CATEGORIES AS MATHEMATICAL MODELS

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ABSTRACT. Category theory is presented as a mathematical modeling framework that highlights the relationships between objects, rather than the objects in themselves. A working definition of model is given, and several examples of mathematical objects, such as vector spaces, groups, and dynamical systems, are considered as categorical models.

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Mathematicians do not study objects, but relations between objects. Thus, they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevant: they are interested in form only.

– Henri Poincaré

Yet, I hope that I managed to convey the message: the mathematical language developed by the end of the 20th century by far exceeds in its expressive power anything, even imaginable, say, before 1960. Any meaningful idea coming from science can be fully developed in this language.

– Mikhail Gromov

1. INTRODUCTION

In the sciences, most of the prominent methods for incorporating mathematics involve setting up stochastic processes, dynamical systems, or statistical models that capture the relevant processes in the scientific subject. At bottom, these techniques all involve interplays of numbers. However, a biologist’s sophisticated understanding of the non-mechanistic nature of life—heredity, reproduction, hierarchical nesting, symbiosis, metabolism, etc.—remains trapped in the realm of ideas. Such ideas can often be reduced to numerical models, but the ideas themselves are

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confined to the background. When an idea is mathematically formalized, it gains clarity, systematicity, and falsifiability.

My hope for category theory is that it can be used to model many of the actual ideas and ways of thinking that exist within science, such as those of the working biologist. Modeling a phenomenon allows us to examine, and interact with, a simplified version of it, and involving mathematics generally provides an additional level of rigor and communicability. A category-theoretic model of the ideas, rather than of the mere quantities, may be able to formally capture a whole conceptual framework, say, ideas about the hierarchical nature of heredity. With such a formal description, one could apply rigorous conceptual—rather than numerical—tests to the ideas themselves. For example, one might check whether the ideas satisfy various internal consistencies, or one might ask about the nature of their integration with the other major conceptual frameworks that exist in the field.

Category theory holds promise for a conceptual integration of science because it has achieved such an integration of mathematics to a remarkable degree. It does so, not by finding a single syntax or format that can encode any structure, but by doing the opposite: leaving the encoding entirely absent. Rather than modeling a given object in itself, category theory models only the relationships between objects.

As mentioned above, there is a plethora of mathematical modeling techniques for scientific subjects, most of them numerical in nature. And quite a few of these, including calculus, dynamical systems, logic, and probability theory, have well-established categorical descriptions. The working scientist can hope to use functors to integrate her ideas with those of others, as well as with familiar modeling techniques, by casting all of them in terms of categories.

1.1. Category theory as modeling language. In this paper I cast category theory (CT) as a universal modeling language. More precisely, I claim that CT is a mathematical model of mathematical models. Before I explain this assertion, I will ground the discussion with a prototypical example of a model to fix ideas, and then I will define what I mean by model.

The radar system in a submarine models the distances between various objects in the physical environment of the submarine. It does so by plotting dots, which correspond to these objects, on a screen using polar coordinates (distance and angle). A representation of the information gathered by the radar system is displayed to the submarine’s pilot in a familiar “self-centric” way, i.e., with the submarine (and hence the pilot) shown at the center of the display. This allows the radar system to become transparent equipment for the pilot. In other words, she can seamlessly integrate the representation into her personal repertoire and thereby use familiar methods to cope appropriately with a situation as it unfolds, say to evade or pursue another object.

In this example, the radar system of the submarine is the model and the pilot is its user. The physical environment of the submarine is the subject of the model; it is the thing being modeled by the radar system. Because the locations of objects in the physical environment are emphasized, rather than suppressed, we refer to them as the foregrounded aspects of the subject. The distances and angles between these locations are the relationships between foregrounded aspects. We refer to the

\[\text{See Heidegger, Being and Time, as well as Andy Clark, Supersizing the Mind.}\]
Let us tie these ideas together by making some observations. The value of the radar system (the model) is measured by the extent to which the pilot’s (the user’s) interaction with submarine’s physical environment (the subject) is successfully mediated by the radar system. This in turn depends on the extent to which the locations of the physical objects in the environment (foregrounded aspects) are propitious for successful interaction with the subject and the extent to which the polar coordinate representation of these objects (the formalization) is faithful.

With this exemplar in mind, we make the following two philosophical postulates, which will help to organize the ideas in this paper.

(1) Modeling a subject is foregrounding certain observed aspects of the subject, and then formalizing these aspects and the observed relationships between them.

(2) The value of a model is measured by the extent to which the user’s interactions with the subject are successfully mediated by the model. This in turn depends on the propitiousness of the foregrounded aspects and the faithfulness of their formalization.

Note that mathematical models seem to put more emphasis on, and care into, the formalism than do other types of models.

1.2. Using models is connecting models. In his *Critique of Pure Reason*, Immanuel Kant makes an important assertion:

> Everything intuited or perceived in space and time, and therefore all objects of a possible experience, are nothing but phenomenal appearances, that is, mere representations, which in the way in which they are represented to us, as extended beings, or as series of changes, have no independent, self-subsistent existence apart from our thoughts.

In other words, our interactions with the subject, which is ostensibly *out there*, are actually interactions with our own familiar models of it. Our mind is an economy of representations, and our thinking consists of negotiations within it. Thus the value of a model is measured by the ease with which it negotiates or interfaces with the other models in our repertoire.

Using models is all about translating between models. To say it another way, the observable aspects of a model are known only by its relationships with other models. It follows that, in order to objectively understand our interactions with a model, it is useful to understand the more general question of how models relate to other models. We add to our list a third and final philosophical postulate, which will be clarified throughout the paper:

(3) A model is known only by its relations to other models.

With our three modeling postulates in hand, we unpack the statement that category theory is a mathematical model of mathematical modeling. We are claiming, then, that category theory is about mathematically foregrounding, and formalizing, certain observable aspects of the subject of mathematical modeling.

To make this claim, we must answer the question, *What observable aspect of mathematical models does category theory (CT) foreground?* The answer is, roughly, that CT foregrounds a sense in which each mathematical model is known by its
relationship with other mathematical models. That is, CT foregrounds the third postulate as an observable aspect of modeling, and formalizes this postulate in terms of morphisms.

Clarifying the above statements will be the subject of the present paper. I will explain how notions in pure mathematics, such as vector spaces or groups, can be viewed as mathematical models, say of linearity or symmetry. I will show how models of linearity, symmetry, and action are all known by the interactions that exist between them.

While many mathematicians would agree with the statement that category theory is valuable for understanding and working with mathematical subjects, this paper does not attempt to prove it. However, our second postulate characterizes what such a statement would mean: the value of category theory should be measured by the extent to which it successfully mediates our interactions with mathematical models. If it is valuable, this would imply that category theory foregrounds and faithfully formalizes a propitious aspect of modeling: namely, that to gainfully use models, it is useful to be able to translate between them.

1.3. Plan of the paper. Our first goal will be to gently introduce categories in Section 2 using what we hope is a familiar mathematical subject, that of matrix arithmetic. We will then discuss linearity as a model in Section 3, where we will emphasize the categorical perspective, i.e., how the mathematical model of linear objects is reflected in (and determined by) the rules defining relationships between linear objects. In Section 4, we define the sort of relationship between categories that captures their structure, namely the functorial relationship. This enables us to consider symmetry and action in Section 5. Finally, we give a few concluding remarks in Section 6.

We will also continually return to our three postulates about modeling. In this way, we will be able to view category theory as a mathematical model of mathematical modeling.

2. Matrices: from groups to enriched categories

Our goal in this section is to introduce category theory by considering the case of matrix arithmetic. We will see that all the usual issues regarding dimension and invertibility are actually information about the structure of a category hidden behind the scenes.

Before we begin, it should be noted that matrices are among the most important tools in mathematical modeling. For example MATLAB, a highly popular technical computing program used by engineers of all kinds, is based primarily on matrix arithmetic. Thus, considering matrices is certainly fair game for thinking about mathematical modeling in the usual sense. Note that the problems usually considered in creating these tools, e.g., speed or accuracy issues, are not being addressed here. Instead, we are considering the abstract idea of matrices.

2.1. Matrices: a review of relevant aspects. A student who is new to linear algebra must learn a few things regarding when matrices can be added and multiplied, the properties of additive and multiplicative identity matrices, the issue of invertibility and non-invertibility, and so on. We now review these because they are precisely what is encoded in the single fact that matrices form a group-enriched
category. That is, we will introduce category theory by explaining how it models the subject of matrices.

For any natural numbers \( m, n \in \mathbb{N} \), let \( \text{Mat}_{m \times n} \) denote the set of \( m \times n \) matrices; as usual, \( m \) is the number of rows and \( n \) is the number of columns. For a matrix \( M \in \text{Mat}_{m \times n} \), we refer to \( (m, n) \) as the dimension of \( M \) and denote this fact by \( \dim(M) = (m, n) \). It is well-known that to add two matrices, say \( M + P \), they must have the same dimension. However, to multiply two matrices, say \( MP \), there is a different kind of restriction: the middle numbers must agree. More precisely, if \( \dim(M) = (m, n) \) and \( \dim(P) = (p, q) \), we require \( n = p \) in order for the product \( MP \) to make sense. In this case, \( \dim(MP) = (m, q) \).

Among the \( n \times n \) matrices, only some are said to be invertible. That is, there is a certain matrix \( I_n \), called the \( n \times n \) identity matrix,

\[
I_n = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix},
\]

and a matrix \( M \) is called invertible if there exists some \( N \) such that \( MN = I_n \) and \( NM = I_n \). Not every matrix is invertible; for example a matrix of all 0’s, as seen below on the left, is not invertible, but neither is the more average-looking matrix on the right:

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad \begin{bmatrix}
1 & 2 & 0 \\
0 & 2 & -2 \\
-2 & -3 & -1
\end{bmatrix}
\]

Note that for any \( m \times n \) matrix \( Q \), we have \( QI_n = Q \), and for any \( n \times p \) matrix \( R \) we have \( I_n R = R \).

2.2. The group of invertible \( n \times n \) matrix multiplication. Many scientists and engineers, from physicists modeling the dynamics of elementary particles, to 3D-animationists modeling the changes of camera-angle perspectives on a physical scene, use (either explicitly or implicitly) a group of invertible \( n \times n \) matrices as part of their toolset. That is, group theory is used in mathematical modeling. We will see below that group theory is a special case of category theory, and we will explain why groups are showing up in the theory of matrix multiplication. First, however, let us recall what a group is, using matrices as the working example.

Let \( \text{InvMat}_n \) denote the set of all invertible \( n \times n \)-matrices. Here are the rules that hold in \( \text{InvMat}_n \), which make it a group in the sense of abstract algebra.

1. There is an established multiplication formula for \( \text{InvMat}_n \). In other words, every two elements \( M, N \in \text{InvMat}_n \) can be multiplied, and the result is again in the group, i.e., \( MN \in \text{InvMat}_n \).
2. Multiplying is associative: for any \( M, N, P \in \text{InvMat}_n \), we have \( (MN)P = M(NP) \).
3. There is an established identity element in \( \text{InvMat}_n \). In other words, there is an element \( I_n \in \text{InvMat}_n \), such that \( I_n M = M = MI_n \) for every \( M \in \text{InvMat}_n \).
(4) There is an established inverse operation in $\text{InvMat}_n$. In other words, for every element $M$, there is an established element $N$, often denoted $N = M^{-1}$, such that $MN = I_n = NM$.

These rules encode a notion of symmetry that we will return to later in Section 5. By symmetry, we only mean an action that can be undone. Each $n \times n$ matrix $M$ encodes an action in the sense that it can be multiplied by any $n \times 1$ matrix $V$, also called an $n$-vector, and the result will be another $n$-vector, $MV$. This action can be undone if we then multiply by the inverse of $M$, as shown by the following argument:

$$M^{-1}(MV) = (M^{-1}M)V = I_nV = V.$$ 

2.3. The monoid of $n \times n$ matrix multiplication. Let $\text{Mat}_{n \times n}$ denote the set of all $n \times n$-matrices, including but not limited to the invertible ones. Rather than being a group, $\text{Mat}_{n \times n}$ is a monoid. As such, three out of the four rules for groups, as enumerated above, are true of $\text{Mat}_{n \times n}$. Namely,

(1) There is an established multiplication formula for $\text{Mat}_{n \times n}$. In other words, every two elements $M, N \in \text{Mat}_{n \times n}$ can be multiplied, and the result is again in the monoid, i.e., $MN \in \text{Mat}_{n \times n}$.

(2) Multiplying is associative: for any $M, N, P \in \text{Mat}_{n \times n}$, we have $(MN)P = M(NP)$.

(3) There is an established identity element in $\text{Mat}_{n \times n}$. In other words, there is an element $I_n \in \text{Mat}_{n \times n}$, such that $I_nM = M = MI_n$ for every $M \in \text{Mat}_{n \times n}$.

A monoid is like a group—elements can be multiplied, multiplication is associative, and there is an identity element—but there is no need for every element of a monoid to be invertible. Invertibility may be nice, but the fact is that not all matrices are invertible. If we want to think about more general matrix multiplication, we need to use monoids.

A group is thus a special kind of monoid, one in which every element is invertible. In the same way, a monoid is a special kind of category, one in which every two elements can be multiplied. Just as we broadened our view from the set of invertible matrices to the set of all $n \times n$ matrices, it is now time to further broaden our view, to consider all $m \times n$ matrices, i.e., matrices that are not necessarily square.

2.4. The category of matrix multiplication. The set of all matrices forms neither a group nor a monoid, but an $\mathbb{N}$-category, which we will denote $\text{Mat}_{-, -}$. Read the symbol $\text{Mat}_{-, -}$ as “blank-by-blank matrices.” In terms of sets, it is the union

$$\text{Mat}_{-, -} := \bigcup_{m,n\in \mathbb{N}} \text{Mat}_{m \times n}.$$ 

The three rules for $\mathbb{N}$-categories are like those for monoids, but with a slight relaxation in the multiplication rule.\(^2\)

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\(^2\) By $\mathbb{N}$-category, I mean a category with a position, or object, for every natural number $n \in \mathbb{N}$. For the definition of a general category, first add the following rule before the listed three:

(0) There is an established set $\text{Ob}$, whose elements are called objects.

Then, replace every occurrence of $\mathbb{N}$ with an occurrence of $\text{Ob}$. In other words, an $\mathbb{N}$-category is a category in which $\text{Ob} = \mathbb{N}$. In the standard definition of a category, $\text{Mat}_{-, -}$ is playing the role of the set of morphisms.
(1) There is an established multiplication formula for $\text{Mat}_{\_ \times \_}$, which is defined as long as the middle terms agree. In other words, matrices $M \in \text{Mat}_{m \times n}$ and $N \in \text{Mat}_{p \times q}$ can be multiplied if and only if $n = p$, and the result is again in the category, i.e., $MN \in \text{Mat}_{m \times q}$.

(2) Multiplying is associative: for any $M, N, P \in \text{Mat}_{\_ \times \_}$, if $MN$ and $NP$ can be multiplied then $(MN)P = M(NP)$.

(3) For each $n \in \mathbb{N}$ there is an established identity element in $\text{Mat}_{n \times n}$. In other words, there is an element $I_n \in \text{Mat}_{n \times n}$, such that $MI_n = M$ for every $M \in \text{Mat}_{m \times n}$ and $I_nN = N$ for every $N \in \text{Mat}_{n \times q}$.

Let’s consider each dimension $n \in \mathbb{N}$ to be a kind of context. Then groups are about actions which do not change context and which are reversible; monoids are about actions which do not change context but which may be irreversible; and categories are about actions which may change context and which may be irreversible. While groups and monoids are said to have elements, the elements in a category (the elements of $\text{Mat}_{\_ \times \_}$ in the above case) are usually called morphisms.

Note that although our definition of category looks very much tuned to matrices, it is actually quite general. When someone speaks of a category, they mean nothing more than an establishment of the structures and rules shown in (0)–(3) above.

2.5. The group-enriched category of matrix arithmetic. We still have not grappled with the fact that matrices can be added. For every pair of natural numbers $m, n \in \mathbb{N}$ the set $\text{Mat}_{m \times n}$ can be given the structure of a new group, which encodes the addition of $m \times n$ matrices. It is a group because one can check that the rules for groups, found in Section 2.2 are satisfied for $\text{Mat}_{m \times n}$, as long as one changes the notion of multiplication to that of addition.

(1) There is an established addition formula for $\text{Mat}_{m \times n}$. In other words, every two elements $M, N \in \text{Mat}_{m \times n}$ can be added, and the result is again in the group, i.e., $M + N \in \text{Mat}_{m \times n}$.

(2) Adding is associative: for any $M, N, P \in \text{Mat}_{m \times n}$, we have $(M + N) + P = M + (N + P)$.

(3) There is an established (additive) identity element in $\text{Mat}_{m \times n}$. In other words, there is an element $Z_{m, n} \in \text{Mat}_{m \times n}$, such that $Z_{m, n} + M = M = M + Z_{m, n}$ for every $M \in \text{Mat}_{m \times n}$.

(4) There is an established (additive) inverse operation in $\text{Mat}_{m \times n}$. In other words, for every element $M$, there is an established element $N$, often denoted $N = -M$, such that $M + N = Z_n = N + M$.

Of course, $Z_{m, n}$ is the $m \times n$ matrix of zeros, and $-M$ is the matrix obtained by multiplying each entry in $M$ by -1.

Thus we see two types of group structures arising in the story of matrix arithmetic: a group encoding matrix addition for $m \times n$ matrices, for any $m, n \in \mathbb{N}$, and a group encoding matrix multiplication for invertible $n \times n$ matrices, for any $n \in \mathbb{N}$. And there is further interaction between the additive and multiplicative operations in the category of matrices; namely, multiplication of matrices distributes over addition in the sense that $M(N + P) = MN + MP$.

The entire addition and multiplication story for matrices, discussed so far, is subsumed in a single category-theoretic statement: Matrices form a group-enriched category with objects $\text{Ob} = \mathbb{N}$. This articulates:

- the dimensionality requirements for multiplication and addition,
Let us clarify the final point. From the perspective that matrices form a category, the notion of invertible matrices comes for free. That is, for every category $\mathcal{C}$, and for every object $n$ in it, there is a group of invertible morphisms from $n$ to itself, called the automorphism group of $n$. When $\mathcal{C}$ is the category $\text{Mat}_{\times \times}$, in which each object is a natural number $n \in \mathbb{N}$, the automorphism group of $n$ is $\text{InvMat}_n$.

In the next section, we will discuss these ideas in the context of mathematical modeling.

3. Modeling linearity

In Section 2 we showed how category theory models the subject of matrix arithmetic. But matrices themselves are one aspect of a highly-valued mathematical modeling framework, namely that of vector spaces. Vector spaces are the mathematical model of linearity, as we will discuss in Section 3.1. The category of vector spaces models this model by foregrounding the sense in which each vector space is known by its relationship with other vector spaces. We will discuss this in Section 3.2.

3.1. Vector spaces: the mathematical models of linearity. The notion of linearity shows up in our visual, linguistic, and cognitive interactions with the world. Indeed our visual system is hardwired to highlight straight lines. In our language, simplicity and goodness are often equated with flatness and straightness; English words such as plain, straightforward, right, direct, correct and true invoke straightness. And linearity also appears to be inherent in our best scientific understanding of the physical universe. For example, general relativity postulates that the universe is locally linearizable (i.e., that close enough to any point, the curved space of the universe can be laid flat), and the predictions founded on that assumption match astoundingly with experiments. Even in mathematics we find that linearity often goes hand in hand with simplicity, where many of our most successful techniques work by reducing a difficult case to a linear one.

As may by now be clear, when I speak of linearity I am referring not only to lines, but to the general notion of flatness, e.g., to planes and higher-dimensional flat spaces. Unlike in curved spaces, such as soap bubbles, we find that in flat spaces a line can point in any direction without having to curve. In mathematics, an abstract flat space is called a vector space, and there is an inherent notion of line in any vector space.

But what are these vector spaces, and what are their relation with linearity in our visual perception, our language, and our cognition? The relationship here is that vector spaces are valuable models of our notions of linearity and flatness. That is,

(1) Vector spaces foreground and formalize certain observed aspects of linearity and relationships between them.

(2) Our visual, linguistic, and cognitive interactions with linearity are successfully mediated by the vector space model.

(3) Vector spaces are known by the relationships between them.
Let’s begin by considering the observable aspects of linearity. Newton’s first law of motion is that the velocity of any object remains constant unless a force is applied to the object. In other words, time acts as a scalar multiplier for the motion of objects: doubling the time doubles the distance traveled but does not alter the direction. Scalar multiplication is at the heart of our notion of linearity: a line is the set of scalar multiples of a given ray.

But what about higher-dimensional linear spaces? In his thought experiments about motion, Galileo imagined a flat plane on which objects could move unobstructed? A plane can be imagined as a 2-dimensional analogue of a line; it is in some sense the simplest 2-dimensional space. A plane has enough structure to discuss not just distance but also direction. Both the angle between lines in a plane and the degree of inclination of a plane embedded in space were necessary for the laws of motion Galileo wished to discuss.

There are other aspects to planes and flat spaces that are important in modeling. Namely, in a flat space, the different directions do not interact. That is, moving forward in \( x \) does not cause any change to occur in \( y \) (compare with a parabola or sphere). An \( n \)-dimensional vector space is a space in which there are \( n \) degrees of freedom, which do not interact with one another. That is, there is a well-defined notion of coordinate system, whereby every point in the space is uniquely determined by \( n \) numbers. This does not hold on a sphere: while it is the case that every point is determined by its latitude and longitude \((\text{lat}, \text{long})\), this determination is not unique. For example, we have \((90^\circ, \text{long}_1) = (90^\circ, \text{long}_2)\) for every pair of longitudes \(\text{long}_1, \text{long}_2\). In other words, the coordinates of latitude and longitude are not free from one another; they interact at the north and south poles. This issue may seem unimportant, but the point is that such caveats cannot be rectified on the sphere precisely because it is not flat.

As mentioned above, the established mathematical models of flat spaces are vector spaces. A (real) vector space is a collection of rays, called vectors, including a ray of length 0, such that each ray can be scaled by any real number, e.g., doubled or tripled in length, and such that any two rays can be added together to form a new ray. When made precise, these ideas are sufficient to define a coordinate system, or basis, which is a minimal set of vectors that span the whole space. For example, if the basis consists of three vectors, \(x, y, z\), then every ray \(r\) in the space can be uniquely obtained by adding together scalar multiples of these three vectors, say \(r = 4x + 3y - 1.5z\).

Thus the observable aspects of linear spaces foregrounded by the mathematical model of vector spaces are:

- the existence of a zero vector,
- scalar multiplication of vectors, and
- addition of vectors.

These aspects and the relationships between them, e.g. commutativity of addition, distributivity of scalar multiplication, existence of additive inverses, etc., are precisely the content of the formal definition of vector space.

René Descartes (and simultaneously Pierre de Fermat) developed the notion of axes, whose coordinates specify any point in a plane (or 3D space). So coordinate systems were invented long before the abstract notion of vector spaces was.

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3Whenever we speak of vector spaces, we mean finite-dimensional vector spaces over the field \(\mathbb{R}\) of real numbers.
However, a major contribution of vector spaces is that they formalize the ability to change between coordinate systems, e.g. by scaling or adding axes to form new axes. Since one can change coordinates at will, perhaps one does not need them at all. In fact, the vector space concept formalizes the idea, due to Lagrange, that the flat space exists, and calculations can take place, a priori—without need for choosing coordinates.

Geometric considerations, such as how two intersecting lines span a plane and two intersecting planes (in space) form a line, are all completely captured by the vector space model. Moreover, various unexpected exceptions—such as the case where the two intersecting planes happen to be the same (and hence the intersection is not a line but the plane), or the case in which the two planes inhabit a space of more than three dimensions and hence intersect in a point rather than a line—are exposed in the mathematical model.

We have shown, then, that the vector space model of linearity successfully mediates our interactions with flat spaces. We have also discussed the system of relationships (commutativity, associativity, distributivity) between various aspects of linearity (zero, scalars, sums). This justifies our first two postulates in this setting, so it remains to explain how vector spaces are known by the relationships between them. We can kill two birds with one stone, because our overarching goal is to explain how category theory foregrounds and formalizes a sense in which each mathematical model is known by its relationship with other models.

3.2. **\textit{Vect}_R**: the categorical model of vector spaces. We will thus consider the observable aspects of vector spaces (zero, scalars, sums) using the mathematical model of category theory. We will highlight how these aspects of a given vector space can be known by the relationships between it and other vector spaces. Category theory formalizes relationships as morphisms (see Section 2.4). The tried-and-true morphisms between vector spaces are called linear transformations, which we now recall.

Let \( V \) and \( W \) be vector spaces. A linear transformation \( f : V \rightarrow W \) is a method for converting vectors in \( V \) to vectors in \( W \). But \( f \) cannot be an arbitrary function: in order to be called a linear transformation, it must preserve the structures we have formalized in our model, namely zero, scalars, and sums. That is, it must satisfy

- \( f(0) = 0 \),
- \( f(r \cdot v) = r \cdot f(v) \), and
- \( f(v + v') = f(v) + f(v') \)

for any \( v, v' \in V \) and \( r \in \mathbb{R} \). In fact, there is a tight connection between the set of linear transformations \( V \rightarrow W \) and the set \( \text{Mat}_{|V| \times |W|} \) of \( |V| \times |W| \) matrices, where \( |V|, |W| \in \mathbb{N} \) are the dimensions of \( V \) and \( W \). Once one chooses a basis for \( V \) and \( W \), this connection becomes a one-to-one correspondence: each matrix represents a single linear transformation. There is an equivalence between the category \( \text{Vect}_\mathbb{R} \) and the category \( \text{Mat}_{- \times -} \) described in Section 2.1; we will henceforth elide the difference.

The point so far is that the morphisms in \( \text{Vect}_\mathbb{R} \) take seriously the structures (zero, scalars, sums) we have formalized in each individual model of linearity. In general, the morphisms in a category are designed to reflect (by preserving) the
structures that define each object. Thus, an object in a category is not only (epistemologically) known by its relationships with other objects; it (ontologically) is what it is by virtue of its relationships with other objects. In a categorical model, the knowing of an object and the being of an object are essentially identical.

In the case of a vector space $V$, pure category theoretic reasoning only allows one to consider those aspects of $V$ that can be defined in terms of morphisms into or out of it. For example, the most important aspect of a vector space is the notion of vector. Any individual vector $v \in V$ defines a ruled line in $V$; that is, all scalar multiples of $v$ lie on a single line, ruled by tick-marks at every integer multiple of $v$. But there is a vector space that represents ruled line-hood, the one-dimensional vector space $\mathbb{R}$ of all scalar multiples of 1. We can now say how the notion of vector is itself defined in terms of morphisms in $\text{Vect}_\mathbb{R}$: for any vector $v \in V$ there is a unique morphism $\mathbb{R} \to V$ for which $v$ is the image of the vector $1 \in \mathbb{R}$.

So if one wishes to think categorically about vectors in a vector space, one can think only in terms of morphisms.\(^4\) In fact, even zero, scalar multiplication, and addition of vectors can be understood in terms of linear transformations. There is a unique map from the 0-dimensional vector space $\mathbb{R}^0$ to any vector space $V$, and it picks out the zero-vector in $V$. Any linear transformation $V \to W$ can be scaled by any real number $r \in \mathbb{R}$; for example if $v: \mathbb{R} \to V$ is a vector represented by a linear transformation as above, then scaling the linear transformation $rv: \mathbb{R} \to V$ represents the scaled vector. There is a similar way to understand addition of vectors in terms of morphisms.

The upshot is that the linear transformations between vector spaces can account for all the formal structure that makes a vector space a vector space. But what about the informal, cognitive aspects of linearity, namely ideas like rays, lines, inclined planes, projections, intersections, and coordinate systems? We have seen that lines in $V$ are actually the same as linear transformations $\mathbb{R} \to V$. But similarly the inclusion of an inclined plane in 3D space is a linear transformation $\mathbb{R}^2 \to \mathbb{R}^3$. The projection of 3D space onto a line or plane is given by a linear transformation $\mathbb{R}^3 \to \mathbb{R}$ or $\mathbb{R}^3 \to \mathbb{R}^2$. Any given coordinate system on a vector space of dimension $n$ is given by a unique linear isomorphism $\mathbb{R}^n \to V$. The notion of intersecting various lines and planes are also beautifully and completely described by a kind of interplay between morphisms, known as pullback.

It is something of a miracle that so many of our intuitive ideas about vector spaces are describable simply in terms of the morphisms between them in $\text{Vect}_\mathbb{R}$. However, to a category-theorist, this is routine. Every well-studied category has this property, because this is precisely the observable aspect of modeling that category theory foregrounds. In other words, in order for a subject to be model-able by category theory, its objects of study must be determined by their morphisms to and from other objects. There is a general theorem by Nobuo Yoneda to this effect: any object $c$ in any category $\mathcal{C}$ is essentially determined by the morphisms into (or out of) $c$.

In other words, categories can only model “relationally-determined” subjects, subjects in which each object is ontologically determined by its relationships to other objects. In this case, what is surprising is how extensive the reach of category theory is. The fact that category theory is consistently used to describe so many

\(^4\)Here is another way to put this fact, in the case $V = \mathbb{R}^n$. Each vector in $\mathbb{R}^n$ can be written uniquely as an $n \times 1$ matrix, which can be identified with a unique morphism $\mathbb{R} \to \mathbb{R}^n$. 
parts of mathematics, from topological spaces to groups to ordered sets to measure spaces, means that all of these subjects are relationally-determined. This justifies our third postulate, at least for these specific cases.

In the next section we define functors, which relate different categories like morphisms relate objects. This will prepare us to discuss symmetry and action in Section 5.

4. Relationships between high-level models

Throughout this paper, we have roughly been interpreting models and relationships, in the context of category theory, as follows. Each object in a category is a model. For example, each vector space is a model of linearity: \( \mathbb{R} \) is a model of linehood, \( \mathbb{R}^2 \) is a model of plane-hood, etc. But each entire category is also a model, albeit one of a higher level. For example, \( \text{Vect}_\mathbb{R} \) is our model of linearity itself.

Each low-level model (object) is defined by its relationships (morphisms) to other low-level models, where these relationships are formalized in the higher-level model (the category). For example, we showed how the linearity—the vector space-ness, as formalized by zero, scalars, and sums—of each vector space \( V \in \text{Vect}_\mathbb{R} \) is defined in terms of morphisms between \( V \) and other vector spaces.

But if low-level models are known by the relationships between them, it may be that high-level models are as well. We now discuss the categorical model of higher-level models, i.e., the category of categories.

4.1. \( \text{Cat} \): the category of categories.\footnote{I apologize for the vagueness with respect to the size issue, but it is not relevant to our discussion.} There is a larger-sized category, denoted \( \text{Cat} \), which includes every normal-sized category as an object. If \( \mathcal{C} \) and \( \mathcal{D} \) are categories, a morphism between them in \( \text{Cat} \) is called a functor, and can be denoted \( F : \mathcal{C} \to \mathcal{D} \).

Just like any morphism, a functor is a relationship between two categories that preserves the defining structure of categories. Recall from Section 2.4 that the defining structure of a category \( \mathcal{C} \) is a set of objects, a set of morphisms, an identity morphism for each object, and a formula for composing morphisms. These satisfy some laws, namely that composing any morphism \( g \) with an identity morphism gives back \( g \) and that composition is associative. To say that functors preserve the structures that define categories is shorthand for the following more formal statement. A functor \( F : \mathcal{C} \to \mathcal{D} \) must:

- assign to each object \( c \in \text{Ob}(\mathcal{C}) \) an object \( F(c) \in \text{Ob}(\mathcal{D}) \),
- assign to each morphism \( g : c \to c' \) in \( \mathcal{C} \) a morphism \( F(g) : F(c) \to F(c') \) in \( \mathcal{D} \),
- ensure that the morphism assigned to each identity in \( \mathcal{C} \) is an identity in \( \mathcal{D} \), i.e., for all \( c \in \text{Ob}(\mathcal{C}) \), a functor \( F \) must ensure that \( F(\text{id}_c) = \text{id}_{F(c)} \), and
- ensure that the composition formula in \( \mathcal{C} \) is compatible with the composition formula in \( \mathcal{D} \), i.e., \( F(g \circ_C h) = F(g) \circ_D F(h) \).

There is a very basic category, which represents object-hood in \( \text{Cat} \) the way that \( \mathbb{R} \) represents ruled line-hood in \( \text{Vect}_\mathbb{R} \). Let \( O \in \text{Ob}(\text{Cat}) \) denote the category with one object, \( \text{Ob}(O) = \{ * \} \); only one morphism, \( \text{id}_*: * \to * \); and the composition formula says \( \text{id}_* \circ \text{id}_* = \text{id}_* \). This category \( O \) might be pictured

\[
(2)
\]

[\( \text{Cat} \): the category of categories.\footnote{I apologize for the vagueness with respect to the size issue, but it is not relevant to our discussion.}]
For any category $\mathcal{C}$, the objects of $\mathcal{C}$ are the same as the functors $\mathbf{O} \rightarrow \mathcal{C}$. Similarly, there is a category $\mathbf{M}$, which might be pictured

(3) \[ \bullet \rightarrow \bullet \]

that represents morphism-choo in $\mathbf{Cat}$. That is, for any category $\mathcal{C}$, the morphisms of $\mathcal{C}$ are the same as functors $\mathbf{M} \rightarrow \mathcal{C}$. There is also a functor that represents identity morphisms as well as a functor that represents composition formulas.

In other words, the morphisms in $\mathbf{Cat}$ can account for all the formal structure that makes a category a category: objects, morphisms, identities, and compositions. This justifies our third postulate in the case of categories: the structure of any given category $\mathcal{C}$ is completely determined by the system of functors that map to it from other categories.

4.2. Two formal justifications for the third postulate. Our goal has been to show that category theory is a mathematical model of mathematical modeling. We discussed the rough meaning of this statement by introducing three postulates about modeling in Section 1, and we said that category theory foregrounds and formalizes the third: that each model is known by its relationships with other models.

In a single category, Yoneda's lemma offers one explication and formal justification of this statement, given the interpretation in the first paragraph of Section 4. However, any functor between two categories provides another way to relate these high-level models, namely by something reminiscent of analogy. That is, if $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor, it relates each object $c \in \mathbf{Ob}(\mathcal{C})$ with an object $F(c) \in \mathbf{Ob}(\mathcal{D})$. The morphisms between $c$ and its neighbors in $\mathcal{C}$ are preserved by $F$, which makes $F$ act as a sort of analogy, as observed by Brown and Porter.\(^6\) Our third postulate was intended as a ordinary-language hybrid of the Yoneda concept and the functorial analogy concept.

We now return to a few of the most important areas of mathematical modeling: symmetry and dynamics. The Yoneda interpretation of our third postulate is always at work, but we will briefly explore the second: how high-level analogies bring out the features of a subject.

5. Symmetry and action

Consider the reflective symmetry of the visible human body or the rotational symmetry of Escher's \textit{Drawing Hands}. The question of symmetry is about reversible action-ability, e.g., the ability to reflect an object and the ability to rotate an object by $180^\circ$ are both reversible. To say that an object is symmetrical with respect to an act is to say that it does not change when it undergoes this act; e.g., the visible human body is unchanged as it undergoes mirror reflection and \textit{Drawing Hands} is unchanged as it undergoes $180^\circ$ rotation.

We begin in Sections 5.1 and 5.2 by considering various questions of symmetry and action-abilities in the above sense, but this is mainly to set up our main point. Namely, a group $G$ of symmetries (reversible action-abilities) exists independently of any thing that is so-symmetrical (unchanged under corresponding actions). But what is the connection of the symmetry to the symmetrical? It is captured simply

\(^6\)See Brown, R.; Porter, T. \textit{Category theory: an abstract setting for analogy and comparison}, \url{http://pages.bangor.ac.uk/~mas010/pdffiles/Analogy-and-Comparison.pdf}, whose aim is similar to, but whose scope is more ambitious than, the present paper.
by a functorial connection between one category (the symmetry category $G$) and the space category $\text{Vect}_\mathbb{R}$. The functorial connection between a category $G$ and another category $C$ is called an action of $G$ on an object of $C$, and we will study actions in Section 5.3. Finally in Section 5.4, we will consider dynamical systems, which are classical exemplars of mathematical modeling, and which again are nothing but functorial connections (roughly, between time and space).

5.1. Modeling reversible action-ability with groups. A square $S$, with unit-length sides, centered at the origin in $\mathbb{R}^2$ is symmetrical with respect to eight action-abilities: it can be rotated and reflected in any combination, it can be:

- Do nothing
- Rotate $90^\circ$
- Rotate $180^\circ$
- Rotate $270^\circ$
- Vertical flip
- $90^\circ$ & Flip
- $180^\circ$ & Flip
- $270^\circ$ & Flip

Neither the word Square, the gray dot, nor the labels $A, B, C, D$ are symmetrical with respect to the same eight-element group that the square itself is. They are drawn in (4) to show that unlike the square, they undergo change when we act on them in these eight ways.

Every one of these eight actions on the square is reversible, and its reverse is another one of the eight actions. The actions are also serializable, in the sense that if each of $a_1, \ldots, a_n$ is one of the eight, then so is the process obtained by doing them in series, called their composition; we denote it $a_1 a_2 \cdots a_n$. The eight element set, which has an identity element and is closed under inverses and compositions, is a group (see Section 2.2), called the dihedral group of order 8, and denoted $D_8$.

Note that the group $D_8$ acts on the square, but in fact $D_8$ exists independently of the square. That is, $D_8$ could just as well act on an octagon or on a single point at the origin. We refer to the elements of $D_8$ as action-abilities because each is able to act on a variety of things. The elements only become actions when they are applied to something, such as a square. We will discuss the notion of action itself in Section 5.3.

Every group can be modeled as a category $G$ with many morphisms, each corresponding to an action-ability, but with only one object. The unique object of $G$ stands for “the abstract thing that is unchanged under these actions.” The identity morphism corresponds to the ability not to act, and the composition formula in $G$ corresponds to the requirement that the serialization of action-abilities is an action-ability. Groups are categories that encode symmetries.

The group $D_8$ is one type of symmetry; there are many others. For example, consider the line that goes through the origin in $\mathbb{R}^2$ at an angle of $30^\circ$. If you take each point and multiply its distance from the origin by a non-zero number $k$, but
do not change the angle, the total result is again the same line: it is unchanged by non-zero scaling. Thus there is a different group, $\mathbb{R}_{\neq 0}$, of non-zero scaling abilities. It is a different type of symmetry than $D_8$, but just as much able to encode a kind of reversible ability to act.

Before we move on, note that we now know that each group is a category. A functor between two groups is called a group homomorphism. The category of all groups and group homomorphisms, denoted $\text{Grp}$, is the high-level model of symmetry itself, just like the category $\text{Vect}_\mathbb{R}$ is the high-level model of linearity.

5.2. **Modeling action-ability with monoids.** A monoid is like a group, except that not all its elements need be invertible. For example, there is a monoid $M$ consisting of four action-abilities, which we can depict using their action on a windowpane.

\[
\begin{align*}
I &= \text{Do nothing} & V &= \text{Vertical Crush} & H &= \text{Horizontal crush} & T &= \text{Total crush} \\
\begin{array}{c|c}
\text{A} & \text{B} \\
\hline
\text{C} & \text{D}
\end{array} & AC \xrightarrow{\text{BD}} & BD & AC \xrightarrow{\text{ABCD}}
\end{align*}
\]

Any sequence of these action-abilities can be serialized, and the result is also an action-ability. Also, there is an identity, “do nothing” action-ability. Thus we have a monoid, and it is not a group because three of the action-abilities are irreversible. Each element of $M$ acts on the windowpane: it sends every point in the windowpane to another point in the windowpane.

A monoid is a category with one object, but with possibly many morphisms from that object to itself, as depicted here:

\[
\begin{array}{c}
\text{\textcircled{}}
\end{array}
\]

The fact that there is only one object means that any two morphisms can be composed; this is the serializability.

In passing we note that the connection between groups and monoids (every group is a monoid but not vice versa) is captured by a high-level analogy, i.e. a functor $\text{Grp} \to \text{Mon}$, where $\text{Mon}$ is the category of all monoids. There are even functors going the other way, $\text{Mon} \to \text{Grp}$, but we will not discuss any of them here.

5.3. **Modeling action with outgoing functors.** The four elements of the monoid $M$ from the previous section (5.2) can be represented by the following matrices:

\[
\begin{align*}
I &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & V &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & H &\mapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & T &\mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{align*}
\]

But how exactly can we enunciate this connection between elements of our monoid $M$ and these matrices, i.e., morphisms in $\text{Vect}_\mathbb{R}$.

The most concise and straightforward way I know is to use functors. A functor $F : M \to \text{Vect}_\mathbb{R}$ assigns to each object in $M$ an object in $\text{Vect}_\mathbb{R}$. Since there is only one object in $M$, we get only one vector space; in the above case, it is $\mathbb{R}^2$. A functor $F$ also assigns to each morphism in $M$ a morphism in $\text{Vect}_\mathbb{R}$; the four elements $\{I, V, H, T\}$ of the monoid are then sent to four linear transformations
\[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

In mathematics, if \( M \) is a monoid (or a group), we define an \( M \)-action on a set to be a functor \( M \to \text{Set} \); we define an \( M \)-action on a vector space to be a functor \( M \to \text{Vect}_\mathbb{R} \); and so on. Thus in (6) we have established an \( M \)-action on \( \mathbb{R}^2 \). The study of functors from a group to \( \text{Vect}_\mathbb{R} \) is often called representation theory in mathematics.

There is a natural category structure on the class of functors between any two categories. These functor categories are often important, as they are in representation theory. In the next section we briefly describe dynamical systems as such.

5.4. **Dynamical systems.** To model the behavior of a system that changes in time, one must decide whether to formalize its change as continuous or discrete. For example, the internal states of a computer may be best modeled with a discrete dynamical system, if it has an internal clock that dictates discrete times at which changes can occur. On the other hand, the concentrations of chemicals in a reaction process are changing continuously, so the reaction is often modeled by a continuous dynamical system.

A **discrete dynamical system** is defined to be a set \( S \) and a where-to-go-next function \( f: S \to S \). A case where \( S \) has eleven elements may be depicted

![Diagram of a discrete dynamical system]

In fact a discrete dynamical system is the same thing as a functor \( \mathbb{N} \to \text{Set} \), where \( \mathbb{N} \) is the monoid of natural numbers under addition. In other words, \( \mathbb{N} \) is a one-object category with morphisms \( 0, 1, 2, \ldots \), and the composite of \( i \) with \( j \) is \( i + j \). A functor \( \mathbb{N} \to \text{Set} \) consists of a set \( S \) and a function \( f^n: S \to S \) for every natural number \( n \). The fact that a functor must preserve the composition formula implies that \( f^i(f^j(s)) = f^{i+j}(s) \) for every \( s \in S \).

A **continuous dynamical system** (sometimes called a flow) is a topological space \( S \) and a continuous function \( f: S \times \mathbb{R}_{\geq 0} \to S \), such that

\[ f(s, 0) = s \quad \text{and} \quad f(f(s, t_1), t_2) = f(s, t_1 + t_2). \]

Here \( f(s, 3.14) \) would tell us where the point \( s \) will be after 3.14 units of time. In fact, a continuous dynamical system can be modeled categorically as a topologically-enriched functor \( \mathbb{R}_{\geq 0} \to \text{Top} \). We do not expect the reader to understand this statement exactly, but the idea is that we can concisely capture the definition of dynamical systems using functors.

We have now shown that the class of dynamical systems (either discrete or continuous) is itself a category of functors from a time category to a space category. For discrete dynamical systems, time and space are modeled discretely (with time as \( \mathbb{N} \) and space as a set). For continuous dynamical systems, they are modeled...
continuously (with time as $\mathbb{R}_{\geq 0}$ and space as a topological space). However, the
dynamical system itself is a functor between these categories.

Time and space can be modeled as independent categories, but part of the human
concept of time is that it acts on objects in space. That is, we understand our model
of time and space by connecting them. This is an example of the third postulate,
and we have modeled it formally in this section using functors.

6. Conclusion

Karl Popper said, “A theory that explains everything, explains nothing.” If cat-
egory theory models algebra, geometry, logic, computer science, probability, and
more, is it not trying to be a model of everything? The point is a bit subtle.

Category theory is not a theory of everything. It is more like, as topologist
Jack Morava put it,7 “a theory of theories of anything”. In other words, a model
of models. It leaves each subject alone to solve its own problems, to sharpen and
refine its toolset in the ways it sees fit. That is, CT does not micromanage in the
affairs of any discipline. However, describing any discipline categorically tends to
bring increased conceptual clarity, because conceptual clarity is CT’s main concern,
its domain of expertise.

Category theory has continually sharpened and refined its own toolset, i.e., its
ability to articulate the various objects, relationships, properties, structures, and
methods that show up throughout mathematics. It consistently shows itself as a
powerful mode of mathematical thinking, and there is no a priori reason it cannot
be similarly successful in science more broadly.

However, a question remains, one which may have been interesting to Edmund
Husserl: what of the modeler? Who is the one that decides that a certain cate-
gory adequately models a certain subject? Who is the one that finds value in the
category-theoretic mode of thought? My hope for category theory is that it can
aid in a mathematically rigorous form of phenomenological reduction,8 in which the
process of thinking is itself elucidated. Category theory could be considered a truly
profound model of modeling if it could model the cognitive apparatus itself, i.e., if
it could mathematically communicate the relationship between subject, model, and
modeler.

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7See Morava, J. “Theories of anything”, http://arxiv.org/abs/1202.0684.
8See Cogan, J., “The phenomenological reduction”, http://www.iep.utm.edu/phen-red/.

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