Adaptive pseudo-rigid-body model for generalized cross-spring pivots under combined loads

Zhongzhou Wang¹,²,#, Haixuan Sun¹,³,#, Bidou Wang¹,# and Peng Wang¹,#

Abstract
Generalized cross-spring pivots (CSPs) are widely used as revolute joints in precision machinery. However, pseudo-rigid-body (PRB) models cannot capture the parasitic motions of a generalized CSP exactly under combined loads; moreover, the characteristic parameters used in PRB methods must be recomputed using optimization techniques. In this study, we develop two simple and accurate PRB models for generalized CSPs. First, a PRB method for a beam is developed based on the beam constraint model and the instantaneous center model, where the beam is modeled as two rigid links joined at a pivot via a torsion spring. Subsequently, two PRB models of the generalized CSP, comprising a four-bar model for accuracy and a pin-joint model for stiffness, are constructed based on a kinematic analysis using the proposed PRB method. A deflection characteristic analysis is then conducted to determine the relationship between the proposed model and the existing models. Finally, the PRB models for the pivot under the action of combined loads are validated via finite element analysis. The error evaluation indicates that the proposed PRB models are more accurate than the results from existing methods. The PRB models proposed here can be used in parametric design of compliant mechanisms.

Keywords
Pseudo-rigid-body model, generalized cross-spring pivot, beam constraint model, compliant mechanism, instantaneous center model, parametric design

Introduction
Flexural pivots transform both motions and energies through elastic deformations.¹ These pivots are used widely in precision engineering because they offer a variety of advantages, including zero backlash, zero friction, no clearance requirements, no assembly requirements and high precision.²,³ Conventional notch-type flexural pivots exhibit high stiffness and small parasitic motions; however, these pivots experience small strokes due to stress concentration. Spring-type flexural pivots provide a wide range of motion and exhibit reduced stress because of elastic averaging effects.

¹Suzhou Institute of Biomedical Engineering and Technology, Chinese Academy of Sciences, China
²University of Chinese Academy of Sciences, China
³Institute for Stem Cell and Regeneration, Chinese Academy of Sciences, China
# These authors contributed equally to this work.

Corresponding author:
Zhongzhou Wang, Suzhou Institute of Biomedical Engineering and Technology, Chinese Academy of Sciences, No. 88, Keling Road, Suzhou New District, Suzhou, Jiangsu 215163, China.
Email: wangle@siib.ac.cn

Creative Commons CC BY: This article is distributed under the terms of the Creative Commons Attribution 4.0 License (https://creativecommons.org/licenses/by/4.0/) which permits any use, reproduction and distribution of the work without further permission provided the original work is attributed as specified on the SAGE and Open Access pages (https://us.sagepub.com/en-us/nam/open-access-at-sage).
The generalized cross-spring pivot (CSP)\(^4\)\(^-\)\(^6\) is formed by crossing two symmetrical beams at an arbitrary position (Figure 1). The CSP can be regarded as a revolute joint with fixed and moving ends. The moving end rotates around the instantaneous center, which nearly coincides with the initial intersection of the two beams for small deflections. In terms of the processing method required, generalized CSPs can be divided into monolithic and nonmonolithic configurations.\(^6\) The nonmonolithic pivot consists of two beams located in different planes and is called the cross-axis flexural pivot (CAFP).\(^7\) In contrast, the monolithic pivot comprises two beams located in the same plane that intersect at a virtual center and is termed the leaf-type isosceles-trapezoidal flexural pivot (LITFP).\(^8\) Generalized CSPs can be combined to form compound pivots to achieve large strokes, small parasitic motions and quasi-constant rotational stiffness.\(^9\)\(^,\)\(^10\) They are commonly used in near-zero-stiffness flexural pivots,\(^11\) precision positioning modules,\(^12\) biaxial cartwheel hinges\(^13\) and butterfly pivots.\(^14\)

The distributed compliance of the generalized CSP results in parasitic motions and nonlinear stiffness performances for these pivots, particularly for large deflections. An accurate and concise parametric model can serve as an important reference for accuracy analysis and motion control. At present, the main techniques used to analyze the generalized CSP include elliptic integrals, finite element analysis (FEA), screw theory, pseudo-rigid-body (PRB) methods and the beam constraint model (BCM).\(^15\) The nonlinear stiffness and stress characteristics of the CAFP and the LITFP were derived using a comprehensive elliptic integral; this process involved solution of complex transcendental equations.\(^16\)\(^,\)\(^17\) Furthermore, optimal design of the CAFP with minimal center shift and stiffness variations was investigated via nonlinear FEA.\(^18\) A parametric model of a cylindrical CAFP was established based on the FEA results to investigate the pivot’s stiffness and stress characteristics.\(^19\) Although the FEA approach provides high accuracy and wide applicability, it offers only limited parametric insights to designers. Screw theory was also used to analyze the center shift in terms of the instantaneous center;\(^20\)\(^,\)\(^21\) however, the assumption of a linear beam restricts the application of screw theory.\(^22\)

The BCM was proposed to simplify transcendental solutions obtained through polynomial approximations for intermediate deflections (i.e. \(< 10\%\) of the beam length).\(^23\)\(^,\)\(^24\) Analytical models for the stiffnesses and center shifts of generalized CSPs were derived from the BCM within a rotation angle range of up to \(\pm 15^\circ\).\(^4\)\(^-\)\(^6\) However, complex load equilibrium and geometric compatibility equations have to be solved when using these models. Furthermore, a chained BCM (CBCM)\(^25\) was used to investigate a CAFP with a contact pair in terms of its stiffness, center shift and stress characteristics and large-scale numerical calculations were required for this purpose.\(^7\)

Various PRB methods have been derived to reproduce paths for compliant mechanisms. The 1R (R: revolute) PRB model regards a cantilever beam as two rigid links joined at a revolute pivot via a torsion spring\(^26\)\(^,\)\(^27\) and thus achieves a trade-off between simplicity and accuracy. Chen et al.\(^28\) and Ding et al.\(^29\) analyzed the performances of notch-type mechanisms and optimized their structural parameters using the 1R PRB method. The PRB method can also be expanded to analyze spatial compliant mechanisms and predict their dynamic characteristics.\(^30\)\(^,\)\(^31\) Jensen and Howell\(^32\) established PRB models to investigate the stiffness and stress characteristics of CAFPs using nonlinear FEA. In addition, PRB methods with multiple degrees of freedom (DOFs) were developed to enhance the accuracy of the models. The 2R PRB model,\(^33\) the 3R PRB model,\(^34\) the 6-DOF PRB model,\(^35\) the dynamic spline\(^36\) and the second-order approximation PRB model\(^37\)
were all proposed for high calculation accuracy; however, these models cannot describe axial deformation under combined loads accurately. To overcome this problem, Tang et al.38 and Yu et al.39 proposed the spline PRB model and the prismatic-revolute-revolute PRB model, respectively, to simulate large deflections with combined end moment and force loads. However, the characteristic parameters for these PRB methods must be determined via a numerical optimization process and these parameters must be recomputed for each different load case. Furthermore, the calculations of the deflections and parasitic motions under combined loads are complex, which caused efficiency problems for the two methods.40 In addition, increasing the number of DOFs can realize high accuracy but can also lead to increased calculation complexity. In contrast, to avoid the need for numerical fittings, Pei et al.8,41,42 developed an analytical PRB method and used it to analyze the stiffness and accuracy characteristics of an LITFP that was subject to a pure bending moment. Verotti43 developed a PRB method based on the Euler–Bernoulli beam equation that focused on the initial and final configurations of a deflected beam. Šalinic and Nikolić44 proposed a 3-DOF PRB approach to modeling of notch-type flexure-hinge mechanisms. However, the total parasitic motion that occurs along a beam has not been expressed explicitly in these PRB models. In summary, the PRB methods described above cannot capture the center-shift and stiffness characteristics of generalized CSPs under the action of combined force and moment loads analytically.

The main original contribution of this article is the establishment of adaptive and analytical PRB models for the generalized CSP under the action of combined moment and force loads. The proposed PRB models are restricted to within a deformation range of 30% of the beam length. Unlike the characteristic parameters that were determined previously by using optimization solutions32 or by ignoring the axial force,8,32,41,42 the parameters of the proposed PRB models can be obtained analytically by considering the influence of the axial force. In contrast to earlier PRB models,8,32,42 the proposed PRB models can predict the rotational stiffness and the parasitic motions of the generalized CSP when it is subjected to different combined loads. Additionally, the generalized CSP can be regarded as being equivalent to a four-bar mechanism, thus allowing the kinematic analysis techniques that are available for rigid-body mechanisms to be applied to compliant mechanisms.

The remainder of this paper is organized as follows. In Section 2, a PRB method for a fixed-guided beam under the action of combined loads is proposed based on the BCM and instantaneous center constraints. In Section 3, two PRB models are constructed for the generalized CSP through kinematic analysis using the proposed PRB method. In Section 4, deflection characteristic analysis is conducted to establish the relationship between the proposed PRB models and existing results. In Section 5, the validation of the PRB models of the pivots is presented based on nonlinear FEA results and the advantages of the proposed models when compared with the existing results are demonstrated via error evaluations. Finally, Section 6 presents the conclusions drawn from the study.

**Pseudo-rigid-body method for a fixed-guided beam**

To investigate the deformation characteristics of generalized CSPs, a PRB method for a beam is established based on the beam constraint model and the instantaneous center model. First, two equivalent models for a fixed-guided beam, comprising an instantaneous center model and a PRB model, are proposed. The characteristic radius factor and the equivalent stiffness factor are then derived analytically to determine the pin-joint location and the load–deflection relationship, respectively.

**Equivalent model**

As indicated in Figure 1, each beam in the pivots can be assumed to be a fixed-guided beam with its free end rotating around the instantaneous center.4,42 This assumption can then be used to develop parametric approximations of the deflection path.

The two equivalent models of a fixed-guided beam are shown in Figure 2. Figure 2(a) shows the instantaneous center model for a beam subjected to a bending moment \( M \), a transverse force \( F \) and an axial force \( P \), which result in the end displacements \( U_X \), \( U_Y \) and \( \theta \), respectively. The \( x \) and \( y \) axes represent the axial and transverse directions of the beam, respectively. End A of the beam AB is fixed, while the free end B follows a near-circular path around the instantaneous center \( O \). The instantaneous center radius factor \( \lambda \in (-\infty, +\infty) \), i.e. the ratio between the vectors \( BO \) and \( BA \), determines the location of the instantaneous center. Figure 2(b) depicts the PRB model of the deflected beam. The near-circular path can be approximated using two rigid links, \( AB' \) and \( AA' \), with two pin-joints. One pin-joint joins the two links using a nonlinear torsion spring and the other joint is located at the free end of the beam. The PRB link \( AB' \) moves to \( A'B' \), thus producing the PRB angle \( \Theta = \angle A'B'B' \). The characteristic radius factor \( \gamma \) is defined as the ratio between the vectors \( A'B' \) and \( AB' \).

Awtar and Sen23,24 proposed the BCM for intermediate deflections (i.e., within 10% of the beam
length) to describe the characteristics of a deflected beam accurately. Their BCM is described as follows:

$$f = \left( \begin{array}{c} k_{11}^{(0)} & k_{12}^{(0)} & 0 \\ k_{12}^{(0)} & k_{22}^{(0)} & 0 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c} u_x \\ u_y \\ \theta \end{array} \right) + p \left( \begin{array}{c} k_{11}^{(1)} & k_{12}^{(1)} & 0 \\ k_{12}^{(1)} & k_{22}^{(1)} & 0 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c} u_x \\ u_y \\ \theta \end{array} \right),$$

$$u_x = \frac{p}{k_{33}} \frac{1}{2} (u_x', \theta) \left( \frac{k_{11}}{k_{12}} \right) \left( \frac{k_{12}}{k_{22}} \right) \left( u_x' \theta \right),$$

$$u_y = \frac{p}{k_{33}} \left[ u_x' + \frac{1}{2} (k_{11} u_x'^2 + 2k_{12} u_x \theta + k_{22} \theta^2)^2 ight] + \frac{1}{2} \left( k_{11} u_x'^2 + 2k_{12} u_x \theta + k_{22} \theta^2 \right).$$

The nondimensional parameters in equations (1) to (3) are defined as (Appendix A)

$$m = \frac{M L}{EI}, \quad f = \frac{F L^2}{EI}, \quad p = \frac{P L^2}{EI}, \quad u_x = \frac{U_x}{L},$$

$u_y = \frac{U_y}{L}$, and $k_{33} = 12 \left( \frac{L}{T} \right)^2$,

where $I = \frac{W T^3}{12}$ is the cross-sectional moment of inertia; $L$, $T$ and $W$ represent the length, thickness and width of the beam, respectively; and $E$ is the Young’s modulus of the material. The values of the characteristic coefficients in equations (1) to (3) are listed in Table 1.

Table 1. Characteristic coefficients for a beam.

| $k_{11}^{(0)}$ | $k_{12}^{(0)}$ | $k_{22}^{(0)}$ | $k_{11}^{(1)}$ | $k_{12}^{(1)}$ | $k_{22}^{(1)}$ |
|----------------|----------------|----------------|----------------|----------------|----------------|
| 12             | -6             | 4              | 6/5            | -1/10          | 2/15           |
| 1/700          | -1/1400        | 11/6300        |

Equation (1) represents the primary motions $u_x$ and $u_y$ in the DOF direction. The first and second matrix terms represent the linear elastic stiffness and the load stiffening, respectively. Equation (2) represents the parasitic motion $u_x$ in the degree of constraint (DOC) direction. The first term describes the pure elastic deformation that is independent of the axial force. The second term captures the kinematic component that results from the constant beam arc-length. The third term captures an elastokinematic component that arises from the axial force and the kinematic term. Equation (3) represents the strain energy formulation. The first and second terms denote the contributions of the stretching and bending of the beam to the total strain energy, respectively.

**Characteristic radius factor**

It is necessary to solve for the characteristic radius factor $\gamma$ to determine the pin-joint location. As shown in Figure 2(a), when the center shift is neglected, the free end of the beam follows a near-circular path around the instantaneous center. The instantaneous center constraints, denoted by $u_x$ and $u_y$, can be expressed as:

$$u_x \approx \lambda \theta,$$

As illustrated in Figure 2(b), the deflections $u_x$ and $u_y$ for the PRB model are given as:

$$u_y = \gamma \theta.$$
\[ u_x = -\gamma(1 - \cos \theta) \approx -\frac{\gamma \theta^2}{2}. \quad (7) \]

For a given value of the PRB angle of \( \theta = 0.5 \) rad, the truncation error between the BCM and the proposed PRB method is less than \( \frac{\gamma \theta^2}{2} = 0.26\% \), as introduced by equation (7).

By substituting equation (7) into equation (2), the following equation is obtained:

\[
-\frac{\gamma \theta^2}{2} = \frac{p}{k_{33}} - \frac{18\lambda^2 - 3\lambda + 2\lambda^2}{30} \theta^2 + \frac{9\lambda^2 - 9\lambda + 11}{6300} \rho \theta^2.
\quad (8)
\]

Based on equations (4) and (7), the PRB angle can be expressed easily using the following equation:

\[
\theta = \frac{\lambda \theta}{\gamma}. \quad (9)
\]

Substitution of equation (9) into equation (8) then yields the following expression:

\[
-\frac{\lambda^2}{2\gamma} = \frac{p}{k_{33}\theta^2} - \frac{18\lambda^2 - 3\lambda + 2\lambda^2}{30} \theta^2 + \frac{9\lambda^2 - 9\lambda + 11}{6300} \rho \theta^2.
\quad (10)
\]

The expression for \( \gamma \) can be calculated as follows:

\[
\gamma = \frac{15\lambda^2}{18\lambda^2 - 3\lambda + 2 - 30\eta \rho}, \quad (11)
\]

where

\[
\eta = \frac{1}{k_{33}\theta^2} + \frac{9\lambda^2 - 9\lambda + 11}{6300}, \quad \eta > 0. \quad (12)
\]

Equations (11) and (12) show that the characteristic radius factor \( \gamma \) is a function of \( \lambda \), \( \rho \) and \( \theta \). Consideration of the special case in which the beam is subjected to a pure moment without axial force allows equation (11) to be reduced to:

\[
\gamma = \frac{15\lambda^2}{18\lambda^2 - 3\lambda + 2}. \quad (13)
\]

Equation (13) has also been derived previously by Pei et al.\(^{41}\) Equation (11) can capture the axial deformation without numerical optimization and is thus more adaptive than equation (13).

Consider a typical beam with the parameters \( k_{33} = 120,000 \) and \( \theta = 0.1 \) rad. Figure 3 shows the characteristic radius factor \( \gamma \) versus various values of \( \lambda \) and \( \rho \). At a given instantaneous center factor \( \lambda \), the axial force applied affects the location of the pin-joint along the beam. When \( \rho = 0 \), the curve reaches the peak value of \( \gamma = 8/9 \) at \( \lambda = 0.75 \). When \( |\lambda| \) approaches infinity, the trend for the characteristic radius factor \( \gamma \) is closer to 5/6.

**Equivalent stiffness factor**

The equivalent stiffness factor of the torsion spring determines the load–deflection relationship. Equation (3) indicates that the equivalent stiffness factor \( k_\theta \) and the stiffness \( k_\theta \) of the beam in the PRB method satisfy the following equations:

\[
\delta \nu = \delta \left( \frac{k_\theta \theta^2}{2} \right), \quad (14)
\]

\[
\delta \nu = \delta \left( \frac{k_\theta \theta^2}{2} \right). \quad (15)
\]

Using equation (2), the following solution for \( \rho \) can be derived:

\[
\rho = \frac{k_{33} \left[ u_x + \frac{1}{2} \left( k_{11}^{(1)} u_x^2 + 2k_{12}^{(1)} u_x \theta + k_{22}^{(1)} \theta^2 \right) \right]}{1 + k_{33} \left( k_{11}^{(2)} u_x^2 + 2k_{12}^{(2)} u_x \theta + k_{22}^{(2)} \theta^2 \right)}. \quad (16)
\]

Combination of equations (3), (5) and (14) yields the following equation:

---

*Wang et al.*
Substitution of equation (9) into equation (18) allows the expression for the solution for the following expression to be obtained:

\[ k_3 \left[ u_x + \frac{1}{2} \left( k_{11} u_x^2 + 2k_{12} u_x \theta + k_{22} \theta^2 \right) \right]^2 \left( \frac{1}{1 + k_{33} \left( k_{11} u_x^2 + 2k_{12} u_x \theta + k_{22} \theta^2 \right)} \right) \]

\[ + p \left[ -\lambda + \left( k_{11} \lambda^2 + 2k_{12} \lambda \theta + k_{22} \theta^2 \right) \right] \]

\[ + \left( k_{11} \lambda^2 + 2k_{10} \lambda + k_{22} \right) = k_\theta \frac{\lambda^2}{\gamma^2}. \]  

Substitution of equation (9) into equation (18) allows the solution for \( k_\theta \) to be written as (Appendix A):

\[ k_\theta = \frac{\gamma^2}{\lambda^2} \left[ \frac{29\lambda^2 - 9\lambda + 1p}{15} + 43\lambda^2 - 3\lambda + 1 \right]. \]  

For the special case where \( p = 0 \), the dimensional torsional stiffness derived from equation (19) can be reduced to (Appendix A):

\[ K_{\theta} = \frac{E I k_\theta}{L} = \frac{4EI \gamma^2 (3\lambda^2 - 3\lambda + 1)}{\lambda^2 L}. \]  

Equation (20) has also been obtained previously. Using equations (14) and (15), the following expression can be formulated:

\[ k_\theta \theta d\theta = k_\theta \Theta d\Theta. \]  

By combining equations (9) and (21), the expression for \( k_\theta \) can be calculated as:

\[ k_\theta = \frac{\lambda^2 k_\theta}{\gamma^2}. \]  

Substitution of equation (19) into equation (22) allows the solution for \( k_\theta \) to be computed as:

\[ k_\theta = \frac{29\lambda^2 - 9\lambda + 1p}{15} + 43\lambda^2 - 3\lambda + 1. \]  

### Two PRB models for the generalized CSP

Two PRB models are established for the generalized CSP under combined loads. First, an LITFP is transformed into a rigid-body mechanism using the PRB method for a beam. Subsequently, a kinematic analysis is performed to derive the four-bar model for accuracy and the pin-joint model for the stiffness.

#### Generalized CSP configurations

Generalized CSPs can be classified into three configurations based on the locations of their instantaneous centers (Figure 4). Two beams are arranged symmetrically about the vertical centerline and their axes cross at the intersection angle \( 2\alpha \). The fixed and moving ends are denoted by \( A_1A_2 \) and \( B_1B_2 \), respectively.

According to the Kennedy principle, the link \( B_1B_2 \) can rotate around the instantaneous center for small deflections. The instantaneous center radius factor \( \lambda \) represents the ratio between the vectors \( B_1\hat{O} \) (or \( B_2\hat{O} \)) and the vector from the instantaneous center to the intersection point of the vectors. This ratio is used to determine the link length, which is required for the four-bar model.
and $\bar{B}_1\bar{A}_1$ (or $\bar{B}_2\bar{A}_2$). The range of $\lambda$ for a monolithic LITFP is $(-\infty, 0) \cup (1, +\infty)$, whereas the corresponding range for a nonmonolithic CAFP is $[0, 1]$. As a special case, the value of $\lambda$ for an isosceles triangular flexure pivot is either 0 or 1.

**Kinematic analysis**

The position vectors for the different configurations must all satisfy the same vector loop equations. Therefore, without loss of generality, the kinematic analysis is performed on the LITFP with $\lambda \in (1, +\infty)$. Similar to the beam depicted in Figure 1, the LITFP is imitated using the four-bar mechanism shown in Figure 5(a). The free-body diagrams for the four-bar model with the external loads are shown in Figure 5(b). By neglecting the center shift for smaller deflections, the generalized CSP can also be modeled as a pin-joint located at the intersection point shown in Figure 5(c). The instantaneous center $O$ is defined as the intersection point of the tangents to the axis lines of the beams at the moving ends.

When the combined loads (i.e., the bending moment $M$, the horizontal force $F$ and the vertical force $P$) are applied to the midpoint $C(x, y)$ (Appendix A), link $B_1B_2$ moves to $B'_1B'_2$. The end of each beam rotates by the same angle $\theta$ because link $B_1B_2$ is assumed to be an ideal rigid body. The PRB angles of the two beams are $\angle B_1A_1B'_1 = \Theta_1$ and $\angle B_2A_2B'_2 = \Theta_2$. Point $O$ moves to $O'$, which results in the center shift characterized by the vector $\bar{d} = OO'$.

$XOY$ represents the global coordinate system used for the LITFP and $X_1O_1Y_1$ and $X_2O_2Y_2$ denote the local coordinate systems for the fixed end of each beam. The signs of the counterclockwise angles are positive and the external loads remain at the same angle during rotation. To simplify the discussion here, the geometrical parameters and the beam loads are both assumed to be consistent with the BCM. Only the case where $\theta > 0$ is discussed here because of the symmetry of the motion.

Consider the free-body diagrams of the four-bar model $A_1B_1B_2A'_2$, which are shown in Figure 5(b). The load equilibrium equations for link $B_1B_2$ can be formulated as:

$$ (f_1 + f_2)\cos \alpha + (p_2 - p_1)\sin \alpha = f, \quad (24) $$

$$ (f_1 - f_2)\sin \alpha + (p_1 + p_2)\cos \alpha = p. \quad (25) $$

It can be found easily from equation (1) that $f_1, f_2$ and $f$ are all of the order of $\theta$. Assuming that $\theta$ is small, combination of equations (24) and (25) gives:

$$ p_1 = p_2 = \frac{p}{2\cos \alpha}. \quad (26) $$

Substitution of equation (26) into equation (11) shows that the lengths of links $A_1B_1$ and $A_2B_2$ are equal:

$$ \| A_1B_1 \| = \| A_2B_2 \| = \gamma. \quad (27) $$

The vector loops shown in Figure 5(a) result in the following equations:

$$ \bar{A}_1B'_1 + \bar{B}_1B'_2 = A'_1A'_2 + A'_2B'_2, \quad (28) $$

$$ \bar{O}C' = OA_1 + A_1B_1 + \bar{B}_1C'. \quad (29) $$

The analytical equations represented by equations (28) and (29) can be written as follows:

$$ -\gamma \sin(\alpha - \Theta_1) + 2\lambda \cos \alpha = 2(\lambda - \gamma)\cos \alpha $$

$$ + \gamma \sin(\alpha + \Theta_2), \quad (30) $$

$$ \gamma \cos(\alpha - \Theta_1) - 2\lambda \cos \alpha \sin \theta = \gamma \cos(\alpha + \Theta_2), \quad (31) $$

$$ x = - (\lambda - \gamma)\sin \alpha - \gamma \sin(\alpha - \Theta_1) + \lambda \cos \alpha \cos \theta, \quad (32) $$
\[ y = (\lambda - \gamma) \cos \alpha + \gamma \cos (\alpha - \Theta_1) - \lambda \cos \alpha \sin \theta. \]  

Equations (30) to (33) can be solved simultaneously for \( \Theta_1, \Theta_2, x \) and \( y \) in terms of \( \theta \). The transcendental expressions can be calculated as follows:

\[
\frac{\Theta_1}{\theta} = \frac{\lambda}{\gamma} \sin \left( \frac{\lambda - \gamma}{2} \right),
\]

\[
\frac{\Theta_2}{\theta} = \frac{-\lambda}{\gamma} \sin \left( \frac{\lambda - \gamma}{2} \right),
\]

\[
x = \beta_1 \lambda \cos \alpha \sin \theta, \tag{36}
\]

\[
y = \beta_1 \left( \cos \theta - 1 + \frac{\gamma}{\lambda} \right) \lambda \cos \alpha + \left( 1 - \frac{\gamma}{\lambda} \right) \lambda \cos \alpha. \tag{37}
\]

Here, \( \beta_1 = \tan \alpha \sqrt{\frac{a}{2} - 1}, \beta_2 = \gamma^2 - 2\lambda \gamma - \lambda \) \((1 - \cos \theta)\) and \( \beta_3 = \gamma^2 \csc^2 \alpha \).

Simplification of equations (34) to (37) using a Taylor series expansion yields the following polynomial expressions:

\[
\frac{\Theta_1}{\theta} = \frac{\lambda}{\gamma}, \tag{38}
\]

\[
\frac{\Theta_2}{\theta} = \frac{-\lambda}{\gamma}, \tag{39}
\]

\[
x = \lambda \cos \alpha \left( \theta - \frac{\theta^3}{6} \right) - \frac{\lambda^2 (\lambda - \gamma) \theta^3}{2 \gamma^2 \cos \alpha}, \tag{40}
\]

\[
y = \lambda \cos \alpha \left( 1 - \frac{\theta^2}{2} \right) - \frac{\lambda (\lambda - \gamma) \theta^2}{2 \gamma \cos \alpha}. \tag{41}
\]

Equations (40) and (41) show that the midpoint \( C \) rotates approximately about the intersection point. The displacements of point \( C \) comprise a component generated by the rotation around the instantaneous center and another component produced by the center shift.

**Four-bar model for accuracy**

As shown in Figure 5(a), the center shift \( \delta \) can be defined using the two components \( dx \) and \( dy \) in the \( x \)- and \( y \)-directions (Appendix A), respectively:

\[
dx = x - \lambda \cos \alpha \sin \theta, \tag{42}
\]

\[
dy = y - \lambda \cos \alpha \cos \theta, \tag{43}
\]

\[
d = \sqrt{dx^2 + dy^2}. \tag{44}
\]

Substitution of equations (36) and (37) into equations (42) and (43) allows the transcendental solution for the center shift components to be expressed as:

\[
dx = (\beta_1 - 1) \lambda \cos \alpha \sin \theta, \tag{45}
\]

\[
dy = (\beta_1 - 1) \left( \cos \theta - 1 + \frac{\gamma}{\lambda} \right) \lambda \cos \alpha. \tag{46}
\]

Consideration of the special case where \( p = 0 \) allows equations (45) and (46) to be reduced to the results obtained by Pei et al.\(^{41}\)

Substitution of equations (40) and (41) into equations (42) and (43) allows the polynomial solution for the center shift components to be expressed as:

\[
dx = \frac{\lambda \theta}{\gamma} dy, \tag{47}
\]

\[
dy = -\frac{\lambda (\lambda - \gamma) \theta^2}{2 \gamma \cos \alpha}. \tag{48}
\]

According to equations (47) and (48), \( dx \) can be determined using \( dy \) and the PRB angle \( \Theta_1 \) (or \( \Theta_2 \)). The \( x \)- and \( y \)-components, i.e. \( dx \) and \( dy \), are of the orders of \( \theta^3 \) and \( \theta^2 \), respectively. For a small rotational angle, \( d_y \gg dx \); therefore, the dominant term in \( d \) is \( dy \).

**Pin-joint model for stiffness**

When the center shift is neglected, the four-bar model can degenerate to give the pin-joint model shown in Figure 5(c). According to the principle of virtual work, the system has a total virtual work of zero in static equilibrium:

\[ f \delta x + p \delta y + m \delta \theta - k_{\theta_1} \Theta_1 \delta \Theta_1 - k_{\theta_2} \Theta_2 \delta \Theta_2 = 0. \tag{49}\]

Use of equations (19), (26) and (27) allows the equation \( k_{\theta_1} = k_{\theta_2} \) to be obtained. Let \( k_\theta = k_{\theta_1} = k_{\theta_2} \). By substituting equations (40) and (41) into equation (49) and neglecting the center shift, the following expression is obtained:

\[ (f \lambda \cos \alpha - p \lambda \cos \alpha \theta + m) \delta \theta - \frac{2 \lambda^2 k_\theta}{\gamma^2} = 0. \tag{50}\]

According to equation (50), the expression for \( \theta \) can be calculated as follows:

\[ \theta = \frac{m + f \lambda \cos \alpha}{2 \lambda^2 k_\theta / \gamma^2 + p \lambda \cos \alpha}. \tag{51}\]

The four-bar model and the pin-joint model can capture the accuracy and the stiffness characteristics, respectively, of generalized CSPs under combined loads. In addition, the derivation process performs a traditional kinematic analysis rather than attempting to solve the deflection compatibility equations. The form of the expressions used here is independent of the specific value of \( \lambda \); therefore, the proposed models can also be applied to other generalized CSP configurations.

**Deflection characteristic analysis**

A deflection characteristic analysis is conducted to provide a better understanding of the relationship between
the proposed model and the existing solutions. Additionally, the linear relationship between the deflections of the pivot and its beams is revealed.

**Center shift analysis**

The center shift of a fixed-guided beam is equal to the difference between the deflections and the instantaneous center constraints. This shift can be calculated from equations (4) to (7), as follows:

\[ dx_i = -\frac{\lambda(\lambda - \gamma)\theta^2}{2\gamma}. \]  

(52)

Here, the subscripts \( i = 1 \) and \( 2 \) refer to beams 1 and 2, respectively.

Comparison of equation (48) with equation (52) shows that a linear relationship between the center shift component of the generalized CSP and its beams can be derived:

\[ dy = \frac{dx_i}{\cos \alpha}. \]  

(53)

The center shift of the generalized CSP in the \( y \)-direction, i.e., \( dy \), is directly proportional to the center shift of each beam in the axial direction; however, it is inversely proportional to the cosine value of the half-intersection angle \( \alpha \).

Substitution of equations (11), (12), and (26) into equation (48) yields the following expression for the center shift component \( d_j \):

\[ dy = -\frac{9\lambda^2 - 9\lambda + 10^2}{15\cos \alpha} + \frac{p}{2\cos^2 \alpha} \left( \frac{1}{k_{35}} + \frac{(9\lambda^2 - 9\lambda + 11)\theta^2}{6300} \right). \]  

(54)

Equation (54) is consistent with the order of the result for the CAFP (\( \lambda \in [0, 1] \)) given by Zhao and Bi.\(^5\). In the case where \( p = 0 \) and \( \lambda = 0.5 \), the result \( dy = \sqrt{2}0^2/12 \) agrees with the solution given by Pei et al.\(^41\).

**Stiffness analysis**

By combining equations (22) and (51), the solution for \( \theta \) can be rewritten as:

\[ \theta = \frac{m + f\lambda \cos \alpha}{2k_\theta + p\lambda \cos \alpha}. \]  

(55)

Equation (55) indicates that the stiffness of a generalized CSP is equal to the sum of the stiffnesses of the two beams and the load stiffening effect of the vertical force.

Substitution of equation (23) into equation (55) produces the following solution:

\[ \theta = \frac{m + f\lambda \cos \alpha}{2(9\lambda^2 - 9\lambda + 1)p/15\cos \alpha} + p\lambda \cos \alpha + 8(3\lambda^2 - 3\lambda + 1). \]  

(56)

Equation (56) can also be obtained from the work of Zhao and Bi.\(^4\)

The rational approach to use of the deflection characteristics involves studying the effects of the various typical loads separately. The moment stiffness \( k_m \) and the force stiffness \( k_f \) are defined as (Appendix A):

\[ k_m = \frac{2(9\lambda^2 - 9\lambda + 1)p}{15\cos \alpha} + 8(3\lambda^2 - 3\lambda + 1) + p. \]  

(57)

\[ k_f = \frac{2(9\lambda^2 - 9\lambda + 1)p}{15\lambda \cos^2 \alpha} + 8(3\lambda^2 - 3\lambda + 1) + p. \]  

(58)

Equations (57) and (58) indicate that the moment and force stiffnesses are both proportional to the vertical force. In the case where \( p = 0 \), the dimensional beam stiffness derived from equation (58) can be reduced to (Appendix A):

\[ K_m = \frac{2k_\mu EI}{L} = \frac{8EI(3\lambda^2 - 3\lambda + 1)}{L}. \]  

(59)

Equation (59) is also identical to the results obtained for the LITFP (\( \lambda \in (1, +\infty) \)) and the CAFP (\( \lambda \in [0, 1] \)) by Pei et al.\(^8,41\) Equation (58) considers the effects of the axial force; therefore, it is more adaptive to the combined loads than equation (59).

As mentioned above, the PRB method presented in Ref. 41, the accuracy model for the CAFP given in Ref. 42 and the pin-joint model for the LITFP and the CAFP proposed in Ref. 8 can be regarded as special loading cases of the proposed PRB models without vertical loads. It should be noted here that the accuracy model for the LITFP in Ref. 42 and the pin-joint model for the LITFP in Ref. 8 were verified experimentally, which validates the effectiveness of the proposed PRB models indirectly. The PRB models proposed herein and the results presented in Refs. 4 and 5 share certain factors because they were all derived based on the BCM. Furthermore, the relationships between the pivot and the beams can be revealed easily using the PRB models.

**Model verification and error evaluation**

In this section, FEA simulations and error evaluations are performed to demonstrate the effectiveness and the advantages of the proposed PRB models, respectively.
FEA verification

FEA simulations are performed to verify the proposed models of the generalized CSP using ANSYS Workbench 16.2 (Ansys Inc., Pittsburgh, PA, USA). The stiffness and accuracy characteristics of the two typical generalized CSPs listed in Table 2 are studied here. The Young’s modulus and Poisson’s ratio values are $E = 73$ GPa and $\mu = 0.3$, respectively. The Beam188 element, which is suitable for analysis of slender beam structures, is selected for meshing in this case. The bending moment and the horizontal force are determined based on the transverse displacement with a deformation of 30% of the beam length without the application of a vertical force. The corresponding rotational angles of the CAFP and LITFP are within approximately 0.6 rad and 0.15 rad, respectively. The applied vertical load is determined by altering the compliance by 10%.

Figure 6 shows the results of total deformation simulations of the generalized CSPs when subjected to a pure moment. The moving ends of pivots 1 and 2 rotate approximately about the initial intersection point at angles of $\theta = 0.6012$ rad and $\theta = 0.1542$ rad, respectively. The stiffness characteristics can be investigated using the relationship between the rotational angles and the combined loads applied to the midpoint of the moving end. To calculate the center shift intuitively, a rigid link is fixed on the moving end and its free end coincides with the instantaneous center. The center shift can then be characterized based on the displacement of the instantaneous center.

Figure 7 shows the rotational stiffness, which corresponds to the inverse of the slopes of the curves, for pivots 1 and 2 under the combined loads. The proposed pin-joint model mainly follows the FEA results. As stated previously, the rotational stiffness is proportional to the vertical force. A positive vertical force $p$ can result in a stiffening effect, while a negative vertical force can reduce the stiffness. The rotational angles of pivots 1 and 2 reach maximum values of 0.6671 rad and 0.1767 rad, respectively, under a negative vertical load. The curves deviate gradually from the FEA results for larger deflections because the pin-joint model is derived by ignoring the parasitic motions.

Figure 8 shows the center shift results, denoted by $dy$ and $d$, for pivots 1 and 2. The moment $m$ and the horizontal force $f$ are selected as the $x$-coordinates rather than use the rotational angle to demonstrate that the four-bar model is suitable for use with different combined loads. Note that the polynomial solution and

| CSP No. | $\lambda$ | $\alpha$ | $L$ (mm) | $T$ (mm) | $W$ (mm) |
|---------|-----------|-----------|----------|----------|----------|
| 1       | 0.5       | $\pi/4$   | 50       | 0.5      | 5        |
| 2       | 2         | $\pi/6$   | 50       | 0.5      | 5        |

Table 2. Parameters of the generalized CSPs.
reference model curves overlap each other (Figure 8(a)). The four-bar model results agree well with the FEA results. The vertical force has a significant effect on the center shift because of the load stiffening effect. Comparisons of $dy$ and $d$ indicate that $dy$ is the dominant term of $d$. The four-bar model is derived by neglecting the effects of the horizontal force on the internal axial force of the beam for simplicity. Therefore, the deviation between the model curves and the FEA simulation results increases for larger deflections.

Figures 7 and 8 show that the proposed PRB models agree reasonably well with the results of the nonlinear FEA. As noted in Section 4.2, the expressions from Refs. 4, 8 and 41 are included in the proposed PRB models. This new PRB method can capture the effects of the vertical force on the stiffness and the center shift, thus demonstrating improved applicability. In addition, the stiffness and the center shift can be either enhanced or reduced by appropriate selection of the characteristic radius factor and the external loads.

**Error evaluation**

The relative error is evaluated to assess the deviations between the results from the analytical models and the FEA results quantitatively, as follows:

$$\omega_x = \left| \frac{x_M - x_{FEA}}{x_{FEA}} \right| \times 100\%,$$

where $\omega_x$ denotes the relative error (Appendix A); $x$ represents the data for $dy$, $d$ and $\theta$; and the subscripts $M$ and $FEA$ indicate the data obtained from the analytical models and the FEA results, respectively.

Figures 9 and 10 describe the maximum and average relative errors in the stiffnesses for pivots 1 and 2 under combined loads, respectively. These average and maximum errors follow the same variation tendencies. Figure 9 shows that the maximum values of all relative errors of the pin-joint model for pivots 1 and 2 are 6.0% and 9.5%, respectively. The proposed pin-joint model for the stiffness is more accurate than the
Figure 8. Accuracy characteristics for pivots 1 and 2. (a) $d_y$ for pivot 1; (b) $d$ for pivot 1; (c) $d_y$ for pivot 2; and (d) $d$ for pivot 2.
previous reference model. In addition, the proposed model is more efficient because it avoids use of numerical optimization methods.

Figures 11 and 12 depict the maximum and average relative errors in the center shift, respectively, for pivots 1 and 2 under combined loads. These average errors and maximum errors follow the same variation tendencies. Figure 11 shows that the maximum values of the relative errors for pivots 1 and 2 are 5.3% and 1.5% for the transcendental solution, 11.4% and 4.2% for the reference model given in Ref. 5, and 13.8% and 11.2% for the polynomial solution, respectively. The accuracies of the three models, ranked from highest to lowest, are listed as follows: the transcendental solution, the reference model, and the polynomial solution. When compared with the polynomial solution, the reference model is more accurate because the higher-order terms are retained in the latter. The transcendental solution is more accurate than the reference model, particularly for the LITFPs, because it is obtained directly from the vector loop equations. However, the polynomial solution can offer parametric insights into the deflection mechanism and can be selected to provide a balance between simplicity and accuracy during the preliminary design stage.

When compared with the results obtained from the existing methods, the proposed PRB models can improve both accuracy and efficiency and are adaptive
to use of combined loads. These models can be applied to the analysis and synthesis of compound pivots.

Conclusions

Adaptive PRB models have been developed to enhance the efficiency and accuracy of the models used for generalized CSPs under combined loads. The main conclusions of this study are as follows. (1) The deflection characteristic analysis indicates that the existing PRB models, when ignoring the force in the DOC direction described in Refs. 8, 41 and 42, can be regarded as special loading cases of the proposed PRB models; the stiffness model in Ref. 4 is equivalent to the proposed pin-joint model; and the proposed PRB models can easily show the linear relationship between the deflections of the pivots and the beams. (2) The FEA validation indicates that the proposed PRB models can predict the rotational stiffness and parasitic motion characteristics of the generalized CSPs when subjected to different combined loads and are more adaptive than the PRB models proposed in Refs. 8, 41 and 42; the stiffness and the center shift can be either enhanced or reduced by appropriate selection of the characteristic radius factor and the external loads. (3) The error evaluation shows that the proposed PRB models can achieve higher accuracy than existing solutions. The pin-joint model is more accurate than the reference model derived using optimization methods in Ref. 32; the characteristic parameters of the proposed PRB method for a beam can be derived directly and can also be adjusted for changes in the axial load, thus improving the computational efficiency. In contrast to the analytical models presented in Ref. 5, the transcendental solution is more accurate, particularly for the LITFPs; the polynomial solution can be selected to provide a balance between simplicity and accuracy during the preliminary design stage.

This study offers an analytical PRB method for modeling of generalized CSPs that is derived directly from the BCM and the instantaneous center.

Figure 11. Maximum relative errors in the center shift for pivots 1 and 2. (a) Maximum $\omega_{dy}$ for pivot 1; (b) maximum $\omega_d$ for pivot 1; (c) maximum $\omega_{dy}$ for pivot 2; and (d) maximum $\omega_d$ for pivot 2.
This PRB method can be extended to establish PRB models for other flexural mechanisms that satisfy the instantaneous center constraints. The proposed PRB models can be used as optimization design tools for accuracy compensation and stiffness synthesis of compound pivots. For example, by considering that the stiffness and the center shift can be reduced by optimizing the characteristic radius factor, both cartwheel hinges and butterfly pivots can be constructed using the generalized CSPs in series to achieve high precision with low stiffness.

The proposed PRB models can maintain their accuracy at a deformation of 30% of the beam length. However, they are also useful for typical practical applications because the deflections in these applications are often small as a result of the stress concentrations and parasitic motions.

The goal in future work is to improve the accuracy of the proposed PRB models over larger deflection ranges by considering the effects of the horizontal force and the parasitic motions. Additional experimental research will also be performed to validate the effectiveness of the proposed models in actual applications.

**Declaration of conflicting interests**
The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

**Funding**
The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This study was supported by the National Key Research and Development Plan [grant number 2017YFC0110400]; and the Strategic Priority Research Program of the Chinese Academy of Sciences [grant number XDA16020704].
**ORCID iD**
Zhongzhou Wang  https://orcid.org/0000-0002-8479-7787

**References**

1. Liu CH, Huang GF and Chen T-L. An evolutionary soft-add topology optimization method for synthesis of compliant mechanisms with maximum output displacement. *J Mech Robot* 2017; 9: 054502.
2. Howell LL, Magleby SP and Olsen BM. *Handbook of compliant mechanisms*. Hoboken: Wiley, 2013, pp. 1–2.
3. Smith ST. *Flexures: elements of elastic mechanisms*. Boca Raton: CRC Press, 2014, pp. 1–5.
4. Zhao HZ and Bi SS. Stiffness and stress characteristics of the generalized cross-spring pivot. *Mech Mach Theory* 2010; 45: 378–391.
5. Zhao HZ and Bi SS. Accuracy characteristics of the generalized cross-spring pivot. *Mech Mach Theory* 2010; 45: 1434–1448.
6. Zhao HZ, Bi SS and Yu JJ. Nonlinear deformation behavior of a beam-based flexural pivot with monolithic arrangement. *Precis Eng* 2011; 35: 369–382.
7. Bilancia P, Berselli G, Magleby S, et al. On the modeling of a contact-aided cross-axial flexural pivot. *Mech Mach Theory* 2020; 143: 103618.
8. Pei X, Yu JJ, Zong GH, et al. The stiffness model of leaf-type isosceles-trapezoidal flexural pivots. *J Mech Des* 2008; 130: 082303.
9. Xu QS. Design of a large-range compliant rotary micro-positioning stage with angle and torque sensing. *IEEE Sens J* 2015; 15: 2419–2430.
10. Bi SS, Zhang SQ and Zhao HZ. Quasi-constant rotational stiffness characteristic for cross-spring pivots in high precision measurement of unbalance moment. *Precis Eng* 2016; 43: 328–334.
11. Zhao HZ, Zhao CX, Ren SY, et al. Analysis and evaluation of a near-zero stiffness rotational flexural pivot. *Mech Mach Theory* 2019; 135: 115–129.
12. Yang M, Du ZJ, Chen FX, et al. Kinetostatic modelling of a 3-PRR planar compliant parallel manipulator with flexure pivots. *Precis Eng* 2017; 48: 323–330.
13. Li Z, Chen X and Jin G. Dimensionless design model for biaxial cartwheel flexure hinges. *Mech Based Des Struct Mach* 2017; 46: 401–409.
14. Pei X, Yu J, Zong G, et al. A family of butterfly flexural joints: Q-LITF pivots. *J Mech Des* 2012; 134: 121005.
15. Hao GB, Yu JJ and Li HY. A brief review on nonlinear modeling methods and applications of compliant mechanisms. *Front Mech Eng* 2016; 11: 119–128.
16. Zhang A, Gou Y and Yang X. Predicting nonlinear stiffness, motion range, and load-bearing capability of leaf-type isosceles-trapezoidal flexural pivot using comprehensive elliptic integral solution. *Math Probl Eng* 2020; 2020: 1390692.
17. Zhang A, Chen G and Jia J. Large deflection modeling of cross-spring pivots based on comprehensive elliptic integral solution. *J Mech Eng* 2014; 50: 80–85.
18. Markovic K and Zelenika S. Optimized cross-spring pivot configurations with minimized parasitic shifts and stiffness variations investigated via nonlinear FEA. *Mech Based Des Struct Mach* 2017; 45: 380–394.
19. Deerden J, Grames C, Orr J, et al. Cylindrical cross-axis flexural pivots. *Precis Eng* 2018; 51: 604–613.
20. Lai LJ and Zhu ZN. Modeling and analysis of a compliance model and rotational precision for a class of remote center compliance mechanisms. *Appl Sci* 2016; 6: 388.
21. Li SZ and Yu JJ. Design principle of high-precision flexure mechanisms based on parasitic-motion compensation. *Chin J Mech Eng-En* 2014; 27: 663–672.
22. Su HJ, Shi H and Yu J. A symbolic formulation for analytical compliance analysis and synthesis of flexure mechanisms. *J Mech Des* 2012; 134: 051009.
23. Awtar S and Sen S. A generalized constraint model for two-dimensional beam flexures: nonlinear load-displacement formulation. *J Mech Des* 2010; 132: 081008.
24. Awtar S and Sen S. A generalized constraint model for two-dimensional beam flexures: nonlinear strain energy formulation. *J Mech Des* 2010; 132: 081009.
25. Chen GM, Ma FL, Hao GB, et al. Modeling large deflections of initially curved beams in compliant mechanisms using chained beam constraint model. *J Mech Robot* 2019; 11: 011002.
26. Howell LL, Midha A and Norton TW. Evaluation of equivalent spring stiffness for use in a pseudo-rigid-body model of large-deflection compliant mechanisms. *J Mech Des* 1996; 118: 126–131.
27. Howell LL and Midha A. Parametric deflection approximations for end-loaded, large-deflection beams in compliant mechanisms. *J Mech Des* 1995; 117: 156–165.
28. Chen X, Deng Z, Hu S, et al. Research on three-stage amplified compliant mechanism-based piezo-driven microgripper. *Adv Mech Eng* 2020; 12: 1–12.
29. Ding B, Yang Z-X, Zhang G, et al. Optimum design and analysis of flexure-based mechanism for non-circular diamond turning operation. *Adv Mech Eng* 2017; 9: 1–10.
30. Bilancia P, Berselli G, Bruzzone L, et al. A CAD/CAE integration framework for analyzing and designing spatial compliant mechanisms via pseudo-rigid-body methods. *Robot Com-Int Manuf* 2019; 56: 287–302.
31. She Y, Meng D, Su H-J, et al. Introducing mass parameters to pseudo-rigid-body models for precisely predicting dynamics of compliant mechanisms. *Mech Mach Theory* 2018; 126: 273–294.
32. Jensen BD and Howell LL. The modeling of cross-axis flexural pivots. *Mech Mach Theory* 2002; 37: 461–476.
33. Yu YQ, Feng ZL and Xu QP. A pseudo-rigid-body 2R model of flexural beam in compliant mechanisms. *Mech Mach Theory* 2012; 55: 18–33.
34. Su HJ. A pseudorigid-body 3R model for determining large deflection of cantilever beams subject to tip loads. *J Mech Robot* 2009; 1: 021008.
35. Venkiteswaran VK, Sikorski J and Misra S. Shape and contact force estimation of continuum manipulators using pseudo rigid body models. *Mech Mach Theory* 2019; 139: 34–45.
36. Valentini PP and Pennestrı` E. Modeling elastic beams using dynamic splines. *Multibody Syst Dyn* 2011; 25: 271–284.
37. Valentini PP and Pennestrì E. Second-order approximation pseudo-rigid model of leaf flexure hinge. *Mech Mach Theory* 2017; 116: 352–359.

38. Tang HY, Zhang D, Guo S, et al. A novel model to simulate flexural complements in compliant sensor systems. *Sensors (Basel)* 2018; 18: 1029.

39. Yu YQ, Zhu SK, Xu QP, et al. A novel model of large deflection beams with combined end loads in compliant mechanisms. *Precis Eng* 2016; 43: 395–405.

40. Yu YQ, Li Q and Xu QP. Pseudo-rigid-body dynamic modeling and analysis of compliant mechanisms. *Proc Inst Mech Eng C-J Mech Eng Sci* 2018; 232: 1665–1678.

41. Pei X, Yu JJ, Zong GH, et al. An effective pseudo-rigid-body method for beam-based compliant mechanisms. *Precis Eng* 2010; 34: 634–639.

42. Pei X, Yu JJ, Zong GH, et al. Analysis of rotational precision for an isosceles-trapezoidal flexural pivot. *J Mech Des* 2008; 130: 052302.

43. Verotti M. A pseudo-rigid body model based on finite displacements and strain energy. *Mech Mach Theory* 2020; 149: 103811.

44. Šalinic S and Nikolić A. A new pseudo-rigid-body model approach for modeling the quasi-static response of planar flexure-hinge mechanisms. *Mech Mach Theory* 2018; 124: 150–161.

**Appendix A**

**Nomenclature**

- $A$: cross-sectional area of the beam
- $E$: Young’s modulus of the material
- $I$: second moment of inertia
- $L$: beam length
- $W$: beam width
- $T$: beam thickness
- $U_X$: translational displacement in the degree of constraint direction
- $u_x$: translational displacement $u_x = \frac{U_X}{T}$ in the degree of constraint direction
- $\theta$: rotational angle in the degree of freedom direction
- $U_Y$: translational displacement in the degree of freedom direction
- $u_y$: translational displacement $u_y = \frac{U_Y}{L}$ in the degree of freedom direction
- $\nu$: strain energy
- $M$: bending moment
- $m$: nondimensional bending moment $m = \frac{ML}{EI}$
- $F$: force in the degree of freedom direction
- $f$: nondimensional force $f = \frac{FL}{EI}$ in the degree of freedom direction
- $P$: force in the degree of constraint direction
- $p$: nondimensional force $p = \frac{PL}{EI}$ in the degree of constraint direction
- $\lambda$: instantaneous center radius factor
- $\gamma$: characteristic radius factor
- $\Theta$: pseudo-rigid-body angle
- $K_\Theta$: equivalent rotational stiffness factor for a beam
- $k_\Theta$: nondimensional equivalent rotational stiffness factor for a beam
- $k_\theta$: nondimensional rotational stiffness for a beam
- $K_m$: moment stiffness for a pivot
- $k_m$: nondimensional moment stiffness for a pivot
- $k_f$: nondimensional force stiffness for a pivot
- $x$: nondimensional displacement of a pivot’s midpoint in the $x$-direction
- $y$: nondimensional displacement of a pivot’s midpoint in the $y$-direction
- $d$: nondimensional total center shift
- $d_x$: nondimensional center shift component in the $x$-direction
- $d_y$: nondimensional center shift component in the $y$-direction
- $\omega$: relative error between the analytical models and the reference results
- $\delta$: variational operator