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Combinatorial study of colored Hurwitz polyzêtas

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Jean-Yves Enjalbert, Hoang Ngoc Minh

1 Université Paris 13, Sorbonne Paris Cit, LIPN, CNRS, UMR 7030, F-93430, Villette-noise, France.
2 Université Lille II, 1 place Délion, 59024 Lille, France

Email addresses: jean-yves.enjalbert@lipn.univ-paris13.fr, hoang@lipn.univ-paris13.fr

Abstract

A combinatorial study discloses two surjective morphisms between generalized shuffle algebras and algebras generated by the colored Hurwitz polyzêtas. The combinatorial aspects of the products and co-products involved in these algebras will be examined.

1 Introduction

Classically, the Riemann zêta function is \( \zeta(s) = \sum_{n>0} n^{-s} \), the Hurwitz zêta function is \( \zeta(s; t) = \sum_{n>0} (n-t)^{-s} \) and the colored zêta function is \( \zeta(s; q) = \sum_{n>0} q^n n^{-s} \), where \( q \) is a root of unit. The three previous functions are defined over \( \mathbb{Z}_{>0} \) but can be generalized over any composition (sequence of positive integers) \( s = (s_1, \ldots, s_r) \), like, respectively, the Riemann polyzêta function \( \zeta(s) = \sum_{n_1>\ldots>n_r>0} n_1^{-s_1} \cdots n_r^{-s_r} \), the Hurwitz polyzêta function \( \zeta(s; t) = \sum_{n_1>\ldots>n_r>0} (n_1-t_1)^{-s_1} \cdots (n_r-t_r)^{-s_r} \) and the colored polyzêta function \( \zeta(s; \xi, t) = \sum_{n_1>\ldots>n_r>0} q^{i_1 n_1} \cdots q^{i_r n_r} n_1^{-s_1} \cdots n_r^{-s_r} \), with \( q \) a root of unit and \( i = (i_1, \ldots, i_r) \) a composition. These sums converge when \( s_1 > 1 \).

To study simultaneously these families of polyzêtas, the colored Hurwitz polyzêtas, for a composition \( s = (s_1, \ldots, s_r) \) and a tuple of complex numbers \( \xi = (\xi_1, \ldots, \xi_r) \) and a tuple of parameters in \( ]-\infty; 1[ \), \( t = (t_1, \ldots, t_r) \), are defined by [6]

\[
D_l(F_{\xi, t}; s) = \sum_{n_1>\ldots>n_r>0} \frac{\xi_1^{n_1} \cdots \xi_r^{n_r}}{(n_1-t_1)^{s_1} \cdots (n_r-t_r)^{s_r}}. \tag{1}
\]

Note that, for \( l = 1, \ldots, r \), the numbers \( \xi_l \) are not necessary roots of unity \( q^i \). We are working, in this note, with the condition
∀i, | \prod_{k=1}^{i} \xi_k | \leq 1 \text{ and } t_i \in ]-\infty; 1[.

Hence, \( D_i(\mathbf{F}_\xi, t; s) \) converges if \( s_1 > 1 \). We note \( E \) the set of \( C \)-tuples verifying \((E)\).

These polyz\(\hat{e}\)tas are obtained as special values of iterated integrals\(^1\) over singular differential 1-forms introduced in [10]. As iterated integrals, they are encoded by words or by non commutative formal power series [10] and are used to construct bases for asymptotic expanding [14] or symbolic integrating fuchian differential equations [11] exactly or approximatively [8]. The meromorphic continuation of the colored Hurwitz polyz\(\hat{e}\)tas\(^2\) is already studied in [5, 6]. In our studies, we constructed an integral representation\(^3\) of colored Hurwitz polyz\(\hat{e}\)tas and a distribution treating simultaneously two singularities and our methods permit to make the meromorphic continuation commutatively over the variables \( s_1, \ldots, s_r \) [5, 6]. Moreover, [6] gives another way to obtain the meromorphic continuation thanks to translation equations [4]. Our methods give the structure of multi-poles [5] (Theorem 4.2) and two ways to calculate algorithmically the multi-residus\(^4\).

In this note, in continuation with our previous works [10, 11, 12, 13, 5, 6], we are focusing on Hopf algebra, for a class of products as minusstuffle (\( uv \)), mulstuffle (\( uv \)), \ldots, and in particular for the new product duffle (\( uv \)), obtained as “tensorial product” of \( uv \) and the well known stuffle (\( uv \)), of symbolic representations of these polyz\(\hat{e}\)tas (see Definition 2.1 and Proposition 2.1 below).

## 2 Combinatorial objects

### 2.1 Some products and their algebraic structures

Let \( X \) be an encoding alphabet and the free monoid over \( X \) is denoted by \( X^* \). The length of any word \( w \in X^* \) is denoted by \( |w| \) and the unit of \( X^* \) is denoted by \( 1_{X^*} \). For any unitary commutative algebra \( A \), a formal power series \( S \) over \( X \) with coefficients in \( A \) can be written as the infinite sum \( \sum_{w \in X^*} \langle S|w \rangle w \). The set of polynomials (resp. formal power series) over \( X \) with coefficients in \( A \) is denoted by \( A\langle X \rangle \) (resp. \( A\langle \langle X \rangle \rangle \)).

The set of degree 1 monomials is \( AX = \{ ax / a \in A, x \in X \} \).

**Definition 2.1** We note \( \mathcal{P} \) the set of products \( * \) over \( A\langle X \rangle \) verifying the conditions:

\(^1\)They are presented as generalized Nielsen polylogarithms in [10] (Definition 2.3) and as generalized Lerch functions in [12] (Definition 3).

\(^2\)See also references and a discussion about meromorphic continuation of Riemann polyz\(\hat{e}\)tas in [5].

\(^3\)This integral representation is obtained by applying successively the polylogarithmic transform [10]. It is an application of non commutative convolution as shown in [9] (Section 2.4). Other integral representations can be also deduced easily by change of variables, for example \( t = zr \) and then \( r = e^{-u} \) [5].

\(^4\)Other meromorphic continuations can also be obtained by Mellin transform as already done in [17] or by classical estimation on the imaginary part [7] but these later work recursively, depth by depth, and the commutativity of this process over the variables \( s_1, \ldots, s_r \) must be proved. Unfortunately, the structure of multi-poles as well as multi-residus are missing in both works [7, 17]. In [16], to make the meromorphic continuation (giving the expression of non positive integers multi-residus via a generalization of Bernoulli numbers – but not of all multi-residus) of the specialization at roots of unity of colored Hurwitz polyz\(\hat{e}\)tas \( D_i(\mathbf{F}_\xi, t; s) \), the author bases on the integral representation, on the contours, of the multiple Hurwitz-Lerch which corresponds mutatis mutandis to the integral representation of generalized Lerch functions introduced earlier in [5] (Corollary 3.3).
(i) the map $\star : A(X) \times A(X) \to A(X)$ is bilinear,
(ii) for any $w \in X^*$, $1_{X^*} \star w = w \star 1_{X^*} = w$,
(iii) for any $a, b \in X$ and $u, v \in X^*$,

$$au \star bv = a(u \star bv) + b(au \star v) + [a, b](u \star v),$$

where $[., .] : AX \times AX \to AX$ is a function verifying:

(S1) $\forall a \in AX$, $[a, 0] = 0$,
(S2) $\forall (a, b) \in (AX)^2$, $[a, b] = [b, a]$,
(S3) $\forall (a, b, c) \in (AX)^3$, $[[a, [b, c]]] = [a, [b, c]]$.

**Example 1 (see [18])** Product of iterated integrals.

The shuffle is a bilinear product such that:

$$\forall w \in X^*, \quad w \shuffle 1_{X^*} = 1_{X^*} \shuffle w = w \quad \text{and} \quad \forall (a, b) \in X^2, \forall (u, v) \in X^{*2}, \quad au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v).$$

For example, for any letter $x_0, x$ and $x'$ in $X$,

$$x_0x' \shuffle x_0^2x = x_0x'x_0^2x + 2x_0x'x_0x + 3x_0^3x'x + 3x_0^2x_0x' + x_0^3x_0x'.$$

**Example 2 (see [15])** Product of quasi-symmetric functions.

Let $X$ be an alphabet indexed by $\mathbb{N}$.

The stuffle is a bilinear product such that:

$$\forall w \in X^*, \quad w \shuffle 1_{X^*} = 1_{X^*} \shuffle w = w \quad \text{and} \quad \forall (x_i, x_j) \in X^2, \forall (u, v) \in X^{*2}, \quad x_iu \shuffle x_jv = x_i(u \shuffle x_jv) + x_j(x_iu \shuffle v) + x_{i+j}(u \shuffle x_jv).$$

In particular, with the alphabet $Y = \{y_1, y_2, y_3, \ldots\}$,

$$\langle y_3y_1 \shuffle y_2 \rangle = y_3y_1y_2 + y_3y_2y_1 + y_3y_3 + y_2y_3 + y_2y_1 + y_3y_1.$$

**Example 3 ([3])** Product of large multiple harmonic sums.

Let $X$ be an alphabet indexed by $\mathbb{N}$.

The minus-stuffle is a bilinear product such that:

$$\forall w \in X^*, \quad w \shuffle 1_{X^*} = 1_{X^*} \shuffle w = w \quad \text{and} \quad \forall (x_i, x_j) \in X^2, \forall (u, v) \in X^{*2}, \quad x_iu \shuffle x_jv = x_i(u \shuffle x_jv) + x_j(x_iu \shuffle v) - x_{i+j}(u \shuffle x_jv).$$

**Example 4 ([16])** Product of colored sums.

Let $X$ be an alphabet indexed by a monoid $(\mathcal{I}, \times)$.

The multishuffle is a bilinear product such that:

$$\forall w \in X^*, \quad w \shuffle 1_{X^*} = 1_{X^*} \shuffle w = w \quad \text{and} \quad \forall (x_i, x_j) \in X^2, \forall (u, v) \in X^{*2}, \quad x_iu \shuffle x_jv = x_i(u \shuffle x_jv) + x_j(x_iu \shuffle v) + x_{i+j}(u \shuffle x_jv).$$

For example, with $X$ indexed by $\mathbb{Q}^*$,

$$x_\frac{3}{2}x_{-1} \shuffle x_{\frac{3}{2}} = x_\frac{3}{2}x_{-1}x_\frac{3}{2} + x_\frac{5}{2}x_{-1} + x_\frac{3}{2}x_{-1} + x_\frac{5}{2}x_{-1} + x_\frac{7}{2}x_{-1} + x_\frac{9}{2}x_{-1}.$$
Remark 2.1 Thanks to the one-to-one correspondence \((i_1, \ldots, i_r) \mapsto x_{i_1} \cdots x_{i_r}\) between tuples of \(I\) and word over \(X\), the calculus of \(x_{\frac{1}{2}} x_{\frac{1}{2}}^{-1} \sqcup x_{\frac{1}{2}}^{-1}\) can be written as 
\[
\left(\frac{1}{2}, -1\right) \sqcup \left(\frac{1}{2}, \frac{1}{2}\right) + \left(\frac{1}{2}, \frac{1}{2}\right) + \left(\frac{1}{2}, -1\right) + \left(\frac{1}{2}, -1\right).
\]

Example 5 (61) Product of colored Hurwitz polyzetas.
Let \(Y\) and \(E\) be two alphabets and consider the alphabet \(A = Y \times E\) with the concatenation defined recursively by \((y, e).(w_Y, w_E) = (yw_Y, ew_E)\) for any letters \(y \in Y\), \(e \in E\), and any word \(w_Y \in Y^*, w_E \in E^*\). The unit of the monoid \(A^*\) is given by \(1_A = (1_Y, 1_E)\). If \(Y\) is indexed by \(N\) and \(E\) by a monoid \((I, \times)\), the duffle is a bilinear product such that \(\forall w \in A^\ast, w \sqcup A^\ast w = w\), \(\forall (y_i, y_j) \in Y^2, \forall (e_i, e_k) \in E^2, \forall (v, u) \in A^{2^\ast}, \ (y_i, e_i).u \sqcup (y_j, e_k).v = (y_i, e_i). (u \sqcup (y_j, e_k).v) + (y_j, e_k). (u \sqcup (y_i, e_i).v)\).

Proposition 2.1 The shuffle, the stuffle, the minus-stuffle and the multistuffe are elements of \(P\), with respectively, \([x_i, x_j] = 0, [x_i, x_j] = x_{i+j}, [x_i, x_j] = -x_{i+j}, [x_i, x_j] = x_{i \times j}\) for any letters \(x_i\) and \(x_j\) of \(X\).
The duffle is in \(P\), with \([y_i, e_i], (y_j, e_j)] = (y_{i+j}, e_{i \times k})\) for all \(y_i, y_j\) in \(Y\), \(e_i, e_k\) in \(E\).

Proposition 2.2 Let \(* \in P\), then \(\langle A(X), * \rangle\) is a commutative algebra.

Proof. We just have to show the commutativity and the associativity of *.
To obtain \(w_1 * w_2 = w_2 * w_1\) for all \(w_1, w_2\) in \(X^\ast\), we use an induction on \(|w_1| + |w_2|\).
It is true when \(|w_1| + |w_2| \leq 1\) thanks to (i) since \(w_1\) or \(w_2\) is \(1_X^\ast\). The equality (iii), the condition (S2) and the commutative of + give the induction. In the same way, an induction on \(|w_1| + |w_2| + |w_3|\) gives \(w_1 * (w_2 * w_3) = (w_1 * w_2) * w_3\) thanks to (iii) and (S3).
If we associate to each letter of \(X\) an integer number called weight, the weight of a word is the sum of the weight of its letters. In this case \(X\) is graduated.
In [15], Hoffman works over \(\overline{X} = X \cup \{0\}\) with \([,] : \overline{X} \times \overline{X} \rightarrow \overline{X}\) and call quasi-product any product in \(P\) with the additional condition :

(S4) Either \([a, b] = 0\) for all \(a, b\) in \(X\); or the weight of \([a, b]\) is the sum of the weight of \(a\) and the weight of \(b\) for all \(a, b\) in \(X\).

Example 6 1. The shuffle is a quasi-product.

2. Let \(X\) be an alphabet indexed by \(N\) and define the weight of \(x_i\), \(i \in N\), by \(i\). Then the stuffle is a quasi-product.

Theorem 2.1 (15) If \(X\) is graduated and has a quasi-product * , then \(\langle A(X), * \rangle\) is a commutative graded \(A\)-algebra.

We can define
(i) a comultiplication \(\Delta : A(X) \rightarrow A(X) \otimes A(X),\)
(ii) a counit \(\epsilon : A(X) \rightarrow A,\)
by:
\[
\forall w \in X^\ast, \Delta w = \sum_{u+v=w} u \otimes v \quad \text{and} \quad \epsilon(w) = \begin{cases} 1 & \text{if } w = 1_X, \\ 0 & \text{otherwise}. \end{cases}
\]
The coproduct \(\Delta\) is coassociative so \(\langle A(X), \Delta, \epsilon \rangle\) is a coalgebra.
Lemma 2.1 For any \( w \in X^* \) and \( x \in X, (x \otimes 1_{X^*}) \Delta w + 1_{X^*} \otimes xw = \Delta xw. \)

Proof. \( \forall w \in X^*, \forall x \in X, \Delta xw = \sum_{u,v=w \otimes w} u \otimes v = \sum_{u',v'=w \otimes w} xu' \otimes v + 1_{X^*} \otimes xw \) so \( \Delta xw = x \otimes 1_{X^*} \left( \sum_{u'=w \otimes w} u' \otimes v \right) + 1_{X^*} \otimes xw = (x \otimes 1_{X^*}) \Delta w + 1_{X^*} \otimes xw. \) \( \square \)

Proposition 2.3 If \( \ast \in \mathcal{P}, \) then \( (A(X), \ast, \Delta, \epsilon) \) is a bialgebra.

Remember that \( \ast \) acts over \( A(X) \otimes A(X) \) by \( (u \otimes v) \ast (u' \otimes v') = (u \ast u') \otimes (v \ast v'). \)

Proof. \( \epsilon \) is obviously a \( \ast \)-homomorphism. It still has to be show \( \Delta(w_1) \ast \Delta(w_2) = \Delta(w_1 \ast w_2) \) over \( X^* \). This equality is true if \( w_1 \) or \( w_2 \) is equal to \( 1_{X^*}. \)

Assume now that \( \Delta(u) \ast \Delta(v) = \Delta(u \ast v) \) for any word \( u \) and \( v \) such that \( |u| + |v| \leq n, \) \( n \in \mathbb{N}, \) and let \( w_1 \) and \( w_2 \) be in \( X^* \) with \( |w_1| + |w_2| = n + 1. \) We note \( w_1 = auw \) and \( w_2 = bvw \), with \( a \) and \( b \) two letters of \( X, \) \( u \) and \( v \) two words of \( X^*. \) Thus, by definition, \( \Delta w_1 = \sum_{u_1u_2=au} au \otimes u_2 + 1_{X^*} \otimes au \) and \( \Delta w_2 = \sum_{v_1v_2=bv} bv \otimes v_2 + 1_{X^*} \otimes bv. \)

\( \Delta(w_1) \ast \Delta(w_2) = \sum_{u_1u_2=au, v_1v_2=bv} (au_1 \ast bv_1) \otimes (u_2 \ast v_2) + \sum_{u_1u_2=au} (au_1) \otimes (u_2 \ast bv) + \sum_{v_1v_2=bv} (bv_1) \otimes (au \ast v_2) + 1_{X^*} \otimes (au \ast bv) \)

\( +\sum_{v_1v_2=bv} (bv_1) \otimes (au \ast v_2) + 1_{X^*} \otimes (au \ast bv) \)

Using the induction hypothesis then the lemma 2.1 (since \( [a,b] \in AX \)) gives

\[ \Delta(w_1) \ast \Delta(w_2) = \Delta((a \ast w_2) + \Delta([a,b](u \ast v))) + \Delta([a,b](u \ast v)) \]

\[ = \Delta(a \ast w_2) + \Delta([a,b](u \ast v)) \]

\[ = \Delta(a \ast w_2) + b(u \ast v) + [a,b](u \ast v) \]

\[ = \Delta(w_1 \ast w_2). \] \( \square \)
Remark 2.2 In particular, $\Delta$ is a $\omega$-homomorphism, a $\mathfrak{L}$-homomorphism and a $\mathfrak{Q}$-homomorphism.

Let $C_n$ be the set of positive integer sequences $(i_1, \ldots, i_k)$ such that $i_1 + \ldots + i_k = n$.

**Theorem 2.2** Define $a_*$ by, for all $x_1, \ldots, x_n$ in $X$,

$$a_* (x_1 \ldots x_n) = \sum_{(i_1, \ldots, i_k) \in C_n} (-1)^k x_1 \ldots x_{i_1} \ast x_{i_1+1} \ldots x_{i_1+i_2} \ast \ldots \ast x_{i_1+\ldots+i_{k-1}+1} \ldots x_n$$

then, if $\ast \in \mathcal{P}$, $(A\langle X \rangle, \ast, \Delta, \epsilon, a_*)$ is a Hopf algebra.

**Proof.** With the applications:

$$\mu : A \rightarrow A\langle X \rangle \quad \text{and} \quad \lambda : A\langle X \rangle \otimes A\langle X \rangle \rightarrow A\langle X \rangle$$

$$u \otimes v \mapsto u \ast v,$$

the antipode must verify $m \circ (a_* \otimes Id) \circ \Delta = \mu \circ \epsilon$, or, in equivalent terms

$$\sum_{u \in X^*} a_*(u) \ast v = \langle w| 1_{X^*} \rangle 1_{X^*}.$$

i.e. $\left\{ a_*(1_{X^*}) = 1_{X^*} \right\}$

and, if $w = x_1 \ldots x_n$ with $n \geq 2, x_1, \ldots, x_n \in X$,

$$a_*(w) = - \sum_{k=1}^{n-1} a_*(x_1 \ldots x_k) \ast x_{k+1} \ldots x_n.$$

An induction over the length $n$ shows that $a_*$ defined in theorem verifies these equalities, and, in the same way, $a_*$ verifies $m \circ (Id \otimes a_*) \circ \Delta = \mu \circ \epsilon$. \qed

**Corollary 2.1** If $\ast$ is $\omega$ or $\mathfrak{L}$ or $\mathfrak{Q}$ or $\mathfrak{U}$, then this construction gives an Hopf algebra. Moreover, for $\omega$ or $\mathfrak{U}$, we obtain a graduated Hopf algebra.

### 2.2 Iterated integral

Let us associate to each letter $x_i$ in $X$ a 1-differential form $\omega_i$, defined in some connected open subset $U$ of $\mathbb{C}$. For all paths $z_0 \sim z$ in $U$, the *Chen iterated integral* associated to $w = x_{i_1} \cdots x_{i_k}$ along $z_0 \sim z$, noted is defined recursively as follows

$$\alpha^z_{z_0} (w) = \int_{z_0 \sim z} \omega_{i_1}(z_1) \alpha^z_{z_0} (z_{i_2} \cdots x_{i_k}) \quad \text{and} \quad \alpha^z_{z_0} (1_{X^*}) = 1, \quad \text{(2)}$$

verifying the *rule of integration by parts* [2]:

$$\alpha^z_{z_0} (u \otimes v) = \alpha^z_{z_0} (u) \alpha^z_{z_0} (v). \quad \text{(3)}$$

We extended this definition over $A\langle X \rangle$ (resp. $A\llangle X \rrangle$) by

$$\alpha^z_{z_0} (S) = \sum_{w \in X^*} \langle S| w \rangle \alpha^z_{z_0} (w). \quad \text{(4)}$$
2.3 Shuffle relations

2.3.1 First encoding for colored Hurwitz polyzetas

Let \( \xi = (\xi_n) \) be a sequence of complex numbers and \( T \) a family of parameters. Put \( X' \) an alphabet indexed over \( \mathbb{N}^* \times \mathbb{C}^n \times T \) and \( X = \{x_0\} \cup X' \). To each \( x \) in \( X \) we associate the differential form:

\[
\begin{align*}
\omega_0(z) &= \frac{dz}{z} & \text{if } x = x_0 \\
\omega_{i,\xi,t}(z) &= \frac{1}{\prod_{k=1}^{i} \xi_k} \left( \frac{dz}{z^i} \right) & \text{if } x = x_{i,\xi,t} \text{ with } i \geq 1.
\end{align*}
\]

For any \( T \)-tuple \( t = (t_1, \ldots, t_r) \) we associate the \( T \)-tuple \( \mathbf{t} = (\mathbf{t_1}, \ldots, \mathbf{t_r}) \) given by

\[
\begin{align*}
\mathbf{t_1} &= t_1 - t_2, \\
\mathbf{t_2} &= t_2 - t_3, & \text{in this way} \\
\vdots \\
\mathbf{t_r} &= t_{r-1} - t_r \\
\end{align*}
\]

We choose the sequence \( \xi \) and the family \( t \) such that the condition \((E)\) is satisfied.

**Proposition 2.4** For any \( s = (s_1, \ldots, s_r) \) with \( s_1 > 1 \) if \( \xi = (\xi_1, \ldots, \xi_r) \in \mathcal{E}' \) and \( t = (t_1, \ldots, t_r) \in T' \), then \( \text{Di}(\mathbf{F}_{\xi,t}; s) = \alpha_0^{s_1}(x_0^{s_1-1}x_{1,\xi,t}^{s_2-1} \ldots x_0^{s_r-1}x_{r,\xi,t}^{s_r-1}) \).

**Proof.** Since \( \omega_{i,\xi,t}(z) = \sum_{n>0} \prod_{k=1}^{i} \xi_k^n \frac{dz}{z^{i+1}} \) then \( \alpha_0^s(x_{r,\xi,t}^{s_r-1}) = \sum_{n>0} \prod_{k=1}^{r} \xi_k^n \frac{z^{n-s_r}}{n-s_r} \).

Hence, \( \alpha_0^s(x_0^{s_1-1}x_{1,\xi,t}^{s_2-1} \ldots x_0^{s_r-1}x_{r,\xi,t}^{s_r-1}) \) gives

\[
\sum_{n_1 > \ldots > n_r > 0} \frac{\prod_{j=1}^{r} \xi_j^{n_j}}{(n_1-t_1)^{s_1} \ldots (n_r-t_r)^{s_r}}.
\]

\( \square \)

**Theorem 2.3** Let \( T \) be the group of parameters generated by \( \langle T; + \rangle \), \( C \) be a subgroup of \( \langle \mathbb{C}^*, \cdot \rangle \) and \( A \) a sub-ring of \( \mathbb{C} \). Put \( \mathcal{C}' = \mathbb{C}^n \cap \mathcal{E} \) and \( T' \) the set of finite tuple with elements in \( \mathcal{C}' \). Then the algebra generated by \( \{ \text{Di}(\mathbf{F}_{\xi,t}; s) \} \in \mathcal{C}' \times T' \) is the \( A \) modulus generated by \( \{ \text{Di}(\mathbf{F}_{\xi,t}; s) \} \in \mathcal{C}' \times T' \).

**Proof.** We have express the product \( \text{Di}(\mathbf{F}_{\xi,t}; s) \text{Di}(\mathbf{F}_{\xi',t'; s'}) \), with \( s = (s_1, \ldots, s_r), s' = (s'_1, \ldots, s'_r), \xi, \xi' \in \mathcal{C}' \) and \( t = (t_1, \ldots, t_r), t' = (t'_1, \ldots, t'_r) \in T' \), as linear combination of colored Hurwitz polyzetas. This is an iterated integral associated to \( x_0^{s_1-1}x_{1,\xi,t}^{s_2-1} \ldots x_0^{s_r-1}x_{r,\xi,t}^{s_r-1} = x_0^{s_1-1}x_{1,\xi,t}^{s_2-1} \ldots x_0^{s_r-1}x_{r,\xi,t}^{s_r-1} \) which is a sum of
terms of the form $x_0^{s_1-1}x_1,\xi(1),\underline{z_1} \cdots x_0^{s_n-1}x_{j_n,\xi(n),\underline{z_{n+1}}} \cdots x_0^{s_{t'-1}}x_{j_{t'},\xi(t'),\underline{z_{t'+1}}},$ with $s_i \in \mathbb{N}, \xi(i)$ is $\xi$ or $\xi'$ and $t(i)$ is $t_j$ or $t'_j$ for all $i$; and $r'' = r + r''$. Note that

$$
\alpha_{\xi}(x_i,\xi,\underline{z_i})^{x_0^{s-1}}x_{j',\xi',\underline{z'}_{s+1}} = \int_0^z \sum_{m>0} \prod_{k=1}^m \zeta_{x_i,\xi,\underline{z_i}}^{x_0^{s-1}}x_{j',\xi',\underline{z'}_{s+1}} = \sum_{m, n>0} (m + n - t_i - t'_j)(n - t'_j)^s,
$$

$$
\alpha_{\xi}(x_0^{s_i-1}x_1,\xi(1),\underline{z_1}) \cdots x_0^{s_{t'-1}}x_{j_{t'},\xi(t'),\underline{z_{t'+1}}} = \sum_{m, n>0} \prod_{i=1}^m \zeta_{x_0^{s_1-1}x_1,\xi(1),\underline{z_1}} \cdots x_0^{s_{t'-1}}x_{j_{t'},\xi(t'),\underline{z_{t'+1}}} = \sum_{n_i = m_i + \ldots + m_{t''}, \xi''} \prod_{i=1}^m \frac{z_i^{s_i}}{(n_i - t'_j)^{s_i}},
$$

with $n_i = m_i + \ldots + m_{t''}$ and $\xi'' = \xi''(1)$ for $i > 1$ so $\xi'' \in C$; we can express each term of the shuffle product as $\text{Di}(F_{\xi'',t''};s')$. Note that the shuffle product over two words of $X^*X'$ acts separately over $(C',\cdot)$, $(T',+)$ and the convergent compositions. We can describe the situation with the shuffle algebra$^3$:

**Theorem 2.4** Let $H$ be the $Q$-algebra generated by the colored Hurwitz polyzetas. The map $\zeta : (Q(\langle x_i^r x_j,\xi,\iota \rangle),\iota) \to (H, \cdot)$, $x_0^{s_i}x_1,\xi,\underline{z_i} \cdots x_0^{s_{t'-1}}x_{j',\xi',\underline{z_{t'+1}}}$ is a surjective algebra morphism.

**Example 7** Since $\text{Di}(F_{\xi,t};3) = \alpha_1(x_0^2x_1,\xi,t)$ and $\text{Di}(F_{\xi',t';2}) = \alpha_0(x_0x_1,\xi',t')$, then $\text{Di}(F_{x,\xi';t';3}) \text{Di}(F_{x',\xi,t';2}) = \alpha_0(x_0x_1,\xi',t'') \alpha_1(x_0^2x_1,\xi,t)$.

Example 1 with $x = x_1,\xi,\iota$ and $x' = x_1,\xi',\iota$ gives the expression of $\text{Di}(F_{\xi,\xi';t'})$.

But the first term obtained is

$$
\alpha_1(x_0x_1,\xi',t'',x_0^2x_1,\xi,t) = \int_0^z \frac{dz_1}{z_1} \int_0^{z_1} \sum_{m>0} \xi^{t''}z_2m-t'-1dz_2 \int_0^{z_2} \sum_{n>0} \xi^n z_5n-t-1dz_5 = \frac{(x_0^2x_1,\xi,t)}{(x_0x_1,\xi',t')}.
$$

$^3$Working in $Q(\langle x_i^r x_j,\xi,\iota \rangle)$ implies working in the graduated Hopf algebra $(Q(X^*),\iota,\Delta,\epsilon,\text{a}_{\text{a}}).$
We can make similar calculus for the other terms and find:

\[ \text{Di}(F_{\xi'; t}; 3) \text{Di}(F_{\xi'; t}; 2) \]
\[ = \text{Di}(F_{\xi'; \xi'; t'}, (t + v, 1); (2, 3)) + 2 \text{Di}(F_{\xi'; \xi'; t'}, (t + v, 1); (3, 2)) \]
\[ + 3 \text{Di}(F_{\xi'; \xi'; t'}, (t + v, 1); (4, 1)) + 3 \text{Di}(F_{\xi'; \xi'; t'}, (t + v, 1); (4, 1)) \]
\[ + \text{Di}(F_{\xi'; \xi'; t'}, (t + v, 1); (3, 2)). \]

### 2.3.2 Second encoding for colored Hurwitz polyzetas

For the Hurwitz polyzetas, we can obtain an encoding indexed by a finite alphabet. Let the alphabet \( X = \{x_0; x_1\} \) and associate to \( x_0 \) the form \( \omega_0(z) = z^{-1}dz \) and at \( x_1 \) the form \( \omega_1(z) = (1 - z)^{-1}dz \).

For each \( x \in X \) and \( \lambda \in \mathbb{C} \), we note \( (\lambda x)^* = \sum_{k \geq 0} (\lambda x)^k \). Then, (see [10], [11]),

\[ \alpha_0^1 (x_0^{a_1-1} (t_1 x_0)^{s_1} x_1 \ldots x_0^{a_r-1} (t_r x_0)^{s_r} x_1) = \zeta(s, t). \]

**Theorem 2.5** Let \( H' \) be the \( \mathbb{Q} \)-algebra generated by the Hurwitz polyzetas and \( X \) the \( \mathbb{Q} \)-algebra generated by \( (t_1 x_0)^{s_1} x_1 \ldots (t_r x_0)^{s_r} x_r \). Then, \( \zeta : (X, \omega) \twoheadrightarrow (H', \cdot) \) is a surjective morphism of algebras.

Note that we can apply the idea of encoding of “simple” colored Hurwitz zetas functions (with depth one: \( r = 1 \)). Let \( \xi = (\xi_n) \) be a sequence of complex numbers in the unit ball \( B(0; 1) \) and \( T \) a family of parameters. Let \( X = \{x_0, x_1, \ldots\} \) be a alphabet indexed by \( \mathbb{N} \). Associate to \( x_0 \) the differential form \( \omega_0(z) = z^{-1}dz \) and to \( x_i, i \geq 1 \), the differential form \( \omega_i(z) = \xi_i (1 - \xi z)^{-1}dz \).

**Proposition 2.5** With this notation, \( \alpha_0^1 ((t x_0)^* x_0)^{s-1} (t x_0)^* x_1) = \sum_{n \geq 0} \frac{\xi_n^n}{(n - t)^{s}}. \)

**Proof.** Since \( \frac{\xi d z_0}{1 - \xi z_0} = \xi \sum_{n \geq 0} (\xi z_0)^n d z_0, \) we can write

\[ \alpha_0^1 ((t x_0)^k x_i) = t^k \int_0^z \frac{d z_0}{z_0^k} \int_0^{z_0} \frac{d z_1}{z_1} \ldots \int_0^{z_1} \frac{d \xi}{\xi} \sum_{n \geq 0} (\xi z_0)^n d z_0 = \sum_{n \geq 0} t^k \frac{\xi^n z^n}{n^{k+1}}, \]

for \( z \in B(0; 1) \) and for \( k \in \mathbb{N} \). Thanks to the absolute convergence,

\[ \alpha_0^1 ((t x_0)^* x_1) = \sum_{n \geq 0} \frac{\xi^n z^n}{n} \sum_{k \geq 0} \left( \frac{t}{n} \right)^k = \sum_{n \geq 0} \frac{\xi^n z^n}{n - t}. \]

In the same way, if \( z \in B(0; 1) \):

\[ \forall k \in \mathbb{N}, \quad \alpha_0^1 ((t x_0)^k x_0 (t x_0)^* x_i) = \sum_{n \geq 0} \frac{\xi^n z^n}{n - t} \frac{t^n}{n^{k+1}}, \]

so \( \alpha_0^1 ((t x_0)^* x_0 (t x_0)^* x_i) = \sum_{n \geq 0} \frac{\xi^n z^n}{(n - t)^2} \).
Remark 2.3 Note that, with the same notation,
\[ \alpha_0^z \left( (t x_0)^* x_0 \right)^{s-1} (t x_0)^* x_0 \right) = \sum_{n>0} \frac{\xi^n}{(n-t)^s}. \]

\[ \square \]

In other words, this encoding appears to be widespread only as couples of the type \( \xi = (1, 1, \ldots, 1, \xi_r) \) : with \( \xi_1 = 1 \) and \( \omega_1 = (1 - z)^{-1}dz \),
\[ \alpha_0^1 \left( x_0^{s_1-1} (t_1 x_0)^{s_1} x_1 \cdots x_0^{s_r-1} (t_r x_0)^{s_r} x_r \right) = \sum_{n_1>n_r>0} \xi_0^{n_1} \xi_0^{n_2} \cdots \xi_0^{n_r} (n_1 - t_1)^{s_1} \cdots (n_r - t_r)^{s_r}. \]

2.4 Duffle relations

Let \( \lambda = (\lambda_n) \) be a set of parameters, \( s = (s_1, \ldots, s_r) \) a composition, \( \xi \in \mathbb{C}^r \). Then
\[ \forall n \in \mathbb{Z}_{>0}, \quad M^n_{ss, (\lambda)}(\lambda) = \sum_{n>n_1>\ldots>n_r>0} \prod_{i=1}^r \xi_i^{n_i} \lambda_i^{s_i} \quad \text{and} \quad M^n_{11, (\lambda)}(\lambda) = 1. \quad (7) \]

We can export the duffle over the tuples \( s = (s_1, \ldots, s_r) \in \mathbb{Z}_{>0}^r \) and \( \xi \in \mathbb{C}^r \) with :
\[ (s, \xi) \mathbb{I} \mathbb{I} ((1, 1), 1) = ((1, 1), 1) \mathbb{I} \mathbb{I} (s, \xi) = (s, \xi) \quad \text{and} \]
\[ (s_1, s; \xi_1, \xi) \mathbb{I} \mathbb{I} (r_1, r; \rho_1, \rho) = (s_1; \xi_1) \cdot ((s; \xi) \mathbb{I} \mathbb{I} (r_1, r; \rho_1, \rho)) + (r_1; \rho_1) \cdot ((s_1, s; \rho_1) \mathbb{I} \mathbb{I} (r; \rho)) + (s_1 + r_1; \xi_1 \rho_1) \cdot ((s; \xi) \mathbb{I} \mathbb{I} (r; \rho)). \quad (8) \]

Proposition 2.6 Let \( s = (s_1, \ldots, s_l) \) and \( r = (r_1, \ldots, r_k) \) be two compositions, \( \xi \in \mathbb{C}^l \), \( \rho \in \mathbb{C}^k \). Then
\[ \forall n \in \mathbb{N}, \quad M^n_{ss, (\lambda)}(\lambda) M^n_{rr, (\rho)}(\lambda) = M^n_{s(s, (\xi)) \mathbb{I} \mathbb{I} (r; (\rho))}(\lambda). \]

Proof. Put the compositions \( s' = (s_2, \ldots, s_l), r' = (r_2, \ldots, r_k) \), the tuples of complex numbers \( \xi' = (\xi_2, \ldots, \xi_l) \) and \( \rho' = (\rho_2, \ldots, \rho_k) \), then
\[ M^n_{ss, (\xi)}(\lambda) M^n_{rr, (\rho)}(\lambda) = \sum_{n>n_1, n>n_1'} \xi_1^{n_1} \lambda_1^{s_1} \lambda_1^{s_1} \xi_1^{n_1} \lambda_1^{s_1} M^n_{rr, (\rho)}(\lambda) \]
\[ = \sum_{n>n_1} \xi_1^{n_1} \lambda_1^{s_1} M^n_{ss, (\xi)}(\lambda) M^n_{rr, (\rho)}(\lambda) + \sum_{n>n_1'} \rho_1^{n_1'} \lambda_1^{r_1} M^n_{ss, (\xi)}(\lambda) M^n_{rr, (\rho)}(\lambda) \]

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+ \sum_{n>m} (\xi_1 \rho_1)^m \lambda_m^{s_1 + r_1} M_{s_1}^{r_1} \ell_{\xi,\ell}(\lambda) M_{r_1 \rho_1}^{s_1}(\lambda).

A recurrence ended the demonstration.

\[ \square \]

**Theorem 2.6** Let \( s = (s_1, \ldots, s_t) \) and \( r = (r_1, \ldots, r_k) \) be two compositions, \( \xi \) a \( t \)-tuple and \( \rho \) a \( k \)-tuple of \( E \), \( t = (t_1, \ldots, t) \) a \( t \)-tuple and \( t' = (t_1, \ldots, t) \) a \( k \)-tuple, both formed by the same parameter \( t \) diagonally. Then

\[ \text{Di}(F_{\xi t'; s}) \text{ Di}(F_{\xi t'; s'}) = \text{Di}(F_{\xi t; t'; s''}), \]

with \((s''; \xi'') = (s; \xi) \boxplus (s'; \xi').\)

**Proof.** With \( \lambda_n = \frac{1}{n-t} \) for all \( n \in \mathbb{N} \), \( M_{s_t \xi}^n(\lambda) = \sum_{n>n_1>\cdots>n_k} \prod_{i=1}^{r} \frac{\xi_{s_i}}{(n_i-t)^{n_i}}. \) So

\[ \lim_{n \to \infty} M_{s_t \xi}^n(\lambda) = \text{Di}(F_{\xi t; s}) \]

and taking the limit of Proposition 2.6 gives the result.

\[ \square \]

**Example 8** The use of examples 2 and 4 gives

\[ \text{Di}(F_{(\frac{3}{4}, -1); t; (3, 1)}) \text{ Di}(F_{(\frac{1}{2}); t; (2)}) = \text{Di}(F_{(\frac{3}{4}, -1); (t, t, t); (3, 1, 2)}) + \text{Di}(F_{(\frac{3}{4}, -1); (t, t, t); (3, 2, 1)}) \]

+ \text{Di}(F_{(\frac{3}{4}, -1); t; (3, 3)}) + \text{Di}(F_{(\frac{3}{4}, -1); (t, t, t); (2, 3, 1)}) + \text{Di}(F_{(\frac{3}{4}, -1); t; (5, 1)})

**Remark 2.4** Extend the duffle product to triplets \((s, t, \xi) \in \cup_{r \in \mathbb{N}^*} \mathbb{N}^r \times \{t\}^r \times \mathbb{C}^r\) by

\[ (s_1, r_1; t_1; \xi_1) \boxplus (r_1, t_1; \xi_1; \rho_1) = (s_1; t_1; \xi_1) \boxplus (r_1; t_1; \xi_1; \rho_1), \]

and define the function \( F \) over \( \mathbb{I} = \cup_{r \in \mathbb{N}^*} \mathbb{N}^r \times \{t\}^r \times \mathbb{C}^r \) by \( F(s, t, \xi) = \text{Di}(F_{\xi t; s}). \)

Then, by Theorem 2.6, the function \( F : (\mathbb{I}, \boxplus) \to (\mathbb{C}, .) \) is morphism of algebras.

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