Remarks on local controllability for the Boussinesq system with Navier boundary condition

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Abstract

This note deals with the local exact controllability to a particular class of trajectories for the Boussinesq system with nonlinear Navier–slip boundary conditions and internal controls having vanishing components. Briefly speaking, in two dimensions, the local exact controllability property is obtained using only one control in the heat equation, meanwhile two scalar controls are required in three dimensions.

Résumé

Remarque sur la contrôlabilité locale du système de Boussinesq avec la condition de frontière de Navier. Cette note concerne la contrôlabilité locale d’une classe particulière de trajectoires, ceci pour le système de Boussinesq avec la condition de Navier non linéaire et certains contrôles internes. Brève-ment, la propriété de contrôlabilité exacte locale s’obtient en dimension deux n’utilisant que le contrôle associé à l’équation de la chaleur. Tandis que, deux contrôles scalaires sont nécessaires pour obtenir notre résultat dans le cas de dimension trois.

1. Introduction

The interaction of incompressible fluids with a diffusion process can be modeled by a coupled system between the Navier–Stokes and heat equations, usually called Boussinesq system. On bounded domains, both heat and the velocity field can show a different behaviour on its boundary. In this paper, nonlinear Navier–type boundary conditions for the fluid flow and homogeneous Neumann conditions for the diffusion equation are considered in order to study the local exact controllability for the Boussinesq system with few scalar controls.

Henceforth, let $\Omega$ be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N = 2$ or $N = 3$) of class $C^\infty$. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which is the control domain. Here, we will use the notation $Q := \Omega \times (0, T)$, $\Sigma := \partial \Omega \times (0, T)$ and $n$ the outward unit normal vector to $\Omega$. Moreover, $C$ denotes a generic positive constant which may depend on $\Omega$ and $\omega$.

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In this Note, we will consider the Boussinesq system with Navier–slip and Neumann conditions

\[
\begin{align*}
\begin{cases}
  y_t - \nabla \cdot (Dy) + (y, \nabla)y + \nabla p = u\chi_{\omega} + \theta e_N, & \nabla \cdot y = 0 \quad \text{in} \quad Q, \\
  \theta_t - \Delta \theta + y \cdot \nabla \theta = v\mathbf{1}_\omega & \quad \text{in} \quad Q, \\
  y \cdot n = 0, \quad (\sigma(y, p) \cdot n)_{tg} + f(y)_{tg} = 0, & \nabla \theta \cdot n = 0 \quad \text{on} \quad \Sigma, \\
  y(\cdot, 0) = y_0(\cdot), \quad \theta(\cdot, 0) = \theta_0(\cdot) \quad \text{in} \quad \Omega,
\end{cases}
\end{align*}
\]  

(1)

as well as the linearized Boussinesq system (around a target flow of the form \((0, \overline{p}, \overline{f})\))

\[
\begin{align*}
\begin{cases}
  y_t - \nabla \cdot (Dy) + \nabla p = h_1 + u\chi_{\omega} + \theta e_N, & \nabla \cdot y = 0 \quad \text{in} \quad Q, \\
  \theta_t - \Delta \theta + y \cdot \nabla \theta = h_2 + v\mathbf{1}_\omega & \quad \text{in} \quad Q, \\
  y \cdot n = 0, \quad (\sigma(y, p) \cdot n)_{tg} + (A(x, t)y)_{tg} = 0, & \nabla \theta \cdot n = 0 \quad \text{on} \quad \Sigma, \\
  y(\cdot, 0) = y_0(\cdot), \quad \theta(\cdot, 0) = \theta_0(\cdot) \quad \text{in} \quad \Omega,
\end{cases}
\end{align*}
\]  

(2)

where \(y = y(x, t)\) is the velocity field of the fluid, \(\theta = \theta(x, t)\) their temperature, \(v\) and \(u = (u_1, \ldots, u_N)\) stands for the controls, which are acting in an arbitrary fixed domain \(\omega \times (0, T)\), where \(\chi_\omega\) is a smooth positive function such that \(\chi_\omega = 1\) in \(\omega'\), \(\omega' \subset \omega\), and \(1_\omega\) is the indicator function. Here, the gravity vector field is given by \(e_N = (0, 1)\) for \(N = 2\), or \(e_N = (0, 0, 1)\) for \(N = 3\). Moreover, \(f : \mathbb{R}^N \to \mathbb{R}^N\) is a nonlinear regular function given, \(\sigma(y, p) := -p\mathbf{I} + Dy\) is the stress tensor, \(A\) is a \(N \times N\) matrix–valued function in a suitable space, and \(tg\) stands for the tangential component of the corresponding vector field, i.e., \(y_{tg} = y - (y \cdot n)n\).

In the context of controllability, the first results for the Boussinesq system were made by Fursikov and Imanuvilov in [9] and [10]. The work by S. Guerrero [12] shows the local exact controllability to the trajectories of the Boussinesq system with Dirichlet boundary conditions and \(N + 1\) distributed scalar controls supported in small sets.

Additionally, recent works have been developed for controllability problems with reduced number of controls. For instance, N. Carreño and S. Guerrero in [1] have proven the local null controllability for the Navier–Stokes with Dirichlet conditions and \(N - 1\) scalar controls. The recent work made by S. Guerrero and C. Montoya shows that the local null controllability property is achieved for the \(N\)–dimensional Navier–Stokes system with Navier–slip conditions and \(N - 1\) scalar controls [13]. The methodology in the previous articles are Carleman estimates. In the three dimensional case of the Navier–Stokes system with Dirichlet conditions, J-M. Coron and P. Lissy developed in [4] a new strategy to prove the local null controllability using only one scalar control.

Concerning the \(N\)–dimensional Boussinesq system with Dirichlet conditions, in [7] the authors proved that the local exact controllability to the trajectories can be achieved with \(N - 1\) scalar controls, under certain geometric assumption on the control domain. N. Carreño showed the local controllability of the \(N\)–Boussinesq system using \(N - 1\) scalar controls, without conditions on the control domain [2]. Finally, this Note improves the results of [1] and [13].

Our results extend the results of [1] and [13]. Taking into account the relation between the observability and controllability property, it will be appropriate to consider the following adjoint system related to (2):
\[
\begin{aligned}
\left\{ 
\begin{array}{ll}
-\varphi_t - \nabla \cdot (D\varphi) + \nabla \pi = g - \psi \nabla \vartheta, & \nabla \cdot \varphi = 0 \\
-\psi_t - \Delta \psi = g_0 + \varphi \cdot e_N \\
\varphi \cdot n = 0, & \text{for any } j
\end{array}
\right. \\
\text{in } Q,
\end{aligned}
\]

where \( g, \varphi^T, g_0 \) and \( \psi^T \) satisfying adequate regularity assumptions. We will introduce several spaces and hypotheses over \( \vartheta \) which will be needed in order to have suitable Carleman estimates for the solution of (3):

\[
W = \{ u \in H^1(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega, \ u \cdot n = 0 \text{ on } \partial \Omega \},
\]

\[
H = \{ u \in L^2(\Omega)^N : \nabla \cdot u = 0, \text{ in } \Omega \ u \cdot n = 0 \text{ on } \partial \Omega \},
\]

\[
P_\varepsilon^2 = H^{5/4+\varepsilon}(0, T; L^2(\partial \Omega)^{N \times N}), \quad P^2 = L^2(0, T; H^{5/2}(\partial \Omega)^{N \times N}), \quad \forall \varepsilon > 0,
\]

\[
Y_m := L^2(0, T; H^{2m}(\Omega)^N) \cap H^m(0, T; L^2(\Omega)^N), \quad m = 1, 2.
\]

and

\[
\bar{\vartheta} \in L^\infty(0, T; W^{3, \infty}(\Omega)), \quad \nabla \bar{\vartheta}_t \in L^\infty(Q)^N.
\]

Here, the target flow \((0, \overline{\varphi}, \overline{\theta})\) satisfies the problem

\[
\begin{aligned}
\nabla \overline{\varphi} = \overline{\theta} e_N, & \quad \overline{\vartheta}_t - \Delta \overline{\vartheta} = 0 \quad \text{in } Q, \\
\nabla \overline{\vartheta} \cdot n = 0 & \quad \text{on } \Sigma, \\
\overline{\vartheta}(\cdot, 0) = \overline{\vartheta}_0(\cdot) & \quad \text{in } \Omega.
\end{aligned}
\]

Our first main result is a new Carleman estimate for the solution of (3). Several weight functions are needed:

\[
\begin{aligned}
\alpha(x, t) &= \frac{e^{2\lambda \|\eta\|_{\infty}} - e^{\lambda \eta(x)}}{(t(T - t))^{11}}, \quad \xi(x, t) = \frac{e^{\lambda \eta(x)}}{(t(T - t))^{11}}, \quad \alpha^*(t) = \max_{x \in \Omega} \alpha(x, t), \\
\xi^*(t) &= \min_{x \in \Omega} \xi(x, t), \quad \widetilde{\alpha}(t) = \min_{x \in \Omega} \alpha(x, t), \quad \widetilde{\xi}(t) = \max_{x \in \Omega} \xi(x, t).
\end{aligned}
\]

Here, \( \eta \in C^2(\Omega) \) and satisfies that

\[
|\nabla \eta| > 0 \text{ in } \overline{\Omega} \setminus \omega_0, \quad \eta > 0 \text{ in } \Omega \quad \text{and} \quad \eta \equiv 0 \text{ on } \partial \Omega,
\]

where \( \omega_0 \subset \omega_1 \subset \omega \subset \Omega \) is a nonempty open set. The existence of such a function \( \eta \) is proved in [8].

**Theorem 1.1** Assume \( A \in P_1^2 \cap P^2 \) and \((0, \overline{\varphi}, \overline{\theta})\) satisfying (4)-(5). There exists a constant \( \lambda_0 \), such that for any \( \lambda \geq \lambda_0 \) there exist two constants \( C(\lambda) > 0 \) increasing on \( \|A\|_{P_1^2 \cap P^2} \) and \( s_0(\lambda) > 0 \) such that for any \( j \in \{1, 2\} \), any \( a > 0 \), any \( g \in L^2(Q)^3 \), any \( g_0 \in L^2(Q) \), any \( \varphi^T \in H \) and any \( \psi^T \in L^2(\Omega) \), the solution of (3) satisfies

\[
s^3 \int_Q e^{-2(1+a)s\alpha^*} \xi^3 |\varphi|^2 dxdt + s^5 \int_Q e^{-2(1+a)s\alpha^*} |\varphi|^5 |\psi|^2 dxdt
\]

\[
\leq C \left( \int_Q e^{-2a\alpha^*} |g|^2 + |g_0|^2 dxdt + (N-2)s^7 \int_0^T \int_{\omega^c} e^{-4s\alpha + 2(1-a)s\alpha^*} (\xi)^{12} |\varphi_j|^2 dxdt \right.
\]

\[
+ s^{13} \int_0^T \int_{\omega} e^{-8s\alpha + (6-2a)s\alpha^*} (\xi)^{24} |\psi|^2 dxdt \right)
\]

3
for every \( s \geq s_0 \).

The second main result in this Note concerns the local controllability to a particular class of trajectories of (1). This result is presented as follows:

**Theorem 1.2** Assume \( f \in C^4(\mathbb{R}^N; \mathbb{R}^N) \) with \( f(0) = 0 \) and \( i \in \{1, \ldots, N-1\} \) fixed. Let \((0, p, \overline{\theta}) \) be a solution to (5) satisfying (4). Then, for every \( T > 0 \) and \( \omega \subset \Omega \), there exists \( \delta > 0 \) such that, for every \((y_0, \theta_0) \in [H^3(\Omega)^N \cap \mathcal{W}] \times H^1(\Omega) \) satisfying

\[
(Dy_0 \cdot n)_{tg} + (f(y_0))_{tg} = 0 \quad \text{on} \quad \partial \Omega \quad \text{and} \quad \|(y_0, \theta_0) - (0, \overline{\theta}_0)\|_{[H^3(\Omega)^N \cap \mathcal{W}] \times H^1(\Omega)} \leq \delta,
\]

we can find controls \( v \in L^2(\omega \times (0, T)) \) and \( u \in L^2(0, T; H^2(\omega)^N) \cap H^1(0, T; L^2(\omega)^N) \) with \( u_i \equiv 0 \) and \( u_N \equiv 0 \) such that the corresponding solution \((y, p, \theta)\) to (1) satisfies

\[
y(\cdot, T) = \overline{y}(\cdot, T) \quad \text{and} \quad \theta(\cdot, T) = \overline{\theta}(\cdot, T) \quad \text{in} \quad \Omega.
\]

In the following sections, we will indicate the main ideas of the proof of Theorem 1.1 and Theorem 1.2.

2. A new Carleman inequality

In this section, we give the proof of Theorem 1.1. Our arguments are based in \([1,3,7,13]\). From (3) and using the decomposition \( \rho \varphi = w + z \), \( \rho \pi = \pi_z + \pi_\theta \) and \( \rho \psi = \tilde{\psi} \), where \( \rho(t) = e^{-a s \alpha t} \) and \( a > 0 \), it is very easy to verify that \((w, \pi_w, z, \pi_z)\) and \( \tilde{\psi} \) are solutions to the systems

\[
\begin{aligned}
-w_t - \nabla \cdot (Dw) + \nabla \pi_w &= \rho g; \\
\nabla \cdot w &= 0; \\
\pi_z &= -\rho \varphi - \tilde{\psi} \nabla \theta \\
\end{aligned}
\quad \begin{aligned}
-z_t - \nabla \cdot (Dz) + \nabla \pi_z &= -\rho \varphi - \tilde{\psi} \nabla \theta \\
\nabla \cdot z &= 0 \\
\end{aligned}
\quad \begin{aligned}
w \cdot n &= 0; \\
A(t, x)w \cdot n &+ (A(t, x)z)_{tg} = 0; \\
z \cdot n &= 0; \\
A(t, x)z \cdot n &+ (A(t, x)z)_{tg} = 0 \\
w(\cdot, T) &= 0; \\
z(\cdot, T) &= 0
\end{aligned}
\quad \text{in} \quad Q, \quad \text{in} \quad Q, \quad \text{in} \quad \Sigma, \quad \text{in} \quad \Sigma, \quad \text{in} \quad \Omega, \quad \text{in} \quad \Omega,
\]

and

\[
\begin{aligned}
-\tilde{\psi}_t - \Delta \tilde{\psi} &= \rho g_0 + \rho \varphi_3 - \rho \tilde{\psi} \\
\nabla \tilde{\psi} \cdot n &= 0 \\
\tilde{\psi}(\cdot, T) &= 0
\end{aligned}
\quad \text{in} \quad Q, \quad \text{on} \quad \Sigma, \quad \text{in} \quad \Omega.
\]

We will use the Carleman inequality for parabolic equations with Neumann conditions \([8]\) for the system (10) in order to estimate the global terms associated to \( \tilde{\psi} \). Thus, there exists \( \lambda > 0 \) such that for any \( \lambda > \overline{\lambda} \) there exists a positive constant \( C \) depending on \( \lambda, \Omega, \omega_2, \|\overline{\theta}\|_{L^\infty(0, T; W^\infty(\Omega))} \) such that

\[
\frac{1}{Q} \int e^{-2s a} (s \xi |\tilde{\psi}_t|^2 + s \xi \sum_{m=1}^3 |\partial_{\ell m} \tilde{\psi}|^2 + s^3 \xi^2 |\nabla \tilde{\psi}|^2 + s^5 \xi^5 |\tilde{\psi}|^2) dx dt \\
\leq C \left( \frac{1}{Q} \int e^{-2s a} s^3 \xi^2 (|\rho g_0|^2 + |\varphi_3|^2 + |\rho |^2 |\rho |^{-2} |\tilde{\psi}|^2) dx dt + s^5 \int_0^T \omega_1 \int e^{-2s a} \xi^5 |\tilde{\psi}|^2 dx dt \right),
\]

for every \( s \geq C \).

The arguments below are given for the case \( N = 3 \). For \( k = 1, 3 \), we can deduce the inequality
\[ I(s, z) + J(s, \psi) \leq C \left( \| \rho \|_{L^2(Q)}^2 + \| \rho g_0 \|_{L^2(Q)} + s^5 \int_0^T e^{-2s\alpha \xi^5} |\psi|^2 dxdt \right. \\
+ \left. \sum_{k=1,3} \int_0^T \int e^{-2s\alpha} (s^5 \xi^5)|z_k|^2 + s^3 |\nabla z_k|^2 dxdt \right) \\
+ \left. \int_0^T e^{-2s\alpha} \xi^2 |\nabla \partial_t \pi_s|^2 dxdt \right], \\
\text{where } J(s, \psi) \text{ denotes the left–hand side of (11), and for } k = 1, 3, I(s, z) \text{ is defined by} \\
\begin{align*}
I(s, z) &:= \sum_{k=1,3} s^5 \int_Q e^{-2s\alpha} \xi^5 |z_k|^2 dxdt + s^3 \int_Q e^{-2s\alpha} \xi^3 |\nabla z_k|^2 dxdt + s^3 \int_Q e^{-2s\alpha} \xi^3 |z_k|^2 dxdt \\
&+ \|s^{1/2} e^{-s\alpha} (\xi^*)^{9/22} z\|^2_{L^2(0,T;H^3(\Omega))} + \|s^{-1/2} e^{-s\alpha} (\xi^*)^{-15/22} z\|^2_{L^2(0,T;H^3(\Omega))} \\
&+ \|s^{1/2} e^{-s\alpha} (\xi^*)^{9/22} \pi_s\|^2_{L^2(0,T;H^3(\Omega))}. 
\end{align*}
\]
Here, \( \omega_1 \) and \( \omega_2 \) are open sets such that \( \omega_1 \subseteq \omega_2 \subseteq \omega \). The rest of the proof is oriented towards the absorption of the local pressure term in (12). However, we have omitted these details since analogous arguments can be found in [13], Section 3. Let us remark that the regularity over \( \overline{\theta} \) given in (4) is used in several estimates associated to the pressure term. The other local terms can be estimated in an easier way. Therefore, those local estimates lead to the desired Carleman inequality (7).

3. Local controllability for the Boussinesq system

The proof of Theorem 1.2 follows the ideas in [1] and [13]. Thus, in a first step a null controllability result for (2) with an appropriate right–hand side \( h_1, h_2 \). Here, the idea is to look for a solution in an appropriate weighted functional space. Let us 
\[ L_1 w := w_t - \nabla \cdot Dw \quad \text{and} \quad L_2 w := w_t - \Delta w \]
and let us define the space \( E \) as follows:
\[ \{ (y, p, u, \theta, v) : e^{as\beta} y, e^{as\beta-(1-a)s\beta} (\gamma) - 6 (u_1, 0, 0, \chi_\omega, \rho(\partial_t u_1, 0, 0) \in L^2(\Omega)^3, e^{as\beta} \theta \in L^2(\Omega), \\
e^{4s\beta-(3-a)s\beta} (\gamma) - 12 v_1 L_1 \rho \in L^2(\Omega), \rho u_1 \in L^2(0,T;H^2(\Omega)), \text{supp } u_1 \subset \omega \times (0,T), \\
e^{as\beta} (\gamma^*)^{-12/11} y \in L^1_{\gamma} \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)), \\
e^{(a+1)s\beta} (\gamma^*)^{-3/2} (L_1 y + \nabla p - (u_1, 0, 0) \chi_\omega - \theta e_3) \in L^2(\Omega)^3, \\
e^{(a+1)s\beta} (\gamma^*)^{-5/2} (L_2 y + y \cdot \nabla \theta - v_1 \omega) \in L^2(\Omega) \} =: E, \]
where \( \bar{\rho} := e^{as\beta+2(1-a)s\beta} (\gamma) - 12 e^{-1(1+a)s\beta} (\gamma^*)^{9/22} \) and whose weight functions are given by
\[ \beta(x,t) = \frac{e^{2\lambda} \| \rho \|_\infty - e^{\lambda \eta(x)}}{\ell^{11}(t)}, \quad \gamma(x,t) = \frac{e^{\lambda \eta(x)}}{\ell^{11}(t)}, \quad \beta^*(t) = \max_{x \in \mathbb{H}} \beta(x,t), \]
\[ \gamma^*(t) = \min_{x \in \mathbb{H}} \gamma(x,t), \quad \bar{\beta}(t) = \min_{x \in \mathbb{H}} \beta(x,t), \quad \bar{\gamma}(t) = \max_{x \in \mathbb{H}} \gamma(x,t). \]
(13)

In this case, \( \ell \in C^2([0,T]) \) is a positive function in \([0,T]\) such that \( \ell(t) > t(T-t) \) for all \( t \in [0,T/4] \) and \( \ell(t) = t(T-t) \) for all \( t \in [T/2,T] \).

Proposition 3.1 Let \( s \) and \( \lambda \) be like in Theorem 1.1 and (0, \( \overline{\rho}, \overline{\theta} \)) satisfy (5). Assume that
\[ y_0 \in W, \theta_0 \in H^1(\Omega), \quad e^{(a+1)s\beta} (\gamma^*)^{-3/2} h_1 \in L^2(\Omega)^3 \quad \text{and} \quad e^{(a+1)s\beta} (\gamma^*)^{-5/2} h_2 \in L^2(\Omega). \]
(14)
Then, there exists controls $u_1$ and $v$ such that, if $(y, p, \theta)$ is the associated solution to (2), we have

$$(y, p, u_1, \theta, v) \in E. \text{ In particular } y(\cdot, T) = 0 \text{ and } \theta(\cdot, T) = 0 \text{ in } \Omega.$$ 

The rest of the proof of Theorem 1.2 relies on two fixed point theorems, namely, one for the nonlinearity posed on the boundary condition, and another one, for the convective term in (1). We will mention only these results since the methodology given in [13] can be adapted to (1). Thus, for $N = 3$, we consider the nonlinear system

\[
\begin{align*}
&y_t - \nabla \cdot (Dy) + \nabla p = h_1 + (u_1, 0, 0)\chi_{\omega} + \theta e_3, \quad \nabla \cdot y = 0 \quad \text{in } Q, \\
&\theta_t - \Delta \theta + y \cdot \nabla \theta = h_2 + v_1 \omega,
\end{align*}
\]

\begin{equation}
\begin{align*}
y(\cdot, 0) = y_0(\cdot), \quad \theta(\cdot, 0) = \theta_0(\cdot) \quad \text{in } Q,
\end{align*}
\end{equation}

\begin{equation}
\begin{align*}
y(\cdot, 0) = y_0(\cdot), \quad \theta(\cdot, 0) = \theta_0(\cdot) \quad \text{in } \Omega.
\end{align*}
\end{equation}

**Theorem 3.2** Let us assume that $f \in C^4(\mathbb{R}^3; \mathbb{R}^3)$ with $f(0) = 0$. Then, for every $T > 0$ and $\omega \subset \Omega$, there exists $\delta > 0$ such that, for every $a > 0$ and for every $(y_0, \theta_0) \in H^2(\Omega)^2 \cap W \times H^1(\Omega)$, $h_1 \in Y_1, h_2 \in L^2(Q)$ satisfying $e(a+1)\delta^\gamma(\gamma^*)^{-3/2}h_1 \in L^2(Q)^3$ and $e(a+1)\delta^\gamma(\gamma^*)^{-5/2}h_2 \in L^2(Q)$,

\[
\|h_1\|_{Y_1} + \|h_2\|_{L^2(Q)} + \|y_0\|_{H^2(\Omega)^2 \cap W} + \|\theta_0\|_{H^1(\Omega)} \leq \delta
\]

and (8), there exists controls $v \in L^2(0, T; L^2(\omega))$ and $u_1 \in L^2(0, T; H^2(\omega)) \cap H^1(0, T; L^2(\omega))$ and an associated solution $(y, p, \theta)$ of (15) satisfying $(y, \theta) \in Y_2 \times L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$ such that $(y, p, u_1, \theta, v) \in E$.

**Theorem 3.3** Suppose that $B_1, B_2$ are Banach spaces and

\[
A : B_1 \to B_2
\]

is a continuously differentiable map. We assume that for $b_1^0 \in B_1, b_2^0 \in B_2$ the equality

\[
A(b_1^0) = b_2^0
\]

holds and $A'(b_1^0) : B_1 \to B_2$ is an epimorphism. Then there exists $\delta > 0$ such that for any $b_2 \in B_2$ which satisfies the condition

\[
\|b_2^0 - b_2\|_{B_2} < \delta
\]

there exists a solution $b_1 \in B_1$ of the equation

\[
A(b_1) = b_2.
\]

Let us set

\[
y = \tilde{y}, \quad p = \tilde{p}, \quad \theta = \tilde{\theta}^\gamma + \theta.
\]

For $a = 2 > 1$, we apply Theorem 3.3 with the spaces

\[
B_1 := \{(y, p, u_1, \theta, v) \in E : y \in Y_2\},
\]

\[
B_2 := \{(h_1, y_0, h_2, \theta_0) \in Z_1 \times [H^2(\Omega)^3 \cap W] \times Z_2 \times H^1(\Omega) : h_1, h_2, y_0, \theta_0 \text{ satisfies (16)}\},
\]

and where

\[
Z_1 := L^2(e^{3s\delta^\gamma(\gamma^*)^{-3/2}(0, T); L^2(\Omega)^3}), \quad \text{and} \quad Z_2 := L^2(e^{3s\delta^\gamma(\gamma^*)^{-5/2}(0, T); L^2(\Omega)}).
\]

By defining the operator $A : B_1 \to B_2$ by

\[
A \to (L_1 \tilde{y} + (\tilde{y} \cdot \nabla)\tilde{y} + \nabla \tilde{p} - \tilde{\theta} e_3 - (u_1, 0, 0)\chi_{\omega}, \tilde{y}_0, L_2 \tilde{\theta} + \tilde{y} \cdot \nabla \tilde{\theta} + \tilde{y} \cdot \nabla \tilde{\theta} - v_1 \omega, \tilde{\theta}_0),
\]

for every $(\tilde{y}, \tilde{p}, u_1, \tilde{\theta}, \tilde{v}) \in B_1$, one can easily check the conditions for $A$ in order to complete the proof of Theorem 1.2.
Some open problems. It would be interesting to know if the local controllability to the trajectories with \( N - 1 \) scalar controls holds for \( \overline{\gamma} \neq 0 \) and \( \omega \) like in Theorem 1.2. However, is not clear at all and therefore is an open problem even for the Navier–Stokes system.

On the other side, could be reasonable to expect results of the same kind whether one considers nonlinear conditions such as \( \nabla \theta \cdot n + g(\theta) = 0 \), where \( g \) is a suitable function to study.

Recently, Coron et al. have proved a global exact controllability result for the Navier–Stokes and Navier–type conditions (for small time), see [5]. A challenging problem would be to use the Boussinesq system proposed in this Note in order to apply and prove analogous results to [5].

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References

[1] N. Carreño and S. Guerrero. Local null controllability of the n-dimensional Navier–Stokes system with n-1 scalar controls in an arbitrary control domain. Journal of Mathematical Fluid Mechanics, 15(1):139–153, 2013.
[2] N. Carreño. Local controllability of the n-dimensional Boussinesq system with n-1 scalar controls in an arbitrary control domain. arXiv preprint arXiv:1201.1871, 2012.
[3] J.-M. Coron and S. Guerrero. Null controllability of the n-dimensional Stokes system with n-1 scalar controls. Journal of Differential Equations, 246(7):2908–2921, 2009.
[4] J.-M. Coron and P. Lissy. Local null controllability of the three-dimensional Navier–Stokes system with a distributed control having two vanishing components. Inventiones mathematicae, 198(3):833–880, 2014.
[5] Jean–Michel Coron, Frédéric Marbach, and Franck Sueur. Small–time global exact controllability of the Navier-Stokes equation with Navier slip–with–friction boundary conditions arXiv preprint arXiv:1612.08087, 2018.
[6] E. Fernández-Cara, M. González-Burgos, S. Guerrero, and J.-P. Puel. Null controllability of the heat equation with boundary Fourier conditions: the linear case. ESAIM: Control, Optimisation and Calculus of Variations, 12(3):442–465, 2006.
[7] E. Fernández-Cara, S. Guerrero, O. Y. Imanuvilov, and J.-P. Puel. Some controllability results for the n-dimensional Navier–Stokes and Boussinesq systems with n-1 scalar controls. SIAM journal on control and optimization, 45(1):146–173, 2006.
[8] A. V. Fursikov and O. Y. Imanuvilov. Controllability of evolution equations. Number 34. Seoul National University, 1996.
[9] A. V. Fursikov and O. Y. Imanuvilov. Local exact boundary controllability of the Boussinesq equation. SIAM Journal on Control and optimization, 36(2):391–421, 1998.
[10] A. V. Fursikov and O. Y. Imanuvilov. Exact controllability of the Navier-Stokes and Boussinesq equations. Russian Mathematical Surveys, 54(3):565–618, 1999.
[11] S. Guerrero. Local exact controllability to the trajectories of the Navier-Stokes system with nonlinear Navier-slip boundary conditions. ESAIM: Control, Optimisation and Calculus of Variations, 12(3):484–544, 2006.
[12] S. Guerrero. Local exact controllability to the trajectories of the Boussinesq system. In Annales de l’IHP Analyse non linéaire, volume 23, pages 29–61, 2006.
[13] S. Guerrero and C. Montoya. Local null controllability of the n-dimensional Navier–Stokes system with nonlinear Navier-slip boundary conditions and n-1 scalar controls. Journal de Mathématiques Pures et Appliquées, 113:37–69, 2018.