USE OF AN OPTIMIZED SPATIAL PRIOR IN D-BAR RECONSTRUCTIONS OF EIT TANK DATA

MELODY ALSAKER*
Department of Mathematics
Gonzaga University
MSC 2615
Spokane, WA 99258, USA

JENNIFER L. MUELLER
Department of Mathematics and School of Biomedical Engineering
Colorado State University
1874 Campus Delivery
Fort Collins, CO 80523-1874, USA

(Communicated by Matti Lassas)

Abstract. The aim of this paper is to demonstrate the feasibility of using spatial a priori information in the 2-D D-bar method to improve the spatial resolution of EIT reconstructions of experimentally collected data. The prior consists of imperfectly known information about the spatial locations of inclusions and the assumption that the conductivity is a mollified piecewise constant function. The conductivity values for the prior are constructed using a novel method in which a nonlinear constrained optimization routine is used to select the values for the piecewise constant function that give the best fit to the scattering transform computed from the measured data in a disk. The prior is then included in the high-frequency components of the scattering transform and in the computation of the solution of the D-bar equation, with weights to control the influence of the prior. In addition, a new technique is described for selecting regularization parameters to truncate the measured scattering data, in which complex scattering frequencies for which the values of the scattering transform differ greatly from those in the scattering prior are omitted. The effectiveness of the method is demonstrated on EIT data collected on saline-filled tanks with agar heart and lungs with various added inhomogeneities.

1. Introduction. In 2-D electrical impedance tomography (EIT), the internal electrical properties of a slice of a body are estimated from boundary measurements of electrical fields propagating through the body. In practice, low-frequency current is injected on electrodes on the body’s surface, the resulting voltages are measured on the electrodes, and an inverse problem is solved to reconstruct the interior electrical conductivity distribution. There is an extensive list of potential applications of EIT in medical as well as industrial imaging settings, and in particular EIT has shown great promise in human thoracic imaging (see, for example, [3, 8, 16, 23, 25, 32, 37] and the survey articles [9, 29] and the references therein). However, due to the

2010 Mathematics Subject Classification. Primary: 35R30; Secondary: 65N21, 94A08.

Key words and phrases. Electrical impedance tomography, D-bar methods, reconstruction algorithm, a priori data, nonlinear optimization.

*Corresponding author: Melody Alsaker.
extreme ill-posedness of the inverse problem, which is highly sensitive to measurement noise and modeling errors, EIT reconstructions tend to suffer from low spatial resolution. The acquisition of high-resolution static EIT images is, therefore, one of the biggest current challenges in EIT research.

Many reconstruction methods have been developed for EIT and can be generally categorized as based on iterative least-squares, linearization, Bayesian statistics, neural networks, and direct methods. Since the literature is too vast to survey here, the reader is referred to [19, 6, 26] for further reading, and this work focuses on the direct D-bar method based on [28, 33]. Regularized non-iterative D-bar algorithms have been shown to be computationally fast and provide high-quality, noise-robust EIT images [12, 18, 27, 20]. These methods rely on special exponentially-growing Complex Geometrical Optics (CGO) solutions, as well as a custom nonlinear Fourier transform known as the scattering transform, which encodes the EIT data. In D-bar methods, regularization is performed by truncating unstable higher-frequency scattering data, thus stabilizing the reconstruction in the presence of noisy measurements [22, 26]. However, these higher scattering frequencies tend to encode the sharper details and edges of the image, and so regularization results in a loss of spatial resolution. Another challenge inherent to this method is how to best choose the truncation threshold for the scattering data so as to preserve as much detail as possible while maintaining noise-robustness. Background for the D-bar method used in our implementation is provided in Section 2.

The inclusion of a priori estimates into iterative reconstruction methods has been demonstrated to provide improved spatial resolution in EIT images (see, for example, [4, 5, 7, 10, 11, 14, 15, 21, 35, 36]). More recently, a method for including a priori information into direct D-bar reconstruction methods was proposed in [2] and extended in [17] and [1]. These methods, which have been demonstrated to provide improved spatial resolution in reconstructions of simulated EIT data, begin with an estimated prior conductivity distribution. This conductivity prior is used to compute a priori CGO solutions and scattering data, and these estimates are then incorporated directly into the equations for D-bar. In these works, a variety of techniques were proposed for obtaining conductivity estimates for the prior, including using “blind” estimates based on published values from literature, or extracting average, max, or min values from “standard” D-bar reconstructions with no prior.

In this work, we present a novel method for selecting conductivity estimates for the prior. The prior consists of reasonably accurate approximate locations and boundaries of inclusions, and the assumption that the conductivity is a mollified piecewise constant function. A nonlinear constrained optimization routine is used to find the conductivity values for the mollified piecewise constant function that results in an a priori scattering transform that is a good match for the scattering transform computed from the measured data. We additionally present in Section 4 a new method for selecting the truncation threshold for the scattering transform, in which scattering frequencies are omitted if the values of the scattering transform computed from the data depart too greatly from those in the scattering transform computed from the prior. We explain how to use the prior and the truncation of the scattering transform in tandem in the algorithm.

In Section 5, we demonstrate the method’s effectiveness by applying our new techniques to reconstructions of experimental tank data, collected using the ACE1 EIT system at Colorado State University [24].
representing a human heart and lungs, and we include various simulated pathologies. In all of these experiments, only information known with high confidence (e.g. approximate boundaries for the heart and lungs) are included in the priors. This corresponds, for example, to a scenario in which a prior CT or other scan is available for a patient. No information concerning the added pathologies are included in the priors, and all prior conductivity estimates are computed using the new optimization routine. We show that the method provides improved spatial resolution in reconstructions of realistic tank phantoms, allowing for easier identification of interior structures and enhanced overall appearance over standard D-bar reconstructions, while allowing for automatic construction of the prior once spatial estimates for organ boundaries have been determined. This represents the first systematic presentation of the application of a priori D-bar techniques to experimentally collected data.

2. Background. In this section we include a brief review of the EIT problem and the fully nonlinear D-bar reconstruction method used in our implementation. For more detail concerning both the theoretical background and practical implementation steps, see \[28, 26, 18\].

In EIT, the electric potential \(u(x, y)\) supported on a bounded domain \(\Omega \subset \mathbb{R}^2\) is modeled by the generalized Laplace equation

\[
\nabla \cdot (\sigma(x, y) \nabla u(x, y)) = 0, \quad (x, y) \in \Omega,
\]

where the conductivity distribution \(\sigma(x, y)\) on \(\Omega\) is to be reconstructed based on surface voltage and current measurements. This boundary data is represented by the Dirichlet-to-Neumann (DN) map, which takes boundary voltages to boundary current densities:

\[
\Lambda_{\sigma} : u|_{\partial \Omega} \mapsto \sigma \frac{\partial u}{\partial \nu}|_{\partial \Omega},
\]

where \(\nu\) is the outward normal to the boundary. It was established in [28] that for \(\sigma \in C^2(\Omega)\), the DN map \(\Lambda_{\sigma}\) uniquely determines \(\sigma\).

The 2-D D-bar method used here is based on quantum scattering theory developed in [13, 31], and the global uniqueness proof in [28]. The Complex Geometrical Optics (CGO) solutions for the D-bar method are special solutions to the Schrödinger equation obtained through the change of variables \(\tilde{u} = \sqrt{\sigma}u\) and \(q = \sigma^{-1/2} \Delta \sigma^{1/2}\):

\[
(-\Delta + q(x, y)) \tilde{u}(x, y) = 0, \quad (x, y) \in \Omega.
\]

Under the assumption \(\sigma \equiv 1\) in a neighborhood of \(\partial \Omega\), equation (3) can be smoothly extended to all of \(\mathbb{R}^2\). Associating points \((x, y)\) in \(\mathbb{R}^2\) with points \(z = x + iy \in \mathbb{C}\), and introducing the nonphysical complex frequency parameter \(k \in \mathbb{C}\), the CGO solutions \(\psi(z, k)\) satisfy the Schrödinger equation

\[
(-\Delta + q(z)) \psi(z, k) = 0, \quad z \in \mathbb{R}^2,
\]

with the asymptotic property \(e^{-ikz} \psi(z, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2), \tilde{p} > 2\). Defining

\[
\mu(z, k) := e^{-ikz} \psi(z, k),
\]

the conductivity \(\sigma\) can be obtained pointwise from \(\mu(z, k)\) via

\[
\mu^2(z, 0) = \sigma(z).
\]

From (4), one readily sees that \(\mu(z, k)\) satisfies

\[
[-\Delta - 4ik\partial_z + q(z)]\mu(z, k) = 0.
\]
Differentiating (6) with respect to $\bar{k}$ results in the D-bar equation for $\mu$ [28]:

\[
\bar{\partial}_k \mu(z, k) = \frac{1}{4\pi \bar{k}} t(k) e^{-i(kz + \bar{k}\bar{z})} \mu(z, k), \quad k \in \mathbb{C} \setminus \{0\}.
\]

It was shown in [28] that (7) is uniquely solvable for each $z \in \Omega$ and satisfies the following integral equation,

\[
\mu(z, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{t(k')}{k'(k - k')} e^{-k'(z)} \mu(z, k') dk',
\]

where $e_k(z) := e^{i(kz + \bar{k}\bar{z})}$, and where $t(k)$ is the scattering transform—a nonphysical, nonlinear Fourier transform of $q$—defined by

\[
t(k) := \int_{\Omega} e^{i\bar{k}\bar{z}} q(z) \psi(z, k) dz.
\]

The method requires a connection between the scattering transform and the measured data $\Lambda_\sigma$, which is provided by [28, 31]:

\[
t(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{z}} (\Lambda_\sigma - \Lambda_1) \psi(z, k) ds,
\]

where $\Lambda_1$ denotes the DN map corresponding to $\sigma \equiv 1$ in $\Omega$. The function $\psi|_{\partial\Omega}$ can be computed from

\[
\psi(z, k)|_{\partial\Omega} = e^{ikz}|_{\partial\Omega} - \int_{\partial\Omega} G_k(z - \zeta)(\Lambda_\sigma - \Lambda_1) \psi(\cdot, k) ds,
\]

where $G_k(z)$ is the Faddeev’s Green’s function for the Laplacian, so that $-\Delta G_k(z) = \delta(z)$ (see [13, 34]), defined by:

\[
G_k(z) := e^{ikz} g_k(z), \quad g_k(z) := \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{iz\xi}}{\xi^2 + 2k^2} d\xi.
\]

The basic steps of the method may be summarized by:

1. Collect measured data and form a discrete matrix approximation to the DN map $\Lambda_\sigma$.
2. Compute $\psi|_{\partial\Omega}$ by solving (11), and form $t(k)$ from (10).
3. Solve (8) for $\mu(z, k)$, and recover $\sigma(z) = \mu^2(z, 0)$.

For step (1), in which the DN map (continuum data) is approximated by a finite-dimensional matrix, the reader is referred to [26] for details on the finite-dimensional approximation.

Throughout the rest of this paper, we will let $t(k)$ denote the approximate scattering transform computed from the measured data using (10). Due to the ill-posedness of the inverse problem, $t(k)$ becomes unstable in the presence of noise, with instability increasing exponentially as $|k| \to \infty$. In practical computations, it is therefore necessary to perform a low-pass filtering in the nonlinear Fourier (scattering) domain, which may be accomplished by setting $t(k) = 0$ for $|k| > R$, where the truncation radius $R$ is empirically selected. In [22], this filtering was proven to be a nonlinear regularization strategy. However, in a direct analogue to image processing techniques performed in the linear Fourier domain, the higher frequencies of the scattering domain tend to encode the sharper details and edges of the reconstruction $\sigma$, so we therefore sacrifice some spatial resolution in favor of computational stability.
3. **The experimental test problems.** The five test problems are described here, since it will be convenient to refer to one of the cases in the description of the optimization method for choosing the conductivity values in the prior.

Data was collected on a tank 30 cm in diameter filled with saline, on 32 electrodes, using the ACE1 EIT system at Colorado State University [24], applying adjacent current patterns with current amplitude 3.3 mA at 125 kHz. The effectiveness of the use of a prior to improve spatial resolution was tested on five experimental configurations: (i) agar targets simulating lungs and a heart with conductivities 190 mS/m for the saline, 136 mS/m for the lungs, and 238 mS/m for the heart (ii) the same agar heart and lungs with a copper pipe 1.6 cm in diameter inserted into the lower part of the lung on the right (iii) the same agar heart and lungs with a PVC pipe 2.2 cm in diameter inserted into the lower part of the lung on the right (iv) a second set of agar heart and lungs collected on a different day with conductivities 200 mS/m for the saline, 85 mS/m for the lungs, and 460 mS/m for the heart (v) the same agar heart and lungs used in (iv) with the upper half of the lung on the left removed. These cases are shown in Figure 1.

![Figure 1. The five experimental data sets considered in this paper.](image)

4. **The a priori method.** The starting point for inclusion of prior information in the D-bar method is the computation of a prior conductivity distribution, $\sigma_{pr}(z)$, $z \in \Omega$, which is computed using approximate spatial locations for organ boundaries with assigned approximate regional conductivity values. The main novelty in the method proposed here is in an optimized and automated computation of the regional conductivity values for $\sigma_{pr}$, which will be discussed in this section. From $\sigma_{pr}$, which is mollified to satisfy $\sigma_{pr} \in C^2(\Omega)$, we compute the corresponding Schrödinger potential $q_{pr}$, and then the CGO solutions $\mu_{pr}$ (and therefore $\psi_{pr}$) by solving (6).
We then form a scattering prior \( t_{pr} \) from (9). As in [2, 17, 1], this information is inserted into the D-bar algorithm in two distinct ways, which are described as follows.

We first append \( t_{pr} \) to the truncated scattering data \( t \) to create a piecewise-defined scattering transform \( t_{pw} \). In [2], this was done by selecting regularization parameters \( R_1, R_2 \) with \( R_1 \leq R_2 \), and simply computing the piecewise function

\[
(12) \quad t_{pw}(k) = t_{R_1, R_2}(k) := \begin{cases} 
    t(k), & |k| \leq R_1 \\
    t_{pr}(k), & R_1 < |k| \leq R_2 \\
    0, & |k| > R_2
\end{cases}
\]

In [17, 1], the computation (12) was accomplished with a slight refinement, in which a thresholding parameter was used to detect blow-up in \( t(k) \) for \( |k| \leq R_1 \), and this unstable scattering data \( t(k) \) was further truncated and replaced with values \( t_{pr}(k) \). In either case, since the computation of \( t_{pr} \) is noise-free and in general much more numerically robust than the computation of \( t \), \( R_2 \) may be selected to be significantly larger than \( R_1 \). The larger the value of \( R_2 \), the stronger the influence of the \emph{a priori} information.

A secondary innovation in our proposed method is the use of an optimization technique to find the truncation criteria for \( t(k) \) and therefore compute \( t_{pw}(k) \), in which values of \( t(k) \) are replaced if they deviate too far from the corresponding values in \( t_{pr}(k) \). This will be discussed in further detail within this section. In our method, as in [2, 17, 1], we may control the influence of the prior by adjusting \( R_2 \).

The second way the prior information is incorporated is the following. We use \( \mu_{pr} \) to compute a replacement \( \mu_{int} \) for the asymptotic approximation \( \mu \sim 1 \) used in (8):

\[
(13) \quad \mu_{int}(z) := \frac{1}{\pi R_2^2} \int_{|k| \leq R_2} \mu_{pr}(z, k) dk.
\]

For this step, we follow the work in [2], and the interested reader is directed there for further information.

The end result of these two steps is that we solve a modified D-bar integral equation:

\[
(14) \quad \mu_{R_2, \alpha}(z, k) = \alpha + (1 - \alpha)\mu_{int}(z) + \frac{1}{(2\pi)^2} \int_{|k| \leq R_2} \frac{t_{pw}(k')}{k'(k - k')} e^{-k'\mu_{R_2, \alpha}(z, k')} dk',
\]

where the weighting parameter \( \alpha \) is used to control the influence of the term \( \mu_{int} \) in the resulting reconstruction. Upon solving (14), we may compute the modified reconstruction \( \sigma_{R_2, \alpha}(z) = \mu_{R_2, \alpha}^2(z, 0) \). As was proven in [2], this method constitutes a true nonlinear regularization strategy, where \( \sigma_{R_2, \alpha} \) converges to the true conductivity as the noise level goes to zero, when \( \sigma_{pr} \) represents the true conductivity.

4.1. Computing the conductivity prior \( \sigma_{pr} \). When the true conductivity distribution is unknown, determining useful conductivity values for the prior poses a significant challenge. Various methods have been suggested in [2, 17, 1], which use either “blind” estimates for conductivity values, or use regional average, max, or min values extracted directly from preliminary D-bar reconstructions. The novel approach proposed here is motivated by our observation that when the scattering prior \( t_{pr} \) is a poor match for \( t \), artifacts tend to develop in the resulting reconstructions, as illustrated in Fig. 2. In this figure, we use the data from experimental test case (iv) to illustrate how a poor choice for \( \sigma_{pr} \) will result in introduced artifacts.
in the reconstructions. All reconstructions in Fig. 2 were computed using the a priori method described here, with \( R_2 = 10 \) and \( \alpha = 0 \). The figures are depicted in order according to how well \( t_{pr} \) matched \( t \), from worst match (A) to best match (F). Goodness of fit was quantified by computing the value of the objective function \( J(c) \), which is defined later in this section. As one can see from the figure, a poor choice of \( \sigma_{pr} \), resulting in a poor match between \( t_{pr} \) and \( t \), leads to significant artifacts and distortions in the reconstruction. The appearance of the images improves greatly with improved match between \( t_{pr} \) and \( t \).

With this motivation, we wish to assign regional conductivity values to \( \sigma_{pr} \) that yield scattering data \( t_{pr}(k) \) that is the best fit to the values \( t(k) \) computed from measurements, in the disk \( |k| = R \), which is the region in which a reliable preliminary reconstruction with no prior information is computed. The selection of the parameter \( R \) is empirical, as is typical in truncation of scattering data in D-bar methods, but we selected our \( R \) to be slightly smaller than the radius on which \( t \) is typically stable (we shall see that a larger choice of \( R \) will increase the compute time for the optimization routine).

Using a selection method such as image segmentation—from an auxiliary modality, in our case a photo—we determine \( n \) organ regions within the domain \( \Omega \) such that each region has roughly homogeneous conductivity. Let \( c = [c_1, \ldots, c_n]^T \) be the vector of constant conductivity values \( c_j \) to be assigned to these \( n \) regions, which will used to form \( \sigma_{pr} \). Given a computational \( k \)-grid, truncated so that \( |k| \leq R \), enumerate the resulting finite set of \( N \) computation points \( \{k_1, \ldots, k_N\} \). Denote by \( t^{vec} \) and \( t_{pr}^{vec} \) the \( N \)-dimensional vectors defined by \( t^{vec} := [t(k_1), \ldots, t(k_N)]^T \) and \( t_{pr}^{vec} := [t_{pr}(k_1), \ldots, t_{pr}(k_N)]^T \). Using the D-bar algorithm previously described, for any fixed set of values \( \{k_1, \ldots, k_N\} \), the vector \( t^{vec} \) is completely determined from the measured data. On the other hand, the conductivity prior \( \sigma_{pr} \), and therefore the vector \( c \), completely determines the vector \( t_{pr}^{vec} \), so we may write \( t_{pr}^{vec} = t_{pr}^{vec}(c) \). We define the objective function

\[
J(c) := \|t_{pr}^{vec}(c) - t^{vec}\|^2_2,
\]

and consider the constrained minimization problem

\[
\text{minimize } J(c) \text{ subject to } \ell \leq c \leq u,
\]

where \( \ell \) and \( u \) are \( n \)-vectors of lower and upper bounds. In our implementation, we selected \( \ell = [\sigma_{\text{min}}, \sigma_{\text{min}}, \ldots, \sigma_{\text{min}}]^T \), and \( u = [\sigma_{\text{max}}, \sigma_{\text{max}}, \ldots, \sigma_{\text{max}}]^T \), where \( \sigma_{\text{min}} \) and \( \sigma_{\text{max}} \) respectively denote the minimum and maximum values of \( \sigma \) in the preliminary reconstruction. To solve (16), we employed Matlab’s \texttt{fmincon} SQP (sequential quadratic programming) algorithm for nonlinear constrained minimization, with default tolerances for all stopping criteria. In general, SQP algorithms generate iterates \( c^j \) by modeling (16) on each step with a quadratic programming subproblem, and then use the minimizer of each subproblem to define the next iterate. For a detailed description of the basic SQP algorithm, see Chapter 18 of [30], for example. It should be noted that use of a finer \( k \)-grid or a larger value of \( R \) will result in a larger value for \( N \) and therefore a longer compute time in the optimization routine.

In computations using data from the thoracic phantoms shown in Figure 1, \( n = 4 \) regions were used for computing optimized conductivity estimates. Denoting the conductivity estimates for the background \( c_1 = c_b \), the heart \( c_2 = c_h \), and right and
left lungs, \( c_3 = c_r \) and \( c_4 = c_l \), we optimized \( \mathbf{c} = [c_b, c_h, c_r, c_l]^T \). For all reconstructions presented in Section 5, we chose initial guess \( \mathbf{c}^0 = [\sigma_{\text{avg}}, \sigma_{\text{max}}, \sigma_{\text{min}}, \sigma_{\text{min}}]^T \), where \( \sigma_{\text{avg}} \) is the mean value of all conductivity values in the preliminary reconstruction \( \sigma \). This is in general a reasonable starting guess for tank phantoms representing human thoracic data, as the heart will typically have high conductivity and the lungs will have low conductivity in comparison to the background. However, to test the method’s robustness to changes in initial conditions, with respect to both convergence and the final output \( \mathbf{c} \), we ran a series of experiments in which the components \( c_j^0 \) of the initial guess vector were varied. In this paper, we present the results from these experiments conducted using the dataset for case (iv), and consider four main experimental cases:

1. Fix \( c_b^0 = \sigma_{\text{max}} \), \( c_r^0 = c_l^0 = \sigma_{\text{min}} \); let \( c_h^0 \) vary.

**Figure 2.** Illustration of artifacts and distortions in reconstructions of experimental case (iv) when \( t_{pr} \) is a poor match for \( t \). These images are reconstructions using the \textit{a priori} method described here, with the scattering prior \( t_{pr} \) computed from output \( c^j \) of six iterative steps of the optimization routine. The reconstruction with the initial guess \( \mathbf{c}^0 \) is depicted in (a) and the reconstruction using the final result of the optimization routine is depicted in (f). The value of the objective function \( J(\mathbf{c}) \) is given below each reconstruction. Smaller \( J(\mathbf{c}) \) indicates increased goodness of fit between \( t_{pr} \) and \( t \).
2. Fix $c_0^j = \sigma^{\text{max}}$, $c_0^j = \sigma^{\text{avg}}$; let $c_0^j = c_0^j$ vary.
3. Fix $c_0^j = \sigma^{\text{avg}}$, $c_0^j = c_0^j = \sigma^{\text{min}}$; let $c_0^j$ vary.
4. Hold $c_0^j = c_0^j = c_0^j = c_0^j$; vary all values simultaneously.

In each of the above cases, the values $c_0^j$ were varied so that $\sigma^{\text{min}} \leq c_0^j \leq \sigma^{\text{max}}$ for each $j$. We will denote by $c_0^j$ the “control” initial guess, $c_0^j = [\sigma^{\text{avg}}, \sigma^{\text{max}}, \sigma^{\text{min}}]^T$, corresponding to optimization result $c_\ast$, which was used in the reconstructions depicted in Fig. 6. We performed a total of 43 experiments over the four cases listed above, including the control experiment. For each experiment we recorded the resulting final output vector $c$, the value of the objective function $J(c)$, and the number of iterations required to achieve convergence. We furthermore computed the resulting vector $t_{\text{pr}}^\ast(c)$ for each experiment, and then computed the quantity $D(c) := \|t_{\text{pr}}^\ast(c) - t_{\text{pr}}^\ast(c_{\ast})\|_\infty$ to measure the variation in the resulting scattering data. The optimization routine converged in all cases, and a statistical summary of the results is displayed in Table 1.

The average number of iterations required to converge was 14. The minimal number of iterations to achieve convergence was 9, and the maximum was 19. Statistical analysis revealed a slight—yet statistically significant—linear correlation between the initial value of the objective function $J(c^0)$ and the number of iterations required, with correlation coefficient $r = 0.66$ and $p$-value $p = 1.4 \times 10^{-6}$, indicating that in general a poor initial guess may require more iterations (although this was not always the case). The final optimized value of the objective function $J(c)$ remained extremely consistent across all experiments, with coefficient of variation $3.9 \times 10^{-9}\%$. The resulting conductivity vector $c$ varied only slightly with changing initial guess values, with coefficient of variation $0.74\%$ for each $c_j$. The values for the quantity $D(c)$, which indicates variation in the resulting scattering data across the experiments, were also quite small, with a maximum value of $D(c) = 7.15 \times 10^{-6}$. Furthermore, analogous experiments with other datasets (not presented here, for sake of brevity) have indicated similarly consistent results using the optimization routine. These results together indicate that the method is extremely robust with respect to the initial guess, so that a wide range of methods may be used to assign initial guesses, and a regularization term is not required in the objective function.

| Table 1. Results from experiments with the optimization routine, in which values in the initial guess vector $c^0$ were varied. We include statistical summary values for the initial objective function $J(c^0)$, the number of iterations required (NumIter), the resulting output vector $c$, the optimized value of the objective function $J(c)$, and the quantity $D(c) := \|t_{\text{pr}}^\ast(c) - t_{\text{pr}}^\ast(c_{\ast})\|_\infty$ where $c_{\ast}$ corresponds to the “control experiment” in our tests, which is a measure of variation between test cases of the resulting scattering data. Statistical values presented include the mean, max, min, range, standard deviation, and coefficient of variation of each quantity. |
|---|---|---|---|---|---|---|---|---|---|
| Initial Guess Vector $c^0$ | $c_1^0$ | $c_2^0$ | $c_3^0$ | $c_4^0$ | $J(c^0)$ | NumIter | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $J(c)$ | $D(c)$ |
| Mean | 0.9978 | 1.1116 | 0.9312 | 0.9312 | 62.19 | 14,140 | 1.0181 | 1.1879 | 0.9883 | 0.9515 | 9.8517867 | 1.34E-06 |
| Max | 1.1769 | 1.1769 | 1.1769 | 1.1769 | 145.41 | 19 | 1.0547 | 1.1769 | 0.9145 | 0.9580 | 9.8517867 | 7.15E-06 |
| Min | 0.8658 | 0.8658 | 0.8658 | 0.8658 | 11.78 | 9 | 1.0288 | 1.1480 | 0.8021 | 0.9344 | 9.8517867 | 4.31E-07 |
| Range | 0.3112 | 0.3112 | 0.3112 | 0.3112 | 133.63 | 10 | 0.0259 | 0.0290 | 0.0225 | 0.0235 | 1.470E-09 | 6.32E-06 |
| StDev | 0.0753 | 0.0904 | 0.0904 | 0.0904 | 46.17 | 2.406 | 0.0078 | 0.0087 | 0.0067 | 0.0071 | 3.63E-10 | 1.51E-06 |
| CoefVar | 7.55% | 8.13% | 9.70% | 9.70% | 74.24% | 17.02% | 0.74% | 0.74% | 0.74% | 0.74% | 3.69E-09 | 45.39% |
4.2. Computing the piecewise scattering transform $t_{pw}$. One of the challenges in all regularized D-bar methods is how best to apply the filtering in the Fourier domain so as to appropriately truncate the scattering data $t(k)$. We wish to keep scattering data that is stable, and eliminate the data that is unstable, but making this distinction can pose challenges. In the scheme (12), there may be some values of $t(k)$ inside the radius $|k| \leq R_1$ that are unstable, while some values outside this radius may in fact be stable. One possible solution is to truncate the scattering data based on some simple thresholding criteria, in which “large” values of the scattering data are removed within the radius $R_1$. However, in general, the true noise-free scattering data is unknown, so it is difficult to determine which values of $t(k)$ are problematic, and which values are accurate approximations of the true scattering values. This motivates the following novel scheme for the computation of $t_{pw}$, in which we assume the scattering prior $t_{pr}$ to be an approximation for the true scattering data, and truncate values of $t$ which deviate too far from $t_{pr}$. We note that this technique may be used not only in a priori methods, but also to compute stable regularized reconstructions without the inclusion of prior data in the D-bar equations.

We select truncation parameters $R_1$ and $R_2$, where $R_1$ is chosen so that the disk of radius $R_1$ is slightly larger than the region where we suspect $t(k)$ is stable, and $R_2 \geq R_1$. We further select a computational grid $\mathcal{G}$ of $k$-values that encompasses the disk of radius $R_2$, and define $\mathcal{K}_{R_1} := \mathcal{G} \cap \{ |k| \leq R_1 \}$, and $\mathcal{K}_{R_2} := \mathcal{G} \cap \{ |k| \leq R_2 \}$. We then compute $t(k)$ for $k \in \mathcal{K}_{R_1}$ and $t_{pr}(k)$ for $k \in \mathcal{K}_{R_2}$. For each $k \in \mathcal{K}_{R_1}$, we compute the quantities

$$d^\text{Re} (k) := | \text{Re} (t(k) - t_{pr}(k)) |,$$

$$d^\text{Im} (k) := | \text{Im} (t(k) - t_{pr}(k)) |,$$

and form the sets

$$\mathcal{K}_{R_1}^\text{Re} := \{ k \in \mathcal{K}_{R_1} : d^\text{Re} (k) \leq p^\text{Re} \},$$

$$\mathcal{K}_{R_1}^\text{Im} := \{ k \in \mathcal{K}_{R_1} : d^\text{Im} (k) \leq p^\text{Im} \},$$

where the values $p^\text{Re}$ and $p^\text{Im}$ are empirically selected thresholding parameters. We then form the scattering data $t_{pw}$ by computing real and imaginary parts separately:

(17) \hspace{1cm} \text{Re} (t_{pw})(k) := \begin{cases} \text{Re} (t)(k), & k \in \mathcal{K}_{R_1}^\text{Re}, \\ \text{Re} (t_{pr})(k), & k \in \mathcal{K}_{R_2} \setminus \mathcal{K}_{R_1}^\text{Re}, \\ 0, & k \in \mathcal{G} \setminus \mathcal{K}_{R_2}, \end{cases}$$

(18) \hspace{1cm} \text{Im} (t_{pw})(k) := \begin{cases} \text{Im} (t)(k), & k \in \mathcal{K}_{R_1}^\text{Im}, \\ \text{Im} (t_{pr})(k), & k \in \mathcal{K}_{R_2} \setminus \mathcal{K}_{R_1}^\text{Im}, \\ 0, & k \in \mathcal{G} \setminus \mathcal{K}_{R_2}, \end{cases}$$

so that $t_{pw}(k) = \text{Re} (t_{pw})(k) + i \text{Im} (t_{pw})(k)$. We note that selecting $R_2 = R_1$ will result in a slight expression of the a priori data in $t_{pw}$, since the sets $\mathcal{K}_{R_1} \setminus \mathcal{K}_{R_1}^\text{Re}$ and $\mathcal{K}_{R_1} \setminus \mathcal{K}_{R_1}^\text{Im}$ will in most cases be nonempty. If desired, we may compute a reconstruction with no prior by using only the real and imaginary parts of $t(k)$ truncated as in (17) and (18), with all other scattering values set to 0.

In the experiments presented here, we used $R_1 = 5$, and we selected $p^\text{Re}$ and $p^\text{Im}$ to be the 65th percentiles of all $d^\text{Re} (k)$ and $d^\text{Im} (k)$ values, $k \in \mathcal{K}_{R_1}$, respectively. These choices produced good results across all the experiments, and we furthermore
were able to obtain good results (not presented here, for the sake of brevity) using other choices for $R_1, p^{\text{Re}},$ and $p^{\text{Im}},$ indicating that the method is fairly robust with respect to these parameters.

**Algorithm 1** A priori D-bar method with optimized prior

1. Compute a discrete approximation $L_\sigma$ to the DN map $\Lambda_\sigma.$
2. Choose a preliminary truncation radius $R_1$ for the scattering transform $t,$ and compute $\psi$ for $|k| \leq R_1.$
3. Compute $\sigma$ from equations (11), (10), (8) and (5).
4. Compute spatial organ boundaries for $\sigma_{pr}$.
5. Solve the constrained optimization problem (15), (16) for conductivities $c.$
6. Define $q_{pr}, \psi_{pr},$ and $t_{pr}$ from the functions computed as part of Step 4 corresponding to the selected minimum $c.$
7. Check the fidelity of the scattering transform $t$ against $t_{pr}$ and discard values of poor fidelity using (17) and (18).
8. Form $t_{pw}$ and compute the final reconstruction.

5. Results and discussion. The prior for each data set was constructed from the agar heart and lungs experiment without simulated pathology (i.e. cases (i) and (iv)) by first using MATLAB to construct a polygonal approximation to the organ boundaries by opening the photograph in MATLAB and clicking on the image to store the points. Conductivity values were assigned, and the piecewise constant function constructed from the polygonal boundary approximations was mollified using convolution with a smoothing function to form the prior used in (12). Difference images, wherein the DN map $\Lambda_1$ in (10) is replaced with measurements computed from a tank containing a homogeneous saline solution, were computed for each case using the method in Section 4. It should be noted that in the case of human data, difference images can be obtained by replacing $\Lambda_1$ by measured data from one frame in a sequence of collected frames; alternatively, we may numerically simulate $\Lambda_1$ to obtain absolute images.

In Figure 3, difference images of the agar heart and lungs in case (i) are computed with no prior and with priors of various weights. Recall that increasing $R_2$ increases the weight of the prior by increasing the outer radius of the annulus in which the prior is used in the scattering transform, while decreasing $\alpha$ increases the weight of the prior by increasing the relative emphasis of $\mu_{\text{int}}$ in the reconstruction. The D-bar method with no prior has $\alpha = 1,$ while setting $\alpha = 0$ corresponds to using only the $\mu_{\text{int}}$ term in the integral form of the D-bar equation (14). Therefore, in Figures 3 – 7 the influence of the prior becomes stronger as one moves to the right in each row. Reconstructions in (f) in each figure correspond to a moderate weight in the scattering transform and in the $\mu_{\text{int}}$ term. It is evident in Figure 3 that the stronger the prior, the sharper the organ boundaries become, as is to be expected.

In Figure 4 difference images of case (ii) with and without a spatial prior are presented. While the spatial prior does not include the copper pipe, its presence is evident in all of the reconstructions as a highly conductive region in the lower half of the lung on the right. Due to the very high conductivity of copper and the fact that the method is regularizing and results in smooth approximations to the actual conductivity distribution [2], it is not so surprising that the boundary of the copper pipe is not sharply recovered. This may be attributed both to the blurring caused by
regularization and by the fact that the prior does not include the object. An example of using a prior in the D-bar method without the optimization strategy that does include the inhomogeneity for experimental data with high contrast targets can be found in [2]. The copper pipe does influence the conductivity values chosen for the prior in the right lung because they are chosen through the optimization method to provide the best fit to the scattering data constructed from the measured voltages. This causes the prior in the right lung to have a higher conductivity than that of the left lung and it is evident in the reconstructions that as the influence of the prior increases, the conductivity of the right lung also increases, resulting in some loss of spatial resolution of that lung boundary as well.

In Figure 5 reconstructions with a resistive PVC pipe in the same location as the copper pipe of case (ii) are presented. While the presence of the resistive target is more difficult to distinguish from the rest of the lung, its size and shape are resolved more accurately than those of the copper pipe. Its presence serves to lower the conductivity of the right lung in the prior, which does not include its presence spatially since the lung is assumed to have constant conductivity in the prior, but the spatial boundaries of that lung are resolved more accurately as the influence of the prior increases, contrary to case (ii). The low value of the conductivity of the right lung in the reconstructions causes the scale of the displayed images to make the conductivity of the left lung to be closer to that of the background saline, and so it appears fainter in the images. However, as the influence of the prior increases, its spatial resolution also improves.

The reconstructions in Figure 6 are analogous to those of case (i), but were collected as part of a different experiment on a different day, and so different conductivity values of the agar were used, and it can be seen that the reconstructions differ from those of Figure 3, but with the same property that the increased influence of the prior sharpens the images. In Figure 7 reconstructions of the agar heart and lungs of case (iv) with the top half of the left lung removed are provided. This “pathology” is clearly visible in all of the reconstructions. As is the case with the copper pipe, the conductivity estimate of the left lung is higher than that of the right in the prior, and as the influence of the prior increases, it starts to appear as if the top half of the left lung is present, but has a somewhat higher conductivity than the lower half.

6. Conclusions. This work provides a method of including a priori information in the D-bar algorithm that is especially suited for experimental data where little may be known about the actual conductivity values in the region of interest. The prior consists of imperfect knowledge of the spatial locations and boundaries of inclusions and the assumption that the conductivity is a mollified piecewise constant function. The conductivity values for the prior are chosen using a novel method in which a nonlinear constrained optimization routine is used to select the values for the piecewise constant function that give the best fit to the scattering transform computed from the measured data in a disk. The CGO forward problem is solved at each step in the iterative solution of the optimization routine to compute the values of the objective function at each iteration. The optimal set of conductivity values effectively minimizes the misfit between the scattering transform computed from the data, $t$, and the scattering transform $t_{pw}$, which is pieced together roughly in the annulus $R_1 \leq |k| \leq R_2$ to form the scattering transform $t_{pw}$ used in the computations of the final estimate for the conductivity. The effectiveness of this
approach was demonstrated on experimental data consisting of agar heart and lungs in a saline-filled tank with three types of inhomogeneities: a conductive copper pipe in one “lung” to simulate a tumor, a resistive PVC pipe in one “lung,” and a case in which half of one agar lung is removed. The results show that inclusion of a prior improves the spatial resolution of the images, even when the prior does not include the inhomogeneity, and the optimization method provides an effective method for choosing conductivity values for the prior with no a priori knowledge of those values.

Acknowledgments. The project described was supported by Award Number 1R21EB016869-01A1 from the National Institute Of Biomedical Imaging And
Figure 4. Data Collection 1, case (ii) agar heart and lungs with a conductive copper pipe added to right lung. The influence of the prior increases from left to right in each row.
Figure 5. Data Collection 1, case (iii) agar heart and lungs with a resistive PVC pipe added to right lung. The influence of the prior increases from left to right in each row.
Figure 6. Data Collection 2, case (iv) agar heart and lungs. The influence of the prior increases from left to right in each row.
Figure 7. Data Collection 2, case (v) agar heart and lungs with top half of left lung removed. The influence of the prior increases from left to right in each row.
REFERENCES

[1] M. Alsaker, S. Hamilton and A. Hauptmann, A direct D-bar method for partial boundary data electrical impedance tomography with a priori information, Inverse Problems and Imaging, 11 (2017), 427–454.
[2] M. Alsaker and J. Mueller, A D-bar algorithm with a priori information for 2-dimensional electrical impedance tomography, SIAM J. Imaging Sci, 9 (2016), 1619–1654.
[3] M. Arad, S. Zlochiver, T. Davidson, Y. Shoenfeld, A. Adunsky and A. Abboud, The detection of pleural effusion using a parametric eit technique, Physiol. Meas., 30 (2009), 421–428.
[4] N. J. Avis and D. C. Barber, Incorporating a priori information into the Sheffield filtered backprojection algorithm, Physiol. Meas., 16 (1995), A111–A122.
[5] U. Baysal and B. M. Eyüboglu, Use of a priori information in estimating tissue resistivities - a simulation study, Phys. Med. and Biol., 43 (1998), 3589–3606.
[6] B. H. Brown, Electrical impedance tomography (EIT): A review, Journal of medical engineering & technology, 27 (2003), 97–108.
[7] E. D. L. B. Camargo, Development of an Absolute Electrical Impedance Imaging Algorithm for Clinical Use, PhD thesis, University of São Paulo, 2013.
[8] E. Costa, C. Chaves, S. Gomes, M. Beraldo, M. Volpe, M. Tucci, I. Schettino, S. Bohm, C. Carvalho, H. Tanaka, L. R.G. and M. Amato, Real-time detection of pneumothorax using electrical impedance tomography, Critical Care Medicine, 36 (2008), 1230–1238.
[9] E. Costa, R. Gonzalez Lima and M. Amato, Electrical impedance tomography, in Intensive Care Medicine (ed. J. Vincent), Springer, New York, 2009, 394–404.
[10] H. Dehghani, D. C. Barber and I. Basarab-Horwath, Incorporating a priori anatomical information into image reconstruction in electrical impedance tomography, Physiol. Meas., 20 (1999), 87–102.
[11] D. C. Dobson and F. Santosa, An image-enhancement technique for electrical impedance tomography, Inverse Probl., 10 (1994), 317–334.
[12] M. Dodd and J. Mueller, A real-time D-bar algorithm for 2-D electrical impedance tomography data, Inverse Probl. Imag., 8 (2014), 1013–1031.
[13] L. D. Faddeev, Increasing solutions of the Schroedinger equation, Fifty Years of Mathematical Physics, (2016), 34–36.
[14] D. Ferrario, B. Grychtol, A. Adler, J. Sola, S. H. Bohm and M. Bodenstein, Toward morphological thoracic EIT: Major signal sources correspond to respective organ locations in CT, IEEE T. Med. Imaging, 59 (2012), 3000–3008.
[15] D. Flores-Tapia and S. Pistorius, Electrical impedance tomography reconstruction using a monotonicity approach based on a priori knowledge, in Engineering in Medicine and Biology Society (EMBC), 2010 Annual International Conference of the IEEE, 2010, 4996–4999.
[16] I. Freichs, S. Pulletz, G. Elke, F. Reifferscheid, D. Schädler, J. Scholz and N. Weiler, Assessment of changes in distribution of lung perfusion by electrical impedance tomography, Respiration, 77 (2009), 282–291.
[17] S. Hamilton, J. Mueller and M. Alsaker, Incorporating a spatial prior into nonlinear d-bar eit imaging for complex admittivities, IEEE T. Med. Imaging, 36 (2017), 457–466.
[18] C. N. L. Herrera, M. F. M. Vallejo, J. L. Mueller and R. G. Lima, Direct 2-D reconstructions of conductivity and permittivity from EIT data on a human chest, IEEE T. Med. Imaging, 34 (2015), 267–274.
[19] D. S. Holder, Electrical Impedance Tomography: Methods, History and Applications, CRC Press, 2004.
[20] D. Isaacson, J. L. Mueller, J. C. Newell and S. Siltanen, Reconstructions of chest phantoms by the D-bar method for electrical impedance tomography, IEEE T. Med. Imaging, 23 (2004), 821–828.
[21] J. P. Kaipio, V. Kolehmainen, M. Vauhkonen and E. Somersalo, Inverse problems with structural prior information, Inverse Probl., 15 (1999), 713–729.
[22] K. Knudsen, M. Lassas, J. L. Mueller and S. Siltanen, Regularized D-bar method for the inverse conductivity problem, Inverse Probl. Imag., 3 (2009), 599–624.
[23] K. Lowhagen, S. Lundin and O. Stenqvist, Regional intratidal gas distribution in acute lung injury and acute respiratory distress syndrome - assessed by electric impedance tomography, Minerva Anestesiologica, 76 (2010), 1024–1035.
[24] M. Mellenthin, J. Mueller, E. de Camargo, F. de Moura, T. Santos, R. Lima, S. Hamilton, P. Muller and M. Alsaker, The ACE1 electrical impedance tomography system for thoracic imaging, In review.
[25] M. Muders, H. Luepschen and C. Putensen, Impedance tomography as a new monitoring technique, *Curr Open Crit Care*, 16 (2010), 269–275.
[26] J. L. Mueller and S. Siltanen, *Linear and Nonlinear Inverse Problems with Practical Applications*, SIAM, Philadelphia, PA, 2012.
[27] E. K. Murphy and J. L. Mueller, Effect of domain shape modeling and measurement errors on the 2-D D-bar method for EIT, *IEEE T. Med. Imaging*, 28 (2009), 1576–1584.
[28] A. I. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, *Ann. Math.*, 143 (1996), 71–96.
[29] D. Nguyen, J. C., Thiagingam and A. A. McEwan, A review on electrical impedance tomography for pulmonary perfusion imaging, *Physiol. Meas.*, 33 (2012), 695–706.
[30] J. Nocedal and S. J. Wright, *Numerical Optimization*, 2nd edition, Springer Verlag, 2006.
[31] R. G. Novikov, Multidimensional inverse spectral problem for the equation $-\delta \psi + (v(x) - eu(x))\psi = 0$, *Functional Analysis and Its Applications*, 22 (1988), 263–272.
[32] H. Reinius, J. B. Borges, F. Fredén, L. Jideus, E. D. Camargo, M. B. Amato, G. Hedenstierna, A. Larsson and F. Lennmyr, Real-time ventilation and perfusion distributions by electrical impedance tomography during one-lung ventilation with capnothorax, *Acta Anaesthesiol Scand.*, 59 (2015), 354–368.
[33] S. Siltanen, J. Mueller and D. Isaacson, An implementation of the reconstruction algorithm of A Nachman for the 2D inverse conductivity problem, *Inverse Probl.*, 16 (2000), 681–699.
[34] S. Siltanen, *Electrical Impedance Tomography and Faddeev Green’s Functions*, PhD thesis, Helsinki University of Technology, 1999.
[35] M. Soleimani, Electrical impedance tomography imaging using a priori ultrasound data, *BioMed. Eng. OnLine*, 5.
[36] M. Vauhkonen, D. Vadasz, P. A. Karjalainen, E. Somersalo and J. P. Kaipio, Tikhonov regularization and prior information in electrical impedance tomography, *IEEE T. Med. Imaging*, 17 (1998), 285–293.
[37] J. Victorino, J. Borges, V. Okamoto, G. Matos, M. Tucci, M. Caraman, H. Tanaka, F. Sippmann, D. Santos, C. Barbas, C. Carvalho and M. P. Amato, Imbalances in regional lung ventilation: a validation study on electrical impedance tomography, *American Journal of Respiratory and Critical Care Medicine*, 169 (2004), 791–800.

Received March 2017; revised December 2017.

E-mail address: alsaker@ Gonzaga.edu
E-mail address: mueller@ math.colostate.edu