A Revisit of Symmetry Analysis and Group Classifications of Boiti–Leon–Pempinelli System in (2+1)-Dimensions

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Abstract
In this paper, the Boiti–Leon–Pempinelli system in (2+1)-dimensions is revisited for Lie symmetries and invariant solutions. An infinite dimensional Lie algebra is obtained using the Lie invariance criterion and is further classified into one, two and three-dimensional optimal list of subalgebras. We obtain new explicit exact solutions involving arbitrary functions that have been missed in previous work (Kumar et al. in Comput Math Appl 70(3):212–221, 2015; Int J Appl Comput Math 7(1):1–17, 2021).

Keywords Lie symmetries · Similarity solutions · Optimal system · BLP equation

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Introduction
The Lie symmetry analysis is important in the study of partial differential equations for exact solutions and conservation laws [2, 13–16, 19, 21, 23, 27]. Much research has been conducted in the field of Lie symmetries, and with the development of powerful computational tools such as Maple and Mathematica, it is now possible to investigate partial differential equations for Lie symmetries with much ease. Over the last three decades, researchers have not left a single PDE for Lie symmetry analysis. However, where infinite-dimensional Lie algebra is concerned, there is still a lot of scope in Lie symmetry analysis. Because of the presence of arbitrary functions in the Lie algebra, Lie symmetry analysis for group classifications and invariant solutions becomes more difficult. The classic example is following Boiti–Leon–Pempinelli system (BLP system):

\[ L_1 \equiv u_{ty} - 2u_y u_x - 2uu_{xy} + uu_{xy} + 2v_{xxx} = 0, \]
\[ L_2 \equiv v_t - 2uv_x - v_{xx} = 0, \]

where \( u = u(x, y, t) \), \( v = v(x, y, t) \) and subscripts denote partial differentiation with respect to corresponding variables in the subscript. The system (1) has been investigated for Lie
symmetries in Refs. [9–12, 28, 29]. However, the symmetry analysis in all of these papers is incomplete and inadequate. In Ref. [28], for example, an infinite Lie group involving arbitrary functions is obtained, and invariant solutions are constructed by fixing the arbitrary functions. In Refs. [10, 11], the arbitrary functions in an infinite Lie group are again fixed and group invariant solutions are obtained, and the authors used the same technique to fix arbitrary functions in their recent work [12]. Furthermore, in Ref. [29], the authors made the same error of restricting arbitrary functions to polynomials. Similar deficiencies can be found in Ref. [9] in addition to sloppy group classifications. Moreover, in all these papers the authors have not emphasized on group classifications of Lie algebra of BLP system. Although in Ref. [9] in addition to sloppy group classifications. Moreover, in all these papers the authors

can be obtained by restricting arbitrary functions to first-degree polynomials only. Therefore, randomly fixing arbitrary functions in the Lie algebra is completely incorrect and has no physical significance. So in this work, the arbitrary functions in the infinite Lie group are kept intact and Lie symmetry analysis is performed without loss of generalization.

For Lie group of point transformations that leave system (1) invariant, the following differential operator must be introduced

\[ X = \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u} + \eta^v \frac{\partial}{\partial v}, \quad (2) \]

here each \( \xi \) and \( \eta \) depend on independent variables \( x, y, t \) and dependent variables \( u, v \). These are called infinitesimals that can be constructed from third order prolongation \( X^{(3)} \),

\[ X^{(3)} = X + \eta^{(1)u}_i \frac{\partial}{\partial u^i} + \eta^{(1)v}_i \frac{\partial}{\partial v^i} + \cdots + \eta^{(3)u}_{i_1i_2i_3} \frac{\partial}{\partial u^{i_1i_2i_3}} + \eta^{(3)v}_{i_1i_2i_3} \frac{\partial}{\partial v^{i_1i_2i_3}}. \quad (3) \]

This prolonged operator will helps to extend all the derivatives appearing in the system (1) and the Lie invariance criterion requires the system to be invariant under this prolonged operator on the solution surface of the system (1), that is,

\[ X^{(3)}L_i \big|_{L_i=0} = 0, \quad i = 1, 2, \quad (4) \]

and the most general solution to the invariance criterion (4) is obtained as follow:

\[ \xi^x = 2xf_1'(t) + 2f_2(t), \quad \xi^y = h_1(y), \quad \xi^t = 4f_1(t), \]
\[ \eta^u = -2uf_1'(t) - xf_1''(t) - f_2'(t), \quad \eta^v = -vh_1'(t) + h_2(t), \quad (5) \]

where the arbitrary functions \( f_1(t), f_2(t), h_1(y) \) and \( h_2(y) \) will generate following infinite-dimensional Lie algebra \( g \):

\[ X_1 = 2xf_1'(t) \frac{\partial}{\partial x} + 4f_1(t) \frac{\partial}{\partial t} - \left( 2uf_1'(t) + xf_1''(t) \right) \frac{\partial}{\partial u}, \]
\[ X_2 = 2f_2(t) \frac{\partial}{\partial x} - f_2'(t) \frac{\partial}{\partial u}, \]
\[ X_3 = h_1(y) \frac{\partial}{\partial y} - vh_1'(t) \frac{\partial}{\partial v}, \]
\[ X_4 = h_2(y) \frac{\partial}{\partial v}. \quad (6) \]
Table 1  Commutation relations for Lie algebra (6)

| [X_i, X_j] | X_1 | X_2 | X_3 | X_4 |
|------------|-----|-----|-----|-----|
| X_1        | 0   | -X_2(f_2 f_1' - 2 f_1 f_2') | 0   | 0   |
| X_2        | X_2(f_2 f_1' - 2 f_1 f_2') | 0   | 0   | 0   |
| X_3        | 0   | 0   | 0   | X_4(h_1' h_2 + h_1 h_2') |
| X_4        | 0   | 0   | -X_4(h_1' h_2 + h_1 h_2') | 0   |

Table 2  Commutation relations for Lie algebra (6)

| Ad_{exp(\epsilon X_i)} (X_j) | X_1 | X_2 | X_3 | X_4 |
|-------------------------------|-----|-----|-----|-----|
| X_1                           | e^\epsilon X_2 | X_3 | X_4 |
| X_2                           | X_1 - \epsilon X_2 | X_2 | X_3 | X_4 |
| X_3                           | X_1 | X_2 | X_3 | e^{-\epsilon} X_4 |
| X_4                           | X_1 | X_2 | X_3 + \epsilon X_4 | X_4 |

The Lie algebra g is solvable with following chain of ideals:

\{X_2\} \subset \{X_1, X_2\} \subset \{X_1, X_2, X_4\} \subset \{X_1, X_2, X_3, X_4\},

and it is closed under Lie commutation [X_i, X_j] = X_i X_j - X_j X_i and the Jacobi’s identity [X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0 is also satisfied. The results of all Lie commutations are listed in the Table 1.

Remark 1.1  It is worth noting that the Lie algebra obtained at (6) is more compact than the one obtained in Ref. [29]. The determining equations do not appear to be solved judiciously in Ref. [29].

Remark 1.2  A conclusion can be drawn from the structure (see Lie commutations in the Table 1) of infinite-dimensional Lie algebra (6) that it contains Virasoro subalgebra which is typical characteristics of integrable systems [3–5, 18] (see also [24]). It is therefore very likely that the Boiti–Leon–Pempinelli system (1) may be completely integrable, though it is not quite rigorous way to claim for the integrability.

Multi-dimensional Group Classifications

We define adjoint transformation

\[ \text{Ad}_{\exp(\epsilon X_i)}(X_j) = e^{-\epsilon X_i} X_j e^{\epsilon X_i} = \tilde{X}_j(\epsilon). \]  

(7)

The adjoint transformation (7) can be written through Lie brackets using Campbell-Hausdorff formula as

\[ \text{Ad}_{\exp(\epsilon X_i)}(X_j) = X_j - \epsilon [X_i, X_j] + \frac{\epsilon^2}{2} [X_i, [X_i, X_j]] - \ldots, \]  

(8)

where [\ldots] is Lie bracket defined in Table 1. The relation (8) helps to compile a table of the adjoint actions among each element in Lie algebra (6). All such adjoint actions are listed in the Table 2.
For general element \( X = \sum_{i=1}^{4} X_i \) in the Lie algebra (6), the successive application of the formula (7) yields full adjoint action as follows:

\[
\text{Ad}_{\exp(\epsilon_1 X_1)} \text{Ad}_{\exp(\epsilon_3 X_3)} \text{Ad}_{\exp(\epsilon_4 X_4)} \text{Ad}_{\exp(\epsilon_2 X_2)}(X) = \sum_{i=1}^{4} \tilde{a}_i X_i, \tag{9}
\]

where the coefficient \( \tilde{a}_i \) are given as follows:

\[
\begin{align*}
\tilde{a}_1 &= a_1, \\
\tilde{a}_2 &= e^{\epsilon_1}(-a_1 \epsilon_2 + a_2), \\
\tilde{a}_3 &= a_3, \\
\tilde{a}_4 &= e^{-\epsilon_3}(a_3 \epsilon_4 + a_4).
\end{align*}
\tag{10}
\]

As described in the Ref. [24], the Killing form and the invariant function can be calculated as \( K(X, X) = a_1^2 + a_2^2 \) and \( \phi = f(a_1, a_3) \) (for arbitrary function \( f \)) respectively. Both agree with the fact that \( a_1 \) and \( a_3 \) are invariants of full adjoint action as given in (10).

**One-Dimensional Optimal System**

The procedure for the constructing of an optimal system is quite wild, but the construction becomes simple and straightforward when invariants of full-adjoint action are known [1, 20, 22]. For example, in the present case, \( a_1 \) and \( a_3 \) are invariants of full-adjoint action, by restricting their values the simplification of (9) becomes quite easy. The detailed simplification of (9) with the help of four different cases is described below.

**Case 2.1** When \( a_1 \neq 0, a_3 \neq 0 \), we set \( \epsilon_1 = \epsilon_3 = 0 \) (this means in (9), the adjoint actions \( \text{Ad}_{\exp(\epsilon_1 X_1)} \), \( \text{Ad}_{\exp(\epsilon_3 X_3)} \) are being inactivated), and on setting \( \epsilon_2 = \frac{a_2}{a_1} \) and \( \epsilon_4 = -\frac{a_4}{a_3} \), the coefficients \( \tilde{a}_2, \tilde{a}_4 \) vanish. The general element \( X \) is thus simplifies to \( X_1 + \alpha X_3 \) for \( \alpha = \frac{a_3}{a_1} \).

**Subcase 211** When \( a_1 \neq 0, a_3 = 0 \), and repeating adjoint actions used in case 2.1 will simplify general element \( X \) into \( X_1 \).

**Case 2.2** When \( a_1 = 0, a_3 \neq 0 \), on setting \( a_3 = 1, \epsilon_3 = 0, \epsilon_4 = -a_4 \), the X reduces to \( a_2 e^{\epsilon_1} X_2 + X_3 \) and the coefficient of \( X_2 \) can be scaled to \( \pm 1 \) by taking \( \epsilon_1 = \log \left| \frac{1}{a_2} \right| \), and hence the final simplification would be \( X_3 \pm X_2 \).

**Subcase 221** When \( a_1 = 0, a_2 = 0, a_3 \neq 0 \), the obvious simplification of \( X \) would be \( X_3 \).

**Case 2.3** When \( a_1 \neq 0, a_3 = 0 \), on setting \( a_1 = 1, \epsilon_1 = 0, \epsilon_2 = a_2 \), the X reduces to \( X_1 + a_4 e^{-\epsilon_3} X_4 \) and the coefficient of \( X_4 \) can be scaled to \( \pm 1 \) by taking \( \epsilon_3 = \log |a_4| \), and hence the final simplification would be \( X_1 \pm X_4 \).

**Case 2.4** When \( a_1 = 0, a_3 = 0 \), for \( \epsilon_1 = 0 \), the X reduces to \( a_2 X_2 + a_4 e^{-\epsilon_3} X_4 \), and the final simplification can be achieved as \( X_2 \pm X_4 \) by taking \( \epsilon_3 = \log \left| \frac{a_4}{a_2} \right| \).

**Subcase 241** When \( a_1 = 0, a_2 = 0, a_3 = 0 \), the obvious simplification of \( X \) would be \( X_4 \).

**Subcase 242** When \( a_1 = 0, a_3 = 0, a_4 = 0 \), the obvious simplification of \( X \) would be \( X_2 \).

The above cases and sub-cases can be summed up to one-dimensional optimal list of sub-algebras \( \Theta_1 \) as follow:

\[
X_1 + \alpha X_3, X_1, X_3 + \epsilon X_2, X_3, \\
X_1 + \epsilon X_4, X_2 + \epsilon X_4, X_4, X_2, \quad \alpha = \frac{a_3}{a_1}, \epsilon = \pm 1. \tag{11}
\]
Two-Dimensional Optimal System

All two-dimensional subalgebras are of type \((X_i, X)\), where \(X_i\) runs through optimal list of subalgebras (11) and \(X\) is general element from Lie algebra (6) such that

\[
\left[ X_i, \sum_{i=1}^{4} a_i X_i \right] = \alpha X_i,
\]

where \(\alpha\) is constant and the constants \(a_i\) can be determined by equating coefficients of \(X_i\) on the both sides of (12). In addition, a detailed procedure for the construction of a two-dimensional optimal subalgebra can be seen in the Ref. [8] and a recent work by the author himself [24]. The following is the optimal list of two-dimensional subalgebras \(\Theta_2\) formed by this construction:

\[
\begin{align*}
G_1 & (X_1 + \alpha X_3, X_2), \\
G_2 & (X_1 + \alpha X_3, X_4), \\
G_3 & (X_3 + \epsilon X_2, X_4), \\
G_4 & (X_3, X_1), \\
G_5 & (X_1 + \epsilon X_4, X_2), \\
G_6 & (X_2 + \epsilon X_4, X_1 + X_3), \\
G_7 & (X_4, X_1), \\
G_8 & (X_2, X_3), \\
G_9 & (X_2, X_4).
\end{align*}
\]

Three-Dimensional Optimal System

The optimal list of three-dimensional subalgebras \(\Theta_3\) can be constructed in analogous manner as described in 2.2. The following list of three-dimensional optimal subalgebras is obtained:

\[
\begin{align*}
\mathcal{H}_1 & (X_1, X_3, X_2), \\
\mathcal{H}_2 & (X_1, X_4, X_2), \\
\mathcal{H}_3 & (X_1, X_3, X_4), \\
\mathcal{H}_4 & (X_2, X_3, X_4), \\
\mathcal{H}_5 & (X_1 + X_3, X_2, X_4), \\
\mathcal{H}_6 & (X_1 - X_3, X_2, X_4), \\
\mathcal{H}_7 & (X_1 + \alpha X_3, X_2, X_4).
\end{align*}
\]

In following sections, we shall present similarity reductions corresponding to optimal list of subalgebras (11), (13) and (14).

Similarity Reductions and Invariant Solutions

The number of independent variables in BLP-equation (1) can be reduced by performing similarity reduction, the procedure is well described in the references [2, 6, 7, 16, 17, 25, 26]. As we shall see, the each subalgebra in the list (11) can reduce the number of variables
by one, each subalgebra in the list (13) can reduce the number of variables by two, and each subalgebra in the list (14) can reduce the number of variables by three. The reduction using three-dimensional subalgebra is very interesting as the final reduction shall be the analytical solution for the BLP-equation.

Reductions Under One-Dimensional Subalgebra

In this section, the similarity reductions under one-dimensional optimal list of subalgebras (11) is given. To illustrate the reduction, we consider subalgebra \( X_1 + \alpha X_3 \), and the following characteristics equations may be written:

\[
\begin{align*}
\frac{dx}{2xf_1'(t)} &= \frac{dy}{\alpha h_1(y)} = \frac{dt}{4f_1(t)} = \frac{du}{-(2uf_1'(t) + xf_1''(t))} = \frac{dv}{-\alpha vh_1'(y)}. \\
\end{align*}
\]

On solving these characteristics equations the similarity variables and reduction fields may be quickly derived as follow:

\[
\begin{align*}
\xi &= \frac{x^2}{f_1(t)}, \eta = \int \frac{1}{\alpha h_1(y)}dy - \int \frac{1}{4f_1(t)}dt, \\
u &= -\frac{xf_1'(t)}{4f_1(t)} + \frac{1}{\sqrt{f_1(t)}} \cdot F(\xi, \eta), v = \frac{1}{h_1(y)} \cdot G(\xi, \eta). \\
\end{align*}
\]  

(15)

Substituting the reduction fields \( u \) and \( v \) from (15) into the system (1) shall provide the following reduced system:

\[
\begin{align*}
F_{\eta\eta} + 16\xi^\frac{1}{2} F_{\xi} F_{\eta} + 16\xi^\frac{1}{2} F F_{\xi\eta} - 16\xi^\frac{1}{2} F_{\xi} F_{\xi\eta} - 8F_{\xi\eta} + 64\alpha \xi^\frac{3}{2} G_{\xi\xi\xi} + 96\alpha \xi^\frac{1}{2} G_{\xi\xi} &= 0, \\
G_{\eta} + 16\xi G_{\xi\xi} + 8G_{\xi} + 16\xi^\frac{1}{2} FG_{\xi} &= 0. \\
\end{align*}
\]  

(16a)  

(16b)

The natural solution for the system (16) may be obtained as follows:

\[
\begin{align*}
F &= -\frac{1}{2} \xi^{-\frac{1}{2}}, G = c_1 \xi^\frac{1}{2} + c_2 \xi - 16\xi \eta + c_3. \\
\end{align*}
\]

The final solution of the system (1) is obtained as follow:

\[
\begin{align*}
u^1 &= -\frac{xf_1'(t)}{4f_1(t)} - \frac{1}{2x}, \\
v^1 &= \frac{1}{h_1(y)} \left( -\frac{c_1 x}{\sqrt{f_1(t)}} + \frac{c_2 x^2}{f_1(t)} - 16\xi^\frac{1}{2} \left( \int \frac{1}{\alpha h_1(y)}dy - \int \frac{1}{4f_1(t)}dt \right) + c_3 \right). \\
\end{align*}
\]  

(17a)  

(17b)

The rest of the similarity reductions corresponding to remaining subalgebras in the list (11) are given in the Table 3. The invariant solutions for remaining reduced systems are listed into following five cases:

Case 3.1 Under \( X_1 \). The reduced system can be explicitly solved, and reduced fields are obtained as follow:

\[
\begin{align*}
F &= -\frac{1}{2} \frac{c_1}{c_1 \xi^\frac{1}{2} + c_2}, G = (c_1 \xi + 2c_2 \xi^\frac{1}{2}) \psi_1(y) + \psi_2(y). \\
\end{align*}
\]
Table 3 Similarity reductions corresponding to each subalgebra in the one-dimensional optimal list of subalgebras (11)

| Subalgebra | Similarity transformations | Reduced equations |
|------------|---------------------------|-------------------|
| X₁         | $\xi = \frac{x^2}{f_1(t)}$, $\eta = y$ | $2\xi \frac{1}{2} F_\xi F_\xi + 2\xi \frac{3}{2} F F_\xi \eta - 2\xi F_\xi \xi \eta - F_\xi \eta$ |
|            | $u = -\frac{x f_1'(t)}{4 f_1(t)} + \frac{1}{\sqrt{f_1(t)}} \cdot F(\xi, \eta)$, | $+ 8\xi \frac{3}{2} G_{\xi \xi \eta} + 12\xi \frac{1}{2} G_{\xi \xi} = 0$, |
|            | $v = G(\xi, \eta)$ | $2\xi G_{\xi \xi} + G_\xi + 2\xi \frac{1}{2} F G_\xi = 0$ |
| X₂         | $\xi = x$, $\eta = t$ | $F_\xi \eta + \frac{1}{f_2(\eta)} \frac{d}{d\eta} (f_2(\eta)) F_\xi = 0$, $G_\eta = 0$ |
| X₃         | $u = F(\xi, \eta)$, $v = \frac{1}{h_1(y)} G(\xi, \eta)$ | $G_{\xi \xi \xi} = 0$, $G_\eta - 2 F G_\xi - G_{\xi \xi} = 0$ |
| X₃ + $\epsilon$ X₂ | $\xi = \frac{x}{2\epsilon f_2(t)} - \int \frac{1}{h_1(y)} dy$, $\eta = t$ | $4\epsilon^3 f_2^3(\eta) F_{\xi \eta} + 4\epsilon^3 f_2^3(\eta) f_2'(\eta) F_{\xi} - 4\epsilon^2 f_2^2(\eta) F_{\xi}^2$ |
|            | $u = -\frac{x f_2'(t)}{2 f_2(t)} + F(\xi, \eta)$, | $- 4\epsilon^2 f_2^2(\eta) F F_{\xi} + \epsilon f_2(\eta) F_{\xi \xi \xi} + G_{\xi \xi \xi} = 0$, |
|            | $v = \frac{1}{h_1(y)} G(\xi, \eta)$ | $G_{\xi \xi} - 4\epsilon^2 f_2^2(\eta) G_\eta + 4\epsilon f_2(\eta) F G_\xi = 0$ |
| X₁ + $\epsilon$ X₄ | $\xi = \frac{x^2}{f_1(t)}$, $\eta = y$ | $4\epsilon^\frac{1}{2} F F_{\xi \eta} - 4\epsilon F_{\xi \xi \eta} + 4\epsilon^\frac{1}{2} F F_{\xi} + 16\xi^\frac{3}{2} G_{\xi \xi \xi}$ |
|            | $u = -\frac{x f_1'(t)}{4 f_1(t)} + \frac{1}{\sqrt{f_1(t)}} F(\xi, \eta)$, | $- 2 F_{\xi \eta} + 24\xi^\frac{1}{2} G_{\xi \xi} = 0$, |
|            | $v = \frac{e h_2(y)}{4} \int \frac{1}{f_1(t)} dt + G(\xi, \eta)$ | $16\xi^\frac{1}{2} F G_{\xi} - \epsilon h_2(\eta) + 16\xi G_{\xi \xi} + 8G_\xi = 0$ |
| X₂ + $\epsilon$ X₄ | $\xi = y$, $\eta = t$, | $F_{\xi \eta} + \frac{f_2'(\eta)}{f_2(\eta)} F_\xi = 0$, |
|            | $u = -\frac{x f_2'(t)}{2 f_2(t)} + F(\xi, \eta)$, | $f_2(\eta) G_\eta - \epsilon h_2(\xi) F = 0$ |
|            | $v = \frac{e x h_2(y)}{2 f_2(t)} + G(\xi, \eta)$ | |
The final solution of the system (1) is obtained as follow:

\[ u_2 = -\frac{xf_1'(t)}{4f_1(t)} - \frac{1}{2\sqrt{f_1(t)}} \cdot \frac{c_1}{c_1 \xi + c_2}, \]  
(18a)

\[ v_2 = (c_1 \xi - 2c_2 \xi^2) \psi_1(y) + \psi_2(y). \]  
(18b)

**Case 3.2** Under \( X_2 \). The reduced system can be explicitly solved, and reduced fields are obtained as follow:

\[ F = \psi_1(\eta) + \frac{\phi_1(\xi)}{f_2(\eta)}, \quad G = \phi_2(\xi). \]

The final solution of the system (1) is obtained as follow:

\[ u_3 = -\frac{xf_1'(t)}{f_2(t)} + \psi_1(t) + \frac{\phi_1(y)}{f_2(t)}, \]  
(19a)

\[ v_3 = \phi_2(y). \]  
(19b)

**Case 3.3** Under \( X_3 \). The reduced system can be explicitly solved, and reduced fields are obtained as follow:

\[ F = \left( \frac{d}{d\eta} \psi_1(\eta) \right) \xi^2 + 2 \left( \frac{d}{d\eta} \psi_2(\eta) \right) \xi + 2 \frac{d}{d\eta} \psi_3(\eta) - 2 \psi_1(\eta) \]  
\[ \quad \frac{4 \psi_1(\eta) \xi + 4 \psi_2(\eta)}{4 \psi_1(\eta) x + 4 \psi_2(\eta)}, \]  
\[ G = \frac{1}{2} \psi_1(\eta) \xi^2 + \psi_2(\eta) \xi + \psi_3(\eta). \]

The final solution of the system (1) is obtained as follow:

\[ u_4 = \frac{\frac{d}{dt} \psi_1(t)}{4 \psi_1(t) x + 4 \psi_2(t)}, \]  
(20a)

\[ v_4 = \frac{1}{h_1(y)} \left( \frac{1}{2} \psi_1(t) x^2 + \psi_2(t) x + \psi_3(t) \right). \]  
(20b)

**Case 3.4** Under \( X_3 + \epsilon X_2 \). The reduced system corresponding to this subalgebra can be further reduced using symmetries, yielding two sets of invariant solutions:

\[ \left\{ F = \epsilon f_2(\eta) \frac{d}{d\eta}(\psi_1(\eta)), \quad G = \xi + \psi_1(\eta) \right\}, \]

\[ \left\{ F = -\frac{\epsilon}{f_2(\eta)}, \quad G = c_1 \left[ \xi - \int \frac{1}{f_2^2(\eta)} d\eta \right] + c_2 \right\}. \]

The final solution of the system (1) is obtained as follow:

\[ u_5 = -\frac{xf_1'(t)}{2f_2(t)} + \epsilon f_2(t) \frac{d}{dt}(\psi_1(t)), \]  
(21a)

\[ v_5 = \frac{1}{h_1(y)} \left( \frac{x}{2\epsilon f_2(t)} - \int \frac{1}{h_1(y)} dy + \psi_1(t) \right). \]  
(21b)
\[ u_6 = -\frac{xf_2'(t)}{2f_2(t)} - \frac{\epsilon}{f_2(t)}, \]  
\[ v_6 = \frac{1}{h_1(y)} \left( c_1 \left[ \frac{x}{2e f_2(t)} - \int \frac{1}{h_1(y)} dy - \int \frac{1}{f_2^2(t)} dt \right] + c_2 \right). \]  

(22a)  

(22b)

**Case 3.5** Under \( X_1 + \epsilon X_4 \). The reduced system corresponding to this subalgebra admits four sets of invariant solutions which are given below:

\[ \left\{ \begin{array}{l}
F = \frac{\epsilon h_2 (\eta)}{16}, \quad G = 2\sqrt{\xi} + \phi_1 (\eta) \\
\end{array} \right\}, \]

\[ \left\{ \begin{array}{l}
F = \frac{\epsilon h_2 (\eta)}{16\eta}, \quad G = 2\sqrt{\xi}\eta + \phi_1 (\eta) \\
\end{array} \right\}, \]

\[ \left\{ \begin{array}{l}
F = \frac{-8c_2 + \epsilon}{16c_2\sqrt{\xi} + 8c_3}, \quad G = h_2(\eta) \left( \sqrt{\xi}c_3 + c_2\xi + c_1 + \eta \right) \\
\end{array} \right\}, \]

\[ \left\{ \begin{array}{l}
F = \frac{-8c_2 + \epsilon}{16c_2\sqrt{\xi} + 8c_3}, \quad G = \frac{h_2(\eta)}{2} \left( 2\sqrt{\xi}c_3 + 2c_2\xi + \eta^2 + 2c_1 \right) \\
\end{array} \right\}. \]

The final solutions of the system (1) is obtained as follow:

\[ u_7 = \frac{xf_1'(t)}{4f_1(t)} + \frac{eh_2(y)}{16\sqrt{f_1(t)}}, \]  
\[ v_7 = \frac{eh_2(y)}{4} \int \frac{1}{f_1(t)} dt + \frac{2x}{\sqrt{f_1(t)}} + \phi_1 (y), \]  

(23a)  

(23b)

\[ u_8 = \frac{xf_1'(t)}{4f_1(t)} + \frac{eh_2(y)}{16y\sqrt{f_1(t)}}, \]  
\[ v_8 = \frac{eh_2(y)}{4} \int \frac{1}{f_1(t)} dt + \frac{2xy}{\sqrt{f_1(t)}} + \phi_1 (y), \]  

(24a)  

(24b)

\[ u_9 = \frac{xf_1'(t)}{4f_1(t)} + \frac{-8c_2 + \epsilon}{16c_2x + 8c_3\sqrt{f_1(t)}}, \]  
\[ v_9 = \frac{eh_2(y)}{4} \int \frac{1}{f_1(t)} dt + h_2 (y) \left( \frac{c_3x}{\sqrt{f_1(t)}} + \frac{c_2x^2}{f_1(t)} + c_1 + y \right), \]  

(25a)  

(25b)

\[ u_{10} = \frac{xf_1'(t)}{4f_1(t)} + \frac{-8c_2 + \epsilon}{16c_2x + 8c_3\sqrt{f_1(t)}}, \]  
\[ v_{10} = \frac{eh_2(y)}{4} \int \frac{1}{f_1(t)} dt + h_2 (y) \left( \frac{c_3x}{\sqrt{f_1(t)}} + \frac{c_2x^2}{f_1(t)} + c_1 + \frac{y^2}{2} \right), \]  

(26a)  

(26b)
Case 3.6 Under $X_2 + \epsilon X_4$. The reduced system can be explicitly solved, and reduced fields are obtained as follow:

\[
F = \psi_1(\eta) + \frac{\phi_1(\xi)}{f_2(\eta)},
\]
\[
G = \epsilon h_2(\xi) \int \frac{\psi_1(\eta) f_2(\eta) + \phi_1(\xi)}{f_2^2(\eta)} d\eta + \phi_2(\xi).
\]

The final solution of the system (1) is obtained as follow:

\[
u_{11} = -\frac{xf'_2(t)}{2f_2(t)} + \psi_1(t) + \frac{\phi_1(y)}{f_2(t)},
\]
\[
v_{11} = \epsilon xh_2(y) \int \frac{\psi_1(t) f_2(t) + \phi_1(y)}{f_2^2(t)} dt + \phi_2(y).
\]

Remark 3.1 The solutions $\{u_i, v_i\}, i = 1 \ldots 11$, given in (17)–(27) are highly explicit as each of the solution includes several arbitrary functions, and could be used to describe physical phenomena with greater degrees of freedom. And we believe these solutions have never been documented before not even in the recent work on BLP system [9–12, 28, 29].

Reductions Under Two-Dimensional Subalgebra

The importance of two-dimensional subalgebra is that the reduced equation under one subalgebra will also be invariant under the second subalgebra and thus two-variable reduction can be achieved at one time. We have tried a reduction corresponding to each subalgebra in the list (13) and most can be discarded offhand as they are too trivial or due to the presence of arbitrary functions. Only the following possible reductions can be found that are significant.

Case 3.7 Reduction under subalgebra $G_1(X_1 + \alpha X_3, X_2)$. The invariants of $X_2$ are

\[
\xi = y, \eta = t, u = -\frac{xf'_2(t)}{2f_2(t)} + F(\xi, \eta), v = G(\xi, \eta).
\]

The second generator in term of these invariants can be written in following manner,

\[
X_1 + \alpha X_3 = \alpha h_1(\xi) \frac{\partial}{\partial \xi} + 4 \frac{\partial}{\partial \eta} + 0 \frac{\partial}{\partial F} - \alpha \frac{d}{d\xi} (h_1(\xi))G \frac{\partial}{\partial G},
\]

where the arbitrary functions had to be fixed as $f_1(t) = 1$, $f_2(t) = e^{-2t}$. The corresponding auxiliary equations may written as,

\[
\frac{d\xi}{\alpha h_1(\xi)} = \frac{d\eta}{4} = \frac{dF}{0} = \frac{dG}{-\alpha h'_1(\xi)}.
\]

Upon solving above auxiliary equations, a new set of similarity transformations for reduced equation under $X_2$ may be obtained as follows:

\[
\rho = \int \frac{1}{\alpha h_1(\xi)} d\xi - \frac{1}{4} \eta, F = \tilde{F}(\rho), G = \frac{1}{h_1(\xi)} \tilde{G}(\rho),
\]

and in term of $u$ and $v$, the above similarity transformations can be written as

\[
\rho = \int \frac{1}{\alpha h_1(y)} dy - \frac{1}{4} t, u = x + \tilde{F}(\rho), v = \frac{1}{h_1(y)} \tilde{G}(\rho).
\]

Remark 3.1
The new similarity transformations (28) when substituted into system (1) will reduce it into a system of ODEs,

\[
\frac{d^2}{d\rho^2}(\tilde{F}(\rho)) + 8 \frac{d}{d\rho}(\tilde{F}(\rho)) = 0, \quad \frac{d}{d\rho}(\tilde{G}(\rho)) = 0.
\]

The solution of above system shall provide a final solutions for the system (1), that may be written as follow:

\[
u^{11} = c_1 + x + \exp\left(-8 \int \frac{1}{\alpha h_1(y)} dy + 2 t\right), \quad v^{11} = \frac{c_2}{h_1(y)}.
\]  

Case 3.8 Reduction under subalgebra \(G_4(X_3, X_1)\). The invariance under this subalgebra tells that the solution of the system (1) must be of the form

\[
u = -\frac{x f_1'(t)}{4 f_1(t)} + \frac{1}{\sqrt{f_1(t)}} \tilde{F}(\rho), \quad v = \frac{1}{h_1(y)} \tilde{G}(\rho), \quad \rho = \frac{x^2}{f_1(t)}.
\]  

Substituting similarity transformations (30) back into the system (1), we obtain following system of ODEs

\[
2 \rho \frac{d^3}{d\rho^3}(\tilde{G}(\rho)) + 3 \frac{d^2}{d\rho^2}(\tilde{G}(\rho)) = 0, \quad (31a)
\]
\[
2 \rho \frac{d^2}{d\rho^2}(\tilde{G}(\rho)) + \frac{d}{d\rho}(\tilde{G}(\rho)) + 2 \rho^\frac{3}{2} \frac{d}{d\rho}(\tilde{G}(\rho)) = 0. \quad (31b)
\]

Immediate solution for the system (31)

\[
\tilde{F}(\rho) = \frac{c_2}{2 c_2 \rho^2 + c_3}, \quad \tilde{G}(\rho) = c_1 - c_2 \rho + c_3 \rho^\frac{3}{2}.
\]

The final solution of the system (1) is obtained as follow:

\[
u^{12} = -\frac{x f_1'(t)}{4 f_1(t)} + \frac{c_2}{2 c_2 x + c_3 \sqrt{f_1(t)}}, \quad (32a)
\]
\[
v^{12} = \frac{1}{h_1(y)} \left( c_1 - c_2 \frac{x^2}{\sqrt{f_1(t)}} + \frac{c_3 x}{\sqrt{f_1(t)}} \right). \quad (32b)
\]

Remark 3.2 The symmetry reductions for the three-dimensional optimal (14) have also been attempted, but in most cases, the reductions are too trivial or, in some cases, impossible.

Conclusion

The presence of Virasoro subalgebra in the infinite-dimensional Lie algebra (6) suggests that the Boiti–Leon–Pempinelli system (1) is completely integrable. To cover up the inadequacies in the earlier works [9–12, 28, 29], a detailed groups classification of the infinite-dimensional Lie algebra (6) is obtained, and that too without assigning particular values to the arbitrary functions. One, two and three-dimensional groups classifications have been obtained at (11), (13), (14) respectively.

Further, a class of explicit solutions that involves several arbitrary functions of \(y\) and \(t\), is obtained in (17)–(27). Using two-dimensional symmetry reduction, some more explicit solutions which involve arbitrary functions are also obtained in (29) and (32).
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