Subharmonic and chaotic resonances in solar activity

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Abstract – It is shown that the wavelet regression detrended fluctuations of the monthly sunspot number for 1749–2009 years exhibit strong periodicity with a period approximately equal to 3.7 years. The wavelet regression method detrends the data from the approximately 11-years period. Therefore, it is suggested that the one-third subharmonic resonance can be considered as a background for the 11-years solar cycle. It is also shown that the broad-band part of the wavelet regression detrended fluctuations spectrum exhibits an exponential decay that, together with the positive largest Lyapunov exponent, are the hallmarks of chaos. Using a complex-time analytic approach the rate of the exponential decay of the broad-band part of the spectrum has been theoretically related to the Carrington solar rotation period.

Introduction. – The solar activity is chaotic but has a well-defined mean period of about 11 years. The 11-year cycle is well known for more than a century and a half. Despite this, the nature of the 11-year cycle is still a subject of vigorous investigations. The most popular point of view is a “dynamo-wave” mechanism. It is assumed that a magnetic dynamo, generated by the solar differential rotation and the helicity of turbulent convective flows, produces this propagating wave. Helicity, through the $\alpha$-effect, plays crucial role in this mechanism. Recently, observational data of physical quantities associated with the $\alpha$-effect became available and a considerable progress in this direction was achieved (see, for instance, refs. [1–3]). The $\alpha$-effect has two contributors: one related to helicity of convective vortices and another related to the helicity of magnetic field. Intrinsically the nonlinear character of the problem makes it especially difficult for theoretical investigation. The nonlinear solar dynamo has to be saturated in order to get a quasi-stationary wave. The magnetic part of the $\alpha$-effect can play a crucial role in such saturation, while a certain modification of the turbulent diffusivity and other transport coefficients is unavoidable at this process (see refs. [1–3] and section “Chaotic dynamo”). Results of the above-mentioned simulations show that the dynamo model leads to a steadily oscillating magnetic configuration. The cyclic behavior is typical for moderate dynamo action whereas for the stronger dynamo action a chaotic behavior is usually observed in the dynamo simulations.

It is believed that the dynamo mechanism is mainly operational deep inside the convective zone (or even in the overshoot layer). However, in the upper layers of the convection zone (“at surface”) a strong hydromagnetic activity has been also observed. The presence of large-scale meandering flow fields (like jet streams), banded zonal flows and evolving meridional circulations produces a very complex picture. In this situation one can expect that more than one nonlinear mechanisms can be at background of the observed 11-yearsolar cycle. Namely, it could be inferred from the data analysis presented in this letter that a dynamo mechanism, directly affected by the solar rotation, may generate the basic chaotic oscillations (with a fundamental period different from the 11-year cycle), while another nonlinear mechanism amplifies these oscillations to the observed 11-year chaotic oscillations. In this two-stage picture the mirror asymmetry of the solar magnetic field can be still a key driver of the 11-year activity cycle through an additional nonlinear amplifying mechanism.

Wavelet regression. – Most of the regression methods are linear in responses and statistical analyses of the experimental sunspot data was dominated by linear stochastic methods, while it was recently rigorously shown in ref. [4] that a nonlinear dynamical mechanism (presumably a driven nonlinear oscillator, see also ref. [5]) determines the sunspot cycle. Figure 1 shows the monthly sunspot number (dashed line) for the period 1749–2009...
Fig. 1: (Colour on-line) The monthly sunspot number (dashed line) for the period 1749–2009 years. The data were taken from ref. [6]. The solid curve (trend) corresponds to a wavelet (symmlet) regression of the data.

Fig. 2: The wavelet regression detrended fluctuations from the data shown in fig. 1.

Fig. 3: Spectrum of the wavelet regression detrended fluctuations shown in fig. 2.

Subharmonic resonance. – The wavelet regression method detrends the data from the approximately 11-years period (cf. fig. 1). Therefore, it is plausible that the one-third subharmonic resonance [10] can be considered as a background for the 11-years solar cycle: $11/3 \approx 3$. Indeed, it is known [11] that the interaction of the Alfvén waves (generated in a highly magnetized plasma by a cavity’s moving boundaries) with slow magnetosonic waves can be described using Duffing oscillators (see also refs. [4,12]). Let us imagine a forced excitable system with a large amount of loosely coupled adimensional parameters range) with a wide range of the natural frequencies $\omega_0$:

$$\ddot{x} + \omega_0^2 x + \gamma \dot{x} + \beta x^3 = F \sin \omega t,$$

where $\dot{x}$ denotes the temporal derivative of $x$, $\beta$ is the strength of nonlinearity, and $F$ and $\omega$ are characteristic of a driving force (see also for a nonlinear delay the Duffing dynamo equation (7) in the next section). It is known (see, for instance, ref. [10]) that when $\omega \approx 3\omega_0$ and $\beta \ll 1$ eq. (1) has a resonant solution

$$x(t) \approx a \cos \left(\frac{\omega}{3} t + \varphi\right) + \frac{F}{(\omega^2 - \omega_0^2)} \cos \omega t,$$

where the amplitude $a$ and the phase $\varphi$ are certain constants. This is so-called one-third subharmonic resonance with the driving frequency $\omega$ corresponding approximately to 3.7-years period (the peak in fig. 3 corresponds to the second term in the right-hand side of the eq. (2) while the first term has been detrended).

For the considered system of the oscillators an effect of synchronization can take place and, as a consequence of this synchronization, the characteristic peaks in the spectra of partial oscillations coincide [14].
Fig. 4: The same as in fig. 3 but in semi-logarithmical scales. The dashed straight line indicates an exponential decay.

Chaotic dynamo. – In order to understand the appearance of the 3.7-years period let us represent the spectrum shown in fig. 3 in semi-logarithmical scales: fig. 4. In these scales an exponential behavior corresponds to a straight line. It is known that both stochastic and deterministic processes can result in the broad-band part of the spectrum, but the decay in the spectral power is different for the two cases. The exponential decay indicates that the broad-band spectrum for these data arises from a deterministic rather than a stochastic process. For a wide class of deterministic systems a broad-band spectrum with exponential decay is a generic feature of their chaotic solutions refs. [15–19]. A wavy exponential decay (see fig. 4) is a characteristic of a chaotic behavior generated by time delay differential equations [16]. A classic example of a time delay differential equation with chaotic solutions is the Mackey-Glass equation:

$$\frac{du(t)}{dt} = \frac{0.2 \cdot u(t-\tau)}{(1 + u(t-\tau)^10)} - 0.1 \cdot u(t).$$  \tag{3}

Figure 5 shows the spectrum of a solution of this equation for the time delay $\tau = 30$. The dashed straight line indicates an exponential decay (cf. fig. 4).

In the Parker dynamo, which was generalized in refs. [1–3] (cf. introduction), a time delay in the back influence of the magnetic field on the $\alpha$-effect [20,21]

$$\alpha^\prime(\theta, t, \tau) = \frac{\alpha_0(\theta)}{1 + B^2(\theta, t - \tau)}$$  \tag{4}

(where $B$ is the azimuthal component of the magnetic field) can significantly change the evolution of the magnetic field even for a small time delay. In particular, this non-linear delay can result in the appearance of processes with periods much longer than the fundamental period through a parametric resonance [21].

In the dynamo models that have physically distinct source layers the finite time is required in order to transport magnetic flux from one layer to another (a time delay involved in the $\alpha$-quenching mechanism due to the Lorentz feedback, cf. introduction). Even the Duffing equations for $B$ can be obtained for the dynamo models using the delay idea in this case (cf. eq. (4) in ref. [22]):

$$\ddot{B} + \omega_t^2 B + \gamma \dot{B} + \beta B(t - \tau)f(B(t - \tau)) = 0$$  \tag{5}

where $f(B(t - \tau))$ is a quenching factor, which can be approximated by a nonlinear function. In particular, in ref. [22] (see also ref. [23]) this function has been approximated as

$$f(B) = \frac{1}{4} (1 + \text{erf}(B^2 - B_{\text{min}}^2)) (1 - \text{erf}(B^2 - B_{\text{min}}^2))$$  \tag{6}

and for small $(B^2 - B_{\text{min}}^2)$ we obtain a nonlinear delayed Duffing equation

$$\ddot{B} + \Omega_0^2 B + \gamma \dot{B} + \beta' B^3(t - \tau) = 0,$$  \tag{7}

where $\Omega_0^2$, $\gamma$ and $\beta'$ are certain constants. The subharmonic resonances and chaotic regimes are also known for the delay Duffing equations with a periodic forcing.

It is also significant for those dynamo models that have spatially segregated source regions for the poloidal and toroidal magnetic-field components (such as, for instance, the Babcock-Leighton dynamo mechanism [23]). In the global dynamo models that include meridional circulation the time delay related to the circulation should be comparable to the global rotation period (see below).

The nature of the exponential decay of the power spectra of the chaotic systems (figs. 4 and 5) is still an unsolved mathematical problem. A progress in the solution of this problem has been achieved by the use of the analytical continuation of the equations in the complex domain (see, for instance, [18]). In this approach the exponential decay of chaotic spectrum is related to a singularity in the plane of complex time, which lies nearest
to the real axis (see the insert in fig. 5). The distance between this singularity and the real axis determines the rate of the exponential decay. For many interesting cases chaotic solutions are analytic in a finite strip around the real time axis. This takes place, for instance, for attractors bounded in the real domain (the Lorentz attractor, for instance). In this case the radius of convergence of the Taylor series is also bounded (uniformly) at any real time.

Let us consider, for simplicity, solution \( u(t) \) with simple poles only, and to define the Fourier transform as follows:

\[
\tilde{u}(f) = \frac{(2\pi)^{-1/2}}{T_c/2} \int_{-T_c/2}^{T_c/2} dt \ e^{-i2\pi ft} u(t). \tag{8}
\]

Then using the theorem of residues

\[
\tilde{u}(f) = i(2\pi)^{-1/2} \sum_j R_j \exp(i2\pi fx_j - |2\pi fy_j|), \tag{9}
\]

where \( R_j \) are the poles residue and \( x_j + iy_j \) are their location in the relevant half plane, one obtains the asymptotic behavior of the spectrum \( E(f) = |\tilde{u}(f)|^2 \) at large \( f \)

\[
E(f) \sim \exp(-4\pi |y_{\text{min}}| f), \tag{10}
\]

where \( y_{\text{min}} \) is the imaginary part of the location of the pole which lies nearest to the real axis. In the case of the symmetric analytic strip with a width \( \Delta = 2|y_{\text{min}}| \):

\[
E(f) \sim \exp(-2\pi \Delta f) \tag{11}
\]

(cf. the insert in fig. 5).

The chaotic spectrum provides two different characteristic time-scales for the chaotic system: a period corresponding to the fundamental frequency of the system, \( f_{\text{fund}} \), and a period corresponding to the exponential decay rate, \( 2\pi \Delta \) (cf. eq. (11)). The fundamental period can be estimated using the position of the low-frequency peak (cf. figs. 4 and 5), while the exponential decay rate period \( 2\pi \Delta \) can be estimated using the slope of the straight line of the broad-band part of the spectrum in the semilogarithmic representation. In the case of the global solar dynamo the width of the analytic strip \( \Delta \) can be theoretically estimated using the Carrington solar rotation period:

\[
\Delta \simeq T_c \simeq 25.38 \text{ days}. \tag{12}
\]

This period roughly corresponds to the solar rotation at a latitude of 26°, which is consistent with the typical latitude of sunspots (cf. fig. 4).

Additionally to the exponential spectrum (fig. 4), let us check the chaotic character of the wavelet regression detrended fluctuations calculating the largest Lyapunov exponent: \( \lambda_{\text{max}} \). A strong indicator for the presence of chaos in the examined time series is the condition \( \lambda_{\text{max}} > 0 \). If this is the case, then we have so-called exponential instability. Namely, two arbitrary close trajectories of the system will diverge apart exponentially, that is the hallmark of chaos. To calculate \( \lambda_{\text{max}} \) we used a direct algorithm developed by Wolf et al. [24]. Figure 6 shows the pertaining average maximal Lyapunov exponent at the pertaining time, calculated for the data set shown in fig. 2. The largest Lyapunov exponent converges very well to a positive value \( \lambda_{\text{max}} \simeq 0.286 \text{ mon}^{-1} > 0 \).

**Discussion.** – It should be noted that the same period \( \sim 3.7 \text{ years} \) was recently found for the so-called flip-flop phenomenon of the active longitudes in solar activity [25,26]. Sunspots tend to pop up preferably in certain latitudinal domains and move toward the equator due to the 11-year cycle. Recently, strong indications of non-uniform longitudinal distribution of sunspots (active longitudes) was reported and analyzed in a dynamic frame related to the mean latitude of sunspot formation, in which the active longitudes persist for the last eleven solar 11-years cycles (see refs. [25,26] and references therein). At any given time, one of the two active longitudes (approximately 180° apart) exhibits a stronger activity dominance. The observed alternation of the active longitudes dominance in 3.7 years on average was called as flip-flop phenomenon [25]. It seems rather plausible that the observed flip-flop period and the fundamental period of the wavelet regression detrended fluctuations of solar activity (fig. 4) have the same origin. In this vein, the observation [25,27] that the period of the flip-flop phenomenon follows to variations of the real length of the sunspot cycle (which has the 11-years period on average only) supports the idea of the one-third subharmonic resonance as a background of the 11-years cycle of solar activity.

Another relevant example of the 3.7-years period appearance is the interplanetary magnetic-field polarity variations [28]. A close value (~3.5 years) of the period of the geomagnetic \( aa \) index was reported in ref. [29].

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