KHOVANOV-LIPSHITZ-SARKAR HOMOTOPY TYPE FOR LINKS IN THICKENED HIGHER GENUS SURFACES

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Abstract. We discuss links in thickened surfaces. We define the Khovanov-Lipshitz-Sarkar stable homotopy type and the Steenrod square for the homotopical Khovanov homology of links in thickened surfaces with genus $> 1$.

A surface means a closed oriented surface unless otherwise stated. Of course, a surface may or may not be the sphere. A thickened surface means a product manifold of a surface and the interval. A link in a thickened surface (respectively, a 3-manifold) means a submanifold of a thickened surface (respectively, a 3-manifold) which is diffeomorphic to a disjoint collection of circles.

Our Khovanov-Lipshitz-Sarkar stable homotopy type and our Steenrod square of links in thickened surfaces with genus $> 1$ are stronger than the homotopical Khovanov homology of links in thickened surfaces with genus $> 1$.

It is the first meaningful Khovanov-Lipshitz-Sarkar stable homotopy type of links in 3-manifolds other than the 3-sphere.

We point out that our theory has a different feature in the torus case.

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1. Introduction

In this paper, we discuss links in thickened surfaces. We define the Khovanov-Lipshitz-Sarkar stable homotopy type and the Steenrod square for the homotopical Khovanov homology of links in thickened surfaces with genus $> 1$.

A surface means a closed oriented surface unless otherwise stated. Of course, a surface may or may not be the sphere.

A thickened surface means a product manifold of a surface and the interval. A link in a thickened surface means a submanifold of a thickened surface (respectively, a 3-manifold) which is diffeomorphic to a disjoint collection of circles. If $\mathcal{L}$ is a link in a thickened surface, then a link diagram $L$ which represents $\mathcal{L}$ is in the surface.

Our theory has a special behavior at genus one as explained in §5. In this paper, a higher genus surface means a surface with genus greater than one unless otherwise stated. We will discuss the torus case in a sequel of this paper [9].

In [10], Khovanov defined the Khovanov homology for links in $S^3$, and proved that its graded Euler characteristic is the Jones polynomial of the link. In [2], Bar-Natan proved
that the Khovanov homology is stronger than the Jones polynomial as invaraints of links in $S^3$.

In [12], Lipshitz and Sarkar defined the Khovanov-Lipshitz-Sarkar stable homotopy type for links in $S^3$, and proved that the cohomology group of the Khovanov-Lipshitz-Sarkar stable homotopy type of any link $L$ in $S^3$ is the Khovanov homology of $L$.

**Note.** Khovanov-Lipshitz-Sarkar stable homotopy type is sometimes abbreviated to Khovanov-Lipshitz-Sarkar homotopy type or Khovanov homotopy type, in this paper and in other papers, when it is clear from the context.

In [14], Lipshitz and Sarkar found a method to calculate the second Steenrod square operator on the Khovanov homology for links in $S^3$. In [20], Seed made a computer program of the above method, used it, and showed that the second Steenrod square operator and the Khovanov-Lipshitz-Sarkar stable homotopy type are stronger than the Khovanov homology as invaraints of links in $S^3$.

In [1], Asaeda, Przytycki, and Sikora extended Khovanov homology for links in $S^3$ to thickened surfaces. In [16], Manturov and Nikonov made an alternative definition of that in [1], and obtained a new result by using it. There, the homology is called the *homotopical Khovanov homology*. We review the definition in §2.

In this paper, we constuct Khovanov-Lipshitz-Sarkar stable homotopy type for the homotopical Khovanov homology for links in thickened surfaces with genus $> 1$. It is our main result, Main Theorem [1.1] below. It is the first meaningful example of Khovanov-Lipshitz-Sarkar stable homotopy type for links in other 3-manifolds than the 3-sphere.

**Main Theorem 1.1.** (1) We define Khovanov-Lipshitz-Sarkar stable homotopy type for the homotopical Khovanov homology for links in thickened surfaces with genus $> 1$.

(2) We define the second Steenrod square acting on the homotopical Khovanov homology for links in thickened surfaces with genus $> 1$ by using the Khovanov-Lipshitz-Sarkar stable homotopy type in (1).

(3) Each of the Khovanov-Lipshitz-Sarkar stable homotopy type in (1) and the second Steenrod square in (2) is stronger than the homotopical Khovanov homology as invariants of links in thickened surfaces with genus $> 1$. That is, there is a pair of links in a thickened surface with genus $> 1$ with the following properties: They have different Steenrod squares. They have different Khovanov-Lipshitz-Sarkar stable homotopy types. They have the same homotopical Khovanov homology.

**Note 1.2.** There are other ways to extend Khovanov homology of links in the 3-sphere and the second Steenrod square on Khovanov homology of links in the 3-sphere into thickened surfaces. Manturov (2006 arXiv) [15], Rushworth [19], Tubbenhauer [21], and
Viro (unpublished) [22] introduced other ways to define Khovanov homology for links in thickened surfaces by using virtual links. Dye, Kaestner, and Kauffman [3], Nikonov [17], Kauffman and Ogasa [8] wrote alternative definitions of [15]. Kauffman and Ogasa [8] extended the second Steenrod square of link in $S^3$ to thickened surfaces by using virtual links. See [5, 6, 7] for Virtual links. However, the readers need not know virtual links or the result of any paper in this Note 1.2 in order to read this paper. The techniques for defining Khovanov homology for virtual links are not used in the present paper. Our paper is self-contained, and uses the techniques of Lifshitz and Sarkar.

On the other hand, for the sake of those who are familiar with virtual links, we will sometimes comment on virtual links in this paper. Of course, the readers may skip the parts if they do not know virtual links.

2. The homotopical Khovanov homology for links in thickened surfaces

We review the definition of the homotopical Khovanov homology for links in thickened surfaces introduced in [16]. In this paper we define Khovanov homotopy type for thickened higher genus surfaces, but the homotopical Khovanov homology is defined for links in thickened surfaces with any genus. In this paper, when we say just a surface, its genus may zero, one, or greater than one.

We translate the definition into terminologies in Lipshitz and Sarkar’s paper [12], because we generalize the results about Khovanov-Lipshitz-Sarkar stable homotopy type there.

2.1. Labeled resolution configurations.

**Definition 2.1.** Let $F$ be a closed oriented surface. A resolution configuration $D$ is a pair $(Z(D), A(D))$, where $Z(D)$ is a set of pairwise-disjoint embedded circles in $F$, and $A(D)$ is a totally ordered collection of disjoint arcs embedded in $F$, with $A(D) \cap Z(D) = \partial A(D)$. We call the number of arcs in $A(D)$ the index of the resolution configuration $D$, and denote it by $\text{ind}(D)$. We sometimes abuse notation and write $Z(D)$ to mean $\cup_{Z \in Z(D)} Z$ and $A(D)$ to mean $\cup_{A \in A(D)} A$. Occasionally, we will describe the total order on $A(D)$ by numbering the arcs: a lower numbered arc precedes a higher numbered one.

We changed [12, Definition 2.1] into Definition 2.1 by replacing ‘$S^2$ in [12, Definition 2.1]’ by ‘$F$ in Definition 2.1’.

**Definition 2.2.** Let $F$ be a surface. Let $L$ be a link in $F \times [-1, 1]$. Let $L$ be a link diagram of $L$. Note that $L$ is in $F$. Assume that the link diagram $L$ has $n$ crossings, an ordering of the crossings in $L$, and a vector $v \in \{0, 1\}^n$. There is an associated resolution configuration $D_L(v)$ obtained by taking the resolution of $L$ corresponding to $v$ (that is, taking the 0-resolution at the $i$-th crossing if $v_i = 0$, and the 1-resolution otherwise) and
then placing arcs corresponding to each of the crossings labeled by 0’s in $v$ (that is, at the $i$-th crossing if $v_i = 0$). See Figure 2.1.

Therefore, $n \text{-ind}(D_L(v)) = |v| = \sum v_i$, the (Manhattan) norm of $v$. (Note that $\sum v_i = \sum (v_i)^2$ in this situation, since $0^2 = 0$ and $1^2 = 1$.)

Note that (if $L$ in $F$ denotes a classical link diagram in $S^2$ or virtual link diagram) resolution configurations are the same as what many people often call Kauffman states, which Kauffman first introduced in [4]. (The way to draw arcs in [12] is different from that in [4].)

Note. We draw resolution configurations on the plane $\mathbb{R}^2$ in this paper. Our way of drawing is similar to that in virtual knot theory which is introduced in [3][6][7]. However, in this paper, the strict definition of resolution configurations is Definition 2.1. (On the other hand, the way of drawing resolution configurations on $\mathbb{R}^2$ (respectively, $S^2$) is defined strictly in virtual knot theory.)

We draw a part of a surface $F$ in the upper figure of Figure 2.2. We depict a non-contractible circle in $Z(D)$ on $F$ in the middle and lower figures of Figure 2.2. These kinds of non-contractible circles are drawn as in the left figure of Figure 2.3. We call this circle a circle with (H). We omit drawing a part of the surface $F$ when it is clear from the context. If we need to explain some property of a non-contractible circle, we write it in the right lower side where $x$ is written in the right one of Figure 2.3.

See Figure 2.4, $Z(D) \cup A(D)$ has a neighborhood $N$ such that $N$ is a compact surface and such that the inclusion map of $Z(D) \cup A(D)$ to $N$ is a homotopy type equivalence map. For a given $Z(D) \cup A(D)$, there are many homeomorphism types of $N$ in general.

Definition 2.3. ([12] Definition 2.3.) Given resolution configurations $D$ and $E$, there is a new resolution configuration $D - E$ defined by $Z(D - E) = Z(D) - Z(E)$, $A(D - E) = \{A \in A(D) | \forall Z \in Z(E) : \partial A \cap Z = \emptyset\}$.

Let $D \cap E = D - (D - E)$.

Note that $Z(D \cap E) = Z(E \cap D)$ and $A(D \cap E) = A(E \cap D)$; however, the total orders on $A(D \cap E)$ and $A(E \cap D)$ could be different.
Figure 2.2. The upper figure is a part of a surface \( F \). In each of the middle and lower figures, there is a non-contractible circle on \( F \).
Figure 2.3. Non-contractible circles

Figure 2.4. Two Kauffman states with a single circle and a single arc: The two are different but make the same abstract graph.
**Definition 2.4.** ([12] Definition 2.4.) The core \( c(D) \) of a resolution configuration \( D \) is the resolution configuration obtained from \( D \) by deleting all the circles in \( Z(D) \) that are disjoint from all the arcs in \( A(D) \). A resolution configuration \( D \) is called basic if \( D = c(D) \), that is, if every circle in \( Z(D) \) intersects an arc in \( A(D) \).

**Definition 2.5.** ([12] Definition 2.5.) Given a resolution configuration \( D \) and a subset \( A' \subseteq A(D) \) there is a new resolution configuration \( s_{A'}(D) \), the surgery of \( D \) along \( A' \), obtained as follows. The circles \( Z(s_{A'}(D)) \) of \( s_{A'}(D) \) are obtained by performing embedded surgery along the arcs in \( A' \); in other words, \( Z(s_{A'}(D)) \) is obtained by deleting a neighborhood of \( A' \) from \( Z(D) \) and then connecting the endpoints of the result using parallel translates of \( A' \). The arcs of \( s_{A'}(D) \) are the arcs of \( D \) not in \( A' \), i.e., \( A(s_{A'}(D)) = A(D) - A' \).

Let \( s(D) = s_{A(D)}(D) \) denote the maximal surgery on \( D \).

**Definition 2.6.** Let \( D \) be a resolution configuration. Suppose that, when we carry out a surgery along one arc of \( A(D) \) on circles of \( Z(D) \), the number of the elements of \( Z(D) \) is not changed. Then we call this surgery a single cycle surgery. We call this arc a scs arc.

There is a scs arc in the right figure of Figure 2.4.

See another example in Figure 2.5. Each of these three figures are a part of a surface \( F \). The upper is a link diagram in \( F \). The middle is obtained from the upper by the 0-resolution. The lower is obtained from the upper by the 1-resolution. The lower is obtained from the middle by a surgery along the arc in the middle. This surgery is a single cycle surgery.

See Figure 2.6. The left figure is a part of a closed oriented surface \( F \) with a part of a link diagram in \( F \). We draw it as the right one for convenience when we discuss a single cycle surgery. The right one includes \((H)\). This \((H)\) represents not only the fact in Figure 2.5 but also the fact that the circle is a non-contractible circle as written in Definition 2.2.

Figure 2.7 is an example of drawing a single cycle surgery by using \((H)\) of Figure 2.6.

**Definition 2.7.** If a surgery along an arc increases (respectively, decreases) the number of circles by one, the surgery is called a comultiplication (respectively, multiplication). There are just three kinds of surgeries along an arc: a single cycle surgery, a multiplication, a comultiplication. If an arc produces a multiplication (respectively, comultiplication), the arc is called a \( m \)-arc (respectively, \( c \)-arc). If an arc is an \( m \)-arc or a \( c \)-arc, that is, it is not a scs arc, then the arc is called a mc arc.

**Definition 2.8.** ([12] Definition 2.9). A labeled resolution configuration is a pair \((D, x)\) of a resolution configuration \( D \) and a labeling \( x \) of each element of \( Z(D) \) by either \( x_+ \) or \( x_- \).
Note that (if they are associated with classical diagrams in $S^2$ or virtual link diagrams,) labeled resolution configurations are the same as what many people often call *enhanced Kauffman states* or *enhanced states*. Some people use $v_+$ (respectively, $v_-$) for $x_+$ (respectively, $x_-$).

Let $\{A_i\}_{i \in \Lambda}$ be the set of all labeled resolution configurations made from an arbitrary link diagram $L$ in a surface. Note that $\Lambda$ is a finite set. $\{A_i\}_{i \in \Lambda}$ composes a basis of the
Khovanov homology for $L$ as in $[11] [2] [3] [8] [10] [12]$. We call $\{A_i\}_{i \in \Lambda}$ the \textit{Khovanov basis}. We call each $A_i$ a \textit{Khovanov basis element}.

We will define a partial order on $\{A_i\}_{i \in \Lambda}$. After that, by using the partial order, we will define the differential acting on each $A_i$, and introduce the Khovanov homology for $L$ as in $[12] [8]$. See the definitions in the following subsections for the detail. In order to define moduli spaces and to construct Khovanov-Lipshitz-Sarkar stable homotopy type, we need the partial ordered set.

\textbf{Review.} Comparing $[2] [10]$ with $[12]$, a differential of the Khovanov complex and a partial order on the set of labeled resolution configurations are essentially the same thing. We explain it below.

Let $\{X_i\}_{i \in \Theta}$ be a set of all labeled resolution configurations of an arbitrary link diagram in $S^2$. We want to let the discussion in this Review separated from the one above here, so we use different notation $\{X_i\}_{i \in \Theta}$.

(1) The Khovanov differential acting on $\{X_i\}_{i \in \Theta}$ in $[10] [2]$ induces a partial order on $\{X_i\}_{i \in \Theta}$ as below. We use the following notation

\begin{equation}
\delta X_i = \sum_{j \in \Theta} [X_i; X_j] \cdot X_j,
\end{equation}

and $[X_i; X_j]$ is an integer coefficient. Recall that $[X_i; X_j] \in \{-1, 0, 1\}$. The partial order $\prec$ on $\{X_i\}_{i \in \Theta}$ is induced as follows: If $[X_i; X_j] = \pm 1$, then $X_i \prec X_j$. It follows from this definition that if $u \prec v$ and $v \prec w$ then $u \prec w$. 

\begin{figure}[h]
    \centering
    \includegraphics[width=0.8\textwidth]{figure27.png}
    \caption{A single cycle surgery drawn by using $(H)$.}
\end{figure}
(2) In [12, Definition 2.10], Lipshitz and Sarkar introduced a partial order $\prec$ on the set $\{X_i\}_{i\in\Theta}$ before they define the Khovanov differential acting on each $X_i$.

Furthermore, they use the following method: Let $Y, Z \in \{X_i\}_{i\in\Theta}$. Let $Y \prec Z$. Assume that $Y \prec X \prec Z$ does not hold for any $X \in \{X_i\}_{i\in\Theta} - \{Y, Z\}$. They define an explicit way to assign $+1$ or $-1$ to the pair, $Y$ and $Z$.

By using this partial order and this method to give a sign, they induce the Khovanov differential, acting on $\{X_i\}_{i\in\Theta}$ ([12, Definition 2.15]).

We follow Lipshitz and Sarkar’s method above, in the following subsections.

Lipshitz and Sarkar used the partial ordered set, defined moduli spaces, and constructed Khovanov-Lipshitz-Sarkar stable homotopy type.

### 2.2. A partial order on the set of labeled resolution configurations.

**Definition 2.9.** Let $(D, x)$ and $(E, y)$ be labeled resolution configurations. Then there is a natural labeling $x|_D$ (respectively, $y|_E$) on $D - E$ (respectively, $E - D$), say the *induced labeling*. We often consider induced labelings in the following case: if we restrict each of the labelings $x$ and $y$ to $D \cap E = E \cap D$, we obtain the same labeling from $x$ and $y$.

We define a partial order on $\{A_i\}_{i\in\Lambda}$ in the following definition. By using the partial order, we will induce the differential (Definition 2.10 cited from [16 §2]). One can induce this partial order from the differential: Use [16 (2.1), (2.3)-(2.6)]. (One can do it in a similar way in the Review in the last part of §2.1.)

**Definition 2.10.** There is a partial order $\prec$ on labeled resolution configurations defined as follows. Note that $\alpha \prec \alpha$ holds. Then we declare that $(E, y) \prec (D, x)$ if:

1. The labelings $x$ and $y$ induce the same labeling on $D \cap E = E \cap D$.
2. $D$ is obtained from $E$ by surgering along a single arc of $A(E)$.
   $Z(E - D)$ (respectively, $Z(D - E)$) has an induced labelling $y|_D$ (respectively, $x|_E$) from $y$ (respectively, $x$). There are two sub-cases (i) and (ii).
   (i) $(E - D, y|_D)$ has just one circle $P$, and $(D - E, x|_E)$ has just two circles, $Q$ and $R$. These two labeled resolution configurations satisfy the conditions in Table 2.1. Here, $c$ means a contractible circle, and $n$ means a non-contractible circle. Examples are the upper three figures in Figure 2.8 and the upper six figures in Figure 2.9.

   (ii) $(E - D, y|_D)$ has just two circles, $P$ and $Q$, and $(D - E, x|_E)$ has just one circle $R$. These two labeled resolution configurations satisfy the conditions in Table 2.2. Examples are the lower three figures in Figure 2.8 and the lower six figures in Figure 2.9.
| $P$  | $\prec$ | $Q$  | $\&$ | $R$  |
|-----|--------|-----|------|-----|
| $c$, $x_+$ | $\prec$ | $c$, $x_+$ | $\&$ | $c$, $x_-$ |
| $c$, $x_+$ | $\prec$ | $c$, $x_-$ | $\&$ | $c$, $x_+$ |
| $c$, $x_-$ | $\prec$ | $c$, $x_-$ | $\&$ | $c$, $x_-$ |
| $n$, $x_+$ | $\prec$ | $c$, $x_-$ | $\&$ | $n$, $x_+$ |
| $n$, $x_+$ | $\prec$ | $n$, $x_+$ | $\&$ | $c$, $x_-$ |
| $n$, $x_-$ | $\prec$ | $c$, $x_-$ | $\&$ | $n$, $x_-$ |
| $n$, $x_-$ | $\prec$ | $n$, $x_-$ | $\&$ | $c$, $x_-$ |
| $c$, $x_+$ | $\prec$ | $n$, $x_-$ | $\&$ | $n$, $x_+$ |
| $c$, $x_+$ | $\prec$ | $n$, $x_-$ | $\&$ | $n$, $x_-$ |

Table 2.1. $P$, $Q$, $R$ in Definition 2.10 (2).(i).

| $P$  | $\&$ | $Q$  | $\prec$ | $R$  |
|-----|------|-----|--------|-----|
| $c$, $x_+$ | $\&$ | $c$, $x_-$ | $\prec$ | $c$, $x_-$ |
| $c$, $x_-$ | $\&$ | $c$, $x_+$ | $\prec$ | $c$, $x_-$ |
| $c$, $x_+$ | $\&$ | $c$, $x_+$ | $\prec$ | $c$, $x_+$ |
| $c$, $x_+$ | $\&$ | $n$, $x_+$ | $\prec$ | $n$, $x_+$ |
| $n$, $x_+$ | $\&$ | $c$, $x_+$ | $\prec$ | $n$, $x_+$ |
| $c$, $x_+$ | $\&$ | $n$, $x_-$ | $\prec$ | $n$, $x_-$ |
| $n$, $x_+$ | $\&$ | $c$, $x_+$ | $\prec$ | $n$, $x_-$ |
| $n$, $x_-$ | $\&$ | $n$, $x_-$ | $\prec$ | $c$, $x_-$ |
| $n$, $x_-$ | $\&$ | $n$, $x_-$ | $\prec$ | $c$, $x_-$ |

Table 2.2. $P$, $Q$, $R$ in Definition 2.10 (2).(ii).

Now, $\prec$ is defined to be the transitive closure of this relation.

**Note.** (1) The upper three relations in each of Tables 2.1 and 2.2 are the same as those in [12, Definition 2.10]. The lower six in each is associated with the case where non-contractible circles appear. These cases conform to the conditions for defining the boundary in [16, (2.1), (2.3)-(2.6)].

(2) In Definition 2.9 the relations for homotopy classes of circles are determined naturally. (When we consider a homotopy class of the circles, we let them orient appropriately.)

(3) No surgery in Definition 2.10 is a single cycle surgery although we now consider links in thickened surfaces.
(4) In Figure 2.10, we draw abstractly $Z(E - D) \cup A(E - D)$ in the case where $A(E - D)$ has a single arc. There are only two cases.

**Definition 2.11.** ([12, Definition 2.11].) A *decorated resolution configuration* is a triple $(D, x, y)$ where $D$ is a resolution configuration and $x$ (respectively, $y$) is a labeling of each component of $Z(s(D))$ (respectively, $Z(D)$) by an element of $\{x_+, x_-\}$. Associated to a decorated resolution configuration $(D, x, y)$ is the poset $P(D, x, y)$ consisting of all labeled resolution configurations $(E, z)$ with $(D, y) \prec (E, z) \prec (s(D), x)$. We call $P(D, x, y)$ the poset for $(D, x, y)$.

If we never have $(D, y) \prec (s(D), x)$, we say that $(D, x, y)$ is empty.
Figure 2.9. The partial order of the set of labeled resolution configurations in the lower six of Table 2.1 and in the lower six of Table 2.2. There appear non-contractible circles. Circles with (H) denote non-contractible circles. See Figures 2.11 and 2.12.
Note: In [12, Definition 2.11], Lipshitz and Sarkar dealt with only non-empty decorated resolution configurations. It means that, in their paper, they only consider labelings $x$ on $s(D)$ such that $(D, y) \prec (s(D), x)$. On the other hand, in this paper, we define the case where $(D, x, y)$ is empty, for convenience.

**Definition 2.12.** ([12, A part of Definition 2.15], and [16, Definition 2.2].) Let $L$ be an oriented link diagram in a surface $F$. Let $n$ (respectively, $n_+, n_-$) be the number of crossing (respectively, positive crossing, negative crossing) points of $L$.

For labeled resolution configurations, homological grading $\text{gr}_h$, a quantum grading $\text{gr}_q$, and a homotopical grading $\text{gr}_H$ are defined as follows:

\begin{align}
\text{gr}_h((D_L(u), x)) &= -n_- + |u|, \\
\text{gr}_q((D_L(u), x)) &= n_+ - 2n_- + |u| + \sharp \{Z \in Z(D_L(u)) | x(Z) = x_+ \} - \sharp \{Z \in Z(D_L(u)) | x(Z) = x_- \}.
\end{align}

We consider the set $\mathcal{L} = [S^1; F]$ of all the homotopy classes of free oriented loops in $F$. Let $\bigcirc \in \mathcal{L}$ be the homotopy class of contractible loops. For any closed curve $\gamma$, one can consider the curve $-\gamma$ obtained from $\gamma$ by the orientation change. Let $\mathcal{H}$ be the quotient group of the free abelian group with generator set $\mathcal{L}$ modulo the relations $\bigcirc = 0$ and $[\gamma] = [-\gamma]$ for all free loops $\gamma$. Let $Z(D_L(u)) = \{C_1, C_2, ..., C_\nu\}$. Let $C_i$ equip $x_i$, where $x_i \in \{x_+, x_-\}$ and $i \in \{1, ..., \nu\}$. Define

\begin{align}
\text{gr}_q((D_L(u), x)) &= n_+ - 2n_- + |u| + \sharp \{Z \in Z(D_L(u)) | x(Z) = x_+ \} - \sharp \{Z \in Z(D_L(u)) | x(Z) = x_- \}.
\end{align}
\[ \text{gr}_H((D_L(u), x)) = \sum_{i=1}^{\nu} \deg(x_i) \cdot [C_i] \in \mathcal{H}, \]

where \( \deg(x_{\pm}) = \pm 1. \)

**Note 2.13.** If \( L \) has only one component, then \( n, n_+, n_-, \text{gr}_h, \text{gr}_q, \) and \( \text{gr}_H, \) we do not depend on the orientation of \( L. \) If \( L \) has greater than one component, we use that of \( L \) when we define \( n_+ \) and \( n_. \) Note that, if we change the orientation of \( L \) into the opposite one, then neither \( n_+ \) nor \( n_- \) changes.

We suppose \([\gamma] = [-\gamma]\) when we define \( \mathcal{H}. \) Hence, whichever orientation we give to a disjoint union of circles \( Z(D) \) in a labeled resolution configuration \((D, x), \) \( \text{gr}_H(D, x) \) is the same.

Note the following facts. Let \( \gamma \) and \( \gamma' \) be circles embedded in a surface \( F, \) and \( \gamma \cap \gamma' = \emptyset. \) Let \( A \) be an arc which connects \( \gamma \) and \( \gamma'. \) Assume that we do not consider the orientation of \( \gamma \) nor that of \( \gamma'. \) Let \( \sigma \) be an embedded circle in \( F \) which is obtained from \( \gamma \) and \( \gamma' \) by the surgery along \( A. \) Suppose that we do not consider the orientation of \( \sigma. \) Once we are given \( A, \gamma \) and \( \gamma', \) then \( \sigma \) is determined, and henceforth \([\sigma] \in \mathcal{H} \) is determined.

Let \( \gamma \) be a circle embedded in a surface \( F. \) Let \( A \) be a \( c-\)arc both of whose endpoints are in \( \gamma. \) Assume that we do not consider the orientation of \( \gamma. \) Let \( s \) be a disjoint union of two embedded circles in \( F \) which is obtained from \( \gamma \) by the surgery along \( A. \) We do not consider the orientation of \( s. \) Note that \( s \) determines one or two elements in \( \mathcal{H}. \) Once we are given \( A \) and \( \gamma, \) then \( s \) is determined. Furthermore we can know how many elements of \( \mathcal{H} \) are determined, and what the elements are (respectively, the element is).

Let \( \gamma \) be a circle embedded in a surface \( F. \) Let \( A \) be a \( scs-\)arc both of whose endpoints are in \( \gamma. \) Note that we do not use a \( scs-\)arc when we define the partial order in Definition 2.10.

Each of \( \text{gr}_h(D, x), \text{gr}_q(D, x), \) and \( \text{gr}_H(D, x) \) is independent of which orientation we give \( Z(D), \) and is independent of which we choose \( L \) or \( -L. \) Here, \( -L \) is the link made from \( L \) by reversing the orientation of \( L. \)

By Definition 2.12 we have the following.

**Fact 2.14.** Let \((E, y)\) and \((D, x)\) be labelled resolution configurations of a link diagram in a surface. Let \((E, y) \prec (D, x). \) Then \((E, y) \) and \((D, x)\) have the same quantum grading and the same homotopical grading.

We have the following.
Fact 2.15. Assume that a single cycle surgery changes a (non-labeled) resolution configuration \(D_L(u)\) into \(D_L(v)\). Let \(A_i\) (respectively, \(A_j\)) be a labeled resolution configuration defined on \(D_L(u)\) (respectively, \(D_L(v)\)). Then \(A_i\) and \(A_j\) have different quantum gradings.

Proof of Fact 2.15. Recall the definition of quantum gradings, \(\text{gr}_q((D_L(u), x))\), above. Since \(n_+\) and \(n_-\) are determined by a given link diagram, a single cycle surgery does not change \(n_+ - 2n_-\). By the definition of a (single cycle) surgery and that of \(|z|\), a single cycle surgery changes the parity of \(|u|\). Since a single cycle surgery does not change the number of circles in a labeled resolution configuration, a single cycle surgery does not change the parity of \(\sharp\{Z \in Z(D_L(u)) | x(Z) = x_+\} - \sharp\{Z \in Z(D_L(u)) | x(Z) = x_-\}\). Therefore a single cycle surgery always changes the quantum grading \(\text{gr}_q((D_L(u), x))\). □

We use the partial order in Definition 2.12, and define the integral (\(\mathbb{Z}\)-coefficient) Khovanov chain complex for a link diagram in a thickened surface in the following subsection.

2.3. The differential, and the homotopical Khovanov homology for link diagrams in surfaces.

Our definition is the same as that in [10].

Definition 2.16. The differential \(\delta\). Given an oriented link diagram \(L\) with \(n\) crossings and an ordering of the crossings in \(L\), the Khovanov chain complex is defined as follows. The Khovanov chain group \(KC(L)\) is the \(\mathbb{Z}\)-module freely generated by labeled resolution configurations of the form \((D_L(u), x)\) for \(u \in \{0, 1\}^n\). (The set of all labeled resolution configurations of \(L\) is a basis of \(KC(L)\).) The differential preserves the quantum grading and the homotopical grading, increases the homological grading by 1, and is defined as

\[
\delta(D_L(v), y) = \sum \text{For all}(D_L(u), x)\text{ such that}|u|=|v|+1\text{, and such that}(D_L(v), y)\prec(D_L(u), x)\text{ }(-1)^{s_0(C_{u,v})}(D_L(u), x),
\]

where \(s_0(C_{u,v}) \in \mathbb{Z}_2\) is defined as follows: if \(u = (\epsilon_1, \ldots, \epsilon_{i-1}, 1, \epsilon_{i+1}, \ldots, \epsilon_n)\) and \(v = (\epsilon_1, \ldots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \ldots, \epsilon_n)\), then \(s_0(C_{u,v}) = (\epsilon_1 + \cdots + \epsilon_{i-1})\).

Example 2.17. In Figures 2.11 and 2.12 examples of how the differential acts. Note: Let \(x\) be a labeled resolution configuration such that \(x\) has two circles and one arc which connects the two circles. If \(x\) is different from the left hand side of all identities in this figure and Figure 2.12 then \(\delta x = 0\). Let \(y\) be a labeled resolution configuration such that \(y\) has one circle and one arc. If \(y\) is different from the left hand side of all identities in this figure and Figure 2.12, then \(\delta y = 0\).
Note 2.18. Recall that \( \{ A_i \}_{i \in \Lambda} \) denotes the set of all labeled resolution configurations made from an arbitrary link diagram \( L \) in a surface. We use the following notation

\[
\delta A_i = \sum_{j \in \Lambda} [A_i; A_j] \cdot A_j,
\]

like the identity (2.1). Here, \([A_i; A_j]\) is an integral coefficient. Note that, by Definition 2.16, \([A_i; A_j]\) \( \in \{ -1, 0, 1 \} \).

By Definition 2.16, the following facts hold.

1. If \( A_i \) and \( A_j \) have different quantum gradings, then \([A_i; A_j] = 0\). (This is related to Fact 2.15.) Note: This condition holds in the case of the Khovanov homology for links in \( S^3 \).

2. If \( \text{gr}_h(A_i) + 1 \neq \text{gr}_h(A_j) \), then \([A_i; A_j] = 0\).

3. \([A_i; A_j] \neq 0\) only if \( \text{gr}_q(A_i) = \text{gr}_q(A_j) \), \( \text{gr}_h(A_i) = \text{gr}_h(A_j) \), and

**Figure 2.11.** Examples of how the differential acts. See other examples in Figure 2.12.
Figure 2.12. Examples of how the differential acts. See other examples in Figure 2.11.

\[ \delta \left( \begin{array}{c} x_+ \\ x_- \end{array} \right) = \left( \begin{array}{c} x_- \\ x_+ \end{array} \right) \]

Note that the above facts (1) and (2) follows from this fact (3).

\[ \text{gr}_h(A_i) + 1 = \text{gr}_h(A_j). \]

Note that the above facts (1) and (2) follows from this fact (3).
(4) If \( \delta A_i = 0 \), then \( A_i \) is a maximal element in \( \{A_i\}_{i \in \Lambda} \).

Therefore we have the following. Let \( A \) (respectively \( A' \)) be a labeled resolution configuration. Assume that a (non-labeled) resolution configuration under \( A \) is changed into that under \( A' \) by just one surgery. Suppose that \( \delta A = 0 \). Then \( A \) and \( A' \) are not related by \( \prec \).

By Definition 2.16, we have the following. We use notations in the identities (2.5) and (2.6).

**Proposition 2.19.** Let \((D_L(v), y)\) and \((D_L(u), x)\) be labeled resolution configurations.

1. We have \([ (D_L(v), y) ; (D_L(u), x) ] \neq 0 \) only if we have the following: \((D_L(v) - D_L(u), y)\) is the left hand side of one of the relations by \( \prec \) in the tables in Definition 2.10. \((D_L(u) - D_L(v), x)\) is the right hand side of the above relation by \( \prec \).

2. If \([ (D_L(v), y) ; (D_L(u), x) ] \neq 0 \), then \([ (D_L(v), y) ; (D_L(u), x) ] = (-1)^{s_0(c_{u,v})} \).

By Fact 2.15 and Note 2.18 (1), we have the following.

**Proposition 2.20.** Assume that a single cycle surgery changes a (non-labeled) resolution configuration \( D_L(u) \) into \( D_L(v) \). Let \( A_i \) (respectively, \( A_j \)) be a labeled resolution configuration defined on \( D_L(u) \) (respectively, \( D_L(v) \)). Then \([A_i; A_j] = 0 \).

2.4. The well-definedness of the homotopical Khovanov homology for links in thickened surfaces.

**Theorem 2.21.** ([16]) Let \( L \) be a link diagram of a link \( \mathcal{L} \) in a thickened surface. For \( \delta \) in Definition 2.16

\[(2.7) \quad \delta^2 = 0.\]

**Note.** (1) The above homology uses integer coefficients.

2. By Note 2.13 this homology of \( L \) is the same as that of \(-L\). This homology is independent of which orientation on the disjoint union of all circles in each labeled resolution configuration we choose.

Let \( G \) and \( G' \) be resolution configurations. Let \( R \) be an arc in \( G \). By definition, \( A(G - G') \) and \( A(s_A(G)) \) are subsets of \( A(G) \). Assume that \( A(G - G') \) or \( A(s_A(G)) \) includes an arc which was \( R \) in \( A(G) \). Then we also call this arc in \( A(G - G') \) or \( A(s_A(G)) \), \( R \).

**Proof of Theorem 2.21.** Let \((D, x)\) be any labeled resolution configuration of \( L \). Take two arbitrary arcs, \( A \) and \( A' \), in \( D \). Let \( x \) be labelings on \( D - s_{A,A'}(D) \) induced by \( x \). If \( \delta^2(D - s_{A,A'}(D), x) = 0 \), then Theorem 2.21 holds. Let \((E, z) = (D - s_{A,A'}(D), x)\).

Note that \( E \) has only two arcs \( A \) and \( A' \). We check all cases of \((E, z)\) and prove \( \delta \cdot \delta(E, z) = 0 \) below.

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The union of all arcs $A$ and $A'$ and all circles in $E$ is a topological space, say $P$. If $A$ and $A'$ are included in different connected components of $P$, then the proof is easy. Note that $E$ is a basic resolution configuration. Assume that $P$ above is connected.

In Figure 2.13, we draw all cases of $Z(E) \cup A(E)$ abstractly.

We check all $(E, z)$ as follows.

**Case 1.** If only two arcs, $A$ and $A'$, in $E$, only one arc $A'$ in $s_A(E)$, and only one arc $A$ in $s_{A'}(E)$ are mc arcs, we have $\delta^2(E, z) = 0$ by applying the rule in Figures 2.8 and 2.9 of Definition 2.10.

**Note.** Let $y$ be a labeling on $s(E)$. There is a case where the number of the elements in $P(E, y, z)$ is six. This case is important. We will discuss it in §3 and §5.

**Case 2.** If only two arcs, $A$ and $A'$, in $E$ are scs arcs, $\delta(E, z) = 0$ by Proposition 2.20. Hence $\delta^2(E, z) = 0$. 

---

**Figure 2.13. Connected graphs of the resolution configurations of index 2.**
Case 3. Suppose that only two arcs, $A$ and $A'$, in $E$ are mc arcs, and that only one arc $A'$ in $s_A(E)$ is a scs arc. (See two examples in Figure 2.14.) Then we have the following: The surgery from $s_A(E)$ to $s_{A,A'}(E)$ is a single cycle surgery. Hence the number of circles in $E$ has a different parity of that in $s_{A,A'}(E)$. Therefore the surgery from $s_{A'}(E)$ to $s_{A,A'}(E)$ is a single cycle surgery. By Proposition 2.20, we have $\delta^2(E, z) = 0$.

Assume that only two arcs, $A$ and $A'$, in $E$ are mc arcs, and that only one arc $A$ in $s_{A'}(E)$ is a scs arc. We can prove $\delta^2(E, z) = 0$ by the same method as the above one.

Case 4. Assume that one of only two arcs, $A$ and $A'$, in $E$ is a mc arc, and the other a scs arc. We can let $A$ be a mc arc and $A'$ a scs arc without loss of generality.

Note that, under this setting, there is a case such that only one arc $A'$ in $s_A(E)$ is a mc arc. The other possibility, when $A'$ in $s_A(E)$ is a scs arc, is easy.
In order to show such examples, we need to introduce a terminology.

A shell configuration or shell Kauffman state is a resolution configuration of a link diagram in a surface $F$ drawn in Figure 2.15. We draw only a neighborhood $N$ of the shell configuration in $F$. We assume that $N$ is a compact surface and that the inclusion map of the shell configuration to $N$ is a homotopy type equivalence map.

Two examples of the above case are drawn in Figure 2.16. The upper of each side is a shell configuration.

We must take care of the cases in Figures 2.17 and 2.18, which are associated with Figure 2.16. The other cases are easy.

In the case in Figure 2.17, we have $\delta \cdot \delta = 0$ because of Figure 2.19.

In the case in Figure 2.18, we have $\delta \cdot \delta = 0$ because of Figure 2.20.

Therefore $\delta^2(E, z) = 0$ in all cases. This completes the proof of Theorem 2.21.

\[ \square \]

**Note.** When we consider this case, we should take care of the following facts.

Let $(E, z)$ be a labeled resolution configuration which is obtained by giving a labeling to the left upper shell configuration in Figure 2.17 (respectively, 2.18). Then there is only one element in the set \{p|p is a labeled resolution configuration. $(E, z) \prec p, (E, z) \neq p$\}.

In the identities in Figures 2.11 and 2.12 only the most upper identity in Figure 2.11 and the third identity in Figure 2.12 have two labeled resolution configurations in the right hand side. In both cases, the left hand side has only one labeled resolution configuration and the circle in it is a contractible circle.

Shell configurations, Figures 2.17 and 2.18 are important.

By using the same method in [2 [10]], we have the following.
**Theorem 2.22.** ([16].) Let $L$ and $L'$ be link diagrams of a link $\mathcal{L}$ in a thickened surface. Then the homotopical Khovanov homology of $L$ and that of $L'$ are equivalent.

By Theorem 2.22, the following definition is well-defined.

**Definition 2.23.** Let $\mathcal{L}$ be a link in a thickened surface. Let $L$ be a link diagram of a link $\mathcal{L}$. We define the homotopical Khovanov homology of $\mathcal{L}$ to be that of $L$.

3. **The ladybug configuration for link diagrams in $S^2$**

We review the ladybug configuration for link diagrams in $S^2$, which is introduced in [12, section 5.4]. Lipshitz and Sarkar introduced it to define a CW complex for any link diagram in $S^2$. We cite the definition of it, that of the right pair, and that of the left pair associated with it from [12, section 5.4.2].
Figure 2.17. A link diagram $L$ in a thickened surface, and the relation, which is made by surgeries, among all (non-labeled) resolution configurations made from $L$. One circle in the left lower labeled resolution configurations is non-contractible, and the other contractible.

**Definition 3.1.** ([12, Definition 5.6]). An index 2 basic resolution configuration $D$ in $S^2$ is said to be a ladybug configuration if the following conditions are satisfied (See Figure 3.1).

- $Z(D)$ consists of a single circle, which we will abbreviate as $Z$;
- The endpoints of the two arcs in $A(D)$, say $A_1$ and $A_2$, alternate around $Z$ (that is, $\partial A_1$ and $\partial A_2$ are linked in $Z$).

**Definition 3.2.** ([12, section 5.4.2]). Let $D$ be as above. Let $Z$ denote the unique circle in $Z(D)$. The surgery $s_{A_1}(D)$ (respectively, $s_{A_2}(D)$) consists of two circles; denote these $Z_{1,1}$ and $Z_{1,2}$ (respectively, $Z_{2,1}$ and $Z_{2,2}$); that is, $Z(s_{A_i}(D)) = \{Z_{i,1}, Z_{i,2}\}$. Our main goal is to find a bijection between $\{Z_{1,1}, Z_{1,2}\}$ and $\{Z_{2,1}, Z_{2,2}\}$; this bijection will then
Figure 2.18. A link diagram $L$ in a thickened surface, and the relation, which is made by surgeries, among all (non-labeled) resolution configurations made from $L$. The two circles in the left lower labeled resolution configurations are non-contractible.

tell us which points in $\partial_{\exp} \mathcal{M}(x, y)$ to identify. See [12, (RM-2) in section 5.1] for the notation $\partial_{\exp}$.

As an intermediate step, we distinguish two of the four arcs in $Z - (\partial A_1 \cup \partial A_2)$. Assume that the point $\infty \in S^2$ is not in $D$, and view $D$ as lying in the plane $S^2 - \{\infty\} \cong \mathbb{R}^2$. Then one of $A_1$ or $A_2$ lies outside $Z$ (in the plane) while the other lies inside $Z$. Let $A_i$ be the inside arc and $A_o$ the outside arc. The circle $Z$ inherits an orientation from the disk it bounds in $\mathbb{R}^2$. With respect to this orientation, each component of $Z - (\partial A_1 \cup \partial A_2)$ either runs from the outside arc $A_o$ to an inside arc $A_i$ or vice-versa. The right pair is the pair of components of $Z - (\partial A_1 \cup \partial A_2)$ which run from the outside arc $A_o$ to the inside arc $A_i$. The other pair of components is the left pair. See [12, Figure 5.1].

We explain why the ladybug configuration is important: By each ladybug configuration, there are many possibilities of homotopy type of Khovanov CW complex. So we
Figure 2.19. We put a labeling $x_+$ (respectively, $x_-$) on the left (respectively, right) upper resolution configuration in Figure 2.17. We check what labelled resolution configurations are made from it by the differential.

must check whether such homotopy types are stable homotopy type equivalent or not. See below and Fact 4.2.

**Proposition 3.3.** Let $x$ (respectively, $y$) be a labelled resolution configuration in $S^2$ of homological grading $n$ (respectively, $n+2$). Then the cardinality of the set

$\{p | p$ is a labelled resolution configuration. $x \prec p, p \prec y, p \neq x, p \neq y\}$

is 0, 2, or 4, where $\prec$ is defined in Definition 2.10.

Let $D$ be the ladybug configuration in $S^2$. Since each of $D$ and $s(D)$ has only one circle, we can let $x_+$ or $x_-$ denote a labeling on it. Give $D$ (respectively, $s(D)$) a labeling $x_+$ (respectively, $x_-).$ We call the resultant labeled resolution configuration $(D, x_+)$ (respectively, $(s(D), x_-))$. We obtain a decorated resolution configuration $(D, x_-, x_+)$ as drawn in Figure 3.2.

**Fact 3.4.** The case of 4 in Proposition 3.3 occurs in the above case $(D, x_-, x_+)$. 

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The natural dual of the Khovanov basis is called the dual Khovanov basis. See [8, §3] for the explanation of this kind of dual.

Let $\mathcal{L}$ be a link in $S^3$. Let $L$ be a link diagram in $S^2$ which represents $\mathcal{L}$. In [12] Lipshitz and Sarkar made a consistent method to construct a CW complex for $L$ whose singular homology is the homology of the dual Khovanov chain complex of $\mathcal{L}$. 

4. **Khovanov-Lipshitz-Sarkar stable homotopy type and Steenrod square for links in $S^3$**

The natural dual of the Khovanov chain group is called the dual Khovanov chain group. The natural dual of the Khovanov basis is called the dual Khovanov basis. See [3] §3 for the explanation of this kind of dual.
Note that, by the definition of the Khovanov homology, the cohomology, not homology, of Khovanov homotopy type is the Khovanov homology.

Let $L'$ be a link diagram in $S^2$ which represents $L$. They also proved that the stable homotopy type of the CW complex for $L$ and that for $L'$ are the same. We call this stable homotopy type, Khovanov-Lipshitz-Sarkar stable homotopy type for $L$. Thus this stable homotopy type has Steenrod squares. In [14] Lipshitz and Sarkar found how to calculate the second Steenrod square associated with Khovanov-Lipshitz-Sarkar stable homotopy type by using labeled resolution configurations. In [20] Seed proved the following fact by making a computer program of Lipshitz and Sarkar’s calculation of the second Steenrod square: There is a pair of links in $S^3$ with the following properties. Their second Steenrod squares and their Khovanov-Lipshitz-Sarkar stable homotopy types are different. Their Khovanov homologies are the same.

We review Lipshitz and Sarkar’s method below.

Let $L$ in $S^2$ and $L$ in $S^3$ be as above. In [12] §5 and §6, in particular, Definition 5.3], the Khovanov-Lipshitz-Sarkar stable homotopy type of $L$ and that of $L$ is defined as follows.

Let $(D, x, y)$ be any index $n$ basic decorated resolution configuration. In [12] §5 and §6] Lipshitz and Sarkar associate to $(D, x, y)$ an $(n - 1)$-dimensional $< n - 1 >$-manifold $\mathcal{M}(D, x, y)$ together with an $(n - 1)$-map

$$\mathcal{F} : \mathcal{M}(D, x, y) \to \mathcal{M}_{\bar{\epsilon}(n)}(\bar{1}, \bar{0}).$$

See [12 §3.1] for $< m >$-manifolds and $m$-maps, where $m$ is an integer. See [12 Definition 3.3] for the definition of $\bar{0}, \bar{1}$. 
Figure 3.2. The poset for the decorated resolution configuration associated with a ladybug configuration in $S^2$
Note that we regard the poset for each index \( n \) basic decorated resolution configuration as a flow category. See [12, Definition 3.12] for the definition of flow category.

The **Khovanov flow category** \( \mathcal{C}_K(L) \) has one object for each Khovanov basis element. That is, an object of \( \mathcal{C}_K(L) \) is a labeled resolution configuration of the form \( \mathbf{x} = (D_L(u), x) \) with \( u \in \{0, 1\}^n \). The grading on the objects is the homological grading \( \text{gr}_h \); the quantum grading \( \text{gr}_q \) is an additional grading on the objects. We need the orientation of \( L \) in order to define these gradings, but the rest of the construction of \( \mathcal{C}_K(L) \) is independent of the orientation. Consider objects \( \mathbf{x} = (D_L(u), x) \) and \( \mathbf{y} = (D_L(v), y) \) of \( \mathcal{C}_K(L) \). The space \( \mathcal{M}_{\mathcal{C}_K(L)}(\mathbf{x}, \mathbf{y}) \) is defined to be empty unless \( y \prec x \) with respect to the partial order from Definition 2.10. So, assume that \( y \prec x \). Let \( x| \) denote the restriction of \( x \) to \( s(D_L(v) - D_L(u)) = D_L(u) - D_L(v) \) and let \( y| \) denote the restriction of \( y \) to \( D_L(v) - D_L(u) \). Therefore, \( (D_L(v) - D_L(u), x|, y|) \) is a basic decorated resolution configuration. We define \( \mathcal{M}(D_L(v) - D_L(u), x|, y|) \) as above. Use it, and define

\[
\mathcal{M}_{\mathcal{C}_K(L)}(\mathbf{x}, \mathbf{y}) = \mathcal{M}(D_L(v) - D_L(u), x|, y|),
\]

as smooth manifolds with corners.

In [12, §5], it is proved that, if \( \text{gr}_h \mathbf{x} - \text{gr}_h \mathbf{y} = n \), \( \mathcal{M}_{\mathcal{C}_K(L)}(\mathbf{x}, \mathbf{y}) \) is a disjoint union of some copies of the \( n \)-dimensional cube moduli \( \mathcal{M}_{\varphi(n)}(\overline{1}, \overline{0}). \)

**Definition 4.1.** Let \( L \) in \( S^2 \) and \( \mathcal{L} \) in \( S^3 \) be as above. We define a CW complex \( Y(L) \) for \( L \) below.

Let \( \{a_p\}_{p \in \Lambda} \) be the dual Khovanov basis for \( L \). Note that \( \Lambda \) is a finite set.

Fix \( n \in \mathbb{Z} \). Let \( g_i^n \) be all dual Khovanov basis elements whose homological grading is \( n \) in \( \{a_p\}_{p \in \Lambda} \). We assign to \( g_i^n \) a \((n + N)\)-cell \( e_i^{n+N} \), where \( N \) is a large integer.

We attach the cells, \( e_i^{n+N} \), for all \( n \): We use the moduli spaces defined above, with an arbitrary set of framings which satisfy [12, Definition 3.20], according to the method in [12, Definition 3.23] (note Proposition 4.4 below). The result is \( Y(L) \).

The stable homotopy type of the CW complex \( Y(L) \) is called the **Khovanov-Lipshitz-Sarkar stable homotopy type** for the link diagram \( L \). More precisely, since we use an arbitrary large integer \( N \) to construct \( Y(L) \), we must handle \( N \) as follows. \( N \) times of the formal desuspension of \( Y(L) \) is called the **Khovanov-Lipshitz-Sarkar spectrum** for the link diagram \( L \).

Note: Following Lipshitz and Sarkar [12, Definition 5.5], the Khovanov homology is the reduced cohomology of the Khovanov space shifted by \((-C)\) for some positive integer \( C \). The Khovanov spectrum \( \chi_{Kh}(L) \) is the suspension spectrum of the Khovanov space,
Note. The dual Khovanov chain complex is made from Khovanov chain complex uniquely, and vice versa. So the following two sentences (1) and (2) have the same meaning.

(1) We associate the framed flow category \( \mathcal{C} \) to Khovanov chain complex.
(2) We associate the framed flow category \( \mathcal{C} \) to the dual Khovanov chain complex.
(Here, suppose that the above Khovanov chain complex and the above dual Khovanov chain complex are dual each other.)

When we make \( \mathcal{F} \) above, it is important to analyze the ladybug configuration \cite{12} §5.4.2.

Fact 4.2. (\cite{12} §6.) The stable homotopy type of Khovanov-Lipshitz-Sarkar stable homotopy type for link diagrams in \( S^2 \) does not depend on whether we use the right pair or the left pair of each ladybug configuration.

Fact \ref{fact:4.3} is used in the proof of Fact \ref{fact:4.2}.

Fact 4.3. (This is written in \cite{12} Proof of Proposition 6.5.) Fix a classical link diagram \( L \) in \( S^2 \), and let \( L' \) be the result of reflecting \( L \) across the \( y \)-axis, say, and reversing all of the crossings. Then \( L \) and \( L' \) represent the same link in \( S^3 \).

Proposition 4.4. (This follows from results in \cite{12}. See the comments below.)

The stable homotopy type of the Khovanov-Lipshitz-Sarkar stable homotopy type for link diagrams in \( S^2 \) does not depend on the choice of the coherent framing, which is defined in \cite{12} Definition 3.18.

Proposition \ref{prop:4.4} is the same as \cite{12} (4) in the first part of section six, which is proved in the proof of \cite{12} Proposition 6.1: In three lines above \cite{14} Definition 3.4, it is written, “all such framings lead to the same Khovanov homotopy type \cite{12} Proposition 6.1”. See also \cite{12} Lemma 4.13, which is cited in the proof of Proposition 6.1. In short, in the case of Khovanov homotopy type for links in \( S^3 \), all modulis are contractible so we do not need to check framings on them.

Of course, the same set of modulis and different set of framings construct CW complexes of different stable homotopy types, in general. See Example \ref{example:4.5} below.

Example 4.5. Let \( S^2 \vee S^4 \) denote the one point union of \( S^2 \) and \( S^4 \). Both \( \Sigma^k(S^2 \vee S^4) \) and \( \Sigma^k(CP^2) \), where \( \Sigma^k \) denotes the \( k \)-times suspension and \( k \) is large, have a natural CW decomposition (the base point) \( \partial e^{2+k} \cup e^{4+k} \). Consider a set of moduli spaces associated with \( \Sigma^k(S^2 \vee S^4) \) (respectively, \( \Sigma^k(CP^2) \)). In \( \partial e^{2+k} \), there is no moduli space. In \( \partial e^{4+k} \), take an embedded circle. It is a moduli space. Take the normal bundle of the circle in \( \partial e^{4+k} \),
and take the trivial (respectively, nontrivial) framing. It is a framing on the moduli space.

**Definition 4.6.** In [12] it is proved that if two link diagrams $L$ and $L'$ in $S^2$ represent the same link in $S^3$, then $Y(L)$ and $Y(L')$ (see Definition 4.1) are stable homotopy type equivalent. Thus we obtain a unique stable homotopy type, \textit{Khovanov-Lipshitz-Sarkar stable homotopy type}, and the \textit{Khovanov-Lipshitz-Sarkar spectrum} for links in $S^3$. They are link type invariants.

It is an outstanding property of the Khovanov chain complex and Khovanov stable homotopy type for link diagrams in $S^2$ that, if $\mathcal{M}_{c_k(L)}(x,y) \neq \emptyset$, each connected component of $\mathcal{M}_{c_k(L)}(x,y)$ is determined only by $\text{gr}_b x - \text{gr}_b y$. Chain complexes in other cases do not have this property in general.

5. Ladybug and quasi-ladybug configurations for link diagrams in surfaces

In this section we explain why the torus case is special, and why we concentrate in the higher genus case in this paper. We will discuss the torus case in a sequel of this paper [9].

**Definition 5.1.** Let $D$ be a resolution configuration which is made of one circle and two m-arcs.

Stand at a point in the circle where you see an arc to your right. Go ahead along the circle. Go around one time. Assume that you encounter the following pattern: In the order of travel you next touch the other arc. Then you touch the first arc. Then you touch the other arc again. Finally, you come back to the point at the beginning.

Since both arcs are m-arcs, both satisfy the following property: At both endpoints of each arc, you see the arc in the same side – either on the right hand side or on the left hand side.

If you see the arcs both in the right hand side and in the left hand side (respectively, only in the right hand side) while you go around one time, we call $D$ a \textit{ladybug configuration} (respectively, \textit{quasi-ladybug configuration}).

If $F$ is the 2-sphere, our definition of ladybug configurations is the same as that in §3.

Let $D$ be a ladybug (respectively, quasi-ladybug) configuration. Then $Z(D)$ have only one circle and $A(D)$ have only two arcs. Make $s(D)$. Give $D$ (respectively, $s(D)$) a labeling $x$ (respectively, $y$). We call the decorated resolution configuration $(D, y, x)$ a \textit{decorated resolution configuration associated with the ladybug (respectively, quasi-ladybug) configuration $D$}.

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Note that \((D, y, x)\) may be empty as explained below.

Since each of \(D\) and \(s(D)\) has only one circle, we can let \(x_+\) or \(x_-\) denote \(x\) (respectively, \(y\)).

Proposition 5.2 below explains what difference there is between the torus case and the higher genus surface case. In the higher genus surface case, all modulis associated with quasi-ladybug configurations are the empty set. In the torus case, it does not hold. Therefore the construction of CW complexes in the torus case is more complicated than the higher genus surface case. See also ‘comments below Theorem 7.5’ and Notes 7.7 and 7.12.

**Proposition 5.2.** (1) Let \(F\) be the torus. Let \(D\) be a quasi-ladybug configuration in \(F\). Assume that the only one circle in \(D\) is contractible. Then there is a non-vacuous decorated resolution configuration \((D, x_-, x_+)\) associated with \(D\).

(2) Let \(F\) be a higher genus surface. Let \(D\) be a quasi-ladybug configuration in a surface \(F\). Assume that the only one circle in \(D\) is contractible. Let \((D, y, x)\) be a decorated resolution configuration associated with \(D\). Then \((D, y, x)\) is empty for arbitrary \(x\) and \(y\).

(3) Let \(F\) be an arbitrary surface. Let \(D\) be a ladybug (respectively, quasi-ladybug) configuration in \(F\). Let \((D, y, x)\) be a decorated resolution configuration associated with \(D\). Assume that the only one circle in \(D\) is non-contractible. Then \((D, y, x)\) is empty for arbitrary \(x\) and \(y\).

(4) Let \(F\) be an arbitrary surface. There is a ladybug configuration \(D\) in \(F\) such that a decorated resolution configuration \((D, y, x)\) associated with \(D\) is non-empty.

**Proof of Proposition 5.2.** The proof of Proposition 5.2 (1). In Figure 5.1 we draw an example.

The proof of Proposition 5.2 (2). Let \(D\) be a quasi-ladybug configuration in \(F\). Let \((D, y, x)\) be the decorated resolution configuration associated with \(D\).

Let \(C\) be only one circle in \(D\). Let \(A\) and \(A'\) be just two arcs in \(A(D)\).

Let \(A_p\) and \(A_q\) (respectively, \(A'_p\) and \(A'_q\)) be the endpoints of \(A\) (respectively, \(A'\)). We can suppose that, when we go around \(C\) one time in an orientation, we meet \(A_p, A'_p, A_q,\) and \(A'_q\) in this order, without loss of generality. We obtain four circles in \(F\), as below:

A circle made of the following two: The arc \(A\). An arc which is a part of \(C\), whose boundary is \(A_p \sqcup A_q\), and which includes \(A'_p\).

A circle made of the following two: The arc \(A\). An arc which is a part of \(C\), whose boundary is \(A_p \sqcup A_q\), and which includes \(A'_q\).
Figure 5.1. The poset for a decorated resolution configuration $(D, x_-, x_+)$ associated with a quasi-ladybug configuration on $T^2$: We envelope $T^2$ along two circles as usual, and draw six labeled resolution configurations. Here, we have $[\xi; a] \cdot [a; \eta] = [\xi; b] \cdot [b; \eta] = -[\xi; c] \cdot [c; \eta] = -[\xi; d] \cdot [d; \eta]$. 
A circle made of the following two: The arc $A'$. An arc which is a part of $C$, whose boundary is $A_p \amalg A'_q$, and which includes $A_p$.

A circle made of the following two: The arc $A'$. An arc which is a part of $C$, whose boundary is $A_p \amalg A'_q$, and which includes $A_q$.

Since $D$ is a quasi-ladybug configuration, the four circles above are non-contractible.

The two circles in $s_A(D)$ (respectively, $s_{A'}(D)$) divide $F$ into two connected compact surfaces, $W_A$ and $W_A'$ (respectively, $W_{A'}$ and $W'_{A'}$), with boundary.

Note that the only one circle in $D$ is contractible, and that the genus of $F$ is greater than one. Therefore one of $W_A$ and $W_A'$ (respectively, $W_{A'}$ and $W'_{A'}$) is an annulus and the other has the genus greater than one.

Since the four circles above are non-contractible, $A'$ (respectively, $A$) is not included in the annulus.

Therefore the only one circle in $s(D)$ divided $F$ into two components: a compact surface with boundary $S^1$ and with genus one, and a compact surface with boundary $S^1$ and with genus greater than zero.

Therefore the only one circle in $s(D)$ is non-contractible.

Use the rule in Definitions 2.10 and 2.16 again. Since the two circles in $s_A(D)$ (respectively, $s_{A'}(D)$) and the only one circle in $s(D)$ are non-contractible, $(D, y, x)$ is empty.

This completes the proof of Proposition 5.2(2).

The proof of Proposition 5.2(3). By the rules in Definitions 2.10 and 2.16

The proof of Proposition 5.2(4). The ladybug configuration in Figure 3.2 is put in a 2-disc in $S^2$. Put it in a 2-disc embedded in $F$.

Note that, of course, there is a ladybug configuration which is not put in a 2-disc in $F$: Let $D$ be a ladybug configuration in a surface $F$. Let $A$ be an arc in $A(D)$. Note that $s_A(D)$ may include a non-contractible circle. However there is a ladybug configuration which is put in a 2-disc in $F$.

Therefore we must divide our discussion into three cases: $F = S^2$, $F = T^2$, and $F$ is a higher genus surface. In [12], Lipshitz and Sarkar did the $S^2$ case. In this paper, we obtain new results mainly about the higher genus surface case, and point out the difficulties of the $T^2$ case. In a sequence [9] of this paper, we will write the detail of the $T^2$ case.

**Definition 5.3.** Let $D$ be a ladybug configuration. Let $C$ be only one circle in $Z(D)$. Recall the round trip of $C$, used above when we define ladybug configurations. Cut the circle at the four points where the arcs meet the endpoints. The circle is then divided
into four pieces. Recall that, at the beginning point, you see an arc on the right hand side. We call the first and third pieces of the four, which you are in while your trip, the right pair, and call the other two the left pair. Note that the orientation of your trip and the place where you stand at the beginning of your trip do not change the right and the left pair. Note also that, if $F$ is the 2-sphere, this definition is the same as the one in \cite{12} §5.4.2 and in §3.

It is important that we cannot determine the right and left pair in the case of quasi-ladybug configurations by this method. (We pose the question: Can one find a method to define the right and the left pair for quasi-ladybug configurations, to be compatible with the construction of Khovanov homotopy type?)

By using the right and left pairs introduced above, we determine the right and left pair of the labeled resolution configurations in the middle row of the poset for a given decorated resolution configuration associated with a ladybug configuration (See Figure 3.2 for an example.) The determination is explained in \cite{12} Figure 5.1 and its explanation in §5.4.2.

In this paper, we take the right pair when we construct a CW complex if there is a ladybug configuration. (If we take the left pair, we can construct a CW complex in a parallel method.)

However, in the case of quasi-ladybug configurations, we cannot distinguish the two cases.

It means that, in general, in the case of link diagrams in $T^2$, we may associate more than one CW complex to a single link diagram.

In the case of the higher genus surfaces, we give only one CW complex to a single link diagram. *Reason.* By Proposition 5.2 the decorated resolution configuration associated with an arbitrary quasi-ladybug configuration is empty. Therefore the moduli associated with it is the empty set.

6. A moduli in the case of link diagrams in the torus, which never appears in the case of those in $S^2$ nor in the case of those in higher genus surfaces

In Figure 6.1 we draw the poset for a decorated resolution configuration. Each labeled resolution configuration in it is put in $T^2$. Assume that we give more than one moduli for a single decorated resolution configuration in general, and make many CW complexes for a single link diagram, as explained in §5. Then one of moduli spaces for the decorated resolution configuration in Figure 6.1 is a dodecagon. It is not the 3-dimensional cube moduli. Of course, it is also not a trivial covering of the 3-dimensional cube moduli. This
is a new phenomenon which we do not have in the $S^2$ case. We also do not have it in the higher genus case. That is a reason why the torus case is difficult.

**Review.** See [12, Proposition 5.2]: In the $S^2$ case, each moduli space is the empty set or a trivial covering of the $n$-dimensional cube moduli.

7. The Khovanov-Lipshitz-Sarkar stable homotopy type for links in thickened higher genus surfaces

In this section, we define the Khovanov-Lipshitz-Sarkar stable homotopy type for links in thickened higher genus surfaces (§7.7).

7.1. Moduli spaces for decorated resolution configurations.

**Definition 7.1.** Take an $n$-dimensional cube in a coordinate space $\mathbb{R}^n$ whose vertices are points with coordinates $(a_1, \ldots, a_n)$, where each $a_i$ is 0 or 1. Let $C_n$ be the set of all of these vertices. Elements of $C_n$ can be regarded as vectors. $C_n$ has a partial order $\prec$ as follows: Let $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in C_n$. Let $k \in \{1, \ldots, n\}$. If $u_k = 0, v_k = 1$ and $u_i = v_i$ for $i \neq k$, then $u \prec v$. It follows from this definition that if $u \prec v$ and $v \prec w$ then $u \prec w$.

We define the norm $|u|$ of $u$ to be $u_1 + \ldots + u_n$. (Note that $u_1 + \ldots + u_n = (u_1)^2 + \ldots + (u_n)^2$ in this situation, since $0^2 = 0$ and $1^2 = 1$.)

Recall the norm for resolution configurations in Defintion 2.2. The $n$-dimensional cube flow category $C_C(n)$ in [12, §4] is associated with $C_n$. We use the moduli $M_{C(n)}(\bar{0}, \bar{1})$ for $C_C(n)$, defined there.

Let $(D, x, y)$ be an index $n$ basic decorated resolution configuration in a higher genus surface. Let $y = (D, y)$ and $x = (s(D), x)$. We associate to each non-vacuous $(D, x, y)$ an $(n - 1)$-dimensional $< n - 1 >$-manifold, $M(D, x, y)$ or $M(x, y)$, together with an $(n - 1)$-map

$$\mathcal{F} : M(D, x, y) \to M_{C(n)}(\bar{0}, \bar{1})$$

below, as done in [12] all of §5. In particular, see §5.1 and Proposition 5.2. If $(D, x, y)$ is empty, we associate to $M(x, y)$ the empty set. We suppose that $(D, x, y)$ is not empty below.

This means that we make a moduli space for any pair of Khovanov basis elements in the Khovanov chain complex of $L$. 

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Figure 6.1. The poset for the decorated resolution configuration associated with a quasi-ladybug configuration in $T^2$: We use a net of $T^2$ with a labeled resolution configuration. Each one in the second row includes just two circles. Each one in the third row includes only one circle. The lowest one includes two circles.
Definition 7.2. Take any element \( r \in P(D, x, y) \). Let \( v \) be the vector of \( r \). Define \( \pi \) to be the map \( P(D, x, y) \to C_n \) such that \( \pi(r) = v \).

This map \( \pi: P(D, x, y) \to C_n \) keeps the partial order because \( \text{gr}_h((D_L(u), x)) = -n_- + |u| \).

We define the map \( F \) to be associated with \( \pi \), as done in \cite{12} all of \( §5 \). In particular, see \( §5.1 \) and Proposition 5.2.

Proposition 7.3. The map \( \pi \) in Definition 7.2 is onto.

Theorem 7.4. The map \( F \) is a trivial covering map.

Theorem \( 7.4 \) corresponds to \cite{12} Proposition 5.2 and \( §5.1 \). See \cite{12} \( §3.4.1 \) for the definition of trivial covering maps in this case.

While we prove Proposition 7.3 and Theorem 7.4 in the following subsections, we prove Theorem 7.5 together. In the proof of Theorem 7.4, Theorem 7.5 plays a crucial role.

Theorem 7.5. Let \((D, x, y)\) be an index \( n \) basic decorated resolution configuration in a higher genus surface above. Suppose that \((D, x, y)\) is non-vacuous, as above. Take any element \( A \) in \( P(D, x, y) \). (Recall that \( A \) is a labelled resolution configuration.) Then all arcs in \( A \) are mc arcs.

In the case of links in \( S^3 \), the map \( \pi \) is an epimorphism for all natural numbers \( n \), and the above theorems are true \( (12) \).

In the case of links in the thickened torus and in the case of virtual links, we have different features. See \( §6 \) and \( [8] \).

We prove Proposition 7.3, Theorems 7.4 and 7.5 below.

7.2. The case \( n = 1 \).

Proof of the case \( n = 1 \) of Theorem 7.4 and that of the case \( n = 1 \) of Proposition 7.3. We associate to \( M(x, y) \) one point because the coefficient of the right hand side of Equality \( 2.5 \) is \( +1, 0 \), and \( -1 \). Therefore Theorem 7.4 and Proposition 7.3 hold in this case.

Proof of the \( n = 1 \) case of Theorem 7.5. Suppose that only one arc in \( D \) is a scs arc. By the rule in Definitions 2.10 and 2.16 the decorated resolution configuration is empty. We arrived at a contradiction. \( \square \)
7.3. The case \( n = 2 \).

**Proof of the case \( n = 2 \) of Theorem 7.4** and that of the case \( n = 2 \) of Proposition 7.3. Let \( P \) be a partial ordered set. Let \( \alpha, \beta \in P \). Define \( Q(\alpha, \beta) \) associated with \( P \) to be \( \{ q \in P, q \neq \alpha, q \neq \beta, \alpha < q, q < \beta \} \). We omit the words, associated with \( P \), when it is clear from the context. Let \( A \) be a finite set. Let \( \#A \) be the number of all elements of \( A \).

Take \( Q(x, y) \) associated with \( P(D, x, y) \). Then \( \#Q(x, y) \) is 2 or 4 because of Definitions 2.10 and 2.16 since we assume that \( (D, x, y) \) is non-vacuous. If \( \#Q(x, y) \) is 2, we associate to \( M(x, y) \) one segment. Suppose that \( \#Q(x, y) \) is 4, \( (D, x, y) \) is a ladybug configuration and \( D \) includes only one contractible circle because of Proposition 5.2. We associate to \( M(x, y) \) a disjoint union of two segments. We use the right pair in §5.

Therefore Theorem 7.4 and Proposition 7.3 hold in this case.

**Note 7.6.** If we choose the left pair, we can also construct the Khovanov stable homotopy type. In the case of links in \( S^3 \), both choices give the same Khovanov stable homotopy type (Fact 4.2). We pose the question: Do both choices give the same Khovanov stable homotopy type in the case of links in thickened (higher genus) surfaces?

**Proof of the \( n = 2 \) case of Theorem 7.5.** Suppose that there is a scs arc in a labeled resolution configuration in \( P(D, x, y) \). Then the decorated resolution configuration is empty. Reason. Consider index 2 decorated resolution configurations associated with Figures 2.17-2.20. Other cases are easy.

We arrived at a contradiction. \( \square \)

**Note 7.7.** In the case of virtual links in \([3, 8, 15]\), Theorem 7.5 is not true and the map \( \pi \) is not an epimorphism, even if \( n = 2 \). See [8]. See Note 7.12.

7.4. The first part of the case \( n \geq 3 \).

**Proof of Proposition 7.3.** The case \( n = 1, 2 \) holds by §7.2 and §7.3. We prove the case \( n \geq 3 \). Take two arbitrary labeled resolution configurations, \( w = (D_\xi(L), w) \) and \( z = (D_\zeta(L), z) \), in \( P(D, x, y) \) such that \( |\xi| + 2 = |\zeta| \), and such that \( w < z \). Since \( (D, x, y) \) is not empty, there is such a pair. Assume that \( Q(\xi, \zeta) \) associated with \( C_n \) is \( \{ \eta, \eta' \} \).

Take \( Q(w, z) \) associated with \( P(D, x, y) \). By §7.3 and [12] §5.4, we have that \#\( Q(w, z) \) is 2 or 4. If it is 2, one element of \( Q(w, z) \) is over \( D_\eta(L) \) and the other is over \( D_{\eta'}(L) \) by the rule in Definitions 2.10 and 2.16. If it is 4, two elements of \( Q(w, z) \) are over \( D_\eta(L) \) and the other two are over \( D_{\eta'}(L) \) by the rule in Definitions 2.10 and 2.16. Repeat this procedure. Hence \( \pi \) is onto. \( \square \)
Proof of Theorem 7.5. We have proved the $n < 3$ case in the previous subsections. We prove the $n \geq 3$ case. We prove by reductio ad absurdum. Assume that $g \in P(D, x, y)$ has a scs arc $A$. Let $G$ be a (non-labeled) resolution configuration under $g$. Carry out a surgery along $A$, and obtain a (non-labeled) resolution configuration $G'$. Let $v$ be the vector of $G'$. Then $v \in C_n$. By Proposition 7.3, we have $\pi^{-1}(v) \neq \emptyset$.

Any labeled resolution configuration on $G'$ has a different quantum degree from that of $g$ (See Proposition 2.20.). (Note that $\text{gr}_q g = \text{gr}_q x = \text{gr}_q y$.) By the definition of decorated resolution configurations, all labeled resolution configurations in $P(D, x, y)$ have the same quantum degree. Hence $\pi^{-1}(v) = \emptyset$. We arrived at a contradiction. □

Review. In [12, Proof of Proposition 5.2], the case of three arcs is more complicated than the case of greater than three arcs. (In our way of description in this paper, the case of three arcs is the case $n = 3$.) The reason is as follows: Let $f$ be a local diffeomorphism map from $X$ to $S^m$. If $m > 1$, $f : X \to S^m$ is a trivial covering map. If $m = 1$, $X$ is not a trivial covering map in general. An example is the connected double covering map $S^1 \to S^1$. (Here, $m + 2$ is the above $n$.)

The case of three arcs in [12, Proof of Proposition 5.2] is proved in [12, §5.5] by checking all resolution configurations with three arcs.

In this paper, the case $n = 3$ is also more complicated than the case $n > 3$.

We split the case $n \geq 3$ into the case $n = 3$ and the case $n \geq 4$ below.

7.5. The proof of Theorem 7.4 in the case $n = 3$.

There are just two cases.

Case 1. Suppose that all circles in all labeled resolution configurations in $P(D, x, y)$ are contractible.

By Theorem 7.5 all arcs in all labeled resolution configurations in $P(D, x, y)$ are mc arcs. Hence we can prove Theorem 7.4 in this case, as proved in [12, §5.1, especially Proposition 5.2, and 5.5].

Case 2. Assume that there is a non-contractible circle in a labeled resolution configuration in $P(D, x, y)$.

Recall that \( \partial \mathcal{M}_{c(3)}(\bar{1}, \bar{0}) \) is a circle. By Proposition 7.3 and [12, Definition 3.12, especially (M-3) in it, and Proposition 5.2], we have the following: $\partial \mathcal{M}(x, y)$ is a disjoint union of circles. Furthermore, $\partial \mathcal{M}(x, y)$ is a covering space of $\partial \mathcal{M}_{c(3)}(\bar{1}, \bar{0})$ by $\mathcal{F}$.

Fact 7.8. Any circle which is a connected component of $\partial \mathcal{M}(x, y)$ covers $\partial \mathcal{M}_{c(3)}(\bar{1}, \bar{0})$ by a degree one map.
**Proof of Fact 7.8.** By Theorem 7.5 all arcs in all labeled resolution configurations in $P(D, x, y)$ are mc arcs. We check all basic resolution configurations with three mc arcs. We can use almost the same methods as those in [12, §5.5].

We must take care of the difference between the rule of the partial order in Definition 2.10 and that in [12, Definition 2.10].

The result ([12, Lemma 2.14]) on leaves and co-leaves is used in [12, §5.5]. This result is generalized easily, and also holds in our case. Therefore we only have to concentrate on resolution configurations without a leaf or a co-leaf, as in [12, §5.5].

The duality theorem, [12, Lemma 2.13], is used in [12, §5.5]. This result is generalized easily, and also holds in our case. It helps our purpose below.

If the union of all arcs and circles in a basic resolution configuration is disconnected, there is a leaf or a co-leaf. Note that it is basic. Hence we assume that it is connected.

Let $D$ be any resolution configuration with three arcs such that $Z(D) \cup A(D)$ is connected. We draw all such cases of $Z(D) \cup A(D)$ like abstract graphs in Figure 7.1.

We choose $Z(D) \cup A(D)$ without a leaf or a co-leaf from Figure 7.1 and draw them in Figure 7.2.

Recall the following facts associated with Figures 7.1 and 7.2. In a surface, each $Z(D) \cup A(D)$ has a neighborhood $N$ such that $N$ is a compact surface and such that the inclusion map of $Z(D) \cup A(D)$ to $N$ is a homotopy type equivalence map. There are many homeomorphism types of $N$ in general. We can assume that no scs arc appears.

Let $D$ be a resolution configuration made from one diagram of Figure 7.2. Assume that, if we choose two arcs and one circle from $D$, then they make a ladybug configuration (or a quasi-ladybug configuration). Then we say that $D$ includes a ladybug configuration (or a quasi-ladybug configuration).

Let $D$ induce a quasi-ladybug configuration. By Proposition 5.2 any decorated resolution configuration starting from $D$ is empty.

The following two conditions are equivalent.

1. $D$ does not include a ladybug configuration. Let $E$ be any resolution configuration obtained from $D$ by a single surgery. $E$ is not a ladybug configuration.
2. Neither $D$ nor the dual resolution configuration $D^*$ includes a ladybug configuration.

If we have the above condition (1) (respectively, (2)), the moduli of any decorated resolution configuration starting from $D$ is the empty set or the single 3-dimensional cube moduli.

Note that, even if $D$ does not include a ladybug configuration, $D^*$ may include a ladybug configuration. An example is the case where $D$ is [12 Figure 5.3.g].
Figure 7.1. Connected graphs of the resolution configurations of index 3: The segments denote arcs. We do not use dotted segments here. We draw only $Z(D) \cup A(D)$ abstractly, like abstract graphs.
Let $D$ be a resolution configuration made from one figure of Figure 7.2. Then $D^*$ is also made from one figure of Figure 7.2. *Reason.* Let $G$ be a resolution configuration in Figure 7.1. If $G$ has a leaf (respectively, co-leaf), then $G^*$ has a co-leaf (respectively, leaf).

Apply Propositions 5.2. If the moduli of a decorated resolution configuration starting from $D$ is not the empty set nor a single 3-dimensional cube moduli, $D$ or $D^*$ satisfies the condition: It includes a ladybug configuration, and does not include a quasi-ladybug configuration. The circle in the ladybug configuration is contractible.

Therefore we only have to check decorated resolution configurations starting from $D$ made from the left upper figure in Figure 7.2. The moduli of each is a disjoint union of the 3-dimensional cube moduli or the empty set.

This completes the proof of Fact 7.8. 

Henceforth we have Theorem 7.4 in this case.

7.6. **The proof of Theorem 7.4 in the case $n \geq 4$.**

The case $n \leq 3$ is true by §7.2-7.5. Therefore the case $n \geq 4$ is proved by the same method as one in [12, Proposition 5.2].
7.7. Definition of the Khovanov-Lipshitz-Sarkar stable homotopy type for links in thickened surfaces.
Definition 7.9 is made by adding the fact about the quantum grading to [12, Definition 5.3].

Definition 7.9. Let $L$ be a link diagram in a higher genus surface. The Khovanov flow category $\mathcal{C}_K(L)$ has one object for each Khovanov basis element. That is, an object of $\mathcal{C}_K(L)$ is a labeled resolution configuration of the form $x = (D_L(u), x)$ with $u \in \{0, 1\}^n$. The grading on the objects is the homological grading $\text{gr}_h$, the quantum grading $\text{gr}_q$ and the homotopical grading $\text{gr}_h$ are additional gradings on the objects. We need the orientation of $L$ in order to define these gradings, but the rest of the construction of $\mathcal{C}_K(L)$ is independent of the orientation.

Consider objects $x = (D_L(u), x)$ and $y = (D_L(v), y)$ of $\mathcal{C}_K(L)$. The space $\mathcal{M}_{\mathcal{C}_K(L)}(x, y)$ is defined to be empty unless $y < x$ with respect to the partial order from Definition 2.10. So, assume that $y < x$. Let $x|y$ denote the restriction of $x$ to $s(D_L(v) - D_L(u)) = D_L(u) - D_L(v)$ and let $y|$ denote the restriction of $y$ to $D_L(v) - D_L(u)$. Therefore, $(D_L(v) - D_L(u), x|y)$ is a basic decorated resolution configuration. Recall that we defined $\mathcal{M}(D_L(v) - D_L(u), x|y)$ in §7.1. Define
\[
\mathcal{M}_{\mathcal{C}_K(L)}(x, y) = \mathcal{M}(D_L(v) - D_L(u), x|y)
\]
as smooth manifolds with corners. The composition maps for the resolution configuration moduli spaces (see [12, (RM-1) in §5.1]) induce composition maps
\[
\mathcal{M}_{\mathcal{C}_K(L)}(z, y) \times \mathcal{M}_{\mathcal{C}_K(L)}(x, z) \to \mathcal{M}_{\mathcal{C}_K(L)}(x, y)
\]

Given a flow category $\mathcal{C}$ and an integer $n$, let $\mathcal{C}(n)$ be the flow category obtained from $\mathcal{C}$ by increasing the grading of each object by $n$.

The Khovanov flow category $\mathcal{C}_K(L)$ is equipped with a functor $F$ to $\mathcal{C}(n)[-n_-]$, which is a cover in the sense of [12, Definition 3.28].

On the objects, $F : \text{Ob}_{\mathcal{C}_K(L)} \to \text{Ob}_{\mathcal{C}(n)}$ is defined as
\[
F : \mathcal{M}_{\mathcal{C}_K(L)}((D_L(u), x), (D_L(v), y)) \to \mathcal{M}_{\mathcal{C}(n)}(u, v)
\]
is defined to be composition
\[
\mathcal{M}(D_L(v) - D_L(u), x|y) \xrightarrow{F} \mathcal{M}_{\mathcal{C}(n)}(u, v).
\]

We can say that $F$ is associated with $\pi$ in Definition 7.2.

As explained in §4, use the moduli spaces which are defined above. Thus we construct the Khovanov-Lipshitz-Sarkar stable homotopy type for $L$ and the Khovanov-Lipshitz-Sarkar spectrum $\mathcal{X}_{\mathcal{K}h}(L) = \bigvee_{q, b} \mathcal{X}_{\mathcal{K}h}(L)$ for $L$. 

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Note. In general, we need framings over moduli spaces when we construct a CW complex. However, in the case of Khovanov homotopy type, we do not need to take care of framings. See Proposition 4.4 and the comment below it where it is pointed out that the construction is independent of the choice of framing.

Definition 7.10. Let \( L \) be a link diagram of a link \( L \) in a thickened higher genus surface. We define the Khovanov-Lipshitz-Sarkar homotopy type (respectively, the Khovanov-Lipshitz-Sarkar spectrum \( \mathcal{X}_{Kh}(L) = \bigvee_{q,h} \mathcal{X}_{Kh}^q(L) \)) for \( L \), to be the Khovanov-Lipshitz-Sarkar homotopy type (respectively, the Khovanov-Lipshitz-Sarkar spectrum \( \mathcal{X}_{Kh}(L) = \bigvee_{q,h} \mathcal{X}_{Kh}^q(L) \)) for \( L \), defined in Definition 7.9.

We show examples in §8.

Theorem 7.11. Definition 7.10 above is well-defined.

Proof of Theorem 7.11. Use almost the same method as that in [12, §5]. Note, in particular, that [12, the comment between the end of the proof Proposition 6.3, and Proposition 6.4] also holds in our case: use the rule of partial order in Definition 2.10 instead of that in [12, Definition 2.10].

Proof of Main theorem 1.1. In [20], it is proved that there is a pair of knots, \( J \) and \( J' \), in \( S^3 \) (respectively, \( D^3 \)) with the following properties: The second Steenrod square of \( J \) and that of \( J' \) are different. The Khovanov homotopy type of \( J \) and that of \( J' \) are different. The Khovanov homology of \( J \) and that of \( J' \) are the same.

Take the knots in \( D^3 \) in thickened higher genus surfaces. Therefore Main theorem 1.1 holds.

Furthermore we have the following. Let \( F \) be a surface. Let \( C \) be a circle in \( F \) which represents a nontrivial element of \( H_1(F; \mathbb{Z}) \). Regard \( C \) as a knot in \( F \times [-1,1] \). Take \( J \) and \( J' \) in a 3-ball \( B \) embedded in \( F \times [-1,1] \). Assume that \( C \cap B = \emptyset \). Make a disjoint 2-component link which is made from \( C \) and \( J \) (respectively, \( J' \)). By the above result in [20], these two links have different Steenrod squares and the same Khovanov homology.

Note 7.12. Our case is different from Lipshitz and Sarkar’s case [12] in that the circles in \( D \) may be non-contractible circles, and arcs in \( D \) may be scs arcs. However, as we saw above, by the property on the sign convention in Definition 2.16, the Khovanov chain complexes for links in thickened surfaces have similar theorems in [12, section 5]. More precisely we have Theorem 7.13.
8. Examples of Khovanov homotopy type

We show examples of Khovanov homotopy type.

8.1. **links whose link diagrams have one crossing.** Consider an oriented link diagram $U$ with one crossing, see Fig. 8.1.

The diagram has two resolution configurations, see Fig. 8.2. The resolution configurations correspond to six labeled resolution configurations, see Table 8.1. The homotopical grading is expressed using the homotopical classes of the left and the right loops of the diagram, see Fig. 8.1.

The partial order on the set of labeled resolution configurations depends on the layout of the knot $U$ in the surface. If the loops $\alpha$ and $\beta$ are contractible then the situation does not differ from the classical case, cf. [12, section 9.1]. We have $\mathcal{X}_{kh}(U) = S^0 \vee S^0$ here.

There are three homotopically nontrivial cases.

Case 1 (see Fig. 8.3 upper left). Let one of the loops of $U$ (say $\alpha$) be non-contractible and the other be contractible. Then $[\beta] = 0$ and $[\alpha \beta] = [\alpha] \neq 0$. Hence, we have $x < b$, $y < d$, and the other generators are incomparable because they have different quantum
Table 8.1. Labeled resolution configurations of the diagram U.

| Name | Generator | $\text{gr}_h$ | $\text{gr}_q$ | $\text{gr}_\partial$ |
|------|-----------|---------------|---------------|---------------------|
| a    | $(D_U(1), x_+ x_+)$ | 0 | 1 | $[\alpha] + [\beta]$ |
| b    | $(D_U(1), x_+ x_-)$ | 0 | -1 | $[\alpha] - [\beta]$ |
| c    | $(D_U(1), x_- x_+)$ | 0 | -1 | $- [\alpha] + [\beta]$ |
| d    | $(D_U(1), x_- x_-)$ | 0 | -3 | $- [\alpha] - [\beta]$ |
| x    | $(D_U(0), x_+)$ | -1 | -1 | $[\alpha \beta]$ |
| y    | $(D_U(0), x_-)$ | -1 | -3 | $- [\alpha \beta]$ |

Figure 8.3. Possible layouts of the knot $U$ in the surface.

and homotopical gradings. We can treat $\tilde{X}_{Kh}$ as a desuspension of the cell complex consisting of the basepoint $\ast$, two 0-cells $x$ and $y$, and four 1-cells $a, b, c, d$ where both ends of $a$ and $c$ are the basepoint $\ast$, $b$ is attached to $\ast$ and $x$, $d$ is attached to $\ast$ and $y$.

Thus, $\tilde{X}_{Kh}(U) = \Sigma^{-1}(S^1_a \lor S^1_c \lor D^1_b \lor D^1_d) = S^0 \lor S^0$. In the splitting

$$\tilde{X}_{Kh}(U) = \bigvee_{q, h} \tilde{X}_{Kh}^{q, h}(U)$$

we have $\tilde{X}_{Kh}^{1, [\alpha]}(U) = \tilde{X}_{Kh}^{-1, -[\alpha]}(U) = S^0$, the other $\tilde{X}_{Kh}^{q, h}(U)$ are trivial.

Case 2 (see Fig. 8.3, upper left). Let the composition $\alpha \beta$ be contractible and $\alpha$ be non-contractible. Then $[\beta] = [\alpha^{-1}] = [\alpha] \neq 0$ and $[\alpha \beta] = 0$. Hence, $x \prec b$, $x \prec c$, and the other generators are incomparable. Then we have

$$\tilde{X}_{Kh}^{1, 2[\alpha]}(U) = \tilde{X}_{Kh}^{-1, 0}(U) = \tilde{X}_{Kh}^{-3, -2[\alpha]}(U) = S^0, \tilde{X}_{Kh}^{-3, 0}(U) = \Sigma^{-1}(S^0),$$

the other $\tilde{X}_{Kh}^{q, h}(U)$ are trivial. Thus, $\tilde{X}_{Kh}(U) = S^0 \lor S^0 \lor S^0 \lor \Sigma^{-1}(S^0)$.
Case 3 (see Fig. 8.3 lower). Let the $\alpha$, $\beta$ and $\alpha\beta$ be non-contractible. Then all the generators have different gradings and are incomparable. Hence, each generator yields a nontrivial space in the bouquet decomposition of $X_{Kh}(U) = (\Sigma^{-1}S^0)^{\vee 2} \vee (S^0)^{\vee 4}$:

$$X_{Kh}^{1, [\alpha]+[\beta]}(U) = X_{Kh}^{-1, [\alpha]-[\beta]}(U) = X_{Kh}^{-1, -[\alpha]+[\beta]}(U) = X_{Kh}^{-3, -[\alpha]-[\beta]}(U) = S^0,$$

$$X_{Kh}^{-1, [\alpha\beta]}(U) = X_{Kh}^{-3, -[\alpha\beta]}(U) = \Sigma^{-1}(S^0).$$

8.2. links whose link diagrams have three crossings. Consider the oriented link in a surface of genus 2 drawn in Fig. 8.4. The diagram has two positive and one negative crossings.

The link as an embedded graph consists of a black segment which can be contracted to a point, and four arcs, that generate the fundamental group of the surface, see Fig. 8.5. We denote these arcs as $\alpha$, $\beta$, $\gamma$, $\delta$.

Consider the resolution cube of the diagram (Fig. 8.6). The homotopy classes of the circles in the resolution configurations are all different. Hence, all the labeled resolution
configurations are incomparable by the partial order, and there are no nontrivial decorated resolution configurations. Then the differential in the homotopical Khovanov complex is zero, and the homotopical Khovanov homology coincides with the chain complex. The Khovanov–Lipshitz–Sarkar homotopy type is a bouquet of spheres corresponding to the labeled resolution configurations

$$\mathcal{X}_{Kh} = (\Sigma^{-1} S^0)^{\vee 2} \vee (S^0)^{\vee 8} \vee (S^1)^{\vee 6} \vee (S^2)^{\vee 4}.$$  

The dimension of the spheres is determined by the homological grading of the resolution configuration.
The example above is opposite to the classical case in some sense. For classical links, the homotopical grading does not matter because all circles are contractible. And in this case the homotopical grading brakes all connections between resolution configurations. Note that among the twelve surgeries in the resolution cube six are single circle surgeries.

9. Open questions

Links in thickened surfaces are regarded as virtual links ([5, 6, 7]). Recalling Note 1.2 it is natural to ask a question: Compare the strength of the following invariants.

1. The Khovanov homology for virtual links,
2. The second Steenrod square for virtual links in conjunction with Khovanov homology for virtual links
3. The Khovanov homology for links in thickened surfaces
4. The second Steenrod square for links in thickened surfaces in conjunction with Khovanov homology for links in thickened surfaces
5. The Khovanov-Lipshitz-Sarkar homotopy type for links in thickened surfaces

In [8], it is proved that (2) is stronger than (1).

In this paper we prove the following: (4) is stronger than (3) if the genus is greater than one. (5) is stronger than (3) if the genus is greater than one.

In the case of the thickened torus, we shall deal with it in a separate paper [9]. We have not defined Khovanov-Lipshitz-Sarkar stable homotopy type for virtual links. How does one introduce them?

Can we combine the Steenrod square (and Khovanov homotopy theory) in a single theory that would apply to both links in thickened surfaces and to virtual links?

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\[ L = \]

\[
\begin{align*}
n_+ &= 1 \\
n_- &= 1
\end{align*}
\]
