Nonlinear Dynamics of MicroResonators

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Abstract. The variation of effective parameters in Micro Electro Mechanical Systems strongly affects their performance, design, and control. Hence, it is essential to understand and model the effective parameters in MEMS devices to optimize their designs. Typical MEMS (Microresonator) employ a parallel-plate capacitor, in which one plate is actuated electrically and its motion is detected by capacitive changes. In this paper we investigate nonlinear modelling of microresonators. The nonlinearities from capacitor (quadratic) and midplane stretch were considered for this purpose. The achieved microbeam's equations are nondimensionalized and by using the multiple scales method received to the equations which identified the relations between system dynamics and effective parameters of system. In other part of this paper we expand the Fuzzy Generalized Cell Mapping (FGCM) for multiparameter systems then apply FGCM to the microresonator and see how the uncertainties can affect the working domain of dynamical systems. It can be seen FGCM is so useful method for detecting the working region with variation of parameters.

Keywords: MEMS, Microresonator, Nonlinear, Dynamic, Midplane Stretch, Dynamic, Fuzzy Generalized Cell Mapping, Perturbation, Poincare Map, Stochastic Systems.

1. Introduction

Most MicroElectroMechanical Systems (MEMS) are inherently nonlinear and the micro scale effects and the coupled fields give rise to complete nonlinearities in MEMS. There exist intrinsic nonlinearities and exterior nonlinearities arising from coupling of different domains [1, 2].

The nonlinearities from capacitive (second order) and midplane stretching have been considered in this paper. The transient and the steady state frequency-amplitude dependency of the system will be derived utilizing the multiple scales perturbation method. The developed analytic equation describing the frequency response of the system around resonance can be utilized to explain the dynamics of the system, as well as resonance frequency and peak amplitude.

In another part of this paper the Fuzzy Generalized Cell Mapping (FGCM) [3, 4] will be expanded for multiparameter systems and show that how much this method useful for systems which have uncertainties. This method is applying and it will be shown how much it is useful, for system which has uncertainties, for identification of working domain.
2. Reduced Order Model of Microresonators

The microresonator is composed of a beam resonator, a ground plane underneath in contact with the beam, and one (or more) capacitive transducer electrode(s). The one dimensional electrostatic force, \( f_e \), between two electrodes is

\[
f_e = \frac{\varepsilon_0 A (v - v_p)^2}{2(d - w)^2}
\]

(1)

where, \( \varepsilon_0 = 8.85 \times 10^{-12} \) (\( \text{As/Vm} \)) is permittivity in vacuum, \( A \) is the area of the microplate, \( w = w(x, t) \) is the lateral displacement of the microbeam, and \( v_p \) and \( v = v, \sin(\omega t) \) is the electric load composed of a DC polarization voltage, \( v_p \), and an AC actuating voltage, \( v = v, \sin(\omega t) \).

The equation describing lateral vibrations of the microbeam can be summarized and simplified to the following equation when the beam’s geometry is uniform [5,6].

\[
\rho \frac{\partial^4 w}{\partial t^2} + \varepsilon \frac{\partial^2 w}{\partial t} + E I \frac{\partial^4 w}{\partial x^4} = \frac{\varepsilon_0 A (v - v_p)^2}{2(d - w)^2} + \left( N + \frac{E A_0}{2L} \int \left( \frac{\partial w}{\partial x} \right)^2 \right) \frac{\partial^2 w}{\partial x^2}
\]

(2)

To make the equation of motion dimensionless the following variables are defined. The parameter \( n \) is a constant depending on mode shape of the microbeam.

\[
\tau = \omega_1 t, \quad \omega_1 = \frac{\sqrt{E I}}{L^3} \quad , \quad z = \frac{x}{L}, \quad y = \frac{w}{d}, \quad Y = \frac{w_0}{d}, \quad r = \frac{\omega}{\omega_1}
\]

\[
a_1 = \frac{\varepsilon_0 A L^4}{2n^2d^3EI}, \quad a_2 = \frac{cL^2}{n\sqrt{EI}}, \quad \alpha_1 = 3 \left( \frac{nd}{h} \right)^2 = \frac{A_0}{4I} \left( \frac{nd}{h} \right)^2, \quad N = \frac{N_l^2}{EI}
\]

(3)

Using these parameters, the equation of motion transforms to the following dimensionless equation.

\[
\frac{\partial^4 Y}{\partial \tau^2} + N + \alpha_1 \int \left( \frac{\partial w}{\partial x} \right)^2 \frac{\partial^2 Y}{\partial x^2} = \frac{\varepsilon_0 A (v - v_p)^2}{2(1 - y)^2} + \left( N + \frac{E A_0}{2L} \int \left( \frac{\partial w}{\partial x} \right)^2 \right) \frac{\partial^2 Y}{\partial x^2}
\]

(4)

We apply a separation solution

\[
y = Y(\tau) \phi(z)
\]

(5)

where the spatial function \( \phi(z) \) is called mode shape function, and must satisfy the boundary conditions. By accepting a first harmonic shape function, the temporal function \( Y(\tau) \) would then represent the maximum deflection of the microbeam, which is the middle point for symmetric boundary conditions, and the tip point for microcantilever[7,8].

A microcantilever is a microbeam with the following boundary conditions.

\[
y(0, r) = 0, \quad \frac{\partial y}{\partial z}(0, r) = 0, \quad \frac{\partial^2 y}{\partial z^2}(1, r) = 0, \quad \frac{\partial^3 y}{\partial z^3}(1, r) = 0
\]

(6)
The first harmonic mode shape satisfying the required boundary conditions, is $\phi(x) = \cos\left(\frac{nx}{2}\right)$ and the mode shape parameter $n = \frac{\pi^2}{4}$ [7,8].

Therefore, the required differential equation for the temporal function $Y(\tau)$ related to a microcantilever, would be

$$Y' + h Y + Y = \frac{1}{(1-y)^2} \left[ (\alpha + \beta) + 2\sqrt{2\alpha\beta} \sin(r\tau) - \beta \cos(r\tau) \right] - NY - \alpha Y^3 \quad (7)$$

where $h = a_2, \alpha = a_\gamma r^2, \sqrt{2\alpha\beta} = a_\gamma r, \beta = a_2 r^2$.

The second order expansion will be used to examine the thermal effects on microresonators,

$$Y' + h Y + Y = \left(1 + 2Y + 3Y^2\right) \left[ (\alpha + \beta) + 2\sqrt{2\alpha\beta} \sin(r\tau) - \beta \cos(r\tau) \right] - NY - \alpha Y^3 \quad (8)$$

3. Mathematical Analysis

The microresonators were analysed mathematically in following parts, first with using the perturbation method we achieve to the system dynamics equation and then using the Fuzzy Generalized Cell Mapping and investigating the uncertain microresonator's working and attraction domain.

3.1. Perturbation Analysis

We use the method of multiple scales to find the uniform solutions of Eq.(8) in the following form

$$Y(\tau, \varepsilon) = Y_0(T_0,T_1) + \varepsilon Y_1(T_0,T_1) \quad (9)$$

Then, we have the differential operators

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \frac{\partial}{\partial T_1} + \ldots = D_0 + \varepsilon D_1 + \ldots \quad (10)$$

$$\frac{d^2}{dt^2} = \left(D_0 + \varepsilon D_1 + \ldots\right)^2 = D_0^2 + 2\varepsilon D_0 D_1 + \ldots \quad (11)$$

where $D_k = \frac{\partial}{\partial T_k}, k = 0,1$.

Because actual quantity of $h, \alpha, \beta, \alpha, N, \alpha$ are small, they have been replaced by $\varepsilon h, \varepsilon \alpha, \varepsilon \beta, \varepsilon \alpha$,$\varepsilon N, \varepsilon \alpha_1$ respectively and the stiffness replaced by $1 - \varepsilon \alpha_\gamma$, we investigate the case of $r \approx 1$ ($r \approx 1$ is primary resonance and $r \approx \frac{1}{2}, \frac{1}{4}, \frac{1}{5}$ are secondary resonance) because of that $r$ will be replaced by

$$r = 1 + \varepsilon \sigma \quad (12)$$

where $\sigma$ is detuning parameter. It must be known if the detuning parameter don't be considered in Eq.(8) then only the resonance time response can be achieved (because the variation of frequency doesn't be considered.)

The solution of Eq.(8) in the complex form can be expressed as

$$Y_0 = A(T_0)e^{iT_0} + \overline{A}(T_0)e^{-iT_0} \quad (13)$$

where $\overline{A}$ is the part of complex conjugate of $A$. 

Eliminating the terms that produce the secular term then the following equation will be obtained in

\[ \frac{1}{2} \left( -\sqrt{2}\beta a \cos(\theta) - a(\alpha + \beta \sin(\theta)) + 3\sqrt{2}\beta a \cos(\theta) a \right) \]

\[ \gamma' = \frac{1}{2} \left( N - 2(\beta + \alpha) + \beta \cos(2\theta) - a - \frac{\sqrt{2}\beta a \sin(\theta)}{a} - 9a\sqrt{2}\beta a \sin(\theta) + 3a, a^2 \right) \quad (14) \]

where \( \theta = T_0 + \sigma T_1 \). Assuming \( \alpha' \) and \( \gamma' - \sigma \) remain zero in steady state response. \( (\gamma - \sigma T_1 \) is argument of sinusoidal term and must invariant in time when \( T_1 \to \infty \)) Therefore the following set of couple equations are provided

\[ \frac{1}{2} \left( -\sqrt{2}\beta a \cos(\theta) - a(\alpha + \beta \sin(\theta)) + 3\sqrt{2}\beta a \cos(\theta) a \right) = 0 \]

\[ \frac{1}{2} \left( N - 2(\beta + \alpha) + \beta \cos(2\theta) - a - \frac{\sqrt{2}\beta a \sin(\theta)}{a} - 9a\sqrt{2}\beta a \sin(\theta) + 3a, a^2 \right) - \sigma = 0 \quad (15) \]

Eliminating \( \gamma(T_1) \) and assuming \( 0 \leq a \leq 1 \) provide a relationship between the parameter of the system to have a periodic steady state response with frequency \( r \). Given the equation that shows the \( a \) as a function of \( h, \alpha, \beta, a, r, N \).

\[-36a^5 \left( 1 - 3a^2 \right) \beta a \left( 2\beta - 3(\alpha + 9a^2) \right)^2 (4N - 4\beta + (10 + \frac{1}{a}) (1 + 81a^2) \alpha - 4(r - 1) - 4a, + \]

\[ + 12a, a^2 \right) \left( \beta^2 (16h^2 a + (1 - 3a^2) \alpha ( -8a^2 \beta + (1 + 9a^2) \alpha) - 2a^2 (a^2 (\alpha - 9a^2 \alpha + \]

\[ + 3a^2 \left( 2\beta - 3(\alpha + 9a^2) \right)^2 (4N - 4\beta + (10 + \frac{1}{a}) (1 + 81a^2) \alpha - 4(r - 1) - 4a, + \]

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\[ + 12a, a^2 \right) \left( \beta^2 (16h^2 a + (1 - 3a^2) \alpha ( -8a^2 \beta + (1 + 9a^2) \alpha) - 2a^2 (a^2 (\alpha - 9a^2 \alpha + \]
It can be seen that the increasing of the DC (See Fig.2.(a)) and AC (See Fig.2.(b)) voltage increase the peak amplitude monotonically but both of them decrease the resonance frequency. Increasing the axial load increases the resonance frequency (Fig.2.(c)) on the other hand it has no much effect on peak amplitude.

3.2. Fuzzy Generalized Cell Mapping Analysis

We first expand the FGCM method for multiparameter nonlinear dynamical systems [3,4]. Consider a dynamical system with a fuzzy parameter

\[
\dot{X} = F(X, t, (s^1, s^2, ..., s^l)) \quad X \in D
\]

where \( X \) is the state vector, \( t \) the time variable, \((s^1, s^2, ..., s^l)\) fuzzy parameters where each fuzzy parameter has a membership function \( \mu_{\alpha}(s^i) \in (0, 1) \), we use fuzzy t-norm (fuzzy product (same as system probability) or minimum fuzzy t-norm or any other fuzzy t-norm can be used) to indicating the membership function of \( F(X, t, (s^1, s^2, ..., s^l)) \)

\[
\mu_{\alpha}(S) = \mu_{\alpha_1, \alpha_2, ..., \alpha_l}(s^1, s^2, ..., s^l) = \mu_{\alpha_1}(s^1) \ast \mu_{\alpha_2}(s^2) \ast ... \ast \mu_{\alpha_l}(s^l)
\]

where \( S = (s^1, s^2, ..., s^l) \), \((Q^i = \text{the domain of } i\text{th parameter}) \) and \( S \in Q_1 \times ... \times Q_l \). and

\[
X^2 (4\beta + 6\alpha)(4N - 4\beta + (10 + \frac{1}{a^2} + 8la^2)\alpha - 4(r - 1) - 4a_1 + 12a_2a_1) + a^3 (4N - 4\beta) +
\]

\[
(10 + \frac{1}{a^2} + 8la^2)\alpha - 4(r - 1) - 4a_1 + 12a_2a_1)^2 = 0
\]

(16)

Fig.3. The map plotted for \( \alpha_1 = 1, \alpha_2 = 10^{-3}, \beta = 10^{-3}, a = 99999 \times 10^{-3} \) has Global phase portrait of the microresonator equation with a fuzzy parameter with \( \mu(h, 10^{-4}, 10^{-3}, 10^{-3}) \) as membership function. The membership distribution of fuzzy attractors is colour-coded with black=1.0, 0.8 < red < 1.0, 0.6 < green < 0.8, 0.4 < purple < 0.6, 0.2 < cyan < 0.4, and 0.0 < yellow < 0.2. Symbol + represent the trajectories which go out the interesting domain and symbol * shows stable cell.

\( Q = Q_1 \times ... \times Q_l \), and \( F \) is a vector-valued nonlinear function of its arguments. It is assumed to be periodic in \( t \) with period \( T \) for all \( S \in Q \) and to satisfy the Lipschitz condition for all \( S \in Q \), and \( D \) is
a bounded domain of interest in the state space. A fuzzy Poincare’ map can be obtained from Eq. (17) as

\[ X(n + 1) = G(X(n), S), \quad n = 0, 1, 2, \ldots \]  \hspace{1cm} (19)

Now the FGCM method which has been presented in Hong and Sun works [3,4] can be used for multiparameter systems.

3.2.1. Using FGCM for MicroCantilever

The FGCM have been applied to Eq.(7) for determining the domain of attraction of chaotic attractor in phase plane. The uncertainties from system damping have been considered and the triangular membership function was considered for this analysis (the desired domain were divided to 60*60 cells and one point sampled from each cell and the membership function were divided to 11 segments). This is so important to indicating the working domain of system because it can be showed which load can be applied to uncertain system that the system not able to work with that in desired domain.

4. Conclusion

We modeled the microresonators with considering the nonlinearity from midplane stretch and capacitive then it was solved by using the method of multiple scales. Ac and Dc voltage affected the peak amplitude and resonance frequency of the system also the axial load increases the resonance frequency drastically. We expand the fuzzy generalized cell mapping for multi parameter systems. The FGCM uses for determining the system domain of attraction and it can be seen how the fuzzy parameter affects the domain of attraction of the chaotic attractor.

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