REGULAR POLYNOMIAL ENDO MORPHISMS OF $\mathbb{C}^k$

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§0 Introduction

Let $f = (f_1, \ldots, f_k) : \mathbb{C}^k \to \mathbb{C}^k$ be a mapping such that each $f_j$ is a polynomial of degree $d$. We consider the behavior of $f$ as a dynamical system. That is, we consider the behavior of the iterates $f^n = f \circ \cdots \circ f$ as $n \to \infty$. The points of most interest are those whose forward orbits show recurrent behavior. Thus we are led to focus on the set $K$ of points whose forward orbits are bounded. The point of this paper is to present an approach to studying $K$ by working instead at the “points at infinity” and then making a descent to $K$ via “external rays.”

To illustrate this, let us consider the case $k = 1$. The recurrent dynamics of a polynomial mapping $p : \mathbb{C} \to \mathbb{C}$ is carried by the set $K$, and the chaotic dynamics take place on the Julia set $J := \partial K$. A polynomial mapping may be characterized as a holomorphic mapping which extends continuously to the Riemann sphere. The point at infinity is completely invariant, and it is possible to find a holomorphic function $\varphi$ defined in a neighborhood of $\infty$, called the Böttcher coordinate, such that $\varphi$ conjugates $p$ to the model mapping $\sigma(\zeta) = \zeta^d$ in a neighborhood of infinity. If the set $K$ is connected, then $\varphi$ has an analytic continuation to a conformal equivalence $\varphi : \mathbb{C} - K \to \mathbb{C} - \overline{D}$ with the complement of the closed unit disk. Thus the dynamics of the restriction $p|_{(\mathbb{C} - K)}$ is conjugate to $\sigma$ on all of $\mathbb{C} - \overline{D}$. If the inverse $\varphi^{-1}$ can be extended continuously to $\partial D$, then $p|J$ is represented as a quotient of $\sigma|\partial D$.

Another point of view is to consider the point at infinity as the pole for the Green function for $\mathbb{C} - K$. This serves as the starting point for the use of potential-theoretic methods in the study of polynomial mappings, as was introduced by Brolin [Bro] and further developed by Sibony (see [CG]) and Tortrat [T]. A natural way to descend from infinity to the set $J$ is to follow the gradient lines of the Green function, which are also known as “external rays.” External rays were introduced by Douady and Hubbard and have been developed into a powerful tool for studying the relationship between the mappings $p|(\mathbb{C} - K)$ and $p|J$.

In our approach, we view $\mathbb{C}^k$ as an affine coordinate chart in $\mathbb{P}^k$. Thus $\Pi := \mathbb{P}^k - \mathbb{C}^k$, the hyperplane at infinity, is isomorphic to $\mathbb{C}^{k-1}$. We study polynomial mappings $f$ of degree $d \geq 2$ which are regular, which means that $f$ extends continuously (and thus holomorphically) to $\mathbb{P}^k$. It follows that $\Pi$ is completely invariant, and we let $f_\Pi$ denote the induced dynamical system at infinity. Further, $\Pi$ is (super)-attracting in the normal direction, so the basin $A$ of points which are attracted to $\Pi$ in forward time is an open set containing $\Pi$. This gives a completely invariant partition $\mathbb{P}^k = K \cup A$.

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We will make use of methods of the dynamics of holomorphic mappings of $\mathbb{P}^k$ and $\Pi \simeq \mathbb{P}^{k-1}$. In particular, we will make use of the approach introduced in [HP] and developed more generally and systematically in [FS1–3]. Namely, there is an invariant current $T$ on $\mathbb{P}^k$, and the exterior powers $T^l := T \wedge \cdots \wedge T$, $1 \leq l \leq k$, are well defined, positive, closed currents of bidegree $(l,l)$. The supports $J_l := \text{supp}(T^l)$ serve as a family of intermediate Julia sets. In the maximal degree we have measures $\mu := T^k$ on $\mathbb{P}^k$ and $\mu_\Pi := T^k_{\Pi} - 1$ (corresponding to $f_\Pi$) on $\Pi$. We will denote their supports by $J$ and $J_\Pi$, respectively. The importance of $\mu$ is shown by the fact that it is ergodic (see Fornæss-Sibony [FS2–3]); it is balanced and has maximal entropy ($= k \log d$); and $\mu$ describes the distribution of periodic points (see Briend [Bri]). The corresponding properties hold for $\mu_\Pi$.

Closely related to the current $T$ is the function $G$, defined in (1.1), which measures the superexponential rate at which an orbit approaches $\Pi$. The function $G$ coincides with the plurisubharmonic Green function in $\mathbb{C}^k$ for the set $K$, and $G$ has a logarithmic singularity along $\Pi$. In our passage from the case $k = 1$ to $k > 1$, we replace the point at infinity by the set $J_\Pi$. We replace the one-dimensional set $C - K$ by the set $A \cap \text{supp}(T^{k-1})$. The usual critical locus of $f$ is the set $C$ where the Jacobian determinant of $f$ vanishes. For the purpose of dynamical study, we consider the critical measure $\mu_c := [C] \wedge T^{k-1}$. This is motivated by another critical measure defined in [BS2]. A major theme of this paper is to explore the interplay between the dynamical measures $\mu$ and $\mu_\Pi$, the critical measure $\mu_c$, and the current $T^{k-1}$.

As a first example we consider the Lyapunov exponents, which measure the rate of expansion of the mapping $f$ with respect to a measure. We concern ourselves here with $\Lambda(f) = \Lambda(f, \mu)$, which is defined as the sum of all the Lyapunov exponents of $f$ with respect to $\mu$, and thus $\Lambda(f)$ measures the infinitesimal exponential rate of Lebesgue volume expansion. We show (Theorem 3.2) that the Lyapunov exponents of $(f, \mu)$ and $(f_\Pi, \mu_\Pi)$ are related by the formula

$$
\Lambda(f) = \Lambda(f_\Pi) + \log d + \int G \mu_c.
$$

This generalizes formulas of Przytycki [Pr] and Jonsson [J1] and is analogous to a formula in [BS5]. We are grateful to B. Berndtsson for showing us how to greatly simplify our previous proof.

The connection between $J_\Pi$ and $J$ is mediated by the stable manifolds

$$
W^s(a) = \{ x \in \mathbb{P}^k : \lim_{n \to +\infty} \text{dist}(f^n a, f^n x) = 0 \}
$$

and

$$
W^s_{\text{loc}}(a) = \{ x \in \mathbb{P}^k : \text{dist}(f^n a, f^n x) < \delta, n \geq 0 \}
$$

for $a \in J_\Pi$. By Pesin Theory there is a $\delta = \delta(a) > 0$ for $\mu_\Pi$ a.e. $a$ such that $W^s_{\text{loc}}(a)$ is a complex disk. In using the potential-theoretic approach, we work with a local stable manifold as a current of integration $[W^s_{\text{loc}}(a)]$. (In general, a component of the global stable manifold is not locally closed and does not have locally bounded area, so $W^s(a)$
does not define a current of integration.) A crucial property of $T^k-1$ is that it has, in a measure-theoretic sense, a laminar structure on $A$, made up of currents of integration over pieces of stable manifolds. The strongest example of laminarity is Theorem 5.1: if $f_{11}$ is uniformly expanding on $J_{11}$, then $T^k-1$ has the uniform laminar structure (5.1) in a neighborhood of $\Pi$. (This result was also obtained by Peng [Pe].) For a general map, we establish two sorts of laminar structure (see Theorems 6.4 and 6.10).

The set $W^s(a)$ is a manifold (with singularities), which contains the Pesin disk $W^s_{loc}(a)$, as well as $W^s_{loc}(b)$ whenever $b \in J_{11}$ and $f_{11}^n a = f_{11}^n b$ for some $n \geq 1$. The restriction $G|W^s(a)$ is harmonic for $\mu_{11}$-a.e $a$ and has no critical points on $W^s_{loc}(a)$. We define the set $E_a$ as the set of gradient lines of $G|W^s(a)$ emanating from $a$ (the point at infinity). Thus $E_a$ is parametrized by the angle a gradient line makes inside $W^s(a)$ at $a$. Let $\mathcal{E}$ be the union of all $E_a$ over $\mu_{11}$-a.e. $a$. The elements of $\mathcal{E}$ are called external rays. It follows that the measure $\nu := \mu_{11} \otimes \frac{da}{2\pi}$ is well defined on $\mathcal{E}$. Using the (non-uniform) laminar structure in the general case, we are able to show that for $\nu$ a.e. ray $\gamma \in \mathcal{E}$, there is a well defined endpoint $e(\gamma) \in J$. The basic connection between $\nu$ and $\mu$ is given by transport along the external rays (Theorem 7.3): $e_*\nu = \mu$.

In case the critical measure $\mu_c$ puts no mass near $\Pi$, and $k = 2$, the family of Pesin disks in fact extends to a Riemann surface lamination near $\Pi$ (Theorem 8.8). To obtain results with better control over the geometry, we assume that $f_{11}$ is uniformly expanding on $J_{11}$; thus by the Stable Manifold Theorem, $W^s_{loc}(J_{II}) = \{W^s_{loc}(a) : a \in J_{II}\}$ is a Riemann surface lamination for a uniform $\delta > 0$. We use this lamination to obtain a local conjugacy with a model mapping. Let $C(J_{II})$ denote the set of complex lines through the origin in $\mathbb{C}^k$ corresponding to points of $J_{II}$. Our canonical model is given by the restriction of $f_{II}$, the homogeneous part of $f$, to $C(J_{II})$. If $f_{II}$ is uniformly expanding on $J_{II}$, then the restriction $f_{II}|C(J_{II})$ is conjugate to the restriction $f|W^s_{loc}(J_{II})$ in a neighborhood of $J_{II}$ (Theorem 4.3). This conjugacy, which was also found by G. Peng [Pe], can be viewed as a local Böttcher coordinate for $f$. A global version is given in Theorem 7.4.

The general approach of starting with $f_{11}$ to study $f$ is dual in spirit to that of Hubbard and Papadopol [HP], who study a superattracting fixed point by working on the fiber $\mathbb{P}^{k-1}$ over the blowup. From this point of view, Theorem 4.3 is analogous to Theorem 9.3 of [HP]. A different approach was given by S. Heinemann [H], who works directly on $K$ without appeal to $\Pi$ or $f_{II}$.

The final part of this paper addresses the question of when the landing map $e : \mathcal{E} \to J$ is continuous. When possible, we would like to replace statements about almost every ray with statements about every ray. In the case $k = 1$, the continuity of $e$ is equivalent to local connectedness of $J$. In dimension $k = 2$, $\Pi$ is the Riemann sphere, and $f_{II}$ is a rational map. If $f_{II}$ is hyperbolic, then $\mathcal{E}$ is homeomorphic to $J_{II} \times S^1$. In order to obtain the continuity of $e$, we assume that $f$ is Axiom A. Thus the nonwandering set is the closure of the set of periodic points and can be written as the union $S_0 \cup S_1 \cup S_2$, where $S_j$ is a (uniformly) hyperbolic set with unstable index $j$. Our main result about the landing map is Theorem 10.2: If $k = 2$, if $f$ is Axiom A, if $f^{-1}S_2 = S_2$, and if $\mu_c(A) = 0$, then $e : \mathcal{E} \to J$ is a continuous surjection.

The contents of this paper are as follows: Sections 1 and 2 recall some basic notation and results, mostly on currents and potential theory. §3 is devoted to the proof of the
formula for the sum of the Lyapunov exponents of $\mu$. In §4 we discuss the Stable Manifold Theorem in the context of stable disks through points of $J_\Pi$. We show that $f|W^s(J_\Pi)$ is conjugate to the canonical model in a neighborhood of $J_\Pi$. In §5 we show that $T^{k-1}$ is laminar in a neighborhood of $\Pi$ when $f_\Pi$ is uniformly expanding on $J_\Pi$. This serves as a prototype for our results in §6, where we discuss laminarity in the general case. The difficulties of §6 arise from two sources. First is the fact that in general (without uniform expansion on $J_\Pi$) we need to use Pesin Theory to obtain our stable manifolds, and the geometry of the Pesin disks is not controlled. The second source of difficulty, which is present already in the uniformly expanding case, comes from working globally in $A$, since in general the global stable manifolds do not define currents of integration. In §7, we define external rays and show that $\nu$ almost every external ray has a well-defined landing point, and the endpoint map pushes the measure $\nu$ forward to $\mu$. We also give a global version (Theorem 7.4) of the local conjugacy result of §4. §8 discusses the structure of the support of $T^{k-1}$ inside $A$ for general maps $f$. First we show that for $\mu_\Pi$ a.e. $a$, the global stable manifold $W^s(a)$ is dense in $A \cap \text{supp}(T^{k-1})$. Then we show (Theorem 8.8) that if $\mu_\pi(A) = 0$, then the family of local Pesin disks in $A$ is contained in a Riemann surface lamination by disks which are proper in $A$. The property Axiom A is discussed for endomorphisms of $C^2$ in §9, and §10 is devoted to the proof of Theorem 10.2, which gives the continuous landing of the external rays. Appendix A analyzes the behavior of the homogeneous (canonical) model. Appendix B serves as a reference for some of the hyperbolicity results that are used in §8 and 9.

**List of notation**

- $f$: regular polynomial endomorphism of $C^k$ of degree $d$.
- $f_h$: homogeneous part of $f$ of degree $d$.
- $\Pi$: hyperplane at infinity.
- $f_\Pi$: restriction of $f$ to $\Pi$.
- $C$: critical set of $f$.
- $C_\Pi$: critical set of $f_\Pi$.
- $C^k_*$: $C^k - \{0\}$.
- $\pi$: projection of $C^k_*$ on $\Pi$ or $C^{k+1}_*$ on $P^k$.
- $A$: basin of $\Pi$.
- $K$: complement of $A$.
- $G$: Green function for $f$.
- $G_h$: homogeneous Green function for $f_h$.
- $\tilde{f}$: homogeneous map on $C^{k+1}_*$.
- $\tilde{G}$: homogeneous Green function for $\tilde{f}$.
- $\rho_G$: Robin function for $G$.
- $T$: invariant current for $f$.
- $T_h$: homogeneous invariant current for $f_h$. 

$T_{Π}$ invariant current for $f_{Π}$.
$µ$ $T^k$.
$µ_{Π}$ $T^k_{Π}$.
$J$ support of $µ$.
$J_{Π}$ support of $µ_{Π}$.
$L_a$ line in $P^k$ through 0 associated with $a ∈ Π$.
$Λ(f)$ sum of Lyapunov exponents of $f$ with respect to $µ$.
$Λ(f_{Π})$ sum of Lyapunov exponents of $f_{Π}$ with respect to $µ_{Π}$.
$µ_c$ critical measure: $µ_c = [C] ∩ T^{k-1}$.
$W^s_{loc}(a)$ local stable manifold at $a$.
$A_0$ subset of $A$ where $G > R_0$.
$A_n$ $f^{-n}(A_0)$.
$W^s(a)$ global stable manifold of $a$.
$W^s(a, f_{Π})$ global stable manifold of $a$ with respect to $f_{Π}$.
$W^s(J_{Π})$ stable set of $J_{Π}$ for $f$.
$W^s(J_{Π}, f_h)$ stable set of $J_{Π}$ for $f_h$.
$W^s_0(a)$ local stable disk for $f$ at $a ∈ J_{Π}$.
$W^s_0(a, f_h)$ local stable disk for $f_h$ at $a$ (subset of complex line).
$W_s(J_{Π})$ stable lamination for $f$.
$C(J_{Π})$ complex homogeneous cone over $J_{Π}$.
$Ψ$ conjugation of $f_h|W^s(J_{Π}, f_h)$ to $f|W^s(J_{Π})$.
$W^s_{-m}(a)$ $W^s_{loc}(a) ∩ A_{-m}$.
$Z_{a,n}$ component of $W^s(a) ∩ A_n$ containing $a$.
$N_n(a)$ number of points in $Z_{a,n} ∩ Π$.
$C_n$ $∪_{0 ≤ j ≤ n-1} f^{-j}(C)$ (critical set of $f^n$).
$C_∞$ $∪_{n ≥ 0} C_n$.
$S_{a,n}$ union of incomplete gradient lines in $Z_{a,n}$.
$W_{a,n}$ component of $Z_{a,n} - S_{a,n}$ containing $a$.
$W_a$ union of $W_{a,n}$ over $n ≥ 0$.
$φ_a$ uniformizing map on $W_a$.
$H_a$ range of $φ_a$ (hedgehog domain).
$ψ_a$ inverse of $φ_a$.
$E_a$ set of external rays in $W_a$.
$E$ set of external rays.
$σ$ map on $E$.
$ν$ invariant measure on $E$. 
§1 Regular polynomial endomorphisms and their Green functions

In the following two sections we summarize several basic results that we will use. Additional details may be found in [HP], [FS1–3], and [U]. We recommend the unified treatment in [FS3]. Throughout this paper, we will let $f$ be a regular polynomial endomorphism of $\mathbb{C}^k$ of degree $d \geq 2$. This means that the components of $f$ are polynomials of degree $d$, and the homogenous part $f_h$ of degree $d$ satisfies $f_h^{-1}(0) = \{0\}$. Alternatively, $f$ is regular if and only if $\liminf |f(z)|/|z|^d > 0$ as $|z| \to \infty$. Such mappings are also called strict polynomials by Heinemann [H].

Let $z = (z_1, \ldots, z_k)$ denote (inhomogeneous) coordinates on $\mathbb{C}^k$, and let $[z : t] = [z_1 : \ldots : z_k : t]$ denote homogeneous coordinates on $\mathbb{P}^k$. We fix the embedding of $\mathbb{C}^k$ into $\mathbb{P}^k$ given by $z \mapsto [z : 1]$. Thus $\Pi = \{t = 0\}$ corresponds to the hyperplane at infinity, and $\Pi$ may be identified with with $\mathbb{P}^{k-1}$ using homogeneous coordinates $[z] = [z_1 : \ldots : z_k]$. We equip $\mathbb{P}^k$ with the Fubini-Study metric and measure distances and volumes in that metric unless otherwise stated.

A regular polynomial endomorphism $f$ extends to a holomorphic endomorphism of $\mathbb{P}^k$, still denoted by $f$, which may be defined by the formula $f[z : t] = [t^d f(z)/t : t^d]$. In fact, a holomorphic endomorphism of $\mathbb{P}^k$ has a completely invariant hyperplane exactly when it is conjugate to a regular polynomial endomorphism of $\mathbb{C}^k$. We let $f_{\Pi}$ denote the restriction of $f$ to $\Pi$. Under the identification $\Pi \cong \mathbb{P}^{k-1}$, $f_{\Pi}$ is given by $[z] \mapsto [f_h(z)]$. Let $C$, $C_{\mathbb{P}^k}$ and $C_{\Pi}$ denote the critical sets of $f$ as a map of $\mathbb{C}^k$, $\mathbb{P}^k$ and $\Pi$, respectively. Then we have $C_{\mathbb{P}^k} = C \cup \Pi$ and $C_{\Pi} = C \pitchfork \Pi$.

The model for our study of regular polynomial automorphisms is the case when $f = f_h$ is a homogeneous mapping of $\mathbb{C}^k$. In this case we write $C_h^* = C_h - \{0\}$, and we let $\pi : C_h^* \to \Pi$ be the projection given by $\pi(z) = [z]$. It is evident that $\pi$ gives a semiconjugacy from $f_h$ to $f_{\Pi}$: $\pi \circ f_h = f_{\Pi} \circ \pi$. In fact, $f_h$ is essentially a skew product over $f_{\Pi}$, as is shown in Appendix A.

In the general case, we let $K$ be the compact set of points in $\mathbb{C}^k$ with bounded forward orbits and define $A := \mathbb{P}^k - K$. Thus $A$ is the basin of attraction of $\Pi$. The function

$$G(z) = \lim_{n \to \infty} d^{-n} \log^+ |f^n(z)|$$

(1.1)
are a normal family, then gives the (super-exponential) rate at which the orbit of \( z \in \mathbb{C}^k \) approaches \( \Pi \). This is continuous and plurisubharmonic (psh) on \( \mathbb{C}^k \) and coincides with the pluri-complex Green function of \( K \). We will therefore also call \( G \) the Green function of \( f \). The homogeneous Green function for the homogeneous part \( f_h \) of \( f \) of maximal degree \( d \) is defined in an analogous way, namely as

\[
G_h(z) = \lim_{n \to \infty} d^{-n} \log |f^n_h(z)|.
\]

The functions \( G \) and \( G_h \) are continuous on \( \mathbb{C}^k \) and \( \mathbb{C}^*_h \), respectively. We use \( \log \) instead of \( \log^+ \) so that \( G_h \) is logarithmically homogeneous.

We may also define a homogeneous map \( \tilde{f} \) on \( \mathbb{C}^{k+1} \) by \( \tilde{f}(z,t) = (t^d f(z/t), t^d) \). The pair \( (\tilde{f}, f) \) has properties analogous to those of \( (f_h, f_\Pi) \). The projection \( \pi : \mathbb{C}^{k+1}_* \to \mathbb{P}^k \) given by \( \pi(z,t) = [z : t] \) semiconjugates \( \tilde{f} \) to \( f \): \( f \circ \pi = \pi \circ \tilde{f} \). We define the homogeneous Green function \( \tilde{G} \) for \( \tilde{f} \) by

\[
\tilde{G}(z,t) := \lim_{n \to \infty} d^{-n} \log |\tilde{f}^n(z,t)|
\]

for \( (z,t) \in \mathbb{C}^{k+1}_* \). The connection between \( \tilde{G} \), \( G \) and \( G_h \) is \( \tilde{G}(z,1) = G(z) \) and \( \tilde{G}(z,0) = G_h(z) \). This leads us to the following asymptotic formulas for \( G \) and \( G_h \) near \( \Pi \). Here \( \rho_G \) denotes the Robin function of \( G \) (cf. [BT2]).

**Lemma 1.1.** The asymptotics of \( G \) and \( G_h \) at \( \Pi \) are given by

\[
G_h(z) = \log |z| + \rho_G[z] = G(z) = \log |z| + \rho_G[z] + o(1),
\]

where \( \rho_G \) is continuous on \( \Pi \). Here \( [z] = \pi(z) \) is the projection of \( z \) on \( \Pi \) defined above.

**Proof.** Since \( G_h \) is homogeneous we have

\[
G_h(z) = \log |z| + G_h(z/|z|).
\]

Here the second term is continuous in \( z \) and depends only on the projection \( [z] \) of \( z \) on \( \Pi \). Hence there exists a continuous function \( \rho_G \) on \( \Pi \) such that \( G_h(z/|z|) = \rho_G[z] \). This proves the first formula. To prove the second we write

\[
G(z) = \tilde{G}(z,1) = \log |z| + \tilde{G}(z/|z|, 0) + (\tilde{G}(z/|z|, 1/|z|) - \tilde{G}(z/|z|, 0)) = \log |z| + \rho_G[z] + o(1),
\]

where the last line follows from the continuity of \( \tilde{G} \) on \( \mathbb{C}^{k+1}_* \).

The next result, implicitly contained in [FS2], is crucial.

**Lemma 1.2.** If \( M \subset \mathbb{C}^k \) is a complex manifold, and if the iterates of \( f \), restricted to \( M \), are a normal family, then \( G|M \) is pluriharmonic on \( M \).

**Proof.** We may assume that \( M \subset A \). Passing to a subsequence of the iterates \( f^n = (f^n_1, \ldots, f^n_k) \), we may assume that on \( M \) we have \( |f^n_{(j)}/f^n_{(1)}| \) bounded and \( f^n_{(1)} \to \infty \) as \( j \to \infty \). Thus \( \log |f^n| = \log |f^n_{(1)}| + \frac{1}{2} \log \sum |f^n_{(j)}/f^n_{(1)}|^2 \), so \( \log |f^n| \) is written as a pluriharmonic function plus something bounded. Dividing by \( d^n \) and letting \( n \to \infty \), we have that \( G|M \) is pluriharmonic.

\[\square\]
§2 Invariant Currents

Here we assemble some basic facts about currents and give the definitions of the invariant currents that are defined in terms of the Green functions. Let $\Omega$ be a complex Hermitian manifold. If $M$ is a positive current of bidimension $(p,p)$ on $\Omega$, then $M$ is representable by integration. This means that there is a total variation measure $\|M\|$ and a measurable family of $(p,p)$-vectors $\vec{m}$ of unit length with respect to the Hermitian metric, such that we have the polar decomposition $M = \vec{m} \|M\|$. In terms of a test form $\phi$, this means that $\langle M, \phi \rangle = \int \langle \vec{m}(x), \phi(x) \rangle_x \|M\|(x)$, where $\langle \cdot, \cdot \rangle_x$ denotes the pointwise pairing between vectors and covectors. This representation allows us to treat positive currents as measures.

Let $\mathcal{Z}$ be a closed subset of $\Omega$, and let $M$ be a positive current of bidimension on $\Omega - \mathcal{Z}$. If the total variation measure $\|M\|$ has locally bounded mass near $\mathcal{Z}$, then we may make the trivial extension of $M$ to $\Omega$ by extending the domain of definition of $\|M\|$ to $\Omega$, and setting $\|M\|(\mathcal{Z}) = 0$.

For a general current $M$, we may define $M \llcorner \beta$, the contraction with a smooth form $\beta$, as the current which acts on a test form $\phi$ according to $\langle M \llcorner \beta, \phi \rangle = \langle M, \beta \wedge \phi \rangle$. For a positive current $M$ and a Borel set $S$, we may also define the restriction $M \llcorner S$ by restricting $\|M\|$ to $S$. This is the same as the contraction of $M$ by the function which is 1 on $S$ and 0 elsewhere.

Let $\Omega$ and $\Omega'$ be complex manifolds and $g : \Omega' \to \Omega$ a holomorphic mapping. We define the pullback $g^*S$ of a positive closed current $S$ on $\Omega$ in two cases. The first is when $g$ is a submersion (eg. $g = \pi$): $g^*S$ is then defined by integrating over the fibers of $g$. The second case is when $g$ is a finite branched cover (eg. $g$ is a regular polynomial endomorphism). If $S$ puts no mass on the critical image $gC$, then $S$ coincides with the trivial extension to $\Omega$ of $S \llcorner (\Omega - gC)$. We define $g^*S$ to be the trivial extension to $\Omega'$ of $(g|_{\Omega-g^{-1}gC})^*(S \llcorner (\Omega - gC))$.

The two sorts of currents for which we will take pullbacks are as follows. First, if $[M]$ is a current of integration over a complex manifold, then our definition gives $g^*[M] = [g^{-1}M]$ both in the case where $g$ is a submersion and when $M \cap gC$ does not contain an open subset of $M$. Second, if locally $S \leq (dd^c u)^j$ for a bounded psh function $u$, then $S$ puts no mass on any complex analytic set and thus no mass on $gC$. Similarly, $(dd^c(u \circ g))^j$ puts no mass on $g^{-1}gC$, and $g^*(dd^c u)^j = (dd^c(u \circ g))^j$. Thus $g^*S$ is well defined.

Before we define currents on $\mathbb{P}^k$, we recall the structure of $\mathbb{P}^k$ as a complex manifold. If $U_j$ is the open subset of $\mathbb{P}^k$ where $z_j \neq 0$, then

$$[z_1 : \ldots : z_k : t] \to (z_1/z_j, \ldots, 1, \ldots, z_k/z_j, t/z_j) = (\xi_1, \ldots, 1, \ldots, \xi_{k+1})$$

is a section of the bundle $\pi : C^{k+1}_* \to \mathbb{P}^k$ over $U_j$ and $\xi = (\xi_1, \ldots, \xi_{k+1})$ (with the $j$th coordinate missing) defines a biholomorphism between $U_j$ and $C^k$.

We define invariant currents on $C^k$ by $T_{C^k} := \frac{1}{2\pi} dd^c G$ and $T_{h,C^k} := \frac{1}{2\pi} dd^c G_h$. To define the invariant currents on $\mathbb{P}^k$, we define $g_j$ on the coordinate chart $U_j$ in terms of the $\xi$-coordinates by

$$G(\xi) = \log \frac{1}{|\xi_{k+1}|} + g_j(\xi).$$

By Lemma 1.1, $g_j$ has a continuous extension from $U_j - \Pi$ to $U_j$. A property of the operator $dd^c$ is that $dd^c g_j$ can put no mass on a pluripolar set if $g_j$ is bounded and psh.
Since \(dd^c G = dd^c g_j\) on \(U_j - \Pi\), it follows that this formula defines a positive, closed current on \(U_j\), which coincides with the trivial extension of \(T_{C^k}\) from \(C^k \cap U_j\) to \(U_j\), and these definitions on \(U_j\) fit together to give \(T_{P^k}\). A similar formula serves to define \(T_{h,P^k}\) as a positive, closed current on the affine coordinate chart \(U_j\), and this coincides with the trivial extension of \(T_{h,C^k}\). Since \(T_{P^k}\) and \(T_{h,P^k}\) are the trivial extensions of \(T_{C^k}\) and \(T_{h,C^k}\), respectively, we will just denote these currents as \(T\) and \(T_h\).

We recall that if \(S = dd^c u\) is a positive closed current of bidegree \((1,1)\) on a complex manifold with continuous potential \(u\), and \(M\) is a complex submanifold, then the slice \(S|_M\) is well-defined and is equal to the current on \(M\) defined by \(S|_M = dd^c uM(u)_M\). We have seen that \(T = T_{P^k}\) has a local, continuous psh potential everywhere on \(P^k\). Thus we may define the current \(T_{\Pi} := T|_\Pi\) as the slice current. By Lemma 1.1, we also see that \(T_{\Pi}\) is also given as the slice of the homogeneous current \(T_{\Pi} = T_{h|\Pi}\).

The positive closed currents of bidegree \((1,1)\) on \(P^k\) are characterized in terms of logarithmically homogeneous psh functions on \(C^{k+1}\). In terms of this characterization, we have \(\pi^*T_{P^k} = \frac{1}{2\pi} dd^c \hat{G}\), with \(\hat{G}\) from (1.2), and \(T_{\Pi}\) is the unique positive, closed current \(T_{\Pi}\) on \(\Pi\) such that \(\pi^*T_{\Pi} = T_h\).

Since the current \(T\) has a locally defined continuous potential, we may define the exterior powers \(T^l\) for \(1 \leq l \leq k\). Using the fact that for a bounded, psh function \(g_j\) on \(U_j\), \((dd^c g_j)^l\) puts no mass on a pluripolar set (and thus no mass on \(\Pi\)) for \(1 \leq l \leq k\), we see that the trivial extension of \(T_{C^k}\) to \(P^k\) is given by \(T_{P^k}^l\). Thus we may denote the exterior powers of our currents simply as \(T^l\) without ambiguity. The currents \(T^l, 1 \leq l \leq k\), are positive and closed on \(P^k\) and satisfy \(f^*T^l = d^lT^l\).

The same arguments and properties, e.g. \(f^*_h T^l_h = d^lT^l_h\), apply to \(T_h\), with the small complication that the potential \(G_h = \log|z| + O(1)\) has a logarithmic singularity at the origin. Let us observe, however, that by a familiar calculation \((dd^c \log|z|)^l\) is equal to \((2\pi)^k l\) times the point mass at the origin if \(l = k\); and is a positive, closed current which is absolutely continuous with respect to Lebesgue measure if \(l \leq k - 1\). Now the Comparison Theorem (see [BT1]) may be applied in the standard way to \(G_h\) and \(\log|z|\), to conclude that \((dd^c G_h)^l\) is \((2\pi)^k l\) times the point mass at the origin if \(l = k\), and puts no mass on the origin if \(l < k\).

Most important for us will be the currents \(T^{k-1}, T_{h}^{k-1}\), of bidimension \((1,1)\), and \(\mu := T^{k}\) and \(\mu_{\Pi} := T_{h\Pi}^{k-1}\), of bidimension \((0,0)\). Note that \(\mu\) and \(\mu_{\Pi}\) are represented by probability measures on \(C^k\) and \(\Pi\), respectively. We will denote their supports by \(J := \text{supp}(\mu)\) and \(J_{\Pi} := \text{supp}(\mu_{\Pi})\).

Remark. In the notation of [HP] the latter two sets would be called \(J_k\) and \(J_{\Pi,k-1}\), respectively. We use \(J\) and \(J_{\Pi}\) for brevity, as we will not be using the other intermediate Julia sets.

Currents that appear in complex dynamics often have a laminar structure. Let \(\Omega\) be a complex manifold with a Hermitian metric. Let \((A, \nu)\) be a measure space, and let \(a \mapsto M_a\) denote a measurable family of positive currents on \(\Omega\) with the property that for every relatively compact domain \(\Omega_0 \subset \Omega\) we have

\[
\int_{a \in A} \nu(a) \|M_a\|(\Omega_0) < \infty.
\] (2.1)
It follows from (2.1) that we may define a positive current \( S = \int_{a \in A} \nu(a) M_a \), where the action on a test form \( \phi \) is given by

\[
\langle S, \phi \rangle := \int_{a \in A} \nu(a) \langle M_a, \phi \rangle.
\]

We refer to \( S \) as \textit{laminar} if for almost every \( a \) and \( b \), either \( M_a = M_b \), or the supports of \( M_a \) and \( M_b \) are disjoint. If in addition \( a \to M_a \) is continuous, then we say that \( S \) is \textit{uniformly laminar}. We note that if the currents \( M_a \) are closed in \( \Omega \), then so is \( S \).

A consequence of positivity is that the total variation measure \( \| M_a \| \) is equivalent to the measure \( \mu_{\Pi}(a) \), where \( \beta \) is any strictly positive \((1,1)\)-form, and \((p,p)\) is the bidimension of \( M_a \). Equivalent here means that the two measures are bounded above and below by each other on compact subsets of \( \Omega \), with the constant depending only on \( \beta \). Since \( S \llcorner \beta^p = \int \nu(a) M_a \llcorner \beta^p \), we conclude that the total variation measure of \( S \) is equivalent to the integral of the total variation measures of the currents \( M_a \):

\[
\| S \| \sim \int \nu(a) \| M_a \|. \tag{2.2}
\]

The case that will appear in the sequel is where \( M_a \) is the current of integration over a 1-dimensional complex variety in \( \Omega \). An example is the following result.

**Proposition 2.1.** The following holds on \( \mathbf{P}^k \):

\[
T_h^{k-1} = \int [L_a] \mu_{\Pi}(a), \tag{2.3}
\]

where \( L_a = \pi^{-1}(a) \) is the complex line in \( \mathbf{P}^k \) passing through \( a \in \Pi \) and the origin in \( \mathbf{C}^k \).

**Proof.** We know that \( T_h^l \) puts no mass on the origin for \( l < k \), and no mass on \( \Pi \) in any case. Since taking the wedge product commutes with taking pullback, the identity \( T_h = \pi^* T_{\Pi} \) gives us \( T_h^l = \pi^* (T_{\Pi}^l) \) on \( \mathbf{C}^k_* \), and hence on \( \mathbf{P}^k \) if \( l < k \). Hence, by the definition of \( \pi^* \) as integration over the fibers of \( \pi \), we have

\[
T_h^{k-1} = \int [\pi^{-1}(a)] \mu_{\Pi}(a).
\]

Since \( \{a\} \) and \( \{0\} \) are sets of measure zero with respect to \( L_a \), it follows that \([L_a]\) and \([\pi^{-1}(a)]\) define the same current on \( \mathbf{P}^k \). Therefore, the equation above yields (2.3). \( \square \)

Next we show that the laminar structure of a current is preserved under wedge products. Let \( S \) and \( X \) be positive closed currents on a complex Hermitian manifold \( \Omega \) and suppose that \( X \) is of bidegree \((1,1)\). For any ball \( \Omega_0 \), there is a psh function \( \psi \) on \( \Omega_0 \) such that we may write \( X = dd^c \psi \) on \( \Omega_0 \). In order to define \( X \wedge S \), it suffices to define the action on any test form \( \phi \) on \( \Omega_0 \) as

\[
\langle X \wedge S, \phi \rangle := \langle \psi S, dd^c \phi \rangle. \tag{2.4}
\]
If we have

$$\int_{\Omega_0} |\psi| \|S\| < \infty,$$

then (2.4) defines a positive, closed current. Note that, in this case, if $\psi_m$ is a sequence of smooth, psh functions decreasing to $\psi$, then $dd^c\psi_m \wedge S$ converges in the sense of currents to $X \wedge S = dd^c \psi \wedge S$.

Let us recall the estimate of Alexander and Taylor [AT]. If $u$ is a bounded, psh function on $\Omega$, then for $1 \leq j \leq k$, the current $(dd^c u)^j$ has the property that

$$\int_{\Omega_0} |\psi| \|(dd^c u)^j\| < \infty$$

for any relatively compact domain $\Omega_0 \subset \Omega$ and any psh function $\psi$ on $\Omega$. Thus, if $S$ is a positive, closed current such that locally there exists a bounded, psh function $u$ such that

$$S \leq (dd^c u)^j,$$  \tag{2.5}

then the integral in formula (2.4) converges and defines $X \wedge S$ as a positive, closed current on $\Omega$.

**Lemma 2.2.** Let $M_a, a \in A,$ be a measurable family of positive, closed currents on $\Omega$, and let $\nu$ be a measure on $A$ such that (2.1) holds. Suppose, too, that the current $S = \int_{a \in A} \nu(a) M_a$ satisfies (2.5). If $X$ is a positive, closed current of bidegree $(1,1)$ with local potential $\psi$, then for $\nu$ almost every $a$, $X \wedge M_a$ is a well-defined positive closed current on $\Omega$. Further, we have

$$X \wedge S = \int_{a \in A} \nu(a) (X \wedge M_a),$$

where $X \wedge S$ is defined according to (2.4).

**Proof.** Let $\Omega' \subset \Omega$ be a domain where there is a psh function $\psi$ with $dd^c \psi = X$. By (2.2) and (2.5), we have

$$\int \nu(a) \left( \int_{\Omega_0} |\psi| \|M_a\| \right) \sim \int_{\Omega_0} |\psi| \|S\| < \infty \tag{2.6}$$

for every relatively compact $\Omega_0 \subset \Omega'$. It follows that for $\nu$ almost every $a$ we have $\int_{\Omega_0} |\psi| \|M_a\| < \infty$ for all relatively compact $\Omega_0$, and thus $X \wedge M_a$ is well defined.

Let $\psi_m$ denote a sequence of smooth, psh functions decreasing to $\psi$. If $\phi$ is a test form on $\Omega_0$, then the smooth current $X_m := dd^c \psi_m$ satisfies

$$\langle X_m \wedge S, \phi \rangle = \langle \psi_m S, dd^c \phi \rangle = \int_{a \in A} \nu(a) \langle \psi_m M_a, dd^c \phi \rangle.$$ 

The left hand side converges to $X \wedge S$ as $m \to \infty$. For fixed $a$, the integrand on the right hand side converges to $\langle \psi M_a, dd^c \phi \rangle = \langle X \wedge M_a, \phi \rangle$ as $m \to \infty$. Further, we have

$$|\langle \psi_m M_a, dd^c \phi \rangle| \leq C_\phi \int |\psi| \|M_a\|,$$

where $C_\phi$ does not depend on $m$ or $a$, so the lemma follows from (2.6) and the Dominated Convergence Theorem. \qed
We observe that taking pullbacks respects the laminarity of a positive current \( S \).

**Lemma 2.3.** Let \( \Omega, \Omega' \) be complex manifolds and let \( g : \Omega' \to \Omega \) be a branched covering. Let \( M_a, a \in A, \) be a measurable family of positive currents on \( \Omega, \) and let \( \nu \) be a measure on \( A \) such that (2.1) holds. Then

\[
g^* S = \int_{a \in A} \nu(a) g^* M_a.
\]

**Proof.** This follows immediately from the definition of \( g^* S \) and \( g^* M_a. \)

The object dual to the pullback is the pushforward. If \( S \) is a positive, closed current on \( \Omega', \) then \( g^* S \) is the current on \( \Omega \) defined by

\[
\langle g^* S, \varphi \rangle = \langle S, g^* \varphi \rangle
\]

for all test forms \( \varphi. \) If \( g \) is a finite branched covering, then the pushforward may be thought of, locally, as the inverse of the pullback so that

\[
g^* S = \sum (g^{-1}_j)^* S,
\]

where the sum is taken over all the branches \( g^{-1}_j \) of \( g^{-1}. \) Applying this to the \( d^k \) local inverses of a regular polynomial endomorphism, we get that

\[
f_* T^l = d^{k-l} T, \quad \text{and} \quad f_* (GT^{k-1}) = GT^{k-1}.
\]

### §3 Lyapunov exponents.

In this section we prove a formula for the sum of the Lyapunov exponents of a regular polynomial endomorphism \( f \) of \( \mathbb{C}^k. \) The proof below, which depends significantly on ideas of Berndtsson [Be], simplifies the argument in a previous version of this paper [J2]. A special case of (3.1) below was proved in [J1].

Let us recall the notion of Lyapunov exponents. For more details we refer to [Y]. The sum of the Lyapunov exponents of \( f \) with respect to \( \mu \) is the number \( \Lambda(f) = \Lambda(f, \mu) \) given by

\[
\lim_{n \to \infty} \frac{1}{n} \log |\det Df^n(x)| = \Lambda(f),
\]

for \( \mu \)-a.e. \( x \in \mathbb{P}^k. \) That this is well-defined is part of the statement of Oseledec’s Theorem. Hence \( \Lambda(f) \) measures asymptotic infinitesimal Lebesgue volume growth of \( f^n \) at \( \mu \)-a.e. point. The individual Lyapunov exponents measure the asymptotic growth of the derivative of \( f^n \) in different directions; we will not give the precise definition since we do not need it.

Our formula for \( \Lambda(f) \) will involve the integral of the Green function against a critical measure so we begin by defining the latter measure as

\[
\mu_c := [\mathcal{C}] \wedge T^{k-1} = \frac{1}{2\pi} dd^c H \wedge T^{k-1},
\]

where \( H = \log |\det Df|. \)

Then \( \mu_c \) is a well-defined positive measure because \( T \) has continuous local potentials, and the mass of \( \mu_c \) is finite (= the degree of the critical locus).
**Lemma 3.1.** Let $f = f_h$ be any homogeneous regular polynomial endomorphism of $\mathbb{C}^k$, and let $|\det(Df)|$ and $|\det(Df|\Pi)|$ be the Jacobians of $f$ and $f|\Pi$ in the Euclidean metric on $\mathbb{C}^k$ and the Fubini-Study metric on $\Pi$, respectively. Then

$$|\det(Df)(z)| = d \cdot \left(\frac{|f(z)|}{|z|}\right)^k |\det(Df|\Pi)(z)|.$$  

**Proof.** Pick any $z_0 \in \mathbb{C}^k$. After pre- and post-composing with dilations and unitary maps, we may assume that $f(z_0) = z_0 = (0, \ldots, 0, 1)$. Since $z_0$ and $[z_0]$ are now fixed points, the choices of metrics are irrelevant when computing the Jacobians. We use local coordinates $(\xi, s)$ on $\mathbb{P}^k$ and $\xi$ on $\Pi$, where $\xi_i = z_i/z_k$ for $1 \leq i \leq k - 1$ and $s = t/z_k$. In these coordinates, the homogeneity of $f$ allows us to write

$$f(\xi, s) = (f_1(\xi, 1)/f_k(\xi, 1), \ldots, f_{k-1}(\xi, 1)/f_k(\xi, 1), s^d/f_k(\xi, 1)),$$

$$f|\Pi(\xi) = (f_1(\xi, 1)/f_k(\xi, 1), \ldots, f_{k-1}(\xi, 1)/f_k(\xi, 1)).$$

Since the first $k - 1$ coordinates in $f(\xi, s)$ do not depend on $s$, we see that

$$\det Df(\xi, s)|_{(\xi, s) = (0, 1)} = d \cdot \det Df|\Pi|_{\xi = 0}.$$  

We introduce the factors $|f(z)|^k$ and $|z|^k$ because of the pre- and post-compositions with dilations. This completes the proof.  

**Theorem 3.2.** If $f$ is any regular polynomial endomorphism of $\mathbb{C}^k$, then

$$\Lambda(f) = \log d + \Lambda(f|\Pi) + \int G \mu.$$  

**Proof.** From the Ergodic Theorem and the definition of $\mu$ we have

$$\Lambda(f) = \int_{\mathbb{C}^k} H \mu$$

$$= \frac{1}{2\pi} \int_{\mathbb{C}^k} Hdd^cG \wedge T^{k-1}. $$  

Extend $G$ to a function $\hat{G}$ on $\mathbb{P}^k$ by declaring $\hat{G} = \infty$ on $\Pi$. Then $\hat{G}$ is locally integrable, and $dd^c\hat{G} = 2\pi(T - [\Pi])$. Note that the current of integration over $\Pi$ appears because $\hat{G}$ has a $+\infty$ logarithmic singularity along $\Pi$. We have $H = k(d - 1) \log |z| + O(1)$ near $\Pi - C_\Pi$, so we define $\hat{H}$ on $\mathbb{C}^k$ by $\hat{H}(z) = H(z) + k(\log^+ |z| - \log^+ |f(z)|)$. It follows that $\hat{H}$ extends to a locally integrable function on $\mathbb{P}^k$ which is continuous outside $\mathcal{C} \cup C_\Pi$. Note that if $x \in \Pi$, then $\hat{H}(x)$ depends only on the homogeneous part of $f$ of degree $d$. Thus $\hat{H} = \log d + \log |\det Df|\Pi|_0$ on $\Pi$ by Lemma 3.1.

By the invariance of $\mu = \frac{1}{2\pi}dd^cG \wedge T^{k-1}$, we have

$$\int_{\mathbb{C}^k} (\log^+ |z| - \log^+ |f(z)|) dd^cG \wedge T^{k-1} = 0.$$
Thus the last integral in (3.2) equals
\[ \frac{1}{2\pi} \int_{C^k} \hat{H} dd^c G \wedge T^{k-1}. \]

Using the formula \( dd^c \hat{G} = 2\pi (T - \Pi) \) and the fact that \( dd^c G \wedge T^{k-1} \) is supported on \( C^k \) we see that this equals
\[ \frac{1}{2\pi} \int_{P^k} \hat{H} dd^c \hat{G} \wedge T^{k-1} + \int_{\Pi} \hat{H} \mu_\Pi. \] (3.3)

By Stokes’ Theorem the first term in (3.3) is
\[ \frac{1}{2\pi} \int_{P^k} \hat{G} dd^c \hat{H} \wedge T^{k-1} = \frac{1}{2\pi} \int_{C^k} G dd^c \hat{H} \wedge T^{k-1} \]
\[ = \frac{1}{2\pi} \int_{C^k} G dd^c (H + k(\log^+|z| - \log^+|f(z)|)) \wedge T^{k-1} \]
\[ = \frac{1}{2\pi} \int_{C^k} G dd^c H \wedge T^{k-1} \]
\[ = \int G \mu_c. \]

The first equality holds because \( dd^c \hat{H} \wedge T^{k-1} \) puts no mass on \( \Pi \), and the third equality follows because \( f_*(GT^{k-1}) = GT^{k-1} \) by (2.7), so
\[ \int_{C^k} (dd^c \log^+|f(z)|) GT^{k-1} = \int_{C^k} (f^* dd^c \log^+|z|) GT^{k-1} \]
\[ = \int_{C^k} (dd^c \log^+|z|) f_*(GT^{k-1}) = \int_{C^k} (dd^c \log^+|z|) GT^{k-1}. \]

Further, by the remark above the second term in (3.3) is equal to
\[ \int_{\Pi} (\log d + \log |\det Df_\Pi|) \mu_\Pi = \log d + \Lambda(f_\Pi). \]

This completes the proof.

\[ \square \]

**Corollary 3.3.** \( \Lambda(f) \geq \frac{k+1}{2} \log d \).

**Proof.** It is a result of Briend [Bri] that the Lyapunov exponents of \( f_\Pi \) with respect to \( \mu_\Pi \) are bounded below by \( \frac{1}{2} \log d \) (see §6). In particular, \( \Lambda(f_\Pi) \geq \frac{k-1}{2} \log d \), so Theorem 3.2 gives \( \Lambda(f) \geq \log d + \frac{k-1}{2} \log d = \frac{k+1}{2} \log d \).

\[ \square \]

§4 Stable manifolds and a local model near \( \Pi \).

In §5–§7 we will show how the current \( T^{k-1} A \) has a laminar structure, and how this allows us to describe \( \mu \) in terms of external rays. The laminar structure is easier to handle—and visualize—in the case when \( f_\Pi \) is uniformly expanding on \( J_\Pi \). The expansion enables
us to invoke the (uniform) stable manifold theorem, and thus provides us with a continuous family of local stable manifolds. These manifolds define currents of integration, which are responsible for the laminar structure of \( T^{k-1} \) in a neighborhood of \( \Pi \).

In this section we prove a version of the stable manifold theorem. We also show that \( f \), restricted to the union of the local stable manifolds, is conjugate to the homogeneous map \( f_h \). The homogeneous map \( f_h \) may be seen as the canonical model for \( f \) near \( \Pi \) and the conjugacy as a generalization of the Böttcher coordinate at infinity for one-dimensional polynomials.

Let us therefore assume that \( f_\Pi \) is uniformly expanding on \( J_\Pi \). This means that there exist constants \( c > 0 \) and \( \lambda > 1 \) such that

\[
|Df^n_x v| \geq c\lambda^n |v| \quad x \in J_\Pi, \; v \in T_x \Pi, \; n \geq 1.
\] (4.1)

If \( f \) is expanding on \( J_\Pi \) and \( a \in J_\Pi \), then the tangent space \( T_a \mathbb{P}^k \) splits into a direct sum \( E^u(a) \oplus E^s(a) \), where \( E^u(a) = T_a \Pi \) and \( E^s(a) \) is the eigenspace of \( Df_a \) associated with the zero eigenvalue. We clearly have \( Df_a(E^{u/s}(a)) \subset E^{u/s}(f_\Pi a) \), and \( E^{u/s}(a) \) depends continuously on \( a \). Therefore, with the definition given in Appendix B, \( f_\Pi \) is hyperbolic on \( J_\Pi \).

The stable manifold theorem (see [Ru, p. 96] or [PS, Theorem 5.2]) asserts that there is a local stable manifold \( W^s_{\text{loc}}(a) \) at each point of \( a \) in \( J_\Pi \). This is defined by

\[
W^s_{\text{loc}}(a) := \{ x \in \mathbb{P}^k : d(f^j x, f^j a) < \delta \text{ for all } j \geq 0 \}
\] (4.2)

for small \( \delta > 0 \), and is an embedded real 2-dimensional disk. In fact, since \( f \) is holomorphic, \( W^s_{\text{loc}}(a) \) is a complex disk, i.e. the image of an injective holomorphic immersion of \( \mathbb{D} \). Moreover, the local stable manifolds depend continuously on \( a \) in the \( C^1 \) topology.

It will be convenient to work with neighborhoods of \( \Pi \) defined in terms of the Green function, so let \( A_0 := \{ G > R_0 \} \) and \( A_n = f^{-n} A_0 = \{ G > d^{-n} R_0 \} \), where \( R_0 > 0 \) and \( n \in \mathbb{Z} \). Thus \( A_n \subset A_{n+1} \), \( \bigcap_n A_n = \Pi \) and \( \bigcup_n A_n = A \).

Since \( W^s_{\text{loc}}(a) \cap \Pi = \{ a \} \), it follows from Lemma 1.2 that \( G | W^s_{\text{loc}}(a) \) is harmonic on the complement of \( a \) and equal to \( +\infty \) at \( a \). If we choose \( R_0 \) greater than the maximum of \( G \) on \( \partial W^s_{\text{loc}}(a) \), it follows from the maximum principle that

\[
W^s_0(a) := W^s_{\text{loc}}(a) \cap A_0
\]

is a properly embedded disk in \( A_0 \) for all \( a \in J_\Pi \). We call \( W^s_0(a) \) the local stable disk at \( a \).

We also define global stable manifolds by

\[
W^s(a) = \{ x \in \mathbb{P}^k : d(f^j x, f^j a) \to 0 \text{ as } j \to \infty \}.
\]

In contrast to the diffeomorphism case, the global stable manifolds may have singular points. Notice also that \( W^s(a) \) contains all the local stable manifolds \( W^s_{\text{loc}}(b) \) for \( b \in J_\Pi \) with \( f_\Pi^n b = f_\Pi^n a \), \( n \geq 0 \). We will in fact prove that \( W^s(a) \) is dense in the support of \( T^{k-1} \mathbb{D} \) (see Corollary 8.5). The global stable manifolds may have infinitely many components or be connected (see the example following Proposition 4.2).
In Theorem 4.3 we will show that $f$ is conjugate to $f_h$. In the proof of this theorem we will use a holomorphic homotopy between $f$ and $f_h$, defined by $f_\tau = f_h + \tau(f - f_h)$ for $\tau \in \mathbb{C}$. Note that $\tau = 0$ and $\tau = 1$ correspond to $f_h$ and $f$, respectively, and that the restriction of $f_\tau$ to $\Pi$ is $f_{\Pi}$ for all $\tau$. Hence there are local stable manifolds for $f_\tau$ for all $\tau$. We will need to control the dependence of these manifolds on $\tau$. To get this control, we prove a version of the Stable Manifold Theorem adapted to our situation.

It will be natural to consider the stable set of $J_{\Pi}$, i.e.

$$W^s(J_{\Pi}) = W^s(J_{\Pi}, f) = \{ x \in \mathbb{P}^k : d(f^n x, J_{\Pi}) \to 0 \text{ as } j \to \infty \}.$$ 

Note that the expansion of $f_{\Pi}$ on $J_{\Pi}$ and the superattracting nature of $\Pi$ implies that $W^s(J_{\Pi})$ is closed in $A$ and that $W^s(J_{\Pi}) \cap \Pi = J_{\Pi}$.

Let $G_\tau$ be the Green function for $f_\tau$, $A_{0, \tau} = \{ G_\tau > R_0 \}$, etc.

**Theorem 4.1.** If $\delta$ is small enough and $R_0$ is large enough, then for all $\tau$ with $|\tau| < 2$ the following holds

1. $W^s_{0, \tau}(a; f_\tau)$ is proper in $A_{0, \tau}$ for $a \in J_{\Pi}$ and $W^s_{0, \tau}(a) := W^s_{0, \tau}(a; f_\tau) \cap A_{0, \tau}$ is a properly embedded disk in $A_{0, \tau}$.
2. $W^s_{0, \tau}(a)$ is the connected component of $W^s(a, f_\tau) \cap A_{0, \tau}$ containing $a$. In particular, $W^s_{0, \tau}(a)$ does not depend on the choice of $\delta$.
3. $W^s_{0, \tau}(a)$ depends continuously on $a$ and holomorphically on $\tau$.
4. $G_\tau$ is harmonic and has no critical points on $W^s_{0, \tau}(a)$.

**Proof.** To avoid cumbersome notation we will write $f$ instead of $f_\tau$. However, it is important that the constructions below hold uniformly in $\tau$ (for $|\tau| < 2$).

Our first task is to define good coordinate charts. Pick $a = [a_1 : \ldots : a_k] \in J_{\Pi}$. After a unitary change of coordinates we may assume that $a = [0 : \ldots : 0 : 1]$. Let $\zeta = (\zeta_1, \ldots, \zeta_{k-1})$, where $\zeta_j = z_j/z_k$ and let $t = 1/z_k$. We denote the ball $|\zeta| < \epsilon_1$ by $U_a = U_a(\epsilon_1)$, the disk $|t| < \epsilon_2$ by $V_a = V_a(\epsilon_2)$ and the box $U_a \times V_a$ by $B_a = B_a(\epsilon) = B_a(\epsilon_1, \epsilon_2)$ for $\epsilon_1, \epsilon_2 > 0$. Note that $\Pi$ corresponds to $\{ t = 0 \}$ and the line $L_a$ to $\{ \zeta = 0 \}$.

The Euclidean metric on $B_a$ and the Fubini-Study metric on $\mathbb{P}^k$ differ by at most a multiplicative constant $C \geq 1$ and $C$ is close to one if $\epsilon_1$ and $\epsilon_2$ are small.

We may find an iterate $f^N$, such that (4.1) holds with $n = N$, $c = 1$ and $\lambda = 3C$. Thus, if $\epsilon_1$ and $\epsilon_2$ small enough, then we have

1. If $a, b \in J_{\Pi}$ and $a \neq b$, then there is an $n > 0$ such that $d(f^n a, f^n b) > 3C\epsilon_1$.
2. If $a, b \in J_{\Pi}$, $a \neq b$ and $f^N_a = f^N_b$, then $\overline{B_a} \cap \overline{B_b} = \emptyset$.
3. If $a \in J_{\Pi}$, then $f^N$ has no critical points in $\overline{B_a} - \Pi$.

We define a vertical disk in $B_a$ to be a disk of the form $\{ \zeta = \text{const} \}$ and a vertical-like disk to be the graph of a holomorphic map $U_a \to V_a$. Similarly we define horizontal and horizontal-like disks (these have codimension 1).

By choosing $1 \gg \epsilon_1 \gg \epsilon_2 > 0$, we get that for all $a \in J_{\Pi}$ and for all $f = f_\tau$ with $|\tau| < 2$:

1. $f^N(B_a) \cap B_{f^N_a} \subset U_{f^N_a} \times \mathbb{R}^N f^N_a$.
2. $f^{-N}(B_{f^N_a}) \cap B_a \subset U_a \times V_a$.
3. If $\Sigma$ is a horizontal disk in $B_a$, then $f^N(\Sigma) \cap B_{f^N_a}$ is a horizontal-like disk in $B_{f^N_a}$ and the restriction of $f^N$ to $\Sigma \cap f^{-N}(B_{f^N_a})$ is a biholomorphism.
Conditions (iv)–(vi) are illustrated in Figure 1 (with N=1).

![Figure 1](image_url)

To produce stable manifolds we have to iterate backwards. We claim that
(vii) If Σ′ is a vertical-like disk in $B^N_{\Pi^N}$, then $f^{-N}(\Sigma') \cap B_a$ is a vertical-like disk in $B_a$.

To see this, note that $f^{-N}(\Sigma') \cap B_a$ is an analytic variety in $B_a$. Let $\Sigma$ be a horizontal disk in $B_a$. We claim that $f^N(\Sigma)$ intersects $\Sigma'$ in exactly one point. Indeed, by (v) we may write $f^N(\Sigma) \cap B^N_{\Pi^N} = \{ t = g(\zeta) \}$ and $\Sigma' = \{ \zeta = h(t) \}$, where $g$ and $h$ are holomorphic. Hence the intersection between these two sets is the unique fixed point of the holomorphic map $g \circ h : U^N_{\Pi^N} \rightarrow \frac{1}{2} U^N_{\Pi^N}$. By (v) it follows that $\Sigma$ intersects $f^{-N}(\Sigma') \cap B_a$ in exactly one point. This proves that the latter set is a vertical-like disk.

Now define $B^n_a = B_a \cap f^{-N}(B^N_{f^N_{\Pi^N}}) \cap \ldots \cap f^{-nN}(B^N_{f^n_{\Pi^N}})$ for $n \geq 0$ and $B^\infty_a = \cap_{n \geq 0} B^n_a$.

Using the Kobayashi metric on $U_a$, it follows from (v) and (vi) that there is a constant $\kappa > 0$, independent of $a$ and $n$, such that the diameter of $\Sigma \cap B^n_a$ is less than $\kappa 2^{-n}$ for every horizontal disk $\Sigma$ in $B_a$. We claim that $B^\infty_a$ is a vertical-like disk in $B_a$. Indeed, the estimate above implies that $\Sigma \cap B^\infty_a$ consists of at most one point for every $\Sigma$. On the other hand, a repeated application of (vii) shows that the set $\gamma_n(a)$, defined inductively by $\gamma_0(a) = \{0\} \times V_a$ and $\gamma_n(a) = f^{-N}(\gamma_{n-1}(f^N_{\Pi^N} a)) \cap B_a$, is a vertical-like disk in $B_a$, contained in $B^n_a$. Hence any limit of a subsequence $\gamma_{n_j}(a)$ is a vertical-like disk in $B_a$. By the remark above, this disk must be exactly $B^\infty_a$. We also see that $\gamma_n(a)$, and hence $B^\infty_a$, depends holomorphically on $\tau$.

Clearly $f(B^\infty_a) \subset B^\infty_{f^1_{\Pi^1}}$ for all $a \in J_{\Pi}$. We claim that the sets $B^\infty_a$ are pairwise disjoint. Suppose $a \neq b$. By (i) there exists an $n \geq 0$ such that $f^N_{\Pi^N} b \not\in 3B^N_{f^N_{\Pi^N} a}$. Hence $B^\infty_{f^1_{\Pi^1} a} \cap B^\infty_{f^1_{\Pi^1} b} = \emptyset$, so $B^\infty_a$ and $B^\infty_b$ are disjoint.

We next show that the disks $B^\infty_a$ depend continuously on $a$. Note that if $\epsilon' \leq \epsilon_1$ and $\epsilon' = (\epsilon_1, \epsilon'_2)$, then $B_{\alpha}(\epsilon')$ is the restriction to $V_a(\epsilon'_2)$ of the vertical-like disk defining $B_{\alpha}(\epsilon)$. Pick any $\epsilon'_2 < \epsilon_2$ and $M$ be larger than the Lipschitz constant for all $f_{\tau}$ on $\mathbb{P}^k$. Assume that $b$ is close to $a$ and choose $n$ maximal so that $M^n C d(a, b) < (\epsilon_1^2 + \epsilon'_2^2)^{1/2} - (\epsilon_1^2 + \epsilon_2^2)^{1/2}$.

Then a simple calculation shows that $B^\infty_b$ is contained in $B^\infty_a$, and the latter set intersects every horizontal disk in a set of diameter at most $\kappa 2^{-n}$. Hence $B^\infty_a$ depends continuously on $a$.

It follows from the superattractive nature of $\Pi$ that if $\epsilon_2$ is small enough, then $d(f(x), fa) < d(x, a)$ whenever $a \in J_{\Pi}$ and $x \in B^\infty_a$. Hence, if $\delta > 0$ is small enough, then $W^s_{\text{loc}}(a) = \{ x \in B^\infty_a : d(x, a) < \delta \}$. Thus $W^s_{\text{loc}}(a)$ is a complex disk, compactly contained in $B_a$, depending continuously on $a$ and holomorphically on $\tau$. Thus $W^s_0(a)$ depends continuously on $a$ and holomorphically on $\tau$. 
For $a \in J_{\Pi}$ in a neighborhood of $a_0 \in J_{\Pi}$, let $\zeta$ denote a local holomorphic coordinate for $W^s_{\text{loc}}(a)$ such that $\zeta = 0$ corresponds to $a$. By Lemma 1.1 and Lemma 1.2, the restriction $G|W^s_{\text{loc}}(a)$ has the form $\log |\zeta| + g_\alpha(\zeta)$, where $g_\alpha$ is bounded and harmonic in a neighborhood of $\zeta = 0$. It follows that for $\zeta$ sufficiently small, $G|W^s_{\text{loc}}(a)$ has no critical points. Thus for $R_0$ sufficiently large, $G|W^s_0(a)$ has no critical points.

We claim that if $a \in J_{\Pi}$, then

$$f^{-1}W^s_0(a) \cap A_0 = \bigcup_{f_{\Pi}^{-n}b=a} W^s_0(b).$$

Indeed, $X := f^{-1}W^s_0(a) \cap A_0$ is a subvariety of $A_0$. Any component of $X$ must meet $\Pi$, and this must happen at a point in $f_{\Pi}^{-1}a$. But then a neighborhood of $\Pi$ in $X$ is contained in the union of $B^\infty_b$, where $b \in f_{\Pi}^{-1}a$. Thus (4.3) holds. Repeating the same argument, we see that

$$f^{-j}W^s_0(a) \cap A_0 = \bigcup_{f_{\Pi}^{-j}b=a} W^s_0(b).$$

It remains to be seen that $W^s_0(a)$ is the connected component of $W^s(a) \cap A_0$ containing $a$. We may assume that $R_0$ is so large that $W^s(J_{\Pi}) \cap A_0$ is contained in the union of the boxes $B_a$. In particular $W^s(a) \cap A_0$ is contained in the union these boxes for all $a \in J_{\Pi}$.

Thus, by the definition of $B^\infty_a$, we see that if $x \in W^s(a) \cap A_0$, then $f_n(x) \in B^\infty_{f_{\Pi}^{-n}a}$ for large $n$. Thus (4.4) implies that $x \in \bigcup_{f_{\Pi}^{-n}b=a} W^s_0(b)$. Hence we have shown that

$$W^s(a) \cap A_0 = \bigcup_{b \in W^s(a, f_{\Pi})} W^s_0(b),$$

where $W^s(a, f_{\Pi}) = \cup_{j \geq 0} f_{\Pi}^{-j} f_{\Pi}^j a$. Since the disks $W^s_0(a)$ are disjoint, it follows that $W^s_0(a)$ is the connected component of $W^s(a) \cap A_0$ containing $a$.

**Proposition 4.2.** For $R_0$ large enough we have

$$W^s(J_{\Pi}) \cap A_0 = \bigcup_{a \in J_{\Pi}} W^s_0(a).$$

We let $W^s(J_{\Pi})$ denote the partition of $W^s(J_{\Pi})$ by global stable manifolds. Proposition 4.2 implies that $W^s(J_{\Pi}) \cap A_0$ is a Riemann surface lamination (see [C] for the definition). Now the iterates of $f$ are local biholomorphisms outside the set $C_\infty := \bigcup_{n \geq 0} f^{-n}(C)$. The expansion of $f_{\Pi}$ on $J_{\Pi}$ implies that $C_\infty \cap W^s(J_{\Pi})$ is closed and nowhere dense in $W^s(J_{\Pi})$. Thus $W^s(J_{\Pi}) - C_\infty$ is also a lamination. In Figure 2 we see how the local stable disks in $A_0$ can join at higher levels and create a lamination whose leaves are not simply connected.
Proof of Proposition 4.2. This can be proved by observing that the natural extension \( \hat{J}_\Pi \) has local product structure (see Proposition B.6), but we will give a direct proof. The inclusion “\( \supset \)" is trivial. After replacing \( f \) by an iterate we may assume that (4.1) holds with \( n = c = 1 \) and \( \lambda = 3 \). Let \( M \geq 1 \) be larger than the Lipschitz constant for \( f \) on \( \mathbb{P}^k \).

Let \( \eta > 0 \) be so small that if \( a \in J_\Pi \), then all branches of \( f^{-1}_\Pi \) are single-valued on the ball \( B(f_\Pi a, 4M\eta) \) in \( \Pi \) and the branch mapping \( f_\Pi a \) to a maps \( B(f_\Pi a, 4M\eta) \) into the ball \( B(a, 2M\eta) \). Now let \( x \in W^s(J_\Pi) \cap A_0 \). Let \( n \) be so large that \( d(f^{n+i}x, J_\Pi) < \eta \) for \( j \geq 0 \) and pick points \( a_j \in J_\Pi \) such that \( d(f^{n+j}x, a_j) < \eta \) for \( j \geq 0 \). Then \( (a_j)_{j \geq 0} \) is an \( 2M\eta \)-pseudoorbit in \( J_\Pi \), i.e. \( d(f_\Pi a_j, a_{j+1}) < 2M\eta \). Let \( g_j \) be the branch of \( f^{-1}_\Pi \) on \( B(f_\Pi a_j, 4M\eta) \) mapping \( f_\Pi a_j \) to \( a_j \). Then \( g_j(a_{j+1}) \in B(f_\Pi a_{j-1}, 4M\eta) \) so the point \( b^{(j)} := g_h \circ \cdots \circ g_j(a_{j+1}) \) is well-defined. Moreover \( d(f_\Pi^{n+i}(b^{(j)}), a_i) < 2M\eta \) for \( 0 \leq i \leq j \). Letting \( j \to \infty \) and using the compactness of \( J_\Pi \) we find a point \( b \in J_\Pi \) such that \( d(f_\Pi^n b, a_i) < 3M\eta \) for all \( i \geq 0 \). Hence \( d(f^{n+i}x, f_\Pi^n b) < 4M\eta \) for all \( i \geq 0 \). Assume that \( 4CM\eta < \epsilon \), with \( C \) and \( \epsilon \) from the proof of Theorem 4.1. It follows that \( f^n x \in W^s_\epsilon(b) \), so by (4.4) we have \( x \in W^s_\epsilon(c) \) for some \( c \in f_\Pi^n b \). This completes the proof. \( \square \)

Remark. Proposition 4.2 holds for \( f_\tau \) for \( |\tau| < 2 \) (with a uniform \( R_0 \)).

The following family of examples shows that the behavior of the lamination \( W^s(J_\Pi) \) can be simple when the leaves are simply connected, and complicated otherwise.

Example. Let \( f(z, w) = (z^2 + c, w^2) \). We use the affine coordinate \( \zeta = w/z \) on \( \Pi \), so that \( f_\Pi(\zeta) = \zeta^2 \), and \( J_\Pi = \{ |\zeta| = 1 \} \). Let \( K_c \) denote the (1-dimensional) filled Julia set of \( p_c(z) = z^2 + c \), and let \( G_c(z) = \log |z| + o(1) \) denote the Green function for \( K_c \). It follows that \( G(z, w) = \max(G_c(z), \log^+ |w|) \), and \( W^s(J_\Pi) = \{ G_c(z) = \log^+ |w| > 0 \} \). We may choose a harmonic conjugate function \( G^*_c(z) \) such that \( \phi_c(z) := \exp(G_c(z) + iG^*_c(z)) \approx z \) is single-valued and analytic for \( z \) large. For \( \zeta \in J_\Pi \), the local stable manifold \( W^s_\zeta(\zeta) \) is given (for \( z \) large) as the graph \( w = \zeta \phi_c(z) \). If \( c \) belongs to the Mandelbrot set, then \( K_c \) is connected, and \( \phi_c \) extends analytically to \( \mathbb{C} - K_c \). The stable manifolds are then countable unions of closed disks in \( A \), each of which is a graph of the form \( \{ w = \zeta \phi_c(z) : z \in \mathbb{C} - K_c \} \), \( \zeta \in J_\Pi \).
In case $K_c$ is not connected, we let $\Phi : \mathbb{C} \times \mathbb{C}^* \to \mathbb{C} \times \mathbb{C}^*$ be the biholomorphic mapping given by $(u, v) = \Phi(z, w) = (z/w, 1/w)$. Thus $h := \Phi \circ f \circ \Phi^{-1}$ is given by $h(u, v) = (u^2 + cv^2, v^2)$, which is a homogeneous mapping. Let $G_h(u, v)$ denote the logarithmically homogeneous Green function associated to $h$. Thus

$$G_h(u, v) = G_h(uw, vw) - \log|w| = G_h(z, 1) - \log|w| = G_c(z) - \log|w|.$$ 

The set $\{G_h < 0\}$ is the basin of attraction of $(0, 0)$ for the mapping $h$, and the boundary of the basin is given by $\{G_h = 0\}$.

The set $\{G_c(z) = G_c(u/v) > 0, G_h(u, v) = 0\} \subset \{G_h = 0\}$ is the image of $W^s(J_\Pi) = \{G_c(z) > 0, G_c(z) - \log|w| = 0\}$ under the mapping $\Phi$. Thus $\Phi$ transfers the Riemann surface lamination of the one set to the lamination of the other. The set $\{G_c(z) > 0, G_h(u, v) = 0\}$ is exactly that part of the boundary of the basin where $G_h$ is pluri-harmonic, and thus it part of the boundary that lies inside the Fatou set. The leaves of the Riemann surface lamination of this set have been studied by Hubbard and Papadopol [HP] and Barrett [Ba] and are shown to be dense and to have infinite topological type as well as other complicated behaviors.

Figure 3.

Figure 3 shows slices of $W^s(J_\Pi)$ by complex lines $\{z = c\}$ for the map $f(z, w) = (z^2 - 0.1, w^2 - z^2 + 0.2z - 0.5i)$. In the coordinate $\zeta = w/z$, we have $f_\Pi(\zeta) = \zeta^2 - 1$. The first picture is the Julia set of the map $\zeta \to \zeta^2 - 1$. By Proposition 4.2, the slices above
converge (suitably scaled) to this picture as $c \to \infty$. The remaining five pictures show the slices by the lines \( \{z = 2\}, \{z = 1.3\}, \{z = 1.2\}, \{z = 1.15\}, \{z = 1.1\} \).

We will now show that $f$, restricted to $W_s(J_{\Pi}) \cap A_0$, is conjugate to the canonical model $f_h$. Note that if $f = f_h$ is homogeneous, then $W_s(J_{\Pi}) = C(J_{\Pi}) \cap A$, where $C(J_{\Pi})$ is the complex cone of lines $L_a$, for $a \in J_{\Pi}$. Let $A_{0,h} := \{G_h > R_0\}$.

**Theorem 4.3.** Suppose that $f_{\Pi}$ is uniformly expanding on $J_{\Pi}$. If $R_0$ is large enough, then there is a homeomorphism $\Psi : W_s(J_{\Pi}, f_h) \cap A_{0,h} \to W_s(J_{\Pi}, f) \cap A_0$ conjugating $f_h$ and $f$. Further, $G \circ \Psi = G_h$ and the restriction of $\Psi$ to the local stable disk $W_0^s(a, f_h)$ is a biholomorphism onto $W_0^s(a, f)$ for all $a \in J_{\Pi}$.

**Remark.** Using the Green functions $G_h$ and $G$, it is easy to construct a biholomorphism of $W_0^s(a, f_h)$ onto $W_0^s(a, f)$ taking $G_h$ to $G$. Such a biholomorphism is unique up to a rotation of the disk $W_0^s(a, f_h)$. The difficulty in constructing $\Psi$ is to chose these rotations in a continuous way. Note that the disk $W_0^s(a, f)$ will not, in general, be tangent to $W_0^s(a, f_h)$ at $a$. On the other hand, if $f = f_h$, then we may use $\Psi = \text{id}$. Our approach will be to use a holomorphically varying homotopy between $f$ and $f_h$.

**Proof.** The idea is to define the conjugacy $\Psi$ as $\lim_{n \to \infty} f^{-n} \circ f^n_h$, the difficulty being to define $f^{-n}$. We will use the notation from the proof of Theorem 4.1.

Fix $a \in J_{\Pi}$, $n \geq 1$ and $\tau$ with $|\tau| < 2$. Write $f_\tau = f_h + \tau(f - f_h)$. Let $\Delta_n$ be the disk defined by

$$\Delta_n = f^n_{\Pi}W_0^s(a, f_h) = W_0^s(f^n_{\Pi}a, f_h) \cap A_{-nN,h}.$$ 

Then $f^{-n}_\tau(\Delta_n)$ is a variety in $f^{-n}_\tauA_{-nN,0}$, all of whose components must meet $\Pi$ at some point in $f^{-n}_\Pi f^n_{\Pi}a$. Hence, we may write

$$f^{-n}_\tau(\Delta_n) = \bigcup_{b \in f^{-n}_\Pi f^n_{\Pi}a} \beta_{b,n},$$ 

where $\beta_{b,n}$ is contained in a vertical-like disk in $B_b$. Thus $f^{-n}_\Pi$ maps $\beta_{b,n}$ onto $\Delta_n$ as a branched covering of degree $d$, branched only at $a$.

Hence there are $d^n$ locally defined branches of $f^{-n}_\tau \circ f^n_{\Pi}$ mapping $W_0^s(a, f_h) - \{a\}$ into $\beta_{b,n} \subset B_a$. These branches depend holomorphically on $\tau$. Let $\psi_{a,\tau,n}$ be the branch obtained by analytic continuation of $\psi_{a,0,n} = \text{id}$. Then $\psi_{a,\tau,n}$ is well-defined on $W_0^s(a, f_h)$, depends continuously on $a$ and holomorphically on $\tau$.

The mappings $\psi_{a,\tau,n}$ map $W_0^s(a, f_h) \times \{|\tau| < 2\}$ into $B_a$, hence they form a normal family. We claim that in fact $\psi_{a,\tau,n}$ converges as $n \to \infty$. To see this, we first note that $d^{-n}G_h \circ f^n_{\tau} \to G_{\tau}$ uniformly on compact subsets of $(A_{0,h} - \Pi) \times \{|\tau| < 2\}$ by Lemma 1.1. Hence any limit point $\psi_{a,\tau}$ of $\psi_{a,\tau,n}$ must have the following properties:

- (i) $\psi_{a,\tau}$ depends holomorphically on $\tau$.
- (ii) $G_{\tau} \circ \psi_{a,\tau} = G_h$.
- (iii) $\psi_{a,\tau}$ maps $W_0^s(a, f_h)$ biholomorphically onto $W_0^s(a, f_{\tau})$.
- (iv) $\psi_{a,0} = \text{id}$.

Now suppose that $\psi'_{a,\tau}$ and $\psi''_{a,\tau}$ are two such limits. By (iii) the mapping $\nu_{a,\tau} := (\psi''_{a,\tau})^{-1} \circ \psi'_{a,\tau}$ is a biholomorphism of the disk $W_0^s(a, f_h)$. By (i) $\nu_{a,\tau}$ depends holomorphically on $\tau$ and by (ii) $\nu_{a,\tau}$ is a rotation for all $\tau$. Hence $\nu_{a,\tau}$ is a constant (not depending on $\tau$) times the identity, so by (iv) $\nu_{a,\tau} = \text{id}$, i.e. $\psi'_{a,\tau} = \psi''_{a,\tau}$ for all $\tau$. 

Thus \( \psi_{a,\tau,n} \) converges to a map \( \psi_{a,\tau} \) having the properties (i)–(iv). Define \( \Psi_\tau \) by
\[
\Psi_\tau = \psi_{a,\tau} \text{ on } W^s_0(a, f_h).
\]
Then \( \Psi_\tau \) is a bijection of \( W^s_0(J_{\Pi}, f_h) \cap A_{0,h} \) onto \( W^s_0(J_{\Pi}, f_\tau) \cap A_{0,\tau} \).

We claim that \( \Psi_\tau \) conjugates \( f_h \) to \( f_\tau \). This amounts to showing that, for all \( a \in J_{\Pi} \),
\[
\psi_{f_{\Pi}a,\tau} \circ f_h = f_\tau \circ \psi_{a,\tau} \text{ on } W^s_0(a, f_h).
\]
Now these two mappings are both branched coverings of \( W^s_0(a, f_h) \) onto \( W^s_0(f_{\Pi}a, f_\tau) \cap A_{-1,\tau} \) of degree \( d \), branched only at \( a \). Moreover, we have
\[
G_\tau \circ \psi_{f_{\Pi}a,\tau} \circ f_h = d \cdot G_h = G_\tau \circ f_\tau \circ \psi_{a,\tau}.
\]

Hence there exists a complex number \( \nu_{a,\tau} \) of unit modulus such that
\[
f_\tau \circ \psi_{a,\tau} \circ \nu_{a,\tau} = \psi_{f_{\Pi}a,\tau} \circ f_h.
\]
Since \( \nu_{a,\tau} \) depends holomorphically on \( \tau \) and \( \nu_{a,0} = 1 \) we see that \( \nu_{a,\tau} = 1 \) for all \( \tau \).

We complete the proof by showing that \( \Psi_\tau \) is continuous for all \( \tau \). Fix \( a \in J_{\Pi} \) and pick parametrizations \( \chi_b : \mathbb{C} - D_{R_0} \rightarrow W^s_0(b, f_h) \) for \( b \in J_{\Pi} \) close to \( a \), such that \( \chi_b \) depends continuously on \( b \) and \( G_h \circ \chi_b = \log | \cdot | \). It suffices to prove that \( \psi_{b,\tau} \circ \chi_b \) converges to \( \psi_{a,\tau} \circ \chi_a \) as \( b \rightarrow a \). Again this follows, using the Green functions and the holomorphic dependence on \( \tau \).

\[\Box\]

Remark. The above result is similar to Theorem 9.3 in [HP].

\section{Uniform laminar structure of \( T^{k-1} \) near \( \Pi \).}

The next two sections will be devoted to the laminarity of the current \( T^{k-1} \) on \( A \). In §6, we cover some rather general situations. It is worthwhile, however, to start with the case where \( f_{\Pi} \) is uniformly expanding on \( J_{\Pi} \), and we obtain (5.1), which is our strongest laminarity property.

\begin{Theorem}
If \( f_{\Pi} \) is expanding on \( J_{\Pi} \) and \( R_0 \) is large enough, then
\[
T^{k-1} A_0 = \int [W^s_0(a)] \mu_{\Pi}(a).
\]

\end{Theorem}

\begin{proof}
It follows from Proposition 2.1 that
\[
\frac{1}{d^{j(k-1)}} (f^j)^* (T^{k-1} A_0) \subseteq A_0
\]
\[
= \frac{1}{d^{j(k-1)}} \left((f^j)^* \int [L_a \cap A_0] \mu_{\Pi}(a)\right) \subseteq A_0,
\]
for all \( j \geq 0 \). We will show: (1) the left hand side tends to \( T^{k-1} A_0 \) as \( j \rightarrow \infty \), and (2) the right hand side tends to \( \int [W^s_0(a)] \mu_{\Pi}(a) \) as \( j \rightarrow \infty \).

To prove (1) it suffices to show that \( d^{-j} G_h \circ f^j \rightarrow G \) uniformly on \( A_0 - \Pi \). But from Lemma 1.1 we know that \( G - G_h \) is bounded on \( A_0 \). Thus
\[
\frac{1}{d^j} G_h \circ f^j - G = \frac{1}{d^j} (G_h - G) \circ f^j
\]
\[
= O(\frac{1}{d^j}).
\]
To show (2), we use Lemma 2.3 and calculate
\[ \frac{1}{d^{(k-1)j}} \left( (f^j)^* \int [L_a \cap A_0] \mu_{\Pi}(a) \right) \ll A_0 \]
\[ = \int \frac{1}{d^{(k-1)j}} \left[ f^{-j} (L_a \cap A_0) \cap A_0 \right] \mu_{\Pi}(a) \]
\[ = \int \frac{1}{d^{(k-1)j}} \left[ f^{-j} (L_{f_{\Pi}^j a}) \cap A_0 \right] \mu_{\Pi}(a), \]
where we have used the invariance of $\mu_{\Pi}$. From the proof of Theorem 4.1 we know that $f^{-j} L_{f_{\Pi}^j a} \cap A_0$ is a union of $d^{(k-1)j}$ disjoint complex disks $\gamma_j(b)$, over $b \in f_{\Pi}^{-j} f_{\Pi}^j a$ (at least if $j$ is a multiple of $N$, with $N$ from the same proof). Hence we get
\[ \int \frac{1}{d^{(k-1)j}} \left[ f^{-j} (L_{f_{\Pi}^j a}) \cap A_0 \right] \mu_{\Pi}(a) = \int \frac{1}{d^{(k-1)j}} \sum_{b \in f_{\Pi}^{-j} f_{\Pi}^j a} [\gamma_j(b)] \mu_{\Pi}(a) \]
\[ = \int [\gamma_j(a)] \mu_{\Pi}(a), \]
since $f_{\Pi}^n \mu_{\Pi} = d^{k-1} \mu_{\Pi}$. Moreover, from the same proof it follows that $\gamma_j(a)$ converges to the local stable disk $W_0^s(a)$ in $C^1$-topology, uniformly in $a$. Hence the last line above converges to $\int [W_0^s(a)] \mu_{\Pi}(a)$ as $j \to \infty$, completing the proof.

Theorem 5.1 allows us to describe the support of $T^{k-1} \ll A$ in dynamical terms.

**Corollary 5.2.** If $f_{\Pi}$ is expanding on $J_{\Pi}$, then $\text{supp}(T^{k-1}) \cap A = W^s(J_{\Pi})$.

**Proof.** It follows from Theorem 5.1, from the continuity of $a \to W_0^s(a)$ and from Proposition 4.2 that the support of $T^{k-1} \ll A_0$ is equal to $W^s(J_{\Pi}) \cap A_0$. This proves the corollary, because the sets $\text{supp}(T^{k-1}) \cap A$ and $W^s(J_{\Pi})$ are both completely invariant and any compact subset of either of them is mapped by some iterate of $f$ into $A_0$.

Another consequence of Theorem 5.1 is that $T^{k-1}$ has a uniform laminar structure on $A_n = f^{-n} A_0$ for every $n \geq 0$, hence on every relatively compact subset of $A$.

**Corollary 5.3.** For every $n \geq 0$ we have
\[ T^{k-1} \ll A_n = \int \frac{1}{d^{(k-1)n}} \left[ f^{-n} W_0^s(f_{\Pi}^n a) \right] \mu_{\Pi}(a). \tag{5.2} \]

**Proof.** This is an easy consequence of Theorem 5.1. Indeed,
\[ T^{k-1} \ll A_n = \frac{1}{d^{(k-1)n}} (f^n)^* (T^{k-1} \ll A_0) \]
\[ = \frac{1}{d^{(k-1)n}} (f^n)^* \left( \int [W_0^s(a)] \mu_{\Pi}(a) \right) \]
\[ = \frac{1}{d^{(k-1)n}} \int [f^{-n} (W_0^s(a))] \mu_{\Pi}(a) \]
\[ = \int \frac{1}{d^{(k-1)n}} [f^{-n} (W_0^s(f^n a))] \mu_{\Pi}(a). \]
§6 Laminar Structure of $T^{k-1}$ on $A$.

The main goal of this section is to show that the current $T^{k-1} A$ has a laminar structure. We have seen in the previous section that if $f_1$ is uniformly expanding on $J_1$, then $T^{k-1} A$ has a uniformly laminar structure in a neighborhood of $\Pi$ with respect to the Riemann surface lamination given by the local stable disks $W^s_0(a)$. Also, $T^{k-1}$ is uniformly laminar on $A_n$ for each $n \geq 0$, hence on each compact subset of $A$.

Here we show that there is a nonuniform laminar structure in general. When the expansion of $f_1$ is not uniform, we still have stable manifolds by Pesin theory. Without uniformity, however, we are not able to bound the topological type of the stable manifolds in a neighborhood of $\Pi$. Despite this, there is a formulation of the laminarity of $E(Df)$ (Theorem 6.4) in terms of currents of integration over subvarieties of $A_n$. This allows us to express the restriction of the critical measure $\mu_c$ to $A_n$ as an intersection product with the critical locus (Corollary 6.5). A more global laminar formulation for $T^{k-1} A$ (Theorem 6.10) is obtained by subdividing the manifolds in the lamination into disks, which is done by cutting along the gradient lines of $G$.

Our starting point is the result by Briend [Bri] that the Lyapunov exponents of $f_1$ with respect to $\mu_1$ are strictly positive. More precisely, for $\mu_1$-almost every $a$ and all $v \in T_a \Pi$, $v \neq 0$, we have

$$\liminf_{j \to \infty} \frac{1}{j} \log |Df_1^j(a)v| \geq \frac{1}{2} |v| \log d.$$ 

Thus, if we set $E^u_a = T_a \Pi$, then $f$ is (nonuniformly) expanding on the subspace $E^u_a$. When we consider the mapping $f$ at a point $a$ of $\Pi - C_\Pi$, there is a unique one-dimensional subspace $E^s_a$ of $T_a \mathbb{P}^k$ such that the restriction of $Df$ to $E^s_a$ is zero. We also have $Df(T_a \Pi) \subset T_{f_1 a} \Pi$. Note that $\mu_1(C_\Pi) = 0$, since $C_\Pi$ is pluripolar in $\Pi$ and $\mu_1$ has continuous local potentials. Hence, for $\mu_1$-a.e. $a$ the tangent space of $\mathbb{P}^k$ at $a$ is the direct sum of two subspaces on which $Df^j$ is asymptotically expanding and contracting, respectively. In other words, $\mu_1$ is a hyperbolic measure for $f$.

By Pesin theory there exists a local stable manifold through almost every point of $J_1$. In general, we write

$$W^{s}_{\text{loc}}(a) = \{x \in \mathbb{P}^k : d(f^j x, f^j a) < \delta \ \forall j \geq 0\}$$

for small $\delta > 0$. Since $\mu_1$ is a hyperbolic measure, it is a consequence of Pesin Theory that for $\mu_1$-almost every $a \in J_1$ there exists a $\delta = \delta(a) > 0$ such that $W^{s}_{\text{loc}}(a)$ is an embedded real 2-dimensional disk in $\mathbb{P}^k$, tangent to $E^s_a$ at $a$. Since $f$ is holomorphic, $W^{s}_{\text{loc}}(a)$ is in fact a complex disk in $\mathbb{P}^k$. We may choose $m = m(a) \geq 0$ such that $W^{s}_{\text{loc}}(a)$ is proper in the neighborhood $A_{-m}$ of $\Pi$.

The precise statement from Pesin Theory that we will need is the following, which is an adaptation of Corollary 5.3 of [PS].

For every $\eta > 0$ there exists a compact subset $F = F_\eta$ of $\Pi$ with $\mu_1(F) \geq 1 - \eta$ and an integer $m = m(\eta) \geq 0$ such that the following holds:

(a) $F$ has no isolated points and does not intersect the set $\bigcup_{j \in \mathbb{Z}} f^j(C_\Pi)$. 

(b) For each \( a \in F \), the local stable manifold \( W^s_{\text{loc}}(a) \) is proper in \( A_{-m} \) and \( W^s_{-m}(a) := W^s_{\text{loc}}(a) \cap A_{-m} \) is a properly embedded disk in \( A_{-m} \). The Green function \( G \) is harmonic on \( W^s_{-m}(a) - \{a\} \) and has no critical point there.

(c) The map \( a \to W^s_{-m}(a) \) is continuous in the \( C^1 \) topology and the set of disks \( \{ W^s_{-m}(a) : a \in F \} \) defines a Riemann surface lamination in \( A_{-m} \).

(d) For each \( a \in F \), we let \( L_a \) denote the complex line in \( \mathbb{P}^k \) defined by \( a \), and we let \( D_j(a) \) denote the component of \( f^{-j}(L_{f^j(a)} \cap A_{-m}) \) containing \( a \). Then \( D_j(a) \) is a complex disk in \( A_{-m} \) for \( j \geq 0 \) and \( D_j(a) \) converges in the \( C^1 \) topology to \( W^s_{-m}(a) \) as \( j \to \infty \). This convergence is uniform in \( a \) for \( a \in F \).

Property (a) is a consequence of the fact that \( \mu_{\Pi} \) does not give mass to pluripolar sets. For (b), we know from Lemma 1.2 that \( G \) is harmonic on the local stable manifolds. Since the local stable manifolds vary continuously over \( F \), we see from Lemma 1.1 that \( G \) has no critical points on \( W^s_{\text{loc}}(a) \) if \( \delta \) is small enough. Thus we may choose \( m \geq 0 \) so that \( W^s_{\text{loc}}(a) \cap A_{-m} \) is a properly embedded complex disk in \( A_{-m} \). Properties (c) and (d) follow from the construction of the local stable manifolds as in [PS].

We will call a set \( F \) satisfying (a)–(d) a Pesin box and the associated disks \( W^s_{-m}(a) \) Pesin disks.

In general we must let \( m(\eta) \to \infty \) as \( \eta \to 0 \) to insure that \( W^s_{\text{loc}}(a) \) is a proper disk in \( A_{-m} \) for all \( a \in F \). Figure 4 illustrates this phenomenon.

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**Pesin Disks at Level -2**

![Figure 4](image)

In §5 we saw that the (uniform) hyperbolicity of \( f \) on \( J_{\Pi} \) implies that \( T^{k-1} \) has a uniform laminar structure in a neighborhood of \( \Pi \). The following lemma is a corresponding result in the nonuniformly hyperbolic case. By restricting to a small neighborhood \( A_{-m} \) of \( \Pi \) we get many Pesin disks and these account for a large part of \( T^{k-1} \) in this neighborhood.

**Lemma 6.1.** Given \( \eta > 0 \) there exists \( m \geq 0 \) and a Pesin box \( F = F_\eta \) having properties (a)–(d) above such that

\[
T^{k-1} \cap A_{-m} = \int_{a \in F} [W^s_{-m}(a)] \mu_{\Pi}(a) + S \cup A_{-m}, \quad (6.1)
\]
where $S$ is a positive closed current on $\mathbb{P}^k$ with $\|S\| \leq \eta$.

Proof. We let $F$ satisfy properties (a)–(d) above. From Proposition 2.1 we have

$$T_h^{k-1} \ll A_{m} = \int_{J} [L_{a} \cap A_{m}] \mu_{\Pi}(a) = \int_{J} [L_{f^j a} \cap A_{m}] \mu_{\Pi}(a) = \int_{F} + \int_{F'},$$

where we have used the invariance of $\mu_{\Pi}$. Now we apply $f^{*j}$ to this equation, divide by $d^{(k-1)j}$, restrict to $A_{-m}$ and let $j \to \infty$. As in Theorem 5.1 we see that the left hand side then converges to $T_h^{k-1} \ll A_{-m}$ and, using (b), $\frac{1}{d^{(k-1)j}} f^{*j}(\int_{F} \ll A_{-m})$ converges to $\int_{\mathcal{F}} [W_{m}^s(a)] \mu_{\Pi}(a)$.

Finally, we consider $\frac{1}{d^{(k-1)j}} f^{*j}(\int_{\mathcal{F}} \ll A_{-m})$. The mass norm of the current of integration over a curve of degree $r$ in $\mathbb{P}^k$ is $r$ (up to a constant only depending on the volume of $\mathbb{P}^k$). In particular, $\|L_{a}\| = 1$. It follows that the currents $\frac{1}{d^{(k-1)j}} f^{*j}(\int_{\mathcal{F}} [L_{f^j a}])$ have mass norms that are bounded by $\eta$. Passing to a subsequence, we obtain a limit $S$ which has the desired properties in (6.1).

Corollary 6.2. Given the assumptions in Lemma 6.1 and $n \geq 0$, we have

$$T_h^{k-1} \ll A_{n} = \int_{\mathcal{F}} \frac{1}{d^{(k-1)(n+m)}} [f^{-(n+m)} W_{-m}^{s}(a)] \mu_{\Pi}(a) + S \ll A_{n}, \quad (6.2)$$

where $S$ is a positive, closed current on $\mathbb{P}^k$ with mass norm bounded by $\eta$.

Proof. We pull (6.1) back by $f^{n+m}$ and divide by $d^{(k-1)(n+m)}$. Note that the proof of Lemma 6.1 shows that the current $S$ in (6.1) satisfies $\|(f^{n+m})*S\| \leq d^{(k-1)(n+m)}$.

Corollary 6.2 shows that we can approximate $T_h^{k-1}$ on $A_n$ by laminar currents over closed varieties in $A_n$. We want to pass to the limit and obtain an exact formula, just as in (5.2).

Let us fix a Pesin box $E_0 \subset J_{\Pi}$ satisfying (a)–(d) above with some $\eta > 0$. We may choose $R_0$ so that $m = m(\eta) = 0$. By Poincaré recurrence, the set

$$E = \{a \in J_{\Pi} : f_{\Pi}^{n} a \in E_0 \text{ for infinitely many } n \geq 0\} \quad (6.3)$$

has full measure. It is clearly invariant under $f_{\Pi}$. Recall the notion of the global stable manifold $W^s(a)$ of a point $a \in J_{\Pi}$. In general, $W^s(a)$ will be a very complicated object, but we do have the following.

Lemma 6.3. If $a \in E$, and $n \geq 0$, then $W^s(a) \cap A_{n}$ is a disjoint countable union of connected varieties in $A_n$, each of which intersects $\Pi$ transversely at finitely many points.

Proof. Let $n \leq n_1 < n_2 < \ldots$ be return times for $a$ to $E_0$, i.e. $f_{\Pi}^{n_j} a \in E_0$. Let $Z_a^{(j)} = f^{-n_j} W_0^s(f_{\Pi}^{n_j} a) \cap A_n$ for $j \geq 1$. Then $Z_a^{(j)}$ is a variety in $A_n$ which intersects $\Pi$ transversely at finitely many points because of property (a). Further, $f^{n_{j+1}-n_j}$ maps $W_0^s(f_{\Pi}^{n_j} a)$ into
Theorem 6.4. Given any regular polynomial endomorphism of $\mathbb{C}^k$ there is a set $E \subset J_\Pi$ with $\mu_\Pi(E) = 1$ such that the following holds. If $a \in E$, $n \geq 0$ and $Z_{a,n}$ is the connected component of $W^s(a) \cap A_n$ containing $a$, then $Z_{a,n}$ is a one-dimensional subvariety of $A_n$ intersecting $\Pi$ in $N_n(a) < \infty$ points, and

$$T^{k-1} \ll A_n = \int_E \frac{1}{N_n(a)} [Z_{a,n}] \mu_\Pi(a). \quad (6.4)$$

Further, the restriction of $G$ to $Z_{a,n}$ is harmonic outside the singular locus of $Z_{a,n}$ for all $a \in E$.

Proof. The first part follows from Lemma 6.3 and the harmonicity of $G$ from Lemma 1.2. It remains to show (6.4). Given $\eta > 0$, let $F = F_\eta$ be a Pesin box satisfying (a)–(d) above with $m = m(\eta) \geq 0$. If $a \in f^{-n+m}_\Pi F_\eta \cap E$, then $f^{-(n+m)}(W^s_m(f^{n+m}_\Pi a))$ is a subvariety of $A_n$, contained in $W^s(a)$. By Lemma 6.3 we may therefore write

$$f^{-(n+m)}(W^s_m(f^{n+m}_\Pi a)) = \bigcup_{b \in f^{-(n+m)}_\Pi f^{n+m}_\Pi a} Z_{b,n},$$

and, by definition of $N_n(b)$,

$$\sum_{b \in f^{-(n+m)}_\Pi f^{n+m}_\Pi a} \frac{1}{d(k-1)(n+m)[f^{-(n+m)}(W^s_m(f^{n+m}_\Pi b))]} = \sum_{b \in f^{-(n+m)}_\Pi f^{n+m}_\Pi a} \frac{1}{N_n(b)} [Z_{b,n}].$$

This, Corollary 6.2, and the invariance of $\mu_\Pi$ yield

$$T^{k-1} \ll A_n = \int_{f^{-(n+m)}_\Pi F \cap E} \frac{1}{N_n(a)} [Z_{a,n}] \mu_\Pi(a) + S \ll A_n.$$

Theorem 6.4 follows by letting $\eta \to 0$. \hfill \Box

Corollary 6.5. With the assumptions and notation of Theorem 6.4, we have

$$\mu_c \ll A_n = \int_E \frac{1}{N_n(a)} [C \cap Z_{a,n}] \mu_\Pi(a). \quad (6.5)$$

Further, $\mu_c \ll A_n = 0$ holds if and only if $C \cap Z_{a,n} = \emptyset$ for almost every $a$ and $\mu_c \ll A = 0$ if and only if $C \cap W^s(a) = \emptyset$ for almost every $a$.

Proof. The formula (6.5) follows from Lemma 2.2 and Corollary 6.4. It follows directly from (6.5) that $\mu_c \ll A_n = 0$ if and only if $C \cap Z_{a,n} = \emptyset$ for almost every $a$. By Lemma 6.3 this happens if and only if $C \cap W^s(a) \cap A_n = \emptyset$. Letting $n \to \infty$ we get $\mu_c \ll A = 0$ if and only if $C \cap W^s(a) = \emptyset$ for almost every $a$. \hfill \Box
Formula (6.4) exhibits $T^{k-1} \mathcal{A}_n$ as a laminar current using currents of integration over (closed) subvarieties $Z_{a,n}$ in $A_n$. These varieties $Z_{a,n}$ are subsets of the global stable manifolds $W^s(a)$. We would like to have a laminar structure for $T^{k-1}$ in the larger set $A$. We could try to do this by attempting to extend the varieties $Z_{a,n}$ analytically to subvarieties of $A$. However, in §4 we gave an example of a map $f$ (with $f_{\Pi}$ uniformly expanding on $J_0$) where the global stable manifolds $W^s(a)$ are connected and have locally infinite area in $A$. The analytic continuations of $Z_{a,n}$ would be $W^s(a)$ in this case, hence would not define currents of integration.

Nevertheless, we will show that $T^{k-1} \mathcal{A}$ does have a laminar structure. We will accomplish this by dividing the global stable manifolds $W^s(a)$ into disks $W_a$. These disks are not in general closed in $A$. The construction of $W_a$ consists of cutting $W^s(a)$ along gradient lines of $G$. These gradient lines will be used in §7 to exhibit $\mu$ as a quotient of the product of $\mu_{\Pi}$ and Lebesgue measure on the circle.

Let $E_0$ and $E$ be the sets defined above. Recall that $G|_{W^s_0(a)}$ has no critical points for $a \in E_0$. Let $\mathcal{C}_n = \mathcal{C} \cup f^{-1} \mathcal{C} \cdots \cup f^{-(n-1)} \mathcal{C}$ be the critical set of $f^n$. Let us fix $a \in E$ and $n \geq 0$ and recall the definition of $Z_{a,n}$ above. Suppose that $Z_{a,n} \subset \mathcal{C}_n$. Since the restriction of $G$ to $Z_a$ is harmonic outside the singular locus, $G|_{Z_a}$ cannot be bounded above, so $Z_{a,n} \cap \Pi$ is nonempty. It follows that $Z_{a,n} \cap \mathcal{C}_n \cap \Pi \neq \emptyset$, which contradicts property (a). Thus $\mathcal{C}_n \cap Z_{a,n}$ is a discrete set. In fact it is finite, because $\mathcal{C}_n \cap Z_{a,n+1}$ is also discrete. Now $f^n$ is a local biholomorphism on $\mathcal{C}^k - \mathcal{C}_n$, so $f^{-n}$ serves to transfer certain properties from the Pesin disks of $E_0$. Specifically, let $n$ be such that $f^n \circ a \in E_0$. Such an $n$ will be called a return time. If $n$ is a return time, then $Z_{a,n} - \mathcal{C}_n$ is a manifold, and the restriction of $G$ to this manifold is a harmonic function without critical points.

We wish to define gradient lines of $G$. There is a unique tangent line to the level sets of $G|_{Z_{a,n}}$ at points off of $\mathcal{C}_n$. By the conformal structure, we may define the gradient vector of $G$ to be the tangent to $Z_{a,n} - \mathcal{C}_n$ which is orthogonal to the level line, and which points in the direction of increasing $G$. A gradient line is then an integral curve of the gradient vector field. Equivalently, a gradient line is, locally, a level set for a harmonic conjugate to $G$. We note that these so-called gradient lines are an artifact of the conformal structure. We could have as well defined $\tau$-gradient lines, which make an angle of $\tau$ with respect to the gradient inside the tangent space to $Z_{a,n} - \mathcal{C}_n$.

We say that a gradient line $\gamma$ is complete if it is a complete orbit of the gradient vector field, and if $\sup \gamma G = +\infty$. We let $S_{a,n}$ denote the set $Z_{a,n} \cap \mathcal{C}_n$, together with points of $Z_{a,n} - \mathcal{C}_n$ which are not contained in complete gradient lines.

**Lemma 6.6.** The set $S_{a,n} - \mathcal{C}_n$ consists of a finite number of (incomplete) gradient lines.

**Proof.** If $\gamma$ is an incomplete gradient line, it follows that the closure of $\gamma$ must contain a singular point of the gradient field. Let us fix a point $p \in Z_{a,n} \cap \mathcal{C}_n$. It suffices to show that there are only finitely many gradient lines whose closures contain $p$. For this, we let $h : \Delta \rightarrow Z_{a,n}$ be a holomorphic mapping with $h(0) = p$, and which is a homeomorphism with its image. It follows that $G \circ h(\zeta) = \Re(a_1 \zeta^1) + \cdots$. By further composing with a conformal map fixing the origin, we may achieve that $G \circ h(\zeta) = \Re(\zeta^1)$. Since the gradient lines are conformally invariant, it follows that they are mapped into the gradient lines of the function $\Re(\zeta^1)$, which are a finite family of straight lines through the origin. \qed
It follows that \( S_{a,n} \) is a closed subset of \( Z_{a,n} \), and thus \( Z_{a,n} - S_{a,n} \) is a manifold and an open subset of the subvariety \( Z_{a,n} \subset A_n \). An alternative definition of \( Z_{a,n} - S_{a,n} \) is that it is the largest open subset of \( Z_{a,n} - C_n \), which is invariant under the (positive) gradient flow. For every point \( x \in Z_{a,n} - S_{a,n} \), the gradient line starting at \( x \) approaches a unique point \( a' \in \Pi \).

Let \( W_{a',n} \) denote the connected component of \( Z_{a,n} - S_{a,n} \) containing \( a' \). Then the gradient lines of all points of \( W_{a',n} \) must approach \( a' \). Thus \( W_{a',n} \cap \Pi = \{a'\} \). From the paragraph above, we see that \( W_{a,n} \) consists of all of the complete gradient lines in \( Z_{a,n} - S_{a,n} \) that emanate from \( a \). It is evident, then, that the gradient lines serve as a sort of exponential map from the tangent space of \( W_{a,n} \) at \( a \) to \( W_{a,n} \).

If \( m \) is the number given in conditions (a)–(d) from Pesin Theory given above, then \( Z_{a,-m} \) contains no critical points, and so for \( a \in E \),

\[
W_{a,-m}^s(a) = W_a \cap A_{-m} = Z_{a,-m}.
\]

**Lemma 6.7.** If \( n \) is a return time for \( a \), then \( Z_{a,n} - S_{a,n} \) consists of \( N_n(a) \) components, each of which is of the form \( W_{a',n} \) for some \( a' \in f_{\Pi}^{-n}f_{\Pi}^n a \). The manifolds \( W_{a',n} \) are simply connected. If \( n' > n \) is another return time, then \( W_{a,n} \subset W_{a,n'} \), and

\[
W_{a,n'} - W_{a,n} \subset A_{n'} - A_n.
\]

**Proof.** The first statement is a consequence from the discussion above. For the simple connectivity of \( W_{a,n} \), we let \( \sigma \) denote a simple, closed curve in \( W_{a,n} \). Moving \( \sigma \) along the gradient flow of \( G \) is a homotopy, which takes \( \sigma \) to a neighborhood of \( a \), where it is contractible. The last statements follow because \( W_{a,n} \) is the union of complete gradient lines emanating from \( a \).

We define \( W_a = \bigcup_n W_{a,n} \), where the union is taken over any sequence of return times \( n \to \infty \).

**Corollary 6.8.** \( W_a \) is a simply connected Riemann surface, and the topology of \( W_a \) as a manifold coincides with the topology of \( W_a \) as a topological subspace of \( A \). If \( n \) is a return time for \( a \), then \( W_{a,n} = W_a \cap A_n \).

**Lemma 6.9.** For any \( n \geq 0 \) and any \( a \in E \) we have

\[
[Z_{a,n}] = \sum_{b \in Z_{a,n} \cap \Pi} [W_b \cap A_n].
\]  \quad (6.6)

**Proof.** First suppose \( n \) is a return time for \( a \). Then we know from Lemma 6.7 that

\[
Z_{a,n} = \bigcup_{b \in Z_{a,n} \cap \Pi} W_{b,n} \cup S_{a,n}.
\]

Thus (6.6) follows from the fact that \( S_{a,n} \) has zero area (Hausdorff 2-dimensional measure) in \( \mathbb{P}^k \). This in turn is a consequence of the fact that each incomplete gradient line is the countable union of real analytic arcs of finite length (and thus zero area).

If \( n \) is not a return time for \( a \), then let \( n' > n \) be a return time. Thus

\[
Z_{a,n'} = \bigcup_{b \in Z_{a,n'} \cap \Pi} W_{b,n'} \cup S_{a,n'},
\]

and we get (6.6) by intersecting this formula with \( Z_{a,n} \) and using the fact that \( S_{a,n'} \) has zero area. \qed
From (6.4), (6.6), and the definition of $N_n(a)$ we get

$$T^{k-1} \cdot A_n = \int_E [W_a \cap A_n] \mu_\Pi(a). \quad (6.7)$$

If we let $n \to \infty$ in (6.7), then we have a sequence which is eventually stationary in the sense that it is constant on the open set $A_j$ if $n \geq j$. This yields the following.

**Theorem 6.10.** If $E$ is the set defined by (6.3), then for $a \in E$, there exists a complex disk $W_a$ in $A$ such that

$$T^{k-1} \cdot A = \int_E [W_a] \mu_\Pi(a).$$

In particular, the disk $W_a$ has finite area for $\mu_\Pi$ almost every $a$.

§ 7 External Rays, Global Model

In this Section, we do two things. First we define the set $E$ of external rays and show that $E$ carries a natural measure $\nu$. Further, there is an endpoint mapping $e : E \to J$ defined almost everywhere, and this mapping satisfies $e_* \nu = \mu$ (Theorem 7.3). Second, we prove Theorem 7.4, which is a more global version of Theorem 4.3: the conjugacy is given between $f|\bigcup_{a \in J} W_a$ and the restriction of $f_h$ to a union of hedgehogs.

Let $a$ be a point of $E$, as defined in (6.3). The set $W_a$, defined in §6, is a simply connected Riemann surface, and $G|_{W_a}$ is harmonic. Let $G_a^*$ be a harmonic conjugate for $G|_{W_a}$, well-defined modulo $2\pi i$ and is unique up to an additive constant. Since $G|_{W_a}$ has a logarithmic pole at $a$, the function

$$\varphi_a := e^{G+iG_a^*} : W_a \to \mathbb{C}$$

is analytic on $W_a$, and $\varphi_a$ is locally injective on $W_a$ near $a$. Note that $\log |\varphi_a| = G_a$, and this condition determines $\varphi_a$ uniquely up to multiplication with a constant of unit modulus. Since $G_a^*$ is constant on the gradient lines of $G$, these gradient lines are taken to radial lines. By the construction of $W_a$, there are at most finitely many gradient lines $\gamma$ with the property that $\inf_\gamma G \geq d^{-n}R_0 > 0$ for any $n \geq 0$. Thus the range $H_a := \varphi_a(W_a)$ is a hedgehog domain of the form

$$H_a = \hat{\mathbb{C}} - \left( \hat{D} \cup \bigcup_{j=1}^N (e^{i\theta_j}, e^{r_j+i\theta_j}) \right)$$

for some $0 \leq N \leq \infty$, and $r_j > 0$ is a sequence of points with $r \to 0$. The case $N = 0$ is interpreted as $H_a = \hat{\mathbb{C}} - \hat{D}$. Since $W_a$ is invariant under the gradient flow, it follows that $\varphi_a$ is injective. We let

$$\psi_a := \varphi_a^{-1} : H_a \to W_a \quad (7.1)$$

denote the inverse. Thus the gradient lines in $W_a$ correspond to the images under $\psi_a$ of the rays in $H_a$. 
Let $\mathcal{E}_a$ denote the set of all gradient lines in $W_a$ for $a \in E$, and let $\mathcal{E}$ be the union of all $\mathcal{E}_a$. For $a \in E$, the gradient lines are naturally parametrized by a choice of argument $\theta$. The function $\psi_a$ represents one assignment of argument. Although $\theta$ is not uniquely defined, the induced measure $\frac{dd\theta}{2\pi}$ is on $\mathcal{E}_a$. Thus the measure $\nu = \mu_\Pi \otimes \frac{dd\theta}{2\pi}$ is well defined on $\mathcal{E}$. Note that $f$ maps gradient lines to gradient lines. Thus $f$ induces a measurable mapping $\sigma : \mathcal{E} \to \mathcal{E}$.

We would like to assign an endpoint to $\nu$ almost every ray. For $\gamma \in \mathcal{E}$ and $r > 0$ we set $e_r(\gamma) = \gamma \cap \{G = r\}$. For each $a \in E$, $e_r$ is defined for all but possibly finitely many rays lying in $W_a$ and we will write $e_{a,r}$ for the restriction of $e_r$ to the rays in $W_a$. The mapping $\psi_a$ represents $e_r$ in the sense that if $\gamma\theta$ is the ray in $W_a$ corresponding to argument $\theta$, then

$$e_r(\gamma\theta) = e_{a,r}(\gamma\theta) = \psi_a(e^{r+i\theta}).$$

**Lemma 7.1.** If $a \in E$, and if $W_a$ has finite area as a subset of $\mathbb{P}^k$, then

$$\lim_{r \to 0^+} e_r(\gamma\theta) = \lim_{r \to 0^+} \psi_a(e^{r+i\theta})$$

exists for almost every $\theta$.

**Proof.** We work in affine coordinates in $\mathbb{C}^k \subset \mathbb{P}^k$. Let $\tilde{a} \in \mathbb{C}^k$ denote a point with $|\tilde{a}| = 1$ such that $\pi(\tilde{a}) = a \in \Pi$. Thus we may write $\psi_a(\zeta) = \zeta^{-1}\tilde{a} + h_a(\zeta)$, where $h_a$ is analytic on $H_a$. Away from the hyperplane at infinity, the Euclidean metric on $\mathbb{C}^k$ is equivalent to the Fubini-Study metric on $\mathbb{P}^k$. The condition that $W_a$ has finite area in $\mathbb{P}^k$ is equivalent to $\int_{H_a} |\nabla h_a|^2 < \infty$. It follows that

$$\int_0^1 |\nabla h_a(re^{i\theta})|^2 r dr = \int_0^1 \left| \frac{\partial h_a(re^{i\theta})}{\partial r} \right|^2 r dr < \infty$$

for almost every $\theta$. Thus radial limits exist for these values of $\theta$. □

It follows that there is a measurable mapping $e : \mathcal{E} \to \partial K$ such that

$$e(\gamma) = \lim_{r \to 0^+} e_r(\gamma)$$

for $\nu$ a.e. $\gamma$. Our next step will be to show that the mapping $e$ pushes $\nu$ forward to $\mu$. We will write $G_r := \max(G, r)$.

**Lemma 7.2.** For $a \in E$,

$$(e_{a,r})_* d\theta = dd^c_{W_a} G_r|_{W_a}.$$

**Proof.** Let us first note that the statement of the Lemma is conformally invariant. Under a conformal transformation, the Green function transforms by composition; the gradient lines and level sets are preserved, and so the map $e_{a,r}$ transforms by composition. Similarly, the operator $dd^c$ on the right hand side is invariant, so the right hand side transforms correctly. Now we transform under the map $\psi_a$. Since $\psi_a$ is nonsingular at infinity, the measure $d\theta$ is preserved. The image of $W_a$ is $H_a$, and $G$ is taken to $\log |\zeta|$. The mapping $e_{a,r}$ then takes the angle $\theta$ to the point $re^{i\theta}$ and the Lemma is reduced to an elementary calculation involving $\log |z|$.

□
Theorem 7.3. The endpoint mapping $e$ satisfies $e_*\nu = \mu$. Further, $f \circ e = e \circ \sigma$, where $\sigma : \mathcal{E} \to \mathcal{E}$ is the map induced by $f$.

Proof. Integrating the formula of Lemma 7.2 with respect to $\mu_{\Pi}$, and dividing by $2\pi$, we obtain

$$(e_r)_*\nu = \int \frac{1}{2\pi} dd^c_{W} G_r|_{W_a} \mu_{\Pi}(a). \quad (7.2)$$

Let us choose $n$ sufficiently large that $\{G \geq r\} \subset A_n$. By Theorem 6.4 we have

$$T^{k-1} \llcorner A_n = \int_{E} \frac{1}{N_n(a)} [Z_{a,n}] \mu_{\Pi}(a),$$

and by Lemma 2.2 with $X = \frac{1}{2\pi} dd^c G_r$, we have

$$\frac{1}{2\pi} dd^c G_r \wedge T^{k-1} = \int_{E} \frac{1}{2\pi N_n(a)} dd^c G_r \wedge [Z_{a,n}] \mu_{\Pi}(a). \quad (7.3)$$

Here we have dropped $\llcorner A_n$ because the support of $dd^c G_r$ is contained in $A_n$. By Lemma 6.9, we have

$$[Z_{a,n}] = \sum_{b \in Z_{a,n} \cap \Pi} [W_b \cap A_n],$$

where the sum has $N_n(a)$ terms. Since $G_r$ is continuous, the wedge product of $dd^c G_r$ with the current of integration over $Z = Z_{a,n}$ is equal to the slice measure $dd^c_{Z} G_r|_{Z}$. Since $G_r$ is bounded, $dd^c G_r|_{Z}$ can put no mass on a polar set. In particular there can be no mass on the intersection of a gradient line of $G$ with $\{G = r\}$, which is an isolated point. Thus we have

$$dd^c G_r \wedge [Z_{a,n}] = \sum_{b \in Z_{a,n} \cap \Pi} dd^c G_r \wedge [W_b \cap A_n]$$

$$= \sum_{b \in Z_{a,n} \cap \Pi} dd^c G_r \wedge [W_b].$$

Again the $\llcorner A_n$ is dropped, since $dd^c G_r$ is supported in $A_n$. Thus (7.3) becomes

$$\frac{1}{2\pi} dd^c G_r \wedge T^{k-1} = \int_{E} \frac{1}{2\pi} dd^c G_r \wedge [W_a] \mu_{\Pi}(a).$$

By the continuity of $G_r$, we have $dd^c G_r \wedge [W_a] = dd^c_{W_a} G_r|_{W_a}$, so that by (7.2) we have

$$\frac{1}{2\pi} dd^c G_r \wedge T^{k-1} = (e_r)_*\nu.$$

Finally, as we let $r \to 0$, the left hand side converges to $\mu = \frac{1}{2\pi} dd^c G \wedge T^{k-1}$, and the right hand side converges to $e_*\nu$, which shows that $e_*\nu = \mu$.

It is evident from the definition of $\sigma$ that $f \circ e_{dr} = e_r \circ \sigma$. Thus $f \circ e = e \circ \sigma$. \qed
Example. Let \( f(z_1, z_2) = (z_1^2 + c_1, z_2^2 + c_2) \). Since \( f \) is a product, \( J = J_{c_1} \times J_{c_2} \), where \( J_c \) is the one-dimensional Julia set of \( p_c(z) = z^2 + c \). In the coordinate \( \zeta = w/z \), \( f_{\Pi}(\zeta) = \zeta^2 \), so \( J_{\Pi} = \{ |\zeta| = 1 \} \) and \( f_{\Pi} \) is uniformly expanding on \( J_{\Pi} \). Using the local canonical model (Theorem 4.3) and Proposition A.2, we see that \( E \) is homeomorphic to \( J_{\Pi} \times S^1 = S^1 \times S^1 \). In the example at hand, the homeomorphism is given as follows. For \( \zeta = e^{2\pi i \phi} \in J_{\Pi} \) and \( e^{2\pi i \theta} \in S^1 \), we associate the external ray \( \gamma = \gamma(\phi, \theta) \in E \) with the properties: the projection of \( \gamma \) to the \( z_1 \) axis is the external ray \( \gamma(\theta) \) for \( p_{c_1} \), which is asymptotic to \( re^{2\pi i \theta} \) at infinity, and the projection of \( \gamma \) to the \( z_2 \) axis is the external ray \( \gamma(\phi + \theta) \) for \( p_{c_2} \). If the endpoint map \( e \) is defined, then we may write it as \( e(\phi, \theta) = (e_{c_1}(\phi), e_{c_2}(\phi + \theta)) \). Any identifications in the quotient by \( e \) (i.e. pairs \( (\phi', \theta'), (\phi'', \theta'') \) with \( e(\phi', \theta') = e(\phi'', \theta'') \)) are thus determined by the identifications in the quotients by \( e_{c_1} \) and \( e_{c_2} \).

We conclude this section by proving a global version of the conjugacy given by Theorem 4.3. First note that if \( f_{\Pi} \) is uniformly expanding on \( J_{\Pi} \), then we can assume that \( E_0 = J_{\Pi} \) and that \( W^s(J_{\Pi}) \cap A_0 \cap C_\infty = \emptyset \), where \( C_\infty = \bigcup_{j \geq 0} f^{-j}C \). Thus \( W^s(J_{\Pi}) \cap C_\infty \) is closed and nowhere dense in \( W^s(J_{\Pi}) \), and \( W^s(J_{\Pi}) - C_\infty \) is a Riemann surface lamination on which the gradient flow induced by \( G \) is defined. For \( x \in W^s(J_{\Pi}) - C_\infty \), let \( \gamma(x) \) be the gradient line containing \( x \). Define a numbers \( r(x) \in (0, \infty] \), \( s(x) \in [0, \infty) \) for \( x \in W^s(J_{\Pi}) \) be declaring \( r(x) = s(x) = G(x) \) if \( x \in C_\infty \) and \( r(x) = \sup \{ G(y) : y \in \gamma(x) \} \), \( s(x) = \inf \{ G(y) : y \in \gamma(x) \} \) for \( x \in W^s(J_{\Pi}) - C_\infty \). Let \( S := \{ x \in W^s(J_{\Pi}) : r(x) < \infty \} \). Then \( S \) is the union of incomplete gradient lines.

**Theorem 7.4.** If \( f_{\Pi} \) is uniformly expanding on \( J_{\Pi} \), then the following hold:

1. The set \( S \) of points contained in incomplete gradient lines is closed and nowhere dense in \( W^s(J_{\Pi}) \). Further, \( W^s(J_{\Pi}) - S \) is a Riemann surface lamination in \( A - S \). The leaves of this lamination are exactly the disks \( W_a \) and these are properly embedded in \( A - S \).

2. There is a closed and nowhere dense subset \( S_h \) of \( W^s(J_{\Pi}, f_h) \) such that \( \Psi \), defined in Theorem 4.3 extends to a homeomorphism

\[
\Psi : W^s(J_{\Pi}, f_h) - S_h \to W^s(J_{\Pi}, f) - S,
\]

conjugating \( f_h \) to \( f \). The set \( S_h \) is a union of real rays through the origin in \( C^k \) and if \( D_a \) is the disk in \( W^s(J_{\Pi}, f_h) \) passing through \( a \in J_{\Pi} \), then \( \Psi \) maps \( D_a - S_h \) biholomorphically onto \( W_a \).

3. We may identify the set \( \mathcal{E} \) of external rays with the boundary of \( W^s(J_{\Pi}, f_h) \), i.e. the union of the circles \( \partial D_a \), \( a \in J_{\Pi} \). With this identification, the maps \( e_r \) are defined by \( e_r(\gamma) = \Psi(e_r \cdot \gamma) \) and \( e(\gamma) = \lim_{r \to 0} \Psi(e_r \gamma) \).

4. We have \( \mu_{\mu}(A) = 0 \) if and only if \( W^s(J_{\Pi}) \cap C = \emptyset \). In this case, \( S = S_h = \emptyset \) and hence there is a conjugacy \( \Psi : W^s(J_{\Pi}, f_h) \to W^s(J_{\Pi}, f) \) conjugating \( f_h \) to \( f \).

**Proof.** (1) We first prove that \( S \) is closed. Note that \( S \cap A_0 = \emptyset \). Take \( x \in W^s(J_{\Pi}) - S \). By definition there is a gradient line \( \gamma = \gamma(x) \) containing \( x \) and ending at a point \( a \in J_{\Pi} \). We have to show that the same is true for all points in a neighborhood of \( x \) in \( W^s(J_{\Pi}) \). We may assume that \( x \notin A_0 \), because otherwise there is nothing to prove. Pick \( t > 0 \) and \( n \geq 0 \) with \( s(x) < t < G(x) \) and \( R_0 < d^n t < R_0 d \). Thus \( f^n \gamma \) is a complete gradient
line in $A_{-1}$. In fact, all gradient lines are complete in $A_0$, so there is a complete gradient line $\gamma'$ in $A_0$ containing $f^n\gamma$. We have $\gamma \cap C_{-n} = \emptyset$. Thus there exists a branch $g$ of $f^{-n}$, defined in a neighborhood of $\gamma' \cap \{G > d^nt\}$, such that $g \circ f^n = \text{id}$ on $\gamma \cap \{G > t\}$. The map $\Psi$ is defined on $W^s(J_{\Pi}, f_h)$ and $\gamma'' := \Psi^{-1}(\gamma')$ is a gradient line (i.e. a real line segment) in $A_{0,h}$. Let $U$ be a simply connected neighborhood of $\gamma'' \cap \{d^nt < G_h < \infty\}$ in $W^s(J_{\Pi}, f_h) \cap \{d^nt < G_h < \infty\}$ consisting of gradient lines for $G_h$. Then $\Psi(U)$ is a simply connected neighborhood of $\gamma' \cap \{d^nt < G < \infty\}$ consisting of gradient lines for $G$. We may assume that $g$ is defined in $\Psi(U)$. Thus $g \circ \Psi(U)$ is an open set in $W^s(J_{\Pi}) \cap \{t < G < \infty\}$ consisting of gradient lines in $\{t < G < \infty\}$. Since $x \in g \circ \Psi(U)$ we have proved that $S$ is closed in $W^s(J_{\Pi})$.

Since $S$ is closed, it is also nowhere dense if it contains no relative interior of $W^s(J_{\Pi})$. But if $S$ were to contain a relative interior point of $W^s(J_{\Pi})$, then it contains relative interior of a global stable manifold $W^s(a)$. However, this is not possible, since $S \cap W^s(a)$ consists of at most a countable number of curves.

(2) Let $D_a$ be the disk in $W^s(J_{\Pi}, f_h)$ associated with $a \in J_{\Pi}$. The discussion in the beginning of §7 shows that there is a uniquely defined closed subset $S_{a,h}$ of $D_a$ such that $G_h > R_0$ on $S_{a,h}$ and such that $\Psi|_{D_a \cap \{G_h > R_0\}}$ extends to a biholomorphism of $D_a - S_{a,h}$ onto $W_a$. In fact, if we identify $D_a$ with $\mathbb{C} - \mathbb{D}$, then $D_a - S_{a,h}$ equals the set $H_a$ and $\Psi|_{D_a - S_{a,h}} = \psi_a$ defined in (7.1). Let $S_h = \bigcup_{a \in J_{\Pi}} S_{a,h}$. Clearly $S_{a,h} \cap A_{0,h} = \emptyset$. By analytic continuation in $D_a$ we get $f \circ \Psi = \Psi \circ f_h$ on $W^s(J_{\Pi}, f_h) - S_h$. We show that $S_h$ is closed in $W^s(J_{\Pi}, f_h)$. Take $x \in W^s(J_{\Pi}) - S_h$. Thus $x \in D_a - S_{a,h}$ for some $a \in J_{\Pi}$. Pick $0 < s < t < 1$ such that the point $sx \in D_a - S_{a,h}$ and let $\gamma$ be the gradient line containing $x$. Choose $n \geq 0$ such that $R_0 < d^nsx < dR_0$ and let $\gamma_1$ be the gradient line in $A_{0,h}$ containing $f^n_h(x)$. Then $\gamma' = \Psi(\gamma)$ is a gradient line for $G$ containing $\Psi(sx)$ and ending at $a$ and $\gamma'_1 = \Psi(\gamma_1)$ is a gradient line in $A_0$ containing $f^n(\Psi(x))$. There is a neighborhood $U$ of $\Psi(tx)$ consisting of gradient lines $\gamma''$ for $G$ such that $\tau(\gamma''') = \infty$ and $s(\gamma''') < t$. Let $h$ be a branch of $f_h^{-n}$ defined near $\gamma_1$ such that $g(f_h^n(x)) = x$. If $U$ is small enough, then $g \circ \Psi^{-1}(U)$ is a neighborhood of $x$ in $W^s(J_{\Pi}, f_h)$ disjoint from $S_h$. Thus $S_h$ is closed in $W^s(J_{\Pi}, f_h)$. Since $S_h$ intersects each disk of $W^s(J_{\Pi}, f_h)$ in a nowhere dense set, it is nowhere dense.

To finish the proof, we note that (3) is a consequence of (1) and (2), and (4) follows from Corollary 6.5, from Propostion 4.2, and from (2).

Remark. Both the external rays and the global model are not “canonical” in the sense that they were not defined by dynamical behaviors. In the definitions of $\mathcal{E}$ and the hedgehog sets $H_a$, it is equally natural to replace the “gradient” lines by the family of curves in $W^s(a)$ which cross the level sets of $G|W^s(a)$ at a constant angle $\tau$. The flexibility of considering values of $\tau$ different from $\pi/2$ has proved useful in the case $k = 1$ (see [Le]).

§8 Properties of the Support of $T^{k-1}$

In this Section we show that under rather general conditions the (local) stable disks given by Pesin theory actually determine the set $A \cap \text{supp}(T^{k-1})$. First we show that $W^s(a)$ is dense in the support of $T^{k-1} \mathbb{L} A$ for $\mu_{\Pi}$-a.e. $a$. As in the case of polynomial automorphisms of $\mathbb{C}^2$ ([BLS, Prop 2.9]) we do this by proving a convergence result for currents. See also [FS4, Cor 5.13]. The other result of this Section is that when the critical measure $\mu_c$ vanishes
on $A_n$, the closure of the Pesin family of disks (and thus support of $T^{k-1} A_n$) has a uniformly laminar structure.

First we prove some convergence results for measures on $J_\Pi$. The main tool is Lemma 8.1 below, due to Fornæss and Sibony. Given $a \in J_\Pi$ and $j \geq 0$, define the measures $\nu_{a,j}$ and $\nu_{a,j}'$ by

$$
\nu_{a,j}' := \frac{1}{d(k-1)j} (f_{\Pi}^j)^* \delta_a = \frac{1}{d(k-1)j} \sum_{b \in f_{\Pi}^{-j} a} \delta_b
$$

and

$$
\nu_{a,j} := \nu_{f_{\Pi}^{-j} a, j}' = \frac{1}{d(k-1)j} \sum_{b \in f_{\Pi}^{-j} f_{\Pi}^j a} \delta_b.
$$

**Lemma 8.1 ([FS3, Lemma 8.3]).** There is a constant $C > 0$ such that if $\phi$ is a $C^2$ test function on $\Pi$ and $s > 0$, and

$$
E'(\phi, s, j) := \{ a \in J_\Pi : |\langle \nu_{a,j}', \phi \rangle - \langle \mu_\Pi, \phi \rangle| > s \},
$$

then

$$
\mu_\Pi(E'(\phi, s, j)) \leq \frac{C|\phi|C^2}{d^j s}.
$$

**Lemma 8.2.** As $j \to \infty$, we have $\nu_{a,j} \to \mu_\Pi$ for $\mu_\Pi$-a.e. $a$. If $f_{\Pi}$ is expanding on $J_\Pi$, then $\nu_{a,j} \to \mu_\Pi$ for every $a \in J_\Pi$.

**Proof.** Fix $\phi \in C^2$. It is sufficient to prove that $\langle \nu_{a,j}, \phi \rangle \to \langle \mu_\Pi, \phi \rangle$ for almost every $a$. Let

$$
E_j := \{ a \in \Pi : |\langle \nu_{a,j}, \phi \rangle - \langle \mu_\Pi, \phi \rangle| > d^{-j/2} \}.
$$

By applying Lemma 8.1 with $s = d^{-j/2}$ and using the invariance of $\mu_\Pi$ we get $\mu_\Pi(E_j) \leq Cd^{-j/2}$ (the industrious reader may check that Lemma 8.3 in [FS3] remains valid if $s$ depends on $j$). It follows that the set of $a$ such that $a \notin E_j$ for sufficiently large $j$ has full measure. Clearly $\langle \nu_{a,j}, \phi \rangle \to \langle \mu_\Pi, \phi \rangle$ for these $a$.

If $f_{\Pi}$ is expanding on $J_\Pi$, then there is an $\epsilon > 0$ such that all branches of $f_{\Pi}^{-j}$ are single-valued on balls in $\Pi$ of radius $\epsilon$ centered at points in $J_\Pi$. Further, the diameters of the preimages under $f^j$ of these balls tend to zero uniformly as $j \to \infty$. It follows that if $d(a, a') < \epsilon$, then $\langle \nu_{a,j}', \phi \rangle - \langle \nu_{a',j}', \phi \rangle \to 0$.

Now let $a \in J_\Pi$ be given. If $j$ is large enough, then there is a point $b_j \notin E_j$ close to $f_{\Pi}^{-j} a$. Thus

$$
\langle \nu_{a,j}, \phi \rangle = \langle \nu_{f_{\Pi}^{-j} a, j}' , \phi \rangle \to \langle \mu_\Pi, \phi \rangle \text{ as } j \to \infty.
$$

**Remark.** The statements of Lemma 8.2 also hold with the measures $\nu_{a,j}$ replaced by $\nu_{a,j}'$.

**Corollary 8.3.** $W^s(a, f_{\Pi})$ is dense in $J_\Pi$ for $\mu_\Pi$-a.e. $a \in J_\Pi$. If $f_{\Pi}$ is expanding on $J_\Pi$, then this holds for every $a \in J_\Pi$.

Now we consider convergence of currents.
Proposition 8.4. For almost every \( a \in J_{\Pi} \) we have
\[
\liminf_{j \to \infty} \frac{1}{d(k-1)j} f^{j*}[W_{f_{\Pi}^j a}] \geq T^{k-1} \mathcal{L} A.
\] (8.1)

If \( f_{\Pi} \) is expanding on \( J_{\Pi} \), then
\[
\lim_{j \to \infty} \frac{1}{d(k-1)j} f^{j*}[W_{f_{\Pi}^j a}] = T^{k-1} \mathcal{L} A
\] (8.2)
for every \( a \in J_{\Pi} \).

Proof. Fix \( n \geq 0 \). It suffices to show (8.1) and (8.2) on \( A_n \).

Fix \( \eta > 0 \) and let \( F = F_{\eta} \subset E \) be a Pesin box with \( \mu_{\Pi}(F) \geq 1 - \eta \) satisfying (a)–(d) above. Since the Pesin disks \( W^*_m(b) = W_b \cap A_{-m} \) depend continuously on \( b \) on \( F \), it follows from Lemma 8.2 that
\[
\frac{1}{d(k-1)j} \sum_{b \in f_{\Pi}^{-1} f_j a \cap F} [W_b \cap A_{-m}] \to \int_F [W_b \cap A_{-m}] \mu_{\Pi}(b)
\]
as \( j \to \infty \) for almost every \( a \). Hence, if we define
\[
X_j(a) = \frac{1}{d(k-1)j} f^{j*}[W_{f_{\Pi}^j a}],
\]
then
\[
\liminf_{j \to \infty} X_j(a) \mathcal{L} A_{-m} \geq \int_F [W_b \cap A_{-m}] \mu_{\Pi}(b)
\]
for almost every \( a \). If we pull this back by \( f^{n+m} \), then we get
\[
\liminf_{j \to \infty} X_j(a) \mathcal{L} A_n \geq \int h[W_b \cap A_n] \mu_{\Pi}(b),
\] (8.3)
where \( h = h_\eta = d^{-(k-1)(m+n)} f^{n+m} \chi_F \). In particular we have \( 0 \leq h \leq 1 \) and \( \int h = \mu_{\Pi}(F) \).

By letting \( \eta \to 0 \) we get \( h_\eta \to 1 \) so by dominated convergence we find that the right hand side of (8.3) converges to \( \int [W_b \cap A_n] \mu_{\Pi}(b) = T^{k-1} \mathcal{L} A_n \) for almost every \( a \).

If \( f_{\Pi} \) is expanding on \( J_{\Pi} \), then we may take \( F = J_{\Pi} \) and use the second part of Lemma 8.2. Thus \( X_j(a) \mathcal{L} A_n \to T^{k-1} \mathcal{L} A_n \) for every \( a \in J_{\Pi} \).

Corollary 8.5. For almost every \( a \in J_{\Pi} \) we have \( \overline{W^s(a)} = \text{supp}(T^{k-1} \mathcal{L} A) \). If \( f_{\Pi} \) is uniformly expanding on \( J_{\Pi} \), then this holds for every \( a \in J_{\Pi} \).

Proof. It is clear that \( W_a \subset W^s(a) \) for all \( a \in E \). We claim that \( W_a \subset \text{supp}(T^{k-1} \mathcal{L} A) \).

By the construction of \( W_a \) it suffices to show that \( W^s_0(a) \) is contained in the support of \( T^{k-1} \mathcal{L} A \) for \( a \in E_0 \). But this follows from the continuity of \( W^s_0(a) \) on \( E_0 \), from the fact that \( E_0 \) has no isolated points and from (6.4).

Thus \( \overline{W^s(a)} \subset \text{supp}(T^{k-1} \mathcal{L} A) \). The reverse inclusion is a consequence of Proposition 8.4.
If $f_{\Pi}$ is uniformly expanding on $J_{\Pi}$, then by increasing $R_0$, we have $\mu_c(A_0) = 0$. We now consider the property

$$\mu_c(A_n) = |C| \land (T^{k-1} \mathbf{L} A_n) = 0$$

for some $n \in \mathbb{Z}$, without assuming uniform expansion on $J_{\Pi}$. By Corollary 6.5, this implies that $C \cap Z_{a,n} = \emptyset$ for $\mu_{\Pi}$ almost every $a \in E$. Using the invariance of $\mu_{\Pi}$ and the fact that $fZ_{a,n} \subset Z_{f_{\Pi} a,n}$ we also get that $Z_{a,n} \cap C_{\infty} = \emptyset$ for almost every $a \in E$, where $C_{\infty} = \bigcup_{j \geq 0} f^{-j} C$.

For such $a$, the construction of $W_a$ involves the removal of no gradient lines in $A_n$. Thus $W_a \cap A_n = Z_{a,n}$, and $W_a \cap A_n$ is a properly embedded disk in $A_n$. Further, for these $a$, the mapping $\psi_a$, defined by (7.1) maps the disk $\Delta_n = \{ |\zeta| > \exp(d^{-n}R_0) \}$ biholomorphically onto $W_a \cap A_n$. Let us summarize this.

**Proposition 8.6.**

(1) If $\mu_c(A_n) = 0$, then for almost every $a$, $W_{a,n} := W_a \cap A_n$ is a properly embedded disk in $A_n$, and the restriction of $\psi_a$ to $\Delta_n$ is a biholomorphism onto $W_{a,n}$.

(2) If $\mu_c(A) = 0$, then for almost every $a$ $W_a$ is a properly embedded disk in $A$, and $\psi_a$ maps $\Delta = \{ |\zeta| > 1 \}$ biholomorphically onto $W_a$. Further, $\psi_a : \Delta \to \mathbb{P}^k - J$ is proper.

**Proof.** Everything except the last statement follows from the discussion above. By Theorem 7.3, it follows that for $\mu_{\Pi}$-almost every $a$ the boundary values of the disk $\psi_a : \Delta \to W_a$ lie inside $J$ for almost every $\theta$. Thus $\psi_a : \Delta \to \mathbb{P}^k - J$ is proper by the theorem of Alexander (see [A] and [Ro]).

Suppose $\mu_c(A_n) = 0$. We want to show that the family of disks $W_{a,n}$ extends to a Riemann surface lamination in $A_n$. Let $\Gamma_n$ denote all the uniformizing mappings $\psi_a : \Delta_n \to W_{a,n}$.

**Lemma 8.7.** Either $\Gamma_n$ is a normal family, or there is a nonconstant holomorphic mapping $h : C \to \Pi$ such that $h(C) \subset \Pi - E$.

**Proof.** If $\Gamma_n$ is not a normal family, there is a sequence $\{ \psi_j \} \subset \Gamma_n$ without a convergent subsequence. By the renormalization technique of Brody [La, pp. 68–71] there is a sequence $r_j \to \infty$ and a sequence of Möbius transformations $\rho_j : \{ |\zeta| < r_j \} \to \Delta_n$ such that $\psi_j \circ \rho_j$ converges to a nonconstant mapping $h : C \to \mathbb{P}^k$. Further, $G \circ \psi_j \circ \rho_j$ is a sequence of positive, superharmonic functions which converge normally on $C$ to $G \circ h$. Since every positive, superharmonic function on $C$ is constant (or $\equiv + \infty$), it follows that either $h(C) \subset \{ G = c \}$ or $h(C) \subset \Pi$. The set $\{ G = c \}$ is a compact subset of $\mathbb{C}^k$, so we cannot have $h(C) \subset \{ G = c \}$. Thus we have $h(C) \subset \Pi$. Finally, the disks $\{ W_a \}$ are pairwise disjoint, so an application of the Hurwitz Theorem shows that either $h(C) \subset W_a$ or $h(C) \cap W_a = \emptyset$. The first case is not possible since $h$ is nonconstant, so we have $h(C) \subset \Pi - \{ a \}$. Thus $h(C) \subset \Pi - E$.

**Remark.** If $k = 2$, there can be no nonconstant mapping of $C$ into $\Pi - E$, so $\Gamma_n$ is necessarily a normal family. For $k > 2$ we let $X_1, X_2, \ldots$ denote the distinct irreducible components in $\bigcup_{j \in \mathbb{Z}} f_{\Pi}^j C_{\Pi}$. Since $\mu_c(A_n) = 0$ we conclude, using the Hurwitz Theorem, that for any $X_j$ we have either $h(C) \subset X_j$ or $h(C) \subset \Pi - X_j$. Thus $\Gamma_n$ is a normal
family if for any finite family \( X_{j_1}, \ldots, X_{j_p} \), there is no nonconstant mapping of \( \mathbb{C} \) into \( X_{j_1} \cup \ldots \cup X_{j_p} - \bigcup' X_i \), where \( \bigcup' \) denotes the union of the remaining components. Since normality of \( \Gamma_n \) is the essential ingredient of the following theorem, it would appear to apply in many cases with \( k > 2 \).

**Theorem 8.8.** If \( \mu_c(A_n) = 0 \) and \( k = 2 \), then for each \( a \in \Pi_\Pi \) there is a complex disk \( W_{a,n} \) which is properly embedded in \( A_n \), such that \( a \mapsto W_{a,n} \) is continuous, and such that \( W_{a,n} \cap W_{b,n} = \emptyset \) if \( a \neq b \). Thus the family \( \{W_{a,n} : a \in \Pi_\Pi \} \) is a Riemann surface lamination in \( A_n \). Further, we have

\[
T \lhd A_n = \int [W_{a,n}] \mu_\Pi(a) \quad (8.4)
\]

and for each \( a \) we have \( W_{a,n} \cap C = \emptyset \) or \( W_{a,n} \subset C \cup \{a\} \). If \( \mu_c(A) = 0 \), then all of the above conclusions hold on \( A \).

**Proof.** By Lemma 8.7 and the remark above, \( \Gamma_n \) is a normal family. Let \( \tilde{\Gamma}_n \) denote the mappings \( \psi : \Delta_n \to A_n \) which are normal limits of \( \Gamma_n \). Note that \( G \circ \psi(\zeta) = \log |\zeta| \) for any \( \psi \in \tilde{\Gamma}_n \). Since the disks \( W_a \) are pairwise disjoint, it follows from the Hurwitz Theorem that if \( \psi', \psi'' \in \tilde{\Gamma}_n \), then either \( \psi'(\Delta_n) = \psi''(\Delta_n) \) or \( \psi'(\Delta_n) \cap \psi''(\Delta_n) = \emptyset \). Thus for each \( a \in \Pi_\Pi \) there is a unique image \( W_{a,n} := \psi(\Delta_n) \), \( \psi \in \tilde{\Gamma}_n \), which contains \( a \). The continuity of \( a \mapsto W_{a,n} \) follows from the normality of \( \Gamma_n \). We have \( Z_{a,n} = W_{a,n} \) and \( N_n(a) = 1 \) for almost every \( a \) and hence (8.4) follows from (6.4). Applying Lemma 2.2 to (8.4) we obtain

\[
\mu_c \lhd A_n = \int [W_{a,n} \cap C] \mu_\Pi(a) = 0. \tag{8.5}
\]

By continuity of \( a \to W_{a,n} \), the property that \( W_{a,n} \cap C \neq \emptyset \) but \( W_{a,n} - (C \cup \{a\}) \neq \emptyset \) is open in \( \Pi_\Pi \). Thus this property never holds by (8.5).

**Example.** Let \( f(z,w) = (z^2, \frac{1}{2} z^2 + w^2) \). In the coordinate \( \zeta = w/z \) on \( \Pi \), we have \( f_\Pi(\zeta) = \zeta^2 + \frac{1}{2} \). Let \( \Pi_\Pi \subset \Pi \) denote the filled Julia set for \( f_\Pi \). The point \( \zeta = \frac{1}{2} \) is a parabolic fixed point, and all points of \( \{\frac{1}{2}\} \cup \text{int}(\Pi_\Pi) \subset \Pi \) approach \( \{\zeta = \frac{1}{2}\} \) in forward time. Thus the stable set \( W^{s}(\frac{1}{2}) \) for \( f \) contains a neighborhood of \( \frac{1}{2} \) inside the cone of complex lines \( C(\{\frac{1}{2}\} \cup \text{int}(\Pi_\Pi)) \), which contains an open set in \( \mathbb{P}^k \).

Since \( f = f_h \) is homogeneous, each (local) Pesin disk \( W^{s}_{-m}(a) \) is the complement of a closed disk (centered at the origin) inside the complex line \( L_a \). Thus the family of Pesin disks has an extension to the lamination inside the complex cone of lines \( C(J_\Pi) \). In the example at hand, the critical locus is \( C = \{z = 0\} \cup \{w = 0\} \), and \( T_h \) is supported on \( C(J_\Pi) \). Since \( \{\zeta = 0, \infty\} \) is disjoint from \( J_\Pi \), and \( A \) is a neighborhood of \( \Pi \) disjoint from \( 0 \in \mathbb{C}^2 \), it follows that \( \mu_c \lhd A = [C] \cap T_h \lhd A = 0 \). Thus this example also satisfies the hypotheses of Theorem 8.8.

**§9 Axiom A in \( \mathbb{C}^2 \).**

In the next section we will impose certain hyperbolicity assumptions (see Definition 10.1) on the dynamics on \( f \) in order to prove that all of the external rays land (and land continuously) on \( J \). Most of these assumptions are related to Axiom A, which was introduced by Smale as a property of a smooth dynamical system which enables the understanding
of its global dynamics. In this Section, we discuss Axiom A in the setting of polynomial endomorphisms of $\mathbb{C}^2$, chiefly to clarify our assumptions in Definition 10.1.

The literature on hyperbolic dynamics is vast, but most expositions consider only diffeomorphisms. A regular polynomial endomorphism of $\mathbb{C}^2$ of degree $d \geq 2$ is not invertible, and the hyperbolic theory is slightly different. There seems to be no general, detailed treatment of exactly the results we need, so we will give further definitions and results in Appendix B. More details can be found in [J2]. We also refer to [FS4], where the authors study hyperbolic endomorphisms of $\mathbb{P}^2$.

Suppose that $f$ is a regular polynomial endomorphism of $\mathbb{C}^2$; as usual we regard $f$ as a holomorphic map of $\mathbb{P}^2$. Since $f$ is not injective, we will often have to work with histories of points instead of the points themselves. Precisely, a history of a point $x \in \mathbb{P}^2$ is a sequence $(x_i)_{i \leq 0}$ of points in $\mathbb{P}^2$ such that $x_0 = x$ and $fx_i = x_{i+1}$ for all $i < 0$. We will use the notation $\hat{x}$ for a history $(x_i)$.

Let $L$ be a compact subset of $\mathbb{P}^2$ with $fL = L$. We refer to Appendix B for a definition of what it means for $f$ to be (uniformly) hyperbolic on $L$. Let us only recall that the definition involves the compact set $\hat{L}$ of histories in $L$. The pair $(\hat{L}, \hat{f})$, where $\hat{f}$ is the left shift on $\hat{L}$, is often called the natural extension of $f|_L$. There is a natural projection $\pi : \hat{L} \to L$ such that $\pi(\hat{x}) = x_0$. We say that $L$ has unstable index $i$ if the stable bundle $E^s$ has constant dimension $2 - i$ on $L$. If $L$ has unstable index 2, then $f$ is said to be (uniformly) expanding on $L$ (see Appendix B for an alternative definition). If $f$ is hyperbolic on $L$, then to every point in $x \in L$ and every history $\hat{x} \in \hat{L}$ there is an associated local stable and unstable manifold respectively, defined by

\begin{align*}
W^s_{\text{loc}}(x) &= \{ y \in \mathbb{P}^2 : d(f^iy, f^ix) < \delta \forall i \geq 0 \} \\
W^u_{\text{loc}}(\hat{x}) &= \{ y \in \mathbb{P}^2 : \exists \hat{y} \in \hat{\mathbb{P}^2}, \pi(\hat{y}) = y, d(y_i, x_i) < \delta \forall i \leq 0 \},
\end{align*}

for small $\delta > 0$. Then $W^s_{\text{loc}}(x)$ and $W^u_{\text{loc}}(\hat{x})$ are complex disks of $\mathbb{P}^2$. If $f$ is uniformly expanding on $L$, then the local stable manifolds are empty and the local unstable manifold at $\hat{x}$ is a neighborhood of $x_0$ in $\mathbb{P}^2$.

We also define global stable and unstable manifolds by declaring

\begin{align*}
W^s(x) &= \{ y \in \mathbb{P}^2 : d(f^iy, f^ix) \to 0 \text{ as } i \to \infty \} \\
W^u(\hat{x}) &= \{ y \in \mathbb{P}^2 : \exists \hat{y} \in \hat{\mathbb{P}^2}, \pi(\hat{y}) = y, d(y_i, x_i) \to 0 \text{ as } i \to -\infty \}.
\end{align*}

Note that if $n \geq 0$, $y \in L$ and $f^n y = f^n x$, then $W^s(x)$ contains $W^s_{\text{loc}}(y)$. Hence the global stable manifolds are in general large and quite complicated objects (compare with Corollary 8.5). Both the stable and unstable manifolds may have singularities; this is in contrast to the case of polynomial automorphisms of $\mathbb{C}^2$, where they are immersed copies of $\mathbb{C}$ [BS1].

We now turn to Axiom A regular polynomial endomorphisms of $\mathbb{C}^2$. A point $x \in \mathbb{P}^2$ is wandering if for every neighborhood $V$ of $x$ there exists an $n \geq 1$ such that $f^n(V) \cap V \neq \emptyset$. The non-wandering set $\Omega$ of $f$ is the set of all non-wandering points; it is a compact set. A regular polynomial endomorphism $f$ of $\mathbb{C}^2$ is Axiom A if the periodic points of $f$ are dense in $\Omega$ and $f$ is hyperbolic on $\Omega$. If $f$ is Axiom A, then Smale’s spectral decomposition
theorem (Theorem B.9) asserts that $\Omega$ can be written in a unique way as a finite union of disjoint compact invariant sets $\Omega_{j}$, called basic sets, such that $f|_{\Omega_{j}}$ is transitive, i.e. has a dense orbit. Thus each basic set has a well-defined unstable index.

Let us investigate what the possible basic sets are for an Axiom A regular polynomial endomorphism $f$ of $\mathbb{C}^{2}$. To do this, we first observe that the four sets $\Pi$, $\mathbb{C}^{2} - K$, $\text{int}(K)$ and $\partial K$ are all completely invariant and see what basic sets each one of them may contain.

To begin with, it is clear that $\Omega(f) \cap \Pi = \Omega(f_{\Pi})$. Now $f_{\Pi}$ is a rational map and from one-dimensional dynamics we know that $f_{\Pi}$ is Axiom A if and only if $f_{\Pi}$ is uniformly expanding on $J_{\Pi}$ (see [M]). Hence, if $f$ is Axiom A, then the basic sets in $\Pi$ are $J_{\Pi}$, which is of unstable index $1$, and a finite union of attracting periodic points, all of whose unstable index is zero.

All the points in the open set $\mathbb{C}^{2} - K$ are attracted to $\Pi$ so $(\mathbb{C}^{2} - K) \cap \Omega$ is empty. It is clear that $\{f^{n}\}$ is normal on the interior of $K$, so if $f$ is Axiom A, then the only basic sets in $\text{int}(K)$ are attracting periodic points, all of whose unstable index are zero.

The boundary of $K$ contains the most complicated dynamics. Clearly, no basic sets in $\partial K$ can have unstable index 0. Let $S_{2}$ and $S_{1}$ be the union of the basic sets in $\partial K$ of index 2 and 1, respectively. We note that $S_{1}$ can be empty, as in the example $f(z, w) = (z^{2} + c, w^{2} + c)$, with $c$ outside the Mandelbrot set. On the other hand, $J$ is a basic set of unstable index 2 (see [FS2, Theorem 7.4]), so $J \subset S_{2}$. The question arises whether this inclusion is ever strict or, equivalently, whether $f$ can have repelling periodic points outside $J$. Hubbard and Papadopol [HP] have in fact given an example of a regular polynomial endomorphism of $\mathbb{C}^{2}$ with a repelling periodic point outside $J$ but is seems difficult to check whether their map can be made Axiom A. In any case we have the following.

**Lemma 9.1.** Let $f$ be an Axiom A regular polynomial endomorphism of $\mathbb{C}^{2}$. Then $f^{-1}S_{2} = S_{2}$ if and only if $S_{2} = J$, i.e. if all repelling periodic points are contained in $J$.

**Remark.** A proof is given in [FS4]. We give it here for the convenience of the reader.

**Proof.** The “only if” part is trivial since $f^{-1}(J) = J$, so suppose that $f$ is Axiom A and $f^{-1}(S_{2}) = S_{2}$ but $S_{2} \neq J$. Let $N$ be an open neighborhood of $J$ such that $f^{-1}(N) \subset N$ and $\bigcap_{n \geq 0} f^{-n}(N) = J$. Then $N - J$ contains only wandering points, so $S_{2} - J$ is at a positive distance from $J$ and is therefore a completely invariant compact set. Let $N'$ be an open neighborhood of $S_{2} - J$ disjoint from $J$ with $f^{-1}(N') \subset N'$. Then $N'$ has positive capacity and if $x \in N'$ then $(f^{n})^{*}\delta_{x}/d^{2n}$ cannot converge to $\mu$ as $n \to \infty$. This contradicts Lemma 8.3 in [FS3].

Let $f$ be an Axiom A regular polynomial endomorphism of $\mathbb{C}^{2}$ with $f^{-1}S_{2} = S_{2}$. It follows from Corollary B.10 and the above discussion that any history of a point $\mathbb{C}^{2}$ which is not an attracting periodic point must converge to either $J$ or $S_{1}$. We define the unstable set of $J$ to be the set of points in $\mathbb{C}^{2}$ all of whose histories converge to $J$, i.e.

$$W^{u}(J) = \{x \in \mathbb{C}^{2} : (\dot{x} \in \mathbb{C}^{2}, \pi(\dot{x}) = x) \Rightarrow x_{i} \to J \text{ as } i \to -\infty\}.$$ 

We note that this definition differs from the one in [FS4], where $W^{u}(J)$ is defined as the set of points having at least one history converging to $J$. On the other hand we define the unstable set of $S_{1}$ as

$$W^{u}(S_{1}) = \{x \in \mathbb{C}^{2} : \exists \dot{x} \in \mathbb{C}^{2}, \pi(\dot{x}) = x, x_{i} \to S_{1} \text{ as } i \to -\infty\}.$$
Let $N$ be a neighborhood of $J$ in $C^2$ as in the proof of Lemma 9.1. Clearly $N \subset W^u(J)$ and every point in $C^2$ which is not an attracting periodic point is contained in precisely one of the sets $W^u(J)$ and $W^u(S_1)$.

**Lemma 9.2.** If $x \in W^u(J)$, then there exists an $n \geq 0$ such that $f^{-n}(x) \subset N$. In particular, $W^u(J)$ is open in $C^2$ and $W^u(S_1)$ is closed in $C^2$ except possibly at some of the attracting periodic points.

**Proof.** Let $Z$ be the set of points $y$ in $C^2$ such that for all $n \geq 0$, there is a point in $f^{-n}(y)$ outside $N$. It is clear that if $y \in Z$, then $y$ has at least one preimage in $Z$, so every point $y \in Z$ has a whole history inside $Z$. Such a history cannot converge to $J$ so it follows that $Z \cap W^u(J) = \emptyset$, which completes the proof.

For the proof of the main result in §10 (Theorem 10.2), we will work with slightly weaker hyperbolicity hypotheses.

**Definition 9.3.** A regular polynomial endomorphism $f$ of $C^2$ satisfies condition $(†)$ if the following four properties hold:

1. $f_{\Pi}$ is uniformly expanding on $J_{\Pi}$.
2. $f$ is uniformly expanding on $J$.
3. The nonwandering set of $f$ in $\partial K$ consists of $J$ and a hyperbolic set $S_1$ of unstable index 1.
4. $W^u(S_1) = \bigcup_{\hat{x} \in \hat{S}_1} W^u(\hat{x})$.

**Proposition 9.4.** Let $f$ be an Axiom A regular polynomial endomorphism of $C^2$ with $f^{-1}S_2 = S_2$. Then $f$ satisfies condition $(†)$.

**Proof.** From the above discussion we know that $f$ satisfies conditions $(†1)$, $(†2)$ and $(†3)$, and $(†4)$ follows from Corollary B.10.

**§10 Continuous landing of rays in $C^2$.**

So far we have been able to understand the dynamics in the set $W^*(J_{\Pi})$, or at least on the support of $T^{k-1} \sqcup A$, for rather general $f$. In this Section, we approach the dynamics of $f$ on $J$ by proving Theorem 10.2, which shows that $e : \mathcal{E} \to J$ is a continuous surjection (under suitable assumptions). Our approach is restricted to the case $k = 2$ because we work with unstable manifolds $W^u(\bar{q})$ as Riemann surfaces; if $k > 2$, the unstable manifolds can have dimension $> 1$. If $k = 2$, then $\Pi$ is the Riemann sphere, and $f_{\Pi}$ is a rational mapping. Up to §8, the only hyperbolicity assumption that we have been concerned with has been uniform expansion on $J_{\Pi}$, which for $k = 2$ means that $f_{\Pi}$ is a hyperbolic rational mapping. The reason for this is that we have dealt with the dynamics on $A$ and not directly on $K$. For the continuity of $e : \mathcal{E} \to J$ we need to consider the dynamics on $K$ (or, rather, $\partial K$). Notice that hyperbolicity of $f_{\Pi}$ does not exclude complicated dynamics on $K$.

To motivate $(†5)$ in Definition 10.1 below, let us revisit the example presented after Theorem 7.3, namely $f(z_1, z_2) = (z_1^2 + c_1, z_2^2 + c_2)$. We have $J = J_{c_1} \times J_{c_2}$, where $J_c$ is the one-dimensional Julia set of $p_c(z) = z^2 + c$. Further, $\mathcal{E} \simeq S^1 \times S^1$, so $\mathcal{E}$ is connected and locally connected. Thus, if $e$ maps $\mathcal{E}$ continuously onto $J$, then $J$ also is connected and
locally connected. This, in turn, is equivalent to $J_c$ being connected and locally connected for $j = 1, 2$. Now $J_c$ is connected if and only if the critical point 0 of $p_c$ is not in the basin of attraction of infinity, i.e. the parameter value $c$ is in the Mandelbrot set. Using this one can see that $J$ is connected if and only if $W^s(J_{II}) \cap C = \emptyset$. The question of whether $J_c$ is locally connected is more delicate, but a sufficient condition is that $J_c$ is connected and $p_c$ is uniformly expanding on $J_c$. Thus $J$ is locally connected if $J$ is connected and $f$ is uniformly expanding on $J$.

**Definition 10.1.** We say that a regular polynomial endomorphism $f$ of $C^2$ satisfies condition $(\dagger)$ if $f$ satisfies condition $(\ddagger)$ in Definition 9.3, and $W^s(J_{II}) \cap C = \emptyset$, i.e. if the following five properties hold:

$(\dagger 1)$ $f_{II}$ is uniformly expanding on $J_{II}$.

$(\dagger 2)$ $f$ is uniformly expanding on $J$.

$(\dagger 3)$ The nonwandering set of $f$ in $\partial K$ consists of $J$ and a (possibly empty) hyperbolic set $S_1$ of unstable index 1.

$(\dagger 4)$ $W^u(S_1) = \bigcup_{\hat{x} \in \hat{S}_1} W^u(\hat{x})$.

$(\dagger 5)$ $W^s(J_{II}) \cap C = \emptyset$ (or, equivalently, $\mu_c(A) = 0$).

Let us comment on these conditions. It follows from Proposition 9.4 that if $f$ is Axiom A, $f^{-1}(S_2) = S_2$, and satisfies $(\ddagger 5)$, then $f$ satisfies $(\dagger)$. Using this, one can show that perturbations of the map $f(z, w) = (z^d, w^d)$ satisfy $(\dagger)$.

Conditions $(\dagger 1)$ and $(\dagger 5)$ guarantee that $e_r : E \to \{G = r\}$ is well defined and continuous for $r > 0$ (in general it is defined almost everywhere on $E$). We have $e = \lim e_r$ as $r \to 0$, and $e_r(E) = f^{-1}e_{dr}(E)$. Thus, once we know that $e_r(E)$ is in a small neighborhood of $J$ for all sufficiently small $r$, then condition $(\dagger 2)$ helps us to show that $e_r$ converges uniformly as $r \to 0$. However, we only know that $e_r$ accumulates on $\partial K$, which in general is a larger set than $J$. In fact, the main difficulty in proving continuity for $e$ is to show that $e_r(E)$ accumulates only at $J$. In order to do this, we use properties $(\dagger 3)$ and $(\dagger 4)$.

Let us make some remarks about the connection between the endpoint map $e$ and the conjugacy $\Psi : W^s(J_{II}, f_h) \to W^s(J_{II}, f)$ between $f_h$ to $f$ given in Theorem 7.4. Let $\Delta_a := L_a \cap A_h$ be the disk in $W^s(J_{II}, f_h)$ corresponding to $a$. We may identify the set $E_a$ of external rays in $W_a$ with $\partial \Delta_a$ and $E$ with the boundary of $W^s(J_{II}, f_h)$, i.e. the union of $E_a$ over $a \in J_{II}$. This defines the topology on $E$. With these identifications we have $e_r(\gamma) = \Psi(e^r\gamma)$ for $r > 0$. It follows that $e_r$ is well defined and continuous for all $r > 0$. Thus $\Psi$ extends continuously to $E$ if and only if $e$ is continuous, and in this case $e$ coincides with the restriction of $\Psi$ to $E$. The selfmap $f_h$ on $W^s(J_{II}, f_h)$ induces the selfmap $\sigma$ of $E$.

Our main goal is to prove the following result.

**Theorem 10.2.** If the regular polynomial endomorphism $f$ of $C^2$ satisfies condition $(\ddagger)$, then $e$ maps $E$ Hölder continuously onto $J$ and $f \circ e = e \circ \sigma$. In particular, the conclusions hold if $f$ is Axiom A, if $W^s(J_{II}) \cap C = \emptyset$, and if the expanding part $S_2$ of the nonwandering set of $f$ satisfies $f^{-1}(S_2) = S_2$.

As mentioned above, the main difficulty in proving Theorem 10.2 is to show that the external rays accumulate only at $J$. In particular, there must be no heteroclinic intersection between $S_1$ and $J_{II}$, i.e. no complete orbit $(x_i)_{i \in \mathbb{Z}}$ such that $x_i \to S_1$ as $i \to -\infty$ and $x_i \to J_{II}$ as $i \to \infty$. 
Lemma 10.3. If $f$ satisfies condition $(\ddagger)$, then $W^s(J_{\Pi}) \cap W^u(S_1) = \emptyset$.

We postpone the proof of Lemma 10.3 for the moment and head towards the proof of Theorem 10.2.

We may identify the disk $\Delta_a$ in $W^s(J_{\Pi}, f_h)$ with $\Delta = \{ |\zeta| > 1 \}$ in such a way that $G_\nu(\zeta) = \log |\zeta|$. The restriction of $\Psi$ to $\Delta_a$ then induces a conformal equivalence $\psi_a : \Delta \rightarrow W_a$ such that $G \circ \psi_a(\zeta) = \log |\zeta|$. This last condition defines $\psi_a$ uniquely up to rotation; the precise choice of rotation will not be important in what follows. Given a choice of $\psi_a$, the conjugacy between $f_h$ and $f$ translates into

$$f \circ \psi_a(\zeta) = \psi_{f_{\Pi a}}(\nu_a \zeta^d), \quad (10.1)$$

where $|\nu_a| = 1$.

We first show that the maps $\psi_a$ are uniformly Hölder continuous.

Lemma 10.4. There exist constants $\alpha > 0$ and $C > 0$ such that

$$d(\psi_a(\zeta), \psi_a(\zeta')) \leq Cd(\zeta, \zeta')^\alpha, \quad (10.2)$$

for all $a \in J_{\Pi}$ and $\zeta, \zeta' \in \Delta$.

Proof. The expansion of $f$ on $J$ implies that there exists a neighborhood $N$ of $J$ with $f^{-1}(N) \subset N$, $\lambda > 1$ and a metric equivalent to the Euclidean metric such that $|Df(x)v| \geq \lambda |v|$ for all $x \in N$ and all $v \in T_x \mathbb{C}^2$ with respect to this metric. By Lemma 9.2 and Lemma 10.3 we know that the set $W^s(J_{\Pi}) \cap \{ 1 \leq G \leq d \}$ is a compact subset of the open set $W^u(J)$ so by pulling back by $f$ we see that there exists an $R > 1$ such that $\psi_a \{ 1 < |\zeta| \leq R \} \subset N$ for all $a$. Let $\alpha > 0$ be so small that $d^\alpha < \lambda$ and assume that $R$ is so small that $R^{d-1}d^\alpha < \lambda$. It is sufficient to prove (10.2) for $1 < |\zeta|, |\zeta'| \leq R$.

By differentiating (10.1) and using the estimates above we get that, for $1 < |\zeta| < R^{d-1}$,

$$|D\psi_a(\zeta)| \leq \lambda^{-1} |D\psi_{f_{\Pi a}}(\nu_a \zeta^d)| d|\zeta|^{d-1}. \quad (10.3)$$

Define

$$m(r) = \sup_{a \in J_{\Pi}} \sup_{|\zeta| = r} |D\psi_a(\zeta)|,$$

for $1 < r \leq R$. Then there exists a constant $C' < \infty$ such that

$$m(r) \leq C'(r - 1)^{\alpha - 1}, \quad (10.4)$$

for $R^{d-1} \leq r \leq R$. Using the estimate (10.3) we prove inductively that (10.4) holds for $1 < r \leq R$. Thus (10.2) follows by integrating (10.4).

Proof of Theorem 10.2. We know that $e_r$ is continuous for each $r > 0$. Lemma 10.4 shows that $e_r$ converges uniformly on $E_a$ for each $a$. Thus $e_r$ converges uniformly to $e$, so $e$ is continuous and it follows from Theorem 7.3 that $e(\mathcal{E}) = J$. We have $f \circ e_r = e_d \circ \sigma$, so be letting $r \to 0$ we get $f \circ e = e \circ \sigma$. It remains to be seen that $e$ is Hölder continuous.
Let $N$ and $\lambda$ be as in the proof of Lemma 10.4 and let $\epsilon > 0$ be small. We may assume that $d(f x, f y) \geq \lambda d(x, y)$ for $x, y \in N$, $d(x, y) \leq \epsilon$, and that $d(\sigma \gamma, \sigma \gamma') \geq \lambda d(\gamma, \gamma')$ for $\gamma, \gamma' \in E$, $d(\gamma, \gamma') \leq \epsilon$. There is a number $M$ so that $d(\sigma \gamma, \sigma \gamma') \leq M d(\gamma, \gamma')$ for all $\gamma, \gamma' \in E$. Pick $\alpha > 0$ such that $M \alpha < \lambda$. Let $C > 0$ be so large that $d(e_r(\gamma), e_r(\gamma')) \leq C d(\gamma, \gamma')^{\alpha}$ if $d(\gamma, \gamma') \geq \epsilon$ and $r > 0$.

Since $e(E) = J$, there exists an $R > 0$ such that $e_r(E) \subset N$ for $0 < r \leq R$. Now suppose $r \leq R d^{-j}$ for some $j \geq 0$ and that $d(\gamma, \gamma') \geq \epsilon/\lambda^j$. Then

$$d(e_r(\gamma), e_r(\gamma')) \leq \lambda^{-j} d(e_{d^j}, \sigma^j \gamma, e_{d^j}, \sigma^j \gamma') \leq \lambda^{-j} C d(\sigma^j \gamma, \sigma^j \gamma')^{\alpha} \leq \lambda^{-j} M^{\alpha j} C d(\gamma, \gamma')^{\alpha} \leq C d(\gamma, \gamma')^{\alpha}$$

The theorem thus follows by letting $r \to 0$. 

We now turn to the proof of Lemma 10.3 and proceed in a number of steps. An intersection between $W_a$ and $W^s(J \Pi)$ is a heteroclinic connection. There are two types, as shown in Figure 5. That is, the intersection between $W_a$ and $W^u(S_1)$ can be either a discrete set or a relatively open set. The next Lemma says that the discrete intersection does not occur.

**Lemma 10.5.** Let $W_a$ be the stable disk of a point $a \in J \Pi$. Then either $W_a \cap W^u(S_1) = \emptyset$ or there exists a point $\hat{x} \in \hat{S}_1$ such that $W_a^* \subset W^u(\hat{x})$, where $W_a^* = W_a - \{a\}$.

The key observation in proving the dichotomy is the following.

**Lemma 10.6.** If $U$ is a simply connected open subset of a punctured stable disk $W_a^*$, then all branches of $f^{-i}|_U$ for all $i > 0$ are well-defined and holomorphic on $U$ and they form a normal family there.

**Proof.** That the branches are well-defined follows from condition (‡5). If $V$ is relatively compact in $U$ then all branches of $f^{-i}$ on $V$ map $V$ into a fixed compact subset of $C^2$. Thus they form a normal family on $U$. 

\[\square\]
Proof of Lemma 10.5. Suppose that $y \in W_a \cap W^u(S_1)$. Then by condition (4) there exists a point $\hat{x} \in \hat{S}_1$ such that $y \in W^u(\hat{x})$, i.e. $y$ has a history $\hat{y}$ such that $d(y_i, x_i) \to 0$ as $i \to -\infty$. Let $U$ be any simply connected open subset of $W^u_a$ containing $y$ and let $g_i$ be the unique sequence of branches of $f^{-i}_{\mid U}$ such that $g_i(y) = y_i$. Then $\{g_i\}$ is equicontinuous by Lemma 10.6, so there is a small neighborhood $V$ of $y$ in $U$ such that the maximal distance from $g_i(V)$ to $x_i$ is uniformly small as $i \to \infty$. Hence $V \subset W^u(\hat{x})$ and, by normality of $\{g_i\}$, $U \subset W^u(\hat{x})$. Since $U$ was arbitrary it follows that $W^u_a \subset W^u(\hat{x})$.

Corollary 10.7. Let $J_{\Pi}^\prime$ be the set $a \in J_{\Pi}$ such that $W^u_a \subset W^u(S_1)$. Then $J_{\Pi}^\prime$ is closed, $f_{\Pi}(J_{\Pi}^\prime) = J_{\Pi}^\prime$ and $J_{\Pi}^\prime \neq J_{\Pi}$.

Proof. If $a \notin J_{\Pi}^\prime$, then $W^u_a \cap W^u(S_1) = \emptyset$ by Lemma 10.5. Hence $W_a \cap \{G = 1\}$ is a compact subset of the open set $W^u(J)$ so by continuity there is an open neighborhood $X$ of $a$ in $J_{\Pi}$ such that $W_b \cap \{G = 1\} \subset W^u(J)$ for all $b \in X$. By Lemma 10.5 it follows that $X \cap J_{\Pi}^\prime = \emptyset$ and we conclude that $J_{\Pi} \neq J_{\Pi}^\prime$ is open. That $f_{\Pi}(J_{\Pi}^\prime) = J_{\Pi}^\prime$ follows from the fact that $f(W^u(S_1)) = W^u(S_1)$.

Finally suppose $J_{\Pi}^\prime = J_{\Pi}$. Then $W^u(J_{\Pi}) \subset J_{\Pi} \cup W^u(S_1)$, so $W^u(J_{\Pi})$ does not intersect $W^u(J)$. This contradicts Theorem 7.3, because $W^u(J)$ contains a neighborhood of $J = \text{supp}(\mu)$.

We say that a stable disk $W_b$ lands on $J$ if $\psi_b$ extends continuously to $S^1$ and $\psi_b(S^1) \subset J$. This does not depend on the specific choice of parametrization $\psi_b$.

Lemma 10.8. There exists a dense set of $b \in J_{\Pi}$ such that $W_b$ lands on $J$.

Proof. Since periodic points are dense in $J_{\Pi}$ and $J_{\Pi} - J_{\Pi}^\prime$ is open and nonempty, we can find a periodic point $b' \in J_{\Pi} - J_{\Pi}^\prime$, say of period $n$. Furthermore, $f$ is expanding on $J$, so there exists a neighborhood $N$ of $J$ and $\lambda > 1$ with $f^{-1}(N) \subset N$ and

$$|Df^n(y)v| \geq \lambda^n|v|, \quad (10.5)$$

for all $y \in N$ and all tangent vectors $v$ (we may have to increase $n$). Now the annulus $\psi_{b'}(\{2^{1/d^n} \leq |\zeta| \leq 2\})$ in $W_{b'}$ is a compact subset of $W^u(J)$, so the inverse images under sufficiently high iterates of $f$ points in this annulus will be in $N$. In particular, since $b'$ is periodic, it follows that there exists an $R > 1$ such that $\psi_{b'}(\{1 < |\zeta| \leq R\}) \subset N$. Then, using the estimate (10.5) above, we may prove that $\psi_{b'}$ extends to a Hölder continuous map of $\Delta$, mapping $S^1$ into $J$. The proof is very similar to the proof of Lemma 10.4 and is therefore omitted.

We conclude that $W_{b'}$ lands on $J$ and so does $W_b$ for all preimages $b$ of $b'$ under iterates of $f$. Such preimages are dense in $J_{\Pi}$.

Figure 6 illustrates the effect of a heteroclinic connection. Here $W^u_a$ is in the unstable set of $S_1$ whereas $W_b$ lands on $J$. The stable disks in the middle are of the form $W_{b_n}$, where $b_n$ are preimages of $b$ converging to $a$. Note that the disks $W_{b_n}$ are very “bent” for large $n$. If the $W_{b_n}$’s were graphs over $W_a$, then this would contradict the maximum principle. We cannot show directly that $W_{b_n}$ are graphs over $W_a$, but we will nevertheless prove, using the maximum principle, that the picture is impossible.
It follows from Lemma 10.5 that for each \( a \in J_{\Pi}' \) there exists a (not necessarily unique) history \( \hat{p}_a \) in \( S_1 \) such that \( W^*_a \subset W^u(\hat{p}_a) \). In general, an unstable manifold \( W^u(\hat{q}) \) of a history \( \hat{q} \) in \( S_1 \) is a complicated object, but, as we will see, the information that \( W^*_a \subset W^u(\hat{p}_a) \) implies that \( W^u(\hat{p}_a) \) is in fact an algebraic subvariety of \( \mathbb{C}^2 \). Recall that the image of a holomorphic map of a compact Riemann surface into \( \mathbb{P}^2 \) is an algebraic variety. The authors thank Jeffrey Diller for useful conversations on the proof of the following result.

**Lemma 10.9.** If \( J_{\Pi}' \neq \emptyset \), then there exists an \( a \in J_{\Pi}' \) such that \( W^u(\hat{p}_a) \) is an algebraic subvariety of \( \mathbb{C}^2 \).

**Proof.** Take any point \( a \in J_{\Pi}' \), a complete orbit \( (a_i)_{i \in \mathbb{Z}} \) with \( a_0 = a \) and a complete orbit \( (p_i)_{i \in \mathbb{Z}} \) be a complete orbit in \( S_1 \) such that \( W^*_a \subset W^u((p_{i+j})_{j \leq 0}) \) for all \( i \). We write \( \hat{p}_i \) for the history \( (p_{i+j})_{j \leq 0} \). If \( \delta > 0 \) is small enough, then the local unstable manifolds \( W^u_{\text{loc}}(\hat{p}_i) \) are complex disks for all \( i \) and there exist biholomorphisms \( \eta_i : D_{\delta} \rightarrow W^u_{\text{loc}}(\hat{p}_i) \) with \( |D\eta_i(0)| = 1 \) and complex numbers \( \lambda_i \neq 0 \) such that

\[
\eta_i(\lambda_{i-1}\zeta) = f(\eta_{i-1}(\zeta)), \tag{10.6}
\]

for all \( i \) and all \( |\zeta| < \delta_{i-1} \). Since \( f \) is hyperbolic on \( S_1 \), the numbers \( \delta_i \) are uniformly bounded from below and \( \lambda_{i-n} \cdots \lambda_{i-1} \rightarrow \infty \) as \( n \rightarrow \infty \) for all \( i \), so (10.6) allows us to extend \( \eta_i \) to maps of \( \mathbb{C} \) into \( W^u(\hat{p}_i) \) by defining

\[
\eta_i(\lambda_{i-n} \cdots \lambda_{i-1}\zeta) = f^n(\eta_{i-n}(\zeta)),
\]

for \( n \geq 0 \).

The maps \( \eta_i \) are surjective by the definition of \( W^u(\hat{p}_i) \) but they need not be injective. However, the global unstable manifolds \( W^u(\hat{p}_i) \) have a natural structure as abstract Riemann surfaces given by the maps \( \eta_i \). More precisely, for each \( i \) we define a Riemann surface \( X_i \) as the quotient \( \mathbb{C}/\sim \), where \( z \sim w \) if there are open sets \( U \ni z \) and \( V \ni w \) such that \( \eta_i(U) = \eta_i(V) \). Then the map \( \eta_i \) factors as \( \eta_i = \eta_i' \circ \pi_i \), where \( \pi_i : \mathbb{C} \rightarrow X_i \) is
surjective, $\eta_i : X_i \to \mathbb{C}^2$ is locally injective and the set of points $(z, w) \in X_i \times X_i$ with $z \neq w$ and $\eta_i'(z) = \eta_i'(w)$ is discrete. We will be sloppy and make no distinction between the unstable manifold $W^u(\tilde{p}_i)$ and the Riemann surface $X_i$. Hence we will sometimes view $W^u(\tilde{p}_i)$ as a subset of $\mathbb{C}^2$ and sometimes as an abstract Riemann surface. The precise meaning should be clear from the context.

Now the Riemann surface $W^u(\tilde{p}_i)$ cannot be hyperbolic, because $\eta_i$ maps $\mathbb{C}$ into it so $W^u(\tilde{p}_i)$ is biholomorphic to $\mathbb{C}^*$, $\mathbb{C}$, $\mathbb{P}^1$ or a torus. The last two cases cannot occur, because then $W^u(\tilde{p}_i)$ would be an algebraic subvariety of $\mathbb{P}^2$, which is impossible (because $W^u(\tilde{p}_i) \cap \Pi = \emptyset$). Hence $W^u(\tilde{p}_i)$ is biholomorphic to $\mathbb{C}^*$ or $\mathbb{C}$ for all $i$.

Write $W_i$ instead of $W_{a_i}$ and note that $W^u(\tilde{p}_i)$ has an open subset biholomorphic to $W_i^*$. Let $\Sigma_i$ be the Riemann surface obtained from $W^u(\tilde{p}_i)$ by filling in the hole at $a_i$. Then $\Sigma_i$ is biholomorphic to $\mathbb{C}$ or $\mathbb{P}^1$ for all $i$. If $\Sigma_i$ is biholomorphic to $\mathbb{P}^1$ for some $i$, then $\Sigma_i$ is an algebraic subvariety of $\mathbb{P}^2$ (in fact a line) and we are done, so assume that $\Sigma_i$ is biholomorphic to $\mathbb{C}$ for all $i$.

Suppose that $(\Sigma_i - W_i) \cap W^s(\Pi) \neq \emptyset$ for some $i$. Then $(\Sigma_i - W_i) \cap W_b \neq \emptyset$ for some $b \in J_{\Pi}, b \neq a_i$. By the dichotomy given in Lemma 10.5 we then have that $W_b^* \subset (\Sigma_i - W_i)$ so by filling in the hole at $b$ we see that the closure of $\Sigma_i$ in $\mathbb{P}^2$ is an algebraic subvariety of $\mathbb{P}^2$, which implies that $W^u(\tilde{p}_i)$ is algebraic in this case too.

Let us now suppose that $\Sigma_i$ is biholomorphic to $\mathbb{C}$ and that $(\Sigma_i - W_i) \cap W^s(\Pi) = \emptyset$ for all $i$. Pick biholomorphisms $\chi_i : \mathbb{C} \to \Sigma_i$ such that $\chi_i(0) = a_i$. Note that $f$ induces holomorphic maps of $\Sigma_i$ onto $\Sigma_{i+1}$. Hence we may define entire maps $h_i$ by $\chi_i \circ h_i = f \circ \chi_{i-1}$ for all $i$. The restriction of $f$ to $W_{i-1}$ is a branched covering of $W_i$ of degree $d$, branched only at $a_i$. This implies that $h_i(\zeta) = \zeta^d \exp(u_i(\zeta))$ where $u_i$ is entire. Moreover, the condition $(\Sigma_i - W_i) \cap W^s(\Pi) = \emptyset$ implies that the inverse image of $W_i$ in $\Sigma_{i-1}$ is exactly $W_{i-1}$. Therefore $\lim \sup |h_i(\zeta)| > 0$ as $|\zeta| \to \infty$ and this is only possible if $u_i$ is constant. Hence we may write $h_i(\zeta) = c_i \zeta^d$ for some constants $c_i \neq 0$.

Define $g_i = G \circ \chi_i$. Then, for each $i$, $g_i \geq 0$ is continuous and subharmonic on $\mathbb{C}^*$. The equation $G \circ f = dG$ translates into $g_i \circ h_i = d g_{i-1}$, i.e. $g_i(c_i \zeta^d) = d g_{i-1}(\zeta)$. Iterating this we see that $g_i(\zeta)$ depends only on $|\zeta|$. Thus, by the maximum principle, for each $i$ there exists an $R_i$ such that either $g_i = 0$ on $|\zeta| > R_i$ or $g_i > 0$ for $|\zeta| > R_i$.

If $g_i = 0$ for $|\zeta| > R_i$, then $\chi_i$ maps $|\zeta| > R_i$ into the bounded set $K$ and therefore extends to a holomorphic map of $\mathbb{P}^1$ into $\mathbb{P}^2$. Hence $W^u(\tilde{p}_i)$ is algebraic.

If $g_i > 0$ for $|\zeta| > R_i$, then $\chi_i$ maps $|\zeta| > R_i$ into $\mathbb{C}^2 - K$, and by our previous assumption, the image does not intersect $W^s(\Pi) = \text{supp}(T \mathbf{L} \mathbf{A})$, so $g_i$ is harmonic on $|\zeta| > R_i$. Hence there exist constants $A_i > 0$ and $B_i$ such that $g_i(\zeta) = A_i \log |\zeta| + B_i$ for $|\zeta| > R_i$. Since $G(x) = \log |x| + O(1)$ as $x \to \Pi$, this implies that $|\chi_i(\zeta)| \leq C |\zeta|^{A_i}$ as $\zeta \to \infty$, so again $\chi_i$ extends to a holomorphic map of $\mathbb{P}^1$ into $\mathbb{P}^2$. Hence $W^u(\tilde{p}_i)$ is algebraic, which completes the proof of Lemma 10.9.

We are now in position to prove Lemma 10.3.

**Proof of Lemma 10.3.** Suppose that $W^s(\Pi) \cap W^u(S_1) \neq \emptyset$. Then $J'^{\prime}_1 \neq \emptyset$ so Lemma 10.9 shows that there exist $a \in J_{\Pi}$, a history $\tilde{p}$ in $S_1$ and an irreducible polynomial $P(z, w)$ such that $W^{a}_a \subset W^u(\tilde{p}) = \{ P = 0 \}$. Clearly $W^u(\tilde{p}) \cap J = \emptyset$ so there exists an $\epsilon > 0$ such that $|P| \geq 2\epsilon$ on $J$. 


By Lemma 10.8 there is a dense set of b’s such that \( W_b \) lands on \( J \). If we choose \( b \) close enough to \( a \), then by continuity \( W_b \) will intersect the open set \( |P| < \epsilon \). Let \( U \) be a component of \( \{ \zeta \in \Delta^* : |P(\psi_b(\zeta))| < \epsilon \} \). Then \( U \) is relatively compact in \( \Delta^* \). Further, \( P \) is a nonzero holomorphic function on \( U \), so \( -\log|P| \) is harmonic on \( U \). But \( |P| < \epsilon \) on \( U \) and \( |P| = \epsilon \) on \( \partial U \), contradicting the maximum principle for \( -\log|P| \) on \( U \). This completes the proof of Lemma 10.3.

\[ \square \]

**Corollary 10.10.** If \( f \) satisfies condition (\( \dagger \)) and \( J_\Pi \) is connected, then \( J \) is connected. If \( J_\Pi \) is also locally connected, then so is \( J \).

\[ \square \]

**Proof.** If \( J_\Pi \) is connected (and locally connected) then \( E \) is connected (and locally connected).

**Remark.** In order to prove that \( e \) maps \( E \) continuously onto \( J \), we could do without the assumption that \( f_\Pi \) is expanding on \( J_\Pi \). It would be sufficient to assume that \( \mu_e(A) = 0 \), and that (\( \ddagger \))–(\( \dagger \)) hold. Indeed, by Theorem 8.8 we still have a disk laminaton \( \{ W_a : a \in J_\Pi \} \) and \( e_r \) is defined and continuous on \( E \) for all \( r > 0 \). The only place where the uniform expansion of \( f_\Pi \) on \( J_\Pi \) is used, is to get H"older continuity of \( e \). However, we always get continuity of \( e \).

**Questions.** Is \( J \) a finite quotient of \( E \), i.e. is \( \# e^{-1}(x) \) uniformly bounded on \( J \)? What are the possible identifications on \( E \) introduced by \( e \)?

**§Appendix A. The homogeneous model**

The model for our study of regular polynomial automorphisms is the case when \( f = f_h \) is a homogeneous mapping of \( \mathbf{C}^k \). Here we show that \( f_h \) is essentially a skew product over \( f_\Pi \). If \( g \) is a homogeneous polynomial of degree \( N \), we let \( V = \mathbf{C}^k \cap \{ g = 0 \} \) and \( V_\Pi = \Pi \cap \{ g = 0 \} \). We let \( \Sigma_N \) denote the multiplicative group of the \( N \)th roots of unity in \( \mathbf{C} \). We let \( \mathbf{C}_* = \mathbf{C}_*/\Sigma_N \) denote the quotient. With this complex structure, the mapping \( \mathbf{C}_* \to \mathbf{C}_* \) given by \( \lambda \mapsto \lambda^N \) descends to an isomorphism \( \mathbf{C}_* \to \mathbf{C}_* \). Similarly, we define the (finite) quotient \( \mathbf{C}_*^k = \mathbf{C}_*/\Sigma_N \). Thus we have a holomorphic mapping

\[ s : \Pi - V_\Pi \to \mathbf{C}_*^k \]

given by \( s(z) = g^{-1/N}(z)z \). It follows that the mapping

\[ \psi : \mathbf{C}_* \times (\Pi - V_\Pi) \to \mathbf{C}_*^k - V \]

given by \( \psi(\lambda, [z]) = \lambda^{1/N}s(z) \) is biholomorphic. The homogeneous mapping descends to a finite quotient mapping

\[ \tilde{f}_h : \mathbf{C}_*^k - (V \cup f_h^{-1}V) \to \mathbf{C}_*^k. \]

Since \( s \) is a section of the bundle \( \pi : \mathbf{C}_*^k \to \Pi \), it follows that \( s \circ f_\Pi(z) \) and \( \tilde{f}_h \circ s[z] \) define the same line in \( \mathbf{C}_*^k \). Thus \( \chi(z) := f_h(s(z))/s(f_\Pi(z)) \) is an \( N \) valued holomorphic function on \( \Pi - V_\Pi \). A short calculation shows that

\[ \psi^{-1} \circ \tilde{f}_h \circ \psi(\lambda, z) = (\chi^N(z)\lambda^d, f_\Pi(z)) \quad (A.1) \]
where $\chi^N$ is a (single-valued) analytic function on $\Pi - V_{1,1}$.

We note that for $a \in \Pi$, $W^s_\text{loc}(a, f_h)$ is contained in the line $L_a$. Let $C(J_{1,1})$ denote the union of $J_{1,1}$ and the complex homogeneous cone in $\mathbb{C}^k_*$ over $J_{1,1}$. Our canonical model in §4 is given by the restriction of $f_h$ to $C(J_{1,1})$. Let $\hat{C}(J_{1,1})$ denote the quotient of $C(J_{1,1})$ in $\mathbb{C}^k_* \cup \Pi$.

**Proposition A.1.** Suppose that $C_{1,1} \cap J_{1,1} = \emptyset$. Then there is a continuous function $\eta$ on $J_{1,1}$ with unit modulus such that the restriction of $f_h$ to $\hat{C}(J_{1,1})$ is conjugate to the self-mapping of $\mathbb{C}^* \times J_{1,1}$ given by

$$(\lambda, z) \mapsto (\eta(z)\lambda^d, f_{1,1}(z)).$$

In addition, if $\eta$ has a continuous logarithm on $J_{1,1}$, then $\hat{f}_h|_{\hat{C}(J_{1,1})}$ is conjugate to $(\lambda, |z|) \mapsto (\lambda^d, f_{1,1}|_{|z|})$.

**Proof.** If we let $g$ denote the Jacobian determinant of $f_h$ on $\mathbb{C}^k$, then $g$ is a homogeneous polynomial of degree $N = k(d - 1)$. In particular, $V = \{g = 0\}$ is the critical locus of $f_h$, and $V_{1,1}$ is disjoint from $J_{1,1}$. Thus we may represent $\hat{f}_h$ as in (A.1).

Let us define $\phi(\lambda, z) = (\alpha(z)\lambda, z)$. Then $\phi^{-1} \circ \psi^{-1} \circ \hat{f}_h \circ \psi \circ \phi$ is given by

$$(\lambda, z) \mapsto (\alpha^d(z)\alpha^{-1}(f_{1,1}z)\chi^N(z)\lambda^d, f_{1,1}z).$$

Thus we wish to find $\alpha : J_{1,1} \to \mathbb{R}$ such that

$$\alpha^d(z)\alpha^{-1}(f_{1,1}z)\chi(z) = 1. \tag{A.2}$$

Taking logarithms, we have

$$\log \alpha(z) = -\frac{1}{d} \log |\chi^N(z)| + \frac{1}{d} \log |\alpha(f_{1,1}z)|.$$

Applying $f_{1,1}^j$, dividing by $d^j$, and summing over $j$, we have that

$$\log \alpha(z) = -\sum_{j=0}^{\infty} \frac{1}{d^j+1} \log |\chi^N(f_{1,1}^jz)| \tag{A.3}$$

is continuous on $J_{1,1}$, and $\alpha$ solves (A.2). Setting $\eta = \chi^N/|\chi^N|$, we have the desired form of $\hat{f}_h$.

Finally, if $\eta$ has a continuous logarithm, we may solve (A.2) without taking absolute value.

**Examples.** We consider two mappings: $f_1(z_1, z_2) = (z_1^2, z_2^2)$ and $f_2(z_1, z_2) = (z_1^2, z_2^2)$. We consider the function $g(z) = z_1$, the coordinate $\zeta = z_2/z_1$ on $\Pi - \{0 : 1\}$, and the mapping $\psi : \mathbb{C}^* \times (\Pi - \{0 : 1\}) \to \mathbb{C}^2_*$ given by $\psi(\lambda, \zeta) = \lambda(1, \zeta) = (\lambda, \lambda \zeta)$. In both cases, $J_{1,1} = \{|| \zeta || = 1\}$. The associated mappings on $J_{1,1}$ are $f_1, f_2$ and $f_2, f_1$ respectively.

The normal forms given in the Proposition are

$$\psi^{-1} f_1(\psi(\lambda, \zeta)) = (\lambda^2, \zeta^2), \quad \psi^{-1} f_2(\psi(\lambda, \zeta)) = (\zeta^2 \lambda^2, \zeta^{-2}).$$

There is no continuous function $\alpha : J_{1,1} \to \mathbb{C}^*$ solving (A.3) for $f_2$. For in this case we would have $\alpha^2(\zeta)\alpha^{-1}(\zeta^{-2}) = 1$. But if $A$ is the (integer) winding number of $\alpha\{|| \zeta || = 1\}$ about 0 in $\mathbb{C}^*$, then the winding numbers of $\alpha^2$ and $\alpha^{-1}$ are each $2A$, so we must have $2A + 2A + 2 = 0$, which is a contradiction.

In dimension $k = 2$ we can often assume that $N = 1$. 


Proposition A.2. Suppose that $k = 2$ and that $J_{\Pi} \neq \Pi$. Then there is a continuous function $\eta$ on $J_{\Pi}$ with unit modulus such that the restriction of $f_h$ to $C(J_{\Pi})$ is conjugate to the self-mapping of $C_* \times J_{\Pi}$ given by

$$(\lambda, z) \mapsto (\eta(z)\lambda^d, f_{\Pi}(z)).$$

In addition, if $\eta$ has a continuous logarithm on $J_{\Pi}$, then $f_h|_{C(J_{\Pi})}$ is conjugate to $(\lambda, [z]) \mapsto (\lambda^d, f_{\Pi}[z])$.

Proof. The proof is the same as for Proposition A.1, except that we use $g(z, w) = a_2 z - a_1 w$, where $a = [a_1 : a_2] \in \Pi - J_{\Pi}$. \qed

§Appendix B. Hyperbolicity for endomorphisms.

In this appendix we present some basic results on hyperbolicity for smooth endomorphisms. More details can be found in [J2]. Our main references are [Ru] and [PS]; see also [FS4]. No proofs are given in this appendix; they can be found in the above references.

Let $f$ be a $C^{\infty}$ endomorphism of a finite-dimensional Riemannian manifold $M$. Let $L$ be a compact subset of $M$ with $f(L) = L$ and define

$$\hat{L} = \{(x_i)_{i \leq 0} : x_i \in L, f x_i = x_{i+1}\}.$$ 

Then $\hat{L}$ is a closed subset of $L^\mathbb{N}$, hence compact. We will use the notation $\hat{x}$ for a point $(x_i)_{i \leq 0}$ in $\hat{L}$. The restriction $f|_L$ lifts to a homeomorphism $\hat{f}$ of $\hat{L}$ given by $\hat{f}((x_i)) = (x_{i+1})$. There is a natural projection $\pi$ from $\hat{L}$ to $L$ sending $\hat{x}$ to $x_0$ and the pullback under $\pi$ of the restriction to $L$ of the tangent bundle of $M$ is a bundle on $\hat{L}$ which we call the tangent bundle $T_L$. Explicitly, a point in $T_L$ is of the form $(\hat{x}, v)$ where $\hat{x} \in \hat{L}$ and $v$ is a tangent vector in $T_{x_0} M$. The derivative $Df$ lifts to a map $D\hat{f}$ of $T_\hat{L}$ in a natural way.

Now $f$ is hyperbolic on $L$ if there exists a continuous splitting $T_L = E^u \oplus E^s$ which is invariant under $D\hat{f}$ and such that $D\hat{f}$ is expanding on $E^u$ and contracting on $E^s$. More precisely, $D\hat{f}(E^{u/s}) \subset E^{u/s}$ and there are constants $c > 0$ and $\lambda > 1$ such that for all $n \geq 1$

$$|D\hat{f}^n v| \geq c\lambda^n |v| \quad v \in E^u$$

$$|D\hat{f}^n v| \leq c^{-1}\lambda^{-n} |v| \quad v \in E^s.$$ 

Remark. Such a map is called prehyperbolic in [Ru].

Remark. It is possible to make a smooth change of metric in a neighborhood of $L$ and obtain $c = 1$ in the equation above.

Note that whereas the fiber of the unstable bundle $E^u$ at a point $\hat{x} \in \hat{L}$ depends on the whole history $\hat{x}$ of $x_0$, the fiber of $E^s$ at $\hat{x}$ depends only on the point $x_0$. Hence the dimension of the fiber of $E^u$ at a point $\hat{x}$ depends only on $x_0$, so the dimensions of the fibers of the bundles $E^u$ and $E^s$ are locally constant.

As a special case of the above we say that $f$ is expanding on $L$ if the bundle $E^s$ is trivial. This means that there exist constants $c > 0$ and $\lambda > 1$ such that $|D\hat{f}^n(x)v| \geq c\lambda^n |v|$ for all $x \in L, v \in T_x M$ and all $n \geq 1$. 
A basic result in hyperbolic dynamics is the stable manifold theorem. For each point \( p \) in \( L \) and each history \( \hat{q} \) in \( \hat{L} \), we define local stable and unstable manifolds by

\[
W_{\text{loc}}^s(p) = \{ y \in M : d(f^s y, f^s p) < \delta \ \forall i \geq 0 \}
\]
\[
W_{\text{loc}}^u(\hat{q}) = \{ y \in M : \exists \hat{y}, \pi(\hat{y}) = y, d(y, q_i) < \delta \ \forall i \leq 0 \},
\]

for small \( \delta > 0 \).

The following theorem asserts that the local (un)stable manifolds are indeed nice objects. [Ru, §15] contains an outline of a proof, whereas [PS, Theorem 5.2] proves a more general theorem.

**Theorem B.1 (Stable Manifold Theorem).** If \( \delta \) is small enough, then

(i) For all \( p \in L \) and all \( \hat{q} \in \hat{L} \), \( W_{\text{loc}}^s(p) \) and \( W_{\text{loc}}^u(\hat{q}) \) are embedded \( C^\infty \) balls of \( M \) tangent to \( E^s(p) \) and \( E^u(\hat{q}) \) at \( p \) and \( q_0 \), respectively.

(ii) \( W_{\text{loc}}^s(p) \) and \( W_{\text{loc}}^u(\hat{q}) \) depend continuously on \( p \) and \( \hat{q} \), respectively.

(iii) If \( x \in W_{\text{loc}}^u(\hat{q}) \), then \( d(f^n x, f^n p) \to 0 \) exponentially fast as \( n \to \infty \). Similarly, every point \( x \) in \( W_{\text{loc}}^u(\hat{q}) \) has a unique history \( \hat{x} \) such that \( x_j \in W_{\text{loc}}^u(\hat{f}^j(\hat{q})) \) for all \( j \leq 0 \) and \( d(x_j, q_j) \to 0 \) exponentially fast as \( j \to -\infty \).

If \( \delta \) is small enough, then by continuity \( W_{\text{loc}}^s(p) \) and \( W_{\text{loc}}^u(\hat{q}) \) are almost flat, i.e. \( C^1 \) close to the tangents at \( p \) and \( q_0 \), respectively for all \( p \in L \) and all \( q \in \hat{L} \). Therefore \( W_{\text{loc}}^s(p) \) and \( W_{\text{loc}}^u(\hat{q}) \) intersect in at most one point.

**Definition B.2.** We say that \( L \) has local product structure if \( \delta \) can be chosen so that \( W_{\text{loc}}^s(p) \cap W_{\text{loc}}^u(\hat{q}) \subset L \) for all \( p \) and \( \hat{q} \).

If \( L \) has local product structure, \( p \in L \), \( \hat{q} \in \hat{L} \) and if \( p, q_0 \) are sufficiently close, then \( W_{\text{loc}}^s(p) \) and \( W_{\text{loc}}^u(\hat{q}) \) intersect in exactly one point \( x \in L \) and \( x \) has a history \( \hat{x} \) such that \( x_j \in W_{\text{loc}}^u(\hat{f}^j(\hat{q})) \) for all \( j \leq 0 \). It is not a priori clear that \( \hat{x} \in \hat{L} \), i.e. that \( x_j \in L \) for all \( j \leq 0 \). We therefore make another definition.

**Definition B.3.** We say that \( \hat{L} \) has local product structure if \( \delta \) can be chosen so that if the intersection \( W_{\text{loc}}^s(p) \cap W_{\text{loc}}^u(\hat{q}) \) is nonempty, then it consists of a unique point \( x \in L \) and the unique history \( \hat{x} \) of \( x \) with \( x_j \in W_{\text{loc}}^u(\hat{f}^j(\hat{q})) \) for all \( j \leq 0 \) is contained in \( \hat{L} \).

**Definition B.4.** Let \( \eta > 0 \). An \( \eta \)-pseudoorbit in \( M \) is a sequence \( (x_i)_{[t_1, t_2]} \), where \( -\infty \leq t_1 < t_2 \leq \infty \), such that \( d(f x_i, x_{i+1}) < \delta \) for \( t_1 \leq i < t_2 \). An \( \eta \)-pseudoorbit \( (x_i)_{[t_1, t_2]} \) is \( \epsilon \)-shadowed by an orbit \((y_i)_{[t_1, t_2]}\) if \( d(y_i, x_i) < \epsilon \) for all \( i \in [t_1, t_2] \).

For proofs of the remaining results in this appendix see [J2].

**Theorem B.5 (Shadowing Lemma).** Suppose that \( L \) is hyperbolic and that \( \hat{L} \) has local product structure. Then for each \( \epsilon > 0 \) there exists an \( \eta > 0 \) such that every \( \eta \)-pseudoorbit in \( L \) can be \( \epsilon \)-shadowed by an orbit in \( L \).

Using shadowing we control the orbits of \( f \) staying near \( L \) in positive or negative time.
Proposition B.6 (Fundamental Neighborhood). Let $L$ be a hyperbolic set for a map $f$. Assume that $\hat{L}$ has local product structure. Then $L$ has a neighborhood $U$ in $M$ such that

(i) If $x \in U$ and $f^j x \in U$ for all $j \geq 0$, then $x \in W^s_{\text{loc}}(p)$ for some $p \in L$.
(ii) If $x \in U$ and $x$ has a history $\hat{x}$ with $x_i \in U$ for all $i \leq 0$, then $x \in W^u_{\text{loc}}(\hat{q})$ for some $\hat{q} \in \hat{L}$. More precisely $d(x_i, q_i) < \delta$ for all $i \leq 0$.
(iii) If $(x_i)_{i \in \mathbb{Z}}$ is a complete orbit in $U$ then $x_i \in L$ for all $i$.

Next we consider Axiom A endomorphisms. A point $x \in M$ is wandering if it has a neighborhood $V$ such that $f^n(V) \cap V = \emptyset$ for all $n \geq 1$; otherwise it is called non-wandering. The non-wandering set $\Omega$ of $f$ is the set of all non-wandering points; it is a closed set.

Definition B.7. A map $f$ is said to be Axiom A if its non-wandering set satisfies

(i) $\Omega$ is compact.
(ii) Periodic points are dense in $\Omega$.
(iii) $f$ is hyperbolic on $\Omega$.

Remark. If $\Omega$ satisfies (ii), then $f(\Omega) = \Omega$, so (iii) makes sense. Also, if $f$ is Axiom A, then periodic points (under $\hat{f}$) are dense in $\hat{\Omega}$.

The next proposition shows that the preceding results apply to open Axiom A endomorphisms.

Proposition B.8. If $f$ is Axiom A and open, then $\hat{\Omega}$ has local product structure.

Theorem B.9 (Spectral decomposition). If $f$ is Axiom A, then $\Omega$ can be written in a unique way as a disjoint union $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, where each $\Omega_i$ is compact, satisfies $f(\Omega_i) = \Omega_i$ and $f$ is transitive on $\Omega_i$. The sets $\Omega_i$ are called the basic sets of $f$. Moreover, each $\Omega_i$ can be further decomposed into a finite disjoint union $\Omega_i = \bigcup_{1 \leq j \leq n_i} \Omega_{i,j}$, where $\Omega_{i,j}$ is compact, $f(\Omega_{i,j}) = \Omega_{i,j+1}$ ($\Omega_{i,n_i+1} = \Omega_{i,1}$) and $f^{n_i}$ is topologically mixing on each $\Omega_{i,j}$.

Our final result in this appendix describes forward and backward orbits for an Axiom A endomorphism.

Corollary B.10. Assume that $f$ is Axiom A and $M$ is compact.

(i) If $x \in M$, then there is a unique basic set $\Omega_i$ such that $f^j x \to \Omega_i$ as $j \to \infty$. Moreover, there is a (not necessarily unique) $p \in \Omega_i$ such that $d(f^j x, f^j p) \to 0$ as $j \to \infty$.
(ii) If $\hat{x} \in \hat{M}$, then there is a unique basic set $\hat{\Omega}_i$ such that $x_j \to \hat{\Omega}_i$ as $j \to -\infty$. Moreover, there is a (not necessarily unique) $\hat{q} \in \hat{\Omega}_i$ such that $d(x_j, q_j) \to 0$ as $j \to -\infty$.

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