Specializations of partial differential equations for Feynman integrals

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Abstract

Starting from the Mellin–Barnes integral representation of a Feynman integral depending on a set of kinematic variables $z_i$, we derive a system of partial differential equations w.r.t. new variables $x_j$, which parameterize the differentiable constraints $z_i = y_i(x_j)$. In our algorithm, the powers of propagators can be considered as arbitrary parameters. Our algorithm can also be used for the reduction of multiple hypergeometric sums to sums of lower dimension, finding special values and reduction equations of hypergeometric functions in a singular locus of continuous variables, or finding systems of partial differential equations for master integrals with arbitrary powers of propagators. As an illustration, we produce a differential equation of fourth order in one variable for the one-loop two-point Feynman diagram with two different masses and arbitrary propagator powers.

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1 Introduction

Within dimensional regularization, the general $L$-loop Feynman integral (FI) with $N$ internal momenta $q_i$ and masses $m_i$, and $E$ external momenta $p_i$ is given by

$$J\left(\{s\}, \{m\}, \{\alpha\}\right) = \int \frac{d^d k_1 \ldots d^d k_L}{(i\pi^{d/2})^L} \prod_{i=1}^N \frac{1}{(q_i^2 - m_i^2)^{\alpha_i}},$$

(1)

where $\{s\}$ is the full set of external Lorentz invariants constructed from the external momenta,

$$s_{ij} = p_i \cdot p_j, \quad i, j = 1, \ldots, E, \quad i \leq j,$$

(2)

and $\{m\}$, $\{\alpha\}$ are the sets of the $N$ masses $m_i$ and the $N$ indices $\alpha_i$, respectively. The internal momenta $q_i$ are linear combinations of the loop momenta $k_j$ and external momenta $p_k$. The indices $\alpha_i$ of the propagators are usually assumed to be integer numbers, both positive and negative. In Section 2, we will allow for $\alpha_i$ to be real numbers.

One of the most powerful approaches to the evaluation of FIs is based on the method of differential equations [1–5]. Using the standard method of integration by parts (IBP) [6], any FI (1) can be reduced to a set of so-called master integrals with fixed sets of indices $\alpha_i$. If we have $m$ master integrals, we can write a linear system of $m$ partial differential equations (PDEs) of first order. Alternatively, we can write a single linear differential equation of $m$-th order. In both cases, the $m$ solutions together with $m$ appropriate boundary conditions provide us with representations of all master integrals.

The conventional way to obtain the system of differential equations is to use IBP reduction. In fact, one can differentiate the master integrals w.r.t. some parameters (masses or external Lorentz invariants) and then reduce these derivatives down to the master integrals themselves. In order for this program to work, one has to provide for each integral an injective of $J$ into a special “complete” topology,\footnote{In the literature, also the terms “auxiliary” and “full” are used.} which is minimal and complete in the following sense. Any scalar product of any of the $L$ loop momenta with any other loop momentum or any of the $E$ external momenta is expressible as a linear combination of the propagators. Such a complete topology has exactly $L(L + 1)/2 + LE$ propagators, which are linearly independent. Then, the particular FIs (1), usually called “sector integrals”, appear as special cases of the complete topology, where some of the propagator indices become non-positive.

In this paper, we pursue the idea that the system of PDEs can be derived from Mellin–Barnes representations without resorting to IBP relations. In this case, we also do not need to construct the complete topologies, but can obtain the differential equations directly in a given sector. Moreover, usually all indices $\alpha_i$ in Eq. (1) are taken to be integers, including negative values, which account for possible numerators of the FIs. In this paper, however, we relax this restriction and consider $\alpha_i$ as real numbers.

Applying the methods described, e.g., in Refs. [7, 8], the expression in Eq. (1) can be
written as a multiple Mellin–Barnes representation in the form

\[ J(\{s\}, \{m\}, \{\alpha\}) = C \int_{-i\infty}^{+i\infty} \prod_{j,l} du_l \frac{\Gamma(\sum_i a_{ij} u_i + b_j)}{\Gamma(\sum_i c_{ij} u_i + d_j)} z_j^{\sum_k f_{kl} u_k}, \]  

(3)

where \( C \) is some constant, which depends on the propagator indices \( \alpha_i \) and the space-time dimension \( d \) [9], and \( z_j \) are ratios of the external kinematic invariants in Eq. (2) and the masses \( m_i \). All other parameters, \( a_{ij}, b_i, c_{ij}, d_i \), and \( f_{kl} \), are linear combinations of the space-time dimension \( d \) and the propagator indices \( \alpha_i \). As usual, the integration is performed over the Mellin–Barnes parameters \( u_k \) along the contours that separate the left and right poles in the complex \( u_k \) planes.

The representation (3) is our starting point. In Refs. [10, 11], it was already noted that, using Eq. (3), we can obtain a system of PDEs in the variables \( z_j \). However, this approach is very restrictive in its practical applications. In fact, it requires keeping all \( z_j \) as independent variables, which is usually not the case in real applications. Typically, we have situations where there are either some relations between these variables or some of them are fixed numbers and not subject to differentiation (for example, in the case of a single-scale diagram, or in the case where all masses are equal).

In this paper, we show how, starting from representation (3), we can obtain systems of PDEs with some constraints, \( f_k(z_j) = 0 \) with \( k = 1, \ldots, h \), which will allow us to consider cases of practical interest.

## 2 Algorithm

In this section, we describe in detail our algorithm for obtaining systems of PDEs from the Mellin–Barnes representations (3).

First, taking the residues of the \( \Gamma(-u_l) \) functions at negative integer points, the Mellin–Barnes integral (3) can be written as a linear combination of Horn-type hypergeometric series [12],

\[ H(\vec{a}, \vec{b}, \vec{c}, \vec{d}; \vec{z}) = \sum_i \prod_j C_j \frac{\Gamma(\sum_i a_{ij} l_i + b_j)}{\Gamma(\sum_i c_{ij} l_i + d_j)} \frac{z_j^{l_i}}{l_i!} \frac{z_j^{l_n}}{l_n!}, \]  

(4)

where \( C_j \) are some constants and we use the notations \( \vec{a} = (a_{ij}), \vec{c} = (c_{ij}), \vec{b} = (b_j), \vec{d} = (d_j) \), etc.

Shifting the integration contours in the Mellin–Barnes representation (3) or, equivalently, shifting the summation indices in the hypergeometric representation (4), we obtain differential contiguous relations. These can be expressed in terms of step-up operators \( L_{b_j}^+ \) and step-down operators \( L_{d_j}^- \) [13–17], which shift the indices \( b_j, d_j \) by one unit. Specifically, we
have

\[
H(a, b + \vec{e}_j, c, \vec{d}; \vec{z}) = L^+_b H(a, \vec{b}, c, \vec{d}; \vec{z}) \propto \left( \sum_i a_{ij} \theta_i + b_j \right) H(a, b, c, \vec{d}; \vec{z}),
\]

\[
H(a, \vec{b}, c, \vec{d} - \vec{e}_j; \vec{z}) = L^-_{d_j} H(a, \vec{b}, c, \vec{d}; \vec{z}) \propto \left( \sum_i c_{ij} \theta_i + d_j - 1 \right) H(a, \vec{b}, c, \vec{d}; \vec{z}),
\]

where \( \theta_i = z_i d/dz_i \) is the Euler differential operator and \( \vec{e}_j \) are orthonormal unit basis vectors, with \( \vec{e}_j \cdot \vec{e}_k = \delta_{jk} \).

From the differential contiguous relations (5), a dynamical symmetry algebra may be constructed. From this Lie algebra, a system of PDEs that is satisfied by the function in Eq. (4) may be constructed [18]. More precisely, combining the operators \( L^+_b \) and \( L^-_{d_j} \) from Eq. (5) and differentiating Eq. (4) w.r.t. variables \( z_j \), we can derive a system of \( n \) PDEs:

\[
\begin{pmatrix}
\prod_{j \in m^+_k, n_j \in \{0, a_{kj}\}} L^+_{b_j + n_j} - \frac{1}{z_k} \prod_{j \in m^-_k, n_j \in \{0, c_{kj} - 1\}} L^-_{d_j - n_j}
\end{pmatrix} H(a, \vec{b}, c, \vec{d}; \vec{z}) = 0,
\]

where set \( m^+_k \) consists of the integers \( j \) for which \( a_{kj} \neq 0 \) and set \( m^-_k \) consists of the integers \( j \) for which \( c_{kj} \neq 0 \). We imply here that variables \( a_{ij} \) and \( c_{ij} \) are natural numbers.

This system of PDEs may be derived directly from the Mellin–Barnes representation (3) [11]. In some cases, however, to obtain the full system of PDEs, a prolongation procedure has to be applied, which consists of applying additional derivatives to the system of PDEs to find one or more new nontrivial equations.

In the system of PDEs (6), it is implied that the variables \( z_j \) are independent, i.e. all masses and external Lorentz invariants are different and not equal to zero, and that the propagator indices are real numbers.

As for multivariate specializations of the PDE system (6), we have to consider several different cases [19]. In the first case, the multivariate specialization falls into a singular locus of the PDE system (6). Then, the rank of the new PDE system will be lower than the initial one, for any combination of the parameters \( a, \vec{b}, \vec{c}, \vec{d} \).

The singular loci of the new PDEs are inherited from the old ones with the old variables \( z_i \) and induced locus of multivariate specialization. For some particular combinations of parameters and variables, the loci of the new PDEs could be diminished.

All this may be directly inferred from the PDE system (6), and the final differential equation(s) is/are satisfied by some hypergeometric functions of lower (simpler) class.

For any other multivariate specialization, the rank of the new PDE system is the same as that of the initial one. Nonetheless, we may observe simplifications in the class of functions that satisfy the new PDE system after the application of projective or more general pull-back transformations of variables.

Finally, in the case when the monodromy group of the PDE system is reduced, which manifests itself in a factorization of the PDEs in Eq. (6) or in new differential equation(s) after multivariate specialization, some of the solutions reduce to rational ones, and the remaining solutions may be expressed through hypergeometric functions of lower order.
As already mentioned above, the variables $z_j$ with $j = 1, \ldots, n$ are all independent in the PDE system (6) and vary in some differential manifold in $\mathbb{C}^n$. Now suppose that we impose $r$ differentiable constraints and parametrize the new manifold in terms of new independent variables $x_i$ with $i = 1, \ldots, k$, that is

$$z_j = y_j(\vec{x}), \quad j = 1, \ldots, n$$
$$\vec{x} = (x_1, \ldots, x_k), \quad k < n.$$  \hspace{1cm} (7)

Our goal is to derive from the system (6) of PDEs w.r.t. the variables $z_1, \ldots, z_n$ a new system of PDEs w.r.t. the new variables $x_i, \ldots, x_k$. In the following, we omit for brevity the indexed arguments of $H$ in Eq. (4) and use the shorthand notations $H(\vec{z}) = H(\vec{a}, \vec{b}, \vec{c}, \vec{d}; \vec{z})$ and $H(\vec{x}) = H(\vec{a}, \vec{b}, \vec{c}, \vec{d}; \vec{z}(\vec{x}))$. First, we note that

$$\frac{dH(\vec{x})}{dx_j} = \sum_{i=1}^{n} \frac{\partial H(\vec{z})}{\partial z_i} \frac{\partial y_i}{\partial x_j}. \hspace{1cm} (8)$$

The rank of the PDE system w.r.t. the new variables $\vec{x}$ must be the same as that of the initial PDE system (6), if the new variables $y_j(\vec{x})$ do not fall into the singular locus of the PDE system. If this is not the case, then the total rank of the new PDE system is lower, and we do not consider such degenerate cases here for simplicity. Furthermore, the order $\chi$ of derivatives w.r.t. the new variables must be higher than or equal to the order $\eta$ in Eq. (6). We do not specify $\chi$ at this point, but consider it as an unknown parameter $\chi > \eta$.

By applying the chain rule (8) $\chi$ times, we construct a PDE system in which various derivatives of order less than or equal to $\chi$ w.r.t. new variables $\vec{x}$ are expressed in terms of derivatives w.r.t. old variables $\vec{z}$ and derivatives of $y_j(\vec{x})$ functions. We solve this system and express some of the high-order derivatives w.r.t. old variables $\vec{z}$ through a mixture of derivatives w.r.t. new $\vec{x}$ variables, old $\vec{z}$ variables, and derivatives of known $y_j(\vec{x})$ functions,

$$\left. \frac{\partial^i H(\vec{z})}{\partial z_{j_1} \ldots \partial z_{j_i}} \right|_{i \leq \chi} = \sum_{k<i} A_{j_1 \ldots j_k} \frac{\partial^k H(\vec{z})}{\partial z_{j_1} \ldots \partial z_{j_k}} + \sum_{l \leq \chi} B_{j_1 \ldots j_l} \frac{\partial^l H(\vec{x})}{\partial x_{j_1} \ldots \partial x_{j_l}}. \hspace{1cm} (9)$$

We now proceed with the derivation of Eq. (6) w.r.t. variables $\vec{z}$ through the combined order $\chi - \eta$. Substituting derivatives w.r.t. old variables using Eq. (9), we construct the
matrix $M$ of the PDE system, in which old and new variables are mixed:

\[
\begin{bmatrix}
M_1 & * \\
B_1 & M_2
\end{bmatrix}
\]

If the rank of $M_1$ is less than that of $M$, then there exists a system of independent equations that involves only derivatives w.r.t. new variables $\vec{x}$. If we now perform the row echelon reduction on $M$, the bottom-left block $B_1$ becomes zero. Then, $M_2$ gives us an explicit form of the PDEs w.r.t. new variables $\vec{x}$:

\[
M_2 \begin{bmatrix}
\frac{\partial^i H(\vec{z})}{\partial x_{j_1} \ldots \partial x_{j_l}} \\
\vdots \\
H(\vec{x})
\end{bmatrix} = 0.
\] (10)

However, if the rank of $M$ equals that of $M_1$, then we have to increase the parameter $\chi$, i.e., the number of derivatives w.r.t. new variables $\vec{x}$, and repeat the above procedure.

Let us now consider a special case, in which the constraints have the following form:

\[
z_1 = y(\vec{x}) = x_1, \quad z_2 = y(\vec{x}) = \text{const}_2, \quad \ldots \quad z_n = y_n(\vec{x}) = \text{const}_n,
\] (12)

i.e., we treat all variables $z_j$, except for the first one $z_1$, as constants. Then the maximum order $\chi$ of derivatives w.r.t. $x_1$ must be equal to the differential rank of PDE system (6). Then the chain rule (8) is trivial, and the PDE system (9) contains $\chi$ different equations,

\[
\left. \frac{\partial^i H(\vec{z})}{\partial z_{j_1} \ldots \partial z_{j_i}} \right|_{i \leq \chi} = \sum_{k<i} A_{j_1 \ldots j_k} \frac{\partial^k H(\vec{z})}{\partial z_{j_1} \ldots \partial z_{j_k}} + \sum_{l \leq \chi} B_{j_1 \ldots j_l} \frac{\partial^l H(\vec{x})}{\partial x_1}.
\] (13)

Applying $\chi - \eta$ differentiations w.r.t. variables $\vec{z}$ to Eq. (6) and substituting $\chi$ derivatives from Eq. (13), we construct the matrix $M$ of PDEs in Eq. (10). In the upper triangular form
of the $M$ matrix, $B_1$ is zero and $M_2 = (Q_1, ..., Q_\chi)^T$ is just a vector. Thus, we arrive at an 
ordinary differential equation of order $\chi$ w.r.t. the single variable $x_1$,

$$
\sum_{i=0}^{\chi} Q_{\chi-i} \frac{\partial^i H(x_1)}{\partial x_1^i} = 0.
$$

3 Example

We now illustrate our algorithm by means of a simple example. Specifically, we derive 
differential equation in one variable for the one-loop two-point Feynman diagram with 
different masses and arbitrary powers of propagators,

$$
J(\alpha_1, \alpha_2, m_1, m_2) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2 - m_1^2)^{\alpha_1} [(k - p)^2 - m_2^2]^{\alpha_2}}.
$$

We can rewrite each of the propagators as [7]

$$
\frac{1}{(k^2 - m^2)^\beta} = \frac{1}{(k^2)^\beta} \frac{1}{\Gamma(\beta)} \int_{-\infty}^{+\infty} ds \left( -\frac{m^2}{k^2} \right)^s \Gamma(-s) \Gamma(\beta + s).
$$

Integrating over the massless one-loop propagator, we then obtain a two-fold Mellin–Barnes 
representation of Eq. (15),

$$
J(\alpha_1, \alpha_2, m_1, m_2) = \pi^{d/2} i^{1-d} (p^2)^{d/2-\alpha-\beta} \frac{1}{(2\pi i)^2} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int ds du \left( \frac{p^2}{m_1^2} \right)^s \left( \frac{p^2}{-m_2^2} \right)^u

\times \frac{\Gamma(d/2 - \alpha + s)\Gamma(d/2 - \beta + u)\Gamma(\alpha + \beta - d/2 - s - u)\Gamma(s)\Gamma(u)}{\Gamma(\alpha - s)\Gamma(\beta - u)\Gamma(d - \alpha - \beta + s + u)}.
$$

By constructing step-up and step-down operators according to Eq. (5) and combining 
them with the differentiations w.r.t. $z_1 = p^2/m_1^2$, $z_2 = p^2/m_2^2$, we obtain the following system 
of PDEs of second order in two variables for $J(\alpha, \beta, m_1, m_2)$ [11]:

$$
\theta_1 \left( -\alpha_1 + \frac{d}{2} + \theta_1 \right) \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \theta_1} \frac{2\alpha_1 + 2\alpha_2 - d - 2\theta_1 - 2\theta_2}{2z_1} (\alpha_1 + \alpha_2 - d - \theta_1 - \theta_2 + 1) = 0,
$$

$$
\theta_2 \left( -\alpha_2 + \frac{d}{2} + \theta_2 \right) \frac{\partial}{\partial z_2} + \frac{\partial}{\partial \theta_2} \frac{2\alpha_1 + 2\alpha_2 - d - 2\theta_1 - 2\theta_2}{2z_2} (\alpha_1 + \alpha_2 - d - \theta_1 - \theta_2 + 1) = 0.
$$

Let us now construct an ordinary differential equation w.r.t. the variable $z_1 = x$. To this 
end, we impose the constraints $z_1 = y_1(x) = x$ and $z_2 = y_2(x) = \text{const}_2$. Notice that the 
PDE system (18) is equivalent to the PDE system

$$
\frac{1}{z_1} \theta_1 (c_1 - 1 + \theta_1) - (a + \theta_1 + \theta_2)(b + \theta_1 + \theta_2) = 0,
$$

$$
\frac{1}{z_2} \theta_2 (c_2 - 1 + \theta_2) - (a + \theta_1 + \theta_2)(b + \theta_1 + \theta_2) = 0
$$

(19)
for the Appell hypergeometric function $F_4(a, b, c_1, c_2, z_1, z_2)$ [17], with $a = \alpha_1 + \alpha_2 - d/2$, $b = 1 + \alpha_1 + \alpha_2 - d$, $c_1 = 1 + \alpha_1 - d/2$, and has four different solutions. The singular locus is $z_1 = 0, z_2 = 0$, the line at infinity, and $z_1^2 + z_2^2 + 1 = 2z_1z_2 + 2z_1 + 2z_2$, and, as the variables $y_1(x), y_2(x)$ do not belong to the locus, the rank of the new PDE system is the same as that of the old one. Thus, we choose the number of derivatives w.r.t. $x$ to be $\chi = 4$. In this case, we need $\chi - \eta = 2$ differentiations of the PDE system (18) w.r.t. variables $z_1, z_2$. The algorithm of Section 2 produces the following differential equation of fourth order in $x$:

$$L_4(x)J(\alpha_1, \alpha_2, m_1, m_2) = 0, \quad (20)$$

where $L_4(x)$ is a differential operator of fourth order, whose expression is too lengthy to be listed here.

As for Eq. (19), the region of the exceptional set of parameters, when the monodromy is reduced, is defined by $\{a, b, c_1 - a, c_1 - b, c_2 - a, c_2 - b, c_1 + c_2 - a, c_1 + c_2 - b\} \subset Z$ [14], and, in the case of Eq. (18), we have $-b + c_1 + c_2 = 1$, so that one solution of the PDE system (19) degenerates to the Puiseux type, and the one-variable differential equation for $F_4, \text{Eq. (20)}$, must factorize in such a way that a first-order differential operator splits off.

$$L_1(x)L_3(x)J(\alpha_1, \alpha_2, m_1, m_2) = 0. \quad (21)$$

In terms of hypergeometric functions, the answer may be written as four independent solutions: three $F_4$ functions with various arguments and one polynomial. By defining suitable constants, we may find that the final answer for the one-loop two-point FI with two different masses and arbitrary powers of propagators has only two $F_4$ terms, in the variables $z_1 = p^2/m_2^2, z_2 = m_1^2/p^2$, and three terms in the variables $z_1 = m_1^2/p^2$, $z_2 = m_2^2/p^2$ [7].

Let us now consider the case when the continuous variables $\vec{z}$ take the same values, $z_1 = z_2$. In this case, $J$ is equivalent to the FI (15) with equal masses $m_1 = m_2$, and we have $z_1 = y_1(x) = x$ and $z_2 = y_2(x) = x$. As this univariate specialization does not belong to the singular locus of Eqs. (18) or (19), the rank of the new PDE system is the same as that of the original one. Indeed, for the PDE system related to the Appell hypergeometric function $F_4$, we obtain the following ordinary differential equation of fourth order in one variable:

$$\tilde{L}_4(x)F_4(x, x) = 0, \quad (22)$$

which has three distinct poles at points $0, 1/4, \infty$. Comparing the singular points and local exponents with the differential equation for the hypergeometric function $4F_3$, we recover the well-known result for the univariate specialization of $F_4$ [20]:

$$F_4 \left( \frac{a, b}{c_1, c_2} \bigg| x, x \right) = 4F_3 \left( \frac{a, b, c_1 + c_2, c_1 + c_2 - 1}{c_1, c_2, c_1 + c_2 - 1} \bigg| 4x \right). \quad (23)$$

Above, we found that the monodromy group of the initial PDE system for the considered FI in Eq. (18) is reduced due to the constraint $-b + c_1 + c_2 = 1$ on its parameters. We thus
find the factorization of $\tilde{L}_4(x)$ by substituting the parameters $a, b, c_1, c_2$ from Eq. (18):

$$L_1(x)L_3(x)J(\alpha_1, \alpha_2, m, m) = 0,$$

$$L_1(x) = \frac{d}{dx} + \frac{(x-4)(-\alpha_1 - \alpha_2 + d) + 3x - 8}{x(x-4)},$$

$$L_3(x) = \frac{d^3}{dx^3} + \frac{-(x-8)(\alpha_1 + \alpha_2 - d - 3) + 2d + 18}{(x-4)x} \frac{d^2}{dx^2} - \frac{4((\alpha_1 + \alpha_2)(5(\alpha_1 + \alpha_2) - 8d + 1) + 3d^2) + x(2\alpha_1 - d - 2)(-2\alpha_2 + d + 2)}{4(x-4)x^2} \frac{d}{dx} + \frac{(\alpha_1 + \alpha_2 - d + 1)(\alpha_1 + \alpha_2 - d + 2)(2(\alpha_1 + \alpha_2) - d)}{2(x-4)x^3}.$$

As a consequence, the final answer for the one-loop two-point FI with equal masses can be expressed through the hypergeometric function $\text{}_3F_2$ and a polynomial expression, which may be found in Eqs. (17) and (18) of Ref. [7].

# 4 Conclusions

In this work, we proposed a systematic method for deriving a system of PDEs for a FI whose initial set of Lorentz invariants and masses may be arbitrarily constrained, down to one or more free parameters. This method does not rely on IBP relations and is applicable also for non-integer propagator indices. It proceeds in two steps. In the first step, we treat all external momenta and masses as independent and derive a prototype system of PDEs from the Mellin–Barnes representation of the FI. In the second step, we implement the constraints among the external momenta and masses through a multivariate specialization and construct a new system of PDEs for the particular FI. This method also enables one to conveniently determine, during the second step, the rank of the final PDE system, the number of its rational solutions, and the simplest class of special functions through which the particular FI may be expressed.

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