Towards a specialization map modulo semi-orthogonal decompositions

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Abstract

We propose a conjecture on the existence of a specialization map for derived categories of smooth proper varieties modulo semi-orthogonal decompositions, and verify it for K3 surfaces and abelian varieties.

1 Introduction

In this paper we study the following question: given a family of smooth projective varieties over, say, a punctured disc, and the knowledge of their bounded derived category of coherent sheaves, what can we say about the derived category of the limit fiber?

One motivation is the well-known conjecture of Dubrovin which predicts that a smooth projective variety has semisimple quantum cohomology if and only if its derived category of coherent sheaves has a full exceptional collection (see [Dub98], and also [Bay04]). Since quantum cohomology is deformation invariant, it suggests that the property of having a full exceptional collection is also invariant under deformations. In [Hu18] we showed that this is true locally; more precisely, given a smooth proper scheme $X$ over a locally noetherian scheme $S$, if for one fiber $X_{s_0}$, $D^b(X_{s_0})$ has a full exceptional collection, then so does the geometric fibers in an open neighborhood. It remains to investigate, with the additional hypothesis that $S$ is connected, whether $D^b(X_s)$ has a full exceptional collection for each fiber $X_s$. This reduces to the following:

Question 1.1. Let $R$ be a discrete valuation ring, $K$ its fraction field, and $k$ its residue field. Denote $S = \text{Spec}(R)$, the generic point of $S$ by $\eta$, and the closed point of $S$ by 0. Let $X$ be a smooth projective scheme over $S$. Suppose $D^b(X_\eta)$ has a full exceptional collection. Then does $D^b(X_0)$ have a full exceptional collection?

Now, given a field $k$, we consider the abelian group freely generated by the equivalence classes of derived categories of coherent sheaves of smooth projective varieties over $k$, and then modulo the relation of the form

$$[T] = [S_1] + ... + [S_n]$$

if there is a semi-orthogonal decomposition

$$T = \langle S_1, ..., S_n \rangle.$$ 

We call the resulting group the Grothendieck group of strictly geometric triangulated categories over $k$, and denote it by $K_0(\text{sGT}_k)$. For brevity we denote the class of $D^b(X)$ in $K_0(\text{sGT}_k)$ by $[X]$. If $D^b(X)$ has a full exceptional collection of length $n$, then

$$[X] = n[\text{Spec}(k)].$$
So a question weaker than 1.1 is, with the same hypothesis, whether \( [X_0] = n[\text{Spec}(k)] \), where \( n \) is the length of the full exceptional collection of \( D^b(X_\eta) \). Furthermore, for this weaker question, one can weaken the hypothesis, i.e., instead of assuming \( D^b(X_\eta) \) has a full exceptional collection, we now only assume \( [X_\eta] = n[\text{Spec}(K)] \). More generally, we propose the following conjecture.

**Conjecture 1.2.** There is a natural group homomorphism

\[
\rho_{\text{sgt}} : K_0(\text{sGT}_K) \rightarrow K_0(\text{sGT}_k).
\]

If such a map \( \rho_{\text{sgt}} \) does exist, we call it the specialization map of Grothendieck group of strictly geometric triangulated categories. The validity of conjecture 1.2 would be an evidence to a positive answer to question 1.1. This conjecture is inspired also by [NS17] and [KT17], where the existence of certain specialization maps are used to show that stable rationaly and rationality are closed properties in a smooth proper family. For example, in [NS17], it is shown that there is a natural group homomorphism

\[
\rho_{\text{Var}} : K_0(\text{Var}_K) \rightarrow K_0(\text{Var}_k).
\]

It is not hard to see that there is a canonical surjective homomorphism (see section 2)

\[
\mu : K_0(\text{Var}_k)/\langle L - 1 \rangle \rightarrow K_0(\text{sGT}_k).
\]

It should be believed that \( \mu \) is not an isomorphism, but this problem seems still open.

In this paper we propose a definition of the map \( \rho_{\text{sgt}} \), and verify the well-definedness for K3 surfaces and abelian varieties.

A more natural object to study than \( K_0(\text{sGT}_K) \) is the group generated by the admissible subcategories of derived categories of coherent sheaves of smooth projective varieties, modulo the same kind of relations (1), and one can propose a conjecture parallel to conjecture 1.2. A closely related notion is the Grothendieck ring of pre-triangulated categories introduced in [BLL04].

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2 Definitions and the conjecture

Throughout this section, denote by \( k \) a field of characteristic zero, \( R = k[[t]] \), and denote by \( K \) the fraction field of \( R \). For a smooth proper scheme \( X \) over \( K \), an **snc model** of \( X \) is a proper scheme \( \mathcal{X} \) over \( R \) with the properties that \( \mathcal{X} \) is regular, \( \mathcal{X}_K \) is isomorphic to \( X \) and the special fiber \( \mathcal{X}_0 = \mathcal{X} \times_R k \) is an snc divisor of \( \mathcal{X} \), which means

\[
\mathcal{X}_0 = \bigcup_{i=1}^{n} m_i D_i
\]

as divisors, where \( D_i \) is an irreducible smooth proper scheme over \( k \), \( m_i \) are positive integers, and if one writes \( D_I = \bigcap_{i \in I} D_i \) for \( I \subset \{1, ..., n\} \), then \( \dim D_I = \dim X + 1 - |I| \) for all subsets \( I \) of \( \{1, ..., n\} \). Our only use of the assumption of the characteristic 0 is that such fields admit resolution of singularities and the weak factorization theorems hold in this case ([AKMW02], [Wlo03], [ATT16]). In particular, snc models always exists.
**Definition 2.1.** A \( k \)-linear triangulated category \( T \) is **geometric**, if there is a smooth proper scheme \( Y \) over \( k \) such that \( T \) is equivalent to an admissible triangulated subcategory of \( D^b(Y) \). A \( k \)-linear triangulated category \( T \) is **strictly geometric**, if there is a smooth proper scheme \( Y \) over \( k \) such that \( T \) is equivalent to \( D^b(Y) \).

Let \( K_0(GT_k) \) (resp., \( K_0(sGT_k) \)) be the quotient of the free abelian group generated by the equivalence classes of geometric (resp., strictly geometric) triangulated categories modulo the relations of the form

\[
[T] = [T_1] + \ldots + [T_n]
\]

if there is a semi-orthogonal decomposition

\[
\langle S_1, \ldots, S_n \rangle
\]

of \( T \) such that \( S_i \) is equivalent to \( T_i \) for \( 1 \leq i \leq n \). In particular, the class of the zero category is equal to zero.

Recall the following two theorems of Orlov on the semi-orthogonal decomposition of projective bundles and blow-ups (see [Orl92], or [Huy06, chapter 8]).

**Theorem 2.2.** Let \( Y \) be a smooth projective variety over \( k \), \( E \) be a vector bundle of rank \( r \) over \( Y \), and \( \pi : \mathbb{P}(E) \to Y \) the projective bundle. Then there is a semi-orthogonal decomposition

\[
D^b(\mathbb{P}(E)) = (\pi^*D^b(Y) \otimes \mathcal{O}(a), \ldots, \pi^*D^b(Y) \otimes \mathcal{O}(a + r - 1))
\]

for every integer \( a \).

**Theorem 2.3.** Let \( X \) be a smooth projective variety over \( k \) and \( Y \) a smooth closed subvariety of \( X \) of codimension \( c \geq 2 \), \( \text{Bl}_Y X \) the blowup of \( X \) along \( Y \). Then there is a semi-orthogonal decomposition

\[
D^b(\text{Bl}_Y X) = \langle D_{-c+1}, \ldots, D_{-1}, D^b(X) \rangle
\]

such that \( D_i \) is equivalent to \( D^b(Y) \) for \(-c+1 \leq i \leq -1\).

Denote by \( K_0(\text{Var}_k) \) the Grothendieck group of varieties over \( k \). Recall that \( K_0(\text{Var}_k) \) is the group generated by the isomorphism classes of smooth schemes over \( k \) modulo the relations

\[
[X] = [Y] + [U]
\]

where \( X \) is a smooth scheme over \( k \), \( Y \) is a closed subscheme of \( X \) which is also smooth over \( k \), and \( U = X - Y \). The following theorem of Bittner [Bit04, theorem 3.1] gives an equivalent definition.

**Theorem 2.4.** Let \( k \) be a field of characteristic zero. Then \( K_0(\text{Var}_k) \) is isomorphic to the group generated by the isomorphism classes of smooth proper schemes over \( k \) modulo the relations

\[
[X] - [Y] = [\text{Bl}_Y X] - [E]
\]

where \( X \) is a smooth proper scheme over \( k \), \( Y \) is a smooth closed subscheme of \( X \), \( \text{Bl}_Y X \) is the blow-up of \( X \) along \( Y \), and \( E \) is the corresponding exceptional divisor on \( \text{Bl}_Y X \).
**Corollary 2.5.** Suppose $k$ is a field of characteristic zero. Then there is a natural surjective homomorphism of groups

$$
\mu_k : K_0(\Var_k) / [L - 1] \rightarrow K_0(\sGT_k)
$$

such that $\mu_k([X]) = [\Db(X)]$.

**Proof:** By theorem 2.4, it suffices to show

$$
[D^b(X)] - [D^b(Y)] = [D^b(\Bl_Y X)] - [D^b(E)]
$$

and

$$
[D^b(\P^n)] = 2[D^b(\Spec(k))].
$$

By (2.2), $[D^b(\P^n)] = (n + 1)[D^b(\Spec(k))]$, thus (6) holds. Suppose the codimension of $Y$ in $X$ is $c$, then by (2.3),

$$
[D^b(\Bl_Y X)] = (c - 1)[D^b(Y)] + [D^b(X)],
$$

and by (2.2),

$$
[D^b(E)] = c[D^b(Y)],
$$

so (5) follows. Since $K_0(\Var_k)$ and $K_0(\sGT_k)$ both are generated by the isomorphism classes of smooth proper schemes over $k$, $\mu_k$ is surjective.

**Remark 2.6.** We have ignored the ring structure of $K_0(\Var_k)$. To obtain a ring structure on something like $K_0(\sGT_k)$ or $K_0(\GT_k)$, one need take into account the DG structures (see [BLL04]), and there is then a map like (4).

Now let $k$ and $K$ be the fields as defined at the beginning of this section. The following theorem is [NS17, prop. 3.2.1].

**Theorem 2.7.** There is a unique group homomorphism

$$
\rho_{\text{var}} : K_0(\Var_K) \rightarrow K_0(\Var_k)
$$

such that for a smooth proper scheme $X$ over $K$, an snc model $\mathcal{X}$ of $X$ over $R$ with

$$
\mathcal{X}_k = \sum_{i \in I} n_i D_i,
$$

one has

$$
\rho_{\text{var}}([X]) = \sum_{\emptyset \neq J \subset I} (1 - L)^{|J|-1}[D^\circ_J],
$$

where $D_J = \bigcap_{j \in J} D_j$, and $D^\circ_J = D_J \setminus \left( \bigcup_{i \in I \setminus J} D_i \right)$.

The homomorphism $\rho_{\text{var}}$ is called the **specialization map** of the Grothendieck group of varieties.
Conjecture 2.8. There are natural maps
\[ \rho_{gt} : K_0(\text{GT}_K) \to K_0(\text{GT}_k) \]
and
\[ \rho_{sgt} : K_0(\text{sGT}_K) \to K_0(\text{sGT}_k). \]

In view of theorem 2.7 and corollary 2.5, the conjecture for \( K_0(\text{GT}) \) means that there is a homomorphism \( \rho_{sgt} \) making the following diagram commutative
\[
\begin{array}{ccc}
K_0(\text{Var}_K) & \xrightarrow{\rho_{\text{var}}} & K_0(\text{Var}_k) \\
\mu_K & & \mu_k \\
K_0(\text{sGT}_K) & \xrightarrow{\rho_{\text{sgt}}} & K_0(\text{sGT}_k),
\end{array}
\]
and since \( \mu_K \) is surjective, such \( \rho_{sgt} \) is unique if it exists.

For a field \( L \), denote by \( M_L \) the abelian group freely generated by the isomorphism classes of smooth proper schemes over \( L \). Set
\[ \mathbb{P}_{D_J} = \mathbb{P}(\mathcal{N}_{D_J/X}). \]
In particular, \( \mathbb{P}_{D_i} = D_i \). We define a map
\[ \rho : M_K \to K_0(\text{sGT}_k) \]
by
\[ \rho([X]) = \sum_{\emptyset \neq J \subset I} (-1)^{|J|-1}[\text{D}^b(\mathbb{P}_{D_J})], \]
or equivalently, by theorem 2.2
\[ \rho([X]) = \sum_{\emptyset \neq J \subset I} (-1)^{|J|-1}|J| \cdot [\text{D}^b(D_J)]. \]

By (7) a simple computation shows that
\[ \rho_{\text{var}}([X]) = \sum_{\emptyset \neq J \subset I} (-1)^{|J|-1}[\mathbb{P}_{D_J}]. \]
Therefore \( \rho \) is a natural candidate for \( \rho_{sgt} \). In other words, conjecture 2.8 for \( \rho_{sgt} \) reduces to the following.

Conjecture 2.9. The homomorphism \( \rho : M_K \to K_0(\text{sGT}_k) \) factors through the canonical surjective homomorphism \( M_K \to K_0(\text{sGT}_K) \):
\[
\begin{array}{ccc}
M_K & \xrightarrow{\rho} & K_0(\text{sGT}_k) \\
& & \downarrow \\
K_0(\text{sGT}_K) & & K_0(\text{sGT}_K)
\end{array}
\]
To prove the conjecture, one need to show:

(i) given \(X\) as a representative of its class \([D^b(X)]\) in \(K_0(s\text{GT}_K)\), \(\rho([X])\) is independent of the choice of the snc model \(X\);

(ii) \(\rho([X])\) is independent of the choice of the representative \(X\).

In fact, (i) is needed for the well-definedness of \(\rho\). We state it as follows.

**Theorem 2.10.** \(\rho([X])\) does not depend on the choice of \(X\).

Proof : One can show this by using the weak factorization theorem [AKMW02, Wlo03] and [AT16]. The quickest way is to apply theorem 2.7 and corollary 2.5.

I have no idea how to do step (ii) at present. In this paper I only provide some evidence for it. More precisely, for some examples of derived equivalent smooth proper \(K\)-schemes \(X\) and \(X'\), I am going to verify

\[
\rho([X]) = \rho([X']).
\]

(10)

The first kind of examples are birational derived equivalent \(X\) and \(X'\).

**Lemma 2.11.** Let \(X\) be a smooth proper scheme over \(K\).

(i) Let \(Y\) be a smooth closed subscheme of \(X\). Denote by \(E\) the exceptional divisor on the blowup \(\text{Bl}_Y X\). Then \(\rho(\text{Bl}_Y X) = \rho([X]) - \rho([Y]) + \rho([E])\).

(ii) Let \(E\) be a vector bundle over \(X\) of rank \(r\). Then \(\rho(\mathbb{P}(E)) = r\rho([X])\).

Proof : Use corollary 2.5 and theorem 2.7.

**Example 2.12** (Standard flips). Let \(X\) be a smooth projective scheme over \(K\) and \(Y\) a smooth closed subscheme of \(X\) of codimension \(l + 1\), such that \(Y \cong \mathbb{P}^m\) and the normal bundle \(N_{Y/X} \cong \mathcal{O}(-1)^{l+1}\). Then one can perform the standard flip and obtain a smooth projective scheme \(X'\). By [BO95 theorem 3.6], \(X\) and \(X'\) are derived equivalent. By lemma 2.11 one deduces that

\[
\rho([X]) + l\rho([\mathbb{P}^m]) = \rho(\text{Bl}_Y X) = \rho([X']) + l\rho([\mathbb{P}^m]),
\]

so

\[
\rho([X]) = \rho([X']).
\]

Similarly, one can also try to check (10) for Mukai flops ([Kaw02, Nam03]), and two non-isomorphic crepant resolutions of a Calabi-Yau 3-fold. In the following sections I will verify (10) for K3 surfaces and abelian varieties, under some additional assumptions.

3 Specialization map K3 surfaces

In this section we verify (10) for derived equivalent K3 surfaces which have semistable degenerations over \(R\). Throughout this section we consider only algebraic K3 surfaces.
3.1 Mukai pairings and period mappings

In this subsection we recall the Mukai pairing on K3 surfaces and its relation to derived equivalences (see e.g., [BBR09 chapter 4] and [Huy06 chapter 10]), and then introduce a corresponding notion of period mapping.

Let \( X \) be a K3 surface over \( \mathbb{C} \). The **Mukai pairing** on \( H^*(X, \mathbb{Z}) \) is defined by

\[
\langle (\alpha_0, \alpha_1, \alpha_2), (\beta_0, \beta_1, \beta_2) \rangle := a_1, \beta_1 - \alpha_0, \beta_2 - \alpha_2, \beta_0 \in \mathbb{Z},
\]

where \( \alpha_i, \beta_i \in H^{2i}(X, \mathbb{Z}) \). The corresponding lattice is

\[
E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}.
\]

Set

\[
\tilde{H}^{2,0}(X) = H^{2,0}(X), \quad \tilde{H}^{0,2}(X) = H^{0,2}(X),
\]

\[
\tilde{H}^{1,1}(X) = H^0(X) \oplus H^4(X) \oplus H^{1,1}(X).
\]

The resulting weight two Hodge structure \( \{H^{\text{even}}(X, \mathbb{Z}), \tilde{H}^{p,q}(X)\} \) is denoted by \( \tilde{H}(X, \mathbb{Z}) \).

The following characterization of derived equivalent K3 surfaces is due to [Muk87, Orl97]. See also [Huy06 corollary 10.7, proposition 10.10].

**Theorem 3.1.** Two algebraic K3 surfaces \( X \) and \( Y \) over \( \mathbb{C} \) are derived equivalent if and only if there is a Hodge isometry between \( \tilde{H}(X, \mathbb{Z}) \) and \( \tilde{H}(Y, \mathbb{Z}) \) with respect to the Mukai pairing. If \( \Phi_P : D^b(X) \to D^b(Y) \) is an equivalence with kernel \( P \in D^b(X \times Y) \), the induced map

\[
\Phi_P^H : \tilde{H}(X, \mathbb{Z}) \to \tilde{H}(Y, \mathbb{Z}), \quad \alpha \mapsto q_*(\text{ch}(P) \cdot \text{td}(X \times Y) \cdot p^* \alpha)
\]

is a Hodge isometry, where \( p : X \times Y \to X \), \( q : X \times Y \to Y \) are the two projections.

As an analogue of the usual period domains, we introduce a notion to study the variation of \( \tilde{H}(X, \mathbb{Z}) \).

**Definition 3.2.** Let \( M \) be the Mukai lattice \( E_8(-1)^{\oplus 2} \oplus U^{\oplus 4} \), \( Q(\cdot, \cdot) \) the corresponding symmetric bilinear pairing on \( H_{\mathbb{C}} \). The **Mukai period domain** \( D_M \) is defined to be the classifying space of the following data:

(i) a filtration of complex subspaces \( 0 = F^3 \subset F^2 \subset F^1 \subset F^0 = M_{\mathbb{C}} \) of \( M_{\mathbb{C}} \), such that \( \dim_{\mathbb{C}}(F^2) = 1 \), \( \dim_{\mathbb{C}}(F^1) = 23 \);

(ii) \( Q(F^p, F^{3-p}) = 0 \) for all \( p \);

(iii) \( Q(v, v) > 0 \) for \( v \in F^2 \).

Notice that the condition (iii) together with condition (ii) implies that \( F^1 \cap \overline{F^2} = 0 \), thus induces a weight two integral Hodge structure on \( M \).

**Proposition 3.3.** (i) \( D_M \) is an open subset (in the analytic topology) of a subvariety of a flag variety;

(ii) For a family of K3 surface \( \mathcal{X} \to S \), where \( S \) is a simply connected complex manifold, and an isomorphism \( \tilde{H}^*(\mathcal{X}_s, \mathbb{Z}) \cong M \) as lattices for some point \( 0 \) of \( S \), there is a canonical holomorphic map \( \phi : S \to D_M \), such that

\[
\tilde{H}(\mathcal{X}_s, \mathbb{Z}) \cong \phi(s),
\]

for any point \( s \) of \( S \).
Proof: Both statements follow from the usual argument for the period map of unpolarized K3 surfaces, see [Huy16, chapter 6]. For example, \( \bigoplus_{s \in S} H^0(X_s, \Omega^2_{X_s}) = R^0 \pi_* \Omega^2_{X/S} \) is a holomorphic subbundle of \( \bigoplus_{i=0}^4 R^i \pi_* \mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_S \), so \( \phi \) is holomorphic.

More generally, for a family of K3 surface \( X \to S \), where \( S \) is a complex manifold which is not necessarily simply connected, and an isomorphism \( \tilde{H}^*(X_0, \mathbb{Z}) \cong M \) as lattices for some \( 0 \in S \), there is a canonical holomorphic map \( \phi : S \to \Gamma \setminus D_M \), where \( \Gamma = \text{Aut}_\mathbb{Z}(M, Q) \), the group of automorphisms of the lattice \((M, Q)\), or even the image of \( \pi_1(S) \) in \( \text{Aut}_\mathbb{Z}(M, Q) \). However, the quotient \( \Gamma \setminus D_M \) is not Hausdorff, as remarked in [Huy16, p. 104].

3.2 Degeneration of K3 surfaces

We first recall the theorem on the degeneration of K3 surfaces due to Kulikov [Kul77] (see also [PPS1], [Fri84]).

**Theorem 3.4.** Let \( \pi : X \to \Delta \) be a semistable degeneration of K3 surfaces. Then there exists a birational modification of this semistable degeneration, such that the restriction of \( \pi \) to \( \Delta^* = \Delta \setminus \{0\} \) remains unchanged, and \( K_X \) becomes trivial. After such a modification, the degenerate fiber \( \pi^{-1}(0) = X_0 \) can be one of the following types:

(I) \( X_0 \) is a smooth K3 surface;

(II) \( X_0 = \bigcup_{i=1}^r V_i \), \( r \geq 1 \), \( V_0 \) and \( V_r \) are rational surface, \( V_i \) are ruled elliptic surfaces for \( 1 \leq i \leq r-1 \), \( V_i \cap V_j = \emptyset \) for \( |i-j| > 1 \), and \( V_i \cap V_j \) is an elliptic curve for \( |j-i| = 1 \) and is a section of the ruling on \( V_i \), if \( V_i \) is a ruled elliptic surface;

(III) \( X_0 = \bigcup V_i \), and each \( V_i \) is a smooth rational surface, with all the double curves rational, and the dual graph is a triangulation of \( S^2 \).

Moreover, the three types of degenerations are characterized by the monodromy action \( T \) on \( H^2(X_t, \mathbb{Z}) \), \( 0 \neq t \in \Delta \):

(I) \( T = \text{id} \);

(II) \( T - \text{id} \neq 0 \), \( (T - \text{id})^2 = 0 \);

(III) \( (T - \text{id})^2 \neq 0 \), \( (T - \text{id})^3 = 0 \).

In the following we say that a semistable degeneration of K3 surfaces with \( K_X \) trivial, is of type (I), (II) or (III), if it is of the corresponding type described above.

**Proposition 3.5.** Let \( \pi : X \to \Delta \) be a type (II) semistable degeneration of K3 surfaces. Denote by \( LH^i(X_0) \) the limit Hodge structure on \( H^i(X_t, \mathbb{Z}) \), \( 0 \neq t \in \Delta \). Then

(i) \( W_1 H^2(X_0) \cong H^1(E)^{\oplus r} \) as integral pure Hodge structures;

(ii) \( W_1 H^2(X_0) \cong W_1 LH^2(X_0) \) as integral pure Hodge structures.
Proof: Notice that in general the pure graded pieces in a mixed Hodge structure are \( \text{rational} \) Hodge structures. However in our case there are natural integral Hodge structures on \( W_1H^2(\mathcal{X}_0) \) and \( W_1LH^2(\mathcal{X}_0) \) inducing the rational ones as we will see. So it suffices to show the isomorphisms as rational Hodge structures.

(i) Let \( \mathcal{X}_0 = \bigcup_{i=0}^r V_i. \) For \( j = 1, \ldots, r, \) let \( U_j = \bigcup_{i=j}^r V_i \) and \( U'_j = \bigcup_{i=j}^r V_i, \) and \( D_j = V_{j-1} \cap V_j. \) The Mayer-Vietoris exact sequence

\[
... \to H^{k-1}(D_j) \to H^k(\mathcal{X}_0) \to H^k(U_j) \oplus H^k(U'_j) \to H^k(D_j) \to ...
\]

provides an extension of pure Hodge structures

\[
0 \to H^1(D_j) \to W_1H^2(\mathcal{X}_0) \to W_1H^2(U_j) \oplus W_1H^2(U'_j) \to 0.
\]

By induction on \( r, \) \( W_1H^2(\mathcal{X}_0) \) is a successive extension of \( H^1(D_1), \ldots, H^1(D_r). \) But we can choose different orders of the cuts of \( \mathcal{X}_0, \) which give the splittings. Hence there is a canonical isomorphism \( W_1H^2(\mathcal{X}_0) \cong \bigoplus_{j=1}^r H^1(D_j) \) of Hodge structures.

(ii) By [Fri84, lemma 3.6], the Clemens-Schmidt sequence

\[
H_4(\mathcal{X}_0) \to H^2(\mathcal{X}_0) \to LH^2(\mathcal{X}_0) \xrightarrow{N=T-1} LH^2(\mathcal{X}_0)
\]

is exact over \( \mathbb{Z}, \) and is an exact sequence of mixed Hodge structures. Since \( N(W_1LH^2(\mathcal{X}_0)) = 0, \) we have \( W_1H^2(\mathcal{X}_0) \cong W_1LH^2(\mathcal{X}_0) \) as Hodge structures.

\[\square\]

### 3.3 The specialization map

**Proposition 3.6.** Let \( R \) be an integral domain, \( K \) the fraction field of \( R. \) Let \( \mathcal{X} \) and \( \mathcal{Y} \) be smooth projective schemes over \( R, \) and \( X = \mathcal{X}_K, \) \( Y = \mathcal{Y}_K. \) Suppose \( \Phi : D^b(X) \to D^b(Y) \) is an exact functor which is an equivalence of triangulated categories. Then there exists \( 0 \neq r \in R \) and \( \mathcal{P} \in D^b(X \times_R Y) \) such that for every point \( s \in \text{Spec}(R(\frac{1}{r})), \) the Fourier-Mukai transform

\[
\Phi_{\mathcal{P}_s} : D^b(\mathcal{X}_s) \to D^b(\mathcal{Y}_s)
\]

induced by \( \mathcal{P}_s \) is an equivalence, where \( \mathcal{X}_s \) and \( \mathcal{Y}_s \) are the fiber over the point \( \iota_s : \text{Spec}(\kappa(s)) \to \text{Spec}(R(\frac{1}{r})), \) and \( \mathcal{P}_s = \mathcal{L} \iota_s^* \mathcal{P}, \) and moreover, \( \Phi_{\mathcal{P}_s} = \Phi. \)

Proof: By [Orl03, theorem 3.2.2], there exists \( P \in D^b(X \times_R Y) \) such that \( \Phi = \Phi_P. \) Shrinking \( \text{Spec}(R) \) if necessary, one can find \( \mathcal{P} \in D^b(X \times_R Y) \) such that \( \mathcal{P}_K = P. \) Let \( L \) be a very ample line bundle over \( X. \) Shrinking \( \text{Spec}(R) \) if necessary, there exists a relatively ample line bundle \( \mathcal{L} \) on \( \mathcal{X} \) over \( R, \) such that \( \mathcal{L} \) restricts to \( L. \) Set

\[
\mathcal{E} = \bigoplus_{i=0}^d \mathcal{L}^{\otimes i},
\]

where \( d = \dim X = \dim Y. \) By [Orl09, theorem 4], \( \mathcal{E}_s \) is a classical generator of \( D^b(\mathcal{X}_s), \) namely, the smallest triangulated subcategory of \( D^b(\mathcal{X}_s) \) containing \( \mathcal{E}_s \) and closed under isomorphisms and taking direct summands, is \( D^b(\mathcal{X}_s). \)

Set

\[
Q = P^\vee \otimes^L p_2^*\omega_Y[d], \quad Q = \mathcal{P}^\vee \otimes^L p_2^*\omega_{Y/R}[d].
\]
Then $Q = Q_K$, and for every point $s \in \text{Spec}(R)$, the Fourier-Mukai transform $\Phi_Q : D^b(Y_s) \to D^b(X_s)$ is a left adjoint of $\Phi_P : D^b(X_s) \to D^b(Y_s)$. Moreover, by hypothesis, $\Phi_Qs = \Phi_Q$ is an inverse of $\Phi_Ps = \Phi_P$. Thus the adjoint map

$$\Phi_Q \circ \Phi_Ps(E_K) \to E_K$$

is an isomorphism. By the semi-continuity theorem (for perfect complexes, [EGAIII 7.7.5]), shrinking $\text{Spec}(R)$ if necessary, for every point $s \in \text{Spec}(R)$, the adjoint map

$$\Phi_Qs \circ \Phi_Ps(E_s) \to E_s$$

is an isomorphism. By induction on the generating time of the objects of $D^b(X_s)$ with respect to $\mathcal{E}_s$, this implies that the adjoint morphism of functors

$$\Phi_Qs \circ \Phi_Ps \to id_{D^b(X_s)}$$

is an isomorphism. Thus $\Phi_Ps$ is fully faithful. Finally starting from a very ample line bundle on $Y$, and shrinking $\text{Spec}(R)$ if necessary, we find that $\Phi_Qs$ is also fully faithful, and we are done. □

**Theorem 3.7.** Let $R$ be the local ring $\mathbb{C}[T]_{(T)}$, and $K = \mathbb{C}(T)$. Let $X$ and $Y$ be smooth projective surfaces over $K$ with trivial canonical bundles. Suppose that $X$ and $Y$ are derived equivalent, and both have semistable degenerations over $R$. Then $\rho([X]) = \rho([Y])$ in $K_0(sGT_C)$.

**Proof:** Let $X_R$ and $Y_R$ be semistable degenerations of $X$ and $Y$ over $R$, respectively. Denote the point $(T)$ of $\text{Spec}(R)$ and the point $(T)$ of $\text{Spec}(\mathbb{C}[T])$, both by 0. Then there exists an affine open subset $U$ of $\text{Spec}(\mathbb{C}[T])$ and schemes $\mathcal{X}$ and $\mathcal{Y}$ over $U$, such that

(i) restricting to $U \setminus \{0\}$, $\mathcal{X}$ and $\mathcal{Y}$ are smooth, and each geometric fiber is a K3 surface;

(ii) the base changes of $\mathcal{X}$ and $\mathcal{Y}$ to $\text{Spec}(R)$ are isomorphic to $X_R$ and $Y_R$, respectively.

By proposition 3.6 there is an open subset $V$ of $U$ containing 0, and

$$P \in D^b(\mathcal{X} \times_U \mathcal{Y} \times_U (V \setminus \{0\}))$$

such that the Fourier-Mukai transform

$$\Phi_Ps : D^b(\mathcal{X}_t) \to D^b(\mathcal{Y}_t)$$

is an equivalence, for all $t \in V - \{0\}$. Without loss of generality we assume $V = U$. Consider the analytic topology of $U$. Taking an open disk $\Delta$ of $U$ containing 0, and consider $\mathcal{X}$ and $\mathcal{Y}$ restricting over $\Delta$, we can apply the result of the previous subsections to study the fiber $\mathcal{X}_0$ and $\mathcal{Y}_0$. By theorem 2.10 birational modifications preserving $\mathcal{X} - \mathcal{X}_0$ does not change $\rho([X])$. So by the first statement of theorem 3.6 we can assume $K_{X_0}$ and $K_{Y_0}$ trivial, such that $\mathcal{X}_0$ and $\mathcal{Y}_0$ are described by theorem loc. cit.

For a point $t \in \Delta - 0$, let $\Phi_P^{H} : H^*(\mathcal{X}_t) \to H^*(\mathcal{Y}_t)$ the map on cohomology induced by $\Phi_Ps$. By theorem 3.1 $\Phi_P^{H} : (\mathcal{H}(\mathcal{X}_t, \mathbb{Z}) \to \mathcal{H}(\mathcal{Y}_t, \mathbb{Z})$ is a Hodge isometry. Recall that

$$\Phi_P^{H}(\alpha) = q_{t*}(ch(P_t) \sqrt{td(\mathcal{X}_t \times \mathcal{Y}_t) \cdot p_t^* \alpha}).$$
Since $\text{ch}P_\Delta \sqrt{\text{td}(\mathcal{X}_t \times \mathcal{Y}_t)}$ is a restriction of an algebraic cohomology class on $\mathcal{X}_\Delta \times_{\Delta^*} \mathcal{Y}_\Delta$, we have a commutative diagram

$$
\begin{array}{c}
H^2(\mathcal{X}_t) \oplus H^0(\mathcal{X}_t) \oplus H^4(\mathcal{X}_t) \xrightarrow{N_X} H^2(\mathcal{X}_t) \oplus H^0(\mathcal{X}_t) \oplus H^4(\mathcal{X}_t) \\
\Phi^H_{\mathcal{X}_t} \downarrow \quad \Phi^H_{\mathcal{Y}_t} \downarrow \\
H^2(\mathcal{Y}_t) \oplus H^0(\mathcal{Y}_t) \oplus H^4(\mathcal{Y}_t) \xrightarrow{N_Y} H^2(\mathcal{Y}_t) \oplus H^0(\mathcal{Y}_t) \oplus H^4(\mathcal{Y}_t).
\end{array}
$$

(11)

So the smallest integer $i$ such that $N_X^i = 0$ is equal to that for $N_Y$. We consider the three cases separately.

(i) $N_X = N_Y = 0$. By theorem 3.4, $\mathcal{X}_0$ and $\mathcal{Y}_0$ are K3 surfaces. By proposition 3.3 there is a Hodge isometry between $\tilde{H}(\mathcal{X}_0, \mathbb{Z})$ and $\tilde{H}(\mathcal{Y}_0, \mathbb{Z})$. So by theorem 3.4, $\mathcal{X}_0$ and $\mathcal{Y}_0$ are derived equivalent, so $\rho([X]) = \rho([Y])$.

(ii) $N_X \neq 0$, $N_Y \neq 0$, $N_X^2 = N_Y^2 = 0$. Then with the notation of theorem 3.4, we have

$$
\rho([X]) = [V_0] + [V_r] + \sum_{i=1}^{r-1} [V_i] - 2r[E] = [V_0] + [V_r] - 2[E].
$$

Since $e(X) = 0$, we have $e(V_0) + e(V_r) - 2e(E) = 0$, thus $e(V_0) + e(V_r) = 0$. Since $V_0$ and $V_r$ are rational surfaces, we have $[V_0] + [V_r] = 0$. Therefore

$$
\rho([X]) = -2[E].
$$

It suffices to show $E_X \cong E_Y$. The diagram (11) induces an isomorphism of Hodge structures

$$
N_X(\tilde{H}(X, \mathbb{Z})) \xrightarrow{\sim} N_Y(\tilde{H}(Y, \mathbb{Z})).
$$

But $N_X(\tilde{H}(\mathcal{X}_t, \mathbb{Z})) = N_X(\tilde{H}(\mathcal{X}_t, \mathbb{Z}))$ and $N_Y(\tilde{H}(\mathcal{Y}_t, \mathbb{Z})) = N_Y(\tilde{H}(\mathcal{Y}_t, \mathbb{Z}))$. By definition of the weight filtration on $LH^2(\mathcal{X}_t)$ and $LH^2(\mathcal{Y}_t)$, $N_X(H^2(\mathcal{X}_t)) = W_1 LH^2(\mathcal{X}_t)$ and $N_Y(H^2(\mathcal{Y}_t)) = W_1 LH^2(\mathcal{Y}_t)$, as pure Hodge structures. So by proposition 3.5 $E_X \cong E_Y$.

(iii) $N_X^2 \neq 0$, $N_Y^2 \neq 0$, $N_X^3 = N_Y^3 = 0$. Since $e(\mathcal{X}_0) = 0$ and all the components of $\mathcal{X}_0$ are rational, $\rho([X]) = 0$. The same holds for $\mathcal{Y}_0$. So we are done.

4 Specialization map for abelian varieties

In this section we verify (10) for derived equivalent abelian varieties. In the final result (corollary 4.13) we need to assume that $k$ is an algebraically closed field, because theorem 4.3 need this assumption. However we still state the intermediate statements in a more general setting.
4.1 Derived equivalences of abelian abelian varieties

In this subsection we collect some theorems on derived equivalent abelian varieties due to Mukai, Polishchuk and Orlov. Our references are [Muk87b], [Pol96], [Orl02], and also [Huy06, Chapter 9].

Theorem 4.1 ([Muk87b]). Let $S$ be a scheme, $p : A \to S$ an abelian scheme, and $q : A^t \to S$ its dual abelian scheme:

\[
\begin{array}{c}
\pi_A & \quad & \pi_{A^t} \\
A \times_S A^t & \quad & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \.lu
**Definition 4.5.** Let $A$ be a abelian variety over $K$. A **semistable degeneration** of $A$ over $S$ is a semiabelian scheme $G$ over $S$ with an isomorphism $G_\eta \cong A$. By definition, there is an extension

$$0 \to T_0 \to G_0 \to A_0 \to 0$$

where $A_0$ is an abelian variety over $k$, and $T_0$ is a torus over $k$. If $T_0$ is a split torus, $G$ is called a **split degeneration** of $A$.

**Definition 4.6.** An **ample degeneration** of $A$ is a pair $(G, \mathcal{L})$ where $G$ is a semiabelian degeneration of $A$ over $S$ and $\mathcal{L}$ is a cubical invertible sheaf on $G$ such that $\mathcal{L}_\eta$ is ample.

In fact the condition implies that $\mathcal{L}$ is relatively ample.

Denote $S_i = \text{Spec}(R/m^i)$. For a semiabelian scheme $G$ over $S$, denote $G_{\text{for}} = \lim G \times_S S_i$, and $\mathcal{L}_{\text{for}}$ the corresponding formal completion of $\mathcal{L}$. For an ample degeneration $(G, \mathcal{L})$, there is the associated Raynaud extension

$$0 \to T \to \tilde{G} \xrightarrow{\pi} \tilde{A} \to 0,$$

such that $\tilde{G}$ is an algebraization of the formal scheme $G_{\text{for}}$, $T$ is a torus over $S$, and $\tilde{A}$ is an abelian scheme over $S$, and there is a cubical ample invertible sheaf $\tilde{\mathcal{L}}$ which is the algebraization of $\mathcal{L}_{\text{for}}$.

**Definition 4.7.** A **Split ample degeneration** of $A$ is a triple $(G, \mathcal{L}, \mathcal{M})$, where $G$ is a split degeneration of $A$, $(G, \mathcal{L})$ is an ample degeneration, and $\mathcal{M}$ is a cubical ample invertible sheaf on $\tilde{A}$ such that $\pi^* \mathcal{M} \cong \tilde{\mathcal{L}}$.

By the rigidity of tori [DG70, X. theorem 3.2], $T_0$ is split implies that $T$ is split. Moreover, the character group of $T$ is a constant abelian sheaf over $S$, and we denote the associated constant group by $X$. There is a notion of dual semiabelian scheme $G^t$ over $S$, and the corresponding torus $T^t$ is also split. We denote the constant character group of $T^t$ by $Y$.

**Definition 4.8.** Consider the cone $\mathcal{C} = (X^+_R \times \mathbb{R}_{>0}) \cup \{0\}$. There is a natural action of $Y$ on $\mathcal{C}$ via addition. A $Y$-admissible polyhedral cone decomposition of $\mathcal{C}$ is a (possibly infinite) rational polyhedral cone decomposition $\{\sigma\}_{\alpha \in I}$ of $\mathcal{C}$ such that the collection of the cones $\sigma_\alpha$ is invariant under the action of $Y$ and there are only finitely many orbits.

**Theorem 4.9.** ([Kunn98, theorem 3.5]) Let $(G, \mathcal{L}, \mathcal{M})$ be a split ample degeneration. Then there is a projective regular model $P$, and an admissible cone decomposition $\{\sigma_\alpha\}_{\alpha \in I}$ of $\mathcal{C}$, and we denote by $I_Y^+$ the corresponding orbit space with the orbit of the zero cone removed, such that

(i) the reduced special fiber $(P_0)_{\text{red}}$ is a strict normal crossing divisor on $P$;

(ii) $(P_0)_{\text{red}}$ has a natural stratification with strata $G_{\sigma_\alpha}$ for $\alpha \in I_Y^+$, where $G_{\sigma_\alpha}$ is a semiabelian scheme fitting into an exact sequence

$$0 \to T_{\sigma_\alpha} \to G_{\sigma_\alpha} \to A_0 \to 0,$$

where $A_0$ is the abelian part of the Raynaud extension, and $T_{\sigma_\alpha}$ is a split torus;

(iii) the closure $P_{\sigma_\alpha}$ of the stratum $G_{\sigma_\alpha}$ is the disjoint union of all $G_{\sigma_\beta}$ such that $\alpha$ is a face of $\beta$, and

$$P_{\sigma_\alpha} = G_{\sigma_\alpha} \times_{T_{\sigma_\alpha}} Z_{\sigma_\alpha},$$

is a contraction product, where $T_{\sigma_\alpha} \to Z_{\sigma_\alpha}$ an open torus imbedding into a smooth projective toric variety.
4.3 Degeneration and derived equivalence

**Proposition 4.10.** Let $A$ be an abelian varieties over $K$, which has a split degeneration over $R$. Then $A$ has a split ample degeneration over $R$.

Proof: By the assumption there is a semi-abelian scheme $G$ over $R$ such that $G_K \cong A$ and $G_0$ fits into an extension

$$0 \to T_0 \to G_0 \to A_0 \to 0$$

such that $T_0$ is a split torus over $k$ and $A_0$ is an abelian variety over $k$. By [MB85, I, 2.6] and [Ray70, XI, 1.13] (see also [Lan13, remark. 3.3.3.9]), there is an ample cubical invertible sheaf $\mathcal{L}$ over $G$. Thus $\mathcal{L} \otimes [-1]^* \mathcal{L}$ is also an ample cubical invertible sheaf over $G$. Let

$$0 \to \tilde{T} \to \tilde{G} \to \tilde{A} \to 0$$

be the corresponding Raynaud extension. Then by [Lan13, cor. 3.3.3.3, prop. 3.3.3.6] and [Ray70, XI, 1.11], the invertible sheaf $\mathcal{L} \otimes [-1]^* \mathcal{L}$ over $\tilde{G}$ for is isomorphic to an ample pullback $\mathcal{M} \otimes [-1]^* \mathcal{M}$ over $\tilde{A}$ for which is algebraizable. This provides a split ample degeneration of $A$. \hfill $\Box$

**Lemma 4.11.** Let

$$0 \to T \to G \to A \to 0$$

be an extension of an abelian variety $A$ by a split torus, over a field $k$, and $T \hookrightarrow Z$ be an open torus embedding of $T$ into a smooth complete toric variety $Z$. Then in $K_0(\text{Var}_k)$ one has

$$[G \times^T Z] = [Z] \cdot [A].$$

Proof: By the assumption, $G$ is an fppf $T$-torsor over $A$. Since $T$ is split, $G$ is a product of fppf $\mathbb{G}_m$-torsors over $A$, thus it is also a product of Zariski $\mathbb{G}_m$-torsors, by Hilbert theorem 90. So there is a locally closed stratification $\{U_\alpha\}$ of $A$ such that $(G \times^T Z)|_{U_\alpha}$ is isomorphic to $Z \times U_\alpha$, hence the conclusion. \hfill $\Box$

**Theorem 4.12.** Let $(R, \mathfrak{m})$ be a complete discrete valuation ring, $K$ the fraction field of $R$, and $k$ the residue field of $R$. Let $A$ and $B$ be two abelian varieties over $K$, which are derived equivalent. Then the following holds.

(i) $A$ has semistable reduction if and only if $B$ has semistable reduction.

(ii) $A$ has a split degeneration if and only if $B$ has a split degeneration.

(iii) In case of (i), denote the abelian part of the special fiber of the semistable reduction of $A$ (resp., $B$) by $A_0$ (resp., $B_0$). Then there is a symplectic isomorphism $A_0 \times_k A_0' \sim B_0 \times B_0'$.

Proof: (i) By theorem 4.4 there is a symplectic isomorphism $A \times_K A^t \cong B \times_K B^t$. Then by [MP17, proposition 2.10], $A$ and $B$ are isogenous. Thus the conclusion (i) follows from [BLR90, §7.3, corollary 3].

(ii) Suppose $A$ has a split ample degeneration over $R$. By [FC90, §2.2], $A^t$ has a split ample degeneration over $R$. Let $\mathcal{M}$ (resp., $\mathcal{M}'$) be the Néron model of $A$ (resp., of $A^t$)
over $R$. By the functoriality of Néron models, $\mathcal{A} \times_R \mathcal{A}'$ is the Néron model of $A \times_K A'$. By theorem 4.4, $A \times_K A'$ is isomorphic to $B \times_K B'$. Let $\mathfrak{B}$ (resp., $\mathfrak{B}'$) be the Néron model of $B$ (resp., of $B'$) over $R$. Thus $\mathcal{A} \times_R \mathcal{A}' \cong \mathfrak{B} \times \mathfrak{B}'$, so their special fibers have isomorphic identity components, i.e. $(\mathcal{A} \times_R \mathcal{A}')_k \cong (\mathfrak{B} \times \mathfrak{B}')_k$. By (i), $A$, $A'$, $B$, $B'$ all have semistable reductions over $R$. Thus $(\mathcal{A}_k)^\circ$, $(\mathcal{A}'_k)^\circ$, $(\mathfrak{B}_k)^\circ$ and $(\mathfrak{B}'_k)^\circ$ are all semi-abelian varieties over $k$, hence are geometrically connected. Thus by [EGAIV 4.5.8] $(\mathcal{A}_k)^\circ \times_k (\mathcal{A}'_k)^\circ$ and $(\mathfrak{B}_k)^\circ \times_k (\mathfrak{B}'_k)^\circ$ are connected and thus are isomorphic to $(\mathcal{A} \times_R \mathcal{A}')_k$. Let $T$ (resp. $T'$) be the torus part of $\mathfrak{B}_k$ (resp. $\mathfrak{B}'_k$). Then $T \times_k T'$ is a split torus. Consider the character group $X(T)$ (resp. $X(T')$) of $T$ (resp. $T'$), which are étale sheaves of torsion free abelian groups of finite type. The product $X(T) \times X(T')$ is the character group of $T \times_k T'$, and is therefore a constant sheaf by the splitness of $T \times_k T'$. Considering the action of $\text{Gal}(k^s/k)$ on $X(T)(k^s)$ and $X(T')(k^s)$, one sees that both $X(T)$ and $X(T')$ are constant sheaves over $k_{et}$, and therefore $T$ and $T'$ are split tori over $k$.

(iii) By theorem 4.4 there is an isomorphism $f : A \times_K A' \to B \times_K B'$ of the form

$$f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

such that

$$\begin{pmatrix} \delta^t & -\beta^t \\ -\gamma^t & \alpha^t \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \text{id.}$$

By the functoriality of Néron models the isomorphism $f$ extends to an isomorphism $F : \mathcal{A} \times_R \mathcal{A}' \to \mathfrak{B} \times \mathfrak{B}'$ of the form

$$F = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix}$$

such that

$$\begin{pmatrix} \tilde{\delta}^t & -\tilde{\beta}^t \\ -\tilde{\gamma}^t & \tilde{\alpha}^t \end{pmatrix} \cdot \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix} = \text{id.}$$

Considering the special fibers and using the proof of (ii), one obtains a symplectic isomorphism between the abelian parts of $\mathcal{A}_k^\circ \times_k (\mathcal{A}'_k)^\circ$ and $\mathfrak{B}_k^\circ \times_k (\mathfrak{B}'_k)^\circ$. \hfill \Box

**Proposition 4.13.** Let $A$ and $B$ be two abelian varieties over $K$, which are derived equivalent, and suppose that $A$ has a split degeneration over $R$. Then $A$ and $B$ have snc models $P$ and $Q$ over $R$, respectively, such that either $P_0$ and $Q_0$ are symplectically isomorphic abelian varieties over $k$, or $[P_0] = [Q_0] = 0$ in $K_0(\text{sGT}_k)$.

**Proof:** By theorem 4.12 (ii) and proposition 4.10 both $A$ and $B$ has split ample degeneration over $R$. By theorem 4.12 (iii), if $A$ has good reduction over $R$, then so does $B$, and $A_0$ and $B_0$ are symplectic isomorphic. If $A$ does not have a good reduction over $R$, then by theorem 4.9 and lemma 4.11

$$[P_0] = \sum_{\alpha \in I^+_\gamma} (-1)^{j_\alpha} j_\alpha [A_0] \times [Z_{\sigma_\alpha}]$$

in $K_0(\text{Var}_k)$, where

$$j_\alpha = \dim A + 1 - \dim A_0 - \dim Z_{\sigma_\alpha} = \dim \mathbb{C} - \dim \mathbb{R} \sigma_\alpha.$$
Since each face of $\sigma_{\alpha}$ appears in the above sum, a simple manipulation shows that

$$\sum_{\alpha \in I^+_F} (-1)^{j_{\alpha}-1} j_{\alpha} [Z_{\sigma_{\alpha}}]$$

is equal to a linear combination of split tori in $K_0(\text{Var}_K)$, so

$$[P_0] \equiv 0 \mod (L - 1).$$

Then by corollary 2.5 and the definition (9) of $\rho$, $[P_0] = 0$ in $K_0(\text{sGT}_K)$. By theorem 4.12 $B$ also has a split ample but not good degeneration over $R$, thus one has $[Q_0] = 0$ in $K_0(\text{sGT}_K)$, too. So we are done.

**Corollary 4.14.** Let $(R, m)$ be a complete discrete valuation ring, $K$ the fraction field of $R$, and $k$ the residue field of $R$. Suppose that $k$ is algebraically closed of characteristic 0. Let $A$ and $B$ be two abelian varieties over $K$, which are derived equivalent. Suppose $A$ has a semistable reduction over $R$. Then $\rho([A]) = \rho([B])$.

**Proof:** By theorem 4.12 (i), both $A$ and $B$ semistable reductions over $R$, which are automatically split degenerations because $k$ is algebraically closed. Applying proposition 4.13 and theorem 4.3 we obtain the conclusion.

**5 Open problems**

1. Although our (conjectural) definition of $\rho_{\text{sdt}}$ does not assume the existence of semistable degeneration over $R$, in the above verifications we need to assume this to apply the results for the degeneration of these varieties. It is natural to make the following conjecture. Theorem 4.12 provides an example for it.

**Conjecture 5.1.** Let $R$ be a DVR, $K$ its fraction field. Let $X$ and $Y$ be derived equivalent smooth projective varieties over $K$. Then $X$ has semistable degeneration (resp., good reduction) over $R$ if and only if $Y$ has semistable degeneration (resp., good reduction) over $R$.

This suggests to take into consideration the Galois action on the derived categories, and ask whether there is a Néron-Ogg-Shafarevich-Grothendieck type criterion for the types of degenerations.

2. Does there exist a smooth projective variety $X$ over $k$ such that $[X] = m[\text{Spec}(k)]$ in $K_0(\text{sGT})$ but $D^b(X)$ does not have a full exceptional collection? If there are such varieties, are their quantum cohomology semisimple? The limit fibers of a family of varieties with full exceptional collections are candidates for this.

**References**

[AKMW02] Abramovich, Dan; Karu, Kalle; Matsuki, Kenji; Włodarczyk, Jarosław. Torification and factorization of birational maps. J. Amer. Math. Soc. 15 (2002), no. 3, 531–572.
[AT16] Abramovich, D; Temkin, M. Functorial factorization of birational maps for qe schemes in characteristic 0. Preprint, arXiv:1606.08414.

[BBR09] Bartocci, Claudio; Bruzzo, Ugo; Hernández Ruipérez, Daniel. Fourier-Mukai and Nahm transforms in geometry and mathematical physics. Progress in Mathematics, 276. Birkhäuser Boston, Inc., Boston, MA, 2009.

[Bay04] Bayer, Arend. Semisimple quantum cohomology and blowups. Int. Math. Res. Not. 2004, no. 40, 2069–2083.

[Bit04] Franziska Bittner. The universal Euler characteristic for varieties of characteristic zero. Compos. Math., 140(4):1011–1032, 2004.

[BLL04] Bondal, Alexey I.; Larsen, Michael; Lunts, Valery A. Grothendieck ring of pretriangulated categories. Int. Math. Res. Not. 2004, no. 29, 1461–1495.

[BO95] Bondal A, Orlov D. Semiorthogonal decomposition for algebraic varieties. arXiv preprint alg-geom/9506012, 1995.

[BLR90] Bosch, Siegfried; Lütkebohmert, Werner; Raynaud, Michel. Néron models. Ergebnisse der Mathematik und ihrer Grenzgebiete, 21. Springer-Verlag, Berlin, 1990.

[DG70] M. Demazure and A. Grothendieck (eds.), Schémas en groupes (SGA 3), II: Groupes de type multiplicatif, et structure des schémas en groupes généraux, Lecture Notes in Mathematics, vol. 152, Springer-Verlag, 1970.

[Dub98] Dubrovin, B. Geometry and analytic theory of Frobenius manifolds. Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 315–326.

[EGAIII] Grothendieck A. Eléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): III. Etude cohomologique des faisceaux cohérents, Premiere partie. Publications Mathématiques de l’IHES, 1961, 11: 5–167.

[EGAIV] Grothendieck, A. Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné). IV. Étude locale des schémas et des morphismes de schémas. II. (French) Inst. Hautes Études Sci. Publ. Math. No. 24 1965.

[FC90] Faltings, Gerd; Chai, Ching-Li. Degeneration of abelian varieties. With an appendix by David Mumford. Ergebnisse der Mathematik und ihrer Grenzgebiete, 22. Springer-Verlag, Berlin, 1990.

[Fri84] Friedman, Robert. A new proof of the global Torelli theorem for K3 surfaces. Ann. of Math. (2) 120 (1984), no. 2, 237–269.

[Hu18] Hu, Xiaowen. Deformation of exceptional collections. Preprint arXiv:1805.04050, 2018.

[Huy06] Huybrechts, Daniel. Fourier-Mukai transforms in algebraic geometry. Oxford University Press, 2006.

[Huy16] Huybrechts, Daniel. Lectures on K3 surfaces. Cambridge Studies in Advanced Mathematics, 158. Cambridge University Press, Cambridge, 2016.
[Kaw02] Kawamata, Yujiro. D-equivalence and K-equivalence. J. Differential Geom. 61 (2002), no. 1, 147–171.

[KT17] Kontsevich M, Tschinkel Y. Specialization of birational types. arXiv preprint arXiv:1708.05699, 2017.

[Kul77] Kulikov, Vik. S. Degenerations of K3 surfaces and Enriques surfaces. Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), no. 5, 1008–1042, 1199.

[Künn98] Künemann, Klaus. Projective regular models for abelian varieties, semistable reduction, and the height pairing. Duke Math. J. 95 (1998), no. 1, 161–212.

[Lan13] Lan, Kai-Wen. Arithmetic compactifications of PEL-type Shimura varieties. London Mathematical Society Monographs Series, 36. Princeton University Press, Princeton, NJ, 2013.

[MP17] López Martín, Ana Cristina; Tejero Prieto, Carlos. Derived equivalences of Abelian varieties and symplectic isomorphisms. J. Geom. Phys. 122 (2017), 92–102.

[MB85] Moret-Bailly, Laurent. Pinceaux de variétés abéliennes. Astérisque No. 129 (1985).

[Muk87] Mukai, S. On the moduli space of bundles on K3 surfaces. I. Vector bundles on algebraic varieties (Bombay, 1984), 341–413, Tata Inst. Fund. Res. Stud. Math., 11, Tata Inst. Fund. Res., Bombay, 1987.

[Muk87b] Mukai, Shigeru. Fourier functor and its application to the moduli of bundles on an abelian variety. Algebraic geometry, Sendai, 1985, 515–550, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[Muk94] Mukai, Shigeru. Abelian variety and spin representation, in: Proceedings of the Symposium Hodge Theory and Algebraic Geometry, Sapporo, 1994, University of Warwick, 1998, pp. 110–135.

[Mum72] D. Mumford. An analytic construction of degenerating abelian varieties over complete rings, Compositio Math. 24 (1972), 239–272.

[Nam03] Namikawa, Yoshinori. Mukai flops and derived categories. J. Reine Angew. Math. 560 (2003), 65–76.

[NS17] Nicole, J., Shinder, E. The motivic nearby fiber and degeneration of stable rationality. arXiv preprint arXiv:1708.02790

[Orl92] Orlov, D. O. Projective bundles, monoidal transformations, and derived categories of coherent sheaves. Izv. Ross. Akad. Nauk Ser. Mat. 56 (1992), no. 4, 852–862; translation in Russian Acad. Sci. Izv. Math. 41 (1993), no. 1, 133–141.

[Orl97] Orlov, D. O. Equivalences of derived categories and K3 surfaces. Algebraic geometry, 7. J. Math. Sci. (New York) 84 (1997), no. 5, 1361–1381.

[Orl02] Orlov, D. O. Derived categories of coherent sheaves on abelian varieties and equivalences between them. Izv. Ross. Akad. Nauk Ser. Mat. 66 (2002), no. 3, 131–158; translation in Izv. Math. 66 (2002), no. 3, 569–594.
[Orl03] Orlov, D. O. Derived categories of coherent sheaves and equivalences between them. (Russian) Uspekhi Mat. Nauk 58 (2003), no. 3(351), 89–172; translation in Russian Math. Surveys 58 (2003), no. 3, 511–591.

[Orl09] Orlov, Dmitri. Remarks on generators and dimensions of triangulated categories. Mosc. Math. J. 9 (2009), no. 1, 153–159.

[PP81] Persson, Ulf; Pinkham, Henry. Degeneration of surfaces with trivial canonical bundle. Ann. of Math. (2) 113 (1981), no. 1, 45–66.

[Pol96] Polishchuk, A. Symplectic biextensions and a generalization of the Fourier-Mukai transform. Math. Res. Lett. 3 (1996), no. 6, 813–828.

[Ray70] Raynaud, Michel. Faisceaux amples sur les schémas en groupes et les espaces homogènes. Lecture Notes in Mathematics, Vol. 119 Springer-Verlag, Berlin-New York 1970.

[Sch73] Schmid, Wilfried. Variation of Hodge structure: the singularities of the period mapping. Invent. Math. 22 (1973), 211–319.

[Wlo03] Włodarczyk, Jarosław. Toroidal varieties and the weak factorization theorem. Invent. Math. 154 (2003), no. 2, 223–331.

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