GLOBAL WELLPOSEDNESS FOR THE ENERGY-CRITICAL ZAKHAROV SYSTEM BELOW THE GROUND STATE

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ABSTRACT. The Cauchy problem for the Zakharov system in the energy-critical dimension $d = 4$ is considered. We prove that global well-posedness holds in the full (non-radial) energy space for any initial data with energy and wave mass below the ground state threshold. The result is based on a Strichartz estimate for the Schrödinger equation with a potential. More precisely, a Strichartz estimate is proved to hold uniformly for any potential solving the free wave equation with mass below the ground state constraint. The key new ingredient is a bilinear (adjoint) Fourier restriction estimate for solutions of the inhomogeneous Schrödinger equation with forcing in dual endpoint Strichartz spaces.

1. Introduction

In 1972 Zakharov introduced a set of equations modeling the dynamics of Langmuir waves in plasma physics [25] [21] [8] [23]. These are rapid oscillations of an electric field in a conducting plasma. A simplified scalar version of this model, known as the Zakharov system, is given by the coupled system of Schrödinger-wave equations

$$i\partial_t u + \Delta u = vu,$$

$$\frac{1}{\alpha} \partial_t^2 v - \Delta v = \Delta |u|^2,$$  \hfill (1.1)

where $u(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is the complex envelope of the electric field, $v(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is the ion density fluctuation, and the ion sound speed $\alpha > 0$ is a fixed constant. The well-posedness of the Zakharov system has attracted considerable attention in the mathematical literature over the past 30 years, see for instance [20] [1] [19] [14] [17] [10] and the reference therein, in part due to its close relationship with the cubic nonlinear Schrödinger equation which formally appears by taking the limit $\alpha \to \infty$.

For the purposes of this paper, it is slightly more convenient to replace (1.1) with the corresponding first order system. More precisely, writing $V := v - i|\nabla|^{-1} \partial_t v$, we see that (1.1) (with $\alpha = 1$) is equivalent to

$$i\partial_t u + \Delta u = \Re(V)u,$$

$$i\partial_t V + |\nabla|V = -|\nabla||u|^2$$  \hfill (1.2)

and our goal is to study the Cauchy problem with initial data

$$u(0) = f \in H^s(\mathbb{R}^d; \mathbb{C}), \quad V(0) = g \in H^\ell(\mathbb{R}^d; \mathbb{R}).$$  \hfill (1.3)

The Zakharov system (1.2) is a Hamiltonian system, and in particular, sufficiently regular solutions preserve the (Schrödinger) mass and energy

$$M(u) := \int_{\mathbb{R}^d} |u|^2 dx, \quad E_Z(u, V) := \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} |V|^2 + \frac{1}{2} \Re(V)|u|^2 dx.$$  \hfill (1.4)

Unfortunately, the energy is sign indefinite, and thus in general does not give any a priori control over the $H^1$ norm of $u$. On the other hand, in dimension $d = 4$, the indefinite term in the energy can be controlled by the critical Sobolev embedding

$$\left| \int_{\mathbb{R}^4} \Re(V)|u|^2 dx \right| \leq \|V(t)\|_{L^2(\mathbb{R}^4)} \|u(t)\|_{L^4(\mathbb{R}^4)}^2 \leq C\|V(t)\|_{L^2(\mathbb{R}^4)} \|u(t)\|_{L^2(\mathbb{R}^4)}^2,$$
see (1.8) below for the sharp constant $C$. In particular, when $d = 4$, the kinetic and potential energy have the same scaling, and thus the Zakharov system is energy critical in dimension $d = 4$ with energy space corresponding to the regularity $(s, \ell) = (1, 0)$.

The energy critical Zakharov system is closely connected to the focusing cubic nonlinear Schrödinger equation (NLS)

$$i\partial_t u + \Delta u = -|u|^2 u,$$

(1.5)

which is also energy-critical in dimension $d = 4$ in the sense of invariant scaling. For instance, the cubic NLS (1.5) arises as the subsonic limit $\alpha \to \infty$ of the Zakharov system (1.1) as rigorously proved in [20, 19, 14, 17]. Furthermore, the ground state for the NLS plays a key role in the global dynamics of the Zakharov system. More precisely, recall the Aubin-Talenti function

$$W(x) = \left(1 + |x|^2/8\right)^{-1},$$

(1.6)

and let $W_\lambda := \lambda W(\lambda x)$ for $\lambda > 0$. The function $W$ is the (non-square integrable) ground state of the focusing cubic NLS (1.5) on $\mathbb{R}^4$, and in particular satisfies

$$-\Delta W_\lambda = W_\lambda^3 \quad \text{in } \mathbb{R}^4.$$  

(1.7)

Thus, taking $u = W_\lambda$ gives a family of static solutions to the cubic NLS (1.5). It is also an extremiser of the energy-critical Sobolev inequality: For any $\varphi \in H^1(\mathbb{R}^4)$, we have

$$\|\varphi\|_{L^4(\mathbb{R}^4)} \leq \frac{\|W\|_{L^4(\mathbb{R}^4)}^4}{\|
abla W\|_{L^2(\mathbb{R}^4)}^2} \|
abla \varphi\|_{L^4(\mathbb{R}^4)}.$$  

(1.8)

In seminal work [13] Kenig and Merle proved that the ground state $W$ defines the sharp threshold for global well-posedness, scattering and blow-up in the case of the focusing cubic NLS on $\mathbb{R}^4$ in the radial setting. This has recently been extended by Dodson [9] to the general case.

Returning to the Zakharov system, we observe that the family of static solutions $W_\lambda$ to the NLS, also gives rise to a family of static solutions for the Zakharov system (1.2) on $\mathbb{R}^4$ by simply taking $(u, V) = (W_\lambda, -W_\lambda^2)$. In particular this shows that there certainly exists non-dispersing solutions to (1.2). Moreover, we have

$$E_Z(W, -W^2) = E_S(W) = \frac{1}{4} \|W^2\|_{L^2}^2,$$

where $E_S(u) := \frac{1}{2} \int_{\mathbb{R}^d} \left(\frac{1}{4} |\nabla u|^2 - \frac{1}{4} |u|^4\right) dx$

is the energy of NLS. We refer to the pairs $(W_\lambda, -W_\lambda^2)$ as the ground states of the Zakharov system, as similar to the focusing cubic NLS, they play a crucial role in the global dynamics of (1.1). Indeed, recently [10] proved the following, which is analogous to [13]: Under the energy constraint

$$E_Z(u, V) < E_Z(W, -W^2),$$

(1.9)

the energy space $(u, V) \in H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ is topologically split into the two domains

$$\{\|V\|_{L^2(\mathbb{R}^4)} < \|W^2\|_{L^2(\mathbb{R}^4)}\} \text{ and } \{\|V\|_{L^2(\mathbb{R}^4)} > \|W^2\|_{L^2(\mathbb{R}^4)}\},$$

although the wave $L^2$ norm is not conserved by the Zakharov flow. Further more, all radial solutions in the former domain are global and scattering, while those in the latter can not be global and bounded in the energy space. We refer the reader to the introduction of [10] for more details on the connection between the NLS and the Zakharov equation (1.2).

Regarding the small data theory, in our recent paper [7] we identified the optimal regularity range for $(s, \ell)$ such that the Cauchy problem for the Zakharov system is well-posed for $d \geq 4$. More precisely, for any $(s, \ell) \in \mathbb{R}^2$ satisfying

$$\ell \geq \frac{d}{2} - 2, \quad \max\left\{\ell - \frac{d}{2} + \frac{d - 2}{4}\right\} \leq s \leq \ell + 2, \quad (s, \ell) \neq \left(\frac{d}{2}, \frac{d}{2} - 2\right), \left(\frac{d}{2}, \frac{d}{2} + 1\right)$$

(1.10)

the Cauchy problem (1.1)–(1.3) for $d \geq 4$ is well-posed. In addition, we proved global well-posedness and scattering provided that the Schrödinger initial datum $\|f\|_{H^{s, \ell}(\mathbb{R}^d)}$ is sufficiently small. Notice that in dimension $d = 4$, the energy space regularity $(s, \ell) = (1, 0)$ is admissible and lies on the boundary of this region. This reflects the fact that the Zakharov system is energy critical in $d = 4$. We refer the reader to the introduction of [7] for a thorough account on the well-posedness problem and references to earlier work.
The aim of this paper is to prove global well-posedness in the former domain for $s \geq 1$, which includes the energy space without the radial symmetry assumption. Our main result is the following

**Theorem 1.1** (GWP below the ground state). Let $d = 4$ and $(s, \ell) \in \mathbb{R}^2$ satisfy $1.10$ and $s \geq 1$. Then for any initial data $(u(0), V(0)) \in H^s(\mathbb{R}^4) \times H^\ell(\mathbb{R}^4)$ satisfying

$$E_Z(u(0), V(0)) < E_Z(W, -W^2), \quad \|V(0)\|_{L^2(\mathbb{R}^4)} \leq \|W^2\|_{L^2(\mathbb{R}^4)}, \quad (1.11)$$

the Zakharov system $1.2$ has a unique global solution $(u, V) \in C(\mathbb{R}; H^s(\mathbb{R}^4) \times H^\ell(\mathbb{R}^4))$ satisfying $u \in L^2_{t,loc} W^s_2$. Moreover, the flow map is locally Lipschitz continuous, the mass $M(u)$ and the energy $E_Z(u, V)$ are conserved, and we have the global bounds

$$\sup_{t \in \mathbb{R}} \|\nabla u(t)\|^2_{L^2} \leq \frac{2\|W^2\|^2_{L^2}}{E_Z(W, -W^2) - E_Z(u(0), V(0))} E_Z(u(0), V(0)),$$

$$\sup_{t \in \mathbb{R}} \|V(t)\|^2_{L^2} \leq 4E_Z(u(0), V(0)).$$

Although no blow-up is known for the Zakharov system in $\mathbb{R}^4$, in view of the existence of blow-up for NLS near the threshold, one may naturally expect that the above condition $\|V(0)\|_{L^2(\mathbb{R}^4)} \leq \|W^2\|_{L^2(\mathbb{R}^4)}$ is optimal (see also $[7, 10]$ for further discussion).

Theorem $1.1$ follows from the variational properties of the ground state $(W, -W^2)$, see $[10]$, and the following local well-posedness result with a lower bound on the existence time:

**Theorem 1.2.** Let $d = 4$, $0 < B < \|W^2\|_{L^2(\mathbb{R}^4)}$, $0 < M < \infty$, and $\frac{1}{2} < s < 2$. Then there exists $T = T(B, M, s) > 0$, such that for any initial data $(u(0), V(0)) \in H^s(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ satisfying

$$\|u(0)\|_{H^s} \leq M, \quad \|V(0)\|_{L^2} \leq B, \quad (1.12)$$

the Zakharov system $1.2$ has a unique solution $(u, V) \in C([0, T]; H^s(\mathbb{R}^4) \times L^2(\mathbb{R}^4))$ with the property that $u \in L^1_t([0, T]; W^s_2) \cap L^2_t([0, T]; W^s_2)$ and the flow map is locally Lipschitz continuous.

Although the endpoint $s = \frac{1}{2}$ is excluded from Theorem $1.2$, at a cost of complicating the statement, an endpoint version is possible. In particular, the condition on the Schrödinger data can be sharpened to instead requiring the weaker condition $\|e^{it\Delta} u(0)\|_{L^2_t W^s_2([0, T]; \mathbb{R}^4)} \ll 1$, see Theorem $7.1$.

Theorem $1.2$ implies that provided the wave $L^2$ norm is below the ground state, the time of existence in fact only depends on the size of the norm $\|V\|_{L^2 \times L^2}$. This is in a sharp constrast to NLS, which is scaling-critical in the energy space.

The main difficulty in the proof of Theorem $1.2$ comes from the interaction of the Schrödinger component with the large free wave component, which is the same as in the radial case $[10]$, and also the major difference from the NLS case $[13]$. In particular, the key step is to prove a uniform Strichartz estimate for the model problem

$$(i\partial_t + \Delta - \Re(V))u = F,$$

where the potential $V = e^{it|\nabla|} g \in L^\infty_t L^2_x$ is a free wave, see Theorem $6.1$. It is a refined extension from the radial version in $[10]$ Theorem $1.5$. By considering the supersonic limit $\alpha \to 0$ in $V = e^{i\alpha t|\nabla|} g$, it is also linked to the static potential case. See $[10]$ and the references therein for more background concerning the Strichartz estimate with potentials.

It is crucial for our purpose that the Strichartz estimate is uniform: the best constant is bounded in terms of $\|g\|_{L^2}$. Otherwise, the estimate depending on the profile $g$ follows from a perturbative argument and without any size constraint, using the dispersive property of the free wave $V$. To prove the uniformity, we follow the same strategy as in $[10]$: the concentration compactness argument, or the profile decomposition for the free wave potential $V$. The main difficulty comes from the concentration of $V$, which is described by scaling limit in the profile decomposition, and corresponding to the supersonic limit in the Schrödinger (parabolic) rescaling. The crucial observation is that the interaction with the Schrödinger component is essentially localized to the same frequency as the free wave (which is going to $\infty$ by concentration). In the radial case $[10]$, the frequency localization is achieved by using the normal form transformation for lower frequencies of the Schrödinger component, and the Strichartz estimate for radial solutions with derivative
gain for higher frequencies. The improved Strichartz is exploited also to treat remainder terms in the profile decomposition and in the supersonic approximation, which are decaying only in negative Sobolev (or Besov) norms. The main idea of the present paper is to replace the radial Strichartz estimate with bilinear Strichartz (or Fourier restriction) estimates, exploiting the transversal interaction property of free Schrödinger solutions in high frequency. Thus, our analysis relies on the non-resonance property of the Schrödinger-wave interactions. To be more precise, we prove new bilinear Fourier restriction estimates for solutions of inhomogeneous wave and Schrödinger equations with forcing terms in dual endpoint Strichartz norms. This is one of the main novelties of this paper, see Section 4 for more details.

Extending the remaining part of [10], namely the scattering and the weak blow-up, to the non-radial setting will require to solve another difficulty: a priori and global control of propagation. Similarly to NLS, translation invariance of the Hamiltonian implies conservation of the momentum, and the scaling covariance implies the virial identity, but the Schrödinger-wave interaction destroys the Galilei invariance (or the Lorentz translation invariance of the Hamiltonian implies conservation of the momentum, and the scaling covariance)

1.1. Organisation of the paper. In Section 2 we introduce notation, in particular Fourier multipliers and function spaces, and provide some preliminary estimates. In Section 3 we prove bilinear estimates for the right hand side of the Schrödinger equation. In Section 4 we discuss bilinear Fourier restriction estimates for products of solutions to inhomogeneous wave and Schrödinger equations. In Section 5 we apply this to prove refined bilinear estimates which involve a weaker Besov norm. Section 6 contains one of the main contributions of this paper, namely a uniform Strichartz estimate for the Schrödinger equation with a solution to the wave equation as a potential term satisfying the ground state constraint. Finally, in Section 7 we first prove (a refined version of) Theorem 1.2 and then derive Theorem 1.1.

2. Preliminaries

2.1. Fourier multipliers. In this subsection we fix notation, where we follow [7] Section 2 as far as possible. We require both homogeneous and inhomogeneous Littlewood-Paley decompositions. Let $\psi(r) \in C_0^\infty(\mathbb{R})$ such that $r^2\psi(r) \leq 0 \leq \psi(r) \leq 1$, $\psi(r) = 1$ on $|r| \leq 1$ and $\psi(r) = 0$ on $|r| \geq 2$. We set $\varphi(r) := \psi(r) - \psi(r/2)$ and $\varphi_\lambda(r) := \varphi(r/\lambda)$ for $\lambda > 0$. Then, we have

$$1 = \psi(r) + \sum_{1 \leq \lambda \in 2^\mathbb{N}} \varphi_\lambda(r) \text{ for all } r \in \mathbb{R}, \text{ and } 1 = \sum_{\lambda \in 2^\mathbb{N}} \varphi_\lambda \text{ for all } r \neq 0.$$ 

For $\lambda \in 2^\mathbb{N}$ with $\lambda > 1$, define the Fourier multipliers

$$P_\lambda = \varphi_\lambda(|\nabla|), \quad P^{(t)}_\lambda = \varphi_\lambda(|\partial_t|), \quad C_\lambda = \varphi_\lambda(i\partial_t + \Delta)$$

and for $\lambda = 1$ we take

$$P_1 = \psi(|\nabla|), \quad P^{(t)}_1 = \psi(|\partial_t|), \quad C_1 = \psi(i\partial_t + \Delta)$$

Thus $P_\lambda$ is a (inhomogeneous) Fourier multiplier localising the spatial Fourier support to the set $\{ \lambda/2 < |\xi| < 2\lambda \}$, $P^{(t)}_\lambda$ localises the temporal Fourier support to the set $\{ \lambda/2 \leq |\tau| \leq 2\lambda \}$, and $C_\lambda$ localises the space-time Fourier support to distances $\approx \lambda$ from the paraboloid (for $\lambda > 1$). We also require homogeneous Fourier decomposition. To this end, we define $\lambda \in 2^\mathbb{N}$ the (homogeneous) multipliers

$$\hat{P}_\lambda = \varphi_\lambda(|\nabla|).$$

To restrict the Fourier support to larger sets, we use the notation

$$P^{(t)}_{\leq \lambda} = \psi\left(\frac{|\nabla|}{\lambda}\right), \quad P^{(t)}_{\gtrless \lambda} = \psi\left(\frac{|\partial_t|}{\lambda}\right), \quad C_{\leq \lambda} = \psi\left(i\frac{\partial_t + \Delta}{\lambda}\right),$$

and define $P^{(t)}_{> \lambda} = 1 - P^{(t)}_{\leq \lambda}$ and $C_{> \lambda} = 1 - C_{\leq \lambda}$. For ease of notation, for $\lambda \in 2^\mathbb{N} \geq 1$ we often use the shorthand $P_\lambda f = f_\lambda$. In particular, note that $u_1 = P_1 u$ has Fourier support in $\{ |\xi| \leq 2 \}$, and we have the identities

$$f = \sum_{\lambda \in 2^\mathbb{N}} f_\lambda, \quad f = \sum_{\lambda \in 2^\mathbb{N}} \hat{P}_\lambda f, \quad \text{for any } f \in L^2(\mathbb{R}^d).$$

We reserve the notation $f_\lambda$ to denote the inhomogeneous Littlewood-Paley decomposition.
For brevity, let us denote the frequently used decompositions in modulation by

\[ P^N_\lambda u := C_{\leq (\frac{1}{\theta})}^2 P_\lambda u, \quad P^F_\lambda u := C_{> (\frac{1}{\theta})}^2 P_\lambda u, \]

and take

\[ P^N_\lambda u := \sum_{\lambda \in 2^{\mathbb{Z}}} P^N_\lambda u, \quad P^F_\lambda u := \sum_{\lambda \in 2^{\mathbb{Z}}} P^F_\lambda u. \tag{2.1} \]

Thus \( P^N \) localises to frequencies near to the paraboloid, \( P^F \) localises to frequencies far from the paraboloid, and we have the identities \( u_\lambda = P^N_\lambda u + P^F_\lambda u \) and \( u = P^N u + P^F u \). Note that the multipliers \( P^N \) and \( P^F \) have the parabolic scaling

\[ (P^N_\lambda u)(t/\lambda^2, x/\lambda) = P^N_2 (u(4t/\lambda^2, 2x/\lambda)), \tag{2.2} \]

where \( P^N_2 \) is a space-time convolution with a Schwartz function, so that we can easily deduce that \( P^N_\lambda \) and \( P^F_\lambda \) are bounded on any \( L^p_t L^q_x \) uniformly in \( \lambda \in 2^{\mathbb{Z}} \), and that \( P^N \) and \( P^F \) are bounded on any \( L^p_t B^s_{q,2} \).

2.2. Function spaces. In this subsection we introduce the function spaces which are used throughout this paper. Some are (special cases of) [7, Section 2], although in addition we require certain refinements in view of the bilinear restriction estimates in Section 4. We define the homogeneous Besov spaces \( \dot{B}^s_{q,x} \) using the norm

\[ \| f \|_{\dot{B}^s_{q,x}} = \left( \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{sq} \| \hat{P}_\lambda f \|_{L^q_x}^q \right)^{1/q}. \]

We use the notation \( 2^* = \frac{2d}{d-2} \) and \( 2^* = (2^*)' = \frac{2d}{d+2} \) to denote the endpoint Strichartz exponents for the Schrödinger equation. Thus, for \( d \geq 3 \) we have

\[ \| e^{it\Delta} f \|_{L^q_t L^r_x \cap L^\infty_t L^2_x} + \| \int_0^t e^{i(t-s)\Delta} F(s) ds \|_{L^q_t L^r_x \cap L^\infty_t L^2_x} \lesssim \| f \|_{L^q_x} + \| f \|_{L^q_t L^r_x}, \tag{2.3} \]

see [12]. To control frequency localised nonlinear terms on the righthand side of the Schrödinger equation, for \( \lambda \in 2^{\mathbb{N}} \) we define

\[ \| F \|_{N^\lambda} = \lambda^{\theta} \| C_{\leq (\frac{1}{\theta})}^2 F \|_{L^q_t L^2_x} + \lambda^{s-1} \| F \|_{L^q_t L^2_x}. \]

For later use, we observe that an application of Bernstein’s inequality gives

\[ \| F_\lambda \|_{N^\lambda} \lesssim \lambda^{s} \| F_\lambda \|_{L^q_t L^2_x}. \tag{2.4} \]

To estimate the frequency localised solution, we define

\[ \| u \|_{S^\lambda} = \lambda^{s} \| u \|_{L^\infty_t L^2_x} + \lambda^{s} \| u \|_{L^q_t L^r_x} + \lambda^{s-1} \| (i\partial_t + \Delta) u \|_{L^q_t L^r_x}. \]

We require the stronger norm

\[ \| u \|_{\underline{S}^\lambda} = \lambda^{s} \| u \|_{L^\infty_t L^2_x} + \| (i\partial_t + \Delta) u \|_{N^\lambda}, \]

in order to obtain the refined estimates involving Besov norms. A computation shows that the norms \( \| \cdot \|_{S^\lambda} \) and \( \| \cdot \|_{\underline{S}^\lambda} \) only differ in the low modulation regime. Moreover, a convexity argument gives control over the Schrödinger admissible Strichartz spaces \( L^p_t L^r_x \).

**Lemma 2.1.** Let \( q, r \geq 2, \frac{1}{q} + \frac{d}{2r} = \frac{1}{2}, \) and \( d \geq 3 \). Then for any \( \lambda \in 2^{\mathbb{N}} \) we have

\[ \lambda^s \| u_\lambda \|_{L^q_t L^r_x} \lesssim \| u_\lambda \|_{S^\lambda} \lesssim \| u_\lambda \|_{\underline{S}^\lambda} \]

and the characterisation

\[ \| u_\lambda \|_{S^\lambda} \approx \| u_\lambda \|_{\underline{S}^\lambda} + \| (i\partial_t + \Delta) P^N_\lambda u \|_{L^q_t L^r_x}. \tag{2.5} \]

**Proof.** To prove the first inequality, we observe that by convexity we have \( 0 \leq \theta \leq 1 \) such that

\[ \| u_\lambda \|_{L^q_t L^r_x} \lesssim \| u_\lambda \|_{L^q_t L^r_x}^\theta \| u_\lambda \|_{L^r_t L^2_x}^{1-\theta}, \]

and hence the bound follows from the definition of the norm \( \| \cdot \|_{S^\lambda} \). Thus it remains to prove \( \| \cdot \|_{\underline{S}^\lambda} \). But this follows by unpacking the definitions of the norms \( \| \cdot \|_{S^\lambda} \) and \( \| \cdot \|_{\underline{S}^\lambda} \), applying the Strichartz estimate \( (2.3) \), and observing that the Fourier localisation of \( P^F_\lambda u \) gives

\[ \| P^F_\lambda u \|_{L^2_t L^r_x} \lesssim \lambda \| P^F_\lambda u \|_{L^q_t L^r_x} = \lambda \| (i\partial_t + \Delta)^{-1} (i\partial_t + \Delta) P^F_\lambda u \|_{L^q_t L^r_x} \lesssim \lambda^{-1} \| (i\partial_t + \Delta) u_\lambda \|_{L^q_t L^r_x}. \]
We sum the dyadic terms in $\ell^2$, and control the full solution using the norms
\[
\|u\|_{S^\mu} = \left( \sum_{\lambda \in 2^N} \|u_\lambda\|_{S^\mu_\lambda}^2 \right)^{1 \over 2}, \quad \|u\|_{N^\mu} = \left( \sum_{\lambda \in 2^N} \|u_\lambda\|_{N^\mu_\lambda}^2 \right)^{1 \over 2}, \quad \|F\|_{N^\mu} = \left( \sum_{\lambda \in 2^N} \|F_\lambda\|_{N^\mu_\lambda}^2 \right)^{1 \over 2}
\]
where we recall that $u_\lambda = P_\lambda u$, $F_\lambda = P_\lambda F$ denotes the inhomogeneous Littlewood-Paley decomposition.

For technical reasons related to the duality and the energy inequality, we introduce the notation
\[
\|u\|_Z = \|u\|_{L^\infty_t L^2_x} + \|(i\partial_x + \Delta)u\|_{L^1_t L^2_x + L^2_t L^2_x}.
\]

To localise any of the normed spaces $F = S^*, S^*, \ldots$ defined above to time intervals $I \subset \mathbb{R}$, we define the usual temporal restriction norm as
\[
\|u\|_{F(I)} = \inf_{u' \in F \text{ and } u'|_I = u} \|u'\|_F,
\]
provided that such an extension $u' \in F$ exists.

2.3. The Duhamel formula. The free Schrödinger group is denoted by $\mathcal{U}_0(t) = e^{it\Delta}$. Fix any $t_0 \in \mathbb{R}$ and any interval $I \ni t_0$. Then, the Duhamel integral from $t_0$ is denoted by
\[
\mathcal{I}_0[F](t) = -i \int_{t_0}^t e^{i(t-s)\Delta} F(s)ds,
\]
which is the unique solution $u$ on $I$ of
\[
i \partial_t u + \Delta u = F, \quad u(t_0) = 0.
\]

Later we will use the more general notation with a time-dependent potential $V(t, x)$ on $I \times \mathbb{R}^d$, where $\mathcal{I}_V F$ denotes the unique solution $u$ of
\[
i \partial_t u + \Delta u - Vu = F, \quad u(t_0) = 0
\]
and $\mathcal{U}_V(t; s)$ to denote the homogeneous solution operator, thus for $t, s \in I$, $\mathcal{U}_V(t; s)f$ solves
\[
i \partial_t u + \Delta u - Vu = 0, \quad u(s) = f.
\]

See Section 5.1 for the precise definition and its usage.

The energy inequality we will use for the Schrödinger equation is the following:

**Lemma 2.2.** Let $t_0 \in \mathbb{R}$. For $u(t) = e^{i(t-t_0)\Delta} f + \mathcal{I}_0[F](t)$ and all $\lambda \in 2^N$ we have
\[
\|u_\lambda\|_{S^\mu_\lambda} \lesssim \lambda^s \|f\|_{L^2} + \|F_\lambda\|_{N^\mu_\lambda},
\]
and
\[
\|u\|_{L^\infty_t L^2_x} + \|u\|_{L^2_t L^2_x} \lesssim \|f\|_{L^2} + \sup_{\|w\|_{L^2} \leq 1} \left| \int_{\mathbb{R}^{1+d}} \overline{w} F dtdx \right|,
\]
and
\[
\sup_{\|w\|_{L^2} \leq 1} \left| \int_{\mathbb{R}^{1+d}} \overline{w} C_{\lambda^2} F_\lambda dtdx \right| \lesssim \lambda^{-1}\|F_\lambda\|_{L^2}.
\]

**Proof.** To prove the energy inequality (2.9), by the definition of the norm $\| \cdot \|_{S^\mu_\lambda}$ and the Strichartz estimate (2.10), it suffices to show that
\[
\|\mathcal{I}_0[P_\lambda^F F]\|_{L^\infty_t L^2_x} \lesssim \lambda^{-1}\|F_\lambda\|_{L^2_x}.
\]

Define $G(t) = e^{-it\Delta} F$ and write $P_\mu^F F(t) = e^{it\Delta} P_{(\mathcal{P})}^\mu G(t)$ where $P_{(\mathcal{P})}^\mu$ restricts the temporal Fourier support to the region $\mu/2 < \tau < 2\mu$. Then an application of Bernstein’s inequality gives
\[
\|\mathcal{I}_0[P_\lambda^F F]\|_{L^\infty_t L^2_x} \lesssim \|P_{(\mathcal{P})}^\mu \partial_t^{-1} G\|_{L^\infty_t L^2_x} \lesssim \lambda^{-1}\|G_\lambda\|_{L^2_{t,x}} \approx \lambda^{-1}\|F_\lambda\|_{L^2_{t,x}}
\]
as required.
To prove (2.10), we apply duality and obtain
\[ \|u\|_{L^r_tL^s_x} + \|u\|_{L^r_tL^s_x'} \lesssim \|f\|_{L^2} + \int_{\mathbb{R}^{1+d}} \mathcal{F}I_0 F dt dx + \int_{\mathbb{R}^{1+d}} \mathcal{F}I_0 F dt dx, \]
where \( \|v_1\|_{L^1_tL^2_x} \lesssim 1, \|v_2\|_{L^2_tL^2_x'} \lesssim 1. \) We compute
\[ \int_{\mathbb{R}^{1+d}} \mathcal{F}I_0 F dt dx = \int_{\mathbb{R}^{1+d}} w F dt dx, \]
for
\[ w_j(s) = \begin{cases} \int_s^{\infty} e^{i(s-t)} \Delta v_j(t) dt \quad (s \geq 0), \\ -\int_{-\infty}^s e^{i(s-t)} \Delta v_j(t) dt \quad (s < 0). \end{cases} \]
The function \( w = w_1 + w_2 \) satisfies \( \|w\|_Z \lesssim 1 \) and the proof of (2.10) is complete.

Finally, by Cauchy-Schwarz, it remains to prove that
\[ \lambda \|C_{\lambda^2} P_\lambda w\|_{L^2} \lesssim \|w\|_Z, \]
which follows from the following two inequalities: Firstly, using Bernstein’s inequality,
\[ \|C_{\lambda^2} P_\lambda w_1\|_{L^2} \lesssim \sum_{\mu \gg \lambda^2} \|C_\mu w_1\|_{L^2} \lesssim \sum_{\mu \gg \lambda^2} \mu^{\frac{1}{2}} \|P_\mu^{(t)}(e^{-it\Delta} w_1)\|_{L^1_tL^2_x} \lesssim \sum_{\mu \gg \lambda^2} \mu^{\frac{1}{2}} \|\partial_t P_\mu^{(t)}(e^{-it\Delta} w_1)\|_{L^1_tL^2_x} \lesssim \lambda^{-1} \|(i\partial_t + \Delta) w_1\|_{L^1_tL^2_x}. \]
Secondly, by a similar argument,
\[ \|C_{\lambda^2} P_\lambda w_2\|_{L^2} \lesssim \lambda^{-2} \|(i\partial_t + \Delta) w_2\|_{L^1_tL^2_x}, \]
which completes the proof of (2.11). \( \square \)

Clearly, in view of Lemma 2.24, the inequality (2.23) can also be used to bound the weaker norm \( S^\ast \) and in fact this suffices for the small data theory. However for the large data theory, we need a refinement to Besov norms in the resonant interactions (when \( u \) has Fourier support close to the parabola) and this requires the sharper bound (2.10).

2.4. An elementary product estimate.

**Lemma 2.3.** There exists \( C > 0 \) such that for any \( s, \epsilon \geq 0 \) and any sequence of functions \( (g^{(\mu)})_{\mu \in 2^\mathbb{N}} \) with \( \sup \hat{g}^{(\mu)} \subset \{|x| \approx \mu\} \) we have
\[ \left( \sum_{\lambda \in 2^\mathbb{N}} \lambda^{2(s-1)} \right) \left( \sum_{\mu \gg \lambda} \|P_\lambda (fg^{(\mu)})\|_{L^2_x}^2 \right)^{\frac{1}{2}} \leq C\|\langle \nabla \rangle^{-\epsilon} f\|_{L^2_x} \left( \sum_{\mu \in 2^\mathbb{N}} \mu^{2(s+\epsilon)} \|g^{(\mu)}\|_{L^2_x}^2 \right)^{\frac{1}{2}}. \]

**Proof.** An application of the continuous embedding \( F^0_{2,\infty} \subset H^{-1} \) implies that
\[ \left( \sum_{\lambda \in 2^\mathbb{N}} \lambda^{2(s-1)} \left\| \sum_{\mu \gg \lambda} P_\lambda(fg^{(\mu)}) \right\|_{L^2_x}^2 \right)^{\frac{1}{2}} \leq \sup_{\lambda \in 2^\mathbb{N}} \lambda^s \sum_{\mu \gg \lambda} \|P_\lambda(fg^{(\mu)})\|_{L^2_x}. \]
Let \( \rho(\lambda y)\lambda^d \) denote the kernel of \( P_\lambda. \) Then for any \( \lambda \in 2^\mathbb{N} \) we have the pointwise a.e. estimate
\[ \lambda^s \left( \sum_{\mu \gg \lambda} P_\lambda(fg^{(\mu)})(x) \right) \leq \lambda^s \int_{\mathbb{R}} |\rho(\lambda y)| \lambda^d \sum_{\mu \gg \lambda} |f_{\approx \mu}(x-y)||g^{(\mu)}(x-y)| dy \leq \int_{\mathbb{R}} |\rho(\lambda y)| \lambda^d \left( \sum_{\mu} \mu^{-2\epsilon} |f_{\mu}(x-y)|^2 \right)^{\frac{1}{2}} \left( \sum_{\mu} \mu^{2(s+\epsilon)} |g^{(\mu)}(x-y)|^2 \right)^{\frac{1}{2}} dy \approx M \left( \sum_{\mu} \mu^{-2\epsilon} |f_{\mu}|^2 \right)^{\frac{1}{2}} \left( \sum_{\mu} \mu^{2(s-\epsilon)} |g^{(\mu)}|^2 \right)^{\frac{1}{2}}(x). \]
where $M$ denotes the Hardy-Littlewood maximal function. Therefore, by the $L^2$-boundedness of the maximal function, the standard square function estimate, and Hölder’s inequality we obtain
\[
\left( \sum_{\lambda \in 2^{\mathbb{N}}} \lambda^{2(s-1)} \left\| \sum_{\mu > \lambda} P_{\lambda}(fg^{(\mu)}) \right\|_{L^2}^2 \right)^{\frac{1}{2}} \lesssim \left\| M \left( \sum_{\mu} \mu^{-2\epsilon} |f_{\mu}|^2 \right) \left( \sum_{\mu} \mu^{2(s+\epsilon)} |g^{(\mu)}|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \lesssim \left\| \left( \sum_{\mu} \mu^{-2\epsilon} |f_{\mu}|^2 \right) \left( \sum_{\mu} \mu^{2(s+\epsilon)} |g^{(\mu)}|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \\
\approx \| (\nabla)^{-\epsilon} f \|_{L^2} \left\| \left( \sum_{\mu \in 2^{\mathbb{N}}} \mu^{2(s+\epsilon)} |g^{(\mu)}|^2 \right)^{\frac{1}{2}} \right\|_{L^2}.
\]

3. Bilinear estimates

In this section we give the basic bilinear estimate that we require to control the Schrödinger nonlinearity. Although this bound is not strong enough to obtain the uniform Strichartz estimate that is needed in the proof of Theorem 1.2 it has the advantage that it can be used, together with the energy inequality Lemma 2.2, to easily upgrade a solution $u \in S^s$, to $u \in \dot{S}^s$. The stronger control provided by the space $\dot{S}^s$ is crucial in obtaining the Besov refinement that we require in later sections.

**Theorem 3.1.** Let $d \geq 4$, $0 \leq s < 1$, and $0 \leq \epsilon < 1 - s$. Then
\[
\| vu \|_{N^s} \lesssim \left( \| (\nabla)^{-\epsilon} v \|_{L^2} \| (\nabla)^{1-\epsilon} v \|_{L^2} \right) \| u \|_{\dot{S}^{s+\epsilon}}. \tag{3.1}
\]

Theorem 3.1 suffices to prove small data global well-posedness in $H^s$ and (at least) the case $\epsilon = 0$ can be seen as a special case of Theorem 3.1. On the other hand, to deal with general data below the ground state, we need a version of Theorem 3.1 with a refinement to Besov norms. This is a much more delicate problem, which requires the use of bilinear restriction estimates to exploit the transversality occurring in the (time) resonant interactions.

**Proof of Theorem 3.1.** Let $d \geq 4$, $0 \leq s < 1$, and $0 \leq \epsilon < 1 - s$. We start by proving that
\[
\| vu \|_{N^s} = \left( \sum_{\lambda_0 \in 2^{\mathbb{N}}} \lambda^{2s} \left\| P_{\lambda_0}(vu) \right\|_{N^s_{\lambda_0}} \right)^{\frac{1}{2}} \lesssim \| v \|_{L^2} \| u \|_{\dot{S}^{s+\epsilon}}. \tag{3.2}
\]

Decompose the product into the interactions
\[
P_{\lambda_0}(vu) = P_{\lambda_0}(vu_{\leq \lambda_0}) + P_{\lambda_0}(vu_{> \lambda_0}) + P_{\lambda_0}(vu_{\approx \lambda_0}). \tag{3.3}
\]

For the first term in (3.3), we have for any $\lambda_1 \ll \lambda_0$
\[
\| P_{\lambda_0}(vu_{\lambda_1}) \|_{N^s_{\lambda_0}} \lesssim \lambda_0 \| v_{\approx \lambda_0} u_{\lambda_1} \|_{L^2 L^2} \lesssim \left( \frac{\lambda_1}{\lambda_0} \right)^{\frac{d}{2} - 1 - s - \epsilon} \| v \|_{L^2} \| u_{\lambda_1} \|_{L^2}.
\]

and so summing up over $\lambda_0 \gg \lambda_1$ gives (3.2). For the second term, we observe that an application of Bernstein’s inequality and (2.4) gives for any $\lambda_0 \gg \lambda_1$
\[
\| P_{\lambda_0}(vu_{\lambda_1}) \|_{N^s_{\lambda_0}} \lesssim \lambda_0 \| P_{\lambda_0}(v_{\approx \lambda_1} u_{\lambda_1}) \|_{L^2 L^2} \lesssim \lambda_0^{s + \frac{d}{2} - 1} \| v_{\approx \lambda_1} \|_{L^2} \| u_{\lambda_1} \|_{L^2} \lesssim \left( \frac{\lambda_0}{\lambda_1} \right)^{s + \frac{d}{2} - 1} \| v \|_{L^2} \| u_{\lambda_1} \|_{S^{s+\epsilon}}.
\]

Clearly, provided that $s + \frac{d}{2} - 1 > 0$, this can be summed up over frequencies to give (3.2) for the second term in (3.3). To bound the third term in (3.3), we note that
\[
\| P_{\lambda_0}(vu_{\approx \lambda_0}) \|_{N^s_{\lambda_0}} \lesssim \lambda_0 \| v_{\approx \lambda_0} u_{\approx \lambda_0} \|_{L^2 L^2} \lesssim \| v_{\approx \lambda_0} \|_{L^2} \lambda_0^{s + \frac{d}{2} - 1} \| u_{\approx \lambda_0} \|_{L^2} \lesssim \| v \|_{L^2} \| u_{\approx \lambda_0} \|_{L^2} \lesssim \| v \|_{L^2} \left( \lambda_0 \right)^{s + \frac{d}{2} - 1} \| u_{\approx \lambda_0} \|_{L^2}. \tag{3.2}
\]

Again, this can be summed up to give (3.2). Therefore, for any fixed $0 < \delta \ll 1$, if $\sup \xi \in \{|\sigma| \geq \delta|\xi|^2 + 4\}$, the required bound (3.1) follows from the inequality
\[
\| P_{\lambda_0}^{(l)} \|_{L^2 L^2} \lesssim \| (i\partial_\xi \pm \nabla)v_{\lambda_0} \|_{L^2} \| u_{\lambda_0} \|_{L^2}.
\]
It only remains to prove that, provided \( v \in L^\infty_t L^\frac{4}{3}_x \) with \( \text{supp} \tilde{v} \subset \{|\tau| \leq 2\delta|\xi|^2 + 8\} \), we have
\[
\left( \sum_{\lambda_0 \in 2^0} \|P_{\lambda_0}(vv_\lambda)\|_{N^s_{\lambda_0}}^2 \right)^{\frac{1}{2}} \lesssim \|\langle \nabla \rangle^{-\epsilon} v\|_{L^\infty_t L^\frac{4}{3}_x} \|u\|_{S^{r+s}}. \tag{3.4}
\]
Again we decompose the product
\[
P_{\lambda_0}(uv) = P_{\lambda_0}(vu_{\leq \lambda_0}) + P_{\lambda_0}(vu_{\gg \lambda_0}) + P_{\lambda_0}(uv_{= \lambda_0}) =: I_1 + I_2 + I_3
\]
and consider each interaction separately.

**Contribution of \( I_1 \).** We in fact show that for any \( 2 \leq r \leq d \) and \( \lambda_0 \gg \lambda_1 \) we have the stronger estimate
\[
\|P_{\lambda_0}(vu_{\lambda_1})\|_{N^s_{\lambda_0}} \lesssim \left( \frac{\lambda_1}{\lambda_0} \right)^{\frac{2}{3}-s-\epsilon} \|v\|_{L^\infty_t B^{\frac{4}{3}-2-\epsilon}_{r,\infty}} \|u_{\lambda_1}\|_{S^{r+s}}. \tag{3.5}
\]
This implies (3.4) in the case \( \lambda_0 \gg \lambda_1 \) in view of our assumptions on \( s \) and \( \epsilon \). We start by dealing with the low modulation output. The Fourier support assumption on \( v \) implies that
\[
P_{\lambda_0}^N(vu_{\lambda_1}) = P_{\lambda_0}^N(v\approx_{\lambda_0} P_{\lambda_1}^{(1)} P_{\lambda_1}^F u).
\]
Hence for any \( 2 \leq r \leq d \), the disposability of the multiplier \( P_{\lambda_1}^F \), and Bernstein’s inequality, gives
\[
\lambda_0^s\|P_{\lambda_0}^N(vu_{\lambda_1})\|_{L^2_t L^2_x} \lesssim \lambda_0^{s-1}\|v\approx_{\lambda_0} \|L^\infty_t L^4_x\|P_{\lambda_0}^F_{\lambda_0} P_{\lambda_1}^F u\|_{L^2_t x}
\]
\[
\lesssim \lambda_0^{s-2} \lambda_1^{\frac{1}{2}} \|v\approx_{\lambda_0} \|L^\infty_t L^4_x\| (i\partial_t + \Delta) P_{\lambda_1}^F u\|_{L^2_t x},
\]
\[
\lesssim \left( \frac{\lambda_1}{\lambda_0} \right)^{s-1-\epsilon} \|v\|_{L^\infty_t B^{\frac{4}{3}-2-\epsilon}_{r,\infty}} \|u_{\lambda_1}\|_{S^{r+s}}
\]
thus (3.5) holds. To bound the high modulation output, we observe that
\[
\lambda_0^{s-1}\|P_{\lambda_0}^F(vu_{\lambda_1})\|_{L^2_t x} \lesssim \lambda_0^{s-1}\|v\approx_{\lambda_0} \|L^\infty_t L^4_x\| P_{\lambda_1}^F u\|_{L^2_t x}
\]
\[
\lesssim \left( \frac{\lambda_1}{\lambda_0} \right)^{\frac{2}{3}-s-\epsilon} \|v\|_{L^\infty_t B^{\frac{4}{3}-2-\epsilon}_{r,\infty}} \|u_{\lambda_1}\|_{S^{r+s}}
\]
and so (3.5) follows.

**Contribution of \( I_2 \).** We start by observing that for the part of \( u \) which is close to the paraboloid, the non-resonant identity
\[
P_{\lambda_0}(v P_{\gg \lambda_0}^N u) = P_{\lambda_0}^F (v P_{\gg \lambda_0}^N u)
\]
and Lemma 2.3 implies that for any \( s, \epsilon \geq 0 \) we have
\[
\left( \sum_{\lambda_0 \in 2^0} \lambda_0^{2s} \|P_{\lambda_0}(v P_{\gg \lambda_0}^N u)\|_{N^s_{\lambda_0}}^2 \right)^{\frac{1}{2}} \lesssim \left( \sum_{\lambda_0 \in 2^0} \lambda_0^{2s} \|P_{\lambda_0}(v P_{\gg \lambda_0}^N u)\|_{L^2_t x}^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \|\langle \nabla \rangle^{-\epsilon} v\|_{L^\infty_t L^\frac{4}{3}_x} \left( \sum_{\lambda_0 \in 2^0} \lambda_0^{2(s+\epsilon)} \|P_{\lambda_0}^N u\|_{L^2_t L^\frac{4}{3}_x}^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \|\langle \nabla \rangle^{-\epsilon} v\|_{L^\infty_t L^\frac{4}{3}_x} \|u\|_{S^{r+s}}.
\]
On the other hand, if \( u_{\lambda_1} \) is supported away from the paraboloid, then applying (2.4) we see that for any \( 2 \leq r \leq d \)
\[
\|P_{\lambda_0}(v P_{\lambda_1}^N u)\|_{N^s_{\lambda_0}} \lesssim \lambda_0^s \|P_{\lambda_0}(v\approx_{\lambda_1} P_{\lambda_1}^N u)\|_{L^2_t L^\infty_x}
\]
\[
\lesssim \lambda_0^{s+\frac{d}{2}-1} \|v\approx_{\lambda_1} \|L^\infty_t L^4_x\| P_{\lambda_1}^N u\|_{L^2_t x} \lesssim \left( \frac{\lambda_0}{\lambda_1} \right)^{\frac{d}{2}-1} \|v\|_{L^\infty_t B^{\frac{4}{3}-2-\epsilon}_{r,\infty}} \|u_{\lambda_1}\|_{S^{r+s}}. \tag{3.6}
\]
Since there is a low-high frequency gain, summing up over frequencies gives (3.3).

**Contribution of \( I_3 \).** We now prove, without the Fourier support assumption on \( v \), that for any \( \lambda_0 \approx \lambda_1 \) we have the resonant bound
\[
\lambda_0^s \|P_{\lambda_0}^N(v P_{\lambda_1}^N u)\|_{L^2_t L^2_x} \lesssim \|v\approx_{\lambda_1} \|L^\infty_t L^\frac{4}{3}_x\| u_{\lambda_1}\|_{S^{r+s}}. \tag{3.7}
\]
and the non-resonant estimates with a weaker norm of $v$
\[
\lambda_0^d \| P_{\lambda_0}^N (v P_{\lambda_0}^F u) \|_{L_t^1 L_x^2} \lambda_0^{-1} \| P_{\lambda_0} (v u_{\lambda_0}) \|_{L_t^1 L_x^2} \lesssim \lambda_1^{-1} \| v \|_{L_t^\infty L_x^4} \| u_{\lambda_0} \|_{S_{\lambda_0}^r}.
\] (3.8)
Applying the definition of the norm $N_{\lambda_0}$ together with Bernstein’s inequality, we see that (3.7) and (3.8) implies (3.4) when $\lambda_0 \approx \lambda_1$. To prove the resonant bound (3.7), we simply apply Hölder’s inequality
\[
\lambda_0^d \| P_{\lambda_0}^N (v P_{\lambda_0}^F u) \|_{L_t^1 L_x^2} \lesssim \lambda_0^d \| v \|_{L_t^\infty L_x^{4/3}} \| P_{\lambda_0}^F u \|_{L_t^1 L_x^4} \lesssim \| v \|_{L_t^\infty L_x^{4/3}} \| u_{\lambda_0} \|_{S_{\lambda_0}^r}.
\]
On the other hand, to prove (3.8), if the output has small modulation, then we again apply Hölder’s inequality and observe that
\[
\lambda_0^{-1} \| P_{\lambda_0}^F (v u_{\lambda_0}) \|_{L_t^1 L_x^2} \lesssim \lambda_0^{-1} \| v \|_{L_t^\infty L_x^4} \| u_{\lambda_0} \|_{L_t^1 L_x^4} \lesssim \lambda_0^{-1} \| v \|_{L_t^\infty L_x^4} \| u_{\lambda_0} \|_{S_{\lambda_0}^r}.
\]
If the output has large modulation, then we instead use
\[
\| P_{\lambda}^N u \|_{S_{\lambda}} \lesssim \lambda_0^d \| P_{\lambda_0}^F u \|_{L_t^1 L_x^2} \lesssim \| v \|_{L_t^\infty L_x^4} \| u_{\lambda_0} \|_{S_{\lambda_0}^r}.
\]
Therefore (3.7) and (3.8) follow.

4. Fourier restriction estimates

Our eventual goal is to improve the result of Theorem 3.1 and include a weaker Besov norm on the right hand side. A close inspection of the proof of Theorem 3.1 shows that we already have a Besov gain for all interactions except the resonant case of $I_2$. To obtain a suitable gain in this region, we require stronger estimates which exploit the fact that resonant interactions can only occur for transverse interactions. In particular, the new input is the use of bilinear restriction estimates to exploit the transversality between free waves and free Schrödinger solutions.

There are two bilinear estimates that we require. The first is an inhomogeneous version of the bilinear restriction estimate for wave-Schrödinger interactions. The key point is that we prove that the bilinear restriction type estimate holds not just for free solutions to the Schrödinger equation, but also inhomogeneous restriction estimate for wave-Schrödinger interactions. The key point is that we prove that the bilinear waves and free Schrödinger solutions.

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There are two bilinear estimates that we require. The first is an inhomogeneous version of the bilinear restriction estimate for wave-Schrödinger interactions. The key point is that we prove that the bilinear restriction type estimate holds not just for free solutions to the Schrödinger equation, but also inhomogeneous solutions satisfying $(i \partial_t + \Delta) u \in L_t^4 L_x^4$. This bilinear restriction estimate is used to obtain a high-low frequency gain which is required to place $v(t) \in B_{t,x}^{4,-2}$. Without a high-low gain we can only place $v(t) \in B_{t,x}^{4,-2}$, i.e. we would require a much stronger $\ell^4$ summation over the dyadic frequencies of $v$. Recall that we have defined
\[
\| u \|_Z = \| u \|_{L_t^\infty L_x^4} + \| (i \partial_t + \Delta) u \|_{L_t^1 L_x^2}.
\]
Clearly we have the inequalities
\[
\| u \|_Z \leq \| u \|_{L_t^\infty L_x^4} + \| (i \partial_t + \Delta) u \|_{L_t^1 L_x^2}
\]
and
\[
\| P_{\lambda}^N u \|_{S_{\lambda}} \lesssim \lambda_0^d \| P_{\lambda_0}^F u \|_Z \lesssim \| u_{\lambda_0} \|_{S_{\lambda_0}^r}.
\]

**Theorem 4.1** (Bilinear $L_t^2 L_x^2$ for inhomogeneous wave-Schrödinger interactions). Let $0 \leq \alpha < \frac{1}{2}$. For all $\lambda_0, \lambda_1 \in 2^\mathbb{N}$ and $\mu \in 2^\mathbb{N}$ with $\min\{\mu, \lambda_0\} \lesssim \lambda_1$, and any free wave $v = e^{k u} v_{\lambda}$ we have
\[
\| P_{\leq \lambda_0} (\hat{P}_\mu v u_{\lambda}) \|_{L_t^2 L_x^{(1+\epsilon)}(\mathbb{R}^{1+4})} \lesssim \left( \frac{\min\{\mu, \lambda_0\}}{\lambda_1} \right)^\alpha \min\{\mu, \lambda_0\} \| \hat{P}_\mu v \|_{L_t^\infty L_x^4} \| u_{\lambda} \|_Z.
\] (4.1)

The second bilinear estimate we exploit is an inhomogeneous version of a bilinear restriction estimate for the paraboloid. The key point is that bilinear restriction estimates give additional spatial integrability, which eventually means that we can replace the $L_t^\infty L_x^4$ norm of the free wave $v$, with the weaker Besov norm $L_t^\infty B_{4,\infty}^{-1}$. This is a crucial ingredient for bounding error terms which arise in the profile decomposition arguments in Subsection 6.2

**Theorem 4.2** (Bilinear restriction for inhomogeneous Schrödinger). Let $r > \frac{2}{3}$. For any $\mu \in 2^\mathbb{N}$ we have
\[
\| \hat{P}_\mu (w u) \|_{L_t^1 L_x^2(\mathbb{R}^{1+4})} \lesssim \mu^2 \frac{1}{r} \| w \|_Z \| u \|_Z.
\] (4.2)
The range \( r \geq \frac{5}{4} \) is sharp, in the sense that (4.2) fails for \( r < \frac{5}{4} \). Note that by taking \( w \) and \( u \) to be free solutions to the Schrödinger equation, (4.2) recovers the bilinear restriction estimates for the paraboloid in \( L_t^1 L_x^r \). Bilinear restriction estimates for the paraboloid were first obtained by Tao [22], this was then extended to the mixed norm case \( L_t^1 L_x^r \) with \( q > 1 \) by Lee-Vargas [16]. The case \( L_t^1 L_x^r \), which corresponds to the homogeneous version of (4.2), can be found in [3]. The key importance of Theorems 4.1 and 4.2 is that they hold for inhomogeneous solutions to the Schrödinger equation and wave equations, and thus are particularly well suited to iterative arguments arising in the study of nonlinear PDE.

In order to simplify the presentation, we do not attempt to state the bilinear estimates in Theorem 4.1 and Theorem 4.2 in the greatest possible generality. However, it is clear from the proof given below, that similar statements hold in general dimensions and for general frequency interactions (provided only that the corresponding estimate for free solutions holds).

The strategy to prove both Theorem 4.2 and Theorem 4.1 is similar. For instance, to prove Theorem 4.1, we start by observing that the estimate is true for free solutions. More precisely, we claim that if \( \lambda_0, \lambda_1 \in 2^N \) and \( \mu \in 2^N \) with \( \min \{ \mu, \lambda_0 \} \lesssim \lambda_1 \) then

\[
\| P_{\leq \lambda_0} (\hat{P}_\mu v e^{it\Delta} f_{\lambda_1}) \|_{L_{t,x}^2(\mathbb{R}^{1+4})} \lesssim \left( \frac{\min \{ \mu, \lambda_0 \}}{\lambda_1} \right)^{\frac{3}{2}} \min \{ \mu, \lambda_0 \} \| \hat{P}_\mu v \|_{L_t^\infty L_x^2} \| f_{\lambda_1} \|_{L_x^2},
\]

where we recall that \( v = e^{it|x|} \) is a solution to the free wave equation. If \( \lambda_1 \approx 1 \), then \( \mu \lesssim \lambda_0 \approx 1 \), and so (4.3) follows from Hölder’s inequality and the \( L_t^2 L_x^{\infty} \) Strichartz estimate for the free wave equation [12]

\[
\| P_{\leq \lambda_0} (\hat{P}_\mu v e^{it\Delta} f_{\lambda_1}) \|_{L_{t,x}^2(\mathbb{R}^{1+4})} \lesssim \| \hat{P}_\mu v \|_{L_t^2 L_x^\infty} \| f_{\lambda_1} \|_{L_x^2} \lesssim \mu^{\frac{3}{2}} \| \hat{P}_\mu g \|_{L_x^2} \| f_{\lambda_1} \|_{L_x^2}.
\]

On the other hand, if \( \lambda_1 \gg 1 \), then we decompose the Fourier support into cubes of size \( \lambda_0 \). More precisely, let \( \mathcal{Q}_{\lambda_0} \) denote a decomposition of \( \mathbb{R}^4 \) into cubes \( q \) of diameter \( \frac{\lambda_0}{100} \), and let \( P_q \) be the corresponding Fourier localisation operators such that

\[
f = \sum_{q \in \mathcal{Q}_\mu} P_q f, \quad \text{supp} \hat{P}_q f \subset q.
\]

Then decomposing \( f \) and \( g \) using the Fourier multipliers \( P_q \), and noting that the Fourier support of the output is constrained to frequencies \( \lesssim \lambda_0 \),

\[
\| P_{\leq \lambda_0} (\hat{P}_\mu v e^{it\Delta} f_{\lambda_1}) \|_{L_{t,x}^2} \lesssim \sum_{q, q \in \mathcal{Q}_{\lambda_0}, \text{dist}(q, q) \leq \lambda_0} \| \hat{P}_\mu P_q v e^{it\Delta} P_q f_{\lambda_1} \|_{L_{t,x}^2}
\]

\[
\lesssim \lambda_1^{\frac{3}{2}} \left( \min \{ \mu, \lambda_0 \} \right)^{\frac{3}{2}} \sum_{q, q \in \mathcal{Q}_{\lambda_0}, \text{dist}(q, q) \leq \lambda_0} \| \hat{P}_\mu P_q g \|_{L_x^2} \| P_q f_{\lambda_1} \|_{L_x^2}
\]

\[
\lesssim \left( \frac{\min \{ \mu, \lambda_0 \}}{\lambda_1} \right)^{\frac{3}{2}} \min \{ \mu, \lambda_0 \} \| \hat{P}_\mu v \|_{L_t^\infty L_x^2} \| f_{\lambda_1} \|_{L_x^2}
\]

where the \( L_{t,x}^2 \) bound follows from a short computation using Plancherel (see, for instance, [3] Lemma 2.6 or [4] Theorem 5.2).

In view of the estimate (4.3), a somewhat standard transference type argument implies that it suffices to prove that

\[
\| P_{\leq \lambda_0} (\hat{P}_\mu v e^{i(t-s)\Delta} u_{\lambda_1}) \|_{L_{t,x}^2} \lesssim \lambda_1^{\frac{3}{2} + \epsilon} \left( \min \{ \mu, \lambda_0 \} \right)^{\frac{3}{2} - \epsilon} \| \hat{P}_\mu v \|_{L_t^\infty L_x^2} \left( \| u_{\lambda_1} \|_{L_t^\infty L_x^2} + \| (i\partial_t + \Delta) u_{\lambda_1} \|_{L_t^2 L_x^{\frac{5}{2}}} \right).
\]

Again applying the estimate for free solutions, the Duhamel formula and the (dual) endpoint Strichartz estimate implies that the first estimate in Theorem 4.1 would then follow from the inhomogeneous estimate

\[
\| P_{\leq \lambda_0} \left[ \hat{P}_\mu v(t) \int_{-\infty}^t e^{i(t-s)\Delta} F_{\lambda_1}(s) ds \right] \|_{L_{t,x}^2} \lesssim \lambda_1^{\frac{3}{2} + \epsilon} \left( \min \{ \mu, \lambda_0 \} \right)^{\frac{3}{2} - \epsilon} \| v \|_{L_t^\infty L_x^2} \| F_{\lambda_1} \|_{L_t^2 L_x^{\frac{5}{2}}},
\]

(4.4)
To prove (4.4), we start by observing that for any intervals $I, J \subset \mathbb{R}$ the estimate for free solutions (4.3) together with the endpoint Strichartz estimate immediately implies that
\[
\left\| P_{\leq \lambda_0} \left[ \hat{P}_\mu v(t) \mathbb{1}_I(t) \int_I e^{i(t-s)\Delta} F_{\lambda_1}(s) ds \right] \right\|_{L^2_{t,x}} \lesssim \lambda_1^{-\frac{1}{2}} \left( \min \{ \mu, \lambda_0 \} \right)^{\frac{1}{2}} \left\| \hat{P}_\mu v \right\|_{L^\infty_t L^2_x} \left\| \int_I e^{-is\Delta} F_{\lambda_1}(s) ds \right\|_{L^2_t} \lesssim \lambda_1^{-\frac{1}{2}} \left( \min \{ \mu, \lambda_0 \} \right)^{\frac{1}{2}} \left\| \hat{P}_\mu v \right\|_{L^\infty_t L^2_x} \left\| F_{\lambda_1} \right\|_{L^2_t L^2_x}. \tag{4.5}
\]
If we instead put $F_{\lambda_1} \in L^a_t L^b_x$ for some non-endpoint Strichartz admissible pair $(a, b)$, then as $a > 2$, Theorem 4.4 would simply follow from an application of the Christ-Kiselev Lemma. This argument was used by Visan [24 Lemma 2.5] to prove a bilinear $L^2_t L^2_x$ estimate for the Schrödinger equation. In the endpoint case $a = 2$ the Christ-Kiselev Lemma does not apply, and we instead need combine the above argument with a Whitney decomposition and an estimate of Keel-Tao [12] used in the proof of the endpoint Strichartz estimate.

**Proof of Theorem 4.1.** For ease of notation, let us set $\sigma = \min \{ \mu, \lambda_0 \}$. A direct application of the estimate for free solutions, (4.3), implies that
\[
\left\| P_{\leq \lambda_0} \left[ \hat{P}_\mu v(t) \mathbb{1}_I(t) \int_{-\infty}^t e^{i(t-s)\Delta} F_{\lambda_1}(s) ds \right] \right\|_{L^2_{t,x}} \lesssim \left\| \hat{P}_\mu v \right\|_{L^\infty_t L^2_x} \left\| \mathbb{1}_I(t) \int_I e^{i(t-s)\Delta} F_{\lambda_1}(s) ds \right\|_{L^2_t L^2_x} \lesssim \sigma^{\frac{1}{2}} \left\| \hat{P}_\mu v \right\|_{L^\infty_t L^2_x} \left\| F_{\lambda_1} \right\|_{L^2_t L^2_x}. \tag{4.6}
\]
Arguing as above, another application of (4.3), together with the Duhamel formula shows that it suffices to prove (4.4). Let $\mathcal{D}_j$ denote a decomposition of $\mathbb{R}$ into left closed and right open intervals of length $2^j$. For intervals $I = [a_1, a_2)$ and $J = [b_1, b_2)$ we write $I \gg J$ if $a_1 \gg b_2$. An application of Bernstein’s inequality together with [12 Lemma 4.1] (and duality) implies that for all $r$ in a neighbourhood of 4 and any $I, J \in \mathcal{D}_j$ such that $\text{dist}(I, J) \approx 2^j$ we have
\[
\left\| P_{\leq \lambda_0} \left[ \hat{P}_\mu v(t) \mathbb{1}_I(t) \int_I e^{i(t-s)\Delta} F_{\lambda_1}(s) ds \right] \right\|_{L^2_{t,x}} \lesssim \sigma^{\frac{1}{2}} \left\| \hat{P}_\mu v \right\|_{L^\infty_t L^2_x} \left\| \int_I e^{i(t-s)\Delta} F_{\lambda_1}(s) ds \right\|_{L^2_t L^2_x} \lesssim \sigma^{\frac{1}{2}} 2^{-j(\frac{1}{2} - \frac{r}{4})} \left\| \hat{P}_\mu v \right\|_{L^\infty_t L^2_x} \left\| F_{\lambda_1} \right\|_{L^2_t L^2_x}.
\]
Hence, taking $\theta \in [0, 1]$ and interpolating with (4.3) shows that for any $I, J \in \mathcal{D}_j$ with $\text{dist}(I, J) \approx 2^j$ and any sufficiently small $\epsilon > 0$, by choosing $r$ close to 4 appropriately, we have
\[
\left\| P_{\leq \lambda_0} \left[ \hat{P}_\mu v(t) \mathbb{1}_I(t) \int_I e^{i(t-s)\Delta} F_{\lambda_1}(s) ds \right] \right\|_{L^2_{t,x}} \lesssim \left( \lambda_1^{-\frac{1}{2}} \sigma^{\frac{1}{2}} \right)^{1-\theta} \left( \sigma^{\frac{1}{2}} 2^{-2j(\frac{1}{2} - \frac{r}{4})} \right)^{\theta} \left\| \hat{P}_\mu v \right\|_{L^\infty_t L^2_x} \left\| F_{\lambda_1} \right\|_{L^2_t L^2_x} \lesssim \left( \frac{\sigma}{\lambda_1} \right)^{1-\theta} \sigma 2^{-\epsilon \theta |j-k|} \left\| \hat{P}_\mu v \right\|_{L^\infty_t L^2_x} \left\| F_{\lambda_1} \right\|_{L^2_t L^2_x}
\]
where we choose $k \in \mathbb{Z}$ such that $2^{-\frac{1}{2} k} \approx \sigma$. To conclude the proof of (4.4), we use the Whitney decomposition
\[
\mathbb{1}_{\{t,x\}}(t, s) = \sum_{j \in \mathbb{Z}, I, J \in \mathcal{D}_j, I \gg J, \text{dist}(I, J) \approx 2^j} \mathbb{1}_I(t) \mathbb{1}_J(s) \quad \text{for a.e. } (t, s) \in \mathbb{R}^2 \tag{4.7}
\]
and observe that for any $0 < \theta < 1$ we have
\[
\left\| P_{\leq \lambda_0} \left[ \hat{P}_\mu v(t) \int_{-\infty}^t e^{i(t-s)\Delta} F_{\lambda_1}(s) ds \right] \right\|_{L^2_{t,x}} \lesssim \sum_{j \in \mathbb{Z}} \left( \sum_{I, J \in \mathcal{D}_j, I \gg J, \text{dist}(I, J) \approx 2^j} \left\| P_{\leq \lambda_0} \left[ \hat{P}_\mu v(t) \mathbb{1}_I(t) \int_I e^{i(t-s)\Delta} F_{\lambda_1}(s) ds \right] \right\|_{L^2_{t,x}} \right)^2 \lesssim \left( \frac{\sigma}{\lambda_1} \right)^{1-\theta} \sigma \left\| \hat{P}_\mu v \right\|_{L^\infty_t L^2_x} \sum_{j \in \mathbb{Z}} 2^{-\epsilon \theta |j-k|} \left( \sum_{I, J \in \mathcal{D}_j, I \gg J, \text{dist}(I, J) \approx 2^j} \left\| \mathbb{1}_I F_{\lambda_1} \right\|_{L^2_{t,x}} \right)^2 \lesssim \left( \frac{\sigma}{\lambda_1} \right)^{1-\theta} \sigma \left\| \hat{P}_\mu v \right\|_{L^\infty_t L^2_x} \left\| F_{\lambda_1} \right\|_{L^2_t L^2_x}. \tag{4.8}
\]
The proof of the remaining bilinear estimate, Theorem 4.2, is more involved, as we are trying to put the product into \( L^4_x L^2_t \). In particular, unlike the proof of Theorem 4.1, we cannot gain an \( \ell^2 \) sum over the intervals \( I \in \mathcal{D}_J \) before using the corresponding bilinear restriction estimate for free solutions. Instead, the key new ingredient is an atomic bilinear restriction estimate from [4]. To state this result precisely, we need some additional notation. A function \( \phi \in L^\infty_x L^2_t \) is an atom if we can write \( \phi(t) = \sum_I 1_I(t) e^{it\Delta} f_I \), with the intervals \( I \subset \mathbb{R} \) forming a partition of \( \mathbb{R} \), and the \( f_I : \mathbb{R}^d \to \mathbb{C} \) satisfy
\[
\left( \sum_I \| f_I \|^2_{L^2_t} \right)^{\frac{1}{2}} \leq 1.
\]
We then take
\[
U^2_\Delta = \left\{ \sum_j c_j \phi_j \mid \phi_j \text{ an atom and } (c_j) \in \ell^1 \right\}
\]
with the induced norm
\[
\| u \|_{U^2_\Delta} = \inf_{u = \sum_j c_j \phi_j} \| c_j \|
\]
where the infimum is over all representations of \( u \) in terms of atoms. These spaces were introduced in unpublished work of Tataru, and studied in detail in [15]. The atomic bilinear restriction estimate we require is the following.

**Theorem 4.3.** Let \( r > \frac{5}{3} \). Then, for any \( \mu \in 2\mathbb{Z}^d \) and \( w, u \in U^2_\Delta \) we have
\[
\| \hat{P}_\mu(\hat{w}u) \|_{L^4_t L^2_x(\mathbb{R}^{d+1})} \leq \mu^{2-\frac{d}{4}} \| w \|_{U^2_\Delta} \| u \|_{U^2_\Delta}^{\frac{1}{2}}.
\]

**Proof.** This is an application of [4] Corollary 1.6] together with an additional orthogonality argument. An application of Bernstein’s inequality shows that it suffices to consider the case \( \frac{5}{3} < r < 2 \). Let \( Q_\mu \) denote a decomposition of \( \mathbb{R}^d \) into cubes \( q \) of diameter \( \frac{1}{100} \), and let \( P_q \) be the corresponding Fourier localisation operators such that
\[
f = \sum_{q \in Q_\mu} P_q f, \quad \text{supp } P_q f \subset q.
\]
A short computation using [4] Corollary 1.6] shows that for any \( q, \tilde{q} \in Q_\mu \) such that \( \text{dist}(q, \tilde{q}) \approx \mu \) we have
\[
\| \hat{P}_\mu(\hat{w}u) \|_{L^4_t L^2_x} \lesssim \sum_{q, \tilde{q} \in Q_\mu, \text{dist}(q, \tilde{q}) \approx \mu} \| \hat{P}_q w \|_{L^2_t} \| \hat{P}_{\tilde{q}} u \|_{L^2_x},
\]
Therefore, noting that \( \hat{P}_\mu(\hat{P}_q w \hat{P}_{\tilde{q}} u) = 0 \) unless \( \text{dist}(q, \tilde{q}) \approx \mu \), we conclude that
\[
\| \hat{P}_\mu(\hat{w}u) \|_{L^4_t L^2_x} \lesssim \mu^{2-\frac{d}{4}} \left( \sum_{q \in Q_\mu} \| P_q \|_{L^2_t}^2 \right)^{\frac{1}{2}} \left( \sum_{\tilde{q} \in Q_\mu} \| P_{\tilde{q}} \|_{L^2_x}^2 \right)^{\frac{1}{2}}
\]
where the last line follows from the fact that square sums of almost orthogonal Fourier multipliers is bounded in \( U^2_\Delta \), see for instance [3] Proposition 4.3].

We now turn to the proof of Theorem 4.2.

**Proof of Theorem 4.2.** We adopt the notation used in the proof of Theorem 4.1 thus \( \mathcal{D}_J \) denotes a set of intervals of length \( 2^J \) which form a partition of \( \mathbb{R} \). The first step in the proof of is to show that
\[
\| \hat{P}_\mu(\hat{w}u) \|_{L^4_t L^2_x} \lesssim \mu^{2-\frac{d}{4}} \| w \|_{U^2_\Delta} \left( \| u \|_{L^4_t L^2_x} + \| (i\partial_t + \Delta) u \|_{L^4_t L^2_x} \right),
\]
thus we can replace one of the \( U^2_\Delta \) norms with the inhomogeneous Strichartz type norm. Similar to the proof of Theorem 4.1 in view of Theorem 4.3, it suffices to prove that
\[
\| \hat{P}_\mu \left[ \tilde{w} \int_{-\infty}^t e^{i(t-s)\Delta} F(s) ds \right] \|_{L^4_t L^2_x} \lesssim \mu^{2-\frac{d}{4}} \| w \|_{U^2_\Delta} \| F \|_{L^4_t L^2_x}.
\]
We start by observing that since \( \sum_I \mathbb{1}_I(t) e^{it\Delta} \int_{\mathbb{R}} e^{-i\alpha} F(s) \, ds \) is a rescaled atom, an application of the endpoint Strichartz estimate gives the bound

\[
\left\| \sum_{I, J \in D_j, I \supseteq J, \text{ dist}(I, J) \approx 2^j} \mathbb{1}_I(t) \int_{\mathbb{R}} e^{i(t-s)\Delta} \mathbb{1}_J(s) F(s) \, ds \right\|_{U^2 L^2_t L^\infty_x} \lesssim \left( \sum_{J \in D_j} \left\| \int_{\mathbb{R}} e^{-is\Delta} \mathbb{1}_J(s) F(s) \, ds \right\|_{L^2_t L^2_x}^2 \right)^{\frac{1}{2}} 
\]

\[
\lesssim \left( \sum_{J \in D_j} \left\| \mathbb{1}_J F \right\|_{L^2_t L^1_x} \right)^\frac{3}{2} = \| F \|_{L^2_t L^3_x}^{\frac{3}{2}}. \quad (4.10)
\]

Consequently, an application of Theorem 4.3 implies that

\[
\left\| \hat{\hat{P}} \right\|_{L^1_t L^2_x} \left\| \sum_{I, J \in D_j, I \supseteq J, \text{ dist}(I, J) \approx 2^j} \mathbb{1}_I(t) \int_{\mathbb{R}} e^{i(t-s)\Delta} \mathbb{1}_J(s) F(s) \, ds \right\|_{L^1_t L^2_x} \lesssim \mu^{2 - \frac{2}{3}} \| w \|_{U^2_x} \left\| \sum_{I, J \in D_j, I \supseteq J, \text{ dist}(I, J) \approx 2^j} \mathbb{1}_I(t) \int_{\mathbb{R}} e^{i(t-s)\Delta} \mathbb{1}_J(s) F(s) \, ds \right\|_{U^2_x} \lesssim \mu^{2 - \frac{2}{3}} \| w \|_{U^2_x} \| F \|_{L^1_t L^2_x}^{\frac{3}{2}}. \quad (4.11)
\]

On the other hand, as in the proof of Theorem 1.1 to gain decay in \( j \), we again use an application of [12 Lemma 4.1] and observe that for every \( n \) in a neighbourhood of 4

\[
\left\| \sum_{I, J \in D_j, I \supseteq J, \text{ dist}(I, J) \approx 2^j} \mathbb{1}_I(t) \int_{\mathbb{R}} e^{i(t-s)\Delta} \mathbb{1}_J(s) F(s) \, ds \right\|_{L^1_t L^2_x} \lesssim \left( \sum_{I, J \in D_j, I \supseteq J, \text{ dist}(I, J) \approx 2^j} \| \mathbb{1}_I(t) \int_{\mathbb{R}} e^{i(t-s)\Delta} \mathbb{1}_J(s) F(s) \, ds \|_{L^2_t L^2_x}^2 \right)^{\frac{1}{2}} 
\]

\[
\lesssim 2^{-\frac{3}{2}(1 - \frac{1}{n})} \| F \|_{L^1_t L^2_x}^{\frac{3}{2}}
\]

and hence for every \( r \) in a neighbourhood of 2 we have

\[
\left\| \hat{\hat{P}} \right\|_{L^1_t L^2_x} \left\| \sum_{I, J \in D_j, I \supseteq J, \text{ dist}(I, J) \approx 2^j} \mathbb{1}_I(t) \int_{\mathbb{R}} e^{i(t-s)\Delta} \mathbb{1}_J(s) F(s) \, ds \right\|_{L^1_t L^2_x} \lesssim \left\| w \right\|_{L^1_t L^2_x} \left\| \sum_{I, J \in D_j, I \supseteq J, \text{ dist}(I, J) \approx 2^j} \mathbb{1}_I(t) \int_{\mathbb{R}} e^{i(t-s)\Delta} \mathbb{1}_J(s) F(s) \, ds \right\|_{L^1_t L^2_x} \lesssim 2^{j(\frac{3}{2} - 1)} \left\| w \right\|_{U^2_x} \| F \|_{L^1_t L^2_x}^{\frac{3}{2}}. \quad (4.12)
\]

Let \( \frac{5}{3} < r_1 \leq r \leq r_2 \) with \( r_2 \) in a neighbourhood of 2, and take \( 0 < \theta < 1 \) such that \( \frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2} \). Interpolating between (4.11) and (4.12), and choosing \( r_2 \) appropriately, we obtain for every sufficiently small \( \epsilon > 0 \)

\[
\left\| \hat{\hat{P}} \right\|_{L^1_t L^2_x} \left\| \sum_{I, J \in D_j, I \supseteq J, \text{ dist}(I, J) \approx 2^j} \mathbb{1}_I(t) \int_{\mathbb{R}} e^{i(t-s)\Delta} \mathbb{1}_J(s) F(s) \, ds \right\|_{L^1_t L^2_x} \lesssim \mu^{\theta(2 - \frac{1}{r})} 2^{j(\frac{3}{2} - 1 - \theta)} \left\| w \right\|_{U^2_x} \| F \|_{L^1_t L^2_x} \lesssim \mu^{2 - \frac{4}{3}} 2^{-\epsilon j - k} \left\| w \right\|_{U^2_x} \| F \|_{L^1_t L^2_x}^{\frac{3}{2}}
\]

\[
\approx \mu^{2 - \frac{4}{3} - \epsilon j - k} \left\| w \right\|_{U^2_x} \| F \|_{L^1_t L^2_x}^{\frac{3}{2}}
\]
where we take $k \in \mathbb{Z}$ such that $2^{-\frac{k}{2}} \approx \mu$. Therefore, applying the Whitney decomposition (4.7), we conclude that
\[
\left\| \hat{P}_\mu \left[ \int_{-\infty}^{t} e^{i(t-s)\Delta} f(s) \, ds \right] \right\|_{L^1_t L^\infty_x} \leq \sum_{j \in \mathbb{Z}} \left\| \hat{P}_\mu \left[ \int_{\mathbb{R}_j} e^{i(t-s)\Delta} f(s) \, ds \right] \right\|_{L^1_t L^\infty_x}
\]
\[
\lesssim \mu^{2-\frac{1}{d}} \sum_{j \in \mathbb{Z}} 2^{-c|j-k|} \left\| u_{\lambda_3} \right\|_{L^\infty_t L^\frac{4}{3}_x} \lesssim \mu^{2-\frac{1}{d}} \left\| u_{\lambda_3} \right\|_{L^\infty_t L^\frac{4}{3}_x}
\]
and hence (4.9) follows.

The next step is to replace the $U_{\lambda_3}^2$ norm of $w$ with the required inhomogeneous norm. This follows by essentially repeating the above argument, but using (4.9) in place of Theorem 4.3. More precisely, an application of (4.9) and the $U_{\lambda_3}^2$ bound (4.10) gives
\[
\left\| \hat{P}_\mu \left[ \sum_{I,J \in \mathcal{D}_j, I \geq J \text{ dist}(I,J) \geq 2^j} 1_I(t) \int_{I} e^{i(t-s)\Delta} G(s) \, ds \right] \right\|_{L^1_t L^\infty_x} \lesssim \mu^{2-\frac{1}{d}} \left\| u \right\|_{L^\infty_t L^\frac{4}{3}_x} \left( \left\| u \right\|_{L^\infty_t L^\frac{4}{3}_x} + \left\| (i\partial_t + \Delta) u \right\|_{L^\infty_t L^\frac{4}{3}_x} \right).
\]

On the other hand, again applying [12, Lemma 4.1] we have for every $r$ in a neighbourhood of 2
\[
\left\| \hat{P}_\mu \left[ \sum_{I,J \in \mathcal{D}_j, I \geq J \text{ dist}(I,J) \geq 2^j} 1_I(t) \int_{I} e^{i(t-s)\Delta} G(s) \, ds \right] \right\|_{L^1_t L^\infty_x} \lesssim \left\| \sum_{I,J \in \mathcal{D}_j, I \geq J \text{ dist}(I,J) \geq 2^j} 1_I(t) \int_{I} e^{i(t-s)\Delta} G(s) \, ds \right\|_{L^2_t L^\infty_x} \left\| u \right\|_{L^\infty_t L^\frac{4}{3}_x}
\]
\[
\lesssim 2^{j\left(\frac{1}{2} - 1\right)} \left\| u \right\|_{L^\infty_t L^\frac{4}{3}_x} \left( \left\| u \right\|_{L^\infty_t L^\frac{4}{3}_x} + \left\| (i\partial_t + \Delta) u \right\|_{L^\infty_t L^\frac{4}{3}_x} \right).
\]

Therefore, as in the proof of (4.9), the required bound now follows by interpolation, together with the Whitney decomposition (4.7). Finally, to replace the norms $\left\| u \right\|_{L^\infty_t L^\frac{4}{3}_x} + \left\| (i\partial_t + \Delta) u \right\|_{L^\infty_t L^\frac{4}{3}_x}$ with the norm $\left\| u \right\|_{Z}$ follows from the trivial $L^1_t L^2_x$ transference type argument outlined in (4.10). \qed

5. Refined bilinear estimates

In this section we prove two estimates. The first is a version of Theorem 3.1 with a Besov refinement, which is used to control the error terms in the profile decomposition. The second estimate in this section is a version of the inhomogeneous endpoint Strichartz estimate with two spatially diverging potentials. Again, this estimate plays a key role in the proof of the uniform Strichartz estimate.

5.1. Refinement of Theorem 3.1. The following estimate is the main goal of this section, and it is also one of the core ingredients of this paper.

**Theorem 5.1.** Let $0 \leq s < 1$ and $d = 4$. There exists $0 < \theta < 1$ and $r > 2$ such that for any $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}^4$, any free solution to the wave equation $v = e^{\pm i|\xi| f} \in L^\infty_t L^2_x$, and any $\chi(t) \in L^\infty \cap C^\infty(\mathbb{R})$ satisfying
\[\text{supp}(\hat{\chi}v) \subset \{ |\tau| \lesssim (\xi) \} \]
we have
\[\| P_{\lambda_0}(\chi v u_{\lambda_1}) \|_{N^s_{2,0}} \lesssim \left( \frac{\min\{\lambda_0, \lambda_1\}}{\max\{\lambda_0, \lambda_1\}} \right)^\theta \| \chi \|_{L^\infty_t} \| v \|_{L^\infty_t L^2_x} \| u_{\lambda_1} \|_{S^s_{1,4}} \] \tag{5.1}
and
\[\| T_0[P_{\lambda_0}(\chi v u_{\lambda_2})] \|_{S^s_{2,0}} \lesssim \left( \frac{\lambda_{\min}}{\lambda_{\max}} \right)^\theta \| \chi v u_{\lambda_2} \|_{L^\infty_t B^s_{r,\infty}} \left( \| \chi \|_{L^\infty_t} \| v_{\lambda_2} \|_{L^\infty_t L^2_x} \right)^{1-\theta} \| u_{\lambda_1} \|_{S^s_{1,4}} \] \tag{5.2}
where $\lambda_{\min} = \min\{\lambda_0, \lambda_1, \lambda_2\}$ and $\lambda_{\max} = \max\{\lambda_0, \lambda_1, \lambda_2\}$. 


Proof. The case \( \lambda_0 \gg \lambda_1 \) of both (5.1) and (5.2) follows from (3.5) and Lemma 2.2. On the other hand, if \( \lambda_0 \ll \lambda_1 \), we have to work a little harder as the estimates in Theorem 3.1 only suffice when \( s > 0 \) in this case. We first observe that in view of the non-resonant identity \( P_{\lambda_0}(\chi vP_{\lambda_1}^N u) = P_{\lambda_0}^F(\chi v\approx_{\lambda_1} P_{\lambda_1}^N u_{\lambda_1}) \), together with the definition of \( N_{\lambda_0} \) and \( S_{\lambda_1} \), Theorem 4.1 implies that for any \( 0 < \alpha < \frac{1}{2} \)

\[
\| P_{\lambda_0}(\chi v\approx_{\lambda_1} P_{\lambda_1}^N u) \|_{N_{\lambda_0}^s} \leq \lambda_0^{s-\frac{1}{2}} \| v \|_{L^\infty} \| P_{\lambda_0}(\chi v\approx_{\lambda_1} P_{\lambda_1}^N u) \|_{L_{t,x}^s} \lesssim \left( \frac{\lambda_0}{\lambda_1} \right)^{s+\alpha} \| v \|_{L^\infty} \| v\approx_{\lambda_1} \|_{L^{\infty}_t} \| u_{\lambda_1} \|_{S_{\lambda_1}}.
\]

On the other hand, an application of Hölder’s inequality gives

\[
\| P_{\lambda_0}(\chi v\approx_{\lambda_1} P_{\lambda_1}^N u) \|_{N_{\lambda_0}^s} \leq \lambda_0^{s-\frac{1}{2}} \| P_{\lambda_0}(\chi v\approx_{\lambda_1} P_{\lambda_1}^N u) \|_{L_{t,x}^s} \lesssim \left( \frac{\lambda_0}{\lambda_1} \right)^{s+\frac{1}{2}} \| v\approx_{\lambda_1} \|_{L^{\infty}_t} \| u_{\lambda_1} \|_{S_{\lambda_1}}.
\]

Combining these bounds with (3.6), both (5.1) and (5.2) follow in the case \( \lambda_0 \equiv \lambda_1 \).

It remains to consider the case \( \lambda_0 \approx \lambda_1 \). The estimate (5.1) follows immediately from (3.7) and (3.8). On the other hand, in view of the non-resonant bound (3.8), to prove (5.2) it suffices to show that for \( \lambda_2 \ll \lambda_0 \approx \lambda_1 \) we have

\[
\lambda_0^\alpha \| P_{\lambda_0}^N J_0[\chi v\approx_{\lambda_1} P_{\lambda_1}^N u_{\lambda_1}] \|_{L_{t,x}^2} \lesssim \left( \frac{\lambda_2}{\lambda_1} \right)^{\alpha} \| v_{\approx_{\lambda_2}} \|_{L^{\infty}_t} \| v\approx_{\lambda_1} \|_{L^{\infty}_t} \| u_{\lambda_1} \|_{S_{\lambda_1}}.
\]

An application of the energy estimate in Lemma 2.2 together with the definition of the norm \( \| \cdot \|_{S_{\lambda_0}} \) shows that it is enough to prove the dual formulation

\[
\left| \int_{\mathbb{R}^{1+4}} \bar{\omega}_\lambda \chi v\approx_{\lambda_2} u_{\lambda_1} \, dx \right| \lesssim \left( \frac{\lambda_2}{\lambda_0} \right)^{\alpha} \| \chi v\approx_{\lambda_2} \|_{L^{\infty}_t} \| v\approx_{\lambda_1} \|_{L^{\infty}_t} \| u_{\lambda_1} \|_{S_{\lambda_1}} \| z \|_{2}.
\]

By multiplying \( u \) and \( w \) with a constant, we may assume that

\[
\| u_{\lambda_1} \|_{Z} = \| w_{\lambda_0} \|_{Z} = 1.
\]

Let \( 0 < p < 1, \frac{2}{3} < q < 2, r > 2, \) and \( 0 < \theta < 1 \) such that

\[
1 = \frac{1}{p} + (1 - \theta) \frac{3}{4}, \quad 1 = \frac{1}{q} + \frac{1}{r}, \quad 4 \left( \frac{1}{r} - \frac{1}{2} \right) \theta + \frac{1 - \theta}{4} > 0,
\]

which is easily obtained by taking \( r > 2 \) close enough to 2 for any fixed \( q \in (5/3, 2) \), with \( p \) and \( \theta \) determined by the above equations (one potential choice is \( 1 = \frac{1}{p} + \frac{1}{24}, \frac{1}{q} = \frac{1}{2} + \frac{1}{3}, \frac{1}{r} = \frac{1}{2} + \frac{1}{100}, \frac{1}{r} - \frac{1}{2} = \frac{1}{2} - \frac{1}{100} \), and \( \theta = \frac{6}{7} \)). The convexity of \( L^p_x \) spaces implies that (recall that \( v_1 \) contains all frequencies less than 1)

\[
\left| \int_{\mathbb{R}^{1+4}} \bar{\omega}_\lambda \chi v\approx_{\lambda_2} u_{\lambda_1} \, dx \right| \leq \sum_{\mu \in \mathbb{Z}^2, \lambda_2 \ll \mu \ll \lambda_2} \left| \int_{\mathbb{R}^{1+4}} \hat{\mu}_\lambda(\mathbb{C}_\lambda u_{\lambda_1}) \chi \hat{\mu}_\lambda v\approx_{\lambda_2} \, dx \right|
\]

\[
\leq \sum_{\mu \in \mathbb{Z}^2, \lambda_2 \ll \mu \ll \lambda_2} \| \hat{\mu}_\lambda(\mathbb{C}_\lambda u_{\lambda_1}) \chi \hat{\mu}_\lambda v\approx_{\lambda_2} \|_{L^1_x L^p_x} \| \chi v\approx_{\lambda_1} \|_{L^{\infty}_t} \| \hat{\mu}_\lambda(\mathbb{C}_\lambda u_{\lambda_1}) \hat{\mu}_\lambda v\approx_{\lambda_2} \|^{1-\theta}_{L^1_x L^q_x}.
\]

Applying Theorem 4.2 we see that

\[
\| \hat{\mu}_\lambda(\mathbb{C}_\lambda u_{\lambda_1}) \chi \hat{\mu}_\lambda v\approx_{\lambda_2} \|_{L^1_x L^p_x} \ll \| \hat{\mu}_\lambda(\mathbb{C}_\lambda u_{\lambda_1}) \|_{L^1_x L^p} \| \chi v\approx_{\lambda_1} \|_{L^{\infty}_t} \| \hat{\mu}_\lambda(\mathbb{C}_\lambda u_{\lambda_1}) \hat{\mu}_\lambda v\approx_{\lambda_2} \|^{1-\theta}_{L^1_x L^q_x}
\]

\[
\ll \mu^{2-\theta} \| \chi v\approx_{\lambda_1} \|_{L^{\infty}_t} \| \hat{\mu}_\lambda(\mathbb{C}_\lambda u_{\lambda_1}) \hat{\mu}_\lambda v\approx_{\lambda_2} \|^{1-\theta}_{L^1_x L^q_x}.
\]

Let \( K_\mu(y) \) denote the kernel of the Fourier multiplier \( \hat{\mu}_\lambda \). Note that \( \| K_\mu \|_{L^1(\mathbb{R}^3)} \lesssim 1 \). Hence Theorem 4.1 together with the translation invariance of \( L^p \) spaces gives

\[
\| \hat{\mu}_\lambda(\mathbb{C}_\lambda u_{\lambda_1}) \hat{\mu}_\lambda v\approx_{\lambda_2} \|_{L^1_x L^p_x} \ll \| K_\mu(y) \|_{L^1_x} \| \chi v\approx_{\lambda_1} \|_{L^{\infty}_t} \| \hat{\mu}_\lambda(\mathbb{C}_\lambda u_{\lambda_1}) \hat{\mu}_\lambda v\approx_{\lambda_2} \|^{1-\theta}_{L^1_x L^q_x}
\]

\[
\lesssim \mu \left( \frac{\mu}{\lambda_1} \right)^{\frac{2}{p}} \| \chi v\approx_{\lambda_1} \|_{L^{\infty}_t} \| \hat{\mu}_\lambda(\mathbb{C}_\lambda u_{\lambda_1}) \hat{\mu}_\lambda v\approx_{\lambda_2} \|^{1-\theta}_{L^1_x L^q_x}.
\]
Since \(4(\frac{1}{4} - \frac{1}{2})\theta + \frac{1}{2} < 0\), we conclude that
\[
\left| \int_{\mathbb{R}^{d+4}} \bar{w}_{\lambda_0} \chi v_{\lambda_2} u_{\lambda_1} \, dx dt \right| \lesssim \sum_{\mu \in \mathbb{Z}^2} \left[ \mu \left( \frac{\mu}{\lambda_1} \right)^{\frac{1}{2}} \| v_{\lambda_2} \|_{L^\infty_t B^\theta_{r,\infty}} \right]^{\frac{1}{2}} \left[ \mu \left( \frac{\mu}{\lambda_1} \right)^{\frac{1}{2}} \| v_{\lambda_2} \|_{L^\infty_t L^{\frac{1}{2}}_x} \right]^{\theta - 1}
\]
and therefore \(\text{Remark 5.3}\) follows.

As an easy corollary, we obtain the following.

**Corollary 5.2.** Let \(d = 4 \) and \(0 \leq s < 1\). There exist \(0 < \theta < 1\) and \(r > 2\) such that for any \(|\ell| < \theta\) and any free wave \(v = e^{\pm i|\nabla| f}\) we have
\[
\|\mathcal{I}_0(vu)\|_{S^s} \lesssim \|v\|_{L^\infty_t B^\frac{1}{2} s_{r,\infty}} \|v\|_{L^\infty_t H^{1+\epsilon}} \|u\|_{S^{s-\epsilon}}.
\]

**Remark 5.3.** Although the statement of Corollary 5.2 gives the crucial Besov gain required in later sections, it has the unfortunate problem that it only gives control over the weaker space \(S^s\), but requires that we have \(u \in S^{s}\) in the stronger space. There are a number ways to resolve this difficulty. One option is to essentially iterate the equation twice, since Theorem 3.1 roughly shows that \(\mathcal{I}_0\) maps \(S^s\) to \(S^{s-\epsilon}\).

An alternative is approach is to exploit complex interpolation. More precisely, note that we can write Theorem 3.1 and Corollary 5.2 in the form
\[
\|\mathcal{I}_0 v\|_{B(S^{s \to \mathbb{S}^s}_r)} \lesssim \|v\|_{L^\infty_t L^{\frac{1}{2}}_x}, \quad \|\mathcal{I}_0 v\|_{B(S^{s \to \mathbb{S}^s}_r)} \lesssim \|v\|_{L^\infty_t B^\frac{1}{2} s_{r,\infty}} \|v\|_{L^\infty_t L^{\frac{1}{2}}_x}^{1-\theta}
\]
for any free wave \(v\). As observed above, we only have the crucial factor \(\|v\|_{L^\infty_t B^\frac{1}{2} s_{r,\infty}}\) when we map the strong space \(S^s\) to the weak space \(S^s\). This deficiency can be resolved with a small twist in the function spaces. More precisely, define the complex interpolation space
\[
S^s_{1/2} := [S^s, \mathbb{S}^s]_{1/2}.
\]
Then the above bounds immediately imply that we have
\[
\|\mathcal{I}_0 v\|_{B(S^{s_{1/2}}_{1/2} \to S^{s_{1/2}}_{1/2})} \lesssim \|v\|_{L^\infty_t L^{\frac{1}{2}}_x}^{\frac{1}{2}} \|v\|_{L^\infty_t L^{\frac{1}{2}}_x}^{1-\theta}.
\]
This has the important advantage that we map \(S^s_{1/2}\) to \(S^s_{1/2}\) but retain a power of the Besov norm.

### 5.2. Decay by spatial separation

For the uniform Strichartz estimate in the non-radial case, we need an extra decay for potentials separating in space. We use the notation \(f_a(x) = f(x - a)\) for translates of \(f\) (only within this subsection).

**Lemma 5.4.** Let \(d \geq 3\), \(f, g \in L^{d/2}(\mathbb{R}^d)\) and \(\varepsilon > 0\). Then there exists \(D > 0\) such that for any \(a, b \in \mathbb{R}^d\) satisfying \(|a - b| \geq D\), and any \(F \in L^2_t L^2_x\), we have
\[
\|f_a \mathcal{I}_0[g_b F]\|_{L^2_t L^2_x} \leq \varepsilon \|F\|_{L^2_t L^2_x}.
\]

**Proof.** First, the double endpoint Strichartz with Hölder implies
\[
\|f_a \mathcal{I}_0[g_b F]\|_{L^2_t L^2_x} \leq \|f\|_{L^{d/2}} \|\mathcal{I}_0[g_b F]\|_{L^2_t L^2_x} \lesssim \|f\|_{L^{d/2}} \|g_b F\|_{L^2_t L^2_x} \lesssim \|f\|_{L^{d/2}} \|g\|_{L^{d/2}} \|F\|_{L^2_t L^2_x},
\]
which allows us to approximate \(f\) and \(g\) in \(L^{d/2}\) by Schwartz functions.

Second, in order to dispose of the high frequency of \(F\), we use the local smoothing estimate in a global form:
\[
\|\nabla \mathcal{I}_0 f\|_{L^2_t L^2_{\alpha, Q_{\alpha}}} \lesssim \|f\|_{L^2_t L^2_{\alpha, Q_{\alpha}}},
\]
\[
\|\nabla \mathcal{I}_0 f\|_{L^2_t L^2_{\alpha, Q_{\alpha}}} \lesssim \|f\|_{L^2_t L^2_{\alpha, Q_{\alpha}}},
\]

(5.7)
Thus we can dispose of the high frequency contribution from \( F \) where the last norm is bounded by using the above smoothing estimate so that we can gain as follows, with \( u \) so that we can make its contribution as small as we like by \( D \). By another approximation of \( f, g \), so that we can dispose of 

\[
\text{supp } F_a I_0 |g_b F_{>\lambda}| \subset \{ |\xi| > \lambda/2 \},
\]

(5.8)

so that we can gain as follows, with \( u := I_0 |g_b F_{>\lambda}|, \)

\[
\| f_au \|_{L^2 L^{2^*}} \lesssim \lambda^{-1} \| \nabla (f_au) \|_{L^2 L^{2^*}} \\
\lesssim \lambda^{-1} \| \nabla f \|_{L^{2^*} \rightarrow 2} \| u \|_{L^2 L^{2^*}} + \lambda^{-1} \| f \|_{L^2 L^{2^*}(Q_\alpha)} \| \nabla u \|_{L^2 L^{2^*}(Q_\alpha)},
\]

(5.9)

where the last norm is bounded by using the above smoothing estimate

\[
\| \nabla u \|_{L^2 L^{2^*}(Q_\alpha)} \lesssim \| g_b F_{>\lambda} \|_{L^2 L^{2^*}(Q_\alpha)} \| F_{>\lambda} \|_{L^2 L^{2^*}(Q_\alpha)} \lesssim \| g \|_{L^2 L^{2^*}(Q_\alpha)} \| F \|_{L^2 L^{2^*}}.
\]

(5.10)

Thus we can dispose of the high frequency contribution from \( F_{>\lambda} \) for some large \( \lambda \).

Third, in order to dispose of long time interactions, we exploit the dispersive decay estimate (in \( L^\infty \)). Decomposing the Duhamel integral by

\[
I_0 f = \int_{-L}^{L} U_0(t-s) f(s) ds + \int_{0}^{t-L} U_0(t-s) f(s) ds =: I_{<L} f + I_{>L} f,
\]

(5.11)

we have, using the dispersive estimate, as well as Young and Hölder,

\[
\| f_a I_{>L} g_b F \|_{L^2 L^{2^*}} \lesssim \| f \|_{L^2} \| U_{-L} \|_{L^2} \| g \|_{L^\infty} \| F \|_{L^2 L^{2^*}} \lesssim L^{-d/2} \| f \|_{L^2} \| g \|_{L^\infty} \| F \|_{L^2 L^{2^*}} \lesssim L^{-d/2+1} \| f \|_{L^2} \| g \|_{L^\infty} \| F \|_{L^2 L^{2^*}},
\]

(5.12)

so that we can dispose of \( I_{>L} \) for some large \( L \) > 1.

Thus the problem is reduced to the decay of

\[
f_a I_{<L} [g_b F_{<\lambda}].
\]

(5.13)

Now that both the traveling time and the frequency (group velocity) are bounded, we can exploit the spatial separation \( |a - b| \rightarrow \infty \). An easy way is to use

\[
x U_0(t) = U_0(t)(x - 2it \nabla).
\]

(5.14)

By another approximation of \( f, g \in L^{d/2} \), we may now assume that \( \text{supp } f, \text{supp } g \subset \{ |x| < S \} \) for some \( S \in (0, \infty) \). If \( |a - b| \geq D \rightarrow \infty \), then we have, with \( u := I_{<L} [g_b F_{<\lambda}], \)

\[
\| f_au \|_{L^2 L^{2^*}} \lesssim D^{-1} \| (x-b)f(x-a)u \|_{L^2 L^{2^*}} = D^{-1} \| f(x-a)I_{<L}[(x-b-2it \nabla)g(x-b)F_{<\lambda}] \|_{L^2 L^{2^*}} \lesssim D^{-1} \| f \|_{L^{2^*} \rightarrow 2} \| xg \|_{L^{2^*}} \| F_{<\lambda} \|_{L^2 L^{2^*}} + D^{-1} \| f \|_{L^{2^*} \rightarrow 2} \| I_{<L}[(t-s)\nabla(g_b F_{<\lambda})] \|_{L^2 L^{2^*}},
\]

(5.15)

where the last norm is bounded by using Sobolev, Young and Hölder

\[
\| \int_{|t-s| < L} | \Delta (g_b F_{<\lambda}(s)) | ds \|_{L^2} \lesssim \| 1 \|_{L^\infty \rightarrow L^2} \| g \|_{W^{2,2} \rightarrow L^\infty} \| F_{<\lambda} \|_{L^2 W^{2,2^*}} \lesssim L \lambda^2 \| g \|_{W^{2,2} \rightarrow L^\infty} \| F \|_{L^2 L^{2^*}},
\]

(5.16)

so that we can make its contribution as small as we like by \( D \rightarrow \infty \). □
6. Uniform Strichartz estimates

In this section, we prove a uniform Strichartz estimate for wave potential in $L^2(\mathbb{R}^4)$ below the ground state threshold $|W^2|_{L^2(\mathbb{R}^4)}$. $W$ is the ground state solution to the nonlinear Schrödinger equation on $\mathbb{R}^4$:

$$W(x) := (1 + |x|^2/8)^{-1}, \quad -\Delta W = W^3,$$  \quad (6.1)

or the Aubin-Talenti function, the unique maximizer of the Sobolev inequality $\|\varphi\|_{L^4(\mathbb{R}^4)} \leq C\|\nabla \varphi\|_{L^2(\mathbb{R}^4)}$. It gives rise to the family of static solutions $(u, v) = (W_\lambda, -W_\lambda^2)$, where $W_\lambda(x) := \lambda W(\lambda x)$ is the invariant scaling in $H^1(\mathbb{R}^4)$ and in $L^4(\mathbb{R}^4)$.

**Theorem 6.1.** Let $d = 4$, $0 \leq s < 1$, and $0 < B < |W^2|_{L^2(\mathbb{R}^4)}$. There is a constant $C \in (0, \infty)$, such that for any $g \in L^2(\mathbb{R}^4)$ with $|g|_{L^2} \leq B$, we have the Strichartz estimate with potential $V := e^{it|\nabla|}g$

$$\|u\|_{S^s} \leq C \left[ \inf_{t \in \mathbb{R}} \|u(t)\|_{H^s} + \|(i\partial_t + \Delta - \nabla V)u\|_{N^s} \right].$$ \quad (6.2)

Roughly, the proof of Theorem 6.1 proceeds as follows. Let $C(g)$ be the optimal constant in (6.2) with potential $V = e^{it|\nabla|}g$ and define the quantities

$$M(B) := \sup \{ C(g) \mid |g|_{L^2} \leq B \}, \quad B^* := \sup \{ B > 0 \mid M(B) < \infty \}.$$

Our goal is to show that $B^* = |W^2|_{L^2(\mathbb{R}^4)}$. It is easy enough to show that $B^* > 0$, this is simply a restatement of the small data theory. Suppose for the sake of contradiction that $0 < B^* < |W^2|_{L^2(\mathbb{R}^4)}$. Then there exists a sequence of potentials $V_n$ such that $\|V_n\|_{L^\infty L^2} \neq B^* < |W^2|_{L^2(\mathbb{R}^4)}$ such that the corresponding constant in (6.2) satisfies $C_n = C(V_n) \to \infty$. We now run a profile decomposition as in [2] on the free waves $V_n$ in $L^\infty L^2$ with error going to zero in $H^1$. The estimates from the previous sections together with the orthogonality of the profiles reduces the problem to considering a single profile. It is in this reduction where the improved bilinear estimates in Theorem 3.1 play a crucial role. Finally, to deal with the single profile case, we can essentially proceed as in [10] and show that it suffices to prove a Strichartz estimate for a static potential below the ground state. But this is a consequence of the general double endpoint Strichartz estimates contained in [18].

In the following, we make the proof sketched above precise. In Subsection 6.1 we iterate the Duhamel formula to obtain a number of key identities that are exploited later. The arguments here are essentially algebraic in nature, but they have useful analytic consequences. In Subsection 6.2 we recall the profile decomposition of Bahourri-Gérard [2]. In Subsection 6.3 we reduce the proof of Theorem 6.1 to proving three key properties: (i) orthogonal profiles only interact weakly, (ii) small $L^\infty B^{-1}$ profiles can be discarded, (iii) the single profile case holds. Finally, in the remainder of this section, we give the proof of the properties (i), (ii), and (iii).

### 6.1. Duhamel formula expansion.

The proof of the uniform Strichartz estimate is based on the profile decomposition applied to a sequence of wave potentials. Here we expand the Duhamel formula with respect to the potential, which allows us to reduce the uniform estimate to the case of single profile or remainder in the decomposition. The argument for expansion is simple and algebraic.

Fix $d \geq 3$ and let $t_0 \in \mathbb{R}$ and $t_0 \in I \subset \mathbb{R}$ be an interval. Let

$$Y_0 := L^2(I; L^2(\mathbb{R}^d)), \quad X_0 := C(I; L^2(\mathbb{R}^d)) \cap L^2(I; L^2(\mathbb{R}^d)).$$ \quad (6.3)

For any space-time functions $V \in L^\infty(I; L^{d/2}(\mathbb{R}^d))$ and $f \in L^2(I; L^{2d/3}(\mathbb{R}^d))$, we define $\mathcal{I}_V f$ be the unique solution $u \in C(I; L^2(\mathbb{R}^d)) \cap L^2(I; L^2(\mathbb{R}^d))$ to

$$u(t_0) = 0, \quad (i\partial_t + \Delta - V)u = f \quad (t \in I)$$ \quad (6.4)

and define $\mathcal{U}_V(t; s)$ to be the corresponding homogeneous solution operator with data at $t = s$. Under reasonable assumptions on the potential $V$, the operator $\mathcal{I}_V : Y_0 \to X_0$ is a bounded linear operator.
Lemma 6.2 (\(I_V\) well-defined). Let \(d \geq 3\). There exists an \(\epsilon > 0\) such that for any \(t_0 \in I \subset \mathbb{R}\) and \(V \in L^\infty_t L^{d/2}_x(I \times \mathbb{R}^d)\), if we can write \(I = \cup_{j=1}^N I_j\) with \(N < \infty\) and
\[
\sup_{j=1,\ldots,N} \|V\|_{(L^\infty_t L^{d/2}_x(I_j \times \mathbb{R}^d))} < \epsilon,
\]
then the linear operators \(I_V \in B(Y_0(I) \to X_0(I))\) and \(I_U \cdot (t_0) \in B(L^2_x \to X_0(I))\) exist and are well-defined.

Proof. The first step is to use a perturbative argument via the endpoint Strichartz estimate to construct a solution on each interval \(I_j\). Since there are only \(N\) intervals, we obtain a solution on the whole interval \(I\) and moreover the required bound holds. \(\square\)

It is easy enough to check that the hypothesis in Lemma 6.2 is always satisfied if \(\|V\|_{L^\infty_t L^{d/2}_x(\mathbb{R}^d)} < \epsilon\). On the other hand, it also suffices to simply assume that \(I \subset \mathbb{R}\) is compact and \(V \in C(I; L^{d/2}_x(\mathbb{R}^d))\), since we can simply approximate the potential \(V\) with a smooth, compactly supported (in \(x\)) potential. Alternative assumptions on \(V \in L^\infty_t L^{d/2}_x\) to ensure the existence of \(I_V \in B(Y_0 \to X_0)\) are possible, but the previous lemma suffices for our purposes.

We now turn to the algebraic component of the argument. The key point is to compare the operators \(I_V\) and \(I_{V'}\). This is extremely useful both when \(V' = 0\), and \(V' \neq 0\). Identifying the potential \(V\) with the multiplication operator,
\[
V : X_0 \to Y_0
\]
we obtain a bounded linear operator via Hölder’s inequality, in other words we obtain an operator \(I\), and \(I\) suffices for our purposes.

Proof. More precisely, it suffices to simply assume that \(I \subset \mathbb{R}\) is compact and \(V \in C(I; L^{d/2}_x(\mathbb{R}^d))\), since we can simply approximate the potential \(V\) with a smooth, compactly supported (in \(x\)) potential. Alternative assumptions on \(V \in L^\infty_t L^{d/2}_x\) to ensure the existence of \(I_V \in B(Y_0 \to X_0)\) are possible, but the previous lemma suffices for our purposes.

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\[
V : X_0 \to Y_0
\]
we obtain a bounded linear operator via Hölder’s inequality, in other words we obtain an operator \(I\), and \(I\) suffices for our purposes.
We now consider potentials of the form $V = V_1 + V_2$. By iterating the above, we arrive at the following key lemma.

**Lemma 6.3.** Let $d \geq 3$, $t_0, s_0 \in \mathbb{R}$, $t_0 \in I$ be an interval, and $V = V_1 + V_2$ with $V_j \in L^\infty(I; L^{d/2}(\mathbb{R}^d))$. Suppose that for some Banach spaces $\mathcal{X} \subset X \subset X_0$ and $Y \subset Y_0$, we have finite numbers $M, \varepsilon$ such that

$$
\|I_0\|_{B(\mathcal{X} \to X)} + \|I_{V_j}\|_{B(\mathcal{X} \to X)} + \|V_j\|_{B(X \to Y)} \leq M,
\|I_0 V_0 I_0 V_1\|_{B(\mathcal{X} \to X)} \leq \varepsilon,
$$

(6.12)

for $j = 1, 2$, where $X_0, Y_0$ and $I_{V_j}$ are as defined above. Let $C := \|I\|_{B(\mathcal{X} \to X)}$ and $\tilde{M} := (1 + CM^2)^6$. If $\tilde{M} \varepsilon < 1$, then we have

$$
\|I_{V_j}\|_{B(\mathcal{X} \to X)} \leq (1 - (\tilde{M} \varepsilon)^{1/2})^{-1} M \tilde{M}.
$$

(6.13)

**Proof.** By iterating the Duhamel expansions (6.7) and (6.8), we obtain the identities

$$
I_{V_j} = I_{V_1} (1 + V_2 I_{V_j}) = I_{V_1} (1 + V_1 I_{V_j})
$$

which we rewrite as

$$
[1 - I_{V_1} V_2 I_{V_2} V_1] I_{V_j} = I_{V_1} + I_{V_1} V_2 I_{V_2}.
$$

In order to invert the bracket on the left, we further expand $I_{V_2}$ twice which gives

$$
I_{V_2} = (1 + I_{V_2} V_2) I_0 = I_0 + (1 + I_{V_2} V_2) I_0 V_2 I_0
$$

and hence

$$
I_{V_1} V_2 I_{V_2} V_1 = I_{V_1} V_2 I_{V_0} V_1 + I_{V_1} V_2 (1 + I_{V_2} V_2) I_0 V_2 I_0 V_1
$$

$$
= (1 + I_{V_1} V_1) I_0 V_2 I_0 V_1 + I_{V_1} V_2 (1 + I_{V_2} V_2) I_0 V_2 I_0 V_1 = A I_0 V_2 I_0 V_1
$$

with

$$
A := 1 + I_{V_1} V_1 + I_{V_1} V_2 (1 + I_{V_2} V_2).
$$

Therefore we conclude that

$$
[1 - A I_0 V_2 I_0 V_1] I_{V_j} = I_{V_1} + I_{V_1} V_2 I_{V_2}.
$$

(6.14)

Hence if both $I_{V_j}$ are bounded and $I_0 V_2 I_0 V_1$ is small enough, then $I_{V_j}$ is a small perturbation of $I_{V_1} (1 + V_2 I_{V_2})$ and so bounded as well. More precisely, an application of the assumed operator bounds together with composition of mappings, gives the bounds

$$
\|I_{V_1} + I_{V_1} V_2 I_{V_2}\|_{B(\mathcal{X} \to X)} \leq \|I_{V_1}\|_{B(\mathcal{X} \to X)} + \|I_{V_1}\|_{B(\mathcal{X} \to X)} \|V_2\|_{B(X \to Y)} \|I_{V_2}\|_{B(Y \to X)} \leq M (1 + CM^2)
$$

and

$$
\|A\|_{B(X \to X)} \leq 1 + \|I_{V_1}\|_{B(Y \to X)} \|V_1\|_{B(Y \to X)} + \|I_{V_1}\|_{B(Y \to X)} \|V_2\|_{B(X \to Y)} \|I_{V_2}\|_{B(Y \to X)} (1 + \|I_{V_2}\|_{B(Y \to X)} \|V_2\|_{B(X \to Y)} )
$$

$$
\leq 1 + CM^2 + CM^2 (1 + CM^2) = (1 + CM^2)^2
$$

and therefore

$$
\|A I_0 V_2 I_0 V_1\|_{B(X \to X)} \leq A I_0 V_2 I_0 V_1 \|I_{V_1}\|_{B(\mathcal{X} \to X)} \|I_{V_1}\|_{B(\mathcal{X} \to X)} \leq (1 + CM^2)^2 \varepsilon.
$$

On the other hand, a short computation also gives

$$
\|A I_0 V_2 I_0 V_1\|_{B(X \to X)} \leq (1 + CM^2)^4
$$

and consequently for every $n \geq 0$

$$
\|(A I_0 V_2 I_0 V_1)^n\|_{B(X \to X)} \leq C ((1 + CM^2)^4 (1 + CM^2)^2 \varepsilon)^{1/2} = C ((1 + CM^2)^6 \varepsilon)^{1/2}.
$$

Therefore, since $(1 + CM^2)^6 \varepsilon < 1$ by assumption, the series $\sum_{n=0}^{\infty} (A I_0 V_2 I_0 V_1)^n$ converges absolutely in $B(\mathcal{X} \to X)$. Consequently, by the Neumann series, we can invert the bracket on the left hand side of (6.14) and define

$$
I_{V_j} = [1 - A I_0 V_2 I_0 V_1]^{-1} (I_{V_1} + I_{V_1} V_2 I_{V_2}) : Y \to X.
with the operator bound
\[ \|I_V\|_{B_t(Y \rightarrow X)} \leq \sum_{n=0}^{\infty} \|A(I_0 V_2 I_0 V_1)^n\|_{B_t(X \rightarrow Y)} \leq \sum_{n=0}^{\infty} C(1 + CM^2)^{\frac{n^2}{2}} (1 + CM) = (1 - (\tilde{M} \epsilon)^{\frac{1}{2}})^{-1} CM(1 + CM^2). \]

The only remaining step is to upgrade \( I_V \) to a map into the smaller space \( X \subset X \). This follows by repeating the argument in (II) above. Namely, the identity \( I_V = I_{V_1} + I_{V_2} \) implies that
\[ \|I_V\|_{B_t(Y \rightarrow X)} \leq \|I_{V_1}\|_{B_t(Y \rightarrow X)} + \|I_{V_2}\|_{B_t(Y \rightarrow X)} \leq \tilde{M} M \]

6.2. The profile decomposition. In the following we state the profile decomposition due to Bahouri-Gérard \[2\]. The version we give below is slightly adjusted to our setting, but the proof is the same. Let \( V_n \) be a sequence of free waves with bounded \( L^2(\mathbb{R}^4) \) norm, and let
\[ V_n = \sum_{j=1}^{J} V_n^j + \Gamma_n \]
be its profile decomposition in \( L^2(\mathbb{R}^4) \), such that (along some subsequence)
\[ \lim_{J \to \infty} \lim_{n \to \infty} \|\Gamma_n^J\|_{L^2(B_{\infty,1})} = 0 \]
and
\[ V_n^j(t, x) = g_n^j V^j := (\sigma_n^j)^2 V^j(\sigma_n^j t - t_n^j, \sigma_n^j x - x_n^j), \]
where the profiles \( V^j = e^{it|\nabla| \phi^j} \) are free waves which are independent of \( n \), and satisfy
\[ \lim_{n \to \infty} \left( \|V_n\|_{L_t^2 L_x^4}^2 - \left( \|\Gamma_n^J\|_{L_t^2 L_x^4}^2 + \sum_{j=1}^{J} \|V^j\|_{L_t^2 L_x^4}^2 \right)^\frac{1}{2} \right) = 0. \]

The group elements \( g_n^j = g[\sigma_n^j, x_n^j, t_n^j] \) are asymptotically orthogonal, in the sense that for each pair \( j \neq k \), one of the following holds as \( n \to \infty \):

(i) \( |\log(\sigma_n^j/\sigma_n^k)| \to \infty \) (scale separation).
(ii) \( \sigma_n^j \equiv \sigma_n^k \) and \( |t_n^j - t_n^k| \to \infty \) (time separation).
(iii) \( \sigma_n^j \equiv \sigma_n^k, t_n^j \equiv t_n^k \) and \( |x_n^j - x_n^k| \to \infty \) (space separation).

We may normalize the parameters such that

(i) \( \sigma_n^j \equiv 1, \sigma_n^j \to +\infty \) or \( \sigma_n^j \to +0 \).
(ii) \( t_n^j \equiv 0 \) or \( t_n^j \to \pm\infty \).
(iii) \( x_n^j \equiv 0 \in \mathbb{R}^d \) or \( |x_n^j| \to \infty \).

6.3. Reductions and the proof of Theorem 6.1. The first step in the proof (6.2) is to observe that the claimed Strichartz bound always holds, but with a constant \( C \) that depends on the potential \( V \). This is a consequence of the more general result \[7\] Theorem 7.1. However, for completeness, we include the special case \( d = 4 \) in the low regularity regime with a short proof here.

Theorem 6.4. Let \( d = 4 \) and \( 0 \leq s < 1 \). If \( V = e^{it|\nabla| g} \in L_t^\infty L_x^2 \), then there exists \( C(V) > 0 \) such that for any \( f \in H^s, F \in N^s \), there exists a unique solution \( u \in C(\mathbb{R}, H^s) \cap L_t^\infty L_x^4 \) to
\[ (i\partial_t + \Delta - \Im(V))u = F, \quad u(0) = f \]
and moreover, we have the bound
\[ \|u\|_{S^s} \leq C(V)(\|f\|_{H^s} + \|F\|_{N^s}). \]
Proof.  An application of Theorem 3.1 gives for any interval \( I \subset \mathbb{R} \)
\[
\| \Re(V)u \|_{N'(I)} \lesssim \| V \|_{L_t^\infty L_x^2} \| u \|_{S'(I)}. 
\]
On the other hand, the standard product inequality for Besov spaces gives
\[
\| \Re(V)u \|_{N'(I)} \lesssim \| \Re(V)u \|_{L_t^2 B_{4/3,2}(I \times \mathbb{R}^4)} 
\lesssim \| V \|_{L_t^2 B_{4/3,2}(I \times \mathbb{R}^4)} \lesssim \| V \|_{L_t^2 B_{4/3,2}(I \times \mathbb{R}^4)} \| u \|_{S'(I)}. 
\] (6.19)
In particular, if we have an interval \( I \subset \mathbb{R} \) such that \( V = V_1 + V_2 \) with
\[
\| V_1 \|_{L_t^\infty L_x^2} + \| V_2 \|_{L_t^2 B_{4/3,2}(I \times \mathbb{R}^4)} < \epsilon, 
\]
then provided \( \epsilon > 0 \) is sufficiently small, the iterates
\[(i \partial_t + \Delta)u_j = \Re(V)u_{j-1} + F, \quad u_j(t_0) = f
\]
converge to a solution \( u \in S^s(I) \). To extend this result to \( \mathbb{R} \), we note that by [3, Lemma 4.1] there exists a
finite partition \( \mathbb{R} = \bigcup_{j=1}^N I_j \) into intervals, and a decomposition \( V = V_0 + \sum_{j=1}^N 1_{I_j}(t)V_1 \) such that \( V_0 \) and \( V_1 \) are free waves, and we have the bounds
\[
\| V_0 \|_{L_t^\infty L_x^2} + \sup_{j=1, \ldots, N} \| V_j \|_{L_t^2 B_{4/3,2}(I_j \times \mathbb{R}^4)} \lesssim \epsilon. 
\]
Hence we can iteratively construct the solution on each interval \( I_j \). Since there are only finitely many
intervals, we obtain a global bound as claimed. \( \square \)

Corollary 6.5. Let \( 0 \leq s < 1 \) and \( d = 4 \).
\( \textbf{I} \) There exists \( C > 0 \) and \( 0 < \theta < 1 \) such that for any \( (\text{real-valued}) \) free wave
\( v = \Re(e^{it|\nabla|} g) \in C(\mathbb{R}; L^2) \) we have
\[
\| I_0 \|_{B(N^s \to \mathcal{S}^s)} \leq C, \quad \| v \|_{B(S^\theta \to \mathcal{S}^s)} \leq C \| v \|_{L_t^\infty L_x^2}, \quad \| I_0 v \|_{B(\mathcal{S}^s \to \mathcal{S}^s)} \leq C \| v \|_{L_t^\infty L_x^2} \| v \|_{L_t^\infty B_{4/3,\infty}^1}. 
\]
\( \textbf{II} \) For any interval \( I \subset \mathbb{R} \) and any \( (\text{real-valued}) \) free wave \( v = \Re(e^{it|\nabla|} g) \in C(I; L^2) \), the Duhamel operator
\( I_0 : N^s(I) \to \mathcal{S}^s(I) \) is bounded.

Proof. The first claim follows from the energy inequality (2.9) and the bilinear estimates contained in
Theorem 3.1 and Theorem 5.1 while the second is a direct consequence of Theorem 6.4. \( \square \)

To simplify the argument to follow, we also give a restatement of Lemma 6.3 in the special case \( X = S^s \),
\( \mathcal{X} = \mathcal{S}^s \), and \( Y = N^s \).

Lemma 6.6. Let \( d = 4 \), \( t_0 \in \mathbb{R} \), \( t_0 \in I \) be an interval, and \( V = V_1 + V_2 \) with \( V_j \in L^\infty(I; L^2(\mathbb{R}^4)) \). Suppose
that we have finite numbers \( M, \varepsilon \) such that
\[
\| I_0 \|_{B(N^s \to \mathcal{S}^s)} + \| I_{V_1j} \|_{B(N^s \to \mathcal{S}^s)} + \| I_{V_2j} \|_{B(S^\theta \to \mathcal{S}^s)} \leq M, \quad \| I_{V_1} I_0 V_2 \|_{B(\mathcal{S}^s \to \mathcal{S}^s)} \leq \varepsilon, \quad (6.20)
\]
for \( j = 1, 2 \). Let \( C := \| I \|_{B(\mathcal{S}^s \to \mathcal{S}^s)} \) and \( \tilde{M} := (1 + CM^2)^2 \). If \( \tilde{M} \varepsilon < 1 \), then we have
\[
\| I_{V} \|_{B(N^s \to \mathcal{S}^s)} \leq (1 - \tilde{M} \varepsilon)^{-1} \tilde{M} \tilde{M}. \quad (6.21)
\]
In the remainder of this subsection, we give the proof of Theorem 6.4, assuming the following key properties:
\( \textbf{A1} \) If \( V^1_n \) and \( V^2_n \) are two asymptotically orthogonal profiles as in (6.17), we have
\[
\| I_0 v^1_n I_0 v^2_n \|_{B(\mathcal{S}^s \to \mathcal{S}^s)} \to 0 \quad (6.22)
\]
where \( v^j_n = \Re(V^j_n) \).
\( \textbf{A2} \) If a sequence of free waves \( \Gamma_n \) is bounded in \( L^2 \) and vanishing in \( L^\infty_t B_{4/3,\infty}^1 \), then letting \( w_n = \Re(\Gamma_n) \) we have
\[
\| I_0 w_n \|_{B(\mathcal{S}^s \to \mathcal{S}^s)} \to 0. \quad (6.23)
\]
(A3) If $B^* < \|W^2\|_2$, then the desired estimate holds in the case of single profile. More precisely, if $V^1_n$ is a profile \((6.17)\), and $v^1_n = \mathcal{R}(V^1_n)$, then
\[
\|V^1_n\|_{L^\infty_t L^2_x} = \|V^1\|_{L^\infty_t L^2_x} < \|W^2\|_2 \quad \implies \quad \sup_n \|I_{v^1_n}\|_{B(N^{* \to \infty})} < \infty. \tag{6.24}
\]

**Proof of Theorem 6.1.** Let $v := \mathcal{R}(V)$ and let $U_v(t; t_0)f$ denote the solution to the homogeneous equation
\[
(i\partial_t + \Delta)u = vu, \quad u(t_0) = f.
\]
In view of the identity
\[
U_v(t; t_0)f = I_v(ve^{i\Delta f}) - e^{i\Delta f}
\]
an application of Theorem 3.1 implies that for any free wave $V = e^{i|\nabla|g}$ we have
\[
\|U_v(t; t_0)f\|_{L^2_x} \leq \|I_v\|_{B(N^{* \to \infty})}\|Ve^{i\Delta f}\|_{L^2_x} + \|e^{i\Delta f}\|_{L^2_x} \lesssim (1 + \|I_v\|_{B(N^{* \to \infty})}\|V\|_{L^\infty_t L^2_x})\|f\|_{H^2}
\]
where the implied constant is independent of $V$. Consequently, to prove Theorem 6.1, it suffices to prove that we have a uniform bound for the Duhamel operators $I_v$. An application of Corollary 6.3 implies that \(\|I_v\|_{B(N^{* \to \infty})} < \infty\) but with a bound potentially depending on $V$. Suppose for contradiction that the uniform estimate claimed in Theorem 6.1 fails at some $s \in (0, 1)$ and $B \in (0, \|W^2\|_2)$. Then we have the threshold mass $B^* \in (0, \|W^2\|_2)$ defined by
\[
M(B) := \sup \{|I_v\|_{B(N^{* \to \infty})} | v = \mathcal{R}(V), \text{ free wave with } \|V\|_{L^\infty_t L^2_x} \leq B\},
\]
and a sequence $V_n$ of free waves satisfying
\[
\|V_n\|_{L^\infty_t L^2_x} \nearrow B^*, \quad \|I_{v_n}\|_{B(N^{* \to \infty})} \to \infty \tag{6.26}
\]
with $v_n = \mathcal{R}(V_n)$. Note that $B^* > 0$ is ensured by the small perturbation of the free case using (6.10) and the bounds (i) in Corollary 6.5.

Let \((6.19)\) be the profile decomposition of $V_n$, after passing to a subsequence if necessary, and define $v_n = \mathcal{R}(V_n)$, $v^1_n = \mathcal{R}(V^1_n)$, and $w^n_1 = \mathcal{R}(\Gamma^1_2)$. We begin by considering the case where all the $L^\infty_t L^2_x$ mass concentrates in one profile, that is when $V_n = V^1_n + \Gamma^1_2$ with
\[
\lim_{n \to \infty} \|V_n\|_{L^\infty_t L^2_x} = \|V^1\|_{L^\infty_t L^2_x}, \quad \iff \quad \lim_{n \to \infty} \|\Gamma^1_2\|_{L^\infty_t L^2_x} = 0.
\]
As we have the uniform bound (6.24) for a single profile, we have $\sup_n \|I_{v^1_n}\|_{B(N^{* \to \infty})} \leq M < \infty$. Noting the identity $I_{v_n} = I_{v^1_n} + I_{w^n_1} w^n_1 I_{v_n}$ we obtain
\[
\|I_{v_n}\|_{B(N^{* \to \infty})} \leq \|I_{v^1_n}\|_{B(N^{* \to \infty})}(1 + \|w^n_1\|_{B(S^{* \to N^*})})\|I_{v_n}\|_{B(N^{* \to \infty})}
\]
and hence Corollary 6.5 together with the fact that the error $w^n_1$ vanishes in $L^2_x$, implies that for all sufficiently large $n$ we have
\[
\|I_{v_n}\|_{B(N^{* \to \infty})} \leq M + \frac{1}{2}\|I_{v_n}\|_{B(N^{* \to \infty})} \quad \lim_{n \to \infty} \sup_n \|I_{v_n}\|_{B(N^{* \to \infty})} \leq 2M.
\]
In other words we have a contradiction to the choice of the sequence $V_n$. Thus the single profile case holds.

We now dispose of the case with no profile, in other words when $V_n = \Gamma^1_2$, hence $\|V_n\|_{L^\infty_t L^2_x} \nearrow B^*$ and $\|V_n\|_{L^\infty_t B^{-1}_4} \to 0$. We argue as in the automatic upgrading property (II). Two applications of the identity (6.39) imply that
\[
I_{v_n} = I_0(1 + v_n I_{v_n}) = I_0 + I_0 v_n I_0 + I_0 v_n I_0 v_n I_{v_n}
\]
and hence an application of the bounds in Corollary 6.5 implies that
\[
\|I_{v_n}\|_{B(N^{* \to \infty})} \leq \|I_0\|_{B(N^{* \to \infty})} + \|I_0\|_{B(N^{* \to \infty})}\|v_n\|_{B(S^{* \to N^*})} + \|I_0\|_{B(N^{* \to \infty})}\|v_n\|_{B(S^{* \to N^*})}\|I_{v_n}\|_{B(N^{* \to \infty})}
\]
\[
\leq 1 + \|I_0 v_n\|_{B(S^{* \to S^*})}\|I_{v_n}\|_{B(N^{* \to \infty})}
\]
with the implied constant independent of $n$. Therefore, we conclude via (A2) that $\lim sup_n \|I_{v_n}\|_{B(N^{* \to \infty})} < \infty$ which again contradicts the definition of the sequence $V_n$. 

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The remaining case is when the initial profile satisfies $0 < \|V\|_{L^\infty_t L^2_x} < B^*$. In this case, the asymptotic orthogonality of the profiles in $L^2$ implies that we have a uniform bound below the threshold $B^*$, namely we have some $B < B^*$ such that for any $J \geq 1$

$$\|V_n\|_{L^\infty_t L^2_x} = \|V_1\|_{L^\infty_t L^2_x} \leq B, \quad \limsup_{n \to \infty} \|V_n^{2,j}\|_{L^\infty_t L^2_x} \leq B, \quad \limsup_{n \to \infty} \|\Gamma_n^J\|_{L^\infty_t L^2_x} \leq B \quad (6.27)$$

where we introduce the notation

$$V_n^{J_1,J_2} := \sum_{j=J_1}^{J_2} V_n^j$$

with the convention that $V_n^{J_1,J_2} = 0$ if $J_1 > J_2$. The definition of the threshold $B^*$ together with (6.27) implies that there exists $M > 0$ such that for any $J \geq 1$ we have the uniform bounds

$$\limsup_{n \to \infty} \|I_0\|_{B(N^s \to S^s)} + \limsup_{n \to \infty} \|I_{v_n}\|_{B(N^s \to S^s)} + \limsup_{n \to \infty} \|I_{w_n}\|_{B(N^s \to S^s)} \leq M < \infty. \quad (6.28)$$

An application of Lemma 6.6 together with (A1) implies that for any $J \geq 1$ we have the bound

$$\limsup_{n \to \infty} \|I_{v_n}\|_{B(N^s \to S^s)} = \limsup_{n \to \infty} \|I_{v_n+\Gamma_n^J}\|_{B(N^s \to S^s)} \leq 2M(1 + M^2)^4 < \infty.$$ 

Moreover, via Corollary 6.3 we obtain

$$\|I_0^{v_n^j} I_0^{w_n^j}\|_{B(S^s \to S^s)} \leq \|I_0\|_{B(N^s \to S^s)} \|I_{v_n^j}\|_{B(S^s \to N^s)} \|I_{w_n^j}\|_{B(S^s \to S^s)} \leq \|I_0\|_{B(S^s \to S^s)}$$

where the implied constant is independent of $J$. In particular, choosing $J$ sufficiently large depending on $M$, the property (A2) together with the uniform bounds (6.28) and another application of Lemma 6.6 implies that

$$\limsup_{n \to \infty} \|I_{v_n^j+\Gamma_n^J}\|_{B(N^s \to S^s)} < \infty.$$ 

Since $V_n = V_n^{1,j} + \Gamma_n^J$, this again contradicts the choice of the sequence $V_n$. Therefore Theorem 6.1 follows. \(\Box\)

Thus we are left with (6.22) to (6.24) to be proven.

6.4. Proof of (A2): Decay for the remainder. Corollary 5.2 gives $0 < \theta < 1$ such that

$$\|I_0 w_n\|_{B(S^s \to S^s)} \lesssim \|\Gamma_n\|_{L^1_t L^2_x}^{1-\theta} \|\Gamma_n\|_{L^\infty_t L^2_x}^\theta B_{4,1}.$$ 

Hence letting $n \to \infty$, (6.23) follows. Note that the $X^{s-1,1}$ component of $S^s$ is treated easily by the product estimate

$$\|V\|_{H^{s-1}} \lesssim \|V\|_{B^{s-\delta}_{\infty,\infty}} \|u\|_{B^{2}_{2,2}}, \quad 0 < \delta < \min(s,1-s), \quad \frac{1}{2(-\delta)} = \frac{1}{2} + \frac{-\delta}{4}, \quad (6.29)$$

where the Besov norm of $V$ is controlled by interpolation between $L^2$ and $H^{-1}_4$. On the other hand, the Strichartz component of $S^s$ is treated by Theorem 5.1 which is indeed the hardest part in the entire proof of the uniform Strichartz estimate.

6.5. Proof of (A1): Decay for orthogonal profiles. Here we prove (6.22). Thanks to the uniform bounds on the operators, namely that Theorem 5.1 gives the uniform bound

$$\|I_0 v_n\|_{B(S^s \to S^s)} \lesssim \|V_n\|_{L^\infty_t L^2_x} = \|V\|_{L^\infty_t L^2_x}, \quad (6.30)$$

we have

$$\|I_0 v_n^j I_0 v_n^j\|_{B(S^s \to S^s)} \lesssim \|V_n^j\|_{L^\infty_t L^2_x} \|V_n^j\|_{L^\infty_t L^2_x} = \|V^j\|_{L^\infty_t L^2_x} \|V^j\|_{L^\infty_t L^2_x}, \quad (6.31)$$

and hence we may assume that $V^j = e^{it|\nabla|^\alpha} \varphi^j$ with the spatial profile $\varphi^j \in L^2(\mathbb{R}^4)$ belonging to any dense subset of nice functions, e.g., smooth and compactly supported in $x \in \mathbb{R}^4$ or in the Fourier space.

If the two profiles are separated by scaling, then we exploit the high-low and low-high gains. If they are separated in space-time, then we can reduce to the case $s = 0$ by the complex interpolation, and use Lemma 8.6 for space separation and the dispersive decay for time separation.

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6.5.1. Low frequency decay. Let us first dispose of expanding profiles, namely the case of $\sigma_n^j \to +0$. Since
\[
\|\sigma^2 V(\sigma t, \sigma x)\|_{L^2_t B^1_{\infty,2}} \lesssim \sigma^{1/2} \|V\|_{L^2_t B^1_{1,2}},
\] (6.32)
an application of (6.19) gives
\[
\|\mathcal{I}_0 v_n^j\|_{B(S' \to S')} \lesssim \sigma_n^{1/2} (\sigma_n) s^j \|V\|_{L^2_t B^1_{1,2}}.
\]
In particular, in view of the uniform bound (6.30), we conclude that
\[
\sigma_n^j \to +0 \implies \forall s \in (0, 1): \|\mathcal{I}_0 v_n^j\|_{B(S' \to S')} \to 0 \quad (n \to \infty)
\] (6.33)
and hence we may discard all the profiles $V_n^j$ with $\sigma_n^j \to 0$. In other words, we may assume that either $\sigma_n^j \equiv 1$ or $\sigma_n^j \to \infty$ for each $j$.

6.5.2. Scale separation. Next we consider the case of scale separation for temporary profiles, namely $\sigma_n^2 / \sigma_n^1 \to \infty$ or $\sigma_n^1 / \sigma_n^2 \to \infty$. Note that $\sigma_n^j \geq 1$ by the previous section. An application of Corollary (5.1) together with Theorem (5.1) implies that for every $0 \leq s < 1$ there exists $\epsilon > 0$ such that
\[
\|\mathcal{I}_0 v_n^1 \mathcal{I}_0 v_n^2\|_{B(S' \to S')} \lesssim \min \left\{ \|V_n^1\|_{L^2_t H^s_z} \|V_n^2\|_{L^2_t H^s_z}, \|V_n^1\|_{L^\infty_t H^{-s}_z} \|V_n^2\|_{L^\infty_t H^{-s}_z} \right\}.
\]
Since
\[
\min \left\{ \|V_n^1\|_{L^2_t H^s_z} \|V_n^2\|_{L^2_t H^s_z}, \|V_n^1\|_{L^\infty_t H^{-s}_z} \|V_n^2\|_{L^\infty_t H^{-s}_z} \right\} \lesssim \left( \min \left\{ \frac{\sigma_1^j}{\sigma_n^1}, \frac{\sigma_2^j}{\sigma_n^2} \right\} \right)^s
\]
we conclude that
\[
|\log(\sigma_n^1 / \sigma_n^2)| \to \infty \quad \implies \quad \|\mathcal{I}_0 v_n^1 \mathcal{I}_0 v_n^2\|_{B(S' \to S')} \to 0.
\]
Thus we finish the case of scale separation.

6.5.3. Space-time separation. It remains to consider the case $\sigma_n^1 = \sigma_n^2 = \sigma_n$. Here we approximate the wave profile by decomposition in time: For any $\varphi \in L^2$ and $\epsilon \in (0, 1)$, there is $\chi \in \mathcal{S}(\mathbb{R})$ such that
\[
\text{supp } \mathcal{F} \chi \subset [-\epsilon, \epsilon], \quad |\chi| \leq 1, \quad \|(1 - \chi(t))e^{it|\nabla|} \varphi\|_{L^\infty_t H^{-s}_z} < \varepsilon.
\] (6.34)
Since $\epsilon < 1$, we have
\[
\text{supp } \mathcal{F} \chi(t)e^{\pm it|\nabla|} \varphi = \text{supp } \chi(t) * \delta(t \pm |\xi|) \hat{\varphi}(\xi) \subset \{|\tau| \leq |\xi| + 1\}
\] (6.35)
and the same inclusion for $\mathcal{F} \chi(t)[(1 - \chi(t))e^{\pm it|\nabla|} \varphi]$, so that we can keep using the same non-resonance property for the free waves after the cut-off by $\chi(t)$. In particular, for any free wave $v = \Re(V) = \Re(e^{it|\nabla|} \varphi)$, an application of (3.3) implies that
\[
\sup_{n} \sup_{\epsilon < 1} \| \mathcal{I}_0 \chi(t)v\|_{B(S' \to S')} \lesssim \|V\|_{L^\infty_t L^2_x}
\] (6.36)
while Theorem (5.1) gives $\theta > 0$ such that
\[
\sup_{n} \|\mathcal{I}_0 (1 - \chi(t))v\|_{B(S' \to S')} \lesssim \|(1 - \chi(t))V\|_{L^\theta_t B^{-1}_{\infty,\infty}} \|V\|_{L^\theta_t L^2_x}.
\] (6.37)
Consequently, decomposing the profiles $V^j = \chi V^j + (1 - \chi) V^j$ using the properties (6.34) imply that it suffices to show that
\[
\|\mathcal{I}_0 [\mathcal{I}_0 \chi v^j] \mathcal{I}_0 [\mathcal{I}_0 \chi v^{2j}]\|_{B(S' \to S')} \to 0 \quad (n \to \infty)
\]
where as above, $v^j = \Re(V^j)$. This can be reduced further after observing that another application of Theorem (5.1) implies that we have the high-low gain
\[
\|P_{\lambda_0} \mathcal{I}_0 [\mathcal{I}_0 \chi v^j] \mathcal{I}_0 [\mathcal{I}_0 \chi v^{2j}]\|_{S'} \lesssim \left( \frac{\min \{\lambda_0, \lambda_1\}}{\max \{\lambda_0, \lambda_1\}} \right)^{\theta} \|\chi\|_{L^\infty_t \|V^j\|_{L^\infty_t L^2_x}} \|V^j\|_{L^\infty_t L^2_x} \|u_{\lambda_1}\|_{S'_{\lambda_1}}
\]
and hence it is enough to prove that
\[
\|\mathcal{I}_0 \chi v^j \mathcal{I}_0 [\mathcal{I}_0 \chi v^{2j}]\|_{B(S' \to S')} \to 0 \quad (n \to \infty).
\] (6.38)
Since $S' \subset X_0$ and $Y_0 \subset N^0$, we may replace the operator norm with $B(X_0 \to Y_0)$. Then by the scaling invariance for $u(t, x) \mapsto \lambda^{d/2} u(\lambda^2 t, \lambda x)$, it suffices to show that
\[
\|\mathcal{I}_0 \chi v^j \mathcal{I}_0 [\mathcal{I}_0 \chi v^{2j}]\|_{B(X_0 \to Y_0)} \to 0,
\] (6.39)
where \( \tilde{g}_n \) is defined by
\[ \tilde{g}_n^j V(t, x) := V(t/\sigma_n - t_n^j, x - x_n^j). \] 

(6.40)

Now we use approximation by step functions in time:
\[ \chi(t)\mathcal{E}^{t/|\nabla|}\varphi(x) = \sum_{k=1}^K I_k(t)\varphi_k(x) + R(t, x), \]

(6.41)
such that \( I_1, \ldots, I_K \) are mutually disjoint bounded intervals, \( \varphi_k \in L^2(\mathbb{R}^4) \) and \( \|R\|_{L^\infty_t L^2} < \varepsilon \). Since \( \mathcal{I}_0 : Y_0 \to X_0 \) and \( \|F\|_{B(X_0 \to Y_0)} \lesssim \|F\|_{L^\infty_t L^2} \), the decomposition (6.41) yields
\[ \|\mathcal{I}_0^1 \tilde{g}_n^j \varphi(x)\|_{B(X_0 \to Y_0)} \lesssim \sum_{k, j} \|\mathcal{I}_0^1 \tilde{g}_n^j \varphi_k(x)\|_{B(X_0 \to Y_0)} + O(\varepsilon), \]

(6.42)
so that we can further reduce to
\[ T_n^j(t, x) := \mathcal{I}_0^1 \tilde{g}_n^j \varphi(x) \implies \|T_n^1 \mathcal{I}_0^1 T_n^2 u\|_{B(X_0 \to Y_0)} \to 0 \quad (n \to \infty), \]

(6.43)
for any bounded intervals \( I_1, I_2 \) and any \( \varphi_1, \varphi_2 \in L^2 \).

In the case of time separation \( |t_n^1 - t_n^2| \to \infty \), we have \( |t_n^1 - t_n^2| > 2(\sup I_1 + \sup I_2) \) for large \( n \), then the dispersive decay estimate yields
\[ \|T_n^1 (t) \mathcal{I}_0^1 T_n^2 u\|_{L^2} \lesssim \|\varphi\|_{L^2} \int |t - s|^{-d/2} I_1(t/\sigma_n - t_n^1) I_2(s/\sigma_n - t_n^2) \|\varphi\|_2 \|u(s)\|_2 ds \lesssim \|\varphi\|_{L^2} \|\varphi\|_1 \|u\|_{L^\infty_t L^2} \|t_n^1 - t_n^2\|^{1 - d/2} \|t_n^1 - t_n^2\|^{1 - d/2} I_1(t/\sigma_n - t_n^1). \]

(6.44)
Hence, using \( d \geq 3 \) and \( \sigma_n \geq 1 \),
\[ \|T_n^1 (t) \mathcal{I}_0^1 T_n^2 u\|_{B(L^\infty_t L^2 \to L^2)} \lesssim \|\varphi\|_{L^2} \|\varphi\|_1 \|t_n^1 - t_n^2\|^{1 - d/2} \sigma_n^{(3 - d)/2} \to 0. \]

(6.45)
In the case of space separation \( |x_n^1 - x_n^2| \to \infty \), Lemma 5.4 yields, after discarding the time restriction by intervals,
\[ \|T_n^1 (t) \mathcal{I}_0^1 T_n^2 u\|_{B(L^\infty_t L^2 \to L^2)} \leq \|\varphi^1 (x - x_n^1) \mathcal{I}_0^2 \varphi^2 (x - x_n^2)\|_{B(L^\infty_t L^2 \to L^2)} \to 0. \]

(6.46)
Therefore, we conclude that
\[ \lim_{n \to \infty} \inf \|\mathcal{I}_0^1 \tilde{g}_n^1 \varphi(x)\|_{B(S^0 \to N^0)} \lesssim \varepsilon. \]

(6.47)
Sending \( \varepsilon \to 0 \), we obtain (6.33) as required. This completes the proof of (6.22).

6.6. Proof of (A3): The uniform Strichartz in the single profile case. The strategy to prove the single profile case is similar to that used in the proof of the radial case [10]. We use a weight in the frequency for \( H^s \) adapted to the profile, so that we can exploit the high-low gains while dealing with the main terms of the same frequency as in the \( L^2 \) case, \( s = 0 \) case, where we obtain the uniform estimate (with respect to concentration of the profile) by approximating the wave potential with a step function in time, thereby reducing the estimate to the static potential case. Having only one profile makes the first step simpler.

Let \( v_n = \mathcal{R}(V_n) \) be a sequence in the case of a single profile, namely
\[ V_n = g_n V = (\sigma_n)^2 V(\sigma_n t, \sigma_n x), \quad \|V\|_{L^\infty_t L^2} \ll \|W\|_2, \]

(6.48)
where \( V \) is a free wave (independent of \( n \)), and the translation parameters are removed, since the desired estimate is obviously translation invariant.

The goal of this subsection is to prove
\[ \lim_{n \to \infty} \|\mathcal{I}_{V_n}\|_{B(N^s \to S^s)} < \infty \]

(6.49)
for \( 0 \leq s < 1 \). Since the uniform bound is trivial if \( \sigma_n = 1 \), we may also assume \( \sigma_n \to \infty \). Moreover, since the estimates are stable under small perturbations in \( L^\infty_t L^2 \), (6.10) together with a density argument implies we may assume that \( V = e^{it|\nabla|} \hat{g} \) with \( \hat{g} \in C^\infty(\mathbb{R}^4) \) and
\[ \xi \in \text{supp} \hat{g} \implies 1/\beta < |\xi| < \beta \]

(6.50)
for some $\beta > 1$. We fix this parameter $\beta > 1$ large enough so that the following arguments work. Following \[10\], we introduce a sequence of weight functions $w_n \colon (0, \infty) \to (0, \infty)$ adapted to the frequencies $\sigma_n \to \infty$ as follows. With a large parameter $\beta > 1$ to be fixed, let $\varepsilon_n := (\log \beta^2) / (\log(\sigma_n/\beta^2)) \to +0$ and
\[
w_n(r) = \begin{cases}
1 & (0 < r \leq 1),
\frac{r^{1+\varepsilon_n}}{\beta^2} & (1 \leq r \leq \sigma_n/\beta^2),
\frac{r}{\beta^2} & (\sigma_n/\beta^2 \leq r \leq \sigma_n \beta^2),
\sigma_n & (\sigma_n \beta^2 \leq r < \infty).
\end{cases}
\] (6.51)

Then $w_n$ is continuous, increasing, and piecewise logarithmic-linear,
\[
r/\beta^2 \leq w_n(r) \leq \beta^2 r, \quad \lim_{n \to \infty} w_n(r)/r = 1.
\] (6.52)

Moreover, by \[10\] Lemma 5.1, for any $0 < s < s'$, provided we choose $n \in \mathbb{N}$ sufficiently large, we have for every $0 < r < r'$ the key inequality
\[
w_n^s(r) \leq w_n^s(r') \leq w_n^s(r) \left( \frac{r'}{r} \right)^{s'}.
\] (6.53)

The weight $w_n(r)$ is needed to reduce to the case $s = 0$.

**Lemma 6.7.** For $V_n$, as above, suppose that (6.49) holds for $s = 0$. Then it holds also for $0 \leq s < 1$.

**Proof.** In the following argument, we often omit writing explicitly the dependence on $n$. Fix any $0 < s < 1$ and any sequence $F_n \in \mathcal{N}^s$ with $\|F_n\|_{\mathcal{N}^s} \leq 1$. Let $u = u_n = \mathcal{I}_{\sigma_n}F_n$. In view of (6.52) it suffices to prove that
\[
\|\mathcal{I}^{s}u\|_{\mathcal{N}^0} := \left( \sum_{\lambda \in 2^\mathbb{N}} w(\lambda)2^{s\lambda} \|\mathcal{I}^{s}u\|_{\mathcal{N}^0}^2 \right)^{\frac{1}{2}} \lesssim \left( \sum_{\lambda \in 2^\mathbb{N}} w(\lambda)2^{s\lambda} \|\mathcal{I}^{s}F\|_{\mathcal{N}^0}^2 \right)^{\frac{1}{2}} \equiv \|\mathcal{I}^{s}F\|_{\mathcal{N}^0}
\] (6.54)

where the implied constant is independent of $n$. We decompose $u$ into the frequencies
\[
u_n = u_{\leq \sigma_n/\beta^2} + u_{\sigma_n/\beta^2 < \sigma_n \beta^2} + u_{> \sigma_n \beta^2} =: u_L + u_M + u_H.
\] (6.55)

Suppose for the moment that there exists $\delta > 0$ such that for all sufficiently large $n \in \mathbb{N}$ we have
\[
\|\mathcal{I}^{s}(v_nu_L)\|_{\mathcal{N}^0} + \|\mathcal{I}^{s}(v_nu_H)\|_{\mathcal{N}^0} \lesssim \beta^{-\delta} \|\mathcal{I}^{s}v_n\|_{L^\infty L^2} \|\mathcal{I}^{s}u\|_{\mathcal{N}^0}
\] (6.56)
\[
\|\mathcal{I}^{s}(v_nu_L)\|_{\mathcal{N}^0} \lesssim \beta^{-\delta} \|\mathcal{I}^{s}v_n\|_{L^\infty L^2} \|\mathcal{I}^{s}u\|_{\mathcal{N}^0}
\] (6.57)
\[
\|\mathcal{I}^{s}(v_nu_H)\|_{\mathcal{N}^0} \lesssim \|\mathcal{I}^{s}v_n\|_{L^\infty L^2} \|\mathcal{I}^{s}u\|_{\mathcal{N}^0} + \beta^{-\delta} \|\mathcal{I}^{s}v_n\|_{L^\infty L^2} \|\mathcal{I}^{s}u\|_{\mathcal{N}^0}
\] (6.58)

where the implied constant is independent of $n$. To bound the low frequency contribution $u_L$, we note that $u_L = \mathcal{I}_0([F_n + v_n]u_L)$ and hence an application of Lemma 2.2 together with (6.56) implies that
\[
\|\mathcal{I}^{s}u_L\|_{\mathcal{N}^0} \lesssim \|\mathcal{I}^{s}F_n + v_nu_L\|_{\mathcal{N}^0} \lesssim \|\mathcal{I}^{s}F_n\|_{\mathcal{N}^0} + \beta^{-\delta} \|\mathcal{I}^{s}v_n\|_{L^\infty L^2} \|\mathcal{I}^{s}u\|_{\mathcal{N}^0}.
\]

Similarly, to bound the high frequency contribution, we write $u_H = \mathcal{I}_0([F_n + v_n]u_H)$ and again apply Lemma 2.2 together with (6.58)
\[
\|\mathcal{I}^{s}u_H\|_{\mathcal{N}^0} \lesssim \|\mathcal{I}^{s}(F_n + v_nu_H)\|_{\mathcal{N}^0} \lesssim (1 + \|\mathcal{I}^{s}F_n\|_{L^\infty L^2}) \|\mathcal{I}^{s}F_n\|_{\mathcal{N}^0} + \beta^{-\delta} \|\mathcal{I}^{s}v_n\|_{L^\infty L^2} \|\mathcal{I}^{s}u\|_{\mathcal{N}^0}.
\]

Finally, to bound the remaining medium frequency contribution $u_M$, we apply the assumed uniform Strichartz bound. Let $C_S > 0$ be the best uniform Strichartz constant at $s = 0$, namely
\[
C_S := \sup_n \|\mathcal{I}_{\sigma_n}\|_{B(\mathcal{N}^0 \to \mathcal{S}^0)}.
\] (6.59)

Since $u_M$ satisfies equation
\[
id_t u_M + \Delta u_M - v_n u_M = (v_n u_n + F) M - v_n u_M = F_M + v_n u_H + (v_n u_L)_M - (v_n u_M)_L
\]
applying the uniform Strichartz estimate, the bounds (6.56), (6.57), and (6.58), the bound (6.53) and the Fourier support assumption on \( V_n \), we conclude that

\[
\|w^s u_M\|_{S_0} = \sigma_n^s \|u_M\|_{S_0} \\
\leq C_S \sigma_n^s (\|F_M\|_{N_0} + \|v_n u_H\|_{N_0} + \|(v_n u)_H\|_{N_0} + \|(v_n u_L)_{LM}\|_{N_0} + \|(v_n u_M)_{LM}\|_{N_0}) \\
\lesssim C_S (\|w^s F\|_{N_0} + \|w^s (v_n u_H)\|_{N_0} + \|w^s (v_n u)_H\|_{N_0} + \|w^s (v_n u_L)\|_{N_0} + \|(v_n u^s \sigma_n u_M)_{LM}\|_{N_0}) \\
\lesssim C_S (1 + \|V_n\|_L^\infty L^2_\sigma) \|w^s F\|_{N_0} + \beta^{-\delta} C_S \|V_n\|_L^\infty L^2_\sigma \|w^s u\|_{S_0}
\]

where again the implied constant is independent of \( n \). Therefore, since \( \sup_n \|V_n\|_L^\infty L^2_\sigma < \|W^2\|_{L^2} \), provided we choose \( \beta > 1 \) sufficiently large so that \( \beta^{-\delta} C_S \ll 1 \), the required bound (6.54) follows.

It only remains to prove the inequalities (6.56), (6.57), and (6.58). Note that all the interaction terms on the left have frequency gaps between the high and low frequencies of lower ratio bound \( \beta \): \( v_n u_L \) is high-low to high, \( (v_n u)_{LM} \) is high-high to low, while \( v_n u_H \) and \( (v_n u)_{H} \) are low-high to high. This frequency gap is exploited via the high-low gain in Theorem 5.1 to give the small factor \( \beta^{-\delta} \). To prove (6.56), we begin by noting that the Fourier support assumption on \( V_n \), together with Theorem 5.1 and (6.58) implies that

\[
\|w^s (v_n u_L)\|_{N_0} \lesssim \sum_{\lambda \leq \frac{\mu}{\beta}} \sum_{\mu \leq \frac{\lambda}{\beta}} w^s (\lambda) \|P_\lambda (v_n u_{\mu})\|_{N_0} \\
\lesssim \sum_{\lambda \leq \frac{\mu}{\beta}} \sum_{\mu \leq \frac{\lambda}{\beta}} w^s (\lambda) \left( \frac{\mu}{\lambda} \right)^{\delta+2\delta} \|V_n\|_{L^\infty L_2^\sigma} \|u_{\mu}\|_{S_0} \\
\lesssim \|V_n\|_{L^\infty L_2^\sigma} \|w^s u\|_{S_0} \sum_{\lambda \leq \frac{\mu}{\beta}} \sum_{\mu \leq \frac{\lambda}{\beta}} \left( \frac{\mu}{\lambda} \right)^{\delta} \approx \beta^{-\delta} \|V_n\|_{L^\infty L_2^\sigma} \|w^s u\|_{S_0}.
\]

Similarly,

\[
\|w^s (v_n u)\|_{N_0} \lesssim \sum_{\lambda \leq \frac{\mu}{\beta}} \sum_{\mu \leq \frac{\lambda}{\beta}} w^s (\lambda) \|P_\lambda (v_n u_{\mu})\|_{N_0} \\
\lesssim \sum_{\lambda \leq \frac{\mu}{\beta}} \sum_{\mu \leq \frac{\lambda}{\beta}} w^s (\lambda) \left( \frac{\lambda}{\mu} \right)^{2\delta} \|V_n\|_{L^\infty L_2^\sigma} \|u_{\mu}\|_{S_0} \\
\lesssim \|V_n\|_{L^\infty L_2^\sigma} \|w^s u\|_{S_0} \sum_{\lambda \leq \frac{\mu}{\beta}} \sum_{\mu \leq \frac{\lambda}{\beta}} \left( \frac{\lambda}{\mu} \right)^{\delta} \approx \beta^{-\delta} \|V_n\|_{L^\infty L_2^\sigma} \|w^s u\|_{S_0}.
\]

Thus (6.56) follows. The proof of (6.57) is identical. Finally, the proof of the remaining bound (6.58) requires a little more work to obtain the gain of \( \beta^{-\delta} \) due to the fact we only have a low-high to high gain in Theorem 5.1 when bounding the Duhamel operator \( I_0 \) in \( S^s \). To deal with this technical issue, one option is to simply plug in the equation once more, and exploit the fact that since the potential \( V_n \) has very low frequencies compared to \( u_H \), the Fourier support condition is essentially preserved. To make this precise, we take

\[ u_{\tilde{H}} = u_{\sigma_n \beta^2}. \]

Thus \( u_{\tilde{H}} \) is a slight widening of the Fourier support of \( u_H \). An application of Theorem 5.1 together with the Fourier support condition on \( V_n \) and (6.53) gives

\[
\|w^s (vu_H)\|_{N_0} + \|w^s (v_n u)_{H}\|_{N_0} \lesssim \left( \sum_{\mu \geq \frac{\lambda_2 \beta^2}{\sigma_n \beta^2}} w^{2\lambda} (\mu) \sum_{\mu \leq \frac{\lambda}{\beta}} \|P_\lambda (v_n u_{\mu})\|_{S_0} \right)^{\frac{1}{2}} \\
\lesssim \|V_n\|_{L^\infty L_2^\sigma} \left( \sum_{\mu \geq \frac{\lambda_2 \beta^2}{\sigma_n \beta^2}} w^{2\lambda} (\mu) \|u_{\mu}\|_{S_0} \right)^{\frac{1}{2}} \\
\approx \|V_n\|_{L^\infty L_2^\sigma} \|w^s u_{\tilde{H}}\|_{S_0}.
\]
On the other hand, since \( u_H = \mathcal{I}_0[(F + v_n)u_H] \), again using the Fourier support assumption on \( V_n \) together with Theorem 5.1 and Lemma 2.2 (to deal with the forcing term \( F_n \)), we see that provided \( \beta \gg 1 \)
\[
\|w^* u_H\|_{S^0} \lesssim \|w^* F_n\|_{N^0} + \sum_{\frac{1}{2} \sigma_n \leq \mu \leq \sigma_n} \sum_{\frac{1}{2} \sigma_n \leq \lambda \leq 4 \lambda} \|w^*(\lambda)\|_{J_0^0} \|\mathcal{I}_0 [P_{\lambda n}(v_n u)]\|_{S^0}
\lesssim \|w^* F_n\|_{N^0} + \|v_n\|_{L^p L^2_x} \sum_{\frac{1}{2} \sigma_n \leq \mu \leq \sigma_n} \sum_{\frac{1}{2} \sigma_n \leq \lambda \leq 4 \lambda} w^*(\lambda) \left( \frac{\lambda}{\delta} \right)^{1} \|u_{\lambda}\|_{\mathcal{S}^0}
\lesssim \|w^* F_n\|_{N^0} + \beta^{-\delta} \|v_n\|_{L^p L^2_x} \|w^* u\|_{S^0}.
\]
Therefore (6.58) follows. \( \square \)

Thus we are left with the proof of the case \( s = 0 \). We start by proving a decomposability lemma.

**Lemma 6.8.** There exists a constant \( C > 0 \) such that for any \( t_0 \in \mathbb{R} \) and any partition \( \mathbb{R} = \bigcup_{j=1}^N I_j \) and any free wave \( v = \Re(e^{it|\nabla|^2} g) \in L^\infty_t L^2_x \) we have
\[
\|\mathcal{I}_v\|_{B(N^0 \to S^0)} \leq C (1 + \|v\|_{L^p L^2_x})^2 \sum_{j=1}^N \|\mathcal{I}_v\|_{B(Y_0(I_j) \to X_0(I_j))}
\]
where we define
\[
\|\mathcal{I}_v\|_{B(Y_0(I_j) \to X_0(I_j))} = \sup_{\|F\|_{Y_0(I_j)} \leq 1} \|\mathcal{I}_v [\mathbb{I}_I F]\|_{X_0(I_j)}.
\]

**Proof.** By the automatic upgrading (6.11), together with the boundedness \( Y_0 \subset N^0, \mathcal{S}^0 \subset X_0, \mathcal{I}_0 : N^0 \to \mathcal{S}^0 \) and the boundedness of the multiplication operator \( v : X_0 \to Y_0 \), it suffices to prove
\[
\|\mathcal{I}_v\|_{B(Y_0 \to X_0)} \lesssim 3 \|C_j\|_{S} \quad (6.60)
\]
where we define \( C_j := \|\mathcal{I}_v\|_{B(Y_0(I_j) \to X_0(I_j))} \). By translation invariance, we may assume that \( t_0 = 0 \) and label the partition as \( I_j = [T_{j-1}, T_j] \). The bound \( \|\mathcal{I}_v\|_{B(Y_0(J) \to X_0(J))} \leq \|\mathcal{I}_v\|_{B(Y_0(I) \to X_0(I))} \) for any intervals \( J \subset I \) implies that we may freely add additional points to the partition, in particular, we may assume that \( I_{N_0} = [0, T_{N_0}) \) for some \( 1 < N_0 \leq N \) (i.e. \( t_0 = 0 \) belongs to the partition). Before proceeding further, let us recall the \( TT^* \) argument on the interval \( I_j \). Let \( (T \varphi)(t) := \mathcal{U}_v(t; 0) \varphi \) on \( t \in I_j \). Then using the unitary property of \( \mathcal{U}_v \) we have (here we use the standard convention that \( \int_a^b = \int_a^b \) if \( a < b \))
\[
T^* f = \int_{I_j} \mathcal{U}_v(0; s) f(s) ds, \quad TT^* f = \int_{I_j} \mathcal{U}_v(t; s) f(s) ds = \int_0^{T_j} \mathcal{U}_v(t; s) \mathbb{1}_{I_j}(s) f(s) ds,
\]
and
\[
\|T^* f\|_{L^2_x}^2 = \int_{I_j} \left\langle \mathcal{U}_v(t; s) (\mathbb{1}_{I_j}(s)) f(t) \right\rangle ds + \int_{I_j} \left\langle f(s) \mathcal{U}_v(t; s) (\mathbb{1}_{I_j}(t)) f(t) \right\rangle ds dt,
\]
and consequently for any \( j = 1, \ldots, N \)
\[
\|T^* f\|_{L^2_x}^2 = \|T^* f\|_{B(Y_0(I_j) \to L^2_x)} \leq 2 \|\mathcal{I}_v\|_{B(Y_0(I_j) \to Y_0(I_j))} = 2C_j,
\]
where \( Y_0' = L^2_y L^2_x \) is the dual space. Therefore, unpacking the definition of the operator \( T \), we see that
\[
\left\| \int_{I_j} \mathcal{U}_v(0; s) f(s) ds \right\|_{L^2_x} \leq (2C_j)^{\frac{1}{2}} \|f\|_{Y_0(I_j)}, \quad \|\mathcal{U}_v(t; 0) \varphi\|_{Y_0'(I_j)} \leq (2C_j)^{\frac{1}{2}} \|\varphi\|_{L^2_x}.
\]
Take any \( f \in Y_0 \) and let \( u \) be the solution of
\[
(i \partial_t + \Delta - v) u = f, \quad u(t_0) = 0
\]
and for \( j = 1, \ldots, N \) let \( u_j \) be the solution to
\[
(i \partial_t + \Delta - v) u_j = \mathbb{1}_{I_j} f, \quad u_j(t_0) = 0.
\]
Clearly we can write \( u = \sum_{j=1}^{N} u_j \) and \( u_j(t) = 0 \) if either \( T_{j-1} \geq 0 \) and \( t \leq T_{j-1} \), or \( T_{j} \leq 0 \) and \( t \geq T_{j} \). On the other hand, if \( t > T_{j} > 0 \) or \( t \leq T_{j-1} < 0 \) then we have the identity

\[
    u_j(t) = \mathcal{U}_e(t; 0) \int_{T_{j-1}}^{T_j} \mathcal{U}_e(0; s) f(s) ds.
\]

Therefore, the fact that \( \mathcal{U}_e \) is a unitary operator, together with an application of \eqref{6.61} gives the bounds

\[
    \|u_j\|_{L^\infty_t L^2_x} = \|u_j\|_{L^\infty_t L^2_x(\mathbb{R}^+ \times \mathbb{R}^4)} \leq C_j \|f\|_{Y_0(I_j)}
\]

and

\[
    \|u_j\|_{Y_0^1(I_j)} = \begin{cases} 
    2C_j^2 C_k^2 \|f\|_{Y_0(I_j)} & (k > j \geq N_0 \text{ or } k < j < N_0), \\
    C_j \|f\|_{Y_0(I_j)} & (k = j), \\
    0 & \text{otherwise}.
\end{cases}
\]

Hence, using the \( L^2 \) structure in \( Y_0 \) and \( Y_0' \), we obtain

\[
    \|u_j\|_{X_0} \leq C_j \|f\|_{Y_0(I_j)} + 2 \left( \sum_k \sum_{j} C_j C_k \|f\|_{Y_0(I_j)}^2 \right)^{1/2} \leq 3 \|f\|_{Y_0(I_j)} \left( \sum_j \|f\|_{Y_0(I_j)}^2 \right)^{1/2} = 3 \|f\|_{Y_0(I_j)}.
\]

and thus summing up over \( j \) we conclude that

\[
    \|u\|_{X_0} \leq \sum_j \|u_j\|_{X_0} \leq 3 \|f\|_{X_0} \left( \sum_j \|f\|_{Y_0(I_j)}^2 \right)^{1/2} = 3 \|f\|_{X_0} \|f\|_{Y_0}.
\]

\[
\Box
\]

**Proof of \eqref{6.49} for \( s = 0 \):** Fix the profile \( V = e^{it|\nabla|} g \) with \( \|g\|_{L^2_x} < \|W^2\|_{L^2_x} \), and let \( v = \Re(V) \). The argument proceeds in the following steps:

(i) Choose a nice partition \( \mathbb{R} = \bigcup_{j=0}^{N} I_j \) depending only on \( V \).

(ii) Let \( \sigma_n \to \infty \) be a sequence of positive real numbers, and define \( v_n = \sigma_n^2 v(\sigma_n t, \sigma_n x) \) and \( I_n^0 = \{ \sigma_n t \in I_j \} \). Show that the choice of partition in the first step implies that

\[
\limsup_{n \to \infty} \|\mathcal{I}_{v_n}\|_{B(Y_0(I_j^1) \to X_0(I_j^1))} < \infty.
\]

\[
(6.62)
\]

(iii) Apply Lemma \ref{6.8} to conclude that

\[
\limsup_{n \to \infty} \|\mathcal{I}_{v_n}\|_{B(N^0 \to X_0)} < \infty.
\]

**Step (i): Construction of the partition \( I_j \):** Let \( \epsilon > 0 \) be a small constant, eventually it will be taken to be smaller than various universal constants, and in particular, \( \epsilon \) will be independent of \( v \) (and \( n \)). We first choose \( I_0 = (-\infty, -T) \) and \( I_N = (T, \infty) \) where \( T > \epsilon^{-1} \) satisfies

\[
\|v\|_{L^\infty_t B^0_{2,\infty}((2T(t) > T) \times \mathbb{R}^4)} \leq \epsilon.
\]

This is always possible by breaking \( v \) into a small part, and a \( C_0^\infty \) part, and using the dispersive decay of the free wave. The assumption that we are below the ground state implies that for every \( t^* \in [-T, T] \) we have by \eqref{1.5}

\[
\|\mathcal{I}_{v(t^*)}\|_{B(Y_0 \to X_0)} < \infty.
\]

On the other hand, by \eqref{6.10} and the \( L^2 \) continuity of the free wave \( v \), for every \( t^* \in [-T, T] \) there exists \( \delta > 0 \) such that

\[
\sup_{|t^* - t| < \delta} \|\mathcal{I}_{v(t^*)}\|_{B(Y_0 \to X_0)} < \infty.
\]

By compactness, we then conclude that

\[
C_* = \sup_{t^* \in [-T, T]} \|\mathcal{I}_{v(t^*)}\|_{B(Y_0 \to X_0)} < \infty.
\]

We now choose the (bounded) intervals \( I_j = [t_j, t_{j+1}] \) for \( j = 1, \ldots, N - 1 \) such that \( -T = t_1 < \cdots < t_N = T \) and

\[
\sup_{t \in I_j} \|v(t) - v(t_j)\|_{L^2_x} \leq \epsilon C_*^{-1}.
\]
Step (ii): The localised bounds. There are two distinct cases to consider, the unbounded intervals $I_0$ and $I_N$, in which the operator $I_\nu$ is a perturbation of the free case $I_0$, and the bounded intervals $I_j$ for $j = 1, \ldots, N - 1$ where $I_\nu$ is a small perturbation of the stationary case $I_{\nu(t_j)}$. We start with the more involved case of the unbounded intervals $I_0$ and $I_N$. Choose a smooth cutoffs $\rho, \chi \in C^\infty$ such that

$$\rho = \begin{cases} 
1 & \text{if } |t| > T \\
0 & \text{if } |t| < T/2,
\end{cases} \quad \text{supp } \chi \subset \{|\tau| \leq 1\}, \quad \|\rho - \chi\|_L^\infty(\mathbb{R}) \lesssim T^{-1} \lesssim \epsilon. $$

Expanding the Duhamel formula twice, gives the identity

$$u := I_{(\rho v)_n} F = I_0 F + I_0[(\rho v)_n u] = I_0 F + I_0[(\rho v)_n I_0 F] + I_0[(\rho v)_n I_0((\rho v)_n u)] \quad (6.63)$$

where $(\rho v)_n(t, x) = \rho(\sigma_n t)\sigma_n^{\theta} v(\sigma_n t, \sigma_n x)$. An application of Theorem 5.1 together with Hölder’s inequality and the usual square function estimate gives

$$\|I_0[(\chi v)_n I_0((\rho v)_n u)]\|_{X_0} \lesssim \|I_0[(\chi v)_n I_0((\rho v)_n u)]\|_{S^0} \lesssim \|\chi v\|_{L^\infty B^{\frac{1}{2} - \frac{1}{2} -}} \|\chi v\|_{L^\infty B^{\frac{1}{2} - \frac{1}{2} -}^\infty} \|v\|_{L^2} \|I_0[(\rho v)_n u]\|_{S^0} \lesssim \|\chi v\|_{L^\infty B^{\frac{1}{2} - \frac{1}{2} -}} \|\chi v\|_{L^\infty B^{\frac{1}{2} - \frac{1}{2} -}^\infty} \|v\|_{L^2} \|\rho\|_{L^\infty L^2} \|u\|_{X_0},$$

where the last line followed by rescaling. On the other hand simply applying Hölder’s inequality and the free Strichartz estimate for $I_0$ implies that

$$\|I_0[((\rho - \chi) v)_n I_0((\rho v)_n u)]\|_{X_0} \lesssim \|((\rho - \chi) v)_n\|_{L^\infty L^2} \|I_0((\rho v)_n u)\|_{X_0} \lesssim \|\rho - \chi\|_{L^\infty} \|v\|_{L^2}^2 \|\rho\|_{L^2} \|u\|_{X_0}.$$

In view of our assumptions on $T$, $v$, and the cutoffs $\chi$ and $\rho$, we conclude that

$$\|\chi v\|_{L^\infty B^{\frac{1}{2} - \frac{1}{2} -}} \lesssim \|\rho v\|_{L^\infty B^{\frac{1}{2} - \frac{1}{2} -}} + \|(\chi - \rho) v\|_{L^\infty L^2} \lesssim \epsilon$$

and thus combining the above bounds we obtain

$$\|I_0[(\rho v)_n I_0((\rho v)_n u)]\|_{X_0} \lesssim \|I_0[((\rho - \chi) v)_n I_0((\rho v)_n u)]\|_{X_0} + \|I_0[(\chi v)_n I_0((\rho v)_n u)]\|_{X_0} \lesssim \epsilon^2 \|u\|_{X_0}$$

with the implied constant depending only on $\|v\|_{L^\infty L^2}$, and in particular is independent of $v$, $n$, and the choice of cutoffs $\rho$ and $\eta$. The identity (6.63) then gives

$$\|u\|_{X_0} \lesssim \|F\|_{Y_0} + \|(\rho v)_n I_0 F\|_{Y_0} + \|I_0[(\rho v)_n I_0((\rho v)_n u)]\|_{X_0} \lesssim \|F\|_{Y_0} + \|\rho\|_{L^\infty} \|v\|_{L^2} \|F\|_{Y_0} + \epsilon^\theta \|u\|_{X_0} \lesssim \|F\|_{Y_0} + \epsilon^\theta \|u\|_{X_0}$$

and therefore, provided that $\epsilon > 0$ is sufficiently small (this choice is independent $T$, $n$, the cutoffs $\rho$ and $\eta$, and $v$), we conclude that

$$\limsup_{n \to \infty} \|I_{(\rho v)_n} F\|_{X_0} \lesssim \|F\|_{Y_0}.$$

Since

$$\|I_{v_n} B(\nu_0(t_j^0) \to \nu_0(t_j^0))\| + \|I_{v_n} B(\nu_0(t_j^0) \to \nu_0(t_j^0))\| \leq 2\|I_{(\rho v)_n} B(\nu_0 \to \nu_0)\|$$

we finally see that

$$\limsup_{n \to \infty} \|I_{v_n} B(\nu_0(t_j^0) \to \nu_0(t_j^0))\| + \limsup_{n \to \infty} \|I_{v_n} B(\nu_0(t_j^0) \to \nu_0(t_j^0))\) < \infty.$$

It remains to deal with the bounded intervals $I_j = [\sigma_n^{-1} t_j, \sigma_n^{-1} t_{j+1}]$ with $j = 1, \ldots, N - 1$. But this a direct application of a (localised version) of the small perturbation observation (6.10) after noting that rescaling gives

$$\|I_{v_n(\sigma_n^{-1} t_j)} B(\nu_0 \to \nu_0)\| = \|I_{v_n(t_j)} B(\nu_0 \to \nu_0)\| \leq \sup_{t \in [-T, T]} \|I_{v_n(t)} B(\nu_0 \to \nu_0)\) = C.$$

1Let $\eta_j \in C^\infty$ with $\eta_j(t) = 1$ on $|t| < 1$ and $\eta_j(t) = 0$ for $|t| > 2$, and $\tilde{\eta}_2(0) = 1$, supp$\tilde{\eta}_2 \subset \{|\tau| \leq 1\}$. Define $\rho(t) = 1 - \eta_1(2t/T)$ and $\chi(t) = \eta_2 * \rho(t)$. 32
and hence
\[
\|I_{v_n(\sigma_n^{-1}t_j)}(v_n - v_n(\sigma_n^{-1}t_j))\|_{B(X(I_j^*) \to X(I_j^*))} \leq \|I_{v_n(\sigma_n^{-1}t_j)}\|_{B(Y_0 \to X_0)} \|v_n - v_n(\sigma_n^{-1}t_j)\|_{L_t^\infty L_x^2(I_j^* \times \mathbb{R}^d)} \\
\leq C_n \|v - v(t_j)\|_{L_t^\infty L_x^2(I_j^* \times \mathbb{R}^d)} \leq \epsilon
\]
where the last line used the choice of the intervals \(I_j\). Therefore
\[
\limsup_{n \to \infty} \|I_{v_n}\|_{B(Y_0(I_j^*) \to X_0(I_j^*))} \leq (1 - \epsilon)^{-1} \limsup_{n \to \infty} \|I_{v_n(\sigma_n^{-1}t_j)}\|_{B(Y_0 \to X_0)} \leq \frac{C_n}{1 - \epsilon} < \infty.
\]
This completes the proof of the localised bounds \([6,0.62]\).

Step (iii): Conclusion of proof. In view of the localised bounds \([6,0.62]\), and the fact that the number of intervals in the partition \(I_j\) is independent of \(n\), an application of Lemma \([6,0.68]\) implies \([6,0.49]\).

Thus we have proven (A1)–(A3), thereby completing the proof of Theorem \([6,1]\). It is worth noting that (A1)–(A3) are independent of the contradiction argument in \([6,0.8]\) so they hold true in general.

## 7. Local and global well-posedness

In this section we give the proof of Theorems \([1.1]\) and \([1.2]\). The key point is the refined local well-posedness result which shows that the time of existence is independent of the initial profile, and thus only depends on the size of the norm.

### 7.1. Local well-posedness

A short computation using Sobolev embedding and the Strichartz estimate gives for any \(s > \frac{1}{2}\)
\[
\|e^{it\Delta}f\|_{L_t^4 W_x^\frac{1}{p}([0,T] \times \mathbb{R}^d)} \lesssim T^{\frac{1}{2} \min\{1, (s - \frac{1}{2})\}} \|f\|_{H^s}.
\]
Consequently Theorem \([1.2]\) follows from the following slightly sharper local well-posedness result.

**Theorem 7.1.** Let \(d = 4\) and take \(s, \ell\) satisfying \([1.10]\). Let \(0 < B < \|W^2\|_{L_x^2(\mathbb{R}^d)}\). Then there exists \(\epsilon > 0\) such that for any \(T > 0\) and data \((f, g) \in H^s \times H^\ell\)
\[
\|e^{it\Delta}f\|_{L_t^4 W_x^\frac{1}{p}([0,T] \times \mathbb{R}^d)} \|f\|_{H^s} \leq \epsilon, \quad \|g\|_{L_x^2(\mathbb{R}^d)} \leq B
\]
(7.1)
there exists a unique solution \((u, V) \in C([0,T]; H^s(\mathbb{R}^4) \times H^\ell(\mathbb{R}^4))\) to \([1.2]\) with \(u \in L_t^4 W_x^\frac{1}{p}([0,T] \times \mathbb{R}^4)\) and \((u, V)(0) = (f, g)\). Moreover the data to solution map is locally Lipschitz continuous.

Note that this is a large data result, since given any \(f \in H^s\) with \(s \geq \frac{1}{2}\) there exists a \(T > 0\) such that \(\|e^{it\Delta}f\|_{L_t^4 W_x^\frac{1}{p} < \epsilon}\). The key advantage of Theorem \([7,1]\) over the results in \([7]\) is that we can take the time of existence \(T\) to be independent of \(V\).

**Proof of Theorem 7.1.** By the persistence of regularity obtained in \([7]\) Theorem 8.1), it suffices to consider the case \((s, \ell) = (\frac{1}{2}, 0)\). Fix data \((f, g) \in H^\frac{1}{2} \times H^\ell\) and an interval \(I = [0, T]\) satisfying \([7,1]\) and define
\[
V_0 = e^{i t |\nabla|} g, \quad \mathcal{J}_0[G] = -i \int_0^t e^{-i(t-s)|\nabla|} G(s) ds.
\]
For ease of notation, we take
\[
\|V\|_W = \|V\|_{L_t^{\infty} L_x^2} + \|(i \partial_t + |\nabla|)V\|_{L_t^{\infty} H_x^{\ell - 1}}, \quad \|u\|_D = \|u\|_{L_t^4 L_x^{\frac{1}{2}(1)}}.
\]
The norm \(\|\cdot\|_W\) is used to control the wave evolution, while \(\|\cdot\|_D\) is simply the endpoint Strichartz space. An application of \([7]\) Proposition 6.1 and Proposition 6.2 gives the bounds\(^2\)
\[
\|v\psi\|_{N_x^{\frac{1}{2}(1)}} \lesssim \|V\|_{W(I)} \left(\|\psi\|_{S_x^{\frac{1}{2}(1)}} \|\psi\|_{D(I)}\right)^{\frac{1}{2}} \tag{7.2}
\]
\[
\|J_0[\varphi]\|_{W(I)} \lesssim \left(\|\varphi\|_{S_x^{\frac{1}{2}(1)}} \|\psi\|_{S_x^{\frac{1}{2}(1)}}\right)^{\frac{1}{2}} \left(\|\varphi\|_{D(I)} \|\psi\|_{D(I)}\right)^{\frac{1}{2}} \tag{7.3}
\]
\(^2\)Note that the \(S_x^{\frac{1}{2}} = S_x^{0.0, 0.0}, N_x = N^{0.0, 0.0}\), and \(W = W^{0.0, 0.0}\) where \(S_x^{0.0, 0.0}, N_x^{0.0, 0.0}\), and \(W^{0.0, 0.0}\) are the spaces in \([7]\).
On the other hand, an application of the uniform Strichartz estimate, Theorem 6.1, implies that if \((i\partial_t + \Delta - \Re(V_0))\psi = F\)
with \(\psi(0) = f\) then
\[
\|\psi\|_{S^\frac{1}{2}(I)} \lesssim B \|f\|_{H^\frac{1}{2}} + \|F\|_{N^\frac{1}{2}}. \tag{7.4}
\]
Writing \(\psi = \mathcal{U}_{0}(V_0) f + \mathcal{I}_{0}(V_0) [F] = e^{it\Delta} f + \mathcal{I}_{0}[\Re(V_0)e^{it\Delta} f] + \mathcal{I}_{0}(V_0) [F]\), an application of Theorem 6.1, Lemma 2.2, and Lemma 2.4 gives
\[
\|\psi\|_{D(I)} \lesssim \|e^{it\Delta} f\|_{D(I)} + \|\mathcal{I}_{0}[\Re(V_0)e^{it\Delta} f]\|_{S^\frac{1}{2}(I)} + \|\mathcal{I}_{0}(V_0) [F]\|_{S^\frac{1}{2}(I)}
\]
\[
\lesssim B \|e^{it\Delta} f\|_{D(I)} + \|g\|_{L^2} \left(\|f\|_{H^\frac{1}{2}} \|e^{it\Delta} f\|_{D(I)}\right)^\frac{1}{2} + \|F\|_{N^\frac{1}{2}}(I), \tag{7.5}
\]
Note that the implied constants in the inequalities (7.4)–(7.5) potentially depend on \(B\) (in using Theorem 6.1), but are otherwise independent of \((f, g) \in H^\frac{1}{2} \times L^2\). We now define a sequence \(u_j\) recursively as
\[
u_j = \mathcal{U}_{0}(V_0) f + \mathcal{I}_{0}(V_0) \left[\mathcal{J}_0[\|\nabla\|u_{j-1}|^2] u_{j-1}\right].
\]
A standard application of the above bounds shows that provided \(\epsilon > 0\) is sufficiently small (depending only on the implied constants in (7.2) – (7.4)), \(u_j\) is a Cauchy sequence, and hence converges to a solution \(u \in S^\frac{1}{2}(I)\). The wave component is then defined as \(V = e^{it\nabla}|g + \mathcal{J}_0[\|\nabla\|u|^2]\). Uniqueness of solutions satisfying \(u \in L^2_{t,loc}(I_T; \mathbb{H}^\frac{1}{2}(\mathbb{R}^4))\) follows from [7, Theorem 7.7 and 8.1]. Local Lipschitz continuity follows in the standard way from the estimates above.

7.2. Global well-posedness below the ground state. The proof of Theorem 1.1 follows from the variational properties of the ground state \(W\) together with the refined local well-posedness result in Theorem 1.2. More precisely, we exploit the following result from [10], slightly adjusted to our sign convention.

**Lemma 7.2 ([10] Section 6).** Let \(f \in \dot{H}^1(\mathbb{R}^4)\) and \(g \in L^2(\mathbb{R}^4)\) with
\[
E_Z(f, g) < E_Z(W; -W^2) = \frac{1}{4}\|W^2\|_{L^2(\mathbb{R}^4)}^2, \quad \|g\|_{L^2(\mathbb{R}^4)} \leq \|W^2\|_{L^2(\mathbb{R}^4)}.
\]
Then we have the bounds
\[
\|g\|_{L^2}^2 \leq 4E_Z(f, g), \quad \|\nabla f\|_{L^2}^2 \leq \frac{\|W^2\|_{L^2}}{\|W^2\|_{L^2} - \|g\|_{L^2}^2} \left(4E_Z(f, g) - \|g\|_{L^2}^2\right) \leq \|W^2\|_{L^2}^2.
\]

**Proof.** We begin by observing that for any \(f \in \dot{H}^1(\mathbb{R}^4)\) and \(g \in L^2(\mathbb{R}^4)\), the properties of the ground state \(W\) imply that
\[
E_Z(f, g) = \frac{1}{2}\|\nabla f\|_{L^2}^2 + \frac{1}{4}\|g\|_{L^2}^2 + \frac{1}{2}\Re(g, |f|^2)
\]
\[
\geq \frac{1}{2}\|\nabla f\|_{L^2}^2 + \frac{1}{4}\|g\|_{L^2}^2 - \frac{1}{2}\|g\|_{L^2}\|f\|_{L^2}^2
\]
\[
\geq \frac{1}{2}\|W^2\|_{L^2}^2 (\|W^2\|_{L^2} - \|g\|_{L^2}) \|\nabla f\|_{L^2}^2 + \frac{1}{4}\|g\|_{L^2}^2. \tag{7.7}
\]
If we assume that in addition \((f, g)\) satisfy the constraints in (7.6), then (7.7) immediately gives
\[
\|g\|_{L^2}^2 \leq 4E_Z(f, g)
\]
and similarly
\[
2\|\nabla f\|_{L^2}^2 \leq \frac{\|W^2\|_{L^2}}{\|W^2\|_{L^2} - \|g\|_{L^2}^2} \left(4E_Z(f, g) - \|g\|_{L^2}^2\right)
\]
\[
\leq \frac{\|W^2\|_{L^2}^2}{\|W^2\|_{L^2} - \|g\|_{L^2}^2} (\|W^2\|_{L^2}^2 - \|g\|_{L^2}^2) \leq \|W^2\|_{L^2}^2 (\|W^2\|_{L^2} + \|g\|_{L^2}^2),
\]
which completes the proof. □
Proof of Theorem 1.1. Suppose now that \((s, \ell) \in \mathbb{R}^2\) satisfy (1.10) and \(s \geq 1\). Let \((u(0), V(0)) \in H^s \times H^\ell\) be given initial data satisfying
\[
E_Z(u(0), V(0)) < \frac{1}{4} \|W^2\|_{L^2}^2 - \varepsilon, \quad \|V(0)\|_{L^2} \leq \|W^2\|_{L^2}.
\]
We may restrict to positive times by reversibility. An application of Lemma 7.2 shows that we must have the strict inequality \(\|V(t)\|_{L^2} < \|W^2\|_{L^2}\). Let \(T^*\) be the supremum of all \(T \geq 0\) such that there exists a solution \((u, V) \in C([0, T], H^s \times H^\ell) \cap L^2([0, T], W^{1,4})\) which conserves mass and energy. Then, we claim that for all \(t \in [0, T]\),
\[
E_Z(u(t), V(t)) < \frac{1}{4} \|W^2\|_{L^2}^2 - \varepsilon, \quad \|V(t)\|_{L^2} < \|W^2\|_{L^2}.
\] (7.8)
Otherwise, let \(t_*\) denote the infimum of all \(t \in [0, T]\) such that \(\|V(t)\|_{L^2} \geq \|W^2\|_{L^2}\). By continuity and energy conservation we must have \(\|V(t_*)\|_{L^2} = \|W^2\|_{L^2}\) and Lemma 7.2 yields the contradiction \(\|V(t_*)\|_{L^2} < \|W^2\|_{L^2}\), so that (7.8) is proved.

Suppose now, for the sake of contradiction, that \(T^* < \infty\). According to [7] Theorem 7.6 there exists a sequence of \(T_n > 0\) with \(T_n 
 T^*\) and the solution \((u, V) \in C([0, T_n], H^s \times H^\ell) \cap L^2([0, T_n], W^{1,4})\) conserves mass and energy. Lemma 7.2 implies
\[
\|V(T_n)\|_{L^2}^2 \leq 4E_Z(u(T_n), V(T_n)) = 4E_Z(u(0), V(0)) < \|W^2\|_{L^2}^2 - 4\varepsilon.
\]
Further,
\[
\|u(T_n)\|_{H^1}^2 \leq \|\nabla u(T_n)\|_{L^2}^2 + \|u(0)\|_{L^2}^2 \leq \|W^2\|_{L^2} + m^2.
\]
Both of these bounds are uniform in \(n\). Therefore, by Theorem 1.2 there exists a time \(\tau = \tau(\varepsilon, m)\), independent of \(n\), such that we can extend the solution to \((u, V) \in C([0, T_n + \tau], H^s \times L^2) \cap L^2([0, T_n + \tau], W^{1,4})\). By [7] Theorem 8.1 we have \((u, V) \in C([0, T_n + \tau], H^s \times H^\ell) \cap \tilde{L}^2([0, T_n + \tau], W^{1,4})\) and mass and energy are conserved. For large enough \(n\) we have \(T_n + \tau > T^*\), which is in contradiction to the definition of \(T^*\), and we conclude that \(T^* = \infty\). In addition, [7] Theorem 8.1 implies that for any \(T < \infty\) the flow map \((u(0), V(0)) \mapsto (u, V)\) is Lipschitz continuous as a map from a small ball of initial data in \(H^s \times H^\ell\) below the ground state to solutions in \(C([0, T], H^s \times H^\ell) \cap \tilde{L}^2([0, T], W^{1,4})\). Also, the argument above and Lemma 7.2 imply that
\[
\sup_{t \in [0, \infty)} \|\nabla u(t)\|_{L^2}^2 \leq 2\varepsilon^{-1}\|W^2\|_{L^2}^2 E_Z(u(0), V(0)), \quad \sup_{t \in [0, \infty)} \|V(t)\|_{L^2}^2 \leq 4E_Z(u(0), V(0)).
\]

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