Ap Weights and quantitative estimates in the Schrödinger setting

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Abstract
Suppose $L = -\Delta + V$ is a Schrödinger operator on $\mathbb{R}^n$ with a potential $V$ belonging to certain reverse Hölder class $RH_\sigma$ with $\sigma \geq n/2$. The aim of this paper is to study the $A_p$ weights associated to $L$, denoted by $A^L_p$, which is a larger class than the classical Muckenhoupt $A_p$ weights. We first prove the quantitative $A^L_p$ bound for the maximal function and the maximal heat semigroup associated to $L$. Then we further provide the quantitative $A^L_{p,q}$ bound for the fractional integral operator associated to $L$. We point out that all these quantitative bounds are known before in terms of the classical $A_{p,q}$ constant. However, since $A_{p,q} \subset A^L_{p,q}$, the $A^L_{p,q}$ constants are smaller than $A_{p,q}$ constant. Hence, our results here provide a better quantitative constant for maximal functions and fractional integral operators associated to $L$. Next, we prove two–weight inequalities for the fractional integral operator; these have been unknown up to this point. Finally we also have a study on the “exp–log” link between $A^L_p$ and $BMO_L$ (the BMO space associated with $L$), and show that for $w \in A^L_p$, $\log w$ is in $BMO_L$, and that the reverse is not true in general.

Keywords Schrödinger operator · Weighted inequalities · Fractional integral operator

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1 Introduction and statement of main results

The theory of Muckenhoupt $A_p$ weights plays an important role in harmonic analysis and partial differential equations. For example, it is well known that $A_p$ weights can be characterized equivalently via the boundedness of Hardy–Littlewood maximal functions and the Hilbert transform, the Riesz transforms in higher dimension. Moreover, $A_p$ weights also connect to the BMO space via the exponential and logarithm mapping, i.e., if $w$ is an $A_p$ weight, then $\log w$ is in BMO, conversely, if $\log w \in \text{BMO}$ then there is a $\gamma > 0$ and $p > 1$ such that $w^\gamma \in A_p$.

In recent years, the sharp $A_p$ bound for Calderón–Zygmund operators has been obtained. The cases of the Hilbert and Riesz transforms were shown by Petermichl [30,31], the case of Haar shifts was proven by Lacey, Petermichl and Reguera [25], for dyadic paraproducts by Beznosova [2], for the Bergman projection on the upper half plane by Pott–Reguera [33] and for general Calderón–Zygmund operators by Hytönen [22].

Besides the $A_p$ class, in [29] Muckenhoupt and Wheeden also introduced the fractional weight class $A^\alpha_{p,q}$ in $\mathbb{R}^n$ as follows: a non-negative locally integrable function $w$ is in $A^\alpha_{p,q}$ if

$$[w]_{A^\alpha_{p,q}} := \sup_Q (\frac{1}{|Q|} \int_Q w(x)^q dx)^{\frac{q}{p'}} < \infty,$$

where $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. When $\alpha = 0$, then the class $A^\alpha_{p,q}$ becomes the classical $A_p$ weight. They showed that

$$\| I_{\alpha} : L^p(w^p) \to L^q(w^q) \| < \infty$$

if and only if $[w]_{A^\alpha_{p,q}} < \infty$, where $I_{\alpha}$ is the standard fractional integral operators defined as

$$I_{\alpha} f(x) := \int_{\mathbb{R}^n} f(y) |x - y|^{\alpha - n} dy.$$

Later, a sharp version of this theorem was given by Lacey, Moen, Pérez, and Torres [24] as follows.

**Theorem A** ([24]). Let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and let $w$ be in $A^\alpha_{p,q}$. There holds

$$\| I_{\alpha} : L^p(w^p) \to L^q(w^q) \| \lesssim [w]_{A^\alpha_{p,q}}^{(1 - \frac{n}{\alpha}) \max\{1, \frac{p'}{q'}\}}$$

and this result is sharp in the sense that there is a family of weights $\{w_\delta\}_{\delta \in \mathcal{A}}$ such that

$$\| I_{\alpha} : L^p(w_\delta^p) \to L^q(w_\delta^q) \| \simeq [w_\delta]_{A^\alpha_{p,q}}^{(1 - \frac{n}{\alpha}) \max\{1, \frac{p'}{q'}\}}.$$

They also showed the sharp weighted bound for the fractional maximal operator (we remark that here and throughout the paper, for a measurable set $E$ we write $E(x)$ to mean the indicator function, i.e. $E(x) = \mathbb{1}_E(x)$)

$$M_{\alpha} f(x) := \sup_{Q \text{ a cube}} \frac{Q(x)}{|Q|^{1 - \frac{n}{\alpha}}} \int_Q |f(y)| dy.$$

**Theorem B** ([24]). Let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and let $w$ be in $A^\alpha_{p,q}$. There holds

$$\| M_{\alpha} : L^p(w^p) \to L^q(w^q) \| \lesssim [w]_{A^\alpha_{p,q}}^{(1 - \frac{n}{\alpha}) \frac{p'}{q'}}.$$
It is well-known that the $A_p$ weights, Hilbert (Riesz) transforms, $A_{p,q}^\alpha$ classes, the fractional integral operators, and the corresponding quantitative estimates mentioned above are associated with the standard Laplacian $\Delta$ in $\mathbb{R}^n$. Changing the differential operator from the standard Laplacian $\Delta$ to other second order differential operators $L$ introduces new challenges and directions to explore, see for example some of the well-known results in the past 15 years [6,12–14,19–21,23].

A natural question arises when changing the standard Laplacian $\Delta$ to another second order differential operator $L$: can we have new $A_p$ weights and $A_{p,q}^\alpha$ classes adapted to $L$ such that the related maximal functions, singular integrals and fractional integral operators have the right quantitative estimates in terms of the new $A_p$ or $A_{p,q}^\alpha$?

In this paper, we focus on the Schrödinger operator $L = -\Delta + V$ in $\mathbb{R}^n$, $n \geq 3$, where the non-negative function $V$ is in the reverse Hölder class. There has already been much work done on one-weight inequalities for these operators. However, there has never been sharp estimates (or any sort of quantitative estimates) for these operators. For the first time, we are able to prove such estimates.

Quantitative bounds for the classical operators from harmonic analysis (e.g. Hilbert transform, Riesz transforms, maximal functions) are a deep reflection of the regularity of the classical Laplacian. Operators of the form, Riesz transforms, maximal functions) are a deep reflection of the regularity of the classical Laplacian. Operators of the form $L$ are based on the assumption that the operators under question possess this regularity that $-\Delta$ possesses. In particular, the presence of the (non–negative) potential $V$ makes $L$ non–local in the sense that it is not invariant under translations and dilations. Of course, many techniques, theorems, and heuristics from classical harmonic analysis are based on the assumption that the operators under question possess this regularity that $L$ lacks.

In this Schrödinger setting, a new class of $A_p$ weights associated to $L$ was introduced in [3], see also [38], which is a larger class, properly containing the classical Muckenhoupt $A_p$ weights. To be more precise, given $p > 1$ we define $A_p^\infty = \cup_{\theta \geq 0} A_p^\theta$, where $A_p^\theta$ is the set of weights $w$ such that:

$$[w]_{A_p^\theta} := \sup_{Q \text{ a cube}} \left( \frac{1}{\psi_\theta(Q) |Q|} \int_Q w(y) dy \right)^\frac{1}{\theta} \left( \frac{1}{\psi_\theta(Q) |Q|} \int_Q w(y) y^{-\frac{q'}{p}} dy \right)^\frac{1}{\theta} < \infty,$$

where for each $\theta > 0$, $\psi_\theta$ on the collection of cubes $\{Q\}$ (with sides parallel to the coordinate axes) is defined by

$$\psi_\theta(Q) := \left(1 + \frac{\ell(Q)}{\rho(Q)}\right)^\theta,$$

with $\rho(Q) := \rho(c_Q)$, $c_Q$ is the center of $Q$ and $\ell(Q)$ is the side-length of $Q$, $\rho(x)$ is the critical function associated to the potential function $V$ (we refer to Sect. 2.2 for a precise definition).

We also have the fractional weight class $A_{p,q}^{\alpha,\theta}$ associated to $L$ defined as follows. Let $p > 1$ and let $q$ be defined by $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. We define $A_{p,q}^{\alpha,\theta}$ as the class of weights $w$ such that:

$$[w]_{A_{p,q}^{\alpha,\theta}} := \sup_{Q \text{ a cube}} \left( \frac{1}{\psi_\theta(Q) |Q|} \int_Q w^q(x) dx \right)^\frac{1}{\theta} \left( \frac{1}{\psi_\theta(Q) |Q|} \int_Q w^{-\frac{q'}{p}}(x) dx \right)^\frac{\alpha}{\theta} < \infty.$$

In [3] and [38], they showed that this new weight class $A_p^\infty$ satisfies most of the properties parallel to the classical Muckenhoupt $A_p$ weights, and they also established the weighted boundedness of $M^\theta$, the Hardy–Littlewood maximal function adapted to $L$ (we refer to Sect. 2.
for the definition), and the Riesz transforms $\nabla L^{-1/2}$ in terms of $A^\infty_p$, and the fractional integral operators $L^{-\alpha/2}$ in terms of $A^\alpha_{p,q}$.

We also note that the BMO space associated to $L$ was introduced in [4], denoted by $BMO_\infty$ (for a precise definition, we refer to Sect. 3). They also studied the boundedness of commutators of functions in $BMO_\infty$ and the singular integrals adapted to $L$.

In this paper, we aim to study the following results regarding the weights $A^\infty_p$ and $A^\alpha_{p,q}$:

1. the quantitative estimates for the Hardy–Littlewood maximal function associated to $L$ in terms of $A^\infty_p$;
2. the quantitative estimates for the fractional integral operator associated to $L$, denoted by $L^{-\alpha/2}$, in terms of $A^\alpha_{p,q}$;
3. the “exp-log” link between $A^\infty_p$ and $BMO_\infty$.

To be more specific, the first main result of this paper consists of quantitative estimates for several versions of maximal functions associated to $L$. Here we mainly consider the Hardy–Littlewood type maximal function, the fractional maximal function, and the maximal function associated to the heat semigroup generated by $L$. For $\theta > 0$, and $0 \leq \alpha < n$, the fractional maximal function $M^{\theta,\alpha}$ associated to $L$ is defined as:

$$M^{\theta,\alpha} f(x) := \sup_{Q} \frac{Q(x)}{(\psi_\theta(Q)|Q|)^{1-\frac{n}{\theta}}} \int_Q |f(y)| \, dy.$$

In particular, when $\alpha = 0$, we denote $M^{\theta} f(x) := M^{\theta,0} f(x)$, which is the Hardy–Littlewood type maximal function associated to $L$. We also recall the heat maximal function $M^L$ associated to $L$:

$$M^L f(x) := \sup_{t \geq 0} |e^{-tL} f(x)|.$$

Then we have the following quantitative estimates for the Hardy–Littlewood type maximal function associated to $L$ and the maximal heat semigroup.

**Theorem 1.1** Suppose $\theta > 0$. Then we have that

1. $w(\{x \in \mathbb{R}^n : M^\theta f(x) > \lambda\}) \leq [w]_{A^\theta_p} \left(\frac{\|f\|_{L^p(w)}}{\lambda}\right)^p$ for all $\lambda > 0$ and for every $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$;
2. There is a $C_\theta$ so that

$$\|M^L : L^p(w) \to L^p(w)\| \leq C_\theta \|M^\theta : L^p(w) \to L^p(w)\|.$$

As a consequence, we see that $M^L$ possesses the same quantitative estimate $M^\theta$ does.

Moreover, we also have the following results regarding the fractional maximal function $M^{\theta,\alpha}$ associated $L$.

**Theorem 1.2** Suppose $0 \leq \alpha < n$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. Let $\gamma = \theta/(1 + \frac{\theta'}{q})$, then

$$\|M^{\theta,\alpha} : L^p(w^p) \to L^{\gamma}(w^\gamma)\| \leq C[w]_{A^\alpha_{p,q}} \left(\frac{1}{\theta}\right).$$

The third main result of this paper is a quantitative estimate of the fractional integral operator $L^{-\alpha/2} f(x)$.
Theorem 1.3 Suppose $0 \leq \alpha < n$. Let $1 < p < \frac{n}{\alpha}$ and $q$ be defined by the equation \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \) and let $K$ be defined by the equation \( \left( \frac{1}{K} + \frac{q}{K p} \right) (1 - \frac{\alpha}{n}) \max\{1, \frac{p'}{q}\} = \frac{1}{2} \). For $w \in A^\alpha_{p,q}^{\alpha,q/3K}$ there holds

\[ \left\| L_\alpha^{-\frac{q}{2}} : L^p(w^p) \rightarrow L^q(w^q) \right\| \lesssim \left[ w \right]_{A^\alpha_{p,q}^{\alpha,q/3K}}^{(1-\frac{q}{2}) \max\{1, \frac{p'}{q}\}}, \]

where the implied constant depends on $p, q, \alpha, n,$ and $\theta$.

Here we point out that the maximal operator associated to the heat semigroup $M^L$ and the Hardy–Littlewood type maximal function $M^\theta$ satisfy the quantitative estimate as in Theorem B for $\alpha = 0$, and that the fractional maximal function $M^{\theta,\alpha}$ satisfies the quantitative estimate as in Theorem B. Moreover, the fractional integral operator $L_\alpha^{-\frac{q}{2}} f(x)$ satisfies the quantitative estimate as in Theorem A.

However, we remark that the class of weights $A^\infty_p$ (resp. $A^\alpha_{p,q}^{\alpha,q}$) is associated to $L$, and can be much larger than the standard $A_p$ (resp. $A^\alpha_{p,q}^{\alpha,q}$) classes. Typical examples are as follows.

Example 1.4 Consider $L := -\Delta + 1$ on $\mathbb{R}^n$. Then we have that $\rho(x) \equiv 1$. Then consider the function $w(x) := 1 + |x|^\gamma$ with $\gamma > n(p - 1)$. We see that $w$ is in $A^\infty_p$, however, $w$ is not in classical $A_p$.

We remark that it might be more precise to decorate the various operators, weight classes and other objects we define in this paper with the letter “$p$”, but to avoid cumbersome notation, we do not do this. From the context at hand it should be clear.

In the end, we have a study on the “exp-log” link of $A^\infty_p$ and $\text{BMO}_\infty$. To be more specific, we show that

Theorem 1.5 (i) If $w \in A^\infty_p$, then we have $\log w \in \text{BMO}_\infty$;

(ii) However, the converse is not true in general.

The outline and structure of the paper is as follows. In Sect. 2 we recall some fundamental facts for Schrödinger operators with non-negative potential $V$.

In Sect. 3, we will develop some of the weighted theory associated to the classes $A^\theta_p$ and $A^\alpha_{p,q}^{\alpha,q}$. We will discuss the operators $M^\theta$ and $M^{\theta,\alpha}$ in more detail and prove Theorems 1.1 and 1.2. A key feature in this section is the introduction of a slightly different critical function that we denote $\tilde{\rho}$. There is also the corresponding $\tilde{\psi}_\theta$ function and $\tilde{A}^\alpha_{p,q}^{\alpha,q}$ classes. These new functions are much less sensitive to the precise location of the cube at which they are evaluated. In particular, if $Q \subset Q'$ then there holds $\tilde{\psi}_\theta(Q)^{-1} \lesssim \tilde{\psi}_\theta(Q')^{-1}$. This is an important modification as it mitigates the non–locality of the Schrödinger operator.

In Sect. 4 we will prove Theorem 1.3. For this section, we will show that $L_\alpha^{-\frac{q}{2}}$ is dominated by an appropriate dyadic operator. An important step in this procedure is organizing the cubes in to sub–collections on which $\tilde{\psi}_\theta(Q)$ is roughly equal to $2^r$ for $r \in \mathbb{N}$. This further mitigates the non–locality of the Schrödinger operator as it allows us to essentially ignore the $\tilde{\psi}_\theta$ function for most of the argument.

In Sect. 5, we recall the definition of BMO spaces associated to $L$ and the related properties. And then we will prove Theorem 1.5. The main technique here for (i) are Jensen inequalities and is similar to the classical case. We also point out that in general, the reverse direction “exp” is not true.

Finally, in Sect. 6 we give some concluding remarks. In particular, we prove some new two weight inequalities for $L_\alpha^{-\frac{q}{2}}$. We also give some potential areas of investigation.
2 Preliminaries

In this section we set some notation and recall the well-known facts and results related to Schrödinger operator $L = -\Delta + V$ on $\mathbb{R}^n$ for $n \geq 3$.

We first recall that for a subset $E$ we will write $E(x)$ for the indicator function of $E$; that is $E(x) := |E(x)|$. If $Q$ is a cube, then $\ell(Q)$ will denote the side-length of $Q$.

2.1 Reverse Hölder class

We say that the function $V$ satisfies a Reverse Hölder property of order $\sigma > n/2$ and write $V \in RH_\sigma$, if there exists a positive constant $C$ such that for all cubes $Q$ there holds

$$\left( \frac{1}{|Q|} \int_Q V(y)^\sigma \, dy \right)^{1/\sigma} \leq C \frac{\int_Q V(y) \, dy}{|Q|}.$$  (2.1)

For $\sigma = \infty$, the left hand side of (2.1) is replaced by the essential supremum over $B$. It is well-known that elements of $RH_\sigma$ are doubling measures, and that $RH_\sigma \subset RH_{\sigma'}$ whenever $\sigma' < \sigma$.

2.2 The critical function $\rho(x)$

Associated to $V$ we have the critical function $\rho$ introduced in [37], defined by

$$\rho(x) := \left( \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) \, dy \leq 1 \right\} \right)^{-1}. \quad (2.2)$$

As an example for the harmonic oscillator with $V(x) = |x|^2$, we have $\rho(x) \sim (1 + |x|)^{-1}$.

We state the following property of $\rho$; for the proof see [37].

Lemma 2.3 Let $\rho$ be the critical radius function associated with $L$ defined in (2.2).

(i) There exist positive constants $k_0 \geq 1$ and $C_0 > 0$ so that

$$\frac{\rho(x)}{C_0[\rho(x) + |x - y|]^{k_0} R^0} \leq \rho(y) \leq C_0 \rho(x)[\rho(x) + |x - y|]^{k_0/(1+k_0)},$$

for all $x, y \in \mathbb{R}^n$. In particular, for any ball $B \subset \mathbb{R}^n$, and any $x, y \in B$, we have $\rho(x) \leq C_0^2(1 + \frac{R}{R_0})^2 \rho(y)$.

(ii) There exists $C > 0$ and $\sigma_0 = \sigma_0(\sigma, n)$ so that

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y) \, dy \leq C \left( \frac{r}{R} \right)^{\sigma_0} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y) \, dy$$

for all $x \in \mathbb{R}^n$ and $R > r > 0$.

(iii) For any $x \in \mathbb{R}^n$, we have

$$\frac{1}{\rho(x)^{n-2}} \int_{B(x,\rho(x))} V(y) \, dy = 1.$$

(iv) There exists $C > 0$ so that for any $r > \rho(x)$

$$r^2 \int_{B(x,\rho(x))} V(y) \, dy \leq C \left( \frac{r}{\rho(x)} \right)^{n_0-n+2}$$

where $n_0$ is the doubling order of $V$. That is, $\int_{2B} V \lesssim 2^{n_0} \int_B V$ for any ball $B$. 

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Remark 2.4 It follows from Lemma 2.3 (ii) and (iii) that for any ball $B$,

$$r_B^2 \int_B V(y) \, dy \lesssim \begin{cases} \left( \frac{r_B}{\rho_B} \right)^{\sigma_0} & r_B \leq \rho_B, \\ \frac{r_B}{\rho_B}^{n_0 + 2 - n} & r_B > \rho_B. \end{cases}$$

2.3 Heat kernel bounds for $L$

We now recall the heat kernel upper bounds for the Schrödinger operator.

Denote by $p_t(L)(x, y)$ the integral kernel of the semigroup $\{e^{-tL}\}_{t>0}$ generated by $-L = \Delta - V$ and by $p_t(x, y)$ the integral kernel of the semigroup $e^{-t\Delta}$ generated by $\Delta$. Then obviously we have

$$0 \leq p_{t,L}(x, y) \leq p_t(x, y) := (4\pi t)^{-n/2} \exp(-|x-y|^2/4t).$$

We recall the well-known heat kernel upper bounds for the Schrödinger operator as well as properties for $V$ and its critical radius function $\rho$ as defined in (2.2). The following estimates on the heat kernel of $L$ are well-known.

**Proposition** 2.5 ([16,17]). Let $L = -\Delta + V$ with $V \in RH_\sigma$ for some $\sigma \geq n/2$. Then for each $N > 0$ there exists $C_N > 0$ and $c > 0$ such that

$$p_{t,L}(x, y) \leq C_N e^{-|x-y|^2/ct} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}$$

and

$$|p_{t,L}(x, y) - p_{t,L}(x', y)| \leq C_N \left(\frac{|x-x'|}{\sqrt{t}}\right)^{\sigma_1} e^{-|x-y|^2/ct} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}$$

whenever $|x - x'| \leq \sqrt{t}$ and for any $0 < \sigma_1 < \sigma$.

3 Maximal operators associated to $L$

In this section, we define some maximal operators associated to $L$ and we give quantitative bounds for their norms as operators acting on $L^p(w)$. The bounds will be in terms of the classes of weights defined and discussed in Sect. 5. Above, in the case $V \equiv 0$, all of these operators will reduce to the classical Hardy–Littlewood maximal function.

3.1 New classes of weights associated to $L$

To define the next two classes of weights, we will make use of the critical radius function $\rho$.

Let $Q$ be a cube and define the following functions:

$$\tilde{\rho}(Q) := \sup_{x \in Q} \rho(x) \quad \text{and} \quad \tilde{\psi}_\theta(Q) := \left(1 + \frac{\ell(Q)}{\tilde{\rho}(Q)}\right)^\theta.$$

(3.1)
Using the function $\tilde{\psi}_\theta$ introduced above, we define $\tilde{A}_p^\infty := \cup_{\theta \geq 0} \tilde{A}_p^\theta$, where $\tilde{A}_p^\theta$ is the set of weights $w$ such that:

$$[w]_{\tilde{A}_p^\infty} := \sup_{Q \text{ a cube}} \left( \frac{1}{\psi_\theta(Q)|Q|} \int_Q w(y)dy \right) \left( \frac{1}{\psi_\theta(Q)|Q|} \int_Q \sigma(y)dy \right)^{\frac{p'}{p}} < \infty.$$ 

The classes $A_p^\infty$ and $A_p^\theta$ were introduced in studied further in (for example) [3,38]. The classes $\tilde{A}_p^\infty$ and $\tilde{A}_p^\theta$ are – to our knowledge – new. We need these classes because the standard classes are are very “non–local” in the sense that the functions $\psi_\theta$ depend on the precise location of $Q$. The functions $\tilde{\psi}_\theta$ are not as sensitive to the precise location of $Q$.

**Proposition 3.2** For all $\theta \geq 0$ and weights $w$ there holds

$$[w]_{\tilde{A}_p^\infty} \simeq [w]_{A_p^\theta}, \quad (3.3)$$

and $A_p^\infty = \tilde{A}_p^\infty$.

**Proof** Clearly $[w]_{A_p^\theta} \leq [w]_{\tilde{A}_p^\theta}$ and so to prove (3.3), it suffices to show that

$$[w]_{\tilde{A}_p^\infty} \lesssim [w]_{A_p^\theta}. \quad (3.4)$$

Let $Q$ be a cube and let $x \in Q$. Using Lemma 2.3 we have $\rho(x) \lesssim \left(1 + \frac{\ell(Q)}{\rho(x)}\right)^2 \rho(c_Q)$ and so

$$\psi_\theta(Q) = \left(1 + \frac{\ell(Q)}{\rho(c_Q)}\right)^\theta \lesssim \left(1 + \frac{\ell(Q)}{\rho(x)} \left(1 + \frac{\ell(Q)}{\rho(x)}\right)^2\right)^\theta \lesssim \left(1 + \frac{\ell(Q)}{\rho(x)}\right)^{3\theta}.$$

Thus $1/\tilde{\psi}_\theta(Q) \lesssim 1/\psi_\theta(Q)$ and so (3.4) holds. \(\square\)

We also have a new generalization of the $A_{p, q}$ classes of Muckenhoupt and Wheeden [29]. Using the new auxiliary function introduced in (3.1), define the $\tilde{A}_{p, q}^{\alpha, \theta}$ characteristic of a weight $w$ by:

$$[w]_{\tilde{A}_{p, q}^{\alpha, \theta}} := \sup_{Q \text{ a cube}} \left( \frac{1}{\psi_\theta(Q)|Q|} \int_Q w(y)dx \right) \left( \frac{1}{\psi_\theta(Q)|Q|} \int_Q w^{q'}(x)dx \right)^{\frac{\alpha}{p'}} \cdot \left( \frac{1}{\psi_\theta(Q)|Q|} \int_Q \sigma(y)dy \right)^{\frac{(1-\alpha)p}{p'}}.$$

Similarly, using the new auxiliary function introduced in (3.1), for $\theta > 0$, and $0 \leq \alpha < n$ we define the maximal function $\tilde{M}_{\theta, \alpha}$ associated $L$:

$$\tilde{M}_{\theta, \alpha} f(x) := \sup_Q \frac{Q(x)}{(\tilde{\psi}_\theta(Q)|Q|)^{1-\frac{\alpha}{n}}} \int_Q |f(y)|dy,$$

and in particular, when $\alpha = 0$, we denote

$$\tilde{M}_\theta f(x) := \tilde{M}_{\theta, 0} f(x).$$

### 3.2 Quantitative bounds

In this section, we will give quantitative bounds for the maximal operators defined above. Ideally, we would like to give quantitative bounds for $M_{\theta}^\theta$ in terms of the $A_p^\theta$ characteristic of the weight, and we would like to prove similar assertions for the other maximal operators.
Quantitative estimates in the Schrödinger setting

defined. However, for some of the operators it seems the bounds must be given in terms of the $A^p_\gamma$ characteristic, where $\gamma < \theta$. This is also true in the qualitative versions of these theorems in [3,38]. However, we are able to give the desired quantitative weak bounds.

The proof of Theorem 1.1 will be by the following two lemmas. The first lemma provides the weak-type quantitative estimates of $M^\theta$ as required, the proof of which follows from the standard Besicovitch covering lemma. As a consequence, the estimate in (1) of Theorem 1.1 will be proven.

The second lemma will establish a pointwise bound that easily implies the estimate in (2) of Theorem 1.1, the proof of which follows from the pointwise upper bound of the heat kernel.

**Lemma 3.5** For $1 < p < \infty$, there holds:

$$w \left( \{ M^\theta f > \lambda \} \right) \lesssim [w]_{A^p_\theta} \left( \| f \|_{L^p(w)} / \lambda \right)^p.$$  

**Proof** Let $\Omega_\lambda = \{ M^\theta f > \lambda \}$ and let $K_\lambda$ be any compact subset of $\Omega_\lambda$. For every $x \in K_\lambda$ there is a cube $Q_x$ containing $x$, such that:

$$\frac{1}{\psi_\theta(Q_x)} \int_{Q_x} |f(y)| dy > \frac{\lambda}{2}.$$  

Since this set is compact, by the Besicovitch covering lemma, there is a number $M = M(n)$ such that there are $M$ collections of sets $Q_1, \ldots, Q_M$ such that each $Q_j = \{ Q_y : y \in K_\lambda \}$ and the sets in each $Q_j$ are pairwise disjoint. Additionally, $K_\lambda \subset \bigcup_{j=1}^M Q_j$. In other words, $K_\lambda$ is covered by $M$ collections of disjoint cubes. Thus it is enough to fix a $1 \leq j \leq M$ and set $Q = Q_j$ and estimate $\sum_{Q \in Q} w(Q)$.

Note that for a $Q \in Q$ there holds:

$$w(Q) \leq 2 \int_{\mathbb{R}^n} \frac{w(Q)}{|Q|} \frac{|f(x)|}{\lambda} Q(x) dx.$$  

Using this we have:

$$\sum_{Q \in Q} w(Q) \leq \int_Q \sum_{Q \in Q} \frac{w(Q)}{\psi_\theta(Q)} |Q|^{1/p'} w(x)^{1/p} dx$$  

$$\leq \left( \int_Q \left\{ \sum_{Q \in Q} \frac{w(Q)}{\psi_\theta(Q)} |Q|^{-1/p'} Q(x) \right\} |\sigma(x)| dx \right)^{1/p'} \cdot \| f \|_{L^p(w)}.$$  

Now, the cubes $Q \in Q$ are maximal and are thus disjoint. So the first term in the right-hand side of the last inequality above is equal to:

$$\frac{1}{\lambda} \left( \sum_{Q \in Q} w(Q)^{p'-1} \frac{\sigma(Q)}{\psi_\theta(Q)^{p'} |Q|^{p'}} w(Q) \right)^{1/p'} \leq \frac{[w]_{A^p_\theta}^{1/p}}{\lambda} \left( \sum_{Q \in Q} w(Q) \right)^{1/p'}.$$  

Thus there holds:

$$\sum_{Q \in Q} w(Q) \leq [w]_{A^p_\theta}^{1/p} \left( \sum_{Q \in Q} w(Q) \right)^{1/p'} \frac{\| f \|_{L^p(w)}}{\lambda},$$  

which is the required estimate. 

\[\square\]
Lemma 3.6 For \( \theta \in (0, \infty) \), there exists a constant \( C_\theta \) such that for any locally integrable function \( f \), and for every \( x \in \mathbb{R}^n \) and \( t > 0 \), we have
\[
|e^{-tL} f(x)| \leq C_\theta M^\theta(f)(x).
\]

**Proof** For any fixed \( x \in \mathbb{R}^n \) and \( t > 0 \),
\[
|e^{-tL} f(x)| \leq \int_{\mathbb{R}^n} p_t(x, y)|f(y)|dy \\
\leq C_\theta \int_{\mathbb{R}^n} e^{-|x-y|^2/ct} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-\theta}|f(y)|dy,
\]
where \( \theta \) is any positive constant.

We now denote by \( B := B(x, \sqrt{t}) \) the ball in \( \mathbb{R}^n \) centered at \( x \) with radius \( \sqrt{t} \). Then we have
\[
|e^{-tL} f(x)| \leq \int_{\mathbb{R}^n} p_t(x, y)|f(y)|dy \\
\leq C_\theta \sum_{j=1}^{\infty} \int_{2^j B \setminus 2^{j-1} B} e^{-|x-y|^2/ct} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-\theta}|f(y)|dy \\
+ C_\theta \int_{B} e^{-|x-y|^2/ct} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-\theta}|f(y)|dy \\
\leq C_\theta \sum_{j=1}^{\infty} 2^{j\theta} e^{-c2^{2(j-1)}} \frac{1}{|B|} \int_{2^j B \setminus 2^{j-1} B} \left(1 + \frac{2^j \sqrt{t}}{\rho(x)}\right)^{-\theta}|f(y)|dy \\
+ C_\theta \frac{1}{\psi_N(B)|B|} \int_{B} |f(y)|dy \\
\leq C_\theta \sum_{j=1}^{\infty} 2^{j(\theta+n)} e^{-c2^{2(j-1)}} \frac{1}{\psi_0(2^j B)|2^j B|} \int_{2^j B \setminus 2^{j-1} B} |f(y)|dy \\
+ C_\theta M^\theta(f)(x) \\
\leq C_\theta M^\theta(f)(x).
\]

To prove Theorem 1.2, we first note that
\[
\|M^{\theta, \alpha} : L^p(w^p) \to L^q(w^q)\| \leq \|\tilde{M}^{\theta, \alpha} : L^p(w^p) \to L^q(w^q)\|
\]
follows easily from the definitions of \( M^{\theta, \alpha} \) and \( \tilde{M}^{\theta, \alpha} \). Hence, only the estimate
\[
\|\tilde{M}^{\theta, \alpha} : L^p(w^p) \to L^q(w^q)\| \lesssim [w]_{A^{\theta, \alpha}_{p,q}}^{(1-q/n)}
\]
needs to be shown. And then, Theorem 1.2 follows from the above quantitative estimates and from Proposition 3.2, which shows that \([w]_{A^{\theta, \alpha}_{p,q}} \simeq [w]_{A^{\theta, \alpha}_{p,q}}^{(1-q/n)}\).

To begin with, we need the following two universal bounds for weighted maximal functions. Let \( \mu \) be a weight and \( 0 \leq \alpha < n \). Define
\[
M^\alpha f(x) := \sup_Q \frac{Q(x)}{\mu(Q)^{1-\frac{\alpha}{n}}} \int_Q |f(y)| \mu(y)dy \quad M f(x) := M^0 f(x).
\]
Applying Hölder’s Inequality with exponents $q_1 − 1$.

The proof of the following theorem follows the corresponding proof in [24]. Let $\gamma = \frac{q}{p} - 1$. Then

$$\left\| M^{(\gamma)}_{\mu} : L^p(\mu) \rightarrow L^q(\mu) \right\| \lesssim 1,$$

where the implied constant does not depend on $\mu$.

When $\alpha = 0$, this is the well–known Doob maximal inequality. For $0 < \alpha < n$ see [24, Lemma 4.1]. We will use these facts to prove the following theorem.

**Theorem 3.8** Let $0 \leq \alpha < n$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. There holds

$$\left\| M^{(\gamma,\alpha)}_{\mu} : L^p(w^p) \rightarrow L^q(w^q) \right\| \lesssim [w]\nu_{\alpha,n}^{p,q}.$$

**Proof** The proof of the following theorem follows the corresponding proof in [24]. Let $u = w^q$ and $\sigma = w^{-p'}$ and $r = 1 + \frac{q}{p}$. There holds $\nu_{\alpha,n}^{p}(1 - \frac{q}{n}) = \nu_{\alpha,n}^{r}$. For any cube $Q$ we have

$$\frac{1}{(\psi_\theta(Q)|Q|)^{1 - \frac{q}{n}}} \int_Q |f(y)| dy = \left( \frac{\sigma(Q)u(Q)}{(\tilde{\psi}_\theta(Q)|Q|^{1 + \frac{p'}{q}})} \right)^{1 - \frac{q}{n}} \left( \frac{\frac{r'}{q}}{(u(Q)^{\frac{p'}{q}})} \right)^{\frac{1}{n}} \left( \frac{|Q|}{\sigma(Q)} \right)^{\frac{r'}{q}(1 - \frac{q}{n})} \frac{1}{\sigma(Q)^{1 - \frac{q}{n}}} \int_Q |f(y)| dy.$$

Let $\gamma$ satisfy $\gamma \frac{p'}{q} + \gamma = \theta$ (i.e. $\gamma = \frac{\theta}{1 + p'/q}$). Then this becomes

$$\left\{ \left( \frac{u(Q)}{\psi_\gamma(Q)|Q|} \right)^{\frac{p'}{q}} \left( \frac{\sigma(Q)}{\tilde{\psi}_\theta(Q)|Q|} \right)^{\frac{1}{n}} \left( \frac{|Q|}{\sigma(Q)^{\frac{r'}{q}(1 - \frac{q}{n})}} \right)^{\frac{1}{\sigma(Q)^{1 - \frac{q}{n}}} \int_Q |f(y)| dy.}

Estimating the first factor from above by $[w]\nu_{\alpha,n}^{p,q}$, this is dominated by

$$[w]\nu_{\alpha,n}^{p,q} \left( \frac{|Q|}{u(Q)^{\frac{p'}{q}}} \right)^{\frac{r'}{q}(1 - \frac{q}{n})} \frac{1}{\sigma(Q)^{1 - \frac{q}{n}}} \int_Q |f(y)| dy.$$

Applying Hölder’s Inequality with exponents $q/r'$ and $(q/r')' = (1 - \frac{r'}{q})^{-1}$ and noting that $1 - \frac{r'}{q} = 1 - \frac{p'}{q}(1 - \frac{q}{n})$, this last expression is then dominated by

$$[w]\nu_{\alpha,n}^{p,q} \left( \frac{1}{u(Q)} \int_Q M^{\alpha}_{\sigma}(f\sigma^{-1})^{q/r'} dx \right)^{r'/q}.$$

Taking a supremum over all cubes centered at $x$ we have the pointwise inequality

$$\tilde{M}^{(\gamma,\alpha)} f(x) \leq [w]\nu_{\alpha,n}^{p,q} M_{\alpha} \left\{ M_{\sigma}(f\sigma^{-1})^{q/r'} u^{-1} \right\}(x)^{r'/q}.$$
Thus, we have
\[ \| \tilde{M}^{\alpha,\gamma} f \|_{L^q(u^{1/q})} = \| \tilde{M}^{\alpha,\gamma} f \|_{L^q(u)} \]
\[ \leq [w]_{\Lambda_{p,q}^{2\alpha,\gamma}} \| M_u \{ M_{\alpha,\sigma} (f^{\gamma/q} r f^{1/q} u) \} \|_{L^{\gamma}(u)} \]
\[ \lesssim [w]_{\Lambda_{p,q}^{2\alpha,\gamma}} \| M_{\alpha,\sigma} (f^{\gamma/q} r f^{1/q} u) \|_{L^{\gamma}(u)} . \]
Let \( s = \frac{p'}{q} \). Then \( \frac{1}{s} - \frac{1}{r} = -\frac{\alpha}{n} \) and so by Lemma 3.7 there holds
\[ \| M_{\alpha,\sigma} (f^{\gamma/q} r f^{1/q} u^{1/q}) \|_{L^{r}(u)} = \| M_{\alpha,\sigma} (f^{\gamma/q} r f^{1/q} u^{r/q}) \|_{L^{r}(u)} \lesssim \| (f^{\gamma/q} r f^{1/q} u^{r/q}) \|_{L^{r}(u^{r/q}).} \]
Observe that \( \sigma^{-s} \frac{u^{1/\gamma}}{r} f^{(s \gamma)} = w^p \) (do this by writing \( \sigma \) and \( u \) in terms of \( w \) and then do some gymnastics with the Hölder exponents), and clearly \( |f|^{s \gamma} = |f|^p \) and so this last line is equal to \( \| f \|_{L^p(u^{r/p})} \) as desired. \( \square \)

4 Fractional integral operator

The goal of this section is to prove Theorem 1.3. We first recall some definitions. The heat semigroup associated to \( L \) is a family of operators given by \( H_t f(x) := e^{-tL} f(x) \). For \( 0 < \alpha < n \) using the functional calculus we can write \( L^{-\alpha} \) as an integral operator:
\[ L^{-\alpha} f(x) = \int_0^\infty e^{-tL} f(x) t^{\alpha/2-1} dt. \]
We will prove a quantitative version of a theorem of Tang [38]. This is a version of the theorem of Lacey–Moen–Torres–Pérez adapted to our setting [24].

**Theorem 4.1** Let \( 1 < p < \frac{n}{\alpha} \) and \( q \) be defined by the equation \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \) and let \( K \) be defined by the equation \( \left( \frac{1}{K} + \frac{1}{K} p \right)(1 - \frac{\alpha}{n}) \max\{1, \frac{p'}{q} \} = \frac{1}{2} \). For \( w \in \Lambda_{p,q}^{\alpha,\gamma} \) there holds
\[ \left\| L^{-\alpha} : L^p(u^p) \to L^q(u^q) \right\| \lesssim \| u \|_{\Lambda_{p,q}^{\alpha,\gamma}}^{(\frac{1}{p} - \frac{\alpha}{n}) \max\{1, \frac{p'}{q} \}}, \]
where the implied constant depends on \( p, q, \alpha, n \), and \( \theta \).

Recalling Lemma 3.2, Theorem 1.3 will follow from the following lemma.

**Lemma 4.2** Let \( 1 < p < \frac{n}{\alpha} \) and \( q \) be defined by the equation \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \) and let \( w \) and \( \sigma \) be weights. There holds
\[ \left\| L^{-\alpha} : L^p(u^p) \to L^q(u^q) \right\| \lesssim \| u \|_{\Lambda_{p,q}^{\alpha,\gamma}}^{(\frac{1}{p} - \frac{\alpha}{n}) \max\{1, \frac{p'}{q} \}}. \]

The rest of this section is devoted to the proof of Lemma 4.2. To prove this lemma, we will first show that \( L^{-\alpha} \) can be dominated by a certain dyadic operator. The dyadic operator will essentially be an infinite sum of dyadic versions of the classical fractional integral operator. In principle, we should be able to apply the results of [24] to each term to deduce the desired bound in Lemma 4.2. However, as we will see, there are some subtleties that must be addressed.
The operator is given by an integral operator with kernel $K(x, y)$. By [38, Lemma 3.3] we know that the kernel satisfies the following bound: for every $\phi > 0$ there $C_\phi$ such that
\[
|K(x, y)| \leq \frac{C_\phi}{(1 + |x - y| (\frac{1}{p(x)} + \frac{1}{p(y)}))^{\phi} |x - y|^{n - \alpha}}.
\]

Given this estimate it is now easy to dominate $L^{-\frac{\alpha}{2}}$ by a dyadic operator. First, fix $x \in \mathbb{R}^n$ and let $\phi > 0$. Below, $Q^{(x)}_k$ is the cube of side–length $2^k$ centered at $x$. For non–negative $f$ there holds
\[
\left| L^{-\frac{\alpha}{2}} f(x) \right| \leq \sum_{k \in \mathbb{Z}} \int_{Q^{(x)}_{k+1} \setminus Q^{(x)}_k} \frac{C_\phi}{(1 + |x - y| (\frac{1}{p(x)} + \frac{1}{p(y)}))^{\phi} |x - y|^{n - \alpha}} |f(y)| dy
\]
\[
\leq \sum_{k \in \mathbb{Z}} \int_{Q^{(x)}_{k+1} \setminus Q^{(x)}_k} \frac{C_\phi}{(1 + |x - y| (\frac{1}{p(y)})^{\phi} |x - y|^{n - \alpha}} |f(y)| dy. \quad (4.3)
\]

Now, for $y \in Q^{(x)}_{k+1} \setminus Q^{(x)}_k$, $|x - y| \simeq \ell(Q^{(x)}_k)$ and $\rho(y) \leq \tilde{\rho}(Q^{(x)}_k)$ and so there holds
\[
\frac{C_\phi}{(1 + |x - y| (\frac{1}{p(y)})^{\phi} |x - y|^{n - \alpha}} \leq \frac{C_\phi}{\psi(y)(Q^{(x)}_k)} \frac{\ell(Q^{(x)}_k)}{\rho(Q^{(x)}_k)}. \]

Inserting this into (4.3) we have
\[
\left| L^{-\frac{\alpha}{2}} f(x) \right| \leq C_\phi \sum_{k \in \mathbb{Z}} \int_{Q^{(x)}_{k+1} \setminus Q^{(x)}_k} \frac{1}{\psi(y)(Q^{(x)}_k)} \frac{\ell(Q^{(x)}_k)}{\rho(Q^{(x)}_k)} |f(y)| dy
\]
\[
\leq C_\phi \sum_{k \in \mathbb{Z}} \int_{Q^{(x)}_{k+1} \setminus Q^{(x)}_k} \frac{1}{\psi(y)(Q^{(x)}_k)} (f)_{Q^{(x)}_k}.
\]

Now setting $\phi = \frac{\alpha}{2}$ and recalling that there is a collection of $M = M(n)$ dyadic lattices such that every cube $Q$ is contained in a cube $P$ from one of these lattices with $\ell(P) \lesssim \ell(Q)$, we deduce that $L^{-\frac{\alpha}{2}} f(x)$ can be dominated by a finite sum of operators of the form
\[
I^{D}_{\alpha, \phi} f(x) := \sum_{Q \in D} \frac{(\ell(Q))^{\alpha}}{\tilde{\psi}_\phi(Q)} (f)_{Q} Q(x). \quad (4.4)
\]

Lemma 4.2 will follow if for every dyadic lattice $D$ we can show
\[
\left\| I^{D}_{\alpha, \phi} : L^p(w^p) \to L^q(w^q) \right\| \lesssim \left[ w \right]_{A^{1 - \frac{\alpha}{2}, \phi}_{p, q}}^{(1 - \frac{\alpha}{2}) \max\{1, \frac{\nu'}{\nu} \}}.
\]

We now divide the cubes into collections in which we hold $\tilde{\psi}_\phi(Q)$ constant. Thus, for $r \in \mathbb{N}$ set $Q_r := \{ Q \in D : \tilde{\psi}_\phi(Q) \simeq 2^{r\phi} \}$. Since $\tilde{\psi}_\phi(Q) > 1$, the sum in (4.4) can be written as:
\[ \mathcal{I}_{\alpha,\theta}^D f(x) = \sum_{r \geq 0} \sum_{Q \in \mathcal{Q}_r} (\ell(Q))^\alpha \psi_\theta (Q) f(x) Q(x) \]
\[ \simeq \sum_{r \geq 0} 2^{-r\theta} \sum_{Q \in \mathcal{Q}_r} (\ell(Q))^\alpha (f) Q(x) \]
\[ =: \sum_{r \geq 0} 2^{-r\theta} \mathcal{I}_{\alpha}^{Q_r} f(x). \]

The operators \( \mathcal{I}_{\alpha}^{Q_r} \) are very similar to the standard dyadic versions of the classical fractional integral operator. Indeed, the only difference is that in the classical case, \( \mathcal{Q}_r = \mathcal{D} \).

For the cubes \( Q \in \mathcal{Q}_r \),
\[ \left( \frac{1}{|Q|} \int_Q w(x)^q dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-p'} dx \right)^{\frac{q}{p'}} \leq [w, \sigma]_{A_{p,q}^{\alpha,\theta/2}} 2^{r \left( \frac{2}{p} + \frac{\alpha}{p} + \frac{\theta q}{p'} \right)}. \]

The point of this computation is that on the cubes in \( \mathcal{Q}_r \), the \( A_{p,q} \) characteristic is finite and so we would like to apply the sharp theorem of [24] to each of the operators. The problem with this approach is that the theorem of Lacey–Moen–Pérez–Torres is for the continuous version of the fractional integral operator. Their proof uses a sharp extrapolation theorem and this cannot be directly applied to an operator like \( \mathcal{I}_{\alpha}^{Q_r} \). On the other hand, purely dyadic versions of this theorem, for example [28], are only valid for certain values of \( p \) and \( q \).

We must therefore prove a version of the theorem of Lacey–Moen–Pérez–Torres for the operators \( \mathcal{I}_{\alpha}^{Q_r} \). That is, we must prove the estimate
\[ \| \mathcal{I}_{\alpha}^{Q_r} : L^p(w^p) \rightarrow L^q(w^q) \| \lesssim [w, \sigma]_{A_{p,q}^{\alpha,\theta/2}} 2^{r \left( \frac{2}{p} + \frac{\alpha}{p} + \frac{\theta q}{p'} \right) \max\{1, \frac{p'}{q}\}}. \tag{4.5} \]

Proving this estimate is the content of the next subsection. We will use a modified version of well-known extrapolation theorems. It is likely that the extrapolation theorem in the next subsection exists in the literature and we are aware of many similar theorems, but we have not been able to find an exact version of what we need. In any case, this will be well-known to experts, but we give some details; see [8,24] for more information.

### 4.1 An extrapolation argument

In this section, we will prove (4.5). We will actually prove something slightly more general.

Let \( \mathcal{Q} \) be a finite collection of dyadic cubes. We will define a class of weights in the following way. We define the \( A_{p,q}^{\mathcal{Q},\alpha,\theta} \) characteristic of a weight \( w \) by
\[ [w]_{A_{p,q}^{\mathcal{Q},\alpha,\theta}} := \sup_{Q \in \mathcal{Q}} \left( \frac{1}{|Q|} \int_Q w(x)^q dx \right) \left( \frac{1}{|Q|} \int_Q w^{-p'}(x) dx \right)^{\frac{q}{p'}} < \infty \]
for \( 1 < p \) and for \( p = 1 \)
\[ [w]_{A_{p,q}^{\mathcal{Q},\alpha,\theta}} := \left( \frac{1}{|Q|} \int_Q w(x)^q dx \right) \left( \inf_Q w^q(x) \right) < \infty. \]

Define the following “\( \mathcal{Q} \)-dyadic” maximal function
\[ M^\mathcal{Q} f(x) := \sup_{Q \in \mathcal{Q}} \frac{Q(x)}{|Q|} \int_{[f(x)]} dx \]
and the “$Q$–dyadic” fractional integral operator

$$I_{Q}^{\alpha} f(x) := \sum_{Q \in Q} |Q|^\alpha/n (f)_{Q} Q(x).$$

Estimate (4.5) will follow from the following theorem.

**Theorem 4.6** Let $1 < p < \frac{n}{\alpha}$ and $q$ be defined by the equation $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. For $w \in A_{p,q}^{Q}$, there holds

$$\| I_{Q}^{\alpha} : L^{p}(w^{p}) \rightarrow L^{q}(w^{q}) \| \lesssim [w]^{(1 - \frac{\alpha}{n}) \max\{1, \frac{p}{q}\}}_{A_{p,q}^{Q}},$$

where the implied constant depends on $p, q, \alpha$ and $n$.

We remark again that, in principle, this theorem is proven in [24]. However, in this setting, we are only considering cubes $Q \in Q$ and it is not clear that their theorem can be quoted directly. However, their proof can be modified (in some portions, the proof can be quoted directly) to the present setting, and this is what we do.

The remainder of this subsection is devoted to the proof of this theorem. We will use the same proof as in [24], modified for our setting. The outline is as follows. We first show that it suffices to prove two weak–type bounds. We then prove an extrapolation theorem for our setting. Finally, we will prove a “base estimate” from which we can extrapolate.

In [35,36] Sawyer shows that for the fractional integral operator, strong–type estimates follow from weak–type estimates. He does this by showing in [36] that the fractional integral operator is bounded between two weighted spaces if and only if “testing” holds (that is, if and only if the norm inequality is satisfied uniformly of indicators of cubes; see [26] for a dyadic version of this theorem). But in [35] he shows that if $T$ is a self–adjoint integral operator, then testing holds if $T$ and its adjoint satisfy a weak–type bound. Thus, we have the following.

**Lemma 4.7** Let $w$ be a weight, $0 < \alpha < n$, and $1 < p \leq q < \infty$. Then the operator norm

$$\| I_{Q}^{\alpha} : L^{p}(w^{p}) \rightarrow L^{q}(w^{q}) \|$$

is controlled by

$$\| I_{Q}^{\alpha} : L^{p}(w^{p}) \rightarrow L^{q,\infty}(w^{q}) \| + \| I_{Q}^{\alpha} : L^{q'}(w^{-q'}) \rightarrow L^{p',\infty}(w^{-p'}) \| \lesssim [w]^{(1 - \frac{\alpha}{n})}_{A_{p,q}^{Q}}.$$

Given Lemma 4.7, we now turn our attention to proving the following lemma.

**Lemma 4.8** Let $1 < p < \frac{n}{\alpha}$ and $q$ be defined by the equation $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. For $w \in A_{p,q}^{Q}$, there holds

$$\| I_{Q}^{\alpha} : L^{p}(w^{p}) \rightarrow L^{q,\infty}(w^{q}) \| + \| I_{Q}^{\alpha} : L^{q'}(w^{-q'}) \rightarrow L^{p',\infty}(w^{-p'}) \| \lesssim [w]^{(1 - \frac{\alpha}{n})}_{A_{p,q}^{Q}},$$

where the implied constant depends on $p, q, \alpha$ and $n$.

We first state the extrapolation theorem. It is our version of the extrapolation theorem (Theorem 2.1 in [24]).

**Theorem 4.9** Suppose that $T$ is an operator defined on $C_{c}^{\infty}$. Suppose that $1 \leq p_{0} \leq q_{0} < \infty$ and that

$$\| T f \|_{L^{q_{0}}(w^{q_{0}})} \lesssim [w]^{p_{0}}_{A_{p_{0},q_{0}}^{Q}} \| f \|_{L^{p_{0}}(w^{p_{0}})}.$$

We then have the following.
for all \( w \in A_{p_0, q_0}^Q \) and some \( \gamma > 0 \). Then

\[
\| Tf \|_{L^q(w^p)} \lesssim [w]_{A_{p, q}^Q}^{\gamma \max \left\{ 1, \frac{q_0}{p_0}, \frac{r}{q} \right\}} \| f \|_{L^p(w^p)}
\]

holds for all \( p, q \) satisfying \( \frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0} \) and all \( w \in A_{p, q}^Q \).

As is familiar to experts, the key to proving Theorem 4.9 is a version of the Rubio de Francia iteration algorithm. Once we have established this iteration algorithm, we can prove the extrapolation theorem. We follow the proof in [18]. Below, \( A_p^Q \) is the \( A_p \) class adapted to \( Q \):

\[
[w]_{A_p^Q} := \sup_{Q \in \mathcal{Q}} \left( \frac{1}{|Q|} \int_Q w(x) dx \right)^{\frac{1}{p}} \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{p'}} (x) dx \right)^{\frac{p'}{p}}.
\]

**Lemma 4.10** Suppose that \( 1 \leq r_0 < r \), \( v \in A_r^Q \), and \( g \) is a non-negative function in \( L^{(r/r_0)'}(v) \). Then there is a function \( G \) such that

(a) \( g \lesssim G \);
(b) \( \|G\|_{L^{(r/r_0)'}(v)} \lesssim \|g\|_{L^{(r/r_0)'}(v)} \);
(c) \( Gv \in A_{r_0}^Q \) with \( [Gv]_{A_{r_0}^Q} \lesssim [v]_{A_r^Q} \).

The implied constants are independent of \( r_0, r, Q, v, G \) and \( g \).

**Proof** Let \( t = \frac{r'}{(r/r_0)'} = \frac{r-r_0}{r} \). Note that since \( 1 \leq r_0 < r \) there holds \( 0 < t \leq 1 \). Define

\[
Rg := \left( M^Q(g^{\frac{1}{r'}} v)^{-1} \right)^{\frac{1}{r'}}.
\]

We compute the norm of \( R \) as an operator from \( L^{r/r_0}(v) \) to itself. Let \( f \in L^{r/r_0}(v) \). There holds

\[
\int_{\mathbb{R}^n} (Rg)(x)^{r/r_0} v(x) dx = \int_{\mathbb{R}^n} \left( M^Q(g^{\frac{1}{r'}} v)^{-1} \right)^{r/r_0} (x) v(x) dx
\]

\[
= \int_{\mathbb{R}^n} [M^Q(g^{\frac{1}{r'}} v)(x)]^{r/r_0} v(x)^{1-t} v^{(r-r_0)/r_0} dx
\]

\[
= \int_{\mathbb{R}^n} [M^Q(g^{\frac{1}{r'}} v)(x)]^{r/r_0} v(x)^{-\frac{r}{r}} dx.
\]

Now \( M^Q \) is bounded from \( L^{r'}(v^{-r'}) \) to itself with norm \( [v^{-r'}]_{A_r^Q}^{1/(r'-1)} \). Thus we can continue the estimate with

\[
\int_{\mathbb{R}^n} [M^Q(g^{\frac{1}{r'}} v)(x)]^{r/r_0} v(x)^{-\frac{r}{r}} dx \leq [v^{-r'}]_{A_r^Q}^{1/(r'-1)} \int_{\mathbb{R}^n} g(x)^{r'/r} v(x)^{r'/r_0} v(x)^{-r'} dx
\]

\[
= [v^{-r'}]_{A_r^Q}^{1/(r'-1)} \int_{\mathbb{R}^n} g(x)^{r/r_0} v(x) dx.
\]

And so we have

\[
\left\| R : L^{r/r_0}(v) \to L^{r/r_0}(v) \right\| \leq [v^{-r'}]_{A_r^Q}^{1/(r'-1)} = [v]_{A_r^Q}^{r'}.
\]
Define

\[ G := \sum_{k=0}^{\infty} \frac{R^k(g)}{2^k \| R \|^k} \]

where \( \| R \| := \left\| R : L^{r,r_0} (v) \to L^{r,r_0} (v) \right\| \), and \( R^0 = \text{Id} \). Then \( g \leq G \) and there holds

\[ \| G \|_{L^{r,r_0} (v)} \leq \sum_{k=0}^{\infty} \frac{\| R \|^k \| g \|_{L^{r,r_0} (v)}}{2^k \| R \|^k} \simeq \| g \|_{L^{r,r_0} (v)} . \]

Noting that \((r/r_0)' = r/(r - r_0)\) we see that (a) and (b) are proven.

We now need to estimate \([Gv]_{A^Q_{r_0}}\). First, by applying \( R \) to \( G \) we have

\[ RG = \sum_{k=0}^{\infty} \frac{R^{k+1}(g)}{2^k \| R \|^k} = 2 \| R \| \sum_{k=0}^{\infty} \frac{R^{k+1}(g)}{2^{k+1} \| R \|^{k+1}} \leq 2 \| R \| G . \]

Thus

\[ \left( M^Q (G^{1/t} v)^{-1} \right)^t \lesssim \| R \| G \lesssim [v]_{A^t'_{r}} G . \]

Taking \( t \)-th roots and rearranging we see that

\[ \left( M^Q (G^{1/t} v) \right) \lesssim (G^{1/t} v) [v]_{A^Q} . \]

Thus for all cubes \( Q \in Q \) we have

\[ \frac{1}{|Q|} \int_Q G^{1/t} (x) v(x) dx \lesssim [v]_{A^Q} G^{1/t} v . \]

Again rearranging this implies

\[ G \gtrsim [v]_{A^Q}^{-\frac{t}{r_0'}} \left( \frac{1}{|Q|} \int_Q G(x) v(x) dx \right)^r . \tag{4.11} \]

We now estimate the \( A^Q_{r_0} \) characteristic of \( Gv \). We need to estimate

\[ \left( \frac{1}{|Q|} \int_Q G(x) v(x) dx \right) \left( \frac{1}{|Q|} \int_Q G(x)^{-\frac{1}{r_0 - t}} v(x)^{-\frac{1}{r_0 - t}} dx \right)^{r_0 - 1} . \tag{4.12} \]

By Hölder’s Inequality with exponents \((1/t)\) and \((1/t)' = 1/t - 1\) it follows that the first factor is dominated by

\[ \left( \frac{1}{|Q|} \int_Q G(x)^{1/t} v(x) dx \right)^t \left( \frac{1}{|Q|} \int_Q v(x) dx \right)^{r - 1} . \tag{4.13} \]

By (4.11) the second factor is controlled by

\[ \left( \frac{1}{|Q|} \int_Q [v]^{-\frac{1}{t}}_{A^Q} \left( \frac{1}{|Q|} \int_Q G(y)^{1/t} v(y) dy \right)^{-\frac{1}{r_0 - t}} v^{\frac{1}{r_0 - t}} (x) v(x)^{-\frac{1}{r_0 - t}} dx \right)^{r_0 - 1} . \tag{4.14} \]
Multiplying (4.13) and (4.14) together, and using the fact that \( \frac{r-1}{r_0-1} = -\frac{1}{r-1} \) we see that (4.12) is controlled by

\[
[v]_{A_r}^{Q} \left( \frac{1}{|Q|} \int_{Q} v(x) dx \right) \left( \frac{1}{|Q|} \int_{Q} v(x)^{-\frac{1}{r-1}} dx \right)^{r_0-1} = [v]_{A_r}^{Q} [v]_{A_r}^{Q_{1-t}} = [v]_{A_r}.
\]

This proves (c).

\[\square\]

**Remark 4.15** We now discuss the proof Theorem 4.9. Given the iteration algorithm Lemma 4.10, the proof of Theorem 4.9 is exactly the same as the proof of [24, Theorem 2.1]. We will not restate the proof, but we will explain why it is true.

It is a general principle that given an iteration algorithm like in Lemma 4.10, the extrapolation theorem will follow. The main idea in a proof of the extrapolation theorem is to factor expressions like \(|g(x)| w(x)^q\) into pieces on which the “base case” bound can be used.

The extrapolation argument is not very sensitive to the operator. For example, we do not need to assume that the operator is linear or even sub-linear; we only need to assume that it is defined on (for example) \(C^\infty\), smooth functions with compact support. The fact that we only know data about \(w\) for the cubes \(Q\) might seem insufficient to deduce the claimed bounds, but we are assuming that the operator is bounded for the base case exponents, and this gives enough information to deduce the claimed bounds.

Using Theorem 4.9 we have the following corollary. The proof is in [24, Corollary 2.2].

**Corollary 4.16** Suppose that for some \(1 \leq p_0 \leq q_0 < \infty\), an operator \(T\) satisfies the weak–type \((p_0, q_0)\) inequality:

\[
\|T : L^{p_0}(w^{p_0}) \to L^{q_0, \infty}(w^{q_0})\| \leq c[w]_{A_{p_0, q_0}}^{V treated by \(w\).}
\]

for every \(w \in A_{p_0, q_0}\) and some \(\gamma > 0\). Then \(T\) also satisfies the weak–type \((p, q)\) inequality

\[
\|T : L^p(w^p) \to L^{q, \infty}(w^q)\| \leq c[w]_{A_{p, q}}^{\gamma \max\{1, \frac{q_0}{p_0}, \frac{q}{p}\}}
\]

for all \(1 < p \leq q < \infty\) that satisfy

\[
\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}
\]

and all \(w \in A_{p, q}\).

We now prove a “base case” weak–type estimate from which we can use Corollary 4.16 to extrapolate to all exponents.

**Lemma 4.17** Let \(q_0 = n/(n - \alpha) = (n/\alpha)'<\). There holds

\[
\left\| I_{Q}^{w} f \right\|_{L^{q_0, \infty}(w^{q_0})} \lesssim [w]_{A_{1, q_0}}^{1-\alpha/n} \|f\|_{L^1(w)}
\]

for any weight \(w\).

**Proof** For convenience let \(u = w^{q_0}\). Let \(Q_M\) denote the maximal cubes in \(Q\). Recall that we assume that \(Q\) is finite so every cube in \(Q\) is contained in a unique cube in \(Q_M\). For every \(Q \in Q_M\) we will prove

\[
\| Q I_{Q}^{w} f \|_{L^{q_0, \infty}(u)} \lesssim \| Q f \|_{L^1(Q_M u)^{1/q_0}}.
\]

\[\square\]
This will imply (4.18) by the following argument. Now, for every $Q \in Q_M$ the $A_{1,q_0}^Q$ condition implies $M^Q u(x) \leq [w]_{A_{1,q_0}^Q} u(x)$. We therefore have

$$
\| I_{\alpha}^Q f \|_{L^{q_0,\infty}(u)} \leq \sum_{Q \in Q_M} \| Q I_{\alpha}^Q f \|_{L^{q_0,\infty}(u)}
$$

$$\leq \sum_{Q \in Q_M} \int_Q |f(x)| (M^Q u)^{1/\theta_0}(x) dx
$$

$$\leq [w]_{A_{1,q_0}^Q}^{1/\theta_0} \int_{\mathbb{R}^d} |f(x)| u(x)^{1/\theta_0}(x) dx
$$

$$= [w]_{A_{1,q_0}^Q}^{1-\alpha/n} \int_{\mathbb{R}^d} |f(x)| w(x) dx.
$$

Now, fix a cube in $P \in Q_M$. Observe that there holds

$$P(x) I_{\alpha}^Q f(x) = \sum_{Q \in Q: Q \subset P} |Q|^{\alpha/n} \langle Pf \rangle_Q Q(x).
$$

Note that $I_{\alpha}^Q$ can be written as an integral operator with kernel $K(x, y) := \sum_{Q \in Q: Q \subset P} \frac{Q(x)Q(y)}{|Q|^{1-\alpha/n}}$. Thus using Minkowski’s inequality for the $L^{q_0,\infty}$ norm, there holds

$$\left\| \sum_{Q \in Q: Q \subset P} |Q|^{\alpha/n} \langle Pf \rangle_Q Q(x) \right\|_{L^{q_0,\infty}(u)}
$$

is dominated by

$$\int_P |f(y)| \left\| \sum_{Q \in Q: Q \subset P} \frac{Q(x)Q(y)}{|Q|^{1-\alpha/n}} \right\|_{L^{q_0,\infty}(u)} dy. \quad (4.20)
$$

Now, we compute the $L^{q_0,\infty}(u)$ norm inside the integral. Let $\lambda > 0$ and let $Q_{\lambda}$ be the maximal cubes in $Q$ with $|Q|^{1-\alpha/n} < \lambda^{-1}$. Now, for a fixed $x$, $\sum_{Q \in Q: Q \subset P} Q(x) |Q|^{\alpha/n-1}$ is a geometric sum. Thus, if $\sum_{Q \in Q: Q \subset P} Q(x) |Q|^{\alpha/n-1} > \lambda$, then $x$ is contained in a unique element of $Q_{\lambda}$. Now, let $Q_{\lambda}(y)$ denote the unique element of $Q_{\lambda}$ that contains $y$ (if there is such an element). Note also that $\lambda < |Q|^{\alpha/n-1} = |Q|^{-\frac{1}{q}}$. Using this notation and these observations there holds

$$\lambda \left( u \left\{ x : \sum_{Q \in Q: Q \subset P} \frac{Q(x)Q(y)}{|Q|^{1-\alpha/n}} > \lambda \right\} \right)^{1/q} = \lambda u(Q_{\lambda}(y))^{1/q}
$$

$$\leq \frac{1}{|Q_{\lambda}(y)|^{1-\alpha/n}} u(Q_{\lambda}(y))^{1/q}
$$

$$= \left( \frac{1}{|Q_{\lambda}(y)|} u(Q_{\lambda}(y)) \right)^{1/q}.
$$

Taking a supremum over $\lambda > 0$ we deduce that

$$\left\| \sum_{Q \in Q: Q \subset P} \frac{Q(x)Q(y)}{|Q|^{1-\alpha/n}} \right\|_{L^{q_0,\infty}(u)} \leq (M^Q u(y))^{1/q}.$$

\[ \text{Springer} \]
Inserting this into (4.20) will give (4.19). □

We are now in a position to prove Lemma 4.8. Using extrapolation, we know that
\[ \| I^Q_\alpha : L^p(w^p) \to L^{q,\infty}(w^q) \| \lesssim [w]^{1-\frac{\theta}{p}}_{A^\theta_{p,q}} \]
and
\[ \| I^Q_\alpha : L^{q'}(w^{-q'}) \to L^{p',\infty}(w^{-p'}) \| \lesssim [w]^{1-\frac{\theta}{p'}}_{A^\theta_{q',p'}}. \]

Now, \([w^{-1}]_{A^\theta_{p',q'}} = [w]^{p'}_{A^\theta_{q,p'}}\) and \([w]_{A^\theta_{p,q}} > 1\) so there holds
\[ \| I^Q_\alpha : L^p(w^p) \to L^{q,\infty}(w^q) \| \lesssim [w]^{1-\frac{\theta}{p}}_{A^\theta_{p,q}} + [w^{-1}]_{A^\theta_{q',p'}}^{1-\frac{\theta}{p'}} \lesssim [w]^{(1-\frac{\theta}{p})}_{A^\theta_{p,q}}. \]
Thus the proof of Lemma of 4.8 is complete and so we have proved Theorem 1.3.

5 Weights associated to \(L\) and connections to BMO space associated to \(L\)

In this section, we recall the definition and properties of the BMO space \(\text{BMO}_\infty(\mathbb{R}^n)\) associated to \(L\). Then we build the exp-log connection of \(A^\infty_p\) and \(\text{BMO}_\infty(\mathbb{R}^n)\).

For any \(\theta \geq 0\) we can define the following \(\text{BMO}_\theta(\mathbb{R}^n)\) space as the set of functions such that
\[ \| f \|_{\text{BMO}_\theta(\mathbb{R}^n)} := \sup_{Q \text{ a cube}} \frac{1}{\psi_\theta(Q)} \left| \int_Q f(y) - (f)_Q \right| dy < \infty. \]
We also have the following \(\text{BMO}_\infty(\mathbb{R}^n)\) space
\[ \text{BMO}_\infty(\mathbb{R}^n) := \bigcup_{\theta \geq 0} \text{BMO}_\theta(\mathbb{R}^n). \]

Based on the definition of \(\text{BMO}_\infty(\mathbb{R}^n)\), we provide the proof of Theorem 1.5.

**Proof of Theorem 1.5** Proof of (i):

Suppose that \(w \in A^\infty_p\). Then there exists a \(\theta > 0\) such that \(w \in A^\theta_p\). Let \(\varphi = \log w\) and \(\mu = \log \left( \frac{1}{w} \right) = -\frac{\varphi}{p-1}\). Then for any cube \(Q\) we have \(e^{\varphi(x)}Q e^{(p-1)(\mu)Q} = 1\) and so we can write the \(A^\theta_p\) condition for \(w\) as follows:
\[ \frac{1}{\psi_\theta(Q)^p} \left( \frac{1}{|Q|} \int_Q e^{\varphi(x)-(\varphi)Q} dx \right) \left( \frac{1}{|Q|} \int_Q e^{\mu(x)-(\mu)Q} dx \right)^{p-1} \leq [w]_{A^\theta_p} < \infty. \]
By Jensen’s inequality we have
\[ \frac{1}{|Q|} \int_Q e^{\varphi(x)-(\varphi)Q} dx \geq 1 \quad \text{and} \quad \frac{1}{|Q|} \int_Q e^{\mu(x)-(\mu)Q} dx \geq 1. \]
Thus, noting that \(\psi_\theta(Q)^p = \psi_{p\theta}(Q)\), we conclude that for any \(w \in A^\theta_p\) we have
\[ \frac{1}{\psi_{p\theta}(Q)} \left( \frac{1}{|Q|} \int_Q e^{\varphi(x)-(\varphi)Q} dx \right) \leq \frac{[w]_{A^\theta_p}}{\left( \frac{1}{|Q|} \int_Q e^{\mu(x)-(\mu)Q} dx \right)^{p-1}} \leq [w]_{A^\theta_p}, \quad (5.1) \]
and similarly,
\[ \frac{1}{|Q|} \left( \int_Q e^{-((\varphi(x) - \langle \varphi \rangle_Q)/(p-1))} dx \right)^{p-1} \leq [w]_{A_p}. \quad (5.2) \]

Now for a cube \( Q \), let \( Q_+ := \{ x \in Q : \varphi - \langle \varphi \rangle_Q \geq 0 \} \) and \( Q_- = Q \setminus Q_+ \). Then we have
\[ \frac{1}{\psi_{p\theta}(Q) Q} \left( \int_Q |\varphi(x) - \langle \varphi \rangle_Q| dx \right) = \frac{1}{\psi_{p\theta}(Q) Q} \left( \int_{Q_+} (\varphi(x) - \langle \varphi \rangle_Q) dx + \int_{Q_-} -(\varphi(x) - \langle \varphi \rangle_Q) dx \right). \quad (5.3) \]

For the first term in the right-hand side of the equality above, using the trivial estimate \( t \leq e^t \), we obtain that
\[ \frac{1}{\psi_{p\theta}(Q) Q} \int_{Q_+} (\varphi(x) - \langle \varphi \rangle_Q) dx \leq \frac{1}{\psi_{p\theta}(Q) Q} \int_{Q_+} e^{\varphi(x) - \langle \varphi \rangle_Q} dx \leq \frac{1}{\psi_{p\theta}(Q) Q} \int_Q e^{\varphi(x) - \langle \varphi \rangle_Q} dx \leq [w]_{A_p}, \]

where the last inequality follows from (5.1).

Now for the second term, we first consider the case \( p - 1 \leq 1 \). Then using the trivial estimate \( t \leq e^t \) again we get
\[ \frac{1}{\psi_{p\theta}(Q) Q} \int_{Q_-} -(\varphi(x) - \langle \varphi \rangle_Q) dx \leq \frac{1}{\psi_{p\theta}(Q) Q} \int_{Q_-} e^{-(\varphi(x) - \langle \varphi \rangle_Q)} dx \leq \frac{1}{\psi_{p\theta}(Q) Q} \int_Q e^{-(\varphi(x) - \langle \varphi \rangle_Q)} dx \leq \frac{1}{\psi_{p\theta}(Q) Q} \left( \int_Q e^{-(\varphi(x) - \langle \varphi \rangle_Q)} dx \right)^{p-1} \leq [w]_{A_p}, \]

where the third inequality follows from Hölder’s inequality and the last inequality follows from (5.2).

We now consider the case \( p - 1 > 1 \). Again we have
\[ \frac{1}{\psi_{p\theta}(Q) Q} \int_{Q_-} -(\varphi(x) - \langle \varphi \rangle_Q) dx = \frac{p - 1}{\psi_{p\theta}(Q) Q} \int_{Q_-} -\frac{(\varphi(x) - \langle \varphi \rangle_Q)}{p - 1} dx \leq \frac{p - 1}{\psi_{p\theta}(Q) Q} \int_{Q_-} e^{-(\varphi(x) - \langle \varphi \rangle_Q)} dx. \quad (5.4) \]

Next, we note that \( \psi_{\theta}(Q) \geq 1 \) for all \( Q \) and \( \theta > 0 \), and that \( p - 1 > 1 \). Thus we have
\[ \psi_{p\theta}(Q) \leq \psi_{p\theta}(Q) \]

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which implies that
\[ \frac{1}{\psi_p^\theta(Q)} \leq \frac{1}{\psi_{p_0}^\theta(Q)^{\frac{1}{p-1}}} . \]

Combing the above estimate and the inequality (5.4), we get
\[ \frac{1}{\psi_{p_0}^\theta(Q)} \int_Q - (\varphi(x) - \langle \varphi \rangle_Q) \, dx \leq \frac{p - 1}{\psi_{p_0}^\theta(Q)^{\frac{1}{p-1}}} |Q| \int Q e^{- (\varphi(x) - \langle \varphi \rangle_Q) / p - 1} \, dx \]
\[ \leq (p - 1)[w]_{A_p^\theta}^{\frac{1}{p-1}} , \]
where the last inequality follows from (5.2).

Now combining the estimates of the first and second terms on the right-hand side of (5.3), we obtain that
\[ \frac{1}{\psi_{p_0}^\theta(Q)} |Q| \int_Q |\varphi(x) - \langle \varphi \rangle_Q| \, dx \leq [w]_{A_p^\theta} \max \{ [w]_{A_p^\theta}, (p - 1)[w]_{A_p^\theta} \} . \]
Hence we obtain that \( \log w \in \text{BMO}_{p_0} \subset \text{BMO}_\infty \), which implies that (i) holds.

Proof of (ii). Consider \( L = -\Delta + 1 \) on \( \mathbb{R}^n \). Then from [4] it is known that \( b(x) = |x_j| \), \( 1 \leq j \leq n \) is in \( \text{BMO}_\infty \). However, \( e^{\delta |x_j|} \) is not in \( A_\infty^p \) for any \( \delta > 0 \) and \( p \in [1, \infty) \).

\( \square \)

6 Conclusion

We briefly mention some two weight inequalities for the fractional integral operator \( L^{-\frac{\alpha}{2}} \).
Recall that \( L^{-\frac{\alpha}{2}} \) is dominated by a finite sum of operators of the form
\[ I_{D,\alpha} := \sum_{Q \in D} \frac{(\ell(Q))^\alpha}{\psi_\theta(Q)} \langle f \rangle_Q Q(x) . \]
And by setting \( Q_r := \{ Q \in D : \hat{\psi}_\theta(Q) \simeq 2^r \} \) we can further decompose \( I_{D,\alpha} \) as
\[ I_{D,\alpha} f(x) = \sum_{r \geq 0} \sum_{Q \in Q_r} \frac{(\ell(Q))^\alpha}{\psi_\theta(Q)} \langle f \rangle_Q Q(x) \]
\[ \simeq \sum_{r \geq 0} 2^{-r\theta} \sum_{Q \in Q_r} (\ell(Q))^\alpha \langle f \rangle_Q Q(x) \]
\[ =: \sum_{r \geq 0} 2^{-r\theta} I_{Q_r,\alpha} f(x) . \]
Therefore, to establish a two weight bound, it will be enough to give a two weight bound for the operators \( I_{Q_r,\alpha} \). We also note that if \( v \) is a weight and \( \sigma := v^{-\frac{\nu}{p'}} \) then there holds
\[ \| T : L^p(v) \to L^q(w) \| = \| T(\sigma \cdot) : L^p(\sigma) \to L^q(w) \| . \]

The following was proven by one of us and Scott Spencer [34]. Below, for a weight \( w \) we define
\[ \rho_w(Q) := \frac{1}{w(Q)} \int_Q (M(w Q))(x) \, dx . \]
Lemma 6.1 Let $1 < p \leq q < \infty$ and $\sigma, w$ be two weights. Let $\epsilon_p$ be a monotonic function on $(1, \infty)$ that satisfies $\int_1^\infty \frac{dt}{t^{\epsilon_p}(t)} = 1$ and similarly for $\epsilon_q$. Define

$$\beta(Q) := \frac{\sigma(Q)^{1/p} w(Q)^{1/q}}{|Q|^{1-\frac{\epsilon}{n}}} \rho_{\sigma}(Q)^{1/p} \epsilon_p(\rho_{\sigma}(Q)) \rho_{\sigma}(Q)^{1/q} \epsilon_q'(\rho_{w}(Q)),$$

and set $[\sigma, w]_{p,q,\alpha,r} := \sup_{Q \in \mathcal{Q}_r} \beta(Q)$. Then $\left\| I^Q_{\alpha} (\cdot) : L^p(\sigma) \to L^q(w) \right\| \lesssim [\sigma, w]_{p,q,\alpha,r}$.

Now, define

$$[\sigma, w]^{(\theta)}_{p,q,\alpha} := \sup_{Q \text{ a cube}} \frac{\sigma(Q)^{1/p} w(Q)^{1/q}}{\psi_{\theta/2}(Q) |Q|^{1-\frac{\epsilon}{n}}} \rho_{\sigma}(Q)^{1/p} \epsilon_p(\rho_{\sigma}(Q)) \rho_{\sigma}(Q)^{1/q} \epsilon_q'(\rho_{w}(Q)).$$

The conclusion in Lemma 6.1 can be stated as

$$\left\| I^Q_{\alpha'} (\cdot) : L^p(\sigma) \to L^q(w) \right\| \lesssim 2^{\theta/2} [\sigma, w]^{(\theta)}_{p,q,\alpha}.$$

Thus using Lemma 6.1 and the decomposition of $I^D\alpha$ we have the following theorem

**Theorem 6.2** With definitions as above, there holds

$$\left\| L^{-\frac{\theta}{2}} (\cdot) : L^p(\sigma) \to L^q(w) \right\| \lesssim [\sigma, w]^{(\theta)}_{p,q,\alpha}.$$

See other results in [7,9,34] to deduce similar two weight results in the present setting.

The condition $[\sigma, w]^{(\theta)}_{p,q,\alpha}$ may seem to be complicated beyond the point of usability. Conditions like this are known as “bump” conditions. These bump conditions were introduced in [39] and studied more in [27,34] and are typically smaller than other bump conditions such as Orlicz norms (this was shown by Treil and Volberg in [39]). For more information about two weight inequalities for the fractional integral operator, see [7,9].

Theorem 6.2 has a deficiency. The quantity $\rho_{w}(Q)$ is related to the $A_\infty$ characteristic of a weight. In particular, $[w]_{A_\infty} := \sup_Q \rho_{w}(Q)$. This is an important characteristic in the classical weighted theory. However, it is too large to capture enough information for weights in our classes. It will be interesting to develop an $A_\infty$ theory adapted to the operator $-\Delta + V$.

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