Entropic dynamics on Gibbs statistical manifolds

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Abstract

Entropic dynamics is a framework in which the laws of dynamics are derived as an application of entropic methods of inference. Its successes include the derivation of quantum mechanics and quantum field theory from probabilistic principles. Here we develop the entropic dynamics of a system the state of which is described by a probability distribution. Thus, the dynamics unfolds on a statistical manifold which is automatically endowed by a metric structure provided by information geometry. The curvature of the manifold has a significant influence. Although the method is of general applicability we focus on the statistical manifold of Gibbs distributions (also known as canonical distributions or the exponential family). The model includes an “entropic” notion of time that is tailored to the system under study; the system is its own clock. As one might expect entropic time is intrinsically directional; there is a natural arrow of time which leads to a simple description of the approach to equilibrium. As illustrative examples we discuss dynamics on a space of Gaussians and the discrete 3-state system.

Keywords: Entropic Dynamics, Maximum Entropy, Information Geometry, Canonical Distributions, Exponential Family, Onsager Reciprocal Relations.

1 Introduction

The original method of Maximum Entropy (MaxEnt) is usually associated with the names of Shannon [1] and Jaynes [2-5] although its roots can be traced to Gibbs [6]. The method was designed to assign probabilities on the basis of partial information in the form of expected value constraints and the central quantity, called entropy, which was interpreted as a measure of uncertainty or as an amount of missing information. In a series of developments starting with Shore and Johnson [7] with further contributions from other authors [8-12] the range of applicability of the method was significantly extended. In its new incarnation the purpose of the method of Maximum Entropy, which will be referred as ME to distinguish it from the older version, is to update probabilities from arbitrary priors when new information in the form of constraints is considered [13]. Highlights of the new method include: (1) A unified treatment of Bayesian and entropic methods which demonstrates their mutual consistency. (2) A new concept of entropy as a tool for reasoning that requires no interpretation in terms of heat, multiplicities, disorder, uncertainty, or amount of information. Indeed, entropy needs no interpretation; we only need to know how to use it. (3) A Bayesian concept of information defined in terms of its effects on the beliefs of rational agents — the constraints are the information. (4) The possibility of information that is not in the form of expected value constraints. (We shall see an example below.)
The old MaxEnt was sufficiently versatile to provide the foundations to equilibrium statistical mechanics [2] and to find application in a wide variety of fields such as economics [14], ecology [15,16], cellular biology [17,18], and opinion dynamics [19,20]. As is the case with thermodynamics, all these applications are essentially static. MaxEnt has also been deployed to non-equilibrium statistical mechanics (see [21,22] and subsequent literature in maximum caliber e.g. [23–25]) but the dynamics is not intrinsic to the probabilities; it is induced by the underlying Hamiltonian dynamics of the molecules. Such dynamical models cannot be generalized beyond physics.

The ME version of the maximum entropy method offers the possibility of developing a true dynamics of probabilities. It is a dynamics driven by entropy — an Entropic Dynamics (ED) — which automatically makes it consistent with the principles for updating probabilities. ED naturally leads to an “entropic” notion of time. Entropic time is a device designed to keep track of the accumulation of changes. Its construction involves three ingredients: one must introduce the notion of an instant, verify that these instants are suitably ordered, and finally one must define a convenient notion of duration, or interval between successive instants. One welcome feature is that entropic time is tailored to the system under study; the system is its own clock. Another welcome feature is that such an entropic time is intrinsically directional — an arrow of time is generated automatically.

Entropic dynamics has been successful in reconstructing dynamical models in physics such as quantum mechanics [26,27], quantum field theory [28], and the renormalization group [29]. Beyond physics, it has been recently applied to the fields of finance [30,31] and neural networks [32].

Here we aim for a different class of applications of entropic dynamics. The macrostate of the system is described by a Gibbs distribution over some generic space of microstates. The goal is to study the ED generated by transitions from one macrostate to another. The main assumptions are that changes happen and that they are not discontinuous. We do not explain why changes happen — this is a mechanics without a mechanism. Our goal is to venture an educated estimate of what changes one expects to happen. The second assumption — that systems evolve along continuous trajectories — receives significant support from the observation of nature. It also implies that the study of motion involves two tasks. The first is to describe how a single infinitesimal step occurs. Then second requires a scheme to keep track of how a large number of these short steps accumulate to produce a finite motion. It is the latter task that involves the introduction of the concept of time.

The fact that the space of macrostates is a statistical manifold — each point in the space is a probability distribution — has a profound effect on the dynamics. The reason is that statistical manifolds are naturally endowed with a Riemannian metric structure given by the Fisher-Rao information metric [33–37]. The particular case of Gibbs distributions leads to additional interesting geometrical properties (see e.g. [38,39]) which have been explored in the extensive work relating statistical mechanics to information geometry [40–44].

In this paper we tackle the more formal aspects of an ED on Gibbs manifolds and offer a couple of illustrative examples. Furthermore, we carefully discuss one emergent feature of our model; that is, the average change in the macrostate of the system being proportional the gradient of entropy with symmetric (reciprocal) factors. The latter resembles the Onsager reciprocal relations [45] known within nonequilibrium statistical mechanics.

It is important to emphasize that the entropic dynamics developed here is not a form of nonequilibrium statistical mechanics. Although both describe the changes of macrostates, nonequilibrium statistical mechanics is driven by the microstate dynamics which is given by classical or quantum mechanics, while entropic dynamics is completely agnostic of any

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1Gibbs distributions are also known as canonical distributions, or Gibbs measures, or as the exponential family.
microstate dynamics. This is particularly relevant because the use of methods once restricted to statistical physics in a broad range of scientific endeavours suggests the need for more general dynamical models.

The article is organized as follows: next section discusses the space of Gibbs distributions and its geometric properties; section 3 considers the ideas of entropic dynamics; section 4 tackles the difficulties associated with formulating ED on the curved space of probability distributions; section 5 introduces the notion of entropic time; section 6 describes the evolution of the system in the form of a differential equation; section 7 we offer two illustrative examples of ED on a Gaussian manifold and on a 2-simplex; and finally in section 8, we discuss the resemblance and differences between the ED developed here and the Onsager reciprocity relations.

2 The statistical manifold of Gibbs distributions

2.1 Gibbs distributions

The canonical or Gibbs probability distributions represent the macrostate of a system. They describe a state of uncertainty about the microstate $x \in \mathcal{X}$ of the macroscopic system. A canonical distribution $\rho(x)$ is assigned by maximizing the entropy

$$ S[\rho|q] = - \int dx \rho(x) \log \frac{\rho(x)}{q(x)} $$

relative to the prior $q(x)$ subject to $n$ expected value constraints

$$ \int dx \rho(x) a^i(x) = A^i \quad \text{with} \quad i = 1 \ldots n \, , $$

and normalization of $\rho(x)$. Typically the prior $q(x)$ is chosen to be a uniform distribution over the space $\mathcal{X}$ so that it is maximally non-informative but this is not strictly necessary. The $n$ constraints, on the other hand, reflect the information that happens to be physically relevant to the problem. The resulting canonical distribution is

$$ \rho(x|\lambda) = \frac{q(x)}{Z(\lambda)} \exp[-\lambda_i a^i(x)] $$

where $\lambda = \{\lambda_1 \ldots \lambda_n\}$ are the Lagrange multipliers associated to the expected value constraints and we adopt the Einstein summation convention. The normalization constant is

$$ Z(\lambda) = \int dx q(x) \exp[-\lambda_i a^i(x)] = e^{-F(\lambda)} $$

where $F = -\log Z$ plays a role analogous to the free energy. The Lagrange multipliers $\lambda_i(A)$ are implicitly defined by

$$ \frac{\partial F}{\partial \lambda_i} = A^i . $$

Evaluating the entropy (1) at its maximum yields

$$ S(A) = - \int dx \rho(x|\lambda(A)) \log \frac{\rho(x|\lambda(A))}{q(x)} = \lambda_i(A) A^i - F(\lambda(A)) . $$

which we shall call the macrostate entropy or (when there is no risk of confusion) just the entropy. Equation (6) shows that $S(A)$ is the Legendre transform of $F(\lambda)$: a small change $dA^i$ in the constraints shows that $S(A)$ is indeed a function of the expected values $A^i$,

$$ dS = \lambda_i dA^i \quad \text{so that} \quad \lambda_i = \frac{\partial S}{\partial A^i} . $$
Table 1: Identification of sufficient statistics, priors and Lagrange multipliers for some well-known probability distributions.

| Distribution            | λ parameter | Suff. Stat. | Prior         |
|-------------------------|-------------|-------------|---------------|
| Exponent Polynomial     | λ = β       | a(x) = x^k  | uniform       |
| Gaussian                | λ = \(-\frac{\mu}{\sigma^2}, \frac{1}{2\sigma^2}\) | a(x) = x, x^2 | uniform       |
| Multinomial (k)         | λ = \(-\log(\theta_1, \theta_2, \ldots, \theta_k)\) | a(x) = (x_1, \ldots, x_k) | q(x) = \prod_{i=1}^k x_i! |
| Poisson                 | λ = \(-\log m\) | a(x) = x | q(x) = 1/x! |
| Mixed power laws        | λ = (α, β)  | a = (log x, x) | uniform       |

One might think that defining a dynamics on the family of canonical distributions might be too restricted to be of interest; however this family has widespread applicability. Here it has been derived using the method of maximum entropy but historically it is also known as the exponential family. The latter is the only family of distributions that possess sufficient statistics \(^2\) which turn out to be the functions \(a_i(x)\) in \((\ref{eq:1})\). In Table 1 we give a short list of the priors \(q(x)\) and the functions \(a_i(x)\) that lead to well known distributions \([39,50]\).

Naturally, the method of maximum entropy assumes that the various constraints are compatible with each other so that the set of multipliers \(\lambda\) exists. It is further assumed that the constraints reflect physically relevant information so that the various functions such as \(A^i(\lambda) = \frac{\partial}{\partial \lambda_i}F\) and \(\lambda_i(A) = \frac{\partial}{\partial A^i}S\) that appear in the formalism are both invertible and differentiable and so that the space of Gibbs distributions is indeed a manifold. However, the manifold may include singularities of various kinds which are of particular interest as it may describe phenomena such as phase transitions \([40,42]\).

### 2.2 Information Geometry

In order to establish the notation and to recall some results that will be needed in later sections we offer a brief review of well known results concerning the information geometry of canonical distributions \([36,38]\).

To each set of expected values \(A = \{A^1, A^2, \ldots, A^n\}\), or to the associated set of Lagrange multipliers \(\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}\), there corresponds a canonical distribution. Therefore the set of distributions \(\rho(x|\lambda)\) or, equivalently \(\rho(x|A)\), is a statistical manifold in which each point can be labelled by the coordinates \(\lambda\) or by \(A\). Whether we choose \(\lambda\) or \(A\) as coordinates is purely a matter of convenience. The change of coordinates is implemented using

\[
\frac{\partial A^i}{\partial \lambda_k} = -\frac{\partial^2 \log Z}{\partial \lambda_k \partial \lambda_i} = A^k A^i - \langle a^k a^i \rangle, \tag{8}
\]

where we recognize the covariance tensor,

\[
C^{ij} = \langle (a^i - A^i)(a^j - A^j) \rangle = -\frac{\partial A^i}{\partial \lambda_j}. \tag{9}
\]

Its inverse is given by

\[
C_{jk} = -\frac{\partial \lambda_j}{\partial A^k} = -\frac{\partial^2 S}{\partial A^j \partial A^k}. \tag{10}
\]

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\(^2\) Interestingly, this was a problem proposed by Fisher \([46]\) in the primordium of statistics and later proved independently by Pitman \([47]\), Darmois \([48]\), and Koopman \([49]\).
so that
\[ C^{ij}C_{jk} = \frac{\partial A^i}{\partial A^k} = \delta^i_k \]  \hspace{1cm} (11)

Statistical manifolds are metric spaces. There is an essentially unique measure of the extent to which two neighboring distributions \( \rho(x|A) \) and \( \rho(x|A + dA) \) can be distinguished from each other. This measure of distinguishability provides a statistical notion of distance which is given by FRIM, \( d\ell^2 = g_{ij}dA^idA^j \) where
\[ g_{ij} = \int dx \rho(x|A) \frac{\partial \log \rho(x|A)}{\partial A^i} \frac{\partial \log \rho(x|A)}{\partial A^j}. \]  \hspace{1cm} (12)

For a broader discussion on existence, derivation and consistency of this metric, as well as its properties see [36,38], it suffices to say here that FRIM is the unique metric structure that preserves desired properties of probability distributions [51,52].

To calculate \( g_{ij} \) for canonical distributions we use
\[ g_{ij} = \frac{\partial \lambda_k}{\partial A^i} \frac{\partial \lambda_l}{\partial A^j} \int dx \rho \frac{\partial \log \rho}{\partial \lambda_k} \frac{\partial \log \rho}{\partial \lambda_l} \]  \hspace{1cm} (13)
and
\[ \frac{\partial \log \rho(x|A)}{\partial \lambda_k} = A^k - a^k(x) \]  \hspace{1cm} (14)
so that, using (8-11), we have
\[ g_{ij} = C_{ik}C_{lj}C_{kl} = C_{ij}. \]  \hspace{1cm} (15)

Therefore the metric tensor \( g_{ij} \) is the inverse of the covariance matrix \( C^{ij} \) which, by (10), is the Hessian of the entropy.

As mentioned above, instead of \( A^i \) we could use the Lagrange multipliers \( \lambda_i \) as coordinates. Then the information metric is the covariance matrix,
\[ g^{ij} = \int dx \rho(x|\lambda) \frac{\partial \log \rho(x|\lambda)}{\partial \lambda_i} \frac{\partial \log \rho(x|\lambda)}{\partial \lambda_j} = C^{ij}. \]  \hspace{1cm} (16)

Therefore the distance \( d\ell \) between neighboring distributions can be written in either of two equivalent forms,
\[ d\ell^2 = g_{ij}dA^idA^j = g^{ij}d\lambda_id\lambda_j. \]  \hspace{1cm} (17)

Incidentally, the availability of a unique measure of volume \( dV = (\det g_{ij})^{1/2}d^nA \) implies that there is a uniquely defined notion of the uniform distribution over the space of macrostates. The uniform distribution \( P_u \) assigns equal probabilities to equal volumes, so that
\[ P_u(A)d^nA \propto g^{1/2}d^nA \quad \text{where} \quad g = \det g_{ij}. \]  \hspace{1cm} (18)

To conclude this overview section we note that the metric tensor \( g_{ij} \) can be used to lower the contravariant indices of a vector to produce its dual covector. Using (10) and (12) the covector \( dA_i \) dual to the infinitesimal vector with components \( dA^i \) is
\[ dA_i = g_{ij}dA^j = -\frac{\partial \lambda_i}{\partial A^j}dA^j = -d\lambda_i. \]  \hspace{1cm} (19)
which shows that not only are the coordinates \( A \) and \( \lambda \) related through a Legendre transformation, which is a consequence of entropy maximization, but also through a vector-covector duality, i.e. \(-d\lambda_i \) is the covector dual to \( dA^i \), which is a consequence of information geometry.
3 Entropic Dynamics

Having established the necessary background we can now develop an entropic framework to describe dynamics on the space of macrostates.

3.1 Change happens

Our starting point is the observation that in nature changes happen continuously. Therefore the dynamics we wish to formulate will assume that the system evolves along continuous paths. This assumption of continuity represents a significant simplification because it implies that a finite motion can be analyzed as the accumulation of a large number of infinitesimally short steps. Thus, our first goal will be to find the probability $P(A'|A)$ that the system takes a short step from the macrostate $A$ to the neighboring macrostate $A' = A + dA$. The transition probability $P(A'|A)$ will be assigned by maximizing an entropy. This requires, first, that we identify the particular entropy that is relevant to our problem. Next, we must decide on the prior distribution: what short steps might we expect before we know the specifics of the motion. Finally, we stipulate the constraints that are meant to capture the information that is relevant to the particular problem at hand.

To settle the first item — the choice of entropy — we note that not only we are uncertain about the macrostate at $A$ but we are also uncertain about the microstates $x \in \mathcal{X}$. This means that the actual universe of discourse is the joint space $A \times \mathcal{X}$ and the appropriate statistical description of the system is in terms of the joint distribution

$$P(x, A) = \rho(x|A)P(A) \quad (20)$$

Where $\rho$ is of form [3] which means that we impose $P(x|A)$ to be canonical and the distribution $P(A)$ represents our lack of knowledge about the macrostates.

Our immediate task is to find the transition probability for a step $P(x', A'|x, A)$ by maximizing the entropy

$$S[P|Q] = -\int dA'dx' P(x', A'|x, A) \log \frac{P(x', A'|x, A)}{Q(x', A'|x, A)} , \quad (21)$$

relative to the prior $Q(x', A'|x, A)$ and subject to constraints to be discussed below. (To simplify the notation in multidimensional integrals we write $d^nA' = dA'$ and $d^n x = dx$.)

Although $S$ in (6) and $S$ in (21) are both entropies, in the information theory sense, they represent two very distinct statistical objects. The $S(A)$ in (6) is what one may be used to from statistical mechanics, which is the entropy of the macrostate, while the $S[P|Q]$ in (21) is the entropy to be maximized so that we find the transition probability that better matches the information at hand, that means $S$ is a tool to select the dynamics of the macrostates.

3.2 The Prior

We adopt a prior that implements the idea that the system evolves by taking short steps $A \rightarrow A + \Delta A$ at the macrostate level but is otherwise maximally uninformative. We write

$$Q(x', A'|x, A) = Q(x'|x, A, A')Q(A'|x, A) \quad (22)$$

Note that what we did in (20) is nothing more than assuming a probability distribution for the macrostates. This description is sometimes referred to as superstatistics [53].
and analyze the two factors in turn. We shall assume that a priori, before we know the relation
between the microstates \(x\) and the macrostate \(A\), the prior distribution for \(x'\) is the same
uniform underlying measure \(q(x')\) introduced in \([1]\),

\[
Q(x', A, A') = q(x') .
\]  

(23)

Next we tackle the second factor \(Q(A'|x, A)\). As shown in appendix A using the method of
maximum entropy the prior that enforces short steps but is otherwise maximally uninformative
in that it is spherically symmetric is

\[
Q(A'|x, A) = Q(A'|A) \propto g^{1/2}(A') \exp \left[ -\frac{1}{2\tau} g_{ij} \Delta A^i \Delta A^j \right] .
\]  

(24)

We see that steps of length

\[
\Delta \ell \sim (g_{ij} \Delta A^i \Delta A^j)^{1/2} >> \tau^{1/2}
\]  

(25)

have negligible probability and eventually we will take the limit \(\tau \to 0\). The prefactor \(g^{1/2}(A')\)
ensures that \(Q(A'|A)\) is a probability density.

### 3.3 The constraints

The piece of information we wish to codify through the constraints is the simple geometric
idea that the dynamics remains confined to the statistical manifold \(A\). This is implemented by
writing

\[
P(x', A'|x, A) = P(x'|x, A, A') P(A'|x, A)
\]  

(26)

and imposing that the distribution for \(x'\) is a canonical distribution

\[
P(x'|x, A, A') = \rho(x'|A') \in A .
\]  

(27)

This means that given \(A'\) the distribution of \(x'\) is independent of the initial microstate \(x\) and
macrostate \(A\). The second factor in \([26]\), \(P(A'|x, A)\), is the transition probability we seek.
Note that this is constraint is not of the form of an expected value.

Depending on the particular system under consideration one could formulate richer forms
of dynamics by imposing additional constraints. To give just one example, one could introduce
some drift relative to the direction specified by a covector \(F_i\) by imposing a constraint of the
form \(\langle \Delta A^i \rangle F_i = \kappa\) (see \([27][28]\)). In this paper however we shall limit ourselves to what is
perhaps the simplest case, the minimal ED described by the single constraint \([27]\).

### 3.4 Maximizing the entropy

Substituting \([24]\) and \([27]\) into \([21]\) and rearranging we find

\[
S[P|Q] = \int dA' P(A'|x, A) \left[ -\log \frac{P(A'|x, A)}{Q(A'|A)} + S(A') \right]
\]  

(28)

where \(S(A')\) is the macrostate entropy given in \([6]\). Maximizing \(S\) subject to normalization
gives

\[
P(A'|x, A) \propto Q(A'|A) e^{S(A')}
\]  

\[
\propto g^{1/2}(A') \exp \left[ -\frac{1}{2\tau} g_{ij} \Delta A^i \Delta A^j + S(A') \right] .
\]  

(29)

7
It is noteworthy that $P(A'|x, A)$ turned out to be independent of $x$ which is not surprising since neither the prior nor the constraints indicate any correlation between $A'$ and $x$.

The exponent in (29) has a quadratic term, as discussed when presenting the prior (24), the transition from $A$ to $A'$ has to be an arbitrarily small continuous change. This allows for a linear approximation of $S$, making so that the exponential factor is quadratic in $\Delta A$

$$P(A'|A) = \frac{g^{1/2}(A')}{\mathcal{Z}} \exp \left[ \frac{\partial S}{\partial A^i} \Delta A^i - \frac{1}{2\tau} g_{ij} \Delta A^i \Delta A^j \right].$$  \hspace{1cm} (30)

Where $e^{S(A)}$ was absorbed in the normalization factor $\mathcal{Z}$. This is the transition probability selected from maximum entropy (21). However some mathematical difficulties arise from the fact that (30) is defined over a curved manifold. We are going to explore these mathematical issues and their consequences to motion in the following section.

4 The transition probability

Since the statistical manifold is a curved space, we must understand how the transition probabilities (30) behave under a change of coordinates. As (24) and (30) describe an arbitrarily small step, we wish to express the transition probability, as well as quantities derived from it, calculated up to the order of $\tau$. (30) has the a quadratic motion of order $\tau^0$ and, in it the squared displacement $\Delta A^i \Delta A^j$ is of order $\tau$ while the linear displacement $\Delta A^i$ is of order $\tau^{1/2}$, therefore, even in the limit $\tau \to 0$, the transition will be affected by curvature effects.

From (30) we can calculate the first two moments of this motion, $\langle \Delta A^i \rangle$ and $\langle \Delta A^i \Delta A^j \rangle$, since they are the only two larger than $o(\tau)$. In particular, the exponent in (30) is manifestly invariant, so that one can complete squares and obtain

$$P(A'|A) = \frac{g^{1/2}(A')}{\mathcal{Z}'} \exp \left[ -\frac{1}{2\tau} g_{ij} \left( \Delta A^i - \tau \frac{\partial S}{\partial A^i} \right) \left( \Delta A^j - \tau \frac{\partial S}{\partial A^j} \right) \right]. \hspace{1cm} (31)$$

This reassembles a Gaussian if $g^{1/2}(A)$ were constant, we could immediately identify $\langle \Delta A^i \rangle = \tau g^{ij} \frac{\partial S}{\partial A^j}$ and $\langle \Delta A^i \Delta A^j \rangle = \tau g^{ij}$. However, even in a flat space with curvilinear coordinates, this is not necessarily true. Moments are calculated directly by

$$\langle \Delta A^i \rangle = \int dA' \Delta A^i P(A'|A) \hspace{1cm} \text{(32)}$$

and

$$\langle \Delta A^i \Delta A^j \rangle = \int dA' \Delta A^i \Delta A^j P(A'|A) \hspace{1cm} \text{(33)}$$

where $V^i = g^{ij} \frac{\partial S}{\partial A^j}$. We can also compute the second moment

$$\langle \Delta A^i \Delta A^j \rangle = \int dA' \Delta A^i \Delta A^j P(A'|A) \hspace{1cm} \text{(33)}$$

To facilitate the calculation of the integrals in (32) and (33) it is convenient to write (30) in normal coordinates (NC) at $A$ – labeled with Greek letter indexes $(\mu, \nu)$. In this coordinate system

$$g_{\mu\nu}(A) = \delta_{\mu\nu} \quad \text{and} \quad \frac{\partial g_{\mu\nu}}{\partial \bar{A}^\rho} \bigg|_A = 0 \hspace{1cm} (34)$$
allowing us to approximate \( g(A') = 1 \) for a short transition. A displacement in these coordinates \( \Delta A^\mu \) can be related to the original coordinates by a Taylor expansion in terms of \( \Delta A^i \) as (see \[54,55\])

\[
\Delta A^\mu = \frac{\partial A^\mu}{\partial A^i} \Delta A^i + \frac{1}{2} \frac{\partial^2 A^\mu}{\partial A^j \partial A^k} \Delta A^j \Delta A^k + o(\tau) .
\] (35)

To proceed it is interesting to recall the Christoffel symbols \( \Gamma^i_{jk} \),

\[
\Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{jl}}{\partial A^k} + \frac{\partial g_{lj}}{\partial A^k} - \frac{\partial g_{jk}}{\partial A^l} \right),
\] (36)

which transform as

\[
\Gamma^i_{jk} = \frac{\partial A^i}{\partial A^\mu} \frac{\partial A^\nu}{\partial A^\sigma} \Gamma^\mu_{\nu\sigma} - \frac{\partial A^i}{\partial A^\mu} \frac{\partial A^k}{\partial A^\nu} \tau \delta^\mu_{\nu} .
\] (37)

Since in normal coordinates we have \( \Gamma^\mu_{\nu\sigma} = 0 \), this allow us to isolate \( \Delta A^i \) up to the order \( \tau \) obtaining

\[
\Delta A^i = \frac{\partial A^i}{\partial A^\mu} \Delta A^\mu - \frac{1}{2} \Gamma^i_{jk} \Delta A^j \Delta A^k ,
\] (38)

By squaring (38) we have

\[
\Delta A^i \Delta A^j = \frac{\partial A^i}{\partial A^\mu} \frac{\partial A^j}{\partial A^\nu} \Delta A^\mu \Delta A^\nu + o(\tau) .
\] (39)

Since the exponent in (32) is invariant and in a coordinate transformation we have

\[ dA P(A) = d\bar{A} P(\bar{A}) \], it separates in two terms.

\[
\langle \Delta A^i \rangle = \frac{\partial A^i}{\partial A^\mu} \frac{1}{Z'} \int dA' \Delta A^\mu \exp \left[ -\frac{\delta_{\nu\sigma}}{2\tau} (\Delta A^\nu - \tau V^\nu) (\Delta A^\sigma - \tau V^\sigma) \right] 
- \frac{1}{2} \Gamma^i_{jk} \frac{\partial A^i}{\partial A^\mu} \frac{\partial A^j}{\partial A^\nu} \frac{1}{Z'} \int dA' \Delta A^\mu \Delta A^\nu \exp \left[ -\frac{\delta_{\nu\sigma}}{2\tau} (\Delta A^\nu - \tau V^\nu) (\Delta A^\sigma - \tau V^\sigma) \right].
\] (40)

The integrals can be evaluated from the known properties of a Gaussian. The integral on the fist term gives \( \langle \Delta A^\mu \rangle = \tau \delta^{\mu\nu} \frac{\partial S}{\partial A^\nu} \) and the integral in the second term gives \( \langle A^\mu A^\nu \rangle = \tau \delta^{\mu\nu} \) so that

\[
\langle \Delta A^i \rangle = \frac{\partial A^i}{\partial A^\mu} \tau \delta^{\mu\nu} \frac{\partial S}{\partial A^\nu} - \frac{1}{2} \Gamma^i_{jk} \frac{\partial A^i}{\partial A^\mu} \frac{\partial A^j}{\partial A^\nu} \tau \delta^{\mu\nu} .
\] (41)

Therefore in natural coordinates the first two moments up to order of \( \tau \) are

\[
\langle \Delta A^i \rangle = \tau g^{ij} \frac{\partial S}{\partial A^j} - \frac{\tau}{2} \Gamma^i \, , \quad \text{and} \quad \langle \Delta A^i \Delta A^j \rangle = \tau g^{ij} ,
\] (42)

where \( \Gamma^i = \Gamma^i_{jk} g^{jk} \).

Note that we used several words such as ‘transitions’, ‘short step’, ‘continuous’ and ‘dynamics’ without any established notion of time. In the following section we will discuss time not as an external parameter, but as an emergent parameter from the maximum entropy transition \( \langle 30 \rangle \) and its moments \( \langle 42 \rangle \).

5 Entropic time

Having described a short step transition the next challenge is to study how these short steps accumulate.
5.1 Introducing time

In order to introduce time we note that $A'$ and $A$ are elements of the same manifold, therefore $P(A')$ and $P(A)$ are two probability distributions over the same space. Our established solution to describe the accumulation of changes (see [26]) is to introduce a “book-keeping” parameter $t$, so that it would distinguish the said distributions as labelled by different parameters, i.e. $P(A') = P_t(A)$ and $P(A) = P_t(A)$.

In this formalism we must call these different labels as a description of the system at particular instants $t$ and $t'$. This allow us to call $P(A'|A)$ a transition probability.

$$P_t(A') = P(A') = \int dA P_{\Delta t}(A'|A)P_t(A) \quad (43)$$

where $\Delta t = t' - t$.

As the system changes from $A$ to $A'$ and then to $A''$. The probability $P(A'')$ will be constructed from $P(A')$, not explicitly dependent on $P(A)$. That means (43) represents a Markovian process – conditioned on the present $P_t(A)$, the “future” $P_{t''}(A), t'' > t'$ is independent of the “past” $P_t(A)$. It is important to notice that under this formalism (43) is not used to show that the process is Markovian in an existing time, but rather the concept of time and dynamics developed here makes it Markovian by design.

It is also important to notice that the parameter $t$ presented here is not necessarily the ‘physical’ time (as it appears in Newton’s laws of motion or the Schrodinger equations). Our parameter $t$,which we call entropic time, is an epistemic well-ordered parameter in which the dynamics is defined.

5.2 The entropic arrow of time

It is important to note that the marginalization process in (20) could also lead to

$$P(A) = \int dA' P(A|A')P(A') \quad , \quad (44)$$

where the conditional probabilities are related by Bayes Theorem,

$$P(A|A') = \frac{P(A)}{P(A')} P(A'|A) \quad , \quad (45)$$

showing that a change “forward” will not happen the same way as a change “backwards” unless the system is in some form of stationary state, $P(A) = P(A')$. Another way to present this is that probability theory alone gives no intrinsic distinction of the change “forward” and “backward”. The fact that we assigned the change “forward” by ME makes it so that, in general, the change “backward” is not an entropy maximum. Therefore, the preferential direction of timely flow arises from entropic dynamics naturally.

5.3 Calibrating the clock

In order to connect the entropic time to the transition probability, one needs to introduce the duration $\Delta t$ with respect to the motion. Just as in physics, time in entropic dynamics is defined so that the motion looks simple. The time interval will be chosen so that the parameter $\tau$ that first appeared in the prior (24) takes the role of a time interval.

$$\tau = \eta \Delta t , \quad (46)$$
where $\eta$ is a constant so that $t$ has the units of time. For the remainder of this article we will adopt $\eta = 1$. In principle any monotonic function $t(\tau)$ serves as an order parameter. Our choice is a matter of convenience, “time is defined so that motion looks simple”. Here this is implemented so that for a short transition we have the dimensionless time interval

$$\Delta t = g_{ij} \langle \Delta A^i \Delta A^j \rangle \ . \quad (47)$$

That means, the entropic time is measured by the systems’ fluctuations. Rather than having the changes in the system to be presented in terms of given time intervals (as measured by an external clock), here the system is its own clock.

The moments in (42) can be written up to the order of $\Delta t$ as

$$\langle \Delta A^i \rangle \Delta t = g_{ij} \frac{\partial S}{\partial A^j} - \frac{1}{2} \Gamma_i \ , \quad \text{and} \quad \langle \Delta A^i \Delta A^j \rangle \Delta t = g^{ij} . \quad (48)$$

The computed $\langle \Delta A^i \rangle$ suggests a connection to established results of fluctuation theory that will be explored in section 8. Before this, with the concept of time established, is convenient to write the trajectory of the expected values in terms of a differential equation.

### 6 Diffusion and the Fokker-Planck equation

Our goal of designing the dynamics from entropic methods is accomplished. The entropic dynamics equation of evolution is written in integral form as a Chapman-Kolmogorov equation (43) with a transition probability given by (30). In this section we will conveniently rewrite it in the differential form. The computed drift $\langle \Delta A^i \rangle$ and the fluctuation $\langle \Delta A^i \Delta A^j \rangle$ in (48) describe the dynamical process as a smooth diffusion [55]. Therefore, for a short transition, it is possible to write the evolution of $P_t(A)$, as a Fokker-Planck (Diffusion) equation,

$$\frac{\partial}{\partial t} P = - \frac{\partial}{\partial A^i} (P v^i) , \quad (49)$$

where

$$v^i = g^{ij} \frac{\partial S}{\partial A^j} - \frac{1}{2} g^{ij} \frac{\partial}{\partial A^j} \left( \log \frac{P}{g^{1/2}} \right) . \quad (50)$$

The derivation of (49) and (50) takes into account the fact that the space in which the diffusion happens is curved and is given in appendix B. In equation (50), we see that the current velocity $v^i$ consists of two components. The first term is the drift velocity guided by the entropy gradient and the second term is an osmotic velocity, that is a term that is driven by differences in probability density. The examples presented in the following section will show how these terms interact and the dynamical properties derived from each.

#### 6.1 Derivatives and divergence

Since both the entropy $S$ is a scalar, the velocity defined in (50) is a contravariant vector. However, (49) is not manifestly invariant equation. To check its consistency it is convenient to write it in terms of the invariant object $p$ defined as

$$p(A) = \frac{P(A)}{g^{1/2}(A)} , \quad (51)$$
in terms of this (49) becomes
\[ \frac{\partial}{\partial t} P = -\frac{1}{g^{1/2}} \frac{\partial}{\partial A^i} \left( g^{1/2} pv^i \right). \] (52)

We can recognize the in the right hand side the covariant divergence of the contravariant vector \( pv^i \). Which can be written in the manifestly covariant form
\[ \frac{\partial}{\partial t} P = -D_i (pv^i), \] (53)
where \( D_i \) is the covariant derivative. We can identify (53) as a covariant continuity equation where the flux, \( j^i = pv^i \), can be written from (50) and (51) as
\[ j^i = pg^{ij} \frac{\partial S}{\partial A^j} - \frac{1}{2} g^{ij} \frac{\partial P}{\partial A^j}. \] (54)

The second term, which is related to the osmotic velocity, is a Fick’s law with diffusion tensor \( D^{ij} = g^{ij}/2 \). Note that this is identified from purely probabilistic arguments, rather than assuming a repulsive interaction from the microstate dynamics.

Having the dynamics fully described we can now study its consequences as it will be done in the two following sections.

7 Examples

We established the entropic dynamics by finding the transition probability (30), presenting it as a differential equation in (49) and (50) and presenting it as invariant equation (53). We want to show some examples on how it would be applied and what are the results achieved. Our present goal is not to search for realistic models, but to search for models which are both mathematically simple and general enough so it can give insight on how to use the formalism.

We will be particularly interested in two properties: the drift velocity,
\[ v^i_D = g^{ij} \frac{\partial S}{\partial A^j}, \] (55)
which is the first term in (50), and the static states, \( v^i = 0 \), which are a particular subset of the dynamical system’s equilibrium \( \partial_t P = 0 \). Obtained from (50) as
\[ v^i = 0 \Rightarrow \frac{\partial S}{\partial A^i} - \frac{1}{2} \frac{\partial}{\partial A^i} \log \left( \frac{P}{g^{1/2}} \right) = 0 \] (56)
allowing one to write the static probability
\[ P(A) \propto g^{1/2}(A) \exp[2S(A)], \] (57)
where the factor of 2 in the exponent comes from the fact that the diffusion tensor \( g^{ij}/2 \) explained in section 6.1. This result shows that the invariant stationary probability density (51) is
\[ p(A) \propto \exp[2S(A)]. \] (58)
7.1 A Gaussian Manifold

The statistical manifold defined by the mean values and correlations of a random variable, the microstates \( x^i \), is the space of Gaussian distribution, which is an example of a canonical distribution. Here we consider the dynamics of a two dimensional spherically symmetric Gaussians with a non-uniform variance, \( \sigma(A) = \sigma(A^1, A^2) \), defined by

\[
\langle x^1 \rangle = A^1, \quad \langle x^2 \rangle = A^2 \quad \text{and} \quad \langle (x^i - A^i)(x^j - A^j) \rangle = \sigma^2(A) \delta^{ij}.
\]

These Gaussians are of the form,

\[
\rho(x \mid A) = \frac{1}{2\pi\sigma^2(A)} \exp\left( -\frac{1}{2\sigma^2(A)} \sum_{i=1}^{2} (x^i - A^i)^2 \right)
\]

The entropy of (60) relative to a uniform background measure given by

\[
S(A) = \log(2\pi\sigma(A)^2)
\]

The space of Gaussians with a uniform variance, \( \sigma(A) = \text{constant} \), is flat and the dynamics turns out to be a rather trivial spherically symmetric diffusion. Choosing the variance to be non-uniform yields a richer and more interesting dynamics. Since this example is pursued for purely illustrative purposes we restrict to two dimensions and to spherically symmetric Gaussians. The generalization is immediate.

The Fisher-Rao Information metric for a Gaussian distribution is found, using (12) (See also [13]), to be

\[
dl^2 = \frac{4}{\sigma^2} (d\sigma)^2 + \delta_{ij} dA^i dA^j
\]

so that, using

\[
d\sigma = \frac{\partial\sigma}{\partial A^i} dA^i,
\]

the induced metric \( dl^2 = g_{ij} dA^i dA^j \) leads to,

\[
g_{ij} = \frac{1}{\sigma^2} \left( 4 \frac{\partial\sigma}{\partial A^i} \frac{\partial\sigma}{\partial A^j} + \delta_{ij} \right)
\]

7.1.1 Gaussian submanifold around an entropy maximum

We present an example of our dynamical model that illustrates the motion around an entropy maximum. A simple way to manifest it includes recognizing that, in (50), \( -S \) plays a role analogous to a potential. A rotationally symmetric quadratic potential can, then be substituted in (61) leading to

\[
\sigma(A) = \exp\left( -\frac{(A^1)^2 + (A^2)^2}{4} \right),
\]

which substituted in (64) yields the metric

\[
g_{ij} = \begin{bmatrix}
(A^1)^2 + \sigma^{-2} & A^1 A^2 \\
A^1 A^2 & (A^2)^2 + \sigma^{-2}
\end{bmatrix},
\]

so that

\[
g^{1/2} = \sqrt{(A^1)^2 + (A^2)^2} \sigma^{-2} + \sigma^{-4}.
\]
and the drift velocity (figure 1) is
\[ v_1^d = -\frac{A^1 \sigma^{-2}}{g} \quad \text{and} \quad v_2^d = -\frac{A^2 \sigma^{-2}}{g} \] (68)

and the static probability (figure 2), eq. (57), is
\[ P(A) \propto 4\pi^2 g^{1/2} \sigma^{-4} . \] (69)

Figure 1: Drift velocity field drives the flux along the entropy gradient.

The static distribution results from the dynamical equilibrium between two opposite tendencies. One is the drift velocity field that drives the distribution along the entropy gradient towards the entropy maximum at the origin. The other is the osmotic diffusive force that we earlier identified as the ED analogue of Fick’s law. This osmotic force drives the distribution against the direction of the probability gradient and prevents it from becoming infinitely concentrated at the origin. In the dynamical system’s equilibrium the cancellation between these two opposing forces results in the Gaussian distribution, eq. (69).

### 7.2 2-Simplex Manifold

Here we discuss an example of discrete microstates. The macrostate coordinates, being expected values, are continuous variables. Our subject matter will be a three-state system, \( x \in \{1, 2, 3\} \), such as, for example, a 3-sided die. The statistical manifold is the 2-dimensional simplex and the natural coordinates are the probabilities themselves,

\[ S_2 = \left\{ \rho(x) \mid \rho(x) \geq 0 , \sum_{x=1}^{3} \rho(x) = 1 \right\} . \] (70)

The distributions on the 2-simplex are Gibbs distributions defined by the sufficient statistics of functions

\[ a^i(x) = \delta^i_x \quad \text{so that} \quad A^i = \langle a^i \rangle = \rho(i) . \] (71)

The entropy relative to the uniform discrete measure is

\[ S = -\sum_{i=1}^{3} \rho(i) \log(\rho(i)) = -\sum_{i=1}^{3} A^i \log(A^i) \] (72)

Figure 2: Equilibrium stationary probability for the entropy maximum example.
and the information metric is given by

\[ g_{ij} = \sum_{k=1}^{3} \rho^k \frac{\partial \log(\rho^k)}{\partial A^i} \frac{\partial \log(\rho^k)}{\partial A^j}. \]  

(73)

The two-simplex arise naturally from probability theory due to normalization when one identifies the macrostate of interested to be the probabilities themselves. The choice of sufficient statistics (71) means that the manifold is a two dimensional surface since due to normalization one can write \( A^3 = 1 - A^1 - A^2 \). We will use the the tuple \((A^1, A^2)\) as our coordinates and \( A^3 \) as a function of them. In this scenario, one finds a metric tensor

\[ g_{ij} = \begin{pmatrix}
\frac{1}{A^3} + \frac{1}{A^1} & \frac{1}{A^3} \\
\frac{1}{A^3} & \frac{1}{A^3} + \frac{1}{A^2}
\end{pmatrix}, \]

(74)

and it induces the volume element

\[ g^{1/2} = \sqrt{\frac{1}{A^1 A^2 A^3}}. \]

(75)

From eq. (55) the drift velocity (figure 3) is

\[ v_1^d = A^1 \left[ A^2 \log \left( \frac{A^2}{A^3} \right) + (A^1 - 1) \log \left( \frac{A^1}{A^3} \right) \right] \]

\[ v_2^d = A^2 \left[ A^1 \log \left( \frac{A^1}{A^3} \right) + (A^2 - 1) \log \left( \frac{A^2}{A^3} \right) \right] \]

(76)

Also, the static probability (figure 4) is

\[ P(A) \propto g^{1/2} \prod_{i=1}^{3} (A^i)^{2A^i}. \]

(77)

Figure 3: Drift velocity field for the three state system example.

Figure 4: Static stationary probability for the three state system example.

Figure 3 shows the drift velocity in the natural coordinates \((A^1, A^2)\). In order to better present the symmetries of the dynamics in the simplex, we will define coordinates \((\epsilon^1, \epsilon^2)\) that
transform as

\[
A^1 = \frac{1}{2} + \frac{\epsilon^1}{\sqrt{2}} - \frac{\epsilon^2}{\sqrt{6}}
\]

\[
A^2 = \frac{1}{2} - \frac{\epsilon^1}{\sqrt{2}} - \frac{\epsilon^2}{\sqrt{6}},
\]

(normalization indicates \( A^3 = 2\epsilon^2 \)). In figure 5 we show how, embedded in coordinates \( (A^1, A^2, A^3) \), the vectors defining \((\epsilon^1, \epsilon^2)\) are parallel to the 2-simplex and perpendicular to each other. This implies that the coordinates \((\epsilon^1, \epsilon^2)\) are isometric to the embedding space \((A^1, A^2, A^3)\) with the Euclidean metric. Figure 6 presents the drift velocity field (figure 3) plotted into these coordinates and converging symmetrically to the central point.

Figure 5: Representation of the simplex embedded into \((A^1, A^2, A^3)\) coordinates and how the generating vectors of \((\epsilon^1, \epsilon^2)\).

Figure 6: Drift velocity field for the three state system example in the coordinates \((\epsilon^1, \epsilon^2)\).

In figure 4 we note that the static distribution diverges at the boundary of the simplex. The divergence, which can be traced to the determinant \( g^{1/2} \), reflects the fact that a 2-state system (say, \( i = 1, 2 \)) is easily distinguishable from a 3-state system \((i = 1, 2, 3)\). Indeed, a single datum \( i = 3 \) will tell us that we are dealing with a 3-state system. However, the divergence is integrable. Indeed, we can see (fig 7) that this divergence is not present in the invariant stationary probability \((51)\).

As in the Gaussian case discussed in the previous section the static equilibrium results from the cancellation of two opposing forces, the entropic force along the drift velocity field towards the center of the simplex is cancelled by the osmotic diffusive force away from the center.

8 Avoiding pitfalls - Linear motion

The motion for the first moment in (48) is composed of a term that mimics Onsager reciprocal relations (ORR) and a second one that accounts for probing curvature. This might lead one to apply the dynamics developed here as an extension of ORR in the non-linear regime. However it is important to take into account particular properties of thermodynamical systems. In this section we are going to comment on the challenges of applying entropic dynamics in the nonequilibrium statistical mechanics.
8.1 The Onsager’s approach to nonequilibrium statistical mechanics

Assume the state of a thermodynamical system can be fully described by a finite number of real parameters \( \xi = \{\xi_1, \xi_2, ..., \xi_n\} \) and that thermodynamic entropy is a function of \( \xi \). In the neighborhood of an equilibrium value \( \xi_0 \), the rate of change of those parameters in time is assumed to be linear to the gradient of entropy, meaning

\[
\frac{d\xi_i}{dt} = \gamma^{ij} \frac{\partial S}{\partial \xi_j},
\]

and the terms of such linear transformation \( \gamma \), known as kinetic coefficients, are unspecified functions of \( \xi \). ORR state that the kinetic coefficients are symmetric, \( \gamma^{ij} = \gamma^{ji} \). The time evolution for \( \xi \) presented in (79) is fundamentally different from entropic dynamics, it supposes a fully deterministic motion for the macrostates, while entropic dynamics creates a stochastic process that does not determine the macrostates but rather their probability distribution. Second, the entropic dynamics presented here was only developed in the coordinates given by the expected values \( A^i \) and the only other coordinates considered are the Lagrange multipliers \( \lambda_i \). That is due to the fact that, from the MaxEnt (1) application, \( A^i \) are the natural variables.
for $S$.

The similarity between (48) and (79) could lead one to identify $\xi^i$ as $A^i$ in a flat space. That means, ORR would describe the trajectory for the expected values $\langle A^i \rangle$ and the unknown coefficients would be the terms of the metric, $\gamma^{ij} = g^{ij}$, that are clearly symmetric. This identification, however, is inappropriate to describe nonequilibrium statistical mechanics. As we see from the derivation of ORR presented in appendix C, this would be equivalent to ORR in a system for which the rates of change are

$$\frac{dA^i}{dt} \propto (A^i_o - A^i), \quad (80)$$

that means, each expected value would move towards the equilibrium value $A^i_o$, with a change rate directly proportional to how far from the equilibrium they are. This is inadequate as we should not expect the thermodynamical parameters of a system, such as internal energy and number of particles, to evolve independently.

The entropic dynamics developed here is extremely simple, the only constraint is to keep the motion within the manifold of Gibbs distributions. If one knows that two variables are correlated and that information was not included in the analysis, the results will, likely, not give a good description of the system of interest. But neither ME nor entropic dynamics are to blame. In physics, the fact that the microstate dynamics yields conservation laws is extremely relevant and have to be taken into account in the inference procedure.

That said, entropic dynamics offers a systematic method to find dynamics aligned with fundamental concepts of probability and statistics, while Onsager’s approach is based solely on calculus considerations around a supposed fixed point. Another way to point this out is to say that ORR is based on an understanding of thermodynamics guided purely from physical considerations, while entropic dynamics is inspired by the information theory approach to statistical physics. Both arrive at symmetric (reciprocal) relations. In the Onsager’s formalism reciprocity follows from the time reversibility of the microstate dynamics, in entropic dynamics reciprocity arises from (information) geometric considerations.

The most important difference between the two methods is that, as previously stated, entropic dynamics is completely agnostic of any microstate dynamics, unlike ORR that assumes a subdynamics that is time reversible. This makes entropic dynamics better suited for an extension to fields beyond physics. Also, since (20) is a completely general way to describe the probabilities for the microstates, the entropic dynamics developed in the previous section is applicable in any state of the system, not only near the equilibrium.

9 Conclusions

We conclude with a summary of the main results. In this paper the entropic dynamics framework has been extended to describe dynamics on a statistical manifold. The ME version of the method of maximum entropy played an instrumental role in that it allowed us impose constraints that not in the standard form of expected values.

The resulting dynamics, which follows from purely entropic considerations, takes the form of a diffusive process on a curved space. The effects of curvature turn out to be significant. We found that the probability flux is the resultant of two components. One describes a flux along the entropy gradient and the other is a diffusive or osmotic component that turns out to be the curved-space analogue of Fick’s law with a diffusion tensor $D^{ij} = g^{ij}/2$ given by information geometry.

A highlight of the model is that it includes an “entropic” notion of time that is tailored to the system under study; the system is its own clock. This opens the door to the introduction of
a notion of time that transcends physics and might be useful for social and ecological systems. The emerging notion of entropic time is intrinsically directional. There is a natural arrow of time which manifests itself in a simple description of the approach to equilibrium.

The model developed here is rather minimal in the sense that the dynamics could be extended by taking additional relevant information into account. For example, it is rather straightforward to enrich the dynamics by imposing additional constraints

\[
\langle \Delta A^i \rangle F_i(A) = \kappa'
\]  

involving system-specific functions \( F_i(A) \) that carry information about correlations. This is the kind of further development we envisage exploring in future work. However, even at this minimal level it is already interesting that from purely geometrical considerations we obtained results that recall Onsager’s reciprocity relations. This suggests that such relations need not be restricted to thermodynamics but might be extended to fields beyond physics. In a future publication we will use the dynamics developed here to address problems within the MaxEnt description of random graphs and networks.

As illustrative examples the dynamics was applied to two general spaces of probability distributions. A submanifold of the space of two-dimensional Gaussians and the space of probability distributions for a 3-state system (2-simplex). In each of these we were able to provide insight on the dynamics by presenting the drift velocity and the equilibrium stationary states.

**Acknowledgments**

We would like to thank N. Caticha, C. Cafaro, S. Ipek, N. Carrara, and M. Abedi for important discussions and questions in the development of this article. We thank N. Caticha for pointing out a possible connection of our dynamics to ORR.

P. Pessoa was partially funded by CNPq – Conselho Nacional de Desenvolvimento Científico e Tecnológico– (scholarship GDE 249934/2013-2).

**Appendix**

**A Obtaining the prior**

In this appendix we derive the prior transition probability from \( A \) to \( A' \) seen in (24). This is achieved by maximizing the entropy

\[
S[Q] = \int dA' Q(A'|x, A) \log \left( \frac{Q(A'|x, A)}{R(A'|x, A)} \right),
\]

(82)

where \( R(A'|x, A) \), the prior for (82) would be an earlier stage of information for the systems’ dynamics. The posterior of (82), \( Q(A'|x, A) \), becomes the prior in (21). At this stage \( A \) could evolve into any \( A' \) and the only assumption is that the assigned prior for (82) would give equal probabilities for equal volumes. That is achieved by a prior proportional to the volume element \( R(A'|x, A) \propto g^{1/2}(A') \), where \( g(A) = \det g_{ij}(A) \). There is no need to address normalization of \( R \) since it will no effect in the posterior.

The constraint is so that the motion will be isotropic and continuous on the manifold. This will be imposed by

\[
\int dA' Q(A'|x, A) g_{ij} \Delta A^i \Delta A^j = K .
\]

(83)
where $K$ is a small quantity that eventually will tend to zero. This is so, due to the expected value of $g_{ij} \Delta A^i \Delta A^j$ being invariant only in the limit for short steps $\Delta A^i \to 0$.

The result of maximizing (82) under (83) and normalization is
\[
\int \mathcal{D}[f] \exp \left( -\int dA \right) Q(A'|x, A) \propto g^{1/2}(A') \exp \left( -\alpha g_{ij} \Delta A^i \Delta A^j \right),
\]
where $\alpha$ is the Lagrange multiplier associated to (83). As the result requires $K \to 0$ to make it geometrically invariant, the conjugated Lagrange multiplier should equally be allowed to be taken to infinity. This allows us to define $\tau = 1/\alpha$, such that the short step limit leads to $\tau \to 0$.

Note that, since no motion in $x$ and no correlation between $x$ and $A'$ is induced by the constraints, the result does not depend on the previous microstate $x$, $Q(A'|x, A) = Q(A'|A)$.

### B Derivation of the Fokker-Planck equation

The goal of this appendix is to show that for a dynamics that is a smooth diffusion \footnote{Here smooth diffusion means, as defined by \cite{55}, a stochastic process in which the first two moments are, calculated to the order of $\Delta t$, $\langle \Delta A^i \rangle = b^i \Delta t$, $\langle \Delta A^i \Delta A^j \rangle = g^{ij} \Delta t$ and $\langle \Delta A^i \Delta A^j \Delta A^k \rangle = 0$.} in a curved space, can be written as a Fokker-Planck equation and to obtain its velocity \footnote{See Chapter 4.} from the moments for the motion \footnote{Chapter 4.}. In order to do so, it is convenient to define the drift velocity from
\[
b^i \equiv \lim_{\Delta t \to 0} \frac{\langle \Delta A^i \rangle}{\Delta t} = g^{ij} \frac{\partial S}{\partial A^j} - \frac{1}{2} \Gamma^i.
\]

First let us analyze the change in a smooth integrable function $f(A)$ as the system transitions from $A$ to $A'$.
\[
\Delta f(A) = \frac{\partial f}{\partial A^i} \Delta A^i + \frac{1}{2} \frac{\partial^2 f}{\partial A^i \partial A^j} \Delta A^i \Delta A^j + o(\Delta t),
\]
Since a cubic term, $\Delta A^i \Delta A^j \Delta A^k$ would be $o(\Delta t)$. In a smooth diffusion we can take an expected value of (86) with respect to $P(A'|x, A)$.
\[
\langle \Delta f(A) \rangle = \int dA' P(A'|x, A)(f(A') - f(A)) = \left( b^i \frac{\partial}{\partial A^i} + \frac{1}{2} g^{ij} \frac{\partial^2}{\partial A^i \partial A^j} \right) f(A) \Delta t.
\]
which (87) can be further averaged in $P(A)$
\[
\int dA P(A) \langle \Delta f(A) \rangle = \int dA' P(A') f(A') - \int dA P(A) f(A)
= \int dA P(A) \left( b^i \frac{\partial}{\partial A^i} + \frac{1}{2} g^{ij} \frac{\partial^2}{\partial A^i \partial A^j} \right) f(A) \Delta t.
\]
As established in section 5, $P(A)$ and $P(A')$ are distributions at the instants $t$ and $t'$ respectively.
\[
\int dA \left( \frac{P_A(A) - P_{A'}(A)}{\Delta t} \right) f(A) = \int dA P(A) \left( b^i \frac{\partial}{\partial A^i} + \frac{1}{2} g^{ij} \frac{\partial^2}{\partial A^i \partial A^j} \right) f(A) \Delta t,
\]
which can be partially integrated in the limit of small steps
\[
\int dA \left( \frac{\partial P(A)}{\partial t} \right) f(A) = \int dA \left( - \frac{\partial}{\partial A^i}(b^i P(A)) + \frac{1}{2} \frac{\partial^2}{\partial A^i \partial A^j}(g^{ij} P(A)) \right) f(A).
\]
Due to the generality of $f$ as test function, we identify the integrants,
\[ \frac{\partial}{\partial t} P(A) = -\frac{\partial}{\partial A^{i}} \left( b^{i} P(A) - \frac{1}{2} \frac{\partial}{\partial A^{j}} (g^{ij} P(A)) \right), \]  
(91)
and substitute $b^{i}$ (85) for general coordinates,
\[ \frac{\partial}{\partial t} P(A) = -\frac{\partial}{\partial A^{i}} \left( g^{ij} \frac{\partial S}{\partial A^{j}} P(A) - \frac{1}{2} \frac{\partial g^{ij}}{\partial A^{j}} P(A) - \frac{1}{2} g^{ij} \frac{\partial P(A)}{\partial A^{i}} \right), \]  
(92)
and the contracted Christoffel symbols can be substituted in the identity
\[ \Gamma^{i} = -\frac{1}{g^{1/2}} \frac{\partial g^{ij}}{\partial A^{j}} (g^{1/2} g^{ij}) = -\frac{\partial g^{ij}}{\partial A^{j}} g^{ij} \frac{\partial \log g^{1/2}}{\partial A^{j}}, \]  
(93)
obtaining
\[ \frac{\partial}{\partial t} P(A) = -\frac{\partial}{\partial A^{i}} \left( g^{ij} \frac{\partial S}{\partial A^{j}} - \frac{1}{2} g^{ij} \frac{\partial}{\partial A^{j}} \left( \log \frac{P(A)}{g^{1/2}} \right) \right) P(A), \]  
(94)
The result is a Fokker-Planck equation that is usefully written in the continuity form
\[ \frac{\partial}{\partial t} P = -\frac{\partial}{\partial A^{i}} (Pv^{i}), \]  
(95)
where
\[ v^{i} = g^{ij} \frac{\partial S}{\partial A^{j}} - \frac{1}{2} g^{ij} \frac{\partial}{\partial A^{j}} \left( \log \frac{P}{g^{1/2}} \right), \]  
(96)
completing the derivation.

C On the derivation of Onsager reciprocal relations

In this appendix we will comment on how the ORR (as in (79) with symmetric coefficients) are derived in a general nonequilibrium statistical mechanics and how they compare to the entropic dynamics developed in the main text. This derivation is largely inspired by the derivation provided by Landau and Lifshitz [56] translated into our covariant notation.

This describes a thermodynamic entropy as a function of a set of parameters $\xi$ that change in time, we will study such a change as a fluctuation around an equilibrium value $\xi_{0}$. This equilibrium value has to both be a local maxima of entropy and a fixed point for the dynamics of $\xi$. That means
\[ \frac{\partial \xi^{i}}{\partial t} \bigg|_{\xi = \xi_{0}} = 0 \quad \text{and} \quad \frac{\partial S}{\partial \xi^{i}} \bigg|_{\xi = \xi_{0}} = 0, \]  
(97)
entropy has also to be concave with respect to each $\xi^{i}$ to guarantee that this will describe a maximum.

In order to show the relationship in (79) we will Taylor expand $d\xi^{i}/dt$ around the equilibrium point
\[ \frac{d\xi^{i}}{dt} = L^{i}_{j} (\xi^{j} - \xi_{0}^{j}), \]  
(98)
where $L^{i}_{j}$ would be identifiable as the gradient of the said rates of change,
\[ L^{i}_{j} = \frac{\partial}{\partial \xi^{j}} \frac{d\xi^{i}}{dt} \bigg|_{\xi = \xi_{0}}. \]  
(99)
The calculation of \( L^i_j \) from the knowledge of the deterministic microstate dynamics is only feasible in simple cases and close to equilibrium.

A second step towards ORR is to Taylor expand the gradients of entropy similarly near the equilibrium

\[
\frac{\partial S}{\partial \xi^i} = \beta_{ij} (\xi^j - \xi^j_0) ,
\]

(100)

where \( \beta \) is the Hessian of entropy

\[
\beta_{ij} = \left. \frac{\partial^2 S}{\partial \xi^i \partial \xi^j} \right|_{\xi = \xi_0} .
\]

(101)

unlike \( L \), \( \beta \) is computable through standard methods and it is symmetric. We can directly put (98) and (100) together achieving

\[
\frac{d\xi^i}{dt} = L^i_j [\beta^{-1}]^{jk} \frac{\partial S}{\partial \xi^k} ,
\]

(102)

by direct comparison to (79) we see that

\[
\gamma^{ij} = L^i_k [\beta^{-1}]^{kj} .
\]

(103)

The proof that \( \gamma \) is symmetric relies on time reversal symmetry. In particular, it means that as the values of \( \xi \) change in time correlations between them should evolve in time so that:

\[
\langle \xi^i(0) \xi^j(t) \rangle = \langle \xi^i(-t) \xi^j(0) \rangle .
\]

(104)

The full proof need not be reproduced here (see [45]). However, we note that the assumption of time-reversal symmetry, which applies to the microstate dynamics of atoms and molecules, will not necessarily hold for systems beyond physics. It may therefore be surprising that entropic dynamics, which does not rely on time reversibility, implies similar symmetric reciprocity relations.

If we restrict (100) to the motion of only the expected values \( \xi^i = A^i \) to match the dynamics presented before we have

\[
\beta_{ij} = \left. \frac{\partial^2 S}{\partial A^i \partial A^j} \right|_{A = A_0} = - g_{ij}(A_0) ,
\]

(105)

the choice of coordinates makes so that we can clearly see that the expansion presented in ORR is intrinsically geometrical. Also in these coordinates we can write the kinetic coefficients as

\[
\gamma^{ij} = - L^i_k g^{kj}(A_0) ,
\]

(106)

as of being a metric \( g^{ik} g_{kj} = \delta^i_j \).

So, for that choice of parameters, we can write (79) as:

\[
\frac{dA^i}{dt} = - L^i_k g^{kj}(A_0) \frac{\partial S}{\partial A^j} .
\]

(107)

One might naively assume that the relation between ED and ORR would be to identify the motion of the expected values of entropic dynamics without taking into account the first-order motion will probe into the curvature naturally given by FRIM. If that were to be correct a direct comparison to (48) would yield

\[
L^i_j = - \delta^i_j ,
\]

(108)
that is better written by checking directly into how we defined $L$ in (98),

$$\frac{d\xi^i}{dt} \propto (\xi^0_o - \xi^i),$$

(109)

which is inadequate as we discuss in section 8.

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