Gauging the Superstring under the Group $SU(4)$

Davoud Kamani

Faculty of Physics, Amirkabir University of Technology (Tehran Polytechnic)
P.O.Box: 15875-4413, Tehran, Iran
e-mail: kamani@aut.ac.ir

Abstract

The superstring theory in the light-cone gauge admits various gauge symmetries. Therefore, we gauge the superstring sigma model in the light-cone gauge under the gauge group $SU(4)$. Some properties of the gauged action and the corresponding current will be studied. Besides, two modified actions for the superstring will be obtained.

PACS: 11.25.-w; 11.15.-q

Keywords: Superstring; $SU(4)$ group; Gauge symmetry.
1 Introduction

The gauged actions in the string theory have been studied from the various point of view. The worldsheet gauge fields have been the object of several investigations and they can be introduced in the various string models [1]-[15]. In other words, the worldsheet gauge fields provide a concise Lagrangian formulation of different string models. The two-dimensional $SU(N)$ Yang-Mills theory is part of these models [16]-[23].

Previously we studied the superstring theory in the presence of a $U(1)$ worldsheet gauge field [15]. Here our attention is on the non-Abelian case. For this, we consider the superstring sigma model in the light-cone gauge. This theory admits a global $SO(8) \subset SO(1,9)$ invariance. One can obviously consider the embedding $SU(4) \subset SO(8)$. Therefore, we gauge this theory under the gauge group $SU(4)$, which is a subgroup of the transversal $SO(8)$ group.

We treat the worldsheet gauge field as an independent degree of freedom. In addition, we impose the complex structure in the target space. The string coordinates form a four-component complex field, which obeys the gauged Klein-Gordon action. In the same way, the fermionic degrees of freedom form an eight-component complex spinor field. This extended spinor field obeys the Dirac action with special operator. However, we shall study some properties of the gauged action and the current associated to the gauge symmetry. We use this gauging to modify the superstring action. Therefore, we obtain a modified action and an effective action for the superstring.

This paper is organized as follows. In section 2, the superstring action in terms of extended variables, appropriate for the $SU(4)$ gauge symmetry, will be given. In section 3, this action under the group $SU(4)$ will be gauged. In section 4, two modified actions for the superstring will be obtained. Section 5 is devoted to the conclusions.

2 The superstring action in terms of the complex fields

The superstring theory in the light-cone gauge is described by 8 scalars (describing the spacetime coordinates) and 8 Majorana spinors (describing their fermionic partners) on the worldsheet. The corresponding action is

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma (\eta^{\alpha\beta} \partial_\alpha X^I \partial_\beta X^I - i \bar{\psi}^I \rho^\alpha \partial_\alpha \psi^I),$$

(1)
where $I \in \{1, 2, ..., 8\}$ and $\alpha, \beta \in \{0, 1\}$. The worldsheet metric is $\eta_{\alpha\beta} = \text{diag}(-1, 1)$. In the Majorana basis, the matrices $\rho^0$ and $\rho^1$ are

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (2)$$

This action obviously has the global $SO(8)$ symmetry.

We build the following complex variable from the string coordinates

$$Y = \begin{pmatrix} X^1 + iX^2 \\ X^3 + iX^4 \\ X^5 + iX^6 \\ X^7 + iX^8 \end{pmatrix}. \quad (3)$$

Let $\psi^I = \begin{pmatrix} \psi^I_- \\ \psi^I_+ \end{pmatrix}$ denote a spinor field of the worldsheet. Thus, in the same way, these fermionic fields also define the variables

$$\Psi^\pm = \begin{pmatrix} \psi^1_\pm + i\psi^2_\pm \\ \psi^3_\pm + i\psi^4_\pm \\ \psi^5_\pm + i\psi^6_\pm \\ \psi^7_\pm + i\psi^8_\pm \end{pmatrix}. \quad (4)$$

In terms of these extended variables, the action (1) takes the form

$$S' = -\frac{1}{4\pi\alpha'} \int d^2\sigma [\partial_\alpha Y^\dagger \partial^\alpha Y - 2i(\Psi^\dagger_- \partial_+ \Psi_- + \Psi^\dagger_+ \partial_- \Psi_+)], \quad (5)$$

where $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. This form of the action enables us to study its $SU(4)$ gauge symmetry.

Consider the matrix $U \in SU(4)$ and let $U$ be independent of the worldsheet coordinates $\sigma$ and $\tau$. Therefore, under the global $SU(4)$ transformations

$$Y \rightarrow UY,$$

$$\Psi_- \rightarrow U\Psi_-,$$

$$\Psi_+ \rightarrow U\Psi_+,$$ 

the action $S'$ is invariant.
The gauged action

Now we introduce local vector gauge symmetry by adding a two-dimensional gauge field as worldsheet dynamical degree of freedom. In other words, we deform the action (5) to obtain the local SU(4) gauge symmetry. Thus, we consider the transformations (6) with the coordinate-dependent $U$, 

$$U(\sigma, \tau) = e^{i\lambda_a(\sigma, \tau)T^a},$$

(7)

where \{ $T^a | a = 1, 2, ..., 15$ \} are generators and \{ $\lambda_a(\sigma, \tau) | a = 1, 2, ..., 15$ \} are local parameters of the group SU(4). This implies replacement of the derivative $\partial_\alpha$ with the covariant derivative $D_\alpha$,

$$\partial_\alpha \rightarrow D_\alpha = I_{4 \times 4} \partial_\alpha + igA_\alpha = I_{4 \times 4} \partial_\alpha + igA^a_\alpha T^a.$$  

(8)

The gauge field $A_\alpha$ is a $4 \times 4$ Hermitean matrix which lives in the string worldsheet, and $g$ is the corresponding coupling constant. According to (8) we should also do the following replacements

$$\partial_\pm \rightarrow D_\pm = I_{4 \times 4} \partial_\pm + igA_\pm,$$

(9)

where $A_\pm = \frac{1}{2}(A_0 \pm A_1)$.

Adding all these together, we obtain the gauged action

$$S_{\text{gauged}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma [(D_\alpha Y)^\dagger D^\alpha Y - 2i(\Psi_+^\dagger D_+ \Psi + \Psi_-^\dagger D_- \Psi) + \pi\alpha' F^a_{\alpha\beta} F_a^{\alpha\beta}].$$

(10)

The field strength of $A^a_\alpha$ is

$$F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha - gf^a_{bc} A^b_\alpha A^c_\beta,$$

(11)

where \{ $f^a_{bc}$ \} are structure constants of the Lie algebra corresponding to the group SU(4).

Since the worldsheet gauge field $A_\alpha$ has been treated as an independent degree of freedom \textit{(i.e., it is not pull-back of a spacetime gauge field on the worldsheet)} we introduced its kinetic term in (10). The action (10) under the transformations (6) with the local SU(4) matrix (7), and the gauge transformation

$$A^a_\alpha \rightarrow A^a_\alpha - \frac{1}{g} \partial_\alpha \lambda^a - f^a_{bc} \lambda^b A^c_\alpha,$$

(12)

is symmetric.
3.1 A new form for the gauged action

Define the 8-component complex spinor $\Psi$ as in the following

$$\Psi = \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix}.$$  \hfill (13)

Thus, we obtain

$$2(\Psi^\dagger A_+ \Psi_- + \Psi_+^\dagger A_- \Psi_+) = \bar{\Psi} \rho^\alpha \otimes A_\alpha \Psi,$$  \hfill (14)

where $\bar{\Psi}$ is defined by

$$\bar{\Psi} = \Psi^\dagger \rho^0 \otimes I_{4\times4},$$  \hfill (15)

in which $I_{4\times4}$ is the $4 \times 4$ unit matrix. In the same way, the other fermionic terms in (10) also can be written as

$$2(\Psi^\dagger \partial_+ \Psi_- + \Psi_+^\dagger \partial_- \Psi_+) = \bar{\Psi} \rho^\alpha \otimes I_{4\times4} \partial_\alpha \Psi.$$  \hfill (16)

In our notation, the direct product of any two matrices $P_{2\times2}$ and $Q_{4\times4}$ is defined by

$$P = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}, \quad P \otimes Q = \begin{pmatrix} p_1 Q & p_2 Q \\ p_3 Q & p_4 Q \end{pmatrix},$$  \hfill (17)

which is a $8 \times 8$ matrix.

Introducing Eqs. (14) and (16) into the action (10) gives an elegant form for the gauged action

$$S_{\text{gauged}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma [(D_\alpha Y)^\dagger D^\alpha Y - i\bar{\Psi} \rho^\alpha \otimes D_\alpha \Psi + \pi \alpha' F^\alpha_{\alpha\beta} F^{\alpha\beta}_{\alpha\beta}].$$  \hfill (18)

The eight Majorana spinors of the worldsheet form one Dirac spinor with eight components. The action of this spinor has the feature of the Dirac action with the special operator $\rho^\alpha \otimes D_\alpha$.

3.2 The equations of motion

The equation of motion of the extended variable $Y$, extracted from the action (18), is

$$D_\alpha D^\alpha Y = 0.$$  \hfill (19)

According to the definition of $Y$, this equation leads to the equations of motion of the string coordinates $\{X'(\sigma, \tau)\}$. 

5
For the worldsheet gauge field $A^\alpha$ the equation of motion is
\[
(D_\beta F^{\alpha\beta})_a = \frac{g}{4\pi\alpha'}(i[Y^\dagger T_a D^\alpha Y - (D^\alpha Y)^\dagger T_a Y] - \bar{\Psi} \rho^\alpha \otimes T_a \Psi).
\] (20)

For a field which transforms as the adjoint representation of $SU(4)$ the covariant derivative is $D_\alpha = \partial_\alpha + ig[A_\alpha, \cdot]$. Therefore, the left-hand-side of (20) is
\[
(D_\beta F^{\alpha\beta})_a = \partial_\beta F^{\alpha\beta}_a - gf_{abc} A_\beta^b F^{\alpha\beta}_c.
\] (21)

Finally, the equation of motion of the extended fermionic field $\Psi$ is
\[
\rho^\alpha \otimes D_\alpha \Psi = 0.
\] (22)

This equation is decomposed into the equations of motion of the components of the worldsheet fermions, i.e. $\{\psi^I_\pm(\sigma, \tau)\}$.

### 3.3 The corresponding current

We have the transformations (6) with the local matrix (7), and also the transformation (12). The fermionic parts of (6) can be written as
\[
\Psi \longrightarrow I_{2\times 2} \otimes e^{i\lambda_\alpha(\sigma, \tau) T^a} \Psi.
\] (23)

Thus, the infinitesimal transformations of $Y$, $\Psi$ and $A^\alpha_a$ are
\[
\begin{align*}
\delta Y &= i\lambda_\alpha T^\alpha Y, \\
\delta \Psi &= i\lambda_\alpha I_{2\times 2} \otimes T^a \Psi, \\
\delta A^\alpha_a &= -\frac{1}{g} (D_\alpha \lambda)^a = -\frac{1}{g} \partial_\alpha \lambda^a - f_{bc}^a \lambda^b A^\alpha_c.
\end{align*}
\] (24)

Introducing these transformations into the variation of the action (18) gives the current
\[
J^\alpha_a = \frac{g}{4\pi\alpha'}(i[Y^\dagger T_a D^\alpha Y - (D^\alpha Y)^\dagger T_a Y] - \bar{\Psi} \rho^\alpha \otimes T_a \Psi).
\] (25)

This is a Hermitean current.

According to the current (25), Eq. (20) takes the form $(D_\beta F^{\alpha\beta})_a = J^\alpha_a$. This implies that the current $J^\alpha = J^\alpha_a T^a$ transforms covariantly under the gauge group $SU(4)$. Therefore, by the equations of motion, it is a covariantly constant quantity
\[
D_\alpha J^\alpha = \partial_\alpha J^\alpha + ig[A_\alpha, J^\alpha] = 0.
\] (26)
3.3.1 The current due to the global gauge symmetry

The current (25) can be decomposed as in the following

$$ J^\alpha_a = J^{(0)\alpha}_a - \frac{g^2}{4\pi\alpha'} A^b_a Y^\dagger \{ T_a, T_b \} Y, \quad (27) $$

where

$$ J^{(0)\alpha}_a = \frac{g}{4\pi\alpha'} [ i(Y^\dagger T_a \partial^\alpha Y - \partial^\alpha Y^\dagger T_a Y) - \bar{\Psi} \rho^\alpha \otimes T_a \Psi ] , \quad (28) $$

is the current $J^\alpha_a$ in terms of $A_\alpha = 0$.

Now consider the action (18) with $A_\alpha = 0$,

$$ S_{\text{un-gauged}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma ( \partial_\alpha Y^\dagger \partial^\alpha Y - i\bar{\Psi} \rho^\alpha \otimes I_{4\times4} \partial_\alpha \Psi ). \quad (29) $$

This is the un-gauged action (5), which is symmetric under the global $SU(4)$ transformations (6). The current, associated to this global symmetry, is given by $J^{(0)\alpha}_a$. In this case the conservation law is $\partial_\alpha J^{(0)\alpha}_a = 0$. For showing this, use the equations of motion (19) and (22) with $A_\alpha = 0$, and their Hermitian conjugates, i.e. $\partial_\alpha \partial^\alpha Y^\dagger = 0$ and $\partial_\alpha \bar{\Psi} \rho^\alpha \otimes I_{4\times4} = 0$. In addition, for the fermionic part apply the identities

$$ \rho^\alpha \otimes T_a = (I_{2\times2} \otimes T_a)(\rho^\alpha \otimes I_{4\times4}) = (\rho^\alpha \otimes I_{4\times4})(I_{2\times2} \otimes T_a). \quad (30) $$

4 Gauge modifications of the superstring action

In this section we impose the assumption that the kinetic term of the gauge field, in the gauged action (18), to be absent. Therefore, we receive the action

$$ I = S + S_1, \quad (31) $$

where

$$ S_1 = -\frac{1}{4\pi\alpha'} \int d^2\sigma \{ ig[\partial_\alpha Y^\dagger A^\alpha Y - Y^\dagger A_\alpha \partial^\alpha Y - i\bar{\Psi} \rho^\alpha \otimes A_\alpha \Psi] + g^2 Y^\dagger A_\alpha A^\alpha Y \}. \quad (32) $$

This implies that $A_\alpha$ is an auxiliary field.

4.1 A modified action

The equation of motion of $A_\alpha$ has been given by (20) without the left-hand-side. Thus, we obtain the gauge field as in the following

$$ A_\alpha^\alpha = \frac{4\pi\alpha'}{g^2} (M^{-1})_{ab} J^{(0)\alpha}_b, \quad (33) $$
where the symmetric matrix $M$ is defined by the elements

$$M_{ab} = Y^\dagger \{T_a, T_b\} Y.$$  \hspace{1cm} (34)

The equation (33) reveals an explicit relation between string and the $SU(4)$ gauge field.

According to (33) the action (32) can be written as

$$S'_1 = \text{Tr} \int d^2 \sigma A_\alpha J_{(0)}^\alpha.$$  \hspace{1cm} (35)

This implies that $J_{(0)}^\alpha$ acts as an external current source for the $SU(4)$ gauge field $A_\alpha$.

Combining (33) and (35) gives

$$S''_1 = \frac{1}{8\pi\alpha'} \int d^2 \sigma J_\alpha^a (M^{-1})_{ab} J_\beta^b,$$  \hspace{1cm} (36)

where $J_\alpha^a = \frac{4\pi\alpha'}{g} J_{(0)}^a$. This action is independent of the gauge coupling constant $g$. Finally, we obtain the action

$$I_{\text{Modified}} = S + S''_1.$$  \hspace{1cm} (37)

In fact, this demonstrates a gauge modification for the superstring action, originated from the $SU(4)$-gauging.

4.2 An effective action

By path integration over $A_\alpha^a$ we obtain an effective action of $I$,

$$e^{-iI_{\text{eff}}} = \int \prod_{a=1}^{15} \prod_{\alpha=0}^{1} D A_\alpha^a e^{-iI}.$$  \hspace{1cm} (38)

For computing $I_{\text{eff}}$, we should introduce the Wick’s rotation $\tau \to it$. After calculations and imposing inverse of the Wick’s rotation we receive the effective action as in the following

$$I_{\text{eff}} = S + S''_1 + S_2 + C,$$  \hspace{1cm} (39)

where $S_2$ is

$$S_2 = -\frac{1}{2} \text{Tr} \int d^2 \sigma \ln M.$$  \hspace{1cm} (40)

The constant $C$ also is

$$C = -\frac{15}{2} V_\Sigma \ln \left( \frac{g^2}{4\pi\alpha'} \right),$$  \hspace{1cm} (41)

in which $V_\Sigma$ is the area of the string worldsheet. In fact, this effective action is another gauge modification of the superstring action, due to the $SU(4)$ gauge field.
5 Conclusions

The superstring sigma model in the light-cone gauge enabled us to gauge it under the group $SU(4)$. We observed that this action can be written in terms of the four-component and eight-component complex variables, which are constructed from the worldsheet fields.

The fermionic part of the gauged action has the feature of the Dirac action with an eight-component complex spinor field and a special matrix-derivative operator. The $SU(4)$ worldsheet gauge field couples to this spinor field.

We demonstrated that this model contains some connections between string and the $SU(4)$ gauge field. This enabled us to obtain a modified action for string, due to the $SU(4)$-gauging. Similarly, removing the gauge field through the path integral gave an effective action for the string.

In a similar fashion one can combine the worldsheet fields to obtain the quaternionic and octanionic variables. The former enables us to gauge the action under the group $SU(2)$ and the latter provide gauging under the group $U(1)$.

Note that consideration of two-dimensional gauge field induces a contribution to the conformal anomaly and hence changes the critical dimension of the string models.

References

[1] A.M. Polyakov and P.B. Wiegmann, Phys. Lett. B131 (1983) 121.
[2] A.M. Polyakov and P.B. Wiegmann, Phys. Lett. B141 (1984) 223.
[3] P. Di Vecchia and P. Rossi, Phys. Lett. B140 (1984) 344.
[4] P. Goddard, W. Nahm and D. Olive, Phys. Lett. B140 (1985) 111.
[5] I. Antoniadis and A. Bachas, Nucl. Phys. B278 (1986) 343.
[6] A.N. Redlich and H.J. Schnitzer, Phys. Lett. B167 (1986) 315.
[7] A.N. Redlich, H.J. Schnitzer and K. Tsokos, Nucl. Phys. B289 (1987) 397.
[8] M. Porrati and E.T. Tombolis, Nucl. Phys. B315 (1989) 615.
[9] K. Bardakci, E. Rabinovici and B. Saring, Nucl. Phys. B299 (1988) 151.
[10] E. Witten, Nucl. Phys. B223 (1983) 422.
[11] O. Alvarez, Nucl. Phys. B238 (1984) 61.

[12] K.D. Rothe, Nucl. Phys. B269 (1986) 269.

[13] D.Z. Freedman and K. Pilch, Phys. Lett. B213 (1988) 331.

[14] J. Quackenbush, Phys. Lett. B234 (1990) 285.

[15] D. Kamani, Can. J. Phys. 87 (2009) 695, hep-th/0610216.

[16] E. Witten, Commun. Math. Phys. 141 (1991) 153.

[17] S. Cordes, G.W. Moore and S. Ramgoolam, Nucl. Phys. Proc. Suppl. 41 (1995) 184, hep-th/9411210.

[18] D.J. Gross, Nucl. Phys. B400 (1993) 161, hep-th/9212149.

[19] A.J. Niemi, Phys. Rev. D67 (2003) 106004, hep-th/0206227.

[20] P. Haggi-Mani and B. Sundborg, JHEP 0004 (2000) 031, hep-th/0002189.

[21] B. Sundborg, Nucl. Phys. Proc. Suppl. 102 (2001) 113, hep-th/0103247.

[22] J. Isberg, U. Lindstrom, B. Sundborg and G. Theodoridis, Nucl. Phys. B411 (1994) 122, hep-th/9307108.

[23] S. Ferrara, Nuovo Cim. Lett. 13 (1975) 629.