Fluctuation Operators and Spontaneous Symmetry Breaking

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Abstract

We develop an alternative approach to this field, which was to a large extent developed by Verbeure et al. It is meant to complement their approach, which is largely based on a non-commutative central limit theorem and coordinate space estimates. In contrast to that we deal directly with the limits of $l$-point truncated correlation functions and show that they typically vanish for $l \geq 3$ provided that the respective scaling exponents of the fluctuation observables are appropriately chosen. This direct approach is greatly simplified by the introduction of a smooth version of spatial averaging, which has a much nicer scaling behavior and the systematic development of Fourier space and energy-momentum spectral methods. We both analyze the regime of normal fluctuations, the various regimes of poor clustering and the case of spontaneous symmetry breaking or Goldstone phenomenon.
1 Introduction

In the past decade in a series of papers Verbeure and coworkers developed a beautiful and ingenious framework to study so-called macroscopic fluctuation phenomena in systems and various regimes of quantum statistical mechanics (see the cited literature). The approach is to a large extent based on a quantum variant of the central limit theorem and is mainly performed in real (i.e. configuration) space. Among other things, the general goal is it, to study the limit behavior of correlation functions of so-called fluctuation observables, i.e. appropriately renormalized averages of microscopic observables, averaged over volumes, \( V \), which approach the whole space, \( \mathbb{R}^n \), say. Typically, one arrives, depending on the type of clustering of the microscopic \( l \)-point functions, at certain simple limit algebras as e.g. \( CCR \).

We approach the field from a slightly different angle. In a first step we choose another averaging procedure, which avoids sharp volume cut-offs and, a fortiori, has a very nice and transparent scaling behavior. This is then exploited in the following analysis which systematically develops so-called Fourier-space and energy-momentum spectral methods of observables and correlation functions. We consider it to be an advantage that the calculations turn out to be relatively transparent and lead in a direct way to the desired results.

We first treat the case of normal fluctuations and \( L^1 \)-clustering. We show that all the truncated \( l \)-point functions vanish for \( l \geq 3 \) while they approach a finite, non-trivial limit for \( l = 2 \). The analysis is done both for the \( (k = 0) \)- and the \( (k \neq 0) \)-modes. We emphasize that the calculations for net-momentum different from zero remain also very simple. A variant of the method is then applied to the case of \( L^2 \)-clustering.

In the second part of the paper we embark on the analysis of fluctuations in the presence of spontaneous symmetry breaking (ssb). In a first step we prove some general results in the context of ssb and the Goldstone phenomenon. We then address the problem of macroscopic fluctuations within this context. Among other things, we give a general and rigorous proof that the limit fluctuations are always classical for temperature states (a phenomenon already observed by Verbeure et al in various simple models). The paper ends with a treatment of extremely poor clustering, which can be controlled by a new method we develop in the last section. To sum up, we think that in our view the two different frameworks seem to neatly complement each other and should lead to further interesting results if being combined.

2 The Scenario of Normal Fluctuations

The following analysis works for statistical equilibrium states and/or for vacuum states in quantum field theory. To avoid constant mentioning of the respective
scenario we are actually working in, we usually treat equilibrium (i.e. KMS-) states, to fix the framework. Now, let \( \Omega \) be the vacuum or equilibrium state (rather its GNS-representation; usually we work within a concrete Hilbert space, \( \mathcal{H} \)). As an abstract state we denote it by \( \omega \). Expectations of observables are written as

\[
\langle A \rangle = \omega(A) = (\Omega, A\Omega)
\]

with \( A \) taken from the local algebra, \( A_0 \subset A \), the latter one being the quasi-local norm closure of \( A_0 \). We assume \( \Omega \) to be cyclic with respect to \( A_0 \) or \( A \). That is, we assume

\[
A_0 \cdot \Omega = \mathcal{H}
\]

There are certain differences as to the (assumed) locality properties of the dynamics between (non-)relativistic statistical mechanics and relativistic quantum field theory (RQFT). Denoting the time evolution (acting on the algebra of observables) by \( \alpha_t \), we are confronted with the following phenomenon.

**Observation 2.1** In RQFT part of the usual framework is the assumption

\[
\alpha_t : A_0 \rightarrow A_0
\]

while in statistical mechanics (due to weaker locality behavior) we have in the generic case only

\[
\alpha_t : A \rightarrow A
\]

with \( A_0 \) usually not left invariant as the observables will typically develop infinitely extended tails.

Furthermore, we assume once for all that our system is in a pure, translation invariant phase, that is \( \Omega \) is extremal translation invariant under the space translations (which can, as in the case of lattice systems, also be a discrete subgroup). There can of course exist several coexisting pure phases at the same external parameters as in the regime below a phase transition threshold. These assumptions imply that we can expect certain cluster properties, i.e. decay of correlations (see e.g. [1]).

### 2.1 Definition of Ordinary Fluctuation Operators

We begin by defining the fluctuation operators in the normal situation as it was done in [2].

We assume, for the time being, \( L^1 \)-clustering for the two-point-function, that is

\[
\int |\langle A(x)B \rangle^T| d^nx < \infty \quad A, B \in A
\]
with $A(x)$ the translate of $A$ and
\[ \langle AB \rangle^T = \langle AB \rangle - \langle A \rangle \cdot \langle B \rangle \] (6)

Once for all we assume, to simplify notation, in our particular context that the occurring observables are normalized to $\langle A \rangle = 0$ unless otherwise stated.

**Definition 2.2** We define the normal (finite volume) fluctuation operators as
\[ A_{ VF} := 1/V^{1/2} \cdot \int_V A(x) d^n x =: 1/V^{1/2} \cdot A_V \] (7)

In a next step one wants to give sense to these objects in the limit $V \to \infty$. From the $L^1$-condition we however infer
\[ |(A_{ VF} \Omega, B \Omega)| \leq 1/V^{1/2} \int_{\mathbb{R}^n} |(A(x) \Omega, B \Omega)| d^n x \to 0 \] (8)

Hence $A_{ VF} \Omega \to 0$ on a dense set. Furthermore we have
\[ (A_{ VF} \Omega, A_{ VF} \Omega) = 1/V \int_V \int_V (A(x) \Omega, A(y) \Omega) dx dy = \frac{1}{V} \int_V dx \left( \int_{V-x} \langle A^* A(y-x) \rangle d(y-x) \right) \] (9)

This is less or equal to
\[ (1/V) \cdot V \cdot \sup_x (\int_{V-x} |(\ldots)|) \leq \int_{\mathbb{R}^n} |F(y-x)| d^n(y-x) < \infty \] (10)

(for convenience we sometimes denote a general two-point function by $F(x-y)$).

This suffices to prove weak convergence to zero for $A_{ VF} \Omega$ on the total Hilbert space $\mathcal{H}$.

**Remark 2.3** We note that this proves also the well-known normal-fluctuation result $\langle A_V \cdot A_V \rangle \lesssim V$ in the $L^1$-case. Under certain well-specified conditions the fluctuations can even be weaker than normal. If e.g. $Q_V$ is the local integral over a conserved quantity we proved a divergence significantly weaker than $\sim V$ (cf. [3]). But in general the local fluctuations will diverge in the limit $V \to \infty$ in contrast perhaps to ordinary intuition, even if the quantity is globally conserved due to quantum fluctuations (see also the section about spontaneous symmetry breaking).

A weaker than normal divergence can occur in the following situation. An asymptotic behavior $\sim V$ does only prevail if $\int_V F(u) du \neq 0$ in the limit $V \to \infty$. On the other side such correlation functions tend to oscillate about zero (for physical
reasons; there are e.g. usually preferred relative positions in, say, a quantum liquid). In other words, while
\[ \int F(u) du = 0 \quad (11) \]
may seem to be rather ungeneric at first glance, it can nevertheless happen in a specific context. The general situation is analyzed in the above reference; certain examples of better than normal fluctuations were also found by Verbeure et al in e.g. [4] (see also [5]).

For the fluctuation operators themselves we have due to locality for \( A, B \in A_0 \):
\[ [A_V, B] \text{ independent of } V \text{ for } V \supset V_0 \supset V_B \quad (12) \]
for some \( V_0 \) which contains the localisation region \( V_B \) for \( B \in A_0 \). We then have
\[ \lim_V (A^F_V \cdot C \Omega, B \Omega) = \lim_V ([A^F_V, C] \Omega, B \Omega) + \lim_V (A^F_V \Omega, C^* B \Omega) \quad (13) \]
We have already shown that the second term goes to zero. In the first term the commutator becomes
\[ [A^F_V, C] = V^{-1/2} \cdot [A_{V_0}, C] \quad (14) \]
and hence the first term goes also to zero. In case we assume only \( A \in A \) a further \( L^1 \)-condition for the three-point function is needed to arrive at the same result. As \( A_0 \Omega \) is assumed to be dense in \( H \) and \( \|A^F_V\| < \infty \) uniformly in \( V \), we have

**Proposition 2.4** \( L^1 \)-clustering implies that
\[ A^F_V \to 0 \text{ weakly on } H, \quad \|A^F_V \Omega\| < \infty \text{ uniformly in } V \quad (15) \]
but \( \|A^F_V \Omega\| \) bounded away from zero in general. That is, \( A^F_V \) does not converge strongly to zero and, a fortiori, there is no convergence in norm.

This clearly shows that, in order to have non-trivial limit operators, one has to leave the original Hilbert-space of microscopic observables and has to define or construct an entirely new representation living on a different state.

### 2.2 A Smoothed Version of Fluctuation Operators

Since we employ in the following so-called *Fourier-methods* and related calculational tools, it is advantageous to change to a smoother version of fluctuation operators. As everybody knows, sharp volume cut-offs are both a little bit artificial and technically nasty, since they may sometimes lead to non-generic or spurious effects. In other branches of rigorous statistical mechanics or axiomatic
quantum field theory volume integrations have therefore frequently been emulated or implemented in a slightly different way (see e.g. [6]).

Two choices have basically been in use with the second version having much nicer properties in several respects as we will explain below. Instead of integrating over a sharp volume, $V$, centered e.g. around the coordinate origin, one integrates the shifted observable, $A(x)$, over a smooth test function localized basically in $V$ but having smooth tails.

Remark: As $V$ we choose in the following a ball centered at the origin with radius $R$ and let $R$ go to infinity.

**Definition 2.5** Two admissible families of test functions are the following ones: $f_R(x) \geq 0$ smooth with

$$f_R(x) := \begin{cases} 1 & \text{for } |x| \leq R \\ 0 & \text{for } |x| \geq R + h \end{cases}$$

(16)

or

$$f_R(x) := f(|x|/R) \quad \text{with} \quad f(s) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2 \end{cases}$$

(17)

Note that the latter choice has much nicer behavior under Fourier transform while working with the Fourier transforms of the former version or e.g. the indicator function of the volume $V$ is quite cumbersome). On the other hand, the latter version has tails which are also scaled.

**Lemma 2.6**

$$\hat{f}_R(k) = \text{const} \cdot R^n \cdot \hat{f}(R \cdot k)$$

(18)

where here and in the following “const” denotes an (in this context) irrelevant numerical factor which, a fortiori, may change in the course of a calculation. With the help of this smearing functions we now define

**Definition 2.7 (Smooth Volume Integration)** We redefine the fluctuation operators in the following way

$$A^F_R := R^{-n/2} \cdot \int A(x) \cdot f_R(x) d^n x$$

(19)

with $f_R$, unless otherwise stated, the family given in the second example above (remember $\langle A \rangle := 0$).
3 The Limiting Case for Normal Fluctuations

In order to arrive at a rigorous definition of fluctuation operators in a certain limit state we will follow a line of arguments which may complement the treatment of Verbeure et al in several respects. We will study directly the macroscopic limit of the n-point functions with the help of certain momentum space methods. As they are perhaps not so common in statistical physics we will give the technical details below.

3.1 Some Generalities

Any n-point (correlation) function of the kind \( \langle A_1(x_1) \cdots A_n(x_n) \rangle \) with the \( A_i(x_i) \) the translates of the observables \( A_i \) (which may also contain an implicit time variable \( t_i \) which is however kept fixed in the following) is written as \( W(x_1, \ldots, x_n) \). With the state \( \Omega \) being translation invariant we have

\[
W(x_1, \ldots, x_n) = W(x_1 - x_2, \ldots, x_{n-1} - x_n)
\]

(20)

To express cluster properties in a clear way, we introduce the so-called truncated correlation functions via the following recursion relation:

\[
W(x_1, \ldots, x_n) = \sum_{\text{part}} \prod_{P_i} W^T(x_{i_1}, \ldots, x_{i_k})
\]

(21)

where the sum extends over all partitions of the set \( \{1, \ldots, n\} \) into subsets \( P_i \) with the elements in each subset ordered as \( i_1 < i_2 \ldots < i_k \). The first elements of the recursion are

\[
W(x) = W^T(x) = 0 \quad \text{in our case}
\]

(22)

\[
W^T(x_1, x_2) = W(x_1, x_2) - W(x_1)W(x_2)
\]

(23)

Observation 3.1 In the truncated correlation functions the vacuum state, ground state or equilibrium state, \( \Omega \), has been eliminated in a symmetric way, so that we have, in a sense to be specified,

\[
W^T(x_1, \ldots, x_n) \to 0 \quad \text{for} \quad \sup |x_i - x_j| \to \infty
\]

(24)

In this section we assume the following cluster property

\[
W^T(x_1, \ldots, x_n) \in L^1 \quad \text{in the variables} \quad \{x_1 - x_2, \ldots, x_{n-1} - x_n\}
\]

(25)

From the above we see that the original hierarchy of n-point functions can be reconstructed from the new hierarchy of truncated n-point functions, which
have more transparent cluster properties. The $L^1$-condition allows us to Fourier transform the $W^T(x_1, \ldots, x_l)$ and we get from translation invariance:

$$\text{const} \cdot \int \hat{W}^T(p_1, \ldots, p_l) \cdot e^{-i \sum p_i x_i} \prod dp_i =$$

$$W^T(x_1, \ldots, x_l) = W^T(x_1 - x_2, \ldots, x_{l-1} - x_l)$$

$$= \text{const} \int \hat{W}^T(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_{l-1}) \cdot \delta(p_1 + \cdots + p_l) e^{-i \sum p_i x_i} \prod dp_i$$

$$= \text{const} \int \hat{W}^T(q_1, \ldots, q_{l-1}) e^{-i \sum_{i=1}^{l-1} q_i y_i} \prod dq_i$$ (26)

with

$$y_i := x_i - x_{i+1}, \quad q_i = \sum_{j=1}^{i} p_j \quad i \leq (l-1)$$ (27)

The functional determinant $\det(\partial q/\partial p)$ is one and we can regard $\hat{W}^T$ either as a function of the $q_i$’s or the $p_i$’s. We hence have

**Lemma 3.2** As a Fourier transform of a $L^1$-function $\hat{W}^T(p_1, \ldots, p_{l-1}) = \hat{W}^T(q_1, \ldots, q_{l-1})$ is a continuous and bounded function which decreases at infinity in the $q$-variables.

### 3.2 The $(k = 0)$-Modes

We now study the limit of truncated $l$-point functions with the entries being fluctuation operators $A^F_R$, more precisely their Fourier transforms, i.e.

$$\langle A^F_R(1) \cdots A^F_R(l) \rangle^T =$$

$$\text{const} \cdot R^{ln/2} \cdot \int \hat{f}(R p_1) \cdots \hat{f}(-R[p_1 + \cdots + p_{l-1}]) \cdot \hat{W}^T(p_1, \ldots, p_{l-1}) \prod dp_i$$

$$= \text{const} \cdot R^{ln/2} R^{-(l-1)n} \cdot \int \hat{f}(p'_1) \cdots \hat{f}(-[p'_1 + \cdots + p'_{l-1}]) \cdot \hat{W}^T(p'_1/R, \ldots, p'_{l-1}/R) \prod dp'_i$$ (28)

$\hat{W}$ is continuous and bounded and the $\hat{f}$’s are of rapid decrease. Hence we can perform the limit $R \to \infty$ under the integral and get

**Theorem 3.3** The expression $\langle A^F_R(1) \cdots A^F_R(l) \rangle^T$ scales as $\sim R^{(2-1)n/2}$. This implies that for $l > 2$ the above limit is zero, for $l = 2$ the limit is a finite number bounded away from zero in general. In other words we have

$$\lim_{R \to \infty} \langle A^F_R(1) \cdots A^F_R(l) \rangle^T = 0 \text{ for } l > 2$$ (29)
\[ \lim_{R \to \infty} \langle A^F_R(1) \cdots A^F_R(l) \rangle = \lim_{R \to \infty} \sum_{\text{part } \{ij\}} \prod_{\{ij\}} \langle A^F_R(i) A^F_R(j) \rangle \]

(30)

The relation between the original microscopic system \((\mathcal{A}, \omega)\) and the coarse-grained system of fluctuation operators is a little bit subtle. Note that \(\omega_F\), the limit state to be constructed, can no longer be considered as a state or something like that on the original algebra nor can the fluctuation operators be considered as a representation of, say, \(\mathcal{A}\). One aspect of the impending problems can perhaps best be seen by realizing that e.g.

\[ (A \cdot B)^F_V \neq A^F_V \cdot B^F_V \]

(31)

which pertains also in the limit. That is, in a sense to be defined, we have

\[ (A \cdot B)^F \neq A^F \cdot B^F \]

(32)

the same holding in general for all the higher products. This is one source of non-uniqueness as there is no invariant discrimination between an observable regarded as a single object to be scaled and as a product of other observables, where now each factor has to be scaled separately. The appropriate point of view has to be a different one (as has also been emphasized by Verbeure et al, cf e.g. [2], second ref. p.540f and private communication).

The picture remains relatively clear for the intermediate scales, \(V < \infty\). We have a start system \((\mathcal{A}, \omega)\), labelled by, say, \(V = 0\). On every scale \(V\) we have a new algebra, \(\mathcal{A}^F_V\), (actually a subalgebra of \(\mathcal{A}\)), generated by the observables \(A^F_V, A \in \mathcal{A}\) (including arbitrary finite products \((A_1 \cdots A_n)^F_V\)). If we prefer to consider this algebra on scale \(V\) as a new abstract algebra (i.e. forgetting about the underlying finer algebra \(\mathcal{A}\)), we get also a new, coarse-grained state via the identification

\[ \omega^F_V(\Pi A^F_{ij}) := \omega(\Pi A^F_{ij}) \]

(33)

(A related philosophy was expounded by Buchholz and Verch in e.g. [10] within the context of the algebraic analysis of ultra-violet behavior in quantum field theory.)

The map

\[ R_V : \mathcal{A} \to \mathcal{A}^F_V \]

(34)

can be viewed as kind of a renormalization map, which does however not preserve the algebraic structure (i.e. the algebras are in general not isomorphic). Furthermore one gets a "new" dynamics on this algebra by defining

\[ \alpha^V_t(A^F_V) := (\alpha_t(A))^F_V \]

(35)
Remark 3.4 In our context $\alpha_t$ is assumed to commute with the space translations or with a corresponding lattice version, that is, we have $\alpha_t(A^F) = (\alpha_t A)^F$. (Furthermore it may turn out to be reasonable to scale the time variable on the lhs also.)

On the other hand, in order to construct the limit theory itself, one can proceed in a slightly different direction. The above limits of n-point functions define a consistent hierarchy of new n-point functions which then allow to define a new limit system via the so-called reconstruction theorem (for a pendant in quantum field theory see e.g. [7]). Put differently, we define limit objects, $\{A^F_i\}$, the so-called fluctuation operators, which live in a new Hilbert space built upon the new state, $\omega_F$, defined by the limits:

$$\omega_F(A^F_1 \cdots A^F_n) := \lim_{R \to \infty} \langle A^F_1 \cdots A^F_n \rangle = \sum_{\text{part } \{ij\}} \prod \omega_F(A^F_i \cdot A^F_j)$$

(36)

Note however that the so-called Gelfand-ideal, $I_F$, is large, that is, there are a lot of elements of $A$ which are mapped to zero by this limit with

$$I_F := \{A; \omega_F((A^F)^* \cdot A^F) = 0\}$$

(37)

This is of course typical for such kind of mean-values, as e.g. all space-translates of $A$ yield the same limit element. Shifting one of the observables in the above l-point functions by, say, $a_i$ yields an extra factor $e^{ip_i a_i}$ in the Fourier transform which after the above coordinate transformation goes over into $e^{ip_i/R} a_i$ which goes to one. Summing up we have

Conclusion 3.5 With the help of equation (36) we construct a new limit system, consisting of the algebra of fluctuation operators, $A_F$, and the limit state $\omega_F$. The well-known GNS-construction (see e.g. [8]) allows to construct the corresponding Hilbert-space representation with

$$\omega_F(A^F_1 \cdots A^F_n) = (\Omega_F, A^F_1 \cdots A^F_n \Omega_F)$$

(38)

(where, by abuse of notation, we do not discriminate between operators and their equivalence classes on the rhs).

As all the n-point functions decay into a product of 2-point functions all the commutators are c-numbers:

$$[A^F, B^F] = \omega_F([A^F, B^F])$$

(39)

The system of fluctuation operators is a quasi-free system (cf. [9])

Taking now self-adjoint elements one can, as in [7], represent the new system as a representation of the $CCR$ over the real vector space of s.a. operators. Our
scalar product, induced by the hierarchy of \( n \)-point functions, can be split in the following way.

\[
(A^F \Omega_F, B^F \Omega_F) = \text{Re} (\ldots) + i \text{Im} (\ldots) =: s_F(A^F, B^F) + (i/2)\sigma_F(A^F, B^F)
\]  

(40)

\[
\omega_F([A^F, B^F]) = \sigma_F(A^F, B^F)
\]  

(41)

where \( \sigma_F \) defines a symplectic form. The Weyl-operators, \( e^{iA^F} \) with \( A^F \) s.a., fulfill the CCR-relations

\[
\omega_F(e^{iA^F}) = e^{-1/2s_F(A^F,A^F)}
\]  

(42)

\[
e^{iA^F} \cdot e^{iB^F} = e^{i(A^F+B^F)} \cdot e^{-i/2\sigma_F(A^F,B^F)}
\]  

(43)

In our context the first equation can e.g. be verified as follows: Only the \( 2n \)-point functions are different from zero. On the lhs we hence have

\[
\omega_F(e^{iA^F}) = \sum (-1)^n/(2n)! \cdot \omega_F([A^F]^{2n})
\]  

(44)

It remains to count the number of partitions of an \( 2n \)-set into 2-sets. This number is \( (2n)!/2^n \cdot n! \). In (44) we now get for \( A^F \) s.a. on the rhs

\[
\sum_n 1/n!(-1/2 \cdot \omega_F(A^F A^F))^n = e^{-1/2s_F(A^F,A^F)}
\]  

\( \square \)  

(45)

The above general cluster result of the limit \( n \)-point functions make the study of the limit time evolution relatively straightforward. In a first step it suffices to study the 2-point functions. We define the time evolution in the limit theory by

\[
\omega_F(A^F(t') \cdot B^F(t)) := \lim \omega(A^F_v(t') \cdot B^F_v(t)) = \lim \omega(A(t')_v \cdot B(t)_v)
\]  

(46)

On the limiting GNS-Hilbert space constructed above we now get a bounded sesquilinear form \( (x, y(t)) \) which, by standard results, yields a bounded operator \( U^F(t) \) implementing the time evolution. Here we use that the limit \( n \)-point functions are products of 2-point functions. Furthermore we infer with the help of the above limit process that

\[
(U^F_t x, U^F_t y) = \omega_F(\ldots) = \lim \omega(\ldots) = (x, y)
\]  

(47)

In other words, we arrive at the following conclusion

**Theorem 3.6** The preceding construction yields a strongly continuous unitary time evolution on the limiting GNS-Hilbert space.
Another point worth to be mentioned (since it might perhaps be overlooked) is the question of the non-triviality of the commutators

\[ [A^F, B^F] = \omega_F([A^F, B^F]) \quad (48) \]

In principle it could happen that all the expectation values on the rhs vanish. In that case the limit algebra would be abelian and the fluctuations classical. In a more general context (cf. e.g. [10]) this problem is more complicated. In our situation this question can however be answered in a rather straightforward way.

We have

\[ \lim_V \omega([A^F_V, B^F_V]) = \lim_V \omega([A_V, V^{-1} \cdot B_V]) \quad (49) \]

For \( A, B \in A_0 \), i.e. local, the rhs equals

\[ \lim_V \omega([A_V, B]) \quad (50) \]

We know candidates which lead to a vanishing of the limit for all \( B \in A_0 \). For \( A \) chosen s.a. these are the generators of conserved symmetries, written

\[ Q := \int A(x) d^n x \quad (51) \]

Usually they are assumed to commute with the time evolution, expressed as \( Q(t) = Q \), hence the above limit would also be zero on the full quasi-local algebra. This situation, more specifically the case of spontaneous symmetry breaking (ssb) and Goldstone phenomenon, will be dealt with in more detail in section [3]. In any case, as conserved symmetries are usually not so numerous, we may presume that, in the generic case, not all of these commutators will be zero.

For \( A, B \) not necessarily strictly local our above more general formalism is useful. With

\[ \omega(A(x)B) = F_{AB}(x) \quad , \quad \omega(BA(x)) = G_{AB}(x) \quad (52) \]

the vanishing of the commutator would imply:

\[
0 = [A^F, B^F] = \lim_R R^n \cdot \int |\hat{f}(Rp)|^2 (\hat{F}_{AB}(p) - \hat{G}_{AB}(p)) d^n p \\
= \lim_R \int |\hat{f}(p)|^2 (\hat{F}_{AB}(p/R) - \hat{G}_{AB}(p/R)) d^n p \\
= (\hat{F}_{AB}(0) - \hat{G}_{AB}(0)) \cdot \int |\hat{f}(p)|^2 d^n p \quad (53)
\]

by the theorem of dominated convergence (note that we are in the \( L^1 \)-situation). Hence we have the result
Proposition 3.7

\[ [A^F, B^F] = 0 \iff \hat{F}_{AB}(0) = \hat{G}_{AB}(0) \] (54)

that is

\[ \int F_{AB}(x)d^n x = \int G_{AB}(x)d^n x \] (55)

or

\[ \int (\Omega, [A(x), B]\Omega)d^n x = 0 \] (56)

which is the same result as in the strictly local case.

3.3 The \((k \neq 0)\)-Modes

Up to now only the \((k = 0)\)-modes of fluctuation operators, i.e.
\[ \lim_{V} V^{-n/2} \int_{V} A(x)d^n x, \] have been studied. For various reasons it is useful to have corresponding formulas at hand for fluctuation observables containing a certain net-momentum. This problem was studied by Verbeure et al in e.g. [19] and the results were applied in e.g. [20] in the analysis of Goldstone modes. In the original (real-space) approach the necessary calculations turned out to be quite involved and far from being simple. This is another case in point to demonstrate the merits of our Fourier space scaling methods.

Instead of the original scaling operators, \(A^F_V\) or \(A^F_R\), we now study their \(k \neq 0\)-variants, \(A^F_R(k)\). We begin with a technical lemma.

Lemma 3.8

\[ \hat{A}(k) := (2\pi)^{-n/2} \int e^{ikx} A(x)d^n x \] (57)

is an operator-valued distribution (We use the convention
\[ \hat{f}(k) = (2\pi)^{-n/2} \int e^{-ikx} f(x)d^n x \])

Remark: For a systematic use and proofs of such energy-momentum techniques in quantum statistical mechanics we refer to e.g. [22] where also some more mathematical background is provided.

Integrating now over \(e^{iqx} \cdot f_R(x)\), we get the \(q\)-mode fluctuation operators.

\[ A^F_R(q) := R^{-n/2} \int A(x)e^{iqx} f_R(x)d^n x = R^{n/2} \int \hat{A}(k + q)\hat{f}(Rk)d^n k \]

\[ = R^{n/2} \int \hat{A}(k)\hat{f}(R(k - q))d^n k \] (58)
We can now proceed in exactly the same way as above in the case of the zero-mode analysis and calculate the truncated l-point functions \( \langle A^F_R(1, q_1) \cdots A^F_R(l, q_l) \rangle^T \) (where the indices 1 to \( l \) label different observables). The only thing that changes are the test functions, i.e. \( f_R(x) \rightarrow e^{iqx} \cdot f_R(x) \). We arrive at the conclusion:

**Theorem 3.9 (q-Mode Fluctuation Operators)**

In the case of \( L^1 \)-clustering all truncated correlation functions vanish for \( l \geq 3 \) and the \( l \)-point functions are again sums of products of 2-point functions. The concrete form of the limit-2-point functions is given in formula (62).

If we calculate the limit-2-point functions explicitly we get:

\[
\langle A^F_R(q_1) \cdot B^F_R(q_2) \rangle^T = R^n \int \langle \hat{A}(k_1 + q_1) \hat{B}(k_2 + q_2) \rangle^T \cdot \delta(k_1 + q_1 + k_2 + q_2) \cdot \hat{f}(Rk_1) \hat{f}(Rk_2) dk_1 dk_2
\]

\[
= R^n \int \langle \hat{A}(k_1 + q_1) \hat{B}(-(k_1 + q_1)) \rangle^T \cdot \hat{f}(R(k_1 + q_1)) \hat{f}(-R(k_1 + q_1)) dk_1
\]

\[
= R^n \int \langle \hat{A}(k) \hat{B}(-k) \rangle^T \cdot \hat{f}(R(k - q_1)) \hat{f}(-R(k + q_2)) dk \quad (59)
\]

With \( k' := R(k - q_1) \) we arrive at

\[
\int \hat{W}^T(k'/R + q_1) \cdot \hat{f}(k') \hat{f}(-k' - R(q_1 + q_2)) dk' \quad (60)
\]

By assumption \( \hat{W}^T \) is in \( L^1 \), \( \hat{f} \) is of rapid decrease, so the limit can again be carried out under the integral and we have

**Observation 3.10** For \( q_1 + q_2 \neq 0 \) it holds

\[
\lim_R \langle A^F_R(q_1) \cdot B^F_R(q_2) \rangle^T = 0 \quad (61)
\]

For \( q = q_1 = -q_2 \) we get on the other side

\[
\lim_R \langle A^F_R(q) \cdot B^F_R(-q) \rangle^T = \hat{W}^T(q) \cdot \int \hat{f}(k) \hat{f}(-k) dk \quad (62)
\]

In other words, the limit tests the spectral momentum of the two-point function.

**4 The Case of \( L^2 \)-Clustering**

Before we embark on an investigation of the situation in the regime where phase transitions, vacuum degeneracy and/or spontaneous symmetry breaking (ssb) prevail, we briefly address the case where the clustering is weaker than \( L^1 \) but still \( L^2 \), say. Our above Fourier-space approach can easily handle also this more
singular situation. We hence assume now that the truncated $l$-point functions cluster only in the $L^2$-sense in the difference variables.

Now we cannot conclude that the Fourier transform is bounded and continuous, but we know it is again an $L^2$-function. We repeat the first steps of the above calculation with, however, another scaling exponent, $\alpha$, which we leave open for the moment.

**Definition 4.1** *In the general case we define fluctuation operators by*

$$A^F_R := R^{-\alpha} \cdot \int A(x) f_R(x) dx$$  \hspace{1cm} (63)

We get

$$\langle A^F_R(1) \cdots A^F_R(l) \rangle_T = \text{const} \cdot R^{l(n-\alpha)} \cdot \int \hat{f}(Rp_1) \cdots \hat{f}(-Rq_{l-1}) \cdot \hat{W}^T(q_1, \ldots, q_{l-1}) \prod dq_i$$ \hspace{1cm} (64)

where the $\{p_i\}$ are linear functions of the $\{q_i\}$ as described above. We now apply the Cauchy-Schwartz inequality

$$|\text{lhs}| \leq \text{const} \cdot R^{l(n-\alpha)} \left[ \int (\hat{f}(Rp_1) \cdots \hat{f}(-Rq_{l-1}))^2 \prod dq_i \right]^{1/2} \cdot \left[ \int (\hat{W}^T(q_1, \ldots, q_{l-1}))^2 \prod dq_i \right]^{1/2}$$ \hspace{1cm} (65)

In the first integral on the rhs we make again a variable transformation from $q_i$ to $q'_i := Rq_i$, yielding an overall scaling factor

$$R^{l(n-\alpha)} \cdot R^{-(l-1)n/2}$$ \hspace{1cm} (66)

We again want the limits of the 2-point functions to be both finite and non-trivial, i.e. different from zero in general.

**Proposition 4.2** *To make the rhs of (65) finite in the limit for $l = 2$ the maximal $\alpha$ to choose is*

$$3n - 4\alpha = 0 \quad \text{i.e.} \quad \alpha = (3/4)n$$ \hspace{1cm} (67)

For a general $l$ this leads to the scaling exponent $(n - (1/2)l \cdot n)/2$, which is negative for $l \geq 3$. Hence, all higher truncated $l$-point functions vanish in the limit.

However, to guarantee that the result is really non-trivial, we have to analyze the situation in more detail as the above estimate is only an inequality. In the case of
$L^1$-clustering $\alpha = n/2$ was appropriate. The largest value which can occur in the $L^2$-case is the above maximal $\alpha = (3/4)n$. If we want to avoid that the 2-point functions vanish in the limit we have to choose in the $L^2$-case

$$(1/2)n < \alpha \leq (3/4)n$$

depending on the concrete decay of the 2-point functions in configuration space. We see that, evidently, the situation is now less canonical as compared to the $L^1$-case.

Remark: A related situation (on a lattice) was analyzed by Verbeure et al in [11], where a clustering weaker than $L^1$ was considered with, however, the additional input that the local algebras, sitting at the points of the lattice, form a finite-dimensional Lie-algebra. In that case, suitable scaling exponents are chosen to render the auto-correlation functions finite and non-vanishing, while, on the other side, the finiteness of the limit 3-point functions has to be imposed as an extra assumption. Under this proviso one gets the existence of a limit Lie-algebra, but nevertheless results are only partial while perhaps, on the other side, being also more interesting.

We do not want to dwell too much on this point at the moment, as progress seems to be to a certain extent model-dependent. Furthermore, we develop a different approach in the last section which is able to cope with any kind of poor cluster behavior.

If we want to guarantee the apriori existence or vanishing of the truncated 3-point functions with the help of our above $L^2$-estimate (65), we have to restrict the chosen $\alpha$ in the following way.

**Corollary 4.3** If the appropriate $\alpha$ fulfills $\alpha > (2/3)n$, we get a negative scaling exponent for $l \geq 3$ as

$$n - (1/3)ln \leq 0 \quad \text{for} \quad l \geq 3$$

For $\alpha = 2/3$ the 3-point functions are finite.

**Remark 4.4** One would get corresponding relations for smaller $\alpha$ but higher correlation functions, beginning from a certain order, $l_0(\alpha)$ say. On the other hand, one cannot guarantee the apriori existence of the $l$-point functions for $2 < l < l_0(\alpha)$ as the general scaling relation reads for $l \geq l_0(\alpha)$:

$$l(2\alpha - n) > n \quad \text{and} \quad \alpha > (1/2)n$$

and $\alpha$ being so chosen that the 2-point functions are non-trivial.
5 Spontaneous Symmetry Breaking (SSB) and the Goldstone Phenomenon

5.1 General Remarks

Before we study fluctuation operators in the regime of vacuum–, ground–, equilibrium–state degeneracy, we want to briefly comment, in order to set the stage, on the (rigorous) implementation of sSB in the various areas with particular emphasis on (quantum) statistical mechanics, i.e. condensed matter physics. As this topic has however been much discussed in the past from various points of views, we do not intend to give an exaustive commentary. We only mention some earlier work being of relevance for our argumentation and sketch the general framework.

We assume that our state, $\omega$ or $\Omega$, is (non-)invariant under some automorphism group of $A_0$ or $A$. Furthermore, and this is important (while frequently not clearly stated), we assume the time evolution, $\alpha_t$, to commute with the automorphism group, $\alpha_g$.

**Definition 5.1** $\alpha_g$ is called a symmetry group if

$$\alpha_g \cdot \alpha_t = \alpha_t \cdot \alpha_g$$

**Definition 5.2** If

$$(\Omega, \alpha_g(A)\Omega) = (\Omega, A\Omega)$$

for all $A \in A$, the symmetry is called conserved and can be implemented by a unitary group of operators in the representation space

$$\alpha_g(A) \rightarrow U(g)AU(g^{-1})$$

On the other side, if

$$(\Omega, \alpha_g(A)\Omega) \neq (\Omega, A\Omega)$$

for some $A$, $A$ the symmetry-breaking observable, the symmetry is called spontaneously broken since it still commutes with the time evolution (i.e. formally: with the Hamiltonian, modulo boundary terms due to long-range correlations).

In most cases the (continuous) symmetry group derives from a clearly identifiable generator (we restrict ourselves, for convenience, to one-parameter groups) which is built from a local operator density, i.e.

$$U(s) = e^{isQ}, Q(t) = \int q(x,t)d^n x, Q(t) = Q(0) := Q$$

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Note that there are a lot of technical subtleties lurking behind these operator identities, all of which we cannot mention in the following (for more details and references see e.g. [12]. A nice review is [13] where many of the widely scattered results have been compiled).

**Remark 5.3** In many situations the generator density is the zero-component of a conserved current. Formally the conservation law encodes the time-independence of the global charge, $Q$. Furthermore, for convenience, we assume the symmetry to commute with the space translations, i.e. $U(x)QU(-x) = Q$. This is in fact frequently the case and simplifies certain calculations.

The most crucial consequence is that in case the symmetry is spontaneously broken some of the above relations do only hold in a formal or algebraic sense. More specifically:

**Theorem 5.4** If $\alpha_g$ is spontaneously broken the global generator $Q$ does only exist in a formal sense as a limit

$$Q = \lim_V Q_V \quad , \quad Q_V := \int_V q(x) d^nx$$

for some $A \in A$ and $Q$ is in that case only definable as a nasty operator (see below).

In the following we will take (77) as the defining relation of $ssb$ (the technical details of the various statements can be found in the literature, mentioned above).

The notion of $ssb$ is closely connected with another phenomenon, the so-called Goldstone-phenomenon. While there exists a clear picture in, say, relativistic quantum field theory, the corresponding picture is a little bit blurred in the non-relativistic regime. In the relativistic context we have sharp zero-mass Goldstone-modes, i.e. true particles due to relativistic covariance. On the other hand, in e.g. condensed matter physics or statistical mechanics the situation is less generic. In general we do no longer have sharp excitation modes; we have rather to expect excitation modes having a finite lifetime for momentum different from zero but becoming infinitely sharply peaked for momentum $k \to 0$. The proper view is it to analyze these excitation branches in the full Fourier-space of energy-momentum as has e.g. been done in ref. four of [12] and earlier in the author’s doctoral thesis, the principal object being the spectral-resolution of the 2-point correlation functions (in a neighborhood of $(E,k) = (0,0)$). $SSB$ or the Goldstone phenomenon manifests itself in this quantity by a singular contribution in the spectral measure. One should mention at this place the work
of Bros and Buchholz (see e.g. [14]) about quantum field theory in temperature (i.e. KMS-) states. In this particular context the residual causality and locality properties of the underlying relativistic theory lead to a, in some respects, more generic behavior as compared to the ordinary non-relativistic condensed matter regime.

In the non-relativistic regime it turns out that the concrete structure of the Goldstone mode depends usually on the details of the microscopic interactions (that means both the so-called energy-momentum dispersion-law which can be, to give an example, quadratic or linear near \( k = 0 \) in the case of magnons or phonons, say, and the \( k \)-dependent width of the branch). This led to the desire to characterize the presence of a Goldstone phenomenon by a simple (if qualitative) property. Sometimes one finds in the literature the saying that the Goldstone phenomenon consists of the vanishing of a mass-gap above the ground state. But this statement is in some sense frequently empty. From [15] we know e.g. that a short-ranged Galilei-covariant theory, with a non-vanishing particle density, cannot have a mass-gap due to phonon-excitations which signal the trivial breaking of the Galilei-boosts. Furthermore, in most cases KMS-Hamiltonians have as spectrum the whole real line.

**Remark 5.5** Models like the famous BCS-model (having a gap) are no case in point as they are implicitly breaking Galilei-invariance as do all such mean-field-models. This becomes apparent when analyzing the interaction part of the corresponding Hamiltonian. The complete fermion- or boson-liquid is, on the other side, again Galilei-invariant, hence has no mass-gap, but may, of course, still display e.g. superfluidity.

In the next subsection we will provide a, as we think, more satisfying and completely general characterization of the Goldstone phenomenon which is independent of the details of the model under discussion.

### 5.2 Some Rigorous Results for the Symmetry Generator in the Presence of SSB

After the above introductory remarks we want to prove a couple of rigorous results which characterize to some extent the presence of ssb in the (non-)relativistic regime. The main observation is that the symmetry generator is no longer defined as a nice operator in the representation (Hilbert- or GNS-) space when ssb is present and that this, at first glance, mathematical result encodes some interesting physics.

Let us work, for simplicity, in the context of temperature states. This has the advantage that \( \Omega \) is separating, i.e.

\[
A\Omega = B\Omega \Rightarrow A = B
\]  

(78)
The first task is to give $Q := \lim_V Q_V$ a rigorous meaning. The standard procedure (see the above mentioned literature) is to define $Q$ via:

$$QA\Omega := \lim_V [Q_V, A]\Omega, \quad Q\Omega := 0$$  \hspace{1cm} (79)

for e.g. $A \in A_0$. For $V$ sufficiently large, the commutator on the rhs becomes independent of $V$, hence there is a chance to get a well-defined $Q$ (at least on a dense set of vectors) as on the lhs we have by separability

$$A\Omega = B\Omega \Rightarrow A = B \Rightarrow [Q_V, A - B] = 0$$  \hspace{1cm} (80)

For $A \in A$ one has to employ cluster properties.

**Observation 5.6** We have already seen above that, while such a $Q$ may exist, the corresponding $\|Q_V\Omega\|$ will nevertheless diverge for $V \to \mathbb{R}^n$! This shows that the connection between the global generator and its local approximations is not that simple. The best one can usually expect, even in the case of symmetry conservation, is a weak convergence on a dense set

$$(B\Omega, QA\Omega) = \lim_V (B\Omega, Q_V\Omega)$$  \hspace{1cm} (81)

but, due to the above divergence of $\|Q_V\Omega\|$, we cannot even have weak convergence on the full Hilbert-space. (For more details see the above cited literature; in particular [12], third ref., where the various possibilities in the respective fields have been compared)

We see from the above that $Q$ can be defined as a densely defined operator but usually we want to have more. A conserved continuous symmetry is given by a s.a. generator. Let us see under what conditions the above $Q$ is at least symmetric provided that the $Q_V$ are symmetric. We assume the symmetry to be conserved, i.e.

$$\lim_V (\Omega, [Q_V, A]\Omega) = 0 \quad \text{for all} \quad A \in A$$  \hspace{1cm} (82)

We then have

$$(B\Omega, QA\Omega) = \lim_V (B\Omega, [Q_V, A]\Omega)$$

$$= \lim_V \left( ([Q_V, B]\Omega, A\Omega) + (Q_v\Omega, B^*A\Omega) - (A^*B\Omega, Q_V\Omega) \right)$$  \hspace{1cm} (83)

**Conclusion 5.7** $Q$ is symmetric if $\lim_V (A\Omega, Q_V\Omega) = 0$ for all $A \in A_0$. Under the same proviso it follows

$$(B\Omega, QA\Omega) = \lim_V (B\Omega, Q_V\Omega)$$  \hspace{1cm} (84)
What is the situation if the symmetry is spontaneously broken? For convenience we replace again the sharp volume-integration by our smooth one, i.e.

\[ Q_V \rightarrow Q_R := \int q(x) f_R(x) d^n x \]  

We know that there exists a symmetry-breaking observable \( A \) s.t.

\[ \lim_R \{} Q_R - [Q_R, A] \{} \neq 0 \Rightarrow QA\Omega = \lim_R \{} [Q_R, A] \{} \Omega \neq 0 \]  

Due to the assumed translation invariance, i.e.

\[ U(a)QU(-a) = Q \quad \text{or, what is the same,} \quad U(a)q(x)U(-a) = q(x + a) \]  

we have

\[ (\Omega, QA\Omega) = (\Omega, Q \cdot V^{-1}A_V\Omega) \]  

and

\[ Q \cdot V^{-1}A_V\Omega = V^{-1} \int_V U(x)d^n x \cdot QA\Omega \]  

\( U(x) \) the unitary representation of the translations.

Remark: As a result of a discussion with Detlev Buchholz, following a seminar talk about the paper, we will give a technically more detailed proof of the above statement in the appendix at the end of the paper. This seems to be advisable since, as we are showing below, the global operator, \( Q \), turns out to be non-closable, which will make certain limit-manipulations more cumbersome.

**Lemma 5.8**

\[ s - \lim_V V^{-1} \int_V U(x)d^n x = P_\Omega \]  

\( P_\Omega \) the projector on the (in our case) unique vacuum-ground-equilibrium-state.

Proof: The result is well-known (see e.g. [1]). We give however a very short and slightly different proof using our smooth volume integration. With \( V_R := \int f_R(x)d^n x \), a spectral resolution yields

\[ V_R^{-1} \cdot \int U(x)f_R(x)d^n x = \text{const} \cdot \left( \int f(x)d^n x \right)^{-1} \cdot \int \hat{f}(Rp)dE_p \]  

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Applied to a vector $\psi$ we can now employ Lebesgue’s theorem of dominated convergence and get

$$\lim_V V^{-1} \int U(x) d^n x \cdot \psi = (\hat{f}(0))^{-1} \cdot \hat{f}(0) P_\Omega \psi = P_\Omega \psi \quad \square$$  \hspace{1cm} (92)

This yields

$$0 \neq P_\Omega QA = \lim_V Q \cdot V^{-1} A_V \Omega$$  \hspace{1cm} (93)

On the other hand

$$\lim_V \| V^{-1} A_V \Omega \| = \| P_\Omega A \Omega \| = 0$$  \hspace{1cm} (94)

by an analogous reasoning (note that we assumed $(\Omega, A \Omega) = 0$).

We have now a sequence of vectors, $V^{-1} A_V \Omega$, converging to zero in norm while $Q \cdot V^{-1} A_V \Omega$ converges to $P_\Omega QA \Omega \neq 0$. Summing up what we have shown we arrive at the following conclusion:

**Conclusion 5.9 (Goldstone Theorem)** If we have ssb and a separating vector, $\Omega$, (representing the ground or temperature state), $Q$ can still be defined as an operator which is however not closable, hence, a fortiori, not symmetric (note that symmetric operators are closable). This abstract result has as a practical consequence the physical property exhibited in the preceding formulas. They express the content of the Goldstone phenomenon in the most general and model independent way. We infer that $Q$ induces transitions from a singular part of the continuous spectrum, passing through $(E, p) = (0, 0)$, to the extremal invariant state $\Omega$. On the other side, a conserved symmetry implies

$$Q \Omega = 0, \quad P_\Omega [Q, A] \Omega = 0 \Rightarrow P_\Omega QA \Omega = 0$$  \hspace{1cm} (95)

We show now that the above result really contains the original Goldstone phenomenon. Let us e.g. assume that we have the above result and, on the other side, a gap in the energy spectrum above the state $\Omega$. We emphasized above that an important ingredient of the notion of ssb is the time independence of, say, the above expression. We employ again the spectral resolution of operators with respect to energy-momentum. We hence have

$$0 \neq c = P_\Omega Q \int \hat{A}(k, E) e^{-itE}dkdE \Omega$$  \hspace{1cm} (96)

with $c$ being independent of $t$. We choose a real testfunction $g(t)$ with $\int g(t) dt = 1$. This yields

$$0 \neq c = P_\Omega Q \int A(t) \cdot g(t) dt \Omega = P_\Omega Q \int \hat{A}(E) \hat{g}(E) dE \Omega$$  \hspace{1cm} (97)
If there is a gap above zero we may choose the support of \( \hat{g} \) so that

\[
\text{supp}(\hat{g}) \cap \text{supp}(\text{spec}(H)) = 0
\]  

(98)

Since, by assumption, \( P_\Omega \) has been extracted in the energy-support of \( A \), we get the result \( c = 0 \), that is, no symmetry breaking. But we can infer more about the nature of the energy-momentum spectrum near \((0, 0)\). We see that \( P_\Omega Q A(g(t))\Omega \) depends only on the value of \( \hat{g}(E) \) in \( E = 0 \), which is one in our case, but not on the shape of \( g \). Inspecting equation (93) we can infer the following: The Fourier transform of the rhs contracts around \( k = 0 \) in the limit \( V \to \infty \). On the other side we learned that in the limit both sides have their energy support concentrated in \( E = 0 \). The lhs shows that the limit vector is parallel to \( \Omega \). Whereas we do not want to go into the partly intricate details of the limiting processes of \textit{non-closable operators} (note that it is e.g. dangerous to use the adjoint, \( Q^* \), in the reasoning as it is not densely defined), the latter part of the above theorem should now be obvious.

This sharp excitation around \((E,k) = (0,0)\) extends into the full energy-momentum plane in form of a (usually) smeared excitation branch (having a finite \( k \)-dependent life-time). For the regime of temperature states the situation was analyzed in some detail in the fourth reference of [12] and already in the authors doctoral thesis. We see from the above that a similar situation prevails in the more general case of a separable \( \Omega \) and, analogously, for ground-state models where \( Q \) can be defined in the above way. Even if the above \( Q \) is not definable as a non-closable limit operator we arrive at a similar result by exploiting the limit-expectation values instead of the strong vector- or operator limits, but we do not want to dwell more into the corresponding details in this paper which deals with a different topic.

6 The Canonical (Goldstone) Pair in the Presence of SSB

As far as we can see, the notion of a \textit{canonical Goldstone pair} was introduced by Verbeure et al. in [20]. In the following section we want to prove only a few general (model-independent) results, whereas much more could be shown by combining the framework, developed above, with the techniques mentioned in the preceding section.

We remarked above that \textit{ssb} is characterized by the non-vanishing (but time-independence) of the following commutator limit

\[
0 \neq c = \lim_{V} (\Omega, [Q_V, A(t)]\Omega) 
\]  

(99)

To fix the notation: usually a pure phase is characterized by the non-vanishing of a so-called \textit{order parameter} in the presence of \textit{ssb}. This is an observable, \( B \)
say, with

\[(\Omega, B\Omega) = \begin{cases} c \neq 0 & \text{in the broken phase} \\ 0 & \text{in the conserved phase (above } T_c, \text{ say)} \end{cases} \]  
(100)

From (99) we see that as order parameter we have to choose

\[B := \lim_{V}[Q_V, A] \]  
(101)

while \(A\) is the symmetry breaking observable.

**Example 6.1** *In the Heisenberg-ferromagnet with spontaneous magnetization in, say, the \(z\)-direction the order parameter is \(S_z\) or \(\langle S_z \rangle\). As generator of the broken symmetry one may take \(\sum S_x\) and as symmetry breaking observables e.g. \(S_y\).*

We have seen that we can write

\[0 \neq c = \lim_V(\Omega, [Q_V, A]\Omega) = \lim_V(\Omega, [Q_V, V^{-1}A_V]\Omega) = \lim_R(\Omega, [Q_R, V_R^{-1}A_R]\Omega) \]  
(102)

where

\[Q_R := \int q(x)f_R(x)d^nx \, , \, A_R := \int_{S_R} A(x)d^nx \]  
(103)

with \(V_R\) the volume of the sphere, \(S_R\), with radius \(R\).

We can now split the scaling exponent among the two observables (the volume of the unit sphere being absorbed in the constant).

\[0 \neq \text{const} = \lim_R(\Omega, [R^{-\alpha}Q_R, R^{-(n-\alpha)}A_R]\Omega) \]  
(104)

This form of scaling may yield something reasonable if the scaling exponents can be so adjusted that also

\[(\Omega, R^{-\alpha}Q_RR^{-\alpha}Q\Omega) \text{ and } (\Omega, R^{-(n-\alpha)}AR^{-(n-\alpha)}A\Omega) \]  
(105)

remain finite in this limit.

In general it does not seem to be easy to get both rigorous and general estimates on the scaling behavior of these quantities. Fortunately, in the case of temperature (KMS) states, such estimates are available. In [14] to [18] the following special (*real-space*) version of the *Bogoliubov-Inequality* has been proved and employed for the observables \(Q_R\) and \(V_R^{-1}A_R\):

\[|\langle[Q_R, V_R^{-1}A_R]\rangle|^2 \leq \langle V_R^{-1}A_RV_R^{-1}A_R\rangle \cdot \langle[Q_R, [Q_R, H]]\rangle \]  
(106)
The delicate term is the double commutator on the rhs. If $Q$ is spontaneously broken, boundary terms will survive in the commutator of $Q_R$ and the Hamiltonian, $H$, when taking the limit $R \to \infty$, while in a formal sense they commute. The double commutator saves us two powers of $R$, so to say. That is we arrive after some cumbersome manipulations at

$$\langle [Q_R, [Q_R, H]] \rangle \sim R^{(n-2)} \text{ for } R \to \infty$$

hence

$$\langle V_R^{-1} A_R V_R^{-1} A_R \rangle \gtrsim R^{(2-n)} \text{ for } R \to \infty$$

as the limit on the lhs is a constant different from zero in the case of ssb.

**Theorem 6.2** For temperature states we have for the symmetry breaking observable

$$\langle A_R A_R \rangle \gtrsim R^{(n+2)}$$

That is, compared with the ordinary, normal scaling behavior ($\sim R^n$), the divergence is worse. From this one infers the following decay of the two-point correlation function itself:

$$|\langle A(x)A \rangle| \gtrsim R^{(n-2)}$$

Putting all the pieces together we now have to make the following identification:

$$n - \alpha \geq (n + 2)/2 \Rightarrow \alpha \leq (n - 2)/2$$

in order that the limit commutator is non-trivial, i.e. non-classical. On the other hand, the divergence behavior of $\langle Q_R Q_R \rangle$ can frequently be inferred either from covariance properties (as in relativistic quantum field theory; see e.g. the third reference in [6]) or from an analysis of the spectral behavior in concrete (non-relativistic) models. Summing up we have:

**Conclusion 6.3 (Canonical Pair)** For a covariant four-current in relativistic quantum field theory the two-point function in Fourier space contains a prefactor $\sim p^2$ which yields (after some calculations) an $\alpha = 1/2$ (for space dimension, $n = 3$). On the other side, if we do not have such nice covariance properties the divergence of $\langle Q_R Q_R \rangle$ is generically much worse than $\sim R$ (in three dimensions). This holds, in particular, for the above temperature states. It follows that for temperature states we cannot find a critical exponent $\alpha$ so that both the auto-correlations remain finite in the limit and the commutator non-trivial. That is, for temperature states the limit fluctuations are classical (an observation already made by Verbeure et al for special models, see e.g. [20]).
The situation seems to be less generic for ground state models, i.e. the temperature-zero case. For one, we do not automatically have an a priori estimate as in the above conclusion, from which we can infer that it is the autocorrelation of $A_R$ which is ill-behaved. For another, in temperature states, as was shown in e.g. the fourth reference of [12] by the author, the spectral weight has to become infinite along the Goldstone excitation branch in a specific way (which is governed by the dispersion law of the Goldstone mode) for energy-momentum approaching zero. This sort of singularity is mainly responsible for the poor decay of the respective auto-correlation function. This phenomenon may be absent in the case of ground states as has also been shown for certain Bose-gas models in [20] where some of these questions have been dealt with in greater detail. Note in particular that a variety of aspects may depend on the precise shape of the Goldstone mode near energy-momentum equal to $(0, 0)$ as was shown in the above mentioned paper of the author or in the unpublished doctoral thesis.

On the other side, there has been some interesting work of Pitaevskii and Stringari (see e.g. [21]), who showed that variants of the uncertainty principle may lead to non-trivial results in certain cases for ground state systems if one can exploit and control certain additional sum rules.

Remark 6.4 Note that the ordinary uncertainty principle (for e.g. hermitean operators and ignoring possible domain questions) reads

$$1/4 \cdot |\langle [A, B]\rangle|^2 \leq \langle AA \rangle \cdot \langle BB \rangle \quad (112)$$

One sees that instead of the double commutator of the local symmetry generator and the hamiltonian now a term like $\langle Q_R Q_R \rangle$ occurs. While we have an a priori estimate of the large-R-behavior of the double commutator, the behavior of $\langle Q_R Q_R \rangle$ is probably less generic (in particular in the ground state situation) and we need some extra information of the kind mentioned above.

7 The Case of SSB or Very Poor Decay of Correlations

In the preceding sections we studied the case of $L^1$- or $L^2$-clustering. In this last section we want to briefly show how we can proceed in the case of extremely poor clustering. We want however, for the sake of brevity and in order to better illustrate the method, to concentrate on the simpler case of a uniformly poor decay of all the correlation functions we are discussing. This is of course not always the case but the scheme can be easily generalized (we discuss this topic in more detail in [23], where we treat this question in the context of the renormalisation group analysis).

We hence assume that the truncated $l$-point functions cluster weaker than $L^2$ or $L^1$, say, in the difference variables, $y_i := x_{i+1} - x_i$, (see section 3.1). The
following reasoning works both in the case of non-\textit{L}^{1} or non-\textit{L}^{2} clustering. In the latter case one would again use the \textit{Cauchy-Schwarz-inequality} (as in section 3.2). To illustrate the method we choose the non-\textit{L}^{1} procedure.

So let us assume

\[ W^{T}(y_{1}, \ldots, y_{l-1}) \not\in \textit{L}^{1} \]  \hspace{1cm} (113)

For each \( l \) we assume the existence of a weight factor with a suitable exponent, \( \alpha_{l} \in \mathbb{R} \):

\[ P_{l}(y) := (1 + \sum y_{i}^{2})^{\alpha_{l}/2} \]  \hspace{1cm} (114)

so that

\[ F(y) := P_{l}(y)^{-1} \cdot W^{T}(y) \in \textit{L}^{1} \, \text{ for } \, \alpha_{l} > \alpha_{l}^{inf} \]  \hspace{1cm} (115)

On the other side, we define the fluctuation operators with the exponent \( \gamma \), which will be adjusted later

\[ A_{R}^{F} := R^{-\gamma} \cdot A_{R} \]  \hspace{1cm} (116)

It follows

\[ W^{T}(y) = P_{l}(y) \cdot F(y) \]  \hspace{1cm} (117)

with \( F(y) \) an (in general, \( l \)-dependent) \textit{L}^{1}-function.

For the limit correlation functions we then get

\[
\langle A_{R}^{F}(1) \cdots A_{R}^{F}(l) \rangle^{T} = R^{ln} \cdot R^{-l\gamma} \cdot \int \hat{F}(q) \cdot \hat{P}_{l}(q) \left[ \hat{f}(R_{p_{1}}) \cdots \hat{f}(-R_{q_{l-1}}) \right] \prod dq_{i} \]
\hspace{1cm} (118)

(cf. section 3.1)

\textbf{Remark 7.1} We write the Fourier transform of \( P_{l}(y) \) formally as

\[ \hat{P}_{l}(q) = (1 + \sum D_{q_{i}}^{2})^{\alpha_{l}/2} \]  \hspace{1cm} (119)

(with \( D_{q_{i}} \) the partial derivations). For non-integer \( \alpha_{l}/2 \) this is a pseudo-differential operator. At the moment, for the sake of brevity, we do not want to say more about the corresponding mathematical framework (see [23] for a complete discussion). What we in fact only need are the scaling properties of the expression. If one wants to be careful one may equally well take the explicit expression for the Fourier transform of the above polynomial in the \( y \)-coordinates applied to the product of the \( f_{R} \)'s and exploit its scaling properties.
In any case, we get (with this proviso) and the usual variable transformation $p'_i := Rp_i$:

$$\langle A^F_R(1) \cdots A^F_R(l) \rangle^T = \left[ R^{n_{\alpha}} \cdot \frac{c}{\gamma} \cdot \left( \sum_{\delta} D^2 \right)^{\alpha_{l+1}} \right] \left[ \hat{f}(p'_1) \cdots \hat{f}(-q'_l) \right] \prod dq'_i \quad (120)$$

Again only the explicit scaling prefactor matters in the limit $R \to \infty$. (Note that for non-minimal $\alpha_l$ we may have $\hat{F}(0) = 0$. Technical intricacies like this one will be discussed at length in [23].) To get a finite result for all correlation functions we have to adjust the scaling parameter, $\gamma$, so that the exponents vanish or are negative. We choose $\alpha_2$ for $l = 2$ so that the limit two-point function is finite and non-vanishing. That is:

$$n - 2\gamma + \alpha_2 = 0 \rightarrow \gamma = (n + \alpha_2)/2 \quad (121)$$

Inserting this $\gamma$ in the general expression for $l \geq 3$, we conclude that the scaling prefactor is finite in the limit provided that

$$\alpha_l \leq l\gamma - n = ((l - 1)n + l\alpha_2)/2 \quad (122)$$

with $\gamma$ fixed by the two-point function. For $\alpha_l < l\gamma - n$ we can even conclude that all(!) higher limit correlation functions vanish and that the resulting theory is (quasi-)free. The latter would, for example, be the case if

$$\alpha_l \leq (l - 1) \cdot \alpha_2 \quad (123)$$

holds, since we then have (with $\alpha_2 < n$):

$$\alpha_l \leq (l - 1) \cdot \alpha_2 < (l - 1/2)\alpha_2 = (2l - 1) \cdot \alpha_2/2 < ((l - 1)n + l\alpha_2)/2 \quad (124)$$

but nothing can be concluded in general for, say, $\alpha_l = l \cdot \alpha_2$.

We see that it is of tantamount importance to better understand the asymptotic behavior of truncated $l$-point functions and, in particular, the rate of decay as a function of $l$. We address this topic in more detail in [23].

**Appendix**

The rigorous implementation of the formula

$$U(a)q(x)U(-a) = q(x + a) \quad (125)$$
is
\[
U(a)Q_RU(-a) = U(a) \int q(x)f_R(x)d^n x U(-a) = \int q(x + a)f_R(x)d^n x
\]
\[
= \int q(y)f_R(y - a)d^n y =: Q_R(a) \quad (126)
\]
The first question is: how does the global $Q$ behave under translations? To answer this question we have to take recourse to the definition of the global $Q$ as a limit of local operations. We have
\[
U(a)QA = U(a) \lim_R [Q_R, A] = \lim_R [Q_R(a), A(a)] \quad (127)
\]
since it holds
\[
\lim_n U(a)\psi_n = U(a) \lim_n \psi_n \quad (128)
\]
as $U(a)$ is bounded. If $A$ is local we have for sufficiently large $R$ (and hence, in the limit):
\[
\lim_R [(Q_R(a) - Q_R(0)), A(a)] = 0 \quad (129)
\]
We hence arrive at
\[
U(a)QA = \lim_R [Q_R, A(a)] = QA = QU(a)A \quad (130)
\]

**Lemma 7.2** On the dense set $A_0\Omega$, $Q$ commutes with the translations.

In a next step we have to analyse the action of $Q$ on integrals or averages like $\int_V U(x)AU(-x)d^n x\Omega$. More specifically, we want to show that $Q$ commutes, so to speak, with the operation of integration. We have
\[
Q \cdot \int_V A(x)d^n x\Omega := \lim_R [Q_R, \int_V A(x)d^n x]\Omega \quad (131)
\]
We approximate the integral by a sum, that is:
\[
\int_V A(x)d^n x\psi := \lim_i \sum_i d^n x_i \cdot A(x_i)\psi \quad (132)
\]
and get (as the $Q_R$ are assumed to be nice, that is, closed operators)
\[
[Q_R, \int_V A(x)d^n x]\Omega = \lim_i [Q_R, \sum_i d^n x_i \cdot A(x_i)]\Omega = \lim_i \sum_i d^n x_i \cdot U(x_i)[Q_R(-x_i), A]\Omega \quad (133)
\]
We again choose $R$ so large that

$$[Q_R(-x), A] = [Q_R, A] \quad \text{for all } x \in V$$

(134)

which leads to

$$[Q_R, \int_V A(x) d^nx] \Omega = \lim_i \sum_i d^n x_i \cdot U(x_i) [Q_R, A] \Omega = \int_V U(x) d^n x \cdot [Q_R, A] \Omega$$

(135)

Taking now the limit $R \to \infty$, we get

**Lemma 7.3**

$$Q \int_V A(x) d^n x \Omega = \int_V U(x) d^n x \cdot QA \Omega$$

(136)

This shows, that our manipulations can be justified.

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