Blow-up of solution of an initial boundary value problem for a generalized Camassa-Holm equation

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Abstract

In this paper, we study the following initial boundary value problem for a generalized Camassa-Holm equation

\[
\begin{aligned}
&\left\{
\begin{array}{l}
u_t - u_{xxx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} + k(u - u_{xx})_x = 0, t \geq 0, x \in [0, 1], \\
u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = 0, t \geq 0, \\
u(0, x) = u_0(x), x \in [0, 1],
\end{array}
\right.
\end{aligned}
\]

Where \( k \) is a real constant. We establish local well-posedness of this closed-loop system by using Kato’s theorem for abstract quasilinear evolution equation of hyperbolic type. Then, by using multiplier technique, we obtain a conservation law which enable us to present a blow-up result.

Keywords: Generalized Camassa-Holm equation; Initial boundary value problem; Blow up
MSC: 35G25; 35G30; 35L05

1 Introduction

Recently, Camassa and Holm [1] derived a nonlinear dispersive shallow water wave equation

\[
u_t - u_{xxx} + 3uu_x = 2u_xu_{xx} + uu_{xxx} \tag{1.1}
\]

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which is called Camassa-Holm equation. Here \( u(x, t) \) denotes the fluid velocity at time \( t \) in the \( x \) direction or, equivalently, the height of the water’s free surface above a flat bottom. Eq. (1.1) has a bi-Hamiltonian structure \([2,3]\) and is completely integrable \([1,4]\). It admits, in addition to smooth waves, a multitude of travelling wave solutions with singularities: peakons, cuspons, stumpons and composite waves \([1,5,6]\). Its solitary waves are stable solitons \([7,8]\), retaining their shape and form after interactions \([9]\). It models wave breaking \([10,11,12]\).

The Cauchy problem for the Camassa-Holm equation has been studied extensively. It has been proved to be locally well-posed \([12,13]\) for initial data \( u_0 \in H^S(R) \) with \( S > \frac{3}{2} \). Moreover, it has strong solutions that are global in time \([14,15]\) as well as solutions that blow up in finite time \([14,16,17,18]\). On the other hand, it has global weak solutions with initial data \( u_0 \in H^1[14,19,20]\).

The initial boundary value problem for the Camassa-Holm equation was also studied by several authors. For example, Kwek etc. \([21]\) obtained the local existence and blow-up for an initial boundary value problem for the Camassa-Holm equation with the homogeneous boundary conditions: \( u(0, t) = u_x(0, t) = u(1, t) = u_x(1, t) = 0 \) on interval \([0, 1]\]. Ma and Ding \([22]\) obtained the existence and uniqueness of the local strong solutions to an initial boundary problem for the Camassa-Holm equation on half axis \( R^+ \) with initial data \( u_0 \in H^2(R^+) \cap H^1_0(R^+) \). They also established the global result of the corresponding solution, provided that the initial data \( u_0 \) satisfies certain positivity condition.

In this Letter we are interested in an initial boundary value problem for the following equation

\[
\frac{\partial u}{\partial t} - u_{xxx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} + k(u - u_{xx}) = 0, \tag{1.2}
\]

where \( k \) is a real constant, \( ku_x \) denotes the dissipative term and \( ku_{xxx} \) denotes the dispersive effect. When \( k = 0 \), Eq.(1.2) is the well known Camassa-Holm equation. The initial boundary value problem for Eq.(1.2) we intend to investigate is

\[
\begin{align*}
& u_t - u_{xxx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} + k(u - u_{xx}) = 0, \quad t \geq 0, \quad x \in [0, 1], \\
& u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = 0, \quad t \geq 0, \\
& u(0, x) = u_0(x), \quad x \in [0, 1],
\end{align*}
\]

The remainder of the paper is organized as follows. In Section 2, we establish the local well-posedness for the closed-loop system (1.3) by Kato’s theorem.
In Section 3, by using multiplier technique, we obtain a conservation law of the closed-loop system (1.3). Using this conservation law we present a blow-up result.

We will use the following notation without further comment. * for convolution; $L(Y, X)$ for all bounded linear operator from Banach space $Y$ to $X(L(X)$ if $X = Y$); $\partial_x = \partial/\partial x$; $\Lambda = (1 - \partial_x^2)^{1/2}$; $H^S$ is the usual Sobolev spaces, with the norm $\| \cdot \|_{H^S} = \| \cdot \|_S$ and the inner product $(\cdot, \cdot)_S$; $L^2 = L^2(0, 1)$ with the norm $\| \cdot \|_0$ and the inner product $(\cdot, \cdot)_0$; $H^S_{0,1} = \{u(x) \in H^S(0,1) : u(0) = u(1) = u_x(0) = u_x(1) = 0\}$; $[A, B] = AB - BA$ denotes the commutator of the linear operators $A$ and $B$; $C^k(I; X)$ for the space of all $k$ times continuously differentiable functions defined on an interval $I$ with values in Banach space $X$.

2 Local well-posedness

In this section, we will apply Kato’s theorem [23] to establish the local well-posedness for the closed-loop system (1.3). For convenience, we state Kato’s theorem in the form suitable for our purpose.

Consider the Cauchy problem associated to a quasilinear evolution equation

\[
\begin{cases}
\frac{du}{dt} + A(u)u = f(u) \in X, & t \geq 0 \\
u(0) = u_0 \in Y
\end{cases}
\] (2.1)

where $A(u)$ is a linear operator depending on the unknown $u$, and $u_0$ the initial value. To study the Cauchy problem (local in the time) associated to (2.1), we will make the following assumptions:

(X) $X$ and $Y$ are reflexive Banach spaces where $Y \subset X$, with the inclusion continuous and dense, and there is an isomorphism $Q$ from $Y$ onto $X$.

($A_1$) Let $W$ be an open ball centered in 0 and contained in $Y$. The linear operator $A(u)$ belongs to $G(X, 1, \beta)$ where $\beta$ is a real number, i.e., $-A(u)$ generates a $C_0$-semigroup such that
\[
\left\| e^{-sA(u)} \right\|_{L(X)} \leq e^{\beta s}.
\]

Note that if $X$ is a Hilbert space, then $A \in G(X, 1, \beta)$ if and only if [24]

(a) $(A\phi, \phi)_X \geq -\beta \|\phi\|^2_X, \forall \phi \in D(A)$,

(b) $(A + \lambda I)$ is onto for some (all) $\lambda > \beta$. 

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Under these conditions $A(u)$ is said to be quasi-m-accretive.

(A2) The map $w \in W \to B(w) = [Q, A(w)]Q^{-1} \in L(X)$ is uniformly bounded and Lipschitz continuous, that is, there exist constants $\lambda_1, \mu_1 > 0$, such that for all $w, y \in W$,

$$\|B(w)\|_{L(X)} \leq \lambda_1,$$
$$\|B(w) - B(y)\|_{L(X)} \leq \mu_1 \|w - y\|_Y,$$

(A3) $X \subseteq D(A(w))$ for each $w \in W$ (so that $A(w)|_Y \in L(Y, X)$ by the Closed Graph theorem). Moreover, the map $w \in W \to A(w) \in L(X)$ satisfies the following Lipschitz condition:

$$\|A(w) - A(y)\|_{L(Y, X)} \leq \mu_2 \|w - y\|_X$$

for all $w, y \in W$, where $\mu_2$ is a non-negative constant.

(f) The function $f : W \to Y$ is bounded, i.e., there is a constant $\lambda_2 > 0$ such that $\|f(w)\|_Y \leq \lambda_2$ for all $w \in W$, and the function $w \in X \to f(w)$ is Lipschitz in $X$ (resp. in $Y$), i.e.,

$$\|f(w) - f(y)\|_X \leq \mu_3 \|w - y\|_X, \quad \forall w, y \in W,$$
$$\|f(w) - f(y)\|_Y \leq \mu_4 \|w - y\|_Y, \quad \forall w, y \in W,$$

where $\mu_3, \mu_4$ is non-negative constant.

We are now in position to state Kato’s local well posedness result.

**Theorem 2.1 (Kato’s theorem)** Assume conditions (X); (A1)-(A3) and (f) hold. Given $u_0 \in Y$, there is $T > 0$ and unique solution $u \in C([0, T]; Y) \cap C^1([0, T]; X)$ to (2.1) with $u(0) = u_0$. Moreover, the map $u_0 \in Y \to u \in C([0, T]; Y)$ is continuous.

We now provide the framework in which we shall reformulate problem (1.3).

Let $m = u - u_{xx}$, then Eq.(1.2) takes the form of a quasi-linear evolution equation of hyperbolic type

$$m_t + um_x + 2u_xm + km_x = 0.$$

By using the operator $G(x) = \frac{\cosh(x-[x])^{-\frac{1}{2}}}{2\sinh^2(\frac{1}{2})}$, where $[x]$ denotes the integer part of $x \in [0, 1]$, then $(1 - \partial_x^2)^{-1}f = G \ast f$, $\forall f \in L^2$ and $G \ast m = u$. Using this identity, we can rewrite Eq.(1.2) as

$$u_t + uu_x + \partial_x(G \ast (u^2 + \frac{1}{2}u_x^2)) + ku_x = 0.$$
Then the closed-loop system (1.3) becomes
\[
\begin{aligned}
&\begin{cases}
  u_t + uu_x + \partial_x(G * (u^2 + \frac{1}{2} u_x^2)) + ku_x = 0, t \geq 0, x \in [0, 1] \\
  u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = 0, t \geq 0 \\
  u(0, x) = u_0(x), x \in [0, 1]
\end{cases} \\
\end{aligned}
\]
(2.2)

For the system (2.2), we have the following result.

**Theorem 2.2** Given \( u_0(x) \in H^2_{0,1} \), there exists a maximal value \( T = T(u_0(x)) > 0 \) and a unique solution \( u(x, t) \) to the closed loop system (2.2) such that \( u = u(\cdot ; u_0) \in C([0, T]; H^2_{0,1}) \cap C^1([0, T]; H^1) \). Moreover, the solution depends continuously on the initial data, i.e., the mapping \( u_0 \rightarrow u(\cdot ; u_0) : H^2 \rightarrow C([0, T]; H^2_{0,1}) \cap C^1([0, T]; H^1) \) is continuous.

Let \( A(u)u = u \partial_x u + ku \partial_x u, f(u) = -\partial_x (G * (u^2 + \frac{1}{2} u_x^2)) \), \( Q = \Lambda, X = H^1, Y = H^2_{0,1} \).

**Remark 2.1** The operator \( Q \) maps \( Y \) to \( X \). In fact, \( \forall u \in Y \),
\[
\|Qu\|_1^2 = \|Qu\|_0^2 + \|(Qu)_x\|_0^2 \\
= (Qu, Qu)_0 + ((Qu)_x, (Qu)_x)_0 \\
= \int_0^1 (u - u_{xx})u_x dx + \int_0^1 (u_x - u_{xxx})u_x dx \\
= \int_0^1 u^2 dx - uu_x\big|_0^1 + \int_0^1 u_x^2 dx + \int_0^1 u_{xx}^2 dx - uu_xu_{xx}\big|_0^1 + \int_0^1 u_{xx}^2 dx \\
= \|u\|_0^2 + 2 \|u_x\|_0^2 + \|u_{xx}\|_0^2,
\]
so \( Qu \in X \). Moreover, the operator \( Q \) is an isomorphism of \( Y \) onto \( X \). One may see the similar argument in [28].

In order to prove Theorem 2.2, by applying Theorem 2.1, we only need to verify \( A(u) \) and \( f(u) \) satisfy the conditions (X); (A1)-(A3) and (f). The following lemmas are useful for our arguments.

**Lemma 2.1** ([25]) Let \( f \in H^S, S > \frac{3}{2} \), then
\[
\|\Lambda^{-r}[\Lambda^{r+t+1}, M_f] \Lambda^{-t}\|_{L(L^2)} \leq c\|f\|_{S}, |r|, |t| \leq S - 1,
\]
where \( M_f \) is the operator of multiplication by \( f \) and \( c \) is a constant depending only on \( r, t \).

**Lemma 2.2** ([26]) Let \( X \) and \( Y \) be two Banach spaces and \( Y \) be continuously and densely embedded in \( X \). Let \( -A \) be the infinitesimal generator of the \( C_0 \)-semigroup \( T(t) \) on \( X \) and \( Q \) be an isomorphism from \( Y \) onto \( X \). \( Y \) is \( -A \)-admissible (ie. \( T(t)Y \subset Y, \forall t \geq 0, \) and the restriction of \( T(t) \) to \( Y \) is
a \( C_0 \)-semigroup on \( Y \).) if and only if \( -A_1 = -QAQ^{-1} \) is the infinitesimal generator of the \( C_0 \)-semigroup \( T_1(t) = QT(t)Q^{-1} \) on \( X \). Moreover, if \( Y \) is \( -A \)-admissible, then the part of \(-A\) in \( Y \) is the infinitesimal generator of the restriction of \( T(t) \) to \( Y \).

Now we divide the proof of the Theorem 2.2 into the following lemmas.

**Lemma 2.3** The operator \( A(u) = u\partial_x + k\partial_x \) with \( u \in Y \) belongs to \( G(L^2, 1, \beta) \).

*Proof.* Due to \( L^2 \) being a Hilbert space, \( A(u) \in G(L^2, 1, \beta) \) if and only if there is a real number \( \beta \) such that

\[
(A(u)v, v)_0 \geq -\beta\|v\|_0^2,
\]

(b) \(-A(u)\) is the infinitesimal generator of a \( C_0 \)-semigroup on \( L^2 \) for some (or all) \( \lambda \geq \beta \).

First, let us prove (a). Due to \( u \in Y \), \( u \) and \( u_x \) belongs to \( L^\infty \). Noting that \( \|u_x\|_{L^\infty} \leq \|u\|_Y \), then we have

\[
\frac{1}{2} \int_0^1 u_x w^2 \, dx \leq \frac{1}{2} \|u_x\|_{L^\infty} \|w\|_X^2.
\]

Setting \( \beta = \frac{1}{2} \|u\|_2 \), we have \( (A(u)v, v)_0 \geq -\beta\|v\|_0^2 \).

Next we prove (b). Obviously \( A(u) = u\partial_x + k\partial_x \) is a closed operator. For \( w \in Y \), if there exists \( z \in Y \), such that \((\lambda I + A(u))^{-1}z = w\). Then \((\lambda I + A(u))w = z\), i.e., \( \lambda w + uw_x + kw_x = z \). Multiplying the both sides of this equality by \( w \), and integrating over \((0, 1)\) by parts, we get

\[
\lambda \|w\|_X^2 - \frac{1}{2} \int_0^1 u_x w^2 \, dx = \int_0^1 zw \, dx. \tag{2.3}
\]

Since \( u \in Y \), \( u \) and \( u_x \) belong to \( L^\infty \). Note that \( \|u_x\|_{L^\infty} \leq \|u\|_Y \), then we have

\[
\frac{1}{2} \int_0^1 u_x w^2 \, dx \leq \frac{1}{2} \|u_x\|_{L^\infty} \|w\|_X^2 \leq \frac{1}{2} \|u\|_Y \|w\|_X^2. \tag{2.4}
\]

On the other hand, by Hölder inequality, we have

\[
\int_0^1 zw \, dx \leq \|z\|_X \|w\|_X. \tag{2.5}
\]
Then combining (2.3)-(2.5), we obtain

\[ \lambda \| w \|_X^2 - \frac{1}{2} \| u \|_Y \| w \|_X^2 \leq \| z \|_X \| w \|_X. \]

That is

\[ (\lambda - \frac{1}{2} \| u \|_Y) \| w \|_X^2 \leq \| z \|_X \| w \|_X. \] (2.6)

Setting \( \beta = \frac{1}{2} \| v \|_Y \), then by (2.6), we have

\[ \| w \|_X \leq \frac{1}{\lambda - \beta} \| z \|_X, \quad \forall \lambda > \beta. \]

i.e.,

\[ \| (\lambda I + A(u))^{-1} \|_{L(X)} \leq \frac{1}{\lambda - \beta}, \quad \forall \lambda > \beta. \]

By Hille-Yosida theorem [27], we conclude that the operator \(-A(u)\) is the infinitesimal generator of a \(C_0\)-semigroup on \(X\). This completes the proof of Lemma 2.3. ■

**Lemma 2.4** The operator \(A(u) = u\partial_x + k\partial_x\) with \(u \in Y\) belongs to \(G(H^1, 1, \beta)\).

**Proof.** Due to \(H^1\) being a Hilbert space, \(A(u) \in G(H^1, 1, \beta)\) [24] if and only if there is a real number \(\beta\) such that

(a) \((A(u)v, v)_1 \geq -\beta\|v\|_1^2\),

(b) \(-A(u)\) is the infinitesimal generator of a \(C_0\)-semigroup on \(H^1\) for some(or all) \(\lambda \geq \beta\).

First, let us prove (a). Let \(u \in Y\), then \(u\) and \(u_x\) belongs to \(L^\infty\) and \(\| u_x \|_{L^\infty} \leq \| u\|_2\). Note that

\[ \Lambda(uv_x) = [\Lambda, u]v_x + u\Lambda v_x = [\Lambda, u]v_x + u\partial_x \Lambda v. \]

Then we have

\[ (A(u)v, v)_1 = (\Lambda(uv_x + kv_x), \Lambda v)_0 = ([\Lambda, u]v_x, \Lambda v)_0 - \frac{1}{2} (u_x \Lambda v, \Lambda v)_0 + k(\Lambda v_x, \Lambda v)_0 \leq \| [\Lambda, u] \|_{L(L^2)} \| \Lambda v \|_0^2 + \frac{1}{2} \| u_x \|_{L^\infty} \| \Lambda v \|_0^2 \leq c \| u \|_2 \| v \|_1^2, \]

where we apply Lemma 2.1 with \(r = 0\) , \(t = 0\). Setting \(\beta = c \| u \|_2\), we have \((A(u)v, v)_1 \geq -\beta\|v\|_1^2\).

Next we prove (b). Note that \(Q = \Lambda\) is an isomorphism of \(Y\) onto \(X\) and \(Y\) is
continuously and densely embedded in $X$. Define

$$A_1(u) := [Q, A(u)]Q^{-1} = [A, A(u)]\Lambda^{-1}, \quad B_1(u) := A_1(u) - A(u),$$

then

$$B_1(u) = [\Lambda, (u + \gamma)\partial_x]\Lambda^{-1} - (u + \gamma)\partial_x$$

$$= [\Lambda, u\partial_x]\Lambda^{-1} + \gamma\Lambda\partial_x\Lambda^{-1} - (u + \gamma)\partial_x$$

$$= [\Lambda, u\partial_x\Lambda^{-1} + u\Lambda\partial_x\Lambda^{-1} - u\partial_x$$

$$= [\Lambda, u]\partial_x\Lambda^{-1}.$$

Let $v \in L^2$ and $u \in Y$. Then we have

$$\|B_1(u)v\|_0 = \|[\Lambda, u]\partial_x\Lambda^{-1}v\|_0$$

$$= \|[\Lambda, u]\Lambda^{-1}\partial_xv\|_0$$

$$\leq \|[\Lambda, u]\|_{L(L^2)}\|\Lambda^{-1}\partial_xv\|_0$$

$$\leq c\|u\|_2\|v\|_0$$

where we apply Lemma 2.1 with $r = 0, t = 0$. Therefore, we obtain that $B_1(u) \in L(L^2)$.

Note that $A_1(u) = A(u) + B_1(u)$ and $A(u) \in G(L^2, 1, \beta)$ in Lemma 2.3. By a perturbation theorem for semigroups (cf. §5.2 Theorem 2.3 in [26]), we obtain that $A_1(u) \in G(L^2, 1, \beta')$. By applying Lemma 2.2 with $Y = H^1_{0,1}, X = L^2$ and $Q = \Lambda$, we conclude that $Y$ is $-A$-admissible. So, $-A(u)$ is the infinitesimal generator of a $C_0$-semigroup on $Y$. This completes the proof of Lemma 2.4.

**Lemma 2.5** For all $u \in Y, A(u) \in L(Y, X)$. Moreover,

$$\|(A(u) - A(z))w\|_X \leq \|u - z\|_X \|w\|_Y, u, z, w \in Y.$$

**Proof.** For $u, z, w \in Y$, we have

$$\|(A(u) - A(z))w\|_X = \|(uw_x + kw_x) - (zw_x + kw_x)\|_X$$

$$\leq \|u - z\|_X \|w_x\|_{L^\infty}$$

$$\leq \|u - z\|_X \|w\|_Y.$$

Taking $z = 0$ in the above inequality, we obtain $A(u) \in L(Y, X)$. This completes the proof of Lemma 2.5. ■
Lemma 2.6  For $u \in Y$, $B(u) = [\Lambda, A(u)]\Lambda^{-1} \in L(X)$, and

$$
\|(B(u) - B(z))w\|_X \leq c \|u - z\|_Y \|w\|_X, \quad \forall u, z \in Y, \quad w \in X.
$$

Proof. Let $u, z \in Y$, $w \in X$, then

$$
\|(B(u) - B(z))w\|_X = \|\Lambda[\Lambda,(u - z)\partial_x]\Lambda^{-1}w\|_0
\leq \|\Lambda[\Lambda,(u - z)]\Lambda^{-1}\|_{L(L^2)} \|w_x\|_0
\leq c \|u - z\|_Y \|w\|_X,
$$

where we apply the Lemma 2.1 with $r = -1$ and $t = 1$. Taking $z = 0$ in the above inequality, we obtain $B(u) \in L(X)$. This completes the proof of Lemma 2.6. \( \blacksquare \)

Lemma 2.7  $f(u) = -\partial_x(G * (u^2 + \frac{1}{2}u_x^2))$ satisfies

(a) \( \|f(u)\|_Y \leq c \|u\|_Y^2, \ u \in Y; \)

(b) \( \|f(u) - f(z)\|_X \leq c \|u - z\|_X, \ u, z \in Y; \)

(c) \( \|f(u) - f(z)\|_Y \leq c \|u - z\|_Y, \ u, z \in Y. \)

Proof. Let $u, z \in Y$. Note that $H^1_{0,1}$ is a Banach algebra. Then we have

$$
\|f(u) - f(z)\|_Y = \| - \partial_x G * (u^2 - z^2 + \frac{1}{2}u_x^2 - \frac{1}{2}z_x^2)\|_2
\leq \|(u - z)(u + z)\|_1 + \frac{1}{2}\|(u - z)_x(u + z)_x\|_1
\leq \|u - z\|_1\|u + z\|_1 + \frac{1}{2}\|(u - z)_x\|_1\|(u + z)_x\|_1
\leq \|u - z\|_2\|u + z\|_2 + \frac{1}{2}\|u - z\|_2\|u + z\|_2
\leq (\frac{3}{2}\|u\|_2 + \frac{3}{2}\|z\|_2)\|u - z\|_2
= (\frac{3}{2}\|u\|_Y + \frac{3}{2}\|z\|_Y)\|u - z\|_Y.
$$

This proves (c). Taking $z = 0$ in the above inequality, we obtain (a). Next, we prove (b).
Let \( v, z \in Y \). Note that \( H_{0,1}^1 \) is a Banach algebra. Then we get

\[
\| f(u) - f(z) \|_X = \| - \partial_x G \ast (u^2 - z^2 + \frac{1}{2} u_x^2 - \frac{1}{2} z_x^2) \|_1 \\
\leq \| (u - z)(u + z) \|_0 + \frac{1}{2} \| (u - z)_x(u + z)_x \|_0 \\
\leq \| u - z \|_0 \| u + z \|_0 + \frac{1}{2} \| (u - z)_x \|_0 \| (u + z)_x \|_0 \\
\leq \| u - z \|_1 \| u + z \|_2 + \frac{1}{2} \| u - z \|_1 \| u + z \|_2 \\
\leq (\frac{3}{2} \| u \|_2 + \frac{3}{2} \| z \|_2) \| u - z \|_1 \\
= (\frac{3}{2} \| u \|_Y + \frac{3}{2} \| z \|_Y) \| u - z \|_X.
\]

This completes the proof of Lemma 2.7. ■

Proof of Theorem 2.2. Combining Theorem 2.1 and Lemma 2.3-2.7, we get the statement of Theorem 2.2. ■

3 Blow up

Firstly, by using multiplier technique, we obtain the following conservation law of the closed-loop system (2.2).

**Theorem 3.1** Let \( T > 0 \) be the maximal time of existence of the solution \( u(x, t) \) to the closed-loop system (2.2) (or (1.3)) with the initial data \( u_0(x) \in H_{0,1}^2 \), then

\[
\| u(\cdot, t) \|_1^2 = \| u_0 \|_1^2. \tag{3.1}
\]

**Proof.** Multiplying the first equation of the closed-loop system (2.2) by \( u - u_{xx} \), and integrating over \((0, 1)\) by parts, we get

\[
\int_0^1 u_t(u - u_{xx})\,dx = -\int_0^1 u u_x(u - u_{xx})\,dx - k \int_0^1 u_x(u - u_{xx})\,dx \\
- \int_0^1 \partial_x(G \ast (u^2 + \frac{1}{2} u_x^2))(u - u_{xx})\,dx. \tag{3.2}
\]
For the LHS of \((3.2)\), using the boundary conditions of the closed-loop system \((2.2)\), we have

\[
\int_0^1 u_t(u - u_{xx})dx = \frac{1}{2} \frac{d}{dt} \int_0^1 u^2dx - u_tu_x \bigg|_0^1 + \frac{1}{2} \frac{d}{dt} \int_0^1 u_x^2dx
\]

\[
= \frac{1}{2} \frac{d}{dt} \left( \int_0^1 u^2dx + \int_0^1 u_x^2dx \right)
\]

(3.3)

For the RHS of \((3.2)\), using the boundary conditions of the closed-loop system \((2.2)\), we obtain

\[
- \int_0^1 uu_x(u - u_{xx})dx - k \int_0^1 u_x(u - u_{xx})dx - \int_0^1 \partial_x(G \ast (u^2 + \frac{1}{2} u_x^2))(u - u_{xx})dx
\]

\[
= - \int_0^1 u^2u_xdx + \int_0^1 uu_xu_{xx}dx - k \int_0^1 uu_xdx + k \int_0^1 u_xu_{xx}dx
\]

\[
- G \ast (u^2 + \frac{1}{2} u_x^2)u \bigg|_0^1 + \int_0^1 G \ast (u^2 + \frac{1}{2} u_x^2)u_xdx
\]

\[
+ \partial_x(G \ast (u^2 + \frac{1}{2} u_x^2))u_x \bigg|_0^1 - \int_0^1 \partial_x^2(G \ast (u^2 + \frac{1}{2} u_x^2))u_xdx
\]

\[
= - \frac{1}{3} u^3 \bigg|_0^1 + \int_0^1 uu_xu_{xx}dx - k \frac{u^2}{2} \bigg|_0^1 + k \frac{u_x^2}{2} \bigg|_0^1 + \int_0^1 (u^2 + \frac{1}{2} u_x^2)u_xdx
\]

\[
= \int_0^1 uu_xu_{xx}dx + (u^2 + \frac{1}{2} u_x^2)u \bigg|_0^1 - \int_0^1 (2uu_x + u_xu_{xx})udx
\]

\[
= - \int_0^1 2u^2u_xdx = - \frac{2}{3} u^3 \bigg|_0^1 = 0
\]

(3.4)

It follows from \((3.2)-(3.4)\) that

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 (u^2 + u_x^2)dx = 0
\]

(3.5)

Integrating \((3.5)\) over \((0, t)\), we obtain \((3.1)\). \(\blacksquare\)

**Remark 3.1** Employing the Agmon’s inequality

\[
\int_0^1 u^2(x)dx \leq 2u^2(0) + 4 \int_0^1 u_x^2(x)dx
\]

and the Poincaré inequality

\[
\max_{x \in [0,1]} u^2(x) \leq u^2(0) + 2 \sqrt{\int_0^1 u^2(x)dx} \sqrt{\int_0^1 u_x^2(x)dx}
\]

with \(u(0) = 0\), we have

\[
\max_{x \in [0,1]} |u(x)| \leq 2 \sqrt{\int_0^1 u_x^2(x)dx}
\]

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It then follows from Theorem 3.1 that

$$\sup_{(x,t)\in[0,1] \times [0,\infty)} |u(x,t)| \leq 2\|u_0\|_1$$

This shows that the solution $u(x,t)$ to the closed-loop system (2.2) or system (1.3) is bounded if the initial data $u_0(x) \in H^2_{0,1}$.

Now we present a blow-up result of solution to the closed-loop system (2.2) (or system (1.3)).

**Theorem 3.2** Assume $u_0(x) \in H^2_{0,1}$ and $T$ is the maximal existence time of the solution $u(x,t)$ to the closed-loop system (2.2) (or (1.3)) guaranteed by Theorem 2.2. If there exists one point $x_0 \in (0,1)$ such that $u_{xx}(x_0,t) = 0$ and $u_0'(x_0) < -\sqrt{2}\|u_0\|_1$, then the corresponding solution blows up in finite time. Moreover, the maximal time of existence is estimated above by

$$\frac{1}{\sqrt{2}\|u_0\|_1} \ln\left(\frac{h(0) - \sqrt{2}\|u_0\|_1}{h(0) + \sqrt{2}\|u_0\|_1}\right)$$

where $h(0) = u_0'(x_0)$.

**Proof.** Differentiating the first equation of the closed-loop system (2.2) with respect to $x$, in view of $\partial_x^2 G * f = G * f - f$, we have

$$u_{tx} = -\frac{1}{2}u_x^2 - u_{xx} + \gamma u_{xx} + u^2 - G * (u^2 + \frac{1}{2}u_x^2) \quad (3.6)$$

Let $x = x_0$ in (3.6) and set $h(t) = u_x(x_0,t)$. Noting that $u_{xx}(x_0,t) = 0$ and $G * (u^2 + \frac{1}{2}u_x^2) \geq 0$, we obtain

$$h'(t) \leq -\frac{1}{2}h^2(t) + u^2(x_0,t). \quad (3.7)$$

In view of (3.1) and Sobolev embedding theorem, we have

$$u^2(x_0,t) \leq \|u\|_{L^\infty}^2 \leq \|u\|_1^2 = \|u_0\|_1^2. \quad (3.8)$$

It follows from (3.7) and (3.8) that

$$h'(t) \leq -\frac{1}{2}h^2(t) + \|u_0\|_1^2.$$ 

Note that if $h(0) \leq -\sqrt{2}\|u_0\|_1$, then $h(t) \leq \sqrt{2}\|u_0\|_1$, for all $t \in [0,T)$. Therefore, from the above inequality we obtain

$$\frac{h(0) + \sqrt{2}\|u_0\|_1}{h(0) - \sqrt{2}\|u_0\|_1} e^{\sqrt{2}\|u_0\|_1 t} - 1 \leq \frac{2\sqrt{2}\|u_0\|_1}{h(t) - \sqrt{2}\|u_0\|_1} \leq 0.$$
Due to $0 < \frac{h(0) + \sqrt{2}\|u_0\|_1}{h(0) - \sqrt{2}\|u_0\|_1} < 1$, then exists

$$T_0 \leq \frac{1}{\sqrt{2}\|u_0\|_1} \ln\left(\frac{h(0) - \sqrt{2}\|u_0\|_1}{h(0) + \sqrt{2}\|u_0\|_1}\right)$$

such that

$$\lim_{t \to T_0} h(t) = -\infty.$$ 

Thus $\lim_{t \to T_0} \|u\|_2 = \infty$ because $|u_x(x, t)| \leq \|u_x(x, t)\|_{L^\infty} \leq \|u\|_2$. That is, the solution $u(x, t)$ to the closed-loop system (2.2) does not exist globally in time in function space $H^{2,1}_{0,1}$.

**Remark 3.2** Theorem 3.2 shows that although $\int_0^1 u_x^2 dx$ is bounded (see Theorem 3.1), it does not guarantee that $u_x(x, t)$ is bounded for all $x \in [0, 1]$.

**References**

[1] R. Camassa, D. D. Holm, Phys. Rev. Lett. 71 (1993) 1661.
[2] A. Fokas, B. Fuchssteiner, Physica D 4 (1981) 47.
[3] J. Lenells, J. Phys. A 38 (2005) 869.
[4] A. Constantin, Proc. R. Soc. London A 457 (2001) 953.
[5] J. Lenells, J. Differential Equations 217 (2005) 393.
[6] Y. Li, P. Olver, J. Differential Equations 162 (2000) 27.
[7] A. Constantin, W. A. Strauss, J. Nonlinear Sci. 12 (2002) 415.
[8] A. Constantin, W. A. Strauss, Commun. Pure Appl. Math. 53 (2000) 603.
[9] R. S. Johnson, Proc. R. Soc. London A 459 (2003) 1687.
[10] A. Constantin, Ann. Inst. Fourier (Grenoble) 50 (2000) 321.
[11] A. Constantin, J. Escher, Acta Math. 181 (1998) 229.
[12] A. Constantin, J. Escher, Math. Z. 233 (2000) 75.
[13] G. Rodriguez-Blanco, Nonlinear Anal. 46 (2001) 309.
[14] A. Constantin, J. Escher, Comm. Pure Appl. Math. 51 (1998) 475.
[15] H. Dai, K. Kwek, H. Gao, C. Qu, Front. Math. China 1 (2006), 144.
[16] A. Constantin, J. Differential Equations 141 (1997) 218.
[17] A. Constantin, J. Nonlinear Sci. 10 (2000) 391.
[18] A. Constantin, J. Escher, Ann. Sci. Norm. Sup. Pisa 26 (1998) 303.
[19] A. Constantin, L. Molinet, Comm. Math. Phys. 211 (2000) 45.
[20] Z. Xin, P. Zhang, Comm. Pure Appl. Math. 53 (2000) 1411.
[21] K. Kwek, H. Gao, W. Zhang, C. Qu, J. Math. Phys. 41 (2000) 8279.
[22] S. Ma, S. Ding, J. Math. Phys. 45 (2004) 3479.
[23] T. Kato: Quasi-linear equations of evolution, with applications to partial differential equations. In: Spectral Theory and Differential Equations, Lecture Notes in Math. 1975, pp. 25-70.
[24] T. Kato, Adv. Math. Suppl. Stud. 8 (1983) 93.
[25] T. Kato, Manuscripta Math. 28 (1979) 89.
[26] A. Pazy, Semigroup of Linear Operators and Applications to Partial Differential Equations, Springer Verlag, New York, 1983.
[27] K. Yosida: Functional Analysis, Berlin/New York, Springer-Verlag, 1966.
[28] X. Zong, Y. Zhao, Nonlinear Analysis 67 (2007) 3167.