Period preserving nonisospectral flows and the moduli space of periodic solutions of soliton equations.

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Abstract

Flows on the moduli space of the algebraic Riemann surfaces, preserving the periods of the corresponding solutions of the soliton equations are studied. We show that these flows are gradient with respect to some indefinite symmetric flat metric arising in the Hamiltonian theory of the Whitham equations. The functions generating these flows are conserved quantities for all the equations simultaneously. We show that for 1+1 systems these flows can be imbedded in a larger system of ordinary nonlinear differential equations with a rational right-hand side. Finally these flows are used to give a complete description of the moduli space of algebraic Riemann surfaces corresponding to periodic solutions of the nonlinear Schrödinger equation.

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Introduction.

In the periodic theory of soliton equations we often meet the following problem. The direct scattering transform is well-defined for purely periodic potentials. In the quasiperiodic case the spectrum may have a much more complicated structure (for example it may be similar to the Cantor set [3]) and in general it is impossible to associate a Riemann surface to a quasiperiodic potential. When we consider the inverse problem, general algebraic Riemann surfaces generate quasiperiodic solutions.

Periods of solutions corresponding to a given algebraic Riemann surface can be expressed in terms of some Abelian integrals. Thus the Riemann surfaces corresponding to periodic solutions with a given period form a transcendental submanifold in the moduli space of Riemann surfaces of given genus. (More precisely we have to consider the moduli space of Riemann surfaces of given genus with a set of marked points and jets of local parameters in these points.)

In the present article we use the following approach to study these submanifolds. It has been shown by one of the authors [21] that to any period preserving deformation corresponds a meromorphic differential on the spectral curve with rather strong restrictions on the singularities. Each such differential generates a flow on the moduli space of Riemann surfaces leaving invariant the isoperiodic submanifold. Earlier such flows were introduced by Krichever [12] for more general meromorphic differentials in connection with the topological quantum field theory and Whitham equations. In our paper the period preserving flows are studied more detailed in connection with the problem of characterization of the periodic solutions, especially for the 1 + 1 systems like KdV and NLS. Flows analogous to those studied in the present paper were introduced also by Ercolani, Forest, McLaughlin and Shina [6] in the theory of nonintegrable periodic perturbations of the soliton equations.

These flows can be naturally extended to general finite-gap solutions. In this case they preserve the group of frequencies.

In the 1 + 1 case these flows can be written as ordinary differential equations for the branch points but the right-hand side of of these equations contains Abelian integrals. We show that these flows can be essentially simplified by extending the configuration space. If we add to the branch points the zeroes of the quasimomentum differential as additional variables we obtain a system of ordinary differential equations with a rational right-hand side. These flows are much more convenient both for analytic studies and numerical simulations. Differential equations on the zeroes of the quasimomentum differential were written earlier in [12]. But our observation is that by combining these two sets of parameters together we get differential equations with a rational right-hand
side which are much simpler.

In some sense this system of ODE’s is completely integrable, i.e. it can be linearized in terms of some Abelian integrals. But it does not mean integrability in the Painlevé sense because the structure of the branch points may be rather complicated.

It is possible to introduce local coordinates on the moduli space given as the values of the quasimomentum function in the stationary points of this function. These coordinates are usually multivalued. Such coordinates were also used by Krichever [12], [13], [15]. In some specific cases like the real KdV and the defocusing nonlinear Schrödinger equation they are single-valued global coordinates. For the real KdV case similar coordinates were used by Marchenko [16]. In this case the moduli space was also investigated by McKean and Van Moerbeke [17] with the help of a different coordinate system.

We show that if we change these coordinates the variations of the corresponding Riemann surface are also described by ODE’s with a rational right-hand side. On the basis of these equations a numerical program for studying isoperiodic deformations has been written.

Isoperiodic flows are rather naturally connected with the Whitham equations in the Flaschka-Forest-McLaughlin form [8]. The Hamiltonian theory of the Whitham equations was constructed in some specific cases by Forest and McLaughlin [9] in terms of action-angle variables and for general situation by Dubrovin, Novikov [5] in the local differential-geometrical form and later developed by scientists from the Novikov’s group. Novikov pointed out to the authors that these flows should be gradient in the same flat Riemann metric which arose in the Hamiltonian theory of the Whitham equations. Here we give the proof of this conjecture. A flow is called gradient if its vector field is defined as the gradient of a function with respect to some symmetric Riemann metric. Let us recall that Hamiltonian flows are defined as gradients of Hamiltonian functions in a skew–symmetric metric.

In our case it is important that this symmetric metric is not positive definite and even degenerate in some points. If we consider a gradient flow in a positive definite metric it never conserves the function generating this flow. But in our case the differentials of the functions generating isoperiodic flows span a maximal isotropic subspace of our metric. Thus all these functions are integrals of motion and our gradient system is similar to an integrable Hamiltonian system. In fact there also exists a Hamiltonian formulation of isoperiodic flows but the corresponding skew-symmetric form is nonlocal with respect to the branch points.

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3 The idea of the program interface was suggested to the authors by A. Bobenko.
We use this technique in the third part of our paper to study the moduli space of all algebraic Riemann surfaces of finite genus generating periodic solutions. We fix attention to the nonlinear Schrödinger equation and show that there exists one-to-one correspondence between such Riemann surfaces and some graphs together with the flow parameters mentioned above. The multivaluedness of these parameters corresponds to different topologies of these graphs. Using these graphs it is convenient to describe the branch points of these coordinates. To formulate the hypotheses about the structure of the moduli space proved later we used some numerical simulations.

1 Flows preserving the periods.

We shall study some nonisospectral flows on the space of the finite-gap solutions of the soliton equations preserving the periods.

Let us recall some basic constructions from the periodic soliton theory [23], [18]. For definiteness let us consider the periodic Korteweg-de Vries equation.

The $L$ operator for the Lax representation for KdV is the one-dimensional Schrödinger operator

$$L = -\partial_x^2 + u(x).$$

If $u(x)$ is periodic with a period $P$ (without loss of generality we shall assume $P = 1$ such that $u(x + 1) = u(x)$), then $L$ commutes with the shift operator $S_x \{ (S_x f)(x) = f(x + 1) \}$. The common eigenfunction $\Psi(\lambda, x)$ is called Bloch-Floquet function

$$L \Psi = \lambda \Psi, \ \Psi(\lambda, x + 1) = \mu \Psi(\lambda, x).$$

For generic $\lambda$ we have two eigenfunctions of $S_x$ with the eigenvalues $\mu_+$ and $\mu_-$. In fact $\mu_+(\lambda)$ and $\mu_-(\lambda)$ are two branches of one function holomorphic on a hyperelliptic Riemann surface $Y$, which is called the spectral curve. Finite-gap potentials [19] correspond to algebraic Riemann surfaces (i.e. surfaces with finite number of branch points $\lambda_0, \lambda_1, \ldots, \lambda_{2g}, \infty$). The function $p(\lambda) = \frac{1}{\sqrt{\lambda}} \ln(\mu)$ is called quasimomentum. It is multivalued. The differential of the quasimomentum $dp$ is uniquely defined by the following properties:

1) It is meromorphic on $Y$ with only one pole of second order in the point $\infty$

$$dp \sim \frac{d\lambda}{2\sqrt{\lambda}}.$$
2) All the periods of $dp$ are multiples of $2\pi$

$$\int_{c_n} dp = 2\pi k_n, \quad k_n \in \mathbb{Z}$$

over any closed cycle $c_n$.

In the inverse scattering problem we can consider the quasiperiodic potentials as well as the periodic ones. We can start from a hyperelliptic Riemann surface with arbitrary finite branch points $\lambda_0, \lambda_1, \ldots, \lambda_{2g}$, ($\infty$ is a branch point too). The potentials corresponding to this Riemann surface are parameterized by divisors on this surface of degree $-g$ (or equivalently by collections of $g$ points $\gamma_1, \ldots, \gamma_g$ in $Y$). In general these solutions are quasiperiodic. The basic frequencies are equal to the periods of the differential $dp$, uniquely defined by the property 1) and

2') All the periods of $dp$ are real:

$$\int_{c_n} dp = 2\pi k_n, \quad k_n \in \mathbb{R}$$

over any close cycle $c_n$.

The potential is periodic with period $P_*$ if and only if for all $n$

$$P_* k_n \in \mathbb{Z}.$$

Thus we have the following characterization of the Riemann surfaces corresponding to periodic potentials (the periodicity properties do not depend on the choice of the divisor):

There exists a differential $dp$ with the properties 1) and 2).

We shall study deformations of Riemann surfaces preserving the periods of $dp$. These deformations change the Riemann surfaces. Therefore we call them nonisospectral, but they preserve all the $x$-frequencies of the solutions. In particular they leave invariant the space of potentials periodic in $x$ with a fixed period. Thus we call them isoperiodic.

For the real KdV condition 2') is equivalent to the to the following:

2'') All the $a$-periods of $dp$ are equal to zero

$$\int_{a_n} dp = 0,$$
where $a_1, \ldots, a_g$ are the basic $a$-cycles, generated by real ovals. We can extend this definition to the complex case and study deformations preserving the $b$-periods of differential $dp$ with normalization $2')$ where $a_1, \ldots, a_g$ are $a$-cycles from some fixed basis. In contrast to $2')$ in the complex case this definition depends on the choice of the basis of cycles.

Let us consider the general situation. We assume that the following data are given:

(i) a nonsingular compact Riemann surface $Y$,  
(ii) a finite number of points $y_1, \ldots, y_n$ on this Riemann surface and  
(iii) local parameters $w_1 = 1/z_1, \ldots, w_n = 1/z_n$ near these points.  

Without loss of generality it is convenient to assume that $w_i$ is equal to zero at the point $y_i$.

Now let $H$ be the group of all non-vanishing holomorphic functions $f$ defined on some set of the form $U \setminus \{\infty\}$, where $U \subset \mathbb{C}P^1$ is some neighbourhood of $\infty$ (which may depend on $f$). The group multiplication is given by multiplication of functions. The Lie algebra of this group contains the vector space $\mathfrak{h}$ of all meromorphic functions on $\mathbb{C}P^1$, which are holomorphic on $\mathbb{C} \subset \mathbb{C}P^1$. With the help of these data $(Y, y_1, \ldots, y_n, z_1, \ldots, z_n)$, we define a group homomorphism

$$L : H^n \rightarrow \text{Picard group of } Y :$$

For this purpose we choose some covering

$$Y = Y \setminus \{y_1, \ldots, y_n\} \cup U_1 \cup \ldots \cup U_n$$

with some small disjoint neighbourhoods $U_1, \ldots, U_n$ of $y_1, \ldots, y_n$. Then $L(f_1, \ldots, f_n)$ is defined with respect to this covering by the cocycle, which is equal to $z_i \mapsto f(z_i)$ on $U_i \setminus \{y_i\}$ for all $i = 1, \ldots, n$. In particular each element $(f_1, \ldots, f_n)$ of $\mathfrak{h}^n$ defines a flow on the Picard group of $Y$, which is given by multiplication with $L(\exp(tf_1), \ldots, \exp(tf_n))$ for all $t \in \mathbb{C}$. These are the isospectral flows of the dynamic systems, i.e. these flows do not change the Riemann surface. In the soliton theory we have two fixed elements of $\mathfrak{h}^n$ generating the spatial shifts and the flows generated by other elements are associated with generalized time shifts. If one or both of the spatial flows are trivial we have a system with a reduced spatial dimension. It is convenient to impose conditions on this flows, and therefore to restrict the space of admissible data. In doing so the dynamic system may become a Hamiltonian system. We will consider three cases:

**finite dimensional case:** The both fixed elements of $\mathfrak{h}^n$ induce trivial flows on the Picard group of $Y$. In this case we have finite-dimensional ODE’s and
we do not need to discuss periodicity properties. In this case the ‘isoperiodic deformations’ simply preserve the structure of these equations.

**simple periodic case:** One fixed element of $h^n$ induces a trivial flow and another fixed element of $h^n$ induces a flow with period 1 on the Picard group of $Y$. In this case we have systems with one spatial variable like the KdV or the nonlinear Schrödinger equation and the isoperiodic deformations preserve the periodicity in this variable. The meromorphic function generating the trivial flow maps our Riemann surface $Y$ to the complex plane and it is natural to treat $Y$ as a ramified covering of $\mathbb{CP}^1$. The positions of the branch points gives us the natural coordinates on the moduli space.

**double periodic case:** The both fixed elements of $h^n$ induce flows with period 1 on the Picard group of $Y$. We have systems with two spatial variables like the Kadomtsev-Petviashvili equations and the isoperiodic deformations preserve the periodicity in both spatial variables.

We assume that at all points $y_1, \ldots, y_n$ at least one of the two fixed elements of $h^n$ has a singularity. Otherwise we may neglect the corresponding point and diminish the number $n$ in our data. It is quite obvious that the flow corresponding to some element $f$ of $h^n$ is trivial, if and only if there exists some meromorphic function on $Y$, which solves the Mittag Leffler distribution defined by the cocycle corresponding to $f$. This means that there exists a function holomorphic on $Y \setminus \{y_1, \ldots, y_n\}$ with prescribed singularities $f_i(z_i) + O(1)$ in the points $y_i$, $i = 1, \ldots, n$. Furthermore, a flow corresponding to some element $f$ of $h^n$ has period 1 if and only if there exists some non-vanishing holomorphic function on $Y \setminus \{y_1, \ldots, y_n\}$, such that the quotient of this function divided by the function $z_i \mapsto \exp(f_i(z_i))$ extends to a holomorphic function near $y_i$ for all $i = 1, \ldots, n$. Hence in the three cases mentioned above we have two holomorphic functions $\lambda$ and $\mu$ on $Y \setminus \{y_1, \ldots, y_n\}$:

**In the finite dimensional case** both functions are meromorphic with fixed Mittag Leffler distributions near $y_1, \ldots, y_n$.

**In the simple periodic case** $\lambda$ is a meromorphic function with fixed Mittag Leffler distribution near $y_1, \ldots, y_n$ and $\mu$ is a non-vanishing holomorphic function on $Y \setminus \{y_1, \ldots, y_n\}$, such that $\ln(\mu)$ has a meromorphic branch with fixed Mittag Leffler distribution near $y_1, \ldots, y_n$.

**In the double periodic case** both functions $\lambda$ and $\mu$ are non-vanishing holomorphic functions on $Y \setminus \{y_1, \ldots, y_n\}$, such that $\ln(\lambda)$ and $\ln(\mu)$ have meromorphic branches with fixed Mittag Leffler distributions near $y_1, \ldots, y_n$.

Hence in all cases these two functions obey some relation of the form

$$ R(\lambda, \mu) = 0, $$

where $R(\cdot, \cdot)$ is some holomorphic function on $\mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\}$. In the finite
dimensional case $R(\cdot, \cdot)$ may be chosen to be a polynomial with respect to $\lambda$ and $\mu$ and in the simple periodic case only with respect to $\mu$. In fact, in both cases the meromorphic function $\lambda$ induces a finite covering of the Riemann surface. Then there exists an unique polynomial in $\mu$ with coefficients depending meromorphically on $\lambda$, which describes the relation between $\lambda$ and $\mu$ (see [10]). In the double periodic case we will explain later how to define this function $R(\cdot, \cdot)$. This relation determines the Riemann surface $\tilde{Y}$. In fact, the transcendental curve $\tilde{Y}$ defined by the equation $R(\lambda, \mu) = 0$ may differ from the Riemann surface $Y$ only by some singularities. If $n$ is greater than 2, it may be convenient to modify our construction and identify the points $y_1, \ldots, y_n$ to one multiple point. The corresponding singular Riemann surface is denoted by $Y'$. An analogous construction defines a group homomorphism from $H^n$ into the Picard group of the singular Riemann surface $Y'$.

Let us now give our main examples:

(1) The completely integrable systems, investigated in the papers [1] and [20] are examples for the finite dimensional case.

(2) The KdV equation with periodic boundary conditions is obtained in the simple periodic case with $n = 1$. The fixed elements are $f(z) = z$ and $f(z) = z^2$, $z^2$ generates the trivial flow and $z$ generates the periodic spatial one (see e.g. [18]).

(3) The non-linear Schrödinger equation with periodic boundary conditions is obtained in the modified simple periodic case with $n = 2$. The fixed flows are $(z_1, z_2) \mapsto (z_1, z_2)$ (the trivial one) and $(z_1, z_2) \mapsto (\sqrt{-1}z_1, -\sqrt{-1}z_2)$ (the periodic one) (see [11] and [21]).

(4) The KP equation with periodic boundary conditions with respect to both space variables is obtained in the double periodic case with $n = 1$ and the two functions corresponding to periodic flows are given by $z \mapsto z$ and $z \mapsto \sigma z^2$ (see [13]).

The subject of this article is to consider the space of all data, which obey the conditions introduced in the three cases above. The methods of this paper can be naturally generalized to infinite genus Riemann surfaces with appropriate analytic properties (see e.g. [21]). Two data $(Y, y_1, \ldots, y_n, z_1, \ldots, z_n)$ and $(\tilde{Y}, \tilde{y}_1, \ldots, \tilde{y}_n, \tilde{z}_1, \ldots, \tilde{z}_n)$ are called equivalent, if they may be mapped biholomorphically onto each other. It is quite obvious, that two equivalent data either both obey the above mentioned conditions or none of them. In fact, we are only interested in the set of equivalence classes of data, which obey one of the above mentioned conditions.

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4 We assume, that at all points $y_1, \ldots, y_n$ the greatest common divisor of the degrees of the two fixed functions in $H^n$ is equal to 1. Otherwise the Riemann surface $Y$ may be a non-trivial covering over the curve defined by the equation $R(\lambda, \mu) = 0$.

5 The definition of the Picard group of a singular Riemann surface is given in [22].
1.1 The finite dimensional case

Let us now consider the finite dimensional case. Then those data, which obey the conditions introduced above are in one to one correspondence with plane algebraic curves defined by the equation $R(\lambda, \mu) = 0$. The highest powers of this polynomial with respect to $\lambda$ and $\mu$ are determined by the two fixed elements of $h^n$. It turns out that the polynomials $R(\cdot, \cdot)$, which correspond to some data obeying the above condition of the finite dimensional case, are exactly of the form

$$R_0(\lambda, \mu) + \sum_{m,l} r_{l,m} \lambda^l \mu^m, \text{ with } r_{l,m} \in \mathbb{C}.$$

Here the sum is taken over some finite convex set of $\mathbb{N}_0 \times \mathbb{N}_0$. In fact let us assume that $R(\lambda, \mu, t)$ is a onedimensional family of polynomials corresponding to some data obeying these conditions. Then the expression

$$\frac{\partial \lambda}{\partial t} d\mu - \frac{\partial \mu}{\partial t} d\lambda = -\frac{\partial R(\lambda, \mu, t)}{\partial t} \left( \frac{\partial R(\lambda, \mu, t)}{\partial \lambda} \right)^{-1} d\mu = \frac{\partial R(\lambda, \mu, t)}{\partial t} \left( \frac{\partial R(\lambda, \mu, t)}{\partial \mu} \right)^{-1} d\lambda$$

is a regular meromorphic 1-form of the (in some cases singular) algebraic curve defined by the equation $R(\lambda, \mu, t) = 0$. This formula shows that this form is regular on the domain, where $\lambda$ and $\mu$ are finite. The order of the pole of our form in the point $y_j$ is not greater than the minimal of the orders of $\lambda$ and $\mu$ in this point. We shall call such forms weakly singular. If we want to compare the different Riemann surfaces corresponding to different values of $t$, we may choose either $\lambda$ not to depend on $t$, such that $\mu$ becomes a multivalued function depending on $\lambda$ and $t$ or we choose $\mu$ not to depend on $t$, such that $\lambda$ becomes a multivalued function depending on $\mu$ and $t$. In the first case $\frac{\partial \mu}{\partial t} d\lambda$ and in the second case $\frac{\partial \lambda}{\partial t} d\mu$ is a weakly singular 1-form of our singular Riemann surface. This allows us to identify the tangent space of all equivalence classes obeying the condition in the modified finite dimensional case with the space of weakly singular 1-forms of the corresponding singular Riemann surface. In fact, the parameters $r_{l,m}$ give a global parameterization of the set of all data obeying the conditions of the modified finite dimensional case. The corresponding weakly singular 1-forms are given by

$$\frac{\partial \lambda}{\partial r_{l,m}} d\mu - \frac{\partial \mu}{\partial r_{l,m}} d\lambda =$$
\[-\lambda^{l}\mu^{m} \left( \frac{\partial R(\lambda, \mu, t)}{\partial \lambda} \right)^{-1} d\mu = \lambda^{l}\mu^{m} \left( \frac{\partial R(\lambda, \mu, t)}{\partial \mu} \right)^{-1} d\lambda\]

**Example 1** Let $n$ be equal to 2 and the two fixed elements of $h^2$ be given by

\[(z_1, z_2) \mapsto (z_1, z_2) \text{ and } (z_1, z_2) \mapsto (z_1^{d}, -z_2^{d})\], respectively.

Then the corresponding polynomials are exactly of the form

\[R(\lambda, \mu) = \mu^{2} - \lambda^{2d} - \sum_{l=0}^{d-1} r_{l} \lambda^{l}, \text{ with } r_{l} \in \mathbb{C}.\]

The weakly singular 1-forms of the plane curves defined by the equation $R(\lambda, \mu) = 0$ are given by

\[\frac{\partial \lambda}{\partial r_{l}} d\mu - \frac{\partial \mu}{\partial r_{l}} d\lambda = -\frac{\lambda^{l}}{2\mu} d\lambda \text{ with } l = 0, \ldots, d-1.\]

1.2 The simple periodic case

Now we consider the modified simple periodic case. It corresponds to solutions of $1+1$ systems, which are periodic in one spatial variable. Due to this condition the function $R(\cdot, \cdot)$ is of the form

\[R(\lambda, \mu) = \mu^{M} + \sum_{m=0}^{M-1} \mu^{m} r_{m}(\lambda)\]

with some entire functions $r_{0}(\cdot), \ldots, r_{M-1}(\cdot)$ of finite order which have essential singularities at $\lambda = \infty$. Hence the transcendental curve defined by the equation $R(\lambda, \mu) = 0$ has infinite algebraic genus. Let us again assume that $R(\lambda, \mu, t)$ is a onedimensional family of entire functions corresponding to some data obeying these conditions. The quasimomentum $p$ is the multivalued function defined by $\frac{1}{\sqrt{-1}} \ln(\mu)$. Then the expression

\[\frac{\partial p}{\partial t} d\lambda - \frac{\partial \lambda}{\partial t} dp = \frac{\partial R(\lambda, \mu, t)}{\partial t} \left( \frac{\partial R(\lambda, \mu, t)}{\partial \lambda} \right)^{-1} dp = \sqrt{-1} \frac{\partial R(\lambda, \mu, t)}{\partial t} \left( \mu \frac{\partial R(\lambda, \mu, t)}{\partial \mu} \right)^{-1} d\lambda\]
is a regular meromorphic 1-form on \( \tilde{Y} \). If the deformation of \( R \) corresponds to a continuous deformation of \( Y \) then this differential is also meromorphic on \( Y(t) \). This formula shows again that this form is regular on the domain where \( \lambda \) and \( \mu \) are finite. The order of the pole of our form in the point \( y_j \) is not greater than the minimal of the orders of \( \lambda \) and \( p \) in this point. We shall call such forms weakly singular. Thus any weakly meromorphic differential \( \omega = \nu d\lambda \) of \( Y \) generates an isoperiodic flow defined by

\[
\frac{\partial p}{\partial t} d\lambda - \frac{\partial \lambda}{\partial t} dp = \omega
\]  

(1)

(This representation is rather similar to the form in which the Whitham equations for the KP equation were written [14]). Let \( \lambda_k \) be a simple branch point of our covering \( Y \to \mathbb{CP}^1 \). Then from (1) it follows:

\[
\frac{\partial \lambda_k}{\partial t} = -\frac{\omega(\lambda_k)}{dp(\lambda_k)}
\]  

(2)

(Similar representation is widely used in the Whitham theory [8]). If all the branch point are simple we get a system of ODE’s but with a rather complicated right-hand side. If we want to compare the different Riemann surfaces corresponding to different values of \( t \), we may choose either \( \lambda \) not to depend on \( t \), such that \( \mu \) becomes a multivalued function depending on \( \lambda \) and \( t \), or we choose \( \mu \) not to depend on \( t \), such that \( \lambda \) becomes a multivalued function depending on \( \mu \) and \( t \). In the first case \( \frac{dp}{dt} d\lambda \) and in the second case \( \frac{\partial \lambda}{\partial t} dp \) is a weakly singular 1-form of our singular Riemann surface. Thus we arrive at the following conclusion (see [21]): The tangent space to all data corresponding to solutions which are periodic in one variable with given period is in one-to-one correspondence with the space of weakly singular 1-forms of the singular Riemann surface \( \tilde{Y}(t) \). If we also admit Riemann surfaces of infinite genus, the set of all functions \( R(\cdot, \cdot) \), which correspond to data obeying the condition of the simple periodic case, is again an affine space. But then we have to take care of the boundary conditions near the points \( y_1, \ldots, y_n \). The normalization \( Y \) of our transcendental curve \( \tilde{Y} \) (the surface obtained by removing all the singularities) is now algebraic and coincides with the algebraic curve which plays the role of the spectral data in the finite-gap theory. Consider the following meromorphic function on \( Y \)

\[
\kappa = \frac{dp}{d\lambda} = \frac{1}{\mu} \frac{d\mu}{d\lambda} = -\frac{1}{\mu} \frac{\partial R(\lambda, \mu)}{\partial \lambda} / \frac{\partial R(\lambda, \mu)}{\partial \lambda}.
\]

We may describe \( Y \) by some equation of the form

\[
Q(\kappa, \lambda) = 0.
\]
Let us remark that this representation differs from the standard one. This function $Q(\cdot, \cdot)$ is again uniquely defined as a polynomial in $\kappa$ with coefficients depending meromorphically on $\lambda$, which is even a meromorphic function on $\mathbb{CP}^1 \times \mathbb{CP}^1$, if the corresponding Riemann surface has finite genus. Let us now assume that some regular 1-form $\omega$ is given by

$$\omega = \nu d\lambda = \nu \kappa^{-1} dp$$

with some meromorphic function $\nu$ on our Riemann surface. Then we may calculate the derivative of the function $Q(\cdot, \cdot)$ with respect to the corresponding flow-parameter $t$. In order to do this we choose $\lambda$ to depend not on $t$, such that both functions $\kappa$ and $\mu$ are multivalued functions depending on $\lambda$ and $t$. By definition we have the equation

$$\frac{\partial p}{\partial t} = \nu.$$

Equation (1) takes the form:

$$\frac{\partial^2 p}{\partial t \partial \lambda} = \frac{\partial \kappa}{\partial t} = \frac{\partial \nu}{\partial \lambda} \quad \text{or equivalently} \quad \frac{\partial dp}{\partial t} = d\nu. \tag{3}$$

This representation is similar to the Flaschka-Forest-McLaughlin representation [8] of the Whitham equations. Equation (3) has been derived in the theory of the nonintegrable periodic perturbations of the soliton equations in [6]. Similar equations for meromorphic $\omega$ (which may be not isoperiodic) were written in [12] formula (7.68)). Since we have chosen $\lambda$ not to depend on $t$ we may calculate with this formula the derivative of $Q$ with respect to $t$:

$$\frac{\partial Q(\kappa, \lambda, t)}{\partial t} = -\frac{\partial \kappa}{\partial t} \frac{\partial Q(\kappa, \lambda)}{\partial \kappa} = -\frac{\partial \nu}{\partial \lambda} \frac{\partial Q(\kappa, \lambda)}{\partial \kappa}.$$

If we now could find

(i) some parameterization of a set of functions $Q(\cdot, \cdot)$, which contains the functions describing the data obeying the conditions in the modified simple periodic case and

(ii) all meromorphic functions $\nu$, such that $\nu d\lambda$ is a weakly singular 1-form of the Riemann surface described by $Q(\kappa, \lambda) = 0$,

then this equation defines some flows on the space of parameters, which leaves invariant the subset of those values of the parameters, which correspond to data obeying our condition. If furthermore we
(iii) find one function $Q(\cdot, \cdot)$, which corresponds to some data obeying our conditions,

then we may determine a whole family of such functions $Q(\cdot, \cdot)$.

In the previous considerations $\omega$ was an arbitrary weakly singular form. We now present a natural set of parameters. Consider a canonical basis of cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$, such that the intersection number between $a_i$ and $b_i$ is equal to 1 for all $i = 1, \ldots, g$ and vanishes on all other combinations (see [10]). Then the first homology group is isomorphic to the free abelian group

$$\mathbb{Z}a_1 \oplus \ldots \oplus \mathbb{Z}a_g \oplus \mathbb{Z}b_1 \oplus \ldots \oplus \mathbb{Z}b_g.$$ 

Since the intersection form is a non-degenerate antisymmetric $\mathbb{Z}$-valued form, we may associate to the function $\mu$ an element $m$ of the first homology group, such that the integral of the 1-form $dp$ is equal to $2\pi$ times the intersection form with $m$. By changing the basis we may achieve that

$$\oint_{a_i} dp = 0 \text{ for all } i = 1, \ldots, g \Leftrightarrow m \in \mathbb{Z}a_1 \oplus \ldots \oplus \mathbb{Z}a_g.$$ 

Then there exists a unique basis $\omega_1, \ldots, \omega_g$ of holomorphic 1-forms of our compact Riemann surface, such that

$$\oint_{a_i} \omega_j = 2\pi \sqrt{-1} \delta_{i,j} \text{ for all } i, j \in \{1, \ldots, g\}. \quad (4)$$ 

Now we can define $g$ flows on the space of compact Riemann surfaces of genus $g$, which obey the condition of the simple periodic case:

$$\frac{\partial p}{\partial t_i} d\lambda - \frac{\partial \lambda}{\partial t_i} dp = \omega_i \text{ for all } i = 1, \ldots, g. \quad (5)$$ 

Of course the canonical basis extends uniquely to some neighbourhood of each of these Riemann surfaces. But globally the space of Riemann surfaces together with some canonical basis will be a non-trivial covering over the space of Riemann surfaces. It is quite easy to see, that the parameters $t_1, \ldots, t_g$ locally may be chosen to be given by

$$t_i = \frac{1}{2\pi \sqrt{-1}} \oint_{a_i} pd\lambda \text{ for all } i = 1, \ldots, g. \quad (6)$$

These parameters depend only on the choice of the canonical basis obeying the foregoing condition. Moreover, this shows that all these flows commute.
Now it is obvious that there is a number which does not change under this flows:

The greatest number \( m \in \mathbb{N} \), such that \( m/m \) is an element of the first homology group. We will see in the example of the nonlinear Schrödinger equation that this number classifies the connected components of the set of data obeying the conditions of the modified periodic case.

**Remark 1.1** These flows can be extended to Riemann surfaces corresponding to general finite-gap solutions of soliton equations with one spatial dimension (in general they are quasiperiodic). In this case the ‘isoperiodic deformations’ preserve the group of the frequencies of these solutions. So we consider all data which obey the following condition in the

**simple quasiperiodic case:** one of the fixed elements in \( H^n \) generates a trivial flow and no conditions on the flow generated by the second fixed element are imposed.

The function \( \lambda \) corresponding to the trivial flow maps \( Y \) to \( \mathbb{C}P^1 \) so \( Y \) is represented as a ramified covering of \( \mathbb{C}P^1 \). The position of the branch points gives us local coordinates on the space of such data.

In this case an analogue of the quasimomentum differential can be defined uniquely as an Abel differential \( dp \) of the second kind such that:

- **diff 1:** \( dp \) is holomorphic in \( Y \setminus (y_1 \cup y_2 \cup \ldots \cup y_n) \).
- **diff 2:** \( dp = \frac{1}{\sqrt{-1}} df_k + \text{reg. terms in the point } y_k \) where \((f_1, \ldots, f_n)\) is the second fixed element of \( H^n \).
- **diff 3:** \( \text{Im} \oint_c dp = 0 \) for any closed cycle \( c \).

\( \kappa = dp/d\lambda \) is again a meromorphic function on \( Y \) and the foregoing discussion of the isoperiodic flows carries over to this more general situation. In fact, the right-hand side of (3) is an exact differential so all the periods of the left-hand side are equal to 0. Thus all the periods of \( dp \) are integrals of motion.

Let us consider concrete equations in more detail.

**Example 2** As we mentioned above, the KdV equation with periodic boundary conditions is an example of the simple periodic case [19]. It is well known (see e.g. [17]), that in this case the function \( R(\cdot, \cdot) \) is of the form

\[
R(\lambda, \mu) = \mu^2 - \mu \Delta(\lambda) + 1.
\]
Hence we may calculate the function $\kappa$ in terms of $\lambda$ and $\mu$:

$$\kappa = \frac{\sqrt{-1} \partial R(\lambda, \mu)}{\mu \frac{\partial R(\lambda, \mu)}{\partial \lambda}} = -\frac{\partial \Delta(\lambda)}{\partial \lambda} \frac{\sqrt{-1}}{2\mu - \Delta(\lambda)}.$$ 

Hence the function $Q(\cdot, \cdot)$ has the form

$$Q(\kappa, \lambda) = \kappa^2 - \left( \frac{\partial \Delta(\lambda)}{\partial \lambda} \right)^2 \frac{1}{\Delta^2(\lambda) - 4}.$$ 

If the Riemann surface has finite genus, there are only a finite number of zeroes of $\frac{\partial \Delta(\lambda)}{\partial \lambda}$, where the denominator $\Delta^2(\lambda) - 4$ is not zero too. Moreover, by definition of the two fixed elements of $h$: to the function $\lambda$ there corresponds the element $z \mapsto z^2$ and to the function $\ln \mu$ there corresponds the function $z \mapsto z$, hence for very large $\lambda$ the function $\kappa^2$ is almost equal to $1/4\lambda$. Then $Q(\cdot, \cdot)$ is of the form

$$Q(\kappa, \lambda) = \kappa^2 - \frac{\prod_{i=1}^g (\lambda - \alpha_i)^2}{4 \prod_{j=0}^{2g} (\lambda - \lambda_j)},$$

where $g$ is the genus, $\alpha_1, \ldots, \alpha_g$ are the values of the function $\lambda$ at the $2g$ zeros of the differential $dp$ and $\lambda_0, \ldots, \lambda_{2g}, \infty$ are the values of the function $\lambda$ at the zeroes of the differential $d\lambda$ (these zeroes are of course the branchpoints of the covering map induced by $\lambda$). The standard representation for this Riemann surface differs from this one and reads as:

$$\zeta^2 = 4(\lambda - \lambda_0) \ldots (\lambda - \lambda_{2g}). \quad (7)$$

Now the holomorphic 1-forms of the Riemann surface defined by the equation $Q(\kappa, \lambda) = 0$ are of the form

$$\omega = \frac{o(\lambda)d\lambda}{2\sqrt{d(\lambda)}} \quad (8)$$

Here $o(\cdot)$ denotes a function $\sum_{i=0}^{g-1} \alpha_i \lambda^i$, $d(\cdot)$ denotes the function $d(\lambda) = \prod_{i=0}^{2g} (\lambda - \lambda_j)$ and $q(\cdot)$ denotes the function $q(\lambda) = \prod_{i=1}^g (\lambda - \alpha_i)$. The flow corresponding to this 1-form is given by the differential equation

$$2d(\lambda) \frac{\partial q(\lambda)}{\partial t} - q(\lambda) \frac{\partial d(\lambda)}{\partial t} = \left( 2d(\lambda) \frac{\partial o(\lambda)}{\partial \lambda} - o(\lambda) \frac{\partial d(\lambda)}{\partial \lambda} \right). \quad (9)$$

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This flow may also be formulated in terms of $\Delta(\cdot)$:

$$\frac{\partial \Delta(\lambda)}{\partial t} = \frac{o(\lambda) \partial \Delta(\lambda)}{q(\lambda) \partial \lambda}.$$ 

The flow is written as a system of ordinary differential equations (9), but this system is rather complicated, because the coefficients of the polynomial $q(\lambda)$ (or, equivalently, the zeroes of $q(\lambda)$) depend on the branch points in a rather complicated way (they can be expressed in terms of some hyperelliptic integrals). But we may consider $\lambda_0, \ldots, \lambda_{2g}, \alpha_0, \ldots, \alpha_g$, as independent variables. This means that the differential

$$dp = \frac{q(\lambda)}{2\sqrt{d(\lambda)}} d\lambda = \kappa d\lambda.$$ 

(10)

will have arbitrary periods. Nevertheless, this bigger set of parameters (9) still defines an ordinary differential equation. Moreover, the coefficients of $o(\cdot)$ may depend on all these parameters in a rather complicated way, which is the case in the choice of (4) and (5). So let us present another choice, in which (9) simplifies to ODE’s with a rational right-hand side. If all the values $\alpha_1, \ldots, \alpha_g, \lambda_0, \ldots, \lambda_{2g}$ are pairwise different, we have a basis of holomorphic 1-forms

$$\omega_i = -\frac{c_i}{\lambda - \alpha_i} dp \text{ for all } i = 1, \ldots, g \text{ with some } c_i \in \mathbb{C} \setminus \{0\}.$$ 

In the periodic case, the foregoing formula shows, that the values of the function $\Delta(\alpha_i)$ depend only on the flow parameter corresponding to the form $\omega_i$. It is well known, that $\Delta(\lambda) = 2 \cos \left( \sqrt{\lambda - \lambda_0} \right)$ describes the Riemann surface (of genus 0) corresponding to constant potentials. This transcendental curve has infinitely many double points at the values $\alpha_i = (\lambda_0 + i\pi)^2$ for all $i \in \mathbb{N}$. So let us choose this curve as the starting point of our flows and normalize the 1-forms by the condition

$$\Delta(\alpha_i) = (-1)^i 2 \cosh(t_i) \text{ for all } i \in \mathbb{N}.$$ 

(11)

So we introduce an infinite sequence of flow parameters. But we are interested in the case, where only finitely many of these parameters are not equal to zero. This is precisely the case of Riemann surfaces of finite genus. In fact, generically the genus is equal to the number of flow parameters, which are not equal to zero. The corresponding differential equations are given by

$$\frac{\partial \lambda_j}{\partial t_i} = -\frac{c_i}{\lambda_j - \alpha_i} \text{ with } j = 0, \ldots, 2g;$$ 

(12)
\[
\frac{\partial \alpha_j}{\partial t_i} = -\frac{c_i}{\alpha_j - \alpha_i} \quad \text{with} \quad j = 1, \ldots, i-1, i+1, \ldots, g;
\]
\[
\frac{\partial \alpha_i}{\partial t_i} = \sum_{j \neq i} \frac{c_i}{\alpha_j - \alpha_i} - \frac{1}{2} \sum_{j=0}^{2g} \frac{c_i}{\lambda_j - \alpha_i}.
\]

(13)

The normalization condition implies
\[
c_i^2 = \frac{\prod_{j=0}^{2g}(\alpha_i - \lambda_j)}{\prod_{j \neq i}(\alpha_i - \alpha_j)^2}.
\]

Locally this uniquely defines the coefficients \(c_1, \ldots, c_g\). Due to the normalization condition all these flows commute locally. It is quite easy to see that the singularity of these flows at the starting point may be removed. The \(i\)-th flow opens the \(i\)-th gap. For real flow parameters these flows have no singularities, but in the complex domain they have very complicated singularities. The foregoing formula implies that \(\sum_{j=0}^{2g} \lambda_j - 2 \sum_{i=1}^{g} \alpha_i\) does not depend on \(t\). It is well known, that this constant may be an arbitrary number in \(\mathbb{C}\) (it is related to the mean value of the periodic potential). The integrals over all generators of the first homology group of the form \(dp\) are also invariant under these flows. Hence we have \(2g + 1\) integrals of motion. Our space of parameters has dimension \(3g + 1\). This shows that the set of data with genus \(g\) obeying the conditions of the periodic KdV equation has dimension \(g + 1\).

Consider now the general finite-gap quasiperiodic KdV solutions. The formulas (7), (9), (12) and (13) are valid in this case too. But in the quasiperiodic case we have no natural analogues of \(\Delta(\lambda)\). Nevertheless, the function \(p(\lambda)\) is locally well-defined and the normalization condition (11) is solved by the local parameters \(\sqrt{-1}p(\alpha_i) + i\pi, i = 1, \ldots, g\) of the moduli space of Riemann surfaces corresponding to the solutions with a given set of frequencies. Similar coordinates were used by Marchenko [16] in the periodic KdV theory and by Krichever in the general situation (private communication). All the periods of \(dp\) are integrals of motion of (9). In the space of all \(\lambda_k, \alpha_k\), we have a subspace defined by:
\[
\text{Im} \oint dp = 0
\]

and the flows (9) with arbitrary \(o(\lambda)\) leave this subspace invariant. Thus it is sufficient for us to know one point of this subspace to construct the whole family. In fact these conditions are integrals of motion of our equations.

**Remark 1.2** H. Babujian has observed that the right-hand side of (13) with \(c_i = 1\) appeared also in the quasiclassical limit of the Bethe ansatz equations, which are the Bethe ansatz equations for the Gaudin magnet [2]. The roots of this similarity are not clear now.
Example 3 Let us consider now the non-linear Schrödinger equation as a second example of the simple periodic case. It is very similar to the KdV equation. So let us only mention the modifications. The function $R(\cdot, \cdot)$ is again of the form

$$R(\lambda, \mu) = \mu^2 - \mu \Delta(\lambda) + 1.$$ 

The corresponding Riemann surfaces of finite genus have two covering points of $\lambda = \infty$, which are identified to one multiple point. They are described again by the equation

$$Q(\kappa, \lambda) = \kappa^2 - \frac{\prod_{i=1}^{g}(\lambda - \alpha_i)^2}{\prod_{j=1}^{2g}(\lambda - \lambda_j)} = 0,$$

where $g$ is the algebraic genus of this singular Riemann surface (this is equal to the genus of the normalization plus one), $\alpha_1, \ldots, \alpha_g$ are the values of the function $\lambda$ at the $2g$ zeros of the differential $dp$ and $\lambda_1, \ldots, \lambda_{2g}$ are the values of the function $\lambda$ at the zeroes of the differential $d\lambda$ (these zeroes are of course the branchpoints of the covering map induced by $\lambda$).

The standard representation for this Riemann surface has the form

$$\zeta^2 = (\lambda - \lambda_1) \ldots (\lambda - \lambda_{2g}) = d(\lambda).$$

Now the regular holomorphic 1-forms of the singular Riemann surface defined by the equation $Q(\kappa, \lambda) = 0$ are of the form

$$\omega = \sum_{i=0}^{g-1} \frac{a_i \lambda^i}{\sqrt{d(\lambda)}} d\lambda \text{ with } a_i \in \mathbb{C} \text{ for all } i = 0, \ldots, g - 1.$$

$$dp = \frac{q(\lambda)}{\sqrt{d(\lambda)}} d\lambda.$$

If all the values $\alpha_1, \ldots, \alpha_g, \lambda_1, \ldots, \lambda_{2g}$ are pairwise different, we have again a basis of regular 1-forms

$$\omega_i = -\frac{c_i}{\lambda - \alpha_i} dp \text{ for all } i = 1, \ldots, g \text{ with some } c_i \in \mathbb{C} \setminus \{0\}.$$ 

The values of the function $\Delta(\alpha_i)$ again only depend on the $i$-th flow parameter. The function $\Delta(\lambda) = 2 \cos(\lambda)$ describes the Riemann surface (of genus 0)
corresponding to the zero potentials. So let us choose this curve as the starting point of our flows and normalize the 1-forms by the condition

$$\Delta(\alpha_i) = (-1)^i 2 \cosh(t_i) \text{ for all } i \in \mathbb{Z}.$$  

Again we introduce an infinite sequence of flow parameters. But mostly we assume all but a finite number of parameters to be equal to zero. This is again the case of Riemann surfaces of finite genus. In fact, generically the algebraic genus is equal to the number of flow parameters, which are not equal to zero. The corresponding differential equations are given by

$$\frac{\partial \lambda_j}{\partial t_i} = -\frac{c_i}{\lambda_j - \alpha_i} \text{ with } j = 1, \ldots, 2g;$$

$$\frac{\partial \alpha_j}{\partial t_i} = -\frac{c_i}{\alpha_j - \alpha_i} \text{ with } j = 1, \ldots, i-1, i+1, \ldots;$$

$$\frac{\partial \alpha_i}{\partial t_i} = \sum_{j \neq i} \frac{c_i}{\alpha_j - \alpha_i} - \frac{1}{2} \sum_{j=1}^{2g} \frac{c_i}{\lambda_j - \alpha_i}.$$  

The normalization condition implies

$$c_i^2 = \frac{\prod_{j=1}^{2g} (\alpha_i - \lambda_j)}{\prod_{j \neq i} (\alpha_i - \alpha_j)^2}.$$  

Due to the normalization condition all these flows commute locally. It is quite easy to see that the singularities of these flows at the starting point may be removed. The $i$-th flow opens the $i$-th gap. For real flow parameters these flows have again no singularities, but in the complex domain they have very complicated singularities. The forgoing formula implies that $\sum_{j=1}^{2g} \lambda_j - 2 \sum_{i=1}^{g} \alpha_i$ does not depend on $t$. Due to the condition that near $\lambda = \infty$ $p$ has near $\lambda = \infty$ two branches of the form $\pm \lambda + O(\lambda^{-1})$ this constant must be equal to zero and furthermore, the integral of the 1-form $dp$ along a path connecting the two covering points of $\lambda = \lambda_0$ must be of the form $\pm (2\lambda_0 + 2m\pi) + O(\lambda_0^{-1})$ near the point $\lambda_0 = \infty$. Moreover, the integrals over $2g - 2$ generators of the first homology group of the form $dp$ are also invariant under these flows. Hence we have $2g$ integrals of motion. Our space of parameters has dimension $3g$. This shows that the set of data with algebraic genus $g$ obeying the conditions of the periodic nonlinear Schrödinger equation has dimension $g$. 

1.3 The double periodic case

Let us now consider the double periodic case. Our main example will be the KP equation (see [13]). In general the function \( R(\cdot, \cdot) \), which describes the relation between the functions \( \lambda \) and \( \mu \), is not (uniquely) defined. We assume, that for all \( \lambda \in \mathbb{C} \setminus \{0\} \) the sum over the sheets of the covering map induced by \( \lambda \) of the function \(|\mu|^{-1}\) converges. In this case we may define the function \( R(\lambda, \mu) \) in the following manner:

\[
R(\lambda, \mu) = \prod_{\text{over the sheets of the covering map induced by } \lambda} \left(1 - \frac{\mu}{\mu(\lambda)}\right).
\]

This choice may be characterized by two properties:

(i) For all fixed \( \lambda_0 \in \mathbb{C} \setminus \{0\} \) \( R(\lambda_0, \cdot) \) is an entire function with genus equal to zero.

(ii) For all \( \lambda \in \mathbb{C} \setminus \{0\} \) \( R(\lambda, 0) = 1 \).

Now let us again assume that \( R(\lambda, \mu, t) \) is a one dimensional family of such functions, corresponding to some data obeying the condition of the double periodic case. Then the expression

\[
\frac{\partial \ln \lambda}{\partial t} d\ln \mu - \frac{\partial \ln \mu}{\partial t} d\ln \lambda =
\]

\[
= - \frac{\partial R(\lambda, \mu, t)}{\partial t} \left( \lambda \frac{\partial R(\lambda, \mu, t)}{\partial \lambda} \right)^{-1} d\ln \mu =
\]

\[
= \frac{\partial R(\lambda, \mu, t)}{\partial t} \left( \mu \frac{\partial R(\lambda, \mu, t)}{\partial \mu} \right)^{-1} d\ln \lambda
\]

is a regular meromorphic 1-form of the (in some cases singular) algebraic curve defined by the equation \( R(\lambda, \mu, t) = 0 \). This formula shows again that this form is regular on the domain, where \( \lambda \) and \( \mu \) are finite. Due to the conditions on the functions \( \lambda \) and \( \mu \) this form has poles of order at most one at all the points \( y_1, \ldots, y_n \). Hence it is a regular form of the singular Riemann surface, where all the points \( y_1, \ldots, y_n \) are identified to one multiple point. If we want to compare the different Riemann surfaces corresponding to different values of \( t \), we may choose either \( \lambda \) not to depend on \( t \), such that \( \mu \) becomes a multivalued function depending on \( \lambda \) and \( t \) or we choose \( \mu \) not to depend on \( t \), such that \( \lambda \) becomes a multivalued function depending on \( \mu \) and \( t \). In the first case \( \frac{\partial \ln \mu}{\partial t} d\ln \lambda \) and in the second case \( \frac{\partial \ln \lambda}{\partial t} d\ln \mu \) is a regular 1-form of our singular Riemann surface. This again allows us to identify the tangent space of all
equivalence classes obeying the condition in the modified finite dimensional case with the space of regular 1-forms of the corresponding singular Riemann surface. Let us now restrict to the case \( n = 1 \). In this case the Riemann surface \( Y \) may be chosen to be the non-singular normalization of the transcendental curve described by the equation \( R(\lambda, \mu, t) = 0 \). Moreover, we restrict ourselves to the case in which this normalization is a compact Riemann surface of finite genus \( g \). Then there exists a canonical basis of cycles \( a_1, \ldots, a_g, b_1, \ldots, b_g \), such that the intersection number of \( a_i \) and \( b_i \) is equal to 1 for all \( i = 1, \ldots, g \) and vanishes on all other combinations (see [10]). Then the first homology group is isomorphic to the free abelian group

\[
\mathbb{Z} a_1 \oplus \ldots \oplus \mathbb{Z} a_g \oplus \mathbb{Z} b_1 \oplus \ldots \oplus \mathbb{Z} b_g.
\]

Since the intersection form is a non-degenerate antisymmetric \( \mathbb{Z} \)-valued form, to both functions \( \lambda \) and \( \mu \) we may associate elements \( l \) and \( m \) of the first homology group, such that the integral of the 1-forms \( d \ln \lambda \) and \( d \ln \mu \) is equal to \( 2\pi \sqrt{-1} \) times the intersection form with \( l \) and \( m \) respectively. By changing the basis we may always achieve that either

(i) \( \oint_{a_i} d \ln \lambda = 0 \) for all \( i = 1, \ldots, g \),

or

(ii) \( \oint_{a_i} d \ln \mu = 0 \) for all \( i = 1, \ldots, g \).

Furthermore, if the intersection number of \( l \) and \( m \) is equal to zero, we may even attain, that both conditions are fulfilled.

**Lemma 1.3** If there exists a double periodic non-singular solution corresponding to the compact Riemann surface \( Y \), then the intersection number of \( l \) and \( m \) is zero.

Proof: Both elements \( l \) and \( m \) can be considered as elements of the Lie algebra of the Picard group of \( Y \). Now it is well known that any double periodic solution corresponding to the compact Riemann surface \( Y \) may be described by the family of divisors

\[
D(s, t) = \exp(st + tm)D \text{ with some divisor } D \text{ of degree } g \text{ and } (s, t) \in \mathbb{R}^2.
\]

If this solution is non-singular, all divisors of this family are non-special and have support inside of \( Y \setminus \{y_1, \ldots, y_n\} \). Hence all these divisors may be described by an unique \( g \)-tuple of points of \( Y \). Since both flows are assumed to be periodic, the two parameters \( (s, t) \) of the family run through \( (\mathbb{R}/\mathbb{Z})^2 \). The summation over the points of the divisors defines a map

\[
H_*(((\mathbb{R}/\mathbb{Z})^2, \mathbb{Z}) \to H_*(Y, \mathbb{Z}).
\]
Under this map the two generators of $H_1((\mathbb{R}/\mathbb{Z})^2, \mathbb{Z})$ are mapped onto $l$ and $m$. Moreover, this map respects the intersection number. If the intersection number of $l$ and $m$ is not zero, the image of the generator of $H_2((\mathbb{R}/\mathbb{Z})^2, \mathbb{Z})$ cannot be equal to zero. Then the union of the paths of all points of the divisors of the whole family covers the whole Riemann surface $Y$. This is a contradiction to the assumption, that all divisors of the whole family are non-special and have support inside of $Y \setminus \{y_1, \ldots, y_n\}$.

In the sequel we assume both conditions (i) and (ii). Then there exists a unique basis $\omega_1, \ldots, \omega_g$ of holomorphic 1-forms of our compact Riemann surface, such that

$$\oint_{a_i} \omega_j = \delta_{i,j} \text{ for all } i, j \in \{1, \ldots, g\}.$$ 

Now we can define $g$ flows on the space of compact Riemann surfaces of genus $g$, which obey the condition of the double periodic case:

$$\frac{\partial \ln \lambda}{\partial t_i} d \ln \mu - \frac{\partial \ln \mu}{\partial t_i} d \ln \lambda = \omega_i \text{ for all } i = 1, \ldots, g.$$ 

The canonical basis of course extends uniquely to some neighbourhood of each of these Riemann surfaces. But globally the space of Riemann surfaces together with some canonical basis will be a non-trivial covering over the space of Riemann surfaces. It is quite easy to see, that locally the parameters $t_1, \ldots, t_g$ may be chosen to be given by

$$t_i = \oint_{a_i} \ln(\mu) d \ln(\lambda) \text{ for all } i = 1, \ldots, g.$$ 

Due to our assumption the multivalued function $\ln \mu$ may be chosen to be single valued on each of the cycles $a_1, \ldots, a_g$. Hence these parameters are locally defined only up to summation of some elements of $\mathbb{Z}2\pi\sqrt{-1}$. If the intersection number of $l$ and $m$ is zero, these parameters depend only on the choice of the canonical basis obeying both conditions (i) and (ii). Now it is obvious that three numbers do not change under these flows:

1. the greatest number $l \in \mathbb{N}$, such that $l/l$ is an element of the first homology group.
2. the greatest number $m \in \mathbb{N}$, such that $m/m$ is an element of the first homology group too, and
3. the intersection number between $l$ and $m$.

Isoperiodic flows can be naturally extended to the general quasiperiodic finite-gap solutions. Let $Y$ be a compact Riemann surface. Then any holomorphic
differential on $Y$ generates a deformation preserving all the $x$ and $y$ frequencies.

2 Gradient flows in the simple periodic case

In the section ‘simple periodic case’ the isoperiodic flows were written in terms of the branch points of $Y \lambda_1, \ldots, \lambda_N$ as a system of ordinary differential equations. These equations are defined for the finite-gap quasiperiodic solutions as well as for the periodic ones. In this case these flows are defined on the space of nonsingular compact Riemann surfaces such that one of the fixed elements is represented by a meromorphic function $\lambda$ and generates a trivial flow. $\lambda$ maps $Y$ to $\mathbb{CP}^1$. Hence the zeroes of the quasimomentum are uniquely defined by the branch points and the branch points are the only independent variables.

The Hamiltonian theory of the Whitham equations [5] is based on the existence of the following Riemann metric on our moduli space:

$$ds^2 = \sum_{i=1}^{N} g_{ii} (d\lambda_i)^2,$$
with

$$g_{ii} = \text{res}_{\lambda = \lambda_i} \left( \frac{(dp)^2}{d\lambda} \right) \quad \text{for all } i = 1, \ldots, N, \quad g_{ij} = 0 \quad \text{for } i \neq j.$$  \hspace{1cm} (14)

(This formula is due to Dubrovin [4]).

In this section we show that isoperiodic flows are gradient flows in the metric (14). This fact was pointed out to the authors by S.P.Novikov. We shall consider the flows (3) where the differential $dp$ is normalized by:

$$\oint_{a_i} dp = 0, \quad i = 1, \ldots, g.$$  \hspace{1cm} (15)

Such flows preserve the $b$-periods of $dp$.

**Theorem 2.1** Let $\omega_j$ be a holomorphic differential from the standard basis (4). Then the corresponding flow reads as:

$$\frac{\partial \lambda_k}{\partial t_j} = g^{kl} \frac{\partial H_j}{\partial \lambda_l},$$  \hspace{1cm} (16)

where

$$H_j = -\frac{1}{2\pi} \oint_{b_j} dp.$$  \hspace{1cm} (17)
(The metric tensor \( g_{kl} \) and the inverse matrix \( g^{kl} \) are diagonal \( g_{kl} = \delta^l_k g_{kk} \), the corresponding Riemann metric is flat \([5]\).)

The isoperiodic property means that the functions \( H_j \) are conserved quantities for all flows

\[
\frac{\partial H_j}{\partial t_k} = 0 \text{ for all } j, k.
\]

The representation (16) is very similar to the standard Hamiltonian representation, but we have a symmetric form \( g^{kl} \) instead of a skew-symmetric one in the Hamilton theory.

In this example it is very important that this metric is not positive definite. For a positive definite metric the function generating the flow can not be a conservation law. But for the flows (16) the differentials of the functions \( dH_j \), \( j = 1, \ldots, g \) lie in the ‘light cone’ of this metric. Thus all of them are conserved quantities for the whole system. From this point of view equation (16) is rather similar to integrable Hamiltonian systems.

**Remark 2.2** Equation (16) can be rewritten in a Hamiltonian form with the same ‘Hamiltonians’ but the corresponding skew-symmetric form has no natural representation like (14).

**Remark 2.3** Introducing the flat coordinates of (14) we explicitly linearize the system (16). But the metric \( g_{kk} \) has singularities and the flat coordinates have rather complicated singularities in these points. The metric \( g_{kk} \) is nonsingular if and only if \( dp \) and \( d\lambda \) have no common zeroes.

**Proof of the Theorem 2.1.** Following the paper \([4]\) we will introduce flat coordinates for the metric (14).

For this purpose denote the order of the pole of the function \( \lambda \) in the point \( y_k \) by \( l_k + 1 \). It is natural to choose the local parameters \( w_k \) in the points \( y_k \) so that

\[
\lambda = w_k^{-l_k - 1}.
\]

Consider the following collection of functions on our moduli space:

\[
H_k = -\frac{1}{2\pi} \oint_{b_k} dp, \quad t_k = \frac{1}{\sqrt{-1}} \oint_{a_k} pd\lambda, \quad k = 1, \ldots, g.
\]

\[
r_k = \lim_{\gamma_1 \to \gamma_2} \left[ \int_{\gamma_1}^{\gamma_2} dp + \frac{1}{\sqrt{-1}} f_1(\gamma_1) - \frac{1}{\sqrt{-1}} f_k(\gamma_2) \right],
\]

\[
24
\]
\[ s_k = -\text{res}_{y_k} pd\lambda, \ k = 2, \ldots, n. \]
\[ t_{\alpha,k} = -\text{res}_{y_\alpha} \left. \frac{dp}{k(w_\alpha)^k} \right|_{\alpha = 1, \ldots, n, \ k = 1, \ldots, l_\alpha}. \]

In the points of general position these functions give us a local coordinate system. The metric (14) in these coordinates takes the form

\[
ds^2 = \sum_{k=1}^{g} (dH_k dt_k + dt_k dH_k) + \sum_{k=2}^{n} (dr_k ds_k + ds_k dr_k) + \\
+ \sum_{\alpha=1}^{n} (l_\alpha + 1) \sum_{k=1}^{l_\alpha} dt_{\alpha,k} dt_{\alpha,l_\alpha+1-k}. \quad (18)
\]

From (3) it follows that the flows generated by holomorphic differentials do not change the variables \( r_k, s_k, t_{\alpha,k} \).

Comparison of (18) and (16) finishes the proof.

3 The moduli space of the periodic NLS equation

In this section we want to give a description of the set of data corresponding to periodic solutions of the non-linear Schrödinger equation. It is quite easy to do the same for the KdV equation. As mentioned above the Riemann surfaces corresponding to such periodic solutions of the non-linear Schrödinger equation are described by the equation

\[
\mu^2 - \mu \Delta(\lambda) + 1 = 0
\]

with some entire function \( \Delta \). This function may be described as an infinite-sheeted covering \( \Delta : \mathbb{C}P^1 \to \mathbb{C}P^1 \) with an essential singularity at infinity. If the Riemann surface is of finite genus\(^6\) the function \( \ln \mu \) is meromorphic on both sheets over some open neighbourhood of \( \lambda = \infty \). Hence the function \( \arccos(\Delta/2) \) describes all sheets of the covering map \( \Delta \) over some neighbourhood of \( \Delta = \infty \). Furthermore, this local coordinate near this branchpoint of infinite order extends to all but a finite number of the sheets over the complete domain \( \Delta \in \mathbb{C}P^1 \). Finally for big \( |\lambda| \) \( \ln \mu \) is of the form \( \pm \sqrt{-1}\lambda + O(1/\lambda) \). Hence the global parameter \( \lambda \) of the covering space which covering is induced

\(^6\) With the help of the analysis of [21] our discussion may be extended to the infinite genus case.
by $\Delta$ determined up to sign and summation of multiples of $2\pi\sqrt{-1}$ by this covering map $\Delta : \mathbb{C}P^1 \to \mathbb{C}P^1$. Later we will label the sheets of this covering by the integers and will give a glueing rule for these sheets. The glueing rule will uniquely determine the parameter $\lambda$. This observation proves the following lemma:

**Lemma 3.1** Let $\lambda \mapsto \Delta(\lambda)$ correspond to a Riemann surface of finite genus of a periodic solution of the non-linear Schrödinger equation. Then this function is completely determined by all the values of $\Delta$ at the branchpoints of the covering map

$$\mathbb{C} \to \mathbb{C}, \lambda \mapsto \Delta(\lambda)$$

together with some glueing rule for the infinite number of sheets.

In the first section we introduced flow parameters $(t_i)_{i \in \mathbb{Z}}$, such that the values of $\Delta$ at all branchpoints of the covering map

$$\mathbb{C}P^1 \to \mathbb{C}P^1, \lambda \mapsto \Delta(\lambda)$$

are equal to $(-1)^i 2 \cosh(t_i)$. In order to apply the previous Lemma we have to add to these values the glueing rule for the covering map. Let us first consider the starting point, when $\Delta(\lambda)$ is equal to $2 \cos(\lambda)$. In this case we choose the following glueing rules: We have infinitely many sheets labeled by some index $i \in \mathbb{Z}$. For all $j \in \mathbb{Z}$ the $2j - 1$-th sheet and the $2j$-th sheet has branchpoints at $\Delta = -2, \infty$ and the $2j$-th sheet and the $2j + 1$-th sheet has branchpoints at $\Delta = 2, \infty$, and the cuts are chosen as indicated in figure 1. The parameter $\lambda$ is uniquely defined by the following choice: On the $2j$-th sheet it is equal to $(2j - 1)\pi\sqrt{-1}$ for $\Delta = -2$ and equal to $2j\pi\sqrt{-1}$ for $\Delta = 2$.

On the $2j + 1$-th sheet it is equal to $(2j + 1)\pi\sqrt{-1}$ for $\Delta = -2$ and equal to $2j\pi\sqrt{-1}$ for $\Delta = 2$.

![Figure 1](image-url)
With this glueing rule it is quite obvious what happens, if two cuts along a common sheet are passing each other. In the following figures we move only both cuts along the $2k$-th sheet for some fixed $k \in \mathbb{Z}$ and assume that $j$ runs through $\mathbb{Z} \setminus \{k\}$. The direction of the movement of the cuts from one figure to the next figure is indicated by arrows:

![Figure 2](image2.png)

![Figure 3](image3.png)

![Figure 4](image4.png)
From the figures 2. to the figure 4. the both cuts along the $2k$-th sheet have moved once around each other. Finally the indexes $(2k - 1, 2k, 2k + 1)$ are permuted to $(2k + 1, 2k - 1, 2k)$. This shows that the coordinate given by the values of $\Delta$ at the branchpoints has a singularity of the form $(\Delta_{2k-1} - \Delta_{2k})^{1/3}$, where $\Delta_{2k} = (-1)^{2k-1}2\cosh(t_{2k})$ denotes at the starting point the value of $\Delta$ at the branchpoint between the $2k-1$-th and the $2k$-th sheet and $\Delta_{2k} = (-1)^{2k}2\cosh(t_{2k})$ denotes at the starting point the value of $\Delta$ at the branchpoint between the $2k$-th and the $2k + 1$-th sheet. This gives a complete picture of the branchpoints of the coordinates $(t_i)_{i \in \mathbb{Z}}$.

Let us now describe the structure of the covering of the moduli space with respect to these coordinates. For reasons of simplicity let us assume that the real parts of the values of $\Delta$ at two branchpoints are different, whenever they belong to a common sheet. In this case all the cuts may be chosen of the form $\Delta_i + \sqrt{-1}[0, \infty]$. Hence we have to associate to the values of $\Delta$ at all the branchpoints the numbers of the two corresponding sheets. Due to the previous Lemma this completely determines the corresponding data. These data may be described by a graph with vertices indexed by $i \in \mathbb{Z}$ corresponding to the sheets and links corresponding to the branchpoints and a function from the links into the complex numbers, which is equal to the value of $\Delta$ at the corresponding branchpoint. These data have to obey the following conditions:

(i) The graph is connected.
(ii) All but finitely many vertices are linked exactly with those vertices, whose index differs by $\pm 1$.
(iii) The number of links is equal to the number of pairs of the form $(i, i + 1)$ with $i \in \mathbb{Z}$. Due to condition (ii) this has a precise meaning.
(iv) For large $|i|$ the number corresponding to the link connecting the $i$-th and the $i + 1$-th vertex is equal to $(-1)^i 2$.

It is quite obvious how this description may be extended to the general case. In the following figures we draw the graphs corresponding to figures 1-4 with $k=0$:

\[
\begin{array}{cccccccc}
\ldots & -2 & -2 & 2 & -2 & 0 & 2 & -2 & 2 & \ldots
\end{array}
\]

Graph corresponding to Figure 1.

\footnote{In the infinite genus case this condition has to be weakened to the condition that the number is almost equal to $(-1)^i 2$ in a sense, which depends on the class of solutions, which are considered (see [21]).}
Any neighbourhood of the point $\lambda = \infty$ may be described by glueing together parts of finitely many sheets over some neighbourhoods of the point $\Delta = \infty$ with all other complete sheets over $\Delta \in \mathbb{CP}^1$. Now $\sqrt{-1\ln \mu} = \arccos(\Delta/2)$ defines a single-valued function on some of these neighbourhoods. Hence the glueing rules of all the parts of the sheets of some of these neighbourhoods must coincide with the corresponding glueing rules described by the standard graph. Now let us consider paths in the $\lambda$-plane, which covers big circles in the $\Delta$-plane moving clockwise around. They fit together to two paths with open ends. The corresponding succesion of those sheets, in which the path crosses the negative imaginary axis of the $\Delta$-plane (and therefore the main part of the corresponding turn), results in two sequences of labels of our sheets:

$$\ldots, 2n + 2, 2n, 2n - 2, \ldots \text{ and } \ldots, 2n - 1, 2n + 1, 2n + 3, \ldots$$

Hence for all graphs, which corresponds to periodic solutions of the non-linear Schrödinger equation, these paths should result in these two sequences of labels of our sheets. It is obvious that each part of these paths, which corresponds to one big circle moving clockwise around from the negative imaginary axis back to the negative imaginary axis in the $\Delta$-plane results in a subgraph of the form indicated in figure 5, such that the following conditions are fulfilled:
(a) The real parts of the values of $\Delta$ at the branchpoints are ordered $\text{Re}(\Delta_1) < \ldots < \text{Re}(\Delta_l)$.

(b) Any sheet with label $k_j, j = 1, \ldots, l-1$ has no branchpoint, such that the real part of the corresponding value of $\Delta$ lies between $\text{Re}(\Delta_j)$ and $\text{Re}(\Delta_{j+1})$.

(c) The subgraph is maximal with respect to conditions (a) and (b). This means that the sheet with label $k_0$ has no branchpoint, such that the real part of the corresponding value of $\Delta$ is less than $\text{Re}(\Delta_1)$ and the sheet with label $k_l$ has no branchpoint, such that the real part of the corresponding value of $\Delta$ is bigger than $\text{Re}(\Delta_l)$.

\[
\begin{array}{cccc}
\Delta_1 & \Delta_l \\
k_0 & k_1 & \ldots & k_{l-1} & k_l
\end{array}
\]

Figure 5.

This gives the following condition on our graphs:

(v) For each subgraph, which obeys condition (a), (b) and (c) the labels of the endpoints obey the equation $k_0 - k_l = 2(-1)^{k_0} = 2(-1)^{k_l}$.

Now we can state the main theorem of this section:

**Theorem 3.2** The graphs described above are in one to one correspondence with the data, which corresponds to periodic solutions of the non-linear Schrödinger equation.

Proof: Let us first prove that to all such graphs there exists some entire function $\mathbb{C} \rightarrow \mathbb{C}, \lambda \mapsto \Delta(\lambda)$. In order to do this we divide the infinite number of cuts in three parts: To any part of the graphs of the form indicated in figure 6. and figure 7. there corresponds the entire function

$$\mathbb{C} \rightarrow \mathbb{C}, \lambda \mapsto (-1)^k 2 \cos(\sqrt{\lambda})$$

\[
\begin{array}{cccc}
(-1)^k & (-1)^{k+1} & (-1)^k & (-1)^{k+1} \\
k & k+1 & k+2 & k+3 & \ldots
\end{array}
\]

Figure 6.
Then due to conditions (ii) and (iv) the total covering space corresponding to some graph may be obtained by glueing in the middle of two such copies of $\mathbb{C}P^1$ a finite number of copies of $\mathbb{C}P^1$ along some cuts described by the glueing rule given by the graph. Due to conditions (i) and (iii) this total covering space is isomorphic to $\mathbb{C}P^1$. This shows that all these graphs define some infinite sheeted covering $\mathbb{C}P^1 \to \mathbb{C}P^1$ induced by some entire function $\lambda \mapsto \Delta(\lambda)$, where the parameter $\lambda$ is defined only up to some rational transformation, which leaves invariant the point $\infty$. The transcendental curve defined by the equation $\mu^2 - \mu \Delta(\lambda) + 1 = 0$ has infinitely many ordinary double points. In fact, all the branchpoints of order 1 of the covering map defined by these graphs, at which $\Delta$ is equal to $\pm 2$ correspond to such an ordinary double point. Hence the normalization of this transcendental curve is a compact Riemann surface. Due to condition (v) the function $1/\arccos(\Delta/2)$ is a local parameter in the covering space near infinity. Hence there exists a unique global parameter $\lambda$ of the covering space, such that $\lambda = \arccos(\Delta/2) + O(1/\lambda)$ near infinity. This shows that the Riemann surface defined by the equation $\mu^2 - \mu \Delta(\lambda) + 1 = 0$ corresponds to periodic solutions of the non-linear Schrödinger equation. The other direction of the one to one correspondence is part of the definition of the graph. 

Corollary 3.3 The domain of the flow parameters $(t_i)_{i \in \mathbb{Z}}$ of the non-linear Schrödinger equation, defined in the first section may be chosen to be equal to

$$\left\{(t_i)_{i \in \mathbb{Z}} \in \mathbb{C}^\mathbb{Z} \mid \text{all but a finite number of } t_i \text{'s are equal to zero}\right\}.$$

More precisely, on this domain these flows have have only branchpoints of finite order.

In the proof we described implicitly how to determine the genus of the Riemann surface from the graph. In fact, to all branchpoints of the covering map induced by $\Delta$ there correspond two branchpoints of the covering map induced by $\mu$ from $Y \to \mathbb{C}P^1$. But those branchpoints, where $\Delta$ is equal to $\pm 2$ correspond to ordinary double points and may be neglected, if they have no common sheet.
The non-linear Schrödinger equation has two natural reality conditions. For both of them the function $\Delta(\lambda)$ is assumed to be real, for all $\lambda \in \mathbb{R}$. This condition implies that $\overline{\Delta(\lambda)}$ is equal to $\Delta(\lambda)$. This gives the existence of an antilinear involution of the covering space of the covering map induced by $\lambda \rightarrow \Delta(\lambda)$ corresponding to the antilinear involution $\Delta \rightarrow \overline{\Delta}$. Hence the cuts must be chosen to be invariant under this involution. For the starting point this may be achieved by turning the cuts:

- cut between the $(2j-1)$-th and the $2j$-th sheet $\Delta = -2$
- cut between the $2j$-th and the $(2j+1)$-th sheet $\Delta = 2$

Figure 1'.

There are more complicated situations, in which this antilinear involution permutes some of the sheets and the values of $\Delta$ are not real. In this case all but a finite number of cuts may be chosen to be invariant under the antilinear involution and therefore are part of the real axis $\Delta \in \mathbb{R}$. But finitely many cuts are interchanged under this antilinear involution. We may choose them to be of the form

$$\Delta = \Delta_0 + \sqrt{-1}\mathbb{R}_+ \quad \text{and} \quad \Delta = \overline{\Delta}_0 - \sqrt{-1}\mathbb{R}_+$$

with $\text{Im}\Delta_0 \geq 0$, respectively.

The antilinear involution is then described by an involution of the sheets and the branchpoints, such that the value of $\Delta$ at two branchpoints, which are interchanged under this involution are the complex conjugate of each other. Let us give an example. In the following figure we describe the cuts by the pairs of the corresponding numbers of sheets. Again some number $k \in \mathbb{Z}$ is fixed:

- $(2j-1, 2j)$ with $j \neq k$
  $\Delta = -2$
- $(2j, 2j + 1)$
  $\Delta = .5\sqrt{-1}$
- $(2j+1, 2k+1)$
  $\Delta = .5\sqrt{-1}$
- $(2k-1, 2k+1)$
  $\Delta = 2$

Figure 8.
The involution of the sheets is given by the map
\[ Z \to Z, \; i \mapsto i \text{ for } i \neq k-1, k; \; k-1 \mapsto k; \; \text{and } k \mapsto k-1. \]

Again we may combine this gluing rule to a graph:

\[ \begin{array}{cccccc}
... & -2 & 2 & 2k & 2k+1 & 2k+2 & 2k-1 \\
2k-2 & 2 & 2k & -2 & 2 & \sqrt{-1} \times \frac{1}{2} & \sqrt{-1} \times \frac{1}{2} \\
Graph \text{ corresponding to Figure 8.}
\end{array} \]

Now we can describe the restrictions of the two reality conditions: The so-called focusing case corresponds to the restriction that the values of \( \Delta \) at the branchpoints are elements of \([ -\infty, -2 ] \cup [2, \infty] \). In this case the links and vertices of the graph is always of the same form as the graph corresponding to figure 1. Then it is obvious that the coordinates \( (t_i)_{i \in \mathbb{Z}} \) are global single valued coordinates in the focusing case. The so-called defocusing case corresponds to the restriction that the function \( \Delta(\lambda) \) takes values in \([-2, 2]\) for all \( \lambda \in \mathbb{R} \). Hence the corresponding graph have to consist of one connected purely horizontal part combining all sheets and branchpoints, which are invariant under the involution describing the antilinear involution of the covering space of \( \Delta \). The corresponding values \( \Delta \) have to be elements of \([-2, 2]\) and have to be alternating with respect to the order. Beside this horizontal part the graph may have finitely many vertical strings of finite length, which are symmetric with respect to the horizontal part. This symmetry implements the antilinear involution. Hence the values of \( \Delta \) above the horizontal part of the graph are assumed to have non-negative imaginary part and to be the complex conjugate of the corresponding values below the horizontal part. In this case the coordinates \( (t_i)_{i \in \mathbb{Z}} \) are multivalued with complicated branchpoints.

We want to remark that \( d\lambda \) has a zero at the points where \( \Delta^2 - 4 \) is equal to zero and \( d\mu \) has a zeros at the branchpoints of the covering map induced by \( \Delta \). But all the ordinary double points may be neglected. Hence on the normalization of the algebraic curve defined by \( \mu^2 - \mu \Delta + 1 = 0 \) the differentials \( d\lambda \) and \( d\mu \) have a common zero, whenever the values of \( \Delta \) at two branchpoints of a common sheet are equal to \( \pm 2 \).

Finally let us mention that the differential equations describing the flows to the flow parameters \( (t_i)_{i \in \mathbb{Z}} \) can be solved numerically. We implemented these algebraic differential equations with the help of standard programs up to genus equal to 6. In particular, we checked the branchpoints of these flow parameters.
numerically. With the help of such a program one can calculate quite quickly all the Riemann surfaces corresponding to the graphs described above.

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