Abstract: We show in the example of a one-dimensional asymmetric exclusion process that stationary states of models with parallel dynamics may be written in a matrix product form. The corresponding algebra is quadratic and involves three different matrices. Using this formalism we prove previous conjectures for the equal-time correlation functions of the model.
During the last few years the study of one-dimensional reaction-diffusion models has been of increasing interest. These models describe dynamical processes far away from thermal equilibrium so that in general their stationary probability distribution cannot be derived from an energy function. Therefore different techniques are needed in order to determine the stationary properties. An exact method which turned out to be very successful is the so-called matrix product formalism \cite{1}-\cite{7}. This formalism can be seen as a generalization of stationary states with a product measure in which products of numbers are replaced by products of non-commutative algebraic objects. By representing these objects in terms of matrices, the stationary state and all equal time correlation functions can be derived exactly. Up to now this technique has been applied mostly to systems with sequential dynamics (continuous time evolution) where the stationarity of the state is related to an additive cancelation mechanism from site to site. However, many systems, for example traffic models \cite{8}, are defined by parallel dynamics rather than sequential updates. Therefore it is of interest to find applications of the matrix product technique to systems with parallel dynamics. The present work discusses this problem in the example of a one-dimensional asymmetric exclusion process with parallel updates \cite{9}. A modified matrix product formalism is presented in which a new multiplicative cancelation mechanism plays an essential role. The corresponding matrix algebra is derived and finite-dimensional representations are given. This allows to prove exact results for the equal-time correlation functions which were already given as conjectures in Ref. \cite{9}.

Let us first recall the matrix product formalism for one-dimensional reaction-diffusion models with sequential dynamics and open boundary conditions. A two-state model with \( L \) sites is said to have a matrix product ground state if the stationary probability distribution

\[
P_0(\tau_1, \tau_2, \ldots, \tau_L) = \frac{Z^{-1}}{\langle W | (D + E)^L | V \rangle} \langle W | \prod_{j=1}^{L} (\tau_j D + (1 - \tau_j) E) | V \rangle
\]

(1)

where \( \tau_j \in \{0, 1\} \) is the occupation number at site \( j \). \( E \) and \( D \) are square matrices acting in an auxiliary space which may be either finite or infinite dimensional. The probabilities are the expectation values \( \langle W | \ldots | V \rangle \) of the matrix products normalized by the constant \( Z = \langle W | (D + E)^L | V \rangle \). Formally we may rewrite Eq. (1) as a tensor product

\[
|P_0\rangle = Z^{-1} \langle W | (E \otimes D)^L | V \rangle
\]

(2)

where the vector \( |P_0\rangle \) represents the stationary probability distribution in configuration space. The matrix product representation is a powerful tool since it allows various physical quantities like the particle density

\[
\langle \tau_j \rangle = \frac{\langle W | C^{j-1}DC^{L-j} | V \rangle}{\langle W | C^L | V \rangle} \quad (C = D + E)
\]

(3)

to be computed directly. Higher correlation functions are given by similar expressions in which \( C \) plays the role of a transfer matrix. However, a special mechanism is needed in order to ensure that the state in Eq. (2) is indeed a stationary one. For models with sequential dynamics this mechanism amounts in an additive cancelation from site to site: Assume that the time evolution of the system is described by a master equation \( \frac{d}{dt}|P\rangle = -H|P\rangle \) with a time evolution operator \( H = \sum_{j=1}^{L-1} h_{j,j+1} + h_{1}^{(L)} + h_{L}^{(R)} \), where \( h_{j,j+1} \) is a \( 4 \times 4 \) interaction matrix and \( h^{(L)} \) and \( h^{(R)} \) are \( 2 \times 2 \) matrices for particle input and output at the ends of the chain. Then the matrices \( E \) and
D have to be chosen such that the application of the interaction matrix \( h_{j,j+1} \) to a pair of sites results in a local divergence-like term on the right hand side

\[
  h \left[ \left( \frac{E}{D} \right) \otimes \left( \frac{E}{D} \right) \right] = \left( \frac{\hat{E}}{\hat{D}} \right) \otimes \left( \frac{E}{D} \right) - \left( \frac{E}{D} \right) \otimes \left( \frac{\hat{E}}{\hat{D}} \right), \tag{4}
\]

where \( \hat{E} \) and \( \hat{D} \) are again matrices in the auxiliary space. By summing up the two-particle interactions, all these contributions cancel in the bulk of the chain. The remaining terms at the boundaries have to be canceled by the surface fields for particle input and output:

\[
  < W | h^{(L)} \left( \frac{E}{D} \right) = - < W | \left( \frac{\hat{E}}{\hat{D}} \right), \quad h^{(R)} \left( \frac{E}{D} \right) | V > = \left( \frac{\hat{E}}{\hat{D}} \right) | V > . \tag{5}
\]

The simplest case \( \hat{E} = \hat{D} = 0 \) and its generalization to spin one problems has been considered in Ref. \cite{2}. Another system which has been investigated in detail is the (asymmetric) exclusion process where \( \hat{E} = -\hat{D} = 1 \) \cite{3}. In both cases one is led to a quadratic algebra of two objects \( E \) and \( D \) \cite{4}. In a similar way matrix product ground states were found for particular three-state models \cite{5}. Also excited states can be described with a matrix ansatz \cite{6} where \( \hat{E} = -\hat{D} \) has to be chosen as a time-dependent matrix leading to a quadratic algebra of three different objects. Taking \( \hat{E} \) and \( \hat{D} \) as independent matrices, it was also possible to find the stationary state of particular models with particle reactions \cite{7}.

So far, the interest has been focused mainly on stochastic models with continuous time evolution. However, similar techniques can be used for systems with parallel dynamics. A first example of this type was given in Ref. \cite{8} where the transfer matrix for a deterministic model of directed animals on a strip was investigated. It is the aim of the present work to point out that there could be a broad spectrum of applications to reaction diffusion models with parallel dynamics. For this purpose we consider a one-dimensional asymmetric exclusion process with parallel updates which was originally introduced by G. M. Schütz in Ref. \cite{9}. In this model particles move on a one-dimensional lattice with \( L = 2N \) sites and open boundaries. The bulk dynamics is deterministic and consists of two half time steps. In the first half time step particles at odd positions move one step to the right provided that the neighboured site to the right is empty. Then in the second half time step the particles at even positions move to the right in the same way. In addition particles are injected (removed) stochastically with rate \( \alpha \) (\( \beta \)) at the left (right) boundary:

\[
\begin{align*}
  \text{first half time step:} & \quad \alpha \\
  \text{second half time step:} & \quad \beta 
\end{align*}
\]

The corresponding transfer matrix therefore consists of two factors \( T = T_2 T_1 \)

\[
  T_1 = \mathcal{L} \otimes T \otimes \ldots \otimes T \otimes \mathcal{R} = \mathcal{L} \otimes T^{\otimes(N-1)} \otimes \mathcal{R} \\
  T_2 = T \otimes T \otimes \ldots \otimes T = T^{\otimes N} \tag{6}
\]

\[
\]
As in Eq. (2), we may also write the probability to find the system in the configuration \( \tau \) one has \( \rho \) written as a matrix product state with alternating pairs of matrices \( \alpha < \beta \). The phase diagram of this model shows two phases. For phase with an average particle density \( \alpha > \beta \) one has \( \rho = 1 - \beta/2 > 1/2 \). The total current in the thermodynamic limit is given by \( j = \min(\alpha, \beta) \). The physical behaviour is closely related to that of asymmetric exclusion models with continuous time evolution [3] (there is an additional phase with maximal density in the latter case). It plays a role in traffic models [8] as well as in polymer physics [10]. Related models with deterministic dynamics can be found in Ref. [11] and the influence of defects has been studied in Ref. [2].

As we are going to show below, the stationary state of the exclusion model (3)-(5) can be written as a matrix product state with alternating pairs of matrices \((E, D)\) and \((\hat{E}, \hat{D})\) so that the probability to find the system in the configuration \( (\tau_1, \tau_2, \ldots, \tau_{2N}) \) is given by

\[
P_0(\tau_1, \tau_2, \ldots, \tau_{2N}) = Z^{-1} \langle W | \prod_{i=1}^{N} \left[ \left( \tau_{2i-1} \hat{D} + (1 - \tau_{2i-1}) \hat{E} \right) \left( \tau_{2i} D + (1 - \tau_{2i}) E \right) \right] | V \rangle. \tag{8}
\]

As in Eq. (2), we may also write

\[
| P_0 \rangle = Z^{-1} \langle W | \left( \hat{E} \right) \otimes \left( E \right) \otimes \left( \hat{E} \right) \otimes \left( E \right) \otimes \ldots \otimes \left( \hat{E} \right) \otimes \left( E \right) | V \rangle \tag{9}
\]

where \( Z = \langle W | \left( (\hat{E} + \hat{D}) \otimes (E + D) \right)^{\otimes N} | V \rangle \). It is obvious that in this case the basic mechanism leading to a stationary state has to be different from the usual one for continuous time evolution operators. Instead of the additive cancelation from site to site we now need a multiplicative mechanism suitable for stationary states states of parallel transfer matrices \( T | P_0 \rangle = | P_0 \rangle \). In the case of the above exclusion model this mechanism turns out to be very simple. Let us assume that in each time step the two pairs of matrices \((E, D)\) and \((\hat{E}, \hat{D})\) are exchanged:

\[
T_1 | P_0 \rangle = \langle W | \left( \left( E \right) \otimes \left( \hat{E} \right) \right)^{\otimes N} | V \rangle, \quad T | P_0 \rangle = T_2 T_1 | P_0 \rangle = | P_0 \rangle \tag{10}
\]

This exchange mechanism can be realized by

\[
\mathcal{T} \left[ \left( E \right) \otimes \left( \hat{E} \right) \right] = \left( \hat{E} \right) \otimes \left( E \right), \tag{11}
\]

\[
\langle W | \mathcal{L} \left( \hat{E} \right) = \langle W | \left( E \right), \quad \mathcal{R} \left( E \right) | V \rangle = \left( \hat{E} \right) | V \rangle,
\]

which is equivalent to the algebra

\[
[E, \hat{E}] = [D, \hat{D}] = 0 \quad E \hat{D} = [\hat{E}, D] \quad \hat{D} E = 0 \tag{12}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 - \alpha & 0 \\
\alpha & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & \beta \\
0 & 1 - \beta
\end{pmatrix}. \tag{7}
\]
and the boundary conditions
\[
\langle W|\hat{E}(1-\alpha) = \langle W|E \quad (1-\beta)D|V\rangle = \hat{D}|V\rangle
\]
(13)

The commutation relations (12) involve four different matrices. However, only three of them are independent since the matrix product in Eq. (9) is invariant under the transformation
\[
E \rightarrow U^{-1}E, \quad D \rightarrow U^{-1}D, \quad \hat{E} \rightarrow \hat{E}U, \quad \hat{D} \rightarrow \hat{D}U.
\]
(14)

Because of \([E + D, \hat{E} + \hat{D}] = 0\) it is possible to choose a basis in which both operators \(E + D\) and \(\hat{E} + \hat{D}\) are diagonal. Taking now \(U = (E + D)^{1/2}(\hat{E} + \hat{D})^{-1/2}\) both operators become identical so that we may add the relation
\[
C = E + D = \hat{E} + \hat{D}.
\]
(15)

Eliminating \(\hat{E}\) and \(E\) we therefore obtain a quadratic algebra of three independent objects defined by three bulk equations
\[
\hat{D}C = \hat{D}D = D\hat{D}, \quad [D - \hat{D}, C] = 0
\]
(16)

and two boundary relations
\[
\langle W|\left(D - \alpha C - (1-\alpha)\hat{D}\right) = 0, \quad \left((1-\beta)D - \hat{D}\right)|V\rangle = 0.
\]
(17)

Matrix product states based on quadratic algebras with three objects were first studied in Ref. [6]. A detailed analysis of algebras with more than two objects and their representations will be given in Ref. [13].

In order to check the consistency of the algebra (15)-(16) let us show that the expectation value of any sequence of operators is given uniquely on an abstract level. For only two operators it can be verified by hand that
\[
\langle W|\hat{D}C|V\rangle = \langle W|\hat{D}D|V\rangle = \frac{\alpha^2(1-\beta)}{(\alpha^2 + \alpha\beta)(1-\beta) + \beta^2} \langle W|CC|V\rangle
\]
(18)
\[
\langle W|CD|V\rangle = \frac{\alpha^2(1-\beta) + \alpha\beta}{(\alpha^2 + \alpha\beta)(1-\beta) + \beta^2} \langle W|CC|V\rangle
\]
(19)

so that any expectation value of two operators is a given number times \(Z = \langle W|CC|V\rangle\). In order to check the consistency of the algebra for products of arbitrary length, it is more convenient to use a different basis of operators which is defined by the invertible transformation
\[
X = \frac{1}{\alpha\beta}\left(D - \alpha C + (\alpha - 1)\hat{D}\right)
\]
\[
Y = \frac{1}{\alpha\beta}\left((1-\beta)D - \hat{D}\right)
\]
\[
S = \frac{1}{\alpha\beta}(D - \hat{D}).
\]
(20)

In this basis, the bulk algebra (16) reads
\[
[X,S] = [Y,S] = 0, \quad YX = (1-\alpha)SY + (1-\beta)XS - (1-\alpha)(1-\beta)S^2
\]
(21)
and the boundary relations (17) become particularly simple:

\[ \langle W|X = 0, \quad Y|V = 0. \]  

(22)

As can be seen easily, the application of the bulk relations (21) allows every product of 2\(N\) matrices \(X, Y\) and \(S\) to be ordered as a linear combination of terms like \(X^m S^{2N-m} Y^n\).

Since the only nonzero expectation values of these terms is \(\langle W|S^{2N}|V \rangle\), the expectation value of any product of 2\(N\) matrices is a well-defined number times \(\langle W|S^{2N}|V \rangle\). The actual value of \(\langle W|S^{2N}|V \rangle\) is irrelevant since it is canceled by the normalization constant \(Z = \langle W|C^{2N}|V \rangle\).

Thus it is obvious that the algebra (21)-(22) determines the ground state \(|P_0\rangle\) uniquely on an abstract level. We should emphasize that the mathematical structure of this algebra is different from that for exclusion models with continuous time evolution where one has linear terms in the bulk algebra (e.g. \(DE = D + E\)). Whereas in the latter case any expectation value can be reduced to the empty bracket \(\langle W|V \rangle\), the algebra (16) does not allow to reduce the number of factors in a matrix product. Instead of this we have shown that by means of the algebraic rules the expectation values of all words with the same number of factors are linearly dependent.

The algebra (16)-(17) can be represented by two-dimensional matrices. For \(\alpha \neq \beta\) a representation in which \(C\) is diagonal is given by

\[ C_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad D_1 = \begin{pmatrix} \alpha & 0 \\ -\alpha\beta & \alpha\beta \end{pmatrix}, \quad \hat{D}_1 = \begin{pmatrix} \alpha(1 - \beta) & 0 \\ -\alpha\beta & 0 \end{pmatrix} \]  

(23)

\[ \langle W_1 | = (\alpha, 1 - \alpha), \quad |V_1 \rangle = \begin{pmatrix} 1 - \beta \\ -\beta \end{pmatrix} \]  

(24)

The normalization constant in this representation can be computed easily and reads

\[ Z_1 = (1 - \beta) \alpha^{2N+1} - (1 - \alpha) \beta^{2N+1}. \]  

(25)

As already mentioned, the matrix \(C\) acts like a transfer matrix between the points of the correlation functions. Therefore the length scales to be expected are essentially given by the quotients of the eigenvalues of \(C\). Thus in the present case the correlation functions involve only a single length scale, namely \(\log(\alpha/\beta)\). This length scale diverges at the phase transition line \(\alpha = \beta\) where the constant \(Z_1\) vanishes so that the above representation becomes singular. It turns out that in this case the operator \(C\) cannot be diagonalized so that one has to use a different representation where \(C\) has a Jordan normal form:

\[ C_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix}, \quad \hat{D}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 - \alpha \end{pmatrix} \]  

(26)

\[ \langle W_2 | = (1, 0), \quad |V_2 \rangle = \begin{pmatrix} 1 \\ 1 - \alpha \end{pmatrix} \]  

(27)

Because of

\[ C_2^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \]  

(28)

the normalization constant \(Z\) is now linear in the system size:

\[ Z_2 = 1 + 2N(1 - \alpha). \]  

(29)
Using the matrix product formalism it is now easy to derive explicit expressions for the equal-time correlation functions. Following the ideas of Ref. [9], we first compute the $n$-point functions of the operators

$$\eta_{2j} = \frac{\tau_{2j} - \alpha}{1 - \alpha} \quad \eta_{2j-1} = \frac{\tau_{2j-1}}{1 - \beta}$$

(30)

Denoting the corresponding matrices by

$$F_j = \begin{cases} (D - \alpha C)/(1 - \alpha) & \text{if } j \text{ even} \\ \hat{D}/(1 - \beta) & \text{if } j \text{ odd} \end{cases}$$

(31)

and assuming that the positions $j_1 \ldots j_n$ are are chosen in increasing order these correlation functions are given by

$$\langle \eta_{j_1} \eta_{j_2} \ldots \eta_{j_n} \rangle = \frac{1}{Z} \langle W| C^{j_1-1} F_{j_1} C^{j_2-j_1-1} F_{j_2} C^{j_3-j_2-1} \ldots C^{j_n-j_{n-1}-1} F_{j_n} C^{2N-j_n}|V \rangle.$$ 

(32)

Using the representations (23)-(27) it is easy to check that

$$F_j C^{k-j-1} F_k = F_j C^{k-j} \quad (j < k)$$

(33)

so that the $n$-point correlation functions reduce to the one-point function $\langle \eta_j \rangle$:

$$\langle \eta_{j_1} \eta_{j_2} \ldots \eta_{j_n} \rangle = \langle \eta_{j_1} \rangle = Z^{-1} \langle W| C^{j_1-1} F_{j_1} C^{2N-j_1}|V \rangle.$$ 

(34)

For $\alpha \neq \beta$ the one-point function reads

$$\langle \eta_{2j} \rangle = \frac{1}{Z_1} \alpha^{2N+1-2j} (1 - \beta) (\alpha^{2j} - \beta^{2j})$$

(35)

$$\langle \eta_{2j-1} \rangle = \frac{1}{Z_1} \alpha^{2N+2-2j} \left( \alpha^{2j-1}(1 - \beta) - \beta^{2j-1}(1 - \alpha) \right)$$

(36)

whereas at the transition line $\alpha = \beta$ we have

$$\langle \eta_{2j} \rangle = \frac{1}{Z_2} 2j (1 - \alpha)$$

(37)

$$\langle \eta_{2j-1} \rangle = \frac{1}{Z_2} \left( \alpha + (2j - 1)(1 - \alpha) \right).$$

(38)

Although we used the two-dimensional matrices at this point, Eqs. (34)-(37) do not depend on the choice of the representation since we have shown that the expectation value of any sequence of operators is uniquely given by the commutation relations of the algebra.

Resubstituting $\tau_j$ into Eq. (34) we obtain an exact expression for the $n$-point density correlation functions $\langle \tau_{j_1} \tau_{j_2} \ldots \tau_{j_n} \rangle$. Denoting $\sigma_j = j \mod 2$, they read

$$\langle \tau_{j_1} \tau_{j_2} \ldots \tau_{j_n} \rangle = \alpha^n \prod_{i=1}^n \sigma_i + \sum_{k=1}^n \left( \prod_{i=1}^{k-1} \sigma_i \right) \alpha^{k-1} (1 + \alpha \sigma_k - \beta - \beta \sigma_k) \langle \eta_{j_k} \rangle.$$ 

(39)

As a special case this formula includes the two-point correlation functions which have been given as conjectures in Ref. [9].

The example of the asymmetric exclusion model shows that the powerful matrix product formalism can be applied successfully to models with parallel dynamics. Since models of this type are widely studied, it would be interesting to find further examples in order to understand under which conditions the matrix product technique can be applied. In particular it would be interesting solve the same model on a ring in the presence of a defect. From this one could learn how to solve the full exclusion process (with stochastic hopping in both directions) on a ring with a defect [13]. Despite of intensive efforts the exact solution to this problem is not yet known.

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