Quantum gray solitons in confining potentials

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We define and study hole-like excitations (the Lieb II mode) in a weakly interacting Bose liquid subject to external confinement. These excitations are obtained by semiclassical quantization of gray solitons propagating on top of a Thomas-Fermi background. Radiation of phonons by an accelerated gray soliton leads to a finite life-time of these excitations. It is shown that, for a large number of trapped atoms, most of the Lieb II levels can be experimentally resolved.

Ultra cold atoms restricted to move in one dimension (1d) are now routinely used for experimental realization and investigation of the physics of strongly interacting many-body systems. Due to advances in measurement techniques, the dynamical response of one-dimensional quantum gases has recently gained a central role in these studies [1–3].

Dynamics is ultimately related to the nature of excitation spectrum. The latter is given by the Bethe Ansatz solution of Lieb-Liniger model [4] for 1d ultra cold bosons with short-range interactions [5, 6]. As shown by Lieb [7] the excitations can be decomposed into a superposition of particle-like (Lieb I) and hole-like (Lieb II) excitations in one-to-one correspondence with elementary excitations of one-dimensional fermions. For weak coupling, the dispersion of Lieb I excitations can be obtained semiclassically by linearization of the Gross-Pitaevskii Equation (GPE) resulting in phonon-like spectrum. Lieb I excitations determine the thermodynamic properties of the system and control the power-law decay of correlation functions.

Later, it was shown by Kulish, Manakov and Faddeev [8] that the Lieb II mode can be associated with another semiclassical object – the gray soliton [9]. The latter is a hole-like solution of GPE representing localized density and supercurrent dip propagating on top of of constant background. It is a mean field picture of many quantum Lieb II excitations forming a coherent wavepacket. This classical description holds as long as quantum fluctuations remain small [10] due to a large negative effective mass resulting from the many particles expelled from the soliton core. The same large parameter is responsible for the exponentially suppressed probability to create gray solitons using local perturbations [11].

Even a shallow axial trapping potential present in the experiments breaks the integrability of the Lieb-Liniger model and the Lieb classification of the excitations is apparently lost. While Lieb I phonons can still be defined by linearization of GPE around the appropriate density profile and quantizing the resulting collective oscillations [12, 13], no analogous procedure exists for the Lieb II mode. Classically, however, gray solitons do survive the harmonic confinement if the latter is sufficiently smooth on the length scale of the soliton [14, 15]. They have recently been observed in several experiments [16–18] in agreement with theoretical studies [19, 20]. The physics of underlying quantum states has not been addressed yet.

The aim of this Letter is to define and study the finite-size analogue of the Lieb II mode in a trapped system as corresponding quantum excitations. This is done by applying Bohr-Sommerfeld quantization rule to the classical dynamics of gray soliton. The obtained quasiparticle levels, shown schematically in Fig. 1 have a finite life-time \( \tau \) and precludes resolution of low-lying levels.

![FIG. 1. Schematic picture of trapped Lieb II modes obtained by quantizing motion of gray solitons in the non-uniform background. The non-uniform density profile leads to the finite life-time \( \tau \) and precludes resolution of low-lying levels.](image-url)
Our main findings are as follows. For a system of \( N_{\text{tot}} \) interacting bosons trapped in a potential \( U(x) \) there are exactly \( N_{\text{tot}} \) quantum states below the energy \( E_{\text{d}} \) corresponding to the energy of stationary classical dark soliton localized at the maximum of the trapped density profile. We associate these states, shown schematically in Fig. 1, with the trapped Lieb II mode. In the case of harmonic confinement \( U(x) = m\omega^2 x^2/2 \) we use the Thomas-Fermi density profile to show that \( E_{\text{d}} = (\hbar \omega/\sqrt{2}) N_{\text{tot}} \) and the energies of the trapped Lieb II mode are given by the descending ladder

\[
E_n = E_{\text{max}} - \frac{\hbar \omega}{\sqrt{2}} \left( n + \frac{1}{2} \right),
\]

where \( n = 0, 1, \ldots, N_{\text{tot}} - 1 \). The energy of the highest quantum state \( n = 0 \) is reduced by zero-point oscillations \( (\hbar \omega/\sqrt{2})/2 \) from the classical value \( E_{\text{d}} \). It is interesting to compare this result with the case of extremely strong interactions, so-called Tonks-Girardeau limit [23], where hard-core bosons can be mapped into non-interacting fermions. In this case the Lieb II mode is obtained by creating holes in the filled Fermi sea of \( N_{\text{tot}} \) particles occupying energies \( E_n = E_F - \hbar \omega (n+1/2) \), below the Fermi energy \( E_F = \hbar \omega N_{\text{tot}} \).

The trapped Lieb II states are not eigenstates but rather quasiparticle resonances with finite life-time. The life-time for a state with energy \( E \) is found to be

\[
\tau(E) = \frac{8\mu}{\hbar \omega} F(x),
\]

where \( \mu \) is the chemical potential of the condensate, \( x^d = E/E_{\text{d}} \), and the function \( F(x) \) is defined in Eq. \((24)\). For high energies, \( E \approx E_{\text{d}} \), we find a logarithmically large life-time \( \omega \tau \approx (4\mu/\hbar \omega) \log [6E_{\text{d}}/(E_{\text{d}} - E)] \), while for low energy states, \( E \ll E_{\text{d}} \) we have \( \omega \tau \approx (8\mu/3\hbar \omega)(E/E_{\text{d}}) \). The last expression defines an energy scale \( E^* = (\hbar \omega/\mu) E_{\text{d}} \) below which \( \omega \tau \lesssim 1 \) and Lieb II quasiparticle states cannot be resolved. This condition coincides with the classical picture of an overdamped soliton decaying before completing one period of oscillations in the trap. Another restriction, \( E \gg E^{**} = E_{\text{d}}/K^{3/2} = \mu/K^{1/2} \), arises from the requirement of the validity of the semiclassical treatment [11]. Here \( K = \pi E_{\text{d}}/\mu = (\pi \hbar \omega/\sqrt{2}\mu)N_{\text{tot}} \) is the Luttinger parameter in the center of the trap. For the experiment in Ref. [18] one has \( \hbar \omega/\mu \sim 10^{-2} \) and \( K \sim 10^3 \), therefore \( E^* \sim 0.01E_{\text{d}} \gg E^{**} \) and Eq. \((1)\) describes accurately the energies of the most of the trapped Lieb II states and most of these are well defined quasiparticles, hence our semiclassical quantization of the soliton is justified \( a \) \( \text{posteriori} \).

**Solitons in uniform background.**—To derive the above results we consider the standard Lagrangian density for weakly interacting 1d bosons \( \mathcal{L} = i\hbar \bar{\psi} \partial_x \psi - \mathcal{E}(\bar{\psi}, \psi) \) with the energy density

\[
\mathcal{E} = \frac{\hbar^2}{2m} |\partial_x \psi|^2 + \frac{g}{2} \left( |\psi|^2 - n \right)^2
\]

Here \( \psi(x,t), \bar{\psi}(x,t) \) are bosonic fields, \( m \) is the particle mass and \( g > 0 \) characterizes repulsive interactions between particles. In [3] we have subtracted the constant contribution of the static background density \( n \) (fixed by the chemical potential \( \mu = gn \)). Variation with respect to the fields leads to GPE,

\[
\frac{i}{\hbar} \partial_t \psi = -\frac{\hbar^2}{2m} \partial_x^2 \psi + g \left( |\psi|^2 - n \right) \psi.
\]

As shown in [9] it allows for a one-parameter family of gray solitons \( \psi_s(x - X(t)) \) well localized around the position \( X \) moving with a constant velocity, \( X = X_0 + Vt \).

According to [9] the form of these solutions is

\[
\psi_s(x)/\sqrt{n} = \cos \frac{\Phi}{2} - i \frac{\Phi}{2} \tanh \left( \frac{mgN^2}{2\hbar^2} x \right).
\]

Parameters \( \Phi \) and \( N \) are integral characteristics of the soliton [24] representing the total phase drop across the soliton and the number of particles expelled from the soliton vicinity. The are both fixed by the velocity \( V \) of the soliton,

\[
\Phi = 2 \arctan \sqrt{c^2 - V^2}/V
\]

\[
N = \int dx \left( n - |\psi_s(x)|^2 \right) = \frac{2\hbar}{g} \sqrt{c^2 - V^2}. \tag{7}
\]

Here \( c = \sqrt{gn/m} \) is the sound velocity limiting the velocity of the soliton, \( V^2 < c^2 \).

Following Ref. [15] we wish to establish a description of a soliton as an effective particle. To this end we substitute the solitonic solution given by Eq. \((5)\) into Eq. \((3)\) and integrate it over the length of the system to obtain the energy of the soliton,

\[
E_s = \frac{4}{3} \frac{\hbar m}{g} \left( c^2 - V^2 \right)^{3/2} = \frac{m g N^3}{6\hbar^2} \tag{8}
\]

The canonical momentum of the soliton is

\[
P_s = \hbar n \Phi - mNV \tag{9}
\]

The contribution \( \hbar n \Phi \) comes from a small background supercurrent [25] which carries no energy but must be introduced to compensate for the phase drop \( \Phi \). The second contribution to the momentum describes the deficit of \( N \) particles moving with velocity \( V \). Such an expression for the momentum is completely general and independent of the details of interactions between the particles [26]. In contrast, the energy \( E_s \) does depend on the form of interactions and the fact that it only depends on the particle deficit \( N \) is a direct consequence of the quartic interaction of the GP energy density in Eq. \((3)\).

For our purposes it is convenient to use the canonical formalism where the momentum \( P_s \), rather than velocity \( V \) serves as a control parameter. By formally inverting the relation \((9)\) we obtain the velocity as a function of momentum,

\[
\dot{X} = V(P_s, n) \tag{10}
\]
Substituting $V(P,n)$ into Eqs. (6), (7) and Eq. (8) yields $\Phi(P,n)$, $N(P,n)$ and $E_n(P,n)$. The latter defines the dispersion of the soliton, or its effective Hamiltonian. Indeed, Eq. (10) constitutes the Hamiltonian equation of motion as we have $V = (\partial E/\partial V)/(\partial P/\partial V) = \partial E/\partial P$ following from Eqs. (8) and (9). Another Hamiltonian equation of motion states the conservation of momentum $\dot{P} = 0$ expected from the translational invariance.

**Solitons in Thomas-Fermi density profile.**—
Translational invariance is broken by the presence of external trapping potential $U(x)$. In this case the equation of motion for the momentum will be altered. It should be stressed, however, that the external potential does not act on the coordinate $X$ of the soliton directly, but rather on the particles of the Bose gas forming the soliton. To describe this situation and derive the equations of motion we follow the method pioneered in Ref.[15]. We assume the non-uniform background density is given by the Thomas-Fermi density profile

$$gn(x) = \mu - U(x).$$

(11)

For symmetric $U(x)$, a nonzero density requires $|x| < R$ where the Thomas-Fermi radius $R$ is found from $U(R) = \mu$. The local sound velocity is $c(x) = \sqrt{gn(x)/m}$.

For a sufficiently smooth potential and large $R$ the density $n(x)$ changes smoothly on the typical length scale $\sqrt{gn(x)}/m$.

The classical orbit which encloses the entire available phase space and on its two branches $\Phi = 0$ and $\Phi = \pm 2\pi$ is assumed to have $I(0) = hN_{\text{tot}}$ to $I(E_{\text{ds}}) = 0$. The classical frequency of oscillations is given by $dE/dI = \Omega(E)$ which is negative as expected for the hole-like excitations.

Applying the standard Bohr-Sommerfeld semiclassical quantization rule, the energy levels are found from the condition $I(E_n)/h = n + 1/2$. Therefore there are exactly $N_{\text{tot}}$ quantum states, for any trapping potential $U(x)$. In the case of an harmonic confining potential $U(x) = m\omega^2 X^2/2$, the effective energy in Eq. (13) describes harmonic motion with energy-independent oscillation frequency $\Omega = -\omega/\sqrt{2}$, hence $E_{\text{ds}} = (\hbar\omega/\sqrt{2})N_{\text{tot}}$ and we obtain equidistant spectrum given by Eq. (4).
Interactions with phonons.—The picture presented above of solitons as Landau quasiparticles must be complemented with their interactions with the phonons. Indeed, a soliton propagating in a smooth non-uniform density profile is analogous to an electron moving in static electric field created by some external sources. It is well known that its deceleration leads to radiation of electromagnetic waves removing momentum and energy from the electron, known as Bremsstrahlung [27]. In the case of the gray soliton the emitted phonons lead to an eventual loss of energy and momentum from the soliton resulting in its acceleration due to its negative effective mass. To describe such a process we consider a combined system of the soliton and phonons

\[ L_{\text{eff}} = L_s + L_{s-\text{ph}} + L_{\text{ph}} \]  

(16)

where \( L_s = P^2 X - H(P, X) \) and the last term describes the low-energy phonons in the quadratic harmonic approximation [28, 29],

\[ L_{\text{ph}} = \int dx \left[ -\hbar p \partial_t \varphi - \frac{m c^2}{2n} \rho^2 - \frac{\hbar^2 n}{2m} (\partial_x \varphi)^2 \right] \]  

(17)

Here we use density-phase representation for slow bosonic fields \( \psi(x, t) = \sqrt{n} + \rho(x, t) \exp(i\varphi(x, t)) \) [30]. The coupling \( L_{s-\text{ph}} \) between soliton and low-energy phononic modes is given by the universal form,

\[ L_{s-\text{ph}} = -\hbar \Phi \partial_x (X, t) / \pi - \hbar N \varphi(x, t), \]  

(18)

where we have introduced displacement field \( \vartheta(x, t) = \pi \int_x^\infty \rho(y, t) dy \). The universal coupling [18] was derived in Ref. [26] directly from the principle of translational and gauge invariance and was shown to remain valid away from weak coupling limit. It shows that it is the soliton’s total phase and the number of particles ejected from the soliton core (or, rather their temporal change) that couple to the phonon fields. The coupling [18] is also local, involving phonon fields at the position of the soliton.

Dissipative dynamics of gray solitons.—The phonon action in Eq (17) is quadratic, thus we are able to directly integrate out the phononic degrees of freedom using the Keldysh formalism [26] to obtain an effective action for the soliton. We obtain the following equations of motion modified by the phonons

\[ \dot{X} = V + \frac{h \kappa}{2} \Phi \partial_P \Phi + \frac{h}{2 \kappa} \dot{N} \partial_P N \]  

(19)

\[ \dot{P} = -\partial_X H - \frac{h \kappa}{2} \dot{\Phi} \partial_P \Phi - \frac{h}{2 \kappa} \dot{N} \partial_X N \]

\[ -\frac{\hbar c}{2c^2 - V^2} \left( \frac{\kappa V}{2c} \dot{\Phi}^2 + \dot{\Phi} \dot{N} + \frac{V}{2 \kappa c} \dot{N}^2 \right). \]  

(20)

Here \( \kappa = h n / m c \) is a large number related to the Luttinger parameter \( K = \pi \kappa \) of the phononic Lagrangian [17] and \( V = \partial_P H \) is the velocity of the soliton in the absence of phonons. These equations hold in the adiabatic limit, assuming slow deviation of soliton parameters. Similar expressions were obtained in Ref. [24].

As is well known from the theory of radiation in QED (see e.g. Refs. [24, 31] the dissipative Eqs. (19) and (20) are plagued with runaway solutions. The only consistent way to treat these equations is to solve for the soliton trajectory \( P(t), X(t), N(t) \), \( \Phi(t) \) in the absence of phonons and calculate the non-adiabatic corrections perturbatively. As the number of particles \( N \) ejected from the soliton is conserved the r.h.s. of Eqs. (19) and (20) simplifies considerably. Turning to the energy dissipation rate, we have

\[ \dot{H} = \dot{P} \partial_P H + \dot{X} \partial_X H = -\frac{\hbar \kappa}{2} \frac{c^2}{c^2 - V^2} \dot{\Phi}^2. \]  

(21)

Using Eqs. (6), (7) one can show that \( \Phi = -g N \dot{\Phi} / c^2 \). Averaging Eq. (21) over one period using harmonic motion \( V/c = \sqrt{1 - (N/2\kappa)^2} \cos(\omega t/\sqrt{2}) \), we find the rate of slow change of energy of the soliton

\[ \dot{E} = -\frac{1}{T} \int_0^T \dot{H} dt = -\frac{2h \kappa}{c^2 \sqrt{2}} \int_0^T \dot{V} \dot{dt}, \]

\[ \frac{h \kappa \omega}{2} \left[ 1 - \left( \frac{N}{2 \kappa} \right)^2 \right] = \frac{mg^2 N^2}{2h \omega^2} \frac{N}{2 \kappa}. \]  

(22)

This allows us to calculate the life-time of the soliton,

\[ \tau = \int_0^N \frac{dN}{N} = \frac{mg^2}{h^3 \kappa^2 \omega^2} \int_0^N \frac{N^2 dN}{1 - (N/2 \kappa)^2} = \frac{8 \mu}{h \omega^2 F \left( \frac{N}{2 \kappa} \right)}, \]  

(23)

where we have defined

\[ F(x) = \int_0^x \frac{dy}{1 - y^2} = \frac{1}{2} \log \frac{1 + x}{1 - x}. \]  

(24)

For small \( x \) it behaves like \( F(x) \approx x^3 / 3 \) while near \( x = 1 \) it diverges logarithmically \( F(x) \approx \log \sqrt{2 / (1 - x)} \). Using the fact that \( (N/2 \kappa)^2 = E/E_{\text{abs}} \) we get the results stated in and above Eq. (2).

Conclusions.—Hole-like Lieb II excitations can be generalized to trapped systems by quantizing dynamics of gray soliton in non-uniform background. Despite their finite life-time, the obtained excitations are well defined for current experimental conditions. This can be verified by studying spectrum of phonon absorption, e.g. using parametric driving proposed recently in Ref. [32]. To reveal the quantum nature of the soliton its Heisenberg uncertainty \( \Delta X = \sqrt{\hbar / 8 m \omega N} \) in the position has to be larger than its width, \( \xi = 2h^2 / mgN \) [33]. Their ratio \( \Delta X / \xi = (\sqrt{\pi} / 4) \sqrt{\mu / h \omega K} \sim (mg^2 / h^3 \omega)^{1 / 3} N_{\text{tot}}^{1 / 6} \) is already 0.14 for experiment in Ref. [18], and can be increased, e.g. by reducing \( \omega \) and making interactions \( g \) stronger by a tighter transverse confinement [5].

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