NONANALYTIC-HYPOELLIPTICITY FOR SOME DEGENERATE ELLIPTIC OPERATORS

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We give here an example, as simple as possible, of a degenerate elliptic operator \( \sum_{j=1}^{r} X_j \) where \( X_1, X_2, \ldots, X_r \) are \( r \) vector fields with analytic coefficients which, with their commutators of order 1, span the whole space, and such that there exists a nonanalytic function \( u \) in the Gevrey class \( G_2 \) with \( \sum_{j=1}^{r} X_j^2 u = 0 \).

1. We consider an operator

\[
A = yP + Q
\]

where \( P \) is a second order elliptic (nondegenerate) operator and \( Q \) is a first order operator; we assume the coefficients of \( P \) and \( Q \) are analytic in some neighborhood \( \mathcal{O} \) of the origin in \( \mathbb{R}^n = \{(x,y); x \in \mathbb{R}^{n-1} \text{ and } y \in \mathbb{R}\} \). For simplicity we suppose \([P,Q] = PQ - QP = 0\) (however it is possible to consider more general situations). We assume \( n > 1 \).

We obtain the following result:

**Proposition 1.** Let \( V \) be a neighborhood of the origin in \( \mathbb{R}^{n-1} \times \mathbb{R}^+ = \{(x,y); x \in \mathbb{R}^{n-1} \text{ and } y \in [0, \infty]\} \) which is relatively compact in \( \mathcal{O} \). There exists a function \( u \in G_2(V) \) whose restriction to any neighborhood of the origin is nonanalytic, such that there exists a constant \( C > 0 \) with

\[
||D^\alpha A^k u||_{L^2(V)} \leq C^{[\alpha]+k+1}(2k)! (2\alpha)!
\]

for each \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^n \).

**Proof.** We note \( \Gamma = \partial \cap \{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R}; y = 0\} \). Let \( g \) be in \( G_2(\Gamma) \) and nonanalytic in any neighborhood of the origin in \( \mathbb{R}^{n-1} \). We construct a function \( u \) in some neighborhood of \( \Gamma \) in \( \mathbb{R}^{n-1} \times \mathbb{R}^+ \) such that

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by solving a Dirichlet problem in some neighborhood of \( V \) in \( \mathbb{R}^{n-1} \times \mathbb{R}_+ \). Then \( u \in G_2(\overline{V}) \) and is nonanalytic in any neighborhood of the origin (see [4]).

We get obviously

\[
A^k u = Q^k u \quad \text{in } V.
\]

Now the proof can be completed using the following result:

For each \( v \in G_2(\overline{V}) \), there exists a constant \( C > 0 \) such that, for every \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^n \),

\[
\|D^2 Q^k v\|_{L^2(V)} \leq C^{|
\alpha|+k+1}(2\alpha)! (2k)!.
\]

2. We consider the operator

\[
B = A + D_t^2 = yP + Q + D_t^2,
\]

in the neighborhood \( \mathcal{O} \times \mathbb{R} \) in \( \mathbb{R}^{n+1} = \{ (x, y, t); x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, t \in \mathbb{R} \} \).

We have the following result:

**PROPOSITION 2.** There exists a neighborhood \( W \) of the origin in \( \mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R} \) and a function \( w \in G_2(\overline{W}) \) whose restriction to any neighborhood of the origin is not analytic, such that

\[
Bw = 0 \quad \text{in } W.
\]

**PROOF.** Let us consider the series

\[
w(x, y, t) = \sum_{m=0}^{\infty} t^{2m} \frac{A^m u(x, y)}{(2m)!},
\]

where \( A = D_t - id/dt \).

\[3\] We denote by \( D_t \) the operator \(-id/dt\).

\[4\] Such a series is also used in [4].
where $u$ is given by Proposition 1. By using (2) it is easily seen that the function $w$ is defined in $W = V \times [-\delta, +\delta]$ where $\delta$ is some suitable strictly positive number, and satisfies

$$Bw = 0 \quad \text{in } W$$

and there exists $M > 0$ such that

$$||D_{x,y}^k w||_{L^2(W)} \leq M^{[\alpha] + k + 1} k!(2\alpha)!$$

for each $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$.

Furthermore we have

$$w(x, y, 0) = u(x, y);$$

then $w$ is nonanalytic in any neighborhood of the origin.

3. Examples and applications. Let us consider, for example, the following simple case (with $n = 2$):

$$P = \partial_x^2 + 4\partial_y^2,$$

$$Q = -2mi\partial_y \quad \text{with } m \text{ integer } \geq 1.$$ 

Then

$$(5) \quad B = y(\partial_x^2 + 4\partial_y^2) - 2mi\partial_y + \partial_t^2.$$ 

We use the change of variables

$$(6) \quad y = z_1^2 + \cdots + z_m^2.$$ 

We denote $\tilde{w}$ by

$$\tilde{w}(x, z_1, \ldots, z_m, t) = w(x, z_1^2 + \cdots + z_m^2, t)$$

where $w$ is given by Proposition 2.

The function $\tilde{w}$ is in the Gevrey class of order 2 in some neighborhood of the origin in $\mathbb{R}^{m+2}$ and nonanalytic. (If $\tilde{w}$ were analytic, the function $(x, z_1, t) \mapsto w(x, z_1^2, t)$ would be analytic too in some neighborhood of the origin in $\mathbb{R}^3$. The latter function is even with respect to $z_1$, so the function $w$ would be also analytic in some neighborhood of the origin in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, which contradicts Proposition 2.)

By the change of variables (6), the operator $B$ defined by (5) becomes

$$H = (z_1^2 + \cdots + z_m^2)\partial_x^2 + D_{z_1}^2 + \cdots + D_{z_m}^2 + D_t^2$$

which can be written also in the form

$$(7) \quad H = \sum_{j=1}^{m} (z_j D_x)^2 + \sum_{j=1}^{m} D_{z_j}^2 + D_t^2.$$
In some neighborhood of the origin in $\mathbb{R}^{m+2}$ we have $H\tilde{w} = 0$. Therefore the following result is proved:

**Theorem.** Let $m$ be an integer $\geq 1$. The following operator

$$H = \sum_{j=1}^{m} (z_j D_{x_j})^2 + \sum_{j=1}^{m} D_{x_j}^2 + D_t^2$$

is not analytic-hypoelliptic in $\mathbb{R}^{m+2}$. More precisely, one can find a function $\tilde{w}$ defined in some neighborhood of the origin, belonging to the Gevrey class of order 2, nonanalytic and such that $H\tilde{w} = 0$.

In fact, we can construct, by the same method used here, a function $\tilde{w}$ which does not belong to any Gevrey class of order $\varepsilon < 2$ and which satisfies $H\tilde{w} = 0$.

The operator $H$ is obviously of the form $\sum X_j^2$ and satisfies the Hörmander condition (see [3]), namely in this case the vector fields $X_j$ and their commutators of order 1 span the whole space.

If, in the example (7), we take $m = 1$, it turns out that the operator $z^2 D_x^2 + D_t^2$ is not analytic-hypoelliptic in $\mathbb{R}^3$; but it is known (see [5]) that the operator

$$z^2 D_x^2 + D_t^2$$

is analytic-hypoelliptic in $\mathbb{R}^2$. Let us point out that M. Derridj and C. Zuily have also announced recently the analytic-hypoellipticity for some classes of operators which can be considered as generalizations of (8).

On the other hand, Proposition 2 gives a negative result of analyticity up to the boundary; positive results were given in [1], [2] for some classes of degenerate elliptic operators apparently not far from those of Propositions 1 and 2.

**Bibliography**

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