Doubly coupled matter fields in massive bigravity

Xian Gao(高显)1,2,3;1) Lavinia Heisenberg4,2)

1 School of Physics and Astronomy, Sun Yat-sen University, Guangzhou 510275, China
2 Research Center for the Early Universe (RESCEU), Graduate School of Science, The University of Tokyo, Tokyo 113-0033, Japan
3 Department of Physics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro, Tokyo 152-8551, Japan
4 Institute for Theoretical Studies, ETH Zurich, Clausiusstrasse 47, 8092 Zurich, Switzerland

Abstract: In the context of massive (bi-)gravity, non-minimal matter couplings have been proposed. These couplings are special in the sense that they are free of the Boulware-Deser ghost below the strong coupling scale and can be used consistently as an effective field theory. Furthermore, they enrich the phenomenology of massive gravity. We consider these couplings in the framework of bimetric gravity and study the cosmological implications for background and linear tensor, vector, and scalar. Previous works have investigated special branches of solutions. Here we perform a complete perturbation analysis for the general background equations of motion, completing previous analyses.

Keywords: modified gravity, cosmological perturbation theory, inflation

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1 Introduction

High precision cosmological observations have made it possible to test the underlying fundamental theory of gravity. Together with the assumption of General Relativity (GR) being the right theory, and the cosmological principle, the universe is well described by the ΛCDM model. It constitutes a predominant amount of dark energy in form of a cosmological constant and dark matter. Aside from negligible reported anomalies [1], the model is still the best fit to current cosmological data [2-4]. In spite of its observational triumph, the model suffers from serious theoretical problems, the most persistent being the cosmological constant problem [5].

An alternative scenario for dark energy can be provided by infrared modifications of gravity. The simplest case corresponds to modifications in the form of an additional scalar field [6-10]. The presence of self-interactions of the scalar field and the non-minimal couplings to gravity yield interesting cosmological scenarios [11-19]. Other interesting dark energy scenarios can be accommodated by considering a vector field as an additional field. The question about the consistent self-interactions of the vector field, or similarly its non-minimal coupling to gravity, has been receiving renewed interest lately [20-28].

An unavoidable question is whether the graviton could be massive, which would correspond to a natural infrared modification of gravity, since the mediated force by a massive graviton would be suppressed at large scales. The weakening of the graviton could be put on an equal footing with recent cosmological acceleration. At the linear level the theory is described by the Fierz and Pauli mass terms [29] without introducing the ghostly sixth mode. This linear model, however, suffers from the vDVZ discontinuity [30, 31] when the mass of the graviton is set to zero, since General Relativity is not recovered in that limit. Actually, very soon after that, Vainshtein realized that the linear approximation breaks down at some distance far from the source and that nonlinear interactions become appreciable close to the source [32]. Usually, these non-linear interactions reintroduce the ghostly six mode, the Boulware-Deser ghost [33], and it was a challenging task to construct potential interactions which would propagate only five physical degrees of freedom [34-38]. This ghost-free theory of massive gravity is also technically natural and does not obtain strong renormalization by quantum corrections [39, 40].

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E-mail: gaoxian@mail.sysu.edu.cn
E-mail: lavinia.heisenberg@eth-its.ethz.ch

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In the context of quantum stability of the theory, new ways of coupling the matter fields have been explored [41–43]. The classical potential interactions had to be tuned in a very specific way to keep the Boulware-Deser ghost absent, and if one wants to keep this property also at the quantum level, only very restricted matter couplings through an effective composite metric are allowed. This effective metric is built out of the two metrics in such a way that the matter quantum loops would only introduce a running of the cosmological constant for the effective metric, which in other words correspond exactly to the allowed potential interactions. These doubly coupled matter fields in massive gravity revealed that ghost and gradient instabilities can be successfully avoided together with the strong coupling issues, since the vector and scalar perturbations maintain their kinetic terms [52]. The application to massive bimetric gravity yielded gradient instability in the vector sector and ghost instability in the scalar sector for one of the branches of solutions, whereas the other branch of solutions was free of any ghost instability. It is still an open question whether this second branch of solutions is also free from any gradient instabilities. The main purpose of the present work is to investigate the perturbation analysis of the doubly coupled matter fields on top of general background equations of motion, without specifying the branch and providing also the full quadratic action for the scalar perturbations. Thus, our work completes the analysis started in Ref. [56].

2 Dynamical composite metric

A consistent coupling of some extra scalar field $\phi$ to both metrics simultaneously was introduced in Ref. [41] through a composite metric $g_{\mu\nu}$,

$$\tilde{g}_{\mu\nu} = \alpha^2 g_{\mu\nu} + 2\alpha\beta g_{\mu\lambda}X^\lambda_\nu + \beta^2 f_{\mu\nu},$$

(1)

with $X^\mu_\nu$ defined by

$$X^\mu_\nu X^\lambda_\nu \equiv g^\mu\lambda f_{\mu\nu}.$$  

(2)

We consider the same action as in Ref. [56],

$$S = S^g + S^f + S^{pot} + S^{com},$$

(3)

with

$$S^g = \int d^4x\sqrt{-g}\left(\frac{M^2}{2}R[g] + L^{matter}[g]\right),$$

(4)

$$S^f = \int d^4x\sqrt{-f}\left(\frac{M^2}{2}R[f] + L^{matter}[f]\right),$$

(5)

$$S^{pot} = \int dt \int d^3x\sqrt{-g}M^2m^4\sum_{n=0}^4c_n\epsilon_n(X),$$

(6)

$$S^{com} = \int d^4x\sqrt{-g}P(\bar{X},\phi),$$

(7)

where $R[g]$ and $R[f]$ are Ricci scalars for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. As in Ref. [56], in this work we consider the matter contents of the $g_{\mu\nu}$ and $f_{\mu\nu}$ metrics to be two cosmological constants: $L^{matter}[g] = -M^2\Lambda_g$ and $L^{matter}[f] = -M^2\Lambda_f$. $S^{pot}$ denotes the non-derivative potential interactions $S^{pot}$ of the two metrics, $X$ stands for $X^\mu_\nu$, and for a matrix $M^\mu_\nu$, $\epsilon_n(M)$ are the elementary symmetric polynomials defined by

$$\epsilon_n(M) \equiv n!M^\mu_{[\nu_1}M^\nu_{\mu_2}...M^\mu_{\nu_n]},$$

(8)

where the antisymmetrization is unnormalized. In Eq. (7), $\bar{X}$ denotes the canonical kinetic term of $\phi$ in terms of the composite metric,

$$\bar{X} \equiv \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi.$$  

(9)

In the following we will study this action on the FLRW background and establish our parametrization for linear perturbations.

3 Cosmological parametrization

We parametrize the two metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ to be

$$g_{\mu\nu} d\tau^\mu d\tau^\nu = -N_s^2 \left( e^{2A} - (e^{-\rho})^\nu \right) B_i B^i d\tau^2$$

$$+ 2N_s a_i B_i d\tau dx^i + \alpha^2 (e^H)^\nu_\nu dx^i dx^i,$$

(10)

$$f_{\mu\nu} d\tau^\mu d\tau^\nu = -N_f^2 \left( e^{2\varphi} - (e^{-\gamma})^\nu \right) \Omega_i \Omega^i d\tau^2$$

$$+ 2N_f a_f \Omega_i d\tau dx^i + \alpha_f^2 (e^F)^\nu_\nu dx^i dx^i,$$

(11)

where $N_s$, $a_s$, $N_f$ and $a_f$ are functions of time only, and the matrix exponentials are defined perturbatively.
as \((e^{H})_{ij}\equiv \delta_{ij} + H_{ij} + \frac{1}{2} H_{i}^{k} H_{kj} + \mathcal{O}(H^{3})\) and \((e^{-H})_{ij} = \delta_{ij} - H_{ij} + \frac{1}{2} H_{i}^{k} H_{kj} + \mathcal{O}(H^{3})\), etc. Throughout this paper, spatial indices are raised and lowered by \(\delta_{ij}\) and \(\delta^{ij}\). We further decompose (with \(\partial^{2}=\delta^{ij}\partial_{i}\partial_{j}\))

\[
B_{i} \equiv \partial_{i} B + S_{i}, \quad H_{ij} \equiv 2 \zeta \delta_{ij} + \left(\partial_{i} \partial_{j} - \frac{1}{3} \delta_{ij} \partial^{2}\right) E + \partial_{i} (F_{j} + h_{ij}),
\]

\[
\Omega_{i} \equiv \partial_{i} \omega + \sigma_{i},
\]

\[
\Gamma_{ij} \equiv 2 \psi \delta_{ij} + \left(\partial_{i} \partial_{j} - \frac{1}{3} \delta_{ij} \partial^{2}\right) \chi + \partial_{i} (\xi_{j} + \gamma_{ij}),
\]

with \(\partial_{i} (F_{j}) \equiv \frac{1}{2} \left(\partial_{i} F_{j} + \partial_{j} F_{i}\right)\), etc., and

\[
\partial^{i} S_{i} = \partial^{i} F_{i} = \partial^{i} \sigma_{i} = \partial^{i} \xi_{i} = 0, h_{ij} = \gamma_{ij} = 0, \partial^{i} h_{ij} = \partial^{i} \gamma_{ij} = 0.
\]

Accordingly, it is convenient to parametrize the composite metric to be

\[
\bar{g}_{\mu \nu} d x^{\mu} d x^{\nu} = - \tilde{N}^{2} \left( e^{2 \bar{A}} - (e^{-H})^{ij} \bar{B}_{i} \bar{B}_{j}\right) d t^{2} + 2 \tilde{N} \bar{a} \bar{B}_{i} d t d x^{i} + a^{2} (e^{H})_{ij} d x^{i} d x^{j},
\]

where \(\tilde{N} \equiv a N + N \bar{a} \), \(\bar{a} \equiv a + \beta a_{i}\).

(18)

Similar to Eqs. (12)-(15), we may also decompose

\[
\bar{B}_{i} \equiv \partial_{i} \bar{B} + \bar{S}_{i}, \quad \bar{H}_{ij} \equiv 2 \zeta \delta_{ij} + \left(\partial_{i} \partial_{j} - \frac{1}{3} \delta_{ij} \partial^{2}\right) \bar{E} + \partial_{i} (\bar{F}_{j} + \bar{h}_{ij}),
\]

with \(\partial^{i} \bar{S}_{i} = \partial^{i} \bar{F}_{i} = \partial^{i} \bar{h}_{ij} = \partial^{i} \bar{h}_{ij} = 0\). Note \(\bar{A}\) etc. are expressed in terms of \(\{A, B_{i}, H_{ij}, \varphi, \Omega_{i}, \Gamma_{ij}\}\) as

\[
\bar{A} = \sum_{n=1}^{n} \bar{A}^{(n)}(A, B_{i}, H_{ij}, \varphi, \Omega_{i}, \Gamma_{ij})
\]

(20)

etc., where \(n\) denotes the order in \(\{A, B_{i}, H_{ij}, \varphi, \Omega_{i}, \Gamma_{ij}\}\). At the linear order, we have, for the scalar modes,

\[
\bar{A}^{(1)} = a \cdot \frac{N_{f} \bar{a} + a_{i} \bar{N}}{N}, \quad \bar{B}^{(1)} = a \cdot r_{1} B + a \cdot r_{2} \omega, \quad \bar{\zeta}^{(1)} = a \cdot \frac{\zeta + \beta a_{i}}{a \cdot \chi},
\]

(21)

\[
\bar{E}^{(1)} = a \cdot \frac{E + \beta a_{i}}{a \cdot \chi},
\]

(22)

with

\[
r_{1} = \frac{a \cdot N_{f} \bar{a} + a_{i} \bar{N}}{N_{f} a + a N_{f}} \bar{N}, \quad r_{2} = \frac{a_{i} N_{f} \left(N_{f} \bar{a} + a \bar{N}\right) \bar{N}}{N a_{i} + a N_{f} \bar{N}},
\]

(23)

(24)

(25)

for the vector modes,

\[
\bar{S}_{i}^{(1)} = a \cdot r_{i} S_{i} + a \cdot r_{j} \sigma_{i}, \quad \bar{F}_{i}^{(1)} = a \cdot \frac{a_{i}}{a} F_{i} + a \cdot \frac{a_{i}}{a} \xi_{i},
\]

(26)

and for the tensor modes,

\[
\bar{h}_{ij}^{(1)} = a \cdot \frac{a_{i}}{a} h_{ij} + \beta \cdot \frac{a_{i}}{a} \gamma_{ij}.
\]

(27)

The background equations of motion can be determined by requiring the vanishing of the first order action of \(A, \zeta, \varphi\) and \(\delta \varphi\), which is given by

\[
S_{1} = \int d t d x^{3} \bar{g}^{a_{i} a_{j}} \left( \bar{E}_{A} A + \bar{E}_{\xi} \zeta + \bar{E}_{\varphi} \varphi + \bar{E}_{\delta \varphi} \delta \varphi\right).
\]

(28)

The set of equations of motion are thus given by

\[
\bar{E}_{A} \equiv M_{f}^{2} (3 H_{f}^{2} - A_{f}) + \bar{E}_{\varphi}^{\text{pot}} + \frac{a_{i}^{3}}{a_{f}^{3}} \left(P - 2 \bar{X} \bar{P}_{X}\right) = 0, \quad \bar{E}_{\xi} \equiv M_{f}^{2} \left(3 H_{f}^{2} + \frac{2}{N_{f}} \frac{d H_{f}}{d t} - A_{f}\right) + \bar{E}_{\varphi}^{\text{pot}} + \frac{a_{i}^{3}}{a_{f}^{3}} N_{f} a_{g} = 0,
\]

(29)

(30)

(31)

with \(P_{X}\) is the shorthand for \(\partial P / \partial X\), and \(H_{g}\) and \(H_{f}\) are the Hubble parameters associated with the two metrics respectively, i.e.,

\[
H_{g} \equiv \frac{1}{N_{g} a_{g} d t}, \quad H_{f} \equiv \frac{1}{N_{f} a_{f} d t}.
\]

(33)

In the above,

\[
\bar{E}_{\varphi}^{\text{pot}} = M_{f}^{2} m^{2} \left(c_{0} + 3 a_{f} c_{1} + 6 a_{f}^{2} c_{2} + 6 a_{f}^{3} c_{3}\right), \quad \bar{E}_{\zeta}^{\text{pot}} = b_{1} + a_{s} N_{f} a_{f},
\]

(34)

(35)

\[
\bar{E}_{\varphi}^{\text{pot}} = M_{f}^{2} m^{2} N_{f} \left(c_{1} + 6 a_{f} c_{2} + 18 a_{f}^{2} c_{3} + 24 a_{f}^{3} c_{4}\right), \quad \bar{E}_{\zeta}^{\text{pot}} = b_{2} + b_{3},
\]

(36)

(37)

where we have introduced

\[
b_{1} \equiv M_{f}^{2} m^{2} \left(c_{0} + 2 a_{f} c_{1} + 2 a_{f}^{2} c_{2}\right) N_{f}, \quad b_{2} \equiv M_{f}^{2} a_{f}^{2} c_{1} \left(c_{0} + 4 a_{f} c_{2} + 6 a_{f}^{2} c_{3}\right), \quad b_{3} \equiv 2 M_{f}^{2} m^{2} N_{f} a_{s} \left(c_{2} + 6 a_{s} c_{3} + 12 a_{s}^{2} c_{4}\right),
\]

(38)

(39)

(40)

for later convenience. The equation of motion for the scalar field is given by

\[
\bar{E}_{\varphi} = P_{\varphi} \frac{1}{\bar{N} a^{3} d t} \frac{\left(a^{3} d \bar{\varphi}\right)}{\left(\bar{N} d t P_{X}\right)}.
\]

(41)
4 Cosmological perturbations

The quadratic action for the two tensor perturbations $h_{ij}$ and $\gamma_{ij}$ is given by

$$S_{2}^{\text{tensor}} = \frac{1}{8} \int dt \frac{d^3k}{(2\pi)^3} \left[ N_s a^3 M_s^2 \left( \frac{1}{N^2} \hat{h}^2_{ij} - \frac{k^2}{a^2} \hat{h}^2_{ij} \right) + N_f a^3 M_f^2 \left( \frac{1}{N^2} \hat{\gamma}^2_{ij} - \frac{k^2}{a^2} \hat{\gamma}^2_{ij} \right) + N_g a^3 M^2 (h_{ij} - \gamma_{ij})(h^{ij} - \gamma^{ij}) \right].$$ \hspace{1cm} (42)

where $\hat{a}$ denotes the derivative with respect to $t$,

$$\mathcal{M}^2 = \frac{a_f}{a_g} \left[ M_s^2 m^2 \left( \frac{1}{a_g} S_i - \frac{1}{2N^2} E_i \right)^2 + \frac{1}{4} N_f a_f M_f^2 k^2 \left( \frac{1}{a_g} \sigma_i - \frac{1}{2N^2} \xi_i \right)^2 - \frac{1}{2} N_g a_g c \left( S_i - \frac{a_g a_f}{N a_f} \sigma_i \right)^2 + \frac{N_g a_g^3}{16} \mathcal{M}^2 k^2 (F_i - \xi_i)^2 \right],$$ \hspace{1cm} (43)

where $\mathcal{M}^2$ is given in Eq. (43) and we also introduce

$$C \equiv \frac{1}{1 + \frac{a_g a_f}{N a_f}} \left[ \hat{N} \hat{a} \right] \left( 1 + \frac{a_f N_f}{N a_f} \right) \left( \frac{1}{a_g} S_i - \frac{1}{2N^2} E_i \right)^2 + \frac{a_f N_f}{N a_f} \left( \frac{1}{a_g} \sigma_i - \frac{1}{2N^2} \xi_i \right)^2 + \frac{N_g a_g^3}{16} \mathcal{M}^2 k^2 (F_i - \xi_i)^2 \right],$$ \hspace{1cm} (44)

where $\mathcal{M}^2$ is given in Eq. (43) and we also introduce

$$C \equiv \frac{1}{1 + \frac{a_g a_f}{N a_f}} \left[ \hat{N} \hat{a} \right] \left( 1 + \frac{a_f N_f}{N a_f} \right) \left( \frac{1}{a_g} S_i - \frac{1}{2N^2} E_i \right)^2 + \frac{a_f N_f}{N a_f} \left( \frac{1}{a_g} \sigma_i - \frac{1}{2N^2} \xi_i \right)^2 + \frac{N_g a_g^3}{16} \mathcal{M}^2 k^2 (F_i - \xi_i)^2 \right],$$ \hspace{1cm} (45)

with $b_2$ given in Eq. (39) for short. Since the vector modes $S_i$ and $\sigma_i$ have no dynamics in Eq. (44), we may solve them in terms of $F_i$ and $\xi_i$ and arrive at the reduced action for $F_i$ and $\xi_i$, which is given by

$$S_{2}^{\text{vector}} = \frac{1}{16} \int dt \frac{d^3k}{(2\pi)^3} N_s a^3 \hat{a}^2 k^2 \left[ \mathcal{G}_v \left( \hat{a} \left( \partial_t (F_i - \xi_i) \right) \right)^2 + \mathcal{M}^2 (F_i - \xi_i)^2 \right],$$ \hspace{1cm} (46)

with

$$\mathcal{G}_v = \left( \frac{a^3 N_f}{a^2_f N_g} \frac{1}{M^2_f} + \frac{1}{M^2_s} - \frac{k^2}{2C a^2_g} \right)^{-1}.$$ \hspace{1cm} (47)

From Eq. (46) it is clear that there are two vectorial degrees of freedom given that $\beta \neq 0$, which can be identified as $F_i - \xi_i$. For the stability condition we have to impose $\mathcal{G}_v > 0$.

We study now the linear stability of the scalar modes in our model. Initially we have 9 scalar modes, of which four ($A, B, \zeta$ and $E$) are from $g_{\mu\nu}$, four ($\varphi, \omega, \psi$ and $\chi$) are from $f_{\mu\nu}$, and one is the perturbation of the scalar field $\delta \phi$. In order to simplify the calculation, we choose a gauge in which $\delta \phi = \chi = 0$. In the residual 7 modes, only 2 modes are dynamical, which can be conveniently chosen to be

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Q \\ E \end{pmatrix},$$ \hspace{1cm} (48)

with

$$Q = -\frac{k^2}{6} E + \frac{\beta H_s}{\alpha H_f} \psi.$$ \hspace{1cm} (49)

After some manipulations, the final quadratic action for these two scalar modes takes the following general structure (in matrix form),

$$S_{2}^{\text{scalar}} = \frac{1}{2} \int dt \frac{d^3k}{(2\pi)^3} \left( \tilde{V}^T \mathcal{G} \tilde{V} + \tilde{V}^T \mathcal{F} \tilde{V} + \tilde{V}^T \mathcal{W} \tilde{V} \right),$$ \hspace{1cm} (50)

where $\mathcal{G}_{mn}$ and $\mathcal{W}_{mn}$ are symmetric while $\mathcal{F}_{mn}$ is anti-symmetric, which are given by

$$\mathcal{G}_{mn} = \Xi_{mn} - \frac{1}{D} A_m A_n,$$ \hspace{1cm} (51)

$$\mathcal{F}_{\mu = 1} \equiv - \mathcal{F}_{\mu = 2} = \frac{1}{D} (D_1 A_2 - D_2 A_1),$$ \hspace{1cm} (52)

$$\mathcal{W}_{mn} = - \frac{1}{D} D_m D_n - \frac{1}{2} d \left[ \frac{1}{D} (D_m A_n + D_n A_m) \right],$$ \hspace{1cm} (53)

with $m, n = 1, 2$. In Eqs. (51)-(52), we have

$$D = \frac{\beta^2 H^2_s}{\alpha^2 H^2_f} \left[ \left( \frac{d}{dt} \ln \frac{H_s}{H_f} \right)^2 \Xi_{11} + \Xi_{44} - \frac{d \Xi_{14}}{dt} \right] - \frac{d \Xi_{16}}{dt} + \frac{d \Xi_{14}}{dt} \frac{d \Xi_{16}}{dt},$$ \hspace{1cm} (54)

$$D_1 = \Xi_{16} - \Xi_{14} \frac{d \Xi_{14}}{dt} + \frac{\beta H_s}{\alpha H_f} \left( \frac{d \Xi_{14}}{dt} - \Xi_{14} \right),$$ \hspace{1cm} (55)

$$D_2 = \frac{d \Xi_{16}}{dt} + \Xi_{15} + \frac{k^2}{6} \left( \frac{d \Xi_{14}}{dt} - \Xi_{14} \right) + \frac{\beta H_s}{\alpha H_f} \left( \frac{k^2}{6} \left( \Xi_{14} \frac{d \Xi_{14}}{dt} - \Xi_{14} \right) \right),$$ \hspace{1cm} (56)

$$A_1 \equiv \Xi_{14} - \Xi_{16} + \frac{\beta}{\alpha} \frac{d}{dt} \left( \frac{H_s}{H_f} \right) \Xi_{11},$$ \hspace{1cm} (57)

$$A_2 \equiv \Xi_{14} - \Xi_{15} + \frac{k^2}{6} \left( \Xi_{14} - \Xi_{15} \right) + \frac{\beta H_s}{\alpha H_f} \left( \Xi_{12} - \Xi_{14} \right) \frac{d}{dt} \left( \frac{H_s}{H_f} \right),$$ \hspace{1cm} (58)
and
\[ B_{11} = B_{21} = \Xi_{44} - \frac{k^2}{6} \Xi_{44} - \frac{1}{2} \frac{d}{dt} \left( \Xi_{15} + \Xi_{24} - \frac{k^2}{3} \Xi_{14} \right) \] (61)
\[ B_{22} = \frac{k^4}{36} \Xi_{44} - \frac{k^2}{3} \Xi_{14} + \Xi_{25} - \frac{d}{dt} \left( \frac{k^4}{36} \Xi_{14} - \frac{k^2}{6} \Xi_{15} - \frac{k^2}{6} \Xi_{24} + \Xi_{25} \right), \] (62)
where \( \Xi_{ij} \) with \( i,j = 1,\ldots,6 \) are given in Appendix. Up to now, no approximation has been made in deriving the above expressions.

Unlike the tensor and vector modes, the lengthy expressions in the above make the analysis for the scalar modes rather cumbersome. In the following, we analyze the instabilities in the small scale limit \( k \to \infty \). For the kinetic terms, we have
\[ \mathcal{G}_{11} = \mathcal{G}_{11} + \mathcal{O}(k^{-2}), \quad \text{and} \quad \mathcal{G}_{22} = k^2 \mathcal{G}_{22} + \mathcal{O}(k^0), \] (63)
where
\[ \mathcal{G}_{11} = -\frac{\alpha^2}{N^2 H^2} \frac{d^2\tilde{\Phi}}{d\tilde{\Phi}}^2 (P_X + 2\tilde{X} \dot{P_X}), \] (64)
and
\[ \mathcal{G}_{22} = -\frac{Ca_g^2}{4N_g} - \frac{1}{\bar{D}} \left( \bar{A}_2 \right)^2, \] (65)
with
\[ \bar{D} = M_g^2 N_g a_M \left[ \frac{2a_g^2}{\alpha} \left( \frac{1}{H_f} \frac{d}{N_g dt} \left( \frac{H_g}{H_f} \right) - \frac{H_g^2}{H_f^2} \right) \right. \]
\[ + \frac{2a_g^2 M_f^2 N_f}{a_M^2 M_g^2 N_g} - \frac{Ca_g^2 N_f}{a_M^2 H_f M_g} \left( 1 + \frac{\beta a_g^2 N_f}{\alpha a_M^2 N_f} \right)^2 \]
\[ - \left. \frac{1}{M_g^2 a_g N_g dt} \frac{d}{dt} \left( \frac{a_g^2 M_f^2}{H_f^2} + \frac{\beta^2 a_M^2 H_g^2}{M_g^2} \right) \right], \] (66)
and
\[ \bar{D}_2 = a_g^2 N_g \left\{ \frac{\beta a_g^2}{\alpha} \left[ (P-2\tilde{X} \dot{P_X}) + b_1 + \mathcal{E}_g \right] \right\} 1 \frac{1}{2H_g N_g dt} \frac{d}{dt} \left( H_g \right) + \frac{1}{2} \left( 1 + \frac{\beta H_g}{\alpha H_f} \right) H_g^2 + \frac{3H_g^2}{2\alpha H_f^2} M_f^2 \left( 3H_g^2 + \frac{2}{N_g} \frac{dH_g}{dt} - A_g \right) \]
\[ + \frac{3\alpha \beta \tilde{a}_N P}{2a_M N_g} \left( \frac{H_g}{H_f} - \frac{a_f}{a_g} \right) + \frac{3}{2} \frac{m^2 M_g}{a_M^2 H_f^2} \left( c_0 + c_1 \frac{a_f}{a_g} + N_f_N_g \right) + 2c_2 \frac{a_f}{a_g} N_f_N_g - \frac{a_f}{a_g} \left( c_1 + 2c_2 \frac{a_f}{a_g} + N_f_N_g \right) + 6c_3 \frac{a_f}{a_g} N_f_N_g \]
\[ - \frac{C}{4} \frac{1}{H_f H_g M_g^2} \left( \frac{\beta a_g^2 N_f}{a_M^2 N_g} \right) \]
\[ \times \left\{ - \frac{a_g^2}{a_f} \left( P-2\tilde{X} \dot{P_X} \right) - \frac{1}{\alpha} \left( b_1 - M_g^2 A_g + 3M_g^2 H_g^2 \right) + \frac{a_g^2 N_f H_g M_f^2}{a_M^2 N_g H_f M_f} \left( \frac{\beta a_g^2}{a_f} \left( P-2\tilde{X} \dot{P_X} \right) - \frac{a_f b_2}{a_M^2 H_f} \right) \right\} - \frac{1}{2} \frac{d}{dt} \frac{a_g^2}{\alpha H_f} \left( b_1 - M_g^2 A_g + 3M_g^2 H_g^2 \right) - \frac{a_f b_2}{a_M^2 H_f} \right]. \] (73)
Thus, in the large $k$ limit, the absence of gradient instability requires
\[ \dot{\mathcal{W}}_{11} > 0, \quad \text{and} \quad \dot{\mathcal{W}}_{22} > 0. \] (74)

The propagating speeds of the two scalar modes are given by the eigenvalues of $\mathbf{G}^{-1} \mathbf{W}$, which correspond to
\[ c_i^2 = \frac{\dot{\mathcal{W}}_{ii}}{\mathcal{G}_{ii}} \quad \text{and} \quad c_2^2 = \frac{\dot{\mathcal{W}}_{22}}{\mathcal{G}_{22}} \] (75)
in the same limit.

5 Conclusion

In this work, we have investigated the cosmological perturbation analysis of the bimetric theory with a scalar field coupled simultaneously to both metrics in terms of a composite metric. The scalar field represents the matter field that lives on both metrics.

The ghost and gradient instabilities of the tensor and vector modes as well as the ghost instabilities of the scalar modes of the same model have been analyzed in Ref. [56] for some concrete background evolution, while in this work we complete the analysis by presenting the full quadratic action for the scalar modes (Eq. (50)) as well as the conditions for the absence of gradient instabilities (Eq. (74)) on general background evolution in the presence of matter fields. Although in this work we focus on the small scale limit $k \rightarrow 0$ due to the lengthy expressions, the results presented in this work enable one to make further analysis in different limits as well as on concrete background solutions.

Moreover, we consider only the coupling of the scalar field to the composite metric in a minimal way, while in principle one may consider non-minimal derivative couplings, as was pointed out in Ref. [64]. This bimetric model with doubly coupled matter fields offers an interesting cosmological framework. In one branch of solutions, in which the Hubble rates are proportional to each other, this interesting phenomenology is plagued by the ghost and gradient instabilities, as was shown in Ref. [56]. However, in the other branch of background cosmology with the algebraical ratio between the scale factors of the two metrics, there are no ghost instabilities associated with the vector and scalar perturbations. Here, we also show the conditions for the absence of the gradient instabilities for the scalar perturbations, which were lacking in the literature. Fulfilling all these instability conditions, this branch of solutions still offers a promising dark energy model, which has a very rich phenomenology [65].

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Appendix A

Expressions of $\Xi_{ab}$

The expressions of $\Xi_{ab}$ with $a, b = 1, \cdots, 6$ are given by:
\[ \Xi_{11} = \frac{1}{\Delta} \left( \frac{16}{N^3} \alpha^2 \right) (\frac{d\bar{\phi}}{dt})^2 \left[ k^6 \bar{\phi} \right] \bar{a}^3 a_f a_j a_g a_\phi \frac{H_f H_j M_f M_j N_f N_j}{N_f} \left[ \bar{a}_3 M_j^2 N_f (3\bar{C}_a^5 - 2k^2 M_j^2) + 3\bar{C}_a^5 M_j^2 N_f \right] \] (A1)
\[ \Xi_{12} = \frac{1}{\Delta} \left( \frac{8}{3N^3} \alpha \right) (\frac{d\bar{\phi}}{dt})^2 k^8 \bar{\phi} \bar{a}_3 a_f a_j a_g a_\phi \frac{H_f M_j^2 M_f^2 N_f^2}{N_f} \left[ 2\alpha k^2 a_j H_f M_f^2 N_f - 3\bar{C}_a^5 N_f (\alpha H_f + \beta H_f) \right], \] (A2)
\[ \Xi_{14} = \frac{1}{\Delta} \left( \frac{8k^6 a_f^2 M_f^2 N_f^2}{N^3 a_j^2} \right) \left\{ \bar{C} \left( \frac{d\bar{\phi}}{dt} \right) \bar{a}_3 a_g a_\phi (\alpha F_{2a} N_g - \beta F_{14} N_f) \left( \bar{a}_3 H_f M_f^2 N_f - 3\bar{C}_a^5 H_f M_f^2 N_f \right) \right\}, \] (A3)
\[ \Xi_{15} = \frac{1}{\Delta} \left( \frac{8k^{10} a_f^5 M_f^4 N_f}{3N^3 a_j^2} \right) \left\{ \bar{C} \left( \frac{d\bar{\phi}}{dt} \right) \bar{a}_3 a_g a_\phi a_j H_f M_f^2 N_f \left( 3\bar{C}_a^5 H_f M_f^2 N_f - \bar{a}_3 H_f M_f^2 N_f \right) \right\}, \] (A4)
\[ \Xi_{16} = \frac{1}{\Delta} \left( \frac{8k^6 a_f^2 M_f^2 N_f^2}{N^3 a_j^2} \right) \left\{ \bar{C} \left( \frac{d\bar{\phi}}{dt} \right) \bar{a}_3 a_g a_\phi (\alpha F_{2a} N_g - \beta F_{16} N_f) \left( \bar{a}_3 H_f M_f^2 N_f - 3\bar{C}_a^5 H_f M_f^2 N_f \right) \right\}, \] (A5)
\[ \Xi_{22} = -\frac{1}{4} \frac{4}{N_a \alpha} k^{10} a_f a^2_H f M^2 f M^2 g N^2 g \left\{ 9c \bar{N} a_f^3 a^3_H f H^2 f M^2 f M^2 g \\
- \alpha \left( \frac{\partial \bar{g}}{\partial t} \right)^2 \bar{a}^3 g_{g g} (a_f^3 H_f M_f^2 N_g (2k^2 M^2_g + 3c a_f^2) - 3c a_f^2 N^2 g (c H_f + 2c H_g) \right\} \right\}, \]  
(A6)

\[ \Xi_{24} = \frac{1}{4} \frac{4}{N^{3} a^2_f} k^{12} a_f a^3_H f M^2 g N^2 g \left\{ c \left( \frac{\partial \bar{g}}{\partial t} \right)^2 \bar{a}^3 a^2 g_{g g} (c F_{24} N_g - \beta F_{14} N_f) (a_f^2 H_f M_f^2 N_g - 3c a_f^2 M^2 g N^2 g \right\} \right\}, \]  
(A7)

\[ \Xi_{25} = \frac{1}{4} \frac{4}{N^{3} a^2_f} k^{12} a_f a^3_H f M^2 g N^2 g \left\{ c \left( \frac{\partial \bar{g}}{\partial t} \right)^2 \bar{a}^3 a^2 g_{g g} N_f (a_f^2 H_f M_f^2 N_g - a_f^2 H_f M_f^2 N_g) \right\} \right\}, \]  
(A8)

\[ \Xi_{26} = \frac{1}{4} \frac{4}{N^{3} a^2_f} k^{12} a_f a^3_H f M^2 g N^2 g \left\{ c \left( \frac{\partial \bar{g}}{\partial t} \right)^2 \bar{a}^3 a^2 g_{g g} (c F_{24} N_g - \beta F_{14} N_f) (a_f^2 H_f M_f^2 N_g - 3c a_f^2 M^2 g N^2 g \right\} \right\}, \]  
(A9)

\[ \Xi_{34} = \frac{1}{4} \frac{8}{N^{3}} k^{6} a_f a^2 a^3_H f M^2 g N^2 g \left\{ F_{24} \bar{N} a_f a^2 a_f H_f H^2 f M^2 g \left( a_f^2 M^2 f N_g (3c a_f^2 - 2k^2 M^2_g) + 3c a_f^2 M^2 g N^2 g \right) \right\}, \]  
(A10)

\[ \Xi_{35} = \frac{1}{4} \frac{8}{N^{3}} \beta c \left( \frac{\partial \bar{g}}{\partial t} \right)^2 k^{10} \bar{a}^3 a_f a^2 g_{g g} M^2 f M^2 g N^2 g \left( a_f^2 H_f M_f^2 N_g - 3c a_f^2 M^2 g N^2 g \right), \]  
(A11)

\[ \Xi_{36} = \frac{1}{4} \frac{8}{N^{3}} k^{6} a_f a^2 a^3_H f M^2 g N^2 g \left\{ F_{24} \bar{N} a_f a^2 a_f H_f H^2 f M^2 g \left( a_f^2 M^2 f N_g (3c a_f^2 - 2k^2 M^2_g) + 3c a_f^2 M^2 g N^2 g \right) \right\}, \]  
(A12)

\[ \Xi_{44} = \frac{1}{4} \frac{4}{N^{3} a^2_f} k^{10} a^2 a^2_H f M^2 g N^2 g \left\{ \bar{N}^3 a^3_H f M^2 f \left( c F_{14}^2 - 4k^2 M^2 f M^2 g N^2 g + 6c M^2 f a^2_H f M^2 f M^2 g \right) \right\}, \]  
(A13)

\[ \Xi_{45} = \frac{1}{4} \frac{4}{N^{3} a^2_f} k^{10} a^2 a^2_H f M^2 g N^2 g \left\{ \bar{N}^3 a^3_H f M^2 f \left[ c a_f^3 H_f M^2 f N^2 g (F_{24} - 6a_f^3 H_f H_g M^2 g N^2 g) \right] \right\}, \]  
(A14)

\[ \Xi_{46} = \frac{1}{4} \frac{4}{N^{3} a^2_f} k^{10} a^2 a^2_H f M^2 g N^2 g \left\{ \bar{N}^3 a^3_H f M^2 f \left[ c a_f^3 H_f M^2 f N^2 g (F_{24} - 6a_f^3 H_f H_g M^2 g N^2 g) \right] \right\}, \]  
(A15)
\[
\Xi_{55} = -\frac{1}{\Delta} \left\{ N^3 \left[ a \gamma^2 H_f M_f^2 M_g^2 \left( a^2 M_f^2 N_g (C k^8 a M_g^2 N_f + H_f^2 (54 C M_{55} a_g^2 - 36 k^2 M_{55} M_g^2)) + 54 C M_{55} a_g^2 H_f M_g^2 N_f \right) - 18 C \left( \frac{d\phi}{dt} \right)^2 M_{55} a^2 g_{\phi\phi} (a^2 H_f M_f^2 N_g - \beta a_g^2 H_g M_g^2 N_f)^2 \right] \right\},
\]
\[
\Xi_{66} = -\frac{1}{\Delta} \left\{ N^3 \left[ a^6 H_f^2 M_f^4 (C F_{16} - 4k^2 M_{66} a_g M_g^2 N_f + 6C M_{66} a_g^2 H_g^2 M_g^2 N_f) + 2C a^2 H_f M_g^2 \left( 3M_{66} a_g^2 H_f M_g^2 N_f - F_{15} F_{25} + CF_{26} a_g H_f M_g^2 \right) \right] \right\},
\]
\[
-2C \left( \frac{d\phi}{dt} \right)^2 M_{66} a^2 g_{\phi\phi} (a^2 H_f M_f^2 N_g - \beta a_g^2 H_g M_g^2 N_f)^2 \right\}
\]

In the above,

\[
g_{\phi\phi} = \frac{1}{2} \left( P_{,\chi} + 2 X P_{,\chi} \right),
\]

\(C\) and \(M\) are given in Eqs. (45) and (43), respectively, and

\[
F_{14} = a_g N_g \left[ 2k^2 M_g^2 + 3a^2 \alpha^2 \left( P - 2\bar{X} P_{,\chi} \right) + 2a^2 (b_1 - M_g^2 A_g + 3M_g^2 H_f^2) \right],
\]
\[
F_{16} = 3N_g \left[ \alpha \beta \alpha^2 a_f \left( P - 2\bar{X} P_{,\chi} \right) + a^2 b_2 \right],
\]
\[
F_{24} = a_g N_g F_{16},
\]
\[
F_{26} = a_f N_f \left[ 2k^2 M_f^2 + 3a^2 \alpha^2 \left( P - 2\bar{X} P_{,\chi} \right) + 2a^2 ( - M_f^2 A_f + 3M_f^2 H_f^2) \right] + 3a_b b_1 N_g,
\]
\[
M_{44} = 2k^2 a_g N_g^2 + 3a^2 \left\{ 3 \left( m^2 a_f M_g^2 (c_1 N_g + 2c_2 N_f) + \alpha^2 \tilde{a} N P \right) + a_g \left[ N_g M_g^2 \left( 3m^2 c_0 - 3A_g + 299 \frac{d H_f}{dt} + 3 \frac{2}{N_g} \frac{d H_f}{dt} \right) + N_g M_f^2 + 3m^2 c_1 M_f^2 N_f \right] \right\},
\]
\[
M_{46} = 3a_f \left[ 3a_f \left( m^2 a_g M_f^2 (c_1 N_g + 2c_2 N_f) + \alpha \beta \alpha \tilde{a} N P + 6m^2 a_f M_f^2 (c_2 N_g + 3c_3 N_f) - a_g N_g M_f^2 \right) \right],
\]
\[
M_{55} = \frac{1}{18} k^8 a_g M_g^2 N_g + \frac{1}{6} k^2 a_g^2 N_g M_f^2,
\]
\[
M_{66} = 2k^2 a_f M_f^2 N_f + 3a^2 N_g M_g^2 + 9a^2 \left( 2m^2 a_g M_g^2 (c_2 N_g + 3c_3 N_f) + \beta^2 \tilde{a} N P \right) + 9a^2 \left( 6m^2 c_3 M_f^2 N_g + N_f M_f^2 \left( 24m^2 c_4 - A_f \right) + N_f M_f^2 \right) \frac{d H_f}{dt} \right\}.\]
