Algebras with homogeneous module category are tame

Zhang Yingbo$^1$ and Xu Yunge$^2$

1 School of Mathematics, Beijing Normal University, 100875 P.R.China, zhangyb@bnu.edu.cn
2 Faculty of Mathematics, Hubei University, 430062 P.R.China, xuy@hubu.edu.cn

Abstract

The celebrated Drozd’s theorem asserts that a finite-dimensional basic algebra $\Lambda$ over an algebraically closed field $k$ is either tame or wild, whereas the Crawley-Boevey’s theorem states that given a tame algebra $\Lambda$ and a dimension $d$, all but finitely many isomorphism classes of indecomposable $\Lambda$-modules of dimension $d$ are isomorphic to their Auslander-Reiten translations and hence belong to homogeneous tubes. In this paper, we prove the inverse of Crawley-Boevey’s theorem, which gives an internal description of tameness in terms of Auslander-Reiten quivers.

Contents

Introduction
1. Matrix bimodule problems
   1.1 Definition of matrix bimodule problems
   1.2 Bi-comodule problems and bocses
   1.3 Representation categories of matrix bimodule problems
   1.4 Formal products and formal equations
2. Reductions on matrix bimodule problems
   2.1 Admissible bimodules and induced matrix bimodule problems
   2.2 Eight Reductions
   2.3 Canonical forms
   2.4 Defining systems
3. Classification of minimal wild bocses
   3.1 An exact structure on representation categories of bocses
   3.2 Almost split conflations in the process of reductions
   3.3 Minimal wild bocses
   3.4 Non-homogeneity in the cases of MW1-MW4
4. One-sided pairs
   4.1 Definition of one-sided pairs
   4.2 Differentials in one-sided pairs
   4.3 Reduction sequences of one-sided pairs
   4.4 Major pairs
   4.5 Further reductions
   4.6 Regularizations on non-effective $a$-class and all $b$-class arrows
5. Non-homogeneity of bipartite matrix bimodule problems of wild type
   5.1 An inspiring example
   5.2 Bordered matrices in bipartite case
   5.3 Non-homogeneity in the case of MW5 and classification (I)
   5.4 Bordered matrices in one-sided case
   5.5 Non-homogeneity in the case of MW5 and classification (II)
   5.6 Proof of Main theorem

References

2010 Mathematics Subject Classification: 15A21, 16G20, 16G60, 16G70
The authors are supported by the National Natural Science Foundation of China, No. 11271318, and 11371186.
Introduction

Throughout the paper, we always assume that $k$ is an algebraically closed field, and that all rings or algebras contain identities. We write our maps either on the left or on the right, but always compose them as if they were written on the right.

We start with the following important definition of “tame” and “wild”:

**Definition 1 [D1, CB1, DS]** A finite-dimensional $k$-algebra $\Lambda$ is of tame representation type, if for any positive integer $d$, there are a finite number of localizations $R_i = k[x, \phi_i(x)^{-1}]$ of $k[x]$ and $\Lambda$-$R_i$-bimodules $T_i$ which are free as right $R_i$-modules, such that all but finitely many iso-classes of indecomposable $\Lambda$-modules of dimension at most $d$ are isomorphic to

$$T_i \otimes_{R_i} R_i/(x - \lambda)^m,$$

for some $\lambda \in k$ with $\phi_i(\lambda) \neq 0$, and some positive integer $m$.

A finite-dimensional $k$-algebra $\Lambda$ is of wild representation type if there is a finitely generated $\Lambda$-$k\langle x,y \rangle$-bimodule $T$, which is free as a right $k\langle x,y \rangle$-module, such that the functor

$$T \otimes_{k\langle x,y \rangle} - : k\langle x,y \rangle\text{-mod} \rightarrow \Lambda\text{-mod}$$

preserves indecomposability and isomorphism classes.

Several authors worked on equivalent definitions of “time” and “wild”, for example in terms of generic modules [CB3].

In 1977 Drozd [D1] showed that a finite-dimensional algebra over an algebraically closed field is either of tame representation type or of wild representation type. This result is known as Drozd’s Tame-Wild Theorem, and has been one of the most fundamental results in the representation theory of finite dimensional algebras. On the other hand, however, the proof of Drozd’s Theorem is highly indirect. Indeed, the argument relies on the notion of a bocs (the abbreviation for “bimodule of coalgebra structure”), introduced first by Rojter in [Ro]. In 1988, Crawley-Boevey [CB1] formalized the theory of bocses and showed that for a tame algebra $\Lambda$, and for each dimension $d$, all but finitely many isomorphism classes of indecomposable $\Lambda$-modules of dimension $d$ are isomorphic to their Auslander-Reiten translations and hence belong to homogeneous tubes.

After the work [CB1], many authors tried to prove the converse of Crawley-Boevey’s theorem, aiming to find infinitely many non-isomorphic indecomposable representations $\{M_i \mid i \in I\}$ of the same dimension in the representation category of a layered bocs, such that $M_i \not\cong DTr(M_i)$. Somewhat surprisingly, in 2000 the authors [BCLZ] constructed a strongly homogeneous wild layered bocs $\mathfrak{B}$ for which each representation is homogeneous (i.e., $DTr(M) \simeq M$), and showed that the converse of the Crawley-Boevey’s theorem does not hold true for general layered bocses. Later on, C.M.Ringel proposed a concept of controlled wild, Y.Han described some classes of controlled wild algebras [H], and H.Nagasy proved that a $\tau$-wild algebra is wild [N]. However, the converse of the Crawley-Boevey’s theorem remains open in the case of finite dimensional $k$-algebras. Our main result in this paper, Theorem 3 below, gives a full answer to this problem.

To state our result, we need the following definition:

**Definition 2 [BCLZ, 2.1 Definitions]** Let $\Lambda$ be a finite-dimensional algebra over an algebraically closed field. An indecomposable $\Lambda$-module $M$ is called homogeneous, if $DTr(M) \cong M$. The category $\text{mod-}\Lambda$ is said to be homogeneous, if for each dimension $d$ all but finitely many isomorphism classes of indecomposable $\Lambda$-modules of dimension $d$ are homogeneous.

We will prove the following main theorem throughout this whole article.

**Theorem 3** Let $\Lambda$ be a finite-dimensional algebra over an algebraically closed field. Then $\Lambda$ is of tame representation type if and only if $\text{mod-}\Lambda$ is homogeneous. □
The necessity of Theorem 3 was previously proved by Crawley-Boevey [CB1]. We only need to prove the sufficiency part. The proof is divided into five sections as shown in the contents. Our proof relies on the notions of matrix bimodule problems, their associated bocses, and reduction techniques. Since the matrix bimodule problems associated to finite-dimensional algebras are bipartite, the key of our argument is to find a full subcategory of representation category of a bipartite matrix bimodule problem which admits infinitely many isomorphism classes of non-homogeneous representations of dimension $d$.

## 1 Matrix bimodule problems

In this section, a notion of matrix bimodule problems over a minimal algebra is introduced, which is a generalization of bimodule problems over a field $k$ defined by [CB2] in terms of matrix. Then the associated bi-comodule problems and bocses of matrix bimodule problems are discussed. Finally, a nice connection between a matrix bimodule problem and its associated bocses is builded via the formal products of two structures.

### 1.1 Definition of matrix bimodule problems

The purpose of this subsection is two folds: 1) construct a $k$-algebra $\Delta$ based on a minimal algebra $R$; 2) define matrix bimodule problems over $\Delta$. The concepts and the results are proposed by S. Liu.

Let $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ be a vertex set, where the subset $\mathcal{T}_0$ consists of trivial vertices, such that $\forall X \in \mathcal{T}_0$, there is a $k$-algebra $R_X \simeq k$ with the identity $1_X$; and $\mathcal{T}_1$ consists of non-trivial vertices, such that $\forall X \in \mathcal{T}_1$, there is an algebra $R_X \simeq k[x,\phi_X(x)^{-1}]$ with the identity $1_X$, the finite localization of the polynomial ring $k[x]$ given by a non-zero polynomial $\phi_X(x) \in k[x]$, and $x$ is said to be the parameter associated to $X \in \mathcal{T}_1$. Now we call the $k$-algebra $R = \Pi_{X \in \mathcal{T}} R_X$ a minimal algebra over $\mathcal{T}$ with a set of orthogonal primitive idempotents $\{1_X \mid X \in \mathcal{T}\}$.

We define a tensor product of $p \geq 1$ copies of $R$ over $k$ as follows:

$$R^{\otimes p} = R \otimes_k \cdots \otimes_k R = \sum_{(X_1, \ldots, X_p) \in \mathcal{T} \times \cdots \times \mathcal{T}} R_{X_1} \otimes_k \cdots \otimes_k R_{X_p}.$$  

There exists a natural left and right $R$-module structure on $R^{\otimes p}$:

$$s \otimes_R \alpha = (s \otimes_R r_1) \otimes_k r_2 \otimes_k \cdots \otimes_k r_p;$$
$$\alpha \otimes_R s = r_1 \otimes_k \cdots \otimes_k r_{p-1} \otimes_k (r_p \otimes_R s);$$  

for any $\alpha = r_1 \otimes_k r_2 \otimes_k \cdots \otimes_k r_{p-1} \otimes_k r_p \in R^{\otimes p}, s \in R$. If $r_i \in R_{X_i}, s \in R_Y$, then $s \otimes_R \alpha = 0$ for $Y \neq X_1$ and $\alpha \otimes_R s = 0$ for $Y \neq X_i$. Thus $R^{\otimes p}$ can be viewed as an $R$-$R$-bimodule, or simply an $R^{\otimes 2}$-module, with the module action for any $r, s \in R$:

$$(r \otimes_k s) \otimes_{R^{\otimes 2}} \alpha = r \otimes_R \alpha \otimes_R s = \alpha \otimes_{R^{\otimes 2}} (r \otimes_k s).$$  

Note that $(\alpha \otimes_{R^{\otimes 2}} (r \otimes_k s)) \otimes_R s' = \alpha \otimes_{R^{\otimes 2}} (r \otimes_k ss'), \forall s' \in R$, and $r' \otimes_R (\alpha \otimes_{R^{\otimes 2}} (r \otimes_k s)) = \alpha \otimes_{R^{\otimes 2}} (r' \otimes_k s), \forall r' \in R$. The direct sum of $R^{\otimes p}$ for $p = 1, 2, \cdots$, is still an $R^{\otimes 2}$-module:

$$\Delta = \oplus_{p=1}^{\infty} R^{\otimes p}, \text{ let } \bar{\Delta} = \oplus_{p=2}^{\infty} R^{\otimes p}, \Delta = R \oplus \bar{\Delta}.$$  

We define a multiplication on $R^{\otimes 2}$-module $\Delta$, given by $\Delta \times \Delta \to \Delta \otimes_R \Delta \subseteq \Delta$:

$$\Delta^{\otimes p} \otimes_R \Delta^{\otimes q} \subseteq \Delta^{(p+q-1)}; \quad \alpha \otimes_R \beta = r_1 \otimes_k \cdots \otimes_k (r_p \otimes s_1) \otimes_k s_2 \cdots \otimes_k s_q.$$  

\[\text{(1.1-4)}\]
where $\beta = s_1 \otimes_k \cdots \otimes_k s_q$. And if $r_i \in R_{X_i}, s_j \in R_{Y_j}$, $\alpha \otimes_R \beta = 0$ for $X_p \neq Y_1$. Thus we obtain an associative non-commutative $k$-algebra $(\Delta, \otimes_R, 1_R)$ with the set of orthogonal primitive idempotents $\{1_X \mid X \in T\}$. Moreover, $\Delta \otimes_R \Delta$ can be viewed as an $R^{\otimes 3}$-module: for any $\alpha, \beta \in \Delta, r, s, w \in R$,
\[
(r \otimes_k s \otimes_k w) \otimes_{R^{\otimes 3}} (\alpha \otimes_R \beta) = (\alpha \otimes_k \beta) \otimes_{R^{\otimes 3}} (r \otimes_k s \otimes_k w) = r \otimes_R \alpha \otimes_R s \otimes_R \beta \otimes_R w.
\]

Denote by $\text{IM}_{m \times n}(\Delta)$ the set of matrices over $\Delta$ of size $m \times n$; and by $\mathbb{T}_n(\Delta), \mathbb{N}_n(\Delta), \mathbb{D}_n(\Delta)$ the sets of upper triangular, strictly upper triangular, and diagonal $\Delta$-matrices of size $n \times n$ respectively. The product of two $\Delta$-matrices is the usual matrix product. If $H = (h_{ij}) \in \text{IM}_{m \times n}(R), U = (u_{ij}) \in \text{IM}_{m \times n}(R \otimes_k R), \alpha \in \Delta$, define
\[
H \otimes_R \alpha = (h_{ij} \otimes_R \alpha) \in \text{IM}_{m \times n}(\Delta),
\]
\[
\alpha \otimes_R H = (\alpha \otimes_R h_{ij}) \in \text{IM}_{m \times n}(\Delta);
\]
\[
U \otimes_{R^{\otimes 2}} \alpha = (\alpha \otimes_{R^{\otimes 2}} u_{ij}) = \alpha \otimes_{R^{\otimes 2}} U \in \text{IM}_{m \times n}(\Delta).
\]

The first two are based on Formula (1.1-1). For the last one, note that $(\alpha \otimes_{R^{\otimes 2}} U) \otimes_R H = \alpha \otimes_{R^{\otimes 2}} (UH), H \otimes_R (\alpha \otimes_{R^{\otimes 2}} U) = \alpha \otimes_{R^{\otimes 2}} (HU)$ by the note stated under Formula (1.1-2). Let $U = (u_{ij}) \in \text{IM}_{m \times n}(R), V = (v_{jl}) \in \text{IM}_{n \times r}(R \otimes_k R)$ and $\alpha, \beta \in \Delta$. Then
\[
(\alpha \otimes_{R^{\otimes 2}} U)(\beta \otimes_{R^{\otimes 2}} V) = \left(\sum_{j=1}^{n} ((\alpha \otimes_{R^{\otimes 2}} u_{ij}) \otimes_R (\beta \otimes_{R^{\otimes 2}} v_{jl}))\right)_{i,l} = \left(\sum_{j=1}^{n} (\alpha \otimes_{R^{\otimes 2}} (u_{ij} \otimes_R v_{jl}))\right)_{i,l} = (\alpha \otimes_{R^{\otimes 2}} \beta) \otimes_{R^{\otimes 3}} (UV).
\]

An $R$-$R$-bimodule $S_1$ is said to be a quasi-free bimodule finitely generated by $U_1, \ldots, U_m$, provided that the morphism
\[
(R_{X_1} \otimes_k R_{Y_1}) + \cdots + (R_{X_m} \otimes_k R_{Y_m}) \to S_1, \quad 1_{X_i} \otimes_k 1_{Y_i} \mapsto U_i
\]
is an isomorphism. In this case, $\{U_1, \ldots, U_m\}$ is called an $R$-$R$-quasi-free basis of $S_1$, or $R$-$R$-quasi-basis of $S_1$ for short.

Let $S_p = R^{\otimes(p+1)} \otimes_{R^{\otimes 2}} S_1$, which possesses an $R$-$R$-bimodule structure:
\[
(r \otimes_k s) \otimes_{R^{\otimes 2}} (\alpha \otimes_{R^{\otimes 2}} U) = (r \alpha s) \otimes_{R^{\otimes 2}} U = \alpha \otimes_{R^{\otimes 2}} ((r \otimes_k s) \otimes_{R^{\otimes 2}} U),
\]
(1.1-8)
for $r, s \in R, \alpha \in R^{p+1}, U \in S$. Thus $S = \sum_{p=1}^{\infty} S_p = \Delta \otimes_{R^{\otimes 2}} S_1$ is an $R$-$R$-bimodule, and $S_p$ is said to have index $p$ in $S$.

**Definition 1.1.1** Let $T = \{1, 2, \ldots, t\}$ be a set of integers, and let $\sim$ be an equivalent relation on $T$, such that the set $T/\sim$ is one-to-one corresponding to the vertex set $T$ of a minimal algebra $R$. It may be written as $T = T/\sim$.

**Definition 1.1.2** (i) Define an $R$-$R$-bimodule:
\[
K_0 = \{ \text{diag}(s_{11}, \ldots, s_{tt}) \mid \text{when } i \in X, s_{ii} \in R_X, \text{ and } s_{ij} = s_{ji}, \forall i \sim j \}.
\]
Let $E_X \in \mathbb{D}_t(R_X)$ with the entry $s_{ii} = 1_X$ if $i \in X$ and $s_{ii} = 0$ if $i \notin X$, then $\{E_X \mid X \in T\}$ is an $R$-quasi-basis of $K_0$, and $E = \sum_{X \in T} E_X$ is the identity matrix of size $t$.

(ii) Define a quasi-free $R$-$R$-bimodule $K_1 \subseteq \mathbb{N}_t(R \otimes_k R)$ with an $R$-$R$-quasi-basis:
\[
V = \cup_{(X,Y) \in T \times T} V_{XY} = \{V_1, V_2, \cdots, V_m\}, \quad V_{XY} \subseteq \mathbb{N}_t(R_X \otimes_k R_Y),
\]
where $V \in V_{XY}$ if and only if $1_X V 1_Y = V$. 


(iii) Suppose $\mathcal{K} = \mathcal{K}_0 \oplus (\Delta \otimes_{R \otimes 2} \mathcal{K}_1)$ possesses an algebra structure, where the multiplication $m : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is the usual matrix product over $\Delta$ consisting of $m_{pq} : \mathcal{K}_p \times \mathcal{K}_q \to \mathcal{K}_{p+q}$, $\forall p, q \geq 0$; the unit is given by the canonical inclusion $e : R \cong \mathcal{K}_0 \to \mathcal{K}$. $\mathcal{K}$ is said to be finitely generated in index $(0,1)$ over $\Delta$.

Clearly $\mathcal{K}_0 \cong R$ as algebras. The multiplication $m_{pq}, \forall p, q > 0$ is determined by $m_{11} : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathcal{K}_2$, i.e. Formula (1.1-7). And for any $r \in R, \alpha \in \Delta, (r \otimes_R X) \otimes_R (\alpha \otimes_{R \otimes 2} V_j) = (r \otimes_R \alpha) \otimes_{R \otimes 2} V_j \forall 1 \leq V_j = V_j > 0, or 0 otherwise, $\otimes_R X$ is similar. $\{X \mid X \in T\}$ is a set of orthogonal primitive idempotents of $\mathcal{K}$, and $E = e(1_R)$ is the identity.

Let $T = \{1, 2, \ldots, t\}$ and $T' = \{1, 2, \ldots, t'\}$ be two sets of integers. An order on $T \times T'$ is defined as follows: $(i,j) \preceq (i',j')$ provided that $i > i'$, or $i = i'$ but $j < j'$.

Thus an order on the index set of the entries of a matrix in $\text{IM}_{k\times 1}(\Delta)$ is obtained. Let $M = (\lambda_{ij}) \in IM_{k\times 1}(\Delta)$. The entry $\lambda_{pq}$ is said to be the leading entry of $M$ if $\lambda_{pq} \neq 0$, and any $\lambda_{ij} \neq 0$ implies that $(p,q) \preceq (i,j)$. Let $M = (C_{ij})$ be a partitioned matrix over $\Delta$, one defines similarly the leading block of $M$. In both cases, the index $(p,q)$ is called the leading position of $M$ resp. $M$.

Let $S$ be a subspace of $\text{IM}_k(R)$. An ordered basis $U = \{U_1, \cdots, U_r\}$ with the leading positions $(p_1,q_1), \ldots, (p_r,q_r)$ respectively is called a normalized basis of $S$ provided that

(i) the leading entry of $U_i$ is 1;
(ii) the $(p_i,q_i)$-th entry of $U_j$ is 0 for $j \neq i$;
(iii) $U_i \preceq U_j$ if and only if $(p_i,q_i) \preceq (p_j,q_j)$

The basis $U$ is a linearly ordered set. It is easy to see that $S$ has a normalized basis by Linear algebra. In fact, if $t^2$ variables $x_{ij}$ under the order of matrix indices defined as above are taken, then $S$ will be the solution space of some system of linear equations $\sum_{(i,j)\in T\times T} a_{ij}^l x_{ij} = 0, a_{ij}^l \in k, 1 \leq l \leq s$ for some positive integer $s$. Reducing the coefficient matrix to the simplest echelon form, we assume that $x_{p_1,q_1}, \ldots, x_{p_s,q_r}$ are all the free variables, and $\{U_1, \cdots, U_r\}$ is a basic system of solutions, whose $(p_i,q_i)$-entry is 1 and $(p,q) \preceq (p_i,q_i)$-entry is 0 for $i = 1, \cdots, r$, a normalized basis $U$ of $S$ is obtained.

**Definition 1.1.3** (i) Define a quasi-free $R-R$-bimodule $\mathcal{M}_1 \subseteq IM(R \otimes_k R)$, such that $E_X\mathcal{M}_1E_Y$ has a normalized quasi-basis $\mathcal{A}_{XY} \subseteq IM(k \otimes_k 1, 1 \otimes_k k)$, $\mathcal{A}_{XY}$, as $k$-vector spaces, where $A \in \mathcal{A}_{XY}$ if and only if $1_X A 1_Y = A$. Thus there is a normalized quasi-basis $\mathcal{A} = \cup_{(X,Y)\in T\times T} \mathcal{A}_{XY} = \{A_1, A_2, \cdots, A_n\}$.

(ii) Let $\mathcal{M} = \Delta \otimes_{R \otimes 2} \mathcal{M}_1$, and the algebra $\mathcal{K}$ be given by Definition 1.1.2. Define a $\mathcal{K}$-$\mathcal{K}$-bimodule structure on $\mathcal{M}$, such that the left module action $1 : \mathcal{K} \times \mathcal{M} \to \mathcal{M}$ consists of $l_{pq} : \mathcal{K}_p \times \mathcal{M}_q \to \mathcal{M}_{p+q}, \forall p, q > 0$, and the right one $r : \mathcal{K} \times \mathcal{M} \to \mathcal{M}$ consists of $r_{pq} : \mathcal{M}_p \times \mathcal{K}_q \to \mathcal{M}_{p+q}, \forall p > 0, q \geq 0$ given by usual matrix product respectively. The $\mathcal{K}$-$\mathcal{K}$-bimodule $\mathcal{M}$ is said to be finitely generated in index $(0,1)$ with $\mathcal{M}_0 = \{0\}$.

**Definition 1.1.4** Let $H = \sum_{X \in T} H_X \in IM(R)$ be a matrix, where $H_X = (h_{ij})_{i \times t} \in E_X IM(R)E_X$ with $h_{ij} \in R_X$ for $i,j \in X$, and $h_{ij} = 0$ otherwise. Define a derivation $d : \mathcal{K} \to \mathcal{M}, U \mapsto UH - HU$, yielded by $H$ consists of $d_p : \mathcal{K}_p \to \mathcal{M}_p, \forall p \geq 0$.

It is not difficult to see that $d_0 = 0$, and $d_p$ is determined by $d_1$ for $p > 0$ according to the note stated under Formula 1.1-6).

**Definition 1.1.5** A quadruple $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ is called a matrix bimodule problem provided

(i) $R$ is a minimal algebra with a vertex set $T$ given by Definition 1.1.1;
(ii) $\mathcal{K}$ is an algebra given by Definition 1.1.2;
(iii) $\mathcal{M}$ is a $\mathcal{K}$-$\mathcal{K}$-bimodule given by Definition 1.1.3;
(iv) There is a derivation $d : \mathcal{K} \to \mathcal{M}$ given by Definition 1.1.4.

In particular, if $\mathcal{M} = 0$, $\mathfrak{A}$ is said to be a minimal matrix bimodule problem.
1.2 Bi-comodule problems and Bocses

We define a notion of bi-comodule problems associated to matrix bimodule problems, which is the transition into bocses. The concepts and the proofs are proposed by Y. Han.

Since \( K_1 \) and \( \mathcal{M}_1 \) are both quasi-free \( R-R \)-bimodules, we have their \( R^{\otimes 2} \)-dual structures \( C_1 \) and \( N_1 \) with \( R-R \)-quasi-basis \( V^* \) and \( A^* \) respectively:

\[
C_1 = \text{Hom}_{R^{\otimes 2}}(K_1, R^{\otimes 2}), \quad V^* = \{v_1, v_2, \ldots, v_m\}; \\
N_1 = \text{Hom}_{R^{\otimes 2}}(\mathcal{M}_1, R^{\otimes 2}), \quad A^* = \{a_1, a_2, \ldots, a_n\}.
\]

(1.2-1)

Write \( v : X \rightarrow Y \) (resp. \( a : X \rightarrow Y \)) provided that \( V \in V_{XY} \) (resp. \( A \in A_{XY} \)).

The quasi-basis \( V \) of \( K_1 \) has a natural partial order, namely, \( V_i \prec V_j \), if their leading positions \( (p_i, q_i) < (p_j, q_j) \). Thus \( V_i V_j = \sum_{l<i,j} \gamma_{ijl} \otimes R^{\otimes 2} V_l \), since \( V \subset N_k(R \otimes R) \). For a fixed pair \( (p, q) \), an order on the set \( \{V_i \mid (p_i, q_i) = (p, q)\} \) may be defined, which gives a linear order on \( V \).

**Definition 1.2.1** Let \( K \) be a \( k \)-algebra as in Definition 1.1.2. We define a quasi-free \( R \)-module \( C_0 = \text{Hom}_R(K_0, R) \simeq \sum V_{X \in T} R \otimes e_x \simeq R \) with an \( R \)-quasi-basis \( \{e_x\}_{X \in T} \) dual to \( \{E_X\}_{X \in T} \) of \( K_0 \); and a quasi-free \( R-R \)-bimodule \( C_1 \) with an \( R-R \)-quasi-basis \( V^* \) defined by the first formula of (1.2-1), which has a linear order yielded from that of \( V \). Write \( C = C_0 \oplus C_1 \), and define a coalgebra structure with a counit \( \varepsilon : C \rightarrow R, e_x \mapsto 1_x, v_i \mapsto 0 \) and a coproduct \( \mu : C \mapsto C \otimes_R C \) dual to \( (m_0, m_0, m_{10}, m_{11}) \):

\[
\mu = \left( \begin{array}{c} \mu_{00} \\ \mu_{01} + \mu_{11} \end{array} \right) : \left( \begin{array}{c} C_0 \\ C_1 \end{array} \right) 
\rightarrow \left( \begin{array}{c} C_0 \otimes_R C_0, \\ C_1 \otimes_R C_0 \oplus C_0 \otimes_R C_1 \oplus C_1 \otimes_R C_1 \end{array} \right)
\]

\[
\mu_{00}(v) = e_x \otimes e_y, \quad \mu_{01}(v) = v_t \otimes e(v_t), \quad \mu_{11}(v) = \sum_{i,j,l} \gamma_{ijl} \otimes_R (v_i \otimes_R v_j).
\]

Since \( A \) is linearly ordered and \( \mathcal{V} \subset N_k(R \otimes R) \), \( V_i A_j = \sum_{l>l} \eta_{ijl} \otimes R^{\otimes 2} A_l \) and \( A_i V_j = \sum_{l<i} \eta_{ijl} \otimes R^{\otimes 2} A_l \).

**Definition 1.2.2** Let \( \mathcal{M} \) be a \( K-K \)-bimodule as in Definition 1.1.3. A quasi free \( R-R \)-bimodule \( N_1 \) with an \( R-R \)-quasi-basis \( A^* \) given by the second formula of (1.2-1) is defined. Write \( N = N_1 \), then \( N \) has a \( C-C \)-bi-comodule structure with the left and right co-module actions dual to \( (l_{01}, l_{11}) \) and \( (r_{10}, r_{11}) \) respectively:

\[
t = (t_{01} + t_{11}) : N \mapsto C \otimes_R N, \quad n = C_0 \otimes_R N \oplus C_1 \otimes_R N,
\]

\[
t_{10}(a_i) = e_{v(a_i)} \otimes q_i, \quad t_{11}(a_i) = \sum_{j<l} \eta_{ijl} \otimes R^{\otimes 3} (v_i \otimes_R q_j);
\]

\[
\tau = (\tau_{10} + \tau_{11}) : N \mapsto N \otimes_R C, \quad \tau = N \otimes_R C_0 \oplus N \otimes_R C_1,
\]

\[
\tau_{10}(a_i) = a_i \otimes_R e_{v(a_i)}, \quad \tau_{11}(a_i) = \sum_{l<i,j} \delta_{ijl} \otimes_R (a_i \otimes_R v_j).
\]

**Definition 1.2.3** Assume \( d_1(V_i) = \sum_l \gamma_{il} \otimes R^{\otimes 2} A_l \) defined in 1.1.4. There is a co-derivation \( \partial = (\partial_0, \partial_1) : N \rightarrow C = C_0 \oplus C_1 \) with \( \partial_0 = 0 \) and \( \partial_1(a_i) = \sum v_i \otimes_R v_i \) dual to \( (d_0, d_1) \), such that \( \mu \partial = (\mathcal{I} \otimes \partial) t + (\partial \otimes \mathcal{I}) \partial \).

**Definition 1.2.4** Let \( \mathfrak{A} = (R, K, M, H) \) be a matrix bimodule problem. A quadruple \( \mathfrak{C} = (R, C, N, \partial) \) is said to be a bi-comodule problem associated to \( \mathfrak{A} \) provided

(i) \( R \) is a minimal algebra with a vertex set \( T \);

(ii) \( C \) is a co-algebra given by Definition 1.2.1;

(iii) \( N \) is a \( C-C \)-bi-comodule given by Definition 1.2.2;

(iv) \( \partial : N \rightarrow C \) is a co-derivation given by Definition 1.2.3.

Now we construct a bocses via the bi-comodule problem \( \mathfrak{C} \) associated to \( \mathfrak{A} \). Write \( N^{\otimes p} = N \otimes_R \cdots \otimes_R N \) with \( p \) copies of \( N \) and \( N^{\otimes 0} = R \). Define a tensor algebra \( \Gamma \) of \( N \) over \( R \), whose multiplication is given by the natural isomorphisms:

\[
\Gamma = \bigoplus_{p=0}^{\infty} N^{\otimes p}, \quad N^{\otimes p} \otimes_R N^{\otimes q} \simeq N^{\otimes (p+q)}.
\]
Let $\Xi = \Gamma \otimes_R \mathcal{C} \otimes_R \Gamma$ be a $\Gamma$-$\Gamma$-bimodule of co-algebra structure induced by $R \hookrightarrow \Gamma$, and denoted by $(\Xi, \mu_\Xi, \epsilon_\Xi)$. Define the following three $R$-$R$-bimodule maps:

$$
\begin{align*}
\kappa_1 : \mathcal{N} &\rightarrow \mathcal{C} \otimes_R \mathcal{N} \rightarrow \Gamma \otimes_R \mathcal{C} \otimes_R \mathcal{N} \hookrightarrow \Gamma \otimes_R \mathcal{C} \otimes_R \Gamma, \\
\kappa_2 : \mathcal{N} &\rightarrow \mathcal{N} \otimes_R \mathcal{C} \rightarrow \mathcal{N} \otimes_R \mathcal{C} \otimes_R R \rightarrow \Gamma \otimes_R \mathcal{C} \otimes_R \Gamma, \\
\kappa_3 : \mathcal{N} &\rightarrow \mathcal{C} \rightarrow \Gamma \otimes_R \mathcal{C} \otimes_R \Gamma.
\end{align*}
$$

Lemma 1.2.5 $\text{Im}(\kappa_1 - \kappa_2 + \kappa_3)$ is a $\Gamma$-coideal in $\Xi$. Thus $\Omega := \Xi / \text{Im}(\kappa_1 - \kappa_2 + \kappa_3)$ is a $\Gamma$-$\Gamma$-bimodule of co-algebra structure.

Proof Recall the law of bi-comodule: $(\mu \otimes \mathbb{1})\chi = (\mathbb{1} \otimes \iota)\iota, (\mathbb{1} \otimes \mu)\tau = (\tau \otimes \mathbb{1})\iota, (\mathbb{1} \otimes \tau)\iota = (\iota \otimes \mathbb{1})\tau$ and $(\mathbb{1} \otimes \partial)\iota - \mu \partial + (\partial \otimes \mathbb{1})\iota = 0$. Thus, for any $b \in \mathcal{N}$, we have

$$
\begin{align*}
\mu_\Xi(\kappa_1 - \kappa_2 + \kappa_3)(b) &= \mu_\Xi(1_l \otimes (\bimodule{1}(b))) - \mu_\Xi(\iota(b) \otimes 1_l) + \mu_\Xi(\mathbb{1} \otimes \partial(b) \otimes 1_l) \\
&= (\mu \otimes \mathbb{1})\chi(b) - (\mathbb{1} \otimes \mu)\tau(b) + \mu(\partial(b)) \\
&= (\mathbb{1} \otimes \iota)\iota(\bimodule{1}(b)) - (\tau \otimes \mathbb{1})\imath(b) + (\mathbb{1} \otimes \partial)\iota(b) \\
&= (\mathbb{1} \otimes \iota(\bimodule{1}(b)) - (\partial \otimes \mathbb{1})\iota(b) + (\partial \otimes \mathbb{1})\iota(b) \\
&= u_1(\mathbb{1}(b(1)) + (\mathbb{1} \otimes \tau + \partial)(b(2)) \otimes u_2) \\
&= u_1(\mathbb{1}(b(1)) + (\mathbb{1} \otimes \tau + \partial)(b(2)) \otimes u_2) \in \Xi \otimes \text{Im}(\kappa_1 - \kappa_2 + \kappa_3) + \text{Im}(\kappa_1 - \kappa_2 + \kappa_3) \otimes \Xi
\end{align*}
$$

where $\iota(b) := u_1 (b(1)), \tau(b) := b(2) \otimes u_2$, and each term in each step is viewed as an element in $\Xi \otimes_\Gamma \Xi$ naturally. 

Recall from [Ro] and [CB1], $\mathcal{B} = (\Gamma, \Omega)$ defined as above is a bocs with a layer

$$
L = (R; \omega; a_1, a_2, \ldots, a_n; v_1, v_2, \ldots, v_m).
$$

Denote by $e_\omega$ and $\mu_\omega$ the induced co-unit and co-multiplication, then $\mathcal{B} = \ker e_\omega$ is a $\Gamma$-$\Gamma$-bimodule freely generated by $v_1, v_2, \ldots, v_m$, and $\Omega = \Gamma \otimes \Omega$ as bimodules. From this, we use the embedding: $\mathcal{C}_0 \otimes \mathcal{C}_1 \otimes \mathcal{N} \otimes \mathcal{C}_1 \otimes \mathcal{C}_1 \otimes \mathcal{N} \hookrightarrow \Gamma \otimes_R (\mathcal{C}_0 \otimes \mathcal{C}_1 \otimes \mathcal{N} \otimes_R \mathcal{C}_1 \otimes \mathcal{C}_1 \otimes \mathcal{N}) \otimes_R \Gamma \subset \Omega$; and the isomorphism: $\Omega \otimes R \sim \mathcal{B} = \Gamma \otimes \Gamma \Omega$.

The group-like $\omega : R \rightarrow \Omega, 1_X \mapsto e_X$ is an $R$-$R$-bimodule map. Recall from [CB1] 3.3 Definition, and note that $(\iota_0(a_i) + \iota_1(a_i)) - (\tau_0(a_i) + \tau_1(a_i)) + \iota_1(a_i) = (\kappa_1 - \kappa_2 + \kappa_3)(a_i) = 0$ in $\Omega$, the pair of the differentials determined by $\omega$ is given by $\delta_1 : \Gamma \rightarrow \Omega$:

$$
\begin{align*}
\delta_1(1_X) &= 1_Xe_X - e_X1_X = 0, \quad X \in \mathcal{T}, \\
\delta_1(a_i) &= a_i \otimes_R e_t(a_i) - e_s(a_i) \otimes_R a_i = \tau_0(a_i) - \tau_0(a_i) \\
&= \iota_1(a_i) - \tau_1(a_i) + \partial_1(a_i), \quad 1 \leq j \leq n;
\end{align*}
$$

and $\delta_2 : \Omega \rightarrow \Omega \otimes_\Gamma \Omega, \delta_2(v_j) = \mu(v_j) - e_{s(v_j)} \otimes_R v_j - v_j \otimes_R e_{t(v_j)} = \mu_1(v_j), 1 \leq j \leq m$. Then the bocs $\mathcal{B}$ is said to be the bocs associated to the matrix bimodule problem $\mathcal{A}$. Denote by $(\mathcal{A}, \mathcal{B})$ the pair of a matrix bimodule problem and its associated bocs, or just the pair $(\mathcal{A}, \mathcal{B})$.

Let $(\mathcal{A}, \mathcal{B})$ be a pair with the associated bi-comodule problem $\mathcal{C}$. Then the module actions

$$
l(\mathcal{K}_1 \times A_i), r(A_i \times \mathcal{K}_1) \subseteq \oplus_{i=1}^n R^{\otimes 3} \otimes_R A_i
$$

by the fact stated before Definition 1.2.2, which is called the triangular property. The left and the right co-module actions also possess the triangular property by 1.2.2:

$$
\iota_1(a_i) \in \oplus_{i=1}^{n-1} \mathcal{C}_1 \otimes_R a_i, \quad \tau_1(a_i) \in \oplus_{i=1}^{n-1} a_i \otimes_R \mathcal{C}_1.
$$
Define a $K$-$K$ sub-bimodule of $\mathcal{M}$, then a $K$-$K$-quotient-bimodule of $\mathcal{M}$:

$$\mathcal{M}^{(h)} = \oplus_{i=1}^{n} A_i \subseteq \mathcal{M}, \quad \mathcal{M}^{[h]} = \mathcal{M} / \mathcal{M}^{(h)}.$$  

$\mathfrak{A}^{[h]} = (R, K, \mathcal{M}^{[h]}, \bar{d})$ with $\bar{d}$ induced from $d$ is said to be a quotient problem of $\mathfrak{A}$, but $\mathfrak{A}^{[h]}$ itself might be no longer a matrix bimodule problem. If $\mathcal{N}^{(h)} = \oplus_{i=1}^{n} R \otimes R A_i \otimes R$, then $\mathcal{C}^{(h)} = (R, \mathcal{C}, \mathcal{N}^{(h)}, \partial |_{\mathcal{N}^{(h)}})$ is a sub-bi-comodule problem of $\mathcal{C}$. If $\Gamma^{(h)}$ is a tensor algebra freely generated by $a_1, \ldots, a_h$, then the bocs $\mathfrak{B} = (\Gamma, \Omega)$ has a sub-bocs $\mathfrak{B}^{(h)} = (\Gamma^{(h)}, \Omega^{(h)} \otimes R \Omega \otimes R \Gamma^{(h)})$.

Note a simple fact: let $(\mathfrak{A}, \mathcal{C}, \mathfrak{B})$ be a triple defined as above, then

$$l(K_1 \times \mathcal{M}_1), r(\mathcal{M}_1 \times K_1), d(K_1) \subseteq \mathcal{M}_1^{(h)} \text{ in } \mathfrak{A}$$

$$\iff C_1 \otimes R \mathcal{N}_1^{(h)} = 0, \mathcal{N}_1^{(h)} \otimes R C_1 = 0, \partial(\mathcal{N}_1^{(h)}) = 0 \text{ in } \mathcal{C}$$

$$\iff \delta(\Gamma^{(h)}) = 0 \text{ in } \mathfrak{B}.$$  

In fact, the condition in $\mathfrak{A}$ is equivalent to $\eta_{jil} = 0, \sigma_{ijl} = 0, \zeta_{jil} = 0$ for $l = 1, \ldots, h$ and any $i, j$, which is equivalent to the conditions in $\mathcal{C}$ and $\mathfrak{B}$.

Recall from [CB1] that a representation of a layered bocs $\mathfrak{B}$ is a left $\Gamma$-module $P$ of dimension vector $\underline{d}$ consisting of three sets:

$$\{P_X = k^{d_X} \mid X \in \mathcal{T}\}, \quad \{P(x) : P_X \rightarrow P_X \mid X \in \mathcal{T}\};$$

$$\{P(a_i) : k^{d_{X_i}} \rightarrow k^{d_{V_i}} \mid a_i : X_i \rightarrow Y_i, \ i = 1, \ldots, n\}.  \quad (1.2-5)$$

A morphism from $P$ to $Q$ is given by a $\Gamma$-map $f : \Omega \otimes \Gamma P \rightarrow Q$. Clearly, $\text{Hom}_\Gamma(\hat{\Omega} \otimes \Gamma P, Q) \simeq \bigoplus_{j=1}^{m} \text{Hom}_\Gamma(\Gamma_{s(v_j)} \otimes_k 1_{t(v_j)} P, Q) \simeq \bigoplus_{j=1}^{m} \text{Hom}_\Gamma(1_{t(v_j)} P, 1_{s(v_j)} Q)$. Write

$$f = \{f_X; f(v_j) \mid X \in \mathcal{T}, 1 \leq j \leq m\},  \quad (1.2-6)$$

then [BK] shows that $f$ is a morphism if and only if for all $a_l \in \mathcal{A}^{*}, 1 \leq l \leq n$:

$$P(a_l)f_{v_j} - f_{X_j}Q(a_l) = \sum_{j < l, i} \eta_{jil} \otimes R \otimes R (f(v_j) \otimes R Q(a_j)) - \sum_{i < l, j} \sigma_{ijl} \otimes R \otimes R (P(a_i) \otimes R f(v_j)) + \sum_{i} \zeta_{il} \otimes R \otimes R f(v_i).  \quad (1.2-7)$$

1.3 Representation categories of matrix bimodule problems

In this subsection, a notion of "$*$-product" and the operations between $*$-products are defined, which will be used frequently throughout the paper. Based on this nation, the representation category of a matrix bimodule problem is defined. It is relatively complicated, but seems to be useful for the proof of the main theorem.

**Definition 1.3.1** Let $J(\lambda) = J_d(\lambda)^{e_d} \oplus J_{d-1}(\lambda)^{e_{d-1}} \oplus \cdots \oplus J_1(\lambda)^{e_1}$ be a Jordan form, where $e_i$ are non-negative integers. Denote $e_d + e_{d-1} + \cdots + e_j$ by $m_j$ for $j = 1, \ldots, d$. The following partitioned matrix $W(\lambda)$ similar to $J(\lambda)$ is called a Weyr matrix of eigenvalue $\lambda$:

$$W(\lambda) = \begin{pmatrix} \lambda I_{m_1} & W_{12} & 0 & \cdots & 0 & 0 \\ \lambda I_{m_2} & W_{23} & 0 & \cdots & 0 & 0 \\ \lambda I_{m_3} & \cdots & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda I_{m_{d-1}} & W_{d-1,d} & \cdots & \cdots & \cdots & \cdots \\ \lambda I_{m_d} \end{pmatrix}_{d \times d}.$$
where $W_{j,j+1} = (I_{m_{j+1}} 0)^T$ of size $m_{j} \times m_{j+1}$ with superscript $T$ denoting transpose. A direct sum $W = W(\lambda_1) \oplus W(\lambda_2) \oplus \cdots \oplus W(\lambda_s)$ with distinct eigenvalues $\lambda_i$ is said to be a Weyr matrix. An order “≤” on the base field $k$ may be defined, so that each Weyr matrix has a unique form. Similarly, let $\{Z_{ij} \mid i, j \in \mathbb{Z}^+\}$ be a finite set of vertices, and $S = \oplus_{i,j} kZ_{ij} \oplus k[z, \phi(z)^{-1}]z$ be a minimal algebra. $W \simeq \oplus J_{ij}(\lambda_i)\phi(z)Z_{ij}$ or $\oplus J_{ij}(\lambda_i)\phi(z)Z_{ij} \oplus (z1Z)$ with $\{e_{ij}\} \subset \mathbb{Z}^+$ is said to be a Weyr matrix over $S$. It is possible that some summands of $W$ are diagonal blocks with the diagonal entries being the primitive idempotents of $S$.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem having a set of integers $T = \{1, 2, \cdots , t\}$ with partition $\mathcal{T}$. A Weyr matrix $W$ over $k$ is called $R_X$-regular for $X \in \mathcal{T}_1$, if all the eigenvalues $\lambda$ of $W$ have the property that $\phi_X(\lambda) \neq 0$. An identity matrix $I$ is also called an $R_X$-regular Weyr matrix for $X \in \mathcal{T}_0$. A vector of non-negative integers is said to be a size vector $\underline{m} = (m_1, m_2, \cdots , m_t)$ over $\mathcal{T}$, if $m_i = m_j, \forall i \neq j$. And $\sum_{i=1}^{t} m_i$ is called the size of $\underline{m}$.

**Definition 1.3.2** Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem, $S$ a minimal algebra, and $\sum = \oplus_{p=1}^{\infty} S_{\otimes p}$, see Formula (1.1-3).

(i) Write $H_X = (h_{ij}(x)11, x_{\mathcal{T}t})$ with $h_{ij}(x) \in k[x]$ for $X \in \mathcal{T}_1$, and $x = 1, h_{ij}(x) \in k$ for $X \in \mathcal{T}_0$. Let $\bar{W}_X$ be a Weyr matrix of size $m_X$ over $S$. There exists an $\underline{m} \times \underline{n}$-partitioned matrix over $S$:

$$H_X(\bar{W}_X) = (B_{ij})_{t \times t}, \quad B_{ij} = \left\{ \begin{array}{ll}
h_{ij}(\bar{W}_X)_{m_i \times m_j}, & i, j \in X, \\
0_{m_i \times m_j}, & i \notin X \text{ or } j \notin X, \\
\end{array} \right.$$ 

(ii) Let $\underline{m} = (m_1, \cdots , m_t)$ and $\underline{n} = (n_1, \cdots , n_t)$ be two size vectors over $\mathcal{T}$, and let $F \in \text{IM}_{m_X \times n_X}(S_{\otimes p}), p = 1, 2$, with an $R_X$-module structure. The star product $*$ of $F_X$ and $E_X$ is defined to be a diagonal $\underline{m} \times \underline{n}$-partitioned matrix:

$$F_X * E_X = \text{diag}(B_{11}, \cdots , B_{tt}), \quad B_{ii} = \left\{ \begin{array}{ll}F_X, & i \in X, \\
0, & i \notin X. \\
\right.$$ 

(iii) Let $U = (u_{ij}) \in V_{\mathcal{XY}} \cup A_{\mathcal{XY}}$, and $\{\bar{W}_X \in \text{IM}_{m_X}(S) \mid X \in \mathcal{T}\}, \{\bar{W'}_Y \in \text{IM}_{m_Y}(S) \mid Y \in \mathcal{T}\}$ be two sets of regular Weyr matrices. Suppose there is an $R_X-R_Y$-bimodule structure on $\text{IM}_{m_X \times n_Y}(S_{\otimes p})$ for $p = 1, 2$, and $C \in \text{IM}_{m_X \times n_Y}(S_{\otimes p})$:

$$C \otimes_{R^\otimes 2} (x \otimes_k y) = W_XC_{\bar{W}_Y}. \quad (1.3-1)$$

The star product $*$ of $C$ and $U$ is defined to be an $\underline{(m \times n)}$-partitioned matrix:

$$C * U = (B_{ij})_{t \times t}, \quad B_{ij} = \left\{ \begin{array}{ll}C \otimes_{R^\otimes 2} u_{ij}, & i \in X, j \in Y; \\
0_{m_i \times n_j}, & i \notin X, or j \notin Y. \\
\right.$$ 

**Lemma 1.3.3** Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem.

(i) If $C \in IM_{m_X \times n_Y}(S_{\otimes 2}), d(V_i) = \sum_{i=1}^{n} \zeta_{ii}A_i$ with $\zeta_{ii} \in R_X \otimes_k R_Y$ are given by Definition 1.2.3, then by the usual product of $\Sigma$-matrices:

$$(C * V_i)H_Y(\bar{W}_Y) - H_X(\bar{W}_X)(C * V_i) = \sum_{i=1}^{n} (C \otimes_{R^\otimes 2} \zeta_{ii}) * A_i.$$

(ii) If $F_X \in IM_{m_X \times n_X}(S_{\otimes p}), p = 1, 2$ and $C \in IM_{m_X \times n_Y}(S_{\otimes q}), q = 1, 2$, then by the usual product of $\Sigma$-matrices:

$$(F_X * E_X)(C * U) = \left\{ \begin{array}{ll}(F_X C) * U, & 1_XU = U; \\
0, & 1_XU = 0. \\
\right.$$ 

Similarly, $(C * U)(F_X * E_X) = (CF_X) * U$ for $1_X = U$ and 0 otherwise. Moreover, $$(F_X * E_X)(F'_X * E_X) = (F_X F'_X) * E_X, \quad F_X, F'_X \in IM_{m_X}(S_{\otimes p}), p = 1, 2.$$
(iii) Let \( U \in E_X M_l(R \otimes_k R)E_Y, V \in E_Y M_l(R \otimes_k R)E_Z, G_l \in E_X M_l(R \otimes_k R)E_Z, \) where \( U V = \sum_{l=1}^n \epsilon_l \otimes_R \epsilon_l \) \( \epsilon_l \in R_X \otimes_k R_Y \otimes_k R_Z. \) Let \( m \), \( n \), \( l \) be size vectors over \( T \), \( C \in \text{IM}_{m_X \times n_Y}^1(S^{\otimes p}), D \in \text{IM}_{n_Y \times l_Z}^1(S^{\otimes q}) \) for \( (p, q) \in \{(2, 2), (1, 2), (2, 1)\}. \) Then by the usual \( \Sigma \)-matrix product:

\[
(C \ast U)(D \ast V) = \sum_{l=1}^n ((C \otimes_R D) \otimes_R \epsilon_l) \ast G_l,
\]

where the tensor product \( \oplus_{(X, Y, Z) \in T \times T \times T} \text{IM}_{m_X \times n_Y}^1(S^{\otimes p}) \otimes_R \text{IM}_{n_Y \times l_Z}^1(S^{\otimes q}) \) has an \( R^{\otimes 3} \)-module structure yielded from the \( R \)-\( R \)-bimodule structures given by Formula (1.3-1).

**Proof** (i) Write \( H_X = (\alpha_{pq}), \alpha_{pq} \in R_X; H_Y = (\beta_{pq}), \beta_{pq} \in R_Y; V_l = (v_{pq}), v_{pq} \in R_X \otimes_k R_Y, \) \( \text{IM} \) for the right side. Let \( U = (u_{pq}), u_{pq} \in R_X \otimes_k R_Y \), the left side\( = (F_X 1_X (C \otimes_R u_{pq})) = ((F_X C) \otimes_R u_{pq}) \) the right side.

(ii) Write \( U = (u_{pq}), u_{pq} \in R_X \otimes_k R_Y, \) the left side\( = (F_X 1_X (C \otimes_R u_{pq})) = ((F_X C) \otimes_R u_{pq}) \) the right side.

(iii) Write \( U = (u_{pq}), u_{pq} \in R_X \otimes_k R_Y, \) the left side\( = (F_X 1_X (C \otimes_R u_{pq})) = ((F_X C) \otimes_R u_{pq}) \) the right side.

\[\square\]

**Definition 1.3.4** Let \( \mathcal{A} = (R, K, M, H) \) be a matrix bimodule problem, and \( m \) a size vector over \( T \). Thus a representation \( \bar{P} \) of \( \mathcal{A} \) can be written as an \( m \times m \)-partitioned matrix over \( k \):

\[
\bar{P} = \sum_{X \in T} H_X(W_X) + \sum_{i=1}^n P(a_i) \ast A_i,
\]

where \( W_X \in \text{IM}_{m_X}^1(k) \) is an \( R_X \)-regular Weyl matrix for any \( X \in T \), and \( P(a_i) \in \text{IM}_{m_X \times m_Y}^1(k). \) Taken \( S = k \), the first summand is defined in 1.3.2 (i), and the second one in (iii).

**Definition 1.3.5** Let \( \bar{P}, \bar{Q} \) be two representations of size vectors \( m, n \) respectively. A morphism \( \bar{f} : \bar{P} \rightarrow \bar{Q} \) is an \( m \times n \)-partitioned matrix obtained from Definition 1.3.2 (ii) and (iii) for \( S = k \):

\[
\bar{f} = \sum_{X \in T} f_X \ast E_X + \sum_{j=1}^m f(v_j) \ast V_j,
\]

where \( f_X \in \text{IM}_{m_X \times n_X}^1(k), f(v_j) \in \text{IM}_{m_i(v_j) \times n_i(v_j)}^1(k), \) such that \( \bar{P} \bar{f} = \bar{f} \bar{Q} \) under the matrix product given according to Lemma 1.3.3 (i)–(iii).

If \( \bar{U} \) is an object and \( f' : \bar{Q} \rightarrow \bar{U} \) a morphism, then \( \bar{f} \bar{f}' : \bar{P} \rightarrow \bar{U} \) calculated according to Lemma 1.3.3 (ii)–(iii) is still a morphism. In fact, \( \bar{f} \bar{f}' \bar{P} = \bar{f} \bar{Q} \bar{f}' = (\bar{U} \bar{f}) \bar{f}' = \bar{U} (\bar{f} \bar{f}') \). We denote by \( R(\mathcal{A}) \) the representation category of the matrix bimodule problem \( \mathcal{A} \).

1.4. Formal Products and Formal Equations

In this subsection, 1) a concept of “formal equation” is introduced to build up a nice connection between a matrix bimodule problem and its associated bocs; and 2) a special class of bipartite matrix bimodule problems is noticed, because of the close relation between such problems and finite dimensional algebras.

Let \( \mathcal{A} = (R, K, M, H) \) be a matrix bimodule problem, with the associated bi-comodule problem \( \mathcal{C} = (R, C, N, \partial) \) and the bocs \( \mathcal{B} \). Recall that \( \{E_X\} \) and \( \{\epsilon_X\} \) are dual bases of \( (K_0, C_0); \{V_1, \ldots, V_m\} \) and \( \{v_1, \ldots, v_m\} \) those of \( (K_1, C_1); \{A_1, \ldots, A_n\} \) and \( \{a_1, \ldots, a_n\} \) those of \( (M_1, N_1) \). Set \( S = R, \Sigma = \Delta, \mathcal{M} = (1, \ldots, 1) = \mathcal{N} \) in Definition 1.3.2 (ii)–(iii), then

\[
\begin{align*}
\Upsilon &= \sum_{X \in T} e_X \ast E_X, \\
\Pi &= \sum_{j=1}^m v_j \ast V_j, \\
\Theta &= \sum_{i=1}^n a_i \ast A_i \tag{1.4-1}
\end{align*}
\]
are called the formal products of \((K_0, C_0), (K_1, C_1)\) and \((M_1, N_1)\) respectively.

**Lemma 1.4.1** Let \(\delta\) be the differential in the bocs \(\mathfrak{B}\). We have
\[
(\sum_{i=1}^m v_i \ast V_i) (\sum_{j=1}^m v_j \ast V_j) = \sum_{i=1}^m \mu_{11}(v_i) \ast V_i;
(\sum_{i=1}^n a_i \ast A_i) (\sum_{j=1}^m v_j \ast V_j) = \sum_{i=1}^n \tau_{11}(a_i) \ast A_i;
(\sum_{j=1}^m v_j \ast V_j) (\sum_{i=1}^n a_i \ast A_i) = \sum_{j=1}^m \nu_{11}(a_j) \ast A_i;
(\sum_{j=1}^m v_j \ast V_j) H - H (\sum_{j=1}^m v_j \ast V_j) = \sum_{j=1}^n \partial_1(a_j) \ast A_i;
(\sum_{i=1}^n a_i \ast A_i) (\sum_{x \in T} e_x \ast E_X) - (\sum_{x \in T} e_x \ast E_X) (\sum_{i=1}^n a_i \ast A_i) = \sum_{i=1}^n \delta(a_i) \ast A_i.
\]

**Proof** 1) The second equality is proved first, and the proofs of the first one and the third one are similar. By Lemma 1.3.3 (iii) for \(S = R, p = q = 2\), the left side = \(\sum_{i=1}^n (\sum_{i,j} \sigma_{ijl} \otimes_{R \otimes 3} (a_i \otimes_R v_j)) \ast A_i = \text{the right side}.

2) For the fourth equality, by Lemma 1.3.3 (i) the left side = \(\sum_{i=1}^n (\sum_{i,j} \zeta_{ij} \otimes_{R \otimes 2} v_j) \ast A_i = \text{the right side}.

3) For the last one, by Lemma 1.3.3 (ii) for \(p = 1, q = 2\), the left side = \(\sum_{i=1}^n (a_i \otimes_R e_{y_i} - e_{x_i} \otimes_R a_i) \ast A_i = \text{the right side}.

The matrix equation \((\Theta + H)(\Upsilon + \Pi) = (\Upsilon + \Pi)(\Theta + H)\), more precisely,
\[
(\sum_{i=1}^n a_i \ast A_i + H) (\sum_{x \in T} e_x \ast E_X + \sum_{j=1}^m v_j \ast V_j)
= (\sum_{x \in T} e_x \ast E_X + \sum_{j=1}^m v_j \ast V_j) (\sum_{i=1}^n a_i \ast A_i + H)
\]

is called the formal equation of the pair \((\mathfrak{A}, \mathfrak{B})\) due to the following theorem.

**Theorem 1.4.2** Let \((p_i, q_i)\) be the leading position of \(A_i\) for \(l = 1, \ldots, n\). Then the \((p_i, q_i)\)-entry of the formal equation is
\[
\delta(a_i) = \nu_{11}(a_i) - \tau_{11}(a_i) + \partial_1(a_i).
\]

**Proof.** According to Formula (1.4-2) and Lemma 1.4.1:
\[
\sum_{i=1}^n \delta(a_i) \ast A_i = \sum_{i=1}^n (a_i e_{p_i(a_i)} - e_{\delta(a_i)} a_i) \ast A_i
= \sum_{j=1}^n (v_j \ast V_j)(a_i \ast A_i) - \sum_{j=1}^n (v_j \ast A_j)(a_i \ast V_j) + \sum_{j=1}^n ((v_j \ast V_j)H - H(v_j \ast V_j))
= \sum_{i=1}^n \nu_{11}(a_i) \ast A_i - \sum_{j=1}^n \tau_{11}(a_i) \ast A_i + \sum_{j=1}^n \partial_1(a_j) \ast A_i
= \sum_{i=1}^n \nu_{11}(a_i) - \tau_{11}(a_i) + \partial_1(a_i) \ast A_i.
\]

The expression at the leading position \((p_i, q_i)\) of the formal equation is obtained.

Moreover, the first formula of Lemma 1.4.1 gives:
\[
(\sum_{x \in T} e_x \ast E_X + \sum_{i=1}^m v_i \ast V_i) (\sum_{x \in T} e_x \ast E_X + \sum_{j=1}^m v_j \ast V_j)
= \sum_{x \in T} (e_x \otimes_R e_x) \ast E_X + \sum_{i=1}^m \mu(v_i) \ast V_i.
\]

Let \((\mathfrak{A}, \mathfrak{B})\) be a pair with an index set \(T\) and a vertex set \(\mathcal{T}\). A size vector \(\underline{m} = (m_1, \ldots, m_t)\) over \(T\), and a dimension vector \(\underline{d} = (d_X | X \in \mathcal{T})\) over \(\mathcal{T}\) are said to be associated, if \(m_i = d_X\) for \(i \in X\).

**Corollary 1.4.3** Let \((\mathfrak{A}, \mathfrak{B})\) be a pair. Then the representation categories \(R(\mathfrak{A})\) and \(R(\mathfrak{B})\) are equivalent.

**Proof** Let \(P \in R(\mathfrak{B})\) with dimension vector \(\underline{d}\). Without loss of generality, the set \(\{P(x) = W_X \mid X \in \mathcal{T}\}\) may be assumed to be a set of regular Weyl matrices. Then \(\bar{P}\) of size vector \(\underline{m}\) associated with \(\underline{d}\) in Definition 1.3.4 and \(P\) in Formula (1.2-5) are one-to-one corresponding;
in Definition 1.3.5 and \( f \) in Formula (1.2-6) are one-to-one corresponding. Moreover, \( \bar{P} \bar{f} = \bar{f} \bar{Q} \) if and only if \( f \) satisfies Formula (1.2-7) by Theorem 1.4.2. \( \square \)

Thanks to Corollary 1.4.3, the representations and morphisms in both categories \( R(\mathfrak{A}) \) and \( R(\mathfrak{B}) \) can be denoted by \( P, f \) in a unified manner. Finally we define a special class of matrix bimodule problems to end the subsection. Let \( \mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0) \) be a matrix bimodule problem with \( R \) trivial. \( \mathfrak{A} \) is said to be bipartite provided that \( T = T' \cup T'' \); \( R = R' \times R'' \) and \( \mathcal{K} = \mathcal{K}' \times \mathcal{K}'' \) are direct products of algebras; and \( \mathcal{M} \) is a \( \mathcal{K}'-\mathcal{K}'' \)-bimodule.

**Remark 1.4.4** Let \( \Lambda \) be a finite-dimensional basic \( k \)-algebra, \( J = \text{rad}(\Lambda) \) be the Jacobson radical of \( \Lambda \) with the nilpotent index \( m \), and the top \( S = \Lambda/J \). Suppose \( \{e_1, \ldots, e_h\} \) is a complete set of orthogonal primitive idempotents of \( \Lambda \). Taking the pre-images of \( e_i \) in \( \mathcal{M} \) for each \( i \), \( J \) becomes a \( \mathcal{M} \)-bimodule problem with \( \Lambda \). Moreover, set \( \mathcal{K} = \text{rad}(\Lambda) \) with the nilpotent index \( m \); \( \mathcal{M} \) is essential in the proof of the main theorem, but makes it easier and more intuitive.

A simple calculation shows that the *row indices* of the leading positions of the base matrices in \( \mathcal{A} \) are pairwise different, and the *column index* of the leading position of \( A \in \mathcal{A} \) equals \( j_Z = \max\{j \in Z\} \) for any \( X \in \mathcal{T} \), they are *concentrated*, and the \( j_Z \)-th column is said to be the *main column over \( Z \)*. Such a condition is denoted by RDCC for short. The condition may not be essential in the proof of the main theorem, but makes it easier and more intuitive.

**Example 1.4.5** \([D1, R1]\) Let \( Q = \begin{array}{ccc} a & \ominus & b \\ \ominus & \ominus & \ominus \\ c & \ominus & \ominus \end{array} \) be a quiver, \( I = \langle a^2, ba-ab, ab^2, b^3 \rangle \) be an ideal of \( kQ \), and \( \Lambda = kQ/I \). Denote the residue classes of \( e, a, b \) in \( \Lambda \) still by \( e, a, b \) respectively. Moreover, set \( c = b^2, d = ab \). Then an ordered \( k \)-basis \( \{d, c, b, a, e\} \) of \( \Lambda \) gives a regular representation \( \Lambda \). A matrix bimodule problem \( \mathfrak{A} \) follows from Remark 1.4.4, we may denote by \( A, B, C, D \) the \( R-R \)-quasi-basis of \( \mathcal{M}_1 \), and by \( a, b, c, d \) the \( R-R \)-dual basis of \( \mathcal{N}_1 \). Then the associated bocs \( \mathfrak{B} \) of \( \mathfrak{A} \) has a layer \( L = (R; \omega; a, b, c, d; u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4) \). The formal equation of the pair \((\mathfrak{A}, \mathfrak{B})\) can be written as:

\[
\begin{pmatrix}
0 & 0 & a & b & d \\
0 & 0 & b & 0 & c \\
0 & 0 & b & 0 & a \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & a & b & d \\
0 & 0 & b & 0 & c \\
0 & 0 & b & 0 & a \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
f & 0 & v_1 & v_2 & v_4 \\
0 & v_2 & 0 & v_3 & f \\
v_1 & 0 & v_2 & 0 & f \\
f & v_1 & 0 & 0 & 0
\end{pmatrix}
\]

with \( e = e_X, f = e_Y \) for simplicity. The differentials of the solid arrows of \( \mathfrak{B} \) can be read off according to Theorem 1.4.2:

**2 Reductions on matrix bimodule problems**

In this section, the reduction theorem and eight reductions on matrix bimodule problems corresponding to those on bocses are stated and proved, thus the induced matrix bimodule
problems are constructed. Finally, a concept of defining systems of pairs is defined in order to help to construct the induced pairs in a sequence of reductions.

### 2.1 Admissible bimodules and induced matrix bimodule problems

In the subsection we prove the reduction theorem on matrix bimodule problems via admissible bimodules; then give the connection to the corresponding admissible functors and the reduction theorem on bocses. Before doing so, the following lemma is mentioned first.

**Lemma 2.1.1** Let $D$ be a commutative algebra, and $\Lambda, \Sigma$ be commutative $D$-algebras. Suppose $\Lambda G$ and $\Sigma S$ are finitely generated projective left $\Lambda$-module and right $\Sigma$-module respectively.

(i) $\Lambda G \otimes D S$ is a projective $\Lambda \otimes D \Sigma$-module.

(ii) There exists a $\Lambda \otimes D \Sigma$-module isomorphism:

\[
\text{Hom}_\Lambda(G, \Lambda) \otimes_D \text{Hom}_\Sigma(S, \Sigma) \cong \text{Hom}_{\Lambda \otimes D \Sigma}(\Lambda G \otimes D S, \Lambda \otimes D \Sigma).
\]

**Proof** (i) Suppose $\Lambda G, \Sigma S$ are both free with the basis $\{u_1, \ldots, u_m\}, \{v_1, \ldots, v_n\}$ respectively. Choose a free $\Lambda \otimes D \Sigma$-module $F$ with basis $\{w_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. For $x = \sum_{i=1}^m \lambda_i u_i \in \Lambda G$ and $y = \sum_{j=1}^n v_j \sigma_j \in \Sigma S$, we define $f : \Lambda G \otimes D S \to F, (x, y) = \sum_{i=1}^m \sum_{j=1}^n (\lambda_i \otimes \sigma_j) w_{ij}$.

Then $\lambda x = \sum_{i=1}^m \lambda_i u_i, \text{ and } \lambda y = \sum_{j=1}^n v_j \sigma_j$ for any $\lambda \in D$, and hence $f(x \otimes y) = f(x, y), f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$ and $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$. Thus there exists a unique $\Lambda \otimes D \Sigma$-linear map $\tilde{f} : \Lambda G \otimes D S \to F$ such that $\tilde{f}(x \otimes y) = f(x, y), \forall x \in \Lambda G, y \in \Sigma S$. In particular, $f(u_i \otimes v_j) = w_{ij}$. Thus $\{u_i \otimes v_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is a $\Lambda \otimes D \Sigma$-basis of $\Lambda G \otimes D S$, and $\Lambda G \otimes D S$ is free. Then both $\Lambda G, \Sigma S$ are projective, then there are some $\Lambda G', \Sigma S'$, such that both $\Lambda G + \Sigma S, \Lambda G' + \Sigma S'$ being free.

The assertion follows.

(ii) It is stressed, that $\Lambda G \otimes D S$ and $\text{Hom}_{\Lambda \otimes D \Sigma}(\Lambda G \otimes D S, \Lambda \otimes D \Sigma)$ are projective $\Lambda \otimes D \Sigma$-modules by (i). Consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_\Lambda(G, \Lambda) \times \text{Hom}_\Sigma(S, \Sigma) & \longrightarrow & \text{Hom}_\Lambda(G, \Lambda) \otimes_D \text{Hom}_\Sigma(S, \Sigma) \\
\psi \downarrow & & \downarrow \bar{\psi} \\
\text{Hom}_{\Lambda \otimes D \Sigma}(\Lambda G \otimes D S, \Lambda \otimes D \Sigma) & \text{ } & \text{ }
\end{array}
\]

Let $f \in \text{Hom}_\Lambda(G, \Lambda)$ and $g \in \text{Hom}_\Sigma(S, \Sigma)$. Since $f$ and $g$ are $D$-linear, there exists a $\Lambda \otimes D \Sigma$-linear map $\psi(f, g) : \Lambda G \otimes D S \to \Lambda \otimes D \Sigma$, such that $(\psi(f, g))(x \otimes y) = f(x) \otimes g(y)$, for $(x, y) \in \Lambda G \otimes D S$.

Now $\psi(fg), \psi(f, rg)$ for $r \in D$, thus there exists a unique $\Lambda \otimes D \Sigma$-linear map $\bar{\psi}$ given by $f \otimes g \mapsto \psi(f, g)$, which is clearly natural in both $\text{Hom}_\Lambda(G, \Lambda)$ and $\text{Hom}_\Sigma(S, \Sigma)$. $\bar{\psi}$ is an isomorphism if $\Lambda G, \Sigma S$ are free, consequently, $\psi$ is an isomorphism if $\Lambda G, \Sigma S$ are projective. \(\square\)

Next we introduce a notion of admissible bimodules which is a module-theory version of admissible functors \cite{CBG}. Some preliminaries are needed.

Let $(A, B)$ be a pair with a minimal algebra $R$. Recall from Formula (1.2-4), that $\delta(a_i) = 0$ for the first $h$ arrows, $a_i, i = 1, \ldots, h$, of $B$, if and only if $l(K_1 \times M_1), r(M_1 \times K_1), d(M_1) \subseteq M_1^h$. The algebra $R = R[a_1, \ldots, a_h]$ is said to be pre-minimal.

Let $d = (n_1, \ldots, n_h) \in T$ be a dimension vector over $T$, $R'$ a minimal algebra with the vector $\Sigma L'$ and the algebra $\Delta' = \sum_{p=1}^{\infty} R'^{\otimes p}$, see Formula (1.1-3). Define an $R'-R'$-bimodule $L$ (or an $R$-module over $R'$) of dimension vector $d$ as follows: $L = \bigoplus_{X \in T} L_X$, where $L_X = \oplus_{p=1}^{n_1} R'^{\otimes p} Z_{(X, p)}$ with $Z_{(X, p)} \in T'$, be an $R^*-R'$-bimodule. Let $L^{*} = \bigoplus_{X \in T} L^{*\otimes p}$ be the $R'$-dual module of $L$, where $L^{*\otimes p} = \text{Hom}_{R'}(L_X, R') = \bigoplus_{p=1}^{n_1} R'^{\otimes p} Z_{(X, p)}$. Clearly, $L^{*}$ is an $R-R'$-bimodule.

Denote by $e_{z_{(X, p)}}$ the $(1 \times n_1)$-matrix with the $p$-th entry $1_{z_{(X, p)}}$ and others zero. Then the set $\{e_{z_{(X, p)}} \mid 1 \leq p \leq n_1\}$ forms an $R'$-quasi-basis of $L_X$. Similarly, the set $\{f_{z_{(X, p)}} = e_{z_{(X, p)}}^* = e_{z_{(X, p)}}^{**}\}$
Define $2.1.2$ With the notations as above. The $R'-\bar{R}$-bimodule $L$ of dimension vector $d$ is said to be admissible provided that

(a1) $L$ is sincere over $R'$;

(a2) $\hat{E}_0 \simeq R'$ with an $R'$-quasi-basis:

$$\mathcal{F}_0 = \{F_Z = (FZX)X \in T | Z \in \mathcal{T}' \}, \quad FZX = \sum_{Z(X,p)} Z \hat{f}_Z(X,p) e_{Z(X,p)};$$

(a3) $\hat{E}_1$ is a quasi-free $R'-R'$-bimodule with a quasi-basis $\mathcal{F}_1$ for $i = 1, \ldots, l$:

$$\mathcal{F}_1 = \{F_i = \sum_{p_i, q_i, \xi, 0 < q_i \leq n_X} f_{Z(X,p_i)} \otimes k e_{Z(X,q_i)} | \xi, p_i, q_i = 0 \text{ or } 1 \},$$

where $\{ (p_i, q_i) | \xi, p_i, q_i = 1 \} \cap \{ (p_j, q_j) | \xi, p_j, q_j = 1 \} = \emptyset, \forall X_i = X_j$;

(a4) $\hat{E}_l = \{0\}$.

The $k$-algebra $\hat{E} = \hat{E}_0 \oplus (\hat{\Delta} \otimes R' \otimes Z \hat{E}_1) \subseteq \prod_{X \in \mathcal{T}} \mathbb{T}_{n_X}(\hat{\Delta})$ may be called a pseudo endomorphism algebra of $L$, which is finitely generated in index $(0, 1)$.

Let $M = \bigoplus_{X \in \mathcal{T}} M_X \subseteq \bigoplus_{X \in \mathcal{T}} N_{n_X}(R' \otimes k R')$ be an $R'\otimes R'$-bimodule, where $M_X$ has an $R'$-quasi-basis $\{E(Xpq) | p < q\}$, the matrix units of size $n_X$ with the $(p,q)$-entry $1_{Z(X,p)} \otimes k 1_{Z(X,q)}$ and others zero. There is an $R'-R'$-isomorphism $\kappa : \hat{E}_u \to M, f_{Z(X,p)} \otimes k e_{Z(X,q)} \mapsto E(Xpq)$.

Furthermore, $M_X, \forall X \in \mathcal{T}$, possesses an $\hat{R}\hat{R}$-bimodule structure as follows: if $b, c \in A$ with $t(b) = X = s(c)$, then $b \otimes \hat{R} E_{Xpq} \otimes \hat{R} c = L(b)E_{Xpq}L(c)$. It is clear that $\kappa$ is also an $\hat{R}\hat{R}$-bimodule isomorphism, and $\hat{E}_u$ may be identified with $M$. Thus $\hat{E}_1$ can be viewed as a submodule of $M$.

Write the quasi-free $R'-R'$-bimodule $L^* \otimes k L = \bigoplus_{(X,Y) \in \mathcal{T} \times \mathcal{T}} L_X^* \otimes k L_Y$. An induced matrix bimodule $\mathfrak{A}'$ of $\mathfrak{A}$ based on an admissible bimodule is described below.

Construction 2.1.3 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem. Suppose $\hat{R}, R', L, d, \hat{E}$ are given as in Definition 2.1.2. Then there exists an induced matrix bimodule problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ in the following sense.

(i) Let $\underline{n} = (n_1, \ldots, n_l)$ be the size vector associated with $d$ stated before Corollary 1.4.3. Define a set of integers $T' = \{1, \ldots, t'\}$ with $t' = \sum_{i \in \mathcal{T}} n_i$. Then the set of vertices of $\hat{R}'$ is the partition $T'$ of $T$, and the matrices in $\mathcal{K}', \mathcal{M}'$ and $H'$ are of size $\underline{n} \times \underline{n}$ partitioned under $\mathcal{T}$.

(ii) The $k$-algebra $\mathcal{K}'$ is given as follows. First, let $\mathcal{F}_0' = \{E_{Z} = \sum_{X \in \mathcal{T}} F_{Z}X*e_{X} \in \mathbb{D}_{\mathfrak{T}'}(R') | Z \in \mathcal{T}'\}$ by Definition 1.3.2 (ii) for $S = R$, $p = 1$, and $\mathcal{K}_0'$ be an algebra generated by $\mathcal{F}_0'$ over $R'$. $\mathcal{F}_0'$ is a quasi-basis of $\mathcal{K}_0'$ via the isomorphism $\hat{E}_0 \cong \mathcal{K}_0', F_Z \mapsto \hat{E} Z$, and $\mathcal{K}_0' \simeq R'$. 

Second, let $F'_1 = \{ F'_i = F_i \ast E_{X_i} \mid i = 1, \ldots, l \}$ by 1.3.2 (ii) for $S = R', p = 2$, and $K'_{10}$ be an $R'-R'$-bimodule generated by $F'_1$, where $F'_1$ is a quasi-basis via the isomorphism $E_1 \not\rightarrow K'_{10}, F_i \not\rightarrow F'_i$. There exists a natural order on $F'_1$ according to the leading positions of matrices. Let $K'_{11} = (L^* \otimes k) \otimes_{R \otimes k} K_1 \subseteq N'_t (R' \otimes_k R')$ be an $R'-R'$-bimodule with a quasi-basis:

$$U' = \{ (f_{Z(x',p)} \otimes_k e_{Z(y',q)}) \ast V_j \mid V_j \in V_{X'j}V_j', 1 \leq p \leq n_{X'_j}, 1 \leq q \leq n_{Y'_j}, 1 \leq j \leq m \}$$

given by Definition 1.3.2 (iii) for $S = R', p = 2$. Finally, set $K'_1 = K'_{10} \oplus K'_{11}$ with a quasi-basis $V' = F'_1 \cup U'$.

(iii) Let $\mathcal{M}'_1 = (L^* \otimes_k L) \otimes_{R \otimes k} \mathcal{M}_1$ be an $R'-R'$-bimodule with a normalized quasi-basis $\mathcal{A}' = \{ (f_{Z(x,p)} \otimes_k e_{Z(y, q)}) \ast A_i \mid A_i \in \mathcal{A}_{X,Y}, 1 \leq p \leq n_{X'_j}, 1 \leq q \leq n_{Y'_j}, h < i \leq n \}$ given by Definition 1.3.2 (iii) for $S = R', p = 2$.

(iv) Let $H' = \sum_{X \in T} H_X(L_X(x)) + \sum_{i=1}^{N} L(a_i) \ast A_i$ be a matrix over $R'$, where $L_X(x) = \tilde{W}_X$. $H_X(W_X)$ is defined in 1.3.2 (i); and * is given by 1.3.2 (iii) for $S = R', p = 1$.

(v) The product $m'_{11} : (K'_{10} \oplus K'_{11}) \times (K'_{10} \oplus K'_{11}) \to K'_2$ is given by

$$\begin{align*}
&\big( (f_{Z(x',p)} \otimes_k e_{Z(y',q)}) \ast V_i ) \big) \big( (f_{Z(x',p)} \otimes_k e_{Z(y',q)}) \ast V_j \big) \\
&= \sum_l \big( (f_{Z(x',p)} \otimes_k e_{Z(y',q)}) \big( f_{Z(x',p)} \otimes_k e_{Z(y',q)} \big) \big) \mod R \otimes \mathbb{Z} \gamma_{ij} \ast V_i \ast V_j,
\end{align*}$$

where $(f_{Z(x',p)} \otimes_k e_{Z(y',q)}) = \tilde{f}_{Z(x',p)} \otimes_k e_{Z(y',q)}$.

Admissible bimodules can be transferred to admissible functors as follows. A minimal algebra $R$ can be viewed as a minimal category $A' = \prod_{X \in T} \text{mod} R_X \times \prod_{X \in T} P(R_X)$, [CB1 2.1] by the one-to-one correspondence between the vertex set of $R$ and the set of indecomposable objects of $A'$, the two sets may be identified for the sake of convenience. Then $\tilde{R}$ determines a pre-minimal category $A''$ by adding some morphisms $a_i : X_i \to Y_i, i = 1, \ldots, h$, into $A'$. A similar transfer holds from $R'$ to $B'$. Thus $L$ can be viewed as a functor $\theta' : A' \to B'$, where

$$\theta'(X) = \oplus_{p=1}^{n_X} Z_{(X,p)}, \quad \theta'(c) = L(c), \forall c \in A.$$  \hfill (2.1-1)

We stress, that the opposite construction is usually impossible. Throughout the paper, the right module structure and upper triangular matrix are mainly used, which is opposite to the left module and lower triangular matrix used in [CB1].

**Proposition 2.1.4** The functor $\theta'$ is admissible in the sense of [CB1 4.3 Definition].

**Proof** 1) (A1) is clear; (a1) implies (A2); the finite set $\Lambda = \{ Z_{(X,p)} \mid 1 \leq p \leq n_X, X \in T \}$ has a partial order: $Z_{(X,p)} < Z_{(X,q)}$ if $p < q$ in (A3); (A4) follows from $e_{Z_{(X,p)}} f_{Z_{(X,q)}} = 1_{Z_{(X,p)}}$ for $(X,p) = (Y,q)$, or 0 otherwise; (A5) from $\sum f_{Z_{(X,p)}} e_{Z_{(X,p)}} = 1_{L_X}$.

2) Let $\tilde{E}'_0 = \text{Hom}_R(\tilde{E}_0, R')$, then $\tilde{E}'_0 \simeq \text{Hom}_R(R' \ast R', R') \simeq R'$ by (a2). Since $(\tilde{E}_d)_X \simeq \oplus_{p=1}^{n_X} R' (f_{Z_{(X,p)}} e_{Z_{(X,p)}}) \simeq \oplus_{p=1}^{n_X} R' (f_{Z_{(X,p)}} \otimes R' e_{Z_{(X,p)}}) \simeq \oplus_{p=1}^{n_X} (f_{Z_{(X,p)}} \otimes R' e_{Z_{(X,p)}})$, Lemma 2.1.1 (ii) shows

$$\begin{align*}
\text{Hom}_R(L^* \otimes_{R'} L, R') &\simeq \text{Hom}_{R^* \otimes_{R'} R}(L^* \otimes_{R'} L, R' \otimes_{R'} R') \\
\simeq \text{Hom}_R(L^* \otimes_{R'} R', R') &\simeq L \otimes_{R'} L^*.
\end{align*}$$
We claim, that $\hat{E}_0^* = \sum_{X \in T, p} R'(e_{z(X,p)} \otimes_R f_{z(X,p)})R'$. In fact, $\hat{E}_d^* \subseteq L \otimes_R L^*$, and $\hat{E}_d^*$ has an $R-R$-bimodule structure given by $b(e_{z(X,p)} \otimes_R f_{z(X,p)}), c = (e_{z(X,p)} L(b)) \otimes_R (L(c) f_{z(X,p)}), \forall b, c \in \Lambda$ with $s(b) = X = t(c)$. The $R-R$-structures on $\hat{E}_d^*$ and $\hat{E}_d^*$ ensure that $\hat{E}_0^* = \{(B_X^T)_{X \in T} \in \hat{E}_d^* \mid cB_X^T = B_Y^T c, \forall c \in S \}$ by [M Proposition 3.4, 3.5] as claimed.

3) Let $\hat{E}_1^* = \text{Hom}_{R^2 \otimes_k R'}(\hat{E}_1, R^2')$. Lemma 2.1.1 (ii) shows:

$$\text{Hom}_{R^2 \otimes_k R'}(L^* \otimes_k L, R' \otimes_k R') \simeq \text{Hom}_{R'}(L^*, R') \otimes_k \text{Hom}_{R'}(L, R') \simeq L \otimes_k L^*.$$  \hspace{1cm} (2.1-2)

By (a3) and a similar argument as in 2), $\hat{E}_1^* \simeq \sum_{X \in T, p < q} R'(e_{z(X,p)} \otimes f_{z(X,q)})R'$. 4) Combining 2), 3) and noting $\hat{E}_1^* = \sum_{X \in T, p > q} R'(e_{z(X,p)} \otimes f_{z(X,q)})R' = \{0\}$ by (a4), $\hat{E}_1^* \otimes \hat{E}_1^*$ is $L \otimes_R L^*$. Recall from Formula (2.1-1) that $L \otimes_R L^*$ corresponds to $B' \otimes_A B'$. There is an exact sequence $0 \rightarrow \hat{E}_1^* \rightarrow \hat{E}_0^* \otimes \hat{E}_1^* \rightarrow \hat{E}_1^* \rightarrow 0$, $p(e_{z(X,p)} \otimes f_{z(X,q)}) = e_{z(X,p)} f_{z(X,q)}$, which corresponds to the map $B' \otimes_A B' \rightarrow B'$ in [CBI] 4.3 (A3)]. Thus the kernel $J'$ of the map corresponds to $\hat{E}_1^*$, and $J'$ is projective from $E^*$ being so. (A3) follows.

5) (A6) concerns the bimodule action on $\hat{E}_0^* \simeq R'$. Since $e_{z(X,p)} \otimes f_{z(X,q)} = 0$ for $p > q$ in $\hat{E}_1^*$, (A7) follows. \hspace{1cm} \text{\hfill \Box}

**Proposition 2.1.5** Let $(\mathfrak{A}, \mathfrak{B})$ be a pair, and let $\mathfrak{A}'$ be given by Construction 2.1.3. Then the associated bocs $\mathfrak{B}'$ of $\mathfrak{A}'$ is the induced bocs of $\mathfrak{B}$ given by [CBI] 4.5 Proposition).

**Proof** Denote by $\mathfrak{C}' = (R', C', N', \partial')$ the associated bi-comodule problem of $\mathfrak{A}'$.

1) $\mathfrak{C}_0' = \text{Hom}_{R'}(K_0', R')$. The isomorphism $\hat{E}_0^* = \text{Hom}_{R'}(\hat{E}_0, R^*) \overset{\nu_0^*}{\longrightarrow} \text{Hom}_{R'}(K_0', R') = \mathfrak{C}_0'$ with $\nu_0^*$ being the $R'$-dual map of $\nu_0$ in 2.1.3 (ii) gives the $R'$-quasi-basis $F_0^*$ = $\{e_i'^* = \nu_0^*(F_i^ Chu) \mid Z \in T'\}$ of $\mathfrak{C}_0'$, see the proof 2) of Proposition 2.1.4. And $F_i^*$ is $R'$-dual to $F_i^*$. Thus the kernel $J'$ of the map corresponds to $\hat{E}_1^*$, and $J'$ is projective from $E^*$ being so. (A3) follows.

2) $\mathfrak{C}_1' = \text{Hom}_{R^2 \otimes_k R'}(K_0'^{\otimes 2} \otimes_k K_0', R^2) = \mathfrak{C}_1'^{\otimes 2} \otimes_k \mathfrak{C}_1'^{\otimes 2}$. Since

$$\hat{E}_1^* = \text{Hom}_{R^2 \otimes_k R'}(\hat{E}_1, R^2) \overset{\nu_0^*}{\longrightarrow} \text{Hom}_{R^2 \otimes_k R'}(K_0'^{\otimes 2}, R^2) \simeq \mathfrak{C}_1'^{\otimes 2} \otimes_k \mathfrak{C}_1'^{\otimes 2},$$

is an isomorphism with $\nu_i^*$ being the $R^2$-dual map of $\nu_0$ in 2.1.3 (ii), $F_i^* = \{F_i'^* = \nu_i^* (F_i^ Chu) \mid i = 1, \ldots, l\}$ forms an $R'$-$R'$-basis of $\mathfrak{C}_0'^{\otimes 2}$. And $F_i'^*$ inherits a linear order from $F_i^*$. Since $R'$ is an $R-R$-bimodule via the isomorphism $R' \simeq K_0'$, by Lemma 2.1.1 (ii) and Formula (2.1-2):

$$\mathfrak{C}_1'^{\otimes 2} \simeq \text{Hom}_{R^2 \otimes_k R'}(K_1'^{\otimes 2}, R^2 \otimes_k R') \otimes_k \text{Hom}_{R^2 \otimes_k R'}(K_1', R^2) \simeq \text{Hom}_{R^2 \otimes_k R'}(L^* \otimes_k L^* \otimes_k R^2 \otimes_k R', R^2).$$

Write $(e_{z(X,p)} \otimes_k f_{z(Y,q)}) \otimes R^2 v_j = e_{z(X,p)} \otimes_R v_j \otimes_R f_{z(Y,q)} = v_j pq$. \hspace{1cm} $U'^* = \{v_j pq \mid 1 \leq p \leq n_s(v_j), 1 \leq q \leq n_t(v_j); 1 \leq j \leq m\}$ is an $R'^*$-$R'^*$-basis of $\mathfrak{C}_1'^{\otimes 2}$ dual to $U$ of $\mathfrak{C}_1'^{\otimes 2}$ given in Construction 2.1.3 (ii). The $R'^*$-$R'$-quasi-ideal of $\mathfrak{C}_1'^{\otimes 2}$ is $V'^* = F_1'^* \cup U'^* \cup U'^* \cup U'^* \cup V'^*$. \hspace{1cm} (2.1-2) (iii) and Formula (2.1-2):

3) $\mathfrak{N}_1' = \mathfrak{N}_1^* \otimes_{R^2} \mathfrak{M}_1$, $\mathfrak{M}_1'^* = \mathfrak{N}_1'^* \otimes_{R^2} \mathfrak{M}_1'$ can be proved in a similar manner as that for $\mathfrak{C}_1'^{\otimes 2}$. Write $(e_{z(X,p)} \otimes_k f_{z(Y,q)}) \otimes R^2 v_j = e_{z(X,p)} \otimes_R v_j \otimes_R f_{z(Y,q)} = v_j pq$. $A'^* = \{a_{pq} \mid 1 \leq p \leq n_s(a_{pq}), 1 \leq q \leq n_t(a_{pq}); h \leq i \leq n\}$ is an $R'^*$-$R'$-quasi-ideal of $\mathfrak{N}_1^*$ dual to $A$ of $\mathfrak{N}_1'$.

4) Formula (1.4-1) shows the formal products of $(K_0', \mathfrak{C}_0'), (K_1', \mathfrak{C}_1'), (\mathfrak{M}_1, \mathfrak{N}_1')$ respectively:
Exhibit the formal equation $\Theta'Y' - Y'\Theta' = (\Pi_1\Theta' - \Theta'\Pi_1 + \Pi_1^2 H' + H'\Pi_1') + \Pi_1'\Theta' - \Theta'\Pi_1'$:

$$
\sum_{l}(a_{ljq} \otimes_{R'} e_{i(a_{ljq})}) * A_l - \sum_{l}(e_{i(a_{ljq})} \otimes_{R'} a_{ljq}) * A_l = \sum_{l(i,j)} \left( (v_{ljq})(a_{ljq}) \otimes_{R_{02}} \eta_{ilj} \right) * A_l - \sum_{l(i,j)} \left( (a_{ljq})(v_{ljq}) \otimes_{R_{02}} \sigma_{ilj} \right) * A_l
$$

$$+ \sum_{l,i,j} \left( \xi_{ilj} \otimes_{R_{02}} (v_{ljq}) \right) * A_l + \sum_{l,i,j} \left( (F_i^a(a_{ljq})) \otimes (F_i^a A_l) - \sum_{l,i} \left( (a_{ljq})(F_i^a) \right) * (A_l F_i^a) \right),$$

where $(v_{ljq})(a_{ljq}) = (\sum k v_{ljq} \otimes_{R'} a_{ljq})$, and other matrix products are similar. Thus Theorem 1.4.2 shows the differential $\delta'$ in $\mathfrak{M}$:

$$
\delta'(a_{ljq}) = e_{Z_{(X_i,p)}} \otimes_R \delta(a_{l}) \otimes_R f_{Z_{(Y_i,q)}}
+ \sum_{p'} \nu^{a_{ljq}}_{i}(e_{Z_{(X_{i,p})}} \otimes_R f_{Z_{(Y_{i,q})}}) \otimes_{R'} a_{p'q} - \sum_{q'} \nu^{a_{ljq}}_{i}(e_{Z_{(X_{i,q})}} \otimes_R f_{Z_{(Y_{i,q})}}).
$$

This coincides with the formula in [CBI 4.5 Proposition], i.e. $\mathfrak{M}'$ is the induced bocs of $\mathfrak{M}$. □

### 2.2 Eight reductions

In this subsection seven reductions of matrix bimodule problems based on Definition 2.1.2 and Construction 2.1.3 are introduced, where the last two do not occur in any references on boses. And finally, a regularization is presented as the eighth reduction.

**Proposition 2.2.1** (Localization) Let $(\mathfrak{A}, \mathfrak{M})$ be a pair with $R_X = k[x, \phi(x)^{-1}]$ and $R'_X = k[x, \phi(x)^{-1} c(x)^{-1}]$ a finitely generated localization of $R_X$. Define two algebras $R = R, R' = R_X' \times \prod_{Y \in \mathcal{T} \setminus \{X\}} R_Y$, and an $R'$-$R'$-bimodule $L = R'$. Then $L$ is admissible.

(i) There exists an induced matrix bimodule problem $\mathfrak{M}' = (R', K', M', H')$ of $\mathfrak{A}$ and a fully faithful functor $\vartheta : R(\mathfrak{M}) \to R(\mathfrak{A})$.

(ii) The induced bocs $\mathfrak{M}'$ of $\mathfrak{M}$ given by localization [CBI 4.8] is the associated bocs of $\mathfrak{M}'$.

**Proposition 2.2.2** (Loop mutation) Let $(\mathfrak{A}, \mathfrak{M})$ be a pair with the first arrow $a_1 : X \to X$, such that $\delta(a_1) = 0, X \in \mathcal{T}_0$. Define a pre-minimal algebra $\bar{R} = R[a_1]$, a minimal algebra $R' = R_X' \times \prod_{Y \in \mathcal{T} \setminus \{X\}} R_Y$ with $R_X' = k[x]$ and an $R'$-$R'$-bimodule $L = R'$. Then $L$ is admissible. (i) There exists an induced matrix bimodule problem $\mathfrak{M}' = (R', K', M', H')$ of $\mathfrak{A}$, and an equivalent functor $\vartheta' : R(\mathfrak{M}') \to R(\mathfrak{A})$.

(ii) The induced bocs $\mathfrak{M}'$ of $\mathfrak{M}$ given by the functor $\theta' : A' \to B'$, with $\theta'(Y) = Y, \forall Y \in \mathcal{T}, \theta'(a_1) = x$, is the associated bocs of $\mathfrak{A}'$ by Proposition 2.1.5.

**Proposition 2.2.3** (Deletion) Let $(\mathfrak{A}, \mathfrak{M})$ be a pair, $\mathcal{T}' \subset \mathcal{T}$. Define two algebras $\bar{R} = R, R' = \prod_{X \in \mathcal{T}' \setminus \{X\}} R_X$, and an $R'$-$R'$-bimodule $L = R'$. Then $L$ is admissible.

(i) There exists an induced matrix bimodule problem $\mathfrak{M}' = (R', K', M', H')$ of $\mathfrak{A}$, and a fully faithful functor $\vartheta : R(\mathfrak{M}) \to R(\mathfrak{A})$.

(ii) The induced bocs $\mathfrak{M}'$ of $\mathfrak{M}$ obtained by deletion of $\mathcal{T} \setminus \mathcal{T}'$ [CBI 4.6] is the associated bocs of $\mathfrak{M}'$.

Let the algebra $R_X = k[x, \phi(x)^{-1}]$, $r \in \mathbb{Z}^+$, and $X_1, \ldots, X_s \in k$ with $\phi(X_i) \neq 0$. Write $g(x) = (x - \lambda_1) \cdots (x - \lambda_s)$.

Define a minimal algebra $S$ and an $S$-$R_X$-bimodule $K$:

$$
S = \left( \prod_{i=1}^s \prod_{j=1}^r k_1 Z_{ij} \right) \times k[z, \phi(z)^{-1} g(z)^{-1}];
$$

$$
K = \left( \oplus_{i=1}^s \oplus_{j=1}^r k_1 Z_{ij} \right) \oplus k[z, \phi(z)^{-1} g(z)^{-1}], Z_{ij} = Z_{ji};
$$

$$
K(x) = W : K \to K, W \simeq \oplus_{i=1}^s \oplus_{j=1}^r J_j \lambda_j)Z_{ij} \oplus (z_1 Z_0),
$$

where $\bar{W}$ is a Weyr matrix over $S$. Let $n = \frac{1}{2}sr(r+1) + 1$. Denote by $\{(i, j, l) \mid 1 \leq j \leq r, 1 \leq l \leq j, 1 \leq i \leq s\} \cup \{n\}$ the index set of the direct summands of $K$. There is a partition given by $Z_{ij} = \{(i, j) \mid l = 1, \ldots, j\}, Z = \{n\}$. An order on the set is defined as

$$(i, j, l) < (\ell', j', \ell') \iff i < i'; \text{ or } i = i', l < l'; \text{ or } i = i', l = l', j > j',$$
and $n< (i,j,l)$. Let $e_{(ij)}$ be a $1 \times n$, (resp. $f_{(i,j,l)}$ an $n \times 1$) matrix with $1 \beta_{ij}$ at the $(i,j,l)$-th component and 0 at others. Then $K$ has an $S$-$S$-quasi-basis $\{e_{(ij)}, e_n \mid 1 \leq j < r, 1 \leq l < j, 1 \leq i \leq s\}$, and $K^* = \text{Hom}_S(K,S)$ has $\{f_{(ij)}, f_n \mid 1 \leq j < r, 1 \leq l < j, 1 \leq i \leq s\}$. The $S$-quasi-free-module $E_0$, and the $S$-$S$-quasi-free bimodule $E_1$ have the quasi-basis respectively:

$$F_{ij} = \sum_{l=1}^j f_{(ij)} \otimes_S e_{(l)}; F_n = f_n \otimes_S e_n \mid 1 \leq j < r, 1 \leq l \leq j, 1 \leq i \leq s,$$

(2.2.2)

$$F_{ijl} = \sum_{k} f_{(ij)} \otimes k e_{(j+l-1)}, l = \begin{cases} 1, \ldots, j', & \text{if } j > j'; \\ 2, \ldots, j', & \text{if } j = j'; \\ j, \ldots, j', & \text{if } j < j'. \end{cases}$$

**Proposition 2.5.4** (Unraveling) Let $(\mathcal{A}, \mathcal{B})$ be a pair with $R_X = k[x, \phi(x)^{-1}]$. Define two algebras $R' = R$, $R'' = S \times \prod_{Z \in \mathcal{T}_X} R_z$, and an $R'$-$R$-bimodule $L = K \oplus (\oplus_{Z \in \mathcal{T}_X} R_z)$ with $S$ and $K$ given by Formula (2.2.1). Then $L$ is admissible.

(i) There exists an induced matrix bimodule problem $\mathcal{A}' = (R', X', \mathcal{M}', H')$ and a fully faithful functor $\vartheta : R(\mathcal{A}) \to R(\mathcal{A}')$.

(ii) The induced bocs $\mathcal{B}'$ of $\mathcal{B}$ given by unraveling [CB1, 4.7] is the associated bocs of $\mathcal{A}'$.

The picture below shows $e_{(ij)} \otimes_R f_{(ij)}$ in $E_1^*$ as dotted arrows for $s = 1$, $r = 3$:

Let an algebra $R_{XY}$, a minimal algebra $S$ and an $S$-$R_{XY}$-module $K$ be defined as follows:

$$R_{XY} : X \overset{a_1}{\to} Y, \quad S = \prod_{i=1}^3 S_{Z_i}, \quad S_{Z_i} = k1_{Z_i}, i = 1, 2, 3;$$

$$K_X = k1_{Z_2} \oplus k1_{Z_3}, \quad K_Y = k1_{Z_3} \oplus k1_{Z_2}, \quad K(a_1) = \begin{pmatrix} 0 & 1_{Z_2} \\ 0 & 0 \end{pmatrix} : K_X \to K_Y.$$ (2.2.3)

Let $Z_{(1,1)} = Z = Z_{(1,2)}$, and $Z_{(1,2)} = Z_1, Z_{(1,3)} = Z_3$, then $\{e_{(x,1)}, e_{(x,2)}\}$ is an $S$-quasi-basis of $K_X$, and $\{f_{(x,1)}, f_{(x,2)}\}$ is that of $K_X^* = \text{Hom}_S(K_X, S)$. There is a similar observation on $K_Y$. The $S$-quasi-free module $E_0$, and the $S$-$S$-quasi-free bimodule $E_1$ have the quasi-basis respectively:

$$F_{Z_1} = f_{(x,1)} \otimes_S e_{(x,2)}; F_{Z_3} = f_{(x,1)} \otimes_S e_{(x,3)}; F_{Z_2} = f_{(x,2)} \otimes_S e_{(x,2)}.$$ (2.2.4)

**Proposition 2.5.5** (Edge reduction) Let $(\mathcal{A}, \mathcal{B})$ be a pair with the first arrow $a_1 : X \to Y$, such that $X, Y \in \mathcal{T}_0, \delta(a_1) = 0$. Define a pre-minimal algebra $R = R[a_1]$, a minimal algebra $R' = S \times \prod_{Z \in \mathcal{T}_X} R_z$, and an $R'$-$R$-bimodule $L = K \oplus (\oplus_{Z \in \mathcal{T}_X} R_z)$ with $S$ and $K$ defined in Formula (2.2.3). Then $L$ is admissible.

(i) There exists an induced matrix bimodule problem $\mathcal{A}' = (R', X', \mathcal{M}', H')$, and an equivalent functor $\vartheta : R(\mathcal{A}) \to R(\mathcal{A}')$.

(ii) The induced bocs $\mathcal{B}'$ of $\mathcal{B}$ given by edge reduction [CB1, 4.9] is the associated bocs of $\mathcal{A}'$.

**Proposition 2.5.6** Let $(\mathcal{A}, \mathcal{B})$ be a pair with the first arrow $a_1 : X \to Y$, such that $X, Y \in \mathcal{T}_0, \delta(a_1) = 0$. Set two algebras $R = R[a_1], R' = R$, and an $R'$-$R$-bimodule $L = K \oplus (\oplus_{U \in \mathcal{T}_X} R_U)$ with $K : R_X \to R_Y$. Then $L$ is admissible
(i) There are an induced matrix bimodule problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', d')$ with $\mathcal{K}' = \mathcal{K}$, $\mathcal{M}' = \mathcal{M}'^{(1)}$, $H' = H$, and an induced fully faithful functors $\theta : R(\mathfrak{A}') \to R(\mathfrak{A})$. The subcategory of $R(\mathfrak{A})$ consisting of representations $P$ with $P(a_1) = 0$ is equivalent to $R(\mathfrak{A}')$.

(ii) The induced bocs $\mathfrak{B}'$ of $\mathfrak{B}$ given by the admissible functor $\theta' : A' \to B'$ with $\theta'(U) = U, \forall U \in T$ and $\theta'(a_1) = 0$ is the associated bocs of $\mathfrak{A}'$.

Let $R_{XY} = R_X \times R_Y$ be a minimal algebra with $R_X = k[x, \phi(x)^{-1}], R_Y = k[y]$. Define an algebra $R_{XY}$ with $a_1 : X \to Y$, an algebra $S = k[z, \phi(z)^{-1}]$, and an $S$-$R_{XY}$-bimodule $K$ with $K_X = S, K_Y = S, K(x) = (z), K(a_1) = (1z)$. Then there are a $S$-module $E_0 = SF_Z$ with $F_Z$ defined below, and a $S$-$S$-bimodule $E_1 = 0$.

\[ R_{XY} : x \bigotimes Y ; \quad S : z \bigotimes Z ; \quad K : z \bigotimes S \bigotimes (1) \quad \text{(2.2-5)} \]

\[ F_Z = (f_{x} \boxtimes e_{x}, f_{y} \boxtimes e_{y}) = (1z, 1z). \]

**Proposition 2.2.7** Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with the first arrow $a_1 : X \to Y$ and $\delta(a_1) = 0$. Define a pre-minimal algebra $R = R[a_1]$, a minimal algebra $R' = S \times \prod_{U \in \mathcal{T}(X, Y)} R_U$, and an $R'$-$R$-bimodule $L = K \otimes (\bigoplus_{U \in \mathcal{T}(X, Y)} R_U)$, where $S$ and $K$ are given by Formula (2.2-5). Then $L$ is admissible.

(i) There are an induced matrix bimodule problem $\mathfrak{A}'$, and an induced fully faithful functors $\theta : R(\mathfrak{A}') \to R(\mathfrak{A})$. The subcategory of $R(\mathfrak{A})$ consisting of representations $P$ with $P(a_1)$ invertible is equivalent to $R(\mathfrak{A}')$.

(ii) The induced bocs $\mathfrak{B}'$ given by the admissible functor $\theta' : A' \to B'$ with $\theta'(X) = Z, \theta'(Y) = Z; \theta'(a_1) = (1),$ is the associated bocs of $\mathfrak{A}'$.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, d)$ be a matrix bimodule problem, $\mathcal{C} = (R, \mathcal{C}, \mathcal{N}, \partial)$ be the associated bi-comodule problem, and $\mathfrak{B}$ the bocs of $\mathcal{C}$. Then

\[ \text{d}(V_1) = A_1 + \sum_{l \geq 2} \zeta_l A_l \quad \text{and} \quad \text{d}(V_j) \in \mathcal{M}_1' \quad \text{for} \quad j \geq 2 \quad \text{in} \quad \mathfrak{A} \]

\[ \Longleftrightarrow \delta(a_1) = v_1 \quad \text{in} \quad \mathcal{C} \Longleftrightarrow \delta(a_1) = v_1 \quad \text{in} \quad \mathfrak{B}. \]

In fact, since $\partial(a_1) = \sum_{j \geq 2} \zeta_j v_1$, we have $\partial(a_1) = v_1$, if and only if $\zeta_{11} = 1, \zeta_1 = 0$ for all $j \geq 2$, if and only if $\text{d}(V_j) \in \mathcal{M}_1'$ for all $j \geq 2$, since $\text{d}(V_j) = \zeta_j A_1 + \sum_{l \geq 2} \zeta_l A_l$. On the other hand, noting $\iota_1(a_1) = 0$ and $\iota_1(a_1) = 0$ by $\text{triangularity of} \mathcal{C}$, thus $\delta(a_1) = v_1$, if and only if $\partial(a_1) = v_1$ in $\mathcal{C}$ by the definition of $\delta: \Gamma \to \Omega$ given below Lemma 1.2.5.

**Remark** Let $\mathfrak{A}, \mathcal{C}$ be given as above with $\partial(a_1) = v_1$. Then

(i) $\mathcal{K}'^{(1)} = \mathcal{K}_0 \otimes (\bigoplus_{l = 2}^{m} \Delta \otimes \bigotimes_{R \otimes 2} \mathfrak{V}_j)$ is a sub-algebra of $\mathcal{K}$, and $\mathcal{M}'^{(1)} = \bigoplus_{n = 1}^{m} \Delta \otimes \bigotimes_{R \otimes 2} a_i$ is a $\mathcal{K}'^{(1)}$-sub-bimodule;

(ii) $\mathcal{C}'^{(1)} = \bigotimes_{R \otimes 2} \mathcal{C}$ is a coideal of $\mathcal{C}$, and $\mathcal{C}'^{(1)} = \mathcal{C}/\mathcal{C}'^{(1)}$ is a quotient coalgebra; $N_1' = \bigotimes_{R \otimes 2} a_1$ is a $\mathcal{C}$-sub-bi-comodule, and $N^{(1)} = N/\mathcal{N}_1^{(1)}$ is a $\mathcal{C}^{(1)}$-quotient bi-comodule.

**Proof** (i) By the triangularity (1.2-2), $\text{d}(V_j) = \text{d}(V_j)$ for all $V_j \in \mathcal{M}_1'$ and $\forall V_j, V_j \in \mathcal{C}$:

\[ \text{d}(V_j) = \text{d}(\bigoplus_{l \geq 2} \gamma_{ijl} \otimes \bigotimes_{R \otimes 2} \mathfrak{V}_l) = \sum_{l \geq 2} \gamma_{ijl} \otimes \bigotimes_{R \otimes 2} \mathfrak{V}_l \
\]

The coefficient of $A_1$ is $\sum_{l \geq 2} \gamma_{ijl} \otimes \mathfrak{V}_l$, where $\gamma_{ijl} = 1, \zeta_l = 0$ for $l > 1$ by hypothesis, so that $\gamma_{ijl} = 0$ for all $1 \leq i, j \leq m$. Therefore $V_j = \sum_{l \geq 2} \gamma_{ijl} \otimes \bigotimes_{R \otimes 2} \mathfrak{V}_l \in \mathcal{K}'^{(1)}$ and hence $\mathcal{K}'^{(1)}$ is a subalgebra of $\mathcal{K}$. $\mathcal{M}'^{(1)}$ is a $\mathcal{K}'^{(1)}$-bimodule deduced from the triangularity (1.2-2) easily.

(ii) Since $\mu(v_1) = \mu(\partial(a_1)) = (\partial \otimes \mathcal{I})(e(a_1)) + (\mathcal{I} \otimes \partial)(\tau(a_1)) = (\partial \otimes \mathcal{I})(e(a_1)) + (\mathcal{I} \otimes \partial)(e(a_1) e(a_1)) = 0$, $\mathcal{C}'^{(1)}$ is a coideal of $\mathcal{C}$. $N^{(1)}$ is a $\mathcal{C}$-$\mathcal{C}$-sub-bi-comodule deduced from (1.2-3) easily. \(\square\)
Proposition 2.2.8 (Regulation) Let \((\mathfrak A, \mathfrak B)\) be a pair with \(\delta(a_1) = v_1\).

(i) There is an induced matrix bimodule problem \(\mathfrak A' = (R, \mathfrak K^{(1)}, \mathfrak M^{(1)}, H)\) of \(\mathfrak A\), and an equivalent functor \(\bar{\vartheta} : R(\mathfrak A') \to R(\mathfrak A)\).

(ii) The induced bimodule \(\mathfrak B'\) of \(\mathfrak B\) given by regularization \([\text{CB}1, 4.2]\) is the associated bimodule of \(\mathfrak A'\).

Proof (i) \(\mathfrak A'\) is a matrix bimodule problem by Remark (i) above. Note that \(R' = R, T' = T\). Taken any \(P \in R(\mathfrak A)\) of size vector \(\vec{m}\), let \(f = \sum_{x \in T} I_{m_x} \ast E_x + P(a_1) \ast V_1\), then \(P' = \bar{f}^{-1} Pf \in R(\mathfrak A')\). Therefore, \(\bar{\vartheta}\) is an equivalent functor.

(ii) \(\mathfrak A' = (R, \mathfrak C^{[1]}, \mathfrak N^{[1]}, \bar{\vartheta})\) with \(\bar{\vartheta}\) induced from \(\vartheta\) is the associated bi-comodule problem of \(\mathfrak A'\) by Remark (ii) above. Thus the associated bimodule \(\mathfrak B'\) of \(\mathfrak A'\) is given by regularization from \(\mathfrak B\). \(\square\)

Let \((\mathfrak A, \mathfrak B)\) be a pair with a layer \(L = (R; \omega; a_1, \ldots, a_n; v_1, \ldots, v_m)\) in \(\mathfrak B\). Suppose the first arrow \(a_1 : X \to Y\) with \(\delta(a_1) = \sum_{j=1}^m f_j(x, y)v_j \neq 0\). In order to obtain \(\delta(a_1) = h(x, y)v'_1\), we make the following base change:

\[
(v'_1, \ldots, v'_m) = (v_1, \ldots, v_m)F(x, y),
\]

where \(F(x, y) \in \text{IM}(R \otimes_k R)\) is invertible. When \(X \in T_0\) or \(Y \in T_0\), \(R\) is preserved; but when \(X, Y \in T_1\), some localization \(R'_{X} = R_X[c(x)^{-1}]\) (resp. \(R'_{Y} = R_Y[c(y)^{-1}]\)) is needed \([\text{CB}1, 5]\). Consequently, we have a base change of \(\mathfrak K_1\) dually given by

\[
(V'_1, \ldots, V'_m) = (V_1, \ldots, V_m)F(x, y)^{-T}.
\]

Finally, a simple fact according to all the reductions defined above is mentioned to end the subsection. Let us start from a matrix bimodule problem \(\mathfrak A^0 = (R^0, \mathfrak K^0, \mathfrak M^0, H = 0)\) with \(T^0\) trivial, and after a series of reductions, an induce matrix bimodule problem \(\mathfrak A' = (R', \mathfrak K', \mathfrak M', H')\) is obtained. Then for any \(X \in T'\), \(H'_X = (h_{ij}(x))\), any entry \(h_{ij}(x) = a_{ij} + b_{ij}x \in k[x]\).

2.3 Canonical forms

In this subsection, a canonical form (cf. \([\text{S}]\)) for each representation of a matrix bimodule problem is calculated; and a notion of reduction blocks is defined.

Convention 2.3.1 Suppose \(\mathfrak A\) is a matrix bimodule problem, \(\mathfrak A'\) an induced matrix bimodule problem and \(\bar{\vartheta} : R(\mathfrak A') \to R(\mathfrak A)\) an induced functor. Let \(m'\) be a size vector over \(T'\) of \(\mathfrak A'\). A size vector \(\vec{m} = (m_1, m_2, \ldots, m_t)\) over \(T\) of \(\mathfrak A\) is defined:

(i) for regularization, loop mutation, localization, and Proposition 2.2.6, set \(m = m'\);

(ii) for deletion, set \(m_i = m'_i\) if \(i \in X, X \in T'\), and \(m_i = 0\) if \(i \in X, X \in T \setminus T'\);

(iii) for edge reduction, set \(m_i = m'_i\) if \(i \in Z, Z \neq X, Y, m_i = m'_{Z_{2}} + m'_{Z_{3}}\) if \(i \in X\), and \(m_i = m'_{Z_{2}} + m'_{Z_{3}}\) if \(i \in Y\); For proposition 2.2.7, set \(m_X = m'_{Z_{2}}\), \(m_Y = m'_{Z_{3}}\);

(iv) for unraveling, set \(m_i = m'_i\) if \(i \notin X\), and \(m_i = \sum_{j=1}^n \sum_{j=1}^m j m'_{z_{ij}} + m'_{z_{0}}\) if \(i \in X\).

Then \(\vec{m}\) is said to be the size vector determined by \(m'\), and is denoted by \(\bar{\vartheta}(m')\).

Let \(\mathfrak A = (R, \mathfrak K, \mathfrak M, H)\) be a matrix bimodule problem with \(T\) being trivial, and \(\vec{m}\) be a size vector. For the sake of simplicity, we write

\[
H_{\vec{m}}(k) = \sum_{X \in T} H_X(I_{m_X}), \quad H(k) = \sum_{X \in T} H_X(1).
\]

Let \(P\) be a representation of size vector \(\vec{m}\) in \(R(\mathfrak A)\). Then Definition 1.3.4 shows:

\[
P = H_{\vec{m}}(k) + \sum_{i=1}^n P(a_i) \ast A_i.
\]
sincere over $\mathcal{T}$ in the sequel. We will find an induced matrix bimodule problem $\mathfrak{A}'$ given by minimal steps of reductions, and an object $P' \in R(\mathfrak{A}')$ of sincere size vector $\mathbf{m}'$ over $\mathcal{T}'$, such that $\vartheta(P') \simeq P$ under the induced functor $\vartheta : R(\mathfrak{A}') \to R(\mathfrak{A})$. Let $\mathfrak{A}$ be the associated bocs of $\mathfrak{A}$ with the first arrow $a_1 : X \to Y$. There are three possibilities.

(i) If $\delta(a_1) = v_1$, we proceed with a regularization, and obtain an induced matrix bimodule problem $\mathfrak{A}'$. Set

$$B = (\emptyset)_{m_X \times m_Y}, \quad G = (\emptyset)_{1 \times 1},$$

where $\emptyset$ indicates a distinguished zero entry or block. Suppose $P'$ is given in the proof (i) of Proposition 2.2.8 with $\mathcal{T}' = \mathcal{T}, m' = m$. Then $P'(a_1) = B$ and $\vartheta(P') \simeq P$ in $R(\mathfrak{A})$.

(ii) If $\delta(a_1) = 0$ and $X = Y$, suppose $P(a_1) \simeq J = \oplus_{i=1}^{r} (\oplus_{j=1}^{s} J_{i} \lambda_{j}(\alpha_{ij})), e_{ij} \geq 0$, a Jordan form over $k$ with the maximal size $r$ of the Jordan blocks. We first proceed with a loop mutation $a_1 \mapsto (x)$, then with an unraveling for the polynomial $g(x) = (x - \lambda_1) \cdots (x - \lambda_s)$ and the integer $r$, thus an induced matrix bimodule problem $\mathfrak{A}_1$ of $\mathfrak{A}$ is obtained. Let $f_X \in \mathbb{I}_{m_X}(k)$ be invertible, such that

$$B = f_X^{-1}P(a_1)f_X = W; \quad G = \hat{W},$$

where $W$ is a Weyr matrix over $k$, $\hat{W}$ is that over $R'$ similar to $\oplus_{i=1}^{r} \oplus_{j=1}^{s} J_{i} \lambda_{j}(\alpha_{ij})$. Deleting a set of vertices $\{Z_{0}\} \cup \{Z_{ij} | e_{ij} = 0\}$ from $\mathfrak{A}_1$, an induced problem $\mathfrak{A}'$ of $\mathfrak{A}$ is obtained. Let $\mathbf{m}' = (m'_{ij})_{ij \in \mathcal{T}'}$ be a size vector over $\mathcal{T}'$ with $m'_{Z} = m_{Z}$ for $Z \in \mathcal{T} \setminus \{X\}$, and $m'_{Z_{ij}} = j e_{ij}$, then $\mathbf{m}'$ is sincere. Let $f = f_{X} \ast E_{X} + \sum_{Z \in \mathcal{T} \setminus \{X\}} I_{m_{Z}} \ast E_{Z}$, and $P' = f^{-1} P f$ with the size vector $\mathbf{m}'$ in $R(\mathfrak{A}')$. Then $P'(a_1) = B$ and $P \simeq \vartheta(P')$ in $R(\mathfrak{A})$.

(iii) If $\delta(a_1) = 0$ and $X \neq Y$, we proceed with an edge reduction for $\mathfrak{A}$ and obtain an induced problem $\mathfrak{A}_1$ with the vertex set $\mathcal{T}_1$. If rank$(P(a_1)) = r$, let $f_X \in \mathbb{I}_{m_X}(k), f_Y \in \mathbb{I}_{m_Y}(k)$ be invertible, such that

$$B = f_X^{-1}P(a_1)f_Y = \begin{pmatrix} 0 & 1_r \\ 0 & 0 \end{pmatrix} m_X \times m_Y,$$

where the five cases of $G$ are obtained by deleting a subset $\mathcal{T} \subset \mathcal{T}_1$ from $\mathfrak{A}_1$: ① $\mathcal{T} = \{Z_2\}$ for $r = 0$; now suppose $r > 0$, ② $\mathcal{T} = \{Z_2, Z_3\}$ for $m_y = m_y = m_y > r$; ③ $\mathcal{T} = \{Z_2\}$ for $m_x > m_x, m_y > r$; ④ $\mathcal{T} = \{Z_3\}$ for $m_x > m_x, m_y = r$; ⑤ $\mathcal{T} = \emptyset$ for $m_x, m_y > r$. An induced matrix bimodule problem $\mathfrak{A}'$ of $\mathfrak{A}$ given by $a_1 \mapsto G$ is obtained. Let $\mathbf{m}' = (m'_{ij})_{ij \in \mathcal{T}'}$ be a size vector over $\mathcal{T}'$, with $m'_{Z} = m_{Z}$ for $Z \in \mathcal{T} \setminus \{X, Y\}, m'_{Z_2} = m_x - r, m'_{Z_3} = m_r - r, m'_{Z_4} = m_y - r, m'_{Z_5} = m_y - r$, thus $\mathbf{m}'$ is sincere over $\mathcal{T}'$. Let $f = f_{X} \ast E_{X} + f_Y \ast E_{Y} + \sum_{Z \in \mathcal{T} \setminus \{X, Y\}} I_{m_{Z}} \ast E_{Z}$, and $P' = f^{-1} P f$ with the size vector $\mathbf{m}'$ in $R(\mathfrak{A}')$. Then $P'(a_1) = B$ and $P \simeq \vartheta(P')$ in $R(\mathfrak{A})$.

Lemma 2.3.2 (cf. [5]) Let $\mathfrak{A} = (R, K, M, H)$ be a matrix bimodule problem with $T$ trivial, and let $P$ be given in Formula (2.3-2). Then there exists an induced matrix bimodule problem $\mathfrak{A}' = (R', K', M', H')$ given by one of the following three reductions:

(i) Regularization,

(ii) Loop reduction: a loop mutation, then a unraveling, followed by a deletion.

(iii) Edge reduction: first an edge reduction, followed by a deletion.

There is a representation $P'$ of sincere size vector $\mathbf{m}'$ over $\mathcal{T}'$ in $R(\mathfrak{A}')$, such that $P \simeq \vartheta(P')$ in $R(\mathfrak{A})$ under the fully faithful functor $\vartheta : R(\mathfrak{A}') \to R(\mathfrak{A})$. According to Formulæ (2.3-3)–(2.3-5):

$$P' = H_{\mathbf{m}}(k) + B \ast A_1 + \sum_{i=1}^{n'} P(a'_i) \ast A'_i.$$
**Theorem 2.3.3** (cf. [S]) Let \( \mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0) \) be a matrix bimodule problem with \( T \) trivial. Let \( P \in R(\mathfrak{A}) \) be a representation of sincere size vector \( \underline{m} \). Then there exists a unique sequence of matrix bimodule problems:

\[
\mathfrak{A} = \mathfrak{A}^0, \mathfrak{A}^1, \ldots, \mathfrak{A}^i, \mathfrak{A}^{i+1}, \ldots, \mathfrak{A}^s
\]

\[
(G^1, \ldots, G^i, G^{i+1}, \ldots, G^s)
\]

where \( \mathfrak{A}^{i+1} \) is obtained from \( \mathfrak{A}^i \) by \( a^i \mapsto G^{i+1} \) defined by one of Formula (2.3-3)–(2.3-5). There is also a unique sequence of representations:

\[
P^0 = P, \ P^1, \ldots, P^i, P^{i+1}, \ldots, P^s
\]

\[
(B^1, \ldots, B^i, B^{i+1}, \ldots, B^s)
\]

where \( P^i(a^i) = B^i \) is defined by one of Formulae (2.3-3)–(2.3-5). Let \( \vartheta^{i,i+1} : R(\mathfrak{A}^{i+1}) \to R(\mathfrak{A}^i) \) be the induced functor. There is a representation \( P^{i+1} \in R(\mathfrak{A}^{i+1}) \) of sincere size vector \( \underline{m}^{i+1} \) with \( \vartheta^{i,i+1}(P^{i+1}) \cong P^i \) for \( 0 \leq i < s \).

Write for \( i < j \) the composition of induced functors \( \vartheta^{i,j} = \vartheta^{i,i+1} \cdots \vartheta^{j-1,j} : R(\mathfrak{A}^j) \to R(\mathfrak{A}^i) \). Denote by \( A_1^i \) the first quasi-basis matrix of \( \mathcal{M}_1^i \) in \( \mathfrak{A}^i \), then \( P^{i+1} = H^{i}_k + B^{i+1} \ast A_1^i + \sum_{j=2}^{s+1} M^{i+1}(a^{i+1}_j) \ast A_1^{i+1} \). Using the formula inductively:

\[
\vartheta^{0s}(H^{s}_{\underline{m}_s^s}(k)) = \sum_{i=0}^{s-1} B^{i+1} \ast A_1^i \in R(\mathfrak{A}).
\]

In particular, if \( \mathfrak{A}^s \) is minimal, then \( P^s = H^{s}_{\underline{m}_s^s}(k) \). In this case, the matrix \( \vartheta^{0s}(P^s) \) is called the **canonical form** of \( P \), and denoted by \( P^\infty \). The entry “1” appearing in \( B^{i+1} \) of \( P^\infty \), which is not an eigenvalue when \( B^{i+1} \) being a Weyr matrix, is called a **link** of \( P^\infty \). And denote by \( l(P^\infty) \) the number of the links in \( P^\infty \).

**Corollary 2.3.4** [S] The canonical form of any representation \( P \) over a matrix bimodule problem \( \mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0) \) with \( R \) trivial is uniquely determined. Moreover,

(i) for any \( P, Q \in R(\mathfrak{A}) \), \( P \simeq Q \) if and only if \( P \) and \( Q \) have the same canonical form;

(ii) \( P \) is indecomposable if and only if \( l(P^\infty) = \dim(P) - 1 \).

**Corollary 2.3.5** Let \( \mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H) \) be a matrix bimodule problem with \( T \) trivial, let \( \mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H') \) be an induced matrix bimodule problem obtained by a series of reductions with an induced functor \( \vartheta : R(\mathfrak{A}') \to R(\mathfrak{A}) \). If \( T' \) is trivial and \( P = \vartheta(H'(k)) \) with a sincere size vector \( \underline{m} \) over \( T \) in \( R(\mathfrak{A}) \), then there is a unique reduction sequence \((*)\) performed for \( P \) by Theorem 2.3.3.

We conclude this subsection with a definition of some reduction block \( G^{i+1}_s(R^s) \) (or \( G^{i+1}_s \) for short) over \( R^s \). Under the hypothesis of Corollary 2.3.5, set \( \mathfrak{A}' = \mathfrak{A}^s \) in the sequence \((*)\). Let \( \underline{m}^s = (1, \ldots, 1) \) and \( \underline{m}^i = \vartheta^{is}(\underline{m}^s) \). Write \( s(a^i_1) = X^i \) and \( t(a^i_1) = Y^i \). A matrix \( G^{i+1}_s \in \mathcal{M}_{m_{X^i} \times m_{Y^i}}(R^s) \) is determined by

(i) \( G^{i+1}_s(k) = G^{i+1}_s \otimes 1 \in \mathcal{M}_{m_{X^i} \times m_{Y^i}}(R^s) \otimes_R k \cong \mathcal{M}_{m_{X^i} \times m_{Y^i}}(k) \) is equal to \( B^{i+1} \) given in Theorem 2.3.3;

(ii) write the matrix \( G^{i+1}_s \ast A^1_1 = (g_{pq}) \in \mathcal{M}_{l^s}(R^s) \), if \( g_{pq} \neq 0 \), and \( p \in X^s \) (or equivalently, \( q \in X^s \)), then \( g_{pq} \in R^s_{X^s} \).

And \( G^{i+1}_s \) for \( i = 0, \ldots, s-1 \) are said to be the **reduction blocks of** \( H^s \). Furthermore,

\[
H^s = \sum_{i=0}^{s-1} G^{i+1}_s \ast A^1_1 \quad \text{and} \quad \vartheta^{0s}(H^s(k)) = \sum_{i=0}^{s-1} G^{i+1}_i(k) \ast A^1_1.
\]

For the sake of convenience, a links of \( \vartheta^{0s}(H^s(k)) \) is also said to be a link of \( H^s \). Thus \( \mathfrak{A}^s \) is local if and only if \( l(H^s) = \dim(\vartheta^{0s}(H^s(k))) - 1 \).
2.4 Defining systems

We introduce a concept of defining systems in this subsection. There exist two sorts of systems used in different situations in order to construct induced matrix bimodule problems in a reduction sequence.

Let $B = (b_{ij})_{t \times t}$ and $C = (c_{ij})_{t \times t}$ be two $t \times t$ matrices over $k$. Given $1 \leq p, q \leq t$, the notation $B \equiv_{x(p,q)} C$ (resp. $B \equiv_{x(p,q)} C$) means that $b_{ij} = c_{ij}$ for any $(i, j) \prec (p, q)$ (resp. $(i, j) = (p, q)$, $(i, j) \prec (p, q)$). One can define the similar notations for partitioned matrices.

Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with $\mathcal{T}$ trivial, $H = 0$, the $R$-$R$-quasi-basis $V = \{V_1, \ldots, V_m\}$ of $\mathcal{K}_1$ and $\mathcal{A} = \{A_1, \ldots, A_n\}$ of $\mathcal{M}_1$. Denote by $(p_j, q_j)$ the leading position of $A_j$ for $j = 1, \ldots, n$. Suppose there exists a sequence of reductions in the sense of Lemma 2.3.2:

$$(\mathfrak{A}, \mathfrak{B}) = (\mathfrak{A}^0, \mathfrak{B}^0), (\mathfrak{A}^1, \mathfrak{B}^1), \ldots, (\mathfrak{A}^i, \mathfrak{B}^i), (\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1}), \ldots, (\mathfrak{A}^r, \mathfrak{B}^r), \ldots, (\mathfrak{A}^s, \mathfrak{B}^s). \quad (2.4-1)$$

For each $0 \leq i \leq s$ in the sequence, a matrix equation is defined by

$$E^i : \Phi_m^i H^i(k) \equiv_{x(p^i,q^i)} H^i(k) \Phi_m^i$$

with $\Phi_m^i = \sum_{X \in \mathcal{T}} Z_X * E_X + \sum_{j=1}^{m} Z_j * V_j,$

where $(p^i, q^i)$ is the leading position of $A^i_1$ of $\mathfrak{M}_1$ in the $i$-th pair, $m^i = \vartheta^0_i(1, \ldots, 1)$ is the size vector of $\vartheta^0_i(H^i(k)) \in R(\mathfrak{A})$ over $\mathcal{T}$, $Z_X = (z^X_{pq})_{m_X \times m_X}$, $\forall X \in \mathcal{T}$, and $Z_j = (z^j_{pq})_{m(\nu_j) \times m(\nu_j)}$ for all quasi-base matrices $V_j$ of $\mathcal{K}_1$ in $\mathfrak{A}$, $\nu^{i}_{pq}, \nu^{j}_{pq}$ are pairwise different variables over $k$. $\Phi_m^i$ is called a variable matrix. The system of linear equations in $E^i$, which consists of equations locating in the $(p_j, q_j)$-th block for $j = 1, \ldots, n$, is said to be a defining system of $\mathfrak{K}^i$, and is denoted still by $E^i$. \hfill (2.4-2)

**Theorem 2.4.1** The solution space of the defining system $E^i$ in Formula (2.4-2) is the $k$-vector space spanned by the quasi-basis of $\mathfrak{K}^i_0 \oplus \mathfrak{K}^i_1$ for $0 \leq i \leq s$ in the sequence (2.4-1).

**Proof** Since $H^0 = H = 0$, our theorem holds true for $i = 0$. Suppose the theorem is true for the defining system $E^i$, now consider the defining system $E^{i+1}$.

In the case of Regularization, $m^{i+1} = m^i$, the $k$-vector space spanned by the quasi-basis of $\mathfrak{K}^{i+1}_0 \oplus \mathfrak{K}^{i+1}_1$ is just the solution space of the equations of $E^i$ and the equation $\Phi_m^i H^i(k) \equiv_{x(p^i,q^i)} H^i(k) \Phi_m^i$, which form the equation system $E^{i+1} : \Phi_m^{i+1} H^{i+1}(k) \equiv_{x(p^{i+1},q^{i+1})} H^{i+1}(k) \Phi_m^{i+1}$.

In the case of Loop or Edge reduction, set $a_1 : X \rightarrow Y$ with $X = Y$ for loop reduction; denote by $n = \vartheta^{i+1}(1, \ldots, 1)$, the size vector of $\vartheta^{i+1}(H^{i+1}(k))$ over $\mathcal{T}$. Then the size vector $m^{i+1} = \vartheta^{i+1}(n)$ over $\mathcal{T}$. Since $G^{i+1} = L(a_1)$, write $G^{i+1}(k) = L(a_1)(k)$. The $k$-vector space spanned by the quasi-basis of $\mathfrak{K}^{i+1}_0 \oplus \mathfrak{K}^{i+1}_1$ is the solution space of the matrix equations partitioned under $\mathcal{T}$:

$$\begin{cases}
(\Phi_m^i)_{n} H_n^i(k) \equiv_{x(p^i,q^i)} H_n^i(k)(\Phi_m^i)_{n}, \\
(\Phi_m^i)_{n, X} L(a_1)(k) = L(a_1)(k)(\Phi_m^i)_{n, X},
\end{cases}$$

where if $\Phi_m = (x_{pq})$, then $(\Phi_m^i)_{n} = (X_{pq})$ with $X_{pq} = (x_{pq, \alpha \beta})_{n_p \times n_q}$; since $\delta(a_1) = 0$, the $(p^i, q^i)$-th equation of $E^i$ is a linear combination of previous equations, “$\equiv_{x(p^i,q^i)}$” can be used in the first formula; in the second one $(\Phi_m^i)_{n, X}$ stands for the $(j, j)$-th block of $(\Phi_m^i)_{n}$ with $j \in X$. Since $(\Phi_m^i)_{n} = \Phi_m^{i+1}$ and $H^{i+1}(k) = H_n^i(k) + L(a_1)(k) A_1$, the equation system above is just $E^{i+1} : \Phi_m^{i+1} H^{i+1}(k) \equiv_{x(p^{i+1},q^{i+1})} H^{i+1}(k) \Phi_m^{i+1}$, where $(p^{i+1}, q^{i+1})$ is the index followed by the biggest index over $\mathcal{T}^{i+1}$ of the $(p^i, q^i)$-block partitioned under $\mathcal{T}^i$. Our theorem is proved by induction. \hfill \Box

For an example, see 2.4.5 (iv) below. The theorem implies the following fact obviously.
Corollary 2.4.2 \( \delta(a_1^i) = 0 \) in \( \mathfrak{B}^i \) if and only if the \((p^i, q^j)\)-equation of \( \mathbb{I}^i \), is a linear combination of the equations of \( \mathbb{I}^i \), namely, the equations locating before \((p^i, q^j)\).

Next, we give a deformed system based on the defining system. Fix some \( 0 < r < s \) in the sequence (2.4-1). Suppose \( \mathfrak{A}^r = (R^r, K^r, M^r, H^r) \), and \( \{ V^r_1, \ldots, V^r_m \} \) is a normalized quasi-basis of \( K^r_1 \). If \( i \geq r \), set a size vector over \( m^i_{ri} = \varphi^r(1, \ldots, 1) \) over \( T^r \), where \((1, \ldots, 1)\) is a size vector over \( T^r \). Let \( Z^r_{ij} \) be variable matrices of size \( m^i_{ri}, \forall Y^r \in T^r \), and \( Z^r_{ij} \) be those of size \( m^r_{s(r^i)} \times m^r_{s(r^j)} \), \( 1 \leq j \leq m^r \), then set a variable matrix \( \Psi_{m^i_{ri}} = \sum_{Y^r \in T^r} Z^r_{ij} \cdot E^r_{Y^r} + \sum_{j=1}^{m^r} Z^r_{ij} \cdot V^r \).

Let \( H^i = H^i_1 + H^i_2 \) with \( H^i_1 = \sum_{j=1}^{i-1} G^i_{j+1} \cdot A^i_j, \ H^i_2 = \sum_{j=i}^{r-1} G^i_{j+1} \cdot A^i_j \). Then the matrix equation \( \Psi_{m^i_{ri}} H^i(k) = \varphi(p^i, q^j) H^i(k) \Psi_{m^i_{ri}} \) can be rewritten as

\[
\begin{align*}
\Psi_{m^i_{ri}} H^i_2(k) & = \varphi(p^i, q^j) \Psi_{m^i_{ri}} H^i_1(k) \\
\Psi_{m^i_{ri}} H^i_1(k) & = H^i_1(k) \Psi_{m^i_{ri}} - \Psi_{m^i_{ri}} H^i_1(k).
\end{align*}
\]

(4.4-3)

Corollary 2.4.3 The equation system \( \mathbb{F}^i \) is equivalent to \( \mathbb{E}^i \). And \( \delta(a_1^i) = 0 \) in \( \mathfrak{B}^i \) if and only if the \((p^i, q^j)\)-th equation \( \mathbb{F}^i \) is a linear combination of equations of \( \mathbb{F}^i \), namely, the equations locating before \((p^i, q^j)\).

The above theorem and corollaries will be used in Subsections 5.2–5.4 to calculate the differentials of boxons given by some bordered matrices. Sometimes, it is difficult to determine the dotted arrows in the induced box after some reductions. Instead, we may consider a system of equations on “dotted elements” (see the definition below), and give explicitly the linear relations on those elements, which will be used in Subsections 4.1 and 4.3–4.5.

Theorem 2.4.4 For each \( 0 \leq i \leq s \) in the sequence (2.4-1), there exists a system of equations \( \mathbb{E}^i \) over \( R^i \otimes_k R^i \), whose general solution can be expressed as the formal product \( \Pi^i = \sum_j v^i_j \cdot V^i_j \), namely

(i) the \( R^i \otimes_k R^i \)-quasi-basis \{ \( V^i_j \) \} \_j forms a basic system of solutions of \( \mathbb{E}^i \);
(ii) the \( R^i \otimes_k R^i \)-quasi-basis \{ \( v^i_j \) \} \_j forms a set of free variables.

Proof For \( i = 0 \), let \( \Phi_{m^i} = \sum_{j=1}^{m^i} v^i_j \cdot V^i_j = \Pi \) and \( \mathbb{E}^0 : \Phi_{m^i} H^0 = \varphi(p^i, q^j) H^0 \Phi_{m^0} \) be a matrix equation with \((p, q)\) being the leading position of \( A^1_1 \) in \( \mathcal{M}^1 \), \( H^0 = 0 \). Then \( \Pi \) is a general solution of \( \mathbb{E}^0 \).

Suppose a system \( \mathbb{E}^i \) of the pair \( (\mathfrak{A}^i, \mathfrak{B}^i) \) satisfying condition (i)–(ii) has been obtained:

\[
\mathbb{E}^i : \Phi_{m^i} H^i = \varphi(p^i, q^j) H^i \Phi_{m^i},
\]

where \( \Phi^i = (u^i_{pq}) \) is strictly upper triangular with \( u^i_{pq} \) being a \( k \)-linear combination of some variables over \( R^i \times R^i \) for \( p \in X, q \in Y, X, Y \in T^i \). We now construct a system \( \mathbb{E}^{i+1} \). For the sake of convenience, \( \mathbb{E}_{<(p^i, q^j)}^i \) is used for the equation system \( \mathbb{E}^i \), and \( \mathbb{E}_{(p^i, q^j)}^i \) stands for the \((p^i, q^j)\)-th equation of \( \mathbb{E}^i \).

1) If \( \mathbb{E}_{(p^i, q^j)}^i \) is not a linear combination of the equations of \( \mathbb{E}_{<(p^i, q^j)}^i \), we proceed with a regularization. Thus \( m^{i+1}_{m^i} = m^i, T^{i+1} = T^i \) and \( T^{i+1} = T^i \). The combination of the equation \( \Phi_{m^i} H^i = \varphi(p^i, q^j) H^i \Phi_{m^i} \), and the equations of \( \mathbb{E}_{(p^i, q^j)}^i \) forms an equation system \( \mathbb{E}_{(p^i, q^j)}^{i+1} : \Phi_{m^i+1} H^{i+1} = \varphi(p^i+1, q^{j+1}) H^{i+1} \Phi_{m^i+1}, \) where \( H^{i+1} = H^i + \emptyset \cdot A^1_1 \) by Proposition 2.2.8. And \( \mathbb{E}_{(p^i, q^j)}^{i+1} \) satisfies assertions (i)–(ii).

2) If \( \mathbb{E}_{(p^i, q^j)}^i \) is a linear combination of the equations of \( \mathbb{E}_{<(p^i, q^j)}^i \), we proceed with a loop or an edge reduction. There are a pre-minimal algebra \( R^i = R^i \) in a loop reduction, or \( R^i = R^i[a_1^i] \) in an edge reduction; a minimal algebra \( R^{i+1} \); an admissible \( R^{i+1} - R^i \)-bimodule \( L^i \). Set a size vector \( n = v^{i+1}(1, \ldots, 1) \) over \( T^i \) with \((1, \ldots, 1)\) being a size vector over \( T^{i+1} \), then \( m^i_{m^i} = v^i m^i \) is
a size vector over $\mathcal{T}$. Denote by $\bar{\Phi}_{XY} = (u'_{pq})$ for any $(X, Y) \in \mathcal{T}^i \times \mathcal{T}^j$ a submatrix of $\bar{\Phi}_{m^i}$, such that $u'_{pq} = u_{pq}$ for $p \in X, q \in Y$, or 0 otherwise. Define a variable matrix over $R^{i+1} \otimes_k R^{j+1}$, and a matrix in $M_m^{i+1}(R^{i+1})$:

$$
(\Phi_{m^{i+1}})_{ij} = \sum_{(X,Y) \in \mathcal{T}^i \times \mathcal{T}^j} \sum_{1 \leq p \leq n_X, 1 \leq q \leq n_Y} (f_{z(X,p)} \otimes_k q_{z(Y,q)}) \ast \bar{\Phi}_{XY};
$$

$$
H^i_{m^{i+1}} = \sum_{X \in \mathcal{T}^i} \sum_{1 \leq p \leq n_X} (f_{z(X,p)} \otimes_k q_{z(X,p)}) \ast H^j_X.
$$

where the vertices $Z$ and the matrices $f \otimes_k e$ are given before Definition 2.1.2. If the $R^i$-$R^j$-quasi-basis $\mathcal{V}^j = \{V^j_{1}, \ldots, V^j_{r}\}$ is a basic solution of $\mathcal{F}^j_{\mathcal{S}(p', q')}$, then the $R^{i+1}$-$R^{j+1}$-quasi-basis $\{(f_{z(\ell(v)))}, p) \otimes_k q_{z(\ell(v)), q}) \ast V^j_{1}, \ldots, V^j_{r}\}$ of $\mathcal{K}_{11}^{i+1}$ is a basic system of solutions of the matrix equation $(\Phi_{m^{i+1}})_{m} H^i_{m^{j+1}} \equiv_{(p',q')} H^i_{m^{j+1}} \Phi_{m^{i+1}} m$ partitioned under $\mathcal{T}^i$, since the $(p', q')$-th block is a $k$-linear combination of the others. In other words, the formal product $\Pi_{m^{i+1}}$ of $(\mathcal{K}_{11}^{i+1}, C_{11}^{i+1})$ is a general solution of the matrix equation.

We may assume that $E_{1}$ has a $R^{i+1}$-$R^{j+1}$-quasi-basis $\{f_{1}, \ldots, f_{l}\}$ by Definition 2.1.2 (a3), where $0 \leq l \leq \frac{1}{2}sr(r+1)$ after some deletion in Formula (2.2.4) for $a_{1}^j : X \to X$, $F_{j} L(a_{1}^j) = L(a_{1}^j) F_{j}$; or $l = 0, 1, 2$ after some deletion in Formula (2.2.4) for $a_{1}^j : X \to Y$, $F_{j} L(a_{1}^j) = 0 = L(a_{1}^j) F_{j}$. Then either $\mathcal{K}_{10}^{i+1} = \{0\}$, or $R^{i+1}$-$R^{j+1}$-quasi-basis of $\mathcal{K}_{10}^{i+1}$ is $\{F_{j} \equiv F_{j} \ast E_{X} \mid j = 1, \ldots, l\}$; or $F_{1} = F_{1} \ast E_{X}^j$, or $F_{2} = F_{2} \ast E_{Y}^j$, or both of them given by Construction 2.1.3 (ii), and that of $C_{10}^{i+1}$ is $\{F_{1}^*, \ldots, F_{l}^*\}$ given by Proof 2 of Proposition 2.1.5. Thus the formal product $\Pi_{m^{i+1}} = \sum_{j=0}^{l} (F_{j} \ast F_{j}) L(a_{1}^j) \equiv_{(p', q')} (L(a_{1}^j) \ast A_{1}^j) \Pi_{m^{i+1}}$ partitioned under $\mathcal{T}^i$, since the $(p', q')$-th block is $\sum_{j=0}^{l} (F_{j} \ast F_{j}) L(a_{1}^j) = F_{j} \ast (L(a_{1}^j) F_{j}) = \sum_{j=1}^{l} (F_{j} \ast F_{j}) L(a_{1}^j)\).$ Define

$$
\Phi_{m^{i+1}} = \Pi_{m^{i+1}} + (\Phi_{m^{i+1}})_{m} ; H^{i+1} = H^j_{m^{i+1}} + L(a) \ast A_{1}^j;
$$

$$
\Psi_{m^{i+1}} = \Phi_{m^{i+1}} H^{i+1} \equiv_{(p',q')} H^{i+1} \Phi_{m^{i+1}},
$$

where $(p', q')$ is the index followed by the biggest index over $\mathcal{T}^{i+1}$ of the $(p', q')$-block partitioned under $\mathcal{T}^i$. We claim that the formal product $\Pi_{m^{i+1}} = \Pi_{m^{i+1}}^{j+1} + \Pi_{m^{i+1}}^{j}$ of $(\mathcal{K}_{11}^{i+1}, C_{11}^{i+1})$ is a general solution of $\Psi_{m^{i+1}}$. First, both left and right sides of each block equation of $(\Phi_{m^{i+1}})_{m} (L(a) \ast A_{1}^j) \equiv_{(p', q')} (L(a) \ast A_{1}^j) \Phi_{m^{i+1}}$ partitioned under $\mathcal{T}^i$ are zero blocks, since $(\Phi_{m^{i+1}})_{m}$ is a strict upper triangular partitioned matrix and the index of the leading block of $L(a_{1}^j) \ast A_{1}^j$ is $(p', q')$. Second, at the left and right sides of each block equation of $\Pi_{m^{i+1}} H^{i+1} \equiv_{(p', q')} H^{i+1} \Pi_{m^{i+1}}$ under $\mathcal{T}^i$ are equal blocks. In fact, suppose $H^j_{X} = (h_{q} \beta 1 X)_{q}$ with $h_{q} \beta = 0$ if $\alpha \notin X$ or $\beta \notin X$, then $(H^j_{X} X = (H_{X, \alpha \beta})$ with $H_{X, \alpha \beta} = h_{q} \beta \diag(1_{Z(X, j)} \cdots, 1_{Z(X, n_X)})$, therefore $F_{j} H_{X, \alpha \beta} = H_{X, \alpha \beta} F_{j}$. And the same assertion is valid for $Y \in \mathcal{T}^{j}$. Our Theorem follows by induction.

For an example, see 2.4.5 (iv) below. $\Psi_{m}^{i}$ in Formula (2.4.4) is also called a defining system of the pair $(\mathfrak{A}^{i}, \mathfrak{B}^{i})$. The matrix $\Phi_{m^{i+1}}$ is called a matrix of dotted elements. The concept of the dotted elements possesses two folds of meanings: 1) as variables in the equation system $\Psi^{i}$; 2) as the elements with a series of linear relations after a sequence of reductions. Different meanings will be used for different cases frequently in Section 4.

Next, we define a deformed system $\Psi^{r}$ for some fixed $0 < r < s$, which is equivalent to $\Psi^{i}$. Like the discussion stated before Formula (2.4.3), a matrix equation and a variable matrix of size vector $\bar{m}^{r}$ over $\mathcal{T}^{r}$ are defined:

$$
\Psi^{r} : \Psi_{m^{r}} H_{r}^{j} = \equiv_{(p', q')} \Psi_{m^{r}} H_{r}^{j}^{0} + H_{r}^{j} \Psi_{m^{r}},
$$

$$
\Psi_{m^{r}}^{0} H_{r}^{j} = H_{r}^{j} \Psi_{m^{r}} \Psi_{m^{r}} H_{r}^{j},
$$

$$
\Psi_{m^{r}} = \sum_{X \in \mathcal{T}^{r}} \bar{w}^{r}_{X} \ast E_{X}^{r} + \sum_{j} \bar{v}_{r}^{j} \ast V_{r}^{j}. \tag{2.4.5}
$$
where the definition of $\bar{r}_j^r = (v_{ji}^r)$ and $\bar{r}_x^r = (w_{x_i}^r)$ is analogous to that of Formula (2.4-4).

At the end of the subsection, we perform reduction procedure for the matrix bimodule problem given in Example 1.4.5 in order to show some concrete calculations.

**Example 2.4.5**

(i) Making an edge reduction for the first arrow $a : X \rightarrow Y$ by $a \mapsto G^1 = (1_Z)$, an induced local pair $(\mathfrak{A}, \mathfrak{B})$ with $R^1 = k1_Z$; $H^1 = (1_Z) \ast A$ is obtained.

(ii) Making a loop reduction for $b : Z \rightarrow Z$ by $b \mapsto G^2 = J_2(0)1_X$, an induced local pair $(\mathfrak{A}, \mathfrak{B}^2)$ with $R^2 = k1_X, H^2 = (1_X 0) \ast A + (0 1_X) \ast B$ is obtained. There are two matrix equalities in the formal equation of the pair $(\mathfrak{A}, \mathfrak{B}^2)$:

\[
\begin{pmatrix}
  e & v \\
 0 & e
\end{pmatrix}
\begin{pmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{pmatrix}
+ \begin{pmatrix}
u_1^2 & \bar{u}_1^2 \\
\bar{u}_1^2 & \nu_2^2
\end{pmatrix}
\begin{pmatrix}
0 & 1_X \\
0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 1_X \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
v_{11}^2 & v_{12}^2 \\
v_{21}^2 & v_{22}^2
\end{pmatrix}
+ \begin{pmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{pmatrix}
\begin{pmatrix}
e & v \\
0 & e
\end{pmatrix},
\]

where $(c_{pq})_{2 \times 2}, (d_{pq})_{2 \times 2}$ are splits from $c, d$ respectively, $e \in C^0$ is dual to $E_X = (1_X 10, 1_X 10) \in \mathcal{K}^0$, and $v \in C^2$ is dual to $V = (0 1_X \hat{1} 1_X) \ast EX \in \mathcal{K}^2$ respectively.

(iii) Making a loop mutation $c_{21} \mapsto (x)$, followed by three regularizations, such that $c_{22} \mapsto \emptyset, u_{21}^2 = x; c_{11} \mapsto \emptyset, u_{21}^2 = x; c_{12} \mapsto \emptyset, u_{22}^2 = v_{22}$, an induced pair $(\mathfrak{A}_3, \mathfrak{B}_3)$ is obtained, and the differentials of the solid arrows in $\mathfrak{B}_3$ are:

\[
\begin{align*}
\delta(d_{21}) &= xx - vx \\
\delta(d_{22}) &= u_{21}^2 + v_{22} - v_{22} - d_{21}v \\
\delta(d_{11}) &= u_{21}^2 - v_{21} + v_{22} + u_{d_{21}} \\
\delta(d_2) &= u_{21}^2 + v_{21} - v_{21}^2 - d_{11}v + ud_{22}
\end{align*}
\]

(iv) Finally, we describe the defining systems of Theorem 2.4.1 and 2.4.4 for the pair $(\mathfrak{A}, \mathfrak{B})$.

Since $(\mathfrak{A}, \mathfrak{B})$ is bipartite, $T = T' \times T''$. Thus $\Phi_{l_2}^x = \Phi_{l_2}^1 \times \Phi_{l_2}^2$, and $\Phi_{l_2}^x = \Phi_{l_2}^1 \times \Phi_{l_2}^2$, where the size vector $l^2 = (2, 2, 2, 2)$ is over $T'$ and $n^2 = (2, 2, 2, 2)$ over $T''$. Suppose the systems are $\Phi_{l_2}^x H^2(k) = H^2(k) \Phi_{l_2}^x$, given by 2.4.1, and $\Phi_{l_2}^x H^2 = H^2 \Phi_{l_2}^x$, given by 2.4.4 respectively, where

\[
\Phi_{l_2}^1 (or \bar{\Phi}_{l_2}^1) = \begin{pmatrix}
0 & \Phi_1 & \Phi_2 & \Phi_4 \\
0 & \Phi_2 & \Phi_3 & \Phi_1
\end{pmatrix}, \quad \Phi_{l_2}^2 (or \bar{\Phi}_{l_2}^2) = \begin{pmatrix}
0 & \Phi'_1 & \Phi'_2 & \Phi'_4 \\
0 & \Phi'_2 & \Phi'_3 & \Phi'_1
\end{pmatrix}.
\]

Then $\Phi^i = \begin{pmatrix}
x_{i1}^j & x_{i2}^j \\
x_{i2}^j & x_{i1}^j
\end{pmatrix}$ in $\Phi_{l_2}^1$, $\bar{\Phi}^i = \begin{pmatrix}v_{i1}^2 & v_{i2}^2 \\
v_{i2}^2 & v_{i1}^2
\end{pmatrix}$ in $\Phi_{l_2}^2$ for $i = 0, 1, 2, 3, 4$ by Theorem 2.4.1; and

$\Phi^0 = \begin{pmatrix}0 & 0 \\
0 & 0
\end{pmatrix}$ in $\Phi_{l_2}^0$, which is obtained from a loop reduction $b \mapsto J_2(0)1_X$; $\Phi^i = \begin{pmatrix}u_{i1}^2 & u_{i2}^2 \\
u_{i2}^2 & u_{i1}^2
\end{pmatrix}$ in $\bar{\Phi}_{l_2}^2$, $\Phi^i = \begin{pmatrix}v_{i1}^2 & v_{i2}^2 \\
v_{i2}^2 & v_{i1}^2
\end{pmatrix}$ in $\bar{\Phi}^2$ for $i = 1, 2, 3, 4$ by Theorem 2.4.4.

### 3 Classification of minimal wild bocses

Based on the well known Drozd’s wild configurations, this section is devoted to classifying so-called minimal wild bocses, which are divided into five classes. Then the non-homogeneity
of bocses in the first four classes is proved. But those in the last class have been proved to be strongly homogeneous. Some preliminaries are stated in subsections 3.1 and 3.2.

3.1 An exact structure on representation categories of bocses

In this subsection the concept on exact structure of categories is recalled, especially the exact structure on representation categories of bocses.

Let $\mathcal{A}$ be an additive category with Krull-Schmidt property. We recall from [GR] and [DRSS] the following notions. A pair $(\iota, \pi)$ of composable morphisms

$$(e) \quad M \xrightarrow{\iota} E \xrightarrow{\pi} N$$

in $\mathcal{A}$ is called exact if $\iota$ is a kernel of $\pi$ and $\pi$ is a cokernel of $\iota$.

Let $\mathcal{E}$ be a class of exact pairs which is closed under isomorphisms. The morphisms $\iota$ and $\pi$ appearing in a pair $(e)$ are called an inflation and a deflation of $\mathcal{E}$ respectively, the pair itself is called a conflation, and is denoted by $(\iota, \pi)$.

**Definition 3.1.1** The class $\mathcal{E}$ is said to be an *exact structure* on $\mathcal{A}$, and $(\mathcal{A}, \mathcal{E})$ an *exact category* if the following axioms are satisfied:

- **E1** The composition of two deflations is a deflation.
- **E2** For each $\varphi$ in $\mathcal{A}(N', N)$ and each deflation $\pi$ in $\mathcal{A}(E, N)$, there are some $E'$ in $\mathcal{A}$, an $\varphi'$ in $\mathcal{A}(E', E)$ and a deflation $\pi' : E' \to N'$ such that $\pi' \varphi = \varphi' \pi$.
- **E3** Identities are deflations. If $\varphi \pi$ is a deflation, then so is $\pi$.

(Or $E^{\text{op}}$ Identities are inflations, if $\varphi \iota$ is an inflation, then so is $\iota$.)

An object $P$ in $\mathcal{A}$ is said to be $\mathcal{E}$-projective (or projective for short) if any conflation ending at $P$ is split. Dually an object $I$ in $\mathcal{A}$ is said to be $\mathcal{E}$-injective (or injective for short) if any conflation starting at $I$ is split.

Let $\mathcal{A}$ be a Krull-Schmidt category. A morphism $\pi : E \to N$ in $\mathcal{A}$ is called right almost split if it is not a retraction and for any non-retraction $\varphi : L \to N$, there exists a morphism $\psi : L \to E$ such that $\varphi = \psi \pi$. It is said that $\mathcal{A}$ has right almost split morphisms if for all indecomposable $N$ there exist right almost split morphisms ending at $N$. Dually, left almost split morphisms are defined. It is said that $\mathcal{A}$ has almost split morphisms if $\mathcal{A}$ has right and left almost split morphisms.

A morphism $\pi : E \to N$ is called right minimal if every endomorphism $\eta : E \to E$ with the property that $\pi = \eta \pi$ is an isomorphism. A left minimal morphism $\iota : M \to E$ is defined dually.

**Proposition 3.1.2** Suppose that the Krull-Schmidt category $\mathcal{A}$ carries an exact structure $\mathcal{E}$. Let $(e)$ given in Formula $(*)$ be a conflation. Then the following assertions are equivalent.

(i) $\iota$ is minimal left almost split;

(ii) $\pi$ is minimal right almost split;

(iii) $\iota$ is left almost split and $\pi$ is right almost split.

The conflation $(e)$ in the above proposition is said to be an *almost split conflation*. The exact category $(\mathcal{A}, \mathcal{E})$ is said to have almost split conflations if (i) $\mathcal{A}$ has almost split morphisms; (ii) for any indecomposable non-projective $N$, there exists an almost split conflation $(e)$ ending at $N$; (iii) for any indecomposable non-injective $M$, there exists an almost split conflation $(e)$ starting at $M$.

Now we turn to the representation category of a bocs. Let $\mathcal{B} = (\Gamma, \Omega)$ be a bocs with a layer $L = (\Gamma^j; \omega; a_1, \ldots, a_n; v_1, \ldots, v_m)$. From now on it is always assumed that $\mathcal{B}$ is triangular on the dotted arrows, i.e. $\delta(v_j)$ involves only $v_1, \ldots, v_{j-1}$. In particular, the bocs $\mathcal{B}$ associated to a matrix bimodule problem $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ is triangular by Definition 1.2.1.
The bocs $\mathfrak{B}_0 = (\Gamma, \Gamma)$ is called a principal bocs of $\mathfrak{B}$. The representation category $R(\mathfrak{B}_0)$ is just the module category of $\Gamma$.

**Lemma 3.1.3** Let $\mathfrak{B} = (\Gamma, \Omega)$ be a layered bocs with a principal bocs $\mathfrak{B}_0$. Suppose $\mathfrak{B}$ is triangular on the dotted arrows.

(i) If $\iota : M \to E$ is a morphism of $R(\mathfrak{B})$ with $\iota_0$ injective, then there exist an isomorphism $\eta$ and a commutative diagram in $R(\mathfrak{B})$, such that the bottom row is exact in $R(\mathfrak{B}_0)$. Dually, if $\pi : E \to N$ is a morphism of $R(\mathfrak{B})$ with $\pi_0$ surjective, then there exist an isomorphism $\eta$ and a commutative diagram in $R(\mathfrak{B})$, such that the bottom row is exact in $R(\mathfrak{B}_0)$.

\[
\begin{array}{ccc}
M & \xrightarrow{\iota} & E \\
id & & \downarrow\eta \\
0 & \to & M \xrightarrow{\iota'} E'
\end{array}
\quad
\begin{array}{ccc}
E & \xrightarrow{\pi} & N \\
\downarrow\eta & & \downarrow\id \\
E' & \xrightarrow{\pi'} & N
\end{array}
\quad
\begin{array}{ccc}
\eta \pi & = & 0 \\
\iota \pi & = & 0
\end{array}
\]

(ii) If $(e) : M \xrightarrow{\iota} E \xrightarrow{\pi} N$ with $\iota \pi = 0$ is a pair of composable morphisms in $R(\mathfrak{B})$ and $(e_0) : 0 \to M \xrightarrow{\iota_0} E \xrightarrow{\pi_0} L \to 0$ is exact in the category of vector spaces, then there exists an isomorphism $\eta$ and a commutative diagram in $R(\mathfrak{B})$: \[
\begin{array}{ccc}
(e) & & \begin{array}{c}
\xrightarrow{\iota} \\
id
\end{array} & & \begin{array}{c}
\xrightarrow{\pi} \\
\downarrow\eta \\
\downarrow\id
\end{array} & & \begin{array}{c}
\xrightarrow{\pi'} \\
\xrightarrow{\pi}
\end{array} \\
0 & \to & M & \xrightarrow{\iota'} & E' & \xrightarrow{\pi'} & N & \to & 0
\end{array}
\]

such that $(e')$ is an exact sequence in $R(\mathfrak{B}_0)$. Moreover, by choosing a suitable basis of $M, E', N$, we are able to obtain $\iota_X' = (0, I)$ and $\pi_X' = (I, 0)^T$ for all $X \in \mathcal{T}$.

**Lemma 3.1.4** Let $\mathfrak{B} = (\Gamma, \Omega)$ be a layered bocs, which is triangular on the dotted arrows.

(i) If $\iota : M \to E$ is monic in $R(\mathfrak{B})$ if $\iota_0 : M \to E$ is injective. Dually, $\pi : E \to N$ is epic in $R(\mathfrak{B})$ if $\pi_0 : E \to N$ is surjective.

(ii) A pair of composable morphisms $(e) : M \xrightarrow{\iota} E \xrightarrow{\pi} N$ with $\iota \pi = 0$ is exact in $R(\mathfrak{B})$, if $(e_0) : 0 \to M \xrightarrow{\iota_0} E \xrightarrow{\pi_0} N \to 0$ is exact as a sequence of vector spaces.

**Proof** (i) If $\iota_0$ is injective, then Lemma 3.1.3 (i) gives a commutative diagram with $\iota' : M \to E'$ in $R(\mathfrak{B}_0)$. Given any morphism $\varphi : L \to M$ with $\varphi \iota = 0$, there is $\varphi \eta = \varphi \iota' = 0$. Then $\varphi_0 \iota_0 = 0$ yields $\varphi_0 = 0$. And for any dotted arrow $v_i : X \to Y$, suppose $\delta(v_i) = \sum u_i \otimes \Gamma v_i \Gamma$. There is inductively, $0 = \varphi(v_i) \iota_Y = \varphi(v_i) \iota_Y' + \varphi X \iota'(v_i) + \sum u_j \varphi(u_j) \iota'(u_j) = \varphi(v_i) \iota_Y'$, which yields $\varphi(v_i) = 0$ by the injectivity of $\iota_Y'$. Thus $\varphi_0 = 0$ and $\iota$ is monic. The second assertion on $\pi$ is proved dually.

(ii) It is proved first that $\iota$ is the kernel of $\pi$. If $(e_0)$ is exact, then (i) shows that $\iota$ is monic.

**Theorem 3.1.5** Let a layered bocs $\mathfrak{B} = (\Gamma, \Omega)$ be triangular on the dotted arrows. A class $\mathcal{E}$ of composable morphisms in $R(\mathfrak{B})$ is defined, such that $M \xrightarrow{\iota_0} E \xrightarrow{\pi_0} L$ in $\mathcal{E}$, provided that $\iota \pi = 0$ and $0 \to M \xrightarrow{\iota_0} E \xrightarrow{\pi_0} L \to 0$ is exact as a sequence of vector spaces. It is clear that $\mathcal{E}$ is closed under isomorphisms.
Proposition 3.1.5 [O Theorem 4.4.1] and [BBP] Suppose a layered bocs $\mathfrak{B}$ is triangular on the dotted arrows. Then the class $\mathcal{E}$ defined by Formula (3.1-1) is an exact structure on $R(\mathfrak{B})$, and $(R(\mathfrak{B}), \mathcal{E})$ is an exact category.

Corollary 3.1.6 ([B1, O Lemma 7.1.1]) Let $\mathfrak{B} = (\Gamma, \Omega)$ be a layered bocs.
(i) For any $M \in R(\mathfrak{B})$ with $\text{dim} M = m$, if $m_X \neq 0$ for some vertex $X \in \mathcal{T}_1$, then $M$ is neither projective nor injective.
(ii) For any positive integer $n$, there are only finitely many iso-classes of indecomposable projectives and injectives in $R(\mathfrak{B})$ of dimension at most $n$.

Remark 3.1.7 ([BCLZ, O Definition 4.4.1]) Let $\mathfrak{B} = (\Gamma, \Omega)$ be a layered bocs, such that $(R(\mathfrak{B}), \mathcal{E})$ is an exact structure. The almost split conflations have been defined in a general exact category, particularly in $R(\mathfrak{B})$.
(i) An indecomposable representation $M \in R(\mathfrak{B})$ is said to be homogeneous if there is an almost split conflation $M \xrightarrow{\alpha} E \xrightarrow{\pi} M$. The isomorphism class of $M$ is also said to be homogeneous.
(ii) The category $R(\mathfrak{B})$ (or bocs $\mathfrak{B}$) is said to be homogeneous if for each positive integer $n$, almost all (except finitely many) iso-classes of indecomposable representations in $R(\mathfrak{B})$ with size at most $n$ are homogeneous. For example, if $\mathfrak{B}$ is of representation tame type, then $R(\mathfrak{B})$ is homogeneous [CB1].
(iii) The category $R(\mathfrak{B})$ (or bocs $\mathfrak{B}$) is said to be strongly homogeneous if there exists neither projective nor injective, and all indecomposable representations in $R(\mathfrak{B})$ are homogeneous. For example, if $\mathfrak{B}$ is a local bocs with a layer $(R; \omega; \alpha; \nu), R = k[x, \phi(x)^{-1}], and the differential \delta(a) = xv - vx$. Then $R(\mathfrak{B})$ is strongly homogeneous and representation wild type [BCLZ]. In particular the induced bocs given in Example 2.4.5 (iii) is strongly homogeneous.

Note that $(R(\mathfrak{B}), \mathcal{E})$ may not have any almost split conflation. For example, set quiver $Q = a\bigcup b$, the path algebra $\Gamma = kQ$, and the principal bocs $\mathfrak{B} = (\Gamma, \Gamma)$. Then $R(\mathfrak{B})$ has no almost split conflation, see [V, ZL] for details.

Recalling from [CB1], let $\mathfrak{B} = (\Gamma, \Omega)$ be a minimal bocs. Then for any $X \in \mathcal{T}_1$ with $R_X = k[x, \phi_X(x)^{-1}], and for any $\lambda \in k$ with $\phi_X(\lambda) \neq 0, there is an almost split conflation:

$$S(X, 1, \lambda) \xrightarrow{(0,1)} S(X, 2, \lambda) \xrightarrow{(1)} S(X, 1, \lambda) \quad \text{in} \quad R(\mathfrak{B}),$$

(3.1-2)

where $S(X, 1, \lambda)$ (resp. $S(X, 2, \lambda)$) is given by $k \bigcup J_1(\lambda)$ (resp. $k^2 \bigcup J_2(\lambda)$) at $X$, and (0) at other vertices.

3.2 Almost split conflations in the process of reductions

In order to prove the non-homogeneity of some wild bocses, we must understand the behavior of almost split conflations during reduction procedures. We study under what conditions the homogeneous property is preserved after a sequence of reductions in this subsection.

Lemma 3.2.1 [B1] Let $\mathfrak{B}' = (\Gamma', \Omega')$ be the induced bocs of a bocs $\mathfrak{B}$ given by one of eight reductions in the subsection 2.2, and $N'$ be an indecomposable representation in $R(\mathfrak{B}')$. If $N'$ is non-projective (resp. non-injective) in $R(\mathfrak{B}')$, then so is $\vartheta(N')$ in $R(\mathfrak{B})$.

Lemma 3.2.2 [B1] Let $\mathfrak{B}' = (\Gamma', \Omega')$ be the induced bocs of $\mathfrak{B}$ given by one of eight reductions in the subsection 2.2.
(i) If $\varphi' : M' \to E'$ is a morphism in $R(\mathfrak{B}')$ with $\vartheta(\varphi') : \vartheta(M') \to \vartheta(E')$ being a left minimal almost split inflation in $R(\mathfrak{B})$, then so is $\varphi'$ in $R(\mathfrak{B}')$. Dually if $\pi' : E' \to N'$ is a morphism in $R(\mathfrak{B}')$ with $\vartheta(\pi') : \vartheta(E') \to \vartheta(N')$ being a right minimal almost split deflation in $R(\mathfrak{B})$, then so is $\pi'$ in $R(\mathfrak{B}')$. 

(ii) If \((e') : M' \overset{q'}{\rightarrow} E' \overset{p'}{\twoheadrightarrow} M'\) is a conflation in \(R(\mathfrak{B}')\) with \(\vartheta(e') : \vartheta(M') \overset{q(e')}{\rightarrow} \vartheta(E') \overset{p(e')}{\twoheadrightarrow} \vartheta(M')\) being an almost split conflation in \(R(\mathfrak{B})\), then so is \((e')\) in \(R(\mathfrak{B}')\).

Let \((\mathfrak{A}, \mathfrak{B})\) be a pair with trivial \(R, M \in R(\mathfrak{A})\) be an indecomposable object of size vector \(m_1\). Set \(\mathcal{T}^1 = \{X \in \mathcal{T} \mid m_X \neq 0\}\), suppose \(\mathfrak{A}^1\) is obtained by deleting \(\mathcal{T} \setminus \mathcal{T}^1\) from \(\mathfrak{A}\), and \(M^1 \in R(\mathfrak{A}^1)\) with \(\vartheta_1(M^1) \simeq M\). Suppose a sequence of reductions in the sense of Lemma 2.3.2 is given by Theorem 2.3.3 with respect to \(M^1\):

\[
(\mathfrak{A}, \mathfrak{B}), (\mathfrak{A}^1, \mathfrak{B}^1), \ldots, (\mathfrak{A}^i, \mathfrak{B}^i), (\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1}), \ldots, (\mathfrak{A}^s, \mathfrak{B}^s).
\]

Then there is some \(M^s \in R(\mathfrak{A}^s)\) of size vector \(m_s\), such that \(\vartheta_0^s(M^s) \simeq M\).

**Theorem 3.2.3** Suppose the first arrow \(a^i_1\) is a loop at \(X^s\) with \(\delta(a^i_1) = 0\); and \(M^s_{X^s} = k, M^s(a^i_1) = (\lambda)\) in the last term \(\mathfrak{B}^s\) of the sequence (3.2-1). If \(M\) is homogeneous and \((e) : M \rightarrow E \rightarrow M\) is an almost split conflation in \(R(\mathfrak{A})\), then for \(i = 1, \ldots, s\) there exists an almost split conflation \((e^i) : M^i \rightarrow E^i \rightarrow M^i\) in \(R(\mathfrak{A}^i)\), such that \(\vartheta_0^i(e^i) \simeq (e)\).

**Proof** Induction is used for the proof. The assertion is obviously true for \(i = 1\), since the size vector of \(E\) is \(2m_1\) by Formula (3.1-1). According to Definition 1.3.4 and Formula (2.3-6):

\[
M^i = H^{m_1}_i(k) + \sum_j M^s(a^j_1) * A^j_1, \quad H^{m_1}_i(k) = \sum_{j=1}^{s-1} B^j * A^j_1.
\]

Suppose the assertion is valid for some \(1 \leq i < s\). The formula below gives as \(k\)-matrices of size vector \(m^s = \vartheta_0^i(M^s)\):

\[
M^i = M^s = H^{m_1}_i(k) + B^{i+1} * A^1_1 + \sum_{j=2}^{n_i} M^i(a^j_1) * A^j_1.
\]

There exists an object \(E^i = H^{m_1}_{2m_1}(k) + \sum_{j=1}^{m_1} E^i(a^j_1) * A^j_1 \in R(\mathfrak{A}^i)\) and an almost split conflation \((e^i) : M^i \overset{q^i}{\rightarrow} E^i \overset{p^i}{\twoheadrightarrow} M^i\) in \(R(\mathfrak{A}^i)\), such that \(\vartheta_0^i(e^i) \simeq (e)\). We now treat the \((i+1)\)-th stage via proving the existence of an isomorphism \(\eta : E^i \rightarrow E^{i+1}\) with \(\vartheta_0^{i+1}(e^{i+1}) = B^{i+1} \oplus B^i\).

If this is the case, suppose \(a^i_1 : X \rightarrow Y\), \(S_X\) and \(S_Y\) are invertible matrices determined by changing certain rows and columns of \(B^{i+1} \oplus B^i\), such that \(S_X^{-1}(B^{i+1} \oplus B^i)S_Y = I_2 \oplus B^i\), the usual Kronecker product of two matrices. Define a matrix \(S = \sum_{Z \in \mathcal{T}} S_Z E_Z\) with \(S_Z = I_{m_Z} \oplus B^i\) for \(Z \in \mathcal{T}^i \setminus \{X, Y\}\). Then there are \(k\)-matrices:

\[
R(\mathfrak{B}^i) \ni \xi(E^i) := S^{-1} E^i S = H^{m_1}_{2m_1}(k) + (I_2 \otimes B^{i+1}) * A^1_1 + \sum_{j=2}^{n_i} S^{-1}_{s(a^j_1)} E^i(a^j_1) S_{l(a^j_1)} * A^j_1
\]

Thus an almost split conflation \((e^{i+1})\) in \(R(\mathfrak{B}^{i+1})\) is obtained by Lemma 3.2.2 (ii), such that \(\vartheta_0^{i+1}(e^{i+1}) \simeq (e^{i+1})\) via the isomorphisms \(E^{i+1} \xrightarrow{\xi^{-1}} E^i \xrightarrow{\eta^{-1}} E^i\) in \(R(\mathfrak{B}^i)\). Consequently \(\vartheta_0^{i+1}(e^{i+1}) \simeq \vartheta_0^i(e^i) \simeq (e)\).

The existence of such an isomorphism \(\eta\) is established below.

If \(\delta(a^1_1) = v^1_1 \neq 0\), then \(B^{i+1} = M^i(a^1_1) = (0)\). By the proof (i) of Proposition 2.1.8, there exists an isomorphism \(\eta\), such that \(\bar{E}^i = \eta(E^i) \in R(\mathfrak{A}^i)\) with \(\bar{E}^i(a^1_1) = (0)\) as desired.

If \(\delta(a^1_1) = 0\) in the case of loop or edge reduction, the proof is divided into three parts.

1) We define an object \(L^s = H^{m_1}_{2m_1}(k) + \sum_{a^j_1} L^s(a^j_1) \in R(\mathfrak{B}^s)\) with \(L^s(a^j_1) = (a^j_1)\). Let \(\varphi^s : L^s \rightarrow M^s\) be a morphism in \(R(\mathfrak{B}^s)\), such that \(\varphi^s_Y = (1_Y^s), \forall Y^s \in \mathcal{T}^s\), and \(\varphi^s(v^s) = 0\) for any dotted arrow \(v^s\) in \(\mathfrak{B}^s\). Clearly, \(\varphi^s\) is not a retraction. Thus \(\vartheta_0^s(\varphi^s) : \vartheta_0^s(L^s) \rightarrow \vartheta_0^s(M^s) = M^i\) is not a retraction, since the functor \(\vartheta_0^s\) is fully faithful.
2) Because $\vartheta^i s(L^i) = H_{2m_i}(k) + (I_2 \otimes B^{i+1}) \ast A^i_1 + \sum_{j=2}^{n_i} \vartheta^i s(L^j)(a^j_1) \ast A^j_1$, it is possible to construct an object $L^i$ with $L^i(a^i_1) = B^{i+1} \oplus B^{i+1}$ by changing certain rows and columns in $I_2 \otimes B^{i+1}$, and an isomorphism $\vartheta^is(L^i) \cong L^i$. Thus there is a lifting $\varphi^i : L^i \to E^i$ of $\varphi^i$ with $\varphi^i = \varphi^i \pi^i$, since $\pi^i : E^i \to M^i$ is right almost split in $R(\mathfrak{A}^i)$ by the assumption on $(e^i)$. The triangle and the square below are both commutative:

3) According to Lemma 3.1.3, it may be assumed that the sequence $(e^i) \in R(\mathfrak{B}^i)$ with $\iota^i_z = (0 I_z), \pi^i_z = (I_x 0), \forall Z \in T^i$, then $E^i(a^i_1) = (M^i(a^i_1))^j_0 (M^i(a^i_1))^j_0$). The commutative triangle forces $\varphi^i Z = (I_x C^i Z)$ for each $Z \in T^i$. The commutative square for $j = 1$ gives an equality

\[
\begin{pmatrix}
I_X & C_X \\
D_X & 1
\end{pmatrix}
\begin{pmatrix}
B^{i+1} & K^{i}_1 \\
B^{i+1} & B^{i+1}
\end{pmatrix}
\begin{pmatrix}
I_Y & C_Y \\
D_Y & 1
\end{pmatrix}
\]

Let $\hat{E}^i = \{(\hat{E}^i_z) \mid \dim(E^i_z) = 2m^i, Z \in T^i\}$ be a set of vector spaces. Define a set of maps $\eta : E^i \to \hat{E}^i$, such that $\eta^i_X = (I^i X, C^i X), \eta^i_Y = (I^i Y, C^i Y)$, and $\eta^i_Z = I_{2m^i Z}$ for $Z \in T^i \setminus \{X, Y\}$; $\eta^i Z = 0$ for any $j = 1, \ldots, m^i$. Let $\hat{E}^i(a^i_1) = \eta^i(a^i_1) E^i(a^i_1) \eta^i(a^i_1)$ for $j = 1, \ldots, n_i$, an object $\hat{E}^i = \eta^i E^i \eta^i \cong E^i$ in $R(\mathfrak{A}^i)$ with $\hat{E}^i(a^i_1) = B^{i+1} \oplus B^{i+1}$ is obtained as desired.

Suppose in the sequence below, the first part from the 0-th pair up to the $s$-pair is given by Formula (3.2-1) with respect to the indecomposable object $M = \vartheta^0 s(M^s) \in R(\mathfrak{A})$:

\[
(\mathfrak{A}, \mathfrak{B}, (\mathfrak{A}^1, \mathfrak{B}^1), \ldots, (\mathfrak{A}^s, \mathfrak{B}^s), (\mathfrak{A}^{s+1}, \mathfrak{B}^{s+1}), \ldots, (\mathfrak{A}^\tau, \mathfrak{B}^\tau)).
\]

Firstly, it is assumed that in the sequence (3.2-3), $\mathfrak{B}^s$ is local, $T^s = \{X\}; \mathfrak{B}^{s+1}$ is induced from $\mathfrak{B}^s$ by a loop mutation; the reduction from $\mathfrak{B}^i$ to $\mathfrak{B}^{i+1}$ is given by a localization followed by a regularization, such that $R^{i+1} = k[x, \phi^{i+1}(x)^{-1}]$ for $s < i < \tau$; and $\mathfrak{B}^\tau$ is minimal.

**Corollary 3.2.4** Suppose $M^s$ is an object of $R(\mathfrak{B}^s)$ with $M^s X = k, M^s(x) = (\lambda), \phi^s(\lambda) \neq 0$. If $\vartheta^0 s(M^s) = M \in R(\mathfrak{B})$ is homogeneous with an almost split conflation $(e)$, then there exists an almost split conflation $(e^s)$ given by Formula (3.1-2) in $R(\mathfrak{B}^s)$, such that $\vartheta^0 s(e^s) \cong (e)$.

**Proof** Set $M^s = \vartheta^s s(M^s)$, then $M^s(a^s_1) = (\lambda)$. Theorem 3.2.3 gives an almost split conflation $(e^s)$ in $R(\mathfrak{B}^s)$ with $\vartheta^0 s(e^s) \cong (e)$. Furthermore, $R(\mathfrak{B}^{s+1})$ is equivalent to $R(\mathfrak{B}^s)$, and for $i > s$, $R(\mathfrak{B}^{i+1})$ is equivalent to a subcategory of $R(\mathfrak{A}^i)$ consisting of the objects $M^i$ with $M^i(x) = (\lambda), \phi^{i+1}(x)$. The assertion follows by the fact that $\phi^{i+1}(x) \mid \phi^i(x)$, and induction on $i = s + 1, \ldots, \tau - 1$.

Secondly, it is assumed that in the sequence (3.2-3) the bocs $\mathfrak{B}^s$ has two vertices $X, Y$, and the first arrow $a^s_1 : X \to X$ with $\delta(a^s_1) = 0$. The bocs $\mathfrak{B}^{s+1}$ is induced from $\mathfrak{B}^s$ by a loop mutation $a^{s+1}_j : X \to X$, such that $a^{s+1}_j$ is either a loop at $X$, or an edge from $X$ to $Y$ for $j = 1, \ldots, \tau - s$. In particular, there exists a certain index $s < \epsilon < \tau$, such that $a^{s+1}_{s+1} : X \to Y$ is an edge. The reduction from $\mathfrak{B}^i$ to $\mathfrak{B}^{i+1}$ is given by one of the following three cases:

(i) when $s < i < \epsilon$, if $a^i_1 : X \to X$, a localization followed by a regularization are made with $R^{i+1} = k[x, \phi^{i+1}(x)^{-1}]$; if $a^i_1 : X \to Y$, a regularization, or a reduction given by proposition 2.2.6 is made;
(ii) when $i = \varepsilon$, a reduction given by proposition 2.2.7 is made, the induced bocs $\mathfrak{B}^{i+1}$ is local with a vertex $Z$;

(iii) when $\varepsilon < i < \tau$, then $a_1^i : Z \to Z$, a localization followed by a regularization are made.

Finally, $\mathfrak{B}^7$ is minimal with $R^7 = k[x, \phi^7(x)^{-1}]$.

**Corollary 3.2.5** Suppose $M^7$ is an object of $R(\mathfrak{B}^7)$, $M^7_Z = k, M^7(z) = (\lambda)$ with $\phi^7(\lambda) \neq 0$. If $\vartheta^0(R(M^7)) = M \in R(\mathfrak{B})$ is homogeneous with an almost split conflation $(e, e')$, then there exists an almost split conflation $(\varepsilon, \varepsilon')$ given by Formula (3.1-2) in $R(\mathfrak{B}^7)$, such that $\vartheta^0(R(\varepsilon, \varepsilon')) \simeq (e, e')$.

**Proof** Set $M^8 = \vartheta^0(M^7)$, then $M^8_Z = k, M^8_\varepsilon = k$, and $M^8(a_1^8) = (\lambda)$. Theorem 3.2.3 gives an almost split conflation $(e^8) : M^8 \to E^8 \to M^8$ in $\mathfrak{B}^8$ with $\vartheta^0(e^8) \simeq (e, e')$. Since $R(\mathfrak{B}^{i+1})$ is equivalent to $R(\mathfrak{B}^i)$, we may suppose for some $i > s$, there exists an almost split conflation $(e^{i+1})$ in $R(\mathfrak{B}^{i+1})$ with $\vartheta^i,i+1(e^{i+1}) \simeq (e^i)$ will be constructed according to cases (i)–(iii) stated before the corollary.

(i) A regularization for an edge gives an equivalence $R(\mathfrak{B}^{i+1}) \simeq R(\mathfrak{B}^i)$. And the proof of a regularization for a loop is similar to that of Corollary 3.2.4. Suppose $\delta(a_1^i) = 0$ in $\mathfrak{B}^i$, and a reduction of Proposition 2.2.6 is made by $a_1^i \to 0$. Then $M^i(a_1^i)$ must be $(0)$. By Lemma 3.1.3 (ii), it may be assumed that $i^i = (0), \pi^i = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, thus $E^i(a_1^i) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. Define an object $L \in R(\mathfrak{B}^i)$ of size $2m^i$, such that $L(x) = J_2(\lambda), L(a_1^i) = 0, \forall j$; and a morphism $g : L \to M^i$, such that $g_\varepsilon = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = g_\varepsilon, g(v) = 0$ for any dotted arrow $v$. Then $g$ is not a split epimorphism. Thus there exists a lifting $\bar{g} : L \to E^i$ with $\bar{g}_\varepsilon = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \bar{g}_Y = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Since $\bar{g}$ is a morphism, $L(a_1^i)\bar{g}_\varepsilon = \bar{g}_Y E(a_1^i)$, which leads to $0 = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. Therefore $b = 0, E^i(a_1^i) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. Set $E^{i+1} \in R(\mathfrak{B}^{i+1})$ with $E^{i+1}(x) = E^i(x), E^{i+1}(a_1^{i+1}) = E^i(a_1^i)$ for $j = 2, \ldots, n^i$, then $\vartheta^{i+1}(E^{i+1}) = E^i$.

(ii) Proposition 2.2.7 ensures a possibility that $M^c(a_1^c) = (1)$, so $E^c(a_1^c) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Define a set of matrices: $\{\xi_i = \left(\begin{smallmatrix} 1 & -b \\ 0 & 1 \end{smallmatrix}\right), \xi_\varepsilon = I_2, \xi(v_\varepsilon) = 0\}$, and an object $E^c \in R(\mathfrak{B}^c)$: $E^c_\varepsilon = k^2 = E^c_y, E^c(x) = E^c(x), E^c(a_1^c) = E^{-1}(a_1^c)E^c(\xi(a_1^c)), \xi(E^c(a_1^c)) = I_2$. Let $E^{c+1} \in R(\mathfrak{B}^{c+1})$ with $E^{c+1}_Z = k^2; E^{c+1}(z) = E^c(x), E^{c+1}(a_1^{c+1}) = E^c(a_1^c)$ for $j > 1$. Then $\vartheta^{c+1}(E^{c+1}) = E^c \simeq E^c$ in $R(\mathfrak{B}^c)$.

(iii) Since $\phi^c(\lambda) \neq 0$ and $\phi^{i+1}(x) \mid \phi^i(x), \phi^{i+1}(\lambda) \neq 0$. There is an object $E^{i+1} \in R(\mathfrak{B}^{i+1})$ with $\vartheta^{i+1}(E^{i+1}) \simeq E^i$.

By induction, there is $(e_\varepsilon^i) \in R(\mathfrak{B}^i)$ with $\vartheta^{c+1}(e_\varepsilon^i) \simeq e_\varepsilon^i$, thus $\vartheta^c(\varepsilon_\varepsilon^i) \simeq (e, e')$. □

**Lemma 3.2.6** (i) Suppose that $f(x, y) = \sum_{i, j \geq 0} \alpha_{ij}x^iy^j \in k[x, y]$ with $f(\lambda, \mu) \neq 0$. Let $W_\lambda, W_\mu$ be Weyr matrices of size $m, n$ and eigenvalues $\lambda, \mu$ respectively, and $V = (v_{ij})_{m \times n}$ with $\{v_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ being $k$-linearly independent. Let $f(W_\lambda, W_\mu)V = \sum_{i, j \geq 0} \alpha_{ij}W^i_\lambda V W^j_\mu = (u_{ij})_{m \times n}$. Then $\{u_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is also $k$-linearly independent.

(ii) Let $\mathfrak{B}$ be a bocs with $R = RX \times RY$, where $RX = k[x, \phi_X(x)^{-1}], RY = k[y, \phi_Y(y)^{-1}]$, and $a_1 : X \to Y$. Define $\delta^0(a_1)$ to be a part of $\delta(a_1)$ without terms involving any solid arrow. It is possible that $X = Y$, in this case $x$ stands for the multiplying a dotted arrow from the left and $y$ from the right. Suppose

\[
\begin{align*}
\delta^0(a_1) &= f_{11}(x, y)v_1 \\
\delta^0(a_2) &= f_{21}(x, y)v_1 + f_{22}(x, y)v_2, \\
& \quad \vdots \\
\delta^0(a_n) &= f_{n1}(x, y)v_1 + f_{n2}(x, y)v_2 + \cdots + f_{nn}(x, y)v_n,
\end{align*}
\]

where $f_{ii}(x, y) \in RX \times RY$ are invertible for $i = 1, 2, \ldots, n$. If $x \to W_X$ of size $m$ with eigenvalue $\lambda$ and $\phi_X(\lambda) \neq 0, y \to W_Y$ of size $n$ with eigenvalue $\mu$ and $\phi_Y(\mu) \neq 0$, then the solid arrows splitting from $a_1, \ldots, a_n$ are all going to $\emptyset$ by regularizations in further reductions.
3.3 Minimal wild bocses

In this subsection five classes of minimal wild bocses is defined in order to prove the main theorem. Our classification relies on the Drozd’s wild configurations with refinements at some last reduction stages.

**Proposition 3.3.1** ([D1],[CB1]) Let $\mathcal{B} = (\Gamma, \Omega)$ be a bocs with a layer $L = (R; \omega; a_1, \ldots, a_n; v_1, \ldots, v_m)$ and suppose $a_1 : X \to Y$. If $\mathcal{B}$ is of representation wild type, then it is bound to meet one of the following configurations at some stage of reductions:

**Case 1** $X \in \mathcal{T}_1$, $Y \in \mathcal{T}_0$ (or dually $X \in \mathcal{T}_0$, $Y \in \mathcal{T}_1$), $\delta(a_1) = 0$.

**Case 2** $X, Y \in \mathcal{T}_1$ (possibly $X = Y$), $\delta(a_1) = f(x, y)v_1$ for some non-invertible $f(x, y)$ in $k[x, y, \phi_X(x)^{-1}, \phi_Y(y)^{-1}]$.

Some notations will be fixed first before the classification. There is a decomposition for any non-zero polynomial $f(x, y) \in k[x, y]$:

$$f(x, y) = \alpha(x)h(x, y)\beta(y), \quad \text{where } \alpha(x) \in k[x], \beta(y) \in k[y];$$

and every irreducible factor of $h(x, y)$ contains both $x$ and $y$, or $h(x, y) \in k^*$ with $k^* = k \setminus \{0\}$. Sometimes $\bar{x}$ is used instead of $y$.

Let $k(x, y, z)$ be the fractional field of the polynomial ring $k[x, y, z]$ of three indeterminates. Consider a vector space $S$ generated by the dotted arrows $\{v_1, \ldots, v_m\}$ of a bocs $\mathcal{B}$ over $k(x, y, z)$. Suppose there is a linear combination:

$$G = f_1(x, y, z)v_1 + \cdots + f_m(x, y, z)v_m, \quad f_i(x, y, z) \in k[x, y, z]. \quad (*)$$

Let $h(x, y, z)$ be the greatest common factor of $f_1, \ldots, f_m$, then $f_i/h, \ldots, f_m/h$ are co-prime, and $G = h\sum_{i=1}^{m}(f_i/h)v_i$. There exists some $s_i \in k[x, y, z]$ for $i = 1, \ldots, m$, such that $\sum_{i=1}^{m}(f_i/h) = c(x, y) \in k[x, y]$. Since $S = k[x, y, z, c(x, y)^{-1}]$ is a Hermite ring, there exists some invertible $F(x, y, z) \in \text{M}_n(S)$ with the first column $(f_1/h, \ldots, f_m/h)$. A base change of the form $(w_1, \ldots, w_m) = (v_1, \ldots, v_m)F$ is made, thus $G = h(x, y, z)w_1$.

**Classification 3.3.2** Let $\mathcal{B}^0$ be a wild bocs given by Proposition 3.3.1. Then we are bound to meet an induced bocs $\mathcal{B}$ with a layer $L = (R; \omega; a_1, \ldots, a_n; v_1, \ldots, v_m)$ in one of the five classes at some stage of reductions. And a bocs in those classes is said to be minimal wild, which might be written briefly by MW.

Suppose the bocs $\mathcal{B}$ has two vertices $T = \{X, Y\}$, such that the induced local bocs $\mathcal{B}_X$ is tame infinite with $R_X = k[x, \phi_X(x)^{-1}]$.

**MW1** $\mathcal{B}_Y$ is finite with $R_Y = k1_Y$, and $\delta(a_1) = 0$:

$$x \bigcirc X \xrightarrow{a_1} Y.$$

**MW2** $\mathcal{B}_Y$ is tame infinite with $R_Y = k[y, \phi_Y(y)^{-1}]$, and $\delta(a_1) = f(x, y)v_1$, such that $f(x, y)$ is non-invertible in $k[x, y, \phi_X(x)^{-1}, \phi_Y(y)^{-1}]$:

$$x \bigcirc X \xrightarrow{a_1} Y \bigcirc y.$$

Suppose now we have a local bocs $\mathcal{B}$ with $R = k[x, \phi(x)^{-1}]$: 

where $f_i(x, x) = \alpha_{ii}(x)h_{ii}(x, x)\beta_{ii}(x)$ by Formula (3.3-1), such that $h_{ii}(x, x)$ is invertible in $k[x, \phi(x)^{-1}]$ for $1 \leq i \leq n$; and there is some minimal $1 \leq s \leq n$, such that $f_{ss}(x, x)$ is non-invertible in $k[x, x, \phi(x)^{-1}, \phi(x)^{-1}]$.

Suppose there exists some $1 \leq n_1 \leq n$, such that:

$$
\begin{align*}
\delta^0(a_{n_1+1}) &= K_{n_1+1} + f_{n_1+1,n_1+1}(x, x, x_1)w_{n_1+1}, \\
\ldots & \ldots \\
\delta^0(a_n) &= K_n + f_{n_1,n_1}(x, x, x_1)w_{n_1} + \ldots + f_{n_1,n_1}(x, x)w_{n_1},
\end{align*}
$$

where $f_{ii}(x, x)$ for $1 \leq i \leq n_1$ are invertible in $k[x, \phi(x)^{-1}]$; $\bar{w} = 0$, or $\bar{w} \neq 0$ but $f_{n_1,n_1}(x, x) = 0$. Denote by $x_1$ the solid arrow $a_{n_1}$, there exists a polynomial $\psi(x, x_1)$ being divided by $\phi(x)$.

Write $\delta^1$ the part of differential $\delta$ by deleting all the monomials involving any solid arrow except $x_1$. Suppose the further unraveling for $x$ is restricted to $x \mapsto (\lambda)$ with $\psi(\lambda, x_1) \neq 0$. Then

$$
\begin{align*}
\delta^1(a_{n_1+1}) &= K_{n_1+1} + f_{n_1+1,n_1+1}(x, x_1, x_1)w_{n_1+1}, \\
\ldots & \ldots \\
\delta^1(a_n) &= K_n + f_{n_1,n_1}(x, x_1, x_1)w_{n_1} + \ldots + f_{n_1,n_1}(x, x)w_{n_1},
\end{align*}
$$

where $K_i = \sum_{j=1}^{n_1} f_{ij}(x, x_1, x_1)w_j$, $w_i$ are given by the base changes described below Formula (3) inductively, and $f_{ii}(x, x_1, x_1)$ are invertible in $k[x, x_1, x_1, x_1, \psi(x_1, x_1)^{-1}, \psi(x_1, x_1)^{-1}]$ for $n_1 < i \leq n$.

**MW4** $\bar{w} = 0$, or $\bar{w} \neq 0$ but $(x - \bar{x})^2 | f_{n_1,n_1}(x, x_1)$ in Formula (3.3-3).

**MW5** $\bar{w} \neq 0$ and $(x - \bar{x})^2 \nmid f_{n_1,n_1}(x, x_1)$ in Formula (3.3-3). \(\square\)

The proof of Classification 3.3.2 depends on Classification 3.3.5 of local boces at the end of the subsection, while the proof of 3.3.5 is based on formuleae (3.3-2)-(3.3-9) and Lemma 3.3.3–3.3.4 below.

Let $\mathfrak{B}$ be a local boces having a layer $L = (R; \omega; a_1, \ldots, a_n; v_1, \ldots, v_m)$. If $R = k1X$ is trivial, then the differentials of the solid arrows have two possibilities. First,

$$
\begin{align*}
\delta^0(a_1) &= f_{11}w_1, \\
\ldots & \ldots \\
\delta^0(a_n) &= f_{n_1}w_1 + \ldots + f_{n_n}w_n,
\end{align*}
$$

where $f_{ij} \in k, f_{ii} \neq 0$ for $1 \leq i \leq n$. Second, there exists some $1 \leq n_0 \leq n$, such that:

$$
\begin{align*}
\delta^0(a_1) &= f_{11}w_1, \\
\ldots & \ldots \\
\delta^0(a_{n_0-1}) &= f_{n_0-1,1}w_1 + \ldots + f_{n_0-1,n_0-1}w_{n_0-1}, \\
\delta^0(a_{n_0}) &= f_{n_0}w_1 + \ldots + f_{n_0,n_0-1}w_{n_0-1},
\end{align*}
$$

where $f_{ij} \in k, f_{ii} \neq 0$ for $1 \leq i < n_0$. Set $a_i \mapsto \emptyset$, $i = 1, \ldots, n_0 - 1$, by a series of regularization, then $a_{n_0} \mapsto (x)$ by a loop mutation, an induced local boces $\mathfrak{B}'$ is obtained.
Without loss of generality, the bocs $\mathfrak{B}'$ may still be denoted by $\mathfrak{B}$ with a layer $L$, but $R = k[x]$. The differentials $\delta^0$ have again two possibilities. The first one is given by Formula (3.3-2), such that $h_{ii}(x, x) \neq 0$, i.e. $\langle \lambda \rangle(\bar{f}_{ii}(x, \bar{x}))$ for $i = 1, \ldots, n$. Define a polynomial:
\[
\phi(x) = \prod_{i=1}^{n} c_i(x)h_{ii}(x, x),
\]
(3.3-7)
where $c_i(x)$ appears at the localization in order to do a base change before the $i$-th step of a regularization.

**Lemma 3.3.3** Let $\mathfrak{B}$ be a bocs given by Formula (3.3-2) with a polynomial $\phi(x)$ given by Formula (3.3-7). There exist two cases:

(i) $f_{ii}(x, \bar{x})$ are invertible in $k[x, \bar{x}, \phi(x)^{-1}\phi(\bar{x})^{-1}]$ for $1 \leq i \leq n$;

(ii) $f_{ss}(x, \bar{x})$ is not invertible in $k[x, \bar{x}, \phi(x)^{-1}\phi(\bar{x})^{-1}]$ for some minimal $1 \leq s \leq n$. \hfill $\square$

The second possibility of the differential $\delta^0$ in the case of $R = k[x]$ is given by Formula (3.3-3) for some fixed $1 \leq n_1 \leq n$, such that $f_{ii}(x, x) \neq 0$ for $1 \leq i < n_1$, and $\bar{w} = 0$, or $\bar{w} \neq 0$ but $f_{n_1,n_1}(x, x) = 0$. Let
\[
\phi(x) = \begin{cases} 
\prod_{i=1}^{n_1-1} c_i(x)f_{ii}(x, x), & \bar{w} = 0; \\
\sum_{i=1}^{n_1-1} c_i(x)f_{ii}(x, x), & \bar{w} \neq 0,
\end{cases}
\]
Thus, under the restriction $x \mapsto (\lambda), \phi(\lambda) \neq 0$, an induced bocs given by regularizations with the first arrow $a_{n_1}$ is obtained. There are two possibilities in the further reductions. The first possibility is given by Formula (3.3-4), such that $f_{ii}(x, x, x_{1}) \neq 0$ for $n_1 < i \leq n$. There is a sequence of localizations given by the polynomials $c_i(x, x_{1})$ appeared before the base changes in order to do regularizations. Let
\[
\psi(x, x_{1}) = \phi(x)\prod_{i=n_1+1}^{n} c_i(x, x_{1})f_{ii}(x, x_{1}, x_{1}),
\]
(3.3-9)

**Lemma 3.3.4** Let the differentials in the bocs $\mathfrak{B}$ be given by Formulae (3.3-3)–(3.3-4) with polynomials $\phi(x)$ in (3.3-8), and $\psi(x, x_{1})$ in (3.3-9). There exist two cases.

(i) There exists some $\lambda \in k$ with $\psi(\lambda, x_{1}) \neq 0$, and a minimal $1 \leq n_1 \leq s \leq n$, such that $f_{ss}(\lambda, x_{1}, \bar{x}_{1})$ is non-invertible in $k[x_{1}, \bar{x}_{1}, \psi(\lambda, x_{1})^{-1}\psi(\lambda, \bar{x}_{1})^{-1}]$, i.e., after making an unraveling $x \mapsto (\lambda)$, followed by a series of regularizations $a_{1} \mapsto \emptyset$, $w_i = 0$ for $i = 1, \ldots, n_1 - 1$, the induced local bocs $\mathfrak{B}(\lambda)$ with $R(\lambda) = k[x_{1}, \psi(\lambda, x_{1})^{-1}]$ is in case (ii) of Lemma 3.3.3.

(ii) For any $\lambda \in k$ with $\psi(\lambda, x_{1}) \neq 0$, $f_{ii}(\lambda, x_{1}, \bar{x}_{1})$ are invertible for $n_1 < i \leq n$ in $k[x_{1}, \bar{x}_{1}, \psi(\lambda, x_{1})^{-1}\psi(\lambda, \bar{x}_{1})^{-1}]$, i.e., the induced bocs $\mathfrak{B}(\lambda)$ with $R(\lambda) = k[x_{1}, \psi(\lambda, x_{1})^{-1}]$ is in case (i) of 3.3.3.

Case (ii) is equivalent to (ii)': $f_{ii}(x, x_{1}, \bar{x}_{1})$ are invertible in $k[x, x_{1}, \bar{x}_{1}, \psi(x, x_{1})^{-1}\psi(x, \bar{x}_{1})^{-1}]$ for $1 < i \leq n$.

**Proof** It is only need to prove the equivalence of (ii) and (ii)'.

(ii) $\Rightarrow$ (ii)' If there exists some $n_1 < s \leq n$ with $f_{ss}(x, x_{1}, \bar{x}_{1})$ non-invertible, then it contains a non-trivial factor $g(x, x_{1}, \bar{x}_{1})$ coprime to $\psi(x, x_{1})\psi(x, \bar{x}_{1})$. Consider the variety $V = \{(\alpha, \beta, \gamma) \in k^3 \mid \psi(\alpha, \beta, \gamma) = 0, \psi(\alpha, \beta)\psi(\alpha, \gamma) = 0\}$. Since $\dim(V) \leq 1$, there exists a co-finite subset $\mathcal{L} \subset k$, such that $\forall \lambda \in \mathcal{L}$, the plane $x = \lambda$ of $k^3$ intersects $V$ at only a finite number of points. Thus $g(\lambda, x_{1}, \bar{x}_{1})\psi(\lambda, x_{1})\psi(\lambda, \bar{x}_{1})$ are coprime. Consequently $g(\lambda, x_{1}, \bar{x}_{1})$, thus $f_{ss}(\lambda, x_{1}, \bar{x}_{1})$ is not invertible in $k[x_{1}, \bar{x}_{1}, \psi(\lambda, x_{1})^{-1}\psi(\lambda, \bar{x}_{1})^{-1}]$.

(ii)' $\Rightarrow$ (ii)' If $f_{ii}(x, x_{1}, \bar{x}_{1})$ is invertible in $k[x, x_{1}, \bar{x}_{1}, \psi(x, x_{1})^{-1}\psi(x, \bar{x}_{1})^{-1}]$, then for any $\lambda \in k$ with $\psi(\lambda, x_{1}) \neq 0$, $f_{ii}(\lambda, x_{1}, \bar{x}_{1})$ is invertible in $k[x_{1}, \bar{x}_{1}, \psi(\lambda, x_{1})^{-1}\psi(\lambda, \bar{x}_{1})^{-1}]$. \hfill $\square$

The second possibility of $\delta^1$ is: there exists some $n_2$ with $n_1 < n_2 \leq n$, such that
\[
\begin{cases} 
\delta^1(a_{n_1+1}) = K_{n_1+1} + f_{n_1+1,n_1+1}(x, x_{1}, \bar{x}_{1})w_{n_1+1}, \\
\vdots \\
\delta^1(a_{n_2-1}) = K_{n_2-1} + \cdots + f_{n_2-1,n_2-1}(x, x_{1}, \bar{x}_{1})w_{n_2-1}, \\
\delta^1(a_{n_2}) = K_{n_2} + \cdots + f_{n_2,n_2-1}(x, x_{1}, \bar{x}_{1})w_{n_2-1} + f_{n_2,n_2}(x, x_{1}, \bar{x}_{1})w',
\end{cases}
\]
(3.3-10)
where $K_i = \sum_{j=1}^{n_i-1} f_{ij}(x, x_1, \bar{x}_1)w_j$ for $n_1 < i \leq n_2$, $f_{ii}(x, x_1, x_1) \neq 0$ for $n_1 < i < n_2$, and $\bar{w}' = 0$, or $\bar{w}' \neq 0$ but $f_{n_2,n_2}(x, x_1, x_1) = 0$. Define a polynomial

$$\psi_1(x, x_1) = \begin{cases} 
\phi(x) \prod_{i=n_1+1}^{n_2-1} c_i(x, x_1)f_{ii}(x, x_1, x_1), & \text{if } \bar{w}' = 0; \\
\prod_{i=n_1+1}^{n_2-1} c_{n_2}(x, x_1)\phi(x)f_{ii}(x, x_1, x_1), & \text{if } \bar{w}' \neq 0.
\end{cases}$$

(3.3-11)

Suppose a bocs $\mathfrak{V}$ is med, the differential of which is given by Formula (3.3-3) and (3.3-10) with a polynomial $\psi_1(x, x_1)$ of (3.3-11). Fix any $\lambda_0 \in \mathfrak{K}$ with $\psi_1(\lambda_0, x_1) \neq 0$, there is an induced bocs $\mathfrak{V}_{(\lambda_0)}$ with $R_{(\lambda_0)} = k[x_1, \psi_1(\lambda_0, x_1)^{-1}]$ given by an unraveling $x \mapsto (\lambda_0)$, and then a series of regularizations $a_i \mapsto \emptyset, w_i = 0$ for $i = 1, \ldots, n_1 - 1$. There exist three cases:

1) $\mathfrak{V}_{(\lambda_0)}$ is in case (i) of Lemma 3.3.4, then there exists some $\lambda_1$ with $\psi(\lambda_0, \lambda_1) \neq 0$, such that after sending $x_1 \mapsto (\lambda_1)$ by an unraveling, followed by a series of regularizations, the induced bocs $\mathfrak{V}_{(\lambda_0, \lambda_1)}$ satisfies Lemma 3.3.3 (ii);

2) $\mathfrak{V}_{(\lambda_0)}$ is in case (ii)' of Lemma 3.3.4;

3) $\mathfrak{V}_{(\lambda_0)}$ is in the case of Formulae (3.3-3) and (3.3-10).

In case 3), the above procedure is repeated once again for $\mathfrak{V}_{(\lambda_0)}$. By induction on the number of the finitely many solid arrows, the case 1) or case 2) is finally reached.

**Classification 3.3.5** Let $\mathfrak{V}$ be a local bocs with $R$ trivial, there exist four cases:

(i) $\mathfrak{V}$ has Formula (3.3-5).

(ii) $\mathfrak{V}$ has Formula (3.3-6). And by a series of regularizations $a_i \mapsto \emptyset, i = 1, \ldots, n_0 - 1$, followed by a loop mutation $a_{n_0} \mapsto (x)$, the induced bocs $\mathfrak{V}'$ with a polynomial (3.3-7) is in case (i) of Lemma 3.3.3.

(iii) $\mathfrak{V}$ has an induced local bocs $\mathfrak{V}_{(\lambda_0, \lambda_1, \ldots, \lambda_l)}$ for some $l < n$ in case (ii) of Lemma 3.3.3.

(iv) $\mathfrak{V}$ has an induced local bocs $\mathfrak{V}_{(\lambda_0, \lambda_1, \ldots, \lambda_{l-1})}$ for some $l < n$ in case (ii) of Lemma 3.3.4.

**The proof of Classification 3.3.2** 1) Suppose a two-point wild bocs is med, if $\mathfrak{V}_X$ or $\mathfrak{V}_Y$ is in case (iii) or (iv) of Classification 3.3.5, the induced local bocs given by deleting $Y$ or $X$ may be considered, which is wild type. Therefore it is assumed that one of bocs $\mathfrak{V}_X$ or $\mathfrak{V}_Y$ has Formula (3.3-5) and another is in case (i) of Lemma 3.3.3, or both of $\mathfrak{V}_X$ and $\mathfrak{V}_Y$ are in case (i) of Lemma 3.3.3. MW1 or MW2 follows.

2) If a local wild bocs in case (iii) of Classification 3.3.5 is med, then there is an induced bocs in case (ii) of Lemma 3.3.3. MW3 is reached.

3) If a local wild bocs in case (iv) of Classification 3.3.5 is med, then there is an induced bocs in case (ii)' of Lemma 3.3.4. MW4 or MW5 is reached. \[\square\]

### 3.4 Non-homogeneity in the cases of MW1-4

Throughout the subsection, $(\mathfrak{A}_0, \mathfrak{A}_0^\circ)$ denotes any pair of matrix bimodule problem and its associated bocs.

**Proposition 3.4.1** If $\mathfrak{A}_0$ has an induced bocs $\mathfrak{V}$ in the case of MW1, then $\mathfrak{A}_0$ is non-homogeneous.

**Proof** 1) Let $\mathfrak{V}_X$ be an induced local bocs of $\mathfrak{V}$. Suppose $\vartheta_1 : R(\mathfrak{V}_X) \to R(\mathfrak{V})$, $\vartheta_2 : R(\mathfrak{V}) \to R(\mathfrak{V}_X^\circ)$ are two induced functors, and $\vartheta = \vartheta_2 \vartheta_1 : R(\mathfrak{V}_X) \to R(\mathfrak{V}_X^\circ)$. For any $\lambda \in \mathfrak{K}$, $\phi(\lambda) \neq 0$, a representation $S_\lambda \in R(\mathfrak{V}_X)$ given by $(S_\lambda)_X = k$, $S_\lambda(x) = (\lambda)$ is defined. If $\mathfrak{A}_0$ is homogeneous, then there is a co-finite subset $\mathcal{L} \subseteq \{\lambda \mid \phi(\lambda) \neq 0\}$, such that $\{\vartheta(S_\lambda) \mid \lambda \in \mathcal{L}\}$ is a family of homogeneous iso-classes of $R(\mathfrak{V}_0^\circ)$. By Corollary 3.2.4, there is an almost split conflation $(\epsilon_\lambda) : S_\lambda' \xrightarrow{\epsilon_\lambda} E_\lambda' \xrightarrow{\pi_\lambda} S_\lambda'$ in $R(\mathfrak{V}_X)$ with $E'(x) = J_2(\lambda)$, such that $\vartheta(\epsilon_\lambda)$ is an almost split conflation in $R(\mathfrak{V}_0^\circ)$. Fix any $\lambda \in \mathcal{L}$, and the conflation $(\epsilon_\lambda) = (\vartheta(\epsilon_\lambda) : S_\lambda \xrightarrow{\epsilon_\lambda} E_\lambda \xrightarrow{\pi_\lambda} S_\lambda$ in $R(\mathfrak{V})$, then $(S_\lambda)_X = k, (S_\lambda)_Y = 0, S_\lambda(x) = (\lambda), S_\lambda(a_i) = 0$ for all solid arrow $a_i$ of $\mathfrak{V}$, and...
\((E_\lambda)_X = k^2, (E_\lambda)_Y = 0, E_\lambda(x) = J_2(\lambda), E_\lambda(a_i) = 0\). Since \(\vartheta_2(e_\lambda) = \vartheta(e'_\lambda)\) is almost split in \(R(\mathfrak{B}^0)\), so is \((e_\lambda)\) in \(R(\mathfrak{B})\) by Lemma 3.2.2 (ii) inductively.

2) Let \(L \in R(\mathfrak{B})\) be an object given by \(L_X = L_Y = k, L(x) = (\lambda), L(a_i) = (1)\) and \(L(a_i) = 0\) for \(i > 1\). Let \(g : L \to S_\lambda\) be a morphism with \(g_X = (1), g_Y = (0)\) and \(g(v) = 0\) for all dotted arrows \(v\) of \(\mathfrak{B}\). It is asserted that \(g\) is not a retraction. Otherwise, if there is a morphism \(h : S_\lambda \to L\) such that \(hg = id_{S_\lambda}\), then \(h_X = (1)\) and \(h_Y = (0)\). But \(h\) being a morphism implies that \((1)(1) = h_X L(a) = S_\lambda(a)h_Y = (0)(0)\), a contradiction.

3) There exists a lifting \(\tilde{g} : L \to E_\lambda\) with \(\tilde{g}\pi = g\). If \(\tilde{g}_X = (a, b), \) then \(\tilde{g}_X \pi_X = g_X, i.e., (a, b)(t_i^0) = (1), a = 1\). But \(\tilde{g}\) being a morphism implies: \(\tilde{g}_X E_\lambda(x) = L(x)\tilde{g}_X, i.e., (1, b)(\lambda^1) = (\lambda)(1, b), (\lambda, 1 + b\lambda) = (\lambda, \lambda b),\) a contradiction. Thus \(\mathfrak{B}^0\) is not homogeneous. \(\square\)

Proposition 3.4.2 [B1] If \(\mathfrak{B}^0\) has an induced bocs \(\mathfrak{B}\) in the case of MW2, then \(\mathfrak{B}^0\) is non-homogeneous.

**Proof** Since \(f(x, y)\) is non-invertible in \(k[x, y, \phi_\lambda^{-1}(x)^{-1}, \phi_\mu^{-1}(y)^{-1}]\), after dividing the dotted arrows \(v_j\) by some powers of \(\phi_\lambda(x)\) and \(\phi_\mu(y)\), it may be assumed that \(f(x, y) \in k[x, y]\). There exist three cases on \(f(x, y) = \alpha(x)h(x, y)\beta(y)\) as in Formula (3.3-1).

Case 1) \(h(x, y) \not\in k^*\), then \(h(x, y)\) and \(\phi_\lambda(x)\phi_\mu(y)\) are coprime. There is an infinite set

\[\mathcal{L'} = \{(\lambda, \mu) \in k \times k \mid h(\lambda, \mu) = 0, \phi_\lambda(\lambda)\phi_\mu(\mu) \neq 0\}\]

by Bezout’s theorem. Clearly, \(\mathcal{L'} = \{\lambda \in k \mid (\lambda, \mu) \in \mathcal{L'}\}\) is an infinite set.

Case 2) \(h(x, y) \in k^*\), but there is an irreducible factor \(\beta'(y)\) of \(\beta(y)\) coprime to \(\phi_\mu(y)\). If \(\beta'(\mu) = 0\), then there is an infinite set \(\mathcal{L_X} = \{\lambda \in k \mid \phi_\lambda(\lambda)\phi_\mu(\mu) \neq 0, \beta(\mu) = 0\}\).

Case 3) \(h(x, y) \in k^*\), and \(\beta(y) \mid \phi(y)^c\) for some \(e \in \mathbb{Z}^+\), then there must exist an irreducible factor \(\alpha'(x)\) of \(\alpha(x)\) coprime to \(\phi_\lambda(x)\). If \(\alpha'(\lambda) = 0\), there is an infinite set \(\mathcal{L_Y} = \{\mu \in k \mid \phi_\lambda(\lambda)\phi_\mu(\mu) \neq 0, \alpha(\lambda) = 0\}\).

The cases 1–2 are dealt with first in the following statement 1–3).

1) The discussion is carried out as in proof 1) of Proposition 3.4.1, then an infinite set

\[\mathcal{L} \subseteq \mathcal{L_X}\]

is obtained.

2) Let \(L \in R(\mathfrak{B})\) be an object given by \(L_X = k = L_Y, L(x) = (\lambda), \lambda \in \mathcal{L}; L(y) = (\mu)\) with \((\lambda, \mu) \in \mathcal{L'}; L(a_i) = (1); L(a_i) = 0\) for \(i > 1\). Let \(g : L \to S_\lambda\) be a morphism in \(R(\mathfrak{B})\) with \(g_X = (1), g_Y = (0)\) and \(g(v) = 0\) for all dotted arrows \(v\) then \(g\) is not a retraction. Otherwise, if there is a morphism \(h : S_\lambda \to L\) such that \(hg = id_{S_\lambda}\), then \(h_X = (1)\) and \(h_Y = (0)\). But \(h\) being a morphism implies that \(-1 = S_\lambda(a_1)h_Y - h_X L(a_1) = h(\delta(a_1)) = f(\lambda, \mu)h(v) = 0\), a contradiction.

3) There exists a lifting \(\tilde{g} : L \to E_\lambda\) with \(\tilde{g}\pi = g\). A contradiction appears as the same as in the proof 3) of Proposition 3.4.1.

If case 3) appears, then a set of homogeneous iso-classes \(\{S_\mu \mid \mu \in \mathcal{L}\}\) is used. Let \(L\) be the same as in 2), and a morphism \(g : S_\mu \to L\) be not a section. There is an extension \(\tilde{g} : E_\mu \to L\), which leads to a contradiction. \(\square\)

Proposition 3.4.3 [B1] If \(\mathfrak{B}^0\) has an induced bocs \(\mathfrak{B}\) in the case of MW3 with \(R = k[x, \phi(x)^{-1}]\), then \(\mathfrak{B}^0\) is non-homogeneous.

**Proof** After a series of regularizations \(a_i \to \emptyset, i = 1, \ldots, s - 1\), it may be assumed that \(s = 1\) in MW3. Since \(f_{11}(x, \bar{x})\) is non-invertible in \(k[x, \bar{x}, \phi_\lambda^{-1}(x)^{-1}\phi_\lambda^{-1}(\bar{x})^{-1}]\), by a similar discussion as the beginning of the proof of Proposition 3.4.2, there are infinite sets:

\[\mathcal{L}_1 = \{\lambda \mid h_{11}(\lambda, \mu) = 0, \phi(\lambda) \neq 0\}\]

\[\mathcal{L}_2 = \{\lambda \mid \beta_{11}(\mu) = 0, \phi(\lambda)\beta_{11}(\lambda) \neq 0\}; \mathcal{L}_3 = \{\mu \mid \alpha_{11}(\lambda) = 0, \phi(\mu)\alpha_{11}(\mu) \neq 0\}\]

according to the cases 1–3 respectively. It is easy to see that \(\lambda \neq \mu\) in \(\mathcal{L}_1 \mathcal{L}_3\).
Define a polynomial \( \psi(x) = \phi(x) \prod_{i=1}^{n} \alpha_i(x) \beta_i(x) \), there is an induced bocs \( \mathcal{B}' \) of \( \mathcal{B} \) with \( R' = k[x, \psi(x)^{-1}] \) given by a localization. Note that \( \mathcal{B}' \) is not necessarily minimal. Set the induced functors \( \vartheta_1 : R(\mathcal{B}') \rightarrow R(\mathcal{B}), \vartheta_2 : R(\mathcal{B}) \rightarrow R(\mathcal{B}'_0) \), and \( \vartheta = \vartheta_2 \vartheta_1 : R(\mathcal{B}') \rightarrow R(\mathcal{B}'_0) \).

The case of \( \mathcal{L}_1 \) or \( \mathcal{L}_2 \) is dealt with first in the following (1)–(3).

1) Let \( S_t \in R(\mathcal{B}'_0), \lambda \in \mathcal{L}_1 \) or \( \mathcal{L}_2 \) be an object given by \( (S_\lambda)_x = k, S_\lambda(x) = (\lambda, \psi(\lambda) \neq 0) \), then \( S_\lambda(a_i) = (0) \) for \( 1 \leq i \leq n \) by Lemma 3.2.6 (ii), since \( f_{i\lambda}(\lambda, \lambda) \neq 0 \). If \( \mathcal{B}'_0 \) is homogeneous, then there is a co-finite subset \( \mathcal{L} \subseteq \mathcal{L}_1 \) or \( \mathcal{L}_2 \), such that \( \{\vartheta(S_\lambda) \in R(\mathcal{B}_0' \mid \lambda \in \mathcal{L}\} \) is a family of homogeneous iso-classes of \( \mathcal{B}_0' \). By Theorem 3.2.3 with respect to \( x \in \mathcal{B}' \), there is an almost split conflation \( (e_\lambda') : S_\lambda \xrightarrow{\iota} E_\lambda \xrightarrow{\pi} S_\lambda \) in \( R(\mathcal{B}'_0) \), such that \( \vartheta(e_\lambda') \) is an almost split conflation in \( R(\mathcal{B}_0) \). By Lemma 3.1.3 (ii) it may be assumed that \( \iota = (01), \pi = (10)^T \), thus \( E_\lambda(x) = (\lambda, c) \). By Lemma 3.2.6 (ii) once again, \( E_\lambda(a_i) = (0) \) for \( i = 1, \ldots, n \), therefore \( E_\lambda(x) = J_2(\lambda) \). In fact, if \( E_\lambda(x) \) was \( \lambda I_2, (e_\lambda') \) would be splittable. Fix any \( \lambda \in \mathcal{L} \), and consider the conflation \( (e_\lambda) = \vartheta_1(e_\lambda') \) in \( R(\mathcal{B}) \), which is almost split by Lemma 3.2.2 (ii).

2) Let \( \mu \in k \) with \( \phi(\mu) \neq 0, h_{11}(\lambda, \mu) = 0 \) or \( \beta_{11}(\mu) = 0 \). Define \( L \in R(\mathcal{B}) \) with \( L_x = k^2, L_x = (\lambda, \mu), L(a_1) = J_2(0) \), and \( L(a_i) = 0 \) for \( 2 \leq i \leq n \). \( L \) is well defined. In fact, if \( \eta = \{\eta_X, \eta(v_j)\} : \mathcal{L} \rightarrow \mathcal{L} \) is an isomorphism, then \( \eta(x) \) forces \( \vartheta_1(\eta(v_j)) = (\lambda, \mu) \), and \( L(a_1) \eta_X - \eta_X L(a_1) = f_{11}(L(x), L(x)) \eta(v_j) \) implies \( (0b - a) \cdot 0 = (f_{11}(\lambda, \lambda) v_{11}, f_{11}(\lambda, \lambda) v_{12}) \), which has a solution \( v_{ij} = 0, a = b \). Let \( g : L \rightarrow S_\lambda \) be a morphism with \( g_x = (\lambda, \mu) \) and \( g(v_j) = (0, 0) \) for all \( j \), then \( g \) is not a retraction. Otherwise, there is a morphism \( h : S_\lambda \rightarrow L \) with \( hg = id_{S_\lambda} \).

Thus \( h_x = (1, b) \). Set \( h(v_1) = (c, d) \). Then
\[
S_\lambda(a_1) h_x - h_x L(a_1) = f_{11}(S_\lambda(x), L(x)) h(v_1) \Rightarrow 0, \]
which leads to \( -/(0, 1) = (\ast, 0) \), a contradiction.

3) There is a lifting \( \tilde{g} : L \rightarrow E_\lambda \) with \( \tilde{g} \pi = g \). Set \( \tilde{g}_x = (a b), \tilde{g}_x \pi_x = g_x \) yields \( \tilde{g}_x = (1 b) \). On the other hand, \( \tilde{g} : L \rightarrow E_\lambda \) being a morphism leads to \( \tilde{g}_x E_\lambda(x) = L(x) \tilde{g}_x \), i.e., \( (1 b)(\lambda, \mu) = \tilde{g}_x \tilde{g}_x \lambda, \) then \( (\lambda, \mu) = (0, 0) \), a contradiction. Therefore \( \mathcal{B}_0' \) is not homogeneous.

If \( \mathcal{L}_3 \) appears, then a set of homogeneous iso-classes \( \{\nu \mid \mu \in \mathcal{L}_3\} \) is used. Let \( L \) be the same as in 2), and a morphism \( g : S_\mu \rightarrow L \) be not a section. There is an extension \( \tilde{g} : E_\mu \rightarrow L \), which leads to a contradiction.

**Proposition 3.4.4** If \( \mathcal{B}_0' \) has an induced bocs \( \mathcal{B} \) in the case of MW4 with two polynomials \( \phi(x) \) and \( \psi(x, x_1) \), then \( \mathcal{B}_0' \) is non-homogeneous.

**Proof** Fix some \( \lambda \in k \) with \( \psi(\lambda, x_1) \neq 0 \), \( \mathcal{L}' = \{\mu \mid \psi(\lambda, \mu) \neq 0\} \subseteq k \) is a co-finite subset. Since \( \phi(\lambda) \neq 0 \), a series of regularizations \( a_i \mapsto \emptyset, w_i = 0 \) for \( i = 1, \ldots, n_1 - 1 \) and a loop mutation \( \alpha_a \mapsto (x_1) \) can be made in Formula (3.3-3). Thus an induced bocs \( \mathcal{B}' \) of \( \mathcal{B} \) is obtained. Furthermore, since \( f_{ij}(\lambda, x_1, x_1) \) are invertible in \( k[x_1, \psi(\lambda, x_1)^{-1} \psi(\lambda, x_1)^{-1}] \) for \( n_1 < i < n \), after regularizations \( a_i \mapsto \emptyset, w_i = 0 \) in Formula (3.3-4), an induced minimal local bocs \( \mathcal{B}_\lambda \) is obtained and an induced functor \( \vartheta_1 \) as well.

\[
\mathcal{B}_\lambda : x_1 \xrightarrow{1} X, \quad R_\lambda = k[x_1, \psi(\lambda, x_1)^{-1}], \quad \vartheta_1 : R(\mathcal{B}_\lambda) \rightarrow R(\mathcal{B}).
\]

Set \( \vartheta_2 : R(\mathcal{B}) \rightarrow R(\mathcal{B}'_0) \) and \( \vartheta = \vartheta_2 \vartheta_1 : R(\mathcal{B}_X) \rightarrow R(\mathcal{B}'_0) \).
Let $S'_μ \in R(\mathfrak{B}_λ)$ for any $μ \in \mathcal{L}'$ be given by $(S'_μ)_X = k$ and $S'_μ(\xi_1) = (μ)$. If $\mathfrak{B}^0$ is homogeneous, then there exists a co-finite subset $\mathcal{L}' \subseteq \mathcal{L}'$, such that $\{φ(S'_μ) : μ \in \mathcal{L}'\}$ is a family of homogeneous iso-classes. By Corollary 3.2.4, there is an almost split conflation $(e'_μ)$ in $R(\mathfrak{B}_λ)$, such that $φ(e'_μ)$ is almost split in $R(\mathfrak{B}^0)$. Fix any $μ \in \mathcal{L}'$, and consider the conflation $(e'_μ) = φ(S'_μ) : S_μ \xrightarrow{i} E_μ \xrightarrow{π} S_μ$ in $R(\mathfrak{B})$, where $(E_μ)_X = k^2, E_μ(\xi_1) = λJ_2, E_μ(\xi_1) = J_2(μ), E_μ(\xi_1) = (0), i \neq n_1$. Then $(e'_μ)$ is almost split by Lemma 3.2.2 (ii).

2) Define a representation $L$ of $\mathfrak{B}$ given by $L_X = k^2, L(\xi_1) = J_2(μ), L(\xi_1) = μI_2$ and $L(\xi_1) = 0$ for $i \neq n_1$. $L$ is well defined, in fact, if we make an unraveling for $x \mapsto J_2(λ)$, then by Lemma 3.2.6 (ii), after a sequence of regularizations $(a_{i11} a_{i12})$ (splitting from $a_1\mapsto 0$) and $(a_{i11} a_{i12})$ (splitting from $w_i = 0$ for $i = 1, \cdots, n_1 - 1$, an induced boces with $s^0((a_{i11} a_{i12})) = 0$ is obtained. Let $g : L \to S_μ$ be a morphism with $g_x = (l_0) \oplus g(v_j) = (0)$ for all possible $j$. It is obvious that $g$ is not a retraction.

3) There exists a lifting $g : L \to E_λ$ with $gπ = g$:

\[
\begin{array}{ccc}
L & \xrightarrow{\muI_2} & E_λ \\
\muI_2 & \downarrow & \downarrow μI_2 \\
J_2(λ) & \xrightarrow{j} & J_2(μ)
\end{array}
\]

Since $g_π = g_π = X, g_π = (l_0)$. On the other hand, $(x - y)^2 \mid f_1(x, y)$ for $w = 0$ and $(J_2(λ) - λI_2)^2 = 0$, hence $f_1(L(λ), λI_2) = 0$. $g : L \to E_λ$ being a morphism implies that $L(a_1)g_π = g_π = 0 \oplus (l_0), (l_0) = (0)$, and $(l_0) = 0 = (0)$. The contradiction shows that $\mathfrak{B}^0$ is non-homogeneous.

**Proposition 3.4.5** Let the boces $\mathfrak{B}^0$ have induced boces $\mathfrak{B}^s, \mathfrak{B}^c, \mathfrak{B}^c$ given in Formula (3.2-3) and satisfying the condition (i)–(iii) stated between Corollary 3.2.4 and 3.2.5. Then $\mathfrak{B}^0$ is non-homogeneous.

**Proof** Suppose $\mathfrak{B} = \mathfrak{B}^c$, where $T = \{X, Y\}, R_X = k[x, φ(x)^{-1}], R_Y = kY$, the layer $L = (R; \omega; a_1, \cdots, a_n; v_1, \cdots, v_m)$ and $a_1 : X \to Y, δ(a_1) = 0$:

\[
\begin{array}{ccc}
X & \xrightarrow{a_1} & Y \\
\end{array}
\]

Making a reduction given by $a_1 \mapsto (1)$ of Proposition 2.2.7, an induced local boces $\mathfrak{B}' = \mathfrak{B}^c$ is obtained by a sequence of regularizations with $T' = \{Z\}, R' = k[z, φ'(z)^{-1}]$. Set the induced functors $φ_1 : R(\mathfrak{B}') \to R(\mathfrak{B}), φ_2 : R(\mathfrak{B}) \to R(\mathfrak{B})$, and $φ = φ2φ1 : R(\mathfrak{B}') \to R(\mathfrak{B})$.

1) For any $λ \in k, φ'(λ) ≠ 0$, there is an object $S'_λ \in R(\mathfrak{B}')$ with $(S'_λ)_Z = k, S'_λ(λ) = (λ)$. If $\mathfrak{B}^0$ is homogeneous, then there exists a co-finite subset $\mathcal{L} \subseteq k \setminus \{μ \mid φ'(μ) = 0\}$, such that $\{φ(S'_μ) : μ \in \mathcal{L}\}$ is a family of homogeneous iso-classes. Fix any $λ \in \mathcal{L}'$, there is an almost split conflation $(e'_λ) : S'_λ \xrightarrow{i} E'_λ \xrightarrow{π} S'_λ$ in $R(\mathfrak{B}')$ with $E'_λ(λ) = J_2(λ)$, such that $φ(e'_λ)$ is an almost split conflation in $R(\mathfrak{B})$ by Corollary 3.2.4. Let $(e_λ) = φ_1(e'_λ) : S_λ \to E_λ \to S_λ$ is an almost split conflation in $R(\mathfrak{B})$ by lemma 3.2.2 (ii) inductively, where $S_λ : (λ) \xrightarrow{l_1} k \xrightarrow{l_1} k, E_λ : J_2(λ) \xrightarrow{k^2} I_2 \xrightarrow{k^2}$.

2) Define an object $L \in R(\mathfrak{B})$ with $L_X = k^2, L_Y = 0, L(\xi_1) = λI_2, L(\xi_1) = 0, 1 ≤ i ≤ n$. Define a morphism $g : S_λ \to L$ with $g_x = (0), g_y = 0$, and $g(v) = 0$ for any dotted arrow $v$. It is claimed that $g$ is not a retraction. Otherwise, if there is a morphism $h : L \to S_λ$ with
\( gh = id_{S_\lambda}, \) then \( h_X = \binom{\nu}{\lambda} \). Since \( h_X S_\lambda(a_1) = L(a_1)h_Y \), there is \( \binom{\nu}{\lambda}(1) = 0 \), a contradiction.

3) There exists an extension \( \tilde{g} : E_\lambda \to L \) with \( \iota \tilde{g} = g \), which implies \( \tilde{g}_X = \binom{a}{b} \). Since \( \tilde{g} \) is a morphism, there is \( E(x)\tilde{g}_X = \tilde{g}_X L(x) \), i.e. \( \binom{01}{00} = \binom{a}{b} \), \( \binom{01}{00} = \binom{0}{0} \), a contradiction. Therefore \( \mathcal{B}^0 \) is not homogeneous. \( \square \)

**Remark 3.4.6** For the sake of convenience, the following notation on MW5 is used. Let \( (\mathfrak{A}, \mathcal{B}) \) be a pair with \( R \) being trivial, and \( (\mathfrak{A}', \mathcal{B}') \) be an induced pair of \( (\mathfrak{A}, \mathcal{B}) \) given by a sequence of reductions in the sense of Lemma 2.3.2. Suppose \( (\mathfrak{A}', \mathcal{B}') \) is local, and by calculating a series of triangular formulae according to Subsection 3.3, an induced pair

\[
(\mathfrak{A}_{(\lambda_0, \lambda_1, \cdots, \lambda_{l-1})}, \mathcal{B}_{(\lambda_0, \lambda_1, \cdots, \lambda_{l-1})})
\]  

(3.4-2)

is obtained. Suppose in addition, the pair (3.4-2) is in the case of MW5, which satisfies Formula (3.3-3)–(3.3-4) with the polynomials \( \phi(x) \) and \( \psi(x, x_1) \) given by Formulae (3.3-8) and (3.3-9).

Denote the pair (3.4-2) by \( (\mathfrak{A}^{s+1}, \mathcal{B}^{s+1}) \). It means that \( (\mathfrak{A}^s, \mathcal{B}^{s}) \), with the first arrow \( a_1^s \) and \( \delta(a_1^s) = 0 \), is obtained by a sequence of reductions in the sense of Lemma 2.3.2 starting from \( (\mathfrak{A}, \mathcal{B}) \). After making a loop mutation \( a_1^s \mapsto (x) \), there is an induced pair \( (\mathfrak{A}^{s+1}, \mathcal{B}^{s+1}) \) with \( R^{s+1} = k[x] \). Removing, by regularizations, the pairs of arrows \( (a_1^s, w_1^s); \cdots; (a_{n_1-1}^s, w_{n_1-1}^s) \) from \( \mathcal{B}^{s+1} \) in Formula (3.3-3), an induced pair say \( (\mathfrak{A}^t, \mathcal{B}^t) \) under the assumption \( x \mapsto (\lambda) \) is obtained. Denote the solid arrows of \( \mathcal{B}^t \) by \( a_{j+1}^t \), \( j = 0, \cdots, n - n_1 \), then \( a_1^t \) obtained from \( a_{n_1} \), is the first arrow of \( \mathcal{B}^t \). The bocs \( \mathcal{B}^t \) is also said to be in the case of MW5.

**4. One-sided pairs**

In this section, some special quotient problems of matrix bimodule problems are defined. The purpose is to deal with the most difficult part of the proof of the main theorem in the subsections 5.4–5.5.

**4.1 Definition of one-sided pairs**

We give the definition of one-sided pairs; then consider the induced pairs after some reductions via pseudo formal equations in this subsection.

Let \( \mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0) \) be a matrix bimodule problem with \( \mathcal{T} \) being trivial, let \( \mathfrak{C}, \mathcal{B} \) be the associated bi-comodule problem and bocs respectively. Suppose a sequence of pairs

\((\mathfrak{A}, \mathcal{B}), (\mathfrak{A}^1, \mathcal{B}^1), \cdots, (\mathfrak{A}^r, \mathcal{B}^r)\)
is given by reductions in the sense of Lemma 2.3.2. Assume that the leading position of the first base matrix $A_i^p$ of $\mathcal{M}_i^p$ is $(p^r, q^r)$ contained in the $(p,q)$-th leading block of of $A_1$ of $\mathcal{M}$ partitioned under $\mathcal{T}$. It is further assumed that $d_1, \cdots, d_m$ are the first $m$ solid arrows of $\mathcal{B}^r$, which locate at the $p^r$-th row of the formal product $\Theta^r$, such that $d_m$ is sitting at the last column of the $(p,q)$-block, see the picture below:

\begin{equation}
\begin{array}{cccc}
d_1 & \cdots & \cdots & d_m \\
\end{array}
\end{equation}

(4.1-1)

Recalling the statement between Formula (1.2-3) and (1.2-4), the quotient problem $(\mathfrak{A}^r)^{[m]} = (R^r, K^r, (\mathcal{M}^r)^{[m]}, H^r)$ of $\mathfrak{A}$; the sub-bi-module problem $(\mathcal{C}^r)^{[m]} = (R^r, C^r, (\mathcal{N}^r)^{[m]}, \partial |_{(\mathcal{N}^r)^{[m]}})$ of $\mathfrak{C}$ with the quasi-basis $d_1, \cdots, d_m$ of $(\mathcal{N}^r)^{[m]}$, and the sub-bocs $(\mathfrak{B}^r)^{[m]}$ of $\mathfrak{B}^r$, a quotient-sub-pair $((\mathfrak{A}^r)^{[m]}, (\mathfrak{B}^r)^{[m]})$ is obtained, and it is denoted by $(\overline{\mathfrak{A}}, \overline{\mathfrak{B}})$ for simplicity.

Denote a set of integers by $\mathcal{T} = \mathcal{T}_R \times \mathcal{T}_C \subseteq \mathcal{T}^r$, where $\mathcal{T}_R = \{0\}$ and $\mathcal{T}_C = \{1, 2, \cdots, m\}$ are the row and column indices of $(d_1, d_2, \cdots, d_m)$ respectively. A representation of size vector $\underline{a} = (n_0; n_1, \cdots, n_m)$ over $\overline{\mathfrak{A}}$ can be written as $\overline{M} = (\overline{M}_1, \cdots, \overline{M}_m)$, where $\overline{M}_i$ is an $n_0 \times n_i$-matrix over $k$. But a morphism between two representations must be discussed returning back to the category $R(\mathfrak{A}^r)$. Moreover, if $\overline{\mathfrak{A}}$ is any induced pair of $\mathfrak{A}$, a pseudo functor $\overline{\mathfrak{f}}$ can be considered acting on the objects over $\overline{\mathfrak{A}}$. Recall the formal equation of the pair $(\overline{\mathfrak{A}}^r, \mathfrak{B}^r)$:

\[(\sum_{Y \in \mathcal{T}^r} e_Y^r \ast E_Y^r)(\sum \alpha_i^r A_i^r) = \left(H^r(\sum_j v_j^r \ast V_j^r) - (\sum_j v_j^r \ast V_j^r)H^r\right) + \left((\sum \alpha_i^r A_i^r)(\sum_{Y \in \mathcal{T}^r} e_Y^r \ast E_Y^r + \sum_j v_j^r \ast V_j^r) - (\sum_j v_j^r \ast V_j^r)(\sum \alpha_i^r A_i^r)\right)\]

Let $d_i : X \to Y_i$ (possibly $Y_i = X$, or $Y_i = Y_j$ for $i \neq j$). Then the $(p^r, q^r), \cdots, (p^r, q^r + m - 1)$-th equations of the formal equation of $(\mathfrak{A}^r, \mathfrak{B}^r)$ can be rewritten as:

\[e_X(d_1, d_2, \cdots, d_m) = \begin{pmatrix} w_{11} & \cdots & w_{1m} \\
\vdots & \ddots & \vdots \\
\end{pmatrix}^t = \begin{pmatrix} e_{Y_1} & \cdots & e_{Y_m} \end{pmatrix}^t, \quad (4.1-2)\]

**Remark 4.1.1** We give some explanation on the notations of Formula (4.1-2).

(i) $e_X$ is the $(p^r, p^r)$-th entry of the formal product $\sum_{Y \in \mathcal{T}^r} e_Y^r \ast E_Y^r$; and $e_{Y_\xi}$ the $(q^r + \xi - 1, q^r + \xi - 1)$-th entry of that for $\xi = 1, \cdots, m$.

(ii) For $\xi = 1, \cdots, m$, $w_\xi$ is the $(p^r, q^r + \xi - 1)$-th entry of $H^r(\sum_j v_j^r \ast V_j^r) - (\sum_j v_j^r \ast V_j^r)H^r$.

In fact, $w_\xi = \sum_j \alpha_\xi^r v_j^r$, where $s(v_j^r) \ni p^r, t(v_j^r) \ni q^r + \xi - 1, \alpha_\xi^r \in k$.

(iii) For $1 \leq \eta < \xi \leq m$, $w_{\eta \xi}$ is the $(q^r + \eta - 1, q^r + \xi - 1)$-th entry of $\sum_j v_j^r \ast V_j^r$. In fact, $w_{\eta \xi} = \sum_j \beta_{\eta \xi} v_j^r$ where $s(v_j^r) \ni q^r + \eta - 1, t(v_j^r) \ni q^r + \xi - 1, \beta_{\eta \xi} \in k$.

The differential of $d_1, \cdots, d_m$ can be read off from Formula (4.1-2). Note that in each monomial of the differential, there exists at most one solid arrow multiplying the the dotted one from the left:

\[\delta(d_i) = w_i + \sum_{j < i} d_j w_{ij}, \quad 1 \leq i \leq m. \quad (4.1-3)\]

**Definition 4.1.2** A bocs $\mathfrak{B}$ with a layer $L = (R; \omega; a_1, \cdots, a_m, b_1, \cdots, b_n; w_j, \bar{w}_j, \bar{u}_j, \bar{v}_j)$ is said to be one-sided, provided that $R$ is trivial and $\mathcal{T} = \{X, Y_1, \cdots, Y_h\}$; the solid arrows
$a : X \to Y, b : X \to X$, and the dotted arrows are divided into four classes: $\tilde{u} : X \to X$, $\tilde{v} : Y \to Y', \bar{u} : X \to X, \bar{v} : X \to Y$; the differentials of the solid arrows are given by

$$
\delta(a_i) = \sum_j \alpha_{ij} u_j + \sum_{i' < i,j} \beta_{i'j} a_{i'} u_j + \sum_{b_i < a_{i,j}} \gamma_{i'j} b_i v_j,
$$

$$
\delta(b_i) = \sum_j \lambda_{ij} v_j + \sum_{a_j < b_{i,j}} \mu_{i'j} a_i v_j + \sum_{i' < i,j} \nu_{i'j} b_i v_j,
$$

with the coefficients $\alpha_{ij}, \beta_{i'j}, \gamma_{i'j}, \lambda_{ij}, \mu_{i'j}, \nu_{i'j} \in k$.

The associated bocs $\mathfrak{B}$ of $\mathfrak{A}$ is one-sided by Formula (4.1-3), and $(\mathfrak{A}, \mathfrak{B})$ is called a one-sided pair.

Let $(\mathfrak{A}, \mathfrak{B})$ be any pair, $(\mathfrak{A}^{[h]}, \mathfrak{B}^{(h)}), h \geq 1$, be a quotient-sub-pair. Note that if the reduction $(\mathfrak{A}, \mathfrak{B})$ is made with respect to an admissible $R^rR$-bimodule $L$ by Proposition 2.2.1–2.2.7, then $L$ is also the admissible bimodule of $(\mathfrak{A}^{[h]}, \mathfrak{B}^{(h)})$. Thus there are two sequences of reductions as below, such that $(\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1})$ and $(\mathfrak{A}^{r+i+1}, \mathfrak{B}^{r+i+1})$ are obtained respectively from $(\mathfrak{A}^i, \mathfrak{B}^i)$ and $(\mathfrak{A}^{r+i}, \mathfrak{B}^{r+i})$ by the same admissible bimodule, or by the same regularization 2.1.8 for $0 \leq i < s$:

$$(\mathfrak{A}, \mathfrak{B}), (\mathfrak{A}^i, \mathfrak{B}^i), \ldots, (\mathfrak{A}^s, \mathfrak{B}^s); (\mathfrak{A}^r, \mathfrak{B}^r), (\mathfrak{A}^{r+1}, \mathfrak{B}^{r+1}), \ldots, (\mathfrak{A}^{r+s}, \mathfrak{B}^{r+s}).$$

Let $\mathfrak{A}^i = (\mathfrak{A}^{i+1}[m^i]) = (R^{r+i}, K^{r+i}, \mathcal{M}^i, F^i)$ be a quotient problem with $m^i$ being the number of the base matrices of $\mathcal{M}^i$, and the associated sub-bi-comodule problem $\mathfrak{C}^i = (\mathfrak{C}^{r+i}[m^i]) = (R^{r+i}, C^{r+i}, \mathcal{N}^i, \hat{\mathcal{O}}^i)$. A formula of the pair $(\mathfrak{A}^i, \mathfrak{B}^i)$ is written as follows for $i = 0, 1, \ldots, s$:

$$
e^i_X (F^i + \hat{\Theta}^i) = (W_1, \ldots, W_m) + (F^i + \hat{\Theta}^i)
$$

$$
\begin{pmatrix}
\varepsilon_{y_1}^i & W_{12} & \cdots & W_{1m} \\
\varepsilon_{y_2}^i & \cdots & W_{2m} \\
\vdots & \ddots & \ddots & \ddots \\
\varepsilon_{y_m}^i & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
$$

where the diagonal parts of $e^i_X$ and the most right matrix in Formula (4.1-6) can be viewed as a reduced formal product $\bar{Y}^i$ of $(\mathcal{K}_0, \mathcal{C}_0)$. $\Theta^i$ is the formal product of $(\mathcal{M}_i^1, \mathcal{N}_i^1)$ containing the solid arrows splitting from $d_1, \ldots, d_m$; $F^i + \hat{\Theta}^i$ is a $(1 \times m)$-partitioned matrix under $\mathcal{T}$ with a size vector $n^i = (n^i_0; n^i_1, \ldots, n^i_m)$, and $F^i$ is sitting in the blank part in the picture:

$F^i + \hat{\Theta}^i =

\begin{pmatrix}
d_0 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \cdots & \ddots & \ddots & \ddots \\
\ddots & \cdots & \cdots & \cdots & \cdots \\
d_m & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
$

Reductions are performed according to Theorem 2.4.4. More precisely, the system $\bar{F}^{ri}$ of Formula (2.4-5) for the pair $(\mathfrak{A}^{r+i}, \mathfrak{B}^{r+i})$ can be written as $\bar{F}^i$ below, which is said to be the reduced defining system of the pair $(\mathfrak{A}^i, \mathfrak{B}^i)$.

$$
\bar{F}^i : \bar{\Psi}^i_{\mu} F^i \equiv (\ell^{p,q}_i) \bar{\Psi}^m_{\mu} + F^i \bar{\Psi}^r_{\mu},
$$

(4.1-7)
where the upper indices \(l, m, r\) are used to show the left, middle and right parts of the variable matrix \(\Psi_m\); \(\Psi_m^l\) is the strict upper triangular part of \(e^l_X\); \(\Psi_m^r\) is that of the most right matrix in Formula (4.1-6); and \(\Psi_m^m = (W_1, \ldots, W_m)\). Namely, \(\tilde{F}\) is obtained from Formula (4.1-6) by removing the term \(\tilde{\Theta}\) and the diagonal parts of the most left and right matrices. The strict upper triangular parts of \(e^l_X\) and \(e^i_{ij}\) for \(1 \leq j \leq m\) are constructed inductively in Proof 2) of Theorem 2.4.4; while \(W_h\) and \(W_{hl}\) are the splitting of \(w_h, w_{hl}\).

Since it is difficult to calculate the dotted arrows after a reduction, the linear relation of the dotted elements of \(\Psi\) appearing in the reduction will be described instead.

On the other hand, \(\Pi^i = \Psi_m^i\) may be said to be a \textit{pseudo formal product} of \((K^1, C^1)\); and Formula (4.1-6) a \textit{pseudo formal equation of the pair} \((\Xi', \Psi')\), since the entries of \(\Pi^i\) are dotted elements with some linear relations. The reason that we borrowed the concept of “formal product” was that it is possible to read off the differential of the solid arrows from Formula (4.1-6) according to Theorem 1.4.2. For example for the first arrow \(d^i_{lpq}\):

\[
-\delta(d^i_{lpq}) = w^i_{lpq} + \sum_{j<l} a^i_{jq} w^j_{lpq} + \sum_{q<p} d^i_{pq} w^i_{pq} - \sum_{q>p} w^i_{pq} d^0_{iqq};
\]

(4.1-8)

where \(d^i_{pq}\) is split from \(d_i : X \to Y\), \(d^i_{pq}\) is the \((p, q)\)-th entry obtained from \(d_j\) in \(F_i\).

\textbf{Remark 4.1.3} Let \((\Xi'', \Psi'')\) be any induced pair of \((\Xi, \Psi)\) after several reductions in the sense of Lemma 2.3.2. And \((\Xi'', \Psi'')\) is an induced pair of \((\Xi', \Psi')\) given by one of three reductions of 2.3.2.

(i) If there is a linear relation of dotted elements \(\sum_j u_j = 0\) in \(\Psi_m''\), then \(\sum_j \bar{u}_j = 0\) in \(\Psi_m''\) with \(\bar{u}_j\) being the split of \(u_j\).

(ii) Suppose \(a'_1\) is the first arrow of \(\Psi_m''\), then \(\delta(a'_1) = v + \sum_j \alpha_j u_j\), where \(v, u_j\) are dotted elements of \(\Psi_m''\). If \(v\) is a dotted arrow, and \(v \notin \{u_j\}\), then \(\delta(a'_1) \neq 0\).

(iii) Set \(a'_1 \mapsto \emptyset\) in (ii) by a regularization, it is said that \(v\) is replaced by \(-\sum_j u_j\) in \(\Psi_m''\).

(iv) If we are able to determine that a dotted element \(v\) of \(\Psi_m''\) is linearly independent of all the others, then \(v\) is said to be a \textit{dotted arrow preserved in} \(\Psi_m''\).

\section*{4.2 Differentials in one-sided pairs}

In this subsection, the classification of local one-sided bocses is discussed; and the differentials of the solid arrows of non-local bocses are calculated.

For the sake of simplicity, the black letter \(\delta\) is used instead of the symbol “\(-\delta\)”, see Formulae (4.1-3) and (4.1-8), up to the end of the whole section 4. First of all, Classification 3.3.5 gives the following Classification on local one-sided bocses. The forms in one-sided case are much simpler than those in general case.

\textbf{Classification 4.2.1} Let \(\tilde{\Xi}\) be a local one-sided boc with a layer (letter \(a\) is changed to the letter \(b\)):

\[L = (R; \omega; b_1, \ldots, b_n; v_1, \ldots, v_m).\]

(i) \(\tilde{\Xi}\) with \(R = k_1X\) has differentials by Formula (3.3-5) after some base changes:

\[\delta^0(b_1) = \bar{u}_1, \delta^0(b_2) = \bar{u}_2, \ldots, \delta^0(b_n) = \bar{u}_n.\]

(ii) \(\tilde{\Xi}\) with \(R = k_1X\) has some integer \(1 \leq n_0 < n\), such that the differentials are

\[\begin{align*}
\delta^0(b_i) &= \bar{u}_i, \quad 1 \leq i < n_0; \\
\delta^0(b_{n_0}) &= \sum_{i=1}^{n_0-1} f_{n_0i} \bar{u}_i, \ f_{n_0i} \in k; \\
\delta^1(b_i) &= \sum_{j \neq n_0} f_{ij}(b_{n_0}) \bar{u}_j + f_{ii} \bar{u}_i, \quad \text{with respect to} \ b_{n_0},
\end{align*}\]

where, \(n_0 < i < n\), \(f_{ij}(b_{n_0}) = \beta_{ij}^0 + \beta_{ij}^1 b_{n_0} \in k[b_{n_0}]\) for \(i < j\), \(f_{ii} \in k^*\). (4.2-1)
with $R = k[x]$ is given by a sequence of regularizations, followed by $a_{n_0} \mapsto x$ in Formula (3.3-6). For the sake of simplicity, Formulae (3.3-2) and (3.3-3) are written in a unified form:

$$
\begin{align*}
\delta^0(b_1) &= f_{11}(x)\bar{u}_1, \\
\delta^0(b_2) &= f_{21}(x)\bar{u}_1 + f_{22}(x)\bar{u}_2, \\
&\quad \vdots \\
\delta^0(b_t) &= f_{t1}(x)\bar{u}_1 + f_{t2}(x)\bar{u}_2 + \cdots + f_{tt-1}(x)\bar{u}_{t-1} + f_{tt}(x)\bar{u}_t,
\end{align*}
$$

(4.2-2)

where $f_{ij}(x) = \alpha_{ij}^0 + \alpha_{ij}^1 x \in k[x]$ for $1 \leq i \leq j \leq t$, and $f_{ii}(x) \neq 0$ for $1 \leq i < t$.

(iii) $\mathfrak{B}$ has an induced bocs $\mathfrak{B}_{(\lambda_0, \ldots, \lambda_t)}$ with Formula (4.2-2), where $t = n, f_{nn}(x) \neq 0$; $\phi(x) = 1$ in Formula (3.3-7), and there is some minimal $1 \leq s \leq n$, such that $f_{ss}(x) \in k[x] \setminus k$ is non-invertible in $k[x]$. It is in the case of MW3.

(iv) $\mathfrak{B}$ has an induced bocs $\mathfrak{B}_{(\lambda_0, \ldots, \lambda_{t-1})}$ with Formula (4.2-2), where $t = n < n$, $f_{n1,n1}(x) = 0$ and $\phi(x) = \prod_{i=1}^{n-1} f_{ii}(x)$ in Formula (3.3-8). Denoting $b_{n1}$ by $x_1$, Formula (3.3-4) shows:

$$
\begin{align*}
\delta^1(b_{n1+1}) &= K_{n1+1} + f_{n1+1,n1+1}(x)\bar{u}_{n1+1}, \\
\delta^1(b_{n2}) &= K_{n1+2} + f_{n1+1,n1+1}(x,x_1)\bar{u}_{n1+1} + f_{n1+2,n1+2}(x)\bar{u}_{n1+2}, \\
&\quad \vdots \\
\delta^1(b_n) &= K_n + h_{n,n-1}(x,x_1)\bar{u}_{n1} + \cdots + f_{nn-1}(x,x_1)\bar{u}_{n-1} + f_{nn}(x)\bar{u}_n,
\end{align*}
$$

(4.2-3)

where $f_{ij}(x,x_1) \in k[x,x_1]$ for $i < j$, $f_{ii}(x) \neq 0$; and the polynomial $\psi(x) = \phi(x) \prod_{i=1}^{n-1} c_i(x) f_{ii}(x)$ is given by Formula (3.3-9). It is in the case of MW4.

Note in particular, that MW5 never occurs in one-sided case.

**Proof** The one-sided bocs of (i) is finite type, and that of (ii) is tame infinite.

(iii) There is no localization needed in Formula (4.2-2), thus $c_i(x) = 1$. It is clear that $h_{ii}(x,x) = 1$ in Formula (3.3-1), therefore $\phi(x) = 1$. Finally, any non-zero and non-invertible polynomial of $k[x]$ belongs to $k[x] \setminus k$.

(iv) Suppose the last term of the formulae in (4.2-3) are $0 \neq f_{ii}(x,x_1) = f_{ii}(x)h_{ii}(x,x_1)$ with $h_{ii}(x,x_1) \in k[x,x_1,\psi(x)^{-1}] \setminus k[x]$ or $h_{ii}(x,x_1) = 1$. If there exists a minimal integer $n_1 < s \leq n$, such that $k_{ss}(x,x_1) \notin k[x]$, then for any $\lambda \in k$ with $\psi(\lambda) \neq 0$, the induced bocs $\mathfrak{B}_\lambda$ given by $x \mapsto (\lambda)$ returns to case (iii). Therefore $f_{ii}(x,x_1) = f_{ii}(x) \in k[x]$ for $n_1 < i \leq n$.

From now on, a general one-sided pair $(\mathfrak{A}, \mathfrak{B})$ with $|T| > 1$ is dealt with. If $\mathfrak{B}_X$, the induced local bocs of $\mathfrak{B}$, is in case (iii) or (iv) of Classification 4.2.1, then $\mathfrak{B}_X$ is wild and non-homogeneous, so is $\mathfrak{B}$. Since $\mathfrak{B}_X$ in case (i) of 4.2.1 is relatively simple, the discussion below is concentrated on $\mathfrak{B}_X$ given by Formula (4.2-1) in case (ii) of 4.2.1. Denote the solid edges of $\mathfrak{B}$ before $b_{n0}$ by $a_1, \ldots, a_h$. The differential $\delta^0$ acting on $a_i$’s has two possible expressions. First,

$$
\delta^0(a_1) = \varpi_1, \ldots, \delta^0(a_h) = \varpi_h.
$$

(4.2-4)

Second, there exists some $1 \leq h_1 < h$, such that $\delta^0(a_1) = \varpi_1, \ldots, \delta^0(a_{h_1-1}) = \varpi_{h_1-1}$, but $\delta^0(a_{h_1}) = \sum_{j=1}^{h_1-1} \alpha_{h_1,j}\varpi_j$. Inductively, there are two subsets $\{h_1, \ldots, h_s\} \subseteq \{1, \ldots, h\}$, and $\Lambda = \{1, \ldots, h\} \setminus \{h_1, \ldots, h_s\}$, such that

$$
\begin{align*}
\delta^0(a_i) &= \varpi_i, & i \in \Lambda; \\
\delta^0(a_{h_1}) &= \sum_{j \in \Lambda, j < h_1} \alpha_{h_1,j}\varpi_j, & l = 1, \ldots, s.
\end{align*}
$$

(4.2-5)

**Convention 4.2.2** Suppose $\mathfrak{B}$ is a one-sided bocs with $\mathfrak{B}_X$ given by Formula (4.2-1). All the loops $b_1, \ldots, b_n$ at $X$ are called $b$-class arrows, where the loop $b = b_{n0}$ is said to be effective or $b$-class, the others are non-effective. The edges $a_1, \ldots, a_h$ before $b$ are called $a$-class arrows, where $\{a_i = a_{h_i} \mid 1 \leq i \leq s\}$ are said to be effective or $\bar{a}$-class, the others are non-effective. Let $c_1, c_2, \ldots, c_l$ be the solid edges after $\bar{b}$, which are called $c$-class arrows, and they are effective.
A solid arrow splitting from one of the classes $a, \bar{a}, b, \bar{b}, c,$ or a dotted element splitting from a dotted arrow of $\bar{u}, v, \bar{u}, \bar{v},$-classes in Picture (4.1-4) are said to be in the same class.

A special case of the differential $δ^1$ with respect to $\bar{b}$ on $c$-class arrows is given by

$$
\begin{align*}
δ^1(c_i) &= \sum_{j \in A} \gamma_{ij}(\bar{b})v_j + \gamma_{l,h+1}(\bar{b})\bar{w}_{h+1}, \\
& \quad \ldots \quad \ldots \\
δ^1(c_t) &= \sum_{j \in A} \gamma_{tj}(\bar{b})v_j + \gamma_{l,h+1}(\bar{b})\bar{w}_{h+1} + \ldots + \gamma_{t,h+t}(\bar{b})\bar{w}_{h+t},
\end{align*}
$$

(4.2-6)

where $\{v_j\}_{j \in A} \cup \{\bar{w}_{j+t}\}_{1 \leq j \leq t}$ are dotted arrows, $\gamma_{i,h+i}(\bar{b}) \neq 0$ for $1 \leq i \leq t$.

**Lemma 4.2.3** Let $\mathcal{B}$ be a one-sided bocs with $\mathcal{B}_X$ being in case (ii) of Classification 4.2.1. If Formula (4.2-6) fails, i.e. there exists some minimal $1 \leq l \leq t$ with $γ_{l,h+l}(\bar{b}) = 0$, then $\mathcal{B}$ is non-homogeneous.

**Proof** It is proceed with a sequence of regularizations: $a_j \mapsto \emptyset, \bar{u}_j = 0$ for $1 \leq j < n_0$, and edge reductions $\bar{a}_{h_i} \mapsto (0)$ for $i = 1, \ldots, s$. Then after a loop mutation $\bar{b} \mapsto (x)$, and defining a polynomial $φ(x) = \prod_{i=1}^{l-1} γ_i,h+i(x)$ by Formula (4.2-6), an induced pair $(\mathcal{A}', \mathcal{B}')$ is obtained, such that $R_X' = k[x, φ(x)^{-1}]$ and $\mathcal{B}'_X$ is minimal. Without loss of generality, suppose $T' = \{X, Y\}$ with $R_Y = k$. Making regularizations $c_i \mapsto \emptyset, c'_{h+i} = 0$ for $1 \leq i < l$, an induced bocs $\bigcup_{i} c_i$ of two vertices with $δ(c_i) = 0$ follows, which is in the case of MW1. Thus $\mathcal{B}'$, consequently $\mathcal{B}$, are wild and non-homogeneous. $\square$

Let $\bar{δ}$ be obtained from the differential $δ$ by removing all the monomial involving any non-effective $a, b$-class solid arrows. Now $\bar{δ}$ acting on all $a, b, c$-class arrows is written in the following three formulæ:

$$
\begin{align*}
\bar{δ}(a_i) &= \bar{v}_i + \sum_{h < i} \bar{a}_i (\sum_{j} e_{ij} \bar{u}_j), \quad i \in \Lambda; \\
\bar{δ}(a_{τ}) &= \sum_{j \in h,τ} \bar{a}_{τj} \bar{v}_j + \sum_{i = τ}^{s} \bar{a}_i (\sum_{j} \epsilon_{ij} \bar{u}_j), \quad 1 \leq τ \leq s. \\
\bar{δ}(b_i) &= \bar{u}_i + \sum_{i < h, b_i} \bar{a}_i (\sum_{j} \epsilon_{ij} \bar{v}_j), \quad i < n_0; \\
\bar{δ}(\bar{b}) &= \sum_{j=1}^{n_0-1} \bar{β}_{ij} \bar{u}_j + \sum_{i = 1}^{s} \bar{a}_i (\sum_{j} \epsilon_{ij} \bar{v}_j) + \sum_{i < h, b_i} c_i (\sum_{j} \epsilon'_{ij} \bar{v}_j), \quad i = n_0; \\
\bar{δ}(c_{τ}) &= \sum_{j=1}^{s} \bar{a}_i (\sum_{j} \epsilon_{τij} \bar{u}_j) + \sum_{j \in h,τ} \gamma_{τj}(\bar{b}) \bar{u}_j + \sum_{j \in h} \gamma_{l,h+j}(\bar{b}) \bar{w}_j + \sum_{i = 1}^{s} c_i (\sum_{j} \epsilon'_{τij} \bar{v}_j), \quad 1 \leq τ \leq t.
\end{align*}
$$

(4.2-7)

(4.2-8)

(4.2-9)

where all the coefficients $ε, \bar{ε}, \bar{α}, \bar{ε}', \bar{α}', \bar{β}, ζ, ξ \in k$, and $β(\bar{b}) = β^n + β_{h} \bar{b}, γ(\bar{b}) = γ^n + γ_{h} \bar{b} \in k[\bar{b}]$.

### 4.3. Reduction sequences of one-sided pairs

The purpose of this subsection is tow folds: 1) present a condition on formal products, which can be preserved after some edge reductions; 2) construct a reduction sequence based on 1) starting from a non-local one-sided pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{B}_X$ given by Formula (4.2-1).

**Condition 4.3.1 (BRC)** Let $(\mathcal{A}, \mathcal{B})$ be any pair with trivial $T$.

(i) Suppose the solid arrows $D = \{d_1, \ldots, d_p\}$ and $E = \{e_1, \ldots, e_p\}$ locate in the lowest non-zero row of $\Theta$ and form the first $p+q$ arrows of $\mathcal{B}$ (not necessarily fulfilling of the whole row), such that $e_1, \ldots, e_{p-1}$ are edges starting from $X$, $e = e_p$ is a loop at $X$, and $d_i < e_p, 1 \leq i \leq q$. There exists a set of dotted arrows $U = \{u_1, \ldots, u_q\}$, whose complement in $V'$ is $W = \{w_1, \ldots, w_q\}$.

(ii) Denote by $\bar{δ}$ the part of the differential of a solid arrow in $D \cup E$ by removing all the monomials containing any solid arrow in $D$.

$$
\begin{align*}
\bar{δ}(d_i) &= u_i + \sum_{j=1}^{t} (\sum_{e_i < d_i} \lambda_{ij} e_i w_j), \quad 1 \leq i \leq q; \\
\bar{δ}(e_i) &= \sum_{d_j < e_i} \alpha_{ij} u_j + \sum_{j=1}^{t} (\sum_{i=1}^{l-1} \mu_{ij} e_i w_j), \quad 1 \leq i \leq p,
\end{align*}
$$

(4.2-6)
where the coefficients $\lambda_{ij}, \alpha_{ij}, \mu_{ij} \in k$. Then it is said that $(\mathfrak{A}, \mathfrak{B})$ satisfies the bottom row condition with respect to $(\mathcal{D}, \mathcal{U})$ and $(\mathcal{E}, \mathcal{W})$, or (BRC) for short.

From now on, we use $G(k)$ instead of the reduction block $G$ given below Lemma 2.3.2 for the sake of simplicity up to the end of Section 4, if there is not any confusion to be caused. Suppose the pair $(\mathfrak{A}, \mathfrak{B})$ satisfies (BRC) with $p, q > 1$, and the first arrow of $\mathfrak{B}$ is $a_1 : X \to Y$. Now the condition (BRC) on the induced pair $(\mathfrak{A}', \mathfrak{B}')$ is discussed.

Case (i) $a_1 = d_1$, and $d_1 \mapsto \emptyset$. Let $\mathcal{D}' = \{d_2, \ldots, d_q\}, \mathcal{U}' = \{u_2, \ldots, u_q\}$ and $\mathcal{E}' = \mathcal{W}' = \mathcal{W}$.

Case (ii) $a_1 = e_1$, and $e_1 \mapsto \emptyset$. Let $\mathcal{D}' = \mathcal{D}, \mathcal{U}' = \mathcal{U}; \mathcal{E}' = \{e_2, \ldots, e_p\}, \mathcal{W}' = \mathcal{W}$.

Denote by $\bullet_i$ a solid or dotted arrow of $\mathfrak{B}$. Suppose after a reduction below, $\bullet_i$ splits into a $2 \times 1$ or $2 \times 2$ matrix in $\mathfrak{B}'$. Then the arrows at the second row of the matrix are denoted by $\bullet_{i2}$, or $\bullet_{i21}, \bullet_{i22}$.

Case (iii) $a_1 = e_1$, and $e_1 \mapsto (1, 0)$. Let $\mathcal{D}' = \{d_{21}, d_{22} \mid t(d_i) = X \cup \{d_{j2} \mid t(d_j) \neq X\}, \mathcal{U}' = \{u_{21}, u_{22} \mid t(d_i) = X \cup \{u_{j2} \mid t(d_j) \neq X\}; \text{and } \mathcal{E}' = \{e_{22}, \ldots, e_{p-1,2}, e_{p21}, e_{p22}\}, \mathcal{W}'$ is obtained by the split of $\mathcal{W}$ and an additional dotted arrow $\nu^*_i(e_{Z(X,1)} \otimes_k f_{Z(X,2)})$ defined in Proof 2 of Proposition 2.1.5.

Case (iv) $a_1 = e_1$ and $e_1 \mapsto (0, 1)$. Let $\mathcal{D}' = \{d_{21}, d_{22} \mid t(d_i) = X \cup \{d_{j2} \mid t(d_j) \neq X, Y\}, \mathcal{U}' = \{u_{21}, u_{22} \mid t(d_i) = X \cup \{u_{j2} \mid t(d_j) \neq X, Y\}; \text{and } \mathcal{E}' = \{e_{22}, e_{p21}, e_{p22}\}, \mathcal{W}$ is obtained by the split of $\mathcal{W}$ and two additional dotted arrows $\nu^*_i(e_{Z(X,1)} \otimes_k f_{Z(X,2)}), \nu^*_1(e_{Z(Y,1)} \otimes_k f_{Z(Y,2)})$ given in Proof 2 of 2.1.5.

Lemma 4.3.2 Suppose the pair $(\mathfrak{A}, \mathfrak{B})$ satisfies (BRC) with $p, q > 1$, and the first arrow of $\mathfrak{B}$ is $a_1 : X \to Y$. Then after making a reduction $a_1 \mapsto G$ as above (i)–(iv), the induced pair $(\mathfrak{A}', \mathfrak{B}')$ satisfies (BRC) with respect to $(\mathcal{D}', \mathcal{U}')$ and $(\mathcal{E}', \mathcal{W}')$.

Proof The cases (i) and (ii) are trivial.

Suppose the reduction block $G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in (iii), resp. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in (iv), then $X, Y$ split into two vertices $X', Y'$, resp. three vertices $X', Y', Y''$:

$$e_x \mapsto e'_x = \begin{pmatrix} e_y' & w \\ e_{x'}' \end{pmatrix}, \quad e_y \mapsto e'_y = e_{y'}, \text{ or } e_y \mapsto e'_y = \begin{pmatrix} e_{y''} & w' \\ e_{y''} \end{pmatrix}. $$

In above-mentioned two cases, by condition (i) of (BRC) on $(\mathfrak{A}, \mathfrak{B})$, $e_{i2}$ or $e_{i21}, e_{i22}$ for $1 < i < p$ start at $X'$, but do not end at $X'$, because $t(e_i) \neq X'$; the edge $e_{p21} : X' \to Y'$, and the loop $e_{p22} : Y' \to X'$. Therefore the pair $(\mathfrak{A}', \mathfrak{B}')$ still satisfies (i) of (BRC). By condition (ii) of (BRC) on $(\mathfrak{A}, \mathfrak{B})$, we have

$$\delta(D_i) = U_i + \sum_{j=1}^t \lambda_{ij} GW_j + \sum_{j=1}^t (\sum_{\alpha_i < d_i} \lambda_{ij} E_i) w_j, $$

$$\delta(E_i) = \sum_{d_j < e_i} \alpha_{ij} U_j + \sum_{j=1}^t \mu_{ij} GW_j + \sum_{j=1}^t (\sum_{\alpha_i < d_i} \mu_{ij} E_i) W_j + \sum_{i=1}^q \sum_{\alpha_i < d_i} \nu^*_i(e_{Z(X,1)} \otimes_k f_{Z(X,2)}) + \sum_{i=1}^q \sum_{\alpha_i < d_i} \nu^*_1(e_{Z(Y,1)} \otimes_k f_{Z(Y,2)}) $$

where $\delta(M) = (\delta(a_{ij}))$ for $M = (a_{ij})$. Since the bottom row of $G$ is $(0)$ or $(0, 0)$, $(\mathfrak{A}', \mathfrak{B}')$ still satisfies condition (ii) of (BRC).

Lemma 4.3.3 Let $(\mathfrak{A}, \mathfrak{B})$ be a one-sided pair with $\mathfrak{B}_X$ given by Formula (4.2-1), $\mathcal{T}$ being trivial and $|\mathcal{T}| > 1$. Then $(\mathfrak{A}, \mathfrak{B})$ satisfies (BRC) with respect to the sets:

$$\mathcal{D} = \{a_i, i \in \Lambda \} \cup \{b_j \mid j < n_0\}, \quad \mathcal{U} = \{\bar{w}_i \mid i \in \Lambda \} \cup \{\bar{u}_j \mid j < n_0\}; $$

$$\mathcal{E} = \{\bar{a}_r, 1 \leq r \leq s \} \cup \{\bar{b}\}, \quad \mathcal{W} = \{\bar{w}_i \mid i \notin \Lambda \} \cup \{\bar{u}_j \mid j > n_0\} \cup \{\bar{w}, \bar{u}\text{-class arrows}. \} $$

Theorem 4.3.4 Let $(\mathfrak{A}, \mathfrak{B})$ be a one-sided pair with $\mathfrak{B}_X$ given by Formula (4.2-1), $\mathcal{T}$ being trivial and $|\mathcal{T}| > 1$. Then there exists a sequence of reductions in the sense of Lemma 2.3.2 as the first part of a sequence towards a pair $(\mathfrak{A}', \mathfrak{B}')$ in the case of $\mathcal{M}5$ given by Remark 3.4.6:

$$(\mathfrak{A}, \mathfrak{B}) = (\mathfrak{A}^0, \mathfrak{B}^0), (\mathfrak{A}^1, \mathfrak{B}^1), \ldots, (\mathfrak{A}^\gamma, \mathfrak{B}^\gamma), (\mathfrak{A}^{\gamma+1}, \mathfrak{B}^{\gamma+1}), \ldots, (\mathfrak{A}^\kappa, \mathfrak{B}^\kappa) \quad (4.3-1)$$
where $\kappa$ is the minimal index, such that the pair $(\mathfrak{A}^\kappa, \mathfrak{B}^\kappa)$ satisfies the following condition (B).

Condition (B). If a row of the formal product $\Theta^\beta$ of $(M^\beta_1, N^\beta_1)$ contains some $b$-class arrows of $\mathfrak{B}^\beta$, then the same row of $F^\beta$ contains one and only one nonzero entry which is a link in some reduction block $G^\beta_1$ of $H^\beta$ obtained by an edge reduction.

(i) For $i = 0, 1, \ldots, (\kappa - 2)$, the reduction from $\mathfrak{A}^i$ to $\mathfrak{A}^{i+1}$ is a composition of a series of reductions $\mathfrak{A}^i = \mathfrak{A}_i^0, \mathfrak{A}_i^1, \ldots, \mathfrak{A}_i^r, \mathfrak{A}_i^{r+1} = \mathfrak{A}^{i+1}$:

1. For $0 \leq j < r_i - 1$, the reduction from $\mathfrak{A}_i^j$ to $\mathfrak{A}_i^{j+1}$ is a sequence of regularizations for non-effective $a, b$-class arrows, and finally an edge reduction of the form (0) for an effective $a$ or $b$-class arrow. The reduction form $\mathfrak{A}_i^{r_i-1}$ to $\mathfrak{A}_i^{r_i}$ is a sequence of regularization for non-effective $a, b$-class arrows.

2. The first arrow $a_1^i : X^i \mapsto Y^i$ of $\mathfrak{B}_1^{i,r_i}$ is an effective $a$ or $b$-class edge with $\delta(a_1^i) = 0$. Making an edge reduction $a_1^i \mapsto (1) \text{ or } (01)$, the last term $\mathfrak{A}_i^{r_i+1} = \mathfrak{A}^{i+1}$ is obtained.

(ii) It is possible that there exist a minimal integer $\gamma$, and an index $1 \leq j \leq r_{\gamma} + 1$, such that the first arrow of $\mathfrak{B}^{\gamma,j}$ locates outside the matrix block coming from $\bar{b}$ under $T$, but the first arrow of $\mathfrak{B}^{\gamma,j+1}$ locates at the first column of the block.

(iii) The reduction from $\mathfrak{A}^{\kappa-1,0}$ to $\mathfrak{A}^{\kappa-1,r_{\kappa-1}}$ is a composition of a series of reductions given by $\text{①}$ of (i). There are two possibilities.

1. The first arrow $a_1^{\kappa-1}$ of $\mathfrak{A}^{\kappa-1,r_{\kappa-1}}$ is an effective $a$ or $b$-class solid edge with $\delta(a_1^{\kappa-1}) = 0$. Making an edge reduction $a_1^{\kappa-1} \mapsto (1)$ or $(01)$, the last term $\mathfrak{A}^{\kappa-1,r_{\kappa-1}+1} = \mathfrak{A}^\kappa$ is obtained.

2. The first arrow $a_1^{\kappa-1}$ is an effective $b$-class loop at the down-right corner of the matrix block coming from $\bar{b}$ under $T$ with $\delta(a_1^{\kappa-1}) = 0$. Making a loop reduction $a_1^{\kappa-1} \mapsto W$, a Weyr matrix over $k$, $\mathfrak{A}^{\kappa-1,r_{\kappa-1}+1} = \mathfrak{A}^\kappa$ is obtained.

Proof If there is not any $\bar{a}$-class edges, i.e. $s = 0$, then after a series of regularizations, the unique effective loop in the induced bocs becomes the first arrow with $\delta(\bar{b}) = 0$. Since the induced pair is not local, but the parameter $x$ appears only in a local pair by Remark 3.4.6, set $\bar{b} \mapsto W$ by a loop reduction of Lemma 2.3.2, the final pair $(\mathfrak{A}^1, \mathfrak{B}^1)$ satisfies $\text{②}$ of (iii) with $\kappa = 1$. Suppose $s > 0$, regularizations are made on $a_i, b_j$ before $\bar{a}_1$, the corresponding $\bar{x}, \bar{y}_j = 0$. Thus $\delta(\bar{a}_1) = 0$ by Formula (4.2-7), if $\bar{a}_1 \mapsto (0)$, $\mathfrak{A}^{0,1}$ given by $\text{①}$ of (i) is obtained. If $r_0 > 1$, repeating the procedure in $\text{①}$ of (i), $\bar{x}_0, \bar{y}_0$ is finally reached with the first arrow $a_1^0$ and $\delta(a_1^0) = 0$. If $a_1^0$ is $\bar{a}$-class and $a_1^0 \mapsto (1), (01)$, $\text{①}$ of (iii) is obtained; if $a_1^0 = b$ then $b \mapsto W$, $\text{②}$ of (iii) is obtained, and $\kappa = 1$ in both cases.

Otherwise, if $a_1^0 \mapsto (1)$ or $(01)$ in case $\text{②}$ of (i), the induced pair $(\mathfrak{A}^1, \mathfrak{B}^1)$ is obtained, which satisfies (BRC) by Lemma 4.3.2–4.3.3.

Suppose $(\mathfrak{A}^i, \mathfrak{B}^i)$ for some $i < \kappa - 1$ given in (i) has been obtained. Now we continue the reductions up to the induced pair $(\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1})$. $(\mathfrak{A}^i, \mathfrak{B}^i)$ satisfies (BRC) by Lemma 4.3.2–4.3.3 inductively. Suppose the first arrow of $\mathfrak{B}^{i,0}$ is $a_1^{i,0} = a_{\tau n^i q}$ or $b_{\tau m^i q}$ splitting from a non-effective arrow $a_r$ or $b_r$ with $n^i$ being the index of the bottom row of $\mathfrak{A}^i$, and $q$ being the column index inside the splitting block partitioned under $T$, then $\delta(a_1^{i,0}) = \bar{x}_{\tau m^i q}$ or $\bar{y}_{\tau m^i q}$ by Formulae (4.2-7)–(4.2-8), (4.1-8) and Remark 4.1.3 (i). Thus $a_1^{i,0} \mapsto (0), \bar{x}_{\tau m^i q} = 0$ or $\bar{y}_{\tau m^i q} = 0$ by Remark 4.1.3 (ii). The regularizations are continue for the non-effective arrows inductively, and finally an effective one is sent to $(0)$, then $\mathfrak{A}^{i+1}$ is obtained by $\text{①}$ of (i). With a similar argument as above, $\mathfrak{A}^{i,r_i}$ is reached, the first arrow of $\mathfrak{B}^{i,r_i}$ has the differential $\delta(a_1^i) = 0$. Let $a_1^i \mapsto (1)$ or $(01)$, the $(i + 1)$-th pair is obtained.
If the procedure in (i) was continued without stop, the reduction sequence would have been infinite. Meanwhile, the \(b\)-class loop at the down-right corner of the matrix block splitting from \(b\) partitioned under \(\mathcal{T}\) has never been reached in (i). Therefore the procedure of (iii) must occur at some stage, say at the stage \(\kappa - 1\).

If the first arrow \(a_1^{\kappa-1}\) of \(\mathcal{B}^{\kappa-1,\kappa-1}\) is an edge, then \(a_1^{\kappa-1} \mapsto (1)\) or \((0 1)\) gives case ① of (iii). If \(a_1^{\kappa-1}\) is a loop, then \(a_1^{\kappa-1} \mapsto W\) by a loop reduction of Lemma 2.3.2 gives case ② of (iii), since \(\mathcal{B}^{\kappa-1,\kappa-1}\) is not local from \(\mathcal{B}^{\kappa-1,0}\) being non-local by (BRC). In both cases, the induced pair \((\mathfrak{A}^\kappa, \mathcal{B}^\kappa)\) has the minimal index \(\kappa\) satisfying Condition (B). \(\square\)

Suppose \(s(a_{i-1}^{\kappa-1}) = X_i\) in case (i) ② of Theorem 4.3.4, the reduction on \(a_{i-1}^{\kappa-1}\) gives \(e_{X_{i-1}} \mapsto \left(\begin{array}{c} e_{Y_1}^\kappa \\ \bar{W}_{\kappa 1}^1 \\ \vdots \\ \vdots \\ e_{Y_{\kappa-1}}^\kappa \\ \bar{W}_{\kappa \kappa-1}^{\kappa-1} \\ e_{X_{\kappa}}^\kappa \end{array}\right)\) for \(1 \leq i < \kappa\). Denote by \(\bar{W}_{i\kappa}\) the split of \(\bar{w}_i\) in \(e_{X_i}^\kappa\) for \(1 \leq i < \kappa\). Then \(\bar{W}_{i\kappa}\) can be divided into \((\kappa - i)\)-blocks \(\bar{W}_{i\kappa}^1, \ldots, \bar{W}_{i\kappa}^{\kappa-1}\). Denote by \(n_i^\kappa\) the size of \(e_{X_i}^\kappa\), and by \(n_{ij}^\kappa\) that of \(e_{X_{ij}}^\kappa\), which is 1 in case (iii) ③ of Theorem 4.3.4, or is the same as that of \(W\) in (iii) ②. Thus \(\bar{W}_{i\kappa}^j\) has the size \(n_i^\kappa \times n_{ij}^\kappa+1\). Write \(n^\kappa = \sum_{i=1}^{\kappa} n_i^\kappa\), which is the size of \(e_{X_\kappa}^\kappa\). We have

\[
e_{X_\kappa}^\kappa = \begin{pmatrix} e_{Y_1}^\kappa & \bar{W}_{\kappa 1}^1 & \cdots & \cdots & \bar{W}_{\kappa \kappa-1}^{\kappa-1} \\ e_{Y_2}^\kappa & \vdots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ e_{Y_{\kappa-1}}^\kappa & \bar{W}_{\kappa \kappa-1}^{\kappa-1} & \cdots & \bar{W}_{\kappa 1}^1 & e_{X_{\kappa}}^\kappa \end{pmatrix}
\]

Corollary 4.3.5 The elements in \(\bar{W}_{i\kappa}\) for \(1 \leq i < \kappa\) are dotted arrows of \(\mathcal{B}^\kappa\).

Proof The assertion is already implied in the proof of Theorem 4.3.4.

When we make an edge or a loop reduction of Lemma 2.3.2, the dotted arrows \(\{F_i^\kappa\mid i = 1, \ldots, l\}\), given in proof 2) of Proposition 2.1.5 are said to be \(w\)-class arrows, where the dotted arrows in \(\bar{W}_{i\kappa}\) for \(1 \leq i < \kappa\) of Formula (4.3-2) are specially said to be \(\bar{w}\)-class. Furthermore the elements splitting from \(w\) or \(\bar{w}\)-class arrows are still said to be in the same class.

4.4 Major pairs

We prove in this subsection that under some further assumption, a one sided pair \((\mathfrak{A}, \mathcal{B})\) with \(\mathcal{B}_X\) given by Formula (4.2-1) is not homogeneous.

Let \((\mathfrak{A}, \mathcal{B})\) be a one-sided pair, where \(\mathcal{B}_X\) is given by Formula (4.2-1), and \(\mathcal{B}\) has \(s\) \(\bar{a}\)-class arrows with \(s \geq 1\). According to the coefficients of the first two Formulae of (4.2-8), \(s\) linear combinations of the \(\bar{v}\)-class arrows are define in \(\mathcal{B}\):

\[
\hat{v}_{\tau} = \sum_j (\bar{v}_{\tau j} - \sum_{\bar{a}_\tau < b_\kappa < \bar{b}} \beta_j \varepsilon_{i_{\tau j}}) v_{j}, \quad \tau = 1, \ldots, s.
\]

Fix any \(1 \leq \tau \leq s\), making reductions according to (iii) ③ of Theorem 4.3.4 for \(\kappa = 1\), such that \(a_1^0 = \bar{a}_{\tau} \mapsto (1)\), the induced pair \((\mathfrak{A}_1, \mathcal{B}_1)\) is reached. Then we continue to do further reductions based on Formulae (4.2-7)–(4.2-8) inductively. For \(\bar{a}_\eta < a_i, b_i < \bar{a}_{\eta+1}, \tau < \eta < s\) and \(\bar{a}_s < a_i, b_i < \bar{b}\), by Remark 4.1.3 (ii):

\[
a_i \mapsto \emptyset, \quad \bar{v}_i + \sum_j \varepsilon_{i_{\tau j}} v_{j} = 0, \quad \bar{b}_i \mapsto \emptyset, \quad \bar{u}_i + \sum_j \varepsilon_{i_{\tau j}} \bar{v}_j = 0.
\]

On the other hand, \(\bar{a}_\eta \mapsto \emptyset\) or \((0)\) for \(\tau < \eta < s\) corresponding to \(\delta(\bar{a}_{\tau+\eta}) \neq 0\) or \(= 0\). the dotted element \(\bar{u}_i\) is replaced by the linear composition of \(\bar{v}\)-class arrows inductively by Remark 4.1.3 (iii), the second formula of (4.2-8) shows the formula below in some induced pair:

\[
\delta(\bar{b}) = \sum_{\bar{a}_\tau < b_\kappa < \bar{b}} \beta_i \bar{u}_i + 1(\sum_j \bar{v}_{\tau j} v_{j}) = \sum_j (\bar{v}_{\tau j} - \sum_{\bar{a}_\tau < b_\kappa < \bar{b}} \beta_i \varepsilon_{ij}) \bar{v}_{j} = \hat{v}_{\tau}.
\]
Lemma 4.4.1 Let $(\mathfrak{A}, \mathfrak{B})$ be a one-sided pair with $\mathcal{T}$ being trivial, $s \geq 1$, and $\mathfrak{B}_X$ given by Formula (4.2-1). If there exists some $1 \leq \tau \leq s$, with $v_\tau = 0$ in Formula (4.4-1), then $\mathfrak{B}$ is wild and non-homogeneous.

Proof If $\bar{a}_\tau : X \to Y$, it may be assumed that $\mathcal{T} = \{X, Y\}$.

1. Since $\mathfrak{B}_X$ is minimal with $R_X = k[x, \phi(x)^{-1}]$, there is an almost split conflation $(\varepsilon'_l) : S'_l \to E'_l \to S'_l$ for any $\lambda \in \varpi'$ in $R(\mathfrak{B}_X)$. Let $\vartheta : R(\mathfrak{B}_X) \to R(\mathfrak{B})$ be the induced functor. If $\mathfrak{B}$ is homogeneous, then there is a co-finite subset $\varpi' \subseteq \varpi''$, and a set of almost split conflation $(\{v_\lambda) = \vartheta(\varepsilon'_l) : S_X \to E_X \to S_X | \lambda \in \varpi'' \}$ by Corollary 3.2.4.

2. According to Formula (4.4-1)-(4.4-3), an induced pair $(\mathfrak{A}', \mathfrak{B}')$ is obtained with $\delta(\bar{b}) = 0$. Thus it is possible to construct an object $L \in R(\mathfrak{B})$ with $L_X = k, L_Y = k, L(\bar{a}_\tau) = (1), L(\bar{b}) = (\lambda)$ and $L(b_i) = 0, i > n_0, L(c_i) = 0, i = 1, \ldots, t$. The same argument given in 2) of the proof of Lemma 3.4.1 shows that $\mathfrak{B}$ is non-homogeneous. And $\mathfrak{B}$ is wild by [OCB1, Theorem A].

Theorem 4.4.2 Let $(\mathfrak{A}, \mathfrak{B})$ be a one-sided pair with $\mathcal{T}$ being trivial, $s > 1$, and $\mathfrak{B}_X$ given by Formula (4.2-1). If the elements $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_s\}$ defined in Formula (4.4-1) are linearly dependent, then $\mathfrak{B}$ is wild and non-homogeneous.

Proof Without loss of generality, it may be assumed $\mathcal{T} = \{X, Y\}$. Suppose there is a minimal linearly dependent subset $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_l\}$ with $l$ vectors. Since the case of $l = 1$ has been treated in Lemma 4.4.1, it is assumed here that $l > 1$. Suppose $\tau_1 < \tau_2 < \cdots < \tau_l$ and

$$\bar{v}_{\tau_1} = \beta_2 \bar{v}_{\tau_2} + \cdots + \beta_l \bar{v}_{\tau_l}, \quad \beta_2, \ldots, \beta_l \in k^*.$$  \hspace{1cm} (4.4-5)

1) Making reductions according to Theorem 4.3.4 (i) and (iii) \(1\) for $\kappa = l$, such that $a^{p-1}_{\bar{a}} \mapsto (\lambda^p)^1 \text{ for } 1 \leq p < l$, and $a_l^{\lambda^l} \mapsto (1)$, an induced pair $(\mathfrak{A}'^l, \mathfrak{B}'^l)$ is obtained. The sum $F^l + \Theta^l$ looks like (with only $\bar{a}, \bar{b}, \bar{c}$-class arrows):

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} & 1 & \cdots & \bar{a}_{r_1} & \cdots & \bar{a}_{r_2} & \cdots & \bar{a}_{r_3} & \cdots & \bar{a}_{r_4} & \cdots & \bar{a}_{r_5} & \cdots & \bar{a}_{r_6} & \cdots & \bar{a}_{r_7} & \cdots \hline 0 & 0 & \cdots & \bar{a}_{r_1} & \cdots & \bar{a}_{r_2} & \cdots & \bar{a}_{r_3} & \cdots & \bar{a}_{r_4} & \cdots & \bar{a}_{r_5} & \cdots & \bar{a}_{r_6} & \cdots & \bar{a}_{r_7} \hline 1 & 0 & \cdots & \bar{a}_{r_1} & \cdots & \bar{a}_{r_2} & \cdots & \bar{a}_{r_3} & \cdots & \bar{a}_{r_4} & \cdots & \bar{a}_{r_5} & \cdots & \bar{a}_{r_6} & \cdots & \bar{a}_{r_7} \hline 0 & 1 & \cdots & \bar{a}_{r_1} & \cdots & \bar{a}_{r_2} & \cdots & \bar{a}_{r_3} & \cdots & \bar{a}_{r_4} & \cdots & \bar{a}_{r_5} & \cdots & \bar{a}_{r_6} & \cdots & \bar{a}_{r_7} \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \hline \end{array}$$  \hspace{1cm} (4.4-6)

Since $\mathfrak{A}$ has four vertices, the dimension of $\vartheta_{\mathfrak{A}}(F^l/k)$ in $R(\mathfrak{A})$ is $l + 1$, and the number of links of $F^l$ is $l$, the pair $(\mathfrak{A}'^l, \mathfrak{B}'^l)$ is local by the assertion below Formula (2.3-7).

2) We make further reductions from $\mathfrak{B}'^l$ inductively for the $\bar{p}$-th row ordered by $\bar{p} = l, l - 1, \ldots, 2$ in the reduced formal product $\Theta^l$. For $\bar{p} = l$, similar to Formulae (4.4-2)-(4.4-3): $a_{\bar{a}}' \mapsto \emptyset, i \in \Lambda$; note that $\bar{v}_j : Y \to X$, the matrix splitting from $\bar{v}_j$ in $\mathfrak{B}'^l$ is $(\bar{v}_{j,1}, \ldots, \bar{v}_{j,l})$ of size $1 \times l$, $b_{i\bar{q}} \mapsto \emptyset, \bar{u}_{i\bar{q}} + 1 \sum_j \varepsilon_{i\tau_j} \bar{v}_{j\bar{q}} = 0, i < n_0, \bar{a}_{\bar{q}} \mapsto (0)$ or $\emptyset, \tau_1 < \tau_l \leq s$:

$$\delta^0(\bar{b}_{i\bar{q}}) = \sum \bar{a}_{\tau_1} b_{i\bar{q}} \bar{b}_{i\bar{q}} + 1(\sum_j \varepsilon_{i\tau_j} \bar{v}_{j\bar{q}}) \quad \text{with} \quad \varepsilon_{i\tau_j} \bar{v}_{j\bar{q}} = \bar{v}_{\tau_1 \bar{q}},$$

thus $\bar{b}_{i\bar{q}} \mapsto \emptyset, \bar{v}_{\tau_1 \bar{q}} = 0$ for $q = 1, \ldots, l$ inductively. Next, $b_{i\bar{q}} \mapsto \emptyset$ for $i > n_0, 1 \leq q \leq l$, by Remark 4.1.3 (ii) and $c_{\bar{q}} \mapsto (0)$ or $\emptyset$. The dotted arrows $\bar{u}_{ipq}$ for all $i$ and $p < l, 1 \leq q \leq l$, $\bar{v}_{i\bar{q}}$ for $i < l$ and $1 \leq q \leq l$ are preserved by 4.1.3 (iv). The induced bocs $\mathfrak{B}^{l+1}$ follows.

3) Suppose an induced bocs $\mathfrak{B}^{2l-\bar{p}}$ is reached for some $\bar{p} < l$. Denote the entries of $F^{2l-\bar{p}}$, which are not the entries of $G^{2l-\bar{p}}$ for $j = 1, \ldots, l$, by $\bullet^0$ coming from $\bullet$, one of the $a, b, c$-class solid arrows. Then $\mathfrak{B}^{2l-\bar{p}}$ satisfies the following two conditions:

1. for any $p > \bar{p}$, $a^0_{ip} = 0, i \in \Lambda; a^0_{ip} = 0$ or $(0)$; $b^0_{ipq} = 0, i \neq n_0, \bar{b}_{ipq} = 0; c_{ip} = 0$ or $(0)$.

2. The dotted arrows $\bar{u}_{ipq}$, $\bar{v}_{ipq}$ for all $i, p$ and $q \leq \bar{p}$; and $\bar{v}_{i\bar{q}}$ for $i \leq \bar{p}$ and $1 \leq q \leq l$ are preserved.
Now we continue with reductions on the solid arrows at the $\tilde{p}$-th row of $\Theta^{2l-\tilde{p}}$. According to assumption ①-② and Remark 4.1.3 (ii), with a similar discussion as in 2): $a^0_{\tilde{p}q} \to \emptyset, i \in \Lambda$; $b_{ipq} \to \emptyset, \bar{u}_{ipq} + 1 \sum_j \epsilon_{i\tau_j} \bar{v}_{jq} = 0, i < n_0; \bar{a}_{\eta\tilde{p}} \to (0) \emptyset, \tau_{\tilde{p}} < \eta \leq s; b_{ipq} \to \emptyset, \bar{v}_{\tau pq} = 0; b_{ipq} \to \emptyset, i > n_0, \bar{c}_{ipq} \to (0) \emptyset$, an induced pair $(\tilde{\mathfrak{A}}^{2l-\tilde{p}+1}, \tilde{\mathfrak{B}}^{2l-\tilde{p}+1})$ satisfying assumption ①-② is reached. Finally a pair $(\tilde{\mathfrak{A}}^{2l-1}, \tilde{\mathfrak{B}}^{2l-1})$ is obtained by induction on $\tilde{p}$.

4) Making reductions on the arrows before $\bar{b}_{11}$ in the first row of (4.4-6): $a_{11} \to \emptyset, i \in \Lambda$; $b_{ijq} \to \emptyset, j < n_0; a_{11} \to \emptyset$ or (0) for $1 \leq i \leq s$, an induced bocs $\tilde{\mathfrak{B}}^{2l}$ is obtained. Formula (4.4-5) gives $\bar{v}_{t_{1q}} = \bar{v}_{t_{2q}} + \cdots + \bar{v}_{t_{1q}} = 0$, thus $\delta^0(\bar{b}_{1q}) = 0$ for $q = 1, \ldots, l$. Since $l \geq 2$, the bocs $\tilde{\mathfrak{B}}^{2l}$ is wild and non-homogeneous by (ii) or (iv) of Classification 4.2.1. And hence so is $\tilde{\mathfrak{B}}$. □

**Definition 4.4.3** A one-sided pair $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{B}})$ with $\tilde{\mathfrak{B}}_X$ given by Formula (4.2-1) is said to be a major pair, provided that $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_s\}$ of Formulas (4.4-1) are linearly independent.

### 4.5 Further reductions

Throughout the subsection let $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{B}})$ be a one-sided pair, such that $\tilde{\mathfrak{B}}_X$ is given by Formula (4.2-1); $\tilde{\mathfrak{F}}$ is trivial and $|\tilde{\mathfrak{F}}| > 1$; the pair is major; and the $\varsigma$-class arrows satisfy Formula (4.2-6). Suppose $(\tilde{\mathfrak{A}}^{\varsigma}, \tilde{\mathfrak{B}}^{\varsigma})$ is an induced pair given by Theorem 4.3.4 (iii), we continue to construct a sequence of reductions of Lemma 2.3.2, which is still the first part of a sequence towards a pair $(\tilde{\mathfrak{A}}^{\varsigma}, \tilde{\mathfrak{B}}^{\varsigma})$ in the case of MW5. Let $(\tilde{\mathfrak{A}}^{\varsigma}, \tilde{\mathfrak{B}}^{\varsigma})$ be an induced pair of $(\tilde{\mathfrak{A}}^{\varsigma}, \tilde{\mathfrak{B}}^{\varsigma})$ in the sequence for some $\varsigma > \kappa$. The explicit linear relation of dotted elements in the induced bocs $\tilde{\mathfrak{B}}^{\varsigma+1}$ will be expressed via calculating the differential of the first arrow of $\tilde{\mathfrak{B}}^{\varsigma}$.

Concern the formal product $\tilde{\Theta}^{\varsigma}$ and the pseudo formal product $\tilde{\Pi}^{\varsigma}$ of the pair $(\tilde{\mathfrak{A}}^{\varsigma}, \tilde{\mathfrak{B}}^{\varsigma})$. Put a solid or dotted arrow in a square box; and a matrix block in a rectangular box with four boundaries. The reduction block $G^{\xi}_j$ of $H^{\varsigma}$ for $1 \leq j \leq \kappa$ is defined before Formula (2.3-7), whose upper boundary is that of $I^1_\varsigma$ and denoted by $m^{\varsigma-1}_j$, the lower boundary is that of $F^{\varsigma}$; the left and right boundaries are given by the dotted lines $l^1_\varsigma$ and $r^1_\varsigma$ respectively, see Picture (4.5-1) below. The set of entries and solid arrows in $(F^{\varsigma} + \Theta^{\varsigma})$ but not in $G^{\xi}_j$ for $1 \leq j \leq \kappa$ is divided into $\kappa$ blocks, where the $j$-th block has the upper boundary $m^{\xi-1}_j$ and the lower one $m^1_j$, the left boundary $r^j$ and the right one being that of $F^{\varsigma} + \Theta^{\varsigma}$. Denote by $A^j_\varsigma, B^j_\varsigma, C^j_\varsigma$ in the $j$-block containing the sets of entries in $F^{\varsigma}$ obtained by reductions for $a, b, c$-class arrows and the sets of $a, b, c$-class solid arrows in $\Theta^{\varsigma}$.

$$
\begin{array}{cccc}
I^1_\varsigma & A^1_\varsigma \cup B^1_\varsigma & B^1_\varsigma \cup C^1_\varsigma \\
I^2_\varsigma & A^2_\varsigma \cup B^2_\varsigma & B^2_\varsigma \cup C^2_\varsigma \\
\vdots & \vdots & \vdots \\
I^{\xi+1}_\varsigma & B^{\xi+1}_\varsigma & B^{\xi+1}_\varsigma \cup C^{\xi+1}_\varsigma \\
\vdots & \vdots & \vdots \\
I^{\kappa-1}_\varsigma & B^{\kappa-1}_\varsigma & B^{\kappa-1}_\varsigma \cup C^{\kappa-1}_\varsigma \\
W^{\varsigma} & B^{\varsigma} \cup C^{\varsigma} \\
\end{array}
$$

(4.5-1)

In $A^j_\varsigma, B^j_\varsigma, C^j_\varsigma$, a solid arrow is denoted by $\bullet_{ipq}^j$ splitting from $a_{i}, b_{i}, c_{i}$ respectively, an entry of $F^{\varsigma}$ by $\bullet_{ci,pq}^{j,0}$ where $(p,q)$ is the index in the $n^\varsigma \times n^{\varsigma}_{t(a_i)}, n^\varsigma \times n^{\varsigma}$ or $n^{\varsigma} \times n^{\varsigma}_{t(c_i)}$ -block matrices. For the sake of convenience, $\Phi^{m}_{\underline{a}_{i}}$ of Formula (4.1-7) is also partitioned by the lines $m^{1}_j, l^{j}_j, r^{j}_j$ in the same way as in $F^{\varsigma} + \Theta^{\varsigma}$. 
Remark From now on, the pseudo formal equation (4.1-7) at the c-th step for presenting the differential of the first arrow is considered. After a loop or an edge reduction, some w-class dotted arrows may be added into the induced bocs, but the linear relations among the splits of the dotted elements in the induced pair can be obtained completely by Remark 4.1.3 (i). Therefore the new relation of \(\tilde{u}, \tilde{v}, \tilde{u}, \tilde{w}, \tilde{w}, w\)-class elements during the regularization from \(B^c\) to \(B^{s+1}\) will be concentrated on.

Throughout the subsection, suppose the first arrow \(a'_1 = \bullet_{\tau pq}^c\) of \(B^c\) belongs to \(A^c_i \cup B^c_i \cup C^c_i\) in Picture (4.5-1). Denote by \(\tilde{u}_{ipq}^c\) or \(\tilde{v}_{ipq}^c\) the dotted element in \(\tilde{\Psi}_{m_k}^c\), which corresponds to the entry \(b_{ipq}^c\) or \(a_{ipq}^c\) in \(F^c\), thus \(j \geq i, p \geq \bar{p}, \text{ or } p = \bar{p}\) but \(q < \bar{q}\). On the other hand, denote by \(\tilde{u}_{jy'q'}^c\) or \(\tilde{v}_{jy'q'}^c\) the dotted element corresponding to the solid arrow \(b_{jy'q'}^c\) or \(a_{jy'q'}^c\) in \(\Theta^c\), thus \(j' \leq i, p' < \bar{p}, \text{ or } p' = \bar{p}, q' > \bar{q}\). All of them are coming from \((\mathcal{D}, \mathcal{U})\) in Lemma 4.3.3.

In the following Lemmas, the index \(n_0\) is defined in Formula (4.2-1), and the number \(n\) and the set \(\Lambda\) are defined in Formula (4.2-5).

**Lemma 4.5.1** Let \((A^c_i, B^c_i, C^c_i)\) be a pair induced from \((A^c_i, B^c_i, C^c_i)\) given by Theorem 4.3.4 (iii). Suppose the first arrow of \(B^c\), \(a'_1 = \bullet_{\tau pq}^c\) belongs to \(B^c_i \cup C^c_i\), (see the second thick line below in Picture (4.5-1) for example). Assume that

(i) all \(b_{ipq}^c = \emptyset, i > n_0\), and the corresponding dotted element \(\tilde{u}_{ipq}^c\) is replaced by a linear combination of some \(\tilde{v}\)-class arrows in \(\tilde{\Psi}\); while the dotted arrows \(\tilde{u}_{jy'q'}^c\) are preserved;

(ii) if \(c_{ipq}^c = \emptyset\), there is a linear relation among some elements \(\tilde{w}_{i1pq}, \\sum_{q < i} \tilde{w}_{i1pq}^c, h < i, h + i, p_1 > p\) and some \(\tilde{u}, \tilde{w}\)-class arrows in \(\tilde{\Psi}\); while all the dotted arrows \(\tilde{u}_{jy'q'}^c\) are preserved.

Then after a regularization, the induced pair \((A^{s+1}_p, B^{s+1}_p, C^{s+1}_p)\) still satisfies (i)-(ii). In particular all the dotted arrows \(\tilde{u}_{jy'q'}^c\) except \(\tilde{u}_{\tau pq}^c\) in case (ii); \(\tilde{u}_{\tau pq}^c\), except \(\tilde{u}_{\tau pq}^c\) in case (i); and all the \(\tilde{w}, \tilde{w}\)-class arrows are preserved.

**Proof** The assumption (i)-(ii) are valid for \(\zeta = \kappa\) by Theorem 4.3.4 and Corollary 4.3.5.

(i) If \(a'_1 = b_{\tau pq}^c\), \(\tau > n_0\), then according to the third formula of (4.2-8) and Formula (4.1.8),

\[
\delta(b_{\tau pq}^c) = \tilde{u}_{\tau pq}^c + \sum_{n_0 < i < \tau} \left( \lambda_{\tau i}^c \tilde{u}_{\tau pq}^c + \sum_{j} \lambda_{\tau j}^c \tilde{u}_{\tau pq}^c \sum_{j < h} \phi_{\tau i}^c \tilde{u}_{\tau pq}^c \right) + \sum_{n_0 < i < \tau} \phi_{\tau i}^c \tilde{u}_{\tau pq}^c \sum_{j < h} \phi_{\tau i}^c \tilde{u}_{\tau pq}^c \sum_{j < h} \phi_{\tau i}^c \tilde{u}_{\tau pq}^c
\]

Since \(W_{\tau pq}^c\) is upper triangular, the index \(\bar{p} \leq q \leq \bar{q}\) in \(b_{\tau pq}^c\). By assumption (i), \(\tilde{u}_{\tau pq}^c\) is a dotted arrow, thus \(b_{\tau pq}^c \rightarrow \emptyset, \tilde{u}_{\tau pq}^c\) is replaced by a linear combination of some \(\tilde{v}\)-class arrows by 4.1.3 (ii)-(iii), since \(\tilde{u}_{\tau pq}^c, \tilde{u}_{\tau pq}^c\) are already replaced by those arrows still by assumption (i).

(ii) If \(a'_1 = c_{\tau pq}^c\), then according to Formula (4.2-9) and (4.1.8),

\[
\delta(c_{\tau pq}^c) = \sum_{h < i < h + \tau} \left( \gamma_{\tau i}^c \tilde{u}_{\tau pq}^c \sum_{j} \gamma_{\tau j}^c \tilde{u}_{\tau pq}^c \sum_{j < h} \phi_{\tau i}^c \tilde{u}_{\tau pq}^c \right) + \sum_{h < i < h + \tau} \phi_{\tau i}^c \tilde{u}_{\tau pq}^c \sum_{j < h} \phi_{\tau i}^c \tilde{u}_{\tau pq}^c \sum_{j < h} \phi_{\tau i}^c \tilde{u}_{\tau pq}^c
\]

In the case of \(\delta(c_{\tau pq}^c) \neq 0, c_{\tau pq}^c \rightarrow \emptyset, c_{\tau pq}^c\), and a linear relation among elements \(\tilde{w}_{\tau pq}, \tilde{w}_{\tau pq}, h < i < h + \tau, q \geq \bar{p}\), and some \(\tilde{w}\)-class elements is added, which is given by the right-hand side of the above formula being equal to 0.

The required \(\zeta, \tilde{u}\)-class and all the \(\tilde{v}, \tilde{w}\)-class dotted arrows are preserved, the pair \((A^{s+1}_p, B^{s+1}_p, C^{s+1}_p)\) still satisfies assumption (i)-(ii).
Lemma 4.5.2 Let \( \hat{\mathfrak{A}}, \hat{\mathfrak{B}} \) be an induced pair of \( \hat{\mathfrak{A}}, \hat{\mathfrak{B}} \) with \( \gamma \) existing in Theorem 4.3.4 (ii). Suppose the first arrow of \( \mathfrak{B}, a_1^i = b_{rpq} \in B^i \cup C^i \) with \( \gamma < i \leq \kappa \) (or \( \kappa \)). Assume that

(i) all \( \tilde{b}_{pq} = 0 \), the corresponding dotted element \( \tilde{w}_{pq} \) is replaced by a linear combination of some \( \tilde{w}_{pq} \) for \( p > q \), and some \( w \)-class elements in \( B^c \); while the dotted arrows \( \tilde{w}_{pq}^{i} \) for \( p' < q' \) are preserved;

(ii) all \( \tilde{b}_{pq}^{j} = 0, i > n_0 \), the corresponding \( \tilde{w}_{pq}^{j} \) is replaced by a linear combination of an element \( \tilde{w}_{pq}^{j} \) with \( n_0 < i_1 < i, j < j_1 < \gamma, p_1 < p \), and some \( w \)-class elements in \( B^c \); while the dotted arrows \( \tilde{w}_{pq}^{j} \) are preserved;

(iii) if \( c_{ij}^{l} = 0, \) there is a linear relation among some elements \( w_{ij}^{l} \), \( h < i_1 < h + i, j < j_1 < \gamma, p_1 < p \), and some \( w \)-class elements in \( B^c \); while the dotted arrow \( \tilde{w}_{pq}^{j} \) is preserved.

Then after a regularization, the induced pair \( (\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1}) \) still satisfies (i)-(iii). In particular all the dotted arrows \( \tilde{w}_{pq}^{j} \), \( p' < q' \) is replaced by a linear combination of some \( w \)-class elements by Remark 4.1.3 (ii)-(iii). Assume that

\( a_1^i = b_{rpq}, r > n_0 \), then \( \tilde{b}_{pq}^{0}, \tilde{b}_{pq}^{0} = (1), \tilde{b}_{pq}^{0} = 0, q' < q < \hat{q} \). By assumption (i), Formula (4.2-8) shows

\[
\delta(\tilde{b}_{pq}^{0}) = 1\tilde{w}_{pq}^{0} - \sum_{q<p,q<\hat{q}} w_{pq}^{0} \tilde{w}_{pq}^{0},
\]

where \( w_{pq}^{0} \) is \( w \) or \( \tilde{w} \)-class. Since \( w_{pq}^{0} \) is a dotted arrow still by (i), \( \tilde{b}_{pq}^{0} \) is replaced by a linear combination of some \( w \)-class elements below the \( q' \)-th row and some \( w \)-class elements in the pair \( (\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1}) \).

(ii) If \( a_1^i = b_{r'pq}, r > n_0 \), then \( \tilde{b}_{pq}^{1}, \tilde{b}_{pq}^{1} = 1, \tilde{b}_{pq}^{1} = 0, q' < q < \hat{q} \). Formula (4.2-8) gives

\[
\delta(\tilde{b}_{r'pq}^{1}) = \tilde{u}_{r'pq}^{1} + \sum_{n_0 < i < \gamma} (\beta_1^{0} \tilde{u}_{r'pq}^{0} + \beta_1^{1} \tilde{u}_{r'pq}^{0}) + \sum_{n_0 < i, q < q'} \tilde{c}_{i}^{0} \tilde{u}_{pq}^{1},
\]

where \( \beta_1^{0} \) exists, otherwise \( \beta_1^{0} = 0 \). Since \( \tilde{u}_{r'pq}^{1} \) is a dotted arrow by assumption (ii), \( \tilde{b}_{r'pq}^{1} \) is replaced by a \( w \)-class element and some \( w \)-class elements by Remark 4.1.3 (ii)-(iii).

(iii) If \( a_1^i = c_{r'pq}, r > n_0 \), then according to Formula (4.2-9):

\[
\delta(\tilde{c}_{r'pq}^{1}) = \sum_{h < i < h + i} (\gamma_1^{0} \tilde{u}_{r'pq}^{0} + \gamma_1^{1} \tilde{u}_{r'pq}^{0}),
\]

where \( \gamma_1^{0} \) exists, otherwise \( \gamma_1^{0} = 0 \). If \( \tilde{c}_{r'pq}^{1} \) is replaced by \( w \)-class element and some \( w \)-class elements with the subscripts being bigger than \( h \) is added.

The required \( \tilde{p}, \tilde{u}, \tilde{w} \)-class and all the \( \tilde{w} \)-class dotted arrows are preserved, and the pair \( (\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1}) \) still satisfies assumption (i)-(iii). \( \square \)

Suppose \( j \leq \gamma \) if \( \gamma \) exists, otherwise \( j \leq \kappa \) in case (iii) \( \mathfrak{A}, \mathfrak{B} \) of Theorem 4.3.4, or \( j < \kappa \) in (iii) \( \mathfrak{A}, \mathfrak{B} \), see the first thick line above in Picture (4.5-1) for example.

Assume \( \bar{I}^j \) intersects the \( p \)-th row of \( F \) at the \( q' \)-th column in the \( n_\chi^j \times n_\chi^j \) obtained from \( \bar{a}_{ij} \) partitioned under \( \bar{F} \) with \( \bar{a}_{ij}^{0} = (1) \). Denote by \( \tilde{v}_{i',q'} \) the \( (q', q)_\chi \)-element in the block of size \( n_\chi^j \times n_\chi^j \) splitting from \( \tilde{v}_{i',q'} \). Denote by \( \tilde{V}^j \) the block in the \( n_0\)-th block-column of \( \tilde{\Psi}_\chi^j \).
partitioned under $\mathcal{T}$, such that the row indices of $\hat{V}_i^j$ coincide with the column indices of $I_i^j$, see $(I^1, \hat{V}_1), (I^2, \hat{V}_2)$ in Picture (4.5-2) below.

**Lemma 4.5.3** Let $(\mathcal{A}^\kappa, \mathcal{B}^\kappa)$ be an induced pair of $(\mathcal{A}^\kappa, \mathcal{B}^\kappa)$. Suppose the first arrow of $\mathcal{B}^\kappa$, $a^\kappa_l = \bullet_{pq} \in A_l^\kappa \cup B_l^\kappa \cup C_l^\kappa$, where $\iota \leq \gamma$ if $\gamma$ exists, otherwise $\iota < \kappa$ in case (iii) $\mathcal{Q}$ of Theorem 4.3.4, or $\iota < \kappa$ in (iii) $\mathcal{Q}$. Assume that

(i) all $a_{pq}^j = \emptyset, i \in A$, the corresponding $\underline{w}_{pq}^j$ is replaced by a linear combination of some $w$-class elements in $\mathcal{B}^\kappa$; while all the dotted arrows $\underline{w}_{pq}^j$ are preserved;

(ii) if $\tilde{a}_{pq}^0 = \emptyset, \tilde{a}_{pq} = \emptyset$, there is a linear relation among $\underline{w}, \underline{w}, w, w$-class elements in $\mathcal{B}^\kappa$; while all the dotted arrows $\underline{w}_{pq}^j$ are replaced;

(iii) all $\bar{b}_{pq}^j = \emptyset, i < n_0$, the corresponding $\bar{u}_{pq}^j$ is replaced by a linear combination of some $\bar{v}$-class elements below and some $\bar{v}, w$-class in $\mathcal{B}^\kappa$; while all the dotted arrows $\bar{u}_{pq}^j$ are preserved;

(iv) all $\tilde{b}_{pq}^0 = \emptyset, \tilde{v}_{pq} \bar{v}_{pq}$ corresponding to $a_{pq}^0$ is replaced by a linear combination of some $\bar{v}$-class elements below $\bar{v}, w$-class in $\mathcal{B}^\kappa$; while the dotted arrows $\bar{v}_{pq}^j, p' \leq q_0$, are preserved;

(v) all $\bar{b}_{pq}^0 = \emptyset, i > n_0$, the corresponding element $\bar{u}_{pq}^i$ is replaced by a linear combination of some $\bar{v}$-class elements in $\mathcal{B}^\kappa$; while all the dotted arrows $\bar{u}_{pq}^j$ are preserved;

(vi) if $a_{pq}^0 = \emptyset$, there is a linear relation among some elements $u_{pq}^j, h < i_1 < h + \tau, p_1 = p$, and some $\underline{w}, \underline{w}, w, w$-class elements in $\mathcal{B}^\kappa$; while all the dotted arrows $\underline{w}_{pq}^j$ are preserved.

Then after a regularization, the induced pair $(\mathcal{A}^{\kappa+1}, \mathcal{B}^{\kappa+1})$ still satisfies (i)-(vi). In particular, all the dotted arrows $\bar{v}_{pq}^j, p' \leq q_0$, except $\bar{v}_{pq}^j, w$-class cases (i), (ii), (vi); $\bar{v}_{pq}^j$ except $\bar{u}_{pq}$ in cases (iii) or (v) and $\bar{v}_{pq}^j, p' < q_0$ in case (iv), are preserved.

**Proof** We claim first, that if $\gamma$ exists, than $\hat{a}_{pq}^j > \bar{a}_{pq}$ given in case (ii) of Lemma 4.5.2 can be replaced inductively by some $\bar{v}$-class arrows, when the reductions inside the ($\gamma + 1$)-th block in Picture (4.5-1) are finished. Thus the assumption (i)-(vi) are valid, if $a^\kappa_l$ has the bottom and right boundaries $(m^\kappa, r^\kappa)$, when $\gamma$ exists, according to Lemma 4.5.2; otherwise $a^\kappa_l$ has those $(m^\kappa, r^\kappa)$ in case (iii) $\mathcal{Q}$ of Theorem 4.3.4; or $(m^\kappa, r^\kappa)$ in (iii) $\mathcal{Q}$ by Lemma 4.5.1.

(i) if $a_{pq}^\kappa = a_{pq}^\kappa, \tau \in A, \mathcal{L}_{pq}$ is a dotted arrow by assumption (i). Formula (4.2-7) tells

$$
\delta(a_{pq}^\kappa) = \bar{v}_{pq}^\kappa + \sum_{i,q} a_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l) \Rightarrow a_{pq}^\kappa \Rightarrow 0, v_{pq}^\kappa = -\sum_{i,q} a_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l).
$$

(ii) if $a_{pq}^\kappa = \bar{a}_{pq}$ is effective, then by substituting $\bar{v}_{pq}^\kappa$ given by the formula above,

$$
\delta(\bar{v}_{pq}^\kappa) = \sum_{i,q} \bar{a}_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l) + \sum_{i,q} \bar{a}_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l) = -\sum_{i,q} \bar{a}_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l).
$$

If $a_{pq}^\kappa \Rightarrow 0$, then a linear relation among some $\underline{w}, \underline{w}, w$ and $\bar{v}$-class elements is added.

(iii) if $a_{pq}^\kappa = b_{pq}^\kappa, \tau \leq n_0$, the dotted arrow by assumption (iii):

$$
\delta(b_{pq}^\kappa) = \tilde{a}_{pq}^\kappa + \sum_{i,q} a_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l) \Rightarrow b_{pq}^\kappa \Rightarrow 0, \tilde{a}_{pq}^\kappa = -\sum_{i,q} a_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l).
$$

(iv) if $a_{pq}^\kappa = \bar{b}_{pq}$ is effective, by substituting $\bar{u}_{pq}$ given in (iii) and Formula (4.4-1),

$$
\delta(\bar{v}_{pq}) = \sum_{i,q} \bar{a}_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l) + \sum_{i,q} \bar{a}_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l) + \sum_{i,q} \bar{a}_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l) + \sum_{i,q} \bar{a}_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l) = v_{pq}^\kappa + \sum_{i,q} \bar{a}_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l) + \sum_{i,q} \bar{a}_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l) + \sum_{i,q} \bar{a}_{ipq}^\kappa (\sum_l \epsilon_{ri}^l u_{pq}^l).$$
Since $\bar{u}_{r,\bar{q}}(0) = 1$, and $\hat{v}_{r,\bar{q}}(0)$ is a dotted arrow by assumption (iv), $b_{\bar{q}} \mapsto \emptyset$, and $\hat{v}_{r,\bar{q}}(0)$ is replaced by some $\hat{v}$-class elements below the $q_{p}$-th row, and some $\hat{w}, \bar{w}$-class elements.

(v) If $a_{1}^{r} = b_{\bar{q}}$, $\tau > n_{0}$, since $b_{\bar{q}} = \emptyset$ for all possible $q$ by (iv) above, Formula (4.2-8) shows

$$
\delta(b_{\bar{q}}) = \bar{u}_{r,\bar{q}} + \sum_{i \neq \tau_{r}, i < \tau} b_{\bar{q}}(0) \bar{u}_{i,\bar{q}} + \sum_{i \neq \tau_{r}, i < \tau} c_{i,\bar{q}}(0) \sum_{i} \epsilon_{r,\bar{q}} \bar{u}_{i,\bar{q}} + \sum_{i} c_{i,\bar{q}}(0) \sum_{\bar{q}} \epsilon_{r,\bar{q}} \bar{u}_{i,\bar{q}}.
$$

Since $\bar{u}_{r,\bar{q}}$ is a dotted arrow by assumption (v), $b_{\bar{q}} \mapsto \emptyset$, $\hat{u}_{r,\bar{q}}$ is replaced by some $\hat{v}$-class elements according to the replacement given in (iii) and assumption (v).

(vi) If $a_{1}^{r} = c_{\bar{q}}$, since $b_{\bar{q}} = \emptyset$ for all possible $q$ by (iv), Formula (4.2-9) gives

$$
\delta(c_{\bar{q}}) = \sum_{i \leq h + \tau} c_{i,r}(0) \bar{u}_{i,\bar{q}}(0) + \sum_{i < \tau_{r}} c_{i,\bar{q}}(0) \sum_{i} \epsilon_{r,\bar{q}} \bar{u}_{i,\bar{q}} + \sum_{q < q} c_{r,\bar{q}}(0) \bar{u}_{q,\bar{q}} - \sum_{j \geq h, p > \bar{p}} w_{p}c_{j,\bar{q}}(0).
$$

If $\delta(c_{\bar{q}}(0)) \neq 0$, then $\overline{\mathbb{S}}^{s+1}$ is given by $c_{\bar{q}} \mapsto \emptyset$, and a linear relation is added among $w_{i,\bar{q}}$, $h < i \leq h + \tau$ and some $w, \bar{w}$-class elements, because $c_{i,\bar{q}}$ for $i \leq h$ have already been replaced by some $\bar{w}$-class arrows given in (i).

The required $w, \bar{w}, \bar{v}$-class dotted arrows are preserved, the pair $(\overline{\mathbb{A}}^{s+1}, \overline{\mathbb{B}}^{s+1})$ still satisfies assumption (i)-(vi).

The following picture shows a pseudo formal equation $\Theta^{s}$ of $(\overline{\mathbb{A}}^{s}, \overline{\mathbb{B}}^{s})$ for $\kappa = 5$, $\gamma = 2$ in case (iii) $\otimes$ of Theorem 4.3.4 only with effective arrows. From this, it is possible to see the correspondences of $(\overline{B}_{1}, \overline{V}_{1})$, $(\overline{B}_{2}, \overline{V}_{2})$, $(\overline{B}_{3}, \overline{W}_{3})$, $(\overline{B}_{4}, \overline{W}_{4})$ respectively.

![Picture (4.5-2)]

**4.6 Regularizations on non-effective a class and all b class arrows**

Let $(\overline{\mathbb{A}}, \overline{\mathbb{B}})$ be a one-sided pair, and the induced pair $(\overline{\mathbb{A}}^{s}, \overline{\mathbb{B}}^{s})$ be given by Theorem 4.3.4. Using the notation of Remark 3.4.6, we may assume that an induced local pair $(\overline{\mathbb{A}}^{s}, \overline{\mathbb{A}}^{s})$ of $(\overline{\mathbb{A}}^{s}, \overline{\mathbb{B}}^{s})$ is obtained by a sequence of reductions in the sense of Lemma 2.3.2. Set $a_{1}^{r} \mapsto (x)$, the induced pair $(\overline{\mathbb{A}}^{s+1}, \overline{\mathbb{B}}^{s+1})$ is obtained with $\mathring{R} = k[x]$, and $(\overline{\mathbb{A}}^{t}, \overline{\mathbb{A}}^{t})$ in the case of MW5 is given by a series of regularizations. We will show in the last subsection, that $x, a_{1}^{r} \in \overline{\mathbb{B}}^{t}$ can be only split from some edges of $\overline{\mathbb{B}}$.

It is clear by Lemma 4.5.1–4.5.3 that the non-effective $a$-class and all the $b$-class solid arrows are regularized during the reductions. Note that the conclusions of Lemmas 4.5.1–4.5.3 are still valid, if we deal with the linear relations over the fractional field $k(x)$ of the polynomial ring $k[x]$, or over the field $k(x, x_{1})$ of two indeterminants instead of the base field $k$. Then the non-effective $a$-class and all the $b$-class solid arrows are regularized, which implies the following theorem.
Theorem 4.6.1 Let \((\mathfrak{A}, \mathfrak{B})\) be a one-sided pair having at least two vertices, such that the induced local bocs \(\mathfrak{B}_X\) is given by Formula (4.2-1), the pair is major, and the c-class arrows satisfy Formula (4.2-6). If \((\mathfrak{A}^c, \mathfrak{B}^c)\) given by Theorem 4.3.4 has an induced pair \((\mathfrak{A}^l, \mathfrak{B}^l)\) in the case of MW5 defined by Remark 3.4.6, then the parameter \(x\) and the first arrow \(a'_1\) of \(\mathfrak{B}^l\) belong to \(\bar{a}\) or c-class.

Finally, let \((\mathfrak{A}, \mathfrak{B})\) be a one-sided pair having at least two vertices, such that \(\mathfrak{B}_X\) is in case (i) of Classification 4.2.1. Then \(\mathfrak{B}\) has only \(a, b\)-class solid arrows, where \(b_1, \ldots, b_n\) are all non-effective, and Formulae (4.2-5) is also suitable for \(a\)-class arrows: \(a_i\) for \(i \in \Lambda\) satisfying the first formula of (4.2-5) are non-effective, while \(\bar{a}_i = a_{\bar{h}_i}\) for \(1 \leq i \leq s\) satisfying the second one are effective. If there is an induced pair \((\mathfrak{A}^l, \mathfrak{B}^l)\) in the case of MW5 according to Remark 3.4.6, there are following observations.

1) Let \((\mathfrak{A}, \mathfrak{B})\) be a pair with \(T\) being trivial. The condition \((BRC)'\) is constructed parallel to \((BRC)\) in Condition 4.3.1 as follows.

(i) Let \(D = \{d_1, \ldots, d_q\}\) be a set of solid arrows, and \(E = \{e_1, \ldots, e_p\}\) be another set of solid edges without any loop, such that \(D \cup E\) forms the lowest non-zero row of the formal product \(\Theta\). And let \(U = \{u_1, \ldots, u_q\}\) be a set of dotted arrows, while \(W = V \setminus U\).

(ii) \(\delta(d_i)\) and \(\delta(e_i)\) satisfy the formulae of 4.3.1 (ii).

Then after a reduction given by Cases (i)–(iv) stated before Lemma 4.3.2, the induced pair still satisfies \((BRC)'\). On the other hand, the original pair \((\mathfrak{A}, \mathfrak{B})\) satisfies \((BRC)'\) parallel to Lemma 4.3.3. The proofs of the two facts are much easier than those of above two lemmas.

2) For constructing a reduction sequence from \((\mathfrak{A}, \mathfrak{B})\) up to \((\mathfrak{A}^c, \mathfrak{B}^c)\), what is needed is only the part (i) and (iii) \(\Phi\) of Theorem 4.3.4. In fact, the reduction block \(G^j\) is \(\binom{j}{1}\) or \(\binom{0}{1}\) for \(j < \kappa\), and \(G^c = (1)\) or \(\binom{0}{1}\).

3) For further reductions, what is needed is only Theorem 4.5.3 (i)-(iii), then an induced pair \((\mathfrak{A}^c, \mathfrak{B}^c)\) is reached, where all the \(b\)-class, and non-effective \(a\)-class arrows are regularized step by step.

Corollary 4.6.2 Let \((\mathfrak{A}, \mathfrak{B})\) be a one-sided pair having at least two vertices, such that the induced local bocs \(\mathfrak{B}_X\) is in case (i) of Classification 4.3.1. If \((\mathfrak{A}^c, \mathfrak{B}^c)\) given in 2) above has an induced pair \((\mathfrak{A}^l, \mathfrak{B}^l)\) in the case of MW5 defined by Remark 3.4.6, then the parameter \(x\) and the first arrow \(a'_1\) of \(\mathfrak{B}^l\) belong to \(\bar{a}\)-class.

5. Non-homogeneity of bipartite matrix bimodule problems of wild type

This section is devoted to proving the non-homogeneous property for a wild bipartite matrix bimodule problem satisfying RDCC condition in the case of MW5. As a consequence, the main theorem 3 is proved in the last subsection.

5.1 An inspiring example

The purpose of this subsection is two folds: 1) classify the positions of the first arrows of bimodules in the case of MW5 at formal products; 2) define a notion of the bordered matrix of a matrix, then prove a preliminary lemma on both matrices.

Let \(\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)\) be a bipartite matrix bimodule problem, which has a trivial vertex set \(\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2\) and satisfies RBDCC condition, see Remark 1.4.4. Suppose \(\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')\) is an induced matrix bimodule problem in the case of MW5 defined by Remark 3.4.6. We show the classification of the position of the first arrow \(a'_1\) in the sum \(H' + \Theta'\):

\[
H' + \Theta' = H' + \sum_{i=1}^{n'} a'_i \ast A'_i.
\] (5.1-1)
There is a reduction sequence \((MW4)\) must be found. More precisely, we will reconstruct a new reduction sequence based on matrix indices. If \(p\in(2.3-7)\) in Formula (5.1-1):

- case \((\text{I})\) \(p<\) the row indices of all the links in the \((p,q)\)-block of \(H'\);
- case \((\text{II})\) \(p\geq\) some row index of at least one link in the \((p,q)\)-block of \(H'\).

It is clear that there is no link above the \((p,q)\)-th block, since \(B'\) is already local.

**Lemma 5.1.2** Let \(p_x\) be the row index of \(x\) in \(H'\), then \(p_x>p\) in Classification 5.1.1.

**Proof.** Since \(x\) appears before the first arrow \(a_1'\) in \(B'\), \(p_x>p\) by the order of reductions according to matrix indices. If \(p_x=p\), then the parameter \(x\) locates at the left side of \(a_1'\) in \(H'+\Theta'\), \(\delta(a_1')\) contains only the terms of the form \(\alpha x\), \(\alpha \in k\), which contradicts to the assumption that \(B'\) is in the case of MW5. Thus \(p_x>p\).

**Example 5.1.3** Let \((A,B)\) be a pair constructed by an algebra defined in Example 1.4.5. There is a reduction sequence \((A,B), (A^1,B^1), (A^2,B^2), (A^3,B^3)\) given in Examples 2.4.5, such that the bocs \(B^3\) is strongly homogeneous in the case of MW5 described in Remark 3.1.7 (iii). In order to prove that \((A,B)\) is not homogeneous, another way different from the proof of MW1–MW4 must be found. More precisely, we will reconstruct a new reduction sequence based on the matrix \(\tilde{M}\) over \(k[x]\) with the size vector \(\tilde{m}=(2,2,2,2,3,3,3,3,3)\):

\[
\tilde{M} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

There is a reduction sequence \((\tilde{A},\tilde{B}), (\tilde{A}^1,\tilde{B}^1), (\tilde{A}^2,\tilde{B}^2), (\tilde{A}^3,\tilde{B}^3)\) corresponding to the steps (i)–(iii) of Example 2.4.5, where the reduction from \(B\) to \(B^1\) is given by \(a \mapsto (0,1)\) in the sense of Lemma 2.3.2. Thus \(b\) splits into \(b_1, b_2\) in \(B^1\), and set \(b_1\mapsto(0), b_2\mapsto(0,1)\) from \(B^1\) to \(B^2\). \(B^3\) is obtained from \(B^2\) by an edge reduction \((0)\), followed by a loop mutation, then four regularizations:

\[
\tilde{H}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + A + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

The \((1,5)\)-th block partitioned under \(T\) in the formal equation of \((\tilde{A}^3,\tilde{B}^3)\) is of the form:

\[
\begin{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\end{pmatrix} + X
\]
and $\tilde{d}(d_{22}) = u_{21}^1 - v_{11}^2 - d_{20}s_{02} - d_{21}v, \tilde{d}(d_{10}) = -v_{10}^2 - v_{20}^1 + v_d d_{20}, \tilde{d}(d_{11}) = u_{11}^2 - v_{11}^1 - v_{21}^1 - d_{10}s_{01}, \tilde{d}(d_{12}) = u_{11}^1 + u_{11}^2 - v_{12}^1 - v_{22}^1$, where $\tilde{d}$ is obtained from $d$ by removing the monomials, which involve a solid arrow $d_{22}, d_{10}$ or $d_{11}$. It is clear that the bocs $\mathfrak{B}^3$ satisfies the hypothesis of Proposition 3.4.5. In fact, as $d_{20} \rightarrow (1)$, the solid loops $d_{21}, d_{22}, d_{10}, d_{11}, d_{12}$ will be regularized step by step, because $s_{01}, s_{02}, v_{10}^2, u_{11}^2, u_{11}^1$ are pairwise different dotted arrows. Therefore $\mathfrak{A}, \mathfrak{B}$ is not homogeneous.

Motivated by Example 5.1.3, the general cases are considered. Since the example satisfies Case (I) of Classification 5.1.1, we start from Case (I) in subsection 5.1–5.3.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a bipartite matrix bimodule problem satisfying RDCC condition. Let $\mathfrak{A}'$ be an induced matrix bimodule problem with trivial $R'$, and let $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$ be the induced functor. Suppose $M = \vartheta(H'(k)) = \sum_j M_j \ast A_j \in R(\mathfrak{A})$ with a size vector $l \times \underline{n}$ over $\mathcal{T}$. Let $q = q_2 \in \mathcal{T}_{2}$ for some $Z \in \mathcal{T}_{2}$. Define a size vector $l \times \underline{n}$ over $\mathcal{T}$, and construct a bordered matrix $M = \sum_j \tilde{M}_j \ast A_j \in R(\mathfrak{A})$ with a zero column as follows:

$$\tilde{n}_j = \begin{cases} n_j, & \text{if } j \notin Z, \\ n_j + 1, & \text{if } j \in Z. \end{cases} \quad \tilde{M}_j = \begin{cases} M_j, & \text{if } A_j1Z = 0, \\ (0M_j), & \text{if } A_j1Z = A_j. \end{cases} \quad \text{(5.1-2)}$$

Denote by $(p, q+1)$ the leading position of $M$, such that $q+1$ is the index of the first column of the $q$-th block-column of $M$ partitioned under $\mathcal{T}$. Denote by $\tilde{T}$ the added column index of $M$, the column is sitting in the $q_2$-th block-column as the first one. Applying Theorem 2.4.1, the defining system $\mathfrak{E}$ of $K_0' \oplus K'_1$ given by Formula (2.4-2), and a matrix equation $\mathfrak{E}$ are considered:

$$\mathfrak{E} : \Phi_1^1 M \equiv \bigwedge_{< (p, q+1)} M \Phi_1^2, \quad \tilde{\mathfrak{E}} : \Phi_1^1 \tilde{M} \equiv \bigwedge_{< (p, q)} \tilde{M} \Phi_2^2, \quad \text{(5.1-3)}$$

where the upper scripts $1, 2$ on $\Phi$ stand for the left and right parts of the bipartite variable matrix $\Phi$, the two sets of variables in two parts do not intersect. Since $\mathfrak{A}$ satisfies RDCC condition, the main block-column in $\Phi^2_1$ determined by $Z \in \mathcal{T}$ can be written as $\Phi^2_1 Z = (\Phi^1_1, \ldots, \Phi^2_1)_{\mathcal{T}}$, such that either $\Phi^2_1 = 0$ or $\Phi^2_1 = (z_{pq}^1) \neq 0$, where $z_{pq}^1$ are variables over $k$.

It is clear that the $\tilde{T}$-th column of $\Phi_1^1 \tilde{M}$ is a zero column, we may define two new matrix equations respectively:

$$\mathfrak{E}_{\tau} : 0 \equiv \bigwedge_{< (p, q+1)} M \Phi_1^2, \quad \tilde{\mathfrak{E}}_{\tau} : 0 \equiv \bigwedge_{< (p, q)} \tilde{M} \Phi_2^2. \quad \text{(5.1-4)}$$

Taken any integer $p' \geq p$ and $1 \leq j \leq n_Z$, the $(p', q + j)$-th entry of the right side of $\mathfrak{E}_{\tau}$ is:

$$\sum_{\Phi^2_{1, Z} \neq 0} \sum_{q'} \alpha_{p'q'}^j z_{q', q+}^j. \quad \text{(5.1-5)}$$

It is easy to see that $z_{q', q+}^j$ for all possible $j$ have the same coefficient $\alpha_{p'q'}^j$, the $(p', q')$-th entry of $H(k)$. In the picture below, $n_Z = 5, p' = p$, those five equations are indicated by five circles, and the five variables at the same row of $\Phi^2_{1, Z}$ have the same coefficients.
Lemma 5.1.4 With the notations as above.

(i) The \((p, q + j)\)-th equation is a linear combination of the previous equations in \( \mathbb{E}_x \) if and only if so is the \((p, q + j_2)\)-th equation. Similarly, the same result is valid in \( \mathbb{E}_x \).

(ii) The equations in the system \( \mathbb{E} \) (resp. \( \mathbb{E}_x \)) and those in \( \mathbb{E}_x \) (resp. \( \mathbb{E}_x \)) are the same at the same positions of each main block column, whenever the added \( \tilde{q} \)-th column has been dropped from \( \mathbb{E} \) (resp. \( \mathbb{E}_x \)).

(iii) A subsystem of \( \mathbb{E} \) (resp. \( \mathbb{E}_x \)) consisting of the \( \tilde{q} \)-column in both sides is \( \mathbb{E}_{r \tilde{q}} : 0 \equiv \tilde{M} \Phi_{\tilde{R} \tilde{q}} \), where \( \Phi_{\tilde{R} \tilde{q}} \) stands for the \( \tilde{q} \)-th column of \( \Phi_{\tilde{R}} \). And \( \mathbb{E}_{r \tilde{q}} \) can be solved independently.

(iv) If the \((p, q + j)\)-th equation is a linear combination of the previous equations of \( \mathbb{E} \), then so is the \( p \)-th equation of \( \mathbb{E}_{r \tilde{q}} \).

Proof (i) The assertion follows from Formula (5.1-5).

(ii) Recall that \( q \in \mathbb{Z} \), denote by 0 the index of the first column (row) in the \( q' \)-th block column (row) for any \( q' \in \mathbb{Z} \). For any \( X \in T_{(1)}, Y \in T_{(2)} \), set \( 1 \leq \alpha \leq n_X \) and \( 1 \leq \beta \leq n_Y \). We claim that the \((\alpha, \beta)\)-th equations in the \((h, q_y)\)-th block of \( \mathbb{E} \) and \( \mathbb{E}_x \) for any \( h \in X \) are the same. In fact, the variable matrix \( \Phi_1 \) in the two systems \( \mathbb{E} \) and \( \mathbb{E}_x \) is common; and the \( \beta \)-th column of the \( h \)-th block row in \( M \) and \( \tilde{M} \) are the same in the left side of two equations. Now consider the right side of two equations. Let \( (M_1, \ldots, M_t) \) be the \( \alpha \)-row of the \( h \)-th block row in \( M \) with \( M_j = (\lambda_{j1}, \ldots, \lambda_{jn_j}) \) and that in \( \tilde{M} \) is \( (\tilde{M}_1, \ldots, \tilde{M}_t) \). Then \( \tilde{M}_j = M_j, \forall j \notin Z \); but \( \tilde{M}_j = (0, \lambda_{j1}, \ldots, \lambda_{jn_j}), \forall j \in Z \). Let \( (\Phi_1, \ldots, \Phi_t)^T \) be the \( \beta \)-column of the \( q_x \)-th block column in \( \Phi_\tilde{R}^2 \) with \( \Phi_j = (x_{j1}, \ldots, x_{jn_j}) \) and that in \( \Phi_{\tilde{R}}^2 \) is \( (\tilde{\Phi}_1, \ldots, \tilde{\Phi}_t)^T \), then \( \tilde{\Phi}_j = \Phi_j, \forall j \notin Z \); but \( \tilde{\Phi}_j = (x_{j0}, x_{j1}, \ldots, x_{jn_j})^T, \forall j \in Z \). Thus the additional variables \( x_{j0}, \forall j \in Z \), are killed by 0 in the right side of the equation of \( \mathbb{E}_x \).

(iii) The \( \tilde{q} \)-column in the left side of the matrix equation \( \mathbb{E} \) is a zero column. The variables of \( \Phi_{\tilde{R} \tilde{q}} \) are different from those in \( \Phi_1 \) and in the main block columns of \( \Phi_{\tilde{R}} \) except the \( \tilde{q} \)-th column.

(iv) If there exists some \( 1 \leq j \leq n_x \), such that the \((p, q+j)\)-th equation is a linear combination of the previous equations in \( \mathbb{E}_x \), then so is the \((p, q+j)\)-th equation in \( \mathbb{E}_x \) after deleting \( \Phi_1 \), since the variables of \( \Phi_1 \) and \( \Phi_{\tilde{R}} \) are different. Thus so is the \((p, \tilde{q}+j)\)-th equation in \( \mathbb{E}_x \) by (ii), and so is the \((p, \tilde{q})\)-th equation in \( \mathbb{E}_x \) by (i), finally, so is the \( p \)-th equation in \( \mathbb{E}_{r \tilde{q}} \).

5.2 Bordered matrices in bipartite case

This subsection is devoted to constructing a reduction sequence based on a given sequence and a bordered matrix, which generalizes Example 5.1.3.

Let \( \mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, \mathcal{H} = 0) \) be a bipartite matrix bimodule problem, which has a trivial \( \mathcal{T} \) and satisfies RDCC condition. Let \( \mathfrak{A}^s = (R^s, \mathcal{K}^s, \mathcal{M}^s, \mathcal{H}^s) \) be an induced matrix bimodule problem with \( R^s \) being local and trivial. Then there is a unique sequence of reductions in the sense of Lemma 2.3.2 by Corollary 2.3.5:

\[ \mathfrak{A}, \mathfrak{A}^1, \ldots, \mathfrak{A}^i, \mathfrak{A}^{i+1}, \ldots, \mathfrak{A}^s. \quad (\ast) \]

Write \( M = \vartheta^0 s(\mathcal{H}^s(k)) = \sum_j M_j * A_j \in R(\mathfrak{A}) \) with a size vector \( l \times n \), where \( M_j = G_j^{s+1}(k) \) are given by Formula (2.3-7). Suppose the size vector \( l \times \tilde{n} \) and the representation \( \tilde{M} = \sum_j \tilde{M}_j * A_j \in R(\tilde{\mathfrak{A}}) \) are defined by Formula (5.1-2).

Theorem 5.2.1 There exists a unique reduction sequence based on the sequence (\( \ast\)):

\[ \mathfrak{A}, \tilde{\mathfrak{A}}^1, \ldots, \tilde{\mathfrak{A}}^i, \tilde{\mathfrak{A}}^{i+1}, \ldots, \tilde{\mathfrak{A}}^s \quad (\tilde{\ast}) \]

where \( \tilde{\mathfrak{A}}^i = (\tilde{R}^i, \tilde{\mathcal{K}}^i, \tilde{\mathcal{M}}^i, \tilde{\mathcal{H}}^i) \), the reduction from \( \tilde{\mathfrak{A}}^i \) to \( \tilde{\mathfrak{A}}^{i+1} \) is a reduction or a composition of two reductions in the sense of Lemma 2.3.2. Moreover, \( \tilde{T}^s \) has two vertices, and

\[ \vartheta^0 s(\tilde{\mathcal{H}}^s(k)) = \tilde{M}. \]
Proof We may assume that \( l \times n \) is sincere over \( T \). Otherwise, it is possible to obtain an induced problem \( \mathcal{A}' \) by a suitable deletion, such that \( M \) has a sincere size vector over \( \mathcal{A}' \). In particular, \( \mathcal{A}' \) is still bipartite and satisfies RDCC condition.

We will construct a sequence \( \tilde{z} \) inductively. The original term in the sequence is \( \tilde{A} = \mathcal{A} \). Suppose that a sequence \( \tilde{A}, \tilde{A}^1, \ldots, \tilde{A}^i \) for some \( 0 \leq i < s \) has been constructed and \( \tilde{\vartheta}_i : R(\tilde{A}^i) \to R(\tilde{A}) \) is the induced functor, such that there exists a representation

\[
\tilde{M}^i = \tilde{H}_{i}^i(k) + \sum_{j=1}^{n_i} \tilde{M}^i_j \ast \tilde{A}^i_j \in R(\tilde{A}^i), \quad \text{with} \quad \tilde{\vartheta}_i(\tilde{M}^i) \simeq \tilde{M} \in R(\tilde{A}^0). \tag{5.2-1}
\]

Write \( M^i = \vartheta_i(H^i(k)) = H_{i}^{m_i}(k) + \sum_{j=1}^{n_i} M^i_j \ast A^i_j \in R(\mathcal{A}^i) \), where \( M^i_1 = G^{i+1}_s(k) \) by Formula (2.3-7), and is denoted by \( B \) for simplicity. The first column in the \( q \)-th main block-column of \( M \) under the partition \( T \) is denoted by \( \beta \). Now we are constructing \( \tilde{A}^{i+1} \).

Case 1 \( \tilde{T}^i = T^i \) and \( \tilde{A}^i = \mathcal{A}^i \).

1.1 \( B \cap \beta \) is empty. Then \( \tilde{G}^{i+1} = G^{i+1}, \tilde{H}^{i+1} = H^{i+1} \) and \( \tilde{A}^{i+1} = \mathcal{A}^{i+1} \).

Before giving the following cases, we claim that if \( B \cap \beta \) is non-empty, \( B \) thus \( G^{i+1} \) can not be Weyr matrices. Otherwise, the first arrow \( a_1^i \) of \( \mathcal{B}^i \) will be a loop. Since \( \tilde{T}^i = T^i, \tilde{a}_1^i \) will also be a loop and hence the numbers of rows and columns of \( \tilde{B} \) are the same. When the matrix \( B \) is enlarged by one column, then \( B \) is also enlarged by one row, which is a contradiction to the construction of \( \tilde{M} \). So \( B \) is either \( \emptyset \) from a regularization or \( (0 \ 0 \ 0) \) from an edge reduction.

1.2 \( B \cap \beta \) is non-empty, and \( B = (\emptyset) \). Then \( \tilde{B} = (\emptyset B) \) with \( \emptyset \) being a distinguished zero column, \( \tilde{H}^{i+1} = H^{i+1} \) and \( \tilde{A}^{i+1} = \mathcal{A}^{i+1} \) by a regularization.

1.3 \( B \cap \beta \) is non-empty, \( B = (0 \ 0 \ 0) \) and \( r < \text{the number of columns of } B \). Then \( \tilde{B} = (0 \ 0 \ 0) \) with \( 0 \) being a zero column, \( \tilde{H}^{i+1} = H^{i+1} \) and \( \tilde{A}^{i+1} = \mathcal{A}^{i+1} \) by edge reduction.

1.4 \( B \cap \beta \) is non-empty, \( B = (0 \ 0 \ 0) \). Then \( \tilde{B} = (0 \ 0 \ 0) \) with \( 0 \) being a zero column. Recall from Formula (2.3-5) and Theorem 2.3.3, the following is defined:

\[
\tilde{G}^{i+1} = \begin{cases} 
(0 \ 1 \ Z_2), & \text{if } G^{i+1} = (1 \ Z_2); \\
(0 \ 0 \ 0), & \text{if } G^{i+1} = (1 \ 0) \end{cases}
\]

Then \( \tilde{H}^{i+1} = \sum_{X \in T^i} I_X \ast H^i_X + \tilde{G}^{i+1} \ast A^i_1 \). Consequently \( \tilde{A}^{i+1} \) is induced from \( \tilde{A}^i \) by an edge reduction in the sense of Lemma 2.3.2.

We stress, that after the edge reduction in the subcase 1.4, \( \tilde{T}^{i+1} = T^{i+1} \cup \{Y\} \), where \( Y \) is an equivalent class consisting of the indices of the added columns in the sum \( \tilde{H}^{i+1} + \tilde{G}^{i+1} \) of the pair \((\tilde{A}^{i+1}, \tilde{A}^{i+1})\), and \((\tilde{A}^{i+1}, \tilde{A}^{i+1}) \neq (\mathcal{A}^{i+1}, \mathcal{A}^{i+1}) \) from this stage. The above \( B \) is show in four cases as a small block in the corresponding leading block partitioned under \( T \) in \( \tilde{M} \):

Case 2. \( \tilde{T}^i = T^i \cup \{Y\} \).

2.1 \( B \cap \beta \) is empty. Then \( \tilde{B} = B, \tilde{G}^{i+1} = G^{i+1} \), and \( \tilde{H}^{i+1} = \sum_{X \in \tilde{T}^i} I_X \ast \tilde{H}^i_X + \tilde{G}^{i+1} \ast \tilde{A}_1 \).

If \( B \cap \beta \) is non-empty. Denote by \( \tilde{a}_0^i \) and \( \tilde{a}_1^i \) the first and the second solid arrows of \( \tilde{A}^i \), which locate at \((p', q_0')\) and \((p', q_0' + 1)\) in the formal product \( \tilde{\vartheta}^i \) respectively.
2.2 \( B \cap \beta \) is non-empty, and there exists some \( 1 \leq j \leq n_Z^i \), such that the \((p^i, q^i + j)\)-th equation is a linear combination of previous equations in \( E^i_Z \). Then \( \delta(\tilde{a}^1_0) = 0 \) by Lemma 5.1.4 (ii) then (i), and Corollary 2.4.2. Two reductions are made: the first one is an edge reduction by \( \tilde{a}^1_0 \mapsto (\emptyset) \); and the second one for \( \tilde{a}^1_i \) is made in the same way as that for \( a^1_i \) by Lemma 5.1.4 (ii). Then an induced problem \( \tilde{A}^{i+1} \) is obtain, and \( \tilde{B} = (\emptyset B) \) with 0 being a zero column.

2.3 \( B \cap \beta \) is non-empty, and for all \( 1 \leq j \leq m_Z^i \), the \((p^i, q^i + j)\)-th equation is not a linear combination of previous equations in \( E^i_Z \). Thus \( \delta(\tilde{a}^1_0) \neq 0 \) by Lemma 5.1.4 (i) and Corollary 2.4.2. And \( \delta(\tilde{a}^1_i) \neq 0 \) for any \( 1 \leq j \leq n_Z^i \) by 5.1.4 (i)-(ii) and Corollary 2.4.2. Then two regularizations \( \tilde{a}^1_0 \mapsto (\emptyset) \), \( \tilde{a}^1_i \mapsto (\emptyset) \) are made, and \( \tilde{B} = (\emptyset B) \) with \( \emptyset \) being a distinguished zero column.

In the cases 2.2 and 2.3, there are two reduction blocks \( \tilde{G}^{i+1,0} = (0) \) or \((\emptyset)\), \( \tilde{G}^{i+1,1} = G^{i+1} \), thus \( \tilde{H}^{i+1} = \sum_{\tilde{X} \in \mathcal{T}^i} \tilde{I}_{\tilde{X}} \ast \tilde{H}^{i+1}_{\tilde{X}} + \tilde{G}^{i+1,0} \ast \tilde{A}^1_0 + \tilde{G}^{i+1,1} \ast \tilde{A}^1_i \).

By summing up all the cases, an induced pair \( (\tilde{A}^{i+1}, \tilde{B}^{i+1}) \) and a representation \( \tilde{M}^{i+1} \) with \( \tilde{G}^{0,i+1}(M^{i+1}) \simeq \tilde{M} \) are obtained. The theorem follows by induction. \( \square \)

**Corollary 5.2.2** The main diagonal block \( \tilde{c}^1_Z, Z \in \mathcal{T} \), of \( \tilde{K}^1_0 \oplus \tilde{K}^1_1 \) is of the form:

\[
\begin{pmatrix}
  s_{00} & s_{01} & \cdots & s_{0m} \\
  s_{11} & s_{12} & \cdots & s_{1m} \\
  s_{22} & \cdots & s_{2m} \\
  & & \ddots & \\
  & & & s_{mm}
\end{pmatrix}
\]

where \( m = n_Z^i, s_{01}, s_{02}, \ldots, s_{0m} \) are dotted arrows of \( \tilde{B}^i \).

**Proof** By the construction of \( \tilde{H}^i \), the added “0-column” can be only 0 or \( \emptyset \). Therefore, except \( s_{00} \), the elements at the 0-th row: \( s_{01}, s_{02}, \ldots, s_{0m} \) do not appear in any equation of the defining system of \( \tilde{A}^i \). Thus they are free. \( \square \)

### 5.3 Non-homogeneity in the case of MW5 and classification (I)

The discussion of this subsection is two folds: 1) extend the reduction sequence \( (\hat{s}) \) of Theorem 5.2.1 into a sequence \( (\hat{s}') \), such that there is a parameter \( x \) appearing from the \( (s + 1)\)-th step; 2) prove that any bipartite pair with an induced minimal wild pair in the case of MW5 and Classification 5.1.1 (I) is not homogeneous.

Suppose we have a reduction sequence ending at \( \tilde{A}^t \) defined by Remark 3.4.6:

\[
\tilde{A}, \tilde{A}^1, \ldots, \tilde{A}^s, \tilde{A}^{s+1}, \ldots, \tilde{A}^t, \ldots, \tilde{A}^t = \tilde{A}^t, \quad (s')
\]

where the reduction from \( \tilde{A}^i \) to \( \tilde{A}^{i+1} \) is in the sense of Lemma 2.3.2 for \( 1 \leq i < s \); \( \tilde{A}^s \) is local with \( \delta(a^1_s) = 0 \), set \( a^1_s \mapsto (x) \), \( \tilde{A}^{s+1} \) has a parameter \( x \) locating at the \((p_x, q_x)\)-position of \( H^{s+1} \) and \( R^{s+1} = k[x] \); \( \tilde{A}^{i+1} \) is obtained from \( \tilde{A}^i \) by a regularization for \( s < i < t \). The pair \( (\tilde{A}^t, \tilde{B}^t) \) is in the case of MW5 given by Remark 3.4.6 and satisfying Classification 5.1.1 (I).

Note that the set of integers \( T^i \) and its partition \( T^i \) are all the same for \( i = s, \ldots, t \). Suppose the first arrow \( a^1_s \) of \( \tilde{B}^s \) locates at the \((p, q^s)\)-th position in the formal product \( \Theta^i \) with \( q^s = q + j \) for some \( 1 \leq j \leq n_Z^i \); the first arrow \( a^1_s \) of \( \tilde{B}^s \) locates at the \((p, q + 1)\)-th position in \( \Theta^i \), where \( q + 1 \) is the index of the first column in the \( q \)-th block-column. The picture below shows the position of the first solid arrows in the formal products \( \Theta^i \) of \( (\tilde{A}^i, \tilde{B}^i) \) for \( i = s, \epsilon, t \) (whenever the added \( q_0 \)-th column is ignored):
Assume $R^i = k[x, \phi^i(x)]$, and $H^i$ has size vector $l^i \times n^i$ over $T$ for $s < i \leq t$. Denote uniformly the same size vector $l^i$ by $l$ over $T_i$ with size $l$, and $n^i$ by $n$ over $T_{i+1}$ with size $n$. Then $H^i \subseteq \text{IM}_{l \times n}(R^i)$ are all in the same form but with different $\phi^i(x)$. Since $k(x)$ is an $R^i$-bimodule, $H^i \otimes_{R^i} 1_{k(x)} \subseteq \text{IM}_{l \times n}(R^i) \otimes_{R^i} k(x) \cong \text{IM}_{l \times n}(k(x)).$

Remark 5.3.1 Lemma 5.1.4 (i)-(iv) are still valid if the matrices $M^i$ and $\tilde{M}^i$ over $k(x)$ instead of over $k$ are considered.

Recall the matrix equation defined in 5.1.4 (iii):

$$\tilde{E}_{\tau \bar{q}} : 0 \equiv \tilde{M} \Phi_{\bar{q}, \bar{q}}. \quad (5.3-2)$$

Suppose $x$ locates in the $(p, q, Z')$-th main block partitioned under $T$, see Picture (5.3-1) for $Z' \neq Z$; and Example 5.1.3 for $Z' = Z$. Thus the equation system $\tilde{E}_{\tau \bar{q}}(p, q, Z)$, consisting of the equations below the $p_x$-th row, is over the base field $k$. Denote by $n$ the size of $\tilde{\Psi}$. The solution space of the system $\tilde{E}_{\tau \bar{q}}(p, q, Z)$ contained in $\text{IM}_{n \times 1}(k)$ is a direct sum of two subspaces by Theorem 5.1.4 and Lemma 5.1.4 (iii):

(i) the first subspace is isomorphic to a space spanned by $E_Y$ in $\tilde{K}_{\bar{q}}$, it has a base matrix with the $\bar{q}$-th entry being $1_Y$ and others zero;

(ii) the second one is isomorphic to a subspace of $\tilde{K}_{\bar{q}}$, and its base matrices have non-zero entries only above the $\bar{q}$-th entry.

The second subspace is denoted by $K_{\tau \bar{q}}^{(p, q)}$ with a polynomial $d_{p, q}^0(x) = 1$; a minimal algebra $R_{\tau \bar{q}}^{(p, q)} = \tilde{R}^s = kI_X \times kI_Y$; and a basis $\{U_0^0, \ldots, U_0^\beta\}$ with the dual basis $\{u_0^0, \ldots, u_0^\beta\} \subseteq \text{Hom}(K_{\tau \bar{q}}^{(p, q)}, R_{\tau \bar{q}}^{(p, q)})$. From now on, we consider the equation $\tilde{E}_{\tau \bar{q}}^{(p, q)} : 0 \equiv \tilde{M} \Psi_{\tau \bar{q}}^{(p, q)}$ with the variable matrix $\Psi_{\tau \bar{q}}^{(p, q)} = \sum_{\zeta = 1}^\beta u_0^0 \ast U_0^\zeta$ according to Theorem 2.4.4 and Formula (2.4-5).

Suppose the system $\tilde{E}_{\tau \bar{q}}^{(h)}$ for some $p < h \leq p_x$ have been solved, the solution space $K_{\tau \bar{q}}^{(h)}$ has a basis $\{U_1, \ldots, U_h\} \subseteq \text{IM}_{n \times 1}(R_{\tau \bar{q}}^{(h)})$, where $R_{\tau \bar{q}}^{(h)} = k[x, (\gamma^{h+1}(x))^{-1}]I_X \times kI_Y$ is a minimal algebra with $\gamma^{h+1}(x) = \prod_{\eta=p_x}^{h+1} d^\eta(x)$. Let $\{u_1, \ldots, u_h\} \subseteq \text{Hom}(K_{\tau \bar{q}}^{(h)}, R_{\tau \bar{q}}^{(h)})$ be the dual basis of $\{U_j\}_j$. According to Formula (2.4-5):

$$\Psi_{\tau \bar{q}}^{(h)} = \sum_{\zeta = 1}^\gamma u_\zeta \ast U_\zeta, \quad \tilde{E}_{\tau \bar{q}}^{(h)} : 0 \equiv \tilde{M} \Psi_{\tau \bar{q}}^{(h)}. \quad (5.3-3)$$

The $h$-th equation of $\tilde{E}_{\tau \bar{q}}^{(h)}$ is $\sum_{\zeta = 1}^\gamma f_\zeta(x) \ast u_\zeta$ with $f_\zeta(x) \in R_{\tau \bar{q}}^{(h)}$. There are two possibilities.

(i) $f_\zeta(x) = 0$ for $\zeta = 1, \ldots, \kappa$. In this case $K_{\tau \bar{q}}^{(h-1)} = K_{\tau \bar{q}}^{(h)}$, and the quasi-basis of $K_{\tau \bar{q}}^{(h)}$ are preserved in $K_{\tau \bar{q}}^{(h-1)}$. Let $d^\eta(x) = 1$. 

Picture (5.3-1)
(ii) There exists some \( f_\zeta(x) \neq 0 \). Without loss of generality, it may be assumed that \( f_\zeta(x) \neq 0 \). Choose a new basis in the dual space \( \text{Hom}_k(x)(K^{(h)}_{r\bar{q}} \otimes R^{(h)}_{r\bar{q}} k(x), k(x)) \) at the first line of the formula below, the corresponding basis of \( K^{(h)}_{r\bar{q}} \) is shown at the second line:

\[
\begin{align*}
  u'_\zeta &= u_\zeta, \\
  U'_\zeta &= U_\zeta - f_\zeta(x)/f_\kappa(x)U_\kappa, \quad \text{for } 1 \leq \zeta < \kappa; \\
  u'_\kappa &= \sum_{\zeta=1}^\kappa f_\zeta(x)u_\zeta, \\
  U'_\kappa &= 1/f_\kappa(x)U_\kappa,
\end{align*}
\]

where \( u'_\kappa = 0 \) is the solution of the \( h \)-th equation in the system (5.3-3). Let \( d^h(x) \in k[x] \) be the numerator of \( f_\kappa(x) \), and \( \gamma^h(x) = d^h(x)\gamma^{h+1}(x) \), then \( R^{(h-1)}_{r\bar{q}} = k[x, (\gamma^h(x))^{-1}]_1 x \times k1_y \). Thus \( K^{(h-1)}_{r\bar{q}} \) has a quasi-basis \( \{U'_\zeta \mid \zeta = 1, \cdots, \kappa - 1\} \) over \( R^{(h-1)}_{r\bar{q}} \). The system \( \tilde{H}^{(p-1)}_{r\bar{q}} \) with the solution space \( K^{(p-1)}_{r\bar{q}} \) and polynomial \( \gamma^p(x) \) is finally reached by inverse-order induction.

Suppose \( R^i = k[x, \phi^i(x)^{-1}] \), and the row index of the first arrow of \( \mathfrak{A}^i \) in the formal product \( \Theta^i \) is \( p^i, p_2 \leq p^i \leq p \) for \( s \leq i \leq e \). Define

\[
\tilde{\phi}^i(x) = \phi^i(x)\gamma^p(x) \in k[x], \quad \text{in particular} \quad \tilde{\phi}^i(x) = \phi^i(x)^{\gamma^p(x)}. \tag{5.3-5}
\]

Now we deal with representations of \( \mathfrak{A}^i \) over the field \( k(x) \). Suppose the matrix \( M^i = \vartheta^0_i(H^i(x, \tilde{\phi}^i(x)^{-1})) = \sum_j M^i_j + A_j \) has a size vector \( l \times \bar{m} \) over \( T \), and a matrix \( \bar{M}^i = \sum_j \bar{M}^i_j + A_j \) of size vector \( \bar{l} \times \bar{m} = \text{size of } l \times \bar{m} \) is defined by Formula (5.1-2). Returning to Theorem 2.4.1, the matrix equations for \( i \geq s \) are defined as follows:

\[
\begin{align*}
  E^i : \Phi^i M^i &\equiv_{(p^i, q^i)} M^i \Phi^i, \\
  E^i_r : \Phi^i \bar{M}^i &\equiv_{(p^i, q^i)} \bar{M}^i \Phi^i, \\
  E^i_l : 0 &\equiv_{(p^i, q^i)} \bar{M}^i \Phi^i.
\end{align*}
\]

Theorem 5.3.2 There exists a unique reduction sequence based on the sequence (\( *' \)):

\[
\mathfrak{A}, \mathfrak{A}^1, \cdots, \mathfrak{A}^s, \mathfrak{A}^{s+1}, \cdots, \mathfrak{A}^e, \cdots, \mathfrak{A}^t = \mathfrak{A}', \quad (\mathfrak{A}')
\]

where the first part of the sequence up to \( \mathfrak{A}^s \) is given by Theorem 5.2.1; the reduction from \( \mathfrak{A}^s \) to \( \mathfrak{A}^{s+1} \) is given by a loop mutation \( a^s_1 \rightarrow (x) \), or an edge reduction \( (0) \) followed by a loop mutation \( (x) \); the reduction from \( \mathfrak{A}^t \) to \( \mathfrak{A}' \) for \( s < i < t \) is given by one regularization, or two regularizations, or a reduction given by Lemma 2.2.6, followed by a regularization.

**Proof** If the first arrow \( a^s_1 \) of \( \mathfrak{A}^s \) does not locate at the \( (q+1) \)-th column of the formal product \( \Theta^s \), a loop mutation from \( \mathfrak{A}^s \) to \( \mathfrak{A}^{s+1} \) is made. Otherwise an edge reduction is made by Remark 5.3.1, see 5.1.4 (iv) and Corollary 2.4.2 for details, then followed by a loop mutation.

Now suppose we have an induced bimodule problem \( \mathfrak{A}^i \) for some \( i > s \). If the first arrow \( a^i_1 \) of \( \mathfrak{B}^i \) does not locate at the \((q+1)\)-th column of \( \Theta^i \), a regularization \( a^i_1 \rightarrow \emptyset \) is made. Otherwise, there are two possibilities. 1 There exists some \( 1 \leq j < n_z \), the \((p^i, q+j)\)-th equation is a linear combination of the previous equations in \( \mathfrak{E}^i_{r_l} \), then \( \delta(\tilde{a}^{q}_{0}) = 0 \) by Remark 5.3.1 and Corollary 2.4.2. Set \( \tilde{a}^q_0 \rightarrow (0) \) by Lemma 2.6.6, \( \tilde{a}^q_1 \rightarrow \emptyset \). 2 Otherwise \( \delta(\tilde{a}^{q}_{0}) \neq 0 \). Set \( \tilde{a}^q_0 \rightarrow \emptyset \) and \( \tilde{a}^q_1 \rightarrow \emptyset \). The sequence (\( *' \)) is completed by induction as desired. \( \Box \)

**Corollary 5.3.3** If the bocs \( \mathfrak{B}^i \) in the sequence (\( *' \)) satisfies MW5 defined by Remark 3.4.6 and Classification 5.1.1 (I), then \( \delta(\tilde{a}^{q}_{0}) = 0 \) in \( \mathfrak{B}^i \) in the sequence (\( *' \)).

**Proof** Since the first arrow \( a^i_1 \) of \( \mathfrak{B}^i \) locates at the \((p, q+j)\)-th position with \( \delta(a^i_1) = 0, \delta^0(a^i_1) = 0 \) in \( \mathfrak{B}^i \). Thus \( \delta(\tilde{a}^{q}_{0}) = 0 \) in \( \mathfrak{B}^i \) by Remark 5.3.1 and Corollary 2.4.2. \( \Box \)

**Proposition 5.3.4** Let \( (\mathfrak{A}, \mathfrak{B}) \) be a pair with \( T \) being trivial, such that \( \mathfrak{A} = (R, K, M, H = 0) \) is a bipartite matrix bimodule problem satisfying RDCC condition. If there exists an induced
pair \((\mathfrak{A}', \mathfrak{B}')\) of \((\mathfrak{A}, \mathfrak{B})\) in the case of MW5 defined by Remark 3.4.6, and the sum \(H' + \Theta'\) of \((\mathfrak{A}', \mathfrak{B}')\) satisfies Classification 5.1.1 (I), then \(\mathfrak{B}\) is not homogeneous.

**Proof** Suppose we have a sequence \((*')\) with \(\mathfrak{B}' = \mathfrak{A}'\), then there is a sequence \((\hat{z}')\) based on \((*')\) by Theorem 5.3.2. Corollary 5.3.3 tells that the first arrow \(\hat{a}_0'\) of \(\hat{\mathfrak{B}}\) is an edge with \(\delta(\hat{a}_0') = 0\), and hence \(\hat{a}_0' \mapsto (1)\) may be set according to Proposition 2.2.7. The induced pair is obviously local. Thus it is possible to use the triangular formulae of Subsection 3.3, and an induced pair \((\hat{\mathfrak{A}}', \hat{\mathfrak{B}}')\) is obtained in one of the cases (ii)-(iv) of Classification 3.3.5.

Case 1 If 3.3.5 (ii) is met, then \(\hat{\mathfrak{B}}\) satisfies the hypothesis of Proposition 3.4.5. It is done.

Case 2 If 3.3.5 (iii) is met, then \(\hat{\mathfrak{B}}\) satisfies MW3, it is done by Proposition 3.4.3.

Case 3 If 3.3.5 (iv) is met, and \(\hat{\mathfrak{B}}\) satisfies MW4, it is done by Proposition 3.4.4.

Case 4 If 3.3.5 (iv) is met, and \(\hat{\mathfrak{B}}\) satisfies MW5, then there is an induced pair \((\hat{\mathfrak{A}}', \hat{\mathfrak{B}}')\) in the case of MW5 defined by Remark 3.4.6. In this case the pair \((\mathfrak{A}', \mathfrak{B}')\) is denoted by \((\hat{\mathfrak{A}}, \hat{\mathfrak{B}})\) in order to unify the notations. Suppose the first arrow \(\hat{a}_1'\) of \(\hat{\mathfrak{B}}\) locates at the \(p^1\)-th row in the formal product \(\Theta^1\). We claim that \(p^1 < p\). In fact, the solid arrows \(\hat{a}_j'\) for \(j = 1, \cdots, n_z\) at the \(p\)-th row of \(\Theta^1\) have differentials \(\delta(\hat{a}_j') = s_{0j} + \cdots\) according to Corollary 5.2.2, and hence those arrows will be regularized step by step.

Repeating the above mentioned procedure for \((\hat{\mathfrak{A}}, \hat{\mathfrak{B}})\), if one of the cases 1–3 is met, the procedure stops. Otherwise, if the case 4 is met repeatedly, there exist a sequence of local pairs and a decreasing sequence of the row indices:

\[
(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}), \quad \hat{\mathfrak{A}}', \hat{\mathfrak{B}}', \quad \hat{\mathfrak{A}}'', \hat{\mathfrak{B}}'', \quad \cdots, \quad \hat{\mathfrak{A}}^\beta, \hat{\mathfrak{B}}^\beta,
\]

Since the number of the rows of \(\hat{H}^i\) for \(i = 1, \cdots, \beta\) is fixed, the procedure must stop at some stage \(\beta\), where one of the cases 1–3 appears. The conclusion follows by induction. \(\square\)

### 5.4 Bordered matrices in one-sided case

In this subsection, a notion of reduced defining systems of Formula (2.4-3) is given for some induced pairs of a one-sided pair, which is different from Formula (4.1-7). Then some reduction sequences are constructed starting from one-sided pairs based on bordered matrices.

Let \((\mathfrak{A}, \mathfrak{B})\) be a bipartite pair satisfying RDCC condition, and \((\mathfrak{A}', \mathfrak{B}')\) be an induced pair with \(R'\) being trivial. Suppose \((\mathfrak{A}', \mathfrak{B}')\) has a quotient-sub pair \(((\mathfrak{A}')^{[m]}, (\mathfrak{B}')^{[m]})\) denoted by \((\hat{\mathfrak{A}}, \hat{\mathfrak{B}})\) given in Formulae (4.1-1) and (4.1-2), where the vertex set \(\hat{T} = \hat{T}_R \times \hat{T}_C \subseteq \hat{T}'\), and \(\hat{\mathfrak{B}}\) has a layer \(L = (R; \omega; d_1, \cdots, d_m; \bar{u}, \overline{w}, \bar{v}, \bar{v}')\) by Definition 4.1.2.

**Remark 5.4.1** Suppose \((\mathfrak{A}', \mathfrak{B}')\) with \(\hat{T}'\) being trivial is an induced pair of \((\hat{\mathfrak{A}}, \hat{\mathfrak{B}})\), then it is a quotient-sub-pair of \((\mathfrak{A}'^+, \mathfrak{B}'^+)\) by Formula (4.1-5), where \(\ell\) stands for some index. Recall Theorem 2.4.1 and the defining system \(F'^{r,r+1}\), the variable matrices \(\Psi_{m^r+r^+}^r, \Psi_{0^r+r^+}^r\) given by Formula (2.4-3), we now define its reduced form consisting of the \((p^r, q^r), \cdots (p^r, q^r + m - 1)\)-th blocks of \(\mathfrak{F}'^{r,r+1}\) according to Remark 4.1.1. Suppose there is a functor \(\hat{\varphi} : \hat{\mathfrak{A}}' \to \hat{\mathfrak{B}}\) acting on objects, \(F\) is defined below Formula (4.1-6), and \(\hat{\varphi}(F'(k))\) has a size vector \(\hat{m} = (n_0; n_1, \cdots, n_m)\) partitioned under \(\hat{T}\). 

(i) Denote by \(\bar{Z}_0\) the \((p^r, p^r+\ell)\)-the square block of \(\Psi_{m^r+r^+}^r\) of size \(n_0\) with \(p^r \in X^r\); by \(\bar{Z}_{\xi\ell}\) the \((q^r + \ell - 1, q^r + \ell - 1)\)-th square block of size \(n_\xi\) with \(q^r + \ell - 1 \in Y^r\) for any \(Y^r\). Set

\[
\bar{Z}_0 = Z_{X^r} = (\hat{\varphi}_{pq}^r)^{n_0 \times n_0}, \quad \bar{Z}_{\xi\ell} = Z_{Y^r} = (\hat{\varphi}_{pq}^r)^{n_\xi \times n_\xi}.
\]

(ii) Denote by \(\bar{Z}_{\eta\xi}\) the \((p^r, q^r + 1 - \eta)\)-block of \(\Psi_{m^r+r^+}^r\) with size \(n_0 \times n_\eta\), and by \(\bar{Z}_{\eta\xi}\) the \((q^r + 1 - \eta, q^r + 1 - 1)\)-block for \(\eta < \xi\) of \(\Psi_{m^r+r^+}^r\) with size \(n_\eta \times n_\xi\). Write the matrices
\[ Z_j = (z_{pq}^j)_{n(x_{iv}^j) \times n(v_{ij})} \] for all the dotted arrows of \( \mathcal{B}^r \), where \( \{z_{pq}^j\}_{(p,q),j} \) are different variables over \( k \). Suppose

\[
\tilde{Z}_z = \sum_j \alpha^j_z Z_j, \quad \text{where } s(v_{ij}^j) \supseteq p^r, \quad t(v_{ij}^j) \supseteq q^r + \xi - 1, \quad \alpha^j_z \in k; \\
\tilde{Z}_{nq} = \sum_j \beta^j_{nq} Z_j, \quad \text{where } s(v_{ij}^j) \supseteq q^r + \eta - 1, \quad t(v_{ij}^j) \supseteq q^r + \xi - 1, \quad \beta^j_{nq} \in k.
\]

Return to the pair \((\tilde{\mathfrak{A}}', \tilde{\mathcal{B}}')\), some indices \( p, q, \ldots \), in the formal product \( \tilde{\Theta}' \) will be used in order to distinguish with indices \( p, q, \ldots \), in the formal product \( \Theta^{r+1} \) of \((\tilde{\mathfrak{A}}^{r+1}, \tilde{\mathcal{B}}^{r+1})\) of Formula (5.4.1). Fix an integer \( l \in \{1, \cdots , m\} \) with \( Y_l \neq X \) in Definition 4.1.2, thus \( d_l : X \rightarrow Y_l \) is a solid edge. Suppose \( (\tilde{q}_l + 1) \) is the index of the first column in the \( l \)-th block-column of the formal product \( \tilde{\Theta}' \), such that \((\tilde{p}, \tilde{q}_l + 1)\) is the leading position of the first base matrix of \( \tilde{M}' \). Write \( \tilde{M} = \tilde{\vartheta}(F(k)) = (\tilde{M}_1, \cdots , \tilde{M}_m) \in R(\tilde{\mathfrak{A}}) \) with the size vector \( \tilde{q} \) over \( \tilde{T} \). Then

\[ \mathcal{F} : \tilde{Z}_0 \tilde{M} \equiv_{<(\tilde{p}, \tilde{q}_l + 1)} (\tilde{Z}_1, \cdots , \tilde{Z}_m) + \tilde{M}(\tilde{Z}_{nq})_{1 \leq \eta \leq \xi \leq m} \quad (5.4.1) \]

is called a reduced defining system based on Theorem 2.4.1, which is different from \( \mathcal{F} \) given by Formula (4.1.7). Similarly as in Equation (5.1-4), there is an equation system:

\[ \mathcal{F}_r : 0 \equiv_{<(\tilde{p}, \tilde{q}_l + 1)} (\tilde{Z}_1, \cdots , \tilde{Z}_m) + \tilde{M}(\tilde{Z}_{nq})_{1 \leq \eta \leq \xi \leq m}. \quad (5.4.2) \]

Define a size vector \( \tilde{n} = (\tilde{n}_0; \tilde{n}_1, \cdots , \tilde{n}_m) \) over \( \tilde{T} \) as follows: \( \tilde{n}_j = n_j \) if \( j \notin Y_l \); \( \tilde{n}_j = n_j + 1 \) if \( j \in Y_l \). Construct a representation of \( \tilde{\mathfrak{A}} \) based on \( \tilde{M} \):

\[ \tilde{M} = (\tilde{M}_1, \cdots , \tilde{M}_m) \in R(\tilde{\mathfrak{A}}), \quad \tilde{M}_j = \begin{cases} \tilde{M}_j, & \text{if } j \notin Y_l, \\ (0 \tilde{M}_j), & \text{if } j \in Y_l, \end{cases} \quad (5.4.3) \]

where 0 is a zero column. Write \( \tilde{Z}_0, \tilde{Z}_\xi \) the variable matrices of size \( \tilde{n}_0 \times \tilde{n}_0, \tilde{n}_\xi \times \tilde{n}_\xi \); and \( \tilde{Z}_\xi = \sum_j \alpha^j_\xi \tilde{Z}_j \) of size \( \tilde{n}_0 \times \tilde{n}_\xi \), \( \tilde{Z}_{nq} = \sum_{nq} \beta^j_{nq} \tilde{Z}_j \) of size \( \tilde{n}_n \times \tilde{n}_\xi \) respectively according to Remark 5.4.1. Denote by \( \tilde{q}_l \) the index of the first column of the \( l \)-th block-column of \( \tilde{M} \), we obtain the following two matrix equations:

\[ \mathcal{F} : \tilde{Z}_0 \tilde{M} \equiv_{<(\tilde{p}, \tilde{q}_l)} (\tilde{Z}_1, \cdots , \tilde{Z}_m) + \tilde{M}(\tilde{Z}_{nq})_{1 \leq \eta \leq \xi \leq m} \\
\mathcal{F}_r : 0 \equiv_{<(\tilde{p}, \tilde{q}_l)} (\tilde{Z}_1, \cdots , \tilde{Z}_m) + \tilde{M}(\tilde{Z}_{nq})_{1 \leq \eta \leq \xi \leq m}. \quad (5.4.4) \]

For any \( l' \in Y_l \), denote by \( \tilde{q}_{l'} + 1 \) the index of the first column of the \( l' \)-th block-column of \( \tilde{M} \). Set any integer \( \tilde{p}' \geq \tilde{p} \) and \( 1 \leq h \leq n_{Y_l} \), the \((\tilde{p}', \tilde{q}_{l'} + h)\)-th entry in the right side of \( \mathcal{F}_r \) of Formula (5.4.2) equals

\[ \sum_{q} \gamma_{\tilde{p}' q} z_{q, \tilde{q}_{l'} + h} + \sum_{q} \nu_{\tilde{p}' q} z_{q, \tilde{q}_{l'} + h}, \quad \gamma_{\tilde{p}' q}, \nu_{\tilde{p}' q} \in k, \quad t(v_{ij}) = Y_l. \quad (5.4.5) \]

The picture below shows four equations (abridged by four circles) of \( \mathcal{F}_r \). There are three solid edges ending at \( Y_l \), i.e. \( |Y_l| = 3 \), and \( n_{Y_l} = 4 \). If \( l' \) is the second index of \( Y_l \), then the equations at the \((\tilde{p}', \tilde{q}_{l'} + h)\)-th positions have the same coefficients for \( h = 1, 2, 3, 4 \).
**Lemma 5.4.2** Being parallel to Lemma 5.1.4, there are following assertions.

(i) For any \(1 \leq h_1, h_2 \leq n_l\) and any \(l' \in Y\), the \((p, \tilde{q}_l + h_1)\)-th equation is a linear combination of the previous equations in \(F_\tau\), if and only if so is the \((p, \tilde{q}_l + h_2)\)-th equation. Similarly, the same result is valid in \(\tilde{F}_\tau\).

(ii) The equations in the system \(\tilde{F}\) (resp. \(\tilde{F}_\tau\)) and those in \(F\) (resp. \(F_\tau\)) are the same at the same positions of each block column, whenever the added \(\tilde{q}_l\)-th columns for all \(l' \in Y\) have been dropped from \(\tilde{F}\) (resp. \(\tilde{F}_\tau\)).

(iii) A subsystem of \(\tilde{F}\) (resp. \(\tilde{F}_\tau\)) consisting of the \(\tilde{q}_l\)-column in both sides is \(\tilde{F}_{\tilde{r}_{\tilde{q}_l}} : 0 \equiv \tilde{M}_\tilde{q}_l\), where \(\tilde{M}_\tilde{q}_l\) stands for the \(\tilde{q}_l\)-th column of \(\tilde{M}_\tilde{q}_l\). And the system \(\{\tilde{F}_{\tilde{r}_{\tilde{q}_l}} \mid \forall l' \in Y\}\) can be solved independently.

(iv) If the \((p, \tilde{q}_l + h)\)-th equation for some \(1 \leq h \leq n_r\) is a linear combination of the previous equations in \(F\), then so is the \((p, \tilde{q}_l)\)-th equation in the system \(\{\tilde{F}_{\tilde{r}_{\tilde{q}_l}} \mid \forall l' \in Y\}\).

**Proof** (i)–(ii) See Proof (i)–(ii) of Lemma 5.1.4.

(iii) Note that \(\forall l' \in Y\) the variables at the \(\tilde{q}_l\)-th column of \((\tilde{Z}_\xi)_{1 \leq \xi \leq m}\) and \((\tilde{Z}_\eta)_{1 \leq n \leq \xi \leq m}\) are different from those at the \(h\)-th column for all \(h \neq \tilde{q}_l, \forall l' \in Y\), and different from those in \(\tilde{Z}_0\).

(iv) See proof (iv) of 5.1.4.

Let \((\tilde{A}^s, \tilde{B}^s)\) be an induced pair of \((\tilde{A}, \tilde{B})\) with \(\tilde{R}^s\) being trivial and local. Then there are two sequences of reduction in the sense of Lemma 2.3.2 according to Formula (4.1-5):

\[
\tilde{A}, \tilde{A}^1, \ldots, \tilde{A}^r-1, \tilde{A}^r, \tilde{A}^r+1, \ldots, \tilde{A}^{r+i}, \tilde{A}^{r+i+1}, \ldots, \tilde{A}^{r+s};
\]

\[
\tilde{A}, \tilde{A}^1, \ldots, \tilde{A}^i, \tilde{A}^{i+1}, \ldots, \tilde{A}^s. \quad (\#)
\]

Set \(\tilde{M} = \varphi^{0s}(F^s(k))\) of size vector \(\tilde{n}\) over \(\tilde{T}\), a bordered matrix \(\tilde{M}\) of size vector \(\tilde{n}\) can be constructed according to Formula (5.4-3).

**Theorem 5.4.3** Being parallel to Theorem 5.2.1, there exists a unique reduction sequence based on the sequence \((\#)\), where \(\tilde{\tilde{A}}\) stands for \(\tilde{A}\) in order to simplify the notation:

\[
\tilde{\tilde{A}}, \tilde{\tilde{A}}^1, \ldots, \tilde{\tilde{A}}^r-1, \tilde{\tilde{A}}^r, \tilde{\tilde{A}}^r+1, \ldots, \tilde{\tilde{A}}^{r+i}, \tilde{\tilde{A}}^{r+i+1}, \ldots, \tilde{\tilde{A}}^{r+s};
\]

\[
\tilde{\tilde{A}}, \tilde{\tilde{A}}^1, \ldots, \tilde{\tilde{A}}^i, \tilde{\tilde{A}}^{i+1}, \ldots, \tilde{\tilde{A}}^s. \quad (\tilde{\#})
\]

(i) \(\tilde{\tilde{A}}^i = A^i\) for \(i = 0, 1, \ldots, r\).

(ii) The reduction from \(\tilde{\tilde{A}}^i\) to \(\tilde{\tilde{A}}^{i+1}\) is a reduction or a composition of two reductions in the sense of Lemma 2.3.2 for \(i = 0, \ldots, s - 1\), such that \(\tilde{T}^s\) has two vertices, and \(\tau^{0s}(\tilde{F}^s) = \tilde{M}\).

(iii) The reduction from \(\tilde{\tilde{A}}^{r+i}\) to \(\tilde{\tilde{A}}^{r+i+1}\) is done in the same way as that from \(\tilde{\tilde{A}}^i\) to \(\tilde{\tilde{A}}^{i+1}\).

(iv) The diagonal block \(\tilde{e}_\chi\) in \(\tilde{T}^s\) of \(\tilde{A}^s\) partitioned under \(\tilde{T}\) is of the form of Corollary 5.2.2.

**Proof** (i) is clear.

(ii) The proof is parallel to that of Theorem 5.2.1, the only difference appears in the item 1.4 of Case 1. Suppose an edge reduction is made from \(\tilde{\tilde{A}}^i\) to \(\tilde{\tilde{A}}^{i+1}\) with the reduction block \(\tilde{G}^{i+1}\) being at the \(l'\)-th block column with \(l' \in Y\). Then \(\tilde{G}^{i+1}\) has a size vector \(\tilde{n}^{i+1} \times \tilde{n}^{i+1}\) over \(\tilde{T}\) with \(\tilde{n}^{i+1} = n^{i+1} + 1\), and a zero column is added into the \(l'\)-th block column from the left hand side for every \(l' \in Y\).

(iii) follows from Formula (4.1-5).

(iv) The proof is parallel to that of Corollary 5.2.2.

Being parallel to \((\#')\) at the beginning of Subsection 5.3, there are following two sequences:

\[
\tilde{\tilde{A}}, \tilde{\tilde{A}}^1, \ldots, \tilde{\tilde{A}}^{r-1}, \tilde{\tilde{A}}^r, \tilde{\tilde{A}}^{r+1}, \ldots, \tilde{\tilde{A}}^{r+s}, \tilde{\tilde{A}}^{r+s+1}, \ldots, \tilde{\tilde{A}}^{r+t};
\]

\[
\tilde{\tilde{A}}, \tilde{\tilde{A}}^1, \ldots, \tilde{\tilde{A}}^s, \tilde{\tilde{A}}^{s+1}, \ldots, \tilde{\tilde{A}}^t, \ldots, \tilde{\tilde{A}}^t. \quad (\tilde{\#}')
\]
The reductions from \( \tilde{A} \) to \( \tilde{A}^x \) is given by (\( \ast \)); from \( \tilde{A}^x \) to \( \tilde{A}^{x+1} \) is a loop mutation and a parameter \( x \) appears; the reduction from \( \tilde{A}^t \) to \( \tilde{A}^{t+1} \) is a regularization for \( i = s + 1, \ldots, t - 1 \). The first arrow of \( \tilde{B}^t \) locates at the \( (\bar{p}, \bar{q} + j) \)-th position in the formal product \( \tilde{\Theta}^t \) for some \( 1 \leq j \leq n_t \), and that of \( \tilde{B}^t \) at \( (\bar{p}, \bar{q} + 1) \)-th position in \( \tilde{\Theta}^t \). The pair \( (\tilde{A}^{r+t}, \tilde{B}^{r+t}) \) is minimal wild in the case of MW5 of Remark 3.4.6 and Classification 5.1.1 (II).

**Remark 5.4.4** (i) If the first arrow \( a^1_t \) of \( \tilde{B}^t \) is splitting from \( d_t \) of the one-sided bocs \( \tilde{B} \), then \( d_t : X \to Y_t \) is an edge by Theorem 4.6.1 and Corollary 4.6.2. Therefore it is possible to apply Theorem 5.4.3 with respect to the vertex \( Y_t \in \mathcal{T} \) for the sequence (\( \tilde{\ast} \)), and obtain the sequence (\( \tilde{\ast} \)).

(ii) We will describe how to determine \( \tilde{A}^r \), thus \( \tilde{A} \), in the next subsection.

(iii) Being parallel to Formula (5.3-2), the equation system \( \{\tilde{F}_{r,q}^t \mid \forall t' \in Y_t \} \) given by Lemma 5.4.2 (iii) is considered. Thus some polynomials \( d^{i,t'}(x) \) are obtained for \( t' \in Y_t, j = \bar{p}_x, \cdots, \bar{p} \) inductively, by an analogous discussion as in the subsection 5.3. If \( \tilde{R}^t = k[x, \phi^t(x)^{-1}] \), define a polynomial similar to Formula (5.3-5):

\[
\tilde{\phi}^t(x) = \phi^t(x) \prod_{j=\bar{p}_x}^{\bar{p}} \prod_{t' \in Y_t} d^{i,t'}(x).
\]

**Theorem 5.4.5** Being parallel to Theorem 5.3.2, there exist two unique reduction sequences based on the sequences (\( \tilde{\ast}^t \)):

\[
\tilde{\ast}, \tilde{\ast}^1, \ldots, \tilde{\ast}^{r-1} \tilde{\ast}^r, \tilde{\ast}^{r+1}, \ldots, \tilde{\ast}^{r+s}, \tilde{\ast}^{r+s+1}, \ldots, \tilde{\ast}^{r+\epsilon}, \ldots, \tilde{\ast}^{r+t};
\]

\[
\tilde{\ast}, \tilde{\ast}^1, \ldots, \tilde{\ast}^{s}, \tilde{\ast}^{s+1}, \ldots, \tilde{\ast}^{t}, \ldots, \tilde{\ast}^t. \tag{\( \tilde{\ast}^t \)}
\]

(i) The first parts of the two sequences up to \( \tilde{\ast}^{r+s} \) and \( \tilde{\ast}^s \) respectively are given by (\( \tilde{\ast} \)).

(ii) \( \tilde{\ast}^{s+1} \) is induced from \( \tilde{\ast}^s \) by a loop mutation \( a^s_{i+1} \to (x) \), or an edge reduction (0) followed by a loop mutation \( (x) \); the reduction from \( \tilde{\ast}^s \) to \( \tilde{\ast}^{s+1} \) for \( i > s \) is given by a regularization, or two regularizations, or a reduction given by Lemma 2.2.6 followed by a regularization.

(iii) The reduction from \( \tilde{\ast}^{r+s} \) to \( \tilde{\ast}^{r+s+1} \) is done in the same way as that from \( \tilde{\ast}^s \) to \( \tilde{\ast}^{s+1} \) for \( s < i < t \).

(iv) If the bocs \( \tilde{B}^t \) in the sequence (\( \tilde{\ast}^t \)) satisfies MW5 of Remark 3.4.6 and Classification 5.1.1 (II), then \( \delta(\tilde{a}^0_t) = 0 \) for the first arrow \( \tilde{a}^0_t \) of \( \tilde{B}^t \) in (\( \tilde{\ast} \)).

**Proof** (i) is obvious. The proof of (ii) is parallel to that of Theorem 5.3.2. (iii) follows from Formula (4.1-5). The proof of (iv) is parallel to that of Corollary 5.3.3 by Corollary 2.4.3. \( \square \)

### 5.5 Non-homogeneity in the case of MW5 and classification (II)

Suppose a bipartite pair \( (\mathfrak{A}, \mathfrak{B}) \) has an induced pair \( (\mathfrak{A}', \mathfrak{B}') \) in the case of MW5 of Remark 3.4.6 and Classification 5.1.1 (II) in this subsection. A one-sided quotient-sub pair is determined according to the position of the first arrow \( a^1_1 \) in the formal product \( H^t + \Theta^t \); then the non-homogeneity of the pair \( (\mathfrak{A}, \mathfrak{B}) \) is proved.

Let \( \mathfrak{A} \) be a bipartite matrix bimodule problem satisfying RDCC condition. The sequence

\[
(\mathfrak{A}, \mathfrak{B}), (\mathfrak{A}^1, \mathfrak{B}^1), \ldots, (\mathfrak{A}^x, \mathfrak{B}^x), (\mathfrak{A}^{x+1}, \mathfrak{B}^{x+1}), \ldots, (\mathfrak{A}^\gamma, \mathfrak{B}^\gamma) = (\mathfrak{A}', \mathfrak{B}') \tag{5.5-1}
\]

satisfies the following conditions: \( R^i \) is trivial for \( i \leq \varsigma \), the reduction from \( \mathfrak{A}^i \) to \( \mathfrak{A}^{i+1} \) is in the sense of Lemma 2.3.2 for \( i < \varsigma \); \( \mathfrak{A}^\varsigma \) is local with \( \delta(a^1_1) = 0 \), and \( R^{\varsigma+1} = k[x] \) in \( \mathfrak{B}^{\varsigma+1} \) after a loop mutation; finally, the reduction from \( \mathfrak{A}^i \) to \( \mathfrak{A}^{i+1} \) is a regularization for \( i > \varsigma \), and \( \mathfrak{B}^\gamma = \mathfrak{B}' \) is in the case of MW5 of Remark 3.4.6 and Classification 5.1.1 (II). Suppose the index of the
first arrow $a^*_i$ of $\mathfrak{B}^r$ is $(p^r, q^r)$ in the formal product $\Theta^r$, which is sitting at the $(p, q)$-th block partitioned under $\mathcal{T}$. According to Formula (2.3-7):

$$H^r = \sum_{i=1}^{\varsigma} G^i + A^i_{a^*_1} = \sum_{i=1}^{\varsigma} G^i + (x) \ast A^i_{a^*_1}. \quad (5.5-2)$$

Following discussion will be focused on the reduction blocks $G^i_j$ of $H^r$.

Let $i < \varsigma$, $\vartheta^r : R(\mathfrak{A}^r) \rightarrow R(\mathfrak{A}^t)$ be the induced functor, and $\vartheta^r_i = \vartheta^r(1, \ldots, 1)$. There is a simple fact, that any row (column) index $\rho$ of $H^i + \Theta^i$ in the pair $(\mathfrak{A}^i, \mathfrak{B}^i)$ determines a row (column) index $n^r_1 + \cdots + n^r_\rho$ of $H^r + \Theta^r$ in the pair $(\mathfrak{A}^r, \mathfrak{B}^r)$. Consequently, if the upper (resp. lower, left or right) boundaries of two reduction blocks $G^i_1, G^i_2$ in $H^i$ are collinear, if and only if the corresponding boundaries of two splitting blocks $G^j_1, G^j_2$ in $H^r$ are collinear. The two blocks $G^j_i$ and $G^j_i(k)$ may not be distinguished for the sake of convenience in the following statements.

**Remark 5.5.1** Consider the reduction blocks inside the $(p, q)$-th block partitioned under $\mathcal{T}$. The relative position of the upper boundaries of $G^i_1$ and $G^i_{i+1}$ in this block has three possibilities according to Formulae (2.3-3)-(2.3-5).

(i) The upper boundaries of $G^i_1$ and $G^i_{i+1}$ are collinear, and this occurs if and only if the reduction from $\mathfrak{A}^{i-1}$ to $\mathfrak{A}^i$ is given by one of the following reduction blocks: $G^i = \emptyset$ in a regularization; $G^i = (\lambda)$ in a loop reduction; $G^i = (0), (1)$ or $(01)$ in an edge reduction, moreover the right boundary of $G^i_1$ is not that of the $(p, q)$-th block. In this case their lower boundaries are also collinear.

(ii) The upper boundary of $G^i_{i+1}$ is strictly lower than that of $G^i_1$, and this occurs if and only if $G^i = W$ of size being strictly bigger than 1 in a loop reduction, or $G^i = (\lambda)$ or $(01)$ in an edge reduction, and the right boundary of $G^i_1$ is not that of the $(p, q)$-th block. In this case, the lower boundaries of $G^i_1$ and $G^i_{i+1}$ are also collinear.

(iii) The lower boundary of $G^r_{i+1}$ is the upper boundary of $G^i_1$, and this occurs if and only if the right boundary of $G^i_1$ coincides with that of the $(p, q)$-th block.

Collect all the reduction blocks of $H^r$ inside the $(p, q)$-th block, such that their upper boundaries are above or at that of $a^*_1$:

$$G^{q_1}_r, G^{q_2}_r, \ldots, G^{q_u}_r, \quad \text{with} \quad 1 \leqslant q_1 < q_2 < \cdots < q_u \leqslant \varsigma. \quad (5.5-3)$$

The set of reduction blocks $\{G^{q_i}_r \mid 1 \leqslant i \leqslant u\}$ in (5.5-3) is divided into $h$ groups according to whether the upper boundaries of blocks are collinear or not, and denoted by $\rho_j$ the common upper boundary of the blocks in the $j$-th group for $j = 1, \ldots, h$, where $\rho_{j+1}$ is strictly lower than $\rho_j$:

$$\{G^{q^1}_r, \ldots, G^{q^{u_1}}_r\}, \ldots, \{G^{q^{h_1}}_r, \ldots, G^{q^{u_h}}_r\}, \quad u_1 + \cdots + u_h = u. \quad (5.5-4)$$

The adjacent blocks $G^{q^1}_r$ and $G^{q^{h+1}}_r$ in the $j$-th group have two possibilities: $1$ $G^{q^1}_r$ is in case (i) of Remark 5.5.1. $2$ $G^{q^{h+1}}_r$ is in case (ii) of Remark 5.5.1. Then $G^{q^1}_r$ comes from the next reduction with $q^1_{j+1} = q^1_j + 1$ in $1$. But in $2$, $G^{q^{h+1}}_r$ follows by a sequence of reductions with the upper boundaries of the reduction blocks lower than that of $a^*_1$, and the sequence includes at least one reduction in case (iii) of Remark 5.5.1. Finally, the sequence reaches $G^{q^1}_r$ with the upper boundary $\rho_j$ as a neighbor of $G^{q^1}_r$. Thus $q^1_{j+1} > q^1_j + 1$.

**Lemma 5.5.2** The last block $G^{q^1}_r$ of the $j$-th group must be as in case (ii) of Remark 5.5.1 for $j = 1, \ldots, h$.

**Proof** If $G^{q^1}_r$ is in case (iii) of 5.5.1, then $\rho_j$ is lower than $\rho_{j+1}$ for $j < h$, which is a contradiction to the grouping of Formula (5.5-4); and $a^*_1$ is sitting above $\rho_h$ for $j = h$, which is a contradiction to the choice of the sequence (5.5-3).
Suppose \( G^q_{\tau, u} \) is in case (i) of 5.5.1. Then for \( j < h \), the upper boundaries of \( G^q_{\tau, u} \) and \( G^q_{\tau, u}+1 \) coincide, which is a contradiction to the grouping of (5.5-4). For \( j = h \), \( G^q_{\tau, u} = 0, I, (0I) \), or \( \lambda, I, or \emptyset \) with the height \( d \geq 1 \). Suppose the next reduction is still in the sense of Lemma 2.3.2 given by \( G^q_{\tau, u}+1 \), which is denoted by \( G'_{\tau} \) for simplicity. If \( G'_{\tau} \) is in the case (i) or (ii) of 5.5.1, then \( G'_{\tau} \) and \( G^q_{\tau, u} \) have the same upper boundary, a contradiction to the grouping of (5.5-4); if \( G'_{\tau} \) is in case of 5.5.1 (iii), then \( a'_1 \) locates above \( \rho_h \), which is a contradiction to the choice of (5.5-3). Therefore the reduction in the sense of Lemma 2.3.2 should not be able to continue, and \( r + s = q_{h, u} \) in the sequence \( (\tilde{s}') \) before Remark 5.4.4. Thus the height \( d = 1 \), the parameter \( x \) appears by a loop mutation. Since \( a'_1 \) locates between the upper and lower boundaries of \( G^q_{\tau, u} \), \( x \) and \( a'_1 \) are sitting at the \( p^r \)-th row simultaneously, a contradiction to Lemma 5.1.2. So \( G^q_{\tau, u} \) is in the case of 5.5.1 (ii) as desired. \( \square \)

**Definition 5.5.3** We define \( h \) rectangles in \( \Theta^r \); for \( j < h \), the \( j \)-th rectangle has the upper boundary \( \rho_j \), the lower one \( \rho_{j+1} \), and the left one is the right boundary of \( G^q_{\tau, u} \), the right one is that of the \( (p, q) \)-th block. While the upper boundary of the \( h \)-th rectangle is \( \rho_h \), lower boundary is that of \( G^q_{\tau, u} \). The rectangle with upper boundary \( \rho_j \) is said to be the \( j \)-th ladder, there are altogether \( h \) ladders.

The picture below shows an example for \( h = 3 \). Three groups of Reduction blocks given in sawtooth patterns with some dots, but the last block in each group is given by a rectangle without dots. The upper boundaries of the three ladders are shown by dotted lines.

![Diagram of ladders and rectangles](image)

**Lemma 5.5.4** Let index \( r = q_{h, u} - 1 \) in the sequence (5.5-1). We define a one-sided quotient-sub pair \((\tilde{A}, \tilde{B}) = ([ \Theta^r]^m), (\Theta^r)^m)\) of the pair \((\Theta^r, \Theta^r)\) consisting of the solid arrows \(d_1, \ldots, d_m\) sitting at the \( p^r \)-row as shown in Picture (4.1-1). Then

(i) \( m \geq 1 \);

(ii) all the reduction blocks in \( H^r \) yielded from some split of \( d_2, \ldots, d_m \) locate below the \( p^r \)-th row;

(iii) \( a'_1 \) is split from \( d_l \) with \( l > 1 \). If \((\tilde{A}, \tilde{B})\) satisfies the hypothesis of Theorem 4.6.1 or Corollary 4.6.2, then \( d_l \) is a solid edge.

(iv) \( \varsigma = r + s \) and \( \tau = r + t \). Therefore the sequence (5.5-1) coincides with the first sequence of \( (\tilde{s}') \) given before Remark 5.4.4.

**Proof** (i) follows from Lemma 5.5.2.

(ii) comes from the choice of the reduction blocks of Formula (5.5-3).

(iii) and (iv) are obvious. \( \square \)

**Remark 5.5.5** (i) \( H^r + \Theta^r \) of the pair \((\Theta^r, \Theta^r)\) has also \( h \) ladders in the \( (p, q) \)-th block.

The boundaries of the \( j \)-th ladder of \( H^r + \Theta^r \) is derived from that of the \( j \)-th ladder of \( H^r + \Theta^r \) for \( j = 1, \ldots, h \) according to the simple fact stated before Remark 5.5.1.

(ii) Let \((\Theta^r_X, \Theta^r_X)\) be the induced local pair at \( X \) of \((\Theta^r, \Theta^r)\), denote by \( h_X \) the number of the inheriting ladders of \( H^r_X + \Theta^r_X \) from \( H^r + \Theta^r \), then \( h_X \leq h \).

(iii) Return to sequence \( (\tilde{s}') \) and \( (\tilde{s}) \) in Subsection 5.4. It is easy to see that \( \tilde{H}^{r+t} + \Theta^{r+t} \) has also \( h \) ladders, and the number of rows in the \( h \)-th ladder in \( \tilde{H}^{r+t} + \Theta^{r+t} \) is the same as that in \( H^{r+t} + \Theta^{r+t} \).
Proposition 5.5.6 Let \((\mathfrak{A}, \mathfrak{B})\) be a pair with \(\mathcal{T}\) trivial, such that \(\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)\) is a bipartite matrix bimodule problem satisfying RDCC condition. If there exists an induced pair \((\mathfrak{A}', \mathfrak{B}')\) of \((\mathfrak{A}, \mathfrak{B})\) in the case of MW5 defined by Remark 3.4.6, and the sum \(H' + \Theta'\) of \((\mathfrak{A}', \mathfrak{B}')\) satisfies Classification 5.1.1 (II), then \(\mathfrak{B}\) is not homogeneous.

Proof Suppose the induced pair \((\mathfrak{A}', \mathfrak{B}')\) is the last term \((\mathfrak{A}^r + t, \mathfrak{B}^r + t)\) in the first sequence of Formula \((\tilde{\delta}')\) given before Remark 5.4.4. Keep the notations in the two sequences of \((\tilde{\delta}')\).

We assume in addition that the number of the ladders in \(H^r + t + \Theta^r + t\) is minimal with respect to the property of Classification 5.1.1 (II).

(I) Let \(X\) be given by Definition 4.1.2. If the local pair \((\mathfrak{A}_X^r, \mathfrak{B}_X^r)\) is wild, using the triangular Formulae of Subsection 3.3, an induced minimal wild local pair \(((\mathfrak{A}_X^r)', (\mathfrak{B}_X^r)')\) with the parameter \(x'\) and the first arrow \(a_1'\) is obtained.

(I-1) If \((\mathfrak{B}_X^r)'\) is in the case of MW3, or MW4, or MW5 with \(H_X^r + \Theta_X^r\) being in the case of Classification 5.1.1 (I), then \((\mathfrak{A}, \mathfrak{B})\) is not homogeneous by Proposition 3.4.3, or 3.4.4, or 5.3.4, it is done.

(I-2) If \((\mathfrak{B}_X^r)'\) is in the case of MW5 and Classification 5.1.1 (II), then the number of the inheriting ladders \(h_X \in H_X^r + \Theta_X^r\) does not exceed \(h\) by Remark 5.5.5 (ii). Suppose \(a_1'\) locates at the \(h\)'-ladder. If \(h' < h_X \leq h\), or \(h' = h_X < h\), then it contradicts to the minimality assumption on the number of ladders. If \(h' = h_X = h\), since this ladder contains only one row by Lemma 5.5.4, \(x'\) and \(a_1'\) must locate at the same row, a contradiction to Lemma 5.1.2.

(II) Suppose \((\mathfrak{A}_X^r, \mathfrak{B}_X^r)\) is tame infinite, and its quotient-sub-pair \((\hat{\mathfrak{A}}_X, \hat{\mathfrak{B}}_X)\) is in case (ii) of Classification 4.2.1.

(II-1) If the one-sided pair \((\hat{\mathfrak{A}}, \hat{\mathfrak{B}})\) satisfies the hypothesis of Lemma 4.2.3 or 4.4.1, then \((\hat{\mathfrak{A}}, \hat{\mathfrak{B}})\) is not homogeneous. In fact the loop \(\tilde{b}\) is the unique effective loop of both \(\hat{\mathfrak{B}}_X\) and \(\mathfrak{B}_X^r\).

(II-2) If \((\hat{\mathfrak{A}}, \hat{\mathfrak{B}})\) satisfies the hypothesis of Theorem 4.4.2, then the triangular formulae given in Subsection 3.3 can be used for the local wild pair \((\mathfrak{A}_X^r + 2, \mathfrak{B}_X^r + 2)\) given in Proof 4) of 4.4.2. If the cases of MW3, MW4, or MW5 and Classification 5.1.1 (I) are reached, it is done. If MW5 and Classification 5.1.1 (II) is med again, the first arrow must be outside of the \(h\)-th ladder, which is a contradiction to the minimal number assumption of the ladders.

(III) Now the following two cases are considered. First, \(\mathfrak{B}_X^r\) is tame infinite, \(\mathfrak{B}_X\) is in the case (ii) of Classification 4.2.1, the pair \((\hat{\mathfrak{A}}, \hat{\mathfrak{B}})\) is major and the \(c\)-class arrows satisfy Formula (4.2-6). Second, \(\mathfrak{B}_X^r\) is tame infinite or finite, and \(\mathfrak{B}_X\) is finite.

Then in both cases \(d_i\) of \(\hat{\mathfrak{B}}\), from which \(a_1'\) is split, is a solid edge by Lemma 5.5.4 (iii). Consequently, Formula \((\tilde{\delta}')\) of Theorem 5.4.3 can be used with respect to \(d_i\). Keep the notations in the two sequences of \((\tilde{\delta}')\) of Theorem 5.4.5.

Since \(\delta(a_0') = 0\) in the pair \((\tilde{\tilde{\mathfrak{A}}}', \tilde{\tilde{\mathfrak{B}}}')\) of \((\tilde{\delta}')\) by 5.4.5 (iv), set the edge \(a_0' \mapsto (1)\) by Proposition 2.2.7. Then all the other arrows splitting from \(d_i\) at the same row are regularized in the further reductions by 5.4.3 (iv). The induced pair is obviously local and tame infinite or wild type. Then it is possible to use the triangular formulæ of Subsection 3.3 once again, and an induced pair \((\tilde{\tilde{\mathfrak{A}}}, \tilde{\tilde{\mathfrak{B}}})\) in the cases (ii)-(iv) of Classification 3.3.5 is obtained.

(III-1) If the induced local pair \((\tilde{\tilde{\mathfrak{A}}}, \tilde{\tilde{\mathfrak{B}}})\) is tame infinite, then the two-point pair \((\tilde{\tilde{\mathfrak{A}}}^{r + \epsilon}, \tilde{\tilde{\mathfrak{B}}}^{r + \epsilon})\) satisfies the hypothesis of Proposition 3.4.5, it is done.

(III-2) If \((\tilde{\tilde{\mathfrak{A}}}, \tilde{\tilde{\mathfrak{B}}})\) is in the case of MW3, or MW4, or MW5 of Remark 3.4.6 and Classification 5.1.1 (I), it is done.

(III-3) If \((\tilde{\tilde{\mathfrak{A}}}, \tilde{\tilde{\mathfrak{B}}})\) is in the case of MW5 of Remark 3.4.6 and classification 5.1.1 (II), and suppose in addition, the first arrow of \(\tilde{\tilde{\mathfrak{B}}}\) locates at the \(h_1\)-th ladder with \(h_1 < h\), then there is a contradiction to the minimality number assumption of the ladders.

(III-4) If \((\tilde{\tilde{\mathfrak{A}}}, \tilde{\tilde{\mathfrak{B}}})\) is in the case of MW5 of Remark 3.4.6 and classification 5.1.1 (II), and suppose in addition, the first arrow of \(\tilde{\tilde{\mathfrak{B}}}\) locates still at the \(h\)-th ladder, it is needed to do induction on some pairs of integers. Denote by \(\sigma\) the number of the rows in the \(h\)-th ladder of
$H^{r+t} + \Theta^{r+t}$, which is a constant after making some bordered matrices by Remark 5.5.5 (iii). And denote by $m$ the number of the solid arrows in the pair $(\mathfrak{A}, \mathfrak{B})$ in Formula (4.1-1), which is also a constant. Define a finite set with $\sigma m$ pairs:

$$S = \{(\varrho, \zeta) \mid 1 \leq \varrho \leq \sigma, \zeta = 1, \ldots, m\},$$

ordered by $(\varrho^1, \zeta^1) \prec (\varrho^2, \zeta^2) \iff \varrho_1 > \varrho_2$, or $\varrho^1 = \varrho^2$, $\zeta^1 < \zeta^2$.

In order to unify notations, the induced minimal wild local pair $(\mathfrak{A}^{r+t}, \mathfrak{B}^{r+t})$ in $(\bar{s}')$ is denoted by $(\hat{\mathfrak{A}}, \hat{\mathfrak{B}})$. Let $(\varrho, \zeta) = (\bar{p}, l) \in S$, where $\bar{p}$ is the row-index of the first arrow $\hat{a}_1 = a^1_t$ in $\hat{\Theta}^t$; $l$ is the subscript of the edge $d_l$, from which $a^1_t$ is split, since $F^t + \bar{\Theta}^t$ is contained in the $h$-th ladder by Lemma 5.5.4. Similarly, let $(\varrho^1, \zeta^1) \in S$ be determined by the first arrow $\hat{a}^1_1$ of $\hat{\mathfrak{B}}^1$. Theorem 5.4.3 (iv) ensures that $(\varrho, \zeta) \prec (\varrho^1, \zeta^1)$.

Now the procedure (III) is started once again from the pair $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ instead of $(\hat{\mathfrak{A}}, \hat{\mathfrak{B}})$. If (III-4) appears repeatedly, then after a finite number of steps, an induced pair of (III-1)–(III-3) is reached by induction on $S$.

$$\square$$

### 5.6 Proof of the main theorem

It is ready to prove Theorem 3 given in the introduction.

**Theorem 5.6.1** Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with $T$ trivial, such that $\mathfrak{A} = (R, K, M, H = 0)$ is a bipartite matrix bimodule problem satisfying RDCC condition. If $\mathfrak{B}$ is of wild type, then $\mathfrak{B}$ is not homogeneous.

**Proof** There exists an induced bocs $\mathfrak{B}'$ of the wild bocs $\mathfrak{B}$, which is in one of the cases of MW1–MW5 according to Classification 3.3.2. Proposition 3.4.1–3.4.4 proved that if $\mathfrak{B}'$ is in the case of MW1–MW4, then $\mathfrak{B}$ is not homogeneous. When $\mathfrak{A}$ is bipartite and satisfies RDCC condition, Proposition 5.3.4 and 5.5.6 proved that if the induced bocs $\mathfrak{B}'$ is in the case of MW5 of Remark 3.4.6, then $\mathfrak{B}$ is not homogeneous.

$$\square$$

**Proof of Theorem 3** Let $\Lambda$ be a finite-dimensional basic algebra over an algebraically closed field $k$. We claim that if $\Lambda$ is of wild representation type, then $\text{mod-}\Lambda$ is not homogeneous.

In fact, let $\mathfrak{A}$ be the matrix bimodule problem associated with $\Lambda$. Then $\mathfrak{A}$ is bipartite and satisfies RDCC condition by Remark 1.4.4, and it is representation wild type. Therefore the associated bocs $\mathfrak{B}$ is not homogeneous by Theorem 5.6.1. Note that there is a one-to-one correspondence between the set of equivalent classes of almost split sequences in $\text{mod-}\Lambda$ and that of almost split conflations in $R(\mathfrak{B})$ except finitely many equivalent classes of such sequences, see [B2] and [ZZ]. Therefore $\text{mod-}\Lambda$ is not homogeneous.

$$\square$$

### Acknowledgement

We would like to express our sincere thanks to S.Liu for his proposal of $\Delta$-algebra, to Y.Han for his suggestion of the concept on co-bimodule problem. We thank K.Wang, F.Dai and X.Tang for correcting the English. Y.Zhang is indebted to R.Bautista for giving the open problem in 1991; and to W.W.Crawley-Boevey, D.Simson, C.M.Ringel, T.Lei, S.Wang, K.Li, C.Zhao for some discussions during the long procedure of solving the problem. In particular, she is grateful to Y.A.Drozd for his invitation to visit the Institute of Mathematics, National Academy of Sciences of Ukraine, and helpful conversations.
References

[AR] M. Auslander and I. Reiten, “Representation theory of artin algebras III”, Comm. Algebra (3) 3 (1975) 239 - 294.

[ARS] M. Auslander, I. Reiten, and S. O. Smalø, “Representation Theory of Artin Algebras”, Cambridge Studies in Advanced Mathematics 36 (Cambridge University Press, 1995).

[B1] R. Bautista, “A characterization of finite-dimensional algebra of tame representation type”, preprint (UNAM, 1989).

[B2] R. Bautista, “The category of morphisms between projectives”, Comm. Algebra (11) 32 (2004) 4303 - 4331.

[BBP] R. Bautista, J. Boza and E. Pérez, “Reduction functors and exact structures for Bocses”, Bol. Soc. Mat. Mex. (3) 9 (2003) 21 - 60.

[BB] W. L. Burt and M. C. R. Butler, “Almost split sequences for bocses”, Canad. Math. Soc. Conf. Proc. II (1991) 89 - 121.

[BCLZ] R. Bautista, W. W. Crawley-Boevey, T. Lei and Y. Zhang, “On homogeneous exact categories”, J. Algebra 230 (2000) 665 - 675.

[BK] R. Bautista and M. Kleiner, “Almost split sequences for relatively projective modules”, J. Algebra (1) 135 (1990) 19 - 56.

[CB1] W. W. Crawley-Boevey, “On tame algebras and bocses”, Proc. London Math. Soc. (3) 56 (1988) 451 - 483.

[CB2] W. W. Crawley-Boevey, “Matrix problems and Drozd’s theorem”, Topics in Algebra, 26 (1990) 199 - 222.

[CB3] W. W. Crawley-Boevey, “Tame algebras and generic modules”, Proc. London Math. Soc. (3) 63 (1991) 241 -165.

[D1] Yu. A. Drozd, “On tame and wild matrix problems”, Matrix problems, (Kiev, 1977) 39 - 74.

[D2] Yu. A. Drozd, “Representations of commutative algebras”, Funct.Analysis and its appl. 6 (1972). Engl. transl. 286-288.

[DRSS] P. Dräxler, I. Reiten, S. Smalø and Ø. Solberg, “Exact categories and vector space categories”, Trans. Amer. Math. Soc. (2) 351 (1999) 647 -6 82.

[DS] P. Dowbor and A. Skowroński, “On the representation type of locally bounded categories”, Tsukuba J. Math. (1) 10 (1986) 63 - 72.

[GR] P. Gabriel and A. V. Roiter, “Representation of Finite-dimensional Algebras”, Algebra VIII, Encycl. Math. Sci. 73 (Springer-Verlag, Berlin, 1992).

[I] Y. Han, “Controlled wild algebras”, Proc. London Math. Soc. (3) 83 (2001) 279 - 298.

[HPR] D. Happel, U. Preiser and C.M. Ringel, “Vinberg’s characterization of Dynkin diagrams using subadditive function with application to DTr-periodic modules”, Lecture Notes in Mathematics 832 (Springer-Verlag, Berlin, 1980) 280 - 294.
[J] N. Jacobson, “Basic algebra II”, W.H. Freeman and Company, San Francisco, 1980.

[K] M. Kleiner, “Induced modules and comodules and representations of bocses and DGC’s”, Lecture Notes in Mathematics 903 (Springer-Verlag, Berlin, 1981) 168 - 185.

[Kr] H. Krause, “Generic modules over artin algebras”, Proc. London. Math. Soc. (3) 76 (1998) 276 - 306.

[N] H. Nagase, “τ-wild algebras”, Representations of Algebras II (Beijing Normal University Press, Beijing, 2002) 365 - 372.

[O] S. A. Ovsienko, “Generic representations of free bocses”, preprint 93-010 (Bielefeld, xxxx).

[PS] P. Dräxler and Ø. Solberg, “Exact factors and exact categories”, Bol. Soc. Mat. Mexicana 7 (2001) 59 - 72.

[Rie] Chr. Riedtmann, “Algebren, darstellungsköcher, ueberlagerungen und zurück”, Comm. Math. Helv. 55 (1980) 199 - 224.

[R1] C. M. Ringel, “The representation type of local algebras”, Lecture Notes in Mathematics 488 (Springer, Berlin, 1975) 282 - 305.

[R2] C. M. Ringel, “Tame algebras and integral quadratic forms”, Lecture Notes in Mathematics 1099 (Springer, Berlin, 1984).

[R3] C. M. Ringel, “The development of the representation theory of finite-dimensional algebras 1968-1975”, London Math. Soc. Lecture Notes 238 (Cambridge University Press, Cambridge, 1997).

[Ro] A. V. Rojter, “Matrix problems and representations of BOCS’s”, Lectures Notes in Mathematics 831 (Springer, Berlin, 1980).

[S] V. V. Sergeichuk, “Canonical matrices for basic matrix problems”, Linear algebra and its applications 317 (2000) 53 - 102.

[XZ] Y. Xu and Y. Zhang, “Indecomposability and the number of links”, Science in China (Series A), (5) 31 (2001) xxx - yyy.

[V] D. Vossieck, “A construction of homogeneous matrix problems”, Bol. Soc. Mat. Mex. III (2) 5 (1999) 301-305.

[Z] Y. Zhang, “The structure of stable components”, Canad. J. Math. (3) 43 (1991) 652 - 672.

[ZL] Y. Zhang and T. Lei, “A matrix descripition of a wild category”, Science in China (Series A), (5) 41 (1998) 461 - 475.

[ZLB] Y. Zhang, T. Lei and R. Bautista, “The representation category of a bocs I - IV”, J. Beijing Nor. Univ. (Sciences series), (3) 31 (1995) 313 - 316; (4) 31 (1995) 440 - 445; (2) 32 (1996) 143 - 148; (3) 329 (1996) 289 - 295.

[XZ] Y. Zhang and Y. Xu, “On tame and wild bocses”, Science in China (Ser. A) (4) 48 (2005) 456-468.

[ZZ] X. Zeng and Y. Zhang, “A correspondence of almost split sequences between some categories”, Comm. Algebra (2) 29 (2001) 1 - 26.