Fair Simulation for Nondeterministic and Probabilistic Büchi Automata: a Coalgebraic Perspective

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Abstract

Notions of simulation, among other uses, provide a computationally tractable and sound (but not necessarily complete) proof method for language inclusion. They have been comprehensively studied by Lynch and Vaandrager for nondeterministic and timed systems; for (nondeterministic) Büchi automata the notion of fair simulation has been introduced by Henzinger, Kupferman and Rajamani. We contribute generalization of fair simulation in two different directions: one for nondeterministic tree automata (this has been studied previously by Bomhard); and the other for probabilistic word automata (with a finite state space), both under the Büchi acceptance condition. The former (nondeterministic) definition is formulated in terms of systems of fixed-point equations, hence is readily translated to parity games and then amenable to Jurdziński’s algorithm; the latter (probabilistic) definition bears a strong ranking-function flavor. These two different-looking definitions are derived from one source, namely our coalgebraic modeling of Büchi automata: the proofs of soundness (i.e. that a simulation indeed witnesses language inclusion) are based on these coalgebraic observations, too.

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1 Introduction

Notions of simulation—typically defined as a binary relation subject to a coinductive “one-step mimicking” condition—have been studied extensively in formal verification and process theory. Sometimes existence of a simulation itself is of interest—taking it as the definition of an abstraction/refinement relationship—but another notable use is as a proof method for language inclusion. Language inclusion is fundamental in model checking but often hard to check itself; looking for a simulation—which witnesses language inclusion, by its soundness property, in a step-wise manner—is then a sound (but generally not complete) alternative. For example, (finite) language inclusion between weighted automata is undecidable while existence of certain simulations is PTIME. See [35].

Simulation notions have been introduced for many different types of systems: nondeterministic [27], timed [28] and probabilistic [23], among others. Conventionally many studies take the trivial acceptance condition (any run that does not diverge, i.e. that does not come to a deadend, is accepted). Recently, however, there have been several works on simulations under the Büchi and parity acceptance conditions [10,11,17]. In such settings a simulation notion is subject to an (inevitable) nonlocal fairness condition (on top of one-step mimicking); and often a fair simulation is characterized as a winning strategy of a suitable parity
game, which is then searched using Jurdziński’s algorithm [24].

**Contributions**  It is in this context (namely fair simulation for word/tree automata with nondeterministic/probabilistic branching) that the current paper contributes:

1. We define fair simulation for *nondeterministic tree* automata with the Büchi acceptance condition. We express the notion using a system of fixed-point equations—with explicit $\mu$’s and $\nu$’s indicating least or greatest—and thus the definition makes sense for infinite-state automata too. We also interpret it in terms of a parity game, which is subject to an algorithmic search when the problem instance is finitary. The resulting parity game essentially coincides with the one in [5].

2. We define fair simulation for *probabilistic word* automata with the Büchi acceptance condition, this time with the additional condition that on the simulated side we have a finite-state automaton. This simulation notion is given by a matrix (instead of a relation); this follows our previous work [35] that uses linear programming for searching for such a matrix simulation. Our current notion also requires suitable approximation sequences for witnessing well-foundedness, with a similar intuition to ranking functions.

For the former *nondeterministic tree* setting, a notion of fair simulation—in addition to direct and delayed simulation—has already been introduced in [5]. Their notion focuses on finite state spaces and is defined directly using a parity game. In contrast, our notion—it is given in terms of fixed-point equations and thus generalizes the $\mu$-calculus characterization (that easily follows e.g. from [25]) from words to trees—makes sense for infinite state systems, too. For the latter *probabilistic word* setting, we introduce fair simulation for the first time (to the best of our knowledge).

In both settings our main technical result is soundness, i.e. existence of a fair simulation implies trace inclusion. We also exhibit nontrivial examples of fair simulations.

**Theoretical Backgrounds**  Our two simulation notions (for nondeterministic tree automata and probabilistic word ones) look rather different, but are derived from the same theoretical perspectives. The latter consist of: 1) the theory of coalgebra [21,31], and in particular the generic Kleisli theory of trace and simulation [7,14,19,34,35]; and 2) our recent work [16] on a lattice-theoretic foundation of nested/alternating fixed points, where we generalize progress measures, a central notion in Jurdziński’s algorithm [24] for parity games. We rely on both of these for soundness proofs, too, where we follow another recent work of ours [32] to characterize the accepted language of a Büchi automaton by an “equational system” of diagrams in a Kleisli category. In this paper we shall briefly describe these general theories behind our current results, focusing on their instances that are relevant.

**Organization of the Paper**  In §2 we introduce equational systems, essentially fixing notations for alternating greatest/least fixed points. These notations—and the idea of the role of fixed-point equations—are used in §3 where we concretely describe: our system models (namely *nondeterministic tree* automata and *probabilistic word* automata, both with the Büchi acceptance condition); their accepted languages; simulation definitions; and soundness results. Up to this point everything is in set-theoretic terms, without category theory.

The rest of the paper is devoted to soundness proofs and the theoretical perspectives behind. In §5 we review the coalgebraic backgrounds: Kleisli coalgebras, trace semantics [7,15,19], simulations (under the trivial acceptance condition) [14,34], and coalgebraic trace for Büchi automata [32]. Finally, in §6 we take a coalgebraic look at simulations under the Büchi condition: our first attempt (fair simulation with dividing) is sound but not practically desirable; we circumvent dividing to obtain the concrete definitions in §3–4.
Many proofs, examples, and others are deferred to the appendix.

Future Work  Generalization from the Büchi condition to the parity one is certainly what we aim next. Already it is not very clear how our coalgebraic definition with dividing (§6.1) would generalize: for example, in case of parity automata, there is little sense in comparing the priority of the challenger’s state with that of the simulator’s. It is even less clear how to circumvent dividing.

Aside from fair simulation, notions of delayed simulation are known for Büchi automata [10, 11]: they are subject to slightly different “fairness” constraints. Accommodating them in the current setting is another future work.

We are also interested in automatic discovery of simulations—e.g. via mathematical programming—and its implementation. There we will be based on our previous work [33, 35]. Another direction is to use the current results for program verification—where the Integer type makes problems infinitary—exploiting our non-combinatorial presentation by equational systems. We could do so automatically (by synthesizing a symbolic simulation) or interactively (on a proof assistant).

2 Preliminaries: Equational Systems

Nested, alternating greatest and least fixed points—as in a μ-calculus formula νu₂,µu₁, (p ∧ u₂)∨□u₁—are omnipresent in specification and verification. For their relevance to the Büchi acceptance condition one can recall the well-known translation of LTL formulas to Büchi automata and vice versa (see e.g. [30]). To express such fixed points we follow [2,8] and use equational systems—we prefer them to the textual μ-calculus-like presentations.

Definition 2.1 (equational system). Let L₁, . . . , Lₘ be posets. An equational system E over L₁, . . . , Lₘ is an expression

\[ u₁ = _{η₁} f₁(u₁, . . . , uₘ), \ldots, uₘ = _{ηₘ} fₘ(u₁, . . . , uₘ) \]  

where: u₁, . . . , uₘ are variables, η₁, . . . , ηₘ \in \{μ, ν\}, and fₖ: L₁ × . . . × Lₘ \to Lₖ is a monotone function. A variable uᵢ is a μ-variable if ηᵢ = μ; it is a ν-variable if ηᵢ = ν.

The solution of the equational system E is defined as follows, under the assumption that Lᵢ’s have enough suprema and infima. It proceeds as: 1) we solve the first equation to obtain an interim solution u₁ = l₁(1)(u₂, . . . , uₘ); 2) it is used in the second equation to eliminate u₁ and yield a new equation u₂ = _{η₂} f₂(u₂, . . . , uₘ); 3) solving it again gives an interim solution u₂ = l₂(2)(u₃, . . . , uₘ); 4) continuing this way from left to right eventually eliminates all variables and leads to a closed solution uₘ = lₘ(s) \in Lₘ; and 5) by propagating these closed solutions back from right to left, we obtain closed solutions for all u₁, . . . , uₘ.

A precise definition is found in Appendix A.

It is important that the order of equations matters: for (u = _μ v, v = _ν u) the solution is u = v = ⊤ while for (v = _ν u, u = _μ v) the solution is u = v = ⊥.

Progress Measure The notion of (lattice-theoretic) progress measure [16], although not explicit, plays an important role in the current paper. Its idea is as follows.

Verification of a fixed-point specification amounts mathematically to underapproximating the fixed point. This is usually done very differently for gfp’s and lfp’s. For a gfp νf one provides an invariant l—a post-fixed point l ⊆ f(l)—and then the Knaster-Tarski theorem yields l ⊆ νf. However for an lfp μf the same argument would give an overapproximation;
instead we should appeal to the Cousot-Cousot theorem and consider the *approximation sequence* \( \bot \subseteq f(\bot) \subseteq \cdots \). The sequence eventually converges to \( \mu f \) (possibly after transfinite induction) \(^1\), hence for every ordinal \( \alpha \), the approximant \( f^\alpha(\bot) \) is an underapproximation of \( \mu f \). This is the underlying principle of proofs by *ranking functions*, e.g. of termination.

Progress measures in \(^{16}\), generalizing the combinatorial notion of the same name in Jurdziński’s algorithm for parity games \(^{24}\), are roughly combination of invariants and ranking functions. The latter two must be combined in a intricate manner so that they respect the order of equations in \( \mathbf{1} \) (i.e. priorities in parity games or \( \mu \)-calculus formulas); we do so with the help of a suitable truncated order.

Use of parity games is nowadays omnipresent, with the study of fair simulations being no exception \(^{10}\). Following those previous works, the basic idea behind our developments (below) is to generalize: parity games to equational systems (Def. \( \mathbf{2.1} \)); and accordingly, Jurdziński’s (combinatorial) progress measure to our lattice-theoretic one \(^{16}\).

The precise definition of progress measure, as well as its soundness and completeness results (against the solution of an equational system), is formally stated in Appendix \( \mathbf{B} \).

### 3 Fair Simulation for Nondeterministic Büchi Tree Automata

A *ranked alphabet* is a set \( \Sigma \) with a function \( \| | : \Sigma \to \mathbb{N} \) that gives an *arity* for each \( \sigma \in \Sigma \).

**Definition 3.1** (NBTA). A *nondeterministic Büchi tree automaton* (NBTA) is given by a quintuple \( \mathcal{X} = (X, \Sigma, \delta, I, Acc) \) consisting of a state space \( X \), a ranked alphabet \( \Sigma \), a *transition function* \( \delta : X \to \mathcal{P}(\prod_{\sigma \in \Sigma} X^{\| \sigma \}) \), a set \( I \subseteq X \) of the *initial states*, and a set \( Acc \subseteq X \) of the *accepting states* (often designated by \( \bigcirc \)).

Here we shall sketch necessary notions for defining accepted (tree) languages of NBTA. They are all as usual; precise definitions are found in Appendix \( \mathbf{C} \).

A \( \Sigma \)-tree is a possibly infinite tree whose nodes are labeled from \( \Sigma \) and each node, say labeled by \( \sigma \), has precisely \( |\sigma| \) children. We let \( \text{Tree}_\Sigma \) denote the set of \( \Sigma \)-trees.

A (possibly infinite) \( (\Sigma \times X) \)-labeled tree \( \rho \) is a *run* of an NBTA \( \mathcal{X} = (X, \Sigma, \delta, I, Acc) \) if: the \( X \)-label of its root is initial \( s \in I \); and for each node with a label \( (\sigma, x) \), it has \( |\sigma| \) children and we have \( (\sigma, (x_1, \ldots, x_{\| \sigma \})) \in \delta(x) \) where \( x_1, \ldots, x_{\| \sigma \} \) are the \( X \)-labels of its children.

A run \( \rho \) of an NBTA \( \mathcal{X} \) is said to be *accepting* if any infinite branch \( \pi \) of the tree \( \rho \) satisfies the Büchi acceptance condition (i.e. it visits accepting states (in \( Acc \)) infinitely often). The sets of runs and accepting runs of \( \mathcal{X} \) are denoted by \( \text{Run}_\mathcal{X}^\rho \) and \( \text{AccRun}_\mathcal{X}^\rho \), respectively.

The map \( \text{DelSt} : \text{Run}_\mathcal{X}^\rho \to \text{Tree}_\Sigma \) takes a run, removes its \( X \)-labels and returns a \( \Sigma \)-tree.

**Definition 3.2** (accepted language \( L(\mathcal{X}) \)). For an NBTA \( \mathcal{X} \), its (Büchi) *language* \( L(\mathcal{X}) \subseteq \text{Tree}_\Sigma \) is defined by \( L(\mathcal{X}) = \{ \text{DelSt}(\rho) \mid \rho \in \text{AccRun}_\mathcal{X}^\rho \} \).

We go on to introduce fair simulation for NBTA. Unlike the notion in \( \mathbf{3} \) that is defined combinatorially via a parity game, ours is expressed by means of equational systems \( \mathbf{2} \), hence is applicable to infinitary settings.

**Definition 3.3** (fair simulation for NBTA). Let \( \mathcal{X} = (X, \Sigma, \delta, I_X, Acc_X) \) and \( \mathcal{Y} = (Y, \Sigma, \delta_Y, I_Y, Acc_Y) \) be NBTA. Let \( X_1 = X \setminus Acc_X = \{ \bigcirc \} \), \( X_2 = Acc_X = \{ \bigcirc \} \), and similarly for \( Y = Y_1 \cup Y_2 \). A *fair simulation* from \( \mathcal{X} \) to \( \mathcal{Y} \) is a relation \( R \subseteq X \times Y \) such that:

1. For all \( x \in I_X \), there exists \( y \in I_Y \) such that \( (x, y) \in R \).

\(^1\) In case \( f \) is continuous the sequence converges after \( \omega \) steps. This is the Kleene fixed-point theorem.
2. Let $R_1, \ldots, R_4$ be the solution of the following equational system (note $\mu$’s vs. $\nu$’s).

\[
\begin{align*}
    u_1 &= \nu \square_{X,1}(\Diamond_{Y,1}(\Lambda_{\Sigma}(u_1 \cup u_2 \cup u_3 \cup u_4))) \subseteq X_1 \times Y_1 \\
    u_2 &= \mu \square_{X,2}(\Diamond_{Y,1}(\Lambda_{\Sigma}(u_1 \cup u_2 \cup u_3 \cup u_4))) \subseteq X_2 \times Y_1 \\
    u_3 &= \nu \square_{X,1}(\Diamond_{Y,2}(\Lambda_{\Sigma}(u_1 \cup u_2 \cup u_3 \cup u_4))) \subseteq X_1 \times Y_2 \\
    u_4 &= \mu \square_{X,2}(\Diamond_{Y,2}(\Lambda_{\Sigma}(u_1 \cup u_2 \cup u_3 \cup u_4))) \subseteq X_2 \times Y_2
\end{align*}
\] (2)

Then $R$ is below the solution, that is, $R \subseteq R_1 \cup \cdots \cup R_4$.

Here the functions $\square_{X,i} : \mathcal{P}(\prod_{\sigma \in \Sigma} X^{[\sigma]} \times Y) \to \mathcal{P}(X_i \times Y)$, $\Diamond_{Y,j} : \mathcal{P}(\prod_{\sigma \in \Sigma} X^{[\sigma]} \times Y) \to \mathcal{P}(\prod_{\sigma \in \Sigma} Y^{[\sigma]} \times \prod_{\sigma \in \Sigma} Y^{[\sigma]})$ and $\Lambda_{\Sigma} : \mathcal{P}(X \times Y) \to \mathcal{P}(\prod_{\sigma \in \Sigma} X^{[\sigma]} \times \prod_{\sigma \in \Sigma} Y^{[\sigma]})$ are defined as follows.

$\square_{X,i}(R') := \{(x, y) \in X_i \times Y \mid \forall a \in \delta_X(x). (a, y) \in R'\}$

$\Diamond_{Y,j}(R') := \{(a, y) \in \prod_{\sigma \in \Sigma} X^{[\sigma]} \times Y_j \mid \exists b \in \delta_Y(y). (a, b) \in R'\}$

$\Lambda_{\Sigma}(R') := \left\{ \left( (\sigma, x_1, \ldots, x_{[\sigma]}), (\sigma', y_1, \ldots, y_{[\sigma]}) \right) \in \prod_{\sigma \in \Sigma} X^{[\sigma]} \times \prod_{\sigma \in \Sigma} Y^{[\sigma]} \mid \sigma = \sigma', \right. \forall i. (x_i, y_i) \in R' \}$

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**Theorem 3.4 (soundness).** In the setting of Def. 3.3, existence of a fair simulation from $X$ to $Y$ implies language inclusion, that is, $L(X) \subseteq L(Y)$. ▪

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**Example 3.5.** Let $X$ and $Y$ be the NBTA's illustrated on the right, where a transition $z \xrightarrow{\sigma} (z_1, z_2)$ is represented by $z \xrightarrow{\square} \sigma \xrightarrow{\Diamond} z_1$. Here the ranked alphabet is given by $\Sigma = \{a, b\}$ where $|a| = |b| = 2$. Let $X$ and $Y$ be the state spaces of $X$ and $Y$ respectively, and define $X_1, X_2$ and $Y_1, Y_2$ as in Def. 3.3.

We can see that $R_1 = X_1 \times Y_1$, $R_2 = X_2 \times Y_1$, $R_3 = X_1 \times Y_2$ and $R_4 = X_2 \times Y_2$ are the solution of the equational system (2) in Def. 3.3 induced by $X$ and $Y$ here. Hence $R = X \times Y$ is a fair simulation from $X$ to $Y$, and this implies language inclusion.

Roughly speaking, a parity game is understood as a combinatorial presentation of an equational system like (2) (over finite lattices $L_1, \ldots, L_m$) [16]. Translating (2) leads to:

**Proposition 3.6 (parity game for NBTA fair simulation).** Let $X = (X, \Sigma, \delta_X, I_X, \text{Acc}_X)$ and $Y = (Y, \Sigma, \delta_Y, I_Y, \text{Acc}_Y)$ be NBTA’s. Let $G_{X,Y}$ be a parity game defined as follows.

| Position | Player | The set of possible moves | Priority |
|----------|--------|---------------------------|----------|
| $\ast$   | Odd    | $I_X \setminus \{(x, y) \mid y \in I_{Y}\}$ | 0        |
| $x \in X$| Even   | $\{(x, y) \mid y \in \delta_{X}(x)\}$ | 0        |
| $(x, y) \in X \times Y$ | Odd | $\left\{ \left( (\sigma, x_1, \ldots, x_{[\sigma]}), \sigma \right) \right\}_{x_{[\sigma]} \neq y} \in \delta_{X}(x)$ | 0        |
| $\left( (\sigma, x_1, \ldots, x_{[\sigma]}), y \right) \in \left( \prod_{\sigma \in \Sigma} X^{[\sigma]} \right) \times Y$ | Even | $\left\{ \left( (x, y_1, \ldots), \sigma \right) \right\}_{y_1 \neq y} \in \delta_{Y}(y)$ | 0        |
| $(p_1, \ldots, p_n) \in (X \times Y)^{\ast}$ | Odd | $\{p_i \mid 1 \leq i \leq n\}$ | 0        |

Here $X_1, X_2$ and $Y_1, Y_2$ are as in Def. 3.3. Then a fair simulation from $X$ to $Y$ exists if and only if the player Even is winning in $G_{X,Y}$; if that is the case we have $L(X) \subseteq L(Y)$. ▪

The last game is essentially the same as the one in [5] used for defining their (combinatorial) notion of fair simulation.
4 Fair Simulation for Finite-State Probabilistic Büchi Word Automata

In what follows we adopt the following conventions. The \((x, y)\)-entry of a matrix \(A \in [0, 1]^{X \times Y}\) is denoted by \(A_{x,y}\); the \(x\)-th entry of a vector \(\epsilon \in [0, 1]^X\) is \(\epsilon_x\). For \(A, B \in [0, 1]^{X \times Y}\), we write \(A \leq B\) if \(A_{x,y} \leq B_{x,y}\) for all \(x, y\).

**Definition 4.1 (PBWA).** A (generative) probabilistic Büchi word automaton (PBWA) is a quintuple \(\mathcal{A} = (X, A, M, \iota, \text{Acc})\) consisting of a countable state space \(X\), a countable alphabet \(A\), transition matrices \(M(a) \in [0, 1]^{X \times X}\) for each \(a \in A\), an initial distribution \(\iota \in [0, 1]^X\), and a set \(\text{Acc} \subseteq X\) of accepting states. We require that the matrices \(M(a)\) and the vector \(\iota\) are substochastic: \(\sum_{x \in X} \sum_{x' \in X} (M(a))_{x,x'} \leq 1\) for each \(x \in X\), and \(\sum_{x \in X} \iota_x \leq 1\).

The initial vector and transition matrices are substochastic: \(\sum_{a \in A} \sum_{x' \in X} (M(a))_{x,x'}\) and \(\sum_{x \in X} \iota_x\) are allowed to be strictly less than 1. The missing probabilities are for abort. We require \(\sum_{a \in A} \sum_{x' \in X} (M(a))_{x,x'} \leq 1\): this means our automaton is generative and it chooses which character \(a \in A\) to output. This is in contrast to a reactive automaton (that reads characters), in which case we would require \(\sum_{x' \in X} (M(a))_{x,x'} \leq 1\) for each \(a\).

We shall sketch the definition of accepted languages of PBWAs. This is rather standard (see [6] for a reactive variant); details are found in Appendix D.

The set \(A^\omega\) of all infinite words over \(A\) carries a canonical “cylindrical” measurable structure generated by \(\{wA^\omega \mid w \in A^*\}\). A run of a PBWA \(\mathcal{A} = (X, A, M, \iota, \text{Acc})\) is an infinite word over \(A \times X\); the set \((A \times X)^\omega\) of runs comes with a cylindrical measurable structure, too. It is then straightforward (like [6]) to define a subprobability measure \(\mu^\text{Run}_X\) on \((A \times X)^\omega\) induced by the PBWA \(\mathcal{A}\) (via Carathéodory’s theorem). Now it can be shown [6] that the set \(\text{AccRun}^\omega_X\) of accepting runs—visiting \(X\) infinitely often—is a measurable subset.

**Definition 4.2 (accepted language \(L(\mathcal{A})\)).** For a PBWA \(\mathcal{A}\), its accepted language is a subprobability measure \(L(\mathcal{A})\) over \(A^\omega\) defined by: \(L(\mathcal{A})(wA^\omega) = \mu^\text{Run}_X^{-1}(\text{DelSt}^{-1}(wA^\omega) \cap \text{AccRun}^\omega_X)\) for each \(w \in A^*\). Here \(\text{DelSt} = (\pi_1)^\omega: (A \times X)^\omega \to A^\omega\) is a projection.

We go on to introduce fair simulation for PBWAs. To the best of our knowledge this is the first one for probabilistic Büchi (word) automata. Note that our simulation is given by a matrix and not by a relation; this follows our previous work [14] [35].

**Definition 4.3 (fair simulation for PBWAs).** Let \(\mathcal{A} = (X, A, M_X, \tau_X, \text{Acc}_X)\) and \(\mathcal{Y} = (Y, A, M_Y, \tau_Y, \text{Acc}_Y)\) be probabilistic Büchi word automata with the same alphabet \(A\). A fair simulation from \(\mathcal{X}\) to \(\mathcal{Y}\) is a matrix \(A \in [0, 1]^{X \times X}\) subject to the following. Here we shall define \(X_1 = X \setminus \text{Acc}_Y\) and \(X_2 = \text{Acc}_Y\) (like in Def. 4.3) and define \(Y_1, Y_2\) likewise. Moreover, \(M_{\mathcal{X},i} \in [0, 1]^{X_i \times X_i}\), \(M_{\mathcal{Y},j} \in [0, 1]^{Y_j \times Y_j}\) and \(A_{i,j} \in [0, 1]^{Y_i \times X_j}\) denote the obvious partial matrices of \(M_{\mathcal{X}} \in [0, 1]^{X \times X}\), \(M_{\mathcal{Y}} \in [0, 1]^{Y \times Y}\) and \(A \in [0, 1]^{X \times Y}\), respectively.

1. The matrix \(A\) is a substochastic matrix: \(\sum_{x \in X} A_{x,y} \leq 1\) for each \(y \in Y\).
2. The matrix \(A\) is a forward simulation matrix [33] [35], that is, \(\tau_X \leq \tau_Y \cdot A\) and \(A \cdot M_X(a) \leq M_Y(a) \cdot A\) for each \(a \in A\).
3. The partial matrices \(A_{11} \in [0, 1]^{Y_1 \times X_1}\) and \(A_{12} \in [0, 1]^{Y_1 \times X_2}\) come with their approximation sequences. They are increasing sequences of length \(\overline{\alpha} \leq \alpha\):

\[
A_{11}^{(0)} \leq A_{11}^{(1)} \leq \cdots \leq A_{11}^{(\overline{\alpha})} \in [0, 1]^{Y_1 \times X_1} \quad \text{and} \quad A_{12}^{(0)} \leq A_{12}^{(1)} \leq \cdots \leq A_{12}^{(\overline{\alpha})} \in [0, 1]^{Y_1 \times X_2}
\]

s.t.

\[a. \text{(Approximate } A_{11} \text{ and } A_{12}) \text{ We have } A_{11}^{(\overline{\alpha})} = A_{11} \text{ and } A_{12}^{(\overline{\alpha})} = A_{12}.\]

\[b. \left(A_{11}^{(\overline{\alpha})}\right) \text{ For each } \alpha \leq \overline{\alpha} \text{ and } a \in A \text{ we have: } A_{11}^{(\alpha)} \cdot M_{X,1}(a) = A_{12}^{(\alpha)} \cdot \left(\frac{A_{11}^{(\alpha)}}{A_{21}}, \frac{A_{12}^{(\alpha)}}{A_{22}}\right).\]
c. \((A_{12}^{(0)}, \text{base})\) The 0-th approximant \(A_{12}^{(0)}\) is the zero matrix \(O\).

d. \((A_{12}^{(\alpha)}, \text{step})\) For each \(\alpha < \pi\) and \(a \in A\): \(A_{12}^{(\alpha+1)} \cdot M_{X,2}(a) \leq M_{Y,1}(a) \cdot \left(A_{11}^{(\alpha)} A_{12}^{(\alpha)} A_{22}\right)\).

e. \((A_{12}^{(\alpha)}, \text{limit})\) \(A_{12}^{(\alpha)}\) for each \(y \in Y_1\) and \(x \in X_2\).

The previous notion is the combination of: 1) Kleisli simulation (see [35] and also Table[c] later) for mimicking one-step behaviors; and 2) progress measure [16] that accounts the invariable 1(c) later) for mimicking one-step behaviors; and 2) progress measure [16] that accounts

The fair simulation notions in §3–4 (for nondeterminism and probability) may look different, but they arise from the same source, namely our coalgebraic study of Büchi automata [32].

5 Coalgebraic Backgrounds

The fair simulation notions in [3][4] (for nondeterminism and probability) may look different, but they arise from the same source, namely our coalgebraic study of Büchi automata [32].

5.1 Modeling a System as a Function \(X \rightarrow \text{TFX}\)

The conventional coalgebraic modeling of systems—as a function \(X \rightarrow \text{FX}\)—is known to capture branching-time semantics (such as bisimilarity) [21][31]. In contrast accepted languages of Büchi automata (with nondeterministic or probabilistic branching) constitute linear-time semantics; see [13] for so-called the linear-time–branching time spectrum.

For the coalgebraic modeling of such linear-time semantics we follow the “Kleisli modeling” tradition [15][19][30]. Here a system is parametrized by a monad \(T\) and an endofunctor \(F\) on \(\text{Sets}\): the former represents the branching type while the latter represents the (linear-time) transition type; and a system is modeled as a function of the type \(X \rightarrow \text{TFX}\).

A monad \(T\) is a construct from category theory [29]: it is a functor \(T: \mathcal{C} \rightarrow \mathcal{C}\) equipped with unit \(\eta_X: X \rightarrow \text{TX}\) and multiplication \(\mu_X: T^2X \rightarrow \text{TX}\), both given by arrows in \(\mathcal{C}\) for each object \(X \in \mathcal{C}\), subject to some axioms. In this paper we use two examples \(T = \mathcal{P}, \mathcal{G}\): the powerset monad \(\mathcal{P}\) (on the category \(\text{Sets}\) of sets and functions) for nondeterminism;

Another eminent approach to coalgebraic linear-time semantics is the Eilenberg-Moore one (see e.g. [1][22] : notably in the latter a system is expressed as \(X \rightarrow \text{FTX}\). The Eilenberg-Moore approach can be seen as a categorical generalization of determinization or the powerset construction. This however makes the approach hard to apply to infinite words or trees, since already for Büchi word automata, it is known that deterministic ones are less expressible than general, nondeterministic ones.
and the sub-Giry monad $\mathcal{G}$ (on $\text{Meas}$ of measurable spaces and measurable functions) for probabilistic branching. The latter is a “sub” variant of the well-known Giry monad \cite{Giry1991}.

\textbf{Definition 5.1} (the monads $\mathcal{P}$ and $\mathcal{G}$). The powerset monad $\mathcal{P}$ on $\text{Sets}$ carries a set $X$ to $\mathcal{P}X = \{ S \subseteq X \}$, and a function $f: X \to Y$ to $\mathcal{P}f: \mathcal{P}X \to \mathcal{P}Y$, $S \mapsto f[S] = \{ f(x) \mid x \in S \}$. For each set $X$, its unit $\eta_X^\mathcal{P}: X \to \mathcal{P}X$ is given by the singleton map $x \mapsto \{ x \}$; and its multiplication $\mu_X^\mathcal{P}: \mathcal{P}^2X \to \mathcal{P}X$ is by union $M \mapsto \bigcup_{A \in M} A$.

The sub-Giry monad $\mathcal{G}$ on $\text{Meas}$ carries a measurable space $(X, \mathcal{F}_X)$ to $(\mathcal{G}X, \mathcal{G}\mathcal{F}_X)$, where $\mathcal{G}X$ is the set of all subprobability measures on $X$ and $\mathcal{G}\mathcal{F}_X$ is the smallest $\sigma$-algebra such that, for each $S \in \mathcal{F}_X$, the function $\text{ev}_S: \mathcal{G}X \to [0,1]$ defined by $\text{ev}_S(P) = P(S)$ is measurable. The action of $\mathcal{G}$ on arrows is given by the pushforward measure: for $f: X \to Y$, $P \in \mathcal{G}X$ and $T \in \mathcal{F}_Y$, $(\mathcal{G}f)(P)(T) = P(f^{-1}(T))$. The unit $\eta_X^\mathcal{G}: X \to \mathcal{G}X$ is given by the Dirac measure $\eta_X^\mathcal{G}(x) = \delta_x$; and $\mu_X^\mathcal{G}: \mathcal{G}^2X \to \mathcal{G}X$ is given by $\Psi \mapsto \left( S \mapsto \int_{\mathcal{F}(X, \mathcal{F}_X)} \text{ev}_S d\Psi \right)$.

Intuitively $\eta_X^\mathcal{G}: X \to TX$ turns an element into a trivial branching while $\mu_X^\mathcal{G}: T^2X \to TX$ suppresses two successive branching into one. See \cite{Baier2008} for further illustration.

For the other parameter $F$—for the type of linear-time behaviors—we use the following.

\textbf{Definition 5.2} (the functors $F_\Sigma$ (on $\text{Sets}$) and $F_\mathcal{A}$ (on $\text{Meas}$)). Let $\Sigma$ be a ranked alphabet. The functor $F_\Sigma: \text{Sets} \to \text{Sets}$ carries a set $X$ to $F_\Sigma X = \bigsqcup_{\sigma \in \Sigma} X[\sigma]$; and a function $f$ to $\prod_{\sigma \in \Sigma} f[\sigma]$. Let $\mathcal{A}$ be a countable alphabet, thought of as a measurable set with the discrete $\sigma$-algebra. The functor $F_\mathcal{A} = \mathcal{A} \times (\_): \text{Meas} \to \text{Meas}$ carries a measurable space $X$ to the product space $\mathcal{A} \times X$; and a measurable map $f$ to $\text{id}_\mathcal{A} \times f$.

Our system models in \cite{BDM+2012} readily allow categorical modeling as arrows $X \to TFX$: the transition function of an NBTA (Def. 3.1) is a function $X \to \mathcal{P}F_\Sigma X$; and the transition matrices of a PBWA (Def. 4.1) collectively give a (measurable) function $X \to \mathcal{G}F_\mathcal{A} X$.

\subsection{5.2 Coalgebras in a Kleisli Category}

Given a monad $T$ on a category $\mathcal{C}$, the standard construction of the Kleisli category $\mathcal{K}(T)$ is defined as follows (see e.g. \cite{Biemann2005}): its objects are those of $\mathcal{C}$; its arrows $f: X \to Y$ are precisely arrows $f: X \to TY$ in $\mathcal{C}$; and its identity and composition $\circ$ are defined with the aid of unit $\eta^T$ and multiplication $\mu^T$. Intuitively a Kleisli arrow $f: X \to Y$ is a function from $X$ to $Y$ with $T$-branching. Then a system dynamics $X \to TFX$—with $T$-branching over linear-time $F$-behaviors—is a Kleisli arrow $X \to TX$, a (proper) $T$-coalgebra in $\mathcal{K}(T)$ \footnote{For distinction we write $\to$ for arrows in $\mathcal{K}(T)$ (not $\to$), and $\circ$ for composition in $\mathcal{K}(T)$ (not $\circ$).}.

Studies of coalgebras $X \to TX$ are initiated in \cite{Biemann2005} and developed henceforth e.g. in \cite{Baier2008,Biemann2005,Biemann2009,Biemann2011,Biemann2012,Klop2009} leading to the following coalgebraic theory of trace and simulation.

\textbf{(Table 1(a))} In \cite{Baier2008} it is shown that, for $T = \mathcal{P}$ (for nondeterminism) and $D$ (the \textit{subdistribution} monad on $\text{Sets}$ for discrete probabilities), and for a suitable functor $F$ on $\text{Sets}$, an initial $F$-algebra $\alpha: FA \Rightarrow A$ in $\text{Sets}$ yields a final $T$-coalgebra $J\alpha^{-1}: A \Rightarrow TX$. Here $J: \text{Sets} \to \mathcal{K}(T)$ is the Kleisli inclusion functor \cite{Biemann2005}. In case $F = F_\Sigma$ an initial algebra is given by the set of all \textit{finite} $\Sigma$-trees; and the unique morphism $\text{tr}(c): X \to A$—i.e. a function $\text{tr}(c): X \to TA$, see Table 1(a)—is nothing but the \textit{finite trace semantics} of the automaton $c: X \to TX$, capturing all the linear-time behaviors that eventually terminate.
For infinitary trace semantics (i.e. all possibly nonterminating linear-time behaviors) its coalgebraic characterization is more involved \[7,19\].

\[
\begin{array}{ccc}
\text{(a) Coalgebraic finite trace:} & \text{(b) Coalgebraic infinitary trace:} & \text{(c) Coalgebraic fwd. and bwd. simulation:} \\
F \alpha A \rightarrow A \text{ is an init. alg. in } \text{Sets} & Z \stackrel{\xi}{\rightarrow} FZ \text{ is a final coalg. in } \text{Sets} & \text{here } c \text{ is simulated by } d \\
\end{array}
\]

\textbf{Table 1} Some known results in the coalgebraic theory of trace and simulation

5.3 Coalgebraic Modeling of Büchi Automata

In the above theory—and in the theory of coalgebra in general—the Büchi acceptance condition has long been considered a big challenge: its nonlocal character (“visit infinitely often”) does not go along with the coalgebraic, local idea of behaviors that is centered around acceptance conditions. Its coalgebraic characterization is more involved \[7,19\]. In the above setting, and also for \(T = G\) on \(\text{Meas}\), it is shown that a final coalgebra \(\xi: Z \rightarrow FZ\)—we have \(Z \rightarrow \text{Tree}_G\) when \(F = FZ\)—yields a weakly final coalgebra \(J\xi\) in \(\mathcal{KL}(T)\). Given \(c\) there is thus at least one morphism from \(c\) to \(J\xi\); there is also a maximal such \(\text{tr}_\infty(c)\), and this is how we capture infinitary trace. In Table 1(b) we indicate this maximality by \(\nu\).

\textbf{(Table 1(b))} In \[14\] it is shown that lax/oplax homomorphisms (Table 1(c)) witness finite trace inclusion \((\text{tr}(c)) \sqsubseteq \text{tr}(d)\). When \(T = \mathcal{P}\) these notions specialize to forward and backward simulation in \[27\], i.e. binary relations that “mimic.” In \[34\] they are shown to witness infinitary trace inclusion too; this is the starting point of the current study of (forward) simulation for Büchi automata. Note that, when \(T = G\), our (fwd.) “simulation” is not a relation but a “function with probabilistic branching” \(f: Y \rightarrow G\mathcal{X}\). The latter is roughly a matrix of dimension \(|Y| \times |X|\); and algorithms to find such are studied in \[35\].

**Definition 5.3** (Büchi \((T, F)\)-system). Let \(T\) be a monad, and \(F\) be an endofunctor, both on some category \(C\) with binary coproducts + and a nullary product 1. Assume also that \(F\) lifts to \(\mathcal{T}\). \(\mathcal{KL}(T) \rightarrow \mathcal{KL}(T)\) with \(FX = F\mathcal{X}\) on objects.

A Büchi \((T, F)\)-system is given by a tuple \(X = (X_1, X_2), c: X \rightarrow F\mathcal{X}, s: 1 \rightarrow X)\) where:

- \(X_1\) and \(X_2\) are objects of \(C\) (with the intuition that \(X_1\) = \{non-accepting states \(\bigcirc\)\} and \(X_2\) = \{accepting states \(\bigcirc\)\}), and we define \(X := X_1 + X_2\);
- \(c: X \rightarrow F\mathcal{X}\) is an arrow in \(\mathcal{KL}(T)\) for dynamics; and
- \(s: 1 \rightarrow X\) is an arrow in \(\mathcal{KL}(T)\) for initial states.

For each \(i \in \{1, 2\}\), we define \(\kappa_i: X_i \rightarrow F\mathcal{X}\) to be the restriction \(c \circ \kappa_i: X_i \rightarrow TFX\) of \(c\) along the coprojection \(\kappa_i: X_i \hookrightarrow X\).
Thus a Büchi \((T,F)\)-system is a (Kleisli) coalgebra \(X \to FX\) augmented with the information on accepting and initial states. The following is straightforward; note that an arrow \(1 \to X\) in \(\mathcal{K}(\mathcal{G})\) is nothing but a probability subdistribution over \(X\).

**Lemma 5.4.** 1. An NBTA (Def. 3.7) over \(\Sigma\) gives rise to a Büchi \((\mathcal{P},F_\Sigma)\)-system.
2. A PBWA (Def. 4.1) over an alphabet \(A\) gives rise to a Büchi \((\mathcal{G},F_\mathcal{G})\)-system.

The next is (a special case of) the main theorem of [32]. Recall that \(\text{Tree}_\Sigma\) is the set of (possibly infinite) \(\Sigma\)-trees (Appendix C); it carries a final coalgebra \(\zeta: \text{Tree}_\Sigma \to F_\Sigma(\text{Tree}_\Sigma)\) in \(\text{Sets}\). We will be using natural orders on the homsets \(\mathcal{K}(\mathcal{P})(X,Y)\) and \(\mathcal{K}(\mathcal{G})(X,Y)\), given by inclusion and pointwise extension of the order on \([0,1]\), respectively.

**Theorem 5.5 ([32]).** 1. Let \(X = ((X_1,X_2),c,s)\) be a Büchi \((\mathcal{P},F_\Sigma)\)-system. Consider an equational system
\[
u_1 =_\mu (J\zeta)^{-1} \circ F_\Sigma[u_1,u_2] \circ c_1 , \quad \nu_2 =_\mu (J\zeta)^{-1} \circ F_\Sigma[u_1,u_2] \circ c_2 \tag{3}
\]
where \(u_i\) ranges over the homset \(\mathcal{K}(\mathcal{P})(X_i,\text{Tree}_\Sigma)\) for \(i \in \{1,2\}\). Diagrammatically:
\[
\begin{array}{ccc}
F_\Sigma X & \xrightarrow{\nu_1} & F_\Sigma(\text{Tree}_\Sigma) \\
\downarrow{c_1} & \Downarrow{J\zeta} \cong & \Downarrow{J\zeta} \\
X_1 & \xrightarrow{u_1} & \text{Tree}_\Sigma \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
F_\Sigma X & \xrightarrow{\nu_2} & F_\Sigma(\text{Tree}_\Sigma) \\
\downarrow{c_2} & \Downarrow{J\zeta} \cong & \Downarrow{J\zeta} \\
X_2 & \xrightarrow{u_2} & \text{Tree}_\Sigma \\
\end{array}
\tag{4}
\]
a. The equational system has a solution, denoted by \(\text{tr}^B(c_i): X_i \to \text{Tree}_\Sigma\) for \(i \in \{1,2\}\).

b. Let \(\text{tr}^B(\mathcal{X}): \{\star\} = 1 \xrightarrow{s} X = X_1 + X_2 \xrightarrow{\{\text{tr}^B(c_1),\text{tr}^B(c_2)\}} \text{Tree}_\Sigma\) be a composite in \(\mathcal{K}(\mathcal{P})\). In case \(\mathcal{X}\) is induced by an NBTA, the set \(\text{tr}^B(\mathcal{X})(\star) \subseteq \text{Tree}_\Sigma\) coincides with the (Büchi) language \(L(\mathcal{X})\) of \(\mathcal{X}\) (Def. 3.3).

2. Let \(\mathcal{X}\) be a Büchi \((\mathcal{G},F_\mathcal{G})\)-system, and consider the same equational system as in (3), but with \(\mathcal{G},F_\mathcal{G},\mathcal{A}^\omega\) replacing \(\mathcal{P},F_\Sigma,\text{Tree}_\Sigma\). Then:

a. The equational system has a solution.

b. Let \(\mathcal{X}\) be induced by a PBWA (Def. 4.1). For the same composite \(\text{tr}^B(\mathcal{X}): 1 \to \mathcal{A}^\omega\) as above we have \(\text{tr}^B(\mathcal{X})(\star) = L(\mathcal{X}) \in \mathcal{G}(\mathcal{A}^\omega)\), the Büchi language of the PBWA (4). ▶

### 6 Coalgebraic Account on Fair Simulations and Soundness Proofs

Here we lay out our coalgebraic study of fair simulations. We are first led to a simulation notion “with dividing,” that is coalgebraically neat but not desirable from a practical viewpoint. Circumventing the dividing construct we obtain the simulation notions in §6.1

The last part of circumventing dividing is different for \(T = \mathcal{P}\) and \(\mathcal{G}\) (hence different definitions of simulation). While one would hope for uniformity, we suspect it to be hard, for the following reason. We observed [35] that the characterization of infinite trace (Table 1b)) is true for \(T = \mathcal{P}\) and \(\mathcal{G}\), but because of different categorical machineries. Since infinite trace is a special case of Büchi acceptance (where every state is accepting) and our soundness proof should rely on its characterization, we expect that this sharp contrast would still stand.

#### 6.1 Coalgebraic Fair Simulation with Dividing

We make the following minimal requirements so that our definitions will make sense.

**Assumption 6.1.** In this section we assume the following conditions on \(T\) and \(F\) on \(\mathcal{C}\).

1. The base category \(\mathcal{C}\) has a final object 1, binary products and countable coproducts.
2. The functor \( F \) has a final coalgebra \( \zeta : Z \cong FZ \) in \( C \).
3. The functor \( F : C \to C \) lifts to \( \overline{F} : \mathcal{KL}(T) \to \mathcal{KL}(T) \) in the sense of \( \overline{F} \circ J = J \circ F \). In particular \( \overline{F}X = FX \) on objects.
4. For each \( X, Y \in \mathcal{KL}(T) \), the homset \( \mathcal{KL}(T)(X, Y) \) carries a partial order \( \sqsubseteq_{X,Y} \).
5. For each \( X, Y \in \mathcal{KL}(T) \), \( \mathcal{KL}(T)(X, Y) \) has the least element \( \bot_{X,Y} : X \to Y \).

The category \( \mathcal{KL}(T) \) comes with two operations, what we call codomain restriction \( g|_{X}^{\alpha} \) and codomain join \( \langle g \rangle_{i} \). The former takes an arrow \( g : V \to \bigsqcup_{i} X_{i} \) and returns \( g|_{X}^{\alpha} : V \to X_{\alpha} \); and the latter is a partial operation that takes a family \( (g_{i} : V \to X_{i})_{i} \in I \) and returns \( \langle g \rangle_{i} : V \to \bigsqcup_{i} X_{i} \). We require the following conditions.

a. The two operations are partially mutually inverse, in the following sense. Given \( g : V \to \bigsqcup_{i} X_{i} \), the codomain join \( \langle g \rangle_{i} \) is always defined and is equal to \( g \).

Conversely, provided that \( \langle g \rangle_{i} \) is defined for a family \( (g_{i})_{i} \), we have \( \langle g \rangle^{X} \rangle_{i} \) is equal to \( g \).

b. Both operators are monotone.

c. (Joinability is downward closed) Let \( f_{i}, g_{i} : V \to X_{i} \) be such that \( f_{i} \subseteq g_{i} \) for each \( i \in I \). If \( \langle g \rangle_{i} \) is defined, then so is \( \langle f \rangle_{i} \).

Cond. \( c \) while it may look unfamiliar, is clearly satisfied by \( T = \mathcal{P}, \mathcal{G} \) by suitably restricting/joining subsets/distributions. Codomain joins are always defined for \( \mathcal{P} ; \) for \( \mathcal{G} \) it is needed that \( \sum_{\alpha} g_{i}(\alpha) \subseteq X_{i} \) for each \( \alpha \in V \). These notions come from that of bicartesian in \( \mathcal{P} \).

With the help of codomain restrictions/joins we define a categorical fair simulation.

\[ \text{Definition 6.2 (forward fair simulation with dividing). Let } T \text{ and } F \text{ be subject to Asm. 6.1. } \]
\[ X = ((X_{1}, X_{2}), c, s) \text{ and } Y = ((Y_{1}, Y_{2}), d, t) \text{ be Büchi } (T, F)\text{-systems; and } \pi \text{ be an ordinal. } \]

A (forward, \( \pi \)-bounded) fair simulation with dividing from \( X \) to \( Y \) is an arrow \( f : Y \to X \) in \( \mathcal{KL}(T) \) subject to the following conditions. Below, for simplicity, a domain-and-codomain restriction \( (f \circ \kappa_{i})|_{X}^{\alpha} \) is denoted by \( f_{i}^{\alpha} \); and we refer to \( f_{11}, f_{12}, f_{21}, f_{22} \) as components of a fair simulation \( f \).

1. The arrow \( f : Y \to X \) is a forward simulation from \( X \) to \( Y \) in the sense of \( \mathcal{P} \) (see also Table \( 1(c) \) on the left). That is: \( c \circ f \subseteq \overline{F}f \circ d \) and \( s \subseteq f \circ t \).

2. The components \( f_{11} : Y_{1} \to X_{1} \) and \( f_{12} : Y_{1} \to X_{2} \) come with dividing of \( d_{1} \) and approximation sequences. The former is a pair \( d_{11}, d_{12} : Y_{1} \to \overline{F}X \) s.t. \( [id_{Y_{1}}, id_{X_{1}}] \circ [d_{11}, d_{12}] = d_{1} \). The latter are (possibly transfinite) increasing sequences of length \( \pi \):

\[ f_{11}^{(0)} \subseteq f_{11}^{(1)} \subseteq \cdots \subseteq f_{11}^{(\pi)} : Y_{1} \to X_{1} \]

and

\[ f_{12}^{(0)} \subseteq f_{12}^{(1)} \subseteq \cdots \subseteq f_{12}^{(\pi)} : Y_{1} \to X_{2} \]

s.t.

a. (Approximate \( f_{11} \) and \( f_{12} \)) We have \( f_{11}^{(\pi)} = f_{11} \) and \( f_{12}^{(\pi)} = f_{12} \).

b. (\( f_{11}^{(\alpha)} \)) For each ordinal \( \alpha \) such that \( \alpha \leq \pi \), the inequality \( 5 \) holds. Note that the arrow \( \langle f_{12}^{(\alpha)} \rangle_{j}, \langle f_{21}, f_{22} \rangle : Y \to X \) therein is one induced by the constructions in Asm. \( 6.1 \). Note also that the required codomain joins do exist.

c. (\( f_{12}^{(\alpha)} \), the base case) For the 0-th approximant, we have \( f_{12}^{(0)} = \bot \).

d. (\( f_{12}^{(\alpha)} \), the step case) For each ordinal \( \alpha \) such that \( \alpha < \pi \), the inequality \( 6 \) holds.

e. (\( f_{12}^{(\alpha)} \), the limit case) If \( \alpha \) is a limit ordinal, then the supremum \( \bigsqcup_{\alpha < \alpha} f_{12}^{(\alpha)} \) exists and \( f_{12}^{(\alpha)} \subseteq \bigsqcup_{\alpha < \alpha} f_{12}^{(\alpha)} \).

\[ \text{Theorem 6.3 (soundness). Let } \pi \text{ be an ordinal. Assume Asm. 6.1 and the following.} \]
7. For an arbitrary Büchi \((T, F)\)-system \(X\), the equational system \([\mathcal{B}]\), with \(F, Z\) replacing \(F_\Sigma, T_{\text{eq}}\), has a (necessarily unique) solution \(\text{tr}^B(c_1), \text{tr}^B(c_2)\).

8. The Kleisli composition \(\circ\), as well as cotupling \([\_\_\_]\) of Kleisli arrows, are monotone with respect to the order \(\subseteq\).

9. The functor \(\mathcal{T}: \mathcal{K}(T) \to \mathcal{K}(T)\) is locally monotone: \(f \subseteq g\) implies \(\mathcal{T}f \subseteq \mathcal{T}g\).

10. The Kleisli composition \(\circ\) is both left and right-strict: \(\bot \circ f = \bot\) and \(f \circ \bot = \bot\).

11. For each \(X, Y\), \(\mathcal{K}(T)(X, Y)\) is \(\omega\)-complete, \(\mathcal{T}\) is locally \(\omega\)-continuous and \(\circ\) is \(\omega\)-continuous.

12. For each limit ordinal \(\alpha \leq \pi\), \(\mathcal{T}\) is locally \(\alpha\)-continuous and \(\circ\) is \(\alpha\)-continuous (in the sense that they preserve whatever \(\alpha\) suprema that exist).

Then a fair \(\pi\)-bounded simulation with dividing, from one Büchi \((T, F)\)-system \(X\) to another \(Y\), witnesses trace inclusion \(\text{tr}^B(X) \subseteq \text{tr}^B(Y)\): \(1 \to T Z\).

We have thus obtained a sound simulation notion that, however, requires dividing of \(d_1: Y_1 \to \mathcal{T}Y\) into \(d_{11}, d_{12}: Y_1 \to \mathcal{T}Y\). Intuitively this is to divide the simulator’s “resources” into two parts, one for simulating challenger’s non-accepting states and the other for accepting states. Finding such “resource allocation” is a challenge in practice; additionally, insistence on such allocation being static is overly restrictive, as Example [E.1] suggests.

The following is a definition that is more desirable in this respect; it indeed yields Def. 3.3 and \([4, 3]\) as its instances. Note that it is not sound in the general sense of Thm. 6.3—the rest of the paper is devoted to finding special cases in which it is sound.

**Definition 6.4** (fair simulation without dividing). In the setting of Def. [6.2], a (forward, \(\pi\)-bounded) fair simulation without dividing is much like in Def. [6.2] but: 1) we do not require dividing \(d_{11}, d_{12}\), and 2) in the diagrams \([6] \) we put \(d_1\) in place of \(d_{11}\) and \(d_{12}\).

### 6.2 Circumventing Dividing: the Nondeterministic Case

For \(T = \mathcal{P}\) we show that a simulation without dividing yields one with dividing. We exploit the idempotency property of \(T = \mathcal{P}\)—one can copy resources as many times as one likes.

**Proposition 6.5** (soundness under idempotency). Under Asm. [6.7], let us assume that each arrow \(f: X \to Y\) in \(\mathcal{K}(T)\) is idempotent, that is, the codomain join \(\llbracket f, f \rrbracket: X \to Y + Y\) necessarily exists and we have \([\text{id}_Y, \text{id}_Y] \circ \llbracket f, f \rrbracket = f\).

1. A simulation without dividing yields one with dividing, with the dividing \(d_{11} = d_{12} = d_1\).
2. In the setting of Thm. [6.3], a simulation without dividing witnesses trace inclusion.

**Lemma 6.6.** Arrows in \(\mathcal{K}(\mathcal{P})\) are idempotent. Hence all the conditions in Prop. [6.5] are satisfied by \(T = \mathcal{P}\) and \(F = F_\Sigma\), where \(\Sigma\) is a ranked alphabet.

There is still a gap between simulation in Def. [6.4] (defined by inequalities) and that in Def. [3.3] (defined by an equational system). The gap is filled by another specific property of \(\mathcal{K}(\mathcal{P})\)—reversibility. It is much like in the “must” predicate transformers.

**Lemma 6.7.** Let \(f: Y \to Z\) be an arrow in \(\mathcal{K}(\mathcal{P})\). We define \(\square f: \mathcal{K}(\mathcal{P})(X, Z) \to \mathcal{K}(\mathcal{P})(X, Y)\) by \(\square f(g)(x) = \{ y \in Y \mid f(y) \subseteq g(x) \}\). Then we have \(f \circ \square f(g) \subseteq g\) and moreover, for all \(h: X \to Y\) such that \(f \circ h \subseteq g\), we have \(h \subseteq \square f(g)\).

The last construction \(\square f\) is used to essentially “reverse” the arrows \(c_1\) and \(c_2\) on the right of the diagrams \([6] \) below, yielding a (proper) equational system.
\textbf{Proposition 6.8.} Let $g_1^{\text{sol}} : Y_1 \rightarrow X_1$, $g_2^{\text{sol}} : Y_1 \rightarrow X_2$, $g_3^{\text{sol}} : Y_2 \rightarrow X_1$ and $g_4^{\text{sol}} : Y_2 \rightarrow X_2$ be the solution of the following equational system.

\begin{align}
  g_1 &= \nu \sqcap_c (\mathcal{F}(\{g_1, g_2\}, \{g_1, g_3\}) \circ d_1) \in \mathcal{K}(\mathcal{P})(Y_1, X_1) \\
  g_2 &= \mu \sqcap_c (\mathcal{F}(\{g_1, g_2\}, \{g_3, g_4\}) \circ d_1) \in \mathcal{K}(\mathcal{P})(Y_1, X_2) \\
  g_3 &= \nu \sqcap_c (\mathcal{F}(\{g_1, g_2\}, \{g_3, g_4\}) \circ d_2) \in \mathcal{K}(\mathcal{P})(Y_2, X_1) \\
  g_4 &= \nu \sqcap_c (\mathcal{F}(\{g_1, g_2\}, \{g_3, g_4\}) \circ d_2) \in \mathcal{K}(\mathcal{P})(Y_2, X_2)
\end{align}

Let $g^{\text{sol}} = ([g_1^{\text{sol}}, g_2^{\text{sol}}, g_3^{\text{sol}}, g_4^{\text{sol}}]) : Y \rightarrow X$. Then $s \subseteq g^{\text{sol}} \circ t$ if and only if there is a fair $\pi$-bounded simulation without dividing (Def. 6.4) from $X$ to $Y$ for some ordinal $\pi$. ◼

Through Prop. 6.8 and that $\mathcal{F}$ and $\circ$ in $\mathcal{K}(\mathcal{P})$ are $\alpha$-continuous for an arbitrary limit ordinal $\alpha$, it is not hard to translate Prop. 6.5 into Thm. 3.4. An explicit proof is in Appendix F.3.2.

### 6.3 Circumventing Dividing: the Probabilistic Case

We turn to the probabilistic setting and prove Thm. 4.4. The strategy for $T = \mathcal{P}$ does not work here, because $\mathcal{G}$ lacks idempotency (cf. Prop. 6.5). We shall rely on other restrictions, from trees to words and a finite state (on the simulated side).

\textbf{Proposition 6.9.} Besides Asm. 6.1 and the assumptions in Thm. 6.3 assume $\text{tr}^{B}(d_1) = \bot$.

Then existence of a fair simulation without dividing from $X$ to $Y$ implies trace inclusion. ◼

The following (non-coalgebraic) lemma states that if $T = \mathcal{G}$, $F = A \times (\_)$ and the state space of $Y$ is finite, then we can modify $Y$ so that the assumption in Prop. 6.9 holds without changing its language. The modification derives from the well-known \textit{fairness} result on Markov chains (see e.g. [4, Chap. 10]); concretely, it states that a nonaccepting state $\sqcap$ from which an accepting state is repeatedly reached in a positive probability can be changed into an accepting state $\sqcup$. The proof uses the notion of \textit{bottom strongly connected component}.

\textbf{Lemma 6.10.} Let $A$ be a countable set and $Y = ((Y_1, Y_2), d, t)$ be a Büchi ($\mathcal{G}, A \times (\_)$)-system s.t. $Y_1$ and $Y_2$ are finite sets. Let $y_{\geq 0} \in Y_1$ be a state s.t. $\text{tr}^{B}(d_1)(y_{\geq 0})(A^\omega) > 0$. We define a Büchi ($\mathcal{G}, A \times (\_)$)-system $Y' = ((Y_1', Y_2'), d', t')$ by: $Y_1' = Y_1 \setminus \{y_{\geq 0}\}$, $Y_2' = Y_2 + \{y_{\geq 0}\}$, $c' = c$ and $s' = s$. Note that $d'$ and $t'$ are well-defined because $Y_1' + Y_2' = Y_1 + Y_2$.

Then we have $[\text{tr}^{B}(d_1), \text{tr}^{B}(d_2)] = [\text{tr}^{B}(d_1'), \text{tr}^{B}(d_2')]$, and moreover, $\text{tr}^{B}(Y) = \text{tr}^{B}(Y')$. ◼

With Lem. 6.10 discharging its assumptions, Prop. 6.9 easily yields Thm. 4.4. The finiteness restriction of $Y$ in Thm. 4.4 is strict: without it soundness fails. See Example 4.2. Whether the restriction to words (not trees) is necessary is still open.

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A Solutions of Equational Systems

Definition A.1 (solution). Let \( E \) be an equational system in Def. 2.1. For each \( i \in [1, m] \) and \( j \in [1, i] \), we define monotone functions \( f_i^1 : L_i \times \cdots \times L_m \to L_i \) and \( l_j^{(i)} : L_{i+1} \times \cdots \times L_m \to L_j \) by induction on \( i \) as follows. When \( i = 1 \),

\[
\begin{align*}
  f_1^1(l_1, \ldots, l_m) &:= f_1(l_1, \ldots, l_m), & \text{and} \\
  l_1^{(1)}(l_2, \ldots, l_m) &:= \begin{cases} 
    l_{\text{lf}} & (\eta_1 = \mu \text{ and } f_1^1(l_2, \ldots, l_m) \text{ has the lfp } l_{\text{lf}} \in L_1) \\
    l_{\text{gfp}} & (\eta_1 = \nu \text{ and } f_1^1(l_2, \ldots, l_m) \text{ has the gfp } l_{\text{gfp}} \in L_1) \\
    \text{undefined} & (\text{otherwise}).
  \end{cases}
\end{align*}
\]

Note here that completeness of \( L_1 \) is not assumed and therefore the monotone function \( f_1^1(l_2, \ldots, l_m) : L_1 \to L_1 \) does not necessarily have the lfp or gfp.

For the step case, the function \( f_{i+1}^1 \) is defined using the \( i \)-th interim solutions \( l_i^{(i)} \) for the variables \( u_1, \ldots, u_i \) obtained so far:

\[
\begin{align*}
  f_{i+1}^1(l_{i+1}, \ldots, l_m) &:= \begin{cases} 
    f_{i+1}(l_i^{(i)}(l_{i+2}, \ldots, l_m), \ldots, f_i^{(i)}(l_{i+1}, \ldots, l_m), l_{i+1}, \ldots, l_m) & (l_i^{(i)}(l_{i+1}, \ldots, l_m) \text{ is defined for each } j \in [1, i]) \\
    \text{undefined} & (\text{otherwise})
  \end{cases}
\end{align*}
\]

For \( j = i + 1 \), \( l_j^{(i+1)} \) is defined by

\[
\begin{align*}
  l_{i+1}^{(i+1)}(l_{i+2}, \ldots, l_m) &:= \begin{cases} 
    l_{\text{lf}} & (\eta_{i+1} = \mu, \text{ and } f_{i+1}^{(i)}(l_{i+2}, \ldots, l_m) \text{ has the lfp } l_{\text{lf}} \in L_{i+1}) \\
    l_{\text{gfp}} & (\eta_{i+1} = \nu, \text{ and } f_{i+1}^{(i)}(l_{i+2}, \ldots, l_m) \text{ has the gfp } l_{\text{gfp}} \in L_{i+1}) \\
    \text{undefined} & (\text{otherwise}).
  \end{cases}
\end{align*}
\]

For \( j \in [1, i] \), \( l_j^{(i+1)} \) is defined using \( f_{i+1}^{(i)}(l_{i+2}, \ldots, l_m) \) as follows.

\[
\begin{align*}
  l_j^{(i+1)}(l_{i+2}, \ldots, l_m) &:= \begin{cases} 
    l_i^{(i)}(l_{i+1}^{(i)}(l_{i+2}, \ldots, l_m), l_{i+2}, \ldots, l_m) & (l_i^{(i)}(l_{i+1}^{(i)}(l_{i+2}, \ldots, l_m) \text{ is defined}) \\
    \text{undefined} & (\text{otherwise})
  \end{cases}
\end{align*}
\]

A family \( (l_1^m, \ldots, l_m^m) \in L_1 \times \cdots \times L_m \) is called the solution of \( E \) if \( l_j^{(m)} : 1 \to L_j \) is defined and \( l_j^{(m)} = l_j^{(i)}(\ast) \) for each \( j \in [1, m] \).

It is easy to see that all the functions \( f_i^1 \) and \( l_j^{(i)} \) involved here are monotone. Whether a solution exists or not depends how “complete” each \( L_i \) is.

B Progress Measures

The following notion embodies the idea of priority.

Definition B.1 (prioritized ordinal, \( \preceq_i \)). Let \( E \) be the equational system in [1] of Def. 2.1. Let us collect the indices of \( \mu \)-variables: \( \{i_1, \ldots, i_k\} = \{i \in [1, m] \mid \eta_i = \mu \text{ in [1]}\} \), and assume that \( i_1 < \cdots < i_k \). A prioritized ordinal for \( E \) is a \( k \)-tuple \( (\alpha_1, \ldots, \alpha_k) \) of ordinals.
For each \( i \in [1, m] \) we define a preorder \( \preceq_i \) between prioritized ordinals—called the \( i \)-th truncated pointwise order—as follows. Let \( a \in [1, k] \) be such that \( i_1 < \cdots < i_{a-1} < i \leq i_a < \cdots < i_k \), that is, \( u_{i_a} \) is the \( \mu \)-variable with the smallest priority above that of \( i \). Then we define \((\alpha_1, \ldots, \alpha_k) \preceq_i (\alpha'_1, \ldots, \alpha'_k)\) if, between the \( i \)-truncations \((\alpha_a, \ldots, \alpha_k)\) and \((\alpha'_a, \ldots, \alpha'_k)\) of the prioritized ordinals, we have \( \alpha_i \leq \alpha'_i \) for each \( i \in [a, k] \).

**Definition B.2** (progress measure for an equational system). Assume the same setting as in Def. 2.1. We further assume that for each \( i \in [1, m] \), \( L_i \) has the smallest element \( \bot \). A progress measure \( p \) for \( E \) is given by a tuple \( p = (\{(\alpha_1, \ldots, \alpha_k)\}, \{p_i(\alpha_1, \ldots, \alpha_k)\}_{i, 0, \ldots, m}) \) that consists of:

- the maximum prioritized ordinal \((\overline{\alpha_1}, \ldots, \overline{\alpha_k})\); and
- the approximants \( p_i(\alpha_1, \ldots, \alpha_k) \in L_i \), defined for each \( i \in [1, m] \) and each prioritized ordinal \((\alpha_1, \ldots, \alpha_k)\) such that \( \alpha_1 \leq \overline{\alpha_1}, \ldots, \alpha_k \leq \overline{\alpha_k} \).

The approximants \( p_i(\alpha_1, \ldots, \alpha_k) \) are subject to:

1. **(Monotonicity)** For each \( i \in [1, m] \), \((\alpha_1, \ldots, \alpha_k) \leq_i (\alpha'_1, \ldots, \alpha'_k)\) implies \( p_i(\alpha_1, \ldots, \alpha_k) \sqsubseteq p_i(\alpha'_1, \ldots, \alpha'_k) \).
2. **(\( \mu \)-variables, base case)** Let \( a \in [1, k] \). Then \( \alpha_a = 0 \) implies \( p_{\alpha_a}(\alpha_1, \ldots, \alpha_a, \ldots, \alpha_k) = \bot \).
3. **(\( \mu \)-variables, step case)** Let \( a \in [1, k] \). Then there exist ordinals \( \beta_1, \ldots, \beta_{a-1} \) such that \( \beta_1 \leq \overline{\alpha_1}, \ldots, \beta_{a-1} \leq \overline{\alpha_{a-1}} \) and

\[
P_{\alpha_a}(\alpha_1, \ldots, \alpha_{a-1}, \alpha_a+1, \alpha_{a+1}, \ldots, \alpha_k) \sqsubseteq f_{\alpha_a} \left( \begin{array}{c}
p_1(\beta_1, \ldots, \beta_{a-1}, \alpha_a, \alpha_{a+1}, \ldots, \alpha_k), \\
\vdots \\
p_m(\beta_1, \ldots, \beta_{a-1}, \alpha_a, \alpha_{a+1}, \ldots, \alpha_k)
\end{array} \right).
\]

(8)

4. **(\( \mu \)-variables, limit case)** Let \( a \in [1, k] \) and let \( \alpha_a \) be a limit ordinal. Then the supremum \( \bigsqcup_{\beta < \alpha_a} p_{\alpha_a}(\alpha_1, \ldots, \beta, \ldots, \alpha_k) \in L_{\alpha_a} \) exists and we have:

\[
P_{\alpha_a}(\alpha_1, \ldots, \alpha_a, \ldots, \alpha_k) \sqsubseteq \bigsqcup_{\beta < \alpha_a} p_{\alpha_a}(\alpha_1, \ldots, \beta, \ldots, \alpha_k).
\]

(9)

5. **(\( \nu \)-variables)** Let \( i \in [1, m] \backslash \{i_1, \ldots, i_k\} \); and let \( a \in [1, k] \) be such that \( i_1 < \cdots < i_{a-1} < i < i_a < \cdots < i_k \). Let \((\alpha_1, \ldots, \alpha_k)\) be a prioritized ordinal. Then there exist ordinals \( \beta_1, \ldots, \beta_{a-1} \) such that \( \beta_1 \leq \overline{\alpha_1}, \ldots, \beta_{a-1} \leq \overline{\alpha_{a-1}} \) and

\[
P_i(\alpha_1, \ldots, \alpha_{a-1}, \alpha_a, \ldots, \alpha_k) \sqsubseteq f_i \left( \begin{array}{c}
p_1(\beta_1, \ldots, \beta_{a-1}, \alpha_a, \alpha_{a+1}, \ldots, \alpha_k), \\
\vdots \\
p_m(\beta_1, \ldots, \beta_{a-1}, \alpha_a, \alpha_{a+1}, \ldots, \alpha_k)
\end{array} \right).
\]

(10)

The definition combines the features of ranking functions (Cond. 3) and invariants (Cond. 5). Note also that in each clause ordinals with smaller priorities can be modified to arbitrary \( \beta_i \).

Note here that the definition of a progress measure in Def. B.2 is slightly different from the one in [16] in the following points.

1. Cond. 1 is given using truncated pointwise order instead of truncated lexicographic order.
2. In Cond. 4, existence of the supremum is explicitly required.

The difference [1] is made for the sake of cleanliness of the soundness proof for our notion of simulation (Thm. 6.3). The difference [2] is made because a homset of \( \mathcal{K}(\mathcal{G}) \) is an \( \omega \)-cpo but not necessarily a dcpo.
Note here that a Kleisli arrow to that Therefore by a dcpo. \( \chi \) is known that there exists \( V \subseteq [0,1] \) such that \( V \notin \mathcal{F}_0[1] \) (see e.g. [18]). Let 1 be a singleton and \( \mathcal{F}_1 \) be the unique measure over 1. It is easy to see that \( G(1, \mathcal{F}_1) \cong ([0,1], \mathcal{F}_0[1]) \).

We define a family \( \mathfrak{A}_V \subseteq \mathcal{K}(\mathcal{G})((0,1], \mathcal{F}_0[1]), (1, \mathcal{F}_1)) \) of Kleisli arrows by

\[
\mathfrak{A}_V := \{ \chi_X : ([0,1], \mathcal{F}_0[1]) \to G(1, \mathcal{F}_1) \mid X \in \mathcal{F}_0[1] \text{ and } X \subseteq V \}.
\]

Here \( \chi_X \) denotes the characteristic function of \( X \), that is,

\[
\chi_X(x) = \begin{cases} 
1 & (x \in X) \\
0 & (x \notin X).
\end{cases}
\]

Note here that a Kleisli arrow \( \chi_X : ([0,1], \mathcal{F}_0[1]) \to G(1, \mathcal{F}_1) \) is identified with a measurable function \( \chi_X : ([0,1], \mathcal{F}_0[1]) \to ([0,1], \mathcal{F}_0[1]) \). It is easy to see that \( X \subseteq X' \) if and only if \( \chi_X \subseteq \chi_{X'} \) where \( \subseteq \) denotes the pointwise extension of the order on \([0,1]\). Hence \( (\mathfrak{A}_V, \subseteq) \) is a dcpo.

Assume that \( \mathcal{K}(\mathcal{G})((0,1], \mathcal{F}_0[1]), (1, \mathcal{F}_1)) \) is a dcpo. Then there exists the least upper bound \( \bigsqcup_{X \subseteq V} \chi_X : ([0,1], \mathcal{F}_0[1]) \to G(1, \mathcal{F}_1) \cong ([0,1], \mathcal{F}_0[1]) \).

Let \( V' := \{ \chi_X : ([0,1], \mathcal{F}_0[1]) \to G(1, \mathcal{F}_1) \mid X \subseteq V \} \). Then we have \( V \subseteq V' \). Therefore by \( V \notin \mathcal{F}_0[1] \), there exists \( v \in V' \) such that \( V \subseteq V' \setminus \{ v \} \). As \( \{ v \} \subseteq \mathcal{F}_0[1] \), we have \( V' \setminus \{ v \} \in \mathcal{F}_0[1] \). It is easy to see that \( \chi_{V'} \) is an upper bound of \( \mathfrak{A}_V \). This contradicts to that \( \bigsqcup_{X \subseteq V} \chi_X \) is the least upper bound of \( \mathfrak{A}_V \). Hence \( \mathcal{K}(\mathcal{G})((0,1], \mathcal{F}_0[1]), (1, \mathcal{F}_1)) \) is not a dcpo.

Hence a homset \( \mathcal{K}(\mathcal{G})(X, Y) \) in the Kleisli category of the sub-Giry monad is not necessarily a dcpo.

In spite of these modifications, the notion of progress measure in Def. 2.2 still satisfy desirable properties that are satisfied by the original definition in [16]—soundness and completeness. The proofs are almost the same as the ones in [16].

**Theorem B.4** (correctness of progress measures). Let \( E \) be the equational system \( \mathcal{E} \). We assume the setting in Def. 2.4 that for each \( i \in [1, m] \), \( L_i \) has the smallest element \( \perp \), and that \( E \) has the solution, \( (l^{sol}_1, \ldots, l^{sol}_m) \).

1. **(Soundness)** For each progress measure \( p \) we have \( p_i(\overline{\alpha_1}, \ldots, \overline{\alpha_k}) \leq l^{sol}_i \) for each \( i \in [1, m] \).

2. **(Completeness)** We assume either of the following conditions.
   a. For each \( i \in [1, m] \), the poset \( L_i \) is directed complete.
   b. For each \( i \in [1, m] \) and \( l_i \in L_1, \ldots, l_i \in L_{i-1}, l_{i+1} \in L_{i+1}, \ldots, l_m \in L_m \), the poset \( L_i \) is \( \omega \)-complete and \( f_i(l_1, \ldots, l_{i-1}, l_{i+1}, \ldots, l_m) : L_i \to L_i \) is \( \omega \)-continuous.

Then there exists a progress measure \( p = ((\overline{\alpha_1}, \ldots, \overline{\alpha_k}), p_i(\overline{\alpha_1}, \ldots, \overline{\alpha_k}))_{i=1}^{k} \) that achieves the solution, that is, \( p_i(\overline{\alpha_1}, \ldots, \overline{\alpha_k}) = l^{sol}_i \) for each \( i \in [1, m] \). Especially if Cond. b is satisfied, we can define \( p \) so that \( \overline{\alpha} \leq \omega \) for each \( i \in [1, m] \). ▲

**C Tree, Run, and Accepting Run**

**Remark C.1.** We let \( \mathbb{N}^* \) and \( \mathbb{N}^\omega \) denote the sets of finite and infinite sequences over natural numbers, respectively. Moreover we let \( \mathbb{N}^\omega := \mathbb{N}^* \cup \mathbb{N}^\omega \). Concatenation of finite/infinite sequences, and/or characters are denoted simply by juxtaposition. Given an infinite sequence \( \pi = \pi_1 \pi_2 \ldots \in \mathbb{N}^\omega \) (here \( \pi_i \in \mathbb{N} \)), its prefix \( \pi_1 \ldots \pi_n \) is denoted by \( \pi_{\leq n} \).
The following formalization of trees and related notions are standard, with its variations used e.g. in [6]. A sequence \(w \in \mathbb{N}^*\) is understood as a *position* in a tree.

**Definition C.2** (\\(\Sigma\)-tree). Let \(\Sigma\) be a ranked alphabet, with each element \(\sigma \in \Sigma\) coming with its arity \(|\sigma| \in \mathbb{N}\). A *\(\Sigma\)-tree* \(\tau\) is given by a nonempty subset \(\text{Dom}(\tau) \subseteq \mathbb{N}^*\) (called the *domain* of \(\tau\)) and a *labeling* function \(\tau : \text{Dom}(\tau) \to \Sigma\) that are subject to the following conditions\(^5\):

1. \(\text{Dom}(\tau)\) is *prefix-closed*: for any \(w \in \mathbb{N}^*\) and \(i \in \mathbb{N}\), \(w_i \in \text{Dom}(\tau)\) implies \(w \in \text{Dom}(\tau)\).
   See Fig. 1
2. \(\text{Dom}(\tau)\) is *lower-closed*: for any \(w \in \mathbb{N}^*\) and \(i,j \in \mathbb{N}\), \(w_j \in \text{Dom}(\tau)\) and \(i \leq j\) imply \(w_i \in \text{Dom}(\tau)\).
   See Fig. 1
3. The labeling function is consistent with arities: for any \(w \in \text{Dom}(\tau)\), let \(\sigma = \tau(w)\). Then \(w0, w1, \ldots, w(|\sigma| - 1)\) belong to \(\text{Dom}(\tau)\), and \(w_i \not\in \text{Dom}(\tau)\) for any \(i\) such that \(|\sigma| \leq i\).
   See Fig. 2

The set of all \(\Sigma\)-trees shall be denoted by \(\text{Tree}_\Sigma\).

The following definitions are standard, too, in the tree-automata literature.

**Definition C.3** (run). A *run* \(\rho\) of an NBTA (Def. B.1) \(\mathcal{X} = (X, \Sigma, \delta, I, \text{Acc})\) is a (possibly infinite) tree whose nodes are \((\Sigma \times X)\)-labeled. That should be consistent with arities of symbols, and compatible with the initial states \((I \subseteq X)\) and the transition \(\delta\) of the automaton \(\mathcal{X}\). Precisely, it is given by the following conditions.

1. A nonempty subset \(\text{Dom}(\rho) \subseteq \mathbb{N}^*\) that is subject to the same conditions (of being prefix-closed and lower-closed) as for \(\Sigma\)-trees (Def. C.2).
2. A labeling function \(\rho : \text{Dom}(\rho) \to \Sigma \times X\) such that, if \(\rho(w) = (\sigma, x)\), then \(w\) has precisely \(|\sigma|\) successors \(w0, w1, \ldots, w(|\sigma| - 1) \in \text{Dom}(\rho)\).
3. Successors are reachable by a transition, in the sense that \((\sigma_w, (x_{w0}, \ldots, x_{w(|\sigma| - 1)})) \in \delta(x_w)\) holds, where \(\rho(w)\) is labeled with \((\sigma_w, x_w)\), and \(\rho(w_i)\) is labeled with \((\sigma_{wi}, x_{wi})\) for any \(0 \leq i < |\sigma|\).
4. The root is in the initial states, that is, \(\rho(\varepsilon) \in I\).

The set of all runs of the NBTA \(\mathcal{X}\) is denoted by \(\text{Run}_\mathcal{X}^\rho\).

The map that takes a run \(\rho \in \text{Run}_\mathcal{X}^\rho\), removes its \(X\)-labels (i.e. applies the first projection to each label), and returns the resulting \(\Sigma\)-labeled tree (that is easily seen to be a \(\Sigma\)-tree, Def. C.2) is denoted by \(\text{DelSt}: \text{Run}_\mathcal{X}^\rho \to \text{Tree}_\Sigma\). We say that a run \(\rho\) is *over* the \(\Sigma\)-tree \(\text{DelSt}(\rho)\).

\(^5\) We shall use the same notation \(\tau\) for a tree itself and its labeling function. Confusion is unlikely.
A branch of a tree is a maximal path from its root $\varepsilon$.

**Definition C.4** (branch). Let $\tau$ be a $\Sigma$-tree. A branch of $\tau$ is either:
- an infinite sequence $\pi = \pi_1\pi_2\ldots \in \omega^\omega$ (where $\pi_i \in \mathbb{N}$) such that any finite prefix $\pi_{\leq n} = \pi_1\ldots\pi_n$ of it belongs to $\text{Dom}(\tau)$; or
- a finite sequence $\pi = \pi_1\ldots\pi_n \in \omega^*$ (where $\pi_i \in \mathbb{N}$) that belongs to $\text{Dom}(\tau)$ and such that $\pi 0 \notin \text{Dom}(\tau)$ (meaning that $\pi$ is a leaf of $\tau$, and that $\tau(\pi)$ is a 0-ary symbol).

The set of all branches of a $\Sigma$-tree $\tau$ is denoted by $\text{Branch}(\tau)$.

The notion of branch is defined similarly for a run, with $\text{Branch}(\rho)$ denoting the set of all branches of $\rho$.

**Definition C.5** (accepting run). A run $\rho$ of an NBTA $\mathcal{X} = (X, \Sigma, \delta, I, \text{Acc})$ is said to be accepting if, any branch $\pi \in \text{Branch}(\rho)$ of it is accepting in the following sense.
- The branch $\pi$ is an infinite sequence $\pi = \pi_1\pi_2\ldots \in \omega^\omega$, and the labels along the branch $(\sigma_0,x_0)(\sigma_1,x_1)(\sigma_2,x_2)\ldots$ where $(\sigma_i,x_i) := \rho(w)$ for each $w \in \omega^*$ visits accepting states infinitely often, that is, there exists a sequence $n_1 < n_2 < \cdots$ of natural numbers such that $x_{\pi_1\ldots\pi_{n_i}} \in F$ for each $i \in \mathbb{N}$; or
- the branch $\pi$ is a finite sequence $\pi = \pi_1\ldots\pi_n \in \omega^*$.

The set of all accepting runs over $\mathcal{X}$ is denoted by $\text{AccRun}_{\mathcal{X}}^\rho$.

## D Accepted Languages of Probabilistic Büchi Word Automata

In this section, for a PBWA $\mathcal{X} = (X, \Sigma, M, \iota, \text{Acc})$, we define its language.

**Definition D.1** (run). For a PBWA $\mathcal{X} = (X, \Sigma, M, \iota, \text{Acc})$, a run over $\mathcal{X}$ is an infinite word $\rho \in (A \times X)^\omega$. The set of all runs over $\mathcal{X}$ is denoted by $\text{Run}_{\mathcal{X}}^\rho$. A partial run over $\mathcal{X}$ is a finite word $\xi \in (A \times X)^* \times X$. A run $\rho = (a_0,x_0)(a_1,x_1)\ldots \in \text{Run}_{\mathcal{X}}^\omega$ is accepting if $x_i \in \text{Acc}$ for infinitely many $i$’s. The set of all accepting runs over $\mathcal{X}$ is denoted by $\text{AccRun}_{\mathcal{X}}^\rho$.

We define the language of $\mathcal{X}$ as a subprobability measure over a set $A^\omega$ of infinite words. To this end, we first fix measurable structures for the set of infinite words and set of runs. Here they are defined in a standard manner, that is, they are defined as the ones induced by cylinder sets (see e.g. 4).

**Definition D.2**. Let $\mathcal{X} = (X, \Sigma, M, \iota, \text{Acc})$ be a PBWA. For $w \in A^*$, the cylinder set generated by $w$ is a set $\text{Cyl}(w) := \{ww' \in A^\omega \mid w \in A^*, w' \in A^\omega\}$.

We write $\mathcal{A}_A^\omega$ for the smallest $\sigma$-algebra over $A^\omega$ that is generated by the cylinder sets $\{\text{Cyl}(w) \mid w \in A^*\}$.

Similarly, for a partial run $\xi = (a_0,x_0)\ldots(a_{i-1},x_{i-1})x_i \in (A \times X)^* \times X$, the cylinder set generated by $\xi$ is a set $\text{Cyl}_{\mathcal{X}}(\xi) := \{(a_0,x_0)\ldots(a_i,x_i)(a_{i+1},x_{i+1})\ldots \in \text{Run}_{\mathcal{X}}^\rho \mid a_i,a_{i+1},\ldots \in A,x_{i+1},x_{i+2},\ldots \in X\}$.

We write $\mathcal{A}_X$ for the $\sigma$-algebra over $\text{Run}_{\mathcal{X}}^\rho$ generated by the cylinder sets $\{\text{Cyl}_{\mathcal{X}}(\xi) \mid \xi \in (A \times X)^* \times X\}$.

We define $\text{DelSt} : \text{Run}_{\mathcal{X}}^\rho \to A^\omega$ by $\text{DelSt}((a_0,x_0)(a_1,x_1)\ldots) := a_0a_1\ldots$.

The following result is much like [6, Lem. 36] and hardly novel.

**Lemma D.3**. The set $\text{AccRun}_{\mathcal{X}}^\rho$ of accepting runs is an $\mathcal{A}_X$-measurable subset of $\text{Run}_{\mathcal{X}}^\rho$. 

Note that as \( P \to \) NoDiv, a PBWA can exhibit divergence. Hence \( X \) does not exhibit divergence. This probability is used to define probabilistic accepted languages of \( \mathcal{X} \). Intuitively, for each partial run \( \xi = (a_0, x_0) \ldots (a_{i-1}, x_{i-1})x_i \in (A \times X)^* \times X \), by

\[
\mu^{\text{Run}_{\mathcal{X}}}_{\mathcal{X}}(\text{Cyl}_\mathcal{X}(\xi)) := \iota_{x_0} \cdot P_X(\xi).
\]

Here \( P_X(\xi) \) is defined inductively as follows.

\[
P_X(\xi) := \begin{cases} 
\text{NoDiv}_X(x_0) & (i = 0) \\
(M(a_0))_{x_0,x_1} \cdot P_X((a_1, x_1) \ldots (a_{i-1}, x_{i-1})x_i) & (i > 0). 
\end{cases}
\]

**Proposition D.6.** Def. [D.5] is well-defined. Namely, there exists a unique subprobability measure \( \mu^{\text{Run}_{\mathcal{X}}}_{\mathcal{X}} \) over \( (\text{Run}_X, \mathcal{F}_X) \) that satisfies equation (13). ▲

**Proof.** We first prove that for each \( i \in \mathbb{N}, a_0, \ldots, a_{i-1} \in A \) and \( x_0, \ldots, x_i \in X \), we have:

\[
P_X((a_0, x_0) \ldots (a_{i-1}, x_{i-1})x_i) = \sum_{a_i \in A} \sum_{x_{i+1} \in X} P_X((a_0, x_0) \ldots (a_{i-1}, x_{i-1})x_i x_{i+1}^+).
\]
We prove it by induction on $i$. If $i = 0$, then $\xi = x_0$ and we have:

$$P_X(\xi) = \text{NoDiv}_x(x_0) = \lim_{k \to \infty} \text{NoDiv}_{X,k}(x)$$

$$= \lim_{k \to \infty} \sum_{a \in A} \sum_{x' \in X} (M(a))_{x_0, x'} \cdot \text{NoDiv}_{X,k-1}(x_1)$$

$$= \lim_{k \to \infty} \sum_{a \in A} \sum_{x' \in X} (M(a))_{x_0, x'} \cdot \text{NoDiv}_{X,k}(x_1)$$

$$= \sum_{a_0 \in A} \sum_{x_1 \in X} (M(a_0))_{x_0, x_1} \cdot (\lim_{k \to \infty} \text{NoDiv}_{X,k}(x_1))$$

$$= \sum_{a_0 \in A} \sum_{x_1 \in X} (M(a_0))_{x_0, x_1} \cdot P_X(x_1)$$

$$= \sum_{a_0 \in A} P_X((a_0, x_0))$$

If $i > 0$, then we have:

$$P_X(\xi) = (M(a_0))_{x_0, x_1} \cdot P_X((a_1, x_1) \ldots (a_{i-1}, x_{i-1}))$$

$$= (M(a_0))_{x_0, x_1} \cdot \sum_{a_i \in A} \sum_{x_1 \in X} P_X((a_1, x_1) \ldots (a_{i-1}, x_{i-1})(a_i, x_i))$$

(by the induction hypothesis)

$$= \sum_{a_i \in A} \sum_{x_1 \in X} (M(a_0))_{x_0, x_1} \cdot P_X((a_1, x_1) \ldots (a_{i-1}, x_{i-1})(a_i, x_i))$$

$$= \sum_{a_i \in A} \sum_{x_1 \in X} P_X((a_0, a_1) \ldots (a_{i-1}, x_{i-1})(a_i, x_i))$$

Hence we have:

$$\mu_X^{\text{Run}_{X}}((a_0, x_0) \ldots (a_{i-1}, x_{i-1})) = \sum_{a_i \in A} \sum_{x_1 \in X} \mu_X^{\text{Run}_{X}}((a_0, x_0) \ldots (a_{i-1}, x_{i-1})(a_i, x_i)).$$

Therefore Prop. [D.6] is immediate from Carathéodory's extension theorem (see e.g. [3]).

Now we can define the language of a PBWA $\mathcal{X}$.

**Definition D.7** (language of PBWA). $\mathcal{X} = (X, A, M, i, \text{Acc})$ be a PBWA. A subprobability measure $L(\mathcal{X})$ over $(A^*, \mathcal{F}_{A^*})$ is defined as follows: for each $w \in A^*$,

$$L(\mathcal{X})(\text{Cyl}(w)) := \mu_X^{\text{Run}_{X}}\left(\text{DelSt}^{-1}(w) \cap \text{AccRun}_{X}\right).$$

(14)

Note here that $\text{DelSt}^{-1}(\text{Cyl}(w)) = \bigcup_{\xi \in \text{DelSt}^{-1}(w)} \text{Cyl}_{\mathcal{X}}(\xi)$. As $w$ is a finite word and the state space $X$ is countable, the union in the above equation is countable one. Hence the set $\text{DelSt}^{-1}(\text{Cyl}(w))$ is measurable.

The following proposition can be proved in a similar manner to Prop. [D.6].

**Proposition D.8.** Def. [D.7] is well-defined. Namely, there exists a unique subprobability measure $L(\mathcal{X})$ over $(\Sigma^*, \mathcal{F}_{\Sigma^*})$ that satisfies equation (14).

The language $L(\mathcal{X})$ — a subprobability measure that tells which words are generated by what probabilities — is essentially the push-forward measure [9] obtained from the one over $\text{Run}_{X}$.  

\[\text{Proposition D.8.} \]
We define a Kleisli arrow systems that are defined as follows.

Example E.1. Let $\mathcal{X} = ((X_1, X_2), c, s)$ and $\mathcal{Y} = ((Y_1, Y_2), d, t)$ be Büchi $(\mathcal{G}, \{a\} \times \_)$-systems illustrated below. Here their state spaces are equipped with discrete $\sigma$-algebras.

We define a Kleisli arrow $f : Y \to X$ by

$$f(y_1)(\{x\}) = \begin{cases} \frac{1}{2} & (x \in \{y_1, y_{22}\}) \\ 0 & \text{(otherwise)} \end{cases} \quad \text{and} \quad f(y_2)(\{x\}) = \begin{cases} \frac{1}{2} & (x \in \{y_{21}, y_{23}\}) \\ 0 & \text{(otherwise)}. \end{cases}$$

Then $f$ is a fair simulation without dividing (Def. 6.4) from $\mathcal{X}$ to $\mathcal{Y}$.

In contrast, $f$ is not a fair simulation with dividing (Def. 6.2). In fact, there exists no fair simulation with dividing from $\mathcal{X}$ to $\mathcal{Y}$.

Example E.2. Let $\mathcal{X} = ((X_1, X_2), c, s)$ and $\mathcal{Y} = ((Y_1, Y_2), d, t)$ be Büchi $(\mathcal{G}, \{a\} \times \_)$-systems that are defined as follows.

- $X_1 = \{x_i \mid i \in \omega\}$, $X_2 = \{x'_i \mid i \in \omega\}$, $Y_1 = \{y_i \mid i \in \omega\}$ and $Y_2 = \{y'_i \mid i \in \omega\}$. Moreover, they are equipped with discrete $\sigma$-algebras.
- $c(x_i)((a, x)) = p_{i+3}$ if $x = x_{i+1}$, $1 - p_{i+3}$ if $x = x'_{i+1}$ and 0 otherwise. Moreover, $c(y_i)((a, y)) = p_{i+2}$ if $y = y_{i+1}$, $1 - p_{i+2}$ if $y = y'_{i+1}$ and 0 otherwise. Here, $p_i \in [0, 1]$ is defined by $p_1 = 1 - \frac{1}{2}$.
- $c(x'_i)((a, x)) = 1$ if $x = x'_{i+1}$ and 0 otherwise. Similarly, $d(y'_i)((a, y)) = 1$ if $y = y'_{i+1}$ and 0 otherwise.
- $s(\ast)(\{x\}) = \frac{1}{2}p_2$ if $x = x_0$, $1 - \frac{1}{2}p_2$ if $x = x'_0$ and 0 otherwise. Moreover, $t(\ast)(\{y\}) = 1$ if $y = y_0$ and 0 otherwise.

We define a family $f = (f_{ij} : Y_i \to X_j)_{i, j \in \{1, 2\}}$ of Kleisli arrows as follows:

- $f_{11}(y_i)(\{x_j\}) = \frac{1}{2}p_{i+2}$ if $j = i$ and 0 otherwise;
- $f_{12}(y_i)(\{x'_j\}) = 1 - \frac{1}{2}p_{i+2}$ if $j = i$ and 0 otherwise;
- $f_{21}(y'_i)(\{x_j\}) = 0$; and
- $f_{22}(y'_i)(\{x'_j\}) = 1$ if $j = i$ and 0 otherwise.

Moreover, for an ordinal $\alpha \in \{0, 1\}$, we define Kleisli arrows $f_{11}(\alpha) : Y_1 \to Y_1$ and
Then the following inequalities hold.

\[
\begin{align*}
    s & \subseteq [\langle f_{11}, f_{12} \rangle, \langle f_{21}, f_{22} \rangle] \circ t \\
    c \circ [\langle f_{11}(0), f_{12}(1) \rangle] & \subseteq F [\langle f_{11}(0), f_{12}(0) \rangle, \langle f_{21}, f_{22} \rangle] \circ d_1 \\
    c \circ [\langle f_{11}(1), f_{12}(1) \rangle] & \subseteq F [\langle f_{11}(1), f_{12}(1) \rangle, \langle f_{21}, f_{22} \rangle] \circ d_1 \\
    c \circ [\langle f_{21}, f_{22} \rangle] & \subseteq F [\langle f_{11}, f_{12} \rangle, \langle f_{21}, f_{22} \rangle] \circ d_2
\end{align*}
\]

Therefore \( f \) is a forward fair simulation without dividing (Def. \([6.1]\)) from \( X \) to \( Y \).

However, it is also easy to see that trace inclusion between them does not hold. Indeed,

\[
\begin{align*}
    \text{tr}(X)(\{a^\omega\}) &= \frac{1}{2} p_2 \cdot (1 - \prod_{i \in \omega} p_{i+3}) + (1 - \frac{1}{2} p_2) \cdot 1 \\
    &= 1 - \frac{1}{2} \prod_{i \in \omega} p_{i+2}
\end{align*}
\]

while

\[
\begin{align*}
    \text{tr}(Y)(\{a^\omega\}) = 1 - \prod_{i \in \omega} p_{i+2}.
\end{align*}
\]

As \( \prod_{i \in \omega} p_{i+2} = \frac{1}{2} > 0 \), we have \( \text{tr}(X) \nsubseteq \text{tr}(Y) \).

## F Omitted Proofs

### F.1 Omitted Proof in \(3\)

\(\triangleright\) Proof of Thm. \([3.4]\): For each \( A, B \in \text{Sets} \), there exists a bijective function \( \Phi_{A,B} : \mathcal{P}(A \times B) \to \mathcal{K}(\mathcal{P}(A) \times \mathcal{P}(B)) \) that is defined by \( \Phi_{A,B}(R)(a) = \{ b \in B \mid (a, b) \in R \} \). Let \( f : A \to A' \) and \( g : B \to B' \). Here \( \circ^{\text{left}} \) and \( \circ^{\text{right}} \) in Def. \([3.3]\) correspond to pre-composition and \( \sqcap_g \) in Lem. \([6.7]\) i.e. \( \Phi_{A,B}(\circ^{\text{left}})(R) = \Phi_{A',B}(R) \circ f \) and \( \Phi_{A,B}(\circ^{\text{right}})(R) = \sqcap_g (\Phi_{A,B'}(R)) \).

Therefore the equational system \([\ref{3.2}]\) in Def. \([3.3]\) is equivalent to the equational system \([\ref{3.3}]\) in Prop. \([6.8]\). Moreover, \( R \subseteq Y \times X \) satisfies condition \([\ref{3.1}]\) in Def. \([3.3]\) if and only if \( s \subseteq \Phi_{Y,X}(R) \circ t \).

Hence immediate from Prop. \([6.8]\) and Prop. \([6.5]\).
F.2 Omitted Proof in §4

Proof of Thm. 4.4 Let $X = ((X_1, X_2), c, s)$ and $Y = ((Y_1, Y_2), d, t)$. By the bijective correspondence between probabilistic matrices and arrows in $\mathcal{Kl}(G)(X, Y)$ where $X$ and $Y$ are equipped with discrete $\sigma$-algebras, a forward fair matrix simulation $A \in [0, 1]^{Y \times X}$ from $X$ to $Y$ (Def. 4.3) exists iff a fair simulation $f : Y \twoheadrightarrow X$ without dividing from $X$ to $Y$ (Def. 6.4) exists (see also Sublemmas F.1, F.2).

We define $Y_{12} \subseteq Y_1$ by $Y_{12} = \{ y \in Y_1 \mid s \mathbf{r}(b)(A^y) > 0 \}$. As $Y_1$ is finite, $Y_{12}$ is also finite. We define a Büchi $(G, A \times (\_))$-system $\mathcal{Y}' = ((Y_1', Y_2'), d', t')$ by $Y_1' = Y_1 \setminus Y_{12}$, $Y_2' = Y_2 + Y_{12}$, $d' = d$ and $t' = t$. As $Y_{12}$ is finite, by repeatedly applying Lem. 6.10 we have $\mathbf{r}(\mathcal{Y}) = \mathbf{r}(\mathcal{Y}')$.

It is easy to see that $f$ is also a fair simulation without dividing from $X$ to $\mathcal{Y}'$. Moreover, by its definition, $\mathcal{Y}'$ satisfies $\mathbf{r}(d_1') = \bot$. Therefore by Prop. 6.9 we have $\mathbf{r}(\mathcal{X}) \not\subseteq \mathbf{r}(\mathcal{Y}')$. Hence $\mathbf{r}(\mathcal{X}) \not\subseteq \mathbf{r}(\mathcal{Y})$ holds.

F.3 Omitted Proofs in §6

F.3.1 Proof of Thm. 6.3

Sublemma F.1. We assume Assm. 6.3 and conditions in Thm. 6.3. We further assume the situation in Def. 6.2. Let $E_\mathcal{X}$ be an equational system that defines $\mathbf{r}(c_1) : X_1 \twoheadrightarrow Z$ and $\mathbf{r}(c_2) : X_2 \twoheadrightarrow Z$ (c.f. §3 in Thm. 5.5). By completeness of progress measure (Thm. 6.2) and Cond. 11 in Thm. 6.3, there exists a progress measure $p_\mathcal{X} = ((\beta_1), (u_1(\beta_1) : X_1 \twoheadrightarrow Z, u_2(\beta_1) : X_2 \twoheadrightarrow Z)_{\beta_1 \leq \overline{\alpha}})$ such that $\beta_1 \leq \omega$, $u_1(\beta_1) = \mathbf{r}(c_1)$ and $u_2 = \mathbf{r}(c_2)$. We define two ordinals $\overline{\gamma_1}$ and $\overline{\gamma_2}$ by $\overline{\gamma_1} = \beta_1$ and $\overline{\gamma_2} = \overline{\alpha}$ where $\overline{\alpha}$ is an ordinal in Def. 6.2. Moreover, for a pair of ordinals $\gamma_1 \leq \overline{\gamma_1}$ and $\gamma_2 \leq \overline{\gamma_2}$, we define three arrows $h_1(\gamma_1, \gamma_2) : X_1 \twoheadrightarrow Z$, $h_2(\gamma_1, \gamma_2) : X_1 \twoheadrightarrow Z$, and $h_3 : X_2 \twoheadrightarrow Z$ by $h_1(\gamma_1, \gamma_2) = u_1(\gamma_1) \circ f_1^{(\gamma_2)}$, $h_2(\gamma_1, \gamma_2) = u_2(\gamma_2) \circ f_2^{(\gamma_2)}$, and $h_3(\gamma_1, \gamma_2) = [u_1(\gamma_1), u_2(\gamma_2)] \circ \langle f_{21}, f_{22} \rangle$. Then, $p := ((\overline{\gamma_1}, \overline{\gamma_2}), (h_1(\gamma_1, \gamma_2), h_2(\gamma_1, \gamma_2), h_3(\gamma_1, \gamma_2))_{\gamma_1 \leq \overline{\gamma_1}, \gamma_2 \leq \overline{\gamma_2}})$ is a progress measure for the following equational system:

$$
\begin{align*}
\nu_1 &= (J_\zeta)^{-1} \circ \nu_1 + \nu_2, \\
\nu_2 &= (J_\zeta)^{-1} \circ \nu_2 + \nu_3, \\
\nu_3 &= (J_\zeta)^{-1} \circ \nu_3 + d_1
\end{align*}
$$

Note that $\nu_{12}, \nu_{21} : X_1 \twoheadrightarrow E_\mathcal{X}$ are the Kleisli arrows in Def. 6.2.

Proof. We check that $p := ((\overline{\gamma_1}, \overline{\gamma_2}), (h_1(\gamma_1, \gamma_2), h_2(\gamma_1, \gamma_2), h_3(\gamma_1, \gamma_2))_{\gamma_1 \leq \overline{\gamma_1}, \gamma_2 \leq \overline{\gamma_2}})$ satisfies the axioms of progress measure (Def. 11): 1. (Monotonicity) We assume $\gamma_1 \leq \gamma'_1$ and $\gamma_2 \leq \gamma'_2$. Then monotonicity of Kleisli composition and that $(f_{11}^{(\alpha)})_{\alpha \leq \overline{\gamma}}$ and $(f_{12}^{(\alpha)})_{\alpha \leq \overline{\gamma}}$ are increasing sequence, we have:

$$
h_1(\gamma_1, \gamma_2) = u_1(\gamma_1) \circ f_1^{(\gamma_2)} \subseteq u_1(\gamma'_1) \circ f_1^{(\gamma'_2)} = h_1(\gamma'_1, \gamma'_2).
$$

Similarly, we also have:

$$
h_2(\gamma_1, \gamma_2) = u_2(\gamma_2) \circ f_2^{(\gamma_2)} \subseteq u_2(\gamma'_2) \circ f_2^{(\gamma'_2)} = h_2(\gamma'_1, \gamma'_2)
$$

and

$$
h_3(\gamma_1, \gamma_2) = [u_1(\gamma_1), u_2(\gamma_2)] \circ \langle f_{21}, f_{22} \rangle = h_3(\gamma'_1, \gamma'_2).
$$
2. (**µ-variables, base case**) By Cond. (10) in Thm. 6.3 and Cond. (2c) in Def. 6.2, we have:

\[ h_1(0, \gamma_2) = u_1(0) \odot f_{11}^{(\gamma_2)} = \bot \odot f_{11}^{(\gamma_2)} = \bot; \] 

and

\[ h_2(\gamma_1, 0) = u_2(\gamma_1) \odot f_{12}^{(0)} = u_2 \odot \bot = \bot. \]

3. (**µ-variables, step case**) For ordinals \( \gamma_1 \leq \gamma_1 \) and \( \gamma_2 \leq \gamma_2 \), we have:

\[
\begin{align*}
h_1(\gamma_1 + 1, \gamma_2) &= u_1(\gamma_1 + 1) \odot f_{11}^{(\gamma_2)} \\
&\subseteq (J \zeta)^{-1} \circ F[u_1(\gamma_1), u_2(\gamma_2)] \circ c_1 \circ f_{11}^{(\gamma_2)} \\
&\subseteq (J \zeta)^{-1} \circ F[u_1(\gamma_1), u_2(\gamma_1)] \circ c_1 \circ f_{11}^{(\gamma_2)} \\
&\subseteq (J \zeta)^{-1} \circ F[u_1(\gamma_1), u_2(\gamma_1)] \odot F[f_{11}^{(\gamma_2)}, f_{12}^{(\gamma_2)}] \odot F[\gamma_1, f_{22}] \circ d_{11} \\
&= (J \zeta)^{-1} \circ F[[u_1(\gamma_1), u_2(\gamma_1)] \odot F[f_{11}^{(\gamma_2)}, f_{12}^{(\gamma_2)}], [u_1(\gamma_1), u_2(\gamma_1)] \odot F[\gamma_1, f_{22}]] \odot d_{11} \\
&= (J \zeta)^{-1} \circ F[[\text{id}_Z, \text{id}_Z] \odot [u_1(\gamma_1) \odot f_{11}^{(\gamma_2)}, u_2(\gamma_1) \odot f_{12}^{(\gamma_2)}] \odot F[f_{21}, f_{22}], [u_1(\gamma_1), u_2(\gamma_1)] \odot F[\gamma_1, f_{22}] \circ d_{11} \\
&\subseteq (J \zeta)^{-1} \circ F[[\text{id}_Z, \text{id}_Z] \odot [u_1(\gamma_1) \odot f_{11}^{(\gamma_2)}, u_2(\gamma_1) \odot f_{12}^{(\gamma_2)}] \odot F[f_{21}, f_{22}], [u_1(\gamma_1), u_2(\gamma_1)] \odot F[\gamma_1, f_{22}] \circ d_{11} \\
&= (J \zeta)^{-1} \circ F[[\text{id}_Z, \text{id}_Z] \odot [h_1(\gamma_1, \gamma_2), h_2(\gamma_1, \gamma_2)], [u_1(\gamma_1), u_2(\gamma_1)] \odot F[\gamma_1, f_{22}] \circ d_{11}.
\end{align*}
\]

Similarly, for some ordinal \( \gamma' \), we have:

\[
\begin{align*}
h_2(\gamma_2 + 1) &= u_2(\gamma_2 + 1) \odot f_{12}^{(\gamma_2 + 1)} \\
&= (J \zeta)^{-1} \circ F[u_1(\gamma'), u_2(\gamma_1)] \odot c_2 \odot f_{12}^{(\gamma_2 + 1)} \\
&\subseteq (J \zeta)^{-1} \circ F[u_1(\gamma'), u_2(\gamma_1)] \odot F[f_{11}^{(\gamma_2)}, f_{12}^{(\gamma_2)}], f_{21}, f_{22}] \circ d_{12} \\
&\subseteq (J \zeta)^{-1} \circ F[[u_1(\gamma'), u_2(\gamma_1)] \odot F[f_{11}^{(\gamma_2)}, f_{12}^{(\gamma_2)}], [u_1(\gamma'), u_2(\gamma_1)] \odot F[\gamma_1, f_{22}] \circ d_{12} \\
&= (J \zeta)^{-1} \circ F[[u_1(\gamma'), u_2(\gamma_1)] \odot F[f_{11}^{(\gamma_2)}, f_{12}^{(\gamma_2)}], [u_1(\gamma'), u_2(\gamma_1)] \odot F[\gamma_1, f_{22}]] \odot d_{12} \\
&\subseteq (J \zeta)^{-1} \circ F[[h_1(\gamma_1, \gamma_2), h_2(\gamma_1, \gamma_2)], h_3] \odot d_{12}.
\end{align*}
\]

4. (**µ-variables, limit case**) Let \( \gamma_1 \) be a limit ordinal. By Cond. (12) of Thm. 6.3, Kleisli composition in \( \mathcal{KT}(T) \) is \( \gamma_1 \)-continuous. Hence for each ordinal \( \gamma_2 \), we have:

\[
\begin{align*}
h_1(\gamma_1, \gamma_2) &= u_1(\gamma_1) \odot f_{11}^{(\gamma_2)} = \bigcup_{\gamma_1' < \gamma_1} u_1(\gamma_1') \odot f_{11}^{(\gamma_2)} = \bigcup_{\gamma_1' < \gamma_1} h_1(\gamma_1', \gamma_2).
\end{align*}
\]

Similarly, for an ordinal \( \gamma_1 \) and a limit ordinal \( \gamma_2 \), we have:

\[
\begin{align*}
h_2(\gamma_1, \gamma_2) &= u_2(\gamma_1) \odot f_{12}^{(\gamma_2)} \subseteq u_2(\gamma_1) \odot \bigcup_{\gamma_1' < \gamma_1} f_{12}^{(\gamma_2)} = \bigcup_{\gamma_1' < \gamma_1} u_2(\gamma_1') \odot f_{12}^{(\gamma_2)} = \bigcup_{\gamma_1' < \gamma_1} h_2(\gamma_1', \gamma_2).
\end{align*}
\]
5. \((\nu\text{-variables})\) For ordinals \(\gamma_1 \leq \gamma_1\) and \(\gamma_2 \leq \gamma_2\), we have:

\[
\begin{align*}
  h_3(\gamma_1, \gamma_2) &= [u_1(\gamma_1), u_2(\gamma_2)] \circ [f_{21}, f_{22}] \\
  &= [(J\zeta)^{-1} \circ F[u_1(\gamma_1), u_2(\gamma_2)] \circ c_1, (J\zeta)^{-1} \circ F[u_1(\gamma_1), u_2(\gamma_2)] \circ c_2] \circ [f_{21}, f_{22}] \\
  &= (J\zeta)^{-1} \circ F[u_1(\gamma_1), u_2(\gamma_2)] \circ [c_1, c_2] \circ [f_{21}, f_{22}] \\
  &= (J\zeta)^{-1} \circ F[u_1(\gamma_1), u_2(\gamma_2)] \circ e \circ [f_{21}, f_{22}] \\
  &= (J\zeta)^{-1} \circ F[u_1(\gamma_1), u_2(\gamma_2)] \circ F[f_{11}, f_{12}] \circ d_2 \\
  &= (J\zeta)^{-1} \circ F[u_1(\gamma_1), u_2(\gamma_2)] \circ [f_{11}, f_{12}], [u_1(\gamma_1), u_2(\gamma_2)] \circ [f_{21}, f_{22}] \circ d_2 \\
  &= (J\zeta)^{-1} \circ F[[u_1(\gamma_1), u_2(\gamma_2)] \circ [f_{11}, f_{12}], [u_1(\gamma_1), u_2(\gamma_2)] \circ [f_{21}, f_{22}] \circ d_2 \\
  &= (J\zeta)^{-1} \circ F[[u_1(\gamma_1) \circ f_{11}, u_2(\gamma_2) \circ f_{12}]], [u_1(\gamma_1), u_2(\gamma_2)] \circ [f_{21}, f_{22}] \circ d_2 \\
  &= (J\zeta)^{-1} \circ F[[h_1(\gamma_1, \gamma_2), h_2(\gamma_1, \gamma_2)], h_3(\gamma_1, \gamma_2)] \circ d_2 \\
\end{align*}
\]

These conclude the proof. 

\textbf{Sublemma F.2.} We assume the situation in Sublem. F.1 and let \((v_1^{\text{sol}}, v_2^{\text{sol}}, v_3^{\text{sol}})\) be the solution of equational system \([15]\). Then we have:

\[
\begin{align*}
  [\text{id}_Z, \text{id}_Z] \circ [v_1^{\text{sol}}, v_2^{\text{sol}}] &\subseteq \text{tr}^B(d_1) \quad \text{and} \quad v_3^{\text{sol}} \subseteq \text{tr}^B(d_2) 
\end{align*}
\]

where \(\text{tr}^B(d_1) : Y_1 \to Z\) and \(\text{tr}^B(d_2) : Y_2 \to Z\) are defined as \(\text{tr}^B(c_1)\) and \(\text{tr}^B(c_2)\) in Thm. 5.5.

\textbf{Proof.} By monotonicity of the functions in the equational system, equational system \([15]\) is equivalent to the following equational system

\[
\begin{align*}
  v'_1 &= \mu \left( (J\zeta)^{-1} \circ F[[\text{id}_Z, \text{id}_Z] \circ [v_1^{\text{sol}}, v_2^{\text{sol}}] \circ d_{11}, (J\zeta)^{-1} \circ F[[\text{id}_Z, \text{id}_Z] \circ [v_1^{\text{sol}}, v_2^{\text{sol}}] \circ d_{12}) \in \mathcal{K}(T)(Y, Z) \times \mathcal{K}(T)(Y, Z) \right) \\
  v'_2 &= \nu (J\zeta)^{-1} \circ F[[\text{id}_Z, \text{id}_Z] \circ [v_1^{\text{sol}}, v_2^{\text{sol}}] \circ d_2 \in \mathcal{K}(T)(Y, Z) \\
\end{align*}
\]

where \(v_{11}^{\text{sol}}\) and \(v_{12}^{\text{sol}}\) are defined by \((v'_{11}, v'_{12}) = v'_1\). More concretely, we have \(v_1^{\text{sol}} = (v_1^{\text{sol}}, v_2^{\text{sol}})\) and \(v_2^{\text{sol}} = v_3^{\text{sol}}\).

By completeness of progress measure (Thm. 5.4), there exists a progress measure \(p' = (\alpha, (v'_1(\alpha), v'_2(\alpha), (\nu^{\text{sol}}(\alpha))_{\alpha \leq \pi})\) for \([17]\) such that \((v'_{11}(\pi), v'_{12}(\pi)) = v_1^{\text{sol}}(\pi) = (v_1^{\text{sol}}, v_2^{\text{sol}})\) and \(v_2^{\text{sol}}(\pi) = v_3^{\text{sol}}\).

For each \(\alpha \leq \pi\), we define \(v'_1(\alpha) : Y_1 \to Z\) and \(v'_2(\alpha) : Y_2 \to Z\) by \(v'_1(\alpha) = [\text{id}_Z, \text{id}_Z] \circ [v'_1(\alpha), v'_2(\alpha)]\) and \(v'_2(\alpha) = v'_2(\alpha)\).

In what follows, we show that \(p'' = (\pi, (v'_1(\alpha), v'_2(\alpha))_{\alpha \leq \pi})\) is a progress measure for the equational system that defines \(\text{tr}^B(d_1) : Y_1 \to Z\) and \(\text{tr}^B(d_2) : Y_2 \to Z\) (c.f. [3] in Thm. 5.5).

1. \textbf{(Monotonicity)} By the monotonicity of \(v'_1(\alpha)\) and \(v'_2(\alpha)\), \(v'_1(\alpha)\) and \(v'_2(\alpha)\) are also monotone.

2. \textbf{(\(\mu\text{-variables, base case})} We have \((v'_{11}(0), v'_{12}(0)) = v'_{11}(0) = \perp\) by definition. Hence

\[
  v''_1(0) = [\text{id}_Z, \text{id}_Z] \circ [v'_{11}(0), v'_{12}(0)] = [\text{id}_Z, \text{id}_Z] \circ [\perp, \perp] = \perp.
\]
3. \((\mu\text{-variables, step case})\) For an ordinal \(\alpha \leq \bar{\alpha}\), we have:

\[
v''_i(\alpha + 1) = [id_Z, id_Z] \circ \{v'_{11}(\alpha + 1), v'_{12}(\alpha + 1)\}
\]

\[
\subseteq [id_Z, id_Z] \circ \{v'_{11}(\alpha), v'_{12}(\alpha)\} \circ d_1,
\]

By Sublem. \(F.1\) and soundness of progress measure (Thm. \(B.4.1\)), we have the inequalities (16).

\[
= (J\zeta)^{-1} \circ F\{[id_Z, id_Z] \circ \{v'_{11}(\alpha), v'_{12}(\alpha)\}, v'_2\} \circ d_1.
\]

Hence \(p'' = ((\bar{\alpha}), (v''_1(\alpha), v''_2(\alpha)))\) is a progress measure and by the soundness of progress measure (Thm. \(B.4.2\)), we have the inequalities (16).

\[
\text{Lemma F.3. We assume the situation in Def. 6.2. Moreover, we define arrows } tr^B(c_1) : X_1 \rightarrow Z, tr^B(c_2) : X_2 \rightarrow Z, tr^B(d_1) : Y_1 \rightarrow Z \text{ and } tr^B(d_2) : Y_2 \rightarrow Z \text{ as in Thm. 5.5. Then we have:}
\]

\[
[tr^B(c_1), tr^B(c_2)] \circ \{f_{11}, f_{12}\} \subseteq tr^B(d_1), \text{ and } [tr^B(c_1), tr^B(c_2)] \circ \{f_{21}, f_{22}\} \subseteq tr^B(d_2).
\]

\[
\begin{align*}
\text{Proof. Let } E_X \text{ be an equational system that defines } tr^B(c_1) \text{ and } tr^B(c_2) \text{ (c.f. (6) in Thm. 5.5). By completeness of progress measure (Thm. B.4.1), there exists a progress measure } p_X = ((\bar{\alpha}), (u_1(\beta_1) : X_1 \rightarrow Z, u_2(\beta_1) : X_2 \rightarrow Z)_{\beta_1 \leq \bar{\alpha}}) \text{ for } E_X \text{ such that:}
\end{align*}
\]

\[
tr^B(c_1) = u_1(\bar{\alpha}) \quad \text{ and } \quad tr^B(c_2) = u_2(\bar{\alpha}).
\]

By Sublem. \(F.3\) and soundness of progress measure (Thm. \(B.4.1\)), we have:

\[
\begin{align*}
\[id_Z, id_Z\] \circ \{v^s_1, v^s_2\} \subseteq tr^B(d_1) \text{ and } v^s_3 \subseteq tr^B(d_1).
\end{align*}
\]
Therefore we have:

\[
[\text{tr}^B(c_1), \text{tr}^B(c_2)] \odot \langle f_{11}, f_{12} \rangle \\
= [\text{tr}^B(c_1), \text{tr}^B(c_2)] \odot \langle f_{11}^{(\alpha)}, f_{12}^{(\alpha)} \rangle \\
= [\text{tr}^B(u_1(\beta_1)), u_2(\beta_1)] \odot \langle f_{11}^{(\alpha)}, f_{12}^{(\alpha)} \rangle \\
= [\text{id}_Z, \text{id}_Z] \odot \langle u_1(\beta_1) \circ f_{11}^{(\alpha)}, u_2 \circ f_{12}^{(\alpha)} \rangle \\
\sqsubseteq [\text{id}_Z, \text{id}_Z] \odot \langle v_{1}^{\text{sol}}, v_{2}^{\text{sol}} \rangle \\
\sqsubseteq \text{tr}^B(d_1) \\
\sqsubseteq \text{tr}^B(d_2)
\]

Similarly, we also have:

\[
[\text{tr}^B(c_1), \text{tr}^B(c_2)] \odot \langle f_{21}, f_{22} \rangle \\
= [u_1(\beta_1), u_2(\beta_1)] \odot \langle f_{21}, f_{22} \rangle \\
\sqsubseteq v_3^{\text{sol}} \\
\sqsubseteq \text{tr}^B(d_2)
\]

These conclude the proof.

\[\blacksquare\]

**Proof of Thm. 6.3** We have:

\[
\begin{align*}
\text{tr}^B(\mathcal{X}) &= [\text{tr}^B(c_1), \text{tr}^B(c_2)] \odot s \\
&\sqsubseteq [\text{tr}^B(c_1), \text{tr}^B(c_2)] \odot [\langle f_{11}, f_{12} \rangle, \langle f_{21}, f_{22} \rangle] \odot t \\
&\sqsubseteq [\text{tr}^B(c_1), \text{tr}^B(c_2)] \odot \langle f_{11}, f_{12} \rangle, [\text{tr}^B(c_1), \text{tr}^B(c_2)] \odot \langle f_{21}, f_{22} \rangle] \odot t \\
&\sqsubseteq [\text{tr}^B(d_1), \text{tr}^B(d_2)] \odot t \\
&= \text{tr}^B(\mathcal{Y})
\end{align*}
\]

This concludes the proof.

\[\blacksquare\]

**F.3.2 Proof of Prop. 6.8**

**Proof of Prop. 6.8** As in Def. 6.2, we write \( f_{ji} : Y_j \to X_i \) for the domain and codomain restriction of \( f : Y \to X \).

\((\Rightarrow)\) Assume \( s \sqsubseteq g^{\text{sol}} \circ t \). By completeness of progress measure (Thm. 6.4.2), there exists a progress measure \( g = ((\overline{\alpha}), (g_i(\alpha)))_{1 \leq i \leq 4, \alpha \leq \overline{\alpha}} \) such that \( g_i^{\text{sol}} = g_i(\overline{\alpha}) \) for each \( i \). We define an ordinal \( \overline{\alpha} \) and two sequences \( (f_{11}^{(\alpha)} : Y_1 \to X_1)_{\alpha \leq \overline{\alpha}} \) and \( (f_{12}^{(\alpha)} : Y_1 \to X_2)_{\alpha \leq \overline{\alpha}} \) by \( \overline{\alpha} = \overline{\beta}, f_{11}^{(\alpha)} = g_1(\alpha) \) and \( f_{12}^{(\alpha)} = g_2(\alpha) \). We define \( f : Y \to X \) by \( f = g^{\text{sol}} \) and show that \( f \) is a fair simulation without dividing from \( \mathcal{X} \) to \( \mathcal{Y} \) where the approximation sequences are given by \( (f_{11}^{(\alpha)})_{\alpha \leq \overline{\alpha}} \) and \( (f_{12}^{(\alpha)})_{\alpha \leq \overline{\alpha}} \).

We first show that \( f \) satisfies Cond. 1 in Def. 6.2. By definition of \( f \), we have the following
inequality.
\[
c \odot f = c \odot [\langle f_{11}, f_{12} \rangle, \langle f_{21}, f_{22} \rangle]
\]

\[
\equiv c \odot [\langle \Box_a (\mathcal{F} f \odot d_1), \Box_a (\mathcal{F} f \odot d_1) \rangle, \langle \Box_a (\mathcal{F} f \odot d_2), \Box_a (\mathcal{F} f \odot d_2) \rangle]
\]

\[
= \{ \langle c_1 \odot \Box_a (\mathcal{F} f \odot d_1), c_2 \odot \Box_a (\mathcal{F} f \odot d_1) \rangle,
\langle c_1 \odot \Box_a (\mathcal{F} f \odot d_2), c_2 \odot \Box_a (\mathcal{F} f \odot d_2) \rangle \}
\]

\[
\equiv \{ \langle \mathcal{F} f \odot d_1, \mathcal{F} f \odot d_1 \rangle, \langle \mathcal{F} f \odot d_2, \mathcal{F} f \odot d_2 \rangle \}
\] (by Lem. 6.7)

\[
= \mathcal{F} f \odot d .
\]

Moreover, by the assumption, we have \( s \sqsubseteq f \odot t \). Therefore Cond. [1] in Def. 6.2 is satisfied.

Next we show that \( f \) satisfies Cond. 2 in Def. 6.4. For each \( \alpha \leq \bar{\pi} \), we define \( f_{11}^{(\alpha)} \) and \( f_{12}^{(\alpha)} \) by \( f_{11}^{(\alpha)} = g_1(\alpha) \) and \( f_{12}^{(\alpha)} = g_2(\alpha) \). As \( g = ((\bar{\beta}), (g_1(\beta)))_{1 \leq i \leq 4, \alpha \leq \bar{\pi}} \) satisfies the axioms of progress measure (Def. 6.2), conditions (2a), (2b) and (2d) in Def. 6.4 are satisfied by \((f_{11}^{(\alpha)})_{\alpha \leq \bar{\pi}}\) and \((f_{12}^{(\alpha)})_{\alpha \leq \bar{\pi}}\). Therefore all the conditions in Def. 6.4 are satisfied by \( f \).

(\(=\)) Conversely, let \( f : Y \to X \) be a fair simulation without dividing from \( X \) to \( Y \) where the approximation sequences are given by \((f_{11}^{(\alpha)})_{\alpha \leq \bar{\pi}}\) and \((f_{12}^{(\alpha)})_{\alpha \leq \bar{\pi}}\). For each \( \alpha \leq \bar{\pi} \), we define arrows \( g_1(\alpha) : Y_1 \to X_1, g_2(\alpha) : Y_2 \to X_2, g_3(\alpha) : Y_2 \to X_1 \) and \( g_4(\alpha) : Y_2 \to X_2, \) by \( g_1(\alpha) = f_{11}^{(\alpha)} \), \( g_2(\alpha) = f_{12}^{(\alpha)} \), \( g_3(\alpha) = f_{21} = f_{11} \) and \( g_4(\alpha) = f_{22} \). Then by conditions in Def. 6.4 and by Lem. 6.7 we have that \( g = ((\bar{\beta}), (g(\beta)))_{1 \leq i \leq 4, \alpha \leq \bar{\pi}} \) is a progress measure for the equational system [7].

F.3.3 Proof of Prop. 6.9

**Proof of Prop. 6.9** Let \((f_{ij} : Y_i \to X_j)_{i,j \in \{1,2\}}\) be a fair simulation without dividing from \( X \) to \( Y \). Moreover, let \( f_{11}^{(\alpha)} \subseteq f_{11}^{(1)} \subseteq \cdots \subseteq f_{11}^{(\alpha)} = f_{11} \) and \( f_{12}^{(\alpha)} \subseteq f_{12}^{(1)} \subseteq \cdots \subseteq f_{12}^{(\alpha)} = f_{12} \) be the approximation sequences.

Let \( E_X \) be the equational system that defines \( \text{tr}^B(c_1) : X_1 \to Z \) and \( \text{tr}^B(c_2) : X_2 \to Z \) (c.f. [9] in Thm. 6.5). By completeness of progress measure (Thm. 6.4.3), there exists a progress measure \( p_{E_X} = ((\bar{\beta}), (u_1(\beta) : X_1 \to Z, u_2(\beta) : X_2 \to Z)_{\beta \leq \bar{\pi}}) \) for \( E_X \) such that \( u_1(\bar{\beta}) = \text{tr}^B(c_1) \) and \( u_2(\bar{\beta}) = \text{tr}^B(c_2) \).

For each pair of ordinals \( \gamma_1 \leq \bar{\beta} \) and \( \gamma_2 \leq \bar{\pi} \), we define three arrows \( h_1(\gamma_1, \gamma_2) : X_1 \to Z, h_2(\gamma_1, \gamma_2) : X_1 \to Z, \) and \( h_3(\gamma_1, \gamma_2) : X_2 \to Z \) by \( h_1(\gamma_1, \gamma_2) := u_1(\gamma_1) \odot f_{11}^{(\gamma_2)}, h_2(\gamma_1, \gamma_2) := u_2 \odot f_{12}^{(\gamma_2)}, \) and \( h_3(\gamma_1, \gamma_2) := [u_1(\bar{\pi}), u_2] \odot \langle f_{21}, f_{22} \rangle \). Moreover, we define ordinals \( \bar{\gamma_1} \) and \( \bar{\gamma_2} \) by \( \bar{\gamma_1} = \bar{\beta} \) and \( \bar{\gamma_2} = \bar{\pi} \).

Then in a similar manner to the proof of Sublem. F.1 we can prove that

\[
p := ((\bar{\gamma_1}, \bar{\gamma_2}), (h_1(\gamma_1, \gamma_2), h_2(\gamma_1, \gamma_2), h_3(\gamma_1, \gamma_2))_{\gamma_1 \leq \bar{\pi}, \gamma_2 \leq \bar{\pi}})
\]

is a progress measure for the following equational system.

\[
\begin{align*}
v_1' &= \mu (J_{\zeta})^{-1} \odot F[\text{id}_Z, \text{id}_Z] \odot \langle v_1', v_2' \rangle, v_3' \odot d_1 \in \mathcal{K}(T)(Y_1, Z) \\
v_2' &= \mu (J_{\zeta})^{-1} \odot F[\text{id}_Z, \text{id}_Z] \odot \langle v_1', v_2' \rangle, v_3' \odot d_1 \in \mathcal{K}(T)(Y_1, Z) \\
v_3' &= \mu (J_{\zeta})^{-1} \odot F[\text{id}_Z, \text{id}_Z] \odot \langle v_1', v_2' \rangle, v_3' \odot d_2 \in \mathcal{K}(T)(Y_2, Z)
\end{align*}
\] (21)
By definition, the trace semantics $\text{tr}^B(d_1) : Y_1 \to Z$ and $\text{tr}^B(d_2) : Y_2 \to Z$ of $\mathcal{Y}$ is defined as the solution of the following equational system.

$$
\begin{aligned}
v_1 &= \mu \left( (J \zeta)^{-1} \circ \mathcal{T}[v_1, v_2] \circ d_1 \right) \in \mathcal{K}(T)(Y_1, Z) \\
v_2 &= \nu \left( (J \zeta)^{-1} \circ \mathcal{T}[v_1, v_2] \circ d_2 \right) \in \mathcal{K}(T)(Y_2, Z)
\end{aligned}
$$

The solution of this equational system is given by $v_1^\text{sol} = \text{tr}^B(d_1) = \bot$ and $v_2^\text{sol} = \text{tr}^B(d_2)$.

Note here that as $\pi_i \circ \text{bc} \circ \bot = \bot = \pi_1 \circ \bot = \pi_4 \circ \text{bc} \circ \bot$

for each $i \in \{1, 2\}$, we have $\text{bc} \circ \bot = \text{bc} \circ \bot$. Therefore by monotonicity of the bicartesian $\text{bc}$, we have $\bot = \bot$.

Hence we can see that the solution of the equational system (21) is given by $v_1^\text{sol} = v_2^\text{sol} = \bot = \text{tr}^B(d_1)$ and $v_3^\text{sol} = \text{tr}^B(d_2)$. Therefore by soundness of progress measure (Thm. B.3.4), we have the following inequalities.

$$
\begin{aligned}
\text{tr}^B(c_1) \odot f_{11} \sqsubseteq \text{tr}^B(d_1), \quad \text{tr}^B(c_2) \odot f_{12} \sqsubseteq \text{tr}^B(d_1) \quad \text{and} \quad [\text{tr}^B(c_1), \text{tr}^B(c_2)] \circ \bot \sqsupseteq [\text{tr}^B(d_1), \text{tr}^B(d_2)] \\
\text{tr}^B(c_1) \odot f_{21} \sqsubseteq \text{tr}^B(d_1), \quad \text{tr}^B(c_2) \odot f_{22} \sqsubseteq \text{tr}^B(d_1) \quad \text{and} \quad [\text{tr}^B(c_1), \text{tr}^B(c_2)] \circ \bot \sqsupseteq [\text{tr}^B(d_1), \text{tr}^B(d_2)]
\end{aligned}
$$

From the first and the second inequalities, and that $\text{tr}^B(d_1) = \bot$, we have

$$
\begin{aligned}
[\text{tr}^B(c_1), \text{tr}^B(c_2)] \circ \bot &= [\text{id}, \text{id}] \circ \bot \\
&= \bot \quad \text{and} \quad [\text{tr}^B(d_1), \text{tr}^B(d_2)] \circ \bot = \bot
\end{aligned}
$$

Hence we have:

$$
\begin{aligned}
\text{tr}^B(\mathcal{Y}) &= [\text{tr}^B(c_1), \text{tr}^B(c_2)] \circ s \\
&\sqsubseteq [\text{tr}^B(c_1), \text{tr}^B(c_2)] \circ [f_{11}, f_{12}] \circ [f_{21}, f_{22}] \circ t \\
&\quad \text{(by definition)} \\
&\sqsubseteq [\text{tr}^B(c_1), \text{tr}^B(c_2)] \circ [f_{11}, f_{12}] \circ [f_{21}, f_{22}] \circ t \\
&\quad \text{(by Cond. 1 in Def. 6.2 and Def. 6.4)} \\
&\sqsubseteq [\text{tr}^B(c_1), \text{tr}^B(c_2)] \circ [f_{11}, f_{12}] \circ [\text{tr}^B(c_1), \text{tr}^B(c_2)] \circ [f_{21}, f_{22}] \circ t \\
&\quad \text{[by conditions in Def. 6.4]} \\
&\sqsubseteq [\text{tr}^B(d_1), \text{tr}^B(d_2)] \circ t \\
&\quad \text{(by definition)}
\end{aligned}
$$

This concludes the proof.

\subsection*{F.3.4 Proof of Lem. 6.10}

\textbf{Proof of Lem. 6.10} Let $Y = Y_1 + Y_2$. Note that $Y$ is a finite set equipped with a discrete $\sigma$-algebra.

The Büchi ($\mathcal{G}, \mathcal{A}, (\bot)$)-system $\mathcal{Y} = (Y_1, Y_2, t, d)$ induces a Markov chain $\mathcal{M}_Y$ such that the state space is defined by $Y_\bot = Y + \{\bot\}$ and the transition function $\tau_Y : Y_\bot \times Y_\bot \to [0, 1]$ is given by

$$
\begin{aligned}
\tau_Y(y, y') &= \begin{cases} 
\sum_{a \in \mathcal{A}} d(y)(\{a, y'\}) & (y, y' \in Y) \\
1 - \sum_{y' \in Y} \sum_{a \in \mathcal{A}} d(y)(\{a, y'\}) & (y \in Y, y' = \bot) \\
1 & (y = y' = \bot) \\
0 & \text{(otherwise)}
\end{cases}
\end{aligned}
$$
A Markov chain $\mathcal{M}_Y$ is defined similarly.

A subset $B \subseteq Y$ is called a strongly connected component (SCC for short) if for all $y, y' \in S$, there exists $y_0, y_1, \ldots, y_n$ such that $y_0 = y$, $y_n = y'$ and $\tau_Y(y_i, y_{i+1}) > 0$ for each $i$. A SCC $B$ is called a bottom strongly connected component (BSCC for short) if $\tau_Y(y, y') = 0$ for each $y \in B$ and $y' \notin B$. For more details, see e.g. [4].

For $Y' \subseteq Y$, we write $\Pr(y \models GFY')$ for the probability where a state in $Y'$ is visited infinitely often on $\mathcal{M}_Y$ from $y \in Y$. By Thm. 5.5.2b, we have:

$$[tr^B(d_1), tr^B(d_2)](y)(A^\omega) = \Pr(y \models GFY_2), \quad \text{and}$$
$$[tr^B(d_1'), tr^B(d_2')](y)(A^\omega) = \Pr(y \models GF(Y_2 + \{y > 0\})).$$

We define $U, U' \subseteq Y$ by

$$U := \bigcup \{B \subseteq Y \mid B \text{ is a BSCC and } B \cap Y_2 \neq \emptyset\}, \quad \text{and}$$
$$U' := \bigcup \{B \subseteq Y \mid B \text{ is a BSCC and } B \cap (Y_2 + \{y > 0\}) \neq \emptyset\}.$$

It is known that $\Pr(y \models GFY_2)$ is given by a probability $\Pr(y \models FU)$ where a state in $U$ is reached in $\mathcal{M}_Y$ (see e.g. [4] Cor. 10.34). Similarly, we have $\Pr(y \models GF(Y_2 + \{y > 0\})) = \Pr(y \models FU')$.

Assume that $y > 0 \in B$ for some BSCC $B$ in $\mathcal{M}_Y$. As $B$ is a BSCC, it has no outgoing transition, on the one hand. On the other hand, by the assumption that $tr^B(d_1)(y_1)(Sigma^\omega) > 0$, we have:

$$\Pr(y > 0 \models FU) = \Pr(y > 0 \models GFY_2) > 0.$$

Hence by the definition of $U'$, we have $B \cap Y_2 \neq \emptyset$ and this implies that $U = U'$.

If $y > 0 \notin B$ for each BSCC $B$, then we have $U = U'$ by definition of $U$ and $U'$.

Therefore in both cases, we have:

$$[tr^B(d_1), tr^B(d_2)](y)(A^\omega) = \Pr(y \models GFY_2)$$
$$= \Pr(y \models FU)$$
$$= \Pr(y \models GF(Y_2 + \{y > 0\}))$$
$$= [tr^B(d_1'), tr^B(d_2')](y)(A^\omega).$$

It remains to prove $[tr^B(d_1), tr^B(d_2)](y)(A) = [tr^B(d_1'), tr^B(d_2')](y)(A)$ for each measurable set $A \subseteq A^\omega$. To this end, by Carathéodory’s extension theorem (see e.g. [8]), it suffices to prove $[tr^B(d_1), tr^B(d_2)](y)(wA^\omega) = [tr^B(d_1'), tr^B(d_2')](y)(wA^\omega)$ for each $w \in A^*$, where $wA^\omega$ denotes the cylinder set $\{ww' \mid w' \in A^\omega\}$.

We inductively define a function $\chi_Y : Y \times A^* \rightarrow \mathcal{G}Y$ by

$$\chi_Y(y, e)(\{y'\}) = \begin{cases} 1 & (y = y') \\ 0 & \text{(otherwise)} \end{cases}$$

and

$$\chi_Y(y, aw)(\{y'\}) = \sum_{y'' \in Y} d(y)((a, y'')) \cdot \chi_Y(y'', w)(\{y'\})$$

where $a \in A$ and $w \in A^*$. 
Then for all \( y \in Y \) and \( w \in A^* \), we have:

\[
[\text{tr}^B(d_1), \text{tr}^B(d_2)](y)(wA^\omega) = \sum_{y' \in Y} \chi_Y(y, w)(y') \cdot [\text{tr}^B(d_1), \text{tr}^B(d_2)](y)(A^\omega) \\
= \sum_{y' \in Y} \chi_Y(y, w)(y') \cdot [\text{tr}^B(d'_1), \text{tr}^B(d'_2)](y)(A^\omega) \\
= [\text{tr}^B(d'_1), \text{tr}^B(d'_2)](y)(wA^\omega).
\]

By Carathéodory’s extension theorem, this implies \([\text{tr}^B(d_1), \text{tr}^B(d_2)](y)(A) = [\text{tr}^B(d'_1), \text{tr}^B(d'_2)](y)(A)\) for each measurable set \( A \), and we have \( \text{tr}^B(Y) = \text{tr}^B(Y') \). ▶