Gaussian fluctuation for spatial average of parabolic Anderson model with Neumann/Dirichlet boundary conditions

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Abstract
Consider the parabolic Anderson model \( \partial_t u = \frac{1}{2} \partial_x^2 u + u \eta \) on the interval \([0, L]\) with Neumann or Dirichlet boundary conditions, driven by space-time white noise \( \eta \). We establish the central limit theorem for the fluctuation of the spatial integral \( \int_0^L u(t, x) \, dx \) as \( L \) tends to infinity, using Malliavin-Stein method. The analysis also relies on uniform estimates on the moments and Malliavin derivative of the solution as \( L \to \infty \). As a byproduct, we prove a result on law of large numbers and obtain that \( L^{-1} \int_0^L \mathbb{E}[u(t, x)^2] \, dx \) converges to the second moment of the solution to parabolic Anderson model on the whole space \( \mathbb{R} \) at time \( t \), as \( L \to \infty \).

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Running head: CLT for PAM on interval.

1 Introduction
Consider the parabolic Anderson model on the interval \([0, L]\)

\[
\begin{align*}
\partial_t u(t, x) &= \frac{1}{2} \partial_x^2 u(t, x) + u(t, x) \eta(t, x), \quad 0 < t \leq T, \ x \in [0, L], \\
u(0) &\equiv 1,
\end{align*}
\]

subject to Neumann or Dirichlet boundary conditions, where \( T > 0 \) is fixed and \( \eta \) denotes space-time white noise on \([0, T] \times \mathbb{R}\), which is a generalized centered Gaussian process with covariance given by

\[
\mathbb{E}[\eta(t, x)\eta(s, y)] = \delta_0(t - s)\delta_0(x - y), \quad t, s \in [0, T], \ x, y \in \mathbb{R}.
\]

Following Walsh [23], the mild solution to the stochastic PDE (1.1) satisfies the following integral equation:

\[
u(t, x) = \int_0^L G_t(x, y) \, dy + \int_0^t \int_0^L G_{t-s}(x, y)u(s, y) \eta(ds \, dy),
\]

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where $G_t(x,y)$ denotes the heat kernel on $[0,L]$ with Neumann or Dirichlet boundary conditions, with expressions given by (A.2) or (A.4), respectively. Here, we omit the dependence on the parameter $L$ of the solution $u$ and the heat kernel $G$ to simplify the notation.

In order to study the Gaussian fluctuation of the spatial average of the solution, we introduce

$$ S_{L,t} := \frac{1}{L} \int_0^L \{u(t,x) - \mathbb{E}[u(t,x)]\} \, dx \quad \text{for all } L \geq 1 \text{ and } t \geq 0. \quad (1.3) $$

The goal of this paper is to prove the following central limit theorem.

**Theorem 1.1.** Fix $T > 0$. Then, as $L \to \infty$,

$$ \sqrt{L} S_{L,\bullet} \xrightarrow{C[0,T]} \int_0^T \sqrt{f(t)} \, dB_t, \quad (1.4) $$

where

$$ f(t) = 2e^{t/4} \int_{-\infty}^{\sqrt{t/2}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy, \quad t \geq 0 \quad (1.5) $$

and $B$ denotes a standard one-dimensional Brownian motion, and $\xrightarrow{C[0,T]}$ denotes the convergence in law in the space of continuous functions $C[0,T]$.

It is well known that strong mixing together with a standard blocking argument can imply a CLT (see Bradley [3]). However, it is not easy to determine the conditions under which the strong mixing holds in the context of SPDEs. Recently, Chen et al [7] have introduced a method to study spatial CLT for a large class of SPDEs, based on Malliavin calculus, Poincaré inequalities, compactness arguments and Paul Lévy’s characterization theorem of Brownian motion. This method has been generalized in [8] and adapted to study the CLT for infinitely-many interacting diffusion processes.

The proceeding two approaches to CLT require stationarity of the process and unfortunately they do not apply to our case since the solution $\{u(t,x) : (t,x) \in [0,T] \times [0,L]\}$ to (1.1) is clearly not stationary in space. In order to prove the CLT in Theorem 1.1, we will appeal to Malliavin-Stein method, which was introduced by Huang et al [13] for the one-dimensional stochastic heat equation driven by a space-time white noise, and later widely extended to multidimensional SPDEs driven by Gaussian noise in [9,11,12,14,16,21]. This approach to CLT provides a convergence rate in terms of total variation distance, using a combination of Malliavin calculus and Stein’s method for normal approximations (see Nourdin and Peccati [17,18]). Also, as we will see, the Malliavin-Stein approach to CLT applies in our non-stationary setting.

Recall that the total variation distance between two random variables $X$ and $Y$ is defined as

$$ d_{TV}(X,Y) = \sup_{B \in \mathcal{B}(\mathbb{R})} |\mathbb{P}(X \in B) - \mathbb{P}(Y \in B)|, \quad (1.6) $$

where $\mathcal{B}(\mathbb{R})$ denotes the family of all Borel subsets of $\mathbb{R}$. We abuse notation and let $d_{TV}(F,N(0,1))$ denote the total variation distance between the law of $F$ and the $N(0,1)$ law.

In the following theorem, we derive the convergence rate for the total variation distance between the normalization of $S_{L,t}$ and standard normal distribution $N(0,1)$.

**Theorem 1.2.** For every $t > 0$ there exists a real number $c = c(t) > 0$ such that for all $L \geq 1$,

$$ d_{TV} \left( \frac{S_{L,t}}{\sqrt{\text{Var}(S_{L,t})}}, N(0,1) \right) \leq \frac{c}{\sqrt{L}}. \quad (1.7) $$
Remark 1.3. The function \( f(t) \) in (1.5) is equal to the second moment of solution (at time \( t \)) to parabolic Anderson model on \( \mathbb{R} \) driven by space-time white noise with constant initial condition. Indeed, let \( \{U(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\} \) solve
\[
\begin{align*}
\partial_t U(t, x) &= \frac{1}{2} \partial^2_x U(t, x) + U(t, x) \eta(t, x), \\
U(0) &= 1.
\end{align*}
\]
Then according to [5, (2.28) and (2.18)], for all \( t \geq 0 \) and \( x \in \mathbb{R} \),
\[
E[U(t, x)^2] = 2e^{t/4} \int_{-\infty}^{\sqrt{t/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = f(t).
\]

Remark 1.4. The limit Gaussian process in (1.4) coincides with the Gaussian fluctuation of the spatial average of \( U \) that solves (1.8). In fact, as a special case of Huang et al [13, Theorem 1.2], they have proved that as \( R \to \infty \),
\[
\frac{1}{\sqrt{R}} \int_0^R [U(\bullet, x) - 1] \, dx \overset{\mathcal{L}(0, T)}{\to} \int_0^\bullet \sqrt{E[U(t, 0)^2]} \, dB_t = \int_0^\bullet \sqrt{f(t)} \, dB_t;
\]
see (1.9) for the identity.

We will prove Theorems 1.1 and 1.2 in Section 4 based on the Malliavin-Stein method; see Propositions 2.1 and 2.2. Here, we point out that unlike the cases considered in the literature mentioned above, in our situation the solution to (1.1) depends on the length of the interval \( L \). We need control the moments of the solution as well as its Malliavin derivative uniformly as \( L \to \infty \); see Lemmas 2.3 and 2.4 in Section 2. As a consequence of these estimates and Poincaré inequality, we obtain a result on law of large numbers; see Corollary 2.6. Section 3 is devoted to the asymptotic behavior of the covariance as \( L \to \infty \), which leads to the expression of the limit Gaussian process in (1.4) and the formula of the function \( f \) in (1.5). And the last section is an Appendix that contains a few technical lemmas on the heat kernel that are used throughout the paper.

We write \( \|Z\|_k \) instead of \( (E[|Z|^k])^{1/k} \), for every \( Z \in L^k(\Omega) \).

2 Preliminaries

2.1 Clark-Ocone formula

Let \( \mathcal{H} = L^2([0, T] \times \mathbb{R}) \). The Gaussian family \( \{W(h)\}_{h \in \mathcal{H}} \) formed by the Wiener integrals
\[
W(h) = \int_{[0, T] \times \mathbb{R}} h(s, x) \eta(ds \, dx)
\]
defines an isonormal Gaussian process on the Hilbert space \( \mathcal{H} \). In this framework we can develop the Malliavin calculus (see Nualart [19]). We denote by \( D \) the derivative operator. Let \( \{\mathcal{F}_s\}_{s \geq 0} \) denote the filtration generated by the space-time white noise \( \eta \).

We recall the following Clark-Ocone formula (see Chen et al [6, Proposition 6.3]):
\[
F = E[F] + \int_{[0, T] \times \mathbb{R}} E[D_{s,y}F | \mathcal{F}_s] \eta(ds \, dz) \quad \text{a.s.},
\]
valid for every random variable \( F \) in the Gaussian Sobolev space \( \mathbb{D}^{1,2} \). Thanks to Jensen’s inequality for conditional expectations, the above Clark-Ocone formula readily yields the following Poincaré-type inequality, which plays an important role throughout the paper:
\[
|\text{Cov}(F, G)| \leq \int_0^T ds \int_\mathbb{R} dz \; \|D_{s,z}F\|_2 \|D_{s,z}G\|_2 \quad \text{for all } F, G \in \mathbb{D}^{1,2}. \tag{2.1}
\]
2.2 The Malliavin-Stein method

Recall the total variation distance between two random variables defined in (1.6). The following bound on $d_{\text{TV}}(F, N(0, 1))$ follows from a suitable combination of ideas from the Malliavin calculus and Stein’s method for normal approximations; see Nualart and Nualart [20, Theorem 8.2.1].

**Proposition 2.1.** Suppose that $F \in \mathbb{D}^{1,2}$ satisfies $E(F^2) = 1$ and $F = \delta(v)$ for some $v$ in the $L^2(\Omega)$-domain of the divergence operator $\delta$. Then,

$$d_{\text{TV}}(F, N(0, 1)) \leq 2\sqrt{\text{Var}((DF, v)_{\mathcal{H}})}.$$ 

The proof of Theorem 1.1 is based on the following generalization of a result of Nourdin and Peccati [18, Theorem 6.1.2].

**Proposition 2.2.** Let $F = (F^{(1)}, \ldots, F^{(m)})$ be a random vector such that, for every $i = 1, \ldots, m$, $F^{(i)} = \delta(v^{(i)})$ for some $v^{(i)} \in \text{Dom}[\delta]$. Assume additionally that $F^{(i)} \in \mathbb{D}^{1,2}$ for $i = 1, \ldots, m$. Let $G$ be a centered $m$-dimensional Gaussian random vector with covariance matrix $(C_{i,j})$ for $1 \leq i, j \leq m$. Then, for every $h \in C^2(\mathbb{R}^m)$ that has bounded second partial derivatives,

$$|E(h(F)) - E(h(G))| \leq \frac{1}{2} \|h''\|_{\infty} \sum_{i,j=1}^{m} E \left( |C_{i,j} - \langle DF^{(i)}, v^{(j)} \rangle_{\mathcal{H}}|^2 \right),$$

where

$$\|h''\|_{\infty} := \max_{1 \leq i,j \leq m} \sup_{x \in \mathbb{R}^m} \left| \frac{\partial^2 h(x)}{\partial x_i \partial x_j} \right|.$$ 

2.3 Moments and Malliavin derivative of $u(t, x)$

In this section, we will give some upper bounds on the moments and Malliavin derivative of $u(t, x)$, uniformly for $L \geq 1$. We first remark that mild form in (1.2) can be understood as

$$u(t, x) = \int_{\mathbb{R}} 1_{[0,L]}(y) G_t(x, y) \, dy + \int_0^t \int_{\mathbb{R}} 1_{[0,L]}(y) G_{t-s}(x, y) u(s, y) \, \eta(ds \, dy).$$

This means that for every $L \geq 1$, the solution $\{u(t, x) : (t, x) \in [0, T] \times [0, L]\}$ can be viewed as a function of the space-time white noise $\eta$ on $[0, T] \times \mathbb{R}$. In what follows, we will always write the spatial integral $\int_0^L$ instead of $\int_{\mathbb{R}} 1_{[0,L]}$, as it is clear in the context.

We now define the Picard iteration for the solution to (1.2). Let $u_0(t, x) = \int_0^L G_t(x, y) \, dy$ for every $(t, x) \in (0, T] \times [0, L]$ and $u_0(0, x) = 1$ for all $x \in [0, L]$. Define iteratively, for every $n \in \mathbb{Z}_+$,

$$u_{n+1}(t, x) := u_0(t, x) + \int_0^t \int_0^L G_{t-r}(x, z) u_n(r, z) \, \eta(dr \, dz).$$

**Lemma 2.3.** Let $\{u(t, x) : (t, x) \in [0, T] \times [0, L]\}$ be the solution to (1.1) and $\{u_n\}_{n=0}^{\infty}$ be defined in (2.3). Then for all $k \geq 2$,

$$c_{T,k} := \sup_{n \geq 0} \sup_{L \geq 1} \sup_{(t,x) \in [0,T] \times [0,L]} \|u_n(t, x)\|_k < \infty$$

and

$$\sup_{L \geq 1} \sup_{(t,x) \in [0,T] \times [0,L]} \|u(t, x)\|_k \leq c_{T,k} < \infty.$$
Proof. It is well known that for every $k \geq 2$ and $(t, x) \in [0, T] \times [0, L]$, $u_n(t, x)$ converges to $u(t, x)$ in $L^k(\Omega)$ as $n \to \infty$. Hence (2.5) follows from (2.4) and we only need to prove (2.4).

It is clear from the definition of heat kernel in (A.2) and (A.4) that

$$
\sup_{L \geq 1} \sup_{(t, x) \in [0, T] \times [0, L]} u_0(t, x) \leq 1. \quad (2.6)
$$

According to (2.3), we see from Burkholder’s inequality and Minkowski’s inequality that for $k \geq 2$,

$$
\|u_{n+1}(t, x)\|_k^2 \leq 2 + 2z_k^2 \int_0^t \int_0^L G_{t-r}(x, z) \|u_n(r, z)\|_k^2 \, dz \, dr,
$$

where $z_k$ denotes the constant in Burkholder’s inequality. The semigroup property for heat kernel (A.7) ensures that

$$
\|u_{n+1}(t, x)\|_k^2 \leq 2 + 2z_k^2 \int_0^t G_{2(t-r)}(x, x) \sup_{z \in [0, L]} \|u_n(r, z)\|_k^2 \, dr,
$$

which implies that

$$
\sup_{z \in [0, L]} \|u_{n+1}(t, z)\|_k^2 \leq 2 + 2z_k^2 K_T \int_0^t \frac{1}{4\pi(t-r)} \sup_{z \in [0, L]} \|u_n(r, z)\|_k^2 \, dr,
$$

where we use the uniform Gaussian upper bound on heat kernel (A.9) and the constant $K_T$ is defined below (A.9). Notice that the constants $z_k$ and $K_T$ do not depend on $L \geq 1$. We now apply [10, Lemma 15 and (56)] with $f_n(t) = \sup_{z \in [0, L]} \|u_n(t, z)\|_k^2$ and $g(s) = \frac{1}{\sqrt{4\pi s}}$ to obtain (2.4). This completes the proof. $\square$

Recently, Chen et al [6] have proved that the moments of Malliavin derivative of the solution to stochastic heat equation have a Gaussian upper bound; see [6, Theorem 6.4]. The following result states that this Gaussian upper bound holds uniformly over $L \geq 1$ for $\|D_{s,y}u(t, x)\|_k$, where $u(t, x)$ is the solution to (1.1).

**Lemma 2.4.** Fix $T > 0$. Then $u(t, x) \in \bigcap_{k \geq 2} \mathcal{D}^{1,k}$ for all $(t, x) \in [0, T] \times [0, L]$. And there exists $C_{T,k} > 0$ such that for all $k \geq 2$, $L \geq 1$, $(t, x) \in [0, T] \times [0, L]$, and for almost every $(s, y) \in (0, t) \times \mathbb{R}$,

$$
\|D_{s,y}u(t, x)\|_k \leq C_{T,k} [0, L](y) p_{t-s}(x - y), \quad (2.7)
$$

where $p_t(x)$ denotes the heat kernel on $\mathbb{R}$, defined in (A.1).

**Proof.** The proof is similar to that of Theorem 6.4 of Chen et al [6]. The main difference is that we need control the moments of Malliavin derivative of $u(t, x)$, uniformly for all $L \geq 1$.

We apply the properties of the divergence operator [19, Prop. 1.3.8] in order to deduce from (2.3) that for almost every $(s, y) \in (0, t) \times \mathbb{R}$,

$$
D_{s,y}u_{n+1}(t, x) = [0, L](y) G_{t-s}(x, y) u_n(s, y) + \int_s^t \int_0^L G_{t-r}(x, z) D_{s,y}u_n(r, z) \eta(dr \, dz) \quad a.s. \quad (2.8)
$$

By induction, we see from (2.8) that for all $n \geq 0$ and $(t, x) \in [0, T] \times [0, L]$, a.s.,

$$
D_{s,y}u_n(t, x) = 0 \quad \text{if } y \notin [0, L]. \quad (2.9)
$$
Moreover, using (2.8), (2.4), Burkholder’s inequality and Minkowski’s inequality, for \((s, y) \in (0, t) \times [0, L],\)

\[
\|D_{s,y}u_{n+1}(t,x)\|^2_k \leq 2c_{T,k}^2 G_{t-s}^2(x,y) + 2z_k^2 \int_s^t dr \int_0^L dz \, G_{t-r}^2(x,z) \|D_{s,y}u_n(r,z)\|^2_k \\
\leq 2c_{T,k}^2 K_T^2 p_{t-s}^2(x-y) + 2z_k^2 K_T^2 \int_s^t dr \int_0^L dz \, p_{t-r}^2(x-z) \|D_{s,y}u_n(r,z)\|^2_k \\
\leq 2c_{T,k}^2 K_T^2 \|D_{s,y}u_{n+1}(t,x)\|_k^2,
\]

(2.10)

where \(z_k\) is the constant in Burkholder’s inequality, and we apply (A.9) in the second inequality. Let \(C_k := (2c_{T,k}^2 K_T^2) \vee (2z_k^2 K_T^2).\) We can iterate (2.10) to find that for \((s, y) \in (0, t) \times [0, L],\)

\[
\|D_{s,y}u_{n+1}(t,x)\|_k^2 \\
\leq C_k \int_s^t dr \int_0^L dz \, p_{t-r}^2(x-z) p_{t-r}^2(z_1-y) \\
+ \cdots + C_k^{n+1} \int_s^t dr \int_0^L dz \, p_{t-r_1}^2(z_1-z_2) p_{t-r_2}^2(z_2-z_3) \cdots p_{t-r_n}^2(z_n-y) \\
\leq \int_0^L dz \, p_{t-s}(x-y) \int_s^t dr \, p_{t-r}^2(z_1-z_2) \cdots p_{t-r_n}^2(z_n-y) \\
\leq \int_0^L dz \, p_{t-s}(x-y) \int_s^t dr \, \int_0^L dz_1 \int_0^L dz_2 \cdots \int_0^L dz_n \\
\leq \Gamma((n+1)/2),
\]

(2.11)

In order to simplify the preceding expression, we need the following two identities

\[
\int_0^L dz \, p_{t-s}(x-y) dy = \sqrt{\frac{t-r}{4\pi(t-s)(s-r)}} \frac{1}{p_{t-s}(x-z)},
\]

(2.12)

and

\[
\int_0 \cdots \int_0 \frac{dr \cdots dr_n}{\sqrt{(1-r_1)(r_1-r_2)\cdots r_n}} = \frac{\Gamma(1/2)^{n+1}}{\Gamma((n+1)/2)},
\]

(2.13)

where \(\Gamma\) denotes the gamma function; see [22, 5.14.2] for (2.13). We see that (2.12) and (2.13) together ensure that

\[
\int_0^L dz \, p_{t-s}(x-y) \int_0^L dz_1 \int_0^L dz_2 \cdots \int_0^L dz_n \\
\leq \int_0^L dz \, p_{t-s}(x-y) \int_0^L dz_1 \int_0^L dz_2 \cdots \int_0^L dz_n \\
\leq \frac{\Gamma(1/2)^{n+1}}{\Gamma((n+1)/2)} \frac{1}{p_{t-s}(x-y)}.\]

(2.14)

Hence, we combine (2.11), (2.14) and (2.9) to obtain that for \((s, y) \in (0, t) \times \mathbb{R},\)

\[
\|D_{s,y}u_{n+1}(t,x)\|_k^2 \leq 1_{[0,L]}(y) \|p_{t-s}(x-y)\|_k^2 \\
= 1_{[0,L]}(y) \sum_{j=0}^n C_{j+1}^j \left(\frac{t-s}{4\pi}\right)^{j/2} \frac{\Gamma(1/2)^{j+1}}{\Gamma((j+1)/2)} \\
\leq 1_{[0,L]}(y) \sum_{j=0}^n C_{j+1}^j \left(\frac{t-s}{4\pi}\right)^{j/2} \frac{\Gamma(1/2)^{j+1}}{\Gamma((j+1)/2)} \\
= \frac{1}{c_{T,k}^2} 1_{[0,L]}(y) \|p_{t-s}(x-y)\|_k^2 \leq c'_{T,k} 1_{[0,L]}(y) \|p_{t-s}(x-y)\|_k^2.
\]

(2.15)
Therefore, (2.15) yields that

\[ \sup_{n \geq 0} \mathbb{E} \left( \left\| D u_n(t, x) \right\|_{H}^2 \right) \leq c'_{T, 2} \int_{0}^{t} ds \int_{-\infty}^{\infty} dy \; p_{t-s}^2(x - y) \]

\[ = c'_{T, 2} \int_{0}^{t} p_{2(t-s)}(0) \; ds = c'_{T, 2} \sqrt{t/\pi} < \infty, \quad (2.16) \]

where we have used the semigroup property of the heat kernel in the first equality.

The reminder of the proof follows from a similar approximation argument as in the proof of [6, Theorem 6.4]. First, we deduce from (2.16) and [19, Lemma 1.2.3] that \( u(t, x) \in \mathbb{D}^{1,2} \) and \( D u_n(t, x) \) converges to \( D u(t, x) \) in the weak topology of \( L^2(\Omega; \mathcal{H}) \) as \( n \to \infty \). Then, we use a smooth approximation \( \{ \psi_{\varepsilon} \}_{\varepsilon > 0} \) to the identity in \( \mathbb{R}_+ \times \mathbb{R} \), and apply Fatou’s lemma and duality for \( L^k \)-spaces, in order to find that for almost every \( (s, y) \in (0, t) \times \mathbb{R} \) and for all \( k \geq 2 \),

\[ \left\| D_{s,y} u(t, x) \right\|_k \leq \limsup_{\varepsilon \to 0} \left\| \int_{0}^{\infty} ds' \int_{-\infty}^{\infty} dy' \; D_{s',y'} u(t, x) \psi_{\varepsilon}(s - s', y - y') \right\|_k \]

\[ \leq \limsup_{\varepsilon \to 0} \sup_{\left\| G \right\|_{k/(k-1)} \leq 1} \left\| \int_{0}^{\infty} ds' \int_{-\infty}^{\infty} dy' \; E \left[ G D_{s',y'} u(t, x) \right] \psi_{\varepsilon}(s - s', y - y') \right\|_k. \]

Choose and fix a random variable \( G \in L^2(\Omega) \) such that \( \left\| G \right\|_{k/(k-1)} \leq 1 \). Because \( D u_n(t, x) \) converges weakly in \( L^2(\Omega; \mathcal{H}) \) to \( D u(t, x) \) as \( n \to \infty \), we can write

\[ \left\| G D_{s,y} u(t, x) \right\|_k \leq \lim_{n \to \infty} \left\| \int_{0}^{\infty} ds' \int_{-\infty}^{\infty} dy' \; E \left[ G D_{s',y'} u(t, x) \right] \psi_{\varepsilon}(s - s', y - y') \right\|_k \]

\[ = \lim_{n \to \infty} \left\| \int_{0}^{\infty} ds' \int_{-\infty}^{\infty} dy' \; \left\| G D_{s',y'} u_n(t, x) \right\|_k \psi_{\varepsilon}(s - s', y - y') \right\|_k \]

\[ \leq \limsup_{n \to \infty} \int_{0}^{\infty} ds' \int_{-\infty}^{\infty} dy' \; \left\| D_{s',y'} u_n(t, x) \right\|_k \psi_{\varepsilon}(s - s', y - y') \]

\[ \leq \sqrt{c'_{T,k}} \int_{0}^{\infty} ds' \int_{-\infty}^{\infty} dy' \; 1_{(0,t)}(s') 1_{[0,L]}(y') \; p_{t-s'}(x - y') \psi_{\varepsilon}(s - s', y - y'), \]

where we have used (2.15) in the last inequality. Let \( \varepsilon \to 0 \) to conclude the proof of (2.7).

Finally, \( u(t, x) \in \cap_{k \geq 2} \mathbb{D}^{1,k} \) follows immediately from the estimate in (2.7). This completes the proof.

**Remark 2.5.** The estimates in Lemmas 2.3 and 2.4 also hold for the solution to stochastic heat equation \( \partial_t u = \frac{1}{2} \partial^2_x u + \sigma(u) \eta \) on the interval \([0, L]\) with Neumann or Dirichlet boundary conditions, where \( \text{Lip}(\sigma) := \sup_{x \neq y} |(\sigma(x) - \sigma(y))/|x - y|| < \infty \). The proof follows along the same lines using the facts that \( |\sigma(x)| \leq |\sigma(0)| + \text{Lip}(\sigma)|x| \) for all \( x \in \mathbb{R} \). However, in the case of periodic boundary conditions, the situation is different since the uniform Gaussian upper bound in (A.9) does not hold for the heat kernel with periodic boundary conditions; see Remark A.2.

We conclude this section by presenting a result on law of large numbers, which is a direct application of Poincaré inequality (2.1) and the estimate on the Malliavin derivative of \( u(t, x) \) in Lemma 2.4.

**Corollary 2.6.** For every \( t > 0 \),

\[ \frac{1}{L} \int_{0}^{L} u(t, x) \, dx \to 1 \quad \text{in } L^2(\Omega) \text{ as } L \to \infty. \]  

(2.17)
Proof. By Poincaré inequality (2.1),

\[
E \left[ \left( \frac{1}{L} \int_0^L \{ u(t, x) - E[u(t, x)] \} \, dx \right)^2 \right] = L^{-2} \int_{[0,L]^2} \text{Cov}(u(t, x), u(t, y)) \, dx \, dy \\
\leq L^{-2} \int_{[0,L]^2} \text{dx} \, \int_0^t \| D_{s,\varepsilon}u(t, x) \| \| D_{s,\varepsilon}u(t, y) \| \, dz \, ds \\
\leq C_{t,2}^2 L^{-2} \int_{[0,L]^2} \text{dx} \, \int_0^t \int_0^L p_{t-s}(x-z)p_{t-s}(y-z) \, dz \, ds \\
\leq C_{t,2}^2 L^{-2} \int_0^t \int_0^L p_{2s}(x-y) \, dx \, dy, \tag{2.18}
\]

where we apply Lemma 2.4 in the second inequality, and in the third inequality we bound \( \int_0^L \text{dz} \) by \( \int_\mathbb{R} \text{dz} \) and then use semigroup property.

Denote

\[
I_L(x) = L^{-1}1_{[0,L]}(x) \quad \text{and} \quad \tilde{I}_L(x) = I_L(-x) \quad \text{for} \ x \in \mathbb{R}. \tag{2.19}
\]

We write

\[
L^{-2} \int_0^t \int_{[0,L]^2} p_{2s}(x-y) \, dx \, dy = \int_0^t \left( I_L * \tilde{I}_L * p_{2s} \right)(0) \, ds \\
\leq L^{-1} \int_0^t \int_{-L}^L p_{2s}(z) \, dz \, ds \leq tL^{-1}, \tag{2.20}
\]

where in the first inequality we use [6, (3.17)]. By (2.18) and (2.20), we let \( L \to \infty \) to obtain

\[
\frac{1}{L} \int_0^L \{ u(t, x) - E[u(t, x)] \} \, dx \to 0 \quad \text{in} \ L^2(\Omega) \quad \text{as} \ L \to \infty,
\]

which together with Lemma A.3 below implies (2.17).

\[\square\]

3 Asymptotic behavior of the covariance

In this section, we will analyze the asymptotic behavior of the covariance of the spatial integral of the solution to (1.1).

Recall from (1.3) that

\[
\mathcal{S}_{L,t} = \frac{1}{L} \int_0^L \{ u(t, x) - E[u(t, x)] \} \, dx.
\]

The following result provides the asymptotic behavior of the covariance function of the renormalized sequence of processes \( \mathcal{S}_{L,t} \) as \( L \) tends to infinity.

Proposition 3.1. For every \( t_1, t_2 > 0 \),

\[
\lim_{L \to \infty} \text{Cov} \left[ \sqrt{L} \mathcal{S}_{L,t_1}, \sqrt{L} \mathcal{S}_{L,t_2} \right] = \int_0^{t_1 \wedge t_2} f(s) \, ds,
\]

where the function \( f \) is defined in (1.5).

In order to prove Proposition 3.1, we need the following supporting lemma.
Lemma 3.2. Denote for \((t, x) \in (0, \infty) \times [0, L] ,\)

\[ I_0(t, x) = \int_0^L G_t(x, y) \, dy \quad \text{and} \quad I_0(0, x) = 1. \tag{3.1} \]

Then

\[ \sup_{L \geq 1} \sup_{(t, x) \in [0, \infty) \times [0, L]} I_0(t, x) \leq 1 \tag{3.2} \]

and for all \(t_1, t_2 > 0 ,\)

\[ \lim_{L \to \infty} \frac{1}{L} \int_0^L I_0(t_1, x) I_0(t_2, x) \, dx = 1. \tag{3.3} \]

Proof. The estimate (3.2) is clear and we need prove (3.3). By Lemma A.1 (1) and the semigroup property (A.7), we write for all \(t_1, t_2 > 0 ,\)

\[ \frac{1}{L} \int_0^L I_0(t_1, x) I_0(t_2, x) \, dx = \frac{1}{L} \int_0^L dx \int_0^L dy_1 dy_2 G_{t_1}(x, y_1) G_{t_2}(x, y_2) \]

\[ = \frac{1}{L} \int_0^L \sup G_{t_1+t_2}(y_1, y_2) \, dy_1 dy_2 \to 1, \quad \text{as} \quad L \to \infty , \]

owing to Lemma A.3. \qed

Proof of Proposition 3.1. Using the mild form in (1.2) and Ito’s isometry, we write

\[ \text{Cov} \left[ \sqrt{L} S_{L, t_1} , \sqrt{L} S_{L, t_2} \right] = \frac{1}{L} \int_0^L \text{Cov}(u(t_1, x), u(t_2, y)) \, dx \, dy \]

\[ = \frac{1}{L} \int_0^L \int_0^L \int_0^{t_1} ds \, dz \, G_{t_1-s}(x, z) G_{t_2-s}(y, z) \text{E}[u(s, z)^2] \]

\[ = \frac{1}{L} \int_0^L \int_0^{t_1} ds \, dz \, I_0(t_1 - s, z) I_0(t_2 - s, z) \text{E}[u(s, z)^2] , \]

where the quantity \(I_0\) is defined in (3.1). Moreover, we have

\[ \text{Cov} \left[ \sqrt{L} S_{L, t_1} , \sqrt{L} S_{L, t_2} \right] = \frac{1}{L} \int_0^L \int_0^{t_1} ds \, dz \, [I_0(t_1 - s, z) I_0(t_2 - s, z) - 1] \text{E}[u(s, z)^2] \]

\[ + \int_0^{t_1} ds \, \frac{1}{L} \int_0^L \text{E}[u(s, z)^2] . \]

By (2.5),

\[ \left| \frac{1}{L} \int_0^{t_1} ds \, \int_0^L \text{d}z \, [I_0(t_1 - s, z) I_0(t_2 - s, z) - 1] \text{E}[u(s, z)^2] \right| \]

\[ \leq c_T^2 \frac{1}{L} \int_0^{t_1} ds \, \int_0^L \text{d}z \, [1 - I_0(t_1 - s, z) I_0(t_2 - s, z)] \to 0 \quad \text{as} \quad L \to \infty , \]

thanks to Lemma 3.2 and dominated convergence theorem.
Therefore, applying (2.5) and dominated convergence theorem, we obtain that
\[
\lim_{L \to \infty} \text{Cov} \left[ \sqrt{L} S_{L,t_1}, \sqrt{L} S_{L,t_2} \right] = \int_0^{t_1 \wedge t_2} ds \lim_{L \to \infty} \frac{1}{L} \int_0^L dz \mathbb{E}[u(s, z)^2] \\
= \int_0^{t_1 \wedge t_2} f(s) ds,
\]
where the second identity follows from Proposition 3.3 below. \qed

**Proposition 3.3.** For every \( t > 0 \),
\[
\lim_{L \to \infty} \frac{1}{L} \int_0^L \mathbb{E}[u(t, x)^2] \, dx = f(t),
\]
where the function \( f \) is defined in (1.5).

In order to prove Proposition 3.3, we need the Wiener chaos expansion for the solution to (1.1): for every \((t, x) \in [0, T] \times [0, L]\),
\[
u(t, x) = \sum_{k=0}^{\infty} \mathcal{I}_k(t, x),
\]
where \( \mathcal{I}_0(t, x) \) is defined in (3.1) and for \( k \geq 1 \),
\[
\mathcal{I}_k(t, x) = \int_0^t \int_0^L \eta(dr_1 \, dz_1) G_{t-r_1}(x, z_1) \cdots \int_0^{r_{k-1}} \eta(dr_k \, dz_k) G_{r_{k-1}-r_k}(z_{k-1}, z_k) \mathcal{I}_0(r_k, z_k).
\]
Moreover, by multiple Ito's isometry,
\[
\|u(t, x)\|_2^2 = \sum_{k=0}^{\infty} \|\mathcal{I}_k(t, x)\|_2^2,
\]
where
\[
\|\mathcal{I}_k(t, x)\|_2^2 = \int_0^t dr_1 \int_0^L dz_1 G_{t-r_1}^2(x, z_1) \cdots \int_0^{r_{k-1}} dr_k \int_0^L dz_k G_{r_{k-1}-r_k}^2(z_{k-1}, z_k) \mathcal{I}_0^2(r_k, z_k).
\]

**Proposition 3.4.** Fix \( T > 0 \). Let \( \mathcal{I}_k \) be as in (3.6). Then for every \( k \in \mathbb{Z}_+ \)
\[
\sup_{L \geq 1} \sup_{(t, x) \in [0,T] \times [0,L]} \|\mathcal{I}_k(t, x)\|_2^2 \leq \frac{K_T^{2k} 4^{-k/2} 2^{k/2}}{1 \Gamma((k+2)/2)},
\]
where \( K_T \) is defined below (A.9). Moreover, for every \( t > 0 \) and \( k \in \mathbb{Z}_+ \)
\[
\lim_{L \to \infty} \frac{1}{L} \int_0^L \|\mathcal{I}_k(t, x)\|_2^2 \, dx = \frac{(t/4)^{k/2}}{\Gamma((k+2)/2)}.
\]
Proof. From (3.2), (3.8) and (A.9),
\[
\|I_k(t, x)\|_2^2 \leq K^2 \int_0^t dr_1 \int_0^L dz_1 p_{r_1}(x-z_1) \ldots \int_0^{r_{k-1}} dr_k \int_0^L dz_k p_{r_k}^2(z_{k-1}-z_k)
\]
\[
= K^2 \int_0^t dr_1 \int_0^L dz_1 \frac{1}{(1-r_1) \times \ldots \times (r_{k-1}-r_k)}
\]
where the first equality follows from the elementary identity (2.12) and change of variables, and the second one holds by the following identity
\[
\int_0^{r_{k-1}} dr_k \int_0^L dz_k G_{r_k-1-r_k}^2(z_{k-1}, z_k) I_0^2(r_k, z_k)
\]
see [22, 5.14.1]. This proves (3.9).

We proceed to prove (3.10). By (3.8) and Lemma A.1 (1), (3),
\[
\frac{1}{L} \int_0^L \|I_k(t, x)\|_2^2 dx
= \frac{1}{L} \int_0^t dr_1 \int_0^L dz_1 G_{2(t-r_1)}^2(z_1, z_1) \ldots \int_0^{r_{k-1}} dr_k \int_0^L dz_k G_{r_k-1-r_k}^2(z_{k-1}, z_k) I_0^2(r_k, z_k)
\]
where, using the expression of heat kernel for \(G_{2(t-r_1)}^2(z_1, z_1)\) in (A.2) and (A.4),
\[
J_{k, 1}^{(1)} = \frac{1}{L} \int_0^t dr_1 p_{2(t-r_1)}^2(0) \int_0^L dz_1 \int_0^{r_1} dr_2 \int_0^L dz_2 G_{r_1-r_2}^2(z_1, z_2)
\]
\[
\times \ldots \times \int_0^{r_{k-1}} dr_k \int_0^L dz_k G_{r_k-1-r_k}^2(z_{k-1}, z_k) I_0^2(r_k, z_k),
\]
\[
J_{k, 1}^{(2)} = \frac{1}{L} \int_0^t dr_1 \sum_{n \neq 0} p_{2(t-r_1)}^2(2n L) \int_0^L dz_1 \int_0^{r_1} dr_2 \int_0^L dz_2 G_{r_1-r_2}^2(z_1, z_2)
\]
\[
\times \ldots \times \int_0^{r_{k-1}} dr_k \int_0^L dz_k G_{r_k-1-r_k}^2(z_{k-1}, z_k) I_0^2(r_k, z_k),
\]
\[
J_{k, 1}^{(3)} = \pm \frac{1}{L} \int_0^t dr_1 \int_0^L dz_1 \sum_{n \in \mathbb{Z}} p_{2(t-r_1)}^2(z_1 + 2n L) \int_0^{r_1} dr_2 \int_0^L dz_2 G_{r_1-r_2}^2(z_1, z_2)
\]
\[
\times \ldots \times \int_0^{r_{k-1}} dr_k \int_0^L dz_k G_{r_k-1-r_k}^2(z_{k-1}, z_k) I_0^2(r_k, z_k),
\]
where, in the definition of \(J_{k, 1}^{(3)}\), the sign “+” (”−” respectively) corresponds to Neumann (Dirichlet) heat kernel.
Now, we apply (3.2) and (A.9) to see that

\[
J_{k,1}^{(2)} \leq K_t^{2(k-1)} L^{-1} \int_0^t dr_1 \sum_{n \neq 0} p_{2(t-r_1)}(2nL) \int_\mathbb{R} dz_1 \int_0^{r_1} dr_2 \int_\mathbb{R} dz_2 p_{r_1-r_2}(z_1-z_2)
\]

\[
\times \ldots \int_0^{r_{k-1}} dr_k \int_0^L dz_k p_{r_k-r_1}(z_{k-1}-z_k)
\]

\[
= K_t^{2(k-1)} \int_{0 < r_k < \ldots < r_1 < t} dr_1 \ldots dr_k \sum_{n \neq 0} p_{2(t-r_1)}(2nL) p_{2(r_1-r_2)}(0) \times \ldots \times p_{2(r_k-r_{k-1})}(0),
\]

where we use semigroup property \( k - 1 \) times in the equality. Since

\[
\sum_{n \neq 0} p_{2(t-r_1)}(2nL) \leq \sum_{n \neq 0} p_{2(t-r_1)}(2n) \leq p_{2(t-r_1)}(0) \sum_{n \neq 0} e^{-n^2} \quad \text{for all } L \geq 1,
\]

we apply dominated convergence theorem to obtain \( \lim_{L \to \infty} J_{k,1}^{(2)} = 0 \).

Moreover, using (3.2), (A.9) and the following two identities

\[
p_t(\sigma x) = \sigma^{-1} p_{t/\sigma^2}(x) \quad \text{and} \quad p_t^2(x) = \frac{1}{\sqrt{4\pi t}} p_{t/2}(x) \quad \text{for all } x \in \mathbb{R}, \sigma > 0,
\]

we see that

\[
|J_{k,1}^{(3)}| \leq K_t^{2(k-1)} L^{-1} \int_0^t dr_1 \int_\mathbb{R} dz_1 \sum_{n \in \mathbb{Z}} p_{2(t-r_1)}(2z_1 + 2nL) \int_0^{r_1} dr_2 \int_\mathbb{R} dz_2 p_{r_1-r_2}(z_1-z_2)
\]

\[
\times \ldots \int_0^{r_{k-1}} dr_k \int_0^L dz_k p_{r_k-r_1}(z_{k-1}-z_k)
\]

\[
= \frac{R_T}{L} \sum_{n \in \mathbb{Z}} \int_{0 < r_k < \ldots < r_1 < t} dr_1 \ldots dr_k \frac{1}{\sqrt{(r_1-r_2) \times \ldots \times (r_{k-1}-r_k)}} \int_\mathbb{R} dz_1 \ldots \int_\mathbb{R} dz_{k-1} \int_0^L dz_k
\]

\[
\times p_{(t-r_1)/2}(z_1+nL) \ldots p_{(r_k-r_{k-1})/2}(z_{k-2}-z_{k-1}) p_{(r_{k-1}-r_k)/2}(z_{k-1}-z_k)
\]

where \( R_T > 0 \) depends only on \( T \). Hence, we apply the semigroup property to obtain that

\[
|J_{k,1}^{(3)}| \leq \frac{R_T}{L} \int_{0 < r_k < \ldots < r_1 < t} dr_1 \ldots dr_k \frac{1}{\sqrt{(r_1-r_2) \times \ldots \times (r_{k-1}-r_k)}} \sum_{n \in \mathbb{Z}} \int_0^L dz_k p_{(t-r_k)/2}(z_k+nL)
\]

\[
= \frac{R_T}{L} \int_{0 < r_k < \ldots < r_1 < t} dr_1 \ldots dr_k \frac{1}{\sqrt{(r_1-r_2) \times \ldots \times (r_{k-1}-r_k)}},
\]

which implies that \( \lim_{L \to \infty} J_{k,1}^{(3)} = 0 \).

The proceeding computation yields that

\[
\frac{1}{L} \int_0^L \|I_k(t,x)\|_2^2 \, dx = J_{k,1}^{(1)} + o(L), \quad \text{as } L \to \infty.
\]

Similarly, using Lemma A.1 (3) to integrate the integral with respect to \( dz_1 \) in the expression of \( J_{k,1}^{(1)} \), we can write

\[
J_{k,1}^{(1)} = J_{k,2}^{(1)} + J_{k,2}^{(2)} + J_{k,2}^{(3)},
\]
where
\[
J_{k,2}^{(1)} = \frac{1}{L} \int_{0}^{t} \int_{0}^{r_{1}} \int_{0}^{r_{2}} \int_{0}^{L} dz_{2} \int_{0}^{L} dz_{3} G_{r_{1} - r_{2}}^{2}(z_{2}, z_{3})
\]
\[\times \ldots \times \int_{0}^{r_{k-1}} dz_{k} G_{r_{k-1} - r_{k}}^{2}(z_{k-1}, z_{k}) I_{0}^{2}(r_{k}, z_{k}),
\]
\[
J_{k,2}^{(2)} = \frac{1}{L} \int_{0}^{t} \int_{0}^{r_{1}} \int_{0}^{r_{2}} \int_{0}^{L} dz_{2} \int_{0}^{L} dz_{3} G_{r_{1} - r_{2}}^{2}(z_{2}, z_{3})
\]
\[\times \ldots \times \int_{0}^{r_{k-1}} dz_{k} G_{r_{k-1} - r_{k}}^{2}(z_{k-1}, z_{k}) I_{0}^{2}(r_{k}, z_{k}),
\]
\[
J_{k,2}^{(3)} = \pm \frac{1}{L} \int_{0}^{t} \int_{0}^{r_{1}} \int_{0}^{r_{2}} \int_{0}^{L} dz_{2} \int_{0}^{L} dz_{3} G_{r_{1} - r_{2}}^{2}(z_{2}, z_{3})
\]
\[\times \ldots \times \int_{0}^{r_{k-1}} dz_{k} G_{r_{k-1} - r_{k}}^{2}(z_{k-1}, z_{k}) I_{0}^{2}(r_{k}, z_{k}),
\]

where, in the definition of \(J_{k,2}^{(3)}\), the sign "+" ("−" respectively) corresponds to Neumann (Dirichlet) heat kernel. Using the same arguments as before, we obtain that \(\lim_{L \to \infty} J_{k,2}^{(2)} = 0\) and \(\lim_{L \to \infty} J_{k,2}^{(3)} = 0\) and hence

\[
\frac{1}{L} \int_{0}^{L} \|I_{k}(t, x)\|_{2}^{2} dx = J_{k,2}^{(1)} + o(L), \quad \text{as } L \to \infty.
\]

Therefore, we can repeat this procedure to conclude that as \(L \to \infty\),

\[
\frac{1}{L} \int_{0}^{L} \|I_{k}(t, x)\|_{2}^{2} dx = o(L) + \int_{0 < r_{k} < \ldots < r_{1} < t} \int_{0}^{r_{1}} \int_{0}^{r_{2}} \ldots \int_{0}^{r_{k-1}} \int_{0}^{L} dz_{k} G_{r_{k-1} - r_{k}}^{2}(z_{k}, z_{k}) I_{0}^{2}(r_{k}, z_{k})
\]
\[\times \frac{1}{L} \int_{0}^{L} dz_{k} G_{r_{k-1} - r_{k}}^{2}(z_{k}, z_{k}) I_{0}^{2}(r_{k}, z_{k}).
\]

Moreover, we use the expression of heat kernel for \(G_{r_{k-1} - r_{k}}(z_{k}, z_{k})\) in (A.2) and (A.4) to decompose the multiple integral above as

\[
\int_{0 < r_{k} < \ldots < r_{1} < t} \int_{0}^{r_{1}} \int_{0}^{r_{2}} \ldots \int_{0}^{r_{k-1}} \int_{0}^{L} dz_{k} G_{r_{k-1} - r_{k}}^{2}(r_{k}, z_{k})
\]
\[+ \int_{0 < r_{k} < \ldots < r_{1} < t} \int_{0}^{r_{1}} \int_{0}^{r_{2}} \ldots \int_{0}^{r_{k-1}} \int_{0}^{L} dz_{k} G_{r_{k-1} - r_{k}}^{2}(z_{k}, z_{k}) I_{0}^{2}(r_{k}, z_{k})
\]
\[\pm \int_{0 < r_{k} < \ldots < r_{1} < t} \int_{0}^{r_{1}} \int_{0}^{r_{2}} \ldots \int_{0}^{r_{k-1}} \int_{0}^{L} dz_{k} G_{r_{k-1} - r_{k}}^{2}(z_{k}, z_{k}) I_{0}^{2}(r_{k}, z_{k})
\]

where in the last line the sign "+" ("−" respectively) corresponds to Neumann (Dirichlet) heat kernel. By (3.2), similar estimate as in (3.13) and dominated convergence theorem, the second term above converges to 0 as \(L \to \infty\). Similarly, the third term above also converges to 0 as \(L \to \infty\) since by (3.2) and (3.14)

\[
\frac{1}{L} \int_{0}^{L} dz_{k} \sum_{n \in \mathbb{Z}} p_{2(r_{k-1} - r_{k})}(2z_{k} + 2nL) I_{0}^{2}(r_{k}, z_{k}) \leq \frac{1}{2L} \int_{0}^{L} dz_{k} \sum_{n \in \mathbb{Z}} p_{2(r_{k-1} - r_{k})/2}(z_{k} + nL) = \frac{1}{2L}.
\]
Therefore, using (3.2), dominated convergence theorem and (3.3), we conclude that

\[
\lim_{L \to \infty} \frac{1}{L} \int_0^L \left\| I_k(t, x) \right\|^2 dx = \frac{(t/4)^{k/2}}{\Gamma((k + 2)/2)},
\]

where the third equality is due to change of variables and (3.12). This proves (3.10). \(\square\)

We are now ready to prove Proposition 3.3.

Proof of Proposition 3.3. By (3.7) and Fubini’s theorem,

\[
\frac{1}{L} \int_0^L E[u(t, x)^2] dx = \sum_{k=0}^\infty \frac{1}{L} \int_0^L \left\| I_k(t, x) \right\|^2 dx.
\]

Since the series \(\sum_{k=0}^\infty \frac{K_k^2}{L^{k/2}} T^{k/2} \Gamma((k + 2)/2)\) converges, by (3.9) and dominated convergence theorem, we have

\[
\lim_{L \to \infty} \frac{1}{L} \int_0^L E[u(t, x)^2] dx = \sum_{k=0}^\infty \lim_{L \to \infty} \frac{1}{L} \int_0^L \left\| I_k(t, x) \right\|^2 dx = \sum_{k=0}^\infty \frac{(t/4)^{k/2}}{\Gamma((k + 2)/2)}
\]

\[
= \sum_{n=1}^\infty \frac{(t/4)^{(n-1)/2}}{\Gamma((n + 1)/2)}
\]

\[
= 2e^{t/4} \int_{-\infty}^{\sqrt{t/2}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = f(t),
\]

where the second equality holds by (3.10), and in the fourth equality we apply the identity (see [4, Lemma 2.3.4])

\[
\sum_{n=1}^\infty \frac{\lambda^{n-1}}{\Gamma((n + 1)/2)} = 2\lambda e\int_{-\infty}^{\sqrt{2\lambda}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy, \quad \text{for all } \lambda \geq 0,
\]

with \(\lambda = \sqrt{t/4}\). This completes the proof of (3.4). \(\square\)

Remark 3.5. The result of Proposition 3.3 can also be seen from the mild form (1.2). Indeed, by Ito’s isometry and Lemma A.1 (1), (3),

\[
\frac{1}{L} \int_0^L E[u(t, x)^2] dx = \frac{1}{L} \int_0^L T_0^2(t, x) dx + \frac{1}{L} \int_0^t ds \frac{1}{L} \int_0^L G_{2(t-s)}(y, y) E[u(s, y)^2] dy
\]

For every \(t > 0\), the existence of \(\lim_{L \to \infty} \frac{1}{L} \int_0^L E[u(t, x)^2] dx\) is justified by Proposition 3.4. Moreover, as in the proof of Proposition 3.4, the dominating term of \(G_{2(t-s)}(y, y)\) is \(p_{2(t-s)}(0)\) as \(L \to \infty\).
Therefore, by (2.5), (A.9), dominated convergence theorem and (3.3), we let \( L \to \infty \) in (3.16) to obtain that \( \lim_{L \to \infty} \frac{1}{L} \int_0^L E[u(t, x)^2] \, dx \) satisfies the renewal equation: for all \( t > 0 \)

\[
f(t) = 1 + \int_0^t \frac{f(s)}{\sqrt{4\pi(t-s)}} \, ds,
\]

which admits a unique solution given by the formula in (1.5).

**Remark 3.6.** It is clear that the renewal equation (3.17) also holds for the function \( t \mapsto E[U(t, 0)^2] \), where \( U \) solves (1.8). In fact, by Ito’s isometry and stationarity

\[
E[U(t, 0)^2] = 1 + \int_0^t \int_\mathbb{R} p_{t-s}(x-y) E[U(t, y)^2] \, dy \, ds
\]

thanks to semigroup property. We refer to [15, Chapter 7] for more information on renewal theory related to stochastic heat equation.

### 4 Proof of Theorems 1.1 and 1.2

In this section, we will apply Propositions 2.1 and 2.2 to prove Theorems 1.1 and 1.2.

Recall \( S_{L,t} \) defined in (1.3). Using stochastic Fubini’s theorem, we write from (1.2) that

\[
S_{L,t} = \int_0^t \int_0^L v_{L,t}(s, y) \eta(ds \, dy) = \delta(v_{L,t}) \quad \text{a.s.,}
\]

where

\[
v_{L,t}(s, y) := L^{-1} 1_{(0,t)}(s) 1_{[0,L]}(y) u(s, y) \int_0^L G_{t-s}(x, y) \, dx
\]

\[
= L^{-1} 1_{(0,t)}(s) 1_{[0,L]}(y) u(s, y) I_0(t-s, y),
\]

where in (4.2) we use Lemma A.1 (1) and the definition of \( I_0(t, x) \) in (3.1).

The key technical result of this section is the following proposition:

**Proposition 4.1.** For every \( T > 0 \) there exists a real number \( A_T > 0 \) such that

\[
\sup_{t, \tau \in [0,T]} \text{Var} \left( \langle DS_{L,t}, v_{L,\tau} \rangle_H \right) \leq \frac{A_T}{L^3} \quad \text{for all } L \geq 1.
\]

**Proof.** According to Proposition 1.3.2 of [19], we see from (4.1) that

\[
D_{r,z} S_{L,t} = 1_{(0,t)}(r) v_{L,t}(r, z) + 1_{(0,t)}(r) \int_0^t \int_0^L D_{r,z} v_{L,t}(s, y) \eta(ds \, dy).
\]

Hence,

\[
\langle DS_{L,t}, v_{L,\tau} \rangle_H = \langle v_{L,t}, v_{L,\tau} \rangle_H + \int_0^\tau dr \int_0^L dz \, v_{L,\tau}(r, z) \left( \int_r^t \int_0^L D_{r,z} v_{L,t}(s, y) \eta(ds \, dy) \right)
\]

\[
= \langle v_{L,t}, v_{L,\tau} \rangle_H + \int_0^t \int_0^L \left( \int_0^{r \wedge s} dr \int_0^L dz \, v_{L,\tau}(r, z) D_{r,z} v_{L,t}(s, y) \right) \eta(ds \, dy),
\]
where in the second equality we use the stochastic Fubini’s theorem. Therefore,
\[
\text{Var} \left( \langle DS_{L,t}, v_{L,\tau} \rangle_H \right) \leq 2 \left( \Phi_{L,t,\tau}^{(1)} + \Phi_{L,t,\tau}^{(2)} \right),
\]
(4.5)
where
\[
\Phi_{L,t,\tau}^{(1)} = \text{Var} \left( \langle v_{L,t}, v_{L,\tau} \rangle_H \right),
\]
(4.6)
\[
\Phi_{L,t,\tau}^{(2)} = \text{Var} \left( \int_0^t \int_0^L \left( \int_0^{T,s} \int_0^L \text{dr} \int_0^L \text{dz} \ v_{L,\tau}(r,z)D_{r,z}v_{L,t}(s,y) \right) \eta(\text{ds} \text{dy}) \right).
\]
(4.7)

We estimate the two quantities \( \Phi_{L,t,\tau}^{(1)} \) and \( \Phi_{L,t,\tau}^{(2)} \) separately. Using the expression in (4.2),
\[
\Phi_{L,t,\tau}^{(1)} = \frac{1}{L^4} \int_{[0,t \wedge \tau]^2} \text{ds}_1 \text{ds}_2 \int_{[0,L]^2} \text{dy}_1 \text{dy}_2 \int_0^{s_1 \wedge s_2} \text{dr} \int \text{dz} \ \times \ \|u(s_1, y_1)D_{r,z}u(s_1, y_1)\|_2 \|u(s_2, y_2)D_{r,z}u(s_2, y_2)\|_2
\]
\[
\leq \frac{4c_{T,4}^2 C_{T,4}^2}{L^4} \int_{[0,t \wedge \tau]^2} \text{ds}_1 \text{ds}_2 \int_{[0,L]^2} \text{dy}_1 \text{dy}_2 \int_0^{s_1 \wedge s_2} \text{dr} \int \text{dz} \ p_{s_1-r}(y_1-z)p_{s_2-r}(y_2-z)
\]
\[
= \frac{4c_{T,4}^2 C_{T,4}^2}{L^4} \int_{[0,t \wedge \tau]^2} \text{ds}_1 \text{ds}_2 \int_{[0,L]^2} \text{dy}_1 \text{dy}_2 \int_0^{s_1 \wedge s_2} \text{dr} \ p_{s_1+s_2-2r}(y_1-y_2),
\]
where we have used Hölder’s inequality, Lemma 2.4 and (2.5) in the second inequality, and the identity holds by the semigroup property of heat kernel. Recalling the functions \( I_L \) and \( \tilde{I}_L \) defined in (2.19), we write
\[
\Phi_{L,t,\tau}^{(1)} \leq \frac{4c_{T,4}^2 C_{T,4}^2}{L^2} \int_{[0,t \wedge \tau]^2} \text{ds}_1 \text{ds}_2 \int_0^{s_1 \wedge s_2} \text{dr} \left( I_L * \tilde{I}_L * p_{s_1+s_2-2r} \right)(0)
\]
\[
\leq \frac{4c_{T,4}^2 C_{T,4}^2}{L^3} \int_{[0,t \wedge \tau]^2} \text{ds}_1 \text{ds}_2 \int_0^{s_1 \wedge s_2} \text{dr} \int_{-L}^{L} p_{s_1+s_2-2r}(z) \text{dz}
\]
\[
\leq \frac{4T^3 c_{T,4}^2 C_{T,4}^2}{L^3},
\]
(4.8)
where, in the second inequality, we use [6, (3.17)].

We proceed to estimate \( \Phi_{L,t,\tau}^{(2)} \). By Ito’s isometry and the expression (4.2), we see from (4.7)
that

\[
\Phi^{(2)}_{L,t,\tau} = \frac{1}{L^4} \int_0^t \int_0^L \left| \int_0^L \int_0^L dr_1 dr_2 \int_0^{[0,\tau]^2} dz_1 dz_2 \mathcal{L}_0(\tau - r_1, z_1) \mathcal{I}_0(\tau - r_2, z_2) \mathcal{I}_0^2(t - s, y)\right|^2 dy ds
\]

\[
= \frac{1}{L^4} \int_0^t ds \int_0^L dy \int_0^{[0,\tau]^2} dr_1 dr_2 \int_0^{[0,L]^2} dz_1 dz_2 \mathcal{E}[u(r_1, z_1)(D_{r_1,z_1}u(s,y)) u(r_2, z_2)D_{r_2,z_2}u(s,y)]
\]

\[
\leq \frac{1}{L^4} \int_0^t ds \int_0^L dy \int_0^{[0,\tau]^2} dr_1 dr_2 \int_0^{[0,L]^2} dz_1 dz_2 \left\|u(r_1, z_1)\right\|_4 \left\|D_{r_1,z_1}u(s,y)\right\|_4 \left\|u(r_2, z_2)\right\|_4 \left\|D_{r_2,z_2}u(s,y)\right\|_4,
\]

thanks to (3.2) and Hölder’s inequality. We apply Lemma 2.4 and (2.5) again in order to obtain that

\[
\Phi^{(2)}_{L,t,\tau} \leq \frac{c_{T,4}^2 C_{T,4}^2}{L^4} \int_0^t ds \int_0^L dy \int_0^{[0,\tau]^2} dr_1 dr_2 \int_0^{[0,L]^2} dz_1 dz_2 p_{s-r_1}(y-z_1)p_{s-r_2}(y-z_2)
\]

\[
\leq \frac{c_{T,4}^2 C_{T,4}^2}{L^4} \int_0^t ds \int_0^L dy \int_0^{[0,\tau]^2} dr_1 dr_2 \int_0^{[0,L]^2} dz_1 dz_2 \sqrt{p_{s-r_1}(y-z_1)p_{s-r_2}(y-z_2)}
\]

\[
= \frac{c_{T,4}^2 C_{T,4}^2}{L^4} \int_0^t ds \int_0^L dy \int_0^{[0,\tau]^2} dr_1 dr_2 \int_0^{[0,L]^2} dz_1 dz_2 \sqrt{p_{2s-r_1-r_2}(z_1-z_2)},
\]

where we use semigroup property in the equality. We use again the functions \( I_L \) and \( \tilde{I}_L \) in (2.19) and write

\[
\Phi^{(2)}_{L,t,\tau} \leq \frac{c_{T,4}^2 C_{T,4}^2}{L^2} \int_0^t ds \int_0^L \left( I_L \ast \tilde{I}_L \ast p_{2s-r_1-r_2} \right)(0)
\]

\[
\leq \frac{c_{T,4}^2 C_{T,4}^2}{L^3} \int_0^t ds \int_0^L \left( I_L \ast \tilde{I}_L \ast p_{2s-r_1-r_2} \right) dz
\]

\[
\leq \frac{T^3 c_{T,4}^2 C_{T,4}^2}{L^3},
\]

(4.9)

where the second inequality follows from [6, (3.17)].

Finally, we combine (4.5), (4.8) and (4.9) to conclude the proof. \( \square \)

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** We apply Proposition 4.1 with \( t = \tau \) to see that for all \( T > 0 \) there exists \( A_T > 0 \) such that for all \( t \in [0, T] \)

\[
\text{Var} \left( \langle DS_{L,t}, v_{L,t} \rangle_h \right) \leq \frac{A_T}{L^3} \quad \text{for all } L \geq 1.
\]

By (4.1) and Proposition 2.1,

\[
d_{TV} \left( \frac{S_{L,t}}{\sqrt{\text{Var}(S_{L,t})}}, N(0, 1) \right) \leq 2 \sqrt{\text{Var} \left( \frac{DS_{L,t}}{\sqrt{\text{Var}(S_{L,t})}}, \frac{v_{L,t}}{\sqrt{\text{Var}(S_{L,t})}} \right)_{\mathcal{H}}}
\]

\[
\leq \frac{2\sqrt{A_T}}{L^{3/2}\text{Var}(S_{L,t})} \quad \text{uniformly for all } t \in [0, T] \text{ and } L \geq 1. \quad (4.10)
\]
Proposition 3.1 ensures that \( \text{Var}(S_{L,t}) \sim \int_0^t f(s) \, ds / L \) as \( L \to \infty \), where the function \( f \) is defined in (1.5). This together with (4.10) implies (1.7).

It remains to prove Theorem 1.1, which consists of the weak convergence of the finite-dimensional distributions and tightness. We establish the tightness in the following proposition.

**Proposition 4.2.** For every \( T > 0 \) and \( k \geq 2 \), there exists \( \alpha_{T,k} > 0 \) such that for all \( t_1, t_2 \in [0, T] \),

\[
\| S_{L,t_2} - S_{L,t_1} \|^2 \leq \alpha_{T,k} |t_2 - t_1|^{1/2} L^{-1/2} \quad \text{uniformly for all } L \geq 1.
\]  

(4.11)

**Proof.** From the express of \( S_{L,t} \) in (4.1),

\[
S_{L,t} = \frac{1}{L} \int_0^t \int_0^L \mathcal{I}_0(t - s, y) u(s, y) \eta(ds \, dy),
\]

(4.12)

where \( \mathcal{I}_0 \) is defined in (3.1).

Assume \( t_1 \leq t_2 \).

**Case 1:** Neumann boundary conditions. In this case, we know that \( \mathcal{I}_0 \equiv 1 \). Hence by (4.12), Burkholder’s inequality and Minkowski’s inequality, for all \( k \geq 2 \),

\[
\| S_{L,t_2} - S_{L,t_1} \|^2 \leq \frac{z_k^2}{L^2} \int_{t_1}^{t_2} \int_0^L \| u(s, y) \|^2_k \, dy \, ds,
\]

where \( z_k \) is the constant in Burkholder’s inequality. Moreover, we apply Lemma 2.3 to obtain that for all \( L \geq 1 \),

\[
\| S_{L,t_2} - S_{L,t_1} \|^2 \leq \frac{z_k^2}{L^2} |t_2 - t_1| \sup_{(s, y) \in [0, T] \times [0, L]} \| u(s, y) \|^2_k \leq \frac{c_{T,k} z_k^2}{L} |t_2 - t_1|,
\]

where \( c_{T,k} \) is the constant in (2.4). This proves (4.11) in the case of Neumann boundary conditions.

**Case 2:** Dirichlet boundary conditions. We write from (4.12) and Burkholder’s inequality

\[
\| S_{L,t_2} - S_{L,t_1} \|^2 \leq \frac{2 z_k^2}{L^2} \left( \int_{t_1}^{t_2} \int_0^L \mathcal{I}_0^2(t_2 - s, y) \| u(s, y) \|^2_k \, dy \, ds 
+ \mathcal{I}_0(t_2 - s, y) \mathcal{I}_0(t_1 - s, y) \| u(s, y) \|^2_k \, dy \, ds \right)
\leq \frac{2 z_k^2}{L^2} \left( \int_{t_1}^{t_2} \int_0^L \mathcal{I}_0^2(t_2 - s, y) \, dy \, ds 
+ \mathcal{I}_0(t_2 - s, y) \mathcal{I}_0(t_1 - s, y) \| u(s, y) \|^2_k \, dy \, ds \right)
\]

\[
:= \frac{2 z_k^2}{L^2} (J_1 + J_2),
\]

(4.13)

where we have used (2.5) in the second inequality.

By (3.2),

\[
J_1 \leq L |t_2 - t_1|.
\]

(4.14)
In order to estimate $J_2$, we appeal to the representation of Dirichlet heat kernel (A.5) and write

$$\mathcal{I}_0(t_2 - t_1 + s, y) - \mathcal{I}_0(s, y) = \frac{2}{L} \sum_{n=1}^{\infty} \sin(n\pi y/L) \int_0^L \sin(n\pi z/L) \, dz \left[ e^{-\frac{n^2\pi^2}{2L^2}(t_2 - t_1 + s)} - e^{-\frac{n^2\pi^2}{2L^2}s} \right].$$

$$= 2 \sum_{n=1}^{\infty} \sin(n\pi y/L) \frac{1 - \cos(n\pi)}{n\pi} \left[ e^{-\frac{n^2\pi^2(t_2 - t_1 + s)}{2L^2}} - e^{-\frac{n^2\pi^2}{2L^2}s} \right].$$

Now, we apply the $L^2([0, L])$-orthogonality of the functions $y \mapsto \sin(n\pi y/L)$ to obtain that

$$J_2 = 4 \int_0^{t_1} ds \sum_{n=1}^{\infty} \int_0^L \sin^2(n\pi y/L) \, dy \left[ \frac{1 - \cos(n\pi)}{n\pi} \right]^2 \left[ e^{-\frac{n^2\pi^2(t_2 - t_1 + s)}{2L^2}} - e^{-\frac{n^2\pi^2}{2L^2}s} \right]^2 \int_0^{t_1} ds \left[ e^{-\frac{n^2\pi^2}{2L^2}L^2} \right]$$

$$\leq 8L \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \left[ 1 \land \frac{n^2\pi^2(t_2 - t_1)}{2L^2} \right] \frac{L^2}{n^2\pi^2} \int_0^{t_1} ds \left[ e^{-\frac{n^2\pi^2}{2L^2}L^2} \right].$$

where, in the first inequality, we use the fact that $1 - e^{-x} \leq 1 \land x$ for all $x \geq 0$. Moreover, we have

$$J_2 \leq \frac{8}{L} \left( \sum_{n=|t_2 - t_1|^{-1/2}L/\pi} \left| t_2 - t_1 \right|^2 + \frac{L^4}{\pi^4} \sum_{n>|t_2 - t_1|^{-1/2}L/\pi} \frac{1}{n^4} \right)$$

$$\leq \frac{8}{L} \left( \frac{L}{\pi} \left| t_2 - t_1 \right|^{3/2} + \frac{L^4}{\pi^4} \int_{|t_2 - t_1|^{-1/2}L/2\pi}^{\infty} \frac{1}{y} \, dy \right)$$

$$= \frac{88}{3\pi} \left| t_2 - t_1 \right|^3.$$  \hspace{1cm} (4.15)

Therefore, we conclude from (4.13), (4.14) and (4.15) that for all $k \geq 2$,

$$\|S_{L,t_2} - S_{L,t_1}\|_k \leq \frac{2\pi^2 c_{L,t_2}}{L^2} \left( L^2 \left| t_2 - t_1 \right| + \frac{88}{3\pi} \left| t_2 - t_1 \right|^{3/2} \right),$$

which implies (4.11) in the case of Dirichlet boundary conditions.

The proof is complete. \hfill \Box

We are now in a position to prove Theorem 1.1.

\textbf{Proof of Theorem 1.1.} The tightness of $\{\sqrt{L} S_L \}_{L \geq 1}$ in the space $C[0, T]$ is a direct consequence of Proposition 4.2. Therefore, according to Billingsley [2], it remains to prove that the finite-dimensional distributions of the process $t \mapsto \sqrt{L} S_{L,t}$ converge to those of $t \mapsto \int_0^t \sqrt{f(s)} \, dB_s$ as $L \to \infty$, where $f$ is defined in (1.5).

Let us choose and fix some $T > 0$ and $m \geq 1$ points $t_1, \ldots, t_m \in (0, T]$. Proposition 3.1 ensures that, for every $i, j = 1, \ldots, m$,

$$\text{Cov} \left( S_{L,t_i}, S_{L,t_j} \right) \sim \frac{1}{L} \int_0^{t_i \land t_j} f(s) \, ds \quad \text{as} \quad L \to \infty. \hspace{1cm} (4.16)$$

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Define the following quantities:

\[ F_i := \frac{S_{L,t_i}}{\sqrt{\text{Var}(S_{L,t_i})}} \quad \text{and} \quad C_{i,j} := \text{Cov}(F_i, F_j) \quad \text{for} \quad i, j = 1, \ldots, m. \]

We will write \( F := (F_1, \ldots, F_m) \), and let \( G = (G_1, \ldots, G_m) \) denote a centered Gaussian random vector with covariance matrix \( C = (C_{i,j})_{1 \leq i,j \leq m} \).

Recall from (4.2) the random fields \( v_{L,t_1}, \ldots, v_{L,t_m} \), and define rescaled random fields \( V_1, \ldots, V_m \) as follows:

\[ V_i := \frac{v_{L,t_i}}{\sqrt{\text{Var}(S_{L,t_i})}} \quad \text{for} \quad i = 1, \ldots, m. \]

According to (4.1), \( F_i = \delta(V_i) \) for all \( i = 1, \ldots, m \), and by duality, \( \text{E} \langle DF_i, V_j \rangle_{\mathcal{H}} = \text{E} [F_i \delta(V_j)] = C_{i,j} \) for all \( i, j = 1, \ldots, m \). Therefore, Proposition 2.2 implies that

\[ |\text{E} h(F) - \text{E} h(G)| \leq \frac{1}{2} \| h'' \|_{\infty} \sqrt{\sum_{i,j=1}^m \text{Var}(DF_i, V_j)_{\mathcal{H}}}, \]

for all \( h \in C^2(\mathbb{R}^m) \). By Proposition 4.1,

\[ \text{Var}(DF_i, V_j)_{\mathcal{H}} = \frac{\text{Var}(DS_{L,t_i}, v_{L,t_j})_{\mathcal{H}}}{\text{Var}(S_{L,t_i}) \text{Var}(S_{L,t_j})} \leq \frac{AT}{L^3 \text{Var}(S_{L,t_i}) \text{Var}(S_{L,t_j})}, \]

which together with (4.16) implies that

\[ \lim_{L \to \infty} |\text{E} h(F) - \text{E} h(G)| = 0, \quad \text{for all} \quad h \in C^2(\mathbb{R}^m). \quad (4.17) \]

On the other hand, owing to (4.16), as \( L \to \infty \),

\[ C_{i,j} = \frac{\text{Cov}(S_{L,t_i}, S_{L,t_j})}{\sqrt{\text{Var}(S_{L,t_i}) \text{Var}(S_{L,t_j})}} \to \frac{\int_{0}^{t_i \wedge t_j} f(s) \, ds}{\sqrt{\int_{0}^{t_i} f(s) \, ds \times \int_{0}^{t_j} f(s) \, ds}}, \]

which yields that as \( L \to \infty \), the random vector \( G \) converges weakly to

\[ \left( \frac{\int_{0}^{t_1} \sqrt{f(s)} \, dB_s}{\sqrt{\int_{0}^{t_1} f(s) \, ds}}, \ldots, \frac{\int_{0}^{t_m} \sqrt{f(s)} \, dB_s}{\sqrt{\int_{0}^{t_m} f(s) \, ds}} \right). \quad (4.18) \]

Therefore, it follows from (4.17) that \( F \) converges weakly to the random vector in (4.18) as \( L \to \infty \). One more appeal to (4.16) shows that as \( L \to \infty \),

\[ \sqrt{L} \left( \frac{S_{L,t_1}}{\sqrt{\int_{0}^{t_1} f(s) \, ds}}, \ldots, \frac{S_{L,t_m}}{\sqrt{\int_{0}^{t_m} f(s) \, ds}} \right) \to \left( \frac{\int_{0}^{t_1} \sqrt{f(s)} \, dB_s}{\sqrt{\int_{0}^{t_1} f(s) \, ds}}, \ldots, \frac{\int_{0}^{t_m} \sqrt{f(s)} \, dB_s}{\sqrt{\int_{0}^{t_m} f(s) \, ds}} \right) \]

in distribution. This completes the proof. \( \square \)
A Appendix

We include in this section a few properties of heat kernel with Neumann/Dirichlet boundary conditions that are used in this paper, some of which could also be found in [23] and [1].

Denote the heat kernel on $\mathbb{R}$ as
\[
p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad t > 0, \; x \in \mathbb{R}.
\]

(A.1)

Recall that we have the following formulas for the heat kernels on $[0, L]$:

- In the case of Neumann boundary conditions,
\[
G_t(x, y) = \sum_{n \in \mathbb{Z}} (p_t(x - y + 2nL) + p_t(x + y + 2nL)), \quad t > 0, \; x, y \in [0, L],
\]

(A.2)

or equivalently,
\[
G_t(x, y) = L^{-1/2} + \frac{2}{L} \sum_{n=1}^{\infty} \cos(n\pi x/L) \cos(n\pi y/L) e^{-\frac{n^2 \pi^2 t}{4L^2}}, \quad t > 0, \; x, y \in [0, L];
\]

(A.3)

and in the case of Dirichlet boundary conditions,
\[
G_t(x, y) = \sum_{n \in \mathbb{Z}} (p_t(x - y + 2nL) - p_t(x + y + 2nL)), \quad t > 0, \; x, y \in [0, L],
\]

(A.4)

or equivalently,
\[
G_t(x, y) = 2L^{-1/2} \sum_{n=1}^{\infty} \sin(n\pi x/L) \sin(n\pi y/L) e^{-\frac{n^2 \pi^2 t}{4L^2}}, \quad t > 0, \; x, y \in [0, L].
\]

(A.5)

Lemma A.1. (1) Symmetry. $G_t(x, y) = G_t(y, x)$ for all $t > 0, \; x, y \in [0, L]$.

(2) In the case of Neumann heat kernel, for all $t > 0$ and $x \in [0, L]$,
\[
\int_0^L G_t(x, y) dy = 1.
\]

(A.6)

(3) Semigroup property. For all $t, s > 0$ and $x, y \in [0, L]$,
\[
\int_0^L G_t(x, z) G_s(z, y) dz = G_{t+s}(x, y).
\]

(A.7)

(4) For every $t > 0$ and $x, y \in [0, L]$,
\[
G_t(x, y) \leq p_t(x - y) \left( 4 + \frac{4}{1 - e^{-L^2/t}} \right).
\]

(A.8)

As a consequence, for all $t \in (0, T], \; L \geq 1$ and $x, y \in [0, L]$,
\[
G_t(x, y) \leq K_T p_t(x - y),
\]

(A.9)

where $K_T = 4 + \frac{4}{1 - e^{-1/T}}$.  

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Proof. The properties (1)-(3) are obvious and we need prove (4). Since the Dirichlet heat kernel is less than the Neumann heat kernel (compare (A.2) and (A.4)), it suffices to prove (4) for Neumann heat kernel.

For every $t > 0$, $n \in \mathbb{Z} \setminus \{-1, 1\}$ and $x, y \in [0, L]$, we write

$$p_t(x - y + 2nL) = p_t(x - y)e^{-\frac{4n^2L^2 + 4(x-y)nL}{2t}} \leq p_t(x - y)e^{-\frac{4n^2L^2 - 4|x-y|^2}{2t}} \leq p_t(x - y)e^{-\frac{|y|^2}{t}}.$$ 

And for $n \in \{-1, 1\}$, we have $p_t(x - y + 2nL) \leq p_t(x - y)$ for all $t > 0$ and $x, y \in [0, L]$. Hence, for every $t > 0$ and $x, y \in [0, L]$,

$$\sum_{n \in \mathbb{Z}} p_t(x - y + 2nL) \leq p_t(x - y) \left(1 + 2 \sum_{n = 0}^{\infty} e^{-\frac{n^2}{t}}\right) = p_t(x - y) \left(1 + \frac{2}{1 - e^{-L^2/t}}\right). \quad (A.10)$$

Similarly, for every $t > 0$, $n \in \mathbb{Z} \setminus \{-1, 1, -2, 2\}$ and $x, y \in [0, L]$, we have

$$p_t(x + y + 2nL) = p_t(x - y)e^{-\frac{4n^2L^2 + 4(x+y)nL + 4xy}{2t}} \leq p_t(x - y)e^{-\frac{4n^2L^2 - 8|x+y|^2}{2t}} \leq p_t(x - y)e^{-\frac{|y|^2}{t}},$$

Clearly, $p_t(x + y + 2nL) \leq p_t(x - y)$ for $n \in \{1, -2, 2\}$ for all $t > 0$ and $x, y \in [0, L]$. Moreover, we observe that

$$p_t(x + y - 2L) = p_t(x - y)e^{-\frac{2(L-x)(L-y)}{t}} \leq p_t(x - y).$$

The proceeding estimates together imply that for every $t > 0$ and $x, y \in [0, L]$,

$$\sum_{n \in \mathbb{Z}} p_t(x + y + 2nL) \leq p_t(x - y) \left(3 + 2 \sum_{n = 0}^{\infty} e^{-\frac{n^2}{t}}\right) = p_t(x - y) \left(3 + \frac{2}{1 - e^{-L^2/t}}\right). \quad (A.11)$$

Therefore, we combine (A.10) and (A.11) to obtain (A.8). Finally, (A.9) is an immediate consequence of (A.8).

□

Remark A.2. The heat kernel with periodic boundary conditions is given by

$$G_t(x, y) = \sum_{n \in \mathbb{Z}} p_t(x - y + nL), \quad t > 0, \ x, y \in [0, L], \quad (A.12)$$

which does not satisfy the property (A.9). To see this, suppose that there exists $C_T > 0$ such that

$$\sum_{n \in \mathbb{Z}} p_t(x - y + nL) \leq C_T p_t(x - y), \quad \text{for all } 0 < t \leq T, \ L \geq 1 \text{ and for all } x, y \in [0, L].$$

Letting $x = L$, $y = 0$ and choosing $n = -1$, it leads to $p_t(0) \leq C_T p_t(L)$ for all $L \geq 1$, which gives a contradiction by letting $L \to \infty$.

Lemma A.3. For all $t > 0$,

$$\lim_{L \to \infty} \frac{1}{L} \int_0^L \int_0^L G_t(x, y) \, dx \, dy = 1. \quad (A.13)$$
Proof. It is clear that (A.13) holds for the heat kernel with Neumann boundary conditions; see (A.6). Let us check it is also true for Dirichlet heat kernel. Recall the alternative representation of Dirichlet heat kernel in (A.5). We write for all $t > 0$,

$$
\frac{1}{L} \int_0^L \int_0^L G_t(x, y) \, dx \, dy = \frac{2}{L^2} \sum_{n=1}^{\infty} \int_0^L \sin(n\pi x/L) \, dx \int_0^L \sin(n\pi y/L) \, dy \, e^{-\frac{n^2 \pi^2 t}{2L^2}} \\
= \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(1 - \cos(n\pi))^2}{n^2} \, e^{-\frac{n^2 \pi^2 t}{2L^2}} \\
= \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \, e^{-\frac{(2k-1)^2 \pi^2 t}{2L^2}}.
$$

By dominated convergence theorem, for all $t > 0$,

$$
\lim_{L \to \infty} \frac{1}{L} \int_0^L \int_0^L G_t(x, y) \, dx \, dy = \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \lim_{L \to \infty} e^{-\frac{(2k-1)^2 \pi^2 t}{2L^2}} = \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1,
$$

where the last identity follows from the fact $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$. This proves (A.13).

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