The solution of the complete nontrivial cycle intersection problem for permutations

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Abstract
We prove the Complete nontrivial cycle \( t \)– intersection problem for permutations of finite set.

I Introduction

Let \( \Omega(n) \) be the set of permutations of \([n]\). We say that two permutations \( p_1, p_2 \in \Omega(n) \) are \( t \)– intersect if they have at least \( t \) common cycles. The main problem we consider here is obtaining the maximal cardinality of \( t \)– cycle intersection family \( \mathcal{A} \) from \( \Omega(n) \), such that \( \mathcal{A} \) contains less than \( t \) common cycles, we call such set nontrivial intersecting. Family of \( t \)– cycle intersecting permutations of \([n]\) we denote \( \Omega(n, t) \), and family of nontrivial \( t \)– cycle intersecting by \( \tilde{\Omega}(n, t) \).

We say that \( i \) is fixed in the permutation \( p \in \Omega(n) \) if it contains singleton cycle \( \{i\} \). Denote \( f(p), p \in \mathcal{P}(n) \) the set of points from \([n]\) fixed by \( p \). Denote also \( G(\mathcal{A}) = \{f(p) : p \in \mathcal{A}\}, \mathcal{A} \in \Omega(n) \).

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Let

\[ M(n, t) = \max \{|A| : A \subset \Omega(n, t)\}, \]
\[ \tilde{M}(n, t) = \max \{|A| : A \in \tilde{\Omega}(n, t)\}. \]

Let

\[ f(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{n!} \]

be the number of permutations on the set \([n]\) that do not have singletons. One can easily show that

\[ \frac{n!}{e} - 1 < f(n) < \frac{n!}{e} + 1. \]

Denote

\[ \gamma(\ell) = \frac{\sum_{i=0}^{n-\ell+1} f\left(n - \frac{\ell+2}{2} + 1 - i\right) \binom{n-\ell+1}{i}}{\sum_{i=0}^{n-\ell} f\left(n - \frac{\ell+2}{2} - i\right) \binom{n-\ell}{i}}. \]

In [9] was proved the following complete cycle \(t\)–intersecting problem for finite permutation.

**Theorem 1** Let \(t \geq 2\), and let \(\ell = t + 2r\) be the largest number not greater that \(n\) satisfying the relation

\[ \frac{\ell - t}{2(\ell - 1)} \gamma(\ell) \leq 1. \tag{1} \]

then

\[ M(n, t) = \sum_{i=t+r}^{t+2r} \binom{t + 2r}{i} \sum_{j=0}^{n-t-2r} \binom{n-t-2r}{j} f(n - i - j). \tag{2} \]

Note that value \(M(n, t)\) in (2) is equal

\[ M(n, t) = \max_{r \leq (n-t)/2} \sum_{i=t+r}^{t+2r} \binom{t + 2r}{i} \sum_{j=0}^{n-t-2r} \binom{n-t-2r}{j} f(n - i - j), \]

where max is taken over \(\ell\), satisfying (1). For \(t = 1\) it is proved in [2],[3], that

\[ M(n, 1) = (n - 1)!. \]

Let \(2^n\) is the family of subsets from \([n]\) and \(\binom{n}{k}\) be the family of \(k\)–element subsets from \([n]\). Let also \(I(n, t)\) be the set of \(t\)–intersecting (in the set
theoretical sense) subsets \(\mathcal{A}\) of \([n]\), \(I(n,k,t)\) be the set of \(t\)-intersecting \(k\)-element subsets of \([n]\) and \(\bar{I}(n,t)\), \(\bar{I}(n,k,t)\) the corresponding families of nontrivial \(t\) intersecting families (\(|\cap_{A \in \mathcal{A}} A| < t\)). Denote
\[
\bar{M}(n,k,t) = \max_{A \in \bar{I}(n,k,t)} |A|.
\]
Hilton and Milner proved in \([6]\)

**Theorem 2**

\[
\bar{M}(n,k,t) = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1, \quad n > 2k.
\]

For \(t > 1\) P.Frankl \([7]\) proved

**Theorem 3** For sufficiently large \(n > n_0(k,t)\)

- If \(t + 1 \leq k \leq 2t + 1\), then \(\bar{M}(n,k,t) = |\nu_1(n,k,t)|\), where
  \[
  \nu_1(n,k,t) = \left\{ V \in \binom{[n]}{k} : |[t+2] \cap V| \geq t + 1 \right\},
  \]

- If \(k > 2t + 1\), then \(\bar{M}(n,k,t) = |\nu_2(n,k,t)|\), where
  \[
  \nu_2(n,k,t) = \left\{ v \in \binom{[n]}{k} : [t] \subset V, \ V \cap [t+1,k+1] \neq \emptyset \right\}
  \cup \left\{ [k+1] \setminus \{i\} : i \in [t] \right\}.
  \]

In \([4]\) problem of determining \(\bar{M}\) was solved completely for all \(n,k,t\):

**Theorem 4** If \(2k - t < n \leq (t + 1)(k - t + 1)\), then
\[
\bar{M}(n,k,t) = M(n,k,t),
\]
if \((t + 1)(k - t + 1) < n\) and \(k \leq 2t + 1\), then
\[
\bar{M}(n,k,t) = |\nu_1(n,k,t)|,
\]

- if \((t + 1)(k - t + 1) < n\) and \(k > 2t + 1\), then
  \[
  \bar{M} = \max\{|\nu_1(n,k,t)|, |\nu_2(n,k,t)|\}.
  \]
Note also that value $M(n, k, t)$ determined for all $n, k, t$ by R. Ahlswede and L. Khachatrian in the paper [5]. Before formulating our main result, let’s make some additional definitions. Denote

$$
\mathcal{H}_i = \left\{ H \in \binom{[t+i]}{t+1} : [t] \subset H \right\} \\
\cup \left\{ H \in \binom{[t+i]}{t+i-1} : [t+1, t+i] \subset H \right\}.
$$

For $C \subset 2^{[n]}$, denote $U(C)$ the minimal upset, containing $C$ and by $\mu(C)$ the set of its minimal elements. Main result of this work is the proof of the following

**Theorem 5**

- If

$$
\max \left\{ \ell = t + 2r : \frac{\ell - t}{2(\ell - 1)} \gamma(\ell) \leq 1 \right\} > t,
$$

then

$$
\tilde{M}(n, t) = M(n, t),
$$

- if

$$
\max \left\{ \ell = t + 2r : \frac{\ell - t}{2(\ell - 1)} \gamma(\ell) \leq 1 \right\} = t,
$$

then

$$
\tilde{M}(n, t) = \max \{ \nu_1(n, t), \nu_2(n, t) \},
$$

where

$$
\nu_i(n, t) = \sum_{S \in U(\mathcal{H}_i)} f(n - |S|).
$$

**II Proof of the main Theorem**

Define fixing procedure $F(i, j, p)$, $i \neq j$ on the set of permutations $p \in \mathcal{P}(n)$:

$$
F(i, j, p) = \begin{cases} 
(p \setminus p_i) \cup \{i\}, p_i \setminus \{i\}, & j = p(i), \\
p, & \text{otherwise},
\end{cases}
$$

where $p_i$ is cycle from $p$ which contains $i$.  


Then fixing operator on the set $\mathcal{A} \subset \Omega(n, t)$ is defined as follows ($p \in \mathcal{A}$)

$$F(i, j, p, \mathcal{A}) = \begin{cases} F(i, j, p), & F(i, j, p) \notin \mathcal{A}, \\ p, & F(i, j, p) \in \mathcal{A}. \end{cases}$$

At last define operator

$$\mathcal{F}(i, j, \mathcal{A}) = \{F(i, j, p, \mathcal{A}); p \in \mathcal{A}\}$$

It is easy to see, that fixing operator $\mathcal{F}(i, j, \mathcal{A})$ preserve volume of $\mathcal{A}$ and its $t$− intersection property. At last note, that making shifting operation finitely number of times for different $i, j$ we obtain from the set $\mathcal{A}$ compressed set, which has the property that for all $i \neq j \in [n]$

$$\mathcal{F}(i, j, \mathcal{A}) = \mathcal{A}$$

and arbitrary pair of permutations $p_1, p_2$ from the compressed $\mathcal{A}$ intersect by at least $t$ fixed points.

Next define (usual) shifting procedure $L(v, w, p)$, $1 \leq v < w \leq n$ as follows. Let $p = \{\{j_1, \ldots, j_{q-1}, v, j_{q+1}, \ldots, \tilde{j}_s\}, \ldots, \{w\}, \pi_1, \ldots, \pi_c\} \in \mathcal{A}$, then

$$L(v, w, p) = \{j_1, \ldots, j_{q-1}, w, j_{q+1}, \ldots, \tilde{j}_s\}, \ldots, \{v\}, \pi_1, \ldots, p_s\}.$$

If $p \in \mathcal{A}$ does not fix $v$ we set

$$L(v, w, p) = p.$$

Now define shifting operator $L(v, w, p, \mathcal{A})$ as follows

$$L(v, w, p, \mathcal{A}) = \begin{cases} L(v, w, p), & L(v, w, p) \notin \mathcal{A}, \\ p, & L(v, w, p) \in \mathcal{A}. \end{cases}$$

At last we define operator $\mathcal{L}(v, w, \mathcal{A})$

$$\mathcal{L}(v, w, \mathcal{A}) = \{L(v, w, \mathcal{A}); p \in \mathcal{A}\}.$$

It is easy to see that operator $\mathcal{L}(v, w, \mathcal{A})$ does not change the volume of $\mathcal{A}$ and preserve $t$− cycle intersection property. Later we will show (Statement [H]) that this operator preserve $t$− cycle nontrivial intersection property. Also it is easy to see that after finite number of operations we come to the compressed $t$− intersection set $\mathcal{A}$ of the volume for which

$$L(v, w, \mathcal{A}) = \mathcal{A}, \ 1 \leq v < w \leq n$$
and each pair of permutations form $\mathcal{A}$ $t-$ intersect by fixed elements. Next we consider only such sets $\mathcal{A}$. We denote the family of fixed compressed $t-$ cycle (nontrivial) intersecting sets of permutations by $L\Omega(n, t)$ ($L\tilde{\Omega}(n, t)$). Also note that such sets $\mathcal{A}$, have the property that all sets from $\mathcal{A}$ have $s$ common cycles iff $|\cap_{p \in \mathcal{A}} f(p)| = s$. Also we assume that all set of permutations considered next are left compressed.

Denote by $\Omega_0(n, t)$ set of partitions $\mathcal{A}$ such that $|\cap_{p \in \Omega_0(n, t)} f(p)| = 0$.

**Statement 1**

$$\tilde{M}(n, k) = \max_{\mathcal{A} \in L\tilde{\Omega}(n, t)} |\mathcal{A}|, \quad (3)$$

$$M_0(n, t) = \max_{\mathcal{A} \in \Omega_0(n, t)} |\mathcal{A}| = \tilde{M}(n, t).$$

And moreover $\mathcal{A} \in \tilde{\Omega}(n, t)$, $|\mathcal{A}| = \tilde{M}(n, t)$ then $\mathcal{A} \in \Omega_0(n, t)$.

Proof. First we prove (3). Let $\mathcal{A} \in \tilde{\Omega}(n, t)$, $|\mathcal{A}| = \tilde{M}(n, k)$. Then either $L(v, w, \mathcal{A}) \in \tilde{\Omega}(n, t)$ or $L(v, w, \mathcal{A}) \in \Omega(n, t) \setminus \tilde{\Omega}(n, t)$. In the first case we continue shiftings. Assume that second case occur. We can assume the $\cap_{p \in \mathcal{A}} f(p) = [t - 1]$, $v = t, w = t + 1, \cap_{p \in L(v, w, \mathcal{A})} f(p) = [t]$. Because $\mathcal{A}$ is maximal, then

$$\{p \in \Omega(n, t) : [t + 1] \subset f(p)\} \subset \mathcal{A}. \quad (4)$$

There are $p_1, p_2 \in \mathcal{A}$ such that

$$f(p_1) \cap [t + 1] = [t]$$

and

$$f(p_2) \cap [t + 1] = [t - 1] \cup \{t + 1\}.$$  

Next we apply shifting $L(v, w, \mathcal{A})$ for $1 \leq v < w \leq n$, $v, w \not\in \{t, t + 1\}$. Then

$$\cap_{p \in L(v, w, \mathcal{A})} f(p) = [t - 1].$$

Thus we can assume that $L(v, w, \mathcal{A}) = \mathcal{A}$ $\forall$ $1 \leq v < w \leq n$, $v, w \not\in \{t, t + 1\}$ and

$$f(p_1) = [a] \setminus \{t + 1\}, \ a \geq t, \ a \neq t + 1,$$

$$f(p_2) = [b] \setminus \{t\}, \ b > t.$$  

From here and (3) it follows that

$$C = U(\{[t - 1] \cup C : \ C \subset [t, \min\{a, b\}]\}) \subset \mathcal{A}$$
and $C$ and for all $1 \leq v < w \leq n$, $L(v, w, C) = C$. Thus $|\cap_{p \in A} f(p)| < t$.

Now we prove second part of the Statement. Assume that $A \subset \hat{\Omega}(n, t) \setminus \Omega_0(n, t)$ and $A = |\hat{M}(n, t)|$. We can suppose that $A$ is shifted and $\{1\} \in f(p), \forall p \in A$. Also we can assume that $A \in L\hat{\Omega}(n, t)$. Consider $p \in \Omega(n, t)$ : $f(p) = \{2, \ldots, n - 1\}$. Next we show that $p \in A$, which leads to the contradiction of the maximality of $A$. Suppose that there exists an $p_1 \in A$ such that $|\{2, n - 1\} \cap f(p_1)| \leq t - 1$.

We can assume that $f(p_1) = [t] \cup \{n\}$. But then $p_2 : f(p_2) = [t - 1] \cup \{n\}$ and hence $p_3 : f(p_3) = [t]$ also belongs to $A$. But then $|f(p_3) \cap f(p_2)| = t - 1$ which is contradicting of the $t-$ intersection property of $A$.

Let $g(A)$ is the family of subsets of $[n]$ such that $A = U(g(A))$. If $A$ is maximal, then we can assume that $g(A)$ is upset and $g^*(A)$ is the set of its minimal elements. Let $G(A)$ be the family of all such $g(A)$. It is easy to see, that $A \in \Omega(n, t)$ ($\hat{\Omega}(n, t)$) iff $g(A) \in I(n, t)$ ($g(A) \in \hat{I}(n, t)$). We can assume that $g(A)$ is left compressed. Denote

$$s^+(a = (a_1 < \ldots < a_j)) = a_j,$$
$$s^+(g(A)) = s^+(\mu(g(A))) = \min_{a = (a_1 < \ldots < a_j) \in g(A)} a_j,$$
$$s_{\min}(A) = \min_{g(A)} s^+(g(A)).$$

It is easy to see that $A \in L\Omega(n, t)$ is a disjoint union

$$A = \bigcup_{f \in g(A)} V(f),$$

where

$$V(f) = \{A \in 2^n : A = f \cup B, B \in [s^+(f), n]\},$$

and if $f \in g(A) : s^+(f) = s^+(g(A))$, then the set of permutations generated by only $f$ is

$$A_f = (U(f) \setminus U(g(A) \setminus \{f\})) = V(f). \tag{5}$$

Note also a simple fact that if $f, f_2 \in g(A)$ and $i \notin f_1 \cup f_2$, $j \in f_1 \cap f_2$, $i < j$, then

$$|f_1 \cap f_2| \geq t + 1.$$
clear we repeat in Lemma all conditions which we consider as default before.

**Lemma 1** Let $A \in L\tilde{\Omega}(n, t)$, $|A| = \tilde{M}(n, t)$ and $g(A) \in G(A)$ is such that $s^+(g(A)) = s_{\min}(G(A))$, then for some $i \geq 2$

$$g(A) = H_i.$$ 

Let $\ell = s^+(g(A))$, $g_0(A) = \{g \in g(A): s^+(g) = \ell\}$, $g_1(A) = g(A) \setminus g_0(A)$. It is easy to see that $\ell > t + 1$. From above it follows that if $f_1, f_2 \in g_0(A)$ and $|f_1 \cap f_2| = t$, then $|f_1| + |f_2| = \ell + t$. Denote

$$\left| \bigcap_{f \in g_1(A)} f \right| = \tau.$$

Consider consequently tow cases $\tau < t$ and $\tau \geq t$.

Let’s assume at first that $\tau < t$. Consider the partition

$$g_0(A) = \bigcup_{t < i < \ell} R_i, R_i = g_0(A) \cap \left(\begin{array}{c} [\ell] \\ i \end{array}\right).$$

Denote

$$R'_i = \{f \subset [\ell - 1]: f \cup \{\ell\} \in R_i\}.$$ 

As above from left compressedness of the set $g(A)$ it follows that for

$$f_i \in R'_i, f_j \in R'_j, i + j \neq \ell + t, |f_i \cap f_j| \geq t.$$

Next we show that $R_i = \emptyset$.

Assume at first that $\forall R_i \neq \emptyset$ we have $R_{\ell+t-i} = \emptyset$, then

$$g' = (g(A) \setminus g_0(A)) \cup \bigcup_{t < i < \ell} R'_i \in \tilde{I}(n, k)$$

and

$$|U(g')| \geq |A|, s^+(g') < s^+(g(A))$$

which contradict our assumptions.

Now assume that $R_i, R_{\ell+t-i} \neq \emptyset$. Let’s at first $i \neq (\ell + t)/2$. Consider new sets

$$\varphi_1 = g_1(A \cup (g_0(A) \setminus (R_i \cup R_{\ell+t-i}))) \cup R'_i,$$

$$\varphi_2 = g_1(A \cup (g_0(A) \setminus (R_i \cup R_{\ell+t-i}))) \cup R'_{\ell+t-i}.$$
We have $\varphi_j \in \tilde{I}(n, k)$. Thus
$$\mathcal{A}_i = U(\varphi_i) \in \tilde{\Omega}(n, t).$$

We will show that under last assumption
$$\max_{j=1,2} |\mathcal{A}_i| > |\mathcal{A}|,$$  \hfill (6)

and come to contradiction. Using (5) it is easy to see that:
\[
|\mathcal{A} \setminus \mathcal{A}_1| = |R_{\ell+t-i}| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} f(n-\ell-t+i-j),
\]
\[
|\mathcal{A}_1 \setminus \mathcal{A}| \geq |R_i| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} f(n-i-j+1),
\]
\[
|\mathcal{A} \setminus \mathcal{A}_2| = |R_i| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} f(n-i-j),
\]
\[
|\mathcal{A}_2 \setminus \mathcal{A}| \geq |R_{\ell+t-i}| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} f(n-\ell-t+i-j+1).
\]

From these equalities follows that if (6) is not valid then
\[
|R_{\ell+t-i}| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} f(n-\ell-t+i-j) \geq |R_i| \sum_{j=0}^{n-\ell} f(n-i-j+1)
\]
and
\[
|R_i| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} f(n-i-j) \geq |R_{\ell+t-i}| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} f(n-\ell-t+i-j+1).
\]

Because $f(n+1) > f(n), n > 0$ last two inequalities couldn’t be valid together. This contradiction shows that $R_i = \emptyset, i \neq (\ell + t)/2$.

Now consider the case $i = (\ell+t)/2$. By pigeon-hole principle there exists a $k \in [\ell - 1]$ and a $\mathcal{S} \subset R'_{(\ell+t)/2}$ such that $k \notin B, B \in \mathcal{S}$ and
\[
|\mathcal{S}| \geq \frac{\ell-t}{2(\ell-1)} |R'_{(\ell+t)/2}|.  \hfill (7)
\]

Hence as before we have $|B_1 \cap B_2| \geq t, B_1, B_2 \in \mathcal{S}$ and
\[
f' = (g(\mathcal{A}) \setminus R_{(\ell+t)/2}) \cup \mathcal{S} \in \tilde{I}(n, k).\]
Next we show that

\[ |U(f')| > |A|. \tag{8} \]

Consider the partition

\[ A = G_1 \cup G_2, \]

where

\[ G_1 = U(g(A) \setminus R(\ell+t)/2), \]
\[ G_2 = U(R(\ell+t)/2) \setminus U(g(A) \setminus R(\ell+t)/2). \]

Consider also partition

\[ U(f') = G_1 \cup G_3, \]

where

\[ G_3 = U(S) \setminus U(g(A) \setminus R(\ell+t)/2). \]

We should show that

\[ |G_3| > |G_2|. \tag{9} \]

We have

\[ |G_2| = |R(\ell+t)/2| \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} f \left( n - \frac{\ell + t}{2} - j \right) \]

and

\[ |G_3| \geq |S| \sum_{j=0}^{n-\ell+1} \binom{n-\ell+1}{j} f \left( n - \frac{\ell + t}{2} - j + 1 \right) . \]

Hence for (9) to be true it is sufficient that

\[ \frac{\ell - t}{2(\ell - 1)} \gamma(\ell) > 1. \]

The last inequality is true since otherwise, if

\[ \frac{\ell - t}{2(\ell - 1)} \gamma(\ell) \leq 1 \]

for some \( \ell > t \), then by \([3]\) \( \tilde{M}(n, k) = M(n, k) \). Hence \( R_{\ell+t}/2 = \emptyset \).

Now consider the case \( \tau \geq t \). We have

\[ \bigcap_{f \in g_1(A)} f = [\tau], \]
\[
\ell = s^+(g(A)) > \tau
\]
and for all \( f \in g_0(A) \),
\[
|F \cap [\tau]| \geq \tau - 1,
\]
if \(|f \cap [\tau]| = \tau - 1\), then \([\tau + 1, \ell] \in f\).

Let’s show that \( \tau \leq t + 1 \).

If \( \tau \geq t + 2 \), then for \( f_1, f_2 \in g(A) \),
\[
|f_1 \cap f_2[\tau]| \geq \tau - 2 \geq t
\]
and thus denoting \( g'_0(A) = \{ f \subset [\ell - 1] : f \cup \{\ell\} \in g(A) \} \), we have
\[
\varphi = (g(A) \setminus g_0(A)) \cup g'_0(A) \in \tilde{I}(n, k)
\]
and
\[
|U(\varphi)| \geq |A|,
\]
\[
s^+(\varphi) < \ell.
\]

This gives the contradiction of minimality of \( \ell \).

Assume now that \( \tau = t + 1 \). Then must be \( \ell = t + 2 \), otherwise using above argument (deleting \( \ell \) from each element of \( g_0(A) \) we come to generating set \( \varphi \in \tilde{I}(n, k) \) for which \(|U(\varphi)| \geq |A| \) and \( s^+(\varphi) < \ell \). It is clear that \( \tau = t + 1 \) and \( \ell = t + 2 \), then \( g(A) = H_2 \).

At last consider the case \( \tau = t \). Denote \( g'_0(A) = \{ f \in g_0(A) : |f \cap [t]| = t - 1 \} \). We have
\[
g'_0(A) \subset \{ f \subset [\ell] : |f \cap [t]| = t - 1, [t + 1, \ell] \subset f \}
\]
and for \( f \in g(A) \setminus g'_0(A) \) we have \([t] \subset f \) and \(|f \cap [t + 1, \ell]| \geq 1 \). Hence
\[
U(g(A)) \subset U(A_{\ell-t}).
\]

and because \( A \) is maximal \( g(A) = \mathcal{H}_{\ell-t} \). Family \( \mathcal{H}_{n-t} \) is trivially \( t \)-intersecting, so we can assume that \( i < n - t \). Denote \( S_i = |U(\mathcal{H}_i)| \). Next prove that if \( S_i < S_{i+1} \), then \( S_{i+1} < S_{i+2} \). We have
\[
S_i = (n - i)! - \sum_{j=0}^{n-t-i} \binom{n-t-i}{j} f(n-t-j) + t \sum_{j=0}^{n-t-i} f(n-t-i-j+1)
\]
and we should show that from inequality
\[
\sum_{j=0}^{n-t-i-1} \binom{n-t-i-1}{j} f(n-t-j+1) \geq t \sum_{j=0}^{n-t-i-1} \binom{n-t-i-1}{j} f(n-t-j-i+1)
\]
(10)
follows
\[
\sum_{j=0}^{n-t-i-2} \binom{n-t-i-2}{j} f(n-t-j+1) \geq t \sum_{j=0}^{n-t-i-2} \binom{n-t-i-2}{j} f(n-t-j-i).
\]
(11)
We rewrite inequality (10) as follows
\[
\sum_{j=0}^{n-t-i-2} \binom{n-t-i-2}{j} f(n-t-j+1) + \sum_{j=0}^{n-t-i-2} \binom{n-t-i-2}{j} f(n-t-j) \geq t \sum_{j=0}^{n-t-i-2} \binom{n-t-i-2}{j} f(n-t-j-i) + t \sum_{j=0}^{n-t-i-2} \binom{n-t-i-2}{j} f(n-t-j-i+1).
\]
From here it is clear that if (11) is true then (10) is also true. From here and expressions for \(S_2\) and \(S_{n-t-1}\) follows the statement of the Theorem. In [9] was shown that
\[
\sum_{j=0}^{n-\ell} \binom{n-\ell}{j} f(n-\ell+1-j) \geq 1 + \ell - t + (n-\ell) \frac{\ell-t}{\ell-t+2}.
\]
For \(\ell = t+2\) rhs of the last inequality is equal to \(1 + \frac{n-t}{2}\). It follows, that for sufficiently large \(n\) and fixed \(t\)
\[
S_2 = (n-t)! - f(n-t) - f(n-t-1) + t > S_{n-t-1} = (n-t)!
\]
\[
- \sum_{j=0}^{n-t-2} \binom{n-t-2}{j} f(n-t-j) + t \sum_{j=0}^{n-t-2} \binom{n-t-2}{j} f(n-t-j-1)
\]
and hence for \(n > n_0(t)\)
\[
\tilde{M}(n,t) = (n-t)! - f(n-t) - f(n-t-1) + t.
\]

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