PARAMETRIC RECTILINEARIZATION FOR $p$-ADIC SEMI-ALGEBRAIC SETS
AND RATIONALITY OF $p$-ADIC INTEGRALS

EVA LEENKNEGT
Department of Mathematics,
Purdue University,
West Lafayette, Indiana 47907, USA

Abstract. We present a rectilinearization theorem for $p$-adic semi-algebraic sets depending on parameters. As an application of our main theorem we present an alternative proof of a rationality result for parametric $p$-adic integrals, due to Denef.

1. Introduction

The technique of rectilinearization was originally used by Cluckers [2], to show that any two infinite $p$-adic semi-algebraic sets are isomorphic if they have the same dimension. To achieve this result, an important step was to show that any semi-algebraic $p$-adic set was isomorphic to a finite number of sets of a specific, simple form. It was already noted by Cluckers at that time that the result might be useful to $p$-adic integration, especially since the used isomorphisms also appeared to be very basic functions.

In a previous paper [3], we presented such an extended rectilinearization result, which states that any semialgebraic $p$-adic set can be partitioned into finitely many parts each of which is semi-algebraically isomorphic to a Cartesian power of basic subsets $\mathbb{Z}_p^{(k)}$ of $\mathbb{Q}_p$, where $\mathbb{Z}_p^{(k)}$ is the set of $p$-adic integers (of any nonnegative order) having coefficients $1, 0, \ldots, 0$ in their $p$-adic expansion, with $k-1$ zeros, see (1) below. Moreover, we could also ensure that the order of a finite number of given semi-algebraic functions, and the order of the Jacobian of the occurring isomorphisms, would equal the order of a monomial with integer powers. We used this rectilinearization result to give a new, very simple proof of the rationality of certain $p$-adic integrals. Rationality of such integrals and the related Poincare series had originally been proved by Denef [4] in two different ways, one of which was based on cell decomposition techniques.

This paper, which was inspired by a similar result of Cluckers for Presburger sets [1], presents a parametric version of the rectilinearization theorem mentioned above. Our motivation here is similar to that in the previous paper, namely that this result allows us to give alternative, simple proofs of the rationality of parametric $p$-adic
integrals. As an application of our main theorem we present an alternative proof of a rationality result for parametric $p$-adic integrals, due to Denef [3, 4].

1.1. Notation and terminology. Let $p$ denote a fixed prime number, $\mathbb{Q}_p$ the field of $p$-adic numbers and $K$ a fixed finite field extension of $\mathbb{Q}_p$. For $x \in K$ let $\text{ord} (x) \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of $x$. Let $R = \{x \in K \mid \text{ord} (x) \geq 0\}$ be the valuation ring, and let $q_K$ denote the cardinality of the residue field of $K$. Put $K^\times = K \setminus \{0\}$ and for $n \in \mathbb{N}_0$ let $P_n$ be the set $\{x \in K^\times \mid \exists y \in K : y^n = x\}$. We recall a form of Hensel’s Lemma and a corollary.

Corollary 2. Let $\text{ord} (\hat{\alpha} \in \alpha)$ be a fixed element of $K$. Prior to that, this result was used in the proofs of [4]. We will also need the following cell decomposition theorem, a proof of which can be found in [2].

We call a subset of $K^n$ semi-algebraic if it is a Boolean combination (i.e. obtained by taking finite unions, complements and finite intersections) of sets of the form 

$$\{x \in K^m \mid f(x) \in P_n\}$$

with $f(x) \in K[X_1, \ldots, X_m]$. The quantifier elimination result by Macintyre [8] implies that the collection of semi-algebraic sets is closed under taking projections $K^m \rightarrow K^{m-1}$. Moreover, also sets of the form $\{x \in K^m \mid \text{ord} (f(x)) \leq \text{ord} (g(x))\}$ with $f(x), g(x) \in K[X_1, \ldots, X_m]$ are semi-algebraic (see Denef’s alternative proof [4] of Macintyre’s theorem). A function $f : A \rightarrow B$ is called semi-algebraic if its graph is a semi-algebraic set; if such a function $f$ is a bijection, we call $f$ an isomorphism. By a finite partition of a semi-algebraic set we mean a partition into finitely many semi-algebraic sets.

Let $\pi$ be a fixed element of $R$ with $\nu(\pi) = 1$, thus $\pi$ is a uniformizing parameter for $R$. For a semi-algebraic set $X \subset K$ and $k > 0$ we write

$$(1) \quad X^{(k)} = \{x \in X \mid x \neq 0 \text{ and } \text{ord} (\pi^{-\text{ord}(x)} x - 1) \geq k\},$$

which is semi-algebraic (see [7], Lemma 2.1); $X^{(k)}$ consists of those points $x \in X$ which have a $p$-adic expansion $x = \sum_{i=s}^{\infty} a_i \pi^i$ with $a_s = 1$ and $a_i = 0$ for $i = s + 1, \ldots, s + k - 1$.

We recall a form of Hensel’s Lemma and a corollary.

Lemma 1 (Hensel). Let $f(t)$ be a polynomial over $R$ in one variable $t$, and let $\alpha \in R$, $e \in \mathbb{N}$. Suppose that $\text{ord} (f(\alpha)) > 2e$ and $\text{ord} (f'(\alpha)) \leq e$, where $f'$ denotes the derivative of $f$. Then there exists a unique $\hat{\alpha} \in R$ such that $f(\hat{\alpha}) = 0$ and $\text{ord} (\hat{\alpha} - \alpha) > e$.

Corollary 2. Let $n > 1$ be a natural number. For each $k > \text{ord} (n)$, and $k' = k + \text{ord} (n)$ the function $K^{(k)} \rightarrow P_n^{(k')} : x \mapsto x^n$ is an isomorphism.

We will also need the following cell decomposition theorem, a proof of which can for example be found in [2]. Prior to that, this result was used in the proofs of [4].

Lemma 3. Let $X \subset K^m$ be semi-algebraic and $b_j : K^m \rightarrow K$ semi-algebraic functions for $j = 1, \ldots, r$. Then there exists a finite partition of $X$ s.t. each part $A$ has the form

$$A = \{x \in K^m \mid \hat{x} \in D, \text{ord} a_1(\hat{x}) \sqcap_1 \text{ord} (x_m - c(\hat{x})) \sqcap_2 \text{ord} a_2(\hat{x}), \quad x_m - c(\hat{x}) \in \lambda P_n\},$$

and such that for each $x \in A$ we have that

$$\text{ord} (b_j(x)) = \frac{1}{n} \text{ord} ((x_m - c(\hat{x}))^{\mu_j} d_j(\hat{x})),\quad$$

where $\hat{x} = (x_1, \ldots, x_{m-1})$, $D \subset K^{m-1}$ is semi-algebraic, the set $A$ projects surjectively onto $D$, $n > 0$, $\mu_j \in \mathbb{Z}$, $\lambda \in K$, $c, d_j : K^{m-1} \rightarrow K$ and $a_i : K^{m-1} \rightarrow K^\times$ are semi-algebraic functions, and each $\sqcap_i$ is either $\leq$ or no condition. If $\lambda = 0$, we use
the conventions that \( \mu_j = 0 \) and \( 0^0 = 1 \), and thus \( \text{ord} (b_j(x)) = \frac{1}{n} \text{ord} (d_j(x)) \).

Moreover, \( D \) has the structure of a \( K \)-analytic manifold and the functions \( c, a_i \) and \( d_j \) are \( K \)-analytic on \( D \).

### 2. A Parametric Version of Rectilinearization

Write \( \pi_m : K^{m+r} \to K^r : (x_1, \ldots, x_{m+r}) \mapsto (x_1, \ldots, x_m) \) for the projection map onto the first \( m \) coordinates. Given a set \( A \subseteq K^{m+n} \) and \( \xi \in \pi_m(A) \), we use the notation \( A_\xi \) to denote the projection

\[
A_\xi := \{ x \in K^n \mid (\xi, x) \in A \}.
\]

We say that a semi-algebraic set \( S \subseteq K^n \) is bounded if there exists a tuple \( a \in (K^\times)^n \), such that for each \( x \in S \) and \( i = 1, \ldots, n \), we have that

\[
-\text{ord } a_i \leq \text{ord } x_i \leq \text{ord } a_i.
\]

The following definitions are central.

**Definition 4.** Let \( A \subseteq K^{r+m} \) be a semi-algebraic set.

We say that a family of functions \( f = \{ f_\xi \}_{\xi \in \pi_r(A)} \) with \( f_\xi : A_\xi \to K^l \) is semi-algebraic if the graph of \( f : A \to K^l : (\xi, x) \mapsto (\xi, f_\xi(x)) \) is a semi-algebraic set.

We say that a family of semi-algebraic functions \( f = \{ f_\xi \}_{\xi \in \pi_r(A)} \) with \( f_\xi : A_\xi \to K \) satisfies condition (2) if there exist constants \( \mu_i \in \mathbb{Z} \) and a semi-algebraic function \( \beta : \pi_r(A) \to K \) such that each \( (\xi, x_1, \ldots, x_m) \in A \) satisfies \( x_i \neq 0 \) if \( \mu_i < 0 \) and

\[
\text{ord}(f_\xi(x)) = \text{ord}(\beta(\xi) \prod_i x_i^{\mu_i}).
\]

We say that a family of semi-algebraic functions \( g = \{ g_\xi \}_{\xi \in \pi_r(A)} \) with \( g_\xi : A_\xi \to K^m \) satisfies condition (3) if each \( g_\xi \) is \( C^1 \) on \( A_\xi \) and there exist constants \( \mu_i \in \mathbb{Z} \) and a semi-algebraic function \( \beta : \pi_r(A) \to K \) such that each \( (\xi, x_1, \ldots, x_m) \in A \) satisfies \( x_i \neq 0 \) if \( \mu_i < 0 \) and

\[
\text{ord}(\text{Jac}(g_\xi(x))) = \text{ord}(\beta(\xi) \prod_i x_i^{\mu_i}).
\]

**Definition 5.** Let \( f : X \to Y \) be a family of semi-algebraic isomorphisms. Say that the family \( f \) is of type \( f_0 \) when \( f \) equals an isomorphism of the following kind:

\[
f_0 : X \subset K^{r+n} \to Y \subset K^{r+n+1} : (\xi, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)
\]

for some \( n \geq 0 \). Say that the family \( f \) is of type \( f_1 \), resp. of type \( f_2 \) or \( t_c \), when \( X \subset K^{r+n} \), \( Y \subset K^{r+n} \) for some \( n \geq 0 \), and \( f \) equals an isomorphism of the following kind:

\[
f_1 : X \to Y : (\xi, x) \mapsto (\xi, \alpha_1 x_1^{a_1}, \ldots, \alpha_n x_n^{a_n}),
\]

\[
f_2 : X \to Y : (\xi, x) \mapsto (\xi, x_1, \ldots, x_{i-1}, x_i \beta(\xi) \prod_{j \neq i} x_j^{b_j}, x_{i+1}, \ldots, x_n), \text{ or}
\]

\[
t_c : X \to Y : (\xi, x) \mapsto (\xi, x_1, \ldots, x_{i-1}, x_i + c(\xi, x_1, \ldots, x_{i-1}), x_{i+1}, \ldots, x_n),
\]

with \( a_i, b_j \in \mathbb{Z} \), \( a_i < 0 \) implies \( x_i \neq 0 \) for \( x \in X \), \( b_j < 0 \) implies \( x_j \neq 0 \) for \( x \in X \), \( \alpha_i \in K \), \( x = (x_1, \ldots, x_n) \), \( \beta : \pi_r(X) \to K \) a semi-algebraic function and...
functions of types \( f \) and \( \Pi \).

Partition Proof.

There exists a finite partition of \( \Sigma \) for \( m > 0 \).

We will need the following lemma:

**Lemma 7.** Let \( S \subseteq K^r \) be a semi-algebraic set, and \( E = \cup_{\xi \in S} E_\xi \subseteq K^m \) be a family of sets, with \( E_\xi \) of the form

\[
E_\xi := \left\{ x \in \Delta_\xi \times \prod_{i=m'}^m R^{(k)} \mid \text{ord } x_m \leq \text{ord } \left( \beta(\xi) \prod_{i=m'}^m x_i^{\nu_i} \right) \right\},
\]

where \( \beta(\xi) \) is a nonzero semi-algebraic function, \( k \in \mathbb{N}_0 \), \( \nu_i \in \mathbb{Z} \), \( \Delta_\xi \) is a bounded subset of \( \prod_{i=m'}^m R^{(k)} \), \( m, m' \in \mathbb{N} \) depending only on \( E \).

There exists a finite partition of \( S \) in semi-algebraic sets \( S_i \) and for each \( E_{S_i} = \cup_{\xi \in S_i} E_\xi \), a finite partition in parts \( A \) such that for each part \( A = \cup_{\xi \in S_i} A_\xi \), there is a family of isomorphisms \( f_\epsilon \) of the form

\[
f_\epsilon : \Sigma_\xi \times \prod_{i=l}^{l'} R^{(k)} \rightarrow A_\xi : x \mapsto (g \circ F_0)(x).
\]

Here \( \Sigma_\xi \) is a bounded set, \( l, l' \leq m \) are independent of \( \xi \), \( g \) is a composition of functions of types \( f_1, f_2 \), and \( F_0 \) is a composition of functions of type \( f_0 \).

**Proof.** Partition \( S \) in \( S_1 = \{ \xi \in S \mid \beta(\xi) = 0 \} \) and \( S_2 = S \setminus S_1 \). The Lemma is trivial for \( E_{S_1} \), so we may assume that \( \beta(\xi) \) is nonzero.

We work by induction on \( m \). The case where \( m' = m \) is trivial, so assume that \( m > m' \).

Note that for \( m' < i < m \), there are no conditions on \( x_i \) if \( \nu_i = 0 \).
We first look at the case where \( \nu_i < 0 \) for all \( i = m', \ldots, m - 1 \). In this case each \( E_\xi \) is a bounded set. Indeed, we have that

\[
0 \leq \text{ord } x_m \leq \text{ord } \beta(\xi) + \sum_{i=1}^{m'-1} \nu_i \text{ord } x_i - \sum_{i=m'}^{m-1} |\nu_i| \text{ord } x_i,
\]

\[
\leq M(\xi) - \sum_{i=m'}^{m-1} |\nu_i| \text{ord } x_i,
\]

where \( M(\xi) \) is a natural number that may depend on \( \xi \). This implies that \( E_\xi \) cannot contain any \( x \) for which \( \text{ord } x_i > \frac{\text{ord } M(\xi)}{|\nu_i|} \) for some \( m' \leq i < m \).

Hence, we may suppose that \( \nu_{m-1} > 0 \). We first prove the proposition when \( \nu_{m-1} = 1 \). We can partition \( E_\xi \) into parts \( E_{\xi,1} \) and \( E_{\xi,2} \), with

\[
E_{\xi,1} = \left\{ x \in E_\xi \mid \text{ord } x_m \leq \text{ord } \left( \beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i} \right) \right\},
\]

\[
E_{\xi,2} = \left\{ x \in E_\xi \mid \text{ord } \left( \beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i} \right) < \text{ord } x_m \right\},
\]

\[
= \left\{ x \in \Delta_\xi \times \prod_{i=m'}^{m} R^{(k)} \mid \text{ord } \left( \beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i} \right) < \text{ord } x_m \right\}.
\]

Since \( \text{ord } (\beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i}) \leq \text{ord } (x_{m-1} \beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i}) \) for \( x \in E_{\xi,1} \), it follows that we have an isomorphism

\[
E'_{\xi,1} \rightarrow E_{\xi,1} : (x_1, \ldots, x_{m-2}, x_{m-1}, x_m) \mapsto (x_1, \ldots, x_{m-2}, x_m, x_{m-1})
\]

(which is a composition of maps of type \( f_1 \) and \( f_2 \)) with \( E'_{\xi,1} \) the set

\[
\left\{ (x_1, \ldots, x_{m-1}) \in \Delta_\xi \times \prod_{i=m'}^{m-1} R^{(k)} \mid \text{ord } (x_{m-1}) \leq \text{ord } \left( \beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i} \right) \right\} \times R^{(k)},
\]

and the lemma follows for \( E'_{\xi,1} \) by the induction hypothesis.

For \( E_{\xi,2} \), let \( D_{\xi, m-1} \) be the set

\[
\left\{ (x_1, \ldots, x_{m-1}) \in \Delta_\xi \times \prod_{i=m'}^{m-1} R^{(k)} \mid \text{ord } \left( \beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i} \right) < \text{ord } x_{m-1} \right\}.
\]

We may suppose that \( \beta(\xi) \in K^{(k)} \). Then, the map

\[
D_{\xi, m-1} \times R^{(k)} \rightarrow E_{\xi,2} : x \mapsto \left( x_1, \ldots, x_{m-2}, \frac{x_{m-1} x_m - \beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i}}{\beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i}}, x_{m-1} \right)
\]

is an isomorphism which is a composition of isomorphisms of type \( f_1 \) and \( f_2 \). Also

\[
\Delta_\xi \times \prod_{i=m'}^{m} R^{(k)} \rightarrow D_{\xi, m-1} \times R^{(k)} : x \mapsto \left( x_1, \ldots, x_{m-2}, \pi \beta(\xi) x_{m-1} \prod_{i=1}^{m-2} x_i^{\nu_i}, x_m \right)
\]

is an isomorphism which is a composition of isomorphisms of type \( f_1 \) and \( f_2 \). This proves the lemma when \( \nu_{m-1} = 1 \).
Suppose now that $\nu_{m-1} > 1$. We prove that we can reduce to the case $\nu_{m-1} = 1$ by partitioning and applying appropriate power maps. Choose $k > \text{ord}(\nu_{m-1})$ and put $k' = k + \text{ord}(\nu_{m-1})$. We may suppose that $k \geq k$, so we have a finite partition $E_\xi = \bigcup_\alpha E_{\xi, \alpha}$, with $\alpha = (\alpha_1, \ldots, \alpha_m) \in K^m$, $\text{ord}(\alpha_1) = 0$, $0 \leq \text{ord}(\alpha_i) < \nu_{m-1}$ for $i = 2, \ldots, m$ and

$$E_{\xi, \alpha} = \{ x \in E_\xi \mid x_{m-1} \in \alpha_{m-1} \cdot R(\tilde{k}), x_i \in \alpha_i \cdot P_{\nu_1}(\tilde{k}) \}
$$

for $i \in \{1, \ldots, m-2\} \cup \{m\}$.

By Corollary 2 we have isomorphisms

$$f_{\xi, \alpha} : C_{\xi, \alpha} \to E_{\xi, \alpha} : x \mapsto (\alpha_1 x_1^{\nu_{m-1}}, \ldots, \alpha_{m-2} x_{m-2}^{\nu_{m-2}}, \alpha_{m-1} x_{m-1}, \alpha_m x_m^{\nu_{m-1}}),$$

with $C_{\xi, \alpha} = \{ x \in \Sigma_\xi \times \prod_{i=m'}^{m} R(\tilde{k}) \mid \text{ord}(x_m) \leq \text{ord}(\beta' x_1 \prod_{i=2}^{m-1} x_i^{\nu_i}) \}$, which are isomorphisms of type $f_1$. Here $\Sigma_\xi$ is a bounded (definable) subset of $\prod_{i=1}^{m-1} R(\tilde{k})$ and $\beta' : S \to K^x$ a semi-algebraic function. This reduces the problem to the case with $\nu_{m-1} = 1$ and thus the lemma is proved.

**Proof of Theorem 7.** We give a proof by induction on $m$. We will show that we can reduce to the case described in equation (4) below. The theorem then follows by Lemma 8. Using Lemma 3 and its notation, we find a finite partition of $X$ such that each part $A$ has the form

$$A = \{ (\xi, x, t) \in K^{\nu+1} \mid \text{ord}(\xi, x, t) \in \mathbb{Z} \},$$

and such that $\text{ord}(t-x, t) = \frac{1}{n} \text{ord}((t-x)\mu_{m,j} d_j(\xi, x))$ for each $x \in A$, with $\mu_{m,j} \in \mathbb{Z}$.

First use a translation $t_{\xi, \alpha}^{(m+1)} : A_{c=0} \to A : x \mapsto (x_1, \ldots, x_m, t + c(\xi, x))$. If we apply the induction hypothesis to the set $D \subset K^{r+1}$ and the functions $a_1, a_2, d_j$, we get a partition of $D$ in parts $D$, such that for each $D$, there exists a family of definable functions $\{f_{\xi, \alpha}\}_{\xi \in \pi_m(D)}$ so that we have isomorphisms of the form

$$f_{\xi, \alpha} : \Delta_\xi \times \prod_{i=l'} R(\tilde{k}) \to D_\xi : (x_1, \ldots, x_l) \mapsto (\tilde{T}_c \circ \hat{g})(x_1, \ldots, x_l),$$

with $\Delta_\xi$ a bounded definable set (of the form described in the formulation of the theorem) and $l, l' \in \mathbb{N}$ depending only on $D$. $\tilde{T}_c$ is a composition of functions of type $t_{\xi, \alpha}$ and $\hat{g}$ a composition of functions of types $f_0, f_1, f_2$. This induces a finite partition of $A_{c=0}$ in parts $\tilde{A}_{c=0}$, such that for each part there is an isomorphism of the form

$$f'_{\xi, \alpha} : B \to \tilde{A}_{c=0} : (\xi, x, t) \mapsto (\xi, f_{\xi, \alpha}(x_1, \ldots, x_l)), t).$$

For each $\xi \in \pi_m(B)$, $B_\xi$ is a set of the form $B_\xi = \{ (x, t) \in \Delta_\xi \times \prod_{i=l'} R(\tilde{k}) \times \mathbb{Z} \mid \phi(\xi, x, t) \}$ with

$$\phi(\xi, x, t) \iff \text{ord}(\alpha_1(\xi) \prod_{i=1}^{l'} x_i^{\eta_i} \square_1 \text{ord}(x_i \square_2 \text{ord}(\alpha_2(\xi) \prod_{i=1}^{l'} x_i^{\nu_i})),$$

where the $\alpha_i : \pi_m(B) \to K$ are semi-algebraic functions. If we compose the functions $f'$ with the translation $t_{\xi, \alpha}^{(m+1)}$, we obtain isomorphisms of the form

$$f = i_{c}^{(m+1)} \circ f' : B \to A_B : (\xi, x, t) \mapsto ((\tilde{T}_c \circ \hat{g})(\xi, x), t + c(\xi, x))$$

for $\xi \in \pi_m(B)$.
between sets $B$ and sets $A_B \subset X$. The sets $A_B$ form a finite partition of $X$. Applying the induction hypotheses, we find that there exist $\mu_{ij} \in \mathbb{Z}$ and semi-algebraic functions $\delta_i : \pi_t(B) \to K$ such that ord $(\delta_j \circ f)(\xi, x, t) = \frac{1}{n} \text{ord}(\delta_i(\xi)^{\mu_{ij+1,j}} \prod_{k=1}^{l} x_i^{\mu_{ijk}})$.

We will show that we only need functions of type $f_0, f_1$ and $f_2$ for our parametric rectilinearization of $B$. Thus the final isomorphisms $\prod R^{(k)} \to A$ will have the form $x \mapsto (T \circ g)(x)$, where $g$ is a function of compositions of types $f_0, f_1$ and $f_2$, and $T$ is a composition of functions of type $t_j$.

If $\lambda = 0$, then $B_{\xi} = \Delta_{\xi} \times \prod_{i=1}^{l} R^{(k)} \times \{0\}$. In this case our isomorphism has the form $(\xi, x) \mapsto (T \circ g \circ f_0)(\xi, x)$. Recall that by Lemma 2, $\alpha_1(\xi) \neq 0 \neq \alpha_2(\xi)$. From now on suppose that $\lambda \neq 0$.

As in the nonparametric case, we may suppose that $\square_2$ is either $\leq$ or no condition and $\square_1$ is the symbol $\leq$ (possibly after partitioning or $\xi, x, t \mapsto (\xi, x, 1/t)$).

Choose $\tilde{k} > \text{ord}(n)$ and put $k' = \tilde{k} + \text{ord}(n)$. We may suppose that $k' > k$, so we have a finite partition $B = \bigcup_{\gamma} B_{\gamma}$ with $\gamma = (\gamma_1, \ldots, \gamma_{l+1}) \in K^{l+1}$, $0 \leq \text{ord}(\gamma_i) < n$ and

$$B_{\gamma} = \{(\xi, x, t) \in B \mid t \in \gamma_{l+1} P_n^{(k')}, x_i \in \gamma_i P_n^{(k')}, \text{ for } i = 1, \ldots, l\}.$$

Now we have isomorphisms

$$f_{\gamma} : C_{\gamma} \to B_{\gamma} : (\xi, x, t) \mapsto (\xi, \gamma_1 x_1^n, \ldots, \gamma_l x_l^n, \gamma_{l+1} x_{l+1}^n).$$

For each $\xi \in \pi_m(B_{\gamma})$, the set $C_{\gamma, \xi}$ is defined as $\{x \in \Delta_{\xi} \times \prod_{i=1}^{l} R^{(k')} \times K^{(k)} \mid \psi(\xi, x, t)\}$, with

$$\psi(\xi, x, t) \mapsto \text{ord} \left( \alpha_1(\xi) \prod_{i=1}^{l} x_i^{\eta_i^*} \right) \leq \text{ord } t \square_2 \text{ ord } \left( \alpha_2(\xi) \prod_{i=1}^{l} x_i^{\eta_i^*} \right),$$

where the $\alpha_i' : \pi_m(C_{\gamma}) \to K$ are semi-algebraic functions (for this we need Lemma 2.4). Put $\bar{f} = f \circ f_{\gamma}$. Then the $b \circ \bar{f}$ satisfy condition (2), since

$$\text{ord } (b_j \circ f(\xi, x, t))) = \frac{1}{n} \text{ord } \left( \beta_j(\xi)^{\mu_{i+1,j}} \prod_{i=1}^{l+1} x_i^{\mu_{ij}} \prod_{i=1}^{l} x_i^{\mu_{ij}} \right)$$

$$= \frac{1}{n} \text{ord } \left( \beta_j(\xi) \prod_{i=1}^{l+1} x_i^{\mu_{ij}} \right) + \text{ord } \left( \beta(\xi) \prod_{i=1}^{l+1} x_i^{\mu_{ij}} \right),$$

and by Lemma 2.4 there exists a semi-algebraic function $\bar{\beta}_j(\xi)$ such that $\text{ord } \bar{\beta}_j(\xi) = \frac{1}{n} \text{ord } \left( \beta_j(\xi) \prod_{i=1}^{l+1} x_i^{\mu_{ij}} \right)$.

Put $\nu_i = \varepsilon_i - \eta_i, \beta(\xi) = \alpha_2(\xi)/\alpha_1(\xi)$. Then the following is an isomorphism

$$D_{\gamma} : C_{\gamma} : (\xi, x, t) \mapsto \left( \xi, x_1, \ldots, x_l, \alpha_1(\xi) t \prod_{i=1}^{l} x_i^{\eta_i^*} \right),$$

with $D_{\gamma, \xi} = \left\{ x \in \Delta_{\xi} \times \prod_{i=1}^{l+1} R^{(k')} \mid \text{ord } t \square_2 \text{ ord } \left( \beta(\xi) \prod_{i=1}^{l} x_i^{\eta_i^*} \right) \right\}$.

The case that $\square_2$ is no condition is now trivial. Summarizing, it follows that we can reduce to the case of an isomorphism $f : E \to X : (\xi, x, t) \mapsto (\xi, f_k(x, t))$, with
functions of types $f$.

**Partition Proof.**

There exists a finite partition of $\Sigma$.

Here $m > m'$ with $m$, $m'$ being natural numbers.

Let $\beta$ be a semi-algebraic function, $\tilde{k} > 0$, and $\nu_i \in \mathbb{Z}$, such that each $b_j \circ f$ satisfies condition (2).

Use Lemma 8 to obtain a partition of $E$ in parts $E_i = \cup_{E_i}$, and families of isomorphisms $\phi_{\xi,i} : \Delta_\xi \times \prod R^{(k)} \rightarrow E_i$. The $\phi_{\xi,i}$ are composed of functions of types $f_0, f_1, f_2$ and the components of $\phi_{\xi,i}$ all satisfy condition (2). Therefore each $b_j \circ f \circ \phi_i$ will satisfy condition (2). That the condition on the Jacobians holds can now be checked in a straightforward way (for a proof, see [3]). Our claim on the form of $\Delta_\xi$ follows immediately from the proof of Lemma 8. This finishes the proof of Theorem 0.

\[\square\]

**Lemma 8.** Let $S \subseteq K^r$ be a semi-algebraic set, and $E = \cup_{E_i} \subseteq K^m$ be a family of sets, with $E_\xi$ of the form

\[
E_\xi := \left\{ x \in \Delta_\xi \times \prod_{i=m'}^m R^{(k)} \mid \text{ord } x_m \leq \text{ord} \left( \beta(\xi) \prod_{i=m'}^{m-1} x_i^{\nu_i} \right) \right\},
\]

where $\beta(\xi)$ is a nonzero semi-algebraic function, $k \in \mathbb{N}_0$, $\nu_i \in \mathbb{Z}$, $\Delta_\xi$ is a bounded subset of $\prod_{i=m'}^{m-1}$, $R^{(k)}$, $m, m' \in \mathbb{N}$ depending only on $E$.

There exists a finite partition of $S$ in semi-algebraic sets $S_i$ and for each $E_{S_i} = \cup_{E_i} E_\xi$, a finite partition in parts $A$ such that for each part $A = \cup_{E_i} A_\xi$, there is a family of isomorphisms $f_\xi$ of the form

\[f_\xi : \Sigma_\xi \times \prod_{i=m'}^l R^{(k)} \rightarrow A_\xi : x \mapsto (g \circ F_i)(x).\]

Here $\Sigma_\xi$ is a bounded set, $l', l \leq m$ are independent of $\xi$, $g$ is a composition of functions of types $f_1, f_2$, and $F_\xi$ is a composition of functions of type $f_0$.

**Proof.** Partition $S$ in $S_1 = \{ \xi \in S \mid \beta(\xi) = 0 \}$ and $S_2 = S \setminus S_1$. The Lemma is trivial for $E_{S_1}$, so we may assume that $\beta(\xi)$ is nonzero.

We work by induction on $m$. The case where $m' = m$ is trivial, so assume that $m > m'$. Note that for $m' < i < m$, there are no conditions on $x_i$ if $\nu_i = 0$.

We first look at the case where $\nu_i < 0$ for all $i = m', \ldots, m - 1$. In this case each $E_\xi$ is a bounded set. Indeed, we have that

\[0 \leq \text{ord } x_m \leq \text{ord } \beta(\xi) + \sum_{i=1}^{m'-1} \nu_i \text{ord } x_i - \sum_{i=m'}^{m-1} \nu_i \text{ord } x_i,
\]

\[\leq M(\xi) - \sum_{i=m'}^{m-1} \nu_i \text{ord } x_i,
\]

where $M(\xi)$ is a natural number that may depend on $\xi$. This implies that $E_\xi$ cannot contain any $x$ for which $\text{ord } x_i > \frac{\text{ord } M(\xi)}{|\nu_i|}$ for some $m' \leq i < m$. 

$\square$
Hence, we may suppose that $\nu_{m-1} > 0$. We first prove the proposition when $\nu_{m-1} = 1$. We can partition $E_\xi$ into parts $E_{\xi,1}$ and $E_{\xi,2}$, with

\[
E_{\xi,1} = \left\{ x \in E_\xi \mid \text{ord } x_m \leq \text{ord } \left( \beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i} \right) \right\},
\]

\[
E_{\xi,2} = \left\{ x \in E_\xi \mid \text{ord } \left( \beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i} \right) < \text{ord } x_m \right\}
\]

\[
= \left\{ x \in \Delta_\xi \times \prod_{i=m'}^{m} R^{(k)} \mid \text{ord } \left( \beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i} \right) < \text{ord } x_m \leq \text{ord } \left( \beta(\xi)x_{m-1} \prod_{i=1}^{m-2} x_i^{\nu_i} \right) \right\}.
\]

Since $\text{ord } (\beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i}) \leq \text{ord } (x_{m-1}\beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i})$ for $x \in E_{\xi,1}$, it follows that we have an isomorphism

\[
E'_{\xi,1} \to E_{\xi,1} : (x_1, \ldots, x_{m-2}, x_{m-1}, x_m) \mapsto (x_1, \ldots, x_{m-2}, x_m, x_{m-1})
\]

(which is a composition of maps of type $f_1$ and $f_2$) with $E'_{\xi,1}$ the set

\[
\left\{ (x_1, \ldots, x_{m-1}) \in \Delta_\xi \times \prod_{i=m'}^{m-1} R^{(k)} \mid \text{ord } (x_{m-1}) \leq \text{ord } \left( \beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i} \right) \right\} \times R^{(k)},
\]

and the lemma follows for $E'_{\xi,1}$ by the induction hypothesis.

For $E_{\xi,2}$, let $D_{\xi,m-1}$ be the set

\[
\left\{ (x_1, \ldots, x_{m-1}) \in \Delta_\xi \times \prod_{i=m'}^{m-1} R^{(k)} \mid \text{ord } \left( \beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i} \right) < \text{ord } x_{m-1} \right\}.
\]

We may suppose that $\beta(\xi) \in K^{(k)}$. Then, the map

\[
D_{\xi,m-1} \times R^{(k)} \to E_{\xi,2} : x \mapsto \left( x_1, \ldots, x_{m-2}, \frac{x_{m-1} x_m}{\beta(\xi) \prod_{i=1}^{m-2} x_i^{\nu_i}}, x_{m-1} \right)
\]

is an isomorphism which is a composition of isomorphisms of type $f_1$ and $f_2$. Also

\[
\Delta_\xi \times \prod_{i=m'}^{m} R^{(k)} \to D_{\xi,m-1} \times R^{(k)} : x \mapsto \left( x_1, \ldots, x_{m-2}, \pi \beta(\xi) x_{m-1} \prod_{i=1}^{m-2} x_i^{\nu_i}, x_m \right)
\]

is an isomorphism which is a composition of isomorphisms of type $f_1$ and $f_2$. This proves the lemma when $\nu_{m-1} = 1$.

Suppose now that $\nu_{m-1} > 1$. We prove that we can reduce to the case $\nu_{m-1} = 1$ by partitioning and applying appropriate power maps. Choose $\hat{k} > \text{ord } (\nu_{m-1})$ and put $\hat{k}' = k + \text{ord } (\nu_{m-1})$. We may suppose that $\hat{k} \geq k$, so we have a finite partition $E_\xi = \bigcup_\alpha E_{\xi,\alpha}$, with $\alpha = (\alpha_1, \ldots, \alpha_m) \in K^m$, $\text{ord } (\alpha_1) = 0$, $0 \leq \text{ord } (\alpha_i) < \nu_{m-1}$ for $i = 2, \ldots, m$, and

\[
E_{\xi,\alpha} = \left\{ x \in E_\xi \mid x_{m-1} \in \alpha_{m-1} R^{(\hat{k})}, \ x_i \in \alpha_i P^{(\hat{k})}_{\nu_i} \right\}
\]

for $i \in \{1, \ldots, m-2\} \cup \{m\}$.

By Corollary 2 we have isomorphisms

\[
f_{\xi,\alpha} : C_{\xi,\alpha} \to E_{\xi,\alpha} : x \mapsto (\alpha_1 x_1^{\nu_{m-1}}, \ldots, \alpha_{m-2} x_{m-2}^{\nu_{m-1}}, \alpha_{m-1} x_{m-1}, \alpha_m x_{m-1}^{\nu_{m-1}}),
\]
with \( C_{\xi,a} = \{ x \in \Sigma_\xi \times \prod_{i=m}^{m-1} R^{(k_i)} | \ \text{ord} (x_m) \leq \text{ord} (\beta' x_1 \prod_{i=2}^{m-1} x_i^{\nu_i}) \} \), which are isomorphisms of type \( f_1 \). Here \( \Sigma_\xi \) is a bounded (definable) subset of \( \prod_{i=m-1}^{m-1} R^{(k_i)} \) and \( \beta': S \to K^\times \) a semi-algebraic function. This reduces the problem to the case with \( \nu_{m-1} = 1 \) and thus the lemma is proved.

\[ \Box \]

2.1. Application to \( p \)-adic integration. Using parametric rectilinearization, we can give a new proof of the rationality of \( p \)-adic integrals with parameters. This was originally proven by Denef, see [5, 6]. The proof uses the rationality result for \( p \)-adic integrals due to Denef, of which we provided an alternative proof in [3].

**Theorem 9** (Rationality, [4]). Let \( S \subset K^m \) be a semi-algebraic set and \( f, g : S \to K \) semi-algebraic functions. If the following integral exists for \( s \in \mathbb{R}, s \gg 0 \) (that is, if the integrand is absolutely integrable for \( s \) sufficiently big), then

\[
I(s) := \int_S |f(x)|^s \cdot |g(x)||dx|
\]

is rational in \( q_K s \) and the denominator of \( I(s) \) is a product of factors of the form \( (1 - q_K^{-a+b}) \) with \( a, b \in \mathbb{Z} \), and \( (a, b) \neq (0, 0) \).

**Proposition 10.** Let \( S \subseteq K^{r+m} \) be a semi-algebraic set and \( f, g : S \to K \) a semi-algebraic function. For every \( \xi \in \pi_r S \), let \( I_\xi(s) \) be the following integral

\[
I_\xi(s) := \int_{S_\xi} |f(\xi, x)|^s |g(\xi, x)||dx|.
\]

If \( I_\xi(s) \) exists for \( s \gg 0 \), then the integral is rational in \( q_K s \). More precisely,

\[
I_\xi(s) = \frac{\beta(\xi)^s \gamma(\xi) P_E(q_K^{-s})}{Q(q_K^s)}
\]

where \( \beta, \gamma \) are semi-algebraic functions \( \pi_r(S) \to K \), and \( Q(T), P_E(T) \in K[T] \). \( Q(T) \) is a product of factors \( (1 - q_K^{-a+b} T^a) \) with \( a, b \in \mathbb{Z} \). These factors do not depend on \( \xi \). The degree of \( P_E(T) \) is bounded by the order of a semi-algebraic function in the variables \( \xi \).

**Proof.** By Theorem [11] and the change of variables formula for \( p \)-adic integrals, \( I_\xi(s) \) is equal to a finite linear combination (with constant coefficients) of integrals

\[
\beta(\xi)^s \gamma(\xi) \int_{\Delta_\xi} \prod_{i=1}^{l'} x_i^{|\mu_i|} \prod_{i=1}^{l} x_i^{|\nu_i|} |dx|.
\]

By Theorem [12] this is equal to

\[
R(q_K^{-s}) \cdot I_\xi(s) := R(q_K^{-s}) \cdot \beta(\xi)^s \gamma(\xi) \int_{\Delta_\xi} \prod_{i=1}^{l'} x_i^{|\mu_i|} \prod_{i=1}^{l} x_i^{|\nu_i|} |dx|,
\]

where \( R(T) \in K(T) \) and the denominator consists of factors \( (1 - q_K^{-a+b} T^a) \) with \( a, b \in \mathbb{Z} \). We know by Theorem [13] that \( \Delta_\xi \) is a set of the form

\[
\left\{ x \in \prod_{i=1}^{l'} R^{(k_i)} | (\text{ord } x_1, \ldots, \text{ord } x_{l'-1}) \in \Gamma_S \right\}
\]
with $\Gamma_S$ a finite subset of $\Gamma^l_{K-1}$ containing elements $\alpha = (\alpha_1, \ldots, \alpha_{l'-1})$ that satisfy relations

$$0 \leq \alpha_j \leq \text{ord} (\beta_j(\xi)) + \sum_{i=1}^{j-1} n_i \alpha_i$$

for non-zero semi-algebraic functions $\beta_i : \pi_r(S) \to K$ and $n_i, m_i \in \mathbb{Z}$ for $i = 1, \ldots, l'-1$. Because of this, $I'_s(\xi)$ is equal to

$$|\beta(\xi)|^s \gamma(\xi) q_K (\sum_{\alpha \in \Gamma_S} q_{\sum_{i=1}^{l'-1} (\mu_i s + \nu_i - 1) \alpha_i}).$$

As this is a finite sum (with the exact number of terms depending on $\xi$), this proves our claim. □

Acknowledgements. I would like to thank Raf Cluckers for suggesting this topic to me.

References

[1] R. Cluckers. Presburger sets and $p$-minimal fields. *J. Symbolic Logic*, 68(1):153–162, 2003, math/0206197.
[2] Raf Cluckers. Classification of semi-algebraic $p$-adic sets up to semi-algebraic bijection. *J. Reine Angew. Math.*, 540:105–114, 2001, math/0311434.
[3] Raf Cluckers and Eva Leenknegt. Rectilinearization of semi-algebraic $p$-adic sets and Denef’s rationality of Poincaré series. *J. Number Theory*, 128(7):2185–2197, 2008.
[4] J. Denef. The rationality of the Poincaré series associated to the $p$-adic points on a variety. *Invent. Math.*, 77:1–23, 1984.
[5] J. Denef. On the evaluation of certain $p$-adic integrals. In *Séminaire de théorie des nombres, Paris 1983–84*, volume 59 of *Progr. Math.*, pages 25–47. Birkhäuser Boston, Boston, MA, 1985.
[6] J. Denef. Arithmetic and geometric applications of quantifier elimination for valued fields. In *Model theory, algebra, and geometry*, volume 39 of *Math. Sci. Res. Inst. Publ.*, pages 173–198. Cambridge Univ. Press, Cambridge, 2000.
[7] Jan Denef. $p$-adic semi-algebraic sets and cell decomposition. *J. Reine Angew. Math.*, 369:154–166, 1986.
[8] A. Macintyre. On definable subsets of $p$-adic fields. *J. Symb. Logic*, 41:605–610, 1976.