UNIFORM PARAMETERIZATION OF SUBANALYTIC SETS AND DIOPHANTINE APPLICATIONS

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Abstract. We prove new parameterization theorems for sets definable in the structure $\mathbb{R}_{an}$ (i.e. for globally subanalytic sets) which are uniform for definable families of such sets. We treat both $C^r$-parameterization and (mild) analytic parameterization. In the former case we establish a polynomial (in $r$) bound (depending only on the given family) for the number of parameterizing functions. However, since uniformity is impossible in the latter case (as was shown by Yomdin via a very simple family of algebraic sets), we introduce a new notion, analytic quasi-parameterization (where many-valued complex analytic functions are used), which allows us to recover a uniform result.

We then give some diophantine applications motivated by the question as to whether the $H^{o(1)}$ bound in the Pila-Wilkie counting theorem can be improved, at least for certain reducts of $\mathbb{R}_{an}$. Both parameterization results are shown to give uniform $(\log H)^{O(1)}$ bounds for the number of rational points of height at most $H$ on $\mathbb{R}_{an}$-definable Pfaffian surfaces. The quasi-parameterization technique produces the sharper result, but the uniform $C^r$-parametrization theorem has the advantage of also applying to $\mathbb{R}_{pow}$-definable families.

1. Introduction

The aim of this section is to give an informal account of the results appearing in this paper. Precise definitions and statements are given in the next section.

Key words and phrases. Rational points of bounded height, $C^r$-parameterizations, quasiparameterizations, subanalytic sets, weakly mild functions, Gevrey functions, Lipschitz cell decomposition, Lipschitz preparation, restricted Pfaffian functions, power maps, entropy, dynamical system theory.

The authors would like to thank EPSRC for supporting parts of the work under grants EP/J019232/1 and EP/N008359/1, and the Leverhulme Trust for the Emeritus Fellowship EM/2015-002. The author R.C. was supported by the European Research Council under the European Community’s Seventh Framework Programme (FP7/2007-2013) with ERC Grant Agreement nr. 615722 MOTMEL-SUM, and thanks the Labex CEMPI (ANR-11-LABX-0007-01).
So, we are concerned with parameterizations of bounded definable subsets of real euclidean space. The definability here is with respect to some fixed (and, for the moment, arbitrary) o-minimal expansion of the real field. By a parameterization of such a set \( X \subseteq \mathbb{R}^n \), we mean a finite collection of definable maps from \((0,1)^m \) to \( \mathbb{R}^n \), where \( m := \dim(X) \), whose ranges cover \( X \). The fact that parameterizations always exist is an easy consequence of the cell decomposition theorem, but the aim is to construct them with certain differentiability conditions imposed on the parameterizing functions together with bounds on their derivatives. The first result in this generality was obtained in [24] (by adapting methods of Yomdin [32] and Gromov [13] who dealt with the semi-algebraic case), where it was shown that for each positive integer \( r \) there exists a parameterization consisting of \( C^r \) functions all of whose derivatives (up to order \( r \)) are bounded by 1. Further, the parameterizing functions may be found uniformly. This means that if \( \mathcal{X} = \{ X_t : t \in T \} \) is a definable family of \( m \)-dimensional subsets of \((0,1)^n \) (say), i.e. the relation “\( t \in T \) and \( x \in X_t \)” is definable in both \( x \) and \( t \), then there exists a positive integer \( N_r \) such that for each \( t \in T \), at most \( N_r \) functions are required to parameterize \( X_t \) and each such function is definable in \( t \). (The bound \( N_r \) does, of course, also depend on the family \( \mathcal{X} \), but we usually suppress this in the notation. The point is that it is independent of \( t \).) Unfortunately, the methods of [24] do not give an explicit bound for \( N_r \) and it is the first aim of this paper to do so in the case that the ambient o-minimal structure is the restricted analytic field \( \mathbb{R}_{an} \) (where the bounded definable sets are precisely the bounded subanalytic sets), or a suitable reduct of it. We prove, in this case, that \( N_r \) may be taken to be a polynomial in \( r \) (which depends only on the given family \( \mathcal{X} \)). While we have only diophantine applications in mind here, this result already gives a complete answer to an open question, raised by Yomdin, coming from the study of entropy and dynamical systems (see e.g. [32], [31], [13], [3]). In fact, even in the case that the ambient structure is just the ordered field of real numbers (which is certainly a suitable reduct of \( \mathbb{R}_{an} \) to which our result applies), the polynomial bound appears to be new and, indeed, gives a partial answer to a question raised in [3] (just below Remark 3.8). Our uniform \( C^r \)-parameterization theorem also holds for the expansion of \( \mathbb{R}_{an} \) by all power functions (i.e. the structure usually denoted \( \mathbb{R}^{pow}_{an} \)) and suitable reducts (to be clarified in section 2) of it.

Next we consider mild parameterizations. Here it is more convenient to consider parameterizing functions with domain \((-1,1)^m \) (where \( m \) is the dimension of the set being parameterized) and we demand that they are \( C^\infty \) and we put a bound on all the derivatives. We shall only
be concerned with functions that satisfy a so called 0-mild condition, namely that there exists an $R > 1$ such that for each positive integer $d$, all their $d$’th derivatives have a bound of order $R^{-d} \cdot d!$ (which in fact forces the functions to be real analytic). It was shown in [15] that any reduct of $\mathbb{R}_{an}$ has the 0-mild parameterization property: every definable subset of $(-1, 1)^n$ has a parameterization by a finite set of 0-mild functions. However, this result cannot be made uniform. For Yomdin showed in [33, Proposition 3.3] (see also [34, page 416]) that the number of 0-mild functions required to parameterize the set \{$(x_1, x_2) \in (-1, 1)^2 : x_1 \cdot x_2 = t$\} necessarily tends to infinity as $t \to 0$. Our second parametrization result recovers uniformity in the 0-mild setting but at the expense of, firstly, covering larger sets than, but ones having the same dimension as, the sets in the given family and secondly, covering not by ranges of 0-mild maps but by solutions to (a definable family of) Weierstrass polynomials with 0-mild functions as coefficients.

In [24] the parameterization theorem is applied to show that any definable subset of $(0, 1)^n$ (the ambient o-minimal structure being, once again, arbitrary) either contains an infinite semi-algebraic subset or else, for all $H \geq 1$, contains at most $H^{o(1)}$ rational points whose coordinates have denominators bounded by $H$. (For the purposes of this introduction we refer to such points as $H$-bounded rational points.) Although this result is best possible in general, and is so even for one dimensional subsets of $(0, 1)^2$ definable in the structure $\mathbb{R}_{an}$, it has been conjectured that the $H^{o(1)}$ bound may be improved to $(\log H)^{O(1)}$ for certain reducts of $\mathbb{R}_{an}$ (specifically, for sets definable from restricted Pfaffian functions), and it is our final aim in this paper is to take a small step towards such a conjecture.

We first observe that the point counting theorem from [24] quoted above follows (by induction on dimension) from the following uniform result (the main lemma of [24] on page 610). Namely, if $m < n$ and $\mathcal{X} = \{X_t : t \in T\}$ is a definable family of $m$-dimensional subsets of $(0, 1)^n$, and $\epsilon > 0$, then there exists a positive integer $d = d(\epsilon, n)$ such that for each $t \in T$ and for all $H \geq 1$, all the $H$-bounded rational points of $X_t$ are contained in the union of at most $O(H^d)$ algebraic hypersurfaces of degree at most $d$, where the implied constant depends only on $\mathcal{X}$ and $\epsilon$. Now, for the structure $\mathbb{R}_{an}^{pow}$ (or any of its suitable reducts), our uniform $C^\epsilon$-parameterization theorem allows us to improve the bound here on the number of hypersurfaces to $O((\log H)^{O(1)})$ (for $H > e$, with the implied constants depending only on the family $\mathcal{X}$) but, unfortunately, their degrees have this order of magnitude too. Actually, the bound on the degrees is completely explicit, namely $[(\log H)^{m/(n-m)}]$. 
but as this tends to infinity with $H$, the inductive argument used in 
[24] (where the degree $d$ only depended on $\epsilon$ and $n$) breaks down at this 
point. Our 0-mild (quasi-) parameterization theorem does give a better 
result for (suitable reducts of) the structure $\mathbb{R}_{an}$ in that the number 
of hypersurfaces is bounded by a constant (depending only on $\mathcal{X}$), but 
the bound for their degrees is the same as above and so, once again, 
the induction breaks down.

We can, however, tease out a uniform result for rational points on one 
and two dimensional sets definable from restricted Pfaffian functions, 
but for the general conjecture a completely new uniform parameter-
ization theorem that applies to the intersection of a definable set of 
constant complexity with an algebraic hypersurfaces of nonconstant 
degree is badly needed.

Note. As this paper was being finalised, the arXiv preprints [1], 
[2] appeared. There is some similarity in the methods used there and 
the complex analytic approach here. There seems to be no inclusion in 
either direction in the parameterisation results nor in the diophantine 
applications; the diophantine result of [2] deals with sets of arbitrary dimension but in a smaller structure.

2. Precise statements

2.1 $C^r$-parameterizations

The largest expansion of the real field (that is, the expansion with 
the most definable sets) to which our uniform $C^r$-parameterization the-
orem applies is the structure $\mathbb{R}_{an}^{pow}$, i.e. the expansion by all restricted 
analytic functions and all power functions $(0, \infty) \to (0, \infty) : x \mapsto x^s$ 
(for $s \in \mathbb{R}$). However, when it comes to applications there is consider-
able advantage to be gained from working in suitable reducts of $\mathbb{R}_{an}^{pow}$ for 
which more effective topological and geometric information is available 
for the definable sets. (For example, for sets definable from restricted 
Pfaffian functions one has, through the work of Khovanskii (17) and 
Gabrielov and Vorobjov (11), good bounds (in terms of natural data) 
on the number of their connected components.) And of course, it is 
important for the inductive arguments involved in such applications 
that the parameterizing functions are definable in the same reduct as 
is the set (or family of sets) being parameterized.

It turns out that for our proof here to go through, the property re-
quired of the ambient o-minimal structure is that it should be a reduct 
of $\mathbb{R}_{an}^{pow}$ in which a suitable version of the Weierstrass Preparation The-
orem holds for definable functions. Now, a large class of such reducts 
has been identified and extensively studied, by D. J. Miller in his Ph.D.
thesis ([20], which was inspired by the subanalytic case obtained by Lion and Rolin in [19] and by Parusinski in [22]). These are based on a language for functions in a Weierstrass system together with a certain class of power functions. There is no need for us to go into precise definitions here—we will quote the relevant results from [20] when needed. Suffice it to say that examples include the real ordered field itself, \( \mathbb{R}_{an} \), \( \mathbb{R}_{an}^{pow} \) or, indeed, the expansion of \( \mathbb{R}_{an} \) by any collection of power functions that is closed under multiplication, inverse and composition (i.e. such that the exponents form a subfield of \( \mathbb{R} \)). Many more examples appear in the literature (see [8], [7] and [20]). We shall assume, in the precise statement of the theorem below and throughout section 4, that all notions of definability are with respect to some such fixed reduct of \( \mathbb{R}_{an}^{pow} \).

2.1.1 Convention We fix a reduct of \( \mathbb{R}_{an}^{pow} \) based on a Weierstrass system \( \mathcal{F} \) and a subfield \( K \) of its field of exponents as described in Definition 2.1 of [20]. As there, we denote its language by \( L_{K}^{\mathcal{F}} \).

Note that, for the smallest possible choice of \( \mathcal{F} \) and \( K = \mathbb{Q} \), the \( L_{K}^{\mathcal{F}} \)-definable sets are precisely the semi-algebraic sets.

2.1.2 Definition. Let \( r \) be a non-negative integer or \( +\infty \). The \( C^r \)-norm \( \| f \|_{C^r} \) of a \( C^r \)-function \( f : U \subset \mathbb{R}^m \rightarrow \mathbb{R} \), with \( U \) open, is defined (in \( \mathbb{R} \cup \{+\infty\} \)) by \( \sup_{x \in U} \sup_{|\alpha| \leq r} |f^{(\alpha)}(x)| \),

where we have used the standard multi-index notation, namely \( f^{(\alpha)} \) stands for \( \partial^{\alpha} f / \partial x^{\alpha} (= f \text{ for } \alpha = 0) \), and \( |\alpha| \) denotes \( \sum_{i=1}^{m} \alpha_i \). By the \( C^r \)-norm of a \( C^r \)-map \( f : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^s \), with \( U \) open, we mean the maximum of the \( C^r \)-norms of the component functions of \( f \).

2.1.3 Theorem (The uniform \( C^r \)-parameterization theorem.) Let \( n, k \) be positive integers and \( m \) be a nonnegative integer with \( m \leq n \). Let \( \mathcal{X} = \{ X_t : t \in T \} \) be an \( L_{K}^{\mathcal{F}} \)-definable family of \( m \)-dimensional subsets of \( (0, 1)^n \), where \( T \) is some \( L_{K}^{\mathcal{F}} \)-definable subset of \( \mathbb{R}^k \). Then there exists a polynomial \( D \), depending only on the family \( \mathcal{X} \), such that for each positive integer \( r \), and for each \( t \in T \), there exist analytic maps \( \phi_{r,i,t} : (0, 1)^m \rightarrow X_t \) for \( i = 1, \ldots, D(r) \), whose \( C^r \)-norms are bounded by 1 and whose ranges cover \( X_t \). Moreover, for each \( i \) and \( r \), \( \{ \phi_{r,i,t} : t \in T \} \) is an \( L_{K}^{\mathcal{F}} \)-definable family of functions.

The proof of 2.1.3 is given in section 4.
2.2 Quasi-parametrization

For the main result of this section we require our ambient o-minimal structure to be a reduct of $R_{an}$: we do not know whether theorem 2.2.3 below (or some version of it) holds if power functions with irrational exponents are admitted. We shall be working with complex valued definable functions of several complex variables where the definability here is via the usual identification of $C$ with $R^2$. Naturally enough we will require there to be enough definable holomorphic functions:

2.2.1 Convention We fix a reduct of $R_{an}$ with the following property. If $f : U \subset R^m \to R$, with $U$ open, is a definable, real analytic function, then for each $a \in U$ there exists an open $V \subset C^m$ with $a \in V \cap R^m \subset U$ and a definable holomorphic function $\tilde{f} : V \to C$ such that for all $b \in V \cap R^m$, $\tilde{f}(b) = f(b)$. For the remainder of this subsection and throughout section 5 (unless otherwise stated) definability will be with respect to this structure.

The main examples are the real field and $R_{an}$ itself. Others may be constructed as follows. Let $F$ be a collection of restricted (real) analytic functions closed under partial differentiation and under the operation implicit in 2.2.1 (i.e. under taking the real and imaginary part functions of the local complex extensions). Then the expansion of the real field by $F$ will be a reduct of $R_{an}$ satisfying 2.2.1. This follows fairly easily from the theorem of Gabrielov ([10]) asserting that such a reduct is model complete. (For a local description of the complex holomorphic functions that are definable in such a structure (at least, in a neighbourhood of a generic point) see [30].)

For $R > 0$ we denote by $\Delta(R)$ the open disc in $C$ of radius $R$ and centred at the origin.

2.2.2 Definition Let $R > 0$, $K > 0$ and let $m$ be a positive integer. Then a definable family $\Lambda = \{F_t : t \in T\}$, where $T$ is a definable subset of $R^k$ for some $k$, is called an $(R, m, K)$-family if for each $t \in T$, the function $F_t : \Delta(R)^m \to C$ is holomorphic and for all $z \in \Delta(R)^m$ we have $|F_t(z)| \leq K$.

We shall develop a considerable amount of theory for such families in section 5. To mention just one result, which is perhaps of independent interest, we will show that if $R > 1$ and, for each $t \in T$,

$$F_t(z) = \sum a^{(t)}_\alpha z_1^{\alpha_1} \cdots z_m^{\alpha_m}$$

is the Taylor expansion of $F_t$ around $0 \in C^m$ (where the summation is over all $m$-tuples $\alpha = \langle \alpha_1, \ldots, \alpha_m \rangle \in N^m$), then there exists $M =$
$M(\Lambda) \in \mathbb{N}$ such that $|a_0^{(\alpha)}|$ achieves its maximum value for some $\alpha$ with $|\alpha| \leq M$. (See 2.1.2 for the multi-index notation.) The fact that $M$ is independent of $t$ here is crucial for all the uniformity results that follow and leads to the following

2.2.3 Theorem (The quasi-parameterization theorem.) Let $n$ and $m$ be non-negative integers with $m < n$ and let $X = \{X_s : s \in S\}$ be a definable family of subsets of $[-1, 1]^n$, each of dimension at most $m$, where $S$ is a definable subset of $\mathbb{R}^k$ for some $k$. Then there exists $R > 1$, $K > 0$, a positive integer $d$ and an $(R, m + 1, K)$-family $\Lambda = \{F_t : t \in T\}$ such that each $F_t$ is a monic polynomial of degree $d$ in its first variable and for all $s \in S$, there exists $t \in T$ such that

$$X_s \subseteq \{x = (x_1, \ldots, x_n) \in [-1, 1]^n : \exists w \in [-1, 1]^m \bigwedge_{i=1}^n F_t(x_i, w) = 0\}.$$ 

The proof of 2.2.3 is given in section 5.

2.3 Diophantine applications The above parameterisation results may be applied to obtain results about the distribution of rational points on definable sets.

The height of a rational number $q = a/b$ where $a, b \in \mathbb{Z}$ with $b > 0$ and $\gcd(a, b) = 1$ is defined to be $H(q) = \max(|a|, b)$ and the height of a tuple $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n$ is $H(q) = \max(H(q_i), i = 1, \ldots, n)$. For $X \subset \mathbb{R}^n$ we set

$$X(\mathbb{Q}, H) = \{x \in X \cap \mathbb{Q}^n : H(x) \leq H\}$$

and define the counting function

$$N(X, H) = \#X(\mathbb{Q}, H).$$

In the following, $X \subset T \times [0, 1]^n$ is a family of sets $X_t \subset [0, 1]^n, t \in T$, which is definable in a suitable o-minimal structure (specified in each result). We assume that each fibre $X_t$ has dimension $m < n$.

Note that “hypersurfaces” as used below may be reducible, that “degree” means total degree, and that $[x]$ denotes the integer part of a real number $x$: $[x] \in \mathbb{Z}$ and $[x] \leq x < x + 1$.

2.3.1. Theorem. Let $X \subset T \times (0, 1)^n$ be a family of sets $X_t \subset [0, 1]^n, t \in T$, definable in $\mathbb{R}^{pow}$. Then there exist positive constants $C_1 = C_1(X), c_1 = c_1(X)$ such that, for $H \geq e$ and $t \in T$, $X_t(\mathbb{Q}, H)$ is contained in the union of at most $C_1(\log H)^{c_1}$ real algebraic hypersurfaces of degree at most $[(\log H)^{m/(n-m)}]$. 


For a definable family in the smaller structure $\mathbb{R}_{an}$ we get a more precise result.

**2.3.2. Theorem.** Let $X \subset T \times [-1, 1]^n$ be a family of sets $X_t, t \in T$, definable in $\mathbb{R}_{an}$. Then there exists a positive constant $C_2 = C_2(X)$ such that, if $H \geq e$ and $t \in T$ then $X_t(\mathbb{Q}, H)$ is contained in the union of at most

$$C_2 \left[ (\log H)^{m/(n-m)} \right].$$

If these results could be iterated on the intersections for we would be able to prove a poly-log bound for rational points. However, as the degrees of the hypersurfaces increase with $H$, even a second iteration would require a result for such non-definable families.

However, for certain families of pfaffian sets of dimension 2 (see the basic definitions below) we can carry this out using estimates due to Gabrielov and Vorobjov [11]. They have the right form of dependencies to give a suitable result for the curves arising when the surface is intersected with algebraic hypersurfaces of growing degree. This idea has been used in several previous papers [4], [15], [16], [26], [27].

**2.3.3. Definition.** A *pfaffian chain* of order $r \geq 0$ and degree $\alpha \geq 1$ in an open domain $G \subset \mathbb{R}^n$ is a sequence of analytic functions $f_1, \ldots, f_r$ on $G$ satisfying differential equations

$$df_j = \sum_{i=1}^{n} g_{ij}(x, f_1(x), \ldots, x f_j(x)) dx_i$$

for $1 \leq j \leq r$, where $g_{ij} \in \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_j]$ are polynomials of degree not exceeding $\alpha$. A function

$$f = P(x_1, \ldots, x_n, f_1, \ldots, f_r)$$

where $P$ is a polynomial in $n + r$ variables with real coefficients of degree not exceeding $\beta \geq 1$ is called a *pfaffian function* of order $r$ and degree $(\alpha, \beta)$. A *pfaffian set* will mean the set of common zeros of some finite set pfaffian functions.

**2.3.4. Definition.** By a *pfaffian surface* we will mean the graph in $\mathbb{R}^3$ of pfaffian functions of two variables with a common pfaffian chain of order and degree $(r, \alpha)$, defined on a “simple” domain $G$ in the sense of [11]. Namely, a domain of the form $\mathbb{R}^2, (-1, 1)^2, (0, \infty)^2$ or $\{(u, v) : u^2 + v^2 < 1\}$. We take the *complexity* of the surface to
be the triple \( (r, \alpha, \beta) \), where \( \beta \) is the maximum of the degrees of the coordinate functions defining the surface.

2.3.5. Definition. Let \( X \subset \mathbb{R}^n \). The algebraic part of \( X \), denoted \( \text{X}^\text{alg} \) is the union of all connected positive dimensional semi-algebraic subsets of \( X \). The complement \( X - \text{X}^\text{alg} \) is called the transcendental part of \( X \) and denoted \( \text{X}^\text{trans} \).

By combining 2.1.3 with the methods of [26], [27] we get a uniform result for a family of pfaffian surfaces definable in \( \mathbb{R}^{\text{pow}} \). For an individual surface definable in the structure \( \mathbb{R}_{\text{resPfaff}} \) such a bound is due to Jones-Thomas [16]. Note that a definable surface in \( \mathbb{R}^{\text{pfaff}} \) is more general than a pfaffian surface. Perhaps a combination of the methods could give uniformity for \( \mathbb{R}^{\text{an}} \)-definable families of restricted-pfaffian-definable sets of dimension 2.

2.3.6. Proposition. Let \( r \) be a non-negative integer and \( \alpha, \beta \) positive integers. Let \( X \subset T \times (0, 1)^3 \) be a family of surfaces \( X_t, t \in T \), definable in \( \mathbb{R}^{\text{pow}} \) such that each fibre \( X_t \) is the intersection of \((0, 1)^3\) with a pfaffian surface with complexity (at most) \( (r, \alpha, \beta) \). Then there exists \( C_3(X), c_3(X) \) such that, for \( H \geq e \) and \( t \in T \),

\[
N(X_t^{\text{trans}}, H) \leq C_3(\log H)^{c_3}.
\]

When the family is definable in \( \mathbb{R}^{\text{an}} \) we can prove a more precise uniform result in which the exponent depends only on the complexity of the pfaffian surfaces.

2.3.7. Proposition. Let \( r \) be a non-negative integer and \( \alpha, \beta \) positive integers. Let \( X \subset T \times [-1, 1]^3 \) be a family of surfaces \( X_t, t \in T \), definable in \( \mathbb{R}^{\text{an}} \) such that each fibre \( X_t \) is the intersection of \([-1, 1]^3\) with a pfaffian surface with complexity (at most) \( (r, \alpha, \beta) \). Then there exists \( C_4(X) \) depending only on \( X \) and \( c_4(r, \alpha, \beta) \) such that, for \( H \geq e \) and \( t \in T \),

\[
N(X_t^{\text{trans}}, H) \leq C_4(\log H)^{c_4}.
\]

The proofs of Theorems 2.3.1 and 2.3.2 and Propositions 2.3.6 and 2.3.7, assuming the parameterisation results, are given in Section 3.

3. Proofs of diophantine applications

3.1. Some preliminaries for 2.3.1 and 2.3.2. For a positive integer \( k \) and non-negative integer \( \delta \) we let

\[
\Lambda_k(\delta) = \{ \mu = (\mu_1, \ldots, \mu_k) \in \mathbb{N}^k : |\mu| = \mu_1 + \ldots + \mu_k = \delta \},
\]
\[ \Delta_k(\delta) = \{ \mu = (\mu_1, \ldots, \mu_k) \in \mathbb{N}^k : |\mu| = \mu_1 + \ldots + \mu_k \leq \delta \}, \]
\[ L_k(\delta) = \# \Lambda_k(\delta), \quad D_k(\delta) = \# \Delta_k(\delta), \]
where \( \mathbb{N} \) is the set of nonnegative integers.

Let \( X = X_t \) be a fibre of our definable family. We will adapt the methods of [27], in which we explore \( X(\mathbb{Q}, H) \) with hypersurfaces of degree
\[ d = \lceil (\log H)^{m/(n-m)} \rceil. \]
This leads us to consider \( D_n(d) \times D_n(d) \) determinants \( \Delta \) whose entries are the monomials of degree \( d \) (indexed by \( \Delta_n(d) \)) evaluated at \( D_n(d) \) points of \( X \). These points lie on some algebraic hypersurface of degree \( d \) if and only if \( \Delta = 0 \).

Given some suitable parameterisation of \( X \) by functions of \( k \) variables, we estimate the above determinant by a Taylor expansion of the monomial functions to a suitable order \( b \) (remainder term order \( b + 1 \)). The order of the Taylor expansion will match the size of the matrix, and so we define \( b(m, n, d) \) as the unique integer \( b \) with
\[ D_k(b) \leq D_n(d) < D_k(b + 1). \]

It is an elementary computation, carried out in [27], that
\[ b = b(m, n, d) = \left( \frac{m!}{n!} \right)^{1/m} d^{n/m} (1 + o(1)) \]
where the \( o(1) \) means, here and below, as \( d \to \infty \) with \( m, n \) fixed. In particular,
\[ b(m, n, d) + 1 \leq 2 \left( \frac{m!d^n}{n!} \right)^{1/m} \]
provided \( d \geq d_0(m, n) \) and hence provided \( H \geq H_0(m, n) \).

The fact that \( b \) is rather larger than \( d \) is crucial to the estimates.

3.2. **Proof of 2.3.1.** In this and subsequent subsections, \( C, c, \ldots \) will denote constants depending on \( \mathcal{X} \), while \( E \) denotes a constant depending only on \( m, n \), and in both cases they may differ at each occurrence.

Let \( X = X_t \) be a fibre of \( \mathcal{X} \). We assume for now that \( H \geq H_0(m, n) \) for some \( H_0(m, n) \) to be specified in the course of the proof.

According to Theorem 2.1.3, we can parameterise \( X \) by functions
\[ \phi : (0, 1)^m \to (0, 1)^n \]
such that all partial derivatives of all component functions up to degree \( b + 1 \) are bounded in absolute value by 1, and we can cover \( X \) using at most
\[ Cb^c \]
such functions, where \(C, c\) depend on \(X\).

Let us fix one such function \(\phi = (\phi_1, \ldots, \phi_n)\), where \(\phi_i : (0, 1)^m \rightarrow (0, 1)\). From now on we deal only with \(\phi\), whose properties of relevance depend only on \(m, n\). Thus, from now on, all constants will depend only on \(m, n\).

We consider a \(D_n(d) \times D_n(d)\) determinant of the form
\[
\Delta = \det \left( (x^{(\nu)})^{(\mu)} \right)
\]
with \(\nu = 1, \ldots, D_n(d)\) indexing rows and \(\mu \in \Delta_d(n)\) indexing columns, where \(x^{(\nu)} = (x^{(\nu)}_1, \ldots, x^{(\nu)}_n) \in X(\mathbb{Q}, H)\) are points in the image of \(\phi\), say \(x^{(\nu)} = \phi(z^{(\nu)})\) where \(z^{(\nu)} \in (0, 1)^m\), later to be taken to be in a small disc in \((0, 1)^m\), and \(x^{\mu} = \prod x_i^{\mu_i}\).

As each \(x^{(\nu)}\) is a rational number with denominator \(\leq H\), we find that there is a positive integer \(K\) such that \(K\Delta \in \mathbb{Z}\) and
\[
K \leq H^{ndD_n(d)}.
\]

If we write
\[
\Phi_{\mu} = \prod_{i=1}^{n} \phi_i^{\mu_i}
\]
for the corresponding monomial function on the \(\phi_i\) then we have
\[
\Delta = \det \left( \Phi_{\mu}(z^{(\nu)}) \right).
\]

We now assume that the \(z^{(\nu)}\) all lie in a small disc of radius \(r\) centred at some \(z^{(0)}\) and expand the \(\Phi_{\mu}\) in Taylor polynomials to order \(b\) (with remainder term of order \(b + 1\)). For \(\alpha \in \Delta_k(b), \beta \in \Lambda_k(b + 1)\) we write
\[
Q_{\nu, \mu}^\alpha = \frac{\partial^\alpha \Phi_{\mu}(z^{(0)})}{\alpha!} (z^{(\nu)} - z^{(0)})^\alpha, \quad Q_{\nu, \mu}^\beta = \frac{\partial^\beta \Phi_{\mu}(z^{(0)})}{\beta!} (z^{(\nu)} - z^{(0)})^\beta
\]
with a suitable intermediate point \(z^{(\nu)}_\mu\) (on the line joining \(z^{(0)}\) to \(z^{(\nu)}\)), and with \(\alpha! = \prod_{j=1}^{k} \alpha_j!\), so that the Taylor polynomial is
\[
\Phi_{\mu}(z^{(\nu)}) = \sum_{\alpha \in \Delta_k(b)} Q_{\nu, \mu}^\alpha + \sum_{\beta \in \Lambda_m(b+1)} Q_{\nu, \mu}^\beta
\]
and we have
\[
\Delta = \det \left( \sum_{\alpha \in \Delta_m(b+1)} Q_{\nu, \mu}^\alpha \right).
\]

We expand the determinant as in [25] (see also [28], eqn (2), p48), using column linearity to get
\[
\Delta = \sum_{\tau} \Delta_{\tau}, \quad \Delta_{\tau} = \det \left( Q_{\nu, \mu}^{\tau(\mu)} \right).
\]
with the summation over \( \{ \tau : \Delta_n(d) \to \Delta_m(b + 1) \} \).

Now if, for some \( k \in \{1, \ldots, b\} \), we have

\[
\#\tau^{-1}(A_m(k)) > L_m(k)
\]

then \( \Delta_\tau = 0 \) as the corresponding columns are dependent (the space of homogeneous forms in \( (z^{(\nu)} - z^{(0)}) \) of degree \( k \) has rank \( L_m(k) \)). Thus all surviving terms have a high number of factors of the form \( (z^{(\nu)} - z^{(0)}) \) and/or \( (\zeta^{(\mu)} - z^{(0)}) \). We quantify this.

The function \( \Phi_\mu \) is a product of \( |\mu| \leq d \) functions \( \phi_i \), which have suitably bounded derivatives. Let us consider more generally a function

\[
\Theta = \prod_{i=1}^{\delta} \theta_i
\]

where \( \theta_i \) have \( |\theta_i^{(\alpha)}(z)| \leq 1 \) for all \( |\alpha| \leq b + 1 \). Then, for \( \alpha \) with \( |\alpha| \leq b + 1 \), we have

\[
\Theta^{(\alpha)} = \sum_{\alpha^{(i)}} \mathrm{Ch}(\alpha^{(1)}, \ldots, \alpha^{(\delta)}) \prod_{i=1}^{\delta} \theta_i^{(\alpha^{(i)})}
\]

with the summation over \( \alpha^{(1)} + \ldots + \alpha^{(\delta)} = \alpha \) where

\[
\mathrm{Ch}(\alpha^{(1)}, \ldots, \alpha^{(\delta)}) = \prod_{j=1}^{m} \left( \frac{\alpha_j!}{\alpha_j^{(1)}! \ldots \alpha_j^{(\delta)}!} \right).
\]

Since \( |\theta_i^{(\alpha)}(z)| \leq 1 \) for all \( |\alpha| \leq b + 1 \) we have

\[
|\Theta^{(\alpha)}(z)| \leq \sum_{\alpha^{(i)}} \left( \frac{\alpha_1!}{\alpha_1^{(1)}! \ldots \alpha_1^{(\delta)}!} \right) \times \ldots \times \sum_{\alpha^{(i)}} \left( \frac{\alpha_m!}{\alpha_m^{(1)}! \ldots \alpha_m^{(\delta)}!} \right)
\]

each summation subject to \( \sum_i \alpha_j^{(i)} = \alpha_j \) hence

\[
|\Theta^{(\alpha)}(z)| \leq \delta^{\alpha_1} \ldots \delta^{\alpha_m} = \delta^{|\alpha|}.
\]

Therefore

\[
|Q_{\nu, \mu}^{\alpha}| \leq \frac{|(\mu|r)^{[\alpha]}|}{\alpha!} \leq e^{m|\mu| |r|^{\alpha}} \leq e^{md|\nu|^{\alpha}},
\]

and for a \( \tau \) which avoids the condition above (under which \( \Delta_\tau = 0 \)) all terms in its expansion are bounded above in size by

\[
e^{mdD_n(d)rB}
\]

where

\[
B = B(m, n, d) = \sum_{k=0}^{b} L_m(k)k + \left( D_n(d) - \sum_{\kappa=0}^{b} L_m(\kappa) \right) (b + 1).
\]
Note that \( \left( D_n(d) - \sum_{\kappa=0}^{b} L_m(\kappa) \right) \geq 0 \) by our choice of \( b \). We have the asymptotic expression (see [27])

\[
B = B(m, n, d) = \frac{1}{(m+1)! (m-1)!} \left( \frac{m!}{n!} \right)^{(m+1)/m} d^{m+n/m} (1 + o(1)).
\]

The number of terms from all the \( \Delta_\tau \) is \( D_m(b+1)^{D_n(d)} D_n(d)! \) and we conclude that

\[
|\Delta| \leq D_m(b+1)^{D_n(d)} D_n(d)! e^{mdD_n(d)} r^B.
\]

Thus we have an integer \( K\Delta \) with

\[
K|\Delta| \leq H^{ndD_n(d)} D_m(b+1)^{D_n(d)} D_n(d)! e^{mdD_n(d)} r^B
\]

and if \( K|\Delta| \geq 1 \) then so is its \( B \)th root. Now with our choice of \( d \) we find that

\[
\frac{dnD_n(d)}{B} = E \frac{d^{n+1}}{d^{n/m}} (1 + o(1)) = Ed^{-(n-m)/m} (1 + o(1))
\]

thus

\[
H^{ndD_n(d)/B} \leq E (1 + o(1))
\]

is bounded (as \( d \to \infty \)).

The remaining terms

\[
\left( D_m(b+1)^{D_n(d)} D_n(d)! e^{mdD_n(d)} \right)^{1/B}
\]

are also bounded as \( d \to \infty \) (see [27]) and so

\[
(K|\Delta|)^{1/B} \leq Er
\]

where \( E \) is a constant depending only on \( n, m \), provided \( H \geq H_0(n, m) \) is sufficiently large. If \( rE < 1 \) then all points of \( X(\mathbb{Q}, H) \) parameterised by \( \phi \) from this disk lie on one algebraic hypersurface of degree \( d \), because the rank of the rectangular matrix formed by evaluating all monomials of degree \( \leq d \) at all such points is less than \( D_n(d) \).

Since \( (0,1)^m \) may be covered by some \( E' \) such discs, and there are \( C(b+1)^c \leq C'(\log H)^c \) maps \( \phi \) which cover \( X \), the required conclusion follows for \( H \geq H_0(m, n) \). However for \( H \leq H_0 \) the number of points is bounded depending only on \( H_0, m, n \). \( \Box \)

Note that, in [27, 3.2 and 3.3] it is tacitly assumed that the mildness parameter \( A \) satisfies \( A \geq 1 \).
3.3. **Setup for 2.3.2.** This and the subsequent two subsections are devoted to the proof of Proposition 2.3.2. After some preliminary results, the proof itself is in 3.5.

We have a family of sets $X_t \subset [0, 1]^n$, of dimension $m$, definable in $\mathbb{R}_{an}$, and hence admitting a quasi-parameterisation of the following form. There exist a positive integer $N$, overconvergent analytic functions $h_{i,j} : [-1, 1]^{m+\eta} \to \mathbb{R}$, $i = 1, \ldots, n$, $j = 0, \ldots, N - 1$, converging on a disc of radius $r_0 > 1$, and functions $u_j : T \to [-1, 1]^\eta$

(which need not be definable) such that, setting $u(t) = (u_1(t), \ldots, u_\eta(t))$, for all $t \in T$ and $x = (x_1, \ldots, x_\eta) \in X_t$ there exists $w = (w_1, \ldots, w_m) \in [-1, 1]^m$ such that

$$x_i^N = h_{i,N-1}(w_i, u(t))x_i^{N-1} + \ldots + h_{i,0}(w_i, u(t)).$$

In fact, we can assume that the $h_{i,j}$ are independent of $i$, but don’t need this.

We keep the previous convention regarding constants.

3.4. **Preliminary estimates.** For each $i$, setting $x = x_i$ and $h_j = h_{i,j}$ and suppressing the subscript $i$ and the arguments of the $h_j$, we have a relation

$$x^N = h_{N-1}x^{N-1} + \ldots + h_0.$$

By means of this relation, all powers $x^\nu, \nu \in \mathbb{N}$ may be expressed as suitable linear combinations of $1, x, \ldots, x^{N-1}$, namely

$$x^\nu = \sum_{j=0}^{N-1} q_{\nu,j}x^j$$

with coefficients $q_{\nu,j} \in \mathbb{Z}[h_0, \ldots, h_{N-1}]$. In particular $q_{N,j} = h_j$. We need an estimate for the degree and integer coefficients of the $q_{\nu,j}$.

Let $H$ be the $N \times N$ matrix of analytic functions

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
h_0 & h_1 & h_2 & h_3 & \ldots & h_{N-1}
\end{pmatrix}
$$

Then $H$ acts as a linear transformation on the vector space with (ordered) basis $\{1, x, \ldots, x^{N-1}\}$ and the $q_{\nu,j}$ are the entries of the column
vector

\[
H^\nu \begin{pmatrix} 
1 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

Inductively, the entry \(H^\nu_{ij}\) is in \(\mathbb{Z}[h_1, \ldots, h_N]\) of degree \(\max(0, j - n + \nu)\) and the sum of at most \(\max(n^\nu - 1, j - n, 1)\) pure (i.e. with coefficient 1) monomials. We have proved the following.

**3.4.1. Lemma.** For all \(\nu \in \mathbb{N}\) and \(j = 0, \ldots, N - 1\), \(q_{\nu,j}\) is a sum of at most \(2^\nu\) monomials of degree at most \(\nu\) in the \(h_j\).

The above is for one variable. We now return to the multivariate setting with \(x = (x_1, \ldots, x_n)\). For \(\nu \in \mathbb{N}\) we then have

\[
x^\nu_i = \sum_{j=0}^{N-1} q_{i,\nu,j} x^j
\]

where \(q_{i,\nu,j}\) is the previously labelled \(q_{\nu,j}\) for the relevant \(h_j = h_{i,j}\).

If \(\nu \in \mathbb{N}^n\) we have

\[
x^\nu = \prod_{i=1}^{n} x_i^{\nu_i} = \prod_{i=1}^{n} \sum_{j=0}^{N-1} q_{i,\nu_i,j} x^j = \sum_{\lambda \in M} q_{\nu,\lambda} x^\lambda
\]

where \(q_{\nu,\lambda} = \prod_{i=1}^{n} q_{i,\nu_i,\lambda_i}\).

We now want to bound the derivatives (in the \(w\) variables) of the \(q_{\mu,\lambda}\). For \(\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m\) we set \(\overline{\alpha} = \max(\alpha_i)\).

**3.4.2. Lemma.** For suitable constants \(C, R\) and all \(\alpha \in \mathbb{N}^m\) we have

\[
\left| \frac{q_{\nu,\lambda}(w,u)}{\alpha!} \right| \leq 2^{\nu |(\overline{\alpha} + 1)\}^{(\nu| - 1)m \cdot C^{\nu|} \cdot R|\alpha|}.
\]

**Proof.** For derivatives (in the \(w\)-variables) of the \(h_{i,j}\) we have a bound of the form

\[
\left| \frac{h_{i,j}^\alpha(w,u)}{\alpha!} \right| \leq C \cdot R^{|\alpha|},
\]

where \(\alpha \in \mathbb{N}^m\), valid for every \(i, j\), by Cauchy’s theorem, since they are analytic on some disc of radius \(r_0 > 1\).

We have that \(q_{i,\nu,j}\) is a sum of at most \(2^{\nu_i}\) monomials in the \(h_{i,j}\), each of degree at most \(\nu_i\). Therefore \(q_{\nu,\lambda}\) is a sum of at most \(2^{\nu|}\) monomials each of degree \(|\nu|\) in the \(h_{i,j}\). Consider one such monomial

\[
g = \prod_{h=1}^{\ell} \phi_h
\]
where each $\phi_h \in \{h_{i,j} : i = 1, \ldots, n, j = 0, \ldots, N - 1\}$ and $\ell \leq |\nu|$. As before

$$\partial^\alpha g = \partial^\alpha \phi_1 \ldots \phi_\ell = \sum_{\alpha^{(1)} + \ldots + \alpha^{(\ell)} = \alpha} \mathrm{Ch}(\alpha^{(1)}, \ldots, \alpha^{(\ell)}) \prod_{h=1}^{\ell} \partial^{\alpha^{(h)}} \phi_h.$$ 

Thus

$$\frac{\partial^\alpha g}{\alpha!} \leq \sum_{\alpha^{(1)} + \ldots + \alpha^{(\ell)} = \alpha} \prod_{h=1}^{\ell} \frac{\partial^{\alpha^{(h)}} \phi_i}{\alpha^{(h)}!} \leq C|\nu| R^{\alpha} \sum_{\alpha^{(1)} + \ldots + \alpha^{(\ell)} = \alpha} 1.$$ 

The number of summands is at most $(\pi + 1)^{|\nu|-1}m$. □

3.4.3. Corollary. For $\alpha \in \mathbb{N}^m$ we have

$$\left| \frac{q_{\mu,\lambda}(w,u)}{\alpha!} \right| \leq 2^{|\nu|(|\alpha| + 1)|\nu|m} C|\nu| R^{\alpha}. \quad \square$$

3.5. Proof of Proposition 2.3.2. We let $X = X_t$ be a fibre of $\mathcal{X}$. We will explore $X(\mathbb{Q}, H)$ with real algebraic hypersurfaces of degree

$$d = \left[ (\log H)^{m/(n-m)} \right],$$

and again consider $D_n(d) \times D_n(d)$ determinants

$$\Delta = \det \left( (x^{(\nu)})^\mu \right) = 0$$

where $\mu \in \Delta_n(d)$ indexes monomials and $x^{(\nu)}, \nu = 1, \ldots, D_n(d)$ are points of $X$.

For each $x^{(\nu)}$ there is some $z^{(\nu)}$ such that the quasiparameetrization conditions hold, i.e. “$x^{(\nu)}$ is parameterised by the point $z^{(\nu)}$”. Later we will assume that all the $z^{(\nu)}$ are in the disc of radius $r$ entered at some $z^{(0)}$.

By our assumptions, there is a positive integer $K \leq H^{ndD_n(d)}$ such that

$$K \Delta \in \mathbb{Z}.$$ 

Now let $M = \{\lambda \in \mathbb{N}^n : \lambda_i < N, i = 1, \ldots, n\}$. Then if $x \in X$ we have

$$x^\mu = \sum_{\lambda \in M} x^\lambda q_{\mu,\lambda}(w, u(t))$$

for some $z \in [-1, 1]^m$ where

$$q_{\mu,\lambda} = \prod_{i=1}^{n} q_{\mu_i, \lambda_i}.$$
There is a unique \( b = b(m, n, d, N) \) such that
\[
N^n D_m(b) \leq D_n(d) \leq N^n D_m(b + 1).
\]
Since \( m < n \) there are fewer monomials in \( m \) variables than in \( n \) variables, and so if \( d \) suitably large in terms of \( N, n \) then \( b \) is somewhat larger than \( d \). Set
\[
B(m, n, d, N) = \sum_{\beta=0}^{b} N^n L_m(\beta) \beta + (D_n(d) - N^n \sum_{\beta=0}^{b} L_m(\beta))(b + 1).
\]
Since \( b \) is somewhat larger than \( d \), we will have that \( B \) is somewhat larger than \( ndD_n(d) \) (as \( d \to \infty \)), as will be crucial.

Now we assume that all the \( z^{(\nu)} \) are all in the disc of radius \( r \) entered at some \( z^{(0)} \). We expand each \( q = q_{\mu,\lambda} \) in a Taylor polynomial with remainder term of order \( b+1 \). For \( \alpha \in \Delta_k(b), \beta \in \Lambda_k(b+1) \) we write
\[
Q_{\nu,\mu}^{\lambda,\alpha} = \frac{\partial^\alpha q_{\mu,\lambda}}{\alpha!} (z^{(0)}) (z^{(\nu)} - z^{(0)})^\alpha, \quad Q_{\nu,\mu}^{\lambda,\beta} = \frac{\partial^\beta q_{\mu,\lambda}}{\beta!} (z^{(0)}) (\zeta^{(\nu)}_{\lambda,\beta} - z^{(0)})^\beta
\]
for some suitable intermediate point \( \zeta^{(\nu)}_{\lambda,\beta} \). Then we have
\[
\Delta = \det \left( \sum_{(\lambda,\beta)} Q_{\nu,\mu}^{\lambda,\alpha} \right),
\]
with the summation over \( (\lambda,\alpha) \in \{0,\ldots,N-1\}^n \times \Delta_k(b+1) \).

We expand the determinant as previously in terms of maps \( \tau : \Delta_n(d) \to \{0,\ldots,N-1\}^n \times \Delta_k(b+1) \), giving
\[
\Delta = \sum_{\tau} \Delta_{\tau}, \quad \Delta_{\tau} = \det (Q_{\nu,\mu}^{\tau(\mu)}).
\]
with the summation over \( \tau \) as above.

Now if for some \( \lambda \in M, k \) with \( 0 \leq k \leq b \) we have
\[
\#\tau^{-1}(\{\lambda\} \times \Lambda_m(k)) > L_m(k)
\]
then \( \Delta_{\tau} = 0 \) because the corresponding columns are dependent (the factors \( (x^{(\nu)})^\lambda \) are constant on the rows in those columns).

Since there are \( N^n \) possibilities for \( \lambda \), we have that the total number of columns from which an expansion term of degree \( k \) may be drawn for a surviving term is \( N^n L_m(k) \).

We now assume that \( rR < 1 \). Then every surviving term is estimated by
\[
[(n(b+1)^m C)^d]^{D_n(d)} (rR)^B
\]
for some \( B' \geq B = B(m, n, d, N) \), and since \( rR < 1 \) every term is estimated by the above with \( B' = B \).
The total number of terms, assuming no cancellation, is
\[ D_n(d)! \left( N^n D_m(b + 1) \right)^{D_n(d)}. \]
Thus we have an integer \( K \Delta \) with
\[ K |\Delta| \leq H^{ndD_n(d)} D_n(d)! \left( N^n D_m(b + 1) \right)^{D_n(d)} \left[ (n(b + 1)^m C)^d \right]^D (rR)^B. \]
And if \( K |\Delta| \geq 1 \) then so is its \( B \)th root.

Now we have (see [27])
\[ L_m(d) = \left( \frac{m-1+d}{m-1} \right) = \frac{d^{m-1}}{(m-1)!}(1 + o(1)) \]
where here and below \( o(1) \) means as \( d \to \infty \) for fixed \( m, n, N \). Thus likewise
\[ D_m(d) = L_{m+1}(d) = \frac{d^m}{m!}(1 + o(1)). \]
We find that
\[ b(n, m, N, d) = \left( \frac{m!d^n}{n!N^n} \right)^{1/m} (1 + o(1)) \]
for \( d \to \infty \) with \( n, m, N \) fixed. Thus (by replacing the sum \( \sum_{\delta=0}^{b} L_m(\delta)\delta \)
by an integral)
\[ B(m, n, N, d) = E(m, n, N)d^{m(m+1)/m}(1 + o(1)), \]
where \( E \) is a suitable combinatorial expression.

With our choice of \( d \) we have as before that \( H^{ndD_n(d)/B} \leq E_1 \) is bounded. We also have
\[ \left( D_n(d)! \left( N^n D_m(b + 1) \right)^{D_n(d)} \right)^{1/B} = 1 + o(1) \]
as \( d \to \infty \), so is bounded by some \( E \).

Finally, we have that
\[ \frac{dD}{B} = \frac{E}{d^{n/m-1}(1 + o(1))} \leq \frac{E}{d^{n/m-1}}, \]
while
\[ \frac{bD}{B} = E(1 + o(1)) \leq E. \]

Therefore
\[ |K\Delta|^{1/B} \leq EC Rr \]
and all the points of \( X(\mathbb{Q}, H) \) in the image of the disc lie on one hypersurface of degree at most \( d \) provided
\[ r < (CER)^{-1}. \]
The box \((0, 1)^m \) may be covered by
\[ C_2 = (c(n)C_2E_5R + 1)^m \]
such discs, where \( c(n) \) is the maximum side of a cube inscribed in a unit \( n \)-sphere. This gives the desired conclusion for \( H \geq H_0(m, n, N) \) and for smaller \( H \) it follows as the number of such rational points is bounded. □

3.6. **Proof of 2.3.3 and 2.3.4.** By Theorem 2.1.3 in the case of 2.3.5, and Theorem 2.3.3 in the case of 2.3.6, we find that \( X_t(\mathbb{Q}, H) \) is contained in the intersection of \( X_t \) with at most \( c_5(\mathcal{N}) \) hypersurfaces of degree \( d = \lfloor \log H \rfloor^{m/(n-m)} \).

If any such intersection has dimension 2, then the pfaffian functions parameterising the surface \( X_t \) identically satisfy some algebraic relation. Then the surface \( X_t \) is algebraic, and \( X^\text{trans} \) is empty.

Thus we may assume that all the intersections have dimension at most 1. We will treat these following the method in [27] by dividing the intersections into graphs of functions with suitable properties, and estimating the rational points on any such graphs which are not semi-algebraic using the Gabrielov-Vorobjov estimates.

Suppose that the fibre \( X_t \) is the intersection of \([-1,1]^3\) with the pfaffian surface defined by the pfaffian functions

\[
x, y, z : G \to \mathbb{R}
\]

of complexity (at most) \((r, \alpha, \beta)\). Write \((p, q)\) for the variables in \( G \). Suppose that the polynomial \( F(x, y, z) \) of degree at most \( d \) defines the hypersurface \( V = V_F \).

The intersection \( X_t \cap V \) is the image of the one-dimensional subset \( W \subset G \) defined by

\[
\phi(p, q) = F(x(p, q), y(p, q), z(p, q)) = 0.
\]

It is thus the zero-set of a pfaffian function of complexity \((r, \alpha, d\beta)\). The singular set \( W_s \subset W \) is defined by \( \phi = \phi_p = \phi_q = 0 \), the zero-set pfaffian functions of complexity \((r, \alpha, \alpha + d\beta - 1)\) (see [11, 2.5]).

At a point of \( W - W_s \), \( W \) is locally the graph of a real-analytic function parameterised by \( p \) if \( \phi_p \neq 0 \), or \( q \) if \( \phi_q \neq 0 \).

Proceeding as in [27], we decompose \( V_F \) into “good” curves, and points. Here a “good” curve is a connected subset whose projection into each coordinate plane of \( \mathbb{R}^3 \) is a “good” graph with respect to one or other of the axes; namely, the graph of a function \( \psi \) which is real analytic on an interval, has slope of absolute value 1 at every point, and such that the derivative of \( \psi \) of each order 1, \ldots, \( \lfloor \log H \rfloor \) is either non-vanishing in the interior of the interval or identically zero.

Using the topological estimates of Gabrielov-Vorobjov [11, 3.3], Zell [35], and estimates for the complexities of the various pfaffian functions
involved as in [27], one shows that $V_F$ decomposes into a union of at most

$$C_6(r, \alpha, \beta)d^{C_7(r, \alpha, \beta)}$$

points and “good” curves $Z$. If such a “good” curve $Z$ is semi-algebraic, then so are its projections to each coordinate plane, and also conversely. On a non-algebraic plane “good” graph $Y$, one has

$$N(Y, H) \leq C_8(r, \alpha, \beta)(d \log H)^{C_9(r, \alpha, \beta)}$$

as in [27], using [26] and estimates for pfaffian complexity. □

4. Proof of the $C^r$-parameterization theorem

In this section we prove Theorem 2.1.3. We will use a combination of the preparation result of [20] with a preparation result of [21] which enables us to have centers of preparation with bounded first derivatives, see Theorem 4.3.3 below. Another ingredient for us is the result in [23] on cell decomposition with Lipschitz continuous cell walls, up to transformations coming just from coordinate permutations, see Theorem 4.2.2 below. Pawłucki’s results in [23] refine Kurdyka’s [18] and Valette’s [29] in the sense that only coordinate permutations are needed, and no more general orthogonal transformations.

In section 4.1 we give some results about derivatives of compositions, related to mild functions, Gevrey functions, Faà di Bruno’s formula, and a notion which we call weakly mild functions.

Let us first fix some terminology. For any language $L$ on $\mathbb{R}$ and any subfield $K$ of $\mathbb{R}$, let us denote by $L^K$ the expansion of $L$ by the power maps

$$x \mapsto \begin{cases} x^r, & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

for $r \in K$. Let $L_{an}$ be the subanalytic language on $\mathbb{R}$. We will use some reducts of $L_{an}^K$, following [20].

Let $F$ be a Weierstrass system and let $L_F$ be the corresponding language as in Definition 2.1 of [20]. (The language $L_F$ is a reduct of $L_{an}$.) By the field of exponents of $F$ is meant the set of real $r$ such that $(0, 1) \to \mathbb{R} : x \mapsto (1 + x)^r$ is $L_F$-definable; this set is moreover a field, see Remark 2.3.5 of [20]. Let $K$ be a subfield of the field of exponents of $F$. From Section 4.3 up to the end of Section 4 we will work with the $L_F^K$-structure on $\mathbb{R}$. 
4.1. Compositions. We refine the notion of mild functions from [27] (which resemble Gevrey functions, [12]).

**Definition 4.1.1.** Let $A > 0$ and $C \geq 0$ be real, and let $r > 0$ be either an integer or $+\infty$. A function $f : U \subset (0,1)^m \to (0,1)$ with $U$ open is called $(A,C)$-mild up to order $r$ if it is $C^r$ and for all $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq r$ and all $x \in U$ one has

$$|f^{(\alpha)}(x)| \leq \alpha!(A|\alpha| C)^{|\alpha|}.$$

Call a map $f : U \subset (0,1)^m \to (0,1)^n$ $(A,C)$-mild up to order $r$ if all of its component functions are.

We introduce a weaker notion, namely that of weakly mild functions.

**Definition 4.1.2.** Let $A > 0$ and $C \geq 0$ be real, and $r > 0$ be either an integer or $+\infty$. A function $f : U \subset (0,1)^m \to (0,1)$ with $U$ open is called weakly $(A,C)$-mild up to order $r$ if it is $C^r$ and for all $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq r$ and all $x \in U$ one has

$$|f^{(\alpha)}(x)| \leq \frac{\alpha!(A|\alpha| C)^{|\alpha|}}{x^\alpha},$$

where $x^\alpha$ stands for $\prod_{j=1}^m x_j^{\alpha_j}$. Call a map $f : U \subset (0,1)^m \to (0,1)^n$ weakly $(A,C)$-mild up to order $r$ if all of its component functions are.

By the theory of Gevrey functions [12], it is known that a composition of mild functions is mild. Here we study some related results about compositions, with proofs based on Faà di Bruno’s formula. (We do no effort to control the bounds beyond what we need.)

The next lemma is obvious by the chain rule for derivation.

**Lemma 4.1.3.** Let $r > 0$ be an integer and let

$$f : U \subset (0,1)^m \to (0,1)$$

be $(A,C)$-mild up to order $r$. Then the function

$$V \to (0,1) : x \mapsto f(x/Ar^{C+1})$$

has $C^r$-norm bounded by 1, where $V \subset (0,1)^m$ is the open consisting of $x$ such that $x/Ar^{C+1} := (x_1/Ar^{C+1}, \ldots, x_m/Ar^{C+1})$ lies in $U$.

We give a basic corollary of a multi-variable version of Faà di Bruno’s formula from e.g. [5].

**Lemma 4.1.4.** Let $m \geq 1$ and $d \geq 1$ be integers. Then there exist $A > 0$ and $C \geq 0$, depending only on $m$ and $d$, such that the following holds. Consider a composition $h = f \circ g$, with $g : U \subset \mathbb{R}^d \to V \subset \mathbb{R}^m$,
Let $f : V \to \mathbb{R}$ and $U$ and $V$ open. Let $\nu \in \mathbb{N}^d$ be a nonzero multi-index, write $|\nu| = n$ and suppose that $f$ and $g$ are $C^\nu$. Then

\[ h(\nu) \]

is the sum of no more than $(An^C)^n$ terms of the form (in multi-index notation)

\[ f(\lambda) \prod_{j=1}^s (g(\ell_j))^{k_j}, \]

where $1 \leq s \leq n$, $\lambda, k_j \in \mathbb{N}^m$, $\ell_j \in \mathbb{N}^d$, $0 < |\lambda| \leq n$, $0 < |k_j| \leq n$, $0 < |\ell_j| \leq n$, and where

\[ \sum_{j=1}^s k_j = \lambda, \quad \text{and} \quad \sum_{j=1}^s |k_j|\ell_j = \nu, \]

where $0^0$ is set equal to $1$, and where $h(\nu)$ and $g(\ell_j)$ are evaluated at $u \in U$ and $f(\lambda)$ at $g(u)$.

**Proof.** The lemma follows from Theorem 2.1 of \[5\], by using

\[ \nu! \prod_{j=1}^s k_j!(\ell_j)!^{k_j} \leq \nu! \leq n^n, \]

and by letting the $s$, $\lambda_i$, $k_{ji}$ and $\ell_{jr}$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$, and $r = 1, \ldots, d$ run independently from 0 to $n$ when estimating the number of terms in (2.1) of Theorem 2.1 of \[5\].

We give some results about compositions.

**Proposition 4.1.5.** Let $A > 0$ and $C \geq 0$ be real numbers and let $m > 0$ be an integer. Let $f : V \to (0,1)$ be a function on some open $V \subset (0,1)^m$. Assume for $\beta \in \mathbb{N}^m$ with $|\beta| \leq 1$, that $f^{(\beta)}$ is weakly $(A,C)$-mild up to order $+\infty$. Then, there is $(A',C')$, depending only on $m$, $A$ and $C$, such that for any integers $r > 0$ and $L_i \geq r$, the composition $h = f \circ g$ of $f$ with

\[ g : x \mapsto x^L := (x_1^{L_1}, \ldots, x_m^{L_m}) \]

on the open $U \subset (0,1)^m$ consisting of $x$ with $x^L \in V$, is $(L_{io}A',C')$-mild up to order $r$, with $i_0$ such that $L_{io} = \max_i L_i$.

**Proof.** By Lemma 4.1.4 with $d = m$, and up to enlarging $A$ and $C$ if necessary, we only have to estimate a single term of the form (1) for $\nu$ with $|\nu| \leq r$. Fix $\nu$ with $1 \leq |\nu| \leq r$ and write $|\nu| = n$. If $n = 1$ the statement follows from the conditions on $f^{(\beta)}$ for $\beta$ with $|\beta| \leq 1$. So let us suppose $n > 1$. Fix $s$ with $1 \leq s \leq n$ and $\lambda \in \mathbb{N}^m$ with $|\lambda| \leq n$. Choose $\lambda'$ and $\beta$ in $\mathbb{N}^m$ with $\lambda' + \beta = \lambda$ and $|\beta| = 1$. (Near the end we
will optimize the choice of $\beta$. By the weak $(A,C)$-mildness of $f^{(\beta)}$, we have

$$|f^{(\lambda)}(x^L)| = |(f^{(\beta)})^{(\lambda)}(x^L)| \leq \frac{\lambda!(A|\lambda|C)^{|\lambda|}}{x^{L\lambda}},$$

where

$$L\lambda = (L_i\lambda_i)_i.$$  

We may and do suppose that $i_0 = 1$ for convenience of notation, so that $L_1 = \max_i L_i$. We have

$$|\prod_{j=1}^s (g(\ell_j)(x))^{k_j}| \leq |\prod_{j=1}^s L_i^{[\ell_j]|k_j|} x^{(L-\ell_j)k_j}|$$

where $k_j$ and $\ell_j$ are as in (2) and where

$$(L - \ell_j)k_j = ((L_i - \ell_{ji})k_{ji})_i.$$  

By (2) it follows that

$$\prod_{j=1}^s L_i^{[\ell_j]|k_j|} x^{(L-\ell_j)k_j} = \prod_{j=1}^s \left(\sum_{j=1}^{s} [\ell_j]|k_j| x^{-L\lambda'} + \sum_{j=1}^{s} (L-\ell_j)k_j \right) x^{-L\beta - \sum_{j=1}^{s} \ell_jk_j},$$

where $L\beta = (L_i\beta_i)_i$, and similarly $\ell_jk_j = (\ell_{ji}k_{ji})_i$. Since this last inequality holds for any choice of $\beta$ with $|\beta| = 1$ (and the corresponding $\lambda'$), given $x$ we can choose $\beta$ with $\beta_{i_0} = 1$ when $x_{i_0} = \min_i x_i$. Now we are done since $|\sum_{j=1}^{s} \ell_jk_j| \leq n$ by (2) and $|L\beta| = L_{i_0} \geq r \geq n$. $\square$

**Proposition 4.1.6.** Let $A > 0$, $A' > 0$, $C \geq 0$, $C' \geq 0$ be real numbers, let $m > 0$ and $d > 0$ be integers, and let $r$ be either a positive integer or $+\infty$. Let $f : V \to \mathbb{R}$ and $g : U \to V$ be $C''$ functions on some open sets $U \subset (0,1)^d$ and $V \subset (0,1)^m$. Assume that $f$ is $(A,C)$-mild up to order $r$ and that $g$ is weakly $(A',C')$-mild up to order $r$. Then, there is $(A'',C''')$, depending only on $m,d$ and $A, A', C, C'$, such that the composition $h = f \circ g$ is weakly $(A'',C''')$-mild up to order $r$.

**Proof.** As above, we only have to estimate a single term of the form (1) for $\nu$ with $|\nu| \leq r$, by Lemma 4.1.4.

Fix $\nu$ with $1 \leq |\nu| \leq r$ and write $|\nu| = n$. Fix $s$ with $1 \leq s \leq n$ and $\lambda \in \mathbb{N}^m$ with $|\lambda| \leq n$. By the $(A,C)$-mildness of $f$ up to order $r$ we have

$$|f^{(\lambda)}(x)| \leq \lambda!(A|\lambda|C)^{|\lambda|}.$$  

By the weak $(A',C')$-mildness of $g$ up to order $r$ we have

$$\prod_{j=1}^s |g^{(\ell_j)}(x)|^{k_j} | \leq \prod_{j=1}^s \ell_j!(A'|\ell_j|C')^{[\ell_j]|k_j|} x^{[\ell_j]|k_j|}.$$  

The product of these terms is bounded by

$$\prod_{j=1}^s (g^{(\ell_j)}(x))^{k_j} \leq \prod_{j=1}^s L_i^{[\ell_j]|k_j|} x^{(L-\ell_j)k_j}.$$  

By the weak $(A',C')$-mildness of $g$ up to order $r$ we have
where \(k_j\) and \(\ell_j\) are as in (2). Now by (2), and assuming \(A' \geq 1\),
\[
|\prod_{j=1}^{s} (g^{(\ell_j)}(x))^{k_j}| \leq \left| \frac{\nu ! (A'nC')^n}{x^{\sum_j \ell_j k_j}} \right| \leq \left| \frac{\nu ! (A'nC')^n}{x^{\nu}} \right|.
\]
Putting these together we find bounds as desired.  

We give one concrete instance of Proposition 4.1.6 with polynomial control on the constants.

Lemma 4.1.7. Let \(\varepsilon > 0\), \(A > 0\) and \(C \geq 0\) be real numbers and let \(m > 0\) be an integer. Let \(f : U \to (0, 1)\) be weakly \((A, C)\)-mild up to order \(+\infty\) on some open \(U\) of \((0, 1)^m\). Assume furthermore that \(f(x) > \varepsilon\) on \(U\). Then there are polynomials \(A_1(\ell)\) and \(C_1(\ell)\) in one variable \(\ell\), depending only on \(\varepsilon\), \(A\), \(C\) and \(m\) such that, for any integer \(\ell > 0\), the function sending \(x\) in \(U\) to \(\sqrt{f(x)}\) is weakly \((A_1(\ell), C_1(\ell))\)-mild up to order \(+\infty\). If moreover \(\partial f/\partial x_i\) is also weakly \((A, C)\)-mild up to order \(+\infty\), for some \(i\), then \(A_1(\ell), C_1(\ell)\) can be taken so that, for any integer \(\ell > 0\),
\[
\varepsilon \cdot \partial \sqrt{f} / \partial x_i
\]
is also weakly \((A_1(\ell), C_1(\ell))\)-mild up to order \(+\infty\).

Proof. The lemma follows easily from the chain rule for derivation.  

We apply our results on compositions to a specific kind of functions, called \(a\)-\(b\)-\(m\) functions, and defined as follows.

Definition 4.1.8. Call a function \(U \subset (0, 1)^m \to \mathbb{R}\) monomial if it is of the form
\[
x \mapsto x^\mu := \prod_{i=1}^{m} x_i^{\mu_i}
\]
for some \(\mu_i\) in \(\mathbb{R}\).

Call a function \(U \to \mathbb{R}\) bounded-monomial if it is monomial and its range is a bounded set in \(\mathbb{R}\).

Call a map \(U \to \mathbb{R}^n\) monomial, resp. bounded-monomial, if all of its component functions are.

Call a function \(U \to \mathbb{R}\) analytic-bounded-monomial, abbreviated by \(a\)-\(b\)-\(m\), if it is of the form
\[
x \mapsto F(b(x))
\]
for some bounded-monomial map \(b : U \subset (0, 1)^m \to \mathbb{R}^n\) for some \(n \geq 0\) and for some function \(F : b(U) \to \mathbb{R}\) which is analytic on some open neighborhood of the closure of \(b(U)\) in \(\mathbb{R}^n\).
Say that a function $f : U \subset (0,1)^m \rightarrow \mathbb{R}$ is 0-prepared if there is an a-b-m function $u : U \rightarrow \mathbb{R}$ which takes values in $(1/S, S)$ for some $S > 1$ and a bounded-monomial function $x \mapsto x^\mu$ on $U$ such that

$$f(x) = x^\mu u(x).$$

Call a map $U \subset (0,1)^m \rightarrow \mathbb{R}^n$ a-b-m, resp. 0-prepared, if all of its component functions are.

In our terminology, a 0-prepared function is automatically a-b-m and has bounded range.

**Lemma 4.1.9.** Let $f : U \subset (0,1)^m \rightarrow (0,1)$ be an a-b-m function. Then the function $f$ is weakly $(A, C)$-mild up to order $+\infty$ for some $A > 0$ and some $C \geq 0$.

**Proof.** Any function $F : V \rightarrow (0,1)$ on an open $V$ in $(0,1)^n$ such that $F$ is analytic on some open neighborhood of the closure of $V$ in $\mathbb{R}^n$ is $(A_0, C_0)$-mild up to order $+\infty$ for some $A_0 > 0$ and some $C_0 \geq 0$, see e.g. [12]. Also, for any real $S \geq 1$ and any bounded-monomial function $b : U \subset (0,1)^m \rightarrow (0,S) : x \mapsto x^\mu$ with $U$ open in $(0,1)^m$, the function $b/S$ is weakly $(A_1, C_1)$-mild up to order $+\infty$ for some $A_1 > 0$ and $C_1 \geq 0$. One may for example take $A_1 = C_1 = \max |\mu_i|$. The lemma now follows from Proposition 4.1.6 and the definition of a-b-m functions. □

### 4.2. Cell decomposition with Lipschitz continuous cell walls.

We use the definition of cells for an o-minimal structure $R$ as in [9], Chapter 3. Recall that this definition uses an ordering of the coordinates. The results in this section will be later used for $R = \mathbb{R}$ with $\mathcal{L}_K^\mathbb{F}$-structure.

**Definition 4.2.1 (Cell walls).** Let $R$ be an o-minimal structure as in [9]. Let $C \subset R^n$ be a definable cell, and write

$$C = \{ x \in R^n \mid \wedge_{i=1}^n \alpha_i(x_{<i}) \square_{i1} x_i \square_{i2} \beta_i(x_{<i}) \}$$

for continuous definable functions $\alpha_i$ and $\beta_i$ with $\alpha_i < \beta_i$, $x_{<i} = (x_1, \ldots, x_{i-1})$, and with $\square_{i1}$ either $=, <,$ or no condition, and with $\square_{i2}$ either $<$ or no condition. If $\square_{i1}$ is $=$ or $<$ then we call $\alpha_i$ a wall of $C$. Likewise, if $\square_{i2}$ is $<$, then we also call $\beta_i$ a wall of $C$, where we use the convention that $\square_{i2}$ is no condition if $\square_{i1}$ is equality.

We give a family version of Theorem 3n of [23, page 1047]. In the following result, some of the coordinates will be reordered in order to make cell walls Lipschitz continuous.
Theorem 4.2.2 ([23], family version). Let $R$ be a real closed field equipped with an o-minimal structure. Let $T \subset R^k$ and $G \subset T \times R^n$ be definable. Then there exist a real constant $M_n \geq 1$ depending only on $n$, and a finite partition of $G$
\[ G = S_1 \cup \ldots \cup S_\ell, \]
such that for every $i$ there exists a reordering of the coordinates on $R^n$ such that $S_i$ is a definable cell (w.r.t. to the new ordering of the coordinates on $R^n$) whose walls $\alpha$ are $C^1$ and satisfy
\[ |\frac{\partial \alpha}{\partial x_j}(t, x)| \leq M_n \]
for $j = 1, \ldots, n$. Here, we write $(t, x)$ for coordinates $t$ running over $R^k$ and $x$ over $R^n$.

Proof. By Theorem 3 of [23, page 1047], the theorem holds when $T$ is a singleton, and up to an extra part in the partition which is a definable subset $\Sigma$ of $G$ of dimension less than $n$. To deal with the part $\Sigma$ is easy and classical by induction on $k + n$. (See the proof of Step 1 in Section 4.3 for a similar such argument.) The family version follows from logical compactness. \(\blacksquare\)

4.3. Preparation with a Lipschitz continuous center. We combine the Main Theorem of [20, page 4396] with Proposition 5.2 of [21], to get preparation of $L_K^K$-definable functions with centers which are moreover Lipschitz continuous.

Let us first recall some definitions, with notation from Section 2.1. Let $C \subset R^n$ be an $L_K^K$-definable cell. Write $\Pi_{<n} : R^n \to R^{n-1}$ for the projection sending $x$ in $R^n$ to $x_{<n} = (x_1, \ldots, x_{n-1})$. Write $B$ for $\Pi_{<n}(C)$. Let $\theta : B \to R$ be a continuous, $L_K^K$-definable function whose graph is disjoint from $C$.

Definition 4.3.1 (Units with center). Let $B, C, \theta, x_{<n}$ be as above. An $L_K^K$-definable function $u : C \to R$ is called an $L_K^K$-unit with center $\theta$ if $u$ can be written in the form $u(x) = F(\varphi(x))$, where $\varphi : C \to R^s$ sends $x$ to
\[ (a_1(x_{<n})|x_n - \theta(x_{<n})|^{r_1}, \ldots, a_s(x_{<n})|x_n - \theta(x_{<n})|^{r_s}) \]
for some integer $s \geq 0$, some $L_K^K$-definable functions $a_1, \ldots, a_s$ on $B$, and some $r_i$ in $K$ (possibly zero), such that moreover $\varphi(C)$ is a bounded subset of $R^s$, and such that $F$ is an $L_K^K$-definable, analytic, non-vanishing function on an open neighborhood of the closure of $\varphi(C)$ in $R^s$. We call the set of the functions $\theta, a_1, \ldots, a_s$ a set of accessory functions of $u$. 
Note that there is no unique set of accessory functions of an $L^F_K$-unit $u$ with unit $\theta$, but rather, a set of accessory functions comes by writing $u$ as a composition $F(\varphi)$ as in Definition 4.3.1.

**Definition 4.3.2** (Preparation with center). Let $B, C, \theta, x_{<n}$ be as above. An $L^F_K$-definable function $f : C \to \mathbb{R}$ is called $L^F_K$-prepared with center $\theta$ if $f$ can be written as

$$f(x) = g(x_{<n})|x_n - \theta(x_{<n})|^r u(x)$$

where $g : B \to \mathbb{R}$ is $L^F_K$-definable, $u$ is an $L^F_K$-unit with center $\theta$, and $r$ is in $K$. Say that an $L^F_K$-definable map $f : C \to \mathbb{R}^\ell$ for $\ell \geq 1$ is $L^F_K$-prepared with center $\theta$ if all its component functions are. We call the collection consisting of $g$ and a set of accessory functions of $u$, a set of accessory functions of $f$.

In the following result, some of the coordinates will be reordered, with reordering depending on the piece in a finite partition, in order to ensure Lipschitz continuity for the occurring centers. We state and will need a family version.

**Theorem 4.3.3.** Let $T \subset \mathbb{R}^k$, $\mathcal{X} \subset T \times \mathbb{R}^n$, and $f : \mathcal{X} \to \mathbb{R}^s$ be $L^F_K$-definable for some $s \geq 0$. Then there exist $M \geq 1$ and a finite partition partition of $\mathcal{X}$ into $L^F_K$-definable parts $S_i$ for $i = 1, \ldots, \ell$ for some $\ell \geq 0$ and an $L^F_K$-definable part $\Sigma$ which is nowhere dense in $\mathbb{R}^{k+n}$ and such that, for each $i$ there exists a reordering of the coordinates on $\mathbb{R}^n$ such that the following holds. The set $S_i$ is an open cell (w.r.t. the reordered coordinates), and, with $\Pi_{<n} : \mathbb{R}^{k+n} \to \mathbb{R}^{k+n-1}$ the projection sending $(t, x)$ to $(t, x_1, \ldots, x_{n-1})$ (still in the reordered coordinates), there is an $L^F_K$-definable $C^1$ function $\theta : \Pi_{<n}(S_i) \to \mathbb{R}$ whose graph is disjoint from $S_i$ and which satisfies

$$|\partial \theta / \partial x_i| \leq M$$

for each $i = 1, \ldots, n - 1$, and such that the restriction $f|_{S_i}$ is $L^F_K$-prepared with center $\theta$.

The proof of Theorem 4.3.3 is similar to the proof of Proposition 5.2 of [21], where one uses the Main Theorem of [20] instead of Theorem 3.5 of [21], and Theorem 4.2.2 instead of Proposition 3 of [29]. We give the details for the convenience of the reader.

**Proof of Theorem 4.3.3** Write $\Pi_{<n}$ for the projection $\mathbb{R}^{k+n} \to \mathbb{R}^{k+n-1}$ sending $(t, x)$ to $(t, x_1, \ldots, x_{n-1})$ for $t \in \mathbb{R}^k$ and $x \in \mathbb{R}^n$. Apply the Main Theorem of [20] page 4396] (in combination with the more classical Lemma 4.4 of [20] when $s > 1$), to find a finite partition of $\mathcal{X}$ into $L^F_K$-definable cells such that, for each occurring cell $S_i$, the restriction
$f|_S$ is $\mathcal{L}_F^K$-prepared with center $\theta$ for some $\mathcal{L}_F^K$-definable $C^1$ function $\theta$ on $\Pi_{\leq n}(S)$. (By a cell decomposition we mean a finite partition into $\mathcal{L}_F^K$-definable cells.) Note that the variable $x_n$ plays a special role in this application of the Main Theorem of [20]. Next we do the same with each of the variables $x_i$ in this special role (which amounts to reordering $(x_1, \ldots, x_n)$ as $(x_1, \ldots, x_{i-1}, x_i, \ldots, x_n, x_i)$), as follows.

Write, for each $i = 1, \ldots, n$, $\Pi^i$ for the coordinate projection $\mathbb{R}^{k+n} \to \mathbb{R}^{k+n-1}$ forgetting the variable $x_i$, where $(t, x_1, \ldots, x_n)$ run over $\mathbb{R}^{k+n}$.

Now, for $i$ subsequently equal to $n-1$, $n-2$, and up to $i = 1$, refine the obtained cell decomposition of $\mathcal{X}$ by applying the Main Theorem of [20] to the restrictions of $f$ to the occurring parts, with the variable $x_i$ in the special role (so that in particular, the obtained centers will be functions defined on subsets of $\Pi^i(\mathcal{X})$).

After having applied the Main Theorem of [20] in total $n$ times, we obtain a finite partition of $\mathcal{X}$ into $\mathcal{L}_F^K$-definable parts. It is enough to prove the theorem for the restriction of $f$ to each part occurring in this partition, and hence, we may suppose that $\mathcal{X}$ is equal to a single part, that is, the partition consists of the part $cX$ only.

Now apply Theorem 4.2.2 to find a finite partition of $\mathcal{X}$ into $\mathcal{L}_F^K$-definable parts $S_i$ and a part $\Sigma$, and for each part $S_i$ an ordering of the coordinates. Similarly as above, we may assume that $\mathcal{X} = S_i$, and, that we have changed our notation so that we have the natural ordering $(x_1, \ldots, x_n)$ on the coordinates on $\mathbb{R}^n$ (the ordering of the coordinates $(t_1, \ldots, t_k)$ on $\mathbb{R}^k$ has never been changed). Recapitulating, we may assume that $\mathcal{X}$ is an open cell with extra information on the cell walls coming from our application of Theorem 4.2.2 and that $f$ is $\mathcal{L}_F^K$-prepared with center $\theta$, with $\theta$ an $\mathcal{L}_F^K$-definable $C^1$ function. We still have to ensure (3) for some $M$, which we will now do.

Let $\alpha(t, x_{<n})$ be a cell wall of $\mathcal{X}$ bounding the variable $x_n$. Note that

$$|\partial \alpha/\partial x_i| < M_n$$

for each $i = 1, \ldots, n-1$, with $M_n$ as given by Theorem 4.2.2. By symmetry (or, up to changing the sign of $x_n$), we may suppose that $\alpha(t, x_{<n}) < x_n$ on $\mathcal{X}$, and, that $\theta \leq \alpha$ on $\Pi_{\leq n}(\mathcal{X})$.

Up to an extra finite further partition (and neglecting a lower dimensional piece), we may assume that, on $\mathcal{X}$, either $x_n > 2\alpha - \theta$, or, $x_n < 2\alpha - \theta$. We will ensure (3) with $M = M_n$, by showing that $f$ is $\mathcal{L}_F^K$-prepared with center $\alpha$ which satisfies (4). To this end, it suffices to show that the function $\mathcal{X} \to \mathbb{R} : (t, x) \mapsto x_n - \theta(t, x_{<n})$ is $\mathcal{L}_F^K$-prepared with center $\alpha$. Let us for the rest of the proof abbreviate $\alpha(t, x_{<n})$ and $\theta(t, x_{<n})$ by $\alpha$, resp. by $\theta$. 
In the case that \( x_n > 2\alpha - \theta \) on \( \mathcal{X} \), one has
\[
x_n - \theta = (x_n - \alpha)(1 + \frac{\alpha - \theta}{x_n - \alpha}),
\]
and \( (1 + \frac{\alpha - \theta}{x_n - \alpha}) \) is clearly an \( L^K_F \)-unit with center \( \alpha \), which finishes this case.

In the case that \( x_n < 2\alpha - \theta \), one has
\[
x_n - \theta = (\alpha - \theta)(1 + \frac{x_n - \alpha}{\alpha - \theta}),
\]
and \( (1 + \frac{x_n - \alpha}{\alpha - \theta}) \) is an \( L^K_F \)-unit with center \( \alpha \), which finishes the second and final case and ends the proof of the theorem. \( \blacksquare \)

4.4. **Proof of Theorem 2.1.3.**

4.4.1. Suppose we want to reparameterize \( \mathcal{X} \) as in Theorem 2.1.3, in the case that \( \mathcal{X} \) is the graph of an \( L^K_F \)-definable function \( f : U \subset (0,1)^m \to \mathbb{R}^{n-m} \), where we omit the parameters space \( T \) for simplicity. Of course \( f \) is not automatically 0-prepared. Nevertheless, after applying Theorem 4.3.3 recursively \( m \) times, and after translating in each coordinate by the center, we can reduce to a situation which is 0-prepared. A great deal of effort will lie in controlling the first derivatives along this process: we will work towards the property that \( f \) as well as the occurring centers of preparation have bounded \( C^1 \)-norm. Having this, a triangular translation by the center in each coordinate becomes not only 0-prepared but still has bounded \( C^1 \)-norm. In fact, we will even want each first partial derivative of \( f \) to become a-b-m after some basic transformations, and, we have similar aims for the walls of the occurring cells and their first partial derivatives. To achieve all these requirements jointly we will use induction on \( m \) and an interplay of Theorems 4.2.2 and 4.3.3 Once we have achieved this we can finish off the proof of Theorem 2.1.3 by the results of Section 4.1 as follows. By Lemma 4.1.9, a-b-m functions are automatically weakly mild. One finishes by applying Proposition 4.1.5 to get \( C^r \)-parameterizations in a simple and controlled way, for any integer \( r > 0 \). We proceed in four steps, each proved separately and implying Theorem 2.1.3.

**Step 1. A reduction.** In this step we show a reduction to the assumptions of Case 1, namely, we show that it is enough to prove Theorem 2.1.3 under the extra conditions of Case 1. This reduction is classical. We give the details for the convenience of the reader.

**Case 1.** Each \( X_t \) is the graph of \( f_t \) for some \( L^K_F \)-definable family of functions \( f_t : U_t \subset (0,1)^m \to (0,1)^{n-m} \) with \( C^1 \)-norm bounded
by 1 and with an $\mathcal{L}_{\mathbb{F}}^k$-definable family $U_t$ such that $\mathcal{U} = \{(t, x) \mid t \in T, \ x \in U_t\}$ is an open cell in $(0, 1)^{k+m}$.

It is harmless to assume that $T$ is a subset of $(0, 1)^k$ instead of a subset of $\mathbb{R}^k$. Up to a finite partition of $\mathcal{X} = \{(t, x) \mid t \in T, \ x \in \mathcal{X}_t\}$ into cells and up to reordering the coordinates we may suppose that $\mathcal{X}$ is the graph of $f$ for some $\mathcal{L}_{\mathbb{F}}^k$-definable function $f : \mathcal{U} \subset (0, 1)^{k+m} \rightarrow (0, 1)^{n-m}$. By induction on $k + m$ we may suppose that $\mathcal{U}$ is an open cell in $(0, 1)^{k+m}$. By further partitioning we may suppose that $|f_t^{(\beta)}|$ is maximal on $\mathcal{X}_t$ for some $\beta$ with $|\beta| = 1$ among all first partial order derivatives of $f_t$, and that either $|f_t^{(\beta)}| > 1$ or $|f_t^{(\beta)}| \leq 1$ holds on each $\mathcal{X}_t$. If $|f_t^{(\beta)}| \leq 1$ then one is done. In the case that $|f_t^{(\beta)}| > 1$, we may suppose that the function $g : x_i \mapsto f_t(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m)$ is injective for each $t$ and each $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$, where $i$ is such that $\beta_i = 1$. Now inverting the role of $x_i$ and $g$ we are done by the chain rule for derivation. This finishes the reduction of Theorem 2.1.3 to the extra conditions of Case 1 and thus Step 1 is done.

The above Case 1 yields an $\mathcal{L}_{\mathbb{F}}^k$-definable family of functions $f_t$. From now on we will write

$$f : \mathcal{U} \rightarrow \mathbb{R}^{n-m}$$

for the map sending $(t, x)$ with $x \in U_t$ and $t \in T$ to $f_t(x)$ and with $\mathcal{U}$ consisting of $(t, x)$ with $t \in T$ and $x \in U_t$. We will also continue to write $\mathcal{X}$ for the $\mathcal{L}_{\mathbb{F}}^k$-definable family of the sets $\mathcal{X}_t$ with parameter $t \in T$, and which are the graphs of the $f_t$.

Step 2. Proof of a special case. In this step we prove Theorem 2.1.3 under the special assumptions summarized as Case 2. We mainly use results from Section 4.1 in this step.

Case 2. The set $\mathcal{X}$ is the graph of $f : \mathcal{U} \rightarrow (0, 1)^{n-m}$ where $f$ is 0-prepared and $\mathcal{U} \subset T \times (0, 1)^m \subset (0, 1)^{k+m}$ is a cell whose walls are 0-prepared. Furthermore, for each wall $\alpha$ of $\mathcal{U}$ and each $i = 1, \ldots, m$, the maps $\partial f / \partial x_i$ and $\partial \alpha / \partial x_i$ are a-b-m. (In particular, for each $t$, the map $f_t$ and the walls of $U_t$ have bounded $C^1$-norm.)

Lemmas 4.1.3, 4.1.9 and Proposition 4.1.5 do the job for this step, as follows. Let $r > 0$ be any integer. For each $t$, let $U'_t$ be the subset of $(0, 1)^m$ such that the map

$$\varphi_t : x \mapsto (x_1^{r_m}, x_2^{r_{m-1}}, \ldots, x_m^{r})$$

is a bijection from $U'_t$ to $U_t$. Note that $U'_t$ is an open cell in $(0, 1)^m$. 

Claim 1. There are polynomials $A(r)$ and $C(r)$ such that for each $t$ and any integer $r$, the function $f_t \circ \varphi_t$ and all the walls of the cell $U'_t$ are $(A(r), C(r))$-mild up to order $r$.

For each $t$, the function $f_t \circ \varphi_t$ satisfies the conditions from the claim by Lemma 4.1.9 Proposition 4.1.5 and since a 0-prepared function is a-b-m. Let us now concentrate on a cell wall of $U'_t$.

To name some of the cell boundaries, let us suppose that $x_m$ runs between $\alpha_t(x_m) < m$ and $\beta_t(x_m) < m$ for $x_m$ in $U_t$, where $x_m < m = (x_1, \ldots, x_m-1)$. Let us further suppose, for $x \in U'_t$, that $x_m$ runs between $a_t(x_m) < m$ and $b_t(x_m) < m$ in $U'_t$.

By induction on $m$ and the similar role of $a_t$ and $b_t$ it is enough to prove the claim for $a_t$. By the assumptions of Case 2, there is $S > 1$, an a-b-m function $\alpha_0$ taking values in $(1/S, S)$ and a bounded-monomial function $x_{<m}^\mu t^\nu$ on $U$ such that, for $x \in U_t$,

$$a_t(x_m) = x_{<m}^\mu t^\nu \alpha_0(t, x_m).$$

But then we can write, for $x \in U'_t$,

$$a_t(x_m) = \sqrt{(\varphi_t(x))_{<m}^\mu t^\nu \alpha_0(t, (\varphi_t(x))_{<m})}$$

$$= (x_1^{r_{m-1}}, x_2^{r_{m-1}}, \ldots, x_{m-1}^r)\sqrt{t^\nu \alpha_0(t, x_1^r, \ldots, x_{m-1}^r)}.$$

When $\mu = 0$ and $\nu = 0$, the claim follows from Proposition 4.1.3 and Lemma 4.1.7. The case for general $\mu$ and $\nu$ is a similar exercise as the proof of Lemma 4.1.7. This proves Claim 1.

Now one makes all the $f_t$ and all the walls of $U_t$ of $C^r$-norm bounded by 1 by parameterizing according to Lemma 4.1.3, say with domain $U''_t$. Note that $U''_t$ is an open cell whose walls have $C^r$-norm bounded by 1. One finishes this case by mapping $(0, 1)^m$ onto $U''_t$ in the obvious way. This finishes the proof of Theorem 2.1.3 under the special assumption of Case 2. Step 2 is done.

The rest of the proof will consist of a reduction from Case 1 to Case 2, using induction on $k + m$.

Step 3. A further reduction. In this step we want to reduce (starting from Case 1) to the following Case 3. Namely, we show that it is enough to prove Theorem 2.1.3 under the extra conditions of Case 3.

Case 3. On top of the assumptions of Case 1, the map $f$ as well as the maps $\partial f / \partial x_i$ for $i = 1, \ldots, m$ and the coordinate function $U \to \mathbb{R}: (t, x) \mapsto x_m$ are $L^K_F$-prepared with center $\theta$. Moreover, one has $|\partial \theta / \partial x_i| \leq 1$ for each $i = 1, \ldots, m$. 
Let $f$ and $U$ be as given by Case 1. Now, up to a reordering of the coordinates $x_1, \ldots, x_m$ and up to a finite partition into $\mathcal{L}_F^n$-definable parts, we can reduce to the conditions of Case 3 by Theorem 4.3.3, and, obviously, by a further finite partitioning and scaling by $M$ to make the $C^1$-norm of the maps $\theta_t$ bounded by 1 instead of by the constant $M$ provided by Theorem 4.3.3. This finishes the proof of Step 3. 

**Step 4. Reduction from Case 3 to Case 2.** In this step we show that it is enough to prove Theorem 2.1.3 under the extra conditions of Case 2. By Step 2 itself, this will complete the proof of Theorem 2.1.3.

Let us start from the assumptions of Case 3. We use a recursive procedure starting from Case 3, which goes roughly by recursively translating with the center and applying Theorem 4.3.3 to sets of accessory functions in less variables. To give names to some cell boundaries, let us suppose that $x_m$ runs between $\alpha_t(x_{<m})$ and $\beta_t(x_{<m})$ for $x$ in $U_t$, where $x_{<m} = (x_1, \ldots, x_{m-1})$. Let $\mathcal{S}$ be the finite set of functions consisting of $\alpha$, $\beta$, (the functions in) a set of accessory functions for the $\mathcal{L}_F^n$-prepared maps $f$, a set of accessory functions for the maps $\partial f/\partial x_i$ for $i = 1, \ldots, m$, and a set of accessory functions for the coordinate function $U \to \mathbb{R}: (t, x) \mapsto x_m$ (each of these maps being prepared with center $\theta$ by Case 3). Up to a finite partition, a translation by a constant in the $x_m$-variable and a basic transformation $x_m \mapsto -x_m + 1$ if needed, we may suppose that $0 \leq \theta(t, x_{<m}) < x_m$ on $U$. Now transform via the map 

$$(t, x) \mapsto (t, x_{<m}, x_m + \theta(t, x_{<m})).$$

Then, up to identifying our notation for $f$ and $X$ with the new situation, we are in a situation satisfying the conditions of Case 1 where moreover $f$ is $\mathcal{L}_F^n$-prepared with center 0, and where the $\partial f/\partial x_i$ are a finite sum of bounded maps which are $\mathcal{L}_F^n$-prepared with center 0. (Here, several case assumptions of Case 3 are used, in particular, it is used that $|\partial \theta/\partial x_i| \leq 1$ on $U$, for $i = 1, \ldots, m$, and, that the coordinate functions $(t, x) \mapsto x_m$ on $U$ was also $\mathcal{L}_F^n$-prepared with center $\theta$.)

Let $\Pi_{<m} : \mathbb{R}^{k+m} \to \mathbb{R}^{k+m-1}$ be the projection sending $(t, x)$ to $(t, x_{<m}) = (t, x_1, \ldots, x_{m-1})$. It is harmless to assume that each function in $\mathcal{S}$ either takes values in $(0, 1)$, or, is identically zero. Now use induction on $k + m$, to obtain the reduction to Case 2 from the set-up of the Theorem where $X$ is the graph $\Pi_{<m}(U) \to (0, 1)^s$ for some $s \geq 0$ of the map whose component functions are the functions from $\mathcal{S}$ which are not identically zero. By nesting this information in a classical way one reduces to a situation where $f$ and the walls of $U$ are 0-prepared, and, by the chain rule for derivation, that the maps $\partial f/\partial x_i$ and $\partial \alpha/\partial x_i$ are finite sums of 0-prepared functions for each $i = 1, \ldots, m$ and each
wall $\alpha$ of $U$. Since finite sums of 0-prepared functions are clearly a-b-m, this finishes the proof of Step 4 and thus of Theorem 2.1.3. □

To conclude this section, note that although the $\phi_{r,i,t}$ of Theorem 2.1.3 are analytic for each $r, i, t$, we do not have that they can be extended to an analytic map on an open neighborhood of the closed box $[0, 1]^m$.

5. The proof of the quasi-parameterization theorem

In this section we prove theorem 2.2.3. First of all, however, we require some general results about definable families of holomorphic functions. The definability here is with respect to an arbitrary polynomially bounded o-minimal expansion of the real field, which we now fix. Let us recall the following definition from subsection 2.2 from which we also recall that $\Delta(R)$ denotes the (open) disc in $\mathbb{C}$ of radius $R$ and centred at the origin.

5.1 Definition A definable family $\Lambda = \{F_t : t \in T\}$ is called an $(R, m, K)$-family, where $R, K$ are positive real numbers and $m$ is a positive integer, if for each $t \in T$, $F_t : \Delta(R)^m \to \mathbb{C}$ is holomorphic and for all $z \in \Delta(R)^m$, $|F_t(z)| \leq K$.

Let us first observe that for $\Lambda$ such a family it follows from the Cauchy inequalities (for all general results from the theory of functions of several complex variables we refer the reader to the first chapter of [14]) that we have the following bounds on the Taylor coefficients of each $F_t$ at $0 \in \mathbb{C}^m$:

5.1.1 For all $\alpha \in \mathbb{N}^m$ and all $t \in T$, $\frac{|F_t^{(\alpha)}(0)|}{\alpha!} \leq \frac{K}{R^{|\alpha|}}$.

In particular, if $R > 1$ then $\frac{|F_t^{(\alpha)}(0)|}{\alpha!} \to 0$ as $|\alpha| \to \infty$ and so for each $t \in T$ there exists some $M_t \in \mathbb{N}$ such that

5.1.2 For all $\alpha \in \mathbb{N}^m$, $\frac{|F_t^{(\alpha)}(0)|}{\alpha!} \leq \max\{\frac{|F_t^{(\alpha)}(0)|}{\alpha!} : \alpha \in \mathbb{N}^m, |\alpha| \leq M_t\}$.

The crucial uniformity result, from which the quasi-parameterization theorem will follow, is that $M_t$ may be chosen to be independent of $t$. This in turn will follow from the maximum modulus theorem and the following simple

5.2 Lemma Let $1 < r < R$, $0 < \lambda \leq \frac{1}{2}$ and let $\{\theta_t : t \in T\}$ be a definable family of functions from $(0, R)$ to $(0, R)$. Then there exists $\epsilon \in (0, \lambda)$ such that for all $t \in T$, there exists $y_t \in (r, R)$ such that $\theta_t(y_t - \epsilon) \geq \frac{1}{2}\theta_t(y_t)$. 

Proof. Suppose not. Then there exists a function $\eta : (0, \lambda) \to T$, which by the principle of definable choice we may take to be definable, such that

5.2.1 for all $x \in (0, \lambda)$ and all $y \in (r, R)$, $\theta_{\eta(x)}(y - x) < \frac{1}{2}\theta_{\eta(x)}(y)$.

Pick some $\gamma \in (r, R)$ and consider the definable function $x \mapsto \theta_{\eta(x)}(\gamma)$ for $x \in (0, \lambda)$. It follows from polynomial boundedness that there exists a positive integer $N$ and $\nu \in (0, \lambda)$ such that

5.2.2 for all $x \in (0, \nu)$, $\theta_{\eta(x)}(\gamma) > x^N$.

Now let $k$ be a positive integer and set $x_0 := \frac{R - \gamma}{2k}$, so that $x_0 \in (0, \nu)$ for large enough $k$. By applying 5.2.1 successively with $x = x_0$ and $y = \gamma + x_0, \ldots, \gamma + kx_0$ we see that $0 < \theta_{\eta(x_0)}(\gamma) < \frac{1}{2}\theta_{\eta(x_0)}(\gamma + x_0) < \cdots < \left(\frac{1}{2}\right)^k \theta_{\eta(x_0)}(\gamma + kx_0) < \left(\frac{1}{2}\right)^k R$.

So by 5.2.2 we obtain $\left(\frac{R - \gamma}{2k}\right)^N = x_0^N < \theta_{\eta(x_0)}(\gamma) < \left(\frac{1}{2}\right)^k R$, which is the required contradiction if $k$ is sufficiently large.  

5.3 Theorem  
Let $\Lambda = \{F_t : t \in T\}$ be an $(R, m, K)$-family with $R > 1$. Then there exists $M = M(\Lambda) \in \mathbb{N}$ such that for all $t \in T$ and all $\alpha \in \mathbb{N}^m$,

$$\frac{|F_t^{(\alpha)}(0)|}{\alpha!} \leq \max\left\{\frac{|F_t^{(\alpha)}(0)|}{\alpha!} : \alpha \in \mathbb{N}^m, |\alpha| \leq M\right\}.$$

Proof. Since the conclusion is trivially true for those $t \in T$ such that $F_t \equiv 0$ (no matter how $M$ is chosen) we may assume that no $F_t$ is identically zero.

For $t \in T$ define $\theta_t : (0, R) \to (0, R)$ by

5.3.1 $\theta_t(y) := \frac{R}{k}\sup\{|F_t(z)| : z \in \Delta(y)^m\}$.

Now let $r := R^\frac{1}{4}$, $\lambda := \min\{\frac{1}{2}, R^\frac{1}{4} - R^\frac{1}{4}\}$ and apply 5.2 to obtain $\epsilon \in (0, \lambda)$ such that

5.3.2 for all $t \in T$, there exists $y_t \in (r, R)$ such that $\theta_t(y_t - \epsilon) \geq \frac{1}{2}\theta_t(y_t)$.

Now choose $D \in \mathbb{N}$ so that

5.3.3 $(1 - \frac{\epsilon}{R})^D < \frac{1}{4}$, and

5.3.4 $5(D + 1)^m \leq 2R^D$.

We show that $M := 2D$ satisfies the required conclusion. So fix some arbitrary $t \in T$ and let $B_t := \max\{|\frac{|F_t^{(\alpha)}(0)|}{\alpha!}| : |\alpha| \leq D\}$. (The $\alpha$’s range over $\mathbb{N}^m$ for the rest of this proof.) It is clearly sufficient to show that

(*) for all $\alpha$ with $|\alpha| \geq 2D$ we have $\frac{|F_t^{(\alpha)}(0)|}{\alpha!} \leq B_t$. 


To this end we consider the truncated Taylor expansion of $F_t$, namely

$$P_t(z) := \sum_{|\alpha| \leq D} \frac{F_t^{(\alpha)}(0)}{\alpha!} z^\alpha.$$  Clearly we have

5.3.5 for all $z \in \Delta(R)^m$, $|P_t(z)| \leq (D + 1)^m B_t R^D$.

Now choose \( w = \langle w_1, \ldots, w_m \rangle \in \Delta(R)^m \) with $|w_i| = y_t - \epsilon$ and $|F_t(w)| = \frac{K}{R} \theta_t(y_t - \epsilon)$ (which is possible by 5.3.1 and the maximum modulus theorem), and let $\eta_i := \frac{w_i}{y_t - \epsilon}$ so that $|\eta_i| = 1$ for $i = 1, \ldots, m$. Consider the function $H_t : \Delta(R) \to \mathbb{C}$ given by

$$H_t(u) := \frac{F_t(w\eta) - P_t(u\eta)}{u D + 1}.$$

This is clearly a well-defined analytic function and by the maximum modulus theorem there exists $u_t \in \Delta(R)$ with $|u_t| = y_t$ such that $|H_t(y_t - \epsilon)| \leq |H_t(u_t)|$. Thus

$$|F_t((y_t - \epsilon)\eta) - P_t((y_t - \epsilon)\eta)| \leq \left( \frac{y_t - \epsilon}{y_t} \right)^{D+1}|F_t((u_t\eta) - P_t((u_t\eta)|.$$

However, by 5.3.3, \( \left( \frac{y_t - \epsilon}{y_t} \right)^{D+1} \leq (1 - \frac{\epsilon}{R})^{D+1} < \frac{1}{4} \) so, upon recalling that $w = (y_t - \epsilon)\eta$ and using 5.3.5, we see that

5.3.6 $|F_t(w)| \leq \frac{1}{4} |F_t(u_t\eta)| + \frac{5}{4} (D + 1)^m B_t R^D$.

But $|F_t(w)| = \frac{K}{R} \theta_t(y_t - \epsilon) \geq \frac{K}{2R} \theta_t(y_t)$ by 5.3.2, and since $|u_t\eta| = y_t$ we obtain from this and 5.3.1 that $|F_t(w)| \geq \frac{1}{2} |F_t(u_t\eta)|$. Putting this into 5.3.6 we obtain

5.3.7 $|F_t(w)| \leq \frac{5}{2} (D + 1)^m B_t R^D$.

Now, with a view to proving (*), let $|\alpha| \geq 2D$. Then by applying the Cauchy inequalities in the polydisk $\Delta(y_t - \epsilon)^m$ and using 5.3.1 and 5.3.7 we obtain

$$\frac{|F_t^{(\alpha)}(0)|}{\alpha!} \leq \frac{K}{R} \theta_t(y_t - \epsilon) |(y_t - \epsilon)^{-|\alpha|}| = |F_t(w)| |(y_t - \epsilon)^{-|\alpha|}| \leq \frac{5}{2} (D + 1)^m B_t R^D (y_t - \epsilon)^{-|\alpha|}.$$

But $y_t - \epsilon \geq r - \lambda \geq R^{\frac{2}{3}}$ (by the definitions of $r$ and $\lambda$), so by 5.3.4

$$\frac{|F_t^{(\alpha)}(0)|}{\alpha!} \leq \frac{5}{2} (D + 1)^m B_t R^D \cdot (R^{\frac{2}{3}})^{-2D} = \frac{5}{2} (D + 1)^m R^{D} B_t \leq B_t$$

as required. \( \square \)

It follows immediately from 5.3 that the function $\kappa_A : T \to \mathbb{R}$ given by

5.3.8 $\kappa_A(t) := \max \{|F_t^{(\alpha)}(0)|/\alpha! : \alpha \in \mathbb{N}^m\}$
is (well-defined and) definable. It also determines the topology on $\Lambda$ in the following sense.

### 5.4 Theorem

Let $\Lambda$ be an $(R, m, K)$-family with $R > 1$ as above and let $r$ be a real number satisfying $0 < r < R$. Then there exists a positive real number $B_\Lambda(r)$ such that for all $t \in T$ and all $z \in \Delta(r)^m$ we have $|F_t(z)| \leq B_\Lambda(r) \cdot \kappa_\Lambda(t)$.

**Proof.** Choose a real number $r_0$ such that $\max\{1, r\} < r_0 < R$ and for each $t \in T$ define $G_t : \Delta(\frac{R}{r_0})^m \to C$ by $G_t(z) := F_t(r_0z)$. Then $\Lambda^\alpha := \{G_t : t \in T\}$ is an $(\frac{R}{r_0}, m, K)$-family and since $\frac{R}{r_0} > 1$, we may apply 5.3 to it and obtain some $M(\Lambda^\alpha) \in \mathbb{N}$ such that $\kappa_{\Lambda^\alpha}(t) = \max \{\frac{|r_0^{(\alpha)(0)}|}{\alpha!} : \alpha \in \mathbb{N}^m, |\alpha| \leq M(\Lambda^\alpha)\}$. Fix $t \in T$. Then since $G_t^{(\alpha)}(0) = r_0^{(\alpha)} F_t^{(\alpha)}(0)$ (for all $\alpha \in \mathbb{N}^m$) it follows that for all $z \in \Delta(r)^m$ we have that $|F_t(z)| = |r_0^{(\alpha)} \cdot G_t^{(\alpha)}(0) \cdot z^\alpha| \leq \kappa_{\Lambda^\alpha}(t) \cdot (\frac{r_0}{r_0 - R})^m \leq \kappa_{\Lambda^\alpha}(t) \cdot r_0^M(\Lambda^\alpha) \cdot (\frac{r_0}{r_0 - R})^m$, which gives the required result upon setting $B_\Lambda(r) := r_0^M(\Lambda^\alpha) \cdot (\frac{r_0}{r_0 - R})^m$.

\[\square\]

The topology we are referring to here is determined by the metrics $\delta_r$ $(0 < r < R)$ where, for any two bounded holomorphic functions $F, G : \Delta(R)^m \to C$, we define $\delta_r(F, G) := \sup\{|F(z) - G(z)| : z \in \Delta(r)^m\}$. It turns out that if $\Lambda$ is any $(R, m, K)$-family and if $0 < r, r' < R$, then the metric spaces $(\Lambda, \delta_r)$ and $(\Lambda, \delta_{r'})$ are quasi-isometric via the identity function on $\Lambda$. (Note that this is certainly not true in general for families of $K$-bounded holomorphic functions on $\Delta(R)^m$, e.g. consider, for $R = 2, m = K = 1$, the family $\{0\} \cup \{(\hat{z})^q : q \in \mathbb{N}\}$.) In fact, as we now explain, they are quasi-isometric to a bounded subset of $\mathbb{C}^N$ for some sufficiently large $N$ (depending only on $\Lambda$) endowed with the metric induced by the usual sup-metric on $\mathbb{C}^N$: $\|\langle w_1, \ldots, w_N\rangle\| := \max\{|w_i| : 1 \leq i \leq N\}$.

### 5.5 Definition

We say that an $(R, m, K)$-family $\Lambda = \{F_t : t \in T\}$ is **well-indexed** if, for some $N \in \mathbb{N}$, $T$ is a bounded subset of $\mathbb{C}^N$ and for each $r$ with $0 < r < R$, there exist positive real numbers $c_r, C_r$ such that for all $t, t' \in T$ we have $c_r \delta_r(F_t, F_{t'}) \leq \|t - t'\| \leq C_r \delta_r(F_t, F_{t'})$. That is, the map $t \mapsto F_t$ is a quasi-isometry from the metric space $\langle T, \|\cdot\|\rangle$ to the metric space $\langle \Lambda, \delta_r\rangle$.

### 5.6 Theorem

Every $(R, m, K)$-family with $R > 1$ can be well-indexed.

**Proof.** Let $\Lambda = \{F_t : t \in T\}$ be an $(R, m, K)$-family with $R > 1$. What we are required to prove is that there exists a well-indexed $(R, m, K)$-family, $\Lambda' = \{G_t : t \in T'\}$ say, such that $\Lambda = \Lambda'$ as sets.
Consider the \((R,m,2K)\)-family \(\Omega := \{F_t - F_{t'} : \langle t, t' \rangle \in T^2\}\) and let \(M(\Omega) \in \mathbb{N}\) be as given by 5.3 (with \(\Omega\) in place of \(\Lambda\)). We take our \(N = N(\Lambda)\) to be the cardinality of the set \(\{\alpha \in \mathbb{N}^m : |\alpha| \leq M(\Omega)\}\).

Define the map \(\omega : T \to \mathbb{C}^N\) by \(\omega(t) := \langle F_{\alpha}(0) : \alpha \in \mathbb{N}^m, |\alpha| \leq M(\Omega) \rangle\) and set \(T^* := \omega[T]\). If \(t, t' \in T\) and \(\omega(t) = \omega(t')\) then \(F_{\alpha}(0) = F_{\alpha}(0)\) holds for all \(\alpha \leq M(\Omega)\) and hence, by 5.3 (and the linearity of the derivatives), it holds for all \(\alpha \in \mathbb{N}^m\). Thus \(F_t = F_{t'}\). This does not necessarily imply that \(t = t'\) but, by the principle of definable choice, we may choose a definable right inverse \(\omega^{-1} : T^* \to T\) of \(\omega\) and setting \(G_t = F_{\omega^{-1}(t)}\) (for \(t \in T^*\)), we have that, as a set, \(\Lambda = \{G_t : t \in T^*\}\). We complete the proof by showing that the \((R, m, K)\)-family \(\{G_t : t \in T^*\}\) is well-indexed.

Firstly, by 5.1.1 we have that \(T^* \subseteq \overline{\Delta(K)}^N\), so \(T^*\) is a bounded subset of \(\mathbb{C}^N\). For the quasi-isometric inequalities consider some \(r \in (0, R)\).

We may take \(C_r = \max\{1, r^{-M(\Omega)}\}\). Indeed, suppose \(t, t' \in T^*\). Let \(s = \omega^{-1}(t)\) and \(s' = \omega^{-1}(t')\). Then by the Cauchy inequalities applied to the function \((F_s - F_{s'})\) restricted to the disk \(\Delta(r)^m\), we have that for all \(\alpha \in \mathbb{N}^m\), \(\|F_s - F_{s'}\|^m \leq \frac{\delta_r(F_s, F_{s'})}{r^m}\). In particular, \(\|t - t'\| = \|\omega(s) - \omega(s')\| \leq C_r \delta_r(F_s, F_{s'}) = C_r \delta_r(G_t, G_{t'})\).

Finally, we take \(c_r\) to be \(B_{\Omega}(r)^{-1}\), where \(B_{\Omega}(r)\) is as in 5.4 (with \(\Omega\) in place of \(\Lambda\)). Then, with \(t, t', s, s'\) as above, and \(z \in \Delta(r)^m\) we have by 5.3 and 5.4, \(\|(F_s - F_{s'})(z)\| \leq B_{\Omega}(r) \cdot \kappa_{\Omega}(\langle s, s' \rangle) = B_{\Omega}(r) \cdot \|\omega(s) - \omega(s')\| = B_{\Omega}(r) \cdot \|t - t'\|\). So \(\|t - t'\| \geq c_r \delta_r(G_t, G_{t'})\), as required.

\[\square\]

This result suggests a natural way of compactifying definable \((R, m, K)\)-families. For let \(\Lambda = \{F_t : t \in T\}\) be such a family with \(R > 1\) and assume, as now we may, that it is well-indexed (with \(T\) a bounded subset of \(\mathbb{C}^N\), say). We wish to extend \(\Lambda\) to a family \(\overline{\Lambda}\) well-indexed by the closure \(\overline{T}\) of \(T\) in \(\mathbb{C}^N\). So for \(t \in \overline{T}\) choose a Cauchy sequence \(\langle t(i) : i \in \mathbb{N}\rangle\) in \(T\) converging to \(t\) (in the space \(\langle \mathbb{C}^N, ||\cdot||\rangle\)). Then by the quasi-isometric property of the indexing it follows that \(\langle F_{t(i)} : i \in \mathbb{N}\rangle\) is a Cauchy sequence in \(\langle \Lambda, \delta_r\rangle\) for every \(r \in (0, R)\). So by Weierstrass' theorem on uniformly convergent sequences, there exists a holomorphic function \(F_t : \Delta(R)^m \to \mathbb{C}\) such that, for each \(r \in (0, R)\), \(\delta_r(F_{t(i)}, F_t) \to 0\) as \(i \to \infty\). It is easy to check that \(F_t\) depends only on \(t\) (and not on the particular choice of Cauchy sequence) and that our notation is consistent if \(t\) happens to lie in \(T\). We have the following
5.7 Theorem The collection $\Lambda := \{F_t : t \in T\}$ as defined above is a well-indexed $(R, m, K)$-family.

Proof. Everything follows from elementary facts on convergence (and we may take the same constants $c_r, C_r$ for the quasi-isometric inequalities) apart from the definability of $\Lambda$. To see that this holds too, let

$$\text{graph}(\Lambda) := \{(t, z, w) \in \mathbb{C}^{N+m+1} : t \in T, z \in \Delta(R)^m, F_t(z) = w\}.$$

Then $\text{graph}(\Lambda)$ is a definable subset of $\mathbb{C}^{N+m+1}$ (this being the definition of what it means for $\Lambda$ to be a definable family). We complete the proof by showing that

(*) for all $(t, z, w) \in \mathbb{C}^{N+m+1}$, $t \in T, z \in \Delta(R)^m$ and $F_t(z) = w$ if and only if $z \in \Delta(R)^m$ and $(t, z, w) \in \text{graph}(\Lambda)$.

So let $(t, z, w) \in \mathbb{C}^{N+m+1}$.

Suppose first that $t \equiv \bar{t}, z \in \Delta(R)^m$ and $F_t(z) = w$. Choose a sequence $(t^{(i)} : i \in \mathbb{N})$ in $T$ converging to $t$. Choose $r$ so that $|z| < r < R$. Then by the construction of $F_t$ we have that $\delta_r(F_{t^{(i)}}(z), F_t(z)) \rightarrow 0$ as $i \rightarrow \infty$. In particular, $|F_{t^{(i)}}(z) - F_t(z)| \rightarrow 0$ as $i \rightarrow \infty$, i.e. $F_{t^{(i)}}(z) \rightarrow w$ as $i \rightarrow \infty$. So $(t^{(i)}, z, F_{t^{(i)}}(z)) \rightarrow (t, z, w)$ as $i \rightarrow \infty$. Since $(t^{(i)}, z, F_{t^{(i)}}(z)) \in \text{graph}(\Lambda)$ for each $i \in \mathbb{N}$, it follows that $(t, z, w) \in \text{graph}(\Lambda)$ as required.

For the converse, suppose that $z \in \Delta(R)$ and that $(t, z, w) \in \text{graph}(\Lambda)$. Then certainly $t \equiv \bar{t}$ and we must show that $F_t(z) = w$, thereby completing the proof of (*).

Let $(t^{(i)}, z^{(i)}, w_i) : i \in \mathbb{N}$ be a sequence in $\text{graph}(\Lambda)$ converging to $(t, z, w)$. Then $z^{(i)} \rightarrow z$ as $i \rightarrow \infty$ and since $z \in \Delta(R)$, we may choose $r < R$ so that $z \in \Delta(r)^m$ and $z^{(i)} \in \Delta(r)^m$ for each $i \in \mathbb{N}$. Since $t^{(i)} \rightarrow t$ as $i \rightarrow \infty$, it follows from the construction of $F_t$ that $\delta_r(F_{t^{(i)}}(z^{(i)}) - F_t(z^{(i)})) \rightarrow 0$ as $i \rightarrow \infty$. But by the definition of $\text{graph}(\Lambda)$, $F_{t^{(i)}}(z^{(i)}) = w_i$ for all $i \in \mathbb{N}$ and hence $|w_i - F_t(z^{(i)})| \rightarrow 0$ as $i \rightarrow \infty$. However, $F_t(z^{(i)}) \rightarrow F_t(z)$ as $i \rightarrow \infty$ (because $F_t$ is certainly continuous on $\Delta(r)^m$) and hence $w_i \rightarrow F_t(z)$ as $i \rightarrow \infty$. Since $w_i \rightarrow w$ as $i \rightarrow \infty$ it now follows that $w = F_t(z)$ as required. \[\Box\]

Having shown how to compactify $(R, m, K)$-families, we now projectivize them.

5.8 Theorem Let $\Lambda = \{F_t : t \in T\}$ be an $(R, m, K)$-family with $R > 1$. Assume that for no $t \in T$ does $F_t$ vanish identically. Let $R_0$ satisfy $1 < R_0 < R$. Then there exists a positive real number $K_0$ and an $(R_0, m, K_0)$-family $\Lambda^! = \{G_t : t \in T^!\}$ such that
5.8.1 $\Lambda^\dagger$ is well-indexed and $T^\dagger$ is closed in its ambient space $\mathbb{C}^N$;

5.8.2 for every $t \in T$, there exists $A_t > 0$ and $t^\dagger \in T^\dagger$ such that $G_t^\dagger = A_t \cdot F_t \upharpoonright \Delta(R_0)^m$;

5.8.3 the (real) dimension of $T^\dagger$ is at most that of $T$;

5.8.4 for no $t \in T^\dagger$ is $G_t$ identically zero.

**Proof.** We consider the $(R_0, m, K_0)$-family $\{ \frac{F_t}{\kappa_\Lambda(t)} \upharpoonright \Delta(R_0)^m : t \in T \}$ (cf. 5.3.6), where $K_0 = B_\Lambda(R_0)$ (cf. 5.4). Using 5.6, let $\Lambda^* = \{ G_t : t \in T^* \}$ be a well-indexing of it. Clearly $\dim(T^*) \leq \dim(T)$. We set $T^\dagger := T^*$ and $\Lambda^\dagger := \Lambda^*$ as in 5.7. Then 5.8.1-3 are clear. For 5.8.4, let us first note that if $t \in T^*$ then for some $s \in T$, $G_t = F_s \upharpoonright \Delta(R_0)^m$ and hence there exists $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq M(\Lambda)$ such that $|G_t(\alpha)|_{\alpha!} = 1$ (see 5.5).

Now let $t^\dagger \in T^*$. We must show that $G_t^\dagger$ does not vanish identically. For this, choose $t \in T^*$ such that $\|t - t^\dagger\| < \frac{\varepsilon_1}{2}$ so that $\delta_1(G_t, G_t^\dagger) < \frac{1}{2}$ (by 5.5 with $r = 1$). It now follows from the Cauchy inequalities applied to the function $G_t - G_t^\dagger$ restricted to the unit polydisk $\Delta(1)^m$, that for all $\alpha \in \mathbb{N}^m$ we have $|G_t^\dagger(\alpha)|_{\alpha!} < \frac{1}{2}$. So choosing $\alpha$ with $|G_t(\alpha)|_{\alpha!} = 1$ as above, we see that $|G_t^\dagger(\alpha)|_{\alpha!} > \frac{1}{2}$. In particular, $G_t^\dagger$ does not vanish identically.

\[\square\]

**Remark** The hypothesis that $R_0 < R$ is necessary here: the reader may easily verify that for the $(2, 1, 1)$-family $\Lambda = \{ g_t : t \in [0, \frac{1}{2}] \}$ where $g_t(z) = \frac{1-2t}{1-tz}$, and for each given $K_0 > 0$, there is no $(2, 1, K_0)$-family $\Lambda^\dagger$ satisfying 5.8.1-4.

We are almost ready for the proof of the quasi-parameterization theorem (2.2.3). This will proceed by induction on the dimension of the given family $\{ X_t : t \in T \}$, i.e. the (minimum, real) dimension of the indexing set $T$. The inductive step will involve a use of the Weierstrass Preparation Theorem (or, rather, a modification of the argument used in the complex analytic proof of the Weierstrass Preparation Theorem) and, as usual, one first has to make a transformation so that the function being prepared is regular in one of its variables. Further, in our case the transformation will have to work uniformly for all members of a certain definable family of functions and for all values of the other variables. Unfortunately, the usual linear change of variables does not have this property. Instead we use a variation of the transformation used by Denef and van den Dries in their proof of quantifier elimination.
for the structure $\mathbb{R}_{an}$ (see [3]). The result we require is contained in the following.

5.9 **Theorem** Let $\Lambda = \{F_t : t \in T\}$ be an $(R, m, K)$-family with $R > 1$ (and, for non-triviality, with $m \geq 2$) such that for no $t \in T$ does $F_t$ vanish identically. Let $R'$ and $R''$ be real numbers satisfying $1 < R'' < R' < R$. Then there exist positive integers $D_1, \ldots, D_{m-1}$ and a positive real number $\eta$ such that the bijection

$$\theta : \mathbb{C}^m \to \mathbb{C}^m : z = (z_1, \ldots, z_m) \mapsto (z_1 + \eta z_1^{D_1}, \ldots, z_m + \eta z_1^{D_{m-1}}, z_m)$$

satisfies

5.9.1 $\theta[\Delta(R')^m] \subseteq \Delta(R)^m$,

5.9.2 $\theta^{-1}[\Delta(1)^m] \subseteq \Delta(R'')^m$, and

5.9.3 for each $t \in T$ and $z' \in \Delta(R')^{m-1}$ the function $z_m \mapsto F_t \circ \theta(z', z_m)$ (for $z_m \in \Delta(R')$) does not vanish identically in $z_m$.

This will follow from the following general

5.10 **Lemma** Let $m \geq 1$ and suppose that $\mathcal{X} = \{X_t : t \in T\}$ is a definable family of subsets of $\mathbb{R}^m$ such that for all $t \in T$, $\dim(X_t) < m$. Then there exist positive integers $D_1, \ldots, D_{m-1}$ such that for all $t \in T$, all $\eta > 0$ and all $w_1, \ldots, w_{m-1} \in \mathbb{R}$, there exists $\epsilon = \epsilon(t, \eta, w_1, \ldots, w_{m-1}) > 0$ such that

$$X_t \cap \{\langle w_1 + \eta x_1^{D_1}, \ldots, w_{m-1} + \eta x_1^{D_{m-1}}, x \rangle \in \mathbb{R}^m : 0 < x < \epsilon \} = \emptyset.$$ 

**Proof.** Induction on $m$. For $m = 1$, each $X_t$ is (uniformly) finite. So obviously we can find, for each $t \in T$, an $\epsilon = \epsilon(t) > 0$ such that $X_t \cap (0, \epsilon) = \emptyset$, which is the required conclusion in this case.

Now assume that the lemma holds for some $m \geq 1$ and that $\{X_t : t \in T\}$ is a definable family of subsets of $\mathbb{R}^{m+1}$ each having dimension at most $m$.

For $t \in T$ define $S_t := \{s \in \mathbb{R}^m : \langle y, s \rangle \in X_t \}$ is infinite$. Then $\{S_t : t \in T\}$ is a definable family of subsets of $\mathbb{R}^m$ and clearly $\dim(S_t) < m$ for each $t \in T$. So we may apply the inductive hypothesis to this family and obtain (with a small shift in notation) positive integers $D_2, \ldots, D_m$ such that for all $t \in T$, all $\eta > 0$ and all $w_2, \ldots, w_m \in \mathbb{R}$, there exists $\epsilon = \epsilon(t, \eta, w_2, \ldots, w_m) > 0$ such that for all $x \in (0, \epsilon)$, we have that $\langle w_2 + \eta x^{D_2}, \ldots, w_m + \eta x^{D_m}, x \rangle \notin S_t$, i.e. there are at most finitely many $y \in \mathbb{R}$ such that $\langle y, w_2 + \eta x^{D_2}, \ldots, w_m + \eta x^{D_m}, x \rangle \in X_t$.

Now, by the principle of definable choice, there exists a definable function $H : T \times (0, \infty) \times \mathbb{R}^m \times \mathbb{R} \to (0, 1]$ such that for all $t \in T$, all $\eta \in \mathbb{R}$, all $w \in \mathbb{R}^m$ and all $x \in \mathbb{R}$, its value $H(t, \eta, w, x)$ is some $y \in (0, 1)$
such that for no \( u \in (0, y) \) do we have \( \langle w_1 + \eta u, w_2 + \eta x^{D_2}, \ldots, w_m + \eta x^{D_m}, x \rangle \in X_t \), if such a \( y \) exists (and is, say, 1 otherwise). Notice that by the discussion above, such a \( y \) does indeed exist whenever \( x \in (0, \epsilon(t, \eta, w_1, \ldots, w_m)) \).

We now apply polynomial boundedness to obtain a positive integer \( D_1 \) such that for all \( t \in T \), all \( \eta > 0 \) and all \( w = \langle w_1, \ldots, w_m \rangle \in \mathbb{R}^m \), there exists \( \gamma = \gamma(t, \eta, w) > 0 \), which we may assume is strictly less than \( \epsilon(t, \eta, w_2, \ldots, w_m) \), such that for all \( x \in (0, \gamma) \), we have \( H(t, \eta, x) > x^{D_1} \). So if \( x \in (0, \gamma) \), then \( x \in (0, \epsilon) \) and hence for all \( u \in (0, H(t, \eta, x)) \) we have that \( \langle w_1 + \eta u, w_2 + \eta x^{D_2}, \ldots, w_m + \eta x^{D_m}, x \rangle \notin X_t \). Since this holds for arbitrary \( x \in (0, \gamma) \), we are done (upon taking \( \epsilon(t, \eta, w_1, \ldots, w_m) := \gamma(t, \eta, w) \)).

\[ \square \]

**Proof of 5.9**

For \( t \in T \) and \( u \in (-R, R)^m \), let \( X_{(t,u)} := \{ w \in (-R', R')^m : w + u \in \Delta(R)^m \text{ and } F_t(w + u) = 0 \} \). Then \( \dim(X_{(t,u)}) < m \) because if \( U \) is some non-empty, open subset of \((-R', R')^m \) such that \( w + u \in \Delta(R)^m \) and \( F_t(w + u) = 0 \) for all \( w \in U \), then \( F_t \), being holomorphic, would vanish identically on \( \Delta(R)^m \) which is contrary to hypothesis. So we may apply 5.10 to the family \( X := \{ X_{(t,u)} : \langle t, u \rangle \in T \times (-R, R)^m \} \) and obtain positive integers \( D_1, \ldots, D_{m-1} \) with the property stated in the conclusion of 5.10. Now choose \( \eta \) so small that the resulting map \( \theta \) satisfies 5.9.1 and 5.9.2.

(The inverse of \( \theta \) is given by \( \theta^{-1}(z_1, \ldots, z_m) = (z_1 - \eta z_1^{D_1}, \ldots, z_{m-1} - \eta z_{m-1}^{D_{m-1}}, z_m). \))

To verify 5.9.3, let \( t \in T \) and let \( z' = (z_1, \ldots, z_{m-1}) \in \Delta(R')^{m-1} \). If \( F_t \circ \theta(z', z_m) = 0 \) for all \( z_m \in \Delta(R') \), then, in particular, \( \langle z_1 + \eta x^{D_1}, \ldots, z_{m-1} + \eta x^{D_{m-1}}, x \rangle \in \Delta(R')^m \) and \( F_t(z_1 + \eta x^{D_1}, \ldots, z_{m-1} + \eta x^{D_{m-1}}, x) = 0 \) for all sufficiently small positive \( x \in \mathbb{R} \). However, if the real and imaginary parts of \( z_i \) are, respectively, \( a_i \) and \( b_i \) (for \( i = 1, \ldots, m-1 \)), this implies that \( \langle a_1 + \eta x^{D_1}, \ldots, a_{m-1} + \eta x^{D_{m-1}}, x \rangle \in X_{(t,b)} \) for all sufficiently small \( x > 0 \), where \( b := \langle b_1, \ldots, b_{m-1}, 0 \rangle \). But this clearly contradicts the conclusion of 5.10.

We now come to the proof of the quasi-parameterization theorem (2.2.3), so definability is now, and henceforth, with respect to a structure as described in 2.2.1.

Recall that we are given a definable family \( \mathcal{X} = \{ X_s : s \in S \} \) of subsets of \([-1, 1]^n \) each of dimension at most \( m \), where \( m < n \). We assume that the indexing set \( S \) has been chosen of minimal dimension.
and we denote this dimension by $\dim(X)$. We are required to find some $R > 1, K > 0$, a positive integer $d$, and an $(R, m + 1, K)$-family $\Lambda^*$, each element of which is a monic polynomial of degree at most $d$ in its first variable, such that

\[(*) \text{ for all } s \in S, \text{ there exists } F \in \Lambda^* \text{ such that } X_s \subseteq \{ x = \langle x_1, \ldots, x_n \rangle \in [-1,1]^n : \exists w \in [-1,1]^m \wedge_{i=1}^n F(x_i, w) = 0 \} \].

Let us first consider the case $\dim(X) = 0$, i.e. the case that $X$ is finite. In fact, it is sufficient to consider the case that $X$ consists of a single set, $X$ say, where $X \subseteq [-1,1]^n$ and $\dim(X) \leq m < n$.

5.11 Remark Indeed, it is obvious that, in general, if the conclusion of the quasi-parameterization theorem holds for the families $\{X_s : s \in S_1\}$ and $\{X_s : s \in S_2\}$, then it also holds for the family $\{X_s : s \in S_1 \cup S_2\}$.

Since we are now assuming that our ambient o-minimal structure is a reduct of $\mathbb{R}_{an}$, we may apply the 0-mild parameterization theorem (Proposition 1.5 of [15]) which tells us that (after routine translation and scaling) there exists a finite set $\{\Phi_j : 1 \leq j \leq l\}$ of definable, real analytic maps $\Phi_j = \langle \phi_{j,1}, \ldots, \phi_{j,n}\rangle : (-3,3)^m \to \mathbb{R}^n$ (say) whose images on $[-1,1]^m$ cover $X$, and are such that $\frac{|\phi_{j,i}(0)|}{|\alpha|} \leq c \cdot 3^{-|\alpha|}$ for some constant $c$, and all $\alpha \in \mathbb{N}^m$, $j = 1, \ldots, l$ and $i = 1, \ldots, n$. If we now invoke our other assumption on the ambient 0-minimal structure, then (again, after translation and scaling at the expense of increasing the number of parameterizing functions) there is no harm in assuming that each function $\phi_{j,i}$ has a definable, complex extension (for which we use the same notation) to the polydisk $\Delta(2)^m$.

We now set

$$F(z_1, z_2, \ldots, z_{m+1}) := \prod_{j=1}^l \prod_{i=1}^n (z_1 - \phi_{j,i}(z_2, \ldots, z_{m+1}))$$

for $z_1, z_2, \ldots, z_{m+1} \in \Delta(2)$.

Then $F$ is a monic polynomial of degree $d = ln$ in its first variable. Further, $\{F\}$ is, for some $K > 0$, a $(2, m + 1, K)$-family which clearly has the required property $(*)$.

We now proceed by induction on $\dim(X)$. So consider some $k, m, n \in \mathbb{N}$ with $k = \dim(X) \geq 1$ and $m < n$, and a definable family $\mathcal{X} = \{X_s : s \in S\}$ of subsets of $[-1,1]^n$ with $\dim(X_t) \leq m$ for each $s \in S$, and assume that the theorem holds for families of dimension $< k$ (for arbitrary $m, n$). Now it is easy to show that we may represent $\mathcal{X}$ in the form $\{X_u : u \in [-1,1]^k\}$.
In order to apply the inductive hypothesis we define the family $\mathcal{Y} := \{Y_u' : u' \in [-1, 1]^{k-1}\}$ of subsets of $[-1, 1]^{n+1}$ where, for each $u' \in [-1, 1]^{k-1}$,

\[ Y_{u'} := \{(x, u_k) \in [-1, 1]^{n+1} : x \in X_u\}. \]

5.12 \quad For each $H_t$ is a monic polynomial of degree at most $d$ in its first variable, and

5.14 \quad for each $u' \in [-1, 1]^{k-1}$ there exists $t = t(u') \in T$ such that

\[ Y_{u'} \subseteq \{(x, x_{n+1}) \in [-1, 1]^{n+1} : \exists w \in [-1, 1]^{m+1} (\bigwedge_{i=1}^{n+1} H_t(x_i, w) = 0)\}. \]

In order to prepare the functions in $\Lambda$ as discussed above, we must first remove those $t$ from $T$ such that for some $z_1$ the function $H_t(z_1, \cdot)$ vanishes identically (in its last $m+1$ variables). To do this, we first note that, by the principle of definable choice, the correspondence $u' \mapsto t(u')$ (for $u' \in [-1, 1]^{k-1}$) may be taken to be a definable function and so the set

\[ E := \{u \in [-1, 1]^{k} : H_{t(u')}(u_k, 0) = 0\} \]

is definable (where $0$ is the origin of $\mathbb{R}^{m+1}$). We have $\dim(E) < k$ because if $E$ contained a non-empty open subset of $[-1, 1]^{k}$, then we could find some $u' \in [-1, 1]^{k-1}$ such that $H_{t(u')}(u_k, 0) = 0$ for all $u_k$ lying in some non-empty open interval, which is impossible as $H_{t(u')}(\cdot, 0)$ is a monic polynomial.

Thus, by another use of the inductive hypothesis, the family $\{X_u : u \in E\}$ satisfies the conclusion of the quasi-parameterization theorem and so, by 5.11, it is sufficient to consider the family $\{X_u : u \in [-1, 1]^{k} \setminus E\}$.

For this we define, for each $u \in [-1, 1]^{k} \setminus E$, the function $H_u^* : \Delta(R)^{m+1} \to \mathbb{C}$ by

\[ H_u^*(z) := H_{t(u')}(u_k, z), \]

so that for all $u \in [-1, 1]^{k} \setminus E$, $H_u^*(0) \neq 0$. Now set

\[ \Lambda_0 := \{H_u^* : u \in [-1, 1]^{k} \setminus E\}. \]

Then $\Lambda_0$ is an $(R, m + 1, K)$-family which does not contain the zero function. So we may apply 5.8 to it with, say, $R_0 = \frac{1+R}{2}$ (and $m + 1$
in place of $m$) and obtain, for some $K_0 > 0$, an $(R_0, m + 1, K_0)$-family $Λ_0^\dagger = \{G_t : t \in T_0^\dagger\}$ having properties 5.8.1-4.

I claim that

5.17 for all $u \in [-1, 1]^k \setminus E$, there exists $t^\dagger \in T_0^\dagger$ such that

$X_u \subseteq \{x \in [-1, 1]^n : \exists w \in [-1, 1]^{m+1}(\bigwedge_{i=1}^n H_t(w)(x_i, w) = 0 \wedge G_t(w) = 0)\}$.

Indeed, let $u \in [-1, 1]^k \setminus E$. By 5.8.2, there is some $t^\dagger \in T_0^\dagger$ and $A > 0$ such that for all $z \in \Delta(R_0)^{m+1}$,

5.17.1 $G_t^\dagger(z) = A \cdot H_u^*(z)$.

Now let $x \in X_u$. Then, by 5.12, $(x, u_k) \in Y_{u'}$. Hence, by 5.14, we may choose $w \in [-1, 1]^{m+1}$ such that $H_t(w'(x_i, w) = 0$ for $i = 1, \ldots, n$ and $H_t(w'(u_k, w) = 0$. Since $[-1, 1]^{m+1} \subseteq \Delta(R_0)^{m+1}$, 5.17 now follows from 5.17.1 and 5.15.

In order to complete the proof we must reduce the range of the $w$-variable in 5.17 from $[-1, 1]^{m+1}$ to $[-1, 1]^m$. The idea is simple: we use the relation $G_t^\dagger(w) = 0$ to express $w_{m+1}$ as a function of $w_1, \ldots, w_m$, and then substitute this function for $w_{m+1}$ in the first conjunct appearing in 5.17. Of course, there are some technical difficulties to be overcome. Firstly, we must ensure that $G_t^\dagger(w)$ really does depend on $w_{m+1}$ and this is achieved by the transformation described in 5.9. Secondly, the argument only works locally. However, the compactness of $T_0^\dagger$ will guarantee that this is sufficient. And finally, the functional dependence of $w_{m+1}$ on $w_1, \ldots, w_m$ will, in general, be a many-valued one. This is precisely why we only obtain quasi-parameterization rather than parameterization.

So, to carry out the first step, we apply 5.9 to the $(R_0, m + 1, K_0)$-family $Λ_0^\dagger = \{G_t : t \in T_0^\dagger\}$ (which is permissible as it satisfies 5.8.4) with $R' = 1 + \frac{2(R_0 - 1)}{3}$ and $R'' = 1 + \frac{R_0 - 1}{3}$ (and $m + 1$ in place of $m$).

Let $θ : C^{m+1} \to C^{m+1}$ be as in 5.9 and, for each $t \in T_0^\dagger$ set

5.18 $\tilde{G}_t := G_t \circ θ | \Delta(R')^{m+1}$.

Then $\{\tilde{G}_t : t \in T_0^\dagger\}$ is an $(R', m + 1, K_0)$-family. Further, since the family $Λ_0^\dagger$ is well indexed (5.8.1), it immediately follows (from 5.5 and the Cauchy inequalities) that for each $α \in N^{m+1}$, the function $G_t^{\dagger(α)}(z)$ is continuous in both $t$ and $z$, for $(t, z) \in T_0^\dagger \times \Delta(R_0)^{m+1}$. Since $θ$ is holomorphic throughout $C^{m+1}$ we obtain
5.19 for each \( \alpha \in \mathbb{N}^{m+1} \), the function \( \tilde{G}^{(\alpha)}_t(z) \) is continuous in both \( t \) and \( z \) for \( \langle t, z \rangle \in T^\dagger_0 \times \Delta(R')^{m+1} \).

Also, it follows from 5.9.3 that

5.20 for all \( t \in T^\dagger_0 \) and all \( z' \in \Delta(R')^m \), the function \( \tilde{G}_t(z', \cdot) \) does not vanish identically on \( \Delta(R') \).

Having modified the functions \( G_t \), we must now adjust the functions \( H_{t(w')} \) in order to preserve 5.17. Accordingly, we define, for each \( u \in [-1,1]^k \setminus E \), the function \( \tilde{H}_u : \Delta(R')^{m+2} \to \mathbb{C} \) (which, in fact, only depends on \( u' \)) by

5.21 \( \tilde{H}_u(z_1, z_2, \ldots, z_{m+2}) := H_{t(w')}(z_1, \theta(z_2, \ldots, z_{m+2})) \).

Then \( \{ \tilde{H}_u : u \in [-1,1]^k \setminus E \} \) is an \( (R', m + 2, K) \)-family and, as we show below, the following version of 5.17 holds.

5.22 for all \( u \in [-1,1]^k \setminus E \), there exists \( t^\dagger \in T^\dagger_0 \) such that

\[ X_u \subseteq \{ x \in [-1,1]^n : \exists v \in (-R''', R''')^{m+1} (\bigwedge_{i=1}^n \tilde{H}_u(x_i, v) = 0 \land G_{t^\dagger}(v) = 0) \} \]

Indeed, let \( u = \langle u', u_k \rangle \in [-1,1]^k \setminus E \) and choose \( t^\dagger \in T^\dagger_0 \) as in 5.17. Suppose \( x \in X_u \) and (by 5.17) choose \( w \in [-1,1]^{m+1} \) such that \( \bigwedge_{i=1}^n H_{t(w')}(x_i, w) = 0 \land G_{t^\dagger}(w) = 0 \). Let \( v = \theta^{-1}(w) \). Then by 5.9.2, \( v \in \Delta(R''')^{m+1} \). But all the coordinates of \( v \) are real, so \( v' \in (-R''', R''')^{m+1} \). Also, for \( i = 1, \ldots, n \) we have, by 5.21, that \( \tilde{H}_u(x_i, v) = H_{t(w')}(x_i, \theta(v)) = H_{t(w')}(x_i, w) = 0 \). Similarly, by 5.18, \( \tilde{G}_{t^\dagger}(v) = G_{t^\dagger}(w) = 0 \) and 5.22 follows.

Let us also record here the fact that in view of 5.13, and since the transformation 5.21 does not affect the variable \( z_1 \), we have

5.23 for each \( u \in [-1,1]^k \setminus E \), the function \( \tilde{H}_u \) is a monic polynomial of degree at most \( d \) in its first variable.

We now carry out the local argument, as sketched above, that expresses \( z_{m+1} \) as a many-valued function of \( z' = \langle z_1, \ldots, z_m \rangle \) via the relation

\( \tilde{G}_{t^\dagger}(z', z_{m+1}) = 0 \).

Firstly, fix some \( R_1 \) with \( R'' < R_1 < R' \) and for each \( r \) with \( R'' < r < R_1 \) let \( C_r \) be the circle in \( \mathbb{C} \) with centre 0 and radius \( r \). Consider the set

5.24 \( V_r := \{ \langle t, z' \rangle \in T^\dagger_0 \times \Delta(R_1)^m : \text{for all } z_{m+1} \in C_r, \tilde{G}_t(z', z_{m+1}) \neq 0 \} \).
It follows from 5.19 that $V_r$ is an open subset of $T_0^\parallel \times \Delta(R_1)^m$ (for the $\|\cdot\|$-metric inherited from $\mathbb{C}^{N+m}$, where $N$ is as in 5.8.1). Further, it follows easily from 5.20 that the collection $\{V_r : R'' < r < R_1\}$ covers the compact space $T_0^\parallel \times \Delta(R_1)^m$.

Now, by the Lebesgue Covering Lemma, there exists $\epsilon > 0$, a positive integer $M$, and points $t^{(1)}, \ldots, t^{(M)} \in T_0^\parallel$, $a^{(1)}, \ldots, a^{(M)} \in [-R'', R'']^m$ such that

5.25 the collection $\{t^{(h)} + \Delta(\epsilon)^n : h = 1, \ldots, M\}$ covers $T_0^\parallel$,

5.26 each set $a^{(j)} + \Delta(2\epsilon)^m$ is contained in $\Delta(R_1)^m$ and the collection $\{a^{(j)} + (-\epsilon, \epsilon)^m : j = 1, \ldots, M\}$ covers $[-R'', R'']^m$, and

5.27 for each $h, j = 1, \ldots, M$, there exists $r_{h,j} \in (R'', R_1)$ such that $\langle(t^{(h)}, a^{(j)}) + \Delta(2\epsilon)^{N+m}\rangle \cap (T_0^\parallel \times \Delta(R_1)^m) \subseteq V_{r_{h,j}}$.

Fix, for the moment, $h, j \in \{1, \ldots, M\}$. Then for each $t \in T_0^\parallel \cap (t^{(h)} + \Delta(2\epsilon)^N)$ and each $z' \in a^{(j)} + \Delta(2\epsilon)^m$ it follows from 5.26, 5.27 and 5.24 that the contour integral

$$\frac{1}{2\pi i} \int_{C_{r_{h,j}}} \frac{\partial \hat{G}_t}{\partial z_{m+1}}(z', z_{m+1}) \cdot (\hat{G}_t(z', z_{m+1}))^{-1} \, dz_{m+1}$$

is well defined. It counts the number of zeros (with multiplicity) of the function $\hat{G}_t(z', \cdot)$ lying within the circle $C_{r_{h,j}}$. Further, by 5.19, 5.24 and 5.27, the integral is a continuous function of $\langle t, z' \rangle$ in the stated domain, and so is constant there. Let its value be $q_{h,j}$ and let $Z(t, z') = \langle \rho_1(t, z'), \ldots, \rho_{q_{h,j}}(t, z') \rangle$ be a listing of the zeros of $\hat{G}_t(z', \cdot)$ lying within the circle $C_{r_{h,j}}$ (each one counted according to its multiplicity).

Now, for $t \in T_0^\parallel \cap (t^{(h)} + \Delta(2\epsilon)^N)$, $(z_2, \ldots, z_{m+1}) \in a^{(j)} + \Delta(2\epsilon)^m$, $u \in [-1, 1]^k \setminus E$, and each $l = 1, \ldots, q_{h,j}$, we have, by 5.26 and the fact that $r_{h,j} < R_1$, that

$$z_2, \ldots, z_{m+1}, \rho_l(t, z_2, \ldots, z_{m+1}) \in \Delta(R_1)$$

and hence that the function

$$L^{h,j}_{l,u} : \mathbb{C} \times (a^{(j)} + \Delta(2\epsilon)^m) \to \mathbb{C}$$

given by

5.28 $L^{h,j}_{l,u}(z_1, z_2, \ldots, z_{m+1}) := \prod_{l=1}^{q_{h,j}} \tilde{H}_u(z_1, z_2, \ldots, z_{m+1}, \rho_l(t, z_2, \ldots, z_{m+1}))$

is well defined and, is a monic polynomial of degree at most $d \cdot q_{h,j}$ in $z_1$ (by 5.23).

Now, since $L^{h,j}_{l,u}(z_1, z_2, \ldots, z_{m+1})$ is symmetric in the $\rho_l(t, z_2, \ldots, z_{m+1})$ (i.e. it does not depend on our particular ordering of the list $Z(t, z_2, \ldots, z_{m+1})$),
it follows easily that it is a definable function of all the variables
\( t, u, z_1, \ldots, z_{m+1} \) (restricted to the stated domain) and, as a standard
argument shows, it is holomorphic in \( z_1, z_2, \ldots, z_{m+1} \).

We now scale and translate the function \( L_{t,u}^{h,j} \) by setting
\[
P_{t,u}^{h,j}(z_1, z_2, \ldots, z_{m+1}) := L_{t,u}^{h,j}(z_1, a_1^{(j)} + \varepsilon z_2, \ldots, a_m^{(j)} + \varepsilon z_{m+1})
\]
so that each \( P_{t,u}^{h,j} \) maps \( \mathbb{C} \times \Delta(2)^m \) to \( \mathbb{C} \), is a monic polynomial of degree
at most \( dq_{h,j} \) in \( z_1 \), and is bounded by \( K^{q_{h,j}} \). (Notice that this holds
true, by our convention concerning the monic polynomial of degree 0,
even if \( q_{h,j} = 0 \).)

We now combine the functions \( P_{t,u}^{h,j} \) as \( h \) and \( j \) vary over \( \{1, \ldots, M\} \).
Firstly, for \( h \in \{1, \ldots, M\}, \ u \in [-1,1]^k \setminus E \) and \( t \in T_0^j \cap (t^{(h)} + \Delta(2\varepsilon)^N) \)
define
\[
P_{t,u}^h := \prod_{j=1}^{M} P_{t,u}^{h,j}
\]
so that each \( P_{t,u}^h \) maps \( \mathbb{C} \times \Delta(2)^m \) to \( \mathbb{C} \), is a monic polynomial of degree
at most \( d_h := \sum_{j=1}^{M} dq_{h,j} \) in \( z_1 \), and is bounded by \( (K + 1)^{Mq_h} \), where
\( q_h := \max\{q_{h,j} : j = 1, \ldots, M\} \).

Finally, we set
\[
\Lambda^* := \bigcup_{h=1}^{M} \{ P_{t,u}^h : u \in [-1,1]^k, t \in T_0^j \cap (t^{(h)} + \Delta(\varepsilon)^N) \}.
\]

Then \( \Lambda^* \) is a \((2, m+1, (K + 1)^{Mq})\)-family, where \( q := \max\{q_h : h = 1, \ldots, M\} \), each element of which is a monic polynomial of degree
at most \( \max\{d_h : h = 1, \ldots, M\} \) in its first variable.

We now verify (*) (stated just before 5.11) which will complete
the proof. In fact, it just remains to show that if \( u \in [-1,1]^k \setminus E \),
then there exists \( F \in \Lambda^* \) such that \( X_u \subseteq \{ x \in [-1,1]^n : \exists w \in [-1,1]^m \ w \in \bigwedge_{i=1}^{n} F(x_i, w) = 0 \} \).

So let such a \( u \) be given. Choose \( t^i \in T_0^j \) as in 5.22. By 5.25 we
may choose \( h \in \{1, \ldots, M\} \) such that \( t^i \in t^{(h)} + \Delta(\varepsilon)^N \). We let our \( F \)
be the function \( P_{t^i,u}^h \) (see 5.30), which of course lies in \( \Lambda^* \) (see 5.31).

Now pick any \( x = (x_1, \ldots, x_n) \in X_u \). By 5.22 we may pick \( v = (v', v_{m+1}) \in (-R', R')^{m+1} \) such that \( \bigwedge_{i=1}^{n} H_u(x, v) = 0 \wedge \tilde{G}_{t^i}(v) = 0 \).
By 5.26, there exists \( j \in \{1, \ldots, M\} \) such that \( v' \in a^{(j)} + (-\varepsilon, \varepsilon)^m \).
Now, since \( v_{m+1} \) lies within the circle \( C_{r_{h,j}} \) and is a zero of the function
\( G_{t^i}(v', \cdot) = 0 \), it follows that \( q_{h,j} > 0 \) and that for some \( l = 1, \ldots, q_{i,j} \),
we have \( v_{m+1} = \rho_l(t^i, v') \). Thus \( \bigwedge_{i=1}^{n} H_u(x, v', \rho_l(t^i, v')) = 0 \), and hence
\( \bigwedge_{i=1}^{n} L_{t^i,u}^{h,j}(x, v') = 0 \) (see 5.28). We now choose \( w \in [-1,1]^m \) such that
\( v' = a^{(j)} + \varepsilon w \). Then \( \bigwedge_{i=1}^{n} P_{t^i,u}^{h,j}(x_i, w) = 0 \) (see 5.29). It follows that
\[ \bigwedge_{i=1}^n P_{t_i,u_i}^h (x_i, w) = 0 \text{ (see 5.30), i.e. } \bigwedge_{i=1}^n F(x_i, w) = 0, \text{ and we are done}. \]

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