SOME REMARKS ON NON PROJECTIVE FROBENIUS ALGEBRAS AND LINEAR CODES

JOSÉ GÓMEZ-TORRECILLAS, ERIK HIETA-AHO, F. J. LOBILLO, SERGIO LÓPEZ-PERMOUTH, AND GABRIEL NAVARRO

Abstract. With a small suitable modification, dropping the projectivity condition, we extend the notion of a Frobenius algebra to grant that a Frobenius algebra over a Frobenius commutative ring is itself a Frobenius ring. The modification introduced here also allows Frobenius finite rings to be precisely those rings which are Frobenius finite algebras over their characteristic subrings. From the perspective of linear codes, our work expands one’s options to construct new finite Frobenius rings from old ones. We close with a discussion of generalized versions of the McWilliam identities that may be obtained in this context.

1. Introduction

The core ingredient of the definition of a Frobenius algebra $A$ finitely generated as a module over its commutative ground ring $K$, as it appears in the literature, is the requirement that there exist a non degenerate associative $K$–bilinear form on $A$; this requirement mimics accurately the definition of a Frobenius algebra over a field. However, that definition also includes the somewhat technical requirement that $A$ be projective as a $K$–module. At first, since all modules over a field are indeed free and hence projective, the extra condition appears innocuous. Furthermore, the extra condition apparently served the purpose of affording the result, mentioned without a detailed proof in [7, p. 434], that every Frobenius algebra over a Frobenius commutative ring is a Frobenius ring. Unfortunately, not every finite Frobenius ring is projective over its characteristic subring.

In this paper we remove the projectivity technical requirement from the usual definition of a Frobenius algebra over a commutative ring and propose that said expression be used instead for a properly modified notion. We show that thus extending the definition affords us a very natural result, namely, that a finite ring $A$ of characteristic $n$ is Frobenius if and only if it is a (not necessarily projective) Frobenius algebra over its characteristic ring $\mathbb{Z}_n$. We, therefore, adopt the expression non projective Frobenius algebra over a ring when all else is satisfied but possibly not the projectivity requirement. Notice that, as is a common practice in mathematics, our expression “non projective”, is a less pedantic option to express “not necessarily projective”.

An additional benefit from our approach is that we may adapt, without using the projectivity of $A$ over $K$, the arguments in the usual proof that Frobenius algebras over a field are Frobenius as rings to the case of non projective Frobenius algebras over a ring (Theorem 16). This result is interesting in its own right and has potential to open many doors in the realm of applications. Note that one may also derive Theorem 16 from the recent analysis of the Frobenius property for Artin algebras in [6]. However, while more general, this approach is significantly more...
technical and we felt that there is value in sharing our proof here in hopes that it would be more accessible for a larger group of interested readers.

One of the motivations behind this extension of the expression non projective Frobenius algebra over a ring arises because of applications of this notion in the algebraic theory of error-correcting codes. Finite Frobenius rings are extensively used as alphabets for ring-linear block codes; Frobenius algebras over a finite field are an important example of Frobenius rings. In contrast, a well known characterization of Frobenius rings [9] may be rephrased by asserting that a finite ring is Frobenius if and only if there exists an non degenerate associative bilinear form over its characteristic subring.

The fundamental problem in the theory of linear codes is the analysis and classification of linear codes over an alphabet, usually a finite field $\mathbb{F}$. Since there is a tremendous number of subspaces for the vector space $\mathbb{F}^n$ the task at hand is quite monumental. A common frequently successful practice has been to filter the codes considered in the following manner: add an $\mathbb{F}$-algebra structure on the vector space $\mathbb{F}^n$ (let us refer to $\mathbb{F}^n$, equipped with the additional structure, as $A_n$ to remind us of the fact that it is now an algebra of dimension $n$) and consider, say, only those subspaces which are (left, right, two-sided) ideals of $A_n$. We refer to the additional algebraic structure on $\mathbb{F}^n$ as the ambient (or the ambient space) of the codes targeted and to the criterion applied to choose the codes we wish to consider as the filter being applied.

As the interest in studying codes in ambients endowed with additional algebraic structure expanded from considering field alphabets to ring alphabets, Frobenius rings have tended to become central to the conversation. The reason for this preference goes back to papers that show that the fundamental structural properties of codes over fields extend to codes that have Frobenius rings as their alphabets. In fact, there is plenty of evidence to show that Frobenius rings are precisely the extent to which this type of considerations can be extended if one wishes to acquire the desired structural properties. For the properties to hold, the alphabet must be Frobenius.

In this paper, we investigate an alternative role that Frobenius rings appear to play very naturally in this context. It seems to be the case that coding-theoretic structural properties are attained, regardless of the structure of the alphabet, when the ambient is a non projective Frobenius algebra over a commutative ring possibly other than the alphabet. For example, it may be that the ambient is a non projective Frobenius algebra over its characteristic subring.

We illustrate our ideas by taking advantage of the fact that every finite Frobenius ring of characteristic $n$ can be seen as a non projective Frobenius $\mathbb{Z}_n$-algebra. Our examples include constructing a family of non projective Frobenius algebras based on factor algebras of skew polynomial algebras and using them as the ambient algebra for a wide class of analogues to skew cyclic block codes.

This note is organized as follows. Section 2 provides, with the generality needed here, the statements and proofs of preliminary results on balanced bilinear forms over bimodules; we only include proofs that we deemed illustrative. Section 3 provides our proposed definition of a non projective Frobenius algebra over a commutative ring and various equivalent characterizations; as mentioned above, projectivity of the algebra over the base ring is not required. One of these equivalent conditions is the existence of a Frobenius functional, which will be seen to generalize the generating character and play a similar role to it. In Section 4, we extend results from [10] on annihilators associated to a non degenerate bilinear form from a finite Frobenius ring to a non projective Frobenius algebra. Section 5 contains our observation that, given an algebra $R$ over a Frobenius commutative ring $K$ such
that $R$ is finitely generated as a $K$–module, then $R$ is a non projective Frobenius algebra over $K$ if and only if $R$ is a Frobenius ring. This, in particular, applies to finite rings, viewed as algebras over their characteristic subrings. We also include a method, based on skew polynomials, to construct new Frobenius algebras from a given one. From the point of view of codes, this gives a way to construct new finite Frobenius rings from old ones; this is further discussed in Section 6. That section also contains a discussion of how the general results on bilinear forms defined on modules over non projective Frobenius algebras, developed in the previous sections, may be applied to get the main results of [10]. We close that section with a discussion of for which bilinear forms the version of McWilliams identities stated in [10] holds.

2. Preliminaries on bilinear forms and annihilators

All rings we consider will be unital and possibly non commutative. Let $A$, $B$ rings and $M$ an $A – B$–bimodule. We will simply say that $AMB$ is a bimodule. We also use the notation $AM$ to declare that $M$ is a left $A$–module, and $MB$ for right $B$–modules. Systematic introductions the theory of modules over non commutative rings are [5], or [1]. Everything in this section follows from [1, §30]. For convenience, however, we state the results here in the form they are needed, and provide their proofs in this more limited scope to keep the paper self-contained. Given a second bimodule $BNA$, the abelian group $N^* = \text{hom}_A(N, A)$ of all homomorphisms of right $B$–modules is endowed with the structure of a bimodule $AN^*_B$ by the rule

$$(afb)(n) = af(bn), \quad a \in A, b \in B, f \in N^*, n \in N.$$

A straightforward argument shows that the formula

$$\langle m, n \rangle = \alpha(m)(n), \quad m \in M, n \in N$$

provides a bijective correspondence between $A$–bilinear maps

$$\langle -, - \rangle : M \times N \to A$$

and homomorphisms of left $A$–modules

$$\alpha : M \to N^*.$$  

Moreover, $\langle -, - \rangle$ is associative, in the sense that $\langle mb, n \rangle = \langle m, bn \rangle$ for all $m \in M, n \in N, b \in B$, if and only if the corresponding $\alpha : M \to N^*$ is a homomorphism of right $B$–modules. Analogously, the additive group $^*M = \text{hom}_A(M, A)$ of all homomorphisms of left $A$–modules is a $B – A$–bimodule via the rule

$$(bfa)(m) = f(mb)a, \quad b \in B, a \in A, f \in ^*M.$$  

The assignment

$$\langle m, n \rangle = \beta(n)(m), \quad m \in M, n \in N$$

gives a bijective correspondence between $A$–bilinear forms

$$\langle -, - \rangle : M \times N \to A$$

and homomorphisms of right $A$–modules

$$\beta : N \to ^*M.$$  

Again, the condition of being the bilinear form associative is equivalent to require that $\beta$ is a homomorphism of left $B$–modules.
Definition 1. Let \( \langle -, - \rangle : M \times N \to A \) be an \( A \)-bilinear form, and \( \alpha : M \to N^* \), \( \beta : N \to {}^*M \) the corresponding homomorphisms of \( A \)-modules defined as before. We say that \( \langle -, - \rangle \) is right (resp. left) non degenerate if \( \alpha \) (resp. \( \beta \)) is injective. When \( \langle -, - \rangle \) is left and right non degenerate, we just say that the bilinear form is non degenerate.

The length of a right \( A \)-module \( X \) will be denoted by \( \operatorname{lt}(X_A) \), for a left \( A \)-module \( Y \), by \( \operatorname{lt}(A Y) \). From now on, unless otherwise stated, we will assume that all \( A \)-modules are of finite length. We also assume in the rest of this section that \( A \) is a Quasi-Frobenius ring (see [5, Ch. 13] for various characterizations of these rings). Therefore, left or right \( A \)-modules of finite length are precisely the finitely generated ones. Over the Quasi-Frobenius ring \( A \), \( \operatorname{lt}(A X^*) = \operatorname{lt}(X_A) \) for every finitely generated right \( A \)-module \( X_A \) and \( \operatorname{lt}({}^*Y_A) = \operatorname{lt}(A Y) \) for every finitely generated left \( A \)-module \( {}^*Y \) (see [5, Theorem 13.3.2]).

Lemma 2. If there is a non degenerate \( A \)-bilinear form \( \langle -, - \rangle : M \times N \to A \), then \( \operatorname{lt}(A M) = \operatorname{lt}(N_A) \).

Proof. The following computation, which uses that both \( \alpha \) and \( \beta \) are injective maps, gives the statement.
\[
\operatorname{lt}(A M) \leq \operatorname{lt}(A N^*) = \operatorname{lt}(N_A) \leq \operatorname{lt}(M_A) = \operatorname{lt}(A M).
\]
\[\square\]

Next lemma, as well as Proposition 4 below, can be deduced from [11, Theorem 30.1].

Lemma 3. Assume that \( \operatorname{lt}(A M) = \operatorname{lt}(N_A) \). An \( A \)-bilinear form \( \langle -, - \rangle : M \times N \to A \) is left non degenerate if and only if it is right non degenerate. In such a case, both \( \alpha \) and \( \beta \) are isomorphisms.

Proof. Assume that \( \langle -, - \rangle \) is right non degenerate, that is, the homomorphism of left \( A \)-modules \( \alpha : M \to N^* \) is injective. Since \( A \) is QF, \( \operatorname{lt}(A N^*) = \operatorname{lt}(N_A) = \operatorname{lt}(A M) \). Therefore, \( \alpha \) is an isomorphism. In order to prove that \( \langle -, - \rangle \) is left non degenerate, let \( n \in N \) such that \( \beta(n) = 0 \). This means that, for every \( m \in M \), \( 0 = \beta(n)(m) = \langle m, n \rangle = \alpha(m)(n) \). Since \( \alpha \) is surjective, we get that \( \varphi(n) = 0 \) for all \( \varphi \in N^* \). This implies that \( n = 0 \), since \( A \) is an injective cogenerator right \( A \)-module. Therefore, \( \beta \) is injective. We so far have proved that if \( \langle -, - \rangle \) is right non degenerate, then it is left non degenerate, and \( \alpha \) is an isomorphism. By symmetry, we get the full statement of the lemma.
\[\square\]

Now, for an \( A \)-bilinear form \( \langle -, - \rangle : M \times N \to A \), and subsets \( S \subseteq N \), \( T \subseteq M \), we define
\[\{m \in M \mid \langle m, s \rangle = 0, \forall s \in S\} = S', \]
\[\{n \in N \mid \langle t, n \rangle = 0, \forall t \in T\} = T'.\]

Clearly, \( S \) is a left \( A \)-submodule of \( M \), while \( T' \) is a right \( A \)-submodule of \( N \).

Proposition 4. Assume that \( \langle -, - \rangle \) is non degenerate. For every \( A \)-submodule \( X \) (resp. \( Y \)) of \( N_A \) (resp. of \( A M \)) we have that \( \langle X' \rangle = X \) and \( \langle Y' \rangle = Y \).

Proof. From Lemma 2 \( \operatorname{lt}(A M) = \operatorname{lt}(N_A) \). Consider the commutative diagram of left \( A \)-modules
\[
\begin{array}{c}
0 \to (N/X)^* \to N^* \to X^* \to 0, \\
\alpha \downarrow \quad \quad \alpha \downarrow \quad \quad \alpha' \downarrow \\
0 \to X' \to M \to M/X \to 0
\end{array}
\]
where $\alpha'(m + X)(n) = \langle m, n \rangle = \alpha''(m)(n + X)$ for all $m \in M, n \in N$. We see that $\alpha'$ is injective, so, being $\alpha$ an isomorphism by Lemma 3 we get that $\alpha''$ is an isomorphism, too. This implies that $\alpha''$ is an isomorphism. Therefore, $\langle X \rangle \cong (N / X)'^*$ and $X^* \cong \langle M / X \rangle$ as $A$–modules. Analogously, we have isomorphisms of $A$–modules $Y' \cong \langle M / Y \rangle$ and $Y \cong \langle N / Y \rangle$.

Now, observe that $\langle X \rangle' \subseteq \langle M / X \rangle'$. Thus, $\langle X \rangle' \subseteq \langle X \rangle'$. Analogously, $\langle Y \rangle' \subseteq \langle Y \rangle'$.

Remark 5. With the notation of Proposition 4 we get from its proof that there are isomorphisms of $A$–modules $M' / X \cong X^*$ and $N / Y' \cong Y^*$, as well as $\langle X \rangle \cong \langle (N / X) \rangle$ and $\langle Y \rangle \cong \langle (M / Y) \rangle$.

Corollary 6. The map $(\cdot \langle X \rangle \cdot)$ gives an anti-isomorphism, with inverse $(\cdot \langle X \rangle \cdot)'$, between the lattices of right $A$–submodules of $N$ and of left $A$–submodules of $M$.

If $(\cdot, \cdot)$ is associative, then we easily get from Proposition 4 the following refinement of Corollary 6

Corollary 7. If $(\cdot, \cdot)$ is associative, then the map $(\cdot \langle X \rangle \cdot)$ gives an anti-isomorphism, with inverse $(\cdot \langle X \rangle \cdot)'$, between the lattices of $B - A$–bisubmodules of $N$ and of $A - B$–bisubmodules of $M$.

3. Non projective Frobenius algebras

Let $R$ be an algebra over a commutative Frobenius ring $K$. Setting $A = K$ and $B = R$ in the framework of Section 2 we may thus consider the bimodules $N = R_k$ and $M \cong R_R$. Therefore, $R^* = \langle R = \text{hom}_k(R, K) \rangle$ is endowed with the right and left $R$–module structures

\[(f b)(b') = f(b b'), (b' f)(b), \quad b, b' \in R, f \in R^*.
\]

This gives, indeed, an $R - R$–bimodule structure on $R$. Next, we materialize the discussion of Section 2 to this framework. Recall that a $K$–bilinear form $\langle \cdot, \cdot \rangle : R \times R \to K$ is associative if $\langle b b', b'' \rangle = \langle b, b' b'' \rangle$ for all $b, b', b'' \in R$.

Proposition 8. Let $R$ be an algebra over a commutative Frobenius ring $K$. Assume $R$ to be finitely generated as a $K$–module. The following structures related to $R$ are in bijective correspondence.

1. Associative non degenerate $K$–bilinear forms

2. Isomorphisms of right $R$–modules $\alpha : R \to R^*$.

3. Isomorphisms of left $R$–modules $\beta : R \to R^*$.

4. $K$–linear forms $\epsilon \in R^*$ such that $\epsilon R = R^*$.

5. $K$–linear forms $\epsilon \in R^*$ such that $\epsilon R = R^*$.

Proof. The equivalence between 1, 2 and 3 follows from Lemma 3 and the discussion previous to Definition 4. We only discuss the equivalence between 2 and 4, since that between 3 and 4 is symmetric. If there is an isomorphism of right $R$–modules $\alpha : R \to R^*$, then $\epsilon = \alpha(1)$ generates $R^*$ as a right $R$–module. Conversely, given $\epsilon \in R^*$ such that $R^* = \epsilon R$, we have the surjective homomorphism of right $R$–modules $\alpha : R \to \epsilon R = R^*$ given by $\alpha(b) = \epsilon b$. Since $K$ is Frobenius, $\text{lt}(R_K) = \text{lt}(R_K^*)$, and we get that $\epsilon$ is an isomorphism, indeed. Let us finally argue that the linear form appearing in 1 coincides with that of 4. Let $b, b' \in R$. Then $\langle b, b' \rangle = \alpha(b)(b') = \epsilon(b b') = \epsilon(b b') = (b' \epsilon)(b) = \beta(b')(b)$. Thus, $R \cong R^*$.
Definition 9. An algebra $R$ over a commutative Frobenius ring $K$ is said to be non projective Frobenius if $R$ is finitely generated as a $K$–module and there exits a non degenerate $K$–bilinear form $(\cdot, \cdot) : R \times R \to K$. The $K$–linear form $\epsilon : R \to K$ provided by Proposition 8 will be referred to as the Frobenius functional. It follows from Proposition 8 that a Frobenius $K$–algebra in the sense of [3] is non projective Frobenius. The converse is not true (see Remark 10).

Remark 10. We have already seen that the additional structure maps that, according to Proposition 8, make a $K$–algebra non projective Frobenius, are related by the following equalities

$$
\begin{align*}
\alpha(b)(b') &= (b, b') = \beta(b')(b), \\
\epsilon &= \alpha(1) = \beta(1), \quad \alpha(b) = eb, \quad \beta(b) = be,
\end{align*}
$$

for all $b, b' \in R$.

Remark 11. As a consequence of Lemma 3 and Proposition 8 and from their proofs, we see that, in the latter, we may replace condition (1) by only requiring from the bilinear form to be either left or right non degenerate. Also, in conditions (2) and (3) we only need to require from $\alpha$ or $\beta$ to be surjective.

Example 12. Every semisimple algebra $R$ over a commutative Frobenius ring $K$ with $R_K$ finitely generated is a non projective Frobenius algebra. By virtue of Wedderburn-Artin Theorem, it suffices by proving this when $R$ is simple. In this case, $\rho R \cong \Sigma^n$ for a simple left $R$–module $\Sigma$. Now, $R^* \cong (\Sigma^*)^n$ as right $R$–modules. Since $K$ is Frobenius, $R$ and $R^*$ have the same length as $K$–modules. Therefore, $\Sigma$ and $\Sigma^*$ have the same length as $K$–modules. This implies (look at $R$ as a matrix ring over a division $K$–algebra) that $\Sigma^*$ is a simple right $R$–module. Since its multiplicity in $R^*$ is $n$, we deduce that $R \cong R^*$ as right $R$–modules.

4. Annihilators in non projective Frobenius extensions

Let us come back to the situation of a pair of bimodules $A \alpha A_B$ and $B \alpha N_A$ with a non degenerate $A$–bilinear map $(\cdot, \cdot) : M \times N \to A$. Assume, in addition, that $A$ is a non projective Frobenius algebra over a Frobenius commutative ring $K$, with Frobenius functional $\epsilon : A \to K$. The corresponding associative non degenerate $K$–bilinear form will be denoted by $(\cdot, \cdot)_\epsilon$, and it obeys the rule

$$
(a, a')_\epsilon = \epsilon(aa'), \quad a, a' \in A.
$$

Define, for each subset $S \subseteq N$,

$$
\begin{align*}
\langle S &= \{m \in M : \epsilon((m, s)) = 0 \forall s \in S\},
\end{align*}
$$

which is a $K$–submodule of $M$. Analogously, for every subset $T \subseteq M$ we get the $K$–submodule of $N$

$$
\begin{align*}
T' &= \{n \in N : \epsilon([t, n]) = 0 \forall t \in T\}.
\end{align*}
$$

Indeed, $\langle S \rangle$ and $\langle T \rangle$ are the left and right orthogonal $K$–submodules of $S$ and $T$ with respect to the $K$–bilinear form

$$
[\cdot, \cdot]_\epsilon : M \times N \to K, \quad [m, n]_\epsilon = \epsilon((m, n)).
$$

Obviously, $\langle S \rangle \subseteq \langle S \rangle$ and $\langle T \rangle \subseteq \langle T \rangle$.

Proposition 13. Let $S$ and $T$ be $A$–submodules of $N_A$ and $A M$, respectively. Then

$$
\langle S \rangle = \langle S \rangle, \quad \langle T \rangle = \langle T \rangle.
$$

Thus, $\langle S \rangle$ and $\langle T \rangle$ are $A$–submodules of, respectively, $A M$ and $N_A$. Moreover,

$$
\langle S \rangle = S, \quad \langle T \rangle = T.
$$
Proof. Let $a \in A, m \in M, s \in S$. If $m \in \langle S \rangle$, then, since $S$ is a submodule of $N_A$, we have
\[ 0 = \epsilon(m, sa) = \epsilon(\langle m, s \rangle, a) = \langle m, s \rangle, a \epsilon. \]
Now $a \in A$ is arbitrary and $(\langle - , - \rangle, \epsilon)$ is non degenerate, we thus get that $\langle m, s \rangle = 0$. For the proof of the second part, observe that the bilinear form $[- , -]_\epsilon$ is non degenerate: if $m \in M$ is such that $[m, n] = 0$ for all $n \in N$, then $m \in \langle N \rangle$. Hence, $[- , -]_\epsilon$ is left non degenerate. The argument on the right is similar. □

For a subset $S \subseteq A$, we have the left (resp. right) ideal of $A$ defined by
\[ \text{lann}_A(S) = \{ a \in A : as = 0 \ \forall s \in S \} \]
(resp. by)
\[ \text{rann}_A(S) = \{ a \in A : sa = 0 \ \forall s \in S \}. \]

Corollary 14. For any non projective Frobenius algebra $A$ with Frobenius functional $\epsilon : A \to K$, every right ideal $S$ of $A$, and every left ideal $T$ of $A$, we have
\[ \text{lann}_A(S) = \langle S \rangle, \text{ rann}_A(T) = \langle T \rangle. \]

Therefore,
\[ \text{rann}_A(\text{lann}_A(S)) = S, \quad \text{lann}_A(\text{rann}_A(T)) = T. \]

Proof. Setting $M = N = A$ in Proposition 13 and the bilinear form $\langle - , - \rangle$ to be the multiplication map of $A$, we get the statement, since $\langle S \rangle = \text{lann}_A(S)$, while $\langle T \rangle = \text{rann}_A(T)$. □

Remark 15. It follows from Corollary 14 that every non projective Frobenius algebra over a Frobenius commutative ring is a Quasi-Frobenius ring. Next section is devoted to sharpen this result.

5. Non projective Frobenius algebras and Frobenius rings

Let $R$ be an algebra over a commutative Frobenius ring $K$ with $R_K$ finitely generated. Let $J$ denote the Jacobson radical of $R$. The socle of a right (or left) $R$-module $X$ is the sum of all its simple $R$-submodules. It is well known that, if $X_R$ is finitely generated, then $\text{Soc}(X_R) = \{ x \in X : xJ = 0 \}$. Therefore, $\text{Soc}(R_R^*)$ may be computed as
\[ (1) \quad \text{Soc}(R_R^*) = \{ f \in R^* : f(J) = 0 \} \cong (R/J)^*_R \cong (R/J)_R, \]
where the last isomorphism of $R/J$–modules (and, hence, of $R$–modules) holds because $R/J$ is semisimple, and Example 12 applies.

Theorem 16. Let $R$ be an algebra over a commutative Frobenius ring $K$ with $R_K$ finitely generated. Then $R$ is a non projective Frobenius $K$–algebra if and only if $R$ is a Frobenius ring.

Proof. Assume first that $R$ is Frobenius as a ring. Then $\text{Soc}(R_R) \cong (R/J)_R$. By 1, $(R/J)_R \cong \text{Soc}(R_R^*)$. Now, $R_R$ is injective because $R$ is Frobenius, and $R_R^*$ is injective because $K$ is Frobenius. Since they have isomorphic essential socles, it follows that $R_R \cong R_R^*$, and $R$ is a non projective Frobenius algebra. Conversely, Corollary 14 implies that $R$ is Quasi-Frobenius. Since $R$ is non projective Frobenius, we have an isomorphism of $R$–modules $R_R \cong R_R^*$. Therefore, taking 1 into account, we get
\[ \text{Soc}(R_R) \cong \text{Soc}(R_R^*) \cong (R/J)_R. \]
Hence, $R$ is a Frobenius ring. □
Remark 17. Theorem 16 can be deduced also deduced from [6, Proposition 1.5], since the commutative Frobenius base ring $K$ is a minimal injective cogenerator of the category of $K$–modules.

Corollary 18. Let $R$ be a finite ring of characteristic $n$. Then $R$ is a Frobenius ring if and only if $R$ is a non projective Frobenius $\mathbb{Z}_n$–algebra.

Remark 19. A non projective Frobenius algebra needs not to be projective over its commutative base Frobenius ring. One of the simplest examples is the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_4$, which is clearly Frobenius and, by Corollary 18, a non projective Frobenius algebra over $\mathbb{Z}_4$. However, $R$ is not projective as a $\mathbb{Z}_4$–module, so it is not a Frobenius algebra in the classical setting.

Next, we will describe a method for constructing Frobenius rings from skew polynomial rings with coefficients in a non projective Frobenius algebra.

Let $A$ be a non projective Frobenius algebra over a Frobenius commutative ring $K$ with Frobenius functional $\epsilon : A \to K$, and associative non degenerate $K$–bilinear form

$$\langle -, - \rangle : A \times A \to K, \quad \langle a, b \rangle = \epsilon(ab).$$

Consider the skew polynomial ring $S = A[x; \sigma]$, where $\sigma$ is a $K$–algebra automorphism of $A$, and let $f = \sum_{i=0}^m f_i x^i \in S$ be a monic twosided polynomial, that is, $Sf = fS$. Since $Sf$ is a twosided ideal of $S$, we get the $K$–algebra $R = S/Sf$, which is finitely generated as a $K$–module because $f$ is monic. Indeed, every $g \in S$ can be written as $g = qf + r$, for suitable $q, r \in S$ with $r$ of degree smaller than $m$. This implies, always with $f$ monic, that $qR \cong A^m$. In other words, we will identify the elements of $R$ with polynomials in $S$ with degree less than $m$, with the operations made modulo $f$. For $g \in R$, the notation $g_0$ stands for its term of degree 0. Finally, we assume that $f_0$ is a unit of $A$.

Theorem 20. $R$ is a non projective Frobenius $K$–algebra with nondegenerate associative bilinear form

$$\langle -, - \rangle : R \times R \to K, \quad \langle g, h \rangle = \epsilon((gh)_0).$$

Therefore, $R$ is a Frobenius ring.

Proof. A straightforward computation modulo $f$ shows that, for $g = \sum_{i=0}^{m-1} g_i x^i, h = \sum_{i=0}^{m-1} h_i x^i \in R$, we have

$$\langle gh \rangle_0 = g_0 h_0 - \sum_{i=1}^{m-1} g_{m-i} \sigma^{m-i}(h_i) f_0.$$  \hfill (2)

Obviously, $\langle -, - \rangle$ is $K$–bilinear. In order to prove that it is non degenerate it suffices, by Remark 14, that $\langle -, - \rangle$ is right non degenerate. So, let $g \in R$ such that $\langle g, h \rangle = 0$ for all $h \in R$. Let $\lambda \in A$. Taking $h = \lambda \in A$ in (2), we get that

$$0 = \epsilon(g_0 \lambda) = \langle g_0, \lambda \rangle = \langle g_0, \lambda \rangle.$$  \hfill (3)

Since $\langle -, - \rangle$ is non degenerate, we deduce that $g_0 = 0$. Now, let $i \in \{1, \ldots, m\}$. Setting $h = \lambda x^i$ in (2), we have

$$0 = -\epsilon(g_{m-i} \sigma^{m-i}(\lambda) f_0) = -\langle g_{m-i}, \sigma^{m-i}(\lambda) f_0 \rangle.$$  \hfill (4)

Now, $\lambda \in A$ is arbitrary, $f_0 \in A$ is a unit, and $\sigma^{m-i}$ is an automorphism of $A$, so, we get from the non degeneracy of $\langle -, - \rangle$ that $g_{m-i} = 0$. Thus, $g = 0$, and $\langle -, - \rangle$ is non degenerate. By Theorem 16 $R$ is a Frobenius ring. \hfill $\square$
6. Codes with a Frobenius alphabet

Let $R$ be a finite ring of characteristic $n$, and consider $\mathbb{Z}_n \subseteq \mathbb{C}^\times$ as the group of $n$-th roots of unit. Recall that the abelian group $\mathbb{Z}_n$ has a unique structure of ring, and that it is Frobenius. Now, we have

$$\hat{R} = \text{hom}_{\mathbb{Z}}(R, \mathbb{C}^\times) = \text{hom}_{\mathbb{Z}}(R, \mathbb{Z}_n) = \text{hom}_{\mathbb{Z}_n}(R, \mathbb{Z}_n) = R^*.$$

Therefore, Corollary 18 in view of Theorem 8 can be rephrased by telling that $R$ is Frobenius if and only if $R$ has a generating character (namely, the Frobenius functional), as already proved in [9].

Let $A$ be a finite Frobenius ring of characteristic $n$, and Frobenius functional (or generating character) $\epsilon : A \to \mathbb{Z}_n$. Let $(-,-) : M \times N \to A$ a non degenerated bilinear form, where $A M$ and $N_A$ are finite $A$–modules. We know that $\text{lt}(A M) = \text{lt}(N_A)$ and, by Lemma $\frac{5}{3}$ $A M \cong A N^*$ and $N_A \cong M_A^*$. Let see that this framework cover the module-theoretical setting considered in [10].

Example 21. Consider an anti-automorphism $\theta : A \to A$, and a left module $A M$. We can consider the right $A$–module $N$ whose underlying additive group is $M$, with the right $A$–module structure defined by $a m = \theta^{-1}(a)m$ for all $a \in A, m \in M$.

Then, as already observed in [10] Remark 4.9 a non degenerate $A$–bilinear form $(-,-) : M \times N \to A$ is, precisely, a non degenerate sesquilinear form in the sense of [10] $\S3$. When $M$ has to be considered as the word ambient space for $A$–linear codes, a canonical choice is to put $M = A^n$ with its canonical left $A$–module structure.

Example 22. Of course, the same finite abelian group $M$ may support both a left $A$–module structure and a right $A$–module structure. This case, considered in [10] $\S4$, is clearly covered by our general formalism. Here, a natural choice is $M = A^n$ with its canonical $A$–bimodule structure.

Next, we will see how the results in [10] are derived from our general theory. So, fix a non degenerate $A$–bilinear form $(-,-) : M \times N \to A$, where $A$ is a finite Frobenius ring of characteristic $n$, and $A M$, $N_A$ are finite modules. Let $\epsilon : A \to \mathbb{Z}_n$ be a Frobenius functional (that, is, a generating character). According to examples 21 and 22 this setting covers all cases considered in [10].

Proposition 23. [10] Proposition 3.7 and Proposition 4.7] If $T \subseteq M$ is a left $A$–submodule and $S \subseteq N$ is a right $A$–submodule, then

$$(\epsilon(S))' = S = (\epsilon(S))^*, \quad (T')^* = T = \epsilon(T^*).$$

Proof. In view of Corollary 18 this proposition is a particular case of Proposition 13.

By $|X|$ we denote the cardinal of a finite set $X$.

Theorem 24. [10] Theorem 3.6 and Theorem 4.6] Let $S \subseteq N_A$ and $T \subseteq A M$ be submodules. Then $|S||S'| = |M| = |N|$ and $|T||T'| = |M| = |N|$.

Proof. Consider the non degenerate $\mathbb{Z}_n$–bilinear form $[-,-]_\epsilon : M \times N \to \mathbb{Z}_n$ defined by $[m,n]_\epsilon = \epsilon((m,n))$ for $m \in M, n \in N$. By Proposition 13 $\epsilon(S) = \epsilon(S)$, the latter being the left orthogonal of $S$ with respect to $[-,-]_\epsilon$. Now, by Remark 5 we have an isomorphism of $\mathbb{Z}_n$–modules $M/\epsilon(S) \cong \text{hom}_{\mathbb{Z}_n}(S, \mathbb{Z}_n)$. Therefore, $|M| = |S||\text{hom}_{\mathbb{Z}_n}(S, \mathbb{Z}_n)| = |\epsilon(S)||\epsilon(S)|$, since $\text{hom}_{\mathbb{Z}_n}(S, \mathbb{Z}_n)$ is nothing but the character group of $S$. Now, $M \cong \text{hom}_{\mathbb{Z}_n}(N, \mathbb{Z}_n)$, which gives that $|M| = |N|$.

Recall that the Hamming weight $wt(x)$ of a vector $x \in A^n$ is defined by the number of nonzero components of $x$. Given an additive code $C \subseteq A^n$, the Hamming
weight enumerator of $C$ is the complex polynomial in two variables $X, Y$

$$W_C(X, Y) = \sum_{x \in C} X^{n - wt(x)} Y^{wt(x)}.$$ 

A general version of McWilliams identity appears in [10, Theorem 5.2]. Unfortunately, it is not valid for every non degenerate bilinear form, as the following example shows.

**Example 25.** Set $A = \mathbb{F}_2$, and let $\langle \cdot, \cdot \rangle : \mathbb{F}_2^2 \times \mathbb{F}_2^2 \rightarrow \mathbb{F}_2$ the non degenerate $\mathbb{F}_2$–bilinear form defined by the non singular matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

that is, $\langle x, y \rangle = x^t Q y$ for all $x, y \in \mathbb{F}_2^2$. Let $C \subseteq \mathbb{F}_2^2$ the linear code $C = \{(0, 0), (1, 0)\}$. Its dual (with respect to $\langle \cdot, \cdot \rangle$) is $C = \{(0, 0), (1, 1)\}$. Therefore,

$$W_C(X, Y) = X^2 + XY,$$

$$W'_C(X, Y) = X^2 + Y^2.$$ 

If the McWilliams identity stated in [10, Theorem 5.2] was applicable to $\langle \cdot, \cdot \rangle$, then we would have

$$W'_C(X, Y) = \frac{1}{2} W_C(X + Y, X - Y) = X^2 + XY.$$ 

Therefore, the identities from [10, Theorem 5.2] do not hold for a general non degenerate bilinear form.

**Theorem 5.2** in [10] is a consequence of [9, Theorem 11.3] whenever the isomorphisms

$$\alpha : A^m \rightarrow \hat{A}^m, \quad \alpha(x)(y) = \epsilon(\langle x, y \rangle)$$

and

$$\beta : A^m \rightarrow \hat{A}^m, \quad \beta(x)(y) = \epsilon(\langle y, x \rangle)$$

are isometries with respect to the Hamming weights in $A^m$ and $\hat{A}^m$. This is the statement of [10, Lemma 5.3], which does not hold for an arbitrary non-degenerate $A$–bilinear form on $A^m$. The argument is valid, however, for suitable bilinear forms.

To be more precise, let $e_i$ denote, for $i = 1, \ldots, m$, the vector of $A^m$ whose only nonzero component is the $i$–th, which is 1.

**Theorem 26.** [10, Theorem 5.2] Let $\langle \cdot, \cdot \rangle : A^m \times A^m \rightarrow A$ a non degenerate $A$–bilinear form such that there is a generating character $\epsilon$ on $A$ such that $\epsilon(\langle e_i, e_j \rangle) = 0$ for all $i \neq j$. Let $C \subseteq A^m$ be a left (resp. right) $A$–linear code $C \subseteq A^m$, and set $D = C^*$ (resp. $D = C$). Then

$$W_D(X, Y) = \frac{1}{|C|} W_C(X + (|A| - 1)Y, X - Y).$$

**Proof.** Under the hypothesis on $\langle \cdot, \cdot \rangle$, the statement of [10, Lemma 5.3] holds true. Thus, proceed as in the proof of [10, Theorem 5.2].

We conclude this section by giving some examples of non projective Frobenius algebras for which the annihilators with respect to their associative bilinear forms are the euclidean duals with respect to certain bases. The notation $X^\perp$ will be used to denote the Euclidean dual of a given code $X$, with respect to the Euclidean product which will be clear in each situation.
Example 27. Let $G$ be a finite group, and consider the group algebra $A = \mathbb{Z}_n G$, which is a basic example of Frobenius $\mathbb{Z}_n$–algebra, with the associative nondegenerate bilinear form defined by

$$\left\langle \sum_{g \in G} \alpha_g g, \sum_{h \in G} \beta_h h \right\rangle = \sum_{g \in G} \alpha_g \beta_g^{-1}.$$ 

Note that, if we denote by $[\cdot, \cdot]$ the obvious euclidean bilinear form on $A$, then $[a, b] = (a, \theta(b))$, for all $a, b \in A$, where $\theta : A \to A$ denotes the involution determined by $\theta(g) = g^{-1}$ for $g \in G$. Hence, $S^\perp = \theta(S')$ for every subset $S$ of $A$, which implies that if $C$ is a left ideal of $A$, then $C^\perp$ is a left ideal of $A$, too.

Let $\sigma$ be an automorphism of a finite Frobenius ring $A$ of characteristic $n$. Consider a Frobenius functional (or generating character) $\epsilon : A \to \mathbb{Z}_n$, and the non degenerate $\mathbb{Z}_n$–bilinear form given by $\langle a, b \rangle = \epsilon(ab)$ for all $a, b \in A$. Let $m$ be a multiple of the order of $\sigma$. The polynomial $x^m - 1 \in S = A[x; \sigma]$ is central, so we may consider the finite ring $A = S/S(x^m - 1)$, which is a Frobenius ring and a non projective Frobenius $\mathbb{Z}_n$–algebra, according to Proposition [20] with the nondegenerate associative $\mathbb{Z}_n$–bilinear form

$$(3) \quad (\cdot, \cdot)_A : A \times A \to \mathbb{Z}_n, \quad (f, g) = \epsilon((fg)_0).$$

On the other hand, looking at $A$ as free $A$–module of rank $n$ with basis $\{1, x, \ldots, x^{m-1}\}$, we have the Euclidean $A$–bilinear form

$$[-, -] : A \times A \to A, \quad [f, g] = \sum_{i=0}^{n-1} f_i g_i.$$

Proposition 28. Let $\theta : A \to A$ be defined by

$$\theta \left( \sum_{i=0}^{m-1} f_i x^i \right) = \sum_{i=0}^{m-1} \sigma^{-i}(f_i)x^{-i}.$$ 

Then $\theta$ is a $\mathbb{Z}_n$–algebra involution. Moreover, for every left $A$–submodule $V$ of $A$, we have

$$V^\perp = \theta(V).$$

Proof. By [4] Lemma 26], $\theta$ is a $\mathbb{Z}_n$–algebra involution. Now, for $f, g \in A$, a straightforward computation gives that $(f \theta(g))_0 = [f, g]$. Therefore, by (3),

$$(4) \quad (f, \theta(g))_A = \epsilon([f, g]).$$

Given $f \in V^\perp$, and $\theta(g) \in \theta(V)$, we get from (4) that $(f, \theta(g))_A = 0$. Hence, $f \in \theta(V)$, and we obtain the inclusion of $\mathbb{Z}_n$–modules $V^\perp \subseteq \theta(V)$. Now, since $(\cdot, \cdot)_A$ and $[-, -]$ are non degenerate, we may apply Theorem [24] to both bilinear forms, and get

$$|\theta(V)| = \frac{|A|}{|\theta(V)|} = \frac{|A|}{|V|} = |V^\perp|$$

which implies the equality $V^\perp = \theta(V)$.

Following [2], left ideals of $A$ are called $\sigma$–cyclic codes. The following consequence generalizes [2] Corollary 18 from fields to finite Frobenius rings.

Corollary 29. If $C$ is a $\sigma$–cyclic code, then $C^\perp$ is a $\sigma$–cyclic code.

Proof. Since $C$ is a left ideal of $A$, we get that $\theta(C)$ is a right ideal of $A$ and, therefore, $\theta(C)$ is a left ideal. By Proposition 28 $C^\perp$ becomes a left ideal of $A$.  \qed
REFERENCES

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, 2nd edition. Springer, New York, 1992.

[2] D. Boucher and F. Ulmer, Coding with skew polynomial rings. Journal of Symbolic Computation 44 (2009), 1644–1656.

[3] S. Ellenberg and T. Nakayama, On the dimension of modules and algebras, II. Frobenius algebras and quasi-Frobenius rings. Nagoya Math. J. 9 (1955), 1-16.

[4] J. Gómez-Torrecillas, F. J. Lobillo and G. Navarro. Dual codes from annihilators: transpose Hamming ring extensions. Contemporary Mathematics, to appear. https://arxiv.org/abs/1803.00239

[5] F. Kasch, Modules and Rings, Academic Press, London, 1982.

[6] M.C. Iovanov. Frobenius–Artin algebras and infinite linear codes. J. Pure Appl. Algebra 220 (2016), 560–576.

[7] T. Y. Lam, Lectures on Modules and Rings, Springer-Verlag, Berlin, 1999.

[8] T. Nakayama. On Frobeniusean Algebras, I. Annals of Mathematics 40 (1939), 611-633.

[9] J. A. Wood, Duality for modules over finite rings and applications to coding theory, Amer. J. Math. 121 (1999), 555-575.

[10] S. Szabo and J. A. Wood, Properties of dual codes defined by nondegenerate forms. Journal of Algebra Combinatorics Discrete Structures and Applications 4 (2017), 105–113.

CITIC and Department of Algebra, University of Granada  
E-mail address: gomezj@ugr.es

Department of Mathematics and Computer Science, University of Puget Sound  
E-mail address: ehietaaho@pugetsound.edu

CITIC and Department of Algebra, University of Granada  
E-mail address: jlobillo@ugr.es

Department of Mathematics, Ohio University  
E-mail address: lopez@ohio.edu

CITIC and Department of Computer Science and AI, University of Granada  
E-mail address: gnavarro@ugr.es