Toroidal & orbifold compactifications at large D

and D-duality

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Abstract

In this paper I will further investigate the spectrum of quantum gravity or string theories at large number of dimensions. We will see that volumes of certain orbifolds shrink at large D. It follows that the mass spectra of the leading Kaluza-Klein towers and also of wrapped brane states on these orbifolds possess a non-trivial dependence on D: KK modes become heavy at large D, whereas wrapped branes become light. This observation can be used either to apply the Large Dimension Conjecture or, as we will do, to investigate the possibility of a D-duality symmetry, which relates in gravity compactifications of different dimensions. We will set up the general rules for D-duality in higher dimensional gravity. However due to existence of the critical dimensions, D-duality is quite restricted in string theory. As simple tests for D-duality in string theory, we will discuss the duality between M-theory on a two-dimensional orbifold, namely the Möbius strip, and an IIB S-fold on a circle, which corresponds to the heterotic CHL string, as well the duality between a (truncated) 12-dimensional theory on a three-dimensional orbifold and another IIB S-fold compactification. Finally I also comment on the possible existence of exotic theories at large D.
I. INTRODUCTION

String theory provides an explicit (and possibly unique) realization for a theory of quantum gravity and the existence of a critical dimension $D = 10(26)$ is an essential and char-
acteristic feature of the (bosonic) super string theory \[1-4\]. The requirement for a critical dimensions originates from the consistency of the underlying world-sheet conformal field theory as well as from Lorentz invariance of the target space theory. In addition it is known \[10, 11\] for many years that local supersymmetry in flat space-time can be only realized in supergravity theory with maximal number of dimensions \(D = 11\). Nevertheless, besides these arguments from string theory or from supersymmetry, it would be very interesting to get bounds on the number of space-time dimensions from general considerations in quantum gravity.

On the other hand, the concept of dimension of space-time appears not completely well-defined in string theory or in quantum gravity. In particular it is known that gravitational theories in different numbers of space-time dimensions can be related by duality symmetries. E.g. the type II A superstring in 10 space-time dimensions theory was shown to be equivalent to 11-dimensional M-theory \[12\] and 10-dimensional IIB superstring theory can be formulated in terms of F-theory \[13\], which possesses several features of a 12-dimensional theory. In addition via the AdS/CFT correspondence \[14\] it was realized that a quantum gravity theory in \(d+1\) dimensions can be equivalently described by a \(d\)-dimensional quantum field theory without gravitational degrees of freedom.

Another interesting aspect, relevant for the question about the number of space-time dimensions, is the possibility of emergent space-time and emergent geometry. This question can be nicely addressed within the socalled swampland scenario \[15-18\]. In quantum gravity or in string theory, objects revealing the geometric nature of the theory, or its stringy nature, can become part of the low-energy modes, as happens for instance in certain infinite distance limits in field space, like the decompactification limit \[19, 20\] or the tensionless string limit \[21\]. It is for example well-known that the presence of towers of Kaluza Klein modes in an effective field theory is a clear signal for the emergence of an additional compact geometry. It has been also argued in the context of the AdS distance conjecture \[22\] that quantum gravity in pure AdS space-time belongs to the swampland, which means that in the limit of large AdS radius always a tower of light states is emerges, which corresponds to the existence of a additional space-time dimensions.

In a recent paper \[23\], also motivated by ideas from the large D expansion of gravity \[24, 25\], we addressed the question is large \(D\) quantum gravity in the swampland? Namely

\footnote{Some non-critical string constructions were considered in \[5-9\].}
we have provided certain swampland arguments to constrain the number of space-time dimensions coming from the spectrum of Kaluza-Klein modes, from black hole physics and from the weak gravity conjecture. Concretely, we proposed in [23] to include the number $D$ of space dimensions in the parameter space of quantum gravity. This amounts to treat $D$, in addition to the geometric parameters like the radius $R$ of a compact space, as new swampland parameter and to include the dependence on $D$ into the distance functionals $\Delta(D)$, which measure the distances between different backgrounds in quantum gravity. Then, the proposed Large-D Conjecture (LDC), together with a negative distance conjecture (NDC) state that $\Delta(D)$ for certain towers of states should stay positive as a function of the number of space-time dimensions, unless there is a dual tower of states at large space-time dimensions. For the Kaluza-Klein states this is equivalent to require that the leading Kaluza-Klein tower is always lighter than the effective Planck scale of the theory. As a general consequence of the LDC, the number of space-time dimensions has to be smaller or equal to some critical value that depends on the typical size of space-time. Furthermore in [26], we considered new geometric flow equations, called D-flow, which describe the variation of space-time geometries under the change of the number of dimensions.

In [23] we also briefly mentioned a possible new D-duality symmetry between large and small dimension.\(^2\) D-duality states that for certain towers that are light in $D$ dimensions and heavy in backgrounds of dual number dimensions, denoted by $\tilde{D}$, there should exist dual towers, which are heavy in $D$ dimensions and light in $\tilde{D}$ dimensions, and vice versa. For example the light tower in D-dimensions and the heavy tower in $\tilde{D}$ dimensions can be given by perturbative KK modes, whereas their dual towers are given in terms of non-perturbative dual winding modes. For D-duality to be a symmetry of gravity, the spectra (and interactions) in $D$ and in $\tilde{D}$ dimensions should agree with each other.

It is the aim of this paper to investigate in more detail the spectrum of gravity theories at large $D$, where in this paper, $D$ will always refer to the number of internal compact dimensions. In [23] we have shown the Kaluza-Klein spectrum of $AdS_d \times S^D$ has a non-trivial dependence on the number of dimensions, since the volume of the $D$-dimensional sphere depends on the number of space-time in a particular way. Namely keeping the diameter $R$ of the sphere, which defines the maximal distance between two points fixed, the

\(^2\) D-duality, although in a different setting, was also discussed in [27]. Moreover c-duality was discussed in [28].
volume of the sphere nevertheless shrinks in the limit of large $D$. This means that the KK modes become heavy for fixed $R$ and large $D$. We have utilized this observation for the LDC in order to constrain the number of dimensions for $AdS_d \times S^D$ geometries by $R$ of the D-dimensional sphere. However for the curved $AdS_d \times S^D$ backgrounds the KK masses in AdS space start to become heavier than the Planck mass in the regime where the space-time curvature is already large.

Here we continue this discussion by analyzing the Kalazu-Klein spectra for flat backgrounds, where there are no curvature corrections, namely for tori $T^{(D)}$ and for certain orbifolds $O^{(D)}$ at arbitrary dimensions $D$. We will see that, whereas the leading KK tower of tori $T^{(D)}$ is independent of $D$, the leading tower of certain D-dimensional orbifolds $O^{(D)}$ indeed possesses an interesting and non-trivial dependence on $D$: like for the higher-dimensional spheres, keeping the diameter $R$ of the orbifold fixed, the orbifold volume is also sensitive to the number of space-time dimensions and shrinks at large $D$. So the KK modes of the orbifold in question become heavy at large $D$, when keeping the diameter $R$ of $O^{(D)}$ fixed. This opens the possibility to apply the LDC for orbifold compactifications and raised the question if one can derive bounds on $D$ for toroidal and orbifold compactifications. In fact we will see that, unlike for the sphere $S^D$, the topology of the orbifolds $O^{(D)}$ is such that it also allows, at least in principle, for a full tower of dual states, which is due branes that are completely wrapped around the orbifold. Then the mass scale of the wrapped brane states on the orbifolds possess a non-trivial dependence on $D$, namely they become light in the large $D$ limit, opposite to the mass scale of the KK states. With this observation we will investigate D-duality symmetries for certain orbifold backgrounds in gravity and in string theory. As we will discuss, an already known example of this kind of D-duality is the duality between type II superstring on $S^1$ and M-theory on $T^2$, or some orbifolds version of it. Also certain F-theory ”compactifications” fall in the class of D-dual theories.

The paper is organized as follows. In the next section, we will briefly review the NDC and the LDC and the motivation for D-duality. Then in section 3, we will extend the discussion of [23] by analyzing the leading KK spectra for toroidal and two classes of orbifold backgrounds at arbitrary $D$. We will also discuss the topological properties of the D-dimensional orbifolds. These will allow the existence of a tower of wrapped brane states, which can at least in principle be dual to the tower of KK states. As necessary conditions for D-duality, We will set up the D-duality transformation rules, namely as relations between $D$ and $\tilde{D}$ as a
function of the radius $R$ of the orbifold. We will also discuss the general rules for D-dual pairs of orbifold spaces. However, as we will discuss, these examples can realized in string theory only in a quite restricted setting, namely as string theory/M-theory dualities, or as string theory/F-theory dualities. In the latter case, a certain truncation of the spectrum is necessary.

II. THE LDC AND D-DUALITY

In this section we recall briefly some of the main aspects of the Large Dimension Conjecture, which was introduced in [23]. The starting point is the Swampland Distance Conjecture (SDC) [16], which states that at large distances $\Delta$ in the field space of a $d$-dimensional (effective) quantum gravity theory there must be an infinite tower of states with mass scale $m$ such that

$$SDC: \quad m = M_P e^{-\Delta}. \quad (1)$$

The mass scale $m$ can be seen as the scale below which the EFT provides a good description of the low-energy physics. Hence the $d$-dimensional EFT breaks down when $\Delta \to \infty$.

In general $\Delta$ determines the geodesic distance in the parameter space of background in quantum gravity, which are characterized by some parameters $\phi$, implying that $\Delta$ is a certain function of $\phi$, i.e. $\Delta = \Delta(\phi)$. In the EFT $\phi$ is in general associated to a canonically normalized scalar field $\Phi(\phi)$ with kinetic energy $L_{EFT} \simeq \frac{1}{2} (\partial \Phi)^2 + \ldots$. Then the distance functional is proportional to $\Phi$:

$$\Delta(\phi) = \lambda \Phi(\phi). \quad (2)$$

For (string) compactifications the SDC is due to the higher dimensional nature of the theory, namely the relevant tower of states is given in terms of KK momentum states with masses $m_{KK} = 1/R$. Then has the form $\Delta_{KK} \simeq \log R$ and becomes large in the decompactification limit $R \to \infty$. Another realization of the SDC is due to the tower of string excitations that become massless in the weak coupling limit, i.e. $\Delta_{\text{string}} \simeq -\log g_s$.

On the other hand in the opposite limit $R \to 0$ the KK modes are heavy and the geometrical picture of a compact space gets lost. However in string theory there is often a duality symmetry, like T-duality with a T-dual tower of fundamental string winding modes (F1-strings) with masses $m_{\text{wind}} = R$ and distance functional $\Delta_{\text{wind}} = -\log R$, which becomes
large for $R \to 0$. More generally, duality symmetries often follow from the SDC, when considering "opposite" large distance limits in field space.

The SDC implicitly also implies that if it happens that there is an infinite distance limit in field space without a corresponding light tower of states that satisfies the SDC, then it cannot be possible to approach this infinite distance point. Based on this statement, we now reverse the logic and define, for a given tower tower of states $|i\rangle$, the quantity $\Delta_i$ (the "distance") as the negative logarithm of its typical mass scale $m_i$, i.e.

$$\Delta_i \sim -\log m_i.$$  

(3)

In addition there possibly can be a dual tower of states $|\tilde{i}\rangle$ with the dual distance $\tilde{\Delta}_i \sim -\log \tilde{m}_i$.

On the basis of these definitions and arguments we now formulate the Negative Distance Conjecture (NDC) demanding that, for a particular leading tower $|i\rangle$, an "infinite distance" limit $\Delta_i \to -\infty$ is obstructed, i.e. not allowed, unless there is a dual tower $|\tilde{i}\rangle$, which becomes light in this limit.

There are three basic arguments in favour of the NDC:

(i) For a tower with negative $\Delta_i$, the associated tower mass scale is above the Planck mass, $m > M_P$. For the leading KK tower this obstructs a proper geometric interpretation of the tower.

(ii) For a tower with negative $\Delta_i$, the coupling constant $g_i$, like $g_{KK}$ or $g_s$, associated to the tower $|i\rangle$ becomes large, i.e. $g_i > 1$. So the EFT becomes strongly coupled and hard to be controlled.

(iii) For a tower with negative $\Delta_i$, the effective Planck mass $M_P^{(d)}$ in $d$ dimensions becomes smaller than the fundamental Planck mass $M_P^{(D)}$ or becomes smaller than the string scale $M_s$, which means the effective gravity theory becomes strongly coupled at energies lower than $M_P^{(d)}$.

So these are three circumstances, which should not happen in a well defined and controllable EFT for quantum gravity, unless there is a weakly coupled dual tower of states, with $\tilde{m} < M_P$ and $\tilde{g}_i < 1$.

Typically the tower masses depend on the geometric parameters $\phi$, i.e. $m = m(\phi)$. A good example for the NDC is given by the KK modes, where the limit $\Delta_{KK} \to -\infty$ implies
φ → −∞ and the SDC resp. NDC says that there must be a second tower of states or φ → −∞ cannot be possible. A further example of the NDC in string theory is provided by the string states (i.e. by the string itself) and the associated tower of states in the limit of weak string coupling, gs → 0. Indeed, the string scale reads

\[ M_s^2 = (g_s)^{1/2} M_p^2 \xrightarrow{g_s \to 0} 0 . \]  

The NDC is then a statement about the the strong coupling limit gs → ∞ limit. Naively, this strong coupling limit is not controllable, however string dualities allow to probe this limit, where a new tower of light states arises, provided e.g. by (wrapped) NS5-branes or by D1-strings in the type IIB string. Actually, it has been argued [21] that those two cases of KK modes and string states may be everything there is at infinite distance limits, as long as a suitable duality frame is chosen to analyse the limit.

In addition the tower masses may also depend on the number D of compact dimensions in a non-trivial way: \( m = m(\phi, D) \). Therefore we want to include D into the swampland distance functional as new parameter:

\[ \Delta_i = \Delta_i(\phi, D) . \]  

This is unusual, since there is no known dynamical theory for D, and also no kinetic term for D in the EFT. But the general idea is that \( \Delta_i(\phi, D) \) determines the distance between space-time geometries of different dimension.

Having introduced D as swampland parameter, the Large Distance Conjecture (LDC) can now be formulated as an extension of the NDC: If there is a leading tower of states \(|i\rangle\) with a non-trivial D-dependence, the corresponding distance must be a positive function of the EFT fields \( \phi \) and of the number of dimensions D,

\[ \Delta_i(\phi, D) \geq 0 , \]  

unless there is a dual u tower \(|\tilde{i}\rangle\), such that \( \tilde{\Delta}_i(\phi, D) \) becomes positive when \( \Delta_i(\phi, D) \), when \( \Delta_i(\phi, D) \) changes its sign. Since \( \Delta_i(\phi, D) \) depends on \( \phi \) and on D, the LDC normally puts a bound on D as a function of \( \phi \), like:

\[ D \leq D_0(\phi) . \]  

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Let us briefly recall that for a compactification of the form $M_d \times K_D$ with arbitrary numbers $d$ and $D$ of non-compact and compact dimensions ($D_{\text{tot}} = d + D$) and metric of the form,

$$ds_{D_{\text{tot}}}^2 = e^{2\phi}ds_{M_d}^2 + e^{2\phi}ds_{K_D}^2,$$

($\beta = -\frac{D}{d-2}$) the KK mass scale is given as [23]

$$m_{KK} \approx \mathcal{V}_D^{-\frac{1}{d}}M_P^{(d+D)} \approx M_P^{(d)}\mathcal{V}_D^{-\frac{d+D-2}{D(d-2)}} \approx M_P^{(d)}\exp\left[-\sqrt{\frac{d + D - 2}{D(d-2)}}\Phi\right].$$

Here $M_P^{(d+D)}$ and $M_P^{(d)}$ are the Planck masses in the (d+D)-dimensional or d-dimensional Einstein frames, respectively, and $\mathcal{V}_D$ is the volume of the compact space measured in units of $1/M_P^{(d+D)}$. It follows that the constant $\lambda(d, D)$ in eq.(2) is given by

$$\lambda(d, D) = \sqrt{\frac{d + D - 2}{D(d-2)}}. \quad (10)$$

For $d = D = D_{\text{tot}}/2$, this fall-off parameters goes to zero in the large D limit: $\lambda \approx \sqrt{\frac{1}{D}} \rightarrow 0$.

In addition also the canonical field $\Phi$ may depend on the number of dimensions and the LDC bound for KK modes reads

$$\Phi(d, D) \geq 0, \quad (11)$$

Let us now give some qualitative arguments under which circumstances we expect the LDC bound to apply and where, on the other hand, dual tower of states are in principle possible. Actually there are two kinds of generic situations:

- there are no fluxes and no potential for the volume modulus: these backgrounds are typically tori, orbifolds or Calabi-Yau spaces without fluxes. As we will discuss, for certain orbifolds the KK masses have a non-trivial D-dependence. Furthermore dual towers of states can be present and can lead to a D-duality-symmetry.

- there are non-zero fluxes: these backgrounds are typically $AdS_d \times S^D$ or warped Calabi-Yau spaces with fluxes. Here the complex structure moduli are typically frozen out and flux quantization does not allow for small Kähler moduli. Dual towers are not present because of the topology of the sphere. As it was discussed [23], in this case the LDC provides an upper bound on D. The microscopic explanation of the bound on D is related to flux compactification of the underlying D-brane model.

In the following we will further discuss the KK spectrum of tori and certain orbifolds at large D and how D-duality symmetries can be possibly realized.
III. TOROIDAL AND ORBIFOLD COMPACTIFICATIONS AT ARBITRARY AND LARGE D

In this section we analyze the spectrum of tori and certain orbifolds at arbitrary D, where this discussion is a priori not restricted to string theory.

A. Circle compactification

Let us consider the well known case of Kaluza-Klein modes on tori. First we will consider the Kaluza-Klein tower of states in flat d-dimensional Minkowski space, which arises from compactification of an additional internal circle of radius $R$. So the total number of dimensions is $D_{\text{tot}} = d + 1$. In the string frame or respectively in the $(d + 1)$-dimensional Einstein frame their masses are given

$$m_{\text{KK}}^2 = \left( \frac{l}{R} \right)^2 = \left( \frac{l}{r} \right)^2 (M_P^{(d+1)})^2,$$

where the dimensionless radius $r = R/M_P^{(d+1)}$ is measured here and in the following in units of the $D_{d+1}$-dimensional Planck mass $M_P^{(d+1)}$. The integer $l$ denotes the KK charge.

When the radius $r$ approaches the critical radius $r_0 = 1$, the KK tower becomes heavy and for $r < 1$ the tower starts above the Planck scale. However in string theory, for $r < 1$ the elementary string, with string tension

$$T^{(1)} = M_s^2 = (g_s^{(10)})^{1/2}(M_P^{(10)})^2,$$

where $g_s^{(10)}$ is the 10-dimensional string coupling constant and $M_s$ is the string mass, can wrap around the circle and the winding modes become light. (Here we assume that $D_{\text{tot}} = 10$.) Specifically their masses are given as

$$m_{\text{wind}}^2 = n^2 R^2 M_s^4 = n^2 r^2 g_s^{(10)} (M_P^{(10)})^2,$$

where $n$ is the winding number of the string.

As it is well know, the T-duality symmetry is exchanging the KK with the winding spectrum, keeping the KK and winding masses invariant and acting in the following way on the other quantities:

$$l \leftrightarrow n, \quad R \leftrightarrow 1/(RM_s^2).$$
Here $R_0 = 1/M_s$ is the fixed point of the T-duality symmetry. T-duality also acts in a non-trivial way on the 10-dimensional string coupling constant,

$$g_s^{(10)} \rightarrow \frac{g_s^{(10)}}{RM_s},$$

and it follows that the Planck mass transforms under T-duality in the following way:

$$M_P^{(10)} \rightarrow M_P^{(10)} (RM_s)^{1/4}.$$  \hspace{1cm} (17)

For $R > R_0$ the KK modes are building-up space-time geometry, whereas for $R < R_0$ the winding modes take over and are building-up a dual geometry. Note that in the bosonic or heterotic string theories, T-duality symmetry acts as self-duality between KK modes and winding modes, and the critical radius $R_0$ is the limiting radius, in the sense that one can restrict $r > R_0$ in the physical moduli space. Here the geometries at large and at small radii are completely equivalent to each other. On the other hand, T-duality is mapping the KK spectrum of type IIA on the winding spectrum of type IIB and vice versa. Moreover the KK and winding spectra in type IIA or, respectively in type IIB, are not equivalent to each other [29]. Therefore the moduli spaces for type IIA and for type IIB are given by all real values for the radii. Staying e.g. within type IIA (or within type IIB), the KK geometry at large radii and the dual winding geometry are not completely equivalent to each other; however the KK geometry geometry of type IIA and the winding geometry of type IIB are indeed equivalent. We will come back to similar issues in section IV, when we discuss some other kind of KK and winding dualities that depend also on the number of space-time dimensions.

At the end of this section, we want to express the above quantities also in terms of the lower-dimensional Planck mass $M_p^{(9)}$ of the effective field theory in $d = 9$ dimensions. As it is well known, $M_p^{(9)}$ and $M_p^{(10)}$ are related as (all these relations can be generalized to arbitrary dimensions):

$$(M_p^{(9)})^7 = R (M_p^{(10)})^8 = r (M_p^{(10)})^7 = r (g_s^{(10)})^{-7/4} M_s^7.$$  \hspace{1cm} (18)

It follows that the KK masses and the winding masses in terms of the lower-dimensional Planck mass $(M_p^{(9)})$ can be expresses as

$$m_{KK}^2 = l^2 r^{-16/7} (M_p^{(9)})^2,$$

$$m_{wind}^2 = n^2 r^{16/7} g_s^{(9)} (M_p^{(9)})^2.$$  \hspace{1cm} (19)
Here the 9-dimensional string coupling constant is defined as

$$ (g^{(9)}_s)^2 = \frac{(g^{(10)}_s)^2}{RM_P^{(9)}}. $$

Note that the 9-dimensional quantities $M_P^{(9)}$ and $g_s^{(9)}$ are invariant under T-duality transformations. Hence T-duality is also manifest in the lower dimensional Einstein frame.

B. Two-torus

Now we consider the different Kaluza-Klein towers of states, which arises from compactification on a two-dimensional torus $T(2)$, i.e. the total number of dimensions is $D_{tot} = d + 2$. A string compactification of a two-torus with metric $G_{ij}$ and antisymmetric tensor field $B$ is described by four moduli fields, which can be grouped into two complex parameters, namely into a Kähler parameter $T$ and a complex structure parameter $U$ in the following way:

$$ T = (\sqrt{G} + iB), $$
$$ U = \frac{1}{G_{11}}(\sqrt{G} - iG_{12}) = \frac{r_2}{r_1}e^{-i\theta}. $$

The metric can be explicitly given in term of two radii $R_1$ and $R_2$ and one additional angle $\theta$ as

$$ G_{11} = r_1^2, \quad G_{22} = r_2^2, \quad G_{12} = r_1r_2 \sin \theta $$

and

$$ V(T^{(2)}) = \sqrt{G} = r_1r_2 \cos \theta $$

denotes the overall volume of $T^{(2)}$.

The corresponding KK and winding spectrum is characterized by two KK numbers $l_i$ and two winding numbers $n_i$ and the corresponding masses are given as

$$ m_{n_i,m_i}^2 = \frac{|l_2 - il_1U + in_1T - n_2TU|^2}{\text{Re} T \text{Re} U}. $$

As it is well known, the KK and winding spectrum is invariant under discrete $O(2, 2, \mathbb{Z})$ duality transformations. These transformations can be written as $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U$, which act separately on the two moduli $T$ and $U$ as

$$ T \rightarrow \frac{aT - ib}{icT + d}, \quad U \rightarrow \frac{aU - ib'}{icU + d'}, $$

where $a, b, c, d$ and $a', b', c', d'$ are integers.
Here the $SL(2, \mathbb{Z})_T$ transformations generate the shifts of $B$ and the T-duality transformations and the $SL(2, \mathbb{Z})_U$ transformations correspond to the geometric transformations that leave the 2-torus invariant. In order to keep the spectrum unchanged, one has to transform the KK numbers $l_i$ and winding numbers $n_i$ in an appropriate way.

Let us first consider the different KK towers, i.e. setting the winding numbers $n_1 = n_2 = 0$. In this case the spectrum is invariant under the geometric $SL(2, \mathbb{Z})_U$ transformations. In the $(d + 2)$-dimensional Einstein frame the KK masses take the following values

$$m_{KK,l_1,l_2}^2 = \frac{|l_2 - l_1|^{2}(\sin \theta - i \cos \theta)|^2}{r_2^2 \cos^2 \theta} (M_P^{(d+2)})^2. \quad (26)$$

For a rectangular torus with $\theta = 0$ and the two radii set to be equal to each other, $r_1 = r_2 = r$, the two leading KK towers $|l_i\rangle$ on the torus are the ones with a single non-vanishing KK charge $l_i$ ($i = 1, 2$); they possess masses

$$|l_i\rangle : \quad m_{KK,l_i}^2 \equiv l_i^2 m_{KK,(i)}^2(M_P^{(d+2)})^2, \quad m_{KK,(i)}^2 = \frac{1}{r^2} = \frac{1}{\mathcal{V}(T^2)}. \quad (27)$$

These two leading towers KK towers have a mass$^2$ scale identical to the inverse volume of the torus, and they transform in the fundamental 2 representation of $SL(2)$. In order for the leading mass scale $m_{KK,(i)}^2 \leq 1$, one has to require that

$$r^2 \geq 1. \quad (28)$$

Note that for $\theta \neq 0$, $m_{KK,(i)}^2$ is not anymore identical to the inverse volume of the torus.

Next, consider the subleading tower of KK states $|l\rangle$ with $l_1 = l_2 = l$. Their masses for $\theta = 0$ are

$$|l\rangle : \quad m_{KK,l}^2 = l^2 m_{KK,(S)}^2(M_P^{(d+2)})^2, \quad m_{KK,(S)}^2 = \frac{2}{r^2} = \frac{2}{\mathcal{V}(T^2)}. \quad (29)$$

This subleading KK tower transforms as a singlet under $SL(2)$. In order for the subleading, singlet mass scale $m_{KK,(S)}^2 \leq 1$, one has to require that

$$r^2 \geq 2. \quad (30)$$

For the elementary string winding states the situation is analogous. On the rectangular torus with $r_1 = r_2 = r$, the $SL(2)$ doublet winding states with a single winding number $n_i \neq 0$ have masses

$$|n_i\rangle : \quad m_{\text{wind},n_i}^2 \equiv n_i^2 r^2 g_s^{(10)}(M_P^{(d+2)})^2. \quad (31)$$

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whereas the $SL(2)$ singlet winding states with two non-vanishing winding number $n_1 = n_2 = n$ have masses

$$|n\rangle : m_{\text{wind},n}^2 \equiv 2n^2 r^2 g_s^{(10)} (M_P^{d+2})^2 .$$

(32)

C. Two dimensional orbifold

In the following we are looking for an orbifold, for which the only the $SL(2)$ KK modes are invariant. This orbifold is defined as

$$\mathcal{O}_{A}^{(2)} = T^{(2)}/\Gamma^{(2)} ,$$

(33)

where $\Gamma^{(2)}_A$ is the $\mathbb{Z}_2$ group, which acts by interchanging the two coordinates of the torus, i.e.

$$\Gamma^{(2)}_A : x_1 \leftrightarrow x_2 .$$

(34)

The group $\Gamma^{(2)}$ acts on the complex structure modulus $U$ as $U \to 1/\bar{U}$. Therefore, diving out by $\Gamma^{(2)}$, one has to set $r_1 = r_2$, i.e. the $U$ modulus is restricted to satisfy $U = e^{-i\theta}$. While the volume of the original torus is $V(T^2) = r^2 \cos \theta$, the orbifolded version $\mathcal{O}^{(2)}$ has a reduced volume:

$$V(\mathcal{O}^{(2)}) = V(T^2)/2 = (r^2 \cos \theta)/2 .$$

(35)

Looking at the induced topology of the orbifold, the first homology group $H_1$ as well as the fundamental group $\pi_1$ of $\mathcal{O}^{(2)}$ are also reduced compared to the two-torus and are given as

$$H_1(\mathcal{O}^{(2)}_A, \mathbb{Z}) = \pi_1(\mathcal{O}^{(2)}) = \mathbb{Z} .$$

(36)

Moreover $\mathcal{O}^{(2)}_A$ has one 1-dimensional boundary, namely the line $x_1 = x_2$, which is the fixed line of $\Gamma^{(2)}$. In fact, $\mathcal{O}^{(2)}_A$ is an non-orientable surface with $S^1$ as boundary, it is just two two-dimensional Möbius strip.

The leading KK tower on the orbifold must be invariant under the $\Gamma^{(2)}_A$ group action; as said before it is the $SL(2)$ singlet tower $|l\rangle$ with $l_1 = l_2 = l$. One can show that the mass scale of this leading KK tower is identical to inverse orbifold volume, only if one chooses $\theta = 0$, i.e. if the underlying torus is rectangular. Then the leading KK mass scale on the orbifold A can be indeed rewritten as

$$m_{KK}^2 = \frac{2}{r^2} = \frac{1}{V(\mathcal{O}^{(2)})} (M_P^{d+2})^2 \quad \text{for} \quad \theta = 0 .$$

(37)
D. D-dimensional torus

Next we generalize this discussion to backgrounds \( \mathbb{R}^{1,d-1} \times T^{(D)} \), with \( \mathbb{R}^{1,d-1} \) being the \( d \)-dimensional Minkowski space and \( T^{(D)} \) a torus of dimension \( D \). It is determined by a constant metric \( G_{ij} \), which is characterized by \( D \) different radii \( r_i \) \( (i = 1, \ldots, D) \) and \( D(D-1)/2 \) different angles \( \theta_{ij} \). The volume of \( T^{(D)} \) is given as
\[
V(T^{(D)}) = \sqrt{\det G_{ij}}.
\]

In general the mass spectrum of the KK states has the following form
\[
m_{KK}^2 = l_i G^{ij} l_j (M_P^{(d+D)})^2,
\]
where \( G^{ij} \) is the inverse metric of the D-dimensional torus. Moreover, certain discrete \( SL(D) \) transformations on the \( r_i \) and the \( \theta_{ij} \), which build a subgroup of the general duality group \( SO(D, D, \mathbb{Z}) \), leave the torus unchanged. In order to keep the KK spectrum invariant, these transformations must be accompanied by corresponding linear transformation among the KK quantum numbers \( l_i \).

The leading KK towers \( |l_i\rangle \) on the torus are the ones with a single non-vanishing KK charge \( l_i \) \( (i = 1, \ldots, D) \) in the \( i \)-th. direction. They transform in the fundamental representation \( D \) of the group \( SL(D) \) and have a mass in terms \((d + D)\)-dimensional Planck mass \( M_P^{(d+D)} \)
\[
|l_i\rangle : \quad m_{KK,i}^2 = l_i^2 g^{ii} (M_P^{(d+D)})^2.
\]

Next let us consider the subleading, \( SL(D) \) singlet KK tower \( |l\rangle \) with all momenta equal to each other, i.e. \( l_i = l \), where the subleading tower mass scale in units of \( M_P^{(d+D)} \) is given as
\[
|l\rangle : \quad m_{KK,l}^2 = l^2 m_{KK,(s)}^2 \quad \text{with} \quad m_{KK,(s)}^2 = \sum_{i,j} G^{ij} (M_P^{(d+D)})^2.
\]
As we will see in the following, these masses depend on the number of dimensions \( D \) in a particular, non-trivial way.

E. D-dimensional orbifold

Now we will consider orbifolds of the form
\[
\mathcal{O}^{(D)} = T^{(D)}/\Gamma^{(D)}.
\]
Here \( \Gamma^{(D)} \) is a certain discrete symmetry group of the D-dimensional torus. The volume of the orbifold is in general given as

\[
\mathcal{V}(\mathcal{O}^{(D)}) = \mathcal{V}(T^{(D)})/|\Gamma_D|,
\]

where \( |\Gamma_D| \) is the order of the group \( \Gamma^{(D)} \). As we will see, only the towers that are subleading on the torus will be invariant under the orbifold group and will survive the orbifold projection. So these towers will become the leading KK towers on the orbifolds and they indeed possess a non-trivial dependence on the number of dimensions.

Let us consider look for the D-dimensional orbifold, which leaves the singlet KK tower invariant. For that we take a torus which is symmetric under the exchange \( i \leftrightarrow j \), which implies that all diagonal metric elements are equal to each other, i.e. \( G_{ii} = r^2 \) and also all non-diagonal metric elements must be equal to each other, i.e. \( G_{ij} = G_{ik} = G_{kj} = r^2 \sin \theta \).

In this case the volume of the torus becomes

\[
\mathcal{V}(T^{(D)}) = \sqrt{\det G_{ij}} = r^D \sqrt{(1 - \sin \theta)^{D-1}(1 + (D - 1) \sin \theta)}
\]

The orbifold group \( \Gamma^{(D)} \) is the symmetric group \( S_D \) of all possible permutations among the torus coordinates:

\[
\Gamma^{(D)} : \quad x_i \leftrightarrow x_j, \quad i, j = 1, \ldots, D.
\]

Its order is given as \( |\Gamma^{(D)}| = D! \) and the orbifold volume becomes

\[
\mathcal{V}(\mathcal{O}^{(D)}) = \mathcal{V}(T^{(D)})/D! = \frac{r^D}{D!} \sqrt{(1 - \sin \theta)^{D-1}(1 + (D - 1) \sin \theta)}
\]

We see that \( \mathcal{V}(\mathcal{O}^{(D)}) \) decreases in the large D limit when keeping the diameter \( r \) fixed, compared to the volume of the covering torus.

The orbifold action leaves only one 1-cycle of the underlying torus invariant and hence the first homology group \( H_1 \) and the fundamental group \( \pi_1 \) of \( \mathcal{O}^{(D)} \) are given as

\[
H_1(\mathcal{O}^{(D)}, \mathbb{Z}) = \pi_1(\mathcal{O}^{(D)}) = \mathbb{Z}.
\]

Furthermore the local structure of \( \mathcal{O}^{(D)} \) is of the form

\[
\mathcal{O}^{(D)} = I_{D-1} \times S^1,
\]
where $I_{D-1}$ is a (D-1)-dimensional space with one isolated point. Hence $O^{(D)}$ possesses one isolated $S^1$, which is fixed by $\Gamma^{(D)}$ with $x_1 = x_2 = \cdots = x_D$ and is a D-dimensional generalization of the two-dimensional M"obius strip.

Now the $SL(D)$ singlet KK tower $|l\rangle$ of eq. (41) with all momenta equal to each other, i.e. $l_i = l$, becomes the leading tower on $O^{(D)}$; it is invariant under the orbifold group $\Gamma^{(D)}$ and survives as invariant KK tower on the orbifold. So let us compare the corresponding KK masses in eq. (41) with the orbifold volume $\mathcal{V}(O^{(D)})$ in eq. (46). For the torus with $G_{ii} = r^2$ and $G_{ij} = G_{ik} = G_{kj} = r^2 \sin \theta$ this mass scale becomes

$$m^{2}_{KK,(S)} = m^{2}_{KK,(S)} = \frac{D}{r^2(1 + (D-1) \sin \theta)}. \quad (49)$$

We can now solve $m^2_{KK,(S)} = \mathcal{V}(O^{(D)})^{-2/D}$ directly for any $D$, and we can check that indeed there exists a solution $\theta_D$ for any $D$, which for large $D$ behaves such that

$$\sin \theta_D \sim -\frac{1}{D}. \quad (50)$$

So for a special value of $\theta_D$ the KK mass scale can be indeed expressed in terms of the orbifold volume:

$$m^{2}_{KK,(S)} = \left( \frac{1}{\mathcal{V}(O^{(D)})} \right)^{\frac{1}{D}} = \frac{1}{r^2} \left( \frac{D!}{\sqrt{(1 - \sin \theta_D)^{D-1}(1 + (D - 1) \sin \theta_D)}} \right)^{\frac{1}{D}}. \quad (51)$$

In terms of the effective d-dimensional Planck mass $M_{P}^{(d)} = \mathcal{V}(O^{(D)})^{1/(d-2)} M_{P}^{(d+D)}$, the masses $m^2_{KK,(S)}$ can then be expressed as

$$m^{2}_{KK,(S)} = \left( \frac{D!}{r^D \sqrt{(1 - \sin \theta_D)^{D-1}(1 + (D - 1) \sin \theta_D)}} \right)^{\frac{2(D+d-2)}{2(d-2)}}. \quad (52)$$

One sees that for large $D$ and fixed $r$, these masses become very big. Note that this behaviour is very similar to the KK masses on the sphere - see [23].

Conversely $m^2_{KK,(S)} \leq 1$ for $D$ being smaller than a critical value, namely

$$D! \leq \sqrt{\det G_{ij}} = r^D \sqrt{(1 - \sin \theta_D)^{D-1}(1 + (D - 1) \sin \theta_D)}. \quad (53)$$

For large $D$ we can approximate $D! \approx D^D$ and $\theta_D = 0$, i.e. $\det G_{ij} \approx r^{2D}$, where $r$ is the typical length scale of the torus, and then $m^2_{KK,(S)}$ grows with $D$ as

$$m^{2}_{KK,(S)} \approx \left( \frac{D}{r} \right)^{\frac{2(D+d-2)}{d-2}}. \quad (54)$$
Constraining $m_{KK,(S)}^2$ to be smaller than the Planck mass, i.e. $m_{KK,(S)} \leq 1$, one derives the following simple bound on $D$:

$$D \leq r.$$  \hspace{1cm}(55)

Note that this is a relation between $D$ and the radius of the original torus, which is indeed a physically meaningful relation, since the one-dimensional length scale of the orbifold is still given by $r$ and only its volume is reduced compared to the torus volume.

IV. LARGE DIMENSION CONJECTURE AND D-DUALITY FOR ORBIFOLDS

As we have seen in the previous section, the leading, invariant KK states on the orbifold $O(D)$ possess are non-trivial dependence on the number of dimensions $D$ and are approximately related to the following mass scale:

$$m_{KK,(S)}^2 \equiv m(D, r) = \left( \frac{D^\alpha}{r^2} \right),$$ \hspace{1cm}(56)

with $\alpha$ being in general a background dependent parameter, $\alpha = 2$ for the orbifold A and $\alpha = 1$ for the orbifold B. As we discussed in the last section, for $D^\alpha > r^2$, \hspace{1cm}(57)

this mass scale of the leading KK tower becomes heavier than the Planck scale and therefore all leading KK states decouple from the theory at scales below $M_P$.\textsuperscript{3}

Now let us assume that the LDC (or the NDC) must hold for the leading KK towers on the orbifolds (or even for the non-leading towers on the torus). Then there are two options: either the relation (57) provides an upper bound on the allowed dimensions or there must be a new kind of (non-perturbative) duality symmetry, called D-duality, that acts on $D$ and on $r$ in a non-trivial way.

In the following we want to investigate the necessary conditions that D-duality in general gravity theories, a priori not restricted to string theory, can be at least in principle realized \textsuperscript{3}In case there are elementary winding states, like on the torus, the T-dual tower of singlet elementary winding states $|n\rangle$ becomes heavier than the Planck scale for $D^\alpha > 1/R^2$. So irrespective of the perturbative T-duality symmetry between the KK and winding states there is always a regime where both towers $|l\rangle$ and $|n\rangle$ become heavier than the Planck scale, namely

$$D^\alpha > \max(r^2, \frac{1}{r^2}).$$  \hspace{1cm}(58)
on the considered orbifolds. However for superstring theory resp. for M-theory the number of dimensions is restricted to be smaller than 11, and for higher-dimensional constructions further truncations of the spectrum appear to be necessary. Concretely we want to investigate the possibility that there exists a new tower of brane states $|\tilde{l}\rangle$ with masses $\tilde{m}$, possibly in a different number of dimensions $\tilde{D}$, which is dual to the KK states in $D$ dimensions. Whereas the KK states become heavy for large $D$ and small $r$, the dual brane states become light for large $\tilde{D}$ and small $\tilde{r}$ and vice versa. Dual to the branes in $D$ dimensions there should be also the corresponding KK states in $\tilde{D}$ dimensions. The wrapped brane states in general correspond to the homotopy class $\pi_D(M)$ ($\pi_{\tilde{D}}(\tilde{M})$) of the $D(\tilde{D})$–dimensional space. A further necessary condition for the existence of a full tower of one-particle wrapped brane states is that the fundamental group of $M$ $(\tilde{M})$ is non-trivial, i.e. $\pi_1(M(D)) \neq 1$ ($\pi_1(\tilde{M}(\tilde{D})) \neq 1$).

As we have seen, for both $D$-dimensional orbifold spaces A and B under consideration we have that

$$\pi_1(O^{(D)}) = \mathbb{Z},$$

and hence one can indeed get a full tower of wrapped membrane states.

Since, as we have just seen, the necessary topological condition for a dual tower of wrapped branes is satisfied, a D-duality symmetry is in principle possible, which then can act as:

$$D - \text{duality}: |l\rangle \longleftrightarrow |\tilde{l}\rangle$$

with the following identification of KK and branes tower mass scales

$$m_{\text{brane}}(D, r) = \tilde{m}_{KK}(\tilde{D}, \tilde{r}),$$

$$m_{KK}(D, r) = \tilde{m}_{\text{brane}}(\tilde{D}, \tilde{r}).$$

Note that these relations are meant to hold in a given number of uncompactified dimensions $d$, which means that the effective theory in $d$ dimensions is invariant under D-duality. From a higher dimensional point of view, D-duality should be seen as duality between two theories in $d + D$ and in $d + \tilde{D}$ number of dimensions. For the same, fixed number of compact dimensions, namely $D = \tilde{D}$, D-duality is relating the radius $r$ with a dual radius $\tilde{r}$ in a particular way. Moreover we will also discuss the possibility that D-duality relates a theory in $D$ dimensions with a theory in a dual, however different number of compact dimensions $\tilde{D}$; so it can possibly act as a large $D$ – small $D$ duality transformation.
V. SOME CONCRETE STRING EXAMPLES OF D-DUAL PAIRS

Since string theory or M-theory is restricted by the existence of the critical dimension, the use of D-duality appears to be rather restricted. Nevertheless we want to consider some dual pairs of theories in two different numbers of dimensions, namely string/M-theory duality and also string/F-theory duality. We will first start in \( d = 9 \) space-time dimensions. Following our general arguments, the KK states from \( 9 + D \) dimensions must correspond to wrapped branes in the dual \((9 + \tilde{D})\)-dimensional geometry. The simplest example is the case with \( D = 2 \) and \( \tilde{D} = 1 \), which corresponds to the well-known M-theory/string theory duality. For higher \( D \) one gets theories, which a priori possess more degrees of freedom than its dual string theory with \( \tilde{D} \) compact dimensions. Therefore a truncation of the degrees of freedoms is necessary. An example for this is a 12-dimensional theory with \( D = 3 \) and a 3-brane being dual to type IIB in \( d = 9 \) and \( \tilde{D} = 1 \). As we will see, the necessary truncation is closely related to the section constrain in exceptional field theory.

A. M-theory \((D = 2)\) and IIB string \((\tilde{D} = 1)\)

As it is well-known \cite{12}, M-theory in 11 dimensions is dual to IIA superstring in 10 space-time dimensions. Here we want to consider M-theory on a 2-torus and also on the 2-dimensional orbifold \( O^{(2)} \). The D-dual theory is IIB on \( S^1 \), which is obtained from type IIA on the circle plus one T-duality transformation. So following our previous notation, we have that \( D = 2 \) and \( \tilde{D} = 1 \) and we will now discuss the duality between IIB on \( S^1 \) and M-theory on \( T^{(2)} \) from the D-duality perspective, possibly with a further \( \mathbb{Z}_2 \) projection on \( O^{(2)} \).

1. Toroidal case

The type IIB U-duality group in 9 dimensions is \( SL(2, \mathbb{R}) \times \mathbb{R} \) and the well-known 9-dimensional bosonic field content is as follows:

(i) three scalar fields, which parametrize the moduli space \( SL(2)/U(1) \times \mathbb{R} \),

(ii) two vector fields \((A^{(1)}_\mu, B^{(1)}_\mu)\), which transform as \( 2 \) under \( SL(2) \) plus one vector field \( C^{(1)}_\mu \), which is a singlet under \( SL(2) \),
(iii) two 2-forms \( (A^{(2)}_{\mu\nu}, B^{(2)}_{\mu\nu}) \), which transform as 2 under \( SL(2) \),
(iv) one 4-form \( A^{(4)}_{\mu\nu\lambda\rho} \), which transform as 1 under \( SL(2) \).

All these fields can be obtained from the reduction of 11-dimensional M-theory metric \( G_{mn} \)
and the M-theory 3-form \( C_{mnp} \) on \( T^2 \) plus one T-duality transformation in the well-known way [30].

The corresponding particles and branes in 9 dimensions are as follows (see also [29]):

(i) One D3-brane: it is the brane which is dual to the un-wrapped M2-brane of M-theory.
(ii) One D2-brane, which is the un-wrapped M2-brane.
(iii) two strings, which transform as 2 under the \( SL(2) \) duality group. They are the IIB D1-
and F1-branes, which the M2-branes wrapped around one direction of the \( T^2 \). Actually the
tension of the general \((n_1, n_2)\)-brane is given as [30]

\[
T^{(n_1,n_2)} = \left( g_s^{(10)} n_1^2 + \frac{n_2^2}{g_s^{(10)}} \right)^{1/2} M_9^2.
\] (62)

(iv) one IIB KK particle: it corresponds to the M2-brane, completely wrapped around the
compact \( T^2 \), and transforming as singlet under the \( SL(2) \) duality group.
(v) two IIB particles, being wrapped F1 and D1 strings: they correspond to the two M-
theory KK particles in the \( x_9 \) and \( x_{10} \) directions, and transforms as 2 under the \( SL(2) \)
duality group.

For illustrative reasons, let us consider some of the well known mass relations for the toroidal
case. They nicely fit into the general D-duality framework. First we compare the masses
of the \( SL(2) \) singlet states, namely the IIB KK modes with the M-theory wrapped M2
modes. For simplicity we will set the two radii of the M-theory 2-torus equal to each other,
i.e. \( R_1 = R_2 = R \). Therefore the IIB string coupling \( g_s^{(10)} = R_2/R_1 \) is equal to one. The
first D-duality relation in (61), expressed in units of the 9-dimensional Einstein frame, then
becomes

\[
m^{(M)}_{M2} = \tilde{m}^{(IIB)}_{KK} \quad \Longleftrightarrow \quad r^{12} = \frac{1}{\tilde{r}^{8/7}}. \] (63)

Second we consider the \( SL(2) \) doublet states, namely the two KK states on the M-theory
side and the two wrapped 1-branes on the IIB side. Setting their masses equal to each other
we get the second D-duality relation in (61):

\[
F1 : \quad m^{(M)}_{\text{KK}} = \tilde{m}^{(IIB)}_{\text{F1}} \iff \frac{1}{r} \left( \frac{1}{r} \right)^{2/7} = \left( g_s^{(10)} \right)^{1/2} \tilde{r}^{\frac{6}{7}},
\]

\[
D1 : \quad m^{(M)}_{\text{KK}} = \tilde{m}^{(IIB)}_{\text{D1}} \iff \frac{1}{r} \left( \frac{1}{r} \right)^{2/7} = \left( g_s^{(10)} \right)^{-1/2} \tilde{r}^{\frac{6}{7}}.
\] (64)

From eq.(63) one gets that \( r = \tilde{r}^{-2/3} \) Furthermore it follows from eqs.(64) that indeed \( g_s^{(10)} = 1 \). One can check that the two D-duality relations (63) and (64) are of course identical to the known relations between IIB and M-theory [12, 30]. So in this case D-duality is completely equivalent to the know duality between M-theory and the type IIB superstring in 9 dimensions.

2. Two-dimensional orbifold

Next we want to consider the two orbifold version of M-theory on \( T^{(2)} \) and of IIB on \( S^1 \).

In M-theory the \( SL(2) \) corresponds to the exchange of the two directions of \( T^{(2)} \), so the \( SL(2) \) is just the geometric modular group of \( T^{(2)} \), which acts as \( x_1 \leftrightarrow x_2 \) and also on the two radii as \( R_1 \leftrightarrow R_2 \). In IIB the group \( SL(2) \) is the S-duality group of the compactified IIB theory, which acts on the type IIB axion-dilaton field \( \tau \) as \( \tau \leftrightarrow -1/\tau \). Therefore on the type IIB side the \( \mathbb{Z}_2 \) orbifold \( O^{(2)} \) corresponds to a non-geometric operation, namely its corresponds to modding with the IIB S-duality symmetry. Namely modding out by \( SL(2) \), and setting \( R = R_1 = R_2 \), means that the IIB string coupling is not a free parameter, i.e. \( g_s^{(10)} = R_1/R_2 = 1 \). So the resulting orbifold theory is a certain S-fold of type II B.

Only the \( SL(2) \) singlet states survive the \( \mathbb{Z}_2 \) orbifold projection. These are first the IIB KK modes on \( S^1 \), whereas in M-theory, the \( SL(2) \) singlet state is the wrapped M2-brane. Because the volume of the M-theory orbifold is reduced by half compared to the 2-torus, the first D-duality relation (63) gets an additional factors of \( \sqrt{D} = \sqrt{2} \) and takes the form

\[
m^{(M)}_{\text{M2}} = \tilde{m}^{(IIB)}_{\text{KK}} \quad \iff \quad \left( \frac{r}{\sqrt{2}} \right)^{12/7} = \frac{1}{r^{8/7}}.
\] (65)

We can also consider the wrapped fundamental F1-string and the wrapped D1-string on the IIB side. The \( SL(2) \) invariant string state is given by the linear combination

\[
|\text{brane}\rangle_{\text{inv}} =
\]

\[\text{4} \quad \text{Lower dimensional heterotic M-theory orbifolds were considered in [33–35].}\]
\( \frac{1}{\sqrt{2}} (|F1\rangle + |D1\rangle) \). On the M-theory side the linear combination of the two KK states, 
\(|KK\rangle_{inv} = \frac{1}{\sqrt{2}} (|l, 0\rangle + |0, l\rangle) \), is \( SL(2) \) invariant. We have to remember that the mass 
of the invariant M-theory KK modes \(|KK\rangle_{inv} \) gets an additional factor of \( \sqrt{D} = \sqrt{2} \) and we 
also have to take into account the reduced orbifold volume when going to the 9-dimensional 
Einstein frame. Then setting the brane mass equal to KK mass we get the second D-duality 
relation

\[
\frac{m^{(M)}_{KK_{inv}}}{m^{(IIB)}_{brane_{inv}}} \quad \iff \quad \frac{\sqrt{2}}{r} \left( \frac{\sqrt{2}}{r} \right)^{2/7} = \frac{1}{\sqrt{2}} \left( \sqrt{g_{s}^{(10)}} + \frac{1}{\sqrt{g_{s}^{(10)}}} \right)^{1/2} \tilde{r} \frac{6}{7}. \quad (66)
\]

Note that due to the orbifold action the string tension in the IIB orbifold is reduced by a 
factor of \( 1/\sqrt{2} \). From eq.\((65)\) we get that \( r = \sqrt{2} \tilde{r}^{-2/3} \) and from eq.\((66)\) one obtains that again \( g_{s}^{(10)} = 1 \).

Let us discuss what this orbifolded version of IIB or, respectively, of M-theory is. From 
the 11-dimensional perspective there is one 10-dimensional boundaries, where additional 
matter fields must be located. As said before the orbifold \( O^{(2)} \) is just the non-orientable 
two-dimensional Möbius strip. Actually M-theory compactified on the Möbius strip was 
already discussed before in [36]. There is a gauge theory with reduced rank=10 located at 
the boundary of the Möbius strip. From the dual IIB perspective it is a strongly coupled 
S-fold with the type IIB coupling set equal to one. To be more precise the dual theory 
corresponds to a heterotic string theory, namely it is the so called CHL string [37, 38] in 
nine dimensions, which possesses one point in moduli space with an unbroken \( E_8 \) gauge 
symmetry. Then, the fixed coupling constant of the type IIB S-fold corresponds in the 
heterotic description to a fixed angle \( \theta \) of the M-theory torus. So we see that the type IIB 
S-fold has a nice interpretation in terms of the CHL heterotic string.

One can also consider analogous D-dual pairs in a smaller number \( d \) of uncompactified 
dimensions by increasing the number of compact dimensions, e.g. with \( d = 4, D = 7 \) and 
\( \tilde{D} = 6 \) or with \( d = 2, D = 9 \) and \( \tilde{D} = 8 \). In this case the orbifold action will lead to some 
exotic S-folds or U-folds, similar to the type II orbifolds, which were constructed in [39, 40]. 
It would be interesting to see, if these S- or U-folds again have a simpler interpretation 
as a compactified CHL-like heterotic string theory. Since the associated orbifolds are non-
orientable and compactifications of this kind lead to orientifolds without vector structure 
[41, 42].
B. **12-dimensional theory** \((D = 3)\) and IIB string \((\tilde{D} = 1)\)

1. **Toroidal case**

Now we want to go towards larger \(D\) by increasing the number of dimensions \(D\) by one and keeping \(\tilde{D} = 1\). For that we consider again type IIB on \(\mathbb{R}^{1,8} \times S^1\). The dual theory with \(D = 3\) is a 12-dimensional theory, being closely related to F-theory, compactified on \(T^3\) or an orbifold version of it. D-duality now requires the existence of a fundamental 3-brane in 12 dimensions. It will turn out to be the D3-brane of the IIB superstring. Due to the close relation to F-theory [13], we will call it the F3-brane and it can be wrapped around the entire 3-dimensional compact space and then leads to a full tower of particles in 9 dimensions. This tower of states transform as a singlet under the \(SL(3)\) duality group. The corresponding, dual KK particles are the KK modes of type IIB on \(S^1\).

Let us investigate in more detail what kind of 12-dimensional theory on \(T^3\) can be D-dual to 9-dimensional type IIB on \(\mathbb{R}^{1,8} \times S^1\). A priori, the duality group of the 12-dimensional theory in 9 dimensions is given by the group \(SL(3)\). Therefore under dimensional reduction on \(T^3\) all 12-dimensional fields are transforming under irreducible representations of \(SL(3)\). In order to identify the compactified 12-dimensional theory with type IIB in 9 dimensions, the group \(SL(3)\) must be broken to \(SL(2)\). Hence we also decompose the irreducible representations of \(SL(3)\) under its \(SL(2)\) subgroup.

First we will assume that the 12-dimensional theory possesses a metric \(G_{MN}\). It decomposes in 9 dimensions as, with internal index \(i = 1, 2, 3\):

- 9-dimensional metric \(G_{\mu\nu}\),
- 3 vector fields \(G_{\mu i}^{(1)}\) \((i = 1, 2, 3)\), which transform as \(3\) of \(SL(3)\) and as \(1 + 2\) under \(SL(2)\),
- 6 scalar fields \(G^{ij}\), which transform as \(2 + 4\) under \(SL(2)\), and which parametrize the geometric moduli space \(\mathcal{M} = (SL(3) \times \mathbb{R})/SU(2)\) of \(T^3\).

Second, the existence of the wrapped F3-branes implies an Abelian 4-form \(D_{MNPQ}^{(4)}\) with corresponding 5-form field strength \(H_{MNPQR}^{(5)}\). In 9 dimensions it decomposes as follows:

- one singlet vector field \(D_{\mu ijk}^{(1)}\),
• three 2-forms $D_{\mu\nu ij}^{(2)}$, which transform as $3$ of $SL(3)$ and as $1 + 2$ under $SL(2)$,
• three 3-forms $D_{\mu\nu\lambda i}^{(3)}$, which transform as $3$ of $SL(3)$ and as $1 + 2$ under $SL(2)$,
• one 4-form $D_{\mu\nu\lambda\rho}^{(4)}$, which is a singlet under $SL(2)$ (and which is dual to a 3-form in 9 dimensions).

Together with the metric these comprise 264 bosonic degrees of freedom.

Let us connect these 12-dimensional fields to the bosonic fields of type IIB in 9 dimensions. Specifically the type IIB fields can be expressed in terms of the F-theory fields in the following way, where we split the internal space index now as $i = (\alpha, s)$ with $\alpha = 1, 2$ and $s = 3$:

• three scalar fields $\phi = G_{2,2}$, $g_{1,1} = G_{1,1}$ and $A = G_{1,2}$, which transform as $1 + 2$ under $SL(2)$, and which parametrize the moduli space $SL(2)/U(1) \times \mathbb{R}$,
• two vector fields $(A_{\mu}^{(1)}, B_{\mu}^{(1)}) = G_{\mu \alpha}$, which transform as $2$ under $SL(2)$ plus one vector field $C_{\mu}^{(1)} = D_{\mu \alpha \beta s}^{(1)}$, which is a singlet under $SL(2)$,
• two 2-forms $(A_{\mu\nu}^{(2)}, B_{\mu\nu}^{(2)}) = D_{\mu\nu \alpha s}^{(2)}$, which transform as $2$ under $SL(2)$,
• one 4-form $A_{\mu\nu\lambda\rho}^{(4)} = D_{\mu\nu\lambda\rho}^{(4)}$, which is dual to a 3-form and which transform as $1$ under $SL(2)$.

These fields indeed comprise the 128 bosonic fields of type IIB. We see that the 12-dimensional theory contains 136 additional bosonic degrees of freedom compared to the type IIB string. In order to match the 12-dimensional field content with the type IIB field content, one can first restrict the three-torus $T^3$ to be a direct product $T^2 \times S^1$. This will reduce the number of massless scalars from 6 to 4. In addition, like in F-theory, the volume modulus of the $T^2$ must be frozen, such that only the complex structure modulus of the 2-torus will survive. Furthermore one has to set to zero the vector field $G_{\mu s}$, the 2-form field $D_{\mu\nu\alpha\beta}^{(2)}$ and the three 3-form fields $D_{\mu\nu\lambda\iota}^{(3)}$.

After this truncation the theory does not allow anymore for a full 12-dimensional geometric interpretation. Actually one ends up with the field content that was already discussed in the context of exceptional $SL(2)$ field theory in 12 dimensions by [43]. As in [43] one can impose a section condition of the following form

$$\partial_\alpha \partial_s = 0 .$$

(67)
This condition has the effect that the remaining fields cannot depend on all coordinates of the $T^3$. It possesses two solutions, namely one, which leads to M-theory or equivalently to type IIA, and a second one which leads to type IIB in nine-dimensions. In our discussion, we have implicitly assumed the type IIB solution $\partial_\alpha = 0$, namely that all fields do not depend on the two coordinates $x_1$.

However note that our field content is different from the one introduced in [43]. Concretely, in [43] three additional vector fields in 9 dimensions were introduced, which transform as $1+2$ under $SL(2)$. In our case these three vector fields originates from the 12-dimensional metric and the 12-dimensional 4-form - see above.

Let us now discuss the branes and the particles of the 9-dimensional type IIB theory. They can be all derived from the 12-dimensional F3-brane and from the momentum (KK) modes in the internal directions in the following way:

- One IIB D3-brane: it corresponds to the unwrapped F3-brane and transforms as singlet under the $SL(3)$ duality group.
- one IIB D2-brane: it corresponds to the F3-brane, wrapped around the compact $x_{12}$ direction, and transforms as singlet under the $SL(2)$ duality group.
- IIB D1- and F1 strings: they correspond to the two double-wrapped F3-branes, wrapped around the compact $(x_9, x_{12})$ and $(x_{10}, x_{12})$ directions, and transform as 2 under the $SL(2)$ duality group.
- one IIB KK particle: it corresponds to the F3-brane, completely wrapped around the compact $T^3$, and transforms as singlet under the $SL(2)$ duality group.
- two IIB particles, being wrapped F1 and D1 strings: they corresponds to the two KK particles in the $x_9$ and $x_{10}$ directions, and transforms as 2 under the $SL(2)$ duality group.

It is clear that the full 12-dimensional theory on $T^3$ before the truncation possesses more branes and particles than its type IIB counterpart. In particular the full 12-dimensional theory leads to a $SL(3)$ triplet of KK particles along all directions of $T^3$, as well as to a $SL(3)$ triplet of double-wrapped F3-branes, which correspond to three 9-dimensional strings, denoted by F1, D1$_1$ and D1$_2$. 

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Now consider the D-duality mass relations between the 12-dimensional theory on $T^3$ and type IIB on $S^1$. First, the F3-brane, wrapped around the entire $T^3$, leads to a tower of particles in 9 dimensions. This tower of states transform as a singlet under the $SL(3)$ duality group. The corresponding, dual KK particles are the KK modes of type IIB on $S^1$. The mass scale of the wrapped F3-branes in 9-dimensional Planck units is given as:

$$m_{F3}^{(F)} = r^{\frac{18}{7}}. \quad (68)$$

Hence the first duality relation in (61) becomes

$$m_{F3}^{(F)} = \tilde{m}_{KK}^{(IIB)} \iff r^{\frac{18}{7}} = \frac{1}{\tilde{r}^{8/7}}. \quad (69)$$

Second we consider the KK states on the F-theory side with the wrapped 1-branes on the IIB side. Setting their masses equal to each other we get the second D-duality relation in (61)

$$m_{KK}^{(F)} = \tilde{m}_{brane}^{(IIB)} \iff \left(\frac{1}{r}\right)^{10/7} = \sqrt{g_s^{(9)} / \tilde{r}^7}. \quad (70)$$

From eq.(69) one has that $r = \tilde{r}^{-4/9}$. Furthermore it follows from eqs.(70) that the 9-dimensional string coupling constant is determined to be $g_s^{(9)} = r^{-64/63}$. Via the relation $g_s^{(9)} = g_s^{(10)} / \tilde{r}^{4/7}$ one then obtains that $g_s^{(10)} = r^{-4/9}$.

2. Orbifold case

Now we want to describe the orbifolded version of the 12-dimensional ($D=3$)/9-dimensional ($\tilde{D}=1$) duality. Namely we consider the following orbifold:

$$\mathcal{O}^{(3)} = T^{(3)}/\Gamma_3, \quad (71)$$

where $\Gamma_3 = S^3$ is the permutation group of six elements, which acts on the three coordinates as:

$$\Gamma^{(3)} : \quad x_i \leftrightarrow x_j, \quad i, j = 1, \ldots, 3. \quad (72)$$

The volume of $\mathcal{O}^{(3)}$ is given as

$$V(\mathcal{O}^{(3)}) = \left(\frac{r}{6^{1/3}}\right)^3, \quad (73)$$

and the local structure of $\mathcal{O}^{(3)}$ is

$$\mathcal{O}^{(3)} = I_2 \times S^1, \quad (74)$$
where $I_2$ is a two-dimensional space with one isolated fixed point and with $H_1(O^{(3)}, \mathbb{Z}) = H_2(O^{(3)}, \mathbb{Z}) = \pi_1 O^{(3)} = \mathbb{Z}$. Hence $O^{(3)}$ possesses one 1-dimensional one fixed circle of $\Gamma^{(3)}$.

Under the orbifold projection only the $SL(3)$ singlet towers are invariant. These are:

- The unwrapped F3-brane, which corresponds to a 3-brane in 9 dimensions.
- A $SL(3)$ invariant single wrapped F3-brane around the fixed $S^1$ circle, which leads to a tower of 2-branes in 9 dimensions.
- A $SL(3)$ invariant, doubled wrapped F3-brane around $I_2$, which leads to one 1-brane in 9 dimensions.
- The completely wrapped F3 around the compact $O^3$, which corresponds to the tower of KK particles in 9 dimensions.
- One tower of a $SL(3)$ invariant linear combination of KK particles. It corresponds to the $SL(3)$ invariant linear combination of wrapped F1, D1 and D1' strings.

The mass relations for the two sets of particles in 9 dimensions are

$$m^{(F)}_{F3} = m^{(IIB)}_{KK} \iff \left( \frac{r}{6^{1/3}} \right)^{18} = \frac{1}{\tilde{r}^{8/7}},$$

and

$$m^{(F)}_{KK} = m_{brane} \iff \left( \frac{6^{1/3}}{r} \right)^{10/7} = \frac{1}{3^{1/3}} \left( 1 + \frac{1}{\sqrt{g_s^{(10)}}} + \frac{1}{\sqrt{g_s^{(10)}}} \right)^{1/3} \frac{r^{8/7}}{\tilde{r}^{8/7}}.$$  \hspace{1cm} (75)

But since we started from 12-dimensional F-theory, a further truncation of the degrees of freedom should be applied. The resulting theory in 9 space-time dimensions is an exotic, S-fold like theory, whose precise relation to string theory is not obvious. The orbifold action possesses one co-dimension-two fixed line, i.e. one (9+1)-dimensional hypersurface. Like for the heterotic M-theory, one expects that certain matter fields are located at on this hypersurface in 9 dimensions. A priori the theory contains additional 3-dimensional and 2-dimensional branes. In the simplest case, they can be eliminated by the standard F-theory truncation, described above. In this case one is ending again at the same theory as M-theory on the Möbius strip which is dual to the heterotic CHL string in nine dimensions. If other consistent for the case of the three-dimensional orbifold $O^{(3)}$ exist is however not clear.
C. Higher dimensions

Let us sketch how one can possibly proceed to higher dimensions, considering a dual pair of compactifications with a compact space $\mathcal{M}$ of dimension $D$, being dual to $\tilde{\mathcal{M}}$ of dimension $\tilde{D}$. $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are both orbifolds of the form

$$\mathcal{O}(D) = T(D) / \Gamma_D; \quad \text{and} \quad \mathcal{O}^{(\tilde{D})} = T^{(\tilde{D})} / \tilde{\Gamma}_{\tilde{D}},$$

(77)

with the same orbifold types on both sides of the D-duality. This is necessary for that the $SL(D)$ and $SL(\tilde{D})$ invariant states match up for the two orbifolds.

D-duality relates the KK modes in $d + D$ dimensions with the branes that are wrapped around the entire compact space $\tilde{\mathcal{M}}$ of dimension $\tilde{D}$, and vice versa. Concretely we assume that in $d + D$ dimensional theory there exist an Abelian $(1 + D)$-form $A^{(1+D)}$ with corresponding $(2 + D)$-form field strength $F^{(2+D)}$. This field strength is then sourced by a $D$-dimensional brane, which can be naturally wrapped our the compact $\mathcal{M}$. From the point of view of the lower dimensional field theory in $\mathbb{R}^{1,d-1}$, the wrapped branes are particles, which couple to a 2-form gauge field strength $F_2$. We assume that the brane tension $T_{\text{brane}}^{(D)}$ is determined by the higher-dimensional Planck mass $M_P^{(d+D)}$ as

$$T_{\text{brane}}^{(D)} = (g^{(d+D)})^\alpha \left( M_P^{(d+D)} \right)^{D+1}.$$  

(78)

Here $(g^{(d+D)})^\alpha$ is an elementary coupling constant in the $(d + D)$-dimensional theory, related to a dilaton field, and $\alpha$ is an a-priori undetermined parameter. The same kind of relations also hold in $d + \tilde{D}$ dimensions.

Now we are ready to employ the relations eq.(61) between the KK modes of $\mathcal{O}(D)$ and the winding modes of the orbifold $\mathcal{O}^{(\tilde{D})}$ as necessary conditions for D-duality. For the orbifolds at large dimensions the duality relations take the form

$$m_{\text{brane}}(D, r) = \tilde{m}_{\text{KK}}(\tilde{D}, \tilde{r}) \iff (g^{(d+D)})^\alpha \left( \frac{r}{D} \right)^{\frac{D(d-3)}{d-2}} = \left( \frac{\tilde{D}}{\tilde{r}} \right)^{\frac{\tilde{D}+d-2}{d-2}},$$

$$m_{\text{KK}}(D, r) = \tilde{m}_{\text{brane}}(\tilde{D}, \tilde{r}) \iff \left( \frac{D}{r} \right)^{\frac{D(d-3)}{d-2}} = \left( \frac{\tilde{D}}{\tilde{r}} \right)^{\frac{\tilde{D}+d-2}{d-2}},$$

(79)

We see that for a given pair of dual dimensions $D$ and $\tilde{D}$ the above equations imply particular relations between $r$ and $\tilde{r}$. Note that the D-dimensional branes respectively the $\tilde{D}$-dimensional branes can be also wrapped around the invariant 1-cycle of the orbifolds,
leading to further branes in the effective theory, with additional conditions on the mass spectra in the dual theories. Furthermore, identification with critical superstring theories will require a truncation of the spectrum, similar to the F-theory cases, considered before.

Concerning string theory, the 12-dimensional case with $D = 3$ and $\tilde{D} = 1$ can be generalized to higher-dimensional F-theory like constructions. E.g. a duality between type II in 8 space-time dimensions on $T^2$ ($\tilde{D} = 2$) and a 13-dimensional S-theory with $D = 5$ was discussed in [44–46]. Following the D-duality proposal, it implies the existence of a 5-dimensional S-brane in 13 dimensions, wrapped around $T^5$, which is dual to the singlet KK states on $T^2$. In order to match this theory with string theory, the $SL(5)$ duality group must be broken to the 8-dimensional U-duality group of the type II string, which is $SL(3) \times SL(2)$. Therefore a section condition must be chosen, which breaks $SL(5)$ to $SL(3) \times SL(2)$. This can be partly achieved by restricting the five-torus $T^5$ to be a direct product $T^3 \times T^2$. The mass relations can be easily worked out using the equations given before.

VI. D-DUALITY AS STRONG-WEAK COUPLING DUALITY

As we will discuss now, D-duality can be also regarded as strong-weak coupling duality, now for a dual pair of theories on an orbifold with a fixed, given number of D compact dimensions. Consider first string theory on a circle circle compactification with Abelian gauge symmetry is $U(1)_L \times U(1)_R$. The KK modes are electrically charged under the diagonal subgroup $U(1)_{L+R}$ with electric gauge coupling constant $g^{(e)} \simeq 1/r$, where the KK quantum number $l$ corresponds to their electric charges. The winding modes are electrically charged under the orthogonal linear combination $U(1)_{L-R}$. Their electric charges are given in terms of the winding numbers $n$, and the electric gauge coupling constant $\tilde{g}^{(e)}$ of the winding states is proportional to the radius, $\tilde{g}^{(e)} \simeq r$.

Generalizing to toroidal compactifications, the effective theory on $\mathbb{R}^{1,d-1}$ possesses an Abelian gauge symmetry with gauge group $G = U(1)^{D_L}_L \times U(1)^{D_R}_R$ with electric 2-form field strengths $F_{2,i}^{(e)}$ coming from the internal part of the metric. The KK modes on $\mathbb{R}^{1,d-1}$ are charged with respect to the gauge symmetry group $G$ and the corresponding electric gauge couplings $g_i^{(e)}$ of the $U(1)$ gauge symmetries are proportional to the corresponding KK mass scales.

Finally consider the orbifold effective field theory, and take for concreteness the A orbifold
\( \mathcal{O}^{(D)} = T^{(D)}/\Gamma^{(D)}_A \). Only one overall \( U(1) \) gauge group with associated field strength \( F_2^{(e)} \) remains as invariant gauge symmetry. The associated charged states are given by the leading KK tower tower of the form \( |1, 1, \ldots, 1\rangle \). The corresponding gauge coupling is determined by the KK mass scale in eq.(52), which for large \( D \) becomes

\[
g^{(e)} \simeq \left( \frac{D}{r} \right)^{\frac{D+d-2}{d-2}}.
\] (80)

We see that the gauge theory becomes strongly coupled for \( D \geq r \). Therefore asking for a weakly coupled gauge theory we derive the same constraint as from demanding that the associated KK tower is starting below the Planck mass.

The Abelian \( U(1) \) gauge theory is expected to possess a strong-weak coupling Olive/Montonen-like duality symmetry \([47–50]\). The corresponding dual magnetically charged states couple to the dual field strength \( \tilde{F}_{d-2}^{(m)} \), which is a \((d-2)\) form on \( \mathbb{R}^{1,d-1} \). Actually, the magnetically charged objects originate from a \((d-4)\)-brane in the effective \( \mathbb{R}^{1,d-1} \) theory, which is the magnetic dual to the KK particles and is the magnetic source of the \( \tilde{F}_{d-2}^{(m)} \). Lifting this form to the \((d+D)\)-dimensional space, it corresponds to a magnetic \( \tilde{F}_{d+D-2}^{(m)} \) form with a gauge potential \( \tilde{A}_{d+D-3}^{(m)} \). It couples to a \((d+D-4)\)-brane, which is wrapped around the orbifold space. From the higher dimensional point of view, the magnetic \((d+D-4)\)-branes are just Kaluza-Klein monopoles, being wrapped around around the compact space. The magnetic coupling \( g_m \) is proportional to its tension \( T_m \), which in units of the \( d \)-dimensional Planck mass is given as

\[
g^{(m)} \simeq T^{(m)} \simeq \left( \frac{r}{D} \right)^{\frac{D(d-3)}{d-2}}.
\] (81)

So from this perspective, D-duality on the orbifold \( \mathcal{O}^{(D)} \) is nothing else than the electric – magnetic duality in the effective gauge theory on \( \mathbb{R}^{1,d-1} \):

\[
D - \text{duality} : \quad g^{(e)} \leftrightarrow g^{(m)}.
\] (82)

However the KK modes and the branes are not self-dual objects, since they have in general different dimension. Therefore, the conjectured D-duality, namely the electric-magnetic duality in \( \mathbb{R}^{1,d-1} \), is in general not a self-duality symmetry but a duality between the electric, point-like KK modes and magnetic \((d-4)\)-branes.
As we have discussed in this paper, the volume of certain D-dimensional compact orbifolds possesses a non-trivial dependence on D, namely it shrinks at large D. This behaviour is then inherited by the mass scale of the leading KK tower in the associated effective field theory. This observation can be utilized to apply the Large Distance Conjecture (LDC) \([23]\) in order to derive a general bound on the number of dimensions of orbifold compactifications, which is typically of the form

\[
\text{LDC bound : } \quad D^\alpha < r^2, \quad (83)
\]

with \(\alpha\) being a background dependent parameter. This bound on \(D\) depends on the length scale \(r\) of the theory. This suggests that in the IR for large scales there is a good notion of geometry, and the concept of a large number of space-time dimensions is well defined. However in the UV at small distances for small \(r\), the notion of geometry breaks down, and the number of space-time dimensions is bound by a number of order one.

As alternative of excluding the large \(D\) regime of quantum gravity by the LDC-bound, one can investigate the possibility of a D-duality symmetry, which acts non-trivially on \(D\) and \(r\). As we have discussed, the non-trivial fundamental homotopy group of the considered orbifolds allow for full towers of particles in the effective theory, which is dual to the KK tower of states and which is due to branes with \((D+1)\)-dimensional world volume, being completely wrapped around the D-dimensional orbifold. M-theory on the Möbius strip with a tower of wrapped M2-branes and its realization as CHL heterotic string in 9 space-time dimension is a non-trivial example of a D-duality based on an orbifold compactification. From the IIB perspective, this orbifold theory is a non-perturbative IIB S-fold, namely it corresponds to an orientifold without vector structure.

Going one step higher in the number of dimensions, we have investigated a D-duality between a 12-dimensional theory, closely related to F-theory and compactified on a 3-dimensional space, and its D-dual theory on a circle from 10 to 9 dimensions. The 12-dimensional theory contains fundamental F3-branes, wrapped around the compact space, being dual to the KK modes of the dual circle. The orbifold version of this dual pair leads to another rather exotic, non-supersymmetric S-fold in 9 dimensions, with additional matter fields on the orbifold fixed plane. It is still not clear, if this orbifold is a consistent quantum gravity theory or if it already belongs to the swampland.
Even much less is known about the existence of possible D-dual pairs originating from gravitational theories in dimensions higher than 12. Space-time supersymmetry is apparently not anymore possible in high number of dimensions, and therefore there are severe restrictions from stability considerations on the quantum consistency of gravitational theories in the large D limit. This is similar to the bosonic string theory, and previous attempts \[51-54\] to derive the superstring via compactification from the bosonic string can be also viewed from the point of view of D-duality, possibly extended to orbifold theories. Finally it would interesting to see, if the swampland cobordism conjecture \[55, 56\] can provide further input for gravitational theories at large D.

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