Solutions to the ultradiscrete Toda molecule equation expressed as minimum weight flows of planar graphs

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Abstract

We define a function by means of the minimum weight flow on a planar graph and prove that this function satisfies the ultradiscrete Toda molecule equation, its Bäcklund transformation and the two-dimensional Toda molecule equation. The method we employ in the proof can be considered as fundamental to the integrability of ultradiscrete soliton equations.

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1. Introduction

The box and ball system (BBS) [1] is a cellular automaton with genuine soliton-like behaviour in spite of its simple time evolution rule.

A first picture one can use to describe the dynamics of the BBS is that obtained by introducing the dependent variable $B_t^j$ to represent the state of the box at site $j$ and time $t$, with $B_t^j = 1$ when there is a ball in the box and $B_t^j = 0$ when not. The time evolution rule is then rewritten as

$$B_t^{j+1} = \min \left(1 - B_t^j, \sum_{l=-\infty}^{j-1} (B_t^l - B_t^{l+1})\right)$$

(1)

where $B_t^j$ is required to satisfy the following boundary conditions:

$$B_t^j = 0 \quad \text{for} \quad |j| \gg 0.$$  

(2)

This difference equation is also called the ultradiscrete KdV equation. By employing the dependent variable transformation:

$$B_t^j = T_t^{j+1} + T_t^{j+1} - T_t^{j+2} - T_t^j,$$

(3)

dynamics (1) is transformed into

$$T_t^{j+2} + T_t^j = \max \left(T_t^{j+2} + T_t^{j+1} - 1, T_t^{j+1} + T_t^{j+1}\right).$$

(4)
Figure 1. An example of a state in the BBS.

There is another depiction of the BBS dynamics obtained by denoting by $E^+_n$ the length of the $n$th block of empty boxes and by $Q^+_n$ the length of the $n$th block of balls [2]. Figure 1 shows an example of the state of the BBS where $E^+_0 = \infty$, $Q^+_1 = 3$, $E^+_1 = 2$, $Q^+_2 = 1$, $E^+_2 = \infty$.

The time evolution rule for $Q^+_n$ and $E^+_n$ is written as

$$Q^+_{n+1} = \min \left( \sum_{l=1}^{n} Q^+_l - \sum_{l=1}^{n-1} Q^+_{l+1}, E^+_n - 1 \right),$$

$$E^+_{n+1} = Q^+_n + E^+_n - Q^+_{n+1},$$

for $1 \leq n \leq N$ where $E^+_n$ are required to satisfy the following boundary conditions:

$$E^+_0 = E^+_N = \infty.$$

These difference equations are called the ultradiscrete Toda molecule equation [3]. By employing the dependent variable transformations:

$$Q^+_n = F^+_n F^+_{n-1} - F^+_{n+1} + F^+_{n+2},$$

$$E^+_n = F^+_{n-1} F^+_n - F^+_{n+1} + F^+_{n+2},$$

dynamics (5) and (6) are rewritten as

$$F^+_n + F^+_{n+2} = \min \left( 2F^+_{n+1}, F^+_n + F^+_{n+2} \right),$$

where the boundary condition for $F^+_n$ is

$$F^+_{-1} = F^+_{N+1} = \infty.$$

The dynamics whose time evolution rules can be written in terms of $\min(\max)$ and $\pm$ operators are called ‘ultradiscrete systems’. They can be obtained from discrete soliton equations by the limiting procedure called ‘ultradiscretization’ [4]. Ultradiscrete soliton equations such as the BBS have rich structures such as the existence of $N$-soliton solutions and an infinite amount of conserved quantities [5], similar to ordinary soliton equations. Therefore, ultradiscrete systems are thought of as preserving the main characteristics of the soliton equations and it is an interesting problem to try to obtain the structure of the solutions to ultradiscrete soliton equations, as opposed to those of ordinary soliton equations.

One approach is to find ultradiscrete analogues of the various concepts that appear in the context of ordinary soliton equations. In previous papers, for example, we proposed a notion of vertex operators [6, 7]. The ultradiscretization of signature-free determinants (called ‘permanents’) is discussed in [8] and the relationship between these types of solutions and ultradiscrete soliton equations in [9, 10].

Another approach is by means of combinatorial methods, in particular, disjoint multiflows in planar graphs, a representation that originated in [11]. Such structured functions defined by a weighted flow on a planar graph and their identities are discussed in [12] and their applications to soliton equations are presented in [13]. The Toda molecule equation is also called a ‘Q-system’ and from the point of view of cluster algebras, exact solutions of
the Q-system can be expressed by some initial values combined with a graph representation [14, 15]. Such a graph-based representation is also presented in the study of numerical algorithms [16] and can be applied to the initial value problem for integrable systems (as discussed in, for example, [17]), especially the initial value problem of the discrete and ultradiscrete Toda equation [18].

In this paper, we introduce another structure of the soliton solutions of ultradiscrete soliton equations. We first define a function expressed as the minimum weight flow in a planar graph. By means of a simple and ultradiscrete(-closed) method, we also prove that this function solves the ultradiscrete Toda molecule equation, its Bäcklund transformation and the two-dimensional Toda molecule equation. Since this structure and method can be applied to several soliton equations, this property can be regarded as crucial to the integrability of ultradiscretized soliton equations.

2. Minimum weight flow on a skewed grid graph

In this section, we introduce a graph with weighted edges and prove some propositions regarding the properties of this graph.

**Definition 1.** We define the graph $\Gamma$ as follows.

- It consists of infinitely many vertices labelled by $(i, j)$ where $i \in \mathbb{Z}$, $j \in \mathbb{Z}_{\geq 0}$.
- For $j > 0$, it consists of directed edges:
  - $(i, j)$ to $(i + 2, j - 1)$, denoted by $d_{i,j}$;
  - $(i, j)$ to $(i, j - 1)$, denoted by $v_{i,j}$.
- The weight $w_{\text{edge}}$ for an edge $e$ is given by
  \[
  w_{\text{edge}}(e) = \begin{cases} 
  \eta_i^j & (e = d_{i,j}) \\
  0 & (e = v_{i,j})
  \end{cases}
  \]
  where $\eta_i^j$ stands for
  \[
  \eta_i^j = i \Omega_j + C_j,
  \]
  and the parameters $\Omega_j$ satisfy the relations
  \[
  \Omega_1 \leq \Omega_2 \leq \cdots \leq \Omega_j \leq \cdots
  \]
  The $C_j$ are constants.

By construction, $\Gamma$ separates into two disjoint planar graphs $\Gamma_{\text{even}}$ and $\Gamma_{\text{odd}}$, according to whether the first component $i$ of the vertex $(i, j)$ is even or odd. Figure 2 shows the planar graph representing $\Gamma_{\text{even}}$ or $\Gamma_{\text{odd}}$.

**Definition 2.** Let $N > 0$ and $0 \leq n \leq N$. The function $F_{N,n}^t$ is defined by

\[
F_{N,n}^t = \begin{cases} 
\min_{p \in P_{N,n}} w(p) & (0 \leq n \leq N) \\
\infty & \text{(otherwise)}
\end{cases}
\]

where $P_{N,n}$ is the set of flows on $\Gamma$ with the source $(t, N)$ and the sink $(t + 2n, 0)$ and $w(p)$ is the sum of the weights of edges that make up the flow $p$, i.e.

\[
w(p) = \sum_{e \in p} w_{\text{edge}}(e).
\]
We also define $F^t_{0,0} = 0$ and $F^t_{0,n} = \infty$ for $n \neq 0$ and denote $F^t_{N,n} = F^t_n$ for brevity, unless this causes confusion.

By construction, the set of flows $\mathcal{P}^t_{N,n}$ and the edges in the flow $p$ are finite. Since the flow $p$ passes through the edge labelled $d_{t+2(1)}$ only once for each $1 \leq i \leq n$, we can obtain the sequence of these edges $(d_{t+1}, d_{t+2(1)}, \ldots, d_{t+2(n-1)}, p_n)$ or a simpler expression $(p_1, p_2, \ldots, p_n)$ from the flow $p$. Here, by construction, the sequence is decreasing, i.e. $N > p_1 > p_2 > \cdots > p_n \geq 1$, and the mapping from the flow $p$ to the decreasing sequence $(p_1, p_2, \ldots, p_n)$ is a bijection. Therefore, we can identify the flow $p$ with the decreasing sequence $(p_1, p_2, \ldots, p_n)$. By means of these decreasing sequences, $F^t_n$ can be rewritten as

$$F^t_n = \min_{1 \leq p_1 < p_2 < \cdots < p_n \leq N} \sum_{k=1}^n \eta^{t+2(k-1)}.$$  \hfill (17)

**Proposition 3.** For the flows $p \in \mathcal{P}^t_n$ and $p' \in \mathcal{P}^{t+2}_{n}$, if there exists a number $j$ such that $p_j \geq p'_j + 1$, $p$ and $p'$ share at least one vertex on $\Gamma$.

Figure 3 shows an example of this proposition. Two flows $p$ and $p'$, expressed by dotted and dashed lines respectively, share the vertex $(t+2, 3)$ for $p_1 = 4, p_1' = 3$.

**Proof.** We consider the first $m$ satisfying the conditions and note that the flow $p$ in $\mathcal{P}^t_n$ must contain the edge $d_{t+2(m-1)}, p_m$, which has the vertex $(t+2m, p_m-1)$. If $m = 1$, the flow $p'$ starts at $(t+2, N)$ and moves on the edge $v_{t+2,N}, v_{t+2,N-1}, \ldots, v_{t+2,p_1}, \ldots, v_{t+2,p_1'+1}$. Then, the edges $d_{t, p_1}$ and $v_{t+2, p_1}$ have the common vertex $(t+2, p_1-1)$. If $m > 1$
Figure 3. An example satisfying proposition 3.

Figure 4. An example illustrating proposition 4. We can replace the dashed path with the dotted path from $q$ to $q'$ without changing the total weight.

(i.e. $p_m' + 1 \leq p_m < p_{m-1} \leq p_m'$), by virtue of the same observation, $p'$ contains the edge $v_{t+2m,p_m}$, which has the vertex $(t + 2m, p_m - 1)$.

Proposition 4. Let $p$ and $p'$ be flows which give the minimum weight flow for each source and sink. If $p$ and $p'$ share two vertices $q$ and $q'$, there exist two flows $\tilde{p}$ and $\tilde{p}'$ which go through the same edges from $q$ to $q'$ and keep each weight for each source and sink.

(See figure 4 depicting an example for this proposition).

Proof. Let $p_{q \rightarrow q'}$ and $p_{q' \rightarrow q''}$ be the edges where the flows $p$ and $p'$ go from $q$ to $q'$ respectively. Without loss of generality, we can set $w(p_{q \rightarrow q'}) \geq w(p_{q' \rightarrow q'})$. If $w(p_{q \rightarrow q'}) > w(p_{q' \rightarrow q'})$, we can obtain the new flow $\tilde{p}$ by replacing $p_{q \rightarrow q'}$ with $p_{q' \rightarrow q''}$ in $p$. The flow $\tilde{p}$ has the same source and sink as $p$ and satisfies $w(\tilde{p}) < w(p)$, which contradicts the definition of $p$. Therefore,
Figure 5. An example illustrating proposition 5. By interchanging flows after they intersect, we can obtain new flows without changing the total weight.

**Proposition 5.** Let \( p \) and \( p' \) be flows which give \( F_t^{N,m} \) and \( F_{t+2l}^{N',n-l} \) and share at least one vertex. One has

\[
F_t^{N,m} + F_{t+2l}^{N',n-l} \geq F_t^{N,n} + F_{t+2l}^{N',m-l}.
\]  

(18)

(See figure 5 depicting an example of this proposition.)

**Proof.** By interchanging each orbit after the vertex where the flows intersect (if there are several such vertices, choose any one of them), we can obtain new paths \( \tilde{p} \in P_t^{N,n} \) and \( \tilde{p'} \in P_{t+2l}^{N',n-l} \). The total weights of the two flows are conserved, before and after the procedure, and we obtain (18) by virtue of the definition of \( F_t^{N,n} \). □

### 3. Ultradiscrete Toda molecule equation

In this section, we prove that the function \( F_t^{N,n} \) defined in the previous section solves the ultradiscrete Toda molecule equation.

**Theorem 6.** For \( 0 \leq n \leq N \), \( F_t^{n} \) solves the ultradiscrete Toda molecule equation:

\[
F_t^{n} + F_{t+2}^{n} = \min \left( 2F_{t+1}^{n}, F_{t+1}^{n} + F_{t+1}^{n-1} \right).
\]  

(19)

When \( n = 0 \) or \( n = N \), \( F_t^{n} \) is linear and \( F_{t+1}^{n} + F_{t+1}^{n-1} \) does not contribute to the minimum because \( F_{t-1}^{n} = F_{N+1}^{n} = \infty \), so \( F \) trivially satisfies equation (19). Henceforth, we consider only the case \( 1 \leq n \leq N - 1 \).

**Proof.** We introduce four (two times two) steps that have to be proven.

(I) Proof of \( F_t^{n} + F_{t+2}^{n} \leq \min \left( 2F_{t+1}^{n}, F_{t+1}^{n} + F_{t+1}^{n-1} \right) \).

(I-i) Proof of \( F_t^{n} + F_{t+2}^{n} \leq \min \left( 2F_{t+1}^{n}, F_{t+1}^{n} + F_{t+1}^{n-1} \right) \).

(I) Proof of \( F_t^{n} + F_{t+2}^{n} \leq 2F_{t+1}^{n} \).

(II) Proof of \( F_t^{n} + F_{t+2}^{n} \leq \min \left( 2F_{t+1}^{n}, F_{t+1}^{n} + F_{t+1}^{n-1} \right) \).

(II-i) Proof of \( F_t^{n} + F_{t+2}^{n} \leq \min \left( 2F_{t+1}^{n}, F_{t+1}^{n} + F_{t+1}^{n-1} \right) \).

(II) Proof of \( F_t^{n} + F_{t+2}^{n} \leq 2F_{t+1}^{n} \).
Let \( p \in P_{t+1} \) be a flow that gives the minimum weight, i.e. \( F_{t+1}^p = w(p) \). Here, there exists \( p' \in P_t \) such that \( w(p') = \sum_{i=1}^n \eta_{p_i}^{t+2(i-1)} \). Then, from the definition of \( F_n^t \) we obtain
\[
F_{t+1}^p - (\Omega_{p_1} + \Omega_{p_2} + \cdots + \Omega_{p_n}) = w(p') \geq F_n^t.
\] (20)

By virtue of the same discussion, we also obtain
\[
F_{t+1}^p + (\Omega_{p_1} + \Omega_{p_2} + \cdots + \Omega_{p_n}) \geq F_{t+2}^n,
\] (21)
and adding these inequalities yields the proof.

(I-i) Proof of \( F_{t+1}^p + F_{t+2}^p \leq F_{t+1}^n + F_{t+2}^{n-1} \).

The flows which give \( F_{t+1}^p \) and \( F_{t+2}^p \) must intersect because the flow \( p \) starts at \((t, N)\) and ends at \((t+2n+2, 0)\) and \( p' \) starts at \((t+2, N)\) and ends at \((t+2n, 0)\). Hence, we obtain the inequality by virtue of proposition 5.

By virtue of the discussions in (I-i) and (I-ii), we obtain the relations
\[
F_{t+1}^p + F_{t+2}^p \leq F_{t+1}^n + F_{t+2}^{n-1}
\] and
\[
F_{t+1}^n + F_{t+2}^n \leq 2F_{t+1}^p,
\] (22)
which is equivalent to the equality we want to prove.

(Ii) Proof of \( F_{t+1}^p + F_{t+2}^p \geq \min (2F_{t+1}^n, F_{n+1}^t + F_{t+2}^{n-1}) \).

Let \( p \) and \( p' \) be the flows which give \( F_{t+1}^n \) and \( F_{t+2}^n \).

(II-i) In case the two flows share at least one vertex.

By virtue of proposition (5), we obtain
\[
F_{t+1}^p + F_{t+2}^p \geq F_{t+1}^n + F_{t+2}^{n-1}.
\] (23)

(II-ii) In case the two flows do not share any vertices.

By a similar discussion as in (II-i), we obtain
\[
F_{t+1}^p + F_{t+2}^p \geq \frac{2F_{t+1}^n}{(\Omega_{p_1} + \cdots + \Omega_{p_n}) + (\Omega_{p'_1} + \cdots + \Omega_{p'_n})}
= 2F_{t+1}^n + (\Omega_{p'_1} - \Omega_{p_1}) + \cdots + (\Omega_{p'_n} - \Omega_{p_n}).
\] (24)

Here, by contraposition of proposition 3, \( p'_j \) must be greater than \( p_j \) for all \( j \) (i.e. \( \Omega_{p'_j} \geq \Omega_{p_j} \)). Then we obtain
\[
F_{t+1}^p + F_{t+2}^p \geq 2F_{t+1}^n.
\] (25)

By virtue of the discussions in (II-i) and (II-ii), we obtain the relation
\[
F_{t+1}^p + F_{t+2}^p \geq F_{t+1}^n + F_{t+2}^{n-1} \quad \text{or} \quad F_{t+1}^n + F_{t+2}^n \geq 2F_{t+1}^p,
\] (26)
which is equivalent to the equality we want to prove. \( \square \)

It should be noted that the initial value problem of this equation (19) is equivalent to that of the BBS (5), (6). For arbitrary initial values, parameters \( \Omega_k \) and \( C_k \) are calculated by employing the ‘10’-elimination method [19] for a BBS state.
4. Bäcklund transformation for the ultradiscrete Toda molecule equation

A Bäcklund transformation is a relation between soliton solutions which have different numbers of solitons and, in general, yields a procedure to obtain a new solution from a given one by solving a simpler set of equations.

**Theorem 7.** For $0 \leq n \leq N$, $F^t_n = F^t_{N,n}$ and $G^t_n = F^t_{N-1,n}$ satisfy the Bäcklund transformation for the ultradiscrete Toda molecule equation:

\[
F^t_n + G^t_{n-1} = \min \left( F^t_{n-1} + G^t_{n-1} + \Omega_N, F^t_{n-1} + G^t_n \right)
\]

(27)

\[
F^t_n + G^t_{n+2} = \min \left( F^t_{n+1} + G^t_{n+1}, F^t_{n+1} + G^t_{n+2} \right).
\]

(28)

**Proof.** As before, we introduce two times two steps for each equation. Figures 6–10 are visual explanations of this proof for $N = 3, n = 2$, which can help to better understand the proof.

(I) Proof of $F^t_n + G^t_{n-1} \leq \min \left( F^t_{n-1} + G^t_{n-1} + \Omega_N, F^t_{n-1} + G^t_n \right)$.

(I-i) Proof of $F^t_n + G^t_{n-1} \leq F^t_{n-1} + G^t_{n-1} + \Omega_N$.

Let $p$ and $p'$ be the flows which give $F^t_{n-1}$ and $G^t_{n-1}$. We obtain

\[
F^t_{n-1} + G^t_{n-1} \geq F^t_n + G^t_{n-1} - \Omega_{p_1} + (\Omega_{p_1} - \Omega_{p_2}) + \cdots + (\Omega_{p_{n-1}} - \Omega_{p_n}).
\]

(29)

Note that the two flows $p$ and $p'$ share the sink $(t + 2n - 1, 0)$. Let us consider the case where there exists a number $m$ such that $\Omega_{p_{m-1}} < \Omega_{p_m}$, and let $m_0$ be the smallest of such $m$. If $m_0 = 2$ (i.e. $N \geq p_1 > p_2 > p_3$), the flow $p$ passes from $(t + 1, p_1 - 1)$ to $(t + 1, p_2)$ and $p'$ passes from $(t + 1, N - 1)$ to $(t + 1, p_3')$. By the same consideration for $m_0 > 2$ (i.e. $p_{m_0-2} > p_{m_0-1} > p_{m_0} > p_{m_0+1}$), the two flows must intersect. Therefore, by employing proposition 4, we can always choose $p$ and $p'$ to satisfy

\[
\Omega_{p_{m-1}} \geq \Omega_{p_m} (2 \leq m \leq n).
\]

(30)

Hence, we obtain

\[
F^t_{n-1} + G^t_{n-1} \geq F^t_n + G^t_{n-1} - \Omega_{p_1} \geq F^t_n + G^t_{n-1} - \Omega_N.
\]

(31)

(Iii) Proof of $F^t_n + G^t_{n-1} \leq F^t_{n-1} + G^t_n$.

We can use proposition 5 since the flows start at $(t, N)$ and $(t, N - 1)$ and terminate at $(t + 2n - 2, 0)$ and $(t + 2n, 0)$ and obtain the inequality.

(II) Proof of $F^t_n + G^t_{n-1} \geq \min \left( F^t_{n-1} + G^t_{n-1} + \Omega_N, F^t_{n-1} + G^t_n \right)$.

Let $p$ and $p'$ the paths which give $F^t_n$ and $G^t_{n-1}$.

(II-i) In case the two flows share at least one vertex.

By virtue of proposition 5, one then has

\[
F^t_n + G^t_{n-1} \geq F^t_{n-1} + G^t_n.
\]

(32)

(II-ii) In case the two flows do not share any vertices.
Figure 6. An example of the proof of (I-i) in theorem 7. The two flows in the left picture satisfy \( p_1' \geq p_2 \) (i.e. \( \Omega_{p_1'} \geq \Omega_{p_2} \)).

Figure 7. Another example of the proof of (I-i) in theorem 7. If \( p_1' < p_2 \) as on the left, we can change the dotted flow so that it satisfies \( \Omega_{p_1'} \geq \Omega_{p_2} \) without changing the total weight, as on the right.

Note that \( p \) and \( p' \) start at \((t, N)\) and \((t, N - 1)\) respectively and \( p \) never has \( v_{t,N} \) as its first edge due to the condition. Thus, \( p \) must choose \( d_{t,N} \), i.e. \( p_1 = N \), and we can employ the same discussion as in proof (II-ii) of theorem 6 for the flow after passing \( d_{t,N} \), i.e. the decreasing sequence \((p_2, p_3, \ldots, p_n) \in \mathcal{P}_{N-1,n-1}^t \) and \( p' \in \mathcal{P}_{N-1,n-1}^t \). Then, we obtain

\[
F^t_n + G^t_{n-1} \geq F^{t-1}_{n} + G^{t+1}_{n-1} + \Omega_{p_1} + (\Omega_{p_2} - \Omega_{p_1'}) + \cdots + (\Omega_{p_n} - \Omega_{p_{n-1}'}) \geq F^{t-1}_{n} + G^{t+1}_{n-1} + \Omega_N. \tag{33}
\]

(III) Proof of \( F^t_n + G^{t+2}_n \leq \min (F^{+1}_{n+1} + G^{+1}_{n+1}, F^{+1}_{n+1} + G^{+1}_{n+1}) \).

(III-i) Proof of \( F^t_n + G^{t+2}_n \leq F^{+1}_{n+1} + G^{+1}_{n+1} \).
Let $p$ and $p'$ be the flows which give $F_{n+1}^t$ and $G_{n+1}^t$. Since they share the sink $(t+2n+1, 0)$, by a discussion similar to (I-i) in this proof, we can choose the flows to satisfy
\[
\Omega_{p_m} \geq \Omega_{p'_m} \quad (1 \leq m \leq n).
\] (34)
Thus, we obtain
\[
F_{n+1}^t + G_{n+1}^t \geq F_{n}^t + G_{n}^t + (\Omega_{p_1} - \Omega_{p'_1}) + \cdots + (\Omega_{p_n} - \Omega_{p'_n})
\]
\[
\geq F_{n}^t + G_{n}^{t+2}.
\] (35)
(III-ii) Proof of $F_{n}^t + G_{n}^{t+2} \leq F_{n+1}^t + G_{n+2}^t$.

We can employ proposition 5 since the flows start at $(t, N)$ and $(t+2, N-1)$ and terminate at $(t+2n+2, 0)$ and $(t+2n, 0)$ and obtain the inequality.
(IV) Proof of $F_t^i + G_t^{i+2} \geq \min \{ F_{n+1}^i + G_{n+1}^i, F_{n+1}^i + G_{n+1}^{i+2} \}$.

Let $p$ and $p'$ be the flows which give $F_t^i$ and $G_t^{i+2}$.

(IV-i) In the case of two flows sharing at least one vertex.

By virtue of proposition 5, one has

$$F_t^i + G_t^{i+2} \geq F_{n+1}^i + G_{n+1}^{i+2}.$$  \hspace{1cm} (36)

(IV-ii) In the case of two flows not sharing any vertices.

One has

$$F_t^i + G_t^{i+2} \geq F_{n+1}^i + G_{n+1}^i + (\Omega_{p_1} - \Omega_{p_1}) + \cdots + (\Omega_{p_1} - \Omega_{p_1}) + \cdots + (\Omega_{p_1} - \Omega_{p_1})$$  \hspace{1cm} (37)

and by the condition, $p$ never has $d_t, N$ as its first edge because it cannot go through $(t + 2, N - 1)$ which is the source of $p'$. Thus, $p$ must choose $v_{t, N}$ and we can employ proposition 3 for the flow after passing $v_{t, N}$ and $p'$ and obtain

$$F_t^i + G_t^{i+2} \geq F_{n+1}^i + G_{n+1}^{i+1}.$$  \hspace{1cm} (38)

□

5. Ultradiscrete two-dimensional Toda molecule equation

The discrete two-dimensional Toda molecule equation [21] is an extension of the discrete Toda molecule equation. This equation is also called T-system [22] in studies of solvable lattice models. In this section, we first introduce an extended graph and the flow over it and prove that the function given by the minimum weight solves the ultradiscretization of the two-dimensional Toda molecule equation. We call this equation the ‘ultradiscrete two-dimensional Toda molecule equation’.

Definition 8. We define the graph $\hat{\Gamma}$ as follows.

• It consists of infinitely many vertices labelled $(i, j, k)$, where $i, j \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}$.
For $k > 0$, it consists of direct edges:

- $(i, j, k)$ to $(i + 1, j - 1, k - 1)$, denoted by $d_{i,j,k}$;
- $(i, j, k)$ to $(i, j, k - 1)$, denoted by $v_{i,j,k}$.

The weight $w_{\text{edge}}$ for the edge $e$ is given by

$$w_{\text{edge}}(e) = \begin{cases} \eta_{i,j,k}^{i,j} & (e = d_{i,j,k}) \\ 0 & (e = v_{i,j,k}) \end{cases}$$  \hspace{1cm} (39)

Here, $\eta_{i,j,k}^{i,j}$ stands for

$$\eta_{i,j,k}^{i,j} = iP_k - jQ_k + C_k,$$  \hspace{1cm} (40)

and the parameters $C_k$ are constants and $P_k, Q_k$ satisfy the relations

$$P_1 \leq P_2 \leq \cdots \leq P_k \leq \cdots$$  \hspace{1cm} (41)

and

$$\sum_{i=1}^{n} P_{p_i} \leq \sum_{i=1}^{n} P_{p'_i} \iff \sum_{i=1}^{n} Q_{p_i} \leq \sum_{i=1}^{n} Q_{p'_i}$$  \hspace{1cm} (42)

for all decreasing sequences $N \geq p_1 > p_2 > \cdots > p_n \geq 1$.

Conditions (41) and (42) lead to the relation

$$Q_1 \leq Q_2 \leq \cdots \leq Q_k \leq \cdots.$$  \hspace{1cm} (43)

Thus, we can replace conditions (42) and (41) with (42) and (43).

By identifying the summation of the first and second components $i+j$ for the vertex $(i, j, k)$, the graph $\Gamma$ is now expressed as an infinite disjoint planar graph. However, we may consider a finite subgraph when treating a specific flow.

**Definition 9.** Let $N > 0$ and $0 \leq n \leq N$. The function $F_{l,m}^{l,m}_{n,n}$ is defined by

$$F_{l,m}^{l,m}_{n,n} = \begin{cases} \min_{p \in \mathcal{P}^{l,m}_{n,n}} w(p) & (0 \leq n \leq N) \\ \infty & (\text{otherwise}) \end{cases}$$  \hspace{1cm} (44)

where $\mathcal{P}^{l,m}_{n,n}$ is the set of flows on $\Gamma$ with the source $(l, m, N)$ and the sink $(l + n, m - n, 0)$ and $w(p)$ is the sum of the weight of the edges in the flow $p$. We denote $F_{l,m}^{l,m}_{n,n} = F_{n,n}^{l,m}$ for brevity unless this is ambiguous.

By the construction of the flow it should be clear that we can also identify the flow $p$ with the decreasing sequence $(p_1, p_2, \ldots, p_n)$ such that

$$F_{l,m}^{l,m}_{n,n} = \min_{1 \leq p_1 < p_{n-1} < \cdots < p_n \leq N} \sum_{k=1}^{n} \eta_{l,m,k}^{l,k-1,m-k}.$$  \hspace{1cm} (45)

**Theorem 10.** The function $F_{n,n}^{l,m}$ satisfies the ultradiscrete two-dimensional Toda molecule equation:

$$F_{n,n}^{l,m} + F_{n+1,n,m-1}^{l+1,m-1} = \min \left( F_{n,n}^{l+1,m} + F_{n,n}^{l,m-1}, F_{n+1,n+1}^{l,m} + F_{n+1,n-1}^{l+1,m-1} \right).$$  \hspace{1cm} (46)

**Proof.** Note that the vertices $(l, m, n)$ and $(l+1, m-1, n)$ are in the same planar graph and their relative positions are the same as that in definition 1. Therefore, we can use the same method as in theorem 6; in particular, steps (I-ii) and (II) are completely the same. Only the
proof of $F_{n}^{l,m} + F_{n}^{l+1,m-1} \leq F_{n}^{l+1,m} + F_{n}^{l,m-1}$ requires some additional discussion because the flows $p$ and $p'$ which give $F_{n}^{l+1,m}$ and $F_{n}^{l,m-1}$ are in different planar graphs. Here, one has

$$\sum_{k=1}^{n} \eta_{k}^{l+1,m-1,k} + \sum_{k=1}^{n} P_{k} + \sum_{k=1}^{n} \eta_{k}^{l,m-1,k} + \sum_{k=1}^{n} Q_{k}. \quad (47)$$

If $\sum_{k=1}^{n} P_{k} \geq \sum_{k=1}^{n} P_{k}'$, then we obtain

$$\sum_{k=1}^{n} \eta_{k}^{l+1,m-1,k} + \sum_{k=1}^{n} P_{k} + \sum_{k=1}^{n} Q_{k} \leq F_{n}^{l,m} + F_{n}^{l-1,m+1}. \quad (49)$$

If $\sum_{k=1}^{n} P_{k} \leq \sum_{k=1}^{n} P_{k}'$, by virtue of condition (42), then we obtain

$$\sum_{k=1}^{n} \eta_{k}^{l+1,m-1,k} + \sum_{k=1}^{n} P_{k} + \sum_{k=1}^{n} Q_{k} \leq F_{n}^{l,m}$$

which reduces to the same inequality.

It should be noted that the cellular automaton corresponding to equation (46) has not been found yet. Therefore, the method to determine $P_k$, $Q_k$, and $C_k$ from the initial values is yet unknown.

To end this section, let us give some examples satisfying the strong restriction (42). Consider the restriction of the parameters $P_j$, $Q_j$ such that $MP_j = Q_j$ for $M > 0$. Now, the parameters satisfy the restriction (42) and the function $F_{n}^{l,m}$ is such that $F_{n}^{l+1,m+M} = F_{n}^{l,m}$ under this restriction, which is nothing but the reduction to the dynamics of multi-kind ball and box systems [20] (in particular, the dynamics reduce to the standard BBS for $M = 1$).

6. Concluding remarks

In this paper, we have introduced functions given by minimal weight flows on planar graphs and have proven that they yield the ultradiscrete Toda molecule equation, its Bäcklund transformation and the ultradiscrete two-dimensional Toda molecule equation, for purely structural reasons.

By virtue of the vertex operator for the ultradiscrete soliton equations introduced in [6] and [7], the $N$-soliton solution $F_{N,n}^{l}$ is expressed as

$$F_{N,n}^{l} = \min \left( F_{N-1,n}^{l+2}, \eta_{N}^{l} + F_{N-1,n-1}^{l+2} \right). \quad (51)$$

which can be interpreted as separating the arguments of ‘min’ in (51) according to whether the first edge of the flow is $d_{N-1,n}$ or $v_{N,n}$. Therefore, the graph representation is clearly more essential to the structure of the solutions of ultradiscrete soliton equations. Furthermore, this representation can be valid only in ultradiscrete systems, because a simple expression such as (51) cannot be obtained for the discrete Toda molecule equation.

The representation is also applicable to the ultradiscrete KdV equation (4), which is written as the minimum weight of the flow depicted in figure 11. However, it is more difficult to prove that the function given by the minimum weight flow solves the equation relying only on its structure. One of the reasons may be the difference of the relation on which each equation is based, i.e. the discrete KdV equation is based on the Plücker relation for Casorati determinants and the discrete Toda molecule equation is based on Jacobi’s formula for a
Figure 11. An example of the flow representation of a solution of the ultradiscrete KdV equation.
The source is the top vertex, and the sink is one of the bottom vertices. Numbers over the diagonal
edges are weights and $\xi_{N,j}^t = t/\Omega_N - j + C_N$.

determinant (though it is a special case of the Plücker relation). It is an interesting problem to
clarify why these equations describe the same dynamics in spite of such differences.

It is also interesting to try to discover the direct relationship between the graph structure
of the ultradiscrete Toda molecule equation and the determinant solutions of the discrete Toda
molecule equation.

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