Ground-state properties of hard-core anyons in one-dimensional optical lattices

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I. INTRODUCTION

The physical systems with fractional statistics have been a subject of great interest in past decades and have been intensively studied for two-dimensional systems [1, 2, 3, 4]. For instance, the elementary excitations of a fractional quantum Hall (FQH) liquid satisfy fractional statistics. This has been observed in two-dimensional electron gas and anyon has become an important concept in studying the FQH effect [1, 2, 3, 4]. As a natural generalization of the Bose and Fermi gas, anyon gas has also found application in various one-dimensional (1D) systems [5, 6, 7, 8]. Despite no explicitly experimental proof of the realization of 1D anyon gas, many theorists have dedicated to study them [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Currently anyons also stimulated intensive research on topological quantum computation because the statistical properties are closely related to the topological order. Particularly, by controlling exchange interaction between pairs of neutral atoms in optical lattice, Anderlini et al. [22] realized the key operation in quantum information processing, i.e., the quantum SWAP gate (which control the states inter-change between two qubits). Cold atoms in optical lattice are also proposed to create, manipulate, and test anyons [23, 24, 25].

By tightly confining the particle motion in two directions to zero point oscillations 1D quantum gas is obtained [26, 27, 28], where the radial degrees of freedom are frozen and the quantum gas is effectively described by a 1D model [29]. Experimentally, by means of anisotropic magnetic trap or two-dimensional optical lattice potentials, a 1D Bose gas in the strongly correlated Tonks-Girardeau (TG) regime can be achieved [27, 28]. The TG gas has been shown to display the "fermionized" character in many aspects. For example, it has the same density distribution and thermodynamic behavior as the free fermion [30, 31, 32, 33, 34]. However, the off-diagonal density matrices and the momentum distributions exhibit quite different behaviors due to the different permutation symmetries of the Bose and Fermi wave functions [35, 36]. The momentum distribution of fermions shows typical oscillations but that of bosons shows the structure of single peak. Both of them are symmetric about the zero momentum. For the anyon gas satisfying fractional statistics, however the momentum distribution is shown to be asymmetric when the statistical parameters deviate from the Bose and Fermi limit [16, 17, 18, 19], and with the change in statistical parameter the distribution evolves from a Bose distribution to a Fermi one and vice versa.

While most of the theoretical works on the 1D anyon gas focus on the continuum system, in this paper we investigate the ground state of a 1D anyonic system confined in optical lattice with weak harmonic trap in the hard core limit. Although the hard-core bosons have been studied extensively and intensively for both homogeneous continuum system and lattice system, no result was given for the hard-core anyons (HCAs) in optical lattice. We shall study how the fractional statistics affect the ground-state properties, such as the momentum distribution. By extending the exact numerical method originally used to treat hard-core bosons by Rigol and Muramatsu [37] to deal with the hard-core anyons, we evaluate the exact one-particle Green’s function of the ground state, with which the reduced one body density matrix (ROBDM) and thus the momentum distribution can be obtained exactly for different statistical parameters. This method is based on Jordan-Wigner transformation and has been applied in the investigation of hard core bosons confined in optical lattice for both ground state and dynamics. It turns out to be very efficient to study the universal behaviors of the system with arbitrary confining potentials combined with one-dimensional optical lattices. The properties of anyonic statistics shall be displayed in the momentum distribution and with the change in statistics parameter the system exhibits the Bose statistics, Fermi statistics, and the fractional statistics in between. The
mathematical origin of the asymmetry can be traced back to the reduced one body density matrix.

The paper is organized as follows. In Sec. II, we give a brief review of 1D anyonic model and introduce the numerical method. In Sec. III, we present the momentum distributions, ROBDM, and occupation for different statistics parameter and filling numbers. A brief summary is given in Sec. IV.

II. FORMULATION OF THE MODEL AND METHOD

The second quantized Hamiltonian of the hard-core anyons confined in optical lattice of $L$ sites with a weak harmonic trap takes the form of

$$H_{\text{HCA}} = -i \sum_{l=1}^{L} \left( a_{l+1}^{\dagger} a_l + \text{H.C.} \right) + \sum_{l=1}^{L} V_l n_l$$

(1)

with the lattice dependent external potential

$$V_l = V_0 (l - (L + 1)/2)^2,$$

where $V_0$ denotes the strength of the harmonic trap. The anyonic creation operator $a_j^{\dagger}$ and annihilation operator $a_j$ satisfy the generalized commutation relations

$$a_j a_l = \delta_{jl} - e^{-i\chi \pi (j-l)} a_l^{\dagger} a_j,$$

$$a_j^{\dagger} a_l = -e^{i\chi \pi (j-l)} a_l a_j^{\dagger}$$

(2)

for $j \neq l$ with the addition of hard-core condition $a_j^2 = a_j^{\dagger 2} = 0$ and $\{a_l, a_l^{\dagger}\} = 1$. The sign function $\epsilon(x)$ gives -1, 0, or 1 depending on whether $x$ is negative, zero, or positive. The parameter $\chi$ is related with fractional statistics and will be restricted in the regime of $[0, 1]$ in the present paper. Particularly, $\chi = 0$ and 1 correspond to Fermi statistics and Bose statistics, respectively. In the Hamiltonian the particle number operator $n_l = a_l^{\dagger} a_l$, and $t$ denotes the hopping between the nearest neighbor sites.

This model can be solved exactly by the generalized Jordan-Wigner transformation,

$$a_j = \exp \left( i\chi \pi \sum_{1 \leq s < j} f_s^{\dagger} f_s \right) f_j,$$

$$a_j^{\dagger} = f_j^{\dagger} \exp \left( -i \chi \pi \sum_{1 \leq s < j} f_s^{\dagger} f_s \right),$$

(3)

where $f_j^{\dagger}$ and $f_j$ are creation and annihilation operators for spinless fermions. The hard-core anyonic Hamiltonian with $N$ anyons can be mapped onto the noninteracting fermionic system for $N_f$ fermions ($N_f = N$),

$$H_F = -i \sum_{l=1}^{L} \left( f_{l+1}^{\dagger} f_l + \text{H.C.} \right) + \sum_{l=1}^{L} V_l n_l$$

(4)

with fermionic particle number operator $n_l = f_l^{\dagger} f_l$. While the eigen problem for Hamiltonian $H_F$ for vanishing harmonic potential $V_l = 0$ can be obtained easily through Fourier transformation, we investigate here the situation with weak harmonic trap and Rigol-Muramatsu method should be a good choice. Using the above transformation the one-particle Green’s function of hard-core anyon is formulated as

$$G_{jl} = \langle \Psi^{G}_{\text{HCA}} | a_j^{\dagger} a_l | \Psi^{G}_{\text{HCA}} \rangle$$

$$= \langle \Psi^{G}_{F} | e^{i \chi \pi \sum_{j=1}^{l} f_j^{\dagger} f_j} e^{-i \chi \pi \sum_{j=1}^{l} f_j^{\dagger} c_j} | \Psi^{G}_{F} \rangle$$

$$= \langle \Psi^{A}_{F} | \Psi^{B}_{F} \rangle$$

(5)

with

$$\langle \Psi^{A}_{F} | = \left[ f_j^{\dagger} \exp \left( -i \chi \pi \sum_{\beta} f_{\beta}^{\dagger} f_{\beta} \right) \right]^{\dagger} | \Psi^{G}_{F} \rangle,$$

$$| \Psi^{B}_{F} \rangle = f_l^{\dagger} \exp \left( -i \chi \pi \sum_{\gamma} f_{\gamma}^{\dagger} f_{\gamma} \right) | \Psi^{G}_{F} \rangle.$$

$| \Psi^{G}_{\text{HCA}} \rangle$ is the ground state of hard core anyonic system and $| \Psi^{G}_{F} \rangle$ is the ground state of free spinless fermionic system. The Green’s function can be obtained by constructing the many-particle ground state of fermions with the eigenstates of single-particle,

$$| \alpha \rangle = c_1^{\dagger} | 0 \rangle = \sum_l \varphi_n (l) f_l^{\dagger} | 0 \rangle,$$

where $\alpha$ means the $\alpha$th state and $l$ means the $l$th site. The many body ground state of $N_f$ free spinless Fermions takes the following form:

$$| \Psi^{G}_{F} \rangle = c_1^{\dagger} c_2^{\dagger} \cdots c_{N_f}^{\dagger} | 0 \rangle = \prod_{l=1}^{N_f} \sum_{l=1}^{N_f} P_{ln} f_l^{\dagger} | 0 \rangle$$

(6)

with $P_{ln} = \varphi_n (l)$, which can be expressed as an $L \times N_f$ matrix $P$. After an easy evaluation the state $| \Psi^{A}_{F} \rangle$ reads

$$| \Psi^{A}_{F} \rangle = \prod_{n=1}^{N_f} \sum_{l=1}^{L} P_{ln}^{T} f_l^{\dagger} | 0 \rangle$$

with

$$P_{ln}^{A} = \exp (-i \chi \pi) P_{ln} \quad \text{for} \quad l \leq j - 1,$$

$$P_{ln}^{A} = P_{ln} \quad \text{for} \quad l \geq j$$

for $n \leq N_f$, and $P_{jN_f}^{A} = 1$ and $P_{jN_f}^{A} = 0$ ($l \neq j$). The state $| \Psi^{B}_{F} \rangle$ has the same form with the replacement of $j$ by $l$. The Green’s function is a determinant dependent on the $L \times (N_f + 1)$ matrices $P^{TA}$ and $P^{TB}$,

$$G_{jl} = \langle \Psi^{A}_{F} | \Psi^{B}_{F} \rangle = \det \left[ (P^{TA})^{T} P^{TB} \right].$$

(7)
The natural orbitals \( \phi \) can be expressed as
\[
\phi \text{ with } \phi \text{ of the one-particle density matrix,
}

and can be understood as the effective single-particle fluid, while in the Fermi limit \((\chi = 1)\) most of bosons populate at the zero momentum state, which corresponds to a Bose superfluid, while in the Fermi limit \((\chi = 0)\) the step-function distribution is shown, which is the characteristic feature of free fermions. Both of them are symmetric about the zero momentum. When the statistical parameter \( \chi \) deviates from these two limits the momentum distribution is asymmetric about the zero momentum. Anyons distribute in the regime of positive momentum with more probability than that in the regime of negative momentum and redistribute between these two regimes with the increase in statistical parameter \( \chi \). Finally the distribution evolves between Fermi and Bose distribution with the change in statistical parameter.

When the harmonic trap is present we have to turn to the numerical diagonal method \([27]\) in order to get the \( N \) lowest single particle eigenstates of free spinless Fermi model and construct the ground state of many body anyonic system. The momentum distributions are displayed in Fig. 2 for filling number \( N = 50 \) and \( N = 150 \) confined in an optical lattice with \( L = 300 \) sites. In this situation the anyons populate in broader regime compared with the confinement-free case as shown in Fig. 1. This is more evidently seen in the Fermi limit. In the absence of harmonic trap anyons (pure Fermions in this limit) occupy the momentum states in the regime of \(-1/2 < k < 1/2\) almost homogeneously while the combined harmonic trap results that higher momentum states are occupied and the distribution is no longer a sharp step-function. For all statistical parameters the half width of momentum distribution becomes wider. This can be understood by uncertainty principle that the presence of harmonic trap reduces the uncertainty in the coordinate space and leads to the increase of uncertainty in the momentum space. When more particles are loaded in the lattice they will populate at higher momentum states with more probability and the peak of momentum profiles will not be so sharp as those in the case of less filling number. Obviously the filling number and the strength of harmonic trap shall not affect the statistical property that with the change in statistical parameter anyon’s momentum distribution evolves continuously from a Bose distribution to...
a Fermi one. In these two limits the distributions are symmetry about the zero momentum and in between the profiles are asymmetric.

In order to clarify the origin of asymmetric momentum distribution of anyons we show the ROBDM for $L = 300$ lattice sites with 50 anyons in Fig. 3, where $j$ and $l$ ranged from 1 to $L$. In the Bose and Fermi limits the ROBDMs are real (top row) whereas in the regime deviating from these two limits the ROBDMs are Hermitian, i.e., possessing symmetric real part ($\text{Re}[\rho_{jl}] = \text{Re}[\rho_{lj}]$) and anti-symmetric imaginary part ($\text{Im}[\rho_{jl}] = -\text{Im}[\rho_{lj}]$), which are exhibited in last four rows for $\chi = 0.2, 0.4, 0.6$, and 0.8. Therefore the formula of momentum distribution can be departed into two parts as below that one part is even function of $k$ and the other one is odd function of $k$,

$$n(k) = \frac{1}{2\pi} \sum_{j,l=1}^{L} e^{-ik(j-l)} \rho_{jl}$$

$$= \frac{1}{2\pi} \sum_{j,l=1}^{L} \{\text{Re}[\rho_{jl}] \cos k(j-l) + \text{Im}[\rho_{jl}] \sin k(j-l)\} .$$

It is because of this that the momentum distribution becomes asymmetric about zero momentum. The imaginary part $\text{Im}\rho_{jl}$ is an odd function of statistical parameter $\chi$ so the peak at positive momentum as shown above will shift to negative momentum if we take $\chi$ as negative ($\chi \rightarrow -\chi$) [16].

In Fig. 4a we display the occupation of natural orbitals for the system with 50 anyons in 300 lattice sites combined with weak harmonic trap ($V_0 = 1.0 \times 10^{-4}t$). In the Fermi limit each anyon occupies one orbital and the occupation distribution is a step function. In the Bose limit most anyons occupy the lower orbitals and the occupation distribution exhibits sharp single-peak structure. In the regime deviating from these two extreme points the evolution from one to the other indicates that with the decrease in statistical parameter more and more anyons occupy higher orbitals and finally distribute in the lowest $N$ orbitals homogeneously in the Fermi limit. The statistical effect on the natural orbital is shown in Figs. 4b-4d for the same system as Fig. 4a, where the modulus of the lowest natural orbitals ($|\phi^1|$) is exhibited for statistical parameter $\chi = 0.0, 0.4$, and 1.0. In the Bose limit the modulus are almost smooth and only weak oscillations appear around the boundary while the oscillations become more and more obvious with the decrease in statistical parameter. In the Fermi limit $N$ peaks display clearly.

IV. CONCLUSIONS

In summary, we have investigated the ground-state properties of 1D anyon gas confined in optical lattices combined with a weak harmonic trap in the hard core...
limit using exact numerical method. With Jordan-Wigner transformation the hard-core anyon model is related to polarized free spinless Fermi model and thus exact ground state can be constructed from that of $N$ free fermions which is composed of the lowest $N$ single-particle eigenstates of free fermion. Then by calculating the one-particle Green’s function we obtain the ROBDM and momentum distribution. It is indicated that the ROBDM is a complex Hermitian matrix and the momentum distributions show properties distinct from the bosons and fermions. In the Bose limit and Fermi limit the ROBDM is real and the momentum distributions are symmetric about the zero momentum while anyons populate in the momentum space asymmetrically and would rather stay in some special regime with large probability. With the change in statistic parameter the system exhibits the Bose statistics, Fermi statistics and the fractional statistics in between. The anyonic system exhibits characteristic feature between Bose and Fermi statistics also in the occupation of natural orbitals and the modulus of natural orbital shows more and more obvious oscillation with the decrease in statistical parameter.

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