CHARACTERIZATION OF PROJECTIVE SPACES BY SESHADRI CONSTANTS

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Abstract. We prove that an $n$-dimensional complex projective variety is isomorphic to $\mathbb{P}^n$ if the Seshadri constant of the anti-canonical divisor at some smooth point is greater than $n$. We also classify complex projective varieties with Seshadri constants equal to $n$.

1. Introduction

It is believed that the projective space $\mathbb{P}^n$ has the most positive anti-canonical divisor among complex projective varieties. Various characterizations of $\mathbb{P}^n$ have been found corresponding to different explanations of the “positivity” of the anti-canonical divisor. Using Kodaira vanishing theorem, Kobayashi and Ochiai [KO73] proved that if an $n$-dimensional projective manifold $X$ with an ample line bundle $H$ satisfies $-K_X \equiv (n+1)H$, then $(X,H) \cong (\mathbb{P}^n,\mathcal{O}(1))$. Kobayashi-Ochiai’s characterization was generalized by Ionescu [Ion86] (in the smooth case) and Fujita [Fuj87] (allowing Gorenstein rational singularities) assuming the weaker condition that $K_X + (n+1)H$ is not ample. Later, Cho, Miyaoka and Shepherd-Barron [CMSB02] (simplified by Kebekus in [Keb02]) showed that a Fano manifold is isomorphic to $\mathbb{P}^n$ if the anti-canonical degree of every curve is at least $n+1$. Their proofs rely on deformation of rational curves which still works if we allow isolated local complete intersection quotient singularities (see [CT07]). Besides, Kachi and Kollár [KK00] gave characterizations of $\mathbb{P}^n$ in arbitrary characteristic that generalized [KO73] and [CMSB02, Keb02] with a volume lower bound assumption.

The purpose of this paper is to provide a characterization of $\mathbb{P}^n$ among complex $\mathbb{Q}$-Fano varieties by the local positivity of the anti-canonical divisor, namely the Seshadri constants. Recall that a complex projective variety $X$ is said to be $\mathbb{Q}$-Fano if $X$ has klt singularities and $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

Definition 1. Let $X$ be a normal projective variety and $L$ an ample $\mathbb{Q}$-Cartier divisor on $X$. Let $p \in X$ be a smooth point. The Seshadri constant of $L$ at $p$, denoted by $\epsilon(L,p)$, is defined as

$$ \epsilon(L,p) := \sup \{ x \in \mathbb{R}_{>0} \mid \sigma^*L - xE \text{ is ample} \}, $$

where $\sigma : \text{Bl}_pX \to X$ is the blow-up of $X$ at $p$, and $E$ is the exceptional divisor of $\sigma$.

It is clear that $\epsilon(-K_{\mathbb{P}^n},p) = n + 1$ for any point $p \in \mathbb{P}^n$. Our main result characterizes $\mathbb{P}^n$ as the only $\mathbb{Q}$-Fano variety with Seshadri constant bigger than $n$:

Theorem 2. Let $X$ be a complex $\mathbb{Q}$-Fano variety of dimension $n$. If there exists a smooth point $p \in X$ such that $\epsilon(-K_X,p) > n$, then $X \cong \mathbb{P}^n$. 

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Note that Theorem 2 only assumes that \( \epsilon(-K_X, p) > n \) for some smooth point \( p \) rather than any smooth point (although the existence of such \( p \) immediately implies the same inequality for a general smooth point). We also remark here that when \( X \) is smooth, Theorem 2 was obtained by Bauer and Szemberg in \( [BS09, \text{Theorem 1.7}] \) using different methods.

Since the Seshadri constant of a quadric hypersurface in \( \mathbb{P}^{n+1} \) is equal to \( n \), the lower bound on the Seshadri constant in Theorem 2 is sharp. It turns out that this is not the only \( \mathbb{Q} \)-Fano varieties achieving such lower bound, and the full list is given by the following theorem.

**Theorem 3.** Let \( X \) be a \( n \)-dimensional complex \( \mathbb{Q} \)-Fano variety. Then there exists a smooth point \( p \in X \) with \( \epsilon(-K_X, p) = n \) if and only if \( X \) is one of the following:

1. a degree \( d+1 \) weighted hypersurface \( X_{d+1} = (x_0x_{n+1} = f(x_1, \cdots, x_n)) \subset \mathbb{P}(1^{n+1}, d) \),
2. a quartic weighted hypersurface \( X_4 = (x_{n+1} + x_n, h(x_0, \cdots, x_{n-1}) = f(x_0, \cdots, x_{n-1})) \) \( (h \neq 0) \) or \( (x_n, x_{n+1} = f(x_0, \cdots, x_{n-1})) \subset \mathbb{P}(1^n, 2, 2) \),
3. the blow-up of \( \mathbb{P}^n \) along the complete intersection of a hyperplane and a hypersurface of degree \( d \leq n \),
4. the quotient of the quadric \( Q_k = (\sum_{i=0}^{k} x_i^2 = 0) \subset \mathbb{P}^{n+1}(2 \leq k \leq n + 1) \) by an involution \( \tau(x_i) = \delta_i x_i (\delta_i = \pm 1) \) that is fixed point free in codimension 1 and such that not all the \( \delta_i(i = 0, \cdots, k) \) are the same,
5. a Gorenstein log Del Pezzo surface of degree \( \geq 4 \) (for the classification of such surfaces, see \[HVS\ §3\]).

When \( X \) is smooth, the condition \( \epsilon(-K_X, p) = n \) implies that \( (-K_X \cdot C) \geq n \) for any curve \( C \subset X \) passing through a very general point \( p \). If in addition \( X \) has dimension at least 3, then by \( [My04] \) and \( [CD15] \), \( X \) is either a quadric hypersurface or the blow-up of \( \mathbb{P}^n \) along a smooth subvariety of codimension 2 and degree \( d \leq n \) contained in a hyperplane. On the other hand, in the surface case some of our results have been proved by \( [San14, \text{Theorem 1.8}] \) under the somewhat restrictive assumption that \( (K_X^2) \in \{4, 5, 6, 7, 8, 9\} \).

Hence the above theorem is a natural generalization of their results to the singular and higher dimensional case, although our proof uses a completely different strategy.

Finally we show that in general the Seshadri constant \( \epsilon(-K_X, p) \) can be any rational number between 0 and \( n \). This is in sharp contrast with Theorem 2 where we have seen that there is a gap between \( n \) and \( n + 1 \) for the possible values of \( \epsilon(-K_X, p) \).

**Theorem 4.** For any rational number \( 0 < c \leq n \), there exists an \( n \)-dimensional \( \mathbb{Q} \)-Fano variety \( X \) with a smooth point \( p \) such that \( \epsilon(-K_X, p) = c \).

The paper is organized as follows. In Section 2 we prove Theorem 2. Denote the blow up of \( X \) at \( p \) by \( \sigma : \hat{X} = \text{Bl}_p X \to X \), then the divisor \( D := \sigma^*(-K_X) - \epsilon(-K_X, p)E \) is nef by the definition of the Seshadri constant. Under the assumption that \( \epsilon(-K_X, p) > n \), we use Kawamata-Viehweg vanishing theorem to show that \( D \) is semiample and \( g = |kD| : \hat{X} \to Y \) maps \( E \) isomorphically onto its image for sufficiently divisible \( k \). A simple computation yields that \( (-K_X \cdot C) = \epsilon(-K_X, p) - (n - 1) > 1 \) for any curve \( C \) contracted by \( g \). We show in Lemma 8 that \( g \) cannot be birational under these assumptions and therefore has to be a morphism of fiber type with target \( Y = g(E) \cong \mathbb{P}^{n-1} \). Then Lemma 6 implies that \( \hat{X} \) is a \( \mathbb{P}^{1} \)-bundle over \( \mathbb{P}^{n-1} \), thus \( X \cong \mathbb{P}^{n} \). The proof of Lemma 8 relies
on a dimension reduction argument and Lemma 5. As an application of Theorem 2 we show that $\mathbb{P}^n$ is the only Ding-semistable $\mathbb{Q}$-Fano variety of volume at least $(n + 1)^n$ (see Theorem 11). This improves the equality case of [Fuj15, Theorem 1.1] where Fujita proved for Ding-semistable Fano manifolds.

In Section 3, we classify all $\mathbb{Q}$-Fano varieties with Seshadri constants equal to $n$. By the same reason as the proof of Theorem 2, we still have that $D$ is semiample. We divide the classification into two parts. In Section 3.1, we study cases when $g$ is birational. We show that $g|_{E}$ is a closed embedding, $-(K_Y + g(E))$ is ample, $g(E)$ is nef (see Lemma 11). We classify such pairs $(Y, g(E))$ in Lemma 13. Then we obtain the partial classification after a detailed study of the structure of the birational morphism $g$ (see Lemma 14). In Section 3.2, we study cases when $g$ is of fiber type. It is not hard to see that every fiber of $g$ has dimension 1, the generic fiber of $g$ is isomorphic to $\mathbb{P}^1$, $g|_E : E \to Y$ is a double cover, and $-K_{\hat{X}}$ is $g$-ample. After pulling back $g$ to $E$ and taking the normalization, we obtain a conic bundle $\tilde{\hat{g}} : \tilde{X} \to E \cong \mathbb{P}^{n-1}$ with two sections (see Lemma 16, Corollary 17 and Lemma 18). From the classification of the conic bundle $\tilde{\hat{g}}$ and the quotient map $g|_{E}$ (see Lemma 19 and 20), we finish the classification of $X$ and hence prove Theorem 3.

Finally in Section 4, we provide examples showing that the Seshadri constant of a $\mathbb{Q}$-Fano variety can be any positive rational number less than $n$.

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2. Proof of Theorem 2

Lemma 5. Let $\pi : S \to T$ be a proper birational morphism between normal surfaces. Let $C \subset S$ be a $K_S$-negative $\pi$-exceptional curve. Then $(-K_S \cdot C) \leq 1$, with equality if and only if $S$ has only Du Val singularities along $C$. (Since $K_S$ is not necessarily $\mathbb{Q}$-Cartier, we use the intersection theory of Weil divisors on surfaces by Mumford [Mum61].)

Proof. Let $\phi : \tilde{S} \to S$ be the minimal resolution of $S$. Denote the exceptional curves of $\phi$ by $E_i$. Then we have

$$K_{\tilde{S}} + \sum_i a_i E_i \equiv \phi^* K_S, \quad \text{where } a_i \geq 0.$$ 

Let $\tilde{C}$ be the birational transform of $C$ under $\phi$. Since $\pi \circ \phi$ contracts $\tilde{C}$, we have $(\tilde{C}^2) < 0$. By the assumption that $C$ is $K_S$-negative, we have

$$(K_{\tilde{S}} \cdot \tilde{C}) = (\phi^* K_S \cdot \tilde{C}) - \sum_i a_i (E_i \cdot \tilde{C}) \leq (K_S \cdot C) < 0.$$ 

Hence $\tilde{C}$ is a $(-1)$-curve on $\tilde{S}$ and $(-K_S \cdot C) \leq (-K_S \cdot \tilde{C}) = 1$.

It is clear that $(-K_S \cdot C) = 1$ if and only if $\sum_i a_i (E_i \cdot \tilde{C}) = 0$, i.e. $a_i = 0$ whenever $\tilde{C}$ intersects $E_i$. By the negativity lemma (cf. [KM98, Lemma 3.41]), this is equivalent
to saying that \( a_i = 0 \) whenever \( E_i \) is connected to \( \tilde{C} \) through a chain of \( \phi \)-exceptional curves. Thus the equality holds if and only if \( S \) has Du Val singularities along \( C \).

Lemma 6. Let \( \pi : S \to T \) be a proper surjective morphism from a normal surface \( S \) to a smooth curve \( T \). Assume that the generic fiber of \( \pi \) is isomorphic to \( \mathbb{P}^1 \), and all fibers of \( \pi \) are generically reduced and irreducible. Then \( \pi \) is a smooth \( \mathbb{P}^1 \)-fibration, i.e. \( S \) is a geometrically ruled surface over \( T \).

Proof. For any closed point \( t \in T \), denote by \( S_t \) the scheme-theoretic fiber of \( \pi \) at \( t \). It is clear that \( \pi \) is flat, so \( \chi(S_t, \mathcal{O}_{S_t}) = \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 1 \). Besides, \( S \) being normal implies that the Cartier divisor \( S_t \) on \( S \) has no embedded points. Then \( S_t \) being generically reduced and irreducible yields that \( S_t \) is an integral curve. Therefore, \( S_t \cong \mathbb{P}^1 \).

7 (Proof of Theorem [2]). Denote by \( \sigma : \tilde{X} = \text{Bl}_p X \to X \) the blow up of \( X \) at \( p \) with exceptional divisor \( E \). Let \( D := \sigma^*(-K_X) - \epsilon(-K_X, p)E \) be the nef divisor. Since \(-K_X = \sigma^*(-K_X) - (n - 1)E \), we know that \( D - K_X \) is ample. Hence Shokurov’s basepoint-free theorem [KM98, Theorem 3.3] implies that \( D \) is semiample.

Let \( g : \tilde{X} \to Y \) be the ample model of \( D \) (i.e. \( g \) is the morphism determined by the complete linear system \([kD]\) for some \( k \gg 0 \)). Let \( m \) be a positive integer such that \( mD \) is Cartier. Notice that \( mD - E - K_X \) is ample by \( \epsilon(-K_X, p) > n \), so Kawamata-Viehweg vanishing implies that \( H^1(\tilde{X}, mD - E) = 0 \). Hence \( H^0(\tilde{X}, mD) \to H^0(\tilde{X}, mD|_E) \) is surjective for \( m \in \mathbb{Z}_{>0} \) with \( mD \) being Cartier. As a result, \( g|_E : E \to Y \) is a closed embedding. Thus any curve \( C \) contracted by \( g \) is not contained in \( E \), which implies that \((C \cdot \sigma^*(-K_X)) > 0 \). Since \( 0 = (C \cdot D) = (C \cdot \sigma^*(-K_X)) - \epsilon(-K_X, p)(C \cdot E) \), we know that \((C \cdot E) > 0 \).

Suppose \( g \) contracts \( C \) to a point \( y \in Y \). Consider the scheme-theoretic fiber \( g^{-1}(y) \) of \( g \). Since \( g|_E \) is a closed embedding, the scheme-theoretic intersection \( E \cap g^{-1}(y) \) is a reduced closed point, say \( q \). If there is another curve \( C' \neq C \) contained in \( g^{-1}(y) \), then \( E \cap g^{-1}(y) \) has multiplicity at least 2 at \( q \), a contradiction! So \( \text{Supp} \ g^{-1}(y) = C \) and \( g^{-1}(y) \) is smooth and transversal to \( E \) at \( q \). In particular, we have \((C \cdot E) = 1 \) for any curve \( C \) contracted by \( g \). Since \( \tilde{X} \) has klt singularities, it is Cohen-Macaulay by [KM98, Theorem 5.22]. In addition we have \(-K_{\tilde{X}} \sim_{g, q} \lambda E \) where \( \lambda = \epsilon(-K_X, p) - n + 1 > 1 \). Hence by the following lemma, \( g \) cannot be birational.

Lemma 8. Let \( g : \tilde{X} \to Y \) be a proper birational morphism between quasi-projective normal varieties and \( E \) a smooth \( g \)-ample Cartier divisor on \( \tilde{X} \) such that \(-K_{\tilde{X}} \sim_{g, q} \lambda E \) for some \( \lambda \geq 1 \). Assume that \( \tilde{X} \) is Cohen-Macaulay and \( g|_E : E \to G = g(E) \) is an isomorphism, then \( \lambda = 1 \) and \( Y \) is smooth along \( G \).

Proof. Let \( H \) be a very ample divisor on \( Y \) such that \( H^0(Y, \mathcal{O}_Y(H)) \to H^0(G, \mathcal{O}_G(H)) \) is surjective. Let \( y \in Y \) be a closed point in the exceptional locus of \( g \) and let \( H_1, \cdots, H_{n-2} \) be general members of \([H]\) containing \( y \). Let \( C = g^{-1}(y) \) and \( S = g^*H_1 \cap \cdots \cap g^*H_{n-2} \). We claim that \( S \) is a normal surface. Since \( E|_S \) is ample and \( g|_E \) is an isomorphism, it is easy to see as above that \( C \) is an irreducible curve and \( E \cap C \) is supported at a single point \( q \). As \( \tilde{X} \) is Cohen-Macaulay, \( S \) is \( S_2 \). By Bertini’s theorem \( S \setminus C \) is smooth in codimension one and \( G \cap H_1 \cap \cdots \cap H_{n-2} \) (scheme-theoretic intersection) is smooth at \( y \). It follows that \( E|_S \) is smooth at \( q \). Since \( E \) is Cartier, we see that \( S \) is also smooth at \( q \in C \), hence \( S \) is smooth in codimension one and it is normal.
It is clear that \( g|_S \) is a birational morphism that contracts \( C \). By adjunction \( K_S = (K_X + g^*H_1 + \cdots + g^*H_{n-2})|_S \), thus \( (-K_S \cdot C) = (-K_X \cdot C) = \lambda(E \cdot C) = \lambda \geq 1 \). On the other hand by Lemma 5 we have \((-K_S \cdot C) \leq 1\). Hence \( \lambda = (-K_S \cdot C) = 1 \) and \( S \) has only Du Val singularities along \( C \). Since contracting a \((-1)\)-curve (i.e. a curve that has anti-canonical degree 1) from a surface with Du Val singularities produces a smooth point, \( g(S) \) and hence \( Y \) is smooth at \( y \). Note that \( y \) is arbitrary in the exceptional locus, so \( Y \) is smooth along \( G \).

**Remark 9.** In fact more is true. Under the same assumptions of the lemma, \( \hat{X} \) is indeed the blowup of \( Y \) along a divisor in \( G \). We postpone its proof to the next section.

Returning to the proof of Theorem 2 we see that \( g \) has to be a fiber type contraction. Since \( g|_E \) is a closed embedding, we know that \( g|_E : E \to Y \) is in fact an isomorphism. In particular, \( E \cong Y \cong \mathbb{P}^{n-1} \). Let us define \( S, H_i \) as in the proof of Lemma 8. By the same argument there, \( S \) is a normal surface. Since the singular set of \( \hat{X} \) has codimension at least 2, by generic smoothness we know that the generic fiber of \( g : \hat{X} \to Y \) is smooth. So the contraction \( g \) being \( K_X \)-negative implies that the generic fiber of \( g \) is a smooth rational curve. In particular, the generic fiber of \( g|_S : S \to g(S) \) is isomorphic to \( \mathbb{P}^1 \). Hence applying Lemma 6 yields that \( C \cong \mathbb{P}^1 \), which means that \( g : \hat{X} \to Y \) is a smooth \( \mathbb{P}^1 \)-fibration.

It is clear that \( s = g|_E^{-1} : Y \to E \) gives a section of \( g \), thus \( \hat{X} = \mathbb{P}_Y(E) \) is a \( \mathbb{P}^1 \)-bundle where \( E \) is a rank 2 vector bundle over \( Y \). Then the section \( E \) corresponds to a surjection \( E \to N \) for some line bundle \( N \) on \( Y \). Denote the kernel of this surjection by \( M \). By the adjunction formula on \( \mathbb{P}^1 \)-bundles, we know that \( \mathcal{O}_Y(-1) \cong s^*\mathcal{N}_{E/X} \cong M^{-1} \otimes \mathcal{N} \). For simplicity we may assume \( M \cong \mathcal{O}_Y \), then we get \( \mathcal{N} \cong \mathcal{O}_Y(-1) \) and hence a short exact sequence

\[
0 \to \mathcal{O}_Y \to E \to \mathcal{O}_Y(-1) \to 0.
\]

Since \( \text{Ext}^1(\mathcal{O}_Y(-1), \mathcal{O}_Y) \cong H^1(\mathbb{P}^{n-1}, \mathcal{O}(1)) = 0 \), the above exact sequence splits. So \( E \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-1) \) and \( E \) corresponds to the second projection \( \mathcal{O}_Y \oplus \mathcal{O}_Y(-1) \to \mathcal{O}_Y(-1) \). As a result, \( \hat{X} \) is isomorphic to the blow up of \( \mathbb{P}^n \) at one point with \( E \) corresponding to the exceptional divisor. Therefore, \( X \cong \mathbb{P}^n \). □

The following is an application of Theorem 2 to Ding-semistable \( \mathbb{Q} \)-Fano varieties with maximal volume (see [Fuj15], [Liu16] for backgrounds). This improves Fujita’s result on the equality case in [Fuj15, Theorem 5.1]. We remark that a different proof is presented in [Liu16, Proof 2 of Theorem 36].

**Theorem 10.** Let \( X \) be a Ding-semistable \( \mathbb{Q} \)-Fano variety of dimension \( n \). If \( ((-K_X)^n) \geq (n + 1)^n \), then \( X \cong \mathbb{P}^n \).

**Proof.** Notice that \( ((-K_X)^n) \leq (n + 1)^n \) by [Fuj15, Corollary 1.3]. Thus we have \( ((-K_X)^n) = (n + 1)^n \). Let \( p \in X \) be a smooth point. From [Fuj15, Proof of 5.1], we see that \( \epsilon(-K_X, p) = n + 1 \). Hence \( X \cong \mathbb{P}^n \) by Theorem 2. □

### 3. Equality case

In this section we prove Theorem 3. Let \( X \) be an \( n \)-dimensional \( \mathbb{Q} \)-Fano variety with a smooth point \( p \in X \). Assume \( \epsilon(-K_X, p) = n \). Following the proof of Theorem 2 we
have that $D = \sigma^*(-K_X) - nE$ is semiample on $\hat{X}$ and induces the morphism $g : \hat{X} \to Y$. We now separate into two cases base on different behavior of $g$.

3.1. $g$ is birational.

**Lemma 11.** If $g : \hat{X} \to Y$ is birational, then $g|_E$ is a closed embedding, $-(K_Y + g(E))$ is ample and $g(E) \cong \mathbb{P}^{n-1}$ is a nef divisor in the smooth locus of $Y$. Moreover, $Y$ is a $\mathbb{Q}$-Fano variety.

**Proof.** We see that $mD - E - K_{\hat{X}} = (m-1)D$ is nef and big, so Kawamata-Viehweg vanishing implies that $g|_E : E \to Y$ is a closed embedding as in the proof of Theorem 2. Hence $g(E) \cong E \cong \mathbb{P}^{n-1}$. By Lemma 8 it lies in the smooth locus of $Y$.

Since $g$ is induced by $D$, $-(K_Y + g(E)) = \pi_*D$ is ample. To show the nefness of $g(E)$ we only need to show that $(L \cdot g(E)) \geq 0$ for a line $L$ in $g(E)$. We may assume $L$ intersects the exceptional locus of $g$. Denote by $L'$ the strict transform of $L$ in $\hat{X}$. Let $W = g^*g(E) - E$, then it is an effective Cartier divisor supported on $\text{Ex}(g)$. Since $W \sim_{g, \mathbb{Q}} -K_{\hat{X}}$ is $g$-ample, we have $\text{Ex}(g) \subseteq W$, hence $(L' \cdot W) \geq 1$ and $(L \cdot g(E)) = (L' \cdot (E + W)) = -1 + (L' \cdot W) \geq 0$. □

According to Lemma 11 we are now in the situation of Lemma 8 with $\lambda = 1$. In order to classify $X$, we first need to study the structure of the birational map $g : \hat{X} \to Y$ in greater detail. This is accomplished by the following lemma.

**Lemma 12.** Under the same notations and assumptions as in Lemma 8, $\hat{X}$ is the blowup of $Y$ along a divisor in $G$.

**Proof.** First note that by Lemma 8 and its proof, $\hat{X}$ has only compound Du Val singularities along $\text{Ex}(g)$, hence after shrinking $\hat{X}$ and $Y$ we may assume that $\hat{X}$ has only klt singularities.

Let $W = g^*G - E$ as above, then $W$ is $g$-exceptional and $-W$ is a $g$-ample Cartier divisor on $\hat{X}$, hence we have $\hat{X} \cong \text{Proj} \oplus_{m=0}^{\infty} \mathcal{J}_m$ where $\mathcal{J}_m = g_*\mathcal{O}_{\hat{X}}(-mW)(m = 0, 1, \ldots)$. It is clear that each $\mathcal{J}_m$ is an ideal sheaf on $Y$. Let $\mathcal{J} = \mathcal{J}_1$, we claim that $\mathcal{J}$ is the ideal sheaf of a hypersurface in $g_*E$ and $\mathcal{J}_m = \mathcal{J}^m$.

To see this, note that since $-mW - K_{\hat{X}} \sim_{g, \mathbb{Q}} (m + 1)E$ is $g$-ample and $\hat{X}$ is klt, we have $R^1g_*\mathcal{O}_{\hat{X}}(-mW) = 0$ for all $m \geq 0$. Hence from the pushforward $g_*$ of

$$0 \to \mathcal{O}_{\hat{X}}(-g^*G - mW) \to \mathcal{O}_{\hat{X}}(-(m + 1)W) \to \mathcal{O}_E(-(m + 1)W) \to 0$$

we obtain an exact sequence

$$0 \to \mathcal{J}_m(-G) \to \mathcal{J}_{m+1} \to \mathcal{O}_E(-(m + 1)W) \to 0$$

Taking $m = 0$, by Nakayama lemma we see that locally $\mathcal{J} = (a, b)$ is the ideal sheaf of $g(W)$ where $a = 0$ (resp. $a = b = 0$) is the local defining equation of $G$ (resp. $g(W)$). Note that the restriction of $g$ to $E$ is an isomorphism, so $g(W) \cong W \cap E$ is a divisor (not necessarily irreducible or reduced) in $G$. Suppose we have shown $\mathcal{J}_m = \mathcal{J}^m$ for some $m \geq 1$ (the case $m = 1$ being clear), then the above exact sequence tells us that $\mathcal{J}_{m+1}$ is generated by $a \cdot \mathcal{J}_m$ and $b^{m+1}$, hence $\mathcal{J}_{m+1} = \mathcal{J}^{m+1}$ as well. The claim then follows by induction on $m$ and the lemma follows immediately from the claim. □
Now we will classify the pairs \((Y, g(E))\) satisfying the statement of Lemma \([11]\). By abuse of notation, we will simply denote the divisor by \(E\) instead of \(g(E)\). We remark that Bonavero, Campana and Wiśniewski classified such pairs in [BCW02] when \(Y\) is smooth.

**Lemma 13.** Let \(Y\) be an \(n\)-dimensional \(\mathbb{Q}\)-Fano variety containing a prime divisor \(E \cong \mathbb{P}^{n-1}\) in its smooth locus.

1. If \(\rho(Y) = 1\), then either \(Y\) is a weighted projective space \(\mathbb{P}(1^n, d)\) for some \(d \in \mathbb{Z}_{>0}\) and \(E\) the hyperplane defined by the vanishing of the last coordinate, or \(n = 2\), \(Y \cong \mathbb{P}^2\) and \(E\) is a smooth conic curve;
2. If \(\rho(Y) \geq 2\) and \(-(K_Y + E)\) is ample, then \(Y\) is a \(\mathbb{P}^1\)-bundle \(\mathbb{P}(O \oplus O(-d))\) over \(\mathbb{P}^{n-1}\) for some \(d \in \mathbb{Z}_{>0}\) and \(E\) is a section. If \(n \geq 3\) and \(d \geq n\) then \(E\) is the only section with negative normal bundle.

**Proof.** Note that in the case \(\rho(Y) = 1\), \(E\) is necessarily an ample divisor on \(Y\). As \(E\) does not intersect the singular locus of \(Y\), \(Y\) has only isolated singularities. By adjunction \(-(K_Y + E)|_E = -K_E\) is ample, hence \(-(K_Y + E)\) is ample as well. Let \(Y^o\) be the smooth locus of \(Y\) and \(i : E \to Y^o\) the inclusion.

First assume \(\rho(Y) = 1\) and \(n \geq 3\). By the generalized version of Lefschetz hyperplane theorem [GM88, Theorem II.1.1], \(H^i(Y^o, E, Z) = H^i(Y^o, E, Z) = 0\) for \(i < n\), hence by the universal coefficient theorem, \(H^n(Y^o, E, Z)\) is torsion free. As \(n \geq 3\), this implies the restriction map \(i^* : H^2(Y^o, Z) \to H^2(E, Z)\) is injective and has torsion free cokernel. But \(H^2(E, Z) \cong \mathbb{Z}\) since \(E \cong \mathbb{P}^{n-1}\), so \(i^*\) is in fact an isomorphism. As \(Y\) is \(\mathbb{Q}\)-Fano we have \(H^1(Y, O_Y) = 0\) by Kawamata-Viehweg vanishing and \(Y\) is Cohen-Macaulay by [KM93, Theorem 5.22]. Since \(Z = \text{Sing}Y\) consists of isolated points and \(n \geq 3\), by the long exact sequence of cohomology with support

\[
\cdots \to H^3_2(Y, O_Y) \to H^4(Y, O_Y) \to H^5(Y^o, O_{Y^o}) \to H^2_2(Y, O_Y) \to \cdots
\]

we get \(H^1(Y^o, O_{Y^o}) = 0\). Combining this with the exponential sequence \(0 \to Z \to O_{Y^o} \to O_{Y^o} \to 0\), we see that the restriction \(i^* : \text{Cl}(Y) = \text{Pic}(Y^o) \to \text{Pic}(E) \cong \mathbb{Z}\) is also an isomorphism.

Let \(H\) be the ample generator of \(\text{Cl}(Y)\), then \(E \sim dH\) for some \(d \in \mathbb{Z}_{>0}\). Let \(\pi : Y' \to Y\) be the (normalization of the) cyclic cover of degree \(d\) of \(Y\) ramified at \(E\) and \(E' = \pi^{-1}(E)_{\text{red}}\). Then \(K_{Y'} + E' = \pi^*(K_Y + E)\) as \(E'\) is the only branched divisor, hence \(Y'\) is also \(\mathbb{Q}\)-Fano and \(E'\) satisfies the same assumptions of the lemma. We also have \((O_{E'}(dE')) \cong O_{E'}(\pi^*E) = \pi^*N_{E/Y} \cong O_{E'}(d)\), hence \(N_{E'/Y^o} \cong O_{E'}(1)\) is the hyperplane class. Note that \(E'\) is ample since it’s the preimage of the ample divisor \(E\). It now follows from the long exact sequence

\[
0 \to H^0(Y', O_{Y'}) \to H^0(Y', O_{Y'}(E')) \to H^0(E', N_{E'/Y^o}) \to H^1(Y', O_{Y'}) = 0
\]

that the linear system \(|E'|\) is base point free, has dimension \(n\) and defines an isomorphism \(Y' \cong \mathbb{P}^n\) such that \(E'\) is mapped to a hyperplane. Our original pair \((Y, E)\) is then obtained by taking a cyclic quotient of degree \(d\) ramified at \(E'\), and is easily seen to be as claimed in the statement of the lemma.

Next assume \(\rho(Y) = 1\) and \(n = 2\). Then \(Y\) has quotient singularity and is \(\mathbb{Q}\)-factorial, hence \(\text{Cl}(Y)\) has rank one. As \(E\) is ample, \(\pi_1(E) \to \pi_1(Y^o)\) is surjective by [GM88, Theorem II.1.1], but \(\pi_1(E) = \pi_1(\mathbb{P}^1) = 0\), so \(Y^o\) is simply connected as well. In particular,
$\text{Cl}(Y) = \text{Pic}(Y^o)$ is torsion-free and thus $\cong \mathbb{Z}$. Let $r$ be the index of $i^*\text{Cl}(Y)$ in $\text{Pic}(E)$. As $-(K_Y + E)|_E = -K_E$ has degree 2, $r = 1$ or 2. Let $H$ be the ample generator of $\text{Cl}(Y)$, then $(H,E) = r$ and $E \sim dH$ for some $d \in \mathbb{Z}_{>0}$. Let $\pi : Y' \to Y$ be the corresponding cyclic cover of degree $d$ and define $E'$ as before. By the same argument as the $n \geq 3$ case, we have $N_{E'/Y'} \cong \mathcal{O}_E(r)$, and if $r = 1$, the linear system $|E'|$ defines an isomorphism $(Y',E') \cong (\mathbb{P}^2, \text{hyperplane})$, while if $r = 2$, then linear system $|E'|$ embeds $Y'$ into $\mathbb{P}^3$ as a quadratic surface. Taking cyclic quotients, we see that the origin $(Y,E)$ is again as claimed.

Finally assume $\rho(Y) \geq 2$ and $-(K_Y + E)$ is ample. Let $l$ be a line in $E$. We claim that there is an extremal ray $\mathbb{R}_{\geq 0}[\Gamma]$ in $\overline{NE}(Y)$ with $\Gamma$ an irreducible reducible curve on $Y$ such that $[\Gamma] \not\in \mathbb{R}_{\geq 0}[l]$ and $(E \cdot \Gamma) > 0$. If $(E \cdot l) = 0$, then such $\Gamma$ exists since $E$ is not numerically trivial. If $(E \cdot l) > 0$, consider the exact sequence

$$0 \to T_E|_l \to T_Y|_l \to N_{E/Y}|_l \to 0.$$ 

It is clear that $T_E|_l$ is ample because $E \cong \mathbb{P}^{n-1}$. On the other hand, deg $N_{E/Y}|_l = (E \cdot l) > 0$ hence $N_{E/Y}|_l$ is ample. Therefore, $T_Y|_l$ is also ample which implies that $l$ is a very free rational curve in $Y$. Since $\rho(Y) \geq 2$, $\mathbb{R}_{\geq 0}[l]$ cannot be an extremal ray of $\overline{NE}(Y)$ (otherwise the contraction of $l$ will contract $Y$ to a single point), which means that such $\Gamma$ exists.

Now let $h : Y \to Z$ be the contraction of $\Gamma$. As we argued in the proof of Theorem 2, $h|_E : E \to Y$ is a closed embedding, hence $(E \cdot \Gamma) = 1$. Since $-(K_Y + E)$ is ample, we have $(-K_Y \cdot \Gamma) > 1$. Then by the same reason in the proof of Theorem 2 we conclude that $h$ has to be a fiber type contraction. Hence $Y$ is a $\mathbb{P}^1$-fibration over $Z \cong \mathbb{P}^{n-1}$ admitting a section $h|_E^{-1} : Z \to E$, so $Y \cong \mathbb{P}^1(Z(\mathcal{O} \oplus \mathcal{O}(-d)))$ with $d \geq 0$. If $n \geq 3$, then $E$ corresponds to either a surjection $\mathcal{O} \oplus \mathcal{O}(-d) \to \mathcal{O}$ or a surjection $\mathcal{O} \oplus \mathcal{O}(-d) \to \mathcal{O}(-d)$. If in addition $d \geq n$, then $-(K_Y + E)$ being ample implies that $E$ is the unique section corresponding to the second projection $\mathcal{O} \oplus \mathcal{O}(-d) \to \mathcal{O}(-d)$. 

Combining the last two lemmas we can give a partial classification of $X$:

**Lemma 14.** If $g$ is birational then $X$ is one of the following:

1. a degree $d+1$ weighted hypersurface $X_{d+1} = (x_0x_{n+1} = f(x_1, \cdots , x_n)) \subset \mathbb{P}^{1^{n+1}}, d$;
2. the blow-up of $\mathbb{P}^n$ along the complete intersection of a hyperplane and a hypersurface of degree $d \leq n$;
3. a Gorenstein log Del Pezzo surface of degree $\geq 5$.

**Proof.** By Lemma [13] we have the following cases:

1. $Y \cong \mathbb{P}^{1^n}, d$ with homogeneous coordinate $[y_0 : \cdots : y_n]$ and $g(E) = (y_n) = 0$. We have $N_{g(E)/Y} \cong \mathcal{O}_E(d)$. By Lemma [12] $X$ is obtained by blowing up a hypersurface $S = (f = 0)$ in $g(E)$ where $f$ is a homogeneous polynomial in $y_0, \cdots , y_{n-1}$. As $N_{E/X} \cong \mathcal{O}_E(-1)$ we see that deg $f = d + 1$. Consider the rational map $\phi : Y \dasharrow \mathbb{P}^{1^{n+1}}, d$ given by

$$[y_0 : \cdots : y_n] \mapsto [x_0 : \cdots : x_{n+1}] = [y_n : y_0 : \cdots : y_{n-1} : \frac{f(y_0, \cdots , y_{n-1})}{y_n}]$$

whose image lies in the weighted hypersurface $X_{d+1}$ define by $x_0x_{n+1} = f(x_1, \cdots , x_n)$. It is clear that $\phi$ is contracts $g(E)$ to the point $[0 : \cdots : 0 : 1 : 0]$ and the indeterminacy locus of $\phi$ is exactly $S$. By inspecting each affine chart $(x_i \neq 0) \subset Y$ it is easy to see
that after blowing up $S$, $\phi$ extends to a birational morphism $\hat{X} \to X_{d+1}$ that contracts $E$, hence $X \cong X_{d+1}$ as in the first case in the statement of the lemma.

(2) $Y$ is a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-d))$ over $\mathbb{P}^{n-1}$ ($n \geq 3$) and $g(E)$ is a section. Since $g(E)$ is nef by Lemma 11, we have $d < n$ by Lemma 13. Going back to the last part of the proof of Lemma 13 we see that the section $g(E)$ corresponds to a surjection $\mathcal{O} \oplus \mathcal{O}(-d) \to \mathcal{O}$ and hence $N_{g(E)/Y} \cong \mathcal{O}(d)$. By Lemma 12 as in previous case, $\hat{X}$ is obtained by blowing up a hypersurface $S$ of degree $d + 1$ in $g(E)$. It is straightforward to see that the elementary transformation of $Y$ with center $S$ is the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1))$ over $\mathbb{P}^{n-1}$, which is isomorphic to the blowup of a point $R$ on $\mathbb{P}^n$, such that the strict transform $E'$ (resp. $H$) of $g(E)$ (resp. the negative section on $Y$) becomes the exceptional divisor over $R$ (resp. a hyperplane in $\mathbb{P}^n$ that is disjoint from $R$). Contracting $E'$ and reversing this procedure we see that $X$ is the blowup of $\mathbb{P}^n$ along a hypersurface of degree $d + 1 \leq n$ in a hyperplane.

(3) $Y \cong \mathbb{P}^2$ and $g(E)$ is a smooth conic, or $Y$ is a ruled surface over $\mathbb{P}^1$ and $g(E)$ is a section. In either case $Y$ is smooth and $\hat{X}$ is obtained by blowing up subschemes of $g(E)$. Locally on $Y$, such a subscheme is defined by $(a = b^k = 0)$ where $a, b$ are local coordinates such that $g(E) = (a = 0)$. $\hat{X}$ then has local equation $at = b^k$ or $a = b^k t$ and it follows that both $\hat{X}$ and $X$ have only Du Val singularities of type $A$. As $D = \sigma^*(-K_X) - 2E$ is big and nef and Cartier in this case we have $(K^2_X) = (D^2) - 4(E^2) = (D^2) + 4 \geq 5$, so $X$ is as described in the third case of the statement of the lemma. □

3.2. $g$ is of fiber type.

**Lemma 15.** If $g$ is of fiber type, then every fiber has dimension 1, $g|_E : E \to Y$ is a double cover and $-K_X \sim_{g, \mathbb{Q}} E$ is $g$-ample.

**Proof.** Since $\epsilon(-K_X, p) > n - 1$, $\hat{X}$ is $\mathbb{Q}$-Fano, so $-K_X \sim_{g, \mathbb{Q}} E$ is $g$-ample. $D|_E$ is ample, so $E \to Y$ is finite and every fiber of $g$ has dimension one. Let $l$ be a general fiber, then $l \cong \mathbb{P}^1$ and $(-K_X \cdot l) = 2 = (E \cdot l)$, so $E$ is a double section. □

Similar to the previous case, we first analyze the local structure of $g$ in a slightly more general setting. For ease of notations, we call $g : \hat{X} \to Y$ (where $\hat{X}$ and $Y$ are normal quasi-projective varieties) a rational conic bundle if $g$ is proper, every fiber of $g$ has dimension 1 and the generic fiber is isomorphic to $\mathbb{P}^1$. If in addition $\hat{X}$ is Cohen-Macaulay and there exists a Cartier divisor $E$ on $\hat{X}$ such that $-K_{\hat{X}} \sim_{g, \mathbb{Q}} E$ is $g$-ample, then we say that the rational conic bundle is Gorenstein. It is clear that a conic bundle is automatically a Gorenstein rational conic bundle.

**Lemma 16.** Let $g : S \to C$ be a Gorenstein rational conic bundle. Assume dim $S = 2$, then $S$ is a conic bundle and in particular has only Du Val singularities.

**Proof.** Let $l$ be an irreducible component of a fiber of $g$, then $(-K_S \cdot l) = (E \cdot l)$ is a positive integer since $E$ is Cartier and $-K_S$ is $g$-ample. On the other hand, if $F$ is a general fiber of $g$ then $(-K_S \cdot F) = 2$. Hence every fiber of $g$ has at most two irreducible components (counting multiplicities), so on the minimal resolution of $S$ (which is a birationally ruled surface over $C$), every fiber over $C$ has one of the following as its dual graph:

$$( -2 ) - ( -1 ) - ( -2 ) ,$$  

$$( -1 ) - ( -2 ) - ( -2 ) - \cdots - ( -2 ) - ( -1 ) ,$$
or
\[
\begin{align*}
-2 & \quad -2 \\
-2 & \quad -2 \\
& \quad \quad \vdots \\
-2 & \quad -1
\end{align*}
\]

As \( S \) is obtained by contracting those \((-2)\)-curves, it has only Du Val singularities and is a conic bundle.

**Corollary 17.** If \( g : \tilde{X} \to Y \) is a Gorenstein rational conic bundle such that \( Y \) is smooth, then \( \tilde{X} \) is a conic bundle over \( Y \). In particular, \( \tilde{X} \) is a hypersurface in \( \mathbb{P}(E) \) for some rank 3 vector bundle \( E \) on \( Y \).

**Proof.** Let \( y \in Y \) and \( C \) a general complete intersection curve on \( Y \) passing through \( y \). Let \( S = \tilde{X} \times_Y C \). Since \( \tilde{X} \) is Cohen-Macaulay, \( S = S_2 \). From the proof of Lemma 16 we know that the fiber \( g^{-1}(y) \) has at most 2 irreducible components (counting multiplicities), hence \( S \) is smooth at every generic point of \( g^{-1}(y) \), hence \( S \) is normal. By adjunction it is easy to see that \( S \) is a Gorenstein rational conic bundle over \( C \), so by Lemma 16, \( S \) has only Du Val singularities and is a conic bundle, hence every fiber of \( g \) is isomorphic to a conic and \( \tilde{X} \) has cDV singularities which is Gorenstein. The lemma then follows from standard arguments (see e.g. [Cut88, Theorem 7]).

Unfortunately in our classification problem, the Gorenstein rational conic bundle \( g : \tilde{X} \to Y \) does not have a smooth base. Nevertheless, there is a smooth double section \( E \). Hence we would like to apply Corollary 17 to \( \tilde{g} : \tilde{X} \to \tilde{Y} \), where \( \tilde{Y} \cong E \) and \( \tilde{X} \) is the normalization of \( \tilde{X} \times_Y \tilde{Y} \). For this purpose, we need to show that \( \tilde{X} \) is Gorenstein rational conic bundle over \( \tilde{Y} \). This is given by the following lemma.

**Lemma 18.** Let \( g : \tilde{X} \to Y \) be a Gorenstein rational conic bundle and \( \phi : \tilde{Y} \to Y \) a finite morphism between normal varieties. Let \( \tilde{X} \) be the normalization of \( \tilde{X} \times_Y \tilde{Y} \). Assume that \( \tilde{X} \) has klt singularities and the branch divisor of \( \phi \) is disjoint from the singular locus of \( \tilde{Y} \) and \( Y \). Then \( \tilde{g} : \tilde{X} \to \tilde{Y} \) is also a Gorenstein rational conic bundle.

**Proof.** By shrinking \( Y \) we may assume either \( \phi \) is unramified in codimension one or both \( Y \) and \( \tilde{Y} \) are smooth. In the first case \( \tilde{X} \) is also klt by [KM98, Proposition 5.20] hence is CM, and the other properties of Gorenstein rational conic bundles are preserved by a finite base change that is étale in codimension one. In the second case \( g \) is a conic bundle by Lemma 17 hence the same holds for \( \tilde{g} \).

The pullback \( E' \) of \( E \) to \( \tilde{X} \) is then a union of two sections \( E_1 \) and \( E_2 \). If they are disjoint, we have a simple description of the conic bundle \( \tilde{g} \):

**Lemma 19.** Let \( \tilde{g} : \tilde{X} \to \tilde{Y} \) be a conic bundle with smooth base. Assume that there are two disjoint sections \( E_1 \) and \( E_2 \) that are Cartier as divisors on \( \tilde{X} \) and such that \( -K_{\tilde{X}} \sim_{\mathbb{Q}} E_1 + E_2 \). Then there is a birational morphism \( u : \tilde{X} \to Z = \mathbb{P}_Y(O \oplus L) \) (where \( L \cong N_{E_1/\tilde{X}} \)) sending \( E_1, E_2 \) to two disjoint sections \( E_1', E_2' \) of \( Z \) such that \( \tilde{X} \) is the blow up of \( Z \) along a divisor in \( E_2 \).
Proof. If every fiber of $\tilde{g}$ is an irreducible $\mathbb{P}^1$ then $\tilde{X} \cong \mathbb{P}_Y(\mathcal{O} \oplus \mathcal{L})$ and there is nothing to prove. So we may assume $l = l_1 + l_2$ is a reducible fiber. We have $(E_1 + E_2 \cdot l_2) = (\mathcal{L}_X \cdot l_2) = 1$ $(j = 1, 2)$. Since the section $E_i$ is Cartier, we have $(E_i \cdot l_j) = \delta_{ij}$ after rearranging indices. Let $u : \tilde{X} \to Z$ be the contraction of the extremal ray $\mathbb{R}_+[l_2]$ and let $E_1', E_2'$ be strict transform of $E_1, E_2$. As $E_i$ is a section of $\tilde{g}$ and $E_i \to \tilde{Y}$ factors through $E_i'$, the restriction $u|_{E_i}$ is an isomorphism. In addition we have $-(K_{\tilde{X}} + E_2) \sim_{u, \mathcal{Q}} 0$ since its intersection number with $l_2$ is zero. Hence the lemma follows by a direct application of Lemma 12. 

Putting everything together and specializing to $E \cong \mathbb{P}^{n-1}$, we now finish the second part of the classification of $X$ with $\epsilon(-K_X, p) = n$.

Lemma 20. If $g$ is of fiber type then $X$ is one of the following:

1. a Gorenstein log del Pezzo surface of degree 4;
2. quotient of a quadric hypersurface in $\mathbb{P}^{n+1}$ by an involution that is fixed point free in codimension 1;
3. a quartic weighted hypersurface in $\mathbb{P}(1^n, 2^2)$.

Proof. If $n = \dim X = 2$ then by Lemma 16, $\tilde{X}$ and hence $X$ has only Du Val singularities. We have $\sigma^*(-K_X) - 2E \sim_{g, \mathcal{Q}} 0$, so $(K_X^2) = -4(E^2) = 4$ and we are in case (1). Hence in the remaining part of the proof we assume that $n \geq 3$.

We keep using the notations introduced in this subsection. Let $\tilde{X} \to \tilde{X}$ be the Stein factorization of the composition $\tilde{X} \to \tilde{X} \to X$, then $\tilde{X} \to X$ is a double cover. The double cover $E \to Y$ is either unramified in codimension one or the quotient $\mathbb{P}^{n-1} \to \mathbb{P}(1^{n-1}, 2)$ in which case the branch divisor is a hyperplane on $\mathbb{P}^{n-1}$, so the conditions and conclusions of Lemma 18 are satisfied and we see that $\tilde{g} : \tilde{X} \to \tilde{Y}$ is a conic bundle over $\tilde{Y} \cong \mathbb{P}^{n-1}$ by Corollary 17.

If $h : \tilde{X} \to \tilde{X}$ is unramified in codimension one, so is $\tilde{X} \to X$ and we have $\text{codim}_{E_1 \cap E_2} E_i \geq 2$. But since $\tilde{X}$ is Cohen-Macaulay and $E' = E_1 + E_2$ is a Cartier divisor, $E_1 \cup E_2$ is $S_2$. It follows that $E_1$ and $E_2$ do not intersect at all, hence they are disjoint smooth Cartier divisors in $\tilde{X}$ with normal bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. As $K_{\tilde{X}} + E_1 + E_2 = h^*(K_{\tilde{X}} + E) \sim_{g, \mathcal{Q}} 0$, it follows from Lemma 19 that $\tilde{X}$ is a blowup of $Z \cong \mathbb{P}_Y(\mathcal{O} \oplus \mathcal{O}(-1)) \cong \text{Bl}_Z \mathbb{P}^n$ along a hypersurface in the strict transform of a hyperplane. For the normal bundle to match, it is the blowup of a quadric hypersurface. As $\tilde{X}$ is obtained by contracting $E_1 \cup E_2$ from $\tilde{X}$, it is a quadric hypersurface in $\mathbb{P}^{n+1}$, and $X$ is the quotient of $\tilde{X}$ by an involution that acts fixed point free in codimension one as in case (2).

If $h : \tilde{X} \to \tilde{X}$ is ramified in codimension one, then it is ramified along $\tilde{g}^*H$ where $H$ is a hyperplane on $\tilde{Y}$. As in the last paragraph $E_1 \cap E_2$ has pure codimension one, so $E'$ is a union of two $\mathbb{P}^{n-1}$ intersecting transversally at a hyperplane. The conic bundle $\tilde{X}$ is a hypersurface in some $\mathbb{P}(\mathcal{E})$ over $\tilde{Y}$. To compute $\mathcal{E}$, first note that $-(K_{\tilde{X}} + E') = \tilde{g}^*M$ for some $M \in \text{Pic}(\mathcal{E})$ since it restricts to a trivial bundle on every fiber of $\tilde{g}$; we also have $-(K_{\tilde{X}} + E')|_{E'} = -(K_{E'}) = (n-1)\tilde{g}^*H$, so $M \sim (n-1)H$. Combining with $N_{\tilde{E}'/\tilde{X}} \cong \tilde{g}^{*}\mathcal{O}_{\tilde{Y}}(-H)$ we have $-K_{\tilde{X}}|_{E'} \cong \tilde{g}^{*}(n-2)H$. Now apply $\tilde{g}_*$ to the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}} - E) \to \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}}) \to \mathcal{O}_{E'}(-K_{\tilde{X}}) \to 0$$
we obtain another exact sequence
\[0 \to \mathcal{O}_X(\alpha) \to 
\] hence \( \tilde{g}_* \mathcal{O}_X(-K_X) \cong \oplus_{k=0}^2 \mathcal{O}_X(n-k) \) and we may choose \( \mathcal{E} \cong \oplus_{k=0}^2 \mathcal{O}_X(kH) \). Let \( \pi \) be the projection \( \mathbb{P}(\mathcal{E}) \to \tilde{Y} \) and \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \) the relative hyperplane class. \( \tilde{X} \) corresponds to section of \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \pi^* \mathcal{O}_X(mH) \) for some \( m \in \mathbb{Z} \) and by adjunction formula we have \( \mathcal{O}_X(-K_X) \cong \mathcal{O}_X(1) \otimes \tilde{g}^* \mathcal{O}_X((n-3-m)H) \), hence \( \tilde{g}_* \mathcal{O}_X(-K_X) \cong \mathcal{E} \otimes \mathcal{O}_X((n-3-m)H) \). Comparing this to the previous formula for \( \tilde{g}_* \mathcal{O}_X(-K_X) \) we see that \( m = 0 \). The surjection \( \mathcal{E} \to \mathcal{O}_X \) defines a section \( S \) of \( \mathcal{O}(\mathbb{P}(\mathcal{E})) \to \tilde{Y} \) that is disjoint with \( \tilde{X} \) (since \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)|_S \cong \mathcal{O}_S \)) and the linear projection from \( S \) makes \( \tilde{X} \) into a double cover over the \( \mathbb{P}^1 \)-bundle \( \mathcal{O}(\mathcal{O}(H) \oplus \mathcal{O}(2H)) \), which is also the blowup of a point \( \mathbb{P}^n \), such that \( E' \) is mapped to the exceptional divisor and \( \tilde{g}^* H \) to the strict transform of a hyperplane passing through the center of blowup. \( \tilde{X} \) is then a double cover of \( \mathbb{P}^n \), and as \( -(K_X + E') \cong \alpha \) \( \tilde{g}^* H \) we have \( K_X \sim (n-1)\tilde{g}^* H \) where \( H' \) is a hyperplane on \( \mathbb{P}^n \) and \( \tau : \tilde{X} \to \mathbb{P}^n \) the double cover. It follows that \( \tilde{X} \) is a weighted hypersurface of degree 4 in \( \mathbb{P}(1^{n+1}, 2) \). The original \( X \) is then obtained as the quotient of \( \tilde{X} \) by an involution that fixes a hyperplane section (i.e. the strict transform of \( \tilde{g}^* H \)), hence is a quartic weighted hypersurface in \( \mathbb{P}(1^n, 2^2) \) as in case (3).

21 (Proof of Theorem 3). By Lemma 14 and 20, we have the following five possibilities for \( X \). Note that by Theorem 2 it suffices to show that \( \epsilon(-K_X, p) \geq n \) in each case.

(1) \( X \cong X_{d+1} = (x_0 x_{n+1} = f(x_1, \ldots, x_n)) \subseteq \mathbb{P}(1^{n+1}, d) \). If \( d = 1 \) then \( X \) is a quadric hypersurface and the result is clear (or see case 4). Otherwise \( d > 1 \) and we have \( q = [0 : \cdots : 0 : 1] \in X \). Let \( p \) be a smooth point on \( X \) and let \( \sigma : Z \to \mathbb{P}(1^{n+1}, d) \) be the blowup of \( \mathbb{P}(1^{n+1}, d) \) at \( p \) with exception divisor \( V \). Let \( H \) be the divisor class \( \mathcal{O}(1) \) on \( \mathbb{P}(1^{n+1}, d) \), then we have \( \sigma^*(-K_X) - nE = n(\sigma^*H - V)|_X \). The base locus of the linear system \( |\sigma^*H - V| \) on \( Z \) is the strict transform of the line \( l \) joining \( p \) and \( q \). For general choice of \( p \) we have \( l \not\subset X \), hence \( \sigma^*(-K_X) - nE \) is nef on \( \tilde{X} \), yielding \( \epsilon(-K_X, p) \geq n \).

(2) \( X \) is a quartic hypersurface in \( \mathbb{P}(1^n, 2^2) \). Up to weighted projective isomorphism we may assume that \( X \) is defined by the equation \( q(x_n, x_{n+1}) + x_n h(x_0, \cdots, x_{n-1}) = f(x_0, \cdots, x_{n-1}) \) where \( \deg q = \deg h = 2 \), \( \deg f = 4 \) and \( h = 0 \) if \( q \neq ax_n^2 \). Let \( p \in X \) be a smooth point and define \( H, V \) in the similar way as in the first case. We have \( \sigma^*(-K_X) - nE = n(\sigma^*H - V)|_X \). The base locus of \( |\sigma^*H - V| \) is the plane \( \Sigma \) spanned by \( p \) and the line \( (x_0 = \cdots = x_{n-1} = 0) \), so \( D \) is nef (i.e. \( \epsilon(-K_X, p) \geq n \)) if and only if for every curve \( C \subseteq \Sigma \cap X \) we have \( (D \cdot C) \geq 0 \). It is easy to see that \( \frac{1}{n}(D \cdot C) = \frac{1}{4} \deg C - \text{mult}_p C \). As \( \deg(\Sigma \cap X) \leq 4 \) we see that \( (D, C) \geq 0 \) if and only if \( \Sigma \cap X \) is an irreducible curve that is smooth at \( p \). Suppose \( p = [c_0 : \cdots : c_{n+1}] \), then \( \Sigma \cap X \) is given by the equation \( q(y_1, y_2) + h(c_0, \cdots, c_{n-1})y_0^2 = f(c_0, \cdots, c_{n-1})y_0^4 \) in \( \Sigma \cong \mathbb{P}(1, 2, 2) \). From this it is clear that \( \epsilon(-K_X, p) \geq n \) for general \( p \in X \) if and only if \( q \) is not a square or \( hq \neq 0 \). After another change of variable we see that \( X \) is a quartic hypersurface of the form \( x_n x_{n+1} = f(x_0, \cdots, x_{n-1}) \) or \( x_{n+1}^2 + x_n h(x_0, \cdots, x_{n-1}) = f(x_0, \cdots, x_{n-1}) \) \( (h \neq 0) \).

(3) \( X \) is the blowup of a hypersurface \( S \) of degree \( d \leq n \) in a hyperplane of \( \mathbb{P}^n \). Let \( V \) be the exceptional divisor over \( S \), \( H \) the pullback of \( \mathcal{O}_{\mathbb{P}^n}(1) \) on \( X \) and \( H' \subset X \) the strict transform of the hyperplane containing \( S \). Let \( p \in X \) be a point outside \( H' \cup V \). We have \( D = \sigma^*(-K_X) - nE = \sigma^*H' + n(\sigma^*H - E) \). We want to show that \( D \) is nef. Since \( \sigma^*H - E \) is already nef, it remains to show that \( (D \cdot l) > 0 \) where \( l \) is a line in \( \sigma^*H' \). Then
a direct computation shows that \((D \cdot l) = (-K_X \cdot l) = (((n + 1)H - V) \cdot l) = n + 1 - d > 0\). Thus \(D\) is nef and \(\epsilon(-K_X, p) \geq n\).

(4) \(X = Q/\tau\) where \(Q\) is a quadric hypersurface and \(\tau \in \text{Aut}(Q)\) an involution that is fixed point free in codimension one. Let \(p_1\) be a smooth point of \(Q\), let \(p_2 = \tau(p_1)\) and \(p\) be their image in \(X\). Let \(\psi : \tilde{Q} \to Q\) be the blowup of \(p_1\) and \(p_2\) with exceptional divisors \(E_1\) and \(E_2\). Since \(h : Q \to X\) is étale in codimension one, the divisor \(D' = \sigma^*(-K_X) - nE\) pulls back to \(D'' = \psi^*(-K_Q) - nE_1 - nE_2 = n(\psi^*H - E_1 - E_2)\) where \(H\) is the hyperplane class on \(Q\). Similar to case (1), \(D''\) is the restriction of a line bundle (also denoted by \(D''\)) on blowup of \(\mathbb{P}^{n+1}\) at \(p_1, p_2\) whose base locus is the strict transform of the line \(l\) joining \(p_1\) and \(p_2\). We also have \((D'' \cdot l) = -n < 0\). Hence \(D\) is nef and \(\epsilon(-K_X, p) \geq n\) if and only if \(l \not\subseteq Q\). We may diagonalize \(\tau\) and choose homogeneous coordinate \(x_i\) so that \(\sigma(x_i) = \delta_i x_i\) where \(\delta_i = \pm 1\). It is then not hard to verify that \(l \not\subseteq Q\) for general choice of \(p\) if and only if \(Q\) is given by the equation \(\sum_{i=0}^{k} x_i^2 = 0\) for some \(2 \leq k \leq n + 1\) such that \(\delta_i\) take different values for \(i = 0, \ldots, k\).

(5) \(X\) is a Gorenstein log Del Pezzo surface of degree \((K_X^2) \geq 4\). We claim that if \(S\) is a Gorenstein log Del Pezzo surface of degree \(d \geq 3\), then there exists an irreducible curve \(C \in | - K_S|\) with a double point \(p\) lying in the smooth locus of \(S\). After blowing up \(d - 3\) general points on \(S\), it suffices to prove the claim when \(d = 3\), in which case \(S\) is an irreducible cubic surface in \(\mathbb{P}^3\) by \([\text{HW}81, \text{Theorem 4.4}]\). But then there are only finitely many lines on \(S\) whereas by dimension count there exists \(C \in | - K_S|\) that is singular at any given \(p \in S\), hence the claim follows immediately. Using such \(C \in | - K_X|\) and take \(p = \text{Sing}(C)\), we have \(\sigma^*(-K_X) - 2E \sim C'\) where \(C'\) is the strict transform of \(C\) and \((C'^2) = (K_X^2) - 4 \geq 0\), hence \(C'\) is nef and \(\epsilon(-K_X, p) \geq n = 2\).

It remains to show that all \(\mathbb{Q}\)-Fano varieties listed in the statement of Theorem 3 have only klt singularities. From the equations there we see that the singularities of \(X\) are always quotients of cA-type singularities that are étale in codimension 1 (hence are klt by \([\text{Kol13}, 1.42]\) and \([\text{KM98, Proposition 5.20}]\)) except when \(X\) is a quartic hypersurface \(x_n^2 + x_nh = f\) in \(\mathbb{P}(1^n, 2^n)\) and \(x \in (x_n = x_{n+1} = 0) \cap X\) satisfies \(\text{mult}_x h = 2\) and \(\text{mult}_x f \geq 3\). In the latter case, we may assume \(x = [1 : 0 : \cdots : 0]\) and locally \(X\) is a double cover of \(\mathbb{C}^n\) ramified along \(D = (x_nh = f)\). If \(h\) is not a perfect square, then the pair \((\mathbb{C}^n, D)\) degenerates to \((\mathbb{C}^n, D_0)\) where \(D_0 = (x_nh = 0)\) (consider the \(\mathbb{C}^*\)-action \((x_1, \ldots, x_n) \mapsto (t^2x_1, \ldots, t^2x_{n-1}, tx_n)\) for \(t \neq 0\)). Clearly \((\mathbb{C}^n, \frac{1}{2}D_0)\) is klt, so it follows from adjunction that \((\mathbb{C}^n, \frac{1}{2}D)\) is also klt which implies that \(X\) is klt by \([\text{KM98, Proposition 5.20}]\). If \(h\) is a perfect square, then by \([\text{KM98, page 168}]\) we know that \(X\) is a cDV singularity which is klt as well. \(\square\)

4. Seshadri constants below \(n\)

In this section, we prove Theorem 4 using the following examples.

Example 22. Let \(X\) be the weighted projective space \(\mathbb{P}(1, a_1, \cdots, a_n)\) where \(a_1 \leq \cdots \leq a_n\) are positive integers satisfying \(\gcd(a_1, \cdots, a_n) = 1\). Let \(p \in X\) be the smooth point with coordinate \([1 : 0 : \cdots : 0]\). We claim that the Seshadri constant of \(-K_X\) at \(p\) is \(\epsilon(-K_X, p) = \frac{1}{a_n}(1 + \sum_{i=1}^{n} a_i)\). As before let \(\sigma : \tilde{X} \to X\) be the blowup of \(X\) at \(p\) and \(E\) the exceptional divisor. Since \(\tilde{X}\) is a toric variety, the torus invariant divisor \(L_x = \sigma^*(-K_X) - xE\) is nef if and only if it has non-negative intersection number with all
torus invariant lines, and as $-K_X$ is ample on $X$ and $E$ has ample conormal bundle, it suffices to check $(L_i \cdot l_i) \geq 0$ where $l_i$ is the strict transform of the line on $X$ joining $p$ and the point whose only nonzero coordinate is at the $i$-th entry ($i > 0$). It is straightforward to compute $(L_x \cdot l_i) = \frac{1}{a_i}(1 + \sum_{i=1}^n a_i) - x$, so $\epsilon(-K_X, p) = \frac{1}{a_n}(1 + \sum_{i=1}^n a_i)$. Taking $a_1 = \cdots = a_{n-1} = 1$, $a_m = r - m$, $a_{m+1} = \cdots = a_n = s$ where $1 \leq m < n$ and $s \geq r > m$ we get $\epsilon(-K_X, p) = n - m + \frac{s}{r}$, hence the Seshadri constant $\epsilon(-K_X, p)$ can be any rational number in the interval $(1, n]$.

**Example 23.** More generally, let $X$ be the weighted projective space $\mathbb{P}(a_0, \cdots, a_n)$ where $a_0 \leq \cdots \leq a_n$ have no common factor and $p \in X$ a smooth point on the line $l : x_2 = \cdots = x_n = 0$ (such $p$ exists exactly when gcd$(a_0, a_1) = 1$). We claim that $\epsilon(-K_X, p)$ is the smaller one of $\frac{1}{a_n} \sum_{i=0}^n a_i$ and $\frac{1}{a_0a_1} \sum_{i=0}^n a_i$. Indeed, since $X$ is toric and $p$ is invariant under an $(n-1)$-dimensional subtorus $T$, the Mori cone of $\hat{X} = Bl_p X$ is generated by a line in $E$ and the strict transform $\hat{C}$ of a curve $C \subseteq X$ containing $p$ that is invariant under the action of $T$. Hence $C$ is the line joining $p$ and a $T$-invariant point. For $D = \sigma^*(\epsilon(-K_X) - \delta E)$, we have $(D \cdot \hat{C}) = \frac{1}{a_0a_1} \sum_{i=0}^n a_i - \delta$ if $C = l$, otherwise $(D \cdot \hat{C}) = \frac{1}{a_j} \sum_{i=0}^n a_i - \delta$ for some $j$. The claim then follows by setting $(D \cdot \hat{C}) \geq 0$. Taking $a_0 = s-1$, $a_1 = \cdots = a_{n-1} = s$, $a_n = (r-1)(s-1) - (n-1)s$ with $s \geq r \gg 0$ we get $\epsilon(-K_X, p) = \frac{s}{r}$, hence the Seshadri constant $\epsilon(-K_X, p)$ can be any rational number in the interval $(0, 1]$ as well.

**Remark 24.** As the previous examples give some possible values of $\epsilon(-K_X, p)$, it is natural to ask whether these are all possible values. When $\epsilon(-K_X, p) \geq n - 1$, the Rationality Theorem [KM93, Theorem 3.5] implies that $\epsilon(-K_X, p)$ is necessarily a rational number.

When $\epsilon(-K_X, p) < n - 1$, it is not clear to us whether $\epsilon(-K_X, p)$ is rational, although there are no known examples of irrational Seshadri constants according to [Laz04, Remark 5.1.13].

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