A SYMPLECTIC FIXED POINT THEOREM
FOR COMPLEX PROJECTIVE SPACES

BY BARRY FORTUNE AND ALAN WEINSTEIN

1. Arnold’s conjecture. An automorphism $\psi$ of a symplectic manifold $(P,\omega)$ is homologous to the identity if there is a smooth family $\psi_t$ ($t \in [0,1]$) of automorphisms such that the time-dependent vector field $\xi_t$ defined by $d\psi_t/dt = \xi_t \circ \psi_t$ is globally hamiltonian; i.e. if there is a smooth family $H_t$ of real-valued functions on $P$ such that $\xi_t \cdot \omega = dH_t$. It was conjectured by Arnold [1], as an extension of the Poincaré-Birkhoff annulus theorem [3, 7], that every automorphism of a compact symplectic manifold $P$, homologous to the identity, has at least as many fixed points as a function on $P$ has critical points.

Arnold’s conjecture was proven by Conley and Zehnder [4] for the torus $T^{2n} \approx \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with its usual symplectic structure. They show that every symplectic automorphism of $T^{2n}$, homologous to the identity, has at least $n + 1$ fixed points, and at least $2^{2n}$ if all are nondegenerate. Their method was extended in [8] to prove a version of Arnold’s conjecture for arbitrary $P$ under the additional assumption that the hamiltonian vector field $\xi_t$ is sufficiently $C^0$ small.

In this note we announce a proof of Arnold’s conjecture for the complex projective space $\mathbb{C}P^n$ with its standard symplectic structure. We prove that a symplectic diffeomorphism of $\mathbb{C}P^n$, homologous to the identity, has at least $n + 1$ distinct fixed points. (By the Lefschetz fixed point theorem, any continuous map from $\mathbb{C}P^n$ to itself, homotopic to the identity, has at least $n + 1$ fixed points counted with multiplicities.) For $n = 1$ ($\mathbb{C}P^1 \approx S^2$) the result was already known [1], but with a proof which worked only in this two-dimensional case.

The proof for $T^{2n}$ in [4] made use of a variational principle in which the fixed points of the map were identified with periodic solutions of a time-dependent hamiltonian system and then identified with critical points of a functional on the space of contractible loops on $T^{2n}$. The corresponding functional in the case of $\mathbb{C}P^n$ is multiple valued, and there are other difficulties connected with the curved geometry of $\mathbb{C}P^n$, so we need a new approach. Our trick is to consider the hamiltonian system on $\mathbb{C}P^n$ as the reduction, in the sense of [6], of a hamiltonian system on $\mathbb{C}^{n+1}$ and then adapt recently developed methods [2] for finding periodic orbits in $\mathbb{C}^{n+1}$. This method is similar to that of Conley and Zehnder in that a problem on a compact manifold is lifted to a problem on euclidean space invariant under a group of transformations.
2. Lifting to $\mathbb{C}^{n+1}$. Consider $\mathbb{C}^{n+1}$ with its usual symplectic structure $\text{Im} \sum dz_i \wedge d\bar{z}_i$. The hamiltonian $K(z) = \sum z_i \bar{z}_i$ generates the periodic flow $T^t(z_1, \ldots, z_{n+1}) = (e^{2\mu}z_1, \ldots, e^{2\mu}z_n)$ with period $\pi$, and hence an action of $S^1 = \mathbb{R}/\pi\mathbb{Z}$ (the Hopf fibration). The reduced manifold $K^{-1}(1)/S^1$ can be identified with $\mathbb{C}P^n$, and any $S^1$-invariant hamiltonian system on $\mathbb{C}^{n+1}$ induces a system on $\mathbb{C}P^n$, called the reduced system. Our idea is to use this procedure in the opposite direction.

Fixed points of $\psi: \mathbb{C}P^n \to \mathbb{C}P^n$ are the same as solution curves $\bar{\sigma}: [0,1] \to \mathbb{C}P^n$ with $\bar{\sigma}(0) = \bar{\sigma}(1)$ for the time-dependent hamiltonian system which generates the family $\bar{\sigma}_t$ connecting the identity to $\bar{\sigma}$. Let $\bar{H}_t$ be the hamiltonian family for this system; since each $\bar{H}_t$ contains an arbitrary constant, we may assume that $\bar{H}_t(x) > 0$ for all $t \in [0,1]$ and all $x \in \mathbb{C}P^n$. Now let $H_t: \mathbb{C}^{n+1} \to \mathbb{R}$ be the unique function which is homogeneous of degree 2 and whose restriction to $K^1 = S^{2n+1}$ is the pullback of $\bar{H}_t$. Then $H_t$ is $S^1$-invariant and defines a time-dependent hamiltonian system on $\mathbb{C}^{n+1}$ whose reduced system is $H_t$.

By the general theory of reduction, we know that $S^{2n+1}$ is an invariant manifold for $H_t$, and the orbits of $\bar{H}_t$ on $\mathbb{C}P^n$ are the images of orbits of $H_t$ on $S^{2n+1}$. Furthermore, if $\bar{\sigma}$ is the image of $\sigma$, then $\bar{\sigma}(1) = \bar{\sigma}(0)$ if and only if $\sigma(1) = T_\mu \sigma(0)$ for some $\mu \in \mathbb{R}/\pi\mathbb{Z}$. If we change the hamiltonian $H_t$ to $H_t + \lambda K$ for some $\lambda \in \mathbb{R}$, then the “flow” of $H_t + \lambda K$ will still project to that of $\bar{H}_t$, but now by choosing $\lambda \equiv \mu \mod \pi$ we can make $\sigma(1) = \sigma(0)$. In other words, to each closed solution curve $\bar{\sigma}$ for $\bar{H}_t$ and, hence, to each fixed point of $\psi$ there corresponds a collection of pairs $(\sigma, \lambda)$ where $\lambda \in \mathbb{R}$ and $\sigma$ is a closed solution curve for $H_t + \lambda K$ on $S^{2n+1}$. The set of all pairs $(\sigma, \lambda)$ corresponding to a given fixed point is diffeomorphic to $S^1 \times \mathbb{Z}$.

By Hamilton’s principle the closed solution curves for $H_t + \lambda K$ on $\mathbb{C}^{n+1}$ are exactly the critical points of the functional

$$g(z) = \int_0^1 -i(z'(t), z(t)) \, dt + \int_0^1 H_t(z(t)) \, dt + \lambda \int_0^1 |z(t)|^2 \, dt = A(z) + H(z) + \lambda K(z).$$

Since we are interested in critical points for all possible values of $\lambda$, we may consider $\lambda$ as a Lagrange multiplier and look for critical points of $f(z) = A(z) + H(z)$ constrained to the infinite-dimensional sphere $K^{-1}(1)$.

We are thus faced with two problems. The first is to do the analysis which shows that $f(z)$ has many critical points on $K^{-1}(1)$, and the second is to show that all these critical points cannot belong to fewer than $n+1$ families of type $S^1 \times \mathbb{Z}$ coming from distinct fixed points of $\psi$.

3. Critical point analysis. The solution of the problems stated at the end of §2 forms the content of [5] and will only be summarized briefly here.

It turns out that the critical point theory developed in [2], based on the notion of relative index, is applicable to our problem, with some modifications made to permit working on the sphere $K^{-1}(1)$ within the space of loops of Sobolev class $H^{1/2}$ in $\mathbb{C}^{n+1}$. The values of the Lagrange multiplier $\lambda$ are then found to be equal to the critical values of the functional $f$ on $K^{-1}(1)$.
The minimax nature of the critical point theory makes it possible to estimate these values by comparison with the action functional $A$. A combinatorial argument then shows that these critical values cannot lie in less than $n + 1$ cosets of $\mathbb{R}$ (mod $\pi \mathbb{Z}$) unless some critical values merge, in which case $\psi$ would have uncountably many fixed points.

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