Semiclassical approach to universality in quantum chaotic transport

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Abstract – The statistics of quantum transport through chaotic cavities with two leads is encoded in transport moments \(M_m = \text{Tr}[ (tt^\dagger)^m ]\), where \(t\) is the transmission matrix, which have a known universal expression for systems without time-reversal symmetry. We present a semiclassical derivation of this universality, based on action correlations that exist between sets of long scattering trajectories. Our semiclassical formula for \(M_m\) holds for all values of \(m\) and an arbitrary number of open channels. This is achieved by mapping the problem into two independent combinatorial problems, one involving pairs of set partitions and the other involving factorizations in the symmetric group.

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Introduction. – A remarkable fact about electronic transport in ballistic systems with chaotic classical dynamics is that they display universal properties like, e.g., conductance fluctuations and weak localization [1]. These properties are well described by random matrix theory (RMT) [2–4], in which the system’s details are neglected and its \(S\) matrix is modeled as a random element from an appropriate matrix ensemble, specified only by the symmetries present. It has always been a central problem to derive such universal results from a semiclassical approximation in which classical dynamics is taken into account. In this formulation, quantum universality must emerge as a result of action correlations that are present in any chaotic system: in essence, the existence of long scattering trajectories that have nearly the same action and interfere constructively in the semiclassical limit [5]. The same relation exists between long periodic orbits and spectral statistics of closed systems [6].

Consider a cavity with two leads attached, with \(N_1\) and \(N_2\) open channels supported in each lead. This is taken as a model of actual situations involving semiconductor quantum dots. Let \(t\) be the transmission block of the unitary \(S\) matrix. The transport moments \(M_m = \text{Tr}[ (tt^\dagger)^m ]\) carry much information about scattering through the system. If time-reversal symmetry is broken by a strong magnetic field, the RMT calculation of these quantities can be carried out for arbitrary values of \(N_1\) and \(N_2\) [7]. However, semiclassical derivations that agree exactly with RMT predictions are available only for the first two moments [8,9], which are related to the conductance and shot-noise of the cavity. Higher moments have been treated perturbatively in \(1/N\), where \(N = N_1 + N_2\), but only to leading [10] and next-to-leading [11] orders.

In this article we remedy this situation, providing a semiclassical derivation which is valid for all transport moments and to all orders in perturbation theory, for broken time-reversal symmetry. Starting from a diagrammatic formulation, we show how the diagrams that are relevant for \(M_m\), which involve scattering trajectories between leads, may be obtained from closed diagrams involving only periodic orbits. Then, we map them into certain factorizations of permutations in the symmetric group. The factorizations required are different from the ones in [10] (see also [12]) and apparently have not been considered before.

Semiclassical transport moments. – In the semiclassical limit \(\hbar \rightarrow 0, N \rightarrow \infty\), the matrix elements of \(t\) may be approximated [13] by

\[
t_{oi} \approx \frac{1}{\sqrt{4N_H}} \sum_{\gamma:i \rightarrow o} A_\gamma e^{iS_\gamma/\hbar},
\]

where the sum is over trajectories starting at incoming channel \(i\) and ending at outgoing channel \(o\), \(S_\gamma\) is the
action of trajectory $\gamma$ and $A_\gamma$ is an amplitude related to its stability. The prefactor contains the Heisenberg time $T_H$, which equals $N$ times the classical average dwell time. Expanding the trace, transport moments become
\[ M_m \approx \frac{1}{T_H^2} \prod_{j=1}^m \sum_{i_j, o_j} A_{\gamma_j} A_{\gamma_j}^* e^{i(S_j - S_{j+1})/\hbar}. \] (2)

The sum involves two sets of $m$ trajectories, the $\gamma$'s and the $\sigma$'s. $A_{\gamma} = \prod_j A_{\gamma_j}$ is a collective stability and $S_j = \sum_j S_{\gamma_j}$ is a collective action, and analogously for $\sigma$. Most importantly, the structure of the trace implies that these two sets of trajectories connect the channels in a different order, and we can arrange it so that $\gamma_j$ goes from $i_j$ to $o_j$, while $\sigma_j$ goes from $i_j$ to $o_{j+1}$.

The result of (2) is in general a strongly fluctuating function of the energy, so a local energy average is introduced. When this averaging is performed in the stationary phase approximation, it selects those sets of $\sigma$'s that have almost the same collective action as the $\gamma$'s. In the past 10 years [14] it has been established that if the encounter was arranged differently, with $\gamma_j$ starting and $\sigma_j$ ending at the same channels, there is freedom in choosing which is which.

We show two examples of correlated sets of trajectories in fig. 1(a), (b), both contributing to $M_3$. Naturally, this is a simplification: in a realistic chaotic system these trajectories would be long and extremely convoluted. The first example, fig. 1(a), has a triple encounter, within which the $\sigma$'s are represented by dashed lines. Before the encounter $\sigma_j$ is indistinguishable from $\gamma_j$; after the encounter it becomes indistinguishable from $\gamma_{j+1}$, i.e. the changing of partners inside the encounter happens in such a way that $\sigma_j$ starts at $i_j$ and ends at $o_{j+1}$. Notice that if the encounter was arranged differently, with $\sigma_j$ pairing up with $\gamma_{j-1}$, this would not lead to an acceptable contribution to (2). The second example, fig. 1(b), has two double encounters. In this case each $\sigma$ ends up along the same $\gamma$ it started with. It is only acceptable as a contribution to (2) when all outgoing channels coincide.

The present work will apply exclusively to systems for which the dynamics is not invariant under time-reversal. This is because we shall not consider situations where a $\sigma$ trajectory runs in the opposite sense with respect to a $\gamma$ trajectory. This is a significant restriction on the possible correlated sets, and analogously simplifies the treatment.

A correlated pair of trajectory sets may be represented by a diagram with a structure. A diagram is a topological entity, a graph where encounters become vertices of even valence and the pieces of trajectories leading from one encounter to another (or to leads) become edges [10]. (Channels are formally also vertices, but they should not be confused with encounters, so we will keep calling them channels.) The structure is a prescription for walking in the graph, i.e. it specifies how the $\sigma$'s change partners at the vertices. Examples are shown in fig. 1(c), (d).

It is known [16] that the contribution of a diagram to transport moments consists of $(-1)^V N^V - E C(N_1, N_2)$, where $V$ is the number of vertices, $E$ is the number of edges and $C(N_1, N_2)$ is the contribution due to the channels. This factorizes into two parts. First, there is the number of ways they can be assigned among the possible ones existing in the leads. If the number of distinct incoming and outgoing channels are $m_1$ and $m_2$, then this is
\[ B(m_1, m_2) = \frac{N_1!}{(N_1 - m_1)!} \frac{N_2!}{(N_2 - m_2)!}. \] (3)

Second, when there are $k$ different $\sigma$'s starting and ending at the same channels, there is a $k!$, since they could be labeled in that many different ways.

The diagram in fig. 1(a) contributes as $-B(3, 3)/N^5$ to $M_3$. On the other hand, the one in fig. 1(b) contributes, also to $M_3$, as $2B(2, 2)/N^5$, because $\sigma_2$ and $\sigma_3$ start and end at the same channels, so there is freedom in choosing which is which.

It was shown in [10] that to leading order in $1/N$ all transport moments $M_m$ are determined by diagrams that have the topology of a tree. This simplifying feature allowed a recursive approach that lead to an explicit solution in agreement with RMT for all $m$. Later, diagrams determining the first corrections were constructed [11] by grafting trees on base diagrams with more complicated sizes.

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topology. We consider a different method, inspired by the treatment of $M_1$ and $M_2$ in [16]. The crucial point is a relation between transport diagrams and certain closed diagrams which have no channels but have one extra vertex.

**Pre-diagrams.** – We now describe our method, which draws on several combinatorial ideas. Details will be presented in a future publication. Let $S_m$ denote the symmetric group of order $m$, i.e. the set of all permutations of $m$ symbols. We denote by $I_m$ the identity in $S_m$ and by $c_m = (12\cdots m)$ the complete cyclic permutation with increasing elements. Note that $c_m$ is the permutation the $\sigma$’s are required to perform on the channel labels, i.e. they should take $i_j$ to $o_{j+1}$. On the other hand, the $\gamma$’s perform $I_m$ by taking $i_j$ to $o_j$.

Let us introduce a class of graphs we call **pre-diagrams**, which have $m$ incoming and $m$ outgoing channels, assumed all distinct. We call $m$ the order of the pre-diagram. Trajectory $\gamma_j$ is still required to go from $i_j$ to $o_j$, but $\sigma_j$ is allowed to go from $i_j$ to any outgoing channel. This means that in a pre-diagram the permutation performed by the $\sigma$’s on the channel labels is generally different from $c_m$. Let $\pi$ denote this permutation. We present an example in fig. 2(a). Clearly, to every given diagram with given structure we can associate a unique pre-diagram (obtained simply by ignoring coincidences between channels).

There are therefore two important differences between a diagram and its pre-diagram. First, in a diagram trajectory $\sigma_j$ start at $i_j$ and end at $o_{j+1}$, while in a pre-diagram it can end anywhere and the set $\sigma$ may implement any permutation $\pi$ on the channel labels.

Second, there may be coinciding channels in a diagram, while they are all different in a pre-diagram. The idea is that true diagrams can be obtained from a pre-diagram by means of those coincidences among channels that make $\pi$ effectively equivalent to $c_m$. For example, if $\pi = I_m$ then several coincidences are needed, such as all incoming or all outgoing channels. In order to count all diagrams, we must count all possible coincidences for all possible pre-diagrams.

**Admissible pairs of partitions.** – Let us formalize the above ideas. Coinciding channels induce partitions of the set $\{1,\ldots,m\}$ on the incoming and outgoing leads. Suppose two sets of $m$ points arranged vertically side by side. Given a partition $L$ of the left set and a partition $R$ of the right set, identify the points according to the blocks of the partitions. Given a permutation $\pi \in S_m$, draw lines going from the left points to their images under $\pi$ on the right. If $L$ and $R$ have $m_1$ and $m_2$ blocks, respectively, this produces a bipartite graph $G(L,R;\pi)$ with $m_1$ vertices on the left and $m_2$ vertices on the right. An example is shown in fig. 3, where the left partition is $L = \{1,2,3,4\}$, the right one is $R = \{1,2,4\}, \{3\}$ and the permutation is $\pi = (12)(34)$.

We say that $L$ and $R$ form a $\pi$-admissible coincidence pair if $G(L,R;\pi) = G(L,R;\pi_{cm})$. Physically, this means that the permutation induced by the $\sigma$’s is “effectively” equal to $c_m$, and the diagram contributes to (2). Let $M(L,R;\pi)$ be the incidence matrix of $G(L,R;\pi)$. This means that $M_{jk}$ is the number of bonds going from block $j$ on the left to block $k$ on the right. The incidence matrix of the graph in fig. 3 is $\begin{pmatrix} 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. As we have discussed, each multiple bond gives rise to a factorial because the $\sigma$ trajectories could be interchanged. We thus define the multiplicity of the pair $(L,R)$ as

$$\mu(L,R;\pi) = \prod_{jk}[M(L,R;\pi)]_{jk}$$

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![Diagram](image-url)
counts, with the correct multiplicity, diagrams with \( m_1 \) distinct channels on the left and \( m_2 \) distinct channels on the right, which have pre-diagram with permutation \( \pi \).

Once it is known, we can compute

\[
F(\pi, N_1, N_2) = \sum_{m_1, m_2} f(\pi, m_1, m_2) B(m_1, m_2),
\]

where \( B(m_1, m_2) \) is given by eq. (3). The quantity above is the total contribution to the sum (2) associated with pre-diagrams for which the permutation performed by the \( \sigma \)'s on channel labels is \( \pi \). Notice that since \( \pi \in S_m \) this depends implicitly on \( m \).

We therefore meet our first combinatorial problem, which is to obtain the function \( F(\pi, N_1, N_2) \), or at least make it computable via a generating function or a recurrence relation. This requires understanding the set \( \mathcal{A}(\pi, m_1, m_2) \), which seems to be a very complicated combinatorial problem. Once we have \( F(\pi, N_1, N_2) \) under control, we are left with the problem of finding all possible pre-diagrams.

**Relation to correlated periodic orbits.** – We now turn to the question of counting pre-diagrams. First, we associate pre-diagrams with certain collections of correlated periodic orbits. Let \( \alpha \) be a single periodic orbit and let \( \beta \) be a set of periodic orbits that is correlated with \( \alpha \), i.e., \( \beta \) differs from \( \alpha \) only in encounters. We show an example in fig. 4, where one periodic orbit is correlated with three others, in a situation with two 2-encounters and one 3-encounter.

Given \( \alpha \) and \( \beta \), suppose we “cut open” an \( m \)-encounter. This produces \( 2m \) endpoints, \( m \) of them corresponding to the “beginning” of trajectories (leaving the encounter) and another \( m \) to “ending” of trajectories (arriving at the encounter). We interpret them as incoming and outgoing channels, respectively. Then, we choose one of the incoming channels to be \( i_1 \), and use \( \alpha \) to label all channels in sequence: the piece of \( \alpha \) that starts in \( i_j \) (and necessarily ends in \( o_j \)) becomes \( \gamma_j \), while the piece of \( \beta \) that starts in \( i_j \) becomes \( \sigma_j \). This obviously produces a pre-diagram, with a certain permutation \( \pi \) to be determined. See fig. 2(b).

There are two caveats in the procedure outlined above. The first is that some of the \( \beta \)'s might not participate in the encounter we chose to cut open, and would not become scattering trajectories. This is remedied simply by demanding that in our correlated pair \( \alpha, \beta \) there must an \( m \)-encounter in which all \( \beta \)'s participate. Second, even though any pre-diagram related to \( M_m \) can be produced in this way, some of them could be produced more than once. We show the simplest possible such example in fig. 5: the same pre-diagram could be produced having three \( \beta \)'s or only one \( \beta \). However, the structure of the encounter that was opened is necessarily different. In order to avoid this overcounting, we have at our disposal the possibility to select the permutation experienced by the \( \sigma \)'s inside the encounter that was opened. We shall make use of this right after eq. (7).

In [15], pairs of correlated periodic orbits were associated with factorizations of permutations. We summarize the idea. Suppose some correlated pair \( \alpha, \beta \). Label the encounter stretches in such a way that the end of stretch \( j \) is followed by the beginning of stretch \( j + 1 \). This produces the permutation \( c_E \), where \( E \) is the number of stretches, acting on the “exit-to-entrance” space (it goes from the exit of an encounter to the entrance of another one). A variant of this construction is shown in fig. 4, where \( \alpha \) is represented by a solid line.

The orbits behave differently at the encounters (the “entrance-to-exit” space). At any encounter \( \alpha \) corresponds...
to the identity permutation, since it takes the entrance of a stretch to the exit of the same stretch. On the other hand, $\beta$ may be represented by a non-trivial permutation $P$, whose number of cycles is equal to $k$, the number of encounters. In the example shown in fig. 4 we have $P = (125)(36)(47)$. The product $c_E P = Q$, acts on “exit-to-exit” space, leading from the exit of an encounter to the exit of another one. It must be a single cycle if there is a single $\beta$. Since $c_E$ is fixed, the total number of correlated pairs equals the number of solutions in $S_E$ to the factorization $c_E = Q P^{-1}$.

In our problem we have a single periodic orbit $\alpha$ which is correlated to a set of any number of periodic orbits, $\beta$. It is possible to adapt this mapping into permutations to aid us in the enumeration of pre-diagrams. We have to remove the condition that the permutation $Q$ be single-cycle, to allow for more than one $\beta$. In the example shown in fig. 4 there are three different $\beta$’s, and we have $Q = (1)(264)(375)$. We must also ensure that all $\beta$’s take part in the encounter we open, which we convention to be the first one. Finally, we have to make sure that when we produce the pre-diagram, the permutation implemented by $\sigma$ on the channel labels is given by $\pi$. When we remove the encounter involving stretches $\{1,2,5\}$ in fig. 4, we produce a pre-diagram for which $\pi$ is the identity in $S_3$.

Some factorizations of permutations. – Let $\{P\}$ denote the set of integers which are not fixed points of the permutation $P$. Let $P_1$ denote the first cycle of $P$, the one that contains the element “1”. Given a set $s$, we define $P|s$, the restriction of $P$ to $s$, to be the permutation obtained by simply erasing from the cycle representation of $P$ all symbols not in $s$. For example, $(123)|{1,3} = (13)$.

We also define a slightly more involved operation we call reduction. Given a permutation $Q$ and a set $s$, the reduction $R_s[Q]$ is obtained by first restricting $Q$ to $s$ and then making each element as small as possible keeping positivity and relative order. For example, $R_{(2,3,4,6)}[264] = (143)$. This is found as follows. First, because we are restricting to $\{2,3,4,6\}$, we must write explicitly the fixed point: $(264)(3)$. Then reduction leads to $(143)(2)$. Finally, we may omit again the fixed point. Another example: $R_{(2,3,4)}[(264)] = (13)$.

Suppose we have correlated orbits described by the equation $c_E P = Q$. The set of elements involved in the first encounter is $\{P_1\}$, assumed to have $m$ elements. In the example of fig. 4 this is $\{1,2,5\}$. The $\gamma$ trajectories start and end at this encounter and, by construction, visit these elements in increasing order, i.e. they implement a permutation which is simply $c_E|\{P_1\}$. This is $(125)$ in fig. 4.

We must determine what is $\pi$, the permutation induced by $\sigma$ on those labels. First, we take account the permutation $Q$ and reduce it to the appropriate space, $Q|\{P_1\}$. In fig. 4 this is $(1)(2)(5)$. This acts on exit-to-exit space, i.e. it takes incoming channels to incoming channels. We must therefore multiply it by the inverse of $P_1$ in order to reverse the permutation effected inside the first encounter. The result, $Q|\{P_1\} P_1^{-1}$, takes incoming channels to outgoing channels. In fig. 4 this is also $(125)$, just like for the $\gamma$’s.

At this point, we have the permutations implemented by both $\gamma$ and $\sigma$ on the channel labels. This first is $c_E|\{P_1\}$ and the second is $Q|\{P_1\} P_1^{-1}$. The permutation $\pi$ corresponds to measuring the second one with respect to the first one. In other words, we must carry out a change of coordinates of sorts. We therefore multiply both quantities by the inverse of the first. This turns the second one into $Q|\{P_1\} P_1^{-1} c_E|\{P_1\}$. In our example of fig. 4, this gives $(1)(2)(5)$. The permutation $\pi \in S_m$ is finally obtained after reduction:

$$
\pi = R_{\{P_1\}} \left[ Q|\{P_1\} P_1^{-1} c_E|\{P_1\} \right].
$$

As we have mentioned, some pre-diagrams can be produced more than once. A simple example is shown in fig. 5. In order to avoid this overcounting we may impose a convention on the permutation experienced by the $\sigma$’s inside the encounter which is cut open. A very convenient choice is to demand that $P_1^{-1} = c_E|\{P_1\}$, which means that $P_1^{-1}$ is increasing (i.e. its elements are ordered increasingly) or, equivalently, that $P_1$ is decreasing (i.e. its elements are ordered decreasingly, as in fig. 5(a)). This leads to a much simpler expression for the permutation $\pi$:

$$
\pi = R_{\{P_1\}} [Q].
$$

In particular, the number of cycles of $\pi$ equals the number of individual periodic orbits in the set $\beta$.

All in all, the combinatorial problem that needs to be solved is the following. One must find the number of solutions in $S_E$, let us denote it by $\Xi(m, \pi, E, V)$, to the factorization equation $c_E = Q P^{-1}$ which satisfy several conditions: i) $P$ has $V+1$ cycles; ii) $P_1^{-1}$ is increasing and of size $m$; iii) all cycles of $Q$ have at least one element in common with $P_1$; iv) the reduction of $Q$ to $\{P_1\}$ is equal to a given $\pi$.

We therefore meet our second combinatorial problem, which is to obtain the function $\Xi(m, \pi, E, V)$, or at least make it computable via a generating function or a recurrence relation. As we have seen, this requires handling a very involved factorization problem in the symmetric group.

Conclusions. – We have shown that the semiclassical calculation for transport moments results in

$$
M_m = \sum_{\pi \in S_m} \sum_{E,V} F(\pi, N_1, N_2) \sum_{E,V} \Xi(m, \pi, E, V) \frac{(-1)^{V}}{N_E - V},
$$

where the quantities involved have been defined in the text. This expression is exact, i.e. valid for arbitrary numbers of channels. It is easy to implement it in symbolic packages and verify that it indeed reproduces RMT results, as far as it can be checked. Unfortunately, at present both combinatorial problems presented here remain open, so that a closed form expression for $M_m$ is
beyond reach. However, we stress that the semiclassical approach is not intended as a computational tool. Its merit is in revealing the dynamical origin of universality.

Exact semiclassical results are somewhat surprising in the regime of finite number of channels, specially for high moments, because the number of channels in a lead is small when its width is of the order of the electronic wavelength, when diffraction effects are expected to be important [17]. It is not clear what role is played by wavelength, when diffraction effects are expected to be small when its width is of the order of the electronic moments, because the number of channels in a lead is theregime offinitenumber of channels, specially for high...

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