A Framework for Constructing Centralized Coded Caching Schemes via Hamming Distance

Xi Zhong, Minquan Cheng, Ruizhong Wei

Abstract

In coded caching system we prefer to design a coded caching scheme with low subpacketization and small transmission load (i.e., the low implementation complexity and the efficient transmission during the peak traffic times). Especially exponentially growing subpacketization is known to be a major issue for practical implementation of coded caching. Placement delivery array (PDA) is a very effective combinatorial structure to design code caching scheme under uncoded placement. In this paper we propose a framework of constructing PDAs via Hamming distance. As applications, two classes of coded caching schemes with linear subpacketization can be obtained directly. In addition, compared with the previously known schemes with linear subpacketization, our scheme have smaller subpacketization and smaller transmission load for some parameters.

Index Terms

Coded caching scheme, Placement delivery array, Hamming distance.

I. INTRODUCTION

The immense growth in wireless data traffic driven by video-on-demand services leads to an enormous pressure on transmission for wireless network. Especially the high temporal variability of network traffic results in congestion during the peak traffic times while underutilization during off-peak times. Caching system effectively allows to shift traffic from peak to off-peak times, thereby smoothing out traffic variability and reducing congestion.

In order to make fully use of the local caching, coded caching system was proposed in [15] which can significantly further reduce the amount of transmission during the peak traffic times. In a centralize coded caching system, a server containing $N$ files with equal size connected to $K$ users, each of which has memory of size $M$ files, through an error-free shared link. An $F$-division $(K, M, N)$ coded caching system consists of two phases, i.e., the placement phase during the off-peak traffic times and the delivery phase during the peak traffic times. In the placement phase, each file is split into $F$ packets with equal size and the server places some packets of all the files into each user’s cache without knowledge of later demands. In the delivery phase, assume that each user requires one file randomly. Then the server transmits some coded signals with the size of at most $R$ files, say transmission rate $R$, satisfying various demands from users.

The first coded caching scheme proposed by Maddah-Ali and Niesen in [15], say Ali-Niesen scheme, achieves the minimum transmission rate when $N > K$. Therefore the Ali-Niesen scheme has been extended to numerous models, such as Device-to-Device (D2D) caching systems [9], private coded caching [24], online caching [17], coded distributed computing [14], hierarchical caching [13], secure caching [20] and so on. While the subpacketization of the Ali-Niesen scheme increases exponentially with the number of users, which leads to high implementing complexity and infeasibility in reality. So apart from small rate, low-subpacketization is also a meaningful problem that needs to be considered. Hence we aim to design coded caching schemes with transmission rate $R$ and subpacketization $F$ both as small as possible.

A. Previously known results

In fact constructing coded caching schemes with small subpacketization and transmission rate is an open problem. There are some existing schemes constructed through various methods at the cost of the increase of transmission rate compared with Ali-Niesen scheme. The authors in [25] proposed a structure, i.e., $(K, F, Z, S)$ placement delivery array (PDA), realizing an $(K, M, N)$ coded caching scheme by means of expressing placement phase and delivery phase where the subpacketization is $F$ and transmission rate is $R = \frac{S}{F}$. By PDA the authors obtained two classes of schemes with lower subpacketization level compared with Ali-Niesen scheme. Apart from PDA, there are many other characterizations of coded caching schemes such as hypergraphs [22], strong edge colored of bipartite graphs [26], Ruzsa-Szemeredi graphs [21], combinatorial design theory [2], [23], line graph [12] and so on.

Zhong and Cheng are with Guangxi Key Lab of Multi-source Information Mining & Security, Guangxi Normal University, Guilin 541004, China, (e-mail: chengqinshi@hotmail.com, dequan.liang@hotmail.com).
R. Wei is with Department of Computer Science, Lakehead University, Thunder Bay, ON, Canada, P7B 5E1, (e-mail: rwei@lakeheadu.ca).
In [21] it was pointed out that all the deterministic coded caching schemes introduced above can be represented by PDAs. Hence constructing appropriate PDAs makes great sense to coded caching. There are some known constructions from view points of combinatorial designs [5]–[8], bipartite graphs [16] and concatenating construction [18], [27] so on. It is worth noting that the framework of constructing coded caching schemes proposed in [8] can include most of the previously known schemes. We list most of the previously known deterministic schemes, which have advantages on the subpacketization or the transmission rate, in Table I.

| References and parameters | Number of Users $K$ | Cache Fraction $\frac{M}{N}$ | Rate $R$ | Subpacketization $F$ |
|--------------------------|---------------------|---------------------------|-----------|---------------------|
| [15] $K, M, N \in \mathbb{Z}^+$ with $M < N$ and $K \frac{M}{N} \in \mathbb{Z}^+$ | $K$ | $\frac{M}{N}$ | $\frac{K(1-\frac{K}{M})}{K \frac{M}{N}+1}$ | $\left(\frac{K}{M}\right)$ |
| [2], $v, k \in \mathbb{Z}^+$, there exists a symmetric $(v, k, 2)$-BIBD with non repeated blocks | $v$ | $1 - \frac{k-1}{v}$ | 1 | $kv$ |
| [7], $m, q, t, z \in \mathbb{Z}^+$ with $t \leq m$ and $z \leq q$ | $(m \choose t)q^t$ | $1 - (2^{q-1} - 1)^t$ | $\frac{(q-1)^t}{2^{q-1}-1}$ | $O((\frac{q-1}{2^{q-1}-1})^{1+k\frac{z^2}{q}})$ |
| $(m+1)q$ | $\frac{z}{q}$ | $\frac{2^{q-1}-1}{z}$ | $\left(\frac{2^{q-1}-1}{q}\right)^{\frac{t}{q}}$ |
| [12], $k, m, t \in \mathbb{Z}^+$ with $m+m+t \leq k$ and $q$ is a prime power | $\left[\frac{k}{t}\right]q$ | $1 - \frac{\frac{m+t}{q}}{\left[\frac{k}{t}\right]}q$ | $\frac{m+t}{q\left[\frac{k}{t}\right]}q$ | $\left[\frac{k}{m+t}\right]q$ |
| [9], $m, q, t \in \mathbb{Z}^+$ with $m+m+t \leq k$ and $q$ is a prime power | $\frac{k+t+1}{1}q$ | $1 - q^{\frac{m+t+1}{q}+1}$ | $q^{m+t+1}+1$ | $O(q^{m+t+1}+q)$ |
| [19], $K, M, N \in \mathbb{Z}^+$ with $M < N$, and $K \frac{M}{N} \frac{1}{g(q-1)} \in \mathbb{Z}^+$ | $K$ | $\frac{M}{N}$ | $\frac{K}{g+1}(1 - \frac{1}{\frac{M}{N}})$ | $O(e^q)$ |
| [26], $m, a, b, \lambda \in \mathbb{Z}^+$ with $0 < a < m$ and $b > \min\{a, b\}$ | $\left(\begin{array}{c} m \\ a \\ \end{array}\right)$ | $\left(\begin{array}{c} \frac{(q-2)}{(2)} \\ \frac{a+b-2a}{(2)} \\ \frac{a-b}{(2)} \end{array}\right)$ | $\left(\begin{array}{c} m \\ a \\ \end{array}\right)$ | $\left(\begin{array}{c} n \\ a \\ \end{array}\right)$ |
| [5], $v, k \in \mathbb{Z}^+$ with $t < k$ | $\left(\begin{array}{c} k \\ t+1 \\ \end{array}\right)$ | $1 - \frac{t+1}{(t)}$ | $\left(\begin{array}{c} t+1 \\ (t) \end{array}\right)$ | $\left(\begin{array}{c} k \\ t+1 \end{array}\right)$ |

### Table I: Summary of some known coded caching schemes

**B. Contributions and arrangement of this paper**

In this paper we prefer to considering linear subpacketization schemes with small transmission rate when $N > K$. We first propose a framework of constructing PDAs via Hamming distance. Secondly, we obtain two classes of coded caching schemes with linear subpacketization. Moreover, the comparison between new schemes obtained by our PDAs and previous known schemes shows that our new schemes have advantages of low subpacketization and small transmission rate for some parameters.

The rest of the paper is organized as follows. In Section III we state some notations used in this paper, then introduce the preliminaries about coded caching and the realization from placement delivery array (PDA) to coded caching scheme. In Section III we introduce the framework of constructing PDAs via Hamming distance. In Section IV and Section V a class of PDAs is obtained respectively. In Section VI we give the comparison between our new schemes and known schemes. Conclusion is drawn in Section VII.

### II. Preliminaries

In this paper, we will use the following notations unless otherwise stated.

- We use bold capital letter, bold lower case letter and curlicue letter to denote array, vector and set respectively.
- For any positive integers $m$ and $t$ with $t < m$, let $\{0, m\} = \{0, 1, \ldots, m-1\}$ and $\left(\begin{array}{c} m \\ t \\ \end{array}\right) = \{T \mid T \subseteq \{0, m\}, |T| = t\}$, i.e., $\left(\begin{array}{c} m \\ t \\ \end{array}\right)$ is the collection of all $t$-sized subsets of $\{0, m\}$.
- Given a $m$ length vector $a$ and a set $T \subseteq \{0, m\}$, $a|_T$ is a vector obtained by deleting the coordinates $j \in \{0, m\} \setminus T$.

#### A. Coded Caching

Let us consider the centralized coded caching system (Fig.1), where a sever containing $N$ files denoted by $W = \{W_n \mid n \in [0, N]\}$ linked to $K$ users denoted by $C = [0, K)$ with $K < N$ through an error-free shared link, and each user has a memory of size $M$ files with $M < N$.

An $F$-division $(K, M, N)$ coded caching scheme operates in two phases which can be sketched as follows:
1) **Placement Phase:** All the files are sub-divided into $F$ equal packets, i.e., $W = \{W_{n,j} \mid j \in [0, F), n \in [0, N]\}$. Each user has access to the files set $W$. $Z_k$ denotes the packets subset of $W$ cached by user $k$. The size of $Z_k$ is less than or equal to the capacity of each user’s cache memory size $M$.

2) **Delivery Phase:** Each user requests one file from $W$ randomly. Denote the requested file numbers by $d = (d_0, d_1, \cdots, d_{K-1})$, i.e., user $k$ requests file $W_{d_k}$, where $k \in K$, $d_k \in [0, N)$. Upon receiving the request $d$, the server broadcasts XOR of packets with size of at most $R_d$ files to users so that each user is able to recover his requested file with help of its caching contents.

In this paper, we focus on the maximum transmission load. The transmission rate of a coded caching scheme is defined as the maximal transmission amount among all the requests during the delivery phase, i.e.,

$$R = \max_{d \in [0, N)^K} \{R_d\}.$$ 

Clearly the efficiency of the data transmission in the delivery phase increases with decreasing transmission rate. Furthermore, the complexity of the implementing an $F$-division coded caching scheme increases as the value of $F$. So we prefer to design a scheme with the subpacketization and transmission rate as small as possible.

**B. Placement delivery array and coded caching scheme**

Yan et al. in [25] proposed the concept of placement delivery array to characterize the placement phase and delivery phase simultaneously.

**Definition 1:** (25) For positive integers $K$ and $F$, an $F \times K$ array $P = (p_{i,j})$, $i \in [0, F)$, $j \in [0, K)$, composed of a specific symbol "*" called star and $S$ nonnegative integers $[0, S)$, is called a $(K, F, S)$ placement delivery array (PDA) if it satisfies $C_1$ in the following conditions:

$C_1$. For any two distinct entries $p_{i_1,j_1}$ and $p_{i_2,j_2}$, $p_{i_1,j_1} = p_{i_2,j_2} = s$ is an integer only if

- $i_1 \neq i_2$, $j_1 \neq j_2$, i.e., they lie in distinct rows and distinct columns; and
- $p_{i_1,j_2} = p_{i_2,j_1} = \ast$, i.e., the corresponding $2 \times 2$ subarray formed by rows $i_1, i_2$ and columns $j_1, j_2$ must be one of the following form

\[
\begin{pmatrix}
  s & \ast \\
  \ast & s
\end{pmatrix}
\text{ or }
\begin{pmatrix}
  \ast & s \\
  s & \ast
\end{pmatrix}.
\]

For any positive integer $Z \leq F$, $P$ is denoted by $(K, F, Z, S)$ PDA if

$C_2$. each column has exactly $Z$ stars.

Using Algorithm 1 the following result can be obtained.

**Theorem 1:** (25) Using Algorithm 1 an $F$-division caching scheme for a $(K, M, N)$ caching system can be realized by a $(K, F, Z, S)$ PDA with $\frac{M}{N} = \frac{Z}{F}$. Each user can decode his requested file correctly for any request $d$ at the rate $R = \frac{S}{F}$.

From Algorithm 1 we have the following realization process from a $(K, F, S)$ PDA to a $F$-$(K, M, N)$ coded caching scheme. In a $(K, F, S)$ PDA $P$, if entry $p_{j,k} = \ast$, it demonstrates that user $k$ has already cached the packets induced by $j$ of all files in server. Otherwise user $k$ hasn’t cached. Moreover, For any integer $s$ occurring in PDA, the server broadcasts XOR symbol of

\[\text{1}^{\text{Memory sharing technique may lead to non equally divided packets}}\]
Algorithm 1 caching scheme based on PDA in [25]

1: procedure PLACEMENT(P, W)
2: Split each file \( W_n \in W \) into \( F \) packets, i.e., \( W_n = \{ W_{n,j} \mid j \in [0, F) \} \).
3: for \( k \in K \) do
4: \( Z_k \leftarrow \{ W_{n,j} \mid p_{j,k} = *, \forall n \in [0, N) \} \)
5: end for
6: end procedure

7: procedure DELIVERY(P, W, d)
8: for \( s = 0, 1, \ldots, S - 1 \) do
9: Server sends \( \bigoplus_{p_{j,k} = s, j \in [0,F), k \in [0,K)} W_{d_k,j} \).
10: end for
11: end procedure

those packets corresponding to all entries having \( s \). In fact the property of the PDA guarantees every user can get his requested file. For the details, the interested reader is referred to [25]. Here we just propose an example to show the relationship between a PDA and its realizing scheme by Algorithm I.

Example 1: It is easy to verify that the following array is a \((6, 4, 2, 4)\) PDA:

\[
P = \begin{pmatrix}
* & * & * & 0 & 1 & 2 \\
0 & * & 1 & * & * & 3 \\
0 & * & 2 & * & * & 3 \\
1 & 2 & * & 3 & * & *
\end{pmatrix}.
\]

Using Algorithm I one can obtain a 4-division \((6, 3, 6)\) coded caching scheme in the following way.

- **Placement Phase:** From Line 2 we have \( W_i = \{ W_{i,0}, W_{i,1}, W_{i,2}, W_{i,3} \} \), \( i \in [0, 6) \). Then by Lines 3-5, the contents in each user are

\[
Z_0 = \{ W_{i,0}, W_{i,1} \mid i \in [0, 6) \}, \quad Z_1 = \{ W_{i,0}, W_{i,2} \mid i \in [0, 6) \},
\]

\[
Z_2 = \{ W_{i,0}, W_{i,3} \mid i \in [0, 6) \}, \quad Z_3 = \{ W_{i,1}, W_{i,2} \mid i \in [0, 6) \},
\]

\[
Z_4 = \{ W_{i,1}, W_{i,3} \mid i \in [0, 6) \}, \quad Z_5 = \{ W_{i,2}, W_{i,3} \mid i \in [0, 6) \}.
\]

- **Delivery Phase:** Assume that the request vector is \( d = (0, 1, 2, 3, 4, 5) \). By Lines 8-10 server sends \( W_{0,2} \oplus W_{1,1} \oplus W_{3,0}, W_{0,3} \oplus W_{2,1} \oplus W_{4,0}, W_{1,3} \oplus W_{2,2} \oplus W_{5,0} \) and \( W_{3,3} \oplus W_{4,2} \oplus W_{5,1} \) to all users. Then all requests are satisfied.

From Theorem I and Example I an \( F \)-division \((K, M, Z, N)\) coded caching scheme can be obtained by constructing an appropriate PDA. Then designing coded caching schemes with small rate and great packet level is turned to constructing PDAs where \( F \) and \( S \) are both as small as possible.

III. NEW CONSTRUCTION VIA HAMMING DISTANCE

In this section, we propose a new construction framework via Hamming distance to generate arrays which satisfy some of the conditions of PDA, and then obtain new PDAs through partitioning the entries of the arrays.

A. The framework of constructing via Hamming distance

Let \( x \) and \( y \) be vectors of length \( m \). The Hamming distance from \( x \) to \( y \), denoted by \( d(x, y) \), is defined to be the number of coordinates at which \( x \) and \( y \) differ. The Hamming weight of \( x \), denoted by \( wt(x) \), is defined to be the number of nonzero coordinates in \( x \).

Construction 1: For any positive integers \( m, \omega, F, K \) and \( q \geq 2 \) with \( \omega < m \), given two subsets \( A \) and \( B \) of \([0, q]^m\) where \( |A| = F \) and \( |B| = K \), an \( F \times K \) array \( P = (p_{a,b}) \), \( a \in A, b \in B \) can be obtained as follows:

\[
p_{a,b} = \begin{cases} 
    a + b & \text{if } d(a, b) = \omega \\
    * & \text{otherwise}
\end{cases}
\]

where \( a \pm b = (a_0 \pm b_0, a_1 \pm b_1, \ldots, a_{m-1} \pm b_{m-1}) \), \( a = (a_0, a_1, \ldots, a_{m-1}) \), \( b = (b_0, b_1, \ldots, b_{m-1}) \). In addition all the operations are carried under mod \( q \) in this paper.

For the sake of simplicity, the vectors in all the examples in this paper are simplified, e.g., \((1, 1, 0, 0)\) is written as \( 1100 \).
**Example 2:** When \( m = 4, \omega = 3, q = 2 \), \( A \) is the collection of \( m \) length binary vectors with Hamming weight 2 and \( B \) is the collection of \( m \) length binary vectors with Hamming weight 1, from Construction [1] the following \( 6 \times 4 \) array can be obtained.

| \( a, b \) | 1100 | 0100 | 0010 | 0001 |
|-----------|------|------|------|------|
| 1100      | *    | *    | 1110 | 1101 |
| 1010      | *    | 1110 | *    | 1011 |
| 1001      | *    | 1101 | 1011 | *    |
| 0110      | 1110 | *    | *    | 0111 |
| 0101      | 1101 | *    | 0111 | *    |
| 0011      | 1011 | 0111 | *    | *    |

It is very interesting that the array is a \((4, 6, 3, 4)\) MN PDA.

**Example 3:** When \( m = 4, \omega = 2, q = 2 \), \( A \) and \( B \) are both the collection of \( m \) length binary vectors with Hamming weight 2, from Construction [1] the following \( 6 \times 6 \) array can be obtained.

| \( a, b \) | 1100 | 1010 | 1001 | 0110 | 0101 | 0011 |
|-----------|------|------|------|------|------|------|
| 1100      | *    | 0110 | 0101 | 1010 | 1001 | *    |
| 1010      | *    | 0110 | 1100 | *    | 1011 | *    |
| 1001      | 0101 | 0011 | *    | *    | 1100 | 1010 |
| 0110      | 1010 | 1100 | *    | *    | 0011 | 0101 |
| 0101      | 1001 | *    | 1100 | 0011 | *    | 0110 |
| 0011      | *    | 1001 | 1010 | 0101 | 0110 | *    |

From Example [3] we can see that no vector occurs more than once in each row and each column. Furthermore if a vector \( e \) occurs in two distinct entries, i.e., \( p_{a_1,b_1} = p_{a_2,b_2} = e \), and \( d(a_1,b_2) \neq \omega \), then we have \( p_{a_1,b_2} = p_{a_2,b_1} = * \). For instance, vector 0110 occurs in entries \((1100, 1010)\) and \((1010, 1100)\) meanwhile \( d(1100, 1100) \neq 2 \), then clearly \( p_{a_1,b_2} = p_{a_1,b_2} = * \). In fact this is not accidental. In general, we have the following proposition.

**Proposition 1:** Let \( P \) be the array generated by Construction [1] if there are two distinct entries being the same vector \( e \), say \( p_{a_1,b_1} = p_{a_2,b_2} = e \), then the following two statements hold:

1) The vector \( e \) occurs in the different columns and different rows, i.e., the condition C1-a in Definition [1] holds.
2) The subarray formed by rows \( a_1, a_2 \) and columns \( b_1, b_2 \) satisfies the condition C1-b in Definition [1] if and only if \( d(a_1,b_2) \neq \omega \).

**Proof.** Suppose that a vector \( e = (e_0, e_1, \ldots, e_{m-1}) \in [0, q]^m \) occurs in two distinct entries, say \((a_1, b_1)\) and \((a_2, b_2)\) where

\[
\begin{align*}
a_1 &= (a_{1,0}, a_{1,1}, \ldots, a_{1,m-1}), \\
b_1 &= (b_{1,0}, b_{1,1}, \ldots, b_{1,m-1}), \\
a_2 &= (a_{2,0}, a_{2,1}, \ldots, a_{2,m-1}), \\
b_2 &= (b_{2,0}, b_{2,1}, \ldots, b_{2,m-1}).
\end{align*}
\]

From Construction [1] we have

\[
e = a_1 + b_1 = a_2 + b_2.
\]

(3)

That is, \( a_1 - a_2 = b_1 - b_2 \). Clearly \( a_1 = a_2 \) if and only if \( b_1 = b_2 \). So vector \( e \) occurs in the different columns and different rows. If \( d(a_1,b_2) \neq \omega \), then \( p_{a_1,b_2} = * \) from Construction [1] Since \( d(a_1,b_2) = wt(a_1 - b_2) \neq \omega \), by (3) we have \( a_2 - b_1 = a_1 - b_2 \), so \( d(a_2,b_1) = d(a_1,b_2) \neq \omega \) holds. This implies that \( p_{a_2,b_1} = * \). Conversely if \( p_{a_1,b_2} = p_{a_2,b_1} = * \), we also have \( d(a_1,b_2) \neq \omega \) similarly.

From Proposition [1] we find the array \( P \) generated by Construction [1] has satisfied the Condition C1-a in Definition [1]. In order to construct PDAs we only need to make any two distinct entries having the same vector satisfy the condition 2) in Proposition [1]. So we should part the collection of all entries having the same vector into several subsets such that any two different entries in each subset satisfy the condition 2) in Proposition [1]. That is the main discussion in the following subsection.

### B. The partitions of the entries

For any positive integers \( m, q, \omega \) and the given subsets \( A, B \in [0, q]^m \) with \( \omega < m \) and \( q \geq 2 \), we can obtain a array \( P \) by Construction [1]. Assume that vector \( e \) occurs \( g_e \) times in \( P \) generated by Construction [1] say \( p_{a_1,b_1} = \ldots = p_{a_g,b_g} = e \). Let \( \mathcal{E}_e = \{(a_1,b_1), (a_2,b_2), \ldots, (a_{g_e},b_{g_e})\} \). We claim that for some positive integer \( h_e \) there always exists a partition \( \mathcal{X}_e = \{\mathcal{X}_{e,0}, \mathcal{X}_{e,1}, \ldots, \mathcal{X}_{e,h_e-1}\} \) for \( \mathcal{E}_e \), i.e., \( \mathcal{E}_e = \bigcup_{i=0}^{h_e-1} \mathcal{X}_{e,i} \), satisfying

**Property 1:** For any two different entries \((a_1,b_1), (a_2,b_2) \in \mathcal{X}_{e,i}, i \in [0,h_e), d(a_1,b_2) \neq \omega \) always holds

for some integer \( 1 \leq h_e \leq g_e \). An trivial partition of \( \mathcal{E}_e \) is \( \mathcal{X}_{e,i} = \{(a_i,b_i)\} \) for each \( i = 1, 2, \ldots, g_e \), i.e., each entry forms a subset of \( \mathcal{E}_e \). Of course we can also use Algorithm [2] to obtain a nontrivial partition of \( \mathcal{E}_e \) by means of greedy method.
Algorithm 2 A greedy algorithm for partitioning $\mathcal{E}_e$

1: $i = 0$
2: while $\mathcal{E}_e \neq \emptyset$ do
3: Pick any $(a, b) \in \mathcal{E}_e$, and let $\mathcal{X}_{e,i} = \{(a, b)\}$.
4: for $(a', b') \in \mathcal{E}_e \setminus (a, b)$ do
5: if $d(a', b') \neq \omega$ then
6: $\mathcal{X}_{e,i} = \mathcal{X}_{e,i} \cup \{(a', b')\}$
7: $\mathcal{E}_e \leftarrow \mathcal{E}_e \setminus \mathcal{X}_{e,i}$
8: end if
9: end for
10: $i \leftarrow i + 1$
11: end while

**Construction 2:** Let $P$ be an $F \times K$ array generated by Construction 1. For each vector $e$ in $P$, given a partition $\mathcal{X}_e = \{X_{e,0}, \ldots, X_{e,h_e-1}\}$ satisfying Property 1, then we can obtain a new array $P' = (p'_{a,b})$, $a \in A$ and $b \in B$, where

$$p'_{a,b} = \begin{cases} (e, i) & \text{if } p_{a,b} = e, (a, b) \in X_{e,i}, i \in [0, h_e) \\ \ast & \text{otherwise}\end{cases}$$

Clearly any two entries, say $p_{a_1, b_1} = p_{a_2, b_2} = (e, i)$, in $P'$ satisfy $d(a_1, b_2) \neq \omega$. So from Proposition 1, $P'$ is a PDA.

**Example 4:** Let us consider the parameters and the $6 \times 6$ array in Example 3 again. By (2), we have $\mathcal{E}_{0110}, \mathcal{E}_{0101}, \mathcal{E}_{1010}, \mathcal{E}_{1001}, \mathcal{E}_{0011}, \mathcal{E}_{1100}$ and their partitions satisfying Property 1 as follows.

$$\begin{align*}
\mathcal{E}_{0110} &= X_{0110,0} \cup X_{0110,1} = \{(0011, 0101), (0101, 0011)\} \cup \{(1010, 1100), (1100, 1010)\} \\
\mathcal{E}_{0101} &= X_{0101,0} \cup X_{0101,1} = \{(0011, 0110), (0110, 0011)\} \cup \{(1001, 1100), (1100, 1001)\} \\
\mathcal{E}_{1010} &= X_{1010,0} \cup X_{1010,1} = \{(0011, 1001), (1001, 0011)\} \cup \{(0110, 1100), (1100, 0110)\} \\
\mathcal{E}_{1001} &= X_{1001,0} \cup X_{1001,1} = \{(0011, 1010), (1010, 0011)\} \cup \{(0110, 1010), (1100, 1001)\} \\
\mathcal{E}_{0011} &= X_{0011,0} \cup X_{0011,1} = \{(0101, 0110), (0110, 0101)\} \cup \{(1001, 1010), (1010, 1001)\} \\
\mathcal{E}_{1100} &= X_{1100,0} \cup X_{1100,1} = \{(0101, 1010), (0110, 1001)\} \cup \{(1001, 1010), (1010, 1001)\}
\end{align*}$$

Based on the above partition and by Construction 2 the following $(6, 6, 2, 12)$ PDA can be obtained.

\[
\begin{array}{cccccccc}
\text{a/b} & 1100 & 1010 & 1001 & 0110 & 0101 & 0011 \\
1100 & \ast & 0110,1 & 0101,1 & 1010,1 & 1001,1 & \ast \\
1010 & 0110,1 & \ast & 0011,1 & 1100,1 & \ast & 1001,0 \\
1001 & 0101,1 & 0011,1 & \ast & \ast & 1100,0 & 1010,0 \\
0110 & 1010,1 & 1100,1 & \ast & \ast & 0011,0 & 0101,0 \\
0101 & 1001,1 & \ast & 1100,0 & 0011,0 & \ast & 0110,0 \\
0011 & \ast & 1001,0 & 1010,0 & 0101,0 & 0110,0 & \ast \\
\end{array}
\]

From Construction 1 and Construction 2 the following result can be obtained.

**Theorem 2:** The $F \times K$ array $P'$ generated from Construction 2 is a $(K, F, S)$ PDA where $S = \sum_{e \in P} h_e$.

From the above introductions, we only need to consider designing appropriate partition $\mathcal{X}_e$ for each set $\mathcal{E}_e$ in $P$ generated by Construction 1. Hence, in the following two sections we will propose a partition for the parameter $q = 2$ and $3$ respectively.

For any vector $e = (e_0, e_1, \ldots, e_{m-1})$ occurring in $P$ obtained by Construction 1 let $\mathcal{C}_e = \{i \in [0, m) | e_i = 0\}$, i.e., the set consists of all the coordinates having 0.

**IV. The PDA with parameter $q = 2$**

In this section, we consider $q = 2$. First, for any sets $A, B \in [0, 2)^m$ we propose a partition $\mathcal{X}_e$ satisfying the Property 1 for each $\mathcal{E}_e$ to construct PDAs. Furthermore, for some special $A$ and $B$ we improve this partition through merging some elements to obtain new partition which can obtain better schemes.
A. The partition for any $A$ and $B$

When $q = 2$ for any vector $e = (e_0, e_1, \ldots, e_{m-1})$ occurring in $P$ obtained by Construction 1, clearly $|C_e| = m - \omega$ always holds. Then the following partition $X_e$ of $E_e$ can be obtained.

**Partition 1:** For any vector $e$ in $P$ obtained by Construction 1 and for each vector $t \in [0, 2)^{m-\omega}$ we define

$$X_{e,t} = \{(a, b) : \exists e, a|C_e = b|C_e = t\}. \quad (4)$$

Let $X_e = \{X_{e,t} | X_{e,t} \neq \emptyset, t \in [0, 2)^{m-\omega}\}$, then $X_e$ is a partition for $E_e$.

**Proposition 2:** Partition 1 satisfies Property 1.

**Proof.** For any vector $e = (e_0, e_1, \ldots, e_{m-1})$ in $P$, let us consider the $X_e$ in Partition 1. For any $X_{e,t} \in X_e$, $t \in [0, 2)^{m-\omega}$, if $|X_{e,t}| = 1$, our statement always holds. When $|X_{e,t}| \geq 2$, for any two different entries $(a_1, b_1), (a_2, b_2) \in X_{e,t}$, let

$$a_1 = (a_{1,0}, a_{1,1}, \ldots, a_{1,m-1}), \quad a_2 = (a_{2,0}, a_{2,1}, \ldots, a_{2,m-1}),$$

$$b_1 = (b_{1,0}, b_{1,1}, \ldots, b_{1,m-1}), \quad b_2 = (b_{2,0}, b_{2,1}, \ldots, b_{2,m-1}).$$

From $q = 2$ and $C_e = \{i \in [0, m] | e_i = 0\}$, for any $j \in C_e$ we have $a_{1,j} = b_{1,j}, a_{2,j} = b_{2,j}$ and $|C_e| = m - \omega$. From 1 we have $a_1|C_e = b_1|C_e = t = a_2|C_e = b_2|C_e$, then for any $j \in C_e$, $a_{1,j} = a_{2,j}$ and $a_{1,j} = b_{2,j}$ always hold. For any $j \in [0, m] \setminus C_e$, we have $a_{2,j} = b_{2,j} + 1$ due to $d(a_2, b_2) = \omega$. If $d(a_1, b_2) = \omega$, then for any $j \in [0, m] \setminus C_e$, we have $a_{1,j} = b_{2,j} + 1$ meanwhile $a_{2,j} = b_{2,j} + 1$, therefore $a_{1,j} = a_{2,j}$. Then for any $j \in [0, m]$, $a_{1,j} = a_{2,j}$, i.e., $a_1 = a_2$, which constrains the condition 1) of Proposition 1. Hence, $d(a_1, b_2) \neq \omega$ holds and partition $X_e$ satisfies the Property 1. The proof is completed.

**Example 5:** Let us consider the parameters and the $6 \times 6$ array in Example 3 again. From Partition 1 we have $E_{0110}, E_{0101}, E_{1010}, E_{1001}, E_{0011}, E_{1100}$ and their partitions as follows.

$$E_{0110} = X_{0110,01} \cup X_{0110,10} = \{(0011, 0101), (0101, 0011)\} \cup \{(1010, 1100), (1100, 1010)\}$$

$$E_{0101} = X_{0101,01} \cup X_{0101,10} = \{(0011, 0110), (0110, 0011)\} \cup \{(1010, 1100), (1100, 1010)\}$$

$$E_{1010} = X_{1010,01} \cup X_{1010,10} = \{(0011, 1010), (1010, 0011)\} \cup \{(0110, 1100), (1100, 0110)\}$$

$$E_{1001} = X_{1001,01} \cup X_{1001,10} = \{(0011, 1010), (1010, 0011)\} \cup \{(0110, 1100), (1100, 0110)\}$$

$$E_{0011} = X_{0011,01} \cup X_{0011,10} = \{(0101, 0110), (0110, 0101)\} \cup \{(1010, 1100), (1100, 1010)\}$$

$$E_{1100} = X_{1100,01} \cup X_{1100,10} = \{(0101, 0110), (0110, 0101)\} \cup \{(1010, 1100), (1100, 1010)\}$$

By Construction 1 the following $(6, 6, 2, 12)$ PDA can be obtained which is as same as the PDA in Example 3.

| $a/b$ | 1100 | 1010 | 1001 | 0110 | 0101 | 0011 |
|-------|------|------|------|------|------|------|
| 1100  | *    | 1010 | 1001 | 1010 | 1001 | 10 * |
| 1010  | 0110 | 10 *  | 0011 | 10  | 1001 | 10 * |
| 1001  | 0101 | 10 | 1001 | 10 | 0110 | 10 | 1010 |
| 0110  | 1010 | 10 | 1100 | 10 | 0011 | 10 | 0101 |
| 0101  | 1001 | 10 | 1100 | 10 | 0110 | 10 | 0110 |
| 0011  | 1001 | 10 | 1010 | 10 | 0101 | 10 | 0110 |

To show the performance of Partition 1, the comparison of parameter $S$ in our PDAs obtained by Partition 1 and the PDAs obtained by partition from Algorithm 2 is as follows.

**TABLE II:** The comparison of parameter $S$ in PDAs generated by Partition 1 and the partition from Algorithm 2

| $A = \{t \in [0, 2)^{\omega(t)} : 2\}$ and $B = \{t \in [0, 2)^{\omega(t)} : 1\}$, $\omega = 3$ | By Partition 1 $S = \omega$ | By Algorithm 2 $S = \omega$ |
|-----------------------------------------------|----------------|----------------|
| $A = B = \{0000, 1100, 1101, 1110, 0001, 1101, 0011, 1111\}$, $\omega = 2$ | 6              | 6              |

From Table I under some parameters Partition 1 achieves same performance of the partition by Algorithm 2. Since the parameters of obtained PDAs by Partition 1 vary from the selection about sets $A$ and $B$, next let us consider the following two special cases as examples.
1) The first case: When \( K = \binom{m}{s} \) and \( F = \binom{m}{t} \) for any positive integers \( m, s, t \) with \( s, t < m \), by Constructions 1 and Partition 1 we have the following result.

**Theorem 3:** For any positive integers \( s, t, m, \omega \) and integer \( \lambda \) with \( \omega = s + t - 2\lambda < m \) and \( \lambda \leq \min\{t, s\} \), there exists a \((\binom{m}{t}, \binom{m}{s}), (\binom{m}{t} - \binom{m}{s}, t + s - 2\lambda)\) PDA which can realize a \((K, M, N)\) coded caching scheme with \( \frac{M}{N} = 1 - \binom{m-t}{s} / \binom{m}{s} \), the subpacketization \( F = \binom{m}{s} \) and the transmission rate \( R = \binom{m}{s} / \binom{m}{t} \).

**Proof.** Let \( P \) consists of all the binary vectors with Hamming weight \( t \) and \( A \) consists of all the binary vectors with Hamming weight \( s \). Then we have a \((m) \times \binom{m}{t}\) array \( P \). For any entry \( p_{a,b} = e \neq * \), \( a \in A \), \( b \in B \), let \( e_{a,b}^{(1)} = \{ i \in [0, m] | a_i = b_i = 1 \} \).

From 1 and \( \lambda = 0.5^{t+s-2\lambda} \), \( \binom{m}{m-s} = \lambda \) always holds. Since for any vector \( b \in B \) there are \( \binom{m}{s} \) vectors \( a \in A \) such that \( d(a, b) = \omega \), we have \( Z = \binom{m}{s} - \binom{m}{s} \binom{m-s}{s} \lambda \). Then the memory ratio is \( \frac{M}{N} = \frac{F}{F} = 1 - \binom{m-t}{s} / \binom{m}{s} \). From 1 and \( q = 2 \), for any vector \( e \in [0, 2)^m \) there exist vectors \( a \in A \) and \( b \in B \) such that \( a + b = e \) and \( \binom{m}{m-s} = \lambda \). So the collection of all vectors occurring in \( P \) is exactly the collection of all binary vectors with Hamming weight \( \omega \), i.e., the number of vectors occurring in \( P \) is \( S = \binom{m}{t} \). By Partition 1 for any vector \( e \), we have

\[
\begin{align*}
\mathcal{E}_e &= |\{ t \in [0, 2)^m | \mathcal{X}_{e,t} \neq \emptyset \}| \\
&= |\{ e_{a,b}^{(1)} | a \in A, b \in B, (a, b) \in \mathcal{E}_e, e_{a-b} = \lambda \}| \\
&= \binom{m-t}{s} - \binom{m-s}{s} \lambda \\
&= \binom{m-t+s-2\lambda}{s} - \binom{m-s}{s} \lambda.
\end{align*}
\]

From Theorem 2 we have \( S = S' h_e = \binom{m-t+s-2\lambda}{s} \). Then transmission rate \( R = \frac{S}{F} = \binom{m-t+s-2\lambda}{s} \).

**Example 6:** When \( m = 4 \), \( \omega = 2 \), \( t = s = 2 \), \( \lambda = 1 \), the array obtained by Construction 1 is the array in Example 3. Based on Partition 1 for each vector \( e \), we have \( h_e = \binom{m-t}{s} = 2 \) and the partition for \( \mathcal{E}_e \) has been already showed in Example 5 where \( \mathcal{E}_e = \mathcal{X}_{e,00} \bigcup \mathcal{X}_{e,01} \). Then from Construction 2, the obtained PDA is also as same as (6, 2, 12) PDA in Example 5.

**Remark 1:** The PDA in Theorem 3 is exactly the result of Theorem 3 in [26]. Furthermore, the authors in [10] showed that the color number of strong coloring of the bipartite graph generated in [10] is minimum. This implies that given the parameter \( K, F, Z \) and the position of stars placed according to the bipartite graph in [10], the PDA obtained by the bipartite graph in [10] has the minimum value of \( S \). So when we take the set \( A \) and \( B \) in Theorem 3 our Partition 1 has the minimum cardinality for each \( e \) in the PDA generated by Construction 1.

2) The second case: Given \( K = F = 2^m \) for any positive integer \( m \), we have the following result.

**Theorem 4:** For any positive integers \( m, \omega \) with \( \omega < m \), there exists a \((2^m, 2^m, 2^m - \binom{m}{s}, 2^m - \binom{m}{t})\) PDA which can realize a \((K, M, N)\) coded caching scheme with \( \frac{M}{N} = 1 - \binom{m}{t}/2^m \), the subpacketization \( F = 2^m \) and the transmission rate \( R = \binom{m}{s}/2^m \).

**Proof.** Let \( A = B = [0, 2)^m \). From Construction 1, we first obtain a \(2^m \times 2^m\) array \( P \). From 1 the number of no-star entries in each column is \( \binom{m}{s} \) so that \( Z = 2^m - \binom{m}{s} \) and then the memory ratio is \( \frac{M}{N} = \frac{Z}{F} = 1 - \binom{m}{t}/2^m \). Since \( A = B = [0, 2)^m \), the collection of all vectors occurring in \( P \) is exactly the collection of all binary vectors with Hamming weight \( \omega \), i.e., the number of vectors occurring in \( P \) is \( S' = \binom{m}{s} \). By Partition 1 for any vector \( e \) we have

\[
\begin{align*}
\mathcal{E}_e &= |\{ t \in [0, 2)^m | \mathcal{X}_{e,t} \neq \emptyset \}| \\
&= |\{ e_{a,b}^{(1)} | a \in A, b \in B, (a, b) \in \mathcal{E}_e \}| \\
&= \binom{m-s}{s} \omega = 2^m - \omega.
\end{align*}
\]

From Theorem 2 we have \( S = S' h_e = \binom{m}{s} 2^m - \omega \). Then transmission rate \( R = \frac{S}{F} = \binom{m}{s}/2^m \).

**Example 7:** When \( m = 3 \), \( \omega = 2 \), and \( A = B = [0, 2)^3 \), the following \( 8 \times 8 \) array \( P \) can be obtained by Construction 1:

| a \ b | 000 010 100 110 001 011 101 111 |
|------|------------------------|
| 000 * | * 110 101 011 11 * |
| 100 * | 110 * 011 * 11 01 |
| 010 * | 110 * 011 101 * 01 |
| 110 110 * | * * 011 101 * |
| 001 * | 101 011 * * 11 |
| 111 111 * | * 110 111 * |
From Partition 1 we have $E_{110, 0} = X_{110, 0} \cup X_{110, 1} = \{(110, 000), (000, 110), (010, 100), (100, 010)\} \cup \{(111, 001), (001, 111), (011, 101), (101, 111)\}

$E_{010, 0} = X_{101, 0} \cup X_{101, 1} = \{(101, 000), (000, 101), (010, 100), (100, 010)\} \cup \{(111, 010), (011, 111), (011, 101), (101, 111)\}

$E_{001, 0} = X_{011, 0} \cup X_{011, 1} = \{(011, 000), (000, 111), (001, 010), (010, 011)\} \cup \{(111, 100), (100, 111), (101, 111), (110, 101)\}$

Then from Construction 2 a $(8, 8, 5, 6)$ PDA $P'$ can be obtained as follows.

| $a \cdot b$ | 000 | 100 | 010 | 110 | 001 | 101 | 011 | 111 |
|--------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 000          | *   | *   | 110,0 | *   | 101,0 | * | 011,0 | * |
| 100          | *   | *   | 110,0 | *   | 011,0 | * | 101,1 | * |
| 010          | *   | 110,0 | *   | 011,0 | *   | * | 101,1 | * |
| 110          | 110,0 | *   | * | 011,1 | 101,1 | * | * | 110,1 |
| 001          | *   | 101,0 | 011,0 | * | * | * | * | 110,1 |
| 101          | 101,0 | *   | 011,1 | * | * | 110,1 | * | * |
| 011          | 011,0 | * | 101,1 | * | 110,1 | * | * | * |
| 111          | * | 011,1 | 101,1 | * | 110,1 | * | * | * |

In fact, when we take $A = B = [0, 2]^m$ in Theorem 4 for each vector $e$ in the PDA generated by Construction 1 we can further merge some elements of Partition 1 to obtain a new partition with smaller cardinality.

**B. The improved partition for $A = B = [0, 2]^m$**

When $A = B = [0, 2]^m$ in Theorem 4 we improve the Partition 1 by merging some elements. It is worth noting that the improved partition proposed in the following can be used to any sets $A$ and $B$. Due to length limitations, we omit the discussions of $A$ and $B$ in general case.

**Partition 2:** For any vector $e$ in $P$, there always exists a $l_e$-partition $D = \{D_0, D_1, \ldots, D_{l_e-1}\}$ for $[0, 2)^{m-\omega}$ such that the distance of any two different vectors in $D_i$, $i \in [0, l_e)$ is at least $\omega + 1$. For $X_e$ in Partition 1 and for each $D_i$, let $Y_e, D_i = \bigcup_{t \in D_i} X_{e, t}, \quad X_{e, t} \subseteq X_e$.

Therefore $E_e = \bigcup_{i=0}^{l_e-1} Y_e, D_i$. Then partition $Y_e = \{Y_e, D_0, Y_e, D_1, \ldots, Y_e, D_{l_e-1}\}$ is an $l_e$-partition for $E_e$ for some integer $l_e$.

Clearly the above $l_e$-partition $Y_e$ for $E_e$ is decided by the $l_e$-partition $D$ for $[0, 2)^{m-\omega}$.

**Proposition 3:** The partition $P$ satisfies Property 1.

**Proof.** From Proposition 2 it is sufficient to consider any two different entries $(a_1, b_1) \in X_{e, t_1}$ and $(a_2, b_2) \in X_{e, t_2}$, $t_1, t_2 \in D_i, i \in [0, l_e), j_1 \neq j_2$. By (4) we have $t_1 = a_1 | c_{e, t_1}$, $t_2 = b_2 | c_{e, t_2}$. From Partition 2 we have $d(a_1, b_2) \geq d(a_1 | c_{e, t_2}, b_2 | c_{e, t_1}) = d(t_1, t_2) \geq \omega + 1$. Clearly $d(a_1, b_2) \neq \omega$ always holds. The proof is complete.

From Theorem 2 the PDA based on Partition 2 has parameter $S = \sum_{e \in P} l_e$. So we only need to consider the value of $l_e$ in partition $D$ for each $e$. By means of the result on the vertex coloring, according to the values of $\omega$ and $m$ the following result can be obtained.

**Lemma 1:** When $A = B = [0, 2]^m$, for each vector $e$ in $P$ generated by Construction 1 the cardinality $l_e$ of Partition 2 can be obtained as follows.

- When $m < 2\omega + 1$, $l_e = 2^{m-\omega}$, i.e., we can not merge any two elements in Partition 1
- When $m = 2\omega + 1$, $l_e = 2^{m-\omega-1}$.
- When $m > 2\omega + 1$, $l_e \leq 1 + \sum_{i=1}^{\omega} (m-\omega)$.

The proof is referred to Appendix A. From the first condition of Lemma 1 the scheme obtained by Partition 2 is exact the scheme in Theorem 4 When $m \geq 2\omega + 1$, from Proposition 3 and Theorem 2 the following results can be obtained.

**Theorem 5:** For any positive integers $m$, $\omega$ with $m \geq 2\omega + 1$, there exists a PDA which can realize a $(K, M, N)$ coded caching scheme with $F = K = 2^m, M = 1 - \binom{m}{\omega}/2^m$ and

- if $m = 2\omega + 1, R = \binom{m}{\omega}/2^{2\omega+1}$,
- if $m > 2\omega + 1, R \leq \binom{m}{\omega}(1 + \sum_{i=1}^{\omega} (m-\omega))/2^m$.

Let us consider the performance of the schemes in Theorem 4 and Theorem 5. For any positive integers $m$ and $\omega$ with $m > 2\omega + 1$, we can obtain two schemes, say Scheme 1 and Scheme 2, from Theorem 4 and Theorem 5 respectively for the
same user number, subpacketization and memory size. Denote the transmission rate of Scheme 1 by $R_{Th}^4$ and the transmission rate of Scheme 2 by $R_{Th}^5$. Then we have

$$\frac{R_{Th}^5}{R_{Th}^4} \leq \frac{(m)(1 + \sum_{i=1}^{m-\omega} (m-\omega))}{2^m} \cdot \frac{2^\omega}{(m)^{\omega}} = \frac{\sum_{i=0}^{m-\omega} (m-\omega)}{2^{m-\omega}} < 1$$

(5)

Clearly we can see that $\frac{R_{Th}^5}{R_{Th}^4}$ decreases with the increasing of the value of $m - \omega - \omega = m - 2\omega$. This implies that compared with the scheme in Theorem 4, the transmission rate of the scheme in Theorem 5 reduces with the increasing of the value $m - 2\omega$. Furthermore when $m > 3\omega + 1$, let us consider 5 by the following inequality in [11]

$$\binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{k-1} < \binom{2n}{n} < \binom{2n}{n}$$

for any even nonnegative integers $n$ and $k$ with $0 \leq k \leq n$. Let $n = \frac{m-\omega}{2}$ and $k = \omega + 1$ with $m > 3\omega + 1$, then we have

$$\sum_{i=0}^{\omega} \binom{m-\omega}{i} < \binom{\omega+1}{\omega+1} \cdot 2^{m-\omega}.$$ 

Since

$$\frac{\binom{m-\omega}{\omega+1}}{\binom{m-\omega}{m-\omega}} = \frac{(m-\omega)!(\omega+1)!}{(m-2\omega-1)!}$$

$$= \frac{(m-\omega)(m-\omega+1)(m-\omega+2)\cdots(\omega+2)}{(m-2\omega-1)(m-2\omega-2)(m-2\omega-3)\cdots(\omega+2)}$$

$$\leq \frac{(m-\omega)^{\omega+1} - 2}{(m-2\omega-1)}$$

$$= \left(1 + \frac{\omega+1}{m-2\omega-1}\right)^{m-3\omega+2}$$

$$= O(e^{\omega+1})(m \rightarrow \infty)$$

(5) can be written as

$$\frac{R_{Th}^5}{R_{Th}^4} < \frac{1}{2^{m-3\omega+2}} \cdot \left(1 + \frac{\omega+1}{m-2\omega-1}\right)^{m-3\omega+2}$$

$$= O\left(e^{\frac{\omega+1}{2}}\right)(m \rightarrow \infty).$$

That is $R_{Th}^5$ is about $2^{m-3\omega}$ times smaller than $R_{Th}^4$. This implies that Partition 2 is very useful when $m > 3\omega + 1$.

V. THE PDA WITH PARAMETER $q = 3$

In this section, we consider $q = 3$. First, for any sets $A, B \subseteq [0, 3)^m$ we propose a partition $\mathcal{X}_e$ satisfying Property 1 for each $\mathcal{E}_e$ to construct PDAs. Furthermore, for parameter $m > \frac{3\omega}{2}$ we obtain a new partition with smaller cardinality by merging some sets of the original partition.

A. The partition for any $A$ and $B$

When $q = 3$, given a vector $e$ in $P$ obtained by Construction 1 for each element of $\mathcal{E}_e$, say $(a, b)$, we have that the cardinality of $C_{a-b}$ is $m - \omega$ by [1]. Then the following partition $\mathcal{X}_e$ of $\mathcal{E}_e$ can be obtained.

Partition 3: For any vector $e$ in $P$ obtained by Construction 1 and for each element $T \in \binom{[0, m)}{m-\omega}$, define

$$\mathcal{X}_{e,T} = \{(a, b) \mid (a, b) \in \mathcal{E}_e, C_{a-b} = T\}$$

(6)

Let $\mathcal{X}_e = \{\mathcal{X}_{e,T} \mid \mathcal{X}_{e,T} \neq \emptyset, T \in \binom{[0, m)}{m-\omega}\}$. It is not difficult to check that $\mathcal{X}_e$ is a partition of $\mathcal{E}_e$.

Proposition 4: Partition 3 satisfies Property 1.

Proof. For any vector $e = (e_0, e_1, \ldots, e_{m-1})$ in $P$, let us consider the $\mathcal{X}_e$ in Partition 3. For any $\mathcal{X}_{e,T} \in \mathcal{X}_e$, $T \in \binom{[0, m)}{m-\omega}$, if $|\mathcal{X}_{e,T}| = 1$, our statement always holds. When $|\mathcal{X}_{e,T}| \geq 2$, any two different entries $(a_1, b_1), (a_2, b_2) \in \mathcal{X}_{e,T}$, let us show they satisfy Property 1. Since $(a_1, b_1), (a_2, b_2) \in \mathcal{E}_e$, by [3] we have

$$e = a_1 + b_1 = a_2 + b_2.$$
From (6), we have
\[ \mathbf{a}_1|_\tau = \mathbf{b}_1|_\tau, \; \mathbf{a}_2|_\tau = \mathbf{b}_2|_\tau \]
holds. Since all the operation under mod \( q = 3 \),
\[ \mathbf{a}_1|_\tau = \mathbf{b}_1|_\tau = \mathbf{a}_2|_\tau = \mathbf{b}_2|_\tau \quad (8) \]
Denote
\[ \mathbf{a}_1 = (a_{1,0}, a_{1,1}, \ldots, a_{1,m-1}), \quad \mathbf{a}_2 = (a_{2,0}, a_{2,1}, \ldots, a_{2,m-1}), \]
\[ \mathbf{b}_1 = (b_{1,0}, b_{1,1}, \ldots, b_{1,m-1}), \quad \mathbf{b}_2 = (b_{2,0}, b_{2,1}, \ldots, b_{2,m-1}). \]
Since \( d(\mathbf{a}_1, \mathbf{b}_1) = d(\mathbf{a}_2, \mathbf{b}_2) = \omega \), for any \( j \in [0, m) \setminus \mathcal{T} \)
\[ a_{1,j} \neq b_{1,j}, \quad a_{2,j} \neq b_{2,j}, \quad (9) \]
holds. Then we can get \( a_{1,j} = b_{2,j} \) or \( a_{1,j} = a_{2,j} \) by (7) and (9). If \( a_{1,j} = a_{2,j} \) holds for each \( j \in [0, m) \setminus \mathcal{T} \), we have \( \mathbf{a}_1 = \mathbf{a}_2 \) by (8). Then \( \mathbf{b}_1 = \mathbf{b}_2 \) by (7). This contradicts our hypothesis \( (\mathbf{a}_1, \mathbf{b}_1) \neq (\mathbf{a}_2, \mathbf{b}_2) \). So there are at least one coordinate, say \( j' \in [0, m) \setminus \mathcal{T} \), such that \( a_{1,j'} = b_{2,j'} \) holds. Then we have \( d(\mathbf{a}_1, \mathbf{b}_2) = d(\mathbf{a}_2, \mathbf{b}_1) < \omega \). So the partition \( \mathcal{X}_e \) satisfies the Property 1. \( \square \)

For any positive integers \( m \), let \( \mathcal{A} = \mathcal{B} = [0, 3)^m \). The following result can be obtained.

**Theorem 6:** For any positive integers \( m, \omega \) with \( \omega < m \), there exists a \( (3^m, 3^m, 3^m - (\omega)^2, (\omega)^3) \) PDA which gives a \( (K, M, N) \) coded caching scheme with \( \frac{M}{N} = 1 - \frac{(\omega)^2}{3^m} \), the subpacketization \( F = 3^m \) and the transmission rate \( R = \frac{m}{3^m} \).

**Proof.** Let \( \mathcal{A} = \mathcal{B} = [0, 3)^m \). From Construction 1 we can obtain a \( 3^m \times 3^m \) array \( \mathbf{P} \). Since for any vector \( \mathbf{b} \in \mathcal{B} \) there are \( (\omega)^2 \) vectors \( \mathbf{a} \in \mathcal{A} \) where \( d(\mathbf{a}, \mathbf{b}) = \omega \) from (1). We have \( Z = 3^m - (\omega)^2 \omega \) and the memory ratio \( \frac{M}{N} = \frac{Z}{F} = 1 - \frac{(\omega)^2}{3^m} \). Since for any vector \( \mathbf{e} \in [0, 3)^m \) there always exists vectors \( \mathbf{a} \in \mathcal{A} \) and \( \mathbf{b} \in \mathcal{B} \) where \( \mathbf{e} = \mathbf{a} + \mathbf{b}, d(\mathbf{a}, \mathbf{b}) = \omega \), the number of vectors occurring in \( \mathbf{P} \) is \( S' = 3^m \). By Partition 3 for any vector \( \mathbf{e} \), we have
\[ h_e = |\mathcal{X}_e| = |\{ \mathbf{T} \in [(0, m)] \mid \mathcal{X}_{e,T} \neq \emptyset \}| = |\{ \mathbf{C}_{a-b} \mid \mathbf{a} \in \mathcal{A}, \mathbf{b} \in \mathcal{B}, (\mathbf{a}, \mathbf{b}) \in \mathcal{E}_e \}| = \frac{m}{\omega} \]

Then from Theorem 2 we have \( S = S'h_e = \frac{(\omega)^3}{3^m} \). So the transmission rate \( R = \frac{S}{F} = \frac{(\omega)^3}{3^m} \). Then the proof is completed. \( \square \)

**Example 8:** When \( m = 3, \omega = 2, \mathcal{A} = \mathcal{B} = [0, 3)^m \), for any vector \( \mathbf{e} \in \mathbf{P} \) obtained by Construction 1, we have a 3-partition \( \mathcal{X}_e = \{ \mathcal{X}_{e,0}, \mathcal{X}_{e,1}, \mathcal{X}_{e,2} \} \) from Partition 3. From Theorem 6 a \( (27, 27, 15, 81) \) PDA \( \mathbf{P}' \) can be obtained. For vector \( \mathbf{e} = 110 \) in \( \mathbf{P} \), the rows and the columns from \( \mathcal{E}_{110} \) in \( \mathbf{P}' \) form the following \( 12 \times 12 \) sub-array. It is easy to check that this sub-array satisfies Condition 2 of a PDA.

| a \ b | 000 | 010 | 100 | 110 | 201 | 211 | 212 | 202 | 021 | 012 | 022 | 122 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 000   | *   | *   | 110,2 | 201,1 | *   | 202,1 | *   | 021,0 | *   | 022,0 | *   |     |
| 100   | *   | 110,2 | 001,1 | *   | 002,1 | *   | *   | 221,0 | *   | 222,0 | *   |     |
| 010   | *   | 110,2 |       | *   | 221,1 | *   | 222,1 | 001,0 | *   | 002,0 | *   |     |
| 110   | 110,2 |       |       | *   |       | *   | 221,1 |       | *   | 201,0 | *   | 202,0 |
| 201   | 201,1 | 001,1 |       | *   |       | *   |       | 110,0 | 222,2 | 022,2 |       | *   |
| 211   |       |       |       |       |       |       |       |       | 110,0 |       |       |       |
| 202   | 202,1 | 002,1 |       | *   |       | *   |       | 110,0 |       |       |       |       |
| 212   | *   | 222,1 | 002,1 | 110,0 | *   |       | *   |       |       |       |       |       |
| 021   | 021,0 |       |       |       |       |       |       |       |       |       |       |       |
| 121   |       |       |       |       |       |       |       |       |       |       |       |       |
| 122   |       |       |       |       |       |       |       |       |       |       |       |       |

In fact, when \( m > \frac{3^2}{2} \) for each vector \( \mathbf{e} \) in the PDA generated by Construction 1 we can further merge the elements of Partition 3 to obtain a new partition with smaller cardinality. That is the contents of the following subsection.
B. The improved partition when \( m > \frac{3\omega}{2} \)

**Partition 4:** When \( m > \frac{3\omega}{2} \), for any vector \( e \) in \( P \), there always exists a \( l_e \)-partition \( D = \{ D_0, D_1, \ldots, D_{l_e-1} \} \) for \( \left( \frac{0,m}{m-\omega} \right) \), such that the cardinality of the intersection of any two elements from \( D_i \), \( i \in [0,l_e) \) is less than \( m - \frac{3\omega}{2} \). For \( X_e \) in Partition 3 and for each \( D_i \), let

\[
\mathcal{Y}_{e,D_i} = \bigcup_{T \in D_i} X_e, T \in X_e.
\]

Therefore \( \mathcal{E}_e = \bigcup_{i=0}^{l_e-1} \mathcal{Y}_{e,D_i} \), then \( \mathcal{Y}_e = \{ \mathcal{Y}_{e,D_0}, \mathcal{Y}_{e,D_1}, \ldots, \mathcal{Y}_{e,D_{l_e-1}} \} \) is a \( l_e \)-partition for \( \mathcal{E}_e \).

Clearly the above \( l_e \)-partition \( \mathcal{Y}_e \) for \( \mathcal{E}_e \) is decided by the \( l_e \)-partition \( D \) for \( \left( \frac{0,m}{m-\omega} \right) \).

**Proposition 5:** Partition 4 satisfies Property 1.

**Proof.** From Proposition 1 we only need to consider any two different entries \((a_1, b_1) \in X_e, T_i\) and \((a_2, b_2) \in X_e, T_j\), \( T_j \in D_i \), \( i \in [0,l_e) \), \( j \neq j \). Let

\[
a_1 = (a_1,0,a_1,\ldots,a_{1,m-1}), \quad a_2 = (a_2,0,a_2,\ldots,a_{2,m-1}),
\]

\[
b_1 = (b_1,0,b_1,\ldots,b_{1,m-1}), \quad b_2 = (b_2,0,b_2,\ldots,b_{2,m-1}).
\]

We have \( T_{j_1} = C_{a_1-b_1} \) and \( T_{j_2} = C_{a_2-b_2} \) due to Partition 3 From Partition 4 we have \(|C_{a_1-b_1} \cap C_{a_2-b_2}| = |T_{j_1} \cap T_{j_2}| < m - \frac{3\omega}{2} \). From (3) we have that

\[
a_1 + b_1 = a_2 + b_2.
\]

Since for any \( s \in C_{a_1-b_1} \ \backslash \ (C_{a_1-b_1} \cap C_{a_2-b_2}) \), \( a_1,s = b_1,s \) and \( a_2,s \neq b_2,s \) always hold, we have 2\( a_1,s = a_2,s + b_2,s \). Hence \( a_1,s \neq b_2,s \). And for any \( s \in C_{a_2-b_2} \ \backslash \ (C_{a_1-b_1} \cap C_{a_2-b_2}) \), \( a_2,s = b_2,s \) and \( a_1,s \neq b_1,s \) always hold, we have \( a_2,s = a_1,s + b_1,s \). Hence \( a_1,s = b_2,s \). So \( d(a_1, b_2) \geq 2(m - \omega - |C_{a_1-b_1} \cap C_{a_2-b_2}|) = 2(m - \omega - |T_{j_1} \cap T_{j_2}|) > \omega \). Hence \( d(a_1, b_2) \neq \omega \) always holds. Then \( \mathcal{Y}_e \) satisfies the Property 1. The proof is completed.

Similarly to the proof of Lemma 1, taking advantage of vertex coloring the following result can be obtained.

**Lemma 2:** For any positive integers \( m, \omega \) with \( m > \frac{3\omega}{2} \), and for any vector \( e \) occurring in \( P \) obtained by Construction 1 in Partition 4 we have

\[
l_e \leq 1 + \sum_{i=0}^{m-\omega-1} \left( \frac{m-\omega}{i} \right) \left( \frac{\omega}{m-\omega-i} \right).
\]

The proof of Lemma 2 is presented in Appendix B.

When \( A = B = [0,3]^m \) in Theorem 6 the following schemes can be obtained by Lemma 2 and Theorem 2.

**Theorem 7:** For any positive integers \( m, \omega \) with \( m > \frac{3\omega}{2} \), there exists a PDA which can realize a \((K,M,N)\) coded caching scheme with \( F = K = 3^m, \ \frac{M}{N} = 1 - \left( \frac{2^\omega}{3^m} \right), \) the subpacketization \( F = 3^m \) and the transmission rate \( R \leq 1 + \sum_{i=0}^{m-\omega-1} \left( \frac{m-\omega}{i} \right) \left( \frac{\omega}{m-\omega-i} \right) \).

**VI. PERFORMANCE ANALYSIS**

In this section, we compare the performance of the proposed schemes in Theorem 5 and Theorem 7 with the Ali-Niesen scheme in [15] and the scheme in [2], which are both listed in Table 1.

**A. The performance of Theorem 5**

1) The comparison with Ali-Niesen Scheme in [15]: It is hard to propose the comparison between the scheme in Theorem 5 and the Ali-Niesen scheme for any positive integers \( m \) and \( \omega \). So we take \( \omega = 1 \) and \( m > 3 \) as an example. From Theorem 5 we have a PDA \( A_1 \) which gives a scheme with

\[
K = 2^m, \quad F_1 = 2^m, \quad \frac{M}{N} = 1 - \frac{m}{2^m}, \quad R_1 = \frac{m^2}{2^m}.
\]

Let \( K = 2^m \) and \( t = 2^m - m \). From [15] we have a Ali-Niesen \((2^m, (2^m), (2^m), (2^m))\) PDA \( B_1 \) which gives a scheme with

\[
K = 2^m, \quad F_{MN} = \left( \frac{2^m}{m} \right), \quad \frac{M}{N} = 1 - \frac{m}{2^m}, \quad R_{MN} = \frac{m}{2^m - m + 1}.
\]
For the same values of \( K \) and \( \frac{M}{N} \) of our obtained PDA \( A_1 \) and Ali-Niesen PDA \( B_1 \), we have
\[
\frac{F_1}{F_{MN}} = \frac{2^m}{(\frac{m}{2m})^m} \geq \frac{2^m}{(\frac{m}{2m})^m} = \frac{1}{(\frac{m}{2m})^m K^{m-1}},
\]
where (10) uses inequality \((\frac{m}{2m})^m < (em/t)^t\).

For the fixed \( K \) and \( \frac{M}{N} \), \( F_1 \) is at least \( \frac{1}{mK} \) times smaller than \( F_{MN} \) by (10) while \( R_1 \) is at most \( \log_2 K \) times larger than \( R_{MN} \) by (11). Clearly \((\frac{m}{2m})^m K^{m-1}\) is exponential with \( K \). Similarly we can also discuss the value of \( \omega \) and \( m \) for the other case. Due to the space limitation, we take some examples listed in the following table.

### TABLE III: The comparisons between the scheme in Theorem 5 and the Ali-Niesen scheme in [15]

| Scheme | K | F | M/N | R |
|--------|---|---|-----|---|
| \( m = 3, \omega = 1 \) in Theorem 5 | 8 | 8 | 0.625 | 0.75 |
| \( K = 8, t = 5 \) in [15] | 8 | 56 | 0.625 | 0.5 |
| \( m = 4, \omega = 1 \) in Theorem 5 | 16 | 16 | 0.750 | 1 |
| \( K = 16, t = 12 \) in [15] | 32 | 1820 | 0.750 | 0.308 |
| \( m = 5, \omega = 1 \) in Theorem 5 | 32 | 32 | 0.844 | \( \leq 0.781 \) |
| \( K = 32, t = 27 \) in [15] | 32 | 201376 | 0.844 | 0.179 |
| \( m = 5, \omega = 2 \) in Theorem 5 | 32 | 6451240 | 0.687 | 1.25 |
| \( K = 32, t = 22 \) in [15] | 32 | 6451240 | 0.687 | 0.435 |

When \( m = 3, \omega = 1 \) in Theorem 5 let \( K = 8, t = 5 \) in [15]. When \( m = 4, \omega = 1 \) in Theorem 5 let \( K = 16, t = 12 \) in [15]. When \( m = 5, \omega = 1 \) in Theorem 5 let \( K = 32, t = 27 \) in [15]. When \( m = 5, \omega = 2 \) in Theorem 5 let \( K = 32, t = 22 \) in [15].

In this table, given same user number \( K \) and memory ratio \( \frac{M}{N} \), we can see that our scheme in Theorem 5 has advantage of subpacketization.

2) The comparison with the scheme in [2]: In [2] it’s worth noting that for any positive integers \( v \) and \( k \) the existence of PDA depends on the existence of symmetric \((v, k, 2)\)-BIBD with non repeated blocks. While for symmetric \((v, k, 2)\)-BIBD, we have the following remark.

#### Remark 2: ([3]) It is well known that the necessary condition of a symmetric \((v, k, 2)\)-BIBD is that \( k = \frac{1+\sqrt{8v-7}}{2} \) is an integer. Furthermore up to now there exists a symmetric \((v, k, 2)\)-BIBD when \( v \) and \( k \) satisfying one of the following conditions

- If \( v \) is even, then \( k - 2 \) is a square.
- If \( v \) is odd, then the equation \( z^2 = (k - 2)x^2 + (-1)^{\frac{v-1}{2}} 2y^2 \) has a solution in integers \( x, y, z \) not all zero.

From the above remark, we can see that there are a few results on the existence of the symmetric BIBDs. This implies that using symmetric BIBDs, a few PDAs can be obtained.

Since the scheme generated by the PDA in [2] has large memory ratio, we consider the case \( m > 2\omega + 1 \) in Theorem 5. Then we have a PDA \( A_2 \) which gives a scheme with
\[
K = 2^m, \ F_1 = 2^m, \ M = 1 - \frac{(m)}{2m}, \ R_1 \leq \frac{(m)}{2m}(1 + \sum_{i=1}^{\omega} \frac{(m-i)}{2m}).
\]

Let \( v = 2^m \) and \( k = \frac{(m)}{\omega} + 1. \) Assume that there exists a \((2^m, \frac{(m)}{\omega} + 1, 2)\) BIBD. Then from [2], we have a \((2^m, 2^m(\frac{(m)}{\omega}) + 1, \frac{(m)}{\omega} + 1)(2^m - (\frac{(m)}{\omega})), 2^m(\frac{(m)}{\omega} + 1)\) PDA \( B_2 \) which gives a scheme with
\[
K = 2^m, \ F_2 = 2^m(\frac{(m)}{\omega} + 1), \ M = 1 - \frac{(m)}{2m}, \ R_2 = 1.
\]

For the fixed \( K \) and \( \frac{M}{N} \) of our obtained PDA \( A_2 \) and PDA \( B_2 \), we have
\[
\frac{F_1}{F_2} = \frac{2^m}{2^m(\frac{(m)}{\omega} + 1)} = \frac{1}{(\frac{(m)}{\omega} + 1)},
\]
\[
\frac{R_1}{R_2} \leq \frac{(m)}{2m}(1 + \sum_{i=1}^{\omega} \frac{(m-i)}{2m}).
\]

By (12) and (13), the following statements hold.
When \((\frac{m}{\omega})(1 + \sum_{i=1}^{\omega} (\frac{m}{i^2}))\) < 1, for the same user number \(K\) and memory ratio \(\frac{M}{N}\), our scheme has both smaller transmission rate and the subpacketization than those of the scheme generalized by \(B_2\), i.e., the PDA in [2].

For example when \(m = 4, \omega = 1\) in Theorem 5 we let \(v = 16, k = 5\) in [2]. When \(m = 5, \omega = 1\) in Theorem 5 we let \(v = 32, k = 6\) in [2]. When \(m = 6, \omega = 1\) in Theorem 5 we let \(v = 32, k = 7\) in [2]. Then we have the following table.

| Scheme | K | F | M/N | R |
|--------|---|---|-----|---|
| \(m = 4, \omega = 1\) in Theorem 5 | 16 | 16 | 0.750 | < \(1\) |
| \(v = 16, k = 5\) in [2] | 16 | 80 | 0.750 | \(1?\) |
| \(m = 5, \omega = 1\) in Theorem 5 | 32 | 32 | 0.844 | \(\leq 0.781\) |
| \(v = 32, k = 6\) in [2] | 32 | 192 | 0.844 | \(1?\) |
| \(m = 6, \omega = 1\) in Theorem 5 | 64 | 64 | 0.906 | \(\leq 0.563\) |
| \(v = 32, k = 7\) in [2] | 64 | 448 | 0.906 | \(1?\) |

The symbol \(?\) in this table means that from Remark 2 there exists the \((\epsilon, kv, k(v - k + 1), kv)\) PDA for the parameters \(v\) and \(k\) in [2].

When \((\frac{m}{\omega})(1 + \sum_{i=1}^{\omega} (\frac{m}{i^2}))\) > 1 we claim that compared with the scheme realized by \(B_2\), the reduction in the packets number of our scheme is at least \(2^\omega\) times larger than the increase in the transmission rate since

\[
\frac{1}{(\frac{m}{\omega}) + 1} \cdot \frac{(\frac{m}{\omega})(1 + \sum_{i=1}^{\omega} (\frac{m}{i^2}))}{2^m} < \frac{1}{(\frac{m}{\omega}) + 1} \cdot \frac{(\frac{m}{\omega})2^{m-\omega}}{2^m} = \frac{1}{(\frac{m}{\omega}) + 1} \cdot \frac{(\frac{m}{\omega})}{2^\omega} \\
\approx \frac{1}{2^\omega}.
\]

B. The performance of Theorem 2

1) The comparison with Ali-Niesen Scheme in [15]: It is hard to propose the comparison between the scheme in Theorem 7 and the Ali-Niesen scheme for any positive integers \(m\) and \(\omega\). So we take \(\omega = 2\) as an example. From Theorem 7, we have a PDA \(A_3\) which gives a scheme with

\[K = 3^m, \quad F_1 = 3^m, \quad \frac{M}{N} = 1 - \frac{2m(m-1)}{3^m}, \quad R_1 \leq 2m - 3.\]

Let \(K = 3^m\) and \(t = 3^m - 2m(m - 1)\). From [15] we have a Ali-Niesen \((3^m, (3^m), (3^m), (3^m), (3^m))\) PDA \(B_3\) which gives a scheme with

\[K = 3^m, \quad F_{MN} = \left(\frac{3^m}{2m(m-1)}\right), \quad \frac{M}{N} = 1 - \frac{2m(m-1)}{3^m}, \quad R_{MN} = \frac{2m(m-1)}{3^m - 2m(m - 1) + 1}.\]

Given the value of \(K\) and \(\frac{M}{N}\) of our obtained PDA \(A_3\) and Ali-Niesen PDA \(B_3\), we have

\[
\frac{F_1}{F_{MN}} = \frac{3^m}{(2m(m-1))} \geq \frac{1}{(2m(m-1)) \cdot 2^m}, \quad R_1 \leq (2m - 3)(3^m - 2m(m - 1) + 1) \approx \frac{K}{m - 1} \quad (\text{14})
\]

\[
\frac{R_{MN}}{R_1} \leq \frac{2m - 3)(3^m - 2m(m - 1) + 1)}{2m(m - 1)} \approx \frac{K}{m - 1} \quad (\text{15})
\]

where (14) refers to inequality \((\frac{m}{\omega}) < (em/t)^t\).

For the fixed \(K\) and \(\frac{M}{N}\), \(F_1\) is at least \(\frac{1}{(2m(m-1))^{2m(m-1)-1}}\) times smaller than \(F_{MN}\) by (14) while \(R_1\) is at most \(\frac{K}{m-1}\) times larger than \(R_{MN}\) by (15). Clearly

\[
\frac{e \cdot 2^m}{(2m(m-1))^{2m(m-1)-1} \cdot 2^{2m(m-1)-1}} = \frac{1}{m-1} \cdot \frac{e \cdot 2^m}{2^{2m(m-1)-1}} \cdot \frac{(K/m-1)^{2m(m-1)-1}}{m-1}
\]

is exponential with \(\frac{K}{m-1}\). This implies that the scheme realized by \(A_3\) significantly reduces the subpacketization while just increases several time transmission rate compared with the Ali-Niesen scheme realized \(B_3\). Similarly we can also discuss the value of \(\omega\) and \(m\) for the other case. Due to the space limitation, we have a table as follows.
TABLE V: The comparisons between the scheme in Theorem 7 and Ali-Niesen Scheme in [15]

| Scheme | $K$ | $F$ | $\frac{M}{N}$ | $R$ |
|--------|-----|-----|-------------|-----|
| $m = 3, \omega = 2$ in Theorem 7 | 27 | 27 | 0.556 | $\leq 3$ |
| $K = 27, t = 15$ in [15] | 27 | $10^7.24$ | 0.556 | 0.75 |
| $m = 4, \omega = 3$ in Theorem 7 | 81 | 81 | 0.605 | $\leq 4$ |
| $K = 81, t = 49$ in [15] | 81 | $10^{22.559}$ | 0.605 | 0.64 |
| $m = 5, \omega = 3$ in Theorem 7 | 243 | 243 | 0.671 | $\leq 4$ |
| $K = 243, t = 163$ in [15] | 243 | $10^{65.604}$ | 0.671 | 0.488 |

In this table, given same user number $K$ and memory ratio $\frac{M}{N}$, we can see that our scheme in Theorem 7 has advantage of subpacketization at the cost of transmission rate compared to Ali-Niesen Scheme.

2) The comparison with the scheme in [2]: From Theorem 7 we have a PDA $A_4$ which gives a scheme with

$$K = 3^m, \quad F_1 = 3^m, \quad M = 1 - \binom{m}{\omega}2^\omega 3^m \quad R_1 \leq 1 + \sum_{i=|m-\omega|}^{m-\omega-1} \binom{m-\omega}{m-\omega-i} \left( \frac{\omega}{m-\omega} \right).$$

Let $v = 3^m$ and $k = \binom{m}{\omega}2^\omega + 1$. Assume there exists a $(3^m, \binom{m}{\omega}2^\omega + 1, 2)$ BIBD. Then from [2], we have a $(3^m, 3^m(\binom{m}{\omega}2^\omega + 1), (\binom{m}{\omega}2^\omega + 1)(3^m - (\binom{m}{\omega}2^\omega)), 3^m(\binom{m}{\omega}2^\omega + 1))$ PDA $B_4$ which gives a scheme with

$$K = 3^m, \quad F_2 = 3^m(\binom{m}{\omega}2^\omega + 1), \quad 1 - \frac{M}{N} = \frac{(\binom{m}{\omega}2^\omega)}{3^m}, \quad R_2 = 1.$$

For the fixed $K$ and $\frac{M}{N}$ of our obtained PDA $A_4$ and PDA $B_4$, we have

$$\frac{R_1}{R_2} \leq 1 + \sum_{i=|m-\omega|}^{m-\omega-1} \binom{m-\omega}{m-\omega-i} \left( \frac{\omega}{m-\omega} \right).$$

From (16) and (17), for the fixed $K$ and $\frac{M}{N}$ we can see that compared with the scheme realized by $B_4$, the reduction in the subpacketization of our scheme is at least $2^\omega$ times larger than the increase in the transmission rate since

$$\frac{1}{(\binom{m}{\omega}2^\omega + 1)} \left( 1 + \sum_{i=|m-\omega|}^{m-\omega-1} \binom{m-\omega}{m-\omega-i} \left( \frac{\omega}{m-\omega} \right) \right) \leq \frac{1 + (\binom{m}{\omega}2^\omega)}{1 + (\binom{m}{\omega}2^\omega)} \approx \frac{1}{2^\omega}.$$

In order to visually compare our scheme in Theorem 7 with the scheme in [2], we have a table as follows.

TABLE VI: The comparisons between the scheme in Theorem 7 and the scheme in [2]

| Scheme | $K$ | $F$ | $\frac{M}{N}$ | $R$ |
|--------|-----|-----|-------------|-----|
| $m = 3, \omega = 2$ in Theorem 7 | 27 | 27 | 0.556 | $\leq 3$ |
| $v = 27, k = 13$ in [2] | 27 | 351 | 0.556 | 1? |
| $m = 4, \omega = 3$ in Theorem 7 | 81 | 81 | 0.605 | $\leq 4$ |
| $v = 81, k = 33$ in [2] | 81 | 2673 | 0.605 | 1? |
| $m = 5, \omega = 3$ in Theorem 7 | 243 | 243 | 0.671 | $\leq 7$ |
| $v = 243, k = 81$ in [2] | 243 | 19683 | 0.671 | 1? |

When $m = 3, \omega = 2$ in Theorem 7 let $v = 27, k = 13$ in [2]. When $m = 4, \omega = 3$ in Theorem 7 let $v = 81, k = 33$ in [2]. When $m = 5, \omega = 3$ in Theorem 7 let $v = 243, k = 81$ in [2]. The symbol ? in this table means that from Remark 2 there does not exists the $(v, k, v(k - 1), kv)$ PDA for the parameters $v$ and $k$ in [2].

VII. Conclusion

In this paper, we proposed a framework of constructing PDAs via Hamming distance. Then constructing PDAs is transformed to constructing appropriate partition. As applications, we obtained two classes of coded caching schemes with linear subpacketization. Moreover, the comparison between new schemes obtained by our PDAs and previous known schemes showed that our new schemes have advantages of low subpacketization and small transmission rate for some parameters.
First the following notations are useful. A graph $G$ consists of a set $V(G)$ of vertexes and a set $E(G) \subseteq \{(u,v) : u,v \in V(G)\}$ of edges. The degree of a vertex $v$ in a graph $G$ is the number of vertices in $G$ that are adjacent to $v$. The largest degree among the vertices of $G$ is called the maximum degree of $G$ and is denoted by $\Delta(G)$. A vertex $k$-coloring of a graph $G$ is an assignment of $k$ colors to the vertices of $G$, one color to each vertex, so that adjacent vertices are colored differently. A graph $G$ is $k$-colorable if there exists a coloring of $G$ from a set of $k$ colors. The minimum positive integer $k$ for which $G$ is $k$-colorable is the chromatic number of $G$ and is denoted by $\chi(G)$.

**Lemma 3:** \cite{1} For every graph $G$, $\chi(G) \leq 1 + \Delta(G)$.

Now let us give the proof of Lemma\cite{1}

**Proof.** From Partition\cite{2} let us consider the following cases according to the values of $m$ and $\omega$.

- When $m < 2\omega + 1$, for any different vectors $t_j$, $t_k \in [0, 2)^{m-\omega}$, assume that $d(t_j, t_k) = x, x \in \{1, 2, \ldots, m-\omega\}$. Since $A = B = [0, 2)^{m}$, for any vector $e$ occurring in $P$ obtained by Construction\cite{1} $e$ occurring in every row and every column of $P$. And the collection of all vectors $a_{\alpha} b_{\omega}$ where $\alpha \in A$ is exactly $[0, 2)^{m}$. The collection of all vectors $b_{\omega} a_{\alpha} b_{\omega}$ where $b \in B$ is also exactly $[0, 2)^m$. Hence there always exists $a_{\alpha_1} b_{\omega_1} \in X^\alpha_{e, t_j}$, $a_{\alpha_2} b_{\omega_2} \in X^\alpha_{e, t_k}$ where $d(a_{\alpha_1} b_{\omega_1} a_{\alpha_2} b_{\omega_2}) = \omega - x$. Due to $d(a_{\alpha_1} b_{\omega_1} b_{\omega_2} a_{\alpha_2}) = d(t_j, t_k) = x$, we have $d(a_{\alpha_1}, b_{\omega_2}) = \omega$, which contradicts Property\cite{1}. Therefore any two elements in Partition\cite{1} can not be merged and $t_e = 2^{m-\omega}$.

- When $m = 2\omega + 1$, i.e., $m - \omega = \omega + 1$, for any vector $e$ occurring in $P$ obtained by Construction\cite{1} and for any two vectors $t_j$, $t_k \in [0, 2)^{m-\omega}$, from Partition\cite{2} we can see that $X^\alpha_{e, t_j}$ can merge with $X^\alpha_{e, t_k}$ if and only if $d(t_j, t_k) = \omega + 1 = m - \omega$.
Then the maximum degree \( \Delta(G) \) of vertices in \( G \) is \( \rho(t) \leq \sum_{i=1}^{\omega} \binom{m-\omega}{i} \). Then the maximum degree \( \Delta(G) \) of vertices in \( G \) is \( \sum_{i=1}^{\omega} \binom{m-\omega}{i} \). From Lemma 3 we have

\[
\chi(G) \leq 1 + \Delta(G) \leq 1 + \sum_{i=1}^{\omega} \binom{m-\omega}{i}.
\]  

(18)

From the definition of \( \chi(G) \), there exists a \( \chi(G) \)-coloring of \( G \). In fact the vertex \( \chi(G) \)-coloring of \( G \) corresponds to a \( \chi(G) \)-partition for \( [0,2]^{m-\omega} \) in Partition 2. In the vertex \( \chi(G) \)-coloring of graph \( G \), we make each collection of vertexes having same color as a subset of vertex set \( [0,2]^{m-\omega} \). Then there exists \( \chi(G) \) subsets \( D_0, D_1, \ldots, D_{\chi(G)} \). Since each vertex in \( G \) has exactly one color, each element in \( [0,2]^{m-\omega} \) is exactly contained in one subset. For any two vertexes \( t_1, t_2 \in T, i \in [0, \chi(G)] \), there exist no edge \( (t_1, t_2) \), i.e., \( d(t_1, t_2) \geq \omega + 1 \) which satisfying Partition 2. Hence \( \{D_0, D_1, \ldots, D_{\chi(G)}\} \) is a \( \chi(G) \)-partition for \( [0,2]^{m-\omega} \).

From the above discussion, for the vertex \( \chi(G) \)-coloring of \( G \), we always have a \( l_e = \chi(G) \)-partition for \( T \). From (18) we have \( l_e \leq 1 + \sum_{i=1}^{\omega} \binom{m-\omega}{i} \).

The proof of Lemma 1 is complete.

**APPENDIX B: THE PROOF OF LEMMA 2**

Proof. From Partition 4, we can see that the \( l_e \)-partition \( Y_e \) of set \( E_e \) is decided by \( l_e \)-partition for set \( \binom{[0,m]}{m-\omega} \). Similar to the proof of Lemma 1 in Appendix A, we turn it to a vertex coloring problem.

Given set \( \binom{[0,m]}{m-\omega} \) in Partition 4, define a graph \( G \) with vertex set \( V(G) = \binom{[0,m]}{m-\omega} \) such that there exists an edge connecting any two different vertices \( t_1, t_2 \in V(G) \) if and only if \( |T_1 \cap T_2| \geq m - \frac{3\omega}{2} \). For any vertex \( t \in V(G) \), the number of vertices in \( G \) that are adjacent to \( t \) is \( \sum_{i=m-\omega}^{m-\omega-1} \binom{m-\omega}{i} \binom{\omega}{\omega-i} \), i.e., the degree \( \rho(t) \leq \sum_{i=m-\omega}^{m-\omega-1} \binom{m-\omega}{i} \binom{\omega}{\omega-i} \). Then the maximum degree \( \Delta(G) \) of vertices in \( G \) is \( \sum_{i=m-\omega}^{m-\omega-1} \binom{m-\omega}{i} \binom{\omega}{\omega-i} \). From Lemma 3 we have

\[
\chi(G) \leq 1 + \Delta(G) \leq 1 + \sum_{i=m-\omega}^{m-\omega-1} \binom{m-\omega}{i} \binom{\omega}{\omega-i}.
\]  

(19)

From the definition of \( \chi(G) \), there exists a \( \chi(G) \)-coloring of \( G \). Similar to the proof of Lemma 1 in Appendix A, the vertex \( \chi(G) \)-coloring of \( G \) corresponds to a \( \chi(G) \)-partition for \( \binom{[0,m]}{m-\omega} \).

In the vertex \( \chi(G) \)-coloring of graph \( G \), we make each collection of vertexes having same color as a subset of vertex set. Then there exists \( \chi(G) \) subsets \( D_0, D_1, \ldots, D_{\chi(G)} \) of \( \binom{[0,m]}{m-\omega} \). Since each vertex in \( G \) has exactly one color, each element in \( \binom{[0,m]}{m-\omega} \) is exactly contained in one subset. For any two vertexes \( T_1, T_2 \in D_i, i \in [0, \chi(G)] \), there exist no edge \( (T_1, T_2) \), i.e., \( |T_1 \cap T_2| < m - \frac{3\omega}{2} \) which satisfying Partition 2. Hence \( \{D_0, D_1, \ldots, D_{\chi(G)}\} \) is a \( \chi(G) \)-partition for \( \binom{[0,m]}{m-\omega} \).

For the vertex \( \chi(G) \)-coloring of \( G \), we always have a \( l_e = \chi(G) \)-partition for \( \binom{[0,m]}{m-\omega} \). From (19) we have \( l_e \leq 1 + \sum_{i=m-\omega}^{m-\omega-1} \binom{m-\omega}{i} \binom{\omega}{\omega-i} \). The proof is complete.