Series Representation of Modified Bessel Functions and Its Application in AF Cooperative systems

Mehdi M. Molu
mehdi.molu@york.ac.uk

Abstract—Using fractional-calculus mathematics, a novel approach is introduced to rewrite modified Bessel functions in series form using simple elementary functions. Then, a statistical characterization of the total receive-SNR at the destination, corresponding to the source-relay-destination and the source-destination link SNR, is provided for a general relaying scenario, in which the destination exploits a maximum ratio combining (MRC) receiver. Using the novel statistical model for the total receive SNR at the destination, accurate and simple analytical expressions for the outage probability, the bit error probability, the bit error probability and the ergodic capacity are obtained.

I. INTRODUCTION

For an analytical investigation of AF cooperative communication systems with a maximum-ratio-combining (MRC) at the destination, a statistical model of the total receive signal-to-noise ratio, SNR\text{tot}, is required, which is equivalent to the sum of SNRs corresponding to the Source-Destination (S-D) and the Source-Relay-Destination (S-R-D) link SNRs. To the best of our knowledge, a theoretical statistical model of SNR\text{tot} is yet unknown (but will be presented in this paper); numerous studies, however, consider the problem of finding a statistical model of SNR\text{rd}. There are two main trends in the literature: either a single-branch system model is assumed, in which the receiver operates only on the relay transmission and, hence, no MRC is employed at the destination (e.g., see [1]), or upper and/or lower performance bounds are considered assuming an MRC receiver at the destination (e.g., see [1]).

To cope with the computations involving complicated mathematical functions, a possible practical solution is to use an equivalent series representation of the functions (e.g. [3]). In fact, the appearance of the modified Bessel function of the second kind, \( K_\nu(\cdot) \), in the PDF of SNR\text{rd} is the main source of trouble in AF-related calculations. We aim to substitute \( K_\nu(\cdot) \) with an equivalent series representation in this paper. However, as there is no appropriate and simple series representation of \( K_\nu(\cdot) \) available in the literature, motivated by [4], we derive a novel series representation of \( K_\nu(\cdot) \) in terms of simple elementary functions (such as \( x^\nu e^{-x} \)) using fractional calculus mathematics. With this result, the complex statistical model of SNR\text{rd} turns out to be simple and easily tractable. Therefore, the Cumulative Distribution Function (CDF) and the Probability Density Function (PDF) of SNR\text{rd} will be derived.

The remainder of the paper is organized as follows: in Section II, the system model is introduced. In Section III the fractional-calculus method is exploited to derive an equivalent series representation of \( K_\nu(x) \). Based on the results derived in Section III the PDF and CDF of SNR\text{tot} are derived in Section IV and in Section V novel closed-form expressions are provided for some performance measures of variable-gain AF cooperative systems. Conclusion remarks are provided in Section VI.

II. SYSTEM MODEL

We consider a two-hop variable-gain AF cooperative system. The source (S) sends data to the destination (D) by the help of an intermediate relay node (R). The destination “hears” both the source and the relay transmissions and employs a maximum-ratio-combining (MRC) receiver to jointly exploit all information available at the destination. Also, motivated by practical hardware constraints, it is assumed that the relay operates in half-duplex mode.

The SNR at the output of the MRC receiver is \( \text{SNR}_{\text{tot}} = \gamma (|h_{\text{sd}}|^2 + |h_{\text{rd}}|^2) = \gamma |h_{\text{eq}}|^2 \) where \( \gamma \) is the transmit power at the source node. \(|h_{\text{sd}}|^2 \) and \(|h_{\text{rd}}|^2 | = \frac{|h_{\text{sd}}|^2 |h_{\text{rd}}|^2}{|h_{\text{sd}}|^2 + |h_{\text{rd}}|^2 + 1/\gamma} \) represent equivalent S-D and S-R-D channel powers, respectively. \(|h_{\text{sd}}|^2 \), \(|h_{\text{rd}}|^2 | \) and \(|h_{\text{eq}}|^2 | \) are distributed exponentially with parameter \( \lambda_{\text{sd}} \), \( \lambda_{\text{ar}} \) and \( \lambda_{\text{rd}} \), respectively. The CDF of \(|h_{\text{rd}}|^2 | \) is

\[
F_{|h_{\text{rd}}|^2}(x) = 1 - 2\zeta e^{-\lambda_{\text{rd}} x} K_1(2\zeta)
\]

where \( \zeta = \sqrt{\lambda_{\text{rd}} (x + 1/\gamma)} \) and \( K_\nu(\cdot) \) the modified Bessel function of the second kind and \( \nu \)-th order, \( \lambda_{\text{sd}} \equiv \lambda_{\text{sr}} \lambda_{\text{rd}} \) and \( \lambda_{\text{ar}} \equiv \lambda_{\text{sr}} + \lambda_{\text{rd}} \). A proof of (1) can be found in [5] and [6]. The PDF of \(|h_{\text{rd}}|^2 | \) is

\[
f_{|h_{\text{rd}}|^2}(x) = 2e^{-\lambda_{\text{rd}} x} \left( \lambda_{\text{rd}} (2x + \frac{1}{\gamma}) K_0(2\zeta) + \lambda_{\text{ar}} \zeta K_1(2\zeta) \right).
\]

The results in [2] and [1] do not, easily, lend themselves to further mathematical calculations (e.g. integration) as modified Bessel functions \( K_\nu(\cdot) \) appear. Hence, no statistical model for \( \text{SNR}_{\text{tot}} \) is available in the literature so far.

In the next section, an equivalent representation of \( K_\nu(\cdot) \) is derived that is based on a series-representation involving simple mathematical functions of the form \( x^\nu e^{-x} \). This novel equivalent representation of \( K_\nu(\cdot) \) is then used instead of the \( K_0(\cdot) \) and \( K_1(\cdot) \) functions appeared in [2] and [1]. It will be clear in the next sections that this approach paves the way for further theoretical analysis of AF relaying systems.

III. EQUIVALENT SERIES REPRESENTATION OF MODIFIED BESSEL FUNCTIONS OF SECOND KIND

The mathematical concept of integration and differentiation of arbitrary (non-integer) order is called “fractional calculus”;

\[
\int_0^x f(t) dt = \text{Fractional integral of order } \alpha \text{ of } f(t)
\]

and

\[
\frac{d^n}{dx^n} f(x) = \text{n-th order fractional derivative of } f(x)
\]
foundations of the theory are discussed, e.g., in [7]. It will be used below to derive a simple novel equivalent representation of $K_p(\beta x)$.

Theorem. Equivalent Representation of $K_p(\beta x)$

A modified Bessel function, $K_p(\beta x)$, of the second kind and $\nu$-th order, with $\nu > 0$, can be represented by the infinite series

$$K_p(\beta x) = e^{-\beta x} \sum_{n=0}^{\infty} \sum_{i=0}^{n} \Lambda(\nu, n, i) \cdot (\beta x)^{1-\nu},$$

with the coefficients

$$\Lambda(\nu, n, i) = \frac{(-1)^i \sqrt{\pi} \Gamma(2\nu) \Gamma(\frac{1}{2} + n - \nu)}{2^{\nu-i} \Gamma(\frac{1}{2} - \nu) \Gamma(\frac{1}{2} + n + \nu) n!}$$

that involve the Lah numbers (e.g., [8]),

$$L(n, i) = \left\{ \binom{n-1}{i-1} \frac{n!}{i!} \right\} \quad \text{for} \quad n, i > 0,$$

and the conventions $L(0, 0) = 1$; $L(n, 0) = 0$.

Proof: Let $s$ be a real non-negative number, i.e., $s > 0$ and $s \in \mathbb{C}$. Let $f(x)$ be continuous on $x \in [0, \infty)$ and integrable on any finite subinterval of $x \geq 0$. Then the Riemann-Liouville operator (e.g., [7]) of fractional integration is defined as

$$I^s \{ f(x) \} = \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} f(t) dt.$$  

(6)

On the other hand, from [9] 3.471.4 we have

$$\int_0^x (x-t)^{s-1} t^{-2} e^{-\beta/t} dt = \frac{\Gamma(s) \beta^{\frac{s}{2}-s}}{\sqrt{\pi} x^{s+1}} e^{-\frac{\beta}{2x}} K_{\frac{s}{2}}(\frac{\beta}{2x}).$$

(7)

Assuming $f(t) = t^{-2} e^{-\beta/t}$, the two integrals in [6] and [7] are identical: this motivates the novel approach to derive an equivalent expression for $K_p(\beta x)$ by use of fractional integration.

It follows from [6] and [7] that

$$I^s \{ x^{-2s} e^{-\beta/x} \} = \frac{\beta^{\frac{s}{2}-s}}{\sqrt{\pi} x^{s+1}} e^{-\frac{\beta}{2x}} K_{\frac{s}{2}}(\frac{\beta}{2x}).$$

(8)

The Leibniz rule for the Riemann-Liouville operator is

$$I^s \{ h(x)g(x) \} = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n + s)}{n! \Gamma(s)} I^{s+n} \{ h \} D^n \{ g \}$$  

(9)

where $n$ is a non-negative integer, $s + n$ is a non-negative fractional number and $D^n = (\frac{d}{dx})^n$. By solving $I^{s+n} \{ h(x) \}$ for $h(x) = x^{-2s}$ and $D^n \{ g(x) \}$ for $g(x) = e^{-\beta/x}$, the equivalent Bessel model ([3]) will be derived.

Let $h(x) = x^p$, then according to [6]

$$I^n \{ x^p \} = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^p dt, \quad (\alpha > 0)$$

(10)

and

$$I^{s+n} \{ x^{-2s} \} = \frac{\Gamma(1-2s)}{\Gamma(1-s+n)} x^{n-s}.$$ 

(12)

With $g(x) = e^{-\beta/x}$ in [6], $D^n \{ e^{-\beta/x} \}$ can be found to equal

$$D^n \{ e^{-\beta/x} \} = e^{-\beta/x} \sum_{i=0}^{\infty} (-1)^i L_n(i)(\beta/x)^i,$$ 

(13)

with $L(n, i)$ defined in [5]; this result is taken from [10]. By substituting (12) and (13) into (8) and (9) it is straightforward to obtain

$$\frac{\beta^{\frac{s}{2}-s}}{\sqrt{\pi} x^{s+1}} e^{-\frac{\beta}{2x}} K_{\frac{s}{2}}(\frac{\beta}{2x}) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n + s)}{n! \Gamma(s)} \Gamma(1-s+n) x^{-s-n} \frac{\beta}{2x}^n L(n, i)(\beta/x)^i.$$  

(14)

Changing the variable $x \rightarrow \frac{1}{x}$, assuming $1 - 2s = 2\nu$ and exploiting $K_{-\nu} = K_{\nu}$, the result is the infinite series in (3).

It should be made clear that the above representation of $K_p(\beta x)$ is not valid for $\nu = \{0, \frac{1}{2}, \frac{3}{2}, \ldots \}$. That is because $\Gamma(2\nu)$ and $\Gamma(\frac{1}{2} + n - \nu)$ in [7] diverge to $\pm \infty$. However, for the case of $\nu = 0$, one can compute $K_0(\beta x)$ using the equivalent representation of $K_1(\beta x)$ and $K_2(\beta x)$ by $K_p(\beta x) = K_{\nu-2}(\beta x) + 2(\nu-1) K_{\nu-1}(\beta x)$ that is obtained from [11] 10.38.4.

Finite Series Representation of $K_p(\beta x)$

The equivalent representation of $K_p(\beta x)$ may significantly simplify computations involving $K_p(\beta x)$, as the series in [3] contains the variable $x$ only in the simple function-template $x^{1-\nu} e^{-\beta/x}$ that can, e.g., be easily integrated. The series representation contains, however, an infinite number of terms that can’t be computed in practical applications.

Fortunately, the series representation of $K_p(\beta x)$ is rather accurate for a finite number of terms as defined as follows:

$$K_p(\beta x) = e^{-\beta x} \sum_{n=0}^{k} \sum_{i=0}^{n} \Lambda(\nu, n, i) \cdot (\beta x)^{1-\nu} + \epsilon$$

(15)
with
\[ \epsilon = e^{-\beta x} \sum_{n=k+1}^{\infty} \sum_{i=0}^{n} A(\nu, n, i) \cdot (\beta x)^{n-i}. \] (16)

The first term on the right-hand side of (15) represents the actual function to approximate \( K_{\nu}(\beta x) \), and \( \epsilon \) represents the truncation error that can be neglected. Fig. 1 illustrates numerical values of the finite series representation of \( K_{\nu}(\beta x) \) (with \( k = 2 \) in (15)) for various values of \( \beta \) (dashed lines) and also the theoretically fully accurate values of \( K_{\nu}(\beta x) \) (solid lines). It is clear from the figure that the finite series for \( K_{\nu}(\beta x) \) with only \( k = 2 \) merges with theoretical \( K_{\nu}(\beta x) \) with high accuracy.

**Convenient Representation for Practical Use of the Truncated Series:** For practical use, it is convenient to re-write (15) such that one of the sum-operators is included in the series coefficients. For this, the inner sum over \( i \) in (13) is evaluated row-wise, with the sum-index \( n \) counting the rows. This structure is then summed up column-wise and the result can be written as
\[ K_{\nu}(\beta x) = \frac{e^{-\beta x}}{(\beta x)^{\nu}} \sum_{q=0}^{k} \left( \sum_{l=q}^{k} A(\nu, l, q) \right) \cdot (\beta x)^{q} + \epsilon \] (17)
\[ \approx a_{\nu, k,q}. \]

As long as \( k \) is limited, the above series representation (17) can always be used to replace (15) without any convergence problems but still the truncation error \( \epsilon \) can be made arbitrarily small. Numerical values for the coefficients \( a_{\nu, k,q} \) are given in Table I for the first-order (\( \nu = 1 \)) modified Bessel function of the second kind \( K_{1}(\cdot) \); it should be noted that the coefficient with \( q \)-index 1 is always found to equal \( a_{1, k, 1} \approx \frac{2k}{k+1} \).

**IV. DISTRIBUTION OF EQUIVALENT CHANNEL POWER**

Assuming an MRC receiver at the destination, the total receive SNR is the sum of SNRs corresponding to the S-D and the S-R-D links, i.e., \( \text{SNR}_{\text{tot}} = \text{SNR}_{\text{sd}} + \text{SNR}_{\text{sr}} \). The equivalent channel power-gain can be written as \( |h_{\text{eq}}|^2 = |h_{\text{sd}}|^2 + |h_{\text{sr}}|^2 \), and so, for the CDF of \( |h_{\text{eq}}|^2 \) we obtain
\[ F_{|h_{\text{eq}}|^2}(x) = \mathbb{P}(|h_{\text{sd}}|^2 + |h_{\text{sr}}|^2 \leq x) \]
\[ = \lambda_{\text{sd}} e^{-\lambda_{\text{sd}}} \int_{0}^{x} e^{\lambda_{\text{sd}} u} \cdot F_{|h_{\text{sd}}|^2}(u)du, \] (18)

with \( F_{|h_{\text{sd}}|^2}(\cdot) \) derived in [1]. For simplicity we will restrict calculations below to the high “transmit-SNR” regime but it will be demonstrated, by simulations (e.g., Fig. 5) that this is justified because it leads to very accurate numerical results, even in the low-SNR region.

By plugging (11) into (18), and assuming “high SNR”, (18) simplifies to
\[ F_{|h_{\text{eq}}|^2}(r) = 1 - e^{-\lambda_{\text{eq}}} - 2 \lambda_{\text{eq}} \sqrt{\lambda_{\text{sr}} \lambda_{\text{rd}}} e^{-\lambda_{\text{eq}}} \]
\[ \times \int_{0}^{r} x e^{-(\lambda_{\text{sd}} + \lambda_{\text{sr}} + \lambda_{\text{rd}}) x} K_{1}(2 \sqrt{\lambda_{\text{sr}} \lambda_{\text{rd}}} x) dx. \] (19)

The integral in (19) is non-trivial and does not seem to have a closed-form solution. However, using the results from Section III the integral can be rewritten as follows
\[ \eta \approx \int_{0}^{r} 2\sqrt{\lambda_{\text{sd}} \lambda_{\text{rd}}} e^{-(\lambda_{\text{sd}} + \lambda_{\text{sr}} + \lambda_{\text{rd}}) x} K_{1}(2 \sqrt{\lambda_{\text{sd}} \lambda_{\text{rd}}} x) dx \]
\[ \approx \sum_{q=0}^{k} \left( \frac{2 \sqrt{\lambda_{\text{sd}} \lambda_{\text{rd}}}}{(\lambda_{\text{sr}} - \lambda_{\text{rd}})^{q+1}} \right) \]
\[ \times \left( 1 - \sum_{c=0}^{q} \frac{(\lambda_{\text{sr}} - \lambda_{\text{rd}})^{c}}{c!} x^{c} e^{-(\lambda_{\text{sd}} + \lambda_{\text{sr}} + \lambda_{\text{rd}}) x} \right) \] (20)

where \( \lambda_{\text{sr}} = (\sqrt{\lambda_{\text{sr}}} + \sqrt{\lambda_{\text{rd}}})^2 \). The second step is obtained by using the series representation of \( K_{1}(2 \sqrt{\lambda_{\text{sd}} \lambda_{\text{rd}}} x) \) derived in (17). As the series is truncated (for \( k \) limited), this is an approximation, indicated by the use of “\( \approx \)” instead of strict equality; the truncation error can, however, be made arbitrarily small by choosing proper \( k \). By substituting (20) into (19) it is straightforward to obtain \( F_{|h_{\text{eq}}|^2}(x) \) as
\[ F_{|h_{\text{eq}}|^2}(x) \approx \text{A} e^{-\lambda_{\text{eq}}} + \sum_{q=0}^{k} \text{B} e^{-\lambda_{\text{eq}}} \] (21)

where coefficients \( \text{A} \) and \( \text{B} \) are independent of \( x \), defined as
\[ \text{A} = 1 + \sum_{q=0}^{k} \frac{\lambda_{\text{sd}} (2 \sqrt{\lambda_{\text{sr}} \lambda_{\text{rd}}})^{q+1} (\lambda_{\text{sr}} - \lambda_{\text{rd}})^{q+1}}{(\lambda_{\text{sd}} - \lambda_{\text{rd}})^{q+1}}, \] (22)
\[ \text{B} = \frac{\lambda_{\text{sd}} (2 \sqrt{\lambda_{\text{sr}} \lambda_{\text{rd}}})^{q+1}}{\lambda_{\text{sd}} - \lambda_{\text{rd}}} (\lambda_{\text{sr}} - \lambda_{\text{rd}})^{q+1}. \] (23)

The PDF of \( |h_{\text{eq}}|^2 \) is the derivative of \( F_{|h_{\text{eq}}|^2}(x) \) in (21) w.r.t \( x \), which is easy to calculate as the series representation involves simple elementary functions only:
\[ f_{|h_{\text{eq}}|^2}(x) \approx \text{A} \lambda_{\text{sd}} e^{-\lambda_{\text{eq}}} + \sum_{q=0}^{k} \sum_{c=0}^{q} \text{B} e^{-\lambda_{\text{eq}} - \lambda_{\text{sr}} x^c} x^c e^{-\lambda_{\text{eq}} x} \]. (24)

Fig. 2 illustrates the accuracy of \( f_{|h_{\text{eq}}|^2}(x) \) using the expression derived in (24) (solid line) by a comparison with histogram-results obtained from Monte Carlo simulations.
In the literature (e.g. [12]), when considering relaying systems with MRC receiver at the destination, a method is proposed for estimating the statistics of $SNR_{ad}$ based on the bounding technique $SNR_{ad} = \min(SNR_{x}, SNR_{ad})$. Consequently, $SNR_{ad}$ has as an exponential distribution with parameter $\lambda_{x}+\lambda_{ad}$. Note that, although this method greatly simplifies the calculations by avoiding modified Bessel functions in the formulations, accuracy is sacrificed; Fig. 2 (dashed line) shows the result for $f_{|h_{eq}|^2}(x)$ when using the bounding method as well. It is clear that the approach presented in this paper results in more accurate solutions in comparison with traditional bounding techniques.

V. PERFORMANCE ANALYSIS

The CDF in (1) corresponds to the outage probability. Due to lack of space no illustration is provided for outage probability but with the accuracy of the PDF in Fig. 2 the accuracy of outage probability is ensured as well.

Given the PDF of $|h_{eq}|^2$ in [24] bit error probability (BEP) can be obtained as $p_b = \frac{1}{2} \int_{0}^{\infty} \text{erfc}(\sqrt{\gamma x}) \int_{h_{eq}} f_{|h_{eq}|^2}(x) dx$. From [13, 7.1.19], $\frac{\pi}{2} \text{erfc}(\sqrt{\gamma x}) = -\sqrt{\frac{\gamma}{\pi x}} e^{-\gamma x}$. Then, using integration by parts, $p_b$ can be written according to

$$p_b = \frac{1}{2} \left( 1 - \sqrt{\frac{\gamma A^2}{\gamma + \lambda_{ad}}} + \sqrt{\frac{\gamma}{\pi}} \sum_{q=0}^{k} \sum_{c=0}^{q} \frac{B\Gamma(c + \frac{1}{2})}{(\gamma + \lambda_{ad})^{c+\frac{1}{2}}} \right). \quad (25)$$

Ergodic capacity of an AF cooperative system can be obtained using the definition of capacity as

$$C_{av} = \frac{1}{2} \int_{0}^{\infty} \log(1 + \gamma x) f_{h_{eq}}(x) dx,$$

can be solved in closed-form by substituting $f_{|h_{eq}|^2}(x)$ from [24] and exploiting [9, 4.228.8]. Due to space limit, the closed form expression is not provided in this letter; however, Fig 3 illustrates the Ergodic capacity obtained from the proposed approach. Also, the proposed approach has been examined for multi relay scenarios as well. The excellent agreement between Monte-Carlo Simulations and closed-form solution proves the correctness of the results.

VI. CONCLUSIONS

Based on fractional calculus mathematics, a novel power series representation of $K_{\nu}(\cdot)$ is introduced that has simple elementary functions of the form $x^{n}e^{-\gamma x}$ as its basic components. Based on that the performance of AF cooperative systems is investigated and accurate results are obtained.

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