Rough Solutions of the Einstein Constraint Equations

David Maxwell
University of Washington
E-mail: dmaxwell@math.washington.edu

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Abstract. We construct low regularity solutions of the vacuum Einstein constraint equations. In particular, on 3-manifolds we obtain solutions with metrics in $H^s_{\text{loc}}$ with $s > \frac{3}{2}$. The theory of maximal asymptotically Euclidean solutions of the constraint equations descends completely the low regularity setting. Moreover, every rough, maximal, asymptotically Euclidean solution can be approximated in an appropriate topology by smooth solutions. These results have application in an existence theorem for rough solutions of the Einstein evolution equations.

1. Introduction

The Cauchy problem in general relativity can be formulated as a second-order system of hyperbolic PDEs known as the Einstein equations. Initial data for this problem is a 3-dimensional Riemannian manifold $(M, g)$, and a symmetric $(0, 2)$-tensor $K$ which plays the role of the time derivative of $g$. More precisely, $K$ specifies the second fundamental form of an isometric embedding of $(M, g)$ into an ambient 4-dimensional Lorentzian manifold to be determined by solving the Cauchy problem. For the vacuum Cauchy problem, the ambient Lorentzian manifold is Ricci-flat, and this imposes a compatibility condition on the initial data known as the Einstein constraint equations. These are

$$R - |K|^2 + \text{tr} K^2 = 0$$

$$\text{div} K - d \text{tr} K = 0,$$

where the scalar curvature $R$ and all other quantities involving a metric in (1) are computed with respect to $g$.

For any evolution problem, a natural question is to determine function spaces in which it is well-posed. The first local well-posedness results for the Einstein equations were established in [CB52] for initial data $(g, K) \in C^5 \times C^4$. Subsequent improvements lead to a local well-posedness result for the Einstein equations [HKM77] that requires initial data with $(g, K) \in H^s_{\text{loc}} \times H^{s-1}_{\text{loc}}$ with $s > 5/2$ (we ignore, for the moment, precise asymptotic conditions at infinity). We will call a solution with this last level of regularity a classical solution. Recent work in the theory of nonlinear hyperbolic PDEs has lowered the amount of regularity required. Smith and Tataru [ST] have obtained local well-posedness
for certain nonlinear wave equations with initial data in $H^s \times H^{s-1}$ with $s > 2$. In the case of the vacuum Einstein equations, Klainerman and Rodnianski [KR] established an a-priori estimate for the time of existence of a classical solution of the Einstein equations in terms of the norm of $(\nabla g, K)$ in $H^{s-1} \times H^{s-1}$, again with $s > 2$. These results should have lead to an existence theorem for rough solutions, but the corresponding low regularity theory of the constraint equations was not sufficiently well developed. It was not known if there existed any rough solutions of the constraint equations. Moreover, to pass from the a-priori estimate to an existence theorem for rough initial data requires the existence of a sequence of classical solutions of the constraints approximating the rough solution; this approximation theorem was also missing.

In this paper we construct a family of solutions in $H^s_{loc} \times H^{s-1}_{loc}$ with $s > 3/2$. We prove moreover that every solution in this family can be approximated by a sequence of classical solutions. To compare the lower bound $s > 3/2$ with previous results, we must keep in mind that the constraint equations have typically been solved either in Hölder spaces or in Sobolev spaces $W^{k,p}_{loc} \times W^{k-1,p}_{loc}$, where $k$ is an integer. The classical lower bound [CBIY00] for the existence of solutions of the constraint equations was $k > 3/p + 1$. These metrics have one continuous derivative and can be thought of as analogous to metrics in $H^s_{loc}$ with $s > 5/2$. This lower bound was improved in the settings of compact manifolds [CB03] and asymptotically Euclidean manifolds [Ma03] to $k \geq 2$ and $k > 3/p$. The restriction $k > 3/p$ ensures that the metric is continuous, while the inequality $k \geq 2$ further implies that the curvature belongs to $L^p_{loc}$. Taking $k = 2$ and $p = 2$, these results provide for $H^2_{loc} \times H^1_{loc}$ solutions of the constraint equations. But they do not construct solutions directly in the spaces of interest ($H^s_{loc} \times H^{s-1}_{loc}$ with $s > 2$), nor do they provide an approximation theorem.

Since $s = 3/2$ is the scaling limit for the Einstein equations, this is a natural lower bound for local well-posedness results. It has been suggested [KR03] that it might not be possible to obtain local well-posedness down to $s > 3/2$. In the case of the constraint equations, however, we have shown here that working in spaces with $s > 3/2$ is feasible. In fact, the restriction $s > 3/2$ is analogous to the condition $k > 3/p$ from [CB03] [Ma03] as these thresholds ensure the metric is continuous. A novel feature of the solutions considered here is that when $3/2 < s < 2$, the curvature of $g$ is in general only a distribution, not necessarily an integrable function.

We restrict our attention to maximal (i.e. $\text{tr} K = 0$), asymptotically Euclidean (AE) solutions. In the classical setting, the the conformal method of Lichnerowicz [Li44], Choquet-Bruhat and York [CBY80] provides a parameterization of all such solutions. We review this construction below, as it forms the basis of our construction of rough solutions.
Hereafter we suppose $M$ is an $n$-manifold with $n \geq 3$. We seek a solution of the form

$$\hat{g} = \phi^\frac{4}{n-2} g$$
$$\hat{K} = \phi^{-2} \sigma,$$

where $(M, g)$ is asymptotically Euclidean, $\sigma$ is a trace-free symmetric $(0, 2)$-tensor, and $\phi$ is a conformal factor tending to 1 at infinity. Writing the constraint equations for $\hat{g}$ and $\hat{K}$ in terms of the conformal data $(g, \sigma)$ we obtain

$$-a \Delta \phi + R \phi - |\sigma|^2 \phi^{-2 \kappa - 3} = 0 \quad (2)$$
$$\text{div } \sigma = 0, \quad (3)$$

where $\kappa = \frac{2}{n-2}$ and $a = 2 \kappa + 4$ are dimensional constants. These equations for $\phi$ and $\sigma$ are decoupled (this is a feature of the hypothesis $\text{tr} \ K = 0$) and can be therefore be treated separately.

To construct solutions of (3), known as transverse-traceless tensors, we consider the conformal Killing operator $L$ and vector Laplacian $\Delta_L$. These are defined by

$$L X = L X g - \frac{2}{n} \text{div } X g$$
$$\Delta_L X = \text{div } L X,$$

where $L X$ is the Lie derivative with respect to $X$. If $S$ is a symmetric, traceless $(0, 2)$-tensor, and if we can solve

$$\Delta_L X = - \text{div } S,$$

then setting $\sigma = S + L X$ we have $\text{div } \sigma = 0$. The fact that this is always possible, and that the set $S$ of symmetric, traceless, $(0, 2)$-tensors can be written $S = \text{im } L \oplus \ker \text{div}$ is known as the York decomposition.

The study the Lichnerowicz equation (2), now reduces to determining the set of metrics $(M, g)$ and transverse-traceless tensors $\sigma$ for which (2) admits a (positive) solution. It was shown by Cantor in [Ca79a] that (2) is solvable if and only if $(M, g)$ is conformally related to a scalar flat metric. Following [Ma03], we define the Yamabe invariant $\lambda_g$ of $(M, g)$ to be

$$\lambda_g = \inf_{f \in C_c^\infty (M) \atop f \neq 0} \frac{\int_M a |\nabla f|^2 + R f^2 \ dV}{\| f \|^2_{L^{2n/2\kappa}}}.$$  

In [CaB81], Cantor and Brill considered a related invariant in their investigation of necessary and sufficient conditions for an asymptotically Euclidean metric to be conformally related to a scalar flat metric. It was shown in [Ma03] that $(M, g)$ is conformally related to a scalar flat metric if and only if $\lambda_g > 0$. This provides a necessary and sufficient condition
for the solvability of (2), which we will call the Cantor-Brill condition. Combining the
York decomposition and the Cantor-Brill condition, we obtain an elegant construction of
the entire family of maximal AE solutions of the constraint equations.

The main results of this paper are that the York decomposition and the Cantor-Brill condition
are valid for rough metrics (i.e. metrics with \( s > n/2 \)), and that rough, maximal, AE
solutions of the constraints admit a sequence of classical approximating solutions. At
the core of each of these problems is a second-order linear elliptic operator with rough
coefficients. For the York decomposition we have the vector Laplacian \( \Delta_L \), and for the
Cantor-Brill condition we have the linearized Lichnerowicz equation

\[-a\Delta + V,\]

where \( V \) is a potential. Both operators have the structure

\[a_2 \partial^2 + a_1 \partial + a_0\]

where \( a_i \in H^{s-2+i}_{\text{loc}} \). In Section 3 we obtain mapping properties and a-priori estimates
for operators with these low regularity coefficients. Once the a-priori estimates have been
achieved, the York decomposition follows in a straight-forward way in Section 4. We also
show in section 4 that there are no conformal Killing fields decaying at infinity in \( H^s \) with
\( s > n/2 \). This fact extends a corresponding result for classical metrics [CO81] and is used
crucially in Section 7 to prove the existence of classical approximating sequences. Section
5 obtains properties of the linearized Lichnerowicz equation, and Section 6 applies these
results to solve the Lichnerowicz equation given the Cantor-Brill property.

In the discussion above, we have suppressed important technical details about the function
spaces which we use. The maximal, asymptotically Euclidean setting has the advantage of
possessing a very clean theory, but also requires the use of weighted \( H^s \) spaces to prescribe
asymptotic behaviour near infinity. These spaces are routinely used in the mathematical
relativity community in the special case where \( s \) is a non-negative integer. There is a
generalized version of these spaces for \( s \in \mathbb{R} \) due to Triebel [Tr76a][Tr76b] which has not
been exploited before to solve the constraint equations. In the following section we define
these spaces and prove some simple but fundamental facts about multiplying functions in
these spaces.

2. Weighted Sobolev Spaces

Let \( A_j \) be the annulus \( B_{2^{j+1}} \setminus B_{2^{j-1}} \), and let \( \{ \phi_j \}_{j=0}^\infty \) be a partition of unity satisfying

1. \( \phi_0 \) is supported in \( B_2 \) and \( \phi_j \) is supported in \( A_j \) for \( j \geq 1 \).
2. \( \phi_j(x) = \phi_1(2^{1-j}x) \) for \( j \geq 1 \).
The weighted Sobolev space $H^s_\delta(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $\delta \in \mathbb{R}$ is then defined as a subset of the tempered distributions $\mathcal{S}^*$,

$$H^s_\delta(\mathbb{R}^n) = \left\{ u \in \mathcal{S}^* : \|u\|^2_{H^s_\delta} = \sum_{j=0}^{\infty} 2^{-2\delta j} \|T_j(\phi_j u)\|^2_{H^s} < \infty \right\}.$$

Here $T_j$ is the rescaling operator defined by $(T_j u)(x) = u(2^j x)$. If $\Omega$ is an open subset of $\mathbb{R}^n$, we define $H^s_\delta(\Omega)$ to be the restriction of $H^s_\delta(\mathbb{R}^n)$ to $\Omega$ with norm

$$\|u\|_{H^s_\delta(\Omega)} = \inf_{v \in H^s_\delta(\mathbb{R}^n)} \|v\|_{H^s_\delta(\mathbb{R}^n)}.$$

For properties of these spaces we refer the reader to [Tr76a][Tr76b], where our spaces $H^s_\delta$ correspond with the spaces $h^{2s-2\delta-n}_{2,2s-2\delta-n}$ in that reference. In particular, $H^s_\delta$ is a reflexive Banach space, and the norms corresponding to different partitions of unity are all equivalent. We use Bartnik’s convention [Ba86] for the growth parameter $\delta$ defined for $\delta < \delta’$. If $\delta < \delta’$, then this embedding is compact.

**Lemma 2.1.**

1. When $k$ is a non-negative integer, an equivalent norm for $H^k_\delta$ is the norm on $W^{k,2}_\delta$.

2. If $\theta \in (0,1)$, $s = (1-\theta)s_1 + \theta s_2$ and $\delta = (1-\theta)\delta_1 + \theta \delta_2$, then $H^s_\delta$ is the interpolation space $[H^{s_1}_{\delta_1}, H^{s_2}_{\delta_2}]_\theta$.

3. $H^s_\delta$ is the dual space of $H^{-s}_{-n-\delta}$.

4. If $s \geq s’$ and $\delta \leq \delta’$ then $H^s_\delta$ is continuously embedded in $H^{s’}_{\delta’}$. If $s > s’$ and $\delta < \delta’$, then this embedding is compact.

5. If $s < n/2$, then $H^s_\delta$ is continuously embedded in $L^q_\delta = W^{0,q}_\delta$, where $\frac{1}{q} = \frac{1}{2} - \frac{s}{n}$.

6. If $s > n/2$ then $H^s_\delta(\mathbb{R}^n)$ is continuously embedded in $C^0_\delta(\mathbb{R}^n)$, where $\|f\|_{C^0_\delta} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^{-\delta} |f|$.

7. If $u \in H^s_\delta$, then $\partial u \in H^{s-1}_{\delta-1}$. 

We now turn to the compactness argument. Let \( \{u_k\}_{k=1}^{\infty} \) be a sequence in \( H^s_\delta \) with \( ||u_k||_{H^s_\delta} \leq 1 \). Then the sequence \( \{\phi_0 u_k\}_{k=1}^{\infty} \) is bounded in \( H^s \) and each element has support contained in the ball \( B_2 \). From Rellich’s theorem, we infer the existence of a subsequence \( \{u_{k_j}\}_{j=0}^{\infty} \) such that \( \{\phi_0 u_{k_j}\}_{j=1}^{\infty} \) is Cauchy in \( H^{s'} \) and such that \( ||\chi_0 (u_{k_j} - u_{l_j})||_{H^{s'}} \leq 1 \) for all \( k, l \geq 1 \). Similarly, the sequence \( \{T_1(\phi_1 u_{k_j})\}_{j=1}^{\infty} \) is bounded in \( H^s \) and has uniformly compact support. So there is a sub-subsequence \( \{u_{k_j}^1\}_{j=1}^{\infty} \) such that \( \{T_1(\phi_1 u_{k_j}^1)\}_{j=1}^{\infty} \) is Cauchy in \( H^{s'} \) and

\[
\sum_{j=0}^{1} 2^{-2\delta' j} ||T_j(\phi_j (u_k^1 - u_l^1))||_{H^{s'}} < \frac{1}{2}
\]

for \( k, l \geq 1 \). Continuing iteratively, we obtain sub-subsequences \( \{u_{k_j}^m\}_{j=1}^{\infty} \) such that for \( k, l \geq 1 \),

\[
\sum_{j=0}^{m} 2^{-2\delta' j} ||T_j(\phi_j (u_k^m - u_l^m))||_{H^{s'}} < \frac{1}{2m}
\]

Let \( v_k = u_{k_j}^1 \). Then if \( k, l \geq N \), since \( \delta' > \delta \),

\[
||v_k - v_l||_{H^{s'}_{\delta'}}^2 = \sum_{j=0}^{\infty} 2^{-2\delta' j} ||T_j(\phi_j (v_k - v_l))||_{H^{s'}}^2
\]

\[
= \sum_{j=0}^{N} 2^{-2\delta' j} ||T_j(\phi_j (v_k - v_l))||_{H^{s'}}^2 + \sum_{j=N+1}^{\infty} 2^{-2\delta' j} ||T_j(\phi_j (v_k - v_l))||_{H^{s'}}^2
\]

\[
\leq 2^{-N} + 2^{-2(\delta' - \delta)(N+1)} \sum_{j=N+1}^{\infty} 2^{-2\delta' j} ||T_j(\phi_j (v_k - v_l))||_{H^{s'}}^2
\]

\[
\leq 2^{-N} + 2^{-2(\delta' - \delta)(N+1)} ||v_k - v_l||_{H^{s'}_{\delta'}}^2.
\]

Since the sequence \( \{v_k\}_{k=1}^{\infty} \) is bounded in \( H^s_\delta \) and since \( \delta' > \delta \), we conclude the sequence is Cauchy in \( H^{s'}_{\delta'} \), which proves the result. ∎

We define asymptotically Euclidean manifolds in the usual way using weighted spaces.

**Definition 2.2** Let \( M \) be a smooth, connected, \( n \)-dimensional manifold and let \( g \) be a metric on \( M \) for which \((M, g)\) is complete. Let \( E_r \) be the exterior region \( \{x \in \mathbb{R}^n : |x| > r\} \). For
s > n/2 and ρ < 0, we say (M, g) is asymptotically Euclidean (AE) of class $H^s_ρ$ if

1. The metric $g \in H^s_{loc}(M)$.

2. There exists a finite collection $\{N_i\}_{i=1}^m$ of open subsets of $M$ and diffeomorphisms $\Phi_i : E_1 \mapsto N_i$ such that $M - \cup_i N_i$ is compact.

3. For each $i$, $\Phi_i^* g - \bar{g} \in H^s_{\rho}(E_1)$, where $\bar{g}$ is the Euclidean metric.

The charts $\Phi_i$ are called end charts and the corresponding coordinates are end coordinates. Suppose $(M, g)$ is asymptotically Euclidean, and let $\{\Phi_i\}_{i=1}^m$ be its collection of end charts. Let $U_0 = M - \cup_i \Phi_i(E_2)$, and let $\{\chi_i\}_{i=0}^m$ be a partition of unity subordinate to the sets $\{U_0, N_1, \cdots, N_n\}$. The weighted Sobolev space $H^s_{\delta}(M)$ is the subset of $H^s_{loc}(M)$ such that

$$||u||_{H^s_{\delta}(M)} = ||\chi_0 u||_{H^s(U_0)} + \sum_{i=1}^m ||\Phi_i^* (\chi_i u)||_{H^s_{\delta}(\mathbb{R}^n)}$$

is finite. An initial data set $(M, g, K)$ is asymptotically Euclidean if $(M, g)$ is asymptotically Euclidean of class $H^s_{\rho}$ for $s > n/2$ and $\rho < 0$, and if $K \in H^{s-1}_{\rho-1}(M)$. We note that when $n = 3$ and $(M, g, K)$ is AE of class $H^s_{\rho}$ with $s > 2$ and $\rho \leq -1/2$, then $(\nabla g, K) \in H^{s-1} \times H^{s-1}$ and hence $(M, g, K)$ is initial data that can be used, for example, in the theorems of [KR].

The remainder of this section concerns multiplication and Sobolev spaces; we need these properties to work with differential operators with rough coefficients. We recall the following multiplication rule for unweighted Sobolev spaces, which is an easy consequence of the well-known fact that $H^s$ is an algebra for $s > n/2$ together with interpolation and duality arguments.

**Lemma 2.3.** Suppose $t \leq \min(s_1, s_2)$, $s_1 + s_2 \geq 0$ and $t < s_1 + s_2 - \frac{n}{2}$. Then pointwise multiplication extends to a continuous bilinear map

$$H^{s_1} \times H^{s_2} \rightarrow H^t.$$

The previous lemma generalizes to weighted spaces. In the following, we use the notation $A \lesssim B$ to mean $A < cB$ for a certain positive constant $c$. The implicit constant is independent of the functions and parameters appearing in $A$ and $B$ that are not assumed to have a fixed value.

**Lemma 2.4.** Suppose $s \leq \min(s_1, s_2)$, $s_1 + s_2 \geq 0$, and $s < s_1 + s_2 - \frac{n}{2}$. For any $\delta_1, \delta_2 \in \mathbb{R}$, pointwise multiplication extends to a continuous bilinear map

$$H^{s_1}_{\delta_1} \times H^{s_2}_{\delta_2} \rightarrow H^s_{\delta_1+\delta_2}.$$
**Proof:** Suppose \( u_i \in H_{s_i}^{\delta_i} \). Taking \( \phi_k = 0 \) for \( k < 0 \) we have

\[
T_j(\phi_j u_1 u_2) = T_j(\phi_j) \sum_{k=j-1}^{j+1} T_j(\phi_k u_1) \sum_{l=j-1}^{j+1} T_j(\phi_l u_2).
\]

From the restrictions on \( s, s_1, \) and \( s_2 \) we know that multiplication is a continuous bilinear map on the corresponding unweighted Sobolev spaces. Noting that \( T_j \phi_j = T_k \phi_k \) for \( j, k \geq 1 \), we find

\[
||T_j(\phi_j u_1 u_2)||_{H^s}^2 \leq \sum_{k,l=j-1}^{j+1} ||T_j(\phi_k u_1)||_{H^{s_1}}^2 ||T_j(\phi_l u_2)||_{H^{s_2}}^2
\]

\[
\leq \sum_{k,l=j-1}^{j+1} ||T_{j-k} T_k(\phi_k u_1)||_{H^{s_1}}^2 ||T_{j-l} T_l(\phi_l u_2)||_{H^{s_2}}^2.
\]

Now \( T_{j-k} \) must be one of \( T_{-1}, T_0, \) or \( T_1 \), and a same result holds for \( T_{j-l} \). These operators are independent of \( j \) and we find

\[
||T_j(\phi_j u_1 u_2)||_{H^s}^2 \leq \sum_{k,l=j-1}^{j+1} ||T_k(\phi_k u_1)||_{H^{s_1}}^2 ||T_l(\phi_l u_2)||_{H^{s_2}}^2.
\]

It follows that

\[
\sum_{j=0}^{\infty} 2^{-2\delta j} ||T_j(\phi_j u_1 u_2)||_{H^s}^2 \leq \sum_{j=0}^{\infty} 2^{-2\delta j} \sum_{k,l=j-1}^{j+1} ||T_k(\phi_k u_1)||_{H^{s_1}}^2 ||T_l(\phi_l u_2)||_{H^{s_2}}^2
\]

\[
\leq \sum_{k=1}^{1} \sum_{j=0}^{\infty} 2^{-2\delta j} \|T_j(\phi_j u_1)\|_{H^{s_1}}^2 \times
\]

\[
\times 2^{-2\delta_2 (j+k)} \|T_{j+k}(\phi_{j+k} u_2)\|_{H^{s_2}}^2
\]

\[
\leq \sum_{j=0}^{\infty} 2^{-2\delta_1 j} \|T_j(\phi_j u_1)\|_{H^{s_1}}^2 \sum_{k=0}^{\infty} 2^{-2\delta_2 k} \|T_k(\phi_k u_2)\|_{H^{s_2}}^2.
\]

This proves \( ||u_1 u_2||_{H^{s}}_{\delta_1 + \delta_2} \leq ||u_1||_{H^{s_1}} ||u_2||_{H^{s_2}} \). \( \square \)

The following results, related to the multiplication lemmas, are useful for working with nonlinearities.

**Lemma 2.5.** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is smooth. If \( u \in H_{\rho}^{s} \) with \( s > n/2 \) and \( \rho < 0 \), and if \( v \in H_{\delta}^{\sigma} \) with \( \sigma \in [-s, s] \) and \( \delta \in \mathbb{R} \), then

\[
f(u)v \in H_{\delta}^{\sigma}.
\]
Moreover, the map taking \((u, v)\) to \(f(u)v\) is continuous from \(H^s_t \times H^s_\rho\) to \(H^s_\delta\).

**Proof:** It is easy to verify with the help of Lemma 2.4 that if \(u \in H^s\) with \(s > n/2\), and if \(\eta\) is smooth and compactly supported, then \(\eta f(u) \in H^s\) and the map taking \(u\) to \(\eta f(u)\) is continuous from \(H^s\) to \(H^s\).

Now suppose \(u\) and \(v\) satisfy the hypotheses of the lemma. Then

\[
||f(u)v||_{H^s_\delta}^2 = \sum_{j=0}^{\infty} 2^{-2\delta j} ||T_j(\phi_j f(u)v)||_{H^s}^2
\]

\[
= \sum_{j=0}^{\infty} 2^{-2\delta j} \left| \sum_{k=j-1}^{j+1} (T_j \phi_k) f \left( \sum_{i=j-1}^{j+1} \phi_i u \right) T_j \phi_j v \right|^2_{H^s}
\]

\[
\leq \sum_{j=0}^{\infty} 2^{-2\delta j} \left| \sum_{k=j-1}^{j+1} (T_j \phi_k) f (R_j u) \right|^2_{H^s} ||T_j \phi_j v||_{H^s}^2,
\]

where \(R_j u = \sum_{i=j-1}^{j+1} T_{j-i} T_i \phi_i u\)

Let \(\eta = \sum_{k=0}^{2} T_k \phi_1\), so that for \(j > 1\) we have \(\sum_{k=j-1}^{j+1} (T_j \phi_k) = \eta\). Since \(\rho > 0\), \(T_i \phi_i u\) converges to \(0\) in \(H^s\). It follows that \(R_j u\) converges to \(0\) in \(H^s\) as well. Hence \(\eta f(R_j u)\) converges in \(H^s\) to \(\eta f(0)\), and we conclude that there exists a bound \(M\) such that

\[
||\eta(T_j \phi_k) f (R_j u)||_{H^s} \leq M
\]

for all \(j > 1\). The cases \(j = 0\) and \(j = 1\) can be treated similarly. We conclude, taking \(M\) sufficiently large, that

\[
||f(u)v||_{H^s_\delta}^2 \leq M^2 ||v||_{H^s_\delta}^2.
\]

This proves \(f(u)v \in H^s_\delta\).

To establish the continuity of the map \((u, v) \mapsto f(u)v\) acting on \(H^s_t \times H^s_\rho\), we consider any sequence \(\{u_k, v_k\}_{k=1}^{\infty}\) converging to \((u, v)\). Then

\[
f(u)v - f(u_k)v_k = f(u)(v - v_k) - (f(u) - f(u_k))v_k.
\]

From (5) we see that \(f(u)(v - v_k) \to 0\) in \(H^s_\delta\). We wish to establish \((f(u) - f(u_k))v_k \to 0\) as well.

Computing as before we find

\[
||f(u) - f(u_k)||v_k||_{H^s_\delta}^2 \leq \sum_{j=0}^{\infty} 2^{-2\delta j} ||T_j \phi_j v_k||_{H^s}^2 \left( \sup_{j \geq 0} \left| \sum_{i=j-1}^{j+1} (T_j \phi_i) [f (R_j u) - f (R_j u_k)] \right|_{H^s}^2 \right)
\]

\[
\leq ||v_k||_{H^s_\delta} \sup_{j \geq 0} \left| \sum_{i=j-1}^{j+1} (T_j \phi_i) [f (R_j u) - f (R_j u_k)] \right|_{H^s}^2.
\]
Since \( \{v_k\}_{k=1}^\infty \) is bounded in \( H^s_\rho \), it is enough to show
\[
\sup_{j \geq 0} \left\| \sum_{l=j-1}^{j+1} (T_j \phi_l) \left[ f(R_j u) - f(R_j u_k) \right] \right\|_{H^s}^2 \to 0 \tag{6}
\]
as \( k \to \infty \).

Since \( ||T_j \phi_j(w)||_{H^s} \leq 2^{\delta j}||w||_{H^s_\delta} \) for all \( w \in H^s_\delta \), we also have \( ||R_j w||_{H^s} \leq 2^{\delta j}||w||_{H^s_\delta} \). Hence
\[
||R_j u||_{H^s} \leq 2^{\delta j}||u||_{H^s_\delta}
\]
and
\[
||R_j u_k||_{H^s} \leq 2^{\delta j} \left( ||u||_{H^s_\delta} + \sup_{k \geq 1} ||u_k||_{H^s_\delta} \right) \tag{7}
\]
Hence
\[
||f(R_j u) - f(R_j u_k)||_{H^s} \leq ||f(R_j u) - f(0)||_{H^s} + ||f(R_j u_k) - f(0)||_{H^s} \tag{8}
\]
where, as before, \( \eta = \sum_{k=0}^2 T_k \phi_1 = \sum_{l=j-1}^{j+1} (T_j \phi_l) \) for \( j > 1 \). Since \( \delta < 0 \) and since the map \( u \mapsto \eta f(u) \) is continuous from \( H^s \) to \( H^s_\delta \), we conclude from (7) and (8) that
\[
\sup_{j > N} \sup_{k \geq 1} ||f(R_j u) - f(R_j u_k)||_{H^s}
\]
can be made arbitrarily small for \( N \) sufficiently large. On the other hand, for any fixed \( N \)
\[
\sup_{j=0}^N \left\| \sum_{l=j-1}^{j+1} (T_j \phi_l) \left[ f(R_j u) - f(R_j u_k) \right] \right\|_{H^s}
\]
can be made as small as we please by taking \( k \) sufficiently large, since each of the finitely many terms in the supremum tends to 0 as \( k \) goes to infinity. We have hence established (6) and therefore also the desired continuity. \( \square \)

To work with conformal changes of an AE metric, we need a variation of Lemma 2.5. If \( g \) is AE of class \( H^s_\rho \), and if \( f \) is smooth with \( f(0) = 1 \), we want to know that \( f(u)g \) is AE of class \( H^s_\rho \) whenever \( u \in H^s_\rho \). Now
\[
(f(u)g - \overline{g}) = f(u)(g - \overline{g}) + (f(u) - 1)\overline{g}.
\]
Since \( g - \overline{g} \in H^s_\rho \), we know from Lemma 2.5 that \( f(u)(g - \overline{g}) \in H^s_\rho \). The following corollary shows that \( (f(u) - 1)\overline{g} \) belongs to \( H^s_\rho \) as well.

**Corollary 2.6.** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is smooth and \( f(0) = 0 \). If \( u \in H^s_\rho \) with \( s > n/2 \) and \( \rho < 0 \) then \( f(u) \in H^s_\rho \) and the map taking \( u \) to \( f(u) \) is continuous from \( H^s_\rho \) to \( H^s_\rho \).

**Proof:** Since \( f(0) = 0 \) we have from Taylor’s theorem that \( f(x) = F(x)x \) where \( F \) is smooth. Hence \( f(u) = F(u)u \in H^s_\rho \) by Lemma 2.5, and the continuity of the map on \( H^s_\rho \) follows similarly. \( \square \)
Remark. In practice we will use an obvious improvement to Lemma 2.5 and Corollary 2.6. It is easy to see that if \( f \) is only smooth on an open interval \( I \), and if \( \inf u, \sup u \subset I \), then \( f(u)v \in H^\sigma_0 \) and the map \( (u, v) \mapsto f(u)v \) is continuous on \( U \times H^\sigma_0 \) for some neighbourhood \( U \) of \( u \). An analogous statement for Corollary 2.6 also holds.

For functions \( u, v \in C^k \) we have a simple estimate on how the \( k^{th} \) derivatives of \( v \) contribute to the \( C^k \) norm of \( uv \), namely \( ||uv||_{C^k} \lesssim ||u||_{C^0} ||v||_{C^k} + ||u||_{C^k} ||v||_{C^k-1} \). We need an analogous fact for Sobolev spaces, which will be derived from the following commutator estimate for the operator \( \Lambda = (1 - \Delta)^{1/2} \).

**Lemma 2.7.** Suppose \( u \in H^s \) with \( s > n/2 \), \( \sigma \in (-s, s) \), and suppose \( \tau \in (0, 1] \) satisfies \( \tau < \min(s - n/2, s + \sigma) \). Then \([\Lambda^{-\sigma}, u]\) is continuous as a map 
\[
\Lambda^{-\sigma}, u : H^{-\sigma} \to H^\tau.
\]

**Proof:** Since \( \Lambda^\sigma \) is a pseudo-differential operator of order \( \sigma \), and since \( u \) is a pseudo-differential operator of order 0 of the type considered in [Mar88], the claim is a consequence of [Mar88] Corollary 3.4. \( \square \)

**Lemma 2.8.** Suppose \( s > n/2 \), \( \sigma \in (-s, s] \), and \( \tau \in (0, 1] \) satisfies \( \tau < \min(s - n/2, s + \sigma) \). Then for all \( u \in H^s \) and \( v \in H^\sigma \) there is a constant \( c(u) \) such that
\[
||uv||_{H^\sigma} \lesssim ||u||_{L^\infty} ||v||_{H^\sigma} + c(u) ||v||_{H^{\sigma-\tau}}. \tag{9}
\]

**Proof:** First suppose \( \sigma \in (-s, s) \). Then for all \( \phi \in C^\infty_0 \),
\[
\langle uv, \phi \rangle = \langle \Lambda^\sigma v, \Lambda^{-\sigma} u \phi \rangle = \langle \Lambda^\sigma v, u \Lambda^{-\sigma} \phi \rangle + \langle v, \Lambda^\sigma [\Lambda^{-\sigma}, u] \phi \rangle \lesssim ||u||_{L^\infty} ||v||_{H^\sigma} ||\phi||_{H^{-\sigma}} + ||v||_{H^\sigma} \| [\Lambda^{-\sigma}, u] \phi \|_{H^\tau}.
\]

From Lemma 2.7,
\[
||[\Lambda^{-\sigma}, u] \phi||_{H^\tau} \lesssim c(u) ||\phi||_{H^{-\sigma}}.
\]
Hence
\[
||uv||_{H^\sigma} \lesssim ||u||_{L^\infty} ||v||_{H^\sigma} + c(u) ||v||_{H^{\sigma-\tau}}. \tag{10}
\]
In the case \( \sigma = s \), we have from [KP88]
\[
||uv||_{H^s} \lesssim ||u||_{L^\infty} ||v||_{H^s} + ||u||_{H^s} ||v||_{L^\infty} \lesssim ||u||_{L^\infty} ||v||_{H^s} + ||u||_{H^s} ||v||_{H^{s-\tau}},
\]
which completes the proof. \( \square \)
In fact, one can show that \( c(u) = ||u||_{H^s} \) in the previous lemma, but we do not need this fact in the sequel.

3. Elliptic Linear Operators with Rough Coefficients

The regularity of the coefficients of the vector Laplacian \( \Delta_L \) and the conformal Laplacian \(-a\Delta + R\) depends on the regularity of the metric \( g \). The following general class of differential operators \( \mathcal{L}_{m,s}^{\rho} \) encodes this relationship.

**Definition 3.1** Let \( A \) be the linear differential operator

\[
A = \sum_{|\alpha| \leq m} a^\alpha \partial_\alpha,
\]

where \( a^\alpha \) is a \( \mathbb{R}^{k \times k} \) valued function. We say that

\[
A \in \mathcal{L}_{m,s}^{\rho}
\]

if \( a^\alpha \in H^{s-m+|\alpha|} \) for all \( |\alpha| \leq m \). Similarly, if \( \rho < 0 \) we say

\[
A \in \mathcal{L}_{\rho}^{m,s}
\]

if \( a^\alpha \in H^{s-m+|\alpha|} \) for all \( |\alpha| < m \) and if there are constant matrices \( a^\alpha_\infty \) such that \( a^\alpha_\infty - a^\alpha \in H^{\rho} \) for all \( |\alpha| = m \). We call \( A_\infty = \Sigma_{|\alpha| = m} a^\alpha_\infty \partial_\alpha \) the principal part of \( A \) at infinity.

One can easily check with the help of Lemma 2.5 and Corollary 2.6 that when \( g \) is an AE metric of class \( H_{\rho}^{s} \) with \( s > n/2 \) and \( \rho < 0 \), then both the vector Laplacian and the conformal Laplacian are in \( \mathcal{L}_{\rho}^{2,s} \). From Lemmas 2.3 and 2.4 we obtain the following simple properties.

**Corollary 3.2.** Suppose

\[
\eta < \sigma + s - m - \frac{n}{2},
\]

\[
\eta \leq \min(\sigma, s) - m
\]

\[
m \leq \sigma + s.
\]

If \( A \in \mathcal{L}_{m,s}^{\rho} \), then \( A \) is a continuous map

\[
A : H^\sigma \rightarrow H^\eta.
\]

If \( \delta \in \mathbb{R}, \rho < 0 \) and if \( A \in \mathcal{L}_{\rho}^{m,s} \), then \( A \) is a continuous map

\[
A : H^\eta_{\delta} \rightarrow H^\eta_{\delta-m}.
\]
If moreover \( A_\infty = 0 \), then \( A \) is a continuous map

\[
A : H_0^r \to H_0^{r - m + \rho}.
\]

When \( s > n/2 \), the highest order coefficients of \( A \in \mathcal{L}^{m,s} \) are continuous. It then makes sense to talk about their pointwise values. We say \( A \) is elliptic if for each \( x \), the constant coefficient operator \( \sum_{|\alpha| = m} a^\alpha(x) \partial_\alpha \) is elliptic. For \( A \in \mathcal{L}^{m,s}_\rho \) we require additionally that \( A_\infty \) is elliptic.

We now prove an interior regularity a-priori estimate for elliptic operators in \( \mathcal{L}^{m,s} \) in two steps. We first we treat the high order terms using coefficient-freezing and Lemma 2.8.

**Proposition 3.3.** Suppose \( s > n/2 \) and suppose \( A \in \mathcal{L}^{m,s} \) is elliptic and has only has terms of order \( m \) (i.e. \( A = \sum_{|\alpha| = m} a^\alpha(x) \partial_\alpha \)). If \( \sigma \in (m - s, s] \), then for all \( u \in H^\sigma \) supported in a compact set \( K \)

\[
||u||_{H^\sigma} \lesssim ||Au||_{H^{s-m}} + ||u||_{H^{m-s}}, \quad (11)
\]

where the implicit constant depends on \( K \) and \( A \) but not on \( u \).

**Proof:** Fix \( x_0 \in K \) and let \( \epsilon \) be a small parameter to be chosen later. Let \( \chi \) be a cutoff function equal to 1 on \( B_1 \) and equal to 0 outside \( B_2 \), and let \( \chi_\epsilon(x) = \chi((x - x_0)/\epsilon) \).

Let \( A = A_m + R \) where \( A_m \) is the constant coefficient operator \( \sum_{|\alpha| = m} a^\alpha(x_0) \partial_\alpha \) and \( R = \sum_{|\alpha| = m} \epsilon^\alpha \partial_\alpha = \sum_{|\alpha| = m} (a^\alpha - a^\alpha(x_0)) \partial_\alpha \).

From elliptic theory for constant coefficient elliptic operators we have

\[
||\chi_\epsilon u||_{H^\sigma} \lesssim ||A_m \chi_\epsilon u||_{H^{s-m}} + ||\chi_\epsilon u||_{H^{m-s}} \\
\lesssim ||A \chi_\epsilon u||_{H^{s-m}} + ||R \chi_\epsilon u||_{H^{s-m}} + ||\chi_\epsilon u||_{H^{m-s}} \\
\lesssim c(\epsilon)||A u||_{H^{s-m}} + ||[A, \chi_\epsilon] u||_{H^{s-m}} + ||R \chi_\epsilon u||_{H^{s-m}} + c(\epsilon)||u||_{H^{m-s}}.
\]

To estimate the term \( ||R \chi_\epsilon u||_{H^{s-m}} \), we have

\[
||r^\alpha \partial_\alpha \chi_\epsilon u||_{H^{s-m}} = ||\chi_\epsilon r^\alpha \partial_\alpha \chi_\epsilon u||_{H^{s-m}} \\
\lesssim ||\chi_\epsilon r^\alpha||_{L^\infty} ||\chi_\epsilon u||_{H^\sigma} + c(\chi_\epsilon r^\alpha)||\chi_\epsilon u||_{H^{\sigma - \tau}},
\]

where \( \tau > 0 \) is a constant given by Lemma 2.8 satisfying \( \sigma - \tau > m - s \). Taking \( \epsilon \) sufficiently small we can make \( ||\chi_\epsilon r^\alpha||_{L^\infty} \) as small as we please and we obtain for an \( \epsilon \) depending only on \( A \) and \( x_0 \)

\[
||\chi_\epsilon u||_{H^\sigma} \lesssim c(\epsilon)||A u||_{H^{s-m}} + ||[A, \chi_\epsilon] u||_{H^{s-m}} + c(\epsilon, A, x_0)||u||_{H^{\sigma - \tau}}.
\]

Now \( [A, \chi_\epsilon] \in \mathcal{L}^{m-1,s} \) and hence from Corollary 3.2

\[
||[A, \chi_\epsilon] u||_{H^{s-m}} \lesssim c(\epsilon)||u||_{H^{s-1}}.
\]
Since $\tau \leq 1$, we obtain
\[
||u||_{H^\sigma(B_{\epsilon/2}(x))} \leq c(\epsilon, A, x_0) [||Au||_{H^{\sigma-m}} + ||u||_{H^{\sigma-\tau}}].
\]
Covering $K$ with finitely many such balls we find
\[
||u||_{H^\sigma} \leq ||Au||_{H^{\sigma-m}} + ||u||_{H^{\sigma-\tau}},
\]
where the implicit constant depends on $K$ and $A$. Since $\sigma - \tau > m - s$, equation (11) then follows from interpolation. \(\Box\)

We now can prove an interior regularity estimate for elliptic operators in $L^{m,s}$.

**Proposition 3.4.** Let $U$ and $V$ be open sets with $U \subset\subset V$, and suppose $s > n/2$ and $\sigma \in (m - s, s]$. If $A \in L^{m,s}$ is elliptic, then for every $u \in H^\sigma$ we have
\[
||u||_{H^\sigma(U)} \lesssim ||Au||_{H^{\sigma-m}(V)} + ||u||_{H^{m-s}(V)}.
\] (12)

**Proof:** Choose an open set $V_0$ such that $U \subset\subset V_0 \subset\subset V$, and let $\chi$ be a cutoff function equal to 1 on $U$ and compactly supported in $V_0$. Let $A = A_m + A_{\text{low}}$ where $A_m$ is the order $m$ operator $\sum_{|\alpha|=m} a^\alpha \partial_\alpha$. From Proposition 3.3 we have
\[
||\chi u||_{H^\sigma} \lesssim ||A_m \chi u||_{H^{\sigma-m}} + ||\chi u||_{H^{m-s}} \leq ||\chi Au||_{H^{\sigma-m}} + ||[A, \chi]u||_{H^{\sigma-m}} + ||A_{\text{low}} \chi u||_{H^{\sigma-m}} + ||\chi u||_{H^{m-s}}. \tag{13}
\]
Let $\chi'$ be a second cutoff function equal to 1 on $\text{supp} \chi$ and also compactly supported in $V_0$. Arguing as in Proposition 3.3 we have
\[
||[A, \chi]u||_{H^{\sigma-m}} \lesssim ||\chi' u||_{H^{\sigma-1}}. \tag{14}
\]
Now $A_{\text{low}} \in L^{m-1,s-1}$. Pick $\tau$ such that $\tau < s - \frac{n}{2}$ and $\tau \leq \min(1, \sigma - (m - s))$. Then from Corollary 3.2 we have
\[
||A_{\text{low}} \chi u||_{H^{\sigma-m}} \lesssim ||\chi u||_{H^{\sigma-\tau}}. \tag{15}
\]
Combining equations (13)–(15), we obtain
\[
||u||_{H^\sigma(U)} \lesssim ||Au||_{H^{\sigma-m}(V_0)} + ||u||_{H^{\sigma-\tau}(V_0)}.
\]
Finally, we obtain (12) by a standard iteration procedure working with an increasing sequence of open sets $U \subset\subset V_0 \subset\subset \cdots \subset\subset V_M \subset\subset V$ for some $M$ sufficiently large depending only on $s - \frac{n}{2}$ and $\sigma - (m - s)$. \(\Box\)
The key to proving elliptic estimates on weighted spaces is the following generalization of Lemma 5.1 of [CBC81].

**Lemma 3.5.** Let $A$ be a homogeneous constant coefficient linear elliptic operator of order $m < n$ on $\mathbb{R}^n$. For $s \in \mathbb{R}$ and $\delta \in (m - n, 0)$ we have $A : H^{s}_\delta \to H^{s-m}_{\delta-m}$ is an isomorphism.

**Proof:** We consider three ranges of $s$: $[m, \infty), [-\infty, 0]$ and $[0, m]$.

Let $A_{s,\delta}$ denote $A$ acting on $H^s_\delta$. From [CBC81] we know that if $k$ is an integer and $k \geq m$, then $A_{k,\delta}$ has an inverse $A^{-1}_{k,\delta}$. For $s \in [k, k+1]$ we find from interpolation that $A^{-1}_{k,\delta}$ restricts to a map $B_{s,\delta} : H^{s-m}_{\delta-m} \to H^s_\delta$, and it easily follows that $B_{s,\delta} = A^{-1}_{s,\delta}$. This establishes the result for $s \in [m, \infty)$.

To obtain the result for $s \in (-\infty, 0]$ we recall that $H^s_\delta = (H^{-s}_{-n-\delta})^*$. Let $A^*$ be the adjoint of $A$. From the above we know that if $s \leq 0$ and if $\delta \in (m - n, 0)$, then $A^*_{-s+m,-\delta+n+m}$ is an isomorphism. For $u \in H^{s-m}_{\delta-m}$ let $B_{s,\delta}u$ be the distribution defined by

$$\langle B_{s,\delta}u, \phi \rangle = \langle u, (A^*_{-s+m,-\delta+n+m})^{-1}\phi \rangle$$

for all $\phi \in C_0^\infty$. Now

$$| \langle B_{s,\delta}u, \phi \rangle | \leq ||u||_{H^{s-m}_{\delta-m}} ||(A^*_{-s+m,-\delta+n+m})^{-1}||_{H^{-s}_{-\delta-n}} ||\phi||_{H^{-s}_{-\delta-n}}.$$

This proves $B_{s,\delta}u \in H^s_\delta$ and we obtain a continuous map from $H^{s-m}_{\delta-m} \to H^s_\delta$. It easily follows from the definition of $B_{s,\delta}$ that $B_{s,\delta} = A^{-1}_{s,\delta}$. So the result holds for $s \in (-\infty, 0]$.

Finally, the result for $s \in [0, m]$ is obtained by interpolation. $\Box$

Combining Proposition 3.4 and Lemma 3.5 we have the following mapping property of elliptic operators in $L^{m,s}_\rho$ on weighted spaces. The approach of the proof is standard [Ca79b] [CBC81][Ba86] with some small changes needed to accommodate the weighted $H^s$ spaces.

**Proposition 3.6.** Suppose $A \in \mathcal{L}^{m,s}_\rho$ where $s > n/2$, $\sigma \in (m - s, s]$, and $\rho < 0$. Then if $m - n < \delta < 0$, and $\delta' \in \mathbb{R}$ we have

$$||u||_{H^\sigma_\delta} \leq ||Au||_{H^{\sigma-m}_{\delta-m}} + ||u||_{H^{\sigma'}_{\delta'}}$$

for every $u \in H^\sigma$. In particular, $A$ is semi-Fredholm as a map from $H^\sigma_\delta$ to $H^{\sigma-m}_{\delta-m}$.

**Proof:** Let $A = A_\infty + R$ where $A_\infty$ is the principal part of $A$ at infinity. Let $\chi$ be a cutoff function such that $1 - \chi$ has support contained in $B_2$ and is equal to 1 on $B_1$. Let $r$ be a fixed dyadic integer to be selected later, let $\chi_r(x) = \chi(x/r)$, and let $u_r = \chi_r u$. From Lemma 3.5 we have

$$||u_r||_{H^\sigma_\delta} \leq ||A_\infty u_r||_{H^{\sigma-m}_{\delta-m}}.$$
Hence
\[ ||u_r||_{H_\delta^\sigma} \lesssim ||Au_r||_{H_\delta^{\sigma-m}} + ||Ru_r||_{H_\delta^{\sigma-m}} \]
where the implicit constant does not depend on \( r \). Now \( R \in \mathcal{L}_{\rho}^{m,s} \) has vanishing principal part at infinity. Hence, from Corollary 3.2 we obtain
\[ ||Ru_r||_{H_\delta^{\sigma-m}} \lesssim ||R||_{H_\delta^{\sigma-\rho}} ||\chi_{r/2}||_{H_\delta^{s-\rho}} ||u_r||_{H_\delta^\sigma} \]
From Lemma 3.7 proved below we have
\[ \lim_{j \to \infty} ||\chi_{2^j}||_{H^{-\rho}} = 0. \]
Fixing \( r \) large enough we obtain
\[ ||u_r||_{H_\delta^\sigma} \lesssim ||Au_r||_{H_\delta^{\sigma-m}} \]
\[ \lesssim ||\chi_r Au||_{H_\delta^{\sigma-m}} + \|[A, \chi_r]u||_{H_\delta^{\sigma-m}} \]
\[ \lesssim ||Au||_{H_\delta^{\sigma-m}} + ||u||_{H^{\sigma}(B_{2r})}. \quad (17) \]
Let \( u_0 = (1 - \chi_r)u \). Then
\[ ||u_0||_{H^\sigma(B_{2r})} \lesssim ||u||_{H^{\sigma}(B_{2r})} \]
and hence
\[ ||u||_{H_\delta^\sigma} \lesssim ||u_r||_{H_\delta^\sigma} + ||u_0||_{H_{B_{2r}}^\sigma} \]
\[ \lesssim ||Au||_{H_\delta^{\sigma-m}} + ||u||_{H^{\sigma}(B_{2r})} \]
From Proposition 3.4 we then obtain
\[ ||u||_{H_\delta^\sigma} \lesssim ||Au||_{H_\delta^{\sigma-m}} + ||u||_{H^{s-m}(B_{3r})}. \]
Equation (16) now follows since for each \( \delta' \in \mathbb{R} \),
\[ ||u||_{H^{s-m}(B_{3r})} \lesssim ||u||_{H^{\delta'-m}}. \]
That \( A \) is semi-Fredholm is an immediate consequence of (16) choosing any \( \delta' > \delta \). \( \square \)

The following scaling lemma now completes the proof of Proposition 3.6.

**Lemma 3.7.** Suppose \( f \in H_\delta^s \) with \( s \in \mathbb{R} \) and \( \delta > 0 \), and suppose \( f \) vanishes in a neighbourhood of the origin. Then
\[ \lim_{i \to \infty} ||T_{-i}f||_{H_\delta^\sigma} = 0. \]

**Proof:** Without loss of generality we can assume that \( f \) vanishes on \( B_2 \). Then \( T_{j-i}f = 0 \) on \( B_2 \) whenever \( j \leq i \). So
\[ ||T_{-i}f||_{H_\delta^\sigma}^2 = \sum_{j=i+1}^{\infty} 2^{-2\delta j} ||T_j(\phi_j T_{-i}f)||_{H^s}^2 \]
\[ = 2^{-2\delta i} \sum_{j=1}^{\infty} 2^{-2\delta j} ||T_j(\phi_j f)||_{H^s}^2 \]
\[ \leq 2^{-2\delta i} ||f||_{H_\delta^s}^2. \]
Since \( \delta > 0 \), the result is proved. \( \square \)
We conclude this section with a result concerning decay properties of elements in the kernel of elliptic operators in $\mathcal{L}_\rho^{m,s}$.

**Lemma 3.8.** Suppose $A$ is an elliptic operator in $\mathcal{L}_\rho^{m,s}$, where $s > n/2$ and $\rho < 0$. If $u \in H^s_\delta$ for some $\delta < 0$ satisfies $Au = 0$, then $u \in H^s_{\delta'}$ for all $\delta' \in (m - n, 0)$.

**Proof:** Let $A = A_\infty + R$ where $A_\infty$ is the homogeneous constant coefficient linear elliptic operator giving the principal part of $A$ at $\infty$. Then

$$A_\infty u = -Ru \in H^{s-m}_{\delta-m+\rho}.$$ 

Since $A$ is an isomorphism acting on $H^s_{\delta'}$ for each $\delta' \in (m - n, 0)$, we conclude that $u \in H^s_{\delta'}$ for each $\delta' \in (\max(m-n, \delta+\rho), 0)$. Iterating this argument yields the desired result. □

**Remark.** Using partition of unity arguments, the results from this section easily extend to asymptotically Euclidean manifolds. In the sequel, we will use these results in the asymptotically Euclidean context without further comment.

### 4. The York Decomposition

In this section we prove that the vector Laplacian is an isomorphism on certain weighted Sobolev spaces. The first step is to show it is a Fredholm operator with index zero, and that its kernel consists of conformal Killing fields decaying at infinity. The second step is to rule out the existence of such conformal Killing fields. These facts are well known for classical metrics; our only concern is the the low regularity of the metrics involved.

To show that the kernel of $\Delta_L$ consists of conformal Killing fields requires integration by parts, and we need to justify this operation in the low regularity setting. Suppose $(M, g)$ is AE of class $H^s_\rho$ with $s > n/2$ and $\rho < 0$. By working in the local charts that define $H^s(M)$, it is easy to see that for every $\sigma \in [0, s]$ and $\delta \in \mathbb{R}$ there is a natural continuous bilinear form $\langle \cdot, \cdot \rangle_{(M,g)} : H^{-\sigma}_\delta(M) \times H^{-n-\delta}_\sigma(M) \to \mathbb{R}$ that satisfies

$$\langle X, Y \rangle_{(M,g)} = \int_M \langle X, Y \rangle \, dV$$

whenever $X$ and $Y$ are smooth compactly supported vector fields. We note that $\langle \cdot, \cdot \rangle_{(M,g)}$ can be defined similarly for scalar functions, and for a fixed pair $u \in H^{-\sigma}_{\text{loc}}(M)$ and $v \in H^\sigma_{\text{loc}}(M)$, the value of $\langle u, v \rangle_{(M,g)}$ is also well defined so long as either $u$ or $v$ is compactly supported.

Now suppose $X, Y \in H^1_\delta(M)$. Since $\Delta_L X \in H^{-1}_{\delta-2}$, the quantity $\langle -\Delta_L X, Y \rangle_{(M,g)}$ is well defined if $Y \in H^1_{2-n-\delta}(M)$ and $s \geq 1$. Since $s > n/2 > 1$, we only have the restriction
\[ \delta \leq (2 - n)/2. \] Under this same condition on \( \delta \) we know \( \nabla X, \nabla Y \in L^2(M) \). For smooth compactly supported vector fields

\[ \int_M \langle L_X, L_Y \rangle \, dV = \langle -\Delta L X, Y \rangle_{(M,g)} , \]

and a density argument extends integration by parts to \( X, Y \in H^1_\delta(M) \) when \( \delta \leq (2 - n)/2 \).

Proposition 4.1. Suppose \((M, g)\) is AE of class \( H^s_\rho \) with \( s > n/2 \) and \( \rho < 0 \). If \( \delta \in (2 - n, 0) \) then \( \Delta L \) acting on \( H^s_\delta(M) \) is Fredholm with index 0. Moreover, the kernel of \( \Delta L \) is the set of conformal Killing fields in \( H^s_\delta \).

**Proof:** From Proposition 3.6 we know \( \Delta L \) is semi-Fredholm, and for smooth metrics it is known that \( \Delta L \) has index 0. Approximating \( g \) with a sequence of smooth metrics \( \{g_k\} \) we have \( \Delta L_k \to \Delta L \) and hence the index of \( \Delta L \) is also 0. Suppose \( X \in H^s_\delta \) and \( X \in \ker \Delta L \). Then from Lemma 3.8 we know \( X \in H^s_{\delta'} \) for all \( \delta' \in (2 - n, 0) \). In particular, we can pick \( \delta' \leq (2 - n)/2 \) and integrate by parts to find

\[ 0 = \langle -\Delta L X, Y \rangle_{(M,g)} = \int_M \langle L_X, L_Y \rangle \, dV. \]

So \( L X = 0 \) in \( L^2 \) and hence \( X \) is a conformal Killing field. \( \square \)

For smooth metrics on asymptotically Euclidean manifolds we know [CO81] that there are no conformal Killing fields vanishing at infinity and hence \( \Delta L \) is an isomorphism. In the case of less regular metrics, it follows from [Ba04] Theorem 3.6 that if \( n = 3 \) and \( g \) is of class \( H^s_\rho \) with \( s \geq 2 \) and \( \rho < 0 \), then there are no conformal Killing fields in \( H^s_\delta \) for any \( \delta < 0 \). We prove now that the same result holds for \( s > 3/2 \).

To do this we consider a boundary value problem. Let \( \Omega \subset \mathbb{R}^n \) be an open set with smooth boundary and outward pointing normal \( \nu \), and let \( \overline{L} \) and \( \overline{\Delta}_L \) be the differential operators computed with respect to the Euclidean metric. The Neumann boundary operator \( \overline{B} \) for \( \overline{\Delta}_L \) takes a vector field \( X \) to the covector field \( \overline{\nabla}X(\nu, \cdot) \). From standard elliptic theory we have the following a-priori estimate.

**Lemma 4.2.** Suppose \( \Omega \) is a bounded open set with smooth boundary in \( \mathbb{R}^n \). For any vector field \( X \in H^s(\Omega) \) with \( s > 3/2 \) we have

\[ ||X||_{H^s(\Omega)} \lesssim ||\overline{\Delta}_L X||_{H^{s-2}(\Omega)} + ||\overline{B} X||_{H^{s-2}(\partial\Omega)} + ||X||_{H^{s-1}(\Omega)}, \]

where the implicit constant depends on \( \Omega \) but not on \( X \).

From Lemmas 4.2 and 3.5 one readily obtains using the techniques of Proposition 3.6 an a-priori estimate in the exterior region \( E_1 \).
Lemma 4.4. Suppose $(M, g)$ is AE of class $H^s_\rho$ with $s > n/2$ and $\rho < 0$. If $X \in H^s_\delta$ with $s > n/2$ and $\delta < 0$ is a conformal Killing field, then it vanishes outside some compact set.

Proof: We assume for simplicity that $M$ has a single end. Working in end coordinates, we define a sequence of metrics $\{g_k\}_{k=1}^\infty$ on the exterior region $E_1$ via $g_k(x) = g(2^k x)$. Since $\rho < 0$, it follows from arguments similar to those of Lemma 3.7 that $g_k - \overline{g}$ converges to 0 in $H^s_\delta(E_1)$. It follows that the associated operators $\Delta^k_\Lambda$, $L^k$, and $B^k$ converge to their Euclidean analogues as operators on $H^s_\delta(E_1)$.

Suppose, to produce a contradiction, that $X$ is not identically 0 outside any exterior region $E_R$. Let $\hat{X}_k(x) = X(2^k x)$ and let $X_k = \hat{X}_k/||\hat{X}_k||_{H^s_\delta(E_1)}$. Since the sequence $\{X_k\}_{k=0}^\infty$ is bounded in $H^s_\delta$, we conclude after reducing to a subsequence that the sequence converges strongly in $H^{s-1}_{\delta'}$ for any $\delta' \in (\delta, 0)$ to some vector field $X_0$.

We can assume without loss of generality that $\delta \in (2 - n, 0)$. So from Lemma 4.3, the $H^s_\delta$ boundedness of the sequence $\{X_k\}_{k=1}^\infty$, and the identities $L^k X_k = 0$, it follows that

$$||X_{k_1} - X_{k_2}||_{H^s_\delta(E_1)} \leq ||\Delta^k_\Lambda - \Delta^{k_2}_\Lambda||_{H^s_\delta(E_1)} + ||\Delta^k_\Lambda - \Delta^{k_2}_\Lambda||_{H^s_\delta(E_1)} + ||B - B^{k_1}||_{H^s_\delta(E_1)} + ||\overline{B} - B^{k_2}||_{H^s_\delta(E_1)} + ||X_{k_1} - X_{k_2}||_{H^{s-1}_{\delta'}(E_1)}.$$

We conclude $\{X_k\}_{k=1}^\infty$ is Cauchy in $H^s_\delta(E_1)$ and hence converges in $H^s_\delta$ to $X_0$. Since $||X_k||_{H^s_\delta(E_1)} = 1$, $X_0$ cannot be identically zero. Moreover, since $X_k$ is a conformal Killing field for $g_k$, it follows that $X_0$ is a conformal Killing field for $\overline{g}$. But $\overline{g}$ does not admit any nontrivial conformal Killing fields in $H^s_\delta$, a contradiction. \square

Using Lemma 4.4 together with the a-priori estimate Lemma 4.2, we now prove that the set of conformal Killing fields vanishing at infinity is trivial.

Proposition 4.5. Suppose $(M, g)$ is AE of class $H^s_\rho$ with $s > n/2$ and $\rho < 0$. If $X \in H^s_\delta$ with $s > n/2$ and $\delta < 0$ is a conformal Killing field, then it vanishes identically.

Proof: Let $U$ be the interior of the region where $X$ vanishes. From Lemma 4.4 we know that $U$ is non-empty. To show that $U = M$ it is enough to show that it has empty boundary.
Suppose, to produce a contradiction, that \( x \) is a boundary point of \( U \). Working in local coordinates about \( x \), we can assume \( M = \mathbb{R}^n \). Hereafter, all balls and distances are computed with respect to the flat background metric. Let \( y \in U \) and let \( r = d(y, \partial U) \). Then \( B_r(y) \subset U \), and there exists some point \( z \in B_r(y) \cap \partial U \). After making an affine change of coordinates, we can assume \( z = 0 \) and \( g(0) = \overline{g} \).

We construct a sequence of metrics \( \{g_k\}_{k=1}^\infty \) on the unit ball by taking \( g_k(x) = g(2^{-k}x) \). Since \( s > n/2 \) and \( g(0) = \overline{g} \), it readily follows that \( g_k - \overline{g} \) converges to 0 in \( H^s(B_1) \). It follows that the associated maps \( \Delta_k^L, L_k, \) and \( B_k^L \) converge to their Euclidean counterparts as operators on \( H^s(B_1) \).

We construct vector fields \( X_k \) on \( B_1 \) by setting \( \hat{X}_k(x) = X(2^{-k}x) \) and letting \( X_k = \hat{X}_k / ||\hat{X}_k||_{H^s(B_1)} \); this normalization is possible since \( X \) is not identically 0 on \( B_{2^{-k}} \).

Since the sequence is bounded in \( H^s \), we conclude that the sequence converges strongly in \( H^s(B_1) \) to some \( X_0 \in H^s(B_1) \). Moreover, from the choice of the point \( z \), it follows that \( X_k \) vanishes on an open cone \( K \) independent of \( k \).

Arguing as in Lemma 4.4, replacing the use of Lemma 4.3 with Lemma 4.2, we conclude \( X_0 \) is a conformal Killing field for \( g \) and \( X_k \) converges in \( H^s(B_1) \) to \( X_0 \). In particular, \( X_0 \) is a nontrivial conformal Killing field for \( g \) on \( B_1 \) that vanishes on an open cone. But any such conformal Killing field must vanish identically, a contradiction. \( \square \)

Combining Propositions 4.1 and 4.5 we immediately obtain the York decomposition of symmetric \((0,2)\)-tensors.

**Theorem 4.6.** Suppose \((M, g)\) is AE of class \( H^s_\rho \) with \( s > n/2 \) and \( \rho < 0 \). Then \( \Delta_L \) is an isomorphism acting on \( H^s_\delta \) for any \( \delta \in (2 - n, 0) \). If \( S \) is a symmetric trace-free \((0,2)\) tensor in \( H^{s-1}_\delta \), then there is a unique vector \( X \in H^s_\delta \) and a unique transverse traceless tensor \( \sigma \in H^{s-1}_{\delta - 2} \) such that \( S = LX + \sigma \).

### 5. The Linearized Lichnerowicz Equation

Our existence theorem for solutions of the Lichnerowicz equation relies on the well-known method of sub and super-solutions. We use a variation of the constructive method used by Isenberg [Is95] to solve the Lichnerowicz equation on compact manifolds with \( C^2 \) metrics. The method has subsequently been extended to weaker classes of metrics [CB03][Ma03].

In this section we establish properties of the linearized Lichnerowicz equation \(-\Delta + V\) needed to extend the method of sub and super-solutions to the low regularity setting.

**Proposition 5.1.** Suppose \((M, g)\) is AE of class \( H^s_\rho \) with \( s > n/2 \) and \( \rho < 0 \). Suppose also that \( V \in H^{s-2}_{\rho-2} \). Then for \( \delta \in (2 - n, 0) \) the operator \(-\Delta + V : H^s_\delta(M) \rightarrow H^{s-2}_{\delta - 2}(M) \) is
Fredholm with index 0. Moreover, if \( V \geq 0 \) then \(-\Delta + V\) is an isomorphism.

We note that if \( v \in H^{-\sigma}_{\text{loc}}(M) \) with \( \sigma \in [0, s] \), we say \( v \geq 0 \) if

\[
\langle v, u \rangle_{(M, g)} \geq 0
\]

for every \( u \in H^\sigma_{\text{loc}}(M) \) with compact support satisfying \( u \geq 0 \). The proof that \(-\Delta + V\) has index 0 follows that of Proposition 4.1. The proof of the injectivity of \(-\Delta + V\) in the case \( V \geq 0 \) follows from the weak maximum principle proved below.

**Lemma 5.2.** Suppose \((M, g)\) is AE of class \( H^s_\rho \) with \( s > n/2 \) and \( \rho < 0 \). Suppose also that \( V \in H^{s-2}_\rho(M) \) and that \( V \geq 0 \). If \( u \in H^\sigma_{\text{loc}} \) satisfies

\[
-\Delta u + Vu \leq 0, \tag{18}
\]

and if \( u^{(+)} = \max(u, 0) \) is \( o(1) \) on each end of \( M \), then \( u \leq 0 \). In particular, if \( u \in H^s_\delta(M) \) for some \( \delta < 0 \) and \( u \) satisfies (18), then \( u \leq 0 \).

**Proof:** Fix \( \epsilon > 0 \), and let \( v = (u - \epsilon)^{(+)} \). Since \( u^{(+)} = o(1) \) on each end, we see \( v \) is compactly supported. Moreover, an easy computation shows \( uv \) is non-negative, compactly supported, and belongs to \( H^1 \). Since \( V \in H^{s-2}_\rho \), and since \( s - 2 \geq -1 \) we can apply \( V \) to \( uv \). Since \( V \geq 0 \), and since \( uv \geq 0 \), we have \( \langle V, uv \rangle_{(M, g)} \geq 0 \). Finally, since \( u \) satisfies (18) we obtain

\[
\langle -\Delta u, v \rangle_{(M, g)} \leq -\langle V, uv \rangle_{(M, g)} \leq 0.
\]

Integrating by parts we conclude \( v \) is constant and compactly supported, and therefore vanishes identically. This proves \( u \leq \epsilon \), and sending \( \epsilon \) to 0 proves \( u \leq 0 \).

The statement for \( u \in H^s_\delta(M) \) now follows from the the above and the embedding \( H^s_\delta \hookrightarrow C^0_{\delta} \). \( \square \)

**Lemma 5.3.** Suppose \((M, g)\) is AE of class \( H^s_\rho \) with \( s > n/2 \) and \( \rho < 0 \). Suppose also that \( V \in H^{s-2}_\delta(M) \) and \( u \in H^\sigma_{\text{loc}}(M) \) is nonnegative and satisfies

\[
-\Delta u + Vu \geq 0
\]

If \( u(x) = 0 \) at some point \( x \in M \), then \( u \) vanishes identically.

**Proof:** Since \( M \) is connected, it is enough to show that the zero set of \( u \) is open and closed. That \( u^{-1}(0) \) is closed is obvious, so we need only verify \( u^{-1}(0) \) is open. We would like to apply the weak Harnack inequality of [Tr73] Theorem 5.2. This inequality holds under quite general conditions, and in particular can be applied to second order elliptic operators of the form

\[
L u = \partial_i \left( a^{ij} \partial_j u + a^i u \right) + b^i u_j + au
\]
where \( a^{ij} \) is continuous, and \( a^i \in L^p_{\text{loc}}, b^j \in L^p_{\text{loc}}, \) and \( a \in L^{p/2}_{\text{loc}} \) for some \( p > n \). These conditions hold for the Laplacian when \((M, g)\) is AE of class \( H^s \) with \( s > n/2 \), and also hold for the potential term \( V \) when \( n > 3 \). When \( n = 3 \), the potential term \( V \) does not necessarily belong to an \( L^p \) space for any \( p \), and is not considered directly by [Tr73]. None-the-less, a simple transformation of the equation converts it into a more amenable form.

We work in local coordinates near some point \( x \) where \( u(x) = 0 \). Let \( B \) be a small ball about \( x \) (computed with respect to the flat background metric) and let \( \Phi \in H^s_{\text{loc}} \) be any solution in \( B \) of

\[
\overline{\Delta} \Phi = V
\]

where \( \overline{\Delta} \) is the Laplacian computed with respect to the flat background metric. It follows that \( u \) satisfies

\[
-\Delta u + \nabla \cdot (\nabla \Phi u) - \nabla \Phi \cdot \nabla u \geq 0
\]

in a neighbourhood of \( x \). Now \( \nabla \Phi \in H^{s-1}_{\text{loc}} \), and \( H^{s-1}_{\text{loc}} \subset L^q_{\text{loc}} \) for any \( q \geq 1 \) such that

\[
\frac{1}{q} > \frac{1}{2} - \frac{s-1}{n}.
\]

In particular we can pick \( q > n \). Since \( u \geq 0 \) and since \( u(x) = 0 \), the weak Harnack inequality applied to (19) implies \( u \) vanishes in a neighbourhood of \( x \). \( \square \)

6. The Cantor-Brill Condition

In this section we show that there exists a solution of the Lichnerowicz equation

\[
-a \Delta \phi + R \phi - |\sigma|^2 \phi^{-2\kappa-3} = 0
\]

if and only if the Cantor-Brill condition holds. We define the Yamabe invariant

\[
\lambda_g = \inf_{f \in C^\infty_c(M) \atop f \not\equiv 0} \frac{\int_M a |\nabla f|^2 + R f^2 \, dV}{\|f\|_{L^{2^*}}^2},
\]

where \( 2^* = \frac{2n}{n-2} \). We note that when \( R \) is not a locally integrable function, but only a distribution, by \( \int_M R f^2 \, dV \) we mean \( \langle R, f^2 \rangle_{(M,g)} \).

The proof of the following proposition gives conditions equivalent to \( \lambda_g > 0 \) and closely follows the analogous result in [Ma03].
Proposition 6.1. Suppose \((M, g)\) is AE of class \(H^s_\delta\) with \(s > n/2\), and \(\delta \in (2 - n, 0)\). Then the following conditions are equivalent:

1. There exists a conformal factor \(\phi > 0\) such that \(1 - \phi \in H^s_\delta(M)\) and such that \((M, \phi^{4/(n-2)} g)\) is scalar flat.
2. \(\lambda_g > 0\).
3. For each \(\eta \in [0, 1]\), \(P_\eta = -a\Delta + \eta R\) is an isomorphism acting on \(H^s_\delta(M)\).

Proof: Suppose condition 1 holds. Since \(\lambda_g\) is a conformal invariant, we can assume that \(R = 0\). By solving the equation

\[-a\Delta v = R\]

for some smooth positive \(R \in H^{s-2}_{\delta-2}\), we can make the conformal change corresponding to \(\phi = 1 + v\) to a metric with continuous positive scalar curvature \(R\). Let \(K\) be the compact core of \(M\). Since

\[\|f\|_{L^2}^2 \leq \|\nabla f\|_{L^2}^2 + \|f\|_{L^2(K)}^2,\]

we find

\[\|f\|_{L^2}^2 \leq \|\nabla f\|_{L^2}^2 + \|R^{1/2} f\|_{L^2}^2.\]

Hence \(\lambda_g > 0\).

Now suppose condition 2 holds. To show condition 3 is true, it is enough to show each \(P_\eta\) has trivial kernel for each \(\eta \in [0, 1]\). When \(\eta = 0\) the result is obvious, so we consider the case \(\eta \in (0, 1]\). Suppose, to produce a contradiction, that \(P_\eta u = 0\). From Lemma 3.8 we have \(u \in H^s_\delta\) for any \(\delta' \in (2 - n, 0)\). Fixing \(\delta' \leq (2 - n)/2\) we can integrate by parts to obtain

\[0 = \int_M -a\Delta u + \eta Ru^2 \, dV = \int_M a \|\nabla u\|^2 + \eta Ru^2 \, dV \geq \eta \int_M a \|\nabla u\|^2 + Ru^2 \, dV.\]

Consider the map \(Q(v) = \int_M a \|\nabla v\|^2 + Ru^2 \, dV\) defined on \(H^s_{\delta'}\). Since \(\delta' \leq (2 - n)/2\), the map \(v \mapsto \nabla v\) is continuous from \(H^s_{\delta'}\) to \(L^2\). On the other hand, the map \(v \mapsto v^2\) is continuous from \(H^s_{\delta'}\) to \(H^2_{\delta'}\). Since \(R \in H^{s-2}_{\delta-2}\), the map \(v \mapsto (R, v^2)_{(M, g)}\) is continuous if \(s > 2 - s\) and \(2\delta' \leq -n - \delta + 2\). These last conditions hold for \(s > 1\) and \(\delta' \leq (2 - n)/2\). We have therefore shown that \(Q\) is continuous on \(H^s_{\delta'}\). From (22) and the inequality \(\delta' \leq (2 - n)/2\) we have the continuous embeddings \(H^s_{\delta'} \rightarrow H^1_{\delta'} \rightarrow L^2\). Thus, approximating \(u\) in \(H^s_{\delta'}\) with smooth compactly supported functions \(\{u_k\}_{k=1}^\infty\), we find

\[\frac{Q(u_k)}{\|u_k\|_{L^2}^2} \rightarrow \frac{Q(u)}{\|u\|_{L^2}^2} \leq 0\]

by (23). Hence \(\lambda_g \leq 0\), a contradiction.
Finally, suppose condition 3 holds. For each \( \eta \in [0, 1] \), let \( v_\eta \) be the unique solution in \( H^s_\delta \) of

\[
-a \Delta v_\eta + \eta R v_\eta = -\eta R.
\]

Setting \( \phi_\eta = 1 + v_\eta \) we see

\[
-a \Delta \phi_\eta + \eta R \phi_\eta = 0
\]

(24)

To show \( \phi_\eta > 0 \) for each \( \eta \in [0, 1] \) we use a continuity argument. Let \( I = \{ \eta \in [0, 1] : \phi_\eta > 0 \} \). The set is open since \( H^s_\delta \subset C^0_\delta \), and is nonempty since \( \phi_0 = 1 \). If \( \eta_0 \in \bar{I} \), then \( \phi_{\eta_0} \geq 0 \). Since \( \phi_\eta \) solves (24), and since \( \phi_{\eta_0} \) tends to 1 at infinity, Lemma 5.3 then implies \( \phi_{\eta_0} > 0 \). Hence \( \eta_0 \in I \) and \( I \) is closed.

Letting \( \phi = \phi_1 \) we have shown \( \phi > 0 \). It follows from (24) that \( (M, \phi^{\frac{1}{n-2}} g) \) is scalar flat. Moreover, since \( \phi - 1 \in H^s_\delta \) it follows from Lemma 2.5 and Corollary 2.6 that \( (M, \phi^{\frac{1}{n-2}} g) \) is also AE of class \( H^s_\delta \). \( \square \)

In preparation for solving the Lichnerowicz equation, we consider the equation

\[
-\Delta u = F(x, u) = \frac{1}{a} |\sigma|^2 (1 + u)^{-2\kappa-3}.
\]

A subsolution of (25) is a function \( u_- \) that satisfies

\[
-\Delta u_- \leq F(x, u_-),
\]

and a supersolution \( u_+ \) is defined similarly with the inequality reversed.

**Proposition 6.2.** Suppose \( (M, g) \) is AE of class \( H^s_\rho \) with \( s > n/2 \) and \( \rho < 0 \), and suppose \( \sigma \in H^{s-1}_{\delta-1} \) with \( \delta \in (2-n, 0) \). If \( u_- \), \( u_+ \in H^s_\delta (M) \) are a subsolution and a supersolution respectively of (25) that satisfy \(-1 < u_- \leq u_+ \), then there exists a solution \( u \in H^s_\delta \) of (25) such that \( u_- \leq u \leq u_+ \).

**Proof:** We first treat the case \( n = 3 \) and \( 3/2 < s \leq 2 \). For notational simplicity, we set \( c(x) = \frac{1}{a} |\sigma|^2 \) and \( f(u) = (1 + u)^{-2\kappa-3} \) so that \( F(x, u) = c(x)f(u) \). Let \( I = [\min u_-, \max u_+] \), and let

\[
V(x) = c(x) \left| \min_I f' \right|.
\]

It follows that \( V \in H^{s-2}_{2\delta-2} \) and is non-negative. Let \( F_V(x, y) = F(x, y) + V(x)y \), so that \( F_V \) is non-decreasing in \( y \), and let \( L_V = -\Delta + V \). Since \( V \) is non-negative, \( L_V \) is an isomorphism acting on \( H^s_\delta \).

We claim that \( ||F_V(x, u)||_{H^{s-2}_{\delta-2}} \) is uniformly bounded for \( u \in H^s_\delta \) with \( u_- \leq u \leq u_+ \). We first consider the term \( F(x, u) \). From Sobolev embedding, \( \sigma \in L^q_\delta \) where

\[
\frac{1}{q} = \frac{1}{2} - \frac{s-1}{3}.
\]
Hence $|\sigma|^2 \in L^{q/2}_{2\delta - 2}$. If we can show $L^{q/2}_{2\delta - 2}$ is continuously embedded in $H^{-2}_{\delta - 2}$, then we will obtain

$$
||F(x, u)||_{H^{-2}_{\delta - 2}} \leq ||c(x)||_{L^{q/2}_{2\delta - 2}} ||f(u)||_{L^\infty} \lesssim 1.
$$

Now the dual space of $H^{-2}_{\delta - 2}$ is $H^{2}_{-1 - \delta}$. From Sobolev embedding, $H^{2}_{-1 - \delta}$ is continuously embedded in $L^{p}_{-1 - \delta}$, where

$$\frac{1}{p} = \frac{1}{2} - \frac{2 - s}{3}.$$

The dual space of $L^{p}_{-1 - \delta}$ is $L^{p'}_{\delta - 2}$ where

$$\frac{1}{p'} = \frac{1}{2} + \frac{2 - s}{3}.$$

Since $q/2 \geq p'$ exactly when $s \geq 3/2$, and since $2\delta - 2 < \delta - 2$, we obtain the continuous embedding of $L^{q/2}_{2\delta - 2} \hookrightarrow H^{-2}_{\delta - 2}$ as desired. A similar analysis shows that $||V(x)u||_{H^s_{\delta - 2}} \leq 1$ and we obtain

$$
||F_V(x, u)||_{H^{-2}_{\delta - 2}} \lesssim 1. \quad (26)
$$

We now construct a monotone decreasing sequence of functions $u_+ = u_0 \geq u_1 \geq u_2 \geq \cdots$ by iteratively solving

$$L_V u_{i+1} = F_V(x, u_i).$$

The monotonicity of the sequence follows from the maximum principle Lemma 5.2 and the monotonicity of $F_V(x, y)$ in $y$. The maximum principle also implies $u_i \geq u_{i-}$.

From Proposition 5.1 we can estimate

$$||u_{i+1}||_{H^s_{\delta}} \lesssim ||F_V(x, u_i)||_{H^{s-2}_{\delta - 2}}. \quad (27)$$

From (26) and (27) we conclude the sequence $\{u_i\}_{i=1}^\infty$ is bounded in $H^s_{\delta}$. In particular, a subsequence of $\{u_i\}_{i=1}^\infty$ (and by monotonicity, the whole sequence) converges weakly in $H^s_{\delta}$ to a limit $u_\infty$.

It remains to see $u_\infty$ is a solution of (25). Now $u_i$ converges strongly to $u_\infty$ in $H^{s'}_{\delta'}$ for any $s' < s$ and $\delta' > \delta$. Hence, on any compact set the sequence converges in $H^1$ and in $C^0$.

We conclude that for any fixed $\phi \in C^\infty_c$, each $f_j(u_i)\phi$ converges in $H^1$ to $f_j(u_\infty)\phi$. Since each $c_j \in H^{s-2}_{\rho-2}$, and since $s - 2 > -1$, we find

$$\int_M (F_V(x, u_i) - V(x)u_{i+1}) \phi \, dV \to \int_M F(x, u_\infty)\phi \, dV.$$

Moreover,

$$\int_M \langle \nabla u_{i+1}, \nabla \phi \rangle \, dV \to \int_M \langle \nabla u_\infty, \nabla \phi \rangle \, dV.$$
Hence
\[ \int_M \langle \nabla u_\infty, \nabla \phi \rangle \, dV = \int_M F(x, u_\infty) \phi \, dV. \]
Since \( \phi \in C_c^\infty \) is arbitrary, we conclude \(-\Delta u_\infty + F(x, u_\infty) = 0\) as a distribution.

To handle the case \( n = 3 \) and \( s > 2 \) we use a bootstrap. First suppose \( 4 \geq s \geq 2 \). From the above we have a solution \( u \) in \( H^2_\delta \). Since \( 2 > n/2 = 3/2 \) and since \( 2 > s - 2 \in [0, 2] \), we know from Lemma 2.5 and the remark following it that \( c(x)f(u) \in H^{s-2}_{\delta-2} \). Since \(-\Delta u \in H^{s-2}_{\delta-2} \), Proposition 5.1 implies \( u \in H^s_\delta \). We obtain the result for all \( s > 3/2 \) by induction.

Finally, we turn to the case \( n > 3 \). Here we cannot start a bootstrap from \( s = 2 \) since \( H^2 \) is not an algebra. Instead we rely on results from [Ma03] for metrics of class \( W^{k,p}_\rho \).

Let \( k \) be the greatest integer such that \( k \leq s \). From Sobolev embedding we find \((M, g)\) is asymptotically Euclidean of class \( W^{k,p}_\rho \) where
\[ \frac{1}{p} = \frac{1}{2} - \frac{s - k}{n}. \]

Since \( n > 3 \) it easily follows that \( 2 \leq p < \infty \) and moreover that \( k \geq 2 \) and \( k > n/p \). An easy computation shows multiplication by \( f(u) \) is a continuous involution of \( H^{k-1}_\delta \) and \( H^{k-2}_\delta \). From interpolation we conclude \( c(x)f(u) \in H^{s-2}_{\delta-2} \). Proposition 5.1 then implies there exists \( \hat{u} \in H^s_\delta \) with \(-\Delta \hat{u} = c(x)f(u) \). But \( \hat{u} \in W^{k,p}_\rho \) and the Laplacian of a \( W^{k,p}_\rho \) metric is an isomorphism of \( W^{k,p}_\rho \). Hence \( \hat{u} = u \) and we obtain a solution in \( H^s_\delta \).

Using Proposition 6.2 we readily obtain the existence of solutions of (20).

**Theorem 6.3.** Suppose \((M, g)\) is AE of class \( H^s_\delta \) with \( s > n/2 \) and \( \delta \in (2-n, 0) \). Let \( \sigma \) be any transverse-traceless tensor in \( H^{s-1}_{\delta-1}(M) \). There exists a conformal factor \( \phi \) solving (20) if and only if \( \lambda_g > 0 \). Moreover, if a solution exists then it is unique.

**Proof:** If a solution exists, then it follows from (20) that \( g \) is conformally related to a metric with non-negative scalar curvature, and from Proposition 6.1 that \( \lambda_g > 0 \).

If \( \lambda_g > 0 \) we can assume without loss of generality that \( R = 0 \). Setting \( \phi = 1 + v \), solving the Lichnerowicz equation is equivalent to solving
\[ -a \Delta v = |\sigma|^2 (1 + v)^{-2\kappa-3} \quad (28) \]
with the constraint \( v > -1 \).

Evidently \( v_- = 0 \) is a subsolution of (28). To find a supersolution, we solve
\[ -a \Delta v_+ = |\sigma|^2. \quad (29) \]
Moreover, the Fréchet derivative of
\[ ||d|| \]
and sequence of traceless tensors for fixed
\[ t \geq 0 \]
bounds on the norm of the inverse of
\[ R \]
To construct a sequence of transverse-traceless tensors, we let
\[ g \]
approximated arbitrarily well by smooth solutions.

7. Approximation by Smooth Solutions

The following theorem shows that a rough, maximal, AE solution of the constraints can be approximated arbitrarily well by smooth solutions.

**Theorem 7.1.** Let \( (M, g_0, K_0) \) be a maximal AE solution of the constraint equations of class \( H^s_\delta \) with \( s > n/2 \) and \( \delta \in (n - 2, 0) \). For any \( \epsilon > 0 \), there exists a maximal AE solution \( (M, g, K) \) of the constraint equations of class \( H^t_\delta \) for every \( t \geq s \) such that
\[ ||g_0 - g||_{H^s_\delta} < \epsilon \quad \text{and} \quad ||K_0 - K||_{H^{s-1}_\delta} < \epsilon. \]

**Proof:** Let \( \{g_k\}_{k=1}^\infty \) be a sequence of metrics on \( M \) in \( H^t_\delta \) for every \( t \geq s \) such that
\[ ||g_k - g_0||_{H^s_\delta} \to 0. \]
We will write \( \Delta^k_L, \mathbb{L}^k, \text{and div}_k \) for the differential operators corresponding to \( g_k \).

To construct a sequence of transverse-traceless tensors, we let \( \{S_k\}_{k=1}^\infty \) be an arbitrary sequence of traceless \((0, 2)\)-tensors in \( H^{t-1}_{\delta-1} \) for every \( t \geq s \) converging to \( K_0 \) in \( H^{s-1}_{\delta-1} \). Let
\[ X_k \in H^s_\delta \]
be the unique solution of \( \Delta^k_L X_k = \text{div}_k S_k \). Since \( \text{div}_k S_k \in H^{t-2}_{\delta-2} \) for every \( t \geq s \), it follows that \( \mathbb{L} X_k \in H^{t-1}_{\delta-1} \) for every \( t \geq s \). Since \( \Delta_L \) is invertible, we have uniform bounds on the norm of the inverse of \( \Delta^k_L \). Hence
\[ ||X_k||_{H^s_\delta} \leq ||\text{div}_k S_k|| \leq ||\text{div}_k||_{H^{s-1}_{\delta-1}} ||S_k - K_0||_{H^{s-1}_{\delta-1}} + ||\text{div}_k - \text{div}||_{H^{s-1}_{\delta-1}} ||K_0||_{H^{s-1}_{\delta-1}}. \]

So \( ||X_k||_{H^s_\delta} \to 0 \). Letting \( \sigma_k = S_k - \mathbb{L}^k X_k \) it follows that \( \sigma_k \) is transverse-traceless with respect to \( g_k \), and is in \( H^{t-1}_{\delta-1} \) for every \( t \geq s \). Moreover,
\[ ||\sigma_k - K_0||_{H^{s-1}_{\delta-1}} \leq ||\mathbb{L}^k X_k||_{H^{s-1}_{\delta-1}} + ||S_k - K_0||_{H^{s-1}_{\delta-1}} \to 0. \]

The correction to \( g_k \) is now accomplished by the implicit function theorem. Let
\[ F(g, \sigma, v) = -a \Delta_g v + R_g (1 + v) - |\sigma|^2_g (1 + v)^{-2\kappa-3}. \]

For fixed \( g \) and \( \sigma \), let \( F_{g,\sigma}(v) = F(g, \sigma, v) \). Using Lemma 7.2 proved below, we find that the Fréchet derivative of \( F_{g,\sigma} \) is
\[ d F_{g,\sigma}(v)(h) = -a \Delta_g h + R_g h + (2\kappa + 3) |\sigma|^2_g (1 + v)^{-2\kappa-4} h, \]
and \( d F_{g,\sigma}(v) \) is hence continuous in a neighbourhood of \((g, \sigma, v)\) for each \( v \) with \( v > -1 \). Moreover,
\[ d F_{g,0, K_0}(0)(h) = L(h) = -a \Delta_{g_0} h + R_{g_0} h + (2\kappa + 3) |K_0|^2 h. \]
Since $R_0$ is nonnegative, $L$ is an isomorphism. Since $F(g_0, K_0, 0) = 0$, and since $g_k \to g_0$ and $\sigma_k \to K_0$, the implicit function theorem (e.g. [AP93] Lemma 2.2.1) implies that for $k$ sufficiently large there exists $v_k \in H_\delta^s$ such that $v_k \to 0$ and such that $F(g_k, \sigma_k, v_k) = 0$. From the equation $F(g_k, \sigma_k, v_k) = 0$ and a bootstrap we have $v_k \in H_\delta^s$ for every $t \geq s$. Letting $g_k = (1 + v_k)^{2\kappa} g_k$ and $\tilde{K}_k = (1 + v_k)^{-2}\sigma_k$, we conclude from Lemma 2.5 and Corollary 2.6 that $(M, g_k, \tilde{K}_k)$ is an AE data set of class $H_\delta^s$ for every $t \geq s$ and $(\hat{g}_k - g_0, \tilde{K}_k - K_0)$ converges to 0 in $H_\delta^s \times H_\delta^{s-1}$. Taking $k$ sufficiently large proves the theorem. □

The following lemma completes the proof of Theorem 7.1.

**Lemma 7.2.** Suppose $(M, g)$ is AE of class $H_\rho^s$ with $s > n/2$ and $\rho < 0$, and suppose $\sigma \in H_\delta^{s-1}$ with $\delta \in (2 - n, 0)$. Let $U$ be the open subset $\{v \in H_\delta^s : v > -1\}$, and let $\mathcal{G} : U \to H_\delta^{s-2}$ be given by

$$\mathcal{G}(v) = |\sigma|^2 (1 + v)^{-2\kappa-3}.$$  

Then $\mathcal{G}$ has a Fréchet derivative $d\mathcal{G}$ given by

$$d\mathcal{G}(v)(h) = -(2\kappa + 3) |\sigma|^2 (1 + v)^{-2\kappa-4} h.$$

**Proof:** We first consider the maps

$$g(v) = (1 + v)^{-2\kappa-3},$$

$$g'(v) = -(2\kappa + 3)(1 + v)^{-2\kappa-4}.$$

Since $1 \in H_\delta^s$ for every $\epsilon > 0$, it follows from Lemma 2.5 that $g$ and $g'$ are continuous as maps from $U$ to $H_\delta^s$. Since $1 + v > 0$, it follows that there exists a ball $B_r$ of radius $r$ in $H_\delta^s$ such that $1 + v + h > 0$ for all $h \in B_r$. Taking $h \in B_r$, it follows from the continuity of $g'$ that the map $t \to g'(v + th)h$ from $[0, 1]$ to $H_\delta^s$ is Bochner integrable. It is easy to see that

$$g(v + h) - g(v) = \int_0^1 g'(v + th)h \, dt,$$

where the integral is a Bochner integral. Now

$$\|g(v + h) - g(v) - g'(v)h\|_{H_\delta^s} = \left\| \int_0^1 (g'(v + th) - g'(v)) h \, dt \right\|_{H_\delta^s} \leq \int_0^1 \|g'(v + th) - g'(v)\|_{H_\delta^s} \, dt \leq \int_0^1 \|g'(v + th) - g'(v)\|_{H_\delta^s} \, dt \|h\|_{H_\delta^s}.$$
Since $|\sigma|^2 \in H^{s-2}_{2\delta}$, we can take $\epsilon < -\delta$ to obtain

$$||G(v + th) - G(v) - dG(v)h||_{H^{s-2}_{2\delta}} \leq ||\sigma^2||_{H^{s-2}_{2\delta}} \int_0^1 \left\| (g'(v + th) - g'(v)) \right\|_{H^\delta} dt \|h\|_{H^\delta}.$$ 

Since $g'$ is continuous in a neighbourhood of $v$, it follows that

$$\int_0^1 \left\| (g'(v + th) - g'(v)) \right\|_{H^\delta} dt$$

can be made arbitrarily small by taking $\|h\|_{H^\delta}$ small. We conclude that $dG(v)$ is the Fréchet derivative of $G$ at $v$. □

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