The Complexity of Pattern Counting in Directed Graphs, Parameterised by the Outdegree*

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ABSTRACT

We study the fixed-parameter tractability of the following fundamental problem: given two directed graphs \(\overline{H}\) and \(\overline{G}\), count the number of copies of \(\overline{H}\) in \(\overline{G}\). The standard setting, where the tractability is well understood, uses only \(|\overline{H}|\) as a parameter. In this paper we adopt as a parameter \(|\overline{H}| + d(\overline{G})\), where \(d(\overline{G})\) is the maximum outdegree of \(\overline{G}\). Under this parameterisation, we completely characterize the fixed-parameter tractability of the problem in both its non-induced and induced versions through two novel structural parameters, the fractional cover number \(\rho^*\) and the source number \(\alpha_s\). On the one hand we give algorithms with running time \(f(|\overline{H}|, d(\overline{G})) \cdot |\overline{G}|^{\rho^*(\overline{H})+O(1)}\) and \(f(|\overline{H}|, d(\overline{G})) \cdot |\overline{G}|^{\alpha_s(\overline{H})+O(1)}\) for counting respectively the copies and induced copies of \(\overline{H}\) in \(\overline{G}\); on the other hand we show that, unless the Exponential Time Hypothesis fails, for any class \(\mathcal{C}\) of directed graphs the restriction of the problem to patterns in \(\mathcal{C}\) is fixed-parameter tractable if and only if \(\rho^*(\mathcal{C})\) is bounded \((\alpha_s(\mathcal{C}))\) for the induced version). These results explain how the orientation of the pattern can make counting easy or hard, and prove that a classic algorithm by Chiba and Nishizeki and its extensions \((\text{Chiba and Nishizeki, SICOMP '85; Bressan, Algorithmica '21})\) are optimal unless ETH fails.

Our proofs consist of several layers of parameterised reductions that preserve the outdegree of the host graph. To start with, we establish a tight connection between counting homomorphisms from \(\overline{H}\) to \(\overline{G}\) to \#CSP, the problem of counting solutions of constraint satisfactions problems, for special classes of patterns that we call canonical DAGs. To lift these results from canonical DAGs to arbitrary directed graphs, we exploit a combination of several ingredients: existing results for \#CSPs \((\text{Marx JACM 13; Grohe, Marx TALG 14})\), an extension of graph motif parameters \((\text{Curticapean, Dell, Marx STOC 17})\) to our setting, the introduction of what we call monotone reversible minors, and a careful analysis of quotients of directed graphs in order to relate their adaptive width and fractional hypertreewidth to our novel parameters. Along the route we establish a novel bound of the integrality gap for the fractional independence number of hypergraphs based on adaptive width, which might be of independent interest.

CCS CONCEPTS

- Theory of computation \(\rightarrow\) Complexity theory and logic; Problems, reductions and completeness; Parameterized complexity and exact algorithms; Mathematics of computing \(\rightarrow\) Graph algorithms.

KEYWORDS

subgraph counting, pattern counting, graph motifs, induced subgraph counting, directed graph counting, parameterized complexity

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1 INTRODUCTION

We study the complexity of the following fundamental counting problem: given two directed graphs \(\overline{H}\) (the “pattern”) and \(\overline{G}\) (the “host”), count the number of occurrences or induced occurrences of \(\overline{H}\) in \(\overline{G}\). This problem, known as subgraph counting, motif counting, or pattern counting, has gained great popularity because of its apparent ubiquity in a diverse selection of fields, from social network analysis \([49]\) to network science \([41, 42]\), and from database theory \([4, 14, 24, 25, 28]\) and data mining \([1, 48]\) to bioinformatics \([3, 45]\), phylogeny \([34]\), and genetics \([46, 47]\). For this reason, subgraph counting in general has received significant attention from the theoretical community in the last two decades, with a flurry of novel techniques and exciting results \([5, 6, 10, 13, 20–22, 26, 29, 33, 35, 40]\).

Since subgraph counting in general is hard (think of counting cliques), it is common to parameterise the problem so as to allow for a “bad” dependence on some quantity that is believed to be small in practice \([21, 26]\). The standard parameterisation is by the size of \(\overline{H}\), that is, \(|\overline{H}| = |V(\overline{H})| + |E(\overline{H})|\). In that case, one says the problem is fixed-parameter tractable, or in the class FPT, if for some (computable) function \(f\) it admits an algorithm that runs in time \(f(|\overline{H}|) \cdot |\overline{G}|^{O(1)}\) for all \(\overline{H}\) and \(\overline{G}\). This means one considers as efficient an algorithm with running time, say, \(2^{|\overline{H}|} \cdot |\overline{G}|\), but not one with running time \(|\overline{G}|^{|\overline{H}|}\). The rationale is that in practice \(\overline{H}\) is often very small compared to \(\overline{G}\), thus a running time of \(2^{|\overline{H}|} \cdot |\overline{G}|\) is better.
than one of $|\tilde{G}|/\tilde{H}$. Under this parameterisation, the tractability of the problem is well understood: for the undirected version, both the induced and non-induced versions are in FPT if and only if certain invariants of $H$ are bounded [17, 21, 22], and it is not hard to show that the same holds for the directed case as well (see Section 3).

While the parameterisation by $|\tilde{H}|$ is standard, it is also quite restrictive. Consider for instance the problem of counting the induced copies of $\tilde{H}$ in $\tilde{G}$, when parameterised by $|\tilde{H}|$, it is well-known that the problem is in FPT if and only if the pattern size $|\tilde{H}|$ is bounded (see [17] and Appendix B in the full version [12]). Thus, under this parameterisation, one can efficiently count the induced copies of just a finite number of patterns. Suppose instead the parameter is $|\tilde{H}| + d(\tilde{G})$, where $d(\tilde{G})$ is the maximum outdegree of $\tilde{G}$; the problem is then considered tractable if for some (computable) function $f$ it admits an algorithm that runs in time $f(|\tilde{H}|, d(\tilde{G}))\cdot |\tilde{G}|^{O(1)}$ for all $\tilde{H}$ and $\tilde{G}$. It is not hard to see that, under this parameterisation, the problem becomes FPT even for infinite families of patterns. Let indeed $\tilde{H}$ be the acyclic orientation of a $k$-clique: since $\tilde{H}$ has only one source $s$ (a vertex of indegree 0), one can first guess the image of $s$ in $\tilde{G}$ and then iterate over all $(k - 1)$-vertex subsets in the out-neighbourhood of $s$, which yields an algorithm with running time $O(d(\tilde{G})|\tilde{H}|\cdot |\tilde{G}|)$. This idea was in fact extended to counting subgraphs in degenerate host graphs, which have orientations with bounded outdegree [6–9, 13, 30] (see Section 5 for a detailed discussion). Thus, adopting $(|\tilde{H}| + d(\tilde{G})$ as a parameter can open the door to a richer landscape of tractability.

The goal of the present work is to understand precisely what that landscape is; that is, to understand when the aforementioned problems, parameterised by $(|\tilde{H}| + d(\tilde{G}))$, are in FPT as a function of the pattern $\tilde{H}$. In addition to the aforementioned example of pattern counting in degenerate graphs, there is another reason to consider $d(\tilde{G})$ as part of the parameter when counting directed subgraphs: several “real-world” directed graphs that are natural “hosts” have small or constant outdegree. This is true for many web graphs or online social network graphs, where the maximum outdegree is much smaller than the average degree or the maximum indegree; and it is true by construction in graphs produced by generative models such as preferential attachment [2]. As is customary, to express the dependence on the structure of $\tilde{H}$, we formulate the problems as a function of a class $\mathcal{C}$ of patterns — for instance, one may let $\mathcal{C}$ be the class of all directed complete graphs, or of all directed trees. Let then $\mathcal{C}$ denote an arbitrary family of directed graphs, and for any $\tilde{H}$ and $\tilde{G}$ let $\text{#Sub}(\tilde{H} \to \tilde{G})$ and $\text{#IndSub}(\tilde{H} \to \tilde{G})$ denote respectively the number of copies and induced copies of $\tilde{H}$ in $\tilde{G}$. Concretely, we study the problems $\text{#DirSub}_B(\mathcal{C})$ and $\text{#DirIndSub}_B(\mathcal{C})$, which represent the parameterisation by $|\tilde{H}| + d(\tilde{G})$. The goal of the present work is to understand which structural properties of the elements of $\mathcal{C}$ determine whether $\text{#DirSub}_B(\mathcal{C})$ and $\text{#DirIndSub}_B(\mathcal{C})$ are in FPT.

The rest of this manuscript is organised as follows. Necessary preliminaries and precise definitions of the most important technical notions are given in Section 2. Section 3 gives a concise overview of our results and their significance. Section 4 gives a detailed overview of our proofs and their key technical insights. Proofs for all of our claims as well as further details and discussion can be found in the full version of this paper [12].

2 PRELIMINARIES

Given a set $S$, we let $S^2 = S \times S$, and we write $S^{(2)}$ for the set of all unordered pairs of distinct elements of $S$. Let $f : A \times B \to C$ be a function and let $a \in A$. We write $f(a, \ast) : B \to C$ for the function that maps $b \in B$ to $f(a, b)$.

2.1 Graphs and Directed Graphs

We denote graphs by $G, F, H$, and directed graphs by $\tilde{F}, \tilde{G}, \tilde{H}$. Graphs and digraphs are encoded via adjacency lists, and we write $|G|$ (resp. $|\tilde{G}|$) for the length of the encoding. In the remainder of the paper we will call directed graphs just “digraphs”. We use $(u, v)$ for undirected edges, and $(u, v)$ for directed edges, which we also call arcs. Furthermore, we will use $C$ to denote classes of graphs, and $\tilde{C}$ to denote classes of digraphs. Our graphs do not contain multiedges; however, we allow forward-backward arcs $(u, v)$ and $(v, u)$ in digraphs. Additionally, our undirected graphs do not contain loops (edges from a vertex to itself) unless stated otherwise. For technical reasons, we will allow loops in digraphs.

A directed acyclic graph (DAG) is a digraph without directed cycles. A source of a DAG is a vertex with in-degree 0, and a sink of a DAG is a vertex with out-degree 0 (and an isolated vertex is simultaneously a source and a sink). Given a directed (not necessarily acyclic) graph $\tilde{H}$ and a vertex $v \in V(\tilde{H})$, we write $R(v)$ for the set of vertices reachable from $v$ by a directed path; this includes $v$ itself. Given a set of vertices $S \subseteq V(\tilde{H})$, we set

$$R(S) = \bigcup_{v \in S} R(v) \, .$$

Let $H$ be a graph and $\sigma$ a partition of $V(H)$. The quotient graph of $H$ w.r.t. $\sigma$, denoted by $H/\sigma$, is defined as follows: $V(H/\sigma)$ consists of the blocks of $\sigma$, and $(B_1, B_2) \in E(H/\sigma)$ if and only if $v_1, v_2 \in E(H)$ for some $v_1 \in B_1$ and $v_2 \in B_2$. If $H/\sigma$ does not contain loops then it is called a spasm [21]. These definitions can be adapted in the obvious way for digraphs.

Definition 1 (The DAG $\tilde{H}/\sim$). Let $\tilde{H}$ be a digraph and let $x,y \in V(\tilde{H})$. We denote by $\sim$ the equivalence relation over $V(\tilde{H})$ whose classes are the strongly connected components of $\tilde{H}$. We denote by $\tilde{H}/\sim$ the DAG obtained by deleting loops from the quotient of $\tilde{H}$ with respect to the partition of $V(\tilde{H})$ given by $\sim$.

Next we introduce some notions that will be used in our classifications.

Definition 2 (Directed split). The directed split $\tilde{H}^2$ of a graph $H$ is the digraph obtained from the 1-subdivision of $H$ by orienting all edges towards $V(H)$.

Homomorphisms and Colourings. A homomorphism from $H$ to $G$ is a map $\varphi : V(H) \to V(G)$ such that $\varphi(u), \varphi(v) \in E(G)$ whenever $(u,v) \in E(H)$. The set of all homomorphisms from $H$ to $G$ is denoted by $\text{Hom}(H \to G)$. An $H$-colouring of $G$ is a homomorphism $c \in \text{Hom}(\tilde{G} \to H)$. An $H$-coloured graph is a pair $(G,c)$.

Footnote 1: We will see and state explicitly, that all of our hardness results will also entail corresponding hardness in the restricted case of digraphs without loops.
where \( G \) is a graph and \( c \) an \( H \)-colouring of \( G \). A homomorphism \( \varphi \in \text{Hom}(H \to G) \) is color-prescribed (by \( c \)) if \( c(\varphi(v)) = v \) for all \( v \in V(H) \). We write \( \text{Hom}(H \to (G, c)) \) for the set of all homomorphisms from \( H \) to \( G \) color-prescribed by \( c \). These definitions can be adapted for digraphs in the obvious way; we emphasise that a homomorphism \( \varphi \) between digraphs must preserve the direction of the arcs, i.e., \((\varphi(u), \varphi(v)) \in E(\tilde{G})\) whenever \((u, v) \in E(\tilde{H})\).

\section{2.2 Hypergraphs}

A hypergraph is a pair \( H = (V, E) \) where \( V \) is a finite set and \( E \subseteq 2^V \setminus \emptyset \). The elements of \( E \) are called hyperedges or simply edges. Given \( X \subseteq V \), the subhypergraph of \( H \) induced by \( X \) is the hypergraph \( H[X] \) with vertex set \( X \) and edge set \( \{e \cap X : e \in E\} \setminus \emptyset \). The \textit{arity} of a hypergraph \( G \) is \( a(G) = \max\{|e| : e \in E(G)\} \). We denote hypergraphs with the symbols \( G, H, \ldots \).

\textbf{Definition 3} (Tree decompositions). Let \( H \) be a hypergraph. A \textit{tree decomposition} of \( H \) is a pair of a tree \( T \) and a set of subsets \( \mathcal{B} = \{B_t \subseteq V(T) \} \) such that the following conditions are satisfied:

1. \( \bigcup_{t \in V(T)} B_t = V(H) \).
2. For every hyperedge \( e \in E(H) \) there is a bag \( B_t \) such that \( e \subseteq B_t \).
3. For every vertex \( v \in V(H) \), the subgraph \( T'[\{t : v \in B_t\}] \) of \( T \) is connected.

\textbf{Definition 4} (\( f \)-width). Let \( H \) be a hypergraph, let \( f : 2^{V(H)} \to \mathbb{R}_+ \), and let \( (T, \mathcal{B}) \) be a tree decomposition of \( H \). The \textit{\( f \)-width} of \( (T, \mathcal{B}) \) is defined as follows:

\[ f \text{-width}(T, \mathcal{B}) := \max_{t \in V(T)} f(B_t). \]

The \( f \)-width of \( H \) is the minimum \( f \)-width of any tree decomposition of \( H \).

For example, the \textit{treewidth} of a (hyper-)graph is just its \( f \)-width for the function \( f(B) := |B| - 1 \).

Given a hypergraph \( H \) and a vertex-subset \( X \) of \( H \), the \textit{edge cover number} \( \rho_H(X) \) of \( X \) is the minimum number of hyperedges of \( H \) required to cover each vertex in \( X \). The \textit{edge cover number} of \( H \), denoted by \( \rho_H \), is defined as \( \rho_H(V(H)) \).

A \textit{fractional} version of the edge cover number of \( H \) is defined similarly: given \( H \) and \( X \) as above, a function \( \gamma : E(H) \to [0, 1] \) is a fractional edge cover of \( X \) if for each \( v \in X \) we have \( \sum_{e \in X} \gamma(e) \geq 1 \). The fractional edge cover number \( \rho_H^*(X) \) of \( X \) is the minimum of \( \sum_{e \in E(H)} \gamma(e) \) among all fractional edge covers \( \gamma \) of \( X \). The \textit{fractional edge cover number} of \( H \) is \( \rho_H^*(H) := \rho_H^*(V(H)) \).

\textbf{Definition 5} (Generalised and Fractional Hypertreewidth). The \textit{generalised hypertreewidth} of \( H \), denoted by \( \text{htw}(H) \), is the \( \rho_H^* \)-width of \( H \). The \textit{fractional hypertreewidth} of \( H \), denoted by \( \text{htw}^*(H) \), is the \( \rho_H^* \)-width of \( H \).

An \textit{independent set} of a hypergraph \( H \) is a set \( I \) of vertices such that, for each \( u, v \in I \) with \( u \neq v \), there is no hyperedge containing \( u \) and \( v \). The \textit{independence number} of \( H \), denoted by \( \alpha(H) \), is the size of a maximum independent set of \( H \). A \textit{fractional independent set} of a hypergraph \( H \) is a mapping \( \alpha^* : V(H) \to [0, 1] \) such that for each \( e \in E(H) \) we have \( \sum_{v \in e} \alpha^*(v) \leq 1 \).

The \textit{fractional independence number} of \( H \), denoted by \( \alpha^*(H) \), is the maximum of \( \sum_{v \in V} \alpha^*(v) \) among all fractional independent sets \( \alpha^* \).

For a subset \( X \) of vertices in \( V(H) \), we set \( \alpha^*(X) = \sum_{v \in X} \alpha^*(v) \). We remark that, by LP duality, the fractional independence number and the fractional edge cover number are equal (see, \cite{44}):

\textbf{Fact 6}. \textit{Let} \( H \) \textit{be a hypergraph}. \textit{We have} \( \alpha^*(H) = \rho_H^*(H) \).

We continue with the notion of adaptive width, which is equivalent\(^2\) to submodular width as shown by Marx \cite{38}.

\textbf{Definition 7} (Adaptive width). Let \( H \) be a hypergraph. The \textit{adaptive width} of \( H \) is

\[ \text{aw}(H) := \sup \{ \text{width}(H) \mid \alpha^* \text{ is a fractional ind. set of } H \} \]

\textbf{Lemma 8} (\cite{38}). \textit{Let} \( C \) \textit{be a class of hypergraphs}. \textit{Then}

\[ \text{aw}(C) = \infty \Rightarrow \text{htw}(C) = \infty \Rightarrow \text{htw}(C) = \infty \]

\textit{Furthermore all of the above implications are strict, that is, there are hypergraph classes} \( C_1 \) \textit{and} \( C_2 \) \textit{such that}\( \text{aw}(C_1) < \infty \text{ but } \text{htw}(C_1) = \infty \) \textit{and that}\( \text{htw}(C_2) < \infty \text{ but } \text{htw}(C_2) = \infty \).

\section{2.3 Relational Structures}

A \textit{signature} \( r \) is a (finite) tuple of relation symbols \( (R_i)_{i \in [r]} \) with arities \( (a_i)_{i \in [r]} \). The \textit{arity} of \( r \), denoted by \( a(r) \), is the maximum of the \( a_i \). A \textit{relational structure} \( \mathcal{A} \) of signature \( r \) is a tuple \((V, R_{\mathcal{A}}^{\mathcal{A}} \ldots, R_{\mathcal{A}}^{\mathcal{A}})\) where \( V \) is a finite set of elements, called the \textit{universe} of \( \mathcal{A} \), and \( R_{\mathcal{A}}^{\mathcal{A}} \) is a relation on \( V \) of arity \( a_i \) for each \( i \in [r] \). We emphasize that \( R_{\mathcal{A}}^{\mathcal{A}} \) is not necessarily symmetric, and that tuples might contain repeated elements. We will mainly use the symbols \( \mathcal{A} \) and \( \mathcal{B} \) to denote relational structures. Further, we assume that a structure \( \mathcal{A} \) is encoded in the standard way, i.e., the universe and the relations are encoded as lists. We denote by \(|\mathcal{A}|\) the length of the encoding of \( \mathcal{A} \).

\(^2\)Here, \textit{"equivalent"} means that a class of hypergraphs has bounded adaptive width if and only if it has bounded submodular width.
Given two relational structures $\mathcal{A}$ and $\mathcal{B}$ over the same signature $\tau$ with universes $U$ and $V$, a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a mapping $\varphi : U \rightarrow V$ such that, for each $i \in [a(\tau)]$ and for each tuple $t \in U^{|\tau|}$ we have

$$t \in R^\mathcal{A}_i \implies \varphi(t) \in R^\mathcal{B}_i.$$  

We let $\text{Hom}(\mathcal{A} \rightarrow \mathcal{B})$ be the set of homomorphisms from $\mathcal{A}$ to $\mathcal{B}$.

The hypergraph $\mathcal{H}(\mathcal{A})$ of $\mathcal{A}$ has as vertices the universe $V$ of $\mathcal{A}$, and for each tuple $t = \langle v_1, \ldots, v_\kappa \rangle$ of elements of $V$, we add an hyperedge $e_t = \{v_1, \ldots, v_\kappa\}$ if and only if $t$ is an element of a relation of $\mathcal{A}$. To avoid notational clutter, we will define the treewidth, the hypertreewidth, the fractional hypertreewidth and the submodular width of a structure as the respective width measure of its hypergraph. Similarly, a tree decomposition of a structure refers to a tree decomposition of its hypergraph.

### 2.4 Parameterised and Fine-Grained Complexity Theory

A parameterised counting problem is a pair $(P, \kappa)$ of a counting problem $P : \{0, 1\}^* \rightarrow \mathbb{N}$ and a computable function $\kappa : \{0, 1\}^* \rightarrow \mathbb{N}$, called the parameterisation. Consider for example the parameterised clique counting problem:

**#CLIQUE**

Input: a pair of a graph $G$ and a positive integer $k$

Output: the number of $k$-cliques in $G$

Parameter: $k$, that is, $\kappa(G, k) := k$

An algorithm for a parameterised (counting) problem is called a fixed-parameter tractable (FPT) algorithm if there is a computable function $f$ such that, on input $x$, its running time can be bounded by $f(\kappa(x)) \cdot |x|^{O(1)}$. A parameterised (counting) problem is called fixed-parameter tractable if it can be solved by an FPT algorithm.

A parameterised Turing-reduction from $(P, \kappa)$ to $(P', \kappa')$ is an FPT algorithm for $(P, \kappa)$ with oracle access to $P'$, additionally satisfying that there is a computable function $g$ such that, on input $x$, the parameter $\kappa'(y)$ of any oracle query is bounded by $g(\kappa(x))$. We write $(P, \kappa) \leq_{\text{IT}}^f (P', \kappa')$ if a parameterised Turing-reduction exists.

We say that $(P, \kappa)$ is $\#W[1]$-hard if $\#\text{CLIQUE} \leq_{\text{fpt}}^f (P, \kappa)$. The class $\#W[1]$ can be considered a parameterised counting equivalent of NP, and we refer the interested reader to Chapter 14 in the standard textbook of Flum and Grohe [27] for a comprehensive introduction. It is known that $\#W[1]$-hard problems are not fixed-parameter tractable unless standard assumptions, such as ETH, fail.

**Definition 9** (The Exponential Time Hypothesis (ETH) [32]). The Exponential Time Hypothesis (ETH) asserts that 3-SAT cannot be solved in time $\exp(o(n))$, where $n$ is the number of variables.

**Theorem 10** (Chen et al. [15, 16]). Assume that ETH holds. Then there is no function $f$ such that $\#\text{CLIQUE}$ can be solved in time $f(k) \cdot |G|^{o(k)}$.

Note that the previous theorem rules out an FPT algorithm for $\#\text{CLIQUE}$ (and thus all $\#W[1]$-hard problems), unless ETH fails.

### Parameterised Counting Problems

The following parameterised problems are central to the present work. In what follows $\mathcal{C}$ denotes a class of directed graphs, and $\mathcal{C}$ denotes a class of hypergraphs.

#### #DirHom$_3(\mathcal{C})$

**Input:** a pair of digraphs $(\tilde{H}, \tilde{G})$ with $\tilde{H} \in \mathcal{C}$

**Output:** $\#\text{Hom}(\tilde{H} \rightarrow \tilde{G})$

**Parameter:** $|\tilde{H}| + d$ where $d$ is the maximum outdegree of $\tilde{G}$

#### #DirSub$_d(\mathcal{C})$

**Input:** a pair of digraphs $(\tilde{H}, \tilde{G})$ with $\tilde{H} \in \mathcal{C}$

**Output:** $\#\text{Sub}(\tilde{H} \rightarrow \tilde{G})$

**Parameter:** $|\tilde{H}| + d$ where $d$ is the maximum outdegree of $\tilde{G}$

#### #DirIndSub$_d(\mathcal{C})$

**Input:** a pair of digraphs $(\tilde{H}, \tilde{G})$ with $\tilde{H} \in \mathcal{C}$

**Output:** $\#\text{IndSub}(\tilde{H} \rightarrow \tilde{G})$

**Parameter:** $|\tilde{H}| + d$ where $d$ is the maximum outdegree of $\tilde{G}$

#### #CP-DirHom$_d(\mathcal{C})$

**Input:** a pair of digraphs $(\tilde{H}, \tilde{G})$ where $\tilde{H} \in \mathcal{C}$ and an $\tilde{H}$-coloured digraph $(\tilde{G}, c)$

**Output:** $\#\text{Hom}(\tilde{H} \rightarrow (\tilde{G}, c))$

**Parameter:** $|\tilde{H}| + d$ where $d$ is the maximum outdegree of $\tilde{G}$

#### #CSP($\mathcal{C}$)

**Input:** a pair of relational structures $(\mathcal{A}, \mathcal{B})$ over the same signature with $\mathcal{H}(\mathcal{A}) \in \mathcal{C}$

**Output:** $\#\text{Hom}(\mathcal{A} \rightarrow \mathcal{B})$

**Parameter:** $|\mathcal{A}|$

It was shown by Grohe and Marx [31] that the decision version of $\#\text{CSP}(\mathcal{C})$ can be solved in polynomial time if the fractional hypertreewidth of $\mathcal{C}$ is bounded. More precisely, they discovered an algorithm that solves the decision problem in time

$$O((|\mathcal{A}| + |\mathcal{B}|)^{\kappa} + O(1)),$$

assuming that a tree decomposition of $\mathcal{A}$ of $\rho^*$-width at most $r$ is given (see Theorem 3.5 and Lemma 4.9 in [31]); recall that the $\rho^*$-width of a tree decomposition is the maximum fractional edge cover number of a bag. In particular, they show that the partial solutions of each bag can be enumerated in time $O((|\mathcal{A}| + |\mathcal{B}|)^{\kappa} + O(1))$. Thus the dynamic programming algorithm that solves the decision version immediately extends to counting. Finally, since computing such an optimal tree decomposition can be done in time only depending on $\mathcal{A}$, we obtain the following overall running time for the counting problem:

**Theorem 11.** Let $\mathcal{A}$ and $\mathcal{B}$ be relational structures over the same signature and let $r = \text{fhtw}(\mathcal{A})$. There is a computable function $f$ such that we can compute $\#\text{Hom}(\mathcal{A} \rightarrow \mathcal{B})$ in time

$$f(|\mathcal{A}|) \cdot |\mathcal{B}|^{\kappa} + O(1).$$

In particular, $\#\text{CSP}(\mathcal{C})$ is fixed-parameter tractable if $\text{fhtw}(\mathcal{C}) < \infty$.

### 3 RESULTS

We give complete complexity classifications for $\#\text{DirSub}_{d}(\mathcal{C})$ and $\#\text{DirIndSub}_{d}(\mathcal{C})$, into FPT versus non-FPT cases, as a function of $\mathcal{C}$.  

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These complexity classifications, which are formally stated below, have the succinct form "The problem is in FPT if and only if \( \rho(\tilde{C}) \) is bounded", where \( p \) is some parameter measuring the structural complexity of the graphs in \( \tilde{C} \). The definition of those parameters is not elementary and requires the introduction of some ancillary notation and definitions, which we are going to do next. In order to understand why those parameters are the right ones, instead, one should take the technical tour of Section 4.

Let us then introduce our structural parameters. First, we need to define reachability hypergraphs and contours. Let \( \tilde{H} \) be a directed graph, and let \( S \) be the set of its strongly connected components. Denote by \( \sim \) the equivalence relation over \( V(\tilde{H}) \) given by \( S \), and let \( \tilde{H}/\sim \) be the quotient of \( \tilde{H} \) w.r.t. \( \sim \), that is, each strongly connected component in \( S \) becomes a vertex, and we add an arc from \( S_1 \) to \( S_2 \) if there is an arc from a vertex in \( S_1 \) to a vertex in \( S_2 \) (see Section 2 for the formal definition). A strongly connected component \( S \in V(\tilde{H}/\sim) \) is a source if it has in-degree 0 in \( \tilde{H}/\sim \). Let \( S_1, \ldots, S_k \) be the set of all such sources. For any \( S \in V(\tilde{H}/\sim) \) let \( R(S) \) be the set of vertices reachable from \( S \) in \( \tilde{H}/\sim \).

**Definition 12.** The reachability hypergraph of \( \tilde{H} \), denoted by \( R(\tilde{H}) \), is the hypergraph with vertex set \( V(\tilde{H}) \) and edge set \( \{ R(S_i) : i \in [k] \} \).

Intuitively, \( R(\tilde{H}) \) measures the complexity of \( \tilde{H} \) in terms of "reachability relationships". However, to state our classifications correctly, we need to consider a slight modification of \( R(\tilde{H}) \).

**Definition 13.** The contour of \( \tilde{H} \), denoted by \( \Gamma(\tilde{H}) \), is the hypergraph \( R(\tilde{H}) \setminus \cup_{i \in [k]} S_i \).

For instance, if \( \tilde{H}_a \) is obtained by orienting the edges of the 1-subdivision of the complete graph \( K_a \) towards the original vertices, then \( \Gamma(\tilde{H}_a) = K_a \).

**Definition 14.** The fractional cover number of \( \tilde{H} \), denoted by \( \rho^*(\tilde{H}) \), is the fractional edge cover number of the contour of \( \tilde{H} \), that is, \( \rho^*(\tilde{H}) = \rho^*(\Gamma(\tilde{H})) \). The source number of \( \tilde{H} \), denoted by \( \alpha_s(\tilde{H}) \), is the number of sources in \( S \).

Note that \( \alpha_s(\tilde{H}) \) is the number of strongly connected components of \( \tilde{H} \) that are not reachable from any other connected component. Intuitively, both \( \rho^* \) and \( \alpha_s \) measure the complexity of covering \( \tilde{H} \) through its sources. Our main result, the following dichotomy theorem, says that such a "covering complexity" determines the fixed-parameter tractability of our problems. For any class \( \tilde{C} \) of directed graphs let \( \rho^*(\tilde{C}) = \sup_{G \in \tilde{C}} \rho^*(H) \) and \( \alpha_s(\tilde{C}) = \sup_{G \in \tilde{C}} \alpha_s(H) \).

**Theorem 15.** If the Exponential Time Hypothesis holds, then:
1. \( \#\text{DirIndSub}_3(\tilde{C}) \in \text{FPT} \) if and only if \( \rho^*(\tilde{C}) < \infty \)
2. \( \#\text{DirIndSub}_4(\tilde{C}) \in \text{FPT} \) if and only if \( \alpha_s(\tilde{C}) < \infty \)

Note that ETH is used only by the "only if" direction. While the statement of Theorem 15 is simple, its proof is nontrivial — virtually all of this manuscript is devoted to it. To put the theorem into perspective, Table 1 compares it to dichotomies for the other variants of the problem. We also observe that Theorem 15 can be slightly strengthened: we can show the hardness direction even for acyclic host graphs.

As a consequence of Theorem 15, we can claim the optimality (in an FPT sense) of the well-known approach to counting the induced copies of a DAG \( \tilde{H} \) in a host \( G \) of bounded outdegree, used in several recent works on counting in hosts of bounded degeneracy [6–9, 13, 30]. This approach consists in guessing the images of the sources of \( \tilde{H} \) in \( G \), and running time \( f(|\tilde{H}|, d(\tilde{G})) \cdot |\tilde{G}|^\alpha_s(\tilde{H}) + O(1) \). By Theorem 15, unless ETH fails the dependence on \( \alpha_s(\tilde{H}) \) at the exponent cannot be avoided, hence that approach is optimal in an FPT sense.

It shall be noted that, for \( \#\text{DirIndSub}_3(\tilde{C}) \), the non-FPT case in Theorem 15 also yields \( \#\text{W}[1]-\text{hardness} \) (see Section 2 for a definition). For \( \#\text{DirSub}_4(\tilde{C}) \) instead we do not prove \( \#\text{W}[1]-\text{hardness} \); the reason is that our proof uses a reduction from certain families of CSP instances which by [38] we know to be not in FPT if ETH holds, but which are not known to be \( \#\text{W}[1]-\text{hard} \).

When the problems in Theorem 15 are in FPT, we can show simple algorithms that solve them in time \( f(|\tilde{H}|, d(\tilde{G})) \cdot |\tilde{G}|^\rho(\tilde{H}) + O(1) \) where \( \rho \in \{ \rho^*, \alpha_s \} \). Formally, we prove:
THEOREM 16. For some computable function \( f \) there is an algorithm solving \( \text{#DirSub}_d(\tilde{C}) \) in time \( f(|\tilde{H}|, d(\tilde{G})) \cdot |\tilde{G}|^{\rho^*(\tilde{H})+O(1)} \). The same holds for \( \text{#DirIndSub}_d(\tilde{C}) \) with \( \alpha_\delta \) in place of \( \rho^* \).

We point out that theorems 15 and 16 remain true in the edge-weighted or vertex-weighted setting, too.

A simple example shows Theorem 15 and Theorem 16 in action. Let \( \Delta_1 \) and \( \Delta_2 \) be respectively the cyclic and acyclic orientations of \( K_3 \), and for each \( k \in \mathbb{N} \) let \( \Delta^k_1 \) and \( \Delta^k_2 \) consist of \( k \) disjoint copies of respectively \( \Delta_1 \) and \( \Delta_2 \). Finally, let \( \tilde{C}_1 = \{ \Delta^k_1 : k \in \mathbb{N} \} \) and \( \tilde{C}_2 = \{ \Delta^k_2 : k \in \mathbb{N} \} \). Although the patterns are rather elementary, establishing the tractability of \( \text{#DirSub}_d(\tilde{C}_1) \) and \( \text{#DirSub}_d(\tilde{C}_2) \) “by hand” can be laborious. Theorem 15 and Theorem 16 answer immediately: \( \rho^*(\Delta^k_1) = 0 \), since in \( \Delta^k_1 \) every vertex belongs to some source, hence \( \text{#DirSub}_d(\tilde{C}_1) \) is fixed-parameter tractable and solvable in time \( f(|\tilde{H}|, d(\tilde{G})) \cdot |\tilde{G}|^{O(1)} \); but \( \rho^*(\Delta^k_2) = k \), since \( \Gamma(\Delta^k_2) \) has \( k \) disjoint hyperedges, hence \( \text{#DirSub}_d(\tilde{C}_2) \) is not fixed-parameter tractable unless ETH fails. One can also see that \( \alpha_\delta(\Delta^k_1) = \alpha_\delta(\Delta^k_2) = k \); therefore, by Theorem 15, under ETH both \( \text{#DirIndSub}_d(\tilde{C}_1) \) and \( \text{#DirIndSub}_d(\tilde{C}_2) \) are not fixed-parameter tractable.

Another example shows the difference between our parameterisation and the standard one, as well as the necessity of \( \rho^* \) being fractional. Let \( H_k \) be the graph defined as follows. The vertices of \( H_k \) are \( U_k \cup D_k \) where \( U_k = \{1, \ldots, 2k\} \) and \( D_k = \{A \subseteq U_k \mid |A| = k\} \), and for each \( i \in U_k \), there is an edge between \( i \) and \( D \in D_k \) if and only if \( i \in D \). Let \( C \) be the class of all \( H_k \). It is not hard to show that \( H_k \) contains the subdivision of the \( k \)-clique as induced subgraph. Thus the vertex-cover number of \( C \) is unbounded and, assuming ETH, Table 1 yields that \( \text{#Sub}(C) \) and \( \text{#DirSub}(\tilde{C}) \) are not fixed-parameter tractable for any class \( \tilde{C} \) obtained by orienting the graphs in \( C \). However, if we parameterise also by the outdegree of the host, then the situation becomes much more subtle. Let \( \tilde{C} \) be the class of digraphs obtained by orienting the edges in the \( H_k \) from \( U_k \) to \( D_k \); an argument similar to [31, Example 4.2] shows that \( \rho^*(\tilde{H}_k) \leq 2 \) for each \( \tilde{H}_k \in \tilde{C} \), thus \( \text{#DirSub}_d(\tilde{C}) \) is fixed-parameter tractable by Theorem 16. Moreover, [31, Example 4.2] show that any non-fractional cover of \( \tilde{H}_k \) has super-constant weight; this proves that considering the fractional cover number \( \rho^* \) is crucial; its integral counterpart cannot work.

We conclude this section with a result of independent interest developed in our proofs. Let \( \mathcal{H} \) be a hypergraph. The independence number \( \alpha(\mathcal{H}) \) of \( \mathcal{H} \) is the size of the largest subset of \( V(\mathcal{H}) \) such that no two of its elements are contained in a common edge. The natural relaxation of this definition yields the fractional independence number \( \alpha^*(\mathcal{H}) \). Our result says that the integrality gap of \( \alpha \), i.e. the ratio between \( \alpha^*(\mathcal{H}) \) and \( \alpha(\mathcal{H}) \), is bounded by the adaptive width of \( \mathcal{H} \).

THEOREM 17. Every hypergraph \( \mathcal{H} \) satisfies \( \alpha(\mathcal{H}) \geq \frac{1}{2} \cdot \frac{\alpha^*(\mathcal{H})}{aw(\mathcal{H})} \).
4 TECHNICAL OVERVIEW

This section gives an overview of the tools and techniques behind the results of Section 3. The overview focuses on #DirSub_d(C), but similar arguments apply to #DirIndSub_d(C). Before digging into the most technical part, let us give the high-level idea of our proof strategy.

At the root of all our results is a standard connection between copies and homomorphisms, explained in Section 4.1. It is well known indeed that #Sub(\overrightarrow{H} \rightarrow \overrightarrow{G}) can be expressed as a linear combination of homomorphism counts, \( \sum_{\overrightarrow{F}} \overrightarrow{a}_\overrightarrow{H}(\overrightarrow{F}) \cdot \#\text{Hom}(\overrightarrow{F} \rightarrow \overrightarrow{G}) \), where \( \overrightarrow{a}_\overrightarrow{H}(\overrightarrow{F}) > 0 \) and \( \overrightarrow{F} \) ranges over a certain set of quotients of \( \overrightarrow{H} \). Here, a quotient of \( \overrightarrow{H} \) is a directed graph obtained from \( \overrightarrow{H} \) by contracting (not necessarily connected) vertex subsets into single vertices (see Section 2 for the formal definition). It is also known that the complexity of computing \( #\text{Sub}(\overrightarrow{H} \rightarrow \overrightarrow{G}) \) equals, up to \( f(|\overrightarrow{H}|) \) factors, that of computing the hardest \( #\text{Hom}(\overrightarrow{F} \rightarrow \overrightarrow{G}) \) term. Therefore we can reduce \#DirSub_d(C) to and from its homomorphism counting version \#DirHom_d(\overrightarrow{C}), where \( \overrightarrow{C} \) consists of certain quotients of \( \overrightarrow{C} \).

Armed with these results, we proceed as follows.

First, in Section 4.2 we prove that \( \rho^*(\overrightarrow{C}) < \infty \) implies that \#DirSub_d(C) \( \in \text{FPT} \). To this end we prove that if \( \overrightarrow{F} \) is a quotient of \( \overrightarrow{H} \) then the fractional hypertreedwidth of the contour of \( \overrightarrow{F} \) satisfies \( \text{fhtw}(\Gamma(\overrightarrow{F})) \leq \rho^*(\overrightarrow{H}) \). Therefore, \( \text{fhtw}(\Gamma(\overrightarrow{C})) \leq \rho^*(\overrightarrow{C}) < \infty \). We then show that computing \( #\text{Hom}(\overrightarrow{F} \rightarrow \overrightarrow{G}) \) can be reduced in FPT time to counting the homomorphisms from \( \Gamma(\overrightarrow{F}) \) to a hypergraph \( \overrightarrow{G} \) obtained from \( \overrightarrow{F} \) and \( \overrightarrow{G} \). As hypergraph homomorphism counting is in FPT when the pattern has bounded fractional hypertreedwidth, this proves the claim.

Next, in Section 4.3 we prove that \( \rho^*(\overrightarrow{C}) = \infty \) implies that \#DirSub_d(C) \( \notin \text{FPT} \), or ETH fails. To start with, we suppose \( \overrightarrow{C} \) contains only canonical DAGs, directed graphs of a particularly simple type. We can prove that the aforementioned problem of counting homomorphisms between hypergraphs can be reduced to \#DirHom_d(\overrightarrow{C}) if the considered hypergraph patterns belong to the contours of \( \overrightarrow{C} \). By existing results this implies that, unless ETH fails, \#DirHom_d(\overrightarrow{C}) \( \notin \text{FPT} \) whenever the contours of \( \overrightarrow{C} \) have unbounded adaptive width [39]. It remains to lift these results from canonical DAGs to arbitrary DAGs and, ultimately, to arbitrary directed graphs. To this end, we introduce what we call monotone reversible minors (MRMs). Intuitively, \( \overrightarrow{H} \) is an MRM of \( \overrightarrow{H} \) if there exists an FPT reduction from counting copies \( \overrightarrow{H} \) to counting copies of \( \overrightarrow{H} \), and if \( \overrightarrow{H} \) preserves some parameters of interest (like \( \rho^* \)). We show that every directed graph \( \overrightarrow{H} \) has an MRM \( \overrightarrow{H}' \) that is a canonical DAG, so counting \( \overrightarrow{H} \) is at least as hard as counting \( \overrightarrow{H}' \). Next, we show that counting copies of \( \overrightarrow{H}' \) is hard. To this end we show that, if \( \rho^*(\overrightarrow{H}') \) is large, then its contour \( \Gamma(\overrightarrow{H}') \) has large adaptive width or large independence number. By employing arguments from the homomorphism connection above and from [13], this implies that counting the copies of \( \overrightarrow{H}' \) is hard unless ETH fails, which concludes our proof.

4.1 The Directed Homomorphism Basis

The first ingredient of our work is the so-called homomorphism basis introduced by Curticapean, Dell, and Marx [21], which establishes a common connection between (undirected) parameterised pattern counting problems. Although the original framework is for undirected graphs, it can be equally well be formulated for the directed case, as we are going to do. Let \( \overrightarrow{H} \) be a digraph. There is a function \( \text{sub}_\overrightarrow{H} \) of finite support from digraphs to rationals such that for each digraph \( \overrightarrow{G} \):

\[
\text{sub}_\overrightarrow{H}(\overrightarrow{G}) = \sum_{\overrightarrow{F}} \text{sub}_\overrightarrow{H}(\overrightarrow{F}) \cdot \#\text{Hom}(\overrightarrow{F} \rightarrow \overrightarrow{G})
\]

This identity follows by well-known transformations based on inclusion-exclusion and Möbius inversion (see e.g. Chapter 5.2.3. in Lovász [36]). It is also well known that \( \text{sub}_\overrightarrow{H}(\overrightarrow{F}) \neq 0 \) if and only if \( \overrightarrow{F} \) is a quotient of \( \overrightarrow{H} \).

These facts allow us to construct a reduction from the parameterised problem of computing \#Sub(\overrightarrow{H} \rightarrow \overrightarrow{G}) to the parameterised problem of computing \#Hom(\overrightarrow{H} \rightarrow \overrightarrow{G}) and vice versa. More precisely, one can show that computing \#Sub(\overrightarrow{H} \rightarrow \overrightarrow{G}) is precisely as hard (in FPT-equivalence terms) as computing the hardest term \#Hom(\overrightarrow{F} \rightarrow \overrightarrow{G}) in the summation of (1). One direction is obvious — the time to compute \#Sub(\overrightarrow{H} \rightarrow \overrightarrow{G}) is the sum of the times to compute all terms \#Sub(\overrightarrow{F} \rightarrow \overrightarrow{G}) whose number is a function of \( \overrightarrow{H} \). The other direction is nontrivial, and was established for multiple variants of subgraph counting over the past years [6, 14, 21, 24, 43]. Rather than extending those results to yet another variant (directed graphs), we observe that the constructive version of Dedekind’s Theorem on the linear independence of characters yields a general interpolation method that subsumes all those results, including the one for directed graphs. We prove what follows (see Theorem 36 in [12] for a more complex but complete version):

**Theorem 18 (Simplified Version).** Let \((G, +) \) be a semigroup. Let furthermore \((\phi_i)_{i \in [k]} \) with \( \phi_i : G \rightarrow \mathbb{Q} \) be pairwise distinct and non-zero semigroup homomorphisms of \((G, +) \) into \((\mathbb{Q}, +) \), that is, \( \phi_i(g_1 + g_2) = \phi_i(g_1) + \phi_i(g_2) \) for all \( i \in [k] \) and \( g_1, g_2 \in G \). Let \( \phi : G \rightarrow \mathbb{Q} \) be a function

\[
\phi : g \mapsto \sum_{i=1}^{k} a_i \cdot \phi_i(g),
\]

where the \( a_i \) are rational numbers. Then there is an efficient algorithm \( \lambda \), which is equipped with oracle access to \( \phi \) and which computes the coefficients \( a_1, \ldots, a_k \).

In our setting, Theorem 18 yields what follows. First, let \( G \) be the set of all digraphs and \(+\) be the directed tensor product; one can check that \((G, +)\) is indeed a semigroup. Second, for any fixed \( \overrightarrow{H} \) consider the function \( \overrightarrow{G} \mapsto #\text{Hom}(\overrightarrow{H} \rightarrow \overrightarrow{G}) \); one can check this is a semigroup homomorphism into \( \mathbb{Q} \). Using Theorem 18, we can prove:

**Lemma 19.** There exists a deterministic algorithm \( \lambda \) with the following specifications:

- The input of \( \lambda \) is a pair \((\overrightarrow{G'}, i)\) where \( \overrightarrow{G'} \) is a digraph and \( i : G \rightarrow \mathbb{Q} \).

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• $A$ is equipped with oracle access to the function

$$G \mapsto \sum_F \iota(F) \cdot |\text{Hom}(F \to G)|,$$

where the sum is over all (isomorphism classes of) digraphs.

• The output of $A$ is the list with elements $(F, |\text{Hom}(F \to G)|)$ for each $F$ with $\iota(F) \neq 0$.

• For some computable function $f$ the running time of $A$ is bounded by $f(|\iota|) \cdot |G|^{O(1)}$.

• The outdegree of every digraph $G$ on which $A$ invokes the oracle is at most $f(|\iota|) \cdot d(G')$ where $d(G')$ is the maximum outdegree of $G'$.

To understand the meaning of Lemma 19, let $\iota(F) = \text{sub}_H(F)$ for all $F \in G$, see (1). Then Lemma 19 says that, if $A$ has oracle access to $\text{#Sub}(H \to G)$, then $A$ can compute $|\text{Hom}(H \to G)|$ efficiently and by computing $\text{#Sub}(H \to G)$ only for $G$ of outdegree not larger than that of $G'$. This yields a parameterised reduction from $\text{DirSub}_d(C)$ to $\text{DirHom}_d(C)$, where $C'$ is the set of all digraphs $F$ such that $\text{sub}_H(F) \neq 0$ for some $H \in C$. As stated above, $\text{sub}_H(F) \neq 0$ if and only if $F$ is a quotient of $H$. We conclude that computing $\text{#Sub}(H \to G)$ is at least as hard as computing $|\text{Hom}(F \to G)|$ for each $F$ that is a quotient of $H$. In other words we have a parameterised reduction from $\text{DirSub}_d(C)$ to $\text{DirHom}_d(C')$ where $C'$ is the set of all quotients of $C$. Together with the converse reduction (see above) this tells us that $\text{DirSub}_d(C)$ is precisely as hard as $\text{DirHom}_d(C)$ where $C'$ is the set of all quotients of $C$. Thus, classifying the complexity of $\text{DirSub}_d(C)$ boils down to understanding the complexity of $\text{DirHom}_d(C')$ where $C'$ is again the set of all quotients of $C$. Answering this question turns out to be the most challenging task in this work.

4.2 Upper Bounds: A Reduction to #CSP

To understand the complexity of $\text{DirHom}_d(C')$ where $C'$ is the set of all quotients of $C$, we take two steps. First, we show that the problem can be reduced to #CSP, the problem of counting the solutions to a constraint satisfaction problem. Second, we show that the fractional cover number of $C'$ bounds the fractional hypertreewidth of the #CSP instances obtained from $C'$, which makes the problem fixed-parameter tractable by existing results.

4.2.1 A Reduction to #CSP. Let $H$ and $G$ be digraphs and let $d$ be the maximum outdegree of $G$. Let furthermore $k = |H|$ and $n = |G|$. Recall that a source $S$ of $H$ is a strongly connected component of $H$ such that $S$ cannot be reached from any other strongly connected component. Let $S_1, \ldots, S_t$ be the sources of $H$, and let $s_j \in S_t$ for each $i \in [t]$. Finally, let $R_i$ be the set of all vertices of $H$ that can be reached from $s_j$ via a directed path — note that $S_j$ is fully contained in $R_i$. Clearly each arc of $H$ is fully contained in at least one of the $R_i$. Writing $H[R_i]$ for the subgraph of $H$ induced by $R_i$, one can see that every map $\varphi : V(H) \to V(G)$ satisfies:

$$\varphi \in \text{Hom}(H \to G) \iff \forall i \in [t] : \varphi|_{R_i} \in \text{Hom}(H[R_i] \to G),$$

where $\varphi|_{R_i}$ is the restriction of $\varphi$ on $R_i$. In other words, $\varphi$ is a homomorphism if and only if it induces a partial homomorphism from $H[R_i]$ for each $i \in [t]$.

The observation above allows us to reduce the computation of $\text{Hom}(H \to G)$ to counting the solutions of a certain constraint satisfaction problem. Start by fixing an arbitrary order over $V(H)$, so that every $R_i$ appears as an ordered tuple. Now, for each $i \in [t]$, we enumerate all partial homomorphisms $\varphi|_{R_i} \in \text{Hom}(H[R_i] \to G)$. It is well known that this can be done in time $f(k, d) \cdot n^{O(1)}$: simply guess the image of $s_i$ in $V(G)$, and perform a brute force search over the $d^{O(k)}$ vertices of $G$ reachable from $s_i$ in $k$ steps [6, 11, 19]. (Note that this strategy crucially depends on the fact that there is only one source $s_i$, hence it cannot be used to compute the whole set $\text{Hom}(H \to G)$ in FPT time.) Now for every $i \in [t]$ consider the set of all (the images of) the maps in $\text{Hom}(H[R_i] \to G)$. This is a set of ordered tuples of vertices of $G$, i.e., a relation over $V(G)$. We denote this relation by $R_i$. It is not hard to see that the homomorphisms from $H$ to $G$ are precisely those maps from $V(H)$ to $V(G)$ that for every $i \in [t]$ send $R_i$ to an element of $R_i$, and that counting those maps is an instance of a counting constraint satisfaction problem (#CSP).

4.2.2 Bounding the Cost of Solving #CSP over Quotients. Recall the reachability hypergraph $\mathcal{R}(H)$: the hypergraph whose vertex set is $V(H)$ and whose edge set is $\{R_i : i \in [t]\}$. A well-known result due to Grohe and Marx [31] states that counting the solutions to the CSP instance above is fixed-parameter tractable whenever $\mathcal{R}(H)$ has bounded fractional hypertreewidth, where the parameter is $|H|$; in fact, [31] shows that there is an algorithm that solves the problem in time $f(k, d) \cdot |V(G)|^{\text{fhw}(\mathcal{R}(H)) + O(1)}$. Now recall from Section 3 the fractional cover number $\rho^*(H)$ of $H$. We prove:

**Lemma 20.** Let $H$ be a digraph, let $\tilde{F}$ be a quotient graph of $H$, and let $\mathcal{R}(\tilde{F})$ be the reachability hypergraph of $\tilde{F}$. Then $\text{fhw}(\mathcal{R}(\tilde{F})) \leq \rho^*(H)$.

The intuition behind the proof of Lemma 20 is that (i) taking the quotient of a digraph cannot increase its fractional cover number, and (ii) the fractional hypertreewidth of a hypergraph is bounded by its fractional edge cover number (which is the fractional cover number of $H$).

Together with the observations above, this implies that we can compute $\text{#Sub}(H \to G)$ in time $f(|H|, d(G)) \cdot |G|^{\rho^*(H)}$, thus proving Theorem 15 and the tractability part of Theorem 16 for #DirSubd. It remains to prove the intractability part of Theorem 15, which we do in the next sections.

Let us again consider as a toy example $H = \Lambda_4^t$ (the disjoint union of $k$ cyclically oriented triangles). We can use the principle of inclusion and exclusion to reduce the computation of $\text{#Sub}(H \to G)$ to the computation of terms $\text{#Hom}(F \to G)$ where $F$ is a quotient of $H$. Now, it can easily be observed that each quotient of $H$ is a disjoint union of strongly connected components $S_1, \ldots, S_t$. Unfolding our general reduction to #CSP, for each of the strongly connected components $S$, we only have to guess the image $v$ of one vertex $s \in S$ in $G$. Then the image of each additional vertex in $S$ must be reachable from $v$ by a directed path of length at most $k$. Since the outdegree
of $\tilde{G}$ is at most $d^{O(k)}$ possibilities for the images of the remaining vertices. Thus, for each strongly connected components $S$, we can compute $\text{#Hom}(\tilde{F}[S] \to \tilde{G})$ in time $d^{O(k)}|\tilde{G}|$. Finally, we have $\text{#Hom}(\tilde{F} \to \tilde{G}) = \prod_{i=1}^{t} \text{#Hom}(\tilde{F}[S_i] \to \tilde{G})$.

### 4.3 Lower Bounds

The goal of this section is to prove that, roughly speaking, if $\rho^*(\tilde{H})$ is large then $\tilde{H}$ has a quotient $\tilde{F}$ such that computing $\text{#Hom}(\tilde{F} \to \tilde{G})$ is hard when parameterised by $|\tilde{F}| + d(\tilde{G})$. To this end we seek a reduction from $\#\text{CSP}$ to $\#\text{DirHom}_d$, i.e., in the opposite direction of Section 4.2. However, while that direction was relatively easy, since every digraph can be easily encoded as a set of relations, the direction we seek here is significantly harder. Indeed, it is not clear at all how an instance of $\#\text{CSP}$ can be "encoded" as a pair of directed graphs $(H, \tilde{G})$ if we can choose $\tilde{H}$ only from the class $\mathcal{C}$ for which we want to prove hardness.

#### 4.3.1 Encoding $\#\text{CSP}$ Instances via Canonical DAGs

To bypass the obstacle above, we start by considering classes of **canonical DAGs**. A digraph $\tilde{H}$ is a canonical DAG if it is acyclic and every vertex is either a source (i.e., it has indegree $0$) or a sink (i.e., it has outdegree $0$). Note that this implies that $\tilde{H}$ is bipartite, with (say) all sources on the left and all sinks on the right. If $\tilde{C}$ is a class of canonical DAGs, then it is easy to reduce $\#\text{CSP}$ to $\#\text{DirHom}_d$ while preserving all parameters. To see why, let $(H, \tilde{G})$ be a pair of hypergraphs (the instance of $\#\text{CSP}$). Define $\tilde{H}$ by letting $V(\tilde{H}) = V(H) \cup \{x_e : e \in E(H)\}$, and adding $(x_e, u)$ to $E(\tilde{H})$ for every $e \in E(H)$ and every $u \in e$. Define $\tilde{G}$ similarly as a function of $\tilde{G}$. One can then show, using the color-prescribed version of homomorphism counting (defined in Section 2), that $\text{#Hom}(H \to \tilde{G})$ can be computed in FPT time with $|\tilde{H}|$ as a parameter if we can compute $\text{#Hom}(\tilde{H} \to \tilde{G})$ in FPT time with $|\tilde{H}| + d(\tilde{G})$ as a parameter.

Recall then the contour $\Gamma(\tilde{H})$ of $\tilde{H}$ from Section 3. It is immediate to see that, if $\tilde{H}$ is a canonical DAG obtained from $H$ as described above, then $\mathcal{R}(\tilde{H}) = \mathcal{H}$. This is precisely the intuitive role of the contour — to encode the structure of the reachability sets of $\tilde{H}$ (ignoring sources). Indeed, using contours we can then state our main reduction. Let $\mathcal{C}$ be a class of canonical DAGs, and let $\text{#CSP}(\Gamma(\tilde{C}))$ be the restriction of $\#\text{CSP}$ to instances whose left-hand hypergraph (i.e., $\mathcal{H}$) is isomorphic to a contour of $\Gamma(\tilde{C})$. Using as a starting point a reduction due to Chen et al. [18], we prove that $\text{#CSP}(\Gamma(\tilde{C}))$ reduces to $\#\text{DirHom}_d(\tilde{C})$ via parameterised Turing Reductions. Now, under ETH, $\text{#CSP}(\Gamma(\tilde{C})) \notin \text{FPT}$ when the adaptive width of $\Gamma(\tilde{C})$ is unbounded [38], unless ETH fails. By the reduction above, then, we obtain:

**Lemma 21.** $\#\text{DirHom}_d(\tilde{C}) \notin \text{FPT}$ for every class $\tilde{C}$ of canonical DAGs such that $\text{aw}(\Gamma(\tilde{C})) = \infty$, unless ETH fails.

We now seek to lift this result from canonical DAGs to arbitrary directed graphs.

#### 4.3.2 Lifting Hardness to Arbitrary Digraphs via Monotone Reversible Minors

Starting from Lemma 21, we prove a hardness result for $\#\text{DirHom}_d(\tilde{C})$ for general classes of digraphs $\tilde{C}$. To this end we need to reduce from $\#\text{DirHom}_d(\tilde{C})$ to $\#\text{DirHom}_d(\tilde{C'})$ where $\tilde{C'}$ is a class of canonical DAGs, so that we can apply Lemma 21; clearly, the reduction must imply that $\#\text{DirHom}_d(\tilde{C'})$ has unbounded adaptive width.

Towards this end we introduce a kind of graph minors for digraphs, which we call **monotone reversible (MR) minors**. A digraph $\tilde{H}'$ is a MR minor of $\tilde{H}$ if it is obtained from $\tilde{H}$ by a sequence of the following operations:

- deleting a sink, i.e., a strongly connected component from which no other vertices can be reached
- deleting a loop
- contracting an arc

Note that, unlike standard minors, deletion of arbitrary vertices and arbitrary arcs are not allowed. This allows us to prove:

**Lemma 22.** Let $\tilde{C}$ be a class of digraphs and let $\tilde{D}$ be a class of MR minors of $\tilde{C}$. Then there exists a parameterised Turing reduction from $\#\text{DirHom}_d(\tilde{D})$ to $\#\text{DirHom}_d(\tilde{C})$.

Lemma 22 explains the "reversible" part of MR minors—we can efficiently "revert" the operations that yielded a MR of a digraph; for the "monotone" see the next section. The heart of the proof of Lemma 22 proves the claim for the color-prescribed version of the problems; this implies the reduction for the original problems via standard interreducibility arguments. As a consequence of Lemma 22 we obtain:

**Lemma 23.** Let $\tilde{C}$ be a recursively enumerable class of digraphs and let $\tilde{C'}$ be a class of canonical DAGs that are MR minors of digraphs in $\tilde{C}$. If $\text{aw}(\Gamma(\tilde{C'})) = \infty$ then $\#\text{DirHom}_d(\tilde{C}) \notin \text{FPT}$ unless ETH fails.

#### 4.3.3 Lifting Hardness from Homomorphisms to Subgraphs

Recall the arguments of Section 4.1: to prove that $\#\text{DirSub}(\tilde{C})$ is hard when $\rho^*(\tilde{C}) = \infty$, we essentially have to prove that every digraph $\tilde{H}$ with high fractional cover number $\rho^*(\tilde{H})$ has a quotient $\tilde{F}$ that is hard. By the arguments of the previous section, to show that such a quotient $\tilde{F}$ is hard it is enough to show that $\tilde{F}$ has an MR minor $\tilde{F}'$ which is a canonical DAG whose contour $\Gamma(\tilde{F}')$ has high adaptive width. We indeed prove that such a quotient exists. To this end, we consider two cases. Recall that $\alpha(\mathcal{H})$ and $\alpha^*(\mathcal{H})$ denote respectively the independence number and the fractional independence number of a hypergraph $\mathcal{H}$.

(a) $\alpha(\Gamma(\tilde{H}))$ is large. In this case we can show that $\tilde{H}$ contains a large matching whose edges are "isolated enough" for us to construct a quotient $\tilde{F}$ that admits, as MR minor, the $1$-subdivision $\tilde{F}'$ of a large complete graph, where the arcs of $\tilde{F}'$ are directed away from the subdivision vertices. It is easy to see that $\tilde{F}'$ is a canonical DAG, and that $\Gamma(\tilde{F}')$ is the complete graph itself, which has large adaptive width.

(b) $\alpha(\Gamma(\tilde{H}))$ is small. We then choose as quotient $\tilde{F}$ the graph $\tilde{H}$ itself. Recall that, by definition, $\rho^*(\tilde{H}) = \rho^*(\mathcal{R}(\tilde{H})).$ By LP duality the fractional cover number equals the fractional independence number, that is, $\rho^*(\mathcal{R}(\tilde{H})) = \alpha^*(\mathcal{R}(\tilde{H})).$ Using Theorem 17, we deduce that the adaptive width of $\mathcal{R}(\tilde{H})$ is within constant factors of $\alpha^*(\mathcal{R}(\tilde{H}))$, and thus of $\rho^*(\tilde{H})$. By carefully exploiting this fact, we can explicitly construct an MR minor $\tilde{F}'$ of $\tilde{H}$ that is both a canonical DAG and has high adaptive width.
Thus, in both cases we can show that if $\rho^*(\widetilde{H})$ is large then $\widetilde{H}$ admits an MR minor that is a canonical DAG of large adaptive width. Formally, we obtain:

**Lemma 24.** Let $\tilde{C}$ be a class of digraphs such that $\rho^*(\tilde{C}) = \infty$. Then the class $\tilde{C}'$ of all canonical DAGs that are MR minors of quotients of $\tilde{C}$ has unbounded adaptive width.

As a consequence, $\#\text{DirSub}_d(\tilde{C}) \notin \text{FPT}$ whenever $\rho^*(\tilde{C}) = \infty$, unless ETH fails. This concludes the overview of the proof of the lower bounds for $\#\text{DirSub}_d(\tilde{C})$.

The proofs in the complete version actually show that, for computing the fractional cover number and the source count of a directed graph $\widetilde{H}$, it is always sufficient to consider the DAG $\widetilde{H}/+$ (see Figure 1). In other words, the complexity of $\rho \widetilde{H}$ has small tractability of counting homomorphisms into degenerate graphs.

5 RELATED WORK AND OPEN PROBLEMS

Our work is closely related to recent works on pattern counting in degenerate graphs [6–9, 13, 30]. In that setting, one wants to compute the number of copies of an undirected pattern $H$ in an undirected graph $G$, parameterised by $|H|$ plus the degeneracy $d(G)$ of $G$ (the smallest integer $d$ such that there is an acyclic orientation of $G$ with outdegree at most $d$). This problem has been completely classified, as a function of the class of patterns, with respect to both linear-time tractability [6] and fixed-parameter tractability [13]. The crucial difference with our work is that, in our setting, the orientations of $H$ and $G$ are given in input, and are not necessarily acyclic. As a consequence, our problem has a larger set of tractable patterns, if by “pattern” we mean $H$ (the input pattern in the degenerate setting, and the undirected version of $H$ in our setting). In fact, in our problem every connected pattern $H$ has an easy orientation — the one obtained by directing all arcs away from a single vertex, so that $H$ has a single source — whereas in the degenerate setting the problem is easy if and only if the induced matching number of $H$ is small [13].

Our work also sheds new light on the problem of counting homomorphisms (rather than subgraphs) into degenerate graphs. Bressan [11] has shown that we can count homomorphisms from $H$ to $G$ in $	ext{FPT}$ time, parameterised by $|H| + d(G)$, if $H$ has small dag treewidth of $H$, which is the maximum hypertreewidth of the reachability hypergraph of any acyclic orientation of $H$. Our reduction to $\text{#CSP}$ implies that it is sufficient that those reachability hypergraphs have small fractional hypertreewidth; in other words, that $H$ has small fractional dag treewidth. Unfortunately, it is not clear that bounded fractional dag treewidth is a condition strictly weaker than bounded dag treewidth; answering this question entails understanding whether fractional and non-fractional dag treewidth are equivalent, a problem that we leave open. Perhaps more importantly, our work leaves open the problem of understanding what is the combinatorial parameter that determines the fixed-parameter tractability of counting homomorphisms into degenerate graphs. The list of possible candidates includes, but may be not limited to, dag treewidth, fractional dag treewidth, and adaptive width of the reachability hypergraphs.

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