SL$_2$-modules of small homological dimension

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Abstract

Let $V_n$ be the SL$_2$-module of binary forms of degree $n$ and let $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$. We consider the algebra $R = O(V)^{SL_2}$ of polynomial functions on $V$ invariant under the action of SL$_2$. The measure of the intricacy of these algebras is the length of their chains of syzygies, called homological dimension $hd R$. Popov gave in 1983 a classification of the cases in which $hd R \leq 10$ for a single binary form ($p = 1$) or $hd R \leq 3$ for a system of two or more binary forms ($p > 1$).

We extend Popov's result and determine for $p = 1$ the cases with $hd R \leq 100$, and for $p > 1$ those with $hd R \leq 15$. In these cases we give a set of homogeneous parameters and a set of generators for the algebra $R$.

1 Introduction

This paper has two goals. First of all, following a suggestion by Popov, we extend the results of Popov [29] and determine all cases where the algebra of simultaneous invariants of a number of binary forms has low homological dimension. Secondly, we determine the minimal degrees of a homogeneous system of parameters (hsop) in these cases. We also give a minimal system of generators, confirming or correcting classical results.

Our base field is the field $\mathbb{C}$ of complex numbers. The group of all complex $2 \times 2$ matrices with determinant 1 is denoted SL$_2$. Let $V_n$ be the set of binary forms (homogeneous polynomials in two variables) of degree $n$. If $V$ is a rational finite-dimensional SL$_2$-module, then there exist $n_1, \ldots, n_p \in \mathbb{N}$ such that $V \cong V_{n_1} \oplus \ldots \oplus V_{n_p}$ as SL$_2$-modules, and the algebra $R := O(V)^{SL_2}$ of polynomial functions on $V$ invariant under the action of SL$_2$ can be identified with the algebra of joint invariants of $p$ binary forms of degrees $n_1, \ldots, n_p$.

The algebra $R$ is finitely generated ([23]), i.e. there exist a finite number of invariants $j_1, \ldots, j_r$ of $V$ such that $R = \mathbb{C}[j_1, \ldots, j_r]$. Denote by $r$ the minimal number of generators of $R$ and by $m$ the size of a system of parameters of $R$ (set of algebraically independent elements $P_1, \ldots, P_m$ of $R$ such that $R$ is integral over $\mathbb{C}[P_1, \ldots, P_m]$). Then $m$ equals $\sum (n_i + 1) - 3$ when this is positive, and the homological dimension $hd R$ of $R$ equals $r - m$ ([29, Corollary 1]).

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Popov [29] classified the modules $V$ with the property that $\text{hd}\ R \leq 3$, and noticed that all of these were known classically.

In the past 25 years some progress was made and sets of generators for $O(V_n)^{\text{SL}_2}$ were found in the cases $n = 7, 9, 10$ ([7, 8, 12]). The difficulty of this problem is reflected by the large homological dimensions of the algebras of invariants in these cases. For $R := O(V_n)^{\text{SL}_2}$ we have:

$$
\begin{array}{cccccccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 \text{hd} \ R & 0 & 0 & 0 & 0 & 1 & 1 & 25 & 3 & 85 & 98 \\
\end{array}
$$

In this paper we extend Popov’s classification to:

**Theorem 1.1.** Let $R := O(V_n)^{\text{SL}_2}$ and suppose that $\text{hd} \ R \leq 100$. Then $n \leq 10$.

**Theorem 1.2.** Let $R := O(V)^{\text{SL}_2}$ where $V = V_{n_1} \oplus \cdots \oplus V_{n_p}$, and suppose that $4 \leq \text{hd} \ R \leq 15$. Then we have one of the following:

| $n_1, \ldots, n_p$ | $\text{hd} \ m$ | $\text{hsop degrees}$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ | $d_7$ | $d_8$ | $d_{10}$ | $d_{11}$ |
|-------------------|----------------|-------------------|------|------|------|------|------|------|------|-------|-------|
| 1, 1, 1, 2        | 1, 1, 1        | 4                 | 6    | 2    | 3    | 10   | 4    | 6    | 5    | 6     | 7     |
| 1, 2, 2, 2        | 1, 2, 2        | 5                 | 8    | 2    | 5    | 2    | 3    | 2    | 4    | 21    | 21    |
| 1, 2, 2, 2        | 1, 2, 2        | 5                 | 9    | 2    | 9    | 14   | 14   | 14   | 3    | 4     | 21    |
| 1, 2, 2, 2        | 1, 2, 2        | 6                 | 7    | 2    | 4    | 13   | 13   | 3    | 4    | 15    | 15    |
| 1, 2, 2, 2        | 1, 2, 2        | 6                 | 9    | 2    | 9    | 15   | 15   | 15   | 3    | 9     | 15    |
| 1, 2, 2, 2        | 1, 2, 2        | 9                 | 8    | 2    | 4    | 15   | 15   | 3    | 9    | 27    | 21    |
| 1, 2, 2, 2        | 1, 2, 2        | 11                | 10   | 2    | 11   | 21   | 21   | 21   | 21   | 21    | 21    |
| 1, 2, 2, 2        | 1, 2, 2        | 11                | 11   | 2    | 11   | 21   | 21   | 21   | 21   | 21    | 21    |
| 1, 2, 2, 2        | 1, 2, 2        | 11                | 2    | 2    | 11   | 21   | 21   | 21   | 21   | 21    | 21    |
| 1, 2, 2, 2        | 1, 2, 2        | 12                | 2    | 12   | 2    | 21   | 21   | 21   | 21   | 21    | 21    |
| 1, 2, 2, 2        | 1, 2, 2        | 12                | 2    | 12   | 2    | 21   | 21   | 21   | 21   | 21    | 21    |

Table 1: Popov’s classification of $\text{SL}_2$-modules with small $\text{hd} \ R$.
Here \( V \) has a minimal set of generators of size \( r \), with \( d_i \) generators of degree \( i \) (\( 2 \leq i \leq 11 \)). The size of any homogeneous system of parameters (hsop) is \( m \), and the degrees for one particular such system are as given. The column \( \text{hd} \) gives \( \text{hd} R \).

The paper is organised as follows: In §2 we describe the classical results and correct them where needed. In §3 we find a lower bound for \( r \) given the Poincaré series. In §4 we determine \( V \). In §5 we describe how to find a set of generators. A prerequisite is a homogeneous system of parameters, found in §6. The actual generators are constructed in §7.

## 2 The classical results

The table below gives the classical (that is, 19th century) results† ([3, 13–21, 28, 31, 33–36]), possibly slightly amended. Here \( aV_s \) stands for the direct sum \( \oplus_{i=1}^a V_s \) of \( a \) copies of \( V_s \). See also §7.7.

| module | \( r \) | module | \( r \) | module | \( r \) |
|--------|--------|--------|--------|--------|--------|
| \( V_1 \) | 0 | \( V_1 \) | 1 | \( 3V_1 \) | 3 | \( 4V_1 \) | 6 |
| \( V_2 \) | 1 | \( V_1 \oplus V_2 \) | 2 | \( 3V_2 \) | 7 | \( V_1 \oplus 3V_2 \) | 13 |
| \( V_3 \) | 1 | \( V_1 \oplus V_3 \) | 4 | \( 3V_3 \) | 28 | \( V_1 \oplus 3V_3 \) | 97 |
| \( V_4 \) | 2 | \( V_1 \oplus V_4 \) | 5 | \( 3V_4 \) | 25 | \( V_1 \oplus 3V_4 \) | 103 |
| \( V_5 \) | 4 | \( V_1 \oplus V_5 \) | 23 | \( 4V_4 \) | 80 | \( V_1 \oplus 4V_4 \) | 305 |
| \( V_6 \) | 5 | \( V_1 \oplus V_6 \) | 26 | \( V_2 \oplus V_3 \) | 5 | \( V_1 \oplus V_2 \oplus V_3 \) | 15 |
| \( V_7 \) | 6 | \( V_1 \oplus V_7 \) | 147 | \( V_2 \oplus V_4 \) | 6 | \( V_1 \oplus V_2 \oplus V_4 \) | 18 |
| \( V_8 \) | 9 | \( V_1 \oplus V_8 \) | 69 | \( V_2 \oplus V_5 \) | 29 | \( V_1 \oplus V_2 \oplus V_5 \) | 92 |
| \( 2V_2 \) | 3 | \( V_1 \oplus 2V_2 \) | 6 | \( V_2 \oplus V_6 \) | 27 | \( V_1 \oplus V_2 \oplus V_6 \) | 99 |
| \( 2V_3 \) | 7 | \( V_1 \oplus 2V_3 \) | 26 | \( V_3 \oplus V_4 \) | 20 | \( V_1 \oplus V_3 \oplus V_4 \) | 63 |
| \( 2V_4 \) | 8 | \( V_1 \oplus 2V_4 \) | 28 | | | |

### Table 2: The classical results

More generally, Gordan [18, 19] gives for \( V = pV_1 \) the value \( r = \binom{p}{1} \), for \( V = pV_2 \) the value \( r = \binom{p+1}{2} + \binom{p-1}{2} \), and for \( V = V_1 \oplus qV_2 \) the value \( r = q\left(q+1\right) + \binom{q}{2} \), cf. §7.8 below. From the generators in case \( V \oplus V_1 \) one can derive those for \( V \oplus pV_1 \) for all \( p > 1 \), cf. [9, §55], [20, §138A] and §7.7 below.

†In more complicated cases the classical techniques were not powerful enough to determine the precise values of \( r \)—the German school found upper bounds only, the English school claimed to find true values, or at least lower bounds, but the former was mistaken (cf. [22]), the latter unproved.

*Gundelfinger found 64, Sylvester 61, it is 63.

†von Gall found 96, then 67, then 70, Sylvester 69. See also Shioda [30] and Bedratyuk [2].

††von Gall found 33, Sylvester 26, Hammond two more, Dixmier & Lazard 30.

‡‡von Gall found 153, Sylvester 124, Cronil [10] and Bedratyuk 117.

§v on Gall found 96, Sinigallia 97. (Peano [27] has partial results on \( pV_3 \), \( V_1 \oplus pV_3 \).)

||Winter found 94, it is 92.

*Young [36] treats \( pV_4 \) and \( V_1 \oplus pV_4 \) for all \( p \).
3 The number of generators

The Poincaré series of a graded $k$-algebra $R = \oplus_i R_i$ is defined as $P(t) = \sum_i a_i t^i$, where $a_i = \dim_k(R_i)$. Here we consider $k = \mathbb{C}$ and $R = O(V)^{SL_2}$, where $V$ is an $SL_2$-module. Formulas for the coefficients $a_i$ were already given by Cayley and Sylvester. A closed expression for $P(t)$ as a rational function in $t$ was given by Springer [32] for the case of $V = V_n$, and by Brion [4] in general. The webpages [6] list some results of computations due to Bedratynuk and Brouwer that we use.

3.1 Tamisage

Suppose $R$ has Poincaré series $P(t) = \sum a_i t^i$. (Then $a_0 = 1$ and $a_1 = 0$.) Determine numbers $m_i$ as follows: As long as there is an $i > 0$ for which $a_i \neq 0$, find the smallest such $i$. If $a_i < 0$, stop. Otherwise put $m_i := a_i$ and replace $P(t)$ by $P(t)(1-t)^{m_i}$ and repeat. Let undefined $m_i$ be zero. This is the process that Sylvester called ‘tamisage’.

Sylvester’s claim. The number of generators of $R$ is at least $\sum_i m_i$. More precisely: the number of generators of $R$ of degree $i$ is at least $m_i$.

So far this claim is unproved. We use a slightly weaker bound in the below, one that has the advantage of having an easy proof. Maintain two numbers $m_i$ and $M_i$ as lower and upper bounds for the number of generators of degree $i$ in a minimal system of generators. Also maintain upper bounds $M_{ij}$ for the dimension of the space of degree $i$ invariants spanned by those having a factor of degree $j$ but no factor of smaller degree, for $j \leq i$. Put $m_i = a_i - \sum_{j < i} M_{i,j}$ and

$$M_{ij} = \min \{ a_{i-j} a_j, \sum_{t \geq 1, t_j \leq 1} \binom{m + t - 1}{t} S_{t-j,j} \}$$

where $m = M_j$ and $S_{0,j} = 1$ and $S_{a,j} = \sum_{k > j} M_{a,k}$ for $a > 0$. Finally put

$$d_1 = \max_{j < i, a_j \neq 0} a_{i-j},$$

$$d_2 = \max_{j < i, a_j \geq 2} (2a_{i-j} - a_{i-2}),$$

$$d_3 = \max_{j < k < i, a_k \neq 0} (a_{i-j} + a_{i-k} - a_{i-j-k}),$$

$$M_i = a_i - \max\{0, d_1, d_2, d_3\}.$$
$x \mapsto ux$ is an injection of $R_{i-j}$ into $R_i$, so that $d_1$ is such a lower bound. Now consider distinct basic invariants $u \in R_i$ and $v \in R_k$. The images of $x \mapsto ux$ and $y \mapsto vy$ (for $y \in R_{i-k}$) have an intersection consisting of invariants with factor $uv$, so that the dimension of the intersection is $a_{i-j-k}$. This shows that also $d_2$ and $d_3$ are lower bounds. The value given for $m_i$ is clear. Concerning $M_{ij}$, if an invariant of degree $i$ has precisely $t$ factors that are basic invariants of degree $j < i$, then the quotient of degree $i - tj$ can be chosen in (at most) $S_{i-tj}$ ways and the product of $t$ factors can be chosen in $\binom{m+t-1}{t}$ ways.

Now the final lower bound for the number of generators is $r \geq \sum_i m_i$.

**Example.** The Poincaré series $P(t)$ of $O(V_{12})^{SL_2}$ starts

$$1 + t^2 + t^3 + 3t^4 + 3t^5 + 8t^6 + 10t^7 + 20t^8 + 28t^9 + 52t^{10} + 73t^{11} + 127t^{12} + \ldots$$

We find

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| $m_i$ | 1 | 2 | 2 | 4 | 5 | 7 | 9 | 14 | 12 | 9 | 0 | 0 | 0 | 0 | 0 | 0 |
| $g_i$ | 1 | 1 | 2 | 2 | 4 | 5 | 7 | 9 | 14 | 15 | 19 | 18 | 12 | 2 | 1 | 1 |
| $M_i$ | 1 | 1 | 2 | 2 | 4 | 5 | 10 | 13 | 25 | 33 | 57 | 76 |

so that $r \geq 66$, $hdR \geq 56$. The row $g_i$ gives the actual number of generators of degree $i$ (known in this case, cf. [5]), so that $r \geq 113$, $hdR \geq 103$.

### 3.2 Bounds

In the table below, we list modules, the Poincaré series, and lower bounds for $r$ and $hdR$. In many cases, better bounds are obtained by taking more terms.

| module | Poincaré series | $r \geq$ | $hdR \geq$ |
|--------|-----------------|--------|-----------|
| $V_{11}$ | $1 + 2t^4 + 13t^5 + 13t^{10} + 73t^{12} + 110t^{14} + \ldots$ | 158 | 149 |
| $V_{13}$ | $1 + 2t^4 + 22t^5 + 33t^{10} + 181t^{12} + 375t^{14} + \ldots$ | 502 | 491 |
| $V_{14}$ | $1 + t^2 + 3t^4 + 10t^6 + 4t^7 + 31t^8 + 27t^9 + 97t^{10} + 110t^{11} + \ldots$ | 182 | 170 |
| $V_{16}$ | $1 + t^2 + 3t^4 + 3t^5 + 4t^7 + 13t^8 + 18t^9 + 47t^{10} + \ldots$ | 425 | 412 |
| $V_{18}$ | $1 + t^2 + 4t^4 + 5t^5 + 16t^6 + 13t^7 + 27t^8 + 99t^9 + \ldots$ | 198 | 184 |
| $V_{20}$ | $1 + t^2 + 3t^4 + 4t^5 + 5t^6 + 20t^7 + 35t^8 + 102t^9 + \ldots$ | 164 | 144 |
| $V_{22}$ | $1 + t^2 + 4t^4 + 5t^5 + 24t^6 + 26t^7 + 144t^8 + \ldots$ | 123 | 105 |
| $V_{24}$ | $1 + t^2 + 3t^4 + 5t^5 + 7t^6 + 29t^9 + 62t^7 + 201t^8 + \ldots$ | 440 | 414 |
| $V_{28}$ | $1 + t^2 + 3t^4 + 5t^5 + 8t^6 + 40t^7 + 97t^8 + 365t^9 + \ldots$ | 156 | 144 |
| $V_{32}$ | $1 + t^2 + 3t^4 + 6t^5 + 10t^7 + 54t^8 + 153t^9 + \ldots$ | 201 | 171 |
| $V_{2} \oplus V_{6}$ | $1 + 2t^4 + t^5 + 5t^9 + 15t^6 + 17t^7 + 11t^8 + 54t^9 + 108t^{10} + \ldots$ | 35 | 26 |
| $V_{6}$ | $1 + t^2 + 3t^4 + 4t^5 + 9t^6 + 16t^7 + 30t^8 + \ldots$ | 37 | 27 |
| $V_{4} \oplus V_{6}$ | $1 + 2t^2 + 4t^3 + 8t^4 + 16t^5 + 35t^6 + 60t^7 + \ldots$ | 42 | 31 |
| $V_{5} \oplus V_{6}$ | $1 + t^2 + 3t^4 + 6t^5 + 15t^6 + 31t^7 + \ldots$ | 43 | 31 |
| $V_{6}$ | $1 + 2t^2 + 2t^3 + 10t^4 + 14t^5 + 46t^6 + 82t^7 + \ldots$ | 88 | 75 |
4 Determining $V$

Consider $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$ with $n_i \geq 1$ for all $i$. Let $R := \mathcal{O}(V)^{SL_2}$ be the algebra of invariants of $V$. We want to determine $V$ if either $p = 1$ and $\text{hd } R \leq 100$, or $p > 1$ and $\text{hd } R \leq 15$.

First consider the case $p = 1$, $V = V_n$. By [29, Proposition 6], if $n$ is even and $\text{hd } R \leq 100$, then $n \leq 24$ or $n \in \{28, 32\}$. By [25, p. 106], if $n$ is odd, then $r \geq p(n - 2) + \phi(n - 2) - 1$, where $p(n)$ is the partition function and $\phi(n)$ is Euler’s totient function. It follows that $\text{hd } R \geq 168$ for odd $n \geq 17$. We know $\text{hd } R$ for $n \leq 10$ (a table was given above), and $\text{hd } R \geq 103$ for $n = 12$ (see the example above), and for the remaining values we found $\text{hd } R \geq 105$ in Table 3. This proves Theorem 1.1.

Now consider the case $p > 1$ and assume $\text{hd } R \leq 15$. By the monotony theorem [29, Theorem 2b] we have if $V = W \oplus W'$, then $\text{hd } R \geq \text{hd } \mathcal{O}(W)^{SL_2} + \text{hd } \mathcal{O}(W')^{SL_2}$. Therefore, all $n_i$ belong to $\{1, 2, 3, 4, 5, 6, 8\}$, and direct summands $W$ have $\text{hd } \mathcal{O}(W)^{SL_2} \leq 15$.

If all $n_i$ are either 1 or 2, so that $V = mV_1 \oplus nV_2$, then we have the explicit formula $r = \binom{m}{2} + \binom{n+1}{2} \binom{n+1}{2} + \binom{m}{2} + \binom{n+1}{2}$ (see §7.8 below), and $\text{hd } R = $
$r - (3n + 2m - 3)$ for $m + n > 1$. A table of $\text{hd} \ R$ shows that $V$ is as claimed.

| $n \setminus m$ | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0               | 0   | 0   | 0   | 0   | 1   | 3   | 6   | 10  | 15  | 21  |
| 1               | 0   | 0   | 1   | 4   | 9   | 16  | 25  |  |
| 2               | 0   | 1   | 6   | 15  | 28  |     |     |     |     |     |
| 3               | 1   | 5   | 16  | 34  |     |     |     |     |     |     |
| 4               | 5   | 13  | 32  |     |     |     |     |     |     |     |
| 5               | 13  | 26  |     |     |     |     |     |     |     |     |
| 6               | 26  |     |     |     |     |     |     |     |     |     |

Now investigate the remaining possibilities.

Case 1: One of the $n_i$, say $n_1$, is equal to 8.
By the discussion in the proof of [29, Theorem 4], $\text{hd} \ R \geq \frac{(b-1)(b-2)}{4}$, where $b = 3 + \frac{[n_2 + 1]}{2} + \cdots + \frac{[n_n + 1]}{2}$. If $b \geq 8$, then $\text{hd} \ R \geq 21$. We check the cases with $b \leq 7$. By monotony it suffices to look at $V_m \oplus V_b$ for $m = 1, 2, 3, 4, 5, 6$. If $V = V_1 \oplus V_5$ then $R$ is the algebra of covariants of $V_6$, generated by 60 elements ([2]), and $\text{hd} \ R = 61$. In the other cases $\text{hd} \ R \geq 26$ by Table 3.

Case 2: One of the $n_i$, say $n_1$, is equal to 2.
By the discussion in the proof of [29, Theorem 4], $\text{hd} \ R \geq (c - 1)^2$, where $c = \frac{[3n + 1]}{2} + \frac{[n_2 + 1]}{2} + \cdots + \frac{[n_n + 1]}{2}$. Since $n_i > 2$ for some $i$, we have $c \geq 2$. We have $\text{hd} \ R \geq 16$ for $c \geq 5$. We therefore check the cases $c \in \{2, 3, 4\}$.

If $c = 2$, then $V$ is $V_2 \oplus V_3$ or $V_2 \oplus V_4$ and $\text{hd} \ R = 1$.
If $c = 3$, then $V$ is one of $V_1 \oplus V_2 \oplus V_3, V_1 \oplus V_2 \oplus V_4, 2V_2 \oplus V_3, 2V_2 \oplus V_4, V_2 \oplus V_5$ or $V_2 \oplus V_6$. In these six cases one has $\text{hd} \ R = 9, 11, 11, 11, 23, 20$, respectively. (For the first two and last two cases, see Table 2. For the other two, see §§7.3, 7.4.)

If $c = 4$, then by monotony and the above $V$ does not have a direct summand $V_5$ or $V_6$, so that $V$ is one of $V_2 \oplus 2V_3, V_2 \oplus V_5 \oplus V_3, V_2 \oplus 2V_4, 2V_1 \oplus V_2 \oplus V_3, 2V_1 \oplus V_2 \oplus V_4, V_1 \oplus V_2 \oplus V_5, V_1 \oplus V_2 \oplus V_6, 3V_2 \oplus V_3, 3V_2 \oplus V_4, 3V_1 \oplus V_4$. If $V$ is $2V_1 \oplus V_2 \oplus V_3$ or $2V_1 \oplus V_2 \oplus V_4$, then $\text{hd} \ R = 27$ or 48 by Table 4. Explicit generation of invariants for $V_2 \oplus 2V_3$ and $3V_2 \oplus V_4$ shows that $r \geq 29, 49$ so that $\text{hd} \ R \geq 19, 39$ in these cases. By Table 3 $\text{hd} \ R \geq 17$ in the remaining five cases.

Case 3: All of the $n_i$ equal 1, 3, 4, 5 or 6.
If $V$ is $V_1 \oplus V_2, V_1 \oplus V_4, V_1 \oplus V_5, V_2 \oplus V_4, V_3 \oplus V_5$, or $V_1 \oplus V_6$, then $\text{hd} \ R$ equals 1, 1, 2, 14, 1, 18, 20, respectively, by Table 2. If $V$ is $V_2 \oplus V_3, V_4 \oplus V_5, V_4 \oplus V_6, V_4 \oplus V_6, V_5 \oplus V_6, 2V_6, 2V_5 \oplus V_1$, or $V_5 \oplus 2V_4$, then $\text{hd} \ R \geq 16$ by Table 3.
If $V$ is $2V_1 \oplus V_3, 2V_1 \oplus V_4, V_1 \oplus V_2, V_1 \oplus V_3, V_1 \oplus V_4, V_1 \oplus 2V_4, 3V_5, 3V_4, 3V_1 \oplus V_5, 3V_4 \oplus V_5, 4V_4$, then $\text{hd} \ R$ equals 8, 14, 19, 55, 19, 19, 13, 23, 55, 63, respectively, by Tables 2 and 4. By monotony we are done.

This finishes the determination of the $V$ with $\text{hd} \ R \leq 15$. 

7
5 Finding generators

Let \( V \) be an \( SL_2 \)-module, and \( R = \mathcal{O}(V)^{SL_2} \) its algebra of invariants. Finding a minimal set of generators of \( R \) is routine, only requiring computational power, if a good upper bound for the maximum degree of these generators is known. Details for \( V_9 \) and \( V_{10} \) were given in [7, 8]. (The cases considered here are much smaller.) An upper bound for the maximum degree of a basic generator follows from the Poincaré series when a hsop (homogeneous system of parameters), or at least the set of degrees of a hsop, is known.

5.1 Hilbert’s Criterion

One way of computing a system of parameters of \( R \) is finding equations for the nullcone of \( V \). The nullcone of \( V \), denoted \( \mathcal{N}(V) \), is the set of all elements of \( V \) on which all invariants vanish. One shows that \( \mathcal{N}(V_{n_1} \oplus ... \oplus V_{n_p}) \) is the set of all \( (f_1, ..., f_p) \in V_{n_1} \oplus ... \oplus V_{n_p} \) such that \( f_1, ..., f_p \) have a common root of multiplicity \( > \frac{1}{2}n_i \) in \( f_i \) for all \( i = 1, ..., p \). This is a consequence of the Hilbert-Mumford criterion. Let \( \mathcal{V}(J) \) stand for the vanishing locus of \( J \).

**Proposition 5.1.** (Hilbert [24]) Let \( V = V_{n_1} \oplus ... \oplus V_{n_p} \), and \( R = \mathcal{O}(V)^{SL_2} \), and \( m = n_1 + ... + n_p + p - 3 > 0 \). A set \( P_1, ..., P_m \) of homogeneous elements of \( R \) is a system of parameters of \( R \) if and only if \( \mathcal{V}(P_1, ..., P_m) = \mathcal{N}(V) \).

5.2 Dixmier’s Criterion

Since we do not actually need the hsop but only the degrees, the following is often easier to apply than Hilbert’s Criterion.

**Proposition 5.2.** (Dixmier [11]) Let \( G \) be a reductive group over \( \mathbb{C} \), with a rational representation in a vector space \( V \) of finite dimension over \( \mathbb{C} \). Let \( \mathcal{O}(V) \) be the algebra of complex polynomials on \( V \), \( R := \mathcal{O}(V)^G \) the subalgebra of \( G \)-invariants, and \( R_d \) the subset of homogeneous polynomials of degree \( d \) in \( R \). Let \( X \) be the affine variety such that \( \mathbb{C}[X] = R \). Let \( m = \text{dim} X \). Let \( (d_1, ..., d_m) \) be a sequence of positive integers. Assume that for each subsequence \( (j_1, ..., j_p) \) of \( (d_1, ..., d_m) \) the subset of points of \( X \) where all elements of all \( \mathcal{R}_j \) with \( j \in \{j_1, ..., j_p\} \) vanish has codimension not less than \( p \) in \( V \). Then \( R \) has a system of parameters of degrees \( d_1, ..., d_m \).

\[ \Box \]

When applying this criterion it is convenient to have a notation for ‘the codimension of the subset of \( X \) defined by the vanishing of all invariants with degree in \( \{j_1, ..., j_p\} \)’. We’ll use \( [j_1, ..., j_p] \).

Note that for \( e \geq 1 \) an invariant \( g^e \) vanishes if and only if \( g \) vanishes. It follows that if \( j_h | j_i, h \neq i \), then \( [j_1, ..., j_p] = [j_1, ..., j_{h-1}, j_{h+1}, ..., j_p] \).

5.3 From hsop degrees to generator degrees

Let \( R := \mathcal{O}(V)^{SL_2} \). This is a graded algebra, and formulas to compute its Poincaré series \( P(t) \) are well-known. If \( (P_1, ..., P_m) \) is a system of parameters
of \( R \), with degree sequence \((d_1, \ldots, d_m)\), then \( P(t) \) can be written as

\[
P(t) = \frac{t^{e_1} + \ldots + t^{e_s}}{(1 - t^{d_1}) \ldots (1 - t^{d_m})}
\]

and there exist homogeneous \( G_1, \ldots, G_s \in R \) with degrees \( e_1, \ldots, e_s \), such that

\[
R = \bigoplus_{i=1}^{s} G_i \mathbb{C}[P_1, \ldots, P_m].
\]

Now \( \{P_1, \ldots, P_m, G_1, \ldots, G_s\} \) is a (not necessarily minimal) system of generators of \( R \), and \( \max\{d_1, \ldots, d_m, e_1, \ldots, e_s\} \) is an upper bound for the degrees of a set of generators of \( R \).

### 5.4 Polarization

Let \( j = j(f) \) be an invariant of degree \( d \) defined on forms \( f \in V_n \). Let \( i = (i_1, \ldots, i_s) \) be a sequence of nonnegative integers with \( \sum i_h = d \). The \( i \)-polarizations \( j_i \) of \( j \) are defined on \( sV_n \) by

\[
j_i(\sum \lambda_i f_i) = \sum_{i} j_i(f_1, \ldots, f_s) \lambda_i^{i_1} \ldots \lambda_s^{i_s}.
\]

Kraft & Wallach [26] showed for \( n > 1 \) that \( N(sV_n) \) is defined by the polarizations of any set of functions defining \( N(V_n) \).

### 5.5 Transvectants

The simplest examples of invariants are obtained using transvectants. Given \( g \in V_m \) and \( h \in V_n \) the expression

\[
(g, h) \mapsto (g, h)_p := \frac{(m-p)! (n-p)!}{m! n!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \frac{\partial^p g}{\partial x^{p-i} \partial y^i} \frac{\partial^p h}{\partial x^p \partial y^{p-i}}
\]

defines a linear and \( \text{SL}_2 \)-equivariant map \( V_m \otimes V_n \to V_{m+n-2p} \), which is classically called the \( p \)-th transvectant (Uberschiebung). The \( (g, h)_p \) are the coefficients of the image of \( g \otimes h \) under the isomorphism (Clebsch-Gordan formula)

\[
V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{m-n}
\]

(for \( m \geq n \)). We have \( (f, g)_0 = fg \) and \( (f, f)_{2i+1} = 0 \) for all integers \( i \geq 0 \).

### 6 Systems of parameters

In this section we give a homogeneous system of parameters (or at least the degrees of a homogeneous system of parameters) for the algebras of invariants in the cases occurring in Theorem 1.2.
Let $\text{discr}(f)$ denote the discriminant of the polynomial $f$. If $f$ has degree $m$, then $\text{discr}(f)$ is an expression of degree $2m - 2$ in the coefficients of $f$. This expression vanishes if and only if $f$ has a root of multiplicity greater than 1.

Let $\text{res}(f, g)$ denote the resultant of the polynomials $f$ and $g$. If $f$ and $g$ have degrees $m$ and $n$, respectively, then $\text{res}(f, g)$ is an expression of degree $m + n$ in the coefficients of $f$ and $g$. It vanishes if and only if $f$ and $g$ have a common root.

Let $\sim$ denote equality up to a nonzero constant.

**Lemma 6.1.**

(i) Let $l, m \in V_1$. Then $\text{res}(l, m) = (l, m)_1$.

(ii) Let $q \in V_2$. Then $\text{discr}(q) \sim (q, q)_2$.

(iii) Let $l \in V_1$ and $q \in V_2$. Then $\text{res}(l, q) = (q, l^2)_2$.

(iv) Let $q, r \in V_2$. Then $\text{res}(q, r) = (q, r)^2 - (q, q)_2 (r, r)_2$.

(v) Let $f \in V_3$. Then $\text{discr}(f) \sim (f, (f, f)_2)_1$.

(vi) Let $l \in V_1$ and $f \in V_3$. Then $\text{res}(l, f) \sim (f, l^3)_3$.

Brion [4] shows for $V = V_n = V_{n_1} \oplus \ldots \oplus V_{n_p}$ with $p > 1$ that a multihomogeneous system of parameters exists in only 13 cases, namely precisely the cases with $p \in \{2, 3\}$ in Popov's classification (Table 1). We give the systems here (improving that for $2V_3$), together with those for $V_2, V_3, V_4$, for later use.

**Proposition 6.2.** The modules $V$ given in the table below, with generic elements as indicated, have a (multi-)homogeneous system of parameters as given.

| $V$   | element | $\text{hsop degrees}$ | $\text{hsop}(V)$ |
|-------|---------|------------------------|------------------|
| $V_2$ | $q$     | 2                      | $(q, q)_2$       |
| $V_3$ | $c$     | 4                      | $\text{discr}(c)$ |
| $V_4$ | $f$     | 2, 3                   | $(f, f)_4, (f, (f, f)_2)_4$ |
| $2V_1$ | $(l, m)$ | 2                      | $(l, m)_1$       |
| $V_1 \oplus V_2$ | $(l, q)$ | 2, 3                   | $(q, q)_2, (q, l^2)_2$ |
| $V_1 \oplus V_3$ | $(l, c)$ | 4, 4, 4                | $\text{hsop}(V_3), (c, l^3)_3, (c, (c, l^2)_1)_3$ |
| $V_1 \oplus V_4$ | $(l, f)$ | 2, 3, 5, 6             | $\text{hsop}(V_4), (f, l^4)_1, ((f, f)_2, l^4)_4$ |
| $2V_2$ | $(q, r)$ | 2, 2                   | $\text{polarized hsop}(V_2)$ |
| $V_2 \oplus V_3$ | $(q, c)$ | 2, 3, 4, 5             | $(q, q)_2, (c, (c, q)_1)_3, \text{hsop}(V_3), \text{res}(q, c)$ |
| $V_2 \oplus V_4$ | $(q, f)$ | 2, 2, 3, 4, 4          | $(q, q)_2, \text{hsop}(V_4), (f, q^2)_4, ((f, f)_2, q^2)_4$ |
| $2V_3$ | $(c, d)$ | 2, 4, 4, 4             | $(c, d)_3, \text{discr}(c), \text{discr}(d)$, $(c, (c, d)_2)_1)_3, (d, (d, (c, d)_2))_1)_3$ |
| $2V_4$ | $(f, g)$ | 2, 2, 3, 3, 3, 3, 3, 3, 3, 3 | $\text{polarized hsop}(V_4)$ |
| $3V_1$ | $(l, m, n)$ | 2, 2, 2                | $(l, m)_1, (l, n)_1, (m, n)_1$ |
| $2V_1 \oplus V_2$ | $(l, m, q)$ | 2, 2, 3, 3             | $(l, m)_1, (q, q)_2, (g, l^2)_2, (q, m^2)_2$ |
| $V_1 \oplus 2V_2$ | $(l, q, r)$ | 2, 2, 2, 3, 3          | $\text{hsop}(2V_2), (q, l^2)_2, (r, l^2)_2$ |
| $3V_2$ | $(q, r, s)$ | 2, 2, 2, 2, 2          | $\text{polarized hsop}(V_2)$ |
6.1 The cases $V = pV_1$ and $V = pV_2$

Let $V = pV_1$ (or $V = pV_2$), where $p > 1$. Let $I$ be the ideal of $R$ generated by the invariants of degree 2. By Lemma 6.1 (i) (or (ii), (iv)) we have $\mathcal{N}(V) = V(I)$. Consider the proof of the Noether Normalization Lemma. In the situation of an algebra where all generators are homogeneous of the same degree, it produces a homogeneous set of parameters of this same degree. Therefore, $I$, and hence $R$, has a homogeneous system of parameters consisting of $2p - 3$ (or $3p - 3$) elements of degree 2. (For generators, see §7.8 below.)

6.2 The case $V = mV_1 \oplus nV_2$

**Proposition 6.3.** Let $V = mV_1 \oplus nV_2$ with $m, n > 0$, and let $R = \mathcal{O}(V)^{\text{SL}_2}$.

If $2n + 1 \geq m$, then $R$ has a system of parameters consisting of $m + 2n - 2$ invariants of degree 2 and $m + n - 1$ invariants of degree 3.

If $2n + 1 < m$, then $R$ has a system of parameters consisting of $m + 2n - 2$ invariants of degree 2 and $3n$ invariants of degree 3, and $m - 2n - 1$ invariants of degree 6.

*Proof.* Invoke Dixmier’s Criterion. We have $\dim X = 2m + 3n - 3$. We have to show that $[2] \geq m + 2n - 2$ and $[3] \geq m + n - 1$ (if $2n + 1 \geq m$) and $[2, 3] \geq 2m + 3n - 3$.

Consider $v = (l_1, \ldots, l_m, q_1, \ldots, q_n) \in V$. According to Lemma 6.1, among the invariants of degree 2 there are $\text{res}(l_i, l_j)$ and $\text{discr}(q_i)$ and $\text{res}(q_i, q_j)$ for all $i, j$ ($j \neq i$), and among the invariants of degree 3 there are the $\text{res}(l_i, q_j)$. If all of these vanish on $v$, then $v \in \mathcal{N}(V)$. Hence $[2, 3] = \dim X$, as desired.

Look at $[2]$. The element $g = (a\ b) \in \text{SL}_2$ acts via $g.x = dx - by$, $g.y = -cx + ay$. Using $\text{SL}_2$ one can move the first nonzero linear form (if there is one) to $x$. Given that all linear forms have a common zero, this means that the remaining at most $m - 1$ linear forms now look like $cx$ for various constants $c$, zero or not. Given that all quadratic forms have a common double root, and that elements $(a\ 0)$ preserve $x$ and act on the $q_i$, we can pick orbit representatives $dy^2$ (for nonzero $d$) for the quadratic forms, unless none of them involve $y$, in which case we are in the nullcone. Altogether the result has dimension at most $m + n - 1$, so that $[2] \geq \dim X - (m + n - 1) = m + 2n - 2$, as desired.

Look at $[3]$. Now we are in a part of $X$ where each of the quadratic forms shares a zero with each of the linear forms. Suppose first that at least one linear form and at least one quadratic form are nonzero. We can pick $x$ as orbit-representative for one linear form. All quadratic forms now look like $x(ax + by)$ for various $a, b$, and we can normalize one to $cx^2$ or $cxy$ with nonzero $c$. All linear forms now look like $dx$ or $dy$ for various constants $d$. Altogether the result has dimension at most $m - 1 + 2(n - 1) + 1 = m + 2n - 2$, as desired.

If all linear forms are zero, then consider the invariants $(q_i, (q_j, q_k))_2$ with $1 \leq i < j < k \leq n$ of degree 3. Let us compute this for the case of the three forms $ax^2 + 2bxy + cy^2$, $dx^2 + 2exy + fy^2$, $gx^2 + 2hxy + iy^2$. The result is (up
to a constant] \( aei - afh - bdi + bfg + cdh - ceg \), the determinant

\[
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{vmatrix}.
\]

If all such determinants vanish, then any three of the quadratic forms are linearly dependent, so that \( n \) quadratic forms involve at most \( 2n + 2 - 3 = 2n - 1 \) constants, after dividing out \( \text{SL}_2 \). Since \( m \geq 1 \), this is not more than \( m + 2n - 2 \) and we also have the desired bound in this case.

If all quadratic forms are zero, then the result has dimension \( 2m - 3 \), so codimension \( 3n \). We showed \([3] \geq \min(m + n - 1, 3n)\). It follows that for \( mV_1 \oplus nV_2 \) with \( 2n \geq m - 1 \geq 0 \) there is a hsop with \( m + 2n - 2 \) elements of degree 2 and \( m + n - 1 \) elements of degree 3. This proves the first claim.

For the second claim we still have to show that \([6] \geq 1 \) and \([2, 6] \geq m + 2n - 1 \) and \([3, 6] \geq m + n - 1 \), but \([6] \geq [2, 3] = 2m + 3n - 3 \).

\[\square\]

6.3 The case \( V = 2V_1 \oplus V_3 \)

Let \( V = 2V_1 \oplus V_3 \). We have \( \dim X = 5 \). The ring \( R \) has a homogeneous system of parameters with degrees 2, 4, 4, 4, 4.

Indeed, pick \((l, m, c) \in V\). By Proposition 6.2 we find \((l, m) \in \mathcal{N}(2V_1)\) and \((l, c), (m, c) \in \mathcal{N}(V_1 \oplus V_3)\) (and hence \((l, m, c) \in \mathcal{N}(V)\)) when all invariants of degree 4 vanish on \((l, m, c)\). This shows that there is a hsop with degrees 4, 4, 4, 4, 4. Since \([2] \geq 1 \) there is also a hsop with degrees 2, 4, 4, 4, 4 by Dixmier’s Criterion.

6.4 The case \( V = V_1 \oplus V_2 \oplus V_3 \)

Let \( V = V_1 \oplus V_2 \oplus V_3 \). We have \( \dim X = 6 \). We show that the ring \( R \) has a system of parameters with degrees 3, 3, 4, 4, 4, 5.

A form on which all invariants of degrees 3, 4, 5 vanish is in the nullcone, so that \([3, 4, 5] = 6 \). We have to check that \([5] \geq 1\), \([3] \geq 2\), \([4] \geq 3\), \([3, 5] \geq 3\), \([4, 5] \geq 4\), \([3, 4] \geq 5\). Let the forms be \( l, q, f \).

If all invariants of degree 4 vanish, then \( f \) has a double root that is also a root of \( l \), and \( q \) has a double root and only 3 variables are left, so \([4] \geq 3\). If moreover \( \text{res}(q, f) \) vanishes, then only pieces of dimension 2 are left, so \([4, 5] \geq 4\).

Or, if moreover \((q, l^2)\) and \((f, (f, q)_{1, 1})\) vanish, then we are in the nullcone or in a piece of dimension at most 1, so that \([3, 4] \geq 5\).

If all invariants of degree 3 vanish (and in particular \( \text{res}(q, l) \) and \((f, lq)\)) and \((f, (f, q)_{1, 1})\), and \( q \neq 0 \), then w.l.o.g. either \( q = x^2 \), \( l = ax, f = bx^3 + 3cx^2y + 3dxy^2 + ey^3 \) with \( ac = 0 \) and \( ce - d^2 = 0 \), or \( q = hxy, l = ax, f = bx^3 + 3cx^2y + 3dxy^2 + ey^3 \) with \( ae = 0 \) and \( be = cd \), of dimension at most 3 (since the torus of \( \text{SL}_2 \) still acts). If \( q = 0 \) then we are in \( V_1 \oplus V_3 \) of dimension 3. Altogether \([3, 5] \geq [3] \geq 3 \). (In fact \([3, 5] = 3 \) because of the part with \( q = 0 \).)

Finally \([5] \geq 1\) is clear.

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Note that in this case the denominator of \( P(t) \) might suggest to look for a hsop with degrees 2, 3, 4, 5, but there is none since \([2, 3, 5] = 3\).

6.5 The case \( V = V_1 \oplus V_2 \oplus V_4 \)

Let \( V = V_1 \oplus V_2 \oplus V_4 \). We have \( \dim X = 7 \). We show that the ring \( R \) has a system of parameters with degrees 2, 3, 4, 5, 6.

Since \((l, q, f) \in \mathcal{N}(V)\) if and only if \((l, q) \in \mathcal{N}(V_1 \oplus V_2)\) and \((l, f) \in \mathcal{N}(V_1 \oplus V_4)\) and \((q, f) \in \mathcal{N}(V_2 \oplus V_4)\), it follows that if the eight invariants \((q, q)_{2}, (f, f)_{4}, (f, (f, f)_{2})_{5}, (q, (f, f)_{2})_{6}, (f, (f, f)_{2})_{7}, ((f, f)_{2})_{8}, q^{2}_{4}, (f, l^{2})_{4}, ((f, f)_{2}, l^{2})_{6}\) (of degrees 2, 3, 4, 5, 6) vanish, then \((l, q, f) \in \mathcal{N}(V)\).

The above five invariants of degrees 2, 4, 5, 6, together with two random linear combinations of the invariants of degree 3 will constitute a hsop. In particular, we find that the two combinations \((f, (f, f)_{2})_{4} + (f, q^{2})_{4}\) and \((q, (f, f)_{2})_{6} - (f, (f, f)_{2})_{4}\) yield such a hsop. (Using Singular one finds that the ideal generated by these seven invariants contains the sixth power of each invariant of degree 3. Now apply Proposition 5.1.)

Note that in this case the denominator of \( P(t) \) suggests the existence of a hsop with degrees 2, 3, 4, 5, 6. But such a hsop does not exist: all invariants of degrees 2 up to 3 vanish on \((x, 0, 4xy^{3})\), which is not in \( \mathcal{N}(V)\).

6.6 The case \( V = 2V_{2} \oplus V_{3} \)

Let \( V = 2V_{2} \oplus V_{3} \). We have \( \dim X = 7 \). The ring \( R \) has a system of parameters with degrees 2, 3, 4, 5, 6.

Indeed, we have to check that \( [3] \geq 1, [2] \geq 2, [5] \geq 2, [4] \geq 3, [2, 3] \geq 3, [3, 5] \geq 3, [6] \geq 4, [2, 5] \geq 4, [3, 4] \geq 4, [4, 5] \geq 5, [4, 6] \geq 5, [2, 3, 5] \geq 5, [5, 6] \geq 6, [3, 4, 5] \geq 6, [4, 5, 6] \geq 7\).

Let the forms be \( q, r, f \). If \( f \neq 0 \), then we can normalize \( f \) to one of \( x^{3}, x^{2}y \) or \( ary(x + y) \). If \( f = 0 \) but \( q \neq 0 \), then we can normalize \( q \) to \( x^{2} \) or \( xy \).

If all invariants of degree 2 vanish, then \( \text{discr}(q) = \text{discr}(r) = \text{res}(q, r) = 0 \) so that \( q \) and \( r \) have a common double zero. Now if \( q \neq 0 \), then \( r \) is determined by a single constant. So dimensions are at most 1 larger than for the corresponding case for \( V_{2} \oplus V_{3} \). Hence \([2] \geq 3, [4] \geq 4, [6] \geq [2, 3] \geq 4, [2, 5] \geq 4, [4, 6] \geq [3, 4] \geq 5, [4, 5] \geq 5, [2, 3, 5] \geq 5, [3, 4, 5] \geq 6\).

If all invariants of degree 3 vanish, then \((f, (f, q))_{3} = (f, (f, r))_{3} = 0\). That \((f, (f, q))_{3} = 0\) says that the three quadratic forms \( \frac{df}{dx}, \frac{df}{dy} \) and \( q \) are linearly dependent. So, here either \( f \) has a triple root, or \( f_{x} \) and \( f_{y} \) are linearly independent and \( q, r \) are determined by two coefficients each. Hence \([3] \geq 2\).

If all invariants of degree 5 vanish, then \( \text{res}(f, q) = \text{res}(f, r) = 0 \) restrict \( q, r \) when \( f \neq 0 \). Hence \([5] \geq 2\).

If all invariants of degrees 3, 5 vanish, then if \( f \) does not have a double root, then \( q, r \) are determined by one coefficient each. If \( f = x^{2}y \) or \( f = x^{3} \), then \( q, r \) are determined by two coefficients each. If \( f = 0 \), then we are in \( 2V_{2} \) which has dimension 3. Hence \([3, 5] \geq 3\).
If all invariants of degrees 2, 3, 5 vanish, then either \( f = axy(x+y), q = r = 0 \), or \( (f = x^2y \text{ or } f = x^3), q = bx^2, r = cx^2 \) and \((q, r, f) \in \mathcal{N}(V)\), or \( f = 0 \) and again \((q, r, f) \in \mathcal{N}(V)\). Hence \([5, 6] \geq [2, 3, 5] \geq 6 \) and \([3, 4, 5] \geq [2, 3, 5] \geq 6 \).

Finally, if all invariants of degrees 2, 3, 4, 5 vanish, then \( \text{disc}(f) = 0 \) and we are in the nullcone.

Note that in this case the denominator of \( P(t) \) might suggest to look for a hsop with degrees 2, 2, 3, 3, 4, 5, 5, but there is none since \([3, 5] = 3\).

6.7 The case \( V = 2V_2 \oplus V_4 \)

Let \( V = 2V_2 \oplus V_4 \). We have \( \dim X = 8 \). We show that the ring \( R \) has a system of parameters with degrees \( 2, 2, 2, 3, 3, 3, 4, 4 \).

Since \((q, r, f) \in \mathcal{N}(V)\) if and only if \((q, r) \in \mathcal{N}(2V_2)\) and \((q, f, r, f) \in \mathcal{N}(V_2 \oplus V_4)\), it follows that if the nine invariants \((q, q)_2, (q, r)_2, (r, r)_2, (f, f)_4, (f, (f, f)_2)_4, (f, q^2)_4, (f, r^2)_4, ((f, f)_2, q^2)_4, \) and \(((f, f)_2, r^2)_4 \) (of degrees 2, 2, 2, 2, 3, 3, 3, 4, 4) vanish, then \((q, r, f) \in \mathcal{N}(V)\).

The above five invariants of degrees 3, 4, together with three random linear combinations of the four invariants of degree 2 will constitute a hsop. In particular, we find that the three combinations \((q, q)_2 + (q, r)_2, (r, r)_2 + (f, f)_4,\) and \((q, q)_2 - (f, f)_4 \) yield such a hsop. (Using Singular one finds that the ideal generated by these eight invariants contains the 7th power of each invariant of degree 2. Now apply Proposition 5.1.)

Note that in this case the denominator of \( P(t) \) might suggest to look for a hsop with degrees 2, 2, 2, 3, 3, 3, 4, but no such hsop exists since all invariants of degree 2 or 3 vanish on \( (ax^2, bx^2, xy^3) \), so that \([2, 3] \leq 6\).

6.8 The case \( V = 3V_4 \)

Let \( V = 3V_4 \). We have \( \dim X = 12 \). We show that the ring \( R \) has a system of parameters consisting of 6 invariants of degree 2 and 6 of degree 3.

Indeed, since \( 2V_4 \) has a hsop consisting of elements of degrees 2 and 3 only, \( \mathcal{N}(3V_4) \) is determined by elements of degrees 2, 3, so that \([2, 3] = 12 \). We have \( \dim R_2 = 6 \), and using Singular we find that \([2] \geq 6 \). Since \([2, 3] = 12 \), and the six invariants of degree 2 can decrease dimensions by not more than 6, we must have \([3] \geq 6 \). By Dixmier’s Criterion there is a hsop as claimed.

On the other hand, using Singular we also find that \([3] \geq 7 \). It follows that there is also a hsop consisting of 5 invariants of degree 2 and 7 of degree 3.

6.9 The case \( V = 2V_1 + V_4 \)

Let \( V = 2V_1 + V_4 \). We have \( \dim X = 6 \). We show that the ring \( R \) has a system of parameters with degrees 2, 3, 5, 5, 6, 6.

Indeed, let the forms be \( l, m, f \). The vanishing of \((f, f)_2, (f, f)_3, (f, t^4)_4, ((f, f)_2, t^4)_4, (f, m^4)_4, ((f, f)_2, m^4)_4\) implies that \( l \) and \( f \), and also \( m \) and \( f \), have a common zero of multiplicity at least 3 for \( f \). If \( f \neq 0 \) then
Given the Poincaré series and the hsop degrees, it suffices to check that these generators generate $R_d$ for $d \leq 14$, and they do.

### 7.2 The case $V = V_1 \oplus V_2 \oplus V_4$

Let $V = V_1 \oplus V_2 \oplus V_4$. Here $r = 18$, cf. [18], [33]. Let the forms be $l, q, f$ of degrees 1, 2, 4, respectively. The table below gives the 18 basic generators with
degree and multidegree.

| dg mdeg form | dg mdeg form | dg mdeg form |
|-------------|-------------|-------------|
| 2 002 \(q,q\) \(2\) | 4 022 \((f,f)_2, q^2\) \(4\) | 6 222 \((f,l_1, (l, q)_1) \cdot (f,q)_2\) \(4\) |
| 2 210 \((q, l)_2\) | 5 401 \((f, l^4)_4\) | 6 033 \((f, f)_2 \cdot (f,q^2)_3\) \(4\) |
| 3 021 \((f, l^2)_2\) \(4\) | 5 221 \((f, l_1, (l, q)_1)_4\) | 7 412 \((f, l_1, (l, l)_3)_4\) |
| 3 003 \((f, f, f)_2\) \(4\) | 6 411 \((f, l^3, (l, q)_1)_4\) | 7 233 \((f, l_1, (f, f)_3)_4\) \(4\) |
| 4 211 \((f, l^2)_2\) \(4\) | 6 402 \((f, f)_2 \cdot l^4\) \(4\) | 8 413 \((f, l^2, (f, l)_1)_4\) \(4\) |
| 5 221 \((f, l^4)_4\) | 6 033 \((f, f)_2 \cdot (f,q^2)_3\) \(4\) | 9 003 \((f, l^3, (f, l^2)_2)_4\) \(4\) |

Given the Poincaré series and the hsop degrees, it suffices to check that these generators generate \( R_d \) for \( d \leq 15 \), and they do.

### 7.3 The case \( V = 2V_2 \oplus V_3 \)

Let \( V = 2V_2 \oplus V_3 \). We find \( r = 18 \). Let the forms be \( q, r, c \), of degrees 2, 2, 3, respectively. Let \( u = (c, q)_3 \) and \( v = (c, qr)_3 \). The table below gives the 18 basic generators with degree and multidegree. (For multidegree \( i,j,k \) only the entries with \( i \geq j \) are given.)

| dg mdeg form | dg mdeg form | dg mdeg form |
|-------------|-------------|-------------|
| 2 200 \((q, q)_2\) | 4 004 \((c, (c, c)_2)_2\) \(3\) | 6 222 \((c, (q, q)_1)_2\) \(3\) |
| 2 110 \((q, r)_2\) | 5 302 \((c, qu)_3\) | 7 304 \((c, u, (c, (c, q)_1)_2\) \(3\) |
| 3 102 \((c, (q)_3\) \(3\) | 5 212 \((c, qv)_3\) | 7 214 \((c, u, (c, (c, r)_1)_2\) \(3\) |
| 4 112 \((c, (q, c)_2)_2\) \(3\) | 6 312 \((c, (q, r)_1)_2\) \(3\) | |

Given the Poincaré series and the hsop degrees, it suffices to check that these generators generate \( R_d \) for \( d \leq 17 \), and they do.

### 7.4 The case \( V = 2V_2 \oplus V_4 \)

Let \( V = 2V_2 \oplus V_4 \). We find \( r = 19 \). Let the forms be \( q, r, f \), of degrees 2, 2, 4, respectively. The table below gives the 19 basic generators with degree and multidegree. (For multidegree \( i,j,k \) only the entries with \( i \geq j \) are given.)

| dg mdeg form | dg mdeg form | dg mdeg form |
|-------------|-------------|-------------|
| 2 200 \((q, q)_2\) | 3 111 \((f, qr)_4\) | 4 211 \((f, q, (q)_1)_2\) \(4\) |
| 2 110 \((q, r)_2\) | 3 003 \((f, (f, f)_2)_4\) | 5 212 \((f, (f, q)_2 \cdot (q, r)_1)_4\) |
| 2 002 \((f, f)_4\) | 4 202 \((f, q, (q, q)_2)_4\) | 6 303 \((f, (f, q)_2 \cdot (f, q^2)_3)_4\) |
| 3 001 \((f, q^2)_4\) | 4 112 \((f, q, (q, r)_2)_4\) | 6 213 \((f, (f, q)_2 \cdot (f, qr)_3)_4\) |

Given the Poincaré series and the hsop degrees, it suffices to check that these generators generate \( R_d \) for \( d \leq 12 \), and they do.

### 7.5 The case \( V = 3V_4 \)

Let \( V = 3V_4 \). We find \( r = 25 \). Let the forms be \( f, g, h \), all of degree 4. The table below gives the 25 basic generators with degree and multidegree. (For
Given the Poincaré series and the hsop degrees, it suffices to check that these generators generate $R_4$ for $d \leq 15$, and they do. For $d = 15$ this required computing the rank (34734) of a matrix with $10^{10}$ entries.

### 7.6 The case $V = V_3 \oplus V_4$

Let $V = V_3 \oplus V_4$. We find $r = 20$. Let the forms be $c, e$. Omit parentheses where that does not introduce ambiguity, so that $(c, (c, e)_3(c, (c, e)_1)_3)_3$ is written $(c, cc_3cc_1e_3)_3$. Write $l = ce_3$. The table below gives the 20 basic generators with degree and multidegree.

| dg mdeg | form        | dg mdeg | form        | dg mdeg | form        |
|---------|-------------|---------|-------------|---------|-------------|
| 2       | 02          | 3       | 210         | 4       | 211         |
| 3       | 03          | 4       | 111         | 5       | 221         |
| 4       | 40          | 7       | 43          | 9       | 63          |
| 5       | 41          | 8       | 62          | 9       | 63          |
| 6       | 42          | 9       | 44          | 10      | 64          |
|         | (c, cc_3cc_1e_3)_3 |         | (c, cc_3cc_1e_3)_3 |         | (c, cc_3cc_1e_3)_3 |

Given the Poincaré series and the hsop degrees, it suffices to check that these generators generate $R_4$ for $d \leq 18$, and they do.

### 7.7 The case $V = W \oplus mV_1$

Given a basic system of covariants for a module $W$, one finds a basic system of covariants for $W \oplus V_1$ by replacing each covariant $j$ (of order $o$) of the system for $W$ by the $o+1$ covariants $(j, l^i)_i$ $(0 \leq i \leq o)$ and adding the covariant $l$, where $l$ is the linear form corresponding to $V_1$. (This is classical. See [9, §55], [20, §138A].) Equivalently, given the invariants of $W \oplus V_1$, one finds the invariants of $W \oplus mV_1$ by polarization.

One finds further results that should be regarded classical:

| module     | $r$ | module     | $r$ | module     | $r$ |
|------------|-----|------------|-----|------------|-----|
| $2V_1 \oplus V_2$ | 5   | $2V_1 \oplus V_4$ | 13  | $2V_1 \oplus V_4$ | 20  |
| $2V_1 \oplus 2V_2$ | 13  | $3V_1 \oplus V_2$ | 24  | $3V_1 \oplus V_4$ | 30  |
| $2V_1 \oplus V_2 \oplus V_3$ | 35  | $2V_1 \oplus 2V_4$ | 103 | $3V_1 \oplus V_4$ | 63  |
| $2V_1 \oplus V_2 \oplus V_4$ | 57  | | | |

Table 4: Some classical results involving multiple $V_1$
7.8 The case $V = mV_1 \oplus nV_2$

Let $V = mV_1 + nV_2$. Then $r = \binom{n}{3} + \binom{m+1}{2} \binom{n+1}{2} + \binom{m}{2} + \binom{n+1}{2}$.

If the forms are $\ell_i$ ($1 \leq i \leq m$) and $q_j$ ($1 \leq j \leq n$), then there are $\binom{n}{2} + \binom{m}{2} \binom{n+1}{2}$ basic invariants of degree 2, namely $(\ell_i, \ell_j)_1$ for $i < j$ and $(q_i, q_j)_2$ for $i \leq j$, and $n\binom{m+1}{2} + \binom{m}{2}$ basic invariants of degree 3, namely $(q_k, \ell_i, \ell_j)_2$ for $i \leq j$ and $(q_i, q_j, q_k)_1)_2$ for $i < j < k$, and $\binom{m+1}{2} \binom{n}{2}$ basic invariants of degree 4, namely $((q_i, q_j)_1, \ell_k \ell_m)_2$ for $i < j, k \leq m$.

In order to show this, we quote the following result [9, §54]:

Proposition 7.1. Let $R$ and $S$ be two $\text{SL}_2$-algebras whose covariants are finitely generated. Then the covariants of $R \oplus S$ are also finitely generated. If $P_1, \ldots, P_r$ are the generators of the covariants of $R$, and $Q_1, \ldots, Q_s$ are the generators of the covariants of $S$, then a finite generating system can be chosen from the set of transvectants $[P, Q]_l$, $l \geq 0$, where $P$ is a monomial in the $P_i$'s and $Q$ a monomial in the $Q_j$'s.

Apply this with $R = mV_1$ and $S = nV_2$, with forms as above. The covariants of $mV_1$ are generated by the $\ell_i$ themselves, and the invariants $(\ell_i, \ell_j)_1$ for $i < j$.

The covariants of $nV_2$ are generated by the $q_i$ themselves, the covariants $(q_i, q_j)_1$ for $i \leq j$, and the invariants $(q_i, q_j)_2$ for $i \leq j$ and $(q_i, q_j, q_k)_1)_2$ for $i < j < k$.

We add to the set of generators of $R$ the invariants of degrees 3 $(q_k, \ell_i, \ell_j)_2$ for $i \leq j$, and the invariants of degree 4 $((q_i, q_j)_1, \ell_k \ell_m)_2$ for $i < j, k \leq m$. Given that

$$(r_1 \ldots r_p, \ell_1 \ldots \ell_{2p})_{2p} \sim \sum (r_1, \ell_1, \ell_2)_2 \ldots (r_p, \ell_{2p-1}, \ell_{2p})_2,$$

there are no other irreducible invariants.

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