Cramér-type Moderate Deviation for Quadratic Forms with a Fast Rate

Xiao Fang*, Song-Hao Liu†, Qi-Man Shao*†

The Chinese University of Hong Kong*, Southern University of Science and Technology†

Abstract: Let $X_1, \ldots, X_n$ be independent and identically distributed random vectors in $\mathbb{R}^d$. Suppose $\mathbb{E}X_1 = 0$, $\text{Cov}(X_1) = I_d$, where $I_d$ is the $d \times d$ identity matrix. Suppose further that there exist positive constants $t_0$ and $c_0$ such that $\mathbb{E} e^{t_0 |X_1|} \leq c_0 < \infty$, where $| \cdot |$ denotes the Euclidean norm. Let $W = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ and let $Z$ be a $d$-dimensional standard normal random vector. Let $Q$ be a $d \times d$ symmetric positive definite matrix whose largest eigenvalue is 1. We prove that for $0 \leq x \leq \varepsilon n^{1/6}$,

\[
\left| \frac{\text{P}(|Q^{1/2}W| > x)}{\text{P}(|Q^{1/2}Z| > x)} - 1 \right| \leq C \left( \frac{1 + x^5}{\text{det}(Q^{1/2})n} + \frac{x^6}{n} \right) \text{ for } d \geq 5
\]

and

\[
\left| \frac{\text{P}(|Q^{1/2}W| > x)}{\text{P}(|Q^{1/2}Z| > x)} - 1 \right| \leq C \left( \frac{1 + x^3}{\text{det}(Q^{1/2})n^{d/2}} + \frac{x^6}{n} \right) \text{ for } 1 \leq d \leq 4,
\]

where $\varepsilon$ and $C$ are positive constants depending only on $d, t_0$, and $c_0$. This is a first extension of Cramér-type moderate deviation to the multivariate setting with a faster convergence rate than $1/\sqrt{n}$. The range of $x = o(n^{1/6})$ for the relative error to vanish and the dimension requirement $d \geq 5$ for the $1/n$ rate are both optimal. We prove our result using a new change of measure, a two-term Edgeworth expansion for the changed measure, and cancellation by symmetry for terms of the order $1/\sqrt{n}$.

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1 Introduction and Main Result

Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) real-valued random variables with $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = 1$, $\mathbb{E}|X_1|^3 < \infty$. Let $W = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$. The well-known Berry–Esseen bound (Berry (1941), Esseen (1942)) states that

\[
\sup_{x \in \mathbb{R}} |\text{P}(W > x) - (1 - \Phi(x))| \leq \frac{\mathbb{E}|X_1|^3}{\sqrt{n}}, \quad (1.1)
\]
where \( \Phi(\cdot) \) is the standard normal distribution function. The rate \( 1/\sqrt{n} \) is optimal given that the distribution function of \( W \) can have jumps of size \( 1/\sqrt{n} \), e.g., when \( X_1 = \pm 1 \) with probability \( 1/2 \), while \( \Phi(\cdot) \) is continuous.

Esseen (1945) first discovered an improved convergence rate in the multivariate normal approximation of sums of i.i.d. random vectors on centered Euclidean balls. Let \( X_1, \ldots, X_n \) be i.i.d. random vectors in \( \mathbb{R}^d, d \geq 2 \), with \( \mathbb{E} X_1 = 0 \), \( \text{Cov}(X_1) = I_d \), \( \mathbb{E}|X_1|^4 < \infty \), where \( I_d \) denotes the \( d \times d \) identity matrix and \( |\cdot| \) denotes the Euclidean norm. Let \( W = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \) and \( Z \sim N(0, I_d) \). Then, we have (see Esseen (1945, Chapter VII, Theorem 1))

\[
\sup_{x > 0} |\mathbb{P}(|W| > x) - \mathbb{P}(|Z| > x)| \leq \frac{C_d}{\sqrt{n}} (\mathbb{E}|X_1|^4)^{3/2},
\]

where \( C_d \) is a constant depending only on \( d \). For \( d \geq 5 \), Götzte and Zaitsev (2014, Corollary 2.3) later proved that

\[
\sup_{x > 0} |\mathbb{P}(|Q^{1/2}W| > x) - \mathbb{P}(|Q^{1/2}Z| > x)| \leq \frac{C_d}{\det(Q^{1/2})} \mathbb{E}|X_1|^4,
\]

where \( Q \) is a \( d \times d \) symmetric positive definite matrix whose largest eigenvalue is 1, and \( C_d \) is a constant depending only on \( d \). Thus, in particular, under a finite fourth moment condition, the rate of convergence for the chi-square \( \chi_d^2 \) approximation of the squared Euclidean norm of a sum of i.i.d. random vectors \( |W|^2 \) can be improved to \( 1/n^{d/4} \) for \( 2 \leq d \leq 4 \) and to \( 1/n \) for \( d \geq 5 \). In (1.3), both the threshold of the dimension, namely, 5, and the \( 1/n \) rate are optimal (Bentkus and Götzte (1997)).

By assuming in addition that the moment generating function of \( X_1 \) exists in a neighborhood of 0, Cramér (1938) and von Bahr (1967) obtained relative error bounds for the approximation in (1.1) and (1.2), respectively. In particular, from von Bahr (1967, Theorem 3), along with an expansion and symmetry argument (see Appendix A), we have, for \( 0 \leq x \leq \varepsilon n^{1/6} \),

\[
\frac{\mathbb{P}(|W| > x)}{\mathbb{P}(|Z| > x)} - 1 \leq C \left( \frac{1 + x}{\sqrt{n}} + \frac{x^6}{n} \right),
\]

where \( \varepsilon \) and \( C \) are unspecified positive constants, which do not depend on \( n \) and \( x \). We refer to results such as (1.4) as Cramér-type moderate deviations.

The range of \( x \) for the relative error in (1.4) to vanish, namely, \( x = o(n^{1/6}) \), is optimal. More precisely, let \( \{X_1, X_2, \ldots\} \) be a sequence of i.i.d. random vectors in \( \mathbb{R}^d \) with zero mean, identity covariance matrix, and \( \mathbb{E}e^{t_0|X_1|} \leq c_0 < \infty \) for some positive constants \( t_0 \) and \( c_0 \). Let \( W_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i, n \geq 1 \). If the mixed third cumulants of \( X_1 \) are not all zero, then, again from von Bahr (1967), we have (see Appendix A), for any fixed positive constant \( c \),

\[
\frac{\mathbb{P}(|W_n| > cn^{1/6})}{\mathbb{P}(|Z| > cn^{1/6})} \to 1, \quad \text{as } n \to \infty.
\]

By comparing (1.2) and (1.4), we observe the following gap: Taking, say, \( x = 1 \), in (1.4), we obtain

\[
|\mathbb{P}(|W| > 1) - \mathbb{P}(|Z| > 1)| \leq \frac{C}{\sqrt{n}},
\]
which does not recover (1.2) for \( d \geq 2 \). Therefore, there is a gap in the rate of convergence between the Berry–Esseen bound (1.2) or (1.3) and the Cramér-type moderate deviation (1.4). This paper aims to establish a refined Cramér-type moderate deviation theorem with a rate of convergence matching that of the Berry–Esseen bound.

The following theorem is our main result.

**Theorem 1.1.** Let \( X_1, \ldots, X_n \) be i.i.d. random vectors in \( \mathbb{R}^d \), where \( d \geq 1 \), and let \( Q \) be a symmetric positive definite matrix whose largest eigenvalue is 1. Suppose \( \mathbb{E}X_1 = 0 \), \( \text{Cov}(X_1) = I_d \), and \( \mathbb{E}e^{t_0|X_1|} \leq c_0 < \infty \) for some positive constants \( t_0 \) and \( c_0 \). Let \( W = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \) and \( Z \sim N(0, I_d) \). Then, for \( 0 \leq x \leq \varepsilon n^{1/6} \), we have

\[
\left| \frac{\mathbb{P}(\{|Q^{1/2}W| > x\})}{\mathbb{P}(\{|Q^{1/2}Z| > x\}) - 1} \right| \leq C \left( \frac{1 + x^3}{\det(Q^{1/2})n} \right)^{1_{\{d \leq 4\}}} + \frac{1 + x^5}{\det(Q^{1/2})n} 1_{\{d \geq 5\}} + \frac{x^6}{n},
\]

(1.6)

where \( \varepsilon \) and \( C \) are positive constants depending only on \( d, t_0 \), and \( c_0 \).

**Remark 1.1.** Theorem 1.1 provides the first extension of Cramér-type moderate deviation to the multivariate setting with a faster convergence rate than \( 1/\sqrt{n} \). The convergence rates in (1.6) match those in (1.2) and (1.3). In particular, the \( 1/n \) rate and the dimension requirement \( d \geq 5 \) for such a rate are optimal. To prove Theorem 1.1, we use a new change of measure, which may be of independent interest.

**Remark 1.2.** We assume \( \text{Cov}(X_1) = I_d \) and \( \|Q\|_{\text{op}} = 1 \), where \( \|\cdot\|_{\text{op}} \) denotes the operator norm, in Theorem 1.1 without loss of generality. Suppose \( \overline{W} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \overline{X}_i \), where \( \{\overline{X}_i\}_{i=1}^n \) are i.i.d., \( \mathbb{E}\overline{X}_1 = 0 \), \( \text{Cov}(\overline{X}_1) = \overline{\Sigma} \) (positive definite), \( \overline{Q} \) is an arbitrary symmetric positive definite matrix, and \( \overline{\pi} \geq 0 \). Then, \( \mathbb{P}(\{|\overline{Q}^{1/2}\overline{W}| > \overline{\pi}\}) \) reduces to the setting in Theorem 1.1 with

\[
Q = \frac{\overline{\Sigma}^{1/2} \cdot \overline{Q} \cdot \overline{\Sigma}^{1/2}}{\|\overline{\Sigma}^{1/2} \cdot \overline{Q} \cdot \overline{\Sigma}^{1/2}\|_{\text{op}}}, \quad x = \frac{\overline{\pi}}{\|\overline{\Sigma}^{1/2} \cdot \overline{Q} \cdot \overline{\Sigma}^{1/2}\|_{\text{op}}^{1/2}}.
\]

However, the condition becomes \( \mathbb{E}e^{t_0|\overline{\Sigma}^{1/2}\overline{X}_1|} \leq c_0 < \infty \), as in the Lyapunov-type bounds in the literature of multivariate normal approximations; see Bentkus (2005) and Götze and Zaitsev (2014).

**Remark 1.3.** The factor \( \frac{1}{\det(Q^{1/2})} \) in the bound (1.6) also appeared in Götze and Zaitsev (2014) (cf. (1.3)). Such a factor prevents the degenerate case: if the problem is essentially lower dimensional, then the \( 1/n \) rate may not be valid.

This paper is organized as follows: In Section 2, we present the details of our new change of measure and postpone the proofs of lemmas to Section 4. The proof of Theorem 1.1 is given in Section 3. We provide a complete proof of (1.4) and (1.5) in Appendix A.

In Sections 2–4, we use \( \varepsilon \) and \( C \) to denote positive constants depending only on \( d, t_0 \) and \( c_0 \). They may differ in different expressions. We use \( O(\cdot) \) to denote a quantity (which can be random) that is bounded in absolute value by the quantity in the parentheses multiplied by a constant depending only on \( d, t_0 \), and \( c_0 \).
Recall our setting: Let $X_1, \ldots, X_n$ be i.i.d. random vectors in $\mathbb{R}^d$, where $d \geq 1$. Suppose $\mathbb{E}X_1 = 0$, $\text{Cov}(X_1) = I_d$, and $\mathbb{E}e^{t_0|X_i|} \leq c_0 < \infty$ for some positive constants $t_0$ and $c_0$. Let $W = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ and $Z \sim N(0, I_d)$. Without loss of generality, we assume $Q$ is a diagonal matrix with diagonal entries $1 = q_1 \geq q_2 \geq \ldots \geq q_d > 0$. Let $D = Q^{1/2}$. In fact, for any symmetric positive definite matrix $Q$, there exists an orthogonal matrix $P$ such that $Q = P^T \Lambda P$, where $\Lambda = \text{diag}(q_1, q_2, \ldots, q_d)$. We then have

$$\mathbb{P}\left(\left|\sum_{i=1}^n X_i / \sqrt{n}\right| > x\right) = \mathbb{P}\left(\left|\sum_{i=1}^n Y_i / \sqrt{n}\right| > x\right),$$

where $Y_i = PX_i$, $\mathbb{E}Y_i = 0$, $\text{Cov}(Y_i) = I_d$, and $|Y_i| = |X_i|$ and the problem reduces to the special case. Moreover, we assume that $x > 1$ without loss of generality. Otherwise, Theorem 1.1 follows from (1.3) for the case $d \geq 5$ and from Lemma 3.2 (with $\Sigma_1 = Q$ and $b = 0$) for the case $1 \leq d \leq 4$. We also assume that $x \leq \varepsilon n^{1/6}$ for a sufficiently small $\varepsilon > 0$ as in the condition of Theorem 1.1.

**Proof Strategy.** Roughly speaking, von Bahr (1967) proved (1.4) by first using a local exponential change of measure for different subsets $S_b$ of $\mathbb{R}^d$ nearby $b \in \mathbb{R}^d$, then using a normal approximation for the changed measure on each subset $S_b$, and finally combining the approximation results of all of the subsets. The $1/\sqrt{n}$ rate comes from the normal approximation step for each $S_b$.

In contrast, we use a new global change of measure. This is motivated by Aleskevičienė and Statulevičius (1997). They considered, for the case $Q = I_d$ and each $x > 1$, an exponentially tilted $\tilde{W}_A$ such that

$$\mathbb{P}(\tilde{W}_A = dy) = \frac{e^{h|y|^2}}{\mathbb{E}e^{h|W|^2}} \mathbb{P}(W = dy), \quad y \in \mathbb{R}^d,$$

where

$$h = h(x) = 1/2 - 1/2x^2 > 0.$$

We note that if $W$ is replaced by the standard normal $Z \sim N(0, I_d)$, then $\mathbb{E}e^{h|Z|^2} = x^d$ and the exponentially tilted $\tilde{Z}_A$ follows $N(0, x^2 I_d)$. Therefore, $\{\tilde{Z}_A > x\}$ becomes a typical event. Because $W$ is close to normal, we may hope that $\tilde{W}_A$ is close to $N(0, x^2 I_d)$ and use this approximation to obtain the desired relative error bound as in the classical change of measure argument. However, under the condition of Theorem 1.1, $\mathbb{E}e^{h|W|^2}$ may be $\infty$. In fact, even if $\mathbb{E}e^{h|W|^2}$ is finite, it is typically too large for $\tilde{W}_A$ to be close to $N(0, x^2 I_d)$.

Observing that $e^{h|y|^2} = e^{\sqrt{2\pi}Z_{y^2}}$ for the case $Q = I_d$, we modify (2.1) by considering, for the case of general diagonal matrix $Q$ and $D = Q^{1/2}$,

$$\mathbb{P}(\tilde{W} = dy) = \frac{e^{\sqrt{2\pi}BZ_{y^2}}}{\mathbb{E}e^{\sqrt{2\pi}BZ_{y^2}}} \mathbb{P}(W = dy), \quad y \in \mathbb{R}^d,$$

where $\langle A, B \rangle$ denotes the inner product, $Z_x$ is an independent standard normal random vector restricted to the centered ball with radius $z_0$, that is,

$$\mathbb{P}(Z_x = dz) = e^{-|z|^2/2}dz, \quad z \in \mathbb{R}^d,$$
\( \kappa \) is the normalizing constant and \( z_0 = z_0(x) = 3x \) (which will be used in (2.51)). Because of the assumption of finite moment generating function, \( \mathbb{E} e^{(\sqrt{2h}Dz_x, W)} \) is finite for \( 1 < x \leq \varepsilon n^{1/6} \) for a sufficiently small \( \varepsilon \).

The rest of the proof is provided in three steps. First, we write \( \mathbb{P}(|D\tilde{W}| > x) \) as a weighted sum of probabilities involving quadratic forms (cf. (2.6)). Second, we approximate each probability using a two-term Edgeworth expansion (cf. (2.10)). We quantify the error in such an approximation using a result of Götze and Zaitsev (2014) for the case \( d \geq 5 \) (cf. Lemma 3.1) and a modification of a result of Esseen (1945) for the case \( 1 \leq d \leq 4 \) (cf. Lemma 3.2). Finally, we show that the terms of the order \( 1/\sqrt{n} \) in the Edgeworth expansion disappear using a symmetry argument (cf. (2.42)).

Now we begin with the formal proof. Assume without loss of generality that \( \{X_i\}_{i=1}^n \), \( Z \) and \( Z_x \) defined above are jointly independent. From (2.2), the characteristic function of \( \tilde{W} \) can be expressed as

\[
\mathbb{E} e^{(it, \tilde{W})} = \frac{\mathbb{E} e^{(\sqrt{2h}DZ_x, it, W)}}{\mathbb{E} e^{(\sqrt{2h}DZ_x, W)}}. \tag{2.4}
\]

When the expectation is with respect to both \( Z_x \) and \( W \), we compute it by first conditioning on \( Z_x \). Let \( \hat{G}(b) = \mathbb{E} e^{(b, X_1)} \) for a complex vector \( b \in \mathbb{C}^d \). We write the characteristic function of \( D\tilde{W} \) (cf. (2.4)) as

\[
\mathbb{E} e^{(it, D\tilde{W})} = \frac{1}{\mathbb{E} \hat{G}^n(\sqrt{2h}DZ_x/\sqrt{n})} \mathbb{E} \left[ \hat{G}^n(\sqrt{2h}DZ_x/\sqrt{n}) \frac{G^n((\sqrt{2h}DZ_x + iDt)/\sqrt{n})}{G^n(\sqrt{2h}DZ_x/\sqrt{n})} \right].
\]

This implies that \( D\tilde{W} \) is a mixture (depending on the value of \( Z_x \)) of sums of i.i.d. random vectors \( \frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{X}_i \), where each \( \tilde{X}_i \) has the characteristic function \( \frac{\hat{G}((\sqrt{2h}DZ_x/\sqrt{n}) + iDt)}{\hat{G}(\sqrt{2h}DZ_x/\sqrt{n})} \), that is,

\[
\mathbb{P}^{Z_x}(\tilde{X}_1 = dy) = \frac{e^{(\sqrt{2h}Z_x, y)/\sqrt{n}}}{\hat{G}(\sqrt{2h}DZ_x/\sqrt{n})} \mathbb{P}(DX_1 = dy). \tag{2.5}
\]

Hereafter, we use \( \mathbb{P}^{Z_x} \) and \( \mathbb{E}^{Z_x} \) to denote the conditional probability and expectation, respectively, given \( Z_x \). Therefore,

\[
\mathbb{P}(|D\tilde{W}| \leq a) = \frac{1}{\mathbb{E} \hat{G}^n(\sqrt{2h}DZ_x/\sqrt{n})} \mathbb{E} \left[ \hat{G}^n(\sqrt{2h}DZ_x/\sqrt{n}) \mathbb{P}^{Z_x} \left( \left| \frac{\sum_{i=1}^n \tilde{X}_i}{\sqrt{n}} \right| \leq a \right) \right]. \tag{2.6}
\]

We will use a two-term Edgeworth expansion to approximate \( \mathbb{P}^{Z_x} \left( \left| \frac{\sum_{i=1}^n \tilde{X}_i}{\sqrt{n}} \right| \leq a \right) \). To express the two-term Edgeworth expansion, let \( \tilde{\mu}_1 = \tilde{\mu}_i = \mathbb{E}^{Z_x} \tilde{X}_i \) and rewrite

\[
\mathbb{P}^{Z_x} \left( \left| \frac{\sum_{i=1}^n \tilde{X}_i}{\sqrt{n}} \right| \leq a \right) = \mathbb{P}^{Z_x} \left( \left| \frac{\sum_{i=1}^n (X_i - \tilde{\mu}_i)}{\sqrt{n}} \right| \in B(-\sqrt{n}\tilde{\mu}_1, a) \right),
\]

where \( B(b, a) \) denotes the Euclidean ball with center \( b \) and radius \( a \). Denote by \( \tilde{\Sigma} \) the conditional covariance matrix of \( \tilde{X}_1 \) given \( Z_x \). It will be shown in Lemma 2.1 that \( \tilde{\Sigma} \) is
positive definite when \( 1 < x \leq \varepsilon n^{1/6} \) for a sufficiently small \( \varepsilon > 0 \). Denote by \( \phi \) the \( d \)-dimensional standard normal density function,

\[
\tilde{p}(y) = \phi(\Sigma^{-1/2}y)/\sqrt{\det \Sigma} \quad \text{ (density of } N(0, \Sigma))
\]  

(2.7)

and

\[
\tilde{p}'''(y)u^3 = \tilde{p}(y) \left( 3\langle \Sigma^{-1}u, \Sigma^{-1}y, u \rangle - \langle \Sigma^{-1}y, u \rangle^3 \right),
\]

(2.8)

which is the third Frechet derivative of \( p \) in direction \( u \). Let

\[
\tilde{\omega}(y) = \tilde{p}(y) + \frac{1}{6\sqrt{n}} \mathbb{E}\tilde{p}'''(y)(\bar{X}_1 - \bar{\mu}_1)^3.
\]

(2.9)

It can be seen from Lemma 2.2 below that, when \( 1 < x \leq \varepsilon n^{1/6} \) for a sufficiently small \( \varepsilon > 0 \), \( \tilde{\omega}(y) \) is absolutely integrable over \( \mathbb{R}^d \). The two-term Edgeworth expansion for

\[
\mathbb{P}Z_x \left( \left| \frac{\sum_{i=1}^n \bar{X}_i}{\sqrt{n}} \right| \leq a \right)
\]

is given by (cf. Bhattacharya and Rao 1986)

\[
\int_{B(-\sqrt{n}\bar{\mu}_1, a)} \tilde{\omega}(y)dy = \int_{B(0, a)} \tilde{\omega}(y - \sqrt{n}\bar{\mu}_1)dy.
\]

(2.10)

According to (2.6)–(2.10), we define \( \tilde{\Phi}(\cdot) \) to be a signed measure as

\[
d\tilde{\Phi}(y) = \mathbb{E}\left[ \tilde{G}^n(\sqrt{2\varepsilon}DZ_x/\sqrt{n}) \left( \tilde{p}(y - \sqrt{n}\bar{\mu}_1) + \frac{1}{6\sqrt{n}} \mathbb{E}Z_x \{ \tilde{p}'''(y - \sqrt{n}\bar{\mu}_1)(\bar{X}_1 - \bar{\mu}_1)^3 \} \right) \right] dy
\]

(2.11)

and we use \( \tilde{\Phi}(B(0, a)) \) to approximate \( \mathbb{P}(|D\tilde{W}| \leq a) \).

The main result in this section is as follows:

**Proposition 2.1.** **Under the conditions of Theorem 1.1, let \( \tilde{W} \) be as in (2.2) and \( \tilde{\Phi} \) be as in (2.11).** **There exists a positive constant \( \varepsilon \) such that for \( 1 < x \leq \varepsilon n^{1/6} \),

\[
\frac{\mathbb{P}(|D\tilde{W}| > x)}{\mathbb{P}(|DZ| > x)} - 1 = O \left( \frac{x^6}{n} \right) + O(x^2) \sup_{a \geq 0} |\mathbb{P}(|D\tilde{W}| \leq a) - \tilde{\Phi}(B(0, a))|.
\]

(2.12)

To prove Proposition 2.1, we need a few lemmas. These lemmas are proved in Section 4. The first lemma estimates \( \bar{\mu}_1 \) and \( \bar{\Sigma} \) defined above, which depend on the value of \( Z_x \).

**Lemma 2.1.** **There exists a positive constant \( \varepsilon \) such that for \( 1 < x \leq \varepsilon n^{1/6} \), we have, given any \( Z_x \),

\[
\bar{\mu}_1 = \frac{\sqrt{2\varepsilon}QZ_x}{\sqrt{n}} + \frac{1}{2n} \mathbb{E}Z_x \{ (\sqrt{2\varepsilon}DZ_x, X_1)^2 X_1 \} + DV,
\]

(2.13)

and each component of the \( d \)-vector \( V \) is \( O \left( \frac{x^3}{n^{1/6}} \right) \).

\[
\bar{\Sigma} = D \left( I_d + \frac{1}{\sqrt{n}} \mathbb{E}Z_x \{ (\sqrt{2\varepsilon}DZ_x, X_1)X_1X_1^T \} + R \right) D,
\]

(2.14)
Lemma 2.2. There exists a positive constant \( \varepsilon \) such that for \( 1 < x \leq \varepsilon n^{1/6} \), we have, given any \( Z_x \),

\[
\tilde{p}(y - \sqrt{n} \mu_1) + \frac{1}{6 \sqrt{n}} \mathbb{E}^Z \left\{ \tilde{p}'''(y - \sqrt{n} \mu_1)(\tilde{X}_1 - \tilde{\mu}_1)^3 \right\} = H_1(y) + H_2(y), \quad y \in \mathbb{R}^d,
\]

where

\[
H_1(y) = \exp \left\{ -h|DZ_x|^2 + \langle \sqrt{2h}Z_x, y \rangle \right\} \phi(D^{-1}y)(\det D)^{-1} \times \left( 1 + B_0 + O \left( \frac{x^2}{n} \right) \right) \left( 1 + B_1 + O \left( \frac{x^4}{n} + \frac{x^2|D^{-1}y|^2}{n} \right) \right) \left( 1 + B_2 + O \left( \frac{x^4}{n} + \frac{x|D^{-1}y|^3}{n} \right) \right),
\]

\[
B_0 = -\frac{1}{2 \sqrt{n}} \sum_{j=1}^d \tilde{\lambda}_j, \quad (\text{cf. } (2.19))
\]

\[
B_1 = \frac{1}{2 \sqrt{n}} \mathbb{E}^Z \left\{ \langle \sqrt{2h}DZ_x, X_1 \rangle (X_1, D^{-1}y)^2 - \langle \sqrt{2h}DZ_x, X_1 \rangle^2 (X_1, D^{-1}y) \right\},
\]

\[
B_2 = \frac{1}{6 \sqrt{n}} \mathbb{E}^Z \left\{ 3(X_1, X_1) (D^{-1}y - \sqrt{2h}DZ_x, X_1) - (D^{-1}y - \sqrt{2h}DZ_x, X_1)^3 \right\},
\]
and

$$|H_2(y)| \leq C(\det D)^{-1} \phi(D^{-1} y) \exp \left\{ -h |DZ_x|^2 + \sqrt{2h} (Z_x, y) + C \left( \frac{x |D^{-1} y|^2 + x^2 |D^{-1} y|}{\sqrt{n}} \right) \right\}$$

\times \left( \frac{x^2 |D^{-1} y|^4}{n} + \frac{x^4 |D^{-1} y|^2}{n} + \frac{x^2}{n} \right) \left( 1 + |B_0| + \frac{C x^2}{n} \right) \left( 1 + |B_2| + C \left( \frac{x^4}{n} + \frac{x |D^{-1} y|^3}{n} \right) \right).$$

(2.23)

Now we are ready to prove Proposition 2.1.

**Proof of Proposition 2.1.** By (2.2) we have

$$\mathbb{P}(|DW| > x) = \mathbb{E}(\sqrt{2h} DZ_x, W) \int_{|y| > x} \left( \mathbb{E}(\sqrt{2h} DZ_x, y) \right)^{-1} d\mathbb{P}(\tilde{W} \leq y)$$

$$= \mathbb{E}(\sqrt{2h} DZ_x, W) \int_{|y| > x} \left( \mathbb{E}(\sqrt{2h} Z_x, y) \right)^{-1} d\mathbb{P}(D\tilde{W} \leq y)$$

$$= \mathbb{E}(\sqrt{2h} DZ_x, W) \int_{|y| > x} \left( \mathbb{E}(\sqrt{2h} Z_x, y) \right)^{-1} d\Phi(y)$$

$$+ \mathbb{E}(\sqrt{2h} DZ_x, W) \int_{|y| > x} \left( \mathbb{E}(\sqrt{2h} Z_x, y) \right)^{-1} d(\mathbb{P}(D\tilde{W} \leq y) - \Phi(y))$$

$$=: I + II,$$

where \( \mathbb{P}(D\tilde{W} \leq y), y \in \mathbb{R}^d, \) denotes the multivariate distribution function of \( D\tilde{W}. \) Then according to (2.11), we have

$$I = \mathbb{E} \left\{ e^{\sqrt{2h} DZ_x, W} \right\} \int_{|y| > x} \left( \mathbb{E}(\sqrt{2h} Z_x, y) \right)^{-1} d\Phi(y)$$

$$= \int_{|y| > x} \left( \mathbb{E}(\sqrt{2h} Z_x, y) \right)^{-1}$$

$$\times \mathbb{E} \left[ \hat{G}^{n}(\sqrt{2h} DZ_x / \sqrt{n}) \left( \hat{p}(y - \sqrt{n} \mu_1) + \frac{1}{6 \sqrt{n}} \mathbb{E}_{Z_x} \left\{ \hat{p}'''(y - \sqrt{n} \mu_1)(\tilde{X}_1 - \tilde{\mu}_1)^3 \right\} \right) \right] dy. \quad (2.25)$$

By Lemma 2.2 we have

$$I = \int_{|y| > x} \left( \mathbb{E}(\sqrt{2h} Z_x, y) \right)^{-1} \mathbb{E} \left[ \hat{G}^{n}(\sqrt{2h} DZ_x / \sqrt{n}) H_1(y) \right] dy$$

$$+ \int_{|y| > x} \left( \mathbb{E}(\sqrt{2h} Z_x, y) \right)^{-1} \mathbb{E} \left[ \hat{G}^{n}(\sqrt{2h} DZ_x / \sqrt{n}) H_2(y) \right] dy \quad (2.26)$$

$$= I_1 + I_2.$$

We will show below that for \( 1 < x \leq \varepsilon n^{1/6} \) with a sufficiently small \( \varepsilon > 0, \)

$$\hat{G}^{n}(\sqrt{2h} DZ_x / \sqrt{n}) = \left( 1 + O \left( \frac{x_6}{n} \right) \right) e^{h|DZ_x|^2} (1 + B_3), \quad (2.27)$$
where
\[ B_3 = \frac{1}{6\sqrt{n}} \mathbb{E}^Z \langle \sqrt{2h} DZ_x, X_1 \rangle^3 = O \left( \frac{x^3}{\sqrt{n}} \right). \] (2.28)

To prove (2.27), we need the following lemma, whose proof is postponed to Section 4.

**Lemma 2.3.** There exist positive constants \( \varepsilon \) and \( C \) such that for \( |a| \leq \varepsilon \sqrt{n} \),
\[
\left| \mathbb{E} e^{\langle a, W \rangle} - \exp \left( \frac{|a|^2}{2} \right) \left( 1 + \frac{\mathbb{E} \langle a, X_1 \rangle^3}{6\sqrt{n}} \right) \right| \leq C \left( \frac{1}{n} |a|^4 + \frac{1}{n} |a|^6 \right) \exp \left( \frac{|a|^2}{2} + \frac{C |a|^3}{\sqrt{n}} \right). \] (2.29)

Replacing \( a \) in Lemma 2.3 with \( \sqrt{2h} DZ_x \), we obtain (2.27).

We first consider \( I_2 \). By (2.23) and (2.27) and recalling that \( 1 < x \leq \varepsilon n^{1/6} \) and \( |Z_x| \leq 3x \),
\[
|I_2| \leq C \int_{|y| > x} \left( \frac{x^2 |D^{-1} y|^4}{n} + \frac{x^4 |D^{-1} y|^2}{n} + \frac{x^2}{n} \right) \left( \mathbb{E}^Z \langle \sqrt{2h} Z_x, y \rangle \right) \phi(D^{-1} y) (\det D)^{-1} dy
\]
\[
\leq C \mathbb{E} \left[ \left( 1 + B_3 \right) \exp \left( \frac{\langle \sqrt{2h} Z_x, y \rangle + C \left( \frac{x|D^{-1} y|^2}{\sqrt{n}} + \frac{x^2 |D^{-1} y|^2}{\sqrt{n}} \right) \right) \right]
\]
\[
\leq C \left( 1 + |B_0| + C \left( \frac{x^2}{n} \right) \right) \left( 1 + |B_2| + C \left( \frac{x^4}{n} + \frac{x |D^{-1} y|^3}{\sqrt{n}} \right) \right) \right] dy
\]
\[
\leq C \left( 1 + \frac{|D^{-1} y|^2}{\sqrt{n}} \right) \exp \left\{ -\frac{|D^{-1} y|^2}{2} + C \frac{|D^{-1} y|^2}{\sqrt{n}} \right\} dy, \] (2.30)

where in the second inequality, we used \( B_0 = O \left( \frac{x}{\sqrt{n}} \right) \), \( B_2 = O \left( \frac{x^3 + |D^{-1} y|^3}{\sqrt{n}} \right) \), \( B_3 = O \left( \frac{x^3}{\sqrt{n}} \right) \), and \( |D^{-1} y| > x \) if \( |y| > x \). It remains to consider the integral in (2.30), and we will use the following lemma, which is proved in Section 4:

**Lemma 2.4.** For any \( r \geq 2 - d \) and \( 1 < x \leq \varepsilon n^{1/6} \) for a sufficiently small \( \varepsilon \),
\[
\int_{|Dy| > x} |y|^r \exp \left\{ -\frac{|y|^2}{2} + C \frac{x |y|^2}{\sqrt{n}} \right\} dy \leq C(r) x^r \mathbb{P}(|DZ| > x), \] (2.31)

where \( C(r) \) denotes positive constants depending only on \( r, d, t_0, \) and \( c_0 \).

Combining (2.30) and (2.31) (the factor \( \frac{1}{\det D} \) disappears after a change of variable), we obtain
\[
I_2 = O \left( \frac{x^6}{n} \right) \mathbb{P} (|DZ| \geq x). \] (2.32)
We now consider $I_1$ in (2.26). By the definition of $H_1(y)$ in (2.22) and (2.27), we have

\[
\begin{align*}
\hat{G}^n(\sqrt{2\hbar D}Z_x/\sqrt{n})H_1(y) & = \hat{G}^n(\sqrt{2\hbar D}Z_x/\sqrt{n}) \exp \left\{ -\hbar |DZ_x|^2 + \langle \sqrt{2\hbar} Z_x, y \rangle \right\} \phi(D^{-1}y)(\det D)^{-1} \\
& \times \left( 1 + B_0 + O\left( \frac{x^2}{n} \right) \right) \left( 1 + B_1 + O\left( \frac{x^4 + x^2|D^{-1}y|^2}{n} \right) \right) \left( 1 + B_2 + O\left( \frac{x^4 + x|D^{-1}y|^3}{n} \right) \right) \\
& = \left( 1 + O\left( \frac{x^6}{n} \right) \right) \exp \left\{ \langle \sqrt{2\hbar} Z_x, y \rangle \right\} \phi(D^{-1}y)(\det D)^{-1}(1 + B_3) \\
& \times \left( 1 + B_0 + B_1 + B_2 + B_3 + O\left( \frac{x^6 + |D^{-1}y|^6}{n} \right) \right) \\
& = \exp \left\{ \langle \sqrt{2\hbar} Z_x, y \rangle \right\} \phi(D^{-1}y)(\det D)^{-1} \left( 1 + B_0 + B_1 + B_2 + B_3 + O\left( \frac{x^6 + |D^{-1}y|^6}{n} \right) \right),
\end{align*}
\]

(2.33)

where we used $1 < x \leq \varepsilon n^{1/6}$,

\[
B_0 = O\left( \frac{x}{\sqrt{n}} \right), \quad B_1 = O\left( \frac{x|D^{-1}y|^2 + x^2|D^{-1}y|}{\sqrt{n}} \right), \quad B_2 = O\left( \frac{x^3 + |D^{-1}y|^3}{\sqrt{n}} \right), \quad B_3 = O\left( \frac{x^3}{\sqrt{n}} \right),
\]

and straightforward simplifications for terms of order $\frac{1}{n}$. By (2.26) and (2.33),

\[
I_1 = \int_{|y|>x} \phi(D^{-1}y)(\det D)^{-1} \left( \mathbb{E}\exp\left\{ \langle \sqrt{2\hbar} Z_x, y \rangle \right\} \right)^{-1} \\
\times \mathbb{E} \left[ \exp \left\{ \langle \sqrt{2\hbar} Z_x, y \rangle \right\} \left( 1 + B_0 + B_1 + B_2 + B_3 + O\left( \frac{x^6 + |D^{-1}y|^6}{n} \right) \right) \right] dy
\]

(2.34)

\[
= I_{11} + I_{12} + I_{13},
\]

where

\[
I_{11} = \mathbb{P}(|DZ| > x),
\]

(2.35)

\[
I_{12} = \int_{|y|>x} \phi(D^{-1}y)(\det D)^{-1} \left( \mathbb{E}\exp\left\{ \langle \sqrt{2\hbar} Z_x, y \rangle \right\} \right)^{-1} \mathbb{E} \left[ \exp \left\{ \langle \sqrt{2\hbar} Z_x, y \rangle \right\} (B_0 + B_1 + B_2 + B_3) \right] dy,
\]

(2.36)

and

\[
I_{13} = \int_{|y|>x} \phi(D^{-1}y)(\det D)^{-1} O\left( \frac{x^6 + |D^{-1}y|^6}{n} \right) dy.
\]

(2.37)

Using Lemma 2.4, we have

\[
I_{13} = O\left( \frac{x^6}{n} \right) \mathbb{P}(|DZ| > x).
\]

(2.38)
For \(i = 0, 1, 2, 3\), let \(f_i(y) = \mathbb{E} \left[ \exp \left\{ \langle \sqrt{2h} Z_x, y \rangle \rangle \right\} B_i \right]\), and we can verify that
\[
f_i(-y) = -f_i(y). \tag{2.39}
\]
For example, for \(f_0(y)\), recalling (2.20), we have
\[
f_0(-y) = \mathbb{E} \left[ \exp \left\{ \langle \sqrt{2h} (-Z_x), (-y) \rangle \rangle \right\} \left( -\frac{1}{2\sqrt{n}} \sum_{j=1}^{d} \tilde{\lambda}_j(-Z_x) \right) \right]
\]
\[
= -\mathbb{E} \left[ \exp \left\{ \langle \sqrt{2h} Z_x, y \rangle \rangle \right\} \left( -\frac{1}{2\sqrt{n}} \sum_{j=1}^{d} \tilde{\lambda}_j(Z_x) \right) \right]
\]
\[
= -f_0(y),
\]
where the second equality holds because \(Z_x\) has a symmetric distribution, that is, \(\mathcal{L}(Z_x) = \mathcal{L}(-Z_x)\). Because
\[
\phi(-D^{-1}y) \left( \mathbb{E} e^{\langle \sqrt{2h} Z_x, y \rangle} \right)^{-1} = \phi(D^{-1}y) \left( \mathbb{E} e^{\langle \sqrt{2h} Z_x, y \rangle} \right)^{-1},
\]
we have
\[
\int_{|y| > x} \phi(D^{-1}y) \left( \mathbb{E} e^{\langle \sqrt{2h} Z_x, y \rangle} \right)^{-1} f_i(y) dy = 0 \tag{2.41}
\]
for \(i = 0, 1, 2, 3\). By (2.41), we have
\[
I_{12} = 0. \tag{2.42}
\]
Using (2.34), (2.35), (2.38) and (2.42), we have
\[
I_1 = \left( 1 + O(1) \frac{x^6}{n} \right) \mathbb{P}(|DZ| \geq x). \tag{2.43}
\]
Combining (2.26), (2.32) and (2.43), we obtain
\[
I = \left( 1 + O(1) \frac{x^6}{n} \right) \mathbb{P}(|DZ| \geq x). \tag{2.44}
\]
Finally, we consider \(II\) in (2.24). Because \(Z_x\) is symmetric with respect to 0, we have
\[
\mathbb{E} e^{\langle \sqrt{2h} Z_x, y \rangle} = m(|y|) \text{ for some function } m(\cdot) : \mathbb{R}^+ \to \mathbb{R}. \text{ Let } e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^d, \text{ then by symmetry,}
\]
\[
m(a) = \frac{\kappa}{(\sqrt{2\pi})^d} \int_{|z| \leq z_0} e^{\langle \sqrt{2h} z, e_1 a \rangle} e^{-\frac{|z|^2}{2}} dz
\]
\[
= \frac{\kappa}{(\sqrt{2\pi})^d} e^{\kappa a^2} \int_{|z| \leq z_0} e^{-\frac{|z+\sqrt{2he_1 a}|^2}{2}} dz
\]
\[
= \frac{\kappa}{(\sqrt{2\pi})^d} e^{\kappa a^2} \int_{|z+\sqrt{2he_1 a}| \leq z_0} e^{-\frac{|z|^2}{2}} dz
\]
\[
= \kappa e^{\kappa a^2} \mathbb{P}(|Z + \sqrt{2he_1 a| \leq z_0), \tag{2.45}
\]
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where $\kappa$ is defined in (2.3). From the first expression of $m(a)$ in (2.45), we determine that it is increasing and $m(a) \to \infty$ as $a \to \infty$ (recall that $h = \frac{1}{2} - \frac{1}{2x} > 0$). Then, for $II$, we have

$$II = \mathbb{E}e^{\sqrt{2h}D Z_x, W} \int_{a > x} (m(a))^{-1}d\left(\mathbb{P}(|D\tilde{W}| \leq a) - \tilde{\Phi}(B(0, a))\right).$$

(2.46)

By the integration by parts formula, we have

$$II = \mathbb{E}e^{\sqrt{2h}D Z_x, W}(m(x))^{-1}\left(\mathbb{P}\left(\tilde{W} \in D^{-1}B(0, x)\right) - \tilde{\Phi}\left(B(0, x)\right)\right) - \mathbb{E}e^{\sqrt{2h}D Z_x, W} \int_{a > x} \left(\mathbb{P}\left(\tilde{W} \in D^{-1}B(0, a)\right) - \tilde{\Phi}\left(B(0, a)\right)\right) d(m(a))^{-1}. \tag{2.47}$$

Furthermore, recalling that $m(a) \uparrow \infty$ as $a \uparrow \infty$,

$$|II| \leq \mathbb{E}e^{\sqrt{2h}D Z_x, W}(m(x))^{-1}\left|\left(\mathbb{P}\left(\tilde{W} \in D^{-1}B(0, x)\right) - \tilde{\Phi}\left(B(0, x)\right)\right)\right| + \mathbb{E}e^{\sqrt{2h}D Z_x, W} \sup_{r > 0} \left|\mathbb{P}\left(\tilde{W} \in D^{-1}B(0, r)\right) - \tilde{\Phi}\left(B(0, r)\right)\right| \int_{a > x} d|m(a)|^{-1}$$

$$\leq 2\mathbb{E}e^{\sqrt{2h}D Z_x, W}(m(x))^{-1} \sup_{a > 0} \left|\mathbb{P}\left(\tilde{W} \in D^{-1}B(0, a)\right) - \tilde{\Phi}\left(B(0, a)\right)\right|. \tag{2.48}$$

From (2.3), $h = \frac{1}{2} - \frac{1}{2x}$ and $z_0 = 3x$, we have

$$\mathbb{E}e^{h|DZ|^2} = \frac{\kappa}{(\sqrt{2\pi})^d} \int_{|z| \leq z_0} e^{-\frac{z^T(I-2hQ)z}{2}} dz$$

$$= \det(I - 2hQ)^{-1/2} \frac{\kappa}{(\sqrt{2\pi})^d} \int_{|z| \leq z_0} e^{-\frac{z^T}{2} z} dz$$

$$= \kappa \det(I - 2hQ)^{-1/2} \mathbb{P}\left(Z^T(I - 2hQ)^{-1} Z \leq z_0^2\right). \tag{2.49}$$

Recalling that $q_i, i = 1, 2, \ldots, d$ are the diagonal values of $Q$ and combining (2.45), (2.48) and (2.49), we have

$$|II| \leq 2 \sup_{a > 0} |\mathbb{P}(|D\tilde{W}| \leq a) - \tilde{\Phi}(B(0, a))| \cdot \frac{\mathbb{E}e^{\sqrt{2h}D Z_x, W}e^{-hx^2}}{\kappa \mathbb{P}(|Z + \sqrt{2h}e_1 x| \leq z_0)}$$

$$\leq 2 \sup_{a > 0} |\mathbb{P}(|D\tilde{W}| \leq a) - \tilde{\Phi}(B(0, a))| \cdot \frac{\mathbb{E}e^{\sqrt{2h}D Z_x, W} \left(\prod_{i=1}^d (1 - 2hq_i)^{-1/2}\right)e^{-hx^2}}{\mathbb{E}e^{h|DZ|^2}} \mathbb{P}(|Z + \sqrt{2h}e_1 x| \leq z_0). \tag{2.50}$$

Recalling that $z_0 = 3x, x > 1$ and $\frac{1}{2} > h = \frac{1}{2} - \frac{1}{2x} > 0$, we have

$$\frac{1}{\mathbb{P}(|Z + \sqrt{2h}e_1 x| \leq z_0)} \leq \mathbb{P}(|Z| \leq 2x)^{-1} = O(1). \tag{2.51}$$
By the definition of \( h \) and by (2.50) and (2.51),
\[
II = O(1) \sup_{a \geq 0} |\mathbb{P}(|D \widetilde{W}| \leq a) - \Phi(B(0, a))| \frac{\mathbb{E}e^{\sqrt{2\epsilon}DZ,W}}{\mathbb{E}e^{\sqrt{2\epsilon}DZ,W}} \left( \prod_{i=1}^{d} (1 - 2h)_{1/2} \right) e^{-x^2/2}. \tag{2.52}
\]
By (2.52), (2.27), (2.28) and recalling that \( 1 < x \leq \epsilon n^{1/6} \), we have
\[
II = O(1) \sup_{a \geq 0} |\mathbb{P}(|D \widetilde{W}| \leq a) - \hat{\Phi}(B(0, a))| \left( \prod_{i=1}^{d} (1 - 2h)_{1/2} \right) e^{-x^2/2}. \tag{2.53}
\]
Suppose that \( q_{i}, i = 1, 2, \ldots, d \) take \( s \) different values, which means that there exist \( 1 = \lambda_{1} > \lambda_{2} > \ldots > \lambda_{s} > 0 \) and positive integers \( v_{1}, v_{2}, \ldots, v_{s} \) such that
\[
q_{i} = \lambda_{j} \text{ for } v_{j-1} + 1 \leq i \leq v_{j} \text{ and } 1 \leq j \leq s, \tag{2.54}
\]
where \( v_{0} = 0 \). Recalling the definition of \( h \), we then have
\[
\prod_{i=1}^{d} (1 - 2h)_{1/2} = \prod_{i=1}^{s} (1 - 2h\lambda_{i})_{-v_{i}/2} = \prod_{i=1}^{s} \left( 1 - \lambda_{i} + \frac{\lambda_{j}}{x^2} \right)^{-v_{i}/2}. \tag{2.55}
\]
Let \( p = \min\{1 \leq i \leq s, (1 - \lambda_{i})x^{2}/\lambda_{i} > 1\} \) (with \( \min\{\emptyset\} : = s + 1, \prod_{i=s+1}^{s} : = 1 \)) and \( r = \sum_{i=1}^{p-1} v_{i} \leq d \). We then have that (2.55) is smaller than or equal to
\[
\prod_{i=1}^{p-1} \left( \frac{\lambda_{i}}{x^2} \right)^{-v_{i}/2} \prod_{j=p}^{s} (1 - \lambda_{j})^{-v_{j}/2} \leq 2^{d/2} \prod_{j=p}^{s} (1 - \lambda_{j})^{-v_{j}/2} x^{r}, \tag{2.56}
\]
where we used the fact that \( 1/2 \leq \lambda_{i} \leq 1 \) for \( i \leq p - 1 \).

The following lemma, proved in Section 4, gives a lower bound for the tail probability of a sum of weighted chi-square random variables. Denote by \( \chi_{v}^{2} \) a chi-square random variable with \( v \) degrees of freedom.

**Lemma 2.5.** Let \( 1 = \lambda_{1} > \lambda_{2} > \ldots > \lambda_{s} > 0 \) be a sequence of constants, and let \( v_{1}, v_{2}, \ldots, v_{s} \) be a sequence of positive integers such that \( \sum_{i=1}^{s} v_{i} = d \). Suppose \( \{\chi_{v_{1}}^{2}, \ldots, \chi_{v_{s}}^{2}\} \) are independent. For any \( x > 1 \), we have
\[
\mathbb{P}\left( \sum_{i=1}^{s} \chi_{v_{i}}^{2} \geq x^{2} \right) \geq C_{d} \left[ \prod_{i=p}^{s} (1 - \lambda_{i})^{-v_{i}/2} \right] x^{r-2} e^{-x^{2}/2}, \tag{2.57}
\]
where \( C_{d} \) is a positive constant depending only on \( d \), \( p = \min\{1 \leq i \leq s, (1 - \lambda_{i})x^{2}/\lambda_{i} > 1\} \) (with \( \min\{\emptyset\} : = s + 1, \prod_{i=s+1}^{s} : = 1 \)) and \( r = \sum_{i=1}^{p-1} v_{i} \leq d \).

Using (2.53), (2.55), (2.56) and applying Lemma 2.5, we have
\[
II = O(1) x^{2} \mathbb{P}\left( \sum_{i=1}^{s} \chi_{v_{i}}^{2} \geq x^{2} \right) \sup_{a \geq 0} |\mathbb{P}(|D \widetilde{W}| \leq a) - \hat{\Phi}(B(0, a))| \tag{2.58}
\]
\[
= O(1) x^{2} \mathbb{P}(|DZ| \geq x) \sup_{a \geq 0} |\mathbb{P}(|D \widetilde{W}| \leq a) - \hat{\Phi}(B(0, a))|. \]
Now, combining (2.24), (2.44) and (2.58), we complete the proof of Proposition 2.1.
3 Proof of Theorem 1.1

Theorem 1.1 immediately follows from Propositions 2.1 and 3.1 given as follows:

**Proposition 3.1.** Under the conditions of Theorem 1.1, let $\tilde{W}$ be as in (2.2) and $\tilde{\Phi}$ be as in (2.11). For $d \geq 5$ and $1 < x \leq \varepsilon n^{1/6}$ with a sufficiently small $\varepsilon > 0$, we have

$$\sup_{a > 0} |\mathbb{P}(|D\tilde{W}| \leq a) - \tilde{\Phi}(B(0, a))| \leq \frac{Cx^3}{\det(Q^{1/2})n}. \tag{3.1}$$

For $1 \leq d \leq 4$ and $1 < x \leq \varepsilon n^{1/6}$ with a sufficiently small $\varepsilon > 0$, we have

$$\sup_{a > 0} |\mathbb{P}(|D\tilde{W}| \leq a) - \tilde{\Phi}(B(0, a))| \leq \frac{Cx}{\det(Q^{1/2})n^{3/4}}. \tag{3.2}$$

**Proof of Proposition 3.1.** We first prove the case $d \geq 5$. We rely crucially on the following lemma:

**Lemma 3.1** (Corollary 2.3 of Götze and Zaitsev (2014)). Let $Y_1, \ldots, Y_n$ be i.i.d. random vectors in $\mathbb{R}^d$ with mean 0, positive definite covariance matrix $\Sigma$, and finite fourth moments. Let $\sigma^2$ denote the summation of the eigenvalues of $\Sigma$. Let $\phi$ denote the standard normal density in $\mathbb{R}^d$, and, for $y, u \in \mathbb{R}^d$, let

$$p(y) = \phi(\Sigma^{-1/2}y)/\sqrt{\det \Sigma},$$

and

$$p''(y)u^3 = p(y) \left( 3\langle \Sigma^{-1}u, u \rangle \langle \Sigma^{-1}y, u \rangle - \langle \Sigma^{-1}y, u \rangle^2 \right).$$

Then,

$$\sup_{a > 0} \left| \mathbb{P} \left( \frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \in B(b, a) \right) - \int_{B(b, a)} \left( p(y) + \frac{1}{6\sqrt{n}} \mathbb{E}p''(y)Y_1^3 \right) dy \right| \leq \frac{C_d\sigma^d}{n \det(\Sigma^{1/2})} \left( 1 + \frac{b^3}{\sigma^4} \right) \mathbb{E}|\Sigma^{-1/2}Y_1|^4, \tag{3.3}$$

where $C_d$ is a constant depending only on $d$.

Using (2.6) and (2.11), we rewrite the target $\mathbb{P}(|D\tilde{W}| \leq a) - \tilde{\Phi}(B(0, a))$ as

$$\mathbb{P}(|D\tilde{W}| \leq a) - \tilde{\Phi}(B(0, a)) = \frac{1}{\mathbb{E}G^n(\sqrt{2h}DZ_x/\sqrt{n})} \mathbb{E}\left\{ G^n(\sqrt{2h}DZ_x/\sqrt{n}) \left[ \mathbb{P}^{Z_x} \left( \left| \frac{\sum_{i=1}^n \tilde{X}_i}{\sqrt{n}} \right| \leq a \right) - \int_{B(-\sqrt{n}\mu_1, a)} \tilde{\omega}(y)dy \right] \right\}, \tag{3.4}$$

where $\tilde{\omega}(y)$ is defined in (2.9). We bound (3.4) uniformly in $a \geq 0$. From (3.3) with $Y_i = \tilde{X}_i - \tilde{\mu}_1$, $b = -\sqrt{n}\tilde{\mu}_1$, and $\Sigma = \tilde{\Sigma}$, we have

$$\sup_{a > 0} \left| \mathbb{P}^{Z_x} \left( \left| \frac{\sum_{i=1}^n \tilde{X}_i}{\sqrt{n}} \right| \leq a \right) - \int_{B(-\sqrt{n}\tilde{\mu}_1, a)} \tilde{\omega}(y)dy \right| \leq \frac{C_d\tilde{\sigma}^d}{n \det(\tilde{\Sigma}^{1/2})} \left( 1 + \frac{\sqrt{n}\tilde{\mu}_1}{\sigma} \right)^3 \mathbb{E}^{Z_x} |\tilde{\Sigma}^{-1/2}(\tilde{X}_1 - \tilde{\mu}_1)|^4, \tag{3.5}$$

where $\tilde{\sigma}$ and $\tilde{\Sigma}$ are the standard deviation and covariance matrix of $\tilde{X}_1$.
where $\bar{\sigma}^2$ denotes the summation of the eigenvalues of $\tilde{\Sigma}$. By Lemma 2.1 and recalling that $1 < x \leq \varepsilon n^{1/6}$, we have

$$\frac{C_d \bar{\sigma}^d}{n \det(\Sigma^{1/2})} \left( 1 + \left| \frac{n \mu_1}{\sigma} \right|^3 \right) \frac{1}{n} n \det(\hat{Q}^{1/2}) \left| X_1 \right|^4 \leq \frac{C}{n \det(\hat{Q}^{1/2})} \left| X_1 \right|^4 \leq \frac{C x^3}{n \det(\hat{Q}^{1/2})}.$$  

(3.6)

By (3.4)–(3.6) we complete the proof of (3.1).

The result (3.2) for $1 \leq d \leq 4$ is proved by the same argument as for $d \geq 5$, except that instead of Lemma 3.1, we use Lemma 3.2 below with $Y_t = \tilde{\Sigma}^{-1/2}(X_t - \tilde{\mu}_1)$, $\Sigma_1 = \tilde{\Sigma}$, and $b = -\sqrt{n} \tilde{\mu}_1$. From (2.14), for $1 < x \leq \varepsilon n^{1/6}$ with a sufficiently small $\varepsilon$, the largest eigenvalue of $\Sigma$ is smaller than 4. Using Lemma 3.2 we have

$$\sup_{a \geq 0} \left| \mathbb{P}^{\mathbb{Z}^d} \left( \frac{\sum_{i=1}^n X_i}{\sqrt{n}} \leq a \right) - \int_{B(0, a)} \tilde{\omega}(y) dy \right| \leq \frac{C_d}{\det(\Sigma^{1/2})} \left( 1 + \left| \frac{n \tilde{\Sigma}^{-1/2} \tilde{\mu}_1}{\sqrt{n}} \right|^3 \right) \left( \mathbb{E} \left| \tilde{\Sigma}^{-1/2}(X_i - \tilde{\mu}_1) \right|^4 \right)^{3/2}.$$  

(3.7)

Similar to (3.6) and using $d \leq 4$, we have

$$\sup_{a \geq 0} \left| \mathbb{P}^{\mathbb{Z}^d} \left( \frac{\sum_{i=1}^n X_i}{\sqrt{n}} \leq a \right) - \int_{B(0, a)} \tilde{\omega}(y) dy \right| \leq \frac{C_d}{\det(\hat{Q}^{1/2})} \left( 1 + \left| \frac{n \hat{\Sigma}^{-1/2} \hat{\mu}_1}{\sqrt{n}} \right|^3 \right) \left( \mathbb{E} \left| \hat{\Sigma}^{-1/2}(X_i - \hat{\mu}_1) \right|^4 \right)^{3/2} \leq \frac{C x^3}{\det(\hat{Q}^{1/2})}.$$  

(3.8)

From (3.4), (3.7) and (3.8), we complete the proof of (3.2).

**Lemma 3.2.** Let $Y_1, \ldots, Y_n$ be i.i.d. random vectors in $\mathbb{R}^d$ with mean $0$, covariance matrix $I_d$, and finite fourth moments. Let $\sigma$ denote the standard normal density in $\mathbb{R}^d$, and, for $y, u \in \mathbb{R}^d$, let $p(y)$ and $p''(y)u^3$ be as defined in Lemma 3.1 with $\Sigma = I_d$. Let $\Sigma_1$ be a symmetric positive definite matrix with $||\Sigma_1||_{op} \leq 4$. Then, for any $b \in \mathbb{R}^d$, we have

$$\sup_{a \geq 0} \left| \mathbb{P} \left( \frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \in \Sigma_1^{-1/2} B(b, a) \right) - \int_{y \in \Sigma_1^{-1/2} B(b, a)} \left( p(y) + \frac{1}{6 \sqrt{n}} \mathbb{E} p''(y) Y_1^3 \right) dy \right| \leq \frac{C_d}{\det(\Sigma_1^{1/2})} \left( 1 + \left| \Sigma_1^{-1/2} b \right| \left| \frac{d}{n^{1/2}} \right| \right) \left( \mathbb{E} \left| Y_1^4 \right| \right)^{3/2},$$

where $C_d$ is a constant depending only on $d$.

**Proof of Lemma 3.2.** In this proof, we denote by $C_d$ positive constants that depend only on $d$. They may differ in different expressions.

If $\Sigma_1 = I_d$ and $b = 0$, then Lemma 3.2 follows from Esseen (1945, Chapter VII, Theorem 1) by observing that $\int_{y \in B(0, a)} \mathbb{E} p''(y) Y_1^3 dy = 0$. The proof for the general case is a straightforward modification (outlined below) of the proof of Esseen (1945, Chapter VII, Theorem 1) . Concerning notation, we use, e.g., (Eq. 60) to denote the equation
(60) in Esseen (1945, Chapter VII). To be consistent with the notation in Esseen (1945), in this proof, we use the symbol $\varepsilon$ to denote a different quantity from that in the rest of the paper. Other notations used in this proof are as follows: $x = (x_1, x_2, \ldots, x_d)^T$, $t = (t_1, t_2, \ldots, t_d)^T$, $r = |t|$, and $J_{d/2}(\cdot)$ denotes the Bessel function of order $d/2$.

We first give a smoothing inequality for the noncentered ellipsoid $\Sigma_1^{-1/2}B(b,a)$ (cf. (3.15) below). For $\varepsilon > 0$, let

$$Q_{\varepsilon}(x_1, x_2, \ldots, x_k) = \begin{cases} 1 & \text{for } |x| \leq \varepsilon \\ 0 & \text{for } |x| > \varepsilon. \end{cases}$$

It has the following Fourier transform (cf. (Eq. 43) and (Eq. 44)):

$$q_{\varepsilon}(t) = \int_{\mathbb{R}^d} e^{i(t,x)} Q_{\varepsilon}(x) dx = \left( \frac{2\pi a}{|t|} \right)^{d/2} J_{d/2}(\varepsilon |t|). \hspace{1cm} (3.9)$$

Let

$$\widetilde{Q}_{a,b}(x_1, x_2, \ldots, x_k) = \begin{cases} 1 & \text{for } |\Sigma_1^{1/2}x - b| \leq a \\ 0 & \text{for } |\Sigma_1^{1/2}x - b| > a \end{cases}$$

be the indicator of the ellipsoid $\Sigma_1^{-1/2}B(b,a)$. From (3.9), it has the following Fourier transform:

$$\widetilde{q}_{a,b}(t) = \int_{\mathbb{R}^d} e^{i(t,x)} \widetilde{Q}_{a,b}(x) dx = \left( \frac{2\pi a}{\Sigma_1^{-1/2}|t|} \right)^{d/2} J_{d/2}(a|\Sigma_1^{-1/2}|t|) e^{i(t,\Sigma_1^{-1/2}b)} \frac{1}{\det(\Sigma_1^{1/2})}.$$ 

Now, consider the convolution function (cf. (Eq. 45)), for $0 < \varepsilon < a$,

$$M(x) = \frac{\Gamma(1 + \frac{d}{2})}{\pi^{d/2} \varepsilon^d} \int_{\mathbb{R}^d} \widetilde{Q}_{a,b}(x_1 - \xi_1, \ldots, x_d - \xi_d) Q_{\varepsilon}(\xi_1, \ldots, \xi_d) d\xi_1 \ldots d\xi_d.$$ 

Let $A(b,a,\varepsilon, \Sigma_1) = \bigcup_{t \in \Sigma_1^{-1/2}B(b,a)} B(t,\varepsilon)$ and

$$\hat{A}(b,a,\varepsilon, \Sigma_1) = \left( \Sigma_1^{-1/2}B(b,a) \right) \setminus \bigcup_{t \in \Sigma_1^{-1/2}B(b,a)^c} B(t,\varepsilon).$$

We observe that $|M(x)| \leq 1$ for all $x$ and (cf. (Eq. 46))

$$M(x_1, x_2, \ldots, x_k) = \begin{cases} 1 & \text{for } x \in \hat{A}(b,a,\varepsilon, \Sigma_1) \\ 0 & \text{for } x \in (A(b,a,\varepsilon, \Sigma_1))^c. \end{cases}$$

The Fourier transform of $M, m(t)$, is (cf. (Eq. 47))

$$m(t) = \left( \frac{2\pi a}{\Sigma_1^{-1/2}|t|} \right)^{d/2} J_{d/2}(a|\Sigma_1^{-1/2}|t|) e^{i(t,\Sigma_1^{-1/2}b)} 2^{d/2} \frac{1}{\det(\Sigma_1^{1/2})},$$

because the Fourier transform of a convolution is equal to the product of the transforms corresponding to the functions in the convolution.
Thus, replacing $a$ by $a + \varepsilon/2$ and $\varepsilon$ by $\varepsilon/4$ in (3.10), the function (cf. (Eq. 48))

$$
\left(\frac{2\pi(a + \varepsilon/2)}{|\Sigma_1^{-1/2}t|}\right)^{d/2} J_{d/2}\left((a + \varepsilon/2)|\Sigma_1^{-1/2}t|\right) e^{i(t,\Sigma_1^{-1/2}b)} 2^{d/2} \Gamma\left(1 + \frac{d}{2}\right) \frac{J_{d/2}(\varepsilon r/4)}{(\varepsilon r/4)^{d/2}} \frac{1}{\det(\Sigma_1^{1/2})}
$$

is the Fourier transform of a function

$$
= \begin{cases} 
1 & \text{for } x \in \Sigma_1^{-1/2}B(b,a) \\
0 & \text{for } x \in \left(\Sigma_1^{-1/2}B(b,a + \varepsilon)\right)^c .
\end{cases} \quad (3.11)
$$

Similarly, the function (cf. (Eq. 49))

$$
\left(\frac{2\pi(a - \varepsilon/2)}{|\Sigma_1^{-1/2}t|}\right)^{d/2} J_{d/2}\left((a - \varepsilon/2)|\Sigma_1^{-1/2}t|\right) e^{i(t,\Sigma_1^{-1/2}b)} 2^{d/2} \Gamma\left(1 + \frac{d}{2}\right) \frac{J_{d/2}(\varepsilon r/4)}{(\varepsilon r/4)^{d/2}} \frac{1}{\det(\Sigma_1^{1/2})}
$$

is the Fourier transform of a function

$$
= \begin{cases} 
1 & \text{for } x \in \Sigma_1^{-1/2}B(b,a - \varepsilon) \\
0 & \text{for } x \in \left(\Sigma_1^{-1/2}B(b,a)\right)^c .
\end{cases} \quad (3.12)
$$

By the well-known properties of Bessel functions (cf. (Eq. 50)):

$$
\begin{align*}
\left|\frac{J_{d/2}(z)}{z^{d/2}}\right| & \leq C_d \text{ for all positive } z \\
\left|J_{d/2}(z)\right| & \leq \frac{C_d}{\sqrt{z}} \text{ for all positive } z,
\end{align*}
$$

and fact that (recall our assumption that $\|\Sigma_1\|_{op} \leq 4$)

$$
\frac{1}{|\Sigma_1^{-1/2}t|} \leq \frac{2}{|t|} = \frac{2}{r},
$$

we have the following lemma:

**Lemma 3.3.** (cf. Lemma 4 of Esseen (1945, Chapter VII)) Let $a$ and $\varepsilon$ be two assigned constants and $0 < \varepsilon < a$. There exists a function $H(x,b,a,\varepsilon)$ such that

$$
H(x,b,a,\varepsilon) = \begin{cases} 
1 & \text{for } x \in \Sigma_1^{-1/2}B(b,a) \\
0 & \text{for } x \in \left(\Sigma_1^{-1/2}B(b,a + \varepsilon)\right)^c , \text{ and } |H(x,b,a,\varepsilon)| \leq 1
\end{cases}
$$

for all $x$.

Furthermore, the Fourier transform of $H$, $h(t,b,a,\varepsilon)$, can be bounded by a function depending on $t$ only through $r = |t|$, i.e.,

$$
|h(t,b,a,\varepsilon)| \leq \frac{C}{\det(\Sigma_1^{1/2})} \frac{a^{k-1}}{r^{k+1}} , \quad (3.13)
$$
\[ |h(t, b, a, \varepsilon)| \leq \frac{C}{\det(\Sigma_{1/2}^1)} \cdot \frac{a^{\frac{k-1}{2}}}{\varepsilon^r r^{\frac{2k+1}{2}}} . \quad (3.14) \]

There also exists a function \( H(x, b, a, -\varepsilon) \) such that

\[
H(x, b, a, -\varepsilon) = \begin{cases} 
1 & \text{for } x \in \Sigma_{1/2}^{-1}B(b, a - \varepsilon) \\
0 & \text{for } x \in \left( \Sigma_{1/2}^{-1}B(b, a) \right)^c , \text{ and } |H(x, b, a, \varepsilon)| \leq 1
\end{cases}
\]

for all \( x \), the Fourier transform of which, \( h(t, b, a, -\varepsilon) \), satisfies the inequalities (3.13) and (3.14).

Let \( \mu_n(b, a) = \mathbb{P}\left( \frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \in \Sigma_{1/2}^{-1}B(b, a) \right) \) and

\[
\psi(b, a) = \int_{\Sigma_{1/2}^{-1}B(b,a)} \left( p(y) + \frac{1}{6\sqrt{n}} \mathbb{E}p''(y)Y_1^3 \right) dy.
\]

We denote by \( \Delta_n \) the difference of the characteristic functions of

\[
\frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \quad \text{and} \quad p(y) + \frac{1}{6\sqrt{n}} \mathbb{E}p''(y)Y_1^3.
\]

Then, by Lemma 3.3 and the same argument as that in Esseen (1945, p.104) leading to (Eq. 56), we have

\[
|\mu_n(b, a) - \psi(b, a)| \leq \max\{A_1, A_2\}, \quad (3.15)
\]

where

\[
A_1 = |\psi(b, a + \varepsilon) - \psi(b, a)| + \frac{1}{(2\pi)^k} \int_{R_k} |\Delta_n(t)h(t, b, a, \varepsilon)| dt \quad (3.16)
\]

and

\[
A_2 = |\psi(b, a) - \psi(b, a - \varepsilon)| + \frac{1}{(2\pi)^k} \int_{R_k} |\Delta_n(t)h(t, b, a, -\varepsilon)| dt.
\]

Similar to (Eq. 59) and (Eq. 60), we make the following assumptions without loss of generality:

\[ 1^\circ \quad a \leq 4|\Sigma_{1/2}^{-1}b| + 4\log(2 + n), \quad (3.17) \]

or else we choose \( \varepsilon = a/8 \) and proceed as in the subsequent estimations.

\[ 2^\circ \quad \frac{1}{n^{d/2}} \left( \mathbb{E}|Y_1|^4 \right)^{3/2} \leq \frac{1}{8} \]

or else Lemma 3.2 is true with a sufficiently large \( C_d \). Choose (cf. (Eq. 61))

\[ \varepsilon = \frac{a}{n^{d/(d+1)}} \left( \mathbb{E}|Y_1|^4 \right)^{3/2}. \quad (3.19) \]
We may confine ourselves to the estimation of $A_1$, $A_2$ being treated similarly. To obtain an upper bound for $|\psi(b, a + \varepsilon) - \psi(b, a)|$, we first consider

$$
\int_{y \in \Sigma_1^{-1/2}B(b, a + \varepsilon) \setminus \Sigma_1^{-1/2}B(b, a)} p(y) dy.
$$

For $a \leq 4|\Sigma_1^{-1/2}b| + \left(4\log(2 + n) \wedge |\Sigma_1^{-1/2}b|\right)$, from Gaussian anti-concentration inequalities (cf. Bhattacharya and Rao (1986, Chapter 1, Section 3)),

$$
\int_{y \in \Sigma_1^{-1/2}B(b, a + \varepsilon) \setminus \Sigma_1^{-1/2}B(b, a)} p(y) dy \leq \frac{C_d\varepsilon}{\sigma_{\min}} \leq \frac{C_d|\Sigma_1^{-1/2}b|}{\det(\Sigma_1^{1/2})^{d/2n}} \left(\mathbb{E}|Y_1|^4\right)^{3/2},
$$

where $\sigma_{\min}$ is the smallest eigenvalue of $\Sigma_1$, and, in the second inequality, we used $\|\Sigma_1\|_{op} \leq 4$ and (3.19). If $|\Sigma_1^{-1/2}b| \leq 4\log(n + 2)$, we must also consider the case $5|\Sigma_1^{-1/2}b| \leq a \leq 4|\Sigma_1^{-1/2}b| + 4\log(2 + n)$. In this situation,

$$
\int_{\Sigma_1^{1/2}y \in B(b, a + \varepsilon) \setminus B(b, a)} p(y) dy \leq \sup_{\Sigma_1^{1/2}y \in B(b, a + \varepsilon) \setminus B(b, a)} \frac{C_d a^d}{\det(\Sigma_1^{1/2})^{d/2n}} \exp\left\{-\frac{|y|^2}{2}\right\} \left(\mathbb{E}|Y_1|^4\right)^{3/2},
$$

where we used the inequality that the volume of $\Sigma_1^{-1/2}(B(b, a + \varepsilon) \setminus B(b, a))$ is smaller than $C_d a^d \left(\mathbb{E}|Y_1|^4\right)^{3/2} / (\det(\Sigma_1^{1/2})^{d/2n})$. In fact,

$$
\text{Vol}\left(\Sigma_1^{-1/2}(B(b, a + \varepsilon) \setminus B(b, a))\right) = \int_{y \in \Sigma_1^{-1/2}(B(b, a + \varepsilon) \setminus B(b, a))} dy = \det(\Sigma_1^{-1/2}) \int_{y \in B(b, a + \varepsilon) \setminus B(b, a)} dy = \frac{\text{Vol}(B(0, a + \varepsilon) \setminus B(0, a))}{\det(\Sigma_1^{1/2})} \leq \frac{C_d a^d \left(\mathbb{E}|Y_1|^4\right)^{3/2}}{\det(\Sigma_1^{1/2})^{d/2n}},
$$

where Vol$(A)$ denotes the volume of $A \subset \mathbb{R}^d$ and in the last inequality, we used (3.18) and (3.19). Furthermore, because $a \leq 5|y|$ (which follows from $\Sigma_1^{1/2}y \in B(b, a + \varepsilon) \setminus B(b, a)$, $5|\Sigma_1^{-1/2}b| \leq a$, and the assumption that $\|\Sigma_1\|_{op} \leq 4$), we have

$$
\sup_{\Sigma_1^{1/2}y \in B(b, a + \varepsilon) \setminus B(b, a)} \frac{C_d a^d}{\det(\Sigma_1^{1/2})^{d/2n}} \exp\left\{-\frac{|y|^2}{2}\right\} \left(\mathbb{E}|Y_1|^4\right)^{3/2} \leq \sup_{\Sigma_1^{1/2}y \in B(b, a + \varepsilon) \setminus B(b, a)} \frac{C_d |y|^d}{\det(\Sigma_1^{1/2})^{d/2n}} \exp\left\{-\frac{|y|^2}{2}\right\} \left(\mathbb{E}|Y_1|^4\right)^{3/2} \leq \frac{C_d}{\det(\Sigma_1^{1/2})^{d/2n}} \left(\mathbb{E}|Y_1|^4\right)^{3/2}.
$$
Therefore,
\[
\int_{y \in \Sigma_1^{-1/2} B(b,a+\varepsilon) \setminus \Sigma_1^{-1/2} B(b,a)} p(y)dy \leq C_d \frac{1 + |\Sigma_1^{-1/2} b|}{\det(\Sigma_1^{1/2}) n^{d+1}} (\mathbb{E}|Y_1|^4)^{3/2}. \tag{3.21}
\]

From \( |\mathbb{E}p''(y)Y_1^3| \leq C_d |\mathbb{E}|Y_1|^3 (|y| + |y|^3) p(y) \), by similar arguments we have
\[
\int_{y \in \Sigma_1^{-1/2} B(b,a+\varepsilon) \setminus \Sigma_1^{-1/2} B(b,a)} \frac{1}{6\sqrt{n}} |\mathbb{E}p''(y)Y_1^3|dy \leq C_d \frac{1 + |\Sigma_1^{-1/2} b|}{\det(\Sigma_1^{1/2}) n^{d+1}} (\mathbb{E}|Y_1|^4)^{3/2}, \tag{3.22}
\]
where we used \( \frac{\mathbb{E}|Y_1|^3}{\sqrt{n}} \leq \sqrt{\frac{(\mathbb{E}|Y_1|^4)^{3/2}}{n}} \) and (3.18). Thus, by (3.21) and (3.22) we have (cf. (Eq. 62))
\[
|\psi(b, a + \varepsilon) - \psi(b, a)| \leq C_d \frac{1 + |\Sigma_1^{-1/2} b|}{\det(\Sigma_1^{1/2}) n^{d+1}} (\mathbb{E}|Y_1|^4)^{3/2}. \tag{3.23}
\]

To bound (3.16), it remains to consider (cf. (Eq. 63))
\[
I := \frac{1}{(2\pi)^k} \int_{\mathbb{R}_k} |\Delta_n(t)h(t, b, a, \varepsilon)| dt \\
= \frac{1}{(2\pi)^k} \int_{0 \leq r < \frac{\sqrt{\pi}}{(d\beta_4)^1/4}} + \frac{1}{(2\pi)^k} \int_{r > \frac{\sqrt{\pi}}{(d\beta_4)^1/4}} = I_1 + I_2, \tag{3.24}
\]
where \( \beta_4 = \mathbb{E}|Y_1|^4 \). For \( I_1 \), by an argument similar to that in (Eq. 64), we have
\[
I_1 \leq C_d \frac{\det(\Sigma_1^{1/2})}{n^{d+1}} \left( \frac{|\Sigma_1^{-1/2} b|^{d/2}}{n} + \frac{1}{n^{d+1}} \right) (\mathbb{E}|Y_1|^4)^{3/2}. \tag{3.25}
\]
By an argument similar to that leading to (Eq. 76), we have
\[
I_2 \leq C_d \frac{\det(\Sigma_1^{1/2})}{n^{d+1}} (\mathbb{E}|Y_1|^4)^{3/2}. \tag{3.26}
\]
Using (3.24)–(3.26), we obtain
\[
I \leq C_d \frac{\det(\Sigma_1^{1/2})}{n^{d+1}} \left( \frac{|\Sigma_1^{-1/2} b|^{d/2}}{n} + \frac{1}{n^{d+1}} \right) (\mathbb{E}|Y_1|^4)^{3/2}. \tag{3.27}
\]
Therefore, by (3.15), (3.23) and (3.27), we have
\[
A_1 \leq C_d \frac{\det(\Sigma_1^{1/2})}{n^{d+1}} \left( \frac{1 + |\Sigma_1^{-1/2} b|}{n^{d+1}} + \frac{|\Sigma_1^{-1/2} b|^{d+1}}{n^2} \right) (\mathbb{E}|Y_1|^4)^{3/2}, \tag{3.28}
\]
and thus we complete the proof of Lemma 3.2.
\[\square\]
Proof of Lemma 2.1. Recall $|Z_x| \leq 3x$. Let $r = \frac{1}{\sqrt{n}}(\sqrt{2h}DZ_x, X_1)$. Then, by (2.5) and Taylor’s expansion,

$$
\bar{\mu}_1 = \frac{\mathbb{E}^Z_x DX_1 e^r}{\mathbb{E}^Z_x e^r} = \frac{\mathbb{E}^Z_x DX_1 \{1 + r + \frac{1}{2}r^2 + R_1\}}{\mathbb{E}^Z_x \{1 + r + R_2\}} = \frac{\mathbb{E}^Z_x \{DX_1 r + \frac{1}{2}DX_1 r^2 + DX_1 R_1\}}{\mathbb{E}^Z_x \{1 + R_2\}},
$$

where $R_1 = \frac{1}{2} \int_0^1 (1-u)^2 r^2 e^r du$, and $R_2 = \int_0^1 (1-u)^2 r^2 e^r du$. We observe that

$$
\mathbb{E}^Z_x(DX_1 r) = D\frac{\sqrt{2h}DZ_x}{\sqrt{n}}, \quad \mathbb{E}^Z_x \left(\frac{1}{2}DX_1 r^2\right) = \frac{1}{2n} D\mathbb{E}^Z_x \{\sqrt{2h}DZ_x, X_1\}^2 X_1\}. \quad (4.1)
$$

Because of the assumption $\mathbb{E}e^{i|x|} \leq c_0 < \infty$ and $|Z_x| \leq 3x$, for $1 < x \leq \varepsilon n^{1/6}$ with a sufficiently small $\varepsilon > 0$, we have, $\mathbb{E}^Z_x(R_2) = O\left(\frac{x^2}{n}\right)$ and each component of $\mathbb{E}^Z_x(X_1 R_1)$ is $O\left(\frac{x^3}{n^{3/2}}\right)$. Thus,

$$
\bar{\mu}_1 = \frac{\sqrt{2h}QZ_x}{\sqrt{n}} + \frac{1}{2n} D\mathbb{E}^Z_x \{\sqrt{2h}DZ_x, X_1\}^2 X_1\} + DV, \quad (4.2)
$$

where each component of the $d$-vector $V$ is $O\left(\frac{x^3}{n^{3/2}}\right)$. Next, for $\tilde{\Sigma}$, by Taylor’s expansion,

$$
\tilde{\Sigma} = \mathbb{E}^Z_x X_1 \tilde{X_1}^T - \bar{\mu}_1 \bar{\mu}_1^T = \frac{D \left(\mathbb{E}^Z_x X_1 X_1^T e^r\right) D}{\mathbb{E}^Z_x e^r} - \bar{\mu}_1 \bar{\mu}_1^T = \frac{D \left(\mathbb{E}^Z_x X_1 X_1^T (1 + r + R_2)\right) D}{\mathbb{E}^Z_x \{1 + r + R_2\}} - \bar{\mu}_1 \bar{\mu}_1^T. \quad (4.3)
$$

Using similar arguments to control error terms as for (4.2), because

$$
\mathbb{E}^Z_x (X_1 X_1^T) = I_d, \quad \mathbb{E}^Z_x \left(X_1 X_1^T e^r\right) = \frac{1}{\sqrt{n}} \mathbb{E}^Z_x \{\sqrt{2h}Z_x, X_1\} X_1 X_1^T, \quad \mathbb{E}^Z_x r = 0,
$$

we have

$$
\tilde{\Sigma} = D \left(I_d + \frac{1}{\sqrt{n}} \mathbb{E}^Z_x \{\sqrt{2h}DZ_x, X_1\} X_1 X_1^T\} + R\right) D, \quad (4.4)
$$

where $R$ is a matrix such that each of its entries is $O\left(\frac{x^2}{n}\right)$ and $DRD$ absorbs $\bar{\mu}_1 \bar{\mu}_1^T$.

From simple calculations similar to those in (4.1)–(4.4), we obtain (2.15) and (2.16).

Let $A = \mathbb{E}^Z_x \{\sqrt{2h}DZ_x, X_1\} X_1 X_1^T$, and let $A_{ij}$ and $R_{ij}$ be the $(i,j)$th element of matrices $A$ and $R$, respectively. Then, from the definition of determinate, $A_{ij} = O(x)$ and $R_{ij} = O\left(\frac{x^2}{n}\right)$, we have

$$
\det \left(D^{-1} \tilde{\Sigma} D^{-1}\right) = \sum_{\sigma} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^d \left(\delta_{i\sigma(i)} + \frac{1}{\sqrt{n}} A_{i\sigma(i)} + R_{i\sigma(i)}\right) = \det \left(I_d + \frac{A}{\sqrt{n}}\right) + O\left(\frac{x^2}{n}\right), \quad (4.5)
$$

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where the sum is over all permutations \( \sigma \) of \( \{1, \ldots, n\} \), \( \text{sgn} \) denotes the sign of a permutation, \( \delta_{ij} = 1 \) for \( i = j \) and \( \delta_{ij} = 0 \) for \( i \neq j \). Moreover, because \( \lambda_j = O(x) \), we have

\[
\det \left( I_d + A \sqrt{n}^{-1} \right) = \prod_{j=1}^{d} \left( 1 + \frac{1}{\sqrt{n}} \lambda_j \right) = 1 + \frac{1}{\sqrt{n}} \sum_{j=1}^{d} \lambda_j + O \left( \frac{x^2}{n} \right). \tag{4.6}
\]

Combining (4.5) and (4.6), we obtain (2.17) for \( \tilde{\Sigma} \).

For small enough \( x^6/n \), the absolute values of eigenvalues of matrices \( A/\sqrt{n} \) and \( R \) are smaller than 1/4, and thus we have

\[
\tilde{\Sigma}^{-1} = D^{-1} \left( I_d - \frac{1}{\sqrt{n}} A + R' \right) D^{-1}, \tag{4.7}
\]

where

\[
R' = \sum_{r=2}^{\infty} (-1)^r \left( \frac{1}{\sqrt{n}} A + R \right)^r - R. \tag{4.8}
\]

For any two vectors \( X \) and \( Y \) with \( |X| = |Y| = 1 \), we have

\[
|X^T R' Y| \leq \left\{ \left( \frac{1}{\sqrt{n}} A + R \right)^2 \right\}^T X \sum_{r=0}^{\infty} \left( \frac{1}{\sqrt{n}} A + R \right)^r Y \right\} + |X^T R Y|
\leq \left\{ \left( \frac{1}{\sqrt{n}} A + R \right)^2 \right\}^T X \sum_{r=0}^{\infty} \left( \frac{1}{2} \right)^r + |X^T R Y|
\leq C \frac{x^2}{n},
\]

which proves (2.18). Finally, (2.20) follows from the definition of eigenvalues.

\[ \square \]

**Proof of Lemma 2.2.** Recalling that \( \tilde{p}(y) = \phi(\tilde{\Sigma}^{-1/2} y)/\sqrt{\det \Sigma} \), by (2.17) and \( \tilde{\lambda}_j = O(x) \), we have, for sufficiently small \( \varepsilon > 0 \) and \( 1 < x \leq \varepsilon n^{1/6} \),

\[
\tilde{p}(y) = \det(Q)^{-1/2} \left[ 1 - \frac{1}{2\sqrt{n}} \sum_{j=1}^{d} \tilde{\lambda}_j + O \left( \frac{x^2}{n} \right) \right] \phi(\tilde{\Sigma}^{-1/2} y)
= \left[ 1 - \frac{1}{2\sqrt{n}} \sum_{j=1}^{d} \tilde{\lambda}_j + O \left( \frac{x^2}{n} \right) \right] \frac{1}{\sqrt{\det Q}(2\pi)^{d/2}} \exp \left\{ -\frac{1}{2} y^T \tilde{\Sigma}^{-1} y \right\}. \tag{4.10}
\]

By (2.18) and (2.13), we have

\[
y^T \tilde{\Sigma}^{-1} y = y^T Q^{-1} y - \frac{1}{\sqrt{n}} \mathbb{E} Z_x (D^{-1} y, X_1)^2 \langle \sqrt{2h} D Z_x, X_1 \rangle + O \left( \frac{x^2 |D^{-1} y|^2}{n} \right), \tag{4.11}
\]

\[
y^T \tilde{\Sigma}^{-1} (\sqrt{n} \mu_1) = \langle \sqrt{2h} Z_x, y \rangle - \frac{1}{2\sqrt{n}} \mathbb{E} Z_x (D^{-1} y, X_1) \langle \sqrt{2h} D Z_x, X_1 \rangle^2 + O \left( \frac{x^3 |D^{-1} y|}{n} \right),
\]

(Two terms of order \( \frac{1}{\sqrt{n}} \) are combined)

\[
(4.12)
\]
\[(\sqrt{n}\bar{\mu}_1)^T\Sigma^{-1}(\sqrt{n}\bar{\mu}_1) = 2h|DZ_x|^2 + O\left(\frac{x^4}{n}\right). \] (Two terms of order \(\frac{1}{\sqrt{n}}\) are cancelled)

(4.13)

From (2.13),
\[\sqrt{n}\bar{\mu}_1 = \sqrt{2hQ_x} + DV', \] where each component of the \(d\)-vector \(V'\) is \(O\left(\frac{x^2}{\sqrt{n}}\right)\).

(4.14)

From (2.18),
\[\Sigma^{-1} = Q^{-1} + D^{-1}R''D^{-1}, \] where each entry of \(d \times d\) matrix \(R''\) is \(O\left(\frac{x}{\sqrt{n}}\right)\).

(4.15)

From (4.14), (4.15) and (2.15), with \(\hat{X}_1 = \bar{X}_1 - \bar{\mu}_1\), we have, by only keeping the main term (recall there will be a factor of \(1/\sqrt{n}\) in front of the second term on the left-hand side of (2.21))

\[
\mathbb{E}^Z_x\left\{3(\Sigma^{-1}\hat{X}_1, \hat{X}_1)(\Sigma^{-1}(y - \sqrt{n}\bar{\mu}_1), \hat{X}_1) - (\Sigma^{-1}(y - \sqrt{n}\bar{\mu}_1), \hat{X}_1)^3\right\}
\]

\[
= \mathbb{E}^Z_x\left\{3\left< D^{-1}(I_d + O\left(\frac{x}{\sqrt{n}}\right))\Sigma^{-1}\left(\hat{X}_1 + D \cdot O\left(\frac{x}{\sqrt{n}}\right)\right), (\hat{X}_1 + D \cdot O\left(\frac{x}{\sqrt{n}}\right)) \right>^3\right\}
\]

\[
- \mathbb{E}^Z_x\left\{3\left< D^{-1}(I_d + O\left(\frac{x}{\sqrt{n}}\right))\Sigma^{-1}\left(\hat{X}_1 + D \cdot O\left(\frac{x}{\sqrt{n}}\right)\right), (\hat{X}_1 + D \cdot O\left(\frac{x}{\sqrt{n}}\right)) \right>^3\right\}
\]

\[
= \mathbb{E}^Z_x\left\{3\langle X_1, X_1 \rangle\langle D^{-1}y - \sqrt{2h}DZ_x, X_1 \rangle - \langle D^{-1}y - \sqrt{2h}DZ_x, X_1 \rangle^3\right\} + O\left(\frac{x^4}{\sqrt{n}} + \frac{x|D^{-1}y|^3}{\sqrt{n}}\right),
\]

(4.16)

where we used (4.14), (4.15) and an abuse of notation (using \(O(\cdot)\) for vectors and matrices to show the magnitude of their entries) in the first equality, and (2.15) and straightforward simplifications of error terms in the second equality. For example, one of the error terms is

\[
\mathbb{E}^Z_x\left\{\langle D^{-1}I_dD^{-1}(y - \sqrt{2h}QZ_x), \hat{X}_1 \rangle^2\langle D^{-1}I_dD^{-1}(y - \sqrt{2h}QZ_x), D \cdot O\left(\frac{x}{\sqrt{n}}\right) \rangle \right\},
\]

which is of the order

\[O(x^3 + |D^{-1}y|^3)\frac{x}{\sqrt{n}}\mathbb{E}^Z_x|D^{-1}\hat{X}_1|^2 = O\left(\frac{x^4 + x|D^{-1}y|^3}{\sqrt{n}}\right), \] (cf. (4.3)&(4.4)).

By (2.7), (2.8) and (4.10), we have

\[\bar{p}(y - \sqrt{n}\bar{\mu}_1) + \frac{1}{6\sqrt{n}}\mathbb{E}^Z_x\left\{\bar{p}''(y - \sqrt{n}\bar{\mu}_1)(\hat{X}_1 - \bar{\mu}_1)^3\right\}
\]

\[= \left[1 - \frac{1}{2\sqrt{n}}\sum_{j=1}^d \lambda_j + O\left(\frac{x^2}{n}\right)\right]\frac{1}{\sqrt{\det Q(2\pi)^{d/2}}}\exp\left\{-\frac{1}{2}(y - \sqrt{n}\bar{\mu}_1)^T(\Sigma^{-1} - (\Sigma^{-1}(y - \sqrt{n}\bar{\mu}_1), \hat{X}_1)^3\right\}
\]

\[\times \left(1 + \frac{1}{6\sqrt{n}}\mathbb{E}^Z_x\left\{3(\Sigma^{-1}\hat{X}_1, \hat{X}_1)(\Sigma^{-1}(y - \sqrt{n}\bar{\mu}_1), \hat{X}_1) - (\Sigma^{-1}(y - \sqrt{n}\bar{\mu}_1), \hat{X}_1)^3\right\}\right). \] (4.17)
Combining (4.11)–(4.13), (4.16) and (4.17), we have
\[
\tilde{p}(y - \sqrt{n}\tilde{\mu}_1) + \frac{1}{6\sqrt{n}} \mathbb{E}^{Z_x} \left\{ \tilde{p}''(y - \sqrt{n}\tilde{\mu}_1)(\tilde{X}_1 - \tilde{\mu}_1)^3 \right\} \\
= \exp\left\{ -h|DZ_x|^2 + \langle \sqrt{2h}Z_x, y \rangle \right\} \phi(D^{-1}y)(\det D)^{-1} \\
\times \left( 1 + B_0 + O\left( \frac{x^2}{n} \right) \right) \exp\left\{ B_1 + O\left( \frac{x^4}{n} + \frac{x^2|D^{-1}y|^2}{n} \right) \right\} \left( 1 + B_2 + O\left( \frac{x^4}{n} + \frac{x|D^{-1}y|^3}{n} \right) \right) \\
= H_1(y) + H_2(y), 
\] (4.18)
where \( H_1(y) \) is defined in (2.22) and
\[
H_2(y) = \exp\left\{ -h|DZ_x|^2 + \langle \sqrt{2h}Z_x, y \rangle \right\} \phi(D^{-1}y)(\det D)^{-1} \\
\times \left( 1 + B_0 + O\left( \frac{x^2}{n} \right) \right) \exp\left\{ B_1 + O\left( \frac{x^4}{n} + \frac{x^2|D^{-1}y|^2}{n} \right) \right\} - B_1 - 1 + O\left( \frac{x^4}{n} + \frac{x^2|D^{-1}y|^2}{n} \right) \\
\times \left( 1 + B_2 + O\left( \frac{x^4}{n} + \frac{x|D^{-1}y|^3}{n} \right) \right). 
\] (4.19)

Next, consider \( H_2(y) \). By \( B_1 = O\left( \frac{|x|D^{-1}y|^2 + x^2|D^{-1}y|^2}{\sqrt{n}} \right) \), the elementary inequality \(|e^x - 1 - x| \leq \frac{1}{2}x^2e^{|x|} \), and recalling that \( 1 < x \leq \varepsilon n^{1/6} \), we have
\[
\left| \exp\left\{ B_1 + O\left( \frac{x^4}{n} + \frac{x^2|D^{-1}y|^2}{n} \right) \right\} - B_1 - O\left( \frac{x^4}{n} + \frac{x^2|D^{-1}y|^2}{n} \right) - 1 \right| \\
\leq \frac{1}{2} \left( B_1 + O\left( \frac{x^4}{n} + \frac{x^2|D^{-1}y|^2}{n} \right) \right)^2 \exp\left\{ B_1 + O\left( \frac{x^4}{n} + \frac{x^2|D^{-1}y|^2}{n} \right) \right\} \right| \\
\leq C\left( \frac{x^2|D^{-1}y|^4}{n} + \frac{x^4|D^{-1}y|^2}{n} + \frac{x^2}{n} \right) \exp\left\{ C\left( \frac{|x|D^{-1}y|^2 + x^2|D^{-1}y|^2}{\sqrt{n}} \right) \right\} 
\] (4.20)
for some positive constant \( C \). By (4.19) and (4.20), we complete the proof of (2.23).

\[ \square \]

**Proof of Lemma 2.3.** Because of the assumption of the finiteness of the moment generating function of \( X_1 \) near 0, the function \( \mathbb{E}e^{\langle a,X_1 \rangle} \) is finite for all \( a \in \mathbb{R}^d \) with \( |a| \leq \varepsilon \) for a sufficiently small \( \varepsilon > 0 \). Recall \( \tilde{G}(b) = \mathbb{E}e^{\langle b,X_1 \rangle} \) and we have
\[
\mathbb{E}e^{\langle a,W \rangle} = \tilde{G}^n\left( \frac{a}{\sqrt{n}} \right). 
\] (4.21)

Furthermore,
\[
\tilde{G}^n\left( \frac{a}{\sqrt{n}} \right) - \exp\left\{ \frac{|a|^2}{2} + \frac{1}{6\sqrt{n}} \mathbb{E}\langle a, X_1 \rangle^3 \right\} \\
= \exp\left\{ \frac{|a|^2}{2} + \frac{1}{6\sqrt{n}} \mathbb{E}\langle a, X_1 \rangle^3 \right\} \\
\times \left( \exp\left\{ n \left( \log \tilde{G}\left( \frac{a}{\sqrt{n}} \right) - \frac{|a|^2}{2n} - \frac{\mathbb{E}\langle a, X_1 \rangle^3}{6n^{5/2}} \right) \right\} - 1 \right) . 
\] (4.22)
By Taylor’s expansion and using $\mathbb{E}X_1 = 0$ and $\text{Cov}(X_1) = I_d$,

$$\log \hat{G}\left(\frac{a}{\sqrt{n}}\right) - \frac{|a|^2}{2n} - \frac{\mathbb{E}(a, X_1)^3}{6n^{3/2}} = \frac{1}{6} \int_0^1 (1 - u)^3 \left(\frac{d^4}{du^4} \log \hat{G}\left(\frac{ua}{\sqrt{n}}\right)\right) du. \quad (4.23)$$

To bound the integration on the right-hand side of (4.23), we need the following lemma:

**Lemma 4.1.** For $a \in \mathbb{R}^d$ such that $|a| \leq t_0 \sqrt{n}/2$ and $\left|\hat{G}\left(\frac{ua}{\sqrt{n}}\right)\right| \geq \frac{1}{2}, \forall u \in [0, 1]$, we have for $u \in [0, 1]$,

$$\left|\frac{d^4}{du^4} \log \hat{G}\left(\frac{ua}{\sqrt{n}}\right)\right| \leq \frac{C}{n^2} (|a|^4), \quad (4.24)$$

where $C$ is a constant depending only on $d, c_0$, and $t_0$ in Theorem 1.1.

**Proof.**

$$\frac{d^4}{du^4} \log \hat{G}\left(\frac{ua}{\sqrt{n}}\right) = \sum c (\{\beta_1, \ldots, \beta_j\}) \frac{d^4}{du^4} \hat{G}\left(\frac{ua}{\sqrt{n}}\right) \frac{d^4}{du^4} \hat{G}\left(\frac{ua}{\sqrt{n}}\right) \hat{G}\left(\frac{ua}{\sqrt{n}}\right)^3, \quad (4.25)$$

where the summation is over all collections of nonnegative integers $\{\beta_1, \ldots, \beta_j\}$ satisfying $\beta_1 + \cdots + \beta_j = 4, \quad 1 \leq j \leq 4,$

and the constant $c (\{\beta_1, \ldots, \beta_j\})$ depends only on the collection $\{\beta_1, \ldots, \beta_j\}$. For any nonnegative integer $\beta \leq 4$, we have

$$\left|\frac{d^\beta}{dw^\beta} \hat{G}\left(\frac{ua}{\sqrt{n}}\right)\right| = \left|\frac{1}{\sqrt{n}} \mathbb{E}\left(\langle a, X_1\rangle^\beta e^{\frac{u}{\sqrt{n}} \langle a, X_1\rangle}\right)\right| \leq \frac{1}{\sqrt{n}} \mathbb{E}\left(|\langle a, X_1\rangle|^\beta e^{\frac{|a|}{\sqrt{n}} |X_1|}\right). \quad (4.26)$$

Therefore, for $|a| \leq t_0 \sqrt{n}/2$ and $u \in [0, 1]$, we have that (4.26) can be bounded by $\frac{|a|^\beta}{(\sqrt{n})^\beta}$ multiplied by a constant depending only on $d, c_0$, and $t_0$ in Theorem 1.1. Combining (4.25), (4.26), and the condition $\left|\hat{G}\left(\frac{ua}{\sqrt{n}}\right)\right| \geq \frac{1}{2}, \forall u \in [0, 1]$ we complete the proof. \qed

By Taylor’s expansion, we have, $\forall u \in [0, 1]$,

$$\left|\hat{G}\left(\frac{ua}{\sqrt{n}}\right) - 1\right| = \left|\mathbb{E} \int_0^1 \frac{1}{\sqrt{n}} \langle ua, X_1\rangle \exp\left\{\frac{v}{\sqrt{n}} \langle ua, X_1\rangle\right\} dv\right| \leq \frac{|a|}{\sqrt{n}} \mathbb{E} |X_1| e^{\frac{|a|}{\sqrt{n}} |X_1|}. \quad (4.27)$$

Therefore, there exists a constant $\varepsilon > 0$ such that for $|a| \leq \varepsilon \sqrt{n}$, (4.27) is less than 1/2 and

$$\left|\hat{G}\left(\frac{ua}{\sqrt{n}}\right)\right| \geq \frac{1}{2}, \forall u \in [0, 1]. \quad (4.28)$$

By (4.23), (4.28), and Lemma 4.1, we have

$$\left|\log \hat{G}\left(\frac{a}{\sqrt{n}}\right) - \frac{|a|^2}{2n} - \frac{\mathbb{E}(a, X_1)^3}{6n^{3/2}}\right| \leq \frac{C}{n^2} |a|^4. \quad (4.29)$$
For the second factor on the right hand side of (4.22), from the elementary inequality
\[ |\exp(x) - 1| \leq |x| \exp(|x|) \]  
and (4.29), we have
\[ \left| \exp\left\{ n \left( \log \hat{G}\left( \frac{a}{\sqrt{n}} \right) - \frac{|a|^2}{2n} - \frac{\mathbb{E}(a, X_1)^3}{6n^{3/2}} \right) \right\} - 1 \right| \leq \frac{C}{n} |a|^4 \exp\left\{ \frac{C}{n} |a|^4 \right\}. \]  
(4.31)

From (4.22) and (4.31), we have
\[ \left| \hat{G}^n\left( \frac{a}{\sqrt{n}} \right) - \exp\left\{ \frac{|a|^2}{2} + \frac{\mathbb{E}(a, X_1)^3}{6\sqrt{n}} \right\} \right| \leq \frac{C}{n} |a|^4 \exp\left\{ \frac{C}{n} |a|^4 + \frac{|a|^2}{2} + \frac{C|a|^3}{6\sqrt{n}} \right\}. \]  
(4.32)

Next, we give the following bound. By Taylor’s expansion
\[
\begin{align*}
&\left| \exp\left\{ \frac{|a|^2}{2} + \frac{\mathbb{E}(a, X_1)^3}{6\sqrt{n}} \right\} - \exp\left\{ \frac{|a|^2}{2} \left( 1 + \frac{\mathbb{E}(a, X_1)^3}{6\sqrt{n}} \right) \right\} \right|
= \exp\left\{ \frac{|a|^2}{2} \right\} \left| \exp\left\{ \frac{\mathbb{E}(a, X_1)^3}{6\sqrt{n}} \right\} - 1 - \frac{\mathbb{E}(a, X_1)^3}{6\sqrt{n}} \right|
= \exp\left\{ \frac{|a|^2}{2} \right\} \left| \int_0^1 (1 - u) \exp\left\{ \frac{u\mathbb{E}(a, X_1)^3}{6\sqrt{n}} \right\} \left( \frac{\mathbb{E}(a, X_1)^3}{6\sqrt{n}} \right)^2 \, du \right|
\leq \frac{C}{n} |a|^6 \exp\left\{ \frac{|a|^2}{2} + \frac{C}{\sqrt{n}} |a|^3 \right\}. 
\end{align*}
\]  
(4.33)

Combining (4.32) and (4.33), we have for $|a| \leq \varepsilon \sqrt{n}$,
\[ \left| \hat{G}^n\left( \frac{a}{\sqrt{n}} \right) - \exp\left\{ \frac{|a|^2}{2} \left( 1 + \frac{\mathbb{E}(a, X_1)^3}{6\sqrt{n}} \right) \right\} \right| \leq \left( \frac{C}{n} |a|^4 + \frac{C}{n} |a|^6 \right) \exp\left\{ \frac{|a|^2}{2} + \frac{C}{\sqrt{n}} |a|^3 + \frac{C}{n} |a|^4 \right\}, \]  
(4.34)

which is the required result. \( \square \)

**Proof of Lemma 2.4.** The case $d = 1$ follows from the integration by parts formula and the asymptotic tail probability of the $\chi_1$ distribution. In the following, we consider the case $d \geq 2$.

First, we have
\[
\int_{|Dy| > x} |y|^r \exp\left\{ -\frac{|y|^2}{2} + C \frac{x|y|^2}{\sqrt{n}} \right\} \, dy
= \int_{y \in \mathbb{R}^d} 1_{\{ |Dy| > x \}} |y|^r \exp\left\{ -\frac{|y|^2}{2} + C \frac{x|y|^2}{\sqrt{n}} \right\} \, dy
= \int_{u \geq 0} \int_{y \in \partial B(0, u)} 1_{\{ |Dy| > x \}} u^r \exp\left\{ -\frac{u^2}{2} + C \frac{xu^2}{\sqrt{n}} \right\} \, dS \, du
= \int_{u \geq 0} S(\{ |Dy| > x \} \cap \{ y \in \partial B(0, u) \}) u^r \exp\left\{ -\frac{u^2}{2} + C \frac{xu^2}{\sqrt{n}} \right\} \, du, \]  
(4.35)
where $S(\cdot)$ is the Lebesgue measure of $(d-1)$-dimensional surface. Let

$$\xi(a) = \frac{S(\{|Dy| > x\} \cap \{y \in \partial B(0,ax)\})}{S(\{y \in \partial B(0,ax)\})}, \quad a > 0. \quad (4.36)$$

We can easily verify that $\xi(a)$ does not depend on $x$ and $\xi(a)$ is a continuous and increasing function such that

$$\xi(a) = 0 \quad \text{for} \quad 0 < a \leq 1, \quad \text{and} \quad \xi(a) = 1 \quad \text{for} \quad a \geq \frac{1}{q_d}x, \quad (4.37)$$

where $q_d$ is the smallest eigenvalue of $Q$. Let $y_1$ be the first component of a vector $y \in \mathbb{R}^d$. There exists an absolute constant $\delta > 0$ (in particular, it does not depend on $Q$) such that

$$\frac{S(\{|y_1 > x\} \cap \{y \in \partial B(0,(1+\delta)x)\})}{S(\{y \in \partial B(0,(1+\delta)x)\})} = \frac{1}{16}. \quad (4.38)$$

Because $\{|Dy| > x\} \supset \{|y_1| > x\}$ (recall the largest eigenvalue of $Q$ is 1), we then have

$$\xi(1+\delta) = \frac{S(\{|Dy| > x\} \cap \{y \in \partial B(0,(1+\delta)x)\})}{S(\{y \in \partial B(0,(1+\delta)x)\})} \geq \frac{1}{8}, \quad (4.39)$$

We now return to (4.35). By (4.35) and (4.36), we observe that

$$\int_{|Dy| > x} |y|^r \exp\left\{-\frac{|y|^2}{2} + C \frac{x|y|^2}{\sqrt{n}}\right\} dy = \frac{2\pi^d/2}{\Gamma(d/2)} \int_{u > x} \xi\left(\frac{u}{x}\right) u^{r+d-1} \exp\left\{-\frac{u^2}{2} + C \frac{xu^2}{\sqrt{n}}\right\} du, \quad (4.40)$$

where we use the fact that the surface area of the $d$-dimensional unit ball is $2\pi^d/2\Gamma(d/2)$. Next, we deal with the integration on the right-hand side of (4.40). By a change of variable and the integration by parts formula, we have, choosing $\varepsilon > 0$ to be sufficiently small such that $1 - \frac{2C\varepsilon}{\sqrt{n}} > \frac{1}{2}$,

$$\int_{a > 1+\delta} \xi(a) (ax)^{r+d-1} \exp\left\{-\frac{(ax)^2}{2} + C \frac{x(ax)^2}{\sqrt{n}}\right\} x da = \int_{a > 1+\delta} \xi(a) (ax)^{r+d-1} \exp\left\{-\frac{(ax)^2}{2} + C \frac{x(ax)^2}{\sqrt{n}}\right\} x da,$$

$$= -\left(1 - \frac{2Cx}{\sqrt{n}}\right)^{-1} \xi(a) (ax)^{r+d-2} \exp\left\{-\frac{(ax)^2}{2} + C \frac{x(ax)^2}{\sqrt{n}}\right\} x da \bigg|_{1+\delta}$$

$$+ \left(1 - \frac{2Cx}{\sqrt{n}}\right)^{-1} \int_{a > 1+\delta} (ax)^{r+d-2} \exp\left\{-\frac{(ax)^2}{2} + C \frac{x(ax)^2}{\sqrt{n}}\right\} d\xi(a)$$

$$+(r+2)(1 - \frac{2Cx}{\sqrt{n}})^{-1} \int_{a > 1+\delta} \xi(a) (ax)^{r+d-3} \exp\left\{-\frac{(ax)^2}{2} + C \frac{x(ax)^2}{\sqrt{n}}\right\} x da$$

$$:= J_1 + J_2 + J_3. \quad (4.41)$$
Recalling that $\xi(a)$ is increasing, $1/8 \leq \xi(a) \leq 1$ for $a \geq 1 + \delta$, and $1 - \frac{2Cx}{\sqrt{n}} > \frac{1}{2}$, we have

$$J_1 + J_2 \leq C\xi(1 + \delta)((1 + \delta)x)^{r + d - 2} \exp\left\{-\frac{(1 + \delta)x^2}{2} + C\frac{x}{\sqrt{n}}((1 + \delta)x)^2\right\}$$

$$\quad + C(r)((1 + \delta)x)^{r + d - 2} \exp\left\{-\frac{(1 + \delta)x^2}{2} + C\frac{x}{\sqrt{n}}((1 + \delta)x)^2\right\} \int_{a>1+\delta} d\xi(a)$$

$$\leq C(r)x^r x^{d - 2} \exp\left\{-\frac{(1 + \delta)x^2}{2}\right\}. \quad (4.42)$$

Repeating (4.41) with $C = 0$ and $r = 0$, we have

$$\int_{u>(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{d-1} \exp\left\{-\frac{u^2}{2}\right\} du$$

$$= \xi(1+\delta)((1+\delta)x)^{d-2} \exp\left\{-\frac{(1+\delta)x^2}{2}\right\}$$

$$\quad + (d - 2) \int_{a>1+\delta} \xi(a)(ax)^{d-3} \exp\left\{-\frac{(ax)^2}{2}\right\} d\xi(a)$$

$$\quad + \int_{a>1+\delta} (ax)^{d-2} \exp\left\{-\frac{(ax)^2}{2}\right\} d\xi(a)$$

$$\geq \frac{1}{8} x^{d - 2} \exp\left\{-\frac{(1 + \delta)x^2}{2}\right\}, \quad (4.43)$$

where we used the fact that the last two integrations in (4.43) are positive and $\xi(1 + \delta) \geq \frac{1}{8}$ (cf. (4.39)). By (4.42) and (4.43), we have

$$J_1 + J_2 \leq C(r)x^r \int_{u>(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{d-1} \exp\left\{-\frac{u^2}{2}\right\} du. \quad (4.44)$$

Combining (4.41) and (4.44), we obtain

$$\int_{u>(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{r+d-1} \exp\left\{-\frac{u^2}{2} + C\frac{xu^2}{\sqrt{n}}\right\} du$$

$$\leq C(r)x^r \int_{u>(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{d-1} \exp\left\{-\frac{u^2}{2}\right\} du$$

$$\quad + (r + d - 2) \left(1 - \frac{2Cx}{\sqrt{n}}\right)^{-1} \int_{u>(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{r+d-3} \exp\left\{-\frac{u^2}{2} + C\frac{xu^2}{\sqrt{n}}\right\} du. \quad (4.45)$$

If $r - 2 \geq 2 - d$, we can apply (4.45) to the last integration. Performing this procedure $p$
times, where \( p \) is the smallest integer that is greater than or equal to \( \frac{r+d}{2} - 1 \), we have

\[
\int_{u>(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{r+d-1} \exp\left\{-\frac{u^2}{2} + C\frac{ux^2}{\sqrt{n}}\right\} du \\
\leq C(r) \left(\sum_{i=0}^{p} x^{r-2i}\right) \int_{u>(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{d-1} \exp\left\{-\frac{u^2}{2}\right\} du \\
+ \left(1 - \frac{2Cx}{\sqrt{n}}\right)^{p-1} \left(\prod_{i=1}^{p+1} (r + d - 2i)\right) \int_{u>(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{r+d-3-2p} \exp\left\{-\frac{u^2}{2} + C\frac{ux^2}{\sqrt{n}}\right\} du.
\]

(4.46)

Because the last term is \( \leq 0 \), we then have

\[
\int_{u>(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{r+d-1} \exp\left\{-\frac{u^2}{2} + C\frac{ux^2}{\sqrt{n}}\right\} du \\
\leq C(r) \left(\sum_{i=0}^{p} x^{r-2i}\right) \int_{u>(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{d-1} \exp\left\{-\frac{u^2}{2}\right\} du \\
\leq C(r) x^r \int_{u>(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{d-1} \exp\left\{-\frac{u^2}{2}\right\} du,
\]

(4.47)

where the last inequality follows from \( x > 1 \). We can easily verify that

\[
\int_{x<u\leq(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{r+d-1} \exp\left\{-\frac{u^2}{2} + C\frac{ux^2}{\sqrt{n}}\right\} du \\
\leq C(r) x^r \int_{x<u\leq(1+\delta)x} \xi\left(\frac{u}{x}\right) u^{d-1} \exp\left\{-\frac{u^2}{2}\right\} du.
\]

(4.48)

By (4.40), (4.47) and (4.48), we have

\[
\int_{|Dy|>x} |y|^r \exp\left\{-\frac{|y|^2}{2} + C\frac{|y|^2}{\sqrt{n}}\right\} dy \\
\leq C(r) x^r \int_{|Dy|>x} \xi\left(\frac{u}{x}\right) u^{d-1} \exp\left\{-\frac{u^2}{2}\right\} du \\
= C(r) x^r \int_{|Dy|>x} \exp\left\{-\frac{|y|^2}{2}\right\} dy \\
= C(r) x^r \mathbb{P}(|DZ| > x).
\]

(4.49)

This proves Lemma 2.4.

\[\square\]

**Proof of Lemma 2.5.** In this proof, we denote by \( C_d \) positive constants that depend only on \( d \). They may differ in different expressions. All of the chi-square random variables below are assumed to be independent. Because \( x > 1 \), we can verify that \( \lambda_{p-1} \geq 1/2 \). Because \( \lambda_i \) decreases with respect to \( i \), we have

\[
\mathbb{P}\left(\sum_{i=1}^{s} \lambda_i \chi_{v_i}^2 \geq x^2\right) \geq \mathbb{P}\left(\lambda_{p-1} \chi_{v_{p-1}}^2 \geq x^2 + \sum_{i=p}^{s} \lambda_i \chi_{v_i}^2 \geq x^2\right).
\]

(4.50)
For any positive integer \( v \leq d \), from chi-square tail probabilities, there exists a positive constant \( C_d \) depending only on \( d \) such that

\[
\Pr \left( \chi_v^2 \geq a^2 \right) \geq C_d a^{v-2} e^{-\frac{a^2}{2}} \quad \text{for all } a^2 \geq \frac{1}{2s-p+2}. \quad (4.51)
\]

From the definition of \( r \) and (4.51), we have

\[
\Pr \left( \lambda_{p-1} \chi_{p-1}^2 \sum_{j=1}^p v_j \geq a^2 \right) = \Pr \left( \lambda_{p-1} \chi_r^2 \geq a^2 \right)
\geq C_d \left( \frac{a}{\lambda_{p-1}} \right)^{r-2} e^{-\frac{a^2}{2\lambda_{p-1}}} \geq C_d a^{r-2} e^{-\frac{a^2}{2\lambda_{p-1}}} \quad \text{for all } a^2 \geq \frac{1}{2s-p+2}, \quad (4.52)
\]

where in the last inequality we used the fact that \( 1/2 \leq \lambda_{p-1} \leq 1 \).

If \( p = s + 1 \), then (2.57) follows from (4.50), (4.52) with \( a = x \), and \( \left( \frac{1}{\lambda_{p-1}} - 1 \right) x^2 \leq 1 \).

Suppose now that \( p \leq s \). We let \( Y = \lambda_{p-1} \chi_r^2 \) and for any positive integer \( v \), let \( f_v(\cdot) \) be the density of \( \chi_v^2 \). Then, for \( x^2/2^{s-p+1} \leq a^2 \leq x^2 \), we have

\[
\Pr \left( Y + \lambda_p \chi_{vp}^2 \geq a^2 \right) \geq \frac{1}{\lambda_p} \int_0^{a^2/2} f_{vp} \left( \frac{y}{\lambda_p} \right) \Pr \left( Y \geq a^2 - y \right) dy. \quad (4.53)
\]

In the above integration, \( y \in [0, a^2/2] \), and thus \( a^2 \geq a^2 - y \geq a^2/2 \). Furthermore, because \( x^2/2^{s-p+1} \leq a^2 \leq x^2 \), we have

\[
\frac{x^2}{2^{s-p+2}} \leq a^2 - y \leq a^2, \quad (4.54)
\]

and we can apply (4.52) to \( \Pr(Y \geq a^2 - y) \). Plugging (4.52) into (4.53) yields

\[
\Pr \left( Y + \lambda_p \chi_{vp}^2 \geq a^2 \right) \geq C_d \lambda_p^{-vp} \int_0^{a^2/2} \left( \frac{y}{\lambda_p} \right)^{vp-1} e^{-\frac{y}{2\lambda_p} (a^2 - y)^{\frac{v}{2}} - 1} e^{-\frac{a^2}{2}} \frac{dy}{2\lambda_{p-1}}
\geq C_d \lambda_p^{-vp} \int_0^{a^2/2} y^{vp-1} e^{-\frac{y}{2} \left( \frac{1}{\lambda_p} - 1 \right)} dy a^{r-2} e^{-\frac{a^2}{2\lambda_{p-1}}} \quad (4.55)
\geq C_d \lambda_p^{-vp} \int_0^{a^2/2} y^{vp-1} e^{-\frac{y}{2} \left( \frac{1}{\lambda_p} - 1 \right)} dy a^{r-2} e^{-\frac{a^2}{2}} \left( \frac{1}{\lambda_p} - 1 \right).
\]

By a change of variable, \( \frac{1}{\lambda_p} - 1 \leq \frac{1}{x} \leq \frac{1}{a^2} \) and \( \frac{1}{\lambda_p} - 1 > \frac{1}{x^2} \geq \frac{1}{a^2} \), we have (4.55) is greater than or equal to

\[
C_d \lambda_p^{-vp} \left( \frac{1}{\lambda_p} - 1 \right)^{-\frac{vp}{2}} \int_0^{a^2/2} y^{vp-1} e^{-\frac{y}{2} \left( \frac{1}{\lambda_p} - 1 \right)} dy a^{r-2} e^{-\frac{a^2}{2}} \quad (4.56)
\]

\[
\geq C_d \lambda_p^{-vp} \left( \frac{1}{\lambda_p} - 1 \right)^{-\frac{vp}{2}} \int_0^{1/2^{d+1}} y^{vp-1} e^{-\frac{y}{2} \left( \frac{1}{\lambda_p} - 1 \right)} dy a^{r-2} e^{-\frac{a^2}{2}}
\geq C_d (1 - \lambda_p)^{-\frac{vp}{2}} a^{r-2} e^{-\frac{a^2}{2}}.
\]

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Repeating procedures (4.52)–(4.56) $s - p + 1$ times for the right-hand side of (4.50), we have, for $a^2 \in [x^2/2, x^2]$,
\[
P \left( \sum_{i=1}^{s} \lambda_i X_{v_i}^2 \geq a^2 \right) \geq C_d \left[ \prod_{i=p}^{1} (1 - \lambda_i)^{-\frac{v_i}{2}} \right] a^{-2} e^{-\frac{a^2}{2}}.
\] (4.57)

Taking $a = x$ yields the desired result.

A Appendix

Proof of (1.4). The result for bounded $x$ follows immediately from (1.1) and (1.2). In the following, we assume $x > 1$. We use $\delta, \epsilon$, and $\varepsilon$ to denote unspecified positive constants, which do not depend on $n$ and $x$. By von Bahr (1967, Theorem 3), we have, for some positive constant $\delta > 0$ and $1 < x \leq \delta \sqrt{n}$,
\[
P(|W| > x) = (2\pi)^{-d/2} \int_{u \in \Omega_0} \exp \left( n \sum_{v=3}^{\infty} \left( \frac{x}{\sqrt{n}} \right)^v Q_v(u) \right) dS
\]
\[
\times \int_x^{\infty} e^{-y^2/2} y^{-1} dy \left( 1 + O \left( \frac{x}{\sqrt{n}} \right) \right),
\] (A.1)

where $dS$ is the surface measure of $\Omega_0 = \{ u \in \mathbb{R}^d : |u| = 1 \}$ and for each $v \geq 3$, $Q_v : \mathbb{R}^d \to \mathbb{R}$ is a homogeneous polynomial of degree $v$ whose coefficients depend on the mixed cumulants up to order $v$ of $X_1$. For example, $Q_3(u) = \frac{1}{6} \sum_{j,k,l=1}^{d} \mathbb{E}[X_{1j}X_{1k}X_{1l}] u_j u_k u_l$, where $j, k, l$ are the indices of vector components. Moreover, $\sum_{v=3}^{\infty} Q_v(u)$ is convergent for $|u| \leq \epsilon$, where $\epsilon$ is a positive constant.

In the remainder of the proof, assume that $x \leq en^{1/4}$, which can be achieved by choosing the positive constant $\varepsilon$ in the range of $1 < x \leq \varepsilon n^{1/6}$ to be sufficiently small. Because $Q_v$ is a polynomial of degree $v$, we have
\[
n \sum_{v=8}^{\infty} \left( \frac{x}{\sqrt{n}} \right)^v Q_v(u) = n \sum_{v=8}^{\infty} Q_v \left( u \frac{x}{\sqrt{n}} \right)
\]
\[
= n \sum_{v=8}^{\infty} Q_v \left( u \frac{x}{n^{1/4}} \right) \left( \frac{1}{n^{1/4}} \right)^v = \sum_{v=8}^{\infty} n^{1-v/4} Q_v \left( u \frac{x}{n^{1/4}} \right) = O \left( \frac{1}{n} \right),
\]
where in the last step, we used the Dirichlet condition for the convergence of series and the fact that $\sum_{v=3}^{\infty} Q_v(u)$ is convergent for $|u| \leq \epsilon$. Therefore,
\[
\exp \left( n \sum_{v=3}^{\infty} \left( \frac{x}{\sqrt{n}} \right)^v Q_v(u) \right) = \exp \left( n \sum_{v=3}^{7} \left( \frac{x}{\sqrt{n}} \right)^v Q_v(u) + n \sum_{v=8}^{\infty} \left( \frac{x}{\sqrt{n}} \right)^v Q_v(u) \right)
\]
\[
= \exp \left( n \left( \frac{x}{\sqrt{n}} \right)^3 Q_3(u) \right) \left( 1 + O \left( \frac{x^4}{n} \right) \right).
\] (A.2)
From (A.1) and (A.2), we have, for $1 < x \leq \min \{ \delta \sqrt{n}, \epsilon n^{1/4} \}$,

\[
\mathbb{P}(|W| > x) = (2\pi)^{-d/2} \int_{u \in \Omega_0} \exp \left( n \left( \frac{x}{\sqrt{n}} \right)^3 Q_3(u) \right) dS \times \int_x^{\infty} e^{-y^2/2} y^{d-1} dy \left( 1 + O \left( \frac{x}{\sqrt{n}} + \frac{x^4}{n} \right) \right).
\]

Therefore, for $1 < x \leq \epsilon n^{1/6}$ for a sufficiently small $\epsilon > 0$,

\[
\mathbb{P}(|W| > x) = (2\pi)^{-d/2} \int_{u \in \Omega_0} \left( 1 + \frac{x}{\sqrt{n}} Q_3(u) + O \left( \frac{x^6}{n} \right) \right) dS \times \int_x^{\infty} e^{-y^2/2} y^{d-1} dy \left( 1 + O \left( \frac{x}{\sqrt{n}} + \frac{x^4}{n} \right) \right).
\]

By symmetry, because $Q_3$ is a polynomial of degree 3,

\[
\int_{u \in \Omega_0} \frac{x}{\sqrt{n}} Q_3(u) dS = 0.
\]

This result, together with the fact that

\[
(2\pi)^{-d/2} \int_{u \in \Omega_0} \left( \frac{x^3}{\sqrt{n}} Q_3(u) \right) dS \int_x^{\infty} e^{-y^2/2} y^{d-1} dy = \mathbb{P}(|Z| > x), \tag{A.3}
\]

proves (1.4).

\[\square\]

**Proof of (1.5).** From (A.1) and (A.2), for $1 < x_n \leq \min \{ \delta \sqrt{n}, \epsilon n^{1/4} \}$ for a sufficiently small constant $\delta > 0$, we have

\[
\mathbb{P}(|W| > x_n) = (2\pi)^{-d/2} \int_{u \in \Omega_0} \exp \left( n \left( \frac{x_n}{\sqrt{n}} \right)^3 Q_3(u) \right) \left( 1 + O \left( \frac{x^4}{n} \right) \right) dS
\]

\[
\times \int_{x_n}^{\infty} e^{-y^2/2} y^{d-1} dy \left( 1 + O \left( \frac{x}{\sqrt{n}} + \frac{x^4}{n} \right) \right). \tag{A.4}
\]

For $x_n = cn^{1/6} \ll n^{1/4}$, from (A.4), we have

\[
\mathbb{P}(|W| > x_n) = (2\pi)^{-d/2} \int_{u \in \Omega_0} \exp \left( n \left( \frac{x_n}{\sqrt{n}} \right)^3 Q_3(u) \right) dS \int_{x_n}^{\infty} e^{-y^2/2} y^{d-1} dy(1 + o(1)). \tag{A.5}
\]

Recall $Q_3(u) = \frac{1}{6} \sum_{j,k,l=1}^d \mathbb{E}[X_{1j} X_{1k} X_{1l}] u_j u_k u_l$. If the mixed third cumulants of $X_1$ are not all zero, then $Q_3(u)$ is a non-zero function. Moreover, $Q_3(u) = -Q_3(-u)$, and thus $\int_{u \in \Omega_0} Q_3(u) dS = 0$. This implies

\[
\int_{u \in \Omega_0} \exp \left( n \left( \frac{x_n}{\sqrt{n}} \right)^3 Q_3(u) \right) dS = \int_{u \in \Omega_0} \exp (e^3 Q_3(u)) dS > \int_{u \in \Omega_0} dS,
\]

which, together with (A.5) and (A.3), proves (1.5).
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