Lambda-proton and lambda-neutron potentials within Gel’fand-Levitan-Marchenko theory

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Abstract. Spin-dependent potentials for lambda-proton and lambda-neutron interactions are recovered from scattering data using Gel’fand-Levitan-Marchenko inversion theory. Due to experimental difficulties arising from the short lifetimes of hyperons, available experimental scattering data has poor statistics. Theoretical data is therefore used as input in the inversion scheme. By using rational-function approximations to extrapolate the data, so as to ensure well-posedness of the inverse problem, new spin-dependent lambda-proton and lambda-neutron potentials are constructed using Gel’fand-Levitan-Marchenko inversion theory. These new potentials retain all the main features of a baryon-baryon interaction, in addition to certain distinctive features whose effects may show up in Schrödinger calculations.

Keywords: lambda-proton potential, lambda-neutron potential, hyperon-nucleon potential, fixed-angular momentum inversion, Gel’fand-Levitan-Marchenko equation, inverse scattering

1. Introduction

The recovery of a Sturm-Liouville operator from some of its spectral properties is a problem that arises in many contexts within the mathematical sciences. In quantum mechanics, this arises in scattering theory applications, where the inverse problem consists of restoring the Schrödinger operator from scattering data. In cases of spherical symmetry, the radial inverse scattering problem is an inverse Sturm-Liouville problem on the half-line $r \in [0, \infty)$ for the Schrödinger operator $[1]$:

$$L^{(\ell)} = -\frac{d^2}{dr^2} + V(r) + \frac{\ell(\ell + 1)}{r^2}$$

where $L^{(\ell)} \psi = E \psi$, with $E = k^2$ being the energy, $p = k$ the momentum and $\psi$ the radial wavefunction. The system of units used is such that $\hbar = 2\mu = 1$.

The foundations of inverse Sturm-Liouville problems and their applications to quantum scattering were developed in the 1940s through the work of Borg [2,3], Povzner [4], Levitan [5], Bargmann [6,7] and Levinson [8,9], among others. The goal of this study is to determine the operator $L^{(\ell)}$ from scattering phases, $\delta_\ell(k)$. This is carried out in the cases where $V(r)$ is lambda-proton and the lambda-neutron potentials.

The commonly used lambda-proton and the lambda-neutron potentials have their roots in meson theory [10], quark theories [11] and chiral effective field theory [12]. The results
from few-body calculations, using these potentials, have shown some significant differences with experimental observations. For example, the computed lifetimes of the lambda hypertriton using existing lambda-nucleon potentials are about 30 - 50% longer [13] than the recently observed values [14]. Furthermore, charge symmetry breaking in the lambda-nucleon interaction has not been satisfactorily incorporated into the existing lambda-nucleon potentials [15, 16]. These discrepancies suggest the need for more theoretical effort towards understanding the hyperon-nucleon interaction. In this project, this is the motivating reason for proposing Gel’fand-Levitan-Marchenko theory as an alternative to existing theories. A stronger and more diversified theoretical base is key towards making progress in understanding the lambda-nucleon force.

The paper has the following organisation: Sections 2 and 3 outline Gel’fand-Levitan-Marchenko theory and the interpolation method for scattering matrix, respectively. Sections 4 and 5 are devoted to lambda-scattering scattering experiments and theory, respectively. Section 6 carries the results of the application of Gel’fand-Levitan-Marchenko theory to lambda-nucleon scattering. Concluding remarks are presented in section 7.

2. Single-channel Gel’fand-Levitan-Marchenko equation

Transformation operators [17, 18] that act on the regular solutions or the Jost solutions of the Schrödinger equation play a central role in inverse scattering. The application of these transformation operators on the solutions of the Schrödinger equation results in a Povzner-Levitan integral representation [4, 5] for these solutions. By making use of the Povzner-Levitan representation for the Jost solutions, one arrives at the single-channel Gel’fand-Levitan-Marchenko (GLM) equation [19, 20]:

\[
K_{\ell}(r,r') + A_{\ell}(r,r') + \int_{r}^{\infty} K_{\ell}(r,s)A_{\ell}(s,r')ds = 0, \quad r \leq r'
\]  

where \(K_{\ell}(r,r')\) is a kernel related to the potential \(V(r)\). The kernel \(A_{\ell}(r,r')\) is computed from the continuous and discrete spectrum as follows [21–24]:

\[
A(r,r') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^{\pm}_{\ell}(k,r) \{1 - S_{\ell}(k)\} \omega^{\pm}_{\ell}(k,r')dk + \sum_{i=1}^{n_{\ell}} M_{i}\omega^{\pm}_{\ell}(k,r)\omega^{\pm}_{\ell}(k,r')
\]

where \(S_{\ell}(k)\) is the scattering matrix, \(\omega^{\pm}_{\ell}(k,r)\) are Ricatti-Hankel functions, \(n_{\ell}\) is the number of bound states and \(M_{i}\) are normalisation constants for each of the bound state. The first term in Equation (3) is the contribution from the positive eigenvalues of the Schrödinger operator while the second term is the contribution from the negatives eigenvalues. Generally, there is no guiding physical law for choosing the norming constants \(M_{i}\) for each bound state. Therefore, in cases where there are negative eigenvalues, this means instead of a unique potential, one ends up with a set of phase-equivalent potentials [23], which may even be isospectral [9]. A unique potential may only be obtained from GLM theory if there are no negative eigenvalues.

The GLM equation is solved to obtain the output kernel \(K_{\ell}(r,r')\). By making use of the boundary condition for the Goursat problem satisfied by \(K_{\ell}(r,r')\) [1, 26], the potential \(V_{\ell}(r)\) is obtained from the diagonal entries in \(K_{\ell}(r,r')\) i.e.

\[
-2\frac{d}{dr}K_{\ell}(r,r') = V_{\ell}(r)
\]

The separability of the kernels \(K_{\ell}(r,r')\) and \(A_{\ell}(r,r')\) determine whether the GLM will have a closed-form solution or a numerical scheme is needed. In the following section, the properties of the S-matrix and how they affect the separability of these kernels are discussed. From this point going forward, the discussion shall be restricted to the case \(\ell = 0\) (the s-waves), which is of interest in this paper.
3. Interpolation of the scattering matrix
As may be seen from Equation (3), GLM theory requires knowledge of \( S_0(k) \) at all momenta. However, due to limitations in experimental set-up, \( S_0(k) \) is only known for a smaller momentum range \( k \in [k_{\text{min}}, k_{\text{max}}] \). With such a limited scattering matrix, the inverse problem becomes ill-posed. In other words, the inverse problem results in an underdetermined linear system, with fewer constraints than degrees of freedom. In order to provide extra constraints, the scattering matrix is interpolated by a function \( \tilde{S}_0(k) \). Using Bargmann rational function representations of the Jost function [6], one suitable function may be shown to be a ratio of two \( N^{th} \) degree polynomials [27–29]:

\[
\tilde{S}_0(k) = \prod_{n=1}^{N} \left( \frac{k + \alpha_0^n}{k - \alpha_0^n} \right) \left( \frac{k - \beta_0^n}{k + \beta_0^n} \right)
\]  

(5)

where \( \alpha_0^n \) and \( \beta_0^n \) are complex numbers representing the zeros and poles of the Jost functions. For uniformity, one may use the same symbol, \( a_0^m \), to represent these zeros and poles, resulting in the following parametrization [27–29]:

\[
\tilde{S}_0(k) = \prod_{m=1}^{M} \left( \frac{k + a_0^m}{k - a_0^m} \right), \quad \text{where} \quad M = 2N
\]  

(6)

The numerical method used in estimating the \( M \) constants \( a_0^m (m = 1, \ldots, M) \) has been outlined in [28,29]. Using \( \tilde{S}_0(k) \), the integral in Equation (3) for the input kernel is computed through the Cauchy Residue Theorem as shown in [30,31]. The GLM equation is then solved as in [30–33]. In the following sections, single-channel GLM theory is applied to lambda-nucleon scattering.

4. Experimental \( \Lambda \)-nucleon scattering data
Scattering experiments in which free \( \Lambda \) baryons are used either as projectiles are very difficult to perform due to the very short lifetime of hyperons. The reverse kinematics is also difficult for the same reason. In the case of elastic scattering both the \( \Lambda \) and the nucleon emerge in final quantum states that are the same as their initial states:

\[
\Lambda + p \rightarrow \Lambda + p
\]  

(7)

\[
\Lambda + n \rightarrow \Lambda + n
\]  

(8)

However, in nonelastic scattering, the particles may emerge in new quantum states or new particles may be formed. For \( \Lambda + p \) scattering above threshold, the following are some nonelastic channels that may sometimes be coupled to the elastic channel in the \( \Lambda \) momentum range \( 1 – 10 \) GeV [34]:

\[
\Lambda + p \rightarrow \Sigma^0 + p
\]  

(9)

\[
\Lambda + p \rightarrow \Sigma^+ + n
\]  

(10)

\[
\Lambda + p \rightarrow \Sigma^+ + p + \pi^-
\]  

(11)

\[
\Lambda + p \rightarrow \Sigma^- + p + \pi^+
\]  

(12)

\[
\Lambda + p \rightarrow \Lambda + p + \pi^+ + \pi^-
\]  

(13)

The difficulty in scattering experiments arises because a free \( \Lambda \) has a mean lifetime of only \( 2.63\times10^{-10} \) seconds [35]. This is a very short lifetime, especially when compared to a free proton that does not decay and a free neutron that has a lifetime of about 881.5 seconds. Whereas the
nucleon-nucleon scattering database has about 4000 data points, the lambda-nucleon database has only about 40 data points and the hyperon-hyperon database is still empty. In addition to the low number of data points, some of the hyperon-nucleon data sets come with large error bars.

The hyperon-nucleon data available are those from Λ-proton [34,36,37], Σ⁺-proton [38,40], Σ⁻-proton [38,40], Ξ⁻-proton [41,42] and Ξ⁰-proton [34,39] scattering experiments. In much of these data, which is available for low and intermediate energies, the number of hyperon-nucleon scattering events is very low; most have less than 600 events. No experimental data has been reported for hyperon-neutron scattering.

5. Theoretical Λ-nucleon scattering data
As a result of the limited experimental scattering data, one therefore has to resort to using theoretical or simulated data, in order to use the powerful theory of quantum inversion to probe the lambda-nucleon force. The use of theoretical data played a very instrumental role in understanding the nucleon-nucleon interaction. For example in [31] phase shifts computed from the Reid soft core potential were used in restoring the nucleon-nucleon potential.

![Figure 1: Λp phase shifts in (a) 1S0 channel and (b) 3S1 channel. The nonelastic threshold is 640 MeV.](image1)

![Figure 2: Λn phase shifts in (a) 1S0 channel and (b) 3S1 channel. The nonelastic threshold is 650 MeV.](image2)

In this project, theoretical 1S0 and 3S1 phase shifts computed by the Nijmegen group [43]...
are used in GLM inversion. These phase shifts were computed using the NSC97f potential [44]. Plots of these Λp and Λn theoretical phase shifts are shown in Figures 1 and 2 respectively.

In preparation for solving the GLM equation, the Levinson theorem was used to examine the scattering phases in Figures 1 and 2 in order to determine if there are any bound states. It may be observed that $\delta_0(k) \to 0$ as $k \to 0$ for all these phase shifts. The behaviour of the phase shift at infinity may be inferred from the nature of the potentials. GLM inversion scheme used here is valid for bounded potentials i.e. potentials that vanish faster than $r^{-2}$. The S-matrix must therefore behave in such a way that $S_0(k) \to 1$ as $k \to \infty$. Or, equivalently, the phase shift must vanish at very high momenta i.e. $\delta_0(k) \to 0$ as $k \to \infty$. Therefore, based on the application of the Levinson theorem on these phase shifts, the lambda-nucleon potentials do not support any bound states.

The aforementioned scattering phases go beyond threshold momenta, implying that there is a nonelastic channel. In this project, the nonelastic channels have been ignored in the inversion procedure. This approach is justified because the nonelasticity may be observed to be quite weak in the $^1S_0$ channels. Furthermore the nonelastic channels occur at a $\Lambda$ momentum above 600 MeV, which is a very high momentum (energy). The potentials constructed here will be used in few-body calculations in the low-energy range, so leaving out this nonelastic channel is not expected to cause any significant inaccuracies.

6. Results and discussion

Generally, the accuracy of the results from inversion theory is determined by the quality of the scattering data and the accuracy of the interpolation of the scattering matrix. For the interpolation function in Equation (6) $M = 20$ poles were used. The accuracy of this interpolation is of the order of $10^{-2}$, which is comparable to $10^{-1}$ obtained in [31] and $10^{-4}$ in [33]. Single-channel GLM theory was applied to the scattering phases in Figures 1 and 2. The potentials obtained for the s-wave Λp interaction are shown in Figures 3(a) and 3(b) while results for Λn interaction are shown in Figures 4(a) and 4(b).

Figure 3: Λp potential in the (a) $^1S_0$ channel and (b) $^3S_1$ channel.

The results reveal the presence of the known features of a baryon-baryon interaction: short-range repulsion and intermediate attraction. Furthermore, the $\Lambda N(^3S_1)$ potential is weaker than the $\Lambda N(^1S_0)$ potential, as expected. A distinctive feature of these potentials is that the strongest attraction occurs at a smaller radial distance than with most other lambda-nucleon potentials.
Figure 4: An potential in the (a) $^1S_0$ channel and (b) $^3S_1$ channel.

Also noticeable in these results is the presence of a small repulsion barrier, which is quite negligible in the spin triplet channels. It is reasonable to conclude that this barrier is not a Coulomb repulsion, since the $\Lambda$ baryon is neutral. As pointed out in [45], the presence of the nonelastic channel in the scattering data is known to cause oscillations in the potentials obtained from GLM theory. This barrier or nonlocality is a consequence of the nonelasticity that is present in the real part of the scattering phases.

In GLM theory, the scattering matrix must be known at momenta beyond the nonelastic threshold. Therefore, nonlocality is a physical feature of the potential obtained from inversion theory. Its effect may only be diminished by using scattering data below threshold. These oscillations or nonlocalities are very evident in potentials from neutron-deuteron doublet scattering with a small nonelastic threshold [33]. The possible effects of these nonlocalities in a few-body calculation were raised in [49]. In [47,48] it was observed that the use of nonlocal potentials for the triton did result in higher binding energies than those from local potentials. Besides these qualitative assessments on the nature of these new potentials, rigorous assessment will be carried out through few-body calculations.

7. Conclusions
Gel’fand-Levitan-Marchenko inversion theory has been applied on theoretical scattering data in restoring $s$-wave spin-dependent potentials for the lambda-proton and lambda-neutron interactions. The potentials energy-independent, spin dependent, isospin dependent. The need to introduce this theory into the hyperon-nucleon sector arose because of the observed disparities in the results obtained with the currently used hyperon-nucleon forces that have their roots in meson theory and quark theories. These potentials will be tested by computing the binding energy and root-mean-square radius of the lambda hypertriton, and by computing the lambda separation energies of isospin doublets such as helium-4-lambda and hydrogen-4-lambda.

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