Secret Key Generation over Noisy Channels with Common Randomness

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Abstract

This paper investigates the problem of secret key generation over a wiretap channel when the terminals observe correlated sources. These sources may or may not be independent of the main channel and the users overhear them before the transmission takes place. A novel achievable scheme for the case of independent sources is proposed, and it is shown to be optimal under certain less noisy conditions. This result improves upon the existing literature where the more stringent condition of degradedness was needed. For the situation where the sources are correlated with the main channel, a second achievable scheme based on joint source-channel coding is introduced, for which special cases of binary and Gaussian channels are studied. Some optimality results are derived, however, these are much less general than for the first setup.

I. INTRODUCTION

The wiretap channel, introduced by Wyner [2], is the basic model for analyzing secrecy in wireless communications. In this model, the transmitter, named Alice, wants to communicate reliably with Bob while keeping the transmitted message –or part of it– secret from an eavesdropper, named Eve, overhearing the communication through another channel. Secrecy is characterized by the amount of information that is not leaked, which can be measured by the equivocation rate –the remaining uncertainty about the message at the eavesdropper.

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The secrecy capacity of the wiretap channel is thus defined as the maximum transmission rate that can be attained with zero leakage. Csiszár and Körner [3] determined the rate-equivocation region of a general broadcast channel with any arbitrary level of security, which also establishes the secrecy capacity of the wiretap channel. These schemes guarantee secrecy at no communication cost by exploiting an artificial random noise that saturates the eavesdropper’s decoding capabilities.

On the other hand, Shannon [4] showed that it is also possible to achieve a positive secrecy rate by means of a secret key. Alice and Bob can safely communicate over a noiseless public broadcast channel as long as they share a secret key. The rate of this key, however, must be at least as large as the rate of the message to attain perfect secrecy. The main question that arises in this scenario is therefore: how do legitimate users safely exchange information and update the key? The answer is that the users should not communicate the key itself, which would then be compromised. Instead, they should only convey enough information to allow themselves to agree upon a key without disclosing, at the same time, any relevant information about it to the eavesdropper (for further discussion the reader is referred to [5]).

In this work, we study the problem of secret key generation over a wiretap channel with correlated sources at each terminal. These sources are assumed to be independent of the main channel and there is no additional public broadcast channel of finite or infinite rate, as seen in Fig. 1. It is assumed that each node acquires its corresponding source’s $n$-sequence observation before the communication begins. Additionally, at the end of this work, we analyze briefly the model where the sources affect the main channel statistics.
A. Related Work

Ahlswede and Csiszár [6] and Maurer [7] were among the first to study the use of correlated observations available at the legitimate users as a means to agree upon a key. In addition to the correlated observations, the terminals communicate over a public broadcast channel of infinite capacity to which the eavesdropper has also access. Two models were proposed in [6]: the “source model”, where the key generation is based on the common randomness underlying the statistical model between sources, and the “channel model”, where the common randomness is created between inputs and outputs of a channel. Csiszár and Narayan [8] later considered the situation where the public broadcast channel has finite capacity and there is a third “helper” node that is not interested in recovering the key but rather helping Alice and Bob.

More recently, Khisti et al. [9] investigated the situation where there is no helper node, the users communicate over a wiretap channel, and a separate public discussion channel may or may not be available. The simultaneous transmission of a secret message along with a key generation scheme using correlated sources was analyzed by Prabhakaran et al. [10]. The authors obtained a simple expression that shows the trade-off between the achievable secrecy rate and the achievable rate of the secret key. The corresponding Gaussian channel with correlated Gaussian sources but independent of the channel components was recently studied in [11]. Closed form expressions for both secret key generation and secret message transmission were derived. On the other hand, Salimi et al. [12] considered simultaneous key generation of two independent users over a multiple access channel with feedback, where each user eavesdrops the other. In addition, the receiver can actively send feedback, through a private noiseless (or noisy) link, to increase the size of the shared keys.

Previous works [9], [10] do not assume interactive communication, i.e., there is only one round of communication. Salimi et al. [12], however, allowed the end user to respond once through the feedback link. Other authors have analyzed key generation schemes that rely on several rounds of transmissions. Tyagi [13] characterized the minimum communication rate required to generate a maximum-rate secret key with $r$ rounds of interactive communication. He showed that this rate is equal to the interactive common information (a quantity he introduced) and the secret key capacity. In his model, two users observe i.i.d. correlated sources and communicate over an error-free channel. Hayashi et al. [14] studied a similar problem but considered general (not necessarily i.i.d.) sources sequences of finite length. Their proposed protocol attains the secret key capacity for general observations as well.
as the second-order asymptotic term of the maximum feasible secret key length for i.i.d. observations. They also proved that the standard one-way communication protocol fails to attain the aforementioned asymptotic result. Courtade and Halford [15] analyzed the related problem of how many rounds of public transmissions are required to generate a specific number of secret keys. Their model assumes there are $n$ terminals connected through an error-free public channel, where each terminal is provided with a number of messages before transmission that it uses to generate the keys.

The problem of secure source transmission with side information [16]–[18] is closely related to the present work, since the reconstructed source may serve as a key as long as it has been reliably and securely transmitted.

As previously mentioned, the primary focus of the present work is on sources that are independent of the main channel; nonetheless, we will discuss the general situation where they are not. Prior work on secrecy for channels with state include Chen and Vinck’s [19] and Liu and Chen’s [20] analysis of the wiretap channel with state. These works employ Gelfand and Pinsker’s scheme [21] to correlate the transmitted codeword with the channel state at the same time it saturates the eavesdropper’s decoding capabilities. A single-letter expression of the secrecy capacity for this model is still unknown, although a multi-letter bound was provided by Muramatsu [22] and a novel lower bound was recently reported in [23]. As a matter of fact, the complexity of this problem also lies in the derivation of an outer bound that can handle simultaneously secrecy and channels with state.

To the best of our knowledge, only a handful of works studied the problem of key generation for channels with state. The previously mentioned result of Prabhakaran et al. [10] is one of these examples. Zibaeenejad [24] analyzed a similar scenario where there is also a public channel of finite capacity between the users. He provides a novel inner bound that is similar to the second scheme we present in this work (referred to Theorem 3). However, some steps of the proof reported in [24] appear to be obscure and there seems a bound is missing in the final expression. As a consequence, we will not be able to compare our inner bound to this previously reported scheme.

The works found in the literature that are closely related to the problem dealt here [9], [10], [12], [24] derive the equivocation of their schemes using a weak secrecy condition. We will use the same method in our proofs although the strong secrecy criterion has become more frequent nowadays. Recent works on the wiretap channel employ this approach, e.g., [25], [26], where in particular [26] does not assume the messages have a uniform distribution.
B. Our Contribution

In this work, we introduce two novel coding schemes for the problem of secret key generation over a wiretap channel with correlated sources at each terminal. The first scheme is useful when the correlated sources are independent of the main channel and, thanks to a previously reported outer bound [27], this scheme is shown to be optimal whenever the channel and/or source components satisfy the specific less noisy conditions described in Table I. In contrast, the proposed schemes in [9], [10], [12] were optimal only when the stronger degradedness condition holds true for the channel and source components.

On the other hand, the second coding scheme is relevant when the correlated sources affect the main channel; in this scenario, the achievable rate region of our first coding scheme is no longer valid. The optimality of this second scheme is shown only for very limited regimes of binary and Gaussian channel models. This shortcoming may also arise due to the difficulty of deriving a tighter outer bound that combines both secrecy and channels with state, as it was previously mentioned.

Notation and Conventions

Throughout this work, we use the standard notation of [28]. Specifically, given two integers $i$ and $j$, the expression $[i:j]$ denotes the set $\{i, i+1, \ldots, j\}$, whereas for real values $a$ and $b$, $[a,b]$ denotes the closed interval between $a$ and $b$. We use the notation $x^j_i = (x_i, x_{i+1}, \ldots, x_j)$ to denote the sequence of length $j - i + 1$ for $1 \leq i \leq j$. If $i = 1$, we drop the subscript for succinctness, i.e., $x^j = (x_1, x_2, \ldots, x_j)$. Lowercase letters such as $x$ and $y$ are mainly used to represent constants or realizations of random variables, whereas capital letters such as $X$ and $Y$ stand for the random variables in itself. Bold capital letters such as $\mathbf{X}$ and $\mathbf{Y}$ represent matrices, while calligraphic letters such as $\mathbb{X}$ and $\mathbb{Y}$ are reserved for sets, codebooks or special functions.

The set of nonnegative real numbers is denoted by $\mathbb{R}_+$. The probability distribution (PD) of the random vector $X^n$, $p_{X^n}(x^n)$, is succinctly written as $p(x^n)$ without subscript when it can be understood from the argument $x^n$. Given three random variables $X$, $Y$, and $Z$, if its joint PD can be decomposed as $p(xyz) = p(y)p(x|y)p(z|y)$, then they form a Markov chain, denoted by $X \not\rightarrow Y \not\rightarrow Z$. Random variable $Y$ is said to be less noisy than $Z$ w.r.t. $X$ if $I(U;Y) \geq I(U;Z)$ for each random variable $U$ such that $U \not\rightarrow X \not\rightarrow (Y,Z)$ form a Markov chain. This relation is denoted by $Y \succeq_X Z$. Entropy is denoted by $H(\cdot)$, differential entropy, $h(\cdot)$, and mutual information, $I(\cdot;\cdot)$. The expression $\mathcal{C}(x) = \frac{1}{2} \log_2(1 + x)$ stands for
the capacity of a Gaussian channel with SNR of value $x$, and $[x]^+$ denotes $\max\{x, 0\}$. We denote typical and conditional typical sets by $T^n_\delta(X)$ and $T^n_\delta(Y|X^n)$, respectively.

II. PRELIMINARIES

A. Problem Definition

Consider the wiretap channel with common randomness at every node $(A, B, E)$, as shown in Fig. 1. The legitimate users (Alice and Bob) want to agree upon a secret key $K \in \mathcal{K}$ while an eavesdropper (Eve) is overhearing the communication. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E}$, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be six finite sets. Alice, Bob, and Eve observe the random sequences (sources) $A^n$, $B^n$, and $E^n$, respectively, drawn according to i.i.d. samples from the joint distribution $p(abe)$ on $\mathcal{A} \times \mathcal{B} \times \mathcal{E}$. Alice communicates with Bob through a discrete memoryless channel with input $X \in \mathcal{X}$ and output $Y \in \mathcal{Y}$. Eve is listening the communication through another channel with input $X \in \mathcal{X}$ and output $Z \in \mathcal{Z}$. This channel is defined by its transition probability $p(yz|x)$ and is independent of the sources.

**Definition 1 (Code):** An $(n, m)$-code for secret key generation is defined by:

- A (stochastic) encoding function at Alice $\varphi: A^n \rightarrow X^m$, defined by some transition probability $P_\varphi(x^m|a^n)$,
- A (stochastic) secret key function at Alice $\psi_a: A^n \rightarrow \mathcal{K}$, defined by some transition probability $P_{\psi_a}(k|a^n)$,
- A secret key function at Bob $\psi_b: B^n \times Y^m \rightarrow \mathcal{K}$.

The rate of such a code is defined as the number of channel uses per source symbol $\frac{m}{n}$.

**Definition 2 (Achievability):** A tuple $(\eta, R_k) \in \mathbb{R}^2_+$ is said to be achievable if, for any $\epsilon > 0$, there exists $\|K\| \geq 2^{n(R_k-\epsilon)}$ and an $(n, m)$-code $(\varphi, \psi_a, \psi_b)$ such that:

\begin{align*}
\frac{m}{n} &\leq \eta + \epsilon, \quad (1a) \\
\Pr\{K \neq \hat{K}\} &\leq \epsilon, \quad (1b) \\
I(K; E^nZ^m) &\leq n\epsilon, \quad (1c)
\end{align*}

where $K = \psi_a(A^n)$, $\hat{K} = \psi_b(B^n, Y^m)$, and the channel input $X^m$ is the output of the corresponding encoder mapping: $\varphi(A^n)$.

The set of all achievable tuples is denoted by $\mathcal{R}^+$ and is referred to as the secret key region.
B. Outer Bound

The following theorem gives an outer bound on $\mathcal{R}^*$, i.e., it defines the region $\mathcal{R}_{\text{out}} \supseteq \mathcal{R}^*$.

**Theorem 1:** An outer bound on the secret key region of this channel model is given by

$$R_k \leq \max_{p \in \mathcal{P}} \{ \eta[I(T; Y) - I(T; Z)] + I(V; B|U) - I(V; E|U) \},$$

subject to

$$I(V; A|B) \leq \eta I(X; Y),$$

where $\mathcal{P}$ is the set of all input probability distributions given by

$$\mathcal{P} = \{ p(txyzuvabe) = p(tx)p(yz|x)p(abe)p(v|a)p(u|v) \},$$

with $|T| \leq |X|$, $|U| \leq |A| + 1$, and $|V| \leq (|A| + 1)^2$.

**Proof:** Refer to Appendix A for details.

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III. Summary of Main Results

In this section, we propose two key generation schemes for the aforementioned model that lead to two novel inner bounds for the secret key region (cf. Theorems 2 and 3). We then study some special cases where the first scheme turns out to achieve the (optimal) secret key region (cf. Propositions 1, 2, and 3).

A. Inner Bound Separate

The following theorem gives an inner bound on $\mathcal{R}^*$, i.e., it defines the region $\mathcal{R}_{\text{in}} \subseteq \mathcal{R}^*$.

**Theorem 2:** A tuple $(\eta, R_k) \in \mathbb{R}_+^4$ is achievable if there exist random variables $U$, $V$, $Q$, $T$, $X$ on finite sets $\mathcal{U}$, $\mathcal{V}$, $\mathcal{Q}$, $\mathcal{T}$, $\mathcal{X}$, respectively, with joint distribution $p(uvqtxyzabe) = p(q|t)p(tx)p(yz|x)p(abe)p(v|a)p(u|v)$, which verify

$$R_k \leq \eta[I(T; Y|Q) - I(T; Z|Q)] + I(V; B|U) - I(V; E|U),$$

subject to

$$I(U; A|B) \leq \eta I(Q; Y),$$

$$I(V; A|B) \leq \eta I(T; Y).$$

Moreover, it suffices to consider sets $\mathcal{U}$, $\mathcal{V}$, $\mathcal{Q}$, and $\mathcal{T}$ such that $|\mathcal{U}| \leq |\mathcal{A}| + 2$, $|\mathcal{V}| \leq (|\mathcal{A}| + 1)(|\mathcal{A}| + 2)$, $|\mathcal{Q}| \leq |\mathcal{X}| + 2$, and $|\mathcal{T}| \leq (|\mathcal{X}| + 1)(|\mathcal{X}| + 2)$.

**Proof:** The proof is deferred to Section V.
Remark 1: The regions $R_{\text{out}}$ and $R_{\text{in}}$ do not coincide in general. This is due to the presence of condition (6a) in the inner bound, and the looser condition (3) in the outer bound with respect to (6b). We present in Section III-C a few special cases where these differences disappear and both regions coincide.

Remark 2: By setting $U = \emptyset$, the region in Theorem 2 recovers the results in [9, Thm. 1], when the eavesdropper has access to a correlated source, and [10, Thm. 2], when there is no secret message to be transmitted. In these works, there was only one layer to encode the source $A^n$ while our coding scheme allows for two layers, introducing considerably difficulty in the derivation of Eve’s equivocation (see e.g. (26) and (37)). The advantage of having two layers of description is that Theorem 2 recovers the result of Csiszár and Narayan [8], as it will be discussed in Remark 6 below.

Remark 3: It is worth mentioning that Theorem 2 improves upon our previous work in [27, Sec. IV-A] since (6) replaces the more stringent condition: $I(V; A|B) \leq \eta I(Q; Y)$.

Remark 4: The problem of key generation dealt with in the present work is intimately connected to the problem of secure source transmission with side information, at both receiver and eavesdropper [17], [18], since the part of the source that can be reliably and securely transmitted serves as key which is a function of the source. It is thus not surprising that Theorem 2 bears a resemblance to our previous result in [18, Thm. 2].

B. Inner Bound Joint

The following theorem gives an inner bound on $R^*$, i.e., it defines the region $R_{\text{joint}} \subseteq R^*$.

Theorem 3: A tuple $(\eta = 1, R_k) \in \mathbb{R}_+^2$ is achievable if there exist random variables $U, V, X$ on finite sets $\mathcal{U}, \mathcal{V}, \mathcal{X}$, respectively, with joint distribution

\[ p(uvabexyz) = p(u|v)p(vx|a)p(abe)p(yz|x), \]

which verify

\[ R_k \leq I(V; BY|U) - I(V; EZ|U), \tag{7} \]

subject to

\[ I(U; A) \leq I(U; BY), \tag{8a} \]

\[ I(V; A|U) \leq I(V; BY|U). \tag{8b} \]

Moreover, it suffices to consider sets $\mathcal{U}$ and $\mathcal{V}$ such that $|\mathcal{U}| \leq |\mathcal{X}| \cdot |A| + 4$ and $|\mathcal{V}| \leq (|\mathcal{X}| \cdot |A| + 2) (|\mathcal{X}| \cdot |A| + 4)$. 

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\[ E \geq_A B \quad \text{or} \quad B \geq_A E \]
\[
\begin{array}{c|c|c}
Z \geq_X Y & R_k = 0 & \text{Proposition 1} \\
Y \geq_X Z & \text{Proposition 2} & \text{Proposition 3}
\end{array}
\]

**TABLE I**

Regimes where Theorem 2 is optimal. No secret key is achievable if \( Z \geq_X Y \) and \( E \geq_A B \).

**Proof:** The proof is deferred to Section VI.

**Remark 5:** The inner bound joint cannot recover the inner bound separate since the latter allows different secrecy levels for the source and channel codewords. While (6b) imposes the condition that the source codeword should “fit” inside the channel codeword, (8b) inflicts a much stringent restriction.

**C. Optimal Characterization of the Secret Key Rate**

The inner bound \( R_{\text{in}} \) is optimal under certain less noisy conditions in channel and/or source components. These special cases are summarized in Table I.

1) **Eve Has a Less Noisy Channel:** If Eve has a less noisy channel than Bob, i.e., \( Z \geq_X Y \), the information transmitted over the channel is compromised. Therefore, the amount of secret key that can be generated only depends on the statistical differences between sources.

**Proposition 1:** If \( Z \geq_X Y \), a tuple \((\eta, R_k) \in \mathbb{R}_+^2\) is achievable if there exist random variables \( U, V, X \) on finite sets \( \mathcal{U}, \mathcal{V}, \mathcal{X} \), respectively, with joint distribution

\[
p(u,v,a,b,e) \equiv p(u)p(v|a)p(b|e)p(x)p(yz|x),
\]
which verify:

\[
R_k \leq I(V;B|U) - I(V;E|U),
\]  

subject to \( I(V;A|B) \leq \eta \; I(X;Y) \).

**Proof:** Given the less noisy condition on Eve’s channel, i.e., \( I(T;Y) \leq I(T;Z) \) for any RV \( T \) such that \( T \not\to X \not\to (YZ) \), the bound (2) is maximized with \( T = \emptyset \). On the other hand, the region (9) is achievable by setting auxiliary RVs \( Q = T = X \) in \( R_{\text{in}} \).

**Remark 6:** The secret key capacity of the wiretap channel with a public noiseless channel of rate \( R \) [8, Theorem 2.6] turns out to be a special case of Proposition 1, where \( X = Y = Z \) and defining \( \eta \; H(X) = \eta \log |\mathcal{X}| \equiv R \).
2) **Eve Has a Less Noisy Source:** If Eve has a less noisy source than Bob, i.e., $E \succeq_A B$, the amount of secret key that can be generated depends on the amount of secure information transmitted through the wiretap channel.

**Proposition 2:** If $E \succeq_A B$, a tuple $(\eta, R_k) \in \mathbb{R}_+^2$ is achievable if there exist random variables $T, X$ on finite sets $\mathcal{T}, \mathcal{X}$, respectively, with joint distribution $p(txyz) = p(tx)p(yz|x)$, verifying:

$$R_k \leq \eta \left[ I(T;Y) - I(T;Z) \right]. \quad (10)$$

**Proof:** Given the less noisy condition on Eve’s source, i.e., $I(V;B) \leq I(V;E)$ for any RV $V$ such that $V \leftrightarrow A \leftrightarrow (BE)$, the bound (2) is maximized with $U = V$ and independent of the sources. The region (10) is achievable by using the same auxiliary RVs in the inner bound as in the outer bound. ■

**Remark 7:** The bound (10) is equal to the secrecy capacity of the wiretap channel.

**Remark 8:** Even though the bound (10) becomes independent of the sources sequences $(A^n, B^n, E^n)$, we assume $n \neq 0$ and thus the rate $\eta$ is finite.

3) **Bob Has a Less Noisy Channel and Source:** If Bob has a less noisy channel and source than Eve, i.e., $Y \succeq_X Z$ and $B \succeq_A E$, the lower layers of the channel and source codewords are not needed any more.

**Proposition 3:** If $Y \succeq_X Z$ and $B \succeq_A E$, a tuple $(\eta, R_k) \in \mathbb{R}_+^2$ is achievable if there exist random variables $V, X$ on finite sets $\mathcal{V}, \mathcal{X}$, respectively, with joint distribution $p(vabe:xyz) = p(v|a)p(abe)p(x)p(yz|x)$, which verify:

$$R_k \leq \eta \left[ I(X;Y) - I(X;Z) \right] + I(V;B) - I(V;E), \quad (11a)$$

subject to $I(V;A|B) \leq \eta I(X;Y). \quad (11b)$

**Proof:** Given the less noisy conditions on Bob’s channel and source, the bound (2) is maximized with $U = \emptyset$ and $T = X$. The region (11) is achievable by also setting auxiliary RVs $U = Q = \emptyset$ and $T = X$ in the inner bound. ■

**Remark 9:** Proposition 3 extends the results from [9, Thm. 4] and [10, Thm. 3] which assumed the more stringent conditions of degradedness: $A \leftrightarrow B \leftrightarrow E$ and $X \leftrightarrow Y \leftrightarrow Z$.

**IV. Numerical Examples**

In this section, we analyze the performance of the inner bounds presented in Section III for several different channel models.
A. Wiretap Channel with BEC/BSC Sources

Consider the communication system depicted in Fig. 2. The main channel consists of a noiseless link from Alice to Bob and a binary symmetric channel (BSC) with crossover probability $\zeta \in [0, \frac{1}{2}]$ from Alice to Eve (see Fig. 2a). Additionally, the three nodes have access to correlated sources; in particular, Alice observes a binary uniformly distributed source, i.e., $A \sim B(\frac{1}{2})$, which is the input of two parallel channels as shown in Fig. 2b. Bob observes the output of a binary erasure channel (BEC) with erasure probability $\beta \in (0, 1]$, and Eve, a BSC with crossover probability $\epsilon \in [0, \frac{1}{2}]$.

Remark 10: The sources $(A, B, E)$ satisfy different properties according to the values of the parameters $(\beta, \epsilon)$ [29], specifically:

- if $0 \leq \beta < 2\epsilon$, $E$ is a degraded version of $B$, i.e., $A \leftrightarrow B \leftrightarrow E$;
- if $2\epsilon \leq \beta < 4\epsilon(1-\epsilon)$, $B$ is less noisy than $E$, i.e., $B \succeq_{A} E$; and,
- if $4\epsilon(1-\epsilon) \leq \beta < h_2(\epsilon)$, $B$ is more capable than $E$.

For simplicity and in order to compare the different inner bounds, we assume $\eta = 1$ in this channel model.

The following proposition provides a simple expression of the outer bound in Theorem 1.

**Proposition 4 (Outer bound):** If $(\eta = 1, R_k)$ is achievable, then there exists $v \in [0, \frac{1}{2}]$ such that

$$R_k \leq (1 - \beta)h_2(v) + h_2(\epsilon) - h_2(v \star \epsilon) + h_2(\zeta).$$ (12)

**Proof:** See Appendix B. \[\square\]

We now evaluate the inner bounds from Theorem 2 and 3 for this system model.

**Proposition 5 (Inner bound separate):** The tuple $(\eta = 1, R_k) \in \mathcal{R}_{in}$ if and only if there
exist $u, v, q \in [0, \frac{1}{2}]$ such that:

$$R_k \leq (1 - \beta) [ h_2(v * u) - h_2(v)] + h_2(v * \epsilon) - h_2(v * u * \epsilon)$$

$$+ h_2(\zeta) + h_2(q) - h_2(\zeta * q),$$

subject to $\beta [1 - h_2(v * u)] \leq 1 - h_2(q).$  \hfill (13b)

Proof: The bounds (13) are directly calculated from (5) and (6a) with the following choice of input random variables: $T = X, Q = X \oplus Q', V = A \oplus V', U = V \oplus U'.$ Here, $X \sim B(\frac{1}{2}), Q' \sim B(q), V' \sim B(v), \text{ and } U' \sim B(u),$ and each random variable is independent of each other and $(A, B, E).$ The condition (6b) in the inner bound becomes redundant with the mentioned choice of input distribution. \hfill \blacksquare

Proposition 6 (Inner bound joint): The tuple $(\eta = 1, R_k) \in \mathcal{R}_{\text{joint}}$ if and only if there exists $v \in [0, \frac{1}{2}]$ satisfying:

$$R_k \leq \max\{ (1 - \beta) h_2(v) + h_2(\epsilon * \zeta) - h_2(v * \epsilon * \zeta), h_2(\zeta) \}. \hfill (14)$$

Proof: The first part of the maximum in (14) follows from (7) by choosing $X = V \oplus A$ and $U = V \oplus V'$ as input random variables, whereas the second part, by choosing $X = V$ and $U = \emptyset.$ In the two cases, $V \sim B(\frac{1}{2})$ and $V' \sim B(v),$ both independent of each other and $(A, B, E).$ The conditions (8) are redundant with this choice of input distribution. \hfill \blacksquare

As a means of comparison with previous works, we provide next the inner bound presented in [9]. This inner bound is similar to Theorem 2 but with only one layer of description for the source $A,$ thus its achievable region is denoted $\mathcal{R}_{\text{sep-1L}}.$

Proposition 7 ([9, Thm. 1] Inner bound separate 1-layer): The tuple $(\eta = 1, R_k) \in \mathcal{R}_{\text{sep-1L}}$ if and only if there exists $v \in [0, \frac{1}{2}]$ such that

$$R_k \leq h_2(v * \epsilon) - (1 - \beta) h_2(v) - \beta + h_2(\zeta). \hfill (15)$$

Proof: Using the definition of auxiliary random variables made in [9], the maximizing distribution is $a = x, b = \emptyset,$ and $t = u \oplus v',$ where $x \sim B(\frac{1}{2})$ and $v' \sim B(v),$ both independent of each other and the sources. \hfill \blacksquare

Remark 11: Proposition 7 is a special case of Proposition 5 with $u = q = \frac{1}{2}.$ As mentioned in Remark 2, the inner bound [9, Thm. 1] is a special case of our Theorem 2 with $U = \emptyset$ (thus $u = \frac{1}{2}.$) Moreover, given that in this model the Markov chain $X \rightarrow Y \rightarrow Z$ holds, the channel codebook of Proposition 7 has only one layer (thus $q = \frac{1}{2}.$) On the other hand,
there are two layers of description in Proposition 5, and whenever $U \neq \emptyset$ (i.e., $u < \frac{1}{2}$), we have that $Q \neq \emptyset$ (i.e., $q < \frac{1}{2}$). This relationship is determined by (13b).

We perform numerical optimization of the bounds (12)–(15) for different values of $\beta$ while fixing $\zeta = 0.01$ and $\epsilon = 0.05$; the results are shown in Fig. 3. We see that the inner bound separate outperforms the inner bound joint for low values of $\beta$. In this region, there is a substantial gain from the use of the parallel source $A$ which the inner bound joint fails to achieve. We conjecture that the inner bound separate will always be equal or better if the sources are independent of the main channel (as it is the case here). Additionally, the inner bound separate is tight in the regions A and B in Fig. 3, as well as for large $\beta$. In the first case, according to Remark 10, $B \succeq_A E$ and Proposition 3 thus holds. On the other hand, when $\beta \rightarrow 1$, the source $B$ becomes useless and the secret key rate is due to the advantage in the main channel (cf. Proposition 2).

We also see in Fig. 3 the advantage of having two layers of description for the source $A$. Our proposed scheme, denoted “IB sep. (2L)”, attains a higher secret key rate than the scheme with only one layer of description (Proposition 7), denoted “IB sep. (1L)”, for intermediate values of $\beta$. It is in this regime, when source $B$ is no longer less noisy than $E$, that two
layers of description are needed. As a final note, we present in Fig. 4 the situation where $X = Y = Z$; Theorem 2 achieves capacity in this scenario according to Proposition 1. The inner bound joint is also tight in this particular case although we cannot show a general statement as Proposition 1. Unsurprisingly, the scheme with only one layer of description is not tight for intermediate values of $\beta$ as previously mentioned.

B. Wiretap Channel with State

Consider the wiretap channel depicted in Fig. 5a, where the main channel is affected by a state $A \sim B(a)$ and Eve observes a degraded version of $Y$ due to the noise $W \sim B(\zeta)$. Bob and Eve have access to side information correlated to the channel state; their observations are the outputs of parallel BECs whose input is the channel state $A$, as shown in Fig. 5b.

The outer bound from Theorem 1 is only valid when the side informations do not affect the main channel. Since this condition does not apply here, we rely on a different outer bound inspired from previous result in [30] which we provide in the following proposition.

**Proposition 8 (Outer bound):** If $(\eta = 1, R_k)$ is achievable, then

$$R_k \leq \epsilon(1 - \beta)h_2(a) + h_2(\zeta).$$

(16)
Proof: The bound (16) is derived from $R_k \leq I(XA; YB | ZE)$ [30, Thm. 2], where $X = X' + A$ and $X' \sim \mathcal{B} \left( \frac{1}{2} \right)$. Details are omitted.

As with the outer bound, the inner bound from Theorem 2 does not hold when the side informations affect the channel. On the other hand, the inner bound from Theorem 3 does hold even though this is not made explicit in the proof. Thus, we only provide the expression for Theorem 3 next.

**Proposition 9 (Inner bound):** The tuple $(\eta = 1, R_k) \in \mathcal{R}_{\text{joint}}$ if and only if

$$R_k \leq \epsilon h_2(a \ast \zeta) - \beta h_2(a) + (1 - \epsilon) h_2(\zeta).$$  \hspace{1cm} (17)

*Proof:* The bound (17) follows from (7) by choosing $X = V$, $V = V' + A$, and $U = \emptyset$ as input random variables. Here, $V' \sim \mathcal{B} \left( \frac{1}{2} \right)$ and independent of $A$. The conditions (8) are redundant with this choice of input distribution.

The bounds from Propositions 8 and 9 only match in the extreme case $\beta = \epsilon = 0$, when $R(k) \leq h_2(\zeta)$. Unfortunately, for other values of channel parameters the bounds are not tight. This may be a consequence of Proposition 8 being loose and probably, not having an optimal input distribution in Proposition 9. Indeed, the derivation of better bounds for this model offers a considerable mathematical difficulty, which up to the present we have not been able to overcome.

**C. Gaussian Wiretap Channel with State**

Consider the Gaussian wiretap channel defined, for each time slot, as,

$$Y = X + S + W_1,$$  \hspace{1cm} (18a)

$$Z = X + S + W_2.$$  \hspace{1cm} (18b)

Fig. 5. Wiretap channel with state $A$. 

(a) Main channel. (b) Side information.
where \( W_i \sim \mathcal{N}(0, N_i), i \in \{1, 2\} \), denotes the additive white Gaussian noise with i.i.d. samples. The state \( S \sim \mathcal{N}(0, Q) \) is also i.i.d. and independent of \( W_i \) for \( i \in \{1, 2\} \). Additionally, Alice’s channel input satisfies the average power constraint \( \mathbb{E}[X^2] \leq P \).

Analogously to the preceding subsection, we provide next an outer bound for this model based on [30, Thm. 2] and an inner bound based on Theorem 3.

**Proposition 10 (Outer bound):** If \((\eta = 1, R_k)\) is achievable, then
\[
R_k \leq \left[C\left(\frac{P + Q + 2\sqrt{PQ}}{N_1}\right) - C\left(\frac{P + Q + 2\sqrt{PQ}}{N_2}\right)\right]^+.
\]
(19)

*Proof:* The bound (19) is derived from \( R_k \leq I(XS; Y|Z) \) [24, Thm. 4] [30, Thm. 2], where we assume \( N_1 \leq N_2 \) and maximum correlation between \( X \) and \( S \). Details are omitted.

**Proposition 11 (Inner bound):** The tuple \((\eta = 1, R_k)\) \( \in \mathcal{R}_{\text{joint}} \) if and only if
\[
R_k \leq \left[C\left(\frac{P + Q + 2\rho\sqrt{PQ}}{N_1}\right) - C\left(\frac{P + Q + 2\rho\sqrt{PQ}}{N_2}\right)\right]^+,
\]
(20)
where
\[
(1 - \rho^2)P = N_1 - \frac{N_1^2}{P + Q + 2\sqrt{PQ} + N_1}.
\]
(21)

*Proof:* See Appendix E for details.

**Remark 12:** The forms of the bounds given in Propositions 10 and 11 were already presented in [30, Prop. 1 and 2]. However, there is an unclear issue with the derivation of the maximum allowed correlation coefficient \( \rho \). The correct expression should be given by (21) as it is shown in Appendix E. In spite of this small inaccuracy, the optimality analysis in [30, Prop. 3] holds true, namely, the gap between inner and outer bound is always smaller than \( \frac{1}{2} \) bits, and the gap is zero for either \( P \to \infty \) or \( Q \to \infty \).

V. PROOF OF THEOREM 2

This section provides a sketch of the proof of Theorem 2.

**A. Codebook Generation**

Let us define the quantity
\[
R_f \leq (\eta + \epsilon)I(T; Z|Q) - \epsilon_f,
\]
(22)
and fix the following joint probability distribution:
\[
p(qtxyzuvabe) = p(q|t)p(t|x)p(y|x)p(u|v)p(v|a)p(abe).
\]
Then, proceed as follows:

1) Randomly pick $2^nS_1$ sequences $u^n(s_1)$ from $T^n_\delta(U)$ and divide them into $2^{nR_1}$ equal-size bins $B_1(r_1)$, $r_1 \in [1:2^{nR_1}]$;

2) For each codeword $u^n(s_1)$, randomly pick $2^{n(S'_2+S''_2)}$ sequences $v^n(s_1,s'_2,s''_2)$ from $T^n_\delta(V|u^n(s_1))$ and divide them into $2^{nR_2}$ equal-size bins $B_2(s_1,r_2)$, $r_2 \in [1:2^{nR_2}]$;

3) Randomly pick $2^{nR_2}$ sequences $q^m(r_c)$ from $T^n_\delta(Q)$;

4) For each codeword $q^m(r_c)$, randomly pick $2^{n(R_p+R_f)}$ sequences $t^m(r_c,r_p,r_f)$ from $T^n_\delta(T|q^m(r_c))$.

B. Encoding

Given a sequence $a^n$, and the index $s'_2$ chosen uniformly at random in $[1:2^nS'_2]$, the encoder proceeds as follows:

1) It looks for an index $s_1 \equiv s_1$ such that $(u^n(s_1),a^n) \in T^n_\delta(UA)$. If more than one index is found, choose the smallest one. The probability of not finding such an index is arbitrarily small as $n \to \infty$ if $\delta' < \epsilon_1$ and

$$S_1 > I(U;A) + \epsilon_1. \quad (23)$$

2) Then, it looks for an index $s''_2 \equiv s''_2$ such that $(v^n(s_1,s'_2,s''_2),a^n) \in T^n_\delta(VA|u^n(s_1))$. If more than one index is found, choose the smallest one. The probability of not finding such an index is arbitrarily small as $n \to \infty$ if $\delta' < \epsilon_2$ and

$$S''_2 > I(V;A|U) + \epsilon_2. \quad (24)$$

3) Let $B_1(r_1)$ and $B_2(s_1,r_2)$ be the bins of $u^n(s_1)$ and $v^n(s_1,s'_2,s''_2)$, respectively.

4) Define $(r_c,r_p) = M(r_1,r_2) \in [1:2^nR_2] \times [1:2^nR_2]$ where $M$ is an arbitrary fixed one-to-one mapping such that $r_1 = M'(r_c)$ for some mapping $M'$. These two functions can be defined if:

$$R_1 + R_2 = R_c + R_p, \quad (25)$$

$$R_1 \leq R_c. \quad (26)$$

5) The encoder chooses uniformly at random a value $r_f \in [1:2^nR_f]$ and selects the codeword $t^m(r_c,r_p,r_f)$. It then transmits the associated jointly typical sequence $x^m \sim \prod_{i=1}^m p(x_i|t_i(r_c,r_p,r_f))$, generated on the fly.
C. Decoding

Given a sequence $b^n$ and the channel output $y^m$, the decoder proceeds as follows:

1) It looks for the unique set of indices $(r_c, r_p, r_f) \equiv (\hat{r}_c, \hat{r}_p, \hat{r}_f)$ such that
   \[(q^m(\hat{r}_c), t^m(\hat{r}_c, \hat{r}_p, \hat{r}_f), y^m) \in T^n_\delta(QTY).\]

   The probability of error in decoding can be made arbitrarily small as $(n, m) \to \infty$ provided that
   \[R_c + R_p + R_f < (\eta + \epsilon)I(T; Y) - \delta,\]  
   \[R_p + R_f < (\eta + \epsilon)I(T; Y|Q) - \delta.\]  
   (27)  
   (28)

2) Compute $(r_1, r_2) = M^{-1}(r_c, r_p)$.

3) The decoder looks for the unique index $s_1 \equiv \hat{s}_1$ such that $u^n(\hat{s}_1) \in B_1(r_1)$ and $(u^n(\hat{s}_1), b^n) \in T^n_\delta(UB)$. The probability of error in decoding can be made arbitrarily small as $n \to \infty$ provided that
   \[S_1 - R_1 < I(U; B) - \delta.\]  
   (29)

4) Then it looks for the unique set of indices $(s'_2, s''_2) \equiv (\hat{s}'_2, \hat{s}''_2)$ such that $v^n(s_1, \hat{s}'_2, \hat{s}''_2) \in B_2(s_1, r_2)$ and $(v^n(s_1, \hat{s}'_2, \hat{s}''_2), b^n) \in T^n_\delta(VB|u^n(s_1))$. The probability of error in decoding can be made arbitrarily small as $n \to \infty$ provided that
   \[S'_2 + S''_2 - R_2 < I(V; B|U) - \delta.\]  
   (30)

D. Key Generation

The secret key is a deterministic function of the indices $s_1, s'_2, \text{ and } s''_2$, therefore its rate satisfies:
   \[R_k \leq S_1 + S'_2 + S''_2.\]  
   (31)
E. Key Leakage

Consider the following chain of inequalities:

\[
H(s_1 s_2' s_2'' | CE^n Z^m) = H(s_2' A^n | CE^n Z^m) - H(A^n | C s_1 s_2' s_2'' E^n Z^m) \tag{32a}
\]

\[
= H(s_2' A^n | CE^n Z^m) - H(A^n | C s_1 s_2'' E^n) \tag{32b}
\]

\[
= H(s_2' A^n | CE^n Z^m) - H(A^n | U^n V^n E^n) \tag{32c}
\]

\[
\geq H(s_2' A^n | CE^n Z^m) - n H(A | UV E) \tag{32d}
\]

\[
\geq H(s_2' A^n | C r E^n Z^m) - n H(A | UV E) \tag{32e}
\]

\[
= H(s_2' A^n | C r r_p E^n Z^m) + I(s_2' A^n; r_p | C r E^n Z^m) - n H(A | UV E), \tag{32f}
\]

where

- (32a) is due to \((s_1 s_2'')\) being a deterministic function of \((s_2' A^n C)\);
- (32b) stems from the Markov chain \((A^n E^n) \Rightarrow (C s_1 s_2'' E) \Rightarrow Z^m\); and,
- (32c) follows from \((U^n V^n)\) being a deterministic function of \((s_1 s_2' s_2'' C)\) and the Markov chain \((A^n E^n) \Rightarrow (U^n V^n) \Rightarrow (s_1 s_2' s_2'' C)\).

In the last step, we split up the equivocation into two terms as in [18].

The “source” term \(E_s\) writes:

\[
E_s = H(s_2' A^n | C r_1 r_2 E^n) \tag{33a}
\]

\[
= H(s_2' A^n | C r_1 E^n) - H(r_2 | C r_1 E^n) \tag{33b}
\]

\[
= H(A^n | C r_1 E^n) + H(s_2') - H(r_2 | C) + I(r_2; r_1 E^n | C) \tag{33c}
\]

\[
\geq H(A^n | U^n E^n) + n S_2' - n R_2 + I(r_2; E^n | C r_1) \tag{33d}
\]

\[
\geq n H(A | U E) - n \varepsilon^{(1)} + n S_2' - n R_2 + I(r_2; E^n | C r_1), \tag{33e}
\]

where

- (33a) follows from the Markov chain \((s_2' A^n E^n) \Rightarrow (C r_2 r_p) \Rightarrow Z^m\) and the identity \((r_c, r_p) = M(r_1, r_2)\) where \(M\) is a one-to-one mapping;
- (33b) is due to the fact that the bin index \(r_2\) is a deterministic function of \((s_2' A^n C)\);
- (33c) is due to \(s_2'\) being chosen independently of \((r_1 A^n E^n C)\);
- (33d) follows from the Markov chain \((A^n E^n) \Rightarrow U^n \Rightarrow (r_1 C)\), the fact that the index \(s_2'\) is chosen uniformly at random in \([1 : 2^n S_2']\), \(r_2 \in [1 : 2^n R_2]\), and the non-negativity of mutual information; and,
(33e) is lower bounded similarly to Lemma 2 in Section VI-D.

The “channel” term $E_c$ writes:

$$ E_c = H(r_p|C_rE^nZ^m) $$

$$ = H(r_p|C_rZ^m) - I(r_p; E^n|C_rZ^m), \tag{34b} $$

where the first step follows from the fact that $r_p$ is a deterministic function of $(s_2 A^n C)$.

The first term of the r.h.s. of (34b) corresponds to the equivocation (of the private message, given the common message and the output of the channel) in the wiretap channel setting.

Following the arguments of [3, Section IV] and [31, Section 2.3], together with constraint (22), we can easily prove the following lower bound:

$$ H(r_p|C_rZ^m) \geq n(R_p + R_f) - mI(T; Z|Q) - 1 - n\varepsilon^{(2)}, \tag{35} $$

for some sufficiently large $(n, m)$.

Gathering (32)–(35), we have that

$$ H(s_1 s_2 s''_2|C E^n Z^m) \geq nH(A|UE) - n\varepsilon^{(1)} + nS_2 - nR_2 + I(r_2; E^n|C_r) + n(R_p + R_f) $$

$$ - mI(T; Z|Q) - 1 - n\varepsilon^{(2)} - I(r_p; E^n|C_rZ^m) - nH(A|UV E). \tag{36} $$

We now study the multi-letter terms of (36):

$$ I(r_2; E^n|C_r) - I(r_p; E^n|C_rZ^m) $$

$$ = I(r_1 r_2; E^n|C) - I(r_1; E^n|C_r) - I(r_c r_p Z^m; E^n|C) + I(r_c Z^m; E^n|C) \tag{37a} $$

$$ = -I(Z^m; E^n|C_r r_2) - I(r_1; E^n|C) + I(r_c; E^n|C) + I(Z^m; E^n|C_r) \tag{37b} $$

$$ \geq -I(r_1; E^n|C) + I(r_c; E^n|C) \tag{37c} $$

$$ \geq 0, \tag{37d} $$

where

- (37b) follows from the identity $(r_c, r_p) = M(r_1, r_2)$ where $M$ is a one-to-one mapping;
- (37c) is due to the non-negativity of mutual information and the Markov chain $Z^m \rightarrow (r_1 r_2) \rightarrow E^n$; and,
- (37d) stems from the fact that $r_1 = M'(r_c)$ for some mapping $M'$.

Inequality (36) then yields

$$ \frac{1}{n} H(s_1 s_2 s''_2|C E^n Z^m) \geq I(V; A|UE) + S_2 - R_2 + R_p + R_f - \frac{m}{n} I(T; Z|Q) - \varepsilon, \tag{38} $$
for sufficiently large \((n, m)\).

Since the key is a deterministic function of the indices \(s_1, s'_2, \) and \(s''_2\), it will be thus concealed from the eavesdropper if

\[
R_k \leq I(V; A|UE) + S'_2 - R_2 + R_p + R_f - (\eta + \epsilon)I(T; Z|Q) - \epsilon.
\]

**F. Sufficient Conditions**

Putting all pieces together, we have proved that the proposed scheme allows the legitimate users to agree upon a key of rate \(R_k\), while keeping it secret from the eavesdropper if

\[
\begin{align*}
R_1 + R_2 &= R_c + R_p, \\
R_1 &\leq R_c, \\
R_1 &\leq S_1, \\
R_2 &\leq S'_2 + S''_2, \\
S_1 &> I(U; A) + \epsilon_1, \\
S''_2 &> I(V; A|U) + \epsilon_2, \\
R_c + R_p + R_f &< (\eta + \epsilon)I(T; Y) - \delta, \\
R_p + R_f &< (\eta + \epsilon)I(T; Y|Q) - \delta, \\
S_1 - R_1 &< I(U; B) - \delta, \\
S'_2 + S''_2 - R_2 &< I(V; B|U) - \delta, \\
R_k &\leq S_1 + S'_2 + S''_2, \\
R_f &< (\eta + \epsilon)I(T; Z|Q) - \epsilon_f, \\
R_k - R_p - R_f &\leq I(V; A|UE) + S'_2 - R_2 - (\eta + \epsilon)I(T; Z|Q) - \epsilon.
\end{align*}
\]

After applying Fourier-Motzkin elimination to this set, taking \((n, m) \to \infty\), and discarding the redundant conditions, we obtain the reduced set (5)–(6).

We have shown thus far that, averaged over all possible codebooks, the probability of error and key leakage become negligible as \((n, m) \to \infty\) if conditions (5)–(6) hold true. Nonetheless, by applying the selection lemma [32, Lemma 2.2], we may conclude that there exists a specific sequence of codebooks such that the probability of error and key leakage tend to zero as \((n, m) \to \infty\).
VI. PROOF OF THEOREM 3

This section provides a sketch of the proof of Theorem 3.

A. Codebook Generation

Fix the following joint probability distribution:

\[ p(uvabexyz) = p(u|v)p(v|x)p(a|e)p(yz|x). \]

Then, proceed as follows:

1) Randomly pick \( 2^n R_1 \) sequences \( u^n(r_1) \) from \( T^n_\delta(U) \);
2) For each codeword \( u^n(r_1) \), randomly pick \( 2^{n(R_2+R_f)} \) sequences \( v^n(r_1, r_2, r_f) \) from \( T^n_\delta(V|u^n(r_1)) \);
3) Randomly distribute the \( 2^{n(R_2+R_f)} \) sequences \( v^n(r_1, r_2, r_f) \) in \( 2^{nR_k} \) equal-size bins \( B(k) \) with \( k \in [1 : 2^{nR_k}] \), each bin containing \( 2^{n[I(V;EZ|U) - \tilde{\epsilon}]} \) codewords. Therefore,

\[ R_2 + R_f = R_k + I(V;EZ|U) - \tilde{\epsilon}. \]  

The reader can be referred to Fig. 6 for a graphical representation of these steps.

B. Encoding

Given the observed sequence \( a^n \), and the index \( r_f \) chosen uniformly at random in \([1 : 2^{nR_f}]\), the encoder proceeds as follows:

1) It looks for an index \( r_1 \equiv \hat{r}_1 \) such that \( (u^n(\hat{r}_1), a^n) \in T^n_\delta(UA) \). If more than one index is found, choose one uniformly at random, whereas if no such index is found, choose
one uniformly at random in $[1 : 2^{nR_1}]$. The probability of not finding such an index is arbitrarily small as $n \to \infty$ if $\delta' < \epsilon_1$ and
\[ R_1 > I(U; A) + \epsilon_1. \]  
(40)

2) Then, it looks for an index $r_2 \equiv \hat{r}_2$ such that $(v^n(r_1, \hat{r}_2, r_f), a^n) \in T^n_{\delta'}(VA|u^n(r_1))$. If more than one index is found, choose one uniformly at random, whereas if no such index is found, choose one uniformly at random in $[1 : 2^{nR_2}]$. The probability of not finding such an index is arbitrarily small as $n \to \infty$ if $\delta' < \epsilon_2$ and
\[ R_2 > I(V; A|U) + \epsilon_2. \]  
(41)

3) The bin index of $v^n(r_1, r_2, r_f)$ provides the key $k$, i.e., $v^n(r_1, r_2, r_f) \in B(k)$.

4) The encoder then transmits the associated jointly typical sequence to $v^n(r_1, r_2, r_f)$ and $a^n$, i.e., $x^n \sim \prod_{i=1}^n p(x_i|v_i(r_1, r_2, r_f), a_i)$.

C. Decoding

Given the observed sequence $b^n$ and the channel output $y^n$, the decoder proceeds as follows:

1) It looks for the unique set of indices $(r_1, r_2, r_f) \equiv (\hat{r}_1, \hat{r}_2, \hat{r}_f)$ such that
\[ (u^n(\hat{r}_1), v^n(\hat{r}_1, \hat{r}_2, \hat{r}_f), b^n, y^n) \in T^n_{\delta'}(UVBY). \]

The probability of error in decoding can be made arbitrarily small as $n \to \infty$ provided that
\[ R_1 + R_2 + R_f < I(UV; BY) - \epsilon', \]  
(42a)
\[ R_2 + R_f < I(V; BY|U) - \epsilon'. \]  
(42b)

This result does not arise from the conventional random coding technique commonly used in the proofs of achievabilities, as noted e.g. in [33], since the transmitted indices are not independent of the codebook. See Appendix C for further details.

2) Compute the key $k \equiv \hat{k}$, i.e., $v^n(r_1, r_2, r_f) \in B(\hat{k})$. 





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D. Key Leakage

Consider,

\[ I(K; E^n Z^n | C) \leq I(K; U^n E^n Z^n | C) \]

\[ \leq I(K; E^n Z^n | C U^n) \]

\[ = H(K | C U^n) + H(E^n Z^n | C U^n) - H(K E^n Z^n | C U^n) \]

\[ \leq n[R_k + H(EZ | U)] - H(K E^n Z^n | C U^n) \]

\[ = n[R_k + H(EZ | U)] - H(K V^n E^n Z^n | C U^n) + H(V^n | C K U^n E^n Z^n) \]

\[ = n[R_k + H(EZ | U)] - H(V^n E^n Z^n | C U^n) + n\varepsilon^{(1)} \]

\[ = n[R_k + H(EZ | U) + \varepsilon^{(1)}] - H(V^n | C U^n) - H(E^n Z^n | C U^n V^n) \]

\[ \leq n[R_k + H(EZ | U) + \varepsilon^{(1)} - R_2 - R_f + \varepsilon^{(2)} - H(EZ | U V) + \varepsilon^{(3)}] \]

\[ = n[R_k + I(V; EZ | U) - (R_2 + R_f) + \varepsilon^{(4)}] \]

\[ = n[\hat{\varepsilon} + \varepsilon^{(4)}], \]

where

- (43b) is due to \( K \) being independent of the sequence \( U^n; \)
- (43d) stems from \( H(K | C U^n) \leq H(K) \leq n R_k \) and
  \[ H(E^n Z^n | C U^n) \leq \sum_{i=1}^{n} H(E_i Z_i | U_i) \leq n H(EZ | U); \]
- (43f) follows from Fano’s inequality and (39), and from \( K \) being a function of \( V^n \) and \( C; \)
- (43h) is due to Lemmas 1 and 2 (see below); and,
- (43j) is due to (39) and \( \varepsilon^{(4)} = \varepsilon^{(1)} + \varepsilon^{(2)} + \varepsilon^{(3)}. \)

Therefore, the eavesdropper cannot obtain the key. We present now the two lemmas used in step (43h).

Lemma 1: Given the encoding procedure of Section VI-B, if \( R_2 < H(A) - 4\delta \), then

\[ \liminf_{n \to \infty} H(V^n | C U^n) \geq n(R_2 + R_f - \varepsilon). \]

Proof: See Appendix D.

Lemma 2: Given the encoding procedure of Section VI-B,

\[ H(E^n Z^n | C U^n V^n) \geq n[H(EZ | UV) - \varepsilon]. \]
**Proof:** Consider the Markov chain $\mathcal{C} \rightarrow (U^n V^n) \rightarrow (E^n Z^n)$, then

$$H(E^n Z^n | CU^n V^n) = H(E^n Z^n | U^n V^n)$$

(47a)

$$= - \sum_{(u^n v^n e^n z^n) \in T^n (UVEZ)} p(u^n v^n e^n z^n) \log p(u^n v^n e^n z^n)$$

(47b)

$$\geq - \sum_{(u^n v^n e^n z^n) \in T^n (UVEZ)} p(u^n v^n e^n z^n) \log p(u^n v^n e^n z^n)$$

(47c)

$$\geq \sum_{(u^n v^n e^n z^n) \in T^n (UVEZ)} p(u^n v^n e^n z^n) n[H(EZ | UV) - \varepsilon (1)]$$

(47d)

$$\geq (1 - \varepsilon (2)) n[H(EZ | UV) - \varepsilon (1)]$$

(47e)

$$\geq n[H(EZ | UV) - \varepsilon (3)],$$

(47f)

where in the last step we choose $\varepsilon (3)$ large enough to have a lower bound.

E. **Sufficient Conditions**

The scheme described thus far allows the legitimate users to generate a shared key of rate $R_k$ that is concealed from the eavesdropper if

$$R_1 > I(U; A) + \epsilon_1,$$

$$R_2 > I(V; A | U) + \epsilon_2,$$

$$R_f > 0,$$

$$R_1 + R_2 + R_f < I(UV; BY) - \epsilon',$$

$$R_2 + R_f < I(V; BY | U) - \epsilon',$$

$$R_2 < H(A) - \delta',$$

$$R_2 + R_f = R_k + I(V; EZ | U) - \hat{\epsilon}.$$

After applying Fourier-Motzkin elimination to this set, taking $n \to \infty$, and manipulating the expressions, we obtain the reduced set (7)–(8) plus

$$I(V; A | U) \leq H(A),$$

(48)

which always holds for discrete distributions.

We have shown thus far that, averaged over all possible codebooks, the probability of error and key leakage become negligible as $n \to \infty$ if conditions (7)–(8) hold true. Nonetheless, by applying the selection lemma [32, Lemma 2.2], we may conclude that there exists a specific
sequence of codebooks such that the probability of error and key leakage tend to zero as $n \to \infty$.

The bound on the cardinality of the alphabets $\mathcal{U}$ and $\mathcal{V}$ follow from Fenchel–Eggleston–Carathéodory’s theorem and the standard cardinality bounding technique [28, Appendix C], therefore their proof is omitted. This concludes the proof of Theorem 3.

VII. SUMMARY AND DISCUSSION

We investigated the problem of secret key generation over noisy channels in presence of sources of common randomness at all terminals. For the case of sources independent of the main channel, a novel coding scheme using separate source and channel components was introduced which shares common roots with our previous works [18], [27]. This scheme improves upon the existing works in the literature by introducing the use of two layers for key generation based on the source observed at the encoder. The corresponding achievable secret key rate was then shown to be optimal for all classes of less noisy sources and channels (cf. Propositions 1, 2 and 3).

We then provided an achievable scheme based on joint source-channel coding that addresses the scenarios in which the source components directly affect the channel statistic. This problem is much more involved and a major shortcoming in this case is the lack of a good outer bound that combines secrecy with state-dependent channels. Consequently, an optimal characterization of the secret key capacity in this case remained open. Nonetheless, specific binary and Gaussian models where analyzed and the scheme is shown to be optimal for some special regimes of channel and source parameters.

Both achievable schemes seem of interest in the different areas of their application. The inner bound separate outperforms the inner bound joint when the sources are not correlated with the main channel. However, its equivocation analysis is not longer valid when the sources affect the communication channel. In this case, the inner bound joint becomes useful.
APPENDIX A
PROOF OF THEOREM 1 (OUTER BOUND)

The outer bound is derived by following similar steps to those in [27, Thm. 4], which assumed \( \eta = 1 \). It is reproduced here for completeness.

Let \((\eta, R_k)\) be an achievable tuple according to Definition 2, and \( \epsilon > 0 \). Then, there exists an \((n, m)\)-code \((\varphi, \psi_a, \psi_b)\) such that

\[
X^m = \varphi(A^n), \tag{49a}
\]
\[
K = \psi_a(A^n), \tag{49b}
\]
\[
\hat{K} = \psi_b(B^n, Y^m), \tag{49c}
\]

that verify

\[
\frac{m}{n} \leq \eta + \epsilon, \tag{50a}
\]
\[
\Pr\{ K \neq \hat{K} \} \leq \epsilon, \tag{50b}
\]
\[
I(K; E^n Z^m) \leq n\epsilon. \tag{50c}
\]

Before continuing, we present the following remark that is useful to establish Markov chains between the random variables.

**Remark 13:** From the fact that random variables \( A_i, B_i, E_i \) are independent across time and the channel \( X \mapsto (Y, Z) \) is memoryless, the joint distribution of \((K, A^n, B^n, E^n, X^m, Y^m, Z^m)\) can be written as follows. For each \( i \in \{1, \ldots, n\} \) and each \( j \in \{1, \ldots, m\} \), we have

\[
p(k, a^n, b^n, e^n, x^m, y^m, z^m) = p(a^{i-1}, b^{i-1}, e^{i-1}) p(a_i, b_i, e_i) p(a_{i+1}^n, b_{i+1}^n, e_{i+1}^n) \times p(k, x^m|a^n) p(y_{j-1}^{j-1}, z_{j-1}^{j-1}|x_{j-1}) p(y_{j+1}^m, z_{j+1}^m|x_{j+1}^m), \tag{51}
\]

where \( P_{\varphi}(x^m|a^n) = \sum_{k} p(k, x^m|a^n) \) and \( P_{\psi_a}(k|a^n) = \sum_{x^m} p(k, x^m|a^n) \).

We may now carry on with the derivation of the outer bound. First consider,

\[
nR_k = H(K) \tag{52a}
\]
\[
= H(K|E^n Y^m) + I(K; E^n Y^m) \tag{52b}
\]
\[
\leq H(K|E^n Y^m) + I(K; E^n Y^m) - I(K; E^n Z^m) + n\epsilon \tag{52c}
\]
\[
= H(K|E^n Y^m) + I(K; Y^m|E^n) - I(K; Z^m|E^n) + n\epsilon \tag{52d}
\]

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\[
\begin{align*}
\leq H(K|E^nY^m) - H(K|B^nY^m) + I(K; Y^m|E^n) - I(K; Z^m|E^n) + 2n\epsilon \\
= \underbrace{I(K; B^n|Y^m)}_{R_s} - \underbrace{I(K; E^n|Y^m)}_{R_c} + I(K; Y^m|E^n) - I(K; Z^m|E^n) + 2n\epsilon,
\end{align*}
\]

where

- (52c) is due to the security condition (50c); and,
- (52e) follows from (49), (50b), and Fano’s inequality, \(H(K|B^nY^m) \leq n\epsilon\).

We now study separately the “source” term \(R_s\) and the “channel” term \(R_c\). Hence,

\[
R_s = \sum_{i=1}^{n} I(K; B_i|Y^m B^{i-1}) - I(K; E_i|Y^m E_{i+1}^n)
\]

\[
= \sum_{i=1}^{n} I(V_i; B_i|U_i) - I(V_i; E_i|U_i)
\]

\[
= n[I(V_j; B_j|U_j J) - I(V_j; E_j|U_j J)]
\]

\[
= n[I(V; B|U) - I(V; E|U)],
\]

where

- (53b) is due to Csiszár sum identity;
- (53c) follows from the definition of the auxiliary RVs \(U_i = (Y^m B^{i-1} E_{i+1}^n)\) and \(V_i = (KU_i)\);
- (53d) introduces the auxiliary RV \(J\) uniformly distributed over \([1 : n]\) and independent of all the other variables; and,
- (53e) stems from the definition of random variables \(U = (U_j J), V = (V_j J), B = B_j,\) and \(E = E_j\).

This establishes the “source” term in (52f) with auxiliary RVs \((U, V)\) that satisfy the following Markov chain

\[
U_i \rightarrow V_i \rightarrow A_i \rightarrow (B_i E_i).
\]

The first part of (54) is trivial given the definition \(V_i = (KU_i)\), whereas the second part follows from the i.i.d. nature of the sources and that they are correlated to the main channel only through the encoder’s input (49a), see (51),

\[
(KY^m B^{i-1} E_{i+1}^n) \rightarrow A_i \rightarrow (B_i E_i).
\]
The “channel” term $R_c$ can be single-letterized similarly, 

$$R_c = \sum_{i=1}^{m} I(K; Y_i | E^n Y^{i-1}) - I(K; Z_i | E^n Z^{m}_{i+1})$$  \hspace{1cm} (56a)$$

$$= \sum_{i=1}^{m} I(K; Y_i | E^n Y^{i-1} Z^{m}_{i+1}) - I(K; Z_i | E^n Y^{i-1} Z^{m}_{i+1})$$  \hspace{1cm} (56b)$$

$$= \sum_{i=1}^{m} I(T_i; Y_i | Q_i) - I(T_i; Z_i | Q_i)$$  \hspace{1cm} (56c)$$

$$= m[I(T_i; Y_i | Q_i) - I(T_i; Z_i | Q_i)]$$  \hspace{1cm} (56d)$$

$$= m[I(T; Y | Q) - I(T; Z | Q)],$$  \hspace{1cm} (56e)$$

where

- (56b) is due to Csiszár sum identity;
- (56c) follows from the definition of the auxiliary RVs $Q_i = (E^n Y^{i-1} Z^{m}_{i+1})$ and $T_i = (K Q_i)$;
- (56d) introduces the auxiliary RV $L$ uniformly distributed over $[1 : m]$ and independent of all the other variables; and,
- (56e) stems from the definition of random variables $Q = (Q_L L)$, $T = (T_L L)$, $Y = Y_L$, and $Z = Z_L$.

The auxiliary RVs in this term, i.e., $(QT)$, satisfy the following Markov chain

$$Q_i \rightarrow T_i \rightarrow X_i \rightarrow (Y_i Z_i),$$  \hspace{1cm} (57)$$

where the nontrivial part is due to the memorylessness property of the channel and (49b), provided the joint probability distribution satisfies (51). Since neither $Q$ nor $T$ appear on other parts of the outer bound, we may expand $R_c$ as

$$R_c = m \sum_{q \in Q} p_Q(q) [I(T; Y | Q = q) - I(T; Z | Q = q)]$$  \hspace{1cm} (58a)$$

$$\leq m \max_{q \in Q} [I(T; Y | Q = q) - I(T; Z | Q = q)]$$  \hspace{1cm} (58b)$$

$$= m [I(T^*; Y) - I(T^*; Z)],$$  \hspace{1cm} (58c)$$

where in the last step we choose auxiliary RV $T^* \sim p_{T|Q}(-|q)$.

Gathering (52), (53), (56), and (58), the rate of the secret key writes

$$R_k \leq I(V; B|U) - I(V; E|U) + \frac{m}{n} [I(T; Y) - I(T; Z)] + 2\epsilon.$$  \hspace{1cm} (59)$$

If we let $(n, m) \to \infty$ and take arbitrarily small $\epsilon$, we obtain the bound (2).
In order to obtain (3), we use the following Markov chain that is a consequence of (49a), provided the joint probability satisfies (51):

\[(B^n E^n) \rightarrow A^n \rightarrow X^m \rightarrow (Y^m Z^m).\]  

(60)

Due to the data processing inequality, we have

\[I(A^n; Y^m) \leq I(X^m; Y^m) \leq m I(X; Y),\]  

(61)

where in the last inequality we use the memoryless property of the channel. Next, consider

\[I(A^n; Y^m) = I(A^n B^n; Y^m) \]  

(62a)

\[\geq I(A^n; Y^m | B^n) \]  

(62b)

\[= I(A^n; KY^m | B^n) - I(A^n; K | B^n Y^m) \]  

(62c)

\[\geq I(A^n; KY^m | B^n) - n \epsilon \]  

(62d)

\[\geq n[I(A; V | B) - \epsilon], \]  

(62e)

where

- (62a) follows from the Markov chain (60); and,
- (62d) stems from \(H(K | B^n Y^m) \leq n \epsilon\) due to (49) and (50b), and \(H(K | A^n B^n Y^m) \geq 0\).

For the last step, i.e., (62e), consider

\[I(KY^m; A^n | B^n) = I(KY^m; A^n E^n | B^n) \]  

(63a)

\[= \sum_{i=1}^{n} I(KY^m; A_i E_i | B^n A_{i+1} E_{i+1}^n) \]  

(63b)

\[\geq \sum_{i=1}^{n} I(KY^m B^{i-1} E_i^n; A_i E_i | B_i) \]  

(63c)

\[= \sum_{i=1}^{n} I(V_i; A_i | B_i) \]  

(63d)

\[\geq \sum_{i=1}^{n} I(V_i; A_i | B_i) \]  

(63e)

\[= n I(V_J; A_J | B_J J) \]  

(63f)

\[= n I(V_J; A_J | B_J) \]  

(63g)

\[= n I(V; A | B), \]  

(63h)

where
• (63a) stems from the Markov chain \((B^n E^n) \rightarrow A^n \rightarrow (KY^m)\);

• (63c) follows from the sources being i.i.d., i.e., \((A_i E_i) \rightarrow B_i \rightarrow (B^{i-1} B_{i+1} A_{i+1} E_{i+1})\);

• (63d) is due to the auxiliary RV \(V_i = (KY^m B^{i-1} E_{i+1})\);

• (63f) introduces the auxiliary RV \(J\) uniformly distributed over \([1 : n]\) and independent of all the other variables;

• (63g) follows from the independence of \(J\) and \((A J B J)\); and,

• (63h) stems from the definition of random variables \(V = (V J), B = B J,\) and \(A = A J.\)

Putting (61) and (62) together, we obtain:

\[ I(A; V | B) \leq \frac{m}{n} I(X; Y) + \epsilon, \]

which gives the condition (3) as we let \((n, m) \to \infty\) and take an arbitrarily small \(\epsilon.\)

Although the definition of the auxiliary RVs \((TUV)\) used in the proof makes them arbitrarily correlated, the bounds (2) and (3) only depend on the marginal PDs \(p(tx)\) and \(p(uv|a)\). Consequently, we can restrict the set of possible joint PDs to (4), i.e., independent source and channel variables, and still achieve the maximum.

The bound on the cardinality of the alphabets \(T, U,\) and \(V\) follow from Fenchel–Eggleston–Carathéodory’s theorem and the standard cardinality bounding technique [28, Appendix C], therefore their proof is omitted. This concludes the proof of Theorem 1.

**APPENDIX B**

**PROOF OF PROPOSITION 4**

The main channel in the system model depicted in Fig. 2a is degraded, thus the first two terms on the r.h.s. of (2) may be upper-bounded as follows

\[ I(T; Y) - I(T; Z) \leq I(T; Y|Z) \leq I(X; Y|Z) = H(X|Z). \]  

Since \(X\) is the input of a BSC of parameter \(\zeta\) and output \(Z,\) it is clear that

\[ H(X|Z) \leq h_2(\zeta), \]

with equality if \(X \sim B \left(\frac{1}{2}\right).\)

The last two terms on the r.h.s. of (2) are upper-bounded by \(V = A,\) thus

\[ I(V; B|U) - I(V; E|U) \leq H(A|UE) - H(A|UB). \]
We bound these two conditional entropies with the aid of parameter \( v = h_2^{-1}(H(A|U)) \). First, consider

\[
H(A|UE) = H(A|U) - I(E; A|U) \\
= H(A|U) - H(E|U) + H(E|A) \\
\leq h_2(v) - h_2(v \epsilon) + h_2(\epsilon),
\]

where the last step arises from the use of Mrs. Gerber’s lemma

\[
H(E|U) \geq H(H^{-1}(H(A|U)) \epsilon) = h_2(v \epsilon)
\]

since \( E \) is the output of a BSC with parameter \( \epsilon \) and input \( A \). Second, consider

\[
H(A|UB) = \beta H(A|U) = \beta h_2(v).
\]

Combining (66), (68), and (70), we obtain the bound (12). On the other hand, the restriction (3) may be written as \( \beta \leq 1 \), which always holds. This concludes the proof of Proposition 4.

\[\blacksquare\]

\section*{Appendix C}

\section*{Error Probability of Joint Decoding}

In the conventional random coding proof technique, it is assumed that the transmitted message is independent of the chosen codebook. Therefore, the probability of error in the decoding is obtained by averaging over all possible codebook realizations. However, in the proposed joint coding scheme, the transmitted indices depend on the actual codebook. In the following, we upper bound the error probability of the joint decoding by following similar arguments to those in [18], [33], [34].

\subsection*{A. Upper Bound on the Probability of Error}

By the symmetry of the random codebook generation and encoding, we may consider the probability of error conditioned on the true indices \( M = \{M_1 = 1, M_2 = 1, M_f = 1\} \). Let us then define the events:

\[
E_1 = \{(U^n(1), V^n(1, m_2, m_f), B^n, Y^n) \in T^n(UBY) \text{ for some } (m_2, m_f) \neq (1, 1)\}, \\
E_2 = \{(U^n(m_1), V^n(m_1, m_2, m_f), B^n, Y^n) \in T^n(UBY) \text{ for some } m_1 \neq 1, m_2, m_f\}.
\]
Thus, the probability of error in decoding can be upper bounded as
\[
\Pr\{\mathcal{E} \mid \mathcal{M}\} \leq \Pr\{\mathcal{E}_1 \mid \mathcal{M}\} + \Pr\{\mathcal{E}_2 \mid \mathcal{M}\}. \tag{71}
\]
We analyze these two terms in the sequel, but let us first introduce the following two lemmas.

**Lemma 3**: Let \(U^n\) and \(V^n\) be drawn according to the codebook generation procedure, and the indices be chosen according to the encoding procedure. Additionally, let \(A = \{U^n(1) = u^n, V^n(1, 1, 1) = \hat{v}^n, A^n = a^n\}\) and \((m_2, m_f) \neq (1, 1)\). Then, there exists \(\epsilon > 0\) such that
\[
\Pr\{V^n(1, m_2, m_f) = v^n \mid \mathcal{M}, A\} \leq (1 + \epsilon)p(v^n | u^n) \tag{72}
\]
for sufficiently large \(n\).

**Proof**: This lemma can be proved using similar steps as [34, Lemma 1]. See Appendix C-B below for details. ■

**Lemma 4**: Let us assume again that the variables are chosen according to the codebook generation and encoding procedure. Let \(A\) be defined as in the previous lemma, and let \(m_1 \neq 1\) for some \((m_2, m_f)\). Then, there exists \(\epsilon > 0\) such that
\[
\Pr\{U^n(m_1) = u^n, V^n(m_1, m_2, m_f) = v^n \mid \mathcal{M}, A\} \leq (1 + \epsilon)p(u^n v^n) \tag{73}
\]
for some sufficiently large \(n\).

**Proof**: See Appendix C-C below for details. ■

We now resume the analysis of (71). First, by the union bound we have
\[
\Pr\{\mathcal{E}_1 \mid \mathcal{M}\} \leq \sum_{m_2=1}^{2^{nR_2}} \sum_{m_f=m_f^*}^{2^{nR_f}} P_A(m_2, m_f, u^n, v^n, b^n, y^n), \tag{74}
\]
where
\[
m_f^* = \begin{cases} 
1 & \text{if } m_2 \neq 1 \\
2 & \text{if } m_2 = 1,
\end{cases} \tag{75}
\]
\[
P_A(m_2, m_f, u^n, v^n, b^n, y^n) = \Pr\{U^n(1) = u^n, V^n(1, m_2, m_f) = v^n, B^n = b^n, Y^n = y^n \mid \mathcal{M}\}. \tag{76}
\]
In order to analyze (74) we introduce \(\hat{V}^n = (V^n(1, 1, 1), A^n)\) and \(\hat{v}^n = (v^n, a^n)\) for succinctness. Then, for given \(m_2, m_f, u^n, v^n, b^n, \) and \(y^n\), we can write
\[
P_A(\cdot) = \sum_{\hat{V}^n} \Pr\{U^n(1) = u^n, V^n(1, m_2, m_f) = v^n, B^n = b^n, Y^n = y^n, \hat{V}^n = \hat{v}^n \mid \mathcal{M}\} \tag{77a}
\]
\[
\begin{align*}
&= \sum_{\forall \tilde{v}^n} \Pr\left\{ U^n(1) = u^n, B^n = b^n, Y^n = y^n, \tilde{V}^n = \tilde{v}^n \mid \mathcal{M} \right\} \\
&\quad \cdot \Pr\left\{ V^n(1, m_2, m_f) = v^n \mid \mathcal{M}, U^n(1) = u^n, \tilde{V}^n = \tilde{v}^n \right\} \\
&\leq \sum_{\forall \tilde{v}^n} \Pr\left\{ U^n(1) = u^n, B^n = b^n, Y^n = y^n, \tilde{V}^n = \tilde{v}^n \mid \mathcal{M} \right\} (1 + \epsilon)p(v^n | u^n) \\
&= \Pr\left\{ U^n(1) = u^n, B^n = b^n, Y^n = y^n \mid \mathcal{M} \right\} (1 + \epsilon)p(v^n | u^n),
\end{align*}
\]

where

- (77b) follows since \( V^n(1, m_2, m_f) \sim (U^n(1), V^n(1, 1, 1), A^n) \sim (B^n, Y^n) \) is a Markov chain given the true indices \( \mathcal{M} \); and,

- (77c) is due to Lemma 3.

Then,

\[
\Pr\{E_1 \mid \mathcal{M}\} \leq \sum_{m_2=1}^{2^n R_2} \sum_{m_f=m_f^*}^{2^n R_f} \sum_{(u^n, v^n, b^n, y^n) \in T^n (UVBY)} (1 + \epsilon)p(v^n | u^n) \\
\quad \cdot \Pr\left\{ U^n(1) = u^n, B^n = b^n, Y^n = y^n \mid \mathcal{M} \right\} \\
\leq (1 + \epsilon)2^{n(R_2 + R_f + H(V \mid UBY) - H(V \mid U) + \epsilon')} \\
\quad \cdot \Pr\left\{ U^n(1) = u^n, B^n = b^n, Y^n = y^n \mid \mathcal{M} \right\} \\
\leq (1 + \epsilon)2^{n(R_2 + R_f - I(V; BY \mid U) + \epsilon')}.
\]

Therefore, \( \Pr\{E_1 \mid \mathcal{M}\} \) tends to zero as \( n \to \infty \) if

\[
R_2 + R_f < I(V; BY \mid U) - \epsilon'.
\]

We may derive a similar bound for the second term in (71)

\[
\Pr\{E_2 \mid \mathcal{M}\} \leq \sum_{m_2=2}^{2^n R_2} \sum_{m_f=m_f^*}^{2^n R_f} \sum_{(u^n, v^n, b^n, y^n) \in T^n (UVBY)} P_B,
\]

where

\[
P_B = \Pr\{ U^n(m_2) = u^n, V^n(m_2, m_f) = v^n, B^n = b^n, Y^n = y^n \mid \mathcal{M} \}.
\]
where

\begin{align}
\text{(82a)} & \quad \text{follows since } (U_n(m_1), V_n(m_1, m_2, m_f)) \rightarrow (U_1, V_1, 1, 1, A) \rightarrow (B^n, Y^n)
\text{is a Markov chain given the true indices } M; \text{ and,} \\
\text{(82b)} & \quad \text{is due to Lemma 4.}
\end{align}

Therefore,

\[ \Pr\{E_2 \mid M\} \leq (1 + \epsilon) 2^{nR_1 + nR_2 + nR_f - I(UV; BY) - H(UV) + \epsilon'} \]

Therefore, \( \Pr\{E_2 \mid M\} \) tends to zero as \( n \to \infty \) if

\[ R_1 + R_2 + R_f < I(UV; BY) - \epsilon'. \] (84)

Therefore, the probability of error in decoding (71) can be made arbitrarily small as \( n \to \infty \) provided that (79) and (84) hold.

\[ \] B. Proof of Lemma 3

Let \( \mathcal{C}' = \{ V^n(m_2', 1) : m_2' \in [1 : 2^{R_2}] \} \setminus \{ V^n(1, 1, 1), V^n(1, m_2, 1) \} \), i.e., the whole conditional codebook on \( U^n(1) \) without the two codewords of interest and knowing that
\( M_f = 1 \), and let \( \mathcal{A} = \{ U^n(1) = u^n, V^n(1, 1, 1) = \tilde{v}^n, \mathcal{A}^n = \alpha^n \} \) for succinctness. Then, by the law of total probability and Bayes’ theorem we have that

\[
\Pr\{ V^n(1, m_2, m_f) = v^n | \mathcal{M}, \mathcal{A} \} = \sum_{\forall c'} \Pr\{ V^n(1, m_2, m_f) = v^n, c' | \mathcal{M}, \mathcal{A} \} \tag{85a}
\]

\[
= \sum_{\forall c'} \Pr\{ c' | \mathcal{M}, \mathcal{A} \} \Pr\{ V^n(1, m_2, m_f) = v^n | M_1 = 1, \mathcal{A}, c' = c' \} \frac{\Pr\{ M_2 = 1, M_f = 1 | M_1 = 1, \mathcal{A}, c' = c', V^n(1, m_2, m_f) = v^n \}}{\Pr\{ M_2 = 1, M_f = 1 | M_1 = 1, \mathcal{A}, c' = c' \}} \tag{85b}
\]

The second term in (85b) is equal to

\[
\Pr\{ V^n(1, m_2, m_f) = v^n | M_1 = 1, \mathcal{A}, c' = c' \} = p(v^n | u^n) \tag{86}
\]

since the codewords \( V^n(1, m_2, m_f) \) are drawn independently of each other and conditionally on \( U^n(1) \), and \( V^n(1, m_2, m_f) \not\in \mathcal{C}' \).

In order to bound the last term in (85b), we need to define some auxiliary functions. For each \((a^n, u^n, v^n, \tilde{v}^n, c')\) such that \( \Pr\{ c' | \mathcal{M}, \mathcal{A} \} > 0 \), let

\[
i(a^n, u^n, v^n) = \mathbb{1}\{ (u^n, a^n) \not\in T^n_{\alpha}(V A | u^n) \}, \tag{87a}
\]

\[
i(a^n, u^n, v^n, c') = \prod_{v^n \in c'} i(a^n, u^n, v^n), \tag{87b}
\]

\[
n(a^n, u^n, c') = \sum_{v^n \in c'} 1 - i(a^n, u^n, v^n), \tag{87c}
\]

where (87a) is the indicator function of the pair \((v^n, a^n)\) not being jointly typical given that both sequences are jointly typical with \( u^n \). Consequently, (87b) is the indicator function that no sequence in \( c' \) is jointly typical with \( a^n \), whereas (87c) counts the number of sequences in \( c' \) that are jointly typical with \( a^n \). Then,

\[
\Pr\{ M_2 = 1, M_f = 1 | M_1 = 1, \mathcal{A}, c' = c', V^n(1, m_2, m_f) = v^n \} \tag{88a}
\]

\[
= 2^{-nR_f} \left[ 2^{-nR_2} i(a^n, u^n, \tilde{v}^n) i(a^n, u^n, v^n) i(a^n, u^n, c') + [1 + n(a^n, u^n, c') + (1 - i(a^n, u^n, v^n))]^{-1}(1 - i(a^n, u^n, \tilde{v}^n)) \right] \tag{88b}
\]

\[
\leq 2^{-nR_f} \left[ 2^{-nR_2} i(a^n, u^n, \tilde{v}^n) i(a^n, u^n, c') + [1 + n(a^n, u^n, c')]^{-1}(1 - i(a^n, u^n, \tilde{v}^n)) \right], \tag{88c}
\]

where
Similarly, we can bound the following term with the introduction of the set of jointly typical indices \( \mathcal{I}(u^n) = \{ m_2' \in [1 : 2^{nR_2}] : (V^n(1, m_2'), A^n) \in T^u_{\delta'}(V A | u^n) \} \).

\[
\Pr\{ M_2 = 1, M_f = 1 \mid M_1 = 1, A, C' = c' \} \\
= \Pr\{M_f = 1\} \Pr\{M_2 = 1 \mid M_1 = 1, M_f = 1, A, C' = c' \} \\
\geq \Pr\{M_f = 1\} \Pr\{M_2 = 1 \mid M_1 = 1, M_f = 1, A, C' = c', m_2 \notin \mathcal{I}(u^n) \} \\
\cdot \Pr\{m_2 \notin \mathcal{I}(u^n) \mid M_1 = 1, M_f = 1, A, C' = c' \} \\
= 2^{-nR_f} [2^{-nR_2} i(a^n, u^n, \tilde{v}^n) i(a^n, u^n, c') + [1 + n(a^n, u^n, c')]^{-1}(1 - i(a^n, u^n, \tilde{v}^n))]
\cdot \Pr\{m_2 \notin \mathcal{I}(u^n) \mid M_1 = 1, M_f = 1, A, C' = c' \},
\]

\[
\geq 2^{-nR_f} [2^{-nR_2} i(a^n, u^n, \tilde{v}^n) i(a^n, u^n, c') + [1 + n(a^n, u^n, c')]^{-1}(1 - i(a^n, u^n, \tilde{v}^n))]
\cdot \left(1 - 2^{-n[I(V; A|U) - \delta']}\right),
\]

where

- in (89b) we introduce the set \( \mathcal{I}(u^n) \);
- (89c) stems from the encoding procedure conditioned to \( m_2 \) not being a possible index; and,
- (89d) is due to the joint typicality lemma and the fact that \( V^n(1, m_2, 1) \notin C' \).

Therefore,

\[
\frac{\Pr\{ M_2 = 1, M_f = 1 \mid M_1 = 1, A, C' = c', V^n(1, m_2, m_f) = v^n \}}{\Pr\{ M_2 = 1, M_f = 1 \mid M_1 = 1, A, C' = c' \}} \leq \left(1 - 2^{-n[I(V; A|U) - \delta']}\right)^{-1} \\
\leq 1 + \epsilon,
\]

for sufficiently large \( n \). By joining (85), (86) and (90), the lemma is proved.
C. Proof of Lemma 4

Consider first,

\[
\Pr\{U^n(m_1) = u^n, V^n(m_1, m_2, m_f) = v^n \mid \mathcal{M}, \mathcal{A}\} \\
= \Pr\{U^n(m_1) = u^n \mid M_1 = 1, U^n(1) = u^n, A^n = a^n\} \\
\cdot \Pr\{V^n(m_1, m_2, m_f) = v^n \mid \mathcal{M}, \mathcal{A}, U^n(m_1) = u^n\} \tag{91a}
\]

\[
\leq (1 + \epsilon) p(u^n) \cdot \Pr\{V^n(m_1, m_2, m_f) = v^n \mid \mathcal{M}, \mathcal{A}, U^n(m_1) = u^n\}, \tag{91b}
\]

where

- (91a) is due to \(U^n(m_1) \rightarrow (M_1, U^n(1), A^n) \rightarrow (M_2, M_f, V^n(1, 1, 1))\) being a Markov chain; and,

- (91b) follows as a special case of Lemma 3.

Now, let \(C' = \{V^n(1, m_2', m_f') : m_2' \in [1 : 2^nR_2], m_f' \in [1 : 2^nR_f]\} \setminus \{V^n(1, 1, 1)\}\), and let \(\mathcal{A}' = \{\mathcal{A}, U^n(m_1) = u^n\}\) for succinctness. Then, by the law of total probability and Bayes’ theorem we have that

\[
\Pr\{V^n(m_1, m_2, m_f) = v^n \mid \mathcal{M}, \mathcal{A}'\} \\
= \sum_{c'} \Pr\{V^n(m_1, m_2, m_f) = v^n, C' = c' \mid \mathcal{M}, \mathcal{A}'\} \tag{92a}
\]

\[
= \sum_{c'} \Pr\{C' = c' \mid \mathcal{M}, \mathcal{A}'\} \cdot \Pr\{V^n(m_1, m_2, m_f) = v^n \mid M_1 = 1, \mathcal{A}', C' = c'\} \\
\cdot \frac{\Pr\{M_2 = 1, M_f = 1 \mid M_1 = 1, \mathcal{A}', C' = c', V^n(m_1, m_2, m_f) = v^n\}}{\Pr\{M_2 = 1, M_f = 1 \mid M_1 = 1, \mathcal{A}', C' = c'\}} \tag{92b}
\]

The second term in (92b) is equal to

\[
\Pr\{V^n(m_1, m_2, m_f) = v^n \mid M_1 = 1, \mathcal{A}', C' = c'\} = p(v^n | u^n) \tag{93}
\]

since the codewords \(V^n(m_1, m_2, m_f)\) are drawn independently of each other and conditionally on \(U^n(m_1)\). The last term in (92b), on the other hand, is equal to one. Given the true index \(M_1 = 1\), \((U^n(m_1), V^n(m_1, m_2, m_f)) \leftrightarrow (M_1, U^n(1), A^n) \leftrightarrow (M_2, M_f, V^n(1, 1, 1), C')\) is a Markov chain for \(m_1 \neq 1\). The numerator is thus equal to the denominator of the fraction. By combining (91)–(93), the lemma is proved. 

APPENDIX D

PROOF OF LEMMA 1

The proof of this lemma follows largely from the proofs of [28, Lemma 22.2] and [32, Lemma 4.1]. Unlike those proofs, however, we analyze here the behavior of the codeword
$V^n$ rather than the bin index associated to the source sequence $A^n$.

Let us first introduce the random variable $\Upsilon$, such that

$$\Upsilon \triangleq 1 \{ A^n \in T^n_\delta (A) \}. \quad (94)$$

Given the codebook $C$, the randomness in the codewords $(U^n V^n)$ comes from the choice of indices made. Then, consider

$$H(V^n | CU^n) = H(M_2 M_f | CM_1) \quad (95a)$$

$$= H(M_f | CM_1) + H(M_2 | CM_1 M_f) \quad (95b)$$

$$= nR_f + H(M_2 | CM_1 M_f) \quad (95c)$$

$$\geq nR_f + H(M_2 | CM_1 M_f \Upsilon) \quad (95d)$$

$$\geq nR_f + \Pr \{ \Upsilon = 1 \} H(M_2 | CM_1 M_f, \Upsilon = 1) \quad (95e)$$

$$\geq nR_f + (1 - \varepsilon) H(M_2 | CM_1 M_f, \Upsilon = 1), \quad (95f)$$

where (95c) is due to $M_f$ being drawn uniformly at random in $[1 : 2^{nR_f}]$, and independently of any other variable.

Now, for a specific codebook $C = c$ and pair of indices $(M_1, M_f) = (m_1, m_f)$, let us define the random variable $M_c$ with distribution:

$$p_{M_c} \triangleq p_{M_2 | C = c, M_1 = m_1, M_f = m_f, \Upsilon = 1}. \quad (96)$$

Thus, $H(M_c) = H(M_2 | C = c, M_1 = m_1, M_f = m_f, \Upsilon = 1)$. Due to the symmetry of the random codebook generation, the probability $p_{M_c}$ is independent of the value of any of the indices. Therefore, we consider them equal to 1 and we obtain

$$H(M_2 | CM_1 M_f, \Upsilon = 1) = \mathbb{E}_C [H(M_c)] \quad (97a)$$

$$= \sum_{m \in M_2} \mathbb{E}_C [-p_{M_c}(m) \log p_{M_c}(m)] \quad (97b)$$

$$= 2^{nR_2} \mathbb{E}_C [-p_{M_c}(1) \log p_{M_c}(1)], \quad (97c)$$

where the expectation is only over the random codebook.

According to the encoding procedure detailed in Section VI-B, the index $M_2$ is chosen uniformly among all the jointly typical codewords or, if there is no jointly typical codeword with $a^n$, uniformly on the whole codebook. We may thus characterize $p_{M_c}(1)$ as

$$p_{M_c}(1) = \sum_{a^n \in T^n_\delta (A)} \frac{p(a^n)}{\Pr \{ \Upsilon = 1 \}} \left( \frac{\nu_1}{1 + \sum_{i=2}^{2^{nR_2}} \nu_i} + 2^{-nR_2} \prod_{i=1}^{2^{nR_2}} (1 - \nu_i) \right), \quad (98)$$
where, with a slight abuse of notation, the events $\nu_i$ are

$$\nu_i \triangleq 1 \{ V^n(1, i, 1) \in T^n_\delta(V|a^n, U^n(1)) | V^n(1, i, 1) \in T^n_\delta(V|U^n(1)) \subset T^n_\delta(U|a^n) \}. \quad (99)$$

For a given codebook $C = c$, we may rewrite $\nu_i$ in a more convenient manner as

$$\nu_i = \sum_{v^n \in T^n_\delta(V|U^n(1))} 1 \{ v^n \in c \} 1 \{ V^n(1, i, 1) = v^n | v^n \in c \} 1 \{ v^n \in T^n_\delta(V|U^n(1), a^n) \}, \quad (100)$$

and its expected value is

$$\mathbb{E}_C[\nu_i] = |T^n_\delta(V|u^n)| \frac{2^{n(R_2 + R_f)}}{|T^n_\delta(V|u^n)|} 2^{-n(R_2 + R_f)} |T^n_\delta(V|u^n)| \quad (101a)$$

$$= \frac{|T^n_\delta(V|u^n, a^n)|}{|T^n_\delta(V|u^n)|} \quad (101b)$$

$$\triangleq \gamma. \quad (101c)$$

The expected value of (98) depends on the behavior of the term in parentheses. Each $\nu_i$ is a Bernoulli RV with $\mathbb{E}_C[\nu_i] = \gamma$ and is independent of the other $\nu_i$'s. Let us define

$$\nu = \sum_{i=2}^{2nR_2} \nu_i, \quad (102)$$

then $\nu$ is a Binomial RV with parameters $n' = 2^{nR_2} - 1$ and $\gamma$, i.e.,

$$p_{\nu}(j) = \binom{n'}{j} \gamma^j (1-\gamma)^{n' - j} \text{ for } j \in [0 : n']. \quad (103)$$

After some manipulations, it is possible to show that

$$\mathbb{E} \left[ \frac{1}{1 + \nu} \right] = \frac{1 - (1 - \gamma)^{n' + 1}}{\gamma(n' + 1)}. \quad (104)$$

We are now ready to calculate the expected value of (98). In the following, we use $n' + 1 = 2^{nR_2}$ for brevity. Hence,

$$\mathbb{E}_C[p_{M_c}(1)] = \sum_{a^n \in T^n_\delta(A)} \frac{p(a^n)}{\Pr\{ \Upsilon = 1 \}} \mathbb{E}_C \left[ \frac{\nu_1}{1 + \nu} + \frac{1}{n' + 1} \prod_{i=1}^{n'+1} (1 - \nu_i) \right] \quad (105a)$$

$$= \sum_{a^n \in T^n_\delta(A)} \frac{p(a^n)}{\Pr\{ \Upsilon = 1 \}} \left[ \gamma \frac{1 - (1 - \gamma)^{n' + 1}}{\gamma(n' + 1)} + \frac{1}{n' + 1} (1 - \gamma)^{n' + 1} \right] \quad (105b)$$

$$= \sum_{a^n \in T^n_\delta(A)} \frac{p(a^n)}{\Pr\{ \Upsilon = 1 \}} \frac{1}{n' + 1} \quad (105c)$$

$$= \frac{1}{n' + 1} \quad (105d)$$

$$= 2^{-nR_2}. \quad (105e)$$
Let us call the term in parentheses in (98) $\Upsilon_{a^n}$, then we have seen in (105) that $\mathbb{E}_c[\Upsilon_{a^n}] = 2^{-nR_2}$. Consider now,

$$
\mathbb{E}_c[(p_{M_c}(1))^2] = \sum_{a^n \in T^n_s(A)} \left( \frac{p(a^n)}{\Pr\{\Upsilon = 1\}} \right)^2 \mathbb{E}_c[(\Upsilon_{a^n})^2] + \sum_{a^n \in T^n_s(A)} \frac{p(a^n)p(a'^n)}{\Pr\{\Upsilon = 1\}} \mathbb{E}_c[\Upsilon_{a^n}\Upsilon_{a'^n}]
$$

(106a)

$$
\leq \sum_{a^n \in T^n_s(A)} \left( \frac{p(a^n)}{\Pr\{\Upsilon = 1\}} \right)^2 \mathbb{E}_c[\Upsilon_{a^n}] + \sum_{a^n \in T^n_s(A)} \frac{p(a^n)p(a'^n)}{\Pr\{\Upsilon = 1\}} \mathbb{E}_c[\Upsilon_{a^n}]\mathbb{E}_c[\Upsilon_{a'^n}]
$$

(106b)

$$
\leq 2^{n[H(A)+\delta]} \frac{2^{-n[H(A)-\delta]^2}}{(1-\varepsilon)^2} 2^{-nR_2} + 2^{-2n^2R_2}
$$

(106c)

$$
\leq 2^{-n[R_2+H(A)-\xi]} + 2^{-n^2R_2},
$$

(106d)

where

- (106b) is due to $(\Upsilon_{a^n})^2 \leq \Upsilon_{a^n}$ since $\Upsilon_{a^n} \in [0,1]$, and that the events $\Upsilon_{a^n}$ and $\Upsilon_{a'^n}$ are independent given different source sequences; and,

- (106d) holds for some $\xi > 0$.

Therefore,

$$
\text{Var}[p_{M_c}(1)] \leq 2^{-n[R_2+H(A)-\xi]},
$$

(107)

and in view of Chebyshev’s inequality,

$$
\Pr\{|p_{M_c}(1) - \mathbb{E}_c[p_{M_c}(1)]| \geq \delta \mathbb{E}_c[p_{M_c}(1)]\} \leq \frac{\text{Var}[p_{M_c}(1)]}{(\delta \mathbb{E}_c[p_{M_c}(1)])^2}
$$

(108a)

$$
\leq \frac{1}{\delta^2} 2^{-n[H(A)-R_2-\xi]},
$$

(108b)

which tends to zero as $n \to \infty$ if $R_2 < H(A) - \xi$.

Let us define

$$
\chi = 1 \{ |p_{M_c}(1) - 2^{-nR_2}| < \delta 2^{-nR_2} \},
$$

(109)

then, according to (108), $\Pr\{\chi = 1\} \geq 1 - \varepsilon'$ for large $n$ if $R_2 < H(A) - \xi$. Continuing from
to (97), we obtain
\[
H(M_2|CM_1M_f, \Upsilon = 1) = 2^{nR_2} \mathbb{E}c[-p_{M_c}(1) \log p_{M_c}(1)] 
\geq 2^{nR_2} \mathbb{E}c[-p_{M_c}(1) \log p_{M_c}(1) | \chi = 1] \Pr\{\chi = 1\} 
\geq 2^{nR_2} (1 - \delta)2^{-nR_2} \log \left( \frac{2^{nR_2}}{1 + \delta} \right) (1 - \epsilon') 
\geq n(R_2 - \xi'), 
\]
and finally
\[
H(V^n|CU^n) \geq n(R_2 + R_f - \xi''), 
\]
which concludes the proof.

\textbf{APPENDIX E}
\textbf{PROOF OF PROPOSITION 11}

For \( \alpha, \beta \in \mathbb{R}, \alpha \leq \beta \), let us consider the auxiliary random variables
\[
T \sim N(0, P - \alpha^2 Q), 
\]
\[
U' \sim N(0, P - (\alpha^2 - \beta^2) Q), 
\]
such that \( T \) and \( U' \) are independent of every other variable. Then, we may construct
\[
X = T + \alpha S, 
\]
\[
V = T + \beta S, 
\]
\[
U = \sqrt{\mu}V + \sqrt{\bar{\mu}}U', 
\]
where \( \mu \in [0, 1] \) and \( \bar{\mu} = 1 - \mu \). It is easy to verify that \( \mathbb{E}[X^2] = P \) and \( \mathbb{E}[V^2] = \mathbb{E}[U^2] = P - (\alpha^2 - \beta^2) Q \).

To simplify the maximization process, we define:
\[
\alpha = \rho \sqrt{\frac{P}{Q}}, 
\]
\[
\gamma = \beta - \alpha, 
\]
where \( \rho \in [0, 1] \) is the correlation coefficient between \( X \) and \( S \), and \( \gamma \in \mathbb{R} \) is a parameter independent of \( \rho \).

Given a value of \( \rho \), the bounds (8) with \( A = S \) and \( B = \emptyset \) impose the condition
\[
a \gamma^2 + b \gamma + c \geq 0, 
\]
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where

\[ a = -Q[(1 - \rho^2)P + N_1], \quad (116a) \]
\[ b = 2[(1 - \rho^2)PQ - \rho\sqrt{PQ}N_1], \quad (116b) \]
\[ c = P[(1 - \rho^2)(P + 2\rho\sqrt{PQ}) - \rho^2N_1]. \quad (116c) \]

Therefore, the correlation coefficient \( \rho \) restricts the possible values of \( \gamma \) to the interval \([\gamma_1, \gamma_2]\), where \( \{\gamma_1, \gamma_2\} \) are the roots of the polynomial (115).

On the other hand, the bound (7) may be written as

\[ R_k \leq \max_{\rho, \gamma, \bar{\mu}} \frac{1}{2} \log^2 \frac{1 + \bar{\mu}\Delta_1}{1 + \bar{\mu}\Delta_2}, \quad (117) \]

where

\[ \Delta_i = \frac{[P + \gamma Q + \rho(1 + \gamma)\sqrt{PQ}]^2}{(1 - \rho^2)(1 - \gamma^2)PQ + N_1[P + \gamma^2Q + 2\rho\gamma\sqrt{PQ}]} \quad (118) \]

The maximization of (117) renders the parameter \( \bar{\mu} \) a binary variable, i.e.,

\[ \bar{\mu} = \begin{cases} 1 & \text{if } \Delta_1 \geq \Delta_2 \\ 0 & \text{otherwise}, \end{cases} \quad (119) \]

and the achievable secret-key rate may be rewritten as

\[ R_k \leq \max_{\rho, \gamma} \frac{1}{2} \left[ \log^2(1 + \Delta_1) - \log^2(1 + \Delta_2) \right]^+, \quad (120) \]

where the values of \( \rho \) and \( \gamma \) must satisfy condition (115). There is no closed form solution for the aforementioned maximization, and thus one should proceed numerically to optimize jointly \( \rho \) and \( \gamma \). We now provide a simple suboptimal solution.

We first note that the bound (120) is maximized by \( \gamma = 1 \) which, in turn, gives (20). Additionally, with \( \gamma = 1 \) and after some manipulations, condition (115) becomes

\[ (1 - \rho^2)P \geq N_1 - \frac{N_1^2}{P + Q + 2\rho\sqrt{PQ} + N_1}. \quad (121) \]

Since (20) is an increasing function of \( \rho \), the value of \( \rho \) obtained from (21) verifies (121) and maximizes (20). This concludes the proof of Proposition 11.

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