Research Article

Analysis of Social Networks by Using Pythagorean Cubic Fuzzy Einstein Weighted Geometric Aggregation Operators

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Pythagorean cubic set (PCFS) is the combination of the Pythagorean fuzzy set (PFS) and interval-valued Pythagorean fuzzy set (IVPFS). PCFShandle more uncertainties than PFS and IVPFS and thus are more extensive in their applications. The objective of this paper is under the PCFS to establish some novel operational laws and their corresponding Einstein weighted geometric aggregation operators. We describe some novel Pythagorean cubic fuzzy Einstein weighted geometric (PCFEWG) operators to handle multiple attribute group decision-making problems. The desirable relationship and the characteristics of the proposed operator are discussed in detail. Finally, a descriptive case is given to describe the practicality and the feasibility of the methodology established.

1. Introduction

Multicriteria decision-making (MCDM) is a process that can give the ranking result of finite alternatives according to the attribute value of different alternatives, and it is an important aspect of decision sciences. A significant part of the decision-making model that has been commonly used in human impacts is MCDM (or MCGDM) [1]. The assessment information is generally fuzzy because the real decision-making issues have always been created from a complicated context. In general, fuzzy data take two models: one quantitatively and one qualitatively. Fuzzy set (FS) [2], intuitionistic fuzzy set (IFS) [3], Pythagorean fuzzy set (PFS) [4], and so on, can express quantitative fuzzy knowledge. The theory of FS suggested by Zadeh [2] was used to explain fuzzy quantitative knowledge containing only a degree of membership. On this basis, Atanassov [5] proposed the idea of IFS as a generalization of FS; the important aspect is that it has two fuzzy values: the first is called membership grade and the second is called nonmembership grade. Sometimes, meanwhile, the two degrees do not satisfy the limit, so the square sum is less than or equal to one. The PFS was introduced by Yager [4] in which the sum of squares of membership and nonmembership is equal to or less than one. In certain conditions, PFS is capable of expressing the fuzzy data compared to the IFS. For instance, PFS improved the concept of IFS by enlarging its domain. To define this decision information, IFS is invalid, but it can be efficiently defined by PFS. In the Pythagorean fuzzy set, Peng et al. [6] introduced some characteristics, which are division, subtraction, and other significant properties.

To understand multicriteria problems in group decision-making in the Pythagorean fuzzy setting, authors are concerned with the methods of dominance and a ranking of dependencies. For multicriteria decision-making based on Pythagorean fuzzy sets, Khan et al. [7] established prioritized aggregation operators in [7]. Peng et al. [8] advanced linguistic Pythagorean fuzzy sets (LPFSSs) and the Pythagorean fuzzy linguistic numbers’ operating laws and score function. An optimizing variance technique was developed by Wei
et al. [9] to clarify problems involving decision-making depending on Pythagorean fuzzy environments valued at intervals. The Pythagorean fuzzy numbers (PFN) subtraction and division acts were intended by Gou et al. [10]. The notion of the obvious concept of the Pythagorean fuzzy distance degree was provided by Pend et al. [11], which is categorized by a Pythagorean fuzzy number that will minimize a drawback of data additionally proceeding to provide imaginative proof. The well-known definition of the novel score function is also well defined. Liang et al. [12] introduced the Bonferroni weighted Pythagorean fuzzy geometric (BWPGF) operator.

In [13], Garg introduced an interval-valued Pythagorean fuzzy geometric (IVPFG) operator and discussed a new precision function. Khan et al. improved the definition of the multiattribute decision-making TOPSIS system as well as established the integral Choquet integral average of TOPSIS on the basis of IVPFNs [14]. In [15], Khan suggested the GRA method for making multicriteria decisions under the Pythagorean fuzzy condition valued at intervals. The authors first developed the Choquet integral average interval-valued Pythagorean operator and then developed a system for making multiattribute decisions dependent on the GRA technique. An Einstein geometric intuitionistic fuzzy (EGIF) operator was introduced by Wang [16] and an ordered weighted Einstein geometric intuitionistic fuzzy (OWEGIF) operator.

The definition of the intuitionistic fuzzy weighted averaging operator was introduced by Wang and Liu [17] and an ordered weighted Einstein average intuitionistic fuzzy (OWEAF) operator. Einstein operations can be divided into two categories: Einstein sum and product. In [18], Garg implemented the Einstein sum definition of the Pythagorean fuzzy mean aggregation operators such as the average operator of Pythagorean fuzzy Einstein, the geometric mean of Pythagorean fuzzy Einstein, and the ordered geometric weighted operator of Pythagorean fuzzy Einstein. For more related work, one may refer to [19–39].

We will use the Einstein product in this article and present the Pythagorean cubic fuzzy Einstein weighted geometric (PCFEWG) operator definition. Under Pythagorean fuzzy data, these two are new decision-making methods, but the Pythagorean fuzzy cubic Einstein weighted geometric (PCFEWG) operator is more reliable than mean aggregation operators.

This paper is composed of nine sections. We begin with a brief overview relevant to the literature review in Section 1. We provide essential concepts and consequences in Section 2 that we can include in the following aspects. In Section 3, we define the Pythagorean cubic fuzzy number and their properties. We propose Pythagorean cubic fuzzy Einstein operations in Section 4 and examine some excellent features of the suggested operations. We present a Pythagorean cubic fuzzy Einstein weighted geometric aggregation operator (PCFEWG) in Section 5. With Pythagorean cubic fuzzy data, we apply the (PCFEWG) operator to MADM in Section 6 and we also give a case of numerical development (PF EWG) operator in Section 7. In Section 8, the comparative analysis is given and the conclusion is in Section 9.

2. Preliminaries

We introduce a basic definition and essential characteristics in this section.

Definition 1 (see [8]). Let \( \tilde{X} \) be a universal set, then the fuzzy set (FS) \( \tilde{P} \) is defined as follows:

\[
\tilde{P} = \left\{ \tilde{x}, \tilde{\mu}_{\tilde{P}}(\tilde{x}) | \tilde{x} \in \tilde{X} \right\},
\]

where \( \tilde{\mu}_{\tilde{P}}(\tilde{x}) \) is a mapping from \( \tilde{X} \) to [0,1] and \( \tilde{\mu}_{\tilde{P}}(\tilde{x}) \) is known as the membership function of \( \tilde{x} \in \tilde{X} \).

Definition 2 (see [3]). Let \( \tilde{X} \) be a universal set, then the intuitionistic fuzzy set (IFS) \( \tilde{I} \) is defined as follows:

\[
\tilde{I} = \left\{ \tilde{x}, \tilde{\mu}_{\tilde{I}}(\tilde{x}), \tilde{\nu}_{\tilde{I}}(\tilde{x}) | \tilde{x} \in \tilde{X} \right\},
\]

where \( \tilde{\mu}_{\tilde{I}}(\tilde{x}) \) and \( \tilde{\nu}_{\tilde{I}}(\tilde{x}) \) are a mapping from \( \tilde{X} \) to [0,1] also satisfying the conditions 0 \( \leq \tilde{\mu}_{\tilde{I}} \leq 1 \), 0 \( \leq \tilde{\nu}_{\tilde{I}} \leq 1 \), and 0 \( \leq \tilde{\mu}_{\tilde{I}}^2 + \tilde{\nu}_{\tilde{I}}^2 \leq 1 \), for all \( \tilde{x} \in \tilde{X} \) and characterize the membership and nonmembership degree to set \( \tilde{P} \).

Definition 3 (see [19]). Let \( \tilde{X} \) be a universal set, then the Pythagorean fuzzy set (PFS) \( \tilde{P} \) is defined as follows:

\[
\tilde{P} = \left\{ \tilde{x}, \tilde{\mu}_{\tilde{P}}(\tilde{x}), \tilde{\nu}_{\tilde{P}}(\tilde{x}) | \tilde{x} \in \tilde{X} \right\},
\]

where \( \tilde{\mu}_{\tilde{P}}(\tilde{x}) \) and \( \tilde{\nu}_{\tilde{P}}(\tilde{x}) \) are a mapping from \( \tilde{X} \) to [0,1] also satisfying the conditions 0 \( \leq \tilde{\mu}_{\tilde{P}} \leq 1 \), 0 \( \leq \tilde{\nu}_{\tilde{P}} \leq 1 \), and 0 \( \leq (\tilde{\mu}_{\tilde{P}}^2 + \tilde{\nu}_{\tilde{P}}^2) \leq 1 \), for all \( \tilde{x} \in \tilde{X} \) and characterize the membership and nonmembership degree to set \( \tilde{P} \). Let \( \tilde{\mu}_{\tilde{P}}(\tilde{x}) = \sqrt{1 - (\tilde{\mu}_{\tilde{P}}^2 + \tilde{\nu}_{\tilde{P}}^2)} \), then it is known as the Pythagorean fuzzy index of \( \tilde{x} \in \tilde{X} \) to set \( \tilde{P} \), representing the degree of indeterminacy of \( \tilde{P} \). Also, for every \( \tilde{x} \in \tilde{X} \), we represent the Pythagorean fuzzy number (PFN) by \( \tilde{B} = \langle \Lambda_{\tilde{B}}, \Gamma_{\tilde{B}} \rangle \).

Definition 4 (see [19]). Let \( \tilde{B}_{1} = \langle \Lambda_{\tilde{B}_{1}}, \Gamma_{\tilde{B}_{1}} \rangle \), \( \tilde{B}_{2} = \langle \Lambda_{\tilde{B}_{2}}, \Gamma_{\tilde{B}_{2}} \rangle \), and \( \tilde{B} = \langle \Lambda_{\tilde{B}}, \Gamma_{\tilde{B}} \rangle \) be three (PFNs) and \( \Lambda > 0 \), then we have

\[
\begin{align*}
(1) \quad & \tilde{B}_{1} \oplus \tilde{B}_{2} = \langle \Lambda_{\tilde{B}_{1}}^2 + \Lambda_{\tilde{B}_{2}}^2 - \Lambda_{\tilde{B}_{1}}^2 \Lambda_{\tilde{B}_{2}}^2, \Gamma_{\tilde{B}_{1}}^2 + \Gamma_{\tilde{B}_{2}}^2 - \Gamma_{\tilde{B}_{1}}^2 \Gamma_{\tilde{B}_{2}}^2 \rangle; \\
(2) \quad & \tilde{B}_{1} \otimes \tilde{B}_{2} = \langle \tilde{\mu}_{\tilde{B}_{1}} \tilde{\mu}_{\tilde{B}_{2}}, \tilde{\nu}_{\tilde{B}_{1}} \tilde{\nu}_{\tilde{B}_{2}} \rangle; \\
(3) \quad & \tilde{\lambda} \tilde{B} = \langle (\tilde{\mu}_{\tilde{B}})^2, (\tilde{\nu}_{\tilde{B}})^2 \rangle; \\
(4) \quad & \tilde{B} \lambda = \langle (\tilde{\mu}_{\tilde{B}})^2, 1 - (1 - (\tilde{\nu}_{\tilde{B}})^2)^2 \rangle; \\
(5) \quad & \tilde{B}^c = \langle (\tilde{\mu}_{\tilde{B}})^2, 1 - (1 - (\tilde{\nu}_{\tilde{B}})^2)^2 \rangle.
\end{align*}
\]

Definition 5 (see [20]). Let \( \tilde{X} \) be a universal set, then the object with the following formulation is an IVPFN set \( \tilde{R} = \langle \tilde{x}, \tilde{\mu}_{\tilde{R}}(\tilde{x}), \tilde{\nu}_{\tilde{R}}(\tilde{x}) | \tilde{x} \in \tilde{X} \rangle \).
Where $\tilde{\mu}_R(\bar{x}) = [\tilde{\mu}_R^L(\bar{x}), \tilde{\mu}_R^U(\bar{x})] \subseteq [0, 1]$ and $\tilde{\nu}_R(\bar{x}) = [\tilde{\nu}_R^L(\bar{x}), \tilde{\nu}_R^U(\bar{x})] \subseteq [0, 1]$ are the intervals, and $\tilde{\mu}_R^E(\bar{x}) = \inf \tilde{\mu}_R(\bar{x})$ and $\tilde{\nu}_R^E(\bar{x}) = \sup \tilde{\nu}_R(\bar{x})$; similarly, $\tilde{\nu}_R(\bar{x}) = \inf \tilde{\nu}_R(\bar{x})$ and $\tilde{\nu}_R(\bar{x}) = \sup \tilde{\nu}_R(\bar{x})$, for all $\bar{x} \in \tilde{X}$. Also, $0 \leq (\tilde{\mu}_R^L(\bar{x}))^2 + (\tilde{\nu}_R^L(\bar{x}))^2 \leq 1$. Let $\tilde{\mu}_R(\bar{x}) = [\tilde{\mu}_R^L(\bar{x}), \tilde{\mu}_R^U(\bar{x})], \forall \bar{x} \in \tilde{X}$, then it is known as the interval-valued Pythagorean fuzzy index of $\bar{x}$ to $\tilde{R}$, where $\tilde{\mu}_R(\bar{x}) = \sqrt{1 - (\tilde{\mu}_R^L(\bar{x}))^2 + (\tilde{\nu}_R^L(\bar{x}))^2}$ and $\tilde{\nu}_R(\bar{x}) = \sqrt{1 - (\tilde{\mu}_R^L(\bar{x}))^2 + (\tilde{\nu}_R^L(\bar{x}))^2}$ which meet the requirements of the following relationship:

1. If $\tilde{\mu}_R^L(\bar{x}) = \tilde{\mu}_R^U(\bar{x})$ and $\tilde{\nu}_R^L(\bar{x}) = \tilde{\nu}_R^U(\bar{x})$, then an IVIFS set becomes a PFS set.

2. If $\tilde{\mu}_R^L(\bar{x}) + \tilde{\nu}_R^L(\bar{x}) \leq 1$, then an IVIFS becomes an IVIFS.

**Definition 6** (see [21]). Let $\tilde{A} = ([\tilde{\mu}_A^L, \tilde{\mu}_A^U], [\tilde{\nu}_A^L, \tilde{\nu}_A^U])$, $\tilde{A}_1 = ([\tilde{\mu}_A^L, \tilde{\mu}_A^U], [\tilde{\nu}_A^L, \tilde{\nu}_A^U])$, and $\tilde{A}_2 = ([\tilde{\mu}_A^L, \tilde{\mu}_A^U], [\tilde{\nu}_A^L, \tilde{\nu}_A^U])$ are three IVPFNs and $\tilde{A}_1 \supset \tilde{A}_2$, then we have the following:

1. $\tilde{\lambda}(\tilde{\mu}_A) = \left(1 - \left(\tilde{\mu}_A^L\right)^2, 1 - \left(\tilde{\mu}_A^U\right)^2\right)$.

2. $\tilde{\gamma}(\tilde{\mu}_A) = \left(\tilde{\nu}_A^L, \tilde{\nu}_A^U\right)$.

3. $\tilde{\lambda}_1 \oplus \tilde{\lambda}_2 = \left(\tilde{\lambda}_1^L \cap \tilde{\lambda}_2^L, \tilde{\lambda}_1^U \cap \tilde{\lambda}_2^U\right)$.

4. $\tilde{\lambda}_1 \ominus \tilde{\lambda}_2 = \left(\tilde{\lambda}_1^L \setminus \tilde{\lambda}_2^L, \tilde{\lambda}_1^U \setminus \tilde{\lambda}_2^U\right)$.

**Definition 7** (see [21]). Let $\tilde{A} = ([\tilde{\mu}_A^L, \tilde{\mu}_A^U], [\tilde{\nu}_A^L, \tilde{\nu}_A^U])$; the score function of $A$ can be defined as follows using the IVPFN $\tilde{A}$:

$$S(\tilde{A}) = \frac{1}{2} \left(\tilde{\mu}_A^L + \tilde{\mu}_A^U\right)^2 - \tilde{\nu}_A^L - \tilde{\nu}_A^U,$$

where $S(\tilde{A}) \in [0, 1]$.

**Definition 8** (see [23]). Let $\tilde{A} = ([\tilde{\mu}_A^L, \tilde{\mu}_A^U], [\tilde{\nu}_A^L, \tilde{\nu}_A^U])$; the accuracy function of $A$ can be defined as follows using the IVPFN $\tilde{A}$:

$$H(\tilde{A}) = \frac{1}{2} \left(\tilde{\mu}_A^L + \tilde{\mu}_A^U\right)^2 - \tilde{\nu}_A^L - \tilde{\nu}_A^U,$$

where $H(\tilde{A}) \in [0, 1]$.

**Definition 9** (see [21]). Let $\tilde{A} = ([\tilde{\mu}_A^L, \tilde{\mu}_A^U], [\tilde{\nu}_A^L, \tilde{\nu}_A^U])$ and $\tilde{A}_1 = ([\tilde{\mu}_A^L, \tilde{\mu}_A^U], [\tilde{\nu}_A^L, \tilde{\nu}_A^U])$ be two IVFNs, then

$$S(\tilde{A}) = \frac{1}{2} \left(\tilde{\mu}_A^L + \tilde{\mu}_A^U\right)^2 - \tilde{\nu}_A^L - \tilde{\nu}_A^U,$$

$$S(\tilde{A}_1) = \frac{1}{2} \left(\tilde{\mu}_{A_1}^L + \tilde{\mu}_{A_1}^U\right)^2 - \tilde{\nu}_{A_1}^L - \tilde{\nu}_{A_1}^U,$$

are the score of $\tilde{A}$ and $\tilde{A}_1$, respectively, while

$$H(\tilde{A}) = \frac{1}{2} \left(\tilde{\mu}_A^L + \tilde{\mu}_A^U\right)^2 + \tilde{\nu}_A^L + \tilde{\nu}_A^U,$$

$$H(\tilde{A}_1) = \frac{1}{2} \left(\tilde{\mu}_{A_1}^L + \tilde{\mu}_{A_1}^U\right)^2 + \tilde{\nu}_{A_1}^L + \tilde{\nu}_{A_1}^U,$$

are the accuracy of $A$ and $A_1$, respectively, which meet the following criteria:

1. If $S(\tilde{A}) < S(\tilde{A}_1)$, then $\tilde{A} < \tilde{A}_1$.

2. If $S(\tilde{A}) > S(\tilde{A}_1)$, then $\tilde{A} > \tilde{A}_1$.

3. If $S(\tilde{A}) = S(\tilde{A}_1)$, we have the following:

   a. If $H(\tilde{A}) = H(\tilde{A}_1)$, then $\tilde{A} = \tilde{A}_1$.

   b. If $H(\tilde{A}) < H(\tilde{A}_1)$, then $\tilde{A} < \tilde{A}_1$.

   c. If $H(\tilde{A}) > H(\tilde{A}_1)$, then $\tilde{A} > \tilde{A}_1$.

**Definition 10** (see [22]). Let $\tilde{X}$ be a universal set. Then, a cubic set can be stated:

$$C = \{\bar{x}, \tilde{\nu}_C(\bar{x}), \tilde{\nu}_C(\bar{x})\mid \bar{x} \in \tilde{X}\},$$

where $\tilde{\nu}_C$ is an interval-valued fuzzy set in $\tilde{X}$ and $\tilde{\nu}_C$ is a fuzzy set in $\tilde{X}$.

**Definition 11** (see [19]). Let $p_1$ and $p_2$ be two PFNs, then the distance between $p_1$ and $p_2$ can be described as

$$D(p_1, p_2) = \frac{1}{2} \left(\left\vert\tilde{\mu}_1 - \tilde{\mu}_2\right\vert^2 + \left\vert\tilde{\nu}_1 - \tilde{\nu}_2\right\vert^2\right),$$

where $\tilde{\mu}_1 = \left(\tilde{\mu}_1^L, \tilde{\mu}_1^U\right)$ and $\tilde{\nu}_1 = \left(\tilde{\nu}_1^L, \tilde{\nu}_1^U\right)$.

**Definition 12** (see [23]). Let $p_i = ([a_i, b_i], [c_i, d_i])$, $i = 1, 2$, be two IVFNs, then the distance between $p_1$ and $p_2$ is defined as follows:

$$D(p_1, p_2) = \frac{1}{4} \left[\left\vert\tilde{\mu}_1 - \tilde{\mu}_2\right\vert^2 + \left\vert\tilde{\nu}_1 - \tilde{\nu}_2\right\vert^2\right],$$

where $\tilde{\mu}_1 = \left(\tilde{\mu}_1^L, \tilde{\mu}_1^U\right)$ and $\tilde{\nu}_1 = \left(\tilde{\nu}_1^L, \tilde{\nu}_1^U\right)$.

**Definition 13** (see [24]). Let $p_i = ([a_i, b_i], [c_i, d_i]), i = 1, 2, 3, \ldots, n$, be the collection of IVFNs, then IVPFWG operator is defined as
where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $p_i (i = 1, 2, 3, \ldots, n)$ and $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

**Definition 15** (see [24]). Let $p_i = (\left\langle [a_i, b_i], [c_i, d_i] \right\rangle)$, $(i = 1, 2, 3, \ldots, n)$, be the collection of IVPFNs, then IVPFHGW operator is defined as

$$
\text{IVPFHGW}_w(p_1, p_2, \ldots, p_n) = \left\langle \left\langle \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i}, \prod_{i=1}^{n} (b_{\sigma(i)})^{w_i} \right\rangle, \left\langle \sqrt{1 - \prod_{i=1}^{n} \left(1 - (c_{\sigma(i)})^{w_i} \right)}, \sqrt{1 - \prod_{i=1}^{n} \left(1 - (d_{\sigma(i)})^{w_i} \right)} \right\rangle \right\rangle,
$$

where $p_{\sigma(i)}$ is the i-th largest value and $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $p_i (i = 1, 2, 3, \ldots, n)$.

**Definition 16** (see [25, 26]). Let $p_i = (\left\langle [a_i, b_i], [c_i, d_i] \right\rangle)$, $(i = 1, 2, 3, \ldots, n)$, be the collection of IVPFNs, and $\delta > 0$, then the following operational laws are satisfied:

1. $\delta p_i = (\sqrt{1 - (a_i)^2}, \sqrt{1 - (b_i)^2}, (c_i)^2, (d_i)^2)$,
2. $(p_i)^\delta = ([a_i]^\delta, [b_i]^\delta, [1 - (c_i)^2]^\delta, \sqrt{1 - (1 - (d_i)^2)^\delta})$,
3. $p_1 \circ p_2 = ([a_1, a_2, b_1, b_2], \sqrt{(c_1)^2 + (c_2)^2 - (c_1)^2} \cdot (c_2)^2, \sqrt{(d_1)^2 + (d_2)^2 - (d_1)^2} \cdot (d_2)^2)$,
4. $p_1 \odot p_2 = ([\sqrt{(a_1)^2 + (a_2)^2 - (a_2)^2} \cdot (a_2)^2, \sqrt{(b_1)^2 + (b_2)^2 - (b_2)^2} \cdot (d_2)^2], [c_1, c_2, d_1, d_2])$.

**3. Pythagorean Cubic Fuzzy Numbers and Their Characteristics**

In this unit, we define some new concepts of the Pythagorean cubic fuzzy set and discuss the characteristics of the Pythagorean cubic fuzzy set that is not an intuitionistic cubic fuzzy set with the help of illustrations. In this article, $p_{\epsilon}$ stands for a Pythagorean cubic fuzzy set.

**Definition 17** (see [27]). Let $X$ be a fixed set, then a Pythagorean cubic fuzzy set can be defined as

$$
p_{\epsilon} = \left\{ (x, \mu_{\epsilon}(x), \nu_{\epsilon}(x)) | x \in X \right\},
$$

where

$$
\mu_{\epsilon}(x) = \left\langle A(x), \lambda(x) \right\rangle,
\nu_{\epsilon}(x) = \left\langle A(x), \mu(x) \right\rangle,
$$

and

$$
0 \leq (\mu_{\epsilon}(x))^2 + (\nu_{\epsilon}(x))^2 \leq [\bar{1}, 1].
$$

The preceding condition may also be written as follows:

$$
0 \leq (\sup (A(x)))^2 + (\sup (A(x)))^2 \leq 1,
0 \leq \lambda^2(x) + \mu^2(x) \leq 1.
$$

For a Pythagorean cubic set, the degree of indeterminacy is classified as

$$
\pi_{p_{\epsilon}} = \left\langle \sqrt{1 - \sup (A(x))^2} - (\sup (A(x)))^2, \sqrt{1 - \lambda^2(x) - \mu^2(x)} \right\rangle.
$$

For simplicity, we call $(\mu_{\epsilon}, \nu_{\epsilon})$ a Pythagorean cubic fuzzy number (PCFN) denoted by $p_{\epsilon} = (\mu_{\epsilon}, \nu_{\epsilon})$. 

$$
\text{IVPFWG}_w(p_1, p_2, \ldots, p_n) = \left\langle \left\langle \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i}, \prod_{i=1}^{n} (b_{\sigma(i)})^{w_i} \right\rangle, \left\langle \sqrt{1 - \prod_{i=1}^{n} \left(1 - (c_{\sigma(i)})^{w_i} \right)}, \sqrt{1 - \prod_{i=1}^{n} \left(1 - (d_{\sigma(i)})^{w_i} \right)} \right\rangle \right\rangle,
$$
Example 1. Let \( X = \{x_1, x_2, x_3\} \) be a fixed set and consider a set in \( X \) by

\[
p_c = \left( \begin{array}{c}
\langle (x_1, [0.5, 0.6], 0.7), ([0.6, 0.7], 0.5) \rangle \\
\langle (x_2, [0.4, 0.7], 0.6), ([0.5, 0.4], 0.6) \rangle \\
\langle (x_3, [0.4, 0.6], 0.7), ([0.5, 0.2], 0.6) \rangle
\end{array} \right).
\]

Then, also \((0.7)^2 + (0.5)^2 = 0.49 + 0.25 = 0.74 < 1\); similarly, we can calculate the other cases. Thus, \( p_c, p_{c_1}, \) and \( p_{c_2} \) are (PCFNs). Therefore, \( p_c \) are PCFS.

Definition 18. Let \( p_{c_1} = \langle (A_1, \lambda_1), \lambda_1, \mu_1 \rangle, \) \( p_{c_2} = \langle (A_2, \lambda_2), \lambda_2, \mu_2 \rangle \) and \( p_c = \langle (A, \lambda), \lambda, \mu \rangle \) be three PCFNS and \( \delta > 0, \) where \( A_1 = [a_1, b_1], \) \( A = [a, b], \) and \( A = [\bar{a}, \bar{b}] \); the operational laws are as follows:

(1) \( p_{c_1} \oplus p_{c_2} = \left\langle \left[ \sqrt{a_1^2 + a_2^2 - a_1^2 a_2^2}, \sqrt{b_1^2 + b_2^2 - b_1^2 b_2^2} \right], \sqrt{\lambda_1^2 + \lambda_2^2 - \lambda_1^2 \lambda_2^2}, \left\langle \lambda_1, \lambda_2, \mu_1, \mu_2, \lambda \right\rangle \right\rangle, \)

(2) \( p_{c_1} \odot p_{c_2} = \left\langle \left[ \sqrt{1 - \frac{(1 - a_1^2 b_2^2)(1 - b_1^2 a_2^2)}{1 - b_1^2 b_2^2}}, \sqrt{1 - \frac{(1 - b_1^2 a_2^2)(1 - a_1^2 b_2^2)}{1 - a_1^2 a_2^2}} \right], \left\langle \lambda_1, \lambda_2, \mu_1, \mu_2, \lambda \right\rangle \right\rangle, \)

(3) \( \delta p_{c_1} = \left\langle \left[ \sqrt{1 - (1 - a_1^2)^2}, \sqrt{1 - (1 - b_1^2)^2} \right], \sqrt{1 - (1 - \lambda_1^2)^2}, \left\langle \lambda_1, \lambda_2, \mu_1, \mu_2, \lambda \right\rangle \right\rangle, \)

(4) \( p_{c_1}^\delta = \left\langle \left[ \sqrt{1 - (1 - a_1^2)^2}, \sqrt{1 - (1 - b_1^2)^2} \right], \left\langle \lambda_1, \lambda_2, \mu_1, \mu_2, \lambda \right\rangle \right\rangle, \)

(5) \( p_{c_1}^\delta = \langle A_1, \lambda_1, \lambda_1 \rangle. \)

Definition 20. Let \( p_{c_1} = \langle (A_1, \lambda_1), \lambda_1, \mu_1 \rangle \) and \( p_{c_2} = \langle (A_2, \lambda_2), \lambda_2, \mu_2 \rangle \) be two PCFNS, \( S(p_c) \) be the score function of \( p_{c_1} \), and \( S(p_{c_2}) \) be the score function of \( p_{c_2} \). Then,

(1) If \( S(p_{c_1}) < S(p_{c_2}) \), then \( p_{c_1} < p_{c_2} \).

(2) If \( S(p_{c_1}) > S(p_{c_2}) \), then \( p_{c_1} > p_{c_2} \).

(3) If \( S(p_{c_1}) = S(p_{c_2}) \), then \( p_{c_1} \sim p_{c_2} \).

Example 2. Let \( p_{c_1} = \langle ([0.5, 0.7], 0.9), ([0.1, 0.5], 0.6) \rangle, \) \( p_{c_2} = \langle ([0.4, 0.7], 0.6), ([0.2, 0.4], 0.6) \rangle, \) and \( p_{c_3} = \langle ([0.03, 0.8], 0.9), ([0.0, 0.3], 0.7) \rangle \) be three PCFNS. Then, by Definition 18, we have \( S(p_{c_1}) = 0.01, S(p_{c_2}) = 0.027 \) and \( S(p_{c_3}) = -0.0173 \). Thus, \( S(p_{c_1}) > S(p_{c_2}) > S(p_{c_3}) \). Let \( p_{c_1} = \langle ([0.5, 0.7], 0.9), ([0.1, 0.5], 0.6) \rangle \) and \( p_{c_2} = \langle ([0.4, 0.7], 0.6), ([0.2, 0.4], 0.7) \rangle \) be two PCFNS. Then by Definition 19, we have \( S(p_{c_1}) = 0.01 \) and \( S(p_{c_2}) = 0.01 \) Thus, \( S(p_{c_1}) = S(p_{c_2}) \).

Therefore, by Definition 20, we cannot get information from \( p_{c_1} \) and \( p_{c_2} \). Usually, such a case grows in preparation. It is clear from Definition 20 that we are unable to consider the requirement that two PCFNS have the same ranking. On the other side, deviancy may be changed. The consistency property of all the components to the average number in a PCFNS returns that they may accept. For the comparison of two PCFNS, we present a definition of accuracy degree.

Definition 21. Let \( p_c = \langle (A_1, \lambda_1), \lambda_1, \mu_1 \rangle \) be a PCF. Then, we define the accuracy degree of \( p_c \), which is denoted by \( \alpha(p_c) \), where \( A_1 = [a_1, b_1], A = [\bar{a}, \bar{b}], \) and \( A = [a, b], \) then the following will hold:

(1) \( p_{c_1} \oplus p_{c_2} = p_{c_1} \oplus p_{c_2} \),

(2) \( p_{c_1} \odot p_{c_2} = p_{c_1} \odot p_{c_2} \),

(3) \( \delta(p_{c_1} \oplus p_{c_2}) = \delta(p_{c_1}) \oplus \delta(p_{c_2}) \),

(4) \( (\delta_1 + \delta_2) p_c = \delta_1 p_c \oplus \delta_2 p_c \),

(5) \( p_{c_1}^\delta = p_{c_1}^\delta \),

(6) \( p_{c_2}^\delta = p_{c_2}^\delta \).

Proof. The proof is obvious.

We describe a score function and its basic properties to equate two PCFNS.

Definition 19. Let \( p_c = \langle (A_1, \lambda_1), \lambda_1, \mu_1 \rangle \) be a PCF, where \( A_1 = [a_1, b_1], A = [\bar{a}, \bar{b}], \) and \( A = [a, b], \) we can introduce the score function of \( p_c \) as

\[
S(p_c) = \left( \frac{a_1 + b_1 - \lambda_1}{3} \right)^2 - \left( \frac{\bar{a}_1 + b_1 - \mu_1}{3} \right)^2,
\]

where \( S(p_c) \in [-1, 1] \).

Example 3. From example 2, since \( S(p_{c_1}) = 0.01 \) and \( S(p_{c_2}) = 0.01 \), thus, \( S(p_{c_1}) = S(p_{c_2}) \). So, we have \( \alpha(p_{c_1}) = 0.01 \) and \( \alpha(p_{c_2}) = 0.044 \). Thus, \( \alpha(p_{c_1}) > \alpha(p_{c_2}) \). Hence, \( p_{c_1} > p_{c_2} \). As a result, the condition when two PCFNS have the same score has been resolved.

Definition 23. Let \( p_c \) and \( p_{c_2} \) be any two PCFNS on a set \( X = \{x_1, x_2, \ldots, x_n\} \). The following is a definition of the distance measure between \( p_c \) and \( p_{c_2} \):

\[
D(p_c, p_{c_2}) = \frac{1}{6} \left| \left[ a_1^2 - a_2^2 \right] + \left[ b_1^2 - b_2^2 \right] + \left[ \bar{a}_1^2 - \bar{a}_2^2 \right] + \left[ b_1^2 - b_2^2 \right] + \left[ \bar{a}_1^2 - \bar{a}_2^2 \right] \right|,
\]

where \( a, b, \) and \( \bar{a}, \bar{b} \) are the basic properties of two PCFNS.
Example 4. Let $p_{c_1} = \langle \langle 0.6, 0.7 \rangle, 0.3 \rangle, \langle 0.5, 0.7, 0.8 \rangle \rangle$ and $p_{c_2} = \langle \langle 0.5, 0.6 \rangle, 0.4 \rangle, \langle 0.4, 0.7, 0.5 \rangle \rangle$ be two PCFNs. Then,

$$D(p_{c_1}, p_{c_2}) = \frac{1}{6} \|[0.36 - 0.25] + [0.49 - 0.36] + [0.25 - 0.16] + [0.49 - 0.49] + [0.9 - 0.16] + [0.64 - 0.25]||,$$

$$D(p_{c_1}, p_{c_2}) = \frac{1}{6} \|[0.11] + [0.09] + [0] + [0.74] + [0.39]|| = 0.2433.$$  

4. Einstein Operations of Pythagorean Cubic Fuzzy Sets

In this section, we defined the Einstein product $(p_{c_1} \otimes p_{c_2})$ and the Einstein sum $(p_{c_1} \oplus p_{c_2})$ on two PCFSs $p_{c_1}$ and $p_{c_2}$, which can be defined in the following forms.

$$p_{c_1} \otimes p_{c_2} = \left\{ \left[ \left( a_1^2, b_1^2 \right), \lambda_1^2 \right] \cdot \left[ \left( a_2^2, b_2^2 \right), \lambda_2^2 \right] \right\} \left( \left[ \left( \frac{a_1^2}{a_2^2}, \frac{b_1^2}{b_2^2} \right), \mu \right] \right) \left( \left[ \left( \frac{a_2^2}{a_1^2}, \frac{b_2^2}{b_1^2} \right), \mu_2 \right] \right) \right\}.$$

$$p_{c_1} \oplus p_{c_2} = \left\{ \left( \left( a_1^2, b_1^2 \right), \lambda_1^2 \right) \cdot \left( \left( a_2^2, b_2^2 \right), \lambda_2^2 \right) \right\} \left( \left( \left( \frac{a_1^2}{a_2^2}, \frac{b_1^2}{b_2^2} \right), \mu \right) \right) \left( \left( \left( \frac{a_2^2}{a_1^2}, \frac{b_2^2}{b_1^2} \right), \mu_2 \right) \right) \right\}.$$

Theorem 2. Let $n$ be any positive integer and $p_{c}$ is a PCFS, then the exponentiation operation $p_{c} \wedge e^n$ is a mapping from $\mathbb{R}^\times \otimes \rightarrow \otimes$.

$$p_{c} \wedge e^n = \left\{ \left( \left( \frac{a_1^2}{a_2^2}, \frac{b_1^2}{b_2^2} \right), \mu \right) \right\} \left( \left( \left( \frac{a_1^2}{a_2^2}, \frac{b_1^2}{b_2^2} \right), \mu_2 \right) \right) \right\}.$$

where $p_{c} \wedge e^n = p_{c_1} \otimes p_{c_2} \otimes p_{c_3} \otimes \ldots \otimes p_{c_n} \otimes p_{c_1} \otimes p_{c_2}$. Moreover, $p_{c} \wedge e^n$ is a Pythagorean cubic fuzzy set (PCFS), even if $n \in \mathbb{R}^\times$.

Proof. We may prove that equation (24) holds for all positive integers $n$ using mathematical induction. First, it holds for $n = 1$.
Taking the left-hand side of the equation above, \[ p_{c_t, x} = p_{c, \{x, \mu_{p_t}, v_{p_t}\}} | x \in X \]. (27)

\[
\frac{\sqrt{2([a^2_i, b^2_i], \lambda^i_1)^n}}{\sqrt{(2 - [a^2_i, b^2_i], \lambda^i_1)^n + ([a^2_i, b^2_i], \lambda^i_1)^n}} \frac{\sqrt{1 + \left(\left([a^2_i, b^2_i], \mu_i^1\right)\right)^n - \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^n}}{\sqrt{1 + \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^n + \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^n}}
\]

Taking the right-hand side of the equation above, \[ p_{c, \{x, \mu_{p_t}, v_{p_t}\}} | x \in X \].

\[ \text{holds for } n = k. \text{ If equation (25) holds for } n = k, \text{ then equation (25) also holds for } n = k + 1. \]

From equations (25) and (27), we have equation (25) which holds for \( n = 1 \). Next, we show that equation (25)

\[
\frac{\sqrt{2([a^2_i, b^2_i], \lambda^i_1)^{k+1}}}{\sqrt{(2 - [a^2_i, b^2_i], \lambda^i_1)^{k+1} + ([a^2_i, b^2_i], \lambda^i_1)^{k+1}}} \frac{\sqrt{1 + \left(\left([a^2_i, b^2_i], \mu_i^1\right)\right)^{k+1} - \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^{k+1}}}{\sqrt{1 + \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^{k+1} + \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^{k+1}}}
\]

\[ \otimes e_{p_{c, \{x, \mu_{p_t}, v_{p_t}\}} | x \in X} \] (29)

Now, we'll show that equation (25) is valid for every positive integer \( n \),

\[ 0 \leq \left( \frac{\sqrt{2([a^2_i, b^2_i], \lambda^i_1)^n}}{\sqrt{(2 - [a^2_i, b^2_i], \lambda^i_1)^n + ([a^2_i, b^2_i], \lambda^i_1)^n}} \right)^{2} \left( \frac{\sqrt{1 + \left(\left([a^2_i, b^2_i], \mu_i^1\right)\right)^n - \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^n}}{\sqrt{1 + \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^n + \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^n}} \right)^{2} \leq 1, \] (30)

even if \( n \in \mathbb{R}^+ \). Since \( 0 \leq \mu_{p_{c, \{x, \mu_{p_t}, v_{p_t}\}}} (x) \leq 1 \), \( 0 \leq \mu_{p_{c, \{x, \mu_{p_t}, v_{p_t}\}}} (x) + v_{p_{c, \{x, \mu_{p_t}, v_{p_t}\}}} (x) \leq 1 \), then \( 1 - \mu_{p_{c, \{x, \mu_{p_t}, v_{p_t}\}}} (x) \geq v_{p_{c, \{x, \mu_{p_t}, v_{p_t}\}}} (x) \geq 0 \), so

\[ 0 \leq \left( \frac{\sqrt{2([a^2_i, b^2_i], \lambda^i_1)^n}}{\sqrt{(2 - [a^2_i, b^2_i], \lambda^i_1)^n + ([a^2_i, b^2_i], \lambda^i_1)^n}} \right)^{2} \left( \frac{\sqrt{1 + \left(\left([a^2_i, b^2_i], \mu_i^1\right)\right)^n - \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^n}}{\sqrt{1 + \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^n + \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^n}} \right)^{2} \leq 1 \] (31)

\[ 0 \leq \left( \frac{\sqrt{2([a^2_i, b^2_i], \lambda^i_1)^n}}{\sqrt{(2 - [a^2_i, b^2_i], \lambda^i_1)^n + ([a^2_i, b^2_i], \lambda^i_1)^n}} \right)^{n} \left( \frac{\sqrt{1 + \left(\left([a^2_i, b^2_i], \mu_i^1\right)\right)^n - \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^n}}{\sqrt{1 + \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^n + \left(\left([a^2_i, b^2_i], \mu_i^n\right)\right)^n}} \right)^{n} \leq 1. \]
Since

\[
\left\{ \begin{array}{l}
\frac{2([a_i^2,b_i^2],\lambda_i^2)^n}{2 - [a_i^2,b_i^2],\lambda_i^2} + ([a_i^2,b_i^2],\lambda_i^2)^n \\
\frac{2([a_i^2,b_i^2],\lambda_i^2)^n}{1 + 1 - [a_i^2,b_i^2],\lambda_i^2} + ([a_i^2,b_i^2],\lambda_i^2)^n \\
\leq \frac{2([a_i^2,b_i^2],\lambda_i^2)^n}{1 + 1 - [a_i^2,b_i^2],\lambda_i^2} + ([a_i^2,b_i^2],\lambda_i^2)^n \\
\end{array} \right.
\]

(32)

again

\[
1 - \nu^2_{P_{1}}(x) \geq \mu^2_{p_1}(x) \geq 0 \left( 1 + [a_i^2,b_i^2],(\mu_i^2) \right)^n - \left( 1 - [a_i^2,b_i^2],(\mu_i^2) \right)^n \\
\left( 1 + [a_i^2,b_i^2],(\mu_i^2) + 1 - [a_i^2,b_i^2],(\mu_i^2) \right)^n \\
= \left( 1 + [a_i^2,b_i^2],(\mu_i^2) \right)^n - \left( 1 - [a_i^2,b_i^2],(\mu_i^2) \right)^n \\
\left( 1 + [a_i^2,b_i^2],(\mu_i^2) + 1 - [a_i^2,b_i^2],(\mu_i^2) \right)^n \\
\leq \frac{\left( 1 + [a_i^2,b_i^2],(\mu_i^2) \right)^n - \left( 1 - [a_i^2,b_i^2],(\mu_i^2) \right)^n}{\left( 1 + [a_i^2,b_i^2],(\mu_i^2) + 1 - [a_i^2,b_i^2],(\mu_i^2) \right)^n} \\
\right.
\]

(33)

Thus,

\[
\left( 1 + [a_i^2,b_i^2],(\mu_i^2) \right)^n - \left( 1 - [a_i^2,b_i^2],(\mu_i^2) \right)^n \leq \frac{(1 + [a_i^2,b_i^2],(\mu_i^2) - (1 - [a_i^2,b_i^2],(\mu_i^2))}{(1 + [a_i^2,b_i^2],(\mu_i^2) + 1 - [a_i^2,b_i^2],(\mu_i^2))} \\
\]

(34)

From equations (14) and (34), we have

\[
\frac{2([a_i^2,b_i^2],\lambda_i^2)^n}{1 + 1 - [a_i^2,b_i^2],\lambda_i^2} + ([a_i^2,b_i^2],\lambda_i^2)^n \\
\left( 1 + [a_i^2,b_i^2],(\mu_i^2) \right)^n - \left( 1 - [a_i^2,b_i^2],(\mu_i^2) \right)^n \\
\left( 1 + [a_i^2,b_i^2],(\mu_i^2) + 1 - [a_i^2,b_i^2],(\mu_i^2) \right)^n \\
\right.
\]
Taking the left side, 

\[
0 \leq \frac{2\left(\left[a_1^2, b_1^2\right], \lambda_1^n\right)^n}{1 + \left(1 - \left[a_1^2, b_1^2\right], \mu_1^n\right)^n + \left(\left[a_1^2, b_1^2\right], \lambda_1^n\right)^n} \leq 1.
\]

Thus, a PCFS \( P_c^\epsilon \) defined above is a PCFS for any \( n \in \mathbb{R}^+ \).

### 5. Pythagorean Cubic Fuzzy Einstein Weighted Geometric Aggregation Operator

**Definition 25.** Let \( P_{c_j} = (\mu_{P_j}, v_{P_j}), (j = 1, \ldots, m) \), be the collection of PCFVs with \( \leq \mu \), then a PCFWGE operator of dimension \( n \) is a mapping PCFWGE: \( \phi^m \rightarrow \phi \), and

\[
PCFWGE_w^c(P_{c_1}, P_{c_2}, \ldots, P_{c_m}) = P_{c_1}^\epsilon \otimes P_{c_2}^\epsilon \otimes \ldots \otimes P_{c_m}^\epsilon,
\]

(37)

\[\begin{align*}
\text{PCFWGE}_w^c(p_1^\epsilon, p_2^\epsilon, \ldots, p_m^\epsilon) &= \left\langle \left( \begin{array}{ccc}
\sqrt{2}\Pi_{m}^n(a_1^\epsilon) & \sqrt{2}\Pi_{m}^n(b_1^\epsilon) & \sqrt{2}\Pi_{m}^n(\lambda_1^\epsilon) \\
\Pi_{m}^n(2 - a_1^\epsilon) + \Pi_{m}^n(a_1^\epsilon) & \Pi_{m}^n(2 - b_1^\epsilon) + \Pi_{m}^n(b_1^\epsilon) & \Pi_{m}^n(2 - \lambda_1^\epsilon) + \Pi_{m}^n(\lambda_1^\epsilon) \\
\Pi_{m}^n(1 + a_1^\epsilon) & \Pi_{m}^n(1 + b_1^\epsilon) & \Pi_{m}^n(1 + \lambda_1^\epsilon)
\end{array} \right) \rightangle.
\end{align*}\]

(38)

where \( \bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m)^T \) is the weighted vector of \( P_{c_j} \) \( (j = 1, \ldots, n) \) such that \( \sum_{j=1}^n \bar{w}_j = 1 \).

**Proof.** Mathematical induction may be used to prove this theorem. To begin, we prove that equation (38) holds for \( m = 1 \). Taking the left side,
From equations (39) and (40), we have equation (38) which holds for $m = 1$. Now, we show that equation (38) holds for $m = k$.

\[
\text{PCFWG}_{\omega}(p_{i_1}, p_{i_2}, \ldots, p_{i_m}) =
\left\langle \frac{\sqrt{2\Pi_{m=1}^{m}(a_{j}^{2\omega})}}{\Pi_{j=1}^{m} (2 - a_{j}^{2})^{\frac{\omega}{2}} + \Pi_{j=1}^{m} (a_{j}^{2})^{\frac{\omega}{2}}}, \frac{\sqrt{2\Pi_{m=1}^{m}(b_{j}^{2\omega})}}{\Pi_{j=1}^{m} (2 - b_{j}^{2})^{\frac{\omega}{2}} + \Pi_{j=1}^{m} (b_{j}^{2})^{\frac{\omega}{2}}}, \frac{\sqrt{2\Pi_{m=1}^{m}(\lambda_{j}^{2\omega})}}{\Pi_{j=1}^{m} (2 - \lambda_{j}^{2})^{\frac{\omega}{2}} + \Pi_{j=1}^{m} (\lambda_{j}^{2})^{\frac{\omega}{2}}} \right\rangle;
\]

(40)

Next, we are going to show that equation (38) holds for $m = k + 1$.

\[
\text{PCFWG}_{\omega}(p_{i_1}, p_{i_2}, \ldots, p_{i_{m+1}}) =
\left\langle \frac{\sqrt{2\Pi_{m=1}^{m+1}(a_{j}^{2\omega})}}{\Pi_{j=1}^{m+1} (2 - a_{j}^{2})^{\frac{\omega}{2}} + \Pi_{j=1}^{m+1} (a_{j}^{2})^{\frac{\omega}{2}}}, \frac{\sqrt{2\Pi_{m=1}^{m+1}(b_{j}^{2\omega})}}{\Pi_{j=1}^{m+1} (2 - b_{j}^{2})^{\frac{\omega}{2}} + \Pi_{j=1}^{m+1} (b_{j}^{2})^{\frac{\omega}{2}}}, \frac{\sqrt{2\Pi_{m=1}^{m+1}(\lambda_{j}^{2\omega})}}{\Pi_{j=1}^{m+1} (2 - \lambda_{j}^{2})^{\frac{\omega}{2}} + \Pi_{j=1}^{m+1} (\lambda_{j}^{2})^{\frac{\omega}{2}}} \right\rangle;
\]

(41)
Let

\[
\begin{align*}
&s_1 = \left[ \frac{\sqrt{2\prod_{j=1}^{k}(d_j^2)^{\bar{w}_j}} + \sqrt{2\prod_{j=1}^{k}(b_j^2)^{\bar{w}_j}}}{\prod_{j=1}^{k}(2 - a_j^2)^{\bar{w}_j} + \prod_{j=1}^{k}(a_j^2)^{\bar{w}_j}} \right], \\
&s_2 = \sqrt{2\prod_{j=1}^{k}(\lambda_j^2)^{\bar{w}_j}} \prod_{j=1}^{k}(2 - \lambda_j)^{\bar{w}_j}, \\
&s_3 = \left[ \frac{\sqrt{2\prod_{j=1}^{k+1}(a_j^2)^{\bar{w}_j}} + \sqrt{2\prod_{j=1}^{k+1}(b_j^2)^{\bar{w}_j}}}{\prod_{j=1}^{k+1}(2 - a_j^2)^{\bar{w}_j} + \prod_{j=1}^{k+1}(a_j^2)^{\bar{w}_j}} \right], \\
&s_4 = \sqrt{2\prod_{j=1}^{k+1}(\lambda_j^2)^{\bar{w}_j}} \prod_{j=1}^{k+1}(2 - \lambda_j)^{\bar{w}_j}.
\end{align*}
\]

Now, putting these values in equation (40), we have

\[
P_{e_{c_{ij}}} = \left( \frac{s_1}{s_2} \right) \otimes \left( \frac{u_1}{u_2} \right).
\]
Now, putting the values in equation (42), we have

\[
\text{PCFWG}^\ell_{\bar{w}}(p_{c_1}, p_{c_2}, \ldots, p_{c_m}) = \left( \left[ \frac{\sqrt{2\Pi_{j=1}^k(a_{j}^2)}}{\Pi_{j=1}^k(2 - a_{j}^2)\bar{w}_j} + \Pi_{j=1}^k(a_{j}^2)\bar{w}_j} \right] \right) \text{ for all } m, k \geq 1.
\]

Equation (38) holds for \( m = k + 1 \). Thus, equation (38) holds for all \( m \).

**Lemma 1.** Let \( p_{c_j} > 0, \bar{w}_j > 0, (j = 1, \ldots, n) \) and \( \sum_{j=1}^m \bar{w}_j = 1 \). Then,

\[
\prod_{j=1}^m (p_{c_j})^{-\bar{w}_j} \leq \sum_{j=1}^m \bar{w}_j p_{c_j},
\]

where the equality holds if and only if \( p_{c_1} = p_{c_2} = \cdots = p_{c_m} \).

**Theorem 4.** Let \( p_{c_j} = (\mu_{p_{c_j}}, \nu_{p_{c_j}}), (j = 1, \ldots, n) \), be the collection of PCFVs with \( \leq L \), then

\[
\text{PCFWG}^\ell_{\bar{w}}(p_{c_1}, p_{c_2}, \ldots, p_{c_m}) \leq \text{PCFWG}^\ell_{\bar{w}}(\bar{p}_{c_1}, \bar{p}_{c_2}, \ldots, \bar{p}_{c_m}),
\]

where \( \bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)^T \) is the weighted vector of \( p_{c_j} \) (\( j = 1, \ldots, n \)) such that \( \bar{w}_j \in [0, 1] \) and \( \sum_{j=1}^m \bar{w}_j = 1 \).

**Proof.** Straightforward.

**Theorem 5.** Let \( p_{c_j} (j = 1, \ldots, n) \) be the collection of PCFVs with \( \leq L \), where \( \bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)^T \) is the weighted vector of \( (j = 1, \ldots, n) \) such that \( \bar{w}_j \in [0, 1] \) and \( \sum_{j=1}^m \bar{w}_j = 1 \). Then,

1. **Idempotency:** if all \( p_{c_j} (j = 1, \ldots, n) \) are equal, i.e., \( p_{c_j} = p_{c_j} \) then

\[
\text{PCFWG}^\ell_{\bar{w}}(p_{c_1}, p_{c_2}, \ldots, p_{c_m}) = p_{c_j}.
\]

2. **Boundary:**

\[
\mu_{\text{min}} \leq \text{PCFWG}^\ell_{\bar{w}}(p_{c_1}, p_{c_2}, \ldots, p_{c_m}) \leq \mu_{\text{max}} \text{ for every } \bar{w}.
\]

3. **Monotonicity:** let \( p_{c_j}^* = (\mu_{p_{c_j}}^*, \nu_{p_{c_j}}^*), (j = 1, \ldots, n) \) be the collection of PCFVs with \( \leq L \) and \( \mu_{p_{c_j}} \leq \mu_{p_{c_j}}^*, \nu_{p_{c_j}} \leq \nu_{p_{c_j}}^* \), for all \( j \), then

\[
\text{PCFWG}^\ell_{\bar{w}}(p_{c_1}, p_{c_2}, \ldots, p_{c_m}) \leq \text{PCFWG}^\ell_{\bar{w}}(p_{c_1}^*, p_{c_2}^*, \ldots, p_{c_m}^*) \text{ for every } \bar{w}.
\]
Proof. (1) Idempotency: since

\[
PCFWG^L_n(p_1, p_2, \ldots, p_n) = \left( \begin{array}{c}
\frac{\sqrt{2} \sqrt{\Pi_{i=1}^{m} (a_i^2)}}{\sqrt{\Pi_{i=1}^{m} (2 - a_i^2) + \Pi_{i=1}^{m} (a_i^2)}}
\frac{\sqrt{2} \sqrt{\Pi_{i=1}^{m} (b_i^2)}}{\sqrt{\Pi_{i=1}^{m} (2 - b_i^2) + \Pi_{i=1}^{m} (b_i^2)}}
\frac{\sqrt{2} \sqrt{\Pi_{i=1}^{m} (1 + b_i^2)}}{\sqrt{\Pi_{i=1}^{m} (1 + b_i^2) + \Pi_{i=1}^{m} (1 - b_i^2)}}
\end{array} \right).
\]

(51)

For simplicity, we use the notation of PCFs. Let

\[p_j = (\mu_{p_j}, v_{p_j})\]

and

\[\mu_{p_j} = (\langle A_1, \lambda_1 \rangle, \langle \tilde{A}_1, \mu_1 \rangle)\]

and

\[\mu_{p_j} = (\langle A_2, \lambda_2 \rangle, \langle \tilde{A}_2, \mu_2 \rangle)\]

where

\[A_1 = [a_1, b_1], \tilde{A}_1 = [\tilde{a}_1, \tilde{b}_1], A_2 = [a_2, b_2], \text{ and } \tilde{A}_2 = [\tilde{a}_2, \tilde{b}_2] \]

then the above equation can be written in the following form:

\[
PCFWG^L_n(p_1, p_2, \ldots, p_n) = \left( \begin{array}{c}
\frac{\sqrt{2} \sqrt{\Pi_{i=1}^{m} (\mu_{p_i}^2)}}{\sqrt{\Pi_{i=1}^{m} (2 - \mu_{p_i}^2) + \Pi_{i=1}^{m} (\mu_{p_i}^2)}}
\frac{\sqrt{2} \sqrt{\Pi_{i=1}^{m} (1 + \mu_{p_i}^2)}}{\sqrt{\Pi_{i=1}^{m} (1 + \mu_{p_i}^2) + \Pi_{i=1}^{m} (1 - \mu_{p_i}^2)}}
\end{array} \right).
\]

(52)

Now, \(p_j(j = 1, \ldots, n) = p_c\). Then, equation (26) can be written as

\[
PCFWG^L_n(p_1, p_2, \ldots, p_n) = \left( \begin{array}{c}
\frac{\sqrt{2} \sqrt{\Pi_{i=1}^{m} (\mu_{p_j}^2)}}{\sqrt{\Pi_{i=1}^{m} (2 - \mu_{p_j}^2) + \Pi_{i=1}^{m} (\mu_{p_j}^2)}}
\frac{\sqrt{2} \sqrt{\Pi_{i=1}^{m} (1 + \mu_{p_j}^2)}}{\sqrt{\Pi_{i=1}^{m} (1 + \mu_{p_j}^2) + \Pi_{i=1}^{m} (1 - \mu_{p_j}^2)}}
\end{array} \right).
\]

(53)

(2) Boundary:

\[p_{\min} \leq PCFWG^L_n(p_1, p_2, \ldots, p_n) \leq p_{\max} \text{ for every } \bar{w}, \]

where \(p_{\min} = \min(p_{j})\) and \(p_{\max} = \max(p_{j})\). Let \(f(x) = \sqrt{(1 - x^2)/(x^2)}, x \in [0, 1]\), then \(f(x) = (-2/x^3)\sqrt{(x^2/2 - x^2)} < 0\). So, \(f(x)\) is the decreasing function on \((0, 1]\). Since \(\mu_{p_j} \leq \mu_{p_j} \leq \mu_{p_j, \max}\) for all \(j\), then \(f(\mu_{p_j}) \leq f(\mu_{p_j}) \leq f(\mu_{p_j})\) \((j = 1, \ldots, n)\)

\[\sqrt{(2 - \mu_{p_j, \max})/2} \leq \sqrt{(2 - \mu_{p_j, \max})/2} \leq \sqrt{(2 - \mu_{p_j, \max})/2}\]

where \(\bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m)^T\) is the weighted vector of \(p_j\) \((j = 1, \ldots, n)\) such that \(\bar{w}_j \in [0, 1]\) and \(\sum_{j=1}^{m} \bar{w}_j = 1\). Then, we have

\[\prod_{j=1}^{m} \frac{2 - \mu_{p_j, \max}}{\mu_{p_j, \max}} \leq \prod_{j=1}^{m} \frac{2 - \mu_{p_j}^2}{\mu_{p_j}^2} \leq \prod_{j=1}^{m} \frac{2 - \mu_{p_j, \min}}{\mu_{p_j, \min}}\]
Again, let $h(y) = \sqrt{(1-y^2)/(1+y^2)}$, $x \in [0, 1]$, then $h(y) = (-2y/(1-y^2))/(1+y^2) < 0$. So, $h(y)$ is the decreasing function on $(0, 1]$. Since $v_{p_i \min} \leq v_{p_i} \leq v_{p_i \max}$ for all $j$, then $h(v_{p_i \max}) \leq h(v_{p_i}) \leq h(v_{p_i \min})$ for all $j$. $\sqrt{(1-v_{p_i \min}^2/v_{p_i \max}^2)}$, where $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_n)$ is the weighted vector of $p_{c_j} (j = 1, \ldots, n)$ such that $\bar{w}_j \in [0, 1]$ and $\sum_{j=1}^n \bar{w}_j = 1$. Then, we have

Let PCFWG\textsuperscript{c}$_C(p_{c_1}, p_{c_2}, \ldots, p_{c_n}) = p_{c_C}$. Then, equations (52) and (55) can be written as $\mu_{p_{c_{\min}}} \leq \mu_{p_{c_{j}}} \leq \mu_{p_{c_{\max}}}$ and $v_{p_{c_{\min}}} \leq v_{p_{c_{j}}} \leq v_{p_{c_{\max}}}$, respectively. Thus, $S(p_{c_{}}) = \mu_{p_{c_{}} - v_{p_{c_{}}} \leq \mu_{p_{c_{\min}}} - v_{p_{c_{\min}}} = S(p_{c_{\min}})$ and $S(p_{c_{}}) = \mu_{p_{c_{}} - v_{p_{c_{}}} \geq \mu_{p_{c_{\min}}} - v_{p_{c_{\min}}} = S(p_{c_{\max}})$.

If $S(p_{c_{}}) < S(p_{c_{\max}})$ and $S(p_{c_{}}) > S(p_{c_{\min}})$, then $PCFWG^c_w(p_{c_1}, p_{c_2}, \ldots, p_{c_n}) < p_{c_{\max}}$ for all $\bar{w}$.

Thus, from equations (55) to (57), we have $p_{c_{\min}} < PCFWG^c_w(p_{c_1}, p_{c_2}, \ldots, p_{c_n}) < p_{c_{\max}}$ for every $\bar{w}$.

(3) Monotonicity:

The proof follows from (2).

6. An Application of the Pythagorean Cubic Fuzzy Einstein Weighted Geometric (PCFWG) Aggregation Operator to Group Decision-Making Problems

In this unit, we develop an application of Pythagorean cubic fuzzy Einstein weighted geometric (PCFWG) operator to multicriteria decision-making problem.

Algorithm. Let $F = \{F_1, F_2, \ldots, F_m\}$ be the set of $n$ alternatives, $H = \{H_1, H_2, \ldots, H_m\}$ be the set of $m$ attributes, and $\bar{D} = \{\bar{D}_1, \bar{D}_2, \ldots, \bar{D}_k\}$ be the set of $k$ decision makers. Let $w = (w_1, w_2, \ldots, w_m)^T$ be the weighted vector of the attributes $H_i (i = 1, 2, \ldots, m)$, such that $w_i \in [0, 1]$ and $\sum_{i=1}^m w_i = 1$. Then, we have
1. Let \( \vec{\eta} = (\vec{\eta}_1, \vec{\eta}_2, \vec{\eta}_3, \ldots, \vec{\eta}_k)^T \) be the weighted vector of the decision makers \( D_s (s = 1, 2, \ldots, k) \), such that \( \sum_{i=1}^{k} \eta_i = 1 \). This method has the following steps:

**Step 1.** In this step, we construct the Pythagorean cubic fuzzy decision-making matrices, \( \tilde{D} = [\tilde{d}_{ij}]_{n \times m} (s = 1, 2, \ldots, k) \). If the criteria have two types, such as benefit criteria and cost criteria, then the Pythagorean cubic fuzzy decision matrices, \( \tilde{D} = [\tilde{d}_{ij}]_{n \times m} \), can be converted into the normalized Pythagorean cubic fuzzy decision matrices, \( \tilde{R} = (\tilde{e}_{ij})_{n \times m} \), where \( \tilde{e}_{ij} \). If all the criteria have the same type, then there is no need of normalization.

**Step 2.** We use the Pythagorean cubic fuzzy Einstein weighted geometric (PCFEWG) operator to aggregate all the individual normalized Pythagorean cubic fuzzy decision matrices, \( \tilde{R} = (\tilde{e}_{ij})_{n \times m} (s = 1, 2, \ldots, k) \), into the single Pythagorean cubic fuzzy decision matrix, \( \tilde{R} = (\tilde{e}_{ij})_{n \times m} \).

**Step 3.** We aggregate all the preference values \( \tilde{e}_{ij} (j = 1, 2, 3, \ldots, n, i = 1, 2, \ldots, m) \) by using the PCFEWG operator and get the overall preference values \( \tilde{e}_j (j = 1, 2, \ldots, n) \) corresponding to the alternatives \( F_j (j = 1, \ldots, n) \).

**Step 4.** We calculate the scores of \( \tilde{e}_j (j = 1, 2, 3, \ldots, n) \). If there is no difference between two or more than two scores, then we must have to find out the accuracy degrees of the collective overall preference values.

**Step 5.** We arrange the scores of all the alternatives in the form of descending order and select that alternative which has the highest score function.

7. **Numerical Example**

In Pakistan’s stock exchange, listed Internet companies play an important role. The performance of listed companies affects capital market resource allocation and has become a common concern of shareholders, creditors, government bodies, and other stakeholders. An investment firm would like to invest a sum of money in stocks on the Internet. So, the investment bank employs three kinds of experts to determine the possible investment value: market maker, dealer, and finder. Three Internet stocks are chosen in which the earnings ratio is higher than other stocks: (1) is PTCL; (2) is NayaTel; (3) is Wi-Tribe out of three characteristics: (1) is the trend in the stock market; (2) is in the course of policy; (3) is the annual results. About the attributes \( A_j (j = 1, 2, 3) \), the three experts test Internet stocks \( x_i (i = 1, 2, 3) \) and create the following three Pythagorean cubic fuzzy decision matrices in Table 1. Tables 2 and 3 display the expert weights and attribute weights, which all take the form of PCFEs, respectively. Then, to get the most desirable alternative(s), which includes the following steps, we use the approach developed in Section 6:

**Step 1.** The decision maker gives his decision in Tables 1 – 3.

**Step 2.** We apply the Pythagorean cubic fuzzy Einstein weighted geometric (PCFEWG) operator to aggregate all the individual normalized Pythagorean cubic fuzzy decision matrices \( \tilde{R} = (\tilde{e}_{ij})_{n \times m} (s = 1, 2, \ldots, k) \), into the single Pythagorean cubic fuzzy decision matrix, \( \tilde{R} = (\tilde{e}_{ij})_{n \times m} \).

**Aggregated Pythagorean cubic fuzzy decision matrix \( D_1 \):**

\[
X_1 = ([.4660, .6120], .7323; [.5760, .7056], .5726),
X_2 = ([.5780, .7306], .5360; [.4810, .6235], .7283),
X_3 = ([.6662, .7908], .5691; [.4661, .5657], .7381).
\]

**Aggregated Pythagorean cubic fuzzy decision matrix \( D_2 \):**

\[
X_1 = ([.5700, .6742], .6364; [.5532, .7443], .5563),
X_2 = ([.5095, .6564], .5333; [.5972, .6994], .5637),
X_3 = ([.6733, .7002], .6155; [.4523, .5812], .5810).
\]

**Aggregated Pythagorean cubic fuzzy decision matrix \( D_3 \):**

\[
X_1 = ([.6204, .7207], .5655; [.4523, .5810], .5703),
X_2 = ([.6784, .7788], .7024; [.4233, .5232], .5434),
X_3 = ([.6291, .7306], .4948; [.4765, .6040], .6994).
\]

**Step 3.** We aggregate all the preference values, which are

\[
X_1 = ([.6808, .7692], .7472; [.4468, .5859], .4777),
X_2 = ([.7041, .8039], .7070; [.4312, .5310], .5221),
X_3 = ([.7591, .8173], .6866; [.3902, .4940], .5743).
\]

**Step 4.** We calculate the scores of \( X_j (j = 1, 2, 3) \).

\[
S(X_1) = .0207,
S(X_2) = .0532,
S(X_3) = .0773.
\]

**Step 5.** We organize the scores of the alternatives in descending order and choose the highest score function. Hence, \( X_3 > X_2 > X_1 \). Thus, the most wanted alternative is \( X_3 \).

8. **Comparison Analysis**

The same numerical example is solved by using other aggregation operators, including IFEWG (intuitionistic fuzzy Einstein weighted geometric) operator, IFEOWG (intuitionistic fuzzy Einstein ordered weighted geometric) operator, PFEWG (picture fuzzy Einstein weighted geometric) operator, PFEOWG (picture fuzzy Einstein ordered weighted geometric) operator, PyFEWG (Pythagorean fuzzy
Einstein weighted geometric) operator, PyFEOWG (Pythagorean fuzzy Einstein weighted geometric) operator, ICFEWG (intuitionistic cubic fuzzy Einstein weighted geometric) operator, ICFEOWG (intuitionistic cubic fuzzy Einstein ordered weighted geometric) operator, CPFEWG (cubic picture fuzzy Einstein weighted geometric) operator, and CPFEOWG (cubic picture fuzzy Einstein ordered weighted geometric) operator to demonstrate the efficiency and eminent benefits of the proven aggregation operators, by ignoring the additional preference matrix in some existing operators. Different aggregation operators have distinct strategic classifications so that, in compliance with their consultation, they may retain a small disparity. By contrast, the appropriate choice developed by any aggregation operator is important and recognizes the proposed solution’s feasibility and effectiveness of aggregation operators. Table 4 gives a comparative study of the final rankings of all aggregation operators.

9. Conclusion

We introduced the Pythagorean cubic fuzzy set, which is a generalization of the interval-valued Pythagorean fuzzy set, in this paper. Einstein’s Pythagorean cubic fuzzy weighted
geometric operator has been described (PCFEWG). We also discussed some of the fundamental properties of this operator, such as idempotency, boundary, and monotonicity. The Pythagorean cubic fuzzy Einstein weighted geometric (PCFEWG) operator was then used to deal with different parameters for decision-making problems under Pythagorean cubic fuzzy details. We developed a multicriteria decision-making algorithm for Pythagorean cubic fuzzy Einstein weighted geometric problems (PCFEWG). Finally, we put together a numerical example of a decision-making problem.

In future, we can extend this concept for spherical cubic fuzzy sets and their application in multicriteria group decision-making, pattern recognition, and cluster analysis. We can also extend Pythagorean cubic fuzzy sets for various aggregation operators such as Hamacher, Dombi, Haronion mean, Bonferroni mean, TOPSIS, and their applications in group decision-making.

Data Availability
No data were used in this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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