TOPOLOGICAL RESTRICTIONS FOR CIRCLE ACTIONS
AND HARMONIC MORPHISMS

RADU PANTILIE AND JOHN C. WOOD

ABSTRACT

Let $M^m$ be a compact oriented smooth manifold which admits a smooth circle action with isolated fixed points which are isolated as singularities as well. Then all the Pontryagin numbers of $M^m$ are zero and its Euler number is nonnegative and even. In particular, $M^m$ has signature zero.

Since a non-constant harmonic morphism with one-dimensional fibres gives rise to a circle action we have the following applications:

(i) many compact manifolds, for example $\mathbb{C}P^n$, $K3$ surfaces, $S^{2n} \times P_g$ ($n \geq 2$) where $P_g$ is the closed surface of genus $g \geq 2$ can never be the total space of a non-constant harmonic morphism with one-dimensional fibres whatever metrics we put on them;

(ii) let $(M^4, g)$ be a compact orientable four-manifold and $\varphi : (M^4, g) \to (N^3, h)$ a non-constant harmonic morphism. Suppose that one of the following assertions holds:

• $(M^4, g)$ is half-conformally flat and its scalar curvature is zero,
• $(M^4, g)$ is Einstein and half-conformally flat,
• $(M^4, g, J)$ is Hermitian-Einstein.

Then, up to homotheties and Riemannian coverings, $\varphi$ is the canonical projection $T^4 \to T^3$ between flat tori.

INTRODUCTION

It is well known that if a compact oriented smooth manifold $M$ admits a smooth free circle action then its Euler number and all its Pontryagin numbers are zero. This follows from the fact that the tangent bundle of $M$ is the Whitney sum of a trivial real line bundle and the pull back of the tangent bundle of the orbit space.

1991 Mathematics Subject Classification. 58E20, 53C43, 57R20.

Key words and phrases. circle action, harmonic morphism.

The first author gratefully acknowledges the support of the O.R.S. Scheme Awards, the School of Mathematics of the University of Leeds, the Tetley and Lupton Scholarships and the Edward Boyle Bursary.
In this paper we generalize this by proving that if $M$ is a compact oriented smooth manifold which admits a smooth circle action with isolated fixed points which are isolated as singularities as well then (i) all the Pontryagin numbers of $M$ are zero (in particular, the signature of $M$ is zero), (ii) the Euler number of $M$ is even and is equal to the number of fixed points (Theorem 1.1). We obtain this by using a well known formula of R. Bott [10] (see also [23]). Also, we apply an idea of J.D.S. Jones to prove (Theorem 2.8) that the signature of a compact oriented 4-manifold endowed with a non-trivial circle action for which each fixed point has equal exponents is equal to the Euler number of the normal bundle of the components of dimension 2 of the fixed point set.

Harmonic morphisms are smooth maps $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds which preserve Laplace’s equation. They are characterised as harmonic maps which are horizontally weakly conformal [17], [21], i.e. for each point $x \in M$ either $d\varphi_x = 0$ or $d\varphi_x$ is surjective and maps $\mathcal{H}_x = (\ker d\varphi_x)^\perp$ conformally onto $T_{\varphi(x)}N$. Classification results for harmonic morphisms with one-dimensional fibres appear in [3], [6], [7], [11], [29], [30]. In [2] it is proved that any non-constant harmonic morphism with one-dimensional fibres defined on a Riemannian manifold of dimension at least five is submersive whilst, for domains of dimension four, only isolated critical points can occur. Moreover, in [2] it is proved that any non-constant harmonic morphism $(M^4, g) \to (N^3, h)$ induces a locally smooth circle action on $M^4$. We show in fact that, at least outside the critical set, the action is smooth and free (consequence of Proposition 3.1). It follows that the result of Theorem 1.1 can be applied to obtain topological restrictions for the total space of a harmonic morphism with one-dimensional fibres. These are obtained in Theorem 3.3 from which it immediately follows that $\mathbb{C}P^n$, $K3$ surfaces, $S^{2n} \times P_g$ (where $P_g$ is the closed (real) surface of genus $g$ and $n \geq 2$) can never be the total space of a non-constant harmonic morphism with one-dimensional fibres whatever metrics we put on them. The result regarding $\mathbb{C}P^2$ answers a question formulated by P. Baird in a conversation.

Further applications are obtained in Theorem 4.1 where we show that, up to homotheties and Riemannian coverings, the canonical projection $T^4 \to T^3$ between flat tori is the only harmonic morphism with one-dimensional fibres which is defined on a compact half-conformally flat 4-manifold which is either Einstein or scalar-flat. We then apply this result to obtain a new proof of a result of [30, Theorem 4.11] in which the same conclusion is proved for a harmonic morphism $\varphi : (M^4, g) \to (N^3, h)$ between compact Einstein manifolds. To obtain this new proof we first show (Proposition 4.4) that if $\varphi : (M^4, g) \to (N^3, h)$ is a non-constant harmonic morphism between orientable Einstein manifolds then $(M^4, g)$...
is half-conformally flat.

In Theorem 5.1, we prove that the same conclusion as that in Theorem 4.1 holds for a harmonic morphism \( \varphi : (M^4, g, J) \rightarrow (N^3, h) \) defined on a compact Hermitian-Einstein four-manifold.

We are grateful to A.L. Edmonds, I. Hambleton, J.D.S. Jones and M. McCooey for useful information on circle actions. We are also grateful to P. Baird for pointing out a mistake in Proposition 3.1.

1. A TOPOLOGICAL RESTRICTION FOR CIRCLE ACTIONS

By a singularity of a group action we mean a point at which the isotropy group is non-trivial. A fixed point is a singularity where the isotropy group is the entire group.

Let \( M^m \) be a compact oriented smooth manifold of dimension \( m \geq 1 \) endowed with a smooth circle action. Let \( F \) denote its fixed point set and \( V \) its infinitesimal generator. Let \( g \) be a Riemannian metric on \( M \) with respect to which \( V \) is a Killing vector field. Such a metric can be obtained by averaging an arbitrary Riemannian metric over the action. Let \( \nabla \) denote the Levi-Civita connection on \((M^m, g)\).

Obviously \( F \) is the zero set of \( V \) and thus, its connected components are totally-geodesic submanifolds of \((M^m, g)\) of even codimension (see [23]).

Let \( x \in F \) and suppose that the connected component \( N \) of \( x \) in \( F \) has codimension \( 2r \). Because \((\nabla V)_x\) is a skew-symmetric endomorphism of \((T_x M, g_x)\), with respect to a suitably chosen orthonormal frame, \((\nabla V)_x\) is represented by the direct sum of the zero square matrix of dimension \( m - 2r \) and \( \oplus_{j=1}^{r} \begin{pmatrix} 0 & -m_j \\ m_j & 0 \end{pmatrix} \) where \( m_j > 0 \). In fact, from [24, Chapter I, Proposition 1.9] it follows that \( m_j \in \mathbb{Z} \) since \( V \) integrates to give an \( S^1 \) action on \( M^m \). Indeed, since \( V \) is Killing its flow commutes with the exponential map; hence, via the exponential map at \( x \), the linear flow induced by \((\nabla V)_x\) on \( T_x M \) is locally equivalent to the flow of \( V \). Following [23] we shall call the (positive) integers \( m_j \) the exponents of the action at the fixed point \( x \). This is, of course, motivated by the fact that the exponential map of \((M^m, g)\) at \( x \) induces a local equivalence between the given \( S^1 \) action and the following \( S^1 \) action on \( \mathbb{R}^m = \mathbb{R}^{m-2r} \oplus \mathbb{C}^r \):

\[
x \cdot (x_1, \ldots, x_{m-2r}, z_1, \ldots, z_r) = (x_1, \ldots, x_{m-2r}, z_1, \ldots, z_r)^{m_1} \cdot (z_1, \ldots, z_r).
\]

(In particular, this shows that the exponents \((m_1, \ldots, m_r)\) do not depend on the metric \( g \).) If \( x \in F \) is an isolated fixed point (equivalently \( m = 2r \)) then the orientation induced by the corresponding orthonormal frame is determined by the \( r \)-tuple \((m_1, \ldots, m_r)\). Let \( \epsilon(x) \) be +1 or −1 according to whether or not this
orientation agrees with the orientation of \( T_x M \) (cf. [22]).

The main result of this section is the following.

**Theorem 1.1.** Let \( M^m \) be a compact oriented smooth manifold which admits a smooth circle action whose fixed points are isolated singularities.

Then (i) all the Pontryagin numbers of \( M^m \) are zero, (ii) the Euler number of \( M^m \) is even and is equal to the number of fixed points. In particular, the signature of \( M^m \) is zero.

**Proof.** Let \( x \in F \). Because \( x \) is an isolated singularity the exponents at \( x \) are all equal to 1. Equivalently, there exists an orthonormal basis of \((T_x M, g_x)\) with respect to which the matrix of \((\nabla V)_x\) is the direct sum of \( n \) copies of \(
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\).

Thus, if \( F \neq \emptyset \), then \( \dim M = 2n \) is even. Let \( f \) be an \( \text{Ad}(\text{SO}(2n)) \)-invariant symmetric polynomial of degree \( p \leq n \). Then, by a result of R. Bott [10, Theorem 2] (see also [23, Theorem II.6.1]) we have

\[
\sum_{x \in F} f((\nabla V)_x) \chi_n \left( \frac{1}{2\pi} (\nabla V)_x \right) = \int_M f(R)
\]

where \( \chi_n \) is the Pfaffian (see [23, p. 68] or [28, p. 309]) and \( f(R) \) is the closed \( 2p \)-form on \( M \) which represents the cohomology class induced by the Chern-Weil morphism applied to \( f \) via the Levi-Civita connection of \((M^{2n}, g)\) (see [24, Chapter XII] or [28, Appendix C]). Note that the right hand side of (1.1) is zero if \( p < n \).

It is easy to prove that

\[
\chi_n \left( \frac{1}{2\pi} (\nabla V)_x \right) = \frac{(-1)^n \epsilon(x)}{(2\pi)^n}.
\]

By taking \( f = 1 \), from (1.1) and (1.2), we obtain

\[
\sum_{x \in F} \epsilon(x) = 0.
\]

By taking \( f = \frac{(-1)^n}{(2\pi)^n} \chi_n \), from (1.1) and the Gauss-Bonnet Theorem we obtain that the Euler number of \( M^{2n} \) is equal to the cardinal of \( F \) (this also follows from the Poincaré-Hopf theorem or from [23, Theorem II.5.5]). But, by (1.3), the cardinal of \( F \) must be even and hence the Euler number of \( M^{2n} \) is even.

By definition, if \( \dim M \) is not divisible by four then all the Pontryagin numbers of \( M \) are zero.

Suppose that \( n = 2p \) and let \( i_1, \ldots, i_r \) be a partition of \( p \). Denote by \( p_{i_k} \) the \( \text{Ad}(\text{SO}(2m)) \)-invariant symmetric polynomial of degree \( 2i_k \) such that \( p_{i_k} \left( \frac{1}{2\pi} R \right) \) represents the \( i_k \)'th Pontryagin class of \( M \).
Let \( x \in F \) and recall that \((\nabla V)_x\) is the direct sum of \(n\) copies of \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\). Then it is obvious that for any \(\text{Ad}(\text{SO}(2n))\)-invariant symmetric polynomial \(f\) we have
\[
(1.4) \quad f((\nabla V)_x) = c(f, n)
\]
where \(c(f, n)\) is a constant which depends just on \(f\) and \(n\) but not on \(x \in F\).

By taking \(f = p_{i_1} \cdots p_{i_r}\) in (1.1) it follows from (1.2), (1.3), (1.4) that all the Pontryagin numbers of \(M^{2n}\) are zero. The fact that \(M^{2n}\) has zero signature follows from Hirzebruch’s Signature Theorem (see [28]). \(\square\)

2. Blowing-up isolated fixed points of circle actions

For \(j = 1, \ldots, r\) let \(m_j\) be positive integers. We consider the following action on \(C^r\):
\[
(2.1) \quad z \cdot (z_1, \ldots, z_r) = (z^{m_1} z_1, z^{m_2} z_2, \ldots, z^{m_r} z_r).
\]
Obviously, the fixed point set of this action is \(\{0\}\).

In what follows we need the notion of equivariant connected sum of two manifolds endowed with circle actions. The idea of the following construction comes from [12].

**Definition 2.1.** Let \(M\) and \(N\) be manifolds, \(\dim M = \dim N = 2r\), both endowed with non-trivial circle actions.

Let \(x \in M\) and \(y \in N\) be isolated fixed points of these actions having the same set of exponents \(\{m_1, \ldots, m_r\}\). Further, assume that \(\epsilon(x) = -\epsilon(y)\).

Then suitably chosen neighbourhoods \(U_x\) and \(U_y\) about \(x\) and \(y\) in \(M\) and \(N\), respectively, are equivariantly diffeomorphic to open balls of radius three about \(0 \in \epsilon(x)C^r\) and \(0 \in \epsilon(y)C^r\), respectively, where \(C^r\) is endowed with the circle action of (2.1).

The **equivariant connected sum of \(M\) and \(N\) (about \(x \in M\) and \(y \in N\))** is the quotient \((M \setminus V_x \sqcup N \setminus V_y)/\sim\) where \(V_x\) and \(V_y\) correspond, via the above equivariant diffeomorphisms, to the open balls \(\epsilon(x)B(1)\) and \(\epsilon(y)B(1)\), of radius 1, whilst \(\sim\) is induced by the identification \((2 + t)u \sim (2 - t)u\) with \(u \in \partial B(1)\) and \(t \in [-1, 1]\).

Thus the actions glue together to give a non-trivial circle action on the connected sum \(M\#N\).

**Example 2.2.** Let \(C^r\) and \(-C^r\) be endowed with the action given by (2.1). It is easy to see that the connected sum \(C^r\#-C^r\), suitably constructed about \(0 \in C^r\) in each term, inherits in a canonical way a circle action (here \(-C^r\) denotes \(C^r\)
considered with the orientation opposite to its usual one). Moreover this action is without fixed points.

The following definition is based on an idea of J.D.S. Jones arisen from a private conversation. We formulate it only for isolated fixed points although it can be given for any connected component of the fixed point set.

**Definition 2.3.** Let \( x \in F \) be an isolated fixed point with exponents \((m_1, \ldots, m_r)\). The blow-up of \( M \) (considered with the given action) at \( x \) is the equivariant connected sum of \( M \) and \(-\epsilon(x)\mathbb{C}P^r\), about \( x \in M \) and \([1, 0, \ldots, 0] \in \mathbb{C}P^r\), where \( \mathbb{C}P^r \) is considered with the \( S^1 \) action

\[
z \cdot [z_0, z_1, \ldots, z_r] = [z_0, z_1^{m_1} z_1, \ldots, z_r^{m_r} z_r].
\]

In what follows the following obvious lemma will play an important role.

**Lemma 2.4.** Let \( M \) be a manifold endowed with a circle action and let \( F \) be its fixed point set.

Let \( x \) be an isolated fixed point whose exponents are equal: \( m_1 = \ldots = m_r \).

Let \( \hat{F} \) be the fixed point set of the induced action on the blow-up of \( M \) at \( x \). Then

\[
\hat{F} = (F \setminus \{x\}) \cup \mathbb{C}P^{r-1}
\]

where \( \mathbb{C}P^{r-1} = \{[z_0, \ldots, z_r] \in \mathbb{C}P^r \mid z_0 = 0\} \).

**Remark 2.5.** From the above lemma it follows that if besides isolated fixed points a circle action has only components of codimension two then after blowing up all the isolated fixed points we obtain a manifold endowed with a circle action whose fixed point set is of codimension two.

In particular, if the starting manifold is of dimension four then after blowing-up the isolated fixed points we obtain a manifold endowed with a circle action whose fixed point set is of dimension two.

For the next lemma recall the \( \text{Ad}(SO(4)) \)-invariant polynomial \( p_1 \) on \( so(4) \) given by \( p_1(A) = \sum_{i<j} (a^i_j)^2 \) where \( A = (a^i_j)_{i,j=1,\ldots,4} \). As is usual, we shall also denote by \( p_1 \) the corresponding \( \text{Ad}(SO(4)) \)-invariant symmetric bilinear form on \( so(4) \).

If \( (M^4, g) \) is a Riemannian 4-manifold then, by the Chern-Weil theorem, \( p_1(\frac{1}{2\pi}R) \) represents the first Pontryagin class of \( M^4 \) where \( R \) is the curvature form of the Levi-Civita connection of \( (M^4, g) \) (see [24], [28]).

Also we need the following definition (see [23, p. 69]).
**Definition 2.6.** Let \((M^{2n}, g)\) be a Riemannian manifold and let \(V\) be a Killing vector field on it. Let \(N\) be a component of codimension \(2r\) of the zero set of \(V\).

For an \(\text{Ad}(\text{SO}(2n))\)-invariant polynomial \(f\) of degree \(n\) on \(\text{so}(2n)\) the residue of \(V\) over \(N^{2n-2r}\) is given by

\[
\text{Res}_f(N) t^{n-r} = \int_N \frac{f\left(\frac{1}{2\pi}(tR + \nabla V)\right)}{\chi_r\left(\frac{1}{2\pi}(tR + (\nabla V)\perp)\right)}
\]

where \(\nabla\) is the Levi-Civita connection of \((M^{2n}, g)\), \(R\) its curvature form, \(\perp\) denotes the components in \(\text{End}(TN\perp)\) and \(\chi_r\) is the Pfaffian. (Here we expand the right hand side of (2.2) as a power series in \(t\).)

**Lemma 2.7.** Let \((M^4, g)\) be an oriented 4-manifold and let \(V\) be a Killing vector field on it. Suppose that the zero set of \(V\) has a component \(N^2\) of dimension two.

Then the residue \(\text{Res}_{p_1}(N)\) is given by

\[
\text{Res}_{p_1}(N) = \chi(TN\perp)[N]
\]

where \(\chi(TN\perp)[N]\) is the Euler number of the normal bundle \(TN\perp\) of \(N\).

**Proof.** From Definition 2.6 it follows that we can write

\[
\text{Res}_{p_1}(N) t = \int_N \frac{p_1\left(\frac{1}{2\pi}(tR + \nabla V)\right)}{\frac{1}{2\pi}(tR + \nabla V)^\mu}
\]

where

\[
\left(\begin{array}{ccc}
 x & y & t \\
 2 & -1 & 0 \\
 0 & 0 & 1
\end{array}\right) \oplus \left(\begin{array}{ccc}
 0 & 0 & t \\
 0 & -1 & 0 \\
 1 & 0 & 0
\end{array}\right)^\nu = a,
\]

and recall that \((\nabla V)|_N\) is a section of \(\text{End}(TN) \oplus \text{End}(TN\perp)\) (see \([23, \text{Chapter II, Theorem 5.3}]\)). In fact \((\nabla V)^\nu|_N : N \to \mathbb{R}\) is the nowhere zero function on \(N\) characterised by

\[
(\nabla V)|_N = \left(\begin{array}{cc}
 0 & 0 \\
 0 & 0 \\
 0 & 0
\end{array}\right) \oplus \left(\begin{array}{cc}
 0 & (\nabla V)^\nu \\
 -(\nabla V)^\nu & 0
\end{array}\right),
\]

then (2.3) becomes

\[
\text{Res}_{p_1}(N) t = \frac{1}{2\pi} \int_N \left(t^2 p_1(R, R) + 2tp_1(R, \nabla V) + p_1(\nabla V, \nabla V)\right) \times 
\]

\[
\times \left(\frac{1}{(\nabla V)^\nu} - t \frac{R^\nu}{((\nabla V)^\nu)^2} + t^2 \frac{(R^\nu)^2}{((\nabla V)^\nu)^3} - \cdots\right)
\]

\[
= \frac{t}{2\pi} \int_N (2p_1(R, U) - R^\nu)
\]

where \(U = \frac{1}{(\nabla V)^\nu} \nabla V|_N\) - note that with respect to a suitably chosen adapted orthonormal frame we have \(U = \left(\begin{array}{ccc}
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 1
\end{array}\right) \oplus \left(\begin{array}{ccc}
 0 & 1 & 0 \\
 1 & 0 & 0 \\
 0 & 0 & 0
\end{array}\right)\). Also note that \(J = U|_N\) is an almost complex structure on the normal bundle \(TN\perp\) (see \([23]\) for the general case).
Because $N$ is a totally-geodesic submanifold of $(M,g)$ we have that $p_1(R,U) = R^\nu$; hence, from (2.4) it follows that

$$\text{Res}_{p_1}(N) = \frac{1}{2\pi} \int_N R^\nu. \quad (2.5)$$

By the same reason we also have that $J \otimes R^\nu = \left( \begin{smallmatrix} 0 & R^\nu \\ -R^\nu & 0 \end{smallmatrix} \right)$ is the curvature form of the connection induced by $\nabla$ on $TN^\perp$.

The proof now follows from the Chern-Weil theorem. \hfill \square

We now state the main result of this section.

**Theorem 2.8.** Let $M^4$ be a compact oriented 4-manifold endowed with a non-trivial circle action.

Let $F = F_0 \cup F_2$ be the fixed point set where $F_0$ is the set of isolated fixed points and $F_2$ is the union of the components of dimension 2 of the fixed point set. Suppose that the exponents of each isolated fixed point $x \in F_0$ are equal. Then the signature $\sigma[M]$ of $M$ is given by

$$\sigma[M] = \sum_{x \in F_0} \epsilon(x) = \chi(TF_2^\perp)[F_2]. \quad (2.6)$$

In particular, the signature of $M$ is given by the Euler number of the normal bundle of the components of dimension 2 of the zero set of $V$.

**Proof.** Let $\hat{M}$ be the manifold (endowed with a circle action) obtained by blowing-up the isolated fixed points, i.e. the points of $F_0$.

Then, since signatures add when taking connected sums, the signature $\sigma[\hat{M}]$ of $\hat{M}$ is given by

$$\sigma[\hat{M}] = \sigma[M] - \sum_{x \in F_0} \epsilon(x). \quad (2.7)$$

Because the exponents of each isolated fixed point are equal by hypothesis, the induced circle action on $\hat{M}$ has no isolated fixed points. This follows from Lemma 2.4 (see also Remark 2.5).

Thus we can apply [22, Theorem 4.2] to obtain that $\hat{M}$ has signature zero. Combining this with (2.7) gives

$$\sigma[M] = \sum_{x \in F_0} \epsilon(x), \quad (2.8)$$

i.e. the first equality of (2.6).

Now, take a metric on $M$ with respect to which $S^1$ acts by isometries. Then by
applying the Bott formula (see [23, Chapter II, Theorem 6.1]) to the infinitesimal generator of this action we obtain
\[ p_1[M] = \text{Res}_{p_1}(F_0) + \text{Res}_{p_1}(F_2). \]

Since \( p_1(A) = \pm 2\chi(A) \) for \( A \in \text{so}(4) \) (because \( A = (a^i_j) \in \text{so}(4) \) if and only if
\[ a^1_2 = \pm a^3_4, \quad a^3_1 = \mp a^2_4 \text{ and } a^1_4 = \pm a^2_3, \]
we have \( \text{Res}_{p_1}(F_0) = 2 \sum_{x \in F_0} \epsilon(x). \) By using this fact and Lemma 2.7 the equation (2.9) becomes
\[ p_1[M] = 2 \sum_{x \in F_0} \epsilon(x) + \chi(TF_2)\] (2.10)

But by Hirzebruch theorem \( p_1[M] = 3\sigma[M] \) which together with (2.8) and (2.10) gives
\[ 3 \sum_{x \in F_0} \epsilon(x) = 2 \sum_{x \in F_0} \epsilon(x) + \chi(TF_2) \] (2.11)

which immediately yields the second equality of (2.6).

\[ \square \]

**Remark 2.9.** 1) Obviously, Theorem 2.8 generalizes the result of [22, Theorem 4.2]. However, note that our proof of Theorem 2.8 uses that result.

Also, Theorem 2.8 generalizes the result of Theorem 1.1 for dimension four.

2) See [25], [22] for some related results on circle actions.

### 3. An Application to Harmonic Morphisms

Let \( \varphi : (M^{n+1}, g) \to (N^n, h), \ n \geq 1, \) be a non-constant harmonic morphism between compact oriented Riemannian manifolds. Then, by a result of P. Baird [2] the set \( \Sigma \) of critical points of \( \varphi \) is empty if \( \dim M \geq 5 \) and is discrete if \( \dim M = 4. \) For \( x \in M \setminus \Sigma \) set \( V_x = \ker d\varphi_x \) and let \( H_x = V_x^\perp. \) The resulting distributions \( V \) and \( H \) on \( M \setminus \Sigma \) shall be called, as usual, the *vertical* distribution and *horizontal* distribution, respectively.

**Proposition 3.1.** Let \( \varphi : (M^{n+1}, g) \to (N^n, h), \) \( n \geq 1, \) be a non-constant harmonic morphism between compact oriented Riemannian manifolds. Let \( \Sigma \) be the set of critical points of \( \varphi. \)

Then \( \varphi|_{M \setminus \varphi^{-1}(\varphi(\Sigma))} \) can be factorised as \( \xi \circ \psi \) where \( \psi : M \setminus \varphi^{-1}(\varphi(\Sigma)) \to P \) is the projection of an \( S^1 \) principal bundle (for which \( H|_{M \setminus \varphi^{-1}(\varphi(\Sigma))} \) is a principal connection) and \( \xi : P \to N \setminus \varphi(\Sigma) \) is a smooth covering projection. Moreover, for \( n = 3, \) the smooth free \( S^1 \) action on \( M \setminus \varphi^{-1}(\varphi(\Sigma)) \) extends to a continuous \( S^1 \) action on \( M \) which is smooth over \( M \setminus \Sigma \) and whose fixed point set is \( \Sigma. \)

Furthermore, if \( k \) is the unique metric on \( P \) with respect to which \( \xi : (P, k) \to (N \setminus \varphi(\Sigma), h|_{N \setminus \varphi(\Sigma)}) \) is a Riemannian covering then \( \psi : (M \setminus \varphi^{-1}(\varphi(\Sigma)), g|_{M \setminus \varphi^{-1}(\varphi(\Sigma))}) \to (P, k) \) is a submersive harmonic morphism with connected fibres.
Proof. Because $M$ and $N$ are oriented, $V$ is orientable. Thus we can choose $V \in \Gamma(V|_{M \setminus \Sigma})$ such that $g(V, V) = \lambda^{2n-4}$. Clearly, $V$ is smooth on $M \setminus \Sigma$. Furthermore, when $n = 3$, since $|V| \to 0$ as we approach a critical point, $V$ extends to a continuous vector field on $M$ whose zero set is $\Sigma$ and the flow of $V$ extends to a continuous flow on $M$ whose fixed point set is $\Sigma$.

For any $n \geq 1$, it is easy to see that $\varphi|_{M \setminus \varphi^{-1}(\varphi(\Sigma))}$ is a proper submersion. Then by a well-known result of C. Ehresmann [16] $\varphi$ restricted to $M \setminus \varphi^{-1}(\varphi(\Sigma))$ is the projection of a locally trivial fibre bundle. In particular, the orbit space $P$ of $V|_{M \setminus \varphi^{-1}(\varphi(\Sigma))}$ is a smooth manifold. Thus, $\varphi|_{M \setminus \varphi^{-1}(\varphi(\Sigma))}$ can be factorised as $\xi \circ \psi$ where $\psi : M \setminus \varphi^{-1}(\varphi(\Sigma)) \to P$ has connected fibres and $\xi : P \to N \setminus \varphi(\Sigma)$ is a covering projection.

Let $k = \xi^*(h)$ be the unique metric on $P$ with respect to which $\xi : (P, k) \to (N \setminus \varphi(\Sigma), h|_{N \setminus \varphi(\Sigma)})$ becomes a Riemannian covering. It is obvious that $\psi : (M \setminus \varphi^{-1}(\varphi(\Sigma)), g|_{M \setminus \varphi^{-1}(\varphi(\Sigma))}) \to (P, k)$ is a submersive harmonic morphism with compact connected fibres. From [29, Theorem 2.9] it follows that $\psi$ is the projection of a circle bundle where the action on the total space $M \setminus \varphi^{-1}(\varphi(\Sigma))$ is induced by the flow of $V|_{M \setminus \varphi^{-1}(\varphi(\Sigma))}$.

**Remark 3.2.** Recall ([17], [21], [8]) that, for $n = 1$ a harmonic morphism from $(M^2, g)$ to $(N^1, h)$ is, essentially, a harmonic function.

When $n = 2$ given a non-constant harmonic morphism $\varphi : (M^3, g) \to (N^2, h)$ with $M^3$ compact the $S^1$ action extends smoothly over the set of critical points and induces on $M^3$ a structure of a smooth Seifert fibre space [7] and the factorisation in Proposition 3.1 extends smoothly to $M^3$. When $n = 3$ the $S^1$ action extends smoothly if $(M^4, g)$ is Einstein [30] and, again, the factorisation extends smoothly to $M^4$.

From Theorem 1.1 and Proposition 3.1 we obtain the following.

**Theorem 3.3.** Let $\varphi : (M^{n+1}, g) \to (N^n, h) \ (n \geq 3)$ be a non-constant harmonic morphism between compact oriented Riemannian manifolds.

Then all the Pontryagin numbers of $M^{n+1}$ are zero. In particular, the signature of $M^{n+1}$ is zero.

Further, if $n \geq 4$ then the Euler number of $M^{n+1}$ is zero. If $n = 3$, then the Euler number of $M^4$ is even and is equal to the number of critical points of $\varphi$.

**Proof.** If $n \geq 4$ then, by Proposition 3.1, there exists a free $S^1$ action on $M^{n+1}$ whose orbits are connected components of the fibres of $\varphi$. Hence, by the Hopf theorem the Euler number of $M^{n+1}$ is zero. Also, as is well-known (immediate consequence of ([14]), all the Pontryagin numbers of $M^{n+1}$ are zero.
Suppose that \( n = 3 \). If the set of critical points \( \Sigma \) is empty then the same argument as above implies that the Euler number and the Pontryagin number of \( M^4 \) are zero.

Suppose that \( \Sigma \neq \emptyset \) and let \( x \in \Sigma \) and \( y = \varphi(x) \). By Proposition 3.1 we can assume that \( \varphi|_{M \setminus \varphi^{-1}(\varphi(\Sigma))} \) has connected fibres. Let \( B^3 \subseteq N^3 \) be a neighbourhood of \( y \in N^3 \) such that \( B^3 \cap \varphi(\Sigma) = \{y\} \) and which is diffeomorphic to the closed ball of radius two centred at zero in \( \mathbb{R}^3 \). Then \( \varphi^{-1}(B^3) \) is a four-dimensional submanifold-with-boundary of \( M^4 \) such that \( \varphi^{-1}(B^3) \cap \Sigma = \{x\} \). Furthermore, by Proposition 3.1, \( \varphi|_{\varphi^{-1}(B^3) \setminus \{x\}} \) is the projection of an \( S^1 \)-bundle over \( B^3 \setminus \{y\} \). Because \( B^3 \) is the cone over \( S^2 \), \( \varphi^{-1}(B^3) \) is the cone over its boundary. It easily follows (cf. \([2]\)) that the boundary of \( \varphi^{-1}(B^3) \) must be simply-connected.

Consider the embedding \( S^2 = \partial B^3 \subseteq N^3 \setminus \Sigma \). Let \( k \in \mathbb{Z} \) be the Chern number of the \( S^1 \) bundle \( (\varphi^{-1}(S^2), S^2, S^1) \). Then, if \( k \neq 0 \), \( \varphi^{-1}(S^2) \cong S^3/\mathbb{Z}_k \) and, in particular, the fundamental group of \( \varphi^{-1}(S^2) \) is \( \mathbb{Z}_k \). (If \( k = 0 \) the bundle \( (\varphi^{-1}(S^2), S^2, S^1) \) is trivial.) But, \( \varphi^{-1}(S^2) \) is diffeomorphic to the boundary of \( \varphi^{-1}(B^3) \) which we have seen is simply-connected. It follows that \( k = \pm 1 \) and, thus, we can suppose that \( \varphi|_{\varphi^{-1}(B^3) \setminus \{x\}} \) is smoothly equivalent to the projection of the cylinder of the Hopf bundle \( (S^3, S^2, S^1) \).

Thus, by taking, if necessary, the equivariant connected sum of \( M \) and \( -k \mathbb{C}^2 \), about \( x \in M \) and \( 0 \in \mathbb{C}^2 \), where \( \mathbb{C}^2 \) is considered with its canonical circle action, we can suppose that on \( \varphi^{-1}(B^3) \) we have a smooth circle action having \( x \) as a fixed point outside of which the action is free. By repeating this procedure about each point of \( \Sigma \) we obtain on \( M^4 \) a smooth circle action whose fixed point set is \( \Sigma \) outside which the action is free.

By Theorem 1.1, the Pontryagin number of \( M^4 \) is zero and its Euler number is even and equal to the cardinal of \( \Sigma \).

\[ \square \]

**Remark 3.4.** With the same notations as in the proofs of Theorem 1.1 and Theorem 3.3 we have that \( \epsilon(x) = k \) for each \( x \in \Sigma \).

In fact, (1.3) applied in the context of the proof of Theorem 3.3 can be proved by applying Stokes’ Theorem and the Chern-Weil Theorem, i.e. by assuming, if necessary, that \( \varphi \) has connected fibres and \( V|_{M \setminus \Sigma} \) has periodicity \( 2\pi \) then

\[
0 = \frac{1}{2\pi} \int_{N \setminus \cup_{x \in \Sigma} B_x} \varphi^* dF = \sum_{x \in \Sigma} \left( -\frac{1}{2\pi} \int_{\partial B_x} F \right) = \sum_{x \in \Sigma} \epsilon(x)
\]

where \( F \in \Gamma(\Lambda^2(T^*(N \setminus \varphi(\Sigma)))) \) is the curvature form of any principal connection on \( \varphi|_{M \setminus \Sigma} \) and \( B_x \subseteq N \) is a closed ball about each \( \varphi(x) \), \( x \in \Sigma \), such that \( B_{x_1} \cap B_{x_2} = \emptyset \) for \( x_1 \neq x_2 \).
Let $P_g$ be the closed oriented surface of genus $g \geq 0$ and let $n \geq 2$. Since the Euler number of $\mathbb{C}P^n$ is $n + 1$, of $S^{2n} \times P_g$ is $4(1 - g)$ and the signature of a $K3$ surface is $-16$ we have the following immediate consequence of Theorem 3.3.

**Corollary 3.5.** $K3$ surfaces, $\mathbb{C}P^n$, $S^{2n} \times P_g$ ($n \geq 2, g \neq 1$) can never be the domain of a non-constant harmonic morphism with one-dimensional fibres whatever metrics we put on them.

**Remark 3.6.** 1) Note that the projections $S^{2n+1} \times P_g \to \mathbb{C}P^n \times P_g$ induced by the Hopf fibrations $S^{2n+1} \to \mathbb{C}P^n$ are Riemannian submersions with totally-geodesic fibres, with respect to suitable multiples of their standard metrics, and are thus harmonic morphisms.

2) Other constructions of even-dimensional compact manifolds which cannot be the total space of a harmonic morphism with one-dimensional fibres whatever metrics we put on them can be easily obtained by using the product and/or the connected sum of manifolds.

4. **Harmonic morphisms with one-dimensional fibres on compact Riemannian four-manifolds**

The main result of this section is the following.

**Theorem 4.1.** Let $(M^4, g)$ be a compact orientable Riemannian four-manifold and let $\varphi : (M^4, g) \to (N^3, h)$ be a non-constant harmonic morphism.

Then the following assertions are equivalent

(i) $(M^4, g)$ is half-conformally flat and its scalar curvature is zero,

(ii) $(M^4, g)$ is Einstein and half-conformally flat,

(iii) $(M^4, g)$ is Einstein and $\varphi$ is submersive,

(iv) $(M^4, g)$ is Ricci-flat.

Furthermore, if one of the assertions (i), (ii), (iii) or (iv) holds then, up to homotheties and Riemannian coverings, $\varphi$ is the canonical projection $T^4 \to T^3$ between flat tori.

**Proof.** Choose one of the orientations of $M^4$ and let $\omega_g$ be the corresponding volume-form with respect to $g$.

Let $p_1[M]$ be the Pontryagin number of $M^4$. By the Chern-Weil Theorem we have

\[
(4.1) \quad p_1[M] = \frac{1}{4\pi^2} \int_M \left( |W^+|^2 - |W^-|^2 \right) \omega_g
\]

where $W$ is the Weyl tensor of $(M^4, g)$ and $W^+, W^-$ are its self-dual and anti-self-dual components, respectively (see [6 13.8]).
By Theorem 3.3, $p_1[M] = 0$ and hence $W^\pm = 0 \iff W^\mp = 0$. Thus, if $(M^4, g)$ is half-conformally flat then it is conformally flat.

Now, recall that, by the Gauss-Bonnet Theorem, the Euler number of $M^4$ is given by (see [9], [27]):

$$\chi[M] = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} - \frac{|M\text{Ricci}_0|^2}{2} + |W^+|^2 + |W^-|^2 \right) \omega_g \tag{4.2}$$

where $s$ is the scalar curvature of $(M^4, g)$ and $M\text{Ricci}_0$ is the trace-free part of the Ricci tensor of $(M^4, g)$.

If $(M^4, g)$ is half-conformal flat and its scalar curvature is zero then (4.2) becomes

$$\chi[M] = -\frac{1}{16\pi^2} \int_M |M\text{Ricci}_0|^2 \omega_g .$$

But, by Theorem 3.3, $\chi[M] \geq 0$ and hence $(M^4, g)$ must be Einstein.

If $(M^4, g)$ is Einstein and half-conformally flat then $(M^4, g)$ has constant sectional curvature $k^M$ (see [9]). By [29], Proposition 3.3(ii) and [30, Proposition 3.6] we cannot have $k^M < 0$. If $k^M > 0$ then, up to homotheties, the universal cover of $(M^4, g)$ is $S^4$, a situation which cannot occur (see [11, Section 3]). Hence $(M^4, g)$ must be flat.

If $(M^4, g)$ is Einstein and $\varphi$ is submersive then, by Theorem 3.3, the Euler number of $M^4$ is zero. Thus, (4.2) implies that $(M^4, g)$ is flat.

If $(M^4, g)$ is Ricci-flat then, as a consequence of [30, Proposition 3.6], $\varphi$ must be submersive, since on a compact Ricci-flat manifold any Killing vector field is parallel (see [23]). Then, by Theorem 3.3, $\chi[M] = 0$. Now, (4.2) implies that $(M^4, g)$ is flat. The last assertion follows from [29, Theorem 3.4] and an argument as in the proof of [30, Theorem 3.8].

**Remark 4.2.** The fact that assertion (iii) implies that, up to homotheties and Riemannian coverings, $\varphi$ is the canonical projection $T^4 \to T^3$ between flat tori is the result of [30, Theorem 3.8].

From the proof of Theorem 4.1 we obtain the following.

**Proposition 4.3.** Let $\varphi : (M^4, g) \to (N^3, h)$ be a non-constant harmonic morphism defined on an orientable compact Riemannian four-manifold.

Then $(M^4, g)$ is half-conformally flat if and only if it is conformally flat.

Next we give a sufficient condition for the total space of a harmonic morphism $\varphi : (M^4, g) \to (N^3, h)$ to be half-conformally flat. Note that this does not require any compactness or completeness assumption.
Proposition 4.4. Let \((M^4, g)\) be an orientable Einstein four-manifold.

Suppose that there exists a non-constant harmonic morphism \(\varphi : (M^4, g) \to (N^3, h)\) to an orientable Einstein three-manifold \((N^3, h)\).

Then \((M^4, g)\) is half-conformally flat.

Proof. Let \(x \in M\) be a regular point of \(\varphi\) and \(y = \varphi(x)\). Let \(Y_0 \in T_yN\) be a unit vector. Because \((N^3, h)\) is of constant curvature, by \([6]\), there exists an open neighbourhood \(U\) of \(y\) and a submersive harmonic morphism \(\psi_{Y_0} : (U, h|_U) \to P^2\) with values in some Riemann surface \(P^2\) such that its fibre through \(y\) is tangent to \(Y_0\). Then for any other unit vector \(Y \in T_yN\) we can compose \(\psi_{Y_0}\) with an isometry to obtain a submersive harmonic morphism \(\psi_Y\) whose fibre through \(y\) is tangent to \(Y\).

Then, \(\psi_Y \circ \varphi : (\varphi^{-1}(U), g|_{\varphi^{-1}(U)}) \to P^2\) is a submersive harmonic morphism from an orientable Einstein four-manifold to a Riemann surface. By \([33, \text{Theorem 1.1}]\), there exists an (integrable) Hermitian structure \(J_Y\) on \((\varphi^{-1}(U), g|_{\varphi^{-1}(U)})\) with respect to which \(\psi_Y \circ \varphi : (\varphi^{-1}(U), J_Y) \to P^2\) is holomorphic. By restricting, if necessary, the family of \(Y\) to an open subset of the unit sphere in \((T_yN, h_y)\) we can suppose that all the \(J_Y\) induce the same orientation \(\sigma\) on \(\varphi^{-1}(U)\). By the Riemannian Goldberg-Sachs Theorem (see \([1]\)) \(W^+\) is degenerate. Now, by a result of A. Derdziński \([14]\) either \(W^+ = 0\) or there is just exactly one pair \(\pm J\) of (oriented) complex structures compatible with \(g|_{\varphi^{-1}(U)}\). But the latter cannot occur because, obviously, if \(Y_1 \neq \pm Y_2\) then \(J_{Y_1} \neq \pm J_{Y_2}\). Hence \(W^+ = 0\) and the proof follows.

By combining Theorem \([4.1]\) and Proposition \([4.4]\) we immediately obtain a new proof for the following result from \([30, \text{Theorem 4.11}]\).

Theorem 4.5. Let \((M^4, g)\) be a compact Einstein four-manifold and let \((N^3, h)\) be a Riemannian three-manifold with constant curvature.

Let \(\varphi : (M^4, g) \to (N^3, h)\) be a non-constant harmonic morphism.

Then, up to homotheties and Riemannian coverings, \(\varphi\) is the canonical projection \(T^4 \to T^3\) between flat tori.

In particular there exists no harmonic morphism with one-dimensional fibres from a compact Einstein manifold of dimension four to \((S^3, \text{can})\).

Remark 4.6. There exists a harmonic morphism from \(S^4\) considered with a metric which is conformally equivalent to the canonical one to \((S^3, \text{can})\) (see \([4]\)). Note that, in \([4]\), there is also constructed a harmonic morphism from \((S^4, \text{can})\) to \((S^3, h)\) where \(h\) is continuous and smooth away from the poles.
5. Non-existence of harmonic morphisms on compact Hermitian-Einstein four-manifolds

In this section we prove the following.

**Theorem 5.1.** Let $(M^4, g, J)$ be a compact Hermitian-Einstein four-manifold and let $\varphi : (M^4, g) \to (N^3, h)$ be a non-constant harmonic morphism to an arbitrary Riemannian manifold.

Then, up to homotheties and Riemannian coverings, $\varphi$ is the canonical projection $T^4 \to T^3$ between flat tori.

**Proof.** If $\varphi$ is submersive then the proof follows from [30, Theorem 3.8].

Suppose that $\varphi$ has critical points. Then, by [30, Proposition 3.6], $(M^4, g)$ has positive scalar curvature and there exists a smooth Killing vector field $V$ tangent to the fibres of $\varphi$ whose zero set is equal to the set of critical points of $\varphi$.

Suppose that $(M^4, g, J)$ is Kähler-Einstein. Its first Chern class is positive. Then, by [20, Proposition 3.2] or [34, Theorem I.1.3.4], $(M^4, J)$ is either $\mathbb{CP}^1 \times \mathbb{CP}^1$ or is obtained from $\mathbb{CP}^2$ by blowing-up $r$ distinct points, $0 \leq r \leq 8$; such a blow-up has signature $1 - r$. But, by Theorem 3.3, $M^4$ has signature zero and hence either $M^4$ is $\mathbb{CP}^1 \times \mathbb{CP}^1$ or is obtained from $\mathbb{CP}^2$ by blowing-up one point.

In the latter case, $(M^4, J)$ is biholomorphic to $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$; but this admits no Kähler-Einstein metric (see [9, 11.54, 11.56]). If $(M^4, J) = \mathbb{CP}^1 \times \mathbb{CP}^1$ then, by [34, Proposition I.1.4.5], $(M^4, g, J)$ is homothetic to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Hence we may suppose that $(M^4, g, J)$ is isometric to $\mathbb{CP}^1 \times \mathbb{CP}^1$. Let $(x, y) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ be a critical point of $\varphi$ (note that by Theorem 3.3, in this case, $\varphi$ must have exactly four critical points). But $(x, y)$ is also an isolated zero of $V$.

Because $V$ is a Killing vector field, $(\nabla V)_{(x,y)}$ preserves each of the summands in the orthogonal decomposition $T_{(x,y)}(\mathbb{CP}^1 \times \mathbb{CP}^1) = T_x\mathbb{CP}^1 \oplus T_y\mathbb{CP}^1$. Now, recall that $(\nabla V)_{(x,y)}$ is an orthogonal complex structure on $T_{(x,y)}(\mathbb{CP}^1 \times \mathbb{CP}^1)$. Hence we can suppose that $(\nabla V)_{(x,y)} = J_{(x,y)}$. It follows that, by composing $\varphi$ with the inverse of the map $\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \{(-x,-y)\} \to \mathbb{C} \times \mathbb{C}$, given by stereographic projection on each factor, we get a harmonic morphism $\psi : (\mathbb{C}^2, \bar{g}) \to (N^3, h)$ where

$$\bar{g} = \frac{4}{(1 + |z|^2)^2} |dz|^2 + \frac{4}{(1 + |w|^2)^2} |dw|^2.$$

Since the stereographic projection is conformal $\psi$ is induced by the canonical circle action on $\mathbb{C}^2$; note that this is an isometric action with respect to $\bar{g}$. Let $\bar{V}$ be its infinitesimal generator and let $\lambda$ be the dilation of $\psi$. Then, up to a multiplicative constant we have $\bar{g}(<\bar{V}, \bar{V}) = \lambda^2$ (see [11], [23, Section 2]) and $\psi^*(h)$ must be equal to the horizontal component of $\lambda^2\bar{g}$. However, it can be checked without difficulty that then $h$ cannot be extended over $\psi(0)$.
We have thus proved that \((M^4, g, J)\) cannot be Kähler-Einstein. Because \(M^4\) has signature zero, from the main result of [26] it follows that \((M^4, g)\) is \(\mathbb{C}P^2\) with one point blown-up endowed with the Page metric. However, from the discussion in [15] it follows that none of the Killing vector fields of the Page metric has isolated fixed points which are isolated as singularities as well so that this case is not possible either.

\[\square\]

**Remark 5.2.** Thus there is no harmonic morphism from \(\mathbb{C}P^2\# - \mathbb{C}P^2\) endowed with the Page metric to any 3-manifold; there is, however a harmonic morphism from \(\mathbb{C}P^2\# - \mathbb{C}P^2\) endowed with the Page metric to \(\mathbb{C}P^1\) (see [8]).

**References**

[1] V. Apostolov, P. Gauduchon, The Riemannian Goldberg-Sachs theorem, *Internat. J. Math.* 8 (1997), 4, 421–439.

[2] P. Baird, Harmonic morphisms and circle actions on 3- and 4-manifolds, *Ann. Inst. Fourier (Grenoble)* 40 (1990), 1, 177–212.

[3] P. Baird, J. Eells, A conservation law for harmonic maps, *Geometry Symposium. Utrecht 1980*, Lecture Notes in Math. 894, Springer-Verlag, Berlin, Heidelberg, New York, 1981, 1–25.

[4] P. Baird, A. Ratto, Conservation laws, equivariant harmonic maps and harmonic morphisms, *Proc. London Math. Soc.* 64 (1992) 197–224.

[5] P. Baird, J.C. Wood, Bernstein theorems for harmonic morphisms from \(\mathbb{R}^3\) and \(S^3\), *Math. Ann.* 280 (1988), 579–603.

[6] P. Baird, J.C. Wood, Harmonic morphisms and conformal foliations by geodesics of three-dimensional space forms, *J. Austral. Math. Soc. (A)* 51 (1991), 118–153.

[7] P. Baird, J.C. Wood, Harmonic morphisms, Seifert fibre spaces and conformal foliations, *Proc. London Math. Soc.* 64 (1992), 170–196.

[8] P. Baird, J.C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monogr. (N.S.). Oxford Univ. Press (in preparation).

[9] A.L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10, Springer-Verlag, Berlin-New York, 1987.

[10] R. Bott, Vector fields and characteristic numbers, *Michigan Math. J.* 14 (1967) 231-244.

[11] R.L. Bryant, Harmonic morphisms with fibres of dimension one, *Comm. Anal. Geom.* 8 (2000), (to appear), (available from \texttt{http://xxx.lanl.gov/abs/dg-ga/9701002}).

[12] P.T. Church, K. Lamotke, Almost free actions on manifolds, *Bull. Austral. Math. Soc.* 10 (1974) 177–196.

[13] P.T. Church, J.G. Timourian, Maps with 0-dimensional critical set, *Pacific J. Math.* 57 (1975) 59–66.
TOPOLOGICAL RESTRICTIONS, CIRCLE ACTIONS, HARMONIC MORPHISMS 17

[14] A. Derdziński, Self-dual Kähler manifolds and Einstein manifolds of dimension four, Compositio Math. 49 (1983) 405–433.

[15] J. Eells, L. Lemaire, Selected topics in harmonic maps, CBMS Regional Conference Series in Mathematics, 50, Amer. Math. Soc., Providence, RI, 1983.

[16] C. Ehresmann, Les connexions infinitésimales dans un espace fibré différentiable, Colloque de topologie (espaces fibrés), Bruxelles, 1950, Georges Thone, Liège; Masson et Cie., Paris, 1951, 29–55.

[17] B. Fuglede, Harmonic morphisms between Riemannian manifolds, Ann. Inst. Fourier (Grenoble) 28 (1978) 107–144.

[18] G.W. Gibbons, S.W. Hawking, Classification of gravitational instanton symmetries, Comm. Math. Phys 66 (1979) 291–310.

[19] S. Gudmundsson, The Bibliography of Harmonic Morphisms, http://www.maths.lth.se/matematiklu/personal/sigma/harmonic/bibliography.html

[20] N. Hitchin, On the curvature of rational surfaces, Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 2, Stanford Univ., Stanford, Calif., 1973), pp. 65–80. Amer. Math. Soc., Providence, R. I., 1975.

[21] T. Ishihara, A mapping of Riemannian manifolds which preserves harmonic functions, J. Math. Kyoto Univ. 19 (1979) 215–229.

[22] J.D.S. Jones, J.H. Rawnsley, Hamiltonian circle actions on symplectic manifolds and the signature, J. Geom. Phys. 23 (1997) 301–307.

[23] S. Kobayashi, Transformation groups in differential geometry, reprint of the 1972 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[24] S. Kobayashi, K. Nomizu, Foundations of differential geometry, I, II, Interscience Tracts in Pure and Applied Math. 15, Interscience Publ., New York, London, Sydney, 1963, 1969.

[25] H.B. Lawson, Jr., M.L. Michelsohn, Spin geometry, Princeton Mathematical Series, 38, Princeton University Press, Princeton, NJ, 1989.

[26] C. LeBrun, Einstein metrics on complex surfaces, Geometry and physics (Aarhus, 1995), pp. 167–176, Lecture Notes in Pure and Appl. Math., 184, Dekker, New York, 1997.

[27] C. LeBrun, Weyl curvature, Einstein metrics, and Seiberg-Witten theory, Math. Res. Lett. 5 (1998) 423–438.

[28] J.W. Milnor, J.D. Stasheff, Characteristic Classes, Annals of Mathematics Studies, No. 76, Princeton University Press, N.J.; University of Tokyo Press, Tokyo, 1974.

[29] R. Pantilie, Harmonic morphisms with one-dimensional fibres, Internat. J. Math. 10 (1999) 457–501.

[30] R. Pantilie, Harmonic morphisms with 1-dim fibres on 4-dim Einstein manifolds, Preprint, University of Leeds, 1999 (available from http://www.amsta.leeds.ac.uk/~pmtrp/).

[31] N.E. Steenrod, The topology of fibre bundles, Princeton Mathematical Series 14, Princeton: Princeton University Press 1951.

[32] J.C. Wood, Harmonic morphisms, foliations and Gauss maps, Complex differential geometry and non-linear differential equations, Contemp. Math. 49, Amer. Math. Soc., Providence, RI, 1986, 145–183.

[33] J.C. Wood, Harmonic morphisms and Hermitian structures on Einstein 4-manifolds, Internat. J. Math. 3 (1992) 415–439.

[34] S.-T. Yau, On the curvature of compact Hermitian manifolds, Invent. Math. 25 (1974), 213–239.
