Tilings, packings and expected Betti numbers in simplicial complexes

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Abstract

Let $K$ be a finite simplicial complex. We prove that the normalized expected Betti numbers of a random subcomplex in its $d$-th barycentric subdivision $S_d(K)$ converge to universal limits as $d$ grows to $+\infty$. In codimension one, we use canonical filtrations of $S_d(K)$ to upper estimate these limits and get a monotony theorem which makes it possible to improve these estimates given any packing of disjoint simplices in $S_d(K)$. We then introduce a notion of tiling of simplicial complexes having the property that skeletons and barycentric subdivisions of tileable simplicial complexes are tileable. This enables us to tackle the problem: How many disjoint simplices can be packed in $S_d(K)$, $d \gg 0$?

Keywords : simplicial complex, tilings, packings, barycentric subdivision, Betti numbers, shellable complex.

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1 Introduction

1.1 Average Betti numbers

Let $K$ be a finite $n$-dimensional simplicial complex, $n > 0$, and $S_d(K)$ be its first barycentric subdivision. In [15] we introduced, for every $k \in \{1, \ldots, n\}$, a construction of codimension $k$ subcomplexes $V_\epsilon$ of $S_d(K)$ parametrized by the $(k-1)$-dimensional simplicial cochains $\epsilon \in C^{k-1}(K;\mathbb{Z}/2\mathbb{Z})$, see Definition 2.1. The finite set $C^{k-1}(K;\mathbb{Z}/2\mathbb{Z})$ being canonically a probability space, or rather equipped with a family of probability measures $(\mu_\nu)_{\nu \in [0, 1]}$, where $\mu_\nu$ is the product measure for which the probability that the cochain $\epsilon$ takes the value 0 on a given $(k-1)$-simplex of $K$ is $\nu$ (see [15]), this construction produces a random variable $\epsilon \in C^{k-1}(K;\mathbb{Z}/2\mathbb{Z}) \mapsto V_\epsilon \subset S_d(K)$ which raises the following questions of random topology: What is the expected topology of $V_\epsilon$, e.g. its expected Betti numbers? How does this expected topology behave under large iterated barycentric subdivisions? For every $p \in \{0, \ldots, n-k\}$, we denote the $p$-th Betti number of $V_\epsilon$ by $b_p(V_\epsilon) = \dim H_p(V_\epsilon;\mathbb{Z}/2\mathbb{Z})$ and its mathematical expectation after $d$ barycentric subdivision by $\mathbb{E}_{\nu,d}(b_p) = \int_{C^{k-1}(S_d^d(K);\mathbb{Z}/2\mathbb{Z})} b_p(V_\epsilon) d\mu_\nu(\epsilon)$, $d \geq 0$. We proved in [15] that there exist universal constants $c_p^\pm(n, k)$, such that

$$c_p^-(n, k) \leq \lim inf_{d \to +\infty} \frac{\mathbb{E}_{\nu,d}(b_p)}{(n + 1)^d f_n(K)} \leq \lim sup_{d \to +\infty} \frac{\mathbb{E}_{\nu,d}(b_p)}{(n + 1)^d f_n(K)} \leq c_p^+(n, k),$$

where $f_n(K)$ denotes the number of $n$-dimensional simplices of $K$. These results produced counterparts in this combinatorial framework to the ones obtained in [5] and [7, 8] on the
expected Betti numbers of random real algebraic submanifolds of random projective manifolds or random nodal sets in smooth manifolds respectively (see also [10, 12, 13]). Our first result is the following:

**Theorem 1.1.** Let \( n > 0 \), \( k \in \{1, \ldots, n\} \) and \( p \in \{0, \ldots, n-k\} \). Then, for every finite \( n \)-dimensional simplicial complex \( K \) and every \( \nu \in [0,1] \), the sequence \( \left( \frac{\mathbb{E}_{p,d}(b_p)}{f_n(K)(n+1)!} \right)_{d \geq 0} \) converges and its limit does not depend on \( K \).

We denote by \( e_{p,\nu}(n,k) \) the limit given by Theorem 1.1. Recall that with the exception of the case \( p = 0 \) (see [13]), the counterpart of Theorem 1.1 in the theory of random polynomials ([5]) or random nodal sets ([12, 13, 8, 7, 10, 16]) is open. In the case of the standard \( n \)-simplex \( \Delta_n \), we likewise set \( \tilde{b}_p(V_\epsilon) = \sum b_p(\Sigma_\epsilon) \), where the sum is taken over all connected components \( \Sigma_\epsilon \) of \( V_\epsilon \) which do not intersect the boundary of \( \Delta_n \). Then we set \( \mathbb{E}_{p,d}(\tilde{b}_p) = \int_{C^{k-1}(S^d(\Delta_n);\mathbb{Z}/2\mathbb{Z})} \tilde{b}_p(V_\epsilon) d\mu_\epsilon(\epsilon) \) and get:

**Theorem 1.2.** Let \( n > 0 \), \( k \in \{1, \ldots, n\} \) and \( p \in \{0, \ldots, n-k\} \). Then, for every \( \nu \in [0,1] \), the sequence \( \left( \frac{\mathbb{E}_{p,d}(\tilde{b}_p)}{(n+1)!} \right)_{d \geq 0} \) is increasing and bounded from above.

We denote by \( \tilde{e}_{p,\nu}(n,k) \) the limit of the sequence given by Theorem 1.2. We do not know whether \( e_{p,\nu}(n,k) = \tilde{e}_{p,\nu}(n,k) \) or not, except when \( p = 0 \). It is related to a problem of percolation which we introduce and discuss in § 2.3 see Theorem 2.7 (or [1] for another related problem of percolation). Nevertheless we get:

**Theorem 1.3.** Under the hypothesis of Theorem 1.2, \( e_{p,\nu}(n,k) \geq \tilde{e}_{p,\nu}(n,k) > 0 \). Moreover, \( e_{0,\nu}(n,k) = \tilde{e}_{0,\nu}(n,k) \).

The codimension one case \( k = 1 \) plays a special role. Namely, for every \( \epsilon \in C^0(K;\mathbb{Z}/2\mathbb{Z}) \), we alternatively define \( V_\epsilon' \subset |K| \) such that for every simplex \( \sigma \in K \), \( V_\epsilon' \cap \sigma \) is the convex hull of the middle points of the edges of \( \sigma \) where \( \epsilon \) is not constant, that is where the exterior derivative \( d\epsilon \) does not vanish. We proved in [15] that the pairs \((K_V, V_\epsilon')\) and \((K, V_\epsilon')\) are always homeomorphic, in fact isotopic, see Proposition 2.2 of [15]. When \( K \) is the moment polytope of some toric manifold equipped with a convex triangulation, the hypersurfaces \( V_\epsilon' \) coincide with the patchwork hypersurfaces introduced by O. Viro (see [17, 18] and Remark 2.4 of [15]). These hypersurfaces \( V_\epsilon' \) inherit the structure of a CW complex, having a \( p \)-cell for every \((p+1)\)-simplex \( \sigma \) of \( K \) on which \( \epsilon \) is not constant, \( p \in \{0, \ldots, n-1\} \), see Corollary 2.5 of [15]. We denote by \( C_*(V_\epsilon';\mathbb{Z}/2\mathbb{Z}) \) the cellular chain complex of \( V_\epsilon' \). Moreover, the \( p \)-cell \( V_\epsilon' \cap \sigma \) is isomorphic to a product of two simplices: the simplex spanned by vertices of \( \sigma \) where \( \epsilon = 0 \) and the simplex spanned by vertices of \( \sigma \) where \( \epsilon = 1 \).

We now observe that each \( \epsilon \in C^0(K;\mathbb{Z}/2\mathbb{Z}) \) induces a filtration \( \emptyset \subset K_0' \subset K_1' \subset \ldots \subset K_{[n+1]}' = K \), where for every \( i \in \{0, \ldots, \lfloor n+1 \rfloor \} \),

\[
K_i' = \{ \sigma \in K \mid \#\epsilon^{-1}_\sigma(0) \leq i \text{ or } \#\epsilon^{-1}_\sigma(1) \leq i \}.
\]

This \( i \)-th subcomplex \( K_i' \) of \( K \) is thus the union of all simplices that \( V_\epsilon' \) meets along the empty set if \( i = 0 \) or along a product of two simplices, one of which being of dimension \( \leq i - 1 \) if \( i > 0 \). Then:

**Theorem 1.4.** For every finite \( n \)-dimensional simplicial complex \( K \), \( n > 0 \), and every \( \epsilon \in C^0(K;\mathbb{Z}/2\mathbb{Z}) \), the relative simplicial chain complex \( C_*(K, K_0';\mathbb{Z}/2\mathbb{Z}) \) and the shifted cellular chain complex \( C_*(V_\epsilon';\mathbb{Z}/2\mathbb{Z})^{[1]} \) are canonically isomorphic.
We set, for every $p \in \{0, \ldots, n-1\}$ and $d \geq 0$, $\mathbb{E}_{p,d}(b_p(K_0^d)) = \int_{C^0(S^d)} b_p(K_0^d) d\mu_0(\epsilon)$ and will write $\mathbb{E}_\nu$ instead of $\mathbb{E}_{\nu,0}$ for simplicity. We deduce:

**Corollary 1.5.** For every finite $n$-dimensional simplicial complex $K$, $n > 0$, and every $p \in \{0, \ldots, n-1\}$ and $\nu \in [0,1]$, $\mathbb{E}_\nu(b_p(K_0^d)) - b_p(K) \leq \mathbb{E}_\nu(b_p(V'_i)) \leq \mathbb{E}_\nu(b_p(K_0^d)) + b_{p+1}(K)$.

In particular,

$$\lim_{d \to +\infty} \frac{\mathbb{E}_{p,d}(b_p(K_0^d))}{f_n(K)(n+1)!} = e_{p,\nu}(n,1).$$

This result makes it possible to improve the upper estimates of $\mathbb{E}_{\nu,d}(b_p)$ or of $e_{p,\nu}(n,k)$ given in [15] in the case $k = 1$ and to relate them with a packing problem in $K$ or $S^d(K), d > 0$. To this end, for every finite simplicial complex $K$, every $p \in \mathbb{N}$ and $\nu \in [0,1]$, we set

$$M_{p,\nu}(K) = (\nu^{p+1} + (1-\nu)^{p+1})b_p(K) + \nu(1-\nu)(\nu^p + (1-\nu)^p) \sum_{i=p+1}^{\dim K} (-1)^{i+1-p}(f_i(K) - b_i(K)),$$

where $f_i(K)$ denotes the number of $i$-dimensional simplices of $K$ and the sum vanishes if $p \geq \dim K$. The positivity of $M_{p,\nu}(K)$ is closely related to the Morse inequalities associated to the simplicial chain complex of $K$. Our key result is then the following monotony theorem, (see Theorem 3.7).

**Theorem 1.6.** Let $p \in \mathbb{N}$ and $\nu \in [0,1]$. For every finite simplicial complex $K$ and every subcomplex $L$ of $K$, $0 \leq M_{p,\nu}(L) - \mathbb{E}_\nu(b_p(L_0^d)) \leq M_{p,\nu}(K) - \mathbb{E}_\nu(b_p(K_0^d)).$

When $L = \emptyset$, Theorem 1.6 combined with Corollary 1.5 already improves the upper estimates given in Corollary 4.2 of [15] for $k = 1$. But these get improved further whenever the left hand side in Theorem 1.6 is positive. This turns out to be the case when $L$ is a packing of disjoint simplices in $K$, or more generally of simplices which intersect along faces of dimensions less than $p-1$, since $\mathbb{E}_\nu(b_p(L_0^d))$ vanishes in this case as soon as $p > 0$, see Proposition 3.6. Theorem 1.6 combined with Proposition 3.6 raises the following packing problem which we tackle in the second part of the paper, independent from the first one: How many disjoint simplices can be packed in the finite simplicial complex $K$? What about the asymptotic of such a maximal packing in $S^d(K), d \gg 0$? Indeed, let us denote by $L^{p,p}_d$ the finite set of packings of simplices in $S^d(K)$ which intersect each other and the boundary of $S^d(K)$ along faces of dimensions less than $p-1$, where $n, d > 0$ and $p \in \{0, \ldots, n-1\}$. We set

$$\lambda_{p,\nu}^d(n) = \frac{1}{(n+1)!} \max_{L \in L^{p,p}_d} M_{p,\nu}(L).$$

This sequence $(\lambda_{p,\nu}^d(n))_{d \geq 0}$ is increasing and bounded from above, see Proposition 3.9, and we denote by $\lambda_{p,\nu}(n)$ its limit as $d$ grows to $+\infty$. We deduce the following asymptotic result.

**Theorem 1.7.** For every $n > 0$, $\nu \in [0,1]$ and $p \in \{1, \ldots, n-1\},$

$$e_{p,\nu}(n,1) \leq \nu(1-\nu)(\nu^p + (1-\nu)^p) \sum_{i=p+1}^{n} (-1)^{i+1-p} q_{i,n} - \lambda_{p,\nu}(n).$$

In Theorem 1.7 $q_{i,n}$ denotes the asymptotic face number $\lim_{d \to +\infty} \frac{f_i(S^d(K))}{(n+1)!}$, see [3, 4, 14]. We do not know the actual value of $\lambda_{p,\nu}(n)$, but the results of the second part of this paper make it possible to estimate $\lambda_{p,\nu}(n)$ from below, see Theorem 1.13. Is it possible to likewise improve the upper estimates of $\Theta, \Theta'$? What would then play the role of these packings?
1.2 Tilings and Packings

For every positive dimension $n$ and every $s \in \{0, \ldots, n+1\}$, we define the tile $T^n_s$ to be the complement of $s$ facets in the standard $n$-simplex $\Delta_n$. In particular, $T^n_0 = \Delta_n$ and $T^n_{n+1} = \Delta_n^\circ$, the interior of $\Delta_n$. An $n$-dimensional simplicial complex $K$ is called tileable when $|K|$ can be covered by disjoint $n$-dimensional tiles. For instance, the boundary $\partial \Delta_{n+1}$ of the standard $(n+1)$-simplex is tileable and it has a tiling which uses each tile $T^n_s$ exactly once, $s \in \{0, \ldots, n+1\}$, see Corollary 4.2. The $h$-vector $h(T) = (h_0(T), \ldots, h_{n+1}(T))$ of a finite tiling $T$ encodes the number of times $h_s(T)$ each tile $T^n_s$ is used in the tiling, $s \in \{0, 1, \ldots, n+1\}$. We observe the following (see Theorem 4.9):

**Theorem 1.8.** Let $K$ be a tileable finite $n$-dimensional simplicial complex. Then, two tilings $T$ and $T'$ of $K$ have the same $h$-vector provided $h_0(T) = h_0(T')$. When $h_0(T) = 1$, it coincides with the $h$-vector of $K$.

Theorem 1.8 thus provides a geometric interpretation of the $h$-vector for tileable finite $n$-dimensional simplicial complexes which have a tiling $T$ such that $h_0(T) = 1$. When $K$ is connected, we call such a tiling regular. Note that such an interpretation was known for shellable simplicial complexes, a closely related notion, see §4.2. Tileable simplicial complexes have the following key property (see Proposition 4.4 and Corollary 4.15).

**Theorem 1.9.** Let $K$ be a tileable $n$-dimensional simplicial complex. Then, all its skeletons and barycentric subdivisions are tileable. Moreover, any tiling of $K$ induces a tiling on its skeletons and barycentric subdivisions.

We denote by $\text{Sd}(T)$ the tiling of $\text{Sd}(K)$ induced by a tiling $T$ of $K$. It was proved by A. Björner [2] that the barycentric subdivision of a shellable simplicial complex is shellable. Our proof seems more geometric. We actually prove that the first barycentric subdivision $\text{Sd}(T^n_s)$ of each tile $T^n_s$ is tileable, $s \in \{0, \ldots, n+1\}$, see Theorem 4.13. We then study the matrix $H_0$ of size $(n+2) \times (n+2)$ whose rows are the $h$-vectors of the tilings of $\text{Sd}(T^n_s)$, $s \in \{0, \ldots, n+1\}$. Let $\rho_n$ be the involution $(h_0, \ldots, h_{n+1}) \in \mathbb{R}^{n+2} \mapsto (h_{n+1}, \ldots, h_1, h_0) \in \mathbb{R}^{n+2}$. We prove the following:

**Theorem 1.10.** For every $n > 0$, $H_n$ is diagonalizable with eigenvalues $s^!, s \in \{0, \ldots, n+1\}$. Moreover, it commutes with $\rho_n$ and the restriction of $\rho_n$ to the eigenspace of $s^!$ is $(-1)^{n+1-s}\text{id}$.

We denote by $h^n = (h^n_0, \ldots, h^n_{n+1})$ the eigenvector of the transposed matrix $H_n^t$ associated to the eigenvalue $(n+1)!$ and normalize it in such a way that $|h^n| = \sum_{s=0}^{n+1} h^n_s = 1$. We prove that $h^n_0 = h^n_{n+1} = 0$ and that $\rho_n(h^n) = h^n$, see Corollary 4.18. Moreover:

**Theorem 1.11.** Let $K$ be a finite $n$-dimensional simplicial complex equipped with a tiling $T$. Then, the sequence $\frac{1}{h(T)(n+1)!} h(\text{Sd}^d(T))$ converges to $h^n$ as $d$ grows to $+\infty$. Moreover, the matrix $\frac{1}{(n+1)!} H_n^d$ converges to $(1, \ldots, 1)(h^n)^t$ as $d$ grows to $+\infty$.

Hence, the asymptotic $h$-vector of a tiled finite $n$-dimensional simplicial complex $K$ does not depend on $K$ and equals $h^n$. This asymptotic result for $h$-vectors has to be compared with the asymptotic of the $f$-vectors obtained in [3] (see also [4] and [14]). This notion of tiling makes it possible to study the packing problem in tileable simplicial complexes or rather in their first barycentric subdivisions. In particular, we prove the following:
Theorem 1.12. Let $K$ be a finite $n$-dimensional simplicial complex equipped with a tiling $T$. Then, it is possible to pack $h_0(T) + h_1(T)$ disjoint $n$-simplices in $\text{Sd}(K)$. Moreover, this packing can be completed by $h_{n+1-j}(T) + 2^{n-1-j} \sum_{s=0}^{n-1-j} \frac{h_s(T)}{2^s}$ disjoint $j$-simplices for every $j \in \{0, \ldots, n-1\}$.

Since all barycentric subdivisions of the standard simplex are tileable, we are finally able to deduce in the limit the following lower estimates.

Theorem 1.13. For every $n \geq 2$ and $p \in \{1, \ldots, n-1\}$,

$$\lambda_{p,\nu}(n) \geq \frac{\nu(1-\nu)(\nu^p + (1-\nu)^p)}{n+1} \left[ \left( h_{p+2}^n + 2^{p-1} \sum_{i=0}^{p-1} \frac{h_{2i}^n}{2^i} \right) \left( \begin{array}{c} n \\ p+1 \end{array} \right) + \sum_{j=p+1}^{n+1} \left( h_j^n + 2^{j-2} \sum_{i=0}^{j-2} \frac{h_{2i}^n}{2^i} \right) \left( \begin{array}{c} n+p-j \\ p+1 \end{array} \right) \right].$$

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2 Asymptotic behavior of the expected Betti numbers

In the sequel, we denote by $K^{[i]}$ the set of $i$-dimensional simplices of a simplicial complex $K$ and by $K^{[i]}$ its $i$-skeleton. We first prove Theorems 1.2 and 1.3 and then Theorem 1.1. In §2.3 we study a problem of percolation related to these results.
2.1 Proofs of Theorems 1.2 and 1.3

Given a simplicial complex $K$, a $q$-simplex of $Sd(K)$ is of the form $[\hat{\sigma}_0, \ldots, \hat{\sigma}_q]$, where $\hat{\sigma}_i$ denotes the barycenter of the simplex $\sigma_i$ of $K$, $i \in \{0, \ldots, q\}$, and where $\sigma_i$ is a proper face of $\sigma_{i+1}, i \in \{0, \ldots, q-1\}$. The block $D(\sigma)$ dual to a simplex $\sigma \in K$ is the union of all open simplices $[\hat{\sigma}_0, \ldots, \hat{\sigma}_q]$ of $Sd(K)$ such that $\sigma_0 = \sigma$. The union of closed such simplices is denoted by $\overline{D}(\sigma)$, see [11]. We recall the following definition from [15].

Definition 2.1. Let $K$ be an $n$-dimensional simplicial complex, $n > 0$, and let $k \in \{1, \ldots, n\}$. For every $\epsilon \in C^{k-1}(K; \mathbb{Z}/2\mathbb{Z})$, we denote by $V_\epsilon$ the subcomplex of $Sd(K)$ dual to the cocycle $d\epsilon$, where $d : C^{k-1}(K; \mathbb{Z}/2\mathbb{Z}) \to C^k(K; \mathbb{Z}/2\mathbb{Z})$ denotes the coboundary operator. Hence, $V_\epsilon$ is the union of the blocks $D(\sigma)$ dual to the $k$-simplices $\sigma \in K$ such that $\langle d\epsilon, \sigma \rangle \neq 0$.

Proof of Theorem 1.2. For every $m \in \{0, \ldots, d\}$, we observe that

$$E_{\nu,d}(\tilde{b}_p) = \int_{C^{k-1}(Sd^d(\Delta_0); \mathbb{Z}/2\mathbb{Z})} \tilde{b}_p(V_\epsilon) d\nu(\epsilon) \geq \int_{C^{k-1}(Sd^d(\Delta_0); \mathbb{Z}/2\mathbb{Z})} \sum_{\sigma \in Sd^{d-m}(\Delta_0)^{[n]}} \tilde{b}_p(V_\epsilon \cap \sigma) d\mu(\epsilon) \geq \sum_{\sigma \in Sd^{d-m}(\Delta_0)^{[n]}} \int_{C^{k-1}(Sd^d(\sigma); \mathbb{Z}/2\mathbb{Z})} \tilde{b}_p(V_\epsilon) d\mu(\epsilon),$$

where $\mu$ is a product measure, and $f_n(Sd^{d-m}(\Delta_0)) = (n+1)^{d-m} \Sigma E_{\nu,m}(\tilde{b}_p)$, as $f_n(Sd^{d-m}(\Delta_0)) = (n+1)^{d-m}$.

The sequence $(E_{\nu,d}(\tilde{b}_p))_{d \geq 0}$ is thus increasing. Moreover, it is bounded from above by the converging sequence $(f_n(Sd^{d}(\Delta_0)))_{d \geq 0}$, since $V_\epsilon$ is a subcomplex of $Sd^{d+1}(\Delta_0)$, see [3, 4] or [14]. Hence the result. □

Remark 2.2. For every $m \geq 1$, let $C(m)$ be the finite set of homeomorphism classes of pairs $(\mathbb{R}^n, \Sigma)$, where $\Sigma$ is a closed manifold of dimension $n-k$ embedded in $\mathbb{R}^n$ by an embedding of complexity $m$, see Definition 5.2 and § 5.3 of [15]. With the notations of § 5.3 of [15], we deduce the following lower estimate.

$$\hat{e}_{p,\nu}(n,k) \geq E_{\nu,d}(\tilde{b}_p) \frac{E_{\nu,d}(\tilde{b}_p)}{(n+1)^d}, \text{ for every } d \geq 0 \text{ by Theorem 1.2}$$

$$= \sum_{m=1}^d \sum_{\Sigma \in C(m)} b_p(\Sigma) \left( \frac{E_{\nu,d}(\tilde{b}_p)}{(n+1)^d} \right) \geq \sum_{m=1}^d \sum_{\Sigma \in C(m)} b_p(\Sigma) c_\Sigma, \text{ by Theorem 5.11 of [15].}$$

By taking the limit as $d$ grows to $+\infty$, we deduce that $\hat{e}_{p,\nu}(n,k) \geq c_{\nu}(n,k)$ by Definition 5.12 of [15].

Proof of Theorem 1.3. By definition, for every $\epsilon \in C^{k-1}(Sd^{d}(\Delta_0); \mathbb{Z}/2\mathbb{Z})$ and $p \in \{0, \ldots, n-k\}$, $\tilde{b}_p(V_\epsilon) \leq b_p(V_\epsilon)$ so that after integration over $C^{k-1}(Sd^{d}(\Delta_0); \mathbb{Z}/2\mathbb{Z})$, we get $E_{\nu,d}(\tilde{b}_p) \leq E_{\nu,d}(b_p)$. The inequality $\hat{e}_{p,\nu}(n,k) \leq e_{p,\nu}(n,k)$ is deduced from Theorems 1.1 and 1.2 by passing to the limits as $d$ tends to $+\infty$, after dividing by $(n+1)^d$.

When $p = 0$, we remark that by definition, for every $\epsilon \in C^{k-1}(Sd^{d}(\Delta_0); \mathbb{Z}/2\mathbb{Z})$, $b_0(V_\epsilon) \leq \tilde{b}_0(V_\epsilon) + b_0(V_\epsilon \cap \partial\Delta_n)$, as the number of connected components of $V_\epsilon$ which meet $\partial\Delta_n$ is bounded from above by $b_0(V_\epsilon \cap \partial\Delta_n)$. We deduce the inequality

$$\frac{E_{\nu,d}(b_0)}{(n+1)^d} \leq \frac{E_{\nu,d}(\tilde{b}_0)}{(n+1)^d} + \frac{E_{\nu,d}(b_0(V_\epsilon \cap \partial\Delta_n))}{(n+1)^d}. \ (1)$$
However, if \( k = n \), \( \mathbb{E}_{\nu, d}(b_0(V_{\epsilon} \cap \partial \Delta_n)) = 0 \) and so the result follows. If \( k < n \),

\[
\mathbb{E}_{\nu, d}(b_0(V_{\epsilon} \cap \partial \Delta_n)) = \begin{cases} 
\int_{C^{k-1}((Sd^d(\Delta_n), \mathbb{Z}^{22})} b_0(V_{\epsilon} \cap \partial \Delta_n) d\mu_\nu(\epsilon) \\
\int_{C^{k-1}((Sd^d(\Delta_n), \mathbb{Z}^{22})} b_0(V_{\epsilon}) d\mu_\nu(\epsilon), \text{ as } \mu_\nu \text{ is a product measure,}
\end{cases}
\]

\( = O(n^d) \), from Theorem 1.2 of [15].

By letting \( d \) tend to \( +\infty \), we thus deduce from (1) that \( e_{0, \nu}(n, k) \leq \tilde{e}_{0, \nu}(n, k) \). Hence the result.

\[\square\]

**Remark 2.3.** The equality \( e_{0, \nu}(n, k) = \tilde{e}_{0, \nu}(n, k) \) is similar to [13]. Moreover, Theorem 1.3 combined with Remark 2.2 provides the lower bound \( e_{p, \nu}(n, k) \geq c_p(n, k) \) given by Corollary 5.13 of [13].

Theorem 1.3 raises the following question: Are the limits \( e_{p, \nu}(n, k) \) and \( \tilde{e}_{p, \nu}(n, k) \) equal in general or not? This question is related to a problem of percolation that we discuss in §2.3.

### 2.2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we first need several preliminary results. For every \( m \in \{0, \ldots, d\} \), we consider

\[
A_{d,m} = \bigcup_{\sigma \in Sd^{d-m}(K)^{|n|}} \tilde{\sigma} \quad \text{and} \quad B_{d,m} = \bigcup_{\tau \in Sd^m(Sd^{d-m}(K)^{(n-1)})} D(\tau),
\]

so that \( A_{d,m}, B_{d,m} \) are two open sets covering the underlying topological space \( |K| \). For every \( q \)-simplex \( \theta = [\hat{\theta}_0, \ldots, \hat{\theta}_q] \in Sd^{d+1}(K) \), let \( l_m(\theta) \in \{-1, \ldots, q\} \) be the greatest integer \( i \) such that \( \theta_i \in Sd^m(Sd^{d-m}(K)^{(n-1)}) \), where \( l_m(\theta) = -1 \) if and only if there is no such integer \( i \). When \( l_m(\theta) \neq -1 \), we moreover set

\[
r_\theta : \begin{cases} 
\theta \setminus [\hat{\theta}_{l_m(\theta)+1}, \ldots, \hat{\theta}_q] \times [0, 1] & \to \theta \setminus [\hat{\theta}_{l_m(\theta)+1}, \ldots, \hat{\theta}_q] \\
(1-\alpha)\gamma + \alpha z, t) & \mapsto (1-t\alpha)\gamma + t\alpha z,
\end{cases}
\]

where \( \alpha \in [0, 1], \gamma \in [\hat{\theta}_0, \ldots, \hat{\theta}_{l_m(\theta)}] \) and \( z \in [\hat{\theta}_{l_m(\theta)+1}, \ldots, \hat{\theta}_q] \).

**Proposition 2.4.** Let \( K \) be a finite simplicial complex of positive dimension \( n \) and let \( \theta = [\hat{\theta}_0, \ldots, \hat{\theta}_q] \) be a \( q \)-simplex of \( Sd^{d+1}(K) \), \( q \in \{0, \ldots, n\}, \) \( d \geq 0 \). Then for every \( m \in \{0, \ldots, d\} \),

1. \( B_{d,m} \cap \theta = \theta \setminus [\hat{\theta}_{l_m(\theta)+1}, \ldots, \hat{\theta}_q] \) where \( [\hat{\theta}_{l_m(\theta)+1}, \ldots, \hat{\theta}_q] \) is the union of the open simplices \( Sd^d(\Delta_n), \mathbb{Z}^{22}) \) such that \( l_m(\theta) = q \).

2. If \( l_m(\theta) \neq -1 \), \( r_\theta \) retracts \( \theta \setminus [\hat{\theta}_{l_m(\theta)+1}, \ldots, \hat{\theta}_q] \) by deformation onto \( [\hat{\theta}_0, \ldots, \hat{\theta}_{l_m(\theta)}] \).

3. For every face \( \sigma \in \theta \) such that \( l_m(\sigma) \neq -1 \), the restriction of \( r_\theta \) to \( \sigma \) is \( r_\sigma \).

**Proof.** By definition, \( B_{d,m} \) is the union of the open simplices \( Sd^d(\Delta_n), \mathbb{Z}^{22}) \) such that \( l_m(\sigma) \neq -1 \). An open face of \( \theta \) is thus included in \( B_{d,m} \) if and only if it contains a vertex \( \hat{\theta}_i \) with \( i \leq l_m(\theta) \). Hence the first part. The second and third parts follow from the definition of \( r_\theta \). \( \square \)

**Corollary 2.5.** Under the hypothesis of Proposition 2.4, for every \( m \in \{0, \ldots, d\} \),
1. $B_{d,m}$ is an open subset of $|S^{d+1}(K)| = |K|$ and $S^{d+1}(K) \setminus B_{d,m}$ is a subcomplex of $S^{d+1}(K)$.

2. For every subcomplex $L$ of $S^{d+1}(K)$, the retractions $(r_\theta)_{\theta \in L}$ glue together to define $r_L : L \cap B_{d,m} \to L \cap B_{d,m}$. Moreover, $r_L$ retracts $L \cap B_{d,m}$ by deformation onto $L \cap S^{m+1}(S^{d-m}(K)(n-1))$.

3. For every subcomplexes $M < L < S^{d+1}(K)$, the restriction of $r_L$ to $M$ is $r_M$.

Proof. From the first part of Proposition 2.4, the complement of $B_{d,m}$ in $S^{d+1}(K)$ is the union of simplices $\theta$ such that $l_m(\theta) = -1$. It is a subcomplex of $S^{d+1}(K)$, which is closed. Hence the first part. The third part of Proposition 2.4 guarantees that the retractions $(r_\theta)_{\theta \in L}$ glue together to define $r_L : L \cap B_{d,m} \to L \cap B_{d,m}$ and the second part of Proposition 2.4 guarantees that the latter is a retraction of $L \cap B_{d,m}$ to $L \cap S^{m+1}(S^{d-m}(K)(n-1))$. Finally, the last part of Corollary 2.5 follows from the last part of Proposition 2.4.

Proposition 2.6. Under the hypothesis of Theorem 1.1 there exists a universal constant $c(n)$ such that for every $m \in \{0, \ldots, d\}$,

$$|E_{\nu,d}(b_p) - E_{\nu,d}(b_p(V \cap A_{d,m}))| \leq (n+1)!f_n(K)\frac{c(n)}{(n+1)^m} + O(n^d).$$

Proof. It follows from Corollary 2.5 that for every $\epsilon \in C^{k-1}(S^d(K); \mathbb{Z}/2\mathbb{Z})$, $(A_{d,m} \cap V_\epsilon) \cup (B_{d,m} \cap V_\epsilon)$ is an open cover of $V_\epsilon$. The long exact sequence of Mayer-Vietoris associated to this open cover (see § 33 of [1]) reads

$$\ldots \to H_p(A_{d,m} \cap B_{d,m} \cap V_\epsilon) \xrightarrow{i_p} H_p(A_{d,m} \cap V_\epsilon) \oplus H_p(B_{d,m} \cap V_\epsilon) \xrightarrow{j_p} H_p(V_\epsilon) \xrightarrow{\partial_p} H_{p-1}(A_{d,m} \cap B_{d,m} \cap V_\epsilon) \to \ldots$$

where the coefficients are in $\mathbb{Z}/2\mathbb{Z}$. We thus deduce that

$$b_p(V_\epsilon) = \dim \text{Im}(\partial_p) + \dim \text{Im}(j_p) = \dim \text{Im}(\partial_p) + b_p(A_{d,m} \cap V_\epsilon) + b_p(B_{d,m} \cap V_\epsilon) - \dim \text{Im}(i_p).$$

After integration over $C^{k-1}(S^d(K); \mathbb{Z}/2\mathbb{Z})$, we get

$$|E_{\nu,d}(b_p) - E_{\nu,d}(A_{d,m} \cap V_\epsilon)| \leq E_{\nu,d}(b_p(B_{d,m} \cap V_\epsilon)) + E_{\nu,d}(b_p(A_{d,m} \cap B_{d,m} \cap V_\epsilon)) + E_{\nu,d}(b_{p-1}(A_{d,m} \cap B_{d,m} \cap V_\epsilon)).$$

Let us now bound each term of the right hand side. We know from Corollary 2.5 that for every $\epsilon \in C^{k-1}(S^d(K); \mathbb{Z}/2\mathbb{Z})$, $r_{V_\epsilon}$ retracts $V_\epsilon \cap B_{d,m}$ by deformation onto $V_\epsilon \cap S^{m+1}(S^{d-m}(K)(n-1))$. Thus,

$$E_{\nu,d}(b_p(V_\epsilon \cap B_{d,m})) = E_{\nu,d}(b_p(V_\epsilon \cap S^{m+1}(S^{d-m}(K)(n-1)))) \leq f_p(S^{m+1}(S^{d-m}(K)(n-1))) \leq \sum_{m+1} \sum_{\sigma \in S^{d-m}(K)(n-1)} f_p(S^{m+1}(\sigma)) \leq f_s(S^{d-m}(K))f_p(S^{m+1}(\Delta_{n-1})).$$

where $f_s(S^{d-m}(K))$ denotes the total face number of $S^{d-m}(K)$. 

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Let $\tilde{K}$ be the subcomplex of $K$ made of simplices which are faces of $n$-simplices of $K$. Then,

$$f_*\left(Sd^{d-m}(K)\right) = f_*\left(Sd^{d-m}(\tilde{K})\right) + f_*\left(Sd^{d-m}(K\setminus \tilde{K})\right)$$

$$\leq f_n(K)f_*(Sd^{d-m}(\Delta_n)) + f_*(Sd^{d-m}(K\setminus \tilde{K}))$$

$$\leq f_n(K)f_*(Sd^{d-m}(\Delta_n)) + O(n!^{d-m}),$$

from [3], see also [4, 14], since the simplices of $K\setminus \tilde{K}$ are of dimension at most $(n-1)$. Likewise, the sequences $\left(\frac{f_*(Sd^{d-m}(\Delta_n))}{(n+1)!^{d-m}}\right)_{d\in\mathbb{N}}$ and $\left(\frac{f_*(Sd^{d+m+1}(\Delta_{n-1}))}{n^{m+1}}\right)_{m\in\mathbb{N}}$ are universal and convergent, see [3, 14]. They are thus bounded by universal constants. We deduce the existence of a universal constant $c_1(n)$ such that $E_{\nu,d}(b_*(B_{d,m}\cap V_\epsilon)) \leq f_n(K)(n+1)!^{d-m}c_1(n) + O(n!^d)$.

Similarly, from Corollary 2.5 we know that for every $\epsilon \in C^{k-1}(Sd^d(K);\mathbb{Z}/2\mathbb{Z})$ and every $\sigma \in Sd^d-m(K)[m]$,

$$r_{V_\epsilon\cap \sigma} \text{ is a deformation retract of } B_{d,m} \cap \sigma \cap V_\epsilon \text{ onto } \partial \sigma \cap V_\epsilon.$$ So we have

$$E_{\nu,d}(b_*(A_{d,m} \cap B_{d,m} \cap V_\epsilon)) = \sum_{\sigma \in Sd^{d-m}(K)[m]} E_{\nu,d}(b_*(B_{d,m} \cap \sigma \cap V_\epsilon))$$

$$\leq \sum_{\sigma \in Sd^{d-m}(K)[m]} f_*(Sd^{d+m+1}(\partial \sigma))$$

$$\leq f_n(K)(n+1)!^{d-m}f_*(Sd^{d+m+1}(\partial \Delta_n)).$$

Once again the sequence $\left(\frac{f_*(Sd^{d+m+1}(\partial \Delta_n))}{n^{m+1}}\right)_{m\in\mathbb{N}}$ is universal and convergent and thus gets bounded by a universal constant. The third term in [2] is analogous to the second one, so it gets bounded by a universal constant as well. Hence the result.

**Proof of Theorem 1.1.** We first prove the result for $K = \Delta_n$ and then deduce it for a general simplicial complex $K$. For every $d \geq 0$, we set

$$u_d = \frac{1}{(n+1)!^d} \int_{C^{k-1}(Sd^d(\Delta_n);\mathbb{Z}/2\mathbb{Z})} b_*(V_\epsilon)d\mu_\nu(\epsilon).$$

For every $\epsilon \in C^{k-1}(Sd^d(\Delta_n);\mathbb{Z}/2\mathbb{Z})$ and every $m \in \{0, \ldots, d\}$,

$$b_*(V_\epsilon \cap A_{d,m}) = \sum_{\sigma \in Sd^{d-m}(\Delta_n)[n]} b_*(V_\epsilon \cap \sigma).$$

After integration over $C^{k-1}(Sd^d(\Delta_n);\mathbb{Z}/2\mathbb{Z})$, we get $E_{\nu,d}(b_*(V_\epsilon \cap A_{d,m})) = (n+1)!^{d-m}E_{\nu,m}(b_*)$ since $\mu_\nu$ is a product measure, $\#Sd^{d-m}(\Delta_n)[n] = (n+1)!^{d-m}$ (see [3]) and $b_*(V_\epsilon \cap \sigma) = b_*(V_\epsilon \cap \sigma)$ by Corollary 2.5. Dividing by $(n+1)!^d$, we deduce from Proposition 2.6 the upper bound

$$|u_d - u_m| \leq \frac{c(n)}{(n+1)^m} + O\left(\frac{1}{(n+1)!^d}\right).$$

The sequence $(u_d)_{d\in\mathbb{N}}$ is thus a Cauchy sequence and so a converging sequence. The result follows for $K = \Delta_n$. Let us now suppose that $K$ is any finite simplicial complex of dimension $n$ and set, for every $d \geq 0$,

$$v_d = \frac{1}{f_n(K)(n+1)!^d} \int_{C^{k-1}(Sd^d(K);\mathbb{Z}/2\mathbb{Z})} b_*(V_\epsilon)d\mu_\nu(\epsilon).$$
We deduce from Proposition 2.6 after dividing by \( f_n(K)(n + 1)^d \), that \( |v_d - u_m| \leq \frac{c(n)}{(n+1)^m} + O\left(\frac{1}{(n+1)^m}\right) \). Since \((u_m)_{m \in \mathbb{N}}\) converges, this implies that \((v_d)_{d \in \mathbb{N}}\) converges as well and that \( \lim_{d \to +\infty} v_d = \lim_{m \to +\infty} u_m \). Hence the result.

\[\square\]

### 2.3 Percolation

Let \( n > 0 \) and \( k \in \{1, \ldots, n\} \). For every \( 0 \leq m \leq d \) and every \( \sigma \in \text{Sd}^{d-m}(\Delta_n)^{[m]} \), we denote by \( P_{d,m}(\sigma) \) the set of \( \epsilon \in C^{k-1}(\text{Sd}^m(\text{Sd}^{d-m}(\Delta_n) \setminus \hat{\sigma})) \) for which there exists a path in \( V_\epsilon \) connecting \( \partial \Delta_n \) to \( \partial \sigma \). In other words, \( \epsilon \in P_{d,m}(\sigma) \) if and only if \( V_\epsilon \setminus \hat{\sigma} \) percolates between the boundaries of \( \Delta_n \) and \( \sigma \).

**Theorem 2.7.** Let \( \nu \in [0, 1] \). If there exist \( n \geq 3 \) and \( p \in \{0, \ldots, n-k\} \) such that \( e_p,\nu(n,k) > \tilde{e}_p,\nu(n,k) \), then, \( n \geq 3 \), \( p \geq 1 \) and \( \lim_{m \to +\infty} \mu_\nu(P_{m+m',m}(\sigma)) = 1 \), for every \( \sigma \in \text{Sd}^{m'}(\Delta_n)^{[m]} \), \( m', m \in \mathbb{N} \).

**Proof.** If \( p = 0 \), \( e_p,\nu(n,k) = \tilde{e}_p,\nu(n,k) \) for every \( k \in \{1, \ldots, n\} \) and \( n > 0 \), by Theorem 1.3.

If \( n = 2 \), we need to prove the equality for \( k = p = 1 \). In this case, for every \( \epsilon \in C^0(\text{Sd}^d(\Delta_2); \mathbb{Z}/2\mathbb{Z}) \), \( V_\epsilon \) is a homological manifold of dimension one by Theorem 1.1 of [15]. The connected components of \( V_\epsilon \) which meet \( \partial \Delta_2 \) are thus homeomorphic to \( \Delta_1 \) so that \( b_1(V_\epsilon) = \tilde{b}_1(V_\epsilon) \). The equality \( e_1(2,1) = \tilde{e}_1(2,1) \) is then obtained by integrating over \( C^0(\text{Sd}^d(\Delta_2); \mathbb{Z}/2\mathbb{Z}) \).

Let us suppose now that there exists \( n \geq 3 \) and \( p \in \{0, \ldots, n-k\} \) such that \( e_p,\nu(n,k) > \tilde{e}_p,\nu(n,k) \). Let \( m' \in \mathbb{N} \) and \( \sigma \in \text{Sd}^{m'}(\Delta_n)^{[m]} \). For every \( m \geq 1 \), we denote by \( \tilde{P}_{m+m}(\sigma) \) the set of \( \tilde{\epsilon} \in C^{k-1}(\text{Sd}^{m'+m}(\Delta_n) \setminus \text{Sd}^m(\sigma)) \) for which there exists \( \epsilon \in P_{m+m}(\sigma) \) whose restriction to the complement of \( \sigma \) is \( \tilde{\epsilon} \) \((\text{Sd}^{m'+m}(\Delta_n) \setminus \text{Sd}^m(\sigma)) \) is not a simplicial complex. It is enough to show that \( \lim_{m \to +\infty} \mu_\nu(\tilde{P}_{m+m}(\sigma)) = 1 \). Indeed, let \( \tilde{\sigma} \in \text{Sd}^2(\sigma) \) which does not meet the boundary of \( \sigma \). Then, for every \( m \geq 1 \) and \( \tilde{\epsilon} \in \tilde{P}_{m+m+2,m}(\tilde{\sigma}) \), the restriction of \( \tilde{\epsilon} \) to \( \text{Sd}^{m+2}(\text{Sd}^{m'+m}(\Delta_n) \setminus \hat{\sigma}) \) lies in \( P_{m+m+2,m+2}(\sigma) \), since a path in \( V_\epsilon \) connecting \( \partial \Delta_n \) to \( \partial \tilde{\sigma} \) has to cross \( \partial \sigma \). Let \( \tilde{P}_{m+m+2,m+2}(\tilde{\sigma}) \) be the image of \( \tilde{P}_{m+m+2,m+2}(\tilde{\sigma}) \) in \( P_{m+m+2,m+2}(\sigma) \) by this restriction map, so that \( \mu_\nu(\tilde{P}_{m+m+2,m+2}(\tilde{\sigma})) \leq \mu_\nu(\tilde{P}_{m+m+2,m+2}(\tilde{\sigma})) \). Since \( \mu_\nu \) is a product measure, \( \mu_\nu(\tilde{P}_{m+m+2,m+2}(\tilde{\sigma})) \leq \mu_\nu(\tilde{P}_{m+m+2,m+2}(\tilde{\sigma})) \) and we deduce the inequality \( \mu_\nu(\tilde{P}_{m+m+2,m+2}(\tilde{\sigma})) \leq \mu_\nu(\tilde{P}_{m+m+2,m+2}(\tilde{\sigma})) \) which is then obtained by integrating over \( C^0(\text{Sd}^d(\Delta_2); \mathbb{Z}/2\mathbb{Z}) \).

Let us now prove that \( \lim_{m \to +\infty} \mu_\nu(\tilde{P}_{m+m+2,m}(\sigma)) = 1 \), a result which does not depend on the choices of \( m' \) and \( \sigma \). For every \( \epsilon \in C^{k-1}(\text{Sd}^{m'+m}(\Delta_n)) \), let \( \Lambda_\epsilon \) be the union of the connected components of \( V_\epsilon \) which meet \( \partial \Delta_n \), so that \( b_p(V_\epsilon) = \tilde{b}_p(V_\epsilon) + b_p(\Lambda_\epsilon) \). Replacing \( V_\epsilon \) by \( \Lambda_\epsilon \) in Proposition 2.6, we deduce that

\[
\lim_{m \to +\infty} \frac{1}{(n+1)!m'+m} \left| E_{m'+m}(b_p(\Lambda_\epsilon \cap A_{m+m',m})) - E_{m'+m}(b_p(\Lambda_\epsilon \cap A_{m+m',m})) \right| = 0.
\]

Under our hypothesis we then deduce that

\[
\lim_{m \to +\infty} \frac{1}{(n+1)!m'+m} E_{m'+m}(b_p(\Lambda_\epsilon \cap A_{m+m',m})) = e_{p,\nu}(n,k) - \tilde{e}_{p,\nu}(n,k) > 0.
\]

However, by definition,
\[ \mathbb{E}_{m+m'}(b_p(\Lambda_\epsilon \cap A_{m+m',m})) = \sum_{\sigma \in \text{Sd}^{m'}(\Delta_n)[n]} \int_{C_{k-1}(\text{Sd}^{m+m'}(\Delta_n) \setminus \text{Sd}^{m}(\sigma))} \int_{C_{k-1}(\text{Sd}^{m}(\sigma))} b_p(\Lambda_\epsilon \cap \sigma) d\mu_\nu(\epsilon) = \sum_{\sigma \in \text{Sd}^{m'}(\Delta_n)[n]} \int_{\tilde{P}_{m+m',m}(\sigma)} \int_{C_{k-1}(\text{Sd}^{m}(\sigma))} b_p(\Lambda_\epsilon \cap \sigma) d\mu_\nu(\epsilon) \leq \sum_{\sigma \in \text{Sd}^{m'}(\Delta_n)[n]} \mu_\nu(\tilde{P}_{m+m',m}(\sigma)) \mathbb{E}_{\nu,m}(b_p - \tilde{b}_p), \]

so that

\[ \frac{1}{(n+1)!m'} \sum_{\sigma \in \text{Sd}^{m'}(\Delta_n)[n]} \mu_\nu(\tilde{P}_{m+m',m}(\sigma)) \geq \frac{\mathbb{E}_{\nu,m+m'}(b_p(\Lambda_\epsilon \cap A_{m+m',m}))}{(n+1)!m' \mathbb{E}_{\nu,m}(b_p - \tilde{b}_p)}. \]

From what precedes, the right hand side converges to 1 as \( m \) grows to \( +\infty \). We thus deduce that for every \( m' \in \mathbb{N} \) and every \( \sigma \in \text{Sd}^{m'}(\Delta_n)[n] \), \( \lim_{m \to +\infty} \mu_\nu(\tilde{P}_{m+m',m}(\sigma)) = 1 \). Hence the result.

Remark 2.8. 1. It would be interesting to prove that for every \( m' \in \mathbb{N} \), the sequence \( \frac{1}{(n+1)!m'} \sum_{\sigma \in \text{Sd}^{m'}(\Delta_n)[n]} \mu_\nu(\tilde{P}_{m+m',m}(\sigma)) \) which appeared at the end of the proof of Theorem 2.7 converges to a limit \( u_{m'} \) as \( m \) grows to \( +\infty \), without assuming that \( e_{p,\nu}(n,k) > \hat{e}_{p,\nu}(n,k) \). Then, the sequence \( (u_{m'})_{m \in \mathbb{N}} \) would be submultiplicative.

2. It would also be interesting to know whether the sequence \( (\mu_\nu(P_{m+m',m}(\sigma)))_{m \in \mathbb{N}} \) converges or not for every \( m' \) and every \( \sigma \in \text{Sd}^{m'}(\Delta_n)[n] \), and whether or not the limit is 1 without assuming that \( e_{p,\nu}(n,k) > \hat{e}_{p,\nu}(n,k) \). Nevertheless, if this condition is necessary by Theorem 2.7 for \( e_{p,\nu}(n,k) \) to be greater than \( \hat{e}_{p,\nu}(n,k) \), it is not sufficient, a priori. This question of percolation seems already of interest in dimension two.

3 Refined upper bounds

3.1 Induced filtrations

Let \( K \) be a finite \( n \)-dimensional simplicial complex. For every \( \epsilon \in C^0(K;\mathbb{Z}/2\mathbb{Z}) \), \( K \) inherits the filtration

\[ \emptyset \subset K_0^\epsilon \subset K_1^\epsilon \subset \ldots \subset K_{n+1}^\epsilon = K, \]

where \( K_i^\epsilon = \{ \sigma \in K | \#(\epsilon|_\sigma)^{-1}(0) \leq i \text{ or } \#(\epsilon|_\sigma)^{-1}(1) \leq i \}, i \in \{0,1,\ldots,[\frac{n+1}{2}]\} \).

We denote by \( C_*(K,K_0^\epsilon;\mathbb{Z}/2\mathbb{Z}) \) the associated relative simplicial chain complex and by \( C_*(V_\epsilon';\mathbb{Z}/2\mathbb{Z}) \) the cellular chain complex of the CW-complex \( V_\epsilon' \), see Corollary 2.5 of [15]. We now prove Theorem 1.4 and Corollary 1.5.

Proof of Theorem 1.4. For every \( p \in \{0,\ldots,n-k\} \), the relative chain complex \( C_p(K,K_0^\epsilon;\mathbb{Z}/2\mathbb{Z}) \) contains a canonical basis given by simplices on which \( \epsilon \) is not constant. Similarly, a basis of \( C_p(V_\epsilon';\mathbb{Z}/2\mathbb{Z}) \) is given by a \( p \)-cell for every \( (p+1) \)-simplex of \( K \) on which \( \epsilon \) is not constant. Hence the canonical isomorphism between \( C_*(K,K_0^\epsilon;\mathbb{Z}/2\mathbb{Z}) \) and \( C_*(V_\epsilon';\mathbb{Z}/2\mathbb{Z})[1] \). Moreover, the boundary operator of \( C_*(V_\epsilon';\mathbb{Z}/2\mathbb{Z})[1] \) gets identified to the composition \( \partial \circ pr \), where \( \partial \) is the boundary operator of \( C_*(K;\mathbb{Z}/2\mathbb{Z}) \) and \( pr_\epsilon \) is the projection on \( C_*(K,K_0^\epsilon;\mathbb{Z}/2\mathbb{Z}) \).

Theorem 1.4 gives new insights on the following corollary which has been established in [15], see also [14]. Recall that for every finite simplicial complex of dimension \( n \), \( q_K(T) \) denotes the face polynomial \( \sum_{p=0}^n f_p(K)T^p \).
Corollary 3.1 (Theorem 1.5 of [15]). For every finite n-dimensional simplicial complex $K$ and every $\nu \in [0, 1]$, $\chi(K) + \mathbb{E}_\nu(K) = \nu q_K(-\nu) + (1 - \nu) q_K(\nu - 1)$.

Proof. For every $\epsilon \in C^0(K; \mathbb{Z}/2\mathbb{Z})$, the short exact sequence

$$0 \to C(K_0') \to C(K) \to C(K, K_0') \to 0,$$

combined with Theorem 1.4 implies $\chi(K_0') - \chi(V_\epsilon) = \chi(K)$. After integration over all $\epsilon$, we get

$$\chi(K) + \mathbb{E}(\chi) = \mathbb{E}(\chi(K_0')) = \sum_{\sigma \in K} \sum_{\nu} \nu \mathbb{E}(\mu(\nu \epsilon |_{\nu \sigma} = \text{constant})) = \sum_{p=0}^n (-1)^{p+1} + (1 - \nu)^{p+1}) f_p(K).$$

Proof of Corollary 1.3 The short exact sequence $0 \to C(K_0') \to C(K) \to C(K, K_0') \to 0$ induces the long exact sequence in homology

$$\ldots \to H_{p+1}(K) \xrightarrow{(\pi_{p+1})_*} H_{p+1}(K, K_0') \xrightarrow{(\partial_{p+1})_*} H_p(K_0') \xrightarrow{(i_p)_*} H_p(K) \to \ldots$$

We deduce,

$$b_{p+1}(K, K_0') = \dim \ker(\partial_{p+1})_* + \dim \text{Im}(\partial_{p+1})_*$$

$$= \dim \text{Im}(\pi_{p+1})_* + \dim(\ker \delta_p)_*$$

$$= b_{p+1}(K) + b_p(K_0') - \dim \text{Im}(\delta_{p+1})_* - \dim \text{Im}(\pi_p)_*$$

$$\leq b_{p+1}(K) + b_p(K_0').$$

The upper estimates follow using the isomorphism $H_{p+1}(K, K_0') \to \tilde{H}_p(V_\epsilon)$ given by Theorem 1.4. We furthermore deduce

$$b_{p+1}(K, K_0') = \dim \ker(\partial_{p+1})_* + b_p(K_0') - \dim \text{Im}(\delta_p)_*$$

$$= \dim \ker(\partial_{p+1})_* + b_p(K_0') - b_p(K) + \dim \text{Im}(\pi_p)_*$$

$$\geq b_p(K_0') - b_p(K).$$

The isomorphism given by Theorem 1.4 now implies the lower estimate after integration over $C^0(K; \mathbb{Z}/2\mathbb{Z})$. Finally, the asymptotic result follows from the definition of $\epsilon_{p, \nu}(n, 1)$ and the invariance of the Betti numbers of $K$ under barycentric subdivisions.

Thanks to Corollary 1.5, in order to estimate $\mathbb{E}_\nu(b_p)$ from above, it suffices to upper estimate $\mathbb{E}_\nu(b_p(K_0'))$. A rough upper estimate is given by Proposition 3.2, but it is going to be improved by Theorem 3.4 and improved further by Corollary 3.8.

Proposition 3.2. Let $K$ be a finite n-dimensional simplicial complex, $n > 0$. For every $\nu \in [0, 1]$ and every $p \in \{0, 1, \ldots, n - 1\}$, $\mathbb{E}_\nu(b_p(K_0')) \leq f_p(K)(\nu^{p+1} + (1 - \nu)^{p+1}) = \mathbb{E}_\nu(f_p(K_0')).$

Proof. The result follows from the fact that $b_p(K_0') \leq f_p(K_0')$ for every $\epsilon \in C^0(K; \mathbb{Z}/2\mathbb{Z})$, combined with

$$\mathbb{E}(f_p(K_0')) = \int_{C^0(K; \mathbb{Z}/2\mathbb{Z})} \# \{ \sigma \in K^{|p|} | \epsilon_{|p} \text{ is constant} \} d\mu(\epsilon)$$

$$= \sum_{p|K^{|p|}} \mu(\epsilon \in C^0(K; \mathbb{Z}/2\mathbb{Z}) | \epsilon_{|p} \text{ is constant}$$

$$= f_p(K)(\nu^{p+1} + (1 - \nu)^{p+1}).$$
Lemma 3.3. Let $K$ be a finite $n$-dimensional simplicial complex, $n \geq 0$. For every $p \in \{0, \ldots, n\}$, the space of $p$-cycles $Z_p(K) \subset C_p(K)$ has a complement spanned by simplices of $K$.

Proof. We proceed by induction on the codimension of $Z_p(K)$. If $Z_p(K)$ contains all the simplices, there is nothing to prove. Otherwise, we choose a simplex $\sigma \notin Z_p(K)$, so that $\langle \sigma \rangle \cap Z_p(K) = \{0\}$. Then we consider the quotient $C_p(K)/\langle \sigma \rangle$ and the image of $Z_p(K)$ in this quotient. Its codimension decreases by one. The result follows by induction. \hfill \Box

The upper estimate given in Proposition 3.2 can now be improved, see §1.1 for the definition of $M_{p,\nu}(K)$.

Theorem 3.4. Let $\nu \in [0,1]$ and $p \in \mathbb{N}$. Then, for every finite simplicial complex $K$, $E_\nu(b_p(K_0^\nu)) \leq M_{p,\nu}(K)$.

Proof. By definition, for every $\epsilon \in C^0(K; \mathbb{Z}/2\mathbb{Z})$,

$$b_p(K_0^\nu) = \dim Z_p(K_0^\nu) - \dim B_p(K_0^\nu) = f_p(K_0^\nu) - \dim B_{p-1}(K_0^\nu) - \dim B_p(K_0^\nu).$$

Let $S_p(K)$ be a subspace complementary to $Z_p(K)$ in $C_p(K)$. Its dimension equals

$$\dim S_p(K) = f_p(K) - \dim Z_p(K) = (f_p(K) - b_p(K)) - \dim B_p(K) = \sum_{i=p}^n (-1)^{k-p}(f_{i}(K) - b_{i}(K)), \text{ by induction.}$$

From Lemma 3.3 we can choose $S_p(K)$ to be spanned by simplices. A linear combination of such simplices belongs to $C_p(K_0^\nu)$ if and only if each simplex belongs to $C_p(K_0^\nu) = \ker pr_\epsilon$. Therefore, $\dim (S_p(K) \cap C_p(K_0^\nu)) = \# \{ \sigma \in B_p | \sigma \in K_0^\nu \}$, where $B_p$ denotes our basis of $S_p(K)$. However, $S_p(K) \cap C_p(K_0^\nu)$ is transverse to $\ker \partial_p^K$ in $C_p(K_0^\nu)$, so that $\dim (S_p(K) \cap C_p(K_0^\nu)) \leq \dim B_{p-1}(K_0^\nu)$. We deduce

$$E_\nu(b_p(K_0^\nu)) \leq E_\nu(f_p(K_0^\nu)) - E_\nu(\# \{ \sigma \in B_p | \sigma \in K_0^\nu \}) - E_\nu(\# \{ \sigma \in B_{p+1} | \sigma \in K_0^\nu \}) \\
\leq (\nu^{p+1} + (1 - \nu)^{p+1})(f_p(K) - \# B_p) - (\nu^{p+2} + (1 - \nu)^{p+2})\# B_{p+1} \\
\leq (\nu^{p+1} + (1 - \nu)^{p+1})(f_p(K) - \sum_{i=p}^{\dim K} (-1)^{i+1-p}(f_{i}(K) - b_{i}(K))) - \\
(\nu^{p+2} + (1 - \nu)^{p+2})\sum_{i=p+1}^{\dim K} (-1)^{i+1-p}(f_{i}(K) - b_{i}(K)) \\
\leq (\nu^{p+1} + (1 - \nu)^{p+1})b_p(K) + \nu(1 - \nu)(\nu^{p} + (1 - \nu)^p)\sum_{i=p+1}^{\dim K} (-1)^{i+1-p}(f_{i}(K) - b_{i}(K)).$$

The third inequality follows from the fact that $\# B_p = \dim S_p(K)$. \hfill \Box

3.2 Monotony theorem

Lemma 3.5. For every $n > 0$ and $p \in \{1, \ldots, n-1\}$, $M_{p,\nu}(\Delta_n) = \nu(1-\nu)(\nu^p+(1-\nu)^p)\binom{n}{p+1}$, while $M_{0,\nu}(\Delta_n) = 1 + \nu(1-\nu)2n$.

Proof. The simplex $\Delta_n$ is contractible, so that $b_0(\Delta_n) = 1$ and $b_i(\Delta_n) = 0$ for every $i > 0$. Moreover, for every $i \in \{0, \ldots, n\}$, $f_i(\Delta_n) = \binom{n+1}{i+1}$ as the $i$-simplices of $\Delta_n$ are in one-to-one correspondence with sets of $i+1$ vertices of $\Delta_n$.

Thus, for $p \in \{1, \ldots, n-1\}$,
\[
\sum_{i=p+1}^{n} (-1)^{i+1-p}(\binom{n+1}{i+1}) = \sum_{i=p+1}^{n-1} (-1)^{i+1-p}\binom{n}{i} + \binom{n}{p+1} + (-1)^{n+1-p}
= \binom{n}{p+1},
\]

and the result follows from the definition of \(M_{p,\nu}(\Delta_n)\), see \(\S 1.1\). \(\square\)

Let us remark that the left hand side of Theorem 3.4 vanishes for a large family of complexes, for instance those given by the following proposition.

**Proposition 3.6.** Let \(p > 0\) and \(L\) be a finite simplicial complex which contains a family of simplices \(\{\sigma_i \in L, i \in I\}\) such that

1. \(\forall \tau \in L, \exists i \in I\) such that \(\tau < \sigma_i\),

2. \(\forall i \neq j \in I, \dim(\sigma_i \cap \sigma_j) < p - 1\).

Then, for every \(l \geq p\), \(b_p(L) = 0 = \mathbb{E}_\nu(b_p(L_0^\nu))\).

**Proof.** Let \(D = \bigsqcup_{i \in I} \sigma_i\) be the disjoint union of the \(\sigma_i, i \in I\), and \(h : D \to L\) be the associated canonical simplicial map. From the hypothesis, for every \(l \geq p - 1\), \(h : D \to L\) provides a bijection between the \(l\)-simplices of \(D\) and \(L\). Thus, \(h^\#: C_l(D) \to C_l(L)\) is an isomorphism of vector spaces and a chain map which induces an isomorphism \(h_* : H_l(D) \to H_l(L)\). As \(H_l(D) = 0\), we deduce that \(b_l(L)\) vanishes. Now, for every \(\epsilon \in C^0(L; \mathbb{Z}/2\mathbb{Z})\), we set \(\epsilon = \epsilon \circ h\). Then, \(h : D_0^\epsilon \to L_0^\epsilon\) induces another isomorphism \(h_* : H_l(D_0^\epsilon) \to H_l(L_0^\epsilon)\) for \(l \geq p\). However, by definition, \(D_0^\epsilon\) is again a disjoint union of simplices, so that \(H_l(D_0^\epsilon) = 0\). Hence the result. \(\square\)

The following monotony theorem completes Theorem 1.6

**Theorem 3.7.** Let \(\nu \in [0, 1]\) and \(p \in \mathbb{N}\). For every finite simplicial complex \(K\) and every subcomplex \(L\) of \(K\), \(M_{p,\nu}(L) - \mathbb{E}_\nu(b_p(L_0^\nu)) \leq M_{p,\nu}(K) - \mathbb{E}_\nu(b_p(K_0^\nu))\). Similarly, \(\mathbb{E}_\nu(f_p(L)) - \mathbb{E}_\nu(b_p(L_0^\nu)) \leq \mathbb{E}_\nu(f_p(K)) - \mathbb{E}_\nu(b_p(K_0^\nu))\).

**Proof.** Let \(S_p(L)\) (resp. \(S_{p+1}(L)\)) be complementary to \(Z_p(L)\) (resp. \(Z_{p+1}(L)\)) in \(C_p(L)\) (resp. \(C_{p+1}(L)\)) and spanned by \(p\)-simplices, see Lemma 3.3. The intersection of \(S_p(L)\) with \(Z_p(K)\) is thus \(\{0\}\) and \(Z_p(K) \oplus S_p(L) = Z_p(K) + C_p(L)\). Let \(S_p(K, L)\) (resp. \(S_{p+1}(K, L)\)) be the complement of this space in \(C_p(K)\) (resp. \(C_{p+1}(K)\)) spanned by \(p\)-simplices of \(K\) in such a way that \(S_p(K, L)\) completes \(S_p(L)\) to a complement \(S_p(K) = S_p(L) \oplus S_p(K, L)\) of \(Z_p(K)\) in \(C_p(K)\) (resp. \(S_{p+1}(K, L)\) completes \(S_{p+1}(L)\) to a complement \(S_{p+1}(K) = S_{p+1}(L) \oplus S_{p+1}(K, L)\) of \(Z_{p+1}(K)\) in \(C_{p+1}(K)\)).

Let \(B_p(K) \subset C_p(K)\) be the image of \(\partial_{p+1} : C_{p+1}(K) \to C_p(K)\). For every \(\epsilon \in C^0(K; \mathbb{Z}/2\mathbb{Z})\), \(L_0^\epsilon = K_0^\epsilon \cap L\), so that \(B_{p-1}(L_0^\epsilon) \subset B_{p-1}(K_0^\epsilon)\) and \(B_p(L_0^\epsilon) \subset B_p(K_0^\epsilon)\). Moreover, \(S_p(K, L) \cap C_p(K_0^\epsilon)\) is complement to \(Z_p(K_0^\epsilon) + C_p(L_0^\epsilon)\) in \(C_p(K_0^\epsilon)\), so that \(\dim(S_p(K, L) \cap C_p(K_0^\epsilon)) + \dim B_{p-1}(L_0^\epsilon) \leq \dim B_{p-1}(K_0^\epsilon)\) and similarly \(\dim(S_{p+1}(K, L) \cap C_{p+1}(K_0^\epsilon)) + \dim B_p(L_0^\epsilon) \leq \dim B_p(K_0^\epsilon)\). As before we deduce

\[
b_p(K_0^\epsilon) = f_p(K_0^\epsilon) - \dim B_{p-1}(K_0^\epsilon) - \dim B_p(K_0^\epsilon)
\leq f_p(K_0^\epsilon) - f_p(L_0^\epsilon) + (f_p(L_0^\epsilon) - \dim B_{p-1}(L_0^\epsilon) - \dim B_p(L_0^\epsilon)) - \dim(S_p(K, L) \cap C_p(K_0^\epsilon)) - \dim(S_{p+1}(K, L) \cap C_{p+1}(K_0^\epsilon))
\leq f_p(K_0^\epsilon) - \dim(S_p(K) \cap C_p(K_0^\epsilon)) - \dim(S_{p+1}(K) \cap C_{p+1}(K_0^\epsilon)) + b_p(L_0^\epsilon) - (f_p(L_0^\epsilon) - \dim(S_p(L) \cap C_p(L_0^\epsilon)) - \dim(S_{p+1}(L) \cap C_{p+1}(L_0^\epsilon))).
\]
We set $\lambda$, containing a family of simplices $\{E\}_{\nu}$ the hypothesis of Proposition 3.6. Then, Corollary 3.8. Given by Theorem 3.4.

From Theorem 3.7, Proof. Let $L$ be a subcomplex of $K$ satisfying the hypothesis of Proposition 3.6. Then,

$$E_\nu(b_p(K_\nu)) \leq (\nu^p+1)(1-\nu)^{p+1}b_p(K)+\nu(1-\nu)(\nu^p+1-\nu)\sum_{i=p+1}^{\dim K} (-1)^{i+1-p}(f_i(K)-f_i(L)-b_i(K)).$$

Proof. From Theorem 3.7 $E_\nu(b_p(K_\nu)) \leq M_{p,\nu}(K) - M_{p,\nu}(L) + E_\nu(b_p(L_\nu))$ and from Proposition 3.6 $E_\nu(b_p(L_\nu)) = 0$ and $M_{p,\nu}(L) = \nu(1-\nu)(\nu^p+1-\nu)\sum_{i=p+1}^{\dim L} (-1)^{i+1-p}f_i(L)$. Hence the result.

For every $d > 0$, let $\mathcal{L}^{n,p}_d$ be the finite set of simplicial subcomplexes $L$ of $\text{Sd}^d(\Delta_n)$ containing a family of simplices $\{\sigma_i \in L, i \in I\}$ such that

1. $\forall \tau \in L, \exists i \in I$ such that $\tau < \sigma_i$,
2. $\forall i \neq j \in I, \dim(\sigma_i \cap \sigma_j) < p - 1$,
3. $\forall i \in I, \dim(\sigma_i \cap \text{Sd}^d(\partial \Delta_n)) < p - 1$.

We set $\lambda_{d,p}(n) = \frac{1}{(n+1)!^p} \max_{L \in \mathcal{L}^{n,p}_d} M_{p,\nu}(L)$.

Proposition 3.9. The sequence $(\lambda_{d,p}(n))_{d \geq 0}$ is increasing and bounded.

Let $\lambda_{p,\nu}(n)$ be the limit of this sequence $(\lambda_{d,p}(n))_{d \geq 0}$.

Proof. Let $d > 0$. The set $\mathcal{L}^{n,p}_d$ being finite, there exists a subcomplex $L_d$ in $\mathcal{L}^{n,p}_d$ which maximize $M_{p,\nu}$ over $\mathcal{L}^{n,p}_d$. Let $m > 0$. For every $n$-simplex $\sigma$ of $\text{Sd}^m(\Delta_n)$, we choose a simplicial isomorphism $\Delta_n \xrightarrow{f_\sigma} \sigma$. Let $L_{d+m} = \bigcup_{\sigma \in \text{Sd}^m(\Delta_n)} f_\sigma \cdot L_d$. It is a subcomplex of $\text{Sd}^{d+m}(\Delta_n)$. Moreover, $L_{d+m}$ belongs to $\mathcal{L}^{n,p}_{d+m}$, so that $\frac{1}{(n+1)!^p} M_{p,\nu}(L_{d+m}) \leq \lambda_{d+m}(n)$. From Proposition 3.6 $M_{p,\nu}(L_{d+m}) = \nu(1-\nu)(\nu^p+1-\nu)\sum_{i=p+1}^n (-1)^{i+1-p}f_i(L_{d+m})$. However, by construction, $L_{d+m} \cap \text{Sd}^d(\text{Sd}^m(\Delta_n)^{(n-1)})$ is of dimension $< p - 1$ so that it does not contribute to the computation of $M_{p,\nu}(L_{d+m})$. As the cardinality of $\text{Sd}^m(\Delta_n)^{[n]}$ is $(n+1)!^m$, we deduce that $M_{p,\nu}(L_{d+m}) = (n+1)!^m M_{p,\nu}(L_d)$. Thus $\lambda_{d,p}(n) \leq \lambda_{d+m}(n)$.

Now, for every $L \in \mathcal{L}^{n,p}_d$,

$$M_{p,\nu}(L) \leq M_{p,\nu}(\text{Sd}^d(\Delta_n)) = \nu(1-\nu)(\nu^p+1-\nu)\sum_{i=p+1}^n (-1)^{i+1-p}f_i(\text{Sd}^d(\Delta_n)).$$

The sequence $\frac{M_{p,\nu}(\text{Sd}^d(\Delta_n))}{(n+1)!^p}$ is convergent, see [3, 4], so that the sequence $\lambda_{d,p}(n)$ is bounded. Hence the result.
We can now deduce Theorem 1.7. Recall that by definition, for every $i \in \{0, \ldots, n\}$, $q_{i,n} = \lim_{d \to +\infty} \frac{f_i(S_d(K))}{f_n(K)(n+1)^d}$.

**Proof of Theorem 1.7.** We proceed as in the proof of Proposition 3.9. For every $n$-simplex $\sigma \in K$, we choose a simplicial isomorphism $f_{T} : \Delta_n \to \sigma$. For every $d > 0$, there exists a subcomplex $L_d \subseteq E^{d-p}_d$ which maximize the function $M_{p,\nu}$ over $E^{d-p}_d$. We set $K_d = \bigcup_{\sigma \in K_n} f_{\sigma}(L_d)$. We deduce as in the proof of Proposition 3.9 that $M_{p,\nu}(K_d) = f_n(K)M_{p,\nu}(L_d)$. Since $b_{p+1}(S_d(K)) = b_{p+1}(K)$, we deduce

$$
E_{n,d}(b_p) \leq E_{\nu}(b_p(S_d(K))_0 + b_{p+1}(K), \text{ from Corollary 1.5}) \\
\leq M_{p,\nu}(S_d(K)) - M_p,\nu(K_d) + b_{p+1}(K), \text{ from Proposition 3.6 and Theorem 3.7} \\
\leq M_{p,\nu}(S_d(K)) - f_n(K)M_{p,\nu}(L_d) + b_{p+1}(K).
$$

However,

$$
M_{p,\nu}(S_d(K)) = b_p(K)(\nu^{p+1} + (1-\nu)^{p+1} + (1-\nu)(\nu^p + (1-\nu)^p) \sum_{i=p+1}^n (-1)^{i+1-\nu}(f_i(S_d(K)) - b_i(K)),
$$

so that

$$
\lim_{d \to +\infty} \frac{M_{p,\nu}(S_d(K))}{f_n(K)(n+1)^d} = \nu(1-\nu)(\nu^p + (1-\nu)^p) \sum_{i=p+1}^n (-1)^{i+1-\nu}q_{i,n}, \text{ see } [3, 4] \text{ or } [15].
$$

By definition, \( \lim_{d \to +\infty} \frac{M_{p,\nu}(L_d)}{(n+1)^d} = \lambda_{p,\nu}(n) \) and thus

$$
\lim_{d \to +\infty} \frac{E_{d}(b_p)}{f_n(K)(n+1)^d} \leq \nu(1-\nu)(\nu^p + (1-\nu)^p) \sum_{i=p+1}^n (-1)^{i+1-\nu}q_{i,n} - \lambda_{p,\nu}(n).
$$

\( \square \)

In the light of Theorem 1.7, it would be great to be able to compute $\lambda_{p,\nu}(n)$. Theorem 1.13 provides an estimate of this number from below.

4 Tilings

4.1 Tiles

For every $n > 0$ and every $s \in \{0, 1, \ldots, n+1\}$, we set $T_s^n = \Delta_n \setminus (\sigma_1 \cup \ldots \cup \sigma_s)$, where $\sigma_i$ denotes a facet of $\Delta_n$, $i \in \{1, \ldots, s\}$. In particular, the tile $T_{n+1}^n$ is the open $n$-simplex $\Delta_n$ and $T_0^n$ is the closed one $\Delta_n$, see Figure 1.

**Proposition 4.1.** For every $n > 0$ and every $s \in \{0, 1, \ldots, n+1\}$, $T_{s+1}^{n+1}$ is a cone over $T_s^n$, deprived of its center if $s \neq 0$. Moreover, $T_{s+1}^{n+1}$ is a disjoint union $T_{s+2}^{n+1} \sqcup T_{s+1}^n \sqcup T_{s+1}^{n-1} \sqcup \ldots \sqcup T_{n+1}^n$. In particular, the cone $T_s^n$ deprived of its base $T_s^n$ is $T_{s+1}^{n+1}$.

**Proof.** If $s = 0$, $T_0^{n+1} = \Delta_n \cup c \ast \Delta_n$, where $c$ denotes a vertex of $\Delta_{n+1}$. If $s > 0$, $T_s^n = \Delta_n \setminus (\sigma_1 \cup \ldots \cup \sigma_s)$ by definition, where $\sigma_i$ is a facet for every $i \in \{1, \ldots, s\}$, and so $(c \ast T_s^n) \setminus \{c\} = (c \ast \Delta_n) \setminus ((c \ast \sigma_1) \cup \ldots \cup (c \ast \sigma_s))$. However, $c \ast \Delta_n = \Delta_{n+1}$ and $\theta_i = c \ast \sigma_i$ is an $n$-simplex, $i \in \{1, \ldots, s\}$. It follows from the definition that $T_s^{n+1} = (c \ast T_s^n) \setminus \{c\}$. It is the cone over $T_s^n$ deprived of its center $c$. The base $T_s^n$ of this cone is the intersection of $T_{s+1}^{n+1}$ with the base $\theta = \Delta_n$ of the cone $\Delta_{n+1} = c \ast \Delta_n$. Thus, $T_s^{n+1} \setminus T_s^n = (c \ast \Delta_n) \setminus (\theta_1 \cup \ldots \cup \theta_s)$.

The result holds true for $s = 0$ as well, since by definition $T_0^n = \Delta_{n+1} \setminus \Delta_n = T_0^{n+1} \setminus T_0^n$.

By induction, we deduce that for every $s \in \{0, n+1\}$, $T_{s+1}^{n+1} \cap \partial \Delta_{n+1}$ is the disjoint union $T_s^n \sqcup \ldots \sqcup T_0^n$. \( \square \)
Corollary 4.2. For every \( n > 0 \), \( \partial \Delta_{n+1} = \bigsqcup_{s=0}^{n+1} T_s^n \).

Proof. By definition \( T_0^{n+1} = \Delta_{n+1} = \Delta_{n+1} \sqcup \partial \Delta_{n+1} = T_{n+2}^n \sqcup \partial \Delta_{n+1} \). It follows from Proposition 4.1 that \( T_0^{n+1} = T_{n+2}^n \sqcup (\bigsqcup_{s=0}^{n+1} T_s^n) \). Hence the result.

Proposition 4.3. For every \( n > 0 \) and every \( s \in \{0, \ldots, n+1\} \),

\[
f_j(T_s^n) = \begin{cases} 
0 & \text{if } 0 \leq j < s - 1, \\
\binom{n+1-s}{n-j} & \text{if } s - 1 \leq j \leq n.
\end{cases}
\]

By face number of a tile, we mean its number of open simplices of the corresponding dimension.

Proof. We proceed by induction on the dimension \( n \). If \( n = 1 \), one checks the result. Now let us suppose that the result holds true for \( n \geq 1 \). By definition, \( T_0^{n+1} = \Delta_{n+1} \), so that for every \( j \in \{0, \ldots, n+1\} \), \( f_j(T_0^{n+1}) = \binom{n+2}{j+1} = \binom{n+2}{n+1-j} \). Hence the result for \( s = 0 \). Similarly, \( T_{n+2}^{n+1} = \Delta_{n+1} \), so that \( f_j(T_{n+2}^{n+1}) = \delta_{n+1j} \). Now, if \( 1 \leq s \leq n+1 \), we know from Proposition 4.1 that \( T_s^{n+1} = (c \ast T_s^n) \setminus \{c\} \). The open faces of \( T_s^{n+1} \) are thus either the faces of the basis \( T_s^n \) of the cone, or the cones over the faces of \( T_s^n \). We deduce from Pascal’s formula that for every \( j \in \{0, \ldots, n+1\} \), \( f_j(T_s^{n+1}) = f_j(T_s^n) + f_{j-1}(T_s^n) = \binom{n+1-s}{n-j} + \binom{n+1-s}{n+1-j} = \binom{n+2-s}{n+1-j} \).

4.2 Tilings

For every \( n > 0 \), let \( T(n) \) be the set of \( n \)-dimensional simplicial complexes that can be tiled by \( T_0^n, \ldots, T_{n+1}^n \).

Proposition 4.4. 1. Every pure finite simplicial complex of dimension one is tileable.
2. For every \( n > 0 \), if \( K \in \mathcal{T}(n) \), then for every \( i \in \{1, \ldots, n\} \), the \( i \)-skeleton \( K^{(i)} \) belongs to \( \mathcal{T}(i) \). Moreover, any tiling of \( K \) induces a tiling on \( K^{(i)} \).

Recall that an \( n \)-dimensional simplicial complex is called pure if each of its simplex is a face of an \( n \)-simplex. The second part of Proposition 4.4 is the first part of Theorem 1.9.

**Proof.** For the first part we proceed by induction on the number of edges. If such a complex \( K \) contains only one edge, it is tiled by \( T^1_0 = \Delta_1 \), since it is pure. Now let us suppose that the result holds true for every pure complex containing \( N \) edges. Let \( K \) be a pure simplicial complex of dimension one with \( N + 1 \) edges and let \( e \) be an edge of \( K \). Thus, \( K = K' \cup e \) where \( K' \) is a pure simplicial complex covered by \( N \) edges and by the hypothesis it can be tiled. We choose a tiling of \( K' \). Then, \( e \) has 0, 1 or 2 common vertices with \( K' \). We then extend the tiling of \( K' \) to a tiling of \( K \) by adding \( T^1_0, T^1_1 \) or \( T^1_2 \), respectively.

And for the second part we proceed by induction on the dimension \( n \). If \( n = 1 \) the result follows from the first part. Let us suppose that the result holds true for the dimension \( n \). Let \( K \) be a tiled finite simplicial complex of dimension \( n + 1 \). The \( n \)-skeleton of \( K \) can be obtained by removing all open \((n + 1)\)-simplices. These are exactly the interiors of the tiles of \( K \). However, from Proposition 4.4 for every \( s \in \{0, \ldots, n + 2\} \), \( T^{n+1}_s \setminus T^{n+1}_{n+2} \) is tiled by the tiles of dimension \( n \). Thus, the \( n \)-skeleton of \( K \) gets an induced tiling of dimension \( n \). The result follows from the fact that \( K^{(i)} = (K^{(n)})^{(i)} \).

Proposition 4.4 raises the following question. Let \( X \) be a triangulated manifold with boundary. If \( X \) belongs to \( \mathcal{T}(n) \), does \( \partial X \) belong to \( \mathcal{T}(n - 1) \)?

**Example 4.5.** 1. Figure 2 shows some examples of one-dimensional tiled simplicial complexes.

![Tiled one-dimensional simplicial complexes](image)

Figure 2: Tiled one-dimensional simplicial complexes.

2. In dimension 2, \( \partial \Delta_2 \times [0, 1] \) can be tiled using six \( T^2_1 \), by gluing three copies of the tiling shown in Figure 3.

![A union of two \( T^2_1 \)](image)

Figure 3: A union of two \( T^2_1 \).

3. The prism \( \partial(\Delta_2 \times [0, 1]) \) can be tiled using six \( T^2_1 \) and two \( T^3_3 \), by gluing on the two boundary components of the previous example the open simplices \( T^2_2 \).

4. The prism \( \partial(\Delta_2 \times [0, 1]) \) can likewise be tiled by six \( T^2_2 \) and two \( T^2_0 \).
5. The cylindrical parts in examples 3 and 4 above can be glued together to produce a tiled two-torus.

**Definition 4.6.** A tiling of an $n$-dimensional simplicial complex is called regular if and only if it uses one tile $T_0^n$ for each connected component.

Recall that a simplicial complex is called shellable if its maximal simplices can be arranged in linear order $\sigma_1, \sigma_2, \ldots, \sigma_t$ in such a way that the subcomplex $(\cup_{i=1}^{k-1} \sigma_i) \cap \sigma_k$ is pure and $(\dim \sigma_k - 1)$-dimensional for all $k = 2, \ldots, t$, see [9] for instance.

**Proposition 4.7.** Let $K$ be a tiled finite $n$-dimensional simplicial complex that has a filtration $K_1 \subset K_2 \subset \ldots \subset K_{N-1} \subset K_N = K$ of tiled subcomplexes such that for every $i \in \{1, \ldots, N\}$, $K_i$ contains $i$ tiles. Then, the tiling of $K$ is regular and its connected components are shellable and homotopy equivalent to $n$-dimensional spheres or balls.

(Shellable simplicial complexes are known to be homotopy equivalent to wedges of spheres, see Theorem 12.3 of [9] for instance).

**Proof.** We proceed by induction on $N$. If $N = 1$, $K = T_0^n = \Delta_n$, so that $K$ has a regular tiling and is homeomorphic to a ball. Let us suppose that the result holds true for $N$ and that $K$ has a filtration $K_1 \subset \ldots \subset K_N \subset K_{N+1} = K$. Then, $K_{N+1} = K_N \cup T$ where $T \in \{T_0^n, \ldots, T_{n+1}^n\}$, and $K_N$ is regular by the hypothesis. If $K_N$ and $K_{N+1}$ do not have the same number of connected components, then $T$ is a connected component of $K$ so that $T = T_0^n$, as in the case $N = 1$ and $K_{N+1}$ is regular. Moreover, $K_N$ being homotopy equivalent to a union of $n$-dimensional spheres or balls, so is $K_{N+1}$, with one more ball component. Otherwise, $K_N$ and $K_{N+1}$ have the same number of connected components. In this case, $T$ is glued to one of the connected components of $K_N$ and thus $T \neq T_0^n$. Hence, $K_{N+1}$ is regular, which proves the first part. Moreover, if $T = T_{n+1}^n = \Delta$, then since $K_N$ is the union of spheres or balls so is $K_{N+1}$, a homotopy ball of $K_N$ becoming a homotopy sphere. If $T \neq T_{n+1}^n$, the boundary of $T$ is not empty so that $T$ intersects the boundary of $K_{N+1}$. In this case, $K_N$ is a deformation retract of $K_{N+1}$ and is homotopy equivalent to $K_{N+1}$. Hence the result. □

**Definition 4.8.** Let $K$ be an $n$-dimensional finite simplicial complex equipped with a tiling $T$. For every $i \in \{0, \ldots, n+1\}$, let $h_i(T)$ be the number of tiles $T_i^n$ of $T$. The vector $(h_0(T), \ldots, h_{n+1}(T))$ is called the $h$-vector of $T$ and the polynomial $h_T(X) = \sum_{i=0}^{n+1} h_i(T)X^i$ its $h$-polynomial.

The following Theorem 4.9 and Corollary 4.10 complete Theorem 1.8

**Theorem 4.9.** Let $K$ be a tileable $n$-dimensional finite simplicial complex. For every tiling $T$ of $K$, its $h$-polynomial satisfies $\sum_{i=0}^{n+1} h_i(T)X^{n+1-i} = \sum_{i=0}^{n+1} j_{i-1}(K)(X-1)^{n+1-i}$ provided $f_{-1}(K)$ is chosen to be equal to $h_0(T)$. In particular, two tilings $T$ and $T'$ have the same $h$-polynomial if and only if $h_0(T) = h_0(T')$.

**Proof.** Let $T$ be a tiling of $K$. From Proposition 4.3

\[
\sum_{j=0}^{n+1} f_{j-1}(K)T^j = \sum_{j=0}^{n+1} T^j \sum_{s=0}^{j} \binom{n+1-s}{j-s} h_s(T) = \sum_{k=0}^{n+1} k \sum_{s=0}^{n+1} \binom{n+1-s}{j-s} T^j = \sum_{j=0}^{n+1} h_s(T) \sum_{s=0}^{n+1-s} \binom{n+1-s}{j-s} T^{n+1-j} = T^{n+1} \sum_{s=0}^{n+1} h_s(T)(1 + \frac{1}{j})^{n+1-s},
\]

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where the second line follows from Proposition 4.3 and the convention \( f_{-1}(K) = h_0(K) \).

By letting \( X = \frac{T+1}{T} \) or equivalently \( T = \frac{1}{X-1} \), we get
\[
\sum_{s=0}^{n+1} h_s(T) X^{n+1-s} = \sum_{j=0}^{n+1} f_{j-1}(K)(X - 1)^{n+1-j}.
\]

Hence the result. \( \square \)

**Corollary 4.10.** Let \( K \) be a connected \( n \)-dimensional finite simplicial complex equipped with a regular tiling \( \mathcal{T} \). Then, the \( h \)-vector of \( \mathcal{T} \) coincides with the \( h \)-vector of \( K \).

**Proof.** Since \( K \) is connected and \( \mathcal{T} \) regular, \( h_0(\mathcal{T}) = 1 \). But the \( h \)-polynomial of \( K \) is by definition the polynomial satisfying the relation in Theorem 4.9 with \( f_{-1} = 1 \). Hence the result. \( \square \)

**Remark 4.11.** Corollary 4.10 provides a geometric interpretation for the \( h \)-vector of tiled simplicial complexes. It is similar to the known one for shellable complexes.

**Corollary 4.12.** Let \( \mathcal{T} \) be a tiling of an \( n \)-dimensional finite simplicial complex \( K \). Then \( \chi(K) = h_0(\mathcal{T}) + (-1)^n h_{n+1}(\mathcal{T}) \).

**Proof.** From Theorem 4.9, \( \sum_{i=0}^{n+1} h_i(T) X^{n+1-i} = \sum_{i=0}^{n+1} f_{i-1}(K)(X - 1)^{n+1-i} \) with \( f_{-1} = h_0(\mathcal{T}) \). By letting \( X = 0 \) we get \( h_{n+1}(\mathcal{T}) = (-1)^{n+1} h_0(\mathcal{T}) + (-1)^n \chi(K) \) as \( \chi(K) = \sum_{i=0}^{n} (-1)^i f_i(K) \). Hence the result. \( \square \)

The Euler characteristic of an even dimensional finite tiled simplicial complex is thus positive. Corollary 4.12 provides a tiling of spheres in any dimensions and Example 5 a tileable triangulation of a two-torus. Which three-manifolds possess tileable triangulations?

### 4.3 Barycentric subdivision

Recall that for every \( n > 0 \) and \( s \in \{0, \ldots, n + 1\} \), \( T^s_n = \Delta_n \setminus (\sigma_1 \cup \ldots \cup \sigma_s) \) where \( \sigma_i \) denotes a facet of \( \Delta_n \). We set \( \text{Sd}(T^s_n) = \Delta_n \setminus \bigcup_{i=1}^{s} \text{Sd}(\sigma_i) \). The remaining part of the paper rely on the following key result.

**Theorem 4.13.** For every \( n > 0 \) and every \( s \in \{0, \ldots, n + 1\} \), \( \text{Sd}(T^s_n) \) is tileable. Moreover, it can be tiled in such a way that only \( \text{Sd}(T^0_n) \) (resp. \( \text{Sd}(T^s_n) \)) contains the tile \( T^0_0 \) (resp. \( T^n_{n+1} \)) in its tiling and it contains exactly one such tile.

**Proof.** We proceed by induction on the dimension \( n > 0 \). If \( n = 1 \), the tilings \( \text{Sd}(T^0_1) = T^0_0 \sqcup T^1_1 \) and \( \text{Sd}(T^1_1) = 2T^1_1 \) and \( \text{Sd}(T^2_1) = T^1_1 \sqcup T^1_2 \) are suitable, see Figure 4.

![Figure 4: Tilings of subdivided one-dimensional tiles.](image-url)
Now, let us assume that the result holds true for \( r \leq n \) and let us prove it for \( r = n + 1 \). From Corollary 4.2, \( \partial \Delta_{n+1} \) has a tiling \( \bigsqcup_{k=0}^{n+1} T^k_s \). We equip \( \text{Sd}(\partial \Delta_{n+1}) = \bigsqcup_{k=0}^{n+1} \text{Sd}(T^k_s) \) with the regular tiling given by the induction hypothesis. Then, \( \text{Sd}(\Delta_{n+1}) \) gets a partition by cones over the tiles of \( \text{Sd}(\partial \Delta_{n+1}) \) centered at the barycenter of \( \Delta_{n+1} \) where all the cones except the one over \( T^0_n \) are deprived of their center. From Proposition 4.1, this partition induces a regular tiling of \( \text{Sd}(\Delta_{n+1}) = T^{n+1}_0 \). For every \( s \in \{1, \ldots, n+2\} \), we equip \( \text{Sd}(T^s_{n+1}) = \text{Sd}(\Delta_{n+1}) \setminus \bigcup_{j=0}^{s-1} \text{Sd}(T^j_s) \) with the tiling induced by removing the bases of all the cones over the tiles \( \bigcup_{j=0}^{s-1} T^j_s \subset \text{Sd}(\partial \Delta_{n+1}) \). From Proposition 4.1, these cones deprived of their bases are tiles so that we get as well a tiling of \( \text{Sd}(T^s_{n+1}) \). Moreover, when \( s > 0 \), the cone over the unique tile \( T^0_n \) of the tiling \( \text{Sd}(\partial \Delta_{n+1}) \) is deprived of its basis, so that the tiling we get does not contain \( T^{n+1}_0 \). Finally, by the induction hypothesis the tiling of \( \text{Sd}(\partial \Delta_{n+1}) \) contains a unique tile \( T^{n+1}_n \) which is contained in the tiling of \( \text{Sd}(T^s_{n+1}) \subset \text{Sd}(\partial \Delta_{n+1}) \). Thus, the tiling of \( \text{Sd}(T^s_{n+1}) \) contains the tile \( T^{n+1}_n \) only when \( s = n + 2 \) and in this case it contains only one such tile, since from Proposition 4.1 \( T^{n+1}_{n+2} \) is the cone over \( T^s_{n+1} \) deprived of its base and its center. Hence the result.

The proof of Theorem 4.13 provides a tiling of all the subdivided tiles \( \text{Sd}(T^s_n) \), \( s \in \{0, \ldots, n+1\} \), in any dimension \( n \). Let \( H_n \) be the \((n+2) \times (n+2)\) matrix whose \((s+1)-\text{st}\) row is the \( h \)-vector of the tiling of \( \text{Sd}(T^s_n) \). It follows from the proof of Theorem 4.13 that the \( s \)-th row of \( H_{n+1} \) is obtained by adding the first \( s-1 \) rows of \( H_n \) shifted by one step to the right to the \( n+3-s \) last rows of \( H_n \).

Example 4.14. The matrices \( H_n \) for \( n \leq 5 \) are the following:

\[
H_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 1 & 11 & 11 & 1 & 0 \\ 0 & 8 & 14 & 2 & 0 \\ 0 & 4 & 16 & 4 & 0 \\ 0 & 2 & 14 & 8 & 0 \\ 0 & 1 & 11 & 11 & 1 \end{pmatrix},
\]

\[
H_4 = \begin{pmatrix} 1 & 26 & 66 & 26 & 1 & 0 \\ 0 & 16 & 66 & 36 & 2 & 0 \\ 0 & 8 & 60 & 48 & 4 & 0 \\ 0 & 4 & 48 & 60 & 8 & 0 \\ 0 & 2 & 36 & 66 & 16 & 0 \\ 0 & 1 & 26 & 66 & 26 & 1 \end{pmatrix}, \quad H_5 = \begin{pmatrix} 1 & 57 & 302 & 302 & 57 & 1 & 0 \\ 0 & 32 & 262 & 342 & 82 & 2 & 0 \\ 0 & 16 & 212 & 372 & 116 & 4 & 0 \\ 0 & 8 & 160 & 384 & 160 & 8 & 0 \\ 0 & 4 & 116 & 372 & 212 & 16 & 0 \\ 0 & 2 & 82 & 342 & 262 & 32 & 0 \\ 0 & 1 & 57 & 302 & 302 & 57 & 1 \end{pmatrix}.
\]

We thus set \( H_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), but do not consider the case \( n = 0 \) throughout the paper.

Corollary 4.15. Every tiling \( T \) (resp. every regular tiling) of a finite simplicial complex \( K \) induces a tiling \( \text{Sd}(T) \) (resp. a regular tiling) on \( \text{Sd}(K) \). Moreover, \( h(\text{Sd}(T)) = h(T)^t H_n \). □

Let \( \Lambda_n = [\lambda_{ij}] \) be the lower triangular \((n+2) \times (n+2)\) matrix where \( \lambda_{ij} \) denotes the number of \( j-2 \) interior faces of \( \text{Sd}(\Delta_{i-2}) \), \( 1 \leq i, j \leq n+2 \). We agree that the standard simplex of dimension \(-1\) has a unique face in dimension \(-1\). The diagonal entries of \( \Lambda_n \) are \((i-1)! \), \( i \in \{1, \ldots, n+2\} \), see [3] [1] [13]. Let \( F_n \) be the \((n+2) \times (n+2)\) matrix whose \( s \)-th row is the face vector \((f_{-1}(T^{n}_{s-1}), \ldots, f_{n}(T^{n}_{s-1}))\) of \( T^{n}_{s-1} \), where we set \( f_{-1}(T^s_n) = 0 \) if \( s \neq 0 \) and \( f_{-1}(T^0_n) = 1 \).
Theorem 4.16. For every \( n > 0 \), \( F_n \) is unipotent upper triangular and \( H_n F_n = F_n \Lambda_n \). Moreover, \( H_n = [h_{ij}]_{1 \leq i,j \leq n+2} \) satisfies the symmetry property \( h_{ij} = h_{n+3-i,n+3-j} \).

Proof. It follows from Proposition 4.3 that \( F_n = [(n+2-i-j)]_{1 \leq i,j \leq n+2} \), thus it is a unipotent upper triangular matrix. Now, by definition of \( H_n \) and \( F_n \), the \( s \)-th row of the product \( H_n F_n \) is the face vector of \( Sd(T_{n-1}^n) \), \( s \in \{1, \ldots, n+2\} \). Likewise, by definition of \( F_n \) and \( \Lambda_n \), the \( s \)-th row of the product \( F_n \Lambda_n \) is the face vector of \( Sd(T_{n-1}^n) \), \( s \in \{1, \ldots, n+2\} \). Thus, the two products coincide.

Finally, to prove the symmetry property of \( H_n \), we proceed by induction. The matrix \( H_1 \) satisfies this symmetry. Let us now assume that \( H_r \) satisfies the symmetry property for every \( r \leq n \). By definition, the \( s \)-th row of \( H_{n+1} \) is obtained by adding up the first \( s-1 \) rows of \( H_n \) shifted once to the right with its \( n+3-s \) last rows. We deduce that for every \( 1 \leq s, j \leq n+3 \),

\[
h_{s,j}^{n+1} = \sum_{i=1}^{s-1} h_{i,j}^n + \sum_{i=s}^{n+2} h_{i,j}^n = \sum_{i=1}^{s-1} h_{n+3-i,n+4-j} + \sum_{i=s}^{n+2} h_{n+3-i,n+3-j},
\]

since \( H_n \) satisfies the symmetry property by induction hypothesis. Thus,

\[
h_{s,j}^{n+1} = \sum_{i=n+4-s}^{n+2} h_{i,n+4-j}^n + \sum_{i=1}^{n+3-s} h_{i,n+3-j}^n
= h_{n+4-s,n+4-j}^{n+1}.
\]

Hence the result. \( \square \)

Let \( \rho_n \) be the involution \( (h_0, \ldots, h_{n+1}) \in \mathbb{R}^{n+2} \rightarrow (h_{n+1}, \ldots, h_0) \in \mathbb{R}^{n+2} \). The symmetry property given by Theorem 4.16 means that the endomorphism \( H_n \) of \( \mathbb{R}^{n+2} \) commutes with \( \rho_n \).

Corollary 4.17. For every \( n > 0 \), \( H_n \) is diagonalizable with eigenvalues \( s! \), \( s \in \{0,1,\ldots,n+1\} \). Moreover, the restriction of \( \rho_n \) to the eigenspace of \( s! \) is \( (−1)^{n+1-s} s! \text{id} \). The vector \((1,1,\ldots,1) = (f_n(T_0^n), \ldots, f_n(T_{n+1}^n))\) spans the eigenspace of \( H_n \) associated to the eigenvalue \( (n+1)! \).

Proof. By Theorem 4.16, \( H_n \) is conjugated to the matrix \( \Lambda_n \) which is diagonalizable with eigenvalues \( s! \), \( s \in \{0, \ldots, n+1\} \), see \([3, 4, 14]\). The first part follows. Again by Theorem 4.16, we know that the last column of \( F_n \) is an eigenvector of \( H_n \) associated to the eigenvalue \( (n+1)! \). It is the vector \((f_n(T_0^n), \ldots, f_n(T_{n+1}^n)) = (1, \ldots, 1)\) which is preserved by \( \rho_n \). The result is thus proved for \( n = 1 \), since \( 1 \) is an eigenvalue of multiplicity two and the signature of \( \rho_n \) vanishes on the corresponding eigenspace. We will prove by induction that if \( \rho_n \) acts as \( -\text{id} \) on the eigenspace associated to the eigenvalue \( s! \) of \( H_n \), then \( \rho_{n+1} \) acts as \( +\text{id} \) on the eigenspace associated to the eigenvalue \( s! \) of \( H_{n+1} \). The result then follows, as the signature of \( \rho_n \) is either 0 or 1 depending on the parity of \( n \). Let \( s \leq n \) be such that \( \rho_n \) acts as \( -\text{id} \) on the eigenspace associated to the eigenvalue \( s! \) of \( H_n \). By Theorem 4.16, the eigenvectors associated to this eigenvalue are linear combination of the \( n+2-s \) last columns of \( F_n \). Indeed, let \( X_{n,s} \) be an eigenvector of \( \Lambda_n \) associated to the eigenvalue \( s! \), then \( F_n X_{n,s} \) is an eigenvector of \( H_n \) corresponding to \( s! \). By hypothesis, \( \rho_n \) acts as \( -\text{id} \) on the eigenspace spanned by this vector, so that \((1, \ldots, 1)^t F_n X_{n,s} = 0\). By definition of \( F_n \) and from Corollary 4.2, \((1, \ldots, 1)^t F_n = \text{the face vector } f_n \) of \( \partial \Delta_{n+1} \). As \( \Lambda_n \) is lower triangular, there exists \( x_{n+1} \in \mathbb{R} \) such that \( X_{n+1,s} = \begin{pmatrix} X_{n,s} \\ x_{n+1} \end{pmatrix} \). The first row of \( F_{n+1} \) is the face vector of \( \Delta_{n+1} = T_{n+1}^{(n+1)!} \)
and thus equals \((f_n^t, 1)\). The first coefficient of \(F_n X_{n+1,s}\) is thus \(f_n^t X_{n,s} + x_{n+1} = x_{n+1}\).

However, the last row of \(F_n\) is the face vector of \(\Delta_n = T_{n+2}^0\) so that the last coefficient of \(F_n X_{n+1,s}\) is \(x_{n+1}\) as well. Similarly, the second row of \(F_n\) is the face vector of \(T_1^{n+1}\). From Proposition 4.1, \(T_0^{n+1} = T_1^{n+1} \cup T_0^0\) so that this second row differs from the first one by the face vector of \(T_0^0\). Let us denote by \(F_i^0\) the rows of \(F_n\), \(i \in \{1, \ldots, n + 2\}\). We deduce that \(F_{n+1}^2 X_{n+1,s} = F_{n+1}^1 X_{n+1,s} - F_n^1 X_{n,s} = x_{n+1} - F_n^1 X_{n,s}\). Since by Proposition 4.1 we have \(T_{n+1}^{n+1} = T_{n+2}^{n+1} \cup T_{n+1}^{n+1}\), we deduce that \(F_n^{n+2} X_{n+1,s} = F_{n+1}^{n+3} X_{n+1,s} + F_n^{n+2} X_{n,s} = x_{n+1} + F_n^{n+2} X_{n,s}\). By the induction hypothesis, the eigenvector \(F_n X_{n,s}\) is reversed by \(\rho_n\) so that \(F_n^1 X_{n,s} = -F_n^{n+2} X_{n,s}\). Therefore, the second coefficient of the eigenvector \(F_n X_{n+1,s}\) coincides with its second to last. Proceeding in the same way by induction, we deduce that the \(i\)-th coefficient of \(F_n X_{n+1,s}\) coincides with the \((n + 4 - i)\)-th one for \(i \in \{1, \ldots, n + 3\}\), which means \(\rho_n(F_n X_{n+1,s}) = F_n X_{n+1,s}\). Hence the result.

Let \(h^n = (h_0^n, \ldots, h_{n+1}^n)\) be the eigenvector of the transposed matrix \(H_n^d\) associated to the eigenvalue \((n + 1)\) and normalized in such a way that \(\sum_{s=0}^{n+1} h_s^n = 1\). For every tiled finite simplicial complex \((K, T)\) of dimension \(n\), we set \(|h(T)| = \sum_{s=0}^{n+1} h_s^n(T)|\).

**Corollary 4.18.** For every tiled finite \(n\)-dimensional simplicial complex \((K, T)\), the sequence \(\frac{1}{|h(T)||(n+1)!} h(Sd^d(T))\) converges to \(h^n\) as \(d\) grows to \(+\infty\). Moreover, \(h_0^n = h_{n+1}^n = 0\) and \(h^n\) is preserved by the symmetry \(\rho_n\). Finally, \(\frac{1}{(n+1)!} h_n^d\) converges to the matrix \((1, 1, \ldots, 1)(h^n)^t\) as \(d\) grows to \(+\infty\).

**Proof.** From Corollary 4.17 we know that \(H_n = PD P^{-1}\), where \(P\) denotes the matrix of eigenvectors of \(H_n\) and \(D\) the diagonal matrix of eigenvalues \(s!\), \(s \in \{0, \ldots, n + 1\}\), so that \((P^{-1})^t\) is a matrix of eigenvectors of \(H_n^d\). Thus, \(\frac{1}{(n+1)!^d} H_n^d = P \frac{D^d}{(n+1)!^d} P^{-1}\) and \(\frac{D^d}{(n+1)!^d}\) converges to \(Diag(0, \ldots, 0, 1)\) as \(d\) grows to \(+\infty\), compare 4. Therefore, \(\frac{H_n^d}{(n+1)!^d}\) converges to the product \((1, \ldots, 1)^t h\), where \(h\) denotes an eigenvector of \(H_n^d\) associated to the eigenvalue \((n + 1)\!\!), since \((1, \ldots, 1)\) is an eigenvector of \(H_n^d\) associated to the eigenvalue \((n + 1)\!\!\) from Corollary 4.17. For every \(d > 0, (1, \ldots, 1)\) is preserved by \(\frac{H_n^d}{(n+1)!^d}\), so that it is also preserved in the limit by \((1, \ldots, 1)^t h\). Thus, \(|h| = 1\) and since the eigenvalue \((n + 1)\!\!) is simple from Corollary 4.17 we deduce that \(h = h^n\). From Corollary 4.15 we know that for every \(d > 0\) \(h(Sd^d(T)) = h(T)^t H_n^d\). We deduce that \(\frac{1}{|h(T)||(n+1)!} h(Sd^d(T))\) converges to \(\frac{1}{(n+1)!^d} (1, \ldots, 1)(h^n)^t = h^n\). The fact that \(h^n\) is preserved by \(\rho_n\) follows from Corollary 4.17 since \(h^n\) is an eigenvector of \(H_n^d\) associated to the eigenvalue \((n + 1)\!\!) and the matrix \(J\) of \(\rho_n\) is symmetric. Indeed, Corollary 4.17 implies that \(JP = PDiag((-1)^{n+2-i})\), so that \(P^{-1} J = Diag((-1)^{n+2-i}) P^{-1}\) and \(J^t (P^{-1})^t = (P^{-1})^t Diag((-1)^{n+2-i})\). As \(J^t = J\) and the last column of \((P^{-1})^t\) is \(h^n\), \(\rho_n\) fixes \(h^n\). (Another way to see that \(\rho_n(h^n) = h^n\) is to consider a basis \((e_0, \ldots, e_{n+1})\) of eigenvectors of \(H_n\). Then the dual basis \((e_0^*, \ldots, e_{n+1}^*)\) is made of eigenvectors of \(H_n^d\). If \(\rho_n e_i = e_i e_i^*\), then \(\rho_n^d e_i^* = e_i e_i^*\), with \(e_i \in \{\pm 1\}\) and \(\rho_n^d = \rho_n\) under canonical identification between \((\mathbb{R}^{n+2})^*\) and \(\mathbb{R}^{n+2}\).) Finally, by induction on \(d\) we deduce from Theorem 4.13 that the number of tiles \(T_0^n\) and \(T_{n+1}^n\) which are in the tiling \(Sd^d(T)\) does not depend on \(d\) so that \(h_0^n = \lim_{d \to +\infty} \frac{1}{|h(T)||(n+1)!} h_0^n(Sd^d(T)) = 0\) and \(h_{n+1}^n = \lim_{d \to +\infty} \frac{1}{|h(T)||(n+1)!} h(n(Sd^d(T))) = 0\).

Recall that for every finite \(n\)-dimensional simplicial complex \(K\) with face polynomial \(q_K(T) = \sum_p f_p(K) T^p\), the polynomial \(\frac{1}{f_n(K)(n+1)!} q_n(Sd^d(K))(T)\) converges to a limit polyno-
mial $q_n^\infty = \sum_{p=0}^n q_{p,n}T^p$, see [3, 4, 14]. The first part of Corollary 4.18 is nothing but this result expressed in terms of $h$-vector and its proof is similar to the one of [4].

**Corollary 4.19.** For every $n > 0$, $h^n(X) = (X - 1)^n q^n(\frac{1}{X-1})$, where $h^n(X) = \sum_{i=0}^n h_i^n X^i$.

**Proof.** Let $K$ be a finite $n$-dimensional simplicial complex equipped with a tiling $T$. For example, $K = \Delta_n$ being tiled with a single $T_0$. From Theorem 1.12 we know that $|h(T)| = h_n(K)$ and deduce

$$\frac{1}{|h(T)|(n+1)!} \sum_{i=0}^{n+1} f_i(Sd^d(T))X^{n+1-i} = \frac{1}{f_n(K)(n+1)!} \sum_{i=0}^{n+1} f_{i-1}(Sd^d(K))(X-1)^{n+1-i}$$

with $f_{-1}(Sd^d(K)) = h_0(Sd^d(T))$. From Theorem 4.13, $h_0(Sd^d(T)) = h_0(T)$. We thus deduce from Corollary 4.18 and [4] by passing to the limit as $d$ grows to $+\infty$ that $\sum_{i=0}^{n+1} h_i^n X^{n+1-i} = \sum_{i=1}^{n+1} q_i-1.n(X-1)^n+1-i = (X-1)^n q^n(\frac{1}{X-1})$. The result now follows from Corollary 4.18 since $h_0^n = h_{n+1}^0 = 0$ and $\rho_n(h^n) = h^n$.

5 Packings

**Theorem 5.1.** For every $n > 0$, $Sd(\Delta_n)$ contains a packing of disjoint simplices with one $n$-simplex and for every $j \in \{0, \ldots, n-1\}$, $2^{-n-j}$ $j$-simplices. Moreover, for every $s \in \{1, \ldots, n+1\}$, $Sd(T^s_0)$ contains a packing of disjoint simplices with one $(n+1-s)$-simplex and if $s \leq n-1$, for every $j \in \{0, \ldots, n-1-s\}$, $2^{n-1-s-j}$ simplices of dimension $j$. The simplex of dimension $n+1-s$ reads $[\hat{\sigma}_{s-1}, \ldots, \hat{\sigma}_m]$, where for $i \in \{s-1, \ldots, n\}$, $\sigma_i$ denotes an $i$-simplex of $\Delta_n$.

**Proof.** We proceed by induction on the dimension $n > 0$. If $n = 1$, one checks the result, $Sd(\Delta_1) \supset \Delta_1 \cup \Delta_0, Sd(T^1_0) \supset \Delta_1$ and $Sd(T^2_0) \supset \Delta_0$. Suppose now that the result holds true for every dimension $\leq n$. From Corollary 4.2, we know that $\partial \Delta_{n+1}$ is tileable and $\partial \Delta_{n+1} = \bigsqcup_{s=0}^{n+1} T^s_n$, so that $Sd(\partial \Delta_{n+1}) = \bigsqcup_{s=0}^{n+1} Sd(T^s_n)$. The union of the packings given by the induction hypothesis provides a packing of the boundary of $Sd(\Delta_{n+1})$ which contains two $n$-simplices. We replace the $n$-simplex contained in $Sd(T^0_n)$ by its cone centered at the barycenter of $\Delta_{n+1}$. We get in this way a packing of disjoint simplices in $Sd(\Delta_{n+1})$ containing a simplex of dimension $n+1$, a simplex of dimension $n$ and $1 + \sum_{s=0}^{n+1-j} 2^{n-1-s-j}$ simplices of dimension $j$, $j \in \{0, \ldots, n-1\}$. Now, $1 + \sum_{s=0}^{n+1-j} 2^{n-1-s-j} = 1 + \sum_{s=0}^{n+1-j} 2^s = 2^{n-j}$.

Likewise, from Proposition 4.1 we deduce that for every $s \in \{1, \ldots, n+1\}$, $Sd(\partial T^s_{n+1}) = \bigsqcup_{l=s}^{n+1} Sd(T^l_n)$. The union of the packings given by the induction hypothesis provides a packing of simplices in $Sd(T^s_{n+1}) \cap Sd(\partial \Delta_{n+1})$ which contains one simplex of dimension $n+1-s$, of the form $[\hat{\sigma}_{s-1}, \ldots, \hat{\sigma}_m]$, that replaces it by its cone centered at the barycenter of $\Delta_{n+1}$. We thus get a packing of disjoint simplices in $Sd(T^s_{n+1})$ which consists of one simplex of dimension $n+2-s$, of the form $[\hat{\sigma}_{s-1}, \ldots, \hat{\sigma}_{s+1}]$, and one simplex of dimension $n-s$ if $s < n+1$ together with, if $s \leq n-1$, $1 + \sum_{l=0}^{n+1-s-j} 2^{n-1-s-j-l} = 2^{n-s-j}$ simplices of dimension $j$ for every $j \in \{0, \ldots, n-1-s\}$. Finally, $Sd(T^1_{n+2})$ contains the barycenter of $\Delta_{n+1}$, which can be written as $\hat{\sigma}_{n+1}$. Hence the result.

We now able to prove Theorem 1.12.

**Proof of Theorem 1.12** By Corollary 4.15, $Sd(K)$ is tiled by $Sd(T)$ so that the union of the packings given by Theorem 5.1 provides the result. □
We may relax the condition to be disjoint in Theorems 5.1 and 1.12, to get the following results.

**Theorem 5.2.** For every \( n > 0 \), every \( p \in \{1, \ldots, n-1\} \) and every \( s \in \{0, \ldots, n+1\} \), \( \text{Sd}(T^n_s) \) contains a packing of simplices with one simplex of dimension \( n + 1 - s + p \) if \( s \geq p + 1 \) or \( 2^{n-s} \) simplices of dimension \( n \) if \( s \leq p \) together with \( 2^{n-1-s} \) simplices of dimension \( j \), \( j \in \{p, \ldots, \min(n-1-s+p, n-1)\} \) if \( s \leq n - 1 \), in such a way that the intersection of two simplices of this collection is of dimension less than \( p \) and the intersection of each simplex with \( \text{Sd}(\partial \Delta_n) \setminus \text{Sd}(T^n_s) \) is of dimension less than \( p \). Moreover, the \( (n + 1 - s + p) \)-simplex is of the form \([\hat{\sigma}_{s-p-1, \ldots, \hat{\sigma}}]_n\), where for every \( i \in \{s-p, \ldots, n\} \), \( \hat{\sigma}_i \) is an \( i \)-simplex of \( \Delta_n \).

**Proof.** We proceed by induction on \( p \in \{1, \ldots, n\} \). If \( p = 1 \), we first check the result for \( n = 1 \). In this case, \( \text{Sd}(\Delta_1) \) contains exactly two simplices of dimension one intersecting each other at the barycenter of \( \Delta_1 \) and intersecting \( \text{Sd}(\partial \Delta_1) \) at a vertex. This provides a suitable packing for \( \text{Sd}(T^n_0), \text{Sd}(T^n_1) \) and \( \text{Sd}(T^n_2) \).

If \( n > 1 \), we know from Proposition 4.1 that for every \( s \in \{1, \ldots, n\} \), \( \text{Sd}(\partial T^n_0) = \bigcup_{i=s}^n \text{Sd}(T_{i-1}^{n-1}) \). The union for \( l \in \{s, \ldots, n\} \) of the packings given by Theorem 5.1 provides a packing of disjoint simplices in \( \text{Sd}(T^n_0) \cap \text{Sd}(\partial \Delta_n) \) which contains an \( (n-s) \)-simplex of the form \([\hat{\sigma}_{s-1, \ldots, \hat{\sigma}}]_n\), a simplex of dimension \( n - 1 - s \) if \( s \leq n - 1 \) and if \( s < n - 1 \), \( 2^{n-1-s} \) simplices of dimension \( j \) for every \( j \in \{0, \ldots, n - 2 - s\} \). By replacing these simplices by their cones centered at the barycenter of \( \Delta_n \), we get a collection of simplices in \( \text{Sd}(T^n_0) \) which contains one \( (n + 1 - s) \)-simplex of the form \([\hat{\sigma}_{s-1, \ldots, \hat{\sigma}}]_n\) and if \( s \leq n - 1 \), \( 2^{n-1-s} \) simplices of dimension \( j \) for every \( j \in \{1, \ldots, n - s\} \). Moreover, two such simplices of this collection intersect at the barycenter of \( \Delta_n \) and these simplices are contained in \( \text{Sd}(T^n_s) \). If \( s \geq 2 \), we remark that an \( (n + 1 - s) \)-simplex is of the form \([\hat{\sigma}_{s-1, \ldots, \hat{\sigma}}]_n\), where \( \hat{\sigma}_i \) is an \( i \)-simplex of \( \Delta_n \). By choosing a facet \( \sigma_{s-2} \) of \( \sigma_{s-1} \) and by replacing this \( (n + 1 - s) \)-simplex by the \( (n + 2 - s) \)-simplex \([\hat{\sigma}_{s-2, \ldots, \hat{\sigma}}]_n\) we get the required packing. This last simplex indeed intersects \( \text{Sd}(\partial \Delta_n) \setminus \text{Sd}(T^n_s) \) at the vertex \( \{\hat{\sigma}_{s-2}\} \). If \( s = 0 \), we likewise know from Proposition 4.1 that \( \text{Sd}(\partial \Delta_n) = \bigcup_{l=0}^n \text{Sd}(T_{l-1}^{n-1}) \). The union for \( l \in \{0, \ldots, n\} \) of the packings given by Theorem 5.1 provides a packing of disjoint simplices in \( \text{Sd}(\partial \Delta_n) \) which contains two simplices of dimension \( n - 1 \) and for every \( j \in \{0, \ldots, n - 2\} \), \( 2^{n-1-j} \) simplices of dimension \( j \). By replacing these simplices by their cones centered at the barycenter of \( \Delta_n \), we get a collection of simplices of \( \text{Sd}(T^n_0) \) that consists of two simplices of dimension \( n \) and of \( 2^{n-1-j} \) simplices of dimension \( j \), \( j \in \{1, \ldots, n - 1\} \). Moreover, two such simplices intersect at the barycenter of \( \Delta_n \). Finally, if \( s = n + 1 \), the 1-simplex \([\hat{\sigma}_{n-1, \hat{\sigma}}]_n\), where \( \hat{\sigma}_i \subset \Delta_n \) is of dimension \( i \), intersects \( \text{Sd}(\partial \Delta_n) \) at the vertex \( \{\hat{\sigma}_{s-1}\} \). This simplex gives the desired collection and the result follows for \( p = 1 \) and every \( n > 0 \).

Now, let us suppose that the result holds true for every \( r \leq p - 1 \) and \( n \) and let us prove it for \( r = p \) and \( n \) and \( p \), \( n \geq p \). Let \( 2 \leq p \leq n \) and \( s \in \{0, \ldots, n\} \). From Proposition 4.1 \( \text{Sd}(\partial T^n_s) = \bigcup_{l=s}^n \text{Sd}(T_{l-1}^{n-1}) \). Let us equip each \( \text{Sd}(T_{l-1}^{n-1}) \) with a packing given by the induction hypothesis applied to \( p - 1 \leq n - 1 \). The union of these packings gives a packing of simplices of \( \text{Sd}(\partial T^n_s) \) such that the intersection between two simplices is of dimension \( \leq p - 1 \). This packing contains \( 2^{n-2-s} \) simplices of dimension \( j \) for every \( j \in \{p - 1, \min(n - 2 - s + p, n - 2)\} \) if \( s \leq n - 1 \) and one simplex of dimension \( n - 2 + s \), one simplex of dimension \( n - 1 - s + p \) if \( s \geq p \) and \( 1 + \sum_{i=s}^{n-p} 2^{n-1-j} \) simplices of dimension \( n - 1 \) if \( s < p \). By replacing all these simplices by their cones centered at the barycenter of \( \Delta_n \), we get a packing of simplices in \( \text{Sd}(T^n_s) \) which contains \( 2^{n-2-s} \) simplices of dimension \( j + 1 \) for every \( j \in \{p - 1, \min(n - 2 - s + p, n - 2)\} \), that is to say \( 2^{n-1-s-j-p} \) simplices of dimension \( j \) for \( j \in \{p, \min(n - 1 - s + p, n - 2)\} \) if
s ≤ n − 1, as well as 2^p−s simplices of dimension n if s < p and one simplex of dimension n − s + p if s ≥ p. If s > p, this last simplex, by construction, is of the form [σ_s−p,...,σ_n], where σ_i ∈ Δ_n is of dimension i. By choosing a facet σ_s−p−1 of σ_s−p we replace this (n−s+p)-simplex by the (n+1−s+p)-simplex [σ_s−p−1,...,σ_n] to get the result for s ≤ n. Indeed, by construction and the induction hypothesis, two disjoint simplices from this packing intersect each other in dimension ≤ p − 1 and each simplex intersect Sd(Δ_n) \ Sd(T_s^p) in dimension ≤ p − 1. If s = n + 1, every simplex of the form [σ_n−p,...,σ_n] ∈ Sd(Δ_n), where σ_i ∈ Δ_n is of dimension i, gives a p-simplex which intersect Sd(∂Δ_n) at the (p−1)-simplex [σ_n−p,...,σ_n−1]. A packing of Sd(T_s^p) reduced to this simplex is suitable. Hence the result. □

Corollary 5.3. Let K be a finite n-dimensional simplicial complex equipped with a tiling T of h-vector (h_0(T),...,h_n+1(T)) and let 1 ≤ p ≤ n − 1. Then, it is possible to pack h_p+1(T) + 2^p ∑_{s=0}^p h_s(T) simplices of dimension n in Sd(K) in such a way that they intersect each other in dimension less than p. Moreover, this packing can be completed by h_{n+1−p}(T) + 2^{n−1−p} ∑_{s=0}^{n−1−p} h_s(T) simplices of dimension j having the same property, j ∈ {p,...,n−1}.

Proof. By Corollary 4.15, Sd(K) is tiled by Sd(T) so that the union of the packings given by Theorem 5.2 provides the result. □

Remark 5.4. 1. Corollary 5.3 remains valid in the case p = 0 using the convention that simplices intersect in negative dimension when they are disjoint. The case p = 0 then gives back Theorem 1.12.

2. For every tiled finite n-dimensional simplicial complex (K,T), e.g. K = Δ_n, and every 0 ≤ p ≤ n, Theorem 1.12 and Corollary 5.3 provide a sequence of packings in Sd^d(K), d > 0. Corollary 4.18 provides the asymptotic of the number of simplices in each dimension of this sequence.

We finally prove Theorem 1.13.

Proof of Theorem 1.13. By definition, λ_{p,ν}(n) = lim_{d→∞} λ_{p,ν}^d(n) and λ_{p,ν}^d(n) = 1_{(n+1)!} \max_{L_ν \in L^d_{n,p}} M_{p,ν}(L), see §3.2. However, ∂n = T_{n+1}^n is tiled by a single tile and this tiling induces a tiling Sd^d(T) of Sd^d(Δ_n) for every d > 0, see Theorem 5.2. Corollary 5.3 then provides a subcomplex L_d ∈ L_{d,p}^d, see §3.2 which contains h_p(Sd^d−1(T)) + 2^p−1 ∑_{i=0}^{p−1} h_i(Sd^d−1(T)) sim of dimension n together with h_{n+p−j}(Sd^d−1(T)) + 2^{n−2+p−j} ∑_{i=0}^{n−2+p−j} h_i(Sd^d−1(T)) sim of dimension j, j ∈ {p−1,..,n−1}.

From Corollary 4.18 we know that for every i ∈ {0,...,n}, h_i(Sd^d−1(T))_{(n+1)!} converges to \frac{h_i^0}{(n+1)!} as d grows to +∞ and that h^0_i = h^0_{n+1} = 0. For every p ∈ {1,...,n−1} and every j ∈ {p+1,...,n}, M_{p,ν}(Δ_j) = ν(1−ν)^j(p^p +(1−ν)p^p)(p+1) by Lemma 3.5. The result follows after the change of variables j → n + p − j. □

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