Stochastic applications of Caputo-type convolution operators with nonsingular kernels

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\textbf{ABSTRACT}

We consider here convolution operators, in the Caputo sense, with nonsingular kernels. We prove that the solutions to some integro-differential equations with such operators (acting on the space variable) coincide with the transition densities of a particular class of Lévy subordinators (i.e. compound Poisson processes with non-negative jumps). We then extend these results to the case where the kernels of the operators have random parameters, with given distribution. This assumption allows greater flexibility in the choice of the kernel's parameters and, consequently, of the jumps' density function.

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1. Introduction

The convolution operators with nonsingular kernels have drawn, in recent years, a wide interest, from both the theoretical and the applied points of view: see, for example, [1–5] and the references therein. Our aim is to analyze their role in the field of stochastic processes: while the random models associated to differential equations with classical fractional derivatives have been extensively studied, the probabilistic applications of this kind of operators are not yet explored.

It should be noted that it is widely debated whether convolution operators with nonsingular kernels can be properly viewed as fractional derivatives (see, for instance, [6–8] and also [9], for more discussion). In particular, it has been shown that, under some assumptions, they could be expressed as finite-order linear combinations of standard derivatives. However, it is not the purpose of this paper to claim any non-local property; on the contrary, our aim is simply to investigate the associated stochastic processes. We believe that these topics have independent interest, regardless of the lack of novelty of the related operators or their local behavior. Finally, as shown in [7], as well as in [10], the operators with nonsingular kernel do not allow the existence of a
corresponding convolution integral for which they provide the left-inverse; however we do not need this characterization for the analysis to follow.

We can now introduce our setting more precisely. Let \( f : \mathbb{R}^+ \to \mathbb{R} \) be a differentiable and absolutely integrable function in \( AC_{\text{loc}}(0, +\infty) \) (i.e. with integrable first derivative); we use here the following operator

\[
     CF D^\alpha_t f(x) := \frac{B(\alpha)}{1 - \alpha} \int_0^x f(z) e^{-\frac{z}{x}(x-z)} dz, \quad x > 0, \ \alpha \in (0, 1),
\]

(1.1)

(where \( B(\alpha) \) is a positive normalizing constant), which is a variant of the so-called “Caputo-Fabrizio fractional derivative” introduced in [11]: in our case the lower limit of integration is zero, since we identify the kernel with the tail of a (bounded) Lévy measure on \((0, +\infty)\).

Analogously, we define the following convolution operator

\[
     D^\alpha_{x, \nu} f(x) := \frac{B(\alpha)}{1 - \alpha} \int_0^x f(z) E_\nu \left( \frac{x}{1 - \alpha} (x-z)^\nu \right) dz, \quad x > 0, \ \alpha \in (0, 1), \ \nu \in (0, 1],
\]

where \( E_\nu(\cdot) \) is the Mittag-Leffler function, i.e. \( E_\nu(x) := \sum_{j=0}^\infty x^j / \Gamma(j+1) \), for \( x \in \mathbb{R} \), \( \text{Re}(\nu) > 0 \). We note that (1.2) reduces, for \( \alpha = \nu \), to the so-called “Atangana-Baleanu fractional derivative” (in the Caputo sense), see [12], while, for \( \nu = 1 \), it coincides with (1.1). As for (1.1), the kernel of (1.2) is nonsingular in the origin, since \( E_\nu(0) = 1 \).

Moreover, we introduce here the following operator

\[
     D^{\alpha, \rho}_{x} f(x) := \frac{1}{\Gamma(\rho)} \int_0^x f(z) \Gamma(\rho; k_x z) dz, \quad x > 0, \ k_x > 0, \ \rho \in (0, 1],
\]

(1.3)

where \( \Gamma(\rho; x) := \int_x^{+\infty} e^{-w} w^{\rho-1} dw \) is the upper-incomplete gamma function. Also (1.3) generalizes (1.1), to which it reduces for \( \rho = 1 \) and \( k_x = x/(1 - x) \). Again the kernel of (1.3) is nonsingular in the origin, since \( \Gamma(\rho; 0) = \Gamma(\rho) \).

Let now recall that \( \psi : (0, +\infty) \to \mathbb{R}^+ \) is a Bernstein function if it is non-negative, of class \( C^\infty \) and such that, for any \( x > 0 \), \( (-1)^k \frac{d^k}{dx^k} \psi(x) \leq 0 \), \( k \in \mathbb{N} \). It is well-known that \( \psi \) admits the following representation (see [13], p. 21)

\[
     \psi(x) = c + bx + \int_0^{+\infty} (1 - e^{-x}) \mu(dz),
\]

(1.4)

where \( c, b \geq 0 \) and \( \mu \) is a measure on \((0, +\infty)\) satisfying \( \int_0^{+\infty} (1 + z) \mu(dz) < \infty \), called Lévy measure. Moreover, the triplet \((c, b, \mu)\) determines uniquely \( \psi \) (and the reverse holds as well). Let us define the stochastic process \( S := S(t), t \geq 0 \), assuming that it is a subordinator, i.e. Lévy and almost surely non-decreasing. Let \( h_S(B, t) := P(S(t) \in B) \) be its transition probabilities, for any \( t \geq 0 \) and Borel interval \( B \subset (0, +\infty) \). Let now \( \bar{\mu}(s) := \int_s^{+\infty} \mu(dz) \), for \( s \geq 0 \), be the so-called tail Lévy measure; if \( \bar{\mu}(\cdot) \) is absolutely continuous on \((0, +\infty)\) and \( \int_0^{+\infty} \mu(z) dz = +\infty \), the corresponding subordinator \( S \) has
It is proved in [16] that, if the Lévy subordinators, the density is infinite for \( x = 0 \), as happens even for well-known processes (such as, for example, the gamma subordinators), the density is infinite for \( x = 0 \). Thus the result in (1.8) must be modified accordingly.

It is well known that, under the assumption that \( \int_0^{+\infty} \mu_\psi(dz) < \infty \), a Lévy process with triplet \( (0, 0, \mu_\psi) \) is a compound Poisson process, i.e.
\[ S_\psi(t) = \sum_{j=1}^{N(t)} X_j^\psi, \]  
\( (1.9) \)

where \( N := N(t), t \geq 0 \) is a Poisson process with parameter \( \lambda = 1 \), independent of \( X_j^\psi \), for any \( j = 1, 2, \ldots \) (see, for example, [17], p. 49). Moreover, the addends \( X_j^\psi \) are, for \( j = 1, 2, \ldots \), non-negative, independent, identically distributed (i.i.d.) random variables with \( F_{X_j^\psi}(x) := P(X_j^\psi \leq x) \) such that

\[ L \{ F_{X_j^\psi}(y) ; \eta \} = \frac{1}{\eta} (1 - \psi(\eta)). \]  
\( (1.10) \)

In this case, the distribution function of \( S_\psi \) reads, for \( y \in \mathbb{R} \),

\[ F_{S_\psi}(y, t) := P\{S_\psi(t) < y\} = e^{-t} \int_0^{\psi(y)} f_{S_\psi}(x, t) dx, \]  
\( (1.11) \)

where

\[ f_{S_\psi}(x, t) = e^{-t} \sum_{n=1}^{\infty} \frac{t^n}{n!} f^{* (n)}_{X_j^\psi}(x), \]  
\( (1.12) \)

is the density of the absolutely continuous component and \( f^{* (n)} \) denotes the \( n \)-fold convolution of the function \( f \). The compound Poisson process has important applications in different fields, ranging from models of insurance risk, to the analysis of statistical behavior in social and biological systems, as well as to the treatment of certain types of random dynamics in physics.

As a consequence of (1.11) and (1.12), the Laplace transform of \( f_{S_\psi} \) is given by

\[ \tilde{f}_{S_\psi}(\eta, t) = e^{-t} \sum_{n=1}^{\infty} \frac{1}{n!} (\tilde{f}_{X_j^\psi}(\eta)t)^n \]

\[ = \left[ \text{by (1.10)} \right] \]

\[ = e^{-t} \sum_{n=1}^{\infty} \frac{1}{n!} ((1 - \psi(\eta)t)^n = e^{-\psi(\eta)t} - e^{-t}, \quad \eta > 0, \; t \geq 0, \]

instead of (1.5). Correspondingly, as we will prove in the next section, the equation satisfied by the density \( f_{S_\psi} \) differs from (1.8) by two additional terms, which depend on the choice of \( \psi \) and whether or not the density of the subordinator is infinite in the origin, for some values of \( t \).

In the last section, we extend these results by generalizing the previous operators to the case of random parameters, thus obtaining distributed-order convolution operators. We provide the explicit solution of the corresponding equations, at least under simplifying assumptions.

Finally, in the concluding remarks, we hint some applications of the obtained results to the risk theory and, in particular, to a continuous-time model, where the surplus process of the insurance company is modeled by a compound Poisson process with non-negative, absolutely continuous claim sizes. For risk theory applications, in the case of standard fractional equations, see, for example, [18] and [19].

We recall the following definitions of well-known special functions that we will apply later: let \( W_{\alpha, \beta}(x) := \sum_{j=0}^{\infty} x^j / j! \Gamma(\alpha j + \beta), \) for \( x, \alpha, \beta \in \mathbb{C}, \) be the Wright function and let
\[ E^\gamma_{a, b}(x) := \sum_{j=0}^{\infty} \frac{x^j (\gamma)_j}{j! \Gamma(xj + \beta)}, \quad \text{Re}(\alpha), \text{Re}(\beta) > 0, \gamma > 0, x \in \mathbb{R}, \]  
\[ (1.14) \]

where \((\gamma)_j := \gamma(\gamma + 1) \ldots (\gamma + j - 1), j = 0, 1, \ldots,\) be the Prabhakar function (or Mittag-Leffler function with three parameters). We will denote, for brevity, the function \((1.14),\) as \(E^\gamma_{a, b}(x)\) when \(c = 1,\) and as \(E^\gamma_a(x),\) when \(c = b = 1:\)

Let us recall the following formula for the Laplace transform of \((1.14):\)

\[ L\left\{ x^{\beta - 1} E^\gamma_{a, b}(\lambda x^\beta); \eta \right\} = \frac{\eta^{\gamma - \beta}}{(\eta^\beta - \lambda)^\gamma}, \]

for \(\text{Re}(\eta) > 0, \text{Re}(\beta) > 0, \lambda \in \mathbb{C} \) and \(|\lambda \eta^{-\beta}| < 1\) (see [20], p. 47).

2. Main results

Let us consider the convolution operator \(\mathcal{D}^\psi_x\) defined in (1.6) under different assumptions on the kernel \(\tilde{\mu}_\psi(\cdot):\) in the exponential case, \(\mathcal{D}^\psi_x\) reduces to the variant of the Caputo-Fabrizio operator defined in (1.1) and the solution is finite in the origin. Then we analyze two cases, where the kernel of the operator is represented by a Mittag-Leffler (with parameter \(\nu \in (0, 1)\)) or an incomplete gamma function (with parameter \(\rho \in (0, 1)\)). These extensions are both quite natural, since the Mittag-Leffler density is the fractional counterpart of the exponential one; on the other hand, the incomplete gamma function coincides with the tail distribution function of a gamma random variable, which generalizes the exponential. As we will see, even though the last two operators reduce to the first one, for \(\nu = 1\) and \(\rho = 1,\) respectively, the exponential case must be treated separately, since the governing equation is not accordingly obtained as special case. This is a consequence of the different behavior of the solutions in the origin.

2.1. Exponential kernel

Theorem 1. Let \(\text{CFD}_x^\alpha\) be the convolution operator defined in (1.1), with \(B(x) = 1 - x.\) Then the solution to the following equation

\[ \frac{\partial}{\partial t} f(x, t) = - \text{CFD}_x^\alpha f(x, t) + k_x(1 - t)e^{-t - k_x}, \quad x \geq 0, t \geq 0, \quad x \in (0, 1), \]

with \(k_x = x/(1 - x)\) and initial condition \(f(x, 0) = 0,\) is given by

\[ f_{S_\psi}(x, t) = k_x t \exp \{-k_x t - t\} W_{1, 2}(k_x t), \quad x \geq 0, \]

(2.2)

and (2.2) is the density of the absolutely continuous component of \(S_\psi\) defined in (1.9), for \(X_j^\psi\) exponentially distributed with parameter \(k_x,\) for \(j = 1, 2,\ldots.\)

Proof. We first prove that the density \(f_{S_\psi}\) of the absolutely continuous component of (1.9) satisfies the following equation

\[ \frac{\partial}{\partial t} f(x, t) = - \mathcal{D}_x^\psi f(x, t) - \tilde{\mu}_\psi(x) f(0, t) + f_{X_j^\psi}(x) e^{-t}, \quad x, t \geq 0. \]

(2.3)
Indeed, by taking the time-derivative of (1.13), we get
\[
\frac{\partial}{\partial t} \tilde{f}_S(\eta, t) = -\psi(\eta)e^{-\psi(\eta)t} + e^{-t},
\]
which coincides with the Laplace transform of the r.h.s. of (2.3), by considering (1.7) and that \( \int_0^{+\infty} e^{-\eta x} \mu_\psi(x)dx = \psi(\eta)/\eta. \) Equation (2.3) obviously holds only when \( f(0, t) < \infty, \) for any \( t. \) By definition (1.1) we can write, in this case, \( f(0, t) < \infty, \) and
\[
\psi(\eta) = \eta \int_0^{+\infty} e^{-\eta x} k_x x dx = \frac{\eta}{\eta + k_x}.
\]
Then equation (2.3) coincides with (2.1) and (1.13) reduces to
\[
\tilde{f}_S(\eta, t) = e^{-\frac{\eta t}{\eta + k_x}} - e^{-t},
\]
whose inverse is equal to (2.2) and is finite also for \( x = 0. \) Moreover, by considering (1.10), we have that \( f_x(x) = \mathcal{L}^{-1}\{k_x/(\eta + k_x); x\} = k_x e^{-k_x x}, \) where \( \mathcal{L}^{-1}\{\cdot; x\} \) denotes the inverse Laplace transform.

**Remark 2.** This result can be alternatively checked directly, by differentiating (2.2) and applying definition (1.1): indeed we have that
\[
\frac{d^\nu}{dx^\nu} f_x(x, t) = k_x e^{-k_x x} - k_x x e^{-k_x x} \sum_{n=1}^{\infty} \frac{(k_x t)^n}{n!(n+1)!} \int_0^x z^n dz = -k_x e^{-k_x x} \sum_{n=1}^{\infty} \frac{(k_x t)^n}{n!(n+1)!} \int_0^x z^n dz
\]
which proves equation (2.1). Formula (2.2) coincides with the special case, for \( \beta = 1, \) of the distribution of the time-fractional compound Poisson process (with exponential addends) obtained in [21]. Moreover, an alternative expression, in terms of modified Bessel functions, is given in [22].

### 2.2. Mittag-Leffler kernel

We now consider the operator defined in (1.2). Even if the latter reduce to (1.1) for \( \nu = 1, \) the following result holds only for \( \nu < 1, \) as explained in the **Remark 4** below.

**Theorem 3.** Let \( D_x^{\alpha, \nu} \) be the convolution operator (1.2), with \( B(\alpha) = 1 - \alpha, \) for \( \alpha, \nu \in (0,1) \) and \( k_x = \alpha/(1-\alpha). \) Then the solution of the following equation
\[
\frac{\partial}{\partial t} f(x, t) = -D_x^{\alpha, \nu} f(x, t) + k_x e^{-t} x^{\nu-1} E_{\nu, \nu}(-k_x x^{\nu}), \quad x \geq 0, t \geq 0,
\]
with initial condition \( f(x, 0) = 0, \) coincides with the density function
\[ f_{S_w}(x, t) = \frac{e^{-t/x}}{x} \sum_{n=1}^{\infty} \frac{(k_2 t x^n)^n}{n!} E_{\nu, \nu n}(-k_2 x^n). \quad (2.7) \]

**Proof.** In this case \( \mu(x) = E_\nu(-k_2 x^\nu) \) and, by considering (1.15), we get

\[ \psi(\eta) = \eta \int_0^{+\infty} e^{-\eta x} E_\nu(-k_2 x^\nu) dx = \frac{\eta^\nu}{\eta^\nu + k_2}. \]

Thus, we have that

\[ \tilde{f}_{S_w}(\eta, t) = e^{-\frac{\eta^\nu}{\eta^\nu + k_2}} - e^{-t} \quad (2.8) \]

The inverse Laplace transform of (2.8) gives (2.7), as can be easily checked by (1.15).

It has been proved in [21] that (2.7) coincides with the density of the absolutely continuous component of the compound Poisson process \( S_w \), under the assumption that the addends \( X^\nu_j \) are i.i.d. random variables, for \( j = 1, 2, \ldots \), with density function

\[ f_{X^\nu_j}(x) = k_2 x^{\nu-1} E_\nu, \nu (-k_2 x^\nu), \quad x \geq 0. \quad (2.9) \]

In this case we must take into account that, for \( \nu < 1 \), the density (2.7) is infinite when \( x = 0 \); thus we can not apply the Laplace transform formula (1.7); however, for a function \( f \) such that the Laplace transform of the derivative exists, we can write that

\[ L\{D_x^{\nu, \nu} f(x); \eta\} = L\left\{ \frac{d}{dx} f(x); \eta\right\} L\{E_\nu(-k_2 x^\nu); \eta\}, \quad (2.10) \]

by considering (1.2). The space-derivative of (2.7) is equal to

\[ \frac{\partial}{\partial x} f_{S_w}(x, t) = e^{-t/x} \sum_{n=1}^{\infty} \frac{(k_2 t)^n}{n!} \sum_{j=0}^{\infty} \frac{(n)_j (-k_2 x^\nu)^j}{j! \Gamma(\nu n + \nu j - 1)} \quad (2.11) \]

and thus, by (2.10) and (1.15), we get

\[ L\{D_x^{\nu, \nu} f_{S_w}(x); \eta\} = \frac{\eta^{\nu-1} e^{-t/x}}{\eta^\nu + k_2} \sum_{n=1}^{\infty} \frac{(k_2 t)^n}{n!} \sum_{j=0}^{\infty} \frac{(n)_j (-k_2)^j}{j! \eta^{\nu j + \nu n - 1}} \]

\[ = \frac{\eta^\nu e^{-t/x}}{\eta^\nu + k_2} \sum_{n=1}^{\infty} \frac{(k_2 t)^n}{n! (n - 1)!} \sum_{j=0}^{\infty} \frac{(n + j - 1)! (-k_2 / \eta^\nu)^j}{j!} \]

\[ = \frac{\eta^\nu e^{-t}}{\eta^\nu + k_2 \left( e^{\frac{k_2 t}{\eta^\nu + k_2}} - 1 \right)} = \frac{\eta^\nu}{\eta^\nu + k_2} \left[ e^{-\frac{k_2 t}{\eta^\nu + k_2}} - e^{-t} \right]. \quad (2.12) \]

Therefore equation (2.6) is proved to hold, by considering (2.12) together with the time-derivative of (2.8).

**Remark 4.** The previous result holds only for \( \nu < 1 \), since formula (2.11) can be rewritten as
\[ \frac{\partial}{\partial x} f_{s_\nu}(x, t) = e^{-t} \sum_{n=1}^{\infty} \frac{(k_2 t)^n}{n!} \sum_{j=1}^{\infty} \frac{(n)_{j!}(-k_2)^j x^{j+n-2}}{j! \Gamma(jv + \nu n - 1)} + e^{-t} \sum_{n=1}^{\infty} \frac{(k_2 t)^n}{n!} \frac{x^{n-2}}{\Gamma(\nu n - 1)} \]

where the first term of the last sum (i.e. for \( n = 1 \)) vanishes when \( \nu = 1 \). Therefore, in this special case, we get

\[ \mathcal{L}\{D^x_{\nu} f_{s_\nu}(x); \nu\} = \frac{\eta e^{-t}}{\eta + k_2} \sum_{n=1}^{\infty} \frac{(k_2 t/\eta)^n}{n! \Gamma(n - \nu + 1)} \sum_{j=0}^{\infty} \frac{(n+j)_{j!}(-k_2)^j x^{j+n-2}}{j! \Gamma(jv + \nu n - 1)} + \frac{\eta e^{-t}}{\eta + \nu} \sum_{n=1}^{\infty} \frac{(k_2 t/\eta)^n}{n!} \frac{x^{n-2}}{\Gamma(\nu n - 1)} \]

which coincides with the Laplace transform of (2.5), while differs from (2.12), with \( \nu = 1 \).

**Remark 5.** As for the exponential case, also Theorem 3 can be, alternatively, proved directly as follows: the time-derivative of (2.7) reads

\[ \frac{\partial}{\partial t} f_{s_\nu}(x, t) = -e^{-t} \sum_{n=1}^{\infty} \frac{(k_2 t x^{\nu})^n}{n!} \left[ E_{\nu, \nu n}^{\nu, \nu n+1} (-k_2 x^{\nu}) - k_2 x^{\nu} E_{\nu, \nu n+1}^{\nu, \nu n+1} (-k_2 x^{\nu}) \right] + k_2 e^{-t} x^{\nu-1} E_{\nu, \nu}^{\nu, \nu} (-k_2 x^{\nu}) \]

\[ = -e^{-t} \sum_{n=1}^{\infty} \frac{(k_2 t x^{\nu})^n}{n!} E_{\nu, \nu n+1}^{\nu, \nu n+1} (-k_2 x^{\nu}) + k_2 e^{-t} x^{\nu-1} E_{\nu, \nu}^{\nu, \nu} (-k_2 x^{\nu}), \]

where, in the last step, we applied formula (3.6) in [23], for \( m = n + 1 \) and \( z = 0 \). By considering (2.11), together with (1.2), we get

\[ D^x_{\nu} f_{s_\nu}(x, t) \]

\[ = e^{-t} \sum_{n=1}^{\infty} \frac{(k_2 t x^{\nu})^n}{n!} \sum_{j=0}^{\infty} \frac{(n)_{j!}(-k_2)^j x^{j+n-1}}{j! \Gamma(jv + \nu n - 1)} \sum_{l=0}^{\infty} \frac{(-k_2)^l x^{l}}{\Gamma(lv + \nu n)} \int_0^1 (1-y)^{jv + n-2} y^l dy \]

\[ = e^{-t} \sum_{n=1}^{\infty} \frac{(k_2 t x^{\nu})^n}{n!} \sum_{m=0}^{\infty} \frac{(-k_2 x^{\nu})^{m}}{\Gamma(n m + \nu m)} \sum_{j=0}^{\infty} \frac{(n+j-1)!}{j!} \]

\[ = e^{-t} \sum_{n=1}^{\infty} \frac{(k_2 t x^{\nu})^n}{n!} \sum_{m=0}^{\infty} \frac{(n+1)_{m!}(-k_2 x^{\nu})^{m}}{\Gamma(m + \nu m)} = e^{-t} \sum_{n=1}^{\infty} \frac{(k_2 t x^{\nu})^n}{n!} E_{\nu, \nu m}^{\nu, \nu m} (-k_2 x^{\nu}), \]
where \((\gamma)_j := \gamma(\gamma + 1) \cdots (\gamma + j - 1)\), \(j = 0, 1, \ldots\) The last step follows by the well-known formula
\[
\sum_{j=0}^{m} \binom{s+j}{j} = \binom{s+m+1}{m};
\] by considering (2.15) with (2.14), we obtain (2.6).

### 2.3. Incomplete-gamma kernel

Let \(\Gamma(\rho; x) := \int_{x}^{+\infty} e^{-w} w^{\rho-1} dw\) be the upper-incomplete gamma function (which is defined for any \(\rho, x \in \mathbb{R}\) and is real-valued for \(x \geq 0\)).

We now consider the operator defined in (1.3), for a differentiable function \(f\) in \(AC_{\text{loc}}(0, +\infty)\):

\[
\frac{\partial}{\partial t} f(x, t) = -D_{k_x}^{\psi, \rho} f(x, t) + \frac{k_x^\rho x^{\rho-1}}{\Gamma(\rho)} e^{-t - k_x x}, \quad x \geq 0, t \geq 0, \quad x \in (0, 1),
\]

with initial condition \(f(x, 0) = 0\), is given by

\[
f_{\mathcal{S}_x}(x, t) = \frac{e^{-t - k_x x}}{x} \sum_{n=1}^{\infty} \frac{(k_x^\rho tx^n)^n}{n!\Gamma(n)}, \quad x, t \geq 0.
\]

Moreover (2.17) is the density of the absolutely continuous component of \(\mathcal{S}_x\) defined in (1.9), for \(X_j^\psi\) i.i.d. random variables with density function

\[
f_{X_j^\psi}(x) = \frac{k_x^\rho x^{\rho-1} e^{-k_x x}}{\Gamma(\rho)} \quad x \geq 0.
\]

**Proof.** We can write, analogously to the previous cases, that \(\tilde{\mu}(x) = \Gamma(\rho; k_x x)/\Gamma(\rho)\) and

\[
\psi(\eta) = \eta \int_{0}^{+\infty} e^{-\eta x} \frac{\Gamma(\rho; k_x x)}{\Gamma(\rho)} dx = \frac{\eta}{\Gamma(\rho)} \int_{0}^{+\infty} e^{-\eta x} \int_{k_x x}^{+\infty} e^{-w} w^{\rho-1} dw dx
\]

\[
= \frac{\eta k_x^\rho}{\Gamma(\rho)} \int_{0}^{+\infty} e^{-k_x y y^{\rho-1}} \int_{0}^{y} e^{-\eta x} dx dy = 1 - \frac{k_x^\rho}{(\eta + k_x)^\rho}.
\]

Therefore we have that

\[
\tilde{f}_{\mathcal{S}_x}(\eta, t) = e^{-t + \frac{k_x^\rho}{(\eta + k_x)^\rho} t} - e^{-t}.
\]

By taking the inverse Laplace transform of (2.20), we get (2.17). The density in (2.18) is correspondingly obtained, in view of (1.10), by inverting.
As in the previous case, in order to prove that (2.17) satisfies equation (2.16), we cannot apply (1.7), since also the function (2.17) is infinite in the origin. From (2.17), we have that

\[ \frac{\partial}{\partial x} f_{S\nu}(x, t) = \frac{k_x}{x} \sum_{n=1}^{\infty} \frac{(k_0 \nu x^0)^n}{n! \Gamma(\nu n)} + \frac{e^{-t-k_x x}}{x^2} \sum_{n=2}^{\infty} \frac{(k_0 \nu x^0)^n}{n! \Gamma(\nu n-1)} \]  

(2.21)

and

\[ \mathcal{L}\left\{ \frac{\partial}{\partial x} f_{S\nu}(x, t); \eta \right\} = \eta \left[ e^{-\frac{\nu x}{\eta+k_x}} - e^{-t} \right]. \]

Then, by taking into account the analogue of (2.10) together with (2.19), we get

\[ \mathcal{L}\{ D_x^\rho f(x, t); \eta \} = \frac{1}{\Gamma(\rho)} \mathcal{L}\left\{ \frac{\partial}{\partial x} f_{S\nu}(x, t); \eta \right\} \mathcal{L}\{ \Gamma(\rho; k_x x); \eta \} = \left( \frac{\eta+k_x}{\eta+k_x} \right)^\rho \left[ e^{-\left( \frac{\eta+k_x}{\eta+k_x} \right)^\rho - t} - e^{-t} \right], \]

(2.22)

so that (2.16) easily follows, by considering the time-derivative of (2.20).

Remark 7. The previous result holds only for \( \rho \) strictly smaller than 1, since formula (2.21) can be rewritten as

\[ \frac{\partial}{\partial x} f_{S\nu}(x, t) = -\frac{k_x e^{-t-k_x x}}{x} \sum_{n=1}^{\infty} \frac{(k_0 \nu x^0)^n}{n! \Gamma(\nu n)} + \frac{e^{-t-k_x x}}{x^2} \sum_{n=2}^{\infty} \frac{(k_0 \nu x^0)^n}{n! \Gamma(\nu n-1)} + \frac{k_0 \nu x^0 e^{-t-k_x x}}{\Gamma(\rho-1)} \]

where the last term vanishes when \( \rho = 1 \). Therefore, in this special case, we get

\[ \mathcal{L}\{ D_x^{\frac{1}{\nu}} f_{S\nu}(x, t); \eta \} = -\frac{k_x e^{-t}}{\eta+k_x} \sum_{n=1}^{\infty} \frac{(k_x t)^n}{n! (\eta+k_x)^n} + e^{-t} \sum_{n=2}^{\infty} \frac{(k_x t)^n}{n! (\eta+k_x)^n} \]

\[ = \frac{\eta}{\eta+k_x} \left[ e^{-\frac{t}{\eta+k_x}} - e^{-t} \right] - \frac{k_x t e^{-t}}{\eta+k_x} \]

which coincides with the Laplace transform of (2.5), while differs from (2.22), with \( \rho = 1 \).

3. The distributed case

We now extend the results of the previous section by generalizing the kernels to the case of random parameters and obtaining a distributed order operator. We can provide the explicit solution of the corresponding equations, at least under simplifying assumptions.

We recall that, in the case of standard fractional equations, the extension to distributed-order derivatives has been treated, for example, in [24] and [25].

We start by considering the exponential kernel in (1.1) and by assuming that \( \alpha \), instead of being a fixed parameter, is a random variable, taking values in (0, 1), with a given distribution.
**Definition 8.** Let $x$ be a random variable, with values in $(0,1)$ almost surely, and let $q : = q(x)$ be its density function. Then, we define the following convolution operator

\[
\mathcal{CFD}_x^{q(x)} f(x) := \int_0^x \frac{d}{dz} f(z) \left( \int_0^1 e^{-\frac{z}{x}(x-z)} q(x)dx \right) dz, \quad x > 0, \tag{3.1}
\]

for a differentiable function $f$, in $\text{AC}_{loc}(0, + \infty)$.

It is immediate to see that, in the special case where $q(z) = \delta(z-x)$ (with $\delta(\cdot)$ denoting the Dirac delta function), formula (3.1) reduces to (1.1) (under the assumption that $B(x) = 1 - x$).

We will assume hereafter that $q$ is such that $\int_0^1 xq(x)/(1-x)dx < \infty$. This assumption implies, for example, in the Beta case, i.e. for $q(x) = x^{r-1}(1-x)^{s-1}/B(r,s)$, $x \in (0,1)$, (where $B(r,s)$ is the Beta function) that we must choose $s > 1$.

**Theorem 9.** Let $\mathcal{CFD}_x^{q(x)}$ be the convolution operator defined in (3.1); let moreover $A_q(x) := \int_0^1 k_x e^{-k_x x} q(x)dx$, for $x \geq 0$, $k_x = \alpha/(1-x)$ and $B_q(x) := \left[ \int_0^1 k_x q(x)dx \right]$ $\left[ \int_0^1 e^{-k_x x} q(x)dx \right]$, for $x \geq 0$. Then the solution to the following equation

\[
\frac{\partial}{\partial t} f(x,t) = - \mathcal{CFD}_x^{q(x)} f(x,t) + e^{-t} A_q(x) - te^{-t} B_q(x), \quad x \geq 0, t \geq 0, \tag{3.2}
\]

(with initial condition $f(x,0) = 0$), is given by the density of the absolutely continuous component of $\Sigma_\varphi$ defined in (1.9), for $X_\varphi^j$ independent and identically distributed r.v.'s, $j = 1, 2, \ldots$, with density $f_{X_\varphi^j}(x) = A_q(x)$, $x \geq 0$.

**Proof.** In this case we have that $\bar{\mu}(x) = \int_0^1 e^{-k_x x} q(x)dx$; thus we get

\[
\psi(\eta) = \eta \int_0^{+\infty} e^{-\eta x} \int_0^1 e^{-k_x x} q(x)dx dx = \int_0^1 \frac{\eta}{k_x + \eta} q(x)dx
\]

and

\[
\tilde{f}_{\Sigma_\varphi^j}(\eta,t) = e^{-t} \int_0^1 \frac{e^{t k_x x}}{k_x + \eta} q(x)dx - e^{-t}.
\]

The density $f_{X_\varphi^j}$ is obtained, in view of (1.10), by the inversion of $1 - \psi(\eta) = \int_0^1 \frac{k_x e^{-k_x x}}{k_x + \eta} q(x)dx$, which gives $f_{X_\varphi^j}(x) = \int_0^1 k_x e^{-k_x x} q(x)dx$. Even if, without any specific assumption on $q(x)$, we can not write an explicit form for the density $f_{\Sigma_\varphi}$ (since (3.3) cannot be inverted), we can nevertheless obtain its value in zero. Indeed, by (1.12) and by considering that $f_{X_\varphi^n}(0) = 0$, for $n > 1$, while $f_{X_\varphi^n}(0) = \int_0^1 k_x q(x)dx$, for $n = 1$, we get

\[
f_{\Sigma_\varphi}(0,t) = te^{-t} \int_0^1 k_x q(x)dx, \quad t \geq 0. \tag{3.4}
\]

By definition (3.1) we have that

\[
\mathcal{L}\left\{ \mathcal{CFD}_x^{q(x)} f(x); \eta \right\} = \mathcal{L}\left\{ \int_0^1 e^{-k_x x} q(x)dx; \eta \right\} \mathcal{L}\left\{ \frac{d}{dx} f(x); \eta \right\} = \int_0^1 \frac{q(x)}{k_x + \eta} dx \left[ \eta \tilde{f}(\eta) - f(0) \right].
\]
In view of (3.3) and (3.4), we can write

\[
\mathcal{L}\{\frac{CFD_x^q f(x,t)}{n!} \} = \int_0^1 \eta_n(x) dx \frac{e^{-t} \int_0^1 \frac{q(x) dx}{k_n + \eta} - e^{-t}}{-t} \int_0^1 \frac{q(x) dx}{k_n + \eta}\left[ \int_0^1 \frac{\eta_n(x) dx}{k_n + \eta} \right].
\]

(3.5)

By comparing (3.5) with the time-derivative of (3.3), we easily prove that (3.2) holds.

For \( q(z) = \delta(z - x) \), we obtain that \( A_q(x) = B_q(x) = k_x e^{-k_x x} \) and the previous result coincides with that presented in Theorem 1. In another special case, i.e. for \( q(z) = q_1 \delta(z - x_1) + q_2 \delta(z - x_2) \), for \( 0 < x_1 < x_2 < 1 \) and \( q_1, q_2 \in [0,1] \) such that \( q_1 + q_2 = 1 \), we can obtain an explicit form of the density \( f_{S_0} \), which generalizes (2.2).

**Corollary 10.** Let \( CF \tfrac{D_x^{x_1,x_2}}{n!} \) be the convolution operator defined in (3.1) under the assumption that \( q(z) = q_1 \delta(z - x_1) + q_2 \delta(z - x_2) \), \( z \geq 0 \), for \( 0 < x_1 < x_2 < 2 \) and \( q_1, q_2 \in [0,1] \) such that \( q_1 + q_2 = 1 \). Then the solution to the following equation

\[
\frac{\partial}{\partial t} f(x,t) = - \frac{CFD_x^{x_1,x_2}}{n!} f(x,t) + \left[ q_1 k_{x_1} e^{-t-k_{x_1} x} + q_2 k_{x_2} e^{-t-k_{x_2} x} \right] + t(q_1 k_{x_1} + q_2 k_{x_2}) \left[ q_1 e^{-t-k_{x_1} x} + q_2 e^{-t-k_{x_2} x} \right],
\]

(3.6)

\( x \geq 0, t \geq 0 \), (with initial condition \( f(x,0) = 0 \), is given by

\[
f_{S_0}(x,t) = \frac{e^{-t-k_{x_2} x}}{x} \sum_{n=1}^{\infty} \frac{(q_2 k_{x_2} x t)^n}{n!} \sum_{j=0}^{n} \binom{n}{j} \left( \frac{q_1 k_{x_1}}{q_2 k_{x_2}} \right)^j E_{1,n}((k_{x_2} - k_{x_1})x), \quad x \geq 0.
\]

(3.7)

Moreover (3.7) is the density of the absolutely continuous component of \( S_0 \) defined in (1.9), for \( X^j \) independent and identically distributed r.v.’s, \( j = 1,2, \ldots \), with density \( f_{X^j}(x) = q_1 k_{x_1} e^{-k_{x_1} x} + q_2 k_{x_2} e^{-k_{x_2} x}, \ x \geq 0 \).

**Proof.** Equation (3.6) is obtained as special case of (3.2). We only need to prove that (3.7) satisfies equation (3.6), by checking that its Laplace transform coincides with (3.3), as follows

\[
\tilde{f}_{S_0}(\eta,t) = e^{-t} \sum_{n=1}^{\infty} \frac{(q_2 k_{x_2} x t)^n}{n!} \sum_{j=0}^{n} \binom{n}{j} \left( \frac{q_1 k_{x_1}}{q_2 k_{x_2}} \right)^j \int_0^\infty e^{-x(k_2 + \eta)) (k_{x_2} - k_{x_1})x} dx
\]

\[
= e^{-t} \sum_{n=1}^{\infty} \frac{(q_2 k_{x_2} x t)^n}{n!} \sum_{j=0}^{n} \binom{n}{j} \left( \frac{q_1 k_{x_1}}{q_2 k_{x_2}} \right)^j \frac{(k_{x_2} + \eta)^{j-n}}{(k_{x_1} + \eta)^j}
\]

\[
= e^{-t} \sum_{n=1}^{\infty} \frac{(q_2 k_{x_2} x t/(k_{x_1} + \eta))^{n}}{n!} \left( \frac{q_1 k_{x_1}}{q_2 k_{x_2}} \frac{k_{x_1} + \eta}{k_{x_2} + \eta} \right)^n
\]

\[
= e^{-t} \left[ \exp \left\{ \frac{k_{x_1} k_{x_2} + \eta(q_1 k_{x_1} + q_2 k_{x_2})}{(k_{x_1} + \eta)(k_{x_2} + \eta)} \right\} - 1 \right].
\]

The previous expression coincides with (3.3), by the assumption on \( q \) and by considering that \( q_1 + q_2 = 1 \).
In the case of the Mittag-Leffler kernel, we generalize the operator (1.2) to the distributed case, as follows.

**Definition 11.** Let \( \nu \) be a random variable, with values in \((0, 1)\) almost surely, and let \( q := q(\nu) \), be its density function. Then, we define the following convolution operator

\[
D^\nu_x,q(\nu)f(x) := \int_0^x \frac{d}{dz} f(z) \left[ \int_0^1 E_\nu(-kz(x-z)\nu)q(\nu)\,d\nu \right] \,dz, \quad x > 0, \ \alpha \in (0, 1), \tag{3.8}
\]

for a differentiable function \( f \) in \( AC_{loc}(0, +\infty) \).

**Theorem 12.** Let \( D^\nu_x,q(\nu) \) be the convolution operator defined in (3.8); then the solution to the following equation

\[
\frac{\partial}{\partial t} f(x,t) = - D^\nu_x,q(\nu) f(x,t) + k_x e^{-t} \int_0^1 x^{\nu-1} E_{\nu,\nu}(-k_x x^\nu) q(\nu) \,d\nu, \quad x \geq 0, t \geq 0, \tag{3.9}
\]

(with initial condition \( f(x, 0) = 0 \)), is given by

\[
f_{S_\nu}(x, t) = e^{-t} \sum_{n=1}^\infty \frac{(k_t)^n}{n!} \int_0^1 x^n E^n_{\nu,\nu}(-k_x x^n) q(\nu) \,d\nu. \tag{3.10}
\]

Moreover, (3.10) is the density of the absolutely continuous component of \( S_\nu \) defined in (1.9), for \( X^\nu_j \) independent and identically distributed r.v.’s, \( j = 1, 2, \ldots \), with density \( f_{X^\nu_j}(x) = k_x \int_0^1 x^{\nu-1} E_{\nu,\nu}(-k_x x^\nu) q(\nu) \,d\nu \), \( x \geq 0 \).

**Proof.** Following the same lines of the non-distributed case, we can write that

\[
\tilde{f}_{S_\nu}(\eta, t) = e^{-t} \int_0^1 \frac{\eta^{\nu-1}}{\eta^{\nu} + k_x} q(\nu) \,d\nu - e^{-t}
\]

and

\[
\mathcal{L}\{D^\nu_x,q(\nu) f_{S_\nu}(x, t); \eta\} = \int_0^1 \frac{\eta^{\nu-1}}{\eta^{\nu} + k_x} q(\nu) \,d\nu \left[ e^{-t} \int_0^1 \frac{\eta^{\nu-1}}{\eta^{\nu} + k_x} q(\nu) \,d\nu - e^{-t} \right],
\]

so that equation (3.9) easily follows. \( \square \)

As far as the incomplete-gamma kernel case is concerned, we can extend the results of section 2.3 by considering the operator defined in the following

**Definition 13.** Let \( \rho \) be a random variable, with values in \((0, 1)\) almost surely, and let \( q := q(\rho) \), be its density function. Then, we define the following convolution operator

\[
D^\rho_x,q(\rho)f(x) := \int_0^x \frac{d}{dz} f(z) \left( \int_0^1 \frac{\Gamma(\rho; k_z z)}{\Gamma(\rho)} q(\rho) \,d\rho \right) \,dz, \quad x > 0, \ \alpha \in (0, 1), \tag{3.11}
\]

for a differentiable function \( f \) in \( AC_{loc}(0, +\infty) \).

Then, in this case, we prove the following

**Theorem 14.** Let \( D^\rho_x,q(\rho) \) be the convolution operator (3.11); then the solution to the following equation


Analogously to the non-distributed case, we can write that
\[
\frac{\partial}{\partial t} f(x, t) = - \mathcal{D}_x^{\psi, q(\rho)} f(x, t) + e^{-t-k_x} \int_0^1 \frac{k_\rho x^{\rho-1}}{\Gamma(\rho)} q(\rho) d\rho, \quad x \geq 0, t \geq 0, \tag{3.12}
\]
(with initial condition \(f(x, 0) = 0\), is given by the density of the absolutely continuous component of \(S_\psi\) defined in (1.9), for \(X^{\psi}_j\) independent and identically distributed r.v.'s, \(j = 1, 2, \ldots, \) with density \(f_{X^{\psi}_j}(x) = e^{-k_x} \int_0^1 \frac{k_\rho x^{\rho-1}}{\Gamma(\rho)} q(\rho) d\rho, \quad x \geq 0.\)

**Proof.** Analogously to the non-distributed case, we can write that
\[
\tilde{f}_{S_\psi}(\eta, t) = e^{-t} \int_0^1 \frac{(\eta + k_\rho)^\rho - k_\rho^\rho}{(\eta + k_\rho)\rho} q(\rho) d\rho \left[ e^{-t} \int_0^1 \frac{(\eta + k_\rho)^\rho - k_\rho^\rho}{(\eta + k_\rho)\rho} q(\rho) d\rho - e^{-t} \right].
\]
and
\[
\mathcal{L}\{\mathcal{D}_x^{\psi, q(\rho)} f_{S_\psi}(x, t); \eta\} = \int_0^1 \frac{(\eta + k_\rho)^\rho - k_\rho^\rho}{(\eta + k_\rho)\rho} q(\rho) d\rho \left[ e^{-t} \int_0^1 \frac{(\eta + k_\rho)^\rho - k_\rho^\rho}{(\eta + k_\rho)\rho} q(\rho) d\rho - e^{-t} \right].
\]
On the other hand,
\[
\frac{\partial}{\partial t} \tilde{f}_{S_\psi}(\eta, t) = -e^{-t} \int_0^1 \frac{(\eta + k_\rho)^\rho - k_\rho^\rho}{(\eta + k_\rho)\rho} q(\rho) d\rho \int_0^1 \frac{(\eta + k_\rho)^\rho - k_\rho^\rho}{(\eta + k_\rho)\rho} q(\rho) d\rho + e^{-t}.
\]
\[\square\]

As for the exponential kernel case, when \(q(z) = q_1 \delta(z - \rho_1) + q_2 \delta(z - \rho_2), \) for \(0 < \rho_1 < \rho_2 < 1,\) we can obtain an explicit form of the density \(f_{S_\psi},\) which generalizes (2.17).

**Corollary 15.** Let \(\mathcal{D}_x^{\psi, q(\rho)}\) be the convolution operator defined in (3.11) under the assumption that \(q(z) = q_1 \delta(z - \rho_1) + q_2 \delta(z - \rho_2), \) for \(0 < \rho_1 < \rho_2 < 1\) and \(q_1, q_2 \in [0, 1]\) such that \(q_1 + q_2 = 1.\) Then the solution to the following equation
\[
\frac{\partial}{\partial t} f(x, t) = - \mathcal{D}_x^{\psi, q(\rho)} f(x, t) + e^{-t-k_x} \frac{q_1 k_\rho^\rho x^{\rho-1}}{\Gamma(\rho)} \frac{q_2 k_\rho^\rho x^{\rho-1}}{\Gamma(\rho)}, \quad x \geq 0, t \geq 0, \] (with initial condition \(f(x, 0) = 0), \) is given by
\[
f_{S_\psi}(x, t) = \frac{e^{-t-k_x}}{x} \sum_{l=0}^{\infty} \frac{(q_1 k_\rho^\rho x^{\rho-1})^l}{l!} \mathcal{W}_{\rho_1, \rho_2,l}(q_2 k_\rho^\rho x^{\rho-1}), \quad x \geq 0. \tag{3.13}
\]

Moreover (3.13) is the density of the absolutely continuous component of \(S_\psi\) defined in (1.9), for \(X^{\psi}_j\) independent and identically distributed r.v.'s, \(j = 1, 2, \ldots, \) with density \(f_{X^{\psi}_j}(x) = \frac{q_1 k_\rho^\rho x^{\rho-1}}{\Gamma(\rho)} + \frac{q_2 k_\rho^\rho x^{\rho-1}}{\Gamma(\rho)}, \quad x \geq 0.\)

**Proof.** We omit the details of the calculations which retrace those of Corollary 10. \[\square\]
4. Risk-theory applications and concluding remarks

So far we have described the interplay between integro-differential equations based on Caputo-like operators (with nonsingular kernels) and the densities of the corresponding stochastic processes. We proved that, ranging from exponential to Mittag-Leffler or incomplete-gamma kernels, we obtain compound Poisson processes with very different jump distributions.

In particular, passing from the Caputo-Fabrizio operator to the Atangana-Baleanu one, we even lose the finiteness of all the moments of the jumps $X_j^\psi$, $j = 1, 2, \ldots$. Indeed, in the exponential case, $\mathbb{E}S_\psi = \mathbb{E}X_j^\psi = 1/k_\psi = (1 - \alpha)/\alpha$, as can be easily checked by applying the Wald formula and considering that the Poisson rate is unitary. Analogously, in the distributed-order exponential case (i.e. with the operator defined in Def.8), the mean value of $S_\psi$ is equal to $\mathbb{E}((1 - \alpha)/\alpha)$.

On the other hand, due to the well-known power-law behavior of the Mittag-Leffler distribution given in (2.9), the expected value of $S_\psi$ and $X_j^\psi$ are both infinite and the same holds for the distributed order counterpart, obtained under Def.11.

Finally, the incomplete-gamma case can be considered intermediate between the previous ones: indeed, the mean value is $\mathbb{E}S_\psi = \mathbb{E}X_j^\psi = \rho/k_\psi$ (or $\mathbb{E}(\rho/k_\psi)$ in the distributed case) and thus differs from the exponential case only by the constant $\rho$. Nevertheless the behavior of the density (2.18) is completely different, in the origin, from the exponential one, since it tends to infinity, as in the Mittag-Leffler case.

All the previous results can be applied to a continuous-time risk model. If we define the risk reserve process

$$R(t) := a + \beta t - \sum_{j=1}^{N(t)} X_j^\psi,$$  \hspace{1cm} (4.1)

where $a \geq 0$ is the initial risk reserve, $\beta > 0$ is the premium collection rate and $X_j^\psi$, for $j = 1, 2, \ldots$, are the claims occurring according to the Poisson process, then we have that $R(t) = a + \beta t - S_\psi(t)$. If we denote the expected claim size by $\mu$ (i.e. $\mu := \mathbb{E}X_j^\psi$), it is evident that, in order to fulfill the net profit condition $\beta > \mu$ (considering that our Poisson rate is equal to 1), we must discard the case of Mittag-Leffler distributed claim sizes, since the expected value of (2.9) is infinite.

In the other cases, by considering (4.1) together with (1.13), we can obtain the differential equation satisfied by the moment generating function of $R(t)$, $t \geq 0$, when the claim size has distribution with Laplace transform given in (1.10). By taking the first derivative (with respect to the time variable) of

$$\Phi_R(\eta, t) := \mathbb{E}e^{\eta R(t)} = e^{\eta(a+\beta t)}\mathbb{E}e^{-\eta \sum_{j=1}^{N(t)} X_j^\psi}$$  \hspace{1cm} (4.2)

we get
\[
\frac{\partial}{\partial t} \Phi_R(\eta, t) = \beta \eta \Phi_R(\eta, t) + \eta e^{\eta(\alpha + \beta t)} \frac{\partial}{\partial t} f_s(\eta, t) \\
= \beta \eta \Phi_R(\eta, t) + \eta e^{\eta(\alpha + \beta)} \left[ e^{-\psi(\eta)} e^{-\psi(\eta) t} \right] \\
= [\beta \eta - \psi(\eta)] \Phi_R(\eta, t) + e^{\eta(\alpha + \beta) - t} [1 - \psi(\eta)],
\]

(4.3)

which is satisfied by (4.2). It is evident from the first line of (4.3) that also the differential equation governing the risk reserve process can be expressed in terms of the convolution operator \(D_x^\psi\) treated here.

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