Liouville type of theorems with weights for the Navier-Stokes equations and the Euler equations

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Abstract
We study Liouville type of theorems for the Navier-Stokes and the Euler equations on \( \mathbb{R}^N \), \( N \geq 2 \). Specifically, we prove that if a weak solution \((v, p)\) satisfies \(|v|^2 + |p| \in L^1(0, T; L^1(\mathbb{R}^N, w_1(x)dx))\) and \( \int_{\mathbb{R}^N} p(x, t)w_2(x)dx \geq 0 \) for some weight functions \( w_1(x) \) and \( w_2(x) \), then the solution is trivial, namely \( v = 0 \) almost everywhere on \( \mathbb{R}^N \times (0, T) \). Similar results hold for the MHD Equations on \( \mathbb{R}^N, N \geq 3 \).

1 Introduction
We are concerned on the Navier-Stokes equations (the Euler equations for \( \nu = 0 \)) on \( \mathbb{R}^N, N \in \mathbb{N}, N \geq 2 \).

\[
\begin{align*}
\left( \text{NS}_\nu \right) & \\
\frac{\partial v}{\partial t} + (v \cdot \nabla)v &= -\nabla p + \nu \Delta v + f & (x, t) \in \mathbb{R}^N \times (0, \infty) \\
\text{div } v &= 0, & (x, t) \in \mathbb{R}^N \times (0, \infty) \\
v(x, 0) &= v_0(x), & x \in \mathbb{R}^N
\end{align*}
\]

where \( v(x, t) = (v^1(x, t), \ldots, v^N(x, t)) \) is the velocity, \( p = p(x, t) \) is the pressure, \( f = (f_1(x, t), \ldots, f^N(x, t)) \) is the external force, and \( \nu \geq 0 \) is the viscosity. Given \( a, b \in \mathbb{R}^N \), we denote by \( a \otimes b \) the \( N \times N \) matrix with \((a \otimes b)_{ij} = a_ib_j\).
For two $N \times N$ matrices $A$ and $B$ we denote $A : B = \sum_{i,j=1}^{N} A_{ij} B_{ij}$. Given $m \in \mathbb{N} \cup \{0\}, q \in [1, \infty]$, we introduce $W_{m,q}^{\sigma}(\mathbb{R}^{N}) := \{v \in [W_{m,q}^{\sigma}(\mathbb{R}^{N})]^{N}, \text{ div } v = 0\}$, where $W_{m,q}^{\sigma}(\mathbb{R}^{N})$ is the standard Sobolev space on $\mathbb{R}^{N}$, and the derivatives in the operation of $\text{div} (\cdot)$ are in the sense of distribution. In particular, $H_{m}^{\sigma}(\mathbb{R}^{N}) := W_{m,2}^{\sigma}(\mathbb{R}^{N})$ and $L_{q}^{\sigma}(\mathbb{R}^{N}) := W_{0,q}^{\sigma}(\mathbb{R}^{N})$. Similarly, given $q \in [1, \infty]$, we use $L_{q}^{\text{loc},\sigma}(\mathbb{R}^{N})$ to denote the class of solenoidal vector fields, which belongs to $[L_{q}^{\text{loc}}(\mathbb{R}^{N})]^{N}$. In $\mathbb{R}^{N}$ we define weak solutions of the Navier-Stokes (Euler) equations as follows.

**Definition 1.1** We say that a pair $(v, p) \in L^{2}(0, T; L_{\text{loc},\sigma}^{2}(\mathbb{R}^{N})) \times L^{1}(0, T; L_{\text{loc}}^{1}(\mathbb{R}^{N}))$ is a weak solution of $(NS)_{\nu}$ on $\mathbb{R}^{N} \times (0, T)$ with $f \in L^{1}(0, T; L_{\text{loc},\sigma}^{1}(\mathbb{R}^{N}))$ if

$$
-\int_{0}^{T} \int_{\mathbb{R}^{N}} v(x, t) \cdot \phi'(x) \xi(t) dxdt - \int_{0}^{T} \int_{\mathbb{R}^{N}} v(x, t) \otimes v(x, t) : \nabla \phi(x) \xi(t) dxdt \\
= \int_{0}^{T} \int_{\mathbb{R}^{N}} p(x, t) \text{div} \phi(x) \xi(t) dxdt + \nu \int_{0}^{T} \int_{\mathbb{R}^{N}} v(x, t) \cdot \Delta \phi(x) \xi(t) dxdt \\
+ \int_{0}^{T} \int_{\mathbb{R}^{N}} f(x, t) \cdot \phi(x) \xi(t) dxdt
$$

(1.1)

for all $\xi \in C_{0}^{1}(0, T)$ and $\phi = [C_{0}^{\infty}(\mathbb{R}^{N})]^{N}$.

In [1] it is proved that if a weak solution $(v, p)$ of the Euler or Navier-Stokes equations satisfy

$$
v \in L^{2}(0, T; L_{\sigma}^{2}(\mathbb{R}^{N})) \quad \text{and} \quad p \in L^{1}(0, T; \mathcal{H}^{q}(\mathbb{R}^{N}))
$$

(1.2)

for some $q \in (0, 1]$, where $\mathcal{H}^{q}(\mathbb{R}^{N})$ denotes the Hardy space on $\mathbb{R}^{N}$, then $v(x, t) = 0$ almost everywhere on $\mathbb{R}^{N} \times (0, T)$. Furthermore, if $p \in L^{1}(0, T; L^{1}(\mathbb{R}^{N}))$, then there happens the *equipartition of energy* over each component([1]),

$$
\int_{\mathbb{R}^{N}} v^{j}(x, t) v^{k} dx = -\delta_{jk} \int_{\mathbb{R}^{N}} p(x, t) dx.
$$

The main purpose of this paper is to further develop the idea initiated in [1] to obtain substantially extended Liouville type of theorems with suitable weight.
functions for the associated integrations for the Navier-Stokes equations, the Euler equations on $\mathbb{R}^N$, $N \geq 2$, and the (both viscous and invicid) MHD equations on $\mathbb{R}^N$, $N \geq 3$. To the author’s knowledge there exist a previous study on the Liouville type of theorems in for the 3D Navier-Stokes equations with \textit{axisymmetry} for $\nu > 0$ (\cite{2}), which is in completely different fashion from that of \cite{1} and from those studied in this paper. In the case of the Euler equations and the MHD equations, in particular, there exist no previous Liouville type of results available in the literature. Our first main theorem is the following.

**Theorem 1.1** Let $w \in L^1_{loc}(0, \infty)$ be given, which is positive almost everywhere on $[0, \infty)$. Suppose $(v, p)$ is a weak solution to $(NS)_\nu$ with $f \in L^1(0, T; L^1_{loc,\sigma}(\mathbb{R}^N))$ and $\nu \geq 0$ on $\mathbb{R}^N \times (0, T)$ such that

$$
\int_0^T \int_{\mathbb{R}^N} \left( |v(x, t)|^2 + |p(x, t)| \right) \times \left[ w(|x|) + \frac{1}{|x|} \int_0^{|x|} w(s) ds + \frac{1}{|x|^2} \int_0^{|x|} \int_0^r w(s) ds dr \right] dx dt < \infty,
$$

(1.3)

and

$$
\int_{\mathbb{R}^N} p(x, t) \left[ w(|x|) + \frac{N - 1}{|x|} \int_0^{|x|} w(s) ds \right] dx \geq 0 \quad \text{for } t \in (0, T). \quad (1.4)
$$

Then, $v(x, t) = 0$ almost everywhere on $\mathbb{R}^N \times (0, T)$.

**Remark 1.1** If we choose $w(s) \equiv 1$ on $[0, \infty)$, then we recover Liouville part of results of Theorem 1.1 (i) in \cite{1}.

**Remark 1.2** Let us set $w^*(r) := \sup_{0 \leq s \leq r} w(s)$. Then, since

$$
w(r) + \frac{1}{r} \int_0^r w(s) ds + \frac{1}{r^2} \int_0^r \int_0^s w(\rho) d\rho ds \leq \frac{5}{2} w^*(r),
$$

we can replace the condition (1.3) by a stronger one,

$$
\int_0^T \int_{\mathbb{R}^N} \left( |v(x, t)|^2 + |p(x, t)| \right) w^*(|x|) dx dt < \infty \quad (1.5)
$$
to get our conclusion of the theorem from (1.4).

The following is a consequence of the above theorem, which we state as a separate theorem.

**Theorem 1.2** Let \((v, p)\) be a weak solution to \((NS)_\nu\) with \(f \in L^1(0,T; L^1_{\text{loc},\sigma}(\mathbb{R}^N))\) and \(\nu \geq 0\) on \(\mathbb{R}^N \times (0,T)\) such that either

\[
\int_0^T \int_{\mathbb{R}^N} \frac{|v(x,t)|^2 + |p(x,t)|}{1 + |x|} dx dt < \infty, \tag{1.6}
\]

or

\[
p(x,t) \to 0 \text{ as } |x| \to \infty \text{ for almost every } t \in (0,T), \text{ and } \tag{1.7}
\]

\[v \in L^2(0,T; L^q(\mathbb{R}^N)) \quad \text{for some } q \text{ with } 2 < q < \frac{2N}{N-1}. \tag{1.8}\]

Suppose there exists \(w \in L^1(0,\infty)\) such that

\[0 < w(r) \leq \frac{C}{1+r} \quad \text{for some } C > 0 \tag{1.9}\]

for almost every \(r \in [0,\infty),\) and

\[
\int_{\mathbb{R}^N} p(x,t) \left[ w(|x|) + \frac{N-1}{|x|} \int_0^{|x|} w(s) ds \right] dx \geq 0 \quad \text{for almost every } t \in (0,T). \tag{1.10}
\]

Then, \(v(x,t) = 0\) almost everywhere on \(\mathbb{R}^N \times (0,T).\)

**Remark 1.3** The main novelty of the above theorem, compared to Theorem 1.1, is that the integrability conditions (1.6) and (1.7) do not involve restriction on the weight function \(w(r).\) Moreover, we do not need any integrability condition on pressure \(p(x,t)\) in (1.7). The price to pay for these relaxations is that we need to select weight functions from a smaller class than of Theorem 1.1.

Since \(H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)\) by the Sobolev embedding, and \(L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-2}}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)\) for \(2 < q < \frac{2N}{N-2}\) by the standard \(L^q(\mathbb{R}^N)\) interpolation inequality, we easily find that Leray’s weak solution (3) to \((NS)_\nu, \nu > 0\) satisfies

\[v \in L^\infty(0,T; L^2_\sigma(\mathbb{R}^N)) \cap L^2(0,T; H^1_\sigma(\mathbb{R}^N)) \subset L^2(0,T; L^2_\sigma(\mathbb{R}^N)) \quad \text{(1.11)}\]
for all $q \in (2, \frac{2N}{N-2})$. Hence, as an immediate corollary of Theorem 1.2 we obtain:

**Corollary 1.1** Let $v \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^N)) \cap L^2(0,T;H^1_\sigma(\mathbb{R}^N))$ be Leray’s weak solution to \((NS)\nu\) with $f \in L^2(0,T;L^2_\sigma(\mathbb{R}^N))$ and $\nu > 0$. Suppose the pressure $p(x,t)$ satisfies \((1.7)\) and \((1.10)\) for a function $w(r)$ satisfying the conditions of Theorem 1.2. Then, $v(x,t) = 0$ almost everywhere on $\mathbb{R}^N \times (0,T)$.

The proofs of Theorem 1.1 and Theorem 1.2 are given in the next section. Further generalized theorems extending them to the MHD equations are stated and proved in Section 3.

### 2 Proof of the Main Theorems

**Proof of Theorem 1.1** Let us consider a radial cut-off function $\sigma \in C^\infty_0(\mathbb{R}^N)$ such that

\[
\sigma(|x|) = \begin{cases} 
1 & \text{if } |x| < 1 \\
0 & \text{if } |x| > 2,
\end{cases}
\]

and $0 \leq \sigma(x) \leq 1$ for $1 < |x| < 2$. We set

\[
W(\rho) := \int_0^\rho \int_0^s w(r) dr ds.
\]

Then, for each $R > 0$, we define

\[
\varphi_R(x) = W(|x|)\sigma\left(\frac{|x|}{R}\right) = W(|x|)\sigma_R(|x|) \in C^\infty_0(\mathbb{R}^N).
\]

Let $\xi \in C^1_0(0,T)$, and we choose the vector test function $\phi$ in \((1.1)\) as

\[
\phi = \nabla \varphi_R(x).
\]

Then, after routine computations \((1.1)\) becomes

\[
0 = \int_0^T \int_{\mathbb{R}^N} \left[ W''(|x|) \frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \sigma_R(|x|)\xi(t) \, dx dt \\
+ \int_0^T \int_{\mathbb{R}^N} W'(|x|)\sigma'\left(\frac{|x|}{R}\right) \frac{(v \cdot x)^2}{R|x|^2} \xi(t) \, dx dt \\
+ \int_0^T \int_{\mathbb{R}^N} \frac{1}{R} \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \sigma'\left(\frac{|x|}{R}\right) W(|x|)\xi(t) \, dx dt
\]
\begin{align*}
+ \int_0^T \int_{\mathbb{R}^N} \frac{(v \cdot x)^2}{R^2|x|^2} \sigma''\left(\frac{|x|}{R}\right) W(|x|)\xi(t) \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^N} p(x, t) \left[ W''(|x|) + (N-1) \frac{W'(|x|)}{|x|} \right] \sigma_R(|x|)\xi(t) \, dx \, dt \\
+ \frac{2}{R} \int_0^T \int_{\mathbb{R}^N} p(x, t) W'(|x|) \sigma'\left(\frac{|x|}{R}\right) \xi(t) \, dx \, dt \\
+ \frac{N-1}{R} \int_0^T \int_{\mathbb{R}^N} p(x, t) \frac{1}{|x|} \sigma'\left(\frac{|x|}{R}\right) W(|x|)\xi(t) \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^N} p(x, t) \frac{1}{R^2} \sigma''\left(\frac{|x|}{R}\right) W(|x|)\xi(t) \, dx \, dt \\
:= I_1 + \cdots + I_8 \quad \text{(2.5)}
\end{align*}

Note that the term involving derivative with respect to time, the viscosity term and the forcing term in (1.1) vanish altogether, since

\[
\int_0^T \int_{\mathbb{R}^N} v(x, t) \cdot \nabla \varphi_R(x) \xi'(t) \, dx \, dt = 0,
\]

\[
\int_0^T \int_{\mathbb{R}^N} v(x, t) \cdot \nabla (\Delta \varphi_R(x)) \xi(t) \, dx \, dt = 0
\]

for \( v \in L^2(0, T; L^2_{\text{loc}, \sigma}(\mathbb{R}^N)) \) and

\[
\int_0^T \int_{\mathbb{R}^N} f(x, t) \cdot \nabla \varphi_R(x) \xi'(t) \, dx \, dt = 0,
\]

for \( f \in L^1(0, T; L^1_{\text{loc}, \sigma}(\mathbb{R}^N)) \) by the divergence free condition in the sense of distribution. In terms of the function \( W \) defined in (2.2) our condition (1.3) can be written as

\[
\int_0^T \int_{\mathbb{R}^N} \left( |v(x, t)|^2 + |p(x, t)| \right) \left[ W''(|x|) + \frac{1}{|x|} W'(|x|) + \frac{1}{|x|^2} W(|x|) \right] \, dx \, dt < \infty.
\]

(2.6)

Since

\[
\int_0^T \int_{\mathbb{R}^N} \left[ W''(|x|) \frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] |\xi(t)| \, dx \, dt \\
\leq 2 \sup_{0 \leq t \leq T} |\xi(t)| \int_0^T \int_{\mathbb{R}^N} |v(x, t)|^2 \left[ W''(|x|) + \frac{W'(|x|)}{|x|} \right] \, dx \, dt < \infty,
\]

6
We can use the dominated convergence theorem to show that
\[ I_1 \to \int_0^T \int_{\mathbb{R}^N} \left[ W''(|x|) \frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \xi(t) \, dx \, dt \quad (2.7) \]
as \( R \to \infty \). Similarly,
\[ I_5 \to \int_0^T \int_{\mathbb{R}^N} p(x, t) \left[ W''(|x|) + (N - 1) \frac{W'(|x|)}{|x|} \right] \xi(t) \, dx \, dt \quad (2.8) \]
as \( R \to \infty \). For \( I_2 \) we estimate
\[ |I_2| \leq \int_0^T \int_{R<|x|<2R} |v(x, t)|^2 \left| \sigma' \left( \frac{|x|}{R} \right) \right| \frac{W'(|x|)}{|x|} \xi(t) \, dx \, dt \]
\[ \leq 2 \sup_{1<s<2} |\sigma'(s)| \sup_{0 \leq t \leq T} |\xi(t)| \int_0^T \int_{R<|x|<2R} |v(x, t)|^2 \frac{W'(|x|)}{|x|} \, dx \, dt \]
\[ \to 0 \quad (2.9) \]
as \( R \to \infty \) by the dominated convergence theorem. Similarly
\[ |I_3| \leq 2 \int_0^T \int_{R<|x|<2R} \frac{|x|}{R} |v(x, t)|^2 \left| \sigma' \left( \frac{|x|}{R} \right) \right| \frac{W(|x|)}{|x|^2} \xi(t) \, dx \, dt \]
\[ \leq 4 \sup_{1<s<2} |\sigma'(s)| \sup_{0 \leq t \leq T} |\xi(t)| \int_0^T \int_{R<|x|<2R} |v(x, t)|^2 \frac{W(|x|)}{|x|^2} \, dx \, dt \to 0, \quad (2.10) \]
and
\[ |I_4| \leq \int_0^T \int_{R<|x|<2R} \frac{|x|^2}{R^2} |v(x, t)|^2 \left| \sigma'' \left( \frac{|x|}{R} \right) \right| \frac{W(|x|)}{|x|^2} \xi(t) \, dx \, dt \]
\[ \leq 4 \sup_{1<s<2} |\sigma''(s)| \sup_{0 \leq t \leq T} |\xi(t)| \int_0^T \int_{R<|x|<2R} |v(x, t)|^2 \frac{W(|x|)}{|x|^2} \, dx \, dt \to 0 \quad (2.11) \]
as \( R \to \infty \). The estimates for \( I_6, I_7 \) and \( I_8 \) are similar to the above, and we find
\[ |I_6| \leq 2 \int_0^T \int_{R<|x|<2R} |p(x, t)| \frac{|x|}{R} \frac{W'(|x|)}{|x|} \left| \sigma' \left( \frac{|x|}{R} \right) \right| |\xi(t)| \, dx \, dt \]
\[ \leq 4 \sup_{1<s<2} |\sigma'(s)| \sup_{0 \leq t \leq T} |\xi(t)| \int_0^T \int_{R<|x|<2R} |p(x, t)| \frac{W'(|x|)}{|x|} \, dx \, dt \to 0, \quad (2.12) \]
\[ |I_7| \leq (N - 1) \int_0^T \int_{R < |x| < 2R} |p(x, t)| \left| \frac{x}{R} \right| \left| \sigma'(\frac{|x|}{R}) \right| \frac{W(|x|)}{|x|^2} |\xi(t)| \, dx \, dt \]
\[ \leq 2 \sup_{1 \leq s \leq 2} |\sigma'(s)| \sup_{0 \leq t \leq T} |\xi(t)| \int_0^T \int_{R < |x| < 2R} |p(x, t)| \frac{W(|x|)}{|x|^2} \, dx \, dt \rightarrow 0, \]
(2.13)

and
\[ |I_8| \leq \int_0^T \int_{\mathbb{R}^N} |p(x, t)| \frac{|x|^2}{R^2} \left| \sigma''\left(\frac{|x|}{R}\right)\right| \frac{W(|x|)}{|x|^2} |\xi(t)| \, dx \, dt \]
\[ \leq 4 \sup_{1 \leq s \leq 2} |\sigma''(s)| \sup_{0 \leq t \leq T} |\xi(t)| \int_0^T \int_{R < |x| < 2R} |p(x, t)| \frac{W(|x|)}{|x|^2} \, dx \, dt \rightarrow 0 \]
(2.14)
as \( R \to \infty \) respectively. Thus passing \( R \to \infty \) in (2.5), we finally obtain
\[ \int_0^T \int_{\mathbb{R}^N} \left[ W''(|x|)\frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \xi(t) \, dx \, dt \]
\[ = - \int_0^T \int_{\mathbb{R}^N} p(x, t) \left[ W''(|x|) + (N - 1) \frac{W'(|x|)}{|x|} \right] \xi(t) \, dx \, dt \]
(2.15)
for all \( \xi \in C^1_0(0, T) \), which can be written, in terms of the function \( w(r) \), as
\[ \int_{\mathbb{R}^N} \left[ w(|x|) \frac{(v \cdot x)^2}{|x|^2} + \frac{1}{|x|} \int_0^{|x|} w(s) \, ds \left( |v|^2 - \frac{(v \cdot x)^2}{|x|^2} \right) \right] \, dx \]
\[ = - \int_{\mathbb{R}^N} p(x, t) \left[ w(|x|) + (N - 1) \frac{1}{|x|} \int_0^{|x|} w(s) \, ds \right] \, dx \leq 0 \]
(2.16)
for almost every \( t \in (0, T) \) by the hypothesis (1.4). Since \( |v|^2 \geq (v \cdot x)^2/|x|^2 \), we need to have
\[ \int_{\mathbb{R}^N} w(|x|) \frac{(v \cdot x)^2}{|x|^2} \, dx = \int_{\mathbb{R}^N} \frac{1}{|x|} \int_0^{|x|} w(s) \, ds \left[ |v|^2 - \frac{(v \cdot x)^2}{|x|^2} \right] \, dx = 0 \]
almost every \( t(0, T) \). By the hypothesis \( w(|x|) > 0 \) and \( \frac{1}{|x|} \int_0^{|x|} w(s) \, ds > 0 \) for almost every \( x \in \mathbb{R}^N \), and we should have \( v(x, t) = 0 \) for almost every
Proof of Theorem 1.2: The conditions \( w \in L^1(0, \infty) \) and (1.9) imply that there exists a positive constant \( C = C\left(\|w\|_{L^1(0, \infty)}\right) \) such that
\[
 w(r) + \frac{1}{r} \int_0^r w(s) \, ds + \frac{1}{r^2} \int_0^r \int_0^r w(\rho) \, d\rho \, ds \leq \frac{C}{1 + r}.
\] (2.17)

Therefore, if (1.6) holds true, then
\[
\int_0^T \int_{\mathbb{R}^N} (|v(x, t)|^2 + |p(x, t)|) \times \left[ w(|x|) + \frac{1}{|x|} \int_0^{|x|} w(s) \, ds + \frac{1}{|x|^2} \int_0^{|x|} \int_0^r w(\rho) \, d\rho \, ds \right] \, dx \, dt
\leq C \int_0^T \int_{\mathbb{R}^N} \frac{|v(x, t)|^2 + |p(x, t)|}{1 + |x|} \, dx \, dt < \infty
\] (2.18)

Next, we suppose (1.7) holds true. In this case we have the well-known pressure-velocity relation
\[
p(x, t) = \sum_{j,k=1}^N R_j R_k(v_j v_k)(x, t)
\]
with \( R_j, j = 1, \cdots N \), the Riesz transforms in \( \mathbb{R}^N \), and thus the Calderon-Zygmund inequality says(5)
\[
\|p(t)\|_{L^q} \leq C_q \|v(t)\|_{L^2}^2 \quad \forall q \in (2, \infty)
\]
for a constant \( C_q \). Hence, for \( 2 < q < \frac{2N}{N-1} \), we can estimate
\[
\int_0^T \int_{\mathbb{R}^N} (|v(x, t)|^2 + |p(x, t)|) \times \left[ w(|x|) + \frac{1}{|x|} \int_0^{|x|} w(s) \, ds + \frac{1}{|x|^2} \int_0^{|x|} \int_0^r w(\rho) \, d\rho \, ds \right] \, dx \, dt
\]
\[
\leq C \int_0^T \int_{\mathbb{R}^N} \frac{|v(x,t)|^2 + |p(x,t)|}{1+|x|} \, dx \, dt
\]
\[
\leq C \int_0^T \left( \|v(t)\|_{L^2}^2 + \|p(t)\|_{L^2} \right) dt \left( \int_{\mathbb{R}^N} \frac{dx}{(1+|x|)^{\frac{n-2}{2}}} \right)^{\frac{q-2}{q}}
\]
\[
\leq C \int_0^T \|v(t)\|_{L^q}^2 \, dt < \infty.
\] (2.19)

Hence, for both of the cases where either (1.6) or (1.7) holds true we can apply Theorem 1.1 to conclude that \(v(x,t) = 0\) for almost every \((x,t) \in \mathbb{R}^N \times (0, \infty)\). \(\Box\)

3 The case of the MHD equations

In this section we extend the previous results on the system \((NS)_\nu\) to the magnetohydrodynamic equations in \(\mathbb{R}^N, N \geq 3\).

\[
(MHD)_{\mu,\nu}
\]
\[
\begin{cases}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = (b \cdot \nabla)b - \nabla(p + \frac{1}{2}|b|^2) + \nu \Delta v + f,
\frac{\partial b}{\partial t} + (v \cdot \nabla)b = (b \cdot \nabla)v + \mu \Delta b + g,
\text{div } v = \text{div } b = 0,
v(x,0) = v_0(x), \quad b(x,0) = b_0(x)
\end{cases}
\]

where \(v = (v_1, \cdots, v_N)\), \(v_j = v_j(x,t), j = 1, \cdots, N\), is the velocity of the flow, \(p = p(x,t)\) is the scalar pressure, \(b = (b_1, \cdots, b_N)\), \(b_j = b_j(x,t)\), is the magnetic field, and \(v_0, b_0\) are the given initial velocity and magnetic field, satisfying \(\text{div } v_0 = \text{div } b_0 = 0\), respectively. We may consider \(f = (f_1(x,t), \cdots, f_N(x,t))\) and \(g = (g_1(x,t), \cdots, g_N(x,t))\) as external forces for the velocity and to the magnetic fields, respectively. If we set \(b = g = 0\), then \((MHD)_{\mu,\nu}\) reduces to \((NS)_\nu\) of the previous sections. Let us begin with the definition of the weak solutions of \((MHD)_{\mu,\nu}\).

**Definition 3.1** We say the triple of functions \((v, b, p) \in [L^2(0, T; L^2_{\text{loc,} \sigma}(\mathbb{R}^N))]^2 \times [L^2(0, T; L^\infty(\mathbb{R}^N))]^{N+1}\) is a weak solution of \((MHD)_{\mu,\nu}\) if

\[
\begin{align*}
\int_0^T \int_{\mathbb{R}^N} & \left( (v \cdot \nabla)v, \right. \\
& \left. \frac{\partial v}{\partial t} + (v \cdot \nabla)v ight) \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} (b \cdot \nabla)b \, dx \, dt \\
& + \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} \nabla(p + \frac{1}{2}|b|^2) \, dx \, dt \\
& + \nu \int_0^T \int_{\mathbb{R}^N} \Delta v \, dx \, dt + \mu \int_0^T \int_{\mathbb{R}^N} \Delta b \, dx \, dt \\
& + \int_0^T \int_{\mathbb{R}^N} f \cdot v \, dx \, dt \\
& + \int_0^T \int_{\mathbb{R}^N} g \cdot b \, dx \, dt = 0,
\end{align*}
\]
\[ L^1(0, T; L_{\text{loc}}^1(\mathbb{R}^N)) \text{ is a weak solution of } (MHD)_{\mu, \nu} \text{ on } \mathbb{R}^N \times (0, T), \text{ if} \]
\[
- \int_0^T \int_{\mathbb{R}^N} v(x, t) \cdot \phi(x) \xi'(t) dx dt - \int_0^T \int_{\mathbb{R}^N} v(x, t) \otimes v(x, t) : \nabla \phi(x) \xi(t) dx dt
\]
\[
= - \int_0^T \int_{\mathbb{R}^N} b(x, t) \otimes b(x, t) : \nabla \phi(x) \xi(t) dx dt + \int_0^T \int_{\mathbb{R}^N} p(x, t) \text{ div } \phi(x) \xi(t) dx dt
\]
\[
+ \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} |b(x, t)|^2 \text{ div } \phi(x) \xi(t) dx dt + \nu \int_0^T \int_{\mathbb{R}^N} v(x, t) \cdot \Delta \phi(x) \xi(t) dx dt,
\]
\[
+ \int_0^T \int_{\mathbb{R}^N} f(x, t) \cdot \phi(x) \xi(t) dx dt,
\]
\[
(3.1)
\]
and
\[
- \int_0^T \int_{\mathbb{R}^N} b(x, t) \cdot \phi(x) \xi'(t) dx dt - \int_0^T \int_{\mathbb{R}^N} v(x, t) \otimes b(x, t) : \nabla \phi(x) \xi(t) dx dt
\]
\[
= - \int_0^T \int_{\mathbb{R}^N} b(x, t) \otimes v(x, t) : \nabla \phi(x) \xi(t) dx dt + \mu \int_0^T \int_{\mathbb{R}^N} b(x, t) \cdot \Delta \phi(x) \xi(t) dx dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^N} g(x, t) \cdot \phi(x) \xi(t) dx dt,
\]
\[
(3.2)
\]
for all \( \xi \in C^1_0(0, T) \) and \( \phi = [C^2_0(\mathbb{R}^N)]^N \).

We have the following theorem.

**Theorem 3.1** We fix \( \mu, \nu \geq 0, N \geq 3 \). Let \( w \in L_{\text{loc}}^1([0, \infty)) \) be given, which is positive, non-increasing on \([0, \infty)\). Suppose \((v, b, p) \in [L^2(0, T; L_{\text{loc}, \sigma}^2(\mathbb{R}^N))]^2 \times L^1(0, T; L_{\text{loc}}^1(\mathbb{R}^N))\) is a weak solution to \((MHD)_{\mu, \nu}\) with \( f, g \in L^1(0, T; L_{\sigma, \text{loc}}^1(\mathbb{R}^N)) \) on \( \mathbb{R}^N \times (0, T) \) such that
\[
\int_0^T \int_{\mathbb{R}^N} (|v(x, t)|^2 + |b(x, t)|^2 + |p(x, t)|) \times
\]
\[
\times \left[ w(|x|) + \frac{1}{|x|} \int_0^{|x|} w(s) ds + \frac{1}{|x|^2} \int_0^{|x|} \int_0^r w(s) ds dr \right] dx dt < \infty,
\]
\[
(3.3)
\]
and

\[
\int_{\mathbb{R}^N} p(x, t) \left[ w(|x|) + \frac{N - 1}{|x|} \int_0^{|x|} w(s) \, ds \right] \, dx \geq 0 \quad \text{for } t \in (0, T). \tag{3.4}
\]

Then, \( b(x, t) = 0 \) and \( v(x, t) = 0 \) almost everywhere on \( \mathbb{R}^N \times (0, T) \).

**Remark 3.1** Similarly to the case of Euler equations if we choose \( w(s) \equiv 1 \) on \([0, \infty)\), then we recover a part of Liouville type of result in Theorem 3.1 in [1].

**Remark 3.2** Similarly to Remark 1.2 for \( w^*(r) := \sup_{0 \leq s \leq r} w(s) \) we can replace (3.3) by a stronger assumption,

\[
\int_0^T \int_{\mathbb{R}^N} \left( |v(x, t)|^2 + |b(x, t)|^2 + |p(x, t)| \right) w^*(|x|) \, dx \, dt < \infty \tag{3.5}
\]

to derive triviality of the solution from (3.3).

**Proof of Theorem 3.1** The method of proof is similar to that of Theorem 1.1, and we will be brief, describing only essential points. Similarly to (2.1)-(2.4) we choose \( \xi \in C_0^1(0, T) \) and the vector test function \( \phi = \nabla \varphi_R \), where

\[
\varphi_R(x) = W(|x|) \sigma \left( \frac{|x|}{R} \right) = W(|x|) \sigma_R(|x|) \tag{3.6}
\]

with \( W(|x|) = \int_0^{|x|} w(r) \, dr \, ds \), and \( \sigma \) is the cut-off function defined in (2.1). Then, we obtain from (3.1) that

\[
0 = \int_0^T \int_{\mathbb{R}^N} \left[ W''(|x|) \frac{(v \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \sigma_R(|x|) \xi(t) \, dx \, dt
- \int_0^T \int_{\mathbb{R}^N} \left[ W''(|x|) \frac{(b \cdot x)^2}{|x|^2} + W'(|x|) \left( \frac{|b|^2}{|x|} - \frac{(b \cdot x)^2}{|x|^3} \right) \right] \sigma_R(|x|) \xi(t) \, dx \, dt
+ \int_0^T \int_{\mathbb{R}^N} \left( p(x, t) + \frac{1}{2} |b|^2 \right) W''(|x|) + (N - 1) \frac{W'(|x|)}{|x|} \right] \sigma_R(|x|) \xi(t) \, dx \, dt
+ o(1), \tag{3.7}
\]
where $o(1)$ denotes the sum of the terms vanishing as $R \to \infty$. Taking $R \to \infty$ in (3.7), and rearranging the non-vanishing terms, we find that

$$
\int_0^T \int_{\mathbb{R}^N} \left[ W''(|x|) \left( \frac{(v \cdot x)^2}{|x|^2} + \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \xi(t) \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} W''(|x|)|b|^2 \xi(t) \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^N} \left[ \frac{1}{|x|} W'(|x|) - W''(|x|) \right] \frac{(b \cdot x)^2}{|x|^2} \xi(t) \, dx \, dt \\
+ \frac{N-3}{2} \int_0^T \int_{\mathbb{R}^N} |b|^2 \frac{W'(|x|)}{|x|} \xi(t) \, dx \, dt \\
= - \int_0^T \int_{\mathbb{R}^N} p(x,t) \left[ W''(|x|) + (N-1) \frac{W'(|x|)}{|x|} \right] \xi(t) \, dx \, dt
$$

(3.8)

for all $\xi \in C^1_0(0,T)$. Hence

$$
\int_{\mathbb{R}^N} \left[ W''(|x|) \left( \frac{(v \cdot x)^2}{|x|^2} + \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^N} W''(|x|)|b|^2 \, dx \\
+ \int_{\mathbb{R}^N} \left[ \frac{1}{|x|} W'(|x|) - W''(|x|) \right] \frac{(b \cdot x)^2}{|x|^2} \, dx \\
+ \frac{N-3}{2} \int_{\mathbb{R}^N} |b|^2 \frac{W'(|x|)}{|x|} \, dx \\
= - \int_{\mathbb{R}^N} p(x,t) \left[ W''(|x|) + (N-1) \frac{W'(|x|)}{|x|} \right] \, dx
$$

(3.9)

for almost every $t \in (0,T)$. Our assumption (3.4) implies that the right hand side of (3.9) is non-positive. Since each integral of the left hand side of (3.9) is non-negative for $N \geq 3$, we need to have that each term of the left hand side of (3.9) vanishes for almost every $t \in (0,T)$. The requirement,

$$
\int_{\mathbb{R}^N} \left[ W''(|x|) \left( \frac{(v \cdot x)^2}{|x|^2} + \frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|^3} \right) \right] \, dx = 0
$$
implies $v(x,t) = 0$ for almost every $(x,t) \in \mathbb{R}^N \times (0,T)$, as we in the proof of Theorem 1.1. Since $w(r)$ is non-increasing on $[0,\infty)$, we have

$$\frac{1}{|x|} W'(|x|) - W''(|x|) = \frac{1}{|x|} \int_0^{|x|} w(r) \, dr - w(|x|) \geq 0,$$

and therefore

$$\int_{\mathbb{R}^N} \left[ \frac{1}{|x|} W'(|x|) - W''(|x|) \right] \frac{(b \cdot x)^2}{|x|^2} \, dx = 0.$$

Hence the condition $w(r) = W''(r) > 0$ together with the fact $\int_{\mathbb{R}^N} W''(|x|) |b|^2 \, dx = 0$ implies $b(x,t) = 0$ for almost every $(x,t) \in \mathbb{R}^N \times (0,T)$. □

Similarly to Theorem 1.2, we can establish the following:

**Theorem 3.2** Let $(v, b, p)$ be a weak solution to $(MHD)_{\mu,\nu}$ with $\mu, \nu \geq 0$ and $f, g \in L^1(0,T; L^1_{\text{loc,}D}(\mathbb{R}^N))$ on $\mathbb{R}^N \times (0,T)$, $N \geq 3$, such that either

$$\int_0^T \int_{\mathbb{R}^N} \frac{(|v(x,t)|^2 + |b(x,t)|^2 + |p(x,t)|)}{1 + |x|} \, dx \, dt < \infty, \quad (3.10)$$

or

$$|p(x,t)| \to 0 \text{ as } |x| \to \infty \text{ for almost every } t \in (0,T), \quad (3.11)$$

$$|v| + |b| \in L^2(0,T; L^q(\mathbb{R}^N)) \text{ for some } q \text{ with } 2 < q < \frac{2N}{N-1}. \quad (3.12)$$

Suppose there exists $w \in L^1(0,\infty)$, which is positive, non-increasing on $[0,\infty)$ such that

$$0 < w(r) \leq \frac{C}{1+r} \text{ for some constant } C > 0 \quad (3.13)$$

almost every $r \in [0,\infty)$, and

$$\int_{\mathbb{R}^N} p(x,t) \left[ w(|x|) + \frac{N-1}{|x|} \int_0^{|x|} w(s) \, ds \right] \, dx \geq 0 \text{ for almost every } t \in (0,T). \quad (3.14)$$

Then, $v(x,t) = b(x,t) = 0$ almost everywhere on $\mathbb{R}^N \times (0,T)$. 14
In the case of $\mu, \nu > 0$ a global in time weak solutions $(v, b) \in [L^\infty(0, T; L^2(\mathbb{R}^N)) \cap L^2(0, T; H^1(\mathbb{R}^N))]^2$ are constructed in [4]. Hence, using the fact (1.11) we have the following:

**Corollary 3.1** Let $(v, b, p) \in [L^\infty(0, T; L^2(\mathbb{R}^N)) \cap L^2(0, T; H^1(\mathbb{R}^N))]^2 \times L^1(0, T; L^1_{loc}(\mathbb{R}^N))$ be a weak solution to $(MHD)_{\mu,\nu}$ with $f, g \in L^2(0, T; L^2(\mathbb{R}^N))$ and $\mu, \nu > 0$, constructed in [4]. Suppose the pressure $p(x, t)$ satisfies (1.7) and (1.10) for a function $w(r)$ satisfying the conditions of Theorem 1.2. Then, $v = b = 0$ almost everywhere on $\mathbb{R}^N \times (0, T)$.

**Proof of Theorem 3.2** Similarly to the proof of Theorem 1.2 the conditions $w \in L^1(0, \infty)$ and (3.13) imply that there exists a positive constant $C = C(\|w\|_{L^1(0, \infty)})$ such that

$$w(r) + \frac{1}{r} \int_0^r w(s) ds + \frac{1}{r^2} \int_0^r \int_0^s w(\rho) d\rho ds \leq \frac{C}{1 + r}.$$ 

Therefore, if (3.10) holds true, then

$$\int_0^T \int_{\mathbb{R}^N} (|v(x, t)|^2 + |b(x, t)|^2 + |p(x, t)|) \times w(|x|) + \frac{1}{|x|} \int_0^{|x|} w(s) ds + \frac{1}{|x|^2} \int_0^{|x|} \int_0^r w(s) ds dr \right] dx dt \leq C \int_0^T \int_{\mathbb{R}^N} \frac{|v(x, t)|^2 + |b(x, t)|^2 + |p(x, t)|}{1 + |x|} dx dt < \infty \quad (3.15)$$

Next, we suppose (3.11)-(3.12) holds true. In order to handle this case we observe that taking the divergence operation of the first equation of $(MHD)_{\mu,\nu}$, we obtain

$$\Delta(p + \frac{1}{2}|b|^2) = \sum_{j,k=1}^N \partial_j \partial_k (b_j b_k) - \sum_{j,k=1}^N \partial_j \partial_k (v_j v_k).$$

Therefore

$$p(x, t) = -\sum_{j,k=1}^N R_j R_j (b_j b_k)(x, t) + \sum_{j,k=1}^N R_j R_k (v_j v_k)(x, t) - \frac{1}{2} |b(x, t)|^2 + h(x, t),$$

where $R_j = \partial_j (-\Delta)^{-\frac{1}{2}}, j = 1, \cdots, N,$ is the Riesz transform, and $h(x, t)$ is a harmonic function on $\mathbb{R}^N$. The condition (3.11) implies that $h(\cdot, t) = 0$
for almost every $t \in (0, T)$. As in the proof of Theorem 1.2, thanks to the Calderon-Zygmund inequality, we have

$$\|p(t)\|_{L^q}^2 \leq C_q(\|v(t)\|_{L^q}^2 + \|b(t)\|_{L^q}^2) \quad \forall q \in (2, \infty) \quad (3.16)$$

for a constant $C_q$. Therefore

$$\int_0^T \int_{\mathbb{R}^N} (|v(x,t)|^2 + |b(x,t)|^2 + |p(x,t)|) \times \left[ w(|x|) + \frac{1}{|x|} \int_{0}^{|x|} w(s) \, ds + \frac{1}{|x|^2} \int_{0}^{|x|} \int_{0}^r w(s) \, ds \, dr \right] \, dx \, dt$$

$$\leq C \int_0^T \int_{\mathbb{R}^N} \frac{|v(x,t)|^2 + |b(x,t)|^2 + |p(x,t)|}{1 + |x|} \, dx \, dt$$

$$\leq C \int_0^T (\|v(t)\|_{L^q}^2 + \|b(t)\|_{L^q}^2 + |p(t)|_{L^q}^2) \, dt \left( \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|)^{\frac{q-2}{q}}} \right)^{\frac{q}{q-2}}$$

$$\leq C \int_0^T (\|v(t)\|_{L^q}^2 + \|b(t)\|_{L^q}^2) \, dt < \infty \quad (3.17)$$

for $2 < q < \frac{2N}{N-1}$, where we used (3.16). Therefore, for both of the cases whether (3.10) or (3.11)-(3.12) holds true, we can apply Theorem 3.1 to conclude that $v(x,t) = b(x,t) = 0$ for almost every $(x,t) \in \mathbb{R}^N \times (0, \infty)$. □

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