A note about combinatorial sequences and Incomplete Gamma function

H. Bergeron\textsuperscript{a}, E.M.F. Curado\textsuperscript{b,c}, J. P. Gazeau\textsuperscript{b,d}, Ligia M.C.S. Rodrigues\textsuperscript{b}\textsuperscript{(*)}

\textsuperscript{a} Univ Paris-Sud, ISMO, UMR 8214, 91405 Orsay, France
\textsuperscript{b} Centro Brasileiro de Pesquisas Fisicas
\textsuperscript{c} Instituto Nacional de Ciência e Tecnologia - Sistemas Complexos
Rua Xavier Sigaud 150, 22290-180 - Rio de Janeiro, RJ, Brazil
\textsuperscript{d} APC, UMR 7164,
Univ Paris Diderot, Sorbonne Paris Cité, 75205 Paris, France

May 22, 2014

Abstract
In this short note we present a set of interesting and useful properties of a one-parameter family of sequences including factorial and subfactorial, and their relations to the Gamma function and the incomplete Gamma function.

Contents
1 Introduction 1
2 The function $T_n(x, y)$ 2
3 The function $T_n(z)$ 2
4 The relation to the incomplete Gamma function 4
5 Comment 5

1 Introduction
The purpose of this note is to show a set of properties of a one-parameter family of sequences $T_n(z)$, $z$ is a complex variable, which are apparently not known in the literature. Among these sequences we have factorials and

\textsuperscript{*}e-mail: herve.bergeron@u-psud.fr, evaldo@cbpf.br, gazeau@apc.univ-paris7.fr, ligia@cbpf.br
subfactorials and they are seen to be related to the incomplete Gamma function. This family was introduced in a recent paper about generalized binomial distribution [1].

2 The function \( T_n(x, y) \)

Let us consider the following two-variable polynomial:

\[
T_n(x, y) := \sum_{m=0}^{n} \binom{n}{m} (m + x)^m (n - m + y)^{n-m},
\]

which is clearly symmetrical in \( x, y \). Actually, we have the stronger result:

**Proposition 2.1.** The polynomial \( T_n(x, y) \) depends on the sum \( z = x + y \) only:

\[
T_n(x, z - x) = \sum_{m=0}^{n} \binom{n}{m} (m + x)^m (n - m + z - x)^{n-m} \equiv T_n(z),
\]

where \( x \) can be arbitrarily chosen with the convention that \( 0^0 \equiv 1 \) in the cases \( x = 0 \) or \( x = z \).

**Proof.** Let us prove by recurrence on \( n \) that \( T_n(x, z - x) \) does not depend explicitly on \( x \) for all \( z \). For that, we take the partial derivative of \( T_n(x, z - x) \) with respect to \( x \). We have successively, by appropriately shift the summation variable,

\[
\frac{\partial}{\partial x} T_n(x, z - x) = \sum_{m=0}^{n} \binom{n}{m} \left[ m(m + x)^{m-1} (n - m + z - x)^{n-m} \right.
\]

\[
- (n - m)(m + x)^m (n - m + z - x)^{n-m-1} \left. + n \sum_{m=0}^{n-1} \binom{n-1}{m} (m + 1 + x)^{m} (n - 1 - m + z - x)^{n-1-m} \right]
\]

\[
- n \sum_{m=0}^{n-1} \binom{n-1}{m} (m + x)^{m} (n - 1 - m + z + 1 - x)^{n-1-m}
\]

\[
= n \left[ T_{n-1}(1 + x, z - x) - T_{n-1}(x, z + 1 - x) \right].
\]

Now, we note that \( T_0(x, z - x) = 1 \) and so does not depend explicitly on \( x \). Suppose that the property "\( T_k(x, z - x) \) does not depend explicitly on \( x \) for all \( z \)" holds true for all integer \( 1 \leq k \leq n - 1 \). Then \( T_{n-1}(1 + x, z - x) = T_{n-1}(x, z + 1 - x) \) and this implies that \( \partial/\partial x T_n(x, z - x) = 0 \), i.e., \( T_n(x, z - x) \) does not depend explicitly on \( x \) for all \( z \). \( \square \)

3 The function \( T_n(z) \)

Thanks to Proposition 2.1, it is now possible to find a convenient alternate expression of \( T_n(x, z - x) \equiv T_n(z) \). First, let us establish a recurrence formula.
Proposition 3.1. The polynomial $T_n(z)$ satisfies the following recurrence formula:
\begin{align*}
T_n(z) &= (z + n)^n + n T_{n-1}(z + 1), \quad T_0(z) = 1. \tag{3}
\end{align*}

Proof. Taking $x = 0$ in (2), let us write that expression as
\begin{align*}
T_n(z) &= \sum_{m=0}^{n} \binom{n}{m} (m)^n (n - m + z)^{n-m} \\
&= (n + z)^n + n \sum_{m=1}^{n} \binom{n-1}{m-1} \times \\
&\quad \times (m - 1 + 1)^{m-1} (n - 1 - (m - 1) + z + 1 - 1)^{n-1-(m-1)} \\
&= (n + z)^n + n T_{n-1}(1, z + 1 - 1) = (n + z)^n + n T_{n-1}(z + 1),
\end{align*}
where we have applied Proposition 2.1 with $x = 1$. □

It is then straightforward to deduce from this formula the following result.

Proposition 3.2. The polynomial $T_n(z)$ admits the following expansion in powers of $(z + n)$:
\begin{align*}
T_n(z) &= \sum_{k=0}^{n} \frac{n!}{k!} (z + n)^k. \tag{4}
\end{align*}

We note in particular the interesting corollary proving that the family $(T_n(z))_{z \in \mathbb{Z}}$ of integer sequences includes the factorial:
\begin{align*}
T_n(-n) &= n!, \tag{5}
\end{align*}
and consequently also the Gamma function as
\begin{align*}
T_n(-n) &= \Gamma(n + 1). \tag{6}
\end{align*}

The expression (4) allows to easily derive two asymptotic behaviors of $T_n(z)$:
\begin{align*}
\text{At large } z \quad &T_n(z) \sim z^n, \tag{7} \\
\text{At large } n \quad &T_n(z) \sim n^n. \tag{8}
\end{align*}

Indeed, in the latter the dominant term of the sum is for $k = n$ (the term $k = 0$ is not dominant due to the decreasing exponential factor in the Stirling’s formula $n! \sim n^n \sqrt{2\pi n e^{-n}}$). This indicates that the dominant term of the sum in (4) is $(z + n)^n$.

A second expression of $T_n(z)$ is also of interest.

Proposition 3.3. The polynomial $T_n(z)$ admits the following expansion in powers of $(z + n + 1)$:
\begin{align*}
T_n(z) &= \sum_{k=0}^{n} a_k (z + n + 1)^k, \tag{9}
\end{align*}
where the coefficients $a_k$ are given by
\begin{align*}
a_k &= \binom{n}{k} T_{n-k}(k-n-1) = \binom{n}{k} d_{n-k}. \tag{10}
\end{align*}
Proof. Applying formula
\[ a_k = \frac{1}{k!} \frac{d^k}{dz^k} T_n(z) \big|_{z = -n-1} \]
to the original expression where we put \( x = 0 \), we easily derive
\[
\frac{1}{k!} \frac{d^k}{dz^k} T_n(z) \big|_{z = -n-1} = \sum_{m=0}^{n-k} (-1)^{n-m-k} \binom{n}{m} \binom{n-m}{k} m^m (1+m)^{n-m-k} \\
= \binom{n}{k} \sum_{m=0}^{n-k} (-1)^{n-m-k} \binom{n-k}{m} m^m (1+m)^{n-m-k} \\
= \binom{n}{k} T_{n-k}(k-n-1).
\]

The sequence of numbers \( (d_n \equiv T_n(-n-1))_{n \in \mathbb{N}} \), for which (4) gives
\[
d_n = \sum_{k=0}^{n} \frac{n!}{k!} (-1)^k,
\]
and whose first terms are 1, 0, 1, 2, 9, 44, ..., is well known for more than 3 centuries [4, 5, 6, 7, 8]. Its OEIS name is A000166. Its numbers are named subfactorial (and then denoted as \( !n \)) or rencontres numbers, or derangements, since \( d_n \) is the number of permutations of \( n \) elements with no fixed points. They obey the recurrence relations (Euler)
\[
d_n = (n-1)d_{n-1} + (-1)^n \\
d_n = n(d_{n-1} + d_{n-2}).
\]
Their generating function is
\[
D(x) = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!} = \frac{e^{-x}}{1-x},
\]
and their asymptotic behavior at large \( n \) is
\[
\lim_{n \to \infty} \frac{d_n}{n!} = \frac{1}{e}.
\]

4 The relation to the incomplete Gamma function

Let us finally establish the relation between the polynomial \( T_n(z) \) and the incomplete Gamma function [9].

Proposition 4.1. The polynomial \( T_n(z) \) is related to the incomplete Gamma function
\[
\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} \, dt, \quad \text{Re}(a) > 0, \tag{12}
\]
as follows:
\[
T_n(z) = e^{z+n} \Gamma(n+1, z+n). \tag{13}
\]

Note that in the particular case that \( z = -n \), from (13) we reobtain (6).
Proof. It is straightforward to prove, by integration by part, the recurrence formula obeyed by the incomplete Gamma function:

\[ \Gamma(a, x) = e^{-x} x^{a-1} + (a-1) \Gamma(a-1, x) \iff e^{x} \Gamma(a, x) = x^{a-1} + (a-1) e^{x} \Gamma(a-1, x). \]

Applied to polynomial \( T_n(z) \) this formula gives

\[ T_n(z) = e^{x} \Gamma(n+1, z+n) = (z+n)^{n} + n e^{z+n} \Gamma(n, z+1+n-1) \]

\[ = (z+n)^{n} + n T_{n-1}(z+1), \]
and this is precisely (3) with the same initial condition \( T_0(z) = e^{z} \Gamma(1, z) = 1. \)

\[ \Box \]

5 Comment

It is probable that \( T_n(z) \) has a combinatorial interpretation (and alternate simpler expression) for each \( z \in \mathbb{Z} \). For instance the elements of the sequence \( (T_n(1))_{n \in \mathbb{N}} \), whose first terms are 1, 3, 17, 142, 1569,... are the numbers of connected functions on \( n \) labeled nodes, and we also have (OEIS number: A001865, see [8] for references)

\[ T_n(1) = \sum_{k=0}^{n} \frac{n!}{k!} (n+1)^k = e^{n+1} \int_{n+1}^{\infty} x^n e^{-x} \, dx. \]

We can present other examples. For \( z = 2 \) the elements of the sequence \( (T_n(2))_{n \in \mathbb{N}} \), whose first terms are 1, 4, 26, 236, 2760,..., are the numbers of normalized total height of rooted trees with \( n \) nodes (OEIS number: A001863, see [9]). The sequence \( (T_n(3)) \), whose first terms are 1, 5, 37, 366, 4553,..., is A129137: \( T_n(3) \) is the number of trees on 1, 2, 3, ..., \( n \equiv [n] \), rooted at 1, in which 2 is a descendant of 3. And so on.

References

[1] H. Bergeron, E.M.F. Curado, J.P. Gazeau, L. M. C. S. Rodrigues, Symmetric generalized binomial distributions, submitted (2013); arXiv:1308.4863 [math-phys]
[2] W. Magnus, F. Oberhettinger and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, Berlin, 3rd Edition, 1996.
[3] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematical Series-55, 10th Edition, Washington, 1972.
[4] P. R. de Montmort, On the Game of Thirteen (1713), reprinted in Annotated Readings in the History of Statistics, ed. H. A. David and A. W. F. Edwards, Springer-Verlag, 2001, pp. 25-29.
[5] L. Euler, Solution quaestionis curiosae ex doctrina combinationum, Mémoires Académie sciences St. Pétersburg 3 (1809/1810), 57-64; also E738 in his Collected Works, series I, volume 7, pp. 435-440.
[6] L. Comtet, Advanced Combinatorics, Reidel, 1974, p. 182.

[7] J. Desarmenien, Une autre interpretation du nombre de derangements Séminaire Lotharingien de Combinatoire, B08b, 1982, 6 pp. [Formerly : Publ. I.R.M.A. Strasbourg, 1984, 229/S-08 Actes 8e Séminaire Lotharingien, p. 11-16].

[8] The On-Line Encyclopedia of Integer Sequences!, https://oeis.org

[9] J. Riordan and N. J. A. Sloane, Enumeration of rooted trees by total height, *J. Austral. Math. Soc.*, 10 278-282 (1969)