ON BESTVINA-MESS FORMULA

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Abstract.

Bestvina and Mess [BM] proved a remarkable formula for torsion free hyperbolic groups

$$\dim_L \partial \Gamma = cd_L \Gamma - 1$$

connecting the cohomological dimension of a group $\Gamma$ with the cohomological dimension of its boundary $\partial \Gamma$. In [Be] Bestvina introduced a notion of $\mathbb{Z}$-structure on a discrete group and noticed that his formula holds true for all torsion free groups with $\mathbb{Z}$-structure. Bestvina’s notion of $\mathbb{Z}$-structure can be extended to groups containing torsion by replacing the covering space action in the definition by the geometric action. Though the Bestvina-Mess formula trivially is not valid for groups with torsion, we show that it still holds in the following modified form: The cohomological dimension of a $\mathbb{Z}$-boundary of a group $\Gamma$ equals its global cohomological dimension for every PID $L$ as the coefficient group

$$\dim_L \partial \Gamma = \gcd_L(\partial \Gamma).$$

Using this formula we show that the cohomological dimension of the boundary $\dim_L \partial \Gamma$ is a quasi-isometry invariant of a group.

For CAT(0) groups and $L = \mathbb{Z}$ these results were obtained in [GO].

§1 Introduction

A closed subset $Z$ of an $AR$-space $\bar{X}$ is called a $Z$-set if there is a homotopy $H : \bar{X} \times [0,1] \to \bar{X}$ such that $H_0$ is the identity and $H_t(\bar{X}) \subset \bar{X} \setminus Z$ for all $t > 0$. An equivalent statement is that $Z$ is a $Z$-set if for every open set $U \subset \bar{X}$ the inclusion $U \setminus Z \subset U$ is a homotopy equivalence.

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The following generalizes Bestvina’s definition from [Be].

**DEFINITION.** A discrete group $\Gamma$ has $\mathbb{Z}$-structure if there is a pair $(\bar{X}, Z)$ of compact spaces satisfying the axioms:

1. $\bar{X}$ is AR;
2. $Z$ is a $Z$-set in $\bar{X}$;
3. $X = \bar{X} \setminus Z$ is a metric space that admits a proper discontinuous action of $\Gamma$ by isometries with a compact quotient;
4. The compactification $\bar{X}$ of $X$ is Higson dominated.

Originally Bestvina required in (3) a covering space action which automatically makes $\Gamma$ torsion free. Note that for torsion free groups this definition of $\mathbb{Z}$-structure coincides with the original one.

The Higson compactification $\bar{X}_H$ [Roe] of a metric space $X$ is defined by means of the ring $C_H(X)$ of bounded continuous functions $f : X \to \mathbb{R}$ with variation tending to zero at infinity. Namely, given $R < \infty$ and $\epsilon > 0$, there is a compact set $K \subset X$ such that $\text{diam}(f(B_R(x))) < \epsilon$ for $x \in X \setminus K$ where $B_R(x)$ denotes the $R$-ball centered at $x$. A compactification $\bar{X}$ of $X$ is called *Higson dominated* if the identity map $id_X$ extends to a continuous map $\alpha : \bar{X}_H \to \bar{X}$.

For a group $\Gamma$ with $\mathbb{Z}$-structure $(\bar{X}, Z)$ we will denote the set $Z$ as $\partial \Gamma$ and call it a *boundary of the group* $\Gamma$.

For a set $A \subset X$ the trace $A'$ of $A$ on the boundary $Z$ is the intersection with $Z$ of the closure of $A$ in $\bar{X}$,

$$A' = \text{Cl}_{\bar{X}} A \cap Z.$$ 

The Axiom 4 can be restated as follows:

4. For every set $A \subset X$ and every $r > 0$ the trace of $A$ in $Z$ coincides with the trace of the $r$-neighborhood $N_r(A)$:

$$A' = (N_r(A))'.$$

We note that the condition (4) is equivalent to the condition that the action of $\Gamma$ is *small at infinity* [DF] i.e., for every $x \in Z$ and a neighborhood $U$ of $x$ in $\bar{X}$, for every compact set $C \subset X$ there is a smaller neighborhood $V$, $x \in V \subset U$, such that $g(C) \cap V \neq \emptyset$ implies $g(C) \subset U$ for all $g \in \Gamma$. So we can reformulate Axiom 4 once more:

4. For every $x \in X$ and every $r > 0$ the sequence $\{B_r(\gamma(x)) \mid \gamma \in \Gamma \}$ is a 0-sequence in $\bar{X}$.

The Bestvina-Mess formula proven first for torsion free hyperbolic groups [BM] as it noted in [Be] is valid for all torsion free groups with $\mathbb{Z}$-structure. It states

$$\dim_L Z = cd_L \Gamma - 1$$
where $L$ is a PID, $\dim_L Z$ is the cohomological dimension of a space $Z$ with coefficients in $L$, and $cd_L \Gamma$ is the cohomological dimension of a group $\Gamma$ with coefficient in $L$. For a torsion free group $\Gamma$ with a $\mathcal{Z}$-structure $(\bar{X}, Z)$ we have [Be]

$$cd_L \Gamma = \max\{n \mid H^n_{\mathcal{C}}(X; L) \neq 0\}.$$ We recall that a global cohomological dimension of a space $X$ with coefficients in an abelian group $G$ is the following number

$$gcd_G(X) = \max\{n \mid H^n(X; G) \neq 0\}.$$ Since $\bar{X}$ is contractible, the exact sequence of the pair $(\bar{X}, Z)$ implies that

$$cd_L \Gamma - 1 = gcd_L(Z).$$ Thus, the Bestvina-Mess formula is equivalent to the equality

$$dim_L Z = gcd_L(Z).$$ The purpose of this paper is to prove this equality for general groups $\Gamma$ with $\mathcal{Z}$-structure. We do it by an adjustment of Bestvina-Mess’ argument. In the case when $\Gamma$ is a CAT(0) group and $L = \mathcal{Z}$ this formula was established by Geoghegan and Ontaneda [GO] by a different method.

§2 Global and local cohomological dimension of a $\mathcal{Z}$-boundary

The main result is the following

**Theorem 1.** Let $(\bar{X}, Z)$ be a $\mathcal{Z}$-structure on a group $\Gamma$. Then for every principle ideal domain $R$ the cohomological dimension of $Z$ coincides with its global cohomological dimension:

$$dim_R Z = gcd_R(Z).$$ We recall that the cohomological dimension with respect to the coefficient group $G$ of a locally compact metric space $Z$ is the following number

$$dim_G Z = \max\{n \mid H^n(U; G) \neq 0, U \subset_{\text{open}} Z\}$$ where $H^*_{\mathcal{C}}$ denotes the cohomology with compact supports.

Let $\dim_G X = r$. We say that a point $x \in X$ is dimensionally essential (with respect to $G$) if there is a neighborhood $U$ of $x$ such that for every smaller neighborhood $V$ of $x$ the image of the inclusion homomorphism $i_{V,U} : H^*_{\mathcal{C}}(V; G) \to H^*_{\mathcal{C}}(U; G)$ is nonzero. Alexandroff called such points as containing an $r$-dimensional obstruction. We refer to [Dr] for more details on cohomological dimension as well for the following Lemma.
Lemma 1. (P.S. Alexandroff) Let $X$ be a compact metric space with $\dim_G X = r$. Then the set of dimensionally essential points in $X$ contains a locally compact subset $Y$ of $\dim_G Y = r$.

Proof. Let $W$ be an open subset of $X$ with $H^r_c(W; G) \neq 0$. Because of the continuity of cohomology there is a closed in $W$ set $Y$ minimal with respect the property: The inclusion homomorphism $H^r_c(W; G) \to H^r_c(Y; G)$ is nonzero. Then, clearly, $\dim_G Y = r$. For every $x \in Y$ we take $U = W$. Let $V \subset U$ be a neighborhood of $x$. Consider the diagram generated by exact sequences of pairs $(U, U \setminus V)$ and $(Y, Y \setminus V)$.

\[
\begin{array}{ccc}
H^r_c(V; G) & \xrightarrow{i_{V,U}} & H^r_c(U; G) \\
\downarrow j_{V,V\cap Y} & & \downarrow j_{U,V} \\
H^r_c(V \cap Y; G) & \xrightarrow{i_{V,Y \cap Y}} & H^r_c(Y; G) & \xrightarrow{j_{Y,Y \setminus V}} & H^r_c(Y \setminus V; G)
\end{array}
\]

Let $\alpha \in H^r_c(U; G)$ such that $j_{U,Y}(\alpha) \neq 0$. Since $Y$ is minimal, $j_{Y,Y \setminus V}(j_{U,Y}(\alpha)) = 0$. The exactness of the bottom row implies that there is $\beta \in H^r_c(Y \cap V; G)$ such that $i_{Y \cap V,Y}(\beta) = j_{U,Y}(\alpha)$. Since $\dim_G V \leq r$, the homomorphism $j_{V,Y \cap V}$ is an epimorphism and hence there is $\gamma \in H^r_c(V; G)$ with $j_{V,Y \cap V}(\gamma) = \beta$. Therefore $j_{U,Y}i_{V,U}(\gamma) \neq 0$ and hence $i_{V,U}(\gamma) \neq 0$. □

The following lemma is contained essentially in [BM] (Proposition 2.6).

Lemma 2. Suppose that $Z$ is a $Z$-set in an AR space $\bar{X}$ and $\dim_R Z = n$ where $R$ is an abelian group. Then for every dimensionally essential point $z \in Z$ and for every neighborhood $W \subset \bar{X}$ there is an open neighborhood $\bar{U} \subset \bar{X}$, $z \in \bar{U} \subset W$, such that the coboundary homomorphism for the pair $(\bar{U}, \bar{U} \cap Z)$

\[H^n_c(V; R) \to H^{n+1}_c(U; R)\]

is nonzero monomorphism where $V = \bar{U} \cap Z$ and $U = \bar{U} \setminus V$.

Proof. Given $W$ we take open in $Z$ set $V$, $Cl(V) \subset W$, with $H^n_c(V; R) \neq 0$. Let $A = Z \setminus V$. In view of the $Z$-set condition there is a homotopy $F : \bar{X} \times [0,1] \to \bar{X}$ such that

1. $F_0$ is the identity;
2. $F_1|_A$ is the inclusion $A \subset Z$;
3. $F_t(\bar{X} \setminus A) \subset \bar{X} \setminus Z$ for all $t > 0$;
4. $\bar{X} \setminus W \subset F_1(\bar{X})$. 
Let \( Y_1 = F_1(X) \) and \( Y_2 = F(Y_1 \times [0,1]) \). Then \( Y_1 \subset Y_2 \), \( Y_1 \cap Z = Y_2 \cap Z = A \). Since the inclusion \((\bar{X},Y_1) \to (\bar{X},Y_2)\) can be homotoped to the map to \((Y_2,Y_2)\), the homomorphism \( H^n(\bar{X},Y_2;R) \to H^n(\bar{X},Y_1;R) \) is zero. Consider the diagram

\[
\begin{array}{ccc}
H^n(\bar{X},Y_2;R) & \longrightarrow & H^n(Z \cup Y_2,Y_2;R) \\
0 & \downarrow & \simeq \\
H^n(\bar{X},Y_1;R) & \longrightarrow & H^n(Z \cup Y_1,Y_1;R)
\end{array}
\]

The diagram chasing shows that \( j \) is a monomorphism. We define \( \bar{U} = X \setminus Y_2 \). Then \( U = \bar{U} \setminus Z \), \( V = \bar{U} \cap Z \), \( H^n(Z \cup Y_2,Y_2;R) = H^n(\bar{V};R) \), \( H^n(\bar{X},Z \cup Y_2) = H^n(\bar{U};R) \), and \( j \) is the coboundary homomorphism from the exact sequence of the pair \((\bar{U},V)\). Since \( j \) is a monomorphism and \( H^n(\bar{V};R) \neq 0 \), it is nonzero. \( \square \)

We recall that a metric space \( X \) is uniformly contractible if there is a function \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \) with the property that every ball \( B_r(x) \) of radius \( r \) centered at \( x \) is contractible inside the ball \( B_{\rho(r)}(x) \). A metric space \( X \) has bounded geometry if there is \( \epsilon > 0 \) such that for every \( R < \infty \) the \( \epsilon \)-capacity of \( R \)-balls \( B_R(x) \) is uniformly bounded from above. The following is obvious.

**Proposition 1.** Let \( \Gamma \) act properly discontinuously and cocompactly by isometries on a proper metric AR-space \( X \) then \( X \) is uniformly contractible with bounded geometry.

**Corollary 1.** Let \( X \) as in Proposition 1, then \( X \) admits a uniformly bounded open cover \( \mathcal{U} \) with finite dimensional nerve \( K = N(\mathcal{U}) \) and a left proper homotopy inverse \( s : K \to H \) to the projection to the nerve \( \psi : X \to K \) with bounded homotopy \( H : X \times I \to X \) joining \( s \circ \psi \) with the identity \( id_X \).

**Proof.** This is a standard fact of a coarse geometry and it can be found in [HR].

We denote by \( \hat{B}_r(x) \) an open ball around \( x \) of radius \( r \).

**DEFINITION.** [GO] Let \( X \) be a locally compact metric space. The group \( H^i_c(X;G) \) is uniformly trivial if there is a function \( \nu : \mathbb{R}_+ \to \mathbb{R}_+ \) such that the inclusion homomorphism

\[
H^i_c(\hat{B}_r(x);G) \to H^i_c(\hat{B}_{\nu(r)}(x);G)
\]

is trivial for all \( x \in X \) and \( r > 0 \).

**Proposition 2.** Let \( \Gamma \) act properly discontinuously and cocompactly by isometries on a proper metric AR-space \( X \) and let \( H^1_c(X;R) = 0 \) for a PID \( R \). Then \( H^1_c(X;R) = 0 \) is uniformly trivial.
Proof. In view of cocompactness of the action, the orbit $\Gamma x_0$ is an $r_0$-dense in $X$. Hence it suffices to prove that for every $r$ there is $\nu > 0$ such that the homomorphism

$$H_c^i(\hat{B}_r(x_0); R) \to H_c^i(\hat{B}_\nu(x_0); R)$$

is trivial. Let $\psi : X \to K$ and $s : K \to X$ be from Corollary 1 and let $d$ be an upper bound on a homotopy between $s \circ \psi$ and $id_X$. For a subset $A \subset K$ we denote by $st(A)$ the union of all simplices that have nonempty intersection with $A$. Let $L = st(\psi(X \setminus B_{r+2d}(x_0)))$. Then the inclusion $X \setminus \hat{B}_{r+2d}(x_0) \to X \setminus \hat{B}_r(x_0)$ is homotopy factored through the maps

$$X \setminus \hat{B}_{r+2d}(x_0) \xrightarrow{\psi} L \xrightarrow{s} X \setminus \hat{B}_r(x_0).$$

Therefore the inclusion homomorphism $H_c^i(\hat{B}_r(x_0); R) \to H_c^i(\hat{B}_{r+2d}(x_0); R)$ is factored through the group $H^i(K, L; R)$ which is a finitely generated $R$-module. Let $a_1, \ldots, a_m \in H^i(K, L; R)$ be a set of generators. For every $a_j$ there is $\nu_j$ such that the element $\psi^*(a_j)$ vanishes in $H_c^i(B_{\nu_j}(x_0); R)$. We take $\nu = \max\{\nu_j\}$. □

The following Lemma is an extension of a claim that appeared in the proof of Corollary 1.4.(a) [BM].

**Lemma 3.** Let $\Gamma$ act properly discontinuously and cocompactly by isometries on a proper metric AR-space $X$ and let $R$ be a PID. Suppose that $H_c^i(X; R) = 0$ for all $i > n$. Then there is a number $r$ such that for every open set $U \subset X$ the inclusion $U \subset N_r(U)$ into the open $r$-neighborhood induces the zero homomorphism $H_c^i(U; R) \to H_c^i(N_r(U); R)$ for all $i > n$.

**Proof.** We consider a sequence of locally finite uniformly bounded open coverings of $X$, $U_0 \succ U_1 \succ U_2 \succ U_3 \ldots$ where $U_{k+1}$ is a refinement of $U_k$, with $\lim_{k \to 0} mesh(U_k) = 0$, and $U_0$ as in Corollary 1. Let $\psi_j : X \to K_j$ denote a projection to the nerve of $U_j$. Let $s : K_0 \to X$ be a proper homotopy lift and let $d > 0$ be the size of a homotopy between $s \circ \psi_0$ and $id_X$.

Let $U \subset X$ be an open set and let $L = st(\psi_0(X \setminus N_{2d}(U)))$. Then the homotopy commutative diagram

$$
\begin{array}{c}
X \setminus N_{2d}(U) \xrightarrow{\psi_0} X \setminus \psi_0^{-1}(L) \xrightarrow{\psi_0} X \setminus U \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \uparrow \\
L \xrightarrow{=} L \xrightarrow{=} L
\end{array}
$$

$$
\begin{array}{c}
X \setminus N_{2d}(U) \xrightarrow{\psi_0} X \setminus \psi_0^{-1}(L) \xrightarrow{\psi_0} X \setminus U \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \uparrow \\
L \xrightarrow{=} L \xrightarrow{=} L
\end{array}
$$
defines the commutative diagram for cohomology

\[
\begin{array}{ccc}
H_c^i(N_{2d}(U); R) & \xleftarrow{\psi_0^{-1}} & H_c^i(K_0 \setminus L; R) \\
\uparrow \psi_0 & & \downarrow s^* \\
H_c^i(K_0 \setminus L; R) & \xleftarrow{s} & H_c^i(K_0 \setminus L; R).
\end{array}
\]

Therefore it suffices to prove Lemma for open sets \( U \) of the type \( \psi_0^{-1}(K_0 \setminus L) \) where \( L \) is a subcomplex in \( K_0 \), and for cohomology classes that came from \( H_c^i(K_0 \setminus L; R) = H^i(K_0, L; R) \).

The sequence of covers \( \{U_j\} \) defines an inverse sequence of simplicial complexes

\[
K_0 \xleftarrow{\psi_0^1} K_1 \xleftarrow{\psi_1^2} K_2 \xleftarrow{\psi_2^3} \ldots
\]

with the limit space \( X \) and with \( \psi_j = \psi_{j+1}^j \circ \psi_{j+1} \) for all \( j \). We may assume that every bonding map \( \psi_{j+1}^j \) is simplicial with respect to an iterated barycentric subdivision of the target \( K_j \). If \( A = \operatorname{lim}\{L_j, \psi_{j+1}^j|\ldots\} \subset_{Cl} X \), then

\[
H_c^*(X \setminus A; R) = H^*(X, A; R) = \operatorname{lim}_{\rightarrow} H^*(K_j, L_j; R).
\]

By \( C_j^i \) we denote the group of simplicial \( R \)-cochains on \( K_j \) with compact supports. If \( L \subset K_j \) is a subcomplex, by \( C_j^i(K_j, L) \) we denote the subgroup of cochains which are zero on simplices from \( L \). For an \( i \)-simplex \( \sigma \subset K_j \) we denote by \( \phi_\sigma \) the cochain that takes \( \sigma \) to 1 \( \in R \) and which is zero on all other simplices. Then the cochains \( \phi_\sigma \) form a basis of \( C_j^i \). Simplicial approximations \( p_{j+1}^j : K_{j+1} \rightarrow K_j \) of the bonding maps \( \psi_{j+1}^j \) induce homomorphisms \( (p_{j+1}^j)^* : C_j^i \rightarrow C_{j+1}^i \). Let \( A = \operatorname{lim}_{\rightarrow}\{L_j, \psi_{j+1}^j|\ldots\} \) be the limit of inverse sequence of subcomplexes \( L_j \subset K_j \). By the construction of a simplicial approximation \( p_{j+1}^j(L_{j+1}) \subset L_j \). The induced homomorphism on cohomology level

\[
(p_{j+1}^j)^* : H^*(K_j, L_j; R) \rightarrow H^*(K_{j+1}, L_{j+1}; R)
\]

coincides with

\[
(\psi_{j+1}^j)^* : H^*(K_j, L_j; R) \rightarrow H^*(K_{j+1}, L_{j+1}; R)
\]

because the maps

\[
p_{j+1}^j, \psi_{j+1}^j : (K_{j+1}, L_{j+1}) \rightarrow (K_j, L_j)
\]
are homotopic. Since the homology of a direct limit of chain complexes equals the direct limit of homologies, the limit group $H^i_c(X \setminus A; R)$ of the direct system

{$\{H^i(K_j, L_j; R), (p^i_j + 1)^*\}$}

of the homology groups of the cochain complexes $C^i_c(K_j, L_j)$ is the homology of the cochain complex $\lim_{\to} C^i_c(K_j, L_j)$. For very element $\phi \in \lim_{\to} C^i_c(K_j, L_j)$ we take a minimal $k = k_\phi$ such that $\phi$ comes from some $\phi_k \in C^i_c(K_k, L_k)$ and fix such $\phi_k$. Note that if $p^i_j + 1$ are surjections, then $\phi_k$ is unique. We define $supp_X \phi = \psi_1^{-1}(supp(\phi_k))$.

Let $\dim K_0 = m$. Then the Lemma is trivial for $i > m$ (for our special setting). So, if $m \leq n$, Lemma holds true. Let $m > n$. By Proposition 2 the group $H^i_c(X; R)$ is uniformly trivial for $n < i \leq m$. Let $\nu : \mathcal{R}_+ \to \mathcal{R}_+$ be a corresponding function from for all $i$ between $n$ and $m$. Let $\rho(t) = \nu(2t)$. We use notation $\rho^k = \rho \circ \cdots \circ \rho$ for the $k$th iteration. Let $p^i_j : C^i_c \to \lim_{\to} C^i_c$ denote the natural homomorphism. Starting from $i = m$ by induction for $i > n$ we construct a sequence of homomorphisms

$A_i : C^i_0 \to \lim_{\to} C^i_c$

such that

1. Being restricted to the subgroup of cocycles $Z^i_0 \subset C^i_0$, the homomorphism $A_i$ is a lift of $p^i_0|Z^i_0 : Z^i_0 \to \lim_{\to} C^i_c$ with respect to $\delta : \lim_{\to} C^i_c \to \lim_{\to} C^i_c$;
2. $supp_X A_i(\phi_\sigma) \subset \hat{B}_{\rho^m_{i+1}(d)}(x)$ where $\psi_0(x) \in \sigma$.

Construction of $A_m$: For every $m$-simplex $\sigma \subset K_0$ we consider the inverse sequence

$K_0 \setminus \hat{\sigma} \leftarrow (\psi_0^1)^{-1}(K_0 \setminus \hat{\sigma}) \leftarrow (\psi_0^2)^{-1}(K_0 \setminus \hat{\sigma}) \leftarrow \ldots$

of simplicial complexes with the limit space $Y \subset X$. Note that the set $X \setminus Y$ is contained in an open $d$-ball $\hat{B}_d(x)$ with $\psi_0(x) \in \sigma$. Since $H^m_c(X; R)$ is uniformly trivial with the function $\nu$, the inclusion $\hat{B}_d(x) \subset B_{\nu(d)}(x)$ induces zero homomorphism. Thus, the cohomology class $[\phi_\sigma] \in H^m_c(K_0, K_0 \setminus \hat{\sigma}; R)$ defines the class $(\psi_0^{-1})^*[\phi_\sigma] \in H^i_c(X \setminus Y; R)$ which goes to zero under this inclusion $X \setminus Y \subset B_{\nu(d)}(x)$. Taking into account the direct limit description of the cohomologies we can say that there is $j$ such that $(\psi_j^2)^*[\phi_\sigma]$ goes to zero in $H^i_c(K_j, K \setminus st(\psi_j(B_{\nu(d)}(x)))); R)$. We can rephrase this on the cochain level as follows: There is a cochain $a \in C^{m-1}_j$ supported in $\psi_j(B_{\nu(d)}(x))$ such that $\delta a = (p^0_j)^*(\phi_\sigma)$. We define $A_m(\phi_\sigma) = p^* j(a)$. Since $A_m$ is defined on the basis and $C^m_0$ is a free $R$-module, it is defined everywhere. Clearly, the conditions (1)-(2) hold.
Assume that $A_{i+1}$ is constructed. Let $\sigma$ be an $i$-dimensional simplex. We define $A_i$ on a basic cochain $\phi_\sigma$ as follows. Note that $b_\sigma = p_1^\ast(\phi_\sigma) - A_{i+1} \delta \phi_\sigma$ is a cocycle in $\lim_{\rightarrow} C^i_j$ and hence in defines a cocycle in $C^i_k$ for some $k$. For this cocycle we use the same notation $b_\sigma$. By the condition (2), $supp_X(b_\sigma) \subset 2_{2^{2^m-i}(d)}(x)$ where $\psi_0(x) \in \sigma$. Hence $b_\sigma$ defines a cohomology class in $H^i_1(2_{2^{2^m-i}(d)}(x); R)$. As above using uniform triviality of the cohomology group $H^i_1(X; R)$ we obtain that for some $j > k$ we have $\delta a = (p_j^\ast)^{\ast}(b_\sigma)$ for some cochain $a \in C^{n-1}_j$ with the support in $\psi_j(B_{\nu}(2_{2^{2^m-i}(d)}(x)))$. We define $A_i(\phi_\sigma) = p_j^\ast(a)$. It’s easy to verify the conditions (1)-(2).

Let $r = 2^{2^m-n}(d)$. If $L \subset K_0$ is a subcomplex and $\alpha \in C^i_0$, $i > n$, is a cocycle that represents a class in $H^i(K_0, L; R)$, then $A_i(\alpha)$ comes from a cochain $\beta \in C^{n-1}_j$ for some $j$ that cobounds $(p_j^\ast)^{\ast}(\alpha)$. By the condition (2) $\phi_j^{-1}(suppA_i(\alpha)) \subset N_r(\psi_0^{-1}(supp(\alpha)))$.

**Proof of Theorem 1.** The inequality $\dim R Z \geq gcd R(Z)$ is obvious.

We prove the reverse inequality. Let $\dim R Z = n$ and assume that $gcd R \Gamma < n$. Apply Lemma 1 and Lemma 2 to obtain $z \in Z$, an open neighborhood $V$ in $Z$ and an open set $\bar{U} \subset \bar{X}$ with $\bar{U} \cap Z = V$ such that the homomorphism $j : H^i_0(V; R) \to H^i_0(\bar{U}; R)$ is a nontrivial monomorphism. We may assume that $V$ is as in the definition of a dimensionally essential point, i.e., for every smaller neighborhood $V', z \in V' \subset V$, the inclusion homomorphism $H^i_0(V'; R) \to H^i_0(V; R)$ is non zero. Let $r$ be as in Lemma 3. We take a neighborhood $W$ of $z$ in $\bar{X}$ such that $Cl_X W \subset \bar{U}$ and $N_2r(X \setminus U) \cap W = \emptyset$. Then by Axiom 4 the $r$-neighborhood $N_r(W \setminus Z)$ is contained in $U$. We apply Lemma 2 to obtain an open neighborhood of $z$, $\bar{U}_0 \subset W$, such that

$$j' : H^i_0(V_0; R) \to H^i_0(U_0; R)$$

is a nonzero monomorphism where $V_0 = \bar{U}_0 \cap Z$ and $U_0 = \bar{U}_0 \setminus Z$. There is an element $\alpha$ which goes to a non zero element under the inclusion homomorphism $H^i_0(V_0; R) \to H^i_0(V; R)$. Since $N_{\nu_0}(W \setminus Z)$ is contained in $U$, by Lemma 3 the inclusion homomorphism $H^{n+1}_0(U_0; R) \to H^{n+1}_0(U; R)$ is trivial. The commutative diagram with injective vertical arrows

$$
\begin{array}{ccc}
H^i_0(V_0; R) & \longrightarrow & H^i_0(V; R) \\
\downarrow j' & & \downarrow j \\
H^{n+1}_0(U_0; R) & \longrightarrow & H^{n+1}_0(U; R)
\end{array}
$$

gives us a contradiction. □
Corollary 2. The cohomological dimension of the boundary $\dim_R \partial \Gamma$ for groups with $\mathcal{Z}$-structure is a quasi-isometry invariant.

Proof. Let $\Gamma_1$ and $\Gamma_2$ be two groups with $\mathcal{Z}$-structures $(\bar{X}_1, Z_1)$ and $(\bar{X}_2, Z_2)$ respectively. By Theorem 1 and the cohomology exact sequence of pairs $(\bar{X}_i, Z_i)$ it suffices to show that $H^*_c(X_1; R) = H^*_c(X_2; R)$. Since the spaces $X_i$, $i = 1, 2$, are uniformly contractible, by a theorem of Roe $H^*_c(X_i; R) = HX^*(X_i; R)$ [Roe], [HR] where $HX^*(X; R)$ denotes the coarse cohomology groups of $X$. The chain of coarse equivalences

$$X_1 \sim \Gamma_1 \sim \Gamma_2 \sim X_2$$

completes the proof. □

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