FRACTIONAL-FEEDBACK STABILIZATION FOR A CLASS OF EVOLUTION SYSTEMS

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Abstract. We study the problem of stabilization for a class of evolution systems with fractional-damping. After writing the equations as an augmented system we prove in this article first that the problem is well posed. Second, using the LaSelle’s invariance principle we show that the energy of the system is strongly stable. Then, based on a resolvent approach we show a lack of uniform stabilization. Next, using multiplier techniques combined with the frequency domain method, we shall give a polynomially stabilization result under some consideration on the stabilization of an auxiliary dissipating system. Finally, we give some applications to the wave equation.

Contents

1. Introduction 1
2. Augmented model 3
3. Well-posedness 4
4. Strong stabilization 7
5. Lack of uniform stabilization 11
6. Non-uniform stabilization 14
7. Applications to the fractional-damped wave equation 21
   7.1. Internal fractional-damped wave equation 21
   7.2. Fractional-Kelvin-Voigt damped wave equation 22
   7.3. Pointwise fractional-damped string equation 26
References 27

1. Introduction

In recent years, fractional calculus has been increasingly applied in different fields of science [17, 23, 28]. Physical phenomena related to electromagnetism, propagation of energy in dissipative systems, thermal stresses, models of porous electrodes, relaxation vibrations, viscoelasticity and thermoelasticity are successfully described by fractional differential equations [11, 16]. Fractional calculus allows for the investigation of the nonlocal response of mechanical systems, this is the main advantage when compared to the classical calculus.

In the literature, a number of definitions of the fractional derivatives have been introduced, namely the Hadamard, Erdelyi-Kober, Riemann-Liouville, Riesz, Weyl, Grünwald-Letnikov,
Junarjie and the Caputo representation. A thorough analysis of fractional dynamical systems is necessary to achieve an appropriate definition of the fractional derivative. For example, the Riemann-Liouville definition entails physically unacceptable initial conditions (fractional order initial conditions); conversely, for the Caputo representation, the initial conditions are expressed in terms of integer-order derivatives having direct physical significance; this definition is mainly used to include memory effects. Recently, Michele Caputo and Mauro Fabrizio in [9] presented a new definition of the fractional derivative without a singular kernel; this derivative possesses very interesting properties, for instance the possibility to describe fluctuations and structures with different scales. Furthermore, this definition allows for the description of mechanical properties related to damage, fatigue and material heterogeneities.

Let $H$ be a Hilbert space equipped with the norm $\| \cdot \|_H$, and let $A : \mathcal{D}(A) \subset H \to H$ be a self-adjoint and strictly positive operator on $H$. We introduce the scale of Hilbert spaces $H_\beta$, $\beta \in \mathbb{R}$, as follows: for every $\beta \geq 0$, $H_\beta = \mathcal{D}(A^\beta)$, with the norm $\| z \|_\beta = \| A^\beta z \|_H$. The space $H_{-\beta}$ is defined by duality with respect to the pivot space $H$ as follows: $H_{-\beta} = H_\beta^*$ for $\beta > 0$. The operator $A$ can be extended (or restricted) to each $H_\beta$, such that it becomes a bounded operator

$$A : H_\beta \to H_{\beta-1}, \quad \forall \beta \in \mathbb{R}.$$ 

Let a bounded linear operator $B : U \to H_{-\frac{1}{2}}$, where $U$ is another Hilbert space which will be identified with its dual.

The system we consider here is described by:

$$\begin{cases} \partial_t^\alpha u(t) + Au(t) + BB^* \partial_t^\alpha u(t) = 0, \quad t > 0, \\ u(0) = u^0, \quad \partial_t u(0) = u^1, \end{cases} \tag{1.1}$$

where $\partial_t^\alpha$ denoted the fractional derivative defined by

$$\partial_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} v'(s) \, ds, \quad 0 < \alpha < 1, \quad \eta \geq 0. \tag{1.2}$$

We define also the following exponentially modified fractional integro-differential operators

$$I^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} v(s) \, ds, \quad 0 < \alpha < 1, \quad \eta \geq 0. \tag{1.3}$$

With these notations we have

$$\partial_t^\alpha v(t) = I^{1-\alpha} v'(t).$$

There are many definitions for fractional derivatives \[10\], among which Riemann-Liouville definition and Caputo definitions are most widely used \[15\]. The latter has the same Laplace transform as the integer order one, so it is widely used in control theory. In this paper, the fractional derivative damping force is regarded as a control force to study the properties of free damped vibration of the system, so the Caputo definition is used here.

Noting that the case of the wave equation with boundary fractional damping have treated in \[18, 19\] where it is proven the strong stability and the lack of uniform stabilization. However, the case of the plate equation or the beam equation with boundary fractional damping was treated in \[1\] where in addition of that using the domain frequency method it was shown that the energy is polynomially stable.

The main result of this paper concerns the precise asymptotic behavior of the solutions of \[2.3\]-\[2.4\]. Our technique is based on a resolvent estimate.

This paper is organized as follows. In section 2 we reformulate problem \[1.1\] into an augmented system. In Section 3 we give the proper functional setting for the augmented model \[2.3\]-\[2.5\], and prove that this system is well-posed. In Section 4 we establish a resolvent estimate which is correspond to the system \[2.3\]-\[2.5\] and by resolvent method we give the explicit
2. Augmented model

In this section we reformulate (1.1) into an augmented system. Our main result is the following.

Proposition 2.1. We set the constant
\[ \gamma = \frac{2 \sin(\alpha \pi) \Gamma(\frac{d}{2} + 1)}{\pi^{\frac{d}{2} + 1}} \]
and for all \( \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d \) with \( |\xi|^2 = \xi_1^2 + \cdots + \xi_d^2 \) we define the function
\[ p(\xi) = |\xi|^{2 - d}. \]
Then the relation between the input \( U \) and the output \( O \) of the following system
\[
\begin{align*}
\partial_t \varphi(t, \xi) + (|\xi|^2 + \eta)\varphi(t, \xi) &= p(\xi)U(t) \quad \forall \xi \in \mathbb{R}^d, \ t > 0 \\
\varphi(0, \xi) &= 0 \quad \forall \xi \in \mathbb{R}^d \\
O(t) &= \gamma \int_{\mathbb{R}^d} p(\xi)\varphi(t, \xi) d\xi, \quad \forall t \geq 0
\end{align*}
\]
where \( U \in C^0([0, +\infty)) \), is given by
\[ O(t) = I^{1-\alpha, \eta}U(t). \]

Proof. Solving equation (2.1), we obtain
\[ \varphi(t, \xi) = p(\xi) \int_0^t e^{-((|\xi|^2 + \eta)(t-s))}U(s) \, ds. \]
If follows from the third line of (2.1) that
\[ O(t) = \gamma \int_0^t U(s) \int_{\mathbb{R}^d} p(\xi)^2e^{-(|\xi|^2 + \eta)(t-s)} \, d\xi \, ds \]
\[ = \frac{2 \sin(\alpha \pi)}{\pi} \int_0^t \int_0^{+\infty} p^{2\alpha - 1}e^{-(\rho^2 + \eta)(t-s)} \, d\rho \, U(s) \, ds. \]
Now using the fact that \( \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} = \frac{\sin(\alpha \pi)}{\pi} \) then a simple change of variable leads to the relation (2.2). This completes the proof. \( \square \)

Using now Proposition 2.1 and relation (1.3), system (1.1) may be recast into the following augmented system
\[
\begin{align*}
\partial_t^2 u(t) + Au(t) + \gamma B \int_{\mathbb{R}^d} p(\xi) \varphi(t, \xi) d\xi &= 0, \ t > 0, \\
\partial_t \varphi(t, \xi) + (|\xi|^2 + \eta)\varphi(t, \xi) &= p(\xi)B^*\partial_t u(t), \ \xi \in \mathbb{R}^d, \ t > 0, \\
u(0) &= u^0, \ \partial_t u(0) = u^1, \ \varphi(0, \xi) = 0,
\end{align*}
\]
where the function \( p(\xi) \) and the constant \( \gamma \) are given in Proposition 2.1.
3. Well-posedness

In this section, we are interested in showing that system (1.1) is well posed in the sense of semigroups.

Let \( V = L^2(\mathbb{R}^d; U) \), we set the Hilbert space \( \mathcal{H} = H^1_d \times H \times V \) with inner product

\[
\left\langle \begin{pmatrix} u_1 \\ v_1 \\ \varphi_1 \\ u_2 \\ v_2 \\ \varphi_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \\ \varphi_1 \\ u_2 \\ v_2 \\ \varphi_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \left\langle A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} u_2 \right\rangle_H + \langle v_1, v_2 \rangle_H + \gamma \int_{\mathbb{R}^d} \langle \varphi_1(\xi), \varphi_2(\xi) \rangle_U \, d\xi.
\]

If we put \( X = \begin{pmatrix} u \\ \partial_t u \\ \varphi \end{pmatrix} \) it is clear that (2.3)-(2.5) can be written as

\[
(3.1) \quad X'(t) = AX(t), \quad X(0) = X_0,
\]

where \( X_0 = \begin{pmatrix} u_0 \\ u_1 \\ 0 \end{pmatrix} \) and \( A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H} \) is defined by

\[
(3.2) \quad A \begin{pmatrix} u \\ v \\ \varphi \end{pmatrix} = \begin{pmatrix} -Au - \gamma B \int_{\mathbb{R}^d} p(\xi) \varphi(\xi) \, d\xi \\ -(\|\xi\|^2 + \eta)\varphi + p(\xi)B^*v \end{pmatrix},
\]

with domain

\[
(3.3) \quad \mathcal{D}(A) = \left\{ (u, v, \varphi) \in \mathcal{H} : \begin{array}{c} v \in H^1, \quad Au + \gamma B \int_{\mathbb{R}^d} p(\xi) \varphi(\xi) \, d\xi \in H, \\
\|\varphi\|_{L^2(\mathbb{R}^d; U)} \leq \gamma \int_{\mathbb{R}^d} (\|\xi\|^2 + \eta)\|\varphi(\xi)\|_U^2 \, d\xi \leq 0. \end{array} \right\}.
\]

Our main result is giving by the following theorem.

**Theorem 3.1.** The operator \( A \) defined by (3.2) and (3.3), generates a \( C_0 \) semigroup of contractions \( e^{tA} \) in the Hilbert space \( \mathcal{H} \).

**Proof.** To prove this result we shall use the Lumer-Phillips’ theorem (see [20, Theorem 4.3]). Since for every \( X = (u, v, \varphi) \in \mathcal{D}(A) \) we have

\[
\text{Re} \langle AX, X \rangle_{\mathcal{H}} = -\gamma \int_{\mathbb{R}^d} (\|\xi\|^2 + \eta)\|\varphi(\xi)\|_U^2 \, d\xi \leq 0.
\]

then the operator \( A \) is dissipative.

Let \( \lambda > 0 \), we prove that the operator \( (\lambda I - A) \) is a surjection. In other words, we shall demonstrate that given any triplet \( Z = (f, g, h) \in \mathcal{H} \), there is an other triplet \( X = (u, v, \varphi) \in \mathcal{D}(A) \) such that \( (\lambda I - A)X = Z \), which can be recast as follow

\[
\begin{cases}
\begin{aligned}
v &= \lambda u - f, \\
(\lambda^2 I + A)u &= \lambda f + g - \gamma B \int_{\mathbb{R}^d} p(\xi)\varphi(\xi) \, d\xi, \\
\varphi(\xi) &= \frac{p(\xi)}{|\xi|^2 + \eta + \lambda} B^*v + \frac{h(\xi)}{|\xi|^2 + \eta + \lambda}.
\end{aligned}
\end{cases}
\]

Since \( A \) is a non-negative operator then according [27, Proposition 3.3.5] then \( -A \) is m-dissipative. Thus the operator \( (\lambda^2 + A) \) is a bijection and we have

\[
\| (\lambda^2 I + A)^{-1} \|_{\mathcal{L} (\mathcal{H})} \leq \frac{1}{\lambda^2}.
\]
Let \((u_n), (v_n)\) and \((\varphi_n)\) are three sequences defined by induction as follow
\[
\begin{align*}
\begin{cases}
u_0 = -f \in H^{\frac{1}{2}} \subset H, \\
\varphi_0(\xi) = \frac{h(\xi)}{|\xi|^2 + \eta + \lambda} \in V,
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
\begin{cases}
u_{n+1} = -\gamma(\lambda^2 I + A)^{-1}B \int_{\mathbb{R}^d} p(\xi)\varphi_n(\xi) \, d\xi, \\
v_{n+1} = \lambda u_n, \\
\varphi_{n+1}(\xi) = \frac{p(\xi)}{|\xi|^2 + \eta + \lambda} B^* v_n.
\end{cases}
\end{align*}
\]
We denote the constants \(C_1, C_2\) and \(C_3\) by
\[
C_1 = \|f\|_{H^{\frac{1}{2}}} + \|g\|_{H^{\frac{1}{2}}}, \quad C_2 = \|f\|_{H^{\frac{1}{2}}}, \quad C_3 = \left(\int_{\mathbb{R}^d} \frac{(1 + |\xi|)^2 \, d\xi}{(|\xi|^2 + \eta + \lambda)^2}\right)^{\frac{1}{2}} \|h\|_V,
\]
and we set the constants \(K_1\) and \(K_2\) by
\[
K_1 = \|B\|_{C(U; H^{\frac{1}{2}})} \left(\int_{\mathbb{R}^d} \frac{p(\xi)^2 \, d\xi}{(1 + |\xi|)^2}\right)^{\frac{1}{2}}, \quad K_2 = \gamma \|B^*\|_{C(U; L^2(\mathbb{R}^d, U))} \left(\int_{\mathbb{R}^d} \frac{p(\xi)(1 + |\xi|) \, d\xi}{(|\xi|^2 + \eta + \lambda)^2}\right)^{\frac{1}{2}},
\]
which it is clear that they are well defined.

We set the sequences \(a_n = \|u_n\|_{H^{\frac{1}{2}}}, b_n = \|v_n\|_{H^{\frac{1}{2}}},\) and \(c_n = \|1 + |\xi|\|_{L^2(\mathbb{R}^d, U)}\). It is clear using the Hölder inequality that
\[
\begin{align*}
a_0 &\leq \lambda^{-1} C_1, \quad b_0 \leq C_2, \quad c_0 \leq C_3, \\
ar_1 &\leq \lambda^{-2} C_1 K_1, \quad b_1 \leq C_1, \quad c_1 \leq C_2 K_2, \\
ar_2 &\leq \lambda^{-2} C_2 K_1 K_2, \quad b_2 \leq \lambda^{-1} C_3 K_1, \quad c_2 \leq C_1 K_2.
\end{align*}
\]
Using the same arguments we can prove by induction that for all \(n \in \mathbb{N}\) we have \(u_n, v_n \in H^{\frac{1}{2}}, \varphi_n, |\varphi_n| \in V\) and
\[
\begin{align*}
a_{3n} &\leq \lambda^{-(n+1)} C_1 K_1^n K_2^n, \quad b_{3n} \leq \lambda^{-n} C_2 K_1^n K_2^n, \quad c_{3n} \leq \lambda^{-n} C_3 K_1^n K_2^n, \\
a_{3n+1} &\leq \lambda^{-(n+2)} C_3 K_1^{n+1} K_2^n, \quad b_{3n+1} \leq \lambda^{-n} C_1 K_1^n K_2^n, \quad c_{3n+1} \leq \lambda^{-n} C_2 K_1^n K_2^{n+1}, \\
a_{3n+2} &\leq \lambda^{-(n+2)} C_2 K_1^n K_2^{n+1}, \quad b_{3n+2} \leq \lambda^{-(n+1)} C_3 K_1^{n+1} K_2^n, \quad c_{3n+2} \leq \lambda^{-n} C_1 K_1^1 K_2^{n+1}.
\end{align*}
\]
So that, for \(\lambda > 0\) large enough the two sums \(\sum u_n\) and \(\sum v_n\) converge uniformly in \(H^{\frac{1}{2}}\) and the sum \(\sum \varphi_n\) converges uniformly in \(V\). Therefore, by setting \(u = \sum_{n=0}^{+\infty} u_n, v = \sum_{n=0}^{+\infty} v_n\) and \(\varphi = \sum_{n=0}^{+\infty} \varphi_n\) we find
\[
\begin{align*}
u &= u_0 + \sum_{n=1}^{+\infty} u_n = (\lambda^2 I + A)^{-1}(\lambda f + g) - \gamma \sum_{n=1}^{+\infty} (\lambda^2 I + A)^{-1}B \int_{\mathbb{R}^d} p(\xi)\varphi_{n-1}(\xi) \, d\xi \\

&= (\lambda^2 I + A)^{-1} \left((\lambda f + g) - \gamma B \int_{\mathbb{R}^d} p(\xi) \sum_{n=0}^{+\infty} \varphi_n(\xi) \, d\xi\right) \\

&= (\lambda^2 I + A)^{-1}(\lambda f + g) - \gamma (\lambda^2 I + A)^{-1}B \int_{\mathbb{R}^d} p(\xi)\varphi(\xi) \, d\xi.
\end{align*}
\]
Since $\varphi \in V$ we follow that $u \in H^1_0$ and we have $(\lambda^2 I + A)u = (\lambda f + g) - \gamma B \int_{\mathbb{R}^d} p(\xi)\varphi(\xi) \, d\xi$. By the same way we have

$$v = \sum_{n=0}^{+\infty} v_n = v_0 + \sum_{n=1}^{+\infty} v_n = f + \lambda \sum_{n=1}^{+\infty} u_{n-1} = \lambda u + f$$

and also

$$\varphi(\xi) = \sum_{n=0}^{+\infty} \varphi_n(\xi) = \varphi_0(\xi) + \sum_{n=1}^{+\infty} \varphi_n(\xi) = \frac{h(\xi)}{\|\xi\|^2 + \eta + \lambda} - \frac{p(\xi)}{\|\xi\|^2 + \eta + \lambda} B^* \sum_{n=1}^{+\infty} v_{n-1}$$

This prove $v \in H^1_0$ and $|\xi|\varphi, \varphi \in L^2(\mathbb{R}^d; U)$. Finally, it is clear that $Au + \gamma B \int_{\mathbb{R}^d} p(\xi)\varphi(\xi) \, d\xi \in H$, $-(|\xi|^2 + \eta)\varphi + p(\xi)B^*v \in L^2(\mathbb{R}^d; U)$. Hence, we proved that the operator $(A - \lambda I)$ is onto. This completes the proof. \qed

As a consequence of Theorem 5.1 the system (2.6)-(2.7) is well-posed in the energy space $\mathcal{H}$ and we have the following proposition.

**Proposition 3.1.** For $(u^0, u^1, 0) \in \mathcal{H}$, the problem (2.6)-(2.7) admits a unique solution

$$(u, \partial_t u, \varphi) \in C([0, +\infty); \mathcal{H}).$$

and for $(u^0, u^1, 0) \in D(A)$, the problem (2.3)-(2.5) admits a unique solution

$$(u, \partial_t u, \varphi) \in C([0, +\infty); D(A)) \cap C^1([0, +\infty); \mathcal{H}).$$

Moreover, from the density of $D(A)$ in $\mathcal{H}$ the energy of $(u(t), \varphi(t))$ at time $t \geq 0$ by

$$E(t) = \frac{1}{2} \left( \|u(t)\|_{H^1_0}^2 + \|\partial_t u(t)\|_{L^2}^2 + \gamma \int_{\mathbb{R}^d} \|\varphi(t, \xi)\|_U^2 \, d\xi \right).$$

decays as follow

$$(3.4) \quad \frac{dE}{dt}(t) = -\gamma \int_{\mathbb{R}^d} (|\xi|^2 + \eta) \|\varphi(t, \xi)\|_U^2 \, d\xi, \ \forall \ t \geq 0.$$

**Proof.** Noting that the regularity of the solution of the problem (2.3)-(2.5) is consequence of the semigroup properties. We have just to prove (3.1). We set

$$E_1(t) = \frac{1}{2} \left( \|u(t)\|_{H^1_0}^2 + \|\partial_t u(t)\|_{L^2}^2 \right) \quad \text{and} \quad E_2(t) = \frac{\gamma}{2} \left( \int_{\mathbb{R}^d} \|\varphi(t, \xi)\|_U^2 \, d\xi \right).$$

A straightforward calculation gives

$$\frac{dE_1}{dt}(t) = -\gamma \text{Re} \left( \int_{\mathbb{R}^d} p(\xi)\varphi(t, \xi) \, d\xi, B^* \partial_t u(t) \right)_U,$$

and

$$\frac{dE_2}{dt}(t) = \gamma \text{Re} \left( \int_{\mathbb{R}^d} p(\xi)\varphi(t, \xi) \, d\xi, B^* \partial_t u(t) \right)_U - \gamma \int_{\mathbb{R}^d} (|\xi|^2 + \eta) \|\varphi(t, \xi)\|_U^2 \, d\xi.$$
4. Strong stabilization

In this section, we prove that the solutions of system (3.1) converge asymptotically to zero. To achieve this result, we shall make use the LaSalle’s invariance principle extended infinite-dimensional systems [29]. According to this principle, all solutions of (3.1) will asymptotically tend to the maximal invariant subset of the set

\[ I = \left\{ X_0 \in \mathcal{H} : \frac{dE}{dt}(t) = 0 \right\}. \]

Provided that these solutions are pre-compact in \( \mathcal{H} \).

**Lemma 4.1.** Let

\[ E(t) = \frac{1}{2} \left( \| \partial_t u \|_{H^1_x}^2 + \| \partial_t^2 u \|_{L^2}^2 + \gamma \int_{\mathbb{R}^d} \| \partial_t \varphi(x) \|_{L^2}^2 \right) \]

Then the function \( t \rightarrow E(t) \) is non-increasing along solutions of the system (3.1) with initial data are in \( \mathcal{D}(A^2) \). In particular, we have

(4.1) \[ \frac{dE}{dt}(t) = -\gamma \int_{\mathbb{R}^d} (|\xi|^2 + \eta) \| \partial_t \varphi(t, \xi) \|_{L^2}^2 \, d\xi. \]

**Proof.** If \( X_0 \in \mathcal{D}(A^2) \) then the \( X(t) = e^{tA}X_0 \) is a solution of (3.1) with the following regularity

\[ X(t) = \begin{pmatrix} u(t) \\ \partial_t u(t) \\ \partial_t^2 u(t) \\ \partial_t \varphi(t, \xi) \end{pmatrix} \in C([0, +\infty[, \mathcal{D}(A^2)) \cap C^1([0, +\infty[, \mathcal{D}(A)). \]

with \( \dot{X}(t) = \begin{pmatrix} \partial_t u(t) \\ \partial_t^2 u(t) \\ \partial_t \varphi(t, \xi) \end{pmatrix} = AX(t) = Ae^{tA}X_0 = e^{tA}AX_0 \). And since \( AX_0 \in \mathcal{D}(A) \) then

\[ \dot{X}(t) \in C([0, +\infty[, \mathcal{D}(A)) \cap C^1([0, +\infty[, \mathcal{H}), \]

then by setting

\[ E_1(t) = \frac{1}{2} \left( \| \partial_t u \|_{H^1_x}^2 + \| \partial_t^2 u \|_{L^2}^2 + \gamma \int_{\mathbb{R}^d} \| \partial_t \varphi(x) \|_{L^2}^2 \right) \]

and \( E_2(t) = \frac{\gamma}{2} \left( \int_{\mathbb{R}^d} \| \partial_t \varphi(x) \|_{L^2}^2 \right) \)

we have

\[ \frac{dE_1}{dt}(t) = -\gamma \text{Re} \left( \int_{\mathbb{R}^d} p(\xi) \partial_t \varphi(t, \xi) d\xi, B^* \partial_t^2 u \right)_U, \]

and

\[ \frac{dE_2}{dt}(t) = \gamma \text{Re} \left( \int_{\mathbb{R}^d} p(\xi) \partial_t \varphi(t, \xi) d\xi, B^* \partial_t^2 u \right)_U - \gamma \int_{\mathbb{R}^d} (|\xi|^2 + \eta) \| \partial_t \varphi(t, \xi) \|_{L^2}^2 \, d\xi. \]

So that, by summing the two last expressions we obtain (4.1) and consequently the non-increasing property of \( E(t) \) holds. This complete the proof. \( \square \)

**Lemma 4.2.** We assume that the only classical solution of the following system

(4.2) \[ \begin{cases} \partial_t^2 u(t) + Au(t) = 0 \\ B^* \partial_t u(t) = 0. \end{cases} \]

is the trivial one, then the only solution of (3.1) is also the zero solution.

**Proof.** Let \( X = (u, v, \varphi) \in I \) be a classical solution of (3.1). Then from (3.4) we have

\[ \int_{\mathbb{R}^d} (|\xi|^2 + \eta) \| \varphi(s, \xi) \|_{L^2}^2 \, d\xi = 0. \]

which imply that

(4.3) \[ \varphi(t, \xi) \equiv 0 \text{ in } L^2(\mathbb{R}^d; U). \]
By using (1.3), it is clear that system (3.1) reduces to the system (1.2). Then by the assumption made in this lemma we deduce that \( u(t) \equiv 0 \) for all \( t \geq 0 \). This complete the proof. \( \square \)

**Proposition 4.1.** Let \( X_0 = (u_0, v_0, \varphi_0) \in D(A^2) \), then the trajectory of \( \varphi(t) \), the third component of the solution of (3.1), is pre-compact in \( L^2(\mathbb{R}^d; U) \).

**Proof.** Since, for \( D(A^2) \), \( \varphi(t) \) is continuous mapping from \([0, +\infty[\) into \( L^2(\mathbb{R}^d, U) \), it is therefore sufficient to show that
\[
\int_{\mathbb{R}^d} \| \varphi(t, \xi) \|_U^2 \, d\xi \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.
\]

From (3.4) and (4.1) together with the fact that both \( E(t) \) and \( \mathcal{E}(t) \) are non-increasing functions we follow
\[
\int_0^{+\infty} \int_{\mathbb{R}^d} (|\xi|^2 + \eta) \| \varphi(t, \xi) \|_U^2 \, d\xi \, dt < +\infty,
\]
and
\[
\int_0^{+\infty} \int_{\mathbb{R}^d} \| \partial_t \varphi(t, \xi) \|_U^2 \, d\xi \, dt < +\infty.
\]

The remainder of the proof will be divided in two cases.

**Case 1:** \( \eta \neq 0 \). Here we get immediately from (4.4) and (4.5) the following relations
\[
\int_0^{+\infty} \int_{\mathbb{R}^d} \| \varphi(t, \xi) \|_U^2 \, d\xi \, dt < +\infty,
\]
and
\[
\int_0^{+\infty} \int_{\mathbb{R}^d} \| \partial_t \varphi(t, \xi) \|_U^2 \, d\xi \, dt < +\infty.
\]

By using these relations together with the well know inequality \( 2 \Re \langle X, Y \rangle \leq \| X \|^2 + \| Y \|^2 \) for all \( X, Y \), we obtain
\[
\left| \int_{\mathbb{R}^d} \| \varphi(t, \xi) \|_U^2 \, d\xi - \int_{\mathbb{R}^d} \| \varphi(s, \xi) \|_U^2 \, d\xi \right| = 2 \left| \Re \left( \int_s^t \int_{\mathbb{R}^d} \langle \partial_t \varphi(t, \xi), \varphi(t, \xi) \rangle_U \, d\xi \, dt \right) \right|
\leq \int_s^t \int_{\mathbb{R}^d} \| \partial_t \varphi(\xi, t) \|_U^2 + \| \varphi(\xi, t) \|_U^2 \, d\xi \, dt,
\]
then we easily see from (4.6) and (4.7) that
\[
\lim_{t \rightarrow +\infty} \int_{\mathbb{R}^d} \| \varphi(t, \xi) \|_U^2 \, d\xi \quad \text{exist and finite}.
\]

But then (4.6) and (4.8) imply that
\[
\lim_{t \rightarrow +\infty} \int_{\mathbb{R}^d} \| \varphi(t, \xi) \|_U^2 \, d\xi = 0.
\]

**Case 2:** \( \eta = 0 \). In this case (4.4) and (4.5) reduce to
\[
\int_0^{+\infty} \int_{\mathbb{R}^d} |\xi|^2 \| \varphi(t, \xi) \|_U^2 \, d\xi \, dt < +\infty,
\]
and
\[
\int_0^{+\infty} \int_{\mathbb{R}^d} |\xi|^2 \| \partial_t \varphi(t, \xi) \|_U^2 \, d\xi \, dt < +\infty.
\]

Again, by using the inequality \( 2 \Re \langle X, Y \rangle \leq \| X \|^2 + \| Y \|^2 \), we have
\[
\lim_{t \rightarrow +\infty} \int_{\mathbb{R}^d} |\xi|^2 \| \varphi(t, \xi) \|_U^2 \, d\xi \quad \text{exist and finite.}
\]
Thus (4.9) imply that

$$\lim_{t \to +\infty} \int_{\mathbb{R}^d} |\xi|^2 \| \varphi(t, \xi) \|^2_{L^2} \, d\xi = 0.$$  

Therefore, in view of (4.11), it is clear that $\int_{\mathbb{R}^d} \| \varphi(t, \xi) \|^2_{L^2} \, d\xi$ will tends to zero as $t$ goes to $+\infty$, if

$$\lim_{t \to +\infty} \int_{B(0,1)} \| \varphi(t, \xi) \|^2_{L^2} \, d\xi = 0,$$

where $B(0,1)$ is the unit ball in $\mathbb{R}^d$. Next, we prove (4.12) by using the dominated convergence theorem whose conditions of applicability, in the case at hand, are established below:

1) By applying Fubini’s theorem to both inequality (4.9) and (4.10) we have

$$\int_0^{+\infty} |\xi|^2 \| \varphi(\xi, t) \|^2_{L^2} \, dt < +\infty \quad \text{a.e. } \xi \in B(0,1)$$

and

$$\int_0^{+\infty} |\xi|^2 \| \partial_t \varphi(\xi, t) \|^2_{L^2} \, dt < +\infty \quad \text{a.e. } \xi \in B(0,1).$$

So that, by the same argument that led us to (4.11), we may conclude that

$$\lim_{t \to +\infty} |\xi|^2 \| \varphi(t, \xi) \|^2_{L^2} = 0 \quad \text{a.e. } \xi \in B(0,1).$$

Hence, we obtain

$$\lim_{t \to +\infty} \| \varphi(t, \xi) \|^2_{L^2} = 0 \quad \text{a.e. } \xi \in B(0,1).$$

1) Now solving (2.21) we have

$$\varphi(t, \xi) = \varphi_0(\xi) e^{-|\xi|^2 t} + p(\xi) B^* \int_0^t \partial_t u(s) e^{-|\xi|^2 (t-s)} \, ds.$$

So that, by applying integration by parts, to the integral in the right hand side of (4.14), we get

$$\varphi(t, \xi) = \varphi_0(\xi) e^{-|\xi|^2 t} + p(\xi) B^* [u(t) - u(0) e^{-|\xi|^2 t}] - |\xi|^2 p(\xi) B^* \int_0^t \partial_t u(s) e^{-|\xi|^2 (t-s)} \, ds.$$

Hence, one gets

$$\| \varphi(t, \xi) \|_{L^2} \leq \| \varphi_0(\xi) \|_{L^2} + p(\xi) \| B^* \|_{L^2(H^1_x, L^2)} \times$$

$$\left[ \| u(0) \|_{H^1_x} + \| u(t) \|_{H^1_x} + |\xi|^2 \int_0^t \| \partial_t u(s) \|_{H^1_x} e^{-|\xi|^2 (t-s)} \, ds \right].$$

Also by (3.4) we can bound $\| u(t) \|_{H^1_x} \leq E(0)$ and we obtain

$$\| \varphi(t, \xi) \|^2_{L^2} \leq C \left( \| \varphi_0(\xi) \|_{L^2}^2 + p(\xi)^2 \| B^* \|^2_{L^2(H^1_x, L^2)} \left[ 2 + E(0) + |\xi|^2 (1 - e^{-|\xi|^2 t}) \right] \right).$$

Since the right hand side of (4.15) is in $L^1_t(B(0,1))$, therefore by combining (4.13) and (4.15) through the dominated convergence theorem, we get (4.12) the desired result.

**Proposition 4.2.** We assume that the embedding $H^1_x \hookrightarrow H$ is a compact embedding. Let $X_0 = (u_0, v_0, \varphi_0) \in D(A^2)$, then the trajectory of the pair $(u(t), v(t))$ of the solution of the system (3.1) is pre-compact in $H^2_x \times H$. 


Proof. Note that if \( X_0 = (u_0, v_0, \varphi_0) \in \mathcal{D}(\mathcal{A}^2) \) then \((u(t), v(t)) \in H_1 \times H_{1/2}^\perp \). Since that, in view of the assumption made in this proposition it is clear that to prove this result we have just to prove that the quantity \( \|u(t)\|_{H_1}^2 + \|v(t)\|_{H_{1/2}^\perp} \) is bounded in \( H_1 \times H_{1/2}^\perp \). We solve the differential equation (2.4), we get

\[
\varphi(t, \xi) = \varphi_0(\xi)e^{-(|\xi|^2+\eta)t} + p(\xi)B^* \int_0^t \partial_t u(s)e^{-(|\xi|^2+\eta)(t-s)} \, ds
\]

(4.16)

Using the differential equation (2.3), Fubini’s theorem and taking account of (4.16) and the fact that \( \|u(t)\|_{H_1}^2 + \|v(t)\|_{H_{1/2}^\perp} \) is bounded by \( E(t) \) is reduces to the boundedness of the following integral

\[
I = \left\| \int_0^t \int_0^\infty \rho^{2\alpha-1} \partial_t u(t-s)e^{-(\rho^2+\eta)s} \, d\rho \, ds \right\|_{H_{1/2}^\perp}^2
\]

Now we set

\[
I = \left\| \int_0^t \int_0^\infty \rho^{2\alpha-1} \partial_t u(t-s)e^{-(\rho^2+\eta)s} \, d\rho \, ds \right\|_{H_{1/2}^\perp}^2
\]

and to establish our result, it is clear that we have just to prove that \( I \) is bounded. To do so we distinguish two cases.

- **Case 1:** \( \eta \neq 0 \). Using again Fubini’s theorem and the fact that \( E(t) \) is non-increasing function, we obtain

\[
I \leq 2E(0) \left( \int_0^\infty \int_0^1 \rho^{2\alpha-1} e^{-(\rho^2+\eta)s} \, d\rho \, ds \right)^2 = 2E(0) \left( \int_0^\infty \rho^{2\alpha-1} \left( 1 - e^{-(\rho^2+\eta)t} \right) \, d\rho \right)^2
\]


\[
\leq 4E(0) \left( \int_0^\infty \frac{\rho^{2\alpha-1}}{\rho^2 + \eta} \, d\rho \right)^2 < +\infty.
\]

which prove that \( I \) bounded.

- **Case 2:** \( \eta = 0 \). It is clear that according to the first case that the problem of the boundedness of \( I \) reduces to the boundedness of the following integral

\[
I_0 = \left\| \int_0^t \int_0^1 \rho^{2\alpha-1} \partial_t u(t-s)e^{-\rho^2 s} \, d\rho \, ds \right\|_{H_{1/2}^\perp}^2,
\]

where we can suppose that \( t \geq 1 \). Integrating by parts with respect to the \( s \) variable and using again the fact that \( E(t) \) is non-increasing function, we have

\[
I_0 \leq 2 \left( \int_0^1 \rho^{2\alpha-1} \left( e^{-\rho^2} u(0) - e^{-\rho^2} u(t-1) \right) \, d\rho \right)^2 + \left\| \int_0^1 \rho^{2\alpha+1} \int_1^t e^{-\rho^2} u(t-s) \, ds \, d\rho \right\|_{H}^2
\]

\[
\leq C \left( E(0) \left( \int_0^1 \rho^{2\alpha-1} \, d\rho \right)^2 + E(0) \left( \int_0^1 \rho^{2\alpha-1} (e^{-\rho^2} - e^{-\rho^2}) \, d\rho \right)^2 \right) \leq CE(0).
\]
This prove the expected estimate and end the proof.

**Theorem 4.1.** The semigroup $e^{tA}$ is strongly stable, it means that for any initial data $X_0 \in \mathcal{H}$,

\[ \|e^{tA}X_0\|_\mathcal{H} \to 0 \quad \text{as} \quad t \to +\infty. \]

**Proof.** For $X_0 \in \mathcal{D}(A^2)$, the theorem is a direct consequence of Lemma 4.2, Propositions 4.1 and 4.1 and the LaSalle’s invariance principle. Finally, since $\mathcal{D}(A^2)$ is dense in $\mathcal{H}$ this result carries over all $X_0 \in \mathcal{H}$. □

5. Lack of uniform stabilization

In this section we shall prove that the system is not uniformly exponentially stable.

**Lemma 5.1.** Let $\omega \in \mathbb{R}^+$ then for any fixed $\eta > 0$ and $0 < \alpha < 1$ we have

\[
\int_0^{\infty} \frac{\rho^{2\alpha - 1}}{\rho^2 + \eta + i\omega} \, d\rho = \begin{cases} 
-\pi \left(1 + e^{-2i\alpha \pi}\right) & \text{if } \alpha \neq \frac{1}{2} \\
\frac{\pi}{2(\eta^2 + \omega^2)} & \text{if } \alpha = \frac{1}{2},
\end{cases}
\]

where we have denoted by $\theta = \arccos \left( -\frac{\sqrt{\eta^2 + \omega^2 - \eta}}{2} \frac{1}{(\eta^2 + \omega^2)^{\frac{1}{4}}} \right)$.

**Proof.** The two case are proven as follow:

**Case 1:** $\eta \neq \frac{1}{2}$. In this case the integral can be evaluated using the method of residues. Integrating along the positive oriented contour depicted in Figure 1. We set the function

**Figure 1.** Contour for evaluating the integral $\int_0^{\infty} \frac{\rho^{2\alpha - 1}}{\rho^2 + \eta + i\omega} \, d\rho$. 

\[ f(z) = \frac{z^{2\alpha-1}}{z^2 + \eta + i\omega}, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}_-, \]

whose poles are \( z_1 = (\eta^2 + \omega^2)^{\frac{1}{2}} e^{i\theta}, \) \( z_2 = (\eta^2 + \omega^2)^{\frac{1}{2}} e^{i(\theta - \pi)} \) and eventually \( z_0 = 0 \) (see Figure 1). Clearly, we have

\[ |zf(z)| \leq \frac{|z|^{2\alpha}}{|z|^2 - (\eta^2 + \omega^2)^{\frac{1}{2}}} \]

which imply that

\[ \lim_{z \to 0} zf(z) = 0, \quad \lim_{|z| \to +\infty} zf(z) = 0. \]

Then by Jordan’s lemmas we follow

\[ \lim_{r \to 0} \int_{\gamma_r} f(z) \, dz = 0 \]

and

\[ \lim_{R \to +\infty} \int_{\gamma_R} f(z) \, dz = 0 \]

where \( \gamma_r = re^{-it} \) and \( \gamma_R = re^{it} \) for \( t \in [-\pi + \varepsilon, \pi - \varepsilon] \) (see Figure 1). On the segment \([AB]\) one has \( z = \gamma_{AB}(t) = [1 - t]R + rt e^{i(\pi - \varepsilon)} \) for \( t \in [0, 1] \) (see Figure 1), whence by Lebegue dominated convergence theorem we have

\[ \int_{\gamma_{AB}} f(z) \, dz = e^{i(\varepsilon + 2\pi)} \int_r^R \frac{\rho^{2\alpha-1}}{\rho^2 + \eta + i\omega} \, d\rho \, \rightarrow \, -e^{2i\pi} \int_r^R \frac{\rho^{2\alpha-1}}{\rho^2 + \eta + i\omega} \, d\rho \text{ as } \varepsilon \searrow 0. \]

On the segment \([CD]\) one has \( z = \gamma_{CD}(t) = te^{i(-\pi + \varepsilon)} \) for \( t \in [r, R] \) (see Figure 1), whence again by Lebegue dominated convergence theorem we have

\[ \int_{\gamma_{CD}} f(z) \, dz = \int_r^R \frac{\rho^{2\alpha-1}e^{2i\alpha(\varepsilon - \pi)}}{\rho^2 e^{2i\varepsilon} + \eta + i\omega} \, d\rho \, \rightarrow \, e^{-2i\pi} \int_r^R \frac{\rho^{2\alpha-1}}{\rho^2 + \eta + i\omega} \, d\rho \text{ as } \varepsilon \searrow 0. \]

By summing (5.3)-(5.6) and taking the limits as \( r \downarrow 0 \) and \( R \uparrow +\infty \), the method of residues leads to

\[ \frac{e^{-2i\alpha\pi} - e^{2i\alpha\pi}}{2i\pi} \int_0^{+\infty} \frac{\rho^{2\alpha-1}}{\rho^2 + \eta + i\omega} \, d\rho = \text{Res}_{z=z_1, z_2} [f(z)] = \frac{z_1^{2\alpha-2} + z_2^{2\alpha-2}}{2} = \frac{e^{2i(\alpha-1)\theta} [1 + e^{-2i\alpha\pi}]}{2(\eta^2 + \omega^2)\frac{1}{2\alpha}} \]

which leads to the second line of (5.1).

**Case 2:** \( \eta = \frac{1}{2} \). Since \( z_1 \) and \( z_2 \) are the unique poles of \( f \) then we can write

\[ \int_0^{+\infty} \frac{d\rho}{\rho^2 + \eta + i\omega} = \frac{1}{2z_1} \int_0^{+\infty} \frac{\rho - z_1}{\rho^2 - 2\text{Re}(z_1)\rho + |z_1|^2} - \frac{\rho - z_2}{\rho^2 - 2\text{Re}(z_2)\rho + |z_2|^2} \, d\rho \]

\[ = \frac{1}{2z_1} \int_0^{+\infty} \frac{\rho - z_1}{(\rho^2 - \text{Re}(z_1))^2 + \text{Im}(z_1)^2} - \frac{\rho - z_2}{(\rho^2 - \text{Re}(z_2))^2 + \text{Im}(z_2)^2} \, d\rho. \]

A straightforward calculation leads to

\[ \int_0^{+\infty} \frac{d\rho}{\rho^2 + \eta + i\omega} = \frac{\pi}{2z_1}, \]

which leads to the first line of (5.1). And this finish the proof. \( \square \)

**Lemma 5.2.** Let \( H_0 \) and \( V_0 \) two Hilbert spaces such that \( V_0 \hookrightarrow H_0 \) with densely and continuous embedding and \( V_0' \) is the dual space of \( V_0 \) with respect to the pivot space \( H_0 \). Then the set \((V_0' \setminus V_0) \cup \{0\}\) is a vector space.

**Proof.** The proof is done in two stages:
• Let $\lambda \in \mathbb{C}$ and $x \in (V'_0 \setminus V_0) \cup \{0\}$. If $\lambda x \notin (V'_0 \setminus V_0) \cup \{0\}$ then $\lambda x \in V_0$ and $\lambda \neq 0$.

Since $x = \frac{1}{\lambda} \times \lambda x$ then $x \in V_0$ which is absurd. So that, $\lambda x \in (V'_0 \setminus V_0) \cup \{0\}$.

• Let $x, y \in (V'_0 \setminus V_0) \cup \{0\}$. Then there exists a sequence $(y_n)$ of $V_0$ such that $y_n \rightarrow y$.

It is clear that $x + y_n \in (V'_0 \setminus V_0) \cup \{0\}$, then since $x + y_n \rightarrow x + y$ and $(V'_0 \setminus V_0) \cup \{0\}$ is a closed set in $V'_0$ one gets that $x + y \in (V'_0 \setminus V_0) \cup \{0\}$.

This completes the proof. \qed

**Theorem 5.1.** The semigroup $e^{tA}$ is not exponentially stable in the Hilbert space $\mathcal{H}$.

**Proof.** To prove this theorem we shall use the frequency theorem method. We recall that a bounded $C_0$ semigroup generated by an operator $A$ is exponentially stable if and only if $i\mathbb{R} \cap \sigma(A) = \emptyset$ and satisfies the following identity

$$
\limsup_{\omega \to +\infty} \| (i\omega I - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} < +\infty.
$$

Since $A$ is a strictly positive operator with compact resolvent then there exist a sequence of eigenvalues $i\omega_n$ corresponding to the orthonormal base of the eigenfunctions $\phi_n = \left( \frac{u_n}{i\omega_n}, \frac{u_n}{\omega_n} \right)$ of the operator $A_0 = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$ such that $\lim_{n \to +\infty} |\omega_n| = +\infty$ where $u_n \in H_{\frac{d}{2}}$. We distinguish now two cases.

**Case 1:** $B^*u_n = 0$ for some $n \in \mathbb{N}$. It is clear in this case that for a such $n \in \mathbb{N}$ we have $X_n = \left( \begin{array}{c} \frac{u_n}{i\omega_n} \\ \frac{u_n}{\omega_n} \\ 0 \end{array} \right) \in D(A)$ and $(i\omega_n I - A)X_n = 0$ which prove that $X_n$ is an eigenfunction corresponding to the eigenvalue $i\omega_n$. Thus, the semigroup $e^{tA}$ is not uniformly stable.

**Case 2:** $B^*u_n \neq 0$ for all $n \in \mathbb{N}$. In this part we shall prove a general result then given in the theorem. In fact, we will show that the following resolvent estimate

$$
\limsup_{\omega \to +\infty} \| \omega^{-1+\varepsilon} (i\omega I - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} < +\infty
$$

is not even satisfied, for $\varepsilon > 0$ small.

Let $\varphi_n(\xi) = \frac{p(\xi)}{\xi^2 + \eta + i\omega_n}B^*u_n$ and $X_n = \left( \begin{array}{c} \frac{u_n}{i\omega_n} \\ \frac{u_n}{\omega_n} \\ \varphi_n \end{array} \right)$. It is clear that the integrals

$$
\int_{\mathbb{R}^d} \frac{p(\xi)^2}{|\xi|^2 + \eta + i\omega_n} \, d\xi \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{|\xi|^2 p(\xi)^2}{(|\xi|^2 + \eta)^2 + \omega_n^2} \, d\xi
$$

are well defined then $|\xi|\varphi_n \in L^2(\mathbb{R}^d; U)$ and $u_n + \gamma B \int_{\mathbb{R}^d} p(\xi)\varphi_n(\xi) \, d\xi \in H_{\frac{d}{2}} \subset H$ (in fact, otherwise since $(H_{-\frac{d}{2}} \setminus H_{\frac{d}{2}}) \cup \{0\}$ is a vector subspace (Lemma 5.2) then $B \int_{\mathbb{R}^d} p(\xi)\varphi_n(\xi) \, d\xi$ should be in $(H_{-\frac{d}{2}} \setminus H_{\frac{d}{2}})$ and consequently $u_n$ belongs to $(H_{-\frac{d}{2}} \setminus H_{\frac{d}{2}}) \cup \{0\}$ which is absurd).

And since we have

$$
\int_{\mathbb{R}^d} p(\xi)B^*u_n - (|\xi|^2 + \eta)\varphi_n^2 \, d\xi = \omega_n^2 \|B^*u_n\|^2 \int_{\mathbb{R}^d} \frac{p(\xi)^2}{(|\xi|^2 + \eta)^2 + \omega_n^2} \, d\xi,
$$

then $p(\xi)B^*u_n - (|\xi|^2 + \eta)\varphi_n \in L^2(\mathbb{R}^d; U)$ and this shows that $X_n \in D(A)$. 
We set now \( Y_n = \begin{pmatrix} f_n \\ g_n \\ h_n \end{pmatrix} \in \mathcal{H} \) such that
\[
(i\omega_n I - A)X_n = Y_n.
\]
Since we have \( f_n = h_n = 0 \) and
\[
g_n = \gamma BB^* u_n \int_{\mathbb{R}^+} \frac{p(\xi)^2}{|\xi|^2 + \eta + i\omega_n} d\xi = \frac{\gamma d\pi^{\frac{\alpha}{2}}}{\Gamma(d + 1)} BB^* u_n \int_0^{+\infty} \frac{\rho^{2\alpha - 1}}{\rho^2 + \eta + i\omega_n} d\rho.
\]
We set now
\[
\kappa_n = \begin{cases} 
\frac{e^{-i\theta_n}}{\pi(1 + e^{-2i\alpha\pi})} & \text{if } \alpha = \frac{1}{2}, \\
\frac{\rho(1 + e^{-2i\alpha\pi})}{\rho + 2\cos(\alpha\pi)} e^{2i(\alpha - 1)\theta_n} & \text{if } \alpha \neq \frac{1}{2},
\end{cases}
\]
where \( \theta_n = \arccos \left( -\frac{\sqrt{\eta^2 + \omega_n^2}}{2} \right) \). According to Lemma 5.1, the function \( g_n \) can be written as follow
\[
g_n = \frac{\kappa_n}{(\eta_n^2 + \omega_n^2)^{\frac{\alpha}{2}}} BB^* u_n.
\]
then
\[
\omega_n^{1 - \alpha - \varepsilon} \| g_n \|_H \leq \frac{C_{\omega_n^{1 - \alpha - \varepsilon}}}{(\eta_n^2 + \omega_n^2)^{\frac{\alpha}{2}}} \| BB^* u_n \|_H^{\frac{1}{2}} \to 0 \text{ as } n \to +\infty.
\]
Hence, by assuming that the imaginary axis is a subset of the resolvent set, we follow
\[
\limsup_{\omega \in \mathbb{R}, |\omega| \to +\infty} \| \omega^{\alpha - 1 + \varepsilon} (i\omega I - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} \geq \sup_{n \in \mathbb{N}} \| \omega_n^{\alpha - 1 + \varepsilon} (i\omega_n I - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} \geq \sup_{n \in \mathbb{N}} \omega_n^{\alpha - 1 + \varepsilon} \| (i\omega_n I - A)^{-1} Y_n \|_{\mathcal{H}} \geq \lim_{n \to +\infty} \omega_n^{\alpha - 1 + \varepsilon} \| X_n \|_{\mathcal{H}} \geq \lim_{n \to +\infty} \omega_n^{\alpha - 1 + \varepsilon} \| g_n \|_{H} = +\infty.
\]
Thus, (15.7) is not satisfied. So that, the semigroup \( e^{tA} \) is not exponentially stable. \( \square \)

**Remark 5.1.** The prove of the previous theorem shows that the semigroup \( e^{tA} \) is at least dissipating over the time as \( t^{-\frac{1}{1-\alpha}} \). In the next section we will show under some assumptions that the semigroup \( e^{tA} \) is decreasing over the time as \( t^{-\frac{1}{1-\alpha}} \). This prove the optimal decay rate rate of the semigroup.

6. Non-uniform stabilization

This section is devoted to study the non uniform stabilization of system \( (1.1) - (2.4) \). Under some assumptions on the behavior of an auxiliary dissipative operator whose dissipation is generated by the classical \( BB^* \) operator we prove a polynomial decay result for the system \( (1.1) - (2.4) \). For this purpose we will use a frequency domain approach.

**Proposition 6.1.** Assume that \( \eta = 0 \), then the operator \(-A\) is not onto and consequently \( 0 \in \sigma(A) \).

**Proof.** Let \( Y = (0, 0, h(\xi)) \in \mathcal{H} \) and assume that there exists \( X = (u, v, \varphi) \in \mathcal{D}(A) \) such that
\[
-A X = Y.
\]
It follows that \( v = 0, \varphi(x) = \frac{h(x)}{|x|^2} \) and \( Au + \gamma B \int_{\mathbb{R}^d} \frac{p(x)h(x)}{|x|^2} \, dx = 0 \). Let \( \psi \in U \) such that \( \psi \neq 0 \) and we set \( h(x) = \frac{|x|^{2+\alpha}}{(1+|x|)^\alpha} \psi \). It is clear that \( h \in L^2(\mathbb{R}^d; U) \). However, \( \varphi \notin L^2(\mathbb{R}^d; U) \). Thus, the operator \(-A\) is not onto. This complete the proof. \( \square \)

Lemma 6.1. Let \( \omega \in \mathbb{R}^+ \) then for any fixed \( \eta > 0 \) and \( 0 < \alpha < 1 \) we have

\[
\int_0^{\infty} \frac{\rho^{2\alpha-1}}{(\rho^2 + \eta^2 + \omega^2)^2} \, d\rho = \begin{cases} \frac{\sin(2(\alpha-1)(\pi-\phi)) - \sin(2(\alpha-1)\phi)}{4} & \text{if } \alpha \neq \frac{1}{2} \\
\frac{3(2\pi - \phi)}{8(\eta^2 + \omega^2)^{\frac{1}{2}}} & \text{if } \alpha = \frac{1}{2}, \end{cases}
\]

where we have denoted by \( \phi = \arccos \left( \frac{\sqrt{\eta^2 + \omega^2 - \eta}}{(\eta^2 + \omega^2)^{\frac{1}{2}}} \right) \).

Proof. This prove is the same as the one of Lemma 5.1. By keeping the same notations here we just sketch the proof.

**Case 1:** \( \eta \neq \frac{1}{2} \). We set the complex function

\[
f(z) = \frac{z^{2\alpha-1}}{(z^2 + \eta^2 + \omega^2)^2}, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}_-,
\]

whose poles are \( z_1 = (\eta^2 + \omega^2)^{\frac{1}{2}} e^{i\phi}, z_2 = (\eta^2 + \omega^2)^{\frac{1}{2}} e^{i(\phi-\pi)} \) and eventually \( z_3 = 0 \). Using the same arguments as Lemma 5.1 we can show that

\[
\int_{\gamma_r} f(z) \, dz \to 0 \quad \text{as} \quad r \to 0,
\]

and

\[
\int_{\gamma_R} f(z) \, dz \to 0 \quad \text{as} \quad R \to +\infty,
\]

where on the segments \([AB]\) and \([CD]\) we have

\[
\int_{\gamma_{AB}} f(z) \, dz = \int_r^R \frac{e^{2i\alpha(\pi-\rho)} \rho^{2\alpha-1}}{(\rho^2 e^{2i\pi} + \eta^2 + \omega^2)^2} \, d\rho \to \int_r^R \frac{e^{2i\alpha\pi} \rho^{2\alpha-1}}{(\rho^2 + \eta^2 + \omega^2)^2} \, d\rho \quad \text{as} \quad \varepsilon \to 0,
\]

and

\[
\int_{\gamma_{CD}} f(z) \, dz = \int_r^R \frac{e^{2i\alpha(\varepsilon-\rho)} \rho^{2\alpha-1}}{(\rho^2 e^{2i\pi} + \eta^2 + \omega^2)^2} \, d\rho \to \int_r^R \frac{e^{-2i\alpha\pi} \rho^{2\alpha-1}}{(\rho^2 + \eta^2 + \omega^2)^2} \, d\rho \quad \text{as} \quad \varepsilon \to 0,
\]

Summing all these integrals and applying the residues theorem we obtain

\[
\int_0^{\infty} \frac{-\sin(2\alpha\pi)\rho^{2\alpha-1}}{(\rho^2 + \eta^2 + \omega^2)^2} \, d\rho = \lim_{\varepsilon \to 0} \left[ f(z) \right]_{z = z_1, z_2} = \frac{(\eta + \omega^2)^{\frac{1}{2}} (\sin(2(\alpha-1)(\pi-\phi)) - \sin(2(\alpha-1)\phi))}{2\cos(\phi)}
\]

which leads obviously to the first line of (6.1).

**Case 2:** \( \eta = \frac{1}{2} \). In this case we have just to remark that

\[
\frac{1}{(\rho^2 + \eta^2 + \omega^2)^2} = \frac{1}{8\pi^2 \cos(\phi)} \left[ \frac{\rho + \tau \cos(\phi)}{\rho^2 + 2\tau \cos(\phi) \rho + \tau^2} - \frac{\rho - \tau \cos(\phi)}{\rho^2 - 2\tau \cos(\phi) \rho + \tau^2} \right]
\]

\[
+ \frac{6\tau \cos(\phi)}{\rho^2 + 2\tau \cos(\phi) \rho + \tau^2} \left[ \frac{1}{\rho^2 + 2\tau \cos(\phi) \rho + \tau^2} + \frac{1}{\rho^2 - 2\tau \cos(\phi) \rho + \tau^2} \right]
\]

where we have denoted by \( \tau = (\eta^2 + \omega^2)^{\frac{1}{2}} \). \( \square \)
Let’s define now \( \mathcal{H}_0 = H^1_\mu \times H \) and let’s consider the operator \( \mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subset \mathcal{H}_0 \rightarrow \mathcal{H}_0 \) defined by
\[
\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ -A & -BB^* \end{pmatrix}
\]
with domain
\[
\mathcal{D}(\mathcal{A}_0) = \{(w,v) \in \mathcal{H}_0 : v \in H^1_\mu, Av + BB^*v \in H\}.
\]

**Proposition 6.2.** The operator \( \mathcal{A}_0 \) generates a \( C_0 \) semigroup of contractions in the Hilbert space \( \mathcal{H}_0 \). Moreover, the following auxiliary problem
\[
\begin{aligned}
\frac{\partial^2 w(t)}{\partial t^2} + Aw + BB^* \partial_t w(t) &= 0 \\
w(0) &= w^0, \quad \partial_t w(0) = w^1.
\end{aligned}
\]
admits a unique solution \( w(t,x) \) in such a way that if \( (w^0, w^1) \in \mathcal{D}(\mathcal{A}_0) \) the solution \( w(t,x) \) of verifying the following regularity
\[
(w, \partial_t w) \in C([0, +\infty); \mathcal{D}(\mathcal{A}_0)) \cap C^1([0, +\infty); \mathcal{H}_0).
\]
and when \( (w^0, w^1) \in \mathcal{H}_0 \), we have
\[
(w, \partial_t w) \in C([0, +\infty); \mathcal{H}_0).
\]

The energy of the system defined as follow
\[
E_0(t) = \frac{1}{2} \left( \|\partial_t w(t)\|^2_H + \|w(t)\|^2_{H^1_\mu} \right),
\]
is decreasing over the time in particular we have
\[
\frac{dE_0}{dt}(t) = -\|B^* \partial_t w(t)\|^2_U.
\]

**Proof.** To show that \( \mathcal{A}_0 \) generates a \( C_0 \) semigroup of contractions we have to prove according to Lumer-Phillips’ theorem (see [20, Theorem 4.3]) that \( \mathcal{A}_0 \) is m-dissipative. First, let \( (w, v) \in \mathcal{D}(\mathcal{A}_0) \) then we have
\[
\text{Re} \left\langle \mathcal{A}_0 \begin{pmatrix} w \\ v \end{pmatrix}, \begin{pmatrix} w \\ v \end{pmatrix} \right\rangle_{\mathcal{H}_0} = -\|B^* v\|^2_U \leq 0,
\]
which proves that \( \mathcal{A}_0 \) is dissipative. It remind now to prove that the range of \( I - \mathcal{A}_0 \) is \( \mathcal{H}_0 \). For this purpose we let \( (f,g) \in \mathcal{H}_0 \) and we look for a couple \( (w,v) \in \mathcal{D}(\mathcal{A}_0) \) such that
\[
(I - \mathcal{A}_0) \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},
\]
or equivalently,
\[
\begin{aligned}
v &= w + f \\
Aw + w + BB^* w &= g - f - BB^* f.
\end{aligned}
\]
We consider now the following bilinear form on \( H^1_\mu \times H^1_\mu \) defined by
\[
L(w, \psi) = \langle w, \psi \rangle_{H^1_\mu} + \langle w, \psi \rangle_H + \langle B^* w, B^* \psi \rangle_U.
\]
It is clear that \( L \) is continuous and coercive form on \( H^1_\mu \times H^1_\mu \) therefore according to Lax-Migram theorem’s there exist a unique \( w \in H^1_\mu \) such that
\[
L(w, \psi) = \langle g - f, \psi \rangle_H - \langle B^* f, B^* \psi \rangle_U, \quad \forall \psi \in H^1_\mu.
\]
Equivalently, this can be written as follows
\[
\langle Aw + BB^*(w + f), \psi \rangle_{H^1_\mu \times H^1_\mu} = \langle g - f - w, \psi \rangle_H, \quad \forall \psi \in H^1_\mu.
\]
In another words \( Aw + BB^*(w + f) \in H \) and we have \( Aw + w + BB^* w = g - f - BB^* f \). Since \( v = w + f \) then \( v \in H^1_\mu \). Hence, system (6.3) admits a unique solution \( (w, v) \in \mathcal{D}(\mathcal{A}_0) \). Thus, the operator \( \mathcal{A}_0 \) is m-dissipative and consequently the existence and the uniqueness of the
solution of problem (6.2) holds with regularity as described above. Finally, a straightforward calculations gives (6.3).

Let $M$ be an increasing function in $\mathbb{R}_+$. We suppose that $\mathcal{A}_0$ satisfies to the following resolvent estimate

$$\text{(6.5)} \quad i\mathbb{R} \subset \rho(\mathcal{A}_0) \quad \text{and} \quad \limsup_{\omega \in \mathbb{R}, |\omega| \to +\infty} \|M(|\omega|)^{-1}(i\omega I - \mathcal{A}_0)^{-1}\|_{\mathcal{L}(\mathcal{H}_0)} < +\infty.$$ 

This means according to Huang-Prüss [13, 21] and Borichev and Tomilov theorem [6, Theorem 2.4] respectively that the semigroup $e^{t\mathcal{A}_0}$ is exponentially stable if $M(|\omega|) = 1$ and polynomially stable if $M(|\omega|) = |\omega|^\ell$ for some $\ell > 0$, namely we have

$$\|e^{t\mathcal{A}_0}\|_{\mathcal{L}(\mathcal{H}_0)} \leq C e^{-\delta t}, \quad \forall t \geq 0$$

for some $\delta > 0$ when $M(|\omega|) = 1$ and

$$\|e^{t\mathcal{A}_0}(w^0, w^1)\|_{\mathcal{H}_0} \leq \frac{C}{(1 + t)^{\frac{\ell}{2}}} \|(w^0, w^1)\|_{\mathcal{D}(\mathcal{A}_0)}, \quad \forall t \geq 0,$$

for all $(w^0, w^1) \in \mathcal{D}(\mathcal{A}_0)$ when $M(|\omega|) = |\omega|^\ell$. However, when $M(|\omega|) = e^{K_0|\omega|}$ for some $K_0 > 0$ imply from Burq [8] that the semigroup $e^{t\mathcal{A}_0}$ is logarithmically stable, namely we have

$$\|e^{t\mathcal{A}_0}(w^0, w^1)\|_{\mathcal{H}_0} \leq \frac{C}{\log^{k}(2 + t)} \|(w^0, w^1)\|_{\mathcal{D}(\mathcal{A}_0^k)}, \quad \forall t \geq 0,$$

for every $(w^0, w^1) \in \mathcal{D}(\mathcal{A}_0^k)$ and $k \in \mathbb{N}^*$.

**Theorem 6.1.** We assume that $i\mathbb{R} \subset \rho(\mathcal{A})$ and the condition (6.5) holds. Let $\eta > 0$, there exists a constant $C > 0$ such that

$$\|e^{(i\omega I - \mathcal{A})^{-1}}\|_{\mathcal{L}(\mathcal{H})} \leq C|\omega|^{1-\alpha} M(|\omega|).$$

Since $i\mathbb{R} \subset \rho(\mathcal{A})$, then according to Borichev and Tomilov theorem [6, Theorem 2.4], we obtain the following corollary.

**Corollary 6.1.** We assume the condition (6.5) holds with $M(|\omega|) = |\omega|^\ell$ for $\ell \geq 0$. Then the semigroup $e^{t\mathcal{A}}$ is polynomially stable, namely there exists a constant $C > 0$ such that

$$\|e^{t\mathcal{A}}(w^0, u^1, \varphi^0)\|_{\mathcal{H}} \leq \frac{C}{(1 + t)^{\frac{\ell}{2}}} \|(w^0, u^1, \varphi^0)\|_{\mathcal{D}(\mathcal{A})}, \quad \forall t \geq 0,$$

for every initial data $(w^0, u^1, \varphi^0) \in \mathcal{D}(\mathcal{A})$. In particular, the energy of the strong solution of (1.1)-(2.4) satisfy the following estimate

$$E(t) \leq \frac{C}{(1 + t)^{1-\frac{\ell}{2}}} \|(u^0, u^1, 0)\|_{\mathcal{D}(\mathcal{A})}^2.$$  

From Burq [8], see also Batty-Duyckaerts [5] for similar results, we obtain the following corollary.

**Corollary 6.2.** We assume the condition (6.5) holds with $M(|\omega|) = e^{K_0|\omega|}$ for some $K_0 > 0$. Then the semigroup $e^{t\mathcal{A}}$ is logarithmically stable, there exists a constant $C > 0$ such that

$$\|e^{t\mathcal{A}}(w^0, u^1, \varphi^0)\|_{\mathcal{H}} \leq \frac{C}{\log^{(1 + t)}} \|(w^0, u^1, \varphi^0)\|_{\mathcal{D}(\mathcal{A})}, \quad \forall t \geq 0,$$

for every initial data $(w^0, u^1, \varphi^0) \in \mathcal{D}(\mathcal{A})$. In particular, the energy of the strong solution of (1.1)-(2.4) satisfy the following estimate

$$E(t) \leq \frac{C}{\log^{2}(1 + t)} \|(u^0, u^1, 0)\|_{\mathcal{D}(\mathcal{A})}^2.$$
Proof. We need just to prove that

\[
(6.6) \quad \limsup_{\omega \in \mathbb{R}, |\omega| \to +\infty} |\omega|^{\alpha - 1} M(|\omega|)^{-1} \|(i\omega I - A)^{-1}\|_{L(\mathcal{H})} < +\infty
\]

is satisfied. For this purpose, we will use an argument of contradiction. We suppose that (6.6) is false, then there exist a real sequence \((\omega_n)\), with \(\omega_n \to +\infty\) and a sequence \((u_n, v_n, \varphi_n) \in \mathcal{D}(A)\), verifying the following condition

\[
(6.7) \quad \|(u_n, v_n, \varphi_n)\|_{\mathcal{H}} = 1
\]

and

\[
(6.8) \quad \omega_n^{1-\alpha} M(\omega_n)(i\omega_n I - A) \begin{pmatrix} u_n \\ v_n \\ \varphi_n \end{pmatrix} = \begin{pmatrix} f_n \\ g_n \\ h_n \end{pmatrix} \to 0 \text{ in } \mathcal{H}.
\]

Multiplying (6.8) by \(\begin{pmatrix} f_n \\ g_n \\ h_n \end{pmatrix}\) and taking the real part of the inner product, we obtain

\[
(6.9) \quad \text{Re} \left\langle \begin{pmatrix} f_n \\ g_n \\ h_n \end{pmatrix}, \begin{pmatrix} u_n \\ v_n \\ \varphi_n \end{pmatrix} \right\rangle = \omega_n^{1-\alpha} M(\omega_n) \gamma \int_{\mathbb{R}^d} (|\xi|^2 + \eta) \|\varphi_n(\xi)\|_{V}^2 \, d\xi \longrightarrow 0.
\]

Detailing equation (6.8), we get

\[
(6.10) \quad \omega_n^{1-\alpha} M(\omega_n)(i\omega_n u_n - v_n) = f_n \to 0 \text{ in } H^\perp,
\]

\[
(6.11) \quad \omega_n^{1-\alpha} M(\omega_n) \left( i\omega_n v_n + Au_n + \gamma B \int_{\mathbb{R}} p(\xi) \varphi_n(\xi) \, d\xi \right) = g_n \to 0 \text{ in } H,
\]

\[
(6.12) \quad \omega_n^{1-\alpha} M(\omega_n)(i\omega_n \varphi_n + (|\xi|^2 + \eta) \varphi_n - p(\xi) B^* v_n) = h_n \to 0 \text{ in } V.
\]

We draw immediately from (6.10) that

\[
(6.13) \quad \omega_n \|u_n\|_H = O(1).
\]

Taking the inner product of (6.11) with \(u_n\) in \(H\) and using (6.10), one has

\[
\|u_n\|^2_{H^\perp} - \omega_n \|u_n\|^2_H = -\gamma \left\langle \int_{\mathbb{R}^d} p(\xi) \varphi_n(\xi) \, d\xi, B^* u_n \right\rangle_U + \omega_n^{\alpha-1} M(\omega_n)^{-1} \left( (g_n, u_n)_H + i\omega_n (f_n, u_n)_H \right).
\]

Using Cauchy-Schwarz inequality, we obtain

\[
\|u_n\|^2_{H^\perp} - \omega_n \|u_n\|^2_H \leq \omega_n^{\alpha-1} M(\omega_n)^{-1} \|u_n\|_H (|\omega_n| \|f_n\|_H + \|g_n\|_H)
\]

\[
+ \gamma \|B^* u_n\|_U \left( \int_{\mathbb{R}^d} \frac{p(\xi)^2}{|\xi|^2 + \eta} \, d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} (|\xi|^2 + \eta) \|\varphi_n(\xi)\|_V^2 \, d\xi \right)^{\frac{1}{2}}.
\]

Then (6.7), (6.9) and (6.13) leads to

\[
(6.14) \quad \|u_n\|_{H^\perp} - \omega_n \|u_n\|_H \longrightarrow 0.
\]

Following to (6.10) equations (6.11) and (6.12) can be recast as follow

\[
\varphi_n(\xi) = i\omega_n \frac{p(\xi)}{|\xi|^2 + \eta + i\omega_n} B^* u_n - \omega_n^{\alpha-1} M(\omega_n)^{-1} \frac{p(\xi)}{|\xi|^2 + \eta + i\omega_n} B^* f_n
\]

\[
+ \omega_n^{\alpha-1} M(\omega_n)^{-1} \frac{h_n(\xi)}{|\xi|^2 + \eta + i\omega_n}.
\]
that the resolvent of $A$

Multiplying (6.15) by $|\xi|^{2-d}/2$ then integrating over $\mathbb{R}^d$ with respect to the $\xi$ variable and using Cauchy-Schwarz inequality, we obtain

Using Lemma 5.1 and Lemma 6.1 we follow

Using Lemma 5.1 and Lemma 6.1 we follow

which imply from (6.9) that

Now we recall that the semigroup generated by the operator $A_0$ is stable (in the sense of condition (5.5)) in the Hilbert space $\mathcal{H}_0$ then there exist a unique couple $(w_n, z_n) \in D(A_0)$ such that

satisfying the following estimate

since the resolvent of $A_0$ satisfies condition (5.5). Next, we take the inner product in $H$ of the first line of (6.15) with $\omega_n w_n$, one gets

Taking the imaginary part of (6.20), using Cauchy-Schwarz inequality, and (6.19), one gets

Taking the inner product of (6.15) with $\omega_n^2 w_n$ in the Hilbert space $H$, we have

(6.22)

where

(6.16)

and

(6.17)

(6.21)

Using Lemma 5.1, Lemma 6.1 and estimates (6.7), (6.8), (6.13), (6.17), (6.19) and (6.21), we obtain

\begin{equation}
\omega_n^3 \langle B^* u_n, B^* w_n \rangle_U \leq \omega_n^2 \| B^* u_n \|_U \| B^* w_n \|_U \leq C \omega_n^2 M(\omega_n)^{1/2} \| B^* u_n \|_U \| u_n \|_H \\
\leq C \omega_n M(\omega_n)^{1/2} \| B^* u_n \|_U \| u_n \|_H \xrightarrow{n \to +\infty} 0,
\end{equation}

\begin{equation}
\omega_n^3 \int_{\mathbb{R}^d} \frac{p(\xi)^2}{|\xi|^2 + \eta + i\omega_n} d\xi \langle B^* u_n, B^* w_n \rangle_U \leq C \omega_n^{2+\alpha} \| B^* u_n \|_U \| B^* w_n \|_U \\
\leq C \omega_n^\alpha M(\omega_n)^{1/2} \| B^* u_n \|_U \| u_n \|_H \xrightarrow{n \to +\infty} 0,
\end{equation}

\begin{equation}
\omega_n^{n+1} M(\omega_n)^{-1} \left| \int_{\mathbb{R}^d} \frac{p(\xi)^2}{|\xi|^2 + \eta + i\omega_n} d\xi \langle B^* f_n, B^* w_n \rangle_U \right| \\
\leq C \omega_n^{2n-2} M(\omega_n)^{-1/2} \| f_n \|_{H^1} \| u_n \|_H \xrightarrow{n \to +\infty} 0,
\end{equation}

\begin{equation}
\omega_n^{n+1} M(\omega_n)^{-1} \left| \int_{\mathbb{R}^d} \frac{p(\xi)^2}{|\xi|^2 + \eta + i\omega_n} d\xi \langle h_n, B^* w_n \rangle_U \right| \\
\leq \omega_n^{-1} M(\omega_n)^{-1} \left( \int_{\mathbb{R}^d} \frac{p(\xi)^2}{|\xi|^2 + \eta + i\omega_n} d\xi \right)^{1/2} \| h_n \|_{L^2} \| u_n \|_H \\
\leq C \omega_n^{-1} \omega_n^{n/2-1} M(\omega_n)^{-1} \| h_n \|_{L^2} \| u_n \|_H \xrightarrow{n \to +\infty} 0,
\end{equation}

and

\begin{equation}
\omega_n^{n+1} M(\omega_n)^{-1} \| g_n \|_{H^1} \langle g_n, w_n \rangle_H \leq C \omega_n^{n+1} M(\omega_n)^{-1} \| g_n \|_{L^2} \| u_n \|_H \xrightarrow{n \to +\infty} 0.
\end{equation}

Taking the inner product of the first equation of (6.18) with $f_n$, we obtain

\[-\omega_n^2 (w_n, f_n)_H + \langle A^* w_n, A^* f_n \rangle_H + i\omega_n \langle B^* w_n, B^* f_n \rangle_U = \langle u_n, f_n \rangle_H.
\]

This with (6.7), (6.8), (6.13), (6.19) and (6.21) give

\begin{equation}
\omega_n^{n+2} M(\omega_n)^{-1} \| \langle w_n, f_n \rangle_H \|_{U} \leq \omega_n^{n} M(\omega_n)^{-1} \left( \| w_n \|_{H^1} \| f_n \|_{H^1} + \| u_n \|_{H^1} \| f_n \|_{H^1} \right) + \omega_n^{n+3} M(\omega_n)^{-1} \| B^* w_n \|_U \| B^* f_n \|_U \\
\leq C \omega_n^{-1} (1 + M(\omega_n)^{-1} + M(\omega_n)^{-\frac{1}{2}}) \| f_n \|_{H^1} \| u_n \|_H \xrightarrow{n \to +\infty} 0.
\end{equation}

It follows from the combination of (6.22) and (6.23)–(6.28) that \( \| \omega_n u_n \|_H \xrightarrow{n \to +\infty} 0 \). Thus, by (6.14) we have \( \| u_n \|_{H^1} \xrightarrow{n \to +\infty} 0 \). Together with (6.16) and (6.17) imply that \( (u_n, v_n, \varphi_n) \xrightarrow{n \to +\infty} 0 \) which contradicts (6.7). This completes the proof. \( \square \)

**Remark 6.1.** In the case where for all $\delta > 0$, \( \sup_{R, \lambda: \delta} \| \lambda B^*(\lambda^2 I + A)^{-1} B \|_{\mathcal{L}(U)} < \infty \), according to [2] (see also [3]), we can replace the hypothesis (6.5) by the following observability inequalities and we obtain the same results:

- for $\ell = 0$, the assumption (6.5) is equivalent to the following exact observability inequality: there exists $T, C > 0$ such that

\[
\int_0^T \left\| (0 \ B^*)e^{\left( \begin{array}{cc} 0 & I \\ -A & 0 \end{array} \right)} \left( \begin{array}{c} u^0 \\ u^1 \end{array} \right) \right\|_{U}^2 \geq C \left\| (u^0, u^1) \right\|_{H_0}^2, \forall (u^0, u^1) \in H_1 \times H,
\]

and
Let Now we have the following lemma (for proof look at those of Lemma 7.2 and Lemma 7.4)

\[
\int_0^T \left\| (0 B^*) e \left( \begin{array}{cc} 0 & I \\ -A & 0 \end{array} \right) \left( \begin{array}{c} u^0 \\ u^1 \end{array} \right) \right\|^2_U dt \geq C \left\| \left( \begin{array}{c} u^0 \\ u^1 \end{array} \right) \right\|^2_{H^{-\frac{1}{2}} \times H^{-\frac{1}{2}}}, \forall \left( \begin{array}{c} u^0 \\ u^1 \end{array} \right) \in H_1 \times H.
\]

7. Applications to the fractional-damped wave equation

7.1. Internal fractional-damped wave equation. We consider a wave equation with an internal fractional-damping in a bounded domain \( \Omega \) of \( \mathbb{R}^n \) with smooth boundary \( \Gamma = \partial \Omega \)

\[
\begin{cases}
\partial^2_t u(x, t) - \Delta u(x, t) + a(x) \partial_t^\alpha u(x, t) = 0 & \text{in } \Omega \times \mathbb{R}_+ \\
u(x, t) = 0 & \text{on } \Gamma \times \mathbb{R}_+ \\
u(x, 0) = \nu^0(x), \quad \partial_t u(x, 0) = \nu^1(x) & \text{in } \Omega,
\end{cases}
\]

where \( a(x) \) is a positive function in \( \Omega \) verifying that there exist a non empty subset \( \omega_0 \subset \Omega \) and a strictly positive constant \( a_0 \) such that

\[ a(x) \geq a_0 \quad \forall x \in \omega_0. \]

System (7.1) can be recast as follow

\[
\begin{cases}
\partial^2_t u(x, t) - \Delta u(x, t) + \gamma \sqrt{a(x)} \int_{\mathbb{R}^d} p(\xi) \varphi(x, t, \xi) \, d\xi = 0 & (x, t) \in \Omega \times \mathbb{R}_+ \\
\partial_t \varphi(x, t, \xi) + (\|\xi\|^2 + \eta) \varphi(x, t, \xi) = p(\xi) \sqrt{a(x)} \partial_t u(x, t) & (x, t, \xi) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \\
u(x, t) = 0 & (x, t) \in \Gamma \times \mathbb{R}_+ \\
\varphi(x, 0, \xi) = 0 & x \in \Omega, \xi \in \mathbb{R}^d.
\end{cases}
\]

The energy of the system is given by

\[
E(t) = \frac{1}{2} \left( \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^2 + \gamma \int_{\mathbb{R}^d} \|\varphi(t, \xi)\|_{L^2(\Omega)}^2 \, d\xi \right).
\]

The operator \( A = -\Delta \) is strictly positive and auto-adjoint operator in \( H = L^2(\Omega) \) and with domain \( \mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega) \). The operator \( A \) corresponding to the Cauchy problem of system (7.2) is given by

\[
A \left( \begin{array}{c} u \\ v \phi \\ \varphi \end{array} \right) = \left( \begin{array}{c}
\Delta u - \gamma \sqrt{a} \int_{\mathbb{R}^d} p(\xi) \varphi(\xi) \, d\xi \\
-(\|\xi\|^2 + \eta) \varphi(\xi) + p(\xi) \sqrt{a} v \\
\end{array} \right)
\]

with domain in the Hilbert space \( \mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\mathbb{R}^d; L^2(\Omega)) \) given by

\[
\mathcal{D}(A) = \left\{ (u, v, \varphi) \in \mathcal{H} : v \in H_0^1(\Omega), \Delta u - \gamma \sqrt{a} \int_{\mathbb{R}^d} p(\xi) \varphi(\xi) \, d\xi \in L^2(\Omega), \right.
\]

\[
\left. |\xi| \varphi \in L^2(\mathbb{R}^d; L^2(\Omega)), (\|\xi\|^2 + \eta) \varphi(\xi) - p(\xi) \sqrt{a} v \in L^2(\mathbb{R}^d; L^2(\Omega)) \right\}.
\]

Since the embedding \( H_0^1(\Omega) \hookrightarrow L^2(\Omega) \) is compact and the only solution of the following problem

\[
\begin{cases}
\partial^2_t u(x, t) - \Delta u(x, t) = 0 & (x, t) \in \Omega \times \mathbb{R}_+ \\
\sqrt{a(x)} \partial_t u(x, t) = 0 & (x, t) \in \Omega \times \mathbb{R}_+ \\
u(x, t) = 0 & (x, t) \in \Gamma \times \mathbb{R}_+,
\end{cases}
\]

Now we have the following lemma (for proof look at those of Lemma 7.2 and Lemma 7.4)

Lemma 7.1. Let \( \eta > 0 \) and for all \( \omega \in \mathbb{R} \) the operator \( (i\omega I - A) \) is injective and surjective.
We assume that the semigroup of the operator $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}) \subset \mathcal{H}_0 \rightarrow \mathcal{H}_0$ defined by

$$\mathcal{A}_0 \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} v \\ \Delta u - av \end{array} \right)$$

where $\mathcal{H}_0 = H^1_0(\Omega) \times L^2(\Omega)$ with domain

$$\mathcal{D}(\mathcal{A}_0) = \{(u, v) \in \mathcal{H}_0 : \Delta u - av \in L^2(\Omega), \ v \in H^1_0(\Omega)\},$$

is uniformly stable in the energy space $\mathcal{H}_0$, which means that the energy of the following system

$$\begin{cases} 
\partial^2_t w(x, t) - \Delta w(x, t) + a(x)\partial_t w(x, t) = 0 & \text{in } \Omega \times \mathbb{R}_+ \\
\partial_t w(x, t) = 0 & \text{on } \Gamma \times \mathbb{R}_+ \\\nw(x, 0) = w^0(x) & \text{in } \Omega.
\end{cases}$$

is exponentially stable. Noting that this can be held if the so called geometric control condition (GCC) is satisfied (see [4]).

**Proposition 7.1.** Under the above assumption and for $\eta > 0$ the operator $\mathcal{A}$ generates a contraction semigroup satisfying

$$\|e^{t\mathcal{A}}X\|_{\mathcal{H}} \leq \frac{C}{(1 + t)^{\alpha_0}} \|X\|_{\mathcal{D}(\mathcal{A})}, \quad \forall X \in \mathcal{D}(\mathcal{A}), \ t \geq 0,$$

for some constant $C > 0$. This means that the energy of system (1.1) is decreasing to zero as $t$ goes to $+\infty$ as $t^{-\alpha_0}$.

**Proof.** Following to Lemma 7.1 the operator $(i\omega I - \mathcal{A})$ is bijective for every $\omega \in \mathbb{R}$, then using the closed graph theorem we follow that $i\mathbb{R} \subset \rho(\mathcal{A})$. The result follow now from Corollary 6.1.

**Remark 7.1.** In the case where $\Omega = (0, 1) \times (0, 1)$ and

$$a(x) = \begin{cases} 1, \forall x \in (0, \varepsilon) \times (0, 1), \\
0, \text{elsewhere},
\end{cases}$$

where $\varepsilon > 0$ is a constant, we have according to [22] that the semigroup generated by the operator $\mathcal{A}_0$ decays as $t^{-\frac{1}{2}}$ (which is optimal). We obtain in this case from Corollary 6.1 that the polynomial decay rate for the semigroup $e^{t\mathcal{A}}$ is given by $t^{-\alpha_0}$.

However, we obtain a logarithm decay rate of the semigroup $e^{t\mathcal{A}}$ as given in Corollary 6.2 without any geometrical condition according to [13] the resolvent of the operator $\mathcal{A}_0$ satisfies the condition (6.3) with $M(|\omega|) = e^{K_0|\omega|}$ for some $K_0 > 0$.

7.2. Fractional-Kelvin-Voigt damped wave equation. We consider the following damped wave system

$$\begin{cases} 
\partial^2_t u(x, t) - \Delta u(x, t) - \text{div}(a(x)\nabla \partial_t^{\alpha_0} u(x, t)) = 0 & (x, t) \in \Omega \times \mathbb{R}_+ \\
u(x, t) = 0 & (x, t) \in \Gamma \times \mathbb{R}_+ \\
u(x, 0) = u^0(x), \quad \partial_t u(x, 0) = u^1(\chi) & x \in \Omega,
\end{cases}$$

where we have made the same notations as the previous subsection. Equivalently, we have

$$\begin{cases} 
\partial^2_t u(x, t) - \Delta u(x, t) - \gamma \text{div} \left( \sqrt{a(x)} \int_{\mathbb{R}^d} p(\xi)\varphi(x, t, \xi) d\xi \right) = 0 & (x, t) \in \Omega \times \mathbb{R}_+ \\
\partial_t \varphi(x, t, \xi) + (|\xi|^2 + \eta) \varphi(x, t, \xi) = p(\xi)\sqrt{a(x)}\nabla \partial_t u(x, t) & (x, t, \xi) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \\
u(x, t) = 0 & (x, t) \in \Gamma \times \mathbb{R}_+ \\
u(x, 0) = u^0(x), \quad \partial_t u(x, 0) = u^1(x) & x \in \Omega, \quad \varphi(x, 0, \xi) = 0 & x \in \Omega, \quad \xi \in \mathbb{R}^d.
\end{cases}$$

The energy of the system is given by

$$E(t) = \frac{1}{2} \left( \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^2 + \gamma \int_{\mathbb{R}^d} \|\varphi(t, \xi)\|_{L^2(\Omega)}^2 d\xi \right).$$
The operator $A = -\Delta$ is strictly positive and auto-adjoint operator in $H = L^2(\Omega)$ and with domain $\mathcal{D}(A) = H^1_0(\Omega) \cap H^2(\Omega)$. The operator $A$ corresponding to the Cauchy problem of system (7.3) is given by

$$A \begin{pmatrix} u \\ v \\ \varphi \end{pmatrix} = \begin{pmatrix} \Delta u + \gamma \text{div} \left( \sqrt{a} \int_{\mathbb{R}^d} p(\xi) \varphi(\xi) \, d\xi \right) \\ -((|\xi|^2 + \eta) \varphi(\xi) + p(\xi) \sqrt{a} \nabla v) \end{pmatrix}$$

with domain in the Hilbert space $\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega) \times L^2(\mathbb{R}^d; (L^2(\Omega))^n)$ is given by

$$\mathcal{D}(A) = \left\{ (u, v, \varphi) \in \mathcal{H} : u \in H^1_0(\Omega), \Delta u + \gamma \text{div} \left( \sqrt{a} \int_{\mathbb{R}^d} p(\xi) \varphi(\xi) \, d\xi \right) \in L^2(\Omega), \right.$$

$$\left. |\xi| \varphi \in L^2(\mathbb{R}^d; (L^2(\Omega))^n), (|\xi|^2 + \eta) \varphi(\xi) - p(\xi) \sqrt{a} \nabla v \in L^2(\mathbb{R}^d; (L^2(\Omega))^n) \right\}.$$

**Proposition 7.2.** System (7.3) is well posed in the energy space $\mathcal{H}$. Moreover, the semigroup generated by the operator $A$ is not uniformly stable but if we suppose that the support of a have a common boundary $\gamma$ with $\Omega$ with $\text{mes}(\gamma) \neq 0$ then the semigroup is strongly stable.

**Proof.** The well-posedness follows from Theorem 3.1 and the non uniform stabilization follows from Theorem 5.1. However, the strong stabilization follows from Theorem 4.1 since the only solution of the problem

$$\begin{align*}
\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) &= 0 & (x,t) \in \Omega \times \mathbb{R}_+ \\
\sqrt{a(x)} \nabla \partial_t u(x,t) &= 0 & (x,t) \in \Omega \times \mathbb{R}_+ \\
u(x,t) &= 0 & (x,t) \in \Gamma \times \mathbb{R}_+,
\end{align*}$$

is the zero solution in fact, using the second line of (7.4) we can see easily that $\nabla u(x) = f(x)$ and $\partial_t u = g(t)$ in $\omega = \text{supp}(a)$, then by putting this into the first equation of (7.4) we find that $\frac{\partial^2 u}{\partial t^2}$ is constant in $\omega$. Then with all this the solution of (7.4) is written in $\omega$ as follow $u(x,t) = \beta t^2 + \delta t + \phi(x)$, where $\beta$ and $\gamma$ are two real numbers. Using the boundary condition and the assumption made in the proposition we follow that $u = \phi$ in $\omega$. We set now $v(x,t) = \partial_t u(x,t)$, then it is clear that $v$ is continuous with respect to time variable, with value in $L^2(\Omega)$ and it satisfies

$$\begin{align*}
\frac{\partial^2 v(x,t)}{\partial t^2} - \Delta v(x,t) &= 0 & (x,t) \in \Omega \times \mathbb{R}_+ \\
v(x,t) &= 0 & (x,t) \in \omega \times \mathbb{R}_+ \\
v(x,t) &= 0 & (x,t) \in \Gamma \times \mathbb{R}_+.
\end{align*}$$

Since $v \in L^2(\Omega)$ then using the unique continuation theorem we find that $v \equiv 0$ in $\Omega$. This means that $u$ is only depends on the $x$ variable and verifying the following system of equations

$$\begin{align*}
-\Delta u(x) &= 0 & x \in \Omega \\
u(x) &= 0 & x \in \Gamma.
\end{align*}$$

Since the Dirichlet Laplacian operator is invertible we follow that $u \equiv 0$ in $\Omega$. And this completes the proof. \qed

**Lemma 7.2.** For all $\omega \in \mathbb{R}$ the operator $(i\omega I - A)$ is injective.

**Proof.** Let $X = \begin{pmatrix} u \\ v \\ \varphi \end{pmatrix} \in \mathcal{D}(A)$ such that

$$AX = i\omega X \quad (7.5)$$

Then the dissipation property of the operator $A$ imply that

$$\text{Re}(AX, X) = -\gamma \int_{\mathbb{R}^d} (|\xi|^2 + \eta) \|\varphi(\xi)\|^2_{L^2(\Omega))^n} \, d\xi = 0.$$
Then we deduce that 
\[ \varphi(\xi) = 0 \quad \text{in} \ (L^2(\Omega))^n \text{ a.e } \xi \in \mathbb{R}^d. \]
Since that problem (7.5) becomes
\[
\begin{align*}
  v &= i\omega u & \text{in } \Omega, \\
  \omega^2 u + \Delta u &= 0 & \text{in } \Omega, \\
  \text{div}(a(x)\nabla u) &= 0 & \text{in } \text{supp}(a), \\
  u &= 0 & \text{on } \Gamma,
\end{align*}
\]
We denote by \( w_j = \partial_{x_j} u \) and we derive the second and the third equation, one gets
\[
\begin{align*}
  \omega^2 w_j + \Delta w_j &= 0 & \text{in } \Omega, \\
  w_j &= 0 & \text{in } \text{supp}(a),
\end{align*}
\]
By unique continuation theorem we find that \( w_j = 0 \) in \( \Omega \) therefore \( u = 0 \) in \( \Omega \) since \( u_j = 0 \) and consequently \( U = 0 \). Thus, the injection of the operator \((i\omega I - A)\) is proven. \( \square \)

**Lemma 7.3.** Assume that \( \eta > 0 \) and \( \omega \in \mathbb{R} \) then for any \( f \in H^{-1}(\Omega) \) the following problem
\[(7.6) \quad \begin{cases} 
  \omega^2 u + \Delta u + (\omega^2 c_1 + i\omega c_2) \text{div}(a\nabla u) = f & \text{in } \Omega, \\
  u = 0 & \text{on } \Gamma,
\end{cases} \]
where
\[ c_1 = \gamma \int_{\mathbb{R}^d} \frac{p(\xi)^2}{|\xi|^2 + \eta^2 + \omega^2} \, d\xi \quad \text{and} \quad c_2 = \gamma \int_{\mathbb{R}^d} \frac{p(\xi)^2(|\xi|^2 + \eta^2) + \omega^2}{|\xi|^2 + \eta^2 + \omega^2} \, d\xi \]
admits a unique solution \( u \in H^1_0(\Omega) \).

**Proof.** First we note that the coefficients \( c_1 \) and \( c_2 \) are well defined. We distinguish two cases:

**Case 1:** \( \omega = 0 \). In this case we use the Lax-Milgram’s theorem to prove the unique solution \( u \in H^1_0(\Omega) \) of (7.6).

**Case 2:** \( \omega \in \mathbb{R}^+ \). Separating the real and the imaginary parts of \( u \) and \( f \) by writing \( u = u_1 + iu_2 \) and \( f = f_1 + if_2 \) and we consider the following mixed system
\[(7.7) \quad \begin{cases} 
  -\Delta u_1 - c_1 \omega^2 \text{div}(a\nabla u_1) + c_2 \omega \text{div}(a\nabla u_2) = f_1 & \text{in } \Omega, \\
  -\Delta u_2 - c_1 \omega^2 \text{div}(a\nabla u_2) - c_2 \omega \text{div}(a\nabla u_1) = f_2 & \text{in } \Omega, \\
  u_1 = u_2 = 0 & \text{on } \Gamma.
\end{cases} \]
Consider the following bilinear form in \((H^1_0(\Omega) \times H^1_0(\Omega))^2\) defined by
\[
L((u_1, u_2); (w_1, w_2)) = \int_{\Omega} \nabla u_1 \nabla w_1 \, dx + \int_{\Omega} \nabla u_2 \nabla w_2 \, dx + c_1 \omega^2 \int_{\Omega} \nabla u_1 \nabla w_2 \, dx + c_1 \omega^2 \int_{\Omega} \nabla u_2 \nabla w_1 \, dx
\]
\[
+ c_2 \omega \int_{\Omega} \nabla u_2 \nabla w_2 \, dx - c_2 \omega \int_{\Omega} \nabla u_2 \nabla w_1 \, dx + c_2 \omega \int_{\Omega} \nabla u_1 \nabla w_2 \, dx.
\]
It is clear that \( L \) is continuous and coercive in \((H^1_0(\Omega) \times H^1_0(\Omega))^2\) then by Lax-Milgram’s theorem there exists a unique couple \((u_1, u_2) \in H^1_0(\Omega) \times H^1_0(\Omega) \) such that
\[
L((u_1, u_2); (w_1, w_2)) = \langle f_1, w_1 \rangle_{H^{-1} \times H^1_0} + \langle f_2, w_2 \rangle_{H^{-1} \times H^1_0}, \quad \forall (w_1, w_2) \in H^1_0(\Omega) \times H^1_0(\Omega).
\]
This leads to the existence and the uniqueness of a solution of the problem (7.7) in \( H^1_0(\Omega) \times H^1_0(\Omega) \).

This prove in particular that the operator \( A_\omega = -\Delta - (\omega^2 c_1 + i\omega c_2) \text{div}(a\nabla .) \) is invertible from \( H^1_0(\Omega) \) into \( H^{-1}(\Omega) \) then the first line of (7.6) is equivalent to the following equation
\[(7.8) \quad (\omega^2 A_\omega^{-1} - I)u = A_\omega^{-1} f. \]
It follows from the compactness of the embedding \( H^1_0(\Omega) \hookrightarrow H^{-1}(\Omega) \) that the inverse operator \( A_\omega^{-1} \) is compact in \( H^{-1}(\Omega) \). Let’s consider the following problem
\[(7.9) \quad \begin{cases} 
  \omega^2 u + \Delta u + (\omega^2 c_1 + i\omega c_2) \text{div}(a\nabla u) = 0 & \text{in } \Omega, \\
  u = 0 & \text{in } \Gamma,
\end{cases} \]
we multiplying the first line of (7.9) by \( \pi \) and integrating over \( \Omega \), one gets
\[
(7.10) \quad \omega^2 \| u \|^2_{L^2(\Omega)} - \| \nabla u \|^2_{L^2(\Omega)} - (\omega^2 c_1 + i \omega c_2) \| \sqrt{a} \nabla u \|^2_{L^2(\Omega)} = 0,
\]
then by taking the imaginary part of (7.10) we obtain \( \nabla u = 0 \) in \( \text{supp} (a) \). Proceeding as the proof of the previous lemma one gets \( u = 0 \) in \( \Omega \). This prove that the operator \( (\omega^2 A_\omega^{-1} - I) \) is injective. Then following to Fredhom's alternative theorem [7, Théorème 6.6], equation (7.8) admits a unique solution and therefore equation (7.6) admits a unique solution. \( \square \)

**Lemma 7.4.** Let \( \eta > 0 \) and a smooth enough then for all \( \omega \in \mathbb{R} \) the operator \((i \omega I - A)\) is surjective.

**Proof.** Let \( Y = (f, g, h) \in \mathcal{H} \) and we look for an \( X = (u, v, \varphi) \in \mathcal{D}(A) \) such that
\[
(i \omega I - A)X = Y.
\]
Equivalently, we have
\[
(7.11) \quad (i \omega I - A)X = Y.
\]
where \( c_1 \) and \( c_2 \) are defined in Lemma 7.3 and \( F \in L^2(\Omega) \) is given by
\[
F = (c_2 - i \omega c_1) \text{div}(a \nabla f) - i \omega f - g - \gamma \text{div} \left( \sqrt{a} \int_{\mathbb{R}^d} p(\xi) \frac{h(x, \xi)}{\xi^2 + \eta + i \omega} d\xi \right).
\]
Since for a smooth enough \( F \in H^{-1}(\Omega) \) then using Lemma 7.3 problem (7.12) has a unique solution \( u \in H^1_0(\Omega) \) and therefore problem (7.11) has a unique solution \( X \in \mathcal{D}(A) \).

We consider now the following auxiliary problem
\[
(7.13) \quad \begin{cases}
\partial_t^2 w(x, t) - \Delta w(x, t) + \text{div}(a(x) \nabla \partial_t w(x, t)) = 0 & \text{in } \Omega \times \mathbb{R}_+ \\
w(x, 0) = 0 & \text{on } \Gamma \times \mathbb{R}_+ \\
w(x, t) = 0 & \text{in } \Omega
\end{cases}
\]
The equation (7.13) is well posed in the Hilbert space \( \mathcal{H}_0 = H^1_0(\Omega) \times L^2(\Omega) \) and its solution is a semigroup generated by the operator \( A_0 : \mathcal{D}(A) \subset \mathcal{H}_0 \rightarrow \mathcal{H}_0 \) defined by
\[
A_0 \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} v \\ \Delta u - \text{div}(a \nabla v) \end{array} \right)
\]
with domain
\[
\mathcal{D}(A_0) = \{(u, v) \in \mathcal{H}_0 : \Delta u - \text{div}(a \nabla v) \in L^2(\Omega), \; v \in H^1_0(\Omega) \}.
\]
We recall that under some geometric considerations and some regularity of the damping coefficient [24, Theorem 1.2] or [25, Theorem 1.2] the semigroup generated by the operator \( A_0 \) is uniformly stable in the energy space \( \mathcal{H}_0 \), in another words the energy of the system (7.13) is exponentially stable.

**Proposition 7.3.** Under the above assumptions (see [24, Theorem 1.2]) and for \( \eta > 0 \) the operator \( A \) generates a \( C_0 \) semigroup of contractions satisfying
\[
\| e^{tA}X \|_\mathcal{H} \leq \frac{C}{(1 + t)^{\eta_0}} \| X \|_\mathcal{D}(A), \quad \forall X \in \mathcal{D}(A), \; t \geq 0,
\]
for some constant \( C > 0 \). This means that the energy of system (7.1) is decreasing to zero as \( t \) goes to \( +\infty \) as \( t \xrightarrow{\to} \infty \).
7.3. Pointwise fractional-damped string equation. We consider the equation of the vibration of a string of length equal to 1 with a pointwise fractional damping modeled by the following equation

\[
\begin{aligned}
&\frac{\partial^2 u(x,t) + u''(x,t) + \partial_t^{\alpha}u(\zeta,t)\delta_\zeta}{\partial_t} = 0 \quad (x,t) \in (0,1) \times \mathbb{R}_+ \\
u(0,t) = u(1,t) = 0 \\
u(x,0) = u^0(x) = \begin{cases} u^1(x) & x \in (0,1), \end{cases}
\end{aligned}
\]

where the prime denotes the space derivative and \(\delta_\zeta\) is the Dirac mass concentrated in the point \(\zeta\) of \((0,1)\) (See [12, 26] for the classical derivative). Equivalently we have\(^{(7.14)}\)

\[
\frac{\partial^2 u(x,t) - u''(x,t) + \gamma \int_{\mathbb{R}^d} p(\xi)\varphi(t,\xi) \, d\xi \delta_\zeta}{\partial_t} = 0 \quad (x,t) \in (0,1) \times \mathbb{R}_+
\]

\[
\begin{aligned}
&\partial_t \varphi(t,\xi) + (|\xi|^2 + \eta)\varphi(t,\xi) = p(\xi)\partial_u u(\zeta,t) \\
u(0,t) = u(1,t) = 0 \\
u(x,0) = u^0(x) = \begin{cases} u^1(x) & x \in (0,1), \end{cases}
\end{aligned}
\]

where we recall here that \(U = \mathbb{C}, H = L^2(0,1), H^1 = H^1_0(0,1), H^{-\frac{1}{2}} = H^{-\frac{1}{2}}(0,1), Bz = z\delta_\zeta\) for all \(z \in L^2(0,1)\) and \(B^\ast u = u(\zeta)\) for all \(u \in H^1_0(0,1)\).

We consider now the operator \(A : D(A) \subset H \longrightarrow H\) defined by

\[
A \begin{pmatrix} u \\ v \\ \varphi \end{pmatrix} = \begin{pmatrix} u'' + \gamma \int_{\mathbb{R}^d} p(\xi)\varphi(\xi) \, d\xi \delta_\zeta \\
-(|\xi|^2 + \eta)\varphi(\xi) + p(\xi)v(\xi) \end{pmatrix},
\]

in the Hilbert space \(H = H^1_0(0,1) \times L^2(0,1) \times L^2(\mathbb{R}^d;\mathbb{C})\) with domain \(D(A) = \{(u,v,\varphi) \in H : v \in H^1_0(0,1), u'' + \gamma \int_{\mathbb{R}^d} p(\xi)\varphi(\xi,\xi) \, d\xi \delta_\zeta \in L^2(0,1), \xi|\varphi \in L^2(\mathbb{R}^d;\mathbb{C}), -(|\xi|^2 + \eta)\varphi(\xi) + p(\xi)v(\xi) \in L^2(\mathbb{R}^d;\mathbb{C})\}\).

The energy of the solution of system \((7.14)\) is given by

\[
E(t) = \frac{1}{2} \left( \|\partial_t u(t)\|_{L^2(0,1)}^2 + \|u'(t)\|_{L^2(0,1)}^2 + \gamma \int_{\mathbb{R}^d} |\varphi(t,\xi)|^2 \, d\xi \right).
\]

**Proposition 7.4.** The semigroup generated by the operator \(A\) is strongly stable, i.e

\[
\lim_{t \to +\infty} \|e^{At}(u^0,v^0,\varphi^0)\|_H = 0, \quad \forall (u^0,v^0,\varphi^0) \in H,
\]

if and only if \(\zeta \notin \mathbb{Q}\).

**Proof.** The prove is done in two stages:

- We consider the following problem

\[
\begin{aligned}
&\frac{\partial^2 u - u''(x,t)}{\partial_t} = 0 \quad (x,t) \in (0,1) \times \mathbb{R}_+ \\
&\partial_t u(\zeta,t) = 0 \\
u(0,t) = u(1,t) = 0
\end{aligned}
\]

Then the solution of \((7.15)\) is given by

\[
\begin{aligned}
u(x,t) &= \sum_{k=1}^{+\infty} \langle u^0, \sin(k\pi \cdot) \rangle_{L^2(0,1)} \cos(k\pi t) \sin(k\pi x) \\
&\quad + 2 \sum_{k=1}^{+\infty} \langle u^1, \sin(k\pi \cdot) \rangle_{L^2(0,1)} \frac{\sin(k\pi t) \sin(k\pi x)}{k\pi}, \quad \forall x \in (0,1), \forall t \in \mathbb{R}_+,
\end{aligned}
\]

\[
\text{(7.16)}
\]
where $u^0$ and $u^1$ are the initial data. In particular, we have
\[
\partial_t u(\zeta, t) = -2\pi \sum_{k=1}^{+\infty} k\langle u^0, \sin(k\pi \cdot) \rangle_{L^2(0,1)} \sin(k\pi t) \sin(k\pi \zeta) \\
+ 2\sum_{k=1}^{+\infty} \langle u^1, \sin(k\pi \cdot) \rangle_{L^2(0,1)} \cos(k\pi t) \sin(k\pi \zeta) = 0, \quad \forall t \in \mathbb{R}_+.
\]
The uniqueness of the Fourier series imply that $k\langle u^0, \sin(k\pi \cdot) \rangle_{L^2(0,1)} \sin(k\pi \zeta) = 0$ and $\langle u^1, \sin(k\pi \cdot) \rangle_{L^2(0,1)} \sin(k\pi \zeta) = 0$ for all $k \in \mathbb{N}^*$. Since $\zeta \notin \mathbb{Q}$ then $\sin(k\pi \zeta) \neq 0$ for all $k \in \mathbb{N}^*$. Therefore, $\langle u^0, \sin(k\pi \cdot) \rangle_{L^2(0,1)} = 0$ and $\langle u^1, \sin(k\pi \cdot) \rangle_{L^2(0,1)} = 0$ for all $k \in \mathbb{N}^*$. Following to Theorem 4.1 we obtain $u = 0$. Thus, the first implication follows from Theorem 4.1.

- We recall that the sequence of eigenfunctions of the Dirichlet Laplacian operator in $(0,1)$ are given by
  \[ u_k(x) = \sin(k\pi x) \quad \forall x \in (0,1) \]
  formed an orthonormal base of $L^2(0,1)$ with the corresponding eigenvalues $-\mu_k = -k^2$ for all $k \in \mathbb{Z}$. Since $\zeta \in \mathbb{Q}$ then $B^* u_k = \sin(k\pi \zeta) = 0$ for some $n \in \mathbb{N}$. Following to the first case of the proof of Theorem 4.1 $ik$ is an eigenvalue of the operator $\mathcal{A}$. Therefore $\sigma(\mathcal{A}) \cap i\mathbb{R} \neq \emptyset$. This prove the second implication.

This completes the proof. \qed

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