Domain Walls are Diamagnetic

M.B. Voloshin
Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455
and
Institute of Theoretical and Experimental Physics, Moscow, 117259

Abstract

It is shown that contrary to a recent claim in the literature\cite{1}, the domain walls made of scalar field are diamagnetic due to presence of massless fermionic modes on the wall. The diamagnetism vanishes at high temperature. Thus the domain walls could produce no effect on a primordial magnetic field in the early universe.
In a recent paper Iwazaki\cite{1} has considered the magnetic properties of the gas of massless fermions bounded to a domain wall in a theory with spontaneously broken discrete symmetry. The result is claimed to be that the free energy of the gas in magnetic field with the strength $B$ contains a negative term proportional to $-B^{3/2}$, thus that the gas should exhibit spontaneous magnetization. If this were true, the domain walls in the early universe could be a source of the primordial magnetic field.

However, the calculations in Ref. 1 are not quite correct, and the purpose of this note is to present a correct calculation of the magnetic properties of the gas of massless fermions on the wall at zero temperature as well as at finite temperature. The result of the present calculation is that in fact the term in the free energy proportional to $B^{3/2}$ is positive and goes to zero at high temperature $T^2 \gg eB$. Thus the domain walls are diamagnetic, and they can not generate a primordial magnetic field in the early universe.

It can be reminded that a domain wall, located for definiteness perpendicular to the $z$ axis, corresponds to transition of a scalar field $\phi(z)$ between two minima $v_1$ and $v_2$ of the scalar field potential: $\phi(z = -\infty) = v_1$, $\phi(z = +\infty) = v_2$. Furthermore, if there is a fermion field $\psi$ with a Yukawa coupling to $\phi$ such that the fermion mass term $m(\phi)$ changes sign between $v_1$ and $v_2$ such fermion has a zero mode with respect to motion in the $z$ direction\cite{2}. The fermion wave function corresponding to this zero mode thus splits into the product $\psi(t, x, y, z) = \chi(t, x, y) u(z)$ with the scalar function $u(z)$ given by $u(z) = \text{const} \cdot \exp(\pm \int^z m dz)$ and the spinor $\chi$ being an eigenvector of the Dirac matrix $i\gamma_3$: $i\gamma_3\chi = \pm\chi$, where the upper (lower) sign corresponds to $m(v_2)$ negative (positive). The spinor condition on $\chi$ leaves only two components independent, thus the motion of the zero-mode fermions along the wall is described by the (2+1) dimensional Dirac equation for a two-component massless spinor. Clearly, the solutions to that equation (on-shell particles) have one degree of freedom for fermions and one for antifermions. In the absence of an external magnetic field the spectrum of the free Dirac operator is obviously labeled by the two components of the two-dimensional momentum $\mathbf{k}$: $E(k) = \pm k$ (with $k = |\mathbf{k}|$). In the presence of a uniform magnetic field $B$ orthogonal to the wall, the spectrum of the Dirac operator with the electric charge $e$ is found following the textbook method of Landau\cite{3} and is parametrized by one integer quantum number $n \geq 0$ with the degeneracy $eB/(2\pi)$ per unit area of the wall and the energy given by $E_n = \pm \sqrt{2eBn}$.

The existence of the non-zero modes of the motion in the $z$ direction, either non-localized or localized\cite{4} at the wall, can be ignored in calculating the magnetic and thermodynamic properties of the gas of the zero-mode fermions as long as the characteristic energy scale in
the problem, i.e. \( \sqrt{eB} \) and/or the temperature \( T \), is much smaller than the mass gap given by the mass of the fermions in a spatially uniform vacuum state \( v_1 \) or \( v_2 \). Throughout this letter this is assumed to be the case and the contribution of non-zero modes to the partition function is completely ignored.

The standard construction of the physical states of the fermion field starts with defining the vacuum state, where the negative energy states are occupied and the positive energy ones are vacant. In the considered here system of the two-dimensional massless fermions this leaves a \( (Z_2)^N \) degeneracy of the vacuum states corresponding to the two possibilities for the occupation number at \( n = 0 \) and the total number of such states is \( N = \frac{eB}{2\pi} \times \text{Area} \). The non-vacuum states, as usual, correspond to fermions: filled states with positive energy, and to antifermions: holes in the states with negative energy. Accordingly, the free energy per unit area of the fermionic gas at a temperature \( T = 1/\beta \) in the magnetic field \( B \) can be written in the following form

\[
F = F_- + F_+ + F_0 + E_{\text{vac}} ,
\]

where

\[
F_\pm = -\beta^{-1} \frac{eB}{2\pi} \sum_{n=1}^{\infty} \ln \left( 1 + e^{-\beta\sqrt{2eBn}} \right)
\]

is the free energy associated with the real gas of fermions and antifermions\(^1\). The term \( F_0 \) in eq.(1) is associated with the degeneracy of the vacuum state:

\[
F_0 = -\beta^{-1} \frac{eB}{2\pi} \ln 2 .
\]

Finally, \( E_{\text{vac}} \) is the energy per unit area of the wall of any of the degenerate vacuum states\(^2\). Clearly, \( E_{\text{vac}} \) gives the free energy at \( T = 0 \).

Ignoring the divergence of the sum, one might write the vacuum energy as the sum over the energies of the occupied negative-energy eigenstates:

\[
E_{\text{vac}} = -\frac{eB}{2\pi} \sum_{n=1}^{\infty} \sqrt{2eBn} = -\frac{eB}{2\pi} \sqrt{2eB} \zeta \left( -\frac{1}{2} \right) \approx 0.04679 \left( eB \right)^{3/2} ,
\]

\(^1\)The fermion number can be freely exchanged with the non-zero modes, hence there is no chemical potential in eq.(1)\(^1\).

\(^2\)It is this vacuum energy term which is missing in the corresponding calculation of Ref. 1. Also the Euler-Maclaurin expansion for the sum in eq.(4) is unjustifiably truncated, which has lead to a wrong result there. In fact this expansion is of little help in calculating the sum because of the root singularity of the summand at \( n = 0 \).
where the finite answer for the divergent sum is written in terms of the standard \( \zeta(s) \) function analytically continued below its pole at \( s = 1 \). It turns out that this formal manipulation gives the correct expression for the dependence of the vacuum energy on the field strength \( B \), while the overall vacuum energy is of course infinite. In view of this divergence a somewhat more elaborate consideration of the vacuum energy is due, and this consideration, presented in the following lines also introduces an appropriate method for calculating the sums involved in this problem\(^3\).

First, in order to make tractable the sum over the infinite number of negative energy states, it has to be regularized. A gauge invariant regulator factor should depend on a gauge invariant quantity, which in this case is naturally the level energy itself. We chose here the regulator factor in the exponential form: \( \exp(-\epsilon E^2_n) \) with \( \epsilon \) being the regulator parameter. Thus we write the regularized vacuum energy in the form

\[
E_{\text{vac}}^{(r)} = -\frac{e B}{2\pi} \sum_{n=1}^{\infty} \sqrt{2} e B n \exp(-\epsilon 2 e B n).
\]  

(5)

The method of summation to be used here is based on the identity valid for any smooth function \( f(x) \):

\[
\sum_{n=1}^{\infty} f(n) = \int_{\delta}^{\infty} f(x) \sum_{n=-\infty}^{\infty} \delta(x - n) \, dx = \sum_{m=-\infty}^{\infty} \int_{\delta}^{\infty} f(x) \exp(2\pi i m x) \, dx ,
\]

where \( \delta \) is any number such that \( 0 < \delta < 1 \). Notice that the sum in the second expression goes over \( n \) from \( -\infty \) to \( +\infty \). The identity is still valid since the terms with \( n \leq 0 \) are identically zero. Applying this identity to the sum in eq.(5) and noting that in this case in fact \( \delta \) can be set equal to zero, since the term in the sum with \( n = 0 \) is vanishing, one can write

\[
E_{\text{vac}}^{(r)} = -\frac{e B}{2\pi} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} \sqrt{2} e B x \exp(-\epsilon 2 e B x) \exp(2\pi i m x) \, dx .
\]

(7)

For each \( m \) the integral in this equation is clearly that for the \( \Gamma \) function: \( \Gamma(3/2) \). The term with \( m = 0 \) diverges in the limit \( \epsilon \to 0 \), while all the rest terms of the sum are finite. Thus writing separately the singular term and grouping together the terms with symmetric values of \( m \) one finds for \( \epsilon \to 0 \)

\[
E_{\text{vac}}^{(r)} = -\frac{\sqrt{\pi}}{8\pi \epsilon^{3/2}} + \frac{e B}{2\pi} \sqrt{2} e B \zeta(3/2) \frac{1}{4\pi} .
\]

(8)

\(^3\)I am thankful to Igor Aleiner for reminding me this summation technique, which in fact was used by Landau in similar problems and which originally goes back to Poisson.
The singular term in this expression does not depend on $B$ and is exactly equal to the energy of the free vacuum regularized in the same way:

$$E^{(r)}_{\text{vac}}(0) = -\int k e^{-\epsilon k^2} \frac{d^2 k}{(2 \pi)^2} = -\frac{\sqrt{\pi}}{8 \pi e^{3/2}}, \quad (9)$$

while the second term in eq.(8), which is finite and describes the dependence of $E_{\text{vac}}$ on the magnetic field is exactly equal to that in eq.(4) due to the identity $\zeta(-1/2) = -\zeta(3/2)/(4\pi)$, which is a consequence of a general relation between $\zeta(s)$ and $\zeta(1-s)$.

The term with $B^{3/2}$ describing the dependence of the vacuum energy on $B$ in eq.(9) or in eq.(4) is positive. This corresponds to diamagnetism of the wall at zero temperature with a singular at $B \to 0$ diamagnetic susceptibility. The temperature dependence is given by the terms $F_\pm$ and $F_0$ in the free energy in eq.(1). Following Ref.1, we consider here the free energy at a large temperature, i.e. in the limit $\hat{\beta} \equiv \beta \sqrt{2 e B} \ll 1$ (but still the temperature is much less than the mass gap in order to ensure the irrelevance of the non-zero modes). According to eqs.(1 - 3), the temperature dependent part of the free energy is given by

$$F_+ + F_- + F_0 = -\beta^{-1} \frac{e B}{\pi} \left[ \sum_{n=1}^{\infty} \ln \left( 1 + e^{-\hat{\beta} \sqrt{n}} \right) + \frac{1}{2} \ln 2 \right]. \quad (10)$$

In order to find the expansion of the sum in this expression in powers of $\hat{\beta}$ at small $\hat{\beta}$ we apply to the sum the identity (8). Writing separately the Fourier harmonic with $m = 0$ and grouping together the harmonics with symmetric $m$, one finds

$$\sum_{n=1}^{\infty} \ln \left( 1 + e^{-\hat{\beta} \sqrt{n}} \right) = \int_\delta^\infty \ln \left( 1 + e^{-\hat{\beta} \sqrt{x}} \right) dx + \sum_{m=1}^{\infty} \int_\delta^\infty \ln \left( 1 + e^{-\hat{\beta} \sqrt{x}} \right) \left( e^{2\pi i m x} + e^{-2\pi i m x} \right) dx. \quad (11)$$

The first integral in the limit $\delta \to +0$ gives the singular in $\hat{\beta}$ term $\frac{3}{2} \frac{\zeta(3)}{\hat{\beta}^2}$, which upon substitution in eq.(11) reproduces the field independent free energy of a free fermion gas. The rest of the terms of the expansion in powers of $\hat{\beta}$ are obtained by the Taylor expansion of the integrand in the second term in eq.(11) with subsequent separate integration and summation over $m$ in each term of the expansion in $\hat{\beta}$. The parameter $\delta$ can be set to zero before integration in all the terms of the Taylor expansion except the first one, where keeping $\delta$ small but finite ensures the convergence of the sum over $m$. The calculation of this first, constant, term is thus reduced to the formula

$$\lim_{\delta \to +0} \sum_{m=1}^{\infty} \int_\delta^\infty \left( e^{2\pi i m x} + e^{-2\pi i m x} \right) dx = -\frac{1}{2}, \quad (12)$$
and the higher terms of the Taylor expansion in \( \hat{\beta} \) are found from the generic formula

\[
\sum_{m=1}^{\infty} \int_0^\infty x^{p/2} \left( e^{2\pi i m x} + e^{-2\pi i m x} \right) dx = -\frac{\sin(\pi p/4)}{\pi} (2\pi)^{-p/2} \Gamma \left( \frac{p}{2} + 1 \right) \zeta \left( \frac{p}{2} + 1 \right)
\]  

(13)

for \( p > 0 \). (Notice that eq.(12) can also be found by taking in this formula the limit \( p \to 0 \).) In deriving these formulas the oscillating behavior of the integrand is dealt with by temporarily introducing the damping factor \( \exp(-\epsilon x) \) and taking \( \epsilon \to +0 \) in the end: this is a legitimate procedure, since it does not alter the original integral in eq.(11).

It is satisfying to see, using eq.(12), that upon substitution in eq.(10) the constant, \( \hat{\beta} \) independent, term in the sum of eq.(10) exactly cancels the contribution \( F_0 \) to the free energy of the vacuum degeneracy factor. Thus, as expected on general grounds, no term linear in \( B \) arises in the free energy at any temperature. Therefore, using the formula in eq.(13) for \( p = 1 \) one finds in the high temperature limit the expansion for the thermal part of the free energy up to the first \( B \) dependent term as

\[
F_+ + F_- + F_0 = -\frac{3}{4} \zeta(3) \beta^2 - \frac{eB}{2\pi} \sqrt{2eB} \frac{\zeta(3/2)}{4\pi} + O \left( \beta \left( eB \right)^2 \right).
\]  

(14)

One can readily see that the term with \( B^{3/2} \) in this expression exactly cancels in the sum in eq.(11) the corresponding contribution of the vacuum energy, given by eq.(8). Thus one concludes that the singular at \( B \to 0 \) diamagnetic behavior of the fermion gas at the wall vanishes at high temperature, as could be expected on general grounds.

It is a simple exercise to calculate numerically the sum in eq.(10) for arbitrary \( \hat{\beta} \). However in view of the absence of any dramatic phenomena, like the previously claimed\[1\] ferromagnetism of the fermion gas, this calculation is of a little practical interest.

As discussed above, the calculation here is restricted to the situation where the temperature is much smaller than the mass gap for the fermions. If this restriction is relaxed, one has to solve the problem for the full spectrum\[4\] of the fermion scattering states in the presence of the domain wall in (3+1) dimensions. In this case however, the effect of the wall on the magnetic properties of the field system is small, and also no dramatic effects can be expected.

Thus we conclude that the domain walls, had they existed in the early universe in spite of their undesirable cosmological effects pointed out in the literature\[6\], would not have produced any significant impact on a large-scale primordial magnetic field.

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