Asymptotic behavior of solutions of a free boundary problem modeling tumor spheroid with Gibbs–Thomson relation

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Abstract

In this paper we study a free boundary problem modeling the growth of solid tumor spheroid. It consists of two elliptic equations describing nutrient diffusion and pressure distribution within tumor, respectively. The new feature is that nutrient concentration on the boundary is less than external supply due to a Gibbs–Thomson relation and the problem has two radial stationary solutions, which differs from widely studied tumor spheroid model with surface tension effect. We first establish local well-posedness by using a functional approach based on Fourier multiplier method and analytic semigroup theory. Then we investigate stability of each radial stationary solution. By employing a generalized principle of linearized stability, we prove that the radial stationary solution with a smaller radius is always unstable, and there exists a positive threshold value \( \gamma^* \) of cell-to-cell adhesiveness \( \gamma \), such that the radial stationary solution with a larger radius is asymptotically stable for \( \gamma > \gamma^* \), and unstable for \( 0 < \gamma < \gamma^* \).

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1 Introduction

In the last several decades, numerous tumor models have been established to explore the mechanism of tumor growth \cite{25,28}. Since the tumor region is evolving with time, Greenspan \cite{24} first pointed out that it is natural to model tumor growth in form of free boundary problems of partial differential equations. Byrne and Chaplain developed Greenspan’s idea and proposed
the following model [5, 6]:

\[
\begin{aligned}
\Delta \sigma &= \sigma \quad \text{in } \Omega(t) \times (0, +\infty), \\
\Delta p &= -\mu(\sigma - \bar{\sigma}) \quad \text{in } \Omega(t) \times (0, +\infty), \\
\sigma &= \bar{\sigma}(1 - \gamma \kappa) \quad \text{on } \partial\Omega(t) \times (0, +\infty), \\
p &= \bar{p} \quad \text{on } \partial\Omega(t) \times (0, +\infty), \\
V &= -\partial_n p \quad \text{on } \partial\Omega(t) \times (0, +\infty), \\
\Omega(0) &= \Omega_0,
\end{aligned}
\]  

(1.1)

where \(\sigma = \sigma(x,t)\) and \(p = p(x,t)\) denote nutrient concentration and internal pressure within the domain \(\Omega(t) \subset \mathbb{R}^3\) occupied by tumor at time \(t > 0\), respectively. It is a free boundary problem, in which \(\sigma(x,t), p(x,t)\) and \(\Omega(t)\) are unknowns and have to be determined together. \(\kappa\) is the mean curvature, \(V\) is the outward normal velocity and \(n\) denotes the unit outward normal of tumor boundary \(\partial\Omega(t)\), respectively. \(\bar{\sigma}, \bar{p}, \bar{\sigma}, \mu\) and \(\gamma\) are positive constants, where \(\bar{\sigma}\) represents constant external nutrient supply, \(\bar{p}\) denotes constant external pressure, \(\bar{\sigma}\) is a critical nutrient concentration for apoptosis, \(\mu\) is the proliferation rate of tumor cells and \(\gamma\) is cell-to-cell adhesiveness on tumor boundary. \(\Omega_0\) is the region occupied by tumor initially.

The above model (1.1), which is usually called tumor spheroid model with Gibbs–Thomson relation, was proposed based on the following considerations [5, 6]. Firstly, it is hypothesized that energy is expended in maintaining the tumor’s compactness by cell-to-cell adhesion on the boundary and the nutrient acts as a source of such energy and satisfies Gibbs–Thomson relation. This relation means that nutrient concentration on the tumor boundary is less than external nutrient concentration and the jump is proportional to local mean curvature. Secondly, the pressure on the boundary is assumed to be equal to external pressure, reflecting that it is continuous across the tumor boundary. For a special simplification of this model by replacing the first equation of (1.1) with \(\Delta \sigma = C\) for a constant \(C\), radial solution was studied and linear stability of radial stationary solution under asymmetric perturbations was obtained in [5]. Byrne also did a further weakly nonlinear analysis and numerical verification in [8]. Recently, Wu [30] proved that for \(0 < \bar{\sigma}/\bar{\sigma} < \theta_*\) with some constant \(\theta_* \in (0,1)\), problem (1.1) has two radial stationary solutions, and there exist infinite many symmetry-breaking branches of stationary solutions bifurcating from radial stationary ones.

If the boundary conditions \(\sigma = \bar{\sigma}(1 - \gamma \kappa)\) and \(p = \bar{p}\) in (1.1) are replaced by the following

\[
\sigma = \bar{\sigma}, \quad p = \gamma \kappa \quad \text{on } \partial\Omega(t) \times (0, +\infty),
\]  

(1.2)

the corresponding model is called tumor spheroid model with surface tension effect. Boundary condition (1.2) assumes that nutrient concentration is continuous across the tumor boundary and the pressure on the tumor surface is proportional to the mean curvature to maintain the cell-to-cell adhesiveness. Note that tumor spheroid model with surface tension effect belongs to category of Hele–Shaw type. In fact, by letting \(\mu = 0\), it becomes the classical Hele–Shaw problem. The cell proliferation rate \(\mu > 0\) is indeed biologically meaningful in tumor spheroid
model with surface tension effect. This novelty has attracted a lot of attention, and many illuminative results have been obtained. For \(0 < \tilde{\sigma}/\bar{\sigma} < 1\), tumor spheroid model with surface tension effect has a unique radial stationary solution, which is globally asymptotically stable under radially symmetric perturbations (cf. [21]). Bifurcation analysis shows that there exist infinite many symmetry-breaking branches of bifurcation stationary solutions (cf. [2, 9, 15, 22]). For asymptotic stability of radial stationary solution under non-radial perturbations, there exists a threshold value \(\gamma_\ast\) of cell-to-cell adhesiveness such that the unique radial stationary solution is asymptotic stable for \(\gamma > \gamma_\ast\), while it is unstable for \(0 < \gamma < \gamma_\ast\) (cf. [7, 10, 17, 18]). It is worthy to mention that there is not such a threshold value in stability result of Hele–Shaw problem. For extended studies of such type of tumor models, we refer readers to [8, 16, 19, 20, 31–33, 35] and references therein.

In this paper, we study asymptotic stability of the two radial stationary solutions of (1.1) under non-radial perturbations. Our motivation is two-fold. First, tumor spheroid model with surface tension effect neglects the discontinuity of nutrient flux across tumor boundary. Problem (1.1) overcomes this disadvantage. As pointed out by Roose, Chapman and Maini, though the boundary condition with Gibbs–Thomson relation seems speculative, it may be possible to check its veracity in experiments; moreover, problem (1.1) raises a number of interesting mathematical and experimental points (see page 194 in [28]). Second, though the form of problem (1.1) seems similar to tumor spheroid model with surface tension effect, the unknowns are coupled in different ways, which induces totally different nonlinearity and some new difficulties arise. In fact, based on classical theory of elliptic boundary value problem, problem (1.1) and tumor spheroid model with surface tension effect can be both reduced into abstract differential equations of the form \(\partial_t \rho + \Psi(\rho) = 0\), where the unknown \(\rho(t)\) is used to describe the free boundary \(\partial \Omega(t)\). For tumor spheroid model with surface tension effect, \(\Psi\) is a three-order quasi-linear differential operator in suitable Banach spaces. It can be regarded as a perturbation of the corresponding reduced operator of Hele–Shaw problem, so that its linearization is a sectorial operator and the corresponding abstract equation is of parabolic type. While for problem (1.1), \(\Psi\) is first-order and possesses a fully nonlinear structure. We have to make a delicate analysis of \(\Psi\) and determine whether its linearization is a sectorial operator or not. By using the technique of localization and Fourier multiplier analysis developed in [13, 14], we overcome this difficulty and prove that the abstract equation of (1.1) is still of parabolic type. Thus local well-posedness follows from analytic semigroup theory. Then with the help of some profound properties of modified Bessel functions, we calculate and analyze spectra of the linearized operator of \(\Psi\). Finally, based on a generalized principle of linearized stability developed in [27], we prove that the radial stationary solution with smaller radius is always unstable, and there exists a positive threshold value \(\gamma_\ast\) of cell-to-cell adhesiveness \(\gamma\), such that the radial stationary solution with larger radius is asymptotically stable for \(\gamma > \gamma_\ast\), and unstable for \(0 < \gamma < \gamma_\ast\).

It is interesting to compare tumor spheroid model with Gibbs–Thomson relation and tumor spheroid model with surface tension effect. For tumor spheroid model with Gibbs–Thomson relation, except for the two radial stationary solutions, the threshold value \(\gamma_\ast\) of cell-to-cell adhesiveness, which is given by (4.25) and (4.26), is independent of proliferation rate \(\mu\). This
indicates that the proliferation rate $\mu$ has no effect on stability of radial dormant tumor. While for tumor spheroid model with surface tension effect, the corresponding threshold value $\gamma_s$ is a linear function of $\mu$ (cf. [10, 18]), so that the proliferation rate $\mu$ indeed affects tumor’s stability. This is a significant difference between these two modeling methods.

To give a precise statement of our main result, we introduce some notations. Let $\Omega$ be a bounded open domain in $\mathbb{R}^3$ with a smooth boundary $\Gamma$. We denote by $BUC^s(\Omega)$ the space of all bounded and uniformly Hölder continuous functions on $\Omega$ of order $s > 0$. Let $h^s(\Omega)$ denote the so-called little Hölder space, i.e., the closure of $BUC^s(\Omega)$ by $\|\cdot\|_{BUC^s(\Omega)}$.

As mentioned before, if $0 < \bar{\sigma}/\sigma < \theta_s$ then problem (1.1) has two radial stationary solutions with radius denoted by $R_{s1}$ and $R_{s2}$, $(R_{s1} < R_{s2})$, respectively. Let $R_s \in \{R_{s1}, R_{s2}\}$. Denote by $(\sigma_s, p_s, \Omega_s)$ the radial stationary solution with radius $R_s > 0$, where $\Omega_s = \{x \in \mathbb{R}^3 : |x| < R_s\}$, $\sigma_s$ and $p_s$ are given in (2.1) and (2.2). For a fixed $\alpha \in (0, 1)$ and sufficiently small $\delta > 0$, let

$$O_\delta := \{\rho \in h^{4+\alpha}(S^2) : \|\rho\|_{h^{4+\alpha}(S^2)} < \delta\}.$$ 

For given $\rho \in O_\delta$, denote by $\Omega_{\rho}$ the domain enclosed by surface $\Gamma_\rho$, which is the image of the mapping $[\omega \rightarrow (R_s + \rho(\omega))\omega]$ for $\omega \in S^2$. Thus solution $(\sigma, p, \Omega)$ can be rewritten as $(\sigma, p, \rho)$ with $\Omega = \Omega_{\rho}$, and $(\sigma_s, p_s, \Omega_s)$ can be rewritten as $(\sigma_s, p_s, \rho_s)$ with $\rho_s = 0$.

Let $T > 0$, a triple $(\sigma, p, \rho)$ is called a classical solution of problem (1.1) if

$$\sigma(\cdot, t), p(\cdot, t) \in h^{2+\alpha}(\Omega_{\rho(t)}) \times h^{4+\alpha}(\Omega_{\rho(t)}), \quad t \in (0, T],$$

$$\rho(\cdot) \in C([0, T], O_\delta) \cap C^1([0, T], h^{3+\alpha}(S^2)),$$

and satisfies (1.1) pointwise with $\Omega(t) = \Omega_{\rho(t)}$ and $\Omega_0 = \Omega_{\rho_0}$.

The first main result is on well-posedness theory of (1.1), and is formulated below.

**Theorem 1.1.** Given any initial data $\rho_0 \in O_\delta$ for a sufficiently small $\delta > 0$, there exists a unique classical solution $(\sigma, p, \rho)$ of problem (1.1) on interval $[0, T]$ for some $T > 0$.

The proof of Theorem 1.1, given in Section 2 and Section 3, is fulfilled by two steps. Firstly, by using Hanzawa transformation we reduce problem (1.1) into an abstract differential equation $\partial_t \rho + \Psi(\rho) = 0$, which contains an unknown function $\rho(t)$ describing the free boundary $\partial\Omega(t)$. Then we establish well-posedness by employing a localization technique, Fourier multiplier method and a delicate analysis of the linearization of $\Psi$.

Then we consider stability of each radial stationary solution $(\sigma_s, p_s, \Omega_s)$. An important character of problem (1.1) is that it is invariant under coordinate translations. For $(\sigma_s, p_s, \rho_s)$ with radius $R_s$ and a point $x_0 \in \mathbb{R}^3$ in a small neighborhood of origin, we denote by $(\sigma_{s_0}, p_{s_0}, \rho_{s_0})$ the translated radial stationary solution induced by $[x \rightarrow x + x_0]$. Our second main result is stated as follows.

**Theorem 1.2.** Let $0 < \bar{\sigma}/\sigma < \theta_s$. There hold the following assertions:
(i) There exists a positive threshold value $\gamma_*$ of cell-to-cell adhesiveness such that for any $\gamma > \gamma_*$, the radial stationary solution $(\sigma_s, p_s, \rho_s)$ with larger radius $R_{s2}$ is asymptotically stable in the following sense: There exists constant $\epsilon \in (0, \delta)$ such that for any $\rho_0 \in \mathcal{O}_\delta$ satisfying $\|\rho_0\|_{h^{4+\alpha}(\mathbb{S}^2)} < \epsilon$, problem (1.1) has a unique global classical solution $(\sigma, p, \rho)$, which converges exponentially fast to a translated radial stationary solution $(\sigma^{[x_0]}_s, p^{[x_0]}_s, \rho^{[x_0]}_s)$ in $h^{2+\alpha}(\Omega_{\rho(t)}) \times h^{4+\alpha}(\mathbb{S}^2)$ for some $x_0 \in \mathbb{R}^3$, as $t \to +\infty$; while for $0 < \gamma < \gamma_*$, it is unstable.

(ii) For all $\gamma > 0$, the radial stationary solution $(\sigma_s, p_s, \rho_s)$ with smaller radius $R_{s1}$ is unstable.

Theorem 1.2 implies that cell-to-cell adhesiveness $\gamma$ plays a crucial role on tumor’s stability and a smaller value of $\gamma$ may make tumor more aggressive. The proof of Theorem 1.2 is realized by first calculating all eigenvalues of the linearized operator at radial stationary solution and then employing the generalized principle of linearized stability developed in [27].

The structure of the rest of this paper is arranged as follows. In the next section, by using Hanzawa transformation and classical theory of elliptic equations we reduce the free boundary problem (1.1) into an abstract differential equation in little Hölder spaces. Section 3 is devoted to establishing local well-posedness theory via localization argument and Fourier multiplier method. In section 4 we study spectrum of linearized operator at radial stationary solution. In the last section we study asymptotic stability of each radial stationary solution and give the proof of Theorem 1.2.

2 Transformation and reduction

In this section we transform free boundary problem (1.1) into an initial-boundary value problem on a fixed domain, and then reduce it into an evolutionary problem in Banach spaces.

Firstly, we recall some results of problem (1.1) from [30]. The radial stationary solution $(\sigma_s, p_s, \Omega_s)$ can be given by

$$
\Omega_s = \{x \in \mathbb{R}^3 : r = |x| < R_s\}, \quad \sigma_s(r) = \bar{\sigma}(1 - \frac{\gamma}{R_s}) \frac{R_s \sinh r}{r \sinh R_s},
$$

$$
p_s(r) = -\mu \bar{\sigma}(1 - \frac{\gamma}{R_s}) \frac{R_s \sinh r}{r \sinh R_s} + \frac{1}{6} \mu \bar{\sigma} r^2 + \bar{p} + \mu \bar{\sigma}(1 - \frac{\gamma}{R_s}) - \frac{1}{6} \mu \bar{\sigma} R_s^2,
$$

where $R_s$ is a positive root of the following equation

$$
f(R_s) := (1 - \frac{\gamma}{R_s}) \frac{R_s \coth R_s - 1}{R_s^2} = \frac{1}{3} \bar{\sigma}.
$$

The proof of Theorem 1.1 in [30] shows that there exists a constant $\theta_* \in (0, 1)$ such that

For $0 < \bar{\sigma}/\bar{\sigma} < \theta_*$, equation (2.3) has two positive roots $R_{s1}$ and $R_{s2}$ with $R_{s1} < R_{s2}$ and $f'(R_{s1}) > 0$, $f'(R_{s2}) < 0$. (2.4)
To economize our notation, in the sequel, we always set \( R_s := R_{si} \) for \( i = 1, 2 \).

Denote \( \omega(x) := x/|x| \in \mathbb{S}^2 \) for \( x \in \mathbb{R}^3 \setminus \{0\} \). Taking \( a_0 > 0 \) sufficiently small, we see that the mapping
\[
\Phi : \mathbb{S}^2 \times (-a_0, a_0) \to \mathbb{R}^3, \quad \Phi(\omega, r) = (R_s + r)\omega,
\]
is a \( C^\infty \)-diffeomorphism from \( \mathbb{S}^2 \times (-a_0, a_0) \) onto its image \( \text{im}(\Phi) \).

Let \( \delta \in (0, a_0/4) \) and \( \alpha \in (0, 1) \), set
\[
O_\delta := \{ \rho \in h^{4+\alpha}(\mathbb{S}^2) : \|\rho\|_{h^{4+\alpha}(\mathbb{S}^2)} < \delta \}.
\]
For each \( \rho \in O_\delta \), define \( \Gamma_\rho := \text{im}(\Phi(\cdot, \rho(\cdot))) = ((R_s + \rho(\omega))\omega : \omega \in \mathbb{S}^2) \), and denote by \( \Omega_\rho \) the domain enclosed by \( \Gamma_\rho \). Clearly for the initial data \( \Gamma_{\rho_0} = \partial \Omega_0 \) of class \( h^{4+\alpha} \), there exists \( \rho_0 \in O_\delta \) such that \( \Gamma_{\rho_0} = \Gamma_0 \), and accordingly, we have \( \Omega_{\rho_0} = \Omega_0 \).

We take a cut-off function \( \chi \in C^\infty(\mathbb{R}) \) such that
\[
0 \leq \chi \leq 1, \quad \chi(\tau) = \begin{cases} 1, & \text{for } |\tau| \leq \delta, \\ 0, & \text{for } |\tau| \geq 3\delta, \end{cases} \quad \text{and} \quad |\chi'(\tau)| \leq \frac{2}{3\delta}.
\]
Given \( \rho \in O_\delta \), we define the Hanzawa transformation
\[
\Theta_\rho(x) := \begin{cases} x + \chi(|x| - R_s)\rho(\omega(x))\omega(x), & x \in \text{im}(\Phi), \\ x, & x \notin \text{im}(\Phi). \end{cases}
\]
Clearly one can verify that \( \Theta_\rho(\Omega_s) = \Omega_\rho, \Theta_\rho(\Gamma_s) = \Gamma_\rho \), where \( \Gamma_s = \partial \Omega_s = R_s \mathbb{S}^2 \), and
\[
\Theta_\rho \in \text{Diff}^{4+\alpha}(\mathbb{R}^3, \mathbb{R}^3) \cap \text{Diff}^{4+\alpha}(\Omega_s, \Omega_\rho).
\]
Denote by \( \Theta^\rho_* \) and \( \Theta^\rho^* \) respectively the push-forward and pull-back operators induced by \( \Theta_\rho \), i.e.,
\[
\Theta^\rho_* u = u \circ \Theta_\rho^{-1} \quad \text{for } u \in \text{BUC}(\Omega_s),
\]
\[
\Theta^\rho^* v = v \circ \Theta_\rho \quad \text{for } v \in \text{BUC}(\Omega_\rho).
\]
By Lemma 2.1 in [14], for \( 0 \leq k \leq 4 \) there holds
\[
\Theta^\rho_* \in \text{Isom}(h^{k+\alpha}(\Omega_\rho), h^{k+\alpha}(\Omega_s)) \cap \text{Isom}(h^{k+\alpha}(\Gamma_\rho), h^{k+\alpha}(\Gamma_s))
\]
with \( \Theta^\rho_* = [\Theta^\rho_*]^{-1} \). Next, we define a function \( \phi_\rho(x) := r(x) - R_s - \rho(\omega(x)) \) for \( x \in \mathbb{R}^3 \setminus \{0\} \), where \( r(x) = |x| \). Clearly, \( \Gamma_\rho = \phi_\rho^{-1}(0) \). We introduce the following transformed operators:
\[
A(\rho)u := \Theta^\rho_* \Delta(\Theta^\rho_* u), \quad B(\rho)u := \Theta^\rho_* (\nabla \phi_\rho|_{\Gamma_\rho} \cdot \nabla(\Theta^\rho_* u)|_{\Gamma_\rho}),
\]
for \( u \in \text{BUC}^2(\Omega_s) \). Observe that \( A(\rho) \) is the Laplace-Beltrami operator on \( \Omega_s \) with respect to the metric induced by \( \Theta_\rho \), and \( B(\rho) \) is the induced outward normal derivative operator. Since \( \Theta_\rho \) depends on \( \rho \) analytically, by Lemma 2.2 in [13], we have
\[
\left\{ \begin{array}{l}
A(\cdot) \in C^\infty(O_\delta, L(h^{4+\alpha}(\Omega_s), h^{2+\alpha}(\Omega_s))), \\
B(\cdot) \in C^\infty(O_\delta, L(h^{4+\alpha}(\Omega_s), h^{3+\alpha}(\Gamma_s))).
\end{array} \right. \quad (2.5)
\]
Let
\[(B(\rho)u)(\omega) = (B(\rho)u)(x) \quad \text{for} \quad x \in \Gamma_s, \quad \omega = x/|x|.\]

We have
\[B(\cdot) \in C^\infty(O_\delta, L(h^{4+\alpha}(\Omega_s), h^{3+\alpha}(\mathbb{S}^2))). \quad (2.6)\]

A direct computation shows that the mean curvature at point \(x\) on \(\Gamma_\rho\) is given by
\[\kappa(\rho) = \frac{1}{2} \left[ \frac{2r - \Delta_\omega \rho}{r(r^2 + |\nabla_\omega \rho|^2)^{1/2}} + \frac{2r|\nabla_\omega \rho|^2 + \nabla_\omega |\nabla_\omega \rho|^2 \cdot |\nabla_\omega \rho|}{2r^2 + |\nabla_\omega \rho|^2} \right] \quad (2.7)\]

where \(\omega = x/\|x\|\) and \(\Delta_\omega\) is the Laplace-Beltrami operator on \(\mathbb{S}^2\). We easily check that
\[\kappa(\cdot) \in C^\infty(O_\delta, h^{2+\alpha}(\Gamma_s)). \quad (2.8)\]

Finally, we set \(u = \Theta_\rho^*\sigma\) and \(v = \Theta_\rho^*\rho\).

Using above notations, free boundary problem (1.1) can be transformed into the following equivalent initial-boundary value problem
\[
\begin{cases}
A(\rho)u = u & \text{in } \Omega_s \times [0, T], \\
A(\rho)v = -\mu(u - \bar{\sigma}) & \text{in } \Omega_s \times [0, T], \\
u = \bar{\sigma}(1 - \gamma\kappa(\rho)) & \text{on } \Gamma_s \times [0, T], \\
v = \bar{\rho} & \text{on } \Gamma_s \times [0, T], \\
\partial_t \rho = -B(\rho)v & \text{on } \mathbb{S}^2 \times [0, T], \\
\rho(0) = \rho_0 & \text{on } \mathbb{S}^2.
\end{cases} \quad (2.9)
\]

Here we used that the unit outward normal field \(n_{\rho(t)}\) and the normal velocity \(V\) of \(\Gamma_{\rho(t)}\) are given by
\[n_{\rho(t)} = \frac{\nabla \phi_{\rho(t)}}{\|\nabla \phi_{\rho(t)}\|} \quad \text{and} \quad V = \frac{\partial_t \rho}{\|\nabla \phi_{\rho(t)}\|},\]

respectively, for \(\rho := \rho(t)(\omega) \in C([0, T], \mathcal{O}_\delta) \cap C^1([0, T], h^{3+\alpha}(\mathbb{S}^2))\) and some \(T > 0\).

A triple \((u, v, \rho)\) is called a classical solution of problem (2.9) if
\[\left(u, v\right) \in C([0, T], h^{2+\alpha}(\Omega_s) \times h^{4+\alpha}(\Omega_s)),\]
\[\rho \in C([0, T]; \mathcal{O}_\delta) \cap C^1([0, T], h^{3+\alpha}(\mathbb{S}^2)),\]

and it satisfies problem (2.9) pointwise.

By above deduction we have

\[\text{Lemma 2.1} \quad \text{A triple } (\sigma, p, \rho) \text{ is a classical solution of problem (1.1) if and only if the triple } (u, v, \rho) \text{ is a classical solution of problem (2.9) with } u = \Theta_\rho^*\sigma \text{ and } v = \Theta_\rho^*p.\]
Next we further reduce problem (2.9) into an abstract evolutionary problem containing free boundary \( \rho \) only. Given a function \( \rho \in \mathcal{O}_\delta \), a constant \( \lambda \geq 0 \) and \((g, h) \in h^{k+\alpha}(\Omega_s) \times h^{k+2+\alpha}(\Gamma_s)\) for \( k = 0, 1, 2 \), by well-known regularity theory of second-order elliptic equations, the elliptic boundary value problem

\[
\begin{aligned}
\lambda u - A(\rho)u &= g \quad \text{in } \Omega_s, \\
u &= h \quad \text{on } \Gamma_s,
\end{aligned}
\]

has a unique solution which can be given by

\[
u = S_\lambda(\rho)g + T_\lambda(\rho)h.
\] (2.10)

It holds that for \( k = 0, 1, 2 \) (see Lemma 2.3 in [13]),

\[
\begin{aligned}
S_\lambda &\in C^\infty(\mathcal{O}_\delta, L(h^{k+\alpha}(\Omega_s), h^{k+2+\alpha}(\Omega_s))), \\
T_\lambda &\in C^\infty(\mathcal{O}_\delta, L(h^{k+2+\alpha}(\Gamma_s), h^{k+2+\alpha}(\Omega_s))).
\end{aligned}
\] (2.11)

Thus for a given \( \rho \in C([0, T], \mathcal{O}_\delta) \cap C^1([0, T], h^{3+\alpha}(S^2)) \), by solving the first four equations of problem (2.9), we have

\[
\begin{aligned}
u &= U(\rho) := T_1(\rho)[\bar{\sigma}(1 - \gamma \kappa(\rho))], \\
v &= V(\rho) := S_0(\rho)T_1(\rho)[\mu \bar{\sigma}(1 - \gamma \kappa(\rho))] - S_0(\rho)\mu \bar{\sigma} + \bar{p},
\end{aligned}
\] (2.12)

where we have used that \( T_0(\rho)\bar{p} = \bar{p} \). Let

\[
\Psi(\rho) := B(\rho)V(\rho) = -\mu \bar{\sigma} \gamma B(\rho)S_0(\rho)T_1(\rho)\kappa(\rho) + B(\rho)S_0(\rho)T_1(\rho)\mu \bar{\sigma} - B(\rho)S_0(\rho)\mu \bar{\sigma}.
\] (2.13)

It follows from (2.5)–(2.8) and (2.11) that

\[
\Psi \in C^\infty(\mathcal{O}_\delta, h^{3+\alpha}(S^2)).
\] (2.14)

By the above reduction, we see that problem (2.9) is equivalent to the following

\[
\begin{aligned}
\partial_t \rho + \Psi(\rho) &= 0 \quad \text{on } S^2 \times [0, T], \\
\rho(0) &= \rho_0 \quad \text{on } S^2.
\end{aligned}
\] (2.15)

Similarly, a function \( \rho \in C([0, T], \mathcal{O}_\delta) \cap C^1([0, T], h^{3+\alpha}(S^2)) \) is called a classical solution of problem (2.15) if it satisfies each equation of problem (2.15) on \([0, T]\) pointwise.

In summary, we have

**Lemma 2.2** The function \( \rho \) is a classical solution of problem (2.15) if and only if the triple \((u, v, \rho)\) is a classical solution of problem (2.9) with \((u, v)\) given by (2.12).

In order to establish local well-posedness of the above problem (2.15), we need to study the Fréchet derivative \( \partial \Psi(\cdot) \) of nonlinear operator \( \Psi \). In the sequel, we compute the Fréchet derivative \( \partial \Psi(0) \) of \( \Psi \) at \( \rho = 0 \).
Note that $\Omega_{\rho}|_{\rho=0} = \Omega_s$, and we have

$$\kappa(0) = 1/R_s, \quad U(0) = \sigma_s, \quad V(0) = p_s, \quad \Psi(0) = p'(R_s) = 0.$$  

By (2.13), $\Psi(\rho) = B(\rho)V(\rho) = \mu B(\rho)S(\rho) [U(\rho) - \tilde{\sigma}]$. For any $\eta \in h^{4+\alpha}(\mathbb{S}^2)$,

$$\partial \Psi(0)\eta = \mu B(0)S_0(0) [\partial U(0)\eta] + B(0) \{ \partial S_0(0)[\eta, \mu(\sigma_s - \tilde{\sigma})] \} + \partial B(0)[\eta, p_s] =: I + II + III \tag{2.16}$$

where we use the notation

$$\partial U(0)\eta := \lim_{\varepsilon \to 0} \frac{U(\varepsilon\eta) - U(0)}{\varepsilon}, \quad \partial B(0)[\eta, v] := \lim_{\varepsilon \to 0} \frac{B(\varepsilon\eta)v - B(0)v}{\varepsilon}$$

for $\eta \in h^{4+\alpha}(\mathbb{S}^2)$ and $v \in h^{2+\alpha}(\Omega_s)$.

Set

$$c_1 := \frac{\mu \gamma \sigma}{2R_s^2}, \quad c_2 := \frac{\mu \gamma \sigma}{R_s^2} - \mu \sigma'(R_s), \quad c_3 := p''(R_s).$$

We have the following lemma:

**Lemma 2.3** For $\eta \in h^{4+\alpha}(\mathbb{S}^2)$,

$$\partial \Psi(0)\eta = c_1 B(0)S_0(0)T_1(0)\Delta_{\omega}\eta + c_2 B(0)S_0(0)T_1(0)\eta + c_3\eta.$$  

**Proof.** Since $\sigma_{\varepsilon\eta} := T_1(\varepsilon\eta)[\tilde{\sigma}(1 - \gamma \kappa(0))]$ satisfies

$$\sigma_{\varepsilon\eta} - A(\varepsilon\eta)\sigma_{\varepsilon\eta} = 0 \quad \text{in } \Omega_s, \quad \sigma_{\varepsilon\eta} = \tilde{\sigma}(1 - \gamma \kappa(0)) \quad \text{on } \Gamma_s, \tag{2.17}$$

and $\sigma_s = T_1(0)[\tilde{\sigma}(1 - \gamma \kappa(0))]$ satisfies

$$\sigma_s - A(0)\sigma_s = 0 \quad \text{in } \Omega_s, \quad \sigma_s = \tilde{\sigma}(1 - \gamma \kappa(0)) \quad \text{on } \Gamma_s, \tag{2.18}$$

we have

$$[\sigma_{\varepsilon\eta} - \sigma_s] - A(0) [\sigma_{\varepsilon\eta} - \sigma_s] - [A(\varepsilon\eta) - A(0)]\sigma_{\varepsilon\eta} = 0 \quad \text{in } \Omega_s. \tag{2.19}$$

Dividing both sides by $\varepsilon$ and letting $\varepsilon \to 0$, we see that $\partial T_1(0)[\eta, \tilde{\sigma}(1 - \gamma \kappa(0))]$ satisfies

$$u - A(0)u - \partial A(0)[\eta, \sigma_s] = 0 \quad \text{in } \Omega_s, \quad u = 0 \quad \text{on } \Gamma_s, \tag{2.20}$$

which implies that

$$\partial T_1(0)[\eta, \tilde{\sigma}(1 - \gamma \kappa(0))] = \mathcal{S}_1(0)\{ \partial A(0)[\eta, \sigma_s] \}. \tag{2.21}$$

Let $\sigma^s_{\varepsilon\eta} := \Theta_{\varepsilon\eta}^s \sigma_s$, then

$$u_0 := \lim_{\varepsilon \to 0} \frac{\sigma^s_{\varepsilon\eta} - \sigma_s}{\varepsilon} = \chi(r - R_s)\sigma'(r)\eta.$$  

Similarly as (2.20) we find that $u_0$ satisfies

$$u_0 - A(0)u_0 - \partial A(0)[\eta, \sigma_s] = 0 \quad \text{in } \Omega_s, \quad u_0 = \sigma'(R_s)\eta \quad \text{on } \Gamma_s.$$
and it follows that
\[ u_0 = S_1(0)\{\partial A(0)[\eta, \sigma_s]\} + T_1(0)(\sigma'(R_s)\eta). \]
Substituting it into (2.21), we have
\[
\partial T_1(0)[\eta, \bar{\sigma}(1 - \gamma\kappa(0))] = u_0 - T_1(0)(\sigma'(R_s)\eta) \\
= \chi(r - R_s)(\sigma'(r)\eta - T_1(0)(\sigma'(R_s)\eta)).
\] (2.22)

Let \( p^s_{\varepsilon\eta} := \Theta^s_{\varepsilon\eta} p_s \), then
\[
v_0 := \lim_{\varepsilon \to 0} \frac{p^s_{\varepsilon\eta} - p_s}{\varepsilon} = \chi(r - R_s)p'_s(r)\eta.
\]
Note that \( p'_s(R_s) = 0 \), by a similar argument as above we can show that
\[
\partial S_0(0)[\eta, \mu(\sigma_s - \tilde{\sigma})] = S_0(0)\{\partial A(0)[\eta, p_s]\}
= v_0 - \mu S_0(0)u_0.
\] (2.23)

By (2.7), we easily have
\[
\partial \kappa(0)\eta = \lim_{\varepsilon \to 0} \frac{\kappa(\varepsilon\eta) - \kappa(0)}{\varepsilon} = -\frac{1}{R_s^2}(\eta + \frac{1}{2}\Delta_\omega\eta).
\] (2.24)
Thus by (2.22) and (2.24) we compute
\[
I = \mu B(0)S_0(0)[\partial \mathcal{H}(0)\eta]
\]
\[
= \mu B(0)S_0(0)\{\partial T_1(0)[\eta, \bar{\sigma}(1 - \gamma\kappa(0))] - \gamma\bar{\sigma}T_1(0)\partial\kappa(0)\eta\}
= \mu B(0)S_0(0)\{u_0 - T_1(0)(\sigma'_s(R_s)\eta) - \gamma\bar{\sigma}T_1(0)\partial\kappa(0)\eta\}
\] (2.25)
\[
= \mu B(0)S_0(0)u_0 + \frac{\mu\gamma\bar{\sigma}}{2R_s^2}B(0)S_0(0)T_1(0)\Delta_\omega\eta + \mu(\frac{\gamma\bar{\sigma}}{R_s^2} - \sigma'_s(R_s))B(0)S_0(0)T_1(0)\eta.
\]

By (2.23) we have
\[
II = B(0)\{\partial S_0(0)[\eta, \mu(\sigma_s - \tilde{\sigma})]\}
= B(0)v_0 - \mu B(0)S_0(0)u_0
= p''_s(R_s)\eta - \mu B(0)S_0(0)u_0.
\] (2.26)
Finally, a direct computation shows that
\[
III = \lim_{\varepsilon \to 0} \frac{B(\varepsilon\eta)p_s - B(0)p_s}{\varepsilon} = p'_s(R_s)\eta = 0.
\] (2.27)
Hence, by (2.16) and adding (2.25)–(2.27), we immediately complete the proof. \( \square \)
3 Local well-posedness

In this section, we prove local well-posedness of problem (2.15). By using a technique of localization and Fourier multiplier method developed in [13, 14], we shall prove $-\partial \Psi(0)$ is an infinitesimal generator of a strongly continuous analytic semigroup on $L^3 +\alpha(S^2)$, and problem (2.15) is of parabolic type for $\delta$ is sufficiently small. Thus by using analytic semigroup theory and applications to parabolic differential problems (cf. [17, 26]), we get local well-posedness.

For giving expression of linear operators introduced in above section by means of local coordinates, we need some notations similarly as in [11, 13, 14]. Given $\omega \in \mathbb{R}^3$, $\Delta$ is the Laplace-Beltrami operator on $\Omega$, $\Sigma$ is symmetric and uniformly positive definite on $\mathbb{R}^3$. For giving expression of linear operators introduced above section by means of local coordinates, we need some notations similarly as in [11, 13, 14]. Given $m \in \mathbb{N}$ and an atlas $\{(U_l, \psi_l); 1 \leq l \leq m\}$ of $\mathcal{R}_a$ such that

$$\psi_l \in C^\infty(U_l, (-a, a)^2 \times [0, a)), \quad \psi_l : U_l \cap \Gamma_s \to (-a, a)^2 \times \{0\}, \quad 1 \leq l \leq m.$$ Given $l \in \{1, \ldots, m\}$, denote by $\psi_l^*$ and $\psi_l^*$ the push-forward and pull-back operators induced by $\psi_l$, respectively, i.e.,

$$\psi_l^* u = u \circ \psi_l^{-1} \quad \text{for} \quad u \in BUC(U_l),$$

$$\psi_l^* v = v \circ \psi_l \quad \text{for} \quad v \in BUC((-a, a)^2 \times [0, a)).$$

Define the local representation $\tilde{A}_l$, $\tilde{B}_l$ and $\tilde{P}_l$ of $A(0)$, $B(0)$ and $\Delta_\omega$, respectively, on $((-a, a)^2 \times [0, a), \psi_l)$ as

$$\tilde{A}_l := \psi_l^* \circ A(0) \circ \psi_l^*, \quad \tilde{B}_l := \psi_l^* \circ B(0) \circ \psi_l^*, \quad \tilde{P}_l := \psi_l^* \circ \Delta_\omega \circ \psi_l^*.$$ Since $A(0)$ is the Laplace operator on $\Omega$, $B(0)$ is the outward normal derivative operator, and $\Delta_\omega$ is the Laplace-Beltrami operator on $S^2$, there exist (cf. Lemma 3.2 in [13] and Lemma 3.1 in [14])

$$a^l_{jk}(\cdot), a^l_j(\cdot) \in C^\infty((-a, a)^2 \times [0, a), \mathbb{R}), \quad b^l_j(\cdot) \in C^\infty((-a, a)^2, \mathbb{R}), \quad 1 \leq j, k \leq 3,$$

$$p^l_{jk}(\cdot), p^l_j(\cdot) \in C^\infty((-a, a)^2, \mathbb{R}), \quad 1 \leq j, k \leq 2,$$

such that

$$[a^l_{jk}] \text{ is symmetric and uniformly positive definite on } (-a, a)^2 \times [0, a),$$

$$[p^l_{jk}] \text{ is symmetric and uniformly positive definite on } (-a, a)^2,$$ and such that

$$\tilde{A}_l = \sum_{j, k=1}^3 a^l_{jk}(\cdot) \partial_j \partial_k + \sum_{j=1}^3 a^l_j(\cdot) \partial_j, \quad \tilde{B}_l = -\sum_{j=1}^3 b^l_j(\cdot) \partial_j,$$

$$\tilde{P}_l = \sum_{j, k=1}^2 p^l_{jk}(\cdot) \partial_j \partial_k + \sum_{j=1}^2 p^l_j(\cdot) \partial_j,$$ (3.2)
where $\Upsilon$ is the trace operator on $\mathbb{R}^2 \times \{0\}$.

Next we fix the localization at $x_0 := \psi^{-1}(0) \in U_l \cap \Gamma_s$ and define the following linear differential operators with constant coefficients:

$$\tilde{A}_{l,0} := \sum_{j,k=1}^{3} a_{jk}^l (0) \partial_j \partial_k, \quad \tilde{B}_{l,0} := -\sum_{j=1}^{3} b_{j}^l (0) \Upsilon \partial_j, \quad \tilde{P}_{l,0} := -1 + \sum_{j,k=1}^{2} p_{jk}^l (0) \partial_j \partial_k.$$

For any given $(g, h) \in h^{k+\alpha}(\mathbb{H}^3) \times h^{k+2+\alpha}(\partial \mathbb{H}^3)$, $k = 0, 1, 2$, we consider the following elliptic boundary value problem in the half space $\mathbb{H}^3 := \mathbb{R}^2 \times (0, +\infty)$:

$$\begin{cases}
u - \tilde{A}_{l,0} \nu = g & \text{in } \mathbb{H}^3, \\
\Upsilon \nu = h & \text{on } \partial \mathbb{H}^3 := \mathbb{R}^2 \times \{0\}. 
\end{cases} \quad (3.3)$$

By Appendix B in [12], there exists a unique solution $\nu \in h^{k+2+\alpha}(\mathbb{H}^3)$ of problem (3.3) which is given by $\nu = \tilde{S}_{l,0} g + \tilde{T}_{l,0} h$ and

$$\tilde{S}_{l,0} \in L(h^{k+\alpha}(\mathbb{H}^3), h^{k+2+\alpha}(\mathbb{H}^3)),$$

$$\tilde{T}_{l,0} \in L(h^{k+2+\alpha}(\partial \mathbb{H}^3), h^{k+2+\alpha}(\mathbb{H}^3)),$$

for $k = 0, 1, 2$. Moreover, $\tilde{S}_{l,0}$ and $\tilde{T}_{l,0}$ can be expressed as Fourier multiplier operators. For this purpose, we let

$$a_{13}^l := (a_{13}^l(0), a_{23}^l(0)), \quad a_{0}^l := \sum_{j,k=1}^{2} a_{jk}^l(0) \xi_1 \xi_2,$$

$$\lambda^l(\xi) := \frac{i \langle a_i^l(\xi), a_{13}^l(0) \rangle}{a_{13}^l(0)} + \frac{1}{a_{33}^l(0)} \sqrt{a_{33}^l(0)(1 + a_{0}^l(\xi)) - \langle a_i^l, \xi \rangle^2},$$

and

$$q^l(\xi, z) := 1 + a_0^l(\xi) + 2i \langle a_i^l, \xi \rangle z - a_{33}^l(0) z^2, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \ z \in \mathbb{C},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$. Due to (3.1), we easily verify that $\lambda^l(\xi)$ is well-defined and for any given $\xi \in \mathbb{R}^2$, $\lambda^l(\xi)$ is the unique root with positive real part of $q^l(\xi, z)$. Denote $\mathcal{F}$ and $\mathcal{F}^{-1}$ by the Fourier transform and inverse Fourier transform in $\mathbb{R}^2$, respectively. We have (cf. (4.2)–(4.4) in [14])

$$\tilde{S}_{l,0} g(x, y) = \left[ \mathcal{F}^{-1} (1 - e^{-\lambda(\cdot)y}) \frac{1}{1 + a_0(\cdot)} \mathcal{F} g \right](x), \quad (3.4)$$

$$\tilde{T}_{l,0} h(x, y) = \left[ \mathcal{F}^{-1} e^{-\lambda(\cdot)y} \mathcal{F} h \right](x), \quad (3.5)$$

for $g \in h^{k+2+\alpha}(\mathbb{H}^3)$, $h \in h^{k+\alpha}(\mathbb{R}^2)$ and $(x, y) \in \mathbb{H}^3$, $k = 0, 1, 2$. Let

$$p_{0}^l(\xi) := \sum_{j,k=1}^{2} p_{jk}^l(0) \xi_j \xi_k \quad \text{for } \xi \in \mathbb{R}^2.$$
It is easy to see that
\[ \tilde{P}_{l,0} h(x) = -\left[ \mathcal{F}^{-1}(1 + p_0(\cdot)) \mathcal{F} h \right](x) \quad \text{for } h^{k+2+\alpha}(\mathbb{R}^2), \] (3.6)
and
\[ \tilde{P}_{l,0} \in L(h^{k+2+\alpha}(\mathbb{R}^2), h^{k+\alpha}(\mathbb{R}^2)) \quad \text{for } k = 0, 1, 2. \]

By (3.4)–(3.6), we have
\[ \tilde{S}_{l,0} \tilde{T}_{l,0} \tilde{P}_{l,0} h(x, y) = \left[ \mathcal{F}^{-1} \left\{ (e^{-\lambda^l(\cdot)y} - 1)e^{-\lambda^l(\cdot)y} \frac{1 + p_0(\cdot)}{1 + a_0(\cdot)} \right\} \mathcal{F} h \right](x), \] (3.7)
for \( h \in h^{k+\alpha}(\mathbb{R}^2) \) and \((x, y) \in \mathbb{H}^3 \). Define
\[ g^l(\xi) := b_3(0)\lambda^l(\xi) \frac{1 + p_0(\xi)}{1 + a_0(\xi)} \quad \text{for } \xi \in \mathbb{R}^2. \]

By (3.7) we immediately have
\[ G_l := \tilde{S}_{l,0} \tilde{T}_{l,0} \tilde{P}_{l,0} = \mathcal{F}^{-1} g^l(\cdot) \mathcal{F}. \] (3.8)

Given two Banach spaces \( E_0 \) and \( E_1 \) such that \( E_1 \) is densely and continuously embedded into \( E_0 \), we denote \( \mathcal{H}(E_1, E_0) \) by the subspace of all linear operators \( A \in L(E_1, E_0) \) such that \(-A\) generates a strongly continuous analytic semigroup on \( E_0 \). By Remark I.1.2.1 (a) in [1], we have that a linear operator \( A \in L(E_1, E_0) \) belongs to \( \mathcal{H}(E_1, E_0) \) if and only if there exist positive constants \( C \) and \( \lambda_* \) such that
\[ \lambda_* + A \in \text{Isom}(E_1, E_0), \]
\[ |\lambda||x||E_0 + ||x||E_1 \leq C||(\lambda + A)x||E_0, \] (3.9)
for all \( x \in E_1 \) and \( \text{Re}\lambda \geq \lambda_* \).

Based on the property of the symbol \( g_l(\xi) \) of \( G_l \) and Fourier multiplier theory, we have

**Lemma 3.1** \( G_l \in \mathcal{H}(h^{k+\alpha}(\mathbb{R}^2), h^{k+\alpha}(\mathbb{R}^2)). \)

**Proof.** Let
\[ \tilde{\lambda}(\xi, \tau) := \frac{i\langle a^l, \xi \rangle}{a^l_{33}(0)} + \frac{1}{a^l_{33}(0)} \sqrt{a^l_{33}(0)(\tau^2 + a_0(\xi)) - \langle a^l, \xi \rangle^2}, \]
and
\[ \tilde{g}(\xi, \tau) := b_3(0)\tilde{\lambda}(\xi, \tau) \frac{\tau^2 + p_0(\xi)}{\tau^2 + a_0(\xi)} \quad \text{for } \xi \in \mathbb{R}^2, \tau \in (0, +\infty). \]

It is easy to verify that
\[ \lambda^l(\xi) = \tilde{\lambda}(\xi, 1) \quad \text{and} \quad g^l(\xi) = \tilde{g}(\xi, 1). \]
Moreover, \( \tilde{g}(\xi, \tau) \) is positively homogeneous of degree 1 and all derivatives of \( \tilde{g}(\xi, \tau) \) are bounded on \( |\xi|^2 + \tau^2 = 1 \).

By (3.1), we see that \( b^j_l(0) > 0 \) and \([a^j_{lk}(0)], [p^j_{lk}(0)]\) are symmetric and positive definite. It implies that there exist positive constants \( a_*, b_*, c_* \) such that for all \( \xi \in \mathbb{R}^2 \) and \( \tau \in (0, +\infty) \),

\[
\text{Re} \tilde{\lambda}(\xi, \tau) \geq a_* \sqrt{|\xi|^2 + \tau^2}, \quad \tau^2 + a^j_0(\xi) \leq b_*(|\xi|^2 + \tau^2), \quad \tau^2 + p^j_0(\xi) \geq c_*(|\xi|^2 + \tau^2).
\]

Thus we have

\[
\text{Re} \tilde{g}(\xi, \tau) = b^j_l(0) \text{Re} \tilde{\lambda}(\xi, \tau) \frac{\tau^2 + p^j_0(\xi)}{\tau^2 + a^j_0(\xi)} \geq d_* \sqrt{\xi^2 + \tau^2},
\]

where \( d_* := a_* b_*^{-1} c_* b^j_l(0) > 0 \). With these properties of \( \tilde{g}(\xi, \tau) \) and by using Mikhlin-Hörmander multiplier theorem, we have \( \tilde{G}_l = \mathcal{F}^{-1} \tilde{g}(\cdot, 1) \mathcal{F} \in \mathcal{H}(h^{4+\alpha}(\mathbb{R}^2), h^{3+\alpha}(\mathbb{R}^2)) \), cf. the proof of Theorem 4.2 in [12] or Theorem A.2 in [12]. The proof is complete. \( \square \)

Introduce an operator \( \partial_\eta : BUC(\mathbb{S}^2) \to BUC(\Gamma_\eta) \) given by

\[
\partial_\eta(\eta(\tau, \omega)) = \eta(\omega) \quad \text{for} \quad \eta \in BUC(\mathbb{S}^2), \ \omega \in \mathbb{S}^2.
\]

Let \( \beta \in (0, \alpha) \) be fixed and

\[
G := \mathcal{B}(0) \mathcal{S}_0(0) \mathcal{T}_1(0) \Delta_\omega.
\]

For any \( \epsilon > 0 \), by the proof of Lemma 5.1 in [14] (with a trivial modification), there exist \( a \in (0, a_0] \) and a partition of unity \( \{(U_l, \psi_l); 1 \leq l \leq m_\alpha\} \) for \( \mathcal{R}_a \), and a positive constant \( C := C(\epsilon, a) \) such that

\[
\|\psi^l_\epsilon(\psi_l \partial_\eta G \eta) - G_l \psi^l_\epsilon(\psi_l \partial_\eta \eta)\|_{h^{3+\alpha}(\mathbb{R}^2)} \leq \epsilon \|\psi^l_\epsilon(\psi_l \partial_\eta \eta)\|_{h^{4+\alpha}(\mathbb{R}^2)} + C \|\eta\|_{h^{4+\beta}(\mathbb{S}^2)},
\]

for all \( \eta \in h^{4+\alpha}(\mathbb{S}^2) \) and \( l = 1, 2, \ldots, m_\alpha \).

With the above preparation, we have the following

**Lemma 3.2** \( \partial \Psi(0) \in \mathcal{H}(h^{4+\alpha}(\mathbb{S}^2), h^{3+\alpha}(\mathbb{S}^2)) \).

**Proof.** (i) Based on Lemma 3.1 and the sharp estimate (3.10), by applying interpolation inequality and (3.9) we can show that

\[
G = \mathcal{B}(0) \mathcal{S}_0(0) \mathcal{T}_1(0) \Delta_\omega \in \mathcal{H}(h^{4+\alpha}(\mathbb{S}^2), h^{3+\alpha}(\mathbb{S}^2)).
\]

We omit the details and refer to the proof of Theorem 5.2 in [14] with a minor modification, see also a similar proof of Theorem 4.1 in [13].

(ii) Recall that \( c_1 = \frac{1}{2R^2} \mu_\gamma \sigma > 0 \), by (3.11) we have \( c_1 G \in \mathcal{H}(h^{4+\alpha}(\mathbb{S}^2), h^{3+\alpha}(\mathbb{S}^2)) \). Let

\[
K \eta := c_2 \mathcal{B}(0) \mathcal{S}_0(0) \mathcal{T}_1(0) \eta + c_3 \eta \quad \text{for} \quad \eta \in h^{4+\alpha}(\mathbb{S}^2).
\]

By (2.6) and (2.11), we see that

\[
K \in L(h^{4+\alpha}(\mathbb{S}^2), h^{4+\alpha}(\mathbb{S}^2)).
\]
Since $h^{4+\alpha}(S^2)$ is compactly embedded into $h^{3+\alpha}(S^2)$, by a standard perturbation result (cf. Theorem I.1.5.1 in [1]), we have $\partial\Psi(0) = c_1 G + K \in \mathcal{H}(h^{4+\alpha}(S^2), h^{3+\alpha}(S^2))$. The proof is complete. □

Since $\mathcal{H}(h^{4+\alpha}(S^2), h^{3+\alpha}(S^2))$ is an open subset of $L(h^{4+\alpha}(S^2), h^{3+\alpha}(S^2))$ (cf. Theorem I.1.3.1 in [1]), by Lemma 3.2 we immediately get

**Corollary 3.3** Let $\delta > 0$ be sufficiently small. For any given $\rho \in \mathcal{O}_\delta$, there holds

$$\partial\Psi(\rho) \in \mathcal{H}(h^{4+\alpha}(S^2), h^{3+\alpha}(S^2)).$$

Corollary 3.3 implies that problem (2.15) in $\mathcal{O}_\delta$ is of parabolic type in the sense of Amann [1] and Lunardi [26]. By applying analytic semigroup theory and applications to parabolic differential problems (cf. Theorem 8.1.1 and Theorem 8.3.4 in [26]), we get the following result:

**Theorem 3.4** Given $\rho_0 \in \mathcal{O}_\delta$. There exists $T > 0$ such that problem (2.15) has a unique classical solution $\rho \in C([0, T], \mathcal{O}_\delta) \cap C^1([0, T], h^{3+\alpha}(S^2))$.

By Lemma 2.1, Lemma 2.2 and Theorem 3.4, we see that Theorem 1.1 follows and local well-posedness of free boundary problem (1.1) has been obtained.

4 **Spectrum analysis**

In this section we study the spectrum of $\partial\Psi(0)$. Since $h^{4+\alpha}(S^2)$ is compactly embedded into $h^{3+\alpha}(S^2)$, by Lemma 3.2, we see that the spectrum of $\partial\Psi(0)$ consists of all eigenvalues, which will be obtained by employing spherical harmonics and modified Bessel functions.

Recall that $\Delta_\omega$ is denoted by the Laplace-Beltrami operator on the unit sphere $S^2$, and

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\omega.$$

By Lemma 2.3, we have for given $\eta \in h^{4+\alpha}(S^2)$,

$$\partial\Psi(0)\eta = c_1 B(0)S_0(0)T_1(0)\Delta_\omega \eta + c_2 B(0)S_0(0)T_1(0)\eta + c_3 \eta,$$

where

$$c_1 = \frac{\mu \gamma \bar{\sigma}}{2R_s^2}, \quad c_2 = \frac{\mu \gamma \bar{\sigma}}{R_s^2} - \mu \sigma_s'(R_s), \quad c_3 = p''_s(R_s).$$

By using (2.1)–(2.3), a direct calculation shows that

$$\sigma_s'(R_s) = \frac{1}{3} \bar{\sigma} R_s \quad \text{and} \quad p''_s(R_s) = -\mu \left[ \bar{\sigma} (1 - \frac{\gamma}{R_s}) - \bar{\sigma} \right].$$
Hence we can rewrite
\[ c_1 = \frac{\mu \gamma \bar{\sigma}}{2R_s^2}, \quad c_2 = \frac{\mu \gamma \bar{\sigma}}{R_s^2} - \frac{1}{3} \mu \bar{\sigma} R_s, \quad c_3 = -\mu \left[ \bar{\sigma} (1 - \frac{\gamma}{R_s}) - \bar{\sigma} \right]. \quad (4.1) \]

Consider the following problem
\[
\begin{cases}
  u - \Delta u = 0 & \text{in } \Omega_s, \\
  u = c_1 \Delta \omega \eta + c_2 \eta & \text{on } \Gamma_s, \\
  -\Delta v = u & \text{in } \Omega_s, \\
  v = 0 & \text{on } \Gamma_s,
\end{cases}
\]
\[
(4.2)
\]
where \( u = u(r, \omega) \) and \( v = v(r, \omega) \) are unknown functions. By (2.10), we see that the solution of problem (4.2) is given by
\[
\begin{align*}
  u &= T_1(0)[c_1 \Delta \omega \eta + c_2 \eta], \\
  v &= S_0(0)T_1(0)[c_1 \Delta \omega \eta + c_2 \eta].
\end{align*}
\]
\[
(4.3)
\]
Note that \( \mathcal{B}(0) v = \frac{\partial v}{\partial r} \big|_{r=R_s} \) for \( v \in \text{BUC}^2(\Omega_s) \), we have
\[
\partial \Psi(0) \eta = \mathcal{B}(0) v + c_3 \eta
\]
\[
= \frac{\partial v}{\partial r} \big|_{r=R_s} - \mu \left[ \bar{\sigma} (1 - \frac{\gamma}{R_s}) - \bar{\sigma} \right] \eta,
\]
\[
(4.4)
\]
where \( v = v(r, \omega) \) is given by (4.3).

Next we solve problem (4.2) for given
\[
\eta(\omega) = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} c_{kl} Y_{k,l}(\omega) \in C^\infty(S^2),
\]
\[
(4.5)
\]
where \( Y_{k,l}(\omega) \) (\( k \geq 0, -k \leq l \leq k \)) denotes spherical harmonic of order \( (k, l) \), and \( c_{kl} \) is rapidly decreasing in \( k \). By regularity theory of elliptic differential equations, we see that there is a unique smooth solution \((u, v)\) of problem (4.2). Let
\[
\begin{align*}
  u(r, \omega) &= \sum_{k=0}^{\infty} \sum_{l=-k}^{k} a_{kl}(r) Y_{k,l}(\omega) \quad \text{and} \quad v(r, \omega) = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} b_{kl}(r) Y_{k,l}(\omega),
\end{align*}
\]
\[
(4.6)
\]
where \( a_{kl}(r) \) and \( b_{kl}(r) \) are unknown functions to be determined later. Recall that the well-known formula
\[
\Delta \omega Y_{k,l}(\omega) = -(k^2 + k) Y_{k,l}(\omega) \quad \text{for } k \geq 0.
\]
By substituting (4.5) and (4.6) into (4.2), and comparing coefficients of each \( Y_{k,l}(\omega) \), we have

\[
a''_{kl}(r) + \frac{2}{r}a'_{kl}(r) - \frac{k^2 + k}{r^2}a_{kl}(r) = a_{kl}(r),
\]

(4.7)

\[
a_{kl}(R_s) = \left[c_2 - c_1(k^2 + k)\right]c_{kt},
\]

(4.8)

\[
b''_{kl}(r) + \frac{2}{r}b'_{kl}(r) - \frac{k^2 + k}{r^2}b_{kl}(r) = -a_{kl}(r),
\]

(4.9)

\[
b_{kl}(R_s) = 0.
\]

(4.10)

Recall modified Bessel functions (cf. [29]):

\[
I_m(r) = \sum_{k=0}^{\infty} \frac{(r/2)^{m+2k}}{k!\Gamma(m + k + 1)} \quad \text{for } m \geq 0,
\]

(4.11)

which satisfies

\[
\begin{align*}
I''_m(r) + \frac{1}{r}I'_m(r) - \left(1 + \frac{m^2}{r^2}\right)I_m(r) &= 0 \quad \text{for } r > 0, \\
I_m(r) \text{ bounded at } r &\sim 0,
\end{align*}
\]

(4.12)

and there hold the following properties:

\[
I'_m(r) + \frac{m}{r}I_m(r) = I_{m-1}(r) \quad \text{for } m \geq 1,
\]

(4.13)

\[
I'_m(r) - \frac{m}{r}I_m(r) = I_{m+1}(r) \quad \text{for } m \geq 0.
\]

(4.14)

By using (4.12) we get the solution of problem (4.7)–(4.10) is given by

\[
a_{kl}(r) = c_{kt}\left[c_2 - c_1(k^2 + k)\right]\frac{R_s^\frac{k}{2}I_{k+\frac{3}{2}}(r)}{r^\frac{k}{2}I_{k+\frac{1}{2}}(R_s)},
\]

(4.15)

\[
b_{kl}(r) = -c_{kt}\left[c_2 - c_1(k^2 + k)\right]\frac{R_s^\frac{k}{2}I_{k+\frac{1}{2}}(r)}{r^\frac{k}{2}I_{k+\frac{1}{2}}(R_s)} - \frac{r^k}{R_s^k}.
\]

(4.16)

By (4.1), (4.14) and (4.16) we compute

\[
\mathcal{B}(0)(b_{kl}(r)Y_{k,l}(\omega)) + c_3c_{kt}Y_{k,l}(\omega)
\]

\[
= \left\{ b_{kl}(R_s) + c_3c_{kt}\right\}Y_{k,l}(\omega)
\]

\[
= \left\{ -\left(c_2 - c_1(k^2 + k)\right)\frac{I_{k+\frac{3}{2}}(R_s)}{I_{k+\frac{1}{2}}(R_s)} + c_3\right\}c_{kt}Y_{k,l}(\omega)
\]

\[
= \left\{ -\mu\left(\frac{\sigma\gamma}{R_s^2}(1 - \frac{1}{2}(k^2 + k)) - \frac{1}{3}\sigma R_s\frac{I_{k+\frac{3}{2}}(R_s)}{I_{k+\frac{1}{2}}(R_s)} - \mu\frac{\gamma}{R_s} + \mu\sigma\right)\right\}c_{kt}Y_{k,l}(\omega)
\]

\[
= \Lambda_k(\gamma)c_{kt}Y_{k,l}(\omega).
\]
Note that $I_{1/2}(r) = \sqrt{\frac{2}{\pi r}} \sinh r$. One can easily verify that (2.3) is equivalent to
\[ f(R_s) = \left(1 - \frac{\gamma}{R_s}\right) \frac{I_{3/2}(R_s)}{R_s I_{1/2}(R_s)} \quad \text{and} \quad \tilde{\sigma} = 3\tilde{\sigma}(1 - \frac{\gamma}{R_s}) \frac{I_{3/2}(R_s)}{R_s I_{1/2}(R_s)}. \] (4.18)

By using (4.18), we can rewrite
\[ \Lambda_k(\gamma) = \frac{\mu\tilde{\sigma}}{R_s} \left[ \gamma(h_k + j_k) - j_k R_s \right] \quad \text{for } \gamma > 0, \ k = 0, 1, 2 \cdots , \] (4.19)
where
\[ h_k := \left(\frac{k^2 + k}{2} - 1\right) \frac{I_{k+\frac{3}{2}}(R_s)}{R_s I_{k+\frac{1}{2}}(R_s)}, \quad j_k := 1 - \frac{3I_{\frac{3}{2}}(R_s)}{I_{\frac{1}{2}}(R_s) I_{k+\frac{1}{2}}(R_s)} - \frac{I_{\frac{3}{2}}(R_s) I_{k+\frac{3}{2}}(R_s)}{I_{\frac{3}{2}}(R_s) I_{k+\frac{1}{2}}(R_s)}. \] (4.20)

By (4.46) in [31] we have
\[ j_0 < 0, \quad j_1 = 0 \quad \text{and} \quad j_k > 0 \quad \text{for } k \geq 2. \] (4.21)

It follows that
\[ \Lambda_1(\gamma) \equiv 0 \quad \text{for } \gamma > 0. \] (4.22)

A direct computation also shows that
\[ f'(R_s) = -\frac{1}{R_s^2} \left[ \gamma(h_0 + j_0) - j_0 R_s \right], \]
so that by (2.4) and (4.19) we have
\[ \Lambda_0(\gamma) = -\mu\tilde{\sigma} R_s f'(R_s) \neq 0 \quad \text{for } \gamma > 0. \] (4.23)

From (4.4) and the deduction (4.17), we have the following result:

**Lemma 4.1** For any $\eta \in C^\infty(S^2)$ with expansion $\eta = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} c_{kl} Y_{k,l}(\omega)$, there holds
\[ \partial\Psi(0)\eta = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} \Lambda_k(\gamma) c_{kl} Y_{k,l}(\omega), \] (4.24)
where $\Lambda_k(\gamma)$ is given by (4.19) with $\Lambda_0(\gamma) = -\mu\tilde{\sigma} R_s f'(R_s)$ and $\Lambda_1(\gamma) \equiv 0$.

**Lemma 4.1** implies that for $k \in \{0, 1, 2, \cdots \}$ and $\gamma > 0$, $\Lambda_k(\gamma)$ is an eigenvalue of the operator $\partial\Psi(0)$. As mentioned before, the spectrum of $\partial\Psi(0)$, denoted by $\sigma(\partial\Psi(0))$, consists of all eigenvalues, hence we have the following

**Corollary 4.2** The spectrum of $\partial\Psi(0)$ is given by
\[ \sigma(\partial\Psi(0)) = \{\Lambda_k(\gamma): k = 0, 1, 2, \cdots \}. \]
Next, we study the property of eigenvalue $\Lambda_k(\gamma)$. We introduce
\[
\gamma_k := \frac{j_k}{h_k + j_k} R_s \quad \text{for } k = 2, 3, \cdots, \tag{4.25}
\]
where $h_k$ and $j_k$ are given by (4.20).

**Lemma 4.3** For each $k \in \{2, 3, \cdots\}$, we have $\gamma_k > 0$. Moreover, $\lim_{k \to +\infty} \gamma_k = 0$.

**Proof.** By (4.20) and (4.21),
\[
h_k > 0 \quad \text{and} \quad 0 < j_k < 1 \quad \text{for } k = 2, 3, \cdots.
\]
Thus we have
\[
0 < \gamma_k = \frac{j_k R_s}{h_k + j_k} < \frac{R_s}{h_k + 1} \quad \text{for } k = 2, 3, \cdots.
\]
Recall the well-known formula (cf. [29])
\[
I_m(r) = \sqrt{\frac{1}{2\pi m}} \left( \frac{e^r}{2m} \right)^m \left( 1 + O\left( \frac{1}{m} \right) \right) \quad \text{as } m \to +\infty.
\]
We have
\[
\frac{I_{k+\frac{3}{2}}(r)}{I_{k+\frac{1}{2}}(r)} = e^{r(2k + 1)^{k+1}} \left( 1 + O\left( \frac{1}{k} \right) \right) = \frac{r}{2k} + O\left( \frac{1}{k^2} \right) \quad \text{as } k \to +\infty,
\]
and
\[
\frac{R_s}{h_k + 1} = \left[ \left( \frac{k^2 + k}{2} \right) - 1 \right] \frac{I_{k+\frac{3}{2}}(R_s)}{R_s I_{k+\frac{1}{2}}(R_s)} + 1 \right]^{-1} R_s = \frac{4}{k} R_s + O\left( \frac{1}{k^2} \right) \quad \text{as } k \to +\infty.
\]
It immediately follows that $\lim_{k \to +\infty} \gamma_k = 0$. \(\square\)

Let
\[
\gamma_* := \sup\{\gamma_k; k = 2, 3, \cdots\}. \tag{4.26}
\]
By Lemma 4.3, we see that $\gamma_* > 0$.

Now, we state the following useful properties of eigenvalues of $\partial \Psi(0)$:

**Lemma 4.4** (i) For any $\gamma > 0$, we have $\lim_{k \to +\infty} \Lambda_k(\gamma) = +\infty$.

(ii) If $\gamma > \gamma_*$, then $\Lambda_k(\gamma) > 0$ for all $k \in \{2, 3, \cdots\}$; Moreover, 0 is an eigenvalue of $\partial \Psi(0)$ with geometric multiplicity 3.

(iii) If $0 < \gamma < \gamma_*$, then there exists at least one integer $k_0 \geq 2$ such that $\Lambda_{k_0}(\gamma) < 0$.

**Proof.** By (4.19) and (4.25), we see that $\Lambda_k(\gamma)$ can be expressed as
\[
\Lambda_k(\gamma) = \frac{\mu \sigma}{R_s}(h_k + j_k)(\gamma - \gamma_k) \quad \text{for } k = 2, 3, \cdots. \tag{4.27}
\]
Recall that $h_k + j_k > 0$ for all $k = 2, 3, \ldots$. By the proof of Lemma 4.3, we can show that
\[
h_k + j_k = \frac{k}{4} + O(1) \quad \text{as } k \to +\infty.
\]
By $\lim_{k \to +\infty} \gamma_k = 0$, we immediately get the assertion (i).

Clearly, the assertions (ii) and (iii) follow from (4.26), (4.27) and Lemma 4.1. \(\square\)

**Remark 4.5** \(0 \in \sigma(\partial \Psi(0))\) is due to the translation invariance of original free boundary problem (1.1). In fact, recall that $(\sigma_s^{[x_0]}, p_s^{[x_0]}, \rho_s^{[x_0]})$ is the translated radial stationary solution induce by $[x \to x + x_0]$ as introduced before. Note that $\rho_s^{[0]} = 0$. By the reduction in section 2, we have $\Psi(\rho_s^{[x_0]}) = 0$ for $|x_0|$ is sufficiently small. Let $r_x := x/|x|$ for $x \neq 0$. By (2.3) in [23], we have $r_x = (e_1, e_2, e_3)$ with $e_1, e_2, e_3 \in \text{span}\{Y_{1,1}, Y_{1,0}, Y_{1,-1}\}$. Since $|x| = R_s + \rho_s^{[x_0]}$ is equivalent to $|x - x_0| = R_s$, we have
\[
\rho_s^{[x_0]} = |x_0|\langle r_{x_0}, r_x \rangle + O(|x_0|^2) \quad \text{as } |x_0| \to 0,
\]
and $\langle r_{x_0}, r_x \rangle \in \text{span}\{Y_{1,1}, Y_{1,0}, Y_{1,-1}\}$. It implies that $\partial \Psi(0)Y_{1,l} = 0$, $l = -1, 0, 1$. Moreover, for given $\delta > 0$ sufficiently small,
\[
\mathcal{M} := \{\rho_s^{[x_0]}, x_0 \in \mathbb{R}^3, |x_0| < \delta\}
\]
is a 3-dimensional Banach manifold with the tangent space at 0 given by $\text{span}\{Y_{1,1}, Y_{1,0}, Y_{1,-1}\}$.

5 **Asymptotic stability**

In this section we study asymptotic behavior of transient solutions of problem (2.15) and give the proof of Theorem 1.2. Since $0 \in \sigma(\partial \Psi(0))$, the standard linearized stability theorem for parabolic differential equations in Banach spaces is not applicable. We have to employ a so-called generalized principle of linearized stability developed in [27] to overcome this difficulty.

**Theorem 5.1** (i) If $f'(R_s) < 0$ and $\gamma > \gamma_*$, then the equilibrium 0 of problem (2.15) is asymptotically stable in the following sense: There exists constant $\epsilon > 0$ such that for any $\rho_0 \in \mathcal{O}_\delta$ satisfying $\|\rho_0\|_{H^{4+\alpha}(\mathbb{S}^2)} < \epsilon$, problem (2.15) has a unique global classical solution $\rho(t)$ in $\mathcal{O}_\delta$, which converges exponentially fast to a translated equilibrium $\rho_s^{[x_0]} \in \mathcal{M}$ in $h^{4+\alpha}(\mathbb{S}^2)$ for some $x_0 \in \mathbb{R}^3$, as $t \to +\infty$.

(ii) If $f'(R_s) > 0$ or $0 < \gamma < \gamma_*$, the equilibrium 0 is unstable.

**Proof.** (i) Set $X_0 := h^{3+\alpha}(\mathbb{S}^2)$ and $X_1 := h^{4+\alpha}(\mathbb{S}^2)$. For given $T > 0$, let $J := [0, T)$ and $E_0(J) := BUC(J, X_0)$, $E_1(J) := BUC^1(J, X_0) \cap BUC(J, X_1)$. 20
Note that $X_1$ is densely embedded into $X_0$ and the little Hölder spaces have the following well-known interpolation property

$$(h^{σ_0}(S^2), h^{σ_1}(S^2))_{θ,∞}^0 = h^{(1-θ)σ_0 + θσ_1}(S^2)$$

if $(1 - θ)σ_0 + θσ_1 ∉ N$, \hspace{1cm} (5.1)

where $θ ∈ (0, 1)$, $0 < σ_0 < σ_1$ and $(·, ·)_{θ,∞}^0$ denotes the continuous interpolation functor (cf. \[1\] \(26\)). We see that $X_1$ is the trace space of $E_1(J)$.

Set $A := ∂Ψ(0)$. By Lemma 3.2, $A ∈ H(X_1, X_0)$. Then by Remark III. 3.4.2 (b) in \[1\] and (5.1), we have

$$(E_0(J), E_1(J))$$

is a pair of maximal continuous regularity for $A$. \hspace{1cm} (5.2)

For $f'(R_σ) < 0$ and $γ > γ_σ$, by \[4.23\] and Lemma 4.4, we have $Λ_0(γ) = -μσR_σf'(R_σ) > 0$, $Λ_k(γ) > 0$, $k ≥ 2$, and 0 is an eigenvalue of geometric multiplicity 3. By Lemma 4.4 (i), we see that there exists constant $Γ > 0$ such that

$$σ(A) \{0\} ⊂ C^+ := \{z ∈ C : Re z > Γ\}. \hspace{1cm} (5.3)$$

Denote the kernel and the range of $A$ by $N(A)$ and $R(A)$, respectively. Then by Lemma 4.1, for $γ > γ_σ$ we have

$$N(A) = \text{span}\{Y_{1,1}, Y_{1,0}, Y_{1,-1}\} \quad \text{and} \quad N(A) ⊕ R(A) = X_0. \hspace{1cm} (5.4)$$

By Remark 4.5, we see that $M = \{ρ^{[x_0]}_s, x_0 ∈ R^3, |x_0| < δ\}$ is a 3-dimensional $C^1$-manifold near the equilibrium 0 and the tangent space for $M$ at 0 is given by $N(A)$. Thus from this observation and together with (5.2)–(5.4), we see that all assumptions of Theorem 3.1 in \[27\] are satisfied, so that the desired assertion (i) follows from this theorem immediately. More precisely, there exists $ε > 0$ such that the solution $ρ(t)$ of problem (2.15) exists globally for any $ρ_0 ∈ O_δ$ satisfying $∥ρ_0∥_{h^{4+α}(S^2)} < ε$, and moreover, there exist constant $M > 0$ and a point $x_0 ∈ R^3$ close to 0, such that

$$∥ρ(t) - ρ^{[x_0]}_s∥_{h^{4+α}(S^2)} ≤ Me^{-ωt}∥ρ_0∥_{h^{4+α}(S^2)} \quad \text{for} \quad t ≥ 0. \hspace{1cm} (5.5)$$

(ii) If $f'(R_σ) > 0$ or $0 < γ < γ_σ$, then by \[4.23\] and Lemma 4.4 (iii), we see that $σ(-A) ∩ \{z ∈ C : Re z > 0\}$ is not empty. It follows from Theorem 9.1.3 in \[26\] that the equilibrium 0 is unstable. The proof is complete. \hspace{1cm} \Box

**Remark 5.2** Theorem 5.1 (i) can be also proved by using a center manifold theorem for quasi-invariant parabolic differential equations in Banach spaces developed in \[7\]. Since original free boundary problem \[1.1\] is invariant under coordinate translations, the reduce differential equation (2.15) possesses quasi-invariance under a natural local Lie group action induced by coordinate translations. With the aid of Theorem 2.1 in \[7\], we can show that $M$ is a center manifold which attracts nearby transient solutions and get the desired assertion. The proof is more complicated though it can obtain more delicate information than the proof given here, we
refer interested readers to the proof of Theorem 1.1 in [7], see also the proof of main result in [32] and [34] for similar free boundary problems.

Now, we give the proof of our main result Theorem 1.2.

Proof of Theorem 1.2. Recall that for $0 \leq \tilde{\sigma}/\bar{\sigma} < \theta^*$, the free boundary problem (1.1) has two radial stationary solution $(\sigma_s, p_s, \rho_s)$ with radius $R_{s1} < R_{s2}$. By (2.4), we have $f'(R_{s1}) > 0$ and $f'(R_{s2}) < 0$. Thus by Lemma 2.1, Lemma 2.2 and Theorem 5.1 (i), we see that for the radial stationary solution $(\sigma_s, p_s, \rho_s)$ with radius $R_{s2}$ in case $\gamma > \gamma_s$, there exists constant $\epsilon > 0$ such that for any $\rho_0 \in O$ satisfying $\|\rho_0\|_{h^{4+\alpha}(S^2)} < \epsilon$, free boundary problem (1.1) has a unique global solution $(\sigma(t), p(t), \rho(t))$ which is given by

$$\sigma(t) = \Theta_{s}(t) U(\rho(t)), \quad p(t) = \Theta_{s}(t) V(\rho(t)),$$

and $\rho(t)$ is the solution of problem (2.15) with $\rho(0) = \rho_0$, where $U$ and $V$ are given by (2.12). By (5.5) and the reduction in section 2, we see that there exists $x_0 \in \mathbb{R}^3$ such that $(\sigma(t), p(t), \rho(t))$ converges exponentially fast to $(\sigma_s[x_0], p_s[x_0], \rho_s[x_0])$ in $h^{2+\alpha}(\Omega_{\rho(t)}) \times h^{4+\alpha}(\Omega_{\rho(t)}) \times h^{4+\alpha}(S^2)$, as time goes to infinity.

Similarly by Lemma 2.1, Lemma 2.2 and Theorem 5.1 (ii), we see that the radial stationary solution $(\sigma_s, p_s, \rho_s)$ with radius $R_{s2}$ is unstable for $0 < \gamma < \gamma_s$, and the radial stationary solution $(\sigma_s, p_s, \rho_s)$ with radius $R_{s1}$ is unstable for all $\gamma > 0$. This completes the proof. \qed

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