A FOUR DIMENSIONAL EXAMPLE OF RICCI FLAT METRIC ADMITTING ALMOST-KÄHLER NON-KÄHLER STRUCTURE

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Abstract

We construct an example of Ricci-flat almost-Kähler non-Kähler structure in four dimensions.

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1. Let $\mathcal{M}$ be a 4-manifold equipped with a metric $g$ of signature $(++++)$. The pair $(\mathcal{M}, g)$ is called a Riemannian 4-manifold.

An almost hermitian structure on $(\mathcal{M}, g)$ is a tensor field $J : T\mathcal{M} \to T\mathcal{M}$ such that $J^2 = -id$ and $g(JX, JY) = g(X, Y)$. An almost hermitian structure $(\mathcal{M}, g, J)$ is called hermitian if $J$ is integrable. Due to the Newlander-Nirenberg theorem this is equivalent to the vanishing of the Nijenhuis tensor $N_J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$ for $J$.

Given an almost hermitian structure $(\mathcal{M}, g, J)$ one defines the fundamental 2-form $\omega$ by $\omega(X, Y) = g(X, JY)$. An almost hermitian structure $(\mathcal{M}, g, J)$ is called almost-Kähler if its fundamental 2-form is closed. If, in addition, $J$ is integrable then such structure is called Kähler.

This paper is motivated by the following conjecture [5].

**Goldberg’s Conjecture**

*The almost Kähler structure of a compact Einstein manifold is necessarily Kähler.*

The conjecture was proven in the case of non-negative scalar curvature of the Einstein manifold by K. Sekigawa in [10].

In this paper we show that the assumption about compactness of the Einstein manifold is essential for the Goldberg conjecture. In particular, we give an explicit example of a Ricci-flat almost-Kähler non-Kähler structure on a noncompact 4-manifold. This result is given by Theorem 1 of paragraph 4.

2. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^4$. Let $\theta^i = (M, \bar{M}, N, \bar{N})$ be four complex-valued 1-forms on $\mathcal{U}$ such that $M \wedge \bar{M} \wedge N \wedge \bar{N} \neq 0$. Using $\theta^i$ we define a metric $g$ on $\mathcal{U}$ by

$$g = 2(M\bar{M} + N\bar{N}) := M \otimes \bar{M} + \bar{M} \otimes M + N \otimes \bar{N} + \bar{N} \otimes N.$$ 

Clearly $(\mathcal{U}, g)$ is a Riemannian 4-manifold.

The Weyl tensor $W$ of the metric $g$ splits onto self-dual ($W^+$) and anti-self-dual ($W^-$) parts. $(\mathcal{U}, g)$ is said to be (anti-)self-dual iff $(W^+ \equiv 0)$ $W^- \equiv 0$. If $(W^+ \neq 0)$ $W^- \neq 0$ then in every point of $\mathcal{U}$ it defines at most two spinor directions ($[\alpha^+, \beta^+]$) $[\alpha^-, \beta^-]$; see e.g. [3, 9]. $(W^+) W^-$ is said to be of type $D$ if $(\alpha^+) \alpha^-$ coincides with $(\beta^+) \beta^-$. 
Let $e_i = (m, \bar{m}, n, \bar{n})$ be a basis dual to $\theta^i = (M, \bar{M}, N, \bar{N})$. For any $\xi \in \mathbb{C}$ it is convenient to consider 1-forms

$$M_\xi = \frac{M - \xi \bar{N}}{\sqrt{1 + \xi \bar{\xi}}} \quad N_\xi = \frac{N + \xi \bar{M}}{\sqrt{1 + \xi \bar{\xi}}}$$

and vector fields

$$m_\xi = \frac{m - \xi \bar{n}}{\sqrt{1 + \xi \bar{\xi}}} \quad n_\xi = \frac{n + \xi \bar{m}}{\sqrt{1 + \xi \bar{\xi}}}$$

The following Lemma is well known (see for example [3, 9]).

**Lemma 1**

**i)** For any value of the complex parameter $\xi \in \mathbb{C} \cup \{\infty\}$ the expressions

$$J_\xi^+ = i(M_\xi \otimes \overline{m_\xi} - M_\xi \otimes m_\xi + N_\xi \otimes \overline{n_\xi} - N_\xi \otimes n_\xi)$$

$$J_\xi^- = i(M_\xi \otimes m_\xi - \overline{M_\xi} \otimes m_\xi + N_\xi \otimes \overline{n_\xi} - N_\xi \otimes n_\xi)$$

define almost hermitian structures on $(\mathcal{U}, g)$.

**ii)** The fundamental 2-forms corresponding to $J_\xi^+$ and $J_\xi^-$ are respectively given by

$$\omega_\xi^+ = i(M_\xi \wedge \overline{M_\xi} + N_\xi \wedge \overline{N_\xi})$$

$$\omega_\xi^- = i(M_\xi \wedge M_\xi + N_\xi \wedge N_\xi).$$

**iii)** Any almost hermitian structure on $(\mathcal{U}, g)$ is given either by one of $J_\xi^+$ or by one of $J_\xi^-$. Structures $J_\xi^+$ are different from $J_\xi^-$; also, different $\xi$s correspond to different structures.

**iv)** If the metric $g$ is not self-dual then among $J_\xi^+$s only at most four structures, corresponding to specific four values of the parameter $\xi$, may be integrable. Analogously, if the metric $g$ is not anti-self-dual then only at most four $J_\xi^-$s may be integrable.
3. Let \((x^1, x^2, x^3, x^4)\) be Euclidean coordinates on \(U\). Define
\[
z_1 = x^1 + ix^2 \quad z_2 = x^3 + ix^4.
\] (1)

Let \(\partial_k = \frac{\partial}{\partial z_k}\) and \(\bar{\partial}_k = \frac{\partial}{\partial \bar{z}_k}\), \(k = 1, 2\).

Consider two 1-forms \(M\) and \(N\) on \(U\) defined by
\[
M = f(dz_1 + hdz_2) \quad N = \overline{f(dz_2)},
\] (2)
where \(f \neq 0\) (real) and \(h\) (complex) are functions on \(U\).

Since \(M \wedge \bar{M} \wedge N \wedge \bar{N} = dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \neq 0\) then the metric \(g = 2(M\bar{M} + N\bar{N})\) equips \(U\) with the Riemannian structure. Consider almost hermitian structures \(J^+\xi\) for such \((U, g)\). It is interesting to note that if \(\xi = e^{i\phi} = \text{const}\) then the corresponding fundamental 2-form \(\omega^+_{e^{i\phi}}\) reads
\[
\omega^+_{e^{i\phi}} = i(e^{i\phi}dz_2 \wedge dz_1 - e^{-i\phi}d\bar{z}_2 \wedge d\bar{z}_1)
\]
and is closed. Thus, for any \(e^{i\phi} \in S^1\) we constructed an almost-Kähler structure \((U, g, J^+_{e^{i\phi}})\). If the functions \(f\) and \(h\) are general enough, then the metric \(g\) has no chance to be self-dual. Moreover, since in such case there is a finite number of hermitian structures among \(J^+\xi\), then most of our structures must be non-Kähler. Summing up we have the following Lemma.

**Lemma 2** Let \((z_1, \bar{z}_1, z_2, \bar{z}_2)\) be coordinates on \(U\) as in (1). Then for each value of the real constant \(\phi \in [0, 2\pi]\) the metric
\[
g = 2f^2(dz_1 + hdz_2)(d\bar{z}_1 + \bar{h}d\bar{z}_2) + \frac{1}{f^2}dz_2d\bar{z}_2
\] (3)
and the almost complex structure
\[
J^+_{e^{i\phi}} = 2\text{Re}\{ie^{i\phi}[f^2(dz_1 + hdz_2) \otimes (\partial_2 - \bar{h}\partial_1) - \frac{1}{f^2}dz_2 \otimes \partial_1]\}
\] (4)
define an almost-Kähler structure on \(U\).
If the functions \(f\) and \(h\) are general enough to prevent the metric of being self-dual then these structures are non-Kähler for almost all values of \(\phi\).
4. We look for not-self-dual Ricci-flat metrics among the metrics of Lemma 2. For this purpose it is convenient to restrict to the metrics (3) whose anti-self-dual part of the Weyl tensor is strictly of type D. Such a restriction guarantees that all structures (4) are non-Kähler [6, 9].

We recall a useful Lemma [7].

**Lemma 3** Let $g$ be a Ricci-flat Riemannian metric in four dimensions. Assume that the anti-self-dual part of the Weyl tensor for $g$ is strictly of type D. Then, locally there always exist complex coordinates $(z_1, z_2)$ and a real function $K = K(v, z_2, \overline{z_2})$, $v = z_1 + \overline{z_1}$ such that the metric can be written as

$$g = \frac{\varepsilon K_{vv}}{(K_v)^{3/2}}(dz_1 + \frac{K_{vz}}{K_v}dz_2)(dz_1 + \frac{K_{v\bar{z}}}{K_v}d\bar{z}_2) + 4e^{-K(K_v)^{1/2}} \varepsilon K_{vv} dz_2 d\bar{z}_2,$$

where $K_v = \frac{\partial^2 K}{\partial v \partial \bar{z}_2}$, etc. The function $K$ satisfies

$$K_{vv}K_{\bar{z}\bar{z}} - K_{v\bar{z}}K_{vv} - 2e^{-K}(K_{vv} + 2(K_v)^2) = 0,$$

$$K_v > 0, \quad \varepsilon K_{vv} > 0$$

where $\varepsilon$ is either plus or minus one.

Also, every function $K = K(v, z_2, \overline{z_2})$ satisfying (4)-(7) defines, via (3), a Ricci-flat metric. This metric has the anti-self-dual part of the Weyl tensor of strictly type D.

We ask when the metric (3) can be written in the form (3). Identifying coordinates $(z_1, z_2)$ in both metrics we see that it is possible if

$$2f^2 = \frac{\varepsilon K_{vv}}{(K_v)^{3/2}} \quad \text{and} \quad \frac{2}{f^2} = 4e^{-K(K_v)^{1/2}} \varepsilon K_{vv}.$$

These two equations are compatible only if $K_v e^K = 1$. It is a matter of straightforward integration that, modulo the coordinate transformations, the general solution of this equation which simultaneously satisfies the equation (3) is $K = \log(v - 2z_2\overline{z_2})$. Using such $K$ we easily find that in the region

$$U' = \{U \ni (z_1, z_2) \quad \text{s.t.} \quad v - 2z_2\overline{z_2} > 0\}$$

\footnote{This solution was already known to Sławomir Bialecki in 1984 [4].}
the metric (3) with
\[ f = \frac{1}{\sqrt{2(v - 2\overline{z}_2)^{1/4}}}, \quad h = -2\overline{z}_2, \]
is Ricci-flat and strictly of type D on the anti-self-dual side of its Weyl tensor. The explicit expression for such \( g \) reads
\[ g = \frac{1}{(v - 2\overline{z}_2)^{1/2}}(dz_1 - 2\overline{z}_2dz_2)(d\overline{z}_1 - 2z_2d\overline{z}_2) + 4(v - 2z_2\overline{z}_2)^{1/2}dz_2d\overline{z}_2, \quad (8) \]
To have a better insight into this metric we choose new coordinates
\[ x = (v - 2\overline{z}_2)^{1/2}, \quad y = z_2 + \overline{z}_2, \quad z = i(\overline{z}_2 - z_2), \quad q = \frac{z_1 - \overline{z}_1}{2i} \]
on \( U' \). These coordinates are real. The metric (8) in these coordinates reads
\[ g = x(dx^2 + dy^2 + dz^2) + \frac{1}{x} (\frac{1}{2} ydz - \frac{1}{2} ydz + dq)^2. \]
This shows that it belongs to the Gibbons-Hawking class [4] and that its self-dual part of the Weyl tensor vanishes.

We also recall [8] that a suitable Lie-Backlund transformation brings equation (6) to the Boyer-Finley-Plebański [2, 3] equation
\[ F_{yy} + F_{zz} + (e^F)_{xx} = 0 \]
for one real function \( F = F(x, y, z) \) of three real variables. It is interesting to note that the metric (8) corresponds to the simplest solution \( F = 0 \) of this equation.

Summing up we have the following theorem.

**Theorem 1** Let \((z_1, \overline{z}_1, z_2, \overline{z}_2)\) be coordinates on \( U \subset R^4 \cong C^2 \). The Riemannian manifold \((U', g)\), where
\[ U' = \{ U \ni (z_1, z_2) \text{ s.t. } v - 2z_2\overline{z}_2 > 0, \quad v = z_1 + \overline{z}_1 \} \]

is also known to describe the SU(∞) Toda lattice.

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and
\[ g = \frac{1}{(v - 2z_2 \overline{z}_2)^{1/2}}(dz_1 - 2\overline{z}_2 dz_2)(d\overline{z}_1 - 2z_2 dz_2) + 4(v - 2z_2 \overline{z}_2)^{1/2}dz_2 \overline{dz}_2, \]

is Ricci-flat, anti-self-dual and has the anti-self-dual part of the Weyl tensor of type \( D \). Moreover, \((U', g)\) admits a circle of almost-Kähler non-Kähler structures
\[ J^+_{\varphi_0} = 2\text{Re}\{ie^{i\varphi}\left[1 \over 2(v - 2z_2 \overline{z}_2)^{1/2}(dz_1 - 2\overline{z}_2 dz_2) \otimes (\partial_2 + 2z_2 \partial_1) - 2(v - 2z_2 \overline{z}_2)^{1/2}dz_2 \otimes \partial_1\right]\}. \]

These structures are parametrized by the real constant \( \varphi \in [0, 2\pi[. \) Their fundamental 2-forms are given by
\[ \omega^+_{\varphi_0} = i(e^{i\varphi} dz_2 \wedge dz_1 - e^{-i\varphi} \overline{dz}_2 \wedge \overline{dz}_1). \]

5. Interestingly, our examples can be globalized. Indeed, the transformation
\[ t = \frac{1}{2} \log(v - 2z_2 \overline{z}_2), \quad y = z_2 + \overline{z}_2, \quad z = i(\overline{z}_2 - z_2), \quad q = \frac{z_1 - \overline{z}_1}{2i} \]
brings the structures \((g, J^+_{\varphi_0}, \omega^+_{\varphi_0})\) of Theorem 1 to a form which is regular for all the values of the real parameters \((t, y, z, q) \in \mathbb{R}^4\).

6. Finally, we observe that the metric (8), as being anti-self-dual, possesses a strictly Kähler structure. This is given by
\[ J = i[(dz_1 - 2\overline{z}_2 dz_2) \otimes \partial_1 - (d\overline{z}_1 - 2z_2 dz_2) \otimes \partial_1 + dz_2 \otimes (\partial_2 + 2z_2 \partial_1) - dz_2 \otimes (\partial_2 + 2\overline{z}_2 \partial_1)] \]
and belongs to the structures of opposite orientation that \( J^+_{\varphi_0} \). It is interesting whether there exist Ricci-flat metrics that admit almost-Kähler non-Kähler structures but do not admit any strictly Kähler structure.

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