**Introduction**

It is known from the literature that the Ising models in one-dimension (1D) [1] there is no phase transition at finite temperature $T$, while two-dimension (2D) presents a continuous phase transition. Pekalski [2] studied the Ising model on a small-world network (SW) constructed from a ring lattice. In the presence of a finite fraction of additional long-range interactions, the model presents a phase transition at finite temperature. However, the exponent $\beta=0$ was found to be very small, $\beta=0.00001$, in contrast to the previous studies suggesting the mean-field nature of the transition [3,4]. Hong et al. [5] presented results of extensive Monte Carlo simulations of the same model as that in Pekalski [1], which reveal that the phase transition is described by the mean-field exponents $a=0$, $b=0.5$, $g=1$, and $\nu=2$.

In this paper we consider the Ising model in 1D (or ring lattice) directed small-world networks (DSW) and perform an extensive computer simulation study of the Ising model. We applied finite-size scaling (FSS) techniques to extract the exponents and the "renormalized charges" $U^a_i$. Monte Carlo simulations of this system were performed using the spin-flip heat bath algorithm to update the spins. Here, there is a second-order phase transition for $0 < p < 1$, where $p$ is the rewiring probability. In addition, the calculated critical exponents for $p$ do not belong to the same universality class as the regular two-dimensional ferromagnetic model.

**Small-World Networks**

In 1967, Jeffrey and Stanley Milgram conducted an experiment with the objective of having Omaha residents send letters to the residents of the city of Boston, either directly or through acquaintances. Initially they believed that the cards would reach their destination by approximately 100 people. The results showed that of the 160 letters sent, 42 arrived at the destination and that these letters passed on average by five to six intermediate persons [4]. Like this experiment they showed that people would be living in a small world.

In 2003 Ducan JW & Steven [4] proposed a small-world network model with the goal of forming few connections between each individual. In this model the population is distributed in regular one-dimensional networks in which each individual contact the other individuals in the network through re-connections made from the value of the probability $p$.

The small-world network in one-dimension is built from a regular network with two closest neighbors, connected to $N$ nodes and $C$ neighbors. In this network each node is randomly reconnected with $n$ edges with probability $p$. When $p=0$ for the network it is regular (received no long-range connection), but for $0 < p < 1$ the network is small world (short-range links) and $p=1$ random network (long-range connections), as shown in Figure 1.

**Model and Simulations**

The time evolution of the system is given by a single spin-flip like dynamics [5] with a probability described by

$$ P_i = 1 \left[ 1 + \exp \left( -2E_i / K_B T \right) \right] $$

(1)

where $T$ is the temperature, $K_B$ is the Boltzmann constant, and $E_i$ is the energy of the $S_i$ obtained of

$$ E_i = -J \sum_{j \neq i} S_j S_i $$

(2)

where the sum runs over all neighbours $k$ of $i$ with $k=3$ on SNs and the variables $S_i$ assume values. In the above equation $J$ is...
the exchange coupling, comprising a number \( L = 1000, 2000, 4000, 8000, \text{and} 16000 \) of sites. For each system size quenched averages over the connectivity disorder are approximated by averaging over independent realizations. For each simulation we have started with a uniform configuration of spins. We ran Monte Carlo steps (MCS) per spin with \( 2 \times 10^6 \) configurations discarded for thermalization using the "perfect" random-number generator [6]. We do not see any significant change by increasing the number of replicas (R) (for example R=50) and MCS. So, we keep these values constant once they seem to give reasonable results for all simulations.

Results and Discussions

![Figure 1: The three networks with the same numbers of nodes, the same number of links and the same placement of nodes. (a). Regular network (ring-lattice), (b) Small-world network (the Watts-Strogatz Model), and (c) Random network.](image)

![Figure 2: Plot of the logarithm of the magnetization (a) and susceptibility \( X \) (b) at \( T_c \) as a function of the logarithm of \( L \). Plot of (c) and ln \( x \) r... (d) as a function of the logarithm of \( L \).](image)

The magnetization per spin, \( m = \sum S_i / N \), were measured. We can derive from the magnetization measurements the average magnetization, the susceptibility, and the fourth-order magnetic cumulant,
\[ m = \left[ \langle m \rangle \right]_{av}, \]  
(3)

\[ \bar{\langle T \rangle} = \frac{1}{N} \left[ m^2 \langle m \rangle - \langle m^2 \rangle \right]_{av}, \]  
(4)

\[ U_s(T) = \frac{1}{3} \left( \frac{m^4}{\langle m \rangle} \right)_{av}, \]  
(5)

In the above equations \( \langle \ldots \rangle \) stands for thermodynamic averages and \( \ldots \) \( \langle \ldots \rangle \rangle \) for averages over different realizations.

In order to calculate the exponents of these models, we apply finite-size scaling (FSS) theory. We then expect, for large system sizes, an asymptotic FSS behavior of the form

\[ m = \bar{a} L^{\gamma}, \]  
(6)

\[ \bar{\langle T \rangle} = \bar{a} L^{\gamma}, \]  
(7)

where \( \bar{a} \) and \( \bar{\langle T \rangle} \) are the usual critical exponents, and \( f_i(x) \) are FSS functions with

\[ x = (T - T_c) L^{-\gamma}, \]  
(8)

being the scaling variable. The dots in the brackets \([1+\ldots]\) indicate corrections-to-scaling terms. We calculated the error bars from the fluctuations among the different realizations. Therefore, from the size dependence of \( m \) and \( \bar{\langle T \rangle} \), we obtain the exponents ratios \( \bar{a} / \bar{\langle T \rangle} \) and \( \bar{\langle T \rangle} / \bar{a} \), respectively. The susceptibility at its maximum also scales as \( L^{\gamma} \). Moreover, the value of \( T = T_c(L) \) for which \( \bar{\langle T \rangle} = \bar{a} L^{\gamma} \) has a maximum scales with the lattice size as \( T_c(L) = T_c + b L^{-\gamma} \). In this way, Eq. 9 may be used to get \( 1 / \gamma \) \( [7, 8] \).

In the Figure 2b, we plot logarithm of the susceptibility \( \chi \) at \( T_c \) versus \( \ln L \). Of the eq. (7), we obtain the exponents ratio \( \gamma / \bar{\langle T \rangle} = 0.65(1), 0.56(2) \), and 0.56(3) for \( p = 0.1, 0.5 \) and \( 0.9 \), respectively.

In the Figure 2c, we plot the log-log of \( \left[ T_c(L) - T_c \right] \) versus \( L \) and the eq. (8), we obtain the exponents ratio \( 1 / \nu = 0.52(3), 0.53(2) \) and 0.53(3) for \( p = 0.1, 0.5 \) and \( 0.9 \), respectively.

The Figure 2d display logarithm of the susceptibility \( \chi \) at \( T_{max} \) versus logarithm of \( L \). The values of the exponents ratio are \( \bar{a} / \bar{T}_{max} = 0.65(1), 0.54(2) \) and 0.54(3) again for \( p = 0.1, 0.5 \) and \( 0.9 \), respectively.

**Conclusion**

In the present work, we have shown that, by considering the ferromagnetic Ising model in one-dimension on DSW networks there is a continuous phase transition. Our results are different from Ising model in one-dimension on a chain that no presents phase transition. We believe this critical behavior is due to the short and long links present in this DSW networks.

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