ARITHMETIC PROPERTIES OF CUBIC AND OVERCUBIC PARTITION PAIRS

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Abstract. Let \( b(n) \) denote the number of cubic partition pairs of \( n \). We give affirmative answer to a conjecture of Lin, namely, we prove that
\[
b(49n + 37) \equiv 0 \pmod{49}.
\]
We also prove two congruences modulo 256 satisfied by \( \pi(n) \), the number of overcubic partition pairs of \( n \). Let \( \pi(n) \) denote the number of overcubic partition of \( n \). For a fixed positive integer \( k \), we further show that \( \pi(n) \) and \( \pi(n) \) are divisible by \( 2^k \) for almost all \( n \). We use arithmetic properties of modular forms to prove our results.

1. Introduction and statement of results

In a series of papers [3, 4, 5], Chan studied the cubic partition function \( a(n) \) with generating function given by
\[
\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_{\infty}(q^2;q^2)_{\infty}}, \quad |q| < 1,
\]
where \( (a;q)_{\infty} := \prod_{n \geq 0} (1-aq^n) \). The partition function \( a(n) \) satisfies many interesting congruences. For example, it satisfies the following Ramanujan-like congruence
\[
a(3n + 2) \equiv 0 \pmod{3}.
\]

Inspired by Chan’s work, Zhao and Zhong [19] studied the cubic partition pair function \( b(n) \) which is defined by
\[
\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_{\infty}^2(q^2;q^2)_{\infty}^2}.
\]
They established several Ramanujan-like congruences for \( b(n) \) as follows:
\[
\begin{align*}
b(5n + 4) &\equiv 0 \pmod{5}, \\
b(7n + i) &\equiv 0 \pmod{7}, \\
b(9n + 7) &\equiv 0 \pmod{9}.
\end{align*}
\]

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where \( i = 2, 3, 4, 6 \). Recently, Lin [12] studied the arithmetic properties of \( b(n) \) modulo 27. He also conjectured the following four congruences:

\[
\begin{align*}
\text{(1.1)} & \quad b(49n + 37) \equiv 0 \pmod{49}, \\
\text{(1.2)} & \quad b(81n + 61) \equiv 0 \pmod{243}, \\
\text{(1.3)} & \quad \sum_{n \geq 0} b(81n + 7)q^n \equiv 9\frac{(q^2; q^3)_{\infty}(q^3; q^3)_{\infty}}{(q^6; q^6)_{\infty}} \pmod{81}, \\
\text{(1.4)} & \quad \sum_{n \geq 0} b(81n + 34)q^n \equiv 36\frac{(q; q)_{\infty}(q^6; q^6)^2}{(q^3; q^3)_{\infty}} \pmod{81}.
\end{align*}
\]

In two recent papers, Lin, Wang, and Xia [13] and Chern [6] independently proved (1.2), (1.3) and (1.4). In both the articles, it was proved that the congruence (1.2) is in fact true modulo 729. Recently, Hirschhorn [8] also proved the congruence (1.2) modulo 729. However, to the best of our knowledge the congruence (1.1) has not been established till date.

In this paper, we prove that the Lin’s conjecture (1.1) is true.

**Theorem 1.1.** For any non-negative integers \( n \), we have

\[
b(49n + 37) \equiv 0 \pmod{49}.
\]

We have \( b(37) = 80832850 \not\equiv 0 \pmod{7^3} \). Hence, unlike to Lin’s conjecture (1.2), the congruence (1.1) is best possible in the sense that the moduli cannot be replaced by higher power of 7 such that the congruence holds for all \( n \geq 0 \).

In [9], Kim introduced a partition function \( \overline{b}(n) \) whose generating function is given by

\[
\overline{B}(q) := \sum_{n=0}^{\infty} \overline{b}(n)q^n = \frac{(-q; q)_{\infty}(-q^2; q^2)_{\infty}}{(q; q)_{\infty}(q^6; q^6)_{\infty}}.
\]

Kim named \( \overline{b}(n) \) as the number of overcubic partition pairs of \( n \). Using arithmetic properties of quadratic forms and modular forms, Kim [9] derived the following two congruences

\[
\begin{align*}
\overline{b}(8n + 7) & \equiv 0 \pmod{64}, \\
\overline{b}(9n + 3) & \equiv 0 \pmod{3}.
\end{align*}
\]

In [11], Lin proved two Ramanujan-like congruences and several infinite families of congruences modulo 3 satisfied by \( \overline{b}(n) \). He also obtained some congruences for \( \overline{b}(n) \) modulo 5.

In this paper, we prove the following two congruences modulo 256 satisfied by \( \overline{b}(n) \).

**Theorem 1.2.** For any non-negative integers \( n \), we have

\[
\overline{b}(72n + t) \equiv 0 \pmod{256},
\]

where \( t \in \{42, 66\} \).

These two congruences are best possible since

\[
\begin{align*}
\overline{b}(72 + 42) = \overline{b}(114) = 173333430318331391232 \not\equiv 0 \pmod{512}, \\
\overline{b}(66) = 407868414339840 \not\equiv 0 \pmod{512}.
\end{align*}
\]
For any fixed positive integer $k$, Gordon and Ono [7] proved that the number of partitions of $n$ into distinct parts is divisible by $2^k$ for almost all $n$. Bringmann and Lovejoy [2] showed that the number of overpartition pairs of $n$ is divisible by $2^k$ for almost all $n$. In [14], Lin proved that the number of overpartition pairs of $n$ into odd parts is also divisible by $2^k$ for almost all $n$.

In this article, we prove that the number of overcubic partition pairs of $n$ is divisible by $2^k$ for almost all $n$.

**Theorem 1.3.** Let $k$ be a positive integer. Then $b(n)$ is almost always divisible by $2^k$, namely,

$$\lim_{X \to \infty} \frac{\# \left\{ n \leq X : b(n) \equiv 0 \pmod{2^k} \right\}}{X} = 1.$$ 

In [10], Kim studied the overpartition analog of cubic partition function. He defined the overcubic partition function $\overline{a}(n)$ whose generating function is given by

$$A(q) := \sum_{n=0}^\infty \overline{a}(n)q^n = \frac{(-q;q)_\infty (-q^2;q^2)_\infty}{(q;q)_\infty (q^2;q^2)_\infty}.$$ 

We also prove that the number of overcubic partitions of $n$ is divisible by $2^k$ for almost all $n$.

**Theorem 1.4.** Let $k$ be a positive integer. Then $\overline{a}(n)$ is almost always divisible by $2^k$, namely,

$$\lim_{X \to \infty} \frac{\# \left\{ n \leq X : \overline{a}(n) \equiv 0 \pmod{2^k} \right\}}{X} = 1.$$ 

### 2. Proof of Theorems 1.1 and 1.2

We define the following matrix groups:

$$\Gamma := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_\infty := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in \Gamma : n \in \mathbb{Z} \right\}.$$ 

For a positive integer $N$, let

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}.$$ 

The index of $\Gamma_0(N)$ in $\Gamma$ is

$$[\Gamma : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}),$$

where $p$ is a prime divisor of $N$.

We prove Theorems 1.1 and 1.2 using the approach developed in [16, 17]. We now recall some of the definitions and results from [16, 17] which will be used to prove our results. Also, see [18]. For a positive integer $M$, let $R(M)$ be the set of integer sequences $r = (r_\delta)_{\delta|M}$ indexed by the positive divisors of $M$. If
$r \in R(M)$ and $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$ are the positive divisors of $M$, we write $r = (r_{\delta_1}, \ldots, r_{\delta_k})$. Define $c_r(n)$ by

$$(2.1) \quad \sum_{n=0}^{\infty} c_r(n)q^n := \prod_{\delta \mid M} (q^{\delta}; q^{\delta})_{\infty}^{r_{\delta}} = \prod_{\delta \mid M} \prod_{n=1}^{\infty} (1 - q^{n\delta})^{r_{\delta}}.$$  

The approach to proving congruences for $c_r(n)$ developed by Radu [16, 17] reduces the number of coefficients that one must check compared with the classical method which uses Sturm’s bound alone.

Let $m$ be a positive integer. For any integer $s$, let $[s]_m$ denote the residue class of $s$ in $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. Let $\mathbb{Z}^*_m$ be the set of all invertible elements in $\mathbb{Z}_m$. Let $\mathcal{S}_m \subseteq \mathbb{Z}_m$ be the set of all squares in $\mathbb{Z}^*_m$. For $t \in \{0, 1, \ldots, m-1\}$ and $r \in R(M)$, we define a subset $P_{m,r}(t) \subseteq \{0, 1, \ldots, m-1\}$ by

$$P_{m,r}(t) := \left\{ t' : [s]_{24m} \in \mathcal{S}_{24m} \text{ such that } t' \equiv ts + \frac{s-1}{24} \sum_{\delta \mid M} \delta r_{\delta} \pmod{m} \right\}.$$  

**Definition 2.1.** Suppose $m, M$ and $N$ are positive integers, $r = (r_{\delta}) \in R(M)$ and $t \in \{0, 1, \ldots, m-1\}$. Let $k = k(m) := \gcd(m^2 - 1, 24)$ and write

$$\prod_{\delta \mid M} \delta^{ir_{\delta}} = 2^s \cdot j,$$

where $s$ and $j$ are nonnegative integers with $j$ odd. The set $\Delta^*$ consists of all tuples $(m, M, N, (r_{\delta}), t)$ satisfying these conditions and all of the following.

1. Each prime divisor of $m$ is also a divisor of $N$.
2. $\delta \mid M$ implies $\delta \mid mN$ for every $\delta \geq 1$ such that $r_{\delta} \neq 0$.
3. $kN \sum_{\delta \mid M} r_{\delta}mN/\delta \equiv 0 \pmod{24}$.
4. $kN \sum_{\delta \mid M} r_{\delta} \equiv 0 \pmod{8}$.
5. $\gcd(\sum_{\delta \mid M} r_{\delta}, 24m)$ divides $N$.
6. If $2 | m$, then either $4 | kN$ and $8 | sN$ or $2 | s$ and $8 | (1 - j)N$.

Let $m, M, N$ be positive integers. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, $r \in R(M)$ and $r' \in R(N)$, set

$$p_{m,r}(\gamma) := \min_{\lambda \in \{0, 1, \ldots, m-1\}} \frac{1}{24} \sum_{\delta \mid M} r_{\delta} \frac{\gcd^2(\delta a + \delta k \lambda c, mc)}{\delta m}$$

and

$$p_{r'}^*(\gamma) := \frac{1}{24} \sum_{\delta \mid N} r_{\delta}^* \frac{\gcd(d, \delta c)}{\delta}.$$  

**Lemma 2.2.** [16 Lemma 4.5] Let $u$ be a positive integer, $(m, M, N, r = (r_{\delta}), t) \in \Delta^*$ and $r' = (r_{\delta}') \in R(N)$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \Gamma$ be a complete set of representatives of the double cosets of $\Gamma_0(N) \backslash \Gamma/\Gamma_\infty$. Assume that $p_{m,r}(\gamma_i) + p_{r'}^*(\gamma_i) \geq 0$ for all $1 \leq i \leq n$. Let $t_{\min} = \min_{t' \in P_{m,r}(t)} t'$ and

$$\nu := \frac{1}{24} \left\{ \left( \sum_{\delta \mid M} r_{\delta} + \sum_{\delta \mid N} r_{\delta}' \right) [\Gamma : \Gamma_0(N)] - \sum_{\delta \mid N} \delta r_{\delta}' \right\} - \frac{1}{24m} \sum_{\delta \mid M} \delta r_{\delta} - \frac{t_{\min}}{m}.$$
If the congruence \( c_r(mn + t') \equiv 0 \pmod u \) holds for all \( t' \in P_{m,r}(t) \) and \( 0 \leq n \leq \lfloor \nu \rfloor \), then it holds for all \( t' \in P_{m,r}(t) \) and \( n \geq 0 \).

To apply the above lemma, we need the following result which gives us a complete set of representatives of the double coset in \( \Gamma_0(N) \backslash \Gamma / \Gamma_\infty \).

**Lemma 2.3.** [18, Lemma 4.3] If \( N \) or \( \frac{1}{2}N \) is a square-free integer, then

\[
\bigcup_{\delta | N} \Gamma_0(N) \left[ \begin{array}{cc} 1 & 0 \\ \delta & 1 \end{array} \right] \Gamma_\infty = \Gamma.
\]

**Proof of Theorem 4.4** We have

\[
\sum_{n=0}^\infty b(n)q^n = \frac{1}{(q; q)_\infty^2 (q^2; q^2)_\infty^2}.
\]

Using binomial theorem, we have

\[
\sum_{n=0}^\infty b(n)q^n = \frac{(q; q)^4_\infty}{(q^4; q^4)_\infty^2 (q^2; q^2)_\infty^2 (q^8; q^8)_\infty^2 (q^7; q^7)_\infty^2}
\]

\[
\equiv \frac{(q; q)^{49}_\infty}{(q^2; q^2)_\infty^2 (q^7; q^7)_\infty^2} \quad \text{(mod 49)}.
\]

We choose \( (m, M, N, r, t) = (49, 14, 14, 47, -2, -7, 0, 37) \) and it is easy to verify that \( (m, M, N, r, t) \in \Delta^* \) and \( P_{m,r}(t) = \{37\} \). By Lemma 2.3, we know that \( \left\{ \left[ \begin{array}{cc} 1 & 0 \\ \delta & 1 \end{array} \right] : \delta | 14 \right\} \) forms a complete set of double coset representatives of \( \Gamma_0(N) \backslash \Gamma / \Gamma_\infty \).

Let \( \gamma_\delta = \left[ \begin{array}{cc} 1 & 0 \\ \delta & 1 \end{array} \right] \). Let \( r' = (12, 0, 0, 0) \in R(14) \) and we use Sage to verify that \( p_{m,r}(\gamma_\delta) + p_{r'}(\gamma_\delta) \geq 0 \) for each \( \delta | N \). We compute that the upper bound in Lemma 2.2 is \( |\nu| = 48 \). Using Mathematica we verify that \( b((49n + 37) \equiv 0 \pmod{49}) \) for any \( n \leq 48 \).

Example:

\[
b(49 \times 48 + 37) = 25470817092268563525750091322454383557126791567591335
\]
\[
339788372343399013917787449842071480
\]
\[
= 519812593719766602566345167847833379005467666855
\]
\[
37455914048415171408447301784690654520.
\]

Thus, by Lemma 2.2 we conclude that \( b((49n + 37) \equiv 0 \pmod{49}) \) for any \( n \geq 0 \).

**Proof of Theorem 1.2** We first recall the following 2-dissection formula from [11, p. 40, Entry 25]:

\[
\frac{1}{(q; q)_\infty^4} = \frac{(q^4; q^4)_\infty^{14}}{(q^2; q^2)_\infty^{12}(q^8; q^8)_\infty^4} + 4q \frac{(q^4; q^4)_\infty^2 (q^8; q^8)_\infty^4}{(q^2; q^2)_\infty^{10}}.
\]
Employing (2.2) into (1.5) and using the fact that $(-q: q)_\infty = \frac{(q^2: q^2)_\infty}{(q: q)_\infty}$, we have

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{(q^4: q^4)_\infty^2}{(q: q)_\infty^2(q^2: q^2)_\infty^2} \left( \frac{(q^4: q^4)_\infty^4}{(q^2: q^2)_\infty^4(q^8: q^8)_\infty^4} + 4q \frac{(q^4: q^4)_\infty^4(q^8: q^8)_\infty^4}{(q^2: q^2)_\infty^{10}} \right).$$

Extracting the terms containing $2n$ and then using (2.2), we obtain

$$\sum_{n=0}^{\infty} b(2n)q^n = \frac{(q^2: q^2)_\infty^{16}}{(q: q)_\infty^{10}(q^4: q^4)_\infty^{8}} + 16q \frac{(q^4: q^4)_\infty^{36}}{(q^2: q^2)_\infty^{30}(q^8: q^8)_\infty^{8}} + 96q^2 \frac{(q^4: q^4)_\infty^{28}}{(q^2: q^2)_\infty^{32}} + 256q^4 \frac{(q^4: q^4)_\infty^{16}(q^8: q^8)_\infty^{8}}{(q^2: q^2)_\infty^{24}} + 256q^4 \frac{(q^4: q^4)_\infty^{16}(q^8: q^8)_\infty^{8}}{(q^2: q^2)_\infty^{24}} (mod 256).$$

Extracting the terms containing $2n+1$, we obtain, modulo 256

$$\sum_{n=0}^{\infty} b(4n+2)q^n = 16 \frac{(q^2: q^2)_\infty^{40}}{(q: q)_\infty^{36}(q^4: q^4)_\infty^{8}} \left( \frac{(q^2: q^2)_\infty^{16}}{(q: q)_\infty^{8}} \right) \left( \frac{(q^4: q^4)_\infty^{24}}{(q^2: q^2)_\infty^{12}(q^4: q^4)_\infty^{8}} \right) + 4q \frac{(q^4: q^4)_\infty^{14}(q^8: q^8)_\infty^{4}}{(q^2: q^2)_\infty^{10}} \left( \frac{(q^4: q^4)_\infty^{14}(q^8: q^8)_\infty^{4}}{(q^2: q^2)_\infty^{10}} \right).$$

Finally, extracting the terms containing $2n$, we obtain

$$(2.3) \quad \sum_{n=0}^{\infty} b(8n+2)q^n = 16 \frac{(q: q)_\infty^{10}(q^2: q^2)_\infty^{6}}{(q^2: q^2)_\infty^{6}} \quad (mod 256).$$

Now, we choose $(m, M, N, r, t) = (9, 8, 12, (10, 6, -4, 0), 5)$ and it is easy to verify that $(m, M, N, r, t) \in \Delta^*$ and $P_{m,r}(t) = \{5, 8\}$. From Lemma 2.3 we know that

$$\left\{ \begin{array}{l} 1 \\ \delta \end{array} \right\} : |\delta| = 12$$

forms a complete set of double coset representatives of $\Gamma_0(N) \backslash \Gamma / \Gamma_0$. Let $r^* = (0, 0, 0, 0, 0, 0) \in R(12)$ and we use Sage to verify that $p_{m,r}(\gamma_\delta) + p_{r^*}(\gamma_\delta) \geq 0$ for each $\delta | N$, where $\gamma_\delta = \left[ \begin{array}{l} 1 \\ \delta \end{array} \right]$. We compute that the upper bound in Lemma 2.2 is $|\nu| = 11$. Using Mathematica we verify that $b(72n+42) \equiv 0 \ (mod \ 256)$ and $b(72n+66) \equiv 0 \ (mod \ 256)$ for $n \leq 11$. For example:
\[ b(72 \times 11 + 42) = 77612033164641263022428157070444521415046269280561371983344 \]
\[ = 256 \times 30317200455025049336813599885564239117775244893771928593099 \]
and
\[ b(72 \times 11 + 66) = 51908576790072477115950747871530891342334377184238781729434112 \]
\[ = 256 \times 202767878086220613734182608731672231805993660875932741130602. \]
Thus, by Lemma 2.2, we conclude that \( b(72n + t) \equiv 0 (\text{mod } 256) \) for any \( n \geq 0 \), where \( t \in \{42, 66\} \). This completes the proof of the theorem. \( \square \)

3. Proof of Theorems 1.3 and 1.4

Recall that the Dedekind's eta-function \( \eta(z) \) is defined by
\[ \eta(z) := q^{1/24} \prod_{n=1}^{\infty} \left(1 - q^n\right), \]
where \( q := e^{2\pi i z} \) and \( z \) is in the upper half complex plane. A function \( f(z) \) is called an eta-quotient if it is of the form
\[ f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}, \]
where \( N \) is a positive integer and \( r_\delta \) is an integer. We now recall two theorems from [15, p. 18] which will be used to prove our result.

**Theorem 3.1.** [15, Theorem 1.64 and Theorem 1.65] If \( f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta} \) is an eta-quotient with \( \ell = \frac{1}{2} \sum_{\delta \mid N} r_\delta \in \mathbb{Z} \), with the additional properties that
\[ \sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24} \]
and
\[ \sum_{\delta \mid N} N \delta r_\delta \equiv 0 \pmod{24}, \]
then \( f(z) \) satisfies
\[ f \left( \frac{az + b}{cz + d} \right) = \chi(d)(cz + d)^\ell f(z) \]
for every \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) \). Here the character \( \chi \) is defined by
\[ \chi(d) := \left( \frac{(-1)^{\ell} \prod_{\delta \mid N} \delta^{r_\delta}}{d^{\ell}} \right). \]
In addition, if \( c, d, \) and \( N \) are positive integers with \( d \mid N \) and \( \gcd(c, d) = 1 \), then the order of vanishing of \( f(z) \) at the cusp \( \frac{a}{d} \) is
\[ \frac{N}{24} \sum_{\delta \mid N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, \delta)} \cdot \frac{1}{d}. \]

Suppose that \( \ell \) is a positive integer and that \( f(z) \) is an eta-quotient satisfying the conditions of the above theorem. If \( f(z) \) is holomorphic at all of the cusps of
Γ₀(N), then \(f(z) \in M_\ell(Γ₀(N), χ)\). We now use this fact for the eta-quotient \(B_k\) defined by
\[
B_k(z) = \frac{η(48z)^{2k-2}}{η(24z)^4η(96z)^{2k-1}-2}.
\]

Using Theorem 3.1, we find that \(B_k(z) \in M_{2k-2-2}(Γ₀(384))\) for \(k \geq 4\).

From [15], we can rewrite \(B(q^{24})\) as the following eta-quotient
\[
B(z) = \frac{η(96z)^2}{η(24z)^4η(48z)^2}.
\]

Let
\[
F_k(z) = \frac{η(48z)^{2k}}{η(96z)^{2k-1}}.
\]

Then, we have
\[
B_k(z) = \frac{η(48z)^{2k-2}}{η(24z)^4η(96z)^{2k-1}-2} = B(z)F_k(z).
\]

It is not hard to establish the fact that \(F_k(z) \equiv 1 \pmod{2^k}\). Thus,
\[
B_k(z) \equiv B(z) \pmod{2^k}.
\]

Now, let \(m\) be a positive integer. From a deep theorem of Serre [15, p. 43], it follows that if \(f(z)\) is an integral weight modular form in \(M_k(Γ₀(N), χ)\) which has a Fourier expansion
\[
f(z) = \sum_{n=0}^{\infty} a(n)q^n \in \mathbb{Z}[q],
\]
then there is a constant \(α > 0\) such that
\[
# \{n \leq X : a(n) \not\equiv 0 \pmod{m} \} = O \left( \frac{X}{(\log X)^α} \right).
\]

Since \(B_k(z) \in M_{2k-2-2}(Γ₀(384))\) for \(k \geq 4\), the Fourier coefficients of \(B_k(z)\) are almost always divisible by \(2^k\) and so are the Fourier coefficients of \(B(z)\). Now,
\[
B(z) = B(q^{24}) = \sum_{n=0}^{\infty} b(n)q^{24n},
\]
and hence \(b(n)\) is a multiple of \(2^k\) for almost all \(n\) and \(k \geq 4\). This also trivially implies that \(b(n)\) is a multiple of \(2^k\) for almost all \(n\) and \(k < 4\). This completes the proof of the Theorem 1.3.

We now prove Theorem 1.4. The generating function of \(π(n)\) is given by
\[
\overline{A}(q) := \sum_{n=0}^{\infty} \overline{a(n)}q^n = \frac{(-q; q)_{∞}(-q^2; q^2)_{∞}}{(q; q)_{∞}(q^2; q^2)_{∞}}.
\]

Let
\[
A_k(z) = \frac{η(48z)^{2k-1}}{η(24z)^4η(96z)^{2k-1}}.
\]

Using Theorem 3.1, we find that \(A_k(z) \in M_{2k-2-1}(Γ₀(768))\) for \(k > 2\). We also have
\[
A_k(z) = \overline{A}(z)F_k(z) \equiv \overline{A}(z) \pmod{2^k}.
\]

Following the proof of Theorem 1.3, we now readily arrive at the desired result. This recompletes the proof of Theorem 1.4.
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