  \det[L(\lambda; \xi) - \lambda \mu I] = 0.
\]

The relation between Stäckel hierarchy (60) and the second KdV stationary hierarchy (25) is as follows.

**Theorem 3** (22). For fixed \( n \in \mathbb{N} \) and identification \( t_1 \) with \( x \), we get the following equivalence between the Stäckel system (60) and the \( n \)th KdV stationary systems (25):

\[
  \xi_{t_r} = X_r, \quad r = 1, \ldots, n
\]  

(63)

\[
  \uparrow
\]

\[
  u_{t_r} = K_r, \quad 0 = \frac{1}{2} \gamma_n(y_n)_{xx} - \frac{1}{4}[(y_n)_x]^2 + uy_n^2 + c, \quad r = 1, \ldots, n,
\]  

(64)

where the transformation between jet and Viéte coordinates is as follows:

\[
  q_k = \frac{1}{2} \gamma_k, \quad p_k = \frac{1}{2} \sum_{j=1}^{n} (G_1^{-1})_{kj}(q_j)_x, \quad k = 1, \ldots, n,
\]

with new metric tensor \( G_1 \) (59). The constraint is encoded in the last component of the first flow (63)

\[
  0 = (p_n)_x - (X_1)^{2n} \iff 0 = \frac{1}{2} \gamma_n(y_n)_{xx} - \frac{1}{4}[(y_n)_x]^2 + uy_n^2 + c.
\]  

(65)

Besides,

\[
  (q_1)_{t_r} = X_1^r \iff u_{t_r} = K_r, \quad r = 1, \ldots, n
\]  

(66)

and remaining components from systems (63) for \( r = 2, \ldots, n \) are differential consequences of (65) and (66). On the level of Lax representation of the Stäckel hierarchy (38) and the Lax representation (27) of the KdV stationary system (25), fixing \( n \) we have the same identities (49) and now

\[
  L = V_{n+1} \quad \text{under constraint} \quad 0 = \frac{1}{2} \gamma_n(y_n)_{xx} - \frac{1}{4}[(y_n)_x]^2 + uy_n^2 + c.
\]  

(67)

Like in the previous case, both systems (63) and (64) share the same solutions.
Example 2. Consider once more the case \( n = 3 \). In \((q, p)\) coordinates, three Hamiltonians \( h_r = p^T K_r G_p + V^{(6)}_r + c V^{(-1)}_r \), metric tensor \( G \), and its inverse are

\[
h_1 = (2p_1 p_2 + q_1 p_2^2 - q_3 p_3^2) + (-q_1^4 + 3q_1^2 q_2 - 2q_1 q_3 - q_2^2) + c q_3^{-1},
\]

\[
h_2 = [p_1^2 + 2q_1 p_1 p_2 - 2q_3 p_2 p_3 + (q_1^2 - q_2)p_2^2 - q_1 q_3 p_3^3]
\]

\[
+ (-q_1^2 q_2 + q_1^2 q_3 + 2q_1 q_2^2 - 2q_2 q_3) + c q_1 q_3^{-1},
\]

\[
h_3 = (-2q_3 p_1 p_3 - q_3 p_2^2 - 2q_1 q_3 p_2 p_3 - 2q_2 q_3 p_3^2) + (-q_1^3 q_3 + 2q_1 q_2 q_3 - q_2^2) + c q_2 q_3^{-1},
\]

\[G_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & q_1 & 0 \\ 0 & 0 & -q_3 \end{pmatrix}, \quad G_1^{-1} = \begin{pmatrix} -q_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{q_3} \end{pmatrix}. \quad (68)\]

The related autonomous evolution equations are

\[
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = X_1 = \begin{pmatrix} 2p_2 \\ 2q_1 p_2 + 2p_1 \\ -2q_3 p_3 \\ -p_2^2 + 4q_1^2 - 6q_1 q_2 + 2q_3 \\ -3q_1^2 + 2q_2 \\ p_2^2 + 2q_1 + c q_3^{-2} \end{pmatrix}, \quad (69)\]

\[
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \big|_{t_2} = X_2 = \begin{pmatrix} 2p_1 + 2q_1 p_2 \\ 2q_1 p_1 - 2q_3 p_3 + 2(q_1^2 - q_2)p_2 \\ -2q_1 q_3 p_3 - 2q_3 p_2 \\ -2p_1 p_2 - 2q_1 p_2^2 + q_3 p_3^2 + 3q_1^2 q_2 - 2q_1 q_3 - 2q_2^2 - c q_3^{-1} \\ p_2^2 + q_1^2 - 4q_1 q_2 + 2q_3 \\ 2p_1 p_3 + q_1 p_3^2 - q_1^2 + 2q_2 + c q_1 q_3^{-2} \end{pmatrix}, \quad (70)\]

\[
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \big|_{t_3} = X_3 = \begin{pmatrix} -2q_3 p_3 \\ -2q_1 q_3 p_3 - 2q_3 p_2 \\ -2q_3 p_1 - 2q_1 q_3 p_2 - 2q_2 q_3 p_3 \\ 2q_3 p_2 p_3 + 3q_1^2 q_3 - 2q_2 q_3 \\ q_3 p_3^2 - 2q_1 q_3 - c q_3^{-1} \\ 2p_1 p_3 + p_2^2 + 2q_1 p_2 p_3 + q_2 p_3^2 + q_1^3 - 2q_1 q_2 + 2q_3 + c q_2 q_3^{-2} \end{pmatrix}. \quad (71)\]

Lax representations (38) of considered equations are as follows:

\[
L = \begin{pmatrix} -p_2 \lambda^2 - (q_1 p_2 + p_1) \lambda + q_3 p_3 \\ \lambda^4 - q_1 \lambda^3 - (q_2 - q_3^2) \lambda^2 - (p_2^2 + q_1^3 - 2q_1 q_2 + q_3^2) \lambda \\ -q_3 p_3^2 + c q_3^{-1} \end{pmatrix} \begin{pmatrix} \lambda^3 + q_1 \lambda^2 + q_2 \lambda + q_3 \\ \lambda^2 + (q_1 p_2 + p_1) \lambda - q_3 p_3 \\ p_2 \lambda^2 \end{pmatrix}, \quad (72)\]
\[
U_1 = \begin{pmatrix}
0 & 1 \\
\lambda & -2q_1 \\
\end{pmatrix},
U_2 = \begin{pmatrix}
-p_2 & \lambda + q_1 \\
\lambda^2 - q_1\lambda + q_2^2 & p_2 \\
\end{pmatrix},
U_3 = \begin{pmatrix}
-p_2\lambda - p_1 - q_1p_2 & \lambda^2 + q_1\lambda + q_2 \\
\lambda^3 - q_1\lambda^2 + (q_2^2 - q_2)\lambda - p_2^2 - q_1^2 + 2q_1q_2 - 2q_3 & p_2\lambda + p_1 + q_1p_2 \\
\end{pmatrix}.
\]

(73)

Substituting \( q_1 = \frac{1}{2} \gamma_1 = \frac{1}{2} u \), we obtain recursively from (69)

\[
(q_1)_x = (X_1)^1 \iff p_2 = \frac{1}{4} u_x,
\]

\[
(p_2)_x = (X_1)^5 \iff q_2 = \frac{1}{8} u_{xx} + \frac{3}{8} u^2 = \frac{1}{2} \gamma_2,
\]

\[
(q_2)_x = (X_1)^2 \iff p_1 = \frac{1}{16} u_{3x} + \frac{1}{4} u u_x,
\]

\[
(p_1)_x = (X_1)^4 \iff q_3 = \frac{1}{32} u_{4x} + \frac{5}{16} u u_{xx} + \frac{5}{32} u_x^2 + \frac{5}{16} u^3 = \frac{1}{2} \gamma_3,
\]

\[
(q_3)_x = (X_1)^3 \iff p_3 = -\frac{1}{2} \frac{(q_3)_x}{q_3} = -\frac{1}{2} \frac{\gamma_3_x}{\gamma_3},
\]

\[
(p_3)_x = (X_1)^6 \iff 0 = \frac{1}{2} q_3(q_3)_{xx} - \frac{1}{4} (q_3^2)_x + u q_3^2 + c \implies 0 = \pi_1 \gamma_3 = K_4,
\]

(74)

which coincide with (46) with new metric tensor (68) and the last equation of (64). From evolution equation (52), we get

\[
(q_1)_t = (X_2)^1 \iff u_t = \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x = K_2,
\]

(75)

and the remaining equations are differential consequences of (74) and (75). Finally, from evolution equation (71), we get

\[
(q_1)_t = (X_3)^1 \iff u_t = \frac{1}{16} u_{5x} + \frac{5}{8} u u_{3x} + \frac{5}{4} u_x u_{xx} + \frac{15}{8} u^2 u_x = K_3,
\]

(76)

and again, the remaining equations are differential consequences of (74) and (76). Moreover, under above substitution, Stäckel Lax matrices \( U_r \) (73) turn into KdV matrices \( V_r \) (18)–(20), while \( L \) matrix (72) is the KdV matrix \( V_4 \) under the constraint \( 0 = \frac{1}{2} \gamma_3(\gamma_3)_{xx} - \frac{1}{4} [(\gamma_n)_x]^2 + u \gamma_3^2 + c \). Thus, indeed, we have the equivalence between both representations (63)–(65).

Summarizing results of this section, the \( n \)th KdV stationary system (25) has the \( n \)-dimensional Stäckel representation (60), generated by the separation curve (56).

What is interesting, both Stäckel representations (33) and (60) of KdV stationary systems (45) and (64) are related by a Miura transformation\(^{22}\) on the extended phase space (stationary manifold) \( M_n = T^*Q \times R \ni (q, p, c) \), and in consequence, both systems have bi-Hamiltonian representation on \( M_n \).\(^5\)
5 | THE PAINlevé DEFORMATION OF STÄCKEL SYSTEMS

In the sequence of papers,\textsuperscript{17–19} we have presented a systematic deformation of an autonomous Stäckel systems generated by separation curves of the form

\[
\sum_{k=-m}^{2n+2-m} \lambda^k + \sum_{r=1}^{n} h_r \lambda^{n-r} = \lambda^m \mu^2, \quad m \in \{0, 1, ..., n + 1\}
\] (77)

to nonautonomous Painlevé-type systems

\[
\frac{d \xi}{dt_r} = Y_r(\xi, t) = \pi dH_r(\xi, t), \quad r = 1, ..., n.
\] (78)

In consequence, new Hamiltonian functions $H_r$ fulfill the Frobenius integrability conditions

\[
\{H_r, H_s\} + \frac{\partial H_r}{\partial t_s} - \frac{\partial H_s}{\partial t_r} = f_{rs}(t_1, ..., t_n), \quad r, s = 1, ..., n,
\] (79)

where $f_{rs}$ are some functions not depending on the phase-space variables $\xi$ only on the parameters $t_j$, and related Hamiltonian vector fields $Y_k(\xi, t)$ satisfy

\[
[Y_s, Y_r] + \frac{\partial Y_r}{\partial t_s} - \frac{\partial Y_s}{\partial t_r} = 0, \quad r, s = 1, ..., n,
\] (80)

as $\pi d\{H_r, H_s\} = -[Y_r, Y_s]$. Therefore, the set of nonautonomous evolution equations (78) has again common (at least local) solutions $\xi(t_1, ..., t_n, \xi_0)$ through each point $\xi_0$ of $M$\textsuperscript{25–27}.

Observe, that it is always possible to choose another Hamiltonians $H_r \rightarrow H_r + \varphi_r(t_1, ..., t_n)$, defining the same dynamical systems (78), which satisfy

\[
\{H_r, H_s\} + \frac{\partial H_r}{\partial t_s} - \frac{\partial H_s}{\partial t_r} = 0, \quad r, s = 1, ..., n.
\] (81)

Besides, the nonautonomous Hamiltonian equations (78) are represented by the so-called isomonodromic Lax representation

\[
\frac{d}{dt_k} L(\lambda; \xi, t) = [U_k(\lambda; \xi, t), L(\lambda; \xi, t)] + \lambda^m \frac{\partial U_k(\lambda; \xi, t)}{\partial \lambda},
\] (82)

being the compatibility condition for a system of linear equations

\[
\lambda^m \frac{\partial \Psi}{\partial \lambda} = L(\lambda; \xi, t)\Psi, \quad \frac{d}{dt_k} \Psi = U_k(\lambda; \xi, t)\Psi.
\] (83)

The deformation procedure for Stäckel systems considered in previous sections, is as follows.\textsuperscript{18} First, we deform geodesic constants of motion $E_r = \mu^T K_r G \mu$ by terms linear in momenta

\[
E_r = E_r + W_r = \mu^T K_r G \mu + \mu^T J_r, \quad r = 2, ..., n, \quad W_1 = 0,
\] (84)
generated by Killing vectors $J_r$ of metric tensors $G_0$ (36) and $G_1$ (59). Actually, in $(q, p)$ coordinates,

$$W_r = \sum_{k=n-r+2}^{n} (n + 1 - k)q_{r+k-n-2} p_k,$$

for metric $G_0$ and

$$W_r = \sum_{k=n-r+1}^{n-1} (n - k)q_{r+k-n-1} p_k,$$

for metric $G_1$, respectively.

The Hamiltonians $\mathcal{E}_r$ in (84) span a Lie algebra $\mathfrak{g} = \text{span}\{\mathcal{E}_r \in C^\infty(M): r = 1, \ldots, n\}$ with the following commutation relations:

$$\{\mathcal{E}_1, \mathcal{E}_r\} = 0, \quad r = 2, \ldots, n,$$

(87)

where

$$\{\mathcal{E}_r, \mathcal{E}_s\} = (s - r)\mathcal{E}_{r+s-n-2}, \quad r, s = 2, \ldots, n,$$

(88)
in the first case and

$$\{\mathcal{E}_r, \mathcal{E}_s\} = (s - r)\mathcal{E}_{r+s-n-1}, \quad r, s = 2, \ldots, n,$$

(89)
in the second case. We use the convention that $\mathcal{E}_r = 0$ as soon as $r \leq 0$ or $r > n$. The algebra $\mathfrak{g}$ has an Abelian subalgebra

$$\mathfrak{a} = \text{span}\{\mathcal{E}_1, \ldots, \mathcal{E}_\kappa\}, \quad \kappa = \left[\frac{n + 3}{2}\right] \text{ and } \kappa = \left[\frac{n + 2}{2}\right], \text{ respectively.}$$

(90)

As the Hamiltonians $\mathcal{E}_r$ in (84) do not commute, they do not constitute a Liouville integrable system. In particular, there is no reason to expect that they will possess a common, multitime solution for a given initial data $\xi_0$. However, in Ref. 17, we found polynomial-in-times deformations $H_r(t_1, \ldots, t_\kappa)$ of the Hamiltonians $\mathcal{E}_r$ such that the Hamiltonians $H_r$ satisfy the Frobenius integrability condition (79). More specifically, the deformed Hamiltonians $H_r$ are given by

$$H_r = \mathcal{E}_r, \text{ for } r = 1, \ldots, \kappa,$$

$$H_r = \sum_{j=1}^{r} \zeta_{r,j}(t_1, \ldots, t_{r-1})\mathcal{E}_j, \zeta_{r,r} = 1, \text{ for } r = \kappa + 1, \ldots, n,$$

(91)

where $\zeta_{r,j}(t)$ are some polynomial functions of evolution parameters, determined uniquely from Frobenius conditions (81).

In the second step of deformation process, we include the appropriate potentials from (30) and (57). To preserve Frobenius conditions, we have supplemented these potentials by extra lower order potentials with time-dependent coefficients. Actually, the deformation of Hamiltonians (30)
takes the form
\[ h_r = E_r + V_r^{(2n+1)} + cV_r^{(n)} \rightarrow h_r^W = h_r + W_r + \sum_{k=n}^{2n-1} c_k(t)V_r^{(k)}, \quad r = 1, \ldots, n \] (92)

and Hamiltonians (57) the respective form
\[ h_r = E_r + V_r^{(2n)} + cV_r^{(-1)} \rightarrow h_r^W = h_r + W_r + \sum_{k=n}^{2n-1} d_k(t)V_r^{(k)}, \quad r = 1, \ldots, n, \] (93)

where \( c_k(t) \) and \( d_k(t) \) are again some polynomial functions of evolution parameters, determined uniquely from Frobenius conditions for functions \( H_r \),
\[ H_r = h_r^W, \text{ for } r = 1, \ldots, \kappa, \]
\[ H_r = \sum_{j=1}^{r} \zeta_{r,j}(t_1, \ldots, t_{r-1})h_j^W, \quad \zeta_{r,r} = 1, \text{ for } r = \kappa + 1, \ldots, n, \] (94)

with the same \( \zeta_{r,j}(t) \) coefficients as in (91). The details of the deformation procedure as well as an appropriate values of coefficients \( \zeta_{r,j}(t) \), \( c_k(t) \), and \( d_k(t) \) the reader can find in Ref. 18.

Both hierarchies of nonautonomous Frobenius integrable systems have isomonodromic Lax representations, so are represented by a Painlevé-type equations.

**Theorem 4** (19). For the first hierarchy, generated by Hamiltonians (94) and (92), isomonodromic Lax representations are of the form
\[ \frac{d}{dt_k}L(\lambda; \xi, t) = [U_k(\lambda; \xi, t), L(\lambda; \xi, t)] + \frac{\partial U_k(\lambda; \xi, t)}{\partial \lambda}, \quad k = 1, \ldots, n. \] (95)
The \( L(\lambda; \xi, t) \) matrix takes the form (40), (41) where now
\[ w(\lambda; q, p, t) = -\left[ \frac{v(\lambda)^2 - \lambda^{2n+1} - \sum_{k=n}^{2n-1} c_k(t)\lambda^k}{u(\lambda)} \right]_+. \] (96)
Moreover,
\[ U_r(\lambda; \xi, t) = \left[ \frac{u_r(\lambda)L(\lambda)}{u(\lambda)} \right]_+, \text{ for } r = 1, \ldots, \kappa, \]
\[ U_r(\lambda; \xi, t) = \sum_{j=1}^{r} \zeta_{r,j}(t_1, \ldots, t_{r-1}) \left[ \frac{u_j(\lambda)L(\lambda)}{u(\lambda)} \right]_+, \quad \zeta_{r,r} = 1, \text{ for } r = \kappa + 1, \ldots, n. \] (97)

For the second hierarchy, generated by Hamiltonians (94) and (93), isomonodromic Lax representations are of the form
\[ \frac{d}{dt_k}L(\lambda; \xi, t) = [U_k(\lambda; \xi, t), L(\lambda; \xi, t)] + \lambda \frac{\partial U_k(\lambda; \xi, t)}{\partial \lambda}. \] (98)
The $L(\lambda; \xi, t)$ matrix takes the form (40) with entries $u(\lambda; q)$ and $v(\lambda; q, p)$ given by (61), where now

$$w(\lambda; q, p, t) = -\lambda \left[ \frac{v(\lambda)^2}{u(\lambda)} - \lambda^{2n} - \sum_{k=n}^{2n-1} c_k(t)\lambda^{-1} - c\lambda^{-1} \right].$$

(99)

$U_r(\lambda; \xi, t)$ matrices are again of the form (97).

The first members of the hierarchy (92), (94) are determined by the following Hamiltonians

\begin{align*}
n = 1: & \quad h_1^W = E_1 + V_1^{(3)} + (t_1 + c) V_1^{(1)}, \quad H_1 = h_1^W, \\
n = 2: & \quad h_r^W = \mathcal{E}_r + V_r^{(5)} + 3t_2 V_r^{(4)} + (t_1 + c) V_r^{(3)}, \quad H_r = h_r^W, \quad r = 1, 2, \\
n = 3: & \quad h_r^W = \mathcal{E}_r + V_r^{(7)} + 5t_3 V_r^{(5)} + 3t_2 V_r^{(4)} + (t_1 + \frac{15}{2} t_3^2 + c) V_r^{(3)}, \quad H_r = h_r^W, \quad r = 1, 2, 3, \\
n = 4: & \quad h_r^W = \mathcal{E}_r + V_r^{(9)} + 7t_4 V_r^{(7)} + 5t_3 V_r^{(6)} + (3t_2 + \frac{35}{2} t_4^2) V_r^{(5)} + (t_1 + 21t_3 t_4 + c) V_r^{(4)}, \quad H_r = h_r^W, \quad r = 1, 2, 3, \\
\vdots
\end{align*}

(100)

while the first members of the second hierarchy (93), (94) are determined by Hamiltonians

\begin{align*}
n = 1: & \quad h_1^W = E_1 + V_1^{(2)} + t_1 V_1^{(1)} + c V_1^{(-1)}, \quad H_1 = h_1^W, \\
n = 2: & \quad h_r^W = \mathcal{E}_r + V_r^{(4)} + 3t_2 V_r^{(3)} + (t_1 + 3t_2^2) V_r^{(2)} + c V_r^{(-1)}, \quad H_r = h_r^W, \quad r = 1, 2, \\
n = 3: & \quad h_r^W = \mathcal{E}_r + V_r^{(6)} + 5t_3 V_r^{(5)} + (3t_2 + 10t_3^2) V_r^{(4)} + (t_1 + 10t_2 t_3 + 10t_3^3) V_r^{(3)} + c V_r^{(-1)}, \quad H_r = h_r^W, \quad r = 1, 2, 3, \\
n = 4: & \quad h_r^W = \mathcal{E}_r + V_r^{(8)} + 7t_4 V_r^{(7)} + (5t_3 + 21t_4^2) V_r^{(6)} + (3t_2 + 28t_3 t_4 + 35t_4^3) V_r^{(5)} \\
& \quad + (t_1 + 14t_2 t_4 + \frac{15}{2} t_2^2 + 63t_3 t_4^2 + 35t_4^4) V_r^{(4)} + c V_r^{(-1)}, \quad H_r = h_r^W, \quad r = 1, 2, 3, 4, \\
n = 5: & \quad h_r^W = \mathcal{E}_r + V_r^{(10)} + 9t_5 V_r^{(9)} + (7t_4 + 36t_5^2) V_r^{(8)} + (5t_3 + 54t_4 t_5 + 84t_5^3) V_r^{(7)} \\
& \quad + (3t_2 + 36t_3 t_5 + \frac{35}{2} t_4^2 + 180t_4 t_5^2 + 126t_5^4) V_r^{(6)} \\
& \quad + (t_1 + 21t_3 t_4 + 18t_2 t_5 + 108t_4^2 t_5 + 108t_3 t_5^2 + 336t_4 t_5^2 + 126t_5^5) V_r^{(5)} + c V_r^{(-1)}, \quad H_r = h_r^W, \quad r = 1, 2, 3, 4, 5, \vdots
\end{align*}
\[ H_r = h_r^W, \quad r = 1, 2, 3, \quad H_4 = h_4^W + t_3 h_1^W, \quad H_5 = h_5^W + t_4 h_3^W + 2t_3 h_2^W + \left(3t_2 - \frac{1}{2}t_4^2\right) h_1^W, \]

(101)

The first hierarchy (100) is the complete hierarchy of Painlevé I \((P_I)\) systems, as its first, one dimensional, system is a famous \(P_I\) equation. Indeed, denoted \(q_1 = q, \ p_1 = p, \) and \(t_1 = t, \) we have

\[ H = h_1^W = p^2 + q^3 + (t + c)q \]

\[ \downarrow \]

\[ q_t = 2p, \quad p_t = -3q^2 - t - c \]

\[ \downarrow \]

\[ q_{tt} + 6q^2 + 2(t + c) = 0. \]  

(102)

The isomonodromic Lax representation (82) is as follows:

\[ L = \begin{pmatrix} -p & \lambda + q \\ \lambda^2 - q\lambda + q^2 + t + c & p \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ \lambda - 2q & 0 \end{pmatrix}. \]  

(103)

On the other hand, Equation (102) is equivalent to \(P_I\) equation

\[ q_{tt} = 3q^2 + t \]  

(104)

after rescaling \(q \to -\frac{1}{5}q, \ t \to 2^{-\frac{2}{5}}t - c. \)

The second, two dimensional system, from that hierarchy is as follows:

\[ H_1 = h_1^W = 2p_1 p_2 + q_1 p_2^2 - q_1^4 + 3q_1^2 q_2 - q_2^3 + 3t_2 (q_2 - q_1^2) + (t_1 + c)q_1, \]

\[ H_2 = h_2^W = p_1^2 + 2q_1 p_1 p_2 + (q_1^2 - q_2)p_2^2 + p_2 - q_1^3 q_2 + 2q_1 q_2^2 - 3t_2 q_1 q_2 + (t_1 + c)q_2, \]

\[ \{H_1, H_2\} + \frac{\partial H_1}{\partial t_2} - \frac{\partial H_2}{\partial t_1} = 3t_2. \]

Thus,

\[
\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_1} = \begin{pmatrix} 2p_2 \\ 2p_1 + 2q_1 p_2 \\ -p_2^2 + 4q_1^3 - 6q_1 q_2 + 6t_2 q_1 - t_1 - c \\ -3q_1^2 + 2q_2 - 3t_2 \end{pmatrix} = Y_1
\]

\[ \downarrow \]
\[(q_1)_{t_1 t_1} = -6q_1^2 + 4q_2 - 6t_2, (q_2)_{t_1 t_1} = \frac{1}{2}[((q_1)_{t_1})^2 + 2q_1^3 - 8q_1q_2 + 6t_2q_1 - 2t_1 - 2c], \tag{105}\]

\[
\begin{pmatrix}
q_1 \\
q_2 \\
p_1 \\
p_2
\end{pmatrix} =
\begin{pmatrix}
2q_1p_2 + 2p_1 \\
2q_1p_1 + 2(q_1^2 - q_2)p_2 + 1 \\
-2q_1p_2^2 - 2p_1p_2 + 3q_1^2q_2 - 2q_2^2 + 3t_2q_2 \\
p_2^2 + q_1^3 - 4q_1q_2 + 3t_2q_1 - t_1 - c
\end{pmatrix}
\]

\[= Y_2\]

and

\[
[Y_2, Y_1] + \frac{\partial Y_1}{\partial t_2} - \frac{\partial Y_2}{\partial t_1} = 0.
\]

Notice that eliminating \(q_2\) from (105), we get the forth-order equation for \(q \equiv q_1\)

\[
\frac{1}{4} q_{t_1 t_1 t_1 t_1} + 5qq_{t_1 t_1} + \frac{5}{2}(q_{t_1})^2 + 10q^3 + 6t_2q + 2t_1 + 2c = 0,
\]

which is the second equation from the standard \(P_I\) hierarchy.

The isomonodromic Lax representation (82) is of the form

\[
L = \begin{pmatrix}
-p_2\lambda - p_1 - q_1p_2 & \lambda^2 + q_1\lambda + q_2 \\
\lambda^3 - q_1\lambda^2 + (q_1^2 - q_2 + 3t_2)\lambda - p_2^2 + 2q_1q_2 - 3t_2q_1 - t_1 + c & p_2\lambda + p_1 + q_1p_2
\end{pmatrix},
\]

\[
U_1 = \begin{pmatrix}
0 & 1 \\
\lambda - 2q_1 & 0
\end{pmatrix}, \quad U_2 = \begin{pmatrix}
-p_2 & \lambda + q_1 \\
\lambda^2 - q_1\lambda + q_1^2 - 2q_2 + 3t_2 & p_2
\end{pmatrix}.
\]

The third, three-dimensional system, from that hierarchy (100), that is, the deformed system from Example 1, is generated by Hamiltonians

\[
H_1 = h_1^W = h_1 + 5t_3V_1^{(5)} + 3t_2V_1^{(4)} + (t_1 + \frac{15}{2}t_3^2)V_1^{(3)},
\]

\[
H_2 = h_2^W = h_2 + p_3 + 5t_3V_2^{(5)} + 3t_2V_2^{(4)} + (t_1 + \frac{15}{2}t_3^2)V_2^{(3)},
\]

\[
H_3 = h_3^W = h_3 + 2p_2 + q_1p_3 + 5t_3V_3^{(5)} + 3t_2V_3^{(4)} + (t_1 + \frac{15}{2}t_3^2)V_3^{(3)}, \tag{106}
\]

where functions \(h_1, h_2,\) and \(h_3\) are given in Example 1 and

\[
V_1^{(4)} = q_2 - q_1^2, \quad V_2^{(4)} = q_3 - q_1q_2, \quad V_3^{(4)} = -q_1q_3,
\]

\[
V_1^{(5)} = q_1^3 - 2q_1q_2 + q_3, \quad V_2^{(5)} = q_1^2q_2 - q_1q_3 - q_2^2, \quad V_3^{(5)} = q_1^2q_3 - q_2q_3.
\]

The related Hamiltonian vector fields \(Y_r = \pi dH_r, r = 1, 2, 3\) fulfill Frobenius conditions (80) and evolution equations \(\xi_{Y_r} = Y_r\) have the following isomonodromic Lax representations:

\[
L = \begin{pmatrix}
-p_3\lambda^2 - (p_2 + q_1p_3)\lambda - (p_1 + q_1p_2 + q_2p_3) & \lambda^3 + q_1\lambda^2 + q_2\lambda + q_3 \\
-p_2\lambda^2 - (p_1 + q_1p_2 + q_2p_3) & \lambda^3 + q_1\lambda^2 + q_2\lambda + q_3
\end{pmatrix},
\]

\[
L_{21} = \lambda^4 - q_1\lambda^3 - (V_1^{(4)} - 5t_3)\lambda^2 - (V_2^{(4)}(p_2^3 - 5q_1t_3 - 3t_2)\lambda - V_1^{(6)}.
\]
\(-q_1 p_3^2 - 2p_2 p_3 - 5t_3 V_1^{(4)} - 3t_2 q_1 + t_1 + \frac{15}{2} t_3^2 + c,\)

\[ V_1^{(6)} = -q_1^4 + 3q_1^2 q_2 - 2q_1 q_3 - q_2^2, \]

\[ U_1 = \begin{pmatrix} 0 & 1 \\ \lambda - 2q_1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} -p_3 & \lambda + q_1 \\ \lambda^2 - q_1 \lambda + q_1^2 - 2q_2 + 5t_3 & p_3 \end{pmatrix}, \]

\[ U_3 = \begin{pmatrix} -p_3 \lambda - q_1 p_3 - p_2 & \lambda^2 + q_1 \lambda + q_2 \\ \lambda^3 - q_1 \lambda^2 - (V_1^{(4)} - 5t_3) \lambda - V_1^{(5)} - q_3 - p_2^2 - 5q_1 t_3 + 3t_2 & p_3 \lambda + q_1 p_3 + p_2 \end{pmatrix}. \]

The explicit form of isomonodromic Lax representations of higher dimensional systems from the \(P_I\) hierarchy (100) can be constructed with the help of (96) and (97).

The first attempt to the hierarchy of \(P_I\) systems was done in Ref. 16 by Takasaki, where the author started from the opposite side, that is, from string equations of KP (KdV in particular). Using such formalism, he was able to construct, for each \(n\), only the first Painlevé equation

\[ \frac{d\xi}{dt_1} = Y_1(\xi, t) = \pi dH_1(\xi, t), \]

from the system (78), together with its isomonodromic Lax representation. He failed in constructing the remaining equations from the \(P_I\) system (78), with \(k = 2, \ldots, n\), as he did not control the perturbation terms \(W_k\) (85) in remaining Hamiltonians. Now we know that they are generated by Killing vectors (85) of the metric tensor \(G_0\).

Now, let us turn to the second hierarchy of Painlevé systems (101). The first, one dimensional, system is as follows:

\[ H = h_1^W = -q p^2 - q^2 + tq + cq^{-1} \]

\[ \downarrow \]

\[ q_t = -2qp, \quad p_t = p^2 + 2q - t + cq^{-2} \]

\[ \updownarrow \]

\[ q q_{tt} = \frac{1}{2} q_t^2 - 4q^3 + 2tq^2 - 2c. \quad (107) \]

The isomonodromic Lax representation takes the form (98), where

\[ L = \begin{pmatrix} qp & \lambda + q \\ \lambda^2 + (-q + t) \lambda + qp^2 + cq^{-1} & -qp \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ \lambda - 2q + t & 0 \end{pmatrix}. \]

One can verify that Equation (107) is the 34th Painlevé (\(P_{34}\)) equation from the Gambier list.

The second, two-dimensional system from the hierarchy, is generated by Hamiltonians

\[ H_1 = h_1^W = p_1^2 - q_2 p_2^2 + q_1^3 - 2q_1 q_2 + 3t_2 (q_2 - q_1) + (3t_2^2 + t_1) q_1 + cq_2^{-1}, \]

\[ H_2 = h_2^W = -2q_2 p_1 p_2 - q_1 q_2 p_2^2 + p_1 + q_1^2 q_2 - q_2^2 - 3t_2 q_1 q_2 + (3t_2^2 + t_1) q_2 + cq_1 q_2^{-1}, \]
\[ \{H_1, H_2\} + \frac{\partial H_1}{\partial t_2} - \frac{\partial H_2}{\partial t_1} = t_1 + 3t_2^2. \]

Thus,
\[
\begin{pmatrix}
q_1 \\
q_2 \\
p_1 \\
p_2_{t_1}
\end{pmatrix} = \begin{pmatrix}
2p_1 \\
-2q_2 p_2 \\
-3q_1^2 + 2q_2 + 6t_2 q_1 - 3t_2^2 - t_1 \\
p_2^2 + 2q_1 - 3t_2 + cq_2^{-2}
\end{pmatrix} = Y_1,
\]

\[ (q_1)_{t_1 t_1} = -6q_1^2 + 4q_2 + 12t_2 q_1 - 6t_2^2 - 2t_1, q_2(q_2)_{t_1 t_1} = \frac{1}{2}[(q_2)_{t_1}]^2 - 4q_1 q_2^2 + 6t_2 q_2^2 - 2c, \quad (108) \]

Notice that eliminating \( q_2 \) from (108), we get the forth-order equation for \( q \equiv q_1 \)
\[
0 = -\frac{1}{32} q_{t_1 t_1} q_{t_1 t_1 t_1 t_1} + \frac{1}{64}(q_{t_1 t_1 t_1})^2 - \frac{1}{16}(3q_2^2 - 6t_2 q + 3t_2^2 + t_1) q_{t_1 t_1 t_1 t_1} + \frac{3}{8}(q - t_2) q_{t_1 t_1 t_1 t_1} - \frac{1}{2}(q - \frac{9}{8} t_2) (q_{t_1})^2 - \frac{3}{8} (q_{t_1})^2 q_{t_1 t_1} - \frac{15}{4} q_3^2 - 12t_2 q^2 + \frac{(51}{4} t_2^2 + \frac{5}{4} t_1) q - \frac{3}{2} t_1 t_2^2 - \frac{9}{2} t_2^3 q_{t_1 t_1 t_1 t_1} - \frac{3}{4} t_1 (q_{t_1})^2 - \frac{9}{2} q_5^2 + \frac{99}{4} t_2^2 q^4 - (54t_2^2 + 3t_1) q^3 + \frac{21}{2} t_1 t_2 + \frac{117}{2} t_2^4 q^2 - (12t_2 t_2 + \frac{63}{2} t_2^4 + \frac{1}{2} t_2^4) q^2 + \frac{27}{4} t_5^5 + \frac{3}{2} t_1^2 t_2^2 + \frac{2}{2} t_1^2 + \frac{1}{16} - c,
\]

which can be considered as the second equation from the standard \( P_{3,4} \) hierarchy.

The isomonodromic Lax representation (98) is as follows:
\[
L = \begin{pmatrix}
\lambda^3 + (-q_1 + 3t_2) \lambda^2 + (q_1^2 - 3q_1 t_2 - q_2 + 3t_2^2 + t_1) \lambda + q_2 p_2^2 + cq_2^{-1} & p_1 \lambda - q_2 p_2 \\
\end{pmatrix},
\]

\[ U_1 = \begin{pmatrix}
0 & 1 \\
-\lambda - 2q_1 + 3t_2 & 0
\end{pmatrix}, \quad U_2 = \begin{pmatrix}
-\lambda^2 + (-q_1 + 3t_2) \lambda + q_1^2 - 2q_2 - 3t_2 q_1 + t_1 + 3t_2^2 & p_1
\end{pmatrix}.
\]

What is interesting, this system in flat coordinates \((x_1, x_2, p_{x_1}, p_{x_2})\) on \( R^4 \), related with \((q, p)\) coordinates by a point transformation
\[ q_1 = -x_1, \quad q_2 = -\frac{1}{4} x_2^2, \]
is the nonautonomous deformation of the famous Henon–Heiles system\(^{28}\) to consider in Ref. 29 and in the complete version in Ref. 30.
The third, three-dimensional system from that hierarchy (101), that is, the deformed system from Example 2, is generated by Hamiltonians

\[
H_1 = h_1^W = h_1 + 5t_3V_1^{(5)} + (3t_2 + 10t_3^2)V_1^{(4)} + (t_1 + 10t_2t_3 + 10t_3^3)V_1^{(3)},
\]

\[
H_2 = h_2^W = h_2 + p_2 + 5t_3V_2^{(5)} + (3t_2 + 10t_3^2)V_2^{(4)} + (t_1 + 10t_2t_3 + 10t_3^3)V_2^{(3)},
\]

\[
H_3 = h_3^W + t_2h_1^W = h_3 + 2p_1 + q_1p_2 + 5t_3V_3^{(5)} + (3t_2 + 10t_3^2)V_3^{(4)} + (t_1 + 10t_2t_3 + 10t_3^3)V_3^{(3)} + t_2h_1^W,
\]

where functions \( h_1, h_2, \) and \( h_3 \) are given in Example 2. The related Hamiltonian vector fields \( Y_r = \pi dH_r, r = 1, 2, 3 \) fulfill Frobenius conditions (80) and evolution equations \( \xi_{t_r} = Y_r \) have the following isomonodromic Lax representations

\[
L = \begin{pmatrix}
-p_2\lambda^2 - (p_1 + q_1p_2)\lambda + q_3p_3 & \lambda^3 + q_1\lambda^2 + q_2\lambda + q_3 \\
L_{21} & L_{11}
\end{pmatrix},
\]

\[
L_{21} = \lambda^4 - (q_1 - 5t_3)\lambda^3 - (V_1^{(4)} + 5t_3q_1 - 3t_2 - 10t_3^2)\lambda^2 - [V_1^{(5)} + p_2^3 + 5t_3V_1^{(4)} + (3t_2 + 10t_3^2)q_1 - (t_1 + 10t_2t_3 + 10t_3^3)]\lambda - q_3p_3^2 + cq_3^{-1},
\]

\[
U_1 = \begin{pmatrix}
0 & 1 \\
\lambda - 2q_1 + 5t_3 & 0
\end{pmatrix},
\]

\[
U_2 = \begin{pmatrix}
\lambda^2 - (q_1 - 5t_3)\lambda + V_1^{(4)} - q_2 - 5t_3q_1 + 3t_2 + 10t_3^2 & p_2 \\
-p_2\lambda - q_1p_2 - p_1 & \lambda + q_1
\end{pmatrix},
\]

\[
U_3 = \begin{pmatrix}
\lambda^3 - (q_1 - 5t_3)\lambda^2 - (V_1^{(5)} + 5t_3q_1 - 4t_2 - 10t_3^2)\lambda & p_2\lambda + q_1p_2 + p_1 \\
-p_2^2 - V_1^{(5)} - q_3 - 5t_3V_1^{(4)} - (5t_2 + 10t_3^2)q_1 + t_1 + 15t_2t_3 + 10t_3^3 & \lambda^2 + q_1\lambda + q_2 + t_2
\end{pmatrix}.
\]

The systematic construction of isomonodromic Lax representations for higher dimensional members (101) is described by (97)–(99).

Also, in that case, some elements of similar \( P_{34} \) hierarchy appeared in Ref. 11, where stationary flows of equations like these from (120), but with time-independent coefficients, were derived.

### 6 NONHOMOGENEOUS KdV HIERARCHIES AND RELATED NONAUTONOMOUS STATIONARY SYSTEMS

In next sections, we have demonstrated how to reconstruct the KdV hierarchy and its stationary systems form the hierarchies of Stäckel systems (30) and (57), respectively. Here, by the same method, we will construct two different nonhomogeneous KdV hierarchies and related nonautonomous stationary systems directly from Painlevé deformations (100) and (101) of considered Stäckel systems.
We begin from $P_1$ hierarchy (100). For one-dimensional equation (102) ($P_1$), after identification $t_1 = x$ and substitution $q = \frac{1}{2}u$, we get $p = \frac{1}{4}u_x$ and

$$0 = \frac{1}{4}u_{xx} + \frac{3}{4}u^2 + x + c,$$

(109) which is the integrated stationary flow of the following PDE:

$$u_{t_2} = \mathcal{K}_2 + \sigma_{-1} = \partial_x(y_2 + x + c) \equiv \pi_0(y_{1,2} + x + c) = \mathcal{K}_{1,2}$$

(110) from the KdV family. By the same substitution, the Painlevé-type equations generated by Hamiltonians $H_1$ from family (100), for $n = 2, 3, 4, \ldots$, are integrated stationary flows of the following hierarchy of nonhomogeneous KdV equations

$$u_{t_3} = \mathcal{K}_3 + \frac{3}{2}t_2\mathcal{K}_1 + \sigma_{-1} \equiv \mathcal{K}_{2,3} = \pi_0(y_{2,3} + x + c),$$

$$u_{t_4} = \mathcal{K}_4 + \frac{5}{2}t_3\mathcal{K}_2 + \frac{3}{2}t_2\mathcal{K}_1 + \sigma_{-1} \equiv \mathcal{K}_{3,4} = \pi_0(y_{3,4} + x + c),$$

$$u_{t_5} = \mathcal{K}_5 + \frac{7}{2}t_4\mathcal{K}_3 + \frac{5}{2}t_3\mathcal{K}_2 + \frac{3}{2}\left(t_2 + \frac{7}{4}t_4^2\right)\mathcal{K}_1 + \sigma_{-1} \equiv \mathcal{K}_{4,5} = \pi_0(y_{4,5} + x + c),$$

$$u_{t_6} = \mathcal{K}_6 + \frac{9}{2}t_5\mathcal{K}_4 + \frac{7}{2}t_4\mathcal{K}_3 + \frac{5}{2}\left(t_3 + \frac{9}{4}t_5^2\right)\mathcal{K}_2 + \frac{3}{2}\left(t_2 + \frac{9}{2}t_4t_5\right)\mathcal{K}_1 + \sigma_{-1}$$

$$\equiv \mathcal{K}_{5,6} = \pi_0(y_{5,6} + x + c),$$

\[\vdots\]

Contrary to the autonomous case, for fixed $n$, the remaining Painlevé-type equations generated by Hamiltonians $H_2, \ldots, H_n$ from (100) do not reconstruct the lower order equations from the hierarchy (111). Actually, the hierarchy of Painlevé-type systems generated by Hamiltonians (100) is equivalent to the following hierarchy of KdV nonautonomous stationary systems:

$$n = 1 : \quad 0 = y_2 + x + c \equiv y_{1,2} + x + c,$$

$$n = 2 : \quad u_{t_2} = \mathcal{K}_2 \equiv \mathcal{K}_{2,2} = \pi_0y_{2,2},$$

$$0 = y_3 + \frac{3}{2}t_2y_1 + x + c \equiv y_{2,3} + x + c,$$

$$n = 3 : \quad u_{t_2} = \mathcal{K}_2 \equiv \mathcal{K}_{3,2} = \pi_0y_{3,2},$$

$$u_{t_3} = \mathcal{K}_3 + \frac{5}{2}t_3\mathcal{K}_1 \equiv \mathcal{K}_{3,3} = \pi_0y_{3,3},$$

$$0 = y_4 + \frac{5}{2}t_3y_2 + \frac{3}{2}t_2y_1 + \frac{5}{4}t_3^2 + x + c \equiv y_{3,4} + x + c,$$

$$n = 4 : \quad u_{t_2} = \mathcal{K}_2 \equiv \mathcal{K}_{4,2} = \pi_0y_{4,2},$$

$$u_{t_3} = \mathcal{K}_3 + \frac{7}{2}t_4\mathcal{K}_1 \equiv \mathcal{K}_{4,3} = \pi_0y_{4,3},$$

$$u_{t_4} = \mathcal{K}_4 + \frac{7}{2}t_4\mathcal{K}_2 + \frac{7}{2}t_3\mathcal{K}_1 \equiv \mathcal{K}_{4,4} = \pi_0y_{4,4},$$

\[\vdots\]
0 = \gamma_5 + \frac{7}{2} t_4 \gamma_3 + \frac{5}{2} t_3 \gamma_2 + \frac{3}{2} \left( t_2 + \frac{7}{4} t_4^2 \right) \gamma_1 + \frac{7}{2} t_3 t_4 + x + c \equiv \gamma_{4,5} + x + c, \\
n = 5: \quad u_{t_2} = \mathcal{K}_2 \equiv \mathcal{K}_{5,2} = \pi_0 \gamma_{5,2}, \\
u_{t_3} = \mathcal{K}_3 + \frac{9}{2} t_5 \mathcal{K}_1 \equiv \mathcal{K}_{5,3} = \pi_0 \gamma_{5,3}, \\
u_{t_4} = \mathcal{K}_4 + \frac{9}{2} t_5 \mathcal{K}_2 + \frac{7}{2} t_4 \mathcal{K}_1 \equiv \mathcal{K}_{5,4} = \pi_0 \gamma_{5,4}, \\
u_{t_5} = \mathcal{K}_5 + \frac{9}{2} t_5 \mathcal{K}_3 + \frac{9}{2} t_4 \mathcal{K}_2 + \frac{9}{2} \left( t_3 + \frac{5}{4} t_5^2 \right) \mathcal{K}_1 \equiv \mathcal{K}_{5,5} = \pi_0 \gamma_{5,5}, \\
0 = \gamma_6 + \frac{9}{2} t_5 \gamma_4 + \frac{7}{2} t_4 \gamma_3 + \frac{5}{2} \left( t_3 + \frac{9}{4} t_5^2 \right) \gamma_2 + \frac{3}{2} \left( t_2 + \frac{9}{2} t_4 t_5 \right) \gamma_1 \\
+ \frac{15}{8} t_5^3 + \frac{9}{2} t_3 t_5 + \frac{7}{4} t_4^2 + x + c \equiv \gamma_{5,6} + x + c, \\
\vdots \\
(112)

Lemma 1. For arbitrary \( n \in \mathbb{N} \), nonautonomous vector fields \( \mathcal{K}_{n,r} \) fulfill Frobenius conditions

\[
[K_{n,s}, K_{n,r}] + \frac{\partial K_{n,r}}{\partial t_s} - \frac{\partial K_{n,s}}{\partial t_r} = 0, \quad r, s = 2, \ldots, n + 1. 
\] (113)

Theorem 5. The nonautonomous stationary system

\[
u_{t_r} = \mathcal{K}_{n,r} = \pi_0 \gamma_{n,r} , \quad 0 = \gamma_{n,n+1} + x + c, \quad r = 2, \ldots, n
\] (114)

has isomonodromic Lax representation

\[
\frac{d}{dt_r} U_{n,n+1} = [U_{n,r}, U_{n,n+1}] + \frac{\partial U_{n,r}}{\partial \lambda}, \quad r = 1, \ldots, n, 
\] (115)

where \( \frac{d}{dt_r} \) is the evolutionary derivative (39) along the \( r \)th flow \( \mathcal{K}_{n,r} \),

\[
U_{n,r} = V_{n,r} \quad r = 1, \ldots, \kappa, \quad U_{n,r} = \sum_{j=1}^{r} \xi_{r,j}(t_1, \ldots, t_{r-1}) V_{n,j}, \quad \xi_{r,r} = 1, \quad \text{for} \ r = \kappa + 1, \ldots, n, 
\] (116)

\[
V_{n,r} = \begin{pmatrix} -\frac{1}{2} (P_{n,r})_x & P_{n,r} \\ P_{n,r}(\lambda - u) - \frac{1}{2} (P_{n,r})_{xx} & 1/2 (P_{n,r})_x \end{pmatrix}, \quad P_{n,r} = \frac{1}{2} \sum_{i=0}^{r-1} \gamma_{n,i} \lambda^{r-i-1}, \quad \gamma_{n,0} = \gamma_0 
\] (117)

and

\[
U_{n,n+1} = V_{n,n+1} \quad \text{under constraint} \quad 0 = \gamma_{n,n+1} + x + c.
\]

The proof follows from the isomonodromic Lax representation (95), (97) of Painlevé representation of (114) and relations (49) and (50) for their autonomous counterparts.
The hierarchy of nonhomogeneous KdV equations (111) has the following nonisospectral zero curvature representation:

\[
\frac{d}{dt_n}V_1 + \frac{\partial}{\partial \lambda}V_1 - \frac{d}{dx}V_{n,n+1} + [V_1, V_{n,n+1}] = 0, \tag{118}
\]
as \(V_{n,1} = V_1\).

Now, let us pass to the second nonautonomous KdV hierarchy of stationary systems, constructed from the Painlevé-type systems (101). Again, for \(n = 1\), differentiation of (107) by \(t\), division by \(2q\), and substitution \(t = x\), \(q = \frac{1}{2}u + \frac{1}{2}x\), we get the stationary flow of the following PDE:

\[
\begin{align*}
    u_t &= \left(\frac{1}{4} \partial_x^3 + \frac{1}{2} u \partial_x + \frac{1}{2} \partial_x \right) (\gamma_1 + x) = \mathcal{K}_2 + \sigma_0 \equiv \pi_1 \gamma_{1,1} = \mathcal{K}_{1,2} \tag{119}
\end{align*}
\]

from the KdV family. For \(n = 2, 3, 4, \ldots\), by the substitution \(t_1 = x\), \(q_1 = \frac{1}{2}u + \frac{2n-1}{4}t_n\), the Painlevé-type equations, generated by Hamiltonians \(H_1\) from (101) hierarchy, are integrated stationary flows (with respect to \(\pi_1\)) of the following hierarchy of nonhomogeneous KdV equations:

\[
\begin{align*}
    u_{t_1} &= \mathcal{K}_3 + \frac{3}{2} t_1 \mathcal{K}_2 + \frac{3}{8} t_1^2 \mathcal{K}_1 + \sigma_0 \equiv \mathcal{K}_{2,3} = \pi_1 \left( \gamma_2 + \frac{3}{2} t_2 \gamma_1 + \frac{3}{8} t_2^2 \gamma_0 + x \right) = \pi_1 \gamma_{2,2}, \\
    u_{t_2} &= \mathcal{K}_4 + \frac{5}{2} t_1 \mathcal{K}_3 + \frac{3}{2} \left( t_2 + \frac{5}{4} t_3^2 \right) \mathcal{K}_2 + \frac{1}{2} \left( \frac{5}{2} t_2 t_3 + \frac{5}{8} t_3^2 \right) \mathcal{K}_1 + \sigma_0 \equiv \mathcal{K}_{3,4}, \\
    u_{t_3} &= \mathcal{K}_5 + \frac{7}{2} t_4 \mathcal{K}_4 + \frac{5}{2} \left( t_3 + \frac{7}{4} t_4^2 \right) \mathcal{K}_3 + \left( \frac{3}{2} t_2 + \frac{21}{4} t_3 t_4 + \frac{35}{16} t_4^2 \right) \mathcal{K}_2 + \left( \frac{5}{8} t_2^2 + \frac{7}{4} t_2 t_4 + \frac{35}{128} t_4^2 \right) \mathcal{K}_1 + \sigma_0 \equiv \mathcal{K}_{4,5}, \\
    u_{t_4} &= \mathcal{K}_6 + \frac{9}{2} t_5 \mathcal{K}_5 + \left( \frac{7}{2} t_4 + \frac{63}{8} t_5^2 \right) \mathcal{K}_4 + \left( \frac{5}{2} t_3 + \frac{45}{4} t_4 t_5 + \frac{105}{16} t_5^2 \right) \mathcal{K}_3 + \left( \frac{3}{2} t_2 + \frac{21}{8} t_4 + \frac{27}{4} t_4 t_5 \right) \mathcal{K}_2 + \left( \frac{189}{16} t_2 t_5^2 + \frac{315}{128} t_5^3 \right) \mathcal{K}_1 + \sigma_0 \equiv \mathcal{K}_{5,6}, \\
    u_{t_5} &= \mathcal{K}_7 + \frac{11}{2} t_6 \mathcal{K}_6 + \left( \frac{7}{2} t_5 + \frac{63}{8} t_6^2 \right) \mathcal{K}_5 + \left( \frac{5}{2} t_4 + \frac{45}{4} t_5 t_6 + \frac{105}{16} t_6^2 \right) \mathcal{K}_4 + \left( \frac{3}{2} t_2 + \frac{21}{8} t_4 + \frac{27}{4} t_4 t_5 \right) \mathcal{K}_3 + \left( \frac{189}{16} t_2 t_5^2 + \frac{315}{128} t_5^3 \right) \mathcal{K}_2 + \left( \frac{189}{16} t_2 t_5^2 + \frac{315}{128} t_5^3 \right) \mathcal{K}_1 + \sigma_0 \equiv \mathcal{K}_{6,7},
\end{align*}
\]

\(\vdots\) \tag{120}

Again, contrary to the autonomous case, for fixed \(n\), the remaining Painlevé-type equations generated by Hamiltonians \(H_2, \ldots, H_n\) from (101) do not reconstruct the lower order equations from the hierarchy (120). Actually, the hierarchy of Painlevé-type systems generated by Hamiltonians (101) is equivalent to the following hierarchy of nonautonomous KdV stationary systems:

\[
\begin{align*}
    n = 1 : \quad 0 &= \frac{1}{2} \gamma_{1,1}(\gamma_{1,1})_{xx} - \frac{1}{4} (\gamma_{1,1})_x^2 + u \gamma_{1,1}^2 + c,
\end{align*}
\]

\[\quad\]
\( n = 2 \) : \( u_t = \mathcal{K}_2 + \frac{3}{2} t_2 \mathcal{K}_1 \equiv \mathcal{K}_{2,2} = \pi_1 y_{2,1}, \)
\[
0 = \frac{1}{2} y_{2,2}(y_{2,2})_{xx} - \frac{1}{4} [(y_{2,2})_x]^2 + uy_{2,2}^2 + c,
\]
\( n = 3 \) : \( u_t = \mathcal{K}_2 + \frac{5}{2} t_3 \mathcal{K}_1 \equiv \mathcal{K}_{3,2} = \pi_1 y_{3,1}, \)
\[
u_t = \mathcal{K}_3 + \frac{5}{2} t_3 \mathcal{K}_2 + \left( \frac{5}{2} t_2 + \frac{15}{8} t_3^2 \right) \mathcal{K}_1 \equiv \mathcal{K}_{3,3} = \pi_1 y_{3,2},
\]
\[
0 = \frac{1}{2} y_{3,3}(y_{3,3})_{xx} - \frac{1}{4} [(y_{3,3})_x]^2 + uy_{3,3}^2 + c,
\]
\( n = 4 \) : \( u_t = \mathcal{K}_2 + \frac{7}{2} t_4 \mathcal{K}_1 \equiv \mathcal{K}_{4,2} = \pi_1 y_{4,1}, \)
\[
u_t = \mathcal{K}_3 + \frac{7}{2} t_4 \mathcal{K}_2 + \left( \frac{7}{2} t_3 + \frac{35}{8} t_4^2 \right) \mathcal{K}_1 \equiv \mathcal{K}_{4,3} = \pi_1 y_{4,2},
\]
\[
u_t = \mathcal{K}_4 + \frac{7}{2} t_4 \mathcal{K}_3 + \left( \frac{7}{2} t_2 + \frac{35}{8} t_4^2 \right) \mathcal{K}_2 + \left( \frac{7}{2} t_2 + \frac{35}{4} t_3 t_4 + \frac{35}{16} t_4^3 \right) \mathcal{K}_1 \equiv \mathcal{K}_{4,4} = \pi_1 y_{4,3},
\]
\[
0 = \frac{1}{2} y_{4,4}(y_{4,4})_{xx} - \frac{1}{4} [(y_{4,4})_x]^2 + uy_{4,4}^2 + c,
\]
\( n = 5 \) : \( u_t = \mathcal{K}_2 + \frac{9}{2} t_5 \mathcal{K}_1 \equiv \mathcal{K}_{5,2} = \pi_1 y_{5,1}, \)
\[
u_t = \mathcal{K}_3 + \frac{9}{2} t_5 \mathcal{K}_2 + \left( \frac{9}{2} t_4 + \frac{63}{8} t_5^2 \right) \mathcal{K}_1 \equiv \mathcal{K}_{5,3} = \pi_1 y_{5,2},
\]
\[
u_t = \mathcal{K}_4 + \frac{9}{2} t_5 \mathcal{K}_3 + \left( \frac{9}{2} t_3 + \frac{63}{8} t_5^2 \right) \mathcal{K}_2 + \left( \frac{9}{2} t_2 + \frac{63}{4} t_4 t_5 + \frac{105}{8} t_5^3 \right) \mathcal{K}_1 \equiv \mathcal{K}_{5,4} = \pi_1 y_{5,3},
\]
\[
u_t = \mathcal{K}_5 + \frac{9}{2} t_5 \mathcal{K}_4 + \left( \frac{9}{2} t_2 + \frac{63}{8} t_5^2 \right) \mathcal{K}_3 + \left( \frac{9}{2} t_2 + \frac{63}{4} t_4 t_5 + \frac{105}{16} t_5^3 \right) \mathcal{K}_2
\]
\[
+ \left( \frac{9}{2} t_2 + \frac{45}{8} t_4^2 + \frac{63}{4} t_3 t_5 + \frac{315}{16} t_4 t_5^2 + \frac{315}{128} t_5^3 \right) \mathcal{K}_1 \equiv \mathcal{K}_{5,5} = \pi_1 y_{5,4},
\]
\[
0 = \frac{1}{2} y_{5,5}(y_{5,5})_{xx} - \frac{1}{4} [(y_{5,5})_x]^2 + uy_{5,5}^2 + c,
\]
\( \vdots \) \hfill (121)
Besides, the hierarchy of nonhomogeneous KdV equations (120) has again the nonisospectral zero curvature representation in the form (118).

7  CONCLUSIONS

For the KdV hierarchy (2), the related stationary systems (22) have two different representations (24) and (25), being particular Stäckel systems. On the other hand, starting from the family of such Stäckel systems, one can reconstruct related stationary systems (22) and then the whole KdV hierarchy (2). In this article, we have performed the same procedure for Painlevé deformations of considered Stäckel systems. In consequence, we have constructed two nonautonomous families of KdV hierarchies

\[ u_{t_n,r} = \mathcal{K}_{n,r}(t) = \pi_0\gamma_{n,r}(t), \quad u_{t_{n,n+1}} = \mathcal{K}_{n,n+1}(t) = \pi_0(\gamma_{n,n+1}(t) + x), \quad r = 2, \ldots, n, \quad n \in \mathbb{N} \]

and

\[ u_{\tau_n,r} = \mathcal{K}_{n,r}(\tau) = \pi_1\gamma_{n,r-1}(\tau), \quad u_{\tau_{n,n+1}} = \mathcal{K}_{n,n+1}(\tau) = \pi_1(\gamma_{n,n}(\tau) + x), \quad r = 2, \ldots, n, \quad n \in \mathbb{N} \]

with related nonautonomous stationary systems

\[ u_{t_n,r} = \mathcal{K}_{n,r}(t) = \pi_0\gamma_{n,r}(t), \quad 0 = \gamma_{n,n+1}(t) + x + c, \quad r = 2, \ldots, n, \quad n \in \mathbb{N} \]

and

\[ u_{\tau_n,r} = \mathcal{K}_{n,r}(\tau) = \pi_1\gamma_{n,r-1}(\tau), 0 = \frac{1}{2}\bar{\gamma}_{n,n}(\bar{\gamma}_{n,n})_{xx} - \frac{1}{4}[(\bar{\gamma}_{n,n})_x]^2 + u_{\bar{\gamma}_{n,n}}^2 + c, \quad r = 2, \ldots, n, n \in \mathbb{N}, \]

where \( \bar{\gamma}_{n,n} = \gamma_{n,n}(\tau) + x \), having respective Painlevé representations, considered in Section.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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