Abstract

A manifestly gauge invariant formulation of 5-dimensional supersymmetric Yang-Mills theories in terms of 4d superfields is derived. It relies on a supersymmetry and gauge-covariant derivative operator in the $x^5$ direction. This formulation allows for a systematic study of higher-derivative operators by combining invariant 4d superfield expressions under the additional restriction of 5d Lorentz symmetry. In cases where the 5d theory is compactified on a gauge-symmetry-breaking orbifold, the formalism can be used for a simple discussion of possible brane operators invariant under the restricted symmetry of the fixed points. This is particularly relevant to recently constructed grand unified theories in higher dimensions (orbifold GUTs). Several applications, including proton decay operators and brane-localized mass terms, are discussed.
1 Introduction

The standard framework for the discussion of physics above the electroweak scale is supersymmetric grand unification. Taking the phenomenological success of traditional gauge coupling unification seriously, the energy range between the GUT scale and the string (or Planck) scale is the natural domain for higher-dimensional field theories. Indeed, starting with the proposal of Kawamura [1], a number of very simple and realistic higher-dimensional GUT models have recently been constructed [2–12]. Of course, numerous other interesting ideas that are based on supersymmetry (SUSY) in higher dimensions exist, for example, the extra-dimensional SUSY breaking scenarios of [13] or the intermediate scale unification models of [14].

Both conceptually and for the discussion of low-energy phenomenology, a 4d superfield description of the higher-dimensional SUSY is desirable. After the early work of [15], this issue has recently been revived in [16, 17]. In the present paper, we develop the formalism of [16, 17] by providing a manifestly gauge-invariant 4d superfield description of the non-abelian 5d theory. Our particularly simple formulation relies on the consistent use of the supersymmetry- and gauge-covariant derivative operator in the $x^5$ direction. Furthermore, we find that, starting from conventional 4d Super Yang-Mills (SYM) theory and introducing $x^5$ (together with the corresponding gauge connection) as an additional parameter, the full 5d supersymmetric theory is unambiguously determined. This approach, which is based on combining invariant 4d superfield expressions under the additional restriction of 5d Lorentz symmetry, can also be used for a systematic study of higher derivative terms in the 5d SYM theory.

The presented gauge covariant formulation allows for a simple discussion of superfield brane operators invariant under the restricted gauge symmetry of orbifold fixed points. This is particularly relevant to the extra-dimensional GUT models mentioned earlier, where the GUT symmetry is broken to the standard model (SM) gauge group by the boundary conditions of an orbifold compactification. More specifically, SU(5) models in 5 dimensions [1, 3, 5] and SO(10) and E$_6$ models in 6 dimensions [7, 8, 10, 11] have recently been constructed. At first sight, possible couplings in theories of this type are severely restricted. On the one hand, the large gauge and supersymmetry of the bulk excludes many couplings. On the other hand, half of the bulk fields are odd under the discrete symmetry defining the orbifold and therefore vanish at the boundary (or brane), where more couplings are allowed. Furthermore, certain degrees of freedom are defined as brane fields and can therefore not participate in bulk interactions. Superfield brane operators lift many of these restrictions since, due to the presence of the $x^5$ derivative operator, bulk fields that are odd under the discrete symmetry (i.e., vanish at the boundary but have non-zero derivative) can participate in brane-localized interactions. We briefly survey the relevance of these operators to 5d SUSY GUTs, emphasizing, in particular, proton decay operators and brane-localized mass terms.

The paper is organized as follows. After defining the 5d SYM theory in Sect. 2, we explicitly derive its 4d superfield formulation in Sect. 3. Here, our main result is the simple and manifestly gauge invariant formulation based on the covariant derivative in the $x^5$ direction (Eqs. (21)–(24)). Section 4 describes the bulk hypermultiplet while
Sect. 5 outlines the classification of brane operators using the now available gauge covariant formalism. Applications to orbifold GUTs are discussed in Sect. 6, followed by the conclusions in Sect. 7.

2 Gauge multiplet in 5 dimensions

We begin by describing the $N = 1$ ($8$ supercharges) 5d gauge multiplet \[ \text{cf. also} \] using conventions which are as close as possible to $\text{[20]}$. This will make the following transition to 4d superfields particularly simple.

Capitalized indices $M, N, ..$ run over $0, 1, 2, 3, 5$; lower case indices $m, n, ..$ run over $0, 1, 2, 3$. The metric is $\eta_{MN} = \text{diag}(-1, 1, 1, 1, 1)$ and the Dirac matrices can be chosen as

$$
\gamma^M = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 1, \vec{\sigma} \end{pmatrix}, \begin{pmatrix} 1, -\vec{\sigma} \end{pmatrix},
$$

where $\sigma^m = (1, \vec{\sigma})$ and $\bar{\sigma}^m = (1, -\vec{\sigma})$. It is convenient to use symplectic Majorana spinors $\psi^i$, where $i = 1, 2$ transforms under an $SU(2)$ $R$-symmetry. The reality condition reads

$$
\psi^i = \varepsilon^{ij} C \bar{\psi}_j^T,
$$

where the 5d charge conjugation matrix $C$ satisfies $C\gamma^M C^{-1} = (\gamma^M)^T$. We use the explicit form $C = \text{diag}(i\sigma^2, i\sigma^2)$. Note that lower indices $i, j, ..$ transform under the $\bar{2}$ of $SU(2)$-$R$ and the $\varepsilon$ tensor ($\varepsilon^{12} = \varepsilon_{21} = 1$) can be used to raise or lower indices.

The 5d gauge multiplet contains a vector $A^M$, a real scalar $\Sigma$, and an $SU(2)$-$R$ doublet of gauginos $\lambda^i$. Furthermore, one requires three real auxiliary fields $X^a$, which form a triplet of $SU(2)$-$R$. These fields are all in the adjoint representation of the gauge group. The SUSY parameter is a symplectic Majorana spinor $\xi^i$, and the transformation laws are given by

$$
\delta_\xi A^M = i\bar{\xi}_i \gamma^M \lambda^i, \quad \delta_\xi \Sigma = i\bar{\xi}_i \lambda^i, \quad \delta_\xi \lambda^i = \left(\gamma^{MN} F_{MN} + \gamma^M D_M \Sigma\right) \xi^i + i (X^a \sigma^a)_j \xi^j, \quad \delta_\xi X^a = \bar{\xi}_i (\sigma^a)_j^i \gamma^M D_M \lambda^j + i \left[\Sigma, \bar{\xi}_i (\sigma^a)_j^i \lambda^j\right],
$$

where $\gamma^{MN} = \frac{1}{2} [\gamma^M, \gamma^N]$ and $D_M = \partial_M + iA_M$, with appropriate adjoint action of $A_M$ implied. The 5d lagrangian, invariant under this SUSY, reads

$$
\mathcal{L} = \frac{1}{g^2} \left(-\frac{1}{2} \text{tr}(F_{MN})^2 - \text{tr}(D_M \Sigma)^2 - \text{tr}\left(\lambda_i i \gamma^M D_M \lambda^i\right) + \text{tr}(X^a)^2 + \text{tr}(\bar{\lambda}_i [\Sigma, \lambda^i])\right). \quad (7)
$$

1 The sign of the last term in Eq. (7) differs from [18] due to our opposite sign choice of $A_M$ in the covariant derivative $D_M$. The signs of the last two terms in Eq. (7) appear to genuinely disagree with [18]. The consistency of the present equations is most easily confirmed by checking the 4d part of this 5d SUSY, which is worked out explicitly below.
3 Formulation in terms of 4d superfields

Orbifold compactifications of the fifth dimension break at least half of the $N = 1$ 5d SUSY (which corresponds to $N = 2$ SUSY from the 4d perspective). This is obvious since the full set of 5d SUSY transformations generates translations in the $x^5$ direction, and the latter are not a symmetry of the orbifold. To make the surviving $N = 1$ 4d SUSY manifest, it is convenient to consider the decomposition of a 5d symplectic Majorana spinor $\psi^i$ into its components (two 4d Weyl spinors $\psi_L$ and $\psi_R$) under the 4d Lorentz group. It reads

$$\psi^1 = \left( \begin{array}{c} (\psi^*_L)_{\alpha} \\ (\psi^*_R)_{\dot{\alpha}} \end{array} \right), \quad \psi^2 = \left( \begin{array}{c} (\psi^*_R)_{\alpha} \\ -(\psi^*_L)_{\dot{\alpha}} \end{array} \right), \quad \bar{\psi}_1 = \left( \begin{array}{c} (\psi^*_L)_{\alpha} \\ (\psi^*_R)_{\dot{\alpha}} \end{array} \right)^T, \quad \bar{\psi}_2 = \left( \begin{array}{c} -(\psi^*_L)_{\dot{\alpha}} \\ (\psi^*_R)_{\alpha} \end{array} \right)^T.$$ (8)

One can now easily work out the 4d Weyl spinor formulation of Eqs. (3)–(7). We assume that the surviving 4d SUSY is generated by a set of parameters $\xi^i$ defined by the Weyl spinor $\xi_L$, with $\xi_R = 0$. For convenience, we explicitly give the transformation rules of the component fields under this smaller SUSY, using 4d Weyl spinors:

$$\delta_{\xi_L} A^m = i\bar{\xi}_L \sigma^m \lambda_L + i\xi_L \sigma^m \bar{\lambda}_L$$ (9)

$$\delta_{\xi_L} A^5 = -\bar{\xi}_L \bar{\lambda}_R - \xi_L \lambda_R$$ (10)

$$\delta_{\xi_L} \Sigma = i\bar{\xi}_L \bar{\lambda}_R - i\xi_L \lambda_R$$ (11)

$$\delta_{\xi_L} \lambda_L = \sigma^{mn} F_{mn} \xi_L - iD_5 \Sigma \xi_L + iX^3 \xi_L$$ (12)

$$\delta_{\xi_L} \lambda_R = i\sigma^{5m} F_{5m} \tilde{\xi}_L - \sigma^m D_m \Sigma \xi_L + i(X^1 + iX^2) \xi_L$$ (13)

$$\delta_{\xi_L} (X^1 + iX^2) = 2\xi_L \sigma^m D_m \lambda_L - 2i\bar{\xi}_L D_5 \bar{\lambda}_L + i[\Sigma, 2\xi_L \bar{\lambda}_L]$$ (14)

$$\delta_{\xi_L} X^3 = \bar{\xi}_L \sigma^m D_m \lambda_L + i\xi_L D_5 \bar{\lambda}_L - \xi_L \sigma^m D_m \lambda_L - i\xi_L D_5 \lambda_R$$

$$+ i[\Sigma, (\xi_L \bar{\lambda}_R + \xi_L \lambda_R)],$$ (15)

where $\sigma^{mn} = \frac{1}{2}(\sigma^m \sigma^n - \sigma^n \sigma^m)$.

Now observe [16, 18] that the fields $A_m$, $\lambda_L$ and $(X^3 - D_5 \Sigma)$ transform precisely as the components of a vector superfield in Wess-Zumino (WZ) gauge:

$$V = -\theta \sigma^m \bar{\theta} A_m + i\theta^2 \bar{\theta} \lambda_L - i\bar{\theta}^2 \theta \lambda_R + \frac{1}{2} \theta^2 \bar{\theta}^2 \left( X^3 - D_5 \Sigma \right).$$ (16)

Here we use the conventions of [20] for the action of the SUSY transformation $\delta_{\xi_L} = \xi_L \bar{Q} + \bar{\xi}_L Q$ on $\theta$ and $\bar{\theta}$. Furthermore, in slight deviation from the conventions of [20], we define super-gauge transformations of vector $(V)$ and fundamental-representation chiral $(\Psi)$ superfields by

$$e^{2V} \rightarrow e^{-\Lambda} e^{2V} e^\Lambda \quad \text{and} \quad \Psi \rightarrow e^{-\Lambda} \Psi,$$ (17)

where $\Lambda$ is a chiral superfield depending, in general, on $x^5$. For what follows, it is important to recall that the transformation rules for the WZ-gauge component fields are obtained by the application of $\delta_{\xi_L}$ to $V$, followed by a super gauge transformation that takes $V$ back to WZ gauge. This gauge transformation is specified by

$$\Lambda = \sqrt{2} \theta \left( \sqrt{2} \sigma^m \bar{\xi}_L A_m \right) + \theta^2 \left( -2i\bar{\xi}_L \bar{\lambda}_L \right)$$ (18)
in the y basis (i.e., with the component fields as functions of $x^5$ and $y^m = x^m + i\theta\sigma^m\bar{\theta}$).

The next essential observation is that the fields $(\Sigma + iA_5)$, $(-i\sqrt{2}\lambda_R)$ and $(X^1 + iX^2)$ transform as the components of a chiral adjoint superfield in the y basis,

$$\Phi = (\Sigma + iA_5) + \sqrt{2}\theta\left(-i\sqrt{2}\lambda_R\right) + \theta^2(X^1 + iX^2),$$

if, at the same time, this field is defined to transform as

$$\Phi \rightarrow e^{-\Lambda}(\partial_5 + \Phi)e^\Lambda$$

under super gauge transformations. More precisely, this means that the transformation rules of Eqs. (9)–(15) are reproduced by first applying $\delta_\xi$ to $\Phi$ and then returning to WZ gauge by a gauge transformation, Eq. (20), with $\Lambda$ specified in Eq. (18). This point is essential in deriving the 4d SUSY transformation rules directly from the 5d SUSY.

Given the transformation rule, Eq. (20), for $\Phi$, it is clear that

$$\nabla_5 \equiv \partial_5 + \Phi$$

represents a super gauge covariant derivative in the $x^5$ direction. Thus, given a superfield $O$, which is in some representation of the gauge group and transforms covariantly under super gauge transformations, the superfield $\nabla_5O$ transforms covariantly in the same way. For this it is essential that the Lie-algebra valued field $\Phi$ contained in $\nabla_5$ acts on $O$ as specified by the representation under which $O$ transforms. In particular, for the real superfield

$$\nabla_5e^{2V} = \partial_5e^{2V} - \Phi^\dagger e^{2V} - e^{2V}\Phi$$

the transformation rule is

$$\nabla_5e^{2V} \rightarrow e^{\Lambda^\dagger}\left(\nabla_5e^{2V}\right)e^\Lambda.$$  

The 5d lagrangian can now be written in 4d superfield language by combining the two lowest-dimension invariant operators that can be built from the superfield $V$ and the covariant derivative operator $\nabla_5$:

$$L = \frac{1}{2g^2}\text{tr}\left\{W^{\alpha}\bar{W}_{\alpha}\big|_{\theta^2} + \bar{W}_\alpha W^{\alpha}\big|_{\bar{\theta}^2} + \left(e^{-2V}\nabla_5 e^{2V}\right)^2\big|_{\theta^2\bar{\theta}^2}\right\}.\quad (24)$$

Here $W_\alpha$ is the field-strength superfield constructed from $V$ in the usual way. The above lagrangian reproduces Eq. (4) up to derivative terms. Note that, to achieve this agreement, it is not necessary to integrate out the auxiliary fields.

It will prove convenient to define, by analogy to $W_\alpha$, the Lie-Algebra valued superfield $Z = e^{-2V}\nabla_5e^{2V}$. Now the lagrangian takes the particularly compact form

$$L = \frac{1}{2g^2}\text{tr}\left\{(W^{\alpha}W_\alpha)\big|_{\theta^2} + \text{h.c.}\right\} + Z^2\big|_{\theta^2\bar{\theta}^2}\right\}.\quad (25)$$

The superfield $Z$ is not hermitian, but it satisfies the simple condition $Z^\dagger = e^{2V}Ze^{-2V}$.

\footnote{Introducing this covariant derivative is crucial for the manifestly gauge invariant formulation of the non-abelian lagrangian, which represents a significant simplification as compared to \cite{16,17}.}
Note that it is also possible to turn the argument around and to consider the construction of the 5d SUSY lagrangian on the basis of the 4d theory. To achieve this, start with a 4d real superfield $V$ with the usual gauge transformation properties. The gauge parameter is the 4d chiral superfield $\Lambda$. Now consider both superfie lds as functions of the additional parameter $x^5$. To be able to take derivatives in the $x^5$ direction, we are forced to introduce the additional gauge connection $\Phi$, which is a 4d chiral superfield depending on $x^5$. The requirement that $\nabla_5 = \partial_5 + \Phi$ be a covariant derivative enforces the gauge transformation property of Eq. (20). It turns out that the two lowest-dimension invariant operators that can be built from $V$ and $\nabla_5^2$ add up to the 5d SUSY lagrangian, Eq. (24), where the relative normalization of the $W^2$ and the $\nabla_5^2$ terms is fixed by the requirement of 5d Lorentz covariance. As expected on the basis of the 4d $N = 1$ SUSY and the full 5d Lorentz covariance, the larger 5d $N = 1$ SUSY emerges as an additional feature.

Pursuing this line of thinking, it is now straightforward to construct higher-derivative operators of the 5d SYM theory in a systematic way. One simply has to write down all 4d-SUSY-invariant superfield expressions of a given (higher) dimension and constrain the coefficients by the requirement of full 5d Lorentz symmetry.

Though motivated by the idea of orbifold compactification, the discussion of this section was so far restricted to the 5d Lorentz invariant theory. Once 5d Lorentz invariance is broken and a brane is introduced, the above arguments concerning the relative normalization of the $W^2$ and $\nabla_5^2$ operators in Eq. (24) cease to apply. We adopt the attitude that, in the orbifold theory, the bulk lagrangian is nevertheless restricted by 5d Lorentz invariance, while brane localized versions of the $W^2$ and $\nabla_5^2$ operators with unconstrained relative normalization become admissible (see Sect. 5 for more details). Strictly speaking, one would have to appeal to supergravity to put the notion of 5d bulk Lorentz symmetry in the presence of a brane on a firm basis.

4 The hypermultiplet

For completeness, we also present the relevant formulae for the 5d matter multiplet (the hypermultiplet). It contains an SU(2)-R doublet of scalar fields $H^i$, a Dirac field $\psi$ and a doublet of auxiliary fields $F_i$. The transformation laws are (for the ungauged case see [18])

$$\delta_\xi H^i = -\sqrt{2} \varepsilon^{ij} \bar{\xi}_j \psi$$
$$\delta_\xi \psi = i \sqrt{2} \gamma^M D_M H^i \varepsilon_{ij} \xi^j - \sqrt{2} \Sigma H^i \varepsilon_{ij} \xi^j + \sqrt{2} F_i \xi^i$$
$$\delta_\xi F_i = i \sqrt{2} \bar{\xi}_i \gamma^M D_M \psi + \sqrt{2} \bar{\xi}_i \Sigma \psi - 2 i \bar{\xi}_i \lambda^j \varepsilon_{jk} H^k .$$

The off-shell 5d lagrangian, invariant under this SUSY, reads

$$\mathcal{L} = -(D_M H^i)^\dagger (D^M H^i) - i \bar{\psi} \gamma^M D_M \psi + F^i \bar{F}_i - \bar{\psi} \Sigma \psi + H_i^i (\sigma^a X^a)^i_j H^j$$
$$+ H^i_i \Sigma^2 H^i + \left( i \sqrt{2} \bar{\psi} \lambda^i \varepsilon_{ij} H^j + \text{h.c.} \right) .$$

In the 4d superfield formulation, the component fields are arranged in the two chiral 4d superfie lds (in the $y$ basis):

$$H = H^1 + \sqrt{2} \theta \psi_L + \theta^2 (F_1 + D_5 H^2 - \Sigma H^2)$$
\[ H^c = H_2^\dagger + \sqrt{2} \theta \psi_R + \theta^2 (-F_{12}^\dagger - D_5 H_1^\dagger - H_1^\dagger \Sigma) \].  

(31)

As in the pure gauge case, the $\xi_L$ part of the transformation laws given in Eqs. (26)–(28) follows in the superfield formulation by acting with $\delta_{\xi_L}$ on $H$ and $H^c$ and then gauge transforming back to WZ gauge. The two superfields gauge transform according to $H \rightarrow e^{-\Lambda} H$ and $H^c \rightarrow e^{\Lambda} H^c$. The 4d superfield expression for the lagrangian, Eq. (29), reads

\[ \mathcal{L} = \left( H^\dagger e^{2V} H + H^c e^{-2V} H^c \right) \bigg|_{\theta^2 \bar{\theta}^2} + \left( H^c \nabla_5 H \big|_{\theta^2} + \text{h.c.} \right) . \]

(32)

5 Superfield brane operators

An obvious application of the above 4d superfield formalism is the classification of brane operators of a 5d SYM theory compactified to 4d on an orbifold. The most general such orbifold is $\mathbb{R}^4 \times I$, where $I$ is an interval parameterized by $x^5 = y$ and limited by two orbifold fixed points. Without loss of generality, we can discuss a fixed point at $y = 0$ which is left invariant by a $Z_2$ symmetry of the 5d theory corresponding to the reflection $y \rightarrow -y$ of the original 5d manifold.

Consider a $Z_2$ action on the 5d gauge multiplet given by

\[ V(y) \rightarrow PV(-y)P^{-1} \quad \text{and} \quad \Phi(y) \rightarrow -P\Phi(-y)P^{-1} . \]

(33)

Here, in the simplest case, $P$ is an element of the gauge group, $P \in G$ (the $Z_2$ acts by inner automorphism). More generally, the transformation $V \rightarrow PVP^{-1}$ can be replaced by any other automorphism of $G = \text{Lie}(G)$ (outer automorphism), under the restriction that the square of this automorphism is the identity.

Equation (33) is a symmetry of the lagrangian, Eq. (24), and the sign change of the superfield $\Phi$ is required since $\Phi$ enters the lagrangian in combination with $\partial_5$. The fields appearing in Eq. (25) transform under the $Z_2$ as $W_\alpha(y) \rightarrow PW_\alpha(-y)P^{-1}$ and $Z(y) \rightarrow -PZ(-y)P^{-1}$.

Given the $Z_2$ action on $V \in \mathcal{G}$, the Lie algebra $\mathcal{G}$ can be decomposed into its even and odd components, $\mathcal{G} = \mathcal{H} \oplus \mathcal{H}'$, where $\mathcal{H}$ generates the subgroup $H \subset G$ preserved by the orbifolding. Let $T^a$ and $\tilde{T}^\dot{a}$ form a basis of $\mathcal{H}$ and $\mathcal{H}'$ respectively. Then the fields $W_\alpha^a$ and $Z^\dot{a}$ are even under the $Z_2$ and can have non-zero values at the fixed point, while the fields $W_\dot{a}^\alpha$ and $Z^a$ are odd and vanish at the fixed point. Furthermore, the gauge connection at the boundary is specified by $\exp(2V)$, which corresponds to a restriction of the gauge symmetry to $H$ since only $V^a$ is non-vanishing. Thus, the lowest-dimension superfields that can appear in brane operators are

\[ W_\alpha^a, \quad (\nabla_5 W_\alpha)^\dot{a}, \quad Z^\dot{a}, \quad (\nabla_5 Z)^a , \]

(34)

where the argument $y = 0$ is suppressed. Brane operators can be constructed from these fields as in a 4d SUSY theory, given the restrictions of Lorentz invariance ($W$ is a spinor) and of the representation content under $H$. 

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We can write down the following quadratic operators:

\[
\begin{align*}
\mathcal{O}_1 &= c_{ab}^1 (W^\alpha)^a (W_\alpha)^b g_2 + \text{h.c.} \\
\mathcal{O}_2 &= c_{ab}^2 (W^\alpha)^a (\nabla_5 W_\alpha)^b g_2 + \text{h.c.} \\
\mathcal{O}_3 &= c_{ab}^3 (\nabla_5 W^\alpha)^a (\nabla_5 W_\alpha)^b g_2 + \text{h.c.} \\
\mathcal{O}_4 &= c_{ab}^4 Z^a Z^b g_{2\tilde{g}2} \\
\mathcal{O}_5 &= c_{ab}^5 Z^a (\nabla_5 Z)^b g_{2\tilde{g}2} \\
\mathcal{O}_6 &= c_{ab}^6 (\nabla_5 Z)^a (\nabla_5 Z)^b g_{2\tilde{g}2},
\end{align*}
\]

where the \(c^i\) are invariant under \(H\). The operators \(\mathcal{O}_1\) and \(\mathcal{O}_4\) have structures that are already present in the 5d lagrangian. Nevertheless, as will be discussed in more detail in the next section, even if they give rise to distinctive new effects if they are included in the action in this brane-localized version. The operators \(\mathcal{O}_2\) and \(\mathcal{O}_5\) depend on the non-trivial condition that invariants \(c^2\) and \(c^3\) with mixed indices exist. To see that this is possible in principle, consider a group \(G = U(1) \times U(1)\), broken by outer automorphism of one of the \(U(1)\)s to \(H = U(1)\). In this case, both \(W\) and \(Z\) are singlets and all of the above operators can be present.

Note also that, if a chiral superfield \(\Psi\) is localized at the fixed point, this field can be gauged under the group \(H\). In this case, the kinetic term of this field has the form \(\Psi^\dagger \exp(2V^a T^a) \Psi\), with \(T^a\) in the representation appropriate to \(\Psi\).

Next, consider the case where hypermultiplets (cf. Sect. 4) are also present in the bulk. For a hypermultiplet \((H, H^c)\), a \(Z_2\) action consistent with Eq. (33) is given by

\[
H(y) \rightarrow PH(-y) \quad \text{and} \quad H^c(y) \rightarrow -H^c(-y)P,
\]

where the prefactor \(-1\) could also be assigned to \(H\) instead of \(H^c\). Choosing a basis in representation space where \(H^r\) is even and \(H^c\) is odd, the lowest-dimension superfields that can be used for brane operators at \(y = 0\) are

\[
H^r, \quad (\nabla_5 H)^r, \quad H^c, \quad (\nabla_5 H^c)^r.
\]

They can now be combined among each other, with the fields of Eq. (34), and with chiral brane fields to form invariant brane operators. Furthermore, chiral brane fields can be coupled to the fields of Eq. (34). We do not attempt a complete listing of even the lowest-dimensional operators but content ourselves with three examples that will be useful in the next section:

\[
\begin{align*}
\mathcal{O}_7 &= H^c \nabla_5 H g_{2} + \text{h.c.} \\
\mathcal{O}_8 &= (\nabla_5 H^c) H g_{2} + \text{h.c.} \\
\mathcal{O}_9 &= \Psi_1^\dagger e^{2V} (Z^a T^a) \Psi_2 g_{2\tilde{g}2} + \text{h.c.}.
\end{align*}
\]

The operators \(\mathcal{O}_7\) and \(\mathcal{O}_8\) are brane-localized versions of an operator appearing in the bulk hypermultiplet lagrangian, while \(\mathcal{O}_9\) couples the gauge fields in the broken directions to chiral brane fields \(\Psi_1\) and \(\Psi_2\) in appropriate representations of the subgroup \(H\).
The existence of brane operators of the type discussed in this section can have significant impact on the low-energy theory emerging below the compactification scale.

6 Applications

Recall first the generic setup of 5d orbifold GUTs. One starts with a gauge theory on $\mathbb{R}^4 \times S^1$ and restricts the field space by the requirement of symmetry under the discrete group $Z_2 \times Z_2'$. The $S^1$ is parameterized by $x^5 = y \in [0, 2\pi R]$ or, equivalently, by $y' = y - \pi R/2$. The $Z_2$ action is given by Eq. (33), while the $Z_2'$ action is given by an analogous equation where $y$ is replaced by $y'$ and $P$ is replaced by $P'$. Choosing the gauge group $SU(5)$ and representation matrices $P = 1_{5}$ and $P' = \text{diag}(1, 1, 1, -1, -1)$, one finds that the surviving symmetries on the $P$ and the $P'$ branes are $SU(5)$ and $G_{SM} = SU(3) \times SU(2) \times U(1)$. The low-energy spectrum of the gauge sector is precisely that of the MSSM since the $P$ reflection removes all zero modes from $\Phi$, and the additional $P'$ reflection removes the zero modes corresponding to $X, Y$ gauge bosons from $V$.

To solve the doublet-triplet splitting problem, the Higgs multiplets have to be localized in the bulk or on the SM brane. Fermions can be placed on the $SU(5)$ brane, on the SM brane, or in the bulk.

We now want to briefly discuss several possible implications of brane operators localized at the two fixed points for the low-energy theory derived from the 5d orbifold GUT.

First, consider proton decay mediated by the $X, Y$ gauge bosons, which have GUT scale masses because their Kaluza-Klein (KK) spectrum does not contain a zero mode. Naively one would think that this type of process is absent in models where the fermions are localized on the SM brane because the $V$ components corresponding to the broken direction vanish at this brane. However, operators of the type $O_9$ in Eq. (45) may be present, in which case all the usual couplings of SM particles to the 5d analogues of $X, Y$ gauge bosons may exist. Of course, the coupling strength is now not any more an unambiguous prediction of the theory. We leave the more detailed investigation of proton decay in this and other scenarios to a future publication.

Next, consider the masses of the Higgs fields. It is one of the most attractive features of the present models that, after appropriate orbifold projections, a bulk hypermultiplet $(H, H^c)$ in the 5 of $SU(5)$ gives rise to one doublet chiral superfield. More specifically, this is realized by using Eq. (41) as it stands for the $Z_2$ transformation and switching the prefactor $-1$ from $H^c$ to $H$ for the $Z_2'$ transformation. However, given the presence of the operator $O_8$ in Eq. (44) on one of the branes, one obtains a mixing between the doublet zero mode from $H$ and the massive KK modes of the doublet from $H^c$.  

\[^3\] Of course, following the early work on symmetry breaking by compactification, orbifolds, and their interrelation, this type of model building was extensively studied in the framework of string theory. Nevertheless, the present, purely field-theoretic constructions are well motivated as attempts to compare the wealth of low-energy data with the many possible GUT structures in a way that is as direct and simple as possible. For a more detailed discussion of the structure of GUT breaking by field-theoretic orbifolding see, e.g., [6].
To see this in more detail, consider explicitly the part of the full action that is quadratic in fermionic fields and does not include derivatives in brane-parallel directions. We integrate the lagrangian (Eq. (29) or Eq. (32)) over $y \in [0, \pi R/2]$ (where $R \sim 1/M_{\text{GUT}}$ is the compactification scale), add $cO_8$ at the $P$ brane, and restrict our attention to the SU(2) doublet components. The relevant terms read:

$$S = \int d^4x \int_0^{\pi R/2} dy \left( [1 - c\delta(y)]\psi_L \partial_5 \psi_R + \text{h.c.} \right) + \cdots. \quad (46)$$

This action has to be varied under the constraints that $\psi_R$ and $\partial_5 \psi_L$ vanish at the boundaries. The resulting equations of motion are

$$\partial_5 \left( [1 - c\delta(y)]\psi_L \right) = 0 \quad (47)$$
$$[1 - c\delta(y)] \partial_5 \psi_R = 0. \quad (48)$$

Thus, for $c \neq 0$, the usual zero mode, $\psi_L = \text{const.}, \psi_R = 0$, is removed. Formally, one finds a modified zero mode, $\psi_L \sim [1 - c\delta(y)]^{-1}, \psi_R = 0$. However, this is a highly singular function which may couple strongly via various higher derivative operators. Therefore, even though we can not exclude the existence of a related zero-mode in the UV completion of the theory, it does not seem to be an unambiguous prediction of the low-energy effective theory. In fact, when integrating out the auxiliary fields to analyse the scalar part of the action, one finds singular contributions reminiscent of the infamous $\delta(0)$ terms discussed in [18]. Note also that solutions arising in the presence of different types of brane-localized operators have been discussed, e.g., in [26].

Thus, given the limitations of our leading-order, purely field-theoretic analysis, we conclude that the operator $O_8$ significantly affects the Higgs zero mode. Even if a modified zero mode should still be present, its strong suppression at the brane may cause problems for the (necessarily brane-localized) Yukawa interactions. Fortunately, this operator is protected from quantum corrections and may therefore safely be set to zero at a technical level. However, one may still be concerned by the fact that, due to the presence of this operator in the bulk action, there is no obvious symmetry argument excluding the brane-localized version.

Now we turn to the pure gauge sector. The operator $O_1$ of Eq. (35) has already been extensively discussed in the context of orbifold GUTs. In the case where $G = SU(5)$ and $H = SU(3) \times SU(2) \times U(1)$, it contains three independent pieces which represent the (so far uncalculable) threshold corrections of the model [3]. The logarithmic running of differences of gauge couplings above the compactification scale [27] can be understood as the running of the coefficients of these operators [3,9] (see [28] for a recent more detailed analysis).

The operator $O_4$ of Eq. (38) has so far not been used in the construction of orbifold GUTs. In fact, without the present, fully gauge-covariant superfield formalism it is difficult to even write this operator down. We now discuss an interesting and, naively, somewhat mysterious feature of this operator. For simplicity, let $c_4^{ab} = c\delta_{ab}$ and focus on the quadratic term that mixes the broken-direction modes of $\Phi$ and $V$:

$$O_4 = -4c_4^{ab}(\Phi + \Phi^\dagger)^a \partial_5 V^b |_{g_2 \bar{g}_2} + \cdots. \quad (49)$$
Now consider a situation where both $P$ and $P'$ act non-trivially in group space and focus on fields $\Phi^\alpha$ and $V^\hat{\alpha}$ which correspond to a Lie algebra generator broken on both boundaries. In this case $\Phi^\alpha$ has a zero mode. On the one hand, Eq. (49) appears to imply that this zero mode is lifted by mixing with the massive KK modes of $V^\hat{\alpha}$. On the other hand, this zero mode corresponds to the freedom one has in choosing the relative orientation of the symmetry groups on the two branes as subgroups of $G$. This modulus can be described by the Wilson line connecting the two boundaries. The latter can clearly take a non-trivial value by having a gauge potential $A_5$ that vanishes near both branes and is non-zero only in the middle of the bulk. Thus, it should be unaffected by brane operators. This apparent contradiction is resolved by recalling the inhomogeneous gauge transformation property of $\Phi$. In fact, $\Phi$ can always be gauged to zero at the brane. An appropriate gauge transformation parameter $\Lambda$ is defined by

$$\partial_5 e^A = -\Phi e^A, \quad \Lambda|_{\text{brane}} = 0,$$

(50)

(which is clearly consistent with broken gauge invariance at the brane). This explains why Eq. (19) can not be used to argue that $\Phi^\alpha$ obtains a mass.

We leave the discussion of other operators and their role in specific models to future, more phenomenologically oriented work.

7 Conclusions

In this paper, we have given a detailed derivation of the 4d superfield formulation of a 5d SYM theory compactified on a field-theoretic orbifold. The 4d SUSY has been explicitly identified as the unbroken part of the larger SUSY of the original 5d theory. An essential ingredient of our treatment is the gauge and supersymmetry covariant derivative in the $x^5$ direction, $\nabla_5 = \partial_5 + \Phi$. The Lie-Algebra valued chiral superfield $\Phi$ represents the gauge connection in the $x^5$ direction. Its action in field space is specified by the usual Lie algebra action on fields in a representation of the gauge group. The recognition of the full covariance of $\nabla_5$ and the resulting simplification of the 4d superfield formulation in the non-abelian case (cf. Eqs. (21)–(24)) represents our main conceptual progress compared to the earlier treatment of [16]. An immediate consequence is the possibility to construct higher-derivative operators in the 5d theory by combining terms with 4d covariant derivatives and $\nabla_5$ under the restriction of full 5d Lorentz invariance.

In our formulation, it is straightforward to write down brane operators localized at orbifold fixed points where the gauge symmetry is broken to a subgroup of the original

\footnote{The vanishing tree-level potential for this degree of freedom is protected by SUSY but can receive radiative corrections in non-SUSY theories [24].}

\footnote{A physical situation where these considerations apply arises if one attempts to construct a 5d $SO(10)$ model by breaking the group to $SU(5) \times U(1)$ and $SU(5)' \times U(1)'$ on the two boundaries. Although the intersection of these groups is $G_{SM} \times U(1)$, the model is plagued by the presence of $\Phi$ zero modes. The above considerations show that this problem is rather fundamental and can not be overcome by brane operators.}
symmetry group (cf. Eqs. (34)–(45)). This has particular relevance for the phenomenology of orbifold GUT models. It is now possible to discuss brane-localized couplings of fields that vanish at the brane. This is achieved using the $\nabla_5$ derivative of those fields, which is in general non-zero at the brane.

One implication is the possibility of proton decay mediated by $X, Y$ gauge bosons even in the case where fermionic matter is localized at a fixed point where the gauge symmetry is restricted to the standard model group. Another implication is the possibility of a brane-localized mass term mixing the light Higgs doublet with the heavy Kaluza-Klein modes from the 5d hypermultiplet. A further potential area of application, which has not been discussed here but where brane operators may play an important role, is the low-energy supersymmetry breaking in models with extra dimensions (see, e.g., [30]).

We hope that the developed framework will prove useful in the detailed phenomenological analysis of different specific orbifold GUT models. It would furthermore be important to generalize the presented gauge-covariant treatment to SYM theories in more than 5 dimensions.

Acknowledgments: I am very grateful to J. March-Russell for numerous detailed discussions at various stages of this project. I would also like to thank S. Ferrara, R. Rattazzi and C. Scrucca for helpful conversations.

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