BLOW-UP BEHAVIOR OF THE SCALAR CURVATURE ALONG THE CONICAL KÄHLER-RICCI FLOW WITH FINITE TIME SINGULARITIES

RYOSUKE NOMURA

Abstract. We investigate the scalar curvature behavior along the normalized conical Kähler-Ricci flow $\omega_t$, which is the conic version of the normalized Kähler-Ricci flow, with finite maximal existence time $T < \infty$. We prove that the scalar curvature of $\omega_t$ is bounded from above by $C/(T - t)^2$ under the existence of a contraction associated to the limiting cohomology class $[\omega_T]$. This generalizes Zhang’s work to the conic case.

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1. Introduction

Let $X$ be a compact Kähler manifold of dimension $n$, $D$ be a smooth divisor on $X$, and $\beta$ be a positive real number satisfying $0 < \beta < 1$. We consider the normalized conical Kähler-Ricci flow $\omega_t$ on $X$ which is a family of cone metrics with cone angle $2\pi \beta$ along $D$ satisfying the following evolution equation:

$$
\begin{align*}
\frac{\partial}{\partial t} \omega_t &= -\text{Ric}(\omega_t) - \omega_t + 2\pi(1-\beta)[D], \\
\omega_t|_{t=0} &= \omega^*,
\end{align*}
$$

where $[D]$ is the current of integration over $D$, and $\omega^*$ is a certain initial cone metric defined later (see (1.2)). In the case of $D = 0$, $\omega_t$ is called the normalized Kähler-Ricci flow. This case has been studied extensively in the past decades (see [TZ06] [ST16b] [ST12] [CW12] [CW14] [BEG13] [CSW15] [CT15] [GWW15] and the references therein).

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The maximal existence time $T$ of the normalized conical Kähler-Ricci flow is characterized by the following cohomological condition:

$$T = \sup\{ t > 0 \mid [\omega_t] = e^{-t}[\omega_0] + (1 - e^{-t})2\pi c_1(K_X + (1 - \beta)D) \text{ is Kähler}\},$$

which is shown by Shen [She14a, She14b]. In particular, the limiting class $[\omega_T]$ is nef but not Kähler. As $t$ tends to $T$, the flow $\omega_t$ forms singularities. The analysis of the singularities, in particular its curvature behavior, is one of the main objects in the study of the geometric flows. Our purpose here is to investigate the scalar curvature behavior of $\omega_t$ with finite time singularities (i.e. $T < \infty$) as $t$ approaches to $T$.

In the infinite time singularities case (i.e. $T = \infty$), the uniform boundedness of the scalar curvature of the normalized Kähler-Ricci flow (i.e. $D = 0$) was proved by Zhang [Zha09] when the canonical bundle $K_X$ is nef and big. This result was extended by Song-Tian [ST16a] when $K_X$ is semi-ample. Furthermore, Edwards [Edw15] generalized these results to the conic setting. In the case of Fano manifolds, Perelman (see [SeT08]) established a uniform bound for the scalar curvature along the normalized Kähler-Ricci flow and Liu-Zhang [LZ14] extended it to the conic case.

On the other hand, in the finite time singularities case (i.e. $T < \infty$), Collins and Tosatti [CT15] proved that the scalar curvature of the Kähler-Ricci flow $\omega_t$ (i.e. $D = 0$) blows up along the null locus of the limiting class $[\omega_T]$. Zhang [Zha10] showed that the scalar curvature $R(\omega_t)$ of the normalized Kähler-Ricci flow $\omega_t$ satisfies

$$R(\omega_t) \leq \frac{C}{(T - t)^2}$$

assuming the semi-ampleness of the limiting class $[\omega_T]$. This condition is natural in terms of the deep relationship between the Kähler-Ricci flow and the minimal model program (see [ST16b, Zha10]). Our main theorem generalizes Zhang’s result to the conic setting.

We assume the following contraction type condition on the limiting cohomology class $[\omega_T]$. Let $f : X \to Z$ be a holomorphic map between compact Kähler manifolds, whose image is contained in a normal irreducible subvariety $Y$ of $Z$. Let $D_Y$ be an effective Cartier divisor on $Y$ such that the pullback of $D_Y$ satisfies $D = f^*D_Y$. Let $h_Y$ be a smooth Hermitian metric on the line bundle $O_Y(D_Y)$ in the sense of [EGZ09, Section 5], and $s_Y$ be a holomorphic section of $O_Y(D_Y)$ whose zero divisor is $D_Y$. We define the initial cone metric $\omega^*$ by

$$\omega^* := \omega_0 + k\sqrt{-1}\partial\bar{\partial}|s|_h^{2\beta},$$

where $\omega_0$ is a smooth Kähler form on $X$, $k \in \mathbb{R}_{>0}$ is a sufficiently small real number, $s := f^*s_Y$ is the holomorphic section of $O_X(D)$, and $h := f^*h_Y$ is the smooth Hermitian metric on $O_X(D)$. We remark that if we take $k$ sufficiently small, $\omega^*$ is actually a cone metric with cone angle $2\pi\beta$ along $D$.

Let $\omega_t$ be the normalized conical Kähler-Ricci flow with initial cone metric $\omega^*$, and $T$ be the maximal existence time of $\omega_t$. We further assume that $T$ is finite and there exists a smooth Kähler form $\omega_Z$ on $Z$ satisfying

$$[f^*\omega_Z] = [\omega_T] \in H^{1,1}(X, \mathbb{R}).$$
Under these assumptions, we have the following theorem.

**Theorem 1.1.** The scalar curvature $R(\omega_t)$ of $\omega_t$ satisfies

$$R(\omega_t) \leq \frac{C}{(T-t)^2} \text{ on } X \setminus D,$$

where $C > 0$ is a constant independent of $t$.

In contrast with Zhang’s result, we need to treat with the singularities of $\omega_t$ along $D$. This is overcome by using the approximation technique used in [CGP13, She14a, LZ14, Edw15].

**Remark 1.2.** If we replace $(1-\beta)D$ by $\sum_{i \in I} (1-\beta_i)D_i$, where $D_i$ are smooth divisors intersecting transversely, the same argument below gives the same conclusion. But for simplicity, we only treat one smooth divisor case.

2. Approximation of the normalized conical Kähler-Ricci flow by the twisted normalized Kähler-Ricci flow

In the following argument, we assume that the conditions in Theorem 1.1 are always satisfied. We first define a family of reference smooth Kähler forms $\tilde{\omega}_t$ whose cohomology classes are equal to $[\omega_t]$. We set $\tilde{\omega}_\infty$ by

$$\tilde{\omega}_\infty := -\frac{e^{-T}}{1-e^{-T}}\omega_0 + \frac{1}{1-e^{-T}}f^*\omega_Z \in -\frac{1}{1-e^{-T}}[\omega_0] + \frac{1}{1-e^{-T}}[\omega_T] = 2\pi c_1(K_X + (1-\beta)D),$$

and $\tilde{\omega}_t$ by

$$\tilde{\omega}_t := e^{-t}\omega_0 + (1-e^{-t})\tilde{\omega}_\infty = a_t\omega_0 + (1-a_t)\tilde{\omega}_T,$$

where $a_t := (e^{-t} - e^{-T})/(1-e^{-T})$. Since $\tilde{\omega}_T = f^*\omega_Z \geq 0$ is semi-positive, $\tilde{\omega}_t$ are smooth Kähler forms for any $t \in [0, T)$. The cohomology class of $\tilde{\omega}_t$ coincide with $[\omega_t]$.

We next define a family of reference smooth Kähler forms $\tilde{\omega}_{\varepsilon,t}$ whose cohomology classes are equal to $[\omega_t]$. We use the approximation method as in [She14a, LZ14, Edw15] originated from [CGP13]. We denote $\rho_{\varepsilon} := \chi(|s|^2_h, \varepsilon^2)$, where

$$\chi(u, \varepsilon^2) := \beta \int_0^u \frac{(r+\varepsilon^2)^3 - \varepsilon^2}{r} dr.$$ 

Then, $\rho_{\varepsilon}$ are smooth functions on $X$ and converge to $|s|\varepsilon^2$ in $C^\infty_{\text{loc}}(X \setminus D)$ as $\varepsilon \to 0$. We define reference smooth Kähler forms $\tilde{\omega}_{\varepsilon,t}$ by

$$\tilde{\omega}_{\varepsilon,t} := \tilde{\omega}_t + k\frac{\sqrt{-1}\partial\bar{\partial} \rho_{\varepsilon}}{\sqrt{-1}\partial\bar{\partial} \rho_{\varepsilon}} = a_t \tilde{\omega}_{\varepsilon,0} + (1-a_t)\tilde{\omega}_{\varepsilon,\infty}.$$ 

We prove that if we take $k$ sufficiently small, $\tilde{\omega}_{\varepsilon,t}$ is positive for all $t \in [0, T)$. Let $C_1 > 0$ be a constant satisfying

$$-C_1 \omega_Z \leq \sqrt{-1}R_{hv} \leq C_1 \omega_Z \text{ on } Y,$$
where $R_{h_Y}$ is the Chern curvature of $h_Y$. Since $h = f^*h_Y$ and $\hat{\omega}_T = f^*\omega_Z$, we have
\begin{equation}
-C_1\hat{\omega}_T \leq \sqrt{-1}R_h \leq C_1\hat{\omega}_T \quad \text{on } X.
\end{equation}
Let $C_2 > 0$ and $C_3 > 1$ be constants such that
\begin{equation}
\sup_Y |s_Y|_{h_Y} \leq C_2,
\end{equation}
\begin{equation}
\hat{\omega}_T = f^*\omega_Z \leq C_3\omega_0 \quad \text{on } X.
\end{equation}
By (2.5), there exists a constant $C_4 > 0$ independent of $\varepsilon$ such that
\begin{equation}
0 \leq \rho_\varepsilon \leq C_4 \quad \text{on } X.
\end{equation}
By the computation in [CGP13, Section 3], we have
\begin{equation}
\sqrt{-1}\partial\overline{\partial}\rho_\varepsilon = \beta^2\sqrt{-1}((\nabla s \wedge \nabla s)_h) - \beta((|s|^2 + \varepsilon^2)^{1-\beta} - \varepsilon^{2\beta})\sqrt{-1}R_h \\
\geq -\beta C_1 C_2^{2\beta} \hat{\omega}_T,
\end{equation}
where $\nabla$ is the Chern connection of the line bundle $(\mathcal{O}_X(D), h)$, $R_h$ is its Chern curvature, and $\sqrt{-1}((\nabla s \wedge \nabla s)_h)$ is a semi-positive closed real $(1,1)$-form combining the wedge product of differential forms with the Hermitian metric $h$ on $\mathcal{O}_X(D)$. By (2.2), (2.8), and (2.6), we obtain the following inequalities:
\begin{align}
\tilde{\omega}_{\varepsilon,T} = \hat{\omega}_T + k\sqrt{-1}\partial\overline{\partial}\rho_\varepsilon \geq (1 - k\beta C_1 C_2^{2\beta})\hat{\omega}_T \geq (1 - k\beta C_1 C_2^{2\beta} C_3)\hat{\omega}_T, \\
\tilde{\omega}_{\varepsilon,0} = \omega_0 + k\sqrt{-1}\partial\overline{\partial}\rho_\varepsilon \geq \omega_0 - k\beta C_1 C_2^{2\beta} \hat{\omega}_T \geq (1 - k\beta C_1 C_2^{2\beta} C_3)\omega_0.
\end{align}
Finally, these inequalities give the positivity of $\tilde{\omega}_{\varepsilon,t}$ for any $t \in [0, T)$:
\begin{equation}
\tilde{\omega}_{\varepsilon,t} = \hat{\omega}_t + k\sqrt{-1}\partial\overline{\partial}\rho_\varepsilon = a_t \tilde{\omega}_{\varepsilon,0} + (1 - a_t)\tilde{\omega}_{\varepsilon,T} \geq (1 - k\beta C_1 C_2^{2\beta} C_3)\hat{\omega}_t > 0.
\end{equation}
By using these reference smooth Kähler forms, we consider the following flow of potentials:
\begin{equation}
\left\{ \frac{\partial}{\partial t} \varphi_{\varepsilon,t} = \log \left( \frac{\tilde{\omega}_{\varepsilon,t} + \sqrt{-1}\partial\overline{\partial}\varphi_{\varepsilon,t}}{\Omega} \right) - \varphi_{\varepsilon,t} + (1 - \beta) \log(|s|^2 + \varepsilon^2) - k\rho_\varepsilon, \\
\varphi_{\varepsilon,t}|_{t=0} = 0, \right.
\end{equation}
where $\Omega$ is a smooth volume form on $X$ satisfying
\begin{equation}
-\text{Ric}(\Omega) + (1 - \beta)\sqrt{-1}R_h = \tilde{\omega}_\infty \in 2\pi c_1(K_X + (1 - \beta)D).
\end{equation}
We set $\omega_{\varepsilon,t}$ by
\begin{equation}
\omega_{\varepsilon,t} := \tilde{\omega}_{\varepsilon,t} + \sqrt{-1}\partial\overline{\partial}\varphi_{\varepsilon,t}.
\end{equation}
Then, $\omega_{\varepsilon,t}$ satisfies the following twisted Kähler-Ricci flow:
\begin{equation}
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \omega_{\varepsilon,t} = -\text{Ric}(\omega_{\varepsilon,t}) - \omega_{\varepsilon,t} + \eta_\varepsilon, \\
\omega_{\varepsilon,t}|_{t=0} = \tilde{\omega}_{\varepsilon,0} := \omega_0 + k\sqrt{-1}\partial\overline{\partial}\rho_\varepsilon, \end{array} \right.
\end{equation}
where $\eta_\varepsilon := (1 - \beta)\sqrt{-1}\partial\overline{\partial}\log(|s|^2 + \varepsilon^2) + (1 - \beta)\sqrt{-1}R_h$. We remark that $\eta_\varepsilon$ converges to $2\pi(1 - \beta)[D]$ in $C^\infty_{\text{loc}}(X \setminus D)$ and as current on $X$ when $\varepsilon$ goes to 0.
The validity of these approximations (2.11), (2.13) is justified by the following theorem due to Shen [She14a].

**Theorem 2.1** ([She14a, Section 2]). There exists a subsequence $\varepsilon_i$ converging to 0 as $i \to \infty$ such that $\omega_{\varepsilon_i,t}$ converges to $\omega_t$ in $C^\infty_{\text{loc}}(X \setminus D)$ and as current on $X$.

Thanks to this theorem, we only need to estimate $\varphi_{\varepsilon,t}$ and $\omega_{\varepsilon,t}$.

3. **Overview of the proof of Theorem 1.1**

In this section, we outline the proof of Theorem 1.1. First, we need the following formulas.

**Proposition 3.1.** The Ricci curvature $\text{Ric}(\omega_{\varepsilon,t})$ and the scalar curvature $R(\omega_{\varepsilon,t})$ satisfy the following formulas:

\[
\begin{align*}
(a) \quad (1 - e^{-T}) (\text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}) &= -\sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} + e^{-T} \omega_{\varepsilon,t} - \hat{\omega}_T, \\
(b) \quad (1 - e^{-T}) (R(\omega_{\varepsilon,t}) - \text{tr}_{\omega_{\varepsilon,t}}(\eta_{\varepsilon})) &= -\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} + ne^{-T} - \text{tr}_{\omega_{\varepsilon,t}}(\hat{\omega}_T),
\end{align*}
\]

where $v_{\varepsilon,t} := (1 - e^{-T}) \dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\rho_{\varepsilon}$.

**Proof.** (b) follows from (a) by taking traces. We prove (a). By (2.13), (2.2), and (2.12), we have

\[
\begin{align*}
\text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon} &= -\frac{\partial}{\partial t} \omega_{\varepsilon,t} - \omega_{\varepsilon,t} \\
&= \left(-\frac{\partial}{\partial t} \hat{\omega}_t + \frac{\partial}{\partial t} \sqrt{-1} \partial \bar{\partial} \varphi_{\varepsilon,t} \right) - \left(\hat{\omega}_t + k\sqrt{-1} \partial \bar{\partial} \rho_{\varepsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_{\varepsilon,t} \right) \\
&= -\sqrt{-1} \partial \bar{\partial} (\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\rho_{\varepsilon}) - \left(\hat{\omega}_t + \frac{\partial}{\partial t} \hat{\omega}_t \right).
\end{align*}
\]

On the other hand, we get

\[
\begin{align*}
-e^{-T} (\text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}) &= -e^{-T} \left(-\frac{\partial}{\partial t} \omega_{\varepsilon,t} - \omega_{\varepsilon,t} \right) \\
&= e^{-T} \left(\frac{\partial}{\partial t} \hat{\omega}_t + \frac{\partial}{\partial t} \sqrt{-1} \partial \bar{\partial} \varphi_{\varepsilon,t} \right) + e^{-T} \omega_{\varepsilon,t} \\
&= \sqrt{-1} \partial \bar{\partial} (e^{-T} \dot{\varphi}_{\varepsilon,t}) + e^{-T} \omega_{\varepsilon,t} + e^{-T} \frac{\partial}{\partial t} \hat{\omega}_t.
\end{align*}
\]

Combining these equalities and (2.11), we obtain (a). \qed

By this proposition, to obtain the upper bound for the scalar curvature $R(\omega_{\varepsilon,t})$, we only need to estimate $u_{\varepsilon,t} := \text{tr}_{\omega_{\varepsilon,t}}(\hat{\omega}_T)$ and $\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}$. We divide our argument into the following 5 steps:

**Step 1.** The $C^0$-estimate for $v_{\varepsilon,t}$ (Section 4).

**Step 2.** The $C^0$-estimate for $u_{\varepsilon,t} := \text{tr}_{\omega_{\varepsilon,t}}(\hat{\omega}_T)$ using Step 1 and the parabolic Schwarz lemma (Section 5).

**Step 3.** The gradient estimate for $v_{\varepsilon,t}$ (Section 6).
Step 4. The Laplacian estimate for $v_{\varepsilon,t}$ (Section 7):

$$\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} \geq -\frac{C}{T-t}.$$ 

Step 5. Proof of Theorem 1.1 (Section 7).

4. The $C^0$-estimate for $v_{\varepsilon,t}$

In this section, we prove the $C^0$-estimates for $v_{\varepsilon,t}$. More precisely, we prove the following proposition.

**Proposition 4.1.** There exists a constant $C_5 > 0$ independent of $\varepsilon$ and $t$ such that

$$\|v_{\varepsilon,t}\|_{C^0} \leq C_5$$

holds.

To apply the maximum principle, we need the following lemmas.

**Lemma 4.2.** $v_{\varepsilon,t}$ satisfies the following evolution equation

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right) v_{\varepsilon,t} = -n + u_{\varepsilon,t},$$

where $u_{\varepsilon,t} := \text{tr}_{\omega_{\varepsilon,t}}(\hat{\omega}_T)$.

**Proof.** Differentiating (2.11) with respect to $t$, we have

$$\frac{\partial}{\partial t} \hat{\varphi}_{\varepsilon,t} = \text{tr}_{\omega_{\varepsilon,t}} \left( \frac{\partial}{\partial t} \left( \hat{\omega}_{\varepsilon,t} + \sqrt{-1} i \partial \varphi_{\varepsilon,t} \right) \right) - \varphi_{\varepsilon,t},$$

(4.1) i.e.

$$\frac{\partial}{\partial t} (\hat{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t}) = \text{tr}_{\omega_{\varepsilon,t}} \left( \frac{\partial}{\partial t} \hat{\omega}_t \right) + \Delta_{\omega_{\varepsilon,t}} \hat{\varphi}_{\varepsilon,t}.$$

On the other hand, by (2.12) and (2.2), we have

$$\Delta_{\omega_{\varepsilon,t}} \varphi_{\varepsilon,t} = \text{tr}_{\omega_{\varepsilon,t}} (\omega_{\varepsilon,t} - \hat{\omega}_{\varepsilon,t}) = n - \text{tr}_{\omega_{\varepsilon,t}} (\hat{\omega}_t) - \Delta_{\omega_{\varepsilon,t}} (k\rho_{\varepsilon}).$$

Combining (4.1) and (4.2), we obtain

$$\frac{\partial}{\partial t} (\hat{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\rho_{\varepsilon}) = \Delta_{\omega_{\varepsilon,t}} (\hat{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\rho_{\varepsilon}) - n + \text{tr}_{\omega_{\varepsilon,t}} \left( \hat{\omega}_t + \frac{\partial}{\partial t} \hat{\omega}_t \right).$$

(4.3) Next, by using (4.1) again, we have

$$\frac{\partial}{\partial t} (-e^{-T} \hat{\varphi}_{\varepsilon,t}) = -e^{-T} \frac{\partial}{\partial t} (\hat{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t})$$

$$= -\text{tr}_{\omega_{\varepsilon,t}} \left( e^{-T} \frac{\partial}{\partial t} \hat{\omega}_t \right) - \Delta_{\omega_{\varepsilon,t}} (e^{-T} \hat{\varphi}_{\varepsilon,t}).$$

(4.4) By (4.3), (4.4), and (2.1), we get the assertion. \hfill \Box

**Lemma 4.3.** There exists a constant $C_6 > 1$ independent of $\varepsilon$ and $t$ satisfying the following inequalities:

(a) $$\frac{1}{C_6 (|s|_h^2 + \varepsilon^2)^{1-\beta}} \leq \hat{\omega}_{\varepsilon,0}^n \leq C_6 \frac{\Omega}{(|s|_h^2 + \varepsilon^2)^{1-\beta}}.$$
(b) \[ \tilde{\omega}_{\varepsilon,t}^n \leq C_3^n C_6^\Omega \frac{\Omega}{(|s|^2 h + \varepsilon^2)^{1-\beta}}. \]

**Proof.** The first inequality follows from (2.8). We prove (b). For \( 0 < k < C_3 \), by (2.2) and (2.6), we have

\[ \tilde{\omega}_{\varepsilon,T} = \hat{\omega}_T + k \sqrt{-1} \partial \bar{\partial} \rho_{\varepsilon} \leq C_3 \omega_0 + k \sqrt{-1} \partial \bar{\partial} \rho_{\varepsilon} \leq C_3 \tilde{\omega}_{\varepsilon,0}. \]

Since \( C_3 > 1 \), we have

\[ \tilde{\omega}_{\varepsilon,t} = a_t \tilde{\omega}_{\varepsilon,0} + (1 - a_t) \tilde{\omega}_{\varepsilon,T} \leq a_t \tilde{\omega}_{\varepsilon,0} + C_3 (1 - a_t) \tilde{\omega}_{\varepsilon,0} \leq C_3 \tilde{\omega}_{\varepsilon,0}. \]

Therefore we get the assertion. \( \square \)

Using these lemmas, we can prove the uniform lower boundedness of \( v_{\varepsilon,t} \).

**Proposition 4.4.** \( v_{\varepsilon,t} \) is uniformly lower bounded. More precisely, there exists a constant \( C_7 > 0 \) independent of \( \varepsilon \) and \( t \) such that

\[ v_{\varepsilon,t} \geq -C_7. \]

**Proof.** By Lemma 4.2 and the semi-positivity of \( \hat{\omega}_T \), we have

\[ \left( \frac{\partial}{\partial t} - \Delta \omega_{\varepsilon,t} \right) (v_{\varepsilon,t} + nt) = u_{\varepsilon,t} = \text{tr} \omega_{\varepsilon,t} (\hat{\omega}_T) \geq 0. \]

Thus, the maximum principle for \( v_{\varepsilon,t} + nt \) gives the following:

\[ v_{\varepsilon,t} + nt \geq \min_{X \times [0,T]} (v_{\varepsilon,t} + nt) = (1 - e^{-T}) \dot{\varphi}_{\varepsilon,0} + k \rho_{\varepsilon} \geq (1 - e^{-T}) \dot{\varphi}_{\varepsilon,0}. \]

Lemma 4.3 (a) and (2.7) give the lower boundedness of right hand side as follows:

\[ \dot{\varphi}_{\varepsilon,0} = \log \frac{\tilde{\omega}_{\varepsilon,0}^n}{\Omega/(|s|^2 h + \varepsilon^2)^{1-\beta}} - \varphi_{\varepsilon,0} - k \rho_{\varepsilon} \geq - \log C_6 - k C_4. \]

Therefore we get the assertion. \( \square \)

The upper bound for \( v_{\varepsilon,t} \) follows from the next proposition.

**Proposition 4.5.** We have the following inequalities:

(a) \( \varphi_{\varepsilon,t} \leq C_8 \),

(b) \( \dot{\varphi}_{\varepsilon,t} \leq C_9 \),

where \( C_8 > 0 \), \( C_9 > 0 \) independent of \( \varepsilon \) and \( t \).

**Proof.** (a) Since \( \varphi_{\varepsilon,0} = 0 \), \( \varphi_{\varepsilon,t} \) takes maximum at \( (x_0, t_0) \in X \times (0, T) \). By (2.11) and Lemma 4.3 (b), we have the following inequality at \( (x_0, t_0) \):

\[ 0 \leq \frac{\partial}{\partial t} \varphi_{\varepsilon,t} \leq \log \frac{\tilde{\omega}_{\varepsilon,t}^n}{\Omega/(|s|^2 h + \varepsilon^2)^{1-\beta}} - \varphi_{\varepsilon,t} - k \rho_{\varepsilon} \leq \log (C_3^n C_6) - \varphi_{\varepsilon,t}. \]

We obtain \( \varphi_{\varepsilon,t}(x_0, t_0) \leq \log (C_3^n C_6) =: C_8 \) since \( (x_0, t_0) \) is arbitrary, \( \varphi_{\varepsilon,t} \leq C_8 \) holds on \( X \times [0, T] \).

(b) We set \( H_{\varepsilon,t} := (1 - e^t) \dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k \rho_{\varepsilon} + nt \). The same computation in Lemma 4.2 gives

\[ \left( \frac{\partial}{\partial t} - \Delta \omega_{\varepsilon,t} \right) H_{\varepsilon,t} = \text{tr} \omega_{\varepsilon,t} (\omega_0) > 0. \]
By the maximum principle for $H_{\varepsilon,t}$, we have
\[ H_{\varepsilon,t} \geq \min_{X \times \{0\}} H_{\varepsilon,t} = k\rho_{\varepsilon} \geq 0. \]

Therefore, combining with (a) and (2.7), we get the upper bound for $\dot{\varphi}_{\varepsilon,t}$:
\[ \dot{\varphi}_{\varepsilon,t} \leq \frac{\varphi_{\varepsilon,t} + k\rho_{\varepsilon} + nt}{e^{t} - 1} \leq \frac{C_{8} + kC_{4} + nT}{e^{t} - 1}. \]

Combining with the uniform local estimate for the parabolic equation, we get the assertion. \[ \Box \]

5. The $C^{0}$-estimate for $u_{\varepsilon,t}$

In this section, we prove the following proposition.

**Proposition 5.1.** There exists a constant $C_{10} > 0$ independent of $\varepsilon$ and $t$ such that
\[ 0 \leq u_{\varepsilon,t} := \text{tr}_{\omega_{\varepsilon,t}}(\tilde{\omega}_{T}) \leq C_{10}. \]

To prove this, we need the estimate $\eta_{\varepsilon}$ and the parabolic Schwarz lemma.

**Lemma 5.2.** We have the following inequalities of $\eta_{\varepsilon}$.

(a) Lower boundedness of $\eta_{\varepsilon}$:
\[ \eta_{\varepsilon} = (1 - \beta)\varepsilon^{2} \frac{s^{2}}{|s_{h}|^{2} + \varepsilon^{2}} \left( \frac{\sqrt{-1}(\nabla s \wedge \nabla s)_{h}}{|s_{h}|^{2} + \varepsilon^{2}} + \sqrt{-1}R_{h} \right) \geq -(1 - \beta)C_{1}\tilde{\omega}_{T}. \]

(b) For any Kähler form $\omega$, we have
\[ -\langle \eta_{\varepsilon}, \tilde{\omega}_{T} \rangle_{\omega} \leq (1 - \beta)C_{1}|\tilde{\omega}_{T}|_{\omega}^{2} \leq (1 - \beta)C_{1}(\text{tr}_{\omega}(\tilde{\omega}_{T}))^{2}. \]

By the fact that $\tilde{\omega}_{T}$ is the pullback of the Kähler form $\omega_{Z}$ by the holomorphic map $f$, we can use the parabolic Schwarz lemma which is obtained by Song-Tian [ST07]. This is the parabolic version of [Yau78]. Combining it with Lemma 5.2 (b), we have the following Lemma.

**Lemma 5.3** (parabolic Schwarz lemma). $u_{\varepsilon,t}$ and $\log u_{\varepsilon,t}$ satisfy the following inequalities.

(a) $\Delta_{\omega_{\varepsilon,t}}u_{\varepsilon,t} \geq -C_{Z}u_{\varepsilon,t}^{2} + \langle \text{Ric}(\omega_{\varepsilon,t}), \tilde{\omega}_{T} \rangle_{\omega_{\varepsilon,t}}$
\[ \geq -C_{11}u_{\varepsilon,t}^{2} + \langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \tilde{\omega}_{T} \rangle_{\omega_{\varepsilon,t}}. \]

(b) $\left( \frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) u_{\varepsilon,t} \leq u_{\varepsilon,t} + C_{Z}u_{\varepsilon,t}^{2} - \langle \eta_{\varepsilon}, \tilde{\omega}_{T} \rangle_{\omega_{\varepsilon,t}} - \frac{|\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^{2}}{u_{\varepsilon,t}}$
\[ \leq u_{\varepsilon,t} + C_{11}u_{\varepsilon,t}^{2} - \frac{|\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^{2}}{u_{\varepsilon,t}}. \]

(c) $\left( \frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \log u_{\varepsilon,t} \leq C_{Z}u_{\varepsilon,t} + 1 - \frac{\langle \eta_{\varepsilon}, \tilde{\omega}_{T} \rangle_{\omega_{\varepsilon,t}}}{u_{\varepsilon,t}}$
\[ \leq C_{11}u_{\varepsilon,t} + 1. \]
Here, $\nabla$ is $(1,0)$-part of the Levi-Civita connection of $\omega_{\varepsilon,t}$, $C_2 > 0$ is an upper bound for the bisectional curvature of $\omega_Z$, and $C_{11} := C_2 + (1 - \beta)C_1 > 0$.

**Proof of Proposition 5.1** We set $G_{\varepsilon,t} := \log u_{\varepsilon,t} - C_{12}v_{\varepsilon,t}$, where $C_{12} := C_{11} + 1 > 0$. The uniform upper boundedness of $G_{\varepsilon,0}$ follows from (2.6), (2.10) and Proposition 4.1. We assume that $G_{\varepsilon,t}$ achieves maximum at $(x_0, t_0) \in X \times (0, T)$. At this point, by Lemma 5.3(c) and Lemma 4.2, we have $u_{\varepsilon,t} > 0$ and

$$
\left( \frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) G_{\varepsilon,t} \leq (C_{11}u_{\varepsilon,t} + 1) - (C_{12}(u_{\varepsilon,t} - n)) = -u_{\varepsilon,t} + (C_{12}n + 1).
$$

By using the uniform boundedness of $v_{\varepsilon,t}$ (Proposition 4.1), we obtain

$$
G_{\varepsilon,t}(x_0, t_0) \leq \log(C_{12}n + 1) - C_{12}v_{\varepsilon,t} \leq \log(C_{12}n + 1) + C_{12}C_5.
$$

Since $(x_0, t_0)$ is arbitrary, we have $G_{\varepsilon,t} \leq C_{13}$ on $X \times [0, T)$. Hence, by using Proposition 4.1 again, we get the assertion.

### 6. The Gradient estimate for $v_{\varepsilon,t}$

In this section, we prove the following gradient estimate for $v_{\varepsilon,t}$.

**Proposition 6.1.** There exists a uniform constant $C_{14} > 0$ which is independent of $\varepsilon$ and $t$ such that

$$
|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} \leq C_{14}.
$$

To prove this proposition, as in [Zha10], we set $\Psi_{\varepsilon,t} := \frac{|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}}{A - v_{\varepsilon,t}}$ and use the maximum principle to $\Psi_{\varepsilon,t} + u_{\varepsilon,t}$. Here $A > C_5 + 1$ is a fixed constant.

**Lemma 6.2.** We have the following formulas.

(a) $$
\left( \frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) |\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} \quad = \quad |\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - \eta_\varepsilon(\nabla v_{\varepsilon,t}, \nabla v_{\varepsilon,t}) + 2\Re \langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - |\nabla \nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - |\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}.
$$

(b) $$
\left( \frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) v_{\varepsilon,t} \quad = \quad \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} + \Delta_{\omega_{\varepsilon,t}} u_{\varepsilon,t} + \langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_\varepsilon, \sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}}.
$$

(c) $$
\left( \frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \Psi_{\varepsilon,t} \quad = \quad \frac{1}{A - v_{\varepsilon,t}} \left( |\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - |\nabla \nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - |\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - \eta_\varepsilon(\nabla v_{\varepsilon,t}, \nabla v_{\varepsilon,t}) + 2\Re \langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \right) + \frac{1}{(A - v_{\varepsilon,t})^2} (u_{\varepsilon,t} - n) |\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - 2\Re \langle \nabla |\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - \frac{2}{(A - v_{\varepsilon,t})^3} |\nabla v_{\varepsilon,t}|^4_{\omega_{\varepsilon,t}}.
$$
Proof of Proposition \ref{prop1} We will apply the maximum principle to $\Psi_{\varepsilon,t} + u_{\varepsilon,t}$. First, we estimate $\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)\Psi_{\varepsilon,t}$. By Lemma \ref{lem5.2} (a) and Proposition \ref{prop5.1}, we have

\begin{equation}
-\eta_{\varepsilon}(\nabla v_{\varepsilon,t}, \nabla v_{\varepsilon,t}) \leq (1 - \beta)C_{1}\Omega_{T}(\nabla v_{\varepsilon,t}, \nabla v_{\varepsilon,t}) \\
\leq (1 - \beta)C_{1}\text{tr}_{\omega_{\varepsilon,t}}(\omega_{\varepsilon,t})(\nabla v_{\varepsilon,t}, \nabla v_{\varepsilon,t}) \\
\leq (1 - \beta)C_{1}C_{10}|\nabla v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}}.
\end{equation}

For sufficiently small constant $\delta > 0$ which will be determined later, we have

\begin{equation}
2\text{Re}(\nabla v_{\varepsilon,t}, \nabla v_{\varepsilon,t})_{\omega_{\varepsilon,t}} \leq 2|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}|\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}} \leq \frac{1}{\delta}|\nabla v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}} + \delta|\nabla u_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}}.
\end{equation}

Since

\begin{equation}
\nabla \Psi_{\varepsilon,t} = \frac{\nabla|\nabla v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}}}{A - v_{\varepsilon,t}} + \frac{|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}}{(A - v_{\varepsilon,t})^{2}}\nabla v_{\varepsilon,t},
\end{equation}

we have

\begin{equation}
-\frac{2 - \delta}{(A - v_{\varepsilon,t})^{2}}\text{Re}(\nabla|\nabla v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}}, \nabla v_{\varepsilon,t})_{\omega_{\varepsilon,t}} \\
= -\frac{2 - \delta}{A - v_{\varepsilon,t}}\text{Re}(\nabla \Psi_{\varepsilon,t}, \nabla v_{\varepsilon,t})_{\omega_{\varepsilon,t}} + (2 - \delta)\frac{|\nabla v_{\varepsilon,t}|^{4}_{\omega_{\varepsilon,t}}}{(A - v_{\varepsilon,t})^{3}}.
\end{equation}

On the other hand, the Cauchy-Schwarz inequality gives

\begin{align*}
|\langle \nabla |\nabla v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}}| \\
= \left|g^{j_{1}}g^{j_{2}}(\partial_{k}\partial_{1}v_{\varepsilon,t})(\partial_{k}\partial_{j_{1}}v_{\varepsilon,t}) + (\partial_{1}v_{\varepsilon,t})(\partial_{k}\partial_{j_{2}}v_{\varepsilon,t})(\partial_{k}v_{\varepsilon,t})\right| \\
\leq |\nabla v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}}(|\nabla \nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}} + |\nabla ^{2}v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}) \\
\leq \sqrt{2}|\nabla v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}}(\nabla |\nabla v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}} + |\nabla ^{2}v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}})^{1/2}.
\end{align*}

By using this inequality, we obtain the following:

\begin{equation}
-\frac{\delta}{(A - v_{\varepsilon,t})^{2}}\text{Re}(\nabla|\nabla v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}}, \nabla v_{\varepsilon,t})_{\omega_{\varepsilon,t}} \\
\leq \frac{\delta}{(A - v_{\varepsilon,t})^{2}}\left(\sqrt{2}|\nabla v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}}(|\nabla \nabla v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}} + |\nabla ^{2}v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}})^{1/2}\right) \\
= \frac{\sqrt{2}\delta}{(A - v_{\varepsilon,t})^{3/2}}\frac{|\nabla v_{\varepsilon,t}|^{4}_{\omega_{\varepsilon,t}}}{(A - v_{\varepsilon,t})^{1/2}} \\
\leq \frac{\delta}{2(A - v_{\varepsilon,t})^{3}} + \frac{|\nabla \nabla v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}} + |\nabla ^{2}v_{\varepsilon,t}|^{2}_{\omega_{\varepsilon,t}}}{A - v_{\varepsilon,t}}.
\end{equation}
Therefore, combining Proposition \ref{prop:6.2} (c) with \eqref{eq:6.1}, \eqref{eq:6.2}, \eqref{eq:6.4}, \eqref{eq:6.5}, Proposition \ref{prop:5.1} and Proposition \ref{prop:4.1} and \( A - C_5 > 1 \), we obtain the following inequality:

\[
\left( \frac{\partial}{\partial t} - \Delta_{\omega_{e,t}} \right) \Psi_{\varepsilon,t} \\
\leq C_{15} |\nabla v_{\varepsilon,t}|_{\omega_{e,t}}^2 + \delta |\nabla u_{\varepsilon,t}|_{\omega_{e,t}}^2 - \frac{2 - \delta}{A - v_{\varepsilon,t}} \Re (\nabla \Psi_{\varepsilon,t}, \nabla v_{\varepsilon,t})_{\omega_{e,t}} - \delta \frac{|\nabla v_{\varepsilon,t}|_{\omega_{e,t}}^4}{2 (A + C_5)^3},
\]

where \( C_{15} := 1 + (1 - \beta)C_1 C_{10} + (1/\delta) + C_{10} > 0 \).

On the other hand, by Lemma \ref{lem:5.3} (a) and Proposition \ref{prop:5.1} we have

\[
\left( \frac{\partial}{\partial t} - \Delta_{\omega_{e,t}} \right) u_{\varepsilon,t} \leq u_{\varepsilon,t} + C_{11} u_{\varepsilon,t}^2 - \frac{|\nabla u_{\varepsilon,t}|_{\omega_{e,t}}^2}{u_{\varepsilon,t}} \leq C_{16} - 2\delta |\nabla u_{\varepsilon,t}|_{\omega_{e,t}}^2,
\]

where \( C_{16} := C_{10} + C_{11} C_{10}^2 \) and \( 0 < \delta < 1/(2C_{10}) \).

Finally, we obtain the following inequality:

\[
(6.6) \quad \left( \frac{\partial}{\partial t} - \Delta_{\omega_{e,t}} \right) (\Psi_{\varepsilon,t} + u_{\varepsilon,t}) \\
= C_{16} + C_{15} |\nabla v_{\varepsilon,t}|_{\omega_{e,t}}^2 - \delta |\nabla u_{\varepsilon,t}|_{\omega_{e,t}}^2 - \frac{2 - \delta}{A - v_{\varepsilon,t}} \Re (\nabla \Psi_{\varepsilon,t}, \nabla v_{\varepsilon,t})_{\omega_{e,t}} - \delta \frac{|\nabla v_{\varepsilon,t}|_{\omega_{e,t}}^4}{2 (A + C_5)^3}
\]

\[
\leq C_{16} + \left( C_{15} + \frac{1}{\delta} \right) |\nabla v_{\varepsilon,t}|_{\omega_{e,t}}^2 \\
- \frac{2 - \delta}{A - v_{\varepsilon,t}} \Re (\nabla (\Psi_{\varepsilon,t} + u_{\varepsilon,t}), \nabla v_{\varepsilon,t})_{\omega_{e,t}} - \delta \frac{|\nabla v_{\varepsilon,t}|_{\omega_{e,t}}^4}{2 (A + C_5)^3}.
\]

Here, we used the following inequality:

\[
\frac{2 - \delta}{A - v_{\varepsilon,t}} \Re (\nabla u_{\varepsilon,t}, \nabla v_{\varepsilon,t})_{\omega_{e,t}} \leq 2 |\nabla v_{\varepsilon,t}|_{\omega_{e,t}} |\nabla u_{\varepsilon,t}|_{\omega_{e,t}} \\
\leq \frac{1}{\delta} |\nabla v_{\varepsilon,t}|_{\omega_{e,t}}^2 + \delta |\nabla u_{\varepsilon,t}|_{\omega_{e,t}}^2.
\]

The uniform boundedness of \( \Psi_{\varepsilon,0} + u_{\varepsilon,0} \) follows from \cite[Section 4]{CGP13}, Proposition \ref{prop:4.1} and Proposition \ref{prop:5.1}. If \( \Psi_{\varepsilon,t} + u_{\varepsilon,t} \) achieves maximum at \((x_0, t_0) \in X \times (0, T)\), by \eqref{eq:6.6}, we have the following estimate:

\[
0 \leq C_{16} + \left( C_{15} + \frac{1}{\delta} \right) |\nabla v_{\varepsilon,t}|_{\omega_{e,t}}^2 - \delta \frac{1}{2 (A + C_5)^3} |\nabla v_{\varepsilon,t}|_{\omega_{e,t}}^4 \quad \text{at } (x_0, t_0).
\]

It follows that there exists a constant \( C_{17} > 0 \) satisfying

\[
|\nabla v_{\varepsilon,t}|_{\omega_{e,t}}^2 \leq C_{17} \quad \text{at } (x_0, t_0),
\]

which does not depend on \( \varepsilon \) and \( t \). By using the definition of \( \Psi_{\varepsilon,t} \), \( A - v_{\varepsilon,t} > 1 \), and Proposition \ref{prop:5.1} we have the uniform upper bound for \( \Psi_{\varepsilon,t} + u_{\varepsilon,t} \) on \( X \times [0, T) \). Therefore, we obtain the uniform upper bound for \( |\nabla v_{\varepsilon,t}|_{\omega_{e,t}}^2 \).
7. The Laplacian Estimate for $v_{\varepsilon,t}$

In this section, we estimate $\Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}$. In order to prove the uniform upper boundedness of $\Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}$, we need the lower bound for the scalar curvature due to Edwards [Edw15, Corollary 4.3].

**Proposition 7.1** ([Edw15 Corollary 4.3]). The scalar curvature $R(\omega_{\varepsilon,t})$ is uniformly bounded from below by

$$R(\omega_{\varepsilon,t}) - \text{tr}_{\omega_{\varepsilon,t}}(\eta_{\varepsilon}) \geq -C_{18},$$

where $C_{18} > 0$ is a constant independent of $\varepsilon$ and $t$.

Using this estimate, we can easily obtain the following upper bound.

**Proposition 7.2.** There exists a uniform constant $C_{19} > 0$ which is independent of $\varepsilon$ and $t$ such that

$$\Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t} \leq C_{19}.$$

**Proof.** By Proposition 3.1, Proposition 7.1, and $u_{\varepsilon,t} \geq 0$, we have

$$\Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t} = ne^{t-T} - u_{\varepsilon,t} - (1 - e^{t-T})(R(\omega_{\varepsilon,t}) - \text{tr}_{\omega_{\varepsilon,t}}(\eta_{\varepsilon})) \leq n + C_{18} =: C_{19},$$

which proves the assertion. \qed

**Proposition 7.3.** There exists a constant $C_{20} > 0$ independent of $\varepsilon$ and $t$ such that

$$\Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t} \geq -\frac{C_{20}}{T-t}.$$

**Proof.** As in [Zha10 Section 3.3], we set

$$\Phi_{\varepsilon,t} := \frac{B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}}{B - v_{\varepsilon,t}},$$

where $B > 0$ is a sufficiently large uniform constant satisfying $B - C_{19} > 0$, and $B - C_{5} > 1$ so that the numerator and the denominator of $\Phi_{\varepsilon,t}$ are positive. Straightforward calculations show that

\begin{align*}
(7.1) & \quad \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)\Phi_{\varepsilon,t} \\
(7.2) & \quad = -\frac{1}{B - v_{\varepsilon,t}}\Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t} + \frac{1}{(B - v_{\varepsilon,t})^{2}}(u_{\varepsilon,t} - n)(B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}) \\
(7.3) & \quad - \frac{1}{B - v_{\varepsilon,t}}(\langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \sqrt{-1}\partial\bar{\partial}v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} + \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}) \\
& \quad - \frac{2}{B - v_{\varepsilon,t}}\text{Re}\langle \nabla\Phi_{\varepsilon,t}, \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}}.
\end{align*}
We first estimate (7.2). By using $B - v_{\varepsilon,t} > 1$, $B - \Delta \omega_{\varepsilon,t} v_{\varepsilon,t} > 0$, and Proposition $5.1$, we have

\[
(7.4) \quad \frac{-1}{B - v_{\varepsilon,t}} \Delta \omega_{\varepsilon,t} v_{\varepsilon,t} + \frac{1}{(B - v_{\varepsilon,t})^2} (u_{\varepsilon,t} - n)(B - \Delta \omega_{\varepsilon,t} v_{\varepsilon,t})
= \left( \frac{B - \Delta \omega_{\varepsilon,t} v_{\varepsilon,t}}{B - v_{\varepsilon,t}} + \frac{-B}{B - v_{\varepsilon,t}} \right) + \frac{1}{(B - v_{\varepsilon,t})^2} (u_{\varepsilon,t} - n)(B - \Delta \omega_{\varepsilon,t} v_{\varepsilon,t})
\leq C_{21} (B - \Delta \omega_{\varepsilon,t} v_{\varepsilon,t}),
\]

where $C_{21} := 1 + C_{10} > 0$.

We next estimate (7.3). By using Lemma $5.3$ (a) and Proposition $3.1$ (a), we obtain

\[
- \langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - \Delta \omega_{\varepsilon,t} u_{\varepsilon,t}
\leq - \langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} + \tilde{\omega}_T \rangle_{\omega_{\varepsilon,t}} + C_{22}
= \frac{1}{1 - e^{t - T}} \left| \sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} + \tilde{\omega}_T \right|_{\omega_{\varepsilon,t}}^2 - \frac{e^{t - T}}{1 - e^{t - T}} (\Delta \omega_{\varepsilon,t} v_{\varepsilon,t} + u_{\varepsilon,t}) + C_{22},
\]

where $C_{22} := C_{11} C_{10}^2 > 0$. By using Proposition $5.1$, the first term is estimated as follows:

\[
\left| \sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} + \tilde{\omega}_T \right|_{\omega_{\varepsilon,t}}^2 \leq (1 + \delta) \left| \nabla \nabla v_{\varepsilon,t} \right|_{\omega_{\varepsilon,t}}^2 + (1 + 1/\delta) \left| \tilde{\omega}_T \right|_{\omega_{\varepsilon,t}}^2
\leq (1 + \delta) \left| \nabla \nabla v_{\varepsilon,t} \right|_{\omega_{\varepsilon,t}}^2 + C_{23},
\]

where $\delta > 0$ is a uniform constant determined later and $C_{23} := (1 + 1/\delta) C_{10}^2 > 0$.

For the second term, we have

\[
- \frac{e^{t - T}}{1 - e^{t - T}} (\Delta \omega_{\varepsilon,t} v_{\varepsilon,t} + u_{\varepsilon,t}) = \frac{e^{t - T}}{1 - e^{t - T}} (B - \Delta \omega_{\varepsilon,t} v_{\varepsilon,t}) - \frac{Be^{t - T}}{1 - e^{t - T}}
\leq \frac{1}{1 - e^{t - T}} (B - \Delta \omega_{\varepsilon,t} v_{\varepsilon,t}).
\]

Finally, we get

\[
(7.5) \quad - \frac{1}{B - v_{\varepsilon,t}} \langle \text{Ric}(\omega_{\varepsilon,t}) - \eta_{\varepsilon}, \sqrt{-1} \partial \bar{\partial} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - \Delta \omega_{\varepsilon,t} u_{\varepsilon,t}
\leq \frac{C_T}{T - t} \left( \frac{1 + \delta}{B - v_{\varepsilon,t}} \left| \nabla \nabla v_{\varepsilon,t} \right|_{\omega_{\varepsilon,t}}^2 + C_{23} \right) + \frac{C_T}{T - t} (B - \Delta \omega_{\varepsilon,t} v_{\varepsilon,t}) + C_{22},
\]

where $C_T > 0$ is a uniform constant satisfying

\[
\frac{1}{1 - e^{t - T}} \leq \frac{C_T}{T - t}
\]

for all $t \in [0, T)$. 
Combining (7.1) with (7.4) and (7.3), we get

\[
\left( \frac{\partial}{\partial t} - \Delta_{\omega,t} \right) \Phi_{\varepsilon,t} \\
\leq C_{21} (B - \Delta_{\omega,t} v_{\varepsilon,t}) \\
+ \frac{C_T}{T-t} \left( \frac{1 + \delta}{B - v_{\varepsilon,t}} \right) |\nabla \nabla v_{\varepsilon,t}|_{\omega,t}^2 + C_{23} \right) + \frac{C_T}{T-t} (B - \Delta_{\omega,t} v_{\varepsilon,t}) + C_{22} \\
- \frac{2}{B - v_{\varepsilon,t}} \text{Re} \langle \nabla \Phi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega,t,t} \\
\leq \frac{C_{24}}{T-t} + \frac{C_{24}}{T-t} (B - \Delta_{\omega,t} v_{\varepsilon,t}) + \frac{C_T}{T-t} B - v_{\varepsilon,t} \frac{1 + \delta}{B - v_{\varepsilon,t}} |\nabla \nabla v_{\varepsilon,t}|_{\omega,t}^2 \\
- \frac{2}{B - v_{\varepsilon,t}} \text{Re} \langle \nabla ((T - t) \Phi_{\varepsilon,t}), \nabla v_{\varepsilon,t} \rangle_{\omega,t,t}.
\]

(7.6) \[
\left( \frac{\partial}{\partial t} - \Delta_{\omega,t} \right) (T-t) \Phi_{\varepsilon,t} = -\Phi_{\varepsilon,t} + (T - t) \left( \frac{\partial}{\partial t} - \Delta_{\omega,t} \right) \Phi_{\varepsilon,t} \\
\leq (T-t) \left( \frac{\partial}{\partial t} - \Delta_{\omega,t} \right) \Phi_{\varepsilon,t} \\
\leq C_{24} + C_{24} (B - \Delta_{\omega,t} v_{\varepsilon,t}) + C_T \frac{1 + \delta}{B - v_{\varepsilon,t}} |\nabla \nabla v_{\varepsilon,t}|_{\omega,t}^2 \\
- \frac{2}{B - v_{\varepsilon,t}} \text{Re} \langle \nabla ((T - t) \Phi_{\varepsilon,t}), \nabla v_{\varepsilon,t} \rangle_{\omega,t,t}.
\]

We set \( \tilde{\Psi}_{\varepsilon,t} := \frac{|\nabla v_{\varepsilon,t}|_{\omega,t}^2}{B - v_{\varepsilon,t}} \). Combining Lemma 6.2 (c) with (6.3), Proposition 5.1 and Proposition 6.1, we have

\[
(7.7) \left( \frac{\partial}{\partial t} - \Delta_{\omega,t} \right) \tilde{\Psi}_{\varepsilon,t} \\
= \frac{1}{B - v_{\varepsilon,t}} \left( |\nabla v_{\varepsilon,t}|_{\omega,t}^2 - |\nabla \nabla v_{\varepsilon,t}|_{\omega,t}^2 - |\nabla \nabla v_{\varepsilon,t}|_{\omega,t}^2 - \eta_{\varepsilon}(\nabla v_{\varepsilon,t}, \nabla v_{\varepsilon,t}) \right) \\
+ \frac{1}{(B - v_{\varepsilon,t})^2} (u_{\varepsilon,t} - n) |\nabla v_{\varepsilon,t}|_{\omega,t}^2 - \frac{2}{B - v_{\varepsilon,t}} \text{Re} \langle \nabla (\tilde{\Psi}_{\varepsilon,t} - u_{\varepsilon,t}), \nabla v_{\varepsilon,t} \rangle_{\omega,t,t} \\
\leq C_{25} - \frac{|\nabla \nabla v_{\varepsilon,t}|_{\omega,t}^2}{B - v_{\varepsilon,t}} - \frac{2}{B - v_{\varepsilon,t}} \text{Re} \langle \nabla (\tilde{\Psi}_{\varepsilon,t} - u_{\varepsilon,t}), \nabla v_{\varepsilon,t} \rangle_{\omega,t,t},
\]

where \( C_{25} := C_{14} (1 + (1 - \beta)C_1 C_{10} + C_{10}) > 0. \)

We next estimate \( u_{\varepsilon,t} \). We first note that the following estimate holds:

\[
(7.8) \frac{4}{B - v_{\varepsilon,t}} \text{Re} \langle \nabla u_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega,t,t} \leq \delta |\nabla u_{\varepsilon,t}|_{\omega,t}^2 + \frac{4}{\delta} |\nabla v_{\varepsilon,t}|_{\omega,t}^2.
\]
By using Lemma 5.3 (b), Proposition 5.1, Proposition 6.1 and (7.8), we get
\begin{equation}
\left( \frac{\partial}{\partial t} - \Delta \omega_{\epsilon,t} \right) u_{\epsilon,t} \leq C_{10} + C_{11} C_{10}^2 - \frac{1}{C_{10}} |\nabla u_{\epsilon,t}|^2_{\omega_{\epsilon,t}} \\
\leq -\frac{4}{B - v_{\epsilon,t}} \text{Re} \langle \nabla u_{\epsilon,t}, \nabla v_{\epsilon,t} \rangle_{\omega_{\epsilon,t}} + C_{26},
\end{equation}
where we take \(0 < \delta < 1/C_{10}\) and \(C_{26} := C_{10} + C_{11} C_{10}^2 + 4C_{14}/\delta > 0\).

Combining (7.6), (7.7), and (7.9), we have
\begin{equation}
\left( \frac{\partial}{\partial t} - \Delta \omega_{\epsilon,t} \right) \left( (T-t)\Phi_{\epsilon,t} + 2C_T \tilde{\Psi}_{\epsilon,t} + 2C_T u_{\epsilon,t} \right) \\
\leq C_{27} + C_{27}(B - \Delta \omega_{\epsilon,t} v_{\epsilon,t}) - C_T \frac{1 - \delta}{B + C_5} |\nabla v_{\epsilon,t}|^2_{\omega_{\epsilon,t}} \\
- \frac{2}{B - v_{\epsilon,t}} \text{Re} \left\langle \nabla \left( (T-t)\Phi_{\epsilon,t} + 2C_T \tilde{\Psi}_{\epsilon,t} + 2C_T u_{\epsilon,t} \right), \nabla v_{\epsilon,t} \right\rangle_{\omega_{\epsilon,t}}.
\end{equation}

The uniform boundedness of \((T-t)\Phi_{\epsilon,t} + 2C_T \tilde{\Psi}_{\epsilon,t} + 2C_T u_{\epsilon,t}\) at \(t = 0\) follows from [CGP13, Section 4], Proposition 4.1, Proposition 5.1 and Proposition 6.1. If \((T-t)\Phi_{\epsilon,t} + 2C_T \tilde{\Psi}_{\epsilon,t} + 2C_T u_{\epsilon,t}\) achieves maximum at \((x_0, t_0) \in X \times (0, T)\), we have the following estimate at this point:
\begin{equation}
0 \leq C_{27} + C_{27}(B - \Delta \omega_{\epsilon,t} v_{\epsilon,t}) - C_T \frac{1 - \delta}{B + C_5} |\nabla v_{\epsilon,t}|^2_{\omega_{\epsilon,t}} \\
\leq C_{27} + C_{27}(B - \Delta \omega_{\epsilon,t} v_{\epsilon,t}) - C_T \frac{1 - \delta}{B + C_5} \left( \frac{1}{n} (B - \Delta \omega_{\epsilon,t} v_{\epsilon,t})^2 - \frac{B^2}{n} \right).
\end{equation}

Therefore, at this point, there exists a constant \(C_{28} > 0\) satisfying
\[-\Delta \omega_{\epsilon,t} v_{\epsilon,t} \leq C_{28} \text{ at } (x_0, t_0)\]
which is independent of \(\epsilon, t, \) and \((x_0, t_0)\). Combining Proposition 4.1, Proposition 6.1 and Proposition 5.1, we obtain the uniform upper boundedness of \((T-t)\Phi_{\epsilon,t} + 2C_T \tilde{\Psi}_{\epsilon,t} + 2C_T u_{\epsilon,t}\) on \(X \times [0, T)\). Therefore we get the lower bound for \(\Delta \omega_{\epsilon,t} v_{\epsilon,t}\).

**Proof of Theorem 1.1:** By Proposition 3.1 and Proposition 7.3, we have
\begin{equation}
R(\omega_{\epsilon,t}) - \text{tr}\omega_{\epsilon,t}(\eta_\epsilon) = \frac{1}{1 - e^{t-T}} \left( -\Delta \omega_{\epsilon,t} v_{\epsilon,t} + ne^{t-T} - u_{\epsilon,t} \right) \\
\leq \frac{C_T}{T-t} \left( \frac{C_{20}}{T-t} + n \right) \leq \frac{C}{(T-t)^2},
\end{equation}
where \(C > 0\) does not depend on \(\epsilon\) and \(t\). Therefore, by taking \(\epsilon_i \to 0\), we get the assertion.

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