Multi-class classification: mirror descent approach

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Abstract

We consider the problem of multi-class classification and a stochastic optimization approach to it. We derive risk bounds for stochastic mirror descent algorithm and provide examples of set geometries that make the use of the algorithm efficient in terms of error in $k$.

1 Introduction

Classification is one of the core machine learning tasks. The problem of multi-class classification arises in various tasks including document classification (Rennie and Rifkin 2001), image (Foody and Mathur 2004; Lee and Seung 1997), gesture (McNeill 1992) and video recognition (Karpathy, Toderici, Shetty, Leung, Sukthankar, and Fei-Fei 2014) and many others. Datasets for the problems are growing in both number of samples and number of classes $k$. As the expected error of classification algorithms also increases, the growth rate in terms of $k$ becomes crucial.

In this paper we consider the classical one-vs-all margin classification approach (Aly 2005), which was shown to be as good as ECOC and all-vs-all approach, at least from the practical point of view (Rifkin and Klautau 2004). There are several generalization ability guarantees known for the class of learners. Distribution-independent ones rely on function class complexity measures such as Natarajan $d_N$ (Natarajan 1989), graph $d_G$ (Natarajan 1989; Dudley 2010) and VC $d_{VC}$ (Guermeur 2007) dimension. If we consider kernel separators $f_k(x, w) = K(x, w_y)$ with PSD $K$, the bounds lead to $O(k/\sqrt{n})$ excess risk (Daniely, Sabato, Ben-David, and Shalev-Shwartz 2013; Guermeur 2007). Covering number based bound presented in (Zhang 2002) gives $O(\sqrt{k}/n)$ rate. While the bounds provided seem tight, they result in large constants and dimension dependence for combinatorial complexity measures, as for distribution-dependent bounds, little is known beyond the general Rademacher complexity based bound, which is in the worst case of order $O(k/\sqrt{n})$ for typical function classes (e. g. finite VC-dimensional).

As the underlying distribution of the pairs object-class is unknown, the problem can be solved by substituting the actual risk by the empirical one or by means of stochastic optimization. While the general problem statement of minimizing the risk allows the implementation of different optimization algorithms, first order methods are preferable to high-order ones for large-scale problems in terms of their generalization ability and computational efficiency (Bousquet and Bottou 2008). We consider an adaptation of stochastic mirror descent (MD) algorithm (Nemirovsky, Yudin, and Dawson 1982; Beck and Teboulle 2003) to solve the problem. MD is a first-order algorithm for convex function minimization, which restricts the method to convex $\Omega$ and convex in $w$ loss-function $\ell(x, y, w)$, allowing dimension-independent excess risk bounds.
The rest of the paper is organized as follows. In section 2 we describe the general assumptions made concerning the input space. In the next section we describe the algorithm and provide upper bounds for the expected error of the classifier. In section 4 we provide the bounds for large deviations probabilities and the examples of good geometry for the problem.

2 Preliminaries

We first describe the framework of multi-class classification used in the paper. Let \( \mathcal{X} \subseteq \mathbb{R}^d \) be a set of instances, \( \mathcal{Y} = \{1 \ldots k\} \) be a set of classes which form a probability space \( (\mathcal{X} \times \mathcal{Y}, \mathcal{A}, \mathbb{P}) \) and the sample \( S = \{x_i, y_i\}_{i=1}^n \) be i.i.d. drawn from the distribution. We denote \( \mathcal{F} = \{f(\cdot, w): \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \mid w \in \mathcal{W}\} \) the class of decision functions – a parametric class of measurable functions and \( \ell: \mathcal{X} \times \mathcal{Y} \times \mathcal{W} \to \mathbb{R}_+ \) the loss function. We consider one-vs-all approach to the problem by setting the predictor to be \( \hat{f}(x, w^*) = \max_{y \in \mathcal{Y}} f(x, y, w^*) \). The loss function is chosen to be a Lipschitz upper bound on the indicator function \( \hat{f}(x_i, w) = y_i \) and to maximize the margin:

\[
m(x, y, w) = f(y, w) - \max_{y' \in \mathcal{Y}, y' \neq y} f(y', w).
\]

We use \( \ell_p(x, y, w) = \max\{0, 1 - m(x, y, w)/\rho\} \) as the loss function. The problem is then to minimize the expected risk \( F(w) = \mathbb{E}_{(x,y) \sim \mathcal{P}} \ell_f(x, y, w) \):

\[
w^* = \arg \min_{w \in \mathcal{W}} \mathbb{E}_{(x,y) \sim \mathcal{P}} \ell_f(m(x, y, w))
\]

To make the application of mirror descent possible, the underlying function has to be convex, the fact that the distribution \( \mathcal{P} \) is unknown leads to \( f(x, y, w) \) needing to be linear in \( w \), so \( \mathcal{F} = \{\langle x, w^k \rangle \mid w \in \mathcal{W}\} \) further in the paper. The case can also be generalized to PSD-kernel classification via linear classifiers in RKHS (Mohri, Rostamizadeh, and Talwalkar 2012).

3 Oracle inequalities

Mirror descent algorithm is similar to stochastic gradient descent, except that it ensures gradient steps to be made in \( E^* \) by mapping there with \( \nabla \psi \), where \( \psi: E \to \mathbb{R} \) is a strongly convex function with gradient field continuous on \( \mathcal{W} \): \( \psi(w^1) - \psi(w^2) - \langle \nabla \psi(w^2), w^1 - w^2 \rangle \geq \frac{\lambda}{2} \|w^1 - w^2\|^2 \). As the number of classes is a factor of the dimension of \( \mathcal{W} \), choosing the right norm to measure the set diameter can effectively lower the error rate.

MD steps are gradient steps with Bregman divergence of \( \psi \) in the role of the norm:

\[
w^1 = \arg \min_{w \in \mathcal{W}} \psi(w)
\]

\[
w^{m+1} = \arg \min_{w \in \mathcal{W}} \{\Delta(w, w^m) + \alpha_m \langle F'(w^m), w - w^m \rangle\},
\]

\[
\Delta(w^1, w^2) = \psi(w^1) - \psi(w^2) - \langle \nabla \psi(w^2), w^1 - w^2 \rangle \geq \frac{\lambda}{2} \|w^1 - w^2\|^2.
\]

In case of expectation minimization gradients are taken at random points \( g_k \in \partial \ell(x_k, y_k, w^k) \), which ensures \( \mathbb{E}g^k \in \partial F(w^k) \) as long as \( (x_k, y_k) \) and \( w^k \) are independent.

**Lemma 1.** (Nemirovski 2004) For all \( w \in \mathcal{W} \)

\[
\Delta(w, w^{m+1}) \leq \alpha_m \langle g^m, w - w^{m+1} \rangle + \Delta(w, w^m) - \Delta(w^{m+1}, w^m)
\]

**Proof.** Set \( h(w) = \Delta(w, w^m) + \alpha_m \langle g^m, w - w^m \rangle \), then \( w^{m+1} = \arg \min_{w \in \mathcal{W}} h(w) \).

Optimality of \( w^{m+1} \) leads to \( \langle h'(w^{m+1}), w - w^{m+1} \rangle \geq 0 \).

As long as \( h'(w^{m+1}) = \nabla \psi(w^{m+1}) - \nabla \psi(w^m) + \alpha_m g^m \), we can rearrange the terms and get

\[
0 \leq \langle \alpha_m g^m + \nabla \psi(w^{m+1}) - \nabla \psi(w^m), w - w^{m+1} \rangle
\]

\[
= \langle \alpha_m g^m, w - w^{m+1} \rangle - \langle \nabla \psi(w^m), w - w^m \rangle
\]

\[
+ \langle \nabla \psi(w^{m+1}), w - w^{m+1} \rangle + \langle \nabla \psi(w^m), w^{m+1} - w^m \rangle
\]

\[
= \langle \alpha_m g^m, w - w^{m+1} \rangle - \Delta(w, w^{m+1}) + \Delta(w, w^m) - \Delta(w^{m+1}, w^m)
\]

\[\]
Lemma 1 and the fact that $x \mathbb{E}g_k \in \partial F(w^k)$ result in an oracle inequality for stochastic MD.

**Corollary 1.** For $U^2 = \arg\max_{w, w' \in \mathcal{W}} (\psi(w) - \psi(w'))$, $G^2 = \arg\max_{w \in \mathcal{W}} \mathbb{E}\|g(x, y, w)\|^2$, with $g(x, y, w) \in \partial \ell(x, y, w)$ and for any $w \in \mathcal{W}$ and $w^{(n)} = \frac{\sum_{m=0}^{n} \alpha_m w^m}{\sum_{m=1}^{n} \alpha_m}$

$$
\mathbb{E} \left( \ell(x, y, w^{(n)}) - \ell(x, y, w) \right) \leq \frac{U^2 + G^2 \sum_{m=1}^{n} \alpha_m^2 / 2}{\sum_{m=1}^{n} \alpha_m} \quad (2)
$$

**Proof.** According to Lemma 1 and by the strong convexity of $\psi(w)$:

$$
\Delta(w, w^{m+1}) \leq \alpha_m \langle g^m, w - w^{m+1} \rangle + \Delta(w, w^m) - \Delta(w^{m+1}, w^m) \\
\leq \langle \alpha_m g^m, w - w^m \rangle + \langle \alpha_m g^m, w - w^{m+1} \rangle + \Delta(w, w^m) - \frac{1}{2} \|w^{m+1} - w^m\|^2 \\
\leq \langle \alpha_m g^m, w - w^m \rangle + \alpha_m \|g^m\| \|w^m - w^{m+1}\| + \Delta(w, w^m) - \frac{1}{2} \|w^{m+1} - w^m\|^2
$$

Summing over $m = 1, \ldots, n$ gives:

$$
0 \leq \sum_{m=0}^{n} \langle \alpha_m g^m, w - w^m \rangle + \frac{1}{2} \sum_{m=1}^{n} \alpha_m^2 \|g_m\|^2 + \Delta(w, w^1) \quad (3)
$$

As $w^1 = \arg\min_{w \in \mathcal{W}} \psi(w)$:

$$
\Delta(w, w^1) = \psi(w) - \psi(w^1) - \langle \psi'(w^1), w^1 - w \rangle \leq \psi(w) - \psi(w^1) \leq U^2,
$$

By convexity of $\ell$ and the independence of $(x_m, y_m)$ and $w^m$:

$$
\sum_{m=0}^{n} \alpha_m (\ell(x_m, y_m, w^m) - \ell(x_m, y_m, w)) \leq \frac{1}{2} \sum_{m=1}^{n} \alpha_m^2 \|g_m\|^2 + U^2 \\
\sum_{m=1}^{n} \alpha_m (F(w^{(n)}) - F(w)) \leq \sum_{m=1}^{n} \alpha_m (\mathbb{E}(\ell(x, y, w^m) - \ell(x, y, w))) \leq \frac{1}{2} \sum_{m=1}^{n} \alpha_m^2 G^2 + U^2 \quad \square
$$

If steps are constant $\alpha_m = \alpha = \frac{\sqrt{3}U}{\sqrt{n}}$ we get

$$
F(w^{(n)}) - F(w^*) \leq \frac{\sqrt{3}UG}{\sqrt{n}}
$$

The dependence in the number of classes in the excess risk bound is hidden in constants $U$ and $G$ and depends on the choice of distance-generating function $\psi(w)$ and the sets $\mathcal{W}, \mathcal{X}$.

### 4 Probability inequalities

An additional assumption on the exponential moments can be used to get probability estimate of the large deviations.

**Corollary 2.** If there exists $\sigma > 0$ s.t. for all $w \in \mathcal{W}$ : $\mathbb{E}_{(x,y)} e^{d(x_i, y_i, w)/\sigma^2} \leq e^1$, $d(x_i, y_i, w) = \|g(x_i, y_i, w) - \mathbb{E}_{(x,y)} g(x, y, w)\|$, then $\forall \theta > 0$, $g = \max_{w \in \mathcal{W}} \|\mathbb{E}f(x, y, w)\|_*$ :

$$
\mathbb{P} \left\{ \left| \mathbb{E}_{(x,y)} \left[ \ell(x, y, w^{(n)}) - \ell(x, y, w^*) \right] \right| > \sum_{m=1}^{n} \alpha_m \gamma_m g^2 + \frac{U^2}{\sum_{m=1}^{n} \alpha_m} \right\} + \theta \left( \sqrt{\frac{U\sigma^2}{\sum_{m=1}^{n} \gamma_m^2} + \sum_{m=1}^{n} \alpha_m \gamma_m \sigma^2} \right) \leq e^{1-\theta} + e^{-\theta^2/4}
$$
Proof. To prove the bound we use Chernoff’s inequality and (3). Denote $\gamma_m = \frac{\alpha m}{\sum_{m=1}^{\alpha m}}, s_m = \langle E_{(x,y)} g(x, y, w^m) - g(x_m, y_m, w^m), w^m - w^* \rangle$. Note that $E s_m / \sigma^2 = 0$, 

$E_{(x,y)}^e s_m / (\sigma^2 \sqrt{2} U) \leq E_{(x,y)}^e d(x, y, w, w^m) / \sigma^2 \leq e^1$.

Consider a random variable $y : Ey = 0, E e y^2 \leq e$. As $e y \leq y + e^{2y/3}$, for $0 < \alpha^2 < 2/3$ by Jensen’s inequality $E e y^2 \leq E e^{2y/3} \leq e^{2y/3} \leq e^2$. As $E e^\gamma y \leq e^{1+\alpha^2/4}, E e^\alpha x \leq e^2$ for $\alpha > 0$. We apply the inequality to $y^2 = s_m^2 / (2 \sqrt{2} U)$:

$$E \exp \left( \alpha \sum_{m=1}^{n} \gamma_m s_m \right) \leq E \left[ \exp \left( \alpha \sum_{m=1}^{n-1} \gamma_m s_m \right) E_{(x,y)} \exp \left( \alpha \gamma_m s_n \right) \right] \leq$$

$$E \left[ \exp \alpha \sum_{m=1}^{n-1} \gamma_m s_m \exp 8 \alpha^2 \gamma_n^2 U^2 \sigma^4 \right] \leq e^{2 \sqrt{2} U \alpha^2 \sum_{m=1}^{n} \gamma_m^2}$$

Then $P \{ \sum_{m=1}^{n} \gamma_m s_m > \theta \} \leq e^{2 \sqrt{2} U \alpha^2 \sum_{m=1}^{n} \gamma_m^2 - \alpha \theta}$ and for $\alpha = \frac{\theta}{4 \sqrt{2} U \sigma^2 \sum_{m=1}^{n} \gamma_m^2}$:

$$P \left\{ \sum_{m=1}^{n} \gamma_m s_m > \theta \right\} \leq e^{-\frac{\theta^2}{8 \sqrt{2} U}}$$

(4)

Now bound the $E \exp \left( \sum_{m=1}^{n} \alpha_m \gamma_m d(x_m, y_m, w^m)^2 \right)$:

$$E \exp \left( \sum_{m=1}^{n} \alpha_m \gamma_m d(x_m, y_m, w^m)^2 \right) \leq e$$

By the convexity of $E e^x$ in $x$:

$$E \exp \left( \sum_{m=1}^{n} \alpha_m \gamma_m d(x_m, y_m, w^m)^2 \right) \leq e$$

(5)

Use (3) to bound the $E_{(x,y)} \left[ \ell(x, y, w(n)) - \ell(x, y, w^*) \right]$:

$$E_{(x,y)} \left[ \ell(x, y, w(n)) - \ell(x, y, w^*) \right] \leq \sum_{m=1}^{n} \gamma_m \langle E_{(x,y)} g(x, y, w^m), w^m - w^* \rangle \leq$$

$$\sum_{m=1}^{n} \gamma_m \langle E_{(x,y)} g(x, y, w^m) - g(x_m, y_m, w^m), w^m - w^* \rangle +$$

$$\sum_{m=1}^{n} \gamma_m \langle g(x_m, y_m, w^m), w^m - w^* \rangle \leq \sum_{m=1}^{n} \gamma_m s_m + \sum_{m=1}^{n} \alpha_m \gamma_m (g^2 + d(x_m, y_m, w^m)^2) +$$

$$+ \frac{U^2}{\sum_{m=1}^{n} \alpha_m}$$

Combining (4) and (5) we get:

$$P \left\{ E_{(x,y)} \left[ \ell(x, y, w(n)) - \ell(x, y, w^*) \right] > \sum_{m=1}^{n} \alpha_m \gamma_m g^2 + \frac{U^2}{\sum_{m=1}^{n} \alpha_m} + \right\} \leq e^{-\theta} + e^{-\theta^2/4}$$

(6)

Proof for the constant stepsize policy $\alpha = \frac{\sqrt{2U}}{G \sqrt{n}}$:

$$P \left\{ E_{(x,y)} \left[ \ell(x, y, w(n)) - \ell(x, y, w^*) \right] > \frac{3UG}{\sqrt{2n}} + \theta \left( \frac{U^2}{n} + \frac{\sqrt{2U} \sigma^2}{G \sqrt{n}} \right) \right\} \leq e^{-\theta} + e^{-\theta^2/4}$$

Examples
1. Consider $W = \{ w \in \mathbb{R}^{d \times k} \mid \max_i \|w_i\|_2 \leq \Omega \}, \mathcal{X} = \{ x \in \mathbb{R}^d \mid \|x\|_2 < X \}$ and $\psi(w) = \frac{1}{2} \sum_{i=1}^{k} \|w_i\|_2^2$. In this case $\Delta(w^1, w^2) = \frac{1}{2} \sum_{i=1}^{k} \|w^1_i - w^2_i\|_2^2$ and $U^2 = k\Omega^2, G^2 = \frac{2\sum_i \|w_i\|_2^2}{\rho}$. This leads to excess risk rate

$$F(w^{(n)}) - F(w^*) \leq \frac{2\Omega X}{\rho} \sqrt{\frac{k}{n}}$$

$$P \left( \mathbb{E}_{(x,y)} \left[ \ell(x, y, w^{(n)}) - \ell(x, y, w^*) \right] > \frac{3X\Omega}{\rho} \sqrt{\frac{k}{n}} + \theta \left( \frac{k\Omega^2}{n} + \frac{\Omega^2\sigma^2}{X} \sqrt{\frac{k}{n}} \right) \right) \leq e^{1-\theta} + e^{-\theta^2/4}$$

2. If the margins are allowed to be different for different classes: $W = \{ w \in \mathbb{R}^{d \times k} \mid \sum_{i=1}^{k} \|w_i\|_2 \leq \Omega \}$ and $\psi(w)$ is chosen to be strongly convex w.r.t. $\ell_1/\ell_2$ norm: $\psi(w) = e^{\frac{\rho}{1+4/k} \sum_{i=1}^{k} \|w^i\|_2} \text{[Juditsky, Nemirovski, et al. 2011]}, which is the case of favorable geometry, then $U^2 = e \ln k\Omega, G = X/\rho$ and the rate can be pushed to

$$F(w^{(n)}) - F(w^*) \leq \frac{X}{\rho} \sqrt{\frac{2e \Omega \ln(k)}{\sqrt{n}}}$$

$$P \left( \mathbb{E}_{(x,y)} \left[ \ell(x, y, w^{(n)}) - \ell(x, y, w^*) \right] > \frac{3X}{\rho} \sqrt{\frac{e \ln k\Omega}{2n}} + \theta \left( \frac{\sqrt{e \ln k\Omega^2}}{n} + \sqrt{\frac{2e \ln k\Omega^2 \sigma^2}{X}} \right) \right) \leq e^{1-\theta} + e^{-\theta^2/4}$$

5 Class probability

We further focus on the Euclidean setup $W = \{ w \in \mathbb{R}^{d \times k} \mid \max_i \|w_i\|_2 \leq \Omega \}, \mathcal{X} = \{ x \in \mathbb{R}^d \mid \|x\|_2 < X \}$ and $\psi(w) = \frac{1}{2} \sum_{i=1}^{k} \|w_i\|_2^2$ as the most common one. First, let’s assume that prior class probabilities are known $p(i) = \mathbb{P}\{y = i\}$. In such a case the objective function can be modified to choose more probable classes more frequently — with weights $\alpha(y) = \hat{\alpha}(p(y)), \|\alpha(y)\|_\infty = 1$:

$$y_{\text{pred}} = \arg \max_{y \in \mathcal{Y}} \alpha(y)(x, w_y).$$

The convex loss function can then be

$$\ell(x, y, w) = \max \left\{ 0; 1 - 1/\rho(\langle x, y, w \rangle - \alpha(y)/\alpha(y')(x, w_y)) \right\}$$

This turns into the loss described in the previous section in case of uniformly distributed classes: $p(i) = p(j)$. Set the distance-generating function to be a weighted sum of functions for each class: $\omega(w) = \sum \gamma(y)\omega_y(w_y)$, then $\Delta(w^1, w^2) = \sum_{y=1}^{k} \gamma(y)\Delta_{\omega_y}(w^1_y, w^2_y)$.

Let $p'(y, y')$ be the probability of choosing $y'$ as the closest class to the object apart from its own class and $y$ to be its class:

$$p'(y, y') = \mathbb{P}_{(x,y)} \{ y' = \arg \max_{y \in \mathcal{Y}\setminus\{y\}} \alpha(y')/\alpha(y)(x, w_y) \} = \mathbb{P}\{A(y, y')\}$$
The upper bound for each step:
\[
\mathbb{E} \Delta(w, w^{m+1}) \leq \alpha_m (F(w) - F(w^m)) + \mathbb{E} \Delta(w, w^m) + \alpha_m \mathbb{E} \langle g^m, w^m - w^{m+1} \rangle \\
- \frac{1}{2} \mathbb{E} \sum_{i=1,k} \gamma(i) \|w^{m+1} - w^m\|^2 = \alpha_m (F(w) - F(w^m)) + \mathbb{E} \Delta(w, w^m) \\
+ \frac{\alpha_m}{\rho} \sum_{y,y'=1}^k \mathbb{E} \left[ \frac{\left| \langle x_m, w^m_y - w^m_{y'} \rangle \right|}{\alpha(y)} \left| \langle x_m, w^m_y - w^m_{y'} \rangle \right| A(y, y') \right] p'(y, y') \\
- \frac{1}{2} \mathbb{E} \left[ \sum_{i=1}^k \gamma(i) \|w^{m+1}_i - w^m_i\|^2 \right] A(y, y') p'(y, y') \\
\leq \alpha_m (F(w) - F(w^m)) + \mathbb{E} \Delta(w, w^m) + \frac{\alpha_m^2 \Omega^2}{2 \rho^2} \sum_{y,y'=1}^k \left( \frac{p'(y, y')}{\gamma(y)} + \frac{\alpha(y)^2 p'(y, y')}{\alpha(y)^2 \gamma(y')} \right)
\]
To bound the last term notice that \( y' \) is a determined function of \((x, y) \) and \( w \):
\[
p'(y, y') = \mathbb{E}_{x,y \in [m]} p(y) \left[ y' = \arg \max_{y'' \in \mathcal{Y} \setminus \{y\}} (\alpha(y'')) \langle x, w_{y''} \rangle \right],
\]
where \( \mathbb{E}_{x,y \in [m]} \) denotes the conditional expectation of \( x \) and \( w \) with respect to \( y \)
\[
A = \sum_{y,y'=1}^k \left( \frac{p'(y, y')}{\gamma(y)} + \frac{\alpha(y)^2 p'(y, y')}{\alpha(y)^2 \gamma(y')} \right) = \sum_{y=1}^k \frac{p(y)}{\gamma(y)} + \sum_{y=1}^k \frac{p(y)}{\alpha(y)^2 \gamma(y)}
\]
Setting \( \alpha(y)^2 = \gamma(y) = \sqrt{p(y)} \), \( B = \sum_{y=1}^k \sqrt{p(y)} \) and summing over \( m = 1, n \) we get:
\[
F(w^{(m)}) - F(w^*) \leq \frac{B \Omega^2}{2} + \frac{2X^2 B}{2 \rho^2} \sum_{m=1}^n \alpha_m^2
\]
If \( \alpha_m = \frac{\Omega \rho}{X \sqrt{2m}} \):
\[
F(w^{(m)}) - F(w^*) \leq \frac{\Omega X \sqrt{2}}{\rho \sqrt{n}} \sum_{y=1}^k \sqrt{p(y)}
\]
\[\sum_{i=1}^k \sqrt{p(i)} \leq \sqrt{k}, \] so the result is not worse than the previous one, but in case of highly unbalanced classes, e. g. power log law \( p(i) = Ci^{-\beta}, i = 1, k, \beta > 2 \)
\[\sum_{i=1}^k \sqrt{p(i)} = \sum_{i=1}^k \frac{1 - i^{-\beta/2}}{\sqrt{\sum_{i=1}^k i^{-\beta}}} = O(1)
\]
As long as the exact prior probabilities are usually not known exactly, we will assume that they are known up to some constant and the estimate is independent of the sample \((x_i, y_i)_{i=1}^n \): \( p(y) \leq (1 + \epsilon) \hat{p}(y) \)
Again, setting \( \alpha(y)^2 = \gamma(y) = \sqrt{p(y)} \) we get the estimate
\[A \leq 2 \sum_{y=1}^k \frac{p(y)}{\gamma(y)} \leq 2(1 + \epsilon) \sum_{y=1}^k \sqrt{p(y)}
\]
\[
F(w^{(m)}) - F(w^*) \leq \frac{\Omega X \sqrt{2(1 + \epsilon)}}{\rho \sqrt{n}} \sum_{y=1}^k \sqrt{p(y)}
\]
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