QUIVER VARIETIES AND PATH REALIZATIONS
ARISING FROM ADJOINT CRYSTALS OF TYPE $A_n^{(1)}$

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Abstract. Let $B(\Lambda_0)$ be the level 1 highest weight crystal of the quantum affine algebra $U_q(A_n^{(1)})$. We construct an explicit crystal isomorphism between the geometric realization $B(\Lambda_0)$ of $B(\Lambda_0)$ via quiver varieties and the path realization $P^{ad}(\Lambda_0)$ of $B(\Lambda_0)$ arising from the adjoint crystal $B^{ad}$.

INTRODUCTION

The theory of perfect crystals developed in [7] has a lot of important and interesting applications to the representation theory of quantum affine algebras and the theory of vertex models in mathematical physics. In particular, the crystal $B(\lambda)$ of an integrable highest weight module over a quantum affine algebra can be realized as the crystal $P^B(\lambda)$ consisting of $\lambda$-paths arising from a given perfect crystal $B$. In [1], Benkart, Frenkel, Kang and Lee gave a uniform construction of level 1 perfect crystals for all quantum affine algebras. These perfect crystals are called the adjoint crystals because, when forgetting 0-arrows, they coincide with the direct sums of the trivial crystals and the crystals of adjoint or little adjoint representations of finite dimensional simple Lie algebras.

On the other hand, for a symmetric Kac-Moody algebra $\mathfrak{g}$, Lusztig gave a geometric construction of $U_q^{-1}(\mathfrak{g})$ in terms of perverse sheaves on quiver varieties and introduced the notion of canonical basis which yields natural bases for all integrable highest weight modules as well [13, 14]. In [12], Kashiwara and Saito defined a crystal structure on the set $B(\infty)$ of irreducible components of Lusztig’s quiver varieties and showed that $B(\infty)$ is isomorphic to the crystal $B(\infty)$ of $U_q^- (\mathfrak{g})$. Moreover, in [16, 17], Nakajima defined a new family of quiver varieties associated with a dominant integral weight $\lambda$ and gave a geometric realization of the integrable highest weight $\mathfrak{g}$-module $V(\lambda)$. In [18], Saito defined a crystal structure on the set $B(\lambda)$ of irreducible components of certain Lagrangian subvarieties of Nakajima’s quiver varieties, and showed that $B(\lambda)$ is isomorphic to the crystal $B(\lambda)$ of $V(\lambda)$.

Therefore, for quantum affine algebras, it is natural to investigate the crystal isomorphism between the geometric realization $B(\lambda)$ and the path realization $P^B(\lambda)$ for various perfect crystals $B$. In this paper, we will focus on the level 1 highest weight crystal $B(\Lambda_0)$ of the quantum affine algebra $U_q(A_n^{(1)})$, and construct an explicit crystal isomorphism between $B(\Lambda_0)$ and $P^{ad}(\Lambda_0)$, where $P^{ad}(\Lambda_0)$ is

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the path realization arising from the adjoint crystal $B^{ad}$. We will also give a geometric interpretation of the fundamental isomorphism theorem for perfect crystals: $B(\Lambda_0) \cong B(\Lambda_0) \oplus B^{ad}$. One of the key ingredients of our construction is the explicit 1-1 correspondence between $\mathcal{B}(\Lambda_0)$ and $\mathcal{Y}(\Lambda_0)$ discovered in [3, 19], where $\mathcal{Y}(\Lambda_0)$ is the crystal consisting of $(n+1)$-reduced Young diagrams. We hope our construction will provide a new insight toward the understanding of the connection between $\mathcal{B}(\Lambda_0)$ and $\mathcal{P}^{ad}(\Lambda_0)$ for all quantum affine algebras.

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1. THE QUANTUM AFFINE ALGEBRA $U_q(A_n^{(1)})$

Let $I = \mathbb{Z}/(n+1)\mathbb{Z}$ be the index set. The affine Cartan datum of $A_n^{(1)}$-type consists of

(i) the affine Cartan matrix

$$A = (a_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & \cdots & -1 & 2 \end{pmatrix},$$

(ii) dual weight lattice $P^\vee = \bigoplus_{i=0}^n \mathbb{Z}h_i \oplus \mathbb{Z}d$,

(iii) affine weight lattice $P = \bigoplus_{i=0}^n \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta \subset \mathfrak{h}^*$, where

$$\mathfrak{h} = \mathbb{C} \otimes P^\vee, \quad \Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0, \quad \delta(h_i) = 0, \quad \delta(d) = 1 \quad (i, j \in I),$$

(iv) the set of simple coroots $\Pi^\vee = \{h_i | i \in I\}$,

(v) the set of simple roots $\Pi = \{\alpha_i | i \in I\}$ given by

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d) = \delta_{0,j} \quad (i, j \in I).$$

The free abelian group $Q = \bigoplus_{i=0}^n \mathbb{Z}\alpha_i$ is called the root lattice and the semigroup $Q^+ = \sum_{i=0}^n \mathbb{Z}_{\geq 0}\alpha_i$ is called the positive root lattice. For $\alpha = \sum_{i\in I} k_i \alpha_i \in Q^+$, the number $\text{ht}(\alpha) = \sum_{i\in I} k_i$ is called the height of $\alpha$. For $\lambda, \mu \in \mathfrak{h}^*$, we define $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q^+$. The elements in $P^+ = \{\lambda \in P | \lambda(h_i) \geq 0, i \in I\}$ are called the dominant integral weights. Note that the minimal imaginary root is given by $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_n \in Q^+$. The element $c = h_0 + h_1 + \cdots + h_n \in P^\vee$ is called the canonical central element.

Given $n \in \mathbb{Z}$ and any symbol $q$, we define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$
and set \([0]_q! = 1, \,[n]_q! = [n]_q[n-1]_q \cdots [1]_q\). For \(m \geq n \geq 0\), let
\[
\begin{bmatrix}
m \\
n
\end{bmatrix}_q = \frac{[m]_q!}{[n]_q! [m-n]_q!}.
\]

**Definition 1.1.** The *quantum affine algebra* \(U_q(\mathfrak{g}) = U_q(A_n^{(1)})\) is an associative algebra over \(\mathbb{C}(q)\) with 1 generated by \(e_i, f_i (i \in I)\) and \(q^h (h \in P^\vee)\) satisfying the defining relations:

(a) \(q^0 = 1, \ q^{h+h'} = q^{h+h'}\) for \(h, h' \in P^\vee\),

(b) \(q^h e_i q^{-h} = q^{\alpha_i(h)} e_i\) for \(h \in P^\vee\),

(c) \(q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i\) for \(h \in P^\vee\),

(d) \(e_i f_j - f_j e_i = \delta_{ij} (q^h_i - q^{-h}_i)/(q - q^{-1})\) for \(i, j \in I\),

(e) \(\sum_{k=0}^{\infty} \left[\frac{1}{q} \right]_q e_i^{1-\alpha_i(h)} - k e_j e_i^k = 0\) for \(i \neq j\),

(f) \(\sum_{k=0}^{\infty} \left[\frac{1}{q} \right]_q f_i^{1-\alpha_i(h)} - k f_j f_i^k = 0\) for \(i \neq j\).

The definition of *category \(\mathcal{O}_q\)\* Kashiwara operators and crystal bases can be found in [10 14], etc. It was shown in [10] that every \(U_q(\mathfrak{g})\)-module in the category \(\mathcal{O}_q\) has a crystal basis. The notion of *abstract crystals* was introduced in [11]. For convenience, we recall some of the basic definitions and properties of abstract crystals.

**Definition 1.2.** An *abstract crystal* associated with the Cartan datum \((A, \Pi, \Pi^\vee, P, P^\vee)\) is a set \(B\) together with the maps \(wt : B \to P, \tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}\), and \(\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\} (i \in I)\) satisfying the following properties:

(a) \(\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b) \rangle\) for all \(i \in I\),

(b) \(wt(\tilde{e}_i b) = wt(b) + \alpha_i\) if \(\tilde{e}_i b \in B\),

(c) \(wt(\tilde{f}_i b) = wt(b) - \alpha_i\) if \(\tilde{f}_i b \in B\),

(d) \(\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \ \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1\) if \(\tilde{e}_i b \in B\),

(e) \(\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \ \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1\) if \(\tilde{f}_i b \in B\),

(f) \(\tilde{f}_i b = b'\) if and only if \(b = \tilde{e}_i b'\) for \(b, b' \in B, i \in I\),

(g) if \(\varphi_i(b) = -\infty\) for \(b \in B\), then \(\tilde{e}_i b = \tilde{f}_i b = 0\).

We often say that \(B\) is a \(U_q(\mathfrak{g})\)\*-crystal. We denote \(B_\lambda = \{b \in B | wt(b) = \lambda\}\) so that \(B = \bigsqcup_{\lambda \in P} B_\lambda\).

**Example 1.3.**

(1) Let \((L, B)\) be a crystal basis of \(M \in \mathcal{O}_q\). Then \(B\) has a crystal structure, where the maps \(\varepsilon_i, \varphi_i\) are given by
\[
\varepsilon_i(b) = \max\{k \geq 0 | \tilde{e}_i b \neq 0\}, \quad \varphi_i(b) = \max\{k \geq 0 | \tilde{f}_i b \neq 0\}.
\]
In particular, we denote by \(B(\lambda)\) the crystal of the irreducible highest weight module \(V(\lambda)\) with highest weight \(\lambda \in P^+\).

(2) Let \((L(\infty), B(\infty))\) be a crystal basis of \(U_q^{-}(\mathfrak{g})\). Then \(B(\infty)\) has a crystal structure, where the maps \(\varepsilon_i, \varphi_i\) are given by
\[
\varepsilon_i(b) = \max\{k \geq 0 | \tilde{e}_i b \neq 0\}, \quad \varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b) \rangle.
\]
(3) For \( \lambda \in P \), let us consider \( T_\lambda = \{ t_\lambda \} \) with the maps:

\[
\begin{align*}
  \text{wt}(t_\lambda) &= \lambda, \\
  e_i t_\lambda &= \tilde{f}_i t_\lambda = 0 & \text{for } i \in I, \\
  \varepsilon_i(t_\lambda) &= \varphi_i(t_\lambda) = -\infty & \text{for } i \in I.
\end{align*}
\]

Then \( T_\lambda \) is a crystal.

(4) Let \( C = \{ c \} \). We define the maps

\[
\begin{align*}
  \text{wt}(c) &= 0, \\
  \varepsilon_i c &= \tilde{f}_i c = 0, \\
  \varepsilon_i(c) &= \varphi_i(c) = 0 & (i \in I).
\end{align*}
\]

Then \( C \) is a crystal.

**Definition 1.4.** Let \( B_1 \) and \( B_2 \) be crystals.

1. A map \( \psi : B_1 \rightarrow B_2 \) is a **crystal morphism** if it satisfies the following properties:
   1. (a) for \( b \in B_1 \), we have
      \[
      \text{wt}(\psi(b)) = \text{wt}(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b) & \text{for all } i \in I, \\
      (b) for \( b \in B_1 \) and \( i \in I \) with \( \tilde{f}_i b \in B_1 \), we have \( \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) \).
      \]
   2. A crystal morphism \( \psi : B_1 \rightarrow B_2 \) is called **strict** if
      \[
      \psi(\tilde{e}_i b) = \tilde{e}_i \psi(b), \quad \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) & \text{for all } i \in I \text{ and } b \in B_1.
      \]
      Here, we understand \( \psi(0) = 0 \).
   3. \( \psi \) is called an **embedding** if the underlying map \( \psi : B_1 \rightarrow B_2 \) is injective.

Let \( B_1 \) and \( B_2 \) be crystals. The **tensor product** \( B_1 \otimes B_2 \) is defined to be the set \( B_1 \times B_2 \) together with the following maps:

\[
\begin{align*}
  \text{(a) } \text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2), \\
  \text{(b) } \varepsilon_i(b_1 \otimes b_2) &= \max\{ \varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle \}, \\
  \text{(c) } \varphi_i(b_1 \otimes b_2) &= \max\{ \varphi_i(b_1), \varphi_i(b_2) + \langle h_i, \text{wt}(b_2) \rangle \}, \\
  \text{(d) } \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \\
  \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\
  b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \\
  \end{cases} \\
  \text{(e) } \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \\
  \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
  b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2).
  \end{cases}
\end{align*}
\]

It was shown in [11] that there is a unique strict crystal embedding

\[ B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda \otimes C \]

sending \( u_\lambda \) to \( 1 \otimes t_\lambda \otimes c \). Here, \( u_\lambda \) is the highest weight element of \( B(\lambda) \). We denote by \( \iota_\lambda \) the composition of the strict embedding and the natural projection:

\[
\iota_\lambda : B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda \otimes C \rightarrow B(\infty).
\]

Note that \( \iota_\lambda \) is injective, but not a crystal morphism.
2. Path realization

Let $U'_q(g)$ be the subalgebra of $U_q(g)$ generated by $e_i, f_i, q^{\pm h_i}$ ($i \in I$) and we set $\overline{P}' = \bigoplus_{i=0}^{n} \mathbb{Z} h_i$, $\overline{h} = \mathbb{C} \otimes_{\mathbb{Z}} \overline{P}'$, $\overline{P} = \bigoplus_{i=0}^{n} \mathbb{Z} \Lambda_i$ and $\overline{P}^+ = \sum_{i=0}^{n} \mathbb{Z}_{\geq 0} \Lambda_i$. Denote by $\text{cl} : P \to \overline{P}$ the natural projection from $P$ to $\overline{P}$. An abstract crystal $B$ associated with $U'_q(g)$ is called a classical crystal. For $b \in B$, we define $\varepsilon(b) = \sum_{i=0}^{n} \varepsilon_i(b) \Lambda_i$, $\varphi(b) = \sum_{i=0}^{n} \varphi_i(b) \Lambda_i$.

**Definition 2.1.** A perfect crystal of level $\ell$ is a finite classical crystal $B$ satisfying the following conditions:

(a) there exists a finite dimensional $U'_q(g)$-module with a crystal basis whose crystal graph is isomorphic to $B$,
(b) $B \otimes B$ is connected,
(c) there exists a classical weight $\lambda_0 \in \overline{P}$ such that $\text{wt}(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\geq 0} \alpha_i$, $\#(B_{\lambda_0}) = 1$,
(d) for any $b \in B$, we have $\langle c, \varepsilon(b) \rangle \geq \ell$,
(e) for each $\lambda \in \overline{P}_\ell^+ := \{ \mu \in \overline{P}^+ | \langle c, \mu \rangle = \ell \}$, there exist unique vectors $b^\lambda$ and $b_\lambda$ in $B$ such that $\varepsilon(b^\lambda) = \lambda$, $\varphi(b_\lambda) = \lambda$.

Given a dominant integral weight $\lambda$ with $\lambda(c) = \ell$ and a perfect crystal $B$ of level $\ell$, it was shown in [7] that there exists a unique crystal isomorphism, called the *fundamental isomorphism theorem for perfect crystals*,

$$\psi : B(\lambda) \xrightarrow{\sim} B(\varepsilon(b_\lambda)) \otimes B$$

(2.1)

sending $u_\lambda$ to $u_{\varepsilon(b_\lambda)} \otimes b_\lambda$. By applying this crystal isomorphism repeatedly, we get a sequence of crystal isomorphisms

$$B(\lambda) \xrightarrow{\sim} B(\lambda_1) \otimes B \xrightarrow{\sim} B(\lambda_2) \otimes B \otimes B \xrightarrow{\sim} \cdots,$$

where $\lambda_0 = \lambda$, $b_0 = b_\lambda$, $\lambda_{k+1} = \varepsilon(b_k)$, $b_{k+1} = b_{\lambda_{k+1}}$ ($k \geq 0$). The sequence $p_\lambda := (b_k)_{k=0}^\infty$ is called the ground-state path of weight $\lambda$ and a sequence $p = (p_k)_{k=0}^\infty$ of elements $p_k \in B$ is called a $\lambda$-path in $B$ if $p_k = b_k$ for all $k \gg 0$. We denote by $\mathcal{P}^B(\lambda)$ the set of $\lambda$-paths in $B$, which gives rise to the path realization of $B(\lambda)$.

**Theorem 2.2 ([7]).** There exists a unique crystal isomorphism $B(\lambda) \xrightarrow{\sim} \mathcal{P}^B(\lambda)$ which maps $u_\lambda$ to $p_\lambda$.

We list some examples of perfect crystals of level 1 and the corresponding ground-state paths (see [1], [8], etc).

**Example 2.3.**
(1) The crystal $B^1$ and its ground-state $p^1_{\Lambda_i}$ of weight $\Lambda_i \ (i \in I)$ are given as

\[
\begin{array}{cccccccc}
\bullet & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow \\
& 1 & & 2 & & \cdots & & n \rightarrow \\
& 0 & & & & & & b_{n+1}
\end{array}
\]

$p^1_{\Lambda_i} = (\cdots, b_1, b_2, \cdots, b_n, b_{n+1}, b_1, b_2, \cdots, b_{i-1}, b_i)$.

We denote by $P^1(\Lambda_i)$ the set of all $\Lambda_i$-paths in $B^1$.

(2) The crystal $B^n$ and its ground-state $p^n_{\Lambda_i}$ of weight $\Lambda_i \ (i \in I)$ are given as

\[
\begin{array}{cccccccc}
\bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow \\
& 1 & & \cdots & & 2 & & n \rightarrow \\
& 0 & & & & & & b_{n+1}
\end{array}
\]

$p^n_{\Lambda_i} = (\cdots, b_{n+1}, b_n, \cdots, b_2, b_1, b_{n+1}, b_n, \cdots, b_{i+2}, b_{i+1})$.

We denote by $P^n(\Lambda_i)$ the set of all $\Lambda_i$-paths in $B^n$.

(3) The adjoint crystal $B^{ad}$ is given as follows.

Let $B^{ad} = \{\emptyset\} \cup \{b_{\pm \alpha_{ij}} \mid 1 \leq i \leq j \leq n\} \cup \{h_i \mid i = 1, \ldots, n\}$, where $\alpha_{ij} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $1 \leq i \leq j \leq n$. We define the $i$-arrows ($i \in I$) by

\[
\begin{align*}
(i \neq 0) & \quad b_{\alpha}^i \rightarrow b_{\beta} \iff \alpha - \alpha_i = \beta, \\
& \quad b_{\alpha_i}^i \rightarrow h_i \iff b_{-\alpha_i}, \\
(i = 0) & \quad b_{\alpha}^0 \rightarrow b_{\beta} \iff \alpha + \theta = \beta \ (\alpha, \beta \neq \pm \theta), \\
& \quad b_{-\theta}^0 \rightarrow \emptyset \rightarrow b_{\theta},
\end{align*}
\]

where $\theta = \alpha_1 + \cdots + \alpha_n$. The crystal $B^{ad}$ is a perfect crystal of level 1 with the ground-state path of weight $\Lambda_0$

$p^{ad}_{\Lambda_0} = (\cdots, \emptyset, \emptyset, \cdots, \emptyset)$.

There is a crystal isomorphism $p^{ad} : B^n \otimes B^1 \rightarrow B^{ad}$ given by

\[
p^{ad}(\overline{b_j} \otimes b_i) = \begin{cases} 
    b_{\text{wt}(\overline{b_j} \otimes b_i)} & \text{if } \text{wt}(\overline{b_j} \otimes b_i) \neq 0, \\
    h_i & \text{if } \text{wt}(\overline{b_j} \otimes b_i) = 0, i \neq n + 1, \\
    \emptyset & \text{otherwise}.
\end{cases}
\]

We denote by $P^{ad}(\Lambda_0)$ the set of all $\Lambda_0$-paths in $B^{ad}$. 
3. Combinatorics of Young Walls

In [4, 6], Kang gave a combinatorial realization of crystal graphs for basic representations of quantum affine algebras of type $A_n^{(1)}$ ($n \geq 1$), $A_{2n-1}^{(2)}$ ($n \geq 3$), $D_n^{(1)}$ ($n \geq 4$), $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$ ($n \geq 2$), and $B_n^{(1)}$ ($n \geq 3$) by using new combinatorial objects called Young walls, which are a generalization of colored Young diagrams used in [2, 5, 15]. In this work, we focus on the quantum affine algebra $U_q(A_n^{(1)})$.

The Young wall $Y^1$ (resp. $Y^n$) is a wall consisting of colored blocks stacked by the following rules:

(1) the colored blocks should be stacked in the pattern $P^1$ (resp. $P^n$) of weight $\Lambda_k$ given below,

(2) except for the right-most column, there should be no free space to the right of any block.

The patterns are given as follows.

\[
\begin{array}{cccccccc}
0 & 1 & & & & & & \\
 & n & 0 & & & & & \\
 & & n-2 & n & & & & \\
 & & & & & & & \\
0 & 1 & 2 & \ldots & k & k+1 & & \\
 & n & 0 & 1 & 2 & \ldots & k-1 & k \\
\end{array}
\]

the pattern $P^1$ of weight $\Lambda_k$

\[
\begin{array}{cccccccc}
0 & n & & & & & & \\
 & 1 & 0 & & & & & \\
 & & 2 & 1 & & & & \\
 & & & & & & & \\
1 & 0 & n & \ldots & k & k-1 & & \\
2 & 1 & 0 & n & \ldots & k+1 & k \\
\end{array}
\]

the pattern $P^n$ of weight $\Lambda_k$

Note that the heights of the columns of a Young wall $Y$ are weakly decreasing from right to left, so we denote it by $Y = (y_i)_{i \geq 0}$, where $y_i$ is the $i$-th column of $Y$.

**Definition 3.1.** Let $Y$ be a Young wall corresponding to the pattern $P^1$ (resp. $P^n$).

1. An $i$-block in $Y$ is called a removable $i$-block if $Y$ remains a Young wall after removing the block.
2. A place in $Y$ is called an admissible $i$-slot if one may add an $i$-block to obtain another Young wall.
3. A column in $Y$ is said to be $i$-removable (resp. $i$-admissible) if the column has a removable $i$-block (resp. an admissible $i$-slot).

Now we define the action of Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ ($i \in I$) on Young walls. Let $Y = (y_k)_{k \geq 0}$ be a Young wall corresponding to the pattern $P^1$ (resp. $P^n$).

1. To each column $y_k$ of $Y$, we assign

\[
\begin{cases}
- & \text{if } y_k \text{ is } i\text{-removable}, \\
+ & \text{if } y_k \text{ is } i\text{-admissible},
\end{cases}
\]
(b) From this sequence of +’s and −’s, cancel out all (+, −) pairs to obtain a finite sequence of −’s followed by +’s. This sequence (−, · · · , −, +, · · · , +) is called the i-signature of Y.

(c) We define ˜e_i Y to be the Young wall obtained from Y by removing the i-block corresponding to the rightmost − in the i-signature of Y. If there is no − in the i-signature, we set ˜e_i Y = 0.

(d) We define ˜f_i Y to be the Young wall obtained from Y by adding an i-block to the column corresponding to the leftmost + in the i-signature of Y. If there is no + in the i-signature, we set ˜f_i Y = 0.

We also define

\[ \text{wt}(y_j) = \sum_{i \in I} k_{ij} \alpha_i \quad (j \in \mathbb{Z}_{\geq 0}), \]
\[ \text{wt}(Y) = \Lambda_k - \sum_{j \geq 0} \text{wt}(y_j), \]
\[ \varepsilon_i(Y) = \text{the number of −’s in the i-signature of } Y, \]
\[ \varphi_i(Y) = \text{the number of +’s in the i-signature of } Y, \]

where \( k_{ij} \) is the number of i-blocks in the j-th column \( y_j \) of Y. Note that the height of \( y_j \) is \( \text{ht}(\text{wt}(y_j)) \).

Let \( \mathcal{Y}^1(\Lambda_k) \) (resp. \( \mathcal{Y}^n(\Lambda_k) \)) be the set of all Young walls \( Y^1 \) (resp. \( Y^n \)) whose shapes are \((n + 1)\)-reduced Young diagrams; i.e., \( Y^1 = (y_j)_{j \geq 0} \in \mathcal{Y}^1(\Lambda_k) \) (resp. \( Y^n = (\mathcal{Y}_j)_{j \geq 0} \in \mathcal{Y}^n(\Lambda_k) \)) if and only if

\[ \text{ht}(\text{wt}(y_j)) - \text{ht}(\text{wt}(y_{j+1})) < n + 1 \quad (\text{resp. } \text{ht}(\text{wt}(\mathcal{Y}_j)) - \text{ht}(\text{wt}(\mathcal{Y}_{j+1})) < n + 1) \]

for \( j \geq 0 \). Then \( \mathcal{Y}^1(\Lambda_k) \) (resp. \( \mathcal{Y}^n(\Lambda_k) \)) has a \( U_q(g) \)-crystal structure, and we have the following theorem.

**Theorem 3.2.** \([4, 6, 15]\) There is a unique crystal isomorphism \( B(\Lambda_k) \to \mathcal{Y}^1(\Lambda_k) \) (resp. \( \mathcal{Y}^n(\Lambda_k) \)) which maps the highest weight element \( u_{\Lambda_k} \) to the empty Young wall \( \emptyset \).

Let \( p^1 = (\ldots, b_{i_2}, b_{i_1}, b_{i_0}) \) be a \( \Lambda_k \)-path in \( B^1 \). Consider a Young wall \( Y^1_k(p^1) = (y_j(p^1))_{j \geq 0} \) such that the j-th column \( y_j(p^1) \) is \( \emptyset \) if \( j > \text{ht}(\Lambda_k - \text{wt}(p^1)) \), otherwise \( y_j(p^1) \) is the smallest j-th column in \( p^1 \) satisfying the following conditions:

(a) the top color of \( y_j(p^1) \) is \( i_j - 1 \),

(b) \( y_{j+1}(p^1) \leq y_j(p^1) \).

One can prove that the Young wall \( Y^1_k(p^1) \) is contained in \( \mathcal{Y}^1(\Lambda_k) \), and the map

\[ Y^1_k : \mathcal{P}^1(\Lambda_k) \to \mathcal{Y}^1(\Lambda_k) \]

is a crystal isomorphism. If we set \( Y^1 = (y_j)_{j \geq 0} \in \mathcal{Y}^1(\Lambda_k) \), then the inverse image \( p^1 \) of \( Y^1 \) under the crystal isomorphism \( Y^1_k \) is

\[ p^1 = (\ldots, b_{a_j}, \ldots, b_{a_1}, b_{a_0}), \]

where \( a_j \equiv \text{ht}(\text{wt}(y_j)) - j + k \) (mod \( n + 1 \)) for all \( j \geq 0 \).
In a similar manner, given a $\Lambda_k$-path $p^n = (\cdots, \overline{\alpha}_3, \overline{\alpha}_1, \overline{\alpha}_0)$ in $B^n$, we have a Young wall $Y_k^n(p^n) = (\overline{j}(p^n))_{j \geq 0}$ such that the $j$-th column $\overline{j}(p^n)$ is $\emptyset$ if $j > \text{ht}(\Lambda_k - \text{wt}(p^n))$, otherwise $\overline{j}(p^n)$ is the smallest $j$-th column in $P^n$ satisfying the following conditions:

(a) the top color of $\overline{j}(p^n)$ is $i_j$,
(b) $\overline{j + 1}(p^n) \leq \overline{j}(p^n)$.

One can prove that the Young wall $Y_k^n(p^n)$ is contained in $\mathcal{Y}^n(\Lambda_k)$, and the map

\[ Y_k^n : \mathcal{P}^n(\Lambda_k) \rightarrow \mathcal{Y}^n(\Lambda_k) \]

is a crystal isomorphism. If we set $Y^n = (\overline{j})_{j \geq 0} \in \mathcal{Y}^n(\Lambda_k)$, then the inverse image $p^n$ of $Y^n$ under the crystal isomorphism $Y_k^n$ is

\[ p^n = (\ldots, \overline{\alpha}_3, \overline{\alpha}_2, \overline{\alpha}_1, \overline{\alpha}_0, \overline{\alpha}_1, \overline{\alpha}_0, \overline{\alpha}_2) \]

where $b_j \equiv 1 - \text{ht}(\text{wt}(\overline{j})) + j + k (\text{mod } n + 1)$ for all $j \geq 0$.

**Example 3.3.** Let $\mathfrak{g}$ be the Kac-Moody algebra of type $A_3^{(1)}$. Fix an element

\[ b = \tilde{f}_0 \tilde{f}_2 \tilde{f}_1 (\tilde{f}_1 \tilde{f}_2 \tilde{f}_3 \tilde{f}_0)^3 u_{\Lambda_0} \in B(\Lambda_0), \]

where $u_{\Lambda_0}$ is the highest weight element of $B(\Lambda_0)$. Then the $\Lambda_0$-paths $p^1 \in \mathcal{P}^1(\Lambda_0), p^n \in \mathcal{P}^n(\Lambda_0)$ and $p^{ad} \in \mathcal{P}^{ad}(\Lambda_0)$ corresponding to $b$ are given as

\[ p^1 = (\ldots, b_3, b_1, b_2, b_3, b_0, b_1, b_2, b_3, b_0, b_2, b_3, b_0, b_3), \]
\[ p^n = (\ldots, \overline{\alpha}_3, \overline{\alpha}_2, \overline{\alpha}_1, \overline{\alpha}_0, \overline{\alpha}_1, \overline{\alpha}_0, \overline{\alpha}_2, \overline{\alpha}_1), \]
\[ p^{ad} = (\ldots, \emptyset, \emptyset, b_{\alpha_1 + \alpha_2 + \alpha_3}, h_1, h_1, b_{-\alpha_1 - \alpha_2}), \]

which yield the Young walls $Y^1 := Y_0^1(p^1)$ and $Y^n := Y_0^n(p^n)$ as follows:

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 0 & 1 \\
\hline
2 & 1 & 1 & 2 & 3 & 0 \\
\end{array}
\qquad
\begin{array}{cccc}
1 & 0 & 3 & 2 \\
2 & 1 & 0 & 3 \\
0 & 3 & 2 & 1 \\
\end{array}
\]

---

4. **Geometric Constructions of Crystal Graphs**

In this section, we review geometric constructions of crystal bases via quiver varieties. See [3, 12, 14, 17, 18, 19] for more details.

Let $I = \mathbb{Z}/(n + 1)\mathbb{Z}$ and $H$ the set of the arrows such that $i \rightarrow j$ with $i, j \in I$, $i - j = \pm 1$. For $h \in H$, we denote by $\text{in}(h)$ (resp. $\text{out}(h)$) the incoming (resp. outgoing) vertex of $h$. Define an involution $\Omega : H \rightarrow H$ to be the map interchanging $i \rightarrow j$ and $j \rightarrow i$. Let

\[ \Omega = \{ h \in H \mid \text{in}(h) - \text{out}(h) = 1 \} \]

so that $H = \Omega \sqcup \overline{\Omega}$; i.e.,
\((I, \Omega) = \begin{array}{c}
1 \\
2 \\
\cdots \\
n-1 \\
n
\end{array} \quad (I, \overline{\Omega}) = \begin{array}{c}
1 \\
2 \\
\cdots \\
n-1 \\
n
\end{array} \).\\

Note that our graph is an affine Dynkin graph of type \(A_n^{(1)}\). We take the map \(\epsilon : H \to \{-1, 1\}\) given by
\[
\epsilon(h) = \begin{cases}
1 & \text{if } h \in \Omega, \\
-1 & \text{if } h \in \overline{\Omega}.
\end{cases}
\]

For \(\alpha = \sum_{i=0}^{n} k_i \alpha_i \in Q^+\), we define the \(I\)-graded vector space
\[
V(\alpha) = \bigoplus_{i=0}^{n} V_i(\alpha),
\]
where \(V_i(\alpha)\) is the \(\mathbb{C}\)-vector space with an ordered basis \(v^i(\alpha) = \{v^i_0, v^i_1, \ldots, v^i_{k_i} - 1\}\) for all \(i\). Fix an ordered basis
\[
v(\alpha) = \{v^0_0, \ldots, v^0_{k_0 - 1}, v^1_0, \ldots, v^1_{k_1 - 1}, \ldots, v^n_0, \ldots, v^n_{k_n - 1}\},
\]
for \(V(\alpha)\) and set
\[
\dim V(\alpha) = \sum_{i=0}^{n} k_i \alpha_i = \alpha.
\]

In a similar manner, for \(\lambda = \sum_{i=0}^{n} w_i \Lambda_i \in P^+\), we define the \(I\)-graded vector space
\[
W(\lambda) = \bigoplus_{i=0}^{n} W_i(\lambda),
\]
where \(W_i(\lambda)\) is a \(\mathbb{C}\)-vector space of dimension \(w_i\).

Given \(\alpha \in Q^+\), we set \(V = V(\alpha)\) (resp. \(V_i = V_i(\alpha) (i = I)\)) and let
\[
E(\alpha) = E_\Omega(\alpha) \oplus E_{\overline{\Omega}}(\alpha),
\]
where
\[
E_\Omega(\alpha) = \bigoplus_{h \in \Omega} \mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}) = \bigoplus_{i \in I} \mathrm{Hom}(V_{i-1}, V_i),
\]
\[
E_{\overline{\Omega}}(\alpha) = \bigoplus_{h \in \overline{\Omega}} \mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}) = \bigoplus_{i \in I} \mathrm{Hom}(V_i, V_{i-1}).
\]

Let us denote by \(\pi_\Omega\) (resp. \(\pi_{\overline{\Omega}}\)) the natural projection from \(E(\alpha)\) to \(E_\Omega(\alpha)\) (resp. \(E_{\overline{\Omega}}(\alpha)\)). For \(\chi \in E(\alpha)\), if there is no danger of confusion, we write \(x = (x_i \in \mathrm{Hom}(V_{i-1}, V_i))_{i \in I}\) (resp. \(\overline{x} = (\overline{x}_i \in \mathrm{Hom}(V_i, V_{i-1}))_{i \in I}\)) for \(\pi_\Omega(\chi)\) (resp. \(\pi_{\overline{\Omega}}(\chi)\)). The matrix representation of \(x \in E_\Omega(\alpha)\) in the ordered
basis \( v(\alpha) \) is given as

\[
x = \begin{pmatrix}
0 & \cdots & 0 & x_0 \\
x_1 & 0 & \cdots & 0 \\
0 & x_2 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & x_n
\end{pmatrix},
\]

where \( x_i \) is the matrix representation of \( x|_{V_{i-1}} : V_{i-1} \to V_i \) in the ordered bases \( v^i(\alpha) \) and \( v^{i-1}(\alpha) \).

We also may consider the matrix representation of \( x \in E_\Omega(\alpha) \) in the same manner:

\[
\mathbf{x} = \begin{pmatrix}
0 & \mathbf{x}_1 & 0 & \cdots & 0 \\
0 & 0 & \mathbf{x}_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \mathbf{x}_n & 0 \\
\end{pmatrix},
\]

where \( \mathbf{x}_i \) is the matrix representation of \( x|_{V_{i-1}} : V_{i-1} \to V_i \) in the ordered bases \( v^i(\alpha) \) and \( v^{i-1}(\alpha) \).

For \( s \in I, \ 0 \leq i < k_{s-1}, \ 0 \leq j < k_s \), define the linear map \( E^s_{ij} : V_{s-1} \to V_s \) by

\[
E^s_{ij}(v_k^{s-1}) = \begin{cases} 
\mathbf{v}^s_j & \text{if } k = i, \\
0 & \text{otherwise.}
\end{cases}
\]

Similarly, for \( s \in I, \ 0 \leq i < k_s, \ 0 \leq j < k_{s-1} \), define the linear map \( \overline{E}^s_{ij} : V_s \to V_{s-1} \) by

\[
\overline{E}^s_{ij}(v_k^s) = \begin{cases} 
\mathbf{v}^{s-1}_j & \text{if } k = i, \\
0 & \text{otherwise.}
\end{cases}
\]

The algebraic group \( G(\alpha) := \prod_{i \in I} \text{Aut}(V_i) \subset \text{Aut}(V) \) acts on \( E(\alpha) \) by \( (g, \chi) = g\chi g^{-1} \) for \( g \in G(\alpha), \ \chi \in E(\alpha) \). Let \( \langle \cdot, \cdot \rangle \) be the nondegenerate, \( G(\alpha) \)-invariant, sympletic form on \( E(\alpha) \) defined by

\[
\langle \chi, \chi' \rangle = \sum_{h \in H} \epsilon(h) \text{tr}(\chi h \chi' h^{-1})
\]

for \( \chi, \chi' \in E(\alpha) \). Note that \( E(\alpha) \) may be viewed as the cotangent bundle of \( E_\Omega(\alpha) \) (resp. \( E_{\overline{T}}(\alpha) \)) under this form. The moment map \( \mu = (\mu_i : E(\alpha) \to \text{End}(V_i))_{i \in I} \) is given by

\[
\mu_i(\chi) = \sum_{h \in H, \text{in}(h) = i} \epsilon(h) \chi h \chi_i = x_i \mathbf{x}_i - \overline{x}_{i+1} \mathbf{x}_{i+1}
\]

for \( \chi = x + \mathbf{x} \in E(\alpha) \). Note that

\[(4.1) \quad \mu_i(\chi) = 0 \text{ for all } i \in I \quad \text{if and only if} \quad [x, \mathbf{x}] = x\mathbf{x} - \mathbf{x}x = 0.
\]

Here, \( x\mathbf{x} = (x_i \mathbf{x}_i : V_i \to V_i)_{i \in I} \) and \( \mathbf{x}x = (\mathbf{x}_{i+1} x_{i+1} : V_i \to V_i)_{i \in I} \).
An element $\chi \in E(\alpha)$ is nilpotent if there exists an $N \geq 2$ such that for any sequence $h_1, \ldots, h_N \in H$ satisfying $\text{in}(h_i) = \text{out}(h_{i+1})$ $(i = 1, \ldots, N - 1)$, the composition map $\chi_{h_N} \cdots \chi_{h_1}$ is zero. We define Lusztig's quiver variety to be

$$\Lambda(\alpha) = \{ \chi \in E(\alpha) \mid \chi : \text{nilpotent, } \mu_i(\chi) = 0 \text{ for all } i \in I \}.$$  

We denote by $\text{Irr}\Lambda(\alpha)$ the set of all irreducible components of $\Lambda(\alpha)$.

For a pair $(k', k)$ of integers, let $V(k', k) = \bigoplus_{i \in I} V_i(k', k)$ be the $I$-graded vector space with basis $\{ e_j | k' \leq j \leq k \}$ such that $V_i(k', k) = \text{span}_C \{ e_j \mid j \equiv i \mod (n+1) \}$ for $i \in I$. Consider the $C$-linear map $x(k', k) : V(k', k) \to V(k', k)$ sending $e_i$ to $e_{i+1}$, where $e_{k+1} = 0$. Then it is clear that the representation $(V(k', k), x(k', k))$ of the quiver $(I, \Omega)$ is indecomposable and nilpotent. Note that the isomorphism class of $(V(k', k), x(k', k))$ does not change when $k'$ and $k$ are simultaneously translated by a multiple $n + 1$. Moreover, any indecomposable nilpotent finite-dimensional representation of the quiver $(I, \Omega)$ is isomorphic to $(V(k', k), x(k', k))$ for some pair $(k', k)$. Let $\mathcal{Z}$ be the set of all pairs $(k' \leq k)$ of integers defined up to simultaneous translation by a multiple of $n + 1$ and let $\overline{\mathcal{Z}}$ be the set of all functions from $\mathcal{Z}$ to $\mathbb{Z}_{\geq 0}$ with finite support. Note that $\overline{\mathcal{Z}}$ naturally corresponds to isomorphism classes of nilpotent finite-dimensional representations of the quiver $(I, \Omega)$. The set of $G(\alpha)$-orbits on the set of nilpotent elements in $E_{\Omega}(\alpha)$ is naturally indexed by the subset $\mathcal{Z}(\alpha)$ of $\overline{\mathcal{Z}}$ such that, for $f \in \mathcal{Z}(\alpha)$,

$$\sum_{k' \leq k} f(k', k) \cdot \# \{ j \mid k' \leq j \leq k, \ j \equiv i \mod (n+1) \} = \dim V_i \ (i \in I).$$

Here the sum is taken over all $k' \leq k$ up to simultaneous translation by a multiple of $n + 1$. An element $f \in \mathcal{Z}(\alpha)$ is aperiodic if, for any $k' \leq k$, not all integers $f(k', k), f(k' + 1, k + 1), \ldots, f(k' + n, k + n)$ are greater than zero. For any $f \in \mathcal{Z}(\alpha)$, let $\mathcal{C}_f$ be the conormal bundle of the $G(\alpha)$-orbit corresponding to $f$, and let $\overline{\mathcal{C}}_f$ be the closure of $\mathcal{C}_f$. Then we have

**Theorem 4.1** ([14]). The map $f \mapsto \overline{\mathcal{C}}_f$ is a 1-1 correspondence between the set of aperiodic elements in $\mathcal{Z}(\alpha)$ and $\text{Irr}\Lambda(\alpha)$.

In a similar manner, for a pair $(k \geq k')$ of integers, let $\pi(k, k') : V(k', k) \to V(k', k)$ be the $C$-linear map sending $e_i$ to $e_{i-1}$, where $e_{k-1} = 0$. Then the representation $(V(k', k), \pi(k, k'))$ of the quiver $(I, \bar{\Omega})$ is indecomposable and nilpotent, and the isomorphism class of $(V(k', k), \pi(k, k'))$ does not change when $k$ and $k'$ are simultaneously translated by a multiple $n + 1$. Any indecomposable nilpotent finite-dimensional representation of the quiver $(I, \bar{\Omega})$ is isomorphic to $\pi(k, k')$ for some pair $(k \geq k')$. Let $\overline{\mathcal{Z}}$ be the set of all pairs $(k \geq k')$ of integers defined up to simultaneous translation by a multiple of $n + 1$ and let $\overline{\mathcal{Z}}$ be the set of all functions from $\overline{\mathcal{Z}}$ to $\mathbb{Z}_{\geq 0}$ with finite support. Then the set of $G(\alpha)$-orbits on the set of nilpotent elements in $E_{\bar{\Omega}}(\alpha)$ is naturally indexed by the subset $\mathcal{Z}(\alpha)$ of $\overline{\mathcal{Z}}$ such that, for $f \in \mathcal{Z}(\alpha)$,

$$\sum_{k \geq k'} f(k, k') \cdot \# \{ j \mid k \geq j \geq k', \ j \equiv i \mod (n+1) \} = \dim V_i \ (i \in I).$$

Here the sum is taken over all $k \geq k'$ up to simultaneous translation by a multiple of $n + 1$. An element $f \in \mathcal{Z}(\alpha)$ is aperiodic if, for any $k \geq k'$, not all integers $f(k, k'), f(k + 1, k' + 1), \ldots, f(k + n, k' + n)$ are
greater than zero. Then one can show that there is a 1-1 correspondence between the set of aperiodic elements in $\mathbb{Z}(\alpha)$ and $\text{Irr}\Lambda(\alpha)$.

Moreover, Kashiwara and Saito [12] gave a crystal structure on $B(\infty) := \bigoplus_{\alpha \in Q^+} \text{Irr}\Lambda(\alpha)$, and proved the following theorem.

**Theorem 4.2 ([12]).** There is a unique crystal isomorphism $B(\infty) \cong B(\infty)$.

Now we introduce a description of Nakajima’s quiver varieties presented in [16]. Given $\alpha \in Q^+$ and $\lambda \in P^+$, we set $W = W(\lambda)$ (resp. $W_i = W_i(\lambda)$) and let

$$E(\lambda, \alpha) = \Lambda(\alpha) \times \sum_{i \in I} \text{Hom}(V_i, W_i).$$

The group $G(\alpha)$ acts on $E(\lambda, \alpha)$ by $(g, (\chi, t)) = (g\chi g^{-1}, tg^{-1})$. For $\chi \in \Lambda(\alpha)$, an $I$-graded subspace $S$ of $V(\alpha)$ is $\chi$-stable if $\chi_h(S_{\text{out}(h)}) \subset S_{\text{in}(h)}$ for all $h \in H$. An element $(\chi, t) \in E(\lambda, \alpha)$ is called a stable point of $E(\lambda, \alpha)$ if it satisfies the following conditions: if $S$ is a $\chi$-stable subspace of $V$ with $t_i(S_i) = 0$ ($i \in I$), then $S = 0$. Let $E(\lambda, \alpha)^{st}$ be the set of all stable points of $E(\lambda, \alpha)$, and define

$$\Lambda(\lambda, \alpha) = E(\lambda, \alpha)^{st}/G(\alpha).$$

Let $\text{Irr}\Lambda(\lambda, \alpha)$ (resp. $\text{Irr}E(\lambda, \alpha)$) be the set of all irreducible components of $\Lambda(\lambda, \alpha)$ (resp. $E(\lambda, \alpha)$). Since $\text{Irr}\Lambda(\lambda, \alpha)$ can be identified with

$$\{Z \in \text{Irr}E(\lambda, \alpha) | Z \cap E(\lambda, \alpha)^{st} \neq \emptyset\},$$

each irreducible component $X$ in $\text{Irr}\Lambda(\lambda, \alpha)$ can be written as

$$X = \left( \left( X_0 \times \sum_{i \in I} \text{Hom}(V_i, W_i) \right) \cap E(\lambda, \alpha)^{st} \right)/G(\alpha)$$

for some irreducible component $X_0$ in $\text{Irr}\Lambda(\alpha)$.

In [18], Saito gave a crystal structure on

$$B(\lambda) := \bigcup_{\alpha \in Q^+} \text{Irr}\Lambda(\lambda, \alpha),$$

and proved the following theorem.

**Theorem 4.3 ([18]).** There is a unique crystal isomorphism $B(\lambda) \cong B(\lambda)$.

In [3], Frenkel and Savage gave an enumeration of $\text{Irr}\Lambda(\lambda, \alpha)$ in terms of Young and Maya diagrams for type $A^{(1)}_n$. Combining Theorem 4.2 and Theorem 4.3 with (1.1), we obtain an injective map

$$\iota_\lambda : B(\lambda) \hookrightarrow B(\infty).$$

For each irreducible component $X_0 \in \iota_\lambda(\mathbb{B}(\Lambda_k))$, Frenkel and Savage constructed a special point in $X_0 \times \sum_{i \in I} \text{Hom}(V_i, W_i)$ which is not killed by the stability condition, and showed that there is a 1-1 correspondence between the set of such special points and the set of $(n+1)$-reduced colored Young
diagrams. Savage later established a crystal isomorphism between $B(\lambda)$ and Young walls for quantum affine algebras of type $A_n^{(1)}$ and $D_n^{(1)}$ in [19].

We briefly recall the result of [3] for type $A_n^{(1)}$ in terms of Young walls. Note that the orientation appeared in [3] is $\Omega$. Take a Young wall $Y^n \in Y^n(\Lambda_k)$ such that $\text{wt}(Y^n) = \alpha$. Let $l_i$ be the length of the $i$-th row of the Young wall $Y^n$ ($i \geq 1$) and let $N$ be the height of $Y^n$. Set

$$A_{Y^n} := \{(l_i - i + k, 1 - i + k) \mid 1 \leq i \leq N\} \subseteq \mathcal{Z}$$

and consider the function $f \in \mathcal{Z}(\alpha)$ given by

$$f(s, s') = \begin{cases} 1 & \text{if } (s, s') \in A_{Y^n}, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $f$ is aperiodic. Let $\mathcal{C}_f$ be the closure of the conormal bundle of the $G(\alpha)$-orbit $\mathcal{O}_f$ in $E_{\mathcal{P}}(\alpha)$ corresponding to $f$, and define the irreducible component

$$X_{Y^n} := \left( \mathcal{C}_f \times \sum_{i \in I} \text{Hom}(V_i, W_i) \right) \cap E(\Lambda_k, \alpha)_{st} / G(\alpha) \in \text{Irr}\Lambda(\Lambda_k, \alpha).$$

By [3] Theorem 5.5, the map $Y^n \mapsto X_{Y^n}$ is a 1-1 correspondence between

$$\{Y^n \in Y^n(\Lambda_k) \mid \text{wt}(Y^n) = \alpha\} \text{ and } \text{Irr}\Lambda(\Lambda_k, \alpha).$$

Moreover, it is proved in [19] Theorem 8.4.] that the map $Y^n \mapsto X_{Y^n}$ from $Y^n(\Lambda_k)$ to $B(\Lambda_k)$ is a crystal isomorphism. We would like to point out that $\mathcal{C}_f = \iota_{\Lambda_k}(X_{Y^n})$.

Now we construct an element in the $G(\alpha)$-orbit $\mathcal{O}_f$ in $E_{\mathcal{P}}(\alpha)$ from the Young wall $Y^n$. Let $\mathbf{b}_{ij}$ be the $i$-th block from bottom in the $j$-th column of $Y^n$. Let $\text{Color}(\mathbf{b}_{ij})$ be the color of $\mathbf{b}_{ij}$, which is an element in $I$. Define

$$o(\mathbf{b}_{ij}) := \# \{ \mathbf{b}_{rs} \in Y^n \mid \text{Color}(\mathbf{b}_{rs}) = \text{Color}(\mathbf{b}_{ij}), (r, s) \prec (i, j) \},$$

where $\prec$ is the lexicographical order; i.e., $(r, s) \prec (i, j)$ if and only if $r < i$ or $(r = i$ and $s < j$). We define

$$\overline{\pi}(Y^n) := \sum_{\mathbf{b}_{ij} \in Y^n, \, j > 0} \mathbf{v}_{\text{Color}(\mathbf{b}_{ij})}^{o(\mathbf{b}_{ij})} \in E_{\mathcal{P}}(\alpha).$$

For $1 \leq i \leq N$, we denote by $J_i$ the subspace of $V(\alpha)$ generated by

$$\{ \mathbf{v}_{o(\mathbf{b}_{ij})} \mid 0 \leq j < l_i \}.$$ 

By construction, one can show that $J_i$ is invariant under $\overline{\pi}(Y^n)$ and the representation $(J_i, \overline{\pi}(Y^n)|_{J_i})$ of the quiver $(I, \overline{\Omega})$ is isomorphic to the representation $(V(1 - i + k, l_i - i + k), \overline{\pi}(l_i - i + k, 1 - i + k))$ of the quiver $(I, \overline{\Omega})$ for $1 \leq i < N$. Here, $\overline{\pi}(Y^n)|_{J_i}$ is the restriction of $\overline{\pi}(Y^n)$ on the invariant subspace $J_i$. Hence $\overline{\pi}(Y^n)$ is contained in the $G(\alpha)$-orbit $\mathcal{O}_f$ corresponding $f$, which yields

$$\iota_{\Lambda_k}(X_{Y^n}) = \text{the closure of the conormal bundle of the } G(\alpha)-\text{orbit of } \overline{\pi}(Y^n).$$
By a direct computation, for \( t \in \mathbb{Z}_{\geq 0} \), we have
\[
\ker(\varphi(Y^n))^t = \bigoplus_{i=1}^{N} \ker(\varphi(Y^n)|_{J_i})^t
= \bigoplus_{i=1}^{N} \text{span}_C \{ v_{o(b_{ij})}^{\text{Color}(b_{ij})} | \ b_{ij} \in \text{the } i\text{-th row of } Y^n,\ j < t \}
= \text{span}_C \{ v_{o(b_{ij})}^{\text{Color}(b_{ij})} | \ b_{ij} \in Y^n,\ j < t \}.
\]
(4.5)

In the same manner, we take a Young wall \( Y^1 \in Y^1(\Lambda_k) \) such that \( \text{wt}(Y^1) = \alpha \). Denote by \( X_{Y^1} \) the image of \( Y^1 \) under the crystal isomorphism \( Y^1(\Lambda_k) \cong \rightarrow B(\Lambda_k) \).

Let \( b_{ij} \) be the \( i \)-th block from bottom in the \( j \)-th column of \( Y^1 \), and \( \text{Color}(b_{ij}) \) the color of \( b_{ij} \). Set \( o(b_{ij}) := \# \{ b_{rs} \in Y^1 | \text{Color}(b_{rs}) = \text{Color}(b_{ij}), (r, s) \prec (i, j) \} \), where \( \prec \) is the lexicographical order, and define
\[
x(Y^1) := \sum_{b_{ij} \in Y^1,\ j > 0} E_{o(b_{ij}), o(b_{i-1,j})}^{\text{Color}(b_{i,j-1})} \in E_\Omega(\alpha).
\]
(4.6)

Then we have
\[
\iota_{\Lambda_k}(X_{Y^1}) = \text{the closure of the conormal bundle of the } G(\alpha)\text{-orbit of } x(Y^1).
\]
(4.7)

Moreover, we obtain
\[
\ker(x(Y^1))^t = \text{span}_C \{ v_{o(b_{ij})}^{\text{Color}(b_{ij})} | \ b_{ij} \in Y^1,\ j < t \}
\]
(4.8)

for \( t \in \mathbb{Z}_{\geq 0} \).

**Example 4.4.** We use the same notations as in Example 3.3. Set
\[
\alpha := \Lambda_0 - \text{wt}(b) = 4\alpha_0 + 4\alpha_1 + 4\alpha_2 + 3\alpha_3,
\]
and let \( X \) be the irreducible component in \( B(\Lambda_0) \) corresponding to \( b \) via the crystal isomorphism given in Theorem 4.3. Then we have
\[
x(Y^1) = (\mathcal{E}_{00}^0 + \mathcal{E}_{11}^0 + \mathcal{E}_{22}^0) + (\mathcal{E}_{10}^1 + \mathcal{E}_{21}^1 + \mathcal{E}_{32}^1) + (\mathcal{E}_{00}^2 + \mathcal{E}_{11}^2 + \mathcal{E}_{22}^2) + (\mathcal{E}_{00}^3 + \mathcal{E}_{11}^3 + \mathcal{E}_{22}^3),
\]
\[
\varphi(Y^n) = (\mathcal{E}_{10}^0 + \mathcal{E}_{21}^0 + \mathcal{E}_{32}^0) + (\mathcal{E}_{00}^1 + \mathcal{E}_{12}^1 + \mathcal{E}_{23}^1) + (\mathcal{E}_{00}^2 + \mathcal{E}_{11}^2 + \mathcal{E}_{22}^2) + (\mathcal{E}_{00}^3 + \mathcal{E}_{11}^3 + \mathcal{E}_{22}^3),
\]
and \( \iota_{\Lambda_0}(X) \) is the closure of the conormal bundle of the \( G(\alpha) \)-orbit of \( x(Y^1) \) (resp. \( \varphi(Y^n) \)). However, we note that
\[
x(Y^1) + \varphi(Y^n) \notin E(\alpha)
\]
since \([x(Y^1), \varphi(Y^n)] \neq 0\).
5. Quiver Varieties and the Perfect Crystals $B^1, B^n$

In this section, we give an explanation of the 1-1 correspondence between the geometric realization $\mathbb{B}(\Lambda_k)$ and the path realization of the crystal $B(\Lambda_k)$ associated with the perfect crystals $B^1$ and $B^n$, and give a geometric interpretation of the fundamental theorem of perfect crystals in the case of the perfect crystals $B^1$ and $B^n$. Let $\alpha \in Q^+$ and let $\lambda$ be a dominant integral weight of level 1. Choose an irreducible component $X$ in $\text{Irr}\Lambda(\lambda, \alpha)$. For a generic point $\chi = x + \pi \in \iota_\Lambda(X)$, we will give an explicit description of the $\lambda$-path in $B^1$ (resp. $B^n$) corresponding to $X$ using the dimensions of the spaces $\ker x^{i+1} / \ker x^i$ (resp. $\ker \pi^{i+1} / \ker \pi^i$) for $i \geq 0$. For this purpose, we need a couple of lemmas.

Lemma 5.1. Let $X_0$ be an irreducible component in $\text{Irr}\Lambda(\alpha)$. Then, for any $\chi = x + \pi \in X_0$ and $k \in \mathbb{Z}_{\geq 0}$, we have

(a) $\ker (x \pi)^k = \ker (\pi x)^k$,
(b) $\ker \chi^k$ and $\ker \pi^k$ are $\chi$-stable,
(c) $\ker (x \pi)^k$ is $\chi$-stable.

Proof. Let $\chi = x + \pi \in X_0$ and $k \in \mathbb{Z}_{\geq 0}$. By (4.1) we have $[x, \pi] = 0$, which yields

$\chi(\ker \chi^k) \subset \ker \chi^k$, $\chi(\ker \pi^k) \subset \ker \pi^k$ and $\chi(\ker (x \pi)^k) \subset \ker (x \pi)^k$.

Our assertion follows from the fact that $\ker (x \pi)^k$, $\ker \chi^k$ and $\ker \pi^k$ are $\chi$-graded vector spaces. $\square$

Lemma 5.2. For each $X_0 \in \text{Irr}\Lambda(\alpha)$, there exists an open subset $U \subset X_0$ such that

\begin{equation}
\ker x^k \cong \ker x^k \quad \text{and} \quad \ker \pi^k \cong \ker \pi^k
\end{equation}

for any $\chi = x + \pi$, $\chi' = x' + \pi' \in U$ and $k \in \mathbb{Z}_{\geq 0}$.

Proof. By Theorem 4.1, there is an open subset $U_1 \subset X_0$ such that $\pi_0(U_1)$ is contained in the $G(\alpha)$-orbit of some element in $E_0(\alpha)$. In the same manner, there is an open subset $U_2 \subset X_0$ such that $\pi\pi(U_2)$ is contained in the $G(\alpha)$-orbit of some element in $E_0(\alpha)$. Set $U = U_1 \cap U_2 \subset X_0$. Then, by construction, for any $\chi = x + \pi$, $\chi' = x' + \pi' \in U$, there exist $g, \eta \in G(\alpha)$ such that

\begin{align*}
x &= gx'g^{-1} \quad \text{and} \quad \pi = g\pi\eta^{-1},
\end{align*}

which yield, for any $k \in \mathbb{Z}_{\geq 0}$,

\begin{align*}
\ker x^k &= \ker (gx'g^{-1})^k = g(\ker x^k) \quad \text{and} \quad \ker \pi^k = \ker (g\pi\eta^{-1})^k = \eta(\ker \pi^k).
\end{align*}

$\square$

An element $\chi \in X_0$ in the open subset $U \subset X_0$ in Lemma 5.2 will be called a generic point. Thanks to Lemma 5.2, we may consider

$\dim(\ker x^k)$ and $\dim(\ker x^{k+1} / \ker x^k)$ (resp. $\dim(\ker \pi^k)$ and $\dim(\ker \pi^{k+1} / \ker \pi^k)$)

for a generic point $\chi = x + \pi$ in an irreducible component $X_0 \in \text{Irr}\Lambda(\alpha)$. Recall the injective map given in (4.2)

$\iota_{\Lambda_k} : \mathbb{B}(\Lambda_k) \hookrightarrow \mathbb{B}(\infty)$
for $0 \leq k \leq n$. Applying Lemma 5.1 and Lemma 5.2 to (1.6) and (4.3), we obtain the following theorem.

**Theorem 5.3.** (cf. [3]) Let 
\[ p_1^k : \mathcal{B}(\Lambda_k) \rightarrow \mathcal{P}^1(\Lambda_k) \]  
(resp. \( p_n^k : \mathcal{B}(\Lambda_k) \rightarrow \mathcal{P}^n(\Lambda_k) \))

be the unique crystal isomorphism given by Theorem 2.2 and Theorem 4.3, and take an irreducible component \( X \in \mathcal{B}(\Lambda_k) \). Then, for a generic point \( \chi = x + \varpi \in \iota_{\Lambda_k}(X) \), we have

(a) 
\[ p_1^k(X) = (\ldots, b_{a_i}, \ldots, b_{a_1}, b_{a_0}), \]

where \( a_i \equiv \dim(\ker x^{i+1} / \ker x^i) - i + k \, (\mod n + 1) \) for all \( i \geq 0 \),

(b) 
\[ p_n^k(X) = (\ldots, \overline{b}_i, \ldots, \overline{b}_{b_i}, \overline{b}_{b_0}), \]

where \( b_i \equiv 1 - \dim(\ker \overline{x}^{i+1} / \ker \overline{x}^i) + i + k \, (\mod n + 1) \) for all \( i \geq 0 \).

**Proof.** Let \( U \) be an open subset of \( \iota_{\Lambda_k}(X) \) as in Lemma 5.2. By Lemma 5.2, it suffices to show that (a) and (b) hold for some \( \chi = x + \varpi \in U \).

Let \( Y^1 \) be the Young wall in \( \mathcal{Y}^1(\Lambda_k) \) corresponding to \( X \) under the crystal isomorphism \( \mathcal{Y}^1(\Lambda_k) \cong \mathcal{B}(\Lambda_k) \), and \( b_{ij} \) the \( i \)-th block from bottom in the \( j \)-th column of \( Y^1 \). By (4.7), there exists \( \chi = x + \varpi \in U \) such that

\[ x = gx(Y^1)g^{-1} \]

for some \( g \in G(\alpha) \). Then, by the equation (4.8), for \( t \in \mathbb{Z}_{\geq 0} \), we have

\[ \ker x^t = \ker(gx(Y^1)g^{-1})^t = g(\ker(x(Y^1)^t)) = g\left(\text{span}_{C}\{v_{\text{Color}(b_{ij})} | b_{ij} \in Y^1, j < t\}\right). \]

Let \( y_t \) be the \( t \)-th column of \( Y^1 \) for \( t \in \mathbb{Z}_{\geq 0} \). Note that \( y_t = \{b_{it} \in Y^1 | i \geq 1 \} \). Then, we have

\[ \text{wt}(y_t) = \sum_{b_{it} \in y_t} \alpha_{\text{Color}(b_{it})} \]

\[ = \dim\left(\text{span}_{C}\{v_{o(b_{it})} | b_{it} \in y_t\}\right) \]

\[ = \dim\left(\text{span}_{C}\{v_{\text{Color}(b_{ij})} | b_{ij} \in Y^1, j < t + 1\}\right) - \dim\left(\text{span}_{C}\{v_{o(b_{ij})} | b_{ij} \in Y^1, j < t\}\right) \]

\[ = \dim(\ker(x(Y^1))^{t+1}) - \dim(\ker(x(Y^1))^t) \]

\[ = \dim(\ker x^{t+1} - \dim(\ker x^t) \]

(5.2) \[ = \dim(\ker x^{t+1} / \ker x^t), \]

which implies that the height of \( y_t \) is

\[ \text{ht}(\text{wt}(y_t)) = \dim(\ker x^{t+1} / \ker x^t). \]

Consequently, the assertion (a) follows from (3.1) and (3.2).

The remaining assertion (b) can be proved in the same manner. \( \square \)
Combining the crystal isomorphisms (3.1) and (3.3):
\[
Y^1_k : \mathcal{P}^1(\Lambda_k) \rightarrow \mathcal{Y}^1(\Lambda_k) \quad \text{and} \quad Y^n_k : \mathcal{P}^n(\Lambda_k) \rightarrow \mathcal{Y}^n(\Lambda_k)
\]
with (4.7) and (4.4), we have the following proposition, which, together with Theorem 5.3, yields an explicit 1-1 correspondence between $B(\Lambda_k)$ and $\mathcal{P}^1(\Lambda_k)$ (resp. $\mathcal{P}^n(\Lambda_k)$).

**Proposition 5.4.** Let
\[
q^1_k : \mathcal{P}^1(\Lambda_k) \rightarrow B(\Lambda_k) \quad \text{(resp. } q^n_k : \mathcal{P}^n(\Lambda_k) \rightarrow B(\Lambda_k))
\]
be the unique crystal isomorphism given by Theorem 2.2 and Theorem 4.3, and take a $\Lambda_k$-path $p^1 \in \mathcal{P}^1(\Lambda_k)$ (resp. $p^n \in \mathcal{P}^n(\Lambda_k)$). Let
\[
\alpha = \Lambda_k - \text{wt}(p^1) \quad \text{and} \quad X_1 = q^1_k(p^1) \quad \text{(resp. } \beta = \Lambda_k - \text{wt}(p^n) \quad \text{and} \quad X_n = q^n_k(p^n)).
\]
Then
(a) $\iota_{\Lambda_k}(X_1)$ is the closure of the conormal bundle of the $G(\alpha)$-orbit of $x(Y^1_k(p^1))$;
(b) $\iota_{\Lambda_k}(X_n)$ is the closure of the conormal bundle of the $G(\beta)$-orbit of $x(Y^n_k(p^n))$.

Recall the fundamental isomorphism theorem of perfect crystals (2.1). From Theorem 4.3, we have the following crystal isomorphisms:
\[
\psi^1_k : B(\Lambda_k) \sim \rightarrow B(\Lambda_{k-1}) \otimes B^1,
\]
\[
\psi^n_k : B(\Lambda_k) \sim \rightarrow B(\Lambda_{k+1}) \otimes B^n
\]
for $0 \leq k \leq n$. We would like to give a geometric interpretation to the crystal isomorphisms $\psi^1_i$, $\psi^n_i$ in terms of quiver varieties. To do that, we need a couple of lemmas.

Let $V$ be an $I$-graded vector space and $\chi$ an element of $\text{Hom}(V, V)$. If $W$ is a $\chi$-invariant $I$-graded subspace of $V$, then $\chi$ can be viewed as an element in $\text{Hom}(V/W, V/W)$ (resp. $\text{Hom}(W/W)$), which is denoted by $\chi|_{V/W}$ (resp. $\chi|_{W/W}$).

**Lemma 5.5.** Let $x \in \bigoplus_{i \in I} \text{Hom}(V_{i-1}, V_i)$ for an $I$-graded vector space $V := \bigoplus_{i \in I} V_i$, and set
\[
W := \ker x \quad \text{and} \quad y := x|_{V/W}.
\]
Take an element
\[
\overline{y} \in \bigoplus_{i \in I} \text{Hom}(V_i/W_i, V_{i-1}/W_{i-1}) \quad \text{with} \quad [y, \overline{y}] = 0,
\]
where $W_i$ is the $i$-subspace of $W$ for $i \in I$. Then there exists an element
\[
\overline{x} \in \bigoplus_{i \in I} \text{Hom}(V_i, V_{i-1})
\]
such that
\[
[x, \overline{x}] = 0 \quad \text{and} \quad \overline{x}|_{V/W} = \overline{y}.
Proof. Let \( r = \dim V - \dim W \) and \( s = \dim W \). Take an ordered basis for \( W \) and extend it to be an ordered basis for \( V \) so that the matrix representations of \( x, y \) and \( \overline{y} \) are given as follows:

\[
x = \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix}, \quad y = B \quad \text{and} \quad \overline{y} = C
\]

for some \( r \times r \) matrices \( B \) and \( C \), and \( s \times r \) matrix \( A \). Note that \([B,C] = 0\). Since the matrix \( \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \) has full rank, the following equation

\[
\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & AC \\ 0 & 0 \end{pmatrix}
\]

has a solution. Since \( x(V_i) \subset V_{i+1} \) and \( \begin{pmatrix} 0 & AC \\ 0 & 0 \end{pmatrix} \) maps \( V_i \) to \( V_i \), we can choose an \( s \times s \) matrix \( X \) and an \( s \times r \) matrix \( Y \) such that \( \begin{pmatrix} X & Y \\ 0 & C \end{pmatrix} \) is a solution of the equation \(5.3\) and maps \( V_i \) to \( V_{i-1} \) for \( i \in I \). Let

\[
\overline{x} = \begin{pmatrix} X & Y \\ 0 & C \end{pmatrix}.
\]

By construction, we have

\[
\overline{x}(V_i) \subset V_{i-1} \quad (i \in I), \quad \overline{x}|_{V/W} = \overline{y},
\]

and

\[
[x, \overline{x}] = x \overline{x} - \overline{x} x = \begin{pmatrix} 0 & AC \\ 0 & BC \end{pmatrix} - \begin{pmatrix} 0 & XA + YB \\ 0 & CB \end{pmatrix} = 0.
\]

\( \square \)

Lemma 5.6. Let \( U \) be an open subset of \( X_0 \in \text{Irr} \Lambda(\alpha) \) as in Lemma \(5.2\). Set

\[
\beta = \dim(\ker x) \quad (\text{resp.} \quad \gamma = \dim(\ker \overline{x}))
\]

for \( \chi = x + \overline{x} \in U \).

(a) There exists an irreducible component \( X_0' \in \text{Irr} \Lambda(\alpha - \beta) \) such that, for \( \chi = x + \overline{x} \in U \),

\[
\phi \circ (\chi|_{V(\alpha)}/\ker x) \circ \phi^{-1} \in X_0',
\]

where \( \phi : V(\alpha)/\ker x \to V(\alpha - \beta) \) is an \( I \)-graded vector space isomorphism.

(b) There exists an irreducible component \( X_0'' \in \text{Irr} \Lambda(\alpha - \gamma) \) such that, for \( \chi = x + \overline{x} \in U \),

\[
\phi \circ (\chi|_{V(\alpha)/\ker \overline{x}}) \circ \phi^{-1} \in X_0'\prime',
\]

where \( \phi : V(\alpha)/\ker \overline{x} \to V(\alpha - \gamma) \) is an \( I \)-graded vector space isomorphism.
Proof. Note that $\beta$ and $\gamma$ are well-defined by Lemma 5.2. We first deal with the case (a). For an element $\chi = x + \mathbf{r} \in U$, let

$$\chi_\phi := \phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in \text{End}(V(\alpha - \beta)),$$

where $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$ is an $I$-graded vector space isomorphism. Since $\chi \in \Lambda(\alpha)$, we have $\chi_\phi \in \Lambda(\alpha - \beta)$.

Take two elements $\chi = x + \mathbf{r}$, $\chi' = x' + \mathbf{r}' \in U$, and choose two $I$-graded vector space isomorphisms $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$ and $\phi' : V(\alpha)/\ker x' \to V(\alpha - \beta)$. From the properties of $U$ described in the proof of Lemma 5.2, we have

$$x = gx' g^{-1}$$

for some $g \in G(\alpha)$, which yields that $\pi_{\Omega}(\chi_\phi)$ and $\pi_{\Omega}(\chi'_{\phi'})$ are in the same $G(\alpha - \beta)$-orbit. Therefore, there exists an irreducible component $X'_0 \in \text{Irr}\Lambda(\alpha - \beta)$ such that

$$\chi_\phi, \chi'_{\phi'} \in X'_0.$$  

Since $\chi, \chi'$ are arbitrary, our assertion follows.

The remaining case (b) can be proved in the same manner. □

Theorem 5.7. Let $X_0 = \iota_{\Lambda_k}(X)$ for an irreducible component $X \in \text{Irr}\Lambda(\Lambda_k, \alpha)$. Set

$$d = \dim(\ker x) \quad \text{and} \quad \beta = \dim(\ker x) \quad (\text{resp.} \quad d' = \dim(\ker \mathbf{r}) \quad \text{and} \quad \gamma = \dim(\ker \mathbf{r}' \mathbf{r}^{-1}))$$

for a generic point $\chi = x + \mathbf{r} \in X_0$.

(a) There exists a unique irreducible component $X' \in \text{Irr}\Lambda(\Lambda_{k-1}, \alpha - \beta)$ satisfying the following conditions:

(i) there is an open subset $U \subset X_0$ such that, for $\chi = x + \mathbf{r} \in U$,

$$\phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in \iota_{\Lambda_{k-1}}(X'),$$

where $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$ is an $I$-graded vector space isomorphism,

(ii) there is an open subset $U' \subset \iota_{\Lambda_{k-1}}(X')$ such that any element $\chi' \in U'$ can be written as

$$\chi' = \phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1},$$

for some $\chi = x + \mathbf{r} \in X_0$ and some $I$-graded vector space isomorphism $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$,

(iii) moreover, we have

$$\psi_k(X) = X' \otimes b_a \quad \text{and} \quad \text{wt}(b_a) = \Lambda_k - \Lambda_{k-1} - \text{cl}(\beta),$$

where $a \equiv d + k \pmod{n + 1}$.

(b) There exists a unique irreducible component $X'' \in \text{Irr}\Lambda(\Lambda_{k+1}, \alpha - \gamma)$ satisfying the following conditions:
We first deal with the case (a) of the crystal isomorphism

\[
\text{Proof.}
\]

Let \( U \) be an open subset of \( X_0 \) such that, for \( \chi = x + \bar{x} \in U \),

\[
\phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in \iota_{\Lambda_k+1}(X'''),
\]

where \( \phi : (\alpha)/\ker x \to V(\alpha) \) is an I-graded vector space isomorphism,

(ii) there is an open subset \( U'' \subset \iota_{\Lambda_k+1}(X''') \) such that any element \( \chi'' \in U'' \) can be written as

\[
\chi'' = \phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1},
\]

for some \( \chi = x + \bar{x} \in X_0 \) and some I-graded vector space isomorphism \( \phi : (\alpha)/\ker x \to V(\alpha) \),

(iii) moreover, we have

\[
\psi^n_b(X) = X'' \otimes B_b \quad \text{and} \quad \mathrm{wt}(B_b) = \Lambda_k - \Lambda_{k+1} - \mathrm{cl}(\gamma),
\]

where \( b \equiv 1 - d' + k \) (mod \( n + 1 \)).

**Proof.** We first deal with the case (a) of the crystal isomorphism \( \psi^\alpha_b : \mathbb{B}(\Lambda_k) \to \mathbb{B}(\Lambda_{k-1}) \otimes B^\alpha \). Let \( Y \) be the Young wall in \( \mathcal{Y}(\Lambda_k) \) corresponding to \( X \) and \( Y' \) the Young wall obtained by removing the 0-th column from \( Y \). Then \( Y' \) can be viewed as an element in \( \mathcal{Y}(\Lambda_{k-1}) \). Take the irreducible component \( X' \in \mathcal{B}(\Lambda_{k-1}) \) corresponding to \( Y' \). By Theorem 5.3 and (5.2), we have

\[
X' \in \operatorname{Irr}\Lambda(\Lambda_{k-1}, \alpha - \beta) \quad \text{and} \quad \psi^\alpha_b(X) = X' \otimes b_a,
\]

where \( a \equiv d + k \) (mod \( n + 1 \)) and \( \mathrm{wt}(b_a) = \Lambda_k - \Lambda_{k-1} - \mathrm{cl}(\beta) \).

Let \( U \) be an open subset of \( X_0 \) as in Lemma 5.6 and take an element \( \chi = x + \bar{x} \in U \). Since \( x(Y)|_{V(\alpha)/\ker x} \) is naturally identified with \( x(Y') \) and \( x \) is contained in the \( G(\alpha) \)-orbit of \( x(Y) \), by Lemma 5.6 we have

\[
\phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in \iota_{\Lambda_{k-1}}(X')
\]

for an I-graded vector space isomorphism \( \phi : (\alpha)/\ker x \to V(\alpha) \).

Take an element \( \chi' = x' + \bar{x}' \) in an open subset \( U' \) of \( \iota_{\Lambda_{k-1}}(X') \) given as in Lemma 5.2. Then \( x' \) can be written as

\[
x' = gx(Y')g^{-1}
\]

for some \( g \in G(\alpha - \beta) \), which yields that there is \( x \) in the \( G(\alpha) \)-orbit of \( x(Y) \) such that

\[
\phi \circ (x|_{V(\alpha)/\ker x}) \circ \phi^{-1} = x'
\]

for some I-graded vector space isomorphism \( \phi : (\alpha)/\ker x \to V(\alpha - \beta) \). The assertion (ii) follows from Lemma 5.4.

The remaining case (b), \( \psi^n_b : \mathbb{B}(\Lambda_k) \to \mathbb{B}(\Lambda_{k+1}) \otimes B^n \), can be proved in the same manner. \( \square \)
Example 5.8. We use the same notations as in Example 4.4. Let \( X_0 = \iota_{\Lambda_0}(X) \in \text{Irr}\Lambda(\alpha) \). By Theorem 4.1, it suffices to consider a generic point in the fiber \( \pi_{\Omega}^{-1}(\overline{x}(Y^n)) \subset X_0 \). By [14, Section 12.8, Proposition 15.5], we have

\[
\pi_{\Omega}^{-1}(\overline{x}(Y^n)) = \{ x + \overline{x}(Y^n) | x \in E_{\Omega}(\alpha), [x, \overline{x}(Y^n)] = 0 \} = \{ x + \overline{x}(Y^n) | x = x(a_1, \ldots, a_{13}), a_1, \ldots, a_{13} \in \mathbb{C} \}.
\]

Here,

\[
x(a_1, \ldots, a_{13}) = \begin{pmatrix}
0 & 0 & 0 & x_0 \\
x_1 & 0 & 0 & 0 \\
0 & x_2 & 0 & 0 \\
0 & 0 & x_3 & 0
\end{pmatrix} \in E_{\Omega}(\alpha),
\]

\[
x_0 = \begin{pmatrix}
a_1 & a_2 & a_3 \\
0 & 0 & 0 \\
a_4 & 0 & a_5 \\
a_6 & 0 & 0
\end{pmatrix}, \quad x_1 = \begin{pmatrix}
0 & a_1 & a_2 & a_3 \\
0 & a_4 & 0 & a_5 \\
0 & a_6 & 0 & 0 \\
0 & a_{10} & 0 & a_7
\end{pmatrix},
\]

\[
x_2 = \begin{pmatrix}
0 & a_2 & a_3 & 0 \\
0 & 0 & a_5 & 0 \\
a_6 & a_8 & a_{11} & a_9 \\
0 & 0 & a_7 & 0
\end{pmatrix}, \quad x_3 = \begin{pmatrix}
0 & a_2 & 0 & 0 \\
a_4 & a_{12} & a_5 & a_{13} \\
0 & a_6 & a_8 & 0 \\
a_9
\end{pmatrix},
\]

for \( a_1, \ldots, a_{13} \in \mathbb{C} \). Let

\[
x := x(a_1, \ldots, a_{13}), \quad \overline{x} := \overline{x}(Y^n),
\]

and consider \( a_1, \ldots, a_{13} \) as indeterminates. Then we have

\[
\dim(\ker x^k) = \begin{cases} 
0 & \text{if } k = 0, \\
2 + k & \text{if } 1 \leq k \leq 12, \\
15 & \text{otherwise},
\end{cases}
\]

and

\[
\dim(\ker \overline{x}^k) = \begin{cases} 
0 & \text{if } k = 0, \\
4 & \text{if } k = 1, \\
8 & \text{if } k = 2, \\
11 & \text{if } k = 3, \\
14 & \text{if } k = 4, \\
15 & \text{otherwise}.
\end{cases}
\]

Hence we obtain

\[
p^1_0(X) = (\ldots, b_3, b_1, b_2, b_3, b_0, b_1, b_2, b_3, b_0, b_1, b_2, b_3, b_0, b_3),
\]

\[
p^0_0(X) = (\ldots, \overline{b}_3, \overline{b}_2, \overline{b}_0, \overline{b}_1, \overline{b}_0, \overline{b}_2, \overline{b}_1).
\]
In this section, we will prove the main theorem of this paper, Theorem 6.3, which shows that there exists an explicit crystal isomorphism between the geometric realization $\mathbb{B}(\Lambda_0)$ and the path realization $\mathcal{P}^{\text{ad}}(\Lambda_0)$ of $B(\Lambda_0)$ arising from the adjoint crystal $B^{\text{ad}}$. By Theorem 2.2 and Theorem 4.3, we have the crystal isomorphism

$$\mathcal{P}^{\text{ad}} : \mathbb{B}(\Lambda_0) \longrightarrow \mathcal{P}^{\text{ad}}(\Lambda_0).$$

Let $\alpha \in Q^+$ and let $X$ be an irreducible component of $\text{Irr} \Lambda(\Lambda_0, \alpha)$. For a generic point $\chi = x + \mathfrak{F} \in \iota_{\Lambda_0}(X)$, we will give an explicit description of the $\Lambda_0$-path $\mathcal{P}^{\text{ad}}(X)$ in terms of dimension vectors of $\ker(x\mathfrak{F})^{k+1}/\ker(x\mathfrak{F})^k$ for $k \geq 0$.

Let $\alpha, \beta \in Q^+$ with $\beta \leq \alpha$. Consider the diagram given in [14]:

$$\Lambda(\alpha - \beta) \xrightarrow{\pi} \Lambda(\alpha - \beta) \times \Lambda(\beta) \xrightarrow{p_1} F'' \xrightarrow{p_2} F'' \xrightarrow{p_3} \Lambda(\alpha),$$

where $F''$ is the variety of all pairs $(\chi, W)$ such that

- (a) $\chi \in \Lambda(\alpha)$,
- (b) $W$ is a $\chi$-stable subspace of $V(\alpha)$ with $\dim W = \beta$,

and $F'$ is the variety of all quadruples $(\chi, W, f, g)$ such that

- (a) $(\chi, W) \in F''$,
- (b) $f = (f_i)_{i \in I}, g = (g_i)_{i \in I}$ give an exact sequence

$$0 \longrightarrow V_i(\beta) \xrightarrow{f_i} V_i(\alpha) \xrightarrow{g_i} V_i(\alpha - \beta) \longrightarrow 0 \quad (i \in I)$$

such that $\text{im} f = W$.

Then we have

$$p_1(\chi, W, f, g) = (\tilde{g} \circ (\chi|_{V(\alpha)/W}) \circ \tilde{g}^{-1}, f^{-1} \circ (\chi|_{W}) \circ f),$$

where $\tilde{g} : V(\alpha)/W \to V(\alpha - \beta)$ is the $I$-graded vector space isomorphism induced by $g$,

$$p_2(\chi, W, f, g) = (\chi, W), \quad p_3(\chi, W) = \chi,$$

and $\pi$ is the natural first projection. Note that $p_2$ is a $G(\alpha - \beta) \times G(\beta)$-principal bundle and an open map.

Let $U$ be an open subset of $X_0 \in \text{Irr}\Lambda(\alpha)$ as in Lemma 5.2 and $\beta = \dim(\ker x)$ for $\chi = x + \mathfrak{F} \in U$. Define the map $\iota : U \to F''$ by

$$\iota(\chi) = (\chi, \ker x)$$

for $\chi = x + \mathfrak{F} \in U$. Note that $p_3 \circ \iota = \text{id}|_U$. By Lemma 5.6 there exists an irreducible component $X'_0 \in \text{Irr}\Lambda(\alpha - \beta)$ such that, for any $\chi = x + \mathfrak{F} \in U$,

$$\phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in X'_0,$$

where $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$ is an $I$-graded vector space isomorphism. Given an open subset $U' \subset \Lambda(\alpha - \beta)$ with $U' \cap X'_0 \neq \emptyset$, by Lemma 5.3,

$$\tilde{U} := \iota^{-1} \circ p_2 \circ p_1^{-1} \circ \pi^{-1}(U')$$
is a nonempty open subset of $X_0$. Therefore, given an open subset $U' \subset X'_0$, there exists an open subset $\tilde{U} \subset X_0$ such that, for any element $\chi = x + \mathfrak{p} \in \tilde{U}$,

$$\phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in U''$$

for some $I$-graded vector space isomorphism $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$.

In the same manner, let $\gamma = \dim(\ker \mathfrak{p})$, and consider the diagram

$$\Lambda(\alpha - \gamma) \xrightarrow{\pi_\gamma} \Lambda(\alpha - \gamma) \times \Lambda(\gamma) \xrightarrow{\phi_{\gamma}} F' \xrightarrow{\phi_\beta} F'' \xrightarrow{\phi_\gamma} \Lambda(\alpha).$$

Define the map $\overline{\gamma} : U \to F''$ by

$$\overline{\gamma}(\chi) = (\chi, \ker \mathfrak{p})$$

for $\chi = x + \mathfrak{p} \in U$, and let $X''_0$ be an irreducible component as in Lemma 5.6. Then one can deduce that, given an open subset $U' \subset X'_0$, there exists an open subset $\tilde{U} \subset X_0$ such that, for any element $\chi = x + \mathfrak{p} \in \tilde{U}$,

$$\phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in U''$$

for some $I$-graded vector space isomorphism $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$. Consequently, we have the following lemma.

**Lemma 6.1.** With the same notations as in Lemma 5.6, we have the following.

(a) Given an open subset $U' \subset X'_0$, there exists an open subset $\tilde{U} \subset X_0$ such that, for any element $\chi = x + \mathfrak{p} \in \tilde{U}$,

$$\phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in U''$$

for some $I$-graded vector space isomorphism $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$.

(b) Given an open subset $U'' \subset X''_0$, there exists an open subset $\tilde{U} \subset X_0$ such that, for any element $\chi = x + \mathfrak{p} \in \tilde{U}$,

$$\phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in U''$$

for some $I$-graded vector space isomorphism $\phi : V(\alpha)/\ker x \to V(\alpha - \gamma)$.

Combining Lemma 6.1 with Lemma 5.6, we have the following lemma.

**Lemma 6.2.** Let $\alpha \in Q^+$. For each $X_0 \in \text{Irr}\Lambda(\alpha)$, there exists an open subset $U \subset X_0$ such that

$$\ker(x \mathfrak{p})^k \cong \ker(x' \mathfrak{p})^k \quad \text{and} \quad \ker x(x \mathfrak{p})^k \cong \ker x'(x' \mathfrak{p})^k$$

for any $\chi = x + \mathfrak{p}, \chi' = x' + \mathfrak{p} \in U$ and $k \in \mathbb{Z}_{\geq 0}$.

**Proof.** Since the case that $\text{ht}(\alpha) = 0$ is trivial, we may assume $\text{ht}(\alpha) > 0$. Let $U_0$ be an open subset of $X_0$ as in Lemma 5.2 and $\beta = \dim(\ker x)$ for $\chi = x + \mathfrak{p} \in U_0$. Take the irreducible component $X'_0 \in \text{Irr}\Lambda(\alpha - \beta)$ given in Lemma 5.6, and choose an open subset $U'_0$ of $X'_0$ satisfying the conditions of Lemma 5.2. Let $\gamma = \dim(\ker \mathfrak{p})$ for $\chi = x + \mathfrak{p} \in U'_0$. By Lemma 6.1, there exists an open subset $\tilde{U}_0 \subset X_0$ such that, for any element $\chi = x + \mathfrak{p} \in \tilde{U}_0$,

$$\phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in U'_0$$
for some $I$-graded vector space isomorphism $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$. Set $\hat{U}_0 = U_0 \cap \hat{U}_0$. Then, for any $\chi = x + \overline{x} \in \hat{U}_0$, we have

$$\dim \ker x\overline{x} = \dim(\ker x) + \dim(\ker x\overline{x}/\ker x)$$

$$= \dim(\ker x) + \dim\{v \in V(\alpha)/\ker x \mid \overline{x}(V(\alpha)/\ker x)v = 0\}$$

$$= \dim(\ker x) + \dim(\ker \overline{x}|_{V(\alpha)/\ker x})$$

$$= \beta + \gamma.$$

Let us take the irreducible component $X'_0 \in \text{Irr}(\alpha - \beta - \gamma)$ associated with $X'_0$, which is given as in Lemma $5.6$. By the induction hypothesis, there exists an open subset $U' \subset X'_0$ satisfying (6.4). Applying Lemma 6.1 to $X''_0$, $X'_0$ and $X_0$, there exists an open subset $\hat{U} \subset X_0$ such that, for any $\chi = x + \overline{x} \in \hat{U}$,

$$\phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in U'$$

for some $I$-graded vector space isomorphism $\phi : V(\alpha)/\ker x \overline{x} \to V(\alpha - \beta - \gamma)$. Let $U = \hat{U} \cap \hat{U}_0$, then, by construction, $U$ holds the condition (6.4).

An element $\chi \in X_0$ in the open subset $U \subset X_0$ satisfying (5.1) and (6.4) is called a generic point. Note that, for $\chi = x + \overline{x}$ in an irreducible component $X_0 \in \text{Irr}\Lambda(\alpha)$, since $[x, \overline{x}] = 0$, we have

$$\dim(\ker(x|_{\ker(x\overline{x})^{k+1}/\ker(x\overline{x})^k})) = \dim(\ker x(x\overline{x})^k) - \dim(\ker(x\overline{x})^k),$$

where $x|_{\ker(x\overline{x})^{k+1}/\ker(x\overline{x})^k}$ is the linear map in $\text{End}(\ker(x\overline{x})^{k+1}/\ker(x\overline{x})^k)$ induced by $x$. Thanks to Lemma 6.2 one can talk about

$$\dim \ker(x\overline{x})^k, \quad \dim \ker(x\overline{x})^{k+1}/\ker(x\overline{x})^k$$

and

$$\dim(\ker(x|_{\ker(x\overline{x})^{k+1}/\ker(x\overline{x})^k}))$$

for a generic point $\chi = x + \overline{x}$ in an irreducible component $X_0 \in \text{Irr}\Lambda(\alpha)$ and $k \in \mathbb{Z}_{\geq 0}$.

Finally, we are ready to state the main theorem in this paper.

**Theorem 6.3.** Let

$$p^{\text{ad}} : \mathbb{B}(\Lambda_0) \longrightarrow \mathcal{P}^{\text{ad}}(\Lambda_0)$$

be the unique crystal isomorphism given by Theorem 2.2 and Theorem 3.3, and take an irreducible component $X \in \mathbb{B}(\Lambda_0)$. For a generic point $\chi = x + \overline{x} \in t\Lambda_0(X)$ and $k \in \mathbb{Z}_{\geq 0}$, let

$$\theta_k = \dim(\ker(x\overline{x})^{k+1}/\ker(x\overline{x})^k),$$

$$c_k \equiv \dim(\ker(x|_{\ker(x\overline{x})^{k+1}/\ker(x\overline{x})^k})) \mod n + 1,$$

where $0 \leq c_k \leq n$. Then we have

$$p^{\text{ad}}(X) = (\ldots, p_k, \ldots, p_1, p_0),$$

where

$$p_k = \begin{cases} b_{-\text{cl}(\theta_k)} & \text{if cl}(\theta_k) \neq 0, \\
 h_{c_k} & \text{if cl}(\theta_k) = 0 \text{ and } c_k \neq 0, \\
 0 & \text{otherwise}. \end{cases}$$
Proof. Let $X_0 = \nu_{\Lambda_0}(X)$ with $\text{wt}(X) = \Lambda_0 - \alpha$ for some $\alpha \in Q^+$. We will use induction on $\text{ht}(\alpha)$. Since the case that $\text{ht}(\alpha) = 0$ is trivial, we may assume $\alpha \neq 0$. Note that

$$\theta_k = \dim(\ker(x^k)) - \dim(\ker(x^{k+1})),$$

$$c_k = \dim(\ker(x^k)) - \dim(\ker(x^{k+1})) \pmod{n + 1}$$

for a generic point $\chi = x + \pi \in X_0$ and $k \in \mathbb{Z}_{\geq 0}$.

Let $\beta = \dim(\ker(x))$ for a generic point $\chi = x + \pi \in X_0$, and choose the irreducible component $X_0' \in \text{Irr}(\alpha - \beta)$ associated with $X_0$ as in Lemma 5.6. Similarly, let $\gamma = \dim(\ker(x))$ for a generic point $\chi = x + \pi \in X_0$, and take the irreducible component $X_0'' \in \text{Irr}(\alpha - \beta - \gamma)$ associated with $X_0$ as in Lemma 5.6. By Theorem 5.7, we have

$$\psi_{\nu}^1(X_0) = X_0' \otimes b_a$$  and  $$\psi_{\nu}^n(X_0) = X_0'' \otimes b_b$$

for some $b_a \in B^1$, $b_b \in B^n$. From (2.1) and Theorem 4.2, we have

$$\psi_{\nu} : \mathbb{B}(\Lambda_0) \longrightarrow \mathbb{B}(\Lambda_0) \otimes B^\text{ad}.$$

Then, it follows from the crystal isomorphism (2.2) and Theorem 5.7 that

$$\psi_{\nu}^\text{ad}(X_0) = X_0'' \otimes p_{\text{ad}}(b_b \otimes b_a).$$

By the induction hypothesis, there is an open subset $U'' \subset X_0''$ satisfying (6.6). By Lemma 6.1 and (6.5), there is an open subset $\tilde{U} \subset X_0$ such that, for any $\chi = x + \pi \in \tilde{U}$,

$$\phi \circ (\chi|_{V(\alpha)/\ker(x\pi)}) \circ \phi^{-1} \in U''$$

for some isomorphism $\phi : V(\alpha)/\ker(x\pi) \to V(\alpha - \beta - \gamma)$.

On the other hand, by Lemma 6.2 there exists an open subset $\hat{U} \subset X_0$ satisfying (6.4). Set

$$U = \hat{U} \cap \tilde{U}$$

and choose an element $\chi = x + \pi \in U$. Suppose that $\text{wt}(b_b \otimes b_a) \neq 0$. Then, by Theorem 5.7 and (2.2), since

$$\text{wt}(b_b \otimes b_a) = \text{wt}(b_b) + \text{wt}(b_a)$$

$$= \Lambda_n - \Lambda_0 - \text{cl}(\gamma) + \Lambda_0 - \Lambda_n - \text{cl}(\beta)$$

$$= -\text{cl}(\beta + \gamma)$$

$$= -\text{cl}(\dim(\ker(x\pi)))$$

$$= -\text{cl}(\theta_0),$$

we obtain

$$p_{\text{ad}}(b_b \otimes b_a) = b_{\text{wt}(b_b \otimes b_a)} = b_{-\text{cl}(\theta_0)},$$

Suppose $\text{wt}(b_b \otimes b_a) = 0$ and $a \neq n + 1$. Then, by Theorem 5.7 and (2.2), we have

$$p_{\text{ad}}(b_b \otimes b_a) = h_a.$$
and \(a \equiv \dim x \pmod{n + 1}\), which implies that \(a = c_0\). In the same manner, if \(\text{wt}(\mathcal{B}_b \otimes b_a) = 0\) and \(a = n + 1\), we have

\[
p^{\text{ad}}(\mathcal{B}_b \otimes b_a) = \emptyset.
\]

Since, for an arbitrary isomorphism \(\phi : V(\alpha)/\ker x\overline{x} \to V(\alpha - \beta - \gamma)\),

\[
\dim(\ker(x\overline{x})^{k+1}) = \dim(\ker(x\overline{x})^{k+1}/\ker(x\overline{x})) + \dim(\ker(x\overline{x}))
= \dim(\ker(x\overline{x})|V(\alpha)/\ker(x\overline{x}))^k + \dim(\ker(x\overline{x}))
= \dim \ker(\phi \circ (x\overline{x})|V(\alpha)/\ker(x\overline{x}) \circ \phi^{-1})^k + \dim(\ker(x\overline{x}))
= \dim \ker((\phi \circ x|V(\alpha)/\ker(x\overline{x}) \circ \phi^{-1})(\phi \circ x\overline{x}|V(\alpha)/\ker(x\overline{x}) \circ \phi^{-1}))^k + \dim(\ker(x\overline{x}))
\]

our assertion follows from a standard induction argument.

The following corollary, which is an immediate consequence of Theorem 6.1 and Theorem 6.3, can be regarded as a geometric interpretation of the fundamental isomorphism theorem for perfect crystals

\[
\psi_0^{\text{ad}} : \mathbb{B}(\Lambda_0) \sim \mathbb{B}(\Lambda_0) \otimes B^{\text{ad}}.
\]

**Corollary 6.4.** Let \(X_0 = \iota_{\Lambda_0}(X)\) for some \(X \in \text{Irr}_\Lambda(\Lambda_0, \alpha)\). For a generic point \(\chi = x + \overline{x} \in X_0\), set

\[
\theta = \dim(\ker(x\overline{x})) \quad \text{and} \quad c = \dim(\ker x).
\]

Then there exists a unique irreducible component \(X' \in \text{Irr}_\Lambda(\Lambda_0, \alpha - \theta)\) satisfying the following conditions:

(a) there is an open subset \(U \subset X_0\) such that, for \(\chi = x + \overline{x} \in U\),

\[
\phi \circ (\chi|V(\alpha)/\ker x\overline{x}) \circ \phi^{-1} \in \iota_{\Lambda_0}(X'),
\]

where \(\phi : V(\alpha - \theta) \to V(\alpha)/\ker x\overline{x}\) is an \(I\)-graded vector space isomorphism,

(b) there is an open subset \(U' \subset \iota_{\Lambda_0}(X')\) such that any element \(\chi' \in U'\) can be written as

\[
\chi' = \phi \circ (\chi|V(\alpha)/\ker x\overline{x}) \circ \phi^{-1}
\]

for some \(\chi = x + \overline{x} \in X_0\) and some \(I\)-graded vector space isomorphism \(\phi : V(\alpha)/\ker x\overline{x} \to V(\alpha - \theta)\),

(c) moreover, we have

\[
\psi_0^{\text{ad}}(X) = X' \otimes p,
\]

where

\[
p = \begin{cases} 
    b_{-\text{cl}(-)} & \text{if } \text{cl}(-) \neq 0, \\
    h_c & \text{if } \text{cl}(0) = 0 \text{ and } c \neq 0, \\
    0 & \text{otherwise.}
\end{cases}
\]
Example 6.5. We use the same notations as in Example 5.8. Let $W_i = \ker((x \tau)^i)$ for $i \in \mathbb{Z}_{\geq 0}$. Then we have

$$
\dim(W_i) = \begin{cases} 
0 & \text{if } i = 0, \\
\alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3 & \text{if } i = 1, \\
2\alpha_0 + 3\alpha_1 + 3\alpha_2 + 2\alpha_3 & \text{if } i = 2, \\
3\alpha_0 + 4\alpha_1 + 4\alpha_2 + 3\alpha_3 & \text{if } i = 3, \\
4\alpha_0 + 4\alpha_1 + 4\alpha_2 + 3\alpha_3 & \text{otherwise},
\end{cases}
$$

and

$$
\dim(\ker(x|_{W_{i+1}/W_i})) = \dim(\ker((x \tau)^i)) - \dim(\ker((x \tau)^{i-1})) = \begin{cases} 
3 & \text{if } i = 0, \\
1 & \text{if } i = 1, \\
1 & \text{if } i = 2, \\
0 & \text{otherwise}.
\end{cases}
$$

By Theorem 6.3 we have

$$
p^{ad}(X) = (\ldots, \emptyset, b_{\alpha_1+\alpha_2}, h_1, h_1, b_{-\alpha_1-\alpha_2}).
$$

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