Homogeneous Finsler spaces with positive flag curvature

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Abstract

In this paper, we give a complete classification of even-dimensional homogeneous manifolds $G/H$ admitting $G$-invariant Finsler metrics with positive flag curvature. It turns out the list coincides with that of the Riemannian coset spaces with positive sectional curvature.

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1 Introduction

One of the central problems in Riemannian geometry is to classify smooth manifolds admitting Riemannian metrics with positive sectional curvature, Ricci curvature or scalar curvature. For Ricci and scalar curvature the problem is fully understood. However, for sectional curvature this problem is extremely involved and is far from being completely solved. In fact, up to now only homogeneous manifolds which admit invariant Riemannian metrics with positive sectional curvature have been completely classified; see [1, 3, 5, 25]. In the inhomogeneous case, there are only some explicit examples of positive sectional curvature constructed in the biquotients of some Lie groups; see for example [2, 10, 13, 14, 15].

In Finsler geometry this problem possesses the same importance and is more difficult. Nevertheless, the second and the fourth authors of this paper have made some significant progress in the homogeneous case. For example, it is proved in [12] that if $G$ is a connected simply connected Lie group which admits a left invariant Finsler metric with positive flag curvature, then $G$ must be isomorphic to SU(2). It is also shown in the same paper that if $G/H$ is an even-dimensional coset space and $F$ is an invariant...
Randers metric on $G/H$ with positive flag curvature, then $F$ must be a Riemannian metric. Moreover, a complete classification (up to isometries) of homogeneous Randers metrics with positive flag curvature and isotropic S-curvature has also achieved in the paper [16]. However, up to now the problem of the classification of homogeneous manifolds which admit invariant Finsler metrics with positive flag curvature is still open.

In this paper we shall prove the following

**Theorem 1.1** Let $G$ be a compact connected simply connected Lie group and $H$ a connected closed subgroup such that the dimension of the coset space $G/H$ is even. Suppose that there exists a $G$-invariant Finsler metric on $G/H$ with positive flag curvature. Then there exists a $G$-homogeneous Riemannian metric on $G/H$ with positive sectional curvature.

This theorem combined with the results of Wallach ([25]) gives the following corollary:

**Theorem 1.2** Let $G$ be a compact connected simply connected Lie group and $H$ a connected closed subgroup of $G$ such that the dimension of the coset space $G/H$ is even. Assume $\mathfrak{h} = \text{Lie}(H)$ does not contain any nonzero ideal of $\mathfrak{g} = \text{Lie}(G)$. If $G/H$ admits a $G$-invariant Finsler metric with positive flag curvature, then the pair $(G,H)$ must be one of the following:

1. Rank one Riemannian symmetric pairs of compact type. In this case, $G/H$ is one of the even dimensional spheres, the complex projective spaces, the quaternion projective spaces or the 16-dimensional Cayley plane. Moreover, any $G$-invariant Finsler metric on $G/H$ must be Riemannian and with positive flag (sectional) curvature.

2. The pair $(G_2, \text{SU}(3))$. In this case, the coset space is equal to $S^6$, and on the coset space $S^6 = G_2/\text{SU}(3)$ any $G$-invariant Finsler metric must be the Riemannian metric of constant positive flag (sectional) curvature.

3. The pair $(\text{Sp}(n), \text{Sp}(n-1) \times U(1))$. In this case, the coset space is $\mathbb{C}P^m$, where $m = 2n - 1$, and on the coset space $\mathbb{C}P^m = \text{Sp}(n)/\text{Sp}(n-1)U(1)$ there exist $G$-invariant Riemannian metrics as well as non-Riemannian Finsler metrics with positive flag (sectional) curvature.

4. The pair $(\text{SU}(3), T^2)$, where $T^2$ is a maximal torus of $\text{SU}(3)$. In this case, on the coset space $F^6 = \text{SU}(3)/T^2$ there exist $G$-invariant Riemannian metrics as well as non-Riemannian Finsler metrics with positive flag (sectional) curvature.

5. The pair $(\text{Sp}(3), \text{Sp}(1) \times \text{Sp}(1) \times \text{Sp}(1))$. In this case, on the coset space

$$F^{12} = \text{Sp}(3)/\text{Sp}(1) \times \text{Sp}(1) \times \text{Sp}(1)$$

there exist invariant Riemannian metrics as well as non-Riemannian Finsler metrics with positive flag (sectional) curvature.
6. The pair $(F_4, \text{Spin}(8))$. In this case, on the coset space $F_4/\text{Spin}(8)$ there exist $G$-invariant Riemannian metrics as well as non-Riemannian metrics with positive flag (sectional) curvature.

Essentially Theorem 1.2 gives a complete local description of homogeneous manifolds of even dimension admitting invariant Finsler metric with positive flag curvature (see the final remark at the end). However, we must point out that it is still very difficult to classify all homogeneous Finsler metrics (not only the spaces) with positive flag curvature under isometries. In fact, this problem is even very hard in the Riemannian case; see [24] for some information on Riemannian metrics on spheres.

In Section 2, we recall some fundamental definitions and known results needed in this paper. In Section 3, we study submersion of Finsler metrics and prove some results which are useful in studying homogeneous Finsler spaces with positive flag curvature. In Section 4, some further properties of flag curvature under submersion are studied. The sections 5 and 6 are devoted to proving Theorems 1.1 and 1.2 respectively.

2 Preliminaries in Finsler geometry

In this section we recall some definitions and results on Finsler spaces. For general Finsler spaces we refer the readers to [1] and [5]; For homogeneous Finsler spaces we refer to [9].

2.1 Minkowski norms and Finsler metrics

A Minkowski norm on a real vector space $V$, dim $V = n$, is a continuous real-valued function $F : V \rightarrow [0, +\infty)$ satisfying the following conditions:

1. $F$ is positive and smooth on $V \setminus \{0\};$

2. $F(\lambda y) = \lambda F(y)$ for any $\lambda > 0;$

3. with respect to any linear coordinates $y = y^i e_i$, the Hessian matrix

$$
(g_{ij}(y)) = \left( \frac{1}{2} [F^2]_{y^i y^j}(y) \right)
$$

is positive definite at any nonzero $y$.

The Hessian matrix $(g_{ij}(y))$ and its inverse $(g^{ij}(y))$ can be used to raise and lower down indices of relevant tensors in Finsler geometry.

At each $y \neq 0$, the Hessian matrix $(g_{ij}(y))$ defines an inner product $\langle \cdot, \cdot \rangle_y$ on $V$ by

$$
\langle u, v \rangle_y = g_{ij}(y) u^i v^j,
$$

where $u = u^i e_i$ and $v = v^i e_i$. Sometimes we denote the above inner product as $\langle \cdot, \cdot \rangle^F_y$ if there are several norms in consideration. This inner product can also be expressed as

$$
\langle u, v \rangle_y = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)]|_{s=t=0}.
$$

(2.2)
It is easy to check that the above definition is independent of the choice of linear coordinates.

Let $M$ be a smooth manifold with dimension $n$. A Finsler metric $F$ on $M$ is a continuous function $F : TM \to [0, +\infty)$ such that it is positive and smooth on the slit tangent bundle $TM \setminus \{0\}$, and its restriction in each tangent space is a Minkowski norm. Then $(M, F)$ is called a Finsler manifold or a Finsler space.

Here are some important examples.

Riemannian metrics are a special class of Finsler metrics such that the Hessian matrices only depend on $x \in M$ and not on $y \in T_x M$. For a Riemannian manifold, the metric is often referred as the global smooth section $g_{ij}dx^i dx^j$ of $\text{Sym}^2(T^*M)$. Unless otherwise stated, we mainly deal with non-Riemannian metrics in Finsler geometry in this paper.

Randers metrics are the simplest and most important class of non-Riemannian metrics in Finsler geometry. They are defined by $F = \alpha + \beta$, in which $\alpha$ is a Riemannian metric and $\beta$ is a 1-form (see [23]). The notion of Randers metrics has been naturally generalized to $(\alpha, \beta)$-metrics. An $(\alpha, \beta)$-metric is a Finsler metric of the form $F = \alpha(\beta/\alpha)$, where $\phi$ is a positive smooth real function, $\alpha$ is a Riemannian metric and $\beta$ is a 1-form. In recent years, there have been a lot of research works concerning $(\alpha, \beta)$-metrics as well as Randers metrics.

Recently, the first two authors of this article have defined and studied $(\alpha_1, \alpha_2)$-metrics in [26], which generalize the $(\alpha, \beta)$-metrics. The using the same idea, we can define $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-metrics. Let $\alpha$ be a Riemannian metric on $M$, such that $TM$ can be $\alpha$-orthogonally decomposed as $TM = V_1 \oplus \cdots \oplus V_k$, in which each $V_i$ is a $n_i$-dimensional linear sub-bundle with $n_i > 0$ respectively. Let $\alpha_i$ be the restriction of $\alpha$ to each $V_i$ and naturally regard them as functions on $TM$ with $\alpha^2 = \alpha_1^2 + \cdots + \alpha_k^2$. Then a metric $F$ is called an $(\alpha_1, \alpha_2, \ldots, \alpha_3)$-metric if $F$ can be presented as $F = \sqrt{L(\alpha_1^2, \ldots, \alpha_3^2)}$, where $L$ is a positive smooth real function on an open subset of $\mathbb{R}^k$ satisfying certain conditions. In the following we will show that on some coset spaces we consider in this paper there exists invariant $(\alpha_1, \alpha_2)$ or $(\alpha_1, \alpha_2, \alpha_3)$-metrics with positive flag curvature. In our opinion, this class of metrics will be of great interest in Finsler geometry.

2.2 Geodesic spray and geodesic

On a Finsler space $(M, F)$, a local coordinate system \( \{x = (x^i) \in M, y = y^j \partial_{x^j} \in T_x M \} \) is called a standard local coordinates system.

The geodesic spray is a vector field $G$ defined on $TM \setminus 0$. In a standard local coordinate system, it can be presented as

$$G = y^i \partial_{x^i} - 2G^i \partial_{y^i},$$

in which

$$G^i = \frac{1}{4}g^{ij}([F^2]_{x^k} y^k - [F^2]_{x^j}).$$

A non-constant curve $c(t)$ on $M$ is called a geodesic if $(c(t), \dot{c}(t))$ is an integration curve of $G$, in which the tangent field $\dot{c}(t) = \frac{d}{dt}c(t)$ along the curve gives the speed. For any standard local coordinates, a geodesic $c(t)$ $(c^i(t))$ satisfies the equations

$$\dot{c}^i(t) + 2G^i(c(t), \dot{c}(t)) = 0.$$
It is well known that $F(\frac{d}{dt}c(t)) \equiv \text{const}$, i.e., the geodesics we are considering are geodesics of nonzero constant speed.

2.3 Riemannian curvature and flag curvature

On a Finsler manifold, we have a similar curvature as in the Riemannian case, which is called the Riemannian curvature. It can be defined either by the Jacobi field or the structure equation for the curvature of the Chern connection.

On a standard local coordinates system, the Riemannian curvature is a linear map

$$R_y = R^i_k(y)\partial_{x^i} \otimes dx^k : T_x M \to T_x M,$$

defined by

$$R^i_k(y) = 2\partial_x G^i - y^j \partial_{x^j} G^i + 2G^j \partial_{y^j} G^i - \partial_{y^j} G^i \partial_{y^k} G^j.$$  (2.6)

The Riemannian curvature $R_y$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_y$.

Using the Riemannian curvature, we can generalize the notion of sectional curvature to the flag curvature in Finsler geometry. Let $y$ be a nonzero tangent vector in $T_x M$ and $P$ a tangent plane in $T_x M$ containing $y$, linearly spanned by $y$ and $v$ for example. Then the flag curvature of the flag $(P, y)$ is given by

$$K(x, y, y \wedge v) = K(x, y, P) = \frac{\langle R_y v, v \rangle_y}{\langle y, y \rangle_y \langle y, v \rangle_y - \langle y, v \rangle_y^2}.$$  (2.7)

The flag curvature in (2.7) does not depend on the choice of $v$ but only depends on $P$ and $y$. Sometimes we will also write the flag curvature of a Finsler metric $F$ as $K^F(x, y, y \wedge v)$ or $K^F(x, y, P)$ to indicate the metric explicitly.

Z. Shen has made the following important observation which relates the Riemannian curvature of a Finsler metric to that of a Riemannian metric.

Let $Y$ be a tangent vector field on an open set $U \subset M$ which is nowhere vanishing. Then the Hessian matrices $(g_{ij}(Y(x)))$ with respect to any standard local coordinates defines a smooth Riemannian metric on $U$ which is independent of the choice of the local coordinates system. We denote this Riemannian metric as $g_Y$, and call $g_Y$ the localization of $F$ at $Y$. The Riemannian curvatures of $g_Y$ is denoted as $\tilde{R}_y$.

If $Y$ is a geodesic field on an open set $U \subset M$, i.e., if each integration curve of $Y$ is a geodesic of nonzero constant speed, then we have the following theorem of Z. Shen.

**Theorem 2.1** Let $Y$ be a geodesic field on an open set $U \subset M$ such that for $x \in U$, we have $y = Y(x) \neq 0$. Then the Riemannian curvature $R_y$ of $F$ coincides with the Riemannian curvature $\tilde{R}_y$ of the localization $g_Y$ of $F$.

It follows immediately that if $P$ is a tangent plane in $T_x M$ containing $y$, then $K^F(x, y, P) = \tilde{K}^g_Y(x, y, P)$.

3 Submersion of Homogeneous Finsler spaces

In this section we recall some definitions and results on submersion of homogeneous Finsler metrics, which will be useful in our study in the next sections.
3.1 Submersion and the subdued metric

A linear map \( \pi : (V_1, F_1) \to (V_2, F_2) \) between two Minkowski spaces is called an isometric submersion (or submersion for simplicity), if it maps the unit ball \( \{ y \in V_1 \mid F_1(y) \leq 1 \} \) in \( V_1 \) onto the unit ball \( \{ y \in V_2 \mid F_2(y) \leq 1 \} \) in \( V_2 \). Obviously a submersion map \( \pi \) must be surjective. The Minkowski norm \( F_2 \) on \( V_2 \) is uniquely determined by the submersion by the following equality,

\[
F_2(w) = \inf \{ F_1(v) \mid \pi(v) = w \}.
\]

Given the Minkowski space \( (V_1, F_1) \) and the surjective linear map \( \pi : V_1 \to V_2 \), there exists a unique Minkowski norm \( F_2 \) on \( V_2 \) such that \( \pi \) is a submersion. We will call \( F_2 \) the subduced norm. For the details of submersion of Finsler metrics, we refer the readers to [22].

To clarify the relationship between the Hessian matrices of \( F_1 \) and \( F_2 \), we need the notion of horizonal lift. Given a nonzero vector \( w \) in \( V_2 \), the infimum in (3.8) can be reached by a unique vector \( v \), which is called the horizontal lift of \( w \) with respect to the submersion \( \pi \). The horizontal lift \( v \) can also be determined by

\[
\langle y, \ker \pi \rangle_v, \text{ and } \pi(v) = w.
\]

Then the Hessian matrix of \( F_2 \) at \( w \) is determined by the following proposition.

**Proposition 3.1** Let \( \pi : (V_1, F_1) \to (V_2, F_2) \) be a submersion between Minkowski spaces. Assume that \( v \) is the horizontal lift of a nonzero vector \( w \) in \( V_2 \). Then the map \( \pi : (V_1, \langle \cdot, \cdot \rangle_v) \to (V_2, \langle \cdot, \cdot \rangle_w) \) is a submersion between Euclidean spaces.

For any vector \( u \in V_1 \setminus \ker \pi \), the horizontal lift of \( \pi(u) \) is called a horizontal shift of \( u \). If \( w = 0 \in V_2 \) or \( u \in \ker \pi \), then we naturally choose \( v = 0 \) as its horizontal lift or horizontal shift respectively.

A smooth map \( \rho : (M_1, F_1) \to (M_2, F_2) \) between two Finsler spaces is called a submersion, if for any \( x \in M_1 \), the induced tangent map \( \rho_* : (T_x M_1, F_1(x, \cdot)) \to (T_{\rho(x)} M_2, F_2(\rho(x), \cdot)) \) is a submersion between Minkowski spaces. Restricted to the image of the submersion \( \rho \), the metric \( F_2 \) is uniquely determined by \( F_1 \) and the submersion. Let \( F_1 \) be a Finsler metric on \( M_1 \) and \( \rho : M_1 \to M_2 \) be a surjective smooth map. If there is a metric \( F_2 \) on \( M_2 \) which makes \( \rho \) a submersion, then we call \( F_2 \) the subduced metric by \( F_1 \) and \( \rho \). Note that the subdued metric must be unique.

For a submersion between Finsler spaces, the horizontal lift (or the horizontal shift) of a tangent vector field can be similarly defined. The corresponding integration curves define the horizontal lift of smooth curves. Horizontal lift provides a one-to-one correspondence between the geodesics on \( M_2 \) the horizontal geodesics on \( M_1 \), so the horizontal lift of a geodesic field is also a geodesic field. Using Theorem 2.1 Proposition 3.1 and the curvature formula of Riemannian submersions, one can prove the following theorem (see [22]).

**Theorem 3.2** Let \( \rho : (M_1, F_1) \to (M_2, F_2) \) be a submersion of Finsler spaces. Assume that \( x_2 = \rho(x_1) \), and that \( y_2, v_2 \in T_{x_2} M_2 \) are two linearly independent tangent vectors. Let \( y_1 \) be the horizontal lift of \( y_2 \), and \( v_1 \) the horizontal lift of \( v_2 \) with respect to the
induced submersion \( \rho_* : (T_{x_1}M_1, \langle \cdot, \cdot \rangle_{y_1}) \to (T_{x_1}M_2, \langle \cdot, \cdot \rangle_2) \). Then the flag curvature of \((M_1, F_1)\) and \((M_2, F_2)\) satisfies the following inequality:
\[
K^{F_1}(x_1, y_1, y_1 \wedge v_1) \leq K^{F_2}(x_2, y_2, y_2 \wedge v_2).
\] (3.10)

3.2 Homogeneous Finsler metrics subduced by a submersion

A compact connected Finsler space \((M, F)\) is called homogeneous, if the full group of isometries \(I(M, F)\) acts transitively on \(M\). In this case, the identity component \(I_0(M, F)\) also acts transitively on \(M\) (see [17]). If \(G\) is a closed connected subgroup of \(I_0(M, F)\) which acts transitively on \(M\), then \(M\) can also be identify with the coset space \(G/H\), in which \(H\) is the isotropy subgroup at some point. In this case, we usually say that \(F\) is a \(G\)-homogeneous Finsler metric on \(M\). Notice that the isometry group of any compact Finsler space must be compact. Hence we only need to consider compact connected \(G\) with a closed subgroup \(H\), endowed with a \(G\)-homogeneous Finsler metric \(F\) on \(G/H\). Denote the natural projection from \(G\) to \(G/H\) as \(\pi\). Then \(H\) can be chosen as the isotropy subgroup at \(o = \pi(e)\).

Let \(g\) and \(h\) be the Lie algebras of \(G\) and \(H\) respectively. Fix an \(\text{Ad}(G)\)-invariant inner product \(\langle \cdot, \cdot \rangle\) on \(g\). Then we have an orthogonal decomposition
\[
g = h + m,
\] (3.11)
where \(m\) is a subspace of \(g\) invariant under \(\text{Ad}(H)\). The vector space \(m\) can be identified with the tangent space \(T_o(G/H)\).

The restriction to \(T_o(G/H)\) defines a canonical one-to-one correspondence between \(G\)-homogeneous Finsler metrics on \(G/H\) and \(\text{Ad}(H)\)-invariant Minkowski norm on \(m\). For simplicity we also use the same \(F\) to denote the corresponding Minkowski norm on \(m\).

The method of submersion can be applied to define homogeneous Finsler metrics on \(G/H\). Let \(\bar{F}\) be a left invariant Finsler metric on \(G\) which is right invariant under \(H\). Then the following lemma indicates that there exists a uniquely defined subduced homogeneous metric on \(G/H\).

**Lemma 3.3** Keep all the above notations. Then there is a uniquely defined \(G\)-homogeneous metric \(F\) on \(G/H\) such that the tangent map
\[
\pi_* : (T_gG, \bar{F}(g, \cdot)) \to (T_{\pi(g)}(G/H), F(\pi(g), \cdot))
\] (3.12)
is a submersion for any \(g \in G\).

**Proof.** From the above section we see that the tangent map \(\pi_* : T_oG \to T_oM\) defines a unique subduced Minkowski norm \(F\) on \(m = T_oM\) from \(\bar{F}(e, \cdot)\). Since \(\bar{F}(e, \cdot)\) is \(\text{Ad}(H)\)-invariant, \(F\) is also \(\text{Ad}(H)\)-invariant. Left translations by \(G\) defines a \(G\)-homogeneous metric on \(M\), which is also denoted as \(F\) for simplicity. Since \(\pi_*|_{T_oG}\) is a submersion, \(\bar{F}\) is bi-invariant, and \(F\) is \(G\)-homogeneous, the map \(\pi_*|_{T_oG} = g_* \circ \pi_*|_{T_oG} \circ (L_g^{-1})_*\) is a submersion between the Minkowski spaces \((T_gG, \bar{F}(g, \cdot))\) and \((T_{\pi(g)}M, F(\pi(g), \cdot))\), for any \(g \in G\). 

On the other hand, the next lemma indicates that any homogeneous Finsler metric can be subduced from a well-chosen Finsler metric \(\bar{F}\) on a Lie group and the natural projection.
Lemma 3.4 Let $F$ be a homogeneous metric on $G/H$. Then there exists a left invariant $\tilde{F}$ on $G$ which is right $H$-invariant such that $\tilde{F}|_m = F$, and $F$ is subduced from by the projection $\pi : G \to G/H$.

Proof. We only need to construct a $\text{Ad}(H)$-invariant Minkowski norm $\tilde{F}$ satisfying the conditions of the lemma. The conditions that $\tilde{F}|_m = F$ and that $F$ is subduced from $\tilde{F}$ by the projection can be equivalently stated as that the indicatrix $I_F$ of $F$ is tangent to the cylinder $\mathfrak{h} \times I_F$ at each point of the indicatrix $I_F \subset m$ of $F$. A Minkowski norm $\tilde{F}$ on $\mathfrak{g}$ which satisfies this statement but may not be $\text{Ad}(H)$-invariant can be constructed inductively by the following observation:

Any Minkowski norm $F$ on $\mathbb{R}^{n-1} = \{(y^1, \ldots, y^{n-1}, 0) | \forall y^1, \ldots, y^{n-1} \} \subset \mathbb{R}^n$ can be extended to a Minkowski norm $\tilde{F}$ on $\mathbb{R}^n$ such that the indicatrix $I_{\tilde{F}}$ is tangent to the cylinder $I_F \times \mathbb{R}$ at each point of the indicatrix $I_F$ of $F$.

To prove the above assertion, we need a non-negative function $\phi \in C[0, 1] \cap C^{\infty}[0, 1)$, such that on $[1/2, 1]$, $\phi(t) = \sqrt{1 - t}$, all derivatives $\phi^{(k)}$ for $k > 0$ vanish at 0, $\phi'(t) > 0$ and $\phi''(t) < 0$ for $0 < t < 1$. We also need a smooth function $\psi$ on $\mathbb{R}^{n-1}$ such that $\psi$ is compactly supported in the closed ball $B_{1/2}^F(0) \subset \mathbb{R}^{n-1}$ with the $F$-radius $1/2$ and the center 0, and 0 is a critical point of $\psi$ with a negative definite Hessian matrix. When a positive number $\lambda$ is sufficiently close to 0, the set

$$\{y = (y', y^n) | y^n = \pm \phi(F^2(y')) + \lambda \psi(y')\}$$

(3.13)

is a smooth hypersurface surrounding 0. Obviously it is tangent to $I_F \times \mathbb{R}$ at each point of $I_F$. To see that it defines a Minkowski norm $\tilde{F}$, we need to check the convexity condition. Notice that when $|y^n| < \frac{\sqrt{2}}{2} \phi(\frac{1}{2}) F(y')$, it defines the function $\sqrt{F^2(y') + (y^n)^2}$ with $y = (y', y^n)$, for which the convexity condition of the Minkowski norm is satisfied at its smooth points. For other points, the hypersurface \[3.13\] coincides with the graphs of $\pm f$, in which $f(y') = \phi(F^2(y')) + \lambda \phi(y')$ for $y' \in B_{3/4}^F(0)$. The Hessian of $\phi(F^2)$ is negative definite everywhere in $B_{3/4}^F(0)$ except the 0 matrix at $y' = 0$, while $\psi$ has a negative definite Hessian near 0, so $f$ has a negative definite Hessian at each point of $B_{3/4}^F(0)$ when $\lambda$ is sufficiently small. This is enough to prove the convexity condition at all points.

With the Minkowski norm $\tilde{F}$ on $\mathfrak{g}$ inductively constructed above, the average

$$\tilde{F}(y) = \sqrt{\frac{\int_H \tilde{F}^2(\text{Ad}(h)y) d\text{vol}_H}{\int_H d\text{vol}_H}}$$

(3.14)

with respect to any bi-invariant volume form $d\text{vol}_H$ defines an $\text{Ad}(H)$-invariant Minkowski norm. Since $F$ is $\text{Ad}(H)$-invariant, the cylinder $\mathfrak{h} \times I_F$ is also $\text{Ad}(H)$-invariant. Moreover, we have $\tilde{F}|_m = F$. Therefore $I_{\tilde{F}}$ is tangent to $\mathfrak{h} \times I_F$ at each point of $I_F$, i.e., it is the Minkowski norm indicated by the lemma.  ■
4 Submersion and the flag curvature of homogeneous Finsler spaces

4.1 A refinement of Theorem 2.1

Lemma 3.4 implies that one could use the method of submersion to study homogeneous Finsler spaces, especially the flag curvature. Unfortunately, although Theorem 3.2 provides much information, there are no convenient explicit flag curvature formula for a submersion in Finsler geometry. To study flag curvature, we need to reduce the discussion to the Riemannian case. Theorem 2.1 of Shen provides a very enlightening hint. However, geodesic fields can not be canonically found in a homogeneous manifold as in the case of the left (or right) invariant vector fields on a Lie group, or Killing vector fields on a homogeneous (Riemannian or Finsler) manifold. Since in most situations we only need to consider the geometric properties of a homogeneous space at one particular point, say the origin \( o = \pi(e) \), we only require that the vector field \( Y \) in Theorem 2.1 generates a geodesic through \( o \), not necessary generates geodesics through any point on an open subset of \( o \). This observation leads to the following refinement of Theorem 2.1.

**Theorem 4.1** Let \( Y \) be a vector field on a Finsler space \((M, F)\), such that \( y = Y(p) \neq 0 \), and \( Y \) generates a geodesic of constant speed through \( p \). Then \( R_y = \tilde{R}_y \), where \( \tilde{R}_y \) is the Riemannian curvature of \( g_Y \) defined on an open neighborhood of \( p \). Furthermore, for any tangent plane \( P \) in \( T_p M \) containing \( y \), we have \( K_F(p, y, P) = K^{g_Y}(p, y, P) \).

**Proof.** Let \( x = (x^i) \) and \( y = y^i \partial_{x^i} \) be a standard local coordinate system defined on an open neighborhood \( U \) of \( p \), such that \( Y = \partial_{x^1} \). In the following, quantities with respect to \( g_Y \) will be denoted with a tilde.

The covariant derivatives with respect to \( F \) can be expressed as

\[
\nabla^F_Y Y = 2G^i(Y)\partial_{x^i} = \frac{1}{2}[g^{il}(2\partial_{x^k}g_{lj} - \partial_{x^l}g_{jk})](Y)y^jy^k|_{y^1=1,y^2=...=y^n=0}. \tag{4.15}
\]

Similarly, the covariant derivatives with respect to \( g_Y \) can be expressed as

\[
\tilde{\nabla}^F_Y Y = 2\tilde{G}^i(Y)\partial_{x^i} = \frac{1}{2}\tilde{g}^{il}(2\partial_{x^k}\tilde{g}_{lj} - \partial_{x^l}\tilde{g}_{jk})y^jy^k|_{y^1=1,y^2=...=y^n=0}\partial_{x^i}, \tag{4.16}
\]

where \( n = \dim M \). Now, for \( Y = \partial_{x^1} \), we have

\[
(\partial_{x^k}g_{lj})(Y) = \partial_{x^k}\tilde{g}_{lj}, (\partial_{x^l}g_{jk})(Y) = \partial_{x^l}\tilde{g}_{jk}, \text{ and } g^{il}(Y) = \tilde{g}^{il}. \tag{4.17}
\]

Then the equalities \( G^i(Y) = \tilde{G}^i(Y) \) hold on the neighborhood \( U \) of \( p, \forall i \). Thus on \( U \) we have

\[
[\partial_{x^k}G^i(Y)] = [\partial_{x^k}\tilde{G}^i(Y)] = [\partial_{x^k}G^i(Y)] = [\partial_{x^k}\tilde{G}^i(Y)], \tag{4.18}
\]

for any \( i \) and \( k \). Since the flow line of \( Y \) through \( o \) is a geodesic, we have \( G^i(Y) = \tilde{G}^i(Y) = 0, \forall i \), on the integration curve of \( Y \) along \( p \).
Now we show that at the point $p$, we have $N^i_j(Y) = \tilde{N}^i_j(Y)$. In fact,

$$N^i_j = \frac{1}{2}g^{il}[\partial_{x^l}g_{jl} + \partial_{x^l}g_{kl} - \partial_{x^l}g_{jk}]y^k - 2g^{il}C_{jkl}G^k. \quad (4.19)$$

Thus at the point $p$ we have

$$N^i_j(Y) = \frac{1}{2}[g^{il}(\partial_{x^l}g_{jl} + \partial_{x^l}g_{kl} - \partial_{x^l}g_{jk}])(Y)$$
$$= \frac{1}{2}[\tilde{g}^{il}(\partial_{x^l}\tilde{g}_{jl} + \partial_{x^l}\tilde{g}_{kl} - \partial_{x^l}\tilde{g}_{jk}]](Y)$$
$$= \tilde{N}^i_j(Y). \quad (4.20)$$

This proves our assertion. A similar calculation then shows that at the point $p$, we also have

$$N^i_j(Y) = \tilde{N}^i_j(Y) - 2g^{il}(Y)C_{ijk}G^k(Y). \quad (4.21)$$

Since for any $i$, $G^i(p, Y) = 0$, we have

$$\partial_{x^i}N^i_j(Y) = \partial_{x^i}\tilde{N}^i_j(Y) - 2g^{il}(Y)C_{ijk}\partial_{x^i}G^k(Y). \quad (4.22)$$

Thus for any $k$, $\partial_{x^i}G^k(p, Y) = 0$. Therefore at $p$, we have $\partial_{x^i}N^i_j(Y) = \partial_{x^i}\tilde{N}^i_j(Y)$.

Now by the formula (2.6) of the Riemannian curvature $\bar{R}^i_k(y)$ and $\tilde{R}^i_k(y)$ for $y = Y(p)$, we have $R_y = \tilde{R}_y$. The equality for the flag curvatures follows immediately. ■

### 4.2 The flag curvature of a subduced homogeneous metric

Let $G$ be a compact connected Lie group, $H$ a closed subgroup of $G$, and $F$ a $G$-homogeneous Finsler metric on $G/H$. Let $\bar{F}$ be the Finsler metric on $G$ as in Lemma 3.4. Then $\bar{F}$ is left $G$-invariant and right $H$-invariant, and $\bar{F}|_m = F$ when $\bar{F}$ and $F$ are viewed as Minkowski norms on $g$ and $m$, respectively. Moreover, $F$ is subduced by $\bar{F}$ and the natural projection $\pi$. We keep all the other notations of the previous section.

Let $u$ and $v$ be two linearly independent tangent vectors in $m = T_o(G/H)$ and assume that $u$ satisfies the condition

$$\langle u, [u, m]\rangle = 0, \quad (4.23)$$

where $F$ is viewed as a Minkowski norm on $m$ and $[\cdot, \cdot]$ is the composition of the Lie bracket and the orthogonal projection from $g$ to $m$. In this section we will present a strategy to calculate the flag curvature $K^F(o, v_1, v_1 \wedge v_2)$.

First note that there is a right invariant vector field $U_1$ on $G$ extending $u \in m \subset g = T_eG$, which defines a Killing vector field $U$ of $(G/H, F)$. Let $U'$ be the horizontal lift of $U$ with respect to the submersion $\pi : (G, \bar{F}) \to (G/H, F)$. Since $U(o) = u \neq 0$, $U'$ is smooth on an open neighborhood of $e$.

Let $V_1$ be the left invariant vector field on $G$ extending $v \in m \subset g = T_eG$. Let $V'$ be the horizontal shift of $V_1$ with respect to the submersion $\pi : (G, g_{V'}) \to (G/H, g_{V'})$. Then $V'$ is smooth and non-vanishing on an open neighborhood of $e$.

We first deduce the following lemma.

**Lemma 4.2** The following assertions holds:
(1) The horizontal lift of \( u \) with respect to the submersion \( \pi_* : (\mathfrak{g}, \bar{F}) \to (\mathfrak{m}, F) \) is exactly \( u \) itself.

(2) The vectors field \( U \) generates a geodesic of \((G/H, F)\) through \( o \). Moreover, each of the vector fields \( U' \) and \( U_1 \) generates the same geodesic of \((G, \bar{F})\) through \( e \).

(3) The vectors \( u \) and \( v \) in \( \mathfrak{m} \) are the horizontal lift of themselves with respect to the submersion \( \pi_* : (\mathfrak{g}, \langle \cdot, \cdot \rangle_{\bar{F}}) \to (\mathfrak{m}, \langle \cdot, \cdot \rangle_{\bar{F}}) \).

**Proof.** (1) By Lemma 3.4 \( F \) is both the restriction and the subduced metric of \( \bar{F} \). From this the assertion follows.

(2) Without losing generality, we can assume that \( u \) is a unit vector for \( \bar{F} \). Then the assumption (1.23) and the \( \text{Ad}(H) \)-invariance of \( F \) implies that the \( \text{Ad}(G) \)-orbit of \( u \) is tangent to the indicatrix of \( \bar{F} \) at \( u \). The \( F \)-length of \( U \) at any \( \pi(u) \) is \( F(\text{pr}_m(\text{Ad}(g)u)) \). If \( g = \exp(tX), X \in \mathfrak{g} \), then we have

\[
F(\text{pr}_m(\text{Ad}(\exp(tX))u)) = F(u) + o(t),
\]

where \( o(t) \) denotes an infinitesimal quantity of higher order then \( t \). Then the \( F \)-length function of \( U \) has a critical point at \( o \). Since \( U \) is a Killing vector field, it generates a geodesic of \((G/H, F)\) through the critical point \( o \). Similarly, the vector field \( U_1 \) generates a geodesic of \((G, \bar{F})\) through \( e \). Since \( U' \) is the smooth horizontal lift of \( U \) on an open neighborhood of \( e \), \( U' \) also generates a geodesic of \((G, \bar{F})\) through \( e \). Along the integration curve \( \exp(\bar{U}t) \) of \( U_1 \) through \( e \), \( U' \) is both left and right invariant. Since \( U_1(e) = U'(e) = u \), \( U_1 \) is equal to the horizontal lift of \( U \) along the curve \( \exp(\bar{U}t) \), that is, \( U_1 \) and \( U' \) generates the same geodesic \( \exp(\bar{U}t) \) through \( e \).

(3) At any point in \( \mathfrak{m} \), the derivative of \( \bar{F} \) is equal to 0 in the directions of \( \mathfrak{h} \). So by (2.2), for any \( w \in \mathfrak{h} \), we have

\[
\langle v, w \rangle_{\bar{F}} = \frac{1}{2} \frac{\partial}{\partial s} \frac{\partial}{\partial t} \bar{F}^2(u + sv + tw)|_{t=0}||_{s=0} = 0,
\]

which proves the statement for \( v \). The statement for \( u \) is obvious. \( \blacksquare \)

There are small open neighborhoods \( U' \subset G \) of \( e \) and \( \mathcal{U} \subset G/H \), such that on these open subsets the Riemannian metrics \( g_{U'} \) and \( g_U \) are well-defined. Moreover, the map

\[
\pi : (\mathcal{U}', g_{U'}) \to (\mathcal{U}, g_U)
\]

is a Riemannian submersion. For the horizontal lifts \( u \) and \( v \) in \( T_eG \), we have

\[
K^{g_{U'}}(o, u, u \wedge v) = K^{g_{U'}}(e, u, u \wedge v) + \frac{3||A(u, v)||^2}{||u \wedge v||^2},
\]

in which the norms of vectors are defined by \( g_{U'} \), and \( A(u, v) \) is the value of the vertical component of \( \frac{1}{2}[U', V'] \), where \( U' \) and \( V' \) are the horizontal vector fields with respect to the submersion (3.24). Now by (2) of Lemma 4.2 \( K^{g_{U'}}(o, u, u \wedge v) = K^F(o, u, u \wedge v) \) and \( K^{g_{U'}}(e, u, u \wedge v) = K^F(e, u, u \wedge v) \). Therefore we have

**Lemma 4.3** Keep all the notations as above. Then

\[
K^F(o, u, u \wedge v) = K^F(e, u, u \wedge v) + \frac{3||A(u, v)||^2}{||u \wedge v||^2}.
\]
5 Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. Let $G$ be a compact connected simply connected Lie group, $H$ a closed subgroup such that $G$ acts effectively on $G/H$. Let $T$ be a maximal torus of $H$ and $F$ is a $G$-homogeneous Finsler metric on $G/H$ with positive flag curvature.

We first prove a simple lemma.

**Lemma 5.1** Keep the notations as above. Let $N_G(T)$ and $N_H(T)$ be the normalizers of $T$ in $G$ and $H$ respectively. Then the orbit $M_T = N_G(T) \cdot o = N_G(T)/N_H(T) \subset G/H$ of $o = \pi(e)$ consists of all the common fixed points of $T$.

**Proof.** For each $x \in M_T$, let $x = f(o)$ for some $f \in N_G(T)$. Then by the definition of normalizer, for any $t \in T$, there exists $\tilde{t} \in T$ such that $tf = f\tilde{t}$, hence we have $t(x) = t(f(o)) = f(\tilde{t}(o)) = f(o) = x$. This shows that $x$ is fixed by all elements in $T$.

Conversely, if $x \in G/H$ is fixed by all elements in $T$, we shall prove that $x \in M_T$. Suppose $x = f(o)$ for some $f \in G$. Then it follows from the equality $t(x) = x$ that $t(f(o)) = f(o)$. This shows that $f^{-1}Tf$ is another maximal torus in $H$. Since different maximal tori are conjugate to each other, there exists $h \in H$ such that $f^{-1}Tf = h^{-1}Th$. Let $\tilde{f} = fh^{-1}$. Then we have $\tilde{f} \in N_G(T)$ and $x = f(o) = f(h^{-1}(o)) = \tilde{f}(o)$. This proves that $x \in M_T$. ■

**Theorem 5.2** Let $G$ be a compact connected simply connected Lie group and $H$ a closed subgroup such that $G$ acts effectively on $G/H$. Let $T$ be a maximal torus of $H$ and $N_G(T)$ the normalizer of $T$ in $G$. If $G/H$ admits a $G$-homogeneous Finsler metric with positive flag curvature, then

1. The orbit of $N_G(T)$ is totally geodesic in $G/H$.
2. If $\dim G/H$ is even, then $T$ is a maximal torus of $G$. In this case, $\text{rk } G = \text{rk } H$.
3. If $\dim G/H$ is odd, then $N_G(T)/T$ is Lie isomorphic to $S^1$, $SU(2)$ or $SO(3)$. In this case $\text{rk } G = \text{rk } H + 1$.

**Proof.** By a theorem in [7], the common fixed points of any set of isometries is the union of totally geodesic submanifolds. Hence, by lemma 5.1, $M_T = N_G(T) \cdot o$ is the union of totally geodesic submanifolds.

Let $C(T)$ be the centralizer of $T$ in $G$. If $C(T) = T$, then $T$ is a maximal torus of $G$. If $C(T) \neq T$, then $\dim G/H$ is odd dimensional [12, Prop 5.3] and $\dim N_G(T) > \dim T$. In this case, $M_T = N_G(T)/T$ is totally geodesic and hence has positive flag curvature. Notice that $N_G(T)/T$ is a Lie group and the induced metric on $N_G(T)/T$ is a left invariant metric, hence by Theorem 5.1 of [12], $N_G(T)/T$ is Lie isomorphic to $S^1$, $SU(2)$ or $SO(3)$. ■

Now we continue to consider the even dimensional case. By theorem 5.2, $\text{rk } G = \text{rk } H$, i.e., there is a Cartan subalgebra $t$ contained in $\mathfrak{h}$, and we have the decomposition

$$\mathfrak{g} = t + \sum_{\alpha \in \Delta^+} \mathfrak{g}_{\pm \alpha}.$$ (5.29)
where each $\mathfrak{g}_{\pm \alpha}$ is 2-dimensional and called a root plane. Since the ranks of $H$ and $G$ are equal, a root plane $\mathfrak{g}_{\pm \alpha}$ is contained either in $\mathfrak{h}$ or in $\mathfrak{m}$. For simplicity, we will just call $\pm \alpha$ roots of $\mathfrak{h}$ or $\mathfrak{m}$ accordingly.

Let $\alpha$ be a root of $\mathfrak{m}$ and $u$ a nonzero vector in $\mathfrak{g}_{\pm \alpha}$. Denote $t' = \ker \mathfrak{h}$ and the group generated by it as $T'$. Let $\bar{F}$ be the $\Ad(h)$-invariant Minkowski norm on $\mathfrak{g}$ in Lemma 3.3 i.e. $F$ is subduced from $\bar{F}$, and $\bar{F}|_{\mathfrak{m}} = F$.

**Lemma 5.3** For any root $\gamma \neq \pm \alpha$, we have $\langle u, \mathfrak{g}_{\pm \gamma} \rangle_{\bar{F}} = 0$. Moreover, for any root $\beta \notin \{ \pm \gamma, \pm \alpha \}$, such that $\beta \pm \alpha$ are not roots, we have $\langle \mathfrak{g}_{\pm \beta}, \mathfrak{g}_{\pm \gamma} \rangle_{\bar{F}} = 0$.

**Proof.** Since the Minkowski norm $\bar{F}$ on $\mathfrak{g}$ is $\Ad(H)$ invariant, and $u$ is $\Ad(T')$-invariant, $\langle \cdot, \cdot \rangle_{\bar{F}}$ is $\Ad(T')$-invariant, or equivalently, for any $h \in t'$, $\ad(h)$ is skew-adjoint with respect to $\langle \cdot, \cdot \rangle_{\bar{F}}$. Now by Theorem 1.3 of [11], we have

$$\langle [h, v], w \rangle_{\bar{F}} + \langle [h, w], v \rangle_{\bar{F}} + 2C_u([h, u], v, w) = 0,$$

$$\forall h \in \mathfrak{h}, \quad u, v, w \in \mathfrak{g}\{0\}.$$  

Therefore for any $h \in t'$ we have $[h, u] = 0$. Hence, for $h \in t'$, the linear map $\ad(u)$ is skew-adjoint.

Since the operators $\ad(h)$, $h \in t'$ commute with each other, they have the same root spaces. We then have the decomposition

$$\mathfrak{g} = (t + \mathfrak{g}_{\pm \alpha}) + \sum \mathfrak{g}_{\pm \delta},$$

where $\mathfrak{g}_{\pm \delta}$ denotes the common root space of all the $\ad(h)$, $h \in t'$, namely, the sum of all the root planes $\mathfrak{g}_{\pm \tau}$ such that $\tau \equiv \delta \pmod{\alpha}$. Each nonzero $\mathfrak{g}_{\pm \delta}$ has the form

$$\mathfrak{g}_{\pm \tau} + \mathfrak{g}_{\pm (\tau + \alpha)} + \cdots + \mathfrak{g}_{\pm (\tau + k\alpha)}.$$  

Since $\ad(h)$ is skew-adjoint with respect to $\langle \cdot, \cdot \rangle_{\bar{F}}$, different root spaces are orthogonal to each other with respect to $\langle \cdot, \cdot \rangle_{\bar{F}}$. In particular, $\mathfrak{g}_{\pm \alpha}$, in the 0-eigenspace, is orthogonal to other root spaces. This proves the first assertion. If $\beta \neq \pm \alpha$ is a root such that $\alpha \pm \beta$ are not roots, there is a factor $\mathfrak{g}_{\pm \delta}$ in (5.31) which only contains $\mathfrak{g}_{\pm \beta}$. For any other root $\gamma \neq \pm \beta$, $\mathfrak{g}_{\pm \beta}$ and $\mathfrak{g}_{\pm \gamma}$ must belong to different root spaces, so $\langle \mathfrak{g}_{\pm \beta}, \mathfrak{g}_{\pm \gamma} \rangle_{\bar{F}} = 0$.

We now recall the definition of Condition (A) of N. Wallach, which is the key point in the deduction of the main results of [25]. Keeping the notations as above, we say that the pair $(G, H)$ (or $(\mathfrak{g}, \mathfrak{h})$) satisfies Condition (A), if for any two roots $\alpha$, $\beta$ of $\mathfrak{m}$ with $\alpha \neq \pm \beta$, either $\alpha + \beta$ or $\alpha - \beta$ is a root. The following lemma is the main step to prove our main theorem.

**Lemma 5.4** If the homogeneous space $G/H$ admits a $G$-homogeneous Finsler metric $F$ of positive flag curvature, then the pair $(G, H)$ satisfies condition (A).

**Proof.** Suppose conversely that $(G, H)$ does not satisfy Condition (A), i.e. there are roots $\alpha$ and $\beta$ of $\mathfrak{m}$ such that $\beta \neq \alpha$ and neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. Choose a nonzero vector $u \in \mathfrak{g}_\alpha \subset \mathfrak{m}$. Then by Lemma 5.3 for any root $\gamma$ of $\mathfrak{m}$ and $\forall w \in \mathfrak{g}_\gamma$, we have $\langle u, [w, u] \rangle_{\bar{F}} = \langle u, [w, u] \rangle_{\bar{F}} = 0$, where we have used the fact that

$$[\mathfrak{g}_\gamma, \mathfrak{g}_\alpha] \subset \mathfrak{g}_{\gamma + \alpha} + \mathfrak{g}_{\gamma - \alpha}.$$
Hence $u$ satisfies (5.24).

Choose a nonzero $v \in g_\beta$. Then we have $[u, v] = 0$. Let $X_1, \ldots, X_N$ be a basis of the space of all left invariant vector fields on $G$ such that $X_1(e) = u$ and $X_2(e) = v$. Denote the smooth vector fields $U'$ and $V'$ around $e$ as $U'(g) = u'(g)X_1$ and $V'(g) = v'(g)X_1$. Then we have

$$[U', V'](e) = [u, v] + \frac{d}{dt} u'(\exp(tv))|_{t=0}X_1 - \frac{d}{dt} v'(\exp(tv))|_{t=0}X_1. \quad (5.33)$$

Since $[u, v] = 0$, $U'(\exp(tv)) \equiv X_1$ and $V'(\exp(tv)) \equiv X_2$, we have $[U', V'](e) = 0$. Thus

$$K^F(o, u, u \wedge v) = K^F(e, u, u \wedge v). \quad (5.34)$$

Let $U_2$ be the left invariant vector field on $G$ extending $u \in \mathfrak{m} \subset g = T_eG$. Then the integration curve of $U_2$ coincides with that of $U_1$ at $e$, i.e., $U_2$ also generates a geodesic of $(G, \bar{F})$ there. The Riemannian localization $g_{U_2}$ of the left invariant $\bar{F}$ is also left invariant. Therefore we have $K^F(o, u, u \wedge v) = K^{g_{U_2}}(e, u, u \wedge v)$, where the right side is 0 according to Lemma 5.1 of [25]. This is a contradiction. ■

This lemma has an important corollary:

**Corollary 5.5** Let $G$ be a compact connected simply connected Lie group and $H$ a closed subgroup of $G$ such that $G$ acts effectively on $G/H$ and the dimension of $G/H$ is even. If $G/H$ admits an invariant Finsler metric, then $G/H$ must be simple.

**Proof.** By Lemma [5.4] the pair $(G, H)$ satisfies the condition (A). Then by a result of Wallach [25], $G/H$ admits an invariant Riemannian metric with positive sectional curvature. This assertion combined with the classification of even dimensional homogeneous Riemannian manifolds with even dimension proves the corollary. ■

**Remark 5.6** In [18], L. Huang introduced the spray vector $\eta$ and the connection operator $N$ associated to a nonzero $y \in \mathfrak{m}$ as follows. The spray vector $\eta$ is the unique vector in $\mathfrak{m}$ such that

$$g_y(\eta, v) = g_y(y, [v, y]_m), \quad \forall v \in \mathfrak{m}.$$  

The connection operator $N$ is a linear operator on $\mathfrak{m}$ defined by

$$2g_y(N(v), u) = g_y([u, v]_m, y) + g_y([u, y]_m, v) + g_y([v, y]_m, u) - 2C_y(u, v, \eta), \quad \forall u, v \in \mathfrak{m}. $$

Using these two notions, L. Huang has proved the following formula for Riemannian curvature $R_y$

$$g_y(R_y(w), w) = g_y([w, y]_h, w, y) + g_y(\tilde{R}(w), v), \quad v \in \mathfrak{m}, \quad (5.35)$$

where $\tilde{R} = D_yN - N^2 + [N, \text{pr}_{\mathfrak{m}}\text{ad}(y)]$. Now suppose $\alpha, \beta$ are roots of $\mathfrak{m}$ such that $\alpha \neq \pm \beta$ and $\alpha \pm \beta$ are not roots. Choose a nonzero $y \in g_\alpha$. Then by Lemma 5.3 we have $g_y(\text{ad}(y)(w) = [y, w] = 0$. Using Lemma 5.5 again one can check that $N(w) = 0$. It follows that $\tilde{R}(w) = 0$. Together with the fact that $[w, y]_h = 0$ we get $g_y(R_y(w), w) = 0$, namely, the flag curvature of the flag $(y, y \wedge w)$ is zero. This gives an alternative proof of the lemma.
We now summarize the above arguments and conclusions. Let $G$ be a connected simply connected Lie group $G$ and $H$ a closed subgroup of $G$. If the homogeneous space $G/H$ is even dimensional and it admits a $G$-homogeneous Finsler metric $F$ of positive flag curvature, then we have

1. $G$ is simple;
2. $\text{rk} \, G = \text{rk} \, H$;
3. $G$ and $H$ satisfies condition (A).

N. Wallach has listed all possible pairs of $(G, H)$ satisfying the above conditions, and showed that in each case, $G$-homogeneous Riemannian metric with positive sectional curvature can be found on the homogeneous manifolds (particularly on the three new examples in his list). This completes the proof of Theorem 1.1.

6 Proof of Theorem 1.2

In this section we prove Theorem 1.2. The proof is just a case by case check on the isotropy representation of the connected simply connected homogeneous manifolds admitting invariant Finsler metrics of positive flag curvature. We will also present some information on the invariant Finsler metrics on these coset spaces, which will be of interest in its own right from the point of view of Finsler geometry. By Theorem 1.1 the list of connected simply connected homogeneous manifolds admitting invariant Finsler metrics of positive flag curvature coincides with that of the Riemannian case. Now by the results of Wallach (see [25]), the list consists of the following coset spaces (we only give the pairs of Lie groups, which satisfy the assumptions in Theorem 1.2):

1. The Riemannian symmetric pairs $(G, H)$ of compact type of rank one;
2. The pair $(\text{Sp}(n), \text{Sp}(n-1)U(1))$, where $n \geq 2$;
3. The pair $(G_2, \text{SU}(3))$;
4. The pair $(\text{SU}(3), T^2)$, where $T^2$ is a maximal torus of $\text{SU}(3)$;
5. The pair $(\text{Sp}(3), \text{Sp}(1) \times \text{Sp}(1) \times \text{Sp}(1))$.
6. The pair $(F_4, \text{Spin}(8))$.

Now we give a case by case study of invariant Finsler metrics on coset spaces corresponding to the above coset pairs.

1. The Riemannian symmetric pairs $(G, H)$ of compact type of rank one. Note that in this case the isotropy subgroup $H$ acts transitively on the unit sphere of $T_o(G/H)$, where $o = H$ s the origin, with respect to standard Riemannian metric. Then any $G$-invariant Finsler metric on $G/H$ must be a positive multiple of the standard metric.

2. The pair $(\text{Sp}(n), \text{Sp}(n-1)U(1))$, where $n \geq 2$. A description of the isotropy representation of the coset space is presented in [27]. The tangent space $T_o(G/H)$ can be decomposed as $T_o(G/H) = \mathbb{R}^2 \oplus \mathbb{H}^{n-1}$. The subgroup $\text{Sp}(n-1)$ of $H$ acts trivially...
on $\mathbb{R}^2$, and the action of $U(1)$ on $\mathbb{R}^2$ is the standard rotation action. Meanwhile, the action of $H$ on $\mathbb{H}^{n-1}$ is $(A,z)(v) = A(v)\bar{z}$. Therefore the subspaces $\mathbb{R}^2$ and $\mathbb{H}^{n-1}$ are both invariant under $H$ and the action of $H$ is both transitively on the unit sphere of $\mathbb{R}^2$ as well as that of $\mathbb{H}^{n-1}$ with respect to the standard metrics. As pointed out by Onísík [21], the coset space $Sp(n)/Sp(n-1)U(1)$ is the complex projective space $\mathbb{C}P^{2n-1}$. The standard metric of $\mathbb{C}P^{2n-1}$ corresponds to the inner product

$$\langle t, t \rangle = \langle \cdot, \cdot \rangle_{\mathbb{R}^2} + \langle \cdot, \cdot \rangle_{\mathbb{H}^{n-1}},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$ (resp. $\langle \cdot, \cdot \rangle_{\mathbb{H}^{n-1}}$) is the standard inner product on $\mathbb{R}^2$ (resp. $\mathbb{H}^{n-1}$). The general form of an invariant Riemannian metric on $Sp(n)/Sp(n-1)U(1)$ is induced by the inner product on $\mathfrak{m}$ of the form

$$\langle t_1, t_2 \rangle = t_1 \langle \cdot, \cdot \rangle_{\mathbb{R}^2} + t_2 \langle \cdot, \cdot \rangle_{\mathbb{H}^{n-1}},$$

where $t_1, t_2$ are positive real numbers. By the continuity, if the ratio $\frac{t_1}{t_2}$ is close enough to 1, then the corresponding Riemannian metric has positive curvature. On the other hand, let $F$ be an invariant Finsler metric on $Sp(n)/Sp(n-1)U(1)$. Denote the corresponding Minkowski norm on $\mathfrak{m}$ also as $F$. Then by the transitivity indicated above, the restriction of $F$ in $\mathbb{R}^2$ (resp. $\mathbb{H}^{n-1}$) must be a positive multiple of $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$ (resp. $\langle \cdot, \cdot \rangle_{\mathbb{H}^{n-1}}$). Therefore $F$ must be of the form $F = \sqrt{L(\alpha_1, \alpha_2)}$, where $L$ is a homogeneous positive smooth function of degree one satisfying some appropriate conditions, and $\alpha_1, \alpha_2$ are quadratic function on $\mathfrak{m}$ defined by

$$\alpha_1(X, Y) = \langle X, X \rangle_{\mathbb{R}^2}, \quad \alpha_2(X, Y) = \langle Y, Y \rangle_{\mathbb{H}^{n-1}}, \quad (X, Y) \in \mathfrak{m}.$$

The corresponding metric is exactly an $(\alpha_1, \alpha_2)$-metric introduced by the first two authors in [20]. By the continuity of the flag curvature on the Finsler metric, it is easily seen that if $F$ is close enough to the standard Riemannian metric, then $F$ has positive flag curvature. For example, consider a family of Minkowski norm $F_\varepsilon$ on $\mathfrak{m}$ defined by

$$F_\varepsilon(X, Y) = \sqrt{\alpha_1(X, Y) + \alpha_2(X, Y) + \varepsilon \sqrt{\alpha_1(X, Y)^2 + \alpha_2(X, Y)^2}}, \quad (X, Y) \in \mathfrak{m},$$

where $\varepsilon$ is a positive real number. Then the corresponding Finsler metric must have positive flag curvature when $\varepsilon$ is small enough.

(3) The pair $(G_2, SU(3))$. Although this is not a Riemannian symmetric pair, the isotropic representation is transitive on the unit sphere of the tangent space at the origin of $S^6 = G_2/SU(3)$ with respect to the standard Riemannian metric of positive constant sectional curvature. This implies that any $G_2$-invariant Finsler metric on $S^6$ must be a positive multiple of the standard Riemannian metric.

(4) The pair $(SU(3), T^2)$, where $T^2$ is a maximal torus of $SU(3)$. The isotropy representation can be described as follows. The Lie algebra $\mathfrak{su}(3)$ of $SU(3)$ is the compact real form of the complex simple Lie algebra $\mathfrak{g}^C = sl(3, \mathbb{C})$ and the Lie algebra $\mathfrak{t}$ of $T^2$ is a maximal commutative subalgebra of $\mathfrak{su}(3)$. Let $\Delta$ be the root system of $\mathfrak{g}^C$ with respect to $\mathfrak{t}^C$ and fix an order of the root system. Denote the corresponding set of positive roots by $\Delta^+$. Then the set of simple roots consists of two root, denoted
as $\beta_1, \beta_2$, and we have $\Delta^+ = \{\beta_1, \beta_2, \beta_1 + \beta_2\}$. The Lie algebra has a direct sum decomposition

$$su(3) = t + m_1 + m_2 + m_3,$$

(6.36)

where $m_1 = su(3) \cap g_{\pm \beta_1}$, $m_2 = su(3) \cap g_{\pm \beta_2}$, $m_3 = su(3) \cap g_{\pm (\beta_1 + \beta_2)}$, and for a root $\beta \in \Delta^+$ we denote $g_{\pm \beta} = g_{\beta}^+ + g_{\beta}^-$. It is easily seen that $m_i$, $i = 1, 2, 3$ are all invariant under the isotropic action of $T^2$ and they are irreducible representations of $T^2$. However, the action of $T^2$ on the unit sphere of an invariant inner product of these spaces is all nontransitive. This means that there exist $T^2$-invariant non-Euclidean Minkowski norms on $m_i$ (see [8, 9]). In particular, there exists a $T^2$-invariant non-Euclidean Minkowski norm $F$ on $m$ whose restriction to any $m_i$ is not Euclidean. The general form of a $T^2$-invariant inner product has the form

$$\langle \cdot, \cdot \rangle_{(t_1, t_2, t_3)} = t_1 \langle \cdot, \cdot \rangle_1 + t_2 \langle \cdot, \cdot \rangle_2 + t_3 \langle \cdot, \cdot \rangle_3,$$

where $\langle \cdot, \cdot \rangle_i$ is a $T^2$-invariant inner product on $m_i$ (which is unique up to a positive scalar by Schur’s Lemma). By Wallach [25], there exists a triple $(t_1, t_2, t_3)$ (hence infinitely many) such that the corresponding Riemannian metric induced by $\langle \cdot, \cdot \rangle_{(t_1, t_2, t_3)}$ has positive sectional curvature. Similarly as in Case (3), we consider the $T^2$-invariant Finsler non-Euclidean Minkowski norm $F_\varepsilon$ on $m$ defined by

$$F_\varepsilon(X, Y, Z) = \sqrt{\hat{t}_1 \langle X, X \rangle_1 + \hat{t}_2 \langle Y, Y \rangle_2 + \hat{t}_3 \langle Z, Z \rangle_3 + \varepsilon \sqrt{\langle X, X \rangle_1^2 + \langle Y, Y \rangle_2^2 + \langle Z, Z \rangle_3^2}},$$

(6.37)

where $(X, Y, Z) \in m$ and $\varepsilon$ is a positive real number. Then by the continuity, the corresponding SU(3)-invariant Finsler metric on SU(3)/$T^2$ has positive flag curvature when $\varepsilon$ is small enough. Note that the Finsler metric in (6.37) is an $(\alpha_1, \alpha_2, \alpha_3)$-metric as we mentioned in Section 2.1.

(5) The situation here is similar to that of the Case (4). The Lie algebra $g = sp(3)$ of the Lie group $Sp(3)$ has a reductive decomposition

$$g = \mathfrak{h} + m = (sp(1) + sp(1) + sp(1)) + (m_1 + m_2 + m_3),$$

where $m_i$ are invariant subspace of the isotropic representation of $H = Sp(1) \times Sp(1) \times Sp(1)$ and the representation of $H$ on these spaces are irreducible. However, the action of $H$ on the unit sphere of $m_i$ with respect to an $H$-invariant inner product (which is unique up to a positive scalar) is not transitive. Therefore there exist $H$-invariant Minkowski norms on $m$ whose restriction to any $m_i$ is non-Euclidean. Similarly as in the Case (4), we can give a complete description of invariant Riemannian metrics on $G/H$ and prove that there are Riemannian ones as well as non-Riemannian ones with positive flag (sectional) curvature.

(6) The pair $(F_4, Spin(8))$. The situation here is similar to Case (4) and Case (5). The Lie algebra $g = f_4$ of $F_4$ has a reductive decomposition

$$g = \mathfrak{h} + m = so(8) + (m_1 + m_2 + m_3),$$

where $m_i$ are invariant subspace of the isotropic representation of $H = Spin(8)$ and the representation of $H$ on these spaces are irreducible. However, the action of $H$ on
the unit sphere of $m_i$ with respect to an $H$-invariant inner product (which is unique up to a positive scalar) is not transitive. Therefore we can similarly prove that here are Riemannian metrics as well as non-Riemannian ones with positive flag (sectional) curvature on the corresponding coset space. We omit the details here.

The proof of Theorem 1.2 is now completed.

Finally, let us remark that the assumptions that $G$ is compact connected simply connected, $H$ is connected, and $\mathfrak{h}$ does not contain any nonzero ideal of $\mathfrak{g}$ does not prevent us from using Theorem 1.2 to describe general homogeneous Finsler spaces of even dimension admitting positive flag curvature. In fact, changing $G$ to its simply connected covering group, changing $H$ to its identity component, or cancelling the common product factor from both does not affect the local geometry of homogeneous Finsler metric in the consideration. Furthermore, assume that $\mathfrak{h}$ does not contain any nonzero ideal of $\mathfrak{g}$ and the $G$-homogeneous Finsler metric $F$ is of positive flag curvature. Then $G$ can not have a positive dimensional center. In fact, otherwise it will be generated by nonzero Killing vector fields of constant length on $(G/H, F)$. Since $G/H$ has positive flag curvature, it must be compact. Moreover, since it is even dimensional, any Killing vector field of $(G/H, F)$ must vanish somewhere (see [20]). This is a contradiction. Thus $G$ is semisimple and has a compact connected simply connected covering group.

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