The Cheeger cut and Cheeger problem in metric graphs

José M. Mazón

Received: 31 January 2022 / Revised: 30 June 2022 / Accepted: 25 August 2022
© This is a U.S. Government work and not under copyright protection in the US; foreign copyright protection may apply 2022

Abstract
For discrete weighted graphs there is sufficient literature about the Cheeger cut and the Cheeger problem, but for metric graphs there are few results about these problems. Our aim is to study the Cheeger cut and the Cheeger problem in metric graphs. For that, we use the concept of total variation and perimeter in metric graphs introduced in Mazón (Math Eng 5(1):1–38, 2023. https://doi.org/10.3934/mine.2023009), which takes into account the jumps at the vertices of the functions of bounded variation. Moreover, we study the eigenvalue problem for the minus 1-Laplacian operator in metric graphs, whereby we give a method to solve the optimal Cheeger cut problem.

Keywords Cheeger problem, Cheeger cut · Metric graphs · Functions of total variation · Total variation flow · The 1-Laplacian

Mathematics Subject Classification 5R02 · 05C21 · 47J35

Contents

1 Introduction .............................................
2 Preliminaries .............................................
  2.1 Metric graphs ...........................................
  2.2 Convex functions and subdifferentials ....................
  2.3 BV functions and integration by parts ....................
  2.4 The 1-Laplacian in metric graphs .....................
3 The Cheeger problem: Γ-Cheeger and Γ-calibrable sets ................................................................
4 The Cheeger cut in metric graphs ................................
References ..................................................

José M. Mazón
mazon@uv.es

1 Departamento de Análisis Matemático, Universitat de València, Dr. Moliner 50, 46100 Burjassot, Spain

Published online: 03 September 2022
1 Introduction

A metric graph is a combinatorial graph where the edges are considered as intervals of the real line with a distance on each one of them and are glued together according to the combinatorial structure. The resulting metric measure space allows to introduce a family of differential operators acting on each edge \( e \) considered as an interval \((0, \ell_e)\) with boundary conditions at the vertices. We refer to the pair formed by the metric graph and the family of differential operators as *quantum graph*. During the last two decades, quantum graphs became an extremely popular subject because of numerous applications in mathematical physics, chemistry and engineering. Indeed, the literature on quantum graphs is vast and extensive and there is no chance to give even a brief overview of the subject here. We only mention a few recent monographs and collected works with a comprehensive bibliography \[6, 7, 23, 25, 31, 40, 43\].

The historical motivation of the Cheeger cut problem is an isoperimetric-type inequality that was first proved by J. Cheeger in [16] in the context of compact, \( n \)-dimensional Riemannian manifolds without boundary. As a consequence, one obtains the validity of a Poincaré inequality with optimal constant uniformly bounded from below by a geometric constant. Let \( \lambda_1(M) \) be the least non-zero eigenvalue of the Laplace-Beltrami operator on \( M \), then Cheeger proved that

\[
\lambda_1(M) \geq \frac{1}{2} h(M)^2, \quad h(M) := \inf_{A \subset M} \frac{P(A)}{\min\{V(A), V(M \setminus A)\}} \tag{1.1}
\]

where \( V(A) \) and \( P(A) \) denote, respectively, the Riemannian volume and perimeter of \( A \).

The first Cheeger estimates on discrete graphs are due to Dodziuk [21] and Alon and Milman [1]. Since then, these estimates have been improved and various variants have been proved. Consider a finite weighted connected graph \( G = (V, E) \), where \( V = \{x_1, \ldots, x_n\} \) is the set of vertices (or nodes) and \( E \) the set of edges, which are weighted by a function \( w_{ij} = w_{ji} \geq 0 \), \((x_i, x_j) \in E\). In this context, the Cheeger cut value of a partition \( \{S, S^c\} \) \((S^c := V \setminus S)\) of \( V \) is defined as

\[
C(S) := \frac{\text{Cut}(S, S^c)}{\min\{\text{vol}(S), \text{vol}(S^c)\}},
\]

where \( \text{Cut}(A, B) = \sum_{x_i \in A, x_j \in B} w_{ij} \) and \( \text{vol}(S) \) is the volume of \( S \), defined as \( \text{vol}(S) := \sum_{x_i \in S} d_{x_i} \), being \( d_{x_i} := \sum_{j=1}^n w_{x_i, x_j} \) the weight at the vertex \( x_i \). Then,

\[
h(G) := \min_{S \subset V} C(S) \tag{1.2}
\]

is called the *Cheeger constant*, and a partition \( \{S, S^c\} \) of \( V \) is called a *Cheeger cut of \( G \)* if \( h(G) = C(S) \). Unfortunately, the Cheeger minimization problem of computing \( h(G) \) is NP-hard [27, 46]. However, it turns out that \( h(G) \) can be approximated by the first positive eigenvalue \( \lambda_1 \) of the graph Laplacian thanks to the following Cheeger
inequality [17]:

$$\frac{\lambda_1}{2} \leq h(G) \leq \sqrt{2\lambda_1}.$$ 

This motivates the spectral clustering method [33], which, in its simplest form, thresholds the least non-zero eigenvalue of the graph Laplacian to get an approximation to the Cheeger constant and, moreover, to a Cheeger cut. In order to achieve a better approximation than the one provided by the classical spectral clustering method, a spectral clustering based on the graph $p$-Laplacian was developed in [10], where it is shown that the second eigenvalue of the graph $p$-Laplacian tends to the Cheeger constant $h(G)$ as $p \to 1^+$. In [46] the idea was further developed by directly considering the variational characterization of the Cheeger constant $h(G)$

$$h(G) = \min_{u \in L^1} \frac{|u|_{TV}}{\|u - \text{median}(u)\|_1}, \quad (1.3)$$

where

$$|u|_{TV} := \frac{1}{2} \sum_{i,j=1}^{n} w_{ij} |u(x_i) - u(x_j)|.$$ 

In [46], it was proved that the solution of the variational problem (1.3) provides an exact solution of the Cheeger cut problem. If a global minimizer $u$ of (1.3) can be computed, then it can be shown that this minimizer would be the indicator function of a set $\Omega$ (i.e. $u = \chi_\Omega$) corresponding to a solution of the NP-hard problem (1.2).

The subdifferential of the energy functional $|\cdot|_{TV}$ is minus the 1-Laplacian in graphs. Using the nonlinear eigenvalue problem $\lambda \text{sign}(u) \in -\Delta_1 u$, the theory of 1-Spectral Clustering is developed in [13–15, 27]. For a generalization of the above results to the framework of random walk spaces see [37, 38].

The only results about the Cheeger cut problem in metric graphs that we know are the ones given by Del Pezzo and Rossi [19] in which they study the first nonzero eigenvalue of the $p$-Laplacian on a quantum graph with Kirchoff boundary conditions on the vertices and study the Cheeger cut problem, taking the limit as $p \to 1$ of the eigenfunctions. Now, as we will see later, their concept of total variation of a function of bounded variation in a metric graph is not clear and, consequently, also their concept of perimeter (see Remark 2.11). Here we use a different concept of total variation for functions in metric graphs, proposed in [34], and consequently of perimeter, that takes into account the jumps of the function at the vertices.

Following the work by Nicaise [41], where a Cheeger inequality in metric graphs is obtained, there have been very few results in this direction (see [29, 30, 43]).

On the other hand, the Cheeger paper [16] also motivated the so-called Cheeger problem. Given a bounded domain $\Omega \subset \mathbb{R}^N$, the Cheeger constant of $\Omega$ is defined as

$$h_1(\Omega) := \inf \left\{ \frac{\text{Per}(E)}{|E|} : E \subset \Omega, \ E \text{ with finite perimeter, } |E| > 0 \right\},$$
where $\text{Per}(E)$ is the perimeter of $E$ and $|E|$ its Lebesgue measure. Any set $E \subset \Omega$ such that

$$\frac{\text{Per}(E)}{|E|} = h_1(\Omega),$$

is called a **Cheeger set** of $\Omega$. Furthermore, we say that $\Omega$ is **calibrable** if it is a Cheeger set of itself, that is, if

$$\frac{\text{Per}(\Omega)}{|\Omega|} = h_1(\Omega).$$

We shall generically refer to the **Cheeger problem**, as far as the computation or estimation of $h_1(\Omega)$, or the characterization of Cheeger sets of $\Omega$, are concerned. In the last year there has been a lot of literature on the Cheeger problem, see [32, 42] for surveys about the Cheeger problem and [35, 36] for the nonlocal Cheeger problem.

It is well known that the Cheeger constant of $\Omega$ is the limit of the sequence of first eigenvalues of the $p$-Laplacian (with Dirichlet conditions) when $p$ tends to 1, see [28]. A similar result has been obtained by Del Pezzo and Rossi in [20], in the context of metric graphs, but here again the problem is their concept of perimeter in metric graphs. For the Cheeger problem in random walk spaces, that has as a particular case the weighted graphs, see [38].

The aim of this paper is to study the Cheeger cut and Cheeger problem in metric graphs. We introduce the concepts of Cheeger and calibrable sets in metric graphs and we also study the eigenvalue problem whereby we give a method to solve the optimal Cheeger cut problem. To do that we work in the framework we developed in [34] to study the total variation flow in metric graphs.

The structure of the paper is as follows. In Sect. 2 we recall the notion of metric graphs and the results about functions of bounded variation in metric graphs that we need. Then, in Sect. 3 we study the Cheeger problem. We introduce the concepts of Cheeger and calibrable sets in metric graphs, we give different characterizations of the Cheeger constant of a set and its relation with the Max-Flow Min-Cut Theorem and, moreover, we characterize the calibrable sets. Section 3 is also devoted to the eigenvalue problem for the 1-Laplacian in metric graphs and its relations with the Cheeger problem. In Sect. 4 we study the Cheeger cut in metric graphs. We obtain a characterization similar to the one obtained in [46] for weighted graphs, which allows us to prove the existence of an optimal Cheeger cut, and its relation with the eigenvalue problem for the 1-Laplacian obtaining similar results to the ones in [13] for weighted graphs, whereby we give a method to solve the optimal Cheeger cut problem. Finally, we also obtained a Cheeger Inequality in metric graphs.

## 2 Preliminaries

In this section, after giving the basic concepts of metric graphs, we recall the results about total variation functions introduced in [34] that is the framework in which we developed our work.
2.1 Metric graphs

We recall here some basic knowledge about metric graphs, see for instance [7] and the references therein.

A graph $\Gamma$ consists of a finite or countably infinite set of vertices $V(\Gamma) = \{v_i\}$ and a set of edges $E(\Gamma) = \{e_j\}$ connecting the vertices. A graph $\Gamma$ is said to be a finite graph if the number of edges and the number of vertices are finite. An edge and a vertex on that edge are called incident. We will denote $v \in e$ when the edge $e$ and the vertex $v$ are incident. We define $E_v(\Gamma)$ as the set of all edges incident to $v$, and the degree of $v$ as $d_v := |E_v(\Gamma)|$. We define the boundary of $V(\Gamma)$ as

$$\partial V(\Gamma) := \{v \in V(\Gamma) : d_v = 1\},$$

and its interior as

$$\text{int}(V(\Gamma)) := \{v \in V(\Gamma) : d_v > 1\}.$$

We will assume the absence of loops, since if these are present, one can break them into pieces by introducing new intermediate vertices. We also assume the absence of multiple parallel edges.

A walk is a sequence of edges $\{e_1, e_2, e_3, \ldots\}$ in which, for each $i$ (except the last), the end of $e_i$ is the beginning of $e_{i+1}$. A trail is a walk in which no edge is repeated. A path is a trail in which no vertex is repeated.

From now on we will deal with a connected, compact and metric graph $\Gamma$:

- A graph $\Gamma$ is a metric graph if
  1. each edge $e$ is assigned with a positive length $\ell_e \in (0, +\infty]$;
  2. for each edge $e$, a coordinate is assigned to each point of it, including its vertices. For that purpose, each edge $e$ is identified with an ordered pair $(i_e, f_e)$ of vertices, being $i_e$ and $f_e$ the initial and terminal vertex of $e$ respectively, which has no sense of meaning when travelling along the path but allows us to define coordinates by means of an increasing function

$$c_e : e \rightarrow [0, \ell_e]$$

such that, letting $c_e(i_e) := 0$ and $c_e(f_e) := \ell_e$, it is exhaustive; $x_e$ is called the coordinate of the point $x \in e$.

- A graph is said to be connected if a path exists between every pair of vertices, that is, a graph which is connected in the usual topological sense.
- A compact metric graph is a finite metric graph whose edges all have finite length.

If a sequence of edges $\{e_j\}_{j=1}^n$ forms a path, its length is defined as $\sum_{j=1}^n \ell_{e_j}$. The length of a metric graph, denoted $\ell(\Gamma)$, is the sum of the length of all its edges.
Sometimes we identify $\Gamma$ with

$$\Gamma \equiv \bigcup_{e \in E(\Gamma)} e.$$ 

Given a set $A \subset \Gamma$, we define its \textit{length} as

$$\ell(A) := \sum_{e \in E(\Gamma), A \cap e \neq \emptyset} |c_e(A \cap e)|,$$

being $\cdot$ the one dimensional Lebesgue measure.

For two vertices $v$ and $\hat{v}$, the distance between $v$ and $\hat{v}$, $d_\Gamma(v, \hat{v})$, is defined as the minimal length of the paths connecting them. Let us be more precise and consider $x$, $y$ two points in the graph $\Gamma$.

- if $x, y \in e$ (they belong to the same edge, note that they can be vertices), we define the distance-in-the-path-e between $x$ and $y$ as

$$\text{dist}_e(x, y) := |y_e - x_e|;$$

- if $x \in e_a$, $y \in e_b$, with $e_a$ and $e_b$ different edges, let $P = \{e_a, e_1, \ldots, e_n, e_b\}$ be a path ($n \geq 0$) connecting them. Let us call $e_0 = e_a$ and $e_{n+1} = e_b$. Following the definition given above for a path, set $v_0$ the vertex that is the end of $e_0$ and the beginning of $e_1$ (note that these vertices need not be the terminal and the initial vertices of the edges that are taken into account), and $v_n$ the vertex that is the end of $e_n$ and the beginning of $e_{n+1}$. We will say that the distance-in-the-path-$P$ between $x$ and $y$ is equal to

$$\text{dist}_e(x, y) := |y_e - x_e|.$$ 

We define the distance between $x$ and $y$, that we will denote by $d_\Gamma(x, y)$, as the infimum of all the distances-in-paths between $x$ and $y$, that is,

$$d_\Gamma(x, y) = \inf \left\{ \text{dist}_{e_0}(x, v_0) + \sum_{1 \leq j \leq n} \ell_{e_j} + \text{dist}_{e_{n+1}}(v_n, y) : \{e_0, e_1, \ldots, e_n, e_{n+1}\} \text{path connecting } x \text{ and } y \right\}.$$ 

We remark that the distance between two points $x$ and $y$ belonging to the same edge $e$ can be strictly smaller than $|y_e - x_e|$. This happens when there is a path connecting them (using more edges than $e$) with length smaller than $|y_e - x_e|$.

A function $u$ on a metric graph $\Gamma$ is a collection of functions $[u]_e$ defined on $(0, \ell_e)$ for all $e \in E(\Gamma)$, not just at the vertices as in discrete models.

Throughout this work, $\int_\Gamma u(x) dx$ or $\int_\Gamma u$ denotes $\sum_{e \in E(\Gamma)} \int_0^{\ell_e} [u]_e(x_e) \, dx_e$. Note that given $\Omega \subset \Gamma$, we have

$$\ell(\Omega) = \int_\Gamma \chi_\Omega dx.$$
Let \( 1 \leq p \leq +\infty \). We say that \( u \) belongs to \( L^p(\Gamma) \) if \([u]_e \) belongs to \( L^p(0, \ell_e) \) for all \( e \in E(\Gamma) \) and
\[
\|u\|_{L^p(\Gamma)}^p := \sum_{e \in E(\Gamma)} \|[u]_e\|_{L^p(0, \ell_e)}^p < +\infty.
\]
The Sobolev space \( W^{1,p}(\Gamma) \) is defined as the space of functions \( u \) on \( \Gamma \) such that \([u]_e \in W^{1,p}(0, \ell_e) \) for all \( e \in E(\Gamma) \) and
\[
\|u\|_{W^{1,p}(\Gamma)}^p := \sum_{e \in E(\Gamma)} \|[u]_e\|_{L^p(0, \ell_e)}^p + \|[u]'_e\|_{L^p(0, \ell_e)}^p < +\infty.
\]
The space \( W^{1,p}(\Gamma) \) is a Banach space for \( 1 \leq p \leq \infty \). It is reflexive for \( 1 < p < \infty \) and separable for \( 1 \leq p < \infty \). Observe that in the definition of \( W^{1,p}(\Gamma) \) we do not assume the continuity at the vertices. Let us point out that the above spaces are unaffected by the orientation of the edges.

A quantum graph is a metric graph \( \Gamma \) equipped with a differential operator acting on the edges together with vertex conditions. In this work, we will consider the \( 1 \)–Laplacian differential operator given formally by
\[
\Delta_1 u(x) := \left( \frac{u'(x)}{|u'(x)|} \right)',
\]
on each edge.

From now on we will assume that \( \Gamma \) is a finite, compact and connected metric graph.

### 2.2 Convex functions and subdifferentials

Let \( H \) be a real Hilbert space with scalar product \( \langle \cdot, \cdot \rangle_H \) and norm \( \|u\|_H = \sqrt{\langle u, u \rangle_H} \). Given a function \( \mathcal{F} : H \to ]-\infty, \infty] \), we call the set \( D(\mathcal{F}) := \{ u \in H : \mathcal{F}(u) < +\infty \} \) the **effective domain** of \( \mathcal{F} \), and \( \mathcal{F} \) is said to be proper if \( D(\mathcal{F}) \) is non-empty. Further, we say that \( D(\mathcal{F}) \) is **lower semi-continuous** if for every \( c \in \mathbb{R} \), the sublevel set
\[
E_c := \{ u \in D(\mathcal{F}) : \mathcal{F}(u) \leq c \}
\]
is closed in \( H \).

Given a convex proper function \( \mathcal{F} : H \to ]-\infty, \infty] \), its **subdifferential** is defined by
\[
\partial_H \mathcal{F} := \{ (u, h) \in H \times H : \mathcal{F}(u + v) - \mathcal{F}(u) \geq \langle h, v \rangle_H \ \forall \ v \in D(\mathcal{F}) \}.
\]
2.3 BV functions and integration by parts

We need to recall the concept of bounded variation functions and their total variation in metric graphs that we introduce in [34] since this is the framework to study the Cheeger problem.

For bounded variation functions of one variable we follow [3]. Let $I \subset \mathbb{R}$ be an interval, we say that a function $u \in L^1(I)$ is of bounded variation if its distributional derivative $Du$ is a Radon measure on $I$ with bounded total variation $|Du|(I) < +\infty$. We denote by $BV(I)$ the space of all functions of bounded variation in $I$. It is well known (see [3]) that given $u \in BV(I)$ there exists $\overline{u}$ in the equivalence class of $u$, called a good representative of $u$, with the following properties. If $Ju$ is the set of atoms of $Du$, i.e., $x \in Ju$ if and only if $Du(\{x\}) \neq 0$, then $\overline{u}$ is continuous in $I \setminus Ju$ and has a jump discontinuity at any point of $Ju$:

$$
\overline{u}(x_-) := \lim_{y \uparrow x} \overline{u}(y) = Du([a, x]), \quad \overline{u}(x_+) := \lim_{y \downarrow x} \overline{u}(y) = Du([a, x]) \quad \forall x \in Ju,
$$

where by simplicity we are assuming that $I = [a, b]$. Consequently,

$$
\overline{u}(x_+) - \overline{u}(x_-) = Du([x]) \quad \forall x \in Ju.
$$

Moreover, $\overline{u}$ is differentiable at $L^1$ a.e. point of $I$, and the derivative $\overline{u}'$ is the density of $Du$ with respect to $L^1$. For $u \in BV(I)$, the measure $Du$ decomposes into its absolutely continuous and singular parts $Du = Da^u + Ds^u$. Then $Da^u = \overline{u}' \, L^1$. We also split $Ds^u$ in two parts: the jump part $D^j u$ and the Cantor part $D^c u$. $Ju$ denotes the set of atoms of $Du$.

It is well known (see for instance [3]) that

$$
D^j u = Du\downharpoonright Ju = \sum_{x \in Ju} \overline{u}(x_+) - \overline{u}(x_-),
$$

and also,

$$
|Du|(I) = |Da^u|(I) + |D^j u|(I) + |D^c u|(I)
= \int_a^b |\overline{u}'(x)| \, dx + \sum_{x \in Ju} |\overline{u}(x_+) - \overline{u}(x_-)| + |D^c u|(I).
$$

Obviously, if $u \in BV(I)$ then $u \in W^{1,1}(I)$ if and only if $Ds^u \equiv 0$, and in this case we have $Du = \overline{u}' \, L^1$.

A measurable subset $E \subset I$ is a set of finite perimeter in $I$ if $\chi_E \in BV(I)$, and its perimeter is defined as

$$
\text{Per}(E, I) := |D\chi_E|(I).
$$

The structure of sets of finite perimeter is very simple in dimension 1, as the following proposition [3, Proposition 3.52] shows.
Proposition 2.1 If $E$ has finite perimeter in $]a, b[$ and $|E \cap ]a, b[| > 0$, there exist and integer $p \geq 1$ and $p$ pairwise disjoint intervals $J_i = [a_{2i-1}, a_{2i}] \subset \mathbb{R}$ such that $E \cap ]a, b[\text{ is equivalent to the union of the intervals } J_i$ and

$$\text{Per} = \mathfrak{z}(\{i \in \{1, 2, \ldots, 2p\} : a_i \in ]a, b[\}).$$

From now on, when we deal with point-wise valued $BV$-functions we shall always use the good representative.

Given $z \in W^{1,2}(]a, b[)$ and $u \in BV(]a, b[)$, by $zDu$ we mean the Radon measure in $]a, b[$ defined as

$$\langle \varphi, zDu \rangle := \int_a^b \varphi zDu \quad \forall \varphi \in C_c(]a, b[).$$

Note that if $\varphi \in D(]a, b[)$, then

$$\langle \varphi, zDu \rangle = -\int_a^b uz\varphi' dx - \int_a^b uz\varphi' dx,$$

which is the definition given by Anzellotti in [5].

Working as in [5, Corollary 1.6], it is easy to see that

$$|zDu|(B) \leq \|z\|_{L^\infty(]a, b[)} |Du|(B) \quad \text{for all Borelian } B \subset ]a, b[. \quad (2.1)$$

Then, $zDu$ is absolutely continuous with respect to the measure $|Du|$. The following result was given in [34, Proposition 2.1]

Proposition 2.2 Let $z_n \in W^{1,2}(]a, b[)$. If

$$\lim_{n \to \infty} z_n = z \text{ weakly* in } L^\infty(]a, b[),$$

and

$$\lim_{n \to \infty} z'_n = z' \text{ weakly in } L^1(]a, b[),$$

then for every $u \in BV(]a, b[)$, we have

$$z_nDu \rightharpoonup zDu \text{ as measures},$$

and

$$\lim_{n \to \infty} \int_a^b z_nDu = \int_a^b zDu.$$
Lemma 2.3 If \( z \in W^{1,2}(a,b) \) and \( u \in BV([a,b]) \), then
\[
\int_a^b z Du + \int_a^b u(x)z'(x) \, dx = z(b)u(b) - z(a)u(a).
\]

Definition 2.4 We define the set of bounded variation functions in \( \Gamma \) as
\[
BV(\Gamma) := \{ u \in L^1(\Gamma) : [u]_e \in BV(0, \ell_e) \text{ for all } e \in E(\Gamma) \}.
\]

Given \( u \in BV(\Gamma) \), for \( e \in E_v \), we define
\[
[u]_e(v) := \begin{cases} [u]_e(0+), & \text{if } v = i_e \\ [u]_e(\ell_e-), & \text{if } v = f_e. \end{cases}
\]

For \( u \in BV(\Gamma) \), we define
\[
|Du|(\Gamma) := \sum_{e \in E(\Gamma)} |D[u]_e|(0, \ell_e).
\]

We also write
\[
|Du|(\Gamma) = \int_{\Gamma} |Du|.
\]

Obviously, for \( u \in BV(\Gamma) \), we have
\[
|Du|(\Gamma) = 0 \iff [u]_e \text{ is constant in } (0, \ell_e), \forall e \in E(\Gamma).
\]

\( BV(\Gamma) \) is a Banach space with respect to the norm
\[
\|u\|_{BV(\Gamma)} := \|u\|_{L^1(\Gamma)} + |Du|(\Gamma).
\]

Remark 2.5 Note that we do not include a continuity condition at the vertices in the definition of the spaces \( BV(\Gamma) \). This is due to the fact that, if we include the continuity at the vertices, then typical functions of bounded variation such as the functions of the form \( \chi_D \) with \( D \subset \Gamma \) such that \( v \in D \), being \( v \) a common vertex to two edges, would not be elements of \( BV(\Gamma) \).

By the Embedding Theorem for \( BV \)-function (cf. [3, Corollary 3.49, Remark 3.30]), we have the following result.

Theorem 2.6 The embedding \( BV(\Gamma) \hookrightarrow L^p(\Gamma) \) is continuous for \( 1 \leq p \leq \infty \), being compact for \( 1 \leq p < \infty \). Moreover, we also have the following Poincaré inequality:
\[
\|u - \bar{u}\|_p \leq C|Du|(\Gamma) \forall u \in BV(\Gamma), \ 1 \leq p \leq \infty,
\]
where

$$
\bar{u} := \frac{1}{\ell(\Gamma)} \int_{\Gamma} u(x) dx.
$$

Let us point out that in metric graphs $|Du|(\Gamma)$ is not the good definition of total variation of $u$ since it does not measure the jumps of the function at the vertices. In [34], in order to give a definition of total variation of a function $u \in BV(\Gamma)$ that takes into account the jumps of the function at the vertices, we gave a Green’s formula like the one obtained by Anzellotti in [5] for $BV$-functions in Euclidean spaces. To do that we start by defining the pairing $zDu$ between an element $z \in W^{1,1}(\Gamma)$ and a BV function $u$. This will be a metric graph analogue of the classic Anzellotti pairing introduced in [5].

**Definition 2.7** For $z \in W^{1,2}(\Gamma)$ and $u \in BV(\Gamma)$, we define $zDu := ([z]_e, D[u_e])_{e \in E(\Gamma)}$, that is, for $\varphi \in C(\Gamma)$,

$$
\langle zDu, \varphi \rangle = \sum_{e \in E(\Gamma)} \int_0^{\ell_e} \varphi_e[z]_e D[u_e].
$$

We have that $zDu$ is a Radon measure in $\Gamma$ and

$$
\int_{\Gamma} zDu := \sum_{e \in E(\Gamma)} \int_0^{\ell_e} [z]_e D[u_e].
$$

By (2.1) applied edgewise, we have

$$
\left| \int_{\Gamma} zDu \right| \leq ||z||_{L^\infty(\Gamma)} |Du|(\Gamma).
$$

Then, $zDu$ is absolutely continuous with respect to the measure $|Du|$. Given $z \in W^{1,2}(\Gamma)$, for $e \in E_v$, we define

$$
[z]_e(v) := \begin{cases} 
[z]_e(\ell_e) & \text{if } v = f_e, \\
-[z]_e(0) & \text{if } v = i_e.
\end{cases}
$$

By Lemma 2.3, we have

$$
\int_{\Gamma} zDu := \sum_{e \in E(\Gamma)} \int_0^{\ell_e} [z]_e D[u_e]
$$

$$
= - \sum_{e \in E(\Gamma)} \int_0^{\ell_e} [u_e(x)](x)dx + \sum_{e \in E(\Gamma)} ([z]_e(\ell_e)[u_e(\ell_e)] - [z]_e(0)[u_e(0)])
$$

$$
= - \int_{\Gamma} uz' + \sum_{v \in V(\Gamma)} \sum_{e \in E_v(\Gamma)} [z]_e(v)[u_e(v)].
$$

Then, if we define

\[ \int_{\partial \Gamma} zu := \sum_{v \in V(\Gamma)} \sum_{e \in E_v(\Gamma)} [z]_e(v) [u]_e(v), \]

for \( z \in W^{1,2}(\Gamma) \) and \( u \in BV(\Gamma) \), we have the following Green’s formula:

\[ \int_{\Gamma} zDu + \int_{\Gamma} u z' = \int_{\partial \Gamma} zu. \tag{2.2} \]

We define

\[ X_0(\Gamma) := \{ z \in W^{1,2}(\Gamma) : z(v) = 0, \quad \forall v \in V(\Gamma) \}. \]

For \( u \in BV(\Gamma) \) and \( z \in X_0(\Gamma) \), we have the following Green’s formula

\[ \int_{\Gamma} zDu + \int_{\Gamma} u z' = 0. \tag{2.3} \]

We consider now the elements of \( W^{1,2}(\Gamma) \) that satisfy a Kirchhoff condition, that is, the set

\[ X_K(\Gamma) := \left\{ z \in W^{1,2}(\Gamma) : \sum_{e \in E_v(\Gamma)} [z]_e(v) = 0, \quad \forall v \in V(\Gamma) \right\}. \]

Note that if \( z \in X_K(\Gamma) \), then \( z(v) = 0 \) for all \( v \in \partial V(\Gamma) \). Therefore, for \( u \in BV(\Gamma) \) and \( z \in X_K(\Gamma) \), we have the following Green’s formula

\[ \int_{\Gamma} zDu + \int_{\Gamma} u z' = \sum_{v \in \text{int}(V(\Gamma))} \sum_{e \in E_v(\Gamma)} [z]_e(v) [u]_e(v). \tag{2.4} \]

Now, for \( v \in \text{int}(V(\Gamma)) \), we have

\[ \sum_{e \in E_v(\Gamma)} [z]_e(v) [u]_e(v) = 0, \quad \text{for all } \hat{e} \in E_v(\Gamma). \]

Hence

\[ \sum_{e \in E_v(\Gamma)} [z]_e(v) [u]_e(v) = \frac{1}{d_v} \sum_{\hat{e} \in E_v(\Gamma)} \sum_{e \in E_v(\Gamma)} [z]_e(v) ([u]_e(v) - [u]_{\hat{e}}(v)). \]

Therefore, we can rewrite Green’s formula (2.4) as

\[ \int_{\Gamma} zDu + \int_{\Gamma} u z' = \sum_{v \in \text{int}(V(\Gamma))} \frac{1}{d_v} \sum_{\hat{e} \in E_v(\Gamma)} \sum_{e \in E_v(\Gamma)} [z]_e(v) ([u]_e(v) - [u]_{\hat{e}}(v)). \]
Remark 2.8 Given a function $u$ in the metric graph $\Gamma$, we say that $u$ is continuous at the vertex $v$, if

$$[u]_{e_1}(v) = [u]_{e_2}(v), \quad \text{for all } e_1, e_2 \in E_v(\Gamma).$$

We denote this common value as $u(v)$. We denote by $C(\text{int}(V(\Gamma)))$ the set of all functions in $\Gamma$ continuous at the vertices $v \in \text{int}(V(\Gamma))$.

Note that if $u \in BV(\Gamma) \cap C(\text{int}(V(\Gamma)))$ and $z \in X_K(\Gamma)$, then by (2.4), we have

$$\int_{\Gamma} z Du + \int_{\Gamma} uz' = 0.$$  \hspace{1cm} \blacksquare

Definition 2.9 For $u \in BV(\Gamma)$, we define its total variation as

$$TV_\Gamma(u) = \sup \left\{ \left| \int_{\Gamma} u(x)z'(x)dx \right| : z \in X_K(\Gamma), \|z\|_{L^\infty(\Gamma)} \leq 1 \right\}.$$

We say that a measurable set $E \subset \Gamma$ is a set of finite perimeter if $\chi_E \in BV(\Gamma)$, and we define its $\Gamma$-perimeter as

$$\text{Per}_\Gamma(E) := TV_\Gamma(\chi_E),$$

that is

$$\text{Per}_\Gamma(E) = \sup \left\{ \left| \int_E z'(x)dx \right| : z \in X_K(\Gamma), \|z\|_{L^\infty(\Gamma)} \leq 1 \right\}. \quad (2.5)$$

Remark 2.10 We have

$$\text{Per}_\Gamma(E) = \text{Per}_\Gamma(\Gamma \setminus E), \quad \text{for all } E \subset \Gamma \text{ of finite perimeter.} \quad (2.6)$$

In fact, given $z \in X_K(\Gamma)$ with $\|z\|_{L^\infty(\Gamma)} \leq 1$, by Green’s formula (2.4), it is easy to see that

$$\int_{\Gamma} \chi_E z' = -\int_{\Gamma} zD\chi_E + \sum_{v \in \text{int}(V(\Gamma))} \sum_{e \in E_v(\Gamma)} [z]_e(v)[\chi_E]_e(v)$$

$$= -\int_{\Gamma} zD\chi_{\Gamma \setminus E} + \sum_{v \in \text{int}(V(\Gamma))} \sum_{e \in E_v(\Gamma)} [z]_e(v)[\chi_{\Gamma \setminus E}]_e(v) = \int_{\Gamma} \chi_{\Gamma \setminus E} z'.$$

Thus, by (2.5), we have that (2.6) holds. \hspace{1cm} \blacksquare

Remark 2.11 In the works by Del Pezzo and Rossi [19, 20] it is not clear what is their concept of functions of bounded variation on $\Gamma$ and their total variation. They refer to the monograph [3] for the precise definition. However, in [3] only the case of functions of bounded variation in the Euclidean space is studied. Now, reading their works it
seems that for them the space of the bounded variation functions in $\Gamma$ coincides with our space $BV(\Gamma)$, but they do not make it clear if they assume continuity at the vertices. Their total variation of $u \in BV(\Gamma)$ is $|Du|(\Gamma)$ which does not take into account the jumps at the vertices. ■

As a consequence of the above definition, we have the following result.

**Proposition 2.12** $TV_\Gamma$ is lower semi-continuous with respect to the weak convergence in $L^1(\Gamma)$.

As in the local case, we have obtained in [34] the following coarea formula relating the total variation of a function with the perimeter of its superlevel sets.

**Theorem 2.13** (Coarea formula) For any $u \in L^1(\Gamma)$, let $E_t(u) := \{ x \in \Gamma : u(x) > t \}$. Then, \[
TV_\Gamma(u) = \int_{-\infty}^{+\infty} \text{Per}_\Gamma(E_t(u)) \, dt. \tag{2.7}
\]

We introduce now

\[JV_\Gamma(u) := \sum_{v \in \text{int}(V(\Gamma))} \frac{1}{d_v} \sum_{e, \hat{e} \in E_v(\Gamma)} |[u]_e(v) - [u]_{\hat{e}}(v)|.\]

Note that $JV_\Gamma(u)$ measures, in a weighted way, the jumps of $u$ at the vertices. The following results were proved in [34].

**Proposition 2.14** For $u \in BV(\Gamma)$, we have

\[|Du|(\Gamma) \leq TV_\Gamma(u) \leq |Du|(\Gamma) + JV_\Gamma(u).\]

If $u \in BV(\Gamma) \cap C(\text{int}(V(\Gamma)))$, then

\[TV_\Gamma(u) = |Du|(\Gamma).\]

If $\Gamma$ is path graph, that is $d_v = 2$ for all $v \in \text{int}(V(\Gamma))$, then

\[TV_\Gamma(u) = |Du|(\Gamma) + JV_\Gamma(u). \tag{2.8}\]

**Corollary 2.15** For $u \in BV(\Gamma)$, we have

\[TV_\Gamma(u) = 0 \iff u \text{ is constant.}\]

Then

\[\text{Per}_\Gamma(E) = 0 \iff E = \Gamma.\]

In [34] we give an example showing that the equality (2.8) does not hold if $u \notin C(\text{int}(V(\Gamma)))$ or there exists $v \in \text{int}(V(\Gamma))$ with $d_v \geq 3$. 
The Cheeger cut and Cheeger problem in metric graphs

2.4 The 1-Laplacian in metric graphs

In [34], in order to study the total variation flow in the metric graph $\Gamma$ we have introduced the energy functional $\mathcal{F}_\Gamma : L^2(\Gamma) \to [0, +\infty]$ defined by

$$\mathcal{F}_\Gamma(u) := \begin{cases} TV(\Gamma)(u), & \text{if } u \in BV(\Gamma), \\ +\infty, & \text{if } u \in L^2(\Gamma) \setminus BV(\Gamma), \end{cases}$$

which is convex and lower semi-continuous, and we have obtained the following characterization of the subdifferential of $\mathcal{F}_\Gamma$.

**Theorem 2.16** Let $u \in BV(\Gamma)$ and $v \in L^2(\Gamma)$. The following assertions are equivalent:

(i) $v \in \partial \mathcal{F}_\Gamma(u)$;
(ii) there exists $z \in X_K(\Gamma)$, $\|z\|_{L^\infty(\Gamma)} \leq 1$ such that

$$v = -z', \quad \text{that is, } [v]_e = -[z]'_e \text{ in } \mathcal{D}'(0, \ell_e) \forall e \in E(\Gamma) \quad (2.9)$$

and

$$\int_{\Gamma} u(x)v(x)dx = \mathcal{F}_\Gamma(u);$$

(iii) there exists $z \in X_K(\Gamma)$, $\|z\|_{L^\infty(\Gamma)} \leq 1$ such that (2.9) holds and

$$\mathcal{F}_\Gamma(u) = \int_{\Gamma} \mathcal{G}(u) - \sum_{v \in \text{int}(V(\Gamma))} \frac{1}{d_v} \sum_{\hat{e} \in E_v(\Gamma)} \sum_{e \in E_v(\Gamma)} [z]_e(v)([u]_e(v) - [u]_{\hat{e}}(v)).$$

Moreover, $D(\partial \mathcal{F}_\Gamma)$ is dense in $L^2(\Gamma)$.

In [34] the following space was introduced

$$G_m(\Gamma) := \{v \in L^2(\Gamma) : \exists z \in X_K(\Gamma), \quad v = -z' \text{ a.e. in } \Gamma\},$$

and the the following norm was considered in $G_m(\Gamma)$

$$\|v\|_{m,*} := \inf\{\|z\|_{\infty} : z \in X_K(\Gamma), \quad v = -z' \text{ a.e. in } \Gamma\}.$$

In the continuous setting this space was introduce in [39].

Note that, for $v \in G_m(\Gamma)$, we have that there exists $z_v \in X_K(\Gamma)$, such that $v = -z'_v$ and $\|v\|_{m,*} = \|z_v\|_{\infty}$ (see the proof of Theorem 2.19 in [34]).

From the proof of Theorem 2.16, for $f \in G_m(\Gamma)$, we have

$$\|f\|_{m,*} = \sup\left\{ \left| \int_{\Gamma} f(x)u(x)dx \right| : u \in BV(\Gamma), \quad TV(\Gamma)(u) \leq 1 \right\},$$
and, moreover,

$$\partial \mathcal{F}_\Gamma(u) = \left\{ v \in L^2(\Gamma) : \|v\|_{m,\ast} \leq 1, \int_\Gamma u(x)v(x)dx = TV_\Gamma(u) \right\}. \tag{2.10}$$

**Definition 2.17** We define the 1-Laplacian operator in the metric graph $\Gamma$ as

$$(u, v) \in \Delta_1^\Gamma \iff -v \in \partial \mathcal{F}_\Gamma(u),$$

that is, if $u \in L^2(\Gamma) \cap BV(\Gamma)$, $v \in L^2(\Gamma)$ and there exists $z \in X_K(\Gamma)$, $\|z\|_{L^\infty(\Gamma)} \leq 1$ such that

$$v = z', \quad \text{that is, } [v]_e = [z]_e' \quad \text{in } D'(0, \ell_e) \ \forall e \in E(\Gamma)$$

and

$$\mathcal{F}_\Gamma(u) = \int_\Gamma zDu - \sum_{v \in \text{int}(V(\Gamma))} \frac{1}{Dv} \sum_{e,e \in E_v(\Gamma)} [z]_e(v)([u]_e(v) - [u]_e'(v)).$$

**Remark 2.18** Let us point out that formally

$$(u, v) \in \Delta_1^\Gamma \iff v = \frac{Du}{|Du|} \iff [v]_e = \frac{D[v]_e}{|D[v]_e|} \quad \text{in } D'(0, \ell_e) \ \forall e \in E(\Gamma).$$

Then, the $z \in X_K(\Gamma)$, $\|z\|_{L^\infty(\Gamma)} \leq 1$, that appear in the characterization, represent $\frac{Du}{|Du|}$. Note that the operator $\Delta_1^\Gamma$ is multivalued.

### 3 The Cheeger problem: $\Gamma$-Cheeger and $\Gamma$-calibrable sets

Given a set $\Omega \subset \Gamma$ with $0 < \ell(\Omega) < \ell(X)$ and $\text{Per}_\Gamma(\Omega) > 0$, we define its $\Gamma$-Cheeger constant of $\Omega$ by

$$h_\Gamma^\Gamma(\Omega) := \inf \left\{ \frac{\text{Per}_\Gamma(E)}{\ell(E)} : E \subset \Omega, \ \ell(E) > 0 \right\}. \tag{3.1}$$

A set $E \subset \Omega$ achieving the infimum in (3.1) is said to be an $\Gamma$-Cheeger set of $\Omega$. Furthermore, we say that $\Omega$ is $\Gamma$-calibrable if it is an $\Gamma$-Cheeger set of itself, that is, if

$$h_\Gamma^\Gamma(\Omega) = \frac{\text{Per}_\Gamma(\Omega)}{\ell(\Omega)}.$$ 

For ease of notation, we will denote

$$\lambda_\Omega^\Gamma := \frac{\text{Per}_\Gamma(\Omega)}{\ell(\Omega)},$$

for any set $\Omega \subset \Gamma$ with $0 < \ell(\Omega)$.
Note that $\Omega$ is $\Gamma$-calibrable if and only if $\Omega$ minimizes of the functional

$$\text{Per}_{\Gamma}(E) - \lambda_{\Omega}^{\Gamma} \ell(E)$$

on the set $E \subset \Omega$, with $\ell(E) > 0$.

It is well known (see for instance [2]) that in $\mathbb{R}^N$, any Euclidean ball is a calibrable set. It is easy to see that this also happen in path graph, that is $d_v = 2$ for all $v \in \text{int}(V(\Gamma))$. Let us see in the next example that this is not true, in general, in metric graphs.

**Example 3.1** Consider the metric graph $\Gamma$ with fourth vertices and three edges, that is $V(\Gamma) = \{v_1, v_2, v_3, v_4\}$ and $E(\Gamma) = \{e_1 := [v_1, v_2], e_2 := [v_2, v_3], e_3 := [v_3, v_4]\}$, with $\ell_{e_1} = 2$, $\ell_{e_i} = 1$, $i = 2, 3$.

Consider the ball $B\left(v, \frac{5}{8}\right)$, being $v = c_{e_1}^{-1}\left(\frac{3}{2}\right)$. Then,

$$\lambda_{B\left(v, \frac{5}{8}\right)}^{\Gamma} := \frac{\text{Per}_{\Gamma}(B\left(v, \frac{5}{8}\right))}{\ell(B\left(v, \frac{5}{8}\right))} = \frac{3}{\frac{5}{8} + \frac{1}{2} + \frac{2}{8}} = \frac{24}{11}.$$

Now, by (2.4), we have

$$\text{Per}_{\Gamma}\left(B\left(v, \frac{1}{2}\right)\right) = TV_{\Gamma}\left(\chi_{B\left(v, \frac{1}{2}\right)}\right) = \sup \left\{ \left| \int_{\Gamma} u z \right| : z \in X_K(\Gamma), \|z\|_{\infty} \leq 1 \right\}$$

$$= \sup \left\{ - \int_{\Gamma} zD\chi_{B\left(v, \frac{1}{2}\right)} + \sum_{e \in E(v_2)} [z]_e(v_2)[u]_e(v_2) : z \in X_K(\Gamma), \|z\|_{\infty} \leq 1 \right\}$$

$$= \sup \left\{ [z]_{e_1}(c_{e_1}^{-1}(1)) + [z]_{e_1}(v_2) : z \in X_K(\Gamma), \|z\|_{\infty} \leq 1 \right\} = 2$$

$$\lambda_{B\left(v, \frac{1}{2}\right)}^{\Gamma} := \frac{\text{Per}_{\Gamma}(B\left(v, \frac{1}{2}\right))}{\ell(B\left(v, \frac{1}{2}\right))} = 2.$$
Therefore, the ball $B \left( v, \frac{5}{8} \right)$ is not calibrable.

It is easy to see that if $E \subset \Omega := B \left( v, \frac{5}{8} \right)$, with $\ell(E) > 0$, then $\text{Per}_\Gamma(E) \geq 2$, being $\text{Per}_\Gamma(E) = 2$ if $E \subset e_i$. Now, the subset $E \subset \Omega$ with greater volume is $E := [c_{e_1}^{-1}(\frac{7}{8}), v_2]$. Therefore,

$$h_1^\Gamma(\Omega) = \frac{\text{Per}_\Gamma(E)}{\ell(E)} = \frac{2}{\frac{9}{8}} = 16 < 2.$$  

Then, we have that $E$ is the $\Gamma$-Cheeger set of $\Omega$.  

\begin{proof}
Let $E_n \subset \Omega$ with $\ell(E_n) > 0$, such that

$$h_1^\Gamma(\Omega) = \lim_{n \to \infty} \frac{\text{Per}_\Gamma(E_n)}{\ell(E_n)}.$$

By the Embedding Theorem (Theorem 2.6), taking a subsequence if necessary, we have that there exists $E \subset \Gamma$, such that

$$\chi_E = \lim_{n \to \infty} \chi_{E_n} \text{ in } L^1(\Gamma) \text{ and a.e.}$$

Then, by the lower semi-continuity of the total variation (Corollary 2.12), we have

$$\text{Per}_\Gamma(E) \leq \liminf_{n \to \infty} \text{Per}_\Gamma(E_n).$$

Therefore

$$h_1^\Gamma(\Omega) = \frac{\text{Per}_\Gamma(E)}{\ell(E)}.$$

\end{proof}

\begin{remark}
Let $\Omega \subset \Gamma$ with $\text{Per}_\Gamma(\Omega) > 0$ and $\ell(\Omega) > 0$. Then, if there exist $\lambda > 0$ and a function $\xi : \Gamma \to \mathbb{R}$ such that $\xi(x) = 1$ for all $x \in \Omega$, satisfying

$$-\lambda \xi \in \Delta_1^\Gamma \chi_\Omega, \text{ in } \Gamma,$$

then

$$\lambda = \lambda_\Omega^\Gamma.$$  

In fact, we have that there exists $z \in X(\Gamma), \|z\|_{L^\infty(\Gamma)} \leq 1$ such that

$$-\lambda \xi = z', \quad F^\Gamma(\chi_\Omega) = \int_{\Gamma} zD\chi_\Omega - \sum_{v \in \text{int}(V(\Gamma))} \frac{1}{d_v} \sum_{e, \hat{e} \in E_v(\Gamma)} [z]_e(v)([\chi_\Omega]_e(v) - [\chi_\Omega]_{\hat{e}}(v)).$$

\end{remark}
Then, applying Green’s formula (2.3), we have
\[ \lambda_\ell(\Omega) = \int_\Gamma \chi_\Omega \lambda_\xi d\gamma = -\int_\Gamma \chi_\Omega \zeta d\gamma = \int_\Gamma zD\chi_\Omega - \sum_{v \in \text{int}(V(\Gamma))} \sum_{e \in E_v(\Gamma)} [z](v) [\chi_\Omega](v) \]
\[ = F(\chi_\Omega) = \text{Per}_\Gamma(\Omega). \]

It is well known (see [28]) that the classical Cheeger constant
\[ h_1(\Omega) := \inf \left\{ \frac{\text{Per}(E)}{|E|} : E \subset \Omega, \ |E| > 0 \right\}, \]
for a bounded smooth domain \( \Omega \subset \mathbb{R}^N \), is an optimal Poincaré constant, namely, it coincides with the first eigenvalue of the 1-Laplacian:
\[ h_1(\Omega) = \Lambda_1(\Omega) := \inf \left\{ \frac{\int_\Omega |Du| + \int_{\partial\Omega} |u|d\mathcal{H}^{N-1}}{\|u\|_{L^1(\Omega)}} : u \in BV(\Omega), \ |u|_\infty = 1 \right\}. \]

In order to get, in our context, a version of this result, we introduce the following constant. For \( \Omega \subset \Gamma \) with \( 0 < \ell(\Omega) < \ell(\Gamma) \), we define
\[ \Lambda_1^\Gamma(\Omega) = \inf \left\{ TV_\Gamma(u) : u \in BV(\Gamma), \ u = 0 \text{ in } \Gamma \setminus \Omega, \ u \geq 0, \ \int_\Gamma u(x)d(x) = 1, \ TV_\Gamma(u) > 0 \right\}. \]
\[ \Lambda_1'(\Omega) = \inf \left\{ \frac{TV_\Gamma(u)}{\int_\Gamma u(x)d(x)} : u \in BV(\Gamma), \ u = 0 \text{ in } \Gamma \setminus \Omega, \ u \geq 0, \ u \neq 0, \ TV_\Gamma(u) > 0 \right\}. \quad (3.2) \]

**Theorem 3.4** Let \( \Omega \subset X \) with \( 0 < \ell(\Omega) < \ell(X) \). Then,
\[ h_1^\Gamma(\Omega) = \Lambda_1^\Gamma(\Omega). \quad (3.3) \]

**Proof** Given a subset \( E \subset \Omega \) with \( \ell(E) > 0 \), we have
\[ \frac{TV_\Gamma(\chi_E)}{\|\chi_E\|_{L^1(\chi, \nu)}} = \frac{\text{Per}_\Gamma(E)}{\ell(E)}. \]
Therefore,
\[ \Lambda_1^\Gamma(\Omega) \leq h_1^\Gamma(\Omega). \]

Suppose the another inequality does not holds. Then, there exists \( u \in BV(\Gamma) \), \( u = 0 \text{ in } \Gamma \setminus \Omega, \ u \geq 0, \ u \neq 0, \ TV_\Gamma(u) > 0 \), such that
\[ \frac{TV_\Gamma(u)}{\int_\Gamma u(x)d(x)} < h_1^\Gamma(\Omega). \]
Then, by the coarea formula (2.7) and the Cavalieri’s Principle, we obtain

\[0 > TV_\Gamma(u) - h_1^\Gamma(\Omega) \int_\Gamma u(x)dx = \int_0^\infty (\text{Per}_\Gamma(E_t(u)) - h_1^\Gamma(\Omega) \ell(E_t(u))) dt \geq 0,\]

which is a contradiction, and consequently \(\Lambda_1^\Gamma(\Omega) = h_1^\Gamma(\Omega).\) □

Let us point out that a the equality (3.3) was obtained in [20, Theorem 6.2], but using a different concept of total variation and therefore of perimeter (see Remark 2.11).

**Remark 3.5** we are going to give a characterization of the solutions of the Euler-Lagrange equation of the variational problem (3.2). We denote by

\[K_\Omega := \left\{ u \in BV(\Gamma), \ u = 0 \text{ in } \Gamma \setminus \Omega, \ u \geq 0, \ \int_\Gamma u(x)d(x) = 1, \ TV_\Gamma(u) > 0 \right\},\]

and \(I_{K_\Omega}\) is the indicator function of \(K_\Omega\), defined by

\[I_{K_\Omega}(u) := \begin{cases} 0, & \text{if } u \in K_\Omega, \\ \infty, & \text{if } u \notin K_\Omega. \end{cases}\]

Then,

\[
\inf \left\{ TV_\Gamma(u) : u \in BV(\Gamma), \ u = 0 \text{ in } \Gamma \setminus \Omega, \ u \geq 0, \ \int_\Gamma u(x)d(x) = 1, \ TV_\Gamma(u) > 0 \right\} \\
= \inf \left\{ F_\Gamma(u) + I_{K_\Omega}(u) : u \in L^2(\Gamma) \right\}.
\]

Therefore, \(u\) is a minimizer of (3.2) if and only if \(0 \in \partial(F_\Gamma + I_{K_\Omega})(u) = \partial F_\Gamma(u) + \partial I_{K_\Omega}(u)\), where the last equality is consequence of [9, Corollary 2.11]. Then, \(u\) is a minimizer of (3.2) if and only if, \(u \in K_\Omega\) and there exists \(v \in \partial F_\Gamma(u)\) such that \(-v \in \partial I_{K_\Omega}(u)\), that is, \(\int_\Gamma uvdx \leq \int_\Gamma wvdx\) for all \(w \in K_\Omega\). Now by Theorem 2.16, we have \(v \in \partial F_\Gamma(u)\) if and only if there exists \(z \in X_K(\Gamma)\), \(\|z\|_{L^\infty(\Gamma)} \leq 1\) such that

\[v = -z' \quad \text{and} \quad \int_\Gamma u(x)v(x)dx = F_\Gamma(u) = \int_\Gamma zDu - \sum_{v \in \text{int}(V(\Gamma))} \frac{1}{d_v} \sum_{\hat{e} \in E_v(\Gamma)} \sum_{e \in E_v(\Gamma)} [z]_e(v) ([u]_e(v) - [u]_{\hat{e}}(v)).\]

Consequently, we have that \(u\) is a minimizer of (3.2) if and only if \(u \in K_\Omega\) and there exists \(z \in X_K(\Gamma)\), \(\|z\|_{L^\infty(\Gamma)} \leq 1\) such that

\[TV_\Gamma(u) \leq \int_\Gamma zDw - \sum_{v \in \text{int}(V(\Gamma))} \frac{1}{d_v} \sum_{\hat{e} \in E_v(\Gamma)} \sum_{e \in E_v(\Gamma)} [z]_e(v) ([w]_e(v) - [w]_{\hat{e}}(v)), \quad \forall w \in K_\Omega.\]
The Max-Flow Min-Cut Theorem on networks due to Ford and Fulkerson [24], in the continuous case was first studied by Strang [44] in the particular case of the plane. Given a bounded, planar domain $\Omega$, and given two functions $F, c : \Omega \to \mathbb{R}$, we want to find the maximal value of $\lambda \in \mathbb{R}$ such that there exists a vector field $V : \Omega \to \mathbb{R}^2$ satisfying
\[
\begin{align*}
\text{div } V & = \lambda F \\
\|V\|_\infty & \leq c.
\end{align*}
\]
The problem can be interpreted as follows: given a source or sink term $F$, we want to find the maximal flow in $\Omega$ under the capacity constraint given by $c$. It turns out that if $F = 1$ and $c = 1$, then the maximal value of $\lambda$ is equal to the Cheeger constant of $\Omega$, while the boundary of a Cheeger set is the associated minimal cut (see [26, 45]). Let us see now that a similar result also holds in metric graphs.

**Theorem 3.6** Let $\Omega \subset X$ with $0 < \ell(\Omega) < \ell(X)$. Then,
\[
h_1^\Gamma(\Omega) = \sup \{ h \in \mathbb{R}^+ : \exists \mathbf{z} \in X_K(\Gamma), \|\mathbf{z}\|_\infty \leq 1, \mathbf{z}' \geq h \text{ in } \Omega \}
\]
\[
= \sup \left\{ \frac{1}{\|\mathbf{z}\|_\infty} : \mathbf{z} \in X_K(\Gamma), \mathbf{z}' = \chi_{\Omega} \right\}
\]
\[
= \sup \left\{ \frac{1}{\|\mathbf{z}\|_\infty} : \mathbf{z} \in X_K(\Gamma), \mathbf{z}' = 1 \text{ in } \Omega \right\}.
\]

**Proof** Let
\[
B := \{ h \in \mathbb{R}^+ : \exists \mathbf{z} \in X_K(\Gamma), \|\mathbf{z}\|_\infty \leq 1, \mathbf{z}' \geq h \text{ in } \Omega \},
\]
and
\[
\alpha := \sup B.
\]
Given $h \in B$ and $E \subset \Omega$ with $\ell(E) > 0$, applying (2.5), we have
\[
h \ell(E) = \int_E h dx \leq \int_E \mathbf{z}' dx \leq \text{Per}(E).
\]
Hence,
\[
h \leq \frac{\text{Per}(\Gamma)}{\ell(E)}.
\]
Then, taking the supremum in $h$ and the infimum in $E$, we obtain that $\alpha \leq h_1^\Gamma(\Omega)$.

On the other hand, by Theorem 3.4, it is easy to see that
\[
\frac{1}{h_1^\Gamma(\Omega)} = \sup \left\{ \int_\Gamma u(x)dx : u \in W^{1,1}(\Gamma), u \equiv 0 \text{ in } \Omega \setminus \Gamma, u \geq 0, u \neq 0, \|u'\|_{L^1(\Gamma)} > 0 \right\}
\]
\[
\begin{align*}
\sup \left\{ \int_{\Omega} u(x)dx : u \in W^{1,1}(\Gamma), \|u\|_{L^1(\Gamma)} \leq 1, \ u = 0 \text{ in } \Gamma \setminus \Omega, \ u \geq 0, \ u \not\equiv 0 \right\} \\
= -\inf \left\{ -\int_{\Omega} u(x)dx : u \in W^{1,1}(\Gamma), \|u\|_{L^1(\Gamma)} \leq 1, \ u = 0 \text{ in } \Gamma \setminus \Omega, \ u \geq 0, \ u \not\equiv 0 \right\}.
\end{align*}
\]

Then,
\[
\begin{align*}
-\frac{1}{h_1^\Gamma(\Omega)} = \inf \left\{ F(u) + G(L(u)) : u \in L^1(\Gamma) \right\},
\end{align*}
\]

being \(L : W^{1,1}(\Gamma) \to L^1(\Gamma)\) the linear map \(L(u) := u'\), \(F(u) := -\int_{\Gamma} u\chi_{\Omega}dx\) and \(G : L^1(\Gamma) \to [0, +\infty)\) the convex function
\[
G(v) := \begin{cases} 0 & \text{if } \|v\|_{L^1(\Gamma)} \leq 1, \\ +\infty & \text{otherwise}. \end{cases}
\]

By the Fenchel–Rockafellar duality Theorem given in [22, Remark 4.2], we have
\[
\begin{align*}
\inf \left\{ F(u) + G(L(u)) : u \in L^1(\Gamma) \right\} &= \sup \left\{ -G^*(-z) - F^*(L^*(z)) : z \in L^\infty(\Gamma) \right\} \\
&= -\inf \left\{ F^*(L^*(z)) + G^*(-z) : z \in L^\infty(\Gamma) \right\}.
\end{align*}
\]

Now, \(L^*(z) = -z'\), \(G^*(z) = \|z\|_{L^\infty(\Gamma)}\) and
\[
F^*(w) = \sup_{u \in L^1(\Gamma)} \left\{ \int wudx + \int_{\Gamma} u\chi_{\Omega}dx : u \in L^1(\Gamma) \right\}.
\]

Hence
\[
F^*(L^*(z)) = \sup_{u \in L^1(\Gamma)} \left\{ -\int z'uudx + \int_{\Gamma} u\chi_{\Omega}dx : u \in L^1(\Gamma) \right\}.
\]

Therefore,
\[
\frac{1}{h_1^\Gamma(\Omega)} = \inf \left\{ \|z\|_{L^\infty(\Gamma)} : z \in L^\infty(\Gamma), \ z' = \chi_{\Omega} \right\},
\]

from where it follows that
\[
\begin{align*}
h_1^\Gamma(\Omega) &= \sup \left\{ \frac{1}{\|z\|_{L^\infty(\Gamma)}} : z \in L^\infty(\Gamma), \ z' = \chi_{\Omega} \right\} \\
&\leq \sup \left\{ \frac{1}{\|z\|_{\infty}} : z \in X_K(\Gamma), \ z' = 1 \text{ in } \Omega \right\} \leq \alpha,
\end{align*}
\]

and we finish the proof. \(\square\)
Let us recall that, in the local case, a set $\Omega \subset \mathbb{R}^N$ is called \textit{calibrable} if
\[
\frac{\text{Per}(\Omega)}{|\Omega|} = h(\Omega) \coloneqq \inf \left\{ \frac{\text{Per}(E)}{|E|} : E \subset \Omega, \; E \text{ with finite perimeter, } |E| > 0 \right\}.
\]

The following characterization of convex calibrable sets is proved in [2].

\textbf{Theorem 3.7} (\textit{[2]}) Given a bounded convex set $\Omega \subset \mathbb{R}^N$ of class $C^{1,1}$, the following assertions are equivalent:

(a) $\Omega$ is calibrable.
(b) $\chi_\Omega$ satisfies $-\Delta_1 \chi_\Omega = \frac{\text{Per}(\Omega)}{|\Omega|} \chi_\Omega$, where $\Delta_1 u \coloneqq \text{div} \left( \frac{Du}{|Du|} \right)$.

\textbf{Remark 3.8} By (2.10), we have
\[
-\lambda_\Gamma \chi_\Omega \in \Delta_1^\Gamma \chi_\Omega \iff \|\lambda_\Gamma \chi_\Omega\|_{m,*} \leq 1, \text{ and } \int_\Gamma \lambda_\Gamma \chi_\Omega \chi_\Omega = TV_\Gamma(\chi_\Omega).
\]

Now
\[
\int_\Gamma \lambda_\Gamma \chi_\Omega \chi_\Omega = \lambda_\Gamma \ell(\Omega) = TV_\Gamma(\chi_\Omega).
\]

Therefore, we have
\[
-\lambda_\Gamma \chi_\Omega \in \Delta_1^\Gamma \chi_\Omega \iff \|\lambda_\Gamma \chi_\Omega\|_{m,*} \leq 1
\]
\[
\iff \sup \left\{ \left\| \int_\Omega u(x)dx \right\| : u \in BV(\Gamma), \; TV_\Gamma(u) \leq 1 \right\} \leq \frac{\ell(\Omega)}{\text{Per}_\Gamma(\Omega)}.
\]

\[\text{(3.4)}\]

In order to get a similar result to Theorem 3.7 we need the following concept of convexity.

\textbf{Definition 3.9} We say that $\Omega \subset \Gamma$ is \textit{path-convex} if for any $E \subset \Gamma$ with $\ell(E) > 0$,
\[
\text{Per}_\Gamma(\Omega \cap E) \leq \text{Per}_\Gamma(E).
\]

We have the following version of Theorem 3.7.

\textbf{Theorem 3.10} Let $\Omega \subset \Gamma$ be with $0 < \ell(\Omega) < \ell(\Gamma)$. We have:

(i) If $\chi_\Omega$ satisfies
\[
-\lambda_\Gamma \chi_\Omega \in \Delta_1^\Gamma \chi_\Omega \quad \text{in } \Gamma,
\]
then $\Omega$ is $\Gamma$-calibrable.
(ii) If $\Omega$ is path-convex and $\Omega$ is $\Gamma$-calibrable, then Eq. (3.5) holds.
Proof (i) For any $E \subset \Omega$ with $\ell(E) > 0$, applying (3.4) with $u := \frac{\chi_E}{\text{Per}_\Gamma(E)}$, we have

$$\int_{\Omega} \frac{\chi_E}{\text{Per}_\Gamma(E)} \leq \frac{\ell(\Omega)}{\text{Per}_\Gamma(\Omega)}.$$ 

Then,

$$\frac{\text{Per}_\Gamma(\Omega)}{\ell(\Omega)} \leq \frac{\text{Per}_\Gamma(E)}{\ell(E)},$$

and consequently $\Omega$ is $\Gamma$-calibrable.

(ii) Let us prove that the function $f := \frac{\lambda}{\text{Per}_\Gamma(\Omega)} \chi_\Omega$ satisfies $\|f\|_{m,*} \leq 1$. Indeed, if $w \in BV(\Gamma) \cap L^2(\Gamma)$ is nonnegative, by the coarea formula, we have

$$\int_{\Gamma} f(x)w(x)dx = \int_0^\infty \int_{\Gamma} \lambda \chi_\Omega \chi_{E_t(w)} dx dt = \int_0^\infty \lambda \text{Per}(\Omega \cap E_t(w)) dt$$

$$\leq \int_0^\infty \text{Per}(\Omega \cap E_t(w)) dt \leq \int_0^\infty \text{Per}(E_t(w)) dt = TV_\Gamma(w).$$

Splitting any function $w \in BV(\Gamma)$ into its positive and negative part, using the above inequality one can prove that

$$\left| \int_{\Gamma} f(x)w(x)dx \right| \leq TV_\Gamma(w),$$

from where it follows that $\|f\|_{m,*} \leq 1$. Then, by (3.4), we have that $\chi_\Omega$ satisfies (3.5)



Remark 3.11 (i) Note that in Eq. (3.5) we can change $\chi_\Omega$ for a function $\xi$ such that $\xi(x) = 1$ for every $x \in \Omega$.

(ii) Let us see that this assumption $\Omega$ path-convex is necessary for (ii). For that let us give an example of a set $\Gamma$-calibrable not path-convex that verifies (3.5) but not (ii).

Consider the metric graph $\Gamma$ with two vertices and one edges, that is $V(\Gamma) = \{v_1, v_2\}$ and $E(\Gamma) = \{e := [v_1, v_2]\}$, with $\ell_e = 5$. Let $\Omega := [c^{-1}_e(1), c^{-1}_e(2)] \cup [c^{-1}_e(3), c^{-1}_e(4)]$

![Graph Diagram]

If $E \subset \Omega$, with $\text{Per}_\Gamma(E) > 0$, $\ell(E) > 0$, then obviously, $\text{Per}_\Gamma(E) \geq \text{Per}_\Gamma(\Omega)$. Hence, $\Omega$ is $\Gamma$-calibrable. On the other hand, if $E := [c^{-1}_e(1), v_2]$, we have

$$\text{Per}_\Gamma(\Omega \cap E) = 4 > \text{Per}_\Gamma(E) = 1.$$
Thus, $\Omega$ is not path-convex. Now by [8, Theorem 2.11] (see also [12]), $\chi_\Omega$ does not satisfy
\[- \lambda_\Omega^1 \chi_\Omega \in \Delta_1^1 \chi_\Omega \quad \text{in} \ \Gamma, \tag{3.6}\]
since if Eq. (3.6) has a solution, then, $\Omega$ must be of the form $\Omega = [a, b]$.

A celebrated result of De Giorgi [18] states that, if $E$ is a set of finite perimeter in $\mathbb{R}^N$, and $E^*$ is a ball such that $|E^*| = |E|$, then $\text{Per}(E^*) \leq \text{Per}(E)$, with equality holding if and only if $E$ is itself a ball. This implies that
\[h(\Omega^*) \leq h(\Omega).\]

In the next example we will see that this isoperimetric inequality is not true in metric graphs.

**Example 3.12** Consider the metric graph $\Gamma$ of the Example 3.1, that is, $V(\Gamma) = \{v_1, v_2, v_3, v_4\}$ and $E(\Gamma) = \{e_1 := [v_1, v_2], e_2 := [v_2, v_3], e_3 := [v_3, v_4]\}$, with $\ell_{e_1} = 2, \ell_{e_i} = 1, i = 2, 3$. If $E := [v_1, c_{e_1}^{-1}(\frac{3}{2})]$, we have $\ell(E) = \frac{3}{2} = \ell B \left(v_2, \frac{1}{2}\right)$.

Now,
\[\text{Per}_\Gamma(E) = 1, \quad \text{and} \quad \text{Per}_\Gamma \left( B \left(v_2, \frac{1}{2}\right) \right) = 3.\]

\[\blacksquare\]

In this section we introduce the eigenvalue problem associated with the operator $-\Delta_1^1$ and its relation with the Cheeger minimization problem.

Recall that
\[\text{sign}(u)(x) := \begin{cases} 1 & \text{if } u(x) > 0, \\ -1 & \text{if } u(x) < 0, \\ [-1, 1] & \text{if } u = 0. \end{cases}\]

**Definition 3.13** A pair $(\lambda, u) \in \mathbb{R} \times BV(\Gamma)$ is called an $\Gamma$-eigenpair of the operator $-\Delta_1^1$ on $X$ if $\|u\|_{L^1(\Gamma)} = 1$ and there exists $\xi \in \text{sign}(u)$ (i.e., $\xi(x) \in \text{sign}(u(x))$ for every $x \in \Gamma$) such that
\[\lambda \xi \in \partial F_\Gamma(u) = -\Delta_1^1 u.\]

The function $u$ is called an $\Gamma$-eigenfunction and $\lambda$ an $\Gamma$-eigenvalue associated to $u$.

Observe that, if $(\lambda, u)$ is an $\Gamma$-eigenpair of $-\Delta_1^1$, then $(\lambda, -u)$ is also an $\Gamma$-eigenpair of $-\Delta_1^1$.

**Remark 3.14** By Theorem 2.16, the following statements are equivalent:

1. $(\lambda, u)$ is an $\Gamma$-eigenpair of the operator $-\Delta_1^1$. 

(2) \( u \in BV(\Gamma), \|u\|_{L^1(\Gamma)} = 1 \) and there exists \( \xi \in \text{sign}(u) \) and \( z \in X_K(\Gamma), \|z\|_{L^\infty(\Gamma)} \leq 1 \) such that
\[
\lambda \xi = -z',
\]
and
\[
\lambda = TV_\Gamma(u);
\]
(3) \( u \in BV(\Gamma), \|u\|_{L^1(\Gamma)} = 1 \) and there exists \( \xi \in \text{sign}(u) \) and there exists \( z \in X_K(\Gamma), \|z\|_{L^\infty(\Gamma)} \leq 1 \) such that (3.7) holds and
\[
TV_\Gamma(u) = \int_\Gamma z Du - \sum_{v \in \text{int}(V(\Gamma))} \frac{1}{d_v} \sum_{e, \hat{e} \in E_v(\Gamma)} [z]_e(v) ([u]_e(v) - [u]_{\hat{e}}(v)).
\]
(3.9)

Proposition 3.15 Let \((\lambda, u)\) be an \(\Gamma\)-eigenpair of \(-\Delta^\Gamma_1\). Then,

(i) \( \lambda = 0 \iff u \) is constant, that is, \( u = \frac{1}{\ell(\Gamma)} \), or \( u = -\frac{1}{\ell(\Gamma)} \).

(ii) \( \lambda \neq 0 \iff \) there exists \( \xi \in \text{sign}(u) \) such that \( \int_\Gamma \xi(x) dx = 0 \).

Proof (i) By (3.8), if \( \lambda = 0 \), we have that \( TV_\Gamma(u) = 0 \) and then, by (2.15), we get that \( u \) is constant. Thus, since \( \|u\|_{L^1(\Gamma)} = 1 \), either \( u = \frac{1}{\ell(\Gamma)} \), or \( u = -\frac{1}{\ell(\Gamma)} \). Similarly, if \( u \) is constant a.e. then \( TV_\Gamma(u) = 0 \) and, by (3.8), \( \lambda = 0 \).

(ii) \((\iff)\) If \( \lambda = 0 \), by (i), we have that \( u = \frac{1}{\ell(\Gamma)} \), or \( u = -\frac{1}{\ell(\Gamma)} \), and this is a contradiction with the existence of \( \xi \in \text{sign}(u) \) such that \( \int_\Gamma \xi(x) dx = 0 \).

\((\implies)\) By Remark 3.14 there exists \( \xi \in \text{sign}(u) \) and \( z \in X_K(\Gamma), \|z\|_{L^\infty(\Gamma)} \leq 1 \) satisfying (3.7), (3.7) and (3.9). Hence, by Green’s formula (2.4), we have
\[
\lambda \int_\Gamma \xi(x) dx = -\int_\Gamma z' = 0
\]
Therefore, since \( \lambda \neq 0 \),
\[
\int_\Gamma \xi(x) dx = 0.
\]

Recall that, given a function \( u : \Gamma \to \mathbb{R}, \mu \in \mathbb{R} \) is a median of \( u \) with respect to a measure \( \ell \) if
\[
\ell(\{ x \in \Gamma : u(x) < \mu \}) \leq \frac{1}{2} \ell(\Gamma), \quad \ell(\{ x \in \Gamma : u(x) > \mu \}) \leq \frac{1}{2} \ell(\Gamma).
\]
We denote by \( \text{med}_\ell(u) \) the set of all medians of \( u \). It is easy to see that
\[
\mu \in \text{med}_\ell(u) \iff \]
\[-\ell(\{u = \mu\}) \leq \ell(\{x \in \Gamma : u(x) > \mu\}) - \ell(\{x \in \Gamma : u(x) < \mu\}) \leq \ell(\{u = \mu\}),\]

from where it follows that

\[0 \in \text{med}_\ell(u) \iff \exists \xi \in \text{sign}(u) \text{ such that } \int_{\Gamma} \xi(x)dx = 0, \quad (3.10)\]

By Proposition 3.15 and relation (3.10), we have the following result that was obtained for finite weighted graphs by Hein and Bühler in [27].

**Corollary 3.16** If \((\lambda, u)\) is an \(\Gamma\)-eigenpair of \(\Delta^m_1\) then

\[\lambda \neq 0 \iff 0 \in \text{med}_\ell(u).\]

**Proposition 3.17** If \(\left(\lambda, \frac{1}{\ell(\Omega)} \chi_\Omega\right)\) is a \(\Gamma\)-eigenpair of \(-\Delta^\Gamma_1\), then \(\Omega\) is \(\Gamma\)-calibrable.

**Proof** By Remark 3.14 there exists \(\xi \in \text{sign}(\chi_\Omega)\) and \(z \in X(\Gamma), \|z\|_{L^\infty(\Gamma)} \leq 1\) satisfying

\[\lambda^\Gamma_{\Omega, \xi} = -z', \quad \mathcal{F}_{\Gamma} \left(\frac{1}{\ell(\Omega)} \chi_\Omega\right) = \lambda^\Gamma_{\Omega}.\]

Then, since \(\xi = 1\) in \(\Omega\) and verifies

\[\lambda^\Gamma_{\Omega, \xi} = -z', \quad \mathcal{F}_{\Gamma}(\chi_\Omega) = \text{Per}_{\Gamma}(\Omega),\]

we have

\[-\lambda^\Gamma_{\Omega, \xi} \in \Delta^\Gamma_1 \chi_\Omega \quad \text{in } \Gamma.\]

Then, by Theorem 3.10 and having in mind Remark 3.11, we get that \(\Omega\) is \(\Gamma\)-calibrable.

In the next example we see that, in the above proposition, the reverse implication is false in general.

**Example 3.18** Consider the metric graph \(\Gamma\) with two vertices and one edge, that is \(V(\Gamma) = \{v_1, v_2\}\) and \(E(\Gamma) = \{e := [v_1, v_2]\}\), with \(\ell_e = 6\). Let \(\Omega := [c_e^{-1}(1), c_e^{-1}(5)]\).

![Diagram of the metric graph](attachment:image)

Obviously, \(\Omega\) is \(\Gamma\)-calibrable. Now, since \(0 \notin \text{med}_\ell(\chi_\Omega)\), by Corollary 3.16, we have \(\left(\lambda^\Gamma_{\Omega}, \frac{1}{\ell(\Omega)} \chi_\Omega\right)\) is not a \(\Gamma\)-eigenpair of \(\Delta^\Gamma_1\).
4 The Cheeger cut in metric graphs

We defined the $\Gamma$-Cheeger constant of $\Gamma$ as

$$h(\Gamma) := \inf \left\{ \frac{\text{Per}_\Gamma(D)}{\min\{\ell(D), \ell(\Gamma \setminus D)\}} : D \subset \Gamma, \ 0 < \ell(D) < \ell(\Gamma) \right\}$$

or, equivalently,

$$h(\Gamma) = \inf \left\{ \frac{\text{Per}_\Gamma(D)}{\ell(D)} : D \subset \Gamma, \ 0 < \ell(D) \leq \frac{1}{2} \ell(\Gamma) \right\}. \quad (4.1)$$

A partition $(D, \Gamma \setminus D)$ of $\Gamma$ is called a Cheeger cut of $\Gamma$ if $D$ is a minimizer of problem (4.1), i.e., if $0 < \ell(D) \leq \frac{1}{2} \ell(\Gamma)$ and $h(\Gamma) = \frac{\text{Per}_\Gamma(D)}{\ell(D)}$.

Note that if $D \subset \Gamma, 0 < \ell(D) \leq \frac{1}{2} \ell(\Gamma)$, we have

$$\frac{\text{Per}_\Gamma(D)}{\ell(D)} \geq \frac{1}{\frac{1}{2} \ell(\Gamma)} = \frac{2}{\ell(\Gamma)},$$

and therefore

$$h(\Gamma) \geq \frac{2}{\ell(\Gamma)}.$$

We will now give a variational characterization of the Cheeger constant which for finite weighted graphs was obtained in [46] (see also [38]). For compact Riemannian manifolds the first result of this type was obtained by Yau in [47].

**Theorem 4.1** We have

$$h(\Gamma) = \lambda_1(\Gamma) := \inf \left\{ TV_\Gamma(u) : u \in \Pi(\Gamma) \right\}, \quad (4.2)$$

where

$$\Pi(\Gamma) := \{ u \in BV(\Gamma) : \|u\|_1 = 1, \ 0 \in \text{med}_\ell(u) \}.$$

Moreover, there exists a minimizer $u$ of the problem (4.2) and also $t \geq 0$, such that $E_\Gamma(u)$ is a Cheeger cut of $\Gamma$.

**Proof** If $D \subset \Gamma, 0 < \ell(D) \leq \frac{1}{2} \ell(\Gamma)$, then $0 \in \text{med}_\ell(\chi_D)$. Thus,

$$\lambda_1(\Gamma) \leq TV_\Gamma \left( \frac{1}{\ell(D)} \chi_D \right) = \frac{1}{\ell(D)} \text{Per}_\Gamma(D)$$

and, therefore,

$$\lambda_1(\Gamma) \leq h(\Gamma).$$
On the other hand, by the Embedding Theorem (Theorem 2.6) and the lower semi-continuity of the total variation (Corollary 2.12), applying the Direct Method of Calculus of Variation, we have that there exists a function $u \in L^1(\Gamma)$ such that $\|u\|_1 = 1$ and $0 \in \text{med}_\ell(u)$, such that $TV_u(\Gamma) = \lambda_1(\Gamma)$. Now, since $0 \in \text{med}_\ell(u)$, we have $\ell(E_t(u)) \leq \frac{1}{2} \ell(\Gamma)$ for $t \geq 0$ and $\ell(\Gamma \setminus E_t(u)) \leq \frac{1}{2} \ell(\Gamma)$ for $t \leq 0$. Then by the Coarea formula (Theorem 2.13), the Cavalieri’s Principle and having in mind that the set $\{t \in \mathbb{R} : \ell(\{u = t\}) > 0\}$ is countable, we have

$$0 \leq \int_0^\infty (\text{Per}_\Gamma(E_t(u)) - h(\Gamma)\ell(E_t(u))) \, dt + \int_{-\infty}^0 (\text{Per}_\Gamma(X \setminus E_t(u)) - h(\Gamma)\ell(X \setminus E_t(u))) \, dt$$

$$= \int_{-\infty}^{+\infty} \text{Per}_\Gamma(E_t(u)) \, dt - h(\Gamma) \left( \int_0^\infty \ell(E_t(u)) \, dt + \int_{-\infty}^0 \ell(X \setminus E_t(u)) \, dt \right)$$

$$= TV(u) - h(\Gamma) \left( \int_\Gamma u^+(x) \, dx + \int_\Gamma u^-(x) \, dx \right) = TV(u) - h(\Gamma)\|u\|_1$$

$$= TV(u) - h(\Gamma) \leq 0.$$

It follows that for almost every $t \geq 0$ (in the sense of the Lebesgue measure on $\mathbb{R}$),

$$\text{Per}_\Gamma(E_t(u)) - h(\Gamma)\ell(E_t(u)) = 0. \quad (4.3)$$

Since $u \neq 0$, there must exist $t \geq 0$ such that $\ell(E_t(u)) > 0$ and for which (4.3) holds. This yields at once

$$\lambda_1(\Gamma) \leq h(\Gamma),$$

as well as that $E_t(u)$ is a Cheeger cut of $\Gamma$. \qed

**Corollary 4.2** We have

$$h(\Gamma) = \min \left\{ \frac{TV(u)}{\|u - \mu\|_1} : u \in L^1(\Gamma), \mu \in \text{med}_\ell(u) \right\}$$

$$= \inf \left\{ \sup_{c \in \mathbb{R}} \frac{TV(u)}{\|u - c\|_1} : u \in L^1(\Gamma) \right\}.$$

**Proof** A simple calculation shows that

$$\lambda_1(\Gamma) = \min \left\{ \frac{TV(u)}{\|u - \mu\|_1} : u \in L^1(\Gamma), \mu \in \text{med}_\ell(u) \right\}.$$

Let

$$\alpha := \inf \left\{ \sup_{c \in \mathbb{R}} \frac{TV(u)}{\|u - c\|_1} : u \in L^1(\Gamma) \right\}.$$
Given \( u \in L^1(\Gamma) \) and \( \mu \in \text{med}_\ell(u) \), we have

\[
\frac{TV_\Gamma(u)}{\|u - \mu\|_1} \leq \sup_{c \in \mathbb{R}} \frac{TV_\Gamma(u)}{\|u - c\|_1},
\]

hence

\[
h(\Gamma) = \lambda_1(\Gamma) \leq \alpha.
\]

To prove the another inequality, let \( D \subset \Gamma \), with \( \ell(D) \leq \ell(\Gamma \setminus D) \), such that

\[
h(\Gamma) = \frac{\text{Per}_\Gamma(D)}{\ell(D)}.
\]

take \( v := \chi_D - \chi_{\Gamma \setminus D} \). Then,

\[
\alpha = \inf \left\{ \sup_{c \in \mathbb{R}} \frac{TV_\Gamma(u)}{\|u - c\|_1} : u \in L^1(\Gamma) \right\} = \inf \left\{ \max_{|c| \leq 1} \frac{TV_\Gamma(u)}{\|u - c\|_1} : u \in L^1(\Gamma) \right\}
\leq \max_{|c| \leq 1} \frac{TV_\Gamma(v)}{\|v - c\|_1} = \frac{2\text{Per}_\Gamma(D)}{(1-c)\ell(D) + (1+c)\ell(\Gamma \setminus D)} \leq \frac{\text{Per}_\Gamma(D)}{\ell(D)} = h(\Gamma).
\]

\[\square\]

For finite weighted graphs, it is well known that the first non–zero eigenvalue coincides with the Cheeger constant (see [13]). This result is not true for infinite weighted graphs (see [38]). In the next result we will see that this is true in metric graphs.

**Theorem 4.3** We have

\[
h(\Gamma) = \inf\left\{ \lambda \neq 0, \text{ such that } \lambda \text{ is a } \Gamma \text{-eigenvalue of } -\Delta_1^\Gamma \right\}.
\]

Moreover, \( h(\Gamma) \) is the first non-zero \( \Gamma \)-eigenvalue of \(-\Delta_1^\Gamma\) and if \( u \) is a minimizer of problem (4.2), then, there exists \( t \geq 0 \), such that \( E_t(u) \) is a Cheeger cut of \( \Gamma \) and

\[
\left( h(\Gamma), \frac{1}{\ell(E_t(u))} \chi_{E_t(u)} \right)
\]

is a \( \Gamma \)-eigenpair of \(-\Delta_1^\Gamma\).

**Proof** By Corollary 3.16, if \( (\lambda, u) \) is an \( \Gamma \)-eigenpair of \(-\Delta_1^\Gamma\) and \( \lambda \neq 0 \) then \( u \in \Pi(\Gamma) \). Now, \( TV_\Gamma(u) = \lambda \), thus, as a consequence of Theorem 4.1, we have the

\[
h(\Gamma) \leq \lambda.
\]
On the other hand, by Theorem 4.1, there exists \( t \geq 0 \), such that \( E_t(u) \) is a Cheeger cut of \( \Gamma \). Then, \( E_t(u) \) is \( \Gamma \)-calibrable. Hence, by Theorem 3.6,

\[
h(\Gamma) = h^1(\Gamma(u)) = \sup \left\{ \frac{1}{\|z\|_\infty} : z \in X_K(\Gamma), \ z' = \chi_{E_t(u)} \right\}.
\]

Then, there exists a sequence \( z_n \in X_K(\Gamma) \) with \( z'_n = \chi_{E_t(u)} \) for all \( n \in \mathbb{N} \), such that

\[
h(\Gamma) = \lim_{n \to \infty} \frac{1}{\|z_n\|_\infty}.
\]

Now, since \( h(\Gamma) > 0 \), we have \( \{\|z_n\|_\infty : n \in \mathbb{N}\} \) is bounded. Thus, we can assume, taking a subsequence if necessary, that

\[
z_n \to z, \text{ weakly }^* \text{ in } L^\infty(\Gamma), \text{ and } z' = \chi_{E_t(u)}.
\]

Let us see now that \( z \in X_K(\Gamma) \). by Proposition 2.2, we have that

\[
\lim_{n \to \infty} \int_{\Gamma} z_n Du = \int_{\Gamma} z Du, \ \forall u \in BV(\Gamma).
\]

Fix \( v \in V(\Gamma) \). Applying Green’s formula (2.4) to \( z_n \) and \( u \in BV(\Gamma) \), we get

\[
\int_{\Gamma} z_n Du + \int_{\Gamma} u z'_n = \sum_{v \in V(\Gamma)} \sum_{e \in E_v(\Gamma)} [z_n](v)[u](v).
\]

Hence, take \( u \) such that \( [u](v) = 1 \) for all \( e \in E_v(\Gamma) \) and \( [u]_e = 0 \) if \( v \not\in E_v(\Gamma) \), we have

\[
\int_{\Gamma} z_n Du + \int_{\Gamma} u z'_n = \sum_{e \in E_v(\Gamma)} [z_n](v)[u](v) = 0.
\]

Then, taking the limit as \( n \to \infty \) and having in mind (2.2), we obtain

\[
0 = \int_{\Gamma} z Du + \int_{\Gamma} u z' = \sum_{e \in E_v(\Gamma)} [z](v)[u](v).
\]

Therefore, \( z \in X_K(\Gamma) \).

If we take \( \tilde{z} := -h(\Gamma)z \), and \( v := \frac{1}{\ell(E_t(u))} \chi_{E_t(u)} \), we have \( \tilde{z} \in X_K(\Gamma) \), \( \|	ilde{z}\|_\infty \leq 1 \) and

\[
-\tilde{z}' = h(\Gamma)\chi_{E_t(u)}, \ TV_\Gamma(v) = h(\Gamma).
\]
Therefore,
\[
\left( h(\Gamma), \frac{1}{\ell(E_t(u))} \chi_{E_t(u)} \right)
\]
is a $\Gamma$-eigenpair of $-\Delta_1^\Gamma$.

\[\square\]

**Remark 4.4** In [19, Theorem 1.3] was proved that if we define
\[
\Lambda_{2,p}(\Gamma) := \inf \left\{ \int_\Gamma |u'(x)|^p dx : u \in W^{1,p}(\Gamma), \int_\Gamma |u(x)|^{p-2}u(x)dx = 0, u \not\equiv 0 \right\},
\]
then if $u_p$ is a minimizer of (4.4), there exists a subsequence $p_j \to 1^+$, and $u \in BV(\Gamma)$, such that
\[
u_p j \to u \text{ in } L^1(\Gamma),
\]
being $u$ is a minimizer of (4.2). Moreover,
\[
\lim_{p \to 1^+} \Lambda_{2,p}(\Gamma) = \Lambda_{2,1}(\Gamma),
\]
where
\[
\Lambda_{2,1}(\Gamma) := \inf \left\{ |D u|(\Gamma) : u \in \Pi(\Gamma) \right\}.
\]

Let us point out that, since for $u \in BV(\Gamma)$, in general, $|D u|(\Gamma) < TV_\Gamma(u)$, we have $\Lambda_{2,1}(\Gamma) < \lambda_1(\Gamma)$. Moreover, even more, with this definition of $\Lambda_{2,1}(\Gamma)$, it is possible that $\Lambda_{2,1}(\Gamma) = 0$, for instance if $V(\Gamma) = \{v_1, v_2, v_3\}$ and $E(\Gamma) = \{e_1 := [v_1, v_2], e_2 := [v_2, v_3]\}$, with $\ell_{e_1} = \ell_{e_2}$, then if $[u_i]_{e_i} = \chi_{(0, \ell_i)}$, we have $u_i \in \Pi(\Gamma)$ and $|D u_i|(\Gamma) = 0$.

Let $A \subset \Gamma$ with $\text{Per}_\Gamma(A) > 0$, $\ell(A) = \frac{1}{2} \ell(\Gamma)$, and $u = \frac{1}{\ell(\Gamma)} \left( \chi_A - \chi_{\Gamma \setminus A} \right)$. It is easy to see that $TV_\Gamma(u) = \frac{2}{\ell(\Gamma)^2} \text{Per}_\Gamma(A) = \frac{\text{Per}_\Gamma(A)}{\ell(A)} > 0$. Hence, since $\|u\|_1 = 1$ and $0 \in \text{med}_v(u)$, we obtain the following result as a consequence of Theorem 4.1.

**Corollary 4.5** Let $A \subset \Gamma$ with $\ell(A) = \frac{1}{2} \ell(\Gamma)$ and $u = \frac{1}{\ell(\Gamma)} \left( \chi_A - \chi_{\Gamma \setminus A} \right)$. Then,
\[
h(\Gamma) = \frac{\text{Per}_\Gamma(A)}{\ell(A)} \iff u = \frac{1}{\ell(\Gamma)} \left( \chi_A - \chi_{\Gamma \setminus A} \right) \text{ is a minimizer of (4.2).}
\]

A similar result was proved in [19, Theorem 1.4], but we have observed in Remark 4.4 that their concept of perimeter is different to the one we used here.

In the next example we will see that there are Cheeger cup $E$ such that $\ell(E) < \frac{1}{2} \ell(\Gamma)$.

**Example 4.6** Consider the metric graph $\Gamma$ with four vertices and three edges, $V(\Gamma) = \{v_1, v_2, v_3, v_4\}$ and $E(\Gamma) = \{e_1 := [v_1, v_2], e_2 := [v_2, v_3], e_3 := [v_2, v_4]\}$. 
If we assume that \( \ell_{e_i} = L \) for \( i = 1, 2, 3 \), then each \( e_i \) is a Cheeger cut of \( \Gamma \). In fact, if \( D \subset \Gamma \) has \( \text{Per}_\Gamma(D) = 1 \), then \( D \subset e_i \). Now, if \( D \neq e_i \), then
\[
\frac{\text{Per}_\Gamma(D)}{\ell(D)} > \frac{\text{Per}_\Gamma(e_i)}{\ell(e_i)} = \frac{1}{L}.
\]
Moreover, if \( D \subset \Gamma \) and \( L < \ell(D) \leq \frac{3L}{2} \), then \( \text{Per}_\Gamma(D) \geq 2 \). Hence
\[
\frac{\text{Per}_\Gamma(D)}{\ell(D)} \geq \frac{2}{\frac{3L}{2}} = \frac{4}{3L} > \frac{1}{L}.
\]
Thus
\[
h(\Gamma) = \frac{\text{Per}_\Gamma(e_i)}{\ell(e_i)} = \frac{1}{L},
\]
and consequently, each \( e_i \) is a Cheeger cut of \( \Gamma \).

Moreover, \( (h(\Gamma), \frac{1}{L} \chi_{e_i}) \) is a \( \Gamma \)-eigenpair of \(-\Delta^1_{\Gamma}\). For instance, for \( e_1 \), if we define \( z \) as
\[
[z]_{e_1}(x) := -\frac{1}{L}x, \quad [z]_{e_1}(x) := \frac{2}{L}x - 2, \quad i = 2, 3, \ x \in (0, L),
\]
then,
\[
-z' = h(\Gamma)\chi_{e_1} \quad \text{in} \ D(\Gamma), \quad \text{and} \quad TV_{\Gamma}(\frac{1}{L} \chi_{e_1}) = h(\Gamma).
\]
Therefore \( (h(\Gamma), \frac{1}{L} \chi_{e_i}) \) is an \( \Gamma \)-eigenpair of \(-\Delta^1_{\Gamma}\).

If we assume now that \( \ell_{e_1} > \ell_{e_2} + \ell_{e_3} \), then it is easy to see that
\[
h(\Gamma) = \frac{2}{\ell_{e_1} + \ell_{e_2} + \ell_{e_3}}, \quad \text{and} \quad \left( v_1, c_{e_1}^{-1} \left( \frac{\ell_{e_1} + \ell_{e_2} + \ell_{e_3}}{2} \right) \right) \text{ is a Cheeger cut of } \Gamma.
\]
Now we are going to get a Cheeger inequality of type (1.1) for metric graphs. For that let us introduce the Laplace operator $\Delta_\Gamma$ on the metric graph $\Gamma$. This is a standard procedure and we refer the interested reader to [7, 11]. For a function $u \in W^{1,1}(\Gamma)$, if $e \in E(\Gamma)$ and $v \in V(\Gamma)$, we define the normal derivative of $u$ at $v$ as

$$\frac{\partial [u]_e}{\partial n_e}(v) := \begin{cases} -[u]_e(0+), & \text{if } v = i_e \\ [u]_e(\ell -), & \text{if } v = f_e. \end{cases}$$

The Sobolev space $W^{2,2}(\Gamma)$ is defined as the space of functions $u$ on $\Gamma$ such that $[u]_e \in W^{2,2}(0, \ell_e)$ for all $e \in E(\Gamma)$. The operator $\Delta_\Gamma$ has domain

$$D(\Delta_\Gamma) := \left\{ u \in W^{2,2}(\Gamma) : u \text{ continuous and } \sum_{e \in E(\Gamma)} \frac{\partial [u]_e}{\partial n_e}(v) = 0 \text{ for all } v \in V(\Gamma) \right\}$$

and it applies to any function $u \in D(\Delta_\Gamma)$ as follows

$$[\Delta_\Gamma u]_e := ([u]_e)_{xx}, \text{ for all } e \in E(\Gamma).$$

The energy functional form associated to $\Delta_\Gamma$ is given by

$$\mathcal{H}_\Gamma(u) := \int_\Gamma (u'(x))^2 \, dx = \sum_{e \in E(\Gamma)} \|[u]_e'\|_{L^2(0, \ell_e)}^2.$$  

We have

$$\mathcal{H}_\Gamma(u) = -\int_\Gamma u(x) \Delta_\Gamma u(x) \, dx, \text{ for } u \in D(\Delta_\Gamma).$$

The operator $-\Delta_\Gamma$ is selfadjoint in $L^2(\Gamma)$ and

$$\sigma(-\Delta_\Gamma) \setminus \{0\} = \{ \mu_1(\Gamma), \mu_2(\Gamma), \ldots \},$$

being

$$\mu_1(\Gamma) = \text{gap}(-\Delta_\Gamma) = \min \left\{ \frac{\mathcal{H}_\Gamma(u)}{\|u\|_{L^2(\Gamma)}^2} : u \in D(\mathcal{H}_\Gamma), \|u\|_{L^2(\Gamma)} \neq 0, \int_\Gamma u(x) \, dx = 0 \right\}.$$  

**Theorem 4.7** We have the following Cheeger Inequality:

$$\frac{1}{4} h(\Gamma)^2 \leq \text{gap}(-\Delta_\Gamma). \quad (4.5)$$
**Proof** Let \( u \in D(\mathcal{H}_\Gamma) \), with \( \int_\Gamma u(x)dx = 0 \), such that

\[
\frac{\mathcal{H}_\Gamma(u)}{\|u\|_{L^2(\Gamma)}^2} = \text{gap}(-\Delta \Gamma).
\]

If \( \alpha \in \text{med}_\ell(u) \) and \( v := u - \alpha \), we have \( 0 \in \text{med}_\ell(v^2) \). Then, by Theorem 4.1, we have

\[
h(\Gamma) \leq \frac{\int_\Gamma (v^2)'(x)dx}{\int_\Gamma v^2(x)dx}.
\]

Now, since \( \int_\Gamma u(x)dx = 0 \), we have

\[
\int_\Gamma v^2(x)dx \geq \int_\Gamma u^2(x)dx.
\]

On the other hand, by Cauchy-Schwartz

\[
\int_\Gamma (v^2)'(x)dx = 2 \int_\Gamma v(x)v'(x)dx \leq 2 \left( \int_\Gamma v^2(x)dx \right)^{\frac{1}{2}} \left( \int_\Gamma (v')^2(x)dx \right)^{\frac{1}{2}}.
\]

Thus

\[
h(\Gamma)^2 \leq \frac{4 \left( \int_\Gamma v^2(x)dx \right) \left( \int_\Gamma (v')^2(x)dx \right)}{\left( \int_\Gamma v^2(x)dx \right)^2}
\]

\[
= \frac{4 \int_\Gamma (u')^2(x)dx}{\int_\Gamma v^2(x)dx} \leq \frac{4 \mathcal{H}_\Gamma(u)}{\|u\|_{L^2(\Gamma)}^2} = 4\text{gap}(-\Delta \Gamma),
\]

and therefore (4.5) holds. \( \square \)

Let us point out that the Cheeger Inequality (4.5) was also prove by Nicaise [41] (see also [30, 43]) but with a different proof and for a different concept of perimeter.

**Acknowledgements** The author have been partially supported by the Spanish MCIU and FEDER, Project PGC2018-094775-B-100 and by Conselleria d’Innovació, Universitats, Ciència y Sociedad Digital, Project AICO/2021/223. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Funding** Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.
Compliance with ethical standards

Conflict of interest  The author declare that he has not conflict of interest.

Open Access  This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Alon, N., Milman, V.D.: $\lambda_1$, Isoperimetric inequalities for graphs, and superconcentrators. J. Combin. Theory Ser. B 38, 73–88 (1985)
2. Alter, F., Caselles, V., Chambolle, A.: A characterization of convex calibrable sets in $\mathbb{R}^N$. Math. Ann. 332, 329–366 (2005)
3. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford Mathematical Monographs (2000)
4. Andreu, F., Caselles, V., Mazón, J.M.: Parabolic Quasilinear Equations Minimizing Linear Growth Functionals. Progress in Mathematics, vol. 223. Birkhäuser (2004)
5. Anzellotti, G.: Pairings between measures and bounded functions and compensated compactness. Annali di Matematica Pura ed Applicata IV 135, 293–318 (1983)
6. Berkolaiko, G., Carlson, R., Fulling, S., Kuchment, P.: Quantum Graphs and Their Applications. Contemporary Mathematics, vol. 415. American Mathematical Society, Providence (2006)
7. Berkolaiko, G., Kuchment, P.: Introduction to Quantum Graphs. Mathematical Surveys and Monographs, vol. 186. American Mathematical Society, Providence (2013)
8. Bonforte, M., Figalli, A.: Total variation flow and sign fast diffusion in one dimension. J. Differ. Equ. 252, 4455–4480 (2012)
9. Brezis, H.: Operateurs Maximaux Monotones. North Holland, Amsterdam (1973)
10. Bühler, T., Hein, M.: Spectral clustering based on the graph $p$-Laplacian. In: Proceedings of the 26th International Conference on Machine Learning, pp. 81–88. Omnipress (2009)
11. Cattaneo, C.: The spectrum of the continuous Laplacian on a graph. Monatsch. Math. 124, 215–235 (1997)
12. Chang, K.C.: Spectrum of the 1-Laplacian operator. Commun. Contemp. Math. 11, 865–894 (2009)
13. Chang, K.C.: Spectrum of the 1-Laplacian and Cheeger’s constant on graphs. J. Graph Theory 81, 167–207 (2016)
14. Chang, K.C., Shao, S., Zhang, D.: The 1-Laplacian Cheeger cut: theory and algorithms. J. Comput. Math. 33, 443–467 (2015)
15. Chang, K.C., Shao, S., Zhang, D.: Cheeger’s cut, maxcut and the spectral theory of 1-Laplacian on graphs. Sci. China Math. 60, 1963–1980 (2017)
16. Cheeger, J.: A lower bound for the smallest eigenvalue of the Laplacian. In: Problems in Analysis: A Symposium in Honor of Salomon Bochner, pp. 195–199. Princeton University Press (1970)
17. Chung, F.: Spectral Graph Theory (CBMS Regional Conference Series in Mathematics, No. 92). American Mathematical Society (1997)
18. De Giorgi, E.: Sulla proprietà isoperimetrica dell’ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita, Atti della Accademia Nazionale dei Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I(5), 33–44 (1958)
19. Del Pezzo, L., Rossi, J.: Clustering for metric graphs using the $p$-Laplacian. Mich. Math. J. 65, 451–472 (2016)
20. Del Pezzo, L., Rossi, J.: The first eigenvalue of the $p$-Laplacian on quantum graphs. Anal. Math. Phys. 6, 365–391 (2016)
21. Dodziuk, J.: Difference equations, isoperimetric inequality and transience of certain random walks. Trans. Am. Math. Soc. 284, 787–794 (1984)
22. Ekeland, I., Temam, R.: Convex Analysis and Variational Problems. North-Holland Publishing Company, Amsterdam (1976)
23. Exner, P., Keating, J.P., Kuchment, P., Sunada, T., Teplyaev, A.: Analysis on graphs and its applications. In: Proceedings of Symposia in Pure Mathematics, vol. 77. American Mathematical Society, Providence (2008)
24. Ford, L.R., Jr., Fulkerson, D.R.: Maximal flow through a network. Can. J. Math. 8, 399–404 (1956)
25. Gnutzmann, S., Smilansky, U.: Quantum graphs: applications to quantum chaos and universal spectral statistics. Adv. Phys. 55, 527–625 (2006)
26. Grieser, D.: The first eigenvalue of the Laplacian, isoperimetric constants, and the max flow min cut theorem. Archiv der Matematik 87, 75–85 (2006)
27. Hein, M., Bühler, T.: An inverse power method for nonlinear eigenproblems with applications in 1-spectral clustering and sparse PCA. Adv. Neural Inf. Process. Syst. 23, 847–855 (2010)
28. Kawohl, B., Fridman, Y.: Isoperimetric estimates for the first eigenvalue of the $p$-Laplace operator and the Cheeger constant. Comment. Math. Univ. Carolin. 44, 659–667 (2003)
29. Kennedy, J.B., Mugnolo, D.: The Cheeger constant of a quantum graph. Proc. Appl. Math. Mech. 16, 875–876 (2016)
30. Kostenko, A., Nicolusi, N.: Spectral estimates for the infinite quantum graphs. Cal. Var. Partial Differ. Equ. 58(1), Paper 15, 40 pp (2019)
31. Kostrykin, V., Schrader, R.: Laplacians on metric graphs: eigenvalues, resolvents and semigroups. In Quantum Graphs and Their Applications. Contemporary Mathematics, vol. 415, pp. 201–225. American Mathematical Society, Providence (2006)
32. Leonardi, G.P.: An overview on the Cheeger problem. In: New Trends in Shape Optimization. International Series in Numerical Mathematics, vol. 166, pp. 117–139. Springer (2015)
33. von Luxburg, U.: A tutorial on spectral clustering. Stat. Comput. 17, 395–416 (2007)
34. Mazón, J.M.: The total variation flow in metric graphs. Math. Eng. 5(1), 1–38 (2023). https://doi.org/10.3934/mine.2023009
35. Mazón, J.M., Rossi, J.D., Toledo, J.: Nonlocal perimeter, curvature and minimal surfaces for measurable sets. J. Anal. Math. 138(1), 235–279 (2019)
36. Mazón, J.M., Rossi, J.D., Toledo, J.: Nonlocal Perimeter. Curvature and Minimal Surfaces for Measurable Sets. Frontiers in Mathematics, Birkhäuser (2019)
37. Mazón, J.M., Solera, M., Toledo, J.: The heat flow on metric random walk spaces. J. Math. Anal. Appl. 483, 123645 (2020)
38. Mazón, J.M., Solera, M., Toledo, J.: The total variation flow in metric random walk spaces. Calc. Var. 59, 29 (2020)
39. Meyer, Y.: Oscillating Patterns in Image Processing and Nonlinear Evolution Equations. University Lecture Series, vol. 22. American Mathematical Society, Providence, RI (2001)
40. Mugnolo, D.: Semigroup Methods for Evolution Equations on Networks. Understanding Complex Systems. Springer, Cham (2014)
41. Nicaise, S.: Spectre des réseaux topologiques finis. Bull. Sci. Math., II. Sér. 111, 401–413 (1987)
42. Parini, E.: An introduction to the Cheeger problem. Surv. Math Appl. 6, 9–22 (2011)
43. Post, O.: Spectral analysis of metric graphs and related spaces. In: Arzhantseva, G., Valette, A. (eds.) Limits of Graphs in Group Theory and Computer Science, pp. 109–140. Presses Polytechniques et Universitaires Romandes, Lausanne (2009)
44. Strang, G.: Maximal flow through a domain. Math. Program. 26, 123–143 (1983)
45. Strang, G.: Maximum flow and minimum cuts in the plane. J. Glob. Optim. 47, 527–535 (2010)
46. Szlam, A., Bresson, X.: Total variation and Cheeger cuts. In: Proceedings of the 27th International Conference on Machine Learning, Haifa, Israel, (2010)
47. Yau, S.-T.: Isoperimetric constants and the first eigenvalue of a compact Riemann Manifold. Ann. Scient. Ec. Norm. Sup., 4e série, t.8, 487–507 (1975)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.