On analyticity with respect to the replica number in random energy models: I. An exact expression for the moment of the partition function

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Abstract. We provide an exact expression for the moment of the partition function for random energy models of finite system size, generalizing an earlier expression for a grand canonical version of the discrete random energy model presented by Ogure and Kabashima in 2004 (Prog. Theor. Phys. 111 661). The expression can be handled either analytically or numerically, which is useful for examining how the analyticity of the moment with respect to the replica numbers, which play the role of powers of the moment, can be broken in the thermodynamic limit. A comparison with a replica method analysis indicates that the analyticity breaking can be regarded as the origin of the one-step replica symmetry breaking. The validity of the expression is also confirmed using numerical methods for finite systems.

Keywords: rigorous results in statistical mechanics, disordered systems (theory), spin glasses (theory)
1. Introduction

The replica method (RM) is one of the few analytical techniques available for dealing with disordered systems [1, 2]. In general, the RM can be regarded as a scheme for evaluating the generating function

\[ \phi_N(n) = \frac{1}{N} \log \langle Z^n \rangle, \]

for real number \( n \in \mathbb{R} \) (or complex \( \mathbb{C} \)), where \( Z \) and \( \langle \cdots \rangle \) represent the partition function and the average over the quenched randomness that governs the objective system, respectively. Unfortunately, direct evaluation of equation (1) is difficult in general. In statistical mechanics, the system size \( N \) is in most cases assumed to be infinitely large, which often makes the analysis possible for \( n = 1, 2, \ldots \in \mathbb{N} \). Therefore, the RM in statistical mechanics usually first evaluates equation (1) for \( n \in \mathbb{N} \) in the limit \( N \to \infty \) as

\[ \phi(n) = \lim_{N \to \infty} \phi_N(n), \]

and then analytically continues the expression from \( n \in \mathbb{N} \) to \( n \in \mathbb{R} \).

This procedure generally includes at least two mathematical problems, although it was recently proved for several examples that the solutions obtained by the RM are mathematically correct [3, 4]. The first problem concerns the uniqueness of the analytic continuation from \( n \in \mathbb{N} \) to \( n \in \mathbb{R} \). Even if all values of \( \phi_N(n) \) are provided for \( n = 1, 2, \ldots \), in general, it is not possible to determine the analytical continuation from \( n \in \mathbb{N} \) to \( n \in \mathbb{R} \) (or \( \mathbb{C} \)) uniquely. Carlson’s theorem guarantees that the continuation in the right half of the complex plane is unique if the growth rate of \( |\langle Z^n \rangle^{1/N}| \) is upper bounded by

\[ |\langle Z^n \rangle^{1/N}| \leq C_{\text{bound}}. \]
However, this sufficient condition is not satisfied even in the case of the famous Sherrington–Kirkpatrick (SK) model that describes a fully connected Ising spin glass system [6], for which \(|\langle Z^n \rangle^{1/N}| \) scales as \(O(e^{C|n|^2})\), where \(C\) is a certain constant. It was conjectured by van Hemmen and Palmer that this might be related to the failure of the replica symmetric (RS) solution of the SK model at low temperatures, although further exploration in this direction is difficult [7].

The second issue is the possible breaking of the analyticity of \(\phi(n)\). Even if the uniqueness of the analytic continuation from \(n \in \mathbb{N}\) to \(n \in \mathbb{R}\) (or \(\mathbb{C}\)) is guaranteed for \(\phi_N(n)\) of finite \(N\), the analyticity of \(\phi(n) = \lim_{N \to \infty} \phi_N(n)\) can be broken. This implies that if the analyticity breaking occurs at a certain point \(n = n_c < 1\), the analytically continued expression based on \(\phi(n)\) will lead to an incorrect solution for \(n < n_c\). In an earlier study, the authors developed an exact expression for \(\langle Z^n \rangle\) for a grand canonical version of the discrete random energy model (GCDREM) [8,9] of finite system size [10]. The expression indicates that the scenario speculated to occur does occur in this model as \(N \to \infty\). In addition, the model satisfies the sufficient condition of Carlson’s theorem, guaranteeing the uniqueness of the analytical continuation for finite \(N\). These points imply that the analyticity breaking with respect to the replica number \(n\) is the origin of the one-step replica symmetry breaking (1RSB) observed in the GCDREM. However, as the GCDREM is a relatively unfamiliar model, the result obtained may be regarded as anomalous and therefore might be taken to be less significant.

The purpose of this paper is to address this view. More precisely, we develop an exact expression for \(\langle Z^n \rangle\) that is applicable to generic random energy models (REMs) [11] of finite system size. The expression, which is a generalization of that developed for the GCDREM, indicates that analyticity breaking with respect to the replica number occurs in standard REMs as well and is the origin of the 1RSB.

This paper is organized as follows. In section 2, we introduce the model. In order to guarantee the uniqueness of analytical continuation, we will mainly consider the canonical version of the discrete random energy model (CDREM). However, a similar approach is also applicable for standard continuous REM, although the uniqueness of the analytical continuation from \(n \in \mathbb{N}\) to \(n \in \mathbb{C}\) is not guaranteed by Carlson’s theorem. This is shown in appendix A. In section 3, the main part of the paper, an exact expression for \(\langle Z^n \rangle\) is developed. In section 4, the expression developed is used to examine how the analyticity of \(\langle Z^n \rangle\) can be broken as \(N \to \infty\). The utility of the expression for assessing the moment of REMs of finite size is also shown. Section 5 is devoted to a summary.

### 2. Model definition

The canonical discrete random energy model (CDREM), which we will focus on here, is defined as follows. Suppose that \(N\) and \(M\) are natural numbers where \(\alpha = M/N\). The energy for each of the \(2^N\) states \(A = 1, 2, \ldots, 2^N\) is determined as an independent sample from an identical distribution

\[
P(E) = 2^{-M} \left( \frac{M}{2} + E \right),
\]

\[(3)\]
where $E$ is limited to $-M/2, -M/2 + 1, \ldots, M/2$. We denote a set of sampled energy values as $\{\epsilon_A\}$. For each realization of $\{\epsilon_A\}$, the partition function is defined as

$$Z = \sum_{A=1}^{2N} \exp(-\beta \epsilon_A),$$

(4)

where $\beta = T^{-1} > 0$ denotes the inverse temperature. Our main objective is to develop an expression for $\langle Z^n \rangle$, the evaluation of which is computationally feasible for $\forall N, \forall M$ and $\forall n \in \mathbb{C}$, where $\langle \cdot \cdot \cdot \rangle$, in this case, represents the average with respect to $\{\epsilon_A\}$.

Before proceeding further, it is worth explaining why this model has been selected. A distinctive property of this model is that the possible energy values are lower bounded by $-M/2$, which makes it possible to upper bound the absolute value of a modified moment as

$$\left| \left\langle \left( e^{-\beta M/2} Z \right)^n \right\rangle^{1/N} \right| \leq \left\langle \left( e^{-\beta M/2} Z \right)^{\text{Re}(n)} \right\rangle^{1/N} = \left\langle \left( \sum_{A=1}^{2N} e^{-\beta (M/2 + \epsilon_A)} \right)^{\text{Re}(n)} \right\rangle^{1/N},$$

(5)

for any finite $N$. Consider an analytic function $\Psi(n; N)$ which satisfies the condition $|\Psi(n; N)| < O(e^{\pi|n|})$. Carlson’s theorem ensures that if $|\Psi(n; N) - \langle (e^{-\beta M/2} Z)^n \rangle^{1/N}| = 0$ holds for $n = 0, 1, 2, \ldots$, $\Psi(n; N)$ is identical to $\langle (e^{-\beta M/2} Z)^n \rangle^{1/N}$ over the right half of the complex plane of $n$. The identity for $n = 0$ trivially holds if $\Psi(0; N) = 1$. Since $e^{-\beta M/2}$ is a non-vanishing constant and $\langle (e^{-\beta M/2} Z)^n \rangle^{1/N}$ is a single-valued function, this implies that the analytical continuation of $\langle Z^n \rangle^{1/N}$ from $n = 1, 2, \ldots$ to $n \in \mathbb{C}$ is uniquely determined in this model as long as $N$ is finite. This property is useful for examining intrinsic mathematical problems of the RM because we can exclude the possibility of multiple analytical continuations when certain anomalous behavior is observed for $\langle Z^n \rangle$.

This is the main reason why we have selected the CDREM as the objective system in the current paper.

The technique developed in section 3 is also applicable to generic REMs, as shown in appendix A. Unfortunately, we cannot use Carlson’s theorem for guaranteeing the uniqueness of the analytical continuation for the original continuous REM [11], since $|\langle Z^n \rangle^{1/2}|$ is not upper bounded by $e^{\pi|n|}$. However, the analysis in appendix A indicates that the behavior of the continuous REM in the thermodynamic limit is qualitatively the same as that of the CDREM; in both systems, $\phi(n)$ for $n = 1, 2, \ldots$ is described by either of two replica symmetric solutions one of which is proportional to $n$ while the other is nonlinear with respect to $n$, and the replica symmetry breaking solution bifurcates from the nonlinear solution at a certain critical number $0 < n_c < 1$ for $n < n_c$ when the temperature is sufficiently low. This implies that, for a wide class of random energy models, the origin of replica symmetry breaking is not the multiple possibilities of analytical continuation but the analyticity breaking with respect to the replica number that occurs in the thermodynamic limit.

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3. The exact expression for $\langle Z^n \rangle$

The expression for the partition function

$$Z = \sum_{A=1}^{2^N} \exp(-\beta \epsilon_A) = \sum_{i=0}^M n_i \exp(-\beta E_i) = \omega^{-(M/2)} \sum_{i=0}^M n_i \omega^i \quad (\omega \equiv e^{-\beta})$$  \hspace{1cm} (6)$$

is the basis of our analysis, where $n_i$  \((i = 0, 1, \ldots, M)\) represents the number of states with energy $E = i - M/2$. A remarkable property of the DREM is that the probability distribution of $\{n_i\}$ has a feasible form as a multinomial distribution

$$P(\{n_i\}) = \delta_{\sum_i n_i, 2^N} \frac{2^N!}{n_0! \cdots n_M!} \prod_{i=0}^M \{P(E_i)\}^{n_i},$$ \hspace{1cm} (7)$$

where $\delta_{m,n} = 1$ if $m = n$ and 0 otherwise. This makes it possible to assess the moments $\langle Z^n \rangle$ directly from $\{n_i\}$ without referring to the full energy configuration $\{\epsilon_A\}$, yielding

$$\langle Z^n \rangle = \sum_{n_0=0}^\infty \sum_{n_1=0}^\infty \cdots \sum_{n_M=0}^\infty P(\{n_i\}) Z^n$$

$$= \sum_{n_0=0}^\infty \sum_{n_1=0}^\infty \cdots \sum_{n_M=0}^\infty \{P(E_0)\}^{n_0} \{P(E_1)\}^{n_1} \cdots \{P(E_M)\}^{n_2}$$

$$\times \delta_{\sum_{i=0}^M n_i, 2^N} \frac{2^N!}{n_0! \cdots n_M!} Z^n. \hspace{1cm} (8)$$

Two identities are useful for evaluating equation (8):

$$c^n = \int_{H} (\rho)^{-n-1} e^{-\rho z} \, d\rho \quad \frac{1}{\Gamma(-n)} \quad (c > 0, \Gamma(n) \equiv -2i \sin n\pi \Gamma(n)), \hspace{1cm} (9)$$

and

$$\delta_{m,n} = \frac{1}{2\pi i} \oint z^{m-n-1} \, dz, \hspace{1cm} (10)$$

where $i = \sqrt{-1}$. In equation (9), $\Gamma(z) = i/(2\sin z\pi) \int_{H} (\rho)^{z-1} e^{-\rho} \, d\rho$ represents the Gamma function and the integration contour $H$ is provided as shown in figure 1. The integration contour of equation (10) is a single closed loop surrounding the origin. Inserting equations (9) and (10) into equation (8) yields

$$\langle Z^n \rangle = \sum_{n_0=0}^\infty \sum_{n_1=0}^\infty \cdots \sum_{n_M=0}^\infty \{P(E_0)\}^{n_0} \{P(E_1)\}^{n_1} \cdots \{P(E_M)\}^{n_2} \frac{2^N!}{n_0! \cdots n_M!} \times \frac{\omega^{-(nM/2)}}{\Gamma(-n)} \int_{H} d\rho \, (\rho)^{-n-1} e^{-\left(\sum_{i=0}^M n_i \omega^i\right)\rho} \frac{1}{2\pi i} \oint dz \sum_{i=0}^M n_i z^{-n-1} \omega^i$$

$$= \frac{\omega^{-(nM/2)}}{\Gamma(-n)} \int_{H} d\rho \, (\rho)^{-n-1} \left(\frac{2^N!}{2\pi i} \oint dz z^{-2^N-1} \sum_{n_0=0}^\infty \frac{1}{n_0!} \{z P(E_0) e^{-\rho}\}^{n_0}\right)$$

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Figure 1. Integration contour.

\[
\times \left( \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \{zP(E_1)e^{-\omega_1}\}^{n_1} \right) \cdots \left( \sum_{n_M=0}^{\infty} \frac{1}{n_M!} \{zP(E_M)e^{-\omega_M}\}^{n_M} \right)
\]

\[
= \frac{\omega^{-(nM/2)}}{\Gamma(-n)} \int_H d\rho (-\rho)^{-n-1} \frac{(2^N)!}{2\pi i} \oint zz^{-2N-1} e^{z\sum_{i=0}^{M_i} P(E_i)} e^{-\omega_i \rho}.
\]

Finally, applying the residue theorem to the integration with respect to \( z \), we obtain

\[
\langle Z^n \rangle = \frac{\omega^{-(nM/2)}}{\Gamma(-n)} \int_H d\rho (-\rho)^{-n-1} \left( \sum_{i=0}^{M_i} P(E_i)e^{-\omega_i \rho} \right)^{2N},
\]

which exactly holds for any complex value of \( n \). This is the main result of this paper.

Several points are noteworthy here. The first issue is a relation to an earlier study. In [12], Gardner and Derrida provided an expression somewhat similar to equation (12) for non-integer moments of the partition function of the continuous REM. Actually, the present expression for equation (12) can be regarded as a generalization of equation (7) of [12], for which, however, tractability for finite \( N \) is not emphasized. The link of the two expressions is shown in appendix A. The second is the computational cost for evaluating equation (12). For each \( \rho \in \mathbb{C} \), the integrand of equation (12) can be evaluated with \( O(N) \) computations provided \( \alpha = M/N \sim O(1) \). On the other hand, the length of the integration contour \( H \) is infinite, which may seem an obstacle for the numerical evaluation. However, for relatively large \( N \), the integrand rapidly vanishes on the contour \( H \) as \( |\rho| \to \infty \), guaranteeing that the numerical deviation is practically negligible even if we approximate \( H \) by a path of a finite length. Therefore, equation (12) can be evaluated with a feasible computational cost. The third point is the similarity to an approach in information theory sometimes termed the method of types [13]. In this method, the performance of various codes consisting of exponentially many codewords is evaluated by classifying possible events when the codes are randomly generated by varieties of empirical distributions, termed types. The key to this method is accounting for the fact that the number of types grows only polynomially with respect to the code length while the number of the codewords increases exponentially. As a consequence, the
relative weight of the types concentrates at a certain typical value at an exponentially fast rate, which considerably simplifies the performance analysis. In the current case, the set of occupation numbers \( \{n_i\} \) is analogous to the types in information theory. However, in spite of this analogy, objective quantities are usually evaluated using upper and lower bounding schemes in information theory. Therefore, the technique developed in the present paper may serve as a novel scheme for analyzing various codes in information theory. The final issue relates to the relationship to a model examined in an earlier study. In [10], the authors offered a formula similar to equation (12) for a grand canonical version of the discrete random energy model (GCDREM). Unlike the CDREM, the GCDREM is defined by independently generating the occupation number \( n_i \) of each energy level from the beginning. In the present framework, this can be characterized by a factorizable distribution of \( \{n_i\} \),

\[
P(\{n_i\}) = e^{-2N} \prod_{i=0}^{M} \frac{(-P(E_i)2^N)^{n_i}}{n_i!},
\]

which offers a feasible expression

\[
\langle Z^n \rangle = \frac{\omega^{-(nM/2)}}{\Gamma(-n)} \int_H d\rho (-\rho)^{-n-1} \exp \left[ -\sum_{i=0}^{M} (1 - e^{-\omega_i\rho})P(E_i)2^N \right].
\]

This is the counterpart of equation (12). Since the distributions of \( n_i \) are independent, the derivation of equation (14) is easier than that of equation (12). In the GCDREM, the total number of states \( N = \sum_{i=0}^{M} n_i \) fluctuates from sample to sample as \( n_i \) is independently generated at each energy level. The ratio between its standard deviation and expectation vanishes as \( 2^{-N/2} \) and therefore the effect of the statistical fluctuation is practically negligible when \( N \) is reasonably large. Nevertheless, equation (13) implies that an ill-posed sample \( n_i = 0 \) \( (i = 0, 1, \ldots, M) \) can be generated with a probability \( 2^{-N} \), which is not adequate for representing REMs of small system size. Therefore, the development of equation (12) beyond equation (14) is important for REMs of finite size.

4. Applications

In this section, we demonstrate the utility of the expression (12) developed, by considering two applications.

4.1. The thermodynamic limit and the breaking of analyticity

The RM indicates that the CDREM exhibits the following behavior in the thermodynamic limit \( N, M \to \infty \) keeping \( \alpha = M/N \) finite. For details, see appendix B.

Under the RS ansatz, the RM yields two solutions

\[
\phi_{RS1}(n) = n \left( \log 2 + \alpha \log \left( \cosh \left( \frac{\beta}{2} \right) \right) \right),
\]

and

\[
\phi_{RS2}(n) = \log 2 + \alpha \log \left( \cosh \left( \frac{n\beta}{2} \right) \right).
\]
For $\forall \alpha$ and $\forall \beta$, these solutions agree at $n = 1$. Let us denote a critical inverse temperature as

$$\beta_c = \begin{cases} \infty, & \alpha \leq 1, \\ \frac{\log(1 - h_2^{-1}(1 - \alpha^{-1})) - \log(h_2^{-1}(1 - \alpha^{-1}))}{\log(1 - h_2^{-1}(1 - \alpha^{-1})) - \log(h_2^{-1}(1 - \alpha^{-1}))}, & \alpha > 1, \end{cases}$$

where $h_2^{-1}(y)$ is the inverse function of the binary entropy $h_2(x) = -x \log_2(1 - x) - (1 - x) \log_2(1 - x)$ for $0 < x < 1/2$. For $\beta < \beta_c$, $\phi(n)$ can be described by $\phi_{RS1}(n)$ and $\phi_{RS2}(n)$. More precisely, we have that $\exists n_{RS} > 1$ such that $\phi(n) = \phi_{RS2}(n)$ for $n > n_{RS}$ and $\phi(n) = \phi_{RS1}(n)$ for $n < n_{RS}$. On the other hand, for $\beta > \beta_c$, $\phi(n)$ cannot be entirely covered by the RS solutions. In this ‘low temperature’ phase, $\phi(n) = \phi_{RS2}(n)$ holds for $n > n_c = \beta_c/\beta$. However, this solution becomes inadequate for $n < n_c$ since the convexity condition $(\partial/\partial n)(n^{-1}\phi_{RS2}(n)) \geq 0$ is not satisfied. Therefore, we have to construct a novel solution for $n < n_c$ taking the breaking of replica symmetry into account, which provides the 1RSB solution

$$\phi_{1RSB}(n) = \frac{n\alpha\beta}{2} \tanh\frac{\beta_c}{2}. \quad (18)$$

The behavior of $\phi(n)$ is shown in figure 2.

As long as $N$ is finite, $\phi_N(n)$ is generally analytic with respect to $n$ and Carlson’s theorem guarantees the uniqueness of the analytical continuation from $n \in \mathbb{N}$ to $n \in \mathbb{C}$ for $\phi_N(n)$ of the CDREM. This implies that the transitions between multiple solutions mentioned above are highly likely to be due to the breaking of analyticity that possibly occurs in the limit $\phi(n) = \lim_{N \to \infty} \phi_N(n)$. The expression for equation (12) can be used to examine the origin of such potential analyticity breaking.

We here focus on the breaking for the transition between $\phi_{RS2}(n)$ and $\phi_{1RSB}(n)$, which can be regarded as the origin of the 1RSB. For this purpose, we assume that $\alpha > 1$ and $\beta > \beta_c$, and consider only the region $n < 1$ in which $n_c = \beta_c/\beta < 1$ is included. Under such conditions, integration by parts transforms equation (12) to

$$\langle Z' \rangle = \frac{2^N \omega^{-(nM/2)}}{n \Gamma(-n)} \int_H d\rho (-\rho)^{-n} g(\rho) f(\rho), \quad (19)$$

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where

\[ f(\rho) = \left( \sum_{i=0}^{M} P(E_i)e^{-\omega^i\rho} \right)^{2N-1} \]  

and

\[ g(\rho) = \sum_{i=0}^{M} P(E_i)\omega^i e^{-\omega^i\rho}. \]  

Next, we deform the contour \( H \) to a set of straight lines along the real half-axis as \( \tilde{H}: \infty + i\epsilon \to -i\epsilon \to \infty - i\epsilon \), where \( \epsilon \) is an infinitesimal number. Since the contribution around the origin vanishes for \( \text{Re}(n) < 1 \), only the contribution due to the difference between the branches of \( \rho - n \) remains as

\[ \langle Z^n \rangle = 2^N\omega^{-(nM/2)} \Gamma(1-n) \int_0^\infty d\rho \rho^{-n}\tilde{\Gamma}(\rho)f(\rho). \]  

Note that the function \( \tilde{\Gamma}(z) \) is replaced with the ordinary gamma function \( \Gamma(z) \) in this expression due to the difference between the two branches. We now evaluate equation (22), converting the relevant variables as

\[ x = iN\alpha, \quad y = \frac{1}{N\alpha} \log \rho, \]  

which is useful for taking the thermodynamic limit \( N,M \to \infty \) keeping \( \alpha = M/N \) finite. This yields an expression for the moment

\[ \langle Z^n \rangle = e^{-\omega N\alpha(y-x)/2} N\alpha e^{(1-n)\alpha(y-x)} \int_{-\infty}^{\infty} \frac{d\epsilon}{\Gamma(1-n)} e^{\epsilon\alpha(y-x)} \int_0^\infty d\rho \rho^{-n}\tilde{\Gamma}(\rho)f(\rho). \]  

The analysis shown below indicates that \( \alpha(x - 1/2) \) and \( \alpha(y - 1/2) \) can be interpreted as possible values of energy density \( N^{-1}E_i \) and free energy density \( -(N\beta)^{-1}\log Z \), respectively.

An asymptotic form of the double-exponential function for \( N \gg 1 \)

\[ e^{-e^{-Nu}} \sim (1 - e^{-Nu}) \Theta(u), \]  

where \( \Theta(u) = 1 \) for \( u > 0 \) and 0 otherwise, plays a key role for assessing equation (24) in the thermodynamic limit. We first evaluate the asymptotic expression for \( f(e^{N\alpha\beta y}) \) using this formula. Equation (25), in conjunction with an assessment by the saddle point method, yields

\[ \sum_{i=0}^{M} P(E_i)e^{-\omega^i\rho} \sim M \int_0^1 dx e^{N\alpha(h(x)-\log 2)} e^{-N\alpha\beta(x-y)} \]

\[ \sim M \int_0^1 dx e^{N\alpha(h(x)-\log 2)} (1 - e^{-N\alpha\beta(x-y)}) \Theta(x - y) \]

\[ \sim \begin{cases} 
1 - e^{N\alpha(\log(1+\omega)+\beta y-\log 2)}, & y < x_c, \\
1 - e^{N\alpha(h(x)-\log 2)}, & x_c < y < 1/2, \\
0, & 1/2 < y,
\end{cases} \]  

(26)
where $x_c = e^{-\beta}/(1 + e^{-\beta}) < 1/2$ and $h(x) \equiv -x \log x - (1 - x) \log (1 - x)$. Therefore, $f(e^{N\alpha\beta y})$ can be evaluated as

$$f(e^{N\alpha\beta y}) = e^{(2N-1)\log \left( \sum_{i=0}^{M} P(E_i) e^{-\alpha_i e^{N\alpha\beta y}} \right)} \sim e^{-e^{N\mathcal{F}(y)}} \Theta \left( \frac{1}{2} - y \right),$$

where

$$\mathcal{F}(y) = \begin{cases} (1 - \alpha) \log 2 + \tilde{h}(\beta) + \alpha \beta y, & y < x_c, \\ (1 - \alpha) \log 2 + \alpha h(y), & x_c < y < 1. \end{cases}$$

Here, $\tilde{h}(\beta)$ represents the Legendre transformation of the function $h(x)$ as $\tilde{h}(\beta) = \log(2 \cosh(\beta/2)) - \beta/2 = \log(1 + \omega)$. These indicate that in the limit $N \to \infty$ equation (27) can be reduced to the expression

$$f(e^{N\alpha\beta y}) \sim (1 - e^{N\mathcal{F}(y)}) \Theta(x^* - y) \sim \Theta(x^* - y),$$

where $x^* = h^{-1}_2(1 - \alpha^{-1}) = e^{-\beta_c}/(1 + e^{-\beta_c})$. Similarly, $g(e^{N\alpha\beta y})$ can be evaluated as

$$g(e^{N\alpha\beta y}) \sim 2^{-N} e^{N(-\alpha\beta y + \mathcal{F}(y))} \Theta(1 - y).$$

From these, the moment can be asymptotically expressed as

$$\langle Z^n \rangle \sim \frac{2^N \omega^{-(nN/2)} N \alpha \beta}{\Gamma(1 - n)} \int_{-\infty}^{x^*} dy e^{(1-n)N\alpha\beta y} g(e^{N\alpha\beta y})$$

$$\sim \frac{\omega^{-(nN/2)} N \alpha \beta}{\Gamma(1 - n)} \int_{-\infty}^{x^*} dy e^{NG(y)};$$

where

$$G(y) = -n\alpha\beta y + \mathcal{F}(y).$$

On the basis of equation (31), the thermodynamic limit $\phi(n) = \lim_{N \to \infty} N^{-1} \log \langle Z^n \rangle$ can be assessed by examining the profile of $G(y)$. For the case of $\beta > \beta_c$ and $\alpha > 1$, which we currently focus on, $x^* = e^{-\beta_c}/(1 + e^{-\beta_c}) > x_c = e^{-\beta}/(1 + e^{-\beta})$ is guaranteed. This implies that the asymptotic behavior can be classified into two cases depending on whether or not $y_c = e^{-n\beta}/(1 + e^{-n\beta})$, which maximizes $G(y)$, is included in the integration range $y < x^*$:

(i) $y_c < x^*$

As $y_c$ and $x^*$ are parameterized as $y_c = e^{-n\beta}/(1 + e^{-n\beta})$ and $x^* = e^{-\beta_c}/(1 + e^{-\beta_c})$, respectively, this holds for $n > n_c \equiv \beta_c/\beta$. In this case, $G(y)$ is maximized at $y = y_c$, as shown in figure 3, which means equation (31) can be assessed as $\langle Z^n \rangle \sim \exp[N(G(y_c) + n\alpha\beta/2)]$. This yields the thermodynamic limit

$$\phi(n) = \lim_{N \to \infty} \frac{1}{N} \log \langle Z^n \rangle = \log 2 + \alpha \log \left( \cosh \left( \frac{n\beta}{2} \right) \right),$$

which coincides with $\phi_{RS2}(n)$.

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Figure 3. Schematic figure showing $G(y)$ for $\beta > \beta_c$ and $\alpha > 1$. The maximum value in the integration range $y < x^*$ determines the asymptotic behavior $\phi(n) = \lim_{N \to \infty} N^{-1} \log \langle Z^n \rangle$.

(ii) $y_c > x^*$

This holds for $n < n_c$. Figure 3 indicates that $G(y)$ is maximized at $y = x^*$, which is the right-hand end point of the integration range. In this case, the asymptotic behavior is given as

$$\phi(n) = \lim_{N \to \infty} \frac{1}{N} \log \langle Z^n \rangle = n \alpha \beta \left( \frac{1}{2} - x^* \right) = \frac{n \alpha \beta}{2} \tanh \frac{\beta_c}{2}.$$  (34)

This is identical to $\phi_{\text{IRSB}}(n)$.

The analyticity breaking at $n = n_c$ for $\phi(n) = \lim_{N \to \infty} \phi_N(n) = \lim_{N \to \infty} N^{-1} \log \langle Z^n \rangle$ can be interpreted as follows [14,10]. In the low temperature phase of $\beta > \beta_c$, it is considered that the REM of each sample is dominated by a few states of minimum energy. This means that the free energy density in this phase can be roughly expressed as $-(N\beta)^{-1} \log Z \simeq N^{-1} \epsilon_{\text{min}}$, where $\epsilon_{\text{min}}$ denotes the minimum value of the $2^N$ energy states $\epsilon_1, \epsilon_2, \ldots, \epsilon_{2^N}$. The theory of extreme value statistics [15] indicates that $y = M^{-1} \epsilon_{\text{min}} + 1/2$, obeys a Gumbel-type distribution, $P(y)$, which in the current system is characterized as

$$\frac{1}{N} \log P(y) = \begin{cases} -\infty, & y > x^*, \\ (1 - \alpha) \log 2 + \alpha h(y), & 0 < y < x^*, \end{cases}$$  (35)

as $N$ tends to infinity, where $x^* = h_2^{-1}(1 - \alpha^{-1})$ represents the typical value of $y$. The physical implication of this behavior is that it is very rare for $y$ to fluctuate in the right direction because $y$ is given by the minimum of exponentially many energy values, while the fluctuation in the left direction obeys normal-type large deviation statistics described by a finite rate function. Equation (35) indicates that if $y_c < x^*$ holds,
which is the case for $n > n_c$, then $y = y_k = e^{-n_\beta}/(1 + e^{-n_\beta})$, which corresponds to a rare low minimum energy, dominates $<Z^n> = N\alpha_0^\beta \omega^{-nM/2} \int dy e^{-N\alpha_0^\beta y} P(y)$, yielding $\phi(n) = \phi_{\text{1RSB}}(n)$. On the other hand, for $n < n_c$, $<Z^n>$ is dominated by $y = x^*$, which corresponds to the typical value of the minimum energy, and the thermodynamic limit is provided as $\phi(n) = \phi_{\text{1RSB}}(n)$. These indicate that the origin of the 1RSB solution is a singularity of the distribution of the minimum energy, which arises in the thermodynamic limit and is characterized as equation (35). The formula (12) makes it possible to more precisely describe the analyticity breaking of $\phi(n)$ brought about by this singularity.

### 4.2. Numerical assessment of the moment

The utility of equation (12) is not limited to analysis in the thermodynamic limit; this formula is also useful for the numerical assessment of the moment for systems of finite size.

A representative approach to evaluating $<Z^n>$ is to numerically average $Z^n$ by sampling $n_0, n_1, \ldots, n_M$ from equation (7). This can be efficiently performed as follows (also shown in appendix C of [10]).

To generate $n_0, n_1, \ldots, n_M$ following equation (7), we first sample $n_0$ from a binomial distribution

$$P(n_0) = \frac{2^N!}{(2^N - n_0)!n_0!} p_0^{n_0}(1-p_0)^{2^N-n_0},$$

where $p_0 = P(E_0)$. For a generated $n_0$, we next sample $n_1$ from a conditional binomial distribution

$$P(n_1|n_0) = \frac{(2^N - n_0)!}{(2^N - n_0 - n_1)!n_1!} p_1^{n_1}(1-p_1)^{2^N-n_0-n_1},$$

where $p_1 = P(E_1)/(1-P(E_0))$. We repeat this process up to $n_{M-1}$; namely, for a generated set of $n_0, n_1, \ldots, n_{i-1}$, $n_i$ is sampled from a conditional binomial distribution

$$P(n_i|n_0, n_1, \ldots, n_{i-1}) = \frac{(2^N - \sum_{k=0}^{i-1} n_k)!}{(2^N - \sum_{k=0}^{i} n_k)!n_i!} p_i^{n_i}(1-p_i)^{2^N-\sum_{k=0}^{i-1} n_k},$$

where $p_i = P(E_i)/(1-\sum_{k=0}^{i-1} P(E_k))$. After generating $n_0, n_1, \ldots, n_{M-1}$ in this manner, $n_M$ is given as $n_M = 2^N - \sum_{i=0}^{M-1} n_i$, which guarantees satisfaction of the strict constraint $\sum_{i=0}^{M} n_i = 2^N$.

Since each step can be performed by an $O(1)$ computation, the necessary computational cost for this generation is $O(N)$ per sample, which is feasible. However, a problem remains in the evaluation of $<Z^n>$ by this approach based on naive sampling.

The analysis shown in section 4.1 indicates that in the low temperature phase $\beta > \beta_c$, $<Z^n>$ of $n > n_c$ is dominated by samples of a rare low minimum energy value in the

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Figure 4. $\langle Z^n \rangle$ assessed by numerical integration of equation (12) and a naive sampling scheme for $N = 10$. The relevant parameters are $\alpha = 2$ and $\beta = 3\beta_c$, which corresponds to the low temperature phase.

Figure 4 shows a comparison of the numerical evaluation of $\langle Z^n \rangle$ between the sampling and numerical integration approaches for the low temperature phase of $\alpha = 2, \beta = 3\beta_c$. In both approaches the system size was set to $N = 10$ while the number of samples was varied as $N_{\text{samples}} = 10, 10^3$ and $10^6$. The figure shows that as $n$ increases, more samples are necessary for accurately evaluating $\langle Z^n \rangle$. This is consistent with the analysis in section 4.1 indicating that $\langle Z^n \rangle$ for $n \gtrsim n_c$ is dominated by samples of a rare low minimum energy. In the sampling approach, $N_{\text{samples}} = 10^6$ achieves an accuracy similar to that of the numerical integration. However, more samples are necessary as $N$ increases because the probability of generating samples that dominate $\langle Z^n \rangle$ decreases exponentially with $N$. On the other hand, the necessary computational cost for evaluating $\langle Z^n \rangle$ is not greatly dependent on $N$ in the numerical integration, demonstrating a point of superiority of equation (12).

Notice that equation (12) holds not only for $n \in \mathbb{R}$ but also for $n \in \mathbb{C}$. This can be used for characterizing the breaking of the analyticity of $\langle Z^n \rangle$ by the convergence of the zeros of $\langle Z^n \rangle$ to the real axis on the complex $n$ plane as $N$ tends to infinity, which is analogous to a description of phase transitions given by Lee and Yang [16] and Fisher [17]. As far as the authors know, the complex zeros with respect to $n$ have been little examined, except for a few examples [18, 19], although zeros of the complex
temperature or external field for typical samples of REMs were studied in preceding literature [8, 20]. A detailed analysis along this direction will be reported in a subsequent paper [21].

5. Summary

In summary, we have developed an exact expression for the moment of the partition function of the CDREM. The expression is valid not only in the thermodynamic limit but also for systems of finite size. This is useful for investigating the breaking of analyticity with respect to the replica number, which plays the role of the power of the moment, in the thermodynamic limit. Our approach demonstrates that the analyticity breaking due to a singularity in the distribution of the minimum energy is the origin of the 1RSB observed in the low temperature phase of the CDREM. We have also shown that the expression developed is useful for numerically evaluating the moments of finite size systems.

Although the uniqueness of analytical continuation from \( n \in \mathbb{N} \) to \( n \in \mathbb{C} \) is not guaranteed by Carlson’s theorem for generic REMs, the behavior predicted by our approach is qualitatively the same as that of the CDREM. This implies that the 1RSB observed in a wide class of REMs originates from the analyticity breaking with respect to the replica number \( n \), which takes place in the thermodynamic limit \( N \to \infty \) after analytical continuation from \( n \in \mathbb{N} \) to \( n \in \mathbb{C} \) is performed for finite \( N \).

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Appendix A. The exact expression for \( \langle Z^n \rangle \) for the continuous random energy model

We here provide an exact expression for \( \langle Z^n \rangle \) for the continuous random energy model, which was originally introduced by Derrida in [11]. In this model, \( 2^N \) states are distributed in energy \( E \) with the following probability:

\[
P(E) = \frac{1}{\sqrt{\pi NJ^2}} \exp \left( -\frac{E^2}{NJ^2} \right) .
\]  
(A.1)

The moment of the partition function \( Z(\{E_i\}) = \sum_{i=1}^{2^N} \exp (-\beta E_i) \) is expressed as

\[
\langle Z^n \rangle = \int_{-\infty}^{\infty} \left( \prod_{i=1}^{2^N} dE_i P(E_i) \right) Z^n(\{E_i\}).
\]  
(A.2)
Analytic properties of random energy models: I

Applying equation (9) to $Z^n(\{E_i\})$ yields

$$
\langle Z^n \rangle = \frac{1}{\Gamma(-n)} \int_H d\rho (-\rho)^{-n-1} \prod_{i=1}^{2N} \left( \int_{-\infty}^{\infty} dE_i P(E_i) e^{-\rho \exp(-\beta E_i)} \right)
= \frac{1}{\Gamma(-n)} \int_H d\rho (-\rho)^{-n-1} \left( \int_{-\infty}^{\infty} dE P(E) e^{-\rho \exp(-\beta E)} \right)^{2N}
= \frac{1}{\Gamma(-n)} \int_H d\rho (-\rho)^{-n-1} \left( \int_{-\infty}^{\infty} dE e^{-\rho \exp(-\beta E)} \right)^{2N}.
$$

(Numerically evaluating this is computationally feasible for $\forall N$.

Deforming the contour $H$ to $\tilde{H}$: $\infty + i\epsilon \rightarrow +i\epsilon \rightarrow -i\epsilon \rightarrow \infty - i\epsilon$ with an infinitesimal number $\epsilon$ (putting $\rho = r e^{i(\theta - \pi)}$, $\theta = -\pi, \pi$), in conjunction with employing integration by parts $p > \text{Re}(n)$ times to remove the singularity at $\rho = 0$, yields

$$
\langle Z^n \rangle = \frac{1}{\Gamma(p - n)} \int_0^{\infty} dr r^{p-n-1} \left( -\frac{\partial}{\partial r} \right)^p \left( \int_{-\infty}^{\infty} \frac{dE}{\sqrt{\pi N J^2}} e^{-(E^2/N J^2) - r \exp(-\beta E)} \right)^{2N},
$$

which is equivalent to equation (7) of [12].

From equation (A.2), the behavior of the moment in the thermodynamic limit can be investigated in a manner similar to that of the analysis of the CDREM. This investigation indicates that the moment behaves as

$$
\phi(n) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \langle Z^n \rangle = \begin{cases} 
\log 2 + n^2 \beta^2 J^2 / 4, & n > n_{RS}, \\
\log 2 + \beta^2 J^2 / 4, & n < n_{RS}
\end{cases}
$$

(A.5)

for $\beta < \beta_c = 2\sqrt{\log 2 / J}$ in the thermodynamic limit, where $n_{RS} = \beta_c^2 / \beta^2 > 1$. On the other hand,

$$
\phi(n) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \langle Z^n \rangle = \begin{cases} 
\log 2 + n^2 \beta^2 J^2 / 4, & n > n_c, \\
n\beta J \sqrt{\log 2}, & n < n_c
\end{cases}
$$

(A.6)

holds for $\beta > \beta_c$, where $n_c = \beta_c / \beta < 1$.

The inequality $Z(\{E_i\}) = \sum_{i=1}^{2N} e^{-\beta E_i} > \exp(\beta E_1)$ gives

$$
\langle Z^n \rangle^{1/N} > \langle e^{-n\beta E_1} \rangle^{1/N} = \exp \left( n^2 \beta^2 J^2 / 4 \right),
$$

(A.7)

for $n > 0$, implying that $\langle Z^n \rangle^{1/N}$ is not upper bounded by $e^{\pi n}$. This means that the uniqueness of the analytical continuation from $n \in \mathbb{N}$ to $n \in \mathbb{C}$ ($\text{Re}(n) \geq 0$) is not guaranteed by Carlson’s theorem for $\phi_N(n) = N^{-1} \log \langle Z^n \rangle$ in the continuous random energy model. However, the RM successfully reproduces the behavior of equations (A.5) and (A.6). This implies that the transitions $\phi(n) = \lim_{N \rightarrow \infty} \phi_N(n)$ between the multiple solutions that arise in the RM, including the 1RSB transition, are not due to multiple varieties of analytical continuation, but are brought about by the breaking of analyticity with respect to the replica number $n$ in a family of REMs.
Appendix B. Replica analysis of the CDREM

We here show how $\phi_{RS1}(n)$, $\phi_{RS2}(n)$ and $\phi_{1RSB}(n)$ are derived for the CDREM by the replica method (RM). In order to evaluate $\langle Z^n \rangle$ for $n \in \mathbb{R}$ (or $n \in \mathbb{C}$), in the RM the moment is first evaluated for $n \in \mathbb{N}$, for which the power series expansion can be used. This yields the expression

$$\langle Z^n \rangle = \left( \sum_{A=1}^{2^N} e^{-\beta A} \right)^n = \sum_{A_1, A_2, \ldots, A_n} e^{-\beta \sum_{a=1}^n A_a}$$

$$= \sum_{(p_1, p_2, \ldots, p_n)} W(p_1, p_2, \ldots, p_n) \prod_{k=1}^n (I(k\beta))^{p_k}, \quad (B.2)$$

where

$$I(\beta) = \sum_E P(E)e^{-\beta E} \sim \exp \left[ N\alpha \left( \log \left( \cosh \left( \frac{\beta}{2} \right) \right) \right) \right], \quad (B.3)$$

and $W(p_1, p_2, \ldots, p_n)$ represents the number of ways of partitioning $n$ replicas $A_1, A_2, \ldots, A_n$ into $p_1$ states (out of $A = 1, 2, \ldots, 2^N$) in ones, into $p_2$ states in twos, \ldots, and into $p_n$ states in groups of $n$. Clearly, $W(p_1, p_2, \ldots, p_n) = 0$ unless $\sum_{k=1}^n k p_k = n$.

Equation (B.3) indicates that each term in equation (B.2) scales exponentially with respect to $N$ since $W(p_1, p_2, \ldots, p_n)$ depends exponentially on $N$ as well. On the other hand, the number of varieties of partition $(p_1, p_2, \ldots, p_n)$ does not depend on $N$. This implies that the summation of equation (B.2) is dominated by a single term labeled by a certain partition $(p_1^*, p_2^*, \ldots, p_n^*)$ and the exponent $\phi(n) = \lim_{N \to \infty} N^{-1} \log \langle Z^n \rangle$ can be accurately evaluated using only the single dominant term. Namely, $(p_1, p_2, \ldots, p_n)$ plays the role of a replica order parameter. It is noteworthy that this role is similar to that of the types in information theory. However, there is a distinct difference between the two notions because $(p_1, p_2, \ldots, p_n)$ represents relative positions of replicas in the replicated system while the types are defined for a single system.

The remaining problem is how to find the dominant partition. Replica symmetry, which implies that equation (B.1) is invariant under any permutation of replica indices $a = 1, 2, \ldots, n$, offers a useful guideline for solving this problem. This leads to a physically natural assumption that the dominant partition is characterized by this symmetry as well, which yields the following two RS solutions:

- **RS1**: Dominated by $(p_1^*, p_2^*, \ldots, p_n^*) = (n, 0, \ldots, 0)$, giving

$$\langle Z^n \rangle \sim 2^{Nn} \times (I(\beta))^n$$

$$\sim \exp \left[ N n \left( \log 2 + \alpha \left( \log \left( \cosh \left( \frac{\beta}{2} \right) \right) \right) \right) \right]. \quad (B.4)$$

This gives $\lim_{N \to \infty} N^{-1} \log \langle Z^n \rangle = \phi_{RS1}(n)$.

- **RS2**: Dominated by $(p_1^*, p_2^*, \ldots, p_n^*) = (0, 0, \ldots, 1)$, giving

$$\langle Z^n \rangle \sim 2^{N} \times I(n\beta)$$

$$\sim \exp \left[ N \left( \log 2 + \alpha \left( \log \left( \cosh \left( \frac{n\beta}{2} \right) \right) \right) \right) \right]. \quad (B.5)$$

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This gives \( \lim_{N \to \infty} N^{-1} \log \langle Z^n \rangle = \phi_{RS2}(n) \).

Equations (B.4) and (B.5) indicate that \( \phi_{RS1}(n) \) and \( \phi_{RS2}(n) \) can be analytically continued from \( n \in \mathbb{N} \) to \( n \in \mathbb{R} \) (or \( n \in \mathbb{C} \)), respectively. The above equations also mean that \( \phi_{RS1}(n) \) and \( \phi_{RS2}(n) \) agree at \( n = 1 \) in general, which makes it difficult to choose the relevant solution for \( n < 1 \). As a practical solution, we select, as the relevant solution in this region, the solution with the larger first derivative, following an empirical criterion given in [10]. This gives

\[
\phi(n) = \phi_{RS1}(n) \quad \text{for} \quad \beta < \beta_c,
\]

\[
\phi(n) = \phi_{RS2}(n) \quad \text{for} \quad \beta > \beta_c.
\]

However, in the latter case, \( \phi_{RS2}(n) \) is inadequate for \( n < n_c = \beta_c/\beta < 1 \) because the convexity condition \( \left( \partial / \partial n \right) (n^{-1} \phi_{RS2}(n)) \geq 0 \) is not satisfied. Therefore, we have to construct another solution taking the breaking of replica symmetry into account, which yields the 1RSB solution:

- 1RSB: Dominated by \( p^*_m = n/m \) for a certain \( m \) and \( p^*_k = 0 \) for \( k \neq m \), giving

\[
\langle Z^n \rangle \sim 2^{n/m} \times (I(m\beta))^{n/m} \sim \exp \left[ N \frac{n}{m} \left( \log 2 + \alpha \left( \log \left( \cosh \left( m\beta \frac{1}{2} \right) \right) \right) \right) \right].
\]

After analytical continuation, \( m \) is determined so as to extremize the right-hand side, which leads to \( m = n_c = \beta_c/\beta \). It may be noteworthy that the extremum is not the maximum but the minimum. This gives \( \lim_{N \to \infty} N^{-1} \log \langle Z^n \rangle = \phi_{1RSB}(n) \).

Figure 2 schematically shows the profiles of the three solutions.

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