Layer entanglement in multiplex, temporal multiplex, and coupled multilayer networks

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Abstract

Complex networks, such as transportation networks, social networks, or biological networks, translate the complex system they model often by representing one type of interactions. In real world systems, there may be many aspects of the entities that connects them together. These can be captured using multilayer networks, which combine different modalities of interactions in one same model. Coupling in multilayer networks and multiplex networks may exhibit different properties, which can be related to the very nature of the data they model (or to events in time-dependant data). We hypothesise that such properties may be reflected on the way layers are intertwined. In this paper, we want to investigate these through the prism of layer entanglement in multilayer networks. We test them on over 30 real-life networks in 6 different disciplines (social, genetic, transport, co-authorship, trade and neuronal networks). We further propose a random generator, displaying comparable patterns of elementary layer entanglement and transition layer entanglement across 1,329,696 synthetic coupled multilayer networks. Our experiments demonstrate difference of layer entanglement across disciplines, and even suggest a link between entanglement intensity and homophily. We additionally study entanglement in 3 real world temporal datasets displaying a potential rise in entanglement activity prior to other network activity.

Keywords: Multiplex networks; layer entanglement; temporal network; network topology; network generator

1 Introduction

A real world complex system often counts multiple interactions between multiple different entities. When these interactions can be regrouped under multiple families of entities, multilayer network modelling becomes a tool of choice to capture the key components of the system. The use of these models emerge in all fields of science from social sciences to finances, through logistics, biology, and many more [1].

With multilayer networks, the study of multiple viewpoints (or aspects [2]) on the same network data becomes possible. This is critical for example in social network analysis, to study the role of users in different networks, and compare them (for example the same individual may behave differently on LinkedIn, Twitter, or Facebook). These different networks form naturally different types of links that may be overlaid.

Motivated by their practical interest, multilayer networks also show interesting structures [3] that could be exploited to mine community structures or study the roles of nodes and edges in the network through centrality for example. These are also possible in a traditional network analysis standpoint but often requires some kind of simplification (such as
one-mode projection) but the recent advances show they may be obtained directly from the multilayer networks [4, 5, 6].

The key concept in multilayer networks are the layers themselves. Since the structure of such networks is driven by the layers and their aspect [1], understanding how the layers organise can reveal properties unique to the multilayer network model and the understanding of its structure [7] [8]. Particularly, the intertwining of edges, or layer entanglement [9, 10], shows how layers overlap to form coherent structures and substructures.

Although recent works have focused on multilayer network analysis and description [11, 12], not many have so far focused on a large scale analysis grouping multilayer networks of different nature and produced in different disciplines, while comparing them to synthetic models. One comparative study of flow analysis [13] has particularly influenced this paper where emerging structures are described, while not comparing them to synthetic models.

In their seminal work of McPherson et al. [14] discuss how ties emerge in social systems. They investigate how people similarity, i.e. homophily, is a strong driver to the formation of ties, with the addition to make them more durable in a dynamic system. They investigate social ties in a multilayer manner, and argue for further research: “in the impact of multiplex ties on the patterns of homophily; [and] the dynamic of network change over time [...]”.

This work extends our original work in [8] which particularly resonates with the first point of McPherson et al., in that we displayed a link between homophily [14, 15] in social networks and high entanglement intensity networks.

We extend [8], which originally contributed with an open source implementation of entanglement homogeneity and intensity for multiplex networks, while evaluating them over 30 real world networks. We proposed also a synthetic multiplex network generator. A generation of over 10k synthetic networks, and their comparison with the real world networks, displayed common patterns of entanglement homogeneity and intensity that could be specific to the families of applications that generated the networks. In this extended work, we contribute with:

- the theoretical extension of the entanglement computation to a fully multiplex model that takes into account coupling edges;
- the extension of our synthetic generator accordingly;
- the computations on a wider range of real and synthetic networks (1,329,696 synthetic networks were considered);
- and the study of entanglement in large, temporal multiplex networks.

2 Multilayer networks

A multilayer network can be defined as a sequence $M = \{G_l\}_{l \in L} = \{(V_l, E_l)\}_{l \in L}$ where $E_l \subseteq N_l \times V_l$ is a set of edges in one network $l \in L$ of the sequence [1]. Multilayer networks are commonly understood as layers comprised of interactions, where each layer corresponds to a specific aspect of the system.

Multiplex networks are specific multilayer networks so that nodes represent the same entity across all layers. We represent a multiplex network as a structure $M' = (V_M, E_M)$, where $V_M$ is the set of nodes and $E_M$ the set of all edges (in all layers). Multilayer networks can also hold some level of node coupling, i.e. some nodes may be shared amongst a subset of layers – a multiplex network being a multilayer network with a maximum node coupling. In both cases, there may exist coupling edges connecting nodes through layers, forming transition layers. These concern, for example, multilayer networks which are modelling
transportation systems [16]. In that case, we can differentiate the elementary layers (holding inner-layer edges) from the transition layers (holding coupling edges). Each transition layer $t = (l, l')$ between layer $l$ and $l'$ is yet another layer, with a set of nodes and edges. If $S \subset L$ represents the subset of all elementary layers, and $T \subset L$ the subset of all transition layers, we may define our multilayer network $M$ as the union of a multilayer network with elementary layers only and of another multilayer network with transition layers only $M = \{G_l\}_{l \in L} = M_S \cup M_T = \{G_s\}_{s \in S} \cup \{G_t\}_{t \in T}$. The coupling can heavily influence the structural behaviour of multilayer networks [17]. It can also influence the resilience of the network against failures [18] and naturally the diffusion phenomena [19] too.

Among other examples of multilayer networks, a biological system can be studied at the protein, RNA or gene level [20], and similarly, social networks can be studied by taking into account a person’s presence on multiple platforms [21]. For computational purposes, such networks are commonly represented in the form of supra-adjacency matrices, where block-diagonal structure, connecting the same node across individual layers emerges [16]. Algorithms can operate on such matrices directly and thus exploit additional information representing multiple aspects.

Algorithms for analysis of multilayer networks can also operate on sparse adjacency data structure of the multilayer network directly, yet need to take into account that a given node is present in multiple layers. Such representation is suitable for this work, as we are focused primarily on how edges co-occur across layers. Hence, this work focuses primarily on the relations between the layers of a given multilayer network. We next discuss the two measures we consider throughout this work.

### 3 Entanglement in multiplex networks

We briefly discuss the entanglement measures definitions from previous work [9].

#### 3.1 Layer interaction network

Recall our multiplex network $M = (V_M, E_M) = \{G_l\}_{l \in L}$. As mentioned earlier, such a network really distinguishes itself from classical graphs through the use of different layers to
connect nodes. These layers may have different patterns and may overlap together. There may even exist latent dependencies among these layers. To investigate this matter, each layer could be abstracted to one single node and form a new graph, the Layer Interaction Network (hereafter LIN) [9]. Visualizing the LIN is a key component for multilayer network visualization such as in Detangler [7]. In the LIN, \( LIN = (L, F) \), each node \( u_l, u_{l'}, u_{l''} \ldots \) corresponds to a layer \( l, l', l'' \ldots \in L \) of the multiplex network \( M \), and each edge \( f \in F \) captures when two layers overlap through edges. More formally, there exist an edge \( f = (u_l, u_{l'}) \) whenever there exists at least two nodes \( v, v' \in V_M \) such that there exists at least one edge connecting these two nodes on each layer \( e_M = (v, v') \in l \) and \( e'_M = (v, v') \in l' \). The LIN can be interpreted as an edge-layer co-occurrence graph, and the weight of an edge \( f = (u_l, u_{l'}) \), denoted as \( n_{l,l'} \), equals the number of times layers \( l \) and \( l' \) co-occur. By extension, \( n_{l,l'} \) is the number of edges on layer \( l \). This process is illustrated in Figure 1b.

3.2 Layer entanglement
The analysis of layer entanglement is inspired by the analysis of relation content in social networks [22]. The idea is to study the redundancy between relation content, each forming in our formalism a different layer. The layer entanglement measures the “influence” of a layer in its neighbourhood.

This measure is recursively defined: the entanglement \( \gamma_l \) of a layer \( l \) is defined upon the entanglement of the layers it is entangled with. Similarly to the eigen centrality [23], this translates into the recursive equation:

\[
\gamma_l \lambda = \sum_{l' \in T} n_{l,l'} \gamma_{l'}.
\]

The entanglement of a layer \( \gamma_l \) can be retrieved from a vector \( \vec{\gamma} \) which corresponds to the right eigenvector (associated to the maximum eigenvalue \( \lambda \)) of the layer overlap frequency matrix with corresponding overlap, defined as:

\[
C = (c_{ll'}), \quad \text{where} \quad c_{ll'} = \frac{n_{l,l'}}{n_{ll'}}.
\]

this metric was initially discussed in [9], and is constructed using the weights in the LIN (see Figures 1 and 2).

3.3 Entanglement intensity and homogeneity
The layer entanglement \( \gamma \) measures the share of layer \( l \) overlapping with other layers, so that nodes of \( M \) are connected. The more a group of layers interacts together, the more the nodes they connect will be cohesive in view of these layers, hence the more \( \gamma \forall l \in L \) values will be similar (their share of entanglement will be similar). This is captured by the entanglement homogeneity [9] which is then defined as the following cosine similarity:

\[
H = \frac{<1_L, \gamma>}{\|1_L\|\|\gamma\|} \in [0, 1].
\]

Optimal homogeneity is not necessarily reached only when all nodes are connected through all layers, but also when all nodes are connected in a very balanced manner between all layers (see Figure 2). Homogeneity thus permits various symmetries in a given LIN.
Figure 2: Two very different cases of maximum homogeneity \( H = 1 \), the multiplex network and the LIN are shown, with matrices and entanglement measures. a) all layers are saturating all edges, so we have maximum intensity \( I = 1 \); b) layers are well balanced, but we may have a lot more interactions possible.

When a maximum overlap is reached through all layers in the network, the frequencies in the matrix \( C \) (of size \(|L| \times |L|\)) are saturated with \( C_{l,l'} = 1 \). This gives us a theoretical limit to measure the amount of layer overlap through the entanglement intensity [9], defined as:

\[
I = \frac{\lambda}{|L|}.
\]

In practice, both entanglement intensity and homogeneity have been used to measure the coherence of clusters of documents [10].

### 3.4 Transition layer entanglement

We have defined the layer entanglement which measures overlap between layers of a multiplex network, but many multiplex networks include another critical parameter which is coupling edges [3]. The coupling often measures the transition of nodes between layers, hence the transition of nodes are captured by edges connecting nodes across layers.

Recall our multiplex graph \( M = (V_M, E_M) \). Suppose \( S \) is the set of elementary layers, we can then have transition between any pair of elementary layers \( l \in S \) and \( l' \in S \). Let \( u_l = (u, l), u \in V_M, l \in S \), the connection of a node \( u \) within a layer \( l \). A transition layer edge \( e \) can be defined as follows: \( e = (u_l, v_{l'}) \in E_M \) such that \( e \) connects nodes \( \{u, v\} \subseteq V_M \) across layers \( l \neq l' \), \( \{l, l'\} \subseteq S \). Coupling edges often connect a same node across two layers and may be used to model a physical transition, such as a change from subway to train in a station of a transportation network. As a consequence, a pair of layers \( (l, l') = t \) forms a transition layer \( t \in T \) when there exists at least one such edge \( e = (u_l, v_{l'}) \in E_M \). Note that taken together, these elementary and transition layer subsets form the set of all layers \( S \cup T = L \), and that the size of \( T \) is bounded by the size of \( S \) such that \(|T| \leq \frac{1}{2} |S|(|S| - 1)| \).

Now, given this definition, nothing limits the computation of entanglement (introduced in previous Sections 3.1 to 3.3) only to the elementary layers part of \( M_S \), as illustrated in Figure 3. Layer entanglement can also be used to characterise the coupling between these elementary layers if applied to the transition layers \( M_T \).
It is also possible to consider both elementary and transition layers in one multiplex network $M$ to compute entanglement (as shown in Figure 4). However, in practice, the intensity and homogeneity greatly differ between them, and often results in separated connected components of the LIN. This is due to the nature of coupling, which often captures a distinct characteristic of the network. Transition layers mostly connect the same node across layers, while elementary layers do not always display loops.

### 4 A coupled multilayer network generator

In this section, we describe an algorithm for generation of coupled multilayer networks, i.e. multilayer networks which share some nodes across some layers, but does not guarantee that all nodes are being shared between all layers. This kind of networks makes the link between general multilayer networks and multiplex networks (for which the assumption is that all nodes are shared through all layers).

The algorithm is based on the following observations. Let $M = (V_M, E_M)$ represent a coupled multilayer network with layer set $L$. Each node is associated to a random number of layers $\{l_1, l_2, \ldots, l_i\} \subseteq L$. Now for each layer $l_i \in L$ there is a set of nodes $V_{l_i} \subseteq V_M$ which forms a potential set of edges of size $|E_{l_i}| = \frac{1}{2} |V_{l_i}|(|V_{l_i}| - 1)$. We introduce $o$, a parameter determining the probability of a node occurring at a given layer. We then introduce the probability $p$ of an edge to be created between any pair of nodes belonging to a layer so we may avoid cliques to form on each layer. We referred in our previous work to the edge dropout [8], which is $d = 1 - p$ as the share of links we drop from the clique model. Intuitively, the more similar a given random multiplex is to a clique over each layer, the higher its elementary layer intensity should be. The generator also accounts for coupling.

![Image](https://example.com/image.png)

Figure 3: Computing entanglement on the transition layer edges. (a) Coupling edges are illustrated in orange ($L_1 - L_2$ edges) and in purple ($L_2 - L_3$ edges). (b) Computing the corresponding LIN and entanglement measures. Coupling edges of a same node resemble loops except they are defined across two layers. We may notice that: the transition layer $L_2 - L_3$ shows a slightly higher index since there are more transitions for this layer; the homogeneity $H$ is (almost) maximal since both layers are (almost) equally intertwined (only 2 layers, actual $H \approx 0.99986$).
Figure 4: Computing entanglement on both inner-layer and coupling edges. (a) Note that in contrast to the example in Figure 3, we have added a loop to node $p_5$ in layer $L_3$ (in red) and an coupling edge connecting nodes $p_3$ of layer $L_2$ to $p_5$ in $L_3$. (b) Computing the corresponding LIN and entanglement measures. We can notice that the transition layers being the most intertwined display the highest entanglement index. Because of the entanglement relies on the limited overlap of elementary layer edges and transition layer edges, the entanglement intensity $I$ is rather low.

by adding transition layer edges. These coupling edges are connecting nodes across two layers. We introduce $q$, the probability for a same node to be connected across two layers. The higher $q$, the more nodes will be connected through layers. Note that in our initial work [8], neither $o$ nor $q$ were considered ($o$ was in fact picked uniformly).

The purpose of this generator is to offer a simple testbed for further exploration, as well as additional evidence of the relation between homogeneity and intensity on many random, synthetic networks. The Algorithm 1 represents the proposed procedure.

The generator first randomly assigns the same node index to the many layers (lines 2-5). Once assigned, the layers are processed by applying sampling on $|V_{\bar{l}}|^2$ possible edges in layer $\bar{l}$. Note that in line 7, this whole clique is virtually generated. The global multiplex is updated during this process (lines 6-10). These steps are then repeated for each transition layers i.e. pairs of elementary layers (lines 11-14). The implementation thus uses a generator with lazy evaluation, avoiding potential combinatorial explosion with a large number of nodes (very large networks).

4.1 Some theoretical properties of the generator
In this section we show two properties of the proposed generator. We denote $n = |V_M|$ the parameter setting the number of nodes of the network, $m = |L|$ the parameter setting the number of edge layers in the network, and $p$ the inner-layer edge probability.

**Proposition 1** (Number of edges in non-coupled multiplex networks ($q = 0$)) Let $\phi \in \mathbb{N}^+$ represent the number of possible edges. Then $\phi \leq m \cdot \binom{n}{2}$. 
Algorithm 1: Multilayer network generator.

Parameters: Number of nodes $n$, number of layers $m$, inner-layer edge probability $p$, coupling edge probability $q$.

Result: A coupled multilayer network $M$

1. $M \leftarrow \text{emptyMultilayerObject}$;
2. for node in $[1 \ldots n]$ do
3.   layerNodes $\leftarrow \text{assignNodeToLayers}(\text{node}, o, m)$;
4.     \text{▷ Nodes are assigned to layers among $m$ with probability $o$.}
5.   update($M$, layerNodes);
6.     \text{▷ Update global network.}
7. for layer $l_i$ with corresponding node set $V_l$ do
8.   nodeClique $\leftarrow \text{generator of node pairs from } V_l$;
9.     \text{▷ With or without possible loops.}
10.  innerLayerEdges $\leftarrow \text{sampleWithProbability}(\text{nodeClique}, p)$;
11.    \text{▷ Sample via $p$.}
12.  update($M$, innerLayerEdges);
13.    \text{▷ Update global network.}
14. end
15. for layers $l_i, l_j$ with shared node set $V_{l_i, l_j}$ do
16.   sameNodeTransitionLayerEdges $\leftarrow \text{sampleWithProbability}(V_{l_i, l_j}, q)$;
17.     \text{▷ Sample via $q$.}
18.   update($M$, sameNodeTransitionLayerEdges);
19.    \text{▷ Update global network.}
20. end
21. return $M$;

Proof Let $o = 1$. Each layer can have at most $n$ nodes. Assuming they form a clique, each layer is thus comprised of $\binom{n}{2}$ edges. As there are $m$ layers, there can be at most $m \cdot \binom{n}{2}$ edges — a clique of $n$ nodes in each layer (assuming $p = 1$). We refer to this bound as $\Phi \leq m \cdot \binom{n}{2}$.

Corollary 1 (Time complexity) In the limit, as $p \to 1$, a full clique needs to be constructed, assuming each node is projected across all layers. The complexity w.r.t. the number of layers and edges is: $\mathcal{O}(m \cdot \binom{n}{2}) = \mathcal{O}(|E_M|)$.

Note that even though, theoretically, the proposed generator creates a clique and then samples from it, current, lazy implementation only generates the edges needed to satisfy a given $p$ percentage. In practice, only when $p \approx 1$, the generator needs larger portions of space (and time). As such, fully connected networks do not represent real systems, we were able to generate a multitude of very diverse networks.

When considering $q > 0$, this directly translates to the increase of $m$ in Proposition 1. Hence, the number increases linearly with the number of coupling layers added. We next discuss the impacts of $q$ parameter.

Proposition 2 (Coupling edges) The number of coupling edges has worst case complexity of $\mathcal{O}(\binom{m}{2} \cdot n)$.

Proof Let $l_a$ and $l_b$ represent a given pair of layers, where each layer consists of all $n$ possible nodes. As each node couples only to itself, there are at most $n$ edges between $l_a$ and $l_b$. As there are $\binom{m}{2}$ possible layer pairs, if nodes are in each pair fully coupled, the network can have at most $\binom{m}{2} \cdot n$ coupling edges.

The consequence of this proposition is that the number of layers drastically increases the number of possible edges, which can result in longer computation times. We next discuss the relation between entanglement intensity and edge probability when considering transi-tional edges. Since entanglement intensity and homogeneity can be computed for arbitrary
sets of edges, let that be the edges from *elementary* layers, as well as the ones from *transition* layers. Intuitively, entanglement intensity should rise with edge probability: the larger the probability that an edge is present in a given layer, the larger the probability that a given pair of edges will overlap\[1\].

However, is that also the case when considering only transitional layers? Consider the following example of a multiplex network without the coupling edges. No matter what $p$ is employed, if $q \approx 0$, transitional intensity will be low – very few coupling edges are introduced. As such edges induce the transitional layers considered by entanglement computation, the observed LIN will be very sparse. Hence, we posit that the distribution of intensity shall be, in fact *constant* with respect to a given $p$. The proof of this claim is by contradiction. Assuming $p$ would indeed influence coupling entanglement intensity. As transitional intensity is defined solely based on the coupling edges, this claim would imply parameter coupling between $p$ and $q$, which is by the definition (and design) not the case.

Even if the nodes are *isolated* in each layer, coupling intensity can be high. Note also that the node positioning, governed by $o$ on the other hand directly impacts both elementary and transition entanglement, as, for example, in very node-scarce networks, there are fewer possible edges than if all nodes are present in each layer. These points are illustrated in our empirical evaluation Section 6 and further in Appendix.

### 5 Layer entanglement in temporal multiplex networks

Analysis of temporal multiplex networks has shown promising results in multiple fields of science, such as for example healthcare and transportation [24].

Since patterns of layer interaction networks result in typical entanglement values, considering temporal entanglement means textit(ising particular topologies of a temporal multiplex network. For example, a high intensity among members in a multiplex social network communicating through different social media corresponds to a synchronization of communications between them. When such a synchronization corresponds to the preparation of a particular event, understanding such synchronization could help forecast the event.

In this section, we first discuss how we define temporal multiplex networks and entanglement time series. We limit the following discussion to the consideration of entanglement between elementary layers only, *i.e.* only inner-layer edges.

#### 5.1 Temporal multiplex networks and entanglement

Real-life networks often evolve in time, making them behave differently at different points. In our current setting, we define the temporal aspect of our network such as each edge $e_t$ is defined at a specific time point $t$. A multiplex network $M_d$ can then be defined for a given time window $d$. A time window $d = [t_0, t_f]$ covers a time frame (beginning at $t_0$ and ending $t_f$), and the multiplex network $M_d$ is defined such as each edge exists within the time window:

$$M_d = (V_M, \{e_t \in E_M \mid t \in d\}).$$

The second scenario we considered is that of *moving time windows*. Here, edges from the $f$ *past* windows are considered when constructing a given network $M$, *i.e.*,

$$M_f = (V_M, \{e_t \in E_M \mid t \in \{d-f, \ldots, d-i\}\}).$$

\[1\] One of the purposes of this work is to quantify this relation exactly.
Figure 5: Converting temporal edges of multiple types into temporal entanglement series. a) Edges of different types are defined over time between $t_0$ and $t_f$. b) Time frames $d_1$, $d_2$, and $d_3$ are defined so we may construct the three corresponding multiplex network slices. c) For each slice, we can compute a LIN and the corresponding entanglement intensity $I$ and homogeneity $H$, which compose the series once taken together among all slices.

Our intuition is to compare the shape of the networks at different moving time windows. For example, we could compare political social networks under different rulers of a country [25, 26]. To do so, we can simply compute entanglement homogeneity and intensity for each time window and compare them. Considering our multiplex setting, with the nodes to be shared across all time frames is not a limitation since the entanglement computation focuses on edges.

Slicing the time windows is a whole different topic and many options are open [4, 27], as it could be achieved manually, with equal time slices, moving window, or with volume of changes. In our context, we consider the identification of time window through slices of equal size in time, but the principle can be extended. We refer to the size $r$ in time of the slices as time resolution.

We may now investigate entanglement homogeneity and intensity properties with respect to time resolution ($r$), and verify if patterns of intensity/homogeneity variation can be predicted. Note that one challenge of slice-based modelling of temporal multiplex networks is the problem of selecting the correct resolution $r$, i.e. how coarse (or fine)-grained the intervals must be in order to capture desired dynamics.

In a system covering a global period of $D$, once a slicing resolution is chosen, we can observe values of homogeneity and intensity at the time series level, i.e. for each slice $d \in D$, and define the intensity time series $S_I = \{I_M\}, \forall d \in D$ and the homogeneity time series as $S_H = \{H_M\}, \forall d \in D$. These intensity and homogeneity time series can now feed further processing. Note that $S_M_I$ and $S_M_H$ are defined analogously (entanglement for the
past $f$ slices, moving in the increments of one slice). The whole processing from temporal edges to time series is illustrated in Figure 5.

In our following evaluation (Section 6.3), we explore $S_I$ and $S_H$ when also considering a moving window of previous $f$ time slices. The rationale for considering past $f$ slices up to the considered time point is that such information only includes past data, and could indicate whether entanglement can be also used for forecasting purposes. The second option considered, where only the current time slice was plotted, can shed insight on whether online monitoring based on $I$ or $H$ is a sensible option.

6 Empirical evaluation

We now study entanglement intensity and homogeneity across different series of networks. We first investigate entanglement measures across different parameters of synthetic settings. We follow with investigations on a large panel of real world networks. We finish our study with the study of entanglement in temporal multiplex networks.

6.1 Entanglement in synthetic networks

In this first study, we compare entanglement measures over a series of synthetic multiplex networks, using our proposed generator.

6.1.1 Multiplex networks without transition layers

A first generation concerns multiplex networks settings in which transition layers are not specified (for example, friendship over different social platforms).

We used the following hyper-parameter ranges to generate 1,329,696 synthetic networks (a couple are illustrated in Figure 6):

- Number of nodes ($n$) from 10 to 200 in increments of 10.
- $m$ (number of layers) in 1,2,3,4,6,7,9,10.
- Layer assignment probability ($o$), from 0 to 1 in increments of 0.05
- Edge probability ($p$) from 0 to 1 in increments of 0.05.
- Transition layer edge probability ($q$) from 0 to 1 in increments of 0.05.

We measure entanglement intensity $I$ and homogeneity $H$ on each generated network (averaged over all connected components). We investigate the role of the different parameters over the entanglement measures, as illustrated in Figures 7, 8 and 9.
There is an obvious dependency between entanglement intensity and homogeneity since we cannot obtain low homogeneity with high intensity values (Figure 7). This is due to the nature of both measures. With a high intensity, most of the layers are overlapping over most of the network. As a consequence, there is little space for permutations in the way layers overlap, this means the entanglement of all individual layers $\gamma$ tends to align, hence high values of homogeneity. This leads to a denser production of high homogeneity networks as illustrated by the density lines in Figure 7.

The number of nodes $n$ and edges $m$ do not show a strong dependency with homogeneity, but a slight one on intensity. Higher values of $n$ and $m$ make it easier to obtain sparser networks, with the consequence of resulting lower values of intensity. We further illustrate these in Figure 8. This effect mitigates quickly with higher numbers of nodes and layers.

We further explore the edge assignment probability of a node $o$, and the inner-layer edge probability $p$ in Figures 9. There is a first dependency appearing on the layer assignment probability $o$, for which higher values tend to produce higher homogeneity (Figure 9b). Higher homogeneity is reached when all layers contribute equally, meaning that a higher $o$ shows more chances for each layer to contain most of the nodes. We may also observe apparent linear trend between the edge probability $p$ (sparseness) and entanglement intensity (Figure 9d). This trend confirms that sparser networks (i.e. lower $p$) are less “intensely” overlapping over edges. As intensity directly measures this property, this result outlines one of the desired properties of the proposed network generator.
Figure 8: Results on synthetic multiplex networks without considering transition layers. Dependency on the number of nodes $n$ (a, b) and layers $m$ (c, d) on the elementary layer entanglement. The intensity (b, d) shows some influence on each parameter.

### 6.1.2 Multiplex networks with transition layers

A second experiment is focusing on multiplex graphs with transition layer, *i.e.* considering coupling edges in our 1,329,696 generated networks (illustrated in Figure 10). This experiment reproduces the previous one, but focusing on the transition layer entanglement. Results are shown in Figure 11 and 12, dependency on the number of nodes and layers is illustrated in Appendix. From Figure 11, the shape is globally the same, with the difference in a skewer density of high-homogeneity without a dense production of very low intensity generated networks (from the density lines).

The profile is sensibly the same than that of the previous experiment, except that the layer assignment probability $o$ appears to have a more diffuse impact, and the direct dependency is this time observed on the coupling edge probability $q$. Comparison with parameter $p$ obviously does not influence entanglement, but can be found in Appendix materials for additional inspection.
Overall, the networks with transition layers are more saturated when compared to the ones without transition. The reason may be that we only consider here transition layer edges that only connect the same node across layers.

For the interested reader, we also illustrate in the Appendix material the independence of parameters $q$ over the elementary layer entanglement and $p$ over the transition layer entanglement. Finally, we also report there the computation of entanglement over the combined elementary and transition layers, which displays a dependency on both $p$ and $q$ parameters.

### 6.2 Multiplex network comparison across disciplines

We now consider real world static networks. All considered networks are summarised with their main characteristics in Table 1\[^2\]. Unfortunately, we have not found a real case with a large number of transition layer edges, so we limit this evaluation to elementary [\[^2\]The networks are hosted at https://comunelab.fbk.eu/data.php]
layer entanglement. For each network, we computed elementary layer homogeneity and intensity, for all connected components.

We first investigate individual results through the distributions of each metric across network types, Figure 13. We then compare individual networks across entanglement intensity and homogeneity Figure 14.
Two main observations are apparent when studying the results on real networks. First, the difference between social and genetic (biological) multiplex networks becomes obvious when both entanglement intensity, as well as homogeneity are considered (Figure 14). To confirm these differences, we further compare their distributions, i.e., the intensity and homogeneity of social vs. genetic networks, in Figure 15.

In addition, from Figure 14, we may observe that many genetic networks sit in relatively low intensity/homogeneity places, whereas social networks sit in the top right corner: the high entanglement homogeneity of social networks is quite noticeable. This suggests a few interpretations:

- genetic networks show in general very little layer overlap;
- some genetic networks are very sparse and could be simulated with low inner-layer edge probability;
- layers in social networks tend to overlap a lot;
- social networks tend to be quite dense and may be simulated by synthetic networks with a high inner-layer edge probability;
Figure 13: Entanglement homogeneity and intensity compare for each category of networks, showing quite diverse set of properties proper to the different families of networks.

Table 1: Real multiplex networks and their properties. The ID in the second column corresponds to Figure 14.

| Dataset ID | Type        | Nodes | Edges | Number of layers | Mean degree | CC  |
|------------|-------------|-------|-------|------------------|-------------|-----|
| arXiv-Netscience [13] | Coauthorship | 26796 | 59026 | 13               | 4.41        | 3660 |
| PierreAuger [13] | Coauthorship | 126 | 137 | 3               | 2.12        | 4 |
| Arabidopsis [28] | Genetic | 8765 | 18655 | 7               | 4.26        | 387 |
| Bos [28] | Genetic | 369 | 322 | 4               | 1.75        | 82 |
| Candi [28] | Genetic | 418 | 398 | 7               | 1.90        | 50 |
| Colog [28] | Genetic | 4557 | 8182 | 6               | 3.59        | 153 |
| DandRene [28] | Genetic | 10 | 18 | 5               | 2.09        | 45 |
| Drosophila [28] | Genetic | 11970 | 43367 | 7               | 7.25        | 346 |
| Galus [28] | Genetic | 367 | 389 | 6               | 2.12        | 54 |
| HepatitisC [28] | Genetic | 129 | 137 | 3               | 2.12        | 4 |
| Homo Sapiens [28] | Genetic | 36194 | 170899 | 7               | 9.44        | 785 |
| HumanHepes [28] | Genetic | 261 | 259 | 4               | 1.98        | 21 |
| HumanIV [28] | Genetic | 1195 | 1355 | 5               | 2.27        | 13 |
| Cryptobug [28] | Genetic | 151 | 144 | 3               | 1.91        | 21 |
| Plasmodium [28] | Genetic | 1206 | 2522 | 3               | 4.18        | 27 |
| Rattus [28] | Genetic | 3263 | 4268 | 6               | 2.62        | 296 |
| SacChCen [28] | Genetic | 27994 | 282755 | 7               | 20.20       | 432 |
| SacChPomB [28] | Genetic | 10178 | 63677 | 7               | 12.51       | 286 |
| Xenopus [28] | Genetic | 582 | 620 | 5               | 2.13        | 109 |
| YeastLandscape [29] | Genetic | 17770 | 8473997 | 4               | 9537.4       | 4 |
| C-Elegans [30] | Neuronal | 791 | 5863 | 14.82           | 6 |
| Cans [31] | Social | 639951 | 991854 | 3               | 3.01        | 48375 |
| CMX-PhysicsRev-Innovation [31] | Social | 674 | 1551 | 3               | 4.60        | 12 |
| CS-Aarhus [32] | Social | 224 | 620 | 5               | 5.54        | 13 |
| Kapermin-TaylorShop [33] | Social | 150 | 1018 | 4               | 13.57       | 5 |
| Krakardt-High-Tech [34] | Social | 63 | 312 | 3               | 9.90        | 3 |
| Lazeza-Law-Firm [35] | Social | 211 | 2571 | 3               | 24.37       | 3 |
| MalmKing2013 [12] | Social | 339242 | 396671 | 3               | 2.02        | 36041 |
| MoscowAthletics2013 [12] | Social | 133819 | 210250 | 3               | 3.15        | 6323 |
| OBananaSocialData2013 [13] | Social | 3457453 | 4061960 | 3               | 2.35        | 651441 |
| Padgett-Florence-Families [36] | Social | 26 | 35 | 2               | 2.69        | 2 |
| Vickers-Chan-7thGraders [37] | Social | 87 | 740 | 3               | 17.01       | 3 |
| FAQ [38] | Trade | 41713 | 318346 | 364             | 15.26       | 571 |
| EUAIE [39] | Transport | 2034 | 3588 | 37               | 3.53        | 41 |
| London [18] | Transport | 388 | 441 | 3               | 2.21        | 3 |

The results on social networks indicate a high level of layer overlap and it may be due to the overall behaviour of people, which is rather similar across different networks, whatever their means of interaction. Simmelian ties, triadic closure, and homophily (which are well studied in social sciences) are probably strong drivers of this layer overlapping.

6.3 Entanglement in temporal multiplex networks

In our last experiment, we investigate entanglement across time slices of three real-life temporal multiplex networks: Malking2013, MoscowAthletics2013, and Cannes2013 (as found in [12]). Each network consists in a collection of Twitter activity related to some event. The
Figure 14: Real networks: $H \times I$. Labels of networks map to Table 1 (ID). Grey dots represent synthetic samples of Figure 7, with Gaussian kernel density estimation over lines over the real world samples. Social networks fall within the high homogeneity/intensity range, coinciding with the high inner-layer edge probability parameter $p$ of synthetic networks.

The networks are comprised of three layers of connection, namely *retweets*, *replies* and *comments*. They can be summarised as follows. The *MLKing2013* data set consists of 421,083 events covering a week of celebration of M.L. King’s speech “*I have a dream*” in 2013, forming 396,671 edges between 327,708 nodes. The *MoscowAthletics2013* data set consists of 303,330 events covering two weeks of the World Championships of Athletics held in Moscow in 2013, forming 210,250 edges between 88,805 nodes. The *Cannes2013* network consists of 1,297,545 events (temporal edges) covering a month of the 2013 Cannes Film Festival, together forming a network of 930,419 edges and 438,538 nodes. Note that the networks are not trivially small, offering additional evidence of entanglement computation scalability.

The networks were analysed following the methodology introduced in Section 5.

We propose two experiments with regard to time segmentation. The first experiment considers fixed time windows of sizes 1h, 3h, 6h, and 12h. We compare with the activity volume in form of a total number of tweets – as found in [12], Figure 1 for a 1h window
size, here reported in Figures 16a, 17a, and 18a. We normalise here this volume so values are in $[0, 1]$.

We selected the coarse windows at their best readability for each dataset (3h for MLKing2013 in Figure 16b, 6h for MoscowAthletics2013 in Figure 17b, and 12h for Cannes2013 in Figure 18b) – each coarsening is further illustrated in Appendix. A second experiment considers a moving window of the size corresponding to these best windows, sliding by the hours (Figures 16c, 17c, and 18c).

Figure 15: Distributions of homogeneity and intensity when genetic networks are compared to social ones.

Figure 16: Visualization of temporal entanglement across MLKing2013. In grey, volume over the period of time (dotted line for the aggregated volume over sliding window (c)). Intensity in blue and homogeneity in yellow.
In the *MLKing2013* data set (Figure 16), we can observe that spikes of intensity surround the main spike of volume activity. A smaller spike of intensity consistently coincides with a smaller spike of volume at the end of the main spike.

In the *MoscowAthletics2013* data set (Figure 17), the 1h-time window does not show a consistent behaviour. However, we can see that spikes in coarser time windows coincide with the spikes in volume. A larger spike in intensity appears before the final spike in volume.

In the *Cannes2013* data set (Figure 18), the 1h-time window shows some spikes in intensity, especially a major by the end of the period of activity in terms of volume. In coarser time windows, we can notice four main spikes: one before the beginning of volume of activity; the next two ones appear just before a slight increase in the daily volume; the last one appears the day before the last day of the volume activity. This last peak appears even more prominent from the sliding window example.

The volume captures Twitter activity, governed by the human activity following the day/night rhythm. Although entanglement intensity is also submitted to it, we see emerging patterns that seem proper to each type of event. The activity of entanglement shows definitely some relationship with volume while telling a different story. The sports event that is *MoscowAthletics2013* may be much more subject to the day-by-day routine in which different disciplines are at play. On the other hand, the speech celebration in *MLKing2013* has some very specific activity before (could it be anticipation?) and after (could it be ripples?) the event. The movie festival in *Cannes2013* may be governed by sub-events of different importance in terms of networking activity.
In accordance with the position of social networks in our evaluation of real-world networks in Section 6.2, we see a decrease in homogeneity whenever we see spiking of intensity. This may indicate that a lot of the network activity suddenly focuses on one specific modality of exchange (such as replies). Entanglement study may help in targeting when this is driven by a particular modality.

Further studies on the nature of the events, and the specific topologies of the LIN networks that gave rise to these entanglement values is necessary for a more in-depth analysis of each case. Since we see some spiking activity of entanglement before actual events took place, we may suspect that, beyond monitoring, there is a predictive power of modelling time series from entanglement in past data (sliding windows).

7 Discussion and conclusions

In this work, we have revisited the notion of layer entanglement and extended it to coupled multilayer networks and temporal networks. To investigate entanglement, we have proposed a random generator for coupled multilayer networks, and generated a large set of synthetic ones. We have evaluated entanglement intensity and homogeneity in all cases, and compared to static and temporal real world networks.

Our analysis of the synthetic networks outlined that entanglement intensity is directly correlated with edge probability parameter – the sparser the network, the lower the intensity. This result indicates the proposed generator indeed emits networks which adhere to this property. We have also observed that large parts of the generated networks are subject to high homogeneity with various degrees of entanglement intensity.
The detailed inspection of the synthetic networks with respect to the parameters $d$ and the number of layers ($m$) reveals that the generative process is more sensitive to edge probability (layered patterns of intensity emerge), than to the number of layers (uniformly distributed w.r.t. homogeneity) with respect to $p$ or $q$ whether considering the elementary layers or the transition layers. This property indicates the model’s parameters could also be investigated theoretically, which we leave for future work.

The high homogeneity observed may be a byproduct of our computations. First, our random generation induced a lot of small connected components, and small components tend to show higher homogeneity since there are not so many degrees of freedom for edges to overlap. Because we are averaging the entanglement intensity and homogeneity over all components, this may go in favour of high homogeneity. Understanding this effect deserves more investigation. Second, entanglement homogeneity is a cosine measure, and the observed values may suffer from the skewness of cosine values when distributed in a linear space, amplifying the effect of having large values. Furthermore, it might also suffer from the curse of dimensionality in the case of a high number of layers. It would be worth considering normalizing this homogeneity with respect to the number of layers involved and the number of edges they cover.

We further demonstrated that the two measures offer interesting insights when computed across a wide array of real-world networks. The observed relationship between the intensity and homogeneity of layer entanglement with the family of dataset was previously reported for clusters of documents (in [10], Figure 5). In this previous experiments, clusters of documents were mostly located at the left frontier of high intensity for a varying homogeneity. Our current experiments showed that real networks cluster based on their type (e.g. biological vs. social), also close to this frontier. We have observed (from Figure 13) that the set of genetic networks tend to match networks with lower edge probability $p$, as opposed to social networks which tend to find their way in the higher probability area. This should be further investigated, but this may be related to homophily [14, 15]. Homophily is the implied similarity of two entities in a social network, and the property of entities to agglomerate when being similar. If the reason of ‘being similar’ could be modelled as a layer of interaction, the result of a group of entities in ‘being similar’ would lead to the formation of a clique in this layer, hence locating social networks in low probability areas.

The proposed work offers at least two prospects of multiplex network study which are in our belief worth exploring further. The difference between the genetic and social networks is possibly subject to very distinct topologies which emerge in individual layers. This claim may further be investigated via other measurements, such as graphlets, communities or other structures. Next, genetic networks are less homogeneous. Future work includes exploration of this fact, as it can be merely a property of the networks considered, empirical methodology used to obtain the networks or some other effect.

We believe that theoretical properties of the proposed network generator can also be further studied, offering potential insights into how multiplex networks behave and whether the human-made aspects are indeed representative of a given system’s state. The model that we are currently exploring only takes into account a probability of linkage through or within layers without guarantee of connectivity. We made this choice to be able to compare between different fields, without prior that would, for example, rule in favour of similarity to social network. Our future work will investigate other generation models including Erdős–Rényi-based [40] or other with preferential attachment [41].
The analysis of the real-life temporal network offers cues on that layer entanglement can happen prior to some other event. Having tested multiple time scales, we observed that entanglement appears as much consistent to the time series as the volume of data it observes. Too small time windows mostly result in noisy time series carrying low amounts of useful information, while higher coarsening shows activity related to volume, but with a different light on the events that are captured. Future work will dive deeper into these events, and consider testing entanglement as a predictor using approaches such as of Prophet [42].

When considering entanglement as a either a monitoring or a predictive variable, its utility largely depends on the time scale at which a given edge stream needs to be considered. We leave extensive, possibly automatic determination of a setting where entanglement would be of practical relevance for future work. To study the parameters driving the dynamics of entanglement in temporal networks, we will consider comparing entanglement measures with synthetic temporal networks in our future investigations.

Availability
The code for reproduction of experiments is freely available at https://gitlab.com/skblaz/entanglement-multiplex. Further, entanglement analysis and the generator were incorporated into Py3plex library for simpler use.

Acknowledgements
The work of the first author was funded by the Slovenian Research Agency through a young researcher grant. The work of other authors was supported by the Slovenian Research Agency (ARRS) core research programme Knowledge Technologies (P2-0103) and ARRS funded research project Semantic Data Mining for Linked Open Data (financed under the ERC Complementary Scheme, N2-0078). We also acknowledge Dagstuhl seminar-19061 where many ideas implemented in this paper emerged.

Competing interests
The authors declare that they have no competing interests.

Author’s contributions
Both authors have contributed equally to the theoretical background, design of the experiments, and the writing of the manuscript. BR contributed to experiments, but the most of the experiments was handled by BS.

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Appendix

Dependency in synthetic networks over nodes and layers

One can predict of course a level of dependency over the number of nodes $n$ and layers $m$ for the transition layer case too. The dependency tends towards lower entanglement values since when increasing the number of nodes and layers, we increase the degree of freedom for layers to overlap. This trend, first illustrated in Figure 8, is confirmed in Figure 19.

Figure 19: Dependency on the number of nodes and layers on the transition layer entanglement.
Independence of parameters

The distribution of parameters of transition layer intensity and homogeneity over parameter \( p \), and elementary layer intensity and homogeneity over parameter \( q \), show no dependency as illustrated in Figure 20.

Figure 20: From the computation of entanglement over elementary layers (a, c, d) and transition layers (b, e, f), we see no dependency on parameters \( p \) and \( q \).
Combining both elementary and transition layers

As we mentioned in Section 3.4, one can compute entanglement over all the network, combining elementary and transition layers (as illustrated in Figure 21). Although we have not identified practical use cases for this entanglement (often both categories of layers tell a different story), we report here the results over our synthetic networks in Figures 22, 23, and 24. As expected, we may observe a strong dependency over both $p$ and $q$ parameters combined (Figure 24). Note that the current generator does not forbid the creation of loops enabling overlap between elementary and transition layers. A generation of transition layer edges that would connect different nodes between layers would create even more overlap between elementary and transition layers. Such a parameter is actually available in the proposed code, but beyond the scope of this paper.

Figure 21: Visualization of both inner-layer and coupling edges in synthetic multilayer networks.
Figure 22: Homogeneity and intensity $H \times I$ results on 1,329,696 synthetic multiplex networks considering their combined elementary and transition layers with density lines (Gaussian kernel density estimation).
Figure 23: Dependency on the number of nodes and layers on the combined layers entanglement.
Figure 24: Dependency on the different probabilities $o$, $p$ and $q$ on the combined layers entanglement.
Choosing the right size of time window

Choosing the right size of time-window fundamentally depends on the dataset we observe. We report all variations of fixed time window coarsening we have explored, among 1h, 3h, 6h, and 12h-long windows for each of the MLKing2013 (Figures 25), MoscowAthletics2013 (Figure 26), and Cannes2013 (Figure 27) events. Too fine selection displays a lot of noise, too coarse eludes most of the content.

Figure 25: Different sizes of time windows for the MLKing2013 data set.
Figure 26: Different sizes of time windows for the *MoscowAthletics2013* data set.

Figure 27: Different sizes of time windows for the *Cannes2013* data set.