Essential Forward Weak KAM Solution for the Convex Hamilton–Jacobi Equation

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Abstract For a convex, coercive continuous Hamiltonian on a closed Riemannian manifold $M$, we construct a unique forward weak KAM solution of

$$H(x, d_x u) = c(H)$$

by a vanishing discount approach, where $c(H)$ is the Mañé critical value. We also discuss the dynamical significance of such a special solution.

Keywords Hamilton–Jacobi equation, discounted equation, weak KAM solution, Aubry–Mather theory, viscosity solution

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1 Introduction

For a compact connected manifold $M$ without boundary, the Hamiltonian is usually mentioned as a continuous function defined on its cotangent bundle $T^*M$. In [8], the authors firstly proposed the ergodic approximation technique, to consider the existence of so called viscosity solutions to the Hamilton–Jacobi equation

$$H(x, d_x u) = c(H), \quad x \in M \quad (HJ_0)$$

for the Mañé critical value

$$c(H) := \inf \{ c \in \mathbb{R} \mid \exists \omega \in C(M, \mathbb{R}) \text{ such that } H(x, d_x \omega) \leq c, \text{ a.e. } x \in M \}.$$ 

The Hamiltonian they concerned satisfies

- (Coercivity) $H(x, p)$ is coercive in $p \in T^*_x M$, uniformly w.r.t. $x \in M$.

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(Convexity) $H(x, p)$ is convex in $p \in T^*_x M$ for all $x \in M$.

They perturbed $(HJ_0)$ by the following discounted equation
\[
\lambda u + H(x, d_x u) = c(H), \quad x \in M, \lambda > 0
\] (1.1)
of which the Comparison Principle is allowed. Therefore, the viscosity solution $u^\lambda_-$ of (1.1) is unique. In [5, 12], they established the convergence of $u^\lambda_-$ as $\lambda \to 0_+$, to a specified viscosity solution $u^\lambda_0$ of $(HJ_0)$ which can be characterized by the combination of subsolutions of $(HJ_0)$, or Peierls barrier
\[
h^\infty : M \times M \to \mathbb{R}
\] w.r.t. the projected Mather measures $\mathcal{M}$ of $(HJ_0)$, see Appendix for the relevant definitions of $\mathcal{M}$, $h^\infty$, subsolutions etc.

In this paper, we consider a negative limit technique and try to find another specified solution of $(HJ_0)$. Precisely, we consider
\[
-\lambda u + H(x, d_x u) = c(H), \quad x \in M, \lambda > 0
\] $(HJ_{\lambda})$
of which a unique forward $\lambda$-weak KAM solution $u^{+}_{\lambda}$ can be found, which is a subsolution of $(HJ_{\lambda})$, such that for any $x \in M$, there exists an absolutely continuous curve $\gamma^{+}_{\lambda,x} : [0, +\infty) \to M$ ending with it and satisfying
\[
e^{-\lambda t} u^{+}_{\lambda}(\gamma^{+}_{\lambda,x}(t)) - e^{-\lambda s} u^{+}_{\lambda}(\gamma^{+}_{\lambda,x}(s))
= \int_{s}^{t} e^{-\lambda \tau} (L(\gamma^{+}_{\lambda,x}(\tau), \dot{\gamma}^{+}_{\lambda,x}(\tau)) + c(H)) d\tau, \quad \forall 0 \leq s \leq t
\] (S$^+_\lambda$)
for the Lagrangian defined by
\[
L(x, v) := \max_{p \in T^*_x M} \{\langle p, v \rangle - H(x, p)\}, \quad (x, v) \in TM.
\] (1.2)
As $\lambda \to 0_+$, we get the following conclusion:

**Theorem 1.1** Let $H : T^* M \to \mathbb{R}, (x, p) \mapsto H(x, p)$ be a continuous Hamiltonian coercive and convex in $p$. For $\lambda > 0$, the unique forward $\lambda$-weak KAM solution $u^{+}_{\lambda}$ of $(HJ_{\lambda})$ uniformly converges as $\lambda \to 0_+$, to a unique forward 0-weak KAM solution $u^{+}_{0}$ of $(HJ_0)$, which can be interpreted as
\[
u^{+}_{0}(x) = \inf \mathcal{F}_{+}
\] (1.3)
with
\[
\mathcal{F}_{+} := \left\{ w \text{ is a subsolution of } (HJ_0) \left| \int_M w d\mu \geq 0, \forall \mu \in \mathcal{M} \right\}
\] (1.4)
and
\[
u^{+}_{0}(x) = - \inf_{\mu \in \mathcal{M}} \int_M h^\infty(x, y) d\mu(y).
\] (1.5)

**Remark 1.2** The novelty of this paper is that we adapt a symmetric Lagrangian skill to our $C^0$-setting. The lack of regularity invalidates a bunch of important properties of the Mather measures, Peierls barrier etc., so we have to find substitutes in the weak sense.

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1) A subsolution of $(HJ_0)$ which satisfies $(S^+_{\lambda})$ with $\lambda = 0$, also see Definition 4.6 for an alternative expression.
Besides, we mention that \(-u_0^+(x)\) is a viscosity solution of the symmetric equation
\[
H(x, -d_x u(x)) = c(H).
\]
Comparing to the backward 0-weak KAM solutions, the notion of viscosity solutions is more familiar to PDE specialists, although both are proved to be equivalent in [7].

1.1 Dynamic Interpretation of \(u_0^\pm\)

Now the vanishing discount approach supplies us with a pair of solutions of (HJ\(_0\)):
\[
\begin{cases}
u_0^-(x) = \inf_{\mu \in M} \int_M h^\infty(y, x) d\mu(y), \\
u_0^+(x) = -\inf_{\mu \in M} \int_M h^\infty(x, y) d\mu(y).
\end{cases}
\]
\[(1.6)\]

**Definition 1.3 (Conjugated Pair)** A backward 0-weak KAM solution \(u^-\) of (HJ\(_0\)) is conjugated to a forward 0-weak KAM solution \(u^+\), if
- \(u^- = u^+\) on the projected Mather set \(M\) (see Definition 4.4).
- \(u^- \geq u^+\) on \(M\).

In [7, 13], the above definition is applied to explore the dynamic behaviors for \(C^2\)-Tonelli Hamiltonians\(^2\)). However, we have no difficulty to reserve this concept to the \(C^0\)-case by the skills in [6]. Moreover, for the following typical Hamiltonians, \((u_0^-, u_0^+)\) indeed forms a conjugated pair:

1) **(Uniquely Ergodic)** Suppose \(M\) consists of a uniquely ergodic projected Mather measure (generic for \(C^2\)-Tonelli Hamiltonians, see [9]), then
\[
d_c(x, y) := h^\infty(x, y) + h^\infty(y, x) \geq 0
\]
for all \(x, y \in M\), and “=” holds for \(x, y \in M\) (due to the definition of the Mather measure in Appendix). So \((u_0^-, u_0^+)\) is a conjugated pair.

2) **(Mechanical System)** For a mechanical Hamiltonian
\[
H(x, p) = \frac{1}{2} \langle p, p \rangle + V(x),
\]
we can easily get \(c(H) = \max_{x \in M} V(x)\), then the associated \(L(x, v) + c(H) \geq 0\) on \(TM\). Consequently, \(h^\infty: M \times M \to \mathbb{R}\) is nonnegative, so \(u_0^- \geq u_0^+\) on \(M\). On the other side, due to the definition of \(M\), all the Mather measures are supported by equilibriums. So \(u_0^- = u_0^+\) on \(M\). In summary \((u_0^-, u_0^+)\) is a conjugated pair.

3) **(Constant Subsolution)** Such a case is also discussed in [5]. If \(H(x, 0) \leq c(H)\) for all \(x \in M\), i.e., constant is a subsolution of (HJ\(_0\)), then due to the Young Inequality we get
\[
L(x, v) + c(H) \geq L(x, v) + H(x, 0) \geq \langle v, 0 \rangle = 0
\]
for all \((x, v) \in TM\). By a similar analysis like the case of mechanical systems \((u_0^-, u_0^+)\) proves to be a conjugated pair.

\(^2\) \(H: (x, p) \in T^* M \to \mathbb{R}\) is called *Tonelli*, if it’s positive definite and superlinear in \(p\).
1.2 Organization of the Article
In Section 2, we prove some variational properties for nonsmooth Lagrangians. In Section 3, we prove the convergence of $u^\lambda$ as $\lambda \to 0_+$ and give a representative formula for the limit. For the consistency and readability of the article, some preliminary materials are moved to Appendix.

2 Nonsmooth Symmetric Lagrangians

With the same adaption as in [5], without loss of generality we can assume $H(x, p)$ is superlinear in $p$, i.e.,

- **(Superlinearity)** $\lim_{|p|\to +\infty} H(x, p)/|p| = +\infty$, for any $x \in M$.

In that case, by Fenchel’s formula (see (1.2)), the Hamiltonian has an associated Lagrangian $L : (x, v) \in TM \to \mathbb{R}$ which is superlinear and convex in the fibers of the tangent bundle. Consequently, we can propose a symmetrical Lagrangian $\hat{L}(x, v) := L(x, -v)$, of which the following fundamental facts hold:

**Lemma 2.1**

(i) The conjugated Hamiltonian $\hat{H} : T^*M \to \mathbb{R}$ of $\hat{L}(x, v)$ satisfies $\hat{H}(x, p) = H(x, -p)$ for all $(x, p) \in T^*M$. Therefore, $\hat{H}$ is also continuous, superlinear and convex.

(ii) $\hat{H}(x, d_x\omega) \leq c \iff H(x, d_x(-\omega)) \leq c$.

(iii) $c(\hat{H}) = c(H)$.

(iv) The projected Mather measure set $\hat{\mathfrak{M}}$ (associated with $\hat{H}(x, p)$) keeps the same with $\mathfrak{M}$.

(v) The Peierls barrier function associated with $\hat{L}(x, v)$ satisfies $\hat{h}^\infty(y, x) = h^\infty(x, y)$ for any $x, y \in M$.

**Proof**

(i) Due to (1.2) and the definition of $\hat{L}$, we have

$$\hat{H}(x, p) = \max_{v \in T_xM} \{\langle p, v \rangle - \hat{L}(x, v)\}$$

$$= \max_{v \in T_xM} \{\langle p, v \rangle - L(x, -v)\}$$

$$= \max_{w \in T_xM} \{-\langle p, w \rangle - L(x, w)\} = H(x, -p),$$

as is desired.

(ii) If $\omega$ is a subsolution of $\hat{H}(x, d_x\omega) \leq c$, then for any absolutely continuous $\gamma : [-T, T] \to M$ connecting $x, y \in M$, we get

$$\omega(y) - \omega(x) \leq \int_{-T}^{T} (\hat{L}(\gamma, \dot{\gamma}) + c) dt.$$
with $\tilde{\gamma}(t) := \gamma(-t)$ for all $t \in [-T, T]$. As $\gamma$ is arbitrarily chosen, so we get $-\omega \prec L + c$ then $H(x, -d_x \omega) \leq c$ for a.e. $x \in M$. Similarly, $\omega \prec L + c$ indicates $-\omega \prec \tilde{L} + c$.

(iii) As $c(\tilde{H}) = \inf\{c \in \mathbb{R} \mid \exists \omega \in C(M, \mathbb{R})$ such that $\omega \prec \tilde{L} + c\}$, then due to (ii), $c(\tilde{H}) = c(H)$.

(iv) Due to Proposition 2-4.3 of [2], for any measure $\tilde{\mu} \in \tilde{\mathcal{M}}$, there exists a sequence of closed measures $\tilde{\mu}_n \in \mathbb{P}_c(TM)$ (defined in Appendix), such that $\tilde{\mu}_n$ weakly converges to $\tilde{\mu}$ and

$$ \lim_{n \to +\infty} \int_{TM} Ld\tilde{\mu}_n = \int_{TM} Ld\tilde{\mu}. $$

Moreover, for each $\tilde{\mu}_n$ there exists an absolutely continuous curve $\gamma_n : t \in [-T_n, T_n] \rightarrow M$ with $T_n \rightarrow +\infty$ as $n \rightarrow +\infty$, such that

$$ \int_{TM} g d\mu_n = \frac{1}{2T_n} \int_{-T_n}^{T_n} g(\gamma_n(t), \dot{\gamma}_n(t))dt, \quad \forall g \in C_c(TM, \mathbb{R}). $$

Therefore, for $\tilde{\gamma}_n(t) := \gamma_n(-t)$, we have

$$ -c(H) = \lim_{n \to +\infty} \frac{1}{2T_n} \int_{-T_n}^{T_n} L(\gamma_n(t), \dot{\gamma}_n(t))dt $$

$$ = \lim_{n \to +\infty} \frac{1}{2T_n} \int_{-T_n}^{T_n} L(\tilde{\gamma}_n(-t), -\dot{\gamma}_n(-t))dt $$

$$ = \lim_{n \to +\infty} \frac{1}{2T_n} \int_{-T_n}^{T_n} \tilde{L}(\tilde{\gamma}_n(-t), \dot{\tilde{\gamma}}_n(-t))dt $$

$$ = \lim_{n \to +\infty} \frac{1}{2T_n} \int_{-T_n}^{T_n} \tilde{L}(\tilde{\gamma}_n(s), \dot{\tilde{\gamma}}_n(s))ds $$

$$ = -c(\tilde{H}). $$

That indicates $S^*\tilde{\mu}$ is a Mather measure for $\tilde{L}(x, v)$, where $S : TM \rightarrow TM$ is a diffeomorphism defined by $S(x, v) = (x, -v)$. Namely, we have

$$ \int_{TM} g(x, v)dS^*\tilde{\mu}(x, v) := \int_{TM} g(x, -v)d\tilde{\mu}(x, v), \quad \forall g \in C_c(TM, \mathbb{R}). $$

Due to (4.3) and $S \circ S = \text{id}$,

$$ \int_M f(x)d\mu(x) = \int_{TM} f \circ \pi(x, v)d\tilde{\mu}(x, v) $$

$$ = \int_{TM} f \circ \pi(x, -v)dS^*\tilde{\mu}(x, v) $$

$$ = \int_M f(x)d\pi^*S^*\tilde{\mu}(x, v) $$

for all $f \in C(M, \mathbb{R})$, then $\pi^*S^*\tilde{\mu} = \mu \in \mathcal{M}$. So $\tilde{\mathcal{M}} = \mathcal{M}$.

(v) Due to the definition of the Peierls barrier function, we calculate

$$ \hat{h}^\infty(y, x) = \lim_{t \to +\infty} \left( \inf_{\xi \in C^{\infty}(\{0, t\}, M)} \int_0^t \tilde{L}(\xi(s), \dot{\xi}(s))ds + c(\tilde{H})t \right) $$

$$ = \lim_{t \to +\infty} \left( \inf_{\gamma \in C^{\infty}(\{0, t\}, M)} \int_0^t \left( \gamma(t - s) - \frac{d\gamma(t - s)}{ds} \right)ds + c(\tilde{H})t \right) $$

3) See Definition 4.5
Due to Appendix 2 of [5],\footnote{Su X. F. and Zhang J. L.} −λ forward (Forward Remark 2.3) Such a curve \( \gamma \) is called a forward calibrated curve of \( \lambda, x \) \footnote{is the viscosity solution of the following symmetrical H-J equation \( \lambda u + \hat{H}(x, \partial_x u) = c(H), \quad \lambda > 0. \) (2.2)}

\[
\gamma_{\lambda, x}^{-} := \inf_{\gamma \in \gamma_{\lambda, x}^{-}} \int_0^t e^{-\lambda t}(L(\gamma(t), \dot{\gamma}(t)) + c(H))dt,
\]

(2.1) For any viscosity solution \( u_{\lambda}^+ \) of \( (\text{HJ}_\lambda) \) if it satisfies Items (3) and (4) of Proposition 2.2, such a forward \( \lambda \)-weak KAM solution is unique. 

\textit{Remark 2.3} (Forward \( \lambda \)-weak KAM solution) A continuous function \( w : M \to \mathbb{R} \) is called a forward \( \lambda \)-weak KAM solution of \( (\text{HJ}_\lambda) \) if it satisfies Property (5) of Proposition 2.2, such a forward \( \lambda \)-weak KAM solution is unique.

\textit{Proof of Proposition 2.2} (5) By a simple transformation, we can see \( -u_{\lambda}^+ \) is a viscosity solution of (2.2), which is unique due to the Comparison Principle.

(1) For any viscosity solution \( u_0(x) \) of \( (\text{HJ}_0) \), we get

\[
\underline{u}_0(x) := u_0(x) - \|u_0\| \leq u_0(x) + \|u_0\| := \bar{u}_0(x), \quad \forall x \in M.
\]

Consequently, \( \underline{u}_0 \) (resp., \( \bar{u}_0 \)) is a subsolution (resp. supersolution) of (2.2). Due to the Comparison Principle again, for any \( \lambda > 0 \) and any viscosity solution \( \omega_{\lambda} \) of (2.2) satisfies \( \underline{u}_0 \leq \omega_{\lambda} \leq \bar{u}_0 \). So we get the equi-boundedness of \( \{u_{\lambda}^+\}_{\lambda \in (0, 1]} \).

Let \( \gamma : [0, d(x, y)] \to M \) be the geodesic joining \( y \) to \( x \) parameterized by the arc-length, where \( d : M \times M \to \mathbb{R} \) is the Euclidean distance. For every absolute continuous curve \( \xi : [0, +\infty) \to M \)
with \( \xi(0) = x \), we define a curve

\[
\eta(t) = \begin{cases} 
\gamma(t), & t \in [0, d(x, y)], \\
\xi(t - d(x, y)), & t \in [d(x, y), +\infty).
\end{cases}
\] (2.3)

Then we have

\[
-u^+(\lambda)(y) \leq \int_0^{+\infty} e^{-\lambda t}(L(\eta(t), \dot{\eta}(t)) + c(H))dt
\]

\[
\leq \int_0^{d(x,y)} e^{-\lambda t}(L(\gamma(t), \dot{\gamma}(t)) + c(H))dt
+ \int_{d(x,y)}^{+\infty} e^{-\lambda t}(L(\xi(t - d(x, y))), \dot{\xi}(t - d(x, y))) + c(H))dt
\]

\[
\leq \int_0^{d(x,y)} e^{-\lambda t}(L(\gamma(t), \dot{\gamma}(t)) + c(H))dt
+ e^{\lambda d(x,y)} \int_0^{+\infty} e^{-\lambda t}(L(\xi(t)), \dot{\xi}(t)) + c(H))dt.
\]

By minimizing with respect to all \( \xi \in C^{ac}([0, +\infty)) \) with \( \xi(0) = x \), we obtain

\[
-u^+(\lambda)(y) \leq -e^{\lambda d(x,y)} u^+_\lambda(x) + \int_0^{d(x,y)} e^{-\lambda t}(L(\gamma(t), \dot{\gamma}(t)) + c(H))dt.
\]

Therefore, we have

\[
u^+_\lambda(x) - u^+_\lambda(y) \leq (1 - e^{\lambda d(x,y)}) u^+_\lambda(x) + \int_0^{d(x,y)} e^{-\lambda t}(L(\gamma(t), \dot{\gamma}(t)) + c(H))dt
\]

\[
\leq \frac{1 - e^{\lambda d(x,y)}}{\lambda} (|\lambda u^+_\lambda(x)| + C_1) \leq (C + C_1) d(x, y),
\]

where

\[
C := \max \left\{ \left| \max_{x \in M} L(x, 0) + c(H) \right|, \left| \min_{(x, v) \in TM} L(x, v) + c(H) \right| \right\}
\]

and

\[
C_1 := \max \{ L(x, v) : x \in M, \|v\| \leq 1 \}.
\]

By exchanging the role of \( x \) and \( y \), we get the other inequality, which shows that \( u^+_\lambda \) is uniformly Lipschitz and the Lipschitz constant is independent of \( \lambda \).

(2), (3) & (4) By a similar analysis as Propositions 6.2–6.3 in [5], all these three items can be easily proved. \( \square \)

### 3 Discounted Limit of Forward \( \lambda \)-weak KAM Solutions

Recall that \( \widehat{u}^-_\lambda := -u^+_\lambda \) is the unique viscosity solution of (2.2),

\[
\widehat{u}^-_\lambda(x) = \inf_{\gamma(0) = x} \int_{-\infty}^0 e^{\lambda t}(\widehat{L}(\gamma, \dot{\gamma}) + c(H))dt.
\]

So the following conclusion holds instantly:

**Lemma 3.1** ([5]) As \( \lambda \to 0_+ \), \( \widehat{u}^-_\lambda \) converges to a unique function \( \widehat{u}^-_0 \) which is a viscosity solution of the following

\[
\widehat{H}(x, \partial_x u) = c(H)
\] (3.1)
with the following two different interpretations:

- \( \hat{u}_0^- \) is the maximal subsolution \( w : M \to \mathbb{R} \) of (3.1) such that for any projected Mather measure \( \hat{\mu} \in \hat{\mathcal{M}} \), \( \int_M w \cdot d\hat{\mu} \leq 0 \).

- \( \hat{u}_0^- \) is the infimum of functions \( \hat{h}_{\mu}^\infty \) defined by
  \[
  \hat{h}_{\mu}^\infty(x) := \int_M \hat{h}^\infty(y,x)d\hat{\mu}(y), \quad \hat{\mu} \in \hat{\mathcal{M}}.
  \]

Due to Lemmas 2.1 and 3.1, we get

\[
\lim_{\lambda \to 0_+} u_\lambda^+ = - \lim_{\lambda \to 0_+} \hat{u}_\lambda^- = - \hat{u}_0^-,
\]

which is uniquely established and interpreted as the following:

**Lemma 3.2** \( u_0^+ \) is a forward 0-weak KAM solution of (HJ\(_0\)).

**Proof** As \( \hat{u}_0^- \) is the viscosity solution of (3.1), then \( \hat{u}_0^- < \hat{L} + c(H) \) due to Proposition 5.3 of [5]. On the other side, due to

\[
U(x,t) := \inf_{\gamma \in C^\infty([0,t],M)} \left\{ \hat{u}_0^- (\gamma(-t)) + \int_{-t}^0 (\hat{L}(\gamma(\tau),\dot{\gamma}(\tau)) + c(H))d\tau \right\}, \quad \forall t \geq 0
\]

is the unique viscosity solution of the Cauchy problem

\[
\begin{cases}
\partial_t u + \hat{H}(x,d_xu) = c(H) \\
u(x,0) = \hat{u}_0^-(x), & t \geq 0,
\end{cases}
\]

whereas \( \hat{u}_0^- (x) \) is also a viscosity solution of the Cauchy problem. So it follows that \( U(x,t) = \hat{u}_0^-(x) \) for all \( x \in M, t > 0 \). Hence, by the same analysis as in the proof of Proposition 6.2 of [5], for any \( x \in M \), there exists a curve \( \gamma_x^- : (-\infty,0] \to M \) absolutely continuous and ending with \( x \), such that

\[
\hat{u}_0^- (\gamma_x^-(t)) - \hat{u}_0^- (\gamma_x^-(s)) = \int_s^t (\hat{L}(\gamma_x^-(\tau),\dot{\gamma}_x^-(\tau)) + c(H))d\tau
\]

for all \( s \leq t \leq 0 \). After all, \( \hat{u}_0^- \) has to be a backward 0-weak KAM solution of (3.1). Consequently, \( u_0^+ = - \hat{u}_0^- \) has to be a forward 0-weak KAM solution of (HJ\(_0\)).

**Proof of Theorem 1.1** It’s a direct corollary from Lemmas 2.1, 3.1 and 3.2.

### 4 Appendix: Aubry–Mather Theory of Nonsmooth Convex Hamiltonians

As is known, the continuous, superlinear, convex \( H(x,p) \) has a dual Lagrangian

\[
L(x,v) := \max_{p \in \Gamma_x^M} \{ \langle p, v \rangle - H(x,p) \}, \quad (x,v) \in TM
\]

which is also continuous, superlinear and convex in \( v \). Consequently, for any \( x,y \in M \) and \( t > 0 \), the action function

\[
h^t(x,y) := \inf_{\gamma \in C^\infty([0,t],M)} \int_0^t (L(\gamma,\dot{\gamma}) + c(H))d\tau \tag{4.1}
\]

always attains its infimum at an absolutely continuous minimizing curve \( \gamma_{\min} : [0,t] \to M \) due to the *Tonelli Theorem*. In [11], the *Peierls barrier* function

\[
h^\infty(x,y) := \lim_{t \to +\infty} h^t(x,y) \tag{4.2}
\]
is proved to be well-defined and continuous on $M \times M$.

**Definition 4.1 ([11])** The projected Aubry set is defined by
\[ A := \{ x \in M : h^\infty(x, x) = 0 \} \]

Consider $TM$ (resp., $M$) as a measurable space and $\mathbb{P}(TM)$ (resp. $\mathbb{P}(M)$) by the set of all Borel probability measures on it. A measure on $TM$ is denoted by $\tilde{\mu}$, and we remove the tilde if we project it to $M$. We say that a sequence $\{\tilde{\mu}_n\}$ of probability measures weakly converges to a probability measure $\tilde{\mu}$ if
\[
\lim_{n \to +\infty} \int_{TM} f(x,v) d\tilde{\mu}_n(x,v) = \int_{TM} f(x,v) d\tilde{\mu}(x,v)
\]
for any $f \in C_c(TM, \mathbb{R})$. Accordingly, the deduced probability measure $\mu_n$ weakly converges to $\mu$, i.e.
\[
\lim_{n \to +\infty} \int_{M} f(x) d\mu_n(x) := \lim_{n \to +\infty} \int_{TM} f(\pi(x,v)) d\tilde{\mu}_n(x,v)
= \int_{TM} f(\pi(x,v)) d\tilde{\mu}(x,v)
= \int_{M} f(x) d\pi^* \tilde{\mu}(x) =: \int_{M} f(x) d\mu(x)
\]
for any $f \in C(M, \mathbb{R})$.

**Definition 4.2** A probability measure $\tilde{\mu}$ on $TM$ is closed if it satisfies:
- $\int_{TM} |v| d\tilde{\mu}(x,v) < +\infty$;
- $\int_{TM} \langle \nabla \phi(x), v \rangle d\tilde{\mu}(x,v) = 0$ for every $\phi \in C^1(M, \mathbb{R})$.

Let’s denote by $\mathbb{P}_c(TM)$ the set of all closed measures on $TM$. Then the following conclusion is proved in [5]:

**Theorem 4.3** $\min_{\tilde{\mu} \in \mathbb{P}_c(TM)} \int_{TM} L(x,v) d\tilde{\mu}(x,v) = -c(H)$. Moreover, the minimizer is called a Mather measure and we denote by $\mathcal{M}$ the set of them. Similarly, we can project $\mathcal{M}$ to $M \subset \mathbb{P}(M)$ w.r.t. $\pi : TM \to M$, which contains all the projected Mather measures.

**Definition 4.4 ([10])** The Mather set is defined by
\[ \tilde{\mathcal{M}} := \bigcup_{\tilde{\mu} \in \mathcal{M}} \text{supp} (\tilde{\mu}) \subset TM \]
and the projected Mather set $\mathcal{M} := \pi(\tilde{\mathcal{M}})$ is accordingly defined.

**Definition 4.5** (subsolution) A function $u : M \to \mathbb{R}$ is called a viscosity subsolution, or subsolution for short of
\[ H(x, du) = c, \quad x \in M \]
(denoted by $u \prec L + c$), if $u(y) - u(x) \leq h^\epsilon(x,y) + (c - c(H))t$ for all $(x, y) \in M \times M$ and $t \geq 0$.

**Definition 4.6** A function $u : M \to \mathbb{R}$ is called a backward (resp., forward) 0-weak KAM solution of $(HJ_0)$ if it satisfies:
• $u < L + c(H)$, i.e., for any two points $(x, y) \in M \times M$ and any absolutely continuous curve $\gamma : [a, b] \to M$ connecting them, we have

$$u(y) - u(x) \leq \int_{a}^{b} (L(\gamma, \dot{\gamma}) + c(H))dt;$$

• for any $x \in M$ there exists a curve $\gamma_{x}^{-} : (-\infty, 0] \to M$ (resp., $\gamma_{x}^{+} : [0, +\infty) \to M$) ending with (resp., starting from) $x$, such that for any $s < t \leq 0$ (resp., $0 \leq s < t$),

$$u(\gamma_{x}^{-}(t)) - u(\gamma_{x}^{-}(s)) = \int_{s}^{t} (L(\gamma_{x}^{-}, \dot{\gamma}_{x}^{-}) + c(H))dt$$

(resp., $u(\gamma_{x}^{+}(t)) - u(\gamma_{x}^{+}(s)) = \int_{s}^{t} (L(\gamma_{x}^{+}, \dot{\gamma}_{x}^{+}) + c(H))dt$).

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