NOVIKOV FUNDAMENTAL GROUP

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Abstract. Given a 1-cohomology class $u$ on a closed manifold $M$, we define a Novikov fundamental group associated to $u$, generalizing the usual fundamental group in the same spirit as Novikov homology generalizes Morse homology to the case of non exact 1-forms.

As an application, lower bounds for the minimal number of index 1 and 2 critical points of Morse closed 1-forms are obtained, that are different in nature from those derived from the Novikov homology.

1. Introduction and main results

Consider a smooth and closed manifold $M$, and a 1-form $\alpha$ in a non trivial cohomology class $u \in H^1(M, \mathbb{R})$.

By analogy with the exact case, a point $p$ in $M$ is called critical for $\alpha$ if $\alpha_p = 0$. Such a critical point $p$ is said to be non degenerate, or Morse, if it is...
non degenerate as the critical point of a primitive $f_\alpha$ of $\alpha$ in a neighborhood of $p$. The 1-form $\alpha$ itself is said to be Morse if all its critical points are non degenerate.

The Novikov theory relates the critical points of (Morse) 1-forms $\alpha$ in the cohomology class $u$ and the “topology” of the pair $(M, u)$.

The case $u = 0$ is addressed by the Morse theory, which exhibits a tight relation between the topology of $M$ and the critical points of a Morse function on it; in particular, Morse functions have to have enough critical points to generate both $H_*(M)$ and $\pi_1(M)$.

Another extreme case is when $M$ fibers over the circle $\pi_1(M) \to S^1$ and $\alpha = \pi^*(d\theta)$: $\alpha$ obviously has no critical point in this case, regardless of the actual topology of $M$. Notice that the converse is also true: if there is a 1-form without critical points in $u$, then it comes from a fibration over the circle [Tischler1970].

The celebrated Novikov homology [Novikov1981] offers some algebraic measurement of the complexity of the topology of $M$ “with respect to $u$” or of the pair $(M, u)$, and the Novikov inequalities [Novikov1981] as well as further numerical invariants developed in the same spirit [Farber2004, Pajitnov2006] give lower bounds for the number of critical points of Morse 1-forms in the cohomology class $u$. Novikov homology also enters in theory of symplectic fixed points on compact symplectic manifolds [LO1995, LO2017] or Lagrangian intersections/embeddings problems ([Sikorav1986, Damian2009, Gadbled2009]).

The object of this paper is to give a Novikov theoretic version of the fundamental group, that plays in Novikov theory the role the usual fundamental group plays in Morse theory.

More precisely, we fix a cohomology class $u \in H^1(M, \mathbb{R})$, and consider the minimal integration cover $\tilde{M}$ associated to $u$.

Remark 1. In fact, any integration cover could be used, from the minimal one to the universal cover, producing as many (different) versions of the invariant.

The situation is similar to Novikov Homology, which was originally defined using the minimal integration cover [Novikov1981] but later has been extended to the universal covering (or any intermediate covering) by Sikorav [Sikorav1987], so Farber suggested to call the associated complex (resp. its homology) the Novikov-Sikorav complex (resp. the Novikov-Sikorav homology) [Farber2004].

We denote by $\tilde{u} : \pi_1(M) \to \mathbb{R}$ the composition of the Hurewicz homomorphism $\pi_1(M) \to H_1(M, \mathbb{Z})$ with the evaluation map $ev : H_1(M, \mathbb{Z}) \to \mathbb{R}$. Since $\pi_1(M) = \ker \tilde{u}$ (recall $\tilde{M}$ is the minimal integration cover; with another choice, we would have $\tilde{M} \subset \ker \tilde{u}$, which is enough), the homomorphism $\tilde{u}$ descends to a homomorphism also denoted by $u : \mathcal{D} = \pi_1(M)/\pi_1(\tilde{M}) \to \mathbb{R}$. Note that $\mathcal{D}$ is the deck transformation group of $\tilde{M}$.
Then to \((M, u)\) is associated a group \(\pi_1(M, u)\) endowed with an action of \(D\), such that the following holds:

**Theorem 1.1.**

1. to each closed 1-form \(\alpha\) in the class \(u\) is associated a group \(\pi_1(\alpha)\) that is isomorphic to \(\pi_1(M, u)\),
2. for a suitable notion of generators of relations (taking deck transformations and a completion process into account), \(\pi_1(M, u)\) is finitely presented,
3. if \(\alpha\) is Morse, then the minimal number of generators of \(\pi_1(M, u)\) is a lower bound for the number of index 1 critical points of \(\alpha\),
4. similarly, if \(\alpha\) is Morse, then the minimal number of relations in \(\pi_1(M, u)\) is a lower bound for the number of index 2 critical points of \(\alpha\).

Moreover, the following example shows that this invariant is non-trivial, and different in nature from the invariants derived from the Novikov Homology:

**Theorem 1.2.** Let \(S\) be the Poincaré homology sphere and \(M = \mathbb{T}^n \# (S \times S^{n-3})\) be the connected sum of a torus and the product of \(S\) with a sphere.

On \(M\), consider the class \(u = \pi_1^*(d\theta_1)\) where \(\pi_1\) is the projection \(M \to \Gamma_1, \mathbb{S}^1\) to the first coordinate \(\theta_1\) on the torus.

Then the Novikov homology associated to the minimal integration cover of \(u\) vanishes in degrees 1 and 2, but the associated Novikov fundamental group \(\pi_1(M, u)\) is non-trivial and has non-trivial relations.

In particular, any Morse 1-form \(\alpha\) in the class \(u\) necessarily has both index 1 and index 2 critical points.

Notice that the Novikov homology associated to the universal cover of \(M\) (i.e. when \(D = \pi_1(M)\)) contains much more information about the fundamental group than that associated to the integration cover. In particular, in this example, \(HN_1(M, u; \pi_1(M))\) does not vanish. However, \(HN_2(M, u; \pi_1(M)) = 0\), and the homology does not predict any index 2 critical points while the Novikov fundamental group does.

Although the construction in this paper seems natural in regard to the construction of the degree 1 Novikov homology as a projective limit of homologies relative to sublevels, as described by J.-C. Sikorav, we are not aware of a similar definition in the literature.

In [Latour1994], Latour defines several spaces that are interesting to compare to the present construction. First, he considers (Latour1994, 5.7, p. 184) the group \(\pi_1^{\text{latour}}(u)\) of loops that can be “slid to \(-\infty\)” in the minimal integration cover. Despite similar notations, this group is somewhat orthogonal to our construction since such loops are automatically trivial in our construction. The closest notion considered by Latour is the \(\pi_0\) of the space of “paths going to \(-\infty\)”. The Novikov fundamental group we define here can be thought of as a Novikov completion of the group generated by this space.
The paper is organized as follows: the first section is this introduction, the second section is devoted to the construction of the Novikov fundamental group and the definition of a suitable notion of generators and relations, the third to the Morse interpretation of it and the proof of theorem 1.1, and finally the last one to the discussion of an example and the proof theorem 1.2.

2. Novikov fundamental group

2.1. Projective limit with respect to sublevels. Let $M$ be a closed smooth manifold, $u \in H^1(M, \mathbb{R})$ a non-trivial cohomology class, and $\alpha \in u$ a closed 1-form in the class $u$.

Let $\tilde{M}$ be the minimal integration cover of $u$ (see Remark 1) and $\mathcal{D} = \pi_1(M)/\pi_1(\tilde{M})$ the associated deck transformation group.

Notice that with this choice of covering space, we have $\pi_1(\tilde{M}) = \ker u$, and $\mathcal{D} = \mathbb{Z}^k$, where $k$ is the irrationality degree of $u$. In particular, $k = 1$ if $u$ is integral, i.e. if $u \in H^1(M, \mathbb{Z})$. But this will not be used in the sequel.

The 1-form $\alpha$ on $\tilde{M}$ is exact, and we pick a primitive $f_\alpha : \tilde{M} \to \mathbb{R}$. For $h \in \mathbb{R}$, we let

$$\tilde{M} \leq h = \{ p \in \tilde{M}, f_\alpha(p) \leq h \}$$

and

$$[\tilde{M}]_h = \tilde{M}/\sim = \tilde{M}/\sim$$

where $\sim$ collapses $\tilde{M} \leq h$ to a point: $p \sim q \iff f_\alpha(p) \leq h$ and $f_\alpha(q) \leq h$. This space comes with a natural base point $*_h$, given by the collapsed sublevel $\tilde{M} \leq h$.

We now consider the family of groups:

$$[\pi_1(f_\alpha)]_h = \pi_1([\tilde{M}]_h, *_h).$$

Inclusions of sublevels induce natural maps for any $h, h' \in \mathbb{R}$ with $h < h'$:

$$[\pi_1(f_\alpha)]_h \xrightarrow{\zeta_h^{h'}} [\pi_1(f_\alpha)]_{h'}$$

Moreover, these maps are compatible with successive inclusions: if $h, h', h'' \in \mathbb{R}$ are such that $h < h' < h''$, then the following diagram is commutative:

$$[\pi_1(f_\alpha)]_h \xrightarrow{\zeta_h^{h'}} [\pi_1(f_\alpha)]_{h'} \xrightarrow{\zeta_{h'}^{h''}} [\pi_1(f_\alpha)]_{h''}.$$  

As a consequence the projective limit when $h$ goes to $-\infty$ is well defined:

**Definition 2.1.** Define the Novikov fundamental group associated to $\alpha$ as the projective limit

$$\pi_1(f_\alpha) = \lim_{h \to -\infty} [\pi_1(f_\alpha)]_h.$$
As a projective limit, this group comes with maps $\zeta^h$ to the “zipped” groups:

$$\pi_1(\alpha) \xrightarrow{\zeta^h} |\pi_1(\alpha)|_h.$$ 

and each element $g$ has a minimal height $h_{f_\alpha}(g)$ with respect to $f_\alpha$, defined as

$$h_{f_\alpha}(g) = \inf\{h \in \mathbb{R}, g \in \ker \zeta^h\}.$$

**Remark 2.** The fact that an element $g$ appears to be trivial above a level $h_0$ does not mean in general that for some other level $h < h_0$, it can be represented by a path that does not go higher than $h_0$.

For instance, seen in $|\pi_1\tilde{M}|_h$, $g$ might appear as $g = aba^{-1}$, where $a$ is a homotopy class that remains non trivial up to a very high level, provided $b$ is a homotopy class in $|\tilde{M}^{|\leq h_0}|_h$.

![Figure 1. An element in $\pi_1(M, u)$ whose representatives above some level does not eventually become constant.](image)

### 2.2. Invariance.

**Theorem 2.2.** The group $\pi_1(f_\alpha)$ only depends on the cohomology class $u$.

More precisely, if $\alpha$ and $\beta$ are two forms in the same cohomology class $u \in H^1(M, \mathbb{R})$, $f_\alpha$ and $f_\beta$ two primitives of $\alpha$ and $\beta$ on $\tilde{M}$, then there is a canonical isomorphism

$$\pi_1(f_\alpha) \sim \pi_1(f_\beta).$$

**Definition 2.3.** The identification of all these groups via these canonical isomorphism defines a group $\pi_1(M, u)$ which we call the Novikov fundamental group associated to $u$.

**Proof.** Since $\alpha - \beta$ is exact on $M$, there is a function $f$ on $M$ such that $f_\beta = f_\alpha + f \circ \pi$, where $\pi : \tilde{M} \to M$ is the projection. Since $M$ is compact, $f$ is bounded and there is a constant $K$ such that

$$\forall p \in \tilde{M}, f_\alpha(p) - K \leq f_\beta(p) \leq f_\alpha(p) + K.$$
As a consequence, for all \( h \in \mathbb{R} \), we have the following inclusions of sublevels
\[
\tilde{M}^{f_\alpha \leq h - K} \subset \tilde{M}^{f_\beta \leq h} \subset \tilde{M}^{f_\alpha \leq h + K}
\]
which induce morphisms among relative fundamental groups, and make the following diagram commutative:
\[
\dot{\cdots} \rightarrow [\pi_1(f_\alpha)]_{h - K} \rightarrow [\pi_1(f_\alpha)]_{h + K} \rightarrow \dot{\cdots}
\]
\[
\dot{\cdots} \rightarrow [\pi_1(f_\beta)]_{h} \rightarrow \dot{\cdots}
\]
which in turn induce an isomorphism on the projective limit of the fundamental groups.

2.3. **Deck transformations.** For any \( x \in \tilde{M} \) and any \( \tau \in D \) we have
\[
(2.1) \quad f_\alpha(\tau \circ x) = f_\alpha(x) + u(\tau),
\]
so that deck transformations send sublevels to sublevels:
\[
\tau \cdot \tilde{M}^{\leq h} = \tilde{M}^{\leq h + u(\tau)}.
\]
Hence \( \tau \) also acts on \( \pi_1(\alpha) \).

As a consequence, for all \( h \in \mathbb{R} \), we have an isomorphism
\[
[\pi_1(\alpha)]_h \xrightarrow{\tau} [\pi_1(\alpha)]_{h + u(\tau)}.
\]
This induces an isomorphism on the projective limit
\[
\pi_1(\alpha) \xrightarrow{\tau} \pi_1(\alpha)
\]
which finally defines an action of \( D \) on \( \pi_1(\alpha) \).

2.4. **Generators and relations.**

2.4.1. **Generators up to deck transformations and completion.** In general, \( \pi_1(M, u) \) need not be finitely generated in the usual sense. The object of this section is to define a suitable notion of generators that takes both deck transformations and projective limits into account, for which \( \pi_1(M, u) \) will be finitely generated.

Given a subset \( A \subset \pi_1(M, u) = \pi_1(f_\alpha) \), consider its orbit \( A^D \) under all possible deck transformations, and the subgroup
\[
< A, D > = < \{ \tau \cdot g, \tau \in D, g \in A \} >
\]
it generates in \( \pi_1(f_\alpha) \). For each \( h \in \mathbb{R} \), the image of this subgroup in \( [\pi_1(f_\alpha)]_h \) defines a group
\[
[< A, D >]_h = \zeta_h(< A, D >) \subset [\pi_1(f_\alpha)]_h
\]
making the following diagrams commutative
\[
[< A, D >]_h \xrightarrow{\zeta_h} [< A, D >]_{h'} \xrightarrow{\zeta_{h'}} [< A, D >]_{h''} .
\]
Definition 2.4. Define the subgroup generated by $A$ up to deck transformations and completion as the group 

$$< A,\mathcal{D} > = \lim_{h \to} < A,\mathcal{D} > |_h.$$ 

Remark 3. Notice that $< A,\mathcal{D} >$ is bigger in general than the subgroup of $< A,\mathcal{D} >$, since the latter involves only (arbitrary long but) finite products of elements of $A^\mathcal{D}$, while the limit process allows for infinite products.

In the sequel, generated subgroups will always be understood as generated up to deck transforms and completion.

Proposition 2.5. The group $< A,\mathcal{D} >$ generated by $A$ in the sense above only depends on the class $\alpha$, and not on the choice of $\alpha$ or $f_\alpha$ used to define it.

Proof. Suppose $\alpha$ and $\beta$ are two 1-forms in the class $\alpha$, $f_\alpha$ and $f_\beta$ two primitives of $\alpha$ and $\beta$ on $\tilde{M}$.

Let $K = \| f_\alpha - f_\beta \|_\infty$. Then the inclusion of sublevels induces, for any level $h \in \mathbb{R}$, the following commutative diagram

$$\ldots \rightarrow < A,\mathcal{D} > |_{f_\alpha \leq h - K} \rightarrow < A,\mathcal{D} > |_{f_\alpha \leq h + K} \rightarrow \ldots$$

which again induces an isomorphism of projective limits. □

2.4.2. Relations up to deck transformations and completion. To define a notion of relation, we need a notion of free group generated by some elements, that still takes the deck transformations and the completion process into account.

Given a finite set $A = \{ g_1, \ldots, g_k \} \subset \pi_1(f_\alpha)$, we consider the product $\mathcal{D} \times A$ and denote by $F_{\mathcal{D} \times A}$ the group freely generated by its elements. Notice it supports an action of the deck transformations group, by multiplication of the letters in a word:

$$\tau \cdot (w_1 \ldots w_k) = (\tau \cdot w_1, \ldots, \tau \cdot w_k)$$

and for each letter $w_i = (\tau_i, g_i)$

$$\tau \cdot (\tau_i, g_i) = (\tau \tau_i, g_i).$$

The notion of height $h_{f_\alpha}(g) = \inf \{ h \in \mathbb{R}, \zeta^h(g) = 1 \in [\pi_1(f_\alpha)]_h \}$ for elements in $\pi_1(f_\alpha)$ extends to $F_{\mathcal{D} \times A}$ at the level of letters by letting

$$h_{f_\alpha}(\tau, g) = u(\tau) + h_{f_\alpha}(g) \quad \text{and} \quad h_{f_\alpha}(w^{-1}) = h_{f_\alpha}(w)$$

and at the level of words by letting

$$h_{f_\alpha}(w_1 \ldots w_k) = \sup \{ h_{f_\alpha}(w_i) \}_{1 \leq i \leq k}$$
Now, for each level \( h \) we select the subset

\[
F_{D \times A}^{\leq h} = \{ w \in F_{D \times A}, h_f(w) \leq h \},
\]

and consider the group \( [F_{D \times A}]_h = F_{D \times A}/F_{D \times A}^{\leq h} \).

Observe now that for \( h < h' < h'' \), we have compatible morphisms

\[
\xrightarrow{\longrightarrow} [F_{D \times A}]_h \xrightarrow{\longrightarrow} [F_{D \times A}]_{h'} \xrightarrow{\longrightarrow} [F_{D \times A}]_{h''}
\]

that consist in letting all the letters whose height is too small to 1.

**Definition 2.6.** Define the group freely generated by \( A \) up to deck transformation and completion as

\[
\overline{F_{D \times A}} = \lim_{\longleftarrow} [F_{D \times A}]_h.
\]

The evaluation of a word in \( \pi_1(f_\alpha) \) then defines a map :

\[
F_{D \times A} \xrightarrow{\longrightarrow} \langle A, D \rangle \subset \pi_1(f_\alpha).
\]

**Definition 2.7.** Define the group \( \mathcal{R}(A) \) of relations associated to a generating family \( A \) of a subgroup \( G \subset \pi_1(f_\alpha) \) as the kernel of

\[
F_{D \times A} \rightarrow \pi_1(f_\alpha).
\]

Again, we have a notion of generators up to deck transformations and completion for the relations, and given a set \( A \), we let

\[
\rho_{DTC}(A) = \inf \{ \sharp B, B \subset \mathcal{R}(A) \text{ with } \mathcal{R}(A) \text{ is generated by } B \} \in \mathbb{R}
\]

**Definition 2.8.** Given a subgroup \( G \subset \pi_1(f_\alpha) \), we let

\[
\mu_{DTC}(G) = \inf \{ \sharp A, A \subset \pi_1(f_\alpha) \text{ such that } G \text{ is generated by } A \}
\]

and

\[
\rho_{DTC}(G) = \inf \{ \rho_{DTC}(A), A \subset \pi_1(f_\alpha) \text{ such that } G \text{ is generated by } A \}.
\]

**Remark 4.** The usual argument shows that the numbers \( \mu_{DTC}(G) \) and \( \rho_{DTC}(G) \) do not depend on the choice of \( \alpha \) nor \( f_\alpha \).

The object of the next section is to prove the following :

**Theorem 2.9.** The group \( \pi_1(M, u) \) is finitely generated in the above sense and for all Morse 1-form \( \alpha \) in the cohomology class \( u \), we have

\[
\sharp(Crit_1(\alpha)) \geq \mu_{DTC}(\pi_1(\alpha)).
\]

Moreover, the relations are also finitely generated in the above sense and

\[
\sharp(Crit_2(\alpha)) \geq \rho_{DTC}(\pi_1(\alpha)).
\]
3. Morse theoretic interpretation

3.1. Morse-Novikov steps as generators. Suppose now that $\alpha$ is a Morse 1-form in the cohomology class $u \in H^1(M, \mathbb{R})$ (with $u \neq 0$).

Pick also

- a metric $\langle , \rangle$ on $M$ such that the pair $(\alpha, \langle , \rangle)$ is Morse-Smale,
- a primitive $f_\alpha$ of $\alpha$ on $\tilde{M}$,
- a preferred lift $\tilde{c}$ to $\tilde{M}$ of each $c \in \text{Crit}(\alpha)$; this allows for the identification $\text{Crit}(f_\alpha) = \{ \tau \cdot \tilde{c}, \tau \in D, c \in \text{Crit}(\alpha) \}$.
- an arbitrary orientation on the unstable manifold of each $c \in \text{Crit}(\alpha)$; this picks a preferred orientation on the unstable manifolds of all the critical points of $f_\alpha$.

For convenience, we suppose $u \neq 0$. This allows us to pick, for each index 0 critical point $x \in \text{Crit}_0(\alpha)$, an arbitrary path $\gamma_{\tilde{x}} : [0, +\infty) \to \tilde{M}$ such that

- $\gamma_{\tilde{x}}(0) = \tilde{x}$
- $\lim_{t \to +\infty} f_\alpha(\gamma_{\tilde{x}}(t)) = -\infty$.

In fact, we can pick a loop $l$ in $M$ based at $\tilde{x}$ such that $u(l) < 0$, and lift the iterations of $l$ to $\tilde{M}$.

Notice that, letting

$$\gamma_{\tau \cdot \tilde{x}} = \tau \cdot \gamma_{\tilde{x}},$$

automatically selects, for each index 0 critical point $x$ of $f_\alpha$, a preferred path $\gamma_x$ in $\tilde{M}$ from $x$ to $-\infty$.

There are finitely many index 0 critical points for $\alpha$, so there is an upper bound $\kappa$ on the height these paths can reach above their starting point:

$$\forall x \in \text{Crit}_0(f_\alpha), \quad \gamma_x \subset \tilde{M} \leq f_\alpha(x) + \kappa$$

To each index 1 critical point $y$ of $f_\alpha$, the unstable manifold of $y$, endowed with its preferred orientation, defines a path $\gamma : (-\infty, +\infty) \to \tilde{M}$ such that near both ends, we have the alternative

$$\lim_{t \to -\infty} f_\alpha(\gamma(t)) = -\infty \quad \text{or} \quad \lim_{t \to -\infty} f_\alpha(\gamma(t)) = x_- \in \text{Crit}_0(f_\alpha)$$

and

$$\lim_{t \to +\infty} f_\alpha(\gamma(t)) = -\infty \quad \text{or} \quad \lim_{t \to +\infty} f_\alpha(\gamma(t)) = x_+ \in \text{Crit}_0(f_\alpha).$$

In the former case, the corresponding flow line is said to be infinite, or unbounded. In the latter case, it is said to be bounded, and a suitable re-parameterisation (using the value of $f_\alpha$ as the new parameter for instance) and concatenation with the path $\gamma_{x \pm}$ turns the flow line into a path going
to $-\infty$. Doing this continuation at one or both ends if needed, we obtain in all cases a continuous path

$$\gamma_y : (-\infty, +\infty) \to \tilde{M} \text{ with } \lim_{t \to \pm \infty} f_\alpha(\gamma_y(t)) = -\infty,$$

which we call the Morse-Novikov step associated to $y$.

![Figure 2. Morse-Novikov steps. Whenever a flow line rooted at $y$ ends at an index 0 critical point $x$, it is prolonged with the path $\gamma_x$.](image)

**Definition 3.1.** Let $G_\alpha$ be the collection of Morse-Novikov steps associated to the index 1 critical points of $\alpha$

$$G_\alpha = \{ \gamma_{\tilde{y}}, y \in \text{Crit}_1(\alpha) \}.$$  

Each such path $\gamma_{\tilde{y}}$ defines a class $g_{\tilde{y}} = [\gamma_{\tilde{y}}] \in \pi_1(f_\alpha)$, and $G_\alpha$ will often be implicitly and abusively considered as a finite subset in $\pi_1(f_\alpha)$, by considering the collection $\{g_{\tilde{y}}\}$ instead of $\{\gamma_{\tilde{y}}\}$.

**Proposition 3.2.** The collection $G_\alpha$ is a finitely generating family (up to deck transformations and completion) for $\pi_1(M, u)$:

$$\pi_1(M, u) = \langle G_\alpha, D \rangle = \langle \{g_{\tilde{y}}, y \in \text{Crit}_1(f_\alpha)\}, D \rangle.$$  

A straightforward corollary of this proposition is the following:

**Corollary 3.3.** For all Morse 1-form $\alpha$ in the class $u$ we have

$$\sharp \text{Crit}_1(\alpha) \geq \mu_{DT}\text{-}C(\pi_1(M, u)).$$  

**Proof.** Fix some regular level $h$ of $f_\alpha$. Let $g \in \pi_1(f_\alpha)$, and $\gamma : [0, 1] \to [\tilde{M}]_{h-h}$ be a path representing $\zeta^{h-h}(g)$ in $[\pi_1(f_\alpha)]_{h-h}$ (recall $\kappa$ is the constant defined in (3.1)).

The path $\gamma$ defines a finite collection of paths $(\gamma_1, \ldots, \gamma_k)$ from $[0, 1]$ to $\tilde{M}$ with ends on $\{f_\alpha = h - \kappa\}$.

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Pushing a component $\gamma_i$ down by the gradient flow of $-f_\alpha$ moves it either below level $h$ or onto unstable manifolds of index 1 critical points. More precisely, there is a time $T$ after which the curve is contained in the union of the sublevel $\tilde{M}^{\leq h}$ and small neighborhoods of the unstable manifolds of some index 1 critical points. A classical retraction argument then turns it into the concatenation $\gamma'_i$ of paths that are (see figure 3)

1. either a path in $\tilde{M}^{\leq h}$,
2. or a piece of unstable manifold of $y \in \text{Crit}_1(f_\alpha)$ whose ends are either in $\tilde{M}^{\leq h}$ or some index 0 critical point $x \in \text{Crit}_0(f_\alpha)$ with $f_\alpha(x) > h - \kappa$. 
Notice that since the initial path \( \gamma_i \) starts and ends in \( \tilde{M}^{\leq h} \), so does \( \gamma'_i \).

To turn \( \gamma'_i \) into a product of Morse-Novikov steps, we finally apply the following modifications at the bottom of each piece of flow line used in \( \gamma'_i \):

1. If the flow line is unbounded, we insert a two way trip down to \(-\infty\) along the flow line.
2. If the flow line is bounded, we insert a two way trip along the remaining part of the flow line if required, and along the preferred path \( \gamma_x \) associated to this critical point down to \(-\infty\).

This operation completes each arc of unstable manifold in \( \gamma'_i \) into the associated Morse-Novikov step.

Each piece \( \eta \) of \( \gamma'_i \) that is contained in \( \tilde{M}^{\leq h} \) is now completed:

- either by a flow line that goes to \(-\infty\), which is necessarily also contained in \( \tilde{M}^{\leq h-\kappa} \),
- or a piece of flow line down to some critical point \( x \in \text{Crit}_0(f_\alpha) \), followed by the path \( \gamma_x \). Since \( x \) has to be on a lower level than the start or end of \( \eta \), we have \( f_\alpha(x) \leq h - \kappa \), and then by definition of \( \kappa \), \( \gamma_x \) cannot go higher than the level \((h - \kappa) + \kappa = h\).

As a consequence we obtain a sequence of paths from and to \(-\infty\), such that

1. each path is either a Morse-Novikov step or lies in \( \tilde{M}^{\leq h-\kappa} \),
2. in \( \lfloor \tilde{M} \rfloor^h \), the concatenation of these paths is homotopic to \( \gamma_i \).

Finally, \( \gamma'_i \) is homotopic in \( \lfloor \tilde{M} \rfloor^h \) the projection of a product of Morse-Novikov steps:

\[
\gamma'_i \sim \gamma_{y_1}^{\pm 1} \cdots \gamma_{y_k}^{\pm 1} \quad \text{in} \quad \lfloor \tilde{M} \rfloor^h
\]

In particular, we obtain that the class \( g \) we started with satisfies

\[
\zeta^h(g) \in \langle \mathfrak{C}_\alpha, \mathcal{D} > \rangle_h.
\]

This proves that \( \forall h \in \mathbb{R}, \lfloor \pi_1(f_\alpha) \rfloor_h \subset \langle \mathfrak{C}_\alpha, \mathcal{D} > \rangle_h \) and hence

\[
\pi_1(f_\alpha) = \langle \mathfrak{C}_\alpha, \mathcal{D} > \rangle.
\]

\[\square\]

### 3.2. Morse-Novikov relations

Similarly, consider an index 2 critical point \( z \in \text{Crit}_2(f_\alpha) \).

Given a level \( h \) with \( h < f_\alpha(z) \), the set \( W^u(z) \cap \{ f_\alpha > h \} \) is a topological disc (notice that once a trajectory enters the sublevel \( \{ f_\alpha \leq h \} \), it will never exit it anymore). Moreover, starting with a small circle around \( z \) inside its unstable manifold, and pushing it down by the flow using the technique described earlier, we obtain a loop \( \rho_{z,h} \) whose projection in \( \lfloor \tilde{M} \rfloor \) is that of a product of Morse-Novikov steps:

\[
\rho_{z,h} \sim \gamma_{y_1}^{\pm 1} \cdots \gamma_{y_k}^{\pm 1} \quad \text{in} \quad \lfloor \tilde{M} \rfloor_h.
\]

In particular, considering the group \( F = \mathcal{F}_{\mathcal{D} \times \text{Crit}_1(\alpha)} \) freely generated by the index 1 critical points of \( \alpha \) (up to deck transformations and completion),
the sequence \((y_1^{±1}, \ldots, y_k^{±1})\), after removal of the eventual critical points \(y\) such that \(h - \kappa < f_\alpha(y) \leq h\), defines a word

\[ w_{z,h} = y_1^{±1} \cdots y_k^{±1} \in [F]_h. \]

Moreover, these words are compatible with inclusion of sublevels, namely, for \(h < h'\), we have :

\[ \zeta_{h'}^h(w_{z,h}) = w_{z,h'}. \]

As a consequence, the words \((w_{z,h})_h\) define a class \(w_z\) in \(F\).

**Definition 3.4.** Define the relation associated to a critical point \(\bar{z} \in \text{Crit}_2(\alpha)\) as the word \(\tilde{w}_{\bar{z}}\), and let

\[ R_\alpha = \{ \tilde{w}_{\bar{z}}, \bar{z} \in \text{Crit}_2(\alpha) \}. \]

**Proposition 3.5.** The relations associated to the generating family given by the index 1 critical points of \(\alpha\) is spanned by the relations associated to the index 2 critical points :

\[ R(G_\alpha) = \langle R_\alpha, D \rangle. \]

A straightforward corollary of this proposition is the following :

**Corollary 3.6.** For all Morse 1-form \(\alpha\) in the class \(u\) we have

\[ \#\text{Crit}_2(\alpha) \geq \rho_{DTC}(\pi_1(M, u)). \]

**Proof.** By construction, the elements in \(\langle R_\alpha, D \rangle\) are indeed relations. On the other hand, let \(w \in F = \mathbb{F}_{D \times \sigma_\alpha}\) and suppose \(w\) evaluates to 1 in all \([\pi_1(f_\alpha)]_h\) (via the map defined in (2.2)).

Fix a level \(h\), and consider \(w_h = \zeta_h^h(w) \in [F]_h\) and \(\gamma_h\) its evaluation in \([\hat{M}]_h\). Since \([\gamma_h] = 1\) in \([\pi_1(f_\alpha)]_h\), there is a disc \(\delta : D^2 \to [\hat{M}]_h\) with boundary on \(\gamma_h\).

Pushing this disc down by the flow, we obtain a new disc \(\delta' : D^2 \to [\hat{M}]_h\), that has the same boundary (since it did already consist of flow lines) and splits as a union of unstable manifolds of index 2 critical points (seen in \([\hat{M}]_h\)), and regions where it evaluates to the base point \([\hat{M}^{\leq h}]\).

In particular, we obtain that \(\gamma_h\) can be written in \([\hat{M}]_h\) as a product of relations associated to index 2 critical points. \(\square\)

### 4. Example

Let \(S\) denote the Poincaré homology sphere, and consider the manifold

\[ M = \mathbb{T}^{n+3}(S \times S^n) \]

obtained as the connected sum of a torus and \(S \times S^n\).

For convenience, the fundamental group of \(S\) will be denoted by \(G:\)

\[ G = \pi_1(S) = \langle a, b | a^5 = b^3, a^5 = (ab)^2 \rangle. \]
Let $T^{n+3} \xrightarrow{\theta} S^1$ be the first coordinate on the torus, and consider the cohomology class $u = \pi^* d\theta$

where $\pi : M \rightarrow T^{n+3}$ is a projection mapping $S$ to a point.

The integration cover of $u$ is then the manifold

$$(\mathbb{R} \times T^{n+2}) \# (S \times S^n)$$

obtained by performing the connected sum of the cylinder with a fresh copy of $S \times S^n$ at each lift of the point where the connected sum took place in $M$.

We denote by $H_{N*}(M, u)$ the Novikov homology associated to this covering of $M$. Algebraically, this comes down to considering the group $\mathcal{D} = \pi_1(M) / \ker u \simeq \mathbb{Z}$ as defining a local coefficients system, and use coefficients in the Novikov completion $\Lambda = \mathbb{Z}(\mathbb{t})$ of the group ring $\mathbb{Z}[\mathcal{D}] = \mathbb{Z}[\mathbb{t}]$. Namely,

$$\Lambda = \{ \sum_{-N}^{+\infty} a_n t^n \}$$

is the set of Laurent power series over $\mathbb{Z}$, and we consider the complex

$$CN_* (M, u) = \Lambda \otimes_{\mathbb{Z}[\mathbb{t}]} C_*(M, \mathbb{Z}[\mathbb{t}])$$

where $C_*(M, \mathbb{Z}[\mathbb{t}])$ is, for instance, the simplicial chain complex of $M$ with local coefficients in $\mathbb{Z}[\mathcal{D}] = \mathbb{Z}[\mathbb{t}]$.

Finally, the Novikov homology $H_{N*}(M, u)$ is the homology of $CN_*$.

**Proposition 4.1.** With the notations above, we have

$$H_{N0}(M, u) = 0, \quad H_{N1}(M, u) = 0, \quad H_{N2}(M, u) = 0.$$

**Proof.** Vanishing of $H_{N0}(M, u) = 0$ whenever $u \neq 0$ is a classical and general feature of Novikov homology.

To compute $H_{Ni}(M, u) (i = 1, 2)$ we use the following formula using the universal coefficient formula and the flatness of the Novikov ring $\mathbb{Z}((t))$ as $\mathbb{Z}[\mathbb{t}]$ module [Pajitnov2006, Theorem 1.8, p. 339], cf. [LO1995, Appendix C]

$$(4.1) \quad H_{Ni}(M, u) = H_i(\tilde{M}) \otimes_{\mathbb{Z}[\mathbb{t}]} \mathbb{Z}((t)).$$

By the Mayer-Vietoris sequence theorem we have for $i = 1, 2$

$$H_i(\tilde{M}) = H_i(\mathbb{R} \times T^{n+2} \setminus \{p_k, \ k \in \mathbb{Z}\}) \simeq H_i(\mathbb{R} \times T^{n+2}),$$

where the $p_k$ are the lifts of the points where the connected sum was performed.

Since $\mathcal{D} = \mathbb{Z}[\mathbb{t}]$ acts on $\mathbb{R} \times T^{n+2} \setminus \{p_i, i \in \mathbb{Z}\}$ by translation along the $\mathbb{R}$ factor, the induced action on $H_i(\tilde{M})$ is trivial.

In particular, for all $x \in H_i(\tilde{M}) (i = 1, 2)$, we have

$$(1 - t)x = 0.$$

Since $(1 - t)$ is invertible in $\Lambda$, we conclude that $x = 0$, and $H_{Ni}(M, u) = 0$. □
In particular, the Novikov homology associated to the integration covering does not predict any index 1 and 2 critical points.

On the other hand, the Novikov fundamental group is non trivial and appears as a completion of the free product of infinitely many copies of $G$. More precisely, consider the infinite sequence of groups $(G_k)_{k \in \mathbb{Z}}$ where all the groups $G_k$ are copies of $G$, and define, for each $h \in \mathbb{Z}$ the group
\[ \Pi_h = \ast_{k \geq h} G_k. \]
Given two integers $h, h'$ with $h < h'$, we have
\[ \Pi_{h'} = \Pi_h/(G_k = 1, h \leq k < h') \]
and hence there are projections
\[ (4.2) \quad \zeta_h^{h'} : \Pi_h \rightarrow \Pi_{h'}, \]
that are compatible in that for $h < h' < h''$, we have:
\[ \zeta_h^{h''} = \zeta_h^{h'} \circ \zeta_h^{h'} \]
Define then
\[ \Pi = \lim_{\leftarrow} \Pi_h. \]

**Theorem 4.2.** With the above notations, $\pi_1(M, u) = \Pi$.

**Proof.** Pick a 1 form $\alpha$ that “separates” the Poincaré spheres, i.e. such that a primitive $f_\alpha$ has a level $h_0$, that can be supposed to be 0, that does not touch any lift of $S \times S^n$ and the region where the connected sum takes place.

Then for any $h \in \mathbb{Z}$, we have
\[ \pi_1([\tilde{M}]_h) = \ast_{k \geq h} \pi_1(S \times S^n) = \ast_{k \geq h} \pi_1(S). \]
In other words, $[\pi_1(f_\alpha)]_h = \Pi_h$, and since the projections $[\tilde{M}]_h \rightarrow [\tilde{M}]_{h'}$ given by inclusion of sub levels induce the same maps $\zeta_h^{h'}$ as in \( \text{(4.2)} \), we derive $\pi_1(M, u) = \Pi$. $\square$

**Remark 5.** For $h \in \mathbb{Z}$, $[\tilde{M}]_h$ has the same fundamental group as the wedge
\[ \bigwedge_{k \in \mathbb{Z}, k \geq h} (S \times S^n)_k \]
of infinitely many copies of $S \times S^n$. From this point of view, the projections $\zeta_h^{h'}$ are then obtained by collapsing the spheres with indices $h \leq k < h'$ to the base point.

**Remark 6.** The deck transformations act on $\Pi$ by shifting the indices $G_k \xrightarrow{\text{Id}} G_{k+k_0}$. To be more explicit, given a deck transformation $\tau$ with $k_0 = u(\tau) \in \mathbb{Z}$, the morphisms
\[ \Pi_h \xrightarrow{\tau} \Pi_{h+k_0} \]
obtained by considering that a letter in a group \( G_k \) belongs to \( G_{k+k_0} \) are compatible with the inclusions of sublevels hence induce an isomorphism

\[
\Pi \xrightarrow{T} \Pi
\]

which is the action of \( D = \mathbb{Z} \) on \( \Pi \).

Since \( \Pi \) is non trivial, we have \( \mu_{DT}(\Pi) > 0 \). Similarly, since \( \Pi \) contains finite order elements (\( a_0^0 = b_0^6 = 1 \)), it has to have non trivial relations, and \( \rho_{DT}(\Pi) > 0 \).

\textbf{Remark 7.} One can expect \( \mu_{DT}(\Pi) = 2 \) and \( \rho_{DT}(\Pi) = 3 \), but these minimal numbers are not as straightforward to compute as one could expect (for instance, if the free product is replaced by the cartesian product, it is not hard to see that the corresponding group \( G^Z \) is generated, \textit{up to shift}, by only one element).

\textbf{Corollary 4.3.} Any Morse 1 form in the class \( u = \pi^*[d\theta] \) has to have both index 1 and index 2 critical points.

This proves that the lower bounds \((3.3)\) and \((3.4)\) derived from Novikov fundamental group are essentially different from the Novikov inequalities derived from the Novikov homology associated to the minimal integration cover.

It turns out that the Novikov homology associated to the universal cover, also called the Novikov-Sikorav homology \([\text{Sikorav1987, Sikorav2016}]\) and which we denote here by \( H\tilde{N}_*(M, u) \), contains much more information about the fundamental group. In particular, it is not hard to see that \( H\tilde{N}_1(M, u) \neq 0 \) in this example.

However, the second homology group does not detect the relations in the fundamental group and in the example under consideration, we have:

\textbf{Claim 4.4.} \( H\tilde{N}_2(M, u) = 0 \),

In particular, since the Novikov fundamental group does have non trivial relations in this example, this shows that the Novikov fundamental group is an invariant that is not recovered by the Novikov homology, even in its version associated to the universal cover.

\textbf{Proof.} Recall the universal cover of the Poincaré sphere is \( \tilde{S} = S^3 \), so that the universal cover \( \tilde{M} \) of \( M \) is a connected sum of (infinitely many) copies of \( S^3 \times S^n \) and \( \mathbb{R}^{3+n} \).

Let \( A = S^{2+n} \times [0,1] \) be the annulus introduced when performing the connected sum. We can choose a cell decomposition of \( M \) such that no 2-cells enter the region \( A \) (do the connected sum at the center of two \( 3+n \)-cells and decompose the annulus \( A \) in a standard way, using only 0, 1, 2 + n and \( 3 + n \) cells, the 0 and 2 + n ones being on the boundary). In particular, the 2-cells split into two disjoint families, \( \{\sigma_i\} \) and \( \{\tau_j\} \), the first lying in \( S \times S^n \), the second in \( \mathbb{T}^{3+n} \).
We pick a preferred lift $\tilde{\sigma}_i$ and $\tilde{\tau}_j$ of each 2-cell in $M$ to its universal cover $\tilde{M}$, so that the collection of 2 cells is now $\{g \cdot \tilde{\sigma}_i\}_{g,i} \cup \{g \cdot \tilde{\tau}_j\}_{g,j}$, and they split into disjoint families according to the components of $\tilde{M}$:

$$\tilde{M} \setminus \{g \cdot A\}_{g \in \pi_1(M)} = \bigsqcup_{\lambda} S_{\lambda} \cup \bigsqcup_{\mu} P_{\mu}$$

where each $S_{\lambda}$ is a copy of the complement of a finite set of points in $S^3 \times S^n$, and each $P_{\mu}$ a copy of the complement of the lattice $Z^{3+n}$ in $R^{3+n}$.

Finally, we pick a 1-form $\alpha$ in the cohomology class $u$ such that the integral levels of a primitive $f_\alpha$ of it on $\tilde{M}$ separates the $S_{\lambda}$:

$$\forall \lambda, \exists n \in \mathbb{Z}, S_{\lambda} \subset \{n < f_\alpha < n + 1\}.$$  \hfill (4.3)

Consider now a Novikov 2-cycle $s$ (using the cellular version of the Novikov complex associated to our cell decomposition of $M$). It can be written as

$$s = \sum_{g,i} n_{g,i} g \cdot \tilde{\sigma}_i + \sum_{g,j} m_{g,j} g \cdot \tilde{\tau}_j$$

such that, $\forall A \in \mathbb{R}$ :

$$\sharp\{(g, i), n_{g,i} \neq 0 \text{ and } u(g) > A\} < +\infty$$  \hfill (4.4)

and

$$\sharp\{(g, j), m_{g,j} \neq 0 \text{ and } u(g) > A\} < +\infty$$  \hfill (4.5)

Gathering the cells according to the components $S_{\lambda}$ and $P_{\mu}$ we write $s$ as

$$s = \sum_{\lambda} s_{\lambda} + \sum_{\mu} s_{\mu}$$

with :

$$s_{\lambda} = \sum_{(g,i)_{\lambda}} n_{g,i} g \cdot \tilde{\sigma}_i \quad \text{and} \quad s_{\mu} = \sum_{(g,j)_{\mu}} m_{g,j} g \cdot \tilde{\tau}_j$$

where $(g,i)_{\lambda}$ denotes the collection of $(g, i)$ such that $g \cdot \tilde{\sigma}_i \subset S_{\lambda}$, and $(g,j)_{\mu}$ that for which $g \cdot \tilde{\tau}_i \subset P_{\mu}$. The condition $\partial s = 0$ now implies that for each $\lambda$ and $\mu$ we have :

$$\partial s_{\lambda} = 0 \quad \text{and} \quad \partial s_{\mu} = 0.$$  \hfill (4.4)

Moreover, the finiteness condition (4.4) and the bounds (4.3) of $f$ on $S_{\lambda}$ imply that the sum in each $s_{\lambda}$ is finite. In particular, $s_{\lambda}$ is a cycle in $S_{\lambda}$, and since $H_2(S_{\lambda}) = 0$, $s_{\lambda}$ is a boundary :

$$\exists v_{\lambda} \in C_3(S_{\lambda}) \subset C_3(\tilde{M}), \partial v_{\lambda} = s_{\lambda}.$$  \hfill (4.6)

Now, the conditions (4.4) and (4.3) also imply that above any level $A$, there are only finitely many $\lambda$ such that $s_{\lambda} \neq 0$.

As a consequence, the sum $v = \sum_{\lambda} v_{\lambda}$ is a Novikov chain in $\tilde{M}$, and it satisfies

$$\partial v = \sum_{\lambda} s_{\lambda}$$

In particular, the homology class of $s$ is that of $s' = \sum_{\mu} s_{\mu}$.
On the other hand, denote by $t_1 \in \pi_1(M)$ the loop associated to the first coordinate in the torus. It acts on each $P_\mu$ as the translation along $(1,0,\ldots,0)$, and any Novikov 2-cycle $\tau$ in $P_\mu$ is homologous to $t_1 \cdot \tau$.

As a consequence, for all $\mu$:

$$t_1 \cdot [s_\mu] = [s_\mu],$$

(where the brackets denote the homology class). We conclude that $t_1 \cdot [s] = [s]$, and since $(1 - t_1)$ is invertible, $[s] = 0$. □

**Acknowledgement.** HVL thanks JFB and Nguyên Tiên Zung, Lê Ngọc Mai for their invitation to Toulouse and their hospitality during her visit in May 2016, when she and JFB started this project aiming to develop ideas in [Barraud2014] further for the Novikov case.

AG thanks JFB and CIMI for the invitation to Toulouse in June 2017, when she joined the project, and the Hausdorff Research Institute for Mathematics (HIM), University of Bonn, for its support and hospitality.

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