Recursive boson system in the Cuntz algebra $\mathcal{O}_\infty$

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Abstract

Bosons and fermions are often written by elements of other algebras. M. Abe gave a recursive realization of the boson by formal infinite sums of the canonical generators of the Cuntz algebra $\mathcal{O}_\infty$. We show that such formal infinite sum always makes sense on a certain dense subspace of any permutative representation of $\mathcal{O}_\infty$. In this meaning, we can regard as if the algebra $\mathcal{B}$ of bosons was a unital $\ast$-subalgebra of $\mathcal{O}_\infty$ on a given permutative representation by keeping their unboundedness. By this relation, we compute branching laws arising from restrictions of representations of $\mathcal{O}_\infty$ on $\mathcal{B}$. For example, it is shown that the Fock representation of $\mathcal{B}$ is given as the restriction of the standard representation of $\mathcal{O}_\infty$ on $\mathcal{B}$.

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1 Introduction

Bosons and fermions are not only important in physics but also interesting in mathematics. Studies of their algebras spurred the development of the theory of operator algebras [8]. Representations of bosons are used to describe representations of several algebras [6, 12, 15]. Bosons and fermions are often written by elements of other algebras and such descriptions are useful for several computations. For example, the boson-fermion correspondence [16, 17] is well-known. It is shown that bosons and fermions are corresponded as operators on the infinite wedge representation of fermions.
1.1 Motivation

In our previous paper [1], we have presented a recursive construction of the CAR (=canonical anticommutation relation) algebra for fermions in terms of the Cuntz algebra \( \mathcal{O}_2 \) and shown that it may provide us a useful tool to study properties of fermion systems by using explicit expressions in terms of generators of the algebra. Let \( s_1, s_2 \) be the canonical generators of \( \mathcal{O}_2 \), that is, they satisfy that

\[
s_i^* s_j = \delta_{ij} I \quad (i, j = 1, 2), \quad s_1 s_1^* + s_2 s_2^* = I.
\]

Let \( \zeta \) be the linear map on \( \mathcal{O}_2 \) defined by

\[
\zeta(x) \equiv s_1 x s_1^* - s_2 x s_2^*
\]

for \( x \in \mathcal{O}_2 \).

We recursively define the family \( \{ a_1, a_2, a_3, \ldots \} \) by

\[
a_1 \equiv s_1 s_2^*, \quad a_n \equiv \zeta(a_{n-1}) \quad (n \geq 2).
\]

Then \( \{ a_n : n \in \mathbb{N} \} \) satisfies that

\[
a_n a_m^* + a_m^* a_n = \delta_{nm} I, \quad a_n a_m + a_m a_n = a_n a_m^* + a_m^* a_n = 0 \quad (n, m \in \mathbb{N})
\]

where \( \mathbb{N} = \{1, 2, 3, \ldots\} \). We call such \( \{ a_n : n \in \mathbb{N} \} \) by a recursive fermion system (=RFS) in \( \mathcal{O}_2 \). From this description, the C*-algebra \( \mathcal{A} \) generated by fermions is embedded into \( \mathcal{O}_2 \) as a C*-subalgebra with common unit:

\[
\mathcal{A} \equiv C^*\langle\{a_n : n \in \mathbb{N}\}\rangle \hookrightarrow \mathcal{O}_2
\]

Furthermore \( \mathcal{A} \) coincides with the fixed-point subalgebra of \( \mathcal{O}_2 \) with respect to the \( U(1) \)-gauge action. Because every \( a_n \) is written as a polynomial in the canonical generators of \( \mathcal{O}_2 \) and their \( * \)-conjugates, their description is very simple and it is easy to compute the restriction \( \pi\vert_{\mathcal{A}} \) of a representation \( \pi \) of \( \mathcal{O}_2 \) on \( \mathcal{A} \). By using the RFS, we obtain several new results about fermions [2, 3, 4, 5]. For example, assume that \( (\mathcal{H}, \pi) \) is a \( * \)-representation of \( \mathcal{O}_2 \) with a cyclic vector \( \Omega \). If \( \Omega \) satisfies \( \pi(s_1)\Omega = \Omega \), then \( \pi\vert_\mathcal{A} \) is equivalent to the Fock representation of \( \mathcal{A} \) with the vacuum \( \Omega \). If \( \Omega \) satisfies \( \pi(s_1 s_2)\Omega = \Omega \), then \( \pi\vert_\mathcal{A} \) is equivalent to the direct sum of the infinite wedge representation and the dual infinite wedge representation of \( \mathcal{A} \) [13]. In this way, well-known results of fermions are explicitly reformulated by the representation theory of \( \mathcal{O}_2 \).

From this, we speculate that the boson can be also simply written by the generators of a certain Cuntz algebra like the RFS, where the boson means a family \( \{ a_n : n \in \mathbb{N} \} \) satisfying that

\[
a_n a_m^* - a_m^* a_n = \delta_{nm} I, \quad a_n a_m - a_m a_n = a_n a_m^* + a_m^* a_n = 0 \quad (1.1)
\]

Furthermore, the boson is written as a polynomial in the canonical generators of \( \mathcal{O}_2 \) and their \( * \)-conjugates, their description is very simple and it is easy to compute the restriction \( \pi\vert_{\mathcal{A}} \) of a representation \( \pi \) of \( \mathcal{O}_2 \) on \( \mathcal{A} \).
for each \( n, m \in \mathbb{N} \). However, the boson is always represented as a family of unbounded operators on a Hilbert space. Hence the \(*\)-algebra generated by \( \{a_n : n \in \mathbb{N}\} \) never be a \(*\)-subalgebra of any C*-algebra. On the other hand, the C*-algebra approach of boson is well-known as the CCR algebra (CCR = canonical commutation relations, see § 5.2 in [8]). Because the CCR algebra is not a separable C*-algebra, it is impossible to embed it into any Cuntz algebra as a C*-subalgebra. From these problems, it seems that a RFS-like description of bosons by any Cuntz algebra is impossible.

### 1.2 Recursive boson system

In spite of such problems, Mitsuo Abe gave a “formal” realization of the boson by the canonical generators of the Cuntz algebra \( \mathcal{O}_\infty \) in 2006 as follows. Let \( \{s_n : n \in \mathbb{N}\} \) be the canonical generators of \( \mathcal{O}_\infty \), that is,

\[
s_i^* s_j = \delta_{ij} I \quad (i, j \in \mathbb{N}), \quad \sum_{i=1}^k s_is_i^* \leq I \quad (\text{for any } k \in \mathbb{N}).
\]

Define the family \( \{a_n : n \in \mathbb{N}\} \) of formal sums by

\[
a_1 = \sum_{m=1}^{\infty} \sqrt{m} s_m s_m^* + 1, \quad a_n = \rho(a_{n-1}) \quad (n \geq 2)
\]

where \( \rho \) is the formal canonical endomorphism of \( \mathcal{O}_\infty \) defined by

\[
\rho(x) = \sum_{n=1}^{\infty} s_n x s_n^* \quad (x \in \mathcal{O}_\infty).
\]

By formal computation, we can verify that \( a_n \)'s satisfy (1.1) where we assume that infinite sums can be freely exchanged. However infinite sums in these equations do not converge in \( \mathcal{O}_\infty \) in general. Hence (1.2) does not make sense as elements of \( \mathcal{O}_\infty \).

We show that the Abe’s formal description (1.2) can be justified as unbounded operators defined on a certain dense subspace of any permutative representation of \( \mathcal{O}_\infty \). Define \( X_N \equiv \{1, \ldots, N\} \) for \( 2 \leq N < \infty \) and \( X_\infty \equiv \mathbb{N} \). Let \( \{s_n : n \in X_N\} \) be the set of canonical generators of \( \mathcal{O}_N \) for \( 2 \leq N \leq \infty \).

**Definition 1.1** [7, 10, 11] A representation \( (\mathcal{H}, \pi) \) of \( \mathcal{O}_N \) is permutative if there exists a complete orthonormal basis \( \{e_n\}_{n \in \Lambda} \) of \( \mathcal{H} \) and a family \( f = \{f_i\}_{i=1}^{N} \) of maps on \( \Lambda \) such that \( \pi(s_i)e_n = e_{f_i(n)} \) for each \( n \in \Lambda \) and \( i = 1, \ldots, N \). We call \( \{e_n\}_{n \in \Lambda} \) and the linear hull \( \mathcal{D} \) of \( \{e_n\}_{n \in \Lambda} \) by the reference basis and the reference subspace of \( (\mathcal{H}, \pi) \), respectively.
Remark that for any permutation representation \((H, \pi)\) of \(\mathcal{O}_N\) with the reference subspace \(D\), \(\pi(s_n)D \subset D\) and \(\pi(s_n^*)D \subset D\) for each \(n\), but \(\pi(x)D \not\subset D\) for \(x \in \mathcal{O}_\infty\) in general.

From Definition 1.1 and (1.2), we immediately obtain the following fact.

**Fact 1.2** For any permutative representation \((H, \pi)\) of \(\mathcal{O}_\infty\), define the family \(\{A_n : n \in \mathbb{N}\}\) of operators on the reference subspace \(D\) of \((H, \pi)\) by

\[
A_1v \equiv \sum_{m=1}^{\infty} \sqrt{m} \pi(s_ms_{m+1}^*)v,
\]

\[
A_nv \equiv \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \sum_{m=1}^{\infty} \sqrt{m} \pi(s_{m_{n-1}} \cdots s_{m_1} s_m s_{m+1}^* s_{m_1} \cdots s_{m_{n-1}}^*)v
\]

for \(v \in D\) and \(n \geq 2\). Then the family \(\{A_n : n \in \mathbb{N}\}\) satisfies (1.1) on \(D\).

Infinite sums in Fact 1.2 are actually finite for each \(v \in D\). By comparing Fact 1.2 and (1.2), we see that (1.2) is well-defined on the reference subspace of any permutative representation of \(\mathcal{O}_\infty\). Furthermore, the mapping

\[
a_n \mapsto A_n \quad (n \in \mathbb{N})
\]

defines a unital \(*\)-representation \(\pi_B\) of the algebra \(B\) of bosons on \(D\), that is, \(\pi_B(a_n) \equiv A_n\) for each \(n\). In consequence, we obtain the operation

\[
(H, \pi) \mapsto (D, \pi_B)
\]

for any permutative representation \((H, \pi)\) of \(\mathcal{O}_\infty\) to the representation \((D, \pi_B)\) of \(B\). We call \((D, \pi_B)\) the restriction of \((H, \pi)\) on \(B\) and often write it by \((H, \pi|_B)\) for convenience in this paper. Strictly speaking, this is not a restriction because \(B\) is neither a subalgebra of \(\mathcal{O}_\infty\) nor \(\pi(\mathcal{O}_\infty)D \subset D\).

**Remark 1.3** If a \(C^*\)-algebra \(A\) irreducibly acts on a Hilbert space \(\mathcal{H}\), then any (unbounded) operator on \(\mathcal{H}\) can be written by the strong operator limit of elements of \(A\) on \(\mathcal{H}\). However such description always depends on the choice of representation. Fact 1.2 claims that the description (1.2) always hold on any permutative representations of \(\mathcal{O}_\infty\) nevertheless there exist infinitely many inequivalent permutative representations of \(\mathcal{O}_\infty\) and they are not always irreducible.

**Definition 1.4** The family \(\{A_n : n \in \mathbb{N}\}\) in Fact 1.2 is called the recursive boson system (=RBS) in \(\mathcal{O}_\infty\) with respect to a permutative representation \((H, \pi)\) of \(\mathcal{O}_\infty\).
We identify $A_n$ in Fact 1.2 with $a_n$.

Remark that $B$ is neither a subalgebra of $O_\infty$ nor that of the double commutations $\pi(O_\infty)$ of $\pi(O_\infty)$. However for any permutative representation $(\mathcal{H}, \pi)$ of $O_\infty$, we obtain a representation of the boson as $B$ by the RBS. In this sense, it seems that $B$ is a subalgebra of $O_\infty$ in special situation:

$$B = \text{Alg}\langle\{a_n, a_n^* : n \in \mathbb{N}\}\rangle \cong \text{subalgebra of } O_\infty.$$  

1.3 Representations of bosons arising from permutative representations of $O_\infty$

We show the significance of the RBS in the representation theory of operator algebras. The algebra $B$ of bosons always appears with a representation in theoretical physics. Especially, the Fock representation plays the most important role among representations of $B$. It has both the mathematical simple structure and the physical meaning. By the RBS, we can understand the Fock representation from a viewpoint of the representation theory of $O_\infty$.

First, we explain the notion of branching law. For a group $G$, if there exists an embedding of $G$ into some other group $G'$, then any representation $\pi$ of $G'$ induces the restriction $\pi|_G$ of $\pi$ on $G$. The representation $\pi|_G$ is not irreducible in general even if $\pi$ is irreducible. If $\pi|_G$ is decomposed into the direct sum of a family $\{\pi_\lambda : \lambda \in \Lambda\}$ of irreducible representations of $G$, then the equation

$$\pi|_G = \bigoplus_{\lambda \in \Lambda} \pi_\lambda$$

is called the branching law of $\pi$. The branching law can be also considered for a pair of subalgebra and algebra. Thanks to the RBS, we can consider (an analogy of) branching laws of permutative representations of $O_\infty$ which are restricted on $B$.

**Theorem 1.5** (i) For $j \geq 1$, let $(\mathcal{H}, \pi_j)$ be a representation of $O_\infty$ with a cyclic vector $\Omega$ satisfying

$$\pi_j(s_j)\Omega = \Omega.$$

Then there exists a dense subspace $\mathcal{D}_j$ of $\mathcal{H}$ and an action $\eta_j$ of $B$ on $\mathcal{D}_j$ such that $\eta_j(B)\Omega = \mathcal{D}_j$ and

$$\eta_j(a_n a_n^*)\Omega = j^\Omega \quad (n \geq 1).$$

In particular, $\eta_1$ is the Fock representation of $B$ with the vacuum $\Omega$.  

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(ii) Let \((\mathcal{H}, \pi_{12})\) be a representation of \(\mathcal{O}_\infty\) with a cyclic vector \(\Omega\) satisfying
\[
\pi_{12}(s_1 s_2)\Omega = \Omega.
\]
Then there exist two subspaces \(V_1\) and \(V_2\) of \(\mathcal{H}\) and two actions \(\eta_{12}\) and \(\eta_{21}\) of \(\mathcal{B}\) on \(V_1\) and \(V_2\), respectively such that \(V_1 \oplus V_2\) is dense in \(\mathcal{H}\), \(V_1 = \pi_{12}(\mathcal{B})\Omega\), \(V_2 = \pi_{21}(\mathcal{B})\Omega'\) for \(\Omega' \equiv \pi(s_2)\Omega\) and the following holds:
\[
\begin{align*}
\eta_{12}(a_{2n-1})\Omega &= \eta_{21}(a_{2n})\Omega' = 0, \\
\eta_{12}(a_{2n}^* a_{2n})\Omega &= \Omega, \\
\eta_{21}(a_{2n-1}^* a_{2n-1})\Omega' &= \Omega'
\end{align*}
\]
\(n \geq 1\).

(iii) Any two of representations in \(\{\eta_j, \eta_{12}, \eta_{21} : j \geq 1\}\) of \(\mathcal{B}\) are not unitarily equivalent.

(iv) All of representations \(\{\eta_j, \eta_{12}, \eta_{21} : j \geq 1\}\) of \(\mathcal{B}\) are irreducible.

Every representations of \(\mathcal{O}_\infty\) in Theorem 1.5 (i) and (ii) are irreducible permutative representations. Hence Theorem 1.5 shows branching laws of representations of \(\mathcal{O}_\infty\) restricted on \(\mathcal{B}\):
\[
\pi_j|_\mathcal{B} = \eta_j \quad (j \geq 1), \quad \pi_{12}|_\mathcal{B} = \eta_{12} \oplus \eta_{21}.
\]

By comparison to the fermion case in §1.1 this result shows that the RBS is very similar to the RFS in a sense of the representation theory of operator algebras. This result shows the naturality of the description in (1.2).

In §2 we show permutative representations of \(\mathcal{O}_\infty\) and several representations of \(\mathcal{B}\). In §2.3 we prove Theorem 1.5. In §3 we show examples. In §3.2 we give an interpretation of representations of bosons in Theorem 1.5 by formal infinite product of operators.

2 Representations and their relations

In order to show Theorem 1.5 we introduce several representations of \(\mathcal{O}_\infty\) and \(\mathcal{B}\). After this preparation, we show their relations as the proof of Theorem 1.5.
2.1 Permutative representation of Cuntz algebras

For $N = 2, 3, \ldots, +\infty$, let $\mathcal{O}_N$ be the Cuntz algebra $[9]$, that is, a C*-algebra which is universally generated by $s_1, \ldots, s_N$ satisfying $s_i^* s_j = \delta_{ij} I$ for $i, j = 1, \ldots, N$ and

$$\sum_{i=1}^{N} s_i s_i^* = I \quad (\text{if } N < +\infty), \quad \sum_{i=1}^{k} s_i s_i^* \leq I, \quad k = 1, 2, \ldots \quad (\text{if } N = +\infty)$$

where $I$ is the unit of $\mathcal{O}_N$. Because $\mathcal{O}_N$ is simple, that is, there is no non-trivial closed two-sided ideal, any homomorphism from $\mathcal{O}_N$ to a C*-algebra is injective. If $t_1, \ldots, t_n$ are elements of a unital C*-algebra $A$ such that $t_1, \ldots, t_n$ satisfy the relations of canonical generators of $\mathcal{O}_N$, then the correspondence $s_i \mapsto t_i$ for $i = 1, \ldots, N$ is uniquely extended to a ∗-embedding of $\mathcal{O}_N$ into $A$ from the uniqueness of $\mathcal{O}_N$. Therefore we call such a correspondence among generators by an embedding of $\mathcal{O}_N$ into $A$.

Define $X_N \equiv \{1, \ldots, N\}$ for $2 \leq N < +\infty$ and $X_\infty \equiv \mathbb{N}$. For $N = 2, \ldots, +\infty$ and $k = 1, \ldots, \infty$, define the product set $X_N^k \equiv (X_N)^k$ of $X_N$. Let $\{s_n : n \in X_N\}$ be the set of canonical generators of $\mathcal{O}_N$ for $2 \leq N \leq +\infty$.

\textbf{Definition 2.1} For $J = (j_l)_{l=1}^k \in X_N^k$ with $1 \leq k < +\infty$, we write $P_N(J)$ the class of representations $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ with a cyclic unit vector $\Omega \in \mathcal{H}$ such that $\pi(s_J) \Omega = \Omega$ and $\{\pi(s_{j_l} \cdots s_{j_k}) \Omega\}_{l=1}^{k}$ is an orthonormal family in $\mathcal{H}$ where $s_J \equiv s_{j_1} \cdots s_{j_k}$.

We call the vector $\Omega$ in Definition 2.1 by the GP vector of $(\mathcal{H}, \pi)$. A representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ is called a cycle if there exists $J \in X_N^k$ for $1 \leq k < +\infty$ such that $(\mathcal{H}, \pi)$ belongs to $P_N(J)$. Any permutative representation is uniquely decomposed into cyclic permutative representations up to unitary equivalence. For any $J$, $P_N(J)$ contains only one unitary equivalence class $[7, 10, 11, 13]$. We show properties of $P_\infty(j)$ $(j \geq 1)$ and $P_\infty(12)$ more closely as follows.

\textbf{Lemma 2.2} Let $T \equiv \{P_\infty(j), P_\infty(12) : j \geq 1\}$.

(i) For each $X \in T$, any two representations belonging to $X$ are unitarily equivalent.

(ii) Any two of representations in $T$ are not unitarily equivalent.

(iii) All of representations in $T$ are irreducible.
The proof of Lemma 2.2 are given in Appendix A. From Lemma 2.2 (i), we use symbols $P_{\infty}(j), P_{\infty}(12)$ as their representatives.

For $2 \leq N < \infty$, let $t_1, \ldots, t_N$ be the canonical generators of $O_N$. Define the representation $(l_2(N), \pi)$ of $O_N$ by

$$\pi(t_i)e_n = e_{N(n-1)+i} \quad (i = 1, \ldots, N, n \in \mathbb{N}).$$

Then $(l_2(N), \pi)$ is $P_N(1)$ of $O_N$. If we identify $O_\infty$ with a $C^*$-subalgebra of $O_N$ by the embedding of $O_\infty$ into $O_N$ defined by

$$s_{(N-1)(k-1)+i} \equiv t_{N}^{k-1}t_i \quad (k \geq 1, i = 1, \ldots, N-1),$$

then $(l_2(N), \pi|_{O_\infty})$ is $P_\infty(1)$ of $O_\infty$.

2.2 Representations of bosons

We summarize several representations of bosons and their properties. We write $B$ the $*$-algebra generated by $\{a_n : n \in \mathbb{N}\}$ which satisfies (1.1). A representation of $B$ is a pair $(H, \pi)$ such that $H$ is a complex Hilbert space with a dense subspace $D$ and $\pi$ is a $*$-homomorphism from $B$ to the $*$-algebra $\{x \in \text{End}_{\mathcal{C}}(D) : x^*D \subset D\}$. A cyclic vector of $(H, \pi)$ is a vector $\Omega \in D$ such that $\pi(B)\Omega = D$.

Definition 2.3  

(i) For $j \geq 1$, we write $F_j$ the class of representations $(H, \pi)$ of $B$ with a cyclic vector $\Omega$ satisfying $\pi(a_n a_n^*)\Omega = j\Omega$ for each $n \in \mathbb{N}$.

(ii) We write $F_{12}$ the class of representations $(H, \pi)$ of $B$ with a cyclic vector $\Omega$ satisfying

$$\pi(a_{2n-1})\Omega = 0, \quad \pi(a_{2n}a_{2n})\Omega = \Omega$$

for each $n \in \mathbb{N}$.

(iii) We write $F_{21}$ the class of representations $(H, \pi)$ of $B$ with a cyclic vector $\Omega$ satisfying

$$\pi(a_{2n})\Omega = 0, \quad \pi(a_{2n-1}^*a_{2n-1})\Omega = \Omega$$

for each $n \in \mathbb{N}$.

A representation $(H, \pi)$ of $B$ is called irreducible if the commutant of $\pi(B)$ in $B(H)$ is the scalar multiples of $I$. 

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Lemma 2.4 Let $S \equiv \{F_j, F_{12}, F_{21} : j \geq 1\}$.

(i) For each $X \in S$, any two representations belonging to $X$ are unitary equivalent. From this, we can identify a representation belonging to $X \in S$ with $X$.

(ii) Any two of representations in $S$ are not unitarily equivalent.

(iii) All of representations in $S$ are irreducible.

Lemma 2.4 is proved in Appendix B. We consider the case $j = 1$ in Definition 2.3 (i). Then $\pi(a_n a_n^*) \Omega = \Omega$ for each $n$. From this, $\pi(a_n^* a_n) \Omega = 0$. This implies that $\pi(a_n) \Omega = 0$ for each $n$. Because $\Omega$ is a cyclic vector, $F_1$ is the Fock representation of $\mathcal{B}$ with the vacuum $\Omega$.

In this study, we became the first to find $F_j, F_{12}, F_{21}$ from the computation of branching laws of permutative representations of $O_\infty$. After finding the equations of bosons and the vector $\Omega$, we found the conditions of $F_j, F_{12}, F_{21}$ without using permutative representations of $O_\infty$.

2.3 Proof of Theorem 1.5

Before the proof, we summarize basic relations of the RBS $\{a_n : n \in \mathbb{N}\}$ and the canonical generators $\{s_n : n \in \mathbb{N}\}$ of $O_\infty$. From (1.2), the following holds on the reference subspace of any permutative representation of $O_\infty$:

$$s_m a_n = a_{n+1} s_m, \quad s_m a_n^* = a_{n+1}^* s_m \quad (n, m \in \mathbb{N}),$$

$$\rho(x) s_i = s_i x \quad (x \in O_\infty, i \in \mathbb{N}).$$

(i) Fix $j \geq 1$. First, we see that $(\mathcal{H}, \pi_j)$ is $P_\infty(j)$ with the GP vector $\Omega$. We simply write $\pi_j(s_n)$ by $s_n$ for each $n$. Define

$$\mathcal{D}_j \equiv \text{Lin}\{s_n \Omega : J \in \mathbb{N}^n\}$$

where $\mathbb{N}^n \equiv \prod_{l \geq 1} \mathbb{N}^l$. Then $\mathcal{D}_j$ is the reference subspace. We simply write $\{a_n : n \in \mathbb{N}\}$ the RBS on $P_\infty(j)$ and $\mathcal{B}$ the algebra generated by them. From (1.2),

$$a_n a_n^* = \sum_{K \in \mathbb{N}^{n-1}} \sum_{m=1}^{\infty} m s_K s_m s_n^* s_m^*.$$  

By definition, $s_j^m \Omega = (s_j^*)^m \Omega = \Omega$ for any $m \geq 1$. From these, we obtain that $a_n a_n^* \Omega = j^2 \Omega$ for any $n \in \mathbb{N}$.
It is sufficient to show $B \Omega = D_j$. By definition of the RBS, $B \Omega \subset D_j$. We write $(a^*_n)^{-1} \equiv a_n$ and $a_n^0 = (a^*_n)^0 = I$ for convenience. Then for any $n \in \mathbb{N}$, there exists $M \in \mathbb{R}$ such that $s_n \Omega = M (a^*_i)^{n-j} \Omega$. From this, we can derive that

$$s_K \Omega \in B \Omega \quad (K \in \mathbb{N}^*).$$

Hence $D_j \subset B \Omega$. Therefore the statement holds.

(ii) We see that $(\mathcal{H}, \pi_{12})$ is $P_{\infty}(12)$ with the GP vector $\Omega$. The relations of $a_n$’s and $\Omega, \Omega'$ are shown by assumption. Let $V_1 \equiv B \Omega$ and $V_2 \equiv B \Omega'$. Then we see that $V_1$ and $V_2$ are $F_{12}$ and $F_{21}$, respectively. By Lemma 2.4 (ii), $V_1$ and $V_2$ are orthogonal in $\mathcal{H}$.

For $m \geq 1$ and $J = (j_1, \ldots, j_n) \in \mathbb{N}^n$,

$$s_J \Omega = \begin{cases} C_n a^{*(J-1)} a_2 a_4 \cdots a_{2m} \Omega & (n = 2m), \\ C_n a^{*(J-1)} a_1 a_3 \cdots a_{2m-1} \Omega' & (n = 2m - 1) \end{cases}$$

where $a^{*(J-1)} \equiv (a^*_i)^{j_1 - 1} \cdots (a^*_k)^{j_k - 1}$ and $C_n \equiv \{(j_1 - 1)! \cdots (j_n - 1)\}^{-1/2}$. From this, $s_J \Omega \in V_1 \oplus V_2$ for any $J \in \mathbb{N}^*$. This implies that the reference subspace of $\mathcal{H}$ is a subspace of $V_1 \oplus V_2$. Hence $V_1 \oplus V_2$ is dense in $\mathcal{H}$.

(iii) From (i), (ii) and Lemma 2.4 (i), we see that $\eta_j$ is $F_j$ ($j \geq 1$), $\eta_{12}$ is $F_{12}$ and $\eta_{21}$ is $F_{21}$. From these and Lemma 2.4 (ii), the statement holds.

(iv) From Lemma 2.4 (iii), the statement holds.

\section{Example}

### 3.1 Fock representation of RBS

From Theorem 1.5 (i), we obtain a correspondence between state vectors in the Bose-Fock space and vectors in the permutative representation $P_{\infty}(1)$ as follows:

$$(a_1^*)^{j_1-1} \cdots (a_k^*)^{j_k-1} \Omega = \{(j_1 - 1)! \cdots (j_k - 1)\}^{1/2} s_J \Omega \quad (3.1)$$

for $J = (j_1, \ldots, j_k) \in \mathbb{N}^k$. This shows that any physical theory with the Bose-Fock space is rewritten by $O_{\infty}$. Furthermore the Fock vacuum is interpreted as the eigenvector of the generator $s_1$ of $O_{\infty}$. For example, the one-particle state is given as follows:

$$a_n^* \Omega = s_1^{n-1} s_2 \Omega \quad (n \geq 1).$$
On the other hand, if the Fock representation of $B$ is given, then it is always extended to the action of $O_\infty$ as follows:

$$s_m \Omega = \{(m-1)!\}^{-1/2}(a_1^*)^{m-1} \Omega,$$

$$s_m(a_{n_1}^*)^{k_1} \cdots (a_{n_p}^*)^{k_p} \Omega = \{(m-1)!\}^{-1/2} (a_1^*)^{m-1}(a_{n_1+1}^*)^{k_1} \cdots (a_{n_p+1}^*)^{k_p} \Omega,$$

$$s_m^* \Omega = \delta_{m,1} \Omega,$$

$$s_m^*(a_{n_1}^*)^{k_1} \cdots (a_{n_p}^*)^{k_p} \Omega = \begin{cases} 
  \delta_{m,1}(a_{n_1}^*)^{k_1} \cdots (a_{n_p}^*)^{k_p} \Omega & (n_1 \geq 2), \\
  \delta_{m,k_1+1}\sqrt{k_1!}(a_{n_2}^*)^{k_2} \cdots (a_{n_p}^*)^{k_p} \Omega & (n_1 = 1) 
\end{cases}$$

for $1 \leq n_1 < \cdots < n_p$ and $k_1, \ldots, k_p \in \mathbb{N}$.

**Example 3.1** Define the representation $(l_2(\mathbb{N}), \pi)$ of $O_\infty$ by

$$\pi(s_n)e_m \equiv e_{2n-1(2m-1)} \quad (n, m \in \mathbb{N}). \tag{3.2}$$

Then this is $P_\infty(1)$ with the GP vector $e_1$. For the representation in (3.2), the vacuum is $e_1$ and the subspace $H_1$ of one-particle states is given by

$$H_1 \equiv \text{Lin}(\{e_{2n-1+1} : n \geq 1\}).$$

We show that the above correspondence holds for $O_N$ for any $2 \leq N < \infty$.

**Proposition 3.2** If we identify $O_\infty$ with a $C^*$-subalgebra of $O_N$ by (2.1), then $P_N(1)|_B = \text{Fock}$.\[11]

**Proof.** Because $P_N(1)|_{O_N} = P_\infty(1)$, $P_N(1)|_B = P_\infty(1)|_B = \text{Fock}$.\[11]

Let $(\mathcal{H}, \pi)$ be $P_N(1)$ of $O_N$ with the GP vector $\Omega$. From (2.1), the following holds for $1 \leq n_1 < n_2 < \cdots < n_m$ and $k_1, \ldots, k_m \in \mathbb{N}$:

$$(a_{n_1}^*)^{k_1} \cdots (a_{n_m}^*)^{k_m} \Omega = \prod_{i=1}^{m} \sqrt{k_i!} \ t_{1}^{n_{i-1} - 1} t_{N}^{c_{i-1} - 1} t_{b_i} T_2 \cdots T_m \Omega \tag{3.3}$$

where $T_i \equiv t_1^{n_{i-1} - 1} t_{N}^{c_{i-1} - 1} t_{b_i}$ for $i = 2, \ldots, m$ and we define $c_i \in \mathbb{N}$ and $b_i \in \{1, \ldots, N-1\}$ by the equation $k_i = (N-1)(c_i - 1) + b_i - 1$.\[11]
Example 3.3 (Fock representation by $O_2$ and $O_3$) From (3.3), the following holds: When $N=2$,

$$(a_{n_1}^*)^{k_1} \cdots (a_{n_m}^*)^{k_m} \Omega = \prod_{i=1}^{m} \sqrt{k_i!} \ t_1^{n_1-1} t_2^{k_1} t_1^{n_2-n_1} t_2^{k_2} \cdots t_1^{n_m-n_{m-1}} t_2^{k_m} \Omega.$$  

When $N=3$,

$$(a_{n_1}^*)^{k_1} \cdots (a_{n_m}^*)^{k_m} \Omega = \prod_{i=1}^{m} \sqrt{k_i!} \ t_1^{n_1-1} t_3^{c_1-1} t_b t_2 \cdots T_m \Omega$$

where $T_i \equiv t_1^{n_i-n_{i-1}} t_3^{c_i-1} t_b$ for $i = 2, \ldots, m$ and we define $c_i \in \mathbb{N}$ and $b_i \in \{1, 2\}$ by $k_i = 2(c_i - 1) + b_i - 1$.

3.2 Interpretation of representations by infinite product

In this subsection, we consider representations $F_j$ ($j \geq 2$), $F_{12}$ and $F_{21}$ of bosons in Definition 2.3 from a viewpoint of Fock representation. Formal infinite products of operators are introduced for this purpose.

3.2.1 $F_j$

For the cyclic vector $\Omega$ of $F_j$ in Definition 2.3 with $j \geq 2$, it seems that the formal vector

$$\Omega' \equiv \left( \prod_{n=1}^{\infty} a_{j_n}^{-1} \right) \Omega$$  \hspace{1cm} (3.4)

is a new vacuum of $F_j$ up to normalization constant. The cyclic subspace by $\Omega'$ is equivalent to the Fock representation because $a_n \Omega' = 0$ for each $n$ by formal computation. However such vector can not be defined in the representation space of $F_j$. Furthermore $F_j$ is not equivalent to the Fock representation $F_1$ when $j \neq 1$ by Lemma 2.4 (ii). However, the formal notation (3.4) often appears in theoretical physics and it excites curiosity. If we regard that (3.4) is justified by $F_j$, then (3.4) obtains a meaning of the operation in the representation theory.

3.2.2 $F_{12}$ and $F_{21}$

According to the case $F_j$, we write the Fock vacuum by the cyclic vector $\Omega$ of $F_{12}$. Then we obtain the formal vector $\Omega'$ as follows:

$$\Omega' \equiv \left( \prod_{n=1}^{\infty} a_{2n} \right) \Omega.$$  \hspace{1cm} (3.5)
Of course, $\Omega'$ never be defined in the representation space $F_{12}$.

In the same way, we write the Fock vacuum by the cyclic vector $\Omega$ of $F_{21}$. Then we obtain the formal vector $\Omega'$ as follows:

$$\Omega' \equiv \left( \prod_{n=1}^{\infty} a_{2n-1} \right) \Omega. \quad (3.6)$$

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**Appendix**

**A Proof of Lemma 2.2**

(i) Fix $j \geq 1$. We introduce an orthonormal basis of a given representation belonging to $P_\infty(j)$. Let $(\mathcal{H}, \pi)$ be $P_\infty(j)$ with the GP vector $\Omega$. We simply denote $\pi(s_n)$ by $s_n$ for each $n$. Define the subset $\Lambda_j$ of $N^* \equiv \bigsqcup_{l \geq 1} N^l$ by

$$\Lambda_j \equiv \{(m), J \cup (n), n, m \geq 1, n \neq j, J \in N^*\}$$

and $v_J \equiv s_J \Omega$ for $J \in N^*$. Because $s_j \Omega = \Omega$, we see that $\{s_J s_K^* \Omega : J, K \in N^*\} = \{s_J \Omega : J \in \Lambda_j\}$. Hence $\text{Lin}(\{v_J : J \in \Lambda_j\})$ is dense in $\mathcal{H}$. Furthermore $\langle v_J | \Omega \rangle = 0$ when $J \neq (j)$. This implies that $\langle v_J | v_K \rangle = \delta_{J,K}$ for $J, K \in \Lambda_j$. In consequence $\{v_J : J \in \Lambda_j\}$ is a complete orthonormal basis of $\mathcal{H}$. The construction of $\{v_J : J \in \Lambda_j\}$ is independent of the choice of $\mathcal{H}$ except the existence of GP vector $\Omega$. Hence $P_\infty(j)$ is uniquely up to unitary equivalence.

Assume that $(\mathcal{H}, \pi)$ is $P_\infty(12)$ with the GP vector of $\Omega$. We identify $\pi(s_n)$ with $s_n$ for each $n$. By definition, we see that $\{s_J \Omega : J \in N^*\}$ spans a dense subspace of $\mathcal{H}$. Define the sequence $\{T_n \in N^* : n \in N\}$ of multiindices by $T_{2k} \equiv (12)^k$ and $T_{2k-1} = (12)^{k-1} \cup (1)$ for each $k \geq 1$. If $J \in N^n$, then

$$\langle v_J | \Omega \rangle = \delta_{J,T_n}.$$ 

From this, the orthonormal basis $\{v_J : J \in \Lambda_{12}\}$ of $\mathcal{H}$ is given by

$$v_J \equiv s_J \Omega \quad (J \in \Lambda_{12})$$

where $\Lambda_{12} \equiv \{(n2), (m), J \cup (k), J \cup (12) : n, m, k, l \in N, k \neq 2, l \neq 1, J \in N^*\}$. Hence the orthonormal basis of $\mathcal{H}$ is determined only by the assumptions of $\Omega$. Hence $P_\infty(12)$ is unique up to unitary equivalence.
(ii) Assume that $P_\infty(i) \sim P_\infty(j)$. Then there exists a representation of $O_\infty$ with two cyclic vectors $\Omega$ and $\Omega'$ satisfying $s_i \Omega = \Omega$ and $s_j \Omega' = \Omega'$. Because $i \neq j$, $\langle \Omega | \Omega' \rangle = 0$. Furthermore we can verify that $\langle v_j | \Omega' \rangle = \delta_{j(j)} \langle \Omega | \Omega' \rangle$ for any $J \in \Lambda_i \cap \mathbb{N}^n$ with respect to the notation in the proof of (i) for $i$. Hence $\langle v_j | \Omega' \rangle = 0$ for any $J \in \Lambda_i$. This implies that $\Omega' = 0$. This contradicts with the choice of $\Omega'$. Therefore $P_\infty(i) \not\sim P_\infty(j)$.

Fix $i \geq 1$. Assume that $P_\infty(12) \sim P_\infty(i)$. Then there exists a representation of $O_\infty$ with two cyclic vectors $\Omega$ and $\Omega'$ satisfying $s_{12} \Omega = \Omega$ and $s_i \Omega' = \Omega'$. Then $\langle \Omega | \Omega' \rangle = \langle s_{12} \Omega | s_i^2 \Omega' \rangle = 0$. For any $J \in \mathbb{N}^n$,

$$\langle v_j | \Omega' \rangle = \delta_{j(i)} \langle \Omega | \Omega' \rangle = 0.$$ 

This implies $\Omega' = 0$. This contradicts with the choice of $\Omega'$. Hence there exist no such cyclic vector. Therefore the statement holds.

(iii) We use the notation in the proof of (i). Assume that $B \in \mathcal{B}(\mathcal{H})$ satisfies $[B, x] = 0$ for any $x \in O_\infty$. Then we can verify that $\langle Bv_j | v_K \rangle = \delta_{j,k} \langle B \Omega | \Omega \rangle$ for each $J, K \in \Lambda_j$. This implies that $B = \langle \Omega | B \Omega \rangle \cdot I \in CI$. Hence the statement holds.

Assume that $O_\infty$ acts on $\mathcal{H}$ and $\Omega \in \mathcal{H}$ is a cyclic vector such that $s_{12} \Omega = \Omega$. Assume that $B \in \mathcal{B}(\mathcal{H})$ satisfies $[B, x] = 0$ for any $x \in O_\infty$. Then we can verify that

$$\langle Bv_J | v_K \Omega \rangle = \delta_{JK} \langle B \Omega | \Omega \rangle \quad (J, K \in \Lambda_{12}).$$

From this, $B = \langle \Omega | B \Omega \rangle I \in CI$. Hence the statement holds.

\[ \blacksquare \]

### B Proof of Lemma 2.4

(i) Fix $j \geq 1$. By definition, the following is derived:

$$a_n^j \Omega = 0, \quad a_n^k (a_n^*)^k \Omega = (j + k - 1) \cdots j \Omega \quad (n, k \in \mathbb{N}).$$

In addition, if $j \geq 2$, then the following holds for $1 \leq l \leq j - 1$:

$$(a_n^*)^l a_n^j \Omega = (j - 1) \cdots (j - l) \Omega.$$ 

If $k \geq j$, then $\langle \Omega | (a_n^*)^k \Omega \rangle = \langle a_n^k \Omega | \Omega \rangle = 0$. If $1 \leq k \leq j - 1$, then

$$\langle \Omega | (a_n^*)^k \Omega \rangle = C \langle \Omega | (a_n^*)^k (a_n^*)^{j-k} a_n^j \Omega \rangle = C \langle a_n^j \Omega | a_n^{j-k} \Omega \rangle = 0.$$
where \( C \equiv \{(j-1)\cdots k\}^{-1/2} \). This implies \( \langle \Omega|a_n^k\Omega \rangle = 0 \) when \( 1 \leq k \leq j-1 \). In consequence,

\[
\langle \Omega|a_n^k\Omega \rangle = \langle \Omega|(a_n^*)^k\Omega \rangle = 0 \quad (k, n \geq 1). \quad (B.1)
\]

From these, the family of the following vectors is an orthonormal basis of the vector space \( \mathcal{B}\Omega \):

\[
v = C \cdot (a_{n_1}^*)^{k_1} \cdots (a_{n_p}^*)^{k_p}a_{m_1}^{l_1} \cdots a_{m_q}^{l_q}\Omega \quad (B.2)
\]

for \( 1 \leq n_1 < \cdots < n_p \) and \( k_1, \ldots, k_p \in \mathbb{N}, 1 \leq m_1 < \cdots < m_q, l_1, \ldots, l_q \in \{1, \ldots, j-1\}, \{n_1, \ldots, n_p\} \cap \{m_1, \ldots, m_q\} = \emptyset \) and \( p, q \geq 0 \) where we use notations \( a_{n_0}^n = a_{m_0}^m = 1 \) and

\[
C = \left[ \prod_{j=1}^{p} \{(j+k_t-1)\cdots j\} \cdot \prod_{r=1}^{q} \{(j-1)\cdots (j-l_r)\} \right]^{-1/2}.
\]

In particular, when \( j = 1 \), we always assume \( q = 0 \). The existence of the canonical basis consisting of \( v \)'s in \( (B.2) \) implies the uniqueness of the representation. Therefore the statement holds for \( F_j \).

For the cyclic vector \( \Omega \) of \( F_{12} \), we see that

\[
a_{2m-1}^1(a_{2m-1}^*)^\dagger \Omega = l!\Omega, \quad a_{2m}^l(a_{2m}^*)^\dagger \Omega = (l+1)!\Omega, \quad a_{2m}^2\Omega = 0
\]

for \( l, m \geq 1 \). Define

\[
v = C \cdot (a_{n_1}^*)^{k_1} \cdots (a_{n_p}^*)^{k_p}(a_{m_1}^*)^{l_1} \cdots (a_{m_q}^*)^{l_q}a_{t_1}^{l_1} \cdots a_{t_r}^{l_r}\Omega \quad (B.3)
\]

for \( 1 \leq n_1 < \cdots < n_p, 1 \leq m_1 < \cdots < m_q, 1 \leq t_1 < \cdots < t_r, \{m_1, \ldots, m_q\} \cap \{t_1, \ldots, t_r\} = \emptyset \) and \( k_1, \ldots, k_p, l_1, \ldots, l_q \in \mathbb{N} \) where

\[
C = \{k_1! \cdots k_p! (l_1+1)! \cdots (l_q+1)!\}^{-1/2}.
\]

Then the set of all such \( v \)'s in \( (B.3) \) is an orthonormal basis of \( \mathcal{B}\Omega \). Hence the uniqueness of \( F_{12} \) holds.

For \( F_{21} \), we can construct an orthonormal basis by replacing the suffixes \( 2n \) and \( 2n-1 \) in the proof of \( F_{12} \). Hence the uniqueness of \( F_{21} \) holds.

(ii) Assume that \( i \neq j \) and \( F_i \sim F_j \). Then there exists a representation of \( \mathcal{B} \) with two cyclic vectors \( \Omega \) and \( \Omega' \) satisfying

\[
a_n a_n^\dagger \Omega = i\Omega, \quad a_n a_n^\dagger \Omega' = j\Omega' \quad (n \geq 1). \quad (B.4)
\]
From this, \( \langle \Omega | \Omega' \rangle = 0 \). Let
\[
x = (a_{n_1}^*)^{k_1} \cdots (a_{n_p}^*)^{k_p} a_{m_1}^{l_1} \cdots a_{m_q}^{l_q}
\]  
for \( 1 \leq n_1 < \cdots < n_p, 1 \leq m_1 < \cdots < m_q, k_1, \ldots, k_p \in \mathbb{N}, l_1, \ldots, l_q \in \{1, \ldots, j-1\} \) and \( \{m_1, \ldots, m_q\} \cap \{m_1, \ldots, m_q\} = \emptyset \). Define \( M \equiv n_p + m_q + 1 \). Because \( a_M a_M^* \) commutes \( x \), and \( i \neq j \), \( \langle x \Omega | \Omega' \rangle = 0 \) from (B.4). From this and (B.2), \( \Omega' = 0 \). This contradicts with the choice of \( \Omega' \). Hence \( F_i \not\sim F_j \) when \( i \neq j \).

Assume that \( F_j \sim F_{12} \) for some \( j \geq 1 \). Then there exists a representation of \( B \) with two cyclic vectors \( \Omega \) and \( \Omega' \) satisfying
\[
a_n a_n^* \Omega = j \Omega, \quad a_{2n-1} \Omega' = 0, \quad a_{2n} a_{2n}^* \Omega' = \Omega' \quad (n \geq 1).
\]  
From this,
\[
a_{2n} a_{2n}^* \Omega' = 2 \Omega', \quad a_{2n-1} a_{2n-1}^* \Omega' = \Omega' \quad (n \geq 1).
\]  
Hence \( \langle \Omega | \Omega' \rangle = 0 \). Let \( x \) be as in (B.5) and \( M \equiv n_p + m_q + 1 \). Because both \( a_{2M-1} a_{2M-1}^* \) and \( a_{2M} a_{2M}^* \) commute \( x \), \( \langle x \Omega | \Omega' \rangle = 0 \) from (B.6) and (B.7). This implies \( \Omega' = 0 \). This contradicts with the choice of \( \Omega' \). Therefore \( F_j \not\sim F_{12} \) for any \( j \geq 1 \). In the same way, we see that \( F_j \not\sim F_{21} \) for any \( j \geq 1 \).

Assume \( F_{12} \sim F_{21} \). Then there exists a representation of \( B \) with two cyclic vectors \( \Omega \) and \( \Omega' \) satisfying \( a_n^* a_n \Omega = \Omega \) and \( a_{2n-1} \Omega \) is the \( n \geq 1 \) and \( a_{2n-1} a_{2n-1} \Omega' = \Omega' \) and \( a_{2n} \Omega' = 0 \) for each \( n \geq 1 \). Then \( \langle \Omega | \Omega' \rangle = \langle a_n^* a_n \Omega | a_2 t_1 a_2 \cdots a_{2r} \rangle = 0 \).
\[
x = (a_{n_1}^*)^{k_1} \cdots (a_{n_p}^*)^{k_p} (a_{m_1}^{l_1}) \cdots (a_{m_q}^{l_q}) a_{2n} \cdots a_{2r}
\]  
and assume the assumption in (B.3) and \( p + q + r \geq 1 \). Let \( L \equiv 2n_p + 1 + 2m_q + 2t_r + 1 \). Then
\[
\langle x \Omega | \Omega' \rangle = \langle x a_{2L}^* a_{2L} \Omega | \Omega' \rangle = \langle a_{2L}^* x a_{2L} \Omega | \Omega' \rangle = \langle x a_{2L} \Omega | a_{2L} \Omega' \rangle = 0.
\]  
This holds for any such \( x \). Hence \( \Omega' = 0 \). This contradicts with the choice of \( \Omega' \). Therefore Assume \( F_{12} \not\sim F_{21} \).

(iii) Fix \( j \geq 1 \). Let \( \Omega \) be the cyclic vector \( F_j \) such that \( a_n a_n^* \Omega = j \Omega \) for each \( n \in \mathbb{N} \). Assume that \( B \in \mathcal{B}(\mathcal{H}) \) satisfies \( [a_n, B] = [a_n^*, B] = 0 \) for each \( n \). Let \( x = (a_{n_1}^*)^{k_1} \cdots (a_{n_p}^*)^{k_p} (a_{m_1}^{l_1}) \cdots (a_{m_q}^{l_q}) a_{2n} \cdots a_{2r} \). Therefore the off-diagonal part of \( B \) with respect to
vectors in $E$ is zero. Furthermore, we obtain that $\langle Bv|v \rangle = \langle B\Omega|\Omega \rangle$ for any $v \in E$. This implies that $B = \langle \Omega|B\Omega \rangle I \in CI$. Hence $F_j$ is irreducible.

Let $S$ be the set of all vectors $v$ in (3.3). We see that $\langle \Omega|Bv \rangle = 0$ for $v \in S \setminus \{\Omega\}$. From this, $\langle v|Bw \rangle = 0$ for $v, w \in S$, $v \neq w$. Furthermore, from the form of $v \in S$, we obtain that $\langle v|Bv \rangle = \langle \Omega|B\Omega \rangle$ for any $v \in S$. Therefore $B = \langle \Omega|B\Omega \rangle I \in CI$. Hence $F_{12}$ is irreducible. We can prove the irreducibility of $F_{21}$ by replacing the suffixes $2n$ and $2n - 1$ in the proof of $F_{12}$. Hence $F_{21}$ is also irreducible.

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