On the Inviscid Burgers Equation and the Axiom of Choice

John M. Noble
Matematiska institutionen,
Linkpings universitet,
58183 LINKÖPING, Sweden

Summary This article gives an example where the use of Tychonov’s theorem, which is equivalent to the Axiom of Choice, yields two contradictory results. That is, by using Tychonov’s theorem, full proofs of contradictory results are obtained. That the Choice Axiom implies Tychonov’s theorem is standard. That Tychonov’s theorem implies the Axiom of Choice is a result of Kelley, found in the article [9].

This paper discusses the first direction; Choice implies Tychonov’s theorem. The example discussed in this article provides a counter example to the assertion that the unit ball in $L^2$ is relatively weakly compact. Tychonov’s theorem states that the unit ball in $L^2$ is relatively weakly compact. The proof of Tychonov’s theorem crucially uses Choice and Kelley shows that Choice is equivalent to Tychonov’s theorem. Therefore, as a corollary of the proof that contradictory results may be obtained by assuming the relative weak compactness of the unit ball in $L^2$, it follows that the Choice Axiom is inadmissible in mathematical analysis.

The article is structured as follows. Section (1) is introductory. Section (2) illustrates the construction of solutions to the inviscid Burgers equation in terms of the velocities of Euler Lagrange trajectories. It also computes a formula for the evolution of downward jumps for solutions to the inviscid Burgers’ equation. Section (3) sketches the large deviations argument (for smooth, bounded potentials) which leads to the representation of the solution to the inviscid Burgers equation in terms of the velocity of the trajectory that minimises the associated action functional, subject to the appropriate constraints. Also outlined are the standard arguments from the Calculus of Variations that prove the existence of a trajectory at which the minimum of the action functional is attained subject to the appropriate constraints, that this trajectory solves the associated Euler Lagrange equations, and that the solutions of the inviscid Burgers’ equation may be constructed using these minimising trajectories.

Section (4) presents an example of a smooth, bounded, space / time periodic potential, for which the viscosity solution to the inviscid Burgers’ equation with that potential may not be constructed using trajectories that minimise the associated action functional. The solution may be constructed using critical points of the associated action functional, but these critical points are not the minimising trajectories.
1 Introductory

Let \( W = C_0(\mathbb{R}_+) \); that is, continuous functions \( f : \mathbb{R}_+ \to \mathbb{R} \) such that \( f(0) = 0 \). Let \( w \) denote a trajectory for standard Brownian motion in \( \mathbb{R} \) satisfying \( w(0) = 0 \). Let

\[
(W, \mathcal{F}, (\mathcal{F}_{s,t})_{0 \leq s \leq t < +\infty}, \mathbb{P})
\]

denote the filtered Gaussian probability space associated with \( w \), where \( \mathcal{F}_{s,t} \) is the Borel sigma algebra generated by the increments of continuous functions between \( s \) and \( t \); \( (w(u) - w(v))_{s \leq v \leq u \leq t} \), \( \mathcal{F} = \bigcup_{0 \leq s \leq t < +\infty} \mathcal{F}_{s,t} \), and \( \mathbb{P} \) is the probability measure associated with standard Brownian motion. That is, under \( \mathbb{P} \), \( w(0) = 0 \) with probability 1, for any collection \( (t_1, \ldots, t_{n+1}) \) with \( 0 \leq t_1 \leq \ldots \leq t_{n+1} < +\infty \) the random variables \( (w(t_{j+1}) - w(t_j))_{j=1}^n \) are independent Gaussian random variables with \( w(t_{j+1}) - w(t_j) \sim N(0, t_{j+1} - t_j) \) and, for all \( s < t \), \( s \leq u \leq v \leq t \), \( w(v) - w(u) \) is \( \mathcal{F}_{s,t} \) measurable.

Let \( E_{\mathbb{P}} \) denote the expectation operator with respect to \( \mathbb{P} \). Let \( V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) be a smooth, bounded (time dependent) potential. Using subscripts to denote derivatives with respect to the subscripted variable, consider the equation

\[
\begin{align*}
U^{(\epsilon)}(t, x) &= E_{\mathbb{P}} \left[ \exp \left\{ -\frac{1}{\epsilon} \left( \phi(x + \sqrt{\epsilon} w(t)) + \int_0^t V(t - s, x + \sqrt{\epsilon} w(s)) ds \right) \right\} \right], \quad (1)
\end{align*}
\]

A Feynman-Kac representation of the solution may be employed:

\[
U^{(\epsilon)}(0, x) = e^{-\frac{1}{\epsilon} \phi(x)}.
\]

Consider now \( v^{(\epsilon)} = -\epsilon \log U^{(\epsilon)} \) and note that \( v^{(\epsilon)} \) satisfies

\[
\begin{align*}
v_t^{(\epsilon)} &= \frac{\epsilon}{2} v_{xx}^{(\epsilon)} - \frac{1}{\epsilon} \left( v_x^{(\epsilon)} \right)^2 + V \\
v^{(\epsilon)}(0, x) &= \phi(x).
\end{align*}
\]

From the Feynman-Kac representation for \( U^{(\epsilon)} \), given in equation (1), \( v \) satisfies

\[
v^{(\epsilon)}(t, x) = -\epsilon \log E_{\mathbb{P}} \left[ \exp \left\{ -\frac{1}{\epsilon} \left( \phi(x + \sqrt{\epsilon} w(t)) + \int_0^t V(s, x + \sqrt{\epsilon} w(t-s)) ds \right) \right\} \right].
\]

Suppose that \( \phi \) and \( V \) are uniformly bounded. Let \( \mathcal{S}_n \) denote the space of functions \( f : \mathbb{R} \to \mathbb{R} \), with bounded derivative \( \dot{f} \), such that \( \dot{f} \) is piecewise constant on intervals \( [\frac{k}{2^n}, \frac{k+1}{2^n}) \); that is, if \( f \in \mathcal{S}_n \) then there exist real numbers \( (\lambda_j)_{j=-\infty}^{\infty} \) such that

\[
\dot{f} = \sum_j \lambda_j \chi_{[\frac{k}{2^n}, \frac{k+1}{2^n})},
\]
where $\chi_A$ denotes the indicator function for a set $A$. This notation will be used throughout. Let $S = \bigcup_n S_n$. Then, by Varadhan’s theorem from large deviations, it follows that

$$v(t, x) := \lim_{\epsilon \to 0} v(\epsilon)(t, x) = \inf_{\xi: \xi(t) = x, \xi \in \mathcal{S}} \left\{ \phi(\xi(0)) + \frac{1}{2} \int_0^t \dot{\xi}^2(s) ds + \int_0^t V(s, \xi(s)) ds \right\}. \tag{2}$$

The result is well known, but a full proof is given in section (3), for completeness of the presentation. The treatment is taken from Dembo and Zeitouni [5], with appropriate simplifications, because only a special case of their setting is required here. This approach, of taking a partition over the time interval and letting the mesh size tend to zero, seems more appropriate in a situation where it is convenient to locate where Choice enters. Choice is used in the proof, but it seems clear that the statement of the result can be modified and the proof modified to produce a version of the result where the proof does not require Choice. Another full and elegant treatment may be found in Deuschel and Stroock [4].

The next question is whether or not there exists a minimiser in the space $W^{1,2}(\mathbb{R})$; namely, a trajectory at which the infimum is attained. For this specific problem, there exists a trajectory at which the minimum of the action functional is attained. This is a consequence of the relative weak compactness of a ball of finite radius in $L^2$. The result was first established by Tonelli in [13] and [14]. It is treated in a more general framework by Cesari in [3]. The relative weak compactness of the unit ball in $L^2$ is crucial here.

The whole point at issue seems to be the existence of the minimising trajectory, because under the conditions in this article, having established existence of minimising trajectories, it is relatively straightforward to show that they satisfy the associated Euler Lagrange equations.

In more general problems within the Calculus of Variations, where the action functional does not satisfy some crucial hypotheses of the framework of Tonelli, there may not exist minimising trajectories, as may be seen, for example, in Ball and Mizel [2] and [1], where examples are given of situations where global minimisers exist, but do not satisfy the Euler Lagrange equations and examples of situations where global minimisers do not exist.

In the example presented in section (4) of this article, the resulting process does indeed satisfy the Euler Lagrange equations in the limit, but it is shown that the trajectory picked out is not the minimiser. Therefore, the minimiser is not necessarily the trajectory that appears in representation of the ‘viscosity’ solution of the inviscid Burger equation using solutions to the associated Euler Lagrange equations.

Observation: It is the countable version of the Choice Axiom that is shown to lead to contradictory results in this article.

Let $u^{(\epsilon)} = v_2^{(\epsilon)}$, then $u^{(\epsilon)}$ satisfies
\[
\begin{align*}
\begin{cases}
 u_t^{(\epsilon)} + \frac{1}{2}(u^{(\epsilon)}_x)^2 &= \frac{\epsilon}{2} u^{(\epsilon)}_{xx} + V_x \\
 u^{(\epsilon)}(0, \cdot) &= \phi_x(\cdot)
\end{cases}
\end{align*}
\] (3)

Let \( u = \lim_{\epsilon \to 0} u^{(\epsilon)} \), where the limit is taken in the relative weak topology in \( L^2 \). Then the limit exists and satisfies

\[
\begin{align*}
\begin{cases}
 u_t + \frac{1}{2}(u^2)_x &= V_x \\
 u(0, \cdot) &= \phi_x(\cdot)
\end{cases}
\end{align*}
\] (4)

Let

\[ A(\xi; t, x) = \frac{1}{2} \int_0^t \xi^2(s) ds + \int_0^t V(s, \xi(s)) ds + \phi(\xi(0)). \] (5)

If \( V \) is infinitely differentiable and uniformly bounded (as it is in the example considered in this article), then it is standard that for each \( (t, x) \) there exists a trajectory \( \xi^{(t, x)} \) that minimises \( A \) subject to the constraint that \( \xi^{(t, x)}(t) = x \). Throughout, the following notation will be used: for a function \( f \) of two arguments, \( \dot{f} \) will be used to denote the derivative with respect to the first argument and \( f' \) will be used to denote the derivative with respect to the second argument. If \( f \) has only one argument, then either \( f' \) or \( \dot{f} \) may be used to denote the derivative with respect to that argument. Usually \( \dot{f} \) will be employed if the argument serves as a ‘time’ variable and \( f' \) will be used if the argument serves as a space variable. It follows that \( V' = V_x \) and \( \phi' = \phi_x \).

For the action functional defined by equation (5), relatively straightforward arguments using the relative weak compactness of the unit ball, show that there exists a trajectory where the global minimum of the action functional is attained and, having established existence, relatively straightforward arguments from the calculus of variations show that any minimiser satisfies the Euler Lagrange equations,

\[
\begin{align*}
\begin{cases}
 \ddot{\xi}^{(t, x)}(s) &= V'(s, \xi^{(t, x)}(s)) \\
 \xi^{(t, x)}(t) &= x, \quad \xi^{(t, x)}(0) = \phi'(\xi^{(t, x)}(0)).
\end{cases}
\end{align*}
\] (6)

In general, solutions to equation (6) will not be unique if \( t \) is sufficiently large. Having established existence of trajectories where the global minimum is attained and that they solve equation (6), it may be shown that the solution to equation (4) has representation in terms of the minimising trajectories. Let \( \eta \) solve

\[
\begin{align*}
\begin{cases}
 \ddot{\eta} &= V'(t, \eta) \\
 \eta(0, x) &= x, \quad \dot{\eta}(0, x) = \phi_x(x),
\end{cases}
\end{align*}
\] (7)

then \( u \), the solution to equation (4) has a representation

\[ u(t, x) = \dot{\eta}(t, \eta^{-1}(t, x)) = \dot{\xi}^{(t, x)}(t), \] (8)
where $\xi$ solves equation (6). It turns out that $\xi$ also minimises the action functional (5) subject to the constraint that $\xi(t) = x$. These results are all standard and their proofs are all outlined in the article.

The counter example in section (4) gives an example where the viscosity solution to equation (4) may be constructed using trajectories which satisfy the Euler Lagrange equations, but that these trajectories necessarily do not minimise the action functional. The strategy is as follows: The potential chosen is

$$V(t, x) = \cos(\sin(t)) - \cos(x + \sin(t)).$$

The time dependence seems to be crucial here to manufacture a counter example. For fixed initial condition, there is uniqueness of solution to equation (3). With this choice of potential, it is shown that there exists exactly one initial condition $\phi_x^{(\epsilon)}$ that yields a periodic solution to equation (3). It is shown that these periodic solutions $u^{(\epsilon)}$ converge (in the relative weak topology) to a limit $u$ and that this is the only periodic solution of equation (4); $\phi_x$, the weak limit of $\phi_x^{(\epsilon)}$ is the only initial condition that yields periodic solutions to equation (4). It is shown that there is necessarily a periodic modulo $2\pi$ solution to the Euler Lagrange equations involved in the construction of this periodic solution to equation (4) and, indeed, that all trajectories used in the construction, when run backwards, converge to a periodic trajectory. But it is shown that the only two periodic modulo $2\pi$ trajectories which solve equation

$$\ddot{\xi} = \sin(\xi + \sin(t))$$

are

$$\xi(t) = t - \sin(t)$$

and

$$\xi(t) = \pi - t - \sin(t).$$

**Note** Periodic modulo $2\pi$ means that $\xi(t) \mod (2\pi)$ is periodic. It is shown that, for sufficiently large $t$, neither of these minimise the action functional

$$A(\xi; t) := \frac{1}{2} \int_0^t \dot{\xi}^2(s) ds + \int_0^t \cos(\sin(s)) ds - \int_0^t \cos(\xi(s) + \sin(s)) ds + \phi(\xi(0)).$$

The problem became apparent following results in two articles by E, Khanin, Mazel and Sinai [6] and [7]. The article [10] presents analysis of the moments of the stochastic inviscid Burgers’ equation, under a special case of the hypotheses considered in E, Khanin, Mazel, Sinai, which are of interest following the invariant measure proved in [7]. The article [7] used crucially the
existence of trajectories where the minimum of the action functional is attained, that minimising trajectories satisfy the associated Euler Lagrange equations \( (7) \), and that it is minimising trajectories that are used to construct the solution given in equation \( (8) \) to equation \( (4) \). Conditioned on this step, every other part of the argument in the article \[ 7 \] is clear.

2 The Inviscid Burgers Equation and the Euler Lagrange Equations

In general, for the inviscid Burgers’ equation with smooth initial condition and smooth potential, there will be uniqueness of solutions to the associated Euler Lagrange equations with appropriate boundary data up to the onset of downward jumps. Up to the onset of downward jumps, there will be exactly one relevant Euler Lagrange trajectory and this will be the global minimiser. After the onset of downward jumps, there will be a family of solutions to the Euler Lagrange equation with relevant boundary data. The downward jumps must evolve in such a way that the inviscid Burgers equation is satisfied.

Since the material in section \( \mathbf{2} \) is standard, the proofs are only sketched.

The Representation in terms of Euler Lagrange Trajectories

Consider the equation

\[
\begin{aligned}
\begin{cases}
    u_t + \frac{1}{2}(u^2)_x &= V_x \\
    u_0 &= \phi_x
\end{cases}
\end{aligned}
\]

where \( \phi \) is periodic and Lipschitz, and \( \phi_{xx} \) is bounded from above, and \( V \) is smooth and periodic, with all derivatives uniformly bounded. Let \( \theta \) satisfy

\[
\begin{aligned}
\dot{\theta} &= u(t, \theta) \\
\theta(0, x) &= x, \quad \dot{\theta}(0, x) = \phi_x(x).
\end{aligned}
\]

Let \( \dot{u} \) denote the function such that \( \dot{u}(t, x) = u_x(t, x) \) and let \( \ddot{u} \) denote the function such that \( \ddot{u}(t, x) = u_{tx}(t, x) \) and \( V' \) the function such that \( V'(t, x) = V_x(t, x) \), \( \phi'' \) such that \( \phi''(x) = \phi_{xx}(x) \) and \( V'' \) such that \( V''(t, x) = V_{xx}(t, x) \). In short, for a function \( f \in C^\infty(\mathbb{R} \times S^1) \), where \( S^1 \) is used to denote the circle \( [0, 2\pi] \) where 0 is identified with \( 2\pi \), \( \dot{f} \) denotes the derivative with respect to the first argument and \( f' \) denotes the derivative with respect to the second argument. If \( f \) is a function of only one variable, then either \( \dot{f} \) or \( f' \) will be used to denote the derivative.

Set \( u(t, x^-) := \lim_{y \to x^-} u(t, y) \) and \( u(t, x^+) = \lim_{y \to x^+} u(t, y) \). Then it follows that for all \( x \),

\[

u(t, x_-) \geq u(t, x^+) \quad \text{and, in particular}, \quad u(t, \theta(t, x^-)) \geq u(t, \theta(t, x^+)).

\]

Let \( \eta \) satisfy

\[
\begin{aligned}
\begin{cases}
    \ddot{\eta} &= V'(t, \eta) \\
    \eta(0, x) &= x, \quad \dot{\eta}(0, x) = \phi_x(x).
\end{cases}
\end{aligned}
\]
Lemma 1. Let $u$ satisfy equation (9). Suppose that $\sup_{s,x} |V'(s,x)| + \sup_{s,x} |V''(s,x)| < +\infty$ and $\sup_x \phi''(x) < +\infty$ and $\sup_x |\phi'(x)| < +\infty$. Then

$$\sup_{0 \leq s \leq t} \sup_x u_x(s,x) < C(t) < +\infty,$$

where

$$C(t) = \sup_x \phi''(x) + t \sup_{s,x} V''(s,x). \tag{12}$$

That is, there is an upper bound on the derivative.

Proof Set $w = u_x$. Then $w$ satisfies

$$\begin{cases} w_t + w^2 + uw_x = V_{xx} \\ w(0, x) = \phi_{xx}. \end{cases}$$

Let $\tilde{\theta}$ satisfy

$$\begin{cases} \dot{\tilde{\theta}}(t)(s, x) = -u(t - s, \tilde{\theta}(t)(s, x)) & s \geq 0 \\ \tilde{\theta}(t)(0, x) = x \end{cases} \tag{13}$$

It is clear that

$$w(t, x) = \phi''(\tilde{\theta}(t)(t, x)) + \int_0^t V''(s, \tilde{\theta}(t)(t - s, x))ds - \int_0^t w^2(s, \tilde{\theta}(t)(t - s, x))ds.$$

From the hypotheses on $\phi$ and $V$, it follows that for any $t < +\infty$,

$$\sup_{0 \leq s \leq t} \sup_{x \in [0, 2\pi)} w(s, x) \leq C(t) < +\infty$$

and the result follows. \qed

Since the derivative is bounded from above, there can be no ‘upward jumps’ in a solution; any discontinuities have to be ‘downward jumps’. The function of the next lemma is to show that when Euler Lagrange trajectories are being used to construct the solution, a trajectory is used until it enters a downward jump. After this, it is no longer used in the construction.

Lemma 2. Recall that $S^1$ denotes the circle $[0, 2\pi)$, with the identification $0 = 2\pi$. Set

$$S(t) = \{x \in S^1 | \theta_x(t, x) = 0 \}.$$

Then, for all $(s, t)$ such that $s \leq t$, $S(s) \subseteq S(t)$. 

7
Proof Recall the notation \( w = u_x \) and set

\[
f(t, x) = w(t, \theta(t, x)).
\]

Recall that \( \dot{\theta} = u(t, \theta) \). Since \( \theta(0, x) = x \), it follows that \( \theta_x(0, x) \equiv 1 \), so that, directly from equation (10), together with the definition of \( f \),

\[
\begin{align*}
\dot{\theta}_x &= f(t, x)\theta_x, \\
\theta_x(0, x) &\equiv 1,
\end{align*}
\]

for all \( t \geq 0 \) and all \( x \in \mathbb{S}^1 \). Set \( \sigma(x) = \inf\{t|\theta_x(t, x) = 0\} \). It now follows directly from the upper bound in lemma (1) that \( \theta_x(t, x) \equiv 0 \) for all \( t > \sigma(t, x) \). Furthermore, for all \( 0 \leq s \leq t < +\infty \),

\[
0 \leq \theta_x(t, x) = \theta_x(s, x) \exp \left\{ \int_s^t f(r, x) \, dr \right\} < \theta_x(s, x) \exp\{(t - s)C(t)\} < +\infty,
\]

from which the result follows directly.

For all \( t > \sigma(x) \), \( \theta_x(t, x) = 0 \). Recall that \( S(t) = \{x|\theta_x(t, x) = 0\} \) and set \( D(t) = \mathbb{S}^1 \setminus S(t) \). For \( y \in \theta(t, D(t)) \), note that

\[
u(t, y) = \dot{\theta}(t, \theta^{-1}(t, y))
\]

and note that \( |\theta(t, D(t))| := \int_{D(t)} \theta_x(t, x) \, dx = 2\pi \). The sets \( D(t) \) and \( \theta(t, D(t)) \) are open. For \( 0 \leq t \leq \sigma(x) \), \( \theta \) satisfies

\[
\dot{\theta}(t, x) = \frac{d}{dt} u(t, \theta(t, x)) = \ddot{u}(t, \theta(t, x)) + \dot{\theta}(t, x)u'(t, \theta(t, x)) = V(t, \theta(t, x)).
\]

It follows that \( \theta(t, x) = \eta(t, x) \) for all \( 0 \leq s \leq t \) where \( \eta \) satisfies equation (11). It follows that, for \( x \in \theta(t, D(t)) \),

\[
u(t, x) = \dot{\eta}(t, \eta^{-1}(t, x)),
\]

where \( \eta \) satisfies equation (11). Furthermore, for \( x \in \theta(t, D(t)) \), the inverse \( \eta^{-1}(t, x) \) is uniquely defined. A cdlg version of the solution is given by equation (16) for \( x \in \theta(t, D(t)) \) and \( u(t, x) = \lim_{y \uparrow x, y \in \theta(t, D(t))} u(t, y) \) for \( x \in [0, 2\pi) \). Therefore, a trajectory \( \eta(t, x) \) which solves equation (11) is used in the construction for all \( t \in (0, \sigma(x)) \), but is not used for any \( t > \sigma(x) \).

The following computation shows how the downward jumps evolve. After the ‘onset of downward jumps’; namely, for \( t > T \) where \( T = \inf_x \sigma(x) \), the set \( S(x) = \{y|\eta(t, y) = x\} \) may contain more than one element. But the choice of trajectories that may be used in the construction of the solution is not arbitrary. The equation (11) determines how the downward jump sites must evolve.
The Onset of Downward Jumps  Consider the Inviscid Burgers’ Equation (equation (4)). Suppose that \( \phi \) is \( 2\pi \) periodic and \( V \) is \( 2\pi \) periodic in both variables, with \( \int_0^{2\pi} \phi(x)dx = 0 \) and \( \int_0^{2\pi} V(t,x)dx = 0 \) for each \( t \geq 0 \). Then it is straightforward to compute that

\[
u(t,x) = \phi_x(\tilde{\theta}^{(t)}(t,x)) + \int_0^t V'(s,\tilde{\theta}^{(t)}(t-s,x))ds,
\]

where \( \tilde{\theta} \) satisfies equation (13). It follows that

\[
sup_{0 \leq s \leq t} sup_{0 \leq x \leq 2\pi} |\nu(t,x)| \leq sup_x |\phi_x(x)| + t sup_{0 \leq s \leq t} sup_{0 \leq x \leq 2\pi} |V_x(s,x)|.
\]

Suppose that \( u(\cdot,\cdot) \) has a ‘downward jump’ at site \( \theta(t) \). The following analysis shows how the position of the downward jump evolves in time. Since \( u \in L^\infty \), consider integration against test functions \( \psi \in L^1 \).

\[
\int_0^\infty \int_\mathbb{R} \psi(s,y)u_s(s,y)dyds + \int_0^\infty \int_\mathbb{R} \frac{1}{2}\psi(s,y)(u^2)_x(s,y)dyds = \int_0^\infty \int_\mathbb{R} \psi(s,y)V_x(s,y)dyds.
\]  

(17)

Lemma 3 (Downward Jump Evolution). If a downward jump develops, found at site \( \theta(t) \) at time \( t \), then \( \theta(t) \) evolves according to the equation

\[
\dot{\theta}(t) = \frac{u(t,\theta(t)+) + u(t,\theta(t)-)}{2}.
\]  

(18)

Proof (sketch) of lemma (3) The calculation leading to formula (18) is now outlined. Assume that a downward jump in the space variable develops at site \( \theta(t) \), time \( t \) and continues along trajectory \( \theta(s) \) for \( s \geq t \). Consider test functions \( \psi^{(\delta)} \in C^\infty \), such that \( \psi^{(\delta)}(s,x) = 0 \) for all \( s < t \), \( sup_{0<\delta<1} sup_{(s,x) \in \mathbb{R} \times \mathbb{R}} |\psi^{(\delta)}(s,x)| < K < +\infty \) for some \( K \) and with support within a tube of radius \( \delta \) around the graph \( (s,\theta(s))_{s\geq t} \). Suppose, furthermore, that \( \psi^{(\delta)} \) are chosen in such a way that there exists a function \( f \in C^\infty(\mathbb{R}) \) such that \( f(s) = 0 \) for \( s \leq t \) and \( \psi^{(\delta)}(s,\theta(s)) = f(s) \) for all \( \delta > 0 \) and \( s \geq t \). Then, from equation (17) it follows that

\[
\int f(s(\theta))(u(s(\theta),\theta-)-u(s(\theta),\theta+))d\theta + \frac{1}{2} \int f(s)(u^2(s,\theta(s)-)-u^2(s,\theta(s)+))ds = 0
\]

so that, for any test function \( f \in C^\infty(\mathbb{R}) \),
\[
\int f(s)(u(s, \theta(s) -) - u(s, \theta(s) +))\dot{\theta}(s)ds
\]

\[
+ \frac{1}{2} \int f(s)(u^2(s, \theta(s) +) - u^2(s, \theta(s) -))ds = 0,
\]

Equation (18) follows, thus proving lemma (3). \qed

3 The Large Deviations and Calculus of Variations Arguments

Suppose \( V \in C^\infty(\mathbb{R}_+ \times \mathbb{R}) \) is smooth, bounded and periodic in both variables. This section outlines the standard and well known arguments to show that the viscosity solution to the inviscid Burgers equation may be represented in terms of the velocity of solutions to the associated Euler Lagrange equations, that these solutions are the critical points of the associated action functional and that they are minimising trajectories of the action functional.

Let \( U^{(\epsilon)}(t, x; \phi) \) denote the solution to the equation

\[
\begin{cases}
U_t^{(\epsilon)} = \frac{\epsilon}{2} U_{xx}^{(\epsilon)} - \frac{1}{\epsilon} U^{(\epsilon)} V \\
U^{(\epsilon)}(0, x; \phi) = \exp \left\{ -\frac{1}{\epsilon} \phi(x) \right\},
\end{cases}
\]

where \( \phi \) is a bounded Lipschitz function. Set \( u^{(\epsilon)} = -\epsilon \frac{\partial}{\partial x} \log U^{(\epsilon)} \), then \( u^{(\epsilon)} \) satisfies

\[
\begin{cases}
u_t^{(\epsilon)} + \frac{1}{2} \nu_x^{(\epsilon)^2} = \frac{\epsilon}{2} u_{xx}^{(\epsilon)} + V_x \\
u(0, \cdot) = \phi_x.
\end{cases}
\]

A weak limit \( u \) in \( L^2 \) of \( (u^{(\epsilon)})_{\epsilon > 0} \) will provide a ‘viscosity’ solution to the equation

\[
\begin{cases}
u_t + \frac{1}{2} \nu_x^2 = V_x \\
u(0, \cdot) = \phi_x.
\end{cases}
\]

Writing out the Feynman-Kac representation of the solution to equation (19), using \( E_P \) as expectation over standard Brownian motion with initial condition \( w_0 = 0 \) gives

\[
U^{(\epsilon)}(t, x) = E_P \left[ \exp \left\{ -\frac{1}{\epsilon} \left( \int_0^t V(t - s, \sqrt{\epsilon} w_s + x) ds + \phi(\sqrt{\epsilon} w_t + x) \right) \right\} \right].
\]

The following presents a special case of Varadhan’s theorem.

10
Theorem 1 (Varadhan). Suppose that $V$ is smooth and bounded and that $\phi$ is Lipschitz and bounded. Let $\mathcal{S}_n$ denote the space of functions $\xi \in C^1(\mathbb{R})$ such that the derivative $\dot{\xi}$ satisfies

$$
\dot{\xi} = \sum_{k=1}^{2^n+1} \lambda_{n,k} \chi_{\left[\frac{(k-1)t}{2^n}, \frac{kt}{2^n}\right)},
$$

where $(\lambda_{n,k})_{k=1}^{2^n+1}$ is a collection of real numbers. Let $\mathcal{S} = \cup_n \mathcal{S}_n$. Then

$$
\lim_{\epsilon \to 0} -\epsilon \log U^{(\epsilon)}(t,x) = \inf_{\xi \in \mathcal{S} : \xi(t) = x} \left\{ \frac{1}{2} \int_0^t \dot{\xi}^2(s)ds + \int_0^t V(s, \xi(s))ds + \phi(\xi(0)) \right\}.
$$

Proof The proof of this basically follows Dembo and Zeitouni [5], with appropriate simplifications because only a special case of their results is required here. The proof is carried out in steps. Firstly, set $t^{(n)}_k = \frac{k}{2^n} t$ and let $(Z_j)_{j=1}^{2^n}$ denote independent random variables, each with distribution $N(0, \frac{\epsilon t}{2^n})$ (normal, expected value 0 and variance $\frac{\epsilon t}{2^n}$). Set $k(s) = \left[\frac{2^n s}{t}\right]$, where $[.]$ denotes the integer part, so that

$$
k(s) = \sup \left\{ k \in \mathbb{Z} : \frac{kt}{2^n} < s \right\}.
$$

For $z \in \mathbb{R}^{2^n}$, set

$$
Y^{(n)}(s,z) = x + \sum_{j=1}^{k(s)} z_j + \left( \frac{2^n s}{t} - k(s) \right) z_{k(s)+1} \quad 0 \leq s \leq t. \tag{22}
$$

Let $Q^{(\epsilon,n)}$ denote the probability measure with respect to $Z = (Z_j)_{j=1}^{2^n}$. Then, using $E^{(n, \epsilon)}$ to denote expectation with respect to the probability measure $Q^{(\epsilon,n)}$, set

$$
U^{(\epsilon,n)}(t,x) = E^{(\epsilon,n)} \left[ \exp \left\{ -\frac{1}{\epsilon} \left( \phi(Y^{(n)}(t,Z)) + \int_0^t V(t - s, Y^{(n)}(s,Z))ds \right) \right\} \right]. \tag{23}
$$

For $\lambda \in \mathbb{R}^{2^n}$, set

$$
\Lambda^{(n)}(\lambda) = \epsilon \log E^{(\epsilon,n)} \left[ e^{\frac{1}{\epsilon} \sum_{j=1}^{2^n} \lambda_j Z_j} \right] = \frac{1}{2} \sum_{j=1}^{2^n} \frac{t}{2^n} \lambda_j^2.
$$

Let $\Lambda^{(n)}$ denote the Fenchel-Legendre transform of $\Lambda^{(n)}$; namely, for $x \in \mathbb{R}^{2^n}$, set

$$
\Lambda^{(n)}(x) = \sup_{\lambda \in \mathbb{R}^{2^n}} \left\{ \sum_{j=1}^{2^n} \lambda_j x_j - \Lambda^{(n)}(\lambda) \right\} \tag{24}
$$

and note that
\[ \Lambda^{(n)*}(x) = \frac{2^n}{2t} \sum_{j=1}^{2^n} x_j^2. \]  

The following result is a simplified version of the Grtner - Ellis theorem, following the presentation in [5].

**Theorem 2** (Grtner - Ellis). For any closed set \( F \subset \mathbb{R}^{2n} \),

\[
\limsup_{\epsilon \to 0} \epsilon \log Q^{(\epsilon, n)} \{ F \} \leq - \inf_{x \in F} \Lambda^{(n)*}(x) \tag{26}
\]

For any open set \( G \subset \mathbb{R}^{2n} \)

\[
\liminf_{\epsilon \to 0} \epsilon \log Q^{(\epsilon, n)} \{ G \} \geq - \inf_{x \in G} \Lambda^{(n)*}(x). \tag{27}
\]

**Proof of Theorem (2)** Part 1: Upper bound.

Consider any closed and bounded set \( F \subset \mathbb{R}^{2n} \).

Let \( \chi_A \) denote the indicator function of a set \( A \). Recall that \( Z_j \) are independent, identically distributed \( N(0, \frac{et}{2^nt}) \) random variables, so that

\[
E[\epsilon^{\epsilon Z_j}] = \int_{-\infty}^{\infty} \frac{2^{n/2}}{\sqrt{2\pi et}} \exp\left\{ -\frac{2^{n-1}x^2}{et} + \epsilon px \right\} dx = \exp\left\{ \frac{p^2t}{2^{n+1}} \right\}.
\]

Then, because the \( (Z_j)_{j=1}^{2^n} \) are independent,

\[
Q^{(\epsilon, n)} \{ F \} = E^{(\epsilon, n)}[\chi_F(Z)] \leq E^{(\epsilon, n)} \left[ \exp \left\{ \sum_{j=1}^{2^n} \lambda_j Z_j - \inf_{x \in F} \sum_{j=1}^{2^n} \lambda_j x_j \right\} \right]
\]

\[
= \exp \left\{ \frac{et}{2^{n+1}} \sum_{j=1}^{2^n} \lambda_j^2 - \inf_{x \in F} \sum_{j=1}^{2^n} \lambda_j x_j \right\}.
\]

Since \( F \) is closed and bounded, there is a point \( \tilde{x} \in F \) at which the infimum is obtained. Then

\[
\epsilon \log Q^{(\epsilon, n)} \{ F \} \leq \frac{\epsilon^2t}{2^{n+1}} \sum_{j=1}^{2^n} \lambda_j^2 - \epsilon \sum_{j=1}^{2^n} \lambda_j \tilde{x}_j.
\]

The inequality holds for all \( \lambda \) and, in particular, for \( \lambda = \frac{2^n}{et} \tilde{x} \). It follows from equation (25) that for any closed bounded set \( F \),

\[
\epsilon \log Q^{(\epsilon, n)} \{ F \} \leq \frac{2^n}{2t} \sum_{j=1}^{2^n} \tilde{x}_j^2 = - \inf_{x \in F} \Lambda^{(n)*}(x). \tag{28}
\]
Now the result is extended to arbitrary closed sets. Consider a closed set \( F \) and let \( \tilde{x} \) denote a point such that \( \Lambda^{(n)*}(\tilde{x}) = \inf_{x \in F} \Lambda^{(n)*}(x) \). It is easy to see that such a point exists, because \( F \) is closed and \( \Lambda^{(n)*} \) is quadratic and convex. Choose a \( \delta > 0 \) and choose a \( \rho \) such that \( \frac{2^n \rho^2}{2t} > \Lambda^{(n)*}(\tilde{x}) + \delta \).

By Chebychev’s inequality, for all \( \alpha > 0 \),

\[
Q^{(\epsilon,n)} \{ Z_j < -\rho \} = Q^{(\epsilon,n)} \{ -Z_j > -\rho \} \leq e^{-\alpha \rho} E \left[ e^{-\alpha Z_j} \right] = \exp \left\{ -\alpha \rho + \frac{\alpha^2}{2} \frac{t}{2^n} \right\},
\]

yielding

\[
Q^{(\epsilon,n)} \{ Z_j < -\rho \} = Q^{(\epsilon,n)} \{ Z_j > \rho \} \leq \exp \left\{ -\frac{2^n \rho^2}{2\epsilon t} \right\},
\]

It follows that, for \( \rho > 0 \),

\[
Q^{(n,\epsilon)} \{ \mathbb{R}^{2n} \setminus [-\rho,\rho]^{2n} \} \leq 2^{n+1} \exp \left\{ -\frac{2^n \rho^2}{2\epsilon t} \right\},
\]

so that

\[
\limsup_{\epsilon \to 0} \epsilon \log Q^{(n,\epsilon)} \{ \mathbb{R}^{2n} \setminus [-\rho,\rho]^{2n} \} \leq -\frac{2^n \rho^2}{2t} \leq -\inf_{x \in F} \Lambda^{(n)*}(x) - \delta.
\]

Equation (28), holds for all closed bounded sets \( F \). This yields

\[
\limsup_{\epsilon \to 0} \epsilon \log Q^{(n,\epsilon)} \{ F \cap [-\rho,\rho]^{2n} \} \leq -\inf_{x \in F} \Lambda^{(n)*}(x).
\]

Choose \( \rho \) such that \( \frac{2^n \rho^2}{2t} > \Lambda^{(n)*}(\tilde{x}) + \delta \). Since

\[
\lim_{\epsilon \to 0} \epsilon \log Q^{(\epsilon,n)} \{ \mathbb{R}^{2n} \setminus [-\rho,\rho]^{2n} \} \leq -\inf_{x \in F} \Lambda^{(n)*}(x),
\]

it therefore follows that

\[
\limsup_{\epsilon \to 0} \epsilon \log Q^{(\epsilon,n)} \{ F \} = \limsup_{\epsilon \to 0} \epsilon \log \left( Q^{(\epsilon,n)} \{ F \cap [-\rho,\rho]^{2n} \} + Q^{(\epsilon,n)} \{ F \setminus [-\rho,\rho]^{2n} \} \right)
\leq \limsup_{\epsilon \to 0} \epsilon \log \left( Q^{(\epsilon,n)} \{ F \cap [-\rho,\rho]^{2n} \} + Q^{(\epsilon,n)} \{ \mathbb{R}^{2n} \setminus [-\rho,\rho]^{2n} \} \right)
\leq -\inf_{x \in F} \Lambda^{(n)*}(x).
\]

The upper bound of theorem (2), given in equation (26), for arbitrary closed sets, has therefore been established.
For the lower bound advertised in equation (27), consider an open set $G \subset \mathbb{R}^{2n}$. For any point $y \in G$, there exists a $\delta(y) > 0$ and a ball $B_{y,\delta(y)}$ such that $B_{y,\delta(y)} \subset G$. It is therefore sufficient to prove that for all $y \in \mathbb{R}^{2n}$,

$$\lim_{\delta \to 0} \left( \lim_{\epsilon \to 0} \epsilon \log Q^{(\epsilon, n)}(B_{y, \delta}) \right) \geq \frac{2n}{2t} \sum_{j=1}^{2n} y_j^2,$$

Let $C(n)$ denote the volume of the unit ball in $\mathbb{R}^{2n}$. Then

$$\epsilon \log Q^{(\epsilon, n)}(B_{y, \delta}) = \epsilon \log \int_{B_{y, \delta}} \frac{2^{2n-1}n}{(2\pi t)^{2n-1}} \exp \left\{ -\frac{2n-1}{\epsilon t} \sum_{j=1}^{2n} z_j^2 \right\} dz \geq \epsilon (2n-1) \log 2 - 2n-1 \log(2\pi \epsilon) + \epsilon \log(C(n)\delta^{2n}) - \inf_{z \in B_{y, \delta}} \frac{2n-1|z|^2}{t},$$

yielding

$$\lim_{\delta \to 0} \left( \lim_{\epsilon \to 0} \epsilon \log Q^{(\epsilon, n)}(B_{y, \delta}) \right) \geq -\frac{2n}{2t} \sum_{j=1}^{2n} y_j^2.$$

The lower bound follows directly. The proof of theorem (2) is therefore complete.

The approach to proving theorem (1) is firstly to prove a discrete version, given by proposition (1) and then take a limit, which is the subject of lemma (4). Theorem (1) then follows from proposition (1) followed by lemma (4).

**Proposition 1 (The Laplace Method).** Let $V$ be smooth and bounded. Let $Y^{(n)}$ be defined as in equation (22) and note that $\dot{Y}^{(n)}(s, z) = \frac{2n}{t} z_{k(s)+1}$, where $\dot{Y}^{(n)}$ denotes derivative of $Y^{(n)}$ with respect to $s$. Recall the definition of $A$; namely, $A : \mathbb{R}_+ \times W^{1,2}(\mathbb{R}_+) \to \mathbb{R}$, where

$$A(t; \xi) = \frac{1}{2} \int_0^t \xi^2(s) ds + \phi(\xi(0)) + \int_0^t V(s, \xi(s)) ds. \tag{29}$$

Note that for $z \in \mathbb{R}^{2n}$,

$$A(t; Y^{(n)}(t - ., z)) = \frac{1}{2} \sum_{j=1}^{2n} \frac{t}{2n} \left( \frac{2n z_j}{t} \right)^2 + \phi(Y^{(n)}(t, z)) + \int_0^t V(t-s, Y^{(n)}(s, z)) ds.$$

Then, with $U^{(\epsilon, n)}$ defined in equation (23),

$$\lim_{\epsilon \to 0} -\epsilon \log U^{(\epsilon, n)}(t, x) = \inf_{z \in \mathbb{R}^{2n}} A(t; Y^{(n)}(t - ., z)).$$
**Proof** To make the notation slightly more convenient, for \( z \in \mathbb{R}^{2n} \), the following is used:

\[
\tilde{A}(n)(z) := A(t; Y^{(n)}(t - ., z)).
\]

Step 1: Upper bound. For any open set \( G \subset \mathbb{R}^{2n} \),

\[
U^{(\epsilon,n)}(t, x) \geq E^{(\epsilon,n)} \left[ e^{-\frac{1}{\epsilon}} (\phi(Y^{(n)}(t,Z)) + \int_0^t V(t-s,Y^{(n)}(s,Z))ds) \chi_G(Z_1, \ldots, Z_{2n}) \right],
\]

from which it follows directly, by theorem (2) equation (27), that

\[
-\liminf_{\epsilon \to 0} \epsilon \log U^{(\epsilon,n)}(t, x) \leq \inf_{z \in G} \tilde{A}(n)(z).
\]

Set \( m = \inf_{y \in \mathbb{R}^{2n}} \tilde{A}(n)(y) \) and, for \( \delta > 0 \), set \( G^\delta = \{ z | \tilde{A}(n)(z) - m < \delta \} \). Using the continuity of \( A \), it is easy to see that \( G^\delta \) is an open subset of \( \mathbb{R}^{2n} \). By taking \( G = G^\delta \) in equation (27) from theorem (2) and letting \( \delta \to 0 \), it follows that

\[
-\liminf_{\epsilon \to 0} \epsilon \log U^{(\epsilon,n)}(t, x) \leq \inf_{z \in \mathbb{R}^{2n}} A(z).
\]

Part 2: Lower bound. Recall that \( m = \inf_{y \in \mathbb{R}^{2n}} A(y) \). Set

\[
M = \sup_x |\phi(x)| + \sup_{x,s} t|V(s, x)|.
\]

Set

\[
F(z) = \phi(Y^{(n)}(t, z)) + \int_0^t V(t-s,Y^{(n)}(s, z))ds.
\]

Set

\[
A = \{ y | A^{(n)*}(y) \geq m + M + 1 \}.
\]

Fix a \( \delta > 0 \) and, for \( 0 \leq j \leq \lfloor (M + 1)/\delta \rfloor + 1 \), set

\[
A^{(\delta)}_j = \{ y | m + j\delta \leq A^{(n)*}(y) \leq m + (j + 1)\delta \}.
\]

Set \( N = \lfloor (M + 1)/\delta \rfloor + 1 \). Note that \( A^{(\delta)}_0, \ldots, A^{(\delta)}_N, A \) are closed sets and that \( \mathbb{R}^{2n} = \bigcup_{j=0}^N A^{(\delta)}_j \cup A \).

It follows that
\[ U^{(\epsilon,n)}(t,x) \leq \sum_{j=0}^{N} E^{(n,\epsilon)}[e^{-\frac{1}{\epsilon} F(Z) \chi_{A_j^{(\beta)}}(Z)}] + E[e^{-\frac{1}{\epsilon} F(Z) \chi_A(Z)}]. \]

Now, since \( A_j^{(\delta)} \) is closed for each \( j \in \{0, 1, \ldots, N\} \), it follows from the upper bound given by equation (26) in theorem (2) that

\[ -\lim_{\epsilon \to 0} \epsilon \log E^{(n,\epsilon)}[e^{-\frac{1}{\epsilon} F(Z) \chi_{A_j^{(\beta)}}(Z)}] \geq \inf_{x \in A_j^{(\beta)}} F(x) + j\delta \geq \inf_{x \in A_j^{(\beta)}} A(x) - \delta \geq m - \delta. \]

Furthermore, since \( A \) is closed, it follows from the upper bound given by equation (26) from theorem (2) that

\[ -\lim_{\epsilon \to +\infty} \epsilon \log E[e^{-\frac{1}{\epsilon} F(Z) \chi_A(Z)}] \geq -M + M + m + 1 = m + 1. \]

Since \( A_0^{(\delta)}, \ldots, A_N^{(\delta)}, A \) is a finite collection of sets, it follows directly that

\[ \liminf_{\epsilon \to +\infty} -\epsilon \log U^{(\epsilon,n)}(t,x) \geq m - \delta \]

for all \( \delta > 0 \) and hence that

\[ \liminf_{\epsilon \to +\infty} -\epsilon \log U^{(\epsilon,n)}(t,x) \geq m. \]

The proof of proposition (11) is complete. \( \Box \)

Recall that

\[ S^{(n)} = \{ y \in C([0,t]) | \dot{y} = z, \frac{tk}{2n} < s < \frac{t(k+1)}{2n}, z_k \in \mathbb{R} \}. \] (30)

Note that proposition (11) has proved that

\[ -\lim_{\epsilon \to 0} \epsilon \log U^{(\epsilon,n)}(t,x) = \inf_{y \in S^{(n)} | y(t) = x} \left\{ 1/2 \int_0^t \dot{y}^2(s)ds + \int_0^t V(s, y(s))ds + \phi(y(0)) \right\}. \] (31)

The aim is to let \( n \to +\infty \).

Lemma 4.

\[ \lim_{\epsilon \to 0} -\epsilon \log U^{(\epsilon)}(t,x) = \lim_{n \to +\infty} \left( \lim_{\epsilon \to 0} -\epsilon \log U^{(\epsilon,n)}(t,x) \right). \]

**Proof** Let \( w^{(\epsilon)} = \sqrt{\epsilon} w \); namely, a one dimensional Brownian motion with diffusion coefficient \( \epsilon \), with \( w^{(\epsilon)}(0) \equiv 0 \). For \( j = 1, \ldots 2^n \), set

\[ Z_j = w^{(\epsilon)} \left( \frac{tj}{2^n} \right) - w^{(\epsilon)} \left( \frac{t(j-1)}{2^n} \right). \] (32)
Note that $Z_j$ are independent identically distributed random variables, each with distribution $N(0, \frac{t}{n\epsilon^2})$. Recall the notation $Z = (Z_j)_{j=1}^n$. Note that $x + w^{(\epsilon)}(\frac{tk}{n\epsilon}) = Y^{(n)}(\frac{tk}{n\epsilon}, Z)$, where $Y^{(n)}$ is defined by equation (22). For $0 \leq s \leq t$, set $\tilde{w}^{(\epsilon)}(s) = x + w^{(\epsilon)}(s)$. Set

$$F(y) = \int_0^t V(t-s, y(s))ds + \phi(y(t)).$$

Then, using $E$ to denote expectation with respect to the Brownian motion $w^{(\epsilon)}$, and using $Z$ as in equation (32), it follows by Hölder’s inequality that

$$\epsilon \log E \left[ \exp \left\{ -\frac{1}{\epsilon} \left( \int_0^t V(t-s, w^{(\epsilon)}(s) + x)ds + \phi(x + w^{(\epsilon)}(t)) \right) \right\} \right]$$

$$= \epsilon \log E \left[ \exp \left\{ -\frac{1}{\epsilon} F(\tilde{w}^{(\epsilon)}) \right\} \right]$$

$$\leq \frac{cn}{n+1} \log E \left[ \exp \left\{ -\frac{n+1}{n\epsilon} F(Y^{(n)}(., Z)) \right\} \right]$$

$$+ \frac{\epsilon}{n+1} \log E \left[ \exp \left\{ -\frac{n+1}{\epsilon} \left( F(\tilde{w}^{(\epsilon)}) - F(Y^{(n)}(., Z)) \right) \right\} \right].$$

(33)

Similarly, by Hölder’s inequality, it follows that

$$\epsilon \log E \left[ \exp \left\{ -\frac{n}{\epsilon(n+1)} F(Y^{(n)}(., Z)) \right\} \right] \leq \frac{cn}{n+1} \log E \left[ \exp \left\{ -\frac{1}{\epsilon} F(\tilde{w}^{(\epsilon)}) \right\} \right]$$

$$+ \frac{\epsilon}{n+1} \log E \left[ \exp \left\{ -\frac{n}{\epsilon} \left( F(Y^{(n)}(s, Z)) - F(\tilde{w}^{(\epsilon)}) \right) \right\} \right].$$

(34)

Now, set $\tilde{C} = t \sup_x \sup_s |V(s, x)| + \sup_x |\phi(x)|$ and note that

$$\epsilon \log E \left[ \exp \left\{ -\frac{n}{\epsilon(n+1)} F(Y^{(n)}(., Z)) \right\} \right] \geq \epsilon \log E \left[ \exp \left\{ -\frac{1}{\epsilon} F(Y^{(n)}(., Z)) \right\} \right] - \frac{\tilde{C}}{n+1}$$

$$= \epsilon \log U^{(\epsilon,n)}(t, x) - \frac{\tilde{C}}{n+1}$$

(35)

and

$$\epsilon \log E \left[ \exp \left\{ -\frac{(n+1)}{\epsilon n} F(Y^{(n)}(., Z)) \right\} \right] \geq \epsilon \log E \left[ \exp \left\{ -\frac{1}{\epsilon} F(Y^{(n)}(., Z)) \right\} \right] + \frac{\tilde{C}}{n}$$

$$= \epsilon \log U^{(\epsilon,n)}(t, x) + \frac{\tilde{C}}{n}.$$  

(36)

Set

$$I_1 = \frac{\epsilon}{n+1} \log E \left[ \exp \left\{ -\frac{n+1}{\epsilon} (F(\tilde{w}^{(\epsilon)}) - F(Y^{(n)}(., Z))) \right\} \right]$$

17
and

\[ I_2 = \frac{\epsilon}{n} \log E \left[ \exp \left\{ -\frac{n}{\epsilon} \left( F(Y^{(n)}(.,Z)) - F(\tilde{w}^{(\epsilon)}) \right) \right\} \right]. \]

Note that inequalities (33) and (34), using inequalities (35) and (36) now yield

\[
\frac{\epsilon n + 1}{n} \log U^{(\epsilon,n)}(t,x) - \frac{1}{n} \hat{C} - I_2 \\
\leq \epsilon \log U^{(\epsilon)}(t,x) \leq \frac{\epsilon n}{n + 1} \log U^{(n,\epsilon)}(t,x) + \frac{1}{n + 1} \hat{C} + I_1.
\]

(37)

Now, set \( C = \sup_t \sup_x |V_x(t,x)| \) and recall that \( C < +\infty \) by the hypotheses on \( V \). Set

\[
X_k = \sup_{0 \leq r \leq 1} \left| (1 - r)w^{(\epsilon)}(\frac{kt}{2^n}) + r\tilde{w}^{(\epsilon)}(\frac{(k + 1)t}{2^n}) - w^{(\epsilon)}(\frac{(k + r)t}{2^n}) \right|.
\]

Set

\[
\eta_k(r) := w^{(\epsilon)}(\frac{(k + r)t}{2^n}) - w^{(\epsilon)}(\frac{kt}{2^n}),
\]

(38)

then

\[
X_k = \sup_{0 \leq r \leq 1} |r\eta_k(1) - \eta_k(r)|.
\]

It is clear, from the basic property of Brownian motion that increments over disjoint time intervals are independent, that \( X_k \) are independent and identically distributed. Since \( x + w^{(\epsilon)}(t) = Y^{(n)}(t,Z) \), Taylor’s expansion theorem, together with the fact that \( \tilde{w}^{(\epsilon)}(\frac{kt}{2^n}) = Y^{(n)}(\frac{kt}{2^n},Z) \) for all \( 0 \leq k \leq 2^n \) yields

\[
|F(Y^{(n)}(.,Z)) - F(\tilde{w}^{(\epsilon)})| \leq C \sum_{k=0}^{2^n-1} X_k,
\]

so that, for a random variable \( X \) with the same distribution as \( X_k \),

\[
\frac{\epsilon}{n + 1} \log E \left[ \exp \left\{ -\frac{n + 1}{\epsilon} \left( F(Y^{(n)}(.,Z)) - F(\tilde{w}^{(\epsilon)}) \right) \right\} \right] \\
\leq \frac{2^n \epsilon}{n + 1} \log E \left[ \exp \left\{ \frac{n + 1}{2^n \epsilon} CX \right\} \right].
\]

Now, using \( \eta_k(r) \) defined in equation (35), let

\[
\tilde{\eta} = \sup_{0 \leq r \leq 1} |\eta_k(r)|
\]

and note that \( 2\tilde{\eta} > X_k \). From Revuz and Yor [12] page 55 proposition 1.8,
Proposition 2. Let $\beta$ denote a standard Brownian motion with $\beta(0) = 0$ and let $S(t) = \sup_{0 \leq s \leq t} \beta(s)$. Let $P$ denote the probability measure under which $\beta$ is a standard Brownian motion. Then

$$P(S(t) \geq a) \leq \exp \left\{ -\frac{a^2}{2t} \right\}.$$ 

From equation (38), it follows that $\tilde{\eta} \overset{(d)}{=} S(\frac{\epsilon t}{2n})$. Using $Q$ to denote the probability measure with respect to the process $w^{(\epsilon)}$, it follows directly that

$$Q(\tilde{\eta} \geq a) \leq 2 \exp \left\{ -\frac{2n-1}{\epsilon t} a^2 \right\}.$$ 

Set $\gamma = \frac{2n-1}{\epsilon t}$. It follows that

$$E[\exp\{\alpha \tilde{\eta}\}] = \int_0^\infty Q(e^{\alpha \tilde{\eta}} \geq x) dx$$

$$= 1 + \int_1^\infty Q(\tilde{\eta} > \frac{\log x}{\alpha}) dx$$

$$= 1 + \int_0^\infty \alpha e^{\alpha y} Q(\tilde{\eta} > y) dy$$

$$\leq 1 + 2 \int_0^\infty \alpha e^{\alpha y} e^{-\gamma y^2} dy$$

$$\leq 1 + 2\alpha \sqrt{\frac{\pi}{\gamma}} e^{\alpha^2/4\gamma}.$$ 

Using the inequality $1 + ae^b \leq e^{a+b}$ for all $a \geq 0$ and all $b \in \mathbb{R}$, it follows that

$$\frac{2^ne}{n+1} \log E[\exp\{\frac{\alpha}{2n+2\epsilon} CX\}] \leq \frac{2^ne}{n+1} \log E[\exp\{\frac{\alpha}{2n+2\epsilon} CY\}]$$

$$\leq \frac{2^ne}{n+1} \log \left( 1 + \sqrt{(n+1)^2 C^2 \pi t} \right)$$

$$\leq \frac{C \sqrt{\pi \epsilon}}{2^{(n-5)/2}} + \frac{(n+1)C^2 t}{2^{2n-5}}.$$ 

It follows that

$$I_1 := \epsilon \log E \left[ \exp \left\{ \frac{-n+1}{\epsilon} \left( F(\tilde{w}^{(\epsilon)}) - F(Y^{(n)}(\cdot, Z)) \right) \right\} \right] \leq \frac{(n+1)C^2 t}{2^{2n-5}} + \frac{C \sqrt{\pi \epsilon}}{2^{(n-5)/2}}$$

and, similarly, that

$$I_2 := \frac{\epsilon}{n} \log E \left[ \exp \left\{ \frac{-n}{\epsilon} \left( F(Y^{(n)}(\cdot, Z)) - F(\tilde{w}^{(\epsilon)}) \right) \right\} \right] \leq \frac{nC^2 t}{2^{2n}} + \frac{C \sqrt{\pi \epsilon}}{2^{(n-5)/2}}.$$
Putting this into the inequalities (37) yields lemma (4) directly.

**Proof of theorem (1)** Recall that

\[
\lim_{\epsilon \to 0} -\epsilon \log E[\epsilon^{-\frac{1}{2}}(\int_0^t V(t-s,x+w(s))ds + \phi(x+w(t)))]
\]

\[
= \lim_{n \to +\infty} \left( \lim_{\epsilon \to 0} -\epsilon \log E[\epsilon^{-\frac{1}{2}}(\int_0^t V(t-s,Y^{(n,s)}(x,s))ds + \phi(Y^{(n,s)}(x,t)))] \right).
\]

Recall that

\[S_n = \left\{ y \in C([0,t]) \mid \exists (z_1, \ldots, z_{2^n}) \in \mathbb{R}^{2^n} : y(s) = y\left(\frac{kt}{2^n}\right) + \left(s - \frac{kt}{2^n}\right)z_{k+1}, \frac{kt}{2^n} \leq s \leq \frac{(k+1)t}{2^n} \right\},\]

and that

\[
\lim_{\epsilon \to 0} -\epsilon \log U^{(\epsilon,n)}(t,x) = \inf_{y \in S_n, y(t) = x} A(y),
\]

where

\[A(y) = \left\{ \frac{1}{2} \int_0^t \dot{y}^2(s)ds + \int_0^t V(t-s,y(s))ds + \phi(y(0)) \right\}.
\]

It follows directly from the fact that \(S_n \subset S_m\) for \(m > n\), together with the analysis given above, that

\[
\lim_{\epsilon \to 0} -\epsilon \log U^{(\epsilon)}(t,x) = \inf_{n} \inf_{y \in S_n} A(y) = \inf_{y \in S} A(y)
\]

and theorem (1) is proved.

It is now shown that, assuming Tychonov's theorem, hence relative weak compactness of the unit ball in \(L^2\), the minimiser exists. It is then shown that if the minimiser exists, then it satisfies the Euler Lagrange equations.

**Theorem 3** (Existence of the Minimiser). Consider the action functional

\[A(y) = \frac{1}{2} \int_0^t \dot{y}^2(s)ds + \int_0^t V(s,y(s))ds + \phi(y(0)).\]

Then, using the fact that a ball of finite radius in \(L^2\) is compact in the relative weak topology, there exists a trajectory \(\tilde{y}\) such that

\[A(\tilde{y}) = \inf_{y \in W^{1,2}([0,t]) \mid y(t) = x} A(y).
\]
Proof of theorem (3) Consider a sequence \((y_n)_{n=1}^{\infty}\) where each \(y_n \in W^{1,2}([0,t])\) and \(y_n(0) = x\), such that \(A(y_1) = C < +\infty\), such that \(A(y_n)\) is decreasing and such that

\[
\lim_{n \to +\infty} A(y_n) = \inf_{y \in W^{1,2}([0,t]) \mid y(0) = x} A(y).
\]

Consider the sequence \((\dot{y}_n)_{n=1}^{\infty}\), and take a subsequence \((\dot{y}_{n_k})_{k \geq 1}\) that is convergent to a limit \(\dot{y}\) in the relative weak topology. That is, for any test function \(g \in L^2([0,t]), \int_0^t g(s)\dot{y}_{n_k}(s)ds \xrightarrow{k \to +\infty} \int_0^t g(s)\dot{y}_k(s)ds\). Such a sequence exists, since

\[
\sup_n \frac{1}{2} \int_0^t \dot{y}_n^2(s)ds \leq C + \|V\|_{\infty}t + \|\phi\|_{\infty} < +\infty
\]  

(42)

and because any ball of finite radius in \(L^2\) is compact in the relative weak topology. Let \(\tilde{y}\) denote the function such that \(\tilde{y}(0) = x\), with derivative \(\dot{\tilde{y}}\). It follows by choosing test functions \(\chi_{[0,s]}\), which are clearly in \(L^2([0,t])\), that

\[
\lim_{k \to +\infty} |y_{n_k}(s) - \tilde{y}(s)| = 0 \quad \forall s \in [0,t]
\]

and, furthermore, using equation (42) and Holder’s inequality,

\[
\sup_k \sup_{0 \leq s \leq t} |y_{n_k}(s) - \tilde{y}(s)| \leq \int_0^t |\dot{y}_{n_k}(s) - \dot{\tilde{y}}(s)|ds \leq 2t^{1/2}(C + \|V\|_{\infty} + \|\phi\|_{\infty})^{1/2}.
\]

and hence, because \(V\) and \(\phi\) are smooth and uniformly bounded with uniformly bounded first derivatives, it follows by the dominated convergence theorem that

\[
\left| \left( \int_0^t V(t-s, y_{n_k}(s))ds + \phi(y_{n_k}(t)) \right) - \left( \int_0^t V(t-s, \tilde{y}(s))ds - \phi(\tilde{y}(t)) \right) \right|
\]

\[
\leq \|V\|_{\infty} \int_0^t |y_{n_k}(s) - \tilde{y}(s)|ds + \|\phi\|_{\infty} |y_{n_k}(t) - \tilde{y}(t)|
\]

\[
\xrightarrow{k \to +\infty} 0.
\]

Since \(\dot{y}_{n_k} \to \dot{y}\) in \(L^2\) with the relative weak topology, it follows from standard results that convergence is almost everywhere and hence, by Fatou’s lemma,

\[
\liminf_{k \to +\infty} \frac{1}{2} \int_0^t \dot{y}_{n_k}^2(s)ds \geq \frac{1}{2} \int_0^t \dot{\tilde{y}}^2(s)ds
\]

It therefore follows that

\[
\inf_{y \in W^{1,2}([0,t]) \mid y(t) = x} A(y) \geq A(\tilde{y})
\]

and hence that the trajectory \(\tilde{y}\) is a minimiser.
\[ A(\tilde{y}) = \inf_{y \in W^{1,2}([0,t]) | y(t) = x} A(y). \]

The proof of theorem (3) is complete. \hfill \Box

**Theorem 4.** For \( 0 \leq s \leq t \), the minimiser \( \tilde{y} \) in theorem (3) satisfies the Euler Lagrange equations

\[
\begin{aligned}
\dot{\tilde{y}}(s) &= V'(s, \tilde{y}(s)) \\
\tilde{y}(t) &= x, \\
\dot{\tilde{y}}(0) &= \phi(\tilde{y}(0)).
\end{aligned}
\]

**Proof of theorem (4)** The proof is sketched; in this case, the fact that the minimiser satisfies the Euler Lagrange equation is standard, following the arguments of Tonelli in [13, 14]. Consider any \( z \in W^{1,2}([0,t]) \) with \( z(t) = 0 \), so that \( \tilde{y} + \varepsilon z \in W^{1,2}([0,t]) \) for all \( \varepsilon \geq 0 \). Then

\[
\lim_{\varepsilon \to 0} \frac{A(\tilde{y} + \varepsilon z) - A(\tilde{y})}{\varepsilon} = \int_0^t \dot{z}(s)\ddot{\tilde{y}}(s)ds + \int_0^t z(s)V_x(s, \tilde{y}(s))ds + z(0)\phi'(\tilde{y}(0)).
\]

Since, for any \( z \in W^{1,2}([0,t]) \) with \( z(t) = 0 \), \( \lim_{\varepsilon \to 0} \frac{A(\tilde{y} + \varepsilon z) - A(\tilde{y})}{\varepsilon} = 0 \), it follows that for all \( z \in W^{1,2}([0,t]) \) with \( z(t) = 0 \),

\[
0 = z(0) \left( -\ddot{\tilde{y}}(0) + \phi'(\tilde{y}(0)) \right) - \int_0^t z(s) \left( \dddot{\tilde{y}}(s) - V_x(s, \tilde{y}(s)) \right) ds,
\]

from which theorem (4) follows directly. \hfill \Box

It now remains to identify \( u(t, x) = \dot{\eta}(t, \eta^{-1}(t, x)) \) where \( \eta \) solves equation (7) and minimises the action functional.

**Theorem 5.** Let \( V \in C^\infty(\mathbb{R}^2) \) be smooth and bounded and let \( \phi \in C^\infty(\mathbb{R}) \) be smooth and bounded. Let \( u^{(\varepsilon)} \) solve

\[
\begin{aligned}
u_t^{(\varepsilon)} + \frac{1}{2}(u^{(\varepsilon)})^2_x &= \frac{\varepsilon}{2}u^{(\varepsilon)}_{xx} + V_x \\
\varepsilon^{(\varepsilon)} &= \phi_x.
\end{aligned}
\]

Let \( u \) denote any weak in \( L^2 \) limit point of \((u^{(\varepsilon)})_{\varepsilon > 0}\). Then there is a representation of the solution \( u, u(t, x) = \dot{\xi}^{(t,x)}(t), \) where \( \xi \) is the trajectory which provides a minimum for the action

\[
A(t, \xi) := \frac{1}{2} \int_0^t \xi^2(s)ds + \int_0^t V(s, \xi(s))ds + \phi(\xi(0))
\]

subject to the constraint \( \xi^{(t,x)}(t) = x, \xi \in W^{1,2}([0,t]) \). For \( 0 \leq s \leq t \), this trajectory \( \xi \) satisfies the Euler Lagrange equation...
\[
\begin{aligned}
\xi(s) &= V_x(s, \xi(s)) \\
\xi(t) &= x, \quad \xi(0) = \phi_x(\xi(0)).
\end{aligned}
\]

(43)

**Proof of theorem (5)** It has already been shown that any limit point \( u \) has the representation

\[
u(t, x) = \frac{\partial}{\partial x} \inf_{\xi(t) = x} \left\{ \phi(\xi(0)) + \int_0^t \left( \frac{1}{2} \xi^2(s) + V(s, \xi(s)) \right) ds \right\}.
\]

Furthermore, it has been shown that the minimising trajectory \( \xi \) exists and satisfies the Euler Lagrange equations (43). The identification that \( u(t, x) = \dot{\xi}(t) \) where \( \xi \) is a minimizing trajectory subject to the constraint that \( \xi(t) = x \) is completed as follows: Let \((\psi(s, t, x))_{s=0}^t\) denote a trajectory that minimises

\[
\phi(\xi(0)) + \int_0^t \left( \frac{1}{2} \psi^2(s) + V(s, \xi(s)) \right) ds
\]

subject to the conditions \( \xi(t) = x \). Then, using the arguments of the proof of theorem (4), the variational calculus yields that \( \psi \) satisfies

\[
\psi_{ss}(s, t, x) = V'(s, \psi(s, t, x))
\]

where \( V' \) denotes (as usual) derivative of \( V \) with respect to the second argument of \( V \) and \( \psi_s(0, t, x) = \phi'(\psi(0, t, x)) \). Furthermore, \( \psi(t, t, x) \equiv x \), so that \( \psi_x(t, t, x) \equiv 1 \). It follows, using integration by parts, that

\[
\begin{aligned}
u(t, x) &= \frac{\partial}{\partial x} \left( \phi(\psi(0, t, x)) + \int_0^t \left( \frac{1}{2} \psi_s(s, t, x)^2 + V(s, \psi(s, t, x)) \right) ds \right) \\
&= \phi'(\psi(0, t, x)) \psi_x(0, t, x) + [\psi_x(s, t, x) \psi_s(s, t, x)]_{s=0}^t \\
&\quad - \int_0^t \psi_x(s, t, x)(\psi_{ss}(s, t, x) - V_x(s, t, x)) ds \\
&= \psi_x(t, t, x) \psi_s(t, t, x) + \psi_x(0, t, x) \left( \phi'(\psi(0, t, x)) - \psi_s(0, t, x) \right) \\
&= \psi_x(t, t, x),
\end{aligned}
\]

which is the advertised result. Theorem (5) is proved.

One final lemma is required to finish this section, which will be used in the sequel. In section (4), periodic solutions to the Burgers equation will be considered, therefore the initial condition will depend on \( \epsilon \).

**Lemma 5.** Let \( w^{(\epsilon)} \) denote a Brownian motion with diffusion \( \epsilon \), with \( w^{(\epsilon)}(0) = 0 \). Set

\[
U^{(\epsilon)}(t, x; \psi) = -\epsilon \log E \left[ \exp \left\{ -\frac{1}{\epsilon} \left( \psi(x + w^{(\epsilon)}(t)) + \int_0^t V(t - s, x + w^{(\epsilon)}(s)) ds \right) \right\} \right].
\]

23
Let \((\phi^{(\epsilon)})_{\epsilon>0}\) denote a family of functions satisfying \(\sup_{x} \sup_{0<\epsilon \leq 1} |\phi^{(\epsilon)}(x)| < C < +\infty\) for some constant \(C\), and such that

\[
\lim_{\epsilon \to 0} \sup_{x} |\phi^{(\epsilon)}(x) - \phi(x)| = 0.
\]

Then, for any \(T < +\infty\),

\[
\lim_{\epsilon \to 0} \sup_{x} \sup_{0 \leq t \leq T} |\epsilon \log U^{(\epsilon)}(t,x;\phi^{(\epsilon)}) - \epsilon \log U^{(\epsilon)}(t,x;\phi)| = 0.
\]

**Proof** This follows directly from noting that

\[
\epsilon \log U^{(\epsilon)}(t,x;\phi^{(\epsilon)}) - \epsilon \log U^{(\epsilon)}(t,x;\phi)
\]

\[
= \epsilon \log \mathcal{E}_{P}
\left[
\left. e^{-\frac{1}{\epsilon} (\phi^{(\epsilon)} - \phi)(x + w_{i}^{(\epsilon)})} \right\vert \frac{e^{-\frac{1}{\epsilon} (\phi(x+w^{(\epsilon)}(t))+\int_{0}^{t} V(t-s, x+w^{(\epsilon)}(s,t)) ds)} U^{(\epsilon)}(t,x,\phi)}
\right]
\]

so that

\[
|\epsilon \log U^{(\epsilon)}(t,x;\phi^{(\epsilon)}) - \epsilon \log U^{(\epsilon)}(t,x;\phi)| \leq \sup_{x} |\phi^{(\epsilon)}(x) - \phi(x)|.
\]

It follows directly that

\[
\sup_{0 \leq t \leq T} \sup_{x} |\epsilon \log U^{(\epsilon)}(t,x,\phi^{(\epsilon)}) - \epsilon \log U^{(\epsilon)}(t,x,\phi)| \leq \sup_{x} |\phi^{(\epsilon)}(x) - \phi(x)|.
\]

and lemma (5) follows directly. \(\square\)

## 4 The Counter Example

The following example provides an example in which solutions to the Euler Lagrange equations

\[
\begin{align*}
\ddot{\eta} &= V'(t, \eta(t)) \\
\eta(0,x) &= x, \quad \dot{\eta}(0,x) = \phi'(x)
\end{align*}
\]

that provide the representation \(u(t,x) = \dot{\eta}(t, \eta^{-1}(t,x))\) to the solution of the inviscid Burgers equation

\[
\begin{align*}
u_{t} + \frac{1}{2}(u^{2})_{x} &= V_{x} \\
u(0,.) &= \phi_{x}
\end{align*}
\]

are necessarily not the global minimisers of the associated action functional, for any \(x \in [0, 2\pi]\), for all \(t > T > 0\) where, in the example given, \(T = 2\pi\).
The potential \( V(t, x) = \cos(\sin t) - \cos(x + \sin t) \) will be used, so that
\[
V_x(t, x) = \sin(x + \sin t)
\]
and \( \phi \) will be chosen as the unique initial condition so that the solution \( u \) is \( 2\pi \) periodic in both variables.

The viscous Burgers’ equation under consideration is therefore
\[
\begin{align*}
\begin{cases}
  u_\epsilon t + \frac{1}{2} (u_\epsilon)^2_x = \frac{\epsilon}{2} u_{\epsilon xx} + \sin(x + \sin t) \\
u_\epsilon(0, x) = \phi_\epsilon(x),
\end{cases}
\end{align*}
\]
where \( \phi_\epsilon \) will be chosen to provide space / time periodic solutions and the Inviscid Burgers’ equation under consideration is the viscosity limit, which satisfies
\[
\begin{align*}
\begin{cases}
  u_t + \frac{1}{2} (u^2)_x = \sin(x + \sin t) \\
u(0, x) = \phi(x),
\end{cases}
\end{align*}
\]
where \( \phi \) is the limit of \( \phi_\epsilon \) and provides space / time periodic solutions. It will be shown later that for all \( \epsilon \geq 0 \), there exists a unique solution to equation (44) and for all \( \epsilon \geq 0 \), there exists exactly one function \( \phi_\epsilon \) that yields periodic solutions of equation (44). The functions \( \phi_\epsilon \) have a limit \( \phi \), such that \( \sup_{x\in[0,2\pi]} \lim_{\epsilon\to 0} |\phi_\epsilon(x) - \phi(x)| = 0 \), and \( \phi_x \) is the unique initial condition that yields periodic solutions to the inviscid Burgers equation (45). For the periodic solutions \( u_\epsilon \), there is a unique viscosity limit \( u \), which is the unique periodic solution to equation (45).

Attention is restricted to periodic solutions to equations (44) and (45).

The associated action functional is
\[
A(\xi; t) = \int_0^t \left\{ \frac{1}{2} \xi^2(s) + \cos(\sin s) - \cos(\xi(s) + \sin s) \right\} ds + \phi(\xi(0)).
\]
Let \( \phi'(x) = \phi_x(x) \). Using subscripts to denote derivatives with respect to the subscripted variable, easy variational calculus arguments yield that the critical points of the action functional with constraint \( \xi(t) = x \) satisfy \( \xi(s) = \xi(s; t, x) \) for \( 0 \leq s \leq t \), where \( \xi(.; t, x) \) satisfies
\[
\begin{align*}
\begin{cases}
  \xi_{ss}(s; t, x) = \sin(\xi(s; t, x) + \sin s) \\
\xi(t; t, x) = x, \ \xi_s(0; t, x) = \phi'(\xi(0; t, x)).
\end{cases}
\end{align*}
\]
Now suppose that equation (45) with initial condition \( \phi = \phi^{(1)} \) where that \( \phi^{(1)} \) is differentiable at 0 and satisfies \( \phi^{(1)}_x(0) = 0 \). Then for \( t = 2n\pi \) and \( x = 0 \),
\[
\xi(s; 2n\pi, 0) = -2n\pi + s - \sin s
\]
yields a solution to equation (47).
Now consider equation (45) with boundary data \( \phi = \phi^{(2)} \), where \( \phi^{(2)} \) is differentiable at \( \pi \) with \( \phi_x^{(2)}(\pi) = -2 \). Then, for \( x = \pi \) and \( t = 2n + 1 \), the function

\[
\xi(s; (2n + 1)\pi, \pi) = (2n + 1)\pi - s - \sin s
\]

(49)
yields a solution to equation (47). That equations (48) and (49) provide solutions to equation (47) for the prescribed boundary conditions can be seen by plugging into both sides.

**Lemma 6.** For all \( n \geq 1 \), \( \xi(s) = \xi(s; 2n\pi; 0) \), where \( \xi \) is given by equation (48) does not minimise the associated action functional (46) with the conditions \( t = 2n\pi, \xi(2n\pi) = 0 \) and \( \phi_x(0) = 0 \). For all \( n \geq 1 \), \( \xi(s) = \xi(s; (2n + 1)\pi, \pi) \) where \( \xi \) is given by equation (49) does not provide a minimiser for the associated action functional (46), with the conditions \( t = (2n + 1)\pi, x = \pi \) and \( \phi_x(\pi) = -2 \).

**Proof of lemma (6)** Solutions (48) and (49) are considered separately. For solution (48), times \( t = 2n\pi \) are chosen for integer \( n \) and final condition \( \xi(2n\pi) = 0 \). This gives

\[
A(\xi; 0, 2\pi n) = \frac{1}{2} \int_0^{2\pi n} (1 - \cos s)^2 ds + \int_0^{2\pi n} \cos(s) ds - \int_0^{2\pi n} \cos(\sin s) ds + \phi(0)
\]

\[
= \frac{3\pi n}{2} \int_0^{2\pi n} \cos(s) ds + \phi(0) > \frac{3\pi n}{2} + \phi(0).
\]

It is easy to see that the trajectory \( \psi \) such that \( \psi(t) \equiv 0 \) \( \forall t \geq 0 \) is not a solution of the Euler Lagrange equation. The action is

\[
A(\psi; 0, 2\pi n) = \int_0^{2\pi n} \cos(\sin s) ds - \int_0^{2\pi n} \cos(s) ds + \phi(0) = \phi(0),
\]

(51)

so that \( A(\psi; 0, 2\pi n) < A(\xi; 0, 2\pi n) \). The statement in lemma (6) connected with equation (48) is now proved. Note that these two trajectories have the same boundary data \( \psi(2\pi n) = \xi(2\pi n) = 0 \) and \( \dot{\psi}(0) = \dot{\xi}(0) = 0 \).

For equation (49), times \( t = (2n + 1)\pi \) are considered for integer \( n \) and final condition \( \xi(2(n + 1)\pi) = 0 \) is considered. The action is

\[
A(\xi; 0, (2\pi + 1)n) = \frac{1}{2} \int_0^{(2n+1)\pi} (1 - \cos s)^2 ds + \int_0^{(2n+1)\pi} \cos(\sin s) ds
\]

\[
= \int_0^{(2n+1)\pi} \cos(s) ds - \phi(\pi)
\]

\[
= \frac{(6n + 3)\pi}{4} + \int_0^{(2n+1)\pi} \cos(\sin s) ds + \phi(\pi)
\]

\[
> \frac{(6n + 3)\pi}{4} + \phi(\pi).
\]

(52)
Note that $\dot{\xi}(0) = -2$. Compare with the trajectory

$$
\psi(t) = \begin{cases} 
\pi - 2t, & 0 \leq t \leq \frac{\pi}{2} \\
0, & t > \frac{\pi}{2}
\end{cases}
$$

Then $\psi((2n + 1)\pi) = \xi((2n + 1)\pi) = 0$ and $\dot{\psi}(0) = \dot{\xi}(0) = -2$ and, for all $n \geq 0$,

$$
\mathcal{A}(\psi; 0, (2n + 1)\pi) = \pi + \int_0^{\pi/2} \cos(\sin s)ds - \int_0^{\pi/2} \cos(\pi - 2s + \sin s)ds + \phi(\pi) \leq 2\pi + \phi(\pi).
$$

The right hand side is bounded independent of $n$ and the action in (52) is increasing linearly in $n$. The statement in lemma (6) for equation (49) also holds. The proof of lemma (6) is complete.

The crucial point is to show that one of the trajectories $\xi$ given by equation (48) or (49) is necessarily a trajectory connected with solutions of the inviscid Burgers equation.

Let $\eta(t; q, p)$ denote the solution to the equation

$$
\begin{align*}
\ddot{\eta} &= \sin(\eta + \sin(t)) \\
\eta(0) &= q, \\n\dot{\eta}(0) &= p.
\end{align*}
$$

Several lemmas are required. The line of approach is as follows: firstly, initial conditions $(q, p)$ which yield periodic (modulo $2\pi$) solutions to the Euler Lagrange equations (53) are considered. It is shown that $(q, p) = (0, 0)$ and $(q, p) = (\pi, -2)$ are the only two, yielding solutions (modulo $2\pi$) $\eta(t) = t - \sin(t)$ and $\eta(t) = \pi - t - \sin(t)$ respectively. Next, it is shown that there exists exactly one space / time periodic solution to the inviscid Burgers equation and that this is obtained as the viscosity limit of periodic solutions to the viscous Burgers equations. Finally, it is shown that a periodic solution to the Euler Lagrange equation is necessarily associated with the periodic solution to the inviscid Burgers equation in the construction given by equation (58). It has been shown in lemma (6) that neither of the periodic solutions minimise the action functional.

Firstly, let $\eta$ solve equation (53) and set $Y(t) = \eta(t) - \sin(t) + t$ and set $X(t) = Y(-t)$. Note that $X$ satisfies

$$
\ddot{X} = \sin(X - t) + \sin(t).
$$
Lemma 7. Let $X$ denote solution to the equation

$$
\begin{align*}
\dot{X} &= \sin(X - t) + \sin(t) \\
X(0) &= x, \quad \dot{X}(0) = y.
\end{align*}
$$

(54)

where $X : \mathbb{R} \to S^1 = [0, 2\pi)$; that is, $x = x + 2\pi$. Only the initial conditions $(x, y) = (0, 0)$ and $(x, y) = (\pi, 2)$ yield periodic solutions of period $2\pi k$ for some $k \in \mathbb{Z}$. The corresponding periodic solutions on $S^1$ are of period $2\pi$ and are

$$
X(t) \equiv 0
$$

and

$$
X(t) = \pi + 2t.
$$

Proof Consider solutions to equation (54) in $\mathbb{R}$. A solution of period $2\pi l$ in $S^1$ will satisfy

$$
X(t) = \frac{C}{l} t + \sum_{k=-\infty}^{\infty} \alpha_k e^{i\frac{kt}{l}}
$$

for some integer $C$ and some collection $(\alpha_k)_{k=-\infty}^{\infty}$ such that $\alpha_k = \alpha^*_k$, where $\alpha^*_k$ is used to denote the complex conjugate of $\alpha_k$. Set

$$
A(t, \alpha) := \sum_{k=-\infty}^{\infty} \alpha_k e^{ikt}.
$$

(55)

Equation (54) yields

$$
- \sum_{k=-\infty}^{\infty} \left( \frac{k}{l} \right)^2 \alpha_k e^{i\frac{kt}{l}} = \sin \left( \left( \frac{C}{l} - 1 \right) t + \sum_{k=-\infty}^{\infty} \alpha_k e^{i\frac{kt}{l}} \right) + \sin(t).
$$

(56)

Using $K$ to denote the Kroneker delta function

$$
K_l(k) = \begin{cases} 1 & l = k \\ 0 & l \neq k, \end{cases}
$$

it follows directly from equation (56) that

$$
- \left( \frac{k}{l} \right)^2 \alpha_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \sin((C - l)t + A(t, \alpha)) dt - \frac{i}{2} (K_l(k) - K_{-l}(k)).
$$

(57)

Set

$$
\phi(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \cos((C - l)t + A(t, \alpha)) dt.
$$

(58)
Let \((\tilde{\alpha})_{k=\infty}^{\infty}\) denote a solution to equation (57). Then an easy differentiation of equation (58) yields

\[
\left(\frac{k}{l}\right)^{2} \tilde{\alpha}_{k} = \frac{\partial}{\partial \alpha_{-k}} \phi(\tilde{\alpha}) + \frac{i}{2} (K_{l}(k) - K_{-l}(k)).
\]  

(59)

Taking a power series expansion of \(\phi\) yields

\[
\phi(\alpha) = \frac{1}{2} \left\{ e^{i\alpha_{0}} \sum_{n=0}^{\infty} \left( \frac{i^{n}}{n!} \right) \sum_{\begin{subarray}{c} k_{1} + \ldots + k_{n} = -(C-l) \end{subarray}} \alpha_{k_{1}} \ldots \alpha_{k_{n}} + e^{-i\alpha_{0}} \sum_{n=0}^{\infty} \left( \frac{(-i)^{n}}{n!} \right) \sum_{\begin{subarray}{c} k_{1} + \ldots + k_{n} = (C-l) \end{subarray}} \alpha_{k_{1}} \ldots \alpha_{k_{n}} \right\}.
\]  

(60)

For \(\tilde{\alpha}\) such that \(X(t) = \frac{C}{t} + \sum_{k=\infty}^{\infty} \tilde{\alpha}_{k} e^{\frac{i}{l}kt}\), where \(X\) is a real solution to equation (54), it is clear (using \(\dddot{X} = \sin(X - t) + \sin(t)\)) that \(|\dddot{X}| \leq 2\), from which it follows directly that

\[
|\dddot{\tilde{\alpha}}_{k}| \leq \frac{2l^{2}}{k^{2}}
\]  

(61)

for each \(k \neq 0\). It also follows easily from equation (58) that

\[
\left| \frac{\partial^{n}}{\partial \alpha_{k_{1}} \ldots \partial \alpha_{k_{n}}} \phi(\tilde{\alpha}) \right| \leq 1.
\]  

(62)

Taylor’s expansion theorem yields

\[
\phi(\alpha) = \phi(\tilde{\alpha}) + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\begin{subarray}{c} k_{1}, \ldots, k_{n} \end{subarray}} (\alpha - \tilde{\alpha})_{k_{1}} \ldots (\alpha - \tilde{\alpha})_{k_{n}} \frac{\partial^{n}}{\partial \alpha_{k_{1}} \ldots \partial \alpha_{k_{n}}} \phi(\tilde{\alpha}).
\]  

(63)

The expansion given by equation (63) is easily justified by considering the a priori bound on the derivatives given by inequality (62) and also the a priori bound on \(\alpha\) and \(\tilde{\alpha}\) given by inequality (61). Expanding this gives

\[
\phi(\alpha) = \phi(\tilde{\alpha}) + \sum_{n=1}^{\infty} \sum_{\begin{subarray}{c} k_{1}, \ldots, k_{n} \end{subarray}} \alpha_{k_{1}} \ldots \alpha_{k_{n}} \left( \frac{1}{n!} \prod_{i=1}^{n} \partial \alpha_{k_{i}} \phi(\tilde{\alpha}) \right) \bigg|_{\begin{subarray}{c} i=1 \end{subarray}}^{i=1} \prod_{i=1}^{m} \partial \alpha_{k_{i}} \prod_{i=1}^{m} \partial \alpha_{j_{i}} \phi(\tilde{\alpha}) \bigg|_{\begin{subarray}{c} i=1 \end{subarray}}^{i=1} \prod_{i=1}^{m} \partial \alpha_{j_{i}}
\]  

(64)

By comparing equations (60) and (64), it follows that for all \((k_{1}, \ldots, k_{n}) : k_{j} \neq 0\),
\[
\frac{\partial^n}{\prod_{l=1}^{n} \partial \alpha_{k_l}} \phi(\tilde{\alpha}) + \sum_{m=1}^{\infty} \frac{(-1)^m n!}{(n+m)!} \sum_{j_1, \ldots, j_m} \tilde{\alpha}_{j_1} \ldots \tilde{\alpha}_{j_m} \prod_{l=1}^{n} \frac{\partial^{m+n}}{\prod_{l=1}^{n} \partial \alpha_{k_l}} \prod_{l=1}^{m} \partial \alpha_{j_l} \phi(\tilde{\alpha}) = \left\{ \begin{array}{ll}
\frac{i^n}{2} & k_1 + \ldots + k_n = -(C-l) \\
\frac{i^n}{2} & k_1 + \ldots + k_n = C-l \\
0 & \text{other } k_1, \ldots, k_n.
\end{array} \right.
\] (65)

From equation (58), note that

\[
\frac{\partial^{2n+1}}{\partial \alpha_{k_1} \ldots \partial \alpha_{k_{2n+1}}} \phi(\alpha) = (-1)^n \frac{\partial}{\partial \alpha_{k_1} \ldots \partial \alpha_{k_{2n+1}}} \phi(\alpha).
\] (66)

Let \( K = (2\pi + 4l^2 \sum_{j=1}^{\infty} \frac{1}{j^2}) \), so that, using bounds (61) and (62), together with \(|\alpha_0| < 2\pi\),

\[
\Theta(n) := \left| \sum_{m=1}^{\infty} \frac{(-1)^m n!}{(n+m)!} \sum_{j_1, \ldots, j_m} \tilde{\alpha}_{j_1} \ldots \tilde{\alpha}_{j_m} \prod_{l=1}^{n} \frac{\partial^{m+n}}{\prod_{l=1}^{n} \partial \alpha_{k_l}} \prod_{l=1}^{m} \partial \alpha_{j_l} \phi(\tilde{\alpha}) \right| \leq \sum_{m=1}^{\infty} \frac{n!}{(n+m)!} K^m.
\]

Stirling’s formula, found in [8] page 327 yields that

\[
\frac{n!}{(n+m)!} = e^m \left( \frac{n}{n+m} \right)^{n+\frac{1}{2}} \frac{1}{(n+m)^{m+\frac{1}{2}}} e^{\theta_{m,n}},
\]

where \(-\frac{1}{12(n+m)} \leq \theta_{m,n} \leq \frac{1}{12n}\), so that

\[
\Theta(n) \leq e \sum_{m=1}^{\infty} \frac{(Ke)^m}{(n+m)^{m+\frac{1}{2}}} n^{-\frac{1}{2}} \to 0.
\]

It follows from equation (65), together with equation (66), that for all \( k \in \mathbb{Z} \),

\[
\frac{\partial}{\partial \alpha_k} \phi(\tilde{\alpha}) = \left\{ \begin{array}{ll}
\frac{i}{2} e^{i\alpha_0} & k = -(C-l) \\
-\frac{i}{2} e^{-i\alpha_0} & k = C-l \\
0 & k \neq \pm(C-l).
\end{array} \right.
\]

Now consider equation (59). For \( C = l \) there is no solution; \( \frac{\partial}{\partial \alpha_0} \phi(\tilde{\alpha}) = \frac{i}{2} e^{i\alpha_0} \) and equation (59) yields (for \( k = 0 \) ) \( 0 = \frac{i}{2} e^{i\alpha_0} \).

For \( C = 0 \), this yields \( \tilde{\alpha}_k = 0 \) for all \( k \neq 0, \pm l \) and

\[
X(t) = \alpha_0 - \frac{i}{2} (e^{i\alpha_0} - 1)e^{it} + \frac{i}{2} (e^{-i\alpha_0} - 1)e^{-it}.
\]

For this to satisfy

\[
\ddot{X} = \sin(X - t) + \sin(t)
\]
requires

\[-\sin(\alpha_0 + t) + \sin(t) = \sin(\alpha_0 - t + \sin(\alpha_0 + t) - \sin(t)) + \sin(t),\]

and the only solution occurs when \(\alpha_0 = 2n\pi\), yielding \(X(t) \equiv 2n\pi\) for \(n \in \mathbb{Z}\), from which it follows that \(X(t) \equiv 0\) on \(S^1\).

For \(C = 2l\), this yields \(\tilde{\alpha}_k = 0\) for all \(k \neq 0, \pm l\) and the equations yield

\[X(t) = \alpha_0 + 2t + \frac{i}{2}(1 + e^{i\alpha_0})e^{it} - \frac{i}{2}(1 + e^{-i\alpha_0})e^{-it}.\]

It follows that \(\alpha_0\) must satisfy

\[\sin(\alpha_0 + t) + \sin(t) = \sin(\alpha_0 + t - \sin(t) - \sin(\alpha_0 + t)) + \sin(t).\]

This requires \(\alpha_0 = (2n + 1)\pi\), yielding

\[X(t) = (2n + 1)\pi + 2t.\]

For \(C \neq 0, l, 2l\), the equations yield

\[X(t) = \alpha_0 + \frac{C}{l}t - \sin(t) - \left(\frac{t}{C - l}\right)^2 \sin(\alpha_0 + \frac{C}{l}t).\]

and it is easy to see that there are no solutions to

\[\sin(t) + \left(\frac{t}{C}\right)^2 \left(\frac{t}{C - l}\right)^2 \sin(\alpha_0 + \frac{C}{l}t)\]

\[= \sin \left(\alpha_0 + \frac{C - l}{l}t - \sin(t) - \left(\frac{l}{C - l}\right)^2 \sin(\alpha_0 + \frac{l}{C}t)\right) + \sin(t)\]

for \(C \neq 0, l, 2l\). The result is established.

Several lemmas connected with the Burgers’ equation are necessary. They are all standard and inserted for completeness. Their aim is to show that, with the potential under consideration, there exists a periodic solution to the inviscid Burgers equation which arises as a viscosity limit and that there necessarily exists one periodic solution to the Euler Lagrange equations used in the construction of any periodic viscosity solution to the inviscid Burgers equation.

Firstly, it is established that \(\int_0^{2\pi} u^{(c)^2}(t, x)dx\) is bounded, where the bound does not depend on \(t\) or \(c > 0\). This is standard; the proof is included for completeness.
Lemma 8. Let $V \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ be uniformly bounded and $2\pi$ periodic in both variables, satisfying $\int_0^{2\pi} V(t,x)dx = 0$ for all $t \geq 0$. Let $u^{(\epsilon)}$ denote the solution to the equation
\[
\begin{cases}
u^{(\epsilon)}_t + \frac{1}{2}(\nu^{(\epsilon)}_x)^2 = \frac{\epsilon}{2} \nu^{(\epsilon)}_{xx} + V(t,x) \\
u^{(\epsilon)}(0,\cdot) = \text{initial condition.}
\end{cases}
\tag{67}
\]

Let $K_1 = \sup_{(s,x)} |V(s,x)|$, $K_2 = \sup_{(s,x)} |V_x(s,x)|$ and let $\|f\| := \left(\frac{1}{2\pi} \int_0^{2\pi} f^2(x)dx\right)^{1/2}$. If $\|\nu^{(\epsilon)}(0)\| \leq 9\pi$, then $\|\nu^{(\epsilon)}(t)\| \leq 6\pi K_2 + \sqrt{6K_1 + 6\pi K_2}$ for all $t \geq 0$.

**Proof** Set $C(t) = \|\nu^{(\epsilon)}(t)\|^2$. Let $v^{(\epsilon)}$ solve
\[
v^{(\epsilon)}_t = \frac{\epsilon}{2} v^{(\epsilon)}_x - \frac{1}{2} (v^{(\epsilon)}_x)^2 + \frac{1}{2} C(t) - V(t,x),
\]
with initial condition $v^{(\epsilon)}(0,\cdot)$ satisfying $\int_0^{2\pi} v^{(\epsilon)}(0,x)dx = 0$ and $v^{(\epsilon)}_x(0,x) = u^{(\epsilon)}(0,x)$. Note that $v^{(\epsilon)}_x = u^{(\epsilon)}$ and that $\int_0^{2\pi} v^{(\epsilon)}(t,x)dx = 0$ for all $t \geq 0$. Furthermore, note that $v^{(\epsilon)} = -\epsilon \log U^{(\epsilon)}$ where $U^{(\epsilon)}$ satisfies
\[
\begin{cases}
u^{(\epsilon)}_t = \frac{\epsilon}{2} u^{(\epsilon)}_xx - \frac{1}{2} u^{(\epsilon)}_x \left\{ \frac{\epsilon}{2} C(t) - V(t,x) \right\} \\
u^{(\epsilon)}(0,x) = \exp\left\{ -\frac{1}{\epsilon} v^{(\epsilon)}(0,x) \right\}.
\end{cases}
\tag{68}
\]

Let $w$ denote a standard Brownian motion, with $w_0 = 0$. Let $w_{t,t} = w_t - w_s$. Let $\mathbf{P}$ denote the probability measure associated with $w$ and let $E_{\mathbf{P}}[\cdot]$ denote expectation with respect to $\mathbf{P}$. The solution to equation (68) has Kac's representation
\[
U^{(\epsilon)}(t,x) = \mathbf{E}_\mathbf{P} \left[ \exp \left\{ -\frac{1}{\epsilon} v^{(\epsilon)}(0,x + \sqrt{\epsilon} w_{0,t}) + \frac{1}{\epsilon} \int_0^t V(s,x + \sqrt{\epsilon} w_{s,t})ds \right\} \right] e^{-\frac{1}{2\epsilon} \int_0^t C(s)ds}
\tag{69}
\]
Using representation (69) of the solution to equation (68), it follows that
\[
v^{(\epsilon)}(t+s,x) = \frac{1}{2} \int_s^{t+s} C(r)dr - \epsilon \log E \left[ \exp \left\{ -\frac{1}{\epsilon} \left( v^{(\epsilon)}(s,x + \sqrt{\epsilon} w_{s,t+s}) - \int_s^{t+s} V(r,x + \sqrt{\epsilon} w_{r,t+s})dr \right) \right\} \right]
\geq \frac{1}{2} \int_s^{t+s} C(r)dr - K_1 t - \epsilon \log E \left[ \exp \left\{ -\frac{1}{\epsilon} v^{(\epsilon)}(s,x + \sqrt{\epsilon} w_{s,t+s}) \right\} \right].
\tag{70}
\]
Now, since $\int_0^{2\pi} v^{(\epsilon)}(t,x)dx = 0$ and $v^{(\epsilon)}(t,\cdot)$ is continuous, it follows that there exists a point $x(t)$ such that $v^{(\epsilon)}(t,x(t)) = 0$. It follows that
\[
\sup_x |v^{(e)}(t, x)| = \sup_x \left| \int_{x(t)}^x u^{(e)}(t, y) dy \right|
\leq \int_0^{2\pi} |u^{(e)}(t, y)| dy
\leq 2\pi \|u^{(e)}(t, \cdot)\|.
\]

Therefore, \(\sup_x |v(s, x)| \leq 2\pi \|u(s)\|\). Using this, together with \(\int_0^{2\pi} v^{(e)}(t, x) dx = 0\), it follows from inequality (70) that

\[
0 \geq \frac{1}{2} \int_s^{t+s} C(r) dr - K_1 t - 2\pi \|u^{(e)}(t, \cdot)\|.
\]

This may be rewritten as

\[
0 \geq \frac{1}{2} \int_s^{t+s} C(r) dr - K_1 t - 2\pi C^{1/2}(t),
\]

so that

\[
\int_s^{t+s} C(r) dr \leq 2K_1 t + 4\pi C(s)^{1/2}.
\tag{71}
\]

Equation (67) yields

\[
\frac{d}{dt} \|u(t)\|^2 = -\epsilon \int_0^{2\pi} |u_x^2(t, x)| dx + \frac{1}{\pi} \int_0^{2\pi} u(t, x) V_x(t, x) dx
\leq 2K_2 \|u(t)\|,
\]

so that

\[
\frac{d}{dt} C(t)^{1/2} \leq K_2. \tag{72}
\]

Directly from equation (72), it follows that \(C^{1/2}(t + s) \leq C^{1/2}(s) + K_2 t\). This may be written, for \(r < t + s\), as

\[
C^{1/2}(r) \geq C^{1/2}(t + s) - K_2(t + s - r).
\]

For \(r > (t + s) - \frac{C^{1/2}(t+s)}{K_2}\), squaring both sides yields

\[
C(r) \geq C(t + s) - 2K_2(t + s - r)C^{1/2}(t + s) + K_2^2(t + s - r)^2
\]

so, for \(s\) such that \(C^{1/2}(t + s) > K_2 t\), integration yields

\[
\int_s^{t+s} C(r) dr \geq tC(t + s) - K_2 t^2 C^{1/2}(t + s) + K_2^2 \frac{t^3}{3},
\]
for all $t \in [0, \frac{C^{1/2}(t+s)}{K_2}]$. The inequality (71) may then be applied, giving

$$tC(t+s) - K_2 t^2 C(t+s)^{1/2} + K_2^2 \frac{t^3}{3} \leq 2K_1 t + 4\pi C(s)$$

for all $t \in [0, \frac{C^{1/2}(t+s)}{K_2}]$. For any fixed $T < +\infty$, set $M(T) = \sup_{0 \leq s \leq T} C(s)$. Choose $\tau \in [0, T]$ such that $C(\tau) = M(T)$. Then for all $t \in [0, \frac{M^{1/2}(\tau)}{K_2}]$, it follows (taking $\tau$ in the place of $t+s$) that

$$tM(\tau) - K_2 M^{1/2}(\tau) t^2 + K_2^2 \frac{t^3}{3} \leq 2K_1 t + 4\pi M(\tau).$$

Now choose $t = \frac{M^{1/2}(\tau)}{K_2}$, so that

$$M(\tau) \leq 6K_1 + 12\pi K_2 M^{1/2}(\tau)$$

yielding

$$M^{1/2}(\tau) \leq 6\pi K_2 + \sqrt{6K_1 + 6\pi K_2}.$$

Since this holds for all $T$, the result follows directly.

The a priori bound in lemma (8) is a useful step for establishing existence of periodic solutions to the Burgers’ equation under consideration.

**Lemma 9.** Let $V$ be smooth and bounded and periodic in the space variable and let $u_0$ be a bounded initial condition such that $\int_0^{2\pi} u_0(x) dx = 0$. Then, for each $\epsilon \geq 0$, there exists a unique solution to the equation

$$\begin{cases}
  u_t^{(\epsilon)} + \frac{1}{2} (u^{(\epsilon)2})_x = \frac{\epsilon}{2} u_{xx} + V_x \\
  u_0 = \text{initial condition}
\end{cases}$$

**Proof** Let $u^{(1)}$ and $u^{(2)}$ denote two solutions, set $S = u^{(1)} + u^{(2)}$ and $D = u^{(1)} - u^{(2)}$. Let $D(t,x) = \sum_n \lambda_n(t)e^{inx}$. Since $\int_0^{2\pi} u_0(x) dx = 0$, it follows that $\lambda_0 \equiv 0$. Let $\tilde{D}(t,x) = -i \sum_n \frac{\lambda_n(t)}{n} e^{inx}$. Then $D = \tilde{D}_x$ and

$$\begin{cases}
  \tilde{D}_t^{(\epsilon)} + \frac{1}{2} S \tilde{D}_x = \frac{\epsilon}{2} \tilde{D}_{xx} \\
  \tilde{D}(0,x) \equiv 0.
\end{cases}$$

Let $w^{(\epsilon)}$ denote Brownian motion with diffusion $\epsilon$ and set

$$X_{s,t}(x) = x + (w_t^{(\epsilon)} - w_s^{(\epsilon)}) - \frac{1}{2} \int_s^t S(r, X_{r,t}(x)) dr.$$

Then, let $E^{(P)}$ denote expectation with respect to $w^{(\epsilon)}$,
\[ \tilde{D}(t, x) = E^{(P)}[\tilde{D}(0, X_{0,t}(x))] \equiv 0 \quad \forall t \geq 0, \quad x \in [0, 2\pi) \]

so that \( D \equiv 0 \). This holds for all \( \epsilon \geq 0 \).

\[ \square \]

**Lemma 10.** Let \( S \) be a bounded function, \( 2\pi \) periodic in both variables, satisfying \( \int_0^{2\pi} S(t, x)dx = 0 \) for all \( t \in [0, 2\pi] \), such that \( 0 < \int_0^{2\pi} \int_0^{2\pi} |S(t, x)|^2dtdx < +\infty \). Let \( Z \) satisfy

\[
\begin{cases}
\frac{\partial}{\partial s} Z_{s,t}(x) = S(s, Z_{s,t}(x)), & (s, t) \in \mathbb{R}^2 \\
Z_{t,t}(x) = x & t \in \mathbb{R}.
\end{cases}
\]

Then there exists a number \( k \in \mathbb{N} \) such that for all \( s \in \mathbb{R} \) and all \( (x, y) \in [0, 2\pi)^2 \)

\[
\lim_{t \to +\infty} |k(Z_{s,t}(x) - Z_{s,t}(y)) \mod (2\pi)| = 0
\]

and

\[
\lim_{t \to -\infty} |k(Z_{s,t}(x) - Z_{s,t}(y)) \mod (2\pi)| = 0.
\]

**Proof** Set

\[
Q_{nm}(s, s + t) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} e^{imx-ny} \delta_y(Z_{s,t}(x)) dydx = \frac{1}{2\pi} \int_0^{2\pi} e^{i(mx-nZ_{s,s+t}(x))} dx. \quad (73)
\]

Note that

\[
Q_{0,m}(s, s + t) = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases}
\]

It follows directly from equation (73) that

\[
e^{-inZ_{s,s+t}(x)} = \sum_m Q_{nm}(s, s + t)e^{-imx}, \quad (74)
\]

from which, for all \( n \in \mathbb{Z} \),

\[
\sum_m |Q_{nm}(s, s + t)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |e^{-inZ_{s,s+t}(x)}|^2 dx = 1.
\]

Furthermore, \( Q_{nm}(s, s + t) = Q_{nm}(s + 2\pi, s + 2\pi + t) \) for all \( (s, t) \in \mathbb{R}^2 \). For any sequence \( t_1, t_2, \ldots, t_k \), with \( t_0 = 0 \) (where multiplication is taken in the sense of matrix multiplication),

\[
Q(s, s + t_1 + \ldots + t_k) = \prod_{j=0}^{k-1} Q(s + \sum_{l=0}^{j} t_l, s + \sum_{l=0}^{j+1} t_l).
\]
That is,

\[ Q_{nm}(s, s + \sum_{j=1}^{k} t_j) = \sum_{p_1, \ldots, p_k} Q_{np_1}(s, s + t_1)Q_{p_1p_2}(s + t_1, s + t_2) \cdots Q_{pkm}(s + \sum_{j=1}^{k-1} t_j, s + \sum_{j=1}^{k} t_j). \]

For fixed \( s \), let \( Q \) denote \( Q(s, s+2\pi) \). It is clear that one may construct a decomposition \( Q = AP \), where \( P \) is orthonormal, \( P_{nm}^* = P_{-n,-m} \), \( A_{nm} = 0 \) for \( |m| \geq |n| + 1 \) and \( A_{nm}^* = A_{-n,-m} \). Since \( \sum_m |Q_{nm}|^2 = 1 \) and \( P \) is orthonormal, it follows that for each \( n \in \mathbb{Z} \),

\[ \sum_m |A_{nm}|^2 = \sum_m |Q_{nm}|^2 = 1. \]

Note that \( A_{00} = 1 \) and \( P_{00} = 1 \).

Set

\[ S = \{ n \in \mathbb{Z} | Q_{n0} = 0 \} \]

and set \( \mathcal{R} = \mathbb{Z} \setminus S \). Set

\[ T^0 = \{ n \in S | Q_{nm} = 0 \ \forall m \in \mathcal{R} \} \]

and, for \( n \geq 0 \),

\[ T^{(n+1)} = \{ n \in S | Q_{nm} = 0 \ \forall m \in \mathbb{Z} \setminus T^{(n)} \}. \]

It is clear that for all \( n \in \mathbb{N} \), \( T^{(n+1)} \subseteq T^{(n)} \). Set \( \hat{S} = \cap_{n \geq 1} T^{(n)} \). Note that \( Q^{(n)}_{n0} = 0 \) for all \( n \in \hat{S} \) and all \( N \geq 1 \). Let \( Q^{(S)} \) denote \( \{ Q_{mn} \mid (m, n) \in \hat{S} \times \hat{S} \} \) and note that \( Q^{(N)}_{mn} = Q^{(S)}_{mn} \) for all \( (m, n) \in \hat{S} \times \hat{S} \). Let \( \mathcal{R} = \mathbb{Z} \setminus \hat{S} \) and let \( Q^{(R)} \) denote \( \{ Q_{mn} \mid (m, n) \in \mathcal{R} \times \mathcal{R} \} \) and note that \( Q^{(N)}_{mn} = Q^{(R)}_{mn} \) for \( (m, n) \in \mathcal{R} \times \mathcal{R} \).

Let \( Q^{(R)}_{N} = A^{(R)}_{N,0} P^{(R)}_{N} \) denote a decomposition where \( P^{(R)}_{N} \) is orthonormal, \( A^{(R)}_{N,00} = 0 \) for \( |m| \geq |n| + 1 \), \( P^{(R)}_{N,0} = P^{(R)}_{N,-n,-m} \) and \( A^{(R)}_{N,0} = A^{(R)}_{N,-n,-m} \). Note that

\[ Q^{(R)}_{N} = A^{(R)}_{N,0} P^{(R)}_{N} = (A^{(R)} P^{(R)})^N. \]

Set \( \hat{A}^{(R)} = A^{(R)-1} A^{(R)}_2 \) and note that \( \hat{A}^{(R)}_{00} = 0 \) for \( |m| \geq |n| + 1 \). Set \( \hat{P}^{(R)} = P^{(R)}_2 P^{(R)-1} \), so that \( \hat{P}^{(R)} \) is (clearly) orthonormal. Then, since

\[ A^{(R)} P^{(R)} A^{(R)} P^{(R)} = A^{(R)}_2 P^{(R)}_2, \]

it follows that

\[ P^{(R)} A^{(R)} = \hat{A}^{(R)} \hat{P}^{(R)}. \]
From this, it follows that

\[ Q^{(R)2N} = (A^{(R)}P^{(R)})^{2N} = (A^{(R)}\tilde{A}^{(R)})N(P^{(R)}\tilde{P}^{(R)})N = A_2^{(R)N}P_2^{(R)N}. \]

By construction, it is clear that for each \( n \in \tilde{\mathcal{R}} \), \( \sum_{m=-|n|+1}^{|n|-1} |A_{2nm}^{(R)}|^2 > 0 \), from which it follows that \( \lim_{N \to +\infty} A_{2nm}^{(R)N} = 0 \) for all \( m \neq 0 \) and \( \lim_{n \to +\infty} |A_{2n0}^{(R)}N| = 1 \) for all \( n \in \tilde{\mathcal{R}} \). By construction, \( Q_{n0}^2 = A_{2n0} \). It follows that, for all \( n \in \tilde{\mathcal{S}} \), \( Q_{n0}^{2N} = 0 \) for all \( N \geq 1 \), and for all \( n \in \tilde{\mathcal{R}} \), \( \lim_{N \to +\infty} |Q_{n0}^{2N}| = 1 \). Choose \( k_1 = \inf\{n \geq 1 | n \in \tilde{\mathcal{R}}\} \).

Since the same arguments work running time ‘backwards’, there exists a \( k_2 \geq 1 \) such that

\[ \lim_{t \to +\infty} |k_1(Z_{s,s+t} - Z_{s,s+t})| \ mod (2\pi) = 0. \]

It follows that

\[ e^{-ik_1Z_{s,s+2N}(x)} = \sum_m Q_{km}e^{-imx}, \]

it follows that for each \( x \in [0,2\pi) \)

\[ \lim_{N \to +\infty} |e^{-ik_1Z_{s,s+4N\pi}(x)} - Q_{n0}^{2N}| = 0 \]

and hence that for all \( x,y \)

\[ \lim_{N \to +\infty} |k_1(Z_{s,s+4N\pi}(x) - Z_{s,s+4N\pi}(y)) \ mod (2\pi)| = 0, \]

from which it is easy to show that

\[ \lim_{t \to +\infty} |k_2(Z_{s,s+t}(x) - Z_{s,s+t}(y)) \ mod (2\pi)| = 0. \]

Take \( K = k_1 \times k_2 \) and the result follows. Now suppose that \( \tilde{\mathcal{R}} = \{0\} \). Then, \( Q_{n0}(s,s+t) = 0 \) for all \( t \geq 0 n \neq 0 \). Note that

\[ Q_{n0}(s,s+t) = \frac{1}{2\pi} \int_0^{2\pi} e^{inZ_{s,s+t}(x)}dx = \frac{1}{2\pi} \int_0^{2\pi} e^{inY} \frac{\partial Z_{s,s+t}^{-1}(y)}{\partial y}dy, \]

so that

\[ \frac{\partial Z_{s,s+t}^{-1}(y)}{\partial y} \equiv 1. \]

Since this holds for all \( t \in \mathbb{R} \), it follows that

\[ Z_{s,s+t}(y) = c(s,t) + y. \]

It follows that
\[ \frac{\partial}{\partial s} c(s, t) = S(s, c(s, t) + y). \]

Since \( \int_0^{2\pi} S(s, y) dy = 0 \) and \( S \) is \( 2\pi \) periodic, it follows that \( \frac{\partial}{\partial s} c(s, t) = 0 \) and hence that \( S \equiv 0 \).

It follows that, if the hypotheses are satisfied, then there exists a \( K \in \mathbb{N} \) such that

\[ \lim_{t \to +\infty} |K(Z_{s,t}(x) - Z_{s,t}(y)) \mod (2\pi)| = 0 \]

and

\[ \lim_{t \to -\infty} |K(Z_{s,t}(x) - Z_{s,t}(y)) \mod (2\pi)| = 0. \]

The result follows.

**Lemma 11.** For each fixed \( \epsilon \geq 0 \), there exists an initial condition \( u^{(\epsilon)}(0,.) \) which provides a solution, \( 2\pi \) periodic in both \( x \) and \( t \), to the equation

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
u_t^{(\epsilon)} + \frac{1}{2}(\nu^{(\epsilon)})_x = \frac{\epsilon}{2} \nu^{(\epsilon)}_x + \sin(x + \sin(t)) \\
u^{(\epsilon)}(0,.) = \text{initial condition.}
\end{array}
\right.
\end{aligned}
\]

satisfying \( \|u^{(\epsilon)}(t)\| \leq 9\pi \) for all \( t \geq 0 \) and \( \int_0^{2\pi} u^{(\epsilon)}(t,x)dx = 0 \) for all \( t \geq 0 \). For all \( \epsilon \geq 0 \), there is exactly one space time periodic solution to equation (75) satisfying \( \int_0^{2\pi} u^{(\epsilon)}(t,x) \equiv 0 \).

**Proof** Note that equation (75) is simply equation (67) with \( V(t, x) = -\cos(x + \sin(t)) \). Here \( K_1 = K_2 = 1 \), so that the \( L^2 \) norm is uniformly bounded. The line of proof is as follows: firstly, existence of a periodic solution for \( \epsilon > 0 \) is established, then uniqueness for \( \epsilon > 0 \). With the uniform bounds on the \( L^2 \) norm, existence of periodic solution for \( \epsilon = 0 \) follows from existence of periodic solution of \( u^{(\epsilon)} \), together with the uniform bounds in \( \epsilon \) on the \( L^2 \) norm. Finally, uniqueness for \( \epsilon = 0 \) is then proved. From the uniqueness results, it therefore follows that \( u^{(\epsilon)} \) converges in \( L^2 \), in the relative weak topology, to \( u \).

**Part 1: Existence for \( \epsilon > 0 \)** Let \( U^{(\epsilon)} \) satisfy the equation

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
U_t^{(\epsilon)} = \frac{\epsilon}{2} U_{xx}^{(\epsilon)} + \frac{1}{\epsilon} U^{(\epsilon)} \cos(x + \sin(t)) \\
U^{(\epsilon)}(0,.,U_0) = U_0
\end{array}
\right.
\end{aligned}
\]

where \( U_0 \) is a non negative, bounded initial condition.

Consider the operator \( T: L^2(S^1) \to L^2(S^1) \) defined by

\[ T(\phi)(x) := E \left[ \phi(x + \omega^{(\epsilon)}_{2\pi}) \exp \left\{ \frac{1}{\epsilon} \int_0^{2\pi} \cos(x + \omega^{(\epsilon)}_{2\pi-s} + \sin(s))ds \right\} \right], \]

where \( S^1 \) here (as stated before) is the circle \([0, 2\pi] \); the real line with the identification \( x = x + 2\pi \). Note that
\[ T(\phi) = U^{(\epsilon)}(2\pi, ., \phi), \]

where \( U^{(\epsilon)} \) is the solution to equation (76). Note that this is a bounded operator;

\[ \|T\|_2 \leq \exp\{2\pi/\epsilon\}. \]

Let \( r(T) = \sup_{\lambda \in \sigma(T)} |\lambda| \), where \( \sigma \) denotes spectrum. By theorem VI.6 page 192 from Reed and Simon [11], \( r(T) = \lim_{n \to +\infty} \|T^n\|^{1/n} \). It follows that \( \exp\{2\pi/\epsilon\} \geq r(T) \geq \exp\{-2\pi/\epsilon\} \). The operator \( T \) is compact. Therefore, by the Riesz - Schauder theorem (Reed and Simon [11] Theorem VI.15 page 203), \( \sigma(T) \) is a discrete set having no limit points except perhaps for 0. Therefore, there exists an eigenvalue \( \lambda \) such that \( |\lambda| = r(T) \) and this eigen value is of finite multiplicity.

Let \( \phi = \alpha + i\beta \) denote an eigenfunction, with \( \|\phi\|_2 = \|\alpha\|_2 + \|\beta\|_2 = 1 \), where \( \alpha \) and \( \beta \) are real functions, such that \( \|T\phi\|_2 = r(T) \). Since \( \phi \) (by hypothesis) maximises \( \|T\phi\|_2 / \|\phi\|_2 \), it is easy to see that both \( \alpha \) and \( \beta \) maximise \( \|T\phi\|_2 / \|\phi\|_2 \) and therefore that \( \alpha \) is either non negative or non positive and \( \beta \) is either non negative or non positive. The corresponding eigenvalue may be written as \( \lambda = r(T)e^{i\theta} \), for \( \theta \in [0, 2\pi) \). Since both \( \alpha \) and \( \beta \) maximise the expression, it follows, using the notation \( \langle \alpha, \beta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \alpha(x)\beta(x)dx \), that

\[ r^2\|\alpha\|_2^2 = \|T\alpha\|_2^2 = r^2 (\cos^2(\theta)\|\alpha\|_2^2 + \sin^2(\theta)\|\beta\|_2^2 - 2\cos(\theta)\sin(\theta)\langle \alpha, \beta \rangle) \]

and

\[ r^2\|\beta\|_2^2 = \|T\beta\|_2^2 = r^2 (\sin^2(\theta)\|\alpha\|_2^2 + \cos^2(\theta)\|\beta\|_2^2 + 2\sin(\theta)\cos(\theta)\langle \alpha, \beta \rangle). \]

From this, \( \theta = 0 \) or \( \pi \), from which it follows that \( \lambda = r \); the eigenvalue is real and the eigenfunction may be taken as real and non negative.

Since \( U^{(\epsilon)}(2\pi, ., \phi) \) is strictly positive for \( \phi \in L^2 \) non negative with \( \|\phi\|_2 = 1 \), it follows, since \( \phi = \frac{1}{r}U^{(\epsilon)}(2\pi, ., \phi) \), that \( \phi \) is strictly positive.

Set

\[ u^{(\epsilon)}(0, .) = -\epsilon \frac{\partial}{\partial x} \log \phi. \]

This initial condition will provide periodic solutions to equation (75).

**Part 2: Uniqueness for \( \epsilon > 0 \).** Suppose that there are two periodic solutions, \( u^{(\epsilon,1)} \) and \( u^{(\epsilon,2)} \), both of period \( 2\pi k \) in the time variable. Set \( D = u^{(1)} - u^{(2)} \) and \( S = u^{(1)} + u^{(2)} \). Firstly, by lemma (8),

\[ \left( \frac{1}{2\pi} \int_0^{2\pi} S^2(t, x)dx \right)^{1/2} \leq 18\pi. \]
Next, for $\epsilon > 0$, $S(t, \cdot) \in C^{\infty}([0, 2\pi])$ for all $t \geq 0$. Since $S$ is periodic in both variables, this implies that there exists a constant $C < +\infty$ such that $\sup_{t,x} |S(t,x)| < C$. Let $(\alpha_n(t))_{n \in \mathbb{Z}}$ denote the Fourier coefficients of $D$; that is $D(t,x) = \sum_n \alpha_n(t) e^{inx}$. Let $\beta_n = \frac{\alpha_n}{2m}$ for $n \neq 0$ and $\beta_0 \equiv 0$. Set $\tilde{D} = \sum_n \beta_n(t)e^{inx}$. Then $\tilde{D}_x = D$. Note that $\tilde{D}$ satisfies

$$\tilde{\partial}_t \tilde{D} = \frac{\epsilon}{2} \tilde{D}_{xx} - \frac{1}{2} S \tilde{D}_x.$$ 

Set $L(t,x) = \frac{\epsilon}{2} \partial_x^2 - \frac{1}{2} S(t,x) \partial_x$ and let $q(t; x, y)$ denote the solution to

$$\begin{cases}
\partial_t q(t; x, y) = L(t, x)q(t; x, y) \\
q(0; x, y) = \delta_0(x - y).
\end{cases}$$ 

Let $Q$ denote the semigroup defined by

$$Qf(x) = \int q(2\pi; x, y)f(y)dy$$

and note that for all integer $N \geq 0$

$$\tilde{D}(2\pi N, x) = Q^N \tilde{D}(0, x).$$

Because there exists a constant $C < +\infty$ such that $\sup_{t,x} |S(t,x)| < +\infty$, it therefore follows by standard and straightforward results, for $\epsilon > 0$ that $\inf_{0 \leq x \leq 2\pi} \inf_{0 \leq y \leq 2\pi} q(2\pi; x, y) > 0$, where the inequality is strict. From this, it follows that $Q$ is the one step transition kernel of an ergodic (discrete time) time homogeneous Markov chain and hence a standard application of the Ergodic theorem yields that $\lim_{N \to +\infty} q(2N\pi, x, y) \to \tilde{q}(y)$, independent of the initial condition $x$. Since $\tilde{D}$ is periodic, it follows that for all $N \geq 0$, $\tilde{D}(0, x) = \tilde{D}(2N\pi, x) = \int \tilde{q}(y)\tilde{D}(0, y)dy \equiv C$ where $C$ is a constant. Since $\int_0^{2\pi} \tilde{D}(0, x)dx = 0$, it follows that $C \equiv 0$ and hence that $\tilde{D} \equiv 0$.

It follows that, for all $\epsilon > 0$, there exists a unique initial condition for equation (75) that provides solutions which are $2\pi$ periodic in the space variable and periodic in the time variable, and that the solution is unique. In the time variable, the periodic solution has period $2\pi$.

**Uniqueness for $\epsilon = 0$** Consider equation (75) with $\epsilon = 0$. Suppose there exist two solutions $2\pi$ periodic in the space variable and $2\pi k$ periodic in the time variable. Denote the solutions by $u^{(1)}$ and $u^{(2)}$. Set $S = u^{(1)} + u^{(2)}$. Then, as before, it is easy to see that there is a function $\tilde{D}$ such that $\tilde{D}_x = u^{(1)} - u^{(2)}$ and such that $\int_0^{2\pi} \tilde{D}(t, x)dx \equiv 0$, satisfying

$$\tilde{\partial}_t \tilde{D} = -\frac{1}{2} S \tilde{D}_x.$$ 

Let $Z$ denote the process defined by the relation

$$Z_{s,t}(x) = x - \frac{1}{2} \int_s^t S(r, Z_{r,t}(x))dr.$$
Then
\[ \tilde{D}(t, x) = \tilde{D}(0, Z_{0,t}(x)). \]

In particular, it holds for all non negative integer \( N \) that
\[ \tilde{D}(0, x) = \tilde{D}(2N\pi, x) = \tilde{D}(0, Z_{0,2\pi N}(x)). \]

From lemma (10), it follows that \( \tilde{D}(0,.) \) is constant on intervals \((x + (j-1)\frac{2\pi}{k}, x + j\frac{2\pi}{k})\) for some positive integer \( k \) and some \( x \in [0, \frac{2\pi}{k}) \). But since \( u^{(1)} - u^{(2)} \) is bounded, it follows that \( \tilde{D}(t,.) \) is Lipschitz in \( x \) for each \( t \) and hence constant (in the \( x \) variable). Since \( \int_0^{2\pi} \tilde{D}(t, x) \, dx \equiv 0 \), it follows that \( \tilde{D}(t, x) \equiv 0 \) and hence uniqueness is established.

It therefore follows directly that, for \( \epsilon = 0 \), there is a unique initial condition for equation (75) which provides solutions to equation (75) that satisfy \( \int_0^{2\pi} u(t, x) \, dx \equiv 0 \), which are \( 2\pi \) periodic in the space variable and periodic in the time variable. The solution to equation (75) with this initial condition is unique and is \( 2\pi \) periodic in the time variable. Uniqueness of periodic solution to equation (75) with \( \epsilon = 0 \) follows.

Using lemma (8), there exists a weakly convergent subsequence of \( u^{(\epsilon)} \) with limit \( u \), which is \( 2\pi \) periodic in both variables. Any such limit \( u \) is a solution to equation (75) for \( \epsilon = 0 \). Since there exists exactly one periodic solution to equation (75) for \( \epsilon = 0 \), it follows that the sequence \( u^{(\epsilon)} \) converges in \( L^2 \), in the relative weak topology, to \( u \). The proof of lemma (11) is now complete.

**Lemma 12.** Let \( u \) denote the periodic solution to equation (75) with \( \epsilon = 0 \). Let \( \tilde{\theta} \) solve
\[
\begin{cases}
\tilde{\theta} = -u(-t, \tilde{\theta}) \\
\tilde{\theta}(0, x) = x
\end{cases}
\]

Then, for all \( x \in [0, 2\pi) \), either
\[
\limsup_{t \to +\infty} |(\tilde{\theta}(t, x) + t - \sin(t)) \mod (2\pi)| = 0,
\]
or
\[
\limsup_{t \to +\infty} |(\tilde{\theta}(t, x) - \pi - t - \sin(t)) \mod (2\pi)| = 0.
\]

**Proof of lemma (12)** Firstly, lemma (10) gives that there exists a positive integer \( k \) such that for all \( x \in [0, 2\pi) \) and \( y \in [0, 2\pi) \),
\[
\lim_{t \to +\infty} |k(\bar{\theta}(t, x) - \bar{\theta}(t, y))| = 0. \tag{78}
\]

Note that \(\bar{\theta}(t, x) = \theta(-t, x)\), where \(\theta\) solves

\[
\begin{align*}
\dot{\theta}(t, x) &= u(t, \theta(t, x)), \\
\theta(0, x) &= x.
\end{align*}
\]

Set

\[
S := \{(q, p) | q \in S^1, p = -u(0, q)\} \tag{79}
\]

and set \(T : S \to S\), defined such that

\[
T(q, p) = (\bar{\theta}(2\pi, q), -u(2\pi, \bar{\theta}(2\pi, q)). \tag{80}
\]

Then it is clear by lemma \(2\), from the construction of the solution to the inviscid Burgers equation, given by the method of characteristics, described in section \(2\) that for all \(n \geq 0\),

\(T^{(n+1)}S \subseteq T^{(n)}S\). Set

\[
\tilde{S} = \bigcap_{n \geq 1} T^{(n)}S. \tag{81}
\]

Then it is clear that \(\tilde{S}\) is non empty and that \(T\tilde{S} = \tilde{S}\). Furthermore, from equation \(78\), it follows that there exists a positive integer \(k\) such that

\[
\tilde{S} = \{(q + \frac{2\pi j}{k}, -u(0, q + \frac{2\pi j}{k}) | j = 0, 1, \ldots, k\}.
\]

It is now shown that \(\tilde{S}\) consists of a single point; either \(\tilde{S} = \{(0, 0)\}\), or \(\tilde{S} = \{\pi, 2\}\).

By construction, the points \(q + \frac{2\pi j}{k}\) provide initial conditions that give periodic solutions to the equation

\[
\dot{\theta} = u(t, \theta).
\]

Note that \(\theta(-t, .) = \bar{\theta}(t, .)\). From the construction, the trajectories \(\theta\) with these initial conditions survive for all time in the construction of the inviscid Burgers equation described in section \(2\).

They are not absorbed into a downward jump and therefore they satisfy

\[
\begin{align*}
\dot{\theta} &= \sin(\theta + \sin(s)) \\
\theta(0) &= q + \frac{2\pi j}{k}, \quad \dot{\theta}(0) = u(0, q + \frac{2\pi j}{k}), \quad j = 0, 1, \ldots, k - 1.
\end{align*}
\]

Since these trajectories do not intersect, it follows from theorem \(7\) that \(k = 1\) and that EITHER \(\tilde{S} = (0, 0)\) OR \(\tilde{S} = (\pi, 2)\). From this, it follows that EITHER
\[
\limsup_{t \to +\infty} |(\tilde{\theta}(t, x) + t - \sin(t)) \mod (2\pi)| = 0 \quad \forall x \in S^1,
\]

OR

\[
\limsup_{t \to +\infty} |(\tilde{\theta}(t, x) - \pi - t - \sin(t)) \mod (2\pi)| = 0 \quad \forall x \in S^1
\]

and lemma (12) is proved.

Let \( \phi^{(\epsilon)}(.) \) denote the function such that \( u^{(\epsilon)}(0,.) := \phi^{(\epsilon)}(.) \) gives a periodic solution to equation (75). Then \( \phi^{(\epsilon)}(.) \) converges in \( L^2 \), in the relative weak topology to \( \phi_x \) and, furthermore, \( \sup_{0 \leq \epsilon \leq 1} \| \phi^{(\epsilon)}(.) \|_2 \leq 6\pi + \sqrt{6(1 + \pi)} \) by lemma (8). It follows by the Ascoli Arzela lemma that \( \phi^{(\epsilon)}(.) \) has a limit \( \phi \) such that \( \lim_{\epsilon \to 0} \sup_{0 \leq x \leq 2\pi} |\phi^{(\epsilon)}(x) - \phi(x)| = 0. \)

**Completing the Counter Example** The arguments given in the previous sections are all standard and yield the following: let \( u^{(\epsilon)}(.,x) \) denote the unique periodic solution to equation (75), then the sequence \( u^{(\epsilon)}(.) \) is convergent in \( L^2 \), in the relative weak topology, to a limit \( u \), which provides the unique periodic solution to equation (75) with \( \epsilon = 0 \). Firstly,

\[
u(t,x) = -\lim_{\epsilon \to 0} \frac{\partial}{\partial x} \log E_P \left[ e^{-\frac{1}{\epsilon} \left( \phi^{(\epsilon)}(x+\sqrt{\epsilon}w) + \int_0^t \cos(sin(s)))ds - \int_0^t \cos(sin(s)+x+\sqrt{\epsilon}w(s))ds \right)} \right],
\]

yielding

\[
u(t,x) = \frac{\partial}{\partial x} \inf_{\xi(t) = x} \left\{ \phi(\xi(t)) + \int_0^t \cos(\sin s)ds - \int_0^t \cos(\xi(s) + \sin s)ds \right\},
\]

by Varadhan’s theorem (1) and the results of section (3). Next, the function \( u \) has representation

\[
u(t,x) = \dot{\eta}(t, \eta^{-1}(t,x)), \quad (82)
\]

where \( \eta \) solves

\[
\begin{cases}
\dot{\eta} = \sin(\eta + \sin t) \\
\eta(0,x) = x, \quad \dot{\eta}(0,x) = \phi_x(x).
\end{cases} \quad (83)
\]

This may be rewritten as

\[
u(t,x) = \xi(t,x)(t) \quad (84)
\]

where \( \xi \) solves

\[
\begin{cases}
\ddot{\xi}(t,x)(s) = \sin(\xi(t,x)(s) + \sin(s)) & 0 \leq s \leq t \\
\dot{\xi}(t,x)(t) = x, \quad \dot{\xi}(t,x)(0) = \phi_x(\xi(t,x)(0)).
\end{cases} \quad (85)
\]
Furthermore, it was also established that the solutions $\xi$ used in representation (84), which solve equation (85), minimise the action functional

$$
A(\xi; t, x) := \left\{ \phi(\xi(t)) + \int_0^t \cos(\sin s) \, ds - \int_0^t \cos(\xi(s) + \sin s) \, ds \right\}
$$

subject to the constraint that $\xi(t) = x$.

Let $\tilde{\theta}(s, x) = \xi^{(0, x)}(-s)$, then

$$\begin{cases}
\ddot{\tilde{\theta}} = \sin(\tilde{\theta} - \sin s) \\
\dot{\tilde{\theta}}(0, x) = x, \quad \dot{\tilde{\theta}}(0, x) = -u(0, x).
\end{cases}$$

Note that

$$\dot{\tilde{\theta}} = -u(-t, \tilde{\theta})$$

for all $t \geq 0$. Recall the definition of $S$ given in equation (79) and recall the definition of $T$ given in equation (80). Recall the definition of $\tilde{S}$ given by equation (81) and recall that $\tilde{S} = \{(0, 0)\} \cup \{((\pi, 2))\}$. The points of $\tilde{S}$ give initial conditions $(\tilde{\theta}(0), \dot{\tilde{\theta}}(0))$ that yield non intersecting periodic solutions to the equation

$$\ddot{\tilde{\theta}} = \sin(\tilde{\theta} - \sin(t)).$$

Lemma (7) showed that, modulo $2\pi$, there existed only two such trajectories;

$$\tilde{\theta}(t) = -t + \sin(t)$$

and

$$\tilde{\theta}(t) = \pi + t + \sin(t).$$

Since these intersect, therefore $\tilde{S}$ can have at most one point. The point $(q, p)$, where $(q, -p) \in \tilde{S}$ represents the initial condition $(\eta(0), \dot{\eta}(0)) = (q, p)$ for trajectories solving equation (83) that survive for all time in the construction (82) of periodic solutions to the inviscid Burgers equation.

It follows that $\tilde{D}$ contains exactly one point, either $(0, 0)$ or $(\pi, 2)$. It follows that there is exactly one Euler Lagrange trajectory in the construction of the solutions to the inviscid Burgers' equation that survives for all time and that the trajectory is EITHER

$$\xi(t) = t - \sin(t)$$

OR

$$\xi(t) = \pi - 2t - \sin(t).$$
But lemma (6) proves that neither of these minimise the associated action functional for $t \geq 2\pi$. It shows that for any periodic solution to the inviscid Burgers’ equation under consideration, there necessarily exist trajectories of the associated Euler Lagrange equation used in the construction which do not minimise the action functional. It follows that the minimising trajectories do not yield periodic solutions to the inviscid Burgers’ equation.

For sufficiently large $t$, the periodic viscosity solution to the equation

$$u_t + \frac{1}{2} (u^2)_x = \sin(x + \sin t)$$

therefore may not be constructed using the associated Euler Lagrange equations which minimise the action functional.

A counterexample has therefore been given to the assertion that the viscosity solution to the inviscid Burgers’ equation in the case of smooth, bounded space / time periodic potentials may always be constructed from the Euler Lagrange trajectories that minimise the associated action functional.

On the other hand, a full proof of this assertion exists, and has been outlined in this note, if Tychonov’s theorem is assumed. The negation of Tychonov’s theorem implies the negation of the Axiom of Choice. It follows that Tychonov’s theorem, and hence the Axiom of Choice lead to contradictions and are therefore inadmissible in mathematical analysis.  

Electronic mail address for correspondence: jonob@mai.liu.se

**Literature Cited**

[1] J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego, P.J. Swart [1991] *On the Dynamics of Fine Structure* J. Nonlinear Sci. 1 (1991) 17 - 90

[2] J.M. Ball and V.J. Mizel [1985] *One-dimensional Variational Problems whose Minimisers do not Satisfy the Euler-Lagrange Equation* Archive for Rational Mechanics and Analysis vol 90, pp 325 - 388.

[3] L. Cesari [1983] *Optimisation : Theory and Applications* Springer

[4] J-D. Deuschel and D. Stroock [1989] *Large Deviations* Academic Press

[5] A. Dembo and O. Zeitouni [1998] *Large Deviations : Techniques and Applications* (second edition) Springer

[6] W. E, K. Khanin, Mazel, Ya. G. Sinai [1997] *Probability Distribution Functions for the Random Forced Burgers Equation* Physical Review Letters, vol. 78 no. 10, March 1997

[7] Weinan E, Khanin, Mazel, Sinai [2000] *Invariant Measure for Burgers Equation with Random Forcing* Annals of Mathematics, vol. 151 pp 877 - 960.

45
[8] Jacek Jakubowski, Rafał Sztencel [2001] *Wstęp do teorii prawdopodobieństwa* wydanie II
SCRIPT, Warsawa, p 327

[9] J.L. Kelley [1950]*The Tychonoff Product Theorem Implies the Axiom of Choice* Fund. Math.
37 pp 75-76

[10] J.M. Noble [2003]*On the Stochastic Burgers Equation* preprint

[11] M. Reed and B. Simon [1980]*Functional Analysis* Academic Press

[12] D. Revuz, M. Yor [1999]*Continuous Martingales and Brownian Motion* (Third Edition)
Springer

[13] L. Tonelli [1923] *Fondamentali di Calcolo delle Variazioni* Zanichelli 2 vols. I, II 1921-23

[14] L. Tonelli [1934]*Sugli integrali del calcolo delle variazioni in forma ordinaria* Ann. Scuola
Norm. Sup. Pisa e (1934) pp 401 - 450
(in L. Tonelli, Opere Scelte vol. III no. 105, Edizioni Cremonese, Roma, 1961)