Quantum Glass Transition in a Periodic Long-Range Josephson Array

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We show that the ground state of the periodic long range Josephson array frustrated by magnetic field is a glass for a sufficiently large Josephson energies despite the absence of a quenched disorder. Like superconductors, this glass state has non-zero phase stiffness and Meissner response; for smaller Josephson energies the glass “melts” and the ground state loses the phase stiffness and becomes insulating. We find the critical scaling behavior near this quantum phase transition: the excitation gap vanishes as \((J - J_c)^2\), the frequency-dependent magnetic susceptibility behaves as \(\chi(\omega) \propto \sqrt{\omega} \ln \omega\).

I. INTRODUCTION

Glass formation in the absence of \textit{intrinsic} disorder is a long-standing problem but the last years saw a rapid progress in the qualitative understanding of this phenomena. Mostly this progress is due to the solutions of periodic models that assume a mapping between the periodic model and the appropriate random model. The validity of this assumption is still an open question in a general case but it was shown that at least one periodic model allows a direct study of a phase transition and non-ergodic behavior below the transition without any reference to a disordered model. This model describes a long-range Josephson array in a magnetic field and another reason for the interest in this model is that it that can be realized experimentally (cf.\cite{1} for the discussion of experimental conditions).

All these results were obtained in the framework of classical statistical mechanics, the glass formation in a \textit{regular} quantum systems has not been addressed. The goal of this paper is to fill this gap. The problem of a glass formation in \textit{disordered} quantum systems was discussed in a number of works\cite{2}; these works studied the critical behavior near the quantum vitrification transition\cite{3} and the properties of the glassy phase itself\cite{4} using the replica approach. They found that the glass phase transition at \(T = 0\) indeed exists; further, it strongly resembles classical (high \(T\)) phase transition in the same system: the main difference is in the critical exponent of correlation function which decays faster than at the classical critical point: \(D(t) = \langle S_j(0)S_j(t) \rangle \sim t^{-1}\) at \(T = 0\) (cf. \(D(t) \sim t^{-1/2}\) at non-zero \(T\)). A surprising result stated in\cite{2} is that at zero temperature no replica-symmetry-breaking (RSB) is needed for the description of the glassy state, i.e. replica-symmetrical solution is stable at \(T = 0\). Since, usually, RSB is believed to be a signature of non-ergodicity, this result means either absence of non-ergodic behavior at \(T = 0\) or violation of the usual relation between RSB and non-ergodicity. We feel that in order to clarify this important question, an approach that is free from the ambiguities of the replica method should be employed.

Understanding of a quantum glass formation in a system with regular Hamiltonian is important for the general problem of Quantum Computation. The reason is that quantum computer is also a quantum system with exponential number of states and the process of computation can be viewed as an almost adiabatic change of the external parameters resulting in a different state. The crucial question is what are the conditions so that such process does not lead to the collapse of the density matrix due to the coupling to the environment. This question can be addressed to the spin glass system as well and one can learn about decoherence in a generic large system with exponential number of states from the answer to it.

Here we study the quantum version the long-range Josephson array in a frustrating magnetic field that was suggested in\cite{5}. We consider here only the problem of glass formation, approaching the glass from the ”liquid” (i.e. insulating) side. We show that the quantum version of this problem is described by the same dynamic equations as the quantum disordered p-spin model studied in\cite{6}. Thus, we explicitly prove that this frustrated quantum system can be mapped onto the quantum disordered system in a complete analogy with the situation for classical problems. Further, we provide a direct numerical proof that the transition in this model is indeed continuous as conjectured in\cite{2} and we calculate the anomaly of the diamagnetic response associated with this transition.

Another, and more physical, justification of the model is the following. It is well established, both experimentally (cf. e.g.\cite{7}) and theoretically\cite{8} that usual nearest-neighbors Josephson arrays made of small superconducting islands demonstrate zero-\(T\) superconductor-insulator transition as the ratio of the Josephson coupling \(E_J\) between the superconductive islands to the Coulomb energy cost \(E_C = (2e)^2/2C\) for the transfer of the Cooper pair between the islands decreases. At small values of \(x = E_J/E_C\) the ground state is an insulator with nonzero Coulomb gap in the excitation...
spectrum. At nearly-critical values of $x \approx x_{cr}$ the transition between insulating and superconductive states can be triggered by application of a weak magnetic field, producing frustration of the Josephson interaction; moreover, this transition can be split into the sequence of two different transitions: superconductor-metal-insulator. Although main qualitative features of these phenomena are understood, there is still no quantitative theory which describes quantum phase transitions in 2-dimensional short-range systems, especially in the presence of frustration. Therefore, in our attempt to study the origin of a quantum glass state, we have to turn to the simplest (theoretically) model of a Josephson array with long-range interaction, which consists of long superconducting wires (instead of islands). It will allow us to employ some version of mean-field-theory and reduce the problem to a zero-dimensional quantum theory with the interaction that is non-local in time.

The system of our study is a stack of two mutually perpendicular sets of $N$ parallel thin superconducting wires with Josephson junctions at each node that is placed in an external transverse field $H$. Macroscopic quantum variables of this array are the $2N$ superconducting phases associated with each wire (e.g. the value of the phase of the superconducting order parameter at the center of each wire); we will always assume that excitations within individual wire can be neglected, so the whole wire is characterized by one phase, $\phi_m$. In the absence of an external field the phase differences would be zero at each junction, but this is not possible for finite $H$, so the phases are frustrated. Here we assume that the Josephson currents are sufficiently small so that the induced fields are negligible in comparison with $H$ (this imposes an important constraint for the experimental realization of this network). Therefore the array is described by the Hamiltonian

$$
\mathcal{H} = \mathcal{H}_J + \mathcal{H}_C = -E_J \sum_{m,n} \cos(\phi_n - \phi_m) - \frac{2e}{\hbar c} \int A dl + \frac{(2e)^2}{2} \sum_{m,n} \tilde{C}_{m,n}^{-1} \frac{\partial}{\partial \phi_m} \frac{\partial}{\partial \phi_n}
$$

where $H_J$ and $H_C$ represent, correspondingly, Josephson and Coulomb parts of the Hamiltonian, and $\tilde{C}_{m,n}$ is the matrix of the capacitances. There are several different contributions to $\tilde{C}$: self-capacitances of the wires $C_l$ (with respect to substrate), the contact capacitances $C_J$ and mutual capacitances of wires $C_{ll'}$. Below we will assume, that the self-capacitance is the largest of all, $C_l \gg C_0, NC_J$ (the factor $N$ accounts for the fact that there are $N$ contacts along each wire). These conditions allow us to neglect all mutual capacitances and consider the matrix $C_{m,n}$ to be diagonal with eigenvalues $C_l$. It is convenient to rewrite the Hamiltonian in terms of “spin” variables $s_m = e^{i\phi_m}$. Choosing the Landau gauge for the vector potential and introducing $J_0$ by $E_J = \frac{J_0}{\sqrt{N}}$ so that the transition temperature remains constant in the limit $N \to \infty$ at fixed $J_0$ we get

$$
\mathcal{H} = -\sum_{m,n} s_m J_{mn} s_n + \frac{E_C}{2} \sum_n Q_n^2
$$

where $Q_n = -i\partial/\partial \phi_n$ is the charge operator conjugated to the phase $\phi_n$, $E_C = \frac{4e^2}{C_l}$, and $J_{mn}$ is the coupling matrix

$$
\hat{J} = \begin{pmatrix}
0 & J^\uparrow \\
J^\downarrow & 0
\end{pmatrix}
$$

with $J_{jk} = \frac{J_0}{\sqrt{N}} \exp(2\pi i\alpha_{jk}/N)$ and $1 \leq (j,k) \leq N$ where $j(k)$ is the index of the horizontal (vertical) wires; $s_m = e^{i\phi_m}$ where the $\phi_m$ are the superconducting phases of the 2N wires. Here $\alpha = NH^2/\Phi_0$ is the flux per unit strip, $l$ is the inter-node spacing and $\Phi_0$ is the flux quantum.

Because every horizontal (vertical) wire is linked to every vertical (horizontal) wire, the connectivity in this model is high ($N$) and it is accessible to a mean-field treatment (its classical version was developed in [3]). For $\frac{1}{N} \ll \alpha < 1$ there exists an extensive number of metastable solutions which minimize the Josephson (“potential”) part of the Hamiltonian [2]; these minima are separated by the barriers that scale with $N$. A similar (classical) long-range network with disorder was previously found to display a spin glass transition [2] for $\alpha \gg \frac{1}{N}$: in the absence of short-range phase coherence between wires ($\alpha > 1$) it is equivalent to the Sherrington-Kirkpatrick model [4]. Physically this glassy behavior occurs because the phase differences associated with the couplings, $J_{jk}$, acquire random values and fill the interval $(0, 2\pi)$ uniformly. For the periodic case, this condition is satisfied in the “incommensurate window” $\frac{\pi}{N} \ll \alpha \leq 1$, for which the magnetic unit cell is larger than the system size so that the simple “crystalline” phase is inaccessible. There are thus no special field values for which the number of minima of the potential energy are not extensive, in contrast to the situation for $\alpha > 1$. Below we will consider the case $1/N \ll \alpha \ll 1$ only. As follows from the previous studies, the characteristic energy scale related to the potential energy $\mathcal{H}_J$ is of the order of the glass transition temperature of the classical system, $T_G \approx J_0/\sqrt{\alpha}$. The zero-$T$ transition we study here is driven by the competition between Josephson and Coulomb energies, the scale of the latter being $E_C = 4e^2/C_l$. Thus, we expect
that the quantum transition occurs at $J_0/\sqrt{\alpha} \sim E_C$. Our goal is to show that such a (continuous) phase transition indeed occurs and to study the critical behavior near the transition point. Below, in the main part of the paper, we measure all energies in units of $E_C$, and return to the physical units only in the final expression for the critical behavior of the ac diamagnetic susceptibility.

II. QUANTUM LOCATOR EXPANSION

We are going to develop a diagram technique for the Hamiltonian (2) which will be very similar to the one employed previously for the classical Langevin dynamics of the same array. The idea is to treat Coulomb part of the Hamiltonian as the zero-level approximation, and construct expansion in powers of Josephson coupling constant $J_0$, keeping all the terms of the lowest order in the coordination number $1/N$. Thus our approach can be considered as quantum version of the Thouless-Anderson-Palmer method.

The diagram technique for the Matsubara Green function

$$G_{m,n}(\tau) = -(T_s s_m(\tau) s_n^\dagger(0)), \ s(\tau) = e^{-\tau H} se^{\tau H}$$

is closely related to the one developed in\textsuperscript{5}. Dyson equation for the frequency-dependent matrix Green function reads (note that in our units $E_C = 1$):

$$G_\omega = \frac{1}{G_\omega^{-1} - (JJ^\dagger)G_\omega}$$

where we introduced the local Green functions $\tilde{G}_\omega$ that is irreducible with respect to the $J_{ij}$ lines. The matrix $(JJ^\dagger)_{ij}$ depends only on the “distance” $i - j$ and acquires a simple form in Fourier space $(JJ^\dagger)_{p} = (J_0^2/\alpha)\theta(\alpha \pi - |p|)$; therefore in this representation

$$G_\omega(p) = \frac{\theta(\alpha \pi - |p|)}{\tilde{G}_\omega^{-1} - \frac{J_0^2}{\alpha} \tilde{G}_\omega} + \frac{\theta(|p| - \alpha \pi)}{\tilde{G}_\omega^{-1}}$$

Diagrammatically Eq.\textsuperscript{[3]} and the equation for the irreducible function $\tilde{G}_\omega$ are represented by the graphs shown in Fig. 1.

Note that the equation for $\tilde{G}$ is written in the lowest nontrivial order in $\alpha$. Indeed, it is seen from Eq.\textsuperscript{[4]} that nontrivial part of the Green function which contains critical slowing down, is of relatively small weight $\sim \alpha$. It is this long-time part of $G_\omega$ which enters 3-line diagram on Fig.1 and makes it proportional to $\alpha^3$; more complicated diagrams either contain even higher powers of $\alpha$, or are small as $1/N$. Since the second diagram on Fig.1 contains single-site functions only, the whole system of equations can be written in the form

$$G(\omega) = (1 - \alpha)\tilde{G}(\omega) + \tilde{G}(\omega); \quad \tilde{G}(\omega) = \frac{\alpha \tilde{G}(\omega)}{1 - J_0^2 \tilde{G}^2(\omega)/\alpha}$$

$$\tilde{G}(\omega) = \tilde{G}_0(\omega) + \Sigma(\omega); \quad \Sigma(\omega) = \chi_3^2 \left(\frac{J_0^2}{\alpha}\right) \int \tilde{G}^2(t) \exp(i\omega t) \, dt$$

Here $\chi_3 \sim 1$, as in\textsuperscript{4}, is a static value of four-point vertex denoted as a square box in Fig.1 (we assume that, like in\textsuperscript{5}, the main critical anomaly is contained in the 2-point Green function alone). Equations (7,8) should be solved with obvious initial condition:

$$G(t = 0) = \int \frac{d\omega}{2\pi} G(\omega) = 1$$
Similar normalization condition in the classical problem was sufficient to determine \( \tilde{G}(\omega = 0) \) exactly. The same calculation seems to be difficult for the present quantum problem and we will not perform it here. Instead, we will use general properties of the function \( \tilde{G}_0(\omega) \), namely: i) \( \tilde{G}_0(0) \approx 1 \), and ii) \( \tilde{G}_0(\omega) \) is analytic at low \( \omega \) and has characteristic frequency scale of the order of 1. In doing so, we will not determine exact position of the phase transition (i.e. critical value \( J_c \) of the coupling strength \( J_0 \)), but we will show the existence of continuous transition and find the form of critical scaling.

Let us first analyze equations (12,13) omitting the term with \( \Sigma \), and using simplest interpolation \( \tilde{G}_0(\omega) = (\lambda + \omega^2)^{-1} \). Then initial condition (9) gives us the equation for \( \alpha \) which do not contain small parameter \( a \)

Thus \( \lambda \sim 1 \) as long as \( g \leq 1 \). On the other hand, at \( g \gg 1 \) the solution is \( \lambda - g \equiv a \approx (\alpha/4)^2 \). The value of \( a \) determines the asymptotic decay rate of the Green function

with \( \Sigma \) being neglected. It will be seen below that \( a \approx \alpha \) and thus \( \lambda \sim 1 \) near the phase transition point \( g = g_c \) (we will see also that \( \Sigma \sim \alpha \) and thus it is much smaller than the \( \omega^2 \) term at high frequencies \( \omega \gg \alpha^{1/2} \)). It means that the parameter \( a \) can be considered as a smooth function of \( g \) in the vicinity of \( g_c \). Clearly, this conclusion does not depend on the model form of \( \tilde{G}_0(\omega) \) used in the above analysis.

Now we re-introduce \( \Sigma(\omega) \) into the equations for \( \tilde{G}(\omega) \) and focus on its low-frequency behavior at \( \omega \leq \sqrt{\alpha} \):

where \( \tilde{g} = g \lambda^{1/3} \sim g \). Strictly speaking, Eqs.(12,13) do not form closed system since \( a \) should be determined with the use of Eq.(1) which contains high-frequency contributions. However, in this high-frequency region (which produces the main contribution to the normalization condition (1) ) the contribution of \( \Sigma(\omega) \) can be neglected and thus \( a \) can be treated as an external control parameter which governs the transition. Green function defined by the Eqs.(12,13) acquires singularity when \( 2\Sigma(0) = a \). To find the form of this singularity, we make use of the scaling Anzats \( \tilde{G}(t) = qt^{-\nu} \) and neglect \( \omega^2 \) term in the denominator of Eq.(12). Then we find \( \nu = 1/2 \) and \( q \sim \tilde{g}^{-1} \alpha^{1/4} \). This critical-point solution matches the short-time asymptotics (11) at \( t \sim \alpha^{-1/2} \). The estimate for \( \Sigma(0) \) which follows from the above scaling Anzats,

where \( q \approx a \) at \( \tilde{g} \sim 1 \) and \( a \approx \alpha \), as it was expected. These estimates show that second-order phase transition with critical slowing down may indeed occur in the above range of parameters. In the next section we will study the vicinity of the critical point in more details.

### III. Green Function near the \( T = 0 \) Transition Point

To study the form of the critical singularity, it is convenient to define universal scaling functions \( G(\omega) \) and \( \sigma(\omega) \) which do not contain small parameter \( \alpha \ll 1 \), and the parameter \( b \) measuring the proximity to the critical point:

Now Eqs.(12,13) acquires the following form:

\[
G(\tilde{\omega}) = \tilde{G}(\tilde{\omega}); \quad \alpha \Sigma(\omega) = \sigma(\tilde{\omega}); \quad \tilde{\omega} = \omega/\sqrt{\alpha}; \quad b = (a - 2\Sigma(0))/\alpha
\]
Exactly at the critical point \( b = 0 \) the solution of Eq. (15) is

\[
G(\tilde{\omega}) = \left( \frac{\pi}{8} \right)^{1/4} \tilde{g}^{-3/2} |\tilde{\omega}|^{-1/2}
\]  

(16)

Consider now the vicinity of the critical point, \( 0 < b \ll 1 \). It is clear from the form of the solution (16) that the similar result should be valid at \( \tilde{\omega} \gg b^2 \). Next we focus on the long-time, low-\( \omega \) region, \( \tilde{\omega} \ll b^2 \) and will look for the purely exponential solution

\[
G(\tilde{t}) = G_1 \exp(-\tilde{t}/\tau_0).
\]  

(17)

This type of asymptotic solution is known to exist in the classical version of the same model (cf. 5, 7). In the present problem, one can show, considering analytic structure of Eqs.(15), that at \( b > 0 \) the singularity of \( G(\tilde{\omega}) \) which is closest to the real axis of \( \omega \), is necessarily a simple pole at some \( \tilde{\omega} = i/\tau_0 \); the next singularity may exist at \( \tilde{\omega} \geq 3i/\tau_0 \). Solution of Eqs.(15) with the Anzats (17) in the region \( \tilde{t} \gg \tau_0 \) determines parameters \( \tau_0 \) and \( G_1 \) as functions of \( b \):

\[
\tau_0 = \left( \frac{32/27}{g^3} \right)^{1/2} \frac{1}{b^2} \quad G_1 = \left( \frac{27}{2} \right)^{1/2} G^3 b
\]  

(18)

This solution is similar to the one found in 5; however, an important difference is that in the present case the prefactor \( G_1 \) scales to zero at the critical point \( b = 0 \).

The full solution in the vicinity of the transition point should contain both (16) and (18) as asymptotic solutions, and can be written in the form

\[
G(\tilde{t}) = \frac{1}{\sqrt{\tilde{t}}} f \left( \frac{\tilde{t}}{\tau_1} \right) + G_1 \exp \left( -\frac{\tilde{t}}{\tau_0} \right)
\]  

(19)

where \( f(x) \) is some scaling function approaching constant at \( x = 0 \) and fast decaying at \( x \to \infty \), and \( \tau_1 \leq \tau_0/3 \). To confirm an existence of this type of solution, we solved Eqs.(15) numerically for several values of \( b \ll 1 \). The results of this computation are shown on Fig. 1. Clearly, all three functions \( G(\tilde{\omega}) \) coincide in the high-\( \omega \) region, there they are close the square-root asymptotic (16). Low-frequency parts (for \( \omega \leq 0.08 \)) of these solutions can be made coinciding by a proper rescaling of their arguments, \( \omega^* = \Lambda \omega \). Fig. 2 demonstrates linear relation between \( b^{-2} \) and the scaling coefficient \( \Lambda \), as it was suggested by Eqs.(18,19).

![Fig. 1. Low-frequency asymptotic behavior of \( \tilde{G}(\omega) \) at different \( b \) at \( T = 0 \).](image-url)
These results confirm the existence of the $T = 0$ critical behavior of the type of Eq.(19).

**IV. CRITICAL BEHAVIOR AT $T > 0$**

The above results refer to the zero-$T$ phase transition controlled by the single parameter $g = J_0 / \sqrt{\alpha}$. We found that this phase transition is a continuous one and the corresponding critical behavior differs considerably from the one found in an analogous classical model. In particular, at $T = 0$ critical point $g = g_c$ there is no “plato” solution with approximately constant $G(t)$ at $t \rightarrow \infty$ which is known to be a peculiar property of regular classical glasses. Now we consider low but non-zero temperatures $T = \beta^{-1}$ and will find how “classical” critical scaling “grow up” from the “quantum” background; we also find the low-temperature shape of the phase transition line on the plane $(T, g)$.

Green function is defined now at discrete frequencies $\omega_n = 2\pi n T$ and the equations (12), (13) can be written as

$$\hat{G}(\omega_n) = \frac{\alpha}{a - 2\Sigma(\omega_n) + \omega^2}, \quad \Sigma_M(\omega_n) = g^6 \int_0^\beta \hat{G}^3(t) \exp(i\omega_n t) \, dt$$

It will be convenient now to perform analytic continuation of Eqs.(20) and rewrite them in terms of real-time correlation function $D(t) = \langle [S(t), S(0)]_+ \rangle$ and response function $\chi(t) = i[S(t), S(0)]_+ \theta(t)$. The functions $G(\omega_n)$, $D(\omega)$ and $\chi(\omega)$ are related as follows:

$$G(-i\omega + \eta) = \chi(\omega), \quad \eta \rightarrow +0, \quad D(\omega) = \text{Im} \chi(\omega) \coth(\omega/2T)$$

After analytic continuation the Eqs.(21) can be written as

$$\chi(\omega) = \frac{\alpha}{\bar{a} - 2\Sigma(\omega)}; \quad \Sigma(\omega) = 8g^6 \int_0^\infty D^2(t) \chi(t) (\exp(i\omega t) - 1) \, dt; \quad \bar{a} = a - 2\Sigma(\omega = 0)$$
Thus, the expression for magnetic susceptibility has the following form:

\[ \chi_M(\omega) = \left( \frac{2e}{\hbar c} \right)^2 t^2 \sum_{mn} S^*_m \tilde{J}_{mn} S_n , \]

where \( \tilde{J}_{mn} = i m n J_{mn} \). Then magnetic susceptibility \( \chi_M \) can be found making use of the Kubo formula: \( \chi_M(t-t') = i \left[ M(t), M^\dagger(t') \right] \theta(t) \) which leads to the expression

\[ \chi_M(\omega) = \left( \frac{2e}{\hbar c} \right)^2 t^2 \int_0^\infty (e^{i\omega t} - 1) \text{Re} \text{Tr} \tilde{J}(t) \tilde{J}(t) dt . \]

Here we omit the term, containing irreducible four-spin correlator (of the order of \( 1/N \)), and take into account that \( M(H = 0) = 0 \). Note, that equation \( (23) \) formally coincides with classical formula for magnetic response.\(^4\) The matrix functions \( \tilde{J}(t) \) and \( \tilde{\chi}(t) \) contain elements (denoted by superscript \( (0) \)) belonging to the same (horizontal or vertical) sublattice of our array, as well as “off-diagonal” elements (with superscript \( (1) \)) which describe correlation of phases on wires of different type (horizontal/vertical). Relation between these functions is as follows: \( \tilde{\chi}(\omega)^{(1)} = J \tilde{G}(\omega) \tilde{\chi}(\omega)^{(0)} \). Thus, the expression for magnetic susceptibility has the following form:

\[ \chi_M(\omega) = \left( \frac{2e}{\hbar c} \right)^2 \left( \frac{t^2}{12} \right)^2 N^5 \frac{J^2}{\alpha^2} I(\omega) \]

where

\[ I(\omega) = \int \left( \delta(t - t_3) - \frac{J_0^2 \tilde{G}_1^2 t - t_1}{\alpha} \right) \delta(t_1) D(t_1) (e^{i\omega t} - 1) \theta(t) dt_1 dt \]

Near the transition point only long-time parts of all the function in \( (27) \) are relevant, and this expression can be reduced to the form

\[ I(\omega) = (\Sigma(\omega) - \Sigma(0)) \int \chi(t) D(t) e^{i\omega t} dt , \]

where the first factor came from the first brackets in \( (27) \); note that it vanishes in the limit \( \omega \to 0 \).

Using the solution \( (16) \) we obtain at the quantum critical point \( J = J_c \):

\[ I(\omega) = \frac{\alpha}{2\pi} \left( \frac{\alpha \pi}{8\theta^6} \right)^{1/4} \sqrt{\omega \ln \omega} \]

\[ \text{V. DIAMAGNETIC RESPONSE NEAR THE TRANSITION POINT} \]

Correlation and response functions \( D(t) \) and \( \chi(t) \) are not directly measurable in our system, but they can be used in order to calculate measurable physical quantities which is dynamic diamagnetic susceptibility \( \chi_{\text{D}}(\omega) \), like it was done previously for the classical problem.\(^1\) Total magnetic moment induced by time-dependent external magnetic field is given by

where we omitted \( \omega^2 \) term which is irrelevant in the vicinity of the critical point. Equations \( (22) \) form (together with the Fluctuation-Dissipation relation (second of Eqs. \( (21) \)) a closed set which determines critical singularity at \( T > 0 \). Formally Eqs.\( (22) \) coincide with the corresponding “classical” equations from\(^4\), the only difference is in the form of the Fluctuation-Dissipation relation.

Let us consider low temperature region \( T \ll \sqrt{\alpha} \). As long as we are interested in the long-time asymptotic \( t \gg 1/T \), the correlation and response functions are related by classical FDT: \( D(\omega) = 2T/\omega \text{Im} \chi(\omega) \). Characteristic times, which is relevant in \( (22) \), are also belong to classical region \( t \gg 1/T \). Therefore the correlation function at the transition point has the same critical behavior as in the classical case: \( \lim_{t \to \infty} D(t) = q \). However parameter \( a \equiv \lambda - g \) is determined by the ”quantum” region of frequencies \( \omega \gg T \), i.e. by the equation \( (10) \). The substitution of this expression to the equations \( (22) \) allows us to find

\[ q \sim \alpha^{1/4} T^{-1/2}; \quad a \sim \alpha^{3/4} T^{-1/2} \]

In the short-time domain \( t \ll T^{-1} \) the zero-\( T \) critical solution with \( D(t) \sim \alpha^{1/4} t^{-1/2} \) is valid. Equation \( (23) \) demonstrates the way the ”classical” solution with nonzero \( \lim_{t \to \infty} D(t) \) grows up with the temperature increase.
Near the $T = 0$ transition point at high enough frequencies $\omega \gg (J/J_c - 1)^2 \alpha^{-3/2}$ equation (29) still holds. In opposite case of low frequencies

$$I(\omega) = \frac{8J_c^2\alpha^3}{81(J_c - J)^3} \frac{\omega^2 \alpha^{1/2}}{g}.$$  

(30)

Note that the parameter $\tilde{g}$ (which is known up to the factors of order 1 only) does not enter the low-$\omega$ asymptotic of $I(\omega)$.

Making use of the Eqs.(28,29,30) and returning to the original units of frequency, we obtain finally ac diamagnetic susceptibility near the quantum transition point:

$$\chi_M(\omega) \approx \left(\frac{2e}{\hbar c}\right)^2 \frac{J_c^2}{2e} \frac{\sqrt{\alpha} \ln(\omega C_\tau / e^2)}{\ln(\omega C_\tau / e^2)}, \quad \omega \gg \frac{C_1(J - J_c)^2}{e^2\alpha^{5/2}}$$

(31)

$$\chi_M(\omega) = \left(\frac{2e}{\hbar c}\right)^2 \frac{J_c^3}{81\sqrt{\alpha} C_\tau} \frac{\sqrt{\alpha} \ln(\omega C_\tau / e^2)}{\ln(\omega C_\tau / e^2)}, \quad \omega \ll \frac{C_1(J - J_c)^2}{e^2\alpha^{5/2}}$$

(32)

The above expressions are valid at the frequencies $\omega \gg T/h$, otherwise ”classical” asymptotic for the Green functions should be used and will lead to the frequency dependencies like those in $\tilde{I}(\omega)$.

**VI. CONCLUSIONS**

We have shown that regularly frustrated long-range Josephson array has a quantum (zero-temperature) phase transition between Coulomb-dominated insulator phase and a superconductive state. This transition happens when the nearest-neighbors Josephson coupling exceeds the critical value, $J_{ij} \sim N^{-1/2}\sqrt{\alpha}/C_\tau$, where $C_\tau$ is the self-capacitance of an individual wire.

We found that quantum critical behavior of the model at $J \to J_c$ is different from that of an analogous classical system $\tilde{I}$: at the quantum critical point $D(t) \sim t^{-1/2}$ while at the classical critical point $q = \lim_{t \to \infty} D(t)$. However, at any non-zero temperature a ”classical” type of asymptotic behavior is recovered at the longest-times, $t \gg h/T$, leading to $q \propto T^{1/2}$. Near the $T = 0$ critical point the gap in the excitation spectrum decreases as $\alpha^{-1} \propto (J_c - J)^2$.

Near the phase transition the effective inductance $\mathcal{L}$ of the array (defined by $\mathcal{L} \propto \frac{\partial^2 \chi_M(\omega)}{\partial \omega^2}|_{\omega \to 0}$) diverges as $(J_c - J)^{-3}$, this shows that the glass state has a macroscopic phase rigidity (cf. also $\tilde{I}$). Right at the critical point we find unusual frequency behavior of the complex diamagnetic susceptibility, $\chi_M(\omega) \propto \frac{\sqrt{\alpha} \ln \omega}{\ln \omega}$.

Frustrated nature of couplings in our array and comparison with the previous results on the classical version of the same model indicates that the high-$J$ state is a quantum glassy superconductor. The $T = 0$ nonergodic properties (irreversibility, ageing) remain an open question; note here that recent studies of nonequilibrium glassy behavior in a p-spin spherical quantum model assumed strongly dissipative (overdamped) dynamics of the whereas dynamics relevant for the Josephson array at $T = 0$ must be underdamped.

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