FUSS-SCHRÖDER PATHS AND ROOTED PLANE FORESTS

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Abstract. We describe a bijection between \((k,k)\)-Fuss-Schröder paths of type \(\lambda\) and certain rooted plane forests with \(n(k+1)+2\) vertices. This yields a recursion which allows us to analytically enumerate the number of large \((k,r)\)-Fuss-Schröder paths of type \(\lambda\), solving an open question posed by An, Jung, and Kim. Furthermore, we generalize the concept of \((k,r)\)-Fuss-Schröder paths to \((k,S)\)-Fuss-Schröder paths, in which \(r\) can take any value in a given set \(S\), and enumerate these paths as well.

1. Introduction

The \((k,r)\)-Fuss-Schröder path was introduced by Eu and Fu ([4]) as a simultaneous generalization of Dyck paths, Schröder paths, and Fuss-Catalan paths. The paths were defined in their relation through enumerative invariants to generalized cluster complexes in a way aimed to specialize to the analogous relation between Schröder paths and simplicial associahedra at \(k=1\) ([5, 6]). Eu and Fu enumerated the total number of small \((k,r)\)-Fuss-Schröder paths for a fixed \(n\) and number of diagonal steps. Recently, An, Jung, and Kim ([2]) enumerated the number of small \((k,r)\)-Fuss-Schröder paths by fixed type and posed the same question for large \((k,r)\)-Fuss-Schröder paths of fixed type. Our central result is finding and proving such a formula.

Schröder paths provide one generalization for Dyck paths (north, east paths from \((0,0)\) to \((n,n)\) remaining nonstrictly above \(y=x\)) by also allowing diagonal steps. A path is small if no diagonal step lies on the line \(y=x\) and large if no such restriction exists. Alternatively, Fuss-Catalan paths generalize Dyck paths by considering paths from \((0,0)\) to \((n,kn)\) nonstrictly above \(y=kn\). In the way Fuss-Catalan paths are a Fuss analogue of Dyck (or Catalan) paths, \((k,r)\)-Fuss-Schröder paths are a Fuss analogue of Schröder paths in that they may use diagonal steps but also start at \((0,0)\) and end at \((n,kn)\), including both generalizations at once. The type of a given path denotes the partition yielded by consecutive runs of \(E\) steps.

Enumeration of each of these paths have been studied in detail, both in total and by type. The number of Dyck paths by type was found by Krewaras ([7]) and proven with a direct bijective argument by Liaw ([8]). An, Eu, and Kim ([1]) enumerated the number of large Schröder paths by type, and Park and Kim ([9]) did the same for small Schröder paths by type. Finally, An, Jung, and Kim ([2]) enumerated the number of small \((k,r)\)-Fuss-Schröder paths by type.

In Section 2 we describe the relevant definitions and past results in more detail. In Section 3, we define a certain class of rooted plane forests which are in bijection with \((k,r)\)-Fuss-Schröder paths. We discuss the recursive nature of these rooted plane forests before...
translating the relations to generating function identities in Section 4. Using the Lagrange
Inversion Formula, we extract the cardinality of several sets, including: small \((k,r)\)-Fuss-
Schröder paths by type, large \((k,r)\)-Fuss-Schröder paths by type, and large \((k,r)\)-Fuss-
Schröder paths ending in a diagonal step by type. Finally, in Section 5, we generalize our
results to \((k,S)\)-Fuss-Schröder paths, in which a diagonal step may occur at any \(r \in S\) for
some \(S \subset [k]\).

We assume the reader is familiar with the theory of composition of generating functions-
for reference, see [10, Chapter 6].

2. Background

Definition 2.1. A Dyck path of length \(n\) is a path consisting of a sequence of east and
north steps \((E = (1, 0)\) and \(N = (0, 1)\), respectively\) from \((0, 0)\) to \((n,n)\) such that the path
stays nonstrictly above the line \(y = x\).

Definition 2.2. The type of a Dyck path (or any path containing east steps) is the partition
formed by parts with size the maximal consecutive runs of east steps in the path.

Given a partition \(\lambda\), we let \(\ell(\lambda)\) denote its length and \(|\lambda|\) denote the sum of its parts. We
define \(m_\lambda = m_1(\lambda)!m_2(\lambda)!m_3(\lambda)! \cdots\), where \(m_i(\lambda)\) denotes the number of parts of \(\lambda\) equal
to \(i\). Note that for a Dyck path of length \(n\) and type \(\lambda\), we have \(|\lambda| = n\).

Example 2.1. The Dyck path shown in Figure 1 has length \(n = 7\). Its type is \(\lambda = (3, 2, 1, 1)\),
which satisfies \(\ell(\lambda) = 4\), \(|\lambda| = 3 + 2 + 1 + 1 = 7\), and \(m_\lambda = 2! \cdot 1! \cdot 1! = 2\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Dyck_path.png}
\caption{The Dyck path \(NNENNNEEENE\) of length 7.}
\end{figure}

It is well known that the number of Dyck paths of a given length \(n\) is \(C_n = \frac{1}{n+1} \binom{2n}{n}\).

However, one can also restrict to paths of a fixed type:

Theorem 2.1 (Krewaras, [7]). The number of Dyck paths of type \(\lambda\) is
\[\frac{n(n-1) \cdots (n - \ell(\lambda) + 2)}{m_\lambda}.\]

A slightly more general type of path allows diagonals.

Definition 2.3. A large Schröder path of length \(n\) is a path with north, east, and diagonal
steps (where a diagonal step is \((1, 1)\)) from \((0,0)\) to \((n,n)\) which stays nonstrictly above the
line \(y = x\).

A small Schröder path of length \(n\) is a large Schröder path of length \(n\) such that no
diagonal step lies on \(y = x\).
The number of large and small Schröder paths by type has much more recently been enumerated.

**Theorem 2.2** (An, Eu, Kim, [1]). The number of large Schröder paths of type \( \lambda \) is

\[
\frac{1}{|\lambda| + 1} \binom{n}{|\lambda|} \frac{(n+1)!}{\ell(\lambda)!} m_\lambda.
\]

**Theorem 2.3** (Park, Kim, [9]). The number of small Schröder paths of type \( \lambda \) is

\[
\frac{1}{n+1} \binom{n-1}{|\lambda|-1} \frac{(n+1)!}{\ell(\lambda)!} m_\lambda.
\]

We may also generalize Dyck paths in an entirely different sense. Instead of altering the types of steps, we alter the region that contains the path.

**Definition 2.4.** A \( k \)-Fuss-Catalan path of length \( n \) is a path with north and east steps from \((0,0)\) to \((n,kn)\) which stays nonstrictly above the line \( y = kx \).

The number of Fuss-Catalan paths has also been counted by type.

**Theorem 2.4** (Armstrong, [3]). The number of \( k \)-Fuss-Catalan paths of type \( \lambda \) is

\[
\frac{(kn)!}{m_\lambda \cdot (kn+1-\ell(\lambda))!}.
\]

Finally, we come to our main object of study, the \((k,r)\)-Fuss-Schröder path. This generalizes all three paths thus far described.

**Definition 2.5.** For integers \( k \) and \( r \) such that \( 1 \leq r \leq k \), a large \((k,r)\)-Fuss-Schröder path of length \( n \) is a path from \((0,0)\) to \((n,kn)\) using east, north, and diagonal steps which stays nonstrictly above the line \( y = kx \) and such that each diagonal step can only go from the line \( y = kj + r - 1 \) to the line \( y = kj + r \) for some \( j \).

A small \((k,r)\)-Fuss-Schröder path of length \( n \) is a large \((k,r)\)-Fuss-Schröder path of length \( n \) such that no diagonal steps touch the line \( y = kx \).

As before, the type of a \((k,r)\)-Fuss-Schröder path is the partition formed by maximal consecutive runs of east steps.

In the way \( k \)-Fuss-Catalan paths are a Fuss analogue for Dyck (or Catalan) paths, \((k,r)\)-Fuss-Schröder paths are a Fuss analogue for Schröder paths.

An, Jung, and Kim ([2]) enumerated the number of small \((k,r)\)-Fuss-Schröder paths of type \( \lambda \), which is independent of \( r \).

**Theorem 2.5.** The number of small \((k,r)\)-Fuss-Schröder paths of type \( \lambda \) is

\[
\frac{1}{kn+1} \binom{n-1}{|\lambda|-1} \frac{(kn+1)!}{\ell(\lambda)!} m_\lambda.
\]

They posed the open question of enumerating large \((k,r)\)-Fuss-Schröder paths of type \( \lambda \), which is our main focus. Note that for \( 1 \leq r \leq k-1 \), all paths are small, so in these cases the number of large \((k,r)\)-Fuss-Schröder paths of type \( \lambda \) has already been enumerated. Thus we only consider the case \( r = k \).
3. A Rooted Plane Forest Bijection

An, Jung, and Kim ([2]) provide bijections between various Schröder and Dyck paths and noncrossing set partitions and conjecture a similar bijection for large \((k,k)\)-Fuss-Schröder paths.

We approach the enumeration problem from the point of view of trees, which we view as a natural way to interpret the data in each path. In particular, we establish a bijection between \((k,k)\)-Fuss-Schröder paths of type \(\lambda\) and rooted plane forests (or ordered forests). The bijective algorithm is analogous to the algorithm given in [2] relating \((k,k)\)-Fuss-Schröder paths and sparse noncrossing partitions.

**Definition 3.1.** A rooted plane tree (or ordered tree) is a rooted tree such that each vertex has a linearly ordered list of children. It can be defined recursively as a root with an ordered list of rooted plane subtrees which are rooted at each child of the root.

**Definition 3.2.** A rooted plane forest (or ordered forest) is a linearly ordered list of rooted plane trees.

Note that both rooted plane trees and rooted plane forests have natural pre-order labelings stemming from the pre-order transversal. (Recall that the labeling algorithm \texttt{pre-order}(r) on a root \(r\) of a rooted plane tree can be defined by the following steps:

1. Give \(r\) the label \(\ell + 1\) if \(\ell\) is the last label which has been given, or otherwise the label 0 if no label has been given yet.
2. If \((c_1,c_2,\ldots,c_k)\) is the ordered list of children of \(r\), then for \(1 \leq i \leq k\), call \texttt{pre-order}(\(c_i\)).

For a rooted plane forest with an ordered list \((r_1,r_2,\ldots,r_k)\) of plane tree roots, call \texttt{pre-order}(\(r_1\)), \texttt{pre-order}(\(r_2\)), \ldots, \texttt{pre-order}(\(r_k\)) in order.) Our labeling convention is 0-indexed so that the \(V\) vertices are labeled with \(0,1,2,\ldots,V-2,\) and \(V-1\).

**Definition 3.3.** An \((n,k)\)-valid forest is a rooted plane forest with \(2 + (k+1)n\) vertices and \((k+1)n\) edges satisfying the following properties:

1. The forest is the union of two rooted plane trees.
2. If a vertex \(v\) has label \(s\) with \(s \equiv 1 \pmod {k+1}\), then \(v\) has either 0 or \(k+1\) children.
3. If a vertex \(v\) has label \(s\) with \(s \not\equiv 1 \pmod {k+1}\), then \(v\) has \(m(k+1)\) children for some nonnegative integer \(m\).

We say the type of an \((n,k)\)-valid forest is the partition formed by all parts \(m\) (with repetition) corresponding to some \(v\) in the latter category with \(m(k+1)\) children.

**Example 3.1.** An example of an \((n,k)\)-valid forest with pre-order labeling is given in Figure 2. In this case \(n = 4\) and \(k = 2\), and so our distinguished vertices are those labeled 1 \((\text{mod } 3)\). Since the vertex labeled 0 has 3 children and the vertex labeled 5 has 6 children, the type of the forest is \(\lambda = (2,1)\).

**Lemma 3.1.** There is a type preserving bijection between large \((k,k)\)-Fuss-Schröder paths and \((n,k)\)-valid forests. Furthermore, small paths correspond exactly to \((n,k)\)-valid forests in which the second tree is a single vertex, and paths ending with a diagonal correspond to \((n,k)\)-valid forests in which the first tree is a single vertex.

**Proof.** As noted in [2], we may first biject a \((k,k)\)-Fuss-Schröder path to a sequence giving the heights of \(D\) and \(E\) steps. In the rectangle from \((0,0)\) to \((n,kn)\) in which the path is contained, we label the \((n-i)k\)th row by \(i(k+1) + 1\) for \(0 \leq i \leq n-1\) and the horizontal
line \( y = (n-i)k + j \) by \( i(k+1) - j \) for \( 0 \leq j < k \). (Note that we differ from the notation of [2] in that we use indices beginning with 0.)

Reading the labels of \( Ds \) and \( Es \) in increasing order (equivalently, from right to left) yields a nondecreasing sequence \( 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \) which completely characterizes the path. Note that \( s_i \leq (i-1)(k+1) + 1 \) for all \( i \) and that if \( s_i \equiv 1 \pmod{k+1} \) for some \( i \), then the value of \( s_i \) is not repeated. These correspond to the conditions that the path stays above the diagonal \( y = kx \) and that \( D \) steps can only occur between the lines \( y = kj + r - 1 \) and \( y = kj + r \), respectively. Conversely, it is not hard to see that from any nondecreasing integer sequence satisfying these two conditions, we may reconstruct its corresponding \((k,k)\)-Fuss-Schröder path.

We can read out the type of the corresponding path by considering sequences of \( m \) consecutive labels \( j \) such that \( j \not\equiv 1 \pmod{k+1} \): each such \( m \) is a part of the partition.

Now we describe a bijection between such sequences \( s_i \) and \((n,k)\)-valid forests. Given a sequence \( s_1 \leq s_2 \leq \cdots \leq s_n \), consider the following algorithm:

1. Begin with two independent vertices, labeled 0 and 1.
2. Successively read elements of the sequence. If the following \( m \) terms of the sequence are equal to \( j \),
   (a) add \( m(k+1) \) children to the vertex labeled \( j \);
   (b) maintain the pre-order labeling by incrementing the label of each \( i > j \) by \( m(k+1) \) and labeling the children of \( j \) with \( j+1, j+2, \ldots, j+m(k+1) \).

Note that at each step of the process, a multiple of \( k+1 \) children are added to a single vertex, which implies that the pre-order labelings \( \pmod{k+1} \) are preserved. This implies that the property of a vertex being \( 1 \pmod{k+1} \) or not is preserved throughout the process, and so there are no issues regarding which vertices can gain greater than \( k+1 \) children. Additionally, if a vertex \( j \) adds a number of children, the vertices \( 0, 1, 2, \ldots, j \) are never altered again in labels or children. Thus at the end of the process, the labels on vertices with positive numbers of children correspond exactly to the sequence \( s_i \), accounting for multiplicity given \( m(k+1) \) children.

It then only suffices to show that if

\[ s_{j-1} < s_j = s_{j+1} = \cdots = s_{j+m-1} \leq (j-1)(k+1) + 1, \]

then the vertices from \( s_{j-1} + 1 \) to \( (j-1)(k+1) + 1 \) already exist when the children of \( s_j \) are created, and conversely if the sequence \( s_j \) is read out in pre-order from the vertices with
children in the \((n,k)\)-valid forest, then \(s_j \leq (j-1)(k+1)+1\). The first assertion holds because there are \((j-1)(k+1)+2\) vertices in the forest immediately before \(s_j\) is read, and in particular the vertices from \(s_{j-1}+1\) to \((j-1)(k+1)+1\) are available to be chosen. For the same reason, in the reverse direction the maximum integer \(s_j\) could be recorded as is \((j-1)(k+1)+1\).

Finally, it is clear that the type of a \((k,k)\)-Fuss-Schröder path corresponds to the type of an \((n,k)\)-valid forest.

\[\square\]

**Example 3.2.** An example of the process illustrating the bijection between an \((n,k)\)-valid forest and a \((k,k)\)-Fuss-Schröder path corresponding to the same sequence is carried out in Figure 3 and Figure 4. In this case, \(n = 4, k = 2, \lambda = (2,1)\), and \((s_1, s_2, s_3, s_4) = (0, 4, 5, 5)\).

Note that \(s_i \leq (i-1)(k+1)+1\) for all \(i\), and equality holds for \(i = 2\), which corresponds to the fact that the \(D\) step touches the diagonal \(y = 2x\).

![Figure 3](image.png)

**Figure 3.** The \((2,2)\)-Fuss-Schröder path of length 4 corresponding to the sequence \((s_1, s_2, s_3, s_4) = (0, 4, 5, 5)\).

Now to enumerate the number of \((n,k)\)-valid forests, note that the only significant aspect of the labeling is the reduction of the labels \((\text{mod } k+1)\). We therefore count the number of trees with root labeled in a specific residue class \((\text{mod } k+1)\), which is a well-defined notion.

**Definition 3.4.** A \(j\)-rooted \((n,k)\)-valid tree is a rooted plane tree which is labeled in preorder, except in that the labeling begins with \(j\) at the root and is only considered \((\text{mod } k+1)\). Furthermore, the latter two conditions of Definition 3.3 hold:

1. If a vertex \(v\) has label \(s\) with \(s \equiv 1 \pmod {k+1}\), then \(v\) has either 0 or \(k+1\) children.
2. If a vertex \(v\) has label \(s\) with \(s \not\equiv 1 \pmod {k+1}\), then \(v\) has \(m(k+1)\) children for some nonnegative integer \(m\).

The type of a \(j\)-rooted \((n,k)\)-valid tree is the partition formed by all parts \(m\) such that some \(v\) in the latter category has \(m(k+1)\) children.

An \((n,k)\)-valid forest consists of a 0-rooted \((n_1,k)\)-valid tree and a 1-rooted \((n_2,k)\)-valid tree for nonnegative integers \(n_1, n_2\) such that \(n_1 + n_2 = n\). The union of the types (as multisets) of the trees yields the type of the forest.
Let $S_j$ denote the set of $j$-rooted $(n,k)$-valid trees. We then note that there is a recursive
definition for elements of $S_j$. If $j \not\equiv 1 \pmod{k+1}$, then for some nonnegative integer $m$,
an element $T \in S_j$ consists of a root labeled $j$ and $m(k+1)$ children with subtrees in $S_{j+1}, S_{j+2}, \ldots, S_k, S_0, S_1, \ldots, S_j$, respectively. If $j \equiv 1 \pmod{k+1}$, then an element $T \in S_1$ consists of a root labeled 1 and either 0 or $(k+1)$ children with subtrees in $S_2, S_3, \ldots, S_{k-1}, S_k, S_0, S_1$, respectively. This relation will allow us to enumerate elements of $S_j$ by type recursively.

4. ENUMERATION OF TREES

To solve the recursion analytically, we appeal to the classical Lagrange Inversion Formula (see, for example, [10, p. 42]), which allows us to find the coefficients of the compositional inverse of a generating function.

**Theorem 4.1** (Lagrange Inversion Formula). Let $K$ be a field of characteristic 0, and suppose $f(x), G(x) \in K[[x]]$ such that $G(0) \neq 0$ and $f(x) = xG(f(x))$. Then for any power series $H(x) \in K[[x]]$ and any positive integer $n$,

$$[x^n]H(f(x)) = \frac{1}{n}[x^{n-1}]H'(x)G(x)^n,$$

where the operator $[x^n]$ gives the coefficient of the $x^n$ term of a power series.
We apply this powerful tool to enumerate the number of Fuss-Schröder paths of given type. Define the field $K = \mathbb{C}(t_1, t_2, t_3, \ldots)$ given by rational functions of finitely many $t_i$ over $\mathbb{C}$. We write $t^\lambda \in K$ to denote $t^\lambda = t_1^{m_1(\lambda)}t_2^{m_2(\lambda)}t_3^{m_3(\lambda)}\cdots$, where again $m_i(\lambda)$ denotes the number of parts of $\lambda$ of size $i$. We also define the function $\Theta(x)$ and prove some of its properties, which will be useful in our analytic approach.

**Definition 4.1.** Let $\Theta(x) \in K[[x]]$ be defined as

$$\Theta(x) = t_1x + t_2x^2 + t_3x^3 + \cdots.$$  

**Lemma 4.1.** We have

$$\Theta(x)^m = \sum_{n \geq 1} \sum_{|\lambda| = n \atop \ell(\lambda) = m} \frac{m!}{m_\lambda} t^\lambda x^n$$

**Proof.** Each term in $\Theta(x)^m$ is the product of an ordered $m$-tuple of terms in the form $t_i x^i$. But each term is in the form $t^\lambda x^n$ for some partition $\lambda$ induced by the product of the $t_i$ and for some $n$. Note $|\lambda| = n$ and $\ell(\lambda) = m$. Conversely, for each partition $\lambda$, there is an ordered $m$-tuple corresponding to $\lambda$ for each ordered arrangement of the $m_1 + m_2 + m_3 + \cdots = m$ parts, where each of the $m_i$ parts of size $i$ are indistinguishable. It isn’t hard to see that there are then $\left(\frac{m!}{m_\lambda}\right)^m$ $m$-tuples corresponding to each $\lambda$, and so the identity holds. \hfill \Box

As a direct consequence, we get the following, which will aid in our computation.

**Lemma 4.2.** Let $a$ and $b$ be nonnegative integers and let $n \geq 1$. Then

$$[x^{n-1}](1 + \Theta(x))^a(1 + x)^b = \sum_{\lambda} \binom{b}{n - 1 - |\lambda|} \binom{a}{\ell(\lambda)} \frac{\ell(\lambda)!}{m_\lambda} t^\lambda$$

and

$$[x^{n-1}]\Theta'(x)(1 + \Theta(x))^a(1 + x)^b = \sum_{\lambda} \frac{|\lambda|}{a + 1} \binom{b}{n - |\lambda|} \binom{a + 1}{\ell(\lambda)} \frac{\ell(\lambda)!}{m_\lambda} t^\lambda.$$  

**Proof.** By Lemma 4.1, we have

$$[x^{n-1}](1 + \Theta(x))^a(1 + x)^b = [x^{n-1}](1 + x)^b \sum_{\lambda} \binom{a}{\ell(\lambda)} \frac{\ell(\lambda)!}{m_\lambda} t^\lambda x^{|\lambda|}$$

which implies the first equation.

Similarly, we have

$$[x^{n-1}]\Theta'(x)(1 + \Theta(x))^a(1 + x)^b = \frac{1}{a + 1}(1 + x)^b \frac{d}{dx}(1 + \Theta(x))^{a + 1}$$

$$= [x^{n-1}] \frac{1}{a + 1}(1 + x)^b \sum_{\lambda} \binom{a + 1}{\ell(\lambda)} \frac{\ell(\lambda)!}{m_\lambda} t^\lambda x^{|\lambda| - 1}$$

$$= \sum_{\lambda} \frac{|\lambda|}{a + 1} \binom{b}{n - |\lambda|} \binom{a + 1}{\ell(\lambda)} \frac{\ell(\lambda)!}{m_\lambda} t^\lambda$$

as desired. \hfill \Box

Having established these computational lemmas, we now return to our enumeration problem.
Definition 4.2. Let \( j \leq k \) be a nonnegative integer. We define
\[
A_j(x) = \sum_{T \in S_j} t^\lambda x^{\nu(T)}
\]
where \( \nu(T) \) denotes the \( n \) such that \( T \) has \((k+1)n\) edges and \((k+1)n+1\) vertices.

Example 4.1. We have
\[
A_0(x) = 1 + t_1 x + (t_1 + kt_1^2 + t_2)x^2 + \cdots
\]
and
\[
A_1(x) = 1 + x + (1 + kt_1)x^2 + \cdots.
\]

The recursive structure of the elements of each \( S_j \) yields the following recursion for \( A_j(x) \):

Lemma 4.3. If \( j \not\equiv 1 \pmod{k+1} \), then
\[
A_j(x) = 1 + t_1 C(x) + t_2 C(x)^2 + t_3 C(x)^3 + \cdots
\]
where \( C(x) = x A_0(x) A_1(x) \cdots A_k(x) \).

If \( j \equiv 1 \pmod{k+1} \), then
\[
A_1(x) = 1 + C(x).
\]

Furthermore, the \( t^\lambda x^n \) coefficients of \( A_0(x), A_1(x), \) and \( A_0(x)A_1(x) \) count the number of small \((k,k)\)-Fuss-Schröder paths, large \((k,k)\)-Fuss-Schröder paths ending in a diagonal, and large \((k,k)\)-Fuss-Schröder paths, respectively, each of type \( \lambda \).

Proof. All statements are direct corollaries of the recursive properties of elements of \( S_j \) and Lemma 3.1 along with the theory of composition of formal power series. \( \square \)

Now the first equality implies \( A_0(x) = A_2(x) = A_3(x) = \cdots = A_k(x) \), so we denote this common power series by \( A(x) \). We also define \( B(x) = A_1(x) \). Then the lemma can be restated in the following way.

Lemma 4.4. We have
\[
A_j(x) = \begin{cases} A(x) & j \not\equiv 1 \pmod{k+1} \\ B(x) & j \equiv 1 \pmod{k+1} \end{cases}
\]
where \( A(x), B(x), \) and \( C(x) \in K[[x]] \) satisfy the relations
\[
A(x) = 1 + \Theta(C(x))
\]
\[
B(x) = 1 + C(x)
\]
\[
C(x) = x A(x)^k B(x).
\]

Given these relations, we can use the Lagrange Inversion Formula to solve for the coefficients of \( A(x), B(x), \) and \( A(x)B(x) \).

Theorem 4.2. We have
\[
A(x) = 1 + \sum_{n \geq 1} \sum_{|\lambda|} \frac{1}{kn + 1} \binom{n}{|\lambda|} \frac{\ell(\lambda)!}{m_\lambda} t^\lambda x^n
\]
\[
B(x) = 1 + \sum_{n \geq 1} \sum_{|\lambda|} \frac{1}{n} \binom{n}{|\lambda| + 1} \frac{\ell(\lambda)!}{m_\lambda} t^\lambda x^n
\]
\[
A(x)B(x) = 1 + \sum_{n \geq 1} \sum_{|\lambda|} \frac{1}{n} \left( |\lambda| + 1 \right) \frac{1}{kn + 1} \binom{n + 1}{|\lambda| + 1} \frac{\ell(\lambda)!}{m_\lambda} t^\lambda x^n.
\]
Proof. We use the Lagrange Inversion Formula along with Lemma 4.4 and Lemma 4.2. Because
\[ C(x) = x A(x)^k B(x) = x(1 + \Theta(C(x)))^k(1 + C(x)), \]
we'll use the Lagrange Inversion Formula with \( f(x) = C(x) \), \( G(x) = x(1 + \Theta(x))^k(1 + x) \), and varying values of \( H(x) \).

Note that \( A(x) = H(C(x)) \) for \( H(x) = 1 + \Theta(x) \), so for \( n \geq 1 \),
\[
[x^n]A(x) = \frac{1}{n} [x^{n-1}] \Theta'(x)(1 + \Theta(x))^{kn}(1 + x)^n = \frac{1}{n} \sum_{\lambda} \frac{1}{kn + 1} \left( \frac{n}{|\lambda|} \right) \left( \frac{kn + 1}{\ell(\lambda)} \right) \frac{\ell(\lambda)!}{m_\lambda} t^\lambda.
\]
Now \( B(x) = H(C(x)) \) for \( H(x) = 1 + x \), so for \( n \geq 1 \),
\[
[x^n]B(x) = \frac{1}{n} \sum_{\lambda} \left( \frac{n}{|\lambda| + 1} \right) \left( \frac{kn}{\ell(\lambda)} \right) \frac{\ell(\lambda)!}{m_\lambda} t^\lambda.
\]
Finally, \( A(x)B(x) = H(C(x)) \) for \( H(x) = (1 + \Theta(x))(1 + x) \), which has derivative \( H'(x) = \Theta'(x)(1 + x) + (1 + \Theta(x)) \). Thus
\[
[x^n]A(x)B(x) = \frac{1}{n} [x^{n-1}] \Theta'(x)(1 + \Theta(x))^{kn}(1 + x)^{n+1} + (1 + \Theta(x))^{kn+1}(1 + x)^n = \sum_{\lambda} \frac{1}{n} \left( \frac{n}{|\lambda| + 1} + \frac{|\lambda|}{kn + 1} \left( \frac{n + 1}{|\lambda| + 1} \right) \right) \left( \frac{kn + 1}{\ell(\lambda)} \right) \frac{\ell(\lambda)!}{m_\lambda} t^\lambda.
\]

In particular, we recover the formula for the number of small \((k, k)\)-Fuss-Schröder paths of type \( \lambda \) and find two new formulæ.

Corollary 4.1. For \( n \geq 1 \), the number of large \((k, k)\)-Fuss-Schröder paths of type \( \lambda \) is
\[
\frac{1}{n} \left( \frac{n}{|\lambda| + 1} + \frac{|\lambda|}{kn + 1} \left( \frac{n + 1}{|\lambda| + 1} \right) \right) \left( \frac{kn + 1}{\ell(\lambda)} \right) \frac{\ell(\lambda)!}{m_\lambda}.
\]
The number of large \((k, k)\)-Fuss-Schröder paths of type \( \lambda \) which end in a diagonal step is
\[
\frac{1}{n} \left( \frac{n}{|\lambda| + 1} \right) \left( \frac{kn}{\ell(\lambda)} \right) \frac{\ell(\lambda)!}{m_\lambda}.
\]

5. Generalizing to Subsets of \([k]\)

In the original definition of \((k, r)\)-Fuss-Schröder paths, a diagonal step is only allowed to go from the line \( y = kj + r - 1 \) to the line \( y = kj + r \) for some \( j \). In this section, we generalize the definition and enumeration results to paths for which diagonal steps are allowed to occur at a specific subset of residues \((\text{mod} \ k)\).

Definition 5.1. For a subset \( S \subseteq [k] \), a large \((k, S)\)-Fuss-Schröder path of length \( n \) is defined to be a path from \((0, 0)\) to \((n, kn)\) using east steps, north steps, and diagonal steps such that the path never crosses below the line \( y = kx \) and the diagonal steps are only allowed to go from the line \( y = kj + r - 1 \) to the line \( y = kj + r \) for some \( j \) if \( r \in S \).

A small \((k, S)\)-Fuss-Schröder path is a large \((k, S)\)-Fuss-Schröder path such that no diagonal step touches the main diagonal \( y = kx \).

The type of such a path is the partition with parts given by maximal consecutive runs of east steps.
We may prove a sequence of analogous results for \((k, S)\)-Fuss-Schröder paths. Because of the similarity, some details are omitted.

**Definition 5.2.** Given a subset \(S \subseteq \{k\}\) such that \(|S| = d \leq k\), we define a subset \(S' \subseteq [k+d]\) as follows:

Write the numbers 0, 1, 2, 3, \ldots, \(k-1\) in order, and for each \(r \in S\), place a marker after \(k - r\) in the list. The subset \(S'\) is given by the set of indices of markers in the length \(k + d\) list (with first index 0).

The subset \(S'\) corresponds to the subset of residue classes the sequence \(s_i\) can take \((\bmod k + d)\) which correspond to diagonal steps.

**Definition 5.3.** For positive integers \(n, k\) and a subset \(S \subseteq [k]\) with \(d = |S|\), a \((n, k, S)\)-valid labeling is a rooted plane forest with pre-order labeling defined as follows:

1. If a vertex \(v\) has label \(s\) with \(s \equiv r \pmod{k + d}\) for some \(r \in S'\), then \(v\) either has 0 or \(k + d\) children.
2. If a vertex \(v\) has label \(s\) such that \(s \not\equiv r\) for all \(r \in S'\), then \(v\) has \(m(k + d)\) children for some nonnegative integer \(m\).
3. • If \(S\) contains \(k\), then the forest has \(n(k + d) + 2\) vertices, \(n(k + d)\) edges, and is the union of two rooted plane trees.
   • If \(S\) does not contain \(k\), then the forest is a rooted plane tree with \(n(k + d) + 1\) vertices and \(n(k + d)\) edges.

The type of an \((n, k, S)\)-valid forest is defined to be the partition formed by all parts \(m\) (with repetition) such that some \(v\) in the latter category has \(m(k + 1)\) children.

**Lemma 5.1.** There is a type preserving bijection between large \((k, S)\)-Fuss-Schröder paths and \((n, k, S)\)-valid forests. Small paths correspond to \((n, k, S)\)-valid forests in which the second tree is a single vertex, and paths ending with a diagonal step correspond to \((n, k, S)\)-valid forests in which the first tree is a single vertex. (In particular, if \(k \not\in S\), the corresponding path is always small and never ends with a diagonal step.)

**Proof.** The proof is almost the same as that of Lemma 3.1. Note that if \(k \not\in S\), the corresponding sequence \(s_i\) satisfies \(s_i \leq (i - 1)(k + d)\), while if \(k \in S\), \(s_i\) satisfies \(s_i \leq (i - 1)(k + d) + 1\). This accounts for the difference in number of beginning vertices. \(\square\)

**Definition 5.4.** A \(j\)-rooted \((n, k, S)\)-valid tree is a rooted plane tree labeled in pre-order \((\bmod k + d)\) and starting with \(j\) at the root, such that

1. If a vertex \(v\) has label \(s\) with \(s \equiv r \pmod{k + d}\) for some \(r \in S'\), then \(v\) has either 0 or \(k + d\) children.
2. If a vertex \(v\) has label \(s\) such that \(s \not\equiv r\) for all \(r \in S'\), then \(v\) has \(m(k + d)\) children for some nonnegative integer \(m\).

The type of a \(j\)-rooted \((n, k, S)\)-valid tree is as in previous definitions.

Now we see that an \((n, k, S)\)-valid forest is the union of a 0-rooted \((n_1, k, S)\)-valid tree and a 1-rooted \((n_2, k, S)\)-valid tree for some \(n_1 + n_2 = n\) when \(k \in S\), and when \(k \not\in S\), it is simply a 0 rooted \((n, k, S)\)-valid tree.

**Definition 5.5.** Let 

\[ A_S(x) = \sum_T t^\lambda x^{\nu(T)} \]
where the sum is over all \( j \)-rooted \((n,k,S)\)-valid trees \( T \) for some fixed \( j \not\in S' \). Let

\[ B_S(x) = \sum_T t^{|\lambda|} x^{\ell(\lambda)} \]

where the sum is over all \( j \)-rooted \((n,k,S)\)-valid trees \( T \) for some fixed \( j \in S' \). (If there is no \( j \in S' \), then \( B_S(x) = 1 \). As before, all elements of \( S' \) and all elements of \([k+d] \setminus S'\) yield equal generating functions.)

Let

\[ C_S(x) = x A_S(x)^k B_S(x)^d. \]

Now by our recursion,

\[ A_S(x) = 1 + \Theta(C_S(x)) \]

and

\[ B_S(x) = 1 + C_S(x). \]

**Theorem 5.1.** We have

\[
A_S(x) = 1 + \sum_{n \geq 1} \frac{\ell(\lambda)!}{n^{|\lambda|}} \binom{dn}{n-|\lambda|} \binom{kn+1}{\ell(\lambda)} \frac{t^{|\lambda|}}{m_\lambda} x^n \\
B_S(x) = 1 + \sum_{n \geq 1} \frac{\ell(\lambda)!}{n^{|\lambda|}} \binom{dn}{n-|\lambda|} \binom{kn}{\ell(\lambda)} \frac{t^{|\lambda|}}{m_\lambda} x^n \\
A_S(x) B_S(x) = 1 + \sum_{n \geq 1} \frac{\ell(\lambda)!}{n^{|\lambda|}} \binom{dn}{n-|\lambda|} + \binom{dn+1}{n-|\lambda|} \binom{kn+1}{\ell(\lambda)} \frac{t^{|\lambda|}}{m_\lambda} x^n.
\]

**Proof.** Apply the Lagrange Inversion Formula and Lemma 4.2 in the same way as before. \( \square \)

**Corollary 5.1.** For a set \( S \) not containing \( k \), the number of large \((k,S)\)-Fuss-Schröder paths of type \( \lambda \) is

\[ \frac{1}{n} \frac{|\lambda|}{kn+1} \binom{dn}{n-|\lambda|} \frac{\ell(\lambda)!}{\ell(\lambda)} \frac{t^{|\lambda|}}{m_\lambda}. \]

For a set \( S \) containing \( k \), the number of small \((k,S)\)-Fuss-Schröder paths of type \( \lambda \) is the same quantity. The number of large paths ending in a diagonal step is

\[ \frac{1}{n} \binom{dn}{n-|\lambda|} \frac{\ell(\lambda)!}{\ell(\lambda)} \frac{t^{|\lambda|}}{m_\lambda}. \]

The total number of large \((k,S)\)-Fuss-Schröder paths of type \( \lambda \) for a set \( S \) containing \( k \) is

\[ \frac{1}{n} \left( \binom{dn}{n-|\lambda|} + \frac{|\lambda|}{kn+1} \binom{dn+1}{n-|\lambda|} \right) \frac{\ell(\lambda)!}{\ell(\lambda)} \frac{t^{|\lambda|}}{m_\lambda}. \]

Interestingly, these expressions depend only on the cardinality of \( S \).

6. Conclusion

In recent years, much work has been done on enumerating paths of a fixed type, including Dyck paths, Schröder paths, and Fuss-Catalan paths ([8, 1, 9, 3]). An, Jung, and Kim ([2]) enumerated the number of small \((k,k)\)-Fuss-Schröder paths of type \( \lambda \) and posed the question of enumerating large \((k,k)\)-Fuss-Schröder paths of type \( \lambda \), which is achieved in the present paper through the use of generating functions. Furthermore, we enumerate the number of large \((k,k)\)-Fuss-Schröder paths of type \( \lambda \) ending on a diagonal step and generalize Fuss-Schröder paths to \((k,k)\)-Fuss-Schröder paths for any subset \( S \subseteq [k] \). The problem of enumeration by type here falls to the same technique.
For future work, we would be interested in a combinatorial or bijective proof of the new formulae presented here, including the number of large \((k,k)\)-Fuss-Schröder paths of type \(\lambda\), the number of large \((k,k)\)-Fuss-Schröder paths of type \(\lambda\) ending on a diagonal step, and the number of \((k,5)\)-Fuss-Schröder paths of type \(\lambda\) of all three varieties (large, small, and ending on a diagonal step).

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