Non-extremal, $\alpha'$-corrected black holes in 5-dimensional heterotic superstring theory

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Abstract: We compute the first-order $\alpha'$ corrections of the non-extremal Strominger-Vafa black hole and its non-supersymmetric counterparts in the framework of the Bergshoeff-de Roo formulation of the heterotic superstring effective action. The solution passes several tests: its extremal limit is the one found in an earlier publication and the effect of a T duality transformation on it is another solution of the same form with T dual charges. We compute the Hawking temperature and Wald entropy showing that they are related by the first law and Smarr formula. On the other hand, these two contain additional terms in which the dimensionful parameter $\alpha'$ plays the role of thermodynamical variable.

Keywords: Black Holes in String Theory, Superstrings and Heterotic Strings, String Duality

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1 Introduction

The microscopic interpretation of the entropy of a 5-dimensional extremal black-hole solution of the 0th-order in $\alpha'$ string effective action by Strominger and Vafa in ref. [1] is widely acknowledged as one of the main successes of Superstring Theory. Since then, extending this result beyond leading order in $\alpha'$ was a must.\(^1\) From the macroscopic side, advances in this front were possible thanks to the construction in [5] of a four-derivative supersymmetric invariant containing the mixed gauge/gravitational Chern-Simons term, $A \wedge \text{tr} (R \wedge R)$, and to the development of the entropy function formalism by Sen [6], which allows to extract the microcanonical form of the entropy from corrections to the near-horizon geometry, which are much easier to study. Making profit of this, refs. [7–9] calculated corrections to the macroscopic entropy of asymptotically-flat 5-dimensional black holes. These were soon later matched from a microscopic analysis [10] for the particular supergravity model arising from type IIB on $K_3 \times S^1$ (or, equivalently, heterotic string theory on $T^5$), extending the validity of the pioneering result of [1] beyond the supergravity regime. Subsequent calculations in the dual heterotic frame were later performed in [11], being also in agreement with the microscopic entropy previously derived in [10].

In spite of these successes, there were some open issues at that time that have not been fully addressed until very recently. One of them was to actually show that there exists a corrected solution extending from the AdS$_2 \times S^3$ near-horizon geometry to the asymptotically-flat region. Such solution was obtained in [12] in a fully analytical form. Having this solution at our disposal allows us to study aspects of the solutions that are not fully accessible within the near-horizon description. An instance of this is the proper identification between the conserved charges of the solution and the parameters that describe the microscopic system. As discussed in [13,14], this can lead to important consequences in some specific cases.

Another important issue that was not put on a solid ground is the macroscopic calculation of the entropy. This is due to the fact that the presence of Chern-Simons terms does not allow for a direct application of Wald’s entropy formula [15–17], as it is well known [18–22].\(^2\) Strictly speaking, this issue is not specific of Wald’s formula, as it also shows up in the entropy function formalism [6, 23]. Lacking a generalization of the Iyer-Wald prescription, an strategy to deal with the Chern-Simons terms present in the heterotic effective action was proposed in [23], being successfully applied to compute the entropy of 5-dimensional 3-charge black holes in [11, 24]. Remarkably, a gauge-invariant formula for the heterotic superstring effective action has been very recently derived in ref. [22]. Making use of it, the macroscopic entropy of the 5-dimensional extremal Strominger-Vafa and its non-supersymmetric analogues was finally computed in ref. [25], further confirming the existing results and putting them on a much more solid ground.

\(^1\) The problem of finding $\alpha'$ corrections to the classical black-hole solutions had been considered before in refs. [2–4].

\(^2\) It is not difficult to see that, due to the Lorentz-Chern-Simons terms present in the Kalb-Ramond field strength, the Iyer-Wald prescription [17] gives a frame-dependent entropy formula. The coefficients are not correct, either.
The entropy formula obtained in ref. [22] is manifestly frame independent. In certain frames it reduces to a formula similar to the one obtained applying the Iyer-Wald prescription, but with different numerical coefficients which lead to a different result. When applied to the $\alpha'$-corrected non-extremal Reissner-Nordström black hole in ref. [26] only the formula obtained in ref. [22] satisfies the thermodynamical relation $\partial S/\partial M = 1/T$ which follows from the first law of black-hole thermodynamics. It is worth stressing that the main justification for the identification of the Noether charge associated to the Killing vector that generates the event horizon with the black-hole entropy is the fact that it satisfies the first law [16, 17] and that it reduces to the Bekenstein-Hawking entropy in absence of higher-order terms. A general proof of the second law of black-hole thermodynamics for the Wald entropy is not yet available, although there has been remarkable progress in this direction [27, 28].

Although the test of the entropy formula of ref. [22] performed in ref. [26] is quite convincing, it would be most desirable to perform a similar test using a non-extremal generalization of the Strominger-Vafa black hole checking also that in the extremal limit one recovers the entropy computed in ref. [25] directly form the extremal solution. The non-extremal generalization of the Strominger-Vafa black hole was found in ref. [29] shortly after the extremal one but its $\alpha'$ corrections have never been computed. The goal of this work is to compute the 1st-order corrections to the non-extremal Strominger-Vafa black hole and its main thermodynamical properties. To this end, as in our previous works refs. [12, 25, 30–33], we are going to use the Bergshoeff-de Roo version of the heterotic superstring effective action [34] because its supersymmetrization is known among other reasons.

Computing $\alpha'$ corrections is a complicated problem: as usual, a good ansatz compatible with the symmetry of the problem is needed, but the number of independent functions that one still has to determine can be quite large. As we pointed out in ref. [25], T duality can be used to constrain the form of the corrections at least in the solution at hands, which is expected to be related to itself (up to redefinitions of the integration constants) under some T duality transformations.

T duality, which arises in string theory in the presence of toroidal compact directions, takes its simplest form when it is expressed in terms of the fields that result from the dimensional reduction of the theory over those compact directions. In this way, as shown in ref. [40] it is easy to recover the well-known Buscher rules [41, 42]. This observation has subsequently been used to extend the Buscher rules to the type II theories [43, 44] and to the heterotic superstring effective action to 1st order in $\alpha'$ [21, 45]. The possibility to constrain some of the corrections follows from this observation: all the fields that descend from the Kalb-Ramond field via dimensional reduction receive explicit $\alpha'$ corrections that can be computed at 1st-order using the 0th-order solution [21]. These explicit corrections are mixed and interchanged with implicit corrections by T duality and, demanding invariance,
one finds relations between the known (explicit) corrections and the unknown (implicit) ones that can be used to determine the latter. In our case, these relations have prove crucial to find all the corrections.

This paper is organized as follows: in section 2 we describe our 10-dimensional ansatz and in section 3 we explain how we have solved them using the T duality constraints mentioned above. In section 4 we compute the basic thermodynamical quantities of the corrected solution and in section 5 we present our conclusions and point to some directions for future work. The appendices contain complementary information: the 1st order in $\alpha'$ heterotic superstring effective action and the equations of motion of the fields to that order are reviewed in appendix A in the conventions we are using. The 0th-order solutions we start from are reviewed in appendix B. In particular, we derive its basic thermodynamical properties which have to be recovered in the $\alpha' \to 0$ limit. The first law of black hole mechanics is checked at 0th order in appendix C.

## 2 The 10-dimensional ansatz

Based on our knowledge of the 0th-order non-extremal solution\textsuperscript{6} and on our knowledge of the first-order extremal solutions [12, 25, 31, 32], an educated ansatz for the 3-charge 5-dimensional black-hole solution that we want to find and study turns out to require the introduction of 7 independent functions of $\rho$

\[ Z_0, Z_+, Z_-, Z_{h0}, Z_{h-}, W_{tt}, W_{\rho\rho}, \]

(2.1)

The new functions $Z_{h0}$ and $Z_{h-}$ are, respectively, identical to $Z_0$ and $Z_-$ at 0th order and they have to be introduced because these functions get different corrections when they occur in different components of the fields of the solutions. The functions $W_{tt}$ and $W_{\rho\rho}$ reduce to $W$ at 0th order and are needed because $W$ gets different corrections when it is part of the $tt$ or the $\rho\rho$ components of the metric.

As for the constants, including the integration constants found at 0th order, we now need 10:

\[ c_\phi, \beta_0, \beta_+, \beta_-, q_0, q_+, q_-, \omega, \hat{\phi}_\infty, k_\infty. \]

(2.2)

We expect the same number of independent physical parameters as in the 0th-order solutions, namely 6, which means that we need 4 relations between these constants. We are going to assume that the three 0th-order relations eqs. (B.8) are satisfied at first order as well, without corrections. A fourth relation involving $c_\phi$ is obtained when one imposes that the asymptotic value of the dilaton is $\hat{\phi}_\infty$.

\textsuperscript{6}This solutions, found in ref. [29], is reviewed in appendix B. Some of the notation used here is introduced there. The reader is encouraged to read that appendix first.
The functions in eq. (2.1) are assumed to have the following form (i = 0, +, −, j = tt, ρρ):

\[ Z_i = 1 + \frac{q_i}{\rho^2} + \alpha' \delta Z_i, \]
\[ Z_{hi} = 1 + \frac{q_i}{\rho^2} + \alpha' \delta Z_{hi}, \]
\[ W_j = 1 + \frac{\omega}{\rho^2} + \alpha' \delta W_j. \]  

(2.3)

Thus, they become the functions of the 0th-order ansatz eqs. (B.6) when \( \alpha' = 0 \) (note that \( Z_{h0} = Z_0, Z_{h-} = Z_- \) and \( W_{tt} = W_{\rho\rho} = W \)).

In terms of these functions and constants, the 10-dimensional fields are assumed to be given by

\[
d\hat{s}^2 = \frac{1}{Z_0 Z_-} W_{tt} dt^2 - Z_0 (W_{\rho\rho}^{-1} d\rho^2 + \rho^2 d\Omega^2_{(3)}) \\
- \frac{k^2 Z_+}{Z_-} \left[ dz + \beta_+ k_{-1} \left( Z_+^{-1} - 1 \right) dt \right]^2 - dy^m dy^m, \quad m = 1, \ldots, 4, \]  
\[
\hat{H}^{(1)} = \beta_- d \left[ k_{-1} \left( Z_-^{-1} - 1 \right) dt \wedge dz \right] + \beta_0 \rho^2 Z_{h0} \omega_{(3)}, \]  
\[
e^{-2\hat{\phi}} = -\frac{2c_{\phi}}{\rho^3} \frac{Z_{h-}^2}{Z_0 Z_- Z_{h-}'} \sqrt{W_{tt}/W_{\rho\rho}}. \]  

(2.4a)

(2.4b)

(2.4c)

This ansatz reduces to the 0th-order one eqs. (B.3) when \( Z_{h0} = Z_0, Z_{h-} = Z_- \) and \( W_{tt} = W_{\rho\rho} = W \).

All we have to do now is to substitute this ansatz into the equations of motion derived in appendix A.1 and solve for the corrections \( \delta Z_i, \delta Z_{hi}, \delta W_j \). Needless to say, this is a very difficult task. In the next section, we explain step by step how we have done it.

3 Solving the equations of motion

We start by the equations which are the simplest to solve: the Kalb-Ramond (KR) equation (A.6c) and Bianchi identity eq. (A.7). As a matter of fact, the ansatz for the dilaton has been chosen in such a way that the KR equation (A.6c) is automatically solved.

On the other hand, the only non-trivial component of the Bianchi identity eq. (A.7) to first order in \( \alpha' \) is the \( r\theta\phi\psi \) and, using the 0th-order solution, it becomes a differential equation for \( \delta Z_{h0} \) which is solved by

\[
\delta Z_{h0} = d_{h0}^{(0)} + \frac{d_{h0}^{(2)}}{\rho^2} + \frac{2q_0^3 + \omega (q_0^3 + 9q_0\rho^2 + 6\rho^4)}{2q_0\rho^2 (q_0 + \rho^2)^2} - \frac{3\omega}{q_0} \log \left( 1 + \frac{q_0}{\rho^2} \right), \]  

where \( d_{h0}^{(0)} \) and \( d_{h0}^{(2)} \) are integration constants. Imposing that \( Z_{h0} \) does not receive \( \alpha' \) corrections at infinity we obtain \( d_{h0}^{(0)} = 0 \).

The remaining equations of motion for the 6 remaining functions form a highly coupled system of 5 independent differential equations. Note that we have more unknown functions.

\footnote{Our ansatz is written in terms of the KR field strength and, therefore, we must impose the KR Bianchi identity.}
than equations, and the reason is that our ansatz contains some gauge freedom: one of
the functions can be chosen at will by means of an infinitesimal transformation of the radial
coordinate. This freedom will prove to be very useful to express the solution in a simple
form. Despite the complexity of the equations, they can be solved by the following procedure:
first, we use as an ansatz for the $\delta Z$s and $\delta W$s the following series with arbitrary coefficients
\begin{equation}
\delta Z_i = \sum_{n=1}^{\infty} \frac{d_i^{(2n)}}{\rho^{2n}} , \quad \delta Z_{hi} = \sum_{n=1}^{\infty} \frac{d_{hi}^{(2n)}}{\rho^{2n}} , \quad \delta W_j = \sum_{n=1}^{\infty} \frac{d_{wj}^{(2n)}}{\rho^{2n}} .
\end{equation}

Notice that we have assumed that all the powers in $1/\rho$ are even and that there is
no correction to the asymptotic values of $Z$s and $W$s. Furthermore, it follows that the
coefficient $c_\hat{\phi}$ that sets $\lim_{r \to \infty} \hat{\phi} = \hat{\phi}_\infty$ is given by
\begin{equation}
c_\hat{\phi} = \left( q_\alpha + \alpha' d_{h-}^{(2)} \right) e^{-2\hat{\phi}_\infty} .
\end{equation}

Plugging this ansatz into the Einstein equations (A.6) and the dilaton equation (A.6b)
and demanding that they are solved order by order in powers of $1/\rho$ one obtains
algebraic equations for the coefficients $d^{(2n)}$s that one can solve for a large value of $n$.
It turns out that not all of these coefficients are determined by these equations: the equations
of motion are solved for arbitrary values of the coefficients $d_i^{(2)}$, $d_{hi}^{(2)}$, $d_{wj}^{(2)}$ and $d_{h-}^{(2n)}$ with $n \geq 3$.
Having determined the coefficients for a large enough value of $n$ we can determine
the functions associated to those expansions resuming the series.\(^8\) Finally, we have to check
that those functions do solve exactly the equations of motion to first order in $\alpha'$.

Before carrying out this program, though, it is convenient to make some considerations
concerning T duality.

As discussed in appendix B.3, the 0\textsuperscript{th}-order solution, understood as a family, is invariant
under the 0\textsuperscript{th}-order Buscher T duality transformations [41, 42]: one member of the
family is transformed into another with different values of the parameters. It is enough to
transform the parameters according to the rules in eq. (B.37). At 0\textsuperscript{th} order in $\alpha'$, the physical
interpretation of the transformations of the parameters is the expected one: momentum
and winding are interchanged and the radius of the compact dimension is inverted.

It is reasonable to expect that the T duality invariance of the family of solutions
is preserved at first order in $\alpha'$ when the $\alpha'$-corrected Buscher T duality transformations
derived in refs. [21,45] are used because we are just dealing with a more accurate description
of exactly the same physical system. We have shown in ref. [25] that, in the extremal limit,
this is what happens and there is no reason to expect otherwise in the non-extremal case.

T duality is best studied in the dimensionally-reduced solution, where it takes the form
of a discrete symmetry transformation of the action [40] that, essentially, interchanges two vector fields and inverts a scalar. At 0\textsuperscript{th} order, these transformations are given in eq. (B.36).

At first order in $\alpha'$ the transformations have the same form:\(^9\)
\begin{equation}
A \leftrightarrow C^{(1)} , \quad k \leftrightarrow 1/k^{(1)} .
\end{equation}
The main difference with the 0th-order ones is that the relation between the winding vector $C^{(1)}$ and the higher-dimensional fields is modified by $\alpha'$ corrections (hence the upper (1) index). The scalar $k^{(1)}$ contains $\alpha'$ corrections as well. Then, the first-order solution can only be self-dual if the $\alpha'$ corrections one finds for $A$ and $k$ are related to the explicit $\alpha'$ corrections of $C^{(1)}$ and $k^{(1)}$ in a very specific way ref. [25]. Thus, the expected T duality invariance of the $\alpha'$-corrected solution can be used to simplify the problem of finding the corrections and also to test them.

In order to exploit this expected T duality invariance of the corrected solutions it is convenient to start by finding their 5-dimensional form using the relations between higher- and lower-dimensional fields found in ref. [21].

### 3.1 The 5-dimensional form

In order to find the 5-dimensional fields we first need the 10-dimensional KR 2-form potential. Its existence is guaranteed by the fact that we have solved the Bianchi identity determining $\delta Z_{h0}$ in eq. (3.1). The KR 2-form is given by

$$
\hat{B}^{(1)} = \beta^{-1} k_{\infty} \left( Z_{h}^{-1} - 1 \right) dt \wedge dz + \frac{1}{4} \left[ c + \alpha ' d + (\beta_0 q_0 + \alpha ') \cos \theta \right] d\phi \wedge d\psi ,
$$

where $c$ and $d$ are integration constants whose values can be chosen in different coordinate patches so as to make the field globally well-defined. Locally, they can be set to zero, and we will do so from now on for the sake of simplicity.

With this result we can compute the 5-dimensional KR 2-form $B^{(1)}$ and winding vector $C^{(1)}$ using the relations [21]

\begin{align}
B^{(1)}_{\mu \nu} &= \hat{B}_{\mu \nu} + \tilde{g}_{\mu [\nu} \hat{B}_{\nu \] z} + \frac{\alpha '}{4} \tilde{g}_{\mu [\nu} \hat{\Omega}^{(0)}_{\nu]} \hat{a} \hat{\Omega}^{(0)} \hat{b} / \hat{g}_{zz} , \\
C^{(1)}_{\mu} &= \hat{B}_{\mu \hat{z}} - \frac{\alpha '}{4} \left[ \hat{\Omega}^{(0)}_{\mu} \hat{a} \hat{\Omega}^{(0)} \hat{b} - \tilde{g}_{\mu z} \hat{\Omega}^{(0)} \hat{b} / \hat{g}_{zz} \right] ,
\end{align}

and the scalar combination

$$
k^{(1)} = k \left[ 1 - \frac{\alpha '}{4} \hat{\Omega}^{(0)} \hat{a} \hat{\Omega}^{(0)} \hat{b} / \hat{g}_{zz} \right] .
$$

The result, including the 5-dimensional KR 2-form potential and the scalar combination $k^{(1)}$, can be expressed in the string frame as follows:

\begin{align}
&\quad ds^2 = \frac{W_{tt}}{Z_+ Z_-} dt^2 - Z_0 \left( W_{\mu \nu}^{-1} d\mu^2 + \rho^2 d\Omega_{(3)}^2 \right) , \\
&\quad H^{(1)} = \beta_0 \rho^3 Z_{h0} \omega_{(3)} , \\
&\quad k = k_{\infty} \sqrt{Z_+/Z_-} , \\
&\quad k^{(1)} = k_{\infty} \sqrt{Z_+/Z_-} \left( 1 + \alpha ' \Delta_k \right) , \\
&\quad A = k^{-1} \beta_+ \left[ -1 + \frac{1}{Z_+} \right] dt , \\
&\quad C^{(1)} = k_{\infty} \beta_- \left[ -1 + \frac{1}{Z_{h-}} \left( 1 + \alpha ' \Delta C \frac{k}{\beta_-} \right) \right] dt , \\
&\quad e^{-2\phi} = -\frac{2 c_\phi}{\rho^3 Z_{h-}^2} \sqrt{\frac{W_{tt}}{Z_0}} \sqrt{\frac{Z_+ Z_-}{W_{\mu \nu}}} ,
\end{align}
where
\[ c_\phi = c_\phi^* k_\infty , \] (3.9)

and where the functions \( \Delta_k \) and \( \Delta_C \) are given by
\[ \Delta_k = \frac{-W (Z_+ Z'_- - Z_- Z'_+)^2 + (\beta_- Z_+ Z'_+ + \beta_+ Z_- Z'_+)^2}{8 Z_0 Z_+ Z'_+} , \] (3.10a)
\[ \Delta_C = \frac{-(\beta_- Z_+ Z'_+ + \beta_+ Z'_- Z_-) W' + 2 (\beta_+ + \beta_-) Z'_+ W Z'_+}{8 Z_0 Z_- Z_+} . \] (3.10b)

These functions contain the explicit \( \alpha' \) corrections to the fields \( C^{(1)} \) and \( k^{(1)} \) we have mentioned before.

We still have to do two additional modifications. First, the modified Einstein-frame metric is given by
\[ ds^2_E = \left[ -2 c_\phi^* e^{2\phi_\infty} Z_{h-} \rho \right]^{2/3} \left( \frac{W_{tt}}{W_{\rho\rho}} \right)^{1/3} \left\{ f^2 W_{tt} dt^2 - f^{-1} \left( W_{\rho\rho}^{-1} d\rho^2 + \rho^2 d\Omega^2 \right) \right\} , \] (3.11)
\[ f^{-3} = Z_+ Z_- Z_0 . \]

Second, since the equation of motion of the KR field takes exactly the same form as in the 0th-order case we can use the same duality relation eq. (B.13) to convert the 3-form field strength into the 2-form field strength
\[ K = d \left[ -\beta_0 \left( Z_{h0}^{-1} - 1 \right) dt \right] . \] (3.12)

The action of a T duality transformation in the direction of the compact coordinate \( z \) on the solution is just given by eq. (3.4) with the rest of the fields being invariant. As we have discussed, the family of solutions that we are after is expected to be invariant under these transformations.

We are now ready to apply the procedure we have outlined above.

### 3.2 A particular solution

First, for the sake of convenience, we want to re-sum the series in the simplest case, setting
\[ d_i^{(2)} = d_{hi}^{(2)} = d_{wj}^{(2)} = 0 \] and \( d_{h-}^{(2n)} = 0 \) for \( n \geq 3 \). We obtain the following values for the
corrections:\textsuperscript{10}

\[
\delta Z_{h_0} = \frac{2q_0^3 + \omega (q_0^2 + 9q_0\rho^2 + 6\rho^4)}{2q_0\rho^2(q_0 + \rho^2)^2} - 3\omega \log Z_0, \quad (3.13a)
\]

\[
\delta Z_\alpha = \frac{-3q_0^3(2\rho^2 + \omega) + 5q_0^3\omega (3\rho^2 + \omega) + \rho^4\omega^2 (\rho^2 + 2\omega)}{4q_0\rho^2(q_0 + \rho^2)^2(q_0 - \omega)^2} + q_0\rho^2\omega (-3\rho^4 - 5\rho^2\omega + 4\omega^2) + q_0^3(2\rho^6 + 3\rho^4\omega - 13\rho^2\omega^2 - 2\omega^3) + \omega^2q_0(2\rho^2 + \omega) - \rho^2\omega^2(\rho^2 + 2\omega) \log Z_0 + \frac{2q_0^2(2\rho^2 + \omega) + 3\rho^2\omega (\rho^2 + 2\omega) - q_0(2\rho^4 + 10\rho^2\omega + 3\omega^2)}{4\rho^2(q_0 - \omega)^2\omega} \log W, \quad (3.13b)
\]

\[
\delta Z_{h_+} = 0, \quad (3.13c)
\]

\[
\delta Z_- = Z_- \left( \Delta_h - \frac{\Delta C}{\beta_+} \right), \quad (3.13d)
\]

\[
\delta Z_+ = -Z_+ \frac{\Delta C}{\beta_+}, \quad (3.13e)
\]

\[
\delta W_{tt} = -\frac{Z_- (W + \beta_+ \beta_-) + W (\beta_+ + \beta_-) Z_-}{8\beta_+ Z_0 Z_-} Z' W', \quad (3.13f)
\]

\[
\delta W_{rr} = -\frac{\rho^2\omega^2 (2\rho^4 + 3\rho^2\omega + \omega^2) + q_0^3 (-4\rho^4 - 4\rho^2\omega + 2\omega^2)}{4q_0\rho^4(q_0 + \rho^2)(q_0 - \omega)^2} + \frac{q_0^2 (-4\rho^6 + 2\rho^4\omega + 6\rho^2\omega^2 - 4\omega^3) + q_0\omega (6\rho^6 + 5\rho^4\omega - \rho^2\omega^2 + 2\omega^3)}{4q_0\rho^4(q_0 + \rho^2)(q_0 - \omega)^2} + \frac{\omega^2 (2\rho^4 + 3\rho^2\omega + \omega^2)}{4q_0^2\rho^2(q_0 - \omega)^2} \log Z_0 + \frac{(2q_0^2 - 3\omega) (2\rho^4 + 3\rho^2\omega + \omega^2)}{4\rho^2(q_0 - \omega)^2\omega} \log W. \quad (3.13g)
\]

This solution, however, turns out to have a singular horizon. This is due to the choice of integration constants we have made, (\(d_{hi}^{(2)} = d_{hi}^{(2)} = d_{wj}^{(2)} = 0\) and \(d_{hi}^{(2n)} = 0\) for \(n \geq 3\)). In the next section we are going to fix this problem using this particular solution and modifying it by allowing for other choices of the integration constants.

### 3.3 The general solution

We now allow the coefficients of the \(1/\rho^2\) terms in the expansions eq. (3.2) to be different from zero and demand that the solution has a regular black hole solution. We are going to consider separately the \(\omega > 0\) and \(\omega < 0\) cases. At \(0^{th}\) order, these two cases are equivalent up to a coordinate transformation but it may happen that this equivalence is broken by the \(\alpha'\) corrections. In that case we would get two inequivalent solutions. While we do not really expect that black-hole uniqueness/no-hair is broken by the \(\alpha'\) corrections, there is no actual theorem forbidding it and it must be explicitly checked.

\textsuperscript{10}In these expressions we have used the functions \(Z_i, Z_{hi}, W_j\) to get simpler expressions. Since the corrections \(\delta Z_i, \delta Z_{hi}, \delta W_j\) are multiplied by \(\alpha'\), only the zeroth-order term of these functions actually contributes to these expressions and at \(0^{th}\) order \(Z_{hi} = Z, W_j = W\).
3.3.1 Horizon regularity for $\omega > 0$

At 0th order the horizon is at $\rho = 0$ when $\omega > 0$. At first order it may be shifted to $\rho_H = \alpha' \delta \rho$. Therefore, we perform an expansion in $z = (\rho - \rho_H)$, obtaining

\[ e^{-2\phi} = y_{\phi}^{(0)} + \alpha' y_{\phi}^{(0,\log)} \log z + \mathcal{O}(z), \]  
\[ k = y_k^{(0)} + \alpha' y_k^{(0,\log)} \log z + \mathcal{O}(z), \]  
\[ g_{\rho t} = \alpha' y_{\rho t}^{(0)} + \alpha' y_{\rho t}^{(1)} z + y_{\rho t}^{(2)} z^2 + \alpha' y_{\rho t}^{(2,\log)} z^2 \log z + \mathcal{O}(z^3), \]  
\[ g_{E\rho \rho} = \alpha' \frac{y_{\rho \rho}^{(-2)}(z)}{z^2} + y_{\rho \rho}^{(0)} + \alpha' y_{\rho \rho}^{(0,\log)} \log z + \mathcal{O}(z), \]  
\[ g_{E\theta \theta} = y_{\theta \theta}^{(0)} + \mathcal{O}(z), \]

and

\[ F_{\rho t} = \alpha' y_F^{(0)} + \mathcal{O}(z), \]  
\[ G^{(1)}_{\rho t} = \alpha' y_G^{(0)} + \mathcal{O}(z), \]  
\[ H^{(1)}_{\psi \theta \varphi} = y_H^{(0)} + \mathcal{O}(z), \]

where the $y_i$ are combinations of $d_i^{(2)}$, $d_{hi}^{(2)}$, $d_{wj}^{(2)}$ and $\delta \rho$. The $\alpha'$ factors indicate the terms that are purely first order corrections. In order to have a regular horizon we ask that the scalars have a finite near-horizon limit, obtaining the conditions

\[ y_{\phi}^{(0,\log)} = 0, \quad y_k^{(0,\log)} = 0. \]  
\[ y_{\rho t}^{(0,\log)} = 0, \quad y_{\rho t}^{(1,\log)} = 0. \]  
\[ y_{\rho \rho}^{(-2)} = 0, \quad y_{\rho \rho}^{(0,\log)} = 0. \]  
\[ y_{\rho t}^{(1)} = 0, \quad y_{\rho t}^{(2,\log)} = 0. \]

The requirement that there is an event horizon at $z = 0$ implies the vanishing of the constant part of the $g_{E\rho t}$ component of the metric, that is

\[ y_{\rho t}^{(0)} = 0. \]

Demanding the regularity of $g_{E\rho \rho}$ in the near-horizon limit (as in the 0th-order solution) we obtain

\[ y_{\rho \rho}^{(-2)} = 0, \quad y_{\rho \rho}^{(0,\log)} = 0. \]

Finally, in order to have a finite Hawking temperature, we also have to impose

\[ y_{\rho t}^{(1)} = 0, \quad y_{\rho t}^{(2,\log)} = 0. \]
Combining all these conditions we obtain the following conditions for \( \delta \rho, d_0^{(2)}, d_t^{(2)}, d_{\rho\rho}^{(2)}, d_{\omega j}^{(2)} \)

\[
\delta \rho = 0, \\
\begin{align*}
    d_0^{(2)} &= -\omega \left(8q_0^2 + 2q_0\omega + 3\omega^2 \right) + 2(q_0 - \omega) d_{h0}^{(2)} \\
                  &+ \frac{q_0(2q_+ - \omega)(\omega - 2q_+)}{(2q_0 - \omega)(q_+ - q_-)} d_t^{(2)} - \frac{2q_0(q_- - \omega)(\omega - 2q_+)}{(2q_0 - \omega)(q_- - q_+)} d_{h_+}^{(2)} \\
                  &+ \frac{q_0(2q_- - \omega)}{(2q_0 - \omega)(q_- - q_+)} d_t^{(2)} , \\
    d_t^{(2)} &= \frac{\omega (d_-^{(2)} - 2d_{h-}^{(2)} + d_+^{(2)}) - 2q_- (d_-^{(2)} - d_{h-}^{(2)})}{q_+ - q_-}, \\
    d_{\rho\rho}^{(2)} &= -\frac{\omega - (2q_+ - \omega)(-2q_+ + \omega)d_-^{(2)}}{(q_- - q_+)} , \\
    d_{\omega j}^{(2)} &= \frac{-3\omega^2(q_- - q_+)(2q_0 + \omega)}{4q_0^2 (4q_+ - 4q_0(q_+ + q_-) - 4(q_0 + q_+ + q_-)\omega + 3\omega^2) , \\
\end{align*}
\]

The solution still has 4 undetermined parameters: \( d_{h0}^{(2)}, d_{h-}^{(2)}, d_+^{(2)} \) and \( d_-^{(2)} \) that can be determined by demanding that the 4 independent physical charges which describe the black hole are not modified by \( \alpha' \) corrections. Imposing these additional conditions we get as our final result

\[
\begin{align*}
    d_0^{(2)} &= -\omega \left(8q_0^2 + 2q_0\omega + 3\omega^2 \right) + 2(q_0 - \omega) d_{h0}^{(2)} \\
                  &+ \frac{q_0(2q_+ - \omega)(\omega - 2q_+)}{(2q_0 - \omega)(q_+ - q_-)} d_t^{(2)} - \frac{2q_0(q_- - \omega)(\omega - 2q_+)}{(2q_0 - \omega)(q_- - q_+)} d_{h_+}^{(2)} \\
                  &+ \frac{q_0(2q_- - \omega)}{(2q_0 - \omega)(q_- - q_+)} d_t^{(2)} , \\
    d_t^{(2)} &= \frac{\omega (d_-^{(2)} - 2d_{h-}^{(2)} + d_+^{(2)}) - 2q_- (d_-^{(2)} - d_{h-}^{(2)})}{q_+ - q_-}, \\
    d_{\rho\rho}^{(2)} &= -\frac{\omega - (2q_+ - \omega)(-2q_+ + \omega)d_-^{(2)}}{(q_- - q_+)} , \\
    d_{\omega j}^{(2)} &= \frac{-3\omega^2(q_- - q_+)(2q_0 + \omega)}{4q_0^2 (4q_+ - 4q_0(q_+ + q_-) - 4(q_0 + q_+ + q_-)\omega + 3\omega^2) , \\
\end{align*}
\]

3.3.2 Horizon regularity for \( \omega < 0 \)

In this case, the horizon is at \( \rho = \sqrt{-\omega} \) at 0th order. At first order it could be shifted by \( \alpha' \delta \rho \). We consider therefore an expansion in \( z = (\rho - \rho_H) \) with \( \rho_H = \sqrt{-\omega} + \alpha' \delta \rho \). We obtain

\[
\begin{align*}
    e^{-2\phi} &= y_{\phi}^{(0)} + \alpha' y_{\phi}^{(0, \log)} \log z + O(z) , \\
    k &= y_k^{(0)} + \alpha' y_k^{(0, \log)} \log z + O(z) , \\
    g_{Ett} &= \alpha' y_{tt}^{(1)} + y_{tt}^{(1, \log)} z \log z + O(z^2) , \\
    g_{E\rho\rho} &= e^{-\frac{4}{z}} \left[ \frac{\alpha' y_{\rho\rho}^{(-2)}}{z^2} + y_{\rho\rho}^{(-1)} + y_{\rho\rho}^{(-1, \log)} \log z + \alpha' y_{\rho\rho}^{(0, \log)} \log z + O(z) \right] , \\
    g_{E\theta\theta} &= y_{\theta\theta}^{(0)} + O(z) , \\
\end{align*}
\]
where the constants $y_i$ are combinations of $d_i$, $d_{hi}$, $d_{wj}$ and $\delta \rho$. The $\alpha'$ factors indicate the terms that are purely first order corrections. In order to have a regular horizon we ask that the scalars have a finite near-horizon limit. This leads to the conditions

$$y^{(0, \log)} = 0, \quad y^{(1, \log)} = 0.$$  \hfill (3.24)

The requirement that there is an event horizon at $z = 0$ implies the vanishing of the constant part of the $g_{Ett}$ component of the metric
d$$y^{(0)}_{tt} = 0.$$  \hfill (3.25)

Demanding $g_{E\rho\rho}$ approaches the horizon at most as $1/z$ we obtain the conditions
d$$y^{(-2)}_{\rho\rho} = 0, \quad y^{(-1, \log)}_{\rho\rho} = 0.$$  \hfill (3.26)

Finally, in order to have a finite Hawking temperature we have to impose
d$$y^{(1, \log)}_{tt} = 0.$$  \hfill (3.27)

Combining all these conditions we obtain the expressions of $\delta \rho$, $d^{(2)}_0$, and $d^{(2)}_{-}$ in terms of the 5 undetermined parameters $d^{(2)}_{tt}$, $d^{(2)}_{\rho\rho}$, $d^{(2)}_{h0}$, $d^{(2)}_{h-}$ and $d^{(2)}_+$. As in the previous case, 4 integration constants can be determined by demanding that the mass and the 3 asymptotic charges do not get $\alpha'$ corrections. The remaining integration constant can be interpreted as the freedom of choosing the position of the horizon, i.e. the value of $\delta \rho$, and it can be eliminated through a change of coordinates. We finally obtain

$$d^{(2)}_0 = \frac{\omega}{8} \left[ \frac{(2q - \omega)(2q + \omega)}{(q - q_+)(2q_+ - \omega)} \right] - \frac{\omega(q_{-} - q_{+})(2q_{0} - \omega)}{(2q_{0} - \omega)},$$  \hfill (3.28a)

$$d^{(2)}_{tt} = \frac{2q_{-} - \omega}{q_{-} - q_{+}},$$  \hfill (3.28b)

$$d^{(2)}_{\rho\rho} = \frac{\omega}{q_{0} - \omega} + \frac{(2q_{-} - \omega)(2q_{+} - \omega)}{\omega(q_{-} - q_{+})},$$  \hfill (3.28c)

$$d^{(2)}_{-} = -\frac{3\omega^2(q_+ - q_+)(2q_0 - 3\omega)}{4(q_0 - \omega)^2 [4q_+q_+ + 4q_{0}(q_- + q_+ - \omega) - 4\omega(q_- + q_+) + 3\omega^2]},$$  \hfill (3.28d)

$$\delta \rho = d^{(2)}_{h0} = d^{(2)}_{h-} = d^{(2)}_+ = 0.$$  \hfill (3.28e)

### 3.4 Regular solutions

The results of the previous section can be summarized by giving the expressions of the corrections of the functions which lead to regular black holes for the $\omega > 0$ and $\omega < 0$ cases.
3.4.1 $\omega > 0$

In this case, we have

\[
\begin{align*}
\delta Z_{h0} &= \frac{2q_0^3 + \omega (q_0^2 + 9q_0\rho^2 + 6\rho^4)}{2q_0^2(q_0 + \rho^2)} - \frac{3\omega}{q_0^2} \log Z_0, \\
\delta Z_0 &= \frac{8q_0^6 - 24q_0^3\omega + 2q_0^2(7\omega - 4\rho^2) + q_0^2\omega (-4\rho^4 - 2\rho^2\omega + \omega^2)}{4q_0^2\rho^2(q_0 + \rho^2)^2} \times \\
&\quad + \frac{q_0^4\rho^2\omega^2(2\rho^2 + 9\omega) + q_0\rho^4\omega^2(2\rho^2 + 3\omega) - \rho^6\omega^3}{4q_0^2\rho^2(q_0 + \rho^2)^2} (2q_0^2 - 3q_0\omega + \omega^2) \\
&\quad - \frac{q_0(2q_0 - \omega)(2q_0 - \omega) q_0^2}{(2q_0 - \omega)\rho^2 d_\perp} \times \\
&\quad + \frac{\omega^2 q_0^2 (2\rho^2 + \omega - \rho^2\omega^2 (\rho^2 + 2\omega)}{4q_0^2\rho^2(q_0 - \omega)^2} \log (Z_0/W), \\
\delta Z_{h-} &= \frac{\omega q_0 - q_0^2 (2q_0 - \omega)}{2q_0\rho^2} + \frac{\omega q_0^2 (2q_0 - \omega)}{(q_0 - q_+)^2} d_\perp, \\
\delta Z_- &= \delta Z_{h-} + Z_- \left[ \Delta_k - \frac{\Delta C}{\beta_+} + \frac{\rho^3}{4} \left( \frac{\delta Z_{h-}'}{q_-} - \frac{T [\delta Z_{h-}']}{q_+} \right) \right], \\
\delta Z_+ &= -Z_+ \frac{\Delta C}{\beta_+} + T [\delta Z_{h-}], \\
\delta W_{tt} &= \frac{\omega^2}{2q_0\rho^4} - \frac{\beta_-(W + \beta_+\beta_-) + W(\beta_+ + \beta_-)}{8\beta_+ Z_0 Z_-} Z'_- W' \\
&\quad + \frac{(2q_0^2 + \rho^2)(2q_0 - \omega)}{(q_0 - q_+)^2} d_\perp, \\
\delta W_{\rho\rho} &= \frac{\omega \left[ -4q_0^3 + q_0^2 (2\omega - 4\rho^2) + q_0\omega (3\rho^2 + \omega) - 2\rho^2\omega (\rho^2 + \omega) \right]}{4q_0^2\rho^2(q_0 + \rho^2)} \times \\
&\quad + \frac{q_0^4\rho^2\omega^2(2\rho^2 + 9\omega) + q_0\rho^4\omega^2(2\rho^2 + 3\omega) - \rho^6\omega^3}{4q_0^2\rho^2(q_0 + \rho^2)^2} (2q_0^2 - 3q_0\omega + \omega^2) \\
&\quad - \frac{q_0(2q_0 - \omega)(2q_0 - \omega) q_0^2}{(2q_0 - \omega)\rho^2 d_\perp} \times \\
&\quad + \frac{\omega^2 q_0^2 (2\rho^2 + \omega - \rho^2\omega^2 (\rho^2 + 2\omega)}{4q_0^2\rho^2(q_0 - \omega)^2} \log (Z_0/W),
\end{align*}
\]

where $T$ is the operator which implements T-duality through the following transformation of the parameters

\[
q_\pm \leftrightarrow q_\mp, \quad \beta_\pm \leftrightarrow \beta_\mp, \quad k_\infty \leftrightarrow 1/k_\infty.
\]

It is easy to verify that under such transformation the solution is self-dual, as expected.
3.4.2 $\omega < 0$

In this case, the corrections to the functions are given by

$$\delta Z_{h0} = \frac{2q_0^3 + \omega (q_0^3 + 9q_0\rho^2 + 6\rho^4)}{2q_0\rho^2 (q_0 + \rho^2)^2} - \frac{3\omega}{q_0^2} \log Z_0,$$

$$\delta Z_0 = \frac{8q_0^6 - 24q_0^5\omega - \rho^4\omega^3 (\rho^2 + 2\omega) + q_0^3\omega (4\rho^4 - 26\rho^2\omega - 7\omega^2)}{4q_0\rho^2 (q_0 + \rho^2)^2 (q_0 - \omega)^2 (2q_0 - \omega)}$$

$$+ \frac{\omega q_0^3 (8\rho^2 + 22\omega) + \rho^2\omega^2 (2\rho^4 + 11\rho^2\omega - 4\omega^2) + q_0\omega^2 (-10\rho^4 + 25\rho^2\omega + 2\omega^2)}{4\rho^2 (q_0 + \rho^2)^2 (q_0 - \omega)^2 (2q_0 - \omega)}$$

$$+ \frac{(q_0 - \omega)(2q_+ - \omega)(2q_+ - \omega) d_{(2)}^2}{\omega(q_+ - q_+)(2q_0 - \omega)\rho^2} + \frac{\omega^2 q_0 (2\rho^2 + \omega) - \omega^2\rho^2 (\rho^2 + 2\omega)}{4q_0^2\rho^2 (q_0 - \omega)^2} \log Z_0,$$

$$\delta Z_{h-} = -\frac{\omega q_-}{2(q_0 - \omega)\rho^4} - \frac{q_- (q_+ - \omega)(2q_+ - \omega)}{\omega(q_+ - q_-)\rho^4} d_{(2)}^2,$$

$$\delta Z_- = \delta Z_{h-} + Z_- \left[ \Delta k - \frac{\Delta C}{\beta_+} + \frac{\rho^2}{4} \left( \frac{\delta Z_{h-}'}{q_-} - \frac{T}{q_+} \delta Z_{h-} \right) \right],$$

$$\delta Z_+ = -Z_+ \frac{\Delta C}{\beta_+} + T [\delta Z_{h-}],$$

$$\delta W_{tt} = -\frac{\omega^2}{2(q_0 - \omega)\rho^4} - \frac{\beta_- (W + \beta_+ \beta_-) + W (\beta_+ + \beta_-)}{8\beta_+ Z_0 Z_-} Z' W'$$

$$+ \frac{(2q_+ - \omega) (\rho^2 + \omega)}{(q_+ - q_-)\rho^4} d_{(2)}^2,$$

$$\delta W_{\rho\rho} = -\frac{\omega (\rho^2 + \omega) [-4q_0^3 + q_0\omega (5\rho^2 - 2\omega) + \rho^2\omega (2\rho^2 + \omega) + q_0^3 (-4\rho^2 + 6\omega)]}{4q_0^2\rho^4 (q_0 + \rho^2) (q_0 - \omega)^2}$$

$$+ \frac{(2q_+ - \omega) (2q_+ - \omega) (\rho^2 + \omega)}{\omega(q_+ - q_-)\rho^4} d_{(2)}^2 + \frac{\omega^2 (2\rho^4 + 3\rho^2\omega + \omega^2)}{4q_0^2\rho^2 (q_0 - \omega)^2} \log Z_0,$$

where, again, $T$ is the operator that implements T duality as in eqs. (3.30).

As we have discussed in appendix B, at 0th order the $\omega > 0$ and $\omega < 0$ solutions are related by the coordinate transformation eq. (B.9) supplemented by the transformations of the parameters eqs. (B.10) and (B.11). This is still true at first order in $\alpha'$, as can be seen by checking the transformations of all the physical fields. However, in the 0th-order case, the two solutions have the same form and the metric is formally invariant under those transformations plus $\omega \to -\omega$, instead in the first-order case the two solutions have different forms and there is no invariance under those transformations. Thus, the mass and asymptotic charges do not need to be invariant under them, as in the 0th-order case. Nevertheless they do happen to be invariant. On the other hand, the realization of these transformations at the level of the $Z$s and $W$s functions is far from evident.
### 3.4.3 \( \omega = 0 \)

As a further check of the solutions we have obtained, we can take the \( |\omega| \to 0 \) limit

\[
\delta Z_{h0} = \delta Z_0 = \frac{q_0}{\rho^2(q_0 + \rho^2)^2}, \tag{3.32a}
\]

\[
\delta Z_+ = -(1 + s_+ s_-) \frac{q_+ q_-}{\rho^2(q_0 + \rho^2)(q_+ + \rho^2)}, \tag{3.32b}
\]

\[
\delta Z_{h-} = \delta Z_- = \delta W_{tt} = \delta W_{\rho\rho} = 0. \tag{3.32c}
\]

and compare with the extremal \( \alpha' \)-corrected solutions of ref. [25].\(^{11}\)

### 3.4.4 The Reissner-Nordström-Tangherlini limit

This is the \( q_0 = q_+ = q_- = q \) case reviewed at 0th order in appendix B.2. Using the \( \omega > 0 \) solution, we get

\[
\delta Z_{h0} = \frac{2q^3 + \omega (q^2 + 9q\rho^2 + 6\rho^4)}{2q\rho^2 (q + \rho^2)^2} - \frac{3\omega}{q^2} \log Z, \tag{3.33a}
\]

\[
\delta Z_0 = \frac{8q^6 - 22q^4\omega + q^4\omega (13\omega - 4\rho^2) - 2q^3\rho^2 \omega (\rho^2 + 2\omega)}{4q^2\rho^2 (q + \rho^2)^2 (2q^2 - 3q\omega + \omega^2)} - \frac{q^2\rho^2 \omega^2 (\rho^2 + 7\omega) + 2q\rho^4\omega^2 (\rho^2 + \omega) - \rho^6\omega^3}{4q^2\rho^2 (q + \rho^2)^2 (2q^2 - 3q\omega + \omega^2)} + \frac{\omega^2 [q (2\rho^2 + \omega) - \rho^2 (\rho^2 + 2\omega)]}{4q^2\rho^2 (q - \omega)^2} \log (Z/W), \tag{3.33b}
\]

\[
\delta Z_{h-} = \frac{\omega(2q - 3\omega)}{4\rho^4(2q - \omega)}, \tag{3.33c}
\]

\[
\delta Z_- = \frac{\omega(2q - 3\omega)}{4\rho^4(2q - \omega)} - (1 + s_+ s_-) \frac{q\omega}{2\rho^4 (q + \rho^2)^2}, \tag{3.33d}
\]

\[
\delta Z_+ = \frac{\omega(2q - 3\omega)}{4\rho^4(2q - \omega)} + (1 + s_+ s_-) \frac{q^2\rho^2 \omega - q^2 (2\rho^2 + \omega)}{2\rho^4 (q + \rho^2)^2}, \tag{3.33e}
\]

\[
\delta W_{tt} = \frac{\omega^2 (2q + \rho^2) (2q^2 - 2q^2\rho^2 - 3q\omega - \rho^2\omega)}{4q^2\rho^4 (q + \rho^2)^2 (2q - \omega)} - (1 + s_+ s_-) \frac{q\omega (\rho^2 + \omega)}{\rho^4 (\rho^2 + q)^2}, \tag{3.33f}
\]

\[
\delta W_{\rho\rho} = -\frac{\omega [q^3 + q^2 (2\rho^2 - \omega) - 2q\rho^2 \omega + \rho^2 \omega (2\rho^2 + 3\omega)]}{4q^2\rho^2 (q + \rho^2)^2 (q - \omega)} + \frac{\omega^2 (2\rho^4 + 3\rho^2 \omega + \omega^2)}{4q^2\rho^2 (q - \omega)^2} \log (Z/W). \tag{3.33g}
\]

To end this section, we present figures 1, 2 and 3 in which we plot several curvature invariants for a typical choice of integration constants \( q_+, q_-, q_0, \omega \) with \( \omega < 0 \) and for several values of \( \alpha' \) as a way to visualize the effect of those corrections which have to be small anyway. The plots do not extend beyond \( \rho^2 = 0 \) for \( \alpha' \neq 0 \) because there is a logarithmic singularity at that point. There seem to be no other curvature singularities for larger values of \( \rho^2 \), which include the position of the inner horizon, which is slightly displaced to the right of \( \rho^2 = 0 \) by the \( \alpha' \) corrections.

\(^{11}\)It has to be taken into account that the \( \tilde{q}_i \) parameters we are using here are equivalent to the \( \tilde{q}_i \)s in ref. [25].
Figure 1. the Ricci scalar as a function of the radial coordinate for $q_+ = 40$, $q_- = 20$, $q_0 = 10$, $\omega = -5$, $s_+ s_- = -1$, for different values of $\alpha'$. Observe that, for these charges, the black hole does not become supersymmetric in the extremal limit.

Figure 2. The $R_{\mu\nu}R^{\mu\nu}$ invariant as a function of the radial coordinate for $q_+ = 40$, $q_- = 20$, $q_0 = 10$, $\omega = -5$, $s_+ s_- = -1$ for different values of $\alpha'$. Observe that, for these charges, the black hole does not become supersymmetric in the extremal limit.

4 Thermodynamics

We are going to study separately the $\omega < 0$ and $\omega > 0$ cases as a further test of our solution.

4.1 $\omega > 0$

The physical charges of these solutions can be computed using the following definitions:

$$ Q_+^{(1)} = \frac{-1}{16\pi G_N^{(5)}} \int_{S_\infty^3} \left\{ e^{-4\phi/3} k^2 \ast F \right\} = \frac{\pi}{4G_N^{(5)}} k_\infty \beta_+ q_+ , $$

$$ Q_-^{(1)} = \frac{-1}{16\pi G_N^{(5)}} \int_{S_\infty^3} \left\{ e^{-4\phi/3} k^{(1)} \ast G^{(1)} \right\} = \frac{\pi}{4G_N^{(5)}} k_\infty^{-1} \beta_- q_- , $$
Figure 3. The Kretschmann invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ as a function of the radial coordinate for $q_+ = 40$, $q_- = 20$, $q_0 = 10$, $\omega = -5$, $s_+ s_- = -1$, for different values of $\alpha'$. Observe that, for these charges, the black hole does not become supersymmetric in the extremal limit.

$$Q_0^{(1)} = \frac{-1}{16\pi G_N^{(5)}} \int_{S^3_{\infty}} \left\{ e^{8\phi/\beta} \star K^{(1)} \right\} = -\frac{\pi}{4G_N^{(5)}} \beta_0 q_0. \quad (4.1c)$$

These definitions are similar to those used at 0th order eqs. (B.17) with the replacements of 0th-order field strengths and fields by the first-order ones, and the result is formally identical. We have ignored the second terms in the integrands because, as we have explained, they vanish for these solutions, but we have ignored many higher-order terms as well because they fall off fast for large values of $\rho$ and they do not contribute at infinity. They would give a finite contribution elsewhere, though. Therefore, these charges are not localized in the language of ref. [46] and they are defined strictly at infinity.

In agreement with the choices made, the mass is given by the 0th-order expression eq. (B.16). The horizon is at $\rho = \rho_H = 0$ in these coordinates. The radius of the horizon, defined by eq. (B.18) is given at first order ($R_H^{(1)}$) by

$$R_H^{(1)} = R_H^{(0)} \left[ 1 - \alpha' \frac{3\omega + 4q_0 \beta_+ \beta_-}{24q_0} \right], \quad (4.2)$$

where the radius of the horizon of the 0th-order solution, $R_H^{(0)}$ is given in eq. (B.19), and we obtain the first-order Bekenstein-Hawking entropy $S_{BH}^{(1)}$ in terms of the 0th-order one in eq. (B.20b)

$$S_{BH}^{(1)} = S_{BH}^{(0)} \left[ 1 - \alpha' \frac{3\omega + 4q_0 \beta_+ \beta_-}{8q_0} \right]. \quad (4.3)$$

The Hawking temperature is now

$$T_H^{(1)} = T_H^{(0)} \left\{ 1 + \alpha' \left[ \frac{\beta_- \beta_+}{2q_0} + \frac{\omega (8q_0^2 + q_0 q_- + q_0 q_+ + 5q_- q_+)}{2q_0 D} \right. \right.$$

$$\left. + \frac{16q_0(q_- q_+ - 2q_0 q_- - 2q_0 q_+)}{8q_0^2 D} - 8\omega^2 (q_0 + q_- + q_+) + 3\omega^3 \right\}, \quad (4.4)$$
where \( T_{H}^{(0)} \) is the Hawking temperature of the 0th-order solution eq. (B.20a) and where we have defined
\[
D \equiv 4q_+ - 4\omega(q_0 + q_+ + q_+ + 4q_0(q_+ + q_+) + 3\omega^2. \tag{4.5}
\]
Since \( T_{H}^{(1)} \) is proportional to \( T_{H}^{(0)} \), it vanishes when \( T_{H}^{(0)} \) vanishes, namely in the extremal limit \( \omega = 0 \).

We can compute the Wald entropy using the gauge-invariant formula of ref. [22]. We obtain
\[
S_{W}^{(1)} = S_{BH}^{(0)} \left[ 1 + \alpha' \frac{8q_0 + \omega + 4q_0\beta_+\beta_-}{8q_0^2} \right], \tag{4.6}
\]
where \( S_{BH}^{(0)} \) is the Bekenstein-Hawking entropy of the 0th-order solution eq. (B.20b).

Notice the relations
\[
S_{BH}^{(1)}T_{H}^{(1)} = \frac{\pi \omega}{4G_N^{(5)}} \left[ 1 + \alpha' \left\{ \frac{\beta_-}{q_0} + \frac{3(2q_+ - \omega)(2q_+ - \omega)(2q_+ + \omega)}{4q_0^2 D} \right\} \right], \tag{4.7a}
\]
\[
S_{W}^{(1)}T_{H}^{(1)} = \frac{\pi \omega}{4G_N^{(5)}} \left[ 1 + \alpha' \left\{ \frac{\beta_-}{q_0} + \frac{3(2q_+ - \omega)(2q_+ - \omega)(2q_+ + \omega)}{4q_0^2 D} \right\} \right]. \tag{4.7b}
\]

Using the \( \omega > 0 \) form of the solution we cannot compute these products on the inner horizon and check the property eq. (B.31) because the inner horizon is not covered by the coordinates we are using. We can, however, use the latter to check the Smarr formula. Using the general definitions eqs. (B.25), we find that, at first order in \( \alpha' \), the electrostatic potentials evaluated on the horizon are given by
\[
\Phi_+ = k_{\infty}^{-1} \beta_+ \left\{ 1 + \alpha' \left[ -\frac{\omega\beta_-}{2q_0\beta_+ q_+} - \frac{3\omega(2q_0 + \omega)(2q_- - \omega)}{4q_0^2 D} \right] \right\}, \tag{4.8a}
\]
\[
\Phi_- = k_{\infty} \beta_+ \left\{ 1 + \alpha' \left[ -\frac{\omega\beta_-}{2q_0\beta_+ q_-} - \frac{3\omega(2q_0 + \omega)(2q_+ - \omega)}{4q_0^2 D} \right] \right\}, \tag{4.8b}
\]
\[
\Phi_0 = -\beta_0 \left\{ 1 + \alpha' \left[ \frac{4q_0\omega(q_+ + q_+ - \omega)}{q_0^2 D} + \frac{2\omega^2(q_+ + q_- - 3\omega^3)}{4q_0^2 D} \right] \right\}. \tag{4.8c}
\]

If we now plug all this information into the 0th-order Smarr formula eq. (B.23), we find that it is only satisfied up to a term of first order in \( \alpha' \). As a matter of fact, this could have been expected: all the dimensionful parameters in the action can be interpreted as thermodynamical variables and come with their own conjugate thermodynamical potentials [47–49]. Thus, the coefficient of \( \alpha' \) in the additional term should be interpreted as the thermodynamical potential conjugate to \( \alpha' \) that we are going to denote by \( \Phi^{\alpha'} \) and which takes the value
\[
\Phi^{\alpha'} = -\frac{\pi \omega}{4G_N^{(5)}} \frac{8q_0 + \omega + 4q_0\beta_+\beta_-}{8q_0^2}. \tag{4.9}
\]

Then, the Smarr formula takes the form
\[
M^{(1)} = \frac{3}{2} S_{W}^{(1)} T_{H}^{(1)} + \Phi^+ Q_+ + \Phi^- Q_- + \Phi^0 Q_0 + \Phi^{\alpha'} \alpha'. \tag{4.10}
\]
If this interpretation is correct, then we must find a term $\Phi'^\alpha \delta \alpha'$ in the first law, which should take the form
\[
\delta M^{(1)} = T_H^{(1)} \delta S_W^{(1)} + \Phi^+ \delta Q_+ + \Phi^- \delta Q_- + \Phi^0 \delta Q_0 + \Phi'^\alpha \delta \alpha'.
\] (4.11)

This is what we are going to check next.

4.1.1 Checking the first law at first order in $\alpha'$

The procedure we are going to follow to check the first law at first order is essentially the same we followed to check it at a 0th order, in appendix C. Now we will also consider the variation of $\alpha'$ to test the proposal made in the previous section, taking into account that the parameters $\beta_i, q_i, \omega$ are independent of $\alpha'$.

The variations of the physical charges can be easily obtained from eqs. (4.1)
\[
\delta Q_i^{(1)} = -\frac{\pi}{4G_N^{(5)}} (-k_\infty) \gamma_i q_i \beta_i \left[ \frac{2q_i - \omega}{2q_i(q_i - \omega)} q_i \delta q_i - \frac{1}{2(q_i - \omega)} \delta \omega \right],
\] (4.12)
with
\[
\gamma_i = \begin{cases} 
  \pm 1 & \text{if } i = \pm 0 \\
  0 & \text{if } i = 0 
\end{cases}
\] (4.13)

The variation of the mass is
\[
\delta M^{(1)} = \frac{\pi}{4G_N^{(5)}} \left[ \delta q_0 + \delta q_+ + \delta q_- \right] - \frac{3\pi}{8G_N^{(5)}} \delta \omega.
\] (4.14)

The variation of the entropy is
\[
\frac{\delta S_W^{(1)}}{S_H^{(0)}} = \frac{1}{2q_0 + \alpha' \left( -\frac{\beta_- \beta_+}{4q_0^2} - \frac{8q_0 + 3\omega}{16q_0^3} \right)} \delta q_0
\]
\[
+ \left[ \frac{1}{2q_- + \alpha' \left( \frac{\beta_+}{4q_0 q_- \beta_-} + \frac{8q_0 + \omega}{16q_0^2 q_-} \right)} \delta q_- \right]
\]
\[
+ \left[ \frac{1}{2q_+ + \alpha' \left( \frac{\beta_-}{4q_0 q_+ \beta_+} + \frac{8q_0 + \omega}{16q_0^2 q_+} \right)} \delta q_+ \right]
\]
\[
+ \left[ \frac{(2\omega - q_+ - q_-)}{4q_0 q_- q_+ \beta_- \beta_-} + \frac{1}{8q_0^2} \right] \alpha' \delta \omega
\]
\[
+ \left[ \frac{8q_0 + \omega + 4q_0 \beta_- \beta_-}{8q_0^2} \right] \delta \alpha'.
\] (4.15)

Expressing $\delta q_i$ and $\delta \omega$ in terms of $\delta Q_i^{(1)}$ and $\delta M^{(1)}$ through eqs. (4.12) and (4.14) and replacing them into (4.15) we obtain after some algebra
\[
\frac{\delta S_W^{(1)}}{S_H^{(0)}} = \frac{1}{T_H^{(1)}} \left[ \delta M^{(1)} - \Phi_+ \delta Q_+^{(1)} - \Phi_- \delta Q_-^{(1)} - \Phi_0 \delta Q_0^{(1)} - \Phi'^\alpha \delta \alpha' \right],
\] (4.16)
where the potentials $\Phi^{\pm,0}$ are given by eqs. (4.8) and $\Phi'^\alpha$ has the expression eq. (4.9) that we proposed to make sense of the Smarr formula. This confirms our identification and interpretation.
4.1.2 The Reissner-Nordström-Tangherlini case with $\omega > 0$

Due to the choices made, the relations between the integration parameters $\omega, q, \beta$ and the physical parameters $M, Q = Q_0 \ (Q_\pm = k_\omega^2 Q_0)$ are the 0th-order ones eqs. (B.32), which, in its turn, makes it easy to express the corrected thermodynamical quantities in terms of the latter.

\begin{align}
R_{H}^{(1)} &= R_{H}^{(0)} \left\{ 1 - \frac{\alpha'}{24q^2} [3\omega + 4s_+s_- (q - \omega)] \right\}, \\
T_{H}^{(1)} &= T_{H}^{(0)} \left\{ 1 + \frac{\alpha'}{8q^2} [\omega - 4q + 4s_+s_- (q - \omega)] \right\}, \\
S_{BH}^{(1)} &= S_{BH}^{(0)} \left\{ 1 - \frac{\alpha'}{8q^2} [3\omega + 4s_+s_- (q - \omega)] \right\}, \\
S_{W}^{(1)} &= S_{BH}^{(0)} \left\{ 1 + \frac{\alpha'}{8q^2} [(\omega + 8q) + 4s_+s_- (q - \omega)] \right\},
\end{align}

4.2 $\omega < 0$

The thermodynamic quantities can be easily obtained from those computed in the previous section for the $\omega > 0$ case through the transformations eqs. (B.10) and (B.11), after which we have to consider $\omega < 0$.

The asymptotic charges are still given by eqs. (4.1) and the mass still takes the same form as in the 0th-order solution eq. (B.16). The horizon radius, the Hawking temperature, and the Bekenstein-Hawking and Wald entropies are given by

\begin{align}
R_{H}^{(1)} &= R_{H}^{(0)} \left\{ 1 + \alpha' \left[ \frac{\omega}{8(q_0 - \omega)^2} - \frac{1}{6(q_0 - \omega)\beta_+\beta_-} \right] \right\}, \\
T_{H}^{(1)} &= T_{H}^{(0)} \left\{ 1 + \alpha' \left[ \frac{1}{2(q_0 - \omega)\beta_+\beta_-} - \frac{8q_0^2(q_- + q_+ - \omega) + 9q_-q_+\omega}{2(q_0 - \omega)^2 D} ight. \\
& \quad \left. + \frac{4q_0 (4q_-q_+ + 11q_-\omega + 11q_+\omega - 12\omega^2) + 9\omega^3}{8(q_0 - \omega)^2 D} \right] \right\}, \\
S_{BH}^{(1)} &= S_{BH}^{(0)} \left\{ 1 + \alpha' \left[ \frac{3\omega}{8(q_0 - \omega)^2} - \frac{1}{2(q_0 - \omega)\beta_+\beta_-} \right] \right\}, \\
S_{W}^{(1)} &= S_{BH}^{(0)} \left\{ 1 + \alpha' \left[ \frac{3q_0 - 9\omega}{8(q_0 - \omega)^2} + \frac{1}{2(q_0 - \omega)\beta_+\beta_-} \right] \right\},
\end{align}

where $D$ is still given by eq. (4.5).

Notice the relations

\begin{align}
S_{BH}^{(1)} T_{H}^{(1)} &= - \frac{\pi\omega}{4G_N^{(5)}} \left\{ 1 + \alpha' \left[ - \frac{(8q_0q_- + 8q_0q_+ - 4q_-q_+ + 3\omega^2)(2q_0 - 3\omega)}{4(q_0 - \omega)^2 D} \\
& + \frac{\omega(4q_0 + q_- + q_+)(2q_0 - 3\omega)}{2(q_0 - \omega)^2 D} \right] \right\}, \\
S_{W}^{(1)} T_{H}^{(1)} &= - \frac{\pi\omega}{4G_N^{(5)}} \left\{ 1 + \alpha' \left[ \frac{1}{(q_0 - \omega)\beta_+\beta_-} + \frac{3(2q_- - \omega)(2q_+ - \omega)(2q_0 - 3\omega)}{4(q_0 - \omega)^2 D} \right] \right\}.
\end{align}
Now the electrostatic potential conjugate to the charges, still defined by eqs. (B.25) are given by

\[
\Phi_+ = k_{+}^{-1} \left\{ 1 + \alpha' \left[ \frac{\omega \beta_+}{2(q_0 - \omega)(q_+ - \omega)\beta_-} + \frac{3\omega(2q_0 - 3\omega)(2q_- - \omega)}{4(q_0 - \omega)^2 D} \right] \right\}, \quad (4.20a)
\]

\[
\Phi_- = k_{-}^{-1} \left\{ 1 + \alpha' \left[ \frac{\omega \beta_-}{2(q_0 - \omega)(q_- - \omega)\beta_+} + \frac{3\omega(2q_0 - 3\omega)(2q_+ - \omega)}{4(q_0 - \omega)^2 D} \right] \right\}, \quad (4.20b)
\]

\[
\Phi_0 = -\frac{1}{\beta_0} \left\{ 1 - \alpha' \left[ \frac{4q_-q_+\omega + 16q_0\omega(q_- + q_+ - \omega) - 22\omega^2(q_- + q_+) + 21\omega^3}{4(q_0 - \omega)^2 D} \right] \right\}, \quad (4.20c)
\]

and the Smarr formula eq. (4.10) is satisfied with

\[
\Phi^{\alpha'} = -\frac{\pi \omega}{4G_N^{(5)}} \frac{(8q_0 - 9\omega)\beta_+\beta_- + 4(q_0 - \omega)}{8(q_0 - \omega)^2 \beta_+\beta_-}. \quad (4.21)
\]

We have checked explicitly that the first law in the form eq. (4.11) is satisfied with the above potentials.

### 4.2.1 The Reissner-Nordström-Tangherlini case with \( \omega < 0 \)

All the physical quantities can be computed by performing the transformations eqs. (B.10) and (B.11) on the expressions we found for the \( \omega > 0 \) case and we will not write them explicitly here.

To end this section, we present figures 4 and 5 in which the Hawking temperature of the 1st-order solutions is plotted versus the mass for two typical choices of charges, one of which leads to a supersymmetric solution in the extremal case and the other to a non-supersymmetric one, and figures 6 and 7 in which the Wald entropy is plotted versus the mass for several values of \( \alpha' \). For smaller values of the charges and larger values of \( \alpha' \) the slope of the curves changes in the near-extremal region, but in those conditions the effective action we are using is not valid and it would be necessary to add higher-order terms.

### 4.3 \( \omega = 0 \)

Setting \( \omega = 0 \) we recover values of the thermodynamical variables computed in ref. [25] for extremal black holes:\[12\]

\[
M^{(1)}_{\text{ext}} = \frac{\pi}{4G_N^{(5)}} (q_0 + q_- + q_+), \quad (4.22a)
\]

\[
T^{(1)}_{H \text{ ext}} = 0, \quad (4.22b)
\]

\[
S^{(1)}_{W \text{ ext}} = \frac{\pi^2}{2G_N^{(5)}} \sqrt{q_+q_-[q_0 + \alpha(2 + s_+s_-)]}. \quad (4.22c)
\]

---

\[\text{It has to be taken into account that the parameters } q_i \text{ in this paper correspond to the parameters } \hat{q}_i \text{ in ref. [25].}\]
Figure 4. The Hawking temperature as a function of the mass for $\eta_+ Q_+ = 40$, $\eta_- Q_- = 20$, $\eta_0 Q_0 = 10$, for different values of $\alpha'$ ($\eta_{(i)}$ are the same of eq. (C.1c)). Observe that, for these charges, the black hole becomes supersymmetric in the extremal limit.

Figure 5. The Hawking temperature as a function of the mass for $\eta_+ Q_+ = 40$, $\eta_- Q_- = -20$, $\eta_0 Q_0 = 10$ for different values of $\alpha'$. Observe that, for these charges, the black hole does not become supersymmetric in the extremal limit.
Figure 6. The Wald entropy as a function of the mass for $\eta_+ Q_+ = 40$, $\eta_- Q_- = 20$, $\eta_0 Q_0 = 10$ for different values of $\alpha'$. Observe that, for these charges, the black hole becomes supersymmetric in the extremal limit.

Figure 7. The Wald entropy as a function of the mass for $\eta_+ Q_+ = 40$, $\eta_- Q_- = -20$, $\eta_0 Q_0 = 10$ for different values of $\alpha'$. Observe that, for these charges, the black hole does not become supersymmetric in the extremal limit.
5 Conclusions

In this paper we have found the 1st-order corrections in α′ to the non-extremal Strominger-Vafa black hole of ref. [29] in the framework of the Bergshoeff-de Roo heterotic superstring effective theory [34]. In order to determine the integration constants we have demanded regularity and that the physical charges (mass and electric charges) are not modified by the corrections. This greatly simplifies many expressions. Furthermore, we have used in a crucial way the invariance of the solution under α′-corrected T duality transformations [21, 45] to simplify the calculations and as a consistency check of the final results.

We have also used the first law as a consistency check of the solution: the relation $\partial S/\partial M = 1/T$ involves quantities which are computed from the metric alone, $M$ and $T$, and a quantity, the Wald entropy $S$, which depends on the metric as well as on the matter fields through a complicated formula. Thus, it provides a non-trivial consistency check of the solution and of the entropy formula itself. The recovery of the entropy computed in ref. [25] adds confidence to it. This is the second instance in which the first law has been explicitly checked for α′-corrected non-extremal heterotic black holes [26].

Another interesting aspect of our results is the emergence of the Regge slope parameter, α′, as a thermodynamical variable. While, on general grounds, this was to be expected [47–49], it is not so easy to argue that there is a $(d-1)$-form potential associated to this parameter that allows us to understand it as just another charge [50]. This aspect of our results needs further research to be properly understood.

Almost by definition, the α′ corrections do not lead to dramatic changes in the solution: if they did, they would typically be associated to large curvatures and, therefore, the calculation and addition of higher order corrections in α′ would be needed to obtain a reliable string solution. This is why the removal or cloaking of singularities by the corrections, a widely expected and interesting quantum or stringy effect which in the case studied in ref. [51] produces a globally regular black hole, requires the computation of higher-order corrections to be fully confirmed.

The corrections to the thermodynamical quantities, though, are important even if small since a microscopic model dual to the black hole should be able to reproduce them. In spite of the efforts devoted to this problem over the years, such a model has not yet been found. There have been some proposals for the near-extremal black holes which we intend to study in forthcoming work, taking into account that, strictly speaking, in that regime the thermodynamical description of black holes might break down [52].\footnote{This point has been recently revisited in refs. [53, 54], which have clarified how the naive breakdown of the thermodynamical description at low temperatures pointed out in [52] is resolved.}

The extension of our results to the 4-dimensional non-extremal stringy black hole of ref. [55], taking into account the signs of charges that lead to non-supersymmetric extremal limits studied in ref. [25] is another direction of research that we intend to pursue in the near future. For equal charges, these solutions give an embedding of the non-extremal Reissner-Nordström black hole different from the one studied in ref. [26] and it will be interesting to see how different these corrections are for the same metric.
Finally, it would be interesting to consider the effect of rotation. Neutral rotating black holes were recently studied by refs. [63, 64], but obtaining charged rotating black hole solutions of the heterotic superstring effective action at first order in $\alpha'$ still remains an open problem.

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A The heterotic superstring effective action

For the sake of completeness, in this appendix we briefly review the effective action of the heterotic superstring theory at first order in $\alpha'$ that we have used in this paper and which is, essentially, that of ref. [34] adapted to the conventions of [56] and with the non-Abelian gauge fields consistently truncated.\(^{14}\)

In order to describe this action is convenient to start by defining the 0th-order 3-form field strength of the Kalb-Ramond 2-form $\hat{B}$

$$\hat{H}^{(0)} = d\hat{B}, \quad (A.1)$$

$\hat{H}^{(0)}$ is combined with the Levi-Civita spin connection $\hat{\omega}^\hat{a}_\hat{b}$ into the definition of the torsionful spin connection

$$\hat{\Omega}(-)_{\hat{a} \hat{b}} \equiv \hat{\omega}^\hat{a}_\hat{b} - \frac{1}{2} \hat{H}^{(0)}_{\hat{c}} \hat{a}_\hat{b} \hat{c}, \quad (A.2)$$

whose curvature 2-form and Chern-Simons 3-form are defined by

$$\hat{R}_{(-)\hat{a}\hat{b}} = d\hat{\Omega}(-)_{\hat{a} \hat{b}} - \hat{\Omega}(-)_{\hat{c} \hat{b}} \wedge \hat{\Omega}(-)_{\hat{a} \hat{c}}, \quad (A.3a)$$

$$\hat{\Omega}_L(-)_{\hat{a} \hat{b} \hat{c}} = d\hat{\Omega}(-)_{\hat{a} \hat{b}} \wedge \hat{\Omega}(-)_{\hat{c}} - \frac{2}{3} \hat{\Omega}(-)_{\hat{a} \hat{c}} \wedge \hat{\Omega}(-)_{\hat{b} \hat{c}} \wedge \hat{\Omega}(-)_{\hat{a} \hat{b}}. \quad (A.3b)$$

The latter is used into the definition of the first-order 3-form KR field strength\(^{15}\)

$$\hat{H} = d\hat{B} + \frac{\alpha'}{4} \hat{\Omega}_L(-). \quad (A.4)$$

\(^{14}\)The $\alpha'$ corrections to that action we first studied in refs. [34–36]. More recent, and relevant, studies of those corrections can be found in refs. [37–39].

\(^{15}\)Sometimes we will use $\hat{H}^{(1)}$ to emphasize the fact that we are dealing with the first-order field strength.
With these definitions, the effective action of the heterotic string to first order in $\alpha'$ is given by\(^{16}\)

\[
\hat{S} = \frac{\hat{g}_s^2}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{\hat{g}} e^{-2\hat{\phi}} \left[ \hat{R} - 4(\partial \hat{\phi})^2 + \frac{1}{2} \cdot 3! \hat{H}^2 - \frac{\alpha'}{8} \hat{R}(-)_{\hat{\mu}} \hat{a} \hat{R}(-)_{\hat{\nu}} \hat{b} \hat{a} \right],
\]

(A.5)

Setting $\alpha' = 0$ one gets the 0th-order action.

### A.1 Equations of motion

In order to derive the first-order equations of motion it is convenient to use the lemma of ref. [34] which implies that the equations of motion can be obtained by varying the action only with respect to explicit occurrences of the fields (i.e., ignoring them when they occur inside the torsionful spin connection). The resulting equations of motion are

\[
\hat{R}_{\hat{\mu}\hat{\nu}} - 2\nabla_{\hat{\mu}} \partial_{\hat{\nu}} \hat{\phi} + \frac{1}{4} \hat{H}_{\hat{\mu}\hat{\rho}\hat{\sigma}} \hat{H}^{\hat{\mu}\hat{\rho}\hat{\sigma}} - \frac{\alpha'}{4} \hat{R}(-)_{\hat{\mu}} \hat{a}_{\hat{b}} \hat{R}(-)_{\hat{\nu}} \hat{b}_{\hat{a}} = 0,
\]

(A.6a)

\[
(\partial \hat{\phi})^2 - \frac{1}{2} \hat{\phi}^2 \hat{\phi} - \frac{1}{4} \cdot 3! \hat{H}^2 + \frac{\alpha'}{32} \hat{R}(-)_{\hat{\mu}} \hat{a}_{\hat{b}} \hat{R}(-)_{\hat{\nu}} \hat{b}_{\hat{a}} = 0,
\]

(A.6b)

\[
d \left( e^{-2\hat{\phi}} \star \hat{H} \right) = 0.
\]

(A.6c)

As usual, if one expresses the KR field in terms of its 3-form field strength, one must also impose the KR Bianchi identity that ensures the local existence of the KR 2-form potential. It takes the form

\[
d \hat{H} = \frac{\alpha'}{4} \hat{R}(-)_{\hat{\mu}} \hat{a}_{\hat{b}} \wedge \hat{R}(-)_{\hat{\nu}} \hat{b}_{\hat{a}}.
\]

(A.7)

### B The non-extremal, 0th-order solutions

The ansatz for the 0th-order 10-dimensional solutions can be expressed in terms of 4 functions of $\rho$

\[
Z_0, \ Z_+, \ Z_-, \ W,
\]

(B.1)

and 5 constants

\[
\beta_0, \ \beta_+, \ \beta_-, \ \hat{\phi}_\infty, \ k_\infty,
\]

(B.2)

plus the integration constants present in the 4 functions.

According to the no-hair conjectures, the independent physical parameters of the solution are expected to be just 6: the mass $M$, 3 Abelian charges that we will denote by $q_0, q_+, q_-$ and the two real moduli $\hat{\phi}_\infty, k_\infty$. The equations of motion will impose three relations between these 5 constants and the 4 integration constants associated to the 4 functions.

\(^{16}\)As it is written, this action implicitly contains terms of second order in $\alpha'$. They are included for the sake of convenience by they should be consistently disregarded in all computations.
In terms of these functions and constants, the ansatz takes the form

\[ d\hat{s}^2 = \frac{1}{Z_+ Z_-} W dt^2 - Z_0 (W^{-1} d\rho^2 + \rho^2 d\Omega^2_{(3)}) \]
\[ - \frac{k_0^2 Z_+}{Z_-} \left[ dz + \beta_+ k_0^{-1} \left( Z_+^{-1} - 1 \right) dt \right]^2 - dy^m dy^m, \quad m = 1, \ldots, 4, \]  
\[ \hat{H}^{(0)} = \beta_- d \left[ k_0 \left( Z_-^{-1} - 1 \right) dt \wedge dz \right] - \beta_0 \rho^2 Z_0' \omega_{(3)}, \]  
\[ e^{-2\hat{\phi}} = e^{-2\hat{\phi}_0} \frac{Z_-}{Z_0}, \]  

where a prime indicates derivation with respect to the radial coordinate \( \rho \),

\[ d\Omega^2_{(3)} = \frac{1}{4} \left[ (d\psi + \cos \theta d\varphi)^2 + d\Omega^2_{(2)} \right], \]  
\[ d\Omega^2_{(2)} = d\theta^2 + \sin^2 \theta d\varphi^2, \]

are, respectively, the metrics of the round 3- and 2-spheres of unit radii and

\[ \omega_{(3)} = \frac{1}{8} d\cos \theta \wedge d\varphi \wedge d\psi, \]

is the volume 3-form of the former.

This ansatz reduces to the one for extremal black holes (in particular, for the Strominger-Vafa black hole [1]) when the so-called “blackening factor” \( W \) is absent or, equivalently, \( W = 1 \).

The equations of motion are solved at 0th order in \( \alpha' \) for [29]

\[ Z_0 = 1 + \frac{q_0}{\rho^2}, \]
\[ Z_- = 1 + \frac{q_-}{\rho^2}, \]
\[ Z_+ = 1 + \frac{q_+}{\rho^2}, \]
\[ W = 1 + \frac{\omega}{\rho^2}, \]  

where asymptotic flatness and the standard normalization of the metric at spatial infinity have already been imposed, leaving just 4 integration constants

\[ q_0, \ q_+, \ q_-, \ \omega, \]  

which are related to the other 5 constants by the following 3 relations

\[ \beta_i = s_i \sqrt{1 - \frac{\omega}{q_i}}, \quad s_i^2 = 1, \quad i = 0, +, - . \]

This leaves us, as expected, with 4 independent physical parameters (3 charges and the mass) plus 2 independent moduli: the asymptotic value of the 10-dimensional dilaton \( \hat{\phi}_0 \) and the radius of the compact dimension parametrized by the coordinate \( z \) measured in string length units \( k_\infty \equiv R_z/\ell_s \).
When $\omega = 0$ the solution describes the extremal, 3-charge black holes considered in ref. [25]. In this limit all the $\beta$ parameters square to one, $\beta_{0,\pm}^2 = 1$, and they are just give the signs of the charges.

When $\omega \neq 0$, the $\beta$ parameters are complicated functions of the physical parameters which are very complicated to determine explicitly.

The $\omega > 0$ case is related to the $\omega < 0$ by the coordinate transformation

$$\rho^2 \to \rho^2 - \omega,$$

and a redefinition of the parameters

$$q_i \to q_i + \omega, \quad \beta_i \to \frac{q_i}{q_i + \beta_i} \beta_i$$

followed by

$$\omega \to -\omega,$$

where now $\omega$ is taken to be negative.

Observe that this transformation preserves the relations eqs. (B.8) and also the products $\beta_{(i)} q_{(i)}$.

Therefore, we may take $\omega < 0$ with no loss of generality. Negative values of the parameters $q_0 \pm$ can also be related to positive ones by similar transformations. However, these transformations shift $\omega$ by positive quantities and we may end up violating the assumed negativity of $\omega$. Thus, we will not make any assumptions concerning the signs of those parameters although it is well known that they have to be strictly positive if we want to obtain regular black hole in the extremal limit. Nevertheless, we will consider both the $\omega > 0$ and $\omega < 0$ cases because we will use the results obtained in each case will be used in the main body of the paper.

In order to study the black-hole spacetime described by this solution, we rewrite it first in 5-dimensional form.

**B.1 The 5-dimensional form of the solutions**

Upon trivial dimensional reduction on $T^4$ (parametrized by the coordinates $y^m$) we get a 6-dimensional solution which is identical to the 10-dimensional one except for the absence of the $-dy^m dy^m$ term in the metric. A non-trivial compactification on the $S^1$ parametrized by the coordinate $z$ using the relations in [21] at 0th order in $\alpha'$ gives the 5-dimensional string-frame action and a further rescaling of the metric

$$g_{\mu\nu} = e^{4(\phi - \phi_{\infty})/3} g_{\mu\nu} \equiv e^{4\tilde{\phi}/3} \tilde{g}_{\mu\nu},$$

(B.12)

gives the (modified) Einstein-frame action [57]. It is also convenient to dualize the 2-form $B$ into another 1-form $D$, with field strength $K = dD$ through the relation

$$H = e^{8\tilde{\phi}/3} * K.$$  

(B.13)

\footnote{We use $C$ for the winding vector coming from the reduction of the 2-form $\hat{B}$ to distinguish it from the 2-form $B$. Its 2-form field strength is $G = dC$.}
In terms of these variables, but removing the tildes of the Einstein-frame metric and dilaton for the sake of simplicity, the action and the solutions we are considering take the form [20]

\[
S[e^a, \phi, k, A, C, D] = \frac{1}{16\pi G_N^{(5)}} \int \left[ (e^\omega \wedge e^b) \wedge R_{ab} + \frac{4}{3} d\phi \wedge *d\phi + \frac{1}{2} k^{-2} dk \wedge *dk - \frac{1}{2} k^2 e^{-4\phi/3} F \wedge *F - \frac{1}{2} k^{-2} e^{-4\phi/3} G \wedge *G - \frac{1}{2} e^{8\phi/3} K \wedge *K \right. \\
\left. - F \wedge G \wedge C \right],
\]

(B.14)

and

\[
\begin{align*}
&ds^2 = f^2 W dt^2 - f^{-1}(W^{-1} d\rho^2 + \rho^2 d\Omega_{(3)}^2), \quad f^{-3} = Z_+ Z_- Z_0, \\
&F = d \left[ \beta_+ k_\infty^{-1} \left( Z_+^{-1} - 1 \right) dt \right], \\
&G = d \left[ \beta_- k_\infty \left( Z_-^{-1} - 1 \right) dt \right], \\
&K = d \left[ -\beta_0 \left( Z_0^{-1} - 1 \right) dt \right], \\
&e^{-2\phi} = \sqrt{Z_+ Z_- / Z_0}, \\
&k = k_\infty \sqrt{Z_+ / Z_-}.
\end{align*}
\]

The mass of these solutions is given by

\[
M = \frac{3\pi}{8G_N^{(5)}} \left\{ -\omega + \frac{2}{3} (q_+ + q_- + q_0) \right\}.
\]

(B.16)

Observe that the positivity of the mass is compatible with a range of negative values of the parameters $q_0 \pm$. Furthermore, this formula, which is valid for the $\omega > 0$ and $\omega < 0$ cases is invariant under the transformations eqs. (B.10) and (B.11).

The physical charges of these solutions can be defined as [18]

\[
\begin{align*}
Q_+ &= -\frac{1}{16\pi G_N^{(5)}} \int_{S^3_\infty} \left\{ e^{-4\phi/3} k^2 *F + G \wedge C \right\} = \frac{\pi}{4G_N^{(5)}} k_\infty \beta_+ q_+, \\
Q_- &= -\frac{1}{16\pi G_N^{(5)}} \int_{S^3_\infty} \left\{ e^{-4\phi/3} k^{-2} *G + F \wedge C \right\} = \frac{\pi}{4G_N^{(5)}} k_\infty^{-1} \beta_- q_-, \\
Q_0 &= -\frac{1}{16\pi G_N^{(5)}} \int_{S^3_\infty} \left\{ e^{8\phi/3} *K + F \wedge B \right\} = -\frac{\pi}{4G_N^{(5)}} \beta_0 q_0.
\end{align*}
\]

These definitions guarantee that the result does not change when the integration surface is displaced across a region where the equations of motion are satisfied and there are no sources. In the language of ref. [46], these are localized charges. In particular, we get the same value when we integrate on the event horizon and when we integrate at spatial infinity ($S^3_\infty$). Furthermore, observe that, once the constants $q_0 \pm$ have been chosen, the

---

18These definitions are equivalent to those in ref. [25] (excluding the $\ell_s$ factor introduced there for the sake of convenience) because the second terms in the integrands do not contribute in these configurations.
signs of the physical charges are essentially determined by the signs of the $\beta$ parameters. Finally, observe that, as the mass eq. (B.16), the physical charges are invariant under the transformations of the parameters eqs. (B.10) and (B.11).

In these coordinates, if none of the parameters $q_0, \pm$ vanishes, the horizons are located at the zeroes of $g_{tt}$, i.e. at $\rho = \rho_H = 0$ (outer, event, horizon) for $\omega > 0$ and at $\rho = \rho_H = 0, \sqrt{-\omega}$ (inner, Cauchy and outer, event, horizons, respectively) for $\omega < 0$. Therefore, the inner horizon is no covered by the coordinate patch in the $\omega > 0$.

If some of the parameters $q_0, \pm$ are negative, $g_{tt}$ blows up at $\rho^2 = |q|$ where $q$ is any of the negative parameters. The event horizon covers all these singularities if $|\omega| > |q|$ for all the negative parameters. This condition does not guarantee the positivity of the mass when there is more than one negative $q$. In that case, it has to be imposed independently, though. We will assume that these two properties hold.

We are going to study the thermodynamics in the $\omega > 0$ and $\omega < 0$ cases separately for the sake of convenience.

**B.1.1 The $\omega > 0$ case**

In this case, the radius of the horizon, defined as

$$ R_H \equiv \left. \sqrt{|g_{EE\theta\theta}|} \right|_{\rho \to \rho_H}, \quad (B.18) $$

is given by

$$ R_H = \left| \frac{\rho}{f^{1/2}} \right|_{\rho \to -|\omega|^{1/2}} = (q + q - q_0)^{1/6}, \quad (B.19) $$

and the Hawking temperature and Bekenstein-Hawking entropy of the black hole are given by the expressions

$$ T_H = \frac{1}{2\pi} \frac{\omega}{\sqrt{q + q - q_0}}, \quad (B.20a) $$

$$ S_{BH} = \frac{\pi^2}{2G_N^{(5)}} \sqrt{q + q - q_0}, \quad (B.20b) $$

so that

$$ T_H S_{BH} = \frac{\pi \omega}{4G_N^{(5)}}. \quad (B.21) $$

We can use this relation to rewrite eq. (B.16) as follows:

$$ M = \frac{3}{2} S_{BH} T_H + \frac{3\pi}{8G_N^{(5)}} \left\{ -2\omega + \frac{2}{3} (q_+ + q_- + q_0) \right\} $$

$$ = \frac{3}{2} S_{BH} T_H + \frac{\pi}{4G_N^{(5)}} \left[ (q_+ - \omega) + (q_- - \omega) + (q_0 - \omega) \right] $$

$$ = \frac{3}{2} S_{BH} T_H + \frac{\pi}{4G_N^{(5)}} \left[ \left( 1 - \frac{\omega}{q_+} \right) q_+ + \left( 1 - \frac{\omega}{q_-} \right) q_- + \left( 1 - \frac{\omega}{q_0} \right) q_0 \right] $$

$$ = \frac{3}{2} S_{BH} T_H + \frac{\pi}{4G_N^{(5)}} \left( k_\infty \beta_+ Q_+ + k_\infty \beta_+ Q_- - \beta_0 Q_0 \right), \quad (B.22) $$

where we have used the definitions of the charges eqs. (B.17) and the relations eqs. (B.8).
Comparing this equation with the Smarr formula that can be derived by homogeneity arguments or via Komar integrals \cite{58}

\[
M = \frac{3}{2} S_{BH} T_H + \Phi^+ Q_+ + \Phi^- Q_- + \Phi^0 Q_0. \tag{B.23}
\]

we conclude that the electric potentials evaluated on the horizon are given by

\[
\Phi^+ = k_{\infty}^{-1} \beta_+ , \quad \Phi^- = k_{\infty} \beta_-, \quad \Phi^0 = -\beta_0 . \tag{B.24}
\]

This identification can be checked by a explicit calculation of the form of the first law, which we perform in appendix C and also using the definitions

\[
\nu_\alpha F \equiv d\Phi^+, \quad \nu_\alpha G \equiv d\Phi^-, \quad \nu_\alpha K \equiv d\Phi^0. \tag{B.25}
\]

The Smarr formula can also be read as a Bogomol’nyi-type bound

\[
M - \left( \Phi^+ Q_+ + \Phi^- Q_- + \Phi^0 Q_0 \right) = \frac{3}{2} S_{BH} T_H \geq 0, \tag{B.26}
\]

which is saturated in the extremal limit \( S_{BH} T_H \sim \omega = 0 \). This suggests the following definition for the central charge

\[
\mathcal{Z} \equiv \left( \Phi^+ Q_+ + \Phi^- Q_- + \Phi^0 Q_0 \right). \tag{B.27}
\]

The occurrence in this expression of the \( \beta \) parameters, which, in general, depend on the mass and charges, is not so surprising: in the extremal limit, they are just signs and the expression is quite standard. In the non-extremal case, we, actually, expect the bound eq. (B.26) to take a more complicated form that allows for different extremal limits associated to the different skew-eigenvalues of the central charge matrix of at supergravity theory with 16 supercharges.\(^{19}\)

### B.1.2 The \( \omega < 0 \) case

In this case, the radius of the event horizon, the Hawking temperature and the Bekenstein-Hawking entropy of the black hole are given by the expressions

\[
R_H = \left[ (q_+ - \omega)(q_- - \omega)(q_0 - \omega) \right]^{1/6}, \tag{B.28a}
\]

\[
T_H = \frac{1}{2\pi} \frac{\omega}{\sqrt{(q_+ - \omega)(q_- - \omega)(q_0 - \omega)}}, \tag{B.28b}
\]

\[
S_{BH} = \frac{\pi^2}{2G_N^{(5)}} \sqrt{(q_+ - \omega)(q_- - \omega)(q_0 - \omega)}, \tag{B.28c}
\]

As expected, these expressions could have been obtained from those of the \( \omega > 0 \) case eqs. (B.19), (B.20a) and (B.20b) applying the transformations eqs. (B.10) and (B.11).\(^{19}\)

\(^{19}\)See, for instance, the discussions in refs. [59–61].
The relation eq. (B.21) is also preserved with the replacement $\omega \rightarrow -\omega$ and we can use it, as we did in the $\omega > 0$ case, in eq. (B.16) to derive the Smarr formula eq. (B.23). Now, the potentials take the form

$$
\Phi^+ = \frac{k_\infty^{-1}}{\beta_+}, \quad \Phi^- = \frac{k_\infty}{\beta_-}, \quad \Phi^0 = -\frac{1}{\beta_0},
$$

(B.29)

in agreement with the definitions eqs. (B.25).

It is interesting to compute the same quantities for the inner horizon at $\rho = 0$ as well, assuming none of the $q$s are negative. The result is

$$
R_{H\text{inner}} = (q_+q_-q_0)^{1/6}, \quad (B.30a)
$$

$$
T_{H\text{inner}} = \frac{1}{2\pi} \frac{\omega}{\sqrt{q_+q_-q_0}}, \quad (B.30b)
$$

$$
S_{BH\text{inner}} = \frac{\pi^2}{2G_N^{(5)}} \sqrt{q_+q_-q_0}, \quad (B.30c)
$$

and we find

$$
-T_{H\text{inner}}S_{BH\text{inner}} = T_HS_{BH} = -\frac{\pi \omega}{4G_N^{(5)}}. \quad (B.31)
$$

Observe that the charges are equal on both horizons, as follows from the definition we have used (the charges are localized).

B.2 The Reissner-Nordström-Tangherlini solution

An important particular case of the solutions we are considering is the one in which the 3 charge parameters are equal $q_0 = q_+ = q_- \equiv q$. In this case, the three 5-dimensional vector fields are proportional, the scalar fields are constant everywhere and the metric becomes that of the 5-dimensional Reissner-Nordström-Tangherlini black hole [62].

In this case it is not difficult to express the integration constants $q, \omega, \beta = \beta_0$ in terms of the physical parameters $M, Q = Q_0 \ (Q_\pm = k_\infty^\pm Q_0)$:

$$
\omega = \pm \frac{8G_N^{(5)}}{3\pi} \sqrt{M^2 - 9Q^2}, \quad (B.32a)
$$

$$
q = \frac{12G_N^{(5)}}{\pi} \frac{Q^2}{M \mp \sqrt{M^2 - 9Q^2}}, \quad (B.32b)
$$

$$
\beta = \frac{3Q}{M \pm \sqrt{M^2 - 9Q^2}}, \quad \beta_\pm = s_\pm \beta. \quad (B.32c)
$$

Observe that

$$
\beta(\omega > 0) = 1/\beta(\omega < 0). \quad (B.33)
$$

This property is necessary for the electrostatic potential $\Phi$, which we have identified with $\beta$ in the $\omega > 0$ case and with $1/\beta$ in the $\omega < 0$ case to be a physical quantity whose expression does not depend on the coordinates chosen.
Using these relations we can express the radius of the horizon, the Hawking temperature, the Bekenstein-Hawking entropy and the electrostatic potential in terms of the physical parameters

\[ R_H = \left( \frac{12G_N^{(5)}}{\pi} \right)^{1/2} \frac{|Q|}{\left( M - \sqrt{M^2 - 9Q^2} \right)^{1/2}}, \tag{B.34a} \]

\[ T_H = \left( \frac{3}{4\pi G_N^{(5)}} \right)^{1/2} \frac{\sqrt{M^2 - 9Q^2}}{\left( M + \sqrt{M^2 - 9Q^2} \right)^{3/2}}, \tag{B.34b} \]

\[ S_{BH} = \frac{2}{3} \left( \frac{4\pi G_N^{(5)}}{3} \right)^{1/2} \left( M + \sqrt{M^2 - 9Q^2} \right)^{3/2}, \tag{B.34c} \]

\[ \Phi = \frac{3Q}{M + \sqrt{M^2 - 9Q^2}}. \tag{B.34d} \]

The extremal limit of this solution is

\[ M = 3|Q|. \tag{B.35} \]

### B.3 T duality

The dimensionally reduced 0th-order action eq. (B.14) is invariant under the transformations

\[ A \leftrightarrow C, \quad k \leftrightarrow 1/k, \tag{B.36} \]

which interchange the KK vector \( A \) with the winding vector \( C \) and invert the KK scalar that measures the radius of the compact dimension. This symmetry is manifestation at the level of the effective-field theory action of the T duality of the heterotic superstring compactified on a circle, which interchanges KK and winding modes and inverts the compactification radius. When the above transformations are reexpressed in terms of the higher-dimensional variables (components of the metric, KR field and dilaton) [40] they take the form of the famous “Buscher T duality rules” [41, 42].

It is not difficult to see that the effect of the above transformations on the 0th-order solutions eqs. (B.3) is equivalent to the following transformation of some of the parameters of the solutions only:

\[ \beta_+ \leftrightarrow \beta_-, \quad q_+ \leftrightarrow q_-, \quad k_{\infty} \leftrightarrow 1/k_{\infty}. \tag{B.37} \]

Thus, eqs. (B.3), understood as a family of solutions with arbitrary values of these parameters, is invariant under T duality (or self-dual) and it cannot be extended any further using it. Since the Einstein metric is invariant under T duality, all the geometric properties of the solution (temperature, entropy, first law) must also be invariant under T duality and it can be readily seen that, indeed, the temperature and entropy are invariant under the above transformations.
Checking the first law at zeroth order in $\alpha'$

We are going to assume $\omega < 0$, for simplicity. In order to check the first law at 0th order in $\alpha'$ we can use the following relations between the parameters of the solution $\omega, q_{0,\pm}, \beta_{0,\pm}$ and the physical parameters $M, Q_{0,\pm}$

\[
\omega = q_{0,\pm}(1 - \beta_{0,\pm}^2), \quad (C.1a)
\]

\[
M = \frac{3\pi}{8G_N^{(5)}} \left\{ -\omega + \frac{2}{3}(q_+ + q_- + q_0) \right\}, \quad (C.1b)
\]

\[
\eta(i)Q(i) = \beta(i)q(i), \quad (C.1c)
\]

where $\eta_{0,\pm}$ are parameters that depend on $G_N^{(5)}$ and $k_\infty$ and where we have chosen $\omega < 0$.

The basic idea is to find how the entropy varies with the physical parameters using the expression

\[
S = \frac{\pi^2}{2G_N^{(5)}} \sqrt{(q_+ - \omega)(q_- - \omega)(q_0 - \omega)}, \quad (C.2)
\]

and the above relations between the physical parameters and those in terms of which the metric and the entropy are given. The coefficient of $\delta M$ must be $1/T$ with

\[
T = \frac{1}{2\pi} \frac{\omega}{\sqrt{(q_+ - \omega)(q_- - \omega)(q_0 - \omega)}}. \quad (C.3)
\]

The coefficient of $\delta Q_i$ must be the $\Phi$'s

\[
\Phi^+ = \frac{k_\infty^{-1}}{\beta^+}, \quad \Phi^- = \frac{k_\infty}{\beta^-}, \quad \Phi^0 = -\frac{1}{\beta_0}. \quad (C.4)
\]

Varying the above entropy formula, we get

\[
\delta S = \frac{\pi^2}{4G_N^{(5)}} \frac{1}{\sqrt{(q_+ - \omega)(q_- - \omega)(q_0 - \omega)}} \times \left\{ -\left[ (q_- - \omega)(q_0 - \omega) + (q_+ - \omega)(q_0 - \omega) + (q_+ - \omega)(q_- - \omega) \right] \delta \omega \\
+ (q_- - \omega)(q_0 - \omega) \delta q_+ + (q_+ - \omega)(q_0 - \omega) \delta q_- + (q_+ - \omega)(q_- - \omega) \delta q_0 \right\}. \quad (C.5)
\]

From eq. (C.1b) we obtain

\[
\delta \omega = -\frac{8G_N^{(5)}}{3\pi} \delta M + \frac{2}{3} \delta q_i. \quad (C.6)
\]

Using the relations eqs. (C.1c) in eqs. (C.1a) we get

\[
\delta q_i = \frac{1}{1 + \beta_i^2} \delta \omega + \frac{2\eta(i)\beta(i)}{1 + \beta_i^2} \delta Q(i), \quad (C.7)
\]

\[\text{It is assumed that the values of the } q_i \text{s are such that the entropy is real.}\]
and substituting this result into eq. (C.6) we get

$$\delta \omega = -\frac{8G_N^{(5)}}{\pi} X \delta M + 4X \sum \frac{\eta_j \beta_j}{1 + \beta_j^2} \delta Q_j,$$

where we have defined

$$X \equiv \frac{1}{3} \left( 1 - \frac{2}{3} \sum \frac{1}{1 + \beta_i^2} \right)^{-1} = -\frac{1}{\omega} \left( \sum \frac{1}{2q_i - \omega} \right)^{-1}. \quad (C.9)$$

Substituting this result back into eqs. (C.7) we find

$$\delta q_i = -\frac{1}{1 + \beta_i^2} \frac{8G_N^{(5)}}{\pi} X \delta M + \frac{1}{1 + \beta_i^2} \left[ 4X \sum \frac{\eta_j \beta_j}{1 + \beta_j^2} \delta Q_j + 2\eta(i) \beta(i) \delta Q(i) \right]. \quad (C.10)$$

Now, we can substitute eqs. (C.8) and (C.10) into eq. (C.5) and identify the coefficients of $\delta M$ and $\delta Q_i$. The first of these coefficients is given by

$$1/T = 2\pi \frac{X}{\sqrt{(q_+ - \omega)(q_- - \omega)(q_0 - \omega)}} \times \left\{ \left( (q_- - \omega)(q_0 - \omega) + (q_+ - \omega)(q_0 - \omega) + (q_+ - \omega)(q_- - \omega) \right) \right. \\
- \left. \left[ (q_- - \omega)(q_0 - \omega) \frac{1}{1 + \beta_+^2} + (q_+ - \omega)(q_0 - \omega) \frac{1}{1 + \beta_-^2} + (q_+ - \omega)(q_- - \omega) \frac{1}{1 + \beta_0^2} \right] \right\} \\
= 2\pi \frac{X}{\sqrt{(q_+ - \omega)(q_- - \omega)(q_0 - \omega)}} \left\{ (q_- - \omega)(q_0 - \omega) \frac{\beta_+^2}{1 + \beta_+^2} \\
+ (q_+ - \omega)(q_0 - \omega) \frac{\beta_-^2}{1 + \beta_-^2} + (q_+ - \omega)(q_- - \omega) \frac{\beta_0^2}{1 + \beta_0^2} \right\}, \quad (C.11)$$

Since

$$\frac{\beta_+^2}{1 + \beta_+^2} = \frac{q_+ - \omega}{2q_i - \omega}, \quad (C.12)$$

$$1/T = 2\pi \frac{X}{\sqrt{(q_+ - \omega)(q_- - \omega)(q_0 - \omega)}} \left\{ (q_+ - \omega)(q_- - \omega)(q_0 - \omega) \sum \frac{1}{2q_i - \omega} \right\} \\
= 2\pi X \sqrt{(q_+ - \omega)(q_- - \omega)(q_0 - \omega)} \sum \frac{1}{2q_i - \omega} \quad (C.13)$$

in agreement with the first law.
Now, let us compute the coefficient of $\delta Q_+$, for instance, which should be equal to $-\Phi^+/T$, according to the first law

$$-\Phi^+/T = \frac{\pi^2}{4G_N^{(5)}} \frac{2\eta_+\beta_+}{(1 + \beta_2^2)(q_+ - \omega)(q_- - \omega)q_0 - \omega)} \times \left\{ - [(q_- - \omega)(q_0 - \omega) + (q_+ - \omega)(q_0 - \omega) + (q_+ - \omega)(q_- - \omega)] 2X 
+ \left[ \frac{(q_- - \omega)(q_0 - \omega)}{1 + \beta_2^2} + \frac{(q_+ - \omega)(q_0 - \omega)}{1 + \beta_2^2} + \frac{(q_+ - \omega)(q_- - \omega)}{1 + \beta_0^2} \right] 2X 
+ (q_- - \omega)(q_0 - \omega) \right\}$$

$$= \frac{\pi^2}{4G_N^{(5)}} \frac{2\eta_+\beta_+}{(1 + \beta_2^2)(q_+ - \omega)(q_- - \omega)q_0 - \omega)} \times \left\{ - 2X \frac{(q_- - \omega)(q_0 - \omega)\beta_+^2}{1 + \beta_2^2} + \frac{(q_+ - \omega)(q_0 - \omega)\beta_2^2}{1 + \beta_2^2} + \frac{(q_+ - \omega)(q_- - \omega)\beta_0^2}{1 + \beta_0^2} \right\}$$

$$= \frac{\pi^2}{4G_N^{(5)}} \frac{2\eta_+\beta_+}{(1 + \beta_2^2)(q_+ - \omega)(q_- - \omega)q_0 - \omega)} \times \left\{ - 2X (q_+ - \omega)(q_- - \omega) \sum q_i \frac{1}{2q_i - \omega} + \frac{(q_+ - \omega)(q_- - \omega)(q_0 - \omega)}{(q_+ - \omega)} \right\}$$

$$= \frac{\pi^2}{2G_N^{(5)}} \frac{\eta_+\beta_+\sqrt{(q_+ - \omega)(q_- - \omega)(q_0 - \omega)}}{(1 + \beta_2^2)} \frac{2q_i - \omega}{\omega(q_+ - \omega)}$$

$$= \frac{k_N^{(1)}}{\beta_+} \frac{2\pi}{\beta_+} \frac{\sqrt{(q_+ - \omega)(q_- - \omega)(q_0 - \omega)}}{\omega}, \quad (C.14)$$

again in agreement with the first law with the identification of the electric potential $\Phi^+$ made before.

Since all the expressions are symmetric in the three parameters $q_i$ and in the three charges $Q_i$ it is not necessary to check the other two terms in the first law.

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