Time-optimal CNOT between indirectly coupled qubits in a linear Ising chain*

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Abstract

We give analytical solutions for the time-optimal synthesis of entangling gates between indirectly coupled qubits 1 and 3 in a linear spin chain of three qubits subject to an Ising Hamiltonian interaction with a symmetric coupling \(J\) plus a local magnetic field acting on the intermediate qubit. The energy available is fixed, but we relax the standard assumption of instantaneous unitary operations acting on single qubits. The time required for performing an entangling gate which is equivalent, modulo local unitary operations, to the \(\text{CNOT}(1, 3)\) between the indirectly coupled qubits 1 and 3 is \(T = \sqrt{3/2}J^{-1}\), i.e. faster than a previous estimate based on a similar Hamiltonian and the assumption of local unitaries with zero time cost. Furthermore, performing a simple Walsh–Hadamard rotation in the Hilbert space of qubit 3 shows that the time-optimal synthesis of the \(\text{CNOT}^{+}(1, 3)\) (which acts as the identity when the control qubit 1 is in the state \(|0\rangle\), while if the control qubit is in the state \(|1\rangle\), the target qubit 3 is flipped as \(|\pm\rangle \rightarrow |\mp\rangle\)) also requires the same time \(T\).

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1. Introduction

Quantum optimal control theory is now a very well-studied subject, both theoretically and experimentally, with increasing applications in the field of quantum computing and information (for an updated review see, e.g., [1] and references therein). Quantum optimal control techniques aim, e.g., at finding either the best quantum state evolution or the best unitary

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operator evolution with respect to some fixed cost, which can be assumed to be a fidelity with respect to a target (a quantum state or a gate), the purity of the target state, etc. Time-optimal quantum computation, where the cost to be optimized is the time to achieve a given quantum evolution, has also recently become a hot topic in quantum control and quantum information theory [2–22]. For example, minimization of the physical time to achieve a given unitary transformation is relevant to the design of fast elementary gates and provides a more physical ground to describe the complexity of quantum algorithms than the standard concept of gate complexity, which gives the number of elementary gates used in a quantum circuit [23]. Also, the literature in the time-optimal quantum control is rapidly growing and it is not the purpose of this paper to give a complete review. For instance, [2–6] discuss the time optimal generation of unitary operations for a small number of qubits using Lie group methods, the theory of sub-Riemannian geometry, the Pontryagin maximum principle and assuming that 1-qubit operations can be performed arbitrarily fast. The time-optimal synthesis of unitary transformations (quantum gates) between two coupled qubits has been thoroughly discussed [7–9]. The time-optimal evolution of quantum states for qubits and qudits has also been investigated, e.g., in [10, 11], that of a two-level dissipative system, e.g., in [12], while the time-optimal generation of cluster states has been considered by [13]. The extension of the Lie algebraic methods to the coupling of slow and fast systems can be found in [14]. Earlier bounds on the time complexity of generating 2-qubit unitary gates can be found in [15], while lower bounds on the time complexity of n-qubit gates are given in [16] and upper bounds on the time complexity of certain n-qubit gates via several coupling topologies are numerically described in [17]. The relationship between time complexity and gate complexity has also been investigated in [18]. Nielsen et al [19–21] proposed a criterion for optimal quantum computation in terms of a certain geometry in Hamiltonian space, and showed that the quantum gate complexity is related to optimal control cost problems. An adiabatic solution to the optimal control problem in holonomic quantum computation has also been found in [22]. Finally, numerical methods for the design of optimal quantum control evolutions have been proposed, based on the gradient ascent algorithm [24] and on the nonlinear Krotov algorithm [25].

Most of the above works share the assumption that 1-qubit gates have zero time cost. The authors described a theoretical framework for time-optimal quantum computing based on the action principle where such an assumption is not necessary, and named it the quantum brachistochrone (hereon abbreviated as QB)\(^5\). The variational principle is formulated for the time-optimal evolution of a quantum system whose Hamiltonian is subject to a set of constraints (e.g., a finite energy or magnetic field, certain qubit interactions are forbidden) and defines a boundary value problem with fixed initial and final quantum states (or unitary transformations). The QB has been studied for the quantum state evolution in the case of pure [26] and mixed states [27], and for the optimal realization of unitary transformations between the identity and a given target quantum gate [28]. The latter is particularly relevant to the standard quantum computation paradigm since a whole algorithm may be reduced to a sequence of unitary transformations between intermediate states and a final measurement. The more realistic situation, where the target quantum state (gate) can be reached within a finite, tolerable error (a fidelity larger than a specified value), has also been addressed in [29]. The QB problem always reduces to solving a fundamental equation, which can be easily written down once the constraints for the Hamiltonian of the quantum system are known, with given initial boundary conditions, and an equation for the Lagrange multiplier which enforces the dynamical law for the quantum system (the Schrödinger equation for closed systems or, e.g., a

\(^5\) From the Greek ‘βραχιστοζ’, i.e. fast, and ‘χρονοζ’, i.e time.
master equation for Markovian open systems), with given final boundary conditions. Two of us also studied numerically the time complexity of generating unitaries acting on $n$ qubits via a Hamiltonian which contains only 1- and 2-qubit interaction terms [30]. Our research also triggered a number of related works [31–39]. For example, the authors of [32–34] considered the problem of the generation of multipartite entanglement during the QB evolution of quantum states and unitaries, while those of [35–39] studied the QB in the context of non-Hermitian quantum mechanics (for a review of the latter works see, e.g., [40]).

More recently, Khaneja et al [41] considered the problem of the efficient synthesis of the controlled-$\text{X}$ gate (CNOT$(1, 3)$) between two qubits indirectly coupled via an Ising-type coupling to a third qubit, and where the single qubits can be separately addressed via instantaneous local unitaries. Their implementation requires a time $T \approx 1.253J^{-1} \approx \sqrt{\pi/2}J^{-1}$, where $J$ is the Ising coupling between the qubits $(1, 2)$ and $(2, 3)$ in a linear coupling topology, and it was related to the computation of a geodesic on the surface of a sphere with a special metric and to the synthesis of a particular entangling gate called $U_{13}$. Subsequently, similar methods were used to extend these results to the case of unequal Ising couplings between the indirectly coupled qubits [42] and to the case of a spin chain with $n$ qubits [43].

The time-optimal synthesis of interactions between qubits which are indirectly coupled via an intermediate qubit is a typical scenario in a wide array of promising (scalable) experimental realizations of quantum information processing. This is the case for solid-state architectures such as, e.g., the Kane quantum computer [44], where the interactions between distant qubits (nuclear spins of donor atoms in chains of implanted phosphorous ions in silicon) are electronically mediated by the donors’ electrons, or the crystal lattice quantum computer [45], where indirect quantum gates may be performed via laser pulses to sequentially swap adjacent qubits (nuclear spins in a crystal lattice of rare earth elements), and for devices based on molecular single crystal, where bus-qubit electrons mediate interactions of two nuclear spins [46]. Another attractive scenario for the practical realization of scalable quantum computing is that based on the technology of superconducting electrodes coupled through Josephson junctions (for a review see, e.g., [47]). Here, for example, designs for the tunable coupling of pairs of flux qubits via the quantum inductance of a third high-frequency qubit have been proposed [48]. Besides, in hardware based on superconducting circuit quantum electrodynamics (with nearest-neighbor coupled two-dimensional arrays of cavities and qubits [49]) indirect gates between qubits in separate transmission line resonators are mediated via an intermediate qubit employing the dispersive interaction inside the two resonators [50]. Finally, in the more-conventional NMR schemes for quantum information processing, several experiments correlating the frequencies of indirectly coupled qubits can be envisaged [51].

The problem with the synthesis of such indirect couplings is that usually it is time costly (e.g., it may require concatenation of two-qubit operations on the directly coupled qubits) and therefore it is also easily prone to decoherence effects and a degradation of the gate fidelity, both of which are critical effects in practical implementations of quantum computing. The importance and the urgency of new benchmarks in the theory of the time-optimal quantum control of indirect gates for the applications of quantum computing is therefore evident.

In this work, we further investigate the problem of the time-optimal generation of the same entangling quantum gates, $U_{13}'$ and CNOT$(1, 3)$, between two indirectly coupled qubits in 3-qubit systems with linear coupling topology and a similar Ising coupling Hamiltonian as that of [41]. We find that, if the available Hamiltonian is made up of an interacting piece (exactly as in the model of [41]) plus a local magnetic field acting on the intermediate qubit, and is subject to the constraint of a finite energy, then the $U_{13}'$ gate can be optimally realized in a time $T = \sqrt{3/2}J^{-1}$, which is faster than the time required via the
construction of [41]. Furthermore, using a slightly modified interaction Hamiltonian (obtained via a simple change of basis for the Hilbert space of one of the indirectly coupled qubits, qubit 3) and the same energy constraint, we find that the gate CNOT±(1, 3) (which acts as the identity when the control qubit 1 is in the state |0⟩, while if the control qubit is in the state |1⟩ the target qubit 3 is flipped as |±⟩ → |∓⟩) can also be optimally generated in the same time $T = \sqrt{3/2}J^{-1}$ as the entangling gate $U_{13}$. The paper is organized as follows. In section 2, we briefly review the main features of the QB formalism for the time-optimal synthesis of unitary quantum gates. In section 3, we discuss the problem of the efficient generation of the gate $U_{13}$ between the indirectly coupled boundary qubits of a three-linear qubit system subject to an Ising interaction and a local control available for the intermediate qubit, when a finite energy is available. Section 4 is devoted to the study of the time-efficient generation of the CNOT±(1, 3) gate with the slightly modified interaction Hamiltonian plus the same local operation on qubit 2 and the same energy available. Finally, section 5 is devoted to the summary and discussion of our results.

2. Quantum brachistochrone

We want to find the time-optimal way to generate a target unitary operation $U_f \in U(N)$ (modulo physically irrelevant overall phases) by controlling a Hamiltonian $H(t)$ and evolving a unitary operator $U(t)$, both obeying the Schrödinger equation. We assume that $H$ is controllable within a certain available set, dictated either by theoretical conditions (e.g., only certain interactions among qubits are allowed) or by experimental requirements (e.g., a finite energy or a finite magnetic field). At least the ‘magnitude’ of the Hamiltonian must be bounded; otherwise, any gate $U_f$ might be realized in an arbitrarily short time simply by rescaling the Hamiltonian [28]. Physically, this corresponds to the fact that one can afford only a finite energy in the experiment.

The time-optimality problem is formulated using the action [27, 28]

$$S(U, H; \alpha, \Lambda, \lambda_j) := \int_0^1 d\tau [\alpha + L_S + L_C]$$

(1)

$$L_S := \left\langle \Lambda, i \frac{dU}{d\tau} U^\dagger - \alpha H \right\rangle,$$

(2)

$$L_C := \alpha \sum_j \lambda_j f_j(H),$$

(3)

where $\langle A, B \rangle := \text{Tr}(A^\dagger B)$, and the Hermitian operator $\Lambda(\tau)$ and the real functions $\lambda_j(\tau)$ are Lagrange multipliers$^6$. The quantity $\alpha$ is the time cost, and it may be interpreted [27] as a positive independent dynamical variable (a ‘lapse’ function) which measures the physical time $t := \int \alpha(\tau) d\tau$ lapsed in each infinitesimal interval $d\tau$ of the parameter time $\tau$. Variation of $L_S$ by $\Lambda$ gives the Schrödinger equation:

$$i \frac{dU}{d\tau} = HU, \quad \text{or} \quad U(t) = \mathcal{T} e^{-i \int_0^t H(\tau) d\tau},$$

(4)

where $\mathcal{T}$ is the time-ordered product. Variation of $L_C$ by $\lambda_j$ leads to the constraints for $H$:

$$f_j(H) = 0.$$  

(5)

$^6$ The action $S$ is invariant under time reparameterizations $\tau \rightarrow f(\tau)$. 

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In particular, the finite energy condition (associated with the Lagrange multiplier $\lambda_0$) for a system of log $N$ qubits can be written as

$$f_0(H) := \frac{1}{2}[\text{Tr}(H^2) - N\omega^2] = 0,$$

where $\omega$ is a constant. Since overall phases are irrelevant in quantum mechanics, it is natural to consider the time-optimal evolution of unitary operators belonging to the group $U(N)/U(1) \simeq SU(N)$ (i.e. the dynamics is generated by a traceless Hermitian Hamiltonian).

Then, we introduce the operator

$$F := \frac{\partial L_C}{\partial H},$$

which, using (6) and (7), can be explicitly written as

$$F = \lambda_0 H + \sum_{j \neq 0} \lambda_j \frac{\partial f_j(H)}{\partial H}.$$  

Furthermore, from the variation of $S$ with respect to $H$ and using (7), we obtain

$$\Lambda = F,$$

while from the variation of $S$ by $\alpha$, upon using (5) and (9), we obtain the normalization condition

$$\text{Tr}(HF) = 1.$$  

Finally, variation of $S$ by $U$, use of equation (9), and some elementary algebra give the QB equation:

$$\frac{dF}{dt} = [H,F].$$

The QB, together with the constraints, define a boundary-value problem for the evolution of the unitary operator $U(t)$ with fixed initial ($U(t=0) = 1$, where 1 is the identity matrix) and final conditions ($U(t = T) = U_f$, where $T$ is the optimal time duration necessary to achieve the target gate $U_f$). The QB is a set of first-order (nonlinear) differential equations which can always be solved in principle, e.g. numerically, and it is universal, as it holds also in the case of the time-optimal evolution of pure [26] and mixed [27] quantum states.

In more detail, for a given target gate $U_f$, the procedure to find the optimal Hamiltonian $H$ and the optimal time duration $T$ consists of the following stages: (i) specify the constraint functions $f_j(H)$ for the available Hamiltonian; (ii) write down and solve the QB equation (11) together with constraints (5) to obtain $H_{\text{opt}}(t)$; (iii) integrate the Schrödinger equation (4) with $U(0) = 1$ to obtain $U_{\text{opt}}(t)$; and (iv) fix the integration constants in $H_{\text{opt}}(t)$ by imposing the condition that $U_{\text{opt}}(T)$ equals $U_f$ modulo a global (physically irrelevant) phase, i.e.

$$U_{\text{opt}}(T) = e^{i\chi} U_f,$$

where $\chi$ is some real number.

### 3. Ising Hamiltonian and time-optimal entangler $U_{13}^*$

Now we apply the general QB formalism summarized in the previous section to the case of a physical system of three qubits represented by three spins (labeled by a superscript $a \in \{1, 2, 3\}$) interacting via an Ising Hamiltonian with time-independent couplings $J_{12}, J_{23}$ and subject to a local and controllable magnetic field $B_i(t)$ ($i = x, y, z$) acting on the
Our goal is to time-optimally generate the symmetric entangler gate $U'_{13}$ (see equation (4) in [41]), i.e. the target is

$$U_f = U'_{13} := e^{-i\frac{\pi}{2}(\sigma_i^x\sigma_j^x+\sigma_i^y\sigma_j^y)}.$$  

(13)

In more detail, we choose the 3-qubit Ising Hamiltonian

$$H(t) := \frac{\pi}{2} [J_{12}(t)\sigma_i^x\sigma_j^x + J_{23}(t)\sigma_j^x\sigma_k^x] + \vec{B}(t) \cdot \vec{\sigma},$$  

(14)

where we have used the simplified notation, e.g., $\sigma_i^j := \sigma_i \otimes \sigma_j \otimes 1$, $\sigma_i^j := 1 \otimes \sigma_i \otimes \sigma_j$, $\sigma_i^j := 1 \otimes \sigma_i \otimes 1$, and $\sigma_i$ are the Pauli operators [23]. We further assume that the Ising couplings in (14) are equal and (a positive) constant, i.e. $J_{12}(t) = J_{23}(t) := J > 0$. This is formally enforced via the constraints (associated with the Lagrange multipliers $\lambda_1, \lambda_2$)

$$f_1 := \text{Tr}(H\sigma_i^1\sigma_j^2) - 4\pi J = 0$$  

(15)

$$f_2 := \text{Tr}(H\sigma_j^2\sigma_k^3) - 4\pi J = 0.$$  

(16)

Moreover, the finite energy condition (6) reads

$$\vec{B}^2 = \omega^2 - \frac{\pi J^2}{2} = \text{const}.$$  

(17)

We note that the interaction part of the Hamiltonian (14) is exactly the same as $H_c$ in equation (2) of [41], and that the local term in (14) corresponds to the terms $H_A$ and $H_B$ of [41]. Finally, the form (14) of the physical Hamiltonian is guaranteed by operator (8), which in our case explicitly reads

$$F(t) = \lambda_0 H + \sum_{i,j,k} \lambda_{ijk}(t)\sigma_i^1\sigma_j^2\sigma_k^3 + \sum_{i,j} \left[ \mu_{ij}(t)\sigma_i^1\sigma_j^3 + \nu_{ij}(t)\sigma_i^1\sigma_j^2 + \rho_{ij}(t)\sigma_i^2\sigma_j^3 \right]$$  

$$+ \sum_{i} \left[ \eta_i(t)\sigma_i^1 + \xi_i(t)\sigma_i^3 \right].$$  

(18)

where $\lambda_{ijk}(t), \mu_{ij}(t), \nu_{ij}(t), \rho_{ij}(t), \eta_i(t)$ and $\xi_i(t)$ are Lagrange multipliers (with the relabeling $\lambda_1(t) \equiv v_z(t)$ and $\lambda_2(t) \equiv \rho_z(t)$), and the indices $\{i,j\} \in \{x,y,z\}$.

Our task is then to solve the QB equation (11). Before proceeding, we recall that the QB should also satisfy the normalization condition (10). In our model, $F$ is given by (18), while $H$ must satisfy the energy condition (6), so that (10) explicitly reads

$$4[2\lambda_0(t)\omega^2 + 4\pi J (v_z(t) + \rho_z(t))] = 1,$$  

which is a normalization condition for the Lagrange multipliers $\lambda_0(t)$ and $v_z(t) + \rho_z(t)$. At this point, in order to simplify the analysis and its exposition, we choose to work in the normalization ansatz where the Lagrange multiplier $\lambda_0$ is a constant, i.e. $\lambda_0 = 1$, which implies, via (10), the other integral of the motion $v_z + \rho_z = \text{const}$.

By choosing the Hamiltonian (14), we are formally adopting the couplings $J_{12}(t)$ and $J_{23}(t)$ as dynamical variables in action (1)–(3), and we make them become constant on shell via the imposition of constraints (15) and (16). On the other hand, one might have chosen to start with $J_{12} = J_{23} = J = \text{const}$, i.e. with the coupling $J$ as a given constant parameter in action (1)–(3) from the beginning (cf. [41]) and without the need to impose constraints (15) and (16). However, a simple calculation shows that both variational methods lead to the same result (20) for the time-optimal magnetic field $\vec{B}_{OPT}(t)$ (with $B_0, B_z$ and $\omega$ given, respectively, by (52), (53) and (54)) and to the same time-optimal duration (51). In other words, our variational principle is essentially the same as that used in [41], with the caveat that we further impose the finite energy constraint (6).

Note that if the target unitary $U_t$ is reachable by a solution $H(t)$ of the QB equations within this ansatz, the control procedure $H(t)$ is optimal for $U_t$, at least locally. In fact, a solution of the QB equations is by definition the fastest among the procedures close to $H(t)$, including those with a time-varying $\lambda_0(t)$.
Then, comparing the coefficients of the generators of $su(8)$ on both sides of (11), with $F$ given by (18) and $H$ by (14), we find that the relevant QB equations are

\begin{align}
\dot{B}_x &= -\pi J(v_{xy} + \rho_{yz}) \\
\dot{B}_y &= \pi J(v_{xz} + \rho_{xz}) \\
\dot{B}_z &= 0 \\
\dot{v}_{zx} &= -[\pi J\lambda_{3yz} + 2(B_zv_{zy} - B_yv_{zz})] \\
\dot{v}_{zy} &= \pi J\lambda_{3xz} + 2(B_cv_{zx} - B_av_{zc}) \\
\dot{v}_{zz} &= 2(B_cv_{zy} - B_yv_{cx}) \\
\dot{\rho}_{xz} &= -[\pi J\lambda_{3yz} + 2(B_z\rho_{yz} - B_y\rho_{zz})] \\
\dot{\rho}_{yz} &= \pi J\lambda_{3xz} + 2(B_z\rho_{xz} - B_x\rho_{zz}) \\
\dot{\rho}_{zz} &= 2(B_z\rho_{yz} - B_y\rho_{xz}) \\
\dot{\lambda}_{3xz} &= -\pi J(v_{xy} + \rho_{yz}) + 2(B_y\lambda_{3zz} - B_z\lambda_{3yz}) \\
\dot{\lambda}_{3yz} &= \pi J(v_{xz} + \rho_{xz}) - 2(B_x\lambda_{3zz} - B_z\lambda_{3xz}) \\
\dot{\lambda}_{3zz} &= 2(B_x\lambda_{3yz} - B_y\lambda_{3xz}).
\end{align}

As recalled in the previous section, the first step in addressing the QB problem is to solve the fundamental equation (11) for the time-optimal Hamiltonian $H_{OPT}(t)$. In other words, we need to solve equations (19) together with constraints (15)–(17) and with fixed parameters $J$ and $\omega$.

From the integral of the motion $B_z = \text{const}$, equations (19), and the energy constraint (17), we immediately obtain that $B_x^2 + B_y^2 = B_0^2 = \text{const}$ while, from equations (19), we further find the integral of the motion $\lambda_{zzz} = \text{const}$. Exploiting these integrals of the motion and after some lengthy but elementary algebra, we find that the general and non-trivial solution of (19) is given by the Hamiltonian (14) with the time-optimal magnetic field

$$
B_{OPT}(t) = \begin{pmatrix} B_0 \cos \theta(t) \\ B_0 \sin \theta(t) \\ B_z \end{pmatrix},
$$

precessing around the $z$-axis with the frequency $\tilde{\Omega}$, where $\theta(t) := \Omega t + \theta(0)$, and $\tilde{\Omega}$ and $\theta(0)$ are integration constants.

The next step is then to integrate the Schrödinger equation (4) for $U_{OPT}(t)$, given that $H_{OPT}(t)$ is expressed by equations (14) and (20). For this purpose, we exploit the following well-known property for the rotation of the Pauli matrices:

$$
e^{-i\frac{\pi J}{2} \sigma_z} e^{i \theta(t) \sigma_y} = \cos \theta(t) \sigma_x + \sin \theta(t) \sigma_y,
$$

and we rewrite the time-optimal Hamiltonian as

$$
H_{OPT}(t) = e^{-i\frac{\pi J}{2} \sigma_z} H_0 e^{i \theta(t) \sigma_y},
$$

where we have introduced the constant operator

$$
H_0 := B_0 \sigma_z^2 + \left[\frac{\pi J}{2} (\sigma_z^1 + \sigma_z^2) + B_z \right] \sigma_z^2.
$$

Furthermore, defining the transformed unitary operator

$$
\tilde{U}(t) := e^{i \frac{\pi J}{2} \sigma_z^1} U(t),
$$
we easily check that since $U(t)$ should obey the Schrödinger equation (4), $\dot{U}(t)$ should also satisfy
\[ i \frac{d\tilde{U}}{dt} = \tilde{H}\tilde{U} \] (25)
with the time-independent Hamiltonian
\[ \tilde{H} := H_0 - \frac{\Omega}{2} \sigma_z^2 = \text{const.}. \] (26)
We note that the constant Hamiltonian $\tilde{H}$ is diagonal in the 1,3 qubit subspace, i.e.
\[ \tilde{H} = B_0 \sigma_z^2 + B_D^{13} \sigma_z^2 = \text{const.} \] (27)
where we have introduced the operator (in the 1,3 qubit subspace)
\[ B_D^{13} := -\frac{1}{2}(\Omega - 2B_z) + \pi J \text{ Diag}[1, 0, 0, -1]. \] (28)
Then, solving equation (25) together with equation (27) for $\tilde{U}(t)$ and finally inverting (24), it is easy to check that the time-optimal unitary operator $U_{\text{OPT}}(t)$ evolves as
\[ U_{\text{OPT}}(t) = e^{-i\frac{\Omega}{2}\sigma_z t} e^{-iHt} e^{i\frac{\Omega}{2}\sigma_z t}. \] (29)
In particular, the exponential of the constant Hamiltonian appearing on the right-hand side of equation (29) is also diagonalized (in the 1,3 qubit subspace) and can be expanded as
\[ e^{-i\tilde{H}t} = C_D^{13}(t) - i S_D^{13}(t) \tilde{H}, \] (30)
where we have introduced the following operators (acting in the 1,3 qubit subspace):
\[ S_D^{13}(t) := \text{Diag}[s_+(t), s_0(t), s_0(t), s_-(t)], \] (31)
\[ C_D^{13}(t) := \text{Diag}[c_+(t), c_0(t), c_0(t), c_-(t)], \] (32)
which depend upon the functions
\[ s_{\pm}(t) := \frac{\sin \omega_{\pm} t}{\omega_{\pm}}; \quad s_0(t) := \frac{\sin \omega_0 t}{\omega_0}, \] (33)
\[ c_{\pm}(t) := \cos \omega_{\pm} t; \quad c_0(t) := \cos \omega_0 t, \] (34)
and the constants
\[ \omega_{\pm}^2 := B_0^2 + \frac{1}{4}(\Omega - 2(B_z \pm \pi J))^2, \] (35)
\[ \omega_0^2 := B_0^2 + \frac{1}{4}(\Omega - 2B_z)^2. \] (36)
Then, inserting equation (30) into equation (29), one obtains the following more explicit expression for the time-optimal evolution of the unitary operator:
\[ U_{\text{OPT}}(t) = \cos \frac{\Omega t}{2} C_D^{13}(t) - \sin \frac{\Omega t}{2} B_D^{13} S_D^{13}(t) \]
\[ -i \left\{ B_0 S_D^{13}(t) \left[ \cos \phi(t) \sigma_z^2 - \sin \phi(t) \sigma_y^2 \right] \right\} \]
\[ + \left\{ \sin \frac{\Omega t}{2} C_D^{13}(t) + \cos \frac{\Omega t}{2} B_D^{13} S_D^{13}(t) \right\} \sigma_z^2, \] (37)
where $\phi(t) \equiv -[\theta(t) + \theta(0)]/2$.
Expression (37) for $U_{\text{OPT}}(t)$ still depends upon the integration constants $B_0, B_z, \Omega$ and $\theta(0)$ and the coupling $J$. These constants, together with the optimal duration time $T$ of the
evolution and the irrelevant global phase $\chi$, can be finally fixed by imposing the target condition (12).

As stated at the beginning of this section, we are interested in the time-optimal realization of the symmetric entangler gate $U_f = U_{13}^f$ given by equation (13). This is diagonal in the 1,3 qubit subspace and explicitly reads

$$U_f = e^{i \pi/4} U_{13}^D,$$

(38)

Imposing the target condition equation (12) with $U_{\text{OPT}}(t)$ given by equation (37) and $U_f$ given by equations (58)–(62), and separately equating the terms multiplying the Pauli operators $\sigma_1$, $\sigma_y$, and $\sigma_z$ and the identity operator $I$, respectively, we obtain the following set of conditions for operators acting on the 1,3 qubit subspace:

$$0 = B_0 \cos \phi(T) S_{13}^D(T),$$

(39)

$$0 = B_0 \sin \phi(T) S_{13}^D(T),$$

(40)

$$0 = \sin \Omega T/2 C_{13}^D(T) + \cos \Omega T/2 B_{13}^D S_{13}^D(T),$$

(41)

$$e^{i \chi/2} U_{13}^D = \cos \Omega T/2 C_{13}^D(T) - \sin \Omega T/2 B_{13}^D S_{13}^D(T).$$

(42)

From equation (42) (imposing the reality of its left-hand side term), we immediately obtain the value of the global phase $\chi = (k - 1/4)\pi$, where $k \in \mathbb{Z}$. Then, the only non-trivial solution9 for equations (39) and (40) is easily seen to be $S_{13}^D(T) = 0$, which is possible, using equation (31), if and only if $s_x(T) = s_y(T) = s_z(T) = 0$ and, upon comparing with equations (33) and (34), provided that

$$\omega_+ T = \pi n_+, \quad \omega_0 T = \pi n_0,$$

(43)

where $n_+$ and $n_0$ are positive integers. Substituting solution (43) into equations (41), we also obtain the time-optimal integral of the motion $\Omega T = 2\pi m$, where $m \in \mathbb{Z}$. Finally, inserting the time-optimal formulas for $\chi$, $\Omega$ and $\omega_+$, $\omega_0$ into equation (42), we obtain the conditions

$$n_+ = k_0 + 2p; \quad n_- = n_+ + 2r - 1; \quad n_0 = n_+ + 2q - 1$$

(44)

with $k_0 := k + m + 1$, $p, q, r \in \mathbb{Z}$, and the parity of $n_0$ and $n_+$ the same and the opposite of that of $n_+$.

At this point we note that formulas (35) and (36) can be easily inverted (using (43)) to write down the parameters $B_0$, $B_z$ and $J$ and the optimal time duration $T$ as the functions of the integers $n_+$ and $n_0$. Introducing the following (odd-integer-valued) functions of the integers $n_+$ and $n_0$ in order to simplify the notation:

$$f_\pm(n_+, n_-, n_0) := n_+^2 + n_-^2 \pm 2n_0^2,$$

(45)

$$f_0(n_+, n_-) := n_+^2 - n_-^2,$$

(46)

from equations (35)–(36) we thus obtain

$$(JT)^2 = \frac{f_-}{2},$$

(47)

$$(B_0 T)^2 = \pi^2 \left[ n_0^2 - \frac{1}{8} \left( \frac{f_0}{f_-} \right)^2 \right].$$

(48)

9 We exclude the trivial case of $B_0 = 0$. 


We immediately see that one can minimize $JT$ from equation (47) by minimizing $f_-$ as a function of the integers $k$, $p$, $q$ and $r$ (via equations (44) and (45)). In other words, we can determine the optimal time duration $T$ of the quantum evolution necessary to realize the target gate $U_{13}^T$ as measured in terms of the coupling $J$. Then, substituting the time-optimal values of $k$, $p$, $q$ and $r$ into equations (45) and (46) and subsequently into equations (48) and (49), we obtain the time-optimal values of the integrals of the motion $B_0$ and $B_z$. Finally, we still have to impose the energy constraint (17) which, in terms of the functions $f_\pm$ and $f_0$, explicitly reads

$$\left(\alpha T\right)^2 = \frac{\pi^2}{4} \left[f_+ - \frac{1}{2} \left(\frac{f_0}{\sqrt{f_-}}\right)^2 + 4 \left[m - \frac{\sqrt{2}}{4} \frac{f_0}{\sqrt{f_-}}\right]^2\right].$$

Substituting for the time-optimal values of $k$, $p$, $q$ and $r$ into equations (45), (46) and (50), this constrains the values of the coupling $J$ as a function of the (given) energy parameter $\omega$. More explicitly, let us look first for the minima of $T$ via equation (47). For a start, we note that, due to the parity properties of $n_+$, $n_-$ and $n_0$ (equations (44)), the function $f_-(n_+, n_-, n_0)$ is always an odd integer. Moreover, its minimum value $f_-$, MIN = 1 in principle may be achieved in two cases, i.e. either when $n_+$ is even ($k_0$ even) or odd ($k_0$ odd). It is immediate to check that the case of $n_+$ even (and therefore, via (44), $n_-$ and $n_0$ odd) is not possible. Furthermore, in the case in which $n_+$ is odd, defining $k_0 := 2k_0 + 1$ (with $k_0 \in \mathbb{Z}$), the condition $f_-$, MIN = 1 translates into $(p + k_0)(2r - 4q + 1) = 2q^2 - r^2$. Substituting $f_-$, MIN = 1 into equation (48), we can rewrite $(B_0 T)^2 = -16\pi^2 [(q + r + 2(p + k_0))^2 - 1/8][(q - r)^2 - 1/8]$. Imposing the reality condition, i.e. that $(B_0 T)^2 \geq 0$, after a simple algebra we finally obtain that the only possible solution for $f_-$, MIN = 1 would be given by $(1 - 2q)[1 + 2q + 4(p + k_0)] = 1$, with $r = q$ or $r = q/(1 - 2q)$ and $q \neq 0, 1$, which no set of integers $\{k_O, p, q, r\}$ can ever satisfy. The next step is to check whether the next-to-minimum value of $f_-$, NM = 3 may be consistent with the reality condition for $(B_0 T)^2 \geq 0$. Again, it is immediate to verify that the case of $n_+$ odd is impossible. However, in the case of $n_+$ even, by defining $k_0 := 2k_E$ (with $k_E \in \mathbb{Z}$), the condition $f_-$, NM = 3 translates into $(p + k_E)(2r - 4q + 1) = 1 - r(r - 1) + 2q(q - 1)$. Calculating (48) for $f_-$, NM = 3 and $n_+$ even, we obtain $(B_0 T)^2 = -(r^2/3)[(4(p + k_E) + 2q - 1)^2 - 3/2]((2q - 1)^2 - 3/2)$. The reality condition $(B_0 T)^2 \geq 0$ now can be satisfied either for the values of the integers $p + k_E = 1$, with $q = 0, -1$ and $r = 0, -1$, or for $p + k_E = -1$, with $q = 1, 2$ and $r = 1, 2$. Summarizing, we have found that the optimal duration time required to realize a $U_{13}^T$ gate with the Ising Hamiltonian (14) is given by

$$T = \sqrt{\frac{r}{2}} J^{-1},$$

and the corresponding time-optimal magnetic fields are

$$|B_0| := \frac{\sqrt{\pi}}{2 \sqrt{3}} \pi J,$$

$$|B_z| := \frac{\sqrt{\pi}}{2 \sqrt{3}} \pi J \left|m - \frac{1}{\sqrt{2}}\right|.$$
while the time-optimal precession frequency of the magnetic field is given by
\[ |\Omega| = 2\sqrt{\frac{3}{2}\pi m J}, \]
where the integer \( m \) and the phase \( \theta(0) \) are still arbitrary. Moreover, the coupling \( J \) and the energy parameter \( \omega \) are constrained (from equation (50)) by the following condition:
\[ \frac{J}{|\omega|} = \frac{2\sqrt{3}}{\pi} \left[ 11 + 8 \left( m - \sqrt{\frac{3}{8}} \right)^2 \right]^{-1/2}. \]
\[ (55) \]

4. Modified Hamiltonian and time-optimal CNOT(1, 3)

An analysis similar to that performed in the last section can be performed in the following two situations. On the one hand, one can assume to be in the physical situation in which the Hamiltonian (14) is available to a given experimentalist \( O \) working with the standard computational basis, which in our 3-qubit model corresponds to the Hilbert space spanned by the states \( \{|0\rangle, |1\rangle\} \otimes \{|0\rangle, |1\rangle\} \otimes \{|0\rangle, |1\rangle\} \). Then, we can think of another experimentalist \( O' \) who performs measurements in the basis which corresponds to the Hilbert space spanned by the states \( \{|0\rangle, |1\rangle\} \otimes \{|0\rangle, |1\rangle\} \otimes \{|+\rangle, |-\rangle\} \), i.e. the basis where the rotated states \( \{|+\rangle, |-\rangle\} \) (with \( |+\rangle := W|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2} \) and \( |-\rangle := W|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2} \) and \( W \) is the Walsh–Hadamard transform) are used for qubit 3, while qubits 1 and 2 use the standard computational basis. More formally, this situation is equivalent to the second experimentalist seeing an effective, rotated Hamiltonian given by
\[ H'(t) := \frac{\pi}{2} \left[ J_{12}(t)\sigma_1^z\sigma_2^z + J_{23}(t)\sigma_2^z\sigma_3^x \right] + \vec{B}(t) \cdot \vec{\sigma}. \]
\[ (56) \]
On the other hand, one may think of the situation in which one experimentalist who can perform measurements in the standard computational basis for the 3-qubit system, but with the Hamiltonian (56) available. In both cases, Hamiltonians (14) and (56) are related by the transformation
\[ H'(t) = V H(t)V, \]
\[ (57) \]
where we have introduced the operator \( V := 1 \otimes 1 \otimes W \). The goal here is to time-optimally synthesize the gate \( \text{CNOT}(1, 3) \), i.e. the new target is
\[ U_f := \text{CNOT}(1, 3) = e^{-\frac{i\pi}{4}((1+\sigma_1^z)\sigma_2^z-\sigma_1^z\sigma_2^z)}. \]
\[ (58) \]
Then, one may formulate a QB problem using an action principle similar to that of the previous section. In other words, one obtains the QB equation (11), and the form of the available Hamiltonian (56) is guaranteed by the operator \( F' \) formally given by equation (18) with the same set of Lagrange multipliers, but with constraint (16) now replaced by
\[ f' := \text{Tr}(H\sigma_2^z\sigma_3^x) - 4\pi J = 0 \]
\[ (59) \]
(while the energy constraint (17) is the same). A lengthy but simple algebra shows that the QB equations (in the didactic ansatz \( \lambda_0 = 1 \)) are again given by (19), with the exception that now we have to replace everywhere
\[ \rho_iz \rightarrow \rho_ix; \quad \lambda_iz \rightarrow \lambda_ix; \quad \forall i \in \{x, y, z\}. \]
\[ (60) \]
Then, we can follow the same procedure as in the previous section and find that equations (19) admit the integral of the motion \( B_2^2 + B_3^2 = (B_0^i)^2 \) and the general solution (20). In particular, the time-optimal Hamiltonian can be seen to become
\[ H'_{\text{OPT}}(t) = V H_{\text{OPT}}(t)V, \]
\[ (61) \]
where \( H_{\text{OPT}}(t) \) is given by equation (22). The corresponding time-optimal evolution operator is then given by \( U'_{\text{OPT}}(t) = VU_{\text{OPT}}(t)V \), with the \( U_{\text{OPT}}(t) \) of equation (37), and with the operators \( B^1_D, S^1_D \) and \( C^1_D \) always given by equations (28), (31) and (32), respectively.

Now the gate CNOT(1,3) which we want to time-optimally synthesize is given by equation (58), which can be diagonalized (in the 1,3 qubit subspace) as
\[
U_f = VU'^{13}_D V,
\]
where we have introduced the operator (acting in the 1,3 qubit subspace) \( U'^{13}_D := \text{Diag}(1,1,1,-1) \). Following the same methods of the previous section, we then find that the target condition (12) is equivalent to impose again the operator conditions (39)–(42), with the only difference that in (42) we have to replace \( \exp(i\chi + \pi/4)U'^{13}_D \rightarrow \exp(i\chi')U'^{13}_D \). This implies that the optimal value of the global phase is given now by \( \chi' = k'\pi \), with \( k' \in \mathbb{Z} \), while the optimal \( \Omega \) is the same as in the previous section. The non-trivial solution for equations (39) and (40) is again given by \( s_s(T) = s_r(T) = s_0(T) = 0 \), and consequently, by equation (43) for the same functions \( \omega_\pm \) and \( \omega_0 \) given by equations (35) and (36). However, the substitution of the optimal values of \( \chi \) and \( \Omega \) and \( \omega_\pm \) into equation (42) now gives, instead of equations (44),
\[
n'_0 = k'_0 + 2p'; \quad n'_+ = n'_+ + 2r' + 1; \quad n'_0 = n'_0 + 2q', \quad (63)
\]
with \( k'_0 := k' + m, p', q', r' \in \mathbb{Z} \), and the parity of \( n'_0 \) and \( n'_+ \) the same and the opposite of that of \( n'_- \). Considerations similar to those made in the previous section lead to expressions (47)–(50), and to the minimization of the evolution time \( T \) at \( f_{\text{NM}} = 3 \) (with \( f_\pm \) and \( f_0 \) always defined by (45)–(46)). In this case, the condition \( f_{\text{NM}} = 3 \) translates into \( n'_0(2r' - 4q' + 1) = 3 - (2r' + 1)^2 + 8q'^2 \), which cannot be satisfied if \( n'_0 \) is even\(^{12}\). When \( n'_0 \) is odd instead, defining \( k'_0 := 2k'_0 (k'_0 \in \mathbb{Z} \) ), the condition \( f_{\text{NM}} = 3 \) becomes \( (p' + k'_0)(2r' - 4q' + 1) = 2q'(q' + 1) - r'(r' + 2) \). Then, we can rewrite \( (B^1_sT)^2 = -(16\pi^2/3)[(2(p' + k'_0) + q' + 1)^2 - 3/8](q'^2 - 3/8) \), and imposing the condition \( (B^1_sT)^2 \geq 0 \) finally gives \( q' = 0 \) or \( q' = -[1+2(p' + k'_0)] \) and \( (2r' + 1)[(p' + k'_0) + 2r' + 3] = 3 \). The latter can be satisfied either if \( p' + k'_0 = 0 \), together with \( r' = 0, -2 \), or if \( p' + k'_0 = -1 \), together with \( r' = \pm 1 \).

In conclusion, the time-optimal duration necessary to realize a CNOT(1,3) gate with the Hamiltonian (56) is given again by (51), while it is easy to check that the time-optimal magnetic fields \( |B^1_s| \) and \( |B^1_r| \), the precession frequency \( \Omega \), and the ratio \( J/|\omega| \) between the coupling \( J \) and the energy parameter \( \omega \) are also still expressed by, respectively, equations (52), (53), (54), and (55).

5. Discussion

We have presented the exact and analytical solution for the time-optimal realization of two entangling gates, the \( U'^1 \) and the CNOT(1,3), between two indirectly coupled qubits, labeled 1 and 3, in a 3-qubit linear spin chain subject to, respectively, an Ising-type interaction or a slightly modified Ising Hamiltonian, where in both cases a local magnetic field can be applied on the intermediary qubit 2 and the constraint of a finite available energy is imposed. No constraints ensuring instantaneous local unitary operations are imposed. In particular, we showed that the \( U'^{13}_1 \) gate can be optimally realized via an Ising Hamiltonian of the same form as that discussed in [41], and that the time required is shorter than that found for the particular decomposition of the unitary evolution considered in [41]. We then presented the analytical

\(^{12}\) Since, for \( k' := 2k'_0 \), one would have \( 2((p' + k'_0)(2r' - 4q' + 1) + r'(r' + 1) - 2q'^2) = 1 \), which is not possible.
solution for the time-optimal realization of the CNOT(1, 3) via the slightly modified Ising Hamiltonian (56), which again is shown to require the same time duration as the $U_{13}^1$ gate. In order to simplify our analysis and its exposition, we showed results for the pedagogical ansatz where the Lagrange multiplier $\lambda_0$ is a constant. It can be easily proven that this is essentially equivalent to assuming that the local magnetic field $B_z$ acting on the intermediate qubit is also time independent. This means that our solutions may be only locally time optimal, and in the more general case where we do not restrict to $B_z = \text{const}$, i.e. the synthesis of the indirect quantum gates $U_{13}^1$ or CNOT(1, 3) may indeed require an even shorter time. In any case, this does not diminish the novelty and the relevance of our present results.\footnote{The extension to the more general case of a time-dependent $B_z(t)$ is straightforward and will be considered in a forthcoming paper.}

Of course the orbits generated by the Ising Hamiltonian (14) cannot directly reach the target CNOT(1, 3), which can instead be reached via the modified Ising Hamiltonian (56). According to the standard paradigm of time-optimal quantum computing (see, e.g., [2]), where 1-qubit unitary operations are assumed to have zero time cost, one might infer that the CNOT(1, 3) can be still generated via the Ising Hamiltonian (14) provided, for example, that a series of additional local operations acting on qubits 1, 2 and 3 are applied in the sequence given by formula (13) of [2]. Somewhat surprisingly, this would appear to take the same time (51), which is shorter than that found in [41]. However, the reason of the better performance of our quantum evolution stems from the fact that the orbit described in [41] is based on a special decomposition of the unitary operator as a product of unitary factors, each of which is indeed time optimal. The product of such single factors in [41] would also be time optimal if these factors commuted which each other, which is not true in general.

On the other hand, in the more physical and general QB framework, where the time cost of local unitaries is not negligible in principle, one can instead think of the new Hamiltonian (56) as generated by a change of basis of the Hilbert space for the 3-qubit system according to $\{|0\rangle, |1\rangle\} \otimes \{|0\rangle, |1\rangle\} \otimes \{|0\rangle, |1\rangle\} \rightarrow \{|0\rangle, |1\rangle\} \otimes \{|0\rangle, |1\rangle\} \otimes \{|+, |−\rangle\}$, as explained in the previous section. This would relate the Ising Hamiltonians (14) and (56) via equation (57).

In other words, let us consider the time-optimal realization of the gate CNOT(1, 3) under the modified Hamiltonian (56) and let us assume that both the control and the target qubits belong to the standard computational basis $\{|0\rangle, |1\rangle\}$. Then, one can easily see that this is equivalent to the time-optimal realization of the gate CNOT$^\pm(1, 3)$ under the Ising Hamiltonian (14), where the control qubits belong to the basis $\{|0\rangle, |1\rangle\}$ while the target qubits belong to the basis $\{|+, |−\rangle\}$. We should note that our problem is slightly more restricted than that of [41], in the sense that we also impose the finite energy condition (6). In fact, the normalization condition (6) constrains not only the amplitude $|B|$ of the local magnetic field in the Hamiltonian (14) (or (56)), via equations (52) and (53), but also the allowed values for the Ising coupling constant $J$, via equation (55). As a consequence, only a (discrete) set of possible values for the coupling constant $J$ are actually allowed, and the optimal value of $J$ is related to the energy available $\omega$ through equation (55). In particular, one can see that, in the case when the energy $\omega$ available increases (for a fixed integer $m$) the coupling $J$ grows and, therefore, via equation (51), the optimal time required to generate the gates decreases, while the strengths of the optimal local magnetic field, via equations (52) and (53), and also the precession frequency, via equation (54), grow (which is what is physically expected). Alternatively one can use equation (55) to express the optimal values of $|B_0|$ and $|B_z|$ (equations (52) and (53)) as a function of the total energy available $\omega$, and then to evaluate the ratio between the amplitude...
of the local magnetic field controlling the time-optimal evolution and the value of the Ising coupling constant $J$ available in the experiment, obtaining

$$\frac{|\vec{B}|}{J} = \frac{\pi}{2\sqrt{3}} \sqrt{\frac{5 + 8 \left( m - \sqrt{\frac{3}{8}} \right)^2}{5 + 8 \left( m - \sqrt{\frac{3}{8}} \right)^2}}. \quad (64)$$

The ratio (64) is a monotone increasing function of the integer $m$, and it tells us that the amplitude of the driving magnetic field cannot be smaller than $|\vec{B}|_{\text{MIN}} = 2\pi\left(1 - \sqrt{\frac{3}{8}}\right)^{1/2}/(2\sqrt{3}J)$ (for $m = 1$). Furthermore, although the fact that the allowed values of $J$ belong to a discrete set (determined by the values of the integer $m$ via equation (55)) might appear as a limitation of our model, we point out that this feature is just the reflection of our simplifying technical choice (made just for ease of presentation) of considering only equality constraints (5). We conjecture that this feature should be mitigated by extending the QB methods to the more general and physical ansatz where, e.g., the total energy available is only bounded from above (i.e. substituting the equality constraint (6) with an inequality constraint, and using the proper variational techniques, see, e.g., [52]).

Furthermore, we would like to comment on the relation of our results with those of [30]. In particular, in [30] the 1-qubit components of the time optimal Hamiltonian were found to be constant, which is not the case in this work. However, in [30], there is no constraint for the 1-qubit or the 2-qubit components of the Hamiltonian, except for the normalization, which results in the absence of such components in $F$ (equation (8)) and thus yields the constancy of the 1-qubit components. Here, we have constraints (15) and (16) on the 1-qubit and the 2-qubit components of the Hamiltonian and therefore the result changes.

Our work was in part motivated by [41], and therefore we specialized our analysis to the case of the Hamiltonians (14) and (56), where the local operations are in principle allowed only on the intermediary qubit. However, there is no particular reason why one should limit to such a case, and one may think instead of more general situations in which also local interactions for (one of) the boundary qubits are available. Similarly, one could extend the analysis to the case of different interaction couplings between the indirectly coupled qubits (i.e. $J_{12} \neq J_{23}$, see [42]) or for different topological couplings among the qubits (see, e.g., [13] and [17]). Another simple and straightforward generalization of our results would be to include the study of how to time-optimally generate quantum gates other than the CNOT$(1, 3)$ (e.g. SWAP$(1, 3)$, $\sqrt{\text{SWAP}}(1, 3)$, etc). A further natural extension of our research will be also to consider our QB methods in the study of linear (Ising) chains of $n$ qubits (see [43]), e.g., in the search for time-efficient ways of creating and propagating spin order and coherences along the chains, or the transfer speed of single spin excitations via external magnetic fields along Heisenberg spin chains (see, e.g., [53] and references therein).

Finally, the most important motivation of pursuing our study has been already given in the introduction, and cannot but be emphasized again. On the one hand, several architectures for the experimental realization of scalable quantum computers are being continuously developed and tested. On the other hand, the presence of indirect interactions between qubits underpinning the hardware seems to become an almost ubiquitous scenario. The synthesis of indirect quantum gates is usually time costly and critically affected by decoherence, which leads to a degradation of the gate fidelity itself. In conclusion, the demand for advances in the theory of time-optimal quantum control of indirect quantum gates, with its crucial application to the most promising settings for the experimental and actual realization of quantum computers, is all the more urgent, and our results should be seen as a useful step in this direction.
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