DISTRIBUTIONAL POINT VALUES AND DELTA SEQUENCES

RICARDO ESTRADA AND KEVIN KELLINSKY-GONZALEZ

Abstract. Recently Sasane [18] defined a notion of evaluating a distribution at a point using delta sequences. In this paper, we explore the relationship between generalizations of his definition and the standard definition of distributional point values. This allows us to obtain a description of distributional point values via delta sequences and a characterization of when a distribution is actually a regular distribution given by bounded function. We also give a characterization of limits in a continuous variable by the existence of the limits of certain sequences.

1. Introduction

Distributional point values were first defined in one variable by Łojasiewicz [11]. His definition is given as a distributional limit over a continuous variable. In other words, if \( f \in \mathcal{D}'(\mathbb{R}) \) and \( x_0 \in \mathbb{R} \) then we say that \( f \) has a distributional point value, equal to \( \gamma \), at \( x_0 \) if

\[
\lim_{\varepsilon \to 0} f(x_0 + \varepsilon x) = \gamma,
\]

in the distributional sense, that is, if for all test functions \( \phi \in \mathcal{D}(\mathbb{R}) \) we have that

\[
\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi(x) \rangle = \gamma \int_{-\infty}^{\infty} \phi(x) \, dx.
\]

Similarly [12] point values in several variables are defined as a distributional limit over a continuous variable. Point values have been studied extensively and are the first step in the study of distributional asymptotic analysis and of the study of local properties of distributions [2, 5, 15, 16, 17, 23].

It is possible to find in the literature other definitions of distributional point values, based on the use of delta sequences. For instance, in a recent study, Sasane [18] uses the alternative definition “\( f(x_0) = \eta \)” if for all positive and even test functions \( \phi \) with \( \int_{-\infty}^{\infty} \phi(x) \, dx = 1 \) one has

\[
\lim_{n \to \infty} \langle f(x_0 + x), \phi_n(x) \rangle = \eta,
\]

where \( \{\phi_n\}_{n=1}^{\infty} \) is the standard delta sequence generated by \( \phi \), namely, \( \phi_n(x) = n\phi(nx) \).

Naturally the question arises if the two definitions are equivalent. More generally, if \( \mathcal{F} \) is a family of test functions, we would like to consider the relationship between the existence of the distributional point value and the existence of the limit (1.3) whenever the sequence \( \{\phi_n\}_{n=1}^{\infty} \) belongs to \( \mathcal{F} \). Interestingly, the two definitions are not equivalent for many classes of delta sequences, in particular for the family considered in [18]. Nevertheless, we are able to show that for some classes they are actually equivalent.

2010 Mathematics Subject Classification. 46F10.

Key words and phrases. Distributional point values, delta sequences.
In order to study this problem, we start by studying a very general question about limits. Indeed, in a metric space $X$, given a function $f : X \setminus \{x_0\} \to \mathbb{R}$, then the limit
\begin{equation}
\lim_{x \to x_0} f(x) = L,
\end{equation}
events if and only if
\begin{equation}
\lim_{n \to \infty} f(x_n) = L,
\end{equation}
for all sequences $\{x_n\}_{n=1}^\infty$ in $X \setminus \{x_0\}$ that converge to $x_0$. The question we would like to consider is whether the existence of the limit $\lim_{n \to \infty} f(x_n)$ for sequences $\{x_n\}_{n=1}^\infty$ of a certain family implies that (1.4) is satisfied. In Section 3, we consider the case where $X = (0, \infty)$, $x_0 = \infty$ and $f$ continuous, showing that in such cases the existence of the limit
\begin{equation}
\lim_{n \to \infty} f(na) = F(a),
\end{equation}
for all $a > 0$ implies that, in fact, $F$ is a constant function, $F(a) = L$, for all $a > 0$, and that $\lim_{x \to \infty} f(x) = L$. We give examples of other families of sequences for which $\lim_{x \to \infty} f(x) = L$ might not hold true. We are also able to present, in Section 4, a corresponding result when the function $f$ in (1.6) is not necessarily continuous but just measurable and the limit holds almost everywhere.

The plan of the rest of the article is as follows. Basic results on distributional point values are briefly discussed in Section 2 while delta sequences are considered in Section 5. Section 6 gives several useful results on the characterization of distributions and functions using normalized positive test functions. The main results are given in Section 7 where we give equivalent conditions to the existence of point values obtained from a given family of delta sequences. In particular, we prove that the existence of the distributional point value is equivalent to the existence of the point value for the family of standard delta sequences generated by a positive normalized test function. Then we show that for the family employed in [18] the equivalence is the existence of the symmetric distributional point value. We also consider radial delta sequences in several variables, and finish by studying the family of all delta sequences of normalized positive test functions.

2. Preliminaries

We refer to the texts for the basic ideas about distributions [11, 8, 19, 22]. Ideas on the local behavior of distributions can be found in [2, 5, 16, 17, 23]. In this article, we will work mainly in the space $\mathcal{D}'(\mathbb{R}^d)$ of distributions on $\mathbb{R}^d$, dual of the space $\mathcal{D}(\mathbb{R}^d)$ of standard test functions, that is, $C^\infty$ functions with compact support, with its inductive limit topology [20].

If $f \in \mathcal{D}'(\mathbb{R})$ and $x_0 \in \mathbb{R}$ then [11] we say that $f$ has a distributional point value, equal to $\gamma$, at $x_0$ if $\lim_{\varepsilon \to 0} f(x_0 + \varepsilon x) = \gamma$, in the strong topology of $\mathcal{D}'(\mathbb{R})$. Equivalently, since a sequence of distributions converges strongly if and only if it converges weakly [20], if for all test functions $\phi \in \mathcal{D}(\mathbb{R})$ we have that
\begin{equation}
\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi(x) \rangle = \gamma \int_{-\infty}^{\infty} \phi(x) \, dx.
\end{equation}
Interestingly, the existence of the distributional limit \( \lim_{\varepsilon \to 0^+} f(x_0 + \varepsilon x) \) implies that this limit is a constant and that the point value exists. On the other hand, if the limit
\[
\lim_{\varepsilon \to 0^+} f(x_0 + \varepsilon x) = g(x),
\]
equals, then \( g \) does not have to be a constant, but it will have the jump behavior \( [21] \), that is, \( g \) is of the form
\[
g(x) = \gamma_- H(-x) + \gamma_+ H(x),
\]
where \( H \) is the Heaviside function and \( \gamma_{\pm} \) are some constants. Distributions of the form \( (2.3) \) are the most general homogeneous distributions of degree 0 in one variable. Alternatively, \( (2.2) \) and \( (2.3) \) hold if and if the lateral limits \( f(x_0 + 0) = \lim_{\varepsilon \to 0^+} f(x_0 + \varepsilon x) = \gamma_{\pm} \), exist in \( \mathcal{D}'(0, \infty) \) and \( f \) does not have delta functions at \( x_0 \).

In several variables, point values are defined similarly \( [12] \), namely, if \( f \in \mathcal{D}'(\mathbb{R}^d) \), then the distributional point value \( f(x_0) \) exists and equals \( \gamma \) if \( \lim_{\varepsilon \to 0} f(x_0 + \varepsilon x) = \gamma \), distributionally. In several variables, the limit \( \lim_{\varepsilon \to 0} f(x_0 + \varepsilon x) \) could exist without being a constant. In fact, if
\[
\lim_{\varepsilon \to 0^+} f(x_0 + \varepsilon x) = g(x),
\]
then \( g \) is homogeneous of degree 0. Homogeneous distributions of degree zero are given by a formula of the type
\[
\langle g(x), \phi(x) \rangle = \int_0^\infty \langle \alpha(w), \phi(rw) \rangle_{\mathcal{D}'(\mathbb{S})} r^{d-1} dr,
\]
for a certain distribution \( \alpha \in \mathcal{D}'(\mathbb{S}) \) \( [3] \) Thm. 2.6.2]. The distribution \( \alpha \) is the thick distributional value \( [3] \) of \( f \) at \( x_0 \), namely, \( f \) has no delta functions at \( x_0 \) and \( \alpha \) is the thick limit
\[
\lim_{\varepsilon \to 0^+} f(x_0 + r\varepsilon w) = \alpha(w),
\]
in the space \( \mathcal{D}'((0, \infty), \mathcal{D}'(\mathbb{S})) \), that is, for all \( \rho \in \mathcal{D}'(0, \infty) \),
\[
\langle \lim_{\varepsilon \to 0^+} f(x_0 + r\varepsilon w), \rho(r) \rangle_{\mathcal{D}'(0,\infty) \times \mathcal{D}(0,\infty)} = \left( \int_0^\infty \rho(r) \ dr \right) \alpha(w),
\]

3. The continuous case

We start with a general known result that will be useful in our analysis.

**Proposition 3.1.** Let \( f : (0, \infty) \to \mathbb{R} \) be continuous. Suppose that for each \( a > 0 \) the sequence \( \{f(an)\}_{n=1}^\infty \) converges, to \( F(a) \). Then \( F(a) \) does not depend on \( a \), that is,
\[
F(a) = L \text{ for all } a > 0,
\]
for some \( L \), and actually
\[
\lim_{x \to \infty} f(x) = L.
\]

**Proof.** Clearly the function \( F \) is constant in each class of the quotient space \( \mathbb{R}/\mathbb{Q} \), \( F(ra) = F(a) \) if \( r \in \mathbb{Q} \). Also, \( F \) is continuous or of the first Baire class \( [9] \) \( [13] \), so that the set \( D \) of points of continuity of \( F \) is dense in \( (0, \infty) \). Let \( \alpha \in D \). Let \( b > 0 \). If \( \varepsilon > 0 \) then there
exists $\delta > 0$ such that $|a - \alpha| < \delta$ implies $|F(a) - F(\alpha)| < \varepsilon$ and there exist $r$ rational such that $|rb - \alpha| < \delta$. Therefore

$$\int_{(3.3)} |F(b) - F(\alpha)| = |F(rb) - F(\alpha)| < \varepsilon,$$

and since $\varepsilon$ is arbitrary, $F(b) = F(\alpha)$.

In order to prove (3.2), observe that if $\varepsilon > 0$, then for each $a \in [1, 2]$ there exists $n_0 = n_0(a)$ such that $|f(ka) - L| < \varepsilon$ for $k \geq n_0(a)$. This means that

$$\int_{(3.4)} [1, 2] = \bigcap_{n=1}^{\infty} \bigcap_{k \geq n} \{a \in [1, 2] : |f(ka) - L| < \varepsilon\}.$$

Therefore there exists $n_0$ such that $\bigcap_{k \geq n_0} \{a \in [1, 2] : |f(ka) - L| < \varepsilon\}$ contains an interval $I = [\alpha, \beta]$ with $\alpha \neq \beta$. Observe now that $\bigcup_{k=n_0}^{\infty} kI$ contains a ray $(B, \infty)$, since in fact $\bigcup_{k=n_0}^{\infty} kI$ is a closed ray if $n_1 > 1/(\beta - \alpha)$. Hence if $x > B$ then $x = ka$ for some $k \geq n_0$ and some $a \in I$ and, consequently, $|f(x) - L| = |f(ka) - L| < \varepsilon$. □

It is interesting that there are sequences $\{\xi_n\}_{n=1}^{\infty}$ with $\lim_{n \to \infty} \xi_n = \infty$ such that for some continuous functions $f : (0, \infty) \to \mathbb{R}$ the limit

$$\int_{(3.5)} \lim_{n \to \infty} f(a\xi_n) = G(a)$$

exists for all $a > 0$, but the function $G$ is not constant. Indeed, let $f(x) = \sin(2\pi \ln x)$ and $\xi_n = e^{(n+1/n)}$. The limit $\lim_{n \to \infty} f(a\xi_n) = \sin(2\pi \ln a)$, exists but it is not constant, of course.

4. The measurable case

We shall now consider an extension of the results of Section 3 to measurable functions.

**Proposition 4.1.** Suppose $f : (0, \infty) \to \mathbb{R}$ is measurable. For $a > 0$, suppose that the function $F$ defined by

$$\int_{(4.1)} F(a) = \lim_{n \to \infty} f(an),$$

is well-defined almost everywhere. Then $F$ is constant almost everywhere.

**Proof.** Let us first suppose that $f \in L^\infty(0, \infty)$. Let $\phi \in \mathcal{D}(0, \infty)$ be a test function. For $\lambda > 0$, let us define

$$\int_{(4.2)} G(\lambda) = \int_{0}^{\infty} f(\lambda x)\phi(x) \, dx.$$ 

The function $G$ is continuous because $\phi \in \mathcal{D}(0, \infty)$. For a fixed $\lambda$, let us consider the sequence $\{G(\lambda n)\}_{n=1}^{\infty}$. Since $f$ is bounded, we can apply the dominated convergence theorem to see that

$$\int_{(4.3)} \lim_{n \to \infty} G(\lambda n) = \int_{0}^{\infty} \lim_{n \to \infty} f(\lambda nx)\phi(x) \, dx = \int_{0}^{\infty} F(\lambda x)\phi(x) \, dx,$$

exists. The Proposition 3.1 then yields that $\int_{0}^{\infty} F(\lambda x)\phi(x) \, dx$ does not depend on $\lambda$,

$$\int_{0}^{\infty} F(\lambda x)\phi(x) \, dx = \int_{0}^{\infty} F(x)\phi(x) \, dx.$$ 

Therefore, the regular distribution $F$ is constant since $F(\lambda x) = F(x)$, $\lambda > 0$, and only the constants are homogeneous of degree 0 in the interval $(0, \infty)$ [5], that is, $F(x) = C$, ...
as distributions. Notice now that the locally integrable function that gives a regular distribution is unique almost everywhere, so that \( F(x) = C \) (a.e.).

Let us now consider the case of a general measurable function \( f \) for which the limit \( \lim_{n \to \infty} f(an) = F(a) \) exists (a.e.). We can then define the bounded function
\[
(4.4) \quad h(x) = \arctan f(x).
\]
Then \( \lim_{n \to \infty} h(an) = \arctan F(a) \) exists (a.e.). Consequently, \( \arctan F(a) \) is constant, and therefore so is \( F(a) \).

In Proposition 3.1, it is shown that in the continuous case not only is \( F(a) = L \) for all \( a > 0 \), but actually \( \lim_{x \to \infty} f(x) = L \). This is no longer true in the measurable case; for example, if \( f = \chi_B \), the characteristic function of a set \( B \) of measure zero such that \( B \cap (x, \infty) \neq \emptyset \) for all \( x > 0 \), then \( \lim_{x \to \infty} f(x) \) does not exist. Of course, this function \( f \) is equal almost everywhere to a function \( \tilde{f} \), the zero function, for which \( \lim_{x \to \infty} \tilde{f}(x) \) exists. An example where (4.1) exists for all \( a > 0 \) but \( \lim_{x \to \infty} \tilde{f}(x) \) does not exist for any function \( f \) such that \( f(x) = \tilde{f}(x) \) (a.e.) can be constructed as follows. The strategy will be to construct an unbounded set, \( A \), with measure 0 such that \( f(x) \neq 0 \) for infinitely many \( x \in A \).

**Example 4.2.** Let \( \{N_k\}_{k=1}^{\infty} \) be a sequence of positive integers such that
\[
(4.5) \quad kN_k < N_{k+1}.
\]
For each \( k \) let us choose a non empty open interval \( B_k \subset (N_k - 1, N_k) \). Then if \( j \in \mathbb{N} \), \( j \left( \frac{1}{k}, 1 \right) \cap B_k \neq \emptyset \) only if \( N_k \leq j < N_{k+1} \). Therefore, if \( x \in \left( \frac{1}{k}, 1 \right) \), then
\[
(4.6) \quad \chi_{B_k}(jx) = 0 \text{ for all } j \in \mathbb{N}, \; x \notin A_k,
\]
where \( A_k = \bigcup_{j=N_k}^{N_k+1} \frac{1}{j}B_k \). Let now \( \{\eta_k\}_{k=1}^{\infty} \) be a sequence of strictly positive numbers such that the series \( \sum_{k=1}^{\infty} \eta_k \) converges and let us further restrict the sets \( B_k \) by requiring that \( \mu(A_k) < \eta_k \), for all \( k \). Let \( A = \limsup_{k \to \infty} A_k = \bigcap_{k=1}^{\infty} \bigcup_{q=k}^{\infty} A_q \). Then \( \mu(A) = 0 \).

Let us now define the function \( f : (0, \infty) \to \mathbb{R} \) by
\[
(4.7) \quad f(x) = \sum_{k=1}^{\infty} \chi_{B_k}(x).
\]
If \( \tilde{f}(x) = f(x) \) almost everywhere, then the limit \( \lim_{x \to \infty} \tilde{f}(x) \) does not exist. On the other hand, \( \lim_{n \to \infty} f(nx) \) exists almost everywhere. Indeed, it is enough to show the existence almost everywhere in \( (0, 1) \), and the limit of \( f(nx) \) exists and equals 0 if \( x \in (0, 1) \setminus A \) since if \( x \notin A \) then there exists \( k_0 \) such that \( x \notin A_k \) for \( k \geq k_0 \) and consequently, \( f(nx) = 0 \) whenever \( n \geq N_{k_0} \).

An example involving continuous functions can be obtained by a slight modification.

**Example 4.3.** Let \( g \) be a continuous function in \( (0, \infty) \) such that \( 0 \leq g(x) \leq f(x) \) and such that there exist points \( \xi_k \in B_k \), for all \( k \), such that \( g(\xi_k) = 1 \). Then \( \lim_{n \to \infty} g(nx) = 0 \) almost everywhere, but not everywhere, since \( \lim_{x \to \infty} g(x) \) does not exist.
The examples show that it is possible for \( \lim_{n \to \infty} f(an) \) to be equal to a constant \( L \) almost everywhere but without \( \lim_{x \to \infty} f(x) \) existing. We do have a convergence in measure type result.

**Proposition 4.4.** Suppose \( f : (0, \infty) \to \mathbb{R} \) is measurable. Suppose that

\[
\lim_{n \to \infty} f(an) = L \quad (a.e.) .
\]

Then for all \( \varepsilon > 0 \) and all \( C > 1 \),

\[
\lim_{x \to \infty} \frac{\mu \left( \{ t \in [x, Cx] : |f(t) - L| > \varepsilon \} \right)}{\mu ([x, Cx])} = 0 ,
\]

where \( \mu \) denotes the Lebesgue measure of a set.

**Proof.** Let us denote by \( G(x) \) the quotient \( \mu \left( \{ t \in [x, cx] : |f(t) - L| > \varepsilon \} \right) / (1 - C)x \). Notice that \( G \) is a continuous function in \( (0, \infty) \).

Let \( a > 0 \) be fixed and consider the sequence of functions \( f_n(x) = f(nx) \) in the interval \( [a, Ca] \).

Since \( f_n \) converges to \( L \) almost everywhere in this finite interval, it converges to \( L \) in measure. This means that for all \( \varepsilon > 0 \) the measure of the set \( \{ s \in [a, Ca] : |f_n(s) - L| > \varepsilon \} \) tends to zero. But the transformation \( t = ns \) gives

\[
\frac{\mu \left( \{ s \in [a, Ca] : |f_n(s) - L| > \varepsilon \} \right)}{\mu ([a, Ca])} = \frac{\mu \left( \{ t \in [na, Cna] : |f(t) - L| > \varepsilon \} \right)}{\mu ([na, Cna])} = G(na) ,
\]

so that \( \lim_{n \to \infty} G(na) = 0 \). Proposition 3.1 then yields (4.9). \( \square \)

5. Delta sequences

A sequence \( \{ f_n \}_{n=1}^{\infty} \) of distributions is called a delta sequence if \( f_n(x) \to \delta(x) \) in either the strong or the weak topology of \( \mathcal{D}'(\mathbb{R}^d) \), since the two notions are equivalent [20]. In other words, \( \{ f_n \}_{n=1}^{\infty} \) is a delta sequence if

\[
\lim_{n \to \infty} \langle f, \phi \rangle = \phi(0) ,
\]

for all \( \phi \in \mathcal{D}(\mathbb{R}^d) \). In this article we will be interested mostly in the case when the distributions \( f_n \) are actually smooth functions, but general delta sequences are also of interest, of course. They have been employed in several problems [1], such as the definitions of point values [18] or the definition of products of distributions [9, 10, 14].

There are many ways to construct delta sequences. A simple one is the following. Let \( f \) be a fixed distribution of rapid decay at infinity, that is, \( f \in \mathcal{K}'(\mathbb{R}^d) \). Then all the moments \( \mu_k = \langle f(x), x^k \rangle \), exist for \( k \in \mathbb{N}^d \) since all polynomials belong to \( \mathcal{K}(\mathbb{R}^d) \), and the moment asymptotic expansion

\[
f(\lambda x) \sim \sum_{q=0}^{\infty} \sum_{|k|=q} \frac{\mu_k \lambda^k |x|^q}{k!} 1_{\lambda^k |x|^{q+d}} , \quad \text{as } \lambda \to \infty ,
\]
holds in $\mathcal{K}'(\mathbb{R}^d)$ \[1\]. Therefore, when $\mu_0 \neq 0$ if \(\{\xi_n\}_{n=1}^\infty\) is any sequence of positive numbers with $\lim_{n \to \infty} \xi_n = \infty$ then

\begin{equation}
(5.3)\quad g_n(x) = \frac{\xi_n^d}{\mu_0} f(\xi_n x),
\end{equation}

is a delta sequence, generated by $f$ and \(\{\xi_n\}_{n=1}^\infty\). When $\xi_n = n$ for all $n$, we call this sequence the standard delta sequence generated by $f$.

Another useful construction of delta sequences is provided by the ensuing well known result.

**Lemma 5.1.** Suppose \(\{\psi_n\}_{n=1}^\infty\) is a sequence of normalized positive test functions in $\mathcal{D}'(\mathbb{R}^d)$ such that $\text{supp} \psi_n \subset \{x: |x| < r_n\}$, where $\lim_{n \to \infty} r_n = 0$. Then \(\{\psi_n\}_{n=1}^\infty\) is a delta sequence.

**Proof.** Let $\phi$ be any test function. Then by the first mean value theorem for integrals,

\begin{equation}
(5.4)\quad \langle \psi_n, \phi \rangle = \int_{\text{supp} \psi_n} \psi_n(x) \phi(x) \, dx = \phi(x_n),
\end{equation}

for some $x_n \in \text{supp} \psi_n$. Since $|x_n| \leq r_n \to 0$, we obtain that $x_n \to 0$, and consequently, $\phi(x_n) \to \phi(0)$. Thus $\psi_n(x) \to \delta(x)$. \[\square\]

We now give a notion of point value of a distribution based on delta sequences. Our definition applies to several spaces of distributions, but the cases $\mathcal{A} = \mathcal{D}$, $\mathcal{E}$, or $\mathcal{S}$ seem the most relevant.

**Definition 5.2.** Let $\mathcal{A}(\mathbb{R}^d)$ be a space of test functions. Let $\mathfrak{F}$ be a family of delta sequences whose elements belong to $\mathcal{A}(\mathbb{R}^d)$. If $f \in \mathcal{A}'(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$ we say that the value $f(x_0)$ exists and equals $\gamma$ with respect to $\mathfrak{F}$ if

\begin{equation}
(5.5)\quad \lim_{n \to \infty} \langle f(x_0 + x), \phi_n \rangle = \gamma,
\end{equation}

for all $\{\phi_n\}_{n=1}^\infty \in \mathfrak{F}$. When this holds we write

\begin{equation}
(5.6)\quad f(x_0) = \gamma \quad \mathfrak{F}.
\end{equation}

The definition of point value employed by Sasane \[18\] corresponds to the case when $d = 1$, $f \in \mathcal{D}'(\mathbb{R})$, and $\mathfrak{F}$ is the family of all delta sequences whose elements are the standard sequences generated from a positive, normalized, and symmetric test function of $\mathcal{D}(\mathbb{R})$.

### 6. Several Lemmas

In this section, we present several results on how positive test functions allow us to study many properties of distributions. In particular, we see how positive test functions tell us if a distribution is a regular distribution given by a bounded measurable function and give us the essential supremum and infimum of such a function. In this section, and only in this section, we will make a notational difference between a regular distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ and the locally integrable function $f$ that generates it as

\begin{equation}
(6.1)\quad \langle f(x), \phi(x) \rangle = \int_{\mathbb{R}^d} f(x) \phi(x) \, dx, \quad \phi \in \mathcal{D}(\mathbb{R}^d).
\end{equation}
In the rest of the article, we will use the same notation, \( f \), for the distribution and the function.

Let us start with following simple result.

**Lemma 6.1.** The set of functions of the form

\[
\phi = c_1 \psi_1 - c_2 \psi_2 ,
\]

where \( c_1 \) and \( c_2 \) are constants and where \( \psi_1 \) and \( \psi_2 \) are normalized positive test functions is the whole space \( \mathcal{D}(\mathbb{R}^d) \).

When \( d = 1 \), the corresponding space with \( \psi_1 \) and \( \psi_2 \) normalized positive symmetric test functions is the space of all even test functions.

**Proof.** It is enough to show that the real valued elements of \( \mathcal{D}(\mathbb{R}^d) \) have the form \( (6.2) \) for some positive constants \( \zeta_1 \) and \( \zeta_2 \) and let \( \zeta_2 = \zeta_1 - \phi \). Then we write \( \zeta_j = c_j \psi_j \) where the \( \psi_j \) are normalized positive test functions and \( c_j = \int_{\mathbb{R}^d} \zeta_j(x) \, dx \). In the symmetric case we just also ask \( \zeta_1 \) to be even. \( \square \)

Our first characterization using positive normalized test functions is the following.

**Lemma 6.2.** Let \( f \in \mathcal{D}'(\mathbb{R}^d) \). Then \( f \) is a regular distribution in an open set \( U \subset \mathbb{R}^d \), given by a bounded function \( f \in L^\infty(U) \) if and only if there exists a constant \( M > 0 \) such that for all positive, normalized test functions \( \phi \in \mathcal{D}(U) \) we have

\[
\langle f(x), \phi(x) \rangle \leq M .
\]

**Proof.** If \( f \in L^\infty(U) \). Then when \( \phi \in \mathcal{D}(U) \),

\[
\|\langle f(x), \phi(x) \rangle\| = \left| \int_U f(x) \phi(x) \, dx \right| \leq \|f\|_{L^\infty(U)} \|\phi\|_{L^1(U)} ,
\]

so that if \( \phi \) is normalized, \( \langle f(x), \phi(x) \rangle \leq \|f\|_{L^\infty(U)} \). Therefore \( (6.3) \) holds with \( M = \|f\|_{L^\infty(U)} \).

Conversely, if \( (6.3) \) is satisfied for some \( M > 0 \) for all positive, normalized test functions of \( U \) then \( \|\langle f(x), \phi(x) \rangle\| \leq 2M \|\phi\|_{L^1(U)} \) for all real test functions \( \psi \in \mathcal{D}(U) \), because of Lemma 6.1 (or \( 4M \) if complex). This means that \( f \) is continuous in \( \mathcal{D}(U) \), a dense subspace of \( L^1(U) \) with the topology induced by \( L^1(U) \) in its subspace. Hence, \( f \) admits an extension \( f \in (L^1(U))^' \simeq L^\infty(U) \), and this means that

\[
\langle f(x), \psi(x) \rangle = \int_U f(x) \psi(x) \, dx ,
\]

for all \( \psi \in \mathcal{D}(U) \). Therefore, \( f \) is a regular distribution given by the bounded function \( f \) in the open set \( U \). \( \square \)

In the proof we can see that \( \inf \{M : (6.3) \text{ holds}\} \leq \|f\|_{L^\infty(U)} \). In fact, we have more.

**Lemma 6.3.** If \( f \in \mathcal{D}'(\mathbb{R}^d) \) is a regular distribution in \( U \), given by a bounded function \( f \in L^\infty(U) \) then

\[
\|f\|_{L^\infty(U)} = \inf \{M : (6.3) \text{ holds for all positive, normalized test functions}\} ,
\]

and

\[
\|f\|_{L^\infty(U)} = \sup \{\|\langle f, \phi \rangle\| : \phi \in \mathcal{D}(U) \text{ positive, normalized test function}\} .
\]
Proof. Clearly \( \inf \{ M : (6.3) \text{ holds for all positive, normalized test functions} \} \) is equal to \( \sup \{ |\langle f, \phi \rangle | : \phi \in \mathcal{D}(U) \text{ positive, normalized test function} \} \); let us call this \( K \). We know that \( K \leq \| f \|_{L^\infty(U)} \). To prove the converse inequality, let \( s < \| f \|_{L^\infty(U)} \). Then there exists \( x_0 \in U \) such that the distributional point value \( f(x_0) \) exists and \( s < |f(x_0)| \). If \( \phi \) is a positive normalized test function, then so are the test functions \( \varphi_\lambda(x) = \lambda^d \phi(x_0 + \lambda x) \) for all \( \lambda > 0 \), and if \( \lambda \) is big enough, \( \varphi_\lambda \in \mathcal{D}(U) \). Since \( \lim_{\lambda \to \infty} \langle f, \varphi_\lambda \rangle = f(x_0) \), we can find \( \lambda \) such that \( |\langle f, \varphi_\lambda \rangle | > s \). Consequently, \( K > s \), and because \( s < \| f \|_{L^\infty(U)} \) is arbitrary, \( K \geq \| f \|_{L^\infty(U)} \). \( \square \)

In fact, the same argument in the proof of Lemma \([6,3]\) allows us to obtain the ensuing.

**Lemma 6.4.** If \( f \in \mathcal{D}^\prime(\mathbb{R}^d) \) is a real regular distribution in \( U \), given by a function \( f \in L^1(U) \) then the essential supremum and infimum of \( f \) are also given as

\[
\text{esssup} \ f(x) = \sup_{x \in U} \sup_{\phi \in \mathcal{D}(U), \phi \geq 0, \| \phi \|_1 = 1} \langle f(x), \phi(x) \rangle,
\]

and

\[
\text{essinf} \ f(x) = \inf_{x \in U} \inf_{\phi \in \mathcal{D}(U), \phi \geq 0, \| \phi \|_1 = 1} \langle f(x), \phi(x) \rangle.
\]

We notice that when \( f \in L^1(U) \) then (6.7) could be \( +\infty \) and (6.8) could be \( -\infty \).

### 7. Comparison of definitions

We will now study whether the existence of the distributional point value \( f(x_0) \) is equivalent to the existence of \( f(x_0) \) \((\mathfrak{G})\) for several families of delta sequences \( \mathfrak{F} \).

#### 7.1. Standard delta sequences generated by a positive normalized test function.

In this section we consider the family \( \mathfrak{F} \) of standard delta sequences generated by a positive normalized test function of \( \mathcal{D}(\mathbb{R}^d) \).

**Proposition 7.1.** Let \( f \in \mathcal{D}^\prime(\mathbb{R}^d) \). Then \( f \) has a thick distributional point value at \( x_0 \) if and only if for all standard delta sequences generated by a positive normalized test function of \( \mathcal{D}(\mathbb{R}^d) \), \( \{ \phi_n \}_{n=1}^\infty \), the limit

\[
\lim_{n \to \infty} \langle f(x_0 + x), \phi_n(x) \rangle = \gamma_{\{\phi_n\}},
\]

exists.

**Proof.** A standard delta sequences generated by a normalized positive test function \( \phi \) is of the form \( \phi_n(x) = n^d \phi(nx) \). If the distributional thick point value \( f_{x_0}(w) = \gamma(w) \) exists, \( \gamma \in \mathcal{D}^\prime(\mathfrak{S}) \), then

\[
\lim_{n \to \infty} \langle f(x_0 + x), \phi_n(x) \rangle = \lim_{n \to \infty} \langle f(x_0 + x), n^d \phi(nx) \rangle
\]

\[
= \lim_{n \to \infty} \langle f(x_0 + (1/n)x), \phi(x) \rangle
\]

\[
= \int_{0}^{\infty} \langle \gamma(w), \phi(rw) \rangle_{\mathcal{D}^\prime(\mathfrak{S}) \times \mathcal{D}(\mathfrak{S})} r^{d-1} dr,
\]

exists. Conversely, let \( \phi \) be a normalized positive test function. If the limit \([7.1]\) exists for all standard delta sequences generated by a positive normalized test function, it will
exist for \( \phi_n^{(a)} (x) = na^d \phi (na x) \) for all \( a > 0 \). Consequently, if the function \( \Phi \) is defined as
\[
\Phi (a) = \left\langle f (x_0 + x), a^d \phi (a x) \right\rangle , \quad a > 0 ,
\]
then
\[
\lim_{n \to \infty} \Phi (na) = \lim_{n \to \infty} \left\langle f (x_0 + x), \phi_n^{(a)} (x) \right\rangle = \gamma_{\phi_n^{(a)}} ,
\]
eexists for all \( a \). Since \( \Phi \) is continuous, Proposition 3.1 yields that \( \gamma_{\phi_n^{(a)}} = \gamma_0 (\phi) \) is independent of \( a \) and actually \( \lim_{\lambda \to \infty} \Phi (\lambda) = \gamma_0 (\phi) \). Hence,
\[
\lim_{\varepsilon \to 0^+} \left\langle f (x_0 + \varepsilon x), \phi (x) \right\rangle = \gamma_0 (\phi) ,
\]
for all normalized positive test functions. Therefore, Lemma 6.1 yields that the limit \( \lim_{\varepsilon \to 0^+} \left\langle f (x_0 + \varepsilon x) , \phi (x) \right\rangle = \gamma_0 (\phi) \) defines a distribution \( \gamma_0 \in \mathcal{D}' (\mathbb{R}^d) , \) and \( \gamma_0 \) is homogeneous of degree 0, that is, \( \gamma_0 (tx) = \gamma_0 (x) , \ t > 0 \). As explained in Section 2 using [5, Thm. 2.6.2] we conclude that \( \gamma_0 \) is obtained from a distribution \( \gamma \in \mathcal{D}' (\mathbb{S}) \) by the formula
\[
\langle \gamma_0 , \phi \rangle = \int_0^\infty \langle \alpha (w) , \phi (rw) \rangle_{\mathcal{D}' (\mathbb{S}) \times \mathcal{D} (\mathbb{S})} r^{d-1} \, dr ,
\]
and that \( \alpha \) is the thick distributional value of \( f \) at \( x_0 \).

Let \( \{ \phi_n \}_{n=1}^\infty \) be a sequence of test functions. If \( T \) is an orthogonal transformation of \( \mathbb{R}^d \), that is, with \( |\det T| = 1 \), then the sequence \( \{ \phi_n^T \}_{n=1}^\infty , \) where \( \phi_n^T (x) = \phi (T x) \), is also a delta sequence. We have then the following result.

**Proposition 7.2.** Let \( f \in \mathcal{D}' (\mathbb{R}^d) \). Then the distributional point value \( f (x_0) \) exists if and only if for all standard delta sequences generated by a positive normalized test function of \( \mathcal{D} (\mathbb{R}^d) , \) \( \{ \phi_n \}_{n=1}^\infty , \) the limit \( \lim_{n \to \infty} \left\langle f (x_0 + x) , \phi_n (x) \right\rangle = \gamma_{\phi_n} \) exists and for all orthogonal transformations \( T \) of \( \mathbb{R}^d \), \( \gamma_{\phi_n^T} = \gamma_{\phi_n} \).

**Proof.** This follows immediately from Proposition 7.1 if we observe that a homogeneous function or distribution of degree 0 is a constant if and only if it is invariant with respect to orthogonal transformations.

Notice that in one variable, Proposition 7.1 says that \( \lim_{n \to \infty} \left\langle f (x_0 + x) , \phi_n (x) \right\rangle = \gamma_{\phi_n} \) exists for all standard delta sequences generated by a positive normalized test function if and only if there are constants \( \gamma_+ \) and \( \gamma_- \) such that
\[
\lim_{\varepsilon \to 0^+} \left\langle f (x_0 + \varepsilon x) , \psi (x) \right\rangle = \gamma_- \int_{-\infty}^0 \psi (x) \, dx + \gamma_+ \int_0^\infty \psi (x) \, dx ,
\]
for all \( \psi \in \mathcal{D} (\mathbb{R}) \). On the other hand, since the only orthogonal transformations in dimension one are the identity and \( x \sim -x \), Proposition 7.2 says that the distributional point value \( f (x_0) \) exists if and only if for all standard delta sequences generated by a positive normalized test function of \( \mathcal{D} (\mathbb{R}) , \) \( \{ \phi_n \}_{n=1}^\infty , \) the limit \( \lim_{n \to \infty} \left\langle f (x_0 + x) , \phi_n (x) \right\rangle = \gamma_{\phi_n} \) exists and \( \gamma_{\phi_n (-x)} = \gamma_{\phi_n (x)} \).

Our results also give the ensuing equivalence.
Proposition 7.3. Let \( f \in \mathcal{D}'(\mathbb{R}^d) \). Then the distributional point value \( f(x_0) \) exists and equals \( \gamma \) if and only if for \( \mathcal{F} \) the family of standard delta sequences generated by a positive normalized test function

\[
(7.7) \quad f(x_0) = \gamma \quad (\mathcal{F}). 
\]

7.2. Standard delta sequences generated by an even positive normalized test function. We now consider the case of symmetric standard delta sequences, the family considered by Sasane [18].

We first need to explain the idea of symmetric point values. Let \( f \in \mathcal{D}'(\mathbb{R}) \) and \( x_0 \in \mathbb{R} \). The symmetric distributional point value of \( f \) exists at \( x_0 \) and equals \( \gamma \) if

\[
(7.8) \quad \lim_{\varepsilon \to 0} \frac{f(x_0 + \varepsilon x) + f(x_0 - \varepsilon x)}{2} = \gamma,
\]

in \( \mathcal{D}'(\mathbb{R}) \). Each distribution can be written as the sum of an even one and an odd one,

\[
(7.9) \quad g = g_e + g_o,
\]

where

\[
(7.10) \quad g_e(x) = \frac{g(x) + g(-x)}{2}, \quad g_o(x) = \frac{g(x) - g(-x)}{2}.
\]

Applying this to \( g(x) = f(x_0 + x) \), we see that the distributional symmetric value \( f(x_0) \) exists and equals \( \gamma \) if and only if the distributional value \( g_e(0) \) exists and equals \( \gamma \).

Notice also that if \( \phi \) is a test function and we write \( \phi = \phi_e + \phi_o \), then

\[
(7.11) \quad \langle g, \phi \rangle = \langle g_e, \phi_e \rangle + \langle g_o, \phi_o \rangle.
\]

Therefore we have the following result.

Lemma 7.4. A distribution \( f \in \mathcal{D}'(\mathbb{R}) \) has a symmetric distributional value \( \gamma \) at \( x_0 \) if and only if

\[
(7.12) \quad \lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi_e(x) \rangle = \gamma \int_{-\infty}^{\infty} \phi_e(x) \, dx,
\]

for all even test functions \( \phi_e \).

Proof. Indeed, if \( (7.3) \) is satisfied, then

\[
\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi_e(x) \rangle = \lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x) - g_o(\varepsilon x), \phi_e(x) \rangle
\]

\[
= \lim_{\varepsilon \to 0} \left\langle \frac{f(x_0 + \varepsilon x) + f(x_0 - \varepsilon x)}{2}, \phi_e(x) \right\rangle
\]

\[
= \gamma \int_{-\infty}^{\infty} \phi_e(x) \, dx.
\]
Conversely, if (7.12) holds, then for any test function \( \phi = \phi_e + \phi_o \),
\[
\lim_{\varepsilon \to 0} \langle g_e (\varepsilon x) , \phi (x) \rangle = \lim_{\varepsilon \to 0} \langle g_e (\varepsilon x) , \phi_e (x) \rangle \\
= \lim_{\varepsilon \to 0} \langle f (x_0 + \varepsilon x) , \phi_e (x) \rangle \\
= \gamma \int_{-\infty}^{\infty} \phi_e (x) \, dx \\
= \gamma \int_{-\infty}^{\infty} \phi (x) \, dx .
\]
Hence \( g_e (0) = \gamma \), so that the symmetric distributional value of \( f \) at \( x_0 \) equals \( \gamma \).

We can now give an equivalence to the existence of the point value \( f (x_0) = \gamma \) (\( \mathcal{F}_{sy} \)), where \( \mathcal{F}_{sy} \) is the family of standard delta sequences generated by a positive normalized even test function of \( \mathcal{D} (\mathbb{R}) \).

**Proposition 7.5.** Let \( f \in \mathcal{D}' (\mathbb{R}) \). Then the following are equivalent:

1. If \( \mathcal{F}_{sy} \) is the family of standard delta sequences generated by a positive normalized even test function then

\[
(7.13) \quad f (x_0) = \gamma \quad (\mathcal{F}_{sy}) .
\]

2. The symmetric distributional point value of \( f \) exists at \( x_0 \) and equals \( \gamma \).

**Proof.** Indeed, if (7.13) holds then
\[
(7.14) \quad \lim_{n \to \infty} \langle f (x_0 + x) , \phi_n (x) \rangle = \gamma ,
\]
for all standard delta sequences \( \{ \phi_n \}_{n=1}^{\infty} \) generated by a positive normalized even test function \( \phi_e \), and use of Proposition 3.1 yields that
\[
(7.15) \quad \lim_{\varepsilon \to 0} \langle f (x_0 + \varepsilon x) , \phi_e (x) \rangle = \gamma ,
\]
for such normalized even test functions \( \phi_e \). This last statement is equivalent to the fact that (7.12) holds for all even test functions because of Lemma 6.1 and Lemma 7.4 yields that in turn this is equivalent to the symmetric distributional point value being equal to \( \gamma \).

Actually, using the same ideas as in the proof of this Proposition we see that the limit \( \lim_{n \to \infty} \langle f (x_0 + x) , \phi_n (x) \rangle = \gamma (\phi_e) \) exists for all standard delta sequences \( \{ \phi_n \}_{n=1}^{\infty} \) generated by a positive normalized even test function \( \phi_e \) if and only if this limit is a constant \( \gamma \) and (7.13) is satisfied.

### 7.3. The family of standard delta sequences generated by a radial positive normalized test function.

We now consider the family \( \mathcal{F}_{rad} \) of standard sequences generated by a radial positive normalized test function.

Let us start with some notation. We denote \( r = |x| \) the radial variable in \( \mathbb{R}^d \). A test function \( \phi \in \mathcal{D} (\mathbb{R}^d) \) is called *radial* if it is a function of \( r \), \( \phi (x) = \varphi (r) \), for some even function \( \varphi \in \mathcal{D} (\mathbb{R}) \); the space of all radial test functions of \( \mathcal{D} (\mathbb{R}^d) \) is denoted as \( \mathcal{D}_{rad} (\mathbb{R}^d) \). Similarly, we denote as \( \mathcal{D}^\prime_{rad} (\mathbb{R}^d) \) the space of all radial distributions; a distribution \( f \in \mathcal{D}^\prime (\mathbb{R}^d) \) is radial if \( f (T x) = f (x) \) for any orthogonal transformation of \( \mathbb{R}^d \), and this actually means \( \mathcal{F}_{rad} \) that \( f (x) = f_1 (r) \) for some distribution of one variable.
Notice, however, that while \( \phi \) is uniquely determined by \( \varphi \), for a given \( f \) there are several possible distributions \( f_1 \).

When \( d = 1 \) then \( D_{\text{rad}}(\mathbb{R}) \) and \( D'_{\text{rad}}(\mathbb{R}) \) become the spaces of even test functions and distributions, respectively, and are also denoted as \( D_{\text{even}}(\mathbb{R}) \) and \( D'_{\text{even}}(\mathbb{R}) \). This was the situation considered in the previous subsection.

Observe that the space \( D'_{\text{rad}}(\mathbb{R}^d) \) is naturally isomorphic to the dual space \((D_{\text{rad}}(\mathbb{R}^d))'\), that is to say, if the action of a radial distribution is known in all radial test functions, then it can be obtained for arbitrary test functions. Indeed, if \( f \in D'_{\text{rad}}(\mathbb{R}^d) \) and \( \phi \in D(\mathbb{R}^d) \), then

\[
\langle f(x), \phi(x) \rangle = \langle \tilde{f}(x), \tilde{\phi}(x) \rangle,
\]

where \( \tilde{\phi} \in D_{\text{rad}}(\mathbb{R}) \) is given as

\[
\tilde{\phi}(x) = \phi^o(|x|),
\]

\( \phi^o \in D_{\text{even}}(\mathbb{R}) \) being defined as

\[
\phi^o(r) = \frac{1}{\omega} \int_{S} \phi(r\theta) \, d\sigma(\theta).
\]

Here we denote by \( S \) the unit sphere of \( \mathbb{R}^d \), \( d\sigma \) is the Lebesgue measure in \( S \) and \( \omega = 2\pi^{d/2}/\Gamma(d/2) \) is the surface area of the sphere.

Equations (7.17) and (7.18) define the radial component of a test function. We can also define the radial component of a distribution \( f \), \( \tilde{f} \in D'_{\text{rad}}(\mathbb{R}^d) \), as

\[
\langle \tilde{f}(x), \phi(x) \rangle = \langle f(x), \tilde{\phi}(x) \rangle.
\]

The distributional analog of (7.18) is not well defined, however [4, 7].

We say that a distribution \( f \) has a radial distributional point value at \( x_0 \) equal to \( \gamma \) if

\[
\tilde{g}(0) = \gamma,
\]

where \( \tilde{g} \) is the radial component of \( g(x) = f(x_0 + x) \). Similar to Lemma 7.4, we have the following characterization.

**Lemma 7.6.** A distribution \( f \in D'(\mathbb{R}^d) \) has a radial distributional value \( \gamma \) at \( x_0 \) if and only if

\[
\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi_{\text{rad}}(x) \rangle = \gamma \int_{\mathbb{R}^d} \phi_{\text{rad}}(x) \, dx.
\]

for all radial test functions \( \phi_{\text{rad}} \).

**Proof.** If \( \tilde{g}(0) = \gamma \), then

\[
\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi_{\text{rad}}(x) \rangle = \lim_{\varepsilon \to 0} \langle \tilde{g}(\varepsilon x), \phi_{\text{rad}}(x) \rangle
\]

\[
= \gamma \int_{\mathbb{R}^d} \phi_{\text{rad}}(x) \, dx.
\]
On the other hand, if (7.21) holds, then for any test function $\phi$,
\[
\lim_{\epsilon \to 0} \langle \tilde{g} (\epsilon x), \phi(x) \rangle = \lim_{\epsilon \to 0} \langle f (x_0 + \epsilon x), \phi(x) \rangle = \gamma \int_{\mathbb{R}^d} \phi(x) \, dx = \gamma \int_{\mathbb{R}^d} \phi(x) \, dx.
\]
Hence $\tilde{g}(0) = \gamma$, that is, the radial distributional value of $f$ at $x_0$ equals $\gamma$. 

Therefore, we obtain the ensuing equivalence for the fact that $f (x_0) = \gamma$ $(\mathfrak{F}_{\text{rad}})$. 

**Proposition 7.7.** Let $f \in \mathcal{D}'(\mathbb{R}^d)$. Then the following are equivalent:

1. If $\mathfrak{F}_{\text{rad}}$ is the family of standard delta sequences generated by a positive normalized radial test function then
   \[
   f (x_0) = \gamma \quad (\mathfrak{F}_{\text{rad}}).
   \]

2. The radial distributional point value of $f$ exists at $x_0$ and equals $\gamma$.

**Proof.** Indeed, if (7.22) holds then
\[
\lim_{n \to \infty} \langle f (x_0 + \epsilon x), \phi_n (x) \rangle = \gamma,
\]
for all standard delta sequences $\{\phi_n\}_{n=1}^{\infty}$ generated by a positive normalized radial test function $\phi_{\text{rad}}$. Use of Proposition 3.1 yields that
\[
\lim_{\epsilon \to 0} \langle f (x_0 + \epsilon x), \phi_{\text{rad}} (x) \rangle = \gamma,
\]
for such normalized radial test functions $\phi_{\text{rad}}$. This last statement is equivalent to the fact that (7.12) holds for all radial test functions because of Lemma 6.1 and Lemma 7.6 yields that, in turn, this is equivalent to the radial distributional point value being equal to $\gamma$. 

We also have the next result, that is obtained from Lemma 6.1.

**Proposition 7.8.** The limit $\lim_{n \to \infty} \langle f (x_0 + \epsilon x), \phi_n (x) \rangle = \gamma_{\{\phi_n\}}$ exists for all standard delta sequences $\{\phi_n\}_{n=1}^{\infty}$ generated by a positive normalized radial test function $\phi_{\text{rad}}$ if and only if this limit is a constant $\gamma$ and $f (x_0) = \gamma$ $(\mathfrak{F}_{\text{rad}})$.

7.4. **The family of all positive normalized test functions.** We saw in Subsection 7.2 that Sasane’s notion of point values was not equivalent to the standard definition, nor, in the next subsection, is the notion based on the family $\mathfrak{F}_{\text{rad}}$ of standard delta sequences generated by a positive normalized radial test function. Nevertheless, for the family $\mathfrak{F}$ of standard delta sequences generated by a positive normalized test function the point value definition is in fact equivalent to the standard Lojasiewicz definition. Of course, Sasane was considering the family of standard delta sequences generated by an even positive normalized test function $\mathfrak{F}_{\text{sy}}$. Both $\mathfrak{F}_{\text{sy}}$ and $\mathfrak{F}_{\text{rad}}$ are subfamilies of $\mathfrak{F}$. We can also consider families larger than $\mathfrak{F}$. For instance, we can consider the family $\mathfrak{F}_{\text{all}}$ of all delta sequences formed with positive normalized test functions. In this next example, we will see that the
distributional point value \( f(\mathbf{x}_0) = \gamma \) is not equivalent to \( f(\mathbf{x}_0) = \gamma \) \((\mathfrak{F}_{\text{all}})\). Later on we shall find an equivalent formulation to \( f(\mathbf{x}_0) = \gamma \) \((\mathfrak{F}_{\text{all}})\).

**Example 7.9.** Let \( f \) be the regular distribution given by \( f(x) = \sin(1/x) \). Then \( f(0) = 0 \) distributionally \([\mathfrak{M}]\). Let \( a_n \) be a positive sequence with \( a_n \to 0 \) and \( f(a_n) = C > 0 \). For instance, we could take \( a_n = 1/(2\pi n + \pi/6) \). For a fixed \( n \), let \( \{\psi_{n,m}\}_{m=1}^{\infty} \) be a sequence of positive test functions such that \( \psi_{n,m} \to \delta(x - a_n) \) as \( m \to \infty \). Then as \( n \to \infty \), we obtain a sequence \( \delta_n(x) = \delta(x - a_n) \) that converges to \( \delta(x) \). For each \( n \), let \( m_n \) be large enough so that
\[
\left| \int_{B_{1/n}(a_n)} f(x) \psi_{n,m}(x) \, dx - f(a_n) \right| < C/2
\]
and \( \text{supp} \psi_{n,m} \subset B_{1/n}(a_n) \) for \( m \geq m_n \). Then we can define the sequence \( \phi_n(x) = \psi_{n,m_n}(x) \). By Lemma 5.11 this is a delta sequence and we have \( \langle f(x), \phi_n(x) \rangle > C/2 \) for all \( n \) and so \( \lim_{n \to \infty} \langle f(x), \phi_n(x) \rangle \) cannot be equal to 0.

The next lemma will be useful momentarily.

**Lemma 7.10.** If \( \{\phi_n\}_{n=1}^{\infty} \) is a delta sequence of positive test functions then
\[
\lim_{n \to \infty} \|\phi_n\|_{L^1(B \setminus U)} = 0,
\]
where \( B \) and \( U \) are both neighborhoods of the origin.

*Proof.* Choose \( \psi \in \mathcal{D}(\mathbb{R}^d) \) such that \( \psi \geq 0 \),
\[
\psi(x) = 1, \quad x \in B \setminus U, \quad \psi(0) = 0,
\]
which is possible because \( 0 \notin B \setminus U \). Then
\[
\|\phi_n\|_{L^1(B \setminus U)} = \int_{B \setminus U} |\phi_n(x)| \, dx = \int_{B \setminus U} \phi_n(x) \, dx
\]
\[
= \int_{B \setminus U} \psi(x) \phi_n(x) \, dx
\]
\[
\leq \int_B \psi(x) \phi_n(x) \, dx \to \psi(0) = 0,
\]
as \( n \to \infty \). \( \square \)

We are now ready to prove the main result of this section.

**Proposition 7.11.** Suppose \( f \in \mathcal{D}'(\mathbb{R}^d) \) and \( \mathbf{x}_0 \in \mathbb{R}^d \). If
\[
\lim_{n \to \infty} \langle f(\mathbf{x}_0 + \mathbf{x}), \phi_n(\mathbf{x}) \rangle = \gamma,
\]
for all positive delta sequences \( \{\phi_n\} \), then the following two conditions hold:

1. There is an \( r^* > 0 \) such that \( f|_{B_{r^*}(\mathbf{x}_0)} \in L^\infty(B_{r^*}(\mathbf{x}_0)) \).

2. \( \lim_{r \to 0} \| f|_{B_r(\mathbf{x}_0)} - \gamma \chi_{B_r(\mathbf{x}_0)} \|_\infty = 0. \)

Here \( \chi_{B_r(\mathbf{x}_0)} \) is the characteristic function of the ball \( B_r(\mathbf{x}_0) \). Conversely, if (1) and (2) are satisfied, then (7.27) holds for all positive delta sequences with support contained in \( B_{r^*}(0) \).
Proof. Suppose that (7.27) holds. Notice that (1) follows from Lemma 6.2. To see that (2) is true, suppose instead that

\[
\limsup_{r \to 0} \left\| f \chi_{B_r(x_0)} - \gamma \chi_{B_r(x_0)} \right\|_\infty = C > 0.
\]

Let \( r_n \) be a decreasing sequence of positive numbers with \( r_n < r^* \) and \( r_n \to 0 \). For each \( n \), there is a positive normalized test function supported in \( B_{r_n}(0) \), say \( \phi_n \), such that

\[
\langle f(x_0 + x), \phi_n(x) \rangle - \gamma > \frac{C}{2}.
\]

By Lemma 5.1, \( \{\phi_n\}_{n=1}^\infty \) forms a delta sequence and so \( \lim_{n \to \infty} \langle f(x_0 + x), \phi_n(x) \rangle = \gamma \), which contradicts (7.29).

For the converse, let \( \{\psi_n\} \) be a delta sequence of positive normalized test functions supported in \( B_{r^*}(0) \). Since by (1) \( f \) is a regular distribution in \( B_{r^*}(0) \) we have

\[
\langle f(x_0 + x), \psi_n(x) \rangle - \gamma = \int_{B_{r^*}(0)} (f(x_0 + x) - \gamma) \psi_n(x) \, dx.
\]

Let \( \varepsilon > 0 \). By condition (2), we can find an open neighborhood \( V \) of the origin that is contained in \( B_{r^*}(0) \) such that if \( W = x_0 + V \), then \( \|f - \gamma\|_{L^\infty(W)} < \varepsilon \) and for this \( V \) we can find \( n_0 \) such that \( \|\psi_n\|_{L^1(B_{r^*}(0) \setminus V)} < \varepsilon \) if \( n \geq n_0 \). If \( M \) is the constant \( \|f - \gamma\|_{L^\infty(B_{r^*}(x_0))} \), then we have

\[
\langle f(x_0 + x), \psi_n(x) \rangle - \gamma = \int_{W} (f(x) - \gamma) \psi_n(x - x_0) \, dx
\]

\[
+ \int_{B_{r^*}(0) \setminus V} (f(x_0 + x) - \gamma) \psi_n(x) \, dx
\]

\[
\leq \|f - \gamma\|_{L^\infty(W)} \|\psi_n\|_{L^1(V)}
\]

\[
+ \|f - \gamma\|_{L^\infty(B_{r^*}(x_0) \setminus W)} \|\psi_n\|_{L^1(B_{r^*}(0) \setminus V)}
\]

\[
< \varepsilon + M \varepsilon,
\]

and consequently \( \lim_{n \to \infty} \langle f(x_0 + x), \psi_n(x) \rangle = \gamma \). \qed

References

[1] Antosik, P., Mikusiński, J. and Sikorski, R., Theory of distributions. The sequential approach, Elsevier Scientific Publishing Co., Amsterdam; PWN—Polish Scientific Publishers, Warsaw, 1973.
[2] Campos Ferreira, J., Introduction to the Theory of Distributions, Longman, London, 1997.
[3] Estrada, R., The set of singularities of regulated functions of several variables, Collectanea Mathematica 63 (2012), 351-359.
[4] Estrada, R., On radial functions and distributions and their Fourier transforms, J. Fourier Anal. Appl. 20 (2013), 301-320.
[5] Estrada, R. and Kanwal, R. P., A Distributional Approach to Asymptotics. Theory and Applications, Second edition, Birkhäuser, Boston, 2002.
[6] Gordon, R. A., The integrals of Lebesgue, Denjoy, Perron, and Henstock, Amer. Math. Soc., Providence, 1994.
[7] Grafakos, L. and Teschl, G., On Fourier transforms of radial functions and distributions, J. Fourier Anal. Appl. 19 (2013), 167-179.
[8] Kanwal, R.P., Generalized Functions: Theory and Technique, Third Edition, Birkhäuser, Boston, 2004.
[9] Koh, E. L. and Li, C. K., On distributions \( \delta^k \) and \( (\delta')^k \), Math. Nachr. 157 (1992), 243-248.
[10] Li, C. K., An approach for distributional products on $\mathbb{R}^n$, *Integ. Trans. Spec. Funct.* **16** (2005), 139-151.

[11] Lojasiewicz, S., Sur la valeur et la limite d’une distribution en un point, *Studia Math.* **16** (1957), 1-36.

[12] Lojasiewicz, S., Sur la fixation de variables dans une distribution, *Studia Math.* **17** (1958), 1-64.

[13] Natanson, I. P., *Theory of Functions of a Real Variable*, vol. 2, Frederick Ungar Publishing, New York, 1960.

[14] Özçağ, E., Gülen, Ü. and Fisher, B., On the distribution $\delta_x^k$, *Integ. Trans. Spec. Funct.* **9** (2000), 57-64.

[15] J. Peetre, On the value of a distribution at a point, *Portugaliae Math.* **27** (1968), 149–159.

[16] Pilipović, S., Stanković, B., and Takači, A., *Asymptotic Behavior and Stieltjes Transformation of Distributions*, Teubner-Texte zur Mathmatik, Leipzig, 1990.

[17] Pilipović, S., Stanković, B., and Vindas, J., *Asymptotic Behavior of Generalized Functions*, World Scientific, Singapore, 2011.

[18] Sasane, A., A summation method based on the Fourier series of periodic distributions and an example arising in the Casimir effect, *Indag. Math.* **31** (2020), 477-504.

[19] Schwartz, L., *Théorie des Distributions*, Second edition, Hermann, Paris, 1966.

[20] Treves, F., *Topological Vector Spaces, Distributions, and Kernels*, Academic Press, New York, 1967.

[21] Vindas, J. and Estrada, R., On the jump behavior of distributions and logarithmic averages, *J. Math. Anal. Appl.* **347** (2008), 597-606.

[22] Vladimirov, V. S., *Methods of the theory of generalized functions*, Taylor & Francis, London, 2002.

[23] Vladimirov, V. S., Drozhzhinov, Y. N., and Zavialov, B. I., *Tauberian theorems for generalized functions*, Kluwer Academic Publishers Group, Dordrecht, 1988.

**Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA**

*Email address: restrada@math.lsu.edu*

**Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA**

*Email address: kkelli201@lsu.edu*