MINIMAX OPTIMAL SEQUENTIAL HYPOTHESIS TESTS FOR MARKOV PROCESSES

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Under mild Markov assumptions, sufficient conditions for strict minimax optimality of sequential tests for multiple hypotheses under distributional uncertainty are derived. First, the design of optimal sequential tests for simple hypotheses is revisited and it is shown that the partial derivatives of the corresponding cost function are closely related to the performance metrics of the underlying sequential test. Second, an implicit characterization of the least favorable distributions for a given testing policy is stated. By combining the results on optimal sequential tests and least favorable distributions, sufficient conditions for a sequential test to be minimax optimal under general distributional uncertainties are obtained. The cost function of the minimax optimal test is further identified as a generalized f-dissimilarity and the least favorable distributions as those that are most similar with respect to this dissimilarity. Numerical examples for minimax optimal sequential tests under different uncertainties illustrate the theoretical results.

1. Introduction. Sequential hypothesis tests are well-known for being highly efficient in terms of the number of required samples and, as a consequence, for minimizing the decision delay in time-critical applications. In his seminal book [64], Wald showed that, compared to fixed sample size tests, sequential tests can reduce the average number of samples by a factor of two. In general, the ability to allow the overall number of samples to depend on the current history makes sequential procedures more flexible and adaptable than procedures whose sample size is chosen a priori. Comprehensive overviews of sequential hypothesis testing and related topics can be found in [64, 56, 23, 49, 59], to name just a few.

A well-established drawback of sequential hypothesis tests is that their higher efficiency depends critically on the assumption that the process generating the observations indeed follows the assumed model. If this is not the case, i.e., if a model mismatch occurs, the number of samples can increase significantly—compare, for example, [59, Figure 3.4], which illustrates the influence of a model mismatch on the expected run-length of a sequential test.

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for the mean of an auto-regressive process. This observation lead Kiefer and Weiss to propose a sequential test that, in addition to meeting the targeted error probabilities under the hypotheses, minimizes the maximum expected run-length over all feasible distributions [13, 35]. Different variations of the corresponding optimization problem are known as Kiefer–Weiss problem or modified Kiefer-Weiss problem and have received considerable attention in the literature [38, 12, 47, 68]. However, to the present day, exact solutions to the Kiefer–Weiss problem have only been shown for special cases of binary hypothesis tests.

A natural generalization of the Kiefer–Weiss problem is to include the error probabilities in the minimax criterion; that is, to design a test whose maximum error probabilities are minimal over the set of feasible distributions. For fixed sample sizes, this minimax approach to the design of statistical tests was pioneered by Huber [29] and is known as robust hypothesis testing or robust detection. In general, robust hypothesis tests sacrifice some efficiency under ideal conditions in order to be less sensitive to deviations from the ideal case [30]. In this sense, robust hypothesis tests, and robust statistics in general, form a middle ground between parametric and non-parametric approaches. For an overview of existing results, recent advances, and applications of robust statistics see, for example, [33, 39, 69, 1, 70].

The idea underlying this paper is to leverage both sequential and robust hypothesis testing. Ideally, a robust sequential test is fast and reliable, i.e., it requires fewer observations than a fixed sample size test and at the same time works reliably under model mismatch. In this sense, both concepts complement each other: by sequentially performing a robust test, the loss in nominal efficiency can be compensated; by robustly performing a sequential test, its sensitivity to model mismatch can be reduced.

The literature on minimax optimal sequential hypothesis testing is rather scarce. To the best of our knowledge, the general design of strictly minimax optimal sequential tests has not been treated in the literature yet. However, preliminary and related works exist. Some of the earliest results in the field can be found in [40], where a minimax test for the mean of two normal distributions is investigated, and [11], where a minimax optimal procedure for the detection of a mean shift in Brownian motion is derived. The latter has been further generalized in [55]. The majority of the existing literature deals with the design of asymptotically minimax sequential tests. Asymptotic results exist, for example, for tests of distributions of the exponential family [28], the presence of a signal in additive noise [14], multiple distributions with unknown parameters [6], and discrete distributions [20]. An approach to robust sequential testing based on adaptive nonlinearities is suggested in
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Closely related to the problem of minimax sequential tests is the problem of minimax quickest change detection, which was studied in [7, 60, 21, 2]. A noteworthy exception from the asymptotic approach to minimax sequential testing is the work presented in [34], which is concerned with the design of strictly minimax optimal tests for discrete distributions. Finally, the minimax optimal binary sequential test is studied, under slightly stricter assumptions, in [15], of which this paper is an extension and generalization.

The main contribution of this paper is a coherent framework for the design of strictly minimax optimal sequential hypothesis tests for Markov processes under mild assumptions. In particular, no assumptions about the nature of the model mismatch are introduced and the number of hypotheses is assumed to be arbitrary, but finite. An exact and general solution to the general Kiefer–Weiss problem is included as a special case. Moreover, the presented framework allows for the design of strictly minimax optimal tests in practice. This is illustrated with numerical examples, which, to the best of our knowledge, are the first implementations of minimax sequential tests in the literature.

2. Notation and Problem Formulation. In this section, the minimax optimal sequential testing problem is defined and some common notations are introduced. Notations not covered here are defined when they occur in the text.

2.1. Notation. Random variables are denoted by upper case letters, their realizations by lower case letters. Analogously, probability distributions are denoted by upper case letters, their densities by the corresponding lower case letters. Blackboard bold is used to indicate product measures. Measurable sets are denoted by tuples \((\Omega, \mathcal{F})\). Boldface lower case letters are used to indicate vectors; no distinction is made between row and column vectors. The inner product of two vectors \(x\) and \(y\) is denoted by \(\langle x, y \rangle\), the element-wise product by \(xy\). All comparisons between vectors are defined element-wise. The indicator function of a set \(A\) is denoted by \(I(A)\). All comparisons between functions are defined point-wise.

The notation \(\partial_{y_k} f(y)\) is used for the subdifferential [52, §23] of a convex function \(f: \mathcal{Y} \subset \mathbb{R}^K \rightarrow \mathbb{R}\) with respect to \(y_k\) evaluated at \(y\), i.e.,

\[
\partial_{y_k} f(y) := \{ c \in \mathbb{R} : f(y') - f(y) \leq c (y'_k - y_k) \ \forall y' \in \mathbb{R}^K \}.
\]

The superdifferential of a concave function is defined analogously. Both are referred to as generalized differentials in what follows. The length of the
interval corresponding to $\partial_{y_k} f(y)$ is denoted by

\begin{equation}
|\partial_{y_k} f(y)| = \sup_{a,b \in \partial_{y_k} f(y)} |a - b|.
\end{equation}

If a function $f_{y_k}$ exists such that $f_{y_k}(y) \in \partial_{y_k} f(y) \forall y \in \mathcal{Y}$, then $f_{y_k}$ is called a partial generalized derivative of $f$ with respect to $y_k$. The set of all partial generalized derivatives $f_{y_k}$ is denoted by $\partial_{y_k} f$.

2.2. Underlying Stochastic Process. Let $(X_n)_{n \geq 1}$ be a discrete-time stochastic process with values in $(\Omega_X, \mathcal{F}_X)$. The joint distribution of $(X_n)_{n \geq 1}$ on the cylinder set

\begin{equation}
(\Omega^N_X, \mathcal{F}^N_X) := \left( \prod_{n \geq 1} \Omega_X, \prod_{n \geq 1} \mathcal{F}_X \right)
\end{equation}

is denoted by $\mathbb{P}$, the conditional or marginal distributions of an individual random variable $X$ on $(\Omega_X, \mathcal{F}_X)$ by $P$, and the natural filtration [8, Definition 2.32] of the process $(X_n)_{n \geq 1}$ by $(\mathcal{F}_X^n)_{n \geq 1}$. In order to balance generality and tractability, the analysis in this paper is limited to stochastic processes that satisfy the following three assumptions.

1. The process $(X_n)_{n \geq 1}$ admits a time-homogeneous Markovian representation. That is, there exists a $(\Omega_\Theta, \mathcal{F}_\Theta)$-valued stochastic process $(\Theta_n)_{n \geq 1}$ adapted to $(\mathcal{F}_X^n)_{n \geq 0}$ such that

\begin{equation}
P(X_{n+1} \in \mathcal{E} \mid X_1 = x_1, \ldots, X_n = x_n) = P(X_{n+1} \in \mathcal{E} \mid \Theta_n = \theta_n) =: P_{\theta_n}(\mathcal{E}).
\end{equation}

for all $n \geq 1$ and all $\mathcal{E} \in \mathcal{F}_X$. The distribution of $X_1$ is denoted by $P_{\theta_0}$, where $\theta_0$ is assumed to be deterministic and known a priori. An extension to randomly initialized $\theta_0$ should not be hard, but will not be entered.

2. There exists a function $\xi: \Omega_\Theta \times \Omega_X \to \Omega_\Theta$ that is measurable with respect to $P_\theta$ for all $\theta \in \Omega_\Theta$ and that satisfies

\begin{equation}
\Theta_{n+1} = \xi(\theta_n, X_{n+1}) =: \xi_{\theta_n}(X_{n+1})
\end{equation}

for all $n \geq 0$.

3. For all $\theta \in \Omega_\Theta$, the probability measure $P_\theta$ defined in Assumption 2 admits a density $p_\theta$ with respect to some $\sigma$-finite reference measure $\mu$. 
The set of distributions $\mathbb{P}$ on $(\Omega_X^N, \mathcal{F}_X^N)$ that satisfy these three assumptions is denoted by $\mathbb{M}$. The set of distributions $P$ on $(\Omega_X, \mathcal{F}_X)$ that admit densities with respect to $\mu$ is denoted by $\mathcal{M}_\mu$.

The above assumptions are rather mild and are introduced primarily to simplify the presentation of the results. In general, the sufficient statistic $\Theta$ can be chosen as a sliding window of past samples, i.e., $\Theta_n = (X_{n-m}, \ldots, X_n)$, where $m$ is a finite positive integer. Hence, the presented results apply to every discrete-time Markov process of finite order. However, in order to implement the test in practice, $\Omega_\Theta$ should be sufficiently low-dimensional—compare the examples in Section 8. As long as the existence of the corresponding densities is guaranteed, the reference measure $\mu$ in Assumption 3 can be chosen arbitrarily. This aspect can be exploited to simplify the numerical design of minimax sequential tests and is discussed in more detail in Sections 7 and 8.

2.3. Uncertainty Model and Hypotheses. For general Markov processes, the question of how to model distributional uncertainties is non-trivial and has far-reaching implications on the definition of minimax robustness. In the most general case, the joint distribution $\mathbb{P}$ is subject to uncertainty. However, defining meaningful uncertainty models for $\mathbb{P}$ is an intricate task and usually neither feasible nor desirable. An approach that is more tractable and more useful in practice is to assume that at any given time instant $n \geq 1$ the marginal or conditional distribution of $X_n$ is subject to uncertainty.

In this paper, it is assumed that the conditional distributions $P_\theta$, as defined in (2.4), are subject to uncertainty. More precisely, for each $\theta \in \Omega_\Theta$ the conditional distribution $P_\theta$ is replaced by an uncertainty set of feasible distributions $\mathcal{P}_\theta \subset \mathcal{M}_\mu$. This model induces an uncertainty set for $\mathbb{P}$, which is given by

$$
\mathcal{P} := \left\{ \mathbb{P} \in \mathbb{M} : \mathbb{P} = \prod_{n \geq 0} P_{\theta_n}, P_{\theta_n} \in \mathcal{P}_{\theta_n} \right\}
$$

and is completely specified by the corresponding family of uncertainty sets for the conditional distributions $\{\mathcal{P}_\theta : \theta \in \Omega_\Theta\}$.

The goal of this paper is to characterize and design minimax optimal sequential tests for multiple hypotheses under the assumption that under each hypothesis the distribution is subject to the type of uncertainties introduced above. That is, each hypothesis is given by

$$
H_k : \mathbb{P} \in \mathcal{P}_k, \quad k = 1, \ldots, K,
$$
where all $P_k$ are of the form (2.6) and are defined by a corresponding family of conditional uncertainty sets $\{P^{(k)}_\theta : \theta \in \Omega_\Theta\}$. Note that the parameter $\theta$, which corresponds to the sufficient statistic in Assumption 2, does not depend on $k$, i.e., the statistic needs to be chosen such that it is sufficient under all hypotheses.

Before proceeding, it is useful to illustrate the assumptions on the underlying stochastic process and the proposed uncertainty model with an example. Consider an exponentially weighted moving-average process, i.e.,

$$X_{n+1} = \sum_{l=1}^{\infty} a^l X_{n+1-l} + W_{n+1}, \quad (2.8)$$

where $a \in (-1, 1)$ is a known scalar and $(W_n)_{n \geq 1}$ is a sequence of independent random variables that are identically distributed according to $P_W$. This process can equivalently be written as

$$X_{n+1} = a\Theta_n + W_n, \quad (2.9)$$

where the sufficient statistic $\Theta_n$ can be updated recursively via

$$\Theta_{n+1} = \xi_n(X_{n+1}) = \theta_n + X_{n+1}. \quad (2.10)$$

In order to introduce uncertainty to this model, it is assumed that with probability $\varepsilon$ the increment $W_n$ is replaced by an arbitrarily distributed outlier. This model yields the following family of conditional uncertainty sets

$$P_\theta = \{P \in \mathcal{M}_\mu : P(\mathcal{E}) = (1 - \varepsilon)P_W(\mathcal{E} - \theta) + \varepsilon H(\mathcal{E}), H \in \mathcal{M}_\mu\}, \quad (2.11)$$

where $\mathcal{E} \in \mathcal{F}$ and $\mathcal{E} - \theta$ is shorthand for $\{x \in \Omega_X : x + \theta \in \mathcal{E}\}$. In Section 8, a variant of this example is used to illustrate the design of a minimax optimal test with dependencies in the underlying stochastic process.

2.4. Testing Policies and Test Statistics. A sequential test is specified via two sequences of randomized decision rules, $(\psi_n)_{n \geq 1}$ and $(\delta_n)_{n \geq 1}$, that are adapted to the filtration $(\mathcal{F}_X^n)_{n \geq 1}$. Each $\psi_n : \Omega_X^n \to [0, 1]$ denotes the probability of stopping at time instant $n$. Each $\delta_n : \Omega_X^n \to \Delta^K$ is a $k$-dimensional vector, $\delta_n = (\delta_{1,n}, \ldots, \delta_{K,n})$, whose $k$th element denotes the probability of deciding for $\mathcal{H}_k$, given that the test has stopped at time instant $n$. The set of randomized $k$-dimensional decision rules defined on $(\Omega_X^n, \mathcal{F}_X^n)$ is denoted by $\Delta^K_n$. The stopping time of the test is denoted by $\tau = \tau(\psi)$. 


For the sake of a more concise notation, let \( \pi = (\pi_n)_{n \geq 1} \), with \( \pi_n = (\psi_n, \delta_n) \in \Delta_n^1 \times \Delta_n^K \), denote a sequence of tuples of stopping and decision rules. In what follows, \( \pi \) is referred to as a *testing policy* and the set of all feasible policies is denoted by \( \Pi := \times_{n \geq 1} (\Delta_n^1 \times \Delta_n^K) \).

A test statistic is a stochastic process \( (T_n)_{n \geq 0} \) that is adapted to the filtration \( (\mathcal{F}_n^X)_{n \geq 1} \) and allows the stopping and decision rules to be defined as functions that map from the codomain of \( T_n \) to the unit interval. Of particular importance for this paper is the case where the sequence of test statistics \( (T_n)_{n \geq 0} \) is itself a time-homogeneous Markov process and the stopping and decision rules are independent of the time index \( n \). The corresponding testing policies are in the following referred to as time-homogeneous. This property is formalized in the following definition and significantly simplifies the derivation of non-truncated, strictly minimax optimal tests.

**Definition 1.** A policy \( \pi \in \Pi \) is referred to as time-homogeneous if there exists a \( (\Omega_T, \mathcal{F}_T) \)-valued stochastic process \( (T_n)_{n \geq 0} \) that is adapted to the filtration \( (\mathcal{F}_n^X)_{n \geq 1} \) and it holds that

\[
\psi_n = \psi(T_n) \quad \text{and} \quad \delta_n = \delta(T_n),
\]

where the functions \( \psi: \Omega_T \to [0, 1] \) and \( \delta: \Omega_T \to [0, 1]^K \) are independent of the index \( n \).

**2.5. Performance Metrics and Problem Formulation.** The performance metrics considered in this paper are the probability of erroneously rejecting the \( k \)th hypothesis, \( \alpha_k \), and the expected run-length of the sequential test, \( \gamma \). Both are defined as functions of the testing policy and the true distribution:

\[
\gamma(\pi, \mathbb{P}) := E_{\mathbb{P}}[\tau(\psi)],
\]

\[
\alpha_k(\pi, \mathbb{P}) := E_{\mathbb{P}}[1 - \delta_k, \tau],
\]

with \( k = 1, \ldots, K \). A generalization to performance metrics that are defined in terms of the pairwise error probabilities should not be difficult, but would considerably complicate notation, while adding little conceptual insight.

It is important to note that for the design of robust sequential tests the error probabilities and the expected run-length need to be treated as equally important performance metrics. On the one hand, reducing the sample size is typically the reason for using sequential tests in the first place. On the other hand, a test whose error probabilities remain bounded over a given uncertainty set, but whose expected run-length can increase arbitrarily, cannot
be considered robust. In other words, a robust test should not be allowed to delay a decision indefinitely in order to avoid making a wrong decision.

The first optimality criterion considered in this paper is the weighted sum cost, i.e.,

$$L_\lambda(\pi, \mathbf{P}) = \gamma(\pi, \mathbf{P}_0) + \sum_{k=1}^{K} \lambda_k \alpha_k(\pi, \mathbf{P}_k),$$

where $\mathbf{P} = (\mathbf{P}_0, \ldots, \mathbf{P}_K)$ denotes a $K + 1$ dimensional vector of distributions and $\lambda = (\lambda_1, \ldots, \lambda_K)$ denotes a $K$ dimensional vector of non-negative cost coefficients. The minimax problem corresponding to the cost function in (2.15) reads as

$$\inf_{\pi \in \Pi} \sup_{\mathbf{P} \in \mathcal{P}} L_\lambda(\pi, \mathbf{P}),$$

where $\mathbf{P} \in \mathcal{P}$ is used as a compact notation for $\mathbf{P}_k \in \mathcal{P}_k$, $k = 0, \ldots, K$.

The second optimality criterion is the expected run-length under constraints on the error probabilities. The corresponding minimax problem reads as

$$\inf_{\pi \in \Pi} \sup_{\mathbf{P}_0 \in \mathcal{P}_0} \gamma(\pi, \mathbf{P}_0) \text{ s.t. } \sup_{\mathbf{P}_k \in \mathcal{P}_k} \alpha_k(\pi, \mathbf{P}_k) \leq \overline{\alpha}_k,$$

where the constraint holds for all $k = 1, \ldots, K$ and $\overline{\alpha}_k$ denotes an upper bound on the probability of erroneously deciding against $\mathcal{H}_k$. The notation for the minimax optimal policies is fixed below and concludes the section.

**Definition 2.** The set of time-homogeneous policies that are optimal in the sense of (2.16) is denoted by $\Pi_\lambda^\star(\mathcal{P})$. The set of time-homogeneous policies that are optimal in the sense of (2.17) is denoted by $\Pi_\alpha^\star(\mathcal{P})$.

**3. Optimal Tests.** Assume that the distributions $\mathbf{P}_0, \ldots, \mathbf{P}_K \in \mathcal{M}$ are given and fixed. In this case, problems (2.16) and (2.17) reduce to the design of an optimal test for $K$ simple hypotheses, i.e.,

$$\inf_{\pi \in \Pi} L_\lambda(\pi, \mathbf{P})$$

and

$$\inf_{\pi \in \Pi} \gamma(\pi, \mathbf{P}_0) \text{ s.t. } \alpha_k(\pi, \mathbf{P}_k) \leq \overline{\alpha}_k.$$  

The notation for the corresponding optimal policies is fixed in the next definition.
Definition 3. The set of time-homogeneous policies that are optimal in the sense of (3.1) is denoted by $\Pi^*_\lambda(\mathcal{P})$. The set of time-homogeneous policies that are optimal in the sense of (3.2) is denoted by $\Pi^*_\alpha(\mathcal{P})$.

The solutions to both the unconstrained problem (3.1) and the constrained problem (3.2) can be found in the literature. The binary case ($K = 2$) was treated in [16] under the same assumptions as stated in Section 2.2. In [43], the general solution for an arbitrary number of hypotheses and arbitrary underlying stochastic processes is derived.

For easier reference, the solution of (3.1) is restated in this section. To this end, the functions $G^{(k)}(\lambda)$, $g(\lambda)$: $\mathbb{R}_{\geq 0}^{K+1}$ → $\mathbb{R}_{\geq 0}$ are introduced. Let

\begin{equation}
G^{(k)}(\lambda)(z) := \sum_{i=1, i\neq k}^{K} \lambda_i z_i = \left(\sum_{i=1}^{K} \lambda_i z_i\right) - \lambda_k z_k
\end{equation}

and

\begin{equation}
g(\lambda)(z) := \min_{k=1, \ldots, K} G^{(k)}(\lambda)(z) = \left(\sum_{i=1}^{K} \lambda_i z_i\right) - \max_{k=1, \ldots, K} \lambda_k z_k,
\end{equation}

where $z = (z_0, \ldots, z_K) \in \mathbb{R}_{\geq 0}^{K+1}$ and $\lambda \in \mathbb{R}_{\geq 0}^K$ is the vector of cost coefficients introduced in Section 2.5. Note that both $G^{(k)}(\lambda)$ and $g(\lambda)$ are independent of $z_0$; defining them as functions of the $K + 1$ dimensional vector $z$ unifies the notation in what follows.

The cost function that characterizes the optimal test is stated in the following theorem. It extends Theorem 2.1 in [16] to multiple hypotheses.

Theorem 1. Let $\lambda \geq 0$, let $\mathcal{P} \in M_{K+1}$, and let $\rho(\lambda, \Omega) : \Omega_\rho \rightarrow \mathbb{R}_{\geq 0}$, where

\begin{equation}
\Omega_\rho := \mathbb{R}_{\geq 0}^{K+1} \times \Omega_\Theta.
\end{equation}

The integral equation

\begin{equation}
\rho(\lambda, \theta) = \min\left\{g(\lambda)(z), z_0 + \int \rho(\lambda(zp(\theta)), \xi(\theta)) \mu(dx)\right\},
\end{equation}

with $g(\lambda)$ defined in (3.4), has a unique solution and it holds that

\begin{equation}
\inf_{\pi \in \Pi} L(\pi, \mathcal{P}) = \rho(\lambda, 1, \theta_0).
\end{equation}
Theorem 1 follows directly from Theorem 5 and Lemma 6 in [43] and the Markov property of the stochastic process \((X_n)_{n \geq 1}\). Also compare Theorem 5 in [15]. The optimal test statistic and testing policies are obtained by comparing the cost for stopping with the expected cost for continuing under the optimal policy.

**Corollary 1.** The optimal test statistic of a test solving (3.1) is given by

\[
T_n(x_1, \ldots, x_n) = (z_n, \theta_n) \in \Omega_{\rho},
\]

where \(\theta_n\) is a sufficient statistic for \((x_1, \ldots, x_n)\) in the sense of (2.4) and \(z_n = (z_{0,n}, \ldots, z_{K,n})\) is the vector of likelihood ratios (Radon–Nikodym derivatives)

\[
z_{k,n} = \prod_{i=1}^{n} \frac{dP_{\theta_{i-1}}^{(k)}}{d\mu}(x_i) = \prod_{i=1}^{n} P_{\theta_{i-1}}^{(k)}(x_i).
\]

By Assumption 2 in Section 2.2, the test statistic can be calculated recursively via

\[
\theta_{n+1} = \xi_{\theta_n}(x_{n+1}), \quad \theta_0: \text{given a priori}
\]

\[
z_{n+1} = z_n \rho_{\theta_n}(x_{n+1}), \quad z_0 = 1,
\]

with \(\xi_{\theta_n}\) is defined in (2.5).

**Corollary 2.** Let \(\lambda \geq 0\), let \(P \in M^{K+1}\), and let \(\rho_\lambda\) be as defined in Theorem 1. A policy \(\pi\) is time-homogeneous in the sense of Definition 1 and optimal in the sense of (3.1), i.e., \(\pi \in \Pi^*_\lambda(P)\), if and only if its stopping and decision rules are of the form

\[
\psi_n = \psi(z_n, \theta_n) \quad \text{and} \quad \delta_{k,n} = \delta_k(z_n),
\]

where

\[
1 - \mathcal{I}\{g_{\lambda}(z) \geq \rho_\lambda(z, \theta)\} \leq \psi(z, \theta) \leq 1 - \mathcal{I}\{g_{\lambda}(z) > \rho_\lambda(z, \theta)\},
\]

\[
\delta_k(z) \leq \mathcal{I}\left\{G_{\lambda}^{(k)}(z) = g_{\lambda}(z)\right\},
\]

and \((z_n, \theta_n)\) is defined in Corollary 1.
Corollary 1 and Corollary 2 follow immediately from Theorem 6 in [43] and characterize the set of optimal time-homogeneous policies under Markov assumptions. The two components of the optimal test statistic \((z_n, \theta_n)\) correspond to the two types of information that are necessary to apply the optimal stopping rule. The likelihood ratios \(z\) are needed to evaluate the cost for stopping, while the state of the Markov process \(\theta\) is needed to evaluate the conditional expectation that determines the cost for continuing.

The policies defined in Corollary 2 are optimal in the sense of the unconstrained problem (3.1). The solution of the constrained problem is closely related, but its statement is deferred to the next section since it relies on properties of the optimal cost function \(\rho\) that need to be established first.

Before turning to the latter, a more compact notation and a useful way of characterizing the performance of time-homogeneous tests of the form (3.8) are introduced.

For policies of the form given in Corollaries 1 and 2, it is convenient to define the expected run-length and the error probabilities of the underlying test as functions of the initial state of the test statistic, i.e.,

\[
\gamma_{\pi,\mathbb{P}}(z, \theta) := E_{\mathbb{P}}[\tau \mid Z_0 = z, \Theta_0 = \theta],
\]

\[
\alpha_{\pi,\mathbb{P}}(z, \theta) := E_{\mathbb{P}}[1 - \delta_{\tau,k} \mid Z_0 = z, \Theta_0 = \theta],
\]

where \(\gamma_{\pi,\mathbb{P}}: \Omega_\rho \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}\) and \(\alpha_{\pi,\mathbb{P}}: \Omega_\rho \rightarrow [0, 1]\). Since \((Z_n, \Theta_n)_{n \geq 0}\) is a time-homogeneous Markov process, \(\gamma_{\pi,\mathbb{P}}\) and \(\alpha_{\pi,\mathbb{P}}^{(k)}\) solve the Chapman–Kolmogorov equations [27]

\[
\gamma_{\pi,\mathbb{P}}(z, \theta) = (1 - \psi(z, \theta))\left(1 + \int \gamma_{\pi,\mathbb{P}}(zp(x), \xi(x)) P_\theta(\mathrm{d}x)\right)
\]

and

\[
\alpha_{\pi,\mathbb{P}}^{(k)}(z, \theta) = \psi(z, \theta) \left(1 - \delta_k(z, \theta)\right) + 
(1 - \psi(z, \theta)) \int \alpha_{\pi,\mathbb{P}}^{(k)}(zp(x), \xi(x)) P_\theta(\mathrm{d}x).
\]

Conditioning on the true initial states of the statistics reduces the conditional performance metrics in (3.15) and (3.16) to the unconditional ones in (2.13) and (2.14), i.e.,

\[
\gamma_{\pi,\mathbb{P}}(1, \theta_0) = \gamma(\pi, \mathbb{P}) \quad \text{and} \quad \alpha_{\pi,\mathbb{P}}^{(k)}(1, \theta_0) = \alpha_k(\pi, \mathbb{P}).
\]

Being able to characterize the performance of a test in terms of solutions of (3.17) and (3.18) is of critical importance for the proofs given in later sections.
In order to simplify the notation of the central integral equations of this section, let \( \{ \mu_{z, \theta} : (z, \theta) \in \Omega_\rho \} \) and \( \{ P_{z, \theta} : (z, \theta) \in \Omega_\rho \} \) be two families of probability measures on \( \Omega_\rho \) that are defined via

\[ \mu_{z, \theta}(E_z \times E_\theta) := \mu\left( \{ x \in \Omega_X : zp_\theta(x) \in E_z, \xi_\theta(x) \in E_\theta \} \right), \]

(3.20)

\[ P_{z, \theta}(E_z \times E_\theta) := P_\theta\left( \{ x \in \Omega_X : zp_\theta(x) \in E_z, \xi_\theta(x) \in E_\theta \} \right), \]

(3.21)

where \( E_z \times E_\theta \in F_\rho \), with \( F_\rho \) denoting the natural \( \sigma \)-algebra on \( \Omega_\rho \). The notation \( P_{z, \theta} \) is used to refer to the probability measure in (3.21) with \( P_\theta \) chosen to be \( P_\theta^{(k)} \). Using this notation, the integral equations in (3.6), (3.17), and (3.18) can be written more compactly as

\[ \rho_\lambda = \min \left\{ g_\lambda, z_0 + \int \rho_\lambda \, d\mu_{z, \theta} \right\} \]

(3.22)

and

\[ \gamma_{\pi, P} = (1 - \psi) \left( 1 + \int \gamma_{\pi, P} \, dP_{z, \theta} \right), \]

(3.23)

\[ \alpha^{(k)}_{\pi, P} = \psi (1 - \delta_k) + (1 - \psi) \int \alpha^{(k)}_{\pi, P} \, dP_{z, \theta}. \]

(3.24)

Both notations are used in what follows.

### 4. Properties of the Cost Function \( \rho_\lambda \). While Theorem 1 is well-known in the literature, the cost function \( \rho_\lambda \) in (3.6) has rarely been studied in detail. The connection between \( \rho_\lambda \) and the properties of sequential tests using policies \( \pi \in \Pi^*_\lambda \) is the subject of this section. It extends and generalizes the results in Section 3 of [16].

**Theorem 2.** Let \( \rho_\lambda \) be as defined in Theorem 1 and let \( \Pi^*_\lambda(P) \) be as defined in Corollary 3. For all \( \pi \in \Pi^*_\lambda(P) \) it holds that

\[ \rho_\lambda(z, \theta) = z_0 \gamma_{\pi, P_0}(z, \theta) + \sum_{k=1}^{K} \lambda_k z_k \alpha^{(k)}_{\pi, P_k}(z, \theta), \]

(4.1)

where \( \gamma_{\pi, P} \) and \( \alpha^{(k)}_{\pi, P} \) are defined in (3.15) and (3.16), respectively.

See Appendix A for a proof. Theorem 2 connects the performance metrics in Section 2.5 to the optimal cost function \( \rho_\lambda \) and is key to solving the unconstrained minimax problem (2.16). However, obtaining a solution to
the constrained minimax problem (2.17) additionally requires control over the individual error probabilities. In what follows, a connection between the latter and the partial generalized derivatives of \( \rho_\lambda \) is established. First, some useful technical properties of \( \rho_\lambda \) are shown.

**Lemma 1.** For all \( \lambda \geq 0 \) and all \( \mathbf{P} \in \mathbb{M}^{K+1} \) the function \( \rho_\lambda \) that solves (3.6) is non-decreasing, concave and homogeneous of degree one in \( z \).

A proof is detailed in Appendix B. Lemma 1 is significant for two reasons. First, it ensures that \( \rho_\lambda \) admits a generalized differential, a property that is used in the next theorem to establish a connection between \( \rho_\lambda \) and the error probabilities \( \alpha_k \). Second, being concave and homogeneous qualifies \( \rho_\lambda \) as an \( f \)-dissimilarity, i.e., a statistical measure for the joint similarity of \( P_{(0)}^\theta, \ldots, P_{(K)}^\theta \). This property of \( \rho_\lambda \) is not used explicitly in the derivations of the minimax optimal sequential test, but it puts the results into context and will later be shown to provide a unified interpretation of minimax optimal sequential and minimax optimal fixed sample size test. A more detailed discussion of this aspect is deferred to Section 7.

**Theorem 3.** Let \( \rho_\lambda \) be as defined in Theorem 1, let \( \Pi^*_\lambda \) be as defined in Corollary 3, and let \( \gamma_{\pi, P_0}^{(k)}, \alpha_{\pi, F_k}^{(k)} \) be as defined in (3.15) and (3.16), respectively.

1. For all \( \pi \in \Pi^*_\lambda \) and all \( k = 1, \ldots, K \) it holds that
   \[
   \gamma_{\pi, P_0}^{(k)} \in \partial z_0 \rho_\lambda,
   \lambda_k \alpha_{\pi, F_k}^{(k)} \in \partial z_k \rho_\lambda.
   \]

2. For all \( \pi \in \Pi^*_\lambda \), all \( (z, \theta) \in \Omega_\rho \), and all \( k = 1, \ldots, K \) it holds that
   \[
   \left\{ \gamma_{\pi, P_0}^{(k)}(z, \theta) : \pi \in \Pi^*_\lambda \right\} = \partial z_0 \rho_\lambda(z, \theta),
   \left\{ \lambda_k \alpha_{\pi, F_k}^{(k)}(z, \theta) : \pi \in \Pi^*_\lambda \right\} = \partial z_k \rho_\lambda(z, \theta).
   \]

Theorem 3 is proven in Appendix C. Its two parts correspond to a global and a local statement about the generalized differentials of \( \rho_\lambda \). The first part states that for all \( \pi \in \Pi^*_\lambda \) the functions \( \gamma_{\pi, P_0} \) and \( \alpha_{\pi, F_k}^{(k)} \) are valid generalized differentials of \( \rho_\lambda \). The second part states that at every point \( (z, \theta) \in \Omega_\rho \), the generalized differential of \( \rho_\lambda \) coincides with the set of all error probabilities that can be realized by policies \( \pi \in \Pi^*_\lambda \). Note that since the optimal policy is deterministic on the interior of the decision regions, the remaining degrees
of freedom in terms of the error probabilities are exclusively due to the randomization on the boundary of the stopping and decision regions. The local statement in Theorem 3 cannot be extended to a global statement since the integral equations (C.6) and (C.7) establish a coupling between the local differentials so that they cannot be chosen independently of each other. This coupling reflects the fact that changing the randomization on the boundaries of a decision region also affects the overall performance of the corresponding sequential test.

Based on Theorem 3, the following optimality result can be obtained for the constrained sequential testing problem in (3.2).

**Theorem 4.** Let $\mathbf{P} \in \mathbb{M}^{K+1}$ and let $\pi^* \in \Pi_{\mathbf{P}}^*(\mathbf{P})$, i.e., $\pi^*$ solves (3.2). If

$$\lambda^* \in \arg \max_{\lambda \in \mathbb{R}_{\geq 0}^K} \left\{ \rho_{\lambda}(1, \theta_0) - \sum_{k=1}^K \lambda_k \alpha_k \right\},$$

with $\rho_{\lambda}$ defined in Theorem 1, then it holds that for all $\pi \in \Pi_{\mathbf{P}}^*(\mathbf{P})$

$$|\gamma(\pi, P_0) - \gamma(\pi^*, P_0)| \leq \left| \partial_{\lambda_0} \rho_{\lambda^*}(1, \theta_0) \right|,$$

$$\lambda^*_k |\alpha_k(\pi, P_k) - \alpha_k| \leq \left| \partial_{\lambda_k} \rho_{\lambda^*}(1, \theta_0) \right|,$$

Moreover, it holds that

$$\Pi_{\mathbf{P}}^*(\mathbf{P}) \cap \Pi_{\mathbf{P}}^{**}(\mathbf{P}) \neq \emptyset,$$

i.e., there exists at least one $\pi \in \Pi_{\mathbf{P}}^*(\mathbf{P})$ that is optimal in the sense of (3.2).

A proof is detailed in Appendix D. The significance of Theorem 4 lies in the fact that it characterizes solutions to the constrained problem (3.2) in terms of the optimal cost function of the unconstrained problem (3.1). From an algorithmic point of view, Theorem 4 makes it possible to design constrained sequential tests via a systematic optimization of the cost coefficients $\lambda$ instead of Monte Carlo simulations or resampling techniques [58, 57].

Theorems 2, 3, and 4 form the basis for the derivation of minimax optimal tests, which will be characterized as optimal tests for least favorable distributions. The latter are introduced and discussed in the next section.

5. Least Favorable Distributions. The counterpart of the optimal testing problems investigated in the previous sections is the problem of determining the least favorable distributions for a given testing policy $\pi$. In
In this case, the unconstrained problem (3.1) reduces to

\[
\sup_{P \in \mathcal{P}} L_{\Lambda}(\pi, P) = \sup_{P \in \mathcal{P}} \left( \gamma(\pi, P_0) + \sum_{k=1}^{K} \lambda_k \alpha_k(\pi, P_k) \right). \tag{5.1}
\]

Since the expected run-length and the error probabilities are coupled only via the policy, the joint problem in (5.1) decouples into \(K + 1\) individual maximization problems

\[
\sup_{P_0 \in \mathcal{P}_0} \gamma(\pi, P_0) \quad \text{and} \quad \sup_{P_k \in \mathcal{P}_k} \alpha_k(\pi, P_k), \quad k = 1, \ldots, K, \tag{5.2}
\]

which can be solved independently.

For arbitrary stopping and decision rules, solving the problems in (5.2) exactly is challenging and, in general, the least favorable distributions depend on the time instant \(n\) as well as on the history of the random process. However, for time-homogeneous policies of the form (3.12), a more elegant solution can be obtained.

**Theorem 5.** Let \(\mathcal{P}\) be an uncertainty set of the form (2.6) and let \(\gamma_{\pi, P} : \Omega_\rho \to \mathbb{R}_0 \cup \{\infty\}\) and \(\alpha_{\pi, P}^{(k)} : \Omega_\rho \to \mathbb{R}_0\). For all testing policies \(\pi\) of the form (3.12), it holds that the integral equations

\[
\gamma_{\pi, P} = (1 - \psi) \left( 1 + \sup_{H \in \mathcal{P}_0} \gamma_{\pi, P}(zp_\theta(x), \xi_\theta(x)) H(dx) \right) \tag{5.3}
\]

\[
\alpha_{\pi, P}^{(k)} = \psi(1 - \delta_k) + (1 - \psi) \left( \sup_{H \in \mathcal{P}_\theta} \alpha_{\pi, P}(zp_\theta(x), \xi_\theta(x)) H(dx) \right) \tag{5.4}
\]

have unique solutions.

**Theorem 6.** Let \(\mathcal{P} = (\mathcal{P}_0, \ldots, \mathcal{P}_K)\) be uncertainty sets of the form (2.6) and let \(\pi\) be of the form (3.12).

- If it holds that for every \((z, \theta) \in \Omega_\rho\)

\[
\mathcal{Q}^{(0)}_{z, \theta} := \arg \max_{H \in \mathcal{P}_\theta^{(k)}} \gamma_{\pi}(zp_\theta(x), \xi_\theta(x)) H(dx) \neq \emptyset, \tag{5.5}
\]

with \(\gamma_{\pi}\) defined in (5.3), then every distribution

\[
\mathcal{Q}_0 \in \mathcal{Q}_0 := \left\{ P \in \mathcal{P} : P = \prod_{n \geq 0} P_{z_n, \theta_n}, P_{z_n, \theta_n} \in \mathcal{Q}^{(0)}_{z_n, \theta_n} \right\} \tag{5.6}
\]
is least favorable with respect to the expected run-length of the test, i.e.,

\begin{equation}
\gamma(\pi, Q_0) = \sup_{P \in P_0} \gamma(\pi, P).
\end{equation}

- If it holds that for every \((z, \theta) \in \Omega_\rho\)

\begin{equation}
Q_{z, \theta}^{(k)} := \arg \max_{H \in P_\rho^{(k)}} \int \alpha^{(k)}_{\pi}(zp_\theta(x), \xi_\theta(x)) H(dx) \neq \emptyset,
\end{equation}

with \(\alpha^{(k)}_{\pi}\) defined in (5.4), then every distribution

\begin{equation}
Q_k \in Q_k := \left\{ P \in M : P = \prod_{n \geq 0} P_{z_n, \theta_n}, P_{z_n, \theta_n} \in Q_{z_n, \theta_n}^{(k)} \right\}
\end{equation}

is least favorable with respect to the probability of erroneously deciding against \(H_k\), i.e.,

\begin{equation}
\alpha_k(\pi, Q_k) = \sup_{P \in P_k} \alpha_k(\pi, P).
\end{equation}

Theorems 5 and 6 are proven in Appendices E and F, respectively. From Theorem 6 it follows that under the least favorable distributions the process \((X_n)_{n \geq 1}\) is a Markov process with sufficient statistic \((z_n, \theta_n)\). That is, the least favorable distributions adapt to the policy of the test as well as to the history of the random process.

It is worth highlighting that even in the case where all \(X_n\) are mutually independent, the least favorable distributions generate a Markov process whose sufficient statistic is given by the vector \(z\). This is in line with the concept of sequential inference, for which the order of the samples is crucial, even under an independence assumption. For fixed sample size tests, in contrast, the outcome does not depend on how the samples are ordered.

Having characterized optimal tests and least favorable distributions, everything is in place for the derivation of minimax optimal sequential tests.

6. Minimax Optimal Sequential Tests. In this section, sufficient conditions for strict minimax optimality of sequential hypothesis tests are given. Following the procedure for the optimal sequential test without uncertainty, the solutions of the unconstrained problem (2.16) are derived first and are then shown to contain a solution of the constrained problem (2.17).

The following three theorems are stated in sequence and constitute the main contribution of the paper. A discussion and an interpretation of the results is deferred to the next section.
Theorem 7. Let \( \lambda \geq 0 \), let \( \mathcal{P} = (\mathcal{P}_0, \ldots, \mathcal{P}_K) \) be uncertainty sets of the form (2.6), and let \( \rho_{\lambda}, d_{\lambda}, D_{\lambda} : \Omega_{\rho} \times \mathcal{M}_{\mu} \to \mathbb{R}_{\geq 0} \). The equation system

\[
\rho_{\lambda}(z, \theta) = \min\{ g_{\lambda}(z), z_0 + d_{\lambda}(z, \theta) \} \\
d_{\lambda}(z, \theta) = \sup_{P \in \mathcal{P}_\theta} D_{\lambda}(z, \theta; P) \\
D_{\lambda}(z, \theta; P) = \int \rho_{\lambda}(zp_{\theta}(x), \xi_{\theta}(x)) \mu(dx)
\]

with \( g_{\lambda} \) defined in (3.4) has a unique solution.

Theorem 8. Let \( \lambda \geq 0 \), let \( \mathcal{P} = (\mathcal{P}_0, \ldots, \mathcal{P}_K) \) be uncertainty sets of the form (2.6), and let \( \rho_{\lambda}, d_{\lambda}, \), and \( D_{\lambda} \) be as defined in Theorem 7. If for all \((z, \theta) \in \Omega_{\rho}\)

\[
Q_{z, \theta} := \arg \max_{P \in \mathcal{P}_{\theta}} D_{\lambda}(z, \theta; P) \neq \emptyset,
\]

every policy \( \pi \in \Pi_{\lambda}(Q) \) with

\[
Q \in \mathcal{Q} = \left\{ P \in \mathcal{M}^{K+1} : P_k = \prod_{n \geq 1} P_{z_n, \theta_n}^{(k)}, P_{z_n, \theta_n} \in Q_{z_n, \theta_n} \right\}
\]

is minimax optimal in the sense of (2.16), i.e.,

\[
\{ (\pi, Q) : \pi \in \Pi_{\lambda}(Q), Q \in \mathcal{Q} \} \subset \Pi_{\lambda}^*(\mathcal{P}).
\]

Theorem 9. Let \( \mathcal{P} = (\mathcal{P}_0, \ldots, \mathcal{P}_K) \) be uncertainty sets of the form (2.6) and let \((\pi^*, \mathcal{P}^*) \in \Pi_{\alpha}^*(\mathcal{P})\), i.e., \((\pi^*, \mathcal{P}^*) \) solves (2.17). If

\[
\lambda^* \in \arg \max_{\lambda \in \mathbb{R}_{\geq 0}^K} \left\{ \rho_{\lambda}(1, \theta_0) - \sum_{k=1}^{K} \lambda_k \alpha_k \right\},
\]

with \( \rho_{\lambda} \) defined in Theorem 7, then it holds that for all \( \pi \in \Pi_{\lambda^*}(Q)\)

\[
|\gamma(\pi, Q_0) - \gamma(\pi^*, \mathcal{P}^*_0)| \leq |\partial_{z_0} \rho_{\lambda^*}(1, \theta_0)|,
\]

\[
\lambda^*_k |\alpha_k(\pi, Q_k) - \alpha_k| \leq |\partial_{z_k} \rho_{\lambda^*}(1, \theta_0)|,
\]
where $Q$ is defined in Theorem 8. Moreover, it holds that

$$\{ (\pi, Q) : \pi \in \Pi^*_\lambda(Q), Q \in \mathcal{Q} \} \cap \Pi^*_\lambda(\mathcal{P}) \neq \emptyset,$$

i.e., there exists at least one $\pi \in \Pi^*_\lambda(Q)$ that is optimal in the sense of (2.17).

Theorems 7, 8, and 9 are proven in Appendices G, H, and I, respectively.

7. Discussion. Theorems 8 and 9 in the previous section provide a sufficient condition for the characterization of minimax optimal tests in terms of optimal testing policies and least favorable distributions. In this section, the question of existence is discussed and an interpretation in terms of statistical similarity measures is given that provides additional insight and establishes a connection to minimax optimal fixed samples size tests.

7.1. Statistical Similarity Measures. In order to obtain a better conceptual understanding of minimax optimal sequential tests, it is helpful to introduce a class of statistical similarity measures known as $f$-dissimilarities. They were first proposed by Győrfi and Nemetz [24, 25, 26] as an extension of $f$-divergences to multiple distributions and play an important role in the theory of statistical decision making. In particular, the connection between $f$-dissimilarities and Bayesian risks has been a topic of high interest in statistics [42], signal processing [61] and machine learning [50].

In this section, it is shown that the function $\rho_\lambda$ in Theorem 7 induces an $f$-dissimilarity and that this $f$-dissimilarity provides a sufficient characterization of the minimax optimal test. For this purpose, a variation on the concept of $f$-dissimilarities is useful, which is defined as follows.

**Definition 4.** Let $P_1, \ldots, P_K$ be probability measures on a measurable space $(\Omega, \mathcal{F})$ and let $f : \mathbb{R}^K_\geq 0 \times \Omega \rightarrow \mathbb{R}$, with $f(y, \omega) = f(y_1, \ldots, y_K, \omega)$, be homogeneous of degree one and concave in $(y_1, \ldots, y_K)$. The functional

$$I_f(P_1, \ldots, P_K) = \int f(p_1(\omega), \ldots, p_K(\omega), \omega) \mu(d\omega)$$

is called $f$-similarity of $P_1, \ldots, P_K$.

$I_f$ in (7.1) is referred to as a similarity measure since for concave and homogeneous functions $f$, the functional $-I_f = L_f$ is a dissimilarity measure in the sense of Győrfy and Nemetz [26, Definition 1]. Allowing $f$ to depend on the integration variable directly is a minor generalization that, in the
context of this paper, allows the similarity measure to depend on the history of the random process. Similar generalizations have been introduced in the literature before [51, 45, 5].

Using Definition 4, an intuitive characterization of the family of least favorable distributions can be given in terms of a corresponding family of \( f \)-similarities.

**Corollary 3.** At every time instant \( n \geq 1 \), the least favorable distributions of \( X_{n+1} \), conditioned on the state \((Z_n, \Theta_n) = (z, \theta)\), are the feasible distributions that are most similar with respect to the \( f \)-similarity defined by

\[
(7.2) \quad f_{z,\theta}(y, x) = \rho_{\lambda}(zy, \xi_{\theta}(x)),
\]

with \( \rho_{\lambda} \) given in Theorem 7.

The family of \( f \)-divergences defined by \( \{f_{z,\theta} : (z, \theta) \in \Omega_{\rho}\} \) can be interpreted as follows. The equation system in Theorem 7 defines the optimal cost function \( \rho_{\lambda} \), which in turn defines the similarity measure in Corollary 3. The sequential aspect of the test is captured by the parameters \((z, \theta)\). The likelihood ratios \( z \) determine the weights of the individual densities. That is, the larger \( z_k \), the larger the influence of the \( k \)th distribution on the similarity measure. In terms of the underlying hypothesis test, a high value of \( z_k \) implies that the test is likely to decide in favor of \( H_k \). Consequently, depending on whether \( H_k \) is true or not, the least favorable distributions need to be as similar or dissimilar to \( H_k \) as possible. On the other hand, a low value of \( z_k \) implies that a decision in favor of \( H_k \) is highly unlikely so that the corresponding distribution contributes little to the overall similarity measure. The influence of the parameter \( \theta \), i.e., the history of the underlying random process, does not affect the relative weighting of the individual distributions, but rather the shape of \( \rho_{\lambda} \) itself. It strongly depends on the underlying random process.

The use of statistical similarity measures for the design of (sequential) tests for composite hypotheses has been suggested in the literature before—see, for example, [48, 3, 46, 67]. However, in most works a suitable similarity measure is chosen beforehand—usually based on asymptotic results, bounds, or approximations—and is used as a surrogate objective whose optimization is easier than solving the actual testing problem. Here, by contrast, it is shown that the formulation of the testing problem induces a similarity measure and that optimizing the latter is equivalent to solving the former.

The connection of minimax optimal tests to \( f \)-similarities also allows for some interesting comparisons to other robust methods. Particularly instructive is the comparison to the binary minimax fixed sample size test when
the observed random variables are independent with identical uncertainty sets. In [31, 17], it is shown that in this case a sufficient condition for a pair of distributions to be least favorable is that it jointly minimizes all $f$-divergences over the sets of feasible distributions. This implies that the least favorable distributions do not depend on the decision rule, i.e., the likelihood ratio threshold. For the minimax sequential test, this decoupling no longer holds. Instead, the least favorable distributions need to minimize a particular $f$-dissimilarity that depends on $\rho_\lambda$ and the current state of the test statistic. On the one hand, this coupling significantly complicates the design of minimax robust sequential tests. On the other hand, it can be seen how the same principles underpin the test design in both cases: minimax optimality is achieved by using a policy that leads to the best separation of the most similar distributions.

7.2. Existence. The results presented so far allow for some statements about the existence of minimax optimal tests. Stronger statements can be made for the unconstrained problem formulation (2.16) than for the constrained formulation (2.17). The former is considered first.

Since

$$\inf_{\pi \in \Pi} \sup_{\mathbf{P} \in \mathbf{P}} L_\lambda(\pi, \mathbf{P}) \leq g_\lambda(1) \leq \sum_{k=1}^{K} \lambda_k,$$

the minimax optimal objective value in (2.16) is guaranteed to be finite for all cost coefficients $\lambda \geq 0$ and all uncertainty sets $\mathbf{P}$. This includes scenarios where two uncertainty sets overlap or are identical. However, it does not imply that a testing policy exists that achieves this value. In order to guarantee (6.4), i.e., that the supremum is attained, the uncertainty sets $\mathcal{P}_1, \ldots, \mathcal{P}_K$ need to be compact [54, Theorem 4.16]. While this assumption seems restrictive in theory, it is automatically fulfilled when the problem is discretized in order to be solved numerically. Moreover, given that the least favorable distributions are maximizers of concave functionals, we conjecture that compactness is not necessary for the existence of minimax solutions. A more thorough discussion would need to go into the technical details of optimization on function spaces, which is beyond the scope of this paper.

For the constrained problem (2.17), the situation is more involved since, in order to be able to satisfy the constraints on the error probabilities, the uncertainty sets $\mathcal{P}_1, \ldots, \mathcal{P}_K$ need to be sufficiently separated, i.e., the distance between the two sets needs to be large enough to statistically separate them using a finite number of samples. The appropriate way to measure
this distance is via the $f$-similarity that is induced by the corresponding cost function $\rho_\lambda$. This leads to the following result.

**Corollary 4.** The minimax optimal objective in (2.17) is finite, if and only if the right hand side of (6.7) is bounded, i.e., if

$$
(7.3) \quad \sup_{\lambda \in \mathbb{R}^{K}_0} \left\{ \rho_\lambda(1, \theta_0) - \sum_{k=1}^{K} \lambda_k \alpha_k \right\} < \infty,
$$

with $\rho_\lambda$ defined in Theorem 7.

A proof of Corollary 4 is detailed in Appendix J. As in the unconstrained case, existence of a finite supremum does not imply that an optimal policy exists, unless the uncertainty sets are compact.

Interestingly, the conditions for the existence of minimax optimal sequential tests are rather mild, especially for the unconstrained problem formulation. This is in contrast to the fixed sample case, for which much stricter sufficient conditions are given in the literature [31, 17]. This leads us to conjecture that these stricter conditions are only necessary to guarantee that the optimal policy and the least favorable distributions are *decoupled* and that minimax optimal fixed sample size tests for arbitrary uncertainty sets exist, but that they require a joint design of the policy and the least favorable distributions. Unfortunately, applying the results presented in this paper to the fixed sample case is not straightforward. First, as mentioned in Section 3, the concept of ordered samples, which is essential to sequential hypothesis testing and induces state-dependent least favorable distributions, in general does not apply to fixed sample size tests. Second, owing to the deterministic stopping rule, the policies of fixed sample size tests are not time-homogeneous in the sense of Definition 1. However, the second numerical example given in the next section indicates that time-homogeneous policies can lead to truncated stopping rules—see Section 8.2 for more details.

Finally, it should be highlighted that the least favorable distributions do not depend on the reference measure $\mu$. More precisely, if $\mu$ is absolutely continuous with respect to an alternative reference measure $\tilde{\mu}$, it follows from the homogeneity of $\rho_\lambda$ that

$$
D_\lambda(z, \theta; P) = \int \rho_\lambda(z \tilde{p}, \xi_\theta) \, d\mu = \int \rho_\lambda(z \hat{p}, \xi_\theta) \frac{d\mu}{d\tilde{\mu}} \, d\hat{\mu} =: \int \rho_\lambda(z \hat{\tilde{p}}, \xi_\theta) \, d\hat{\mu},
$$

where $\hat{\tilde{p}} = (\hat{\tilde{p}}_0, \ldots, \hat{\tilde{p}}_K)$ are probability densities with respect to $\hat{\tilde{\mu}}$. 
8. Example and Numerical Results. In order to illustrate the presented results, two examples for minimax optimal sequential tests are given in this section. First, a test for three hypotheses that is robust with respect to the error probabilities is designed under the assumption that \((X_n)_{n \geq 1}\) is a sequence of independent random variables with identical uncertainty sets. Second, the case when the underlying processes admits dependencies is illustrated by solving the Kiefer–Weiss problem for a variant of the uncertainty model in Section 2.3. In both examples, the targeted error probabilities are set to \(\alpha_k = 0.01\) for all \(k = 1, \ldots, K\).

The test design is based on the following iterative procedure. First, all \(P^{(k)}_\theta\) are initialized with some feasible distribution. Keeping these distributions fixed, an optimal sequential test is designed by solving (4.2) for \(\lambda^*\) and \(\rho_{\lambda^*}\). In the second step, \(\rho_{\lambda^*}\) is kept fixed and the distributions are updated by solving the optimization problem in (6.4). Both steps are iterated until the changes in the function \(\rho_{\lambda^*}\) are small enough to assume convergence. For both examples, convergence was assumed if the relative difference between two consecutive approximations of \(\rho_{\lambda^*}\) fell below \(10^{-3}\). The question whether this procedure is guaranteed to converge in general is certainly worth investigating, but beyond the scope of this paper.

It should be noted that this iterative procedure does not alternate between the design of optimal testing policies and least favorable distributions. Unless the procedure has converged, the distributions that solve (6.4) are not least favorable in the sense of Theorem 6. Moreover, the test statistic, which is part of the optimal policy, depends on the likelihood ratios and is hence affected by the update of the distributions in the second step.

In order to solve (4.2) and (6.4) numerically, both the state space \(\Omega_\rho\) and the sample space \(\Omega_X\) are discretized using a regularly spaced grid and linear interpolation is used to evaluate functions between grid points. This straightforward approach works well for the examples presented here. However, if a larger number of hypotheses or more complex dependencies need to be considered, more sophisticated approximations need to be used [22, 36]. The linear programming algorithm detailed [16] was used to efficiently solve (4.2) jointly for \(\lambda^*\) and \(\rho_{\lambda^*}\). However, in principle, any suitable convex optimization algorithm can be used to solve (4.2) and any method for solving nonlinear integral equations can be used to obtain \(\rho_{\lambda^*}\).

In both examples, the distributional uncertainty is of the density band type [32], i.e.,

\[
P = \{ P \in \mathcal{M}_\mu : p' \leq p \leq p'' \},
\]

where \(0 \leq p' \leq p'' \leq \infty\), \(P'(\Omega_X) \leq 1\), and \(P''(\Omega_X) \geq 1\). Here \(P'\) and \(P''\)
denote the measures corresponding to \( p' \) and \( p'' \), respectively. The reason for using this uncertainty model is twofold. First, it contains several popular uncertainty models as special cases, for example, the \( \varepsilon \)-contamination model \([29]\), the bounded distribution function model \([44]\), and the \( f \)-divergence ball model \([19]\). Second, an efficient iterative algorithm for the minimization of convex functionals of probability distribution under density band constraints exists that makes it possible to obtain accurate numerical solutions with moderate computational efforts. A more detailed discussion of the band model, its properties, and how to obtain the least favorable distributions numerically can be found in \([32]\) and \([17]\).

8.1. IID process under three hypotheses. For the first example, all \( X_n, n \geq 1 \), are assumed to be independent and distributed on the interval \( \Omega_X = [-1, 1] \). Let \( P_n \) denote the distribution of \( X_n \). The task is to decide between the following three hypotheses:

\[
\begin{align*}
H_1: & \quad P_n \in \mathcal{P}_1, \\
H_2: & \quad P_n \in \mathcal{P}_2, \\
H_3: & \quad P_n = \mathcal{U}_{[-1,1]},
\end{align*}
\]

for all \( n \geq 1 \). Here, \( \mathcal{U}_{[a,b]} \) denotes the continuous uniform distribution on the interval \([a, b]\) and the uncertainty sets \( \mathcal{P}_1, \mathcal{P}_2 \) are of the form (8.1) with

\[
\begin{align*}
p_1'(x) & = ae^{-2x} + 0.1, & p_1''(x) & = ae^{-2x} + 0.3, \\
p_2'(x) & = ae^{2x} + 0.1, & p_2''(x) & = ae^{2x} + 0.3,
\end{align*}
\]

where \( a \approx 0.1907 \) was chosen such that \( P_1'(\Omega_X) = P_2'(\Omega_X) = 0.9 \) and \( P_1''(\Omega_X) = P_2''(\Omega_X) = 1.1 \). The expected run-length was minimized under \( H_3 \), i.e., \( P_n = \mathcal{U}_{[-1,1]} \) for all \( n \). Moreover, in order to keep the domain of the cost function two-dimensional, the reference measure \( \mu \) was set to \( \mu = \mathcal{U}_{[-1,1]} \) so that \( z_0 = z_3 = 1 \) and \( \rho_{\lambda} \) becomes a function of \((z_1, z_2)\) only.

In order to solve this example numerically, the likelihood ratio plane \((z_1, z_2)\) was discretized on \([-20, 10] \times [-20, 10]\) using \( 301 \times 301 \) uniformly spaced grid points and the sample space \( \Omega_X = [-1, 1] \) was discretized using \( 100 \) uniformly spaced grid points. The design procedure detailed above converged after five iterations. The optimal weights were found to be \( \lambda^* \approx (133.41, 133.41, 45.41) \). The resulting cost function \( \rho_{\lambda^*} \), as well as the corresponding testing policy, are depicted in Fig. 1. While the cost function as such provides little insight, the testing policy lends itself to an intuitive interpretation. In analogy to the regular sequential probability ratio test (SPRT), the minimax optimal test consists of two corridors that correspond to a binary test between \( \mathcal{H}_{\{1,2\}} \) and \( \mathcal{H}_3 \), respectively. Interestingly, there is a rather sharp intersection of the two corridors so that the test quickly reduces to a quasi-binary scenario.
The expected run-length and the error probabilities as functions of the state of the test statistic can be obtained either via the partial derivatives of $\rho_\lambda^*$ or by solving the integral equations (3.16) and (2.13) and are depicted in Fig. 2. The “blocky” appearance of some of the functions is due to them having being downsampled to a coarser grid for plotting. Moreover, no smoothing was applied in order not to smear the hard transitions between the decision regions. Finally, note that the plots are oriented differently to provide the most suitable visual representation of the respective function.

The stopping and decision rules in Fig. 1 are depicted as functions of the log-likelihood ratios. The latter are in turn defined in terms of the least favorable distributions, i.e., the distributions that solve the maximization in (6.4). Four examples of densities of least favorable distributions are depicted in Fig. 3. As can be seen, the densities change significantly, depending on the state of the test statistic. In the top left plot, the test statistic is in its initial state, meaning that there is no preference for either hypothesis. Consequently, the least favorable densities are chosen such that all three distributions are equally similar to each other, which in this case implies that they are symmetric around the y-axis and that $q_z^{(1)}$ and $q_z^{(2)}$ jointly mimic the uniform distribution $p_z^{(3)}$. Also note that $q_z^{(1)}$ and $q_z^{(2)}$ overlap on an interval around $x = 0$ so that observations in this interval are statistically indistinguishable under $H_1$ and $H_2$. As the test statistic is updated, the least favorable distributions change. In the upper right and the lower left plot of Fig. 3, two cases are depicted where the test has a strong preference for $H_1$.
Fig 2. Performance metrics as functions of the log-likelihood ratios. Clockwise from the top left: expected run-length, error probability of the first, second, and third type.

or \( \mathcal{H}_2 \), respectively—compare the decision regions in Fig. 1. In both cases, the least favorable densities are no longer symmetric, but their probability masses are shifted, their tail-behavior is noticeably different, and the interval of overlap can no longer be observed. Finally, in the lower right plot, there is a strong preference for \( \mathcal{H}_3 \), which leads to \( q^{(1)}_z \) and \( q^{(2)}_z \) both shifting as much probability mass as possible to their tails in order to reduce the significance of the corresponding observations. It is interesting to observe the effect that an imminent decision for \( \mathcal{H}_3 \) has on \( q^{(1)}_z \) and \( q^{(2)}_z \), namely that they become less similar to each other in order to increase the joint similarity to \( p^{(3)}_z \). This is in contrast to the initial state depicted in the upper left plot, where \( q^{(1)}_z \) and \( q^{(2)}_z \) also try to approximate \( p^{(3)}_z \), but at the same time need to be similar to each other as well.

In order to verify the numerical results, 10 000 Monte Carlo simulations were performed using the testing policy depicted in Fig. 1. The observations were drawn from the least favorable distributions, which were calculated on the fly by solving (6.4) for the current weights \((z_1, z_2)\). The resulting confusion matrix as well the average run-length of the tests are shown in Table 1.
8.2. Binomial AR(1) process under two hypotheses. For the second example, a discrete version of the uncertainty model in Section 2.3 is considered, namely a binomial AR(1) process [65], which has applications, for example, in finance and monitoring [66]. A binomial AR(1) process \( \left( X_n \right)_{n \geq 1} \) is a homogeneous Markov process with transition probabilities

\[
P(X_n = m \mid X_{n-1} = \theta) = \sum_{i=\max\{0,m+\theta-M\}}^{\min\{m,\theta\}} b_{i,\theta,m,M} \beta_0^i (1-\beta_0)^{\theta-i} \beta_1^{m-i} (1-\beta_1)^{M-\theta+i-m},
\]

where \( m = 0, \ldots, M \),

\[
b_{i,\theta,m,M} = \binom{M}{i} \binom{m-\theta}{m-i},
\]

and \( (\beta_0, \beta_1) = \beta \in (0,1)^2 \) characterizes the dependence structure of the process. See [65, Definition 1.1, Remark 1.2] for a formal definition and more details on the parameter \( \beta \) and its feasible values. The sufficient statistic

\[
\begin{align*}
\text{pdf} & \quad (\log z_1, \log z_2) = (0, 0) \\
\text{pdf} & \quad (\log z_1, \log z_2) = (-10, 3) \\
\text{pdf} & \quad (\log z_1, \log z_2) = (3, -10) \\
\text{pdf} & \quad (\log z_1, \log z_2) = (-4, -10)
\end{align*}
\]

Fig 3. Examples of least favorable distributions for different states of the test statistic.
Table 1

| True Hypothesis | $\mathcal{H}_1$ | $\mathcal{H}_2$ | $\mathcal{H}_3$ | Average Run-Length |
|-----------------|-----------------|-----------------|-----------------|-------------------|
| $\mathcal{H}_1$ | 99.22           | 0.00            | 0.78            | 25.46             |
| $\mathcal{H}_2$ | 0.00            | 98.83           | 1.17            | 26.84             |
| $\mathcal{H}_3$ | 0.61            | 0.35            | 99.04           | 36.04             |

Results of $10^4$ Monte Carlo runs of the minimax optimal test using the policy depicted in Fig. 1. The target detection probability is 99%, the theoretical expected run-length under $\mathcal{H}_3$ is $E_{\psi_3} \left[ \tau(\psi) \right] \approx 36.31$ samples.

of the binomial AR(1) process is given by $\Theta_n = X_n$ with $\Omega_\theta = \Omega_X = \{0, 1, \ldots, M\}$. In what follows, $M = 7$.

Let $P_{\theta, \beta}$ denote the distribution in (8.5), i.e., the conditional distribution of $X_n$, given $x_{n-1} = \theta$. The following two simple hypotheses are considered in this example:

\begin{equation}
H_1: \beta = \beta_1 = (0.75, 0.25), \quad H_2: \beta = \beta_2 = (0.5, 0.5).
\end{equation}

Note that for $\beta_2 = (0.5, 0.5)$ the binomial AR(1) process reduces to a process of independent binomial random variables with distribution $B(M, 0.5)$. Hence, the two hypotheses in (8.7) correspond to a test for dependencies in the observed data. The aim is to solve the Kiefer-Weiss problem for the hypotheses in (8.7), i.e., to design a sequential test whose worst-case expected run-length over all possible random processes is minimal. Consequently, the uncertainty sets for the conditional distributions $P_\theta$ are chosen as $P_\theta = M_\mu$ for all $\theta \in \Omega_\theta$. Note that this type of uncertainty can be interpreted as a special case of the density band model in (8.1), with $p' = 0$ and $p'' = 1$, or as an outlier model with contamination ratio $\epsilon = 1$. In analogy to the previous example, the reference measure is set to $\mu = P^{(2)}_\theta = B(7, 0.5)$, so that $z_2 = 1$ and $P_A$ becomes a function of $(z_0, z_1)$ only. The initial state of the sufficient statistic is set to $\theta_0 = 3$.

Since the hypotheses in (8.7) are simple, a regular sequential probability ratio test with log-likelihood ratio thresholds $B < 0 < A$ can be applied as well. However, under the above uncertainty model, its worst-case expected run-length is infinite. In order to see this, consider a deterministic process that alternates between two observations, $x^{(1)}$ and $x^{(2)}$, which are chosen such that their log-likelihood ratios satisfy

\begin{equation}
0 < \log \frac{P_{x^{(1)}, \beta_1}(x^{(2)})}{P_{x^{(1)}, \beta_0}(x^{(2)})} < A \quad \text{and} \quad B < \log \frac{P_{x^{(2)}, \beta_1}(x^{(1)})}{P_{x^{(2)}, \beta_0}(x^{(1)})} < 0.
\end{equation}
For this process, the log-likelihood ratio increments keep canceling each other out so that neither of the thresholds is ever crossed. A minimax robust test, by contrast, makes it possible to leverage the increased efficiency of sequential tests while at the same time having a bounded worst-case run-length.

In order to obtain a numerical solution, the likelihood ratio plane \((z_0, z_1)\) was discretized on \([-4, 8] \times [-6, 6]\) using \(241 \times 241\) uniformly spaced grid points. The iterative design procedure converged after four iterations. The optimal weights were found to be \(\lambda^* \approx (1526.38, 1178.24)\). The resulting cost function \(\rho_{\lambda^*}\), as well as the corresponding testing policy, are depicted in Fig. 4. Both are distinctly different from their counterparts in the first example. First, the stopping region is no longer a corridor, but is of a conic shape with no clear upper and lower threshold. Second, it is noteworthy that for \(\log z_0\) greater than approximately five, the sequential test reduces to a single threshold test, which is an indicator for the test being truncated under certain conditions and is in line with the goal of minimizing the worst-case expected run-length. For different values of \(\theta\), slight changes in the location of the decision regions can be observed, but the overall shape remains the same.

Four examples of conditional distributions that are least favorable with respect to the expected run-length are depicted in Fig. 5. Note that a linear interpolation is used to connect the point masses; although this a potentially misleading representation, it helps to make the differences in shape more...
Fig 5. Examples of least favorable distributions in the state \((1, \theta)\) for different values of the sufficient statistic \(\theta\).

recognizable. In the depicted examples, the least favorable distributions are conditioned on \(z = 1\) and on different values for \(\theta\), which corresponds to the previous observation \(x_{n-1}\). It can be observed how the least favorable distribution, depicted in red, adapts to the state of the test statistic in such a way that it is equally similar to both hypotheses. Moreover, it is noteworthy that, even without bounds on the densities, the least favorable distributions do not reduce to a single point mass, meaning that for any given state of the test statistic there is no single least favorable observation. This is a consequence of the fact that the least favorable distributions under density band uncertainty turn out to be equalizers with respect to the respective performance measure, so that in this case all observations lead to the same expected run-length—see [18] for a more detailed discussion.

In analogy to the first example, 10 000 Monte Carlo simulations were performed using the testing policy depicted in Fig. 4 in order to verify the numerical results. The confusion matrix as well the average run-length of the tests are shown in Table 2. The average run-length under the least favorable
distribution $Q_0$ was obtained to be approximately 74.66 samples. Interestingly, a fixed sample size test between $H_1$ and $H_2$ using 75 samples results in error probabilities close to 1% as well. This raises the question whether there is a relation between the worst-case expected run-length of a minimax sequential test and the number of samples required by the equivalent fixed-sample size test. More precisely, since the latter is an upper bound on the former, the question is whether the least favorable distributions attain this bound; this conjecture will be investigated in future work. Regardless of whether or not the conjecture holds, the minimax sequential test still achieves significant reductions in the average sample size under both hypotheses, in particular $H_2$, as is clear by inspection of Table 2.
APPENDIX A: PROOF OF THEOREM 2

In order to prove Theorem 2, it suffices to show that the right hand side of (4.1) solves the integral equation (3.6). Since $\rho_\lambda$ is unique, this implies that both functions are identical.

Two auxiliary results are used in the proof:

(I) For every function of the form $z_k f(z, \theta)$, with $f : \Omega_\rho \to \mathbb{R}_{\geq 0}$, it holds that
\[
\int z_k f(z, \theta) \, d\mu_{z, \theta} = \int z_k p^{(k)}_\theta(x) f(zp_\theta(x), \xi_\theta(x)) \, \mu(dx)
= z_k \int f(zp_\theta(x), \xi_\theta(x)) P^{(k)}_\theta(dx)
= z_k \int f \, dP^{(k)}_{z, \theta}.
\]

(II) For $\pi = (\psi, \delta) \in \Pi^*_\lambda$, it follows from Corollary 2 that
\[
\sum_{k=1}^{K} \lambda_k z_k (1 - \delta_k) = \sum_{k=1}^{K} \lambda_k z_k - \max_{k=1, \ldots, K} \{ \lambda_k z_k \} = g_\lambda.
\]

Using these properties and the Chapman–Kolmogorov equations (3.17) and (3.18), it follows that for all $(\psi, \delta) \in \Pi^*_\lambda$
\[
\rho_\lambda = z_0 \gamma_{\pi, P_0} + \sum_{k=1}^{K} \lambda_k z_k \alpha^{(k)}_{\pi, P_k}
= (1 - \psi) \left(1 + \int \gamma_{\pi, P_0} \, dP^{(0)}_{z, \theta}\right) + \sum_{k=1}^{K} \lambda_k z_k \left(\psi(1 - \delta_k) + (1 - \psi) \int \alpha^{(k)}_{\pi, P_k} \, dP^{(k)}_{z, \theta}\right)
\overset{(I)}{=} (1 - \psi) \left(z_0 + \int z_0 \gamma_{\pi, P_0} \, d\mu_{z, \theta}\right) + \sum_{k=1}^{K} \lambda_k z_k \psi(1 - \delta_k) + \sum_{k=1}^{K} (1 - \psi) \int \lambda_k z_k \alpha^{(k)}_{\pi, P_k} \, d\mu_{z, \theta}
\overset{(II)}{=} \psi g_\lambda + (1 - \psi) \left(z_0 + \int \left(z_0 \gamma_{\pi, P_0} + \sum_{k=1}^{K} \lambda_k z_k \alpha^{(k)}_{\pi, P_k}\right) \, d\mu_{z, \theta}\right)
= \psi g_\lambda + (1 - \psi) \left(z_0 + \int \rho_\lambda \, d\mu_{z, \theta}\right)
= \min \left\{ g_\lambda, z_0 + \int \rho_\lambda \, d\mu_{z, \theta}\right\},
\]
where the last equality follows again from Corollary 2. This concludes the proof.
APPENDIX B: PROOF OF LEMMA 1

Consider the sequence of functions \((\rho^n_\lambda)_{n \geq 0}\) with \(\rho^0_\lambda = g_\lambda\) and

\[
\rho^n_\lambda = \min \left\{ g_\lambda(z), \ 1 + \int \rho^{n-1}_\lambda d\mu_{z,\theta} \right\}.
\]

Assume that \(\rho^n_\lambda \leq \rho^{n-1}_\lambda\) for some \(n \geq 0\). It then holds that

\[
\rho^{n+1}_\lambda = \min \left\{ g_\lambda(z), \ 1 + \int \rho^n_\lambda d\mu_{z,\theta} \right\}
\]

\[
\leq \min \left\{ g_\lambda(z), \ 1 + \int \rho^{n-1}_\lambda d\mu_{z,\theta} \right\} = \rho^n_\lambda
\]

so that \((\rho^n_\lambda)_{n \geq 0}\) is non-increasing. Moreover, the sequence is non-negative and hence bounded from below. This implies that it converges pointwise to a unique limit. Since this result is used repeatedly in the paper, it is fixed in the next lemma.

**Lemma 2.** Let \((\Omega, \mathcal{F})\) be a measurable space and let \((f_n)_{n \geq 0}\), with \(f_n: \Omega \to \mathbb{R}\), be a sequence of functions. If this sequence is non-increasing, i.e., \(f_n \leq f_{n-1}\) for all \(n \geq 0\), and a function \(g > -\infty\) exists such that \(f_n \geq g\) for all \(n \geq 0\), then the pointwise limit

\[
\lim_{n \to \infty} f_n = f
\]

exists and is unique. The same result holds if the sequence is non-decreasing, i.e., \(f_n \geq f_{n-1}\) for all \(n \geq 0\), and a function \(g < \infty\) exists such that \(f_n \leq g\).

Lemma 2 is a well-known result in Real Analysis. It follows from the fact that for every \(\omega \in \Omega\), it holds that \((f_n(\omega))_{n \geq 0}\) is a monotonic and bounded sequence of real numbers. Sequences of this type are guaranteed to converge to a unique limit—see, for example, [54, Theorem 3.14] and [41, Theorem 3.1.4]. This immediately implies the statement in Lemma 2.

From Lemma 2 it follows that the sequence \((\rho^n_\lambda)_{n \geq 0}\) converges pointwise to unique limit \(\rho_\lambda\) for \(n \to \infty\)—also compare [43, Lemma 4, Lemma 5] and [16, Appendix A]. By definition (3.4), \(g_\lambda\) is non-decreasing, concave, and homogeneous of degree one in \(z\). Using this as a basis, it can be shown via induction that these properties carry over to \(\rho_\lambda\). Here, only the concavity of \(\rho_\lambda\) is proven in detail; the other properties can be shown analogously.
Assume that $\rho^n_\lambda$ is concave for some $n \geq 0$, i.e.,

$$\rho^n_\lambda(\kappa z' + (1 - \kappa) z, \theta) \geq \kappa \rho^n_\lambda(z', \theta) + (1 - \kappa) \rho^n_\lambda(z, \theta)$$

for all $z, z' \in \Omega_\rho$ and all $\kappa \in [0, 1]$. From the fact that $g_\lambda$ and the minimum being concave functions it follows that

$$\rho^{n+1}_\lambda(\kappa z' + (1 - \kappa) z, \theta) = \min\left\{ g_\lambda(\kappa z' + (1 - \kappa) z), z_0 + \int \rho^n_\lambda d\mu_{\kappa z' + (1 - \kappa) z, \theta} \right\}$$

so that $\rho^{n+1}_\lambda$ is concave as well. Taking the limit on both sides of (B.5) yields

$$\rho_\lambda(\kappa z' + (1 - \kappa) z, \theta) \geq \kappa \rho_\lambda(z', \theta) + (1 - \kappa) \rho_\lambda(z, \theta).$$

This concludes the proof.

**APPENDIX C: PROOF OF THEOREM 3**

Since $\rho_\lambda$ is concave, its generalized partial derivatives in the sense of (2.1) exist. Moreover, for $\pi \in \Pi^*_\lambda$, $\rho_\lambda$ can be written as (compare Appendix A)

$$\rho_\lambda = \min\{g_\lambda, d_\lambda\} = \psi\left( \sum_{k+1}^K \lambda_k z_k (1 - \delta_k) \right) + (1 - \psi) d_\lambda$$

where $d_\lambda: \Omega_\rho \to \mathbb{R}_{\geq 0}$ is defined as

$$d_\lambda(z, \theta) := z_0 + \int \rho_\lambda d\mu_{z, \theta}.$$
Lemma 3 (Generalized Leibniz integral rule). Let \((\Omega, \mathcal{F})\) be a measurable space and let \(f : \mathbb{R}^K \times \Omega \rightarrow \mathbb{R}\) be a convex or concave function. If \(f(y, \omega)\) is \(\mu\)-integrable for all \(y \in \mathbb{R}^K\), it holds that

\[
\partial_{y_k} \left( \int_{\Omega} f(y, \omega) \, d\mu(\omega) \right) = \int_{\Omega} \partial_{y_k} f(y, \omega) \, \mu(d\omega),
\]

where the integral on the right hand side is a short-hand notation for the set of integrals over all feasible partial derivatives of \(f\), i.e.,

\[
\int_{\Omega} \partial_{y_k} f(y, \omega) \, \mu(d\omega) := \left\{ c \in \mathbb{R} : \exists f_{y_k} \in \partial_{y_k} f : c = \int_{\Omega} f_{y_k}(y, \omega) \, \mu(d\omega) \right\}.
\]

The generalized Leibniz integral rule is proven, for example, in [53, Theorem 23]. Extensions and variations are given in [45] and [9].

Since

\[
\int_{\rho} \lambda \, d\mu_{\theta} \leq \int_{z} g_{\lambda} \, d\mu_{\theta} \leq \sum_{k=1}^{K} \lambda_k z_k < \infty,
\]

\(\rho\lambda\) is \(\mu_{\theta}\)-integrable for all \((z, \theta) \in \Omega_{\rho}\). Hence, Leibniz’s integral rule applies to \(d\lambda\) so that

\[
\partial_{z_k} \rho_{\lambda}(z, \theta) = \partial_{z_k} \left( z_0 + \int \rho_{\lambda} \, d\mu_{\theta} \right)
\]

\[
= \partial_{z_k} z_0 + \int \partial_{z_k} \left( \rho_{\lambda}(zp_{\theta}(x), \xi_{\theta}(x)) \right) \, \mu(dx)
\]

\[
= \partial_{z_k} z_0 + \int \partial_{z_k} \rho_{\lambda}(zp_{\theta}(x), \xi_{\theta}(x))p^{(k)}_{\theta}(x) \, \mu(dx)
\]

\[
= \partial_{z_k} z_0 + \int \partial_{z_k} \rho_{\lambda}(zp_{\theta}(x), \xi_{\theta}(x)) P^{(k)}_{\theta}(dx)
\]

\[
= \partial_{z_k} z_0 + \int \partial_{z_k} \rho_{\lambda} \, dP^{(k)}_{z, \theta}
\]

By expressing \(\rho_{\lambda}\) as in (C.1) and taking the partial generalized derivatives, the following set-valued integral equations [4] are obtained:

\[
\partial_{z_0} \rho_{\lambda}(z, \theta) = (1 - \psi) \left( 1 + \int \partial_{z_0} \rho_{\lambda} \, dP^{(0)}_{z, \theta} \right),
\]

for \(k = 0\) and

\[
\partial_{z_k} \rho_{\lambda}(z, \theta) = \psi \lambda_k (1 - \delta_k) + (1 - \psi) \int \partial_{z_k} \rho_{\lambda} \, dP^{(k)}_{z, \theta}.
\]
for $k = 1, \ldots, K$. Read from “right to left”, (C.6) and (C.7) state that inserting any function $r'_k \in \partial z_k \rho_\lambda$ into the integral on the right hand side yields another function $r_\lambda \rho_\lambda$ on the left hand side. Read from “left to right”, (C.6) and (C.7) state that given any function $r_k \in \partial z_k \rho_\lambda$ on the left hand side, a function $r'_k \in \partial z_k \rho_\lambda$ exists such that the right hand side evaluates to $r_k$.

The above characterization of the generalized differentials, which follows solely from the concavity and integrability of $\rho_\lambda$, already implies both statements in Theorem 3. By inspection of (3.23) and (3.24), it can be seen that $\gamma_{\pi, P_0}$ and $\lambda_\alpha^{(k)}_{\pi, P_k}$ are solutions of (C.6) and (C.7), respectively, for all $\pi \in \Pi_\lambda^*$. This yields the first statement in Theorem 3.

The second part of Theorem 3 is proven by showing that the sets in the statement are subsets of each other. The details are only given for $k \geq 1$ since the proof for $k = 0$ follows analogously. From the above results it is clear that $\lambda_\alpha^{(k)}_{\pi, P_k}(z, \theta) \in \partial z_k \rho_\lambda$ for all $\pi \in \Pi_\lambda^*$. By definition, this implies that

$$\{ \lambda_\alpha^{(k)}_{\pi, P_k}(z, \theta) : \pi \in \Pi_\lambda^* \} \subset \partial z_k \rho_\lambda(z, \theta).$$

(C.8)

In order to show the converse, the following lemma is useful.

**Lemma 4.** Given two policies $\pi, \pi' \in \Pi_\lambda^*$ that satisfy

$$a = \alpha^{(k)}_{\pi, P_k}(z, \theta) \leq \alpha^{(k)}_{\pi', P_k}(z, \theta) = a'$$

for some $(z, \theta) \in \Omega_\rho$, it holds that for every $\tilde{a} \in [a, a']$ there exists a policy $\tilde{\pi} \in \Pi_\lambda^*$ such that

$$\alpha^{(k)}_{\tilde{\pi}, P_k}(z, \theta) = \tilde{a}.$$

(C.9)

The lemma can be shown by considering a randomized policy $\tilde{\pi}$, which, at each time instant, is chosen to be $\pi$ with probability $\kappa$ and $\pi'$ with probability $(1 - \kappa)$, where $\kappa \in [0, 1]$. The conditional probability of erroneously deciding against $\mathcal{H}_k$ when using this mixed policy is given by the integral equation

$$\alpha^{(k)}_{\tilde{\pi}, P_k} = \kappa \psi(1 - \delta_k) + (1 - \kappa) \psi'(1 - \delta'_k) + \int \alpha^{(k)}_{\tilde{\pi}, P_k} dP^{(k)}_{z, \theta},$$

which has the unique solution $\alpha^{(k)}_{\tilde{\pi}, P_k} = \kappa \alpha^{(k)}_{\pi, P_k} + (1 - \kappa) \alpha^{(k)}_{\pi', P_k}$ so that

$$\alpha^{(k)}_{\pi, P_k}(z, \theta) = \kappa \alpha^{(k)}_{\pi, P_k}(z, \theta) + (1 - \kappa) \alpha^{(k)}_{\pi', P_k}(z, \theta) = \kappa a + (1 - \kappa) a'.$$

(C.11)
The Lemma follows.

Using Lemma 4, the desired result can be shown by contradiction. Assume that \( r_k \in \partial_z \rho_\lambda \) exists such that for some \((z, \theta)\) it holds that

\[
(C.13) \quad r_k(z, \theta) \notin \left\{ \lambda_k \alpha_{\pi, \mathcal{P}_k}(z, \theta) : \pi \in \Pi_\lambda \right\}.
\]

By (C.7), a policy \( \pi' \in \Pi_\lambda \) and a function \( r'_k \in \partial_z \rho_\lambda \) exists such that

\[
(C.14) \quad r_k = \psi' \lambda_k (1 - \delta'_k) + (1 - \psi') \int r'_k \, dP^{(k)}_{z, \theta}.
\]

Now consider the sequence of functions defined by

\[
(C.15) \quad \overline{r}_k^n = \sup_{\pi \in \Pi_\lambda} \left\{ \psi \lambda_k (1 - \delta_k) + (1 - \psi) \int \overline{r}_{k-1} \, dP^{(k)}_{z, \theta} \right\}
\]

with \( \overline{r}_k^0 = r'_k \). Assume that \( \overline{r}_k^n \geq \overline{r}_k^{n-1} \) for some \( n \geq 0 \). Via induction, it follows that

\[
(C.16) \quad \overline{r}_k^{n+1} = \sup_{\pi \in \Pi_\lambda} \left\{ \psi \lambda_k (1 - \delta_k) + (1 - \psi) \int \overline{r}_k \, dP^{(k)}_{z, \theta} \right\}
\]

\[
(C.17) \quad \geq \sup_{\pi \in \Pi_\lambda} \left\{ \psi \lambda_k (1 - \delta_k) + (1 - \psi) \int \overline{r}_{k-1} \, dP^{(k)}_{z, \theta} \right\} = \overline{r}_k^{n+1}.
\]

Since the induction basis is satisfied by construction of \( \overline{r}_k^0 \), it follows that \((\overline{r}_k^n)_{n \geq 0}\) is a non-decreasing sequence. Moreover, since \( \rho_\lambda \) is concave in \( z \), it holds that

\[
(C.18) \quad \sup \partial_z \rho_\lambda(z, \theta) \leq \sup \partial_z \rho_\lambda(0, \theta) = \partial_z \lambda_\rho(z) = \lambda_k
\]

so that \( \overline{r}_k^n = r'_k \) is bounded from above by \( \lambda_k \). Again, using this as an induction bases, it follows that

\[
(C.19) \quad \overline{r}_k^n = \sup_{\pi \in \Pi_\lambda} \left\{ \psi \lambda_k (1 - \delta_k) + (1 - \psi) \int \overline{r}_{k-1} \, dP^{(k)}_{z, \theta} \right\}
\]

\[
(C.20) \quad \leq \sup_{\pi \in \Pi_\lambda} \left\{ \psi \lambda_k (1 - \delta_k) + (1 - \psi) \lambda_k \right\} \leq \lambda_k.
\]

so that \((\overline{r}_k^n)_{n \geq 0}\) is bounded by \( \lambda_k \). Hence, by Lemma 2, the limit \( \lim_{n \to \infty} \overline{r}_k^n = \overline{r}_k \) exist and it solves the integral equation

\[
(C.21) \quad \overline{r}_k^n = \sup_{\pi \in \Pi_\lambda} \left\{ \psi \lambda_k (1 - \delta_k) + (1 - \psi) \int \overline{r}_{k-1} \, dP^{(k)}_{z, \theta} \right\}.
\]
By inspection of $\Pi^*_\lambda$ in Corollary 2, the supremum is achieved by the decision and stopping rules defined by the decision functions

\[(C.22) \quad \delta_k(z) = \mathcal{I}\left(\{G^{(k)}_\lambda(z) = g_\lambda(z)\}\right)\]

and

\[(C.23) \quad \overline{\psi}(z, \theta) = \begin{cases} \mathcal{I}\left(\{g_\lambda(z) > \rho_\lambda(z, \theta)\}\right), & \lambda_k(1 - \delta_k(z)) < \int r^{\pi,1}_k - 1 dP^{(k)}_{z, \theta} \\ \mathcal{I}\left(\{g_\lambda(z) \geq \rho_\lambda(z, \theta)\}\right), & \lambda_k(1 - \delta_k(z)) \geq \int r^{\pi,1}_k - 1 dP^{(k)}_{z, \theta} \end{cases}\]

so that (C.21) can equivalently be written as

\[(C.24) \quad r^{\pi,1}_k = \overline{\psi} \lambda_k(1 - \overline{\delta}_k) + (1 - \overline{\psi}) \int r^{\pi,1}_k - 1 dP^{(k)}_{z, \theta}.\]

Consequently, it holds that $\tau_k = \lambda_k \alpha^{(k)}_{\pi, \overline{P}_k} \geq r_k$. Analogously, by replacing the supremum in (C.15) with the infimum, a sequence $(r^{\pi,1}_n)_{n \geq 0}$ can be constructed that is non-increasing, bounded from below by zero, and, therefore, converges to a unique limit $r_k = \lambda_k \alpha^{(k)}_{\pi, \overline{P}_k} \leq r_k$. This implies that

\[(C.25) \quad \lambda_k \alpha^{(k)}_{\pi, \overline{P}_k}(z, \theta) \leq r_k(z, \theta) \leq \lambda_k \alpha^{(k)}_{\pi, \overline{P}_k}(z, \theta).\]

From Lemma 4 it now follows that a policy $\tilde{\pi} \in \Pi^*_\lambda$ exists such that

\[(C.26) \quad \lambda_k \alpha^{(k)}_{\pi, \overline{P}_k} = r_k(z, \theta),\]

which contradicts the assumption (C.13). Hence, it holds that

\[(C.27) \quad \left\{ \alpha^{(k)}_{\pi, \overline{P}_k}(z, \theta) : \pi \in \Pi^*_\lambda \right\} \subset \partial z_r \rho_\lambda(z, \theta),\]

for all $(z, \theta) \in \Omega_\rho$. This concludes the proof of the second statement in Theorem 3.

**APPENDIX D: PROOF OF THEOREM 4**

The outline of the proof is as follows: first, it is shown that if $\lambda^*$ satisfies (4.2), it holds that $\gamma(\pi^*, \overline{P}) \in \partial z_r \rho_\lambda^*(1, \theta_0)$ and that $\pi_k \in \partial z_r \rho_\lambda^*(1, \theta_0)$. Theorem 4 then follows as a consequence of Theorem 3.

In order to show the first part, the following lemma is useful.
Lemma 5. For $\rho_\lambda$ defined in Theorem 1 it holds that for all $(z, \theta) \in \Omega_\rho$ and all $k = 0, \ldots, K$

\begin{equation}
\lambda_k \partial_{\lambda_k} \rho_\lambda(z, \theta) = z_k \partial_{z_k} \rho_\lambda(z, \theta)
\end{equation}

where $\lambda_0 = 1$ is defined to unify notation.

In order to show the lemma, assume that a function $\tilde{\rho}: \Omega_\rho \to \mathbb{R}_{\geq 0}$ exists such that for all $(z, \theta) \in \Omega_\rho$ it holds that

\begin{equation}
\rho_\lambda(z, \theta) = \tilde{\rho}(\lambda z, \theta),
\end{equation}

Given that (D.2) holds, it immediately follows that

\begin{align*}
z_k \partial_{z_k} \rho_\lambda(z, \theta) &= z_k \partial_{z_k} \tilde{\rho}(\lambda z, \theta) \\
&= \lambda_k z_k \partial_{\lambda_k z_k} \tilde{\rho}(\lambda z, \theta) \\
&= \lambda_k \partial_{\lambda_k} \tilde{\rho}(\lambda z, \theta) = \lambda_k \partial_{\lambda_k} \rho_\lambda(z, \theta),
\end{align*}

where $\partial_{\lambda_k z_k} \tilde{\rho}$ denotes the generalized differential of $\tilde{\rho}$ with respect to $\lambda_k z_k$.

The existence of $\tilde{\rho}$ can be shown via induction. Let the sequence $\rho_n$ be as defined in (B.1) and assume that (D.2) holds for some $n \geq 0$, i.e., a function $\tilde{\rho}^n$ exists such that $\rho_n^\lambda(z, \theta) = \tilde{\rho}^n(\lambda z, \theta)$. It then follows that

\begin{align*}
\rho_{\lambda}^{n+1}(z, \theta) &= \min \left\{ g_\lambda(z) , 1 + \int \rho_n^\lambda(z p_\theta(x), \xi_\theta_n(x)) d\mu(x) \right\} \\
&= \min \left\{ \tilde{g}(\lambda z) , 1 + \int \tilde{\rho}^n(\lambda z p_\theta(x), \xi_\theta_n(x)) d\mu(x) \right\} \\
&=: \tilde{\rho}^{n+1}(\lambda z, \theta),
\end{align*}

where $\tilde{g}(\lambda z) = g_\lambda(z)$ and the induction basis is given by $\rho_0^\lambda = \tilde{\rho}^0 = \tilde{g} = g_\lambda$.

A necessary condition for $\lambda^*$ to solve (4.2) is that for all $k = 1, \ldots, K$

\begin{equation}
0 \in \partial_{\lambda_k} \rho_\lambda(1, \theta_0) - \sum_{k=1}^K \lambda_k \partial_{z_k} \rho_\lambda(1, \theta_0)
\end{equation}

By Lemma 5, it holds that

\begin{equation}
\lambda_k \partial_{\lambda_k} \rho_\lambda(1, \theta_0) - \sum_{k=1}^K \lambda_k \partial_{z_k} \rho_\lambda(1, \theta_0) = \partial_{z_k} \rho_\lambda(1, \theta_0) - \lambda_k \partial_{z_k} \rho_\lambda(1, \theta_0)
\end{equation}

so that by definition of $\lambda^*$

\begin{equation}
\lambda_k^* \partial_{z_k} \rho_\lambda^*(1, \theta_0).
\end{equation}
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By Theorem 3, it also holds that for all \( \pi \in \Pi^*_\lambda (\mathcal{P}) \)

\[
\lambda^*_k \alpha_k (\pi, \mathcal{P}_k) = \lambda^*_k \alpha^{(k)}_{\pi, \mathcal{P}_k} (1, \theta_0) \in \partial \lambda^* (1, \theta_0).
\]

This proves (4.4). In order to show (4.3), it suffices to show that \( \gamma (\pi^*, \mathcal{P}) \in \partial \lambda^* (1, \theta_0) \). Lemma 4 guarantees that a policy \( \pi^* \in \Pi^*_\lambda (\mathcal{P}) \) exists that satisfies the error probability constraints in (2.17) with equality, i.e.

\[
\alpha_k (\pi^*, \mathcal{P}_k) = \alpha_k (\pi^*, \mathcal{P}_k) = \alpha_k.
\]

for all \( k = 1, \ldots, K \). It then follows that

\[
\gamma (\pi^*, \mathcal{P}_0) + \sum_{k=1}^{K} \lambda^*_k \alpha_k = \gamma (\pi^*, \mathcal{P}_0) + \sum_{k=1}^{K} \lambda^*_k \alpha_k (\pi^*, \mathcal{P}_k)
\]

\[
= \inf_{\pi \in \Pi} \left\{ \gamma (\pi, \mathcal{P}_0) + \sum_{k=1}^{K} \lambda^*_k \alpha_k (\pi, \mathcal{P}_k) \right\}
\]

\[
\leq \inf_{\pi \in \Pi^*_\lambda} \left\{ \gamma (\pi, \mathcal{P}_0) + \sum_{k=1}^{K} \lambda^*_k \alpha_k (\pi, \mathcal{P}_k) \right\}
\]

\[
= \inf_{\pi \in \Pi^*_\lambda} \gamma (\pi, \mathcal{P}_0) + \sum_{k=1}^{K} \lambda^*_k \alpha_k,
\]

which implies \( \inf_{\pi \in \Pi^*_\lambda} \gamma (\pi, \mathcal{P}_0) = \gamma (\pi^*, \mathcal{P}_0) \). By definition of \( \pi^* \) and Theorem 3 it follows that

\[
\gamma (\pi^*, \mathcal{P}_0) = \gamma (\pi^*, \mathcal{P}_0) \in \left\{ \gamma (\pi, \mathcal{P}_0 (1, \theta_0) : \pi \in \Pi^*_\lambda \right\} = \partial \lambda^* (1, \theta_0).
\]

This implies that \( \pi^* \in \Pi^*_\lambda (\mathcal{P}) \cap \Pi^*_\alpha (\mathcal{P}) \), which concludes the proof.

APPENDIX E: PROOF OF THEOREM 5

The existence and uniqueness of \( \gamma_{\pi, \mathcal{P}} \) and \( \alpha^{(k)}_{\pi, \mathcal{P}} \) can be proven in analogy to the existence and uniqueness of \( \rho_\lambda \) in Theorem 1. Consider the sequence of functions \( \left( \alpha^{(k)}_{\pi, \mathcal{P}} \right)_{n \geq 0} \) that is defined recursively via

\[
\alpha^{(k)}_{\pi, \mathcal{P}, n} = \psi (1 - \delta_k) + (1 - \psi) \left( \sup_{H \in \mathcal{P}_0} \int \alpha^{(k), n-1}_{\pi, \mathcal{P}} (z p_\theta (x), \xi_\theta (x)) dH (x) \right)
\]

with \( \alpha^{(k), 0}_{\pi, \mathcal{P}} = \psi (1 - \delta_k) \). It is not hard to show that this sequence is nondecreasing and bounded for all \( n \geq 0 \). The nondecreasing property can be
shown via induction. Assuming \( \alpha^{(k), n}_{\pi, P} \geq \alpha^{(k), n-1}_{\pi, P} \), it follows that

\[
\alpha^{(k), n+1}_{\pi, P} = \psi(1 - \delta_k) + (1 - \psi) \left( \sup_{H \in P} \int \alpha^{(k), n}_{\pi, P} (zp_\theta(x), \xi_\theta(x)) dH(x) \right)
\]

\[
\geq \psi(1 - \delta_k) + (1 - \psi) \left( \sup_{H \in P} \int \alpha^{(k), n-1}_{\pi, P} (zp_\theta(x), \xi_\theta(x)) dH(x) \right)
\]

\[
= \alpha^{(k), n}_{\pi, P}.
\]

The induction basis is given by

\[
\alpha^{(k), 1}_{\pi, P} = \psi(1 - \delta_k) + (1 - \psi) \left( \sup_{P \in P_\theta} \int \alpha^{(k), 0}_{\pi, P} (zp_\theta(x), \xi_\theta(x)) dP(x) \right) \alpha^{(k), 0}_{\pi, P}
\]

\[
\geq \psi(1 - \delta_k) = \alpha^{(k), 0}_{\pi, P}.
\]

Boundedness can be shown in the same manner. Assuming that \( \alpha^{(k), n}_{\pi, P} \leq 1 \), it follows that

\[
\alpha^{(k), n+1}_{\pi, P} = \psi(1 - \delta_k) + (1 - \psi) \left( \sup_{P \in P_\theta} \int \alpha^{(k), n}_{\pi, P} (zp_\theta(x), \xi_\theta(x)) dP(x) \right)
\]

\[
\leq \psi(1 - \delta_k) + (1 - \psi) = 1 - \psi \delta_k \leq 1,
\]

with induction basis \( \alpha^{(k), 0}_{\pi, P} = \psi(1 - \delta_k) \leq 1 \). Hence, Lemma 2 applies and the sequence \( \alpha^{(k), n}_{\pi, P} \) converges to the unique limit \( \alpha^{(k)}_{\pi, P} \), which satisfies the integral equation (5.4).

The same arguments can be used to show existence of \( \gamma_{\pi, P} \), the only difference being that \( \gamma_{\pi, P} \) is not bounded from above. More precisely, the sequence \( \gamma^{n}_{\pi, P} \) that is defined recursively via

\[
(E.2) \quad \gamma^{n}_{\pi, P} = (1 - \psi) \left( 1 + \sup_{P \in P_\theta} \int \gamma^{n-1}_{\pi, P} (zp_\theta(x), \xi_\theta(x)) dP(x) \right),
\]

with \( \gamma^{n}_{\pi, P} = 0 \), can be shown to be nondecreasing. Hence, for every \( (z, \theta) \in \Omega_\rho \), \( \left( \gamma^{n}_{\pi, P}(z, \theta) \right)_{n \geq 0} \) is a monotonic sequence of real numbers. If this sequence is bounded, the same arguments as before apply and a unique limit \( \gamma_{\pi, P}(z, \theta) \) exists. If the sequence is unbounded, it is guaranteed to diverge to infinity \([37, \text{Theorem 3.12}]\), i.e., \( \lim_{n \to \infty} \gamma^{n}_{\pi, P}(z, \theta) = \gamma_{\pi, P}(z, \theta) = \infty \). Consequently, \( \gamma_{\pi, P}(z, \theta) \in \mathbb{R}_{\geq 0} \cup \{\infty\} \) exists and is unique for every \( (z, \theta) \in \Omega_\rho \). This concludes the proof.
APPENDIX F: PROOF OF THEOREM 6

Theorem 6 follows from Theorem 5 and can be proven via contradiction. The proof is detailed only for $Q_k$, $k = 1, \ldots, K$; for $k = 0$ it follows analogously. Assume that a distribution $P^* \in \mathcal{P}_k$ exists such that $\alpha_k^{(k)}(\pi, P^*) > \alpha_k^{(k)}(\pi, Q_k)$, with $Q_k$ defined in Theorem 6. By (3.19), this implies that $\alpha_{\pi, P^*}^{(k)}(1, \theta_0) > \alpha_{\pi, Q_k}^{(k)}(1, \theta_0)$, where $\alpha_{\pi, P^*}^{(k)}$ solves

$$\alpha_{\pi, P^*}^{(k)} = \psi(1 - \delta_k) + (1 - \psi) \int \alpha_{\pi, P^*}^{(k)}(zp_\theta(x), \xi_\theta(x)) dP^*_\theta(x)$$

and $\{P^*_\theta\}_{\theta \in \Omega}$ denotes the family of conditional distributions corresponding to $P^*$. However, by definition,

$$\alpha_{\pi, P^*}^{(k)} \leq \psi(1 - \delta_k) + (1 - \psi) \sup_{P \in \mathcal{P}_\theta^{(k)}} \int \alpha_{\pi, P^*}^{(k)}(zp_\theta(x), \xi_\theta(x)) dP(x)$$

so that, using the same arguments as in Appendix E, a nondecreasing sequence of functions $\alpha_{\pi, P^*}^{(k), n}$ can be constructed with $\alpha_{\pi, P^*}^{(k), 0} = \alpha_{\pi, P^*}^{(k)}$ that converges to $\alpha_{\pi, P^*}^{(k), \infty}$ for $n \to \infty$. Since by Theorem 5 this limit is unique, it follows that $\alpha_{\pi, P^*}^{(k), \infty} = \alpha_{\pi, Q_k}^{(k)} \geq \alpha_{\pi, P^*}^{(k)}$, which contradicts the assumption that $\alpha_{\pi, P^*}^{(k)}(1, \theta_0) > \alpha_{\pi, Q_k}^{(k)}(1, \theta_0)$. This concludes the proof.

APPENDIX G: PROOF OF THEOREM 7

The proof of Theorem 7 closely follows the proof Theorem 5 in [43]. That is, it is shown that the functions $\rho_\lambda$, $d_\lambda$, and $D_\lambda$ can be defined as pointwise limits of monotonic and bounded sequences. From this, existence and uniqueness follow.

Let $(\rho_\lambda^n)_{n \geq 0}$, $(d_\lambda^n)_{n \geq 1}$, and $(D_\lambda^n)_{n \geq 1}$ be defined recursively via

$$\rho_\lambda^n(z, \theta) = \min \{ g_\lambda(z), z_0 + d_\lambda^{n-1}(z, \theta) \},$$

$$d_\lambda^n(z, \theta) = \sup_{P \in \mathcal{P}_\theta} D_\lambda^{n-1}(z, \theta; P),$$

$$D_\lambda^n(z, \theta; P) = \int \rho_\lambda^{n-1}(zp_\theta(x), \xi_\theta(x)) \mu(dx),$$

with $\rho_\lambda^0 = g_\lambda$. Since $\rho_\lambda^n$ is a nondecreasing function of $d_\lambda^n$, $d_\lambda^n$ is a nondecreasing function of $D_\lambda^n$, and $D_\lambda^n$ is a nondecreasing function of $\rho_\lambda^{n-1}$, it follows that all three sequences are nondecreasing. Moreover, since $\rho_\lambda^n$ is
upper bounded by \( g_\lambda \) for all \( n \geq 0 \), it holds that

\[
D_\lambda^n(z, \theta; P) \leq \int g_\lambda(zp_\theta(x), \xi_\theta(x)) \, d\mu(x) \leq \sum_{k=1}^K \lambda_k z_k
\]

for all \( P \) and, consequently,

\[
d_\lambda^n(z, \theta) \leq \sup_{P \in \mathcal{P}_\theta} \sum_{k=1}^K \lambda_k z_k = \sum_{k=1}^K \lambda_k z_k.
\]

This concludes the proof.

APPENDIX H: PROOF OF THEOREM 8

A pair \((\pi^*, Q)\) is minimax optimal in the sense of (2.16) if it satisfies the saddle point condition

\[
L_\lambda(\pi^*, P) \leq L_\lambda(\pi^*, Q) \leq L_\lambda(\pi, Q)
\]

for all \( \pi \in \Pi \) and all \( P \in \mathcal{M}_K^{K+1} \). That is, \( \pi^* \) is optimal with respect to \( Q \) and \( Q \) is least favorable with respect to \( \pi^* \).

Assume that the pair \((\pi^*, Q)\) satisfies the conditions in Theorem 8. The inequality on the right hand side of (H.1), i.e., optimality of the policy \( \pi^* \), follows immediately from \( \pi^* \in \Pi_\lambda(Q) \). The inequality on the left hand side, i.e., \( Q \) being least favorable, can be shown as follows. Let \( Q_{z, \theta} \in \mathcal{Q}_{z, \theta} \) so that

\[
D_\lambda(z, \theta; Q_{z, \theta}) = \sup_{P \in \mathcal{P}_\theta} D_\lambda(z, \theta; P) = \sup_{P \in \mathcal{P}_\theta} \int \rho_\lambda(zp(x), \xi_\theta(x)) \, d\mu(x).
\]

By Theorem 2, the integral on the right hand side can equivalently be written as

\[
\sup_{P \in \mathcal{P}_\theta} D_\lambda(z, \theta; P) = \sup_{P \in \mathcal{P}_\theta} \int \rho_\lambda(zp(x), \xi_\theta(x)) \, d\mu(x)
\]

\[
= \sup_{P \in \mathcal{P}_\theta} \left\{ \gamma_0 p_0(x) \gamma_{\pi^*, Q}(zp(x), \xi_\theta(x)) \, d\mu(x) + \sum_{k=1}^K \int \lambda_k z_k p_k(x) \alpha^{(k)}_{\pi^*, Q}(zp(x), \xi_\theta(x)) \, d\mu(x) \right\}
\]

\[
= \sup_{P \in \mathcal{P}_\theta} \left\{ \gamma_0 \int \gamma_{\pi^*, Q}(zp(x), \xi_\theta(x)) \, dP_0(dx) \right\}
\]
The trick at this point is to split the optimization over the densities in the argument of the functions $\gamma_{\pi^*,Q}$, $\alpha^{(k)}_{\pi^*,Q}$ and the measures with respect to which the expected values are taken. The idea is as follows: consider a function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$. In general, it holds that

$$
\sup_{P \in \mathcal{P}} \int f(p(x)) P(dx) \leq \sup_{P \in \mathcal{P}} \int f(q(x)) P(dx),
$$

where equality holds if and only if

$$
Q \in \arg \max_{P \in \mathcal{P}} \int f(p(x)) P(dx) \neq \emptyset.
$$

This argument can easily be generalized to functions of the form of $\gamma_{\pi^*,Q}$ and $\alpha^{(k)}_{\pi^*,Q}$. Making use of the fact that, by definition, $Q_{z,\theta}$ solves the maximization on the right hand side of (H.2), it hence follows that

$$
\sup_{P \in \mathcal{P}_0} D_\lambda(z, \theta; P) = \sup_{P \in \mathcal{P}_0} \left\{ z_0 \int \gamma_{\pi^*,Q}(zq_{z,\theta}(x), \xi_\theta(x)) P_0(dx)
+ \sum_{k=1}^{K} \lambda_k z_k \int \alpha^{(k)}_{\pi^*,Q}(zq_{z,\theta}(x), \xi_\theta(x)) P_k(dx) \right\},
$$

where the densities in the arguments of the functions under the integral have been replaced with the true maximizers. Consequently, the joint maximization over $P$ decouples into individual maximizations over $P_0, \ldots, P_K$:

$$
\sup_{P \in \mathcal{P}_0} D_\lambda(z, \theta; P) = z_0 \sup_{P_0 \in \mathcal{P}_0^{(0)}} \int \gamma_{\pi^*,Q}(zq_{z,\theta}(x), \xi_\theta(x)) P_0(dx)
+ \sum_{k=1}^{K} \lambda_k z_k \sup_{P_k \in \mathcal{P}_0^{(k)}} \int \alpha^{(k)}_{\pi^*,Q}(zq_{z,\theta}(x), \xi_\theta(x)) P_k(dx) \right\},
$$

This, in turn, implies that

$$
Q_{z,\theta}^{(0)} \in \arg \max_{H \in \mathcal{P}_0^{(0)}} \int \gamma_{\pi^*,Q}(zq_{z,\theta}(x), \xi_\theta(x)) H(dx)
$$

and

$$
Q_{z,\theta}^{(k)} \in \arg \max_{H \in \mathcal{P}_0^{(k)}} \int \alpha^{(k)}_{\pi^*,Q}(zq_{z,\theta}(x), \xi_\theta(x)) H(dx)
$$
for all $k = 1, \ldots, K$. From (H.5) and (H.6) and the Chapman–Kolmogorov equations (3.17) and (3.18), it further follows that

\[
\gamma_{\pi^*, Q} = (1 - \psi) \left( 1 + \int \gamma_{\pi, Q} (zq_{z, \theta(x)}, \xi_{\theta}(x)) Q_{z, \theta}(dx) \right)
\]

\[
= (1 - \psi) \left( 1 + \sup_{H \in P_{\theta}} \int \gamma_{\pi, Q} (zq_{z, \theta(x)}, \xi_{\theta}(x)) H(dx) \right)
\]

and

\[
\alpha_{\pi, P}^{(k)} = \psi (1 - \delta_k) + (1 - \psi) \left( \int \alpha_{\pi, P}^{(k)} (zp_{\theta}(x), \xi_{\theta}(x)) Q_{z, \theta}(dx) \right)
\]

\[
= \psi (1 - \delta_k) + (1 - \psi) \left( \sup_{H \in P_{\theta}} \int \alpha_{\pi, P}^{(k)} (zp_{\theta}(x), \xi_{\theta}(x)) H(dx) \right).
\]

That is, $\gamma_{\pi^*, Q}$ and $\alpha_{\pi^*, Q}^{(k)}$ solve the integral equations (5.3) and (5.4). By Theorem 6, this implies that $Q_0$ is least favorable with respect to the conditional expected run-length $\gamma_{\pi, P_0}(z, \theta)$ for all $(z, \theta) \in \Omega_{\rho}$ and that $Q_k$ is least favorable with respect to the conditional error probabilities $\alpha_{\pi, P_k}^{(k)} (z, \theta)$ for all $k = 1, \ldots, K$ and all $(z, \theta) \in \Omega_{\rho}$. Finally, using (3.19), it follows that

\[
\sup_{P \in \mathcal{P}} L_{\lambda}(\pi^*, P) = \sup_{P \in \mathcal{P}} \left\{ \gamma(\pi^*, P_0) + \sum_{k=1}^{K} \lambda_k \alpha_{\pi^*, P_k}^{(k)}(1, \theta_0) \right\}
\]

\[
= \sup_{P \in \mathcal{P}} \left\{ \gamma_{\pi^*, P_0}(1, \theta_0) + \sum_{k=1}^{K} \lambda_k \alpha_{\pi^*, P_k}^{(k)}(1, \theta_0) \right\}
\]

\[
= \gamma_{\pi^*, Q_0}(1, \theta_0) + \sum_{k=1}^{K} \lambda_k \alpha_{\pi^*, Q_k}^{(k)}(1, \theta_0)
\]

\[
= \gamma(\pi^*, Q_0) + \sum_{k=1}^{K} \lambda_k \alpha_{\pi^*, Q_k}^{(k)} = L_{\lambda}(\pi^*, Q).
\]

This concludes the proof.

**APPENDIX I: PROOF OF THEOREM 9**

Theorem 9 is proven in two steps. First, by definition, all policies $\pi \in \Pi_{\lambda}^*(Q)$ satisfy Theorem 4, which immediately implies (6.9) as well as the existence of a policy $\pi^* \in \Pi_{\lambda}^*(Q)$ that satisfies the constraints on the error probabilities with equality for a given vector of distributions $Q$. The second
step is to show that the pair \((\pi^\dagger, Q)\) is a saddle point and, hence, minimax optimal. Using the same arguments as in Appendix H, it holds that

\[
\gamma(\pi^\dagger, Q_0) + \sum_{k=1}^K \lambda^*_k \alpha_k = \gamma(\pi^\dagger, Q_0) + \sum_{k=1}^K \lambda^*_k \alpha_k(\pi^\dagger, Q_k)
\]

\[
= \inf_{\pi \in \Pi} \left\{ \gamma(\pi, Q_0) + \sum_{k=1}^K \lambda^*_k \alpha_k(\pi, Q_k) \right\}
\]

\[
\leq \inf_{\pi \in \Pi^\star} \left\{ \gamma(\pi, Q_0) + \sum_{k=1}^K \lambda^*_k \alpha_k(\pi, Q_k) \right\}
\]

\[
= \inf_{\pi \in \Pi^\star} \gamma(\pi, Q_0) + \sum_{k=1}^K \lambda^*_k \alpha_k
\]

\[
= \gamma(\pi^*, Q_0) + \sum_{k=1}^K \lambda^*_k \alpha_k.
\]

Moreover, by Theorem 8, it holds that

\[
\gamma(\pi^\dagger, Q_0) + \sum_{k=1}^K \lambda^*_k \alpha_k = \gamma(\pi^\dagger, Q_0) + \sum_{k=1}^K \lambda^*_k \alpha_k(\pi^\dagger, Q_k)
\]

\[
= \sup_{P \in \mathcal{P}} \left\{ \gamma(\pi^\dagger, P_0) + \sum_{k=1}^K \lambda^*_k \alpha_k(\pi^\dagger, P_k) \right\}
\]

\[
= \sup_{P_0 \in \mathcal{P}_0} \gamma(\pi^\dagger, P_0) + \sum_{k=1}^K \lambda^*_k \sup_{P_k \in \mathcal{P}_k} \alpha_k(\pi^\dagger, P_k)
\]

\[
= \gamma(\pi^\dagger, P^*_0) + \sum_{k=1}^K \lambda^*_k \alpha_k.
\]

Hence, \((\pi^\dagger, Q)\) satisfies

\[
(I.1) \quad \inf_{\pi \in \Pi^\star} \gamma(\pi, Q_0) = \gamma(\pi^*, Q_0) = \gamma(\pi^\dagger, Q_0) = \gamma(\pi^\dagger, P^*_0) = \sup_{P_0 \in \mathcal{M}} \gamma(\pi^\dagger, P_0)
\]

which implies minimax optimality. Finally, (6.9) follows from (I.1) and the fact that \(\pi^\dagger \in \Pi^\star_{\lambda^\star}(Q)\). This concludes the proof.
APPENDIX J: PROOF OF COROLLARY 4

Corollary 4 can be proven by a standard duality argument. Let $\Pi_{\Pi}(\mathcal{P})$ be the set of all policies that satisfy the error probability constraints in (2.17):

$$\inf_{\pi \in \Pi_{\Pi}(\mathcal{P})} \sup_{\mathcal{P} \in \mathcal{P}} \gamma(\pi, \mathcal{P}_0) \geq \sup_{\lambda \geq 0} \inf_{\pi \in \Pi_{\Pi}(\mathcal{P})} \sup_{\mathcal{P} \in \mathcal{P}} \left\{ \gamma(\pi, \mathcal{P}_0) + \sum_{k=1}^{K} \lambda_k (\alpha_k(\pi, \mathcal{P}_k) - \alpha_k) \right\}$$

$$\geq \sup_{\lambda \geq 0} \left\{ \inf_{\pi \in \Pi} \sup_{\mathcal{P} \in \mathcal{P}} \left( \gamma(\pi, \mathcal{P}_0) + \sum_{k=1}^{K} \lambda_k \alpha_k(\pi, \mathcal{P}_k) - \sum_{k=1}^{K} \lambda_k \alpha_k \right) \right\}$$

$$= \sup_{\lambda \geq 0} \left\{ \rho^\lambda(1, \theta_0) - \sum_{k=1}^{K} \lambda_k \alpha_k \right\}.$$

Theorem 4 follows immediately.

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