$S^7$ Current Algebras

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ABSTRACT

We present $S^7$-algebras as generalized Kac-Moody algebras. A number of free-field representations is found. We construct the octonionic projective spaces $\mathbb{O}P^N$.

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1. Introduction

The seven-sphere is intricately connected to supersymmetry in high dimensions. This relation is rather direct via the representation of Gamma-matrices in terms of octonions [1]. The algebra of imaginary octonions in turn carries a natural geometric structure, namely $S^7$. At the same time we know that in the study of large supersymmetries exceptional groups occur at various points and those are also economically described in terms of octonions. It would be quite satisfying to find a symmetry principle that explains those coincidences. A study of the manifestly supersymmetric string in a twistor-like formulation [3,4,5] as well as in the light-cone [6] formulation leads us to believe that such a symmetry principle could also help a lot in the quantization of the superstring. Hence we decided to study in more detail the common denominator, namely the octonions and their geometric structure, in the setting of conformal field theory of $S^7$ and on $S^7$ [9].

2. Octonions and $S^7$

An algebraic way to describe the octonions consists of writing their multiplication table: for

$$x = \sum_{a=0}^{7} x_a e_a \in O$$

we set $e_0 = 1$ and write

$$x = [x] + \sum_{i=1}^{7} x_i e_i = [x] + \{x\}$$

with

$$e_i e_j = -\delta_{ij} + \sigma_{ijk} e_k ,$$

where the structure constants $\sigma$ are completely antisymmetric and defined by

$$e_i e_{i+1} = e_{i+3} .$$
We have set $e_{i+7} = e_i$. One verifies

\[
[e_i, e_j] = 2\sigma_{ijk}e_k \\
[e_i, e_j, e_k] = (e_i e_j)e_k - e_i (e_j e_k) = 2\rho_{ijkl}e_l ,
\]

\[
\rho_{ijkl} = (\ast \sigma)_{ijkl} = -\frac{1}{6cijklmnp}\sigma_{mnp} .
\]

The associator $\rho_{ijkl}$ appears here as a purely algebraic object. Its geometric meaning becomes clear if we define the sevenbein $Xe_i$ on the sphere $S^7 = \{X \in \mathbf{O} \ | \ |X| = 1\}$, where it defines a globally parallelizing connection. This means that the connection has zero curvature, but in our case there is torsion:

\[
\frac{1}{2}[D_i, D_j] = T_{ijk}(X)D_k = [(e_i^*X^*)(Xe_j)e_k^*]D_k ,
\]

where $D_i$ is the covariant derivative associated with the connection we just defined. Obviously $T_{ijk}(1) = -\sigma_{ijk}$. Less obvious is the fact that

\[
D_i T_{jkl}(1) = -2\rho_{ijkl} ,
\]

but it is precisely this circumstance that permits the following construction.

3. Current Algebra

We define

\[
\mathbf{J}_i^{(\lambda)}(z) = [\lambda^*\omega e^i] \\
\mathbf{J}_i^{(S)}(z) = \frac{1}{2}[(X^S)^*(XS)e^i] \\
\mathbf{J}_i^{(\lambda')}(z) = [(X\lambda')^*(X\omega')e^i] \\
\mathbf{J}_i^{(b)}(z) = [(Xb)^*(Xc)e^i]
\]

with free bosonic fields

\[
\lambda_a(z) \omega_b(w) = \frac{\delta_{ab}}{z - w} ,
\]

\[
\lambda'_a(z) \omega'_b(w) = \frac{\delta_{ab}}{z - w} ,
\]

3
\[ \lambda = \lambda_a e_a, \text{ etc., } X = \lambda /|\lambda|, \text{ and free fermions} \]

\[ S_a(z) S_b(w) = \frac{\delta_{ab}}{z - w}, \]
\[ b_a(z) c_b(w) = \frac{\delta_{ab}}{z - w}. \]

In fact, we can be a little more general and give the fields \( S_a(z), \lambda_a'(z), \omega_b'(w) \) and \( b_a(z), c_b(w) \) an extra label, say \( m \), running from 1 to \( N_s, N_{\lambda'}, N_b \), respectively. Then the current

\[ J^i(z) = J^i_\lambda(z) + \sum_{m=1}^{N_s} \left( J^i_{(S)m}(z) + 2[X^* \partial X^i] \right) + \sum_{m=1}^{N_{\lambda'}} \left( J^i_{(\lambda')m}(z) - 4[X^* \partial X^i] \right) + \sum_{m=1}^{N_b} \left( J^i_{(b)m}(z) + 4[X^* \partial X^i] \right) \]

satisfies the algebra

\[ J^i(z) J^j(w) = \frac{8\delta_{ij}}{(z - w)^2} + \frac{2}{z - w} T_{ijk}(X(w)) J^k(w), \]

where \( T_{ijk}(X) \) is the torsion tensor we defined above. The somewhat unusual form of the current (3.4) arises because we demand (3.5) without additional terms built from \( X \)'s and derivatives, which generically arise in such a construction. This condition also fixes the coefficient 8 uniquely. Our construction can be understood as follows: a current such as \( J^i_{(S)} \) at \( X = 1 \) would define a Kac-Moody algebra on \( S^7 \) in exact analogy to the Lie algebra case. We know, however, that \( S^7 \) is not a group manifold, and hence this naive algebra does not close. It misses closing precisely by terms proportional to the associator \( \rho_{ijkl} \) of the octonions. The first part \( J^i_\lambda(z) \) of the total current acts as a covariant derivative on \( J^i_{(S)} \) with respect to the coordinates \( X \) of the seven-sphere. The relative coefficients are arranged such
that those two contributions cancel, by virtue of (2.7). This mechanism provides us with the classical version of the current algebra (3.5). The quantum version arises via a cancellation of undesirable terms from quantum contributions, namely double contractions between, say, \( J_i^j(S) \) and \( J_j^i(S) \), and from classical contributions that stem from a single contraction between \( J^j_\lambda \) and \([X^* \partial X^j] \). This type of mechanism is of course also familiar from the theory of Kac-Moody algebras.

4. Some more complicated Constructions

Let is now consider, for the moment at a purely classical level, currents of the type

\[
J^a(z) = \sum_{m=0}^{N} \left[ \omega^*_m (\lambda^m X^*_0) (X^0 e^a*) \right],
\tag{4.1}
\]

which generalizes (3.4) if we set \( \lambda_0 = \lambda \). Note that \( J^0(z) \) generates real scale transformations. If we restrict ourselves to the real, complex or quaternionic cases and to the zero mode sector of our conformal fields, we may define the respective projective spaces \( \mathbb{K}P^N \) by gauging \( J \), i.e. by considering the space of homogeneous coordinates \( \lambda \), \( \mathbb{K}^{N+1} \), modulo the transformations generated by \( J \). We note that in those cases all the fields \( \lambda^m \) are equivalent, since the \( X^0 \)-factors simply drop out. Octonion multiplication is nonassociative, and hence the general structure of such currents can be much more complicated.

If \( N = 1 \), we have two possibilities,

\[
J^a = \left[ \omega^*_0 (\lambda^0 e^a*) \right] + \left[ \omega^*_1 (\lambda^1 X^*_0) (X^0 e^a*) \right],
\tag{4.2}
\]

or

\[
N^a(z) = \left[ \omega^*_1 (\lambda^1 e^a*) \right] + \left[ \omega^*_0 (\lambda^0 X^*_1) (X^1 e^a*) \right]
= \left[ (J X^*_0) (X^0 X^*_1) (X^1 e^a*) \right],
\tag{4.3}
\]

which we represent by diagrams with points corresponding to the fields \( \lambda^m \) and links representing the order of multiplication with respect to the base field indicated.
by a black dot:

\[
\begin{array}{c}
\text{by a black dot:} \\
\text{J} & \text{N} \\
1 & 0 & 1 & 0
\end{array}
\]

(4.1) then corresponds to the starlike diagram

\[
\begin{array}{c}
\text{N-1} \\
\text{N} \\
\text{1} \\
\text{0} \\
\text{2} \\
\text{3}
\end{array}
\]

We may generalize it to any tree diagram with \(N + 1\) points by associating with the path

\[
\begin{array}{c}
\text{A} \\
\text{k} \\
\text{k-1} \\
\text{1} \\
\text{0} \\
\text{B}
\end{array}
\]

the product

\[
A \circ_{k \to 0} B = \left( AX^*_\text{(k-1)} \left( (X_{(k-1)}X^*_\text{(k-2)}) \cdots \left( (X_{(1)}X^*_\text{(0)}) (X_{(0)}B) \right) \cdots \right) \right). \quad (4.7)
\]

We note that \( |A \circ_{k \to 0} B| = |A||B| \) and that the product collapses in the real, complex
and quaternionic case to $AB$. The total classical current that corresponds to a tree diagram with $N+1$ points and base point 0 reads now

$$J^a = \sum_{m=0}^{N} \left[ \omega^*_m \lambda_{(m)} \circ e^{a*} \right].$$

(4.8)

We may even put fermions at the endpoints of our tree diagram. Remarkably enough, the current

$$J^i = \left[ \omega^*_{(0)} \lambda_{(0)} e^{i*} \right]$$

$$+ \sum_{\{\lambda\}} \left( \left[ \omega^*_{(m)} \lambda_{(m)} \circ e^{i*} \right] - 4 \left[ \partial X^*_{(m-1)} X_{(m-1)} \circ e^{i*} \right] \right)$$

$$+ \sum_{\{S\}} \left( \left[ S^*_m S_m \circ e^{i*} \right] + 2 \left[ \partial X^*_{(m-1)} X_{(m-1)} \circ e^{i*} \right] \right)$$

$$+ \sum_{\{b\}} \left( \left[ b^*_m c_m \circ e^{i*} \right] + 4 \left[ \partial X^*_{(m-1)} X_{(m-1)} \circ e^{i*} \right] \right)$$

(4.9)

still generates the algebra (3.5) at the quantum mechanical level. Given the complexity of the octonion products in $J^i$, the fact that we obtain such a simple operator product may safely be regarded as a miracle.

5. Octonionic Projective Spaces

At the beginning of the previous section we already described how we wish to think about projective spaces. For definiteness we choose our current corresponding to the starlike diagram (4.5). The gauge orbits of $J^a$ in $\mathbf{O}^{N+1}$ are given by

$$\left\{ \left( \lambda_{(0)} \circ \Omega, \cdots, \lambda_{(N)} \circ \Omega \right) \right\} \bigg| \lambda_{(m)}, \Omega \in \mathbf{O}, \ |\Omega| = 1 \bigg\} ,$$

(5.1)

where $A \circ B = (AX^{-1})(XB)$. Unfortunately, they are not well defined at $\lambda_{(0)} = 0$, and hence this way of defining $\mathbf{O}P^N$ is not terribly convenient. Let us, therefore,
fix our gauge freedom and consider a description in terms of $N + 1$ coordinate patches labelled by the index $b \in \{0, \cdots, N\}$, obtained by choosing $\Omega$ such that $x^a_a = 1$ for any $a$. This means

$$x^a_b = \lambda_{(a)} \circ_{X_{(0)}} (\lambda_{(b)})^{-1}, \ a \in \{0, \cdots, N\} \quad (5.2)$$

and implies the transition functions

$$x^a_b = (x^a_c (x^0_c)^{-1}) \ (x^0_c (x^b_c)^{-1}) = x^a_c \circ_{(c)} (x^0_c)^{-1} \quad (5.3)$$

between patch $(b)$ and patch $(c)$. On each patch we obtain a complete set of independent, compatible, gauge invariant coordinates. For a generic diagram we associate with each point $(a)$ one coordinate $x^a_b$ in every patch $(b)$ and the transition functions read:

$$x^a_b = x^a_c \circ_{a \to b} (x^b_c)^{-1} \quad (5.4)$$

where the path dependent multiplication is of course performed in patch $c$. The coordinate patches from different diagrams are obviously mutually compatible if we identify precisely one patch of a particular diagram with one patch of every other diagram. This implies that different diagrams generate the same manifold. The diagram (4.5) is singled out by the especially simple transition functions involving patch $(0)$:

$$x^a_b = x^a_0 \ (x^b_0)^{-1} \quad (5.5)$$

which are identical to the transition functions in the associative case.

For $\mathbb{O}P^1$ and $\mathbb{O}P^2$ we obtain the standard descriptions, e.g. the atlas

$$(1, y_0, z_0)$$

$$(x_1, 1, z_1) \quad (5.6)$$

$$(x_2, y_2, 1)$$
with the transition functions
\[
\begin{align*}
x_1 &= y_0^{-1} \\
x_2 &= x_1 z_1^{-1} \\
y_2 &= z_1^{-1} \\
y_0 &= y_2 x_2^{-1} \\
z_1 &= z_0 y_0^{-1} \\
z_2 &= x_2^{-1}.
\end{align*}
\] (5.7)

We note that these overlap equations are not unique: we could have substituted the product \( A \circ X B = (AX^{-1})(XB) \) for the usual octonion product \( AB \) in (5.7) and still obtained a perfectly fine projective space, which we will call \( \mathbb{O}_P^2(X) \). It is also obvious that due to the small dimensionality of this space the transition functions are considerably simpler than in the general case (5.4). Let us now consider the so far unknown space \( \mathbb{O}_P^3 \), with “atlas”
\[
(1, y_0, z_0, v_0)
\]
\[
(x_1, 1, z_1, v_1)
\]
\[
(x_2, y_2, 1, v_2)
\]
\[
(x_3, y_3, z_3, 1)
\]

and transition functions (5.3). Since these become ill-defined for \( x_a = 0 \), it behooves us to explain what we mean with “atlas” in this context. Let us first write down the transition functions in terms of the coordinates on patch (1):
\[
\begin{align*}
(1, y_0, z_0, v_0) &= (1, x_1^{-1}, z_1 x_1^{-1}, v_1 x_1^{-1}) \\
(x_2, y_2, 1, v_2) &= (x_1 z_1^{-1}, z_1^{-1}, 1, (v_1 x_1^{-1})(x_1 z_1^{-1})) \\
(x_3, y_3, z_3, 1) &= (x_1 v_1^{-1}, v_1^{-1}, (z_1 x_1^{-1})(x_1 v_1^{-1}), 1)
\end{align*}
\] (5.9)

The “sphere at infinity” in chart (0) is described in charts (1), (2) and (3) as
\[
\begin{align*}
(x_1, 1, z_1, v_1) &= (0, 1, z_1, v_1) \\
(x_2, y_2, 1, v_2) &= (0, z_1^{-1}, 1, (v_1 x_1^{-1})(x_1 z_1^{-1})) \\
(x_3, y_3, z_3, 1) &= (0, v_1^{-1}, (z_1 x_1^{-1})(x_1 v_1^{-1}), 1)
\end{align*}
\] (5.10)

where \( \hat{x} = x/|x| \). Evidently the sphere is mapped to \( \mathbb{O}_P^2(\hat{x}_1) \), but strictly speaking \( \hat{x}_1 \) is not well defined at \( x_1 = 0 \). For the “sphere at infinity” all the charts become
singular, chart (0) because some of the coordinates grow without bound, and charts (1), (2) and (3) because they become different projections of the image of this sphere under the transition functions. In other words, from the data $z_1, v_1$ and $v_2$, for example, we can determine where we are on this image, but not from the data of patch (1) alone. In the real, complex and quaternionic cases the factors $\hat{x}_1$ in (5.10) vanish by associativity and we obtain the famous Hopf maps as the maps from the “sphere at infinity” in patch (0) to the submanifold $x_a = 0$, i.e. to $KP^2$. In the octonionic case this sphere is mapped to a submanifold of real dimension 23. The fibering we obtain is only a $\mathbb{Z}_2$-fibration as in the real case, since we can solve $z_1, v_1$ and $v_2 = (v_1 \hat{x}_1^{-1})(\hat{x}_1 z_1^{-1})$ for $\hat{x}_1$ up to factor of (-1). The analog of the equation $KP^3 = K^3 \cup K^2 \cup K^1 \cup K^0$ for associative $K$ is now

$$
\begin{align*}
OP^3 &= O^3 \cup OP^2(S^7) \\
OP^2 &= O^2 \cup O^1 \cup O^0
\end{align*}
$$

(5.11)

with $OP^2(S^7) = \{OP^2(X) \mid X \in S^7\}$.

The generalization of the above construction to $OP^N$ is straightforward. One obtains $OP^N(S^7)$ if one replaces the usual octonion product in (5.4) with an $X$-dependent one and writes instead of (5.3)

$$x^a_b = (x^a_c \circ x^0_c)^{-1} \circ x^0_b \circ x^b_c)^{-1}.$$

(5.12)

This is the image of the “sphere at infinity” in $O^{N+1}$ as seen in the $N+1$ coordinate patches $b = 1, 2, \ldots, N + 1$, each of which provides us only with a projection of this image.
6. Sugawara Construction

The similarity of \( S^7 \) to a group manifold and the results of Osipov [8] for Malcev-Kac-Moody algebras let us expect that a Sugawara construction is possible also for \( \tilde{S}^7 \). This is not difficult to verify for the simplest current \( J^i_\lambda \) defined in (3.1) and the associated energy-momentum tensor

\[
L = -\frac{1}{8} : J_j J_j : \\
= \frac{7}{8} \left[ \lambda^* \partial \omega - \partial \lambda^* \omega \right] + \frac{1}{8} \left[ \lambda^* \omega e_j \right] \left[ \lambda^* \omega e_j \right],
\]

where we have normal-ordered the currents in the standard way:

\[
: J_j(w) J_k(w) : = \lim_{z \to w} \left( J_j(z) J_k(w) - \frac{56}{(z-w)^2} \right).
\]

In order to generalize this result to currents with arbitrary field content, one has to be careful about normal ordering. Up to this point we have implicitly assumed free-field normal ordering on the right-hand side of the current-current commutation relations. Even though the torsion tensor \( T_{ijk}(X) \) commutes with \( J_k \), the product of those two operators obeys

\[
T_{ijk}(X) J_k = : T_{ijk}(X) J_k : + 8 T_{ijk}(X) [X^* \partial X e_k] \\
= : T_{ijk}(X) J_k : - T_{ikl}(X) \partial_z T_{jkl}(X),
\]

where the left-hand side is free-field normal ordered and the first term on the right-hand side is current normal ordered. It is useful to employ the technology described by Bais, Bouwknegt, Surridge and Schoutens to show the following result: for currents that obey

\[
J_i(z) J_j(w) = -\frac{k \delta^{ij}}{(z-w)^2} + \frac{2}{z-w} : T_{ijk}(X)(J_k - 8 [ X^* \partial X e_k ])(w) :, \]

the Sugawara energy-momentum tensor is given by

\[
L = -\frac{1}{2k+24} : J_j J_j :.
\]
and has a central charge of
\[ c = -\frac{7k}{k+12}, \] (6.6)
in accordance with the known results for Kac-Moody algebras. (6.5) and (6.6) are equivalent to Osipov’s formulas [7]. For currents that obey (3.5) we obtain unfortunately a negative value of \( c \), in accordance with the fact that the simplest system satisfying (3.5) is constructed from a pair of bosonic ghosts. We note that in contrast to the Kac-Moody case, the requirement that \( J_i(z) + \alpha[X^*\partial X e_i](z) \) should transform like a dimension 1 current under the action of the Sugawara energy-momentum tensor fixes only the constant \( \alpha \), not the precise form of the stress-energy tensor. There is a three-parameter family of candidate Sugawara tensors which satisfy this condition. The operator product of \( L(z) \) with itself determines the parameters. There are two solutions: (6.5) for any \( k \) and another solution which would requires a complex value of \( k \). The Sugawara construction is in this sense unique.

7. BRST

In order to construct the BRST charge \( Q \) for the gauge symmetry generators \( J^i \), we introduce ghost fields \( B^i \) and \( C^j \) of dimension 1 and 0 respectively, with correlator
\[ B^i(z) C^j(w) = \frac{\delta^{ij}}{z-w}, \] (7.1)
and construct the BRST current in the usual manner:
\[ q(z) = C^i N^i - T^{ijk}(X)C^i C^j B^k, \] (7.2)
where the currents \( N^i = J^i + 8[X^*\partial X e^i] \) have to obey
\[ N^i(z) N^j(w) = \frac{24\delta^{ij}}{(z-w)^2} + \frac{2}{z-w} : T_{ijk}(X(w)) N^k(w) : \] (7.3)
if the BRST charge \( Q = \oint \frac{dz}{2\pi i} q(z) \) is to be nilpotent. Note the current normal-ordering in the above equation, which is satisfied if the currents \( J^i \) generate (3.5),
i.e. for all the quantum currents constructed in the previous sections.

We have to mention one unusual property of the BRST charge $Q$: the current obtained in the canonical way as the anticommutator $\{B^i(z), Q\}$ does not generate the algebra (3.5). In fact, we have not been able to find a representation of (3.5) with seven conjugate fields along the lines described above.

8. Outlook

In this paper we have shown that $S^7$ provides us with similar structures as $S^3$. Clearly much remains to be done: a representation theory of $S^7$ needs to be worked out, octonionic manifolds should be studied and their relation to $N=8$ superconformal algebras should be established. Octonionic superconformal field theory must be explored. On these and other related issues we hope to report in the near future.

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