Dynamical correlation functions of the $XXZ$ spin-$1/2$ chain

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Abstract

We derive a master equation for the dynamical spin-spin correlation functions of the $XXZ$ spin-$1/2$ Heisenberg finite chain in an external magnetic field. In the thermodynamic limit, we obtain their multiple integral representation.

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1 Introduction

The aim of this article is to describe a new method to compute exact representations of time-dependent correlation functions in quantum integrable lattice models. For that purpose, we consider the example of the $XXZ$ spin-$\frac{1}{2}$ Heisenberg chain in an external magnetic field. For simplicity, we will here mainly focus on the dynamical correlation function of the third component of spin. It will be clear however that the method is general and can be applied to other cases as well.

This work is a continuation of [1], where we have derived a master equation for the (time-independent) correlation functions of the $XXZ$ chain. In the end of that article, we have announced a multiple integral representation for the dynamical $\sigma^z$ correlation function. We give here a proof of that result.

The Hamiltonian of the periodical $XXZ$ spin-$\frac{1}{2}$ Heisenberg chain in a magnetic field [2] is given by

$$H = H^{(0)} - hS_z,$$

where

$$H^{(0)} = \sum_{m=1}^{M} \left( \sigma^x_m \sigma^x_{m+1} + \sigma^y_m \sigma^y_{m+1} + \Delta (\sigma^z_m \sigma^z_{m+1} - 1) \right),$$

$$S_z = \frac{1}{2} \sum_{m=1}^{M} \sigma^z_m, \quad [H^{(0)}, S_z] = 0.$$

Here $\Delta$ is the anisotropy parameter, $h$ denotes the external classical magnetic field, and $\sigma^x, \sigma^y, \sigma^z$ are the spin operators (in the spin-$\frac{1}{2}$ representation) associated with each site of the chain. The length of the chain $M$ is chosen to be even. The simultaneous reversal of all spins is equivalent to a change of sign of the magnetic field, therefore it is enough to consider the case $h \geq 0$. The quantum space of states $\mathcal{H}$ is $\mathcal{H} = \otimes_{m=1}^M \mathcal{H}_m$, where $\mathcal{H}_m \sim \mathbb{C}^2$ is called local quantum space. The operators $\sigma^x, \sigma^y, \sigma^z$ act as the corresponding Pauli matrices in the space $\mathcal{H}_m$ and as the identity operator elsewhere.

The time-dependent local spin operators are defined as

$$\sigma^{x,y,z}_m(t) = e^{iHt} \sigma^{x,y,z}_m e^{-iHt}. \quad (1.2)$$

Since $[\sigma^z_m, S_z] = 0$, we have for the local operator of the third components of spin

$$\sigma^z_m(t) = e^{iH^{(0)}t} \sigma^z_m e^{-iH^{(0)}t}. \quad (1.3)$$

Hence, the dynamical two-point $\sigma^z$ correlation function at zero temperature is given as the following mean-value:

$$g_{zz}(m,t) = \langle \psi_g | \sigma^z_1 e^{iH^{(0)}t} \sigma^z_{m+1} e^{-iH^{(0)}t} | \psi_g \rangle, \quad (1.4)$$

where $|\psi_g\rangle$ denotes the ground state of the Hamiltonian [1].
The method to compute eigenstates and energy levels (Bethe ansatz) of the Hamiltonian \( H(0) \) was proposed by Bethe in 1931 in [3] and developed later in [4, 5, 6]. The algebraic version of the Bethe ansatz was created in the framework of the Quantum Inverse Scattering Method by L.D. Faddeev and his school [7, 8, 9]. Different ways to study the time-independent correlation functions of this model were proposed in the series of works (see e.g. [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]). As for the dynamical correlation functions, up to now, the only known exact results concern the case of free fermions \( \Delta = 0 \) [22, 23, 24, 25, 26, 27, 28, 29].

Our aim is to obtain a multiple integral representation of the expectation value (1.4) for arbitrary \( \Delta > -1 \). For this purpose we use two different approaches leading (hopefully) to the same answer. Both of them are based on the algebraic Bethe ansatz.

The first method [13, 14, 16] consists in acting with the operators \( \sigma_z^1 \) and \( \sigma_z^{m+1}(t) \) on the ground state \( \langle \psi_g | \) to produce a new state, say \( \langle \psi(z,m,t) | \). Then one can compute the resulting scalar product \( \langle \psi(z,m,t) | \psi_g \rangle \). In the time-independent case this can be done by the algebraic Bethe ansatz and the explicit solution of the quantum inverse scattering problem for the \( \sigma_z^1 \) operators in site 1 and \( m+1 \) (see [13, 30]). However, for the calculation of the dynamical correlation functions, we also need to compute the action of \( e^{iH(0)t} \) on arbitrary states of the chain. To achieve this we use the fact that the Hamiltonian \( H(0) \) can be constructed as the logarithmic derivative of the quantum transfer matrix \( T(\lambda) \):

\[
H(0) = 2 \sinh \eta \frac{dT(\lambda)}{d\lambda} T^{-1}(\lambda) \bigg|_{\lambda=\frac{\eta}{2}} - 2M \cosh \eta, \quad \text{with} \quad \cosh \eta = \Delta. \tag{1.5}
\]

Hence, from the Trotter type formula [31, 32], we can obtain the evolution operator \( e^{iH(0)t} \) as the following limit:

\[
e^{\pm it(H(0)+2M \cosh \eta)} = \lim_{L \to \infty} \left( T(\frac{\eta}{2} + \varepsilon) \cdot T^{-1}(\frac{\eta}{2}) \right)^{\pm L}, \quad \varepsilon = \frac{1}{L} 2it \sinh \eta. \tag{1.6}
\]

It is moreover possible to express \( T^{-1}(\eta/2) \) in terms of \( T(-\eta/2) \). Therefore the action of \( e^{iH(0)t} \) reduces to the one of some product of transfer matrices. This enables us to apply directly the results of [15] and [11], where we computed the action of any product of transfer matrices on arbitrary states in a compact form. In the present case it leads to a master equation for the dynamical spin-spin correlation function \( g_{zz}(m,t) \), or more precisely for its generating function (see Theorem 3.1).

The second method to handle \( g_{zz}(m,t) \) is to insert between the two \( \sigma_z^1 \) operators a sum over a complete set of eigenstates of the Hamiltonian, leading to

\[
g_{zz}(m,t) = \sum_i \langle \psi_g | \sigma_z^1 | i \rangle \langle i | \sigma_z^{m+1} | \psi_g \rangle e^{i(E_i - E_0)t}, \tag{1.7}
\]

where \( H(0)|i\rangle = E_i|i\rangle \) and \( H(0)|\psi_g\rangle = E_0|\psi_g\rangle \). In [11] we have shown that for the time-independent case the master equation method allows one to define such a sum in a precise
manner. It uses in fact a twisted version of the transfer matrix \( T(\lambda) \) and the knowledge of determinant representations for the form factors (e.g., corresponding to the matrix elements \( \langle \psi_g | \sigma_z^i | i \rangle \)). This method can easily be generalized to the time-dependent case. So, using the technique developed in [1], we are able to re-sum completely the form factor expansion for the dynamical correlation function and to show that it leads indeed to the time-dependent master equation obtained by the first method described above.

This time-dependent master equation gives a representation of the dynamical \( \sigma^z \) correlation function of a finite XXZ chain. In the thermodynamic limit \( M \to \infty \), the model exhibits three different regimes depending on the value of the anisotropy parameter \( \Delta \) and on the magnetic field \( h \). In this limit, following [13, 14, 15, 1], we obtain a multiple integral representation for the dynamical \( \sigma^z \) correlation function in the massive \((\Delta > 1)\) and massless \((-1 < \Delta \leq 1)\) regimes. It is given as a sum of multiple integrals (see [6,16]) involving the time dependence through the factors \( e^{iEt} \), where \( E \) is the bare one-particle energy.

This paper is organized as follows. In Section 2 we introduce some useful definitions and notations and recall briefly the properties of the twisted transfer matrix. The time-dependent master equation (3.8) for the generating function of the dynamical \( \sigma^z \) correlation function is given in Section 3. In the following two sections we present two independent proofs of (3.8). The first one uses the Trotter–Suzuki procedure for the explicit calculation of the action of \( e^{iH(0)t} \) on arbitrary state of the chain (Section 4). In Section 5 we derive the time-dependent master equation via the form factor type expansion. In Section 6 we obtain a multiple integral representation for the dynamical \( \sigma^z \) correlation function in the thermodynamic limit.

Some technical details are gathered in two appendices. In Appendix A we prove several lemmas on the properties of the solutions of the twisted Bethe equations and of the corresponding eigenstates of the twisted transfer matrix. These lemmas are used in Sections 4 and 5. In Appendix B we compare the representation (6.16) for the dynamical \( \sigma^z \) correlation function at \( \Delta = 0 \) with the one obtained in [29] in the case of free fermions.

2 Algebraic Bethe ansatz and twisted transfer matrix

To derive the master equation for the dynamical correlation functions we use the twisted transfer matrix in the framework of algebraic Bethe ansatz. The definition of this operator and related objects together with a brief sketch of their properties is given below. We refer the reader for more details to [1].

2.1 General framework

The central object of algebraic Bethe ansatz is the quantum monodromy matrix. In the case of the XXZ chain (1.1), this is a 2 \( \times \) 2 matrix

\[
T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}
\]  

(2.1)
with operator-valued entries \( A, B, C \) and \( D \) which depend on a complex parameter \( \lambda \) and act in the quantum space of states \( \mathcal{H} \) of the chain. These operators satisfy a set of quadratic commutation relations given by the \( R \)-matrix of the model.

In the framework of algebraic Bethe ansatz, a sector \( \mathcal{H}^{(M/2-N)} \) of the space of states \( \mathcal{H} \) with a fixed number \( N \) of spin down, \( N = 0, 1, \ldots, M \), is spanned by vectors of the form

\[
|\psi\rangle = \prod_{j=1}^{N} B(\lambda_j)|0\rangle, \tag{2.2}
\]

where \( |0\rangle \) is the state with all spins up and \( \lambda_1, \ldots, \lambda_N \) are arbitrary complex numbers. The dual space is constructed in terms of

\[
\langle \psi | = \langle 0 | \prod_{j=1}^{N} C(\lambda_j), \tag{2.3}
\]

where \( \langle 0 | = |0\rangle^+ \).

The Hamiltonian \( H^{(0)} \) of the homogeneous XXZ chain is given by the ‘trace identity’ in terms of the transfer matrix \( T(\lambda) = A(\lambda) + D(\lambda) \). All these transfer matrices commute, \([T(\lambda), T(\mu)] = 0\), and their common eigenstates (which coincide with those of the Hamiltonian) are usually constructed in the form \( (2.2) \) (or \( 2.3 \)) for \( \lambda_1, \ldots, \lambda_N \) satisfying a system of \( N \) algebraic equations first obtained by Bethe in [3].

The local spin operators can also be expressed in terms of the entries of the monodromy matrix, by solving the quantum inverse scattering problem [13, 30]:

\[
\sigma^\alpha_j = T^{j-1}(\eta/2) \cdot \text{tr}(T(\eta/2)\sigma^\alpha) \cdot T^{-j}(\eta/2). \tag{2.4}
\]

Here \( \sigma^\alpha \) in the r.h.s. acts in the auxiliary space of \( T(\lambda) \), while \( \sigma_j^\alpha \) in the l.h.s. acts in the local quantum space \( \mathcal{H}_j \).

### 2.2 Twisted transfer matrix

It is actually convenient to introduce a slightly more general object:

**Definition 2.1.** The operator

\[
T_\kappa(\mu) = A(\mu) + \kappa D(\mu), \tag{2.5}
\]

where \( \kappa \) is a complex parameter, is called the twisted transfer matrix. The particular case of \( T_\kappa(\mu) \) at \( \kappa = 1 \) corresponds to the usual transfer matrix \( T(\mu) \).

Let us consider the action of the operator \( T_\kappa(\mu) \) in the subspace \( \mathcal{H}^{(M/2-N)} \) with fixed (but arbitrary) number of spins down \( N \). The eigenstates of \( T_\kappa(\mu) \) and their dual states in this subspace are denoted as \( |\psi_\kappa(\{\lambda\})\rangle \) (respectively \( \langle \psi_\kappa(\{\lambda\}) | \)). They can be written in the form
where the parameters $\lambda_1, \ldots, \lambda_N$ satisfy the system of twisted Bethe equations

$$\mathcal{Y}_\kappa(\lambda_j|\{\lambda\}) = 0, \quad j = 1, \ldots, N, \quad (2.6)$$

where

$$\mathcal{Y}_\kappa(\mu|\{\lambda\}) = a(\mu) \prod_{k=1}^N \sinh(\lambda_k - \mu + \eta) + \kappa \, d(\mu) \prod_{k=1}^N \sinh(\lambda_k - \mu - \eta), \quad (2.7)$$

and

$$a(\mu) = \sinh^M(\mu + \eta/2), \quad d(\mu) = \sinh^M(\mu - \eta/2). \quad (2.8)$$

The corresponding eigenvalue of $\mathcal{T}_\kappa(\mu)$ on $|\psi_\kappa(\{\lambda\})\rangle$ (or on a dual eigenstate) is

$$\tau_\kappa(\mu|\{\lambda\}) = a(\mu) \prod_{k=1}^N \frac{\sinh(\lambda_k - \mu + \eta)}{\sinh(\lambda_k - \mu)} + \kappa \, d(\mu) \prod_{k=1}^N \frac{\sinh(\mu - \lambda_k + \eta)}{\sinh(\mu - \lambda_k)}. \quad (2.9)$$

Note that

$$\mathcal{Y}_\kappa(\mu|\{\lambda\}) = \prod_{k=1}^N \sinh(\lambda_k - \mu) \cdot \tau_\kappa(\mu|\{\lambda\}). \quad (2.10)$$

**Definition 2.2.** A solution $\{\lambda\}$ of the system $2.6$ is called admissible if

$$d(\lambda_j) \prod_{k=1, k \neq j}^N \sinh(\lambda_j - \lambda_k + \eta) \neq 0, \quad j = 1, \ldots, N, \quad (2.11)$$

and unadmissible otherwise. A solution is called off-diagonal if the parameters $\lambda_1, \ldots, \lambda_N$ are pairwise distinct, and diagonal otherwise.

It is proven in Appendix A (see Theorem 2.1) that the eigenstates corresponding to the admissible off-diagonal solutions of the system $2.6$ form a basis in $\mathcal{H}^{(M/2-N)}$, at least if $\kappa$ is in a punctured vicinity of the origin (i.e. $0 < |\kappa| < \kappa_0$).

**Notations.** Recall that in the particular case $\kappa = 1$ the corresponding twisted transfer matrix is simply denoted by $\mathcal{T}$ in which the subscript $\kappa$ has been omitted. We follow the same agreement for all related objects. Namely, $(\psi, \tau, \mathcal{Y}) = (\psi_{\kappa}, \tau_{\kappa}, \mathcal{Y}_{\kappa})|_{\kappa=1}$.

At $\kappa = 1$, it follows from the trace identity $1.5$ that the eigenstates of the transfer matrix $\mathcal{T}$ coincide with the ones of the Hamiltonian $1.1$. The corresponding eigenvalues can be obtained from $1.5$, $2.9$:

$$H^{(0)}|\psi(\{\lambda\})\rangle = \left( \sum_{j=1}^N E(\lambda_j) \right) \cdot |\psi(\{\lambda\})\rangle, \quad (2.12)$$

where $E(\lambda)$ is called the bare one-particle energy and is equal to

$$E(\lambda) = \frac{2 \sinh^2 \eta}{\sinh(\lambda + \frac{\eta}{2}) \sinh(\lambda - \frac{\eta}{2})}. \quad (2.13)$$
One can similarly define the bare one-particle momentum. It is given by

\[ p(\lambda) = i \log \left( \frac{\sinh(\lambda - \frac{\eta}{2})}{\sinh(\lambda + \frac{\eta}{2})} \right). \]  

(2.14)

Remark. The equation (2.12) holds if the parameters \( \{\lambda\} \) of the eigenstate \( |\psi(\{\lambda\})\rangle \) correspond to an admissible off-diagonal solution of the system (2.6) at \( \kappa = 1 \):

\[ Y(\lambda_j|\{\lambda\}) = a(\lambda_j) N \prod_{k=1}^{N} \sinh(\lambda_k - \lambda_j + \eta) + d(\lambda_j) N \prod_{k=1}^{N} \sinh(\lambda_k - \lambda_j - \eta) = 0, \quad j = 1, \ldots, N. \]  

(2.15)

Unlike for generic \( \kappa \), the eigenstates basis of \( \mathcal{T} \) (i.e. the eigenstates basis of the Hamiltonian) includes also states corresponding to unadmissible solutions of the system (2.15). We do not consider the expectation values of the operators with respect to such states.

2.3 Scalar products

We recall here the expressions for the scalar product of an eigenstate of the twisted transfer matrix with any arbitrary state of the form (2.2) or (2.3).

Let us first introduce some notations. We define, for arbitrary positive integers \( n, n' \) \((n \leq n')\) and sets of variables \( \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n \) and \( \nu_1, \ldots, \nu_n' \) such that \( \{\lambda\} \subset \{\nu\} \), the \( n \times n \) matrix \( \Omega_{\kappa}(\{\lambda\}, \{\mu\}|\{\nu\}) \) as

\[ (\Omega_{\kappa})_{jk}(\{\lambda\}, \{\mu\}|\{\nu\}) = a(\mu_k) t(\lambda_j, \mu_k) \prod_{a=1}^{n'} \sinh(\nu_a - \mu_k + \eta) \]

\[ - \kappa d(\mu_k) t(\mu_k, \lambda_j) \prod_{a=1}^{n'} \sinh(\nu_a - \mu_k - \eta), \]  

(2.16)

with

\[ t(\lambda, \mu) = \frac{\sinh \eta}{\sinh(\lambda - \mu) \sinh(\lambda - \mu + \eta)}. \]  

(2.17)

We also define, for arbitrary sets of variables \( \lambda_1, \ldots, \lambda_n \) and \( \mu_1, \ldots, \mu_n \), the Cauchy determinant \( \mathcal{X}_n(\{\mu\}, \{\lambda\}) \) as

\[ \mathcal{X}_n(\{\mu\}, \{\lambda\}) \equiv \det_n \left( \frac{1}{\sinh(\mu_k - \lambda_j)} \right) = \frac{\prod_{a>b} \sinh(\lambda_a - \lambda_b) \sinh(\mu_b - \mu_a)}{\prod_{a,b=1}^{n} \sinh(\mu_b - \lambda_a)}. \]  

(2.18)
Proposition 2.1. Let $\lambda_1, \ldots, \lambda_N$ satisfy the system (2.6), $\mu_1, \ldots, \mu_N$ be generic complex numbers. Then

\[
\langle 0 | \prod_{j=1}^{N} C(\mu_j) | \psi_\kappa(\{\lambda\}) \rangle = \langle \psi_\kappa(\{\lambda\}) | \prod_{j=1}^{N} B(\mu_j) | 0 \rangle \\
= \prod_{a=1}^{N} d(\lambda_a) \cdot \mathcal{X}_N^{-1}(\{\mu\}, \{\lambda\}) \cdot \det_N \left( \frac{\partial}{\partial \lambda_j} \tau_\kappa(\mu_k|\{\lambda\}) \right) \quad (2.19) \\
= \frac{\prod_{a=1}^{N} d(\lambda_a)}{\prod_{a>b} \sinh(\lambda_a - \lambda_b) \sinh(\mu_b - \mu_a)} \cdot \det_N \Omega_\kappa(\{\lambda\}, \{\mu\}|\{\lambda\}). \quad (2.20)
\]

The eigenstate $|\psi_\kappa(\{\lambda\})\rangle$ is orthogonal to the dual eigenstate $\langle \psi_\kappa(\{\mu\}) |$ if the sets $\{\lambda\}$ and $\{\mu\}$ are different: $\{\lambda\} \neq \{\mu\}$ (see Appendix A). Otherwise

\[
\langle \psi_\kappa(\{\lambda\}) | \psi_\kappa(\{\lambda\}) \rangle = \frac{\prod_{a=1}^{N} d(\lambda_a)}{\prod_{a,b=1 \atop a \neq b} \sinh(\lambda_a - \lambda_b)} \cdot \det_N \left( \frac{\partial}{\partial \lambda_k} \mathcal{Y}_\kappa(\lambda_j|\{\lambda\}) \right). \quad (2.21)
\]

The equations (2.19)–(2.22) are valid for any arbitrary complex parameter $\kappa$, in particular at $\kappa = 1$. In this case we denote $\Omega = \Omega_\kappa|_{\kappa=1}$.

3 Master equation for dynamical correlation functions

Our main goal is to obtain an explicit representation for the time dependent correlation function of the third components of spin,

\[
\langle \sigma_z^{i}(0) \sigma_z^{m+1}(t) \rangle = \frac{\langle \psi(\{\lambda\}) | \sigma_z^{i}(0) \sigma_z^{m+1}(t) | \psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\}) | \psi(\{\lambda\}) \rangle}. \quad (3.1)
\]

In this expression, $|\psi(\{\lambda\})\rangle$ denotes the ground state of the Hamiltonian (1.1) in the subspace $\mathcal{H}(M/2-N)$. Note however that all the results we present here for the finite chain remain valid for $\{\lambda\}$ being more generally any admissible off-diagonal solution of the system (2.15).

Like in (1), it is convenient to derive this correlation function from a special generating function. Indeed, let us consider the following time-dependent operator:

\[
Q_{l+1, m}^\kappa(t) = T_{\kappa}^l \left( \frac{\eta}{2} \right) \cdot T_{\kappa}^{m-l} \left( \frac{\eta}{2} \right) \cdot e^{itH_0^0} \cdot T_{\kappa}^m \left( \frac{\eta}{2} \right) \cdot e^{-itH_0^0}, \quad (3.2)
\]

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with $H^{(0)}_\kappa$ defined similarly as in (3.5):

$$H^{(0)}_\kappa = 2 \sinh \eta \frac{d T_\kappa(\lambda)}{d \lambda} T_\kappa^{-1}(\lambda) \bigg|_{\lambda=\frac{\eta}{2}} - 2M \cosh \eta. \quad (3.3)$$

Using the expression (2.4) for the reconstruction of the local spin operators in terms of the entries of the monodromy matrix and the fact that the twisted transfer matrix $T_\kappa$ commutes with $H^{(0)}_\kappa$, it is easy to see that

$$\frac{1 - \sigma_{i+1}^z(0)}{2} \cdot \frac{1 - \sigma_{m+1}^z(t)}{2} = T^i \left( \frac{\eta}{2} \right) \cdot D \left( \frac{\eta}{2} \right) \cdot T^{-i} \left( \frac{\eta}{2} \right) \cdot e^{itH^{(0)}} \times D \left( \frac{\eta}{2} \right) \cdot T^{-i} \left( \frac{\eta}{2} \right) \cdot e^{-itH^{(0)}}, \quad (3.4)$$

$$= \frac{1}{2} \frac{\partial^2}{\partial \kappa^2} (Q^\kappa_{i+1,m+1} - Q^\kappa_{i+1,m} - Q^\kappa_{i+2,m+1} + Q^\kappa_{i+2,m})(t) \big|_{\kappa=1}. \quad (3.5)$$

Due to the translational invariance of the correlation functions, we can set $l = 0$ and simply consider the following expectation value:

$$Q_\kappa(m, t) = \frac{\langle \psi(\{\lambda\})| Q^\kappa_{1,m}(t) |\psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\})|\psi(\{\lambda\}) \rangle}. \quad (3.6)$$

In terms of this generating function, the time-dependent correlation function (3.1) is thus given as

$$\langle \sigma_i^z(0) \sigma_{m+1}^z(t) \rangle = 2 \langle \sigma_i^z(0) \rangle - 1 + 2D_m^2 \frac{\partial^2}{\partial \kappa^2} Q_\kappa(m, t) \bigg|_{\kappa=1}, \quad (3.7)$$

where $D_m^2$ denotes the second lattice derivative defined as in (3.5).

Like in the time-independent case, it is possible to derive a master equation for the generating function (3.6) in the finite chain. Indeed, we have the following result:

**Theorem 3.1.** Let $\{\lambda_1, \ldots, \lambda_N\}$ be an admissible off-diagonal solution of the system (2.12). Then there exists $\kappa_0 > 0$ such that, for $0 < |\kappa| < \kappa_0$, the generating function $Q_\kappa(m, t)$ (3.6) in the finite XXZ chain (1.1) is given by the multiple contour integral

$$Q_\kappa(m, t) = \frac{1}{N!} \oint_{\Gamma(\pm \frac{\eta}{2}) \cup \Gamma(\lambda)} \prod_{j=1}^N \frac{dz_j}{2\pi i} \cdot \prod_{b=1}^N e^{it(E(z_b) - E(\lambda_b)) + im(p(z_b) - p(\lambda_b))}$$

$$\times \prod_{a,b=1}^N \sinh^2(\lambda_a - z_b) \cdot \det_N \left( \frac{\partial \tau_a(\lambda_j)}{\partial \lambda_j} \right) \cdot \det_N \left( \frac{\partial \tau_z(\lambda_j)}{\partial \lambda_j} \right) \cdot \det_N \left( \frac{\partial \psi(\lambda_j)}{\partial \lambda_j} \right). \quad (3.8)$$

In this expression, $E(\lambda)$ and $p(\lambda)$ denote respectively the bare one-particle energy and momentum (2.13) and (2.14); the integration contour is such that the only singularities of the integrand (3.8) within $\Gamma(\pm \frac{\eta}{2}) \cup \Gamma(\lambda)$ which contribute to the integral are the points $\{\pm \frac{\eta}{2}\}$ and $\{\lambda\}$.
Remark 1. The master equation \(3.8\) gives the expectation value \(Q_\kappa(m,t)\) with respect to an arbitrary eigenstate \(|\psi(\{\lambda\})\rangle\) corresponding to any admissible off-diagonal solution \(\{\lambda_1, \ldots, \lambda_N\}\) of \((2.15)\). In particular one can choose \(\{\lambda\}\) such that \(|\psi(\{\lambda\})\rangle\) is the ground state of the XXZ Hamiltonian.

Remark 2. The master equation \(3.8\) provides an integral representation of \(Q_\kappa(m,t)\) which is valid at least in a punctured vicinity of \(\kappa = 0\). On the other hand, it is clear from \((3.2)\) and \((3.6)\) that \(Q_\kappa(m,t)\) is an analytical function of \(\kappa\) for \(\kappa \in \mathbb{C} \setminus \{0, \infty\}\). Hence, the representation \(3.8\) can be analytically continued from any vicinity of the origin to the whole complex plane except \(\kappa = 0, \infty\). This does not mean, however, that one can set \(\kappa\) to be an arbitrary specific value directly in the integrand of \(3.8\).

Remark 3. The time and space dependencies in the representation \(3.8\) appear in a very suggestive way, namely through the exponent of the bare energy and momentum. Note however that they do not correspond \(a\ priori\) to any eigenstate since here \(z_j\)'s are integration variables; nevertheless, the contribution of each point in the integration domain is measured in particular by the difference between the bare energy and momentum corresponding to the integration variables \(z_j\) and the one corresponding to the ground state parameters \(\lambda_j\).

As we already discussed in the Introduction, there are two possible ways to prove Theorem 3.1.

The first one, which follows the main strategy of \([1]\), consists in acting with \(Q_{1,m}^0(t)\) on the state \(\langle \psi(\{\lambda\}) |\), in computing the resulting scalar products and in rewriting the sum over partitions that follows as a single multiple integral of Cauchy type. In the next section, we explain how this approach, elaborated in \([1]\), can be applied in the time-dependent case in order to prove Theorem 3.1.

The second one was already announced in the conclusion of \([1]\): the master equation \(3.8\) can be obtained directly via a form factor expansion. Details are presented in Section 5.

The reader who is not interested in the technical details of these derivations can skip the next two sections and go directly to Section 6 where a multiple integral representation of \(3.6\) in the thermodynamic limit is deduced from \(3.8\).

4 Master equation via multiple action of transfer matrices

In this section, we prove Theorem 3.1 using the method developed in \([15]\) and \([1]\). In order to apply this method, we need first to reduce the computation of the generating function \(3.6\) to the evaluation of the expectation value of some product of twisted transfer matrices. The idea is to reconstruct the operator \(\exp(itH_\kappa^{(0)}(0))\) in \(3.2\) as a limit similar to \(1.6\): 

\[
e^{it(H_\kappa^{(0)} + 2M \cosh \eta)} = \lim_{L \to \infty} \left( T_\kappa \left( \frac{\eta}{2} + \varepsilon \right) \cdot T_\kappa^{-1} \left( \frac{\eta}{2} \right) \right)^L, \quad \varepsilon = \frac{1}{L} 2it \sinh \eta, \quad (4.1)
\]
and to use the fact that, for the specific value $\lambda = \eta/2$, the inverse operator $\mathcal{T}_\kappa^{-1}(\eta/2)$ is proportional to $\mathcal{T}_\kappa(-\eta/2)$:

$$\mathcal{T}_\kappa^{-1}(\eta/2) = \left(\kappa a(\eta/2)d(-\eta/2)\right)^{-1} \mathcal{T}_\kappa(-\eta/2).$$

This enables us to express the operator $Q^\kappa_{1,m}(t)$ as the limit

$$Q^\kappa_{1,m}(t) = \lim_{L \to \infty} \mathcal{T}_\kappa^L\left(\frac{\eta}{2} + \varepsilon\right) \cdot \mathcal{T}_\kappa^{m-L}\left(\frac{\eta}{2}\right) \cdot \mathcal{T}_\kappa^{L-m}\left(\frac{\eta}{2}\right) \cdot \mathcal{T}_\kappa^{-L}\left(\frac{\eta}{2} + \varepsilon\right),$$

and

$$Q^\kappa_{1,m}(t) = \lim_{L \to \infty} \mathcal{T}_\kappa^{m-L} \cdot \mathcal{T}_\kappa^L\left(\frac{\eta}{2} + \varepsilon\right) \cdot \mathcal{T}_\kappa^{L-m}\left(\frac{\eta}{2}\right) \cdot \mathcal{T}_\kappa^{-L}\left(\frac{\eta}{2} + \varepsilon\right).$$

Acting with the product of (untwisted) inverse transfer matrices on the eigenstate $|\psi(\{\lambda\})\rangle$, we obtain a representation of the generating function \[Q(x, t)\] in terms of an expectation value of a product of twisted transfer matrices:

$$Q^\kappa_{(m, t)} = \lim_{L \to \infty} \mathcal{T}_\kappa^{m-L} \tau^{m-L}(-\eta/2|\{\lambda\}) \tau^{-L}(\eta/2 + \varepsilon|\{\lambda\}) \mathcal{T}_\kappa^L(\eta/2 + \varepsilon) \mathcal{T}_\kappa^{L-m}(-\eta/2).$$

It is convenient at this stage to introduce some arbitrary parameters $\omega_1, \ldots, \omega_{2L-m}$ and to define $(x_1, \ldots, x_{2L-m}) = (\omega_1 + \varepsilon, \ldots, \omega_L + \varepsilon, \omega_{L+1} - \eta, \ldots, \omega_{2L-m} - \eta)$. It follows that

$$Q^\kappa_{(m, t)} = \lim_{L \to \infty} \lim_{\omega \to \frac{\eta}{2}} \mathcal{T}_\kappa^{m-L} \quad \prod_{a=1}^{2L-m} \tau^{-1}(x_a|\{\lambda\}) \prod_{a=1}^{2L-m} \mathcal{T}_\kappa(x_a).$$

As soon as we have the representation \[Q_{(m, t)}\], we can use directly the equation \[Q_{(m, t)}\] in order to obtain a multiple integral representation for the expectation value \[\langle \prod \mathcal{T}_\kappa(x_a) \rangle\]. It leads to

$$Q^\kappa_{(m, t)} = \lim_{L \to \infty} \lim_{\omega \to \frac{\eta}{2}} \mathcal{T}_\kappa^{m-L} \quad \frac{1}{N!} \oint_{\Gamma(x)} \frac{dz_1}{2\pi i} \cdots \frac{dz_N}{2\pi i} \quad \prod_{a=1}^{2L-m} \tau(x_a|\{z\}) \prod_{a=1}^{2L-m} \mathcal{T}_\kappa(x_a|\{\lambda\})$$

$$\times \prod_{a=1}^{N} \frac{1}{\gamma^\kappa(z_{\lambda}|\{z\})} \cdot \det \Omega^\kappa(\{z\}, \{\lambda\}) \det \Omega^\kappa(\{\lambda\}, \{z\}) \frac{\det N \Omega(\{\lambda\}, \{z\}|\{\lambda\}) \det N \Omega(\{\lambda\}, \{z\}|\{\lambda\})}{\det N \Omega(\{\lambda\}, \{z\}|\{\lambda\})},$$

where the closed contour $\Gamma(x) \cup \Gamma(\lambda)$ surrounds the points $x_1, \ldots, x_{2L-m}$ and $\lambda_1, \ldots, \lambda_N$ and does not contain any other singularities of the integrand. Formally the integrand in the equation \[Q_{(m, t)}\] coincides exactly with its time-independent analogue in \[Q_{(m, t)}\]. The difference, however, is that the number of parameters $x_j$ eventually becomes infinite, and that one has to take the

\[^1\text{One can easily obtain the equation} (4.16)\text{by applying the product} \mathcal{T}_\kappa(\eta/2)\mathcal{T}_\kappa(-\eta/2)\text{to an arbitrary state} \phi.\]

\[^2\text{More precisely,} \Gamma(x) \cup \Gamma(\lambda)\text{is the boundary of a set of polydisks} D(x_a, r) \text{and} D(\lambda_b, r)\text{in} \mathbb{C}^N.\text{Namely,} \Gamma(x) = \bigcup_{a=1}^{2L-m} D(x_a, r), \text{where} D(x_a, r) = \{z \in \mathbb{C}^N : |z_k - x_a| = r, \ a = 1, \ldots, N\}. \text{The integration contour} \Gamma(\lambda)\text{is defined in a similar manner. The radius} r\text{is supposed to be small enough.}\]
homogeneous limit \( \omega_j \to \eta/2 \) before the limit \( L \to \infty \). In this limit, some of the \( x_j \) go to \( \eta/2 + \varepsilon \), while the remaining ones tend to \( -\eta/2 \). Thus, to be able to proceed to this limit, one has to verify that all the solutions of the system

\[
Y_\kappa(z_j|\{z\}) \equiv a(z_j) \prod_{k=1}^{N} \sinh(z_k - z_j + \eta) + \kappa d(z_j) \prod_{k=1}^{N} \sinh(z_k - z_j - \eta) = 0 \quad (4.8)
\]

which contribute to the integral (4.7) are actually separated from the points \( \eta/2 + \varepsilon \), \( -\eta/2 \) and the parameters \( \{\lambda\} \). Note that, thanks to the analyticity of \( \langle Q_\kappa^{1, m} \rangle \) for \( \kappa \in \mathbb{C} \setminus \{0, \infty\} \), it is sufficient to evaluate this generating function in the punctured vicinity of \( \kappa = 0 \). Observe also that one can set \( \eta/2 + \varepsilon \) to be as close to \( \eta/2 \) as necessary, since eventually we have to proceed to the limit \( L \to \infty \).

The analysis of the solutions of the system (4.8) performed in the article [1] was based on the results of [34]. However, most of those results were formulated in the case of inhomogeneous Bethe equations

\[
\prod_{a=1}^{M} \sinh^{\kappa}(z_j - \xi_a + \eta) \prod_{k=1}^{N} \sinh(z_k - z_j + \eta) - \kappa \prod_{a=1}^{M} \sinh^{\kappa}(z_j - \xi_a) \prod_{k=1}^{N} \sinh(z_k - z_j - \eta) = 0, \quad (4.9)
\]

for generic\(^3\) parameters \( \xi_1, \ldots, \xi_M \). We assume that all solutions \( z_j(\kappa) \) of the homogeneous system (4.8) can be obtained from the solutions \( z_j(\kappa|\{\xi\}) \) of the inhomogeneous system (4.9) in the limit \( \xi_a \to \eta/2 \). Then, just like in the time-independent case, unadmissible and diagonal solutions of the system (4.8) do not contribute to the integral (4.7), for \( \det \Omega_\kappa \) or \( \det \Omega \) vanishes at these points, and the only solutions that actually contribute are admissible off-diagonal ones. Due to Lemma A.1, all of them are separated from the points \( \pm \eta/2 \) (and hence from \( \eta/2 + \varepsilon \) as well, for \( \varepsilon \) small enough), at least in a vicinity of \( \kappa = 0 \) as far as \( \kappa \neq 0 \). For \( |\kappa| \) small enough, they are also obviously separated from the parameters \( \{\lambda\} \), which correspond to an admissible off-diagonal solution at \( \kappa = 1 \). Thus, we can formulate

**Lemma 4.1.** Let \( \{\lambda\} \) be an admissible off-diagonal solution to (2.15), and \( \omega_j \to \eta/2 \). There exists \( \kappa_0 > 0 \) such that, for \( 0 < |\kappa| < \kappa_0 \), one can define a closed contour \( \Gamma \{\frac{\eta}{2}\} \cup \Gamma \{-\frac{\eta}{2}\} \cup \Gamma \{\lambda\} \) which satisfies the following properties:

1) it surrounds the points \( \eta/2, -\eta/2 \) and \( \{\lambda\} \), while all admissible off-diagonal solutions of the system (4.8) are outside of this contour;

2) for \( L \) large enough, the only poles which are inside and provide non-vanishing contribution to the integral (4.7) are \( z_j = \eta/2 + \varepsilon \), \( z_j = -\eta/2 \) and \( z_j = \lambda_k \);

3) for \( L \) large enough, the only poles which are outside (within a set of strips of width \( i\pi \)) and provide non-vanishing contribution to the integrand of (4.7) are the admissible off-diagonal solutions of the system (4.8).

\(^3\)Or at least ‘well separated’ (see [34] for definition).
Thanks to this Lemma, we can now take the limit \( \omega_j \to \eta/2 \) in (4.7). It is easy to see that
\[
\lim_{\omega_\alpha \to \eta/2} \left( \kappa^{m-L} \prod_{a=1}^{2L-m} \frac{\tau_\kappa(x_a|\{z\})}{\tau(x_a|\{\lambda\})} \right) = \prod_{b=1}^{N} \left( \frac{\sinh(z_b + \frac{\eta}{2} - \varepsilon)}{\sinh(z_b - \frac{\eta}{2} - \varepsilon)} \right)^L \cdot \left( \frac{\sinh(z_b - \frac{\eta}{2} + \varepsilon)}{\sinh(z_b + \frac{\eta}{2} - \varepsilon)} \right)^{L-m} \cdot \left[ 1 + \frac{\sinh^M \varepsilon}{\sinh^M (\eta + \varepsilon)} \prod_{b=1}^{N} \left( \frac{\sinh(\lambda_b - \frac{3\eta}{2} - \varepsilon)}{\sinh(\lambda_b + \frac{3\eta}{2} - \varepsilon)} \right) \right]^{-L} \cdot \left[ 1 + \frac{\kappa \sinh^M \varepsilon}{\sinh^M (\eta + \varepsilon)} \prod_{b=1}^{N} \left( \frac{\sinh(z_b - \frac{3\eta}{2} - \varepsilon)}{\sinh(z_b + \frac{3\eta}{2} - \varepsilon)} \right) \right]^L.
\]
(4.10)

As the contour defined in Lemma 4.1 is independent of \( L \), one can also safely take the limit \( L \to \infty \) in the integrand. Using the definition of the bare one-particle energy \( E(\lambda) \) and momentum \( p(\lambda) \) (2.13), (2.14), one obtains
\[
\lim_{L \to \infty} \lim_{\omega_\alpha \to \eta/2} \left( \kappa^{m-L} \prod_{a=1}^{2L-m} \frac{\tau_\kappa(x_a|\{z\})}{\tau(x_a|\{\lambda\})} \right) = \prod_{b=1}^{N} \exp \left( it(E(z_b) - E(\lambda_b)) + im(p(z_b) - p(\lambda_b)) \right), \quad (4.11)
\]
which leads to the following multiple integral representation for the generating function \( Q_\kappa(m, t) \) in the finite chain:
\[
Q_\kappa(m, t) = \frac{1}{N!} \oint_{\Gamma(\frac{\eta}{2})} \prod_{j=1}^{N} \frac{dz_j}{2\pi i} \prod_{b=1}^{N} e^{it(E(z_b) - E(\lambda_b)) + im(p(z_b) - p(\lambda_b))} \times \prod_{a=1}^{N} \frac{1}{\Omega_\kappa(x_a|\{z\})} \cdot \det \Omega_\kappa(\{z\}, \{\lambda\}|\{\lambda\}) \cdot \frac{\det_N \Omega(\{\lambda\}, \{z\}|\{\lambda\})}{\det_N \Omega(\{\lambda\}, \{\lambda\}|\{\lambda\})}, \quad (4.12)
\]

It remains to use (2.19)–(2.22) and we obtain the master equation (3.8).

## 5 Master equation via form factor expansion

The master equation (3.8) can be obtained via the form factor expansion for the expectation value \( Q_\kappa(m, t) \). This way is of course much shorter, for it is in particular not necessary to construct the generating function as a special limit of a product of twisted transfer matrices. However, the summation over form factors requires the completeness of the set of eigenstates of \( T_\kappa \) (see Theorem A.1).

Inserting in (3.6) the complete set of the eigenstates \( |\psi_\kappa(\{\mu\})\rangle \) of the twisted transfer matrices between the operators \( T_\kappa^m(\frac{\eta}{2}) \cdot \exp(itH_0^{(0)}) \) and \( T^{-m}(\frac{\eta}{2}) \cdot \exp(-itH^{(0)}) \) (see (3.2)), one
obtains
\[ Q_\kappa(m, t) = \sum_{\{\mu\}} \prod_{b=1}^N e^{it(E(\mu_b) - E(\lambda_b)) + im(p(\mu_b) - p(\lambda_b))} \frac{\psi(\{\lambda\}) |\psi_\kappa(\{\mu\})\rangle}{\langle \psi_\kappa(\{\mu\}) |\psi(\{\lambda\})\rangle}, \tag{5.1} \]
where the sum is taken with respect to all admissible off-diagonal solutions \(\{\mu_1, \ldots, \mu_N\}\) (such that \(-\pi/2 < \Im(\mu_j) \leq \pi/2\)) of the system (2.6). As usual, we suppose \(|\kappa|\) small enough, but not zero. We have also used that \(i \sum_{b=1}^N p(\lambda_b) = \log \tau_\kappa(\eta/2|\{\lambda\}) - M \log \sinh \eta, \tag{5.2} \)
\(H_\kappa^{(0)} |\psi_\kappa(\{\lambda\})\rangle = \left( \sum_{j=1}^N E(\lambda_j) \right) |\psi_\kappa(\{\lambda\})\rangle, \tag{5.3} \)
which follows from (2.9), (2.14) and (3.3). The scalar products in (5.1) can be written in terms of Jacobians via (2.19)–(2.22):
\[ Q_\kappa(m, t) = (-1)^N \sum_{\{\mu\}} \prod_{a,b=1}^N \sinh^2(\lambda_a - \mu_b) \cdot \prod_{b=1}^N e^{it(E(\mu_b) - E(\lambda_b)) + im(p(\mu_b) - p(\lambda_b))} \]
\[ \times \frac{\det N \left( \frac{\partial \tau_\kappa(\lambda_j|\{\mu\})}{\partial \lambda_k} \right)}{\det N \left( \frac{\partial \mu_b(\lambda_j|\{\mu\})}{\partial \lambda_j} \right)} \cdot \frac{\det N \left( \frac{\partial \tau_\kappa(\lambda_j|\{\mu\})}{\partial z_k} \right)}{\det N \left( \frac{\partial \lambda_b(\lambda_j|\{\mu\})}{\partial \lambda_j} \right)}. \tag{5.4} \]
This last sum can be presented as a contour integral in \(\mathbb{C}^N\),
\[ Q_\kappa(m, t) = \frac{(-1)^N}{N!} \oint_{\Gamma(\mu)} \prod_{j=1}^N \frac{dz_j}{2\pi i} \prod_{b=1}^N e^{it(E(z_b) - E(\lambda_b)) + im(p(z_b) - p(\lambda_b))} \]
\[ \times \prod_{a,b=1}^N \sinh^2(\lambda_a - z_b) \cdot \frac{\det N \left( \frac{\partial \tau_\kappa(\lambda_j|z)}{\partial z_k} \right)}{\det N \left( \frac{\partial \lambda_b(z)}{\partial \lambda_j} \right)} \cdot \frac{\det N \left( \frac{\partial \tau_\kappa(\lambda_j|z)}{\partial z_k} \right)}{\det N \left( \frac{\partial \lambda_b(z)}{\partial \lambda_j} \right)}. \tag{5.5} \]
where the contour \(\Gamma(\mu)\) surrounds all admissible off-diagonal solutions of the system (4.8) and does not contain the points \(\pm \eta/2\) and \(\{\lambda\}\). The existence of such a contour follows from Lemma 4.1. The factor \(1/N!\) appears due to the invariance of the off-diagonal solutions with respect to the permutations of \(\{\lambda\}\).

Let us observe finally that the integrand in (5.5) is \(i\pi\)-periodic with respect to each \(z_k\), and that it vanishes as soon as any \(z_k \to \pm \infty\). Hence, the sum of the residues of the integrand within a set of strips of width \(i\pi\) is equal to zero. One can therefore evaluate the integral over the contour \(\Gamma(\mu)\) by taking the residues outside this contour within this set of strips, i.e. inside the contour \(\Gamma(\pm \frac{\eta}{2}) \cup \Gamma(\lambda)\). This leads to the master equation (3.5).
6 Thermodynamic limit

Starting from this section the parameters $\lambda_1, \ldots, \lambda_N$ correspond to the ground state $|\psi(\{\lambda\})\rangle$ of the Hamiltonian in the subspace $\mathcal{H}^{(M/2-N)}$. This means in particular that $\Im \lambda_j = 0$ if $\Re \eta = 0$, and $\Re \lambda_j = 0$ if $\Im \eta = 0$. Recall also that it is enough to consider the case $h \geq 0$, what implies $N \leq M/2$.

In [1] we showed how to obtain the thermodynamic limit of the generating function $Q_\kappa(m,0)$ from the master equation by evaluating explicitly the integrals over $\Gamma\{\lambda\}$ and setting $d(z_k) = 0$ in the remaining integrals. A similar method applies to the time-dependent case, although the existence of the essential singularities at $\pm \eta/2$ in the integrand makes this procedure more subtle.

Let us start with the representation (4.12). Due to the symmetry of the integrand with respect to the set $\{z\}$ we have

$$
\mathcal{G}_{\kappa}(m, t) = \sum_{n=0}^{N} \frac{1}{n!} \sum_{(\lambda)_{\left|\alpha_+\right| = n}} \oint_{\Gamma\{\pm \frac{\eta}{2}\}} \prod_{j=1}^{n} dz_j = \sum_{n=0}^{N} C_N^n \oint_{\Gamma\{\pm \frac{\eta}{2}\}} \prod_{j=1}^{n} dz_j \oint_{\Gamma\{\lambda\}} \prod_{j=n+1}^{N} dz_j.
$$

(6.1)

In this expression, the integral over the contour $\Gamma\{\lambda\}$ can be rewritten as the sum over partitions of the set $\{\lambda\}$ into two disjoint subsets $\{\lambda\} = \{\lambda_{\alpha+}\} \cup \{\lambda_{\alpha-}\}$, with $\#\{\lambda_{\alpha+}\} = n$:

$$
Q_\kappa(m, t) = \sum_{n=0}^{N} \frac{1}{n!} \sum_{(\lambda)_{\left|\alpha_+\right| = n}} \oint_{\Gamma\{\pm \frac{\eta}{2}\}} \prod_{j=1}^{n} dz_j \cdot \prod_{b=1}^{n} e^{itE(z_b) + \text{imp}(z_b)} \prod_{b \in \alpha_-} e^{-itE(\lambda_b) - \text{imp}(\lambda_b)}
$$

$$
\times \det_n \Omega\kappa(\{\lambda\}, \{\lambda_{\alpha+}\} \cup \{\lambda_{\alpha-}\}) \cdot \det_{N} \Omega(\{\lambda\}, \{\lambda_{\alpha+}\} \cup \{\lambda_{\alpha-}\} | | \{\lambda\}), \quad \text{det} \Omega(\{\lambda\}, \{\lambda_{\alpha+}\} \cup \{\lambda_{\alpha-}\} | | \{\lambda\}), \quad \text{det} \Omega(\{\lambda\}, \{\lambda_{\alpha+}\} \cup \{\lambda_{\alpha-}\} | | \{\lambda\}),
$$

(6.2)

where the elements in the sets $\{z\} \cup \{\lambda_{\alpha-}\}$ and $\{\lambda_{\alpha+}\} \cup \{\lambda_{\alpha-}\}$ are ordered accordingly. Here we have used that

$$
\text{Res}_{\{z_n, \ldots, z_N\} = \{\lambda_{\alpha-}\}} \left[ \frac{\det \Omega(\{z_1, \ldots, z_N\}, \{\lambda_{\alpha+}\} \cup \{\lambda_{\alpha-}\} | | \{z_1, \ldots, z_N\})}{\prod_{a=1}^{n} \mathcal{Y}_\kappa(\{z\} \cup \{\lambda_{\alpha-}\})} \cdot \det \Omega(\{z_1, \ldots, z_n\}, \{\lambda_{\alpha+}\} | | \{z_1, \ldots, z_n\} \cup \{\lambda_{\alpha-}\}) \right] = \prod_{a \in \alpha_-} \mathcal{Y}_\kappa(\{z\} \cup \{\lambda_{\alpha-}\}) \cdot \det \Omega(\{z_1, \ldots, z_n\}, \{\lambda_{\alpha+}\} | | \{z_1, \ldots, z_n\} \cup \{\lambda_{\alpha-}\}).
$$

(6.3)

Using now the system of Bethe equations (2.15) for variables $\{\lambda\}$ we can present the integral of (6.2) as follows

$$
\frac{\det_n \Omega\kappa(\{\lambda\}, \{\lambda_{\alpha+}\} | | \{\lambda\} \cup \{\lambda_{\alpha-}\})}{\prod_{a=1}^{n} \mathcal{Y}_\kappa(\{z\} \cup \{\lambda_{\alpha-}\})} = \frac{\det_n \Omega(\{\lambda\}, \{\lambda_{\alpha+}\} | | \{\lambda\})}{\det_n \Omega(\{\lambda\}, \{\lambda_{\alpha+}\} \cup \{\lambda_{\alpha-}\} | | \{\lambda\})} = \frac{\prod_{b=1}^{n} \mathcal{M}_\kappa(\{\lambda_{\alpha+}\}, \{z\})}{\det_n \mathcal{Y}_\kappa(\{z\} \cup \{\lambda_{\alpha-}\})}
$$

$$
\times \frac{\prod_{b=1}^{n} \prod_{a \in \alpha_+} \sinh(\lambda_a - z_b + \eta) \sinh(z_b - \lambda_a + \eta)}{\prod_{a \in \alpha_+} \sinh(\lambda_a - \lambda_{b-} + \eta) \prod_{a, b=1}^{n} \sinh(z_a - z_b + \eta)}
$$

$$
\times \frac{\det_n \mathcal{Y}_\kappa(\{z\} \cup \{\lambda_{\alpha-}\})}{\det_n \mathcal{Y}_\kappa(\{\lambda\} \cup \{\lambda_{\alpha-}\})}.
$$

(6.4)
\[(\tilde{M}_k)_{jk}(\{\lambda_{a+}\}, \{z\}) = t(z_k, \lambda_j) + \kappa t(\lambda_j, z_k) \prod_{a \in a_+} \frac{\sinh(\lambda_a - \lambda_j + \eta)}{\sinh(\lambda_j - \lambda_a + \eta)} \prod_{a=1}^n \frac{\sinh(\lambda_j - z_a + \eta)}{\sinh(z_a - \lambda_j + \eta)} \] (6.5)

and
\[
\Phi'_{jk}(\{\lambda\}) = \delta_{jk} \left[ \frac{d}{d\lambda} \log \frac{d(\lambda)}{a(\lambda)} \right]_{\lambda=\lambda_j} - \sum_{a=1}^N K(\lambda_j - \lambda_a) + K(\lambda_j - \lambda_k), \] (6.6)

with
\[
K(\lambda) = \frac{\sinh 2\eta}{\sinh(\lambda - \eta) \sinh(\lambda + \eta)}. \] (6.7)

In \(\Phi'(\{\lambda_{a+}\} \cup \{\lambda_{a-}\})\), the columns are ordered such that the \(n\) first ones correspond to the subset \(\{\lambda_{a+}\}\) and the \(N-n\) last ones to \(\{\lambda_{a-}\}\). The matrix \(\Psi'\) has a more complicated structure. For \(k > n\) its entries \(\Psi'_{jk}\) coincide with the corresponding entries \(\Phi'_{jk}\) in the \(N-n\) last columns of \(\Phi'(\{\lambda_{a+}\} \cup \{\lambda_{a-}\})\). For the first \(n\) columns one has
\[
\Psi'_{jk} = \frac{a(z_k)t(\lambda_j, z_k) - d(z_k)t(z_k, \lambda_j) \prod_{a=1}^n \frac{\sinh(z_k - \lambda_a + \eta)}{\sinh(z_k - \lambda_a - \eta)} \prod_{b \in a_+} \frac{\sinh(z_k - \lambda_b + \eta)}{\sinh(z_k - \lambda_b - \eta)}}{a(z_k) + \kappa d(z_k) \prod_{b=1}^n \frac{\sinh(z_k - z_b + \eta)}{\sinh(z_k - z_b - \eta)} \prod_{b \in a_+} \frac{\sinh(z_k - \lambda_b + \eta)}{\sinh(z_k - \lambda_b - \eta)}}. \quad k \leq n \] (6.8)

In the thermodynamic limit \(M, N \to \infty, M/N = \text{const}\) this expression can be simplified. Indeed, since \(d(z)\) and \(a(z)\) have zeros of order \(M\) at \(z = \eta/2\) and \(z = -\eta/2\) respectively, one can show that the contributions of the corresponding terms to the total result are bounded by \(\cN^N/N!\). Therefore, at \(M, N \to \infty\), one can set \(d(z) = 0\) if \(z\) is in the vicinity of \(\eta/2\) and \(a(z) = 0\) if \(z\) is in the vicinity of \(-\eta/2\). This gives us a simplified representation for the matrix elements \(6.8\) for \(k \leq n\) in the thermodynamic limit:
\[
\lim_{M \to \infty} \Psi'_{jk} = \tilde{\Psi}'_{jk} = \begin{cases} t(\lambda_j, z_k), & \text{if } z_k \sim \frac{\eta}{2}; \\ -\kappa^{-1}t(z_k, \lambda_j) \prod_{a \in a_+} \frac{\sinh(z_k - \lambda_a + \eta)}{\sinh(z_k - \lambda_a - \eta)} \prod_{b=1}^n \frac{\sinh(z_k - \lambda_b + \eta)}{\sinh(z_k - \lambda_b - \eta)}, & \text{if } z_k \sim -\frac{\eta}{2}. \end{cases} \]

In fact we can say that the limiting value \(\tilde{\Psi}'_{jk}\) has a cut between the points \(\eta/2\) and \(-\eta/2\).

The remaining steps are quite standard (see \[11, 13\]). In the thermodynamic limit the distribution of the ground state parameters \(\{\lambda\}\) can be described by the spectral density \(\rho_{\text{tot}}(\lambda)\). In its turn the spectral density is a particular case of the ‘inhomogeneous spectral density’: \(\rho_{\text{tot}}(\lambda) = \rho(\lambda, z)|_{z=\eta/2}\). This inhomogeneous spectral density satisfies an integral equation
\[
-2\pi i \rho(\lambda, z) + \int_C K(\lambda - \mu)\rho(\mu, z)\,d\mu = t(\lambda, z), \] (6.9)

where the integration contour \(C\) in \(6.9\) depends on the phase of the model. In the massless case \(-1 \leq \Delta \leq 1\) the contour \(C\) is an interval of the real axis \([-\Lambda_h, \Lambda_h]\). The boundary \(\Lambda_h\)
depends on the value of the magnetic field, in particular \( \Lambda_h \to \infty \) at \( h \to 0 \). For \( \Delta > 1 \) (\( \eta < 0 \)) the limits \( \pm \Lambda_h \) are purely imaginary, more precisely the integral in \([39]\) is taken over an interval of the imaginary axis. In particular \( \Lambda_h = -i\pi/2 \) at \( h = 0 \).

In the thermodynamic limit, one can compute the ratio of the determinants \( \det \tilde{\Psi}' \) and \( \det \Phi' \) in terms of the inhomogeneous density. Indeed, since the last \( N - n \) columns of the matrix \( \Phi' \) coincide with the ones of the matrix \( \Phi'({\{\lambda_{\alpha_+}\}} \cup {\{\lambda_{\alpha_-}\}}) \), we have

\[
\begin{equation}
\frac{\det_N \tilde{\Psi}'({\{z\}, {\{\lambda_{\alpha_-}\}} | {\{\lambda_{\alpha_+}\}}})}{\det_N \Phi'({\{\lambda_{\alpha_+}\}} \cup {\{\lambda_{\alpha_-}\}})} = \det \left( \sum_{i=1}^{N} (\Phi')^{-1}_{ji} \tilde{\Psi}'_{ik} \right) .
\end{equation}
\]

Using the results of \([14]\) we have, for \( z \) in a vicinity of \( \eta/2 \),

\[
\begin{equation}
\sum_{i=1}^{N} (\Phi')^{-1}_{ji} t(\lambda_i, z) \to \frac{\rho(\lambda_j, z)}{M_{\text{tot}}(\lambda_j)}, \quad M \to \infty.
\end{equation}
\]

If \( z \) is in a vicinity of \(-\eta/2\), we can set \( z = \tilde{z} - \eta \) which gives \( t(z, \lambda_j) = t(\lambda_j, \tilde{z}) \), hence

\[
\begin{equation}
\sum_{i=1}^{N} (\Phi')^{-1}_{ji} t(\lambda_i, \lambda_j) \to \frac{\rho(\lambda_j, \tilde{z})}{M_{\text{tot}}(\lambda_j)} = \frac{\rho(\lambda_j, z + \eta)}{M_{\text{tot}}(\lambda_j)}, \quad M \to \infty.
\end{equation}
\]

Thus, in the thermodynamic limit \( M, N \to \infty, N/M = \text{const} \) one has

\[
\begin{equation}
\frac{\det_N \tilde{\Psi}'_{jk}(\{z\}, \{\lambda_{\alpha_-}\} | {\{\lambda_{\alpha_+}\}})}{\det_N \Phi'_{jk}(\{\lambda_{\alpha_+}\} \cup {\{\lambda_{\alpha_-}\}})} \to \det \left[ \mathcal{R}_n^\kappa(\lambda_j, z_k | \{\lambda_{\alpha_+}\}, \{z\}) \right],
\end{equation}
\]

where the function \( \mathcal{R}_n^\kappa(\lambda, z | \{\lambda_1, \ldots, \lambda_n\}, \{z_1, \ldots, z_n\}) \) is defined differently in the vicinities of \( \eta/2 \) and \(-\eta/2\):

\[
\begin{equation}
\mathcal{R}_n^\kappa(\lambda, z | \{\lambda\}, \{z\}) = \begin{cases} 
\rho(\lambda, z), & z \sim \eta/2; \\
\kappa^{-1} \rho(\lambda, z + \eta) \prod_{b=1}^{n} \frac{\sinh(z - \lambda_b + \eta) \sinh(z_b - z + \eta)}{\sinh(z - \lambda_b + \eta) \sinh(z - z_b + \eta)}, & z \sim -\eta/2.
\end{cases}
\end{equation}
\]

It remains to replace in \([6.2]\) the sum over partitions of \( \{\lambda\} \) by integrals over the support of the spectral density and we arrive at the multiple integral representation for the dynamical correlation function of the third components of spin:

\[
\begin{equation}
\langle \sigma^z_1(0) \sigma^z_{m+1}(t) \rangle = 2\langle \sigma^z_1(0) \rangle - 1 + 2D_m^2 \frac{\partial^2}{\partial \kappa^2} \mathcal{Q}_\kappa(m, t) \bigg|_{\kappa=1},
\end{equation}
\]

where

\[
\mathcal{Q}_\kappa(m, t) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \int_{-\Lambda_h}^{\Lambda_h} d^n \lambda \ \oint \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \ \prod_{a,b=1}^{n} \frac{\sinh(\lambda_a - z_b + \eta) \sinh(z_b - \lambda_a + \eta)}{\sinh(\lambda_a - \lambda_b + \eta) \sinh(z_a - z_b + \eta)}
\]

\[
\times \prod_{b=1}^{n} e^{it(E(z_b) - E(\lambda_b)) + i m(\rho(z_b) - \rho(\lambda_b))} \det \tilde{M}_n(\{\lambda\}, \{z\}) \cdot \det \mathcal{R}_n^\kappa(\lambda_j, z_k | \{\lambda\}, \{z\}).
\]

\[17\]
The contour $\Gamma(\pm \eta/\sqrt{2})$ surrounds the points $\pm \eta/\sqrt{2}$ and does not contain any other singularities of the integrand. The parameter $\kappa$ in (6.16) is an arbitrary complex different from $0, \infty$. The functions entering the integrand are defined in (2.13), (2.14), (6.5), (6.9) and (6.14).

Due to the factors $\exp(itE(z_0))$ the integrand in (6.16) has essential singularities in the points $\pm \eta/\sqrt{2}$. However, in the case $t=0$, these essential singularities disappear and the integrals around $\pm \eta/\sqrt{2}$ vanish. The remaining part of the integrand has poles of order $m$ at $z_j = \eta/\sqrt{2}$. Hence, at $t=0$ the sum over $n$ in (6.16) is actually restricted to $n \leq m$, and we reproduce the result of [15] for the equal-time correlation function of the third components of spin.

In Appendix B we explain how one can deduce from this expression the result obtained in [29] in the case of free fermions.

Using exactly the same method one can obtain an integral representation for the $\sigma^z$ correlation function for the partly inhomogeneous XXZ chain, where we associate a set of generic complex numbers $\xi_1, \ldots, \xi_m$ with the first $m$ sites of the chain. In this case one should replace in the result (6.16) the one-particle momenta by their natural inhomogeneous modifications

$$p_{\text{inh}}(\lambda) = \frac{i}{m} \sum_{k=1}^{m} \log \left( \frac{\sinh(\lambda - \xi_k)}{\sinh(\lambda - \xi_k + \eta)} \right).$$

(6.17)

The homogeneous case then corresponds to the limit $\xi_k = \eta/2$, $k = 1, \ldots, m$.

7 Conclusion

In this article, we have obtained a multiple integral representation for the dynamical $\sigma^z$ correlation function. It is clear, however, that the method based on the master equation can be applied to other dynamical correlation functions as well. In fact we have seen that the time-dependent master equation for the generating function $Q_\kappa(m,t)$ differs from its time-independent analogue only by the factors $\exp(it(E(z) - E(\lambda)))$, which automatically appear in the framework of the form factor expansions for the correlation functions. It is quite natural to expect that the same simple modification holds also for other correlation functions.

One interesting further development would be to obtain a generalization of the multiple integral representations for the dynamical correlation functions at finite temperature. A method to consider temperature correlation functions by algebraic Bethe ansatz was proposed recently in [20]. It is possible that this technique can be successfully combined with the approach used in this paper. In particular one obvious question is whether there exists also a master equation for the temperature-dependent case. It would lead also to the interesting question of the form factor expansion at non-zero temperature.

It is also well known that, for the case of free fermions $\Delta = 0$, the dynamical correlation functions of the XXZ chain satisfy difference-differential classical exactly solvable equations [35, 36, 37]. It is natural to wonder whether this property holds also for general $\Delta$, or at least for some specific cases. We hope that the multiple integral representations for the dynamical correlation functions open a way to study this problem.
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A Admissible solutions of the twisted Bethe equations

Let $e^{2z_j} = x_j$ and $e^n = q$. Then the system (A.1) takes the form

$$Y_\kappa(x_j|\{x\}) \equiv (x_j - q^{-1})^M \prod_{a=1 \atop a \neq j}^N (x_j - q^2 x_a) - \kappa q^{2N-2-M} (x_j - q)^M \prod_{a=1 \atop a \neq j}^N (x_j - q^{-2} x_a) = 0. \quad (A.1)$$

It is clear that, in the limit $\kappa \to 0$, all admissible solutions of (A.1) go to $q^{-1}$ and the Jacobian matrix $(\partial Y_\kappa(x_j|\{x\})/\partial x_k)$ has the rank zero at $\kappa = 0$ and $x_j = q^{-1}$. Our goal, however, is to solve the system (A.1) for $|\kappa|$ small enough, but not zero.

A simple example shows that, unlike in the inhomogeneous case, the solutions $x_j(\kappa)$ of (A.1) are not holomorphic functions at $\kappa = 0$. Indeed, let $q = i$ (free fermions). Since for admissible solutions $x_j + x_k \neq 0$, we obtain

$$\left(\frac{x_j + i}{x_j - i}\right)^M = \kappa e^{i\pi (N-1-M/2)}, \quad \text{for} \quad q = i. \quad (A.2)$$

This system has an obvious solution

$$x_j = \frac{\theta_j + 1}{\theta_j - 1}, \quad \text{where} \quad \theta_j = -i|\kappa|^\frac{1}{M} e^{i\pi (2n_j + N - 1)}, \quad n_j \in \{0, 1, \ldots, M - 1\}. \quad (A.3)$$

Thus, in this case, $x_j = x_j(\kappa^{1/M})$ and different choices of the branch of $\kappa^{1/M}$ correspond to different solutions.

One can treat the general case similarly.

**Lemma A.1.** There exists $\kappa_0 > 0$ such that, for $0 < |\kappa| < \kappa_0$, all admissible solutions of the system (A.1) belong to the vicinity of $q^{-1}$, but are separated from this point.

**Proof.** Let us make in (A.1) the substitution $x_j = q^{-1} + \theta u_j$, where $\theta$ is one of the solutions
of $\theta^M = kq^{2N-2-M}$. Then we arrive at

\[
\tilde{Y}_\theta(u_j\{u\}) \equiv u_j^M \prod_{a=1}^N \left( \theta(u_j - q^2 u_a) + q - q^{-1} \right) - (\theta u_j - q + q^{-1})^M \prod_{a=1 \atop a \neq j}^N \left( \theta(u_j - q^{-2} u_a) + q^{-1} - q^{-3} \right) = 0. \quad (A.4)
\]

At $\theta = 0$ one has

\[
u_j(0) = (q^{-1} - q) \cdot |q|^{2(1-N)/M} \cdot e^{\frac{2\pi i n_j}{M}}, \quad n_j \in \{0, 1, \ldots, M - 1\}, \quad (A.5)
\]

and

\[
\frac{\partial \tilde{Y}_\theta(u_j\{u\})}{\partial u_k} \bigg|_{u_j = u_j(0)} = M\delta_{jk} \cdot u_j^{-1}(0) \cdot (q - q^{-1})^{N-1}. \quad (A.6)
\]

Hence, due to the implicit function theorem in a vicinity of $\theta = 0$, there exists a unique holomorphic solution $u_j(\theta)$ of the system (A.4) which takes the value (A.5) at $\theta = 0$. This implies that

\[
x_j(\kappa) = q^{-1} + \theta u_j(0) + o(\theta). \quad (A.7)
\]

Therefore, for $|\kappa|$ small enough, but not zero, all the admissible solutions of the system (A.1) are in a vicinity of $q^{-1}$, but $x_j(\kappa) \neq q^{-1}$. It is also evident that the replacement $\theta \rightarrow \theta' = e^{\frac{2\pi i k}{M}}$ corresponds simply to a different choice of the integers $n_j$ in (A.5). \(\square\)

Observe that, due to (A.7), the admissible solutions of the system (A.1) for $|\kappa|$ small enough are off-diagonal if the integers $n_j$ in (A.5) are pairwise distinct. Thus, the system (A.1) (and therefore the system (4.8)) has $C_M^N = \dim(\mathcal{H}(M/2-N))$ different admissible off-diagonal solutions. In the following two lemmas we prove that the corresponding eigenstates $|\psi_\kappa(\{z\})\rangle$ are linearly independent.

**Lemma A.2.** Let the sets $\{z\}$ and $\{z'\}$ be two different admissible off-diagonal solutions of the system (4.8) for $\kappa \neq 0$. Then

\[
|\psi_\kappa(\{z'\})|\psi_\kappa(\{z\})\rangle = 0. \quad (A.8)
\]

**Proof.** The scalar product (A.8) is given by (2.20). If $z'_k \neq z_j, \ \forall j, k$, then one can express $\kappa d(z_k)$ in terms of $a(z_k)$ via (4.8) for the parameters $\{z\}$. Then we obtain

\[
|\psi_\kappa(\{z'\})|\psi_\kappa(\{z\})\rangle = \prod_{b=1}^N \left( d(z'_b) a(z_b) \right) \prod_{a,b=1}^N \frac{\sinh(z'_a - z_b + \eta)}{\sinh(z_a - z_b)} \det \tilde{M}(\{z\}, \{z'\}), \quad (A.9)
\]

\�20
where \( \tilde{M}(\{z\}, \{z'\}) = \tilde{M}_\kappa(\{z\}, \{z'\}) \mid_{\kappa = 1} \) (see (6.5)). It was proved in [15] that this matrix has an eigenvector with zero eigenvalue

\[
\sum_{k=1}^{N} \tilde{M}_{jk} v_k = 0, \quad \text{where} \quad v_k = \prod_{a=1}^{n} \sinh(z'_k - z_a) \prod_{a \neq k}^{n} \sinh^{-1}(z'_k - z'_{a}). \quad (A.10)
\]

Thus, in this case, the scalar product vanishes. If several parameters \( z' \) coincide with \( z \), say, \( z'_j = z_j \) for \( j = 1, \ldots, n \), then one should first proceed to this limit in the first \( n \) columns of the matrix \( \Omega_\kappa (2.20) \), and only in a second step use the equations (4.8). One can easily verify that in this case the matrix we obtain has a zero eigenvector of the same type as \( v_k \). ✷

**Lemma A.3.** There exists \( \kappa_0 > 0 \) such that, for \( 0 < |\kappa| < \kappa_0 \),

\[
\langle \psi_\kappa(\{z\}) | \psi_\kappa(\{z\}) \rangle \neq 0. \quad (A.11)
\]

**Proof.** The ‘square of the norm’ of \( |\psi_\kappa(\{z\})| \) is proportional to the Jacobian (2.22). The last one coincides with the Jacobian (2.22) up to a trivial factor which do not vanish at \( \kappa \neq 0 \). Since \( \partial Y_\theta(u|\{u\})/\partial u_k \) is a continuous function of \( \theta \) and due to (A.6), it is also non-vanishing in a vicinity of \( \kappa = 0 \). ✷

Using these two lemmas we prove

**Theorem A.1.** There exists \( \kappa_0 > 0 \) such that, for \( 0 < |\kappa| < \kappa_0 \), the states \( |\psi_\kappa(\{z\})\rangle \) corresponding to the admissible off-diagonal solutions of the system (4.8) form a basis in the subspace \( \mathcal{H}^{(M/2-N)} \).

**Proof.** It follows from Lemmas A.2, A.3 that

1) Different admissible off-diagonal solutions of the system (4.8) correspond to different states \( |\psi_\kappa(\{z\})\rangle \). Hence, the total number of the last ones is \( \dim(\mathcal{H}^{(M/2-N)}) \).

2) These states are linearly independent. ✷

**B  Free fermions**

In the particular case of free fermions \( \Delta = 0 \) (i.e. \( \eta = -i\pi/2 \)), the dynamical correlation function \( \langle \sigma_1^z(0) \sigma_{m+1}^z(t) \rangle \) can be written in the form [29]:

\[
\langle \sigma_1^z(0) \sigma_{m+1}^z(t) \rangle = \left( \frac{2k_F}{\pi} - 1 \right)^2 + \frac{1}{\pi^2} \int_{-k_F}^{k_F} e^{4it \cos p + imp} \, dp \int_{[-\pi, \pi]} e^{-4it \cos q + imq} \, dq. \quad (B.1)
\]

Here \( \cos k_F = h/4 \). This result was obtained by means of a summation over the complete set of excited states. The constant term in (B.1) corresponds to the diagonal contribution \( \langle \sigma^z \rangle^2 \). The integration variable \( p \) belongs to the Fermi sphere \([-k_F, k_F]\) and describes the ground state.
distribution. The integral over the parameter $q$ corresponds to the thermodynamic limit of the sum over the excitations outside the Fermi sphere.

In this appendix, we show how to reproduce this result by the direct use of the representation \[\text{(6.16)}.\]

At $\eta = -i\pi/2$ the integral equation \[\text{(6.9)}\] is explicitly solvable: the inhomogeneous spectral density has the form

$$\rho(\lambda, z) = \frac{i}{\pi \sinh 2(\lambda - z)},$$

\[\text{(B.2)}\]

whereas the boundary of integration $\Lambda_h$ is defined by the relation $\cosh 2\Lambda_h = 4/h$. The function $R_{\kappa}$ at $\eta = -i\pi/2$ becomes

$$R_{\kappa}(\lambda, z|\{\lambda\},\{z\}) = \left\{\begin{array}{ll}
\frac{i}{\pi \sinh 2(\lambda - z)}, & z \sim -i\pi/4; \\
\frac{1}{\pi \sinh 2(\lambda - z)}, & z \sim i\pi/4.
\end{array}\right. \text{(B.3)}$$

The main simplification, however, comes from the matrix $\tilde{M}_{\kappa}$, which at $\Delta = 0$ is proportional to $\kappa - 1$:

$$\tilde{M}_{\kappa})_{jk}(\{\lambda\}|\{z\}) = \frac{2(\kappa - 1)}{\sinh 2(\lambda_j - z_k)}. \text{(B.4)}$$

Thus, after taking the second derivative over $\kappa$ and setting $\kappa = 1$, all the terms of the series \[\text{(6.16)}\] with $n > 2$ vanish.

Let us first consider the term $Q^{(2)}_{\kappa}(m, t)$ corresponding to $n = 2$. After differentiating with respect to $\kappa$ one has

$$\frac{\partial^2}{\partial \kappa^2} Q^{(2)}_{\kappa}(m, t) \bigg|_{\kappa = 1} = \frac{1}{32\pi^4} \int_{-\Lambda_h}^{\Lambda_h} d^2 \lambda \int_{\Gamma\{\mp \frac{i\pi}{4}\}} d^2 z \cdot \det \left( \frac{e^{itE(z_k)+\text{imp}(z_k)}}{\sinh(\lambda_j - z_k)} \right) \det \left( \frac{e^{-itE(z_k)+\text{imp}(z_k)}}{\sinh(\lambda_j - z_k)} \right). \text{(B.5)}$$

Like in the time-independent case the integral over $z_k$ can be taken by the residues outside of the contour $\Gamma\{\mp \frac{i\pi}{4}\}$, i.e. in the points $z_k = \lambda_j$. This gives us

$$\frac{\partial^2}{\partial \kappa^2} Q^{(2)}_{\kappa}(m, t) \bigg|_{\kappa = 1} = -\frac{1}{4\pi^2} \int_{-\Lambda_h}^{\Lambda_h} d^2 \lambda \det \left( \frac{1 - e^{it(E(\lambda_j)-E(\lambda_k))+\text{imp}(\lambda_j)-\text{imp}(\lambda_k)}}{\sinh(\lambda_j - \lambda_k)} \right). \text{(B.6)}$$

It remains to take the second lattice derivative and, after the standard change of variables $cosh 2\lambda_j = cos^{-1} p_j$, we obtain

$$2D_m \frac{\partial^2}{\partial \kappa^2} Q^{(2)}_{\kappa}(m, t) \bigg|_{\kappa = 1} = \frac{4k_F^2}{\pi^2} - \frac{1}{\pi^2} \int_{-k_F}^{k_F} e^{4it\cos p+\text{imp}} dp \bigg|_{-k_F}^{k_F}. \text{(B.7)}$$
The term \( Q^{(1)}_\kappa(m, t) \) corresponding to \( n = 1 \) appears to be more complicated. We have

\[
Q^{(1)}_\kappa(m, t) = \frac{\kappa - 1}{4\pi^2} \int_{-\Lambda_h}^{\Lambda_h} d\lambda \left( \oint_{\Gamma\{ -\frac{i\pi}{4}\}} + \kappa^{-1} \cdot \oint_{\Gamma\{ \frac{i\pi}{4}\}} \right) \frac{dz}{\sinh^2(\lambda - z)} \cdot e^{it(E(z) - E(\lambda)) + im(p(z) - p(\lambda))}.
\]  

(B.8)

Evaluating the integral over \( \Gamma\{ -\frac{i\pi}{4}\} \) we obtain

\[
\oint_{\Gamma\{ -\frac{i\pi}{4}\}} \frac{dz}{\sinh^2(\lambda - z)} \cdot e^{it(E(z) - E(\lambda)) + im(p(z) - p(\lambda))} = -\oint_{\Gamma\{ \frac{i\pi}{4}\}} \frac{dz}{\sinh^2(\lambda - z)} \cdot e^{it(E(z) - E(\lambda)) + im(p(z) - p(\lambda))} + 2\pi [tE'(\lambda) + mp'(\lambda)].
\]  

(B.9)

It is clear that the second lattice derivative of the last term vanishes, and we have

\[
D_m^2 Q^{(1)}_\kappa(m, t) = -\frac{(\kappa - 1)^2}{4\pi^2\kappa} D_m^2 \int_{-\Lambda_h}^{\Lambda_h} \frac{dz}{\sinh^2(\lambda - z)} \cdot e^{it(E(z) - E(\lambda)) + im(p(z) - p(\lambda))}.
\]  

(B.10)

We can now explicitly differentiate this expression with respect to \( \kappa \) and \( m \), which leads to

\[
2D_m^2 \frac{\partial^2}{\partial \kappa^2} Q^{(1)}_\kappa(m, t) \bigg|_{\kappa=1} = \frac{4}{\pi^2} \int_{-\Lambda_h}^{\Lambda_h} \frac{dz}{2\cosh^2 2z} \cdot e^{it(E(z) - E(\lambda)) + im(p(z) - p(\lambda))}.
\]  

(B.11)

It remains to move the contour \( \Gamma\{ \frac{i\pi}{4}\} \) to the boundaries of the strip

\[
\oint_{\Gamma\{ \frac{i\pi}{4}\}} dz = \int d\lambda - \int d\lambda + \int d\lambda,
\]  

(B.12)

and after the same change of variables as in the case \( n = 2 \), we finally obtain

\[
2D_m^2 \frac{\partial^2}{\partial \kappa^2} Q^{(1)}_\kappa(m, t) \bigg|_{\kappa=1} = \frac{1}{\pi^2} \int_{-k_F}^{k_F} e^{4it\cos p + imp} dp \int_{-\pi}^{\pi} e^{-4it\cos q + imq} dq.
\]  

(B.13)

Using now that\(^4\langle \sigma^z \rangle = 1 - 2k_F/\pi \) we reproduce the result (B.1).

\(^4\)In fact one can obtain \( \langle \sigma^z \rangle \) form the same generating function \( Q_\kappa(m, t) \) by taking the first derivatives with respect to \( m \) and \( \kappa \) at \( \kappa = 1 \).
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