Hyperbolic Invariance

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Abstract

Motivated by the study of duality cascades in supersymmetric quiver gauge theories beyond affine models, we develop in this paper the analysis of a class of simply laced hyperbolic Lie algebras. These are specific generalizations of affine ADE symmetries which form a particular subclass of the so-called Indefinite Lie algebras. Because of indefinite signature of their bilinear form, we show that these infinite dimensional invariances have very special features and admit a remarkable link type IIB background with non zero axion. We also show that hyperbolic root system $\Delta_{\text{hyp}}$ has a $\mathbb{Z}_2 \times \mathbb{Z}_3$ gradation containing two specific and isomorphic proper subsets of affine Kac-Moody root systems baptized as $\Delta_{\text{affine}}^\delta$ and $\Delta_{\text{affine}}^\gamma$. We give an explicit form of the commutation relations for hyperbolic ADE algebras and analyze their Weyl groups $W_{\text{hyp}}$. Comments regarding links with Seiberg like dualities and RG cascades are made.

Keywords: Quiver gauge theories, Large N duality and RG cascades, Lie algebras and their classification, Indefinite Lie algebras and Hyperbolic subset, Hyperbolic root systems, Commutation relations, Weyl groups.

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1 Introduction

During last few decades, finite dimensional Lie algebras and affine Kac-Moody (KM) extensions as well as their representations have been intensively used in establishing strong results in quantum field theory. Recently, root systems and Weyl symmetries of these invariances together with their algebraic geometry analog have been shown to be behind the developments made in the study of supersymmetric quiver gauge theories viewed as type II string low energy limit. In fact this remarkable link between Lie algebra roots and quantum fields is not a new thing since it goes back to the early days of discovery of gauge theories and turns out to encode most of their basic features. Roots are explicitly manifest in several quantum field theoretic models such as in the study of exactly solvable hamiltonian systems, integrable Toda field models [1]-[9], conformal field theories (CFTs) in low and higher dimensions [10]-[16] and string theory.

In four dimensional supersymmetric QFTs, the algebraic features of roots of simply laced ADE Lie algebras play an important role in the study of QFT class embedded in type II string compactifications on local Calabi-Yau (CY) threefolds. More precisely, they are used in the geometric engineering of supersymmetric quiver gauge theories [17]-[25] and in the study of their D-brane realizations. These supersymmetric quiver QFTs, which appear as specific QFT limits of type II string on K3 fibered CY manifolds with ordinary and affine ADE geometries [26], are nowadays subject of great interest in connection with large N field and string dualities [27]-[32].

The principal aim of the present study is to extend results on ordinary ADE and their affine KM structures, as used in 4D $\mathcal{N} = 2$ and $\mathcal{N} = 1$ quiver gauge theories, to the case of hyperbolic generalization of ADE Lie algebras. This generalization is the next leading extension of ordinary ADE Lie algebras which, surprisingly enough, haven’t been sufficiently explored in literature. The leading extension of ordinary ADE is naturally the affine Kac-Moody ADE algebras. Their Lie algebra and algebraic geometry properties as well as the role they play in supersymmetric quiver QFTs are now quite well established.

As we know, there are only partial algebraic results on the indefinite sector of Lie algebras, for instance a classification à la Cartan of these algebras is still missing. So to address the objectives of this study, we first have to complete results on hyperbolic Lie algebras and their algebraic geometry analog. Once armed with these mathematical results, we can then consider the physical application. To avoid a presentation with lot of technicalities mixing physics and diverse mathematical methods, we have judged instructive to divide this analysis into two parts. The first part, to be developed in this paper, deals with the study of roots and Weyl symmetries in hyperbolic algebras. In part II [33], we consider the geometric engineering of 4D $\mathcal{N} = 2$ and $\mathcal{N} = 1$ quiver QFTs based on these hyperbolic ADE Lie algebras as well as their D-brane realizations. The results obtained in present article will be also used to study the QFT duals of hyperbolic quiver gauge theories as well as the analog of RG cascades of affine models.

The organization of this paper is as follows. In section 2, we give details on motivations in our interest into hyperbolic quiver gauge theories. In section 3, we review general aspects of Indefinite Lie algebras; in particular their special hyperbolic subset. Main interest is focused on simply laced hyperbolic ADE symmetries seen the role they play in the construction of hyperbolic quiver QFTs. In section 4, we study the root systems $\Delta_{hyp}$ of these hyperbolic ADE Lie algebras and derive the explicit contents of $\Delta_{hyp}$ as well as their closed proper subsets. In section 5, we summarize our results on $\Delta_{hyp}$ into a theorem giving the full set of root contents of hyperbolic ADE algebras and a corollary on the way they may be used. In this regards it is interesting to anticipate on a particular result in hyperbolic ADE extension which
turns out to be very helpful for the building of the hyperbolic structure. As we will show, hyperbolic ADE have two special isomorphic affine ADE subalgebras denoted as $g^\delta_{affine}$ and $g^\gamma_{affine}$ and most of properties of hyperbolic symmetry may be viewed as kinds of interpolations between corresponding properties into these two specific affine subalgebras. In section 6, we derive the explicit form of the commutation relations for hyperbolic ADE Lie algebras using first the interpolation method between the commutation relations of $g^\delta_{affine}$ and $g^\gamma_{affine}$ and second by introducing a covariant construction based on the use of the bi-linear form of the hyperbolic ADE algebras. In this section, we also give the necessary and sufficient conditions for the unitary highest weight representations of hyperbolic ADE algebras. In section 7, we construct Weyl groups $W_{hyp}$ of these hyperbolic algebras using interpolation scenario between the Weyl sub-groups associated with $g^\delta_{affine}$ and $g^\gamma_{affine}$. Last section is devoted to conclusion and comments.

2 Motivations

In this section, we make two specific comments to motivate our interest into hyperbolic ADE extensions of supersymmetric quiver gauge theories. The first one deals with the algebraic geometry interpretation of ordinary and affine quiver gauge theories and the second with their large N dualities. Then, we address the question of building hyperbolic quiver gauge model. More details on this issue as well as their link with type IIB background with non zero axion can be found in second paper [33].

2.1 Roots in quiver gauge models

To begin recall that 4D supersymmetric ADE quiver gauge theories are QFT$_4$ limits of type II strings on CY threefolds with ADE geometries and are remarkably engineered on ADE Dynkin diagrams. Nodes of the Dynkin graphs encode gauge and adjoint matter multiplets and links between the nodes engineer the various kinds of bi-fundamental matters involved in the supersymmetric $\prod_i U(N_i)$ quiver gauge theory. Moreover, in these 4D quiver QFTs, roots $\alpha$ ($\alpha = \pm \sum k_i \alpha_i$, $k_i \in \mathbb{Z}_+$) of ADE Lie algebras generated by the $\alpha_i$ simple ones, generally realized in $\mathbb{R}^n = \sum \mathbb{R} e_i$ ($e_i e_j = \delta_{ij}$) like

$$\alpha_i = e_i - e_{i+1}, \quad i = 1, \ldots,$$

as in case of $su(n)$ Lie algebras, have an algebraic geometry interpretation. They are in one to one correspondence with the holomorphic volumes

$$v_i = \int_{C P^1} \Omega^{(2,0)},$$

of the homological two-cycles $C P^1_i$ involved in the resolution of ADE singularities [34].

$$v_i = t_i - t_{i+1}, \quad i = 1, \ldots,$$

In this relation the $t_i$s are complex moduli and the $v_i$s describe deformations of local ADE geometry and have a nice interpretation in 4D $\mathcal{N} = 2$ supersymmetric quiver $\prod_{i=1}^r U(N_i)$ gauge theories with adjoint matter superfields $\{\Phi_i : i = 1, \ldots, r\}$. There, the $v_i$ moduli appear as FI like couplings generating the following 4D $\mathcal{N} = 1$ linear chiral superspace potential deformation $\delta W$,

$$\delta W = \sum_{i=1}^r v_i \int d^4 x d^2 \theta Tr (\Phi_i).$$

This special superpotential deformation preserves $\mathcal{N} = 2$ supersymmetry and its non linear $\mathcal{N} = 1$ extension is at the basis of the field theoretic representation of the geometric transition $O(-1) \times$
\( O (-1) \times \mathbb{C}P^1 \to T^* S^3 \) of the conifold. Eq(2.4) is also behind the field theoretic analysis of large \( N \) field dualities and in the derivation of exact results in \( \mathcal{N} = 1 \) supersymmetric gauge theories [34, 35]-[27].

The second comment we want to make is about quantum field interpretation of automorphism symmetry of root system of ADE Lie algebras. Like for roots \( \alpha \), Weyl groups \( W_{ADE} \) of Lie algebras play also a crucial role in understanding part of quantum field theoretic dualities; in particular Seiberg like dualities and RG cascades of affine models [23]. At low energies below string scale where the dynamics of matter and gauge fields is governed by supersymmetric Yang Mills model, one disposes of sets of dual ADE quiver gauge theories with a remarkable subclass whose duality symmetries act on previous \( \upsilon_i \) as

\[
\upsilon_i \to \upsilon_i' = A_{ij} \upsilon_j, \tag{2.5}
\]

These duality symmetries were shown to be isomorphic to the usual Weyl group transformations of ADE root system [23]. By help of correspondence (2.5), the \( A_{ij} \) matrix in above relation is isomorphic to the bi-linear product \( \delta_{ij} - \alpha_i \alpha_j \) that appears in Weyl reflections \( \alpha_i' = \alpha_i - 2 \frac{\alpha_i \alpha_j}{\delta_{ij}} \alpha_j \).

In addition to the two above links, there are other basic ties between 4D super quiver QFTs and ADE algebra. For instance ADE root systems and their Weyl symmetries are also used in brane realization of the quiver gauge theories living in the world volume of parallel \( N_0 \) D3 branes and \( N \) D5 ones partially wrapping \( \mathbb{C}P^1 \) two-cycles of CY3 folds with a local ADE geometry. There, \( N_0 \) D3 is roughly speaking associated with the affine simple root \( \alpha_0 \) of affine KM ADE root system and wrapped \( N \) D5s with remaining \( \alpha_i \) simple ones. In this representation, field theoretic scenarios such as higgsings correspond just to special properties of the root system. Other basic relations between roots and their Weyl automorphisms on one hand; and relevant QFT\(_4\) moduli on the other hand can be also written down. Supersymmetric Yang-Mills gauge couplings \( g_{\text{SYM}}^i \) of the quiver gauge sub-group factors \( U(N_i) \) and corresponding beta functions \( b_i \) including supersymmetric affine ADE conformal field models,

\[
\frac{1}{g_s} = \sum_{i=0}^r \varepsilon_i g_i^{-2}; \quad b_D = \sum_{i=0}^r \varepsilon_i b_i, \tag{2.6}
\]

with \( \varepsilon_i \)'s the usual Dynkin weights, obey a similar law as holomorphic volumes \( \upsilon_i \) eq(2.5). For details on this issue as well as other areas of involvement of Weyl symmetries, we refer to [33], see also [36]-[38].

**2.2 Superfield action**

Despite that above quiver gauge field theories based on finite ADE and affine ADE Lie algebras belong to different classes of QFT\(_4\)'s ( affine QFTs are CFTs while ordinary ones are generally not) and though they have different physical interpretations and different D branes realization; they do however share most of basic features. The point is that their fundamentals are quite same and the observed physical disparities are nothing but manifestations of the Lie algebraic deformations.

\[
\text{affine ADE QFT}^4 \sim \text{ordinary ADE QFT}^4 \text{ plus special deformations.} \tag{2.7}
\]

From this vision of things, one clearly see that basic properties shared by these two classes of quiver QFT\(_4\)'s are in fact just a part of a general picture involving larger extensions of ADE symmetries. This behaviour should be also valid for other extensions of ADE symmetries; in particular for hyperbolic ADE symmetry we are interested in here. As such previous correspondence extends naturally to,

\[
\text{hyperbolic ADE QFT}^4 \sim \text{affine ADE QFT}^4 \text{ plus appropriate deformations,} \tag{2.8}
\]

forming together with eq(2.7) a sequence of three terms describing the first two leading deformations of ordinary super quiver ADE QFT\(_4\)'s.
Recall that from Lie algebraic point of view, affine ADE buildings may be viewed as a leading extension of the corresponding ordinary ADE ones. Starting from a rank \( r \) Cartan matrix \( K_{\text{finite}} \) of a given finite dimensional ADE Dynkin diagram, this extension mainly consists to add a special (affine) node to the ordinary Dynkin diagrams as,

\[
K_{\text{finite}} \rightarrow K_{\text{affine}} = \begin{pmatrix} 2 & -1 \\ -1 & K_{\text{finite}} \end{pmatrix},
\]

where the \((-1)\) entry in the first row refers to the line \( r \)-vector \((-1, 0, \ldots, 0)\) and the other to its transpose. Note that for ADE cases, \( K_{\text{finite}} \) matrix is symmetric and can be split in general as \( K_{\text{finite}}^{ij} = 2\delta^{ij} + 2G_f^{ij} \) where we have set \( G_f^{ij} = (G_f^j + G_f^i)/2 \) and \( G_f^{ij} = G_f^{ij} + G_f^{ji} \). Similar decomposition can be also done for generalized affine ADE and hyperbolic ADE Cartan matrices. We have then \( K_{\text{affine}}^{ij} = 2\delta^{ij} + 2G_a^{ij} \) and \( K_{\text{hyp}}^{ij} = 2\delta^{ij} + 2G_h^{ij} \).

In geometric engineering of four dimensional \( \mathcal{N} = 2 \) \( (\mathcal{N} = 1) \) supersymmetric ADE quiver gauge theories with adjoint \( \Phi_i \) matters and bi-fundamental \( Q_{ij} \) ones, the above algebraic affine extension has a superfield theory realization. It corresponds to deforming the (massive) ordinary ADE quiver theory described by the superfield action \( S_{\text{finite}} = S_{\text{finite}}[Q, V, \Phi] \),

\[
S_{\text{finite}} = \int \! d^4x d^4\theta \left[ \sum_{i,j=1}^{r} Tr \left(Q_{ij} \left[ \exp \left(K_{\text{finite}}^{ij} V_j \right) \right] Q_{ji} \right) + \sum_{i=1}^{r} \zeta_i \int \! d^4x d^4\theta Tr (V_i) \right. \\
+ \int \! d^4x d^4\theta \left[ \sum_{j=r+1}^{n} Tr \left(Q_{ij} \left[ \exp \left(q_{\text{finite}}^{ij} V_j \right) \right] Q_{ji} \right) + \sum_{i=r+1}^{n} \zeta_i \int \! d^4x d^4\theta Tr (V_i) \right. \\
+ \left( \int \! d^4x d^2\theta \left[ \sum_{i,j=1}^{r} G_f^{ij} Tr (Q_{ij} \Phi_i Q_{ji}) + \sum_{i=1}^{r} Tr W (\Phi_i) \right] + \text{hc} \right) \\
+ \left( \int \! d^4x d^2\theta \left[ \sum_i Tr \left( \Phi_i e^{K_a^{i} V_i} \Phi_i \right) \right) + \int \! d^4x d^2\theta \left[ \sum_{i=1}^{r} \frac{1}{g_i} Tr \left(W_i W_i^\alpha \right) \right] + \text{hc} \right) ,
\]

to a 4D \( \mathcal{N} = 2 \) \( (\mathcal{N} = 1) \) affine ADE quiver QFT involving more gauge and matter superfields\(^1\). The extra terms with \( \zeta_i \) couplings are the usual FI terms and the \( V_i \) are the gauge superfields; \( (n - r) \) of them are auxiliary as they have no propagating dynamics, they are introduced for technical reasons; in particular in order to ensure the CY condition in string compactification. They are also needed in the study of the critical behaviour of these supersymmetric quiver QFT\(_8\). The corresponding superfield action \( S_{\text{affine}} \) is obtained from previous one by substituting \( K_{\text{finite}} \), \( G_f^{ij} \) and \( q_{\text{finite}}^{ij} \) by their affine \( K_{\text{affine}} \), \( G_a^{ij} \) and \( q_{\text{affine}}^{ij} \) analog. Since the action \( S_{\text{affine}} \) can be usually put in the form,

\[
S_{\text{affine}} = S_{\text{finite}} + \text{ deformations } ,
\]

one can easily identify the superfield operators that capture the deformation from ordinary ADE quiver gauge theories to the affine ADE ones. In this way, one can recover many known results on QFT deformations such for instance supersymmetric Sine Gordon model and affine ADE Toda field theories respectively obtained by deformations of supersymmetric Liouville theory and ordinary ADE Toda models. A similar conclusion is also valid for affine ADE CFTs and their underlying partners.

In D-brane realization of four dimensional QFT\(_8\)s living in the non compact directions of D5 branes wrapping two cycles with the topology of ordinary ADE Dynkin diagrams, the algebraic affine extension

\(^1\) The extra term in \( S_{\text{finite}} \) involving the rectangular matrix \( q_{\text{finite}}^{ij} \) is required by type II string on CY3 with a local ADE geometry. The explicit form of these \( q^{ij} \)’s may be found in [10-13].
has also a D brane interpretation. It corresponds to adding D3 branes to the existing system of wrapped
D5 ones as shown below [23].

\[ \cup_{i=1}^{r} \{ N_i D5/S^2_i \} \to \cup_{i=1}^{r} \{ N_i D5/S^2_i \} \cup N_0 D3. \quad (2.12) \]

Here the \( S^2_i \) are the two cycles of the resolved ADE geometry; they are in one to one with the simple
roots \( \alpha_i \) of ordinary ADE algebras. Extra D3s fill the transverse space to the \( S^2_i \)s and have much to do
with the imaginary root \( \delta = \sum_{i=0}^{r} \epsilon_i \alpha_i \) of affine ADE systems. The homological cycle associated with \( \delta \)
is known to have no proper holomorphic volume.

### 2.3 Beyond affine model

From above quantum field analysis, one sees that the established results we have for ordinary and affine
quiver gauge models are in fact more general. They may be extended for supersymmetric quiver gauge
theories based on those simply laced Lie algebras going beyond affine KM ones. These gauge theories
are also expected to follow as low energy type II string compactification on a particular class of CY3s
and also have specific D-brane realizations.

From Lie algebra view, one also see that infinite dimensional affine KM generalization is not the
unique possible extension of ordinary ADE Lie algebras. The affine KM extension of ordinary Lie
algebras is an interesting generalization; but roughly speaking has nothing special except that it is
the leading one and has a zero eigenvalue. Actually there are many other possible and remarkable
generalizations although only few of them are under control.

One of the objectives of this paper and its continuation [33] is to work the complete picture by
building consistent quantum field theoretical model going beyond the affine ADE ones. To achieve
this goal, it is interesting to recall that many recent results regarding affine ADE field models are
approached through the algebraic properties of affine ADE root systems \( \Delta_{affine} \) and their algebraic
geometry counterpart. Here also we will follow this path to study hyperbolic extension of affine models.
To that purpose, we need first of all basic information on the algebraic properties of extensions of affine
ADE KM algebras. But unfortunately and remarkably these are not fully available in literature. Except
for some examples [39]-[40], to our knowledge the explicit content of root system \( \Delta_{hyp} \) of hyperbolic Lie
algebras, their corresponding commutation relations and associated Weyl symmetries have not been yet
completely identified. This is why we propose to first work out explicitly the aforementioned structures
for hyperbolic ADE Lie algebras and then come back to the analysis of supersymmetric hyperbolic ADE
quiver gauge theories.

In what follows, we focus on the explicit derivation of useful tools on hyperbolic ADE Lie algebras;
in particular their root system, the commutation relations, unitary conditions for highest weight repre-
sentations and Weyl symmetries. We show amongst others:

(a) The root system \( \Delta_{hyp} \) of hyperbolic extension of affine ADE Lie algebras is given by,

\[ \Delta_{hyp} \cup \{0\} = \{ n\gamma + m\delta + l\alpha; \quad l^2 - mn \leq 1 \quad l = 0, \pm 1; \quad n, m \in \mathbb{Z} \quad \alpha \in \Delta_{finite} \}, \quad (2.13) \]

where \( \gamma \) and \( \delta \) are two imaginary roots satisfying \( \gamma^2 = \delta^2 = 0 \) and \( \gamma \delta = -1 \) and where \( \Delta_{finite} \) is the
root system of the underlying ordinary ADE Lie subalgebra.
(b) The commutation relations defining the hyperbolic ADE Lie algebras read as,

\[
[L, K] = [K, \alpha H] = [\alpha H, \beta H] = 0, \quad \alpha, \beta \in \Delta_{finite},
\]
\[
[\alpha H, T_{p,q}^{(j\beta)}] = j(\alpha \beta) \ T_{p,q}^{(j\beta)}; \quad \alpha, \beta \in \Delta_{finite}
\]
\[
[K, T_{p,q}^{(j\beta)}] = qT_{p,q}^{(j\beta)}; \quad \beta \in \Delta_{finite}
\]
\[
[L, T_{p,q}^{(j\beta)}] = pT_{p,q}^{(j\beta)}; \quad \beta \in \Delta_{finite}
\]
\[
[T_{m,n}, T_{p,q}^{(j\beta)}] = \frac{Y (\ell^2 \alpha^2 - 2mn - 2) Y (j^2 \alpha^2 - 2pq - 2)}{Y ((l\alpha + j\beta)^2 - 2(m + p)(n + q) - 2)} \varepsilon_{l\alpha,j\beta} T^{(l\alpha + j\beta)}_{m+p,n+q}; \quad \alpha, \beta \in \Delta_{finite},
\]
\[
[T_{m,n}^{(l\alpha)}, T_{-m,-n}^{(-l\alpha)}] = \frac{2Y (\ell^2 \alpha^2 - 2mn - 2)}{2mn - l^2 \alpha^2} (nL + mK - l\alpha H), \quad \alpha \in \Delta_{finite}.
\]

In these relations, the operators \( L, K, \alpha H \) and \( T_{p,q}^{(j\beta)} \) are the generators of hyperbolic algebra; they will be discussed in details in sub-section 6.3. The function \( Y(x) = 1 \) if \( x \leq 0 \) and zero otherwise, is the Heveaside like distribution. Note that setting \( n = q = 0 \) for instance and taking \( l, j = 0, \pm 1 \), one discovers the usual commutation relations of affine Kac Moody ADE Lie algebras generated by the step operators \( T_{m,0}^{(l\alpha)} \) \( (m \in \mathbb{Z}) \) and the \( \alpha H \) and \( K \) commuting Cartan ones. With this choice, the operator \( K \) becomes a central element of the affine algebra while \( L \) reduces to a scaling operator often interpreted as a derivation or again as the zero mode Virasoso generator \( L_0 \) of 2D conformal algebra. Observe also that due to the indefinite signature of the bilinear form \( \langle x, y \rangle \equiv xy \) of hyperbolic ADE extension, \( su(2) \) subalgebras have a remarkable apparent pole singularity.

(c) The Weyl group of hyperbolic extension of affine ADE Lie algebras is also a semi-direct product generated by reflections and translations and obeys a quite similar law than the corresponding affine ADE Weyl group. More remarkable features of eqs (2.14) will be given at proper time.

In part II, we consider the geometric engineering of 4D \( \mathcal{N} = 2 \) and \( \mathcal{N} = 1 \) quiver QFTs based on these hyperbolic ADE Lie algebras as well as their D brane realization. The results obtained here are also used to study the QFT duals of hyperbolic quiver gauge theories as well as the analog of RG cascades of affine models.

3 Indefinite Lie algebras

By now it established that there exist three principal classes of Lie algebras; the well known finite dimensional Lie algebras \( g_{finite} \), the affine KM algebras \( g_{affine} \) about which we know quite much and Indefinite Lie algebras \( g_{indef} \) which continue to hide their secrets mainly because of the large arbitrariness they contain. Ordinary and affine Lie algebras have sub-classifications essentially given by the ABCDEFG Cartan classification while there is no classification yet for indefinite Lie algebras. From a mathematical view, these three principal classes of Lie algebras are conveniently described in terms of Cartan matrices \( K_{finite}, K_{affine} \text{ and } K_{indef} \) respectively. The existence of above three sectors of Lie algebras is governed by the following central theorem \[11\].

3.1 Classification

We first give the classification theorem and then make a comment.

Theorem: A generalized indecomposable Cartan matrix \( K \) obeys one and only one of the following three statements:

1. Finite type Lie algebras \( \det K > 0 \) characterized by the existence of a real positive definite vector \( u \) \( (u_i > 0; \ i = 1, 2, \ldots) \) such that \( K_{ij}u_j = v_j > 0 \), where \( v_j \) is a positive vector.
(ii) **Affine type KM algebras**, \( \text{corank}(K) = 1, \det K = 0 \), for which there exist a unique, up to a multiplicative factor, positive integer definite vector \( u \) \( (u_i > 0; \ i = 1, 2, ...) \) such that \( K_{ij}u_j = 0 \). This relation means that the generalized Cartan matrix \( K_{affine} \) has a vanishing eigenvalue.

(iii) **Indefinite type Lie algebras**, \( \det K \leq 0 \) and \( \text{corank}(K) \neq 1 \), for which there exist a real positive definite vector \( u \) \( (u_i > 0; i = 1, 2, ...) \) such that \( K_{ij}u_j = -v_i < 0 \), where \( v \) is as before.

Comment: In present study we consider a special subset of indefinite Lie algebras endowed with *symmetrizable* generalized Cartan matrices. By *symmetrizable*, we mean that corresponding Lie algebras have a symmetric bi-linear invariant form \(( , )\) and their Cartan matrices \( K_{ij} \) are realized as,

\[
K_{ij} \sim (a_i, a_j), \tag{3.1}
\]

where the set  \( \{a_i\} \) is the set of simple roots to be discussed later on. For simplicity, we will sometimes refer to the above product \((a_i, a_j)\) as \( a_i a_j \).

From above classification theorem, one may already make an idea on indefinite Lie algebras. For instance, one already feels that roots in indefinite Lie algebras have much to do with the usual classification of vectors \( v \) in pseudo-Euclidean spaces \( \mathbb{R}^{(p,q)} \). There, vectors \( v \) are classified according to their norms. We have vectors with positive definite norms \( v^2 > 0 \), vectors with zero norms \( v^2 = 0 \) and vectors having negative ‘norms’ \( v^2 < 0 \). As such, one expects that root systems \( \Delta_{indef} \) of indefinite Lie algebras and root lattices \( Q_{indef} \) as well as their maximal toric subalgebras \( h_{indef} \) have underlying geometries with indefinite metric \( \eta \)

\[
\eta = \begin{pmatrix}
-1 & & & \\
& -1 & & \\
& & 1 & \\
& & & 1
\end{pmatrix}, \tag{3.2}
\]

so that the norm \( v^2 = \eta_{ij}v^iv^j \) which also reads as \( v^2 = -\sum_{i=1}^{p} v_i^2 + \sum_{i=p+1}^{q} v_i^2 \) has an indefinite signature, a property which makes the study of the hyperbolic structure very interesting. The minus sign in the right hand side of this relation is then an explicit indication of existence of roots with indefinite norms rendering indefinite Lie algebra analysis more subtle and more rich. From above Vinberg classification, one also learns that what we know about Lie algebras is in fact just the top of an iceberg. For instance what we know on finite dimensional Lie algebras corresponds just to the deeply Euclidean region of the underlying indefinite geometry, which naively can be associated with setting \( p = 0 \) in above relation. In what follows, we focus on the particular hyperbolic subset of indefinite algebras, its underlying geometry is Lorentzian type \( (p = 1) \).

### 3.2 Hyperbolic Lie algebras

This is a special subset of indefinite Lie algebras which is intimately related to finite dimensional and affine KM ones. A classification of the Dynkin diagrams of hyperbolic algebras is available but a long path still remains to do for the explicit properties of the hyperbolic structure. The results we will give here concerns hyperbolic algebras in the sense of [43], but might be extended to other kinds of indefinite Lie algebras that go beyond the Wanglai Lie set. Moreover, though closer to ordinary and affine symmetries, it is interesting to note that the way hyperbolic Lie algebras enter in quantum physics is still unclear and the role they may play in the description of QFTs need more explorations. There
has been attempts in this matter during last decade but not enough for a clear picture. For some specific applications, see [44, 45] where hyperbolic algebras are used to characterize a new class of $\mathcal{N} = 2$ supersymmetric conformal field theory in four dimensions. For other applications of hyperbolic algebras, see for instance [46]-[53]. Now, we turn to give useful details for the present work.

The idea behind the derivation of hyperbolic Lie algebras $\mathfrak{g}_{\text{hyp}}$ is based on the same philosophy one uses in building affine Lie algebras $\mathfrak{g}_{\text{affine}}$ from finite ones $\mathfrak{g}_{\text{finite}}$. The Cartan matrix $K_{\text{hyp}}$ of a hyperbolic Lie algebra $\mathfrak{g}_{\text{hyp}}$ is obtained in two ways:

(i) Strictly hyperbolic; by starting from the Cartan matrix $K_{\text{finite}}$ of a finite dimensional Lie algebra $\mathfrak{g}_{\text{finite}}$ and extending it as,

$$K_{\text{finite}} \rightarrow K_{\text{hyp}} = \begin{pmatrix} 2 & * \\ * & K_{\text{finite}} \end{pmatrix}$$

where the ($*$)s stand for some row and column vectors. This kind of extension has no affine KM sub-symmetry; it does not interest us here and forget about it.

(ii) Hyperbolic, by starting from the Cartan matrix $K_{\text{affine}}$ of an affine KM algebra $\mathfrak{g}_{\text{affine}}$ and extend it as,

$$K_{\text{affine}} \rightarrow K_{\text{hyp}} = \begin{pmatrix} 2 & * \\ * & K_{\text{affine}} \end{pmatrix}.$$ 

It is these kinds of hyperbolic algebras that we are interested in. Note by the way that corresponding

Figure 1: Here we have reported the Dynkin graphs of the two categories of hyperbolic algebras namely class I (hyperbolic) and class II (strictly hyperbolic). The box $D_{\text{affine}}$ of fig 1a represents one of the Dynkin diagrams of affine Lie algebras; it is linked to the hyperbolic node associated with the simple root $a_{-1}$. In the second class, $D_{\text{finite}}$ of fig 1b represents a generic Dynkin diagram of $\mathfrak{g}_{\text{finite}}$.

Following [43], there are 238 possible Dynkin diagrams type those described by figures 1. These hyperbolic Dynkin diagrams denoted as $\mathcal{H}_n^i$; $i = 1, ..., n$, and contain obviously, as sub-diagrams of co-order 1, the usual Dynkin graphs associated with $\mathfrak{g}_{\text{affine}}$ and $\mathfrak{g}_{\text{finite}}$ Lie algebras. By cutting the $a_{-1}$ node of the order $n$ hyperbolic Dynkin diagram, the resulting $(n - 1)$-th sub-diagram one gets is either one of the Dynkin graphs of $\mathfrak{g}_{\text{finite}}$ or one of $\mathfrak{g}_{\text{affine}}$ as in above equation. In what follows we comment the list of the subclass of simply laced hyperbolic ADE Lie algebras based on affine ADE. It is this specific list of indefinite algebras which concerned here.
3.3 Hyperbolic ADE Lie algebras

The full list of simply laced hyperbolic ADE Lie algebras that contain simply laced affine ADE as a maximal subalgebra is given by,

\[
\mathcal{H}_2^3, \; \mathcal{H}_3^3, \; \mathcal{H}_4^3, \; \mathcal{H}_5^4, \; \mathcal{H}_6^5, \; \mathcal{H}_7^6, \; \mathcal{H}_8^6, \; \mathcal{H}_9^6, \; \mathcal{H}_10^6.
\] (3.5)

The Dynkin diagrams of these algebras have remarkable topologies. For the class of hyperbolic \(A_r\) algebras, Dynkin diagrams have a loop and look like the Feynman tade pole diagram of quantum field theory. Hyperbolic DE algebras have however open topologies involving trivalent vertices. Sometimes, they are also denoted as \(T_{p,q,r}\) or equivalently as \(DE_s\). For instance we have \(T_{3,2,2} = DE_5\). All these Dynkin graphs are simply laced and obviously of type figure 1a. They turn out to share several feature with the underlying affine ones. Let us give two illustrating examples.

3.3.1 Hyperbolic algebra \(HA_2\)

This hyperbolic algebra \(HA_2\) is a leading extension of affine KM algebra \(\hat{A}_2\) which appears as a particular subalgebra. As we will see, this hyperbolic algebra has four simple roots denoted as \(a_-, a_0, a_1\) and \(a_2\) generating all other roots. To our knowledge, the full set \(\Delta_{hyp}(HA_2)\) of roots of \(HA_2\) was not worked out before; it will be given later on. But for the moment, note that \(\Delta_{hyp}(HA_2)\) contains as a proper subset the roots of \(\hat{A}_2\) namely,

\[
\pm \alpha_1, \; \pm \alpha_2, \; \pm (\alpha_1 + \alpha_2), \\
n\delta, \\
n\delta \pm \alpha_1, \; n\delta + \alpha_2, \; n\delta \pm (\alpha_1 + \alpha_2)
\] (3.6)

where the first line give the usual roots of ordinary \(A_2\) and where \(\delta\) is the familiar imaginary root of affine KM algebras. In eqs (3.6) \(n\) a non zero integer. The \(HA_2\) algebra is a rank four Lie algebra and has a \(4 \times 4\) Cartan matrix given by,

\[
K(HA_2) = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & -1 \\
0 & -1 & 2 & -1 \\
0 & -1 & -1 & 2
\end{pmatrix}.
\] (3.7)

Its Dynkin diagram, which has four nodes given by the two ordinary ones, the affine node and the hyperbolic one, is reported on figure 2. From this construction one notes that this hyperbolic algebra

![Diagram of HA2](image)

Figure 2: This is the Dynkin diagram of \(HA_2\). It has the topology of a Feynman tade pole of QFTs. The node on the left is associated to the hyperbolic simple root and is linked to the affine one.

has a trivalent vertex and may be thought of as a kind of gluing together an affine \(\hat{A}_2\) with an ordinary
A_1 at the affine node. The topology of the graph of HA_2 shows that such structure is just the leading term of a more general series involving gluing of affine hat A_r's with ordinary A_m's. It can be extended to include generalized Dynkin diagram with generalized Cartan matrix type the following one describing gluing of affine hat A_2 KM algebra with an ordinary A_5 Lie algebra.

\[
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & -1 & -1 & 2 & 0
\end{pmatrix}
\]

Aspects of this kind of symmetries have been considered in [19]. Note also that the graph of HA_2 involves a trivalent vertex; which in open topologies has three linear A_i chains as shown on the following representation.

![Diagram](image)

Figure 3: This figure represents a typical vertex in trivalent mirror geometry. To the central node, it is attached three legs; two of them are of Dynkin type. The third leg is an extra chain which has a natural interpretation in T_{p,q,r} Lie algebras. These kinds of topologies are used in the fibration of gauge groups of quiver gauge theories embedded in Type II on CY3s.

According to the values of p, q and r integers; this determinant can have all possible signs. These extended Dynkin graph were used few years ago in the derivation of exact results in N = 2 supersymmetric quiver gauge theories with both fundamental and bi-fundamental matters [10].

### 3.3.2 Hyperbolic algebra HD_4

This is a simply laced hyperbolic based on affine hat D_4 which appears as a particular subalgebra. This hyperbolic algebra has also four simple roots \( a_-, a_0, a_1 \) and \( a_2 \) generating all others. The full set \( \Delta_{hyp} (HD_4) \) of roots of HD_4 will be given in forthcoming section; it contains as a proper subset the roots of hat D_4 namely \( n\delta + \alpha \) with \( n \in \mathbb{Z} \) and \( \alpha \in \Delta_{finite}(D_4) \). The hyperbolic extension HD_4 has rank five and a 5 x 5 generalized Cartan matrix given by,

\[
K (HD_4) = \begin{pmatrix} 2 & -1 \\ -1 & K (\hat{D}_4) \end{pmatrix},
\]
where $K \left( \hat{D}_4 \right)$ is the Cartan matrix of affine $\hat{D}_4$ and where the $(-1)$ in first row stands for $(-1, 0, 0, 0)$ and the other one for the transpose vector. Its Dynkin diagram which is reported on figure 3 has six nodes; four ordinary ones, one affine and one hyperbolic.

![Hyperbolic node](image)

Figure 4: This is the Dynkin diagram of $HD_4$. The node on the left corresponding the hyperbolic extension. This diagram has a tetravalent vertex.

Here also one can make a similar remark as before; $HD_4$ looks as gluing together three ordinary $A_18$ and an ordinary $A_2$ at the same vertex. As far as extension of affine ADE Kac Moody algebras are concerned, this Dynkin diagram can be viewed as just the leading component of a more general graph involving gluing four $A_m, A_p, A_q$ and $A_r$ chains of ordinary Lie algebras.

![Aq Ap An Ar](image)

Figure 5: This figure represents a typical vertex in tetravalent mirror geometry. To the central node, it is attached four $A$ type legs; two of them are of ordinary Dynkin type and the two others are extra ones.

## 4 Roots in Hyperbolic ADE Lie algebras

We start by fixing conventional notations we will be using. To avoid confusion, finite Lie algebras and their affine and hyperbolic extensions will be denoted respectively as $g_{finite}, g_{affine}$ and $g_{hyp}$. Same conventions will be used for corresponding Cartan matrices $K$, root systems $\Delta$, their simple root basis $\Pi$ and Weyl groups $W$. Note also that $g_{finite}$ stands for one of the ADE Lie algebras, special properties of these simply laced algebras are sometimes implicitly used.

### 4.1 Triplet Realization

Let $g_{hyp}$ be a given hyperbolic Lie algebras with an order $n$ Cartan matrix $K_{hyp}$ and a minimal realization involving the following triplet,

$$(h_{hyp}, \Pi_{hyp}, \Pi''_{hyp}).$$

As this triplet plays a crucial role in the present construction, let us comment its contents, fix some useful terminologies and presents the main lines of our strategy for the reminder of this work.

1) $h_{hyp}$ space: First note that viewed as a linear set, $h_{hyp}$ is a complex vector space of dimension $(2n - m)$ with $m = r + 2$ being the rank of $K_{hyp}$. The space $h_{hyp}$ is endowed with the bilinear form $(,)$. 

\[ h_{hyp} = \mathbb{C}^{2n-m} \]
introduced in section 2. A generic vector \( h \) in \( h_{hyp} \) may be then decomposed in terms of some given \( e_i \) vectors basis as

\[
   h = \sum_{i=1}^{2n-m} h_i e_i. 
\]  

(4.2)

The \( h_i \)'s are decomposition coefficients to be interpreted later on as the commuting Cartan generators of hyperbolic Lie algebras; they may be defined as usual as,

\[
   h_i = < e_i^*, h >, 
\]  

(4.3)

where now the \( e_i^* \)'s are the generators of \( h_{hyp}^* \), the dual space of \( h_{hyp} \). Since \( h_{hyp} \) and \( h_{hyp}^* \), spaces are finite dimensional, they are isomorphic and so can be identified. The one to one linear mapping between \( h_{hyp} \) and \( h_{hyp}^* \), denoted \( v \) and the image of \( x \in h_{hyp} \) is \( x^v \equiv x^+ \in h_{hyp}^* \). In the present case we have \( m = n \) and so one can forget about subtleties regarding the general situations; in particular there is no centre in such hyperbolic algebras contrary to the usual affine KM algebras where there exist a central generator \( K \) commuting with everything. For further simplicity, we will focus directly on the physically interesting real subspace,

\[
   h_{hyp} = \sum_{i=-1}^{r} \mathbb{R} e_i, \quad h_{hyp}^* = \sum_{i=-1}^{r} \mathbb{R} e_i^*, 
\]  

(4.4)

with hermitian elements,

\[
   h^+ = h; \quad h_i^+ = h_i. 
\]  

(4.5)

Note that in case \( \{ e_i \} \) is an orthogonal basis (\( (e_i, e_j) = 0 \) for \( i \neq j \)), and seen that \( h_{hyp} \) has to have \( (1, r + 1) \) signature, then the space \( h_{hyp} \) formally looks like a \( \mathbb{R}^{(1,r+1)} \) Lorentzian space with the metric,

\[
   \eta_{ij} = diag(-1, 1, ..., 1), 
\]  

(4.6)

with \( SO(1, r + 1) \) group as underlying homogeneous symmetry. Therefore norms \( x^2 \) (\( (x, y) \equiv xy \)) of vectors \( x \) in the hyperbolic space \( h_{hyp} \) have an indefinite sign and are as follows

\[
   e_{-1}^2 = -1, \quad e_i e_j = \delta_{ij}, \quad i,j = 0, 1, ..., r, 
\]

\[
   h^2 = (-h_{-1}^2 + h_0^2) + \sum_{i=1}^{r} h_i^2. 
\]  

(4.7)

It turns out that for hyperbolic Lie algebras based on affine KM symmetries, it is also convenient to work in light cone frame where \( \mathbb{R}^{(1,r+1)} \) is thought of as \( \mathbb{R}^{(1,1)} \oplus \mathbb{R}^r \) with a \( SO(1,1) \times SO(r) \) homogeneous symmetry. The previous basis is now changed to \( \{ e^\pm = (e_0 \pm e_{-1})/\sqrt{2}; e_i, 1 \leq i \leq r \} \) with,

\[
   (e^\pm)^2 = 0; \quad e^+ e^- = 1; \quad e_i e_j = \delta_{ij}, \quad i,j = 1, ..., r. 
\]  

(4.8)

A generic element \( h \) of \( h_{hyp} \) reads in this basis as \( h = K^+ e^- + K^- e^+ + \sum_{i=1}^{r} h_i e_i \) and its norm is

\[
   h^2 = 2K^+ K^- + \sum_{i=1}^{r} h_i^2. 
\]  

(4.9)

In this way, one recognizes immediately the part \( \sum_{i=1}^{r} h_i e_i \) of eq(4.2) as elements of the usual Cartan subspace \( h_{finite} \) and \( K^- e^+ \) as the familiar affine central extension of KM algebras. The extra term \( K^+ e^- \) is a new term which has no analogue in affine Lie algebras; it captures the hyperbolic extension.

(2)- Root basis: As a hyperbolic ADE Lie algebra, the space \( h_{hyp} \) is generated by root system \( \Pi_{hyp} \) rather than the basis \( \{ e_i \} \). Generic elements \( h \) are then expanded with respect to simple root vectors \( a_i \),

\[
   \Pi_{hyp} = \{ a_{-1}, a_0, a_1, ..., a_r \}, 
\]  

(4.10)
with \((a_i, a_j) = a_i a_j = K_{ij}\), which is just Cartan matrix \(K_{hyp}\) of the hyperbolic ADE algebras. These \((r + 2)\) simple roots \(a_i\) are obviously related to previous \(e_i\)'s by some given linear combinations. A convenient choice for our present study corresponds to take

\[
a_{-1} = \gamma - \delta; \quad a_0 = \delta - \psi, \tag{4.11}
\]

while the \(r\) other roots \(a_i\) are as usual in finite dimensional ADE Lie algebras. Recall for instance that for ordinary \(A_r\), we have \(a_i = e_i - e_{i+1}\). In the above relation, \(\psi = \sum_{i=1}^{r} \epsilon_i a_i\) is the maximal root of the underlying ordinary ADE Lie algebra and \(\gamma\) and \(\delta\) are as follows,

\[
\gamma = -e^-; \quad \delta = e^+; \quad \gamma \delta = -1. \tag{4.12}
\]

Notice the extra minus sign used in defining \(\gamma\); it is not an adhoc choice. We will see later that this is the right way to do in order to have a simpler definition for root positivity in hyperbolic algebras. Note also that the \(\epsilon_i\) appearing in \(\psi\) are the usual Dynkin weights; these are positive integers. Turning around the relations \(a_{-1} = \gamma - \delta\) and \(a_0 = \delta - \psi\), it is not difficult to see that \(\delta\) and \(\gamma\) are given by the following remarkable sums

\[
\gamma = \sum_{i=-1}^{r} \epsilon_i a_i; \quad \delta = \sum_{i=0}^{r} \epsilon_i a_i; \quad \psi = \sum_{i=1}^{r} \epsilon_i a_i, \tag{4.13}
\]

where for the present study we have \(\epsilon_{-1} = \epsilon_0 = 1\). From this relation, one sees that \(\gamma\) is the hyperbolic extension of \(\delta\) in same manner as \(\delta\) is the affine extension of \(\psi\). The above eqs give an idea on how further extension might be done. Moreover as the hyperbolic simple root \(a_{-1}\) should be positive, we clearly see that positivity of \(a_{-1}\) is linked to that of \(\gamma\). Positivity of \(\gamma\) follows obviously from positivity of the \(\epsilon_i\)'s. Later on, we will derive a general algorithm for defining root positivity in hyperbolic algebras.

Using the simple root basis, expansion of generic elements \(h\) of hyperbolic ADE Lie algebras \(h_{hyp}\) reads as,

\[
h = K^+ e^- + K^- e^+ + \sum_{i=1}^{r} h_i a_i = L\gamma + K\delta + \sum_{i=1}^{r} h_i a_i \tag{4.14}
\]

where \(\{h_i; 1 \leq i \leq r\}\) is the usual set of commuting observables generating the Cartan subalgebra of ordinary ADE and where \(K^+\) and \(K^-\) are two extra hermitian operators dealing with the affine and hyperbolic extension. Note in passing that though central element in affine KM subalgebras, the \(K^-\) generator is no longer a central element in hyperbolic algebras. The same is valid for \(K^+\).

**3- Coroot basis:** Note first that, by help of the \(v : h_{hyp} \to h_{hyp}^*\) isomorphism, the coefficients of the developments eq(4.14), can be computed by using the dual light cone basis \(\{\gamma^*, \delta^*, a_i^*\}\) and the pairing \(<,>\). They are as follows,

\[
K^+ = <\gamma^*, h> = (e^+, h); \quad K^- = <\delta^*, h> = (e^-, h),
\]

\[
H_i = <a_i^*, h> = K_{ij} h_j. \tag{4.15}
\]

Like for simple root basis of \(h_{hyp}^*\) eq(4.14), coroot basis \(\Pi_{hyp}^v\) is a free family of \(h_{hyp}\) defined by,

\[
\Pi_{hyp}^v = \{a_{-1}^v, a_0^v, a_1^v, ..., a_r^v\}. \tag{4.16}
\]

It allows to express hyperbolic ADE Cartan matrix in term of the pairing product as \(K_{ij} = <a_i^v, a_j^v>\) where

\[
a_i^v = \frac{2}{a_i^+} v^{-1}(a_i) \tag{4.17}
\]
Since in ADE Lie algebras $a_i^2 = 2$, then $a_i$ can be identified with $a_i$. As such the Cartan matrix reduces to the expression $K_{ij} = a_i a_j$ given before. We end this comment by noting that like in finite dimensional and affine cases, root system $\Delta_{hyp}$ of hyperbolic Lie algebras has in general a $\mathbb{Z}_2$ gradation implying that roots can be classified into positive roots and negative ones; i.e

$$\Delta_{hyp} = \Delta^+_{hyp} \cup \Delta^-_{hyp}$$  \hspace{1cm} (4.18)

Since negative root sub-system $\Delta^-_{hyp}$ is just $\left( -\Delta^+_{hyp} \right)$, the main thing one has to deal with is the subset $\Delta^+_{hyp}$. Moreover, we know from affine KM symmetries that even within $\Delta^+_{hyp}$, one still has to distinguish between two kinds of roots: (i) real positive roots and (ii) imaginary positive ones. This implies that $\Delta^+_{hyp}$ admits the following sub-grading

$$\Delta^+_{hyp} = \Delta^+_{hre} \cup \Delta^+_{him},$$  \hspace{1cm} (4.19)

where the sub-indices $hre$ and $him$ refer respectively to hyperbolic real roots and hyperbolic imaginary ones. Furthermore, we should also have in mind that $\Delta^\pm_{hyp}$, which are spanned by $\Pi_{hyp}$, are in fact specific subsets of larger space namely the hyperbolic root lattice $Q_{hyp}$ and its sub-lattices $Q^k_{hyp}$,

$$Q^\pm_{hyp} = \sum_{i=-1}^{r} \mathbb{Z}^\pm a_i \subset h_{hyp}.$$  \hspace{1cm} (4.20)

The $Q_{hyp}$ lattice contains $\Delta_{hyp}$ as a proper subsystem closed under hyperbolic Weyl transformations to be discussed later on.

**General Strategy:** Since by construction, hyperbolic Lie algebras $g_{hyp}$ we are interested in here contain as subalgebras the usual finite dimensional $g_{finite}$ and affine algebras $g_{affine}$, one may get much information on the structure of hyperbolic symmetries just by looking for adequate extensions of these proper sub-symmetries. This will be our strategy not only for (a) Deriving root system $\Delta_{hyp}$ for hyperbolic ADE algebras which is the main purpose of this section, but also for: (b) Writing down the explicit form for the commutation relations of hyperbolic ADE extension, (c) Deriving the necessary conditions for unitary highest weight representations of these algebras and (d) Building Weyl symmetries of hyperbolic ADE Lie algebras. The three last objectives will be described in the forthcoming sections.

### 4.2 Building root subsystem $\Delta^+_{hyp}$

From previous discussion, it follows that computation of the full root contents of $\Delta_{hyp}$ reduces to the determination of $\Delta^+_{hre}$ and $\Delta^+_{him}$ sub-systems for real and imaginary positive roots\(^2\). These are subsets of $Q^+_{hyp}$,

$$\Delta^+_{hre} = \left\{ a = \sum_{i=-1}^{r} k_i a_i \in Q^+_{hyp} \mid (a,a) > 0; \hspace{0.5cm} k_i (a_i, a_i) \in a^2 \mathbb{Z} \right\},$$

$$\Delta^+_{him} = \left\{ a = \sum_{i=-1}^{r} k_i a_i \in Q^+_{hyp} \mid (a,a) \leq 0 \right\},$$  \hspace{1cm} (4.21)

which up to now we know only parts of them; that is the part of root $a \in \Delta^+_{affine}$ associated with affine sub-symmetry. What remain to determine is then the extra part,

$$\Delta^+_{hyp} \setminus \Delta^+_{affine}.$$  \hspace{1cm} (4.22)

\(^2\)The sub-indices $hre$ and $him$ carried by $\Delta^\pm_{hre}$ and $\Delta^\pm_{him}$ refer to hyperbolic real and hyperbolic imaginary respectively. The upper $\pm$ refer to positive and negative roots. Similar terminology is used for affine root subsystems $\Delta^\pm_{hre}$ and $\Delta^\pm_{him}$.

For finite case we have $\Delta^+_f$ since there is no $\Delta^+_{him}$.
Moreover as $\Delta_{\text{hyp}}^+$ inherits the Lorentzian signature of $Q_{\text{hyp}}^+$, hyperbolic positive roots are then of three types: space like roots $a$ with positive definite norms $(a,a) > 0$, positive light like roots $a$ with zero norms; i.e $(a,a) = 0$ and positive time like roots $a$ with negative definite ‘norms’ $(a,a) < 0$. This means that in hyperbolic Lie algebras $\Delta_{\text{hyp}}^+$ has a $Z_3$ gradation as,

$$\Delta_{\text{hyp}}^+ = \sum_{q=0,\pm 1} \Delta_{\text{hyp}}^{(+q)} ,$$

with,

$$\Delta_{\text{hyp}}^{(+,0)} = \{ a \in \Delta_{\text{hyp}}^+ | (a,a) = 0 \} ,$$

$$\Delta_{\text{hyp}}^{(+,+)} = \{ a \in \Delta_{\text{hyp}}^+ | (a,a) > 0 \} ,$$

$$\Delta_{\text{hyp}}^{(+,-)} = \{ a \in \Delta_{\text{hyp}}^+ | (a,a) < 0 \} .$$

From this gradation, we now know that the usual affine positive roots namely $(0,n\delta,\alpha) \in \Delta_{\text{are}}^\delta$ with $n \geq 0$, $\alpha \in \Delta_{\text{finite}}^+$ and $(0,p\delta,0) \in \Delta_{\text{aim}}^\delta$ with $p > 0$ are distributed in hyperbolic root system as follows,

$$(0,n\delta,\alpha) \in \Delta_{\text{hyp}}^{(+,+)} ,$$

$$(0,p\delta,0) \in \Delta_{\text{hyp}}^{(+,+)} .$$

Furthermore, since $\delta$ and $\gamma$ are both of them positive light like roots and given that they play a perfect symmetric role in the light cone basis (4.3), transformations exchanging $\delta$ and $\gamma$ should be a symmetry of $\Delta_{\text{hyp}}$. This rotation of $\delta$ and $\gamma$ should be an element of the Weyl group as we will later on. This implies that in $\Delta_{\text{hyp}}$ we have not only one affine root sub-system $\Delta_{\text{finite}}^\gamma$, but rather two. These are the affine root sub-systems $\Delta_{\text{finite}}^\delta$ and $\Delta_{\text{finite}}^\gamma$ associated with the second light like root $\gamma$. In addition to the above relations, $(n\gamma,0,\alpha) \in \Delta_{\text{are}}^\gamma$ with $n \geq 0$, $\alpha \in \Delta_{\text{finite}}^+$ and $(p\gamma,0,0) \in \Delta_{\text{aim}}^\gamma$ with $p > 0$ should be also hyperbolic positive roots. As such we have the following result,

$$(0,n\delta,\alpha) ; \quad (n\gamma,0,\alpha) \in \Delta_{\text{hyp}}^{(+,+)} ; \quad \alpha \in \Delta_{\text{finite}}^+, n \geq 0$$

$$(0,p\delta,0) ; \quad (p\gamma,0,0) \in \Delta_{\text{hyp}}^{(+,+)} ; \quad p > 0 .$$

These are not all roots that we may have; one has just to note that the simple root $a_{-1} = \gamma - \delta$ does not figure among these relations. So there are other roots that still have to be identified. To get these remaining roots, we will use a remarkable observation and a necessary and sufficient condition. Let us discuss them separately.

Parameterisation: The observation we refer to above deals with the fact that roots we know eqs (4.20) may in general be parameterized as,

$$a = n\gamma + m\delta + l\alpha; \quad \alpha \in \Delta_{\text{finite}},$$

with $l = 0, \pm 1$ and $m$ and $n$ are some integers which still need to be specified. By appropriate choices of these integers, one recovers the previous results on affine KM systems. Note that as hyperbolic root lattice should also contain the ordinary one as a proper subset for any integers $m$ and $n$; in particular for $m = n = 0$, it follows that $l\alpha$ must be a root of ordinary Lie algebra ($l\alpha \in \Delta_{\text{finite}}$) which is possible only if $l = \pm 1$ as usual; but in this case we should also have $l = 0$. It is worthwhile to note here that this parameterisation is a tricky one which can be used as well for the derivation of root system of non simply laced hyperbolic algebras. Recall also that since $\alpha \in \Delta_{\text{finite}}^+$ can be expanded in terms of the $r$ simple roots $\alpha_i = a_i$, $i = 1, ..., r$, as $\alpha = \sum_{i=1}^r k_i a_i$ ($k_i \in Z_+$); so eq (4.27) can be rewritten as,

$$a = n\gamma + m\delta + l \sum_{i=1}^r k_i a_i$$

(4.28)
Now replacing $\delta$ and $\gamma$ by their explicit expression (4.13), we get the remarkable expression,

$$a = k_{-1}\gamma + (k_0 - k_{-1})\delta + \sum_{i=1}^r [lk_i - (k_{-1} + k_0)] \epsilon_i a_i.$$  

(4.29)

This formula offers a simple algorithm for defining root positivity in hyperbolic Lie algebras. The general rule that follows from this algorithm is: (i) a root type $(0, 0, \alpha)$ is positive if $\alpha$ does; (ii) a root type $a = (0, n\delta, \alpha)$ is positive if $n$ is positive without reference to $\alpha$ and finally (iii) a generic root type $(n\gamma, m\delta, \alpha)$ is positive if the coefficient in front of $\gamma$ does. This feature was the reason behind the choice of the sign in the identity $\gamma = -e^-$. In term of sings of integers in eq(4.29), root positivity is expressed through the condition $k_{-1} > 0$. If $k_{-1} = 0$; it is manifested through the condition $k_0 > 0$ and so on. We believe that this algorithm is general and applies as well for the classification of roots of indefinite Lie algebras. With these partial results at hand, we turn now to the second point.

**Necessary and sufficient conditions:**

To get the remaining hyperbolic roots, we start first by recalling a standard lemma on roots of Lie algebras. Then we use it to identify the remaining roots of $\Delta_{hyp}$.

**Lemma:** For a generic hyperbolic root $a$ with an expansion $a = \sum_{i=-1}^r p_i a_i$, we have the following results:

- The family of all real roots $a = \sum_{i=-1}^r p_i a_i$ are such that $a^2 > 0$ and $p_ia_i^2 \in a^2\mathbb{Z}$.

- If a generic real root $a$ verifies $p_ia_i^2 \in a^2\mathbb{Z}$ for any $i$; then either $a$ or $(-a)$ belongs to $\Delta_{hyp}^+$.

- If $a^2 \leq t$ for some given integer $t$; then either $a$ or $(-a)$ belongs to $\Delta_{hyp}^+$.

Note that for simply laced hyperbolic Lie algebras where real roots usually have $a^2 = a_i^2 = 2$, the first and second conditions are trivially solved. $p_ia_i^2 \in a^2\mathbb{Z}$ requires that all $p_i$s have to be all of them positive or negative integers; but this is not a new thing for us since we already know this feature. The novelty comes then from the third point which translates to our case as,

$$a^2 \leq 2,$$

(4.30)

and turns out to be the necessary and sufficient condition for building the roots of $\Delta_{hyp}$. Now using our parameterisation $a = n\gamma + m\delta + l\alpha$, we can express the above necessary and sufficient condition as a constraint eq on the integer triplet $(n, m, l)$. This yields,

$$a^2 = (\alpha^2l^2 - 2mn) \leq 2.$$  

(4.31)

As expected, this constraint eq has a $\mathbb{Z}_2 \times \mathbb{Z}_3$ grading which is carried by the symmetry $a \rightarrow -a$ and the indefinite sign of the norm $a^2$ following from the Lorentzian nature of the root lattice of hyperbolic algebras.

**Solutions:** Let us explore the solutions of (4.31) sector by sector according to the signs of the norm of roots.

- Space like positive roots: In this sector, all roots are real and have a unique length $a^2 = 2$. Putting this back into eq(4.31), we get the following condition for $(n, m, l)$,

$$l^2 - mn = 1; \quad l = 0, \pm 1.$$  

(4.32)

Solutions of this constraint relation are given by the following infinite set $(m, n, l) = (0, 0, \pm 1); (m, 0, \pm 1); (0, n, \pm 1)$ and $(\mp 1, \pm 1, 0)$ with $m$ and $n$ non zero integers. Real positive roots of simply
Laced hyperbolic ADE algebras read then as
\[ m\delta \pm \alpha; \quad m\gamma \pm \alpha; \quad \gamma - \delta, \]  
(4.33)
where \( \alpha \in \Delta_{+}^{\text{fini}} \) and \( m \in \mathbb{Z}_{+} \). The corresponding negative roots are determined as said before.

**Light like positive roots:** Hyperbolic light like roots have a zero norm; \( a^2 = 0 \). As such the condition eq.(4.31) reduces to \( l^2 - mn = 0 \) and the solutions for \((n, m, l)\) yield \((-1, 1, \pm 1); \quad (1, -1, \pm 1); \quad (0, m, 0) \) and \((n, 0, 0)\). Light like positive roots are then,
\[ \gamma + \delta + \alpha; \quad m\delta; \quad m\gamma; \quad \alpha \in \Delta_{\text{finite}}; \quad m \in \mathbb{Z}_{+}^* \]  
(4.34)
where now \( \alpha \in \Delta_{\text{finite}} \) and \( m \in \mathbb{Z}_{+}^* \).

**Time like positive roots:** Such roots have no analogue in affine KM algebras because they have negative definite norms; i.e \( a^2 = -\|a^2\| < 0 \). Putting back into eq.(4.31), one gets the condition
\[ 2(l^2 - mn) = -\|a^2\| \]  
which depends on the free parameter \( \|a^2\| \) and so has an arbitrariness which can be used to work out various types of solutions and so different kinds of hyperbolic extensions. The solutions for \((n, m, l)\) will naturally depend on this free parameter and we have a comment to make here.

**General solutions:** If all negative lengths are allowed as we are doing in the present study; i.e \( a^2 < t \) for any integer \( t \in \mathbb{Z}_{-} \), then the above condition reads as \( (l^2 - mn) < 0 \) and positive time like root solutions are
\[ \gamma + m\delta + \alpha; \quad m \in \mathbb{Z}_{+} - \{0, 1\} \]
\[ n\gamma + m\delta + \alpha; \quad m \in \mathbb{Z}_{+}^*, \quad n \in \mathbb{Z}_{+} - \{0, 1\} \]  
(4.35)
\[ n\gamma + m\delta; \quad m \in \mathbb{Z}_{+}^*, \quad n \in \mathbb{Z}_{+}^*. \]
where \( \alpha \in \Delta_{\text{finite}} \). This is a double infinite set showing that hyperbolic extension involve an extra infinity with respect to the standard affine one.

**Particular solutions:** Along with the above general solutions, there exist also others that are contained in the previous one as subsets. These correspond to the situations where one replaces the condition \((l^2 - mn) < 0 \) by weaker ones. This is the case for instance for the very special example where there is only one negative length \( a^2 \) and this is equal to \((-2)\). The previous condition
\[ 2(l^2 - mn) = -\|a^2\| \]  
(4.36)
becomes then \( mn = l^2 + 1 \) and corresponding time-like positive root solutions are
\[ \gamma + \delta; \quad \gamma + 2\delta + \alpha; \quad 2\gamma + \delta + \alpha, \quad \alpha \in \Delta_{\text{finite}}. \]  
(4.37)
Note that all solutions we have derived are stable under exchanging \( \delta \) and \( \gamma \). This is not surprising since this property was expected from the beginning and is manifest in the necessary condition \((a^2l^2 - 2mn) = a^2 \) we have used for characterizing hyperbolic roots. This is then a general feature of the full set \( \Delta_{\text{hyp}} \); it reflects just the fact that \( \delta \) and \( \gamma \) are rotated under Weyl reflections with respect to \( a_{-1} \) as we will see later.
5 More Results

Consider the $(r+2)$ dimensional real space $\mathbb{R}^{r+2}$ of linear forms introduced in previous section; together with the $\mathbb{R}^{r}$ affine and $\mathbb{R}^{r}$ finite subspaces and the corresponding lattices $Q^{hyp}$, $Q^{affine}$ and $Q^{finite}$. The following theorem and corollary give the complete structure of root system of hyperbolic ADE Lie algebras.

5.1 Theorem

Let $\Delta^{finite}$, $\Delta^{affine}$ and $\Delta^{hyp}$ be the sequence of root systems of finite ADE Lie algebras, affine and hyperbolic extensions satisfying the natural embedding $\Delta^{finite} \subset \Delta^{affine} \subset \Delta^{hyp}$, then we have the following results:

- The root system $\Delta^{hyp}$ of hyperbolic ADE Lie algebras belongs to a Lorentzian lattice; it has a $\mathbb{Z}_{2} \times \mathbb{Z}_{3}$ gradation and so splits into two principal blocs $\Delta^{hyp} = \Delta^{+hyp} \cup \Delta^{-hyp}$, each one splits in turn into three subsets as

  \[ \Delta^{+hyp} = \Delta^{(+,+)}_{hyp} \cup \Delta^{(+,0)}_{hyp} \cup \Delta^{(+,-)}_{hyp} \]

  and

  \[ \Delta^{-hyp} = \Delta^{(-,+)}_{hyp} \cup \Delta^{(-,0)}_{hyp} \cup \Delta^{(-,-)}_{hyp}, \]

  with $\Delta^{hyp} = (-\Delta^{hyp})$. In these decompositions, the $\Delta^{(+,q)}_{hyp}$ with $q = 1, 0, -1$ are as before; they capture the three regions of the hyperbolic cone. Sub-indices carried by these $\Delta$s are as before.

- Root subset $\Delta^{+hyp}$ is spanned by the positive simple root system $\Pi^{hyp} = \{a_i; -1 \leq i \leq r\}$. In particular the positive light like roots $\gamma$ and $\delta$, which belong to $\Delta^{(+,0)}_{hyp}$, are expanded in terms of simple roots as $\delta = \sum_{i=0}^{r} \epsilon_i a_i$ and

  \[ \gamma = \sum_{i=-1}^{r} \epsilon_i a_i, \]

  where $\epsilon_i$ with $i = 0, ..., r$ are the usual Dynkin numbers of affine ADE Lie algebras. The extra number $\epsilon_{-1}$ is the weight associated with $a_{-1}$.

- The positive light like roots $\gamma$ and $\delta$ are rotated by the hyperbolic Weyl reflection $\omega_{a_{-1}}(x) = x - (x, a_{-1}) a_{-1}$ as

  \[ \omega_{-}(q\gamma + p\delta) = p\gamma + q\delta, \]

  where $p$ and $q$ are integers.

- Root system $\Delta^{hyp}$ contains two isomorphic affine root sub-systems in one to one correspondence with the two light like roots $\delta$ and $\gamma$. These are,

  \[ \Delta^{\delta}_{affine} = \{m\delta + \alpha; \quad p\delta; \quad \alpha \in \Delta^{finite}, \quad m \in \mathbb{Z}, \quad p \in \mathbb{Z}^{*}\} \]

  and

  \[ \Delta^{\gamma}_{affine} == \{n\gamma + \alpha; \quad q\gamma; \quad \alpha \in \Delta^{finite}, \quad n \in \mathbb{Z}, \quad q \in \mathbb{Z}^{*}\} \]

  which clearly are proper subsets of $\Delta^{hyp}$. As such hyperbolic ADE algebras have two proper affine ADE subalgebras in one to one with the $\delta$ and $\gamma$ pair.
• Generic hyperbolic roots \( a \) in \( \Delta_{hyp} \) are generally expanded in terms of simple roots; but with help of the explicit realization of \( a_0 \) and \( a_{-1} \), they may be also represented like,

\[
a = k_{-1}\gamma + (k_0 - k_{-1})\delta + \sum_{i=1}^r (k_i - (k_{-1} + k_0)\epsilon_i)\alpha_i. \tag{5.7}
\]

In this representation, root positivity is captured by the positivity of \( k_{-1} \) whatever the remaining \( k_i \) integers are. For the special case \( k_{-1} = 0 \), this property is transmitted to \( k_0 \) and so on.

• The root system of simply laced hyperbolic ADE Lie algebras plus the zero vector \( (\Delta_{hyp} \cup \{0\}) \) is given by

\[
\Delta_{hyp} \cup \{0\} = \{n\gamma + m\delta + l\alpha\}, \tag{5.8}
\]

where \( \alpha \in \Delta_{finite} \) where \( n \) and \( m \) are integers constrained as \( mn \geq (l - 1)(l + 1) \) with integer \( l \) taking the three values \( l = 0, \pm 1 \).

Now we turn to explore how these relations may be used in the derivation of the commutation relations.

### 5.2 Corollary

Using the parameterisation eq(4.27) and given two roots \( a = n_1\gamma + m_1\delta + l_1\alpha \) and \( b = n_2\gamma + m_2\delta + l_2\beta \) of \( \Delta_{hyp} \), then we have the following results

• Generic roots \( a = n\gamma + m\delta + l\alpha \) of hyperbolic ADE Lie algebras obey the necessary and sufficient condition

\[
a^2 \leq 2, \tag{5.9}
\]

which reads also as \( l^2 - mn \leq 1 \) with \( l = 0, \pm 1 \).

• The sum \( (a + b) = c \) of two generic hyperbolic roots \( a \) and \( b \) is also a hyperbolic root \( c = q\gamma + p\delta + k\sigma \) root if and only if \( a^2 + b^2 + 2ab \leq 2 \). Equivalently, we should have

\[
k\sigma = l_1\alpha + l_2\beta; \quad p\delta = (m_1 + m_2)\delta; \quad q\gamma = (n_1 + n_2)\gamma, \tag{5.10}
\]

with \( k = 0, \pm 1 \) and moreover \( k^2 - pq \leq 1 \).

• The sum \( c = a + b \) is: (i) a space like hyperbolic root if \( a^2 + b^2 + 2ab = 2 \) or equivalently \( k^2 - pq = 1 \); with \( k = 0, \pm 1 \). (ii) It is a light like hyperbolic root if \( a^2 + b^2 + 2ab = 0 \); i.e \( k^2 - pq = 0 \); with \( k = 0, \pm 1 \) and (iii) It is a time like hyperbolic root if \( a^2 + b^2 + 2ab < 0 \); i.e \( k^2 - pq < 0 \); with \( k = 0, \pm 1 \).

• To each root \( a = n\gamma + m\delta + l\alpha \) of \( \Delta_{hyp} \), we associate a step operator \( S^a \) of the hyperbolic ADE Lie algebras,

\[
a = n\gamma + m\delta + l\alpha \iff S^a = S^a_{m,n}, \tag{5.11}
\]

and to the null vector we associate the Cartan Weyl operator,

\[
0 = a - a \iff < a, \hbar >= -nL - mK + l\alpha H \tag{5.12}
\]

where \( \hbar \) stands for a hermitian element of \( \hbar_{hyp} \).
• Since to each root $\alpha \in \Delta_{\text{finite}}$ is associated a hyperbolic root $a = n\gamma + m\delta + l\alpha \in \Delta_{\text{hyp}}$, it follows that to the usual invariant number $\alpha^2$ of ordinary ADE Lie algebras and their affine extension corresponds,

$$\alpha^2 \iff a^2 = \alpha^2 - 2mn$$  \hspace{1cm} (5.13)

With these results at hand we are now ready to build the commutation relations for hyperbolic ADE Lie algebras.

6 Commutation Relations

To write down the commutation relations of hyperbolic extension of ADE algebras, there are at least two ways to follow: an interpolation method and a covariant approach. Let us first comment these ways to do and then come back to our main purpose.

(1) Interpolation: In this explicit way of doing, one thinks about hyperbolic ADE algebra as an interpolating algebra between its two affine KM subalgebras so that the generators associated with the imaginary light like roots $p\delta$ and $q\gamma$ play a complete symmetric role. Put differently, the root system $\Delta_{\text{hyp}}$ of hyperbolic algebras we derived before eq(5.8) is viewed as an interpolating set between its proper subsets $\Delta_{\text{affine}}^\delta$ and $\Delta_{\text{affine}}^\gamma$, eqs(5.5)-(5.6),

$$\Delta_{\text{affine}}^{(p\delta,0)} \iff \Delta_{\text{affine}}^{(0,q\gamma)} \iff \Delta_{\text{hyp}}^{(0,0)} \iff \Delta_{\text{affine}}^{(0,0)} \iff \Delta_{\text{affine}}^{(0,q\gamma)},$$  \hspace{1cm} (6.1)

where $\Delta_{\text{affine}}^{(p\delta,0)}$ and $\Delta_{\text{affine}}^{(0,q\gamma)}$ stand for $\Delta_{\text{affine}}^\delta$, $\Delta_{\text{affine}}^\gamma$ and $\Delta_{\text{hyp}}^{(p\delta,q\gamma)}$, respectively. The set $\Delta_{\text{finite}}^{(0,0)}$ is the usual root system $\Delta_{\text{finite}}$ of ordinary ADE algebras.

(2) Covariant method: In this method, one uses the power of the hyperbolic bilinear form to write down the commutation relations of hyperbolic algebras in a covariant form. Instead of thinking of roots of hyperbolic algebras, their norms and corresponding generators as $n\gamma + m\delta + l\alpha$, $(n\gamma + m\delta + l\alpha)^2 = -2mn + 2l^2$ and $S_{m,n}^{l\alpha}$, one works directly with vectors $a$, norms $a^2$ and step operators $S^{\pm a}$ in same manner one does in finite dimensional Lie algebras. In this way, subalgebras and their root subsystems correspond to appropriate projections in hyperbolic lattice.

To see how these things work in practice, we start by identifying the generators of the of hyperbolic ADE algebras and their various $su(2)$ subsets.

6.1 Generators

As usual, generators of Lie algebras are of two kinds: step operators $S^{\pm a}$ and commuting Cartan generators $\frac{2}{\sqrt{a}} ah$. The step operators $S^a = S_{m,n}^{\alpha}$ and $S^{-a} = S_{-m,-n}^{-\alpha}$ eq(5.11) are associated with roots $\pm a = \pm (n\gamma + m\delta + l\alpha)$ belonging to $\Delta_{\text{hyp}}$. Their number is same as order of $\Delta_{\text{hyp}}$ and so there are infinitely many. In quantum physics, the $S^{-a}$ and $S^a$ operators are interpreted as creation and annihilation operators and are interchanged under adjoint conjugation

$$\left(S^{\pm a}\right)^\dagger = S^\mp a; \quad \left(S_{m,n}^{l\alpha}\right)^\dagger = S_{-m,-n}^{-l\alpha}.$$  \hspace{1cm} (6.2)

These operators satisfy commutation relations that can be read directly from the root contents of the hyperbolic system $\Delta_{\text{hyp}}$. If one forgets for a while about poles generated by light like roots, one can
already write down the commutation relations of the usual $\text{su}(2)$ subalgebras within $\mathfrak{g}_{hyp}$. These are given by,

$$[S^a, S^{-a}] = \frac{2}{a^2} \hbar; \quad \left[ \frac{2}{a^2} \hbar, S^{\pm a} \right] = \pm 2S^{\pm a}, \quad (6.3)$$

Before going ahead, note that the pole problem of above relations is not manifest in Chevalley basis where one is restricted to step operators $S^{\pm a_i}$ associated with simple roots $\pm a_i$. In this representation, one too simply has,

$$[S^{a_i}, S^{-a_i}] = a_i \hbar; \quad [a_i \hbar, S^{\pm a_i}] = \pm 2S^{\pm a_i} \quad i = -1, 0, 1, ..., r. \quad (6.4)$$

and it seems that there is no algebraic singularity. This is however not completely true, the apparent pole difficulty is not really absent but just translated on the generalized Serre relations defining remaining step operators. Note also that in the hyperbolic extension we are studying, commuting Cartan generators are of two types: The $K, \ L$ operators associated with the light like roots $\delta$ and $\gamma$ and the usual $H_\alpha$ spanning the Cartan algebra of underlying ordinary ADE subalgebra. Upon decomposing the root $\alpha$ on simple root basis ($\alpha = \sum k_i \alpha_i$), the $H_\alpha$s can be also written as $\sum k_i H_\alpha_i$ or $k_i H_l$ for simplicity. These hermitian operators, which satisfy obviously,

$$[K, L] = [K, H_\alpha_i] = [H_\alpha_i, L] = 0; \quad \alpha_i \in \Pi_{finite}, \quad (6.5)$$

with $H_\alpha$ stands for $\alpha H$, have more than one way to be handled. They can be handled either separately component by component or collectively in a compact and covariant form. In the covariant description, these commuting Cartan generators can put in a useful condensed form as $\alpha \hbar$; i.e,

$$a \hbar = -n L - mK + l \alpha H. \quad (6.6)$$

The point is that since to each pair $\pm a$ of non zero root $\pm a = \pm (n \gamma + m \delta + l \alpha)$ belonging to $\Delta_{hyp}$, $\ (a^2 \leq 2)$, one associates an operator triplet $\{S^a, S^{-a}, a \hbar\}$, it is then natural that scalar $a \hbar$ which gives the right combination between the $K, \ L$ and $H_\alpha$ Cartan generators. This is also dictated by the bilinear form of the hyperbolic Lie algebras which indicates that the covariantization of $\alpha H$ should be as in eq(6.6). Note finally that as this $a \hbar$ operator acts on step operators $S^{\pm b}$ of the hyperbolic algebra as usual; that is through the adjoint representation,

$$[a \hbar, S^{\pm b}] = \pm ab S^{\pm b}, \quad (6.7)$$

one sees that on the light cone of root system $(a^2 = 0)$, the operator $a \hbar$ has zero eigenvalues. This is clearly seen on above relation by taking $a = b$ and restricting to the light cone of $\Delta_{hyp}$ where $a^2 = 0$. But this is exactly what we need to overpass the pole singularity we have referred to above. This is the reason behind our qualification of the singularity in

$$\frac{2}{a^2} a \hbar = \frac{2}{2mn - l^2 a^2} (nL + mK - l \alpha H), \quad (6.8)$$

is an apparent pole. On light cone of $\Delta_{hyp}$, the eigenvalue of $2a^{-2} a \hbar$ behaves as $0/0$; and the indetermination is lifted by replacing $\frac{2}{a^2} a \hbar$ just by $2$. Observe that for ordinary ADE Lie subalgebras recovered by taking $m = n = 0$, the above relation reduces to the usual one namely,

$$\frac{2}{a^2} a \hbar \rightarrow \frac{2}{a^2} \alpha H, \quad (6.9)$$

and has no pole. The same result is also valid for affine KM subalgebras $\mathfrak{g}_{affine}^\delta$ and $\mathfrak{g}_{affine}^\gamma$ recovered by setting $mn = 0$ and $m + n \neq 0$. The first corresponds to setting $l = \pm 1$, $m = 0$ and the second to
taking \( l = \pm 1, n = 0 \). In both cases, eq(6.8) reduces to eq(6.9). This explains why there is no pole ambiguity in affine KM algebras. Apparent poles are then a special property to hyperbolic extensions.

Now we are in position to write down the commutation relations for hyperbolic Lie algebras. We start by the interpolating method and then we consider the covariant approach.

### 6.2 Interpolating Method

To write down the commutation relations of hyperbolic extension of ADE Lie algebras, we proceed in three steps as follows: (1) Start by identifying the usual commutation relations associated with roots in \( \Delta_{finite} \); finite dimensional Lie algebras as subalgebras of the hyperbolic one. (2) We consider those commutation relations associated with the two special affine \( \Delta^\delta_{affine} \) and \( \Delta^\gamma_{affine} \) subsets. These commutation relations describe the two particular isomorphic affine KM ADE subalgebras \( \mathfrak{g}^\delta_{affine} \) and \( \mathfrak{g}^\gamma_{affine} \) within hyperbolic ADE generalization. (3) Finally we work out the commutation relations defining hyperbolic ADE algebras by using interpolation idea between \( \mathfrak{g}^\delta_{affine} \) and \( \mathfrak{g}^\gamma_{affine} \).

#### 6.2.1 Hyperbolic algebra by gluing pieces

One of the useful things we have learned from the study of root system \( \Delta_{hyp} \) is that step operators \( S^{\pm a}_{l,m,n} \) of hyperbolic ADE Lie algebras carry in general three quantum numbers as

\[
S^{\pm a}_{l,m,n}, \quad (6.10)
\]

with \( l = 0, \pm 1 \) and the two other integers \( m \) and \( n \) are such that \( l^2 - nm \leq 1 \). For convenience, we split these step operators into two subsets \( \{ E^{\pm a}_{l,m,n} \} \) and \( \{ H^i_{m,n} \} \) according to whether \( l = \pm 1 \) or \( l = 0 \). So \( E^{\pm a}_{l,m,n} \) are the step operators associated with the hyperbolic roots \( \pm a = \pm (n\gamma + m\delta + \alpha) \) having an \( \alpha \) dependence (\( \pm \alpha \in \Delta_{finite} \)) and \( H^i_{m,n} \) are step operators with no explicit \( \alpha \) dependence as they are associated with roots type \( a = n\gamma + m\delta \). The extra upper index \( i = 1, \ldots, r \) is related to the rank of simple roots \( \alpha_i \) in the underlying finite dimensional ADE subalgebra. In addition to the \( L, K \) and \( H_a = \alpha H \) commuting Cartan observables, the generators of the hyperbolic ADE algebras we are after are then,

\[
E^{\alpha}_{m,n}; \quad H^i_{m,n}; \quad \Leftrightarrow \quad a = n\gamma + m\delta + \alpha, \quad a = n\gamma + m\delta, \quad (6.11)
\]

where obviously \( \alpha \in \Delta_{finite} \) and where the \( m \) and \( n \) integers are such that \( mn \geq l^2 - 1 \). We also have the following adjoint conjugation condition \( (E^\alpha_{m,n})^\dagger = E^{-\alpha}_{-m,-n} \) and \( (H^i_{m,n})^\dagger = H^i_{-m,-n} \) useful in the study of the unitary highest weight representations of hyperbolic ADE algebras. The above set of operators contain proper subsets that we know quite well as they generate ordinary ADE subalgebras and their affine extensions. These sets constitute particular solutions of the constraint eq \( mn \geq l^2 - 1 \) and are as follows: (i) finite dimensional subset and two affine KM subalgebras.

**Finite dimensional piece** The finite dimensional piece is generated by the usual zero mode subset of \( S^{\pm a}_{m,n} \) corresponding to \( l = \pm 1 \) and \( m = n = 0 \); that is,

\[
\alpha_i H = H^i_{0,0}; \quad E^\alpha_{0,0}; \quad \alpha \in \Delta_{finite}. \quad (6.12)
\]
These operators generate the ordinary ADE Lie subalgebras and their commutation relations are as follow,

\[
\begin{align*}
[H_{0,0}^i, H_{0,0}^j] &= 0, & [H_{0,0}^i, E_{0,0}^\alpha] &= \alpha^i E_{0,0}^\alpha, \\
[E_{0,0}^\alpha, E_{0,0}^\beta] &= \varepsilon_{\alpha\beta} E_{0,0}^{\alpha+\beta}, & \alpha, \beta, \alpha + \beta \in \Delta_{finite}, \\
[E_{0,0}^\alpha, E_{0,0}^{-\alpha}] &= \frac{2}{(\alpha, \alpha)} \alpha H_{0,0} = \alpha H_{0,0}, \\
[H_{0,0}^i, E_{0,0}^\beta] &= 0, & \text{otherwise,}
\end{align*}
\]

where \(\varepsilon_{\alpha\beta}\) is the usual antisymmetric tensor. Observe in passing that the third eq of these relations involves the bilinear form \((\alpha, \alpha) = \alpha^2\), which in a covariant description it should be read as \(\alpha^2 - 2mn\); but this is exactly \(\alpha^2\) since in this case \(mn = 0\). Along with this affine KM symmetry, there is also a second proper affine subset which is isomorphic the above one with roles of integers \(m\) and \(n\) interchanged. It corresponds to taking \(l = 0, \pm 1\) and \(m = 0\) but \(n\) an arbitrary integer. In this case the generators are,

\[
L; \quad mK; \quad \alpha_i H = H_{m,0}^i; \quad E_{m,0}^\alpha; \quad \alpha \in \Delta_{finite}, \quad m \in \mathbb{Z},
\]

and their commutation relations read as follows:

\[
\begin{align*}
[H_{m,0}^i, H_{p,0}^j] &= -mK \delta^{ij} \delta_{m+p}, \\
[H_{m,0}^i, E_{p,0}^\alpha] &= \alpha^i E_{m+p,0}^\alpha, \\
[E_{m,0}^\alpha, E_{p,0}^\beta] &= \varepsilon_{\alpha\beta} E_{m+p,0}^{\alpha+\beta}, & \alpha, \beta, \alpha + \beta \in \Delta_{finite}, \\
[E_{m,0}^\alpha, E_{p,0}^{-\alpha}] &= \frac{2}{(\alpha, \alpha)} (\alpha H_{m+p,0} - mK \delta_{m+p}), \\
[H_{m,0}^i, E_{p,0}^\beta] &= 0, & \text{otherwise,}
\end{align*}
\]

Note that here also the fourth eq of these commutation relations involves the bilinear form \((\alpha, \alpha) = \alpha^2\), which in a covariant description it should be read as \(\alpha^2 - 2mn\); but this is exactly \(\alpha^2\) since in this case \(mn = 0\). Along with this affine KM symmetry, there is also a second proper affine subset which is isomorphic the above one with roles of integers \(m\) and \(n\) interchanged. It corresponds to taking \(l = 0, \pm 1\) and \(m = 0\) but \(n\) an arbitrary integer. In this case the generators are,

\[
nL; \quad K; \quad \alpha_i H = H_{0,n}^i; \quad E_{0,n}^\alpha; \quad \alpha \in \Delta_{finite}, \quad n \in \mathbb{Z},
\]

and the corresponding commutation relations are given by,

\[
\begin{align*}
[H_{0,n}^i, H_{0,q}^j] &= -nL \delta^{ij} \delta_{n+q}, \\
[H_{0,n}^i, E_{0,q}^\alpha] &= \alpha^i E_{0,q+n}^\alpha, \\
[E_{0,n}^\alpha, E_{0,q}^\beta] &= \varepsilon_{\alpha\beta} E_{0,q+n}^{\alpha+\beta}, & \alpha, \beta, \alpha + \beta \in \Delta_{finite}, \\
[E_{0,n}^\alpha, E_{0,q}^{-\alpha}] &= \frac{2}{(\alpha, \alpha)} (\alpha H_{0,q+n} - nL \delta_{q+n}), \\
[H_{0,n}^i, E_{0,q}^\beta] &= 0, & \text{otherwise,}
\end{align*}
\]

\[
\begin{align*}
[L, H_{0,n}^j] &= [L, E_{0,n}^\alpha] = [L, K] = 0, \\
[K, H_{0,n}^i] &= nH_{0,n}^i; \quad [K, E_{0,n}^\alpha] = nE_{0,n}^\alpha.
\end{align*}
\]
Here also $\alpha^2 - 2mn$ reduces to $\alpha^2$ since $mn = 0$.

### 6.2.2 Necessary conditions for unitary HWRs

Necessary conditions for unitary highest weight representations of above hyperbolic algebra may be obtained as usual by looking at the conditions following from the highest weight representations of its $su(2)$ subalgebras,

$$ [I_+, I_-] = 2I_3; \quad [I_3, I_{\pm}] = \pm I_{\pm} $$

Since there are as many $su(2)$ subalgebras as roots $\alpha = n\gamma + m\delta + \alpha$ in the hyperbolic system $\Delta_{hyp}$, we can write down the corresponding unitary conditions. These are be obtained by requiring that the eigenvalues of $2I_3$, on generic weight vectors $|k, l, \mu >$ of the representation space, have to be integral. Here $|\mu >$ is a generic weight vector of the underlying ordinary ADE subalgebras; it is obtained from a highest weight vector $|\lambda >$ by acting by step operators monomials as,

$$ |\mu >= \prod_{\beta \in \Delta_{finite}} E^{-\beta}_{0,0} |\lambda >; \quad E^0_{0,0} |\lambda >= 0; \quad \alpha H|\lambda >= \alpha|\lambda >, $$

where $\Delta_{finite}$ are the finite roots of the hyperbolic system $\Delta_{hyp}$. Here also $\alpha^2 - 2mn$ reduces to $\alpha^2$ since $mn = 0$.
with $\beta \in \Delta_{finite}$. Moreover as there are two kinds of $\text{su}(2)$ subalgebras in eq (6.19), according to whether $l = 0$ or $l = \pm 1$, it follows that one can write down two types of unitary necessary conditions. Unitary conditions coming from the block,

$$[H_{m,n}^+, H_{m,-n}^-] = -(nL + mK), \quad (6.22)$$

interpreted as $[I_+, I_-] = 2I_3$ with no $\alpha H_{0,0}$ term and others coming from

$$[E_{m,n}^\alpha, E_{-m,-n}^{-\alpha}] = \frac{2}{\alpha^2 - 2mn} (\alpha H_{0,0} - mK - nL). \quad (6.23)$$

In the second case, we have $I_+ = E_{m,n}^\alpha$, $I_- = E_{-m,-n}^{-\alpha}$ and $2I_3 = \frac{2}{\alpha^2} \hbar h$. Later on, we will see how both of these relations can be put altogether by help of eq (6.6), but for the moment note that part of necessary conditions for unitary highest weight representations of hyperbolic ADE algebras can be immediately written down. The point is that as affine KM symmetries are subalgebras of hyperbolic ADE extension, we should at least have the usual unitary conditions on the eigenvalues $k, l$ and $\alpha \mu$,

$$k = <k, l, \mu | K|k, l, \mu>, \quad l = <k, l, \mu | L|k, l, \mu>, \quad (6.24)$$

$$\alpha \mu = <k, l, \mu | \alpha H_{0,0}|k, l, \mu>.$$

of the operators $K, L$ and $\alpha H$ respectively,. Indeed considering the case where $l = \pm 1$ and setting $m = -1$ and $n = 0$ (resp $m = 0$ and $n = -1$) in eq (6.23), one sees that the triplet $(E_{-1,0}^\alpha, E_{1,0}^{-\alpha}, 2(K - \alpha H_{0,0})/\alpha^2)$ (resp $(E_{-1,0}^\alpha, E_{1,0}^{-\alpha}, 2(L - \alpha H_{0,0})/\alpha^2)$) form an $\text{su}(2)$ algebra and so the eigenvalues of $2(K - \alpha H_{0,0})/\alpha^2$ (resp $2(L - \alpha H_{0,0})/\alpha^2$) must be integral. Therefore a first set of necessary conditions reads as,

$$2(k - \alpha \mu) \in \alpha^2 \mathbb{Z}; \quad \alpha \in \Delta_{finite},$$

$$2(l - \alpha \mu) \in \alpha^2 \mathbb{Z}; \quad \alpha \in \Delta_{finite}. \quad (6.25)$$

Moreover, for highest weight states $|k, l, \lambda>$ satisfying amongst others $E_{-1,0}^{-\alpha}|k, l, \lambda> = 0$, these conditions can be reduced further. The idea is that since the commutator $[E_{-1,0}^\alpha, E_{1,0}^{-\alpha}]$ on the HW vector $|k, l, \lambda>$ is positive because $<\lambda [E_{-1,0}^\alpha, E_{1,0}^{-\alpha}] | \lambda> = \|E_{-1,0}^\alpha | \lambda> \|^2$, we should also have $k \geq \alpha \lambda \geq 0$ (resp $l \geq \alpha \lambda \geq 0$). Therefore, we have the conditions $2\alpha \lambda \in \alpha^2 \mathbb{Z}$, and

$$2k \in \psi^2 \mathbb{Z}_+; \quad k \geq \psi \lambda,$$

$$2l \in \psi^2 \mathbb{Z}_+; \quad l \geq \psi \lambda, \quad (6.26)$$

where $\psi$ is the usual maximal root. Along with these constraint eqs, there are further constraint eqs coming from the other $\text{su}(2)$s within the hyperbolic ADE algebras. We will complete this discussion by giving the general necessary conditions for unitary highest weight representations after discussing the covariant approach for hyperbolic algebras.

### 6.3 Covariant method

Now that we know that the problem of indefinite signature of the bilinear form is not essential since the pole is just an apparent algebraic singularity at least at the Lie algebraic level, we can now proceed to write down the commutation relations for hyperbolic ADE Lie algebras using a covariant method. Before going into details, it is interesting to note that despite similarities, hyperbolic ADE algebras differ from what we customary have in ordinary and affine symmetries. There, roots $\alpha$ have a finite number of lengths; one length for ordinary ADE and for affine ADE there are two kinds of root norms;
\(a^2\) is either two or zero. In hyperbolic ADE, there is an infinite number of possible norms and as we have seen this is because of the indefinite signature of the bilinear form.

In the covariant method, we will use the generators,

\[
S^a = S^{\alpha}_{m,n}; \quad mn \geq l^2 - 1; \quad l = 0, \pm 1, \quad (6.27)
\]

instead \(H_{m,n}^i\) and \(E_{p,q}^\alpha\) used in the interpolation approach. We fist give the general form of the commutation relations of our hyperbolic algebras by using to different but equivalent ways. Then we complete the discussion on unitary HWRs initiated before.

### 6.3.1 Standard basis

Results from theorem and corollary of section 4 tell us that for any pair of hyperbolic roots \(a = n\gamma + m\delta + l\alpha\) and \(b = q\gamma + p\delta + j\alpha\) belonging to \(\Delta_{hyp}\) system, we have two kinds of Lie algebra generators. The step operators

\[
\begin{align*}
S^a &= S^{\alpha}_{m,n}; \quad S^{-a} = S^{-\alpha}_{m,-n}; \quad l = 0, \pm 1; \\
S^b &= S^{\beta}_{p,q}; \quad S^{-b} = S^{-\beta}_{p,-q} \quad j = 0, \pm 1,
\end{align*}
\]

(6.28)
carrying the familiar three quantum numbers \(n, m\) and \(l\) and the \(q, p\) and \(j\) analogue and the usual commuting Cartan generators \(L, K\) and \(H\) which, by help of the bilinear form, can be rewritten in a compact form as,

\[
\begin{align*}
ah &= -nL - mK + l\alpha H; \quad l = 0, \pm 1, \\
bh &= -qL - pK + j\alpha H; \quad j = 0, \pm 1.
\end{align*}
\]

(6.29)
The commutation relations obeyed by these operators are easily derived; they follow from the structure of the root system \(\Delta_{hyp}\) and are as follows,

\[
\begin{align*}
[a h, b h] &= 0, \quad a, b \in \Delta_{hyp}, \\
[S^a, S^b] &= \varepsilon_{\alpha\beta} S^{a + b}\quad a, b, a + b \in \Delta_{hyp}, \\
[S^a, S^{-a}] &= \frac{2}{a^2} ah, \quad a \in \Delta_{hyp}, \\
[a h, S^b] &= (ab) S^b, \quad a, b \in \Delta_{hyp}, \\
[S^a, S^b] &= 0, \quad a, b \in \Delta_{hyp}, \quad a + b \notin \Delta_{hyp}.
\end{align*}
\]

(6.30)

With help of these eqs, one can go ahead an write down the commutation relations in terms of \(S^{\alpha}_{m,n}\) modes. Using the correspondence eqs(6.29) and substituting \(S^a\) and \(S^b\) by \(S^{\alpha}_{m,n}\) and \(S^{\beta}_{p,q}\) with \((l^2 - mn) \leq 1\) and \((j^2 - pq) \leq 1\), we find,

\[
\begin{align*}
[K, H_{0,0}] &= [H_{0,0}, L] = [K, L] = 0, \\
[S^{\alpha}_{m,n}, S^{\beta}_{p,q}] &= \varepsilon_{\alpha\beta} S^{\alpha + \beta}_{m+n,p+q}; \quad \left(l^2 - mn\right) \leq 1, \left(j^2 - pq\right) \leq 1, (l\alpha + j\beta)^2 - 2(m + p)(n + q) \leq 2, \\
[S^{\alpha}_{m,n}, S^{-\alpha}_{m,-n}] &= \frac{2}{(l^2\alpha^2 - 2mn)} (l\alpha H - nL - mK), \quad \left(l^2 - mn\right) \leq 1, \\
[\alpha H, S^{\beta}_{p,q}] &= j (\alpha \beta) S^{\beta}_{p,q}; \quad \left(j^2 - pq\right) \leq 1, \\
[K, S^{\beta}_{p,q}] &= q S^{\beta}_{p,q}; \quad \left(j^2 - pq\right) \leq 1, \\
[L, S^{\beta}_{p,q}] &= p S^{\beta}_{p,q}; \quad \left(j^2 - pq\right) \leq 1, \\
[S^{\alpha}_{m,n}, S^{\beta}_{p,q}] &= 0, \quad \left(l^2 - mn\right) \leq 1, \left(j^2 - pq\right) \leq 1, (l\alpha + j\beta)^2 - 2(m + p)(n + q) > 2,
\end{align*}
\]

(6.31)
with \( l, j = 0, \pm 1 \) and \( \alpha, \beta \in \Delta_{finite} \). The conditions on the numbers \( n, m \) and \( l \) and the \( q, p \) and \( j \) ensure that roots \( a, b \) and \( a + b \) belong indeed to root system of hyperbolic algebra. By setting \( S_{m,n}^0 = \sqrt{\gamma} \sum_{i=1}^r H_{m,n}^i \) and \( S_{m,n}^0 = E_{m,n}^r \), it is not difficult to see that above commutation relations are same as those in eqs. Before preceding further, we would like to make two comments on this hyperbolic algebra. The first comment concerns unitary conditions for HWRs and the second deals with link with torus fibration of ordinary ADE.

**Unitary HWRs** Viewed as quantum field theoretical symmetry, the above hyperbolic algebra seems to have a rich physical spectrum since it has two remarkable branches; a standard branch and a new one with no analogue in usual affine KM symmetries. Indeed, unitary highest weight representations of this algebras require operators \( a \) and \( S^a \),

\[
(a \lambda) = ah; \quad (S^a) = S^{-a}. \tag{6.32}
\]

Acting by these operators on weight states \( |x\rangle (|x\rangle = |l\gamma + k\delta + \mu\rangle) \) of hyperbolic weight lattice, we have,

\[
ah|x\rangle = ax|x\rangle; \quad S^{-a}|x\rangle = |x + a\rangle, \tag{6.33}
\]

where the real number \( ax \) is expressed in terms of the eigenvalues \( l, k \) and \( \alpha \mu \) as \( ax = (-lm - kn + l\alpha\mu) \). Unitary conditions for highest weight representations of the hyperbolic algebra are obtained as before by considering the unitary conditions for HW representations of its \( su(2) \) subalgebras on highest weight states

\[
|y\rangle = |l\gamma + k\delta + \lambda\rangle. \tag{6.34}
\]

Here the state \(|\lambda\rangle\) is the same as that we have used before; it satisfies \( E_{0,0}^\alpha|\lambda\rangle = 0 \) and \( aH|\lambda\rangle = (a\lambda)|\lambda\rangle \) with \( \alpha \in \Delta_{finite}, -2\alpha a \sum Z_+ \). The state \(|y\rangle\) is then just the generalization of ordinary \(|\lambda\rangle\) to the of the hyperbolic weight lattice. It satisfies then,

\[
\frac{2}{a^2} a\lambda|y\rangle = \frac{2ay}{a^2}|y\rangle; \quad S^{-a}|y\rangle = |y + a\rangle; \quad S^a|y\rangle = 0. \tag{6.35}
\]

Applying the \( su(2) \) subalgebra equation \( [S^a, S^{-a}] = \frac{2}{a^2} a\lambda \) on this highest weight state \(|y\rangle\), unitary conditions for HW representations of hyperbolic ADE Lie algebras read in general as

\[
\frac{2ay}{a^2} \in Z_+ \quad \tag{6.36}
\]

for any root \( a \in \Delta_{hyp} \). Using the explicit expressions of \( a \) and \( y \), the above condition can be also rewritten as follows,

\[
\frac{2ay}{a^2} = \frac{l (\alpha \lambda) - lm - kn}{l^2 \alpha^2 - 2mn} \in Z_+ \quad \text{for any } (n, m, l) \in L. \tag{6.37}
\]

Note that for \( l = 1 \) and according to whether \( m \neq 0, n = 0 \) or \( m = 0, n \neq 0 \), one respectively discovers the usual conditions one has in the case of unitary highest weight representations of finite dimensional ADE Lie algebras and their affine extensions. But this is we know and is not a new thing. Novelty comes rather from the indefinite signature of the bilinear form leading to two branches since positivity of eq. can be solved in two different ways; either by requiring \( ay \geq 0 \) and \( a^2 > 0 \) or \( ay \leq 0 \) and \( a^2 < 0 \). Put differently, eq. is positive if the two following constraint eqs are fulfilled,

\[
l (\alpha \lambda) - lm - kn \in Z_+; \quad \text{and} \quad l^2 \alpha^2 - 2mn \in Z_+^*,
\]

\[
l (\alpha \lambda) - lm - kn \in Z_-; \quad \text{and} \quad l^2 \alpha^2 - 2mn \in Z_-^*. \tag{6.38}
\]

The first eq is expected as it deals with the deeply Euclidean region \((a^2 > 0)\) of the hyperbolic root system. The second eq in above relation has no analogue in affine KM symmetries since \((a^2 < 0)\); it
captures then the signature of the hyperbolic structure. Note that along with these two well defined branches, there is moreover a third special case corresponding to \(a^2 = 0\). This situation is unclear and deserves more attention.

**Torus fibration** The comment we give here deals with the possible link between hyperbolic algebra we have been considering and ordinary ADE algebras fibered on two torus. The point is that our hyperbolic algebra contains two copies of affine KM symmetries. Each one can be viewed as an ordinary ADE algebra fibered on a \(S^1\) cycle. Indeed as we have the habit to do in two dimensional conformal field theory, the \(E^{\pm,\alpha}_{m,0}\) and \(\alpha H_{m,0}\) operators (resp \(E^{\pm,\alpha}_{0,n}\) and \(\alpha H_{0,n}\)) have a nice holomorphic field realization using operator product expansion. There \(E^{\pm,\alpha}_{m,0}\) and \(\alpha H_{m,0}\) appear as just the Laurent modes expansion of holomorphic conserved currents \(E^{\pm,\alpha}(z)\) and \(\alpha H(z)\). These are just a fibration of ordinary ADE generators \(E^{\pm,\alpha}\) and \(\alpha H\) on a \(S^1\) cycle as shown below,

\[
E^{\pm,\alpha}(z) = \sum_{m \in \mathbb{Z}} z^{-m-1} E^{\pm,\alpha}_{m,0}; \quad E^{\pm,\alpha}_{m,0} = \oint_{z} \frac{dz}{2i\pi} z^m E^{\pm,\alpha}(z),
\]

\[
\alpha H(z) = \sum_{m \in \mathbb{Z}} z^{-m-1} \alpha H_{m,0}; \quad \alpha H_{m,0} = \oint_{z} \frac{dz}{2i\pi} z^m \alpha H(z),
\]

(6.39)

where, roughly speaking, the complex \(z\) variable should be thought of as \(|z| = 1\). Similar relations may be written down for \(E^{\pm,\alpha}_{0,n}\) and \(\alpha H_{0,n}\) operators generating the other copy of affine KM subsymmetry within hyperbolic ADE algebra. The corresponding Laurent expansion reads as follows,

\[
E^{\pm,\alpha}(w) = \sum_{n \in \mathbb{Z}} w^{-n-1} E^{\pm,\alpha}_{0,n}; \quad E^{\pm,\alpha}_{0,n} = \oint_{w} \frac{dw}{2i\pi} w^n E^{\pm,\alpha}(w),
\]

\[
\alpha H(w) = \sum_{n \in \mathbb{Z}} w^{-n-1} \alpha H_{0,n}; \quad \alpha H_{0,n} = \oint_{w} \frac{dw}{2i\pi} w^n \alpha H(w),
\]

(6.40)

where now \(w\) parameterizes the second \(S^1\) cycle. Using interpolating ideas, one may be tempted to think about hyperbolic ADE extension as a fibration of ordinary ADE generators \(E^{\pm,\alpha}\) and \(\alpha H\) on a two torus \(T^2\). Unfortunately this is however not true since if it was the case the conserved currents would be bi-holomorphic functions \(S^{\pm,\alpha}(z, w)\) with Laurent modes

\[
S^{\pm,\alpha}_{m,n} = \oint_{z} \frac{dz}{2i\pi} z^m \left( \oint_{w} \frac{dw}{2i\pi} w^n S^{\pm,\alpha}(z, w) \right),
\]

(6.41)

defined whatever the \(m\) and \(n\) integers are. But this is in disagreement with the constraint eq on hyperbolic roots namely,

\[
mn \geq (l^2 - 1); \quad l = 0, \pm 1.
\]

(6.42)

In the expansion (6.41), there are more mode operators than allowed by the above condition. It would be interesting to work out the precise relation between hyperbolic algebras and torus fibration of finite dimensional Lie algebras. Seen the difficulty brought by the root constraint eq \(a^2 \leq 2\) hyperbolic extension of ADE symmetries, we will develop in what follows a way to overpass, at least, the technical aspect of this problem.

**6.3.2 New Basis**

Besides indefinite signature of the bilinear form, the second difficulty in handling hyperbolic algebra comes from the constraint eq \(a^2 \leq 2\) on allowed roots in \(\Delta_{hyp}\). The point is that contrary to affine KM symmetries, where the solutions for affine constraint eq,

\[
a^2 = (l^2 a^2 - 2mn) = 0, 2,
\]

(6.43)
put no limit on \( m \) and \( n \) integers, the situation is different in hyperbolic symmetries. General solutions put however constraints on the allowed values of \( m \) and \( n \). This Lie algebraic property translates in terms of 2D CFTs as a constraint eq on the Laurent expansion of bi-holomorphic functions. A way to insert this behaviour in the hyperbolic game is to implement the constraint eq \( a^2 = (l^2 \alpha^2 - 2mn) \leq 2 \) in the generator basis. Instead of the \( S^a \) covariant generators considered above, we use rather the new following ones,

\[
T^a = Y \left( a^2 - 2 \right) S^a,
\]

(6.44)

where \( Y(x) \) is the Heaviside like distribution defined as,

\[
Y(x) = 1 \quad \text{if} \quad x \leq 0 \quad \text{and} \quad Y(x) = 0 \quad \text{if} \quad x > 0.
\]

(6.45)

In this weighted basis \{\( T^a \}\}, the condition \( a^2 \leq 2 \) is automatically implemented and there is no need to specify at it a each time. In term of modes, the relation \( 6.44 \), reads as,

\[
T^a_{m,n} = Y \left( l^2 \alpha^2 - 2mn - 2 \right) S^a_{m,n},
\]

(6.46)

with \( l = 0, \pm 1 \). Similar relations are also valid for the covariant Cartan generator \( aH \); but for simplicity we will continue to refer to the normalized operator \( Y \left( a^2 - 2 \right) aH \) in same manner before. Now using these new \( T^a \) basis operators, one can write down the covariant expression of the commutation relations of the hyperbolic extension of the ADE Lie algebras. As before, these relations involve the indefinite bilinear form \( (,\) of the hyperbolic algebra; but in addition the \( Y(x) \) distribution as shown below,

\[
[aH, bb] = 0, \\
[aH, T^b] = ab T^b, \\
[T^a, T^b] = Y \left( a^2 - 2 \right) Y \left( b^2 - 2 \right) \varepsilon_{ab} T^{a+b}, \\
[T^a, T^{-a}] = \frac{2Y \left( a^2 - 2 \right)}{(a,a)} aH,
\]

(6.47)

Clearly, these commutation relations are linear, antisymmetric, closed and verify the Jacobi identity. Now replacing the roots by their explicit expressions; i.e \( a = n\gamma + m\delta + l\alpha, b = q\gamma + p\delta + j\beta \) and the sum \( a + b \) by \( (n + q) \gamma + (m + p) \delta + (l\alpha + j\beta) H \) and doing the same for the step operators \( T^a, T^b \) and \( T^{a+b} \) which get replaced by \( T^a_{m,n}, T^b_{p,q} \) and \( T^{(a+j\beta)}_{m+p,n+q} \) as in eqs(6.39), one can rewrite down the above commutation relations in terms of the mode operators \( T^a_{m,n} \) and the Cartan generators namely \( nL, mK \) and \( l\alpha H \). We find,

\[
[L, K] = [K, aH] = [aH, \beta H] = 0, \\
[aH, T^\beta_{p,q}] = j (a\beta) T^{\pm\beta}_{p,q}, \\
[K, T^\beta_{p,q}] = q T^\beta_{p,q}, \\
[L, T^\beta_{p,q}] = p T^\beta_{p,q}, \\
[T^a_{m,n}, T^\beta_{p,q}] = \frac{Y \left( l^2 - mn - 1 \right) Y \left( j^2 - pq - 1 \right) \varepsilon_{a,j\beta}}{Y \left( (l\alpha + j\beta)^2 - 2(m + p) (n + q) - 2 \right)} T^{(a+j\beta)}_{m+p,n+q}, \\
[T^\beta_{m,n}, T^{-a}_{m,n}] = \frac{2Y \left( l^2 - mn - 1 \right)}{2 mn - l^2 \alpha^2} (nL + mK - l\alpha H),
\]

(6.48)
with \( l = 0, \pm 1, \ j = 0, \pm 1, \ \alpha,\beta \in \Delta_{\text{finite}} \) and where \( l\alpha + j\beta \) should be as \( s\eta \) with \( s = 0, \pm 1 \) and \( \eta \in \Delta_{\text{finite}} \). From these relations, one recognizes the above mentioned subalgebras and the unitary conditions for highest weight representations derived before. In this way of doing, eq (6.41) extends as,

\[
T_{m,n}^{l\alpha} = \oint z^m dz \left( \oint w^n Y \left( i^2 - mn - 1 \right) T_{z,w}^{l\alpha} \right). \tag{6.49}
\]

where now \( T_{z,w}^{l\alpha} \) are bi-holomorphic conserved currents. It would be interesting to put this change back in the algebra (6.48) and try to work out the operator product expansion that defines the infinite dimensional hyperbolic ADE algebras. This is might be a way to study field theoretical deformations of two dimensional conformal field theories with hyperbolic symmetries.

7 Weyl Groups

In this section, we want to study the structure of Weyl group \( W_{\text{hyp}} \) of the hyperbolic ADE Lie algebras we constructed above. Our interest to this group comes from recent applications of such kind of structure in the context of supersymmetric field theories embedded in Type II string compactification on CY threefolds with ADE singularities. There, the so called Seiberg like dualities and RG cascades were shown to have a remarkable interpretation in terms of Weyl transformations. RG cascades, which do exist in type II strings on CY with affine ADE singularities, were also shown to be linked with translation symmetries within affine Weyl groups. We suspect therefore that natural extensions of these Weyl symmetries to hyperbolic Weyl groups would also have interpretations in the context of generalized quiver gauge theories such those recently derived in [19, 50].

7.1 Strategy towards \( W_{\text{hyp}} \)

As in affine ADE Lie algebras, \( W_{\text{hyp}} \) groups of hyperbolic ADE Lie algebras are not defined for all roots \( a \) of \( \Delta_{\text{hyp}} \) just because generic Weyl reflections \( \omega_a \) on elements \( x \) of the space \( h^* \) are not usually defined. From the following typical Weyl transformation \( \omega_a \) \( x = - \frac{2}{a(a,a)} a \), we see that this has no sense for light like roots \( a \) having zero norm \( a^2 = 0 \). Thus like in affine case, \( W_{\text{hyp}} \) is partially generated by Weyl reflections of finite dimensional Lie algebras and certainly by translations which still need to be studied.

It also has to leave stable the full hyperbolic root system \( \Delta_{\text{hyp}} \).

To get the complete structure of the Weyl group of hyperbolic ADE Lie algebras, we will follow the philosophy we’ve used in the building of \( \Delta_{\text{hyp}} \): that is by doing things step by step. First by using the known results on \( W_{\text{affine}}^{\delta} \) and \( W_{\text{affine}}^{\gamma} \) respectively associated with the two proper subsets \( \Delta_{\text{affine}}^{\delta} \) and \( \Delta_{\text{affine}}^{\gamma} \) of \( \Delta_{\text{hyp}} \). Then by taking advantage of the natural embeddings,

\[
W_{\text{hyp}} \supset W_{\text{affine}}^{\delta} \supset W_{\text{finite}}, \tag{7.1}
\]

and

\[
W_{\text{hyp}} \supset W_{\text{affine}}^{\gamma} \supset W_{\text{finite}}, \tag{7.2}
\]

as well as symmetries under the inter-change of the imaginary roots \( \delta \) and \( \gamma \). Like in the derivation of root system and the commutation relations, this way of doing allows us to define \( W_{\text{hyp}} \) as a parametric group interpolating between \( W_{\text{affine}}^{\delta} \) and \( W_{\text{affine}}^{\gamma} \).

7.2 Interpolation Method

From Lie algebraic view, Weyl groups are symmetries of root system of underlying Lie algebras generally acting on vectors \( x \) of the dual space \( h^* \) by shifting it as \( x + \mu \) \( x \). The shift vector \( \mu \) \( x \) is linear in \( x \)
and its explicit expression depends on the dimension of the Lie algebra. For the special case of finite dimensional ADE Lie algebras where all roots are space like and have positive definite norms $a^2 = 2$, Weyl group $W_{finite}$ is a discrete and finite order symmetry generated by fundamental reflections $\omega_i$. The latters are associated with simple roots $\alpha_i$ and act on $h^*$ as,

$$\omega_i (x) = x - 2\frac{(a_i, x)}{(a_i, a_i)}a_i; \quad i = 1, ..., r,$$

which can be further simplified because of the identity $(a_i, a_i) = 2$. These particular $\omega_i$s obey $\omega_i^2 = I_{id}$ and exhibit special features amongst which we give the three following:

(i) Non commutative transformations: Though look like translations, transformations (7.3) do not commute since the composition of two reflections $\omega_j$ and $\omega_i$ involves three terms; two of them namely $(a_j, x)a_j + (a_i, x)a_i$ are symmetric as they interchanged under the operation $a_i \leftrightarrow a_j$ but the third one $(a_j, a_i)(a_i, x)a_j$ one does not. This property is due to the fact that the shift $(a_i, x)a_i$ is non linear in roots; a property which yields non commutativity of Weyl transformations.

(ii) Group law: Weyl group of finite Lie algebras has a finite order; reflections, defined for each node of Dynkin diagram, verify in general the law $(R_i R_j)^{n_{ij}} = I_{id}$ for some positive integers $n_{ij}$ depending on the nature of the Lie algebra. It turns out that these $n_{ij}$s are related with the number of link $K_{ij}K_{ji}$ from $\alpha_i$ to $\alpha_j$ nodes of Dynkin diagram. For a given order $n$ Cartan matrix $K_{ij}$ for instance, we have the results: $n_{ij} = 2$ for the case $K_{ij}K_{ji} = 0$, $n_{ij} = 3$ for $K_{ij}K_{ji} = 1$, $n_{ij} = 4$ for $K_{ij}K_{ji} = 2$ and $n_{ij} = 6$ for $K_{ij}K_{ji} = 3$. These imply, amongst others, that Weyl groups $W_{finite}$ ($A_r$) and $W_{finite}$ ($D_r$) are respectively isomorphic to $S_{r+1}$ and $Z^2 \times S_r$; where $S_n$ stands for permutation group.

(iii) Fix points: A third property, which turns out to be useful in the derivation of $W_{affine}$ and $W_{hyp}$, is that Weyl generators $\omega_i$s leave the imaginary light like roots $\delta$ and $\gamma$ as well as their linear combinations $\rho \delta + q \gamma$ invariants,

$$\omega_i (\rho \delta) = \rho \delta; \quad \omega_i (q \gamma) = q \gamma; \quad \omega_i (\rho \delta + q \gamma) = \rho \delta + q \gamma,$$

These kind of roots are obviously absent in finite dimensional Lie algebras; but appear in affine and hyperbolic root system and turn out to be a basic role.

Note that because of nilpotency of the norm of $\delta$ and $\gamma$ ($\delta^2 = \gamma^2 = 0$), there is no corresponding Weyl transformation $\omega_\delta$ nor $\omega_\gamma$. Instead of these, there is rather extra Weyl transformations that have no counterpart in finite dimensional Lie algebras. These extra symmetries are generated by translations in the hyperbolic root lattice or more generally in the real restriction of the space of dual forms $h^*$. Translations distinguish $W_{affine}$ from $W_{finite}$ and allow to factories $W_{affine}$ into a semi direct product as follows,

$$W_{affine}^\delta = W_{finite} \ltimes T_\delta.$$

In this relation $W_{affine}^\delta$ stands for affine Weyl group leaving invariant the affine root subsystem $\Delta_{affine}^\delta \cup \emptyset = \{ m\delta + l\beta; l = 0, \pm 1, m \in Z \}$ with $\beta \in \Delta_{finite}$ and $T_\delta = \{ t_\alpha^\delta; \quad \alpha \in h^*_ finite \}$ is the group of translations in $h^*$ shifting vectors $x$ belonging to $h^*$ as $x + \nu_\delta (x)$ where now $\nu_\delta (x)$ has a component along the imaginary light like direction $\delta$. In addition to previous reflections $\omega_i$ eq(7.3), the elements $t_\alpha^\delta$ generate $W_{affine}$. Generic elements $t_\alpha^\delta$ of the set $T_\delta$ act on $x \in h^*_ hyp$ as,

$$t_\alpha^\delta (x) = x + (\delta, x)\alpha - (\alpha, x)\delta - \frac{\alpha^2}{2}(\delta, x)\delta,$$

showing that $\nu_\delta (x) = (\delta, x)\alpha - (\alpha, x)\delta - \frac{\alpha^2}{2}(\delta, x)\delta$. Before going ahead let us show that these translations are abelian and exhibit what is the feature that do this job. Acting on the above relation by $t_{\delta^2}$ with
\[ t^\delta \circ t^\alpha (x) = x + (\delta, x) [\beta + \alpha] - [(\beta + \alpha, x)] \delta - \frac{(\beta + \alpha)^2}{2} (\delta, x) \delta + (\delta, x)(\delta, \beta) - \frac{\beta^2}{2} (\delta, \alpha) (\delta, x) \delta. \] (7.7)

Moreover as \((\delta, \beta) = 0\) and \((\delta, \alpha) = 0\) since \(\alpha\) and \(\beta\) belong to \(\Delta_{\text{finite}}\), the second term of the above equation vanishes identically and consequently \(t^\delta \circ t^\alpha\) coincides with \(t^\delta \circ t^\beta\) which is also \(t^{\alpha+\beta}\).

\[ t^\delta \circ t^\delta (x) = t^\delta \circ t^\delta (x) \] (7.8)

Commutativity of transformations (7.8) follows then from the orthogonality between \(\delta\) and \(\Delta_{\text{finite}}\). This particular feature gives us the key we need for the derivation of \(W_{\text{hyp}}\).

### 7.3 Hyperbolic Weyl Group \(W_{\text{hyp}}\)

We start by making three comments which we use for the derivation of the hyperbolic extension of ADE Weyl groups.

1. **Transformations \(t^{m\delta}_\alpha\):** Translations eq.\((7.4)\) defining the action of \(t^\alpha\) on root lattice vectors \(x\) may be extended for all imaginary roots \(m\delta\) of the affine root system \(\Delta_{\text{affine}}\). The resulting transformations which we denote as \(t^{(0,m\delta)}\) act like,

\[ t^{(0,m\delta)}_\alpha = x + m(\delta x)\alpha - m(\alpha x) \delta - \frac{\alpha^2}{2} m^2 (\delta x) \delta. \] (7.9)

Note that this transformation may be also rewritten as \(t^{(0,\delta)}_{m\alpha}\); but for later use we will keep the first notation. Composition of two transformations \(t^{(0,m\delta)}_\alpha\) and \(t^{(0,p\delta)}\) follows in same manner as before. Straightforward calculation shows that \(t^{(0,p\delta)} \circ t^{(0,m\delta)}(x)\) is equal to \(x + (\delta, x) (p\beta + m\alpha) - (p\beta + m\alpha, x) \delta - \frac{(p\beta + m\alpha)^2}{2} (\delta, x) \delta\); thanks to linearity which implies \((m\delta, \alpha) = m(\delta, \alpha) = 0\). This composition is in general equal to \(t^{(0,\delta)}_{p\beta + m\alpha}\); but for the special case \(m = p\), we also have,

\[ t^{(0,m\delta)}_\beta \circ t^{(0,m\delta)}_\alpha = t^{(0,m\delta)}_{\beta+\alpha} = t^{(0,m\delta)}_\alpha \circ t^{(0,m\delta)}_\beta \] (7.10)

The transformations \(t^{(0,m\delta)}_\alpha\), with \(m\) fixed, are obviously abelian for all \(\alpha\) belonging to \(\Delta_{\text{finite}}\) and more generally in \(h_{\text{finite}}\). Note that for \(m = 1\), one recovers the usual defining relation of translation in affine Lie algebras eq.\((7.3)\); but for \(m = 0\), the transformation \(t^{(0,m\delta)}_\alpha\) reduces to the identity operator.

This second feature tells us that there is no analogue of \(t^{(0,m\delta)}\) transformations in finite dimensional Lie algebras.

2. **Affine Weyl group \(W_{\text{affine}}\):** What is valid for translations involving imaginary roots \(m\delta\) is also true for \(n\gamma\) since both of them are orthogonal to \(\Delta_{\text{finite}}\): \((m\delta, \alpha) = (n\gamma, \alpha) = 0\). As such one may also define second kind of translation in hyperbolic space \(h_{\text{hyp}}^*\) as

\[ t^{(n\gamma,0)}_\alpha = x + (n\gamma, x)\alpha - n(\alpha, x) \gamma - \frac{n^2 \alpha^2}{2} (\gamma, x) \gamma. \] (7.11)

These transformations include the fundamental one \(t^{(\gamma,0)}_\alpha\) as well as the identity operator \(t^{(0,0)}_\alpha\) of \(W_{\text{finite}}\). They generate a second abelian subgroup \(T_\gamma\) in analogy with \(T_\delta\) of eq.\((7.5)\). Naturally, this deals with the second copy of affine root system \(\Delta_{\text{hyp}}\) contained in \(\Delta_{\text{hyp}}\) and with a second possible affine Weyl sub-symmetry \(W_{\text{affine}}\) evidently contained in hyperbolic Weyl group we are looking for. By similarity with eq.\((7.5)\), we have therefore,

\[ W_{\text{affine}} = W_{\text{finite}} \rtimes T_\gamma, \] (7.12)
with same features as for standard $W_\text{affine}^\delta$.

(3) Hyperbolic Weyl group $W_{\text{hyp}}$: What we have been doing above is in fact just two aspects of a more general issue. Eqs (7.9) and (7.11) are nothing but two special situations of a general picture. Since any vector $\zeta = m\delta + n\gamma$ is orthogonal to roots $\alpha \in \Delta_\text{finite}$ of finite dimensional Lie algebras, the natural translations extending eqs (7.9) and (7.11) one may define, are

$$t^\zeta_\alpha = x + (\zeta, x)\alpha - (\alpha, x)\zeta - \frac{\alpha^2}{2}(\zeta, x)\zeta. \quad (7.13)$$

Using eqs (7.4) and (7.7), one can check without difficulty that these are abelian transformations generating a more general translation set $T_\zeta = \{t^\zeta_\alpha; \alpha \in h^*_\text{finite}; \zeta \in h^*_{\text{hyp}} \setminus h^*_\text{finite}\}$. Previous sets $T_\delta$ and $T_\gamma$ appear as two special situations recovered by taking $\zeta = \delta$ and $\zeta = \gamma$ respectively. As such we end with the following structure of the Weyl group $W_{\text{hyp}}$ of hyperbolic ADE Lie algebras,

$$W_{\text{hyp}} = W_\text{finite} \ltimes T_\zeta. \quad (7.14)$$

As expected, this group is the semi direct product of $W_{\text{finite}}$ with the co-root lattice $Q^*_\text{hyp}$ of hyperbolic ADE algebras. It has quite similar features as $W_{\text{affine}}$ and has two subgroups $W_\text{affine}_\delta$ and $W_\text{affine}_\gamma$.

This construction generalizes naturally to other hyperbolic extensions of Lie algebras containing affine symmetries.

8 Conclusion and Comments

Motivated by the study of generalizations of affine quiver gauge theories, we have constructed in this paper the hyperbolic extension of affine ADE algebras and given necessary conditions of their unitary highest weight representations. These algebras form a special class of indefinite Lie algebras and have very remarkable features; some of them can be compared with their correspondent in affine KM symmetries and many others go beyond. The present study brings more insight for a better understanding of solvability of supersymmetric quiver QFTs and their large N duals. The hyperbolic ADE algebras we have considered here are shown to have no centre, but have a bi-linear form with indefinite signature making the structure of these extensions very rich and physically attractive.

Our interest into these hyperbolic generalized Lie algebras came initially from a tentative to study relevant deformations of supersymmetric quiver gauge theories emerging as QFT limits in type II compactifications on local CY manifolds with ADE geometries. But in dealing with this study, we have noted that except few specific examples, hyperbolic ADE Lie algebras were surprisingly not enough explored in quantum field theoretic literature. For instance no explicit contents of root system of hyperbolic ADE algebras nor the structure of corresponding Weyl symmetries were used before. Except few examples, the explicit structure of the commutation relations of these infinite dimensional algebras as we have the habit to use it in quantum field theory was also lacking. It was then a necessary task to start by addressing first these questions before coming to our initial objective. Among our results in this matter, we mention the five following:

(1) We have derived the explicit contents of root systems $\Delta_{\text{hyp}}$ of hyperbolic ADE symmetries. These are given by the following double infinite set,

$$\Delta_{\text{hyp}} \cup \{0\} = \{n\gamma + m\delta + l\alpha; \quad mn \geq (l^2 - 1), \quad \alpha \in \Delta_\text{finite} \quad l = 0, \pm 1 \quad m, n \in \mathbb{Z}\}, \quad (8.1)$$

where $\Delta_\text{finite}$ stands for the usual root system of finite ADE Lie algebras and where $\gamma$ and $\delta$ are two light like imaginary roots satisfying $(\gamma, \delta) = -1$, $(\gamma, \gamma) = (\delta, \delta) = 0$ and $(\gamma, \alpha) = (\delta, \alpha) = 0$ for $\alpha \in \Delta_\text{finite}$.
The known root systems $\Delta_{\text{finite}}$ of ordinary ADE symmetries and their extensions $\Delta_{\text{affine}}$ appear naturally as proper subsets as shown below,

$$
\Delta_{\text{hyp}} \supset \Delta_{\text{affine}}; \quad \Delta_{\text{hyp}} \supset \Delta_{\text{affine}}^{\gamma},
\Delta_{\text{affine}} \neq \Delta_{\text{affine}}^{\delta}; \quad \Delta_{\text{affine}} \cap \Delta_{\text{affine}}^{\gamma} = \Delta_{\text{finite}}.
$$

In these embeddings, $\Delta_{\text{affine}}^{\delta}$ stands for the usual affine root system $\{Z\delta + l\alpha; \quad \alpha \in \Delta_{\text{finite}} \quad l = 0, \pm 1\}$ and $\Delta_{\text{affine}}^{\gamma}$ is an isomorphic set obtained form $\Delta_{\text{affine}}^{\delta}$ by substituting $\delta$ by $\gamma$. It is remarkable that hyperbolic ADE extensions have two special affine sub-symmetries. The simple root associated with the affine extension is $\delta - \psi$ for $\Delta_{\text{affine}}^{\delta}$ and $\gamma - \psi$ for $\Delta_{\text{affine}}^{\gamma}$. Imaginary roots of these two isomorphic affine ADE subalgebras are $m\delta$ and $n\gamma$ respectively. Here $\psi$ is the maximal root of $\Delta_{\text{finite}}$.

(2) We have worked out explicitly the defining commutation relations of hyperbolic ADE Lie algebras by using two basis: (i) a covariant basis involving manifestly the invariant bi-linear form of the Lorentzian root lattice $Q_{\text{hyp}}$ and (ii) the standard (non covariant) one, we usually use in affine ADE KM algebras. We have found, amongst others, that hyperbolic ADE Lie algebras do indeed have two particular proper affine KM sub-symmetries $g_{\text{affine}}^{\delta}$ and $g_{\text{affine}}^{\gamma}$ with two central extensions K and L,

$$
g_{\text{affine}}^{\delta} \subset g_{\text{hyp}}; \quad g_{\text{affine}}^{\gamma} \subset g_{\text{hyp}},
\quad g_{\text{affine}}^{\delta} \neq g_{\text{affine}}^{\gamma}; \quad g_{\text{affine}}^{\delta} \cap g_{\text{affine}}^{\gamma} = g_{\text{finite}},
$$

opening as a consequence a window on relevance of hyperbolic ADE extensions and their possible applications in the study of 2D critical phenomena. Obviously $g_{\text{affine}}^{\delta}$ and $g_{\text{affine}}^{\gamma}$ are in one to one with the respective root systems $\Delta_{\text{affine}}^{\delta}$ and $\Delta_{\text{affine}}^{\gamma}$. From this view, it is also interesting to note that because of existence of these two remarkable copies of affine sub-symmetries, hyperbolic ADE Lie algebras $g_{\text{hyp}}$ may be interpreted as just the interpolating algebra between $g_{\text{affine}}^{\delta}$ and $g_{\text{affine}}^{\gamma}$. We suspect that this idea may have physical interpretations in what one may baptize as hyperbolic quantum field theoretic systems. It would be interesting to probe this interpolating idea on the example of supersymmetric quiver gauge theories following from low energy limit of type II string compactifications on CY3 with hyperbolic ADE singularities. Axion field of type IIB is suspected to be behind this interpolation feature.

(3) Turning around results on roots in hyperbolic ADE algebras, we have found a tricky root particularization ($a = n\gamma + m\delta + l\alpha$) encoding naturally what we know about ordinary and affine ADE subsystems. This parallelization allows also to solve the constraint eqs required by hyperbolic extension in a nice way. In this regards, we have derived the algorithm for defining positivity of roots in hyperbolic algebras and likely for indefinite Lie algebras. A generic root $a = n\gamma + m\delta + l\alpha$ of the root system $\Delta_{\text{hyp}}$ is said to be positive if $n > 0$ whatever the other $m$ and $l$ integers are. For the special case $n = 0$, this property is transmitted to $m$ which should be positive whatever $l$ integer is. For $m = 0$ the condition is then transmitted to $l$ which again should be positive.

(4) We have also derived the necessary conditions for unitary highest weight representations of hyperbolic ADE Lie algebras. Starting from a generic $\text{su}(2)$ subalgebra $[S^+, S^-] = \frac{2}{\alpha} \alpha \hbar$ with $a \in \Delta_{\text{hyp}}$ and using standard techniques on unitary highest weight representation theory, we have studied what are the necessary conditions for unitary representations of hyperbolic ADE algebra. To do so, we have first shown that from Lie algebraic point of view the pole $a^2 = 0$ in $\frac{2}{\alpha} \alpha \hbar$ is not a true singularity as it is just an apparent difficulty. The point is that the $\alpha \hbar$ observable has also a zero eigenvalue ($ad \alpha \hbar (S^\pm) = \pm \alpha \hbar S^\alpha$) which is exactly what is needed to lift the pole singularity. From representation theory where we have identities type $a\hbar |x > = ax|x >$; see eqs (6.32-6.35), the difficulty $\frac{2}{\alpha \hbar}$ needs however a further study.
before an exact answer. Leaving this point difficulty on margin, and replacing the generic root \( a \) by its explicit expression \( n\gamma + m\delta + l\alpha \), the above \( \text{su}(2) \) subalgebra yields,

\[
[S_{m,n}^{\alpha}, S_{-m,-n}^{-\alpha}] = \frac{2}{(l^2 - 2mn)} (l\alpha H - nL - mK), \quad (l^2 - mn) \leq 1, \quad (8.4)
\]

Acting as usual by these commutation relation on a highest weight vector \( |y\rangle = |k, l, \lambda\rangle \) of a highest weight representation of hyperbolic ADE algebra,

\[
H |k, l, \lambda\rangle = \lambda |k, l, \lambda\rangle; \quad K |k, l, \lambda\rangle = k |k, l, \lambda\rangle; \quad L |k, l, \lambda\rangle = l |k, l, \lambda\rangle; \quad E^a_{m,n} |k, l, \lambda\rangle = 0; \quad n\gamma + m\delta + l\alpha > 0, \quad (8.5)
\]

with \( H, K \) and \( L \) being the commuting Cartan generators and \( S_{m,n}^{\alpha} \) a generic step operator, one can get unitary necessary conditions. As such, unitary highest weight representations of hyperbolic ADE Lie algebras read then as,

\[
\frac{l(\alpha\lambda) - lm - kn}{l^2\alpha^2 - 2mn} \in \mathbb{Z}_+ \text{ for any } (n, m, l) \in \mathbb{L}. \quad (8.6)
\]

These solutions include as particular cases \( k \geq \psi\lambda \) and \( l \geq \psi\lambda \) which one recognizes as the unitary conditions for highest weight representations of the two proper affine sub-symmetries. The above conditions follow naturally by using the same \( \text{su}(2) \) trick one uses in the derivation of unitary representations of finite dimensional and affine Lie algebras. (5) Finally we have constructed the Weyl group of the hyperbolic extension of affine ADE Lie algebras. This group which is shown to be given by,

\[
W_{\text{hyp}} = W_{\text{finite}} \propto T_{\zeta}, \quad (8.7)
\]

is the semi direct product of \( W_{\text{finite}} \) with the co-root lattice \( \mathbb{Q}^r_{\text{hyp}} \) of hyperbolic ADE algebras. In addition to the Weyl group \( W_{\text{finite}} \) of finite ADE Lie algebras, \( W_{\text{hyp}} \) has as expected two special isomorphic proper subgroups \( W_{\text{affine}}^\delta \) and \( W_{\text{affine}}^\gamma \).

\[
W_{\text{affine}}^\delta \subset W_{\text{hyp}}; \quad W_{\text{affine}}^\gamma \subset W_{\text{hyp}}; \quad W_{\text{affine}}^\delta \cap W_{\text{affine}}^\gamma = W_{\text{finite}}. \quad (8.8)
\]

It is these groups which are used in \[33\]; see also \[54\], to study Seiberg like dualities and RG cascades in hyperbolic quiver gauge theories.

Along with the result we have obtained in this paper, we have also a comment to add. Besides the obvious fact that the present work generalizes naturally to other hyperbolic extensions of Lie algebras; in particular to hyperbolic algebras based on non simply laced affine symmetries, we have learned an important lesson which can serve as a guide for dealing with Indefinite Lie algebras and their quantum field theoretical realizations. Through the above analysis we have learnt that complexity due to indefinite signature of the bilinear form is only apparent. Much things on the study of hyperbolic extension of ordinary Lie algebras can be done using a similar philosophy as in special relativity and quantum electrodynamics QED on Minkowski space. In particular roots are of three kinds; space like roots with positive definite norms, light like and time like ones with negative norms. It is then not surprising if structure constants of hyperbolic ADE Lie algebras and their representations capture the above cone details. As far as quantum physics realizations and unitary highest weight representations of hyperbolic algebras are concerned, it is worthwhile to mention that the spectrum following from eq(8.6) is no so strange as one may think. The unique novelty with respect to what we know about ordinary and
affine Lie algebras is that now the spectrum is richer and so more interesting. For instance the unitary conditions eq(8.37)-(8.39),
\[ \frac{ay}{aa} \in \mathbb{Z}_+. \]  
(8.9)

involves now two branches instead of one. Since the bilinear form is indefinite, positivity of this relation implies: (i) an ordinary class of solutions where both \( ay \) and \( aa \) are positive; it contains as particular subsets what we know on finite dimensional ADE Lie algebras and their affine extensions. (ii) a new class of solutions where both of \( ay \) and \( aa \) are negative so that their ratio is a positive integer. This second branch has no analog in ordinary and affine Lie algebras.

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