Abrupt Desynchronization and Extensive Multistability in Globally Coupled Oscillator Simplices

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Collective behavior in large ensembles of dynamical units with non-pairwise interactions may play an important role in several systems ranging from brain function to social networks. Despite recent work pointing to simplicial structure, i.e., higher-order interactions between three or more units at a time, their dynamical characteristics remain poorly understood. Here we present an analysis of the collective dynamics of such a simplicial system, namely coupled phase oscillators with three-way interactions. The simplicial structure gives rise to a number of novel phenomena, most notably a continuum of abrupt desynchronization transitions with no abrupt synchronization transition counterpart, as well as, extensive multistability whereby infinitely many stable partially synchronized states exist. Our analysis sheds light on the complexity that can arise in physical systems with simplicial interactions like the human brain and the role that simplicial interactions play in storing information.

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Research into the macroscopic dynamics of large ensembles of coupled oscillators have extended our understanding of natural and engineered systems ranging from cell cycles to power grids [11]. However, with few exceptions (including [6, 7]), little attention has been paid to the synchronization dynamics of coupled oscillator systems where interactions are not pair-wise, but rather n-way, with $n \geq 3$. Such interactions are called “simplicial”, where an $n$-simplex represents an interaction between $n+1$ units, so 2-simplices describe three-way interactions, etc [8]. Recent advances suggest that simplicial interactions may be vital in general oscillator systems [9] and may play an important role in brain dynamics [10,12] and other complex systems phenomena [13–21]. In fact, coupled oscillator systems that display clustering and multi-branch entrainment have been shown to be useful models for memory and information storage [15,21]. Despite these findings, the general collective dynamics of coupled oscillator simplices and their utility in storing information are poorly understood.

In this work we study large coupled oscillator simplicies, specifically considering the generic case of 2-simplices, i.e., coupled oscillators with three-way interactions. The natural generalization of the classical Kuramoto model [22] with 2-simplex coupling is given by

$$\dot{\theta}_i = \omega_i + \frac{K}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \sin(\theta_j + \theta_k - 2\theta_i),$$

(1)

where $\theta_i$ represents the phase of oscillator $i$ with $i = 1, \ldots, N$, $\omega_i$ is its natural frequency which is assumed to be drawn from the distribution $g(\omega)$, and $K$ is the global coupling strength. While numerical investigations of systems with non-pairwise interactions have uncovered multistability and chaos [6,7], few analytical results exist their collective behavior remains largely unexplored. Here we focus on large systems and obtain an analytical description of the macroscopic dynamics using a partial dimensionality reduction obtained via a variation of the Ott-Antonsen ansatz [23,24]. In particular, the macroscopic dynamics are captured by a combination of two order parameters that capture the degree of synchronization and asymmetry as oscillators organize into two distinct synchronized clusters. We uncover the novel phenomenon where a continuum of abrupt desynchronization transitions emerge, each transition occurs at a different critical coupling strength, depending on the asymmetry of the system. Interestingly, no complementary abrupt synchronization transitions occur [25,26]. This continuum stems from an extensive multistability whereby, for sufficiently strong coupling, an infinite number of distinct partially synchronized states are stable in addition to the incoherent state, which is stable for all finite coupling strengths. This multistability indicates the capability of storing a wide array of possible information as different oscillator arrangements. Serving as a minimal model for memory storage, the system captures the critical properties of easily transitioning from an information storage state (i.e., synchronized) to the resting state (i.e., incoherent) [27] via abrupt desynchronization. The system then may return to another information storage state with an appropriately chosen perturbation. The rich nonlinear dynamics that emerge in this relatively simple extension of pair-wise coupling to three-way coupling highlights the complexity that may arise via simplicial interactions in systems like the human brain and the implications of these behaviors on information storage.

We begin our analysis by introducing the generalized order parameters $z_q = \frac{1}{N} \sum_{j=1}^{N} e^{iq\theta_j}$ for $q = 1$ and 2. Note that $z_1$ is the classical Kuramoto order parameter while $z_2$ typically measures clustering, which we will see in this system. Using the polar decompositions $z_q = r_q e^{i\psi_q}$ we rewrite Eq. (1) as

$$\dot{\theta}_i = \omega_i + K r_1^2 \sin[2(\psi_1 - \theta_i)],$$

(2)

We then consider the continuum limit $N \rightarrow \infty$ where the state of the system can be described by a density function $f(\theta, \omega, t)$ which describes the density of oscillator with phase between $\theta$ and $\theta + \delta\theta$ and natural frequency between $\omega$ and $\omega + \delta\omega$ at time $t$. Because the number of oscillators in the system is conserved $f$ must satisfy the continuity equation

$$\dot{f}(\theta, \omega, t) = \frac{\partial}{\partial \theta} \int_0^\infty \int_{\theta - \delta\theta}^{\theta + \delta\theta} f(\theta', \omega, t) \, d\theta' - \frac{\partial}{\partial \theta} \int_{\theta - \delta\theta}^{\theta + \delta\theta} f(\theta, \omega, t) \, d\theta.$$
\[ 0 = \partial_t f + \partial_\theta(f\dot{\theta}). \]
Moreover, because each oscillator’s natural frequency is fixed and drawn from \(g(\omega)\) the density function \(f(\theta, \omega, t)\) may be expanded into a Fourier series of the form 
\[ f(\theta, \omega, t) = \frac{g(\omega)}{2\pi} \left[ 1 + \sum_{n=1}^{\infty} \hat{f}_n(\omega, t)e^{i\omega t} + \text{c.c.} \right], \]
where \(\hat{f}_n(\omega, t)\) is the \(n\)th Fourier coefficient and c.c. represents the complex conjugate of the previous sum.

We then consider the symmetric and asymmetric parts \(f_s(\theta, \omega, t)\) and \(f_a(\theta, \omega, t)\), respectively, of \(f(\theta, \omega, t)\) which satisfy \(f(\theta, \omega, t) = f_s(\theta, \omega, t) + f_a(\theta, \omega, t)\) with symmetries \(f_s(\theta + \pi, \omega, t) = f_s(\theta, \omega, t)\) and \(f_a(\theta + \pi, \omega, t) = -f_a(\theta, \omega, t)\). Note that the linearity of the continuity equation implies that if both \(f_s\) and \(f_a\) are solutions, then so is \(f\). While the asymmetric part \(f_a\) does not allow for dimensionality reduction, the symmetric part \(f_s\) does. Noting that the Fourier series of \(f_s\) is given by the even terms of the Fourier series of \(f\), i.e., \(f_s(\theta, \omega, t) = \frac{g(\omega)}{2\pi} \left[ 1 + \sum_{m=1}^{\infty} \hat{f}_m(\omega, t)e^{i2m\theta} + \text{c.c.} \right]\), we ansatz that each even Fourier coefficient decays geometrically, i.e., \(\hat{f}_m(\omega, t) = \hat{a}_m(\theta, t)\). Inserting this and Eq. [2] into the continuity equation, we find that each subspace spanned by each even terms \(e^{i2m\theta}\) collapse onto the same low-dimensional manifold characterized by the condition
\[ \dot{\theta}a = -2i\omega a + K \left( z_1^2 - z_2^2 - a^2 \right). \tag{3} \]

Equation [3] describes the evolution of the complex function \(a(\omega, t)\), and thereby \(f_s\), and responds to the order parameter \(z_1\). Moreover, \(a(\omega, t)\) can be linked to the order parameter \(z_2\) as follows. First, note that in the limit \(N \to \infty\) we have that \(z_2 = \int f_s(\theta, \omega, t)e^{i\omega t}d\theta d\omega\), and after inserting the Fourier series for \(f_s\) this reduces to \(z_2 = \int g(\omega)a^*(\omega, t)d\omega\). To further simplify the relationship we make the assumption that the frequency distribution \(g(\omega)\) is Lorentzian with mean \(\bar{\omega}_0\) and spread \(\Delta\), i.e., \(g(\omega) = \Delta / \pi [(\omega - \bar{\omega}_0)^2 + \Delta^2]\), which has two simple poles in the complex plane at \(\omega = \omega_0 \pm i\Delta\). The integral can then be evaluated using Cauchy’s Residue Theorem [23] by closing the integral contour with a semicircle of infinite radius in the lower-half plane and evaluating at the enclosed pole, yielding \(z_2 = a^*(\omega_0 - i\Delta, t)\). We then evaluate Eq. [4] at \(\omega = \omega_0 - i\Delta\) to obtain
\[ \dot{z}_2 = 2i\omega_0 z_2 - 2\Delta z_2 + K \left( z_1^2 - z_2^2 - z_3^2 \right). \tag{4} \]
Introducing the rescaled parameters \(\bar{K} = K / \Delta\) and \(\bar{\omega}_0 = \omega_0 / \Delta\) with rescaled time \(\bar{t} = t / \Delta\) and using the polar decompositions the yields (where we have dropped the notation for convenience)
\[ \dot{r}_2 = -2r_2 + \bar{K}r_2^2(1 - r_2^2) \cos(2\psi_1 - \psi_2), \tag{5} \]
\[ \dot{\psi}_2 = 2\omega_0 + \bar{K}r_2^2 \frac{1 + r_2^2}{r_2} \sin(2\psi_1 - \psi_2). \tag{6} \]

Equations (5) and (6) describe the dynamics of the even part \(f_s\) of the density function \(f\) via the order parameter \(z_2\) which falls onto a low dimensional manifold similar to the Ott-Antonsen manifold. However, these equations do not capture the asymmetric part of the dynamics, and moreover they depend on the asymmetric part via \(z_2\)’s dependence on \(z_1\). As we will see, this reflects the system’s dependence on asymmetry in oscillator arrangements between two clusters.

To close the dynamics we apply a self-consistency analysis to characterize the order parameter \(z_1\). We first note that, by entering the rotating frame \(\theta \rightarrow \theta + \omega_0 t\) we may set \(\omega_0 = 0\) so that \(\psi_1 = \psi_2 = 0\). Moreover, by rotating initial conditions we may set \(\psi_1 = \psi_2 = 0\). Equation [3] then implies that oscillators that become phase-locked satisfy \(|\omega_1| \leq K r_2^2\), in which case they relax to one of the two stable fixed points \(\theta = \theta^*(\omega_1) + \pi\), where \(\theta^*(\omega) = \arcsin(\omega / K r_2^2) / 2\). These two fixed points correspond to the two clusters that the phase-locked oscillators organize into. Specifically, phase-locked oscillators starting near \(\theta = 0\) or \(\pi\) will end up at the fixed points \(\theta^*(\omega)\) or \(\theta^*(\omega) + \pi\), respectively. The phase-locked population is described by the density function
\[ f_{\text{locked}}(\theta, \omega) = \eta\delta(\theta - \theta^*(\omega)) + (1 - \eta)\delta(\theta - \theta^*(\omega) - \pi), \tag{7} \]
where the asymmetry parameter \(\eta\) describes the fraction of phase-locked oscillators in the \(\theta = 0\) cluster. On the other hand, oscillators satisfying \(|\omega| > K r_2^2\) drift for all time and relax to the stationary distribution
\[ f_{\text{drift}}(\theta, \omega) = \frac{\sqrt{\omega^2 - K^2r_2^4}}{2\pi |\omega + K r_2^2 \sin(2\psi_1 - 2\theta)|}. \tag{8} \]

Next, the order parameter \(z_1\) is given by the integral \(z_1 = \int f(\theta, \omega, t)e^{i\omega t}d\theta d\omega\), which after inserting the density \(f\) as defined by Eqs. (7) and (8) reduces to
\[ r_1 = (2\eta - 1) \int_{-K r_2^2}^{K r_2^2} \frac{1 + \sqrt{1 - (\omega / K r_2^2)^2} \sin(2\psi_1 - 2\theta)}{2} g(\omega) d\omega, \tag{9} \]
where the contribution from the drifting oscillators vanishes due to the symmetry of \(f_{\text{drift}}\). Returning to \(r_2\), Eq. (5) implies that at steady state we have
\[ r_2 = \frac{-1 + \sqrt{1 + K^2 r_2^4}}{K r_2^2}. \tag{10} \]

Thus, the macroscopic steady-state is described by Eqs. (9) and (10).

Interpreting these analytical results in the context of numerical simulations allows us to understand novel phenomena that occur in the dynamics of Eq. (11). Beginning with simulations of a system with \(N = 10^5\) oscillators whose natural frequencies are Lorentzian with \(\omega_0 = 0\) and \(\Delta = 1\), we consider initial conditions of varying asymmetry, setting initial phases to \(\theta_i(0) = 0\) with probability \(\eta\) and otherwise \(\theta_i(0) = \pi\). We then begin simulations at \(K = 16\) and after reaching steady-state slowly decrease \(K\) to zero, then restore it slowly to 16.
decreasing $K$ characterized by the asymmetry parameter $\eta$ and coupling strengths. Partially synchronized branches arise (here five different branches are shown, but in the thermodynamic limit an infinite number of such states exist) and a continuum of abrupt desynchronization transitions at different coupling strengths. Partially synchronized branches are characterized by the asymmetry parameter $\eta$, indicating that these complex dynamics arise from different allocations of phase-locked oscillators in the two clusters at $\theta = 0$ and $\pi$. Next, as $K$ is restored to its initial value of 16 no spontaneous transitions back to synchronization occur, indicating no abrupt synchronization to complement the abrupt desynchronization transitions. Importantly, this highlights both a rich extensive multistability (here five different branches are shown, but in the thermodynamic limit an infinite number of such states exist) and a continuum of abrupt desynchronization transitions at different coupling strengths. We therefore propose the ansatz $\sqrt{K(2\eta - 1)}$ as the abrupt desynchronization occurs. We proceed by inverting Eq. (10), obtaining $K^2 = 2r_2/(1 - r_2^2)$, which can be inserted into Eq. (9), yielding

$$\sqrt{\frac{2r_2}{1 - r_2^2}} = \sqrt{K(2\eta - 1)}$$

$$\times \int_{-2r_2/(1 - r_2^2)}^{2r_2/(1 - r_2^2)} \sqrt{1 + \frac{1 - [\omega(1 - r_2^2)]/2r_2^2}{2}} g(\omega) d\omega.$$  

(11)

While Eq. (11) appears more complicated than Eq. (9), we note that the coupling strength $K$ has been entirely scaled out of the integral, appearing outside with $(2\eta - 1)$. We therefore conclude that if the quantities $\sqrt{K}$ and $2\eta - 1$ cancel one another out, i.e., $\sqrt{K(2\eta - 1)}$ is constant, it follows that the solution $r_2$ in Eq. (11) is independent of $K$. We therefore propose the ansatz $\sqrt{K(2\eta - 1)} = \text{const.}$ and use the initial condition $\eta_{\text{min}}(K_{\text{c}}(1)) = 1$, where $K_{\text{c}}(1)$ denotes the very first coupling strength where a synchronized state is possible with $\eta = 1$, yielding

$$\eta_{\text{min}}(K) = \frac{\sqrt{K_{\text{c}}(1)}}{2\sqrt{K}} + \frac{1}{2}.$$  

(12)

Equation (12) implies that along the minimum branch we have that $\sqrt{K(2\eta - 1)} = \sqrt{K_{\text{c}}(1)} \approx 2.034$, which can be used in Eq. (11) to compute the minimum branch of $r_2$, and in turn $r_1$ via Eq. (10), yielding

$$r_1^\text{min}(K) \approx \frac{1.2120}{\sqrt{K}}$$

and

$$r_2^\text{min}(K) \approx 0.5290.$$  

(13)

In Figs. 2(a) and (b) we plot these minimum branches in solid black curves, which agree with the simulation results. Lastly, by inverting Eq. (12) we find the critical coupling strength $K_{\text{c}}$ as a function of $\eta$ where the abrupt desynchronization transi-
tion occurs, namely

\[ K_c(\eta) \approx \frac{4.137}{(2\eta - 1)^2}. \] (14)

In Fig. 2 we plot the theoretical prediction of the abrupt desynchronization point \( K_c(\eta) \) as a solid curve vs observations from direct simulation as black circles, noting excellent agreement. We emphasize that above this curve we observe extensive multistability and below the curve only the incoherent state is stable.

The 2-simplex phase oscillator model studied here serves as a minimal model for memory and information storage, capturing a number of critical properties. First, each distinct synchronized state corresponds to a specific piece of information, differentiated by the clustering arrangement of the oscillators. Then, from any synchronized state the system can quickly and easily transition to the resting state described by incoherence via the abrupt desynchronization transition decreasing the coupling strength. The microscopic properties of this abrupt desynchronization transition is illustrated in Fig. 3, where for \( \eta = 0.9 \) the distribution \( f(\theta) \) of phases is plotted as \( K \) is decreased slowly through the critical value of \( K_c \approx 6.47 \). Before the transition the distribution is asymmetrically clustered about \( \theta = 0 \) and \( \pi \) and changes slowly until at \( K = K_c \) all information is lost as the distribution becomes uniform. Once the incoherent resting state is reached, the system then may return to any synchronized (i.e., information storage) state via an appropriately designed perturbation that depends on the coupling strength and the desired target state. Moreover, this perturbation can be easily implemented by simply “freezing” each oscillator for a certain time interval as it evolves in the incoherent state.

More broadly, the analysis presented here demonstrates how the extension from pair-wise to more general simplicial (specifically, 2-simplex) interactions in coupled oscillator systems can give rise to a host of complex nonlinear phenomena. Moreover, these phenomena can be captured and described using analytical methods. In particular, we have characterized a continuum of abrupt desynchronization transitions that occur at different coupling strengths without any abrupt synchronization transitions. This continuum stems from extensive multistability, whereby for sufficiently strong coupling an infinite number of partially synchronized states are stable. These different stable states represent synchronized states organized via different asymmetries into two clusters of entrained oscillators. In addition to highlighting the possible complexity that may arise in coupled oscillator systems with simplicial interactions, we hypothesize that simplicial interactions may give rise to novel nonlinear phenomena in other complex systems.

Lastly, we note that similar, albeit more complicated dynamics and synchronization patterns emerge in 3- and higher-order simplex interactions, that is, four-way interactions, five-way interactions, etc. Preliminary investigations (not shown here) indicate that in an \( n \)-simplex system oscillators organize into \( n \) distinct, equidistant clusters around the circle. Abrupt desynchronization transitions persist, as does the extensive multistability, however the locations of transitions appears to increase with \( n \) and the effect of asymmetry on the synchronized states themselves is more complicated. Other fruitful avenues of future research include the incorporation of non-trivial network structures and mixed coupling, i.e., the presence of both \( n \)- and \( m \)-simplex interactions with \( n \neq m \).

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