KRULL DIMENSION FOR LIMIT GROUPS II:
ALIGNING JSJ DECOMPOSITIONS

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ABSTRACT. This is the second paper in a sequence on Krull dimension for limit
groups, answering a question of Z. Sela. In it we develop the notion of a reso-
lution of a sequence of limit groups and show how to derive resolutions of low
complexity from resolutions of high complexity.

1. INTRODUCTION

A basic and important fact in algebraic geometry is that varieties have finite
Krull dimension, that is, given a variety $V$, chains of proper inclusions of irre-
ducible subvarieties have length bounded above by the dimension of $V$. Remark-
ably, solutions to systems of equations defined over a free group $F$ can also be
decomposed into irreducible components [Sel01, BMR00, KM06]. Associated to
each irreducible component is a limit group which plays the role of the coordinate
ring of a variety: the points of the component are tautologically identified with
homomorphisms from the limit group to the free group. A chain of irreducible
subvarieties corresponds to a sequence of epimorphisms of limit groups and finiteness
of Krull dimension comes out of this analysis as the supremum of lengths of
chains of epimorphisms of limit groups, beginning with a fixed limit group. Just
as (affine) varieties are subsets of the affine space, varieties over $F$ are contained in
$F^n$ for some $n$. Finite dimension reduces to bounding lengths of chains of epimor-
phisms of limit groups requiring the same number of generators.

As in any inductive proof, a complexity is assigned to sequences of epimor-
phisms of limit groups. This paper gives a means for deriving, given a sequence of
some complexity, uniformly many sequences of strictly lower complexity. More-
over, the derived sequences are bound together in a graph of sequences of groups,
a resolution, of the original sequence, analogous to a graphs of groups decomposi-
tion, modeled on a common JSJ decomposition induced by the original sequence.
Using the main results from [Lou08b] and [Lou08c] we lift a dimension bound for
the derived sequences to a bound on the length of the resolution (Theorem 8.3),
and from there to a bound on the length of original sequence, depending only on
the complexity of the original sequence. An important feature is that the derived
sequences are obtained through a special process of iteratively adjoining roots,

2000 Mathematics Subject Classification. Primary: 20F65; Secondary: 20E05, 20E06.

Key words and phrases. limit group, krull dimension, JSJ, fully residually free.

Most of this research was done while at the University of Utah. The author also gratefully ac-
knowledges support from the National Science Foundation.
passing to limit group quotients, and then passing to further limit group quotients. These sequences are the motivation for the sequences of adjunctions of roots discussed in [Lou08c].

Solution sets of systems of equations defined over the free group have received considerable attention over the last decade, particularly in the positive resolution of Tarski’s question about elementary equivalence of nonabelian free groups by Sela [Sel01, Sel03, Sel05, Sel04, Sela, Selb]. See also [KM06] for an alternative approach by Kharlampovich and Myasnikov.

The object of this sequence (the present paper and [Lou08c, Lou08b, Lou08a]) is to prove that limit groups have finite Krull dimension (Theorem 1.2). In other words, if

\[ F_n \rightarrow L_1 \rightarrow \cdots \rightarrow L_k \]

is a sequence of proper epimorphisms of limit groups, then \( k \) is bounded by a function of \( n \).

The question comes from algebraic geometry and logic. A system of equations over the free group is a collection of words \( \Sigma = \{ \omega_i \} \) in finitely many variables \( \{ x_i \}_{i=1..n} \). The set of solutions to \( \Sigma \) in \( F \) is identified with a subset of \( F^n \):

\[
V_\Sigma = \{ (a_1, \ldots, a_n) \in F^n \mid \omega_i(a) = 1 \text{ for all } \omega_i \in \Sigma \}
\]

Associated to any such system of equations \( \Sigma \) is a finitely generated group \( G_\Sigma := \langle x_i \rangle / \langle \Sigma \rangle \), and there is a tautological one-to-one correspondence between the sets \( \text{Hom}(G_\Sigma, F) \) and \( V_\Sigma \). Let \( g \in \langle x_i \rangle \). The set \( V_{\{g\}} \cap V_\Sigma \) is a Zariski closed subset of \( V_\Sigma \). If \( G_\Sigma \) is residually free, and if \( V_\Sigma \) is irreducible, that is, it isn’t contained in the union of finitely many proper closed subsets, then for all finite collections of elements \( \{ g_i \} \subset G_\Sigma \setminus \{ 1 \} \), \( V_\Sigma \) properly contains the union

\[
\bigcup_i (V_{\{g_i\}} \cap V_\Sigma)
\]

A point in the complement is a homomorphism \( G_\Sigma \rightarrow F \) which doesn’t kill any element of the finite set, i.e., \( G_\Sigma \) is \( \omega \)-residually free. If \( G_\Sigma \) isn’t residually free, we may always pass to the universal residually free quotient \( RF(G_\Sigma) \) by killing all elements of \( G_\Sigma \) which die under every homomorphism to the free group.

The quotient map \( G_\Sigma \rightarrow RF(G_\Sigma) \) induces a bijection \( \text{Hom}(RF(G_\Sigma), F) \rightarrow \text{Hom}(G_\Sigma, F) \).

**Definition 1.1.** Let \( G \) be a finitely generated group. A sequence \( f_n \in \text{Hom}(G, F) \) such that for all \( g \in G \), either \( f_n(g) = 1 \) for sufficiently large \( n \) or \( f_n(g) \neq 1 \) for sufficiently large \( n \), is stable. The **stable kernel** of a stable sequence \( f_n \) is the normal subgroup of \( G \) generated by all elements which have trivial image for sufficiently large \( n \), and is denoted by \( \text{Ker}(f_n) \). A quotient of a finitely generated group by the stable kernel of a stable sequence is a **limit group**.

Sometimes we say that the sequence \( f_n \) **converges to** \( G/\text{Ker}(f_n) \). To justify this terminology, consider that a homomorphism \( f : G \rightarrow F \) can be thought of as a point in the space of marked free groups, and that a convergent sequence of homomorphisms can be thought of as a sequence of points converging to a point in the completion of the space of marked free groups corresponding to the limit group.
associated to the sequence. See [CG05] for a discussion of limit groups from this point of view.

Although \( \omega \)-residually free groups are limit groups, the converse follows from their finite presentability and is more difficult to prove. See [Sel01, Theorem 4.6] or [Gui04] for proofs of finite presentability, and for a proof which doesn’t use finite presentability, see [BF03, Lemma 1.10]. Since a limit group \( L \) is \( \omega \)-residually free there is always a sequence \( f_n : L \to \mathbb{P} \) converging to \( L \).

A sequence of limit groups is a finite sequence

\[ \mathcal{L} = (\ldots, \mathcal{L}(i_j), \mathcal{L}(i_{j+1}), \ldots) \]

of limit groups, denoted by a calligraphic letter, for example \( \mathcal{L} \) or \( \mathcal{R} \), indexed by a strictly increasing sequence of natural numbers \((i_j)\), equipped with homomorphisms

\[ \varphi_{m,n} : \mathcal{L}(m) \to \mathcal{L}(n) \]

such that

\[ \varphi_{n,l} \circ \varphi_{m,n} = \varphi_{m,l} \]

for all \( m < n < l \). Sometimes we ignore the fact that a sequence isn’t necessarily indexed by adjacent natural numbers, and simply refer to the \( i \)-th element in a sequence \( \mathcal{L} \) as \( \mathcal{L}(i) \). A sequence is a chain if all the maps are epimorphisms. We use this terminology because the dual sequence of \( \text{Hom-sets} \)

\[ \cdots \supset Hom(\mathcal{L}(i), \mathbb{P}) \supset Hom(\mathcal{L}(i+1), \mathbb{P}) \supset \cdots \]

is a chain of varieties.

This explains the terminology “Krull dimension.” A limit group \( L \) determines a variety \( V_L = \text{Hom}(L, \mathbb{P}) \), and a chains of epimorphisms of limit groups originating from \( L \) correspond to chains of irreducible subsets of \( V_L \).

The length of a sequence is denoted by \( \| \mathcal{L} \| \). The proper length of a chain \( \mathcal{L} \), denoted by \( \| \mathcal{L} \|_{pl} \), is the number of indices \( n \) such that \( \mathcal{L}(n-1) \to \mathcal{L}(n) \) is not an isomorphism. The rank of a limit group \( L \) is the minimal number of elements needed to generate \( L \), and is denoted by \( \text{rk}(L) \). The rank of a chain of limit groups is the rank of the first group in the chain: \( \text{rk}(\mathcal{L}) := \text{rk}(\mathcal{L}(1)) \). The first betti number of a chain is defined in the same manner and is denoted by \( b_1(\mathcal{L}) \).

**Theorem 1.2** (Krull dimension for limit groups). For all \( N \) there is a constant \( D = D(N) \) such that if \( \mathcal{L} \) is a chain of rank at most \( N \) then \( \| \mathcal{L} \|_{pl} \leq D \).

This paper and its sibling [Lou08c] contain a complete proof of Theorem 1.2. The sequel contains a proof of Theorem 1.4, which is used in Section 8 to lift a dimension bound for sequences of lower complexity to sequences of 1 complexity. [Lou08c] contains the portion of the proof specific to limit groups, and [Lou08b] contains the remainder of the proof. Theorem 1.4 is a “Krull-like” statement about certain sequences of limit groups, each obtained from the last by adjoining roots, passing to a (certain) limit group quotient, and then perhaps passing to a further quotient limit group. First, some definitions and notation.
The centralizer of $E < G$ is denoted by $Z_G(E)$. Let $\Delta$ be a graph of groups decomposition of $G$. If $V$ is a vertex group of $\Delta$, denote the set of images edge groups incident to $V$ by $\mathcal{E}(V)$.

**Definition 1.3** (Adjoining roots). Let $G$ be a limit group, $\mathcal{E}$ a collection of abelian subgroups of $G$. Suppose that for each element $E \in \mathcal{E}$, we are given a finite index free abelian supergroup $F(E)$. A limit group quotient $G'$ of $G \ast_{E \in \mathcal{E}} F(E)$ such that the restriction of the quotient map to $G$ is injective is said to be obtained from $G$ by adjoining roots $F(E)$ to $E$. An adjunction triple is a tuple $(G,H,G')$ such that $G \hookrightarrow H \rightarrow G'$ obtained from $G$ by adjoining roots, such that $G$ embeds in $G'$ under the quotient map.

A sequence of adjunctions of roots is a pair of sequences of limit groups and a family $\mathcal{E}$ of collections of subgroups $\mathcal{E}_i$ of $G(i) = (G,H,E)$, with base sequence $\mathcal{G}$, such that

- $(G(i),H(i+1),G(i+1))$ is an adjunction triple; $H(i+1)$ is obtained from $G(i)$ by adjoining roots to $\mathcal{E}_i$.
- Each $E' \in \mathcal{E}_{i+1}$ in $G(i+1)$ centralizes, up to conjugacy, the image of an element $E$ of $\mathcal{E}_i$. If $E \in \mathcal{E}_i$ is mapped to $E' \in \mathcal{E}_{i+1}$ then the image of $Z_G(E)$ in $Z_{G'}(E')$ must be finite index.

The complexity of $(\mathcal{G},H,\mathcal{E})$ is the triple

$$\text{Comp}((\mathcal{G},H,\mathcal{E})) := (b_1(\mathcal{G}), \text{depth}_{\text{pc}}(H), \|\mathcal{E}\|).$$

See [Lou08c, Definition 2.4] for the definition of $\text{depth}_{\text{pc}}$, the depth of the principle cyclic analysis lattice of a limit group. Complexities are not compared lexicographically: $(b',d',e') \leq (b,d,e)$ if $b' \leq b$, $d' \leq d$, and $e' \leq e + 2(d-d')b$.

Let $(\mathcal{G},H,\mathcal{E})$ be a sequence of adjunctions of roots. The quantity $N\text{Inj}((\mathcal{G},H,\mathcal{E}))$ is the number of indices $i$ such that $H(i) \rightarrow G(i)$ is not an isomorphism.

**Theorem 1.4** (Adjoining roots ([Lou08c]). Let $(\mathcal{G},H,\mathcal{E})$ be a sequence of adjunctions of roots. There is a function $N\text{Inj}(\text{Comp}((\mathcal{G},H,\mathcal{E})))$ such that

$$N\text{Inj}((\mathcal{G},H,\mathcal{E})) \leq N\text{Inj}(\text{Comp}((\mathcal{G},H,\mathcal{E}))).$$

This is [Lou08c, Theorem 1.4].

**Acknowledgments.** I am deeply indebted to my advisor Mladen Bestvina, and extend many thanks to Matt Clay, Mark Feighn, Chloé Perin, and Zlil Sela for listening carefully and critically to my musings on Krull.

2. Preliminaries

In this section we set up some notation, review theorems from the theory of limit groups, and do some basic setup for later sections. We start by giving some basic properties of limit groups and defining generalized abelian decompositions, or GADs.
Theorem 2.1 (Basic properties of limit groups [BF03], [Sel01]). The following properties are shared by all limit groups.

- Commutative transitivity; maximal abelian subgroups are malnormal; every abelian subgroup is contained in a unique maximal abelian subgroup.
- Abelian subgroups are finitely generated and free.
- Finite presentability and coherence.

Definition 2.2 (Generalized abelian decomposition ([BF03])). A generalized abelian decomposition, shortened to GAD, or abelian decomposition, of a freely indecomposable finitely generated group \( G \), is a graph of groups \( \Delta(R_i, A_j, Q_k, E_l) \) of \( G \) over vertex groups \( R_i, A_j, Q_k \), and abelian edge groups \( E_l \) such that:

- \( A_j \) is abelian for all \( j \).
- \( Q_k \) is the fundamental group of a surface \( \Sigma_k \) with boundary and \( \chi(\Sigma_k) \leq -2 \) or \( \Sigma_k \) is a torus with one boundary component. The \( Q_k \) are “quadratically hanging.”
- Any edge group incident to a quadratically hanging vertex group is conjugate into a boundary component.

The subgroups \( R_i \) are the rigid vertex groups of \( \Delta \).

The peripheral subgroup of an abelian vertex group \( A \) of a GAD is the direct summand \( P(A) \) of \( A \) which is the intersection of all kernels of maps \( A \to \mathbb{Z} \) which kill all images of incident edge groups. The peripheral subgroup is primitive in \( A \): if \( \alpha \in A \setminus P(A) \) then no power of \( \alpha \) is in \( P(A) \). This differs slightly from [BF03] in that they use this term to refer to the subgroup generated by incident edge groups.

If \( \Delta \) is a GAD of a finitely generated group \( G \), we say that a splitting \( G = G_1 *_E G_2 \) or \( G = G_1 *_E \) is visible in \( \Delta \) if it is a one edged splitting from an edge of \( \Delta \), a one edged splitting obtained by cutting a QH vertex group along an essential simple closed curve, or a one edged splitting inherited from a one edged splitting of an abelian vertex group in which the peripheral subgroup is elliptic.

Definition 2.3 (The modular group \( \text{Mod} \)). To a GAD \( \Delta \) of \( G \) we associate the restricted modular group \( \text{Mod}(G, \Delta) \), which is the subgroup of the automorphism group of \( G \) generated by inner automorphisms and all Dehn twists in one edged splittings visible in \( \Delta \).

Let \( G \) be a finitely generated group with a Grushko decomposition

\[
G = G_1 * \cdots * G_p * \mathbb{F}_q
\]

and for each \( i \) a GAD \( \Delta_i \) of \( G_i \). We say that an automorphism \( \varphi \) of \( G \) is \( \Delta_i \)-modular on \( G_i \), or simply modular, if there are automorphisms \( i_g \in \text{Inn}(G) \) and \( \varphi' \in \text{Mod}(G_i) < \text{Mod}(G) \) such that the restriction of \( \varphi \) to \( G_i \) agrees with the restriction of \( i_g \circ \varphi' \) to \( G_i \). The set of automorphisms which are \( \Delta_i \)-modular on \( G_i \) for all \( i \) forms the modular group of \( G \) with respect to \( \{ \Delta_i \} \), and is denoted by \( \text{Mod}(G, \{ \Delta_i \}) \). The group of automorphisms of \( G \) generated by all such groups, as \( \{ \Delta_i \} \) varies over all collections of GAD’s of freely indecomposable free factors of \( G \), is denoted by \( \text{Mod}(G) \).
To a finitely presented freely indecomposable limit group one can associate the abelian JSJ decomposition, a canonical GAD such that every other GAD can be obtained from it by folding, sliding, conjugation, cutting QH vertex groups along simple closed curves, and collapsing subgraphs. The JSJ theory is well developed and we assume some familiarity with it. See [DS99] or [FP06] for treatments not specific to limit groups, or [RS97] and [BF03] for treatments more specific to limit groups. We assume the following normalizations on the abelian JSJ of a limit group:

- The JSJ decomposition is 2-acylindrical: if \( T \) is the associated Bass-Serre tree, then stabilizers of segments of length at least three are trivial. Through folding, sliding, and conjugation, ensuring that edges incident to rigid vertex groups have nonconjugate centralizers (in the vertex group in question), and collapsing, to ensure that no two abelian vertex groups are adjacent, we can arrange that if \( g \) fixes a segment of length two with central vertex \( v \), then the stabilizer of \( v \) is abelian.

- Let \( E \) be an edge group of a limit group \( L \). Then there is a unique abelian vertex group \( A \) such that \( A = \mathbb{Z}_L(E) \). This is somewhat awkward as the normal impulse upon encountering a surjective map \((\text{edge group}) \rightarrow (\text{vertex group})\) is to collapse the edge. This hypothesis does, however, make the construction of strict homomorphisms in subsection 5.2 formally somewhat easier.

- Edge groups incident to rigid vertex groups are closed under taking roots in the ambient group. Edge groups incident to a rigid vertex group \( R \) are non-conjugate in \( R \). An edge group is closed under taking roots in all adjacent non-QH vertex groups.

- If \( E \) is incident to a QH vertex group it is conjugate to a boundary component.

- If \( E \) is an edge group and \( A \) is a valence one abelian vertex group such that \( E \) is finite index in \( A \), then \( E \) is attached to a boundary component \( b \) of a QH vertex group and there are no other incident edge groups with image conjugate to \( b \). The index of \( E \) in \( A \) must be at least three. (If it is one, then the splitting is trivial, and if it is two, then the QH vertex group can be enlarged by gluing a Möbius band to the boundary component representing \( E \).)

The *modular group* of a freely indecomposable limit group \( G \) is the group Mod\((L, \text{JSJ}(L))\), or simply Mod\((L)\) for convenience. This definition agrees with the previous definition of Mod\((L)\) since (1) Mod\((L, \Delta)\) is a subgroup of Mod\((L, \text{JSJ}(L))\) and (2) JSJ decompositions exist.

Let \( \Delta \) be a GAD of a limit group \( L \), and let \( L_P \) be the subgroup of \( L \) constructed by replacing \( A_j \) by \( P(A_j) \) for all \( j \). Let \( R \) be a rigid vertex group of \( \Delta \). The *envelope* of \( R \), Env\((R, \Delta)\), is the subgroup of \( L_P \) generated by \( R \) and the centralizers of edge groups incident to \( R \) in \( L_P \). Let \( H < G \) be a pair of groups. An automorphism of \( G \) is *internal on \( H \)* if it agrees with the restriction of an inner automorphism. The envelope of a rigid vertex group of \( L \) can also be thought of as
the largest subgroup of $L$ containing $R$ for which the restriction of every element of $\text{Mod}(L, \Delta)$ is internal.

A limit group is elementary if it is free, free abelian, or the fundamental group of a closed surface.

Let $L$ be a freely indecomposable nonelementary limit group. The cyclic JSJ of $L$ is the JSJ decomposition associated to the family of all one-edged cyclic splittings in which all noncyclic abelian subgroups are elliptic. Since limit groups have cyclic splittings, the cyclic JSJ of a freely indecomposable nonelementary limit groups is nontrivial.

**Definition 2.4** (Cyclic analysis lattice; depth). The cyclic analysis lattice of a limit group $L$ is a tree of groups constructed as follows:

1. Level 0 of the analysis lattice is $L$.
2. Let $L_1 \ast \cdots \ast L_p \ast F_q$ be a Grushko factorization of $L$. Level 1 of the analysis lattice consists of the groups $L_i$ and $F_q$. There is an edge connecting $L_i$ to each group in level 1. The free factor $F_q$ and any surface or free abelian freely indecomposable free factors of $L$ are elementary, and we take them to be terminal leaves of the tree.
3. Let $L_{i,1}, \cdots, L_{i,j}$, vertex groups of the cyclic JSJ of $L$. Level 3 of the lattice consists of the groups $\{L_{i,k}\}$. There is an edge connecting $L_i$ to $L_{i,k}$ for all $i$ and $k$.
4. Inductively, construct the analysis lattice for each group $L_{i,j}$ and graft the root of the tree to the vertex labeled $L_{i,j}$.
5. Terminate only when all resulting leaves are terminal.

The **depth** of a limit group $L$ is the number of levels in its cyclic analysis lattice, and is denoted by $\text{depth}(L)$. The depth of a sequence of limit groups $\mathcal{L}$ is the greatest depth of a group appearing in $\mathcal{L}$:

$$\text{depth}(\mathcal{L}) := \max_i \{\text{depth}(\mathcal{L}(i))\}$$

Lemma 2.11 below follows from the following theorem of Sela’s, but for completeness we provide an alternative proof.

**Theorem 2.5** (cf. [Sel01], Proposition 4.3). There is a function $r$ such that if $L$ is a limit group then $\text{depth}(L) \leq r(b_1(L))$.

Since the depth of $L$ is bounded by $r$, and the width of the analysis lattice is bounded by homological considerations, this proposition gives a bound on the rank of a limit group in terms of its first betti number.

**Remark 2.6.** The cyclic analysis lattice of a limit group is defined in [Sel01], where he shows that it is finite, and in fact has depth bounded quadratically in the first betti number of the given group.

We don’t need the full strength of Theorem 2.5 in this paper, although it would make some steps slightly easier. For this reason we use slightly more roundabout logic for our proof of Theorem 1.2 than is strictly necessary. Specifically, when something can be controlled by $b_1$ alone, we may need to use $b_1$ and depth.
Since all the facts we use about limit groups can be proven independently of Sela’s Theorem 2.5, see the treatment in [BF03], our proof doesn’t secretly rely on Sela’s proposition.

In order to be able to apply the bound on the length of a strict resolution provided by Theorem 2.10 to sequences constructed throughout the course of the paper, especially in section 4, we need to use the next Lemma to gain control over ranks.

**Lemma 2.7.** There is a function $r(b, d)$ such that if $L$ is a limit group then

$$\text{rk}(L) \leq r(b_1(L), \text{depth}(L))$$

**Proof.** The proof is by induction on the pair $(b, d)$, ordered by comparing both coordinates. Suppose the theorem holds for groups of complexity less than $(b_0, d_0)$. For $(b, d) < (b_0, d_0)$ let $\text{rk}(b, d)$ be a function which bounds the rank. Since we assume a bound on the depth of $L$, we only need to show that the number of vertex groups of the cyclic JSJ of $L$ is controlled by $b_1(L)$. Suppose $B(b)$ is such an upper bound. Since the vertex groups have lower depth, their ranks are bounded by $R = \max \{ \text{rk}(b, d) \mid b \leq b_0, d < d_0 \}$

Then

$$\text{rk}(b, d) \leq B(b_0)R + b_0$$

The term $b_0$ is the largest possible contribution to $b_1(L)$ made by stable letters from the Bass-Serre presentation of $L$ in terms of its cyclic JSJ.

To find $B$, let $\Delta(\mathcal{R}, \mathcal{Q}, \mathcal{A}, \mathcal{E})$ be a cyclic JSJ decomposition of $L$. Modify $\Delta$ by choosing, for each QH vertex group $Q \in \mathcal{Q}$, a pair of pants decomposition $P_Q$, and cut the QH vertex groups along the simple closed curves from $P_Q$. The underlying graph of $\Delta$ has first betti number bounded by $b_0$. Since the first betti number of a nonabelian limit group relative to an abelian subgroup is at least one, $\Delta$ has at most $b_0$ valence one vertex groups. The number of abelian vertex groups $A$ such that $A \neq P(A)$ is bounded above by $b_0$ as well. Thus, to bound the complexity of $\Delta$, we only need to bound the number of valence two vertex groups. Since the number of abelian vertex groups which aren’t equal to their peripheral subgroups is bounded by $b_0$, we only need to bound the number and size of sub-graphs of groups of $\Delta$ with the following form:

$$\cdots *_{\mathbb{Z}} R_1 *_{\mathbb{Z}} \mathbb{Z}^2 *_{\mathbb{Z}} R_2 *_{\mathbb{Z}} \cdots$$

There are at most $4b_0$ maximal disjoint subgraphs of this form. Since $b_1(R_i) \geq 2$, the number of valence two nonabelian vertex groups such that the two incident edge groups don’t generate the first homology is bounded above by $b_0$, we may assume that in such a sub-graph of groups, no incident edge group has trivial image in homology. Such vertex groups behave, when computing homology, like valence one vertex groups. Since the map from $L$ to $L^{ab}$ factors through the graph of groups obtained by abelianizing all rigid vertex groups, we only need to find $b_1$ of graphs of the following form

$$*_{\mathbb{Z}} \mathbb{Z}^2 *_{\mathbb{Z}} \cdots *_{\mathbb{Z}} \mathbb{Z}^2 *_{\mathbb{Z}}$$
relative to the copies of $\mathbb{Z}$ at the ends. If a graph of groups of this form has length $n$, then it contributes at least $n - 1$ to $b_1(L)$. □

It is important to see a limit group in terms of some of its quotient limit groups.

**Definition 2.8** (Strict). Let $G$ be a finitely generated group and $L$ a limit group. A homomorphism $\rho: G \to L$ is Mod($G, \Delta$)–strict if, given a sequence of homomorphisms $f_n: L \to F$ converging to $L$, there exists a sequence $\varphi_n \in$ Mod($G, \Delta$) such that $f_n \circ \rho \circ \varphi_n$ is stable and converges to $G$. Since the abelian JSJ “contains” all other GADs, a Mod($G, \Delta$)–strict homomorphisms is *a fortiori* Mod($G$)–strict, or simply strict.

A sequence of epimorphisms $L_0 \twoheadrightarrow L_1 \twoheadrightarrow \cdots \twoheadrightarrow L_n$ such that each map $L_i \twoheadrightarrow L_{i+1}$ is strict is a partial strict resolution of $L_0$. If $L_n$ is free then the sequence is a strict resolution of $L_0$. The height of a (partial) strict resolution is its proper length. A (partial) strict resolution is proper if all the epimorphisms appearing are proper.

[Sel01, Proposition 5.10] asserts that limit groups admit strict resolutions.

**Theorem 2.9** ([Sel01, BF03]). Fix a finitely generated group $G$ with GAD $\Delta$ and a limit group $L$. The following list of conditions is sufficient to ensure that $\pi: G \to L$ is Mod($G, \Delta$)–strict.

- All edge groups inject.
- All QH subgroups have nonabelian image.
- All envelopes of rigid vertex groups inject.
- If $e$ is an edge of $\Delta$ then at least one of the inclusions of $G_e$ into a vertex group of the one edged splitting of $G$ induced by $e$ is maximal abelian.

In particular, if $G$ admits a strict map to a limit group then it is also a limit group.

**Theorem 2.10** ([Lou08a]). Let $\mathbb{F}_N \twoheadrightarrow L_0 \twoheadrightarrow \cdots \twoheadrightarrow L_k$ be a sequence of proper strict epimorphisms of limit groups. Then $k \leq 3N$.

If $\mathbb{F}_N \twoheadrightarrow L_1 \twoheadrightarrow \cdots \twoheadrightarrow L_k = \mathbb{F}_M$ is a sequence of proper strict epimorphisms, then $k \leq 3(N - M)$.

This theorem is also implied by [Hou08, Theorem 0.4], where Ould-Houcine shows that the the Cantor-Bendixon rank of the closure of the space free groups free groups marked by $n$ elements, points of which are the $n$–generated models of the universal theory of $\mathbb{F}$, that is, limit groups of rank $n$, is bounded. The Cantor-Bendixon rank of this space is exactly the length of a longest strict resolution of $\mathbb{F}_n$. Generalizing this fact to hyperbolic groups is one of the principle difficulties in generalizing Theorem 1.2 for limit groups over hyperbolic groups. The aforementioned theorems rely on linearity of $\mathbb{F}$ in an essential way.

To completely avoid reliance on Theorem 2.5 we need to show that the depth of a limit group is controlled by its rank.

**Theorem 2.11.** The depth of a limit group $L$ is bounded by $6 \text{rk}(L)$.

*Proof.* Let $L = G_1 \twoheadrightarrow G_2 \twoheadrightarrow \cdots \twoheadrightarrow G_n \twoheadrightarrow L$ be a strict resolution of $L$. Let $H_i$ be the freely indecomposable free factors of $L$. The restriction $G_i|_{\text{Im}(H_i)} = K_i$ of
the strict resolution to the images of \( H_i \) is a strict resolution of \( H_i \). Let \( V \) be a vertex group of the cyclic JSJ of \( H_i \), and consider a freely indecomposable free factor \( W \) of \( V \). Since the edge groups of JSJ(\( L \)) contained in the induced decomposition of \( W \) are elliptic in JSJ(\( G_2 \)), the envelopes of rigid vertex groups of JSJ(\( W \)) are contained in envelopes of rigid vertex groups of \( G_2 \), and are embedded in \( K_3 \) under the map \( K_2 \rightarrow K_3 \). Thus the sequence \( K_3 \rightarrow \ldots \), restricted to the image of \( W \) is a strict resolution of \( W \). By induction on the length of a shortest strict resolution, \( W \) has depth at most \( 6 \text{rk}(L) - 2 \). Since \( W \) is at level one in the analysis lattice of \( L \), \( L \) has depth at most \( 6 \text{rk}(L) \).

\[ \square \]

3. Complexity classes of sequences

Our proof of Theorem 1.2 is by induction. To do this we need to have a notion of sequence which allows us to attach a suitable complexity. Unfortunately, it doesn’t seem possible to simply induct on the rank, first betti number, or depth of a sequence.

To cope with this we work with pairs of sequences of limit groups with maps between them, rather than just with sequences of limit groups. Let \( \mathcal{L} \) and \( \mathcal{G} \) be sequences of limit groups. Suppose that \( j \mapsto i_j \) is a monotonically increasing function from the index set for \( \mathcal{G} \) to the index set for \( \mathcal{L} \). If, for all \( j \), there is a homomorphism \( \psi_j : \mathcal{G}(j) \rightarrow \mathcal{L}(i_j) \), such that for all \( j \) and \( k \)

\[
\varphi_{i_j, i_k} \circ \psi_j = \psi_k \circ \varphi_{j, k},
\]

we say that \( \mathcal{G} \) maps to \( \mathcal{L} \) and that \( \psi \) is a map of sequences. To express this relationship we use the familiar notation \( \psi : \mathcal{G} \rightarrow \mathcal{L} \). If \( \psi : \mathcal{G} \rightarrow \mathcal{H} \) and \( \psi' : \mathcal{H} \rightarrow \mathcal{L} \) then there is a map \( \psi' \circ \psi : \mathcal{G} \rightarrow \mathcal{L} \). If \( \psi : \mathcal{G} \rightarrow \mathcal{L} \) then the image of \( \mathcal{G} \) is the sequence \( (\text{Im} \mathcal{G}(j)) \), \( \text{Im} \mathcal{G}(j) := \psi_j(\mathcal{G}(j)) \), whose maps are the restrictions of the maps from \( \mathcal{L} \). Any \( \psi : \mathcal{G} \rightarrow \mathcal{L} \) factors as \( \mathcal{G} \rightarrow \text{Im} \mathcal{G} \hookrightarrow \mathcal{L} \). Since the notation is unambiguous, we write “\( \text{Im} \mathcal{G}(j) \)” for “\( \text{Im} \mathcal{G}(j) \)”.

A map \( \psi \) of sequences is an embedding if every map \( \psi_j \) is an embedding. In this case \( \psi \) is written in the normal fashion. Let \( \psi : \mathcal{G} \hookrightarrow \mathcal{L} \) be an inclusion. For each \( j \) let \( \mathcal{L}_\psi(i_j) \) be the lowest node in the cyclic analysis lattice of \( \mathcal{L}(i_j) \) containing a conjugate of \( \psi_j(\mathcal{G}(j)) \), and set \( d_j \) equal to the depth of \( \mathcal{L}_\psi(i_j) \). Now let the depth of \( \psi \) be

\[
d(\psi) := \max_j \{d_j\}
\]

If \( \rho : \mathcal{H} \rightarrow \mathcal{G} \) then \( d(\rho) = d(\text{Im}(\mathcal{H}) \hookrightarrow \mathcal{G}) \).

We will be interested in pairs of sequences which have maps in both directions. There is a self-map of a sequence (where it is defined) given by \( \varphi_{i, i+1} : \mathcal{G}(i) \rightarrow \mathcal{G}(i + 1) \) for all \( i \). This map is a “shift.” We denote it by \( \varphi_+ \). We now define “resolutions,” which are essentially sequences of epimorphisms, to which we add groups, and regard the additional groups as an auxiliary sequence. After this, we define resolutions of sequences of subgroups, which are our main object of interest.

Note that we have used the word “resolution” in two different contexts. Firstly, there are resolutions of limit groups, which are simply sequences of epimorphisms,
there are strict resolutions, which are resolutions in which every map is strict, and now there are resolutions of sequences. A resolution of a group is always in the first sense, and a resolution of a sequence is always a resolution in the third sense.

**Definition 3.1** (Resolution). A resolution of a chain $\mathcal{L}$ of limit groups is a chain $\mathcal{H}$, indexed by $j = 1..n$, equipped with maps $\pi_j: \mathcal{H}(j) \rightarrow \mathcal{L}(i_j)$ for all $j$, and $\psi_j: \mathcal{L}(i_j) \rightarrow \mathcal{H}(j+1)$, $j+1 \leq n$, such that the following diagrams commute:

$$
\begin{array}{ccc}
\mathcal{H}(j) & \xrightarrow{\psi_j} & \mathcal{H}(j+1) \\
\downarrow{\pi_j} & & \downarrow{\pi_{j+1}} \\
\mathcal{L}(i_j) & \xrightarrow{\psi_{j+1}} & \mathcal{L}(i_{j+1})
\end{array}
$$

If $\mathcal{L}'$ is a subsequence of $\mathcal{L}$ obtained by deleting groups and composing maps then we write $\mathcal{L}' \subset \mathcal{L}$. Since the maps $\mathcal{L}'(j) \rightarrow \mathcal{L}(i_j)$ are injective and surjective, we write this suggestively as $\mathcal{L}' \rightarrow \mathcal{L}$. If $\mathcal{L}$ is a chain and $\mathcal{L}' \rightarrow \mathcal{L}$ then $\mathcal{L}$ is finer than $\mathcal{L}'$ and $\mathcal{L}'$ is coarser than $\mathcal{L}$.

If $\mathcal{L}'$ is a sequence of subgroups of $\mathcal{L}$, $\mathcal{L}'(n) < \mathcal{L}(n)$, then this relation is expressed by the notation $\mathcal{L}' < \mathcal{L}$. If $\mathcal{H}$ is a resolution of $\mathcal{L}$, then this relation is expressed by the notation $\mathcal{H} \sqsubset \mathcal{L}$. If the map $\mathcal{H} \rightarrow \mathcal{L}$ is $\rho$, then we indicate it as a subscript on the “$\sqsubset$”: $\mathcal{H} \sqsubset_{\rho} \mathcal{L}$. The notation is supposed to evoke the commutative diagram above. We may also write the reverse $\mathcal{L} \sqsupset \mathcal{H}$ to indicate that $\mathcal{H}$ is a resolution of $\mathcal{L}$. We leave the maps implicit unless there is risk of confusion.

**Definition 3.2.** If $\mathcal{H} \sqsubset_{\rho} \mathcal{L}$, then $\mathcal{L}$ is the base sequence of the resolution. The depth of the resolution is the depth of $\rho$.

Let $\mathcal{G} \hookrightarrow \mathcal{L}$ be an inclusion. If $\mathcal{H} \sqsubset \mathcal{G}$ is a resolution then $\mathcal{H}$ is a resolution of a subsequence of $\mathcal{L}$. We denote this relation by $\mathcal{H} \sqsubset \mathcal{L}$, even though the maps $\mathcal{L} \rightarrow \mathcal{H}$ are only defined on subgroups. The depth of $\mathcal{H} \sqsubset \mathcal{L}$ is the depth of $\mathcal{G} \hookrightarrow \mathcal{L}$. A resolution of a subsequence is simply a pair of maps $\pi: \mathcal{H} \rightarrow \mathcal{L}$, $\psi: \mathrm{Im}(\pi) \rightarrow \mathcal{L}$, such that $\psi \circ \pi = \varphi_+$ and $\pi \circ \psi = \varphi_+$, where the first $\varphi_+$ is the shift map for $\mathcal{H}$ and the second $\varphi_+$ is the shift map for $\mathrm{Im}(\pi)$. Notice that we cannot compose resolutions without changing the sequences. If $\mathcal{H} \sqsubset \mathcal{G}$ and $\mathcal{G} \sqsubset \mathcal{L}$ then there is no resolution $\mathcal{H} \sqsubset \mathcal{G}$, though there is a resolution $\mathcal{H}' \sqsubset \mathcal{L}$, where $\mathcal{H}'$ is the sequence obtained by omitting every other group of $\mathcal{H}$.

Suppose $\mathcal{H} \sqsubset \mathcal{L}$ and $R < \mathcal{H}(n)$. The sequence of images $\mathcal{R}^n$ defined by

$$
\mathcal{R}^n(m):= \varphi_{n,m}(R) < \mathcal{H}(m), \quad m \geq n
$$

is a resolution of a subsequence of $\mathcal{L}$ with the induced maps.

Let $\mathcal{H} \sqsubset_{\rho} \mathcal{L}$ be a resolution of a subsequence. The complexity of $\mathcal{H} \sqsubset_{\rho} \mathcal{L}$ is the quantity

$$
\text{Comp}(\mathcal{H} \sqsubset_{\rho} \mathcal{L}) = (b_1(\mathcal{H}), d(\rho))
$$

The length and proper length of $\mathcal{H} \sqsubset \mathcal{L}$ are the length and proper length of $\mathcal{H}$, respectively.

\footnote{Strictly speaking we must drop the last element from $\mathrm{Im}(\pi)$.}
To compare complexities we use the following partial order:

\[(b, d) \leq (b', d') \text{ if } b \leq b' \text{ and } d \leq d'\]

The inequality is strict if at least one of the coordinate inequalities is strict. Rather than have one base case for the induction, we use a collection of base cases. The complexities \(\{(b, 2) \mid b \in \mathbb{N}\}\) are minimal, and form the base cases for the inductive proof of Theorem 3.5. The groups in such sequences are free products of elementary limit groups. We finish this section by showing that Theorem 1.2 holds for sequences of minimal complexity.

**Lemma 3.3** (Krull dimension: very low depth). Sequences of proper epimorphisms of limit groups with depth at most 2 have length controlled by the first betti number.

*Proof.* A group of depth 2 is a free product of free abelian, surface, and free groups. There are only \(n_b < \infty\) such groups with a given betti number \(b\). Since limit groups are Hopfian, for a fixed value of \(b\), every sequence of proper epimorphisms of limit groups with first betti number \(b\) has length at most \(n_b\). \(\square\)

This is not really necessary since the analysis of sequences of minimal complexity is a sub-case of our more general analysis of sequences of resolutions of limit groups and the observation that any map with nonabelian image from a nonabelian elementary limit group to another limit group is strict.

**Definition 3.4.** \(\text{Seq}(L, b, d)\) is the set of resolutions of subsequences \(\mathcal{H} \boxsupset L\) such that

\[\text{Comp}(\mathcal{H} \boxsupset L) \leq (b, d)\]

Theorem 1.2 follows formally from the following theorem, which is what we aim to prove in this paper.

**Theorem 3.5** (Krull dimension for resolutions of subsequences). There is a function \(D(b, d)\), independent of \(L\), such that if \(\mathcal{H} \boxsupset L \in \text{Seq}(L, b, d)\) then \(\|\mathcal{H}\|_{pl} \leq D\).

Theorem 1.2 is an immediate consequence of Theorem 3.5 since the trivial resolution \(L \boxsupset_{id} L\) is an element of \(\text{Seq}(\text{rk}(L), b_1(L), \text{depth}(L))\).

4. **Degenerate Maps**

This section exploits Theorem 2.10 for our approach to Krull dimension for limit groups. Given a sequence \(L\) of epimorphisms of limit groups, we construct a resolution of \(L\) whose homomorphisms all respect JSJ decompositions. This will occupy sections 4 through 7. The first step in the construction is to find, given \(L\), a resolution \(\mathcal{H}\) of \(L\) such that every map whose range is in \(\mathcal{H}\) is as far from strict as possible.

**Definition 4.1** (Degenerate). An epimorphism \(\varphi: L \twoheadrightarrow L'\) of limit groups has a strict factorization if there is a quotient limit group \(\psi: L \twoheadrightarrow L_s\) and a strict morphism \(\varphi_s: L_s \twoheadrightarrow L'\) such that \(\varphi_s \circ \psi = \varphi\).
An epimorphism $\varphi: L \to L'$ of limit groups is degenerate if it has no proper strict factorizations.

The philosophy here is that a surjection of limit groups can be decomposed into a strict resolution preceded by a degenerate map:

\[
\begin{array}{cccc}
G_k & \text{strict} & \cdots & \text{strict} & G_2 & \text{strict} & G_1 \\
L & \downarrow & \downarrow & \downarrow & \neq & \downarrow & \neq & L' \\
\end{array}
\]

If $L \to L'$ isn’t degenerate, then there is a $G_1$ adding to the diagram $L \to L'$ as in the figure. Likewise, if $L \to G_1$ isn’t degenerate, then there is a strict $G_2$ which can be added to the figure. Proceeding in this way we find a sequence of strict resolutions

\[
G_k = (G_k \to \cdots \to G_1)
\]

Since (partial) strict resolutions have bounded length (Theorem 2.10), this procedure terminates in finite time and the last map constructed, $L \to G_k$, is degenerate.

A resolution of a subsequence $H/\Box L$ is maximal if, for all $j < j'$, the maps $H(j) \to H(j')$ and $\text{Im}(H)(ij) \to H(j')$ are degenerate. A chain is degenerate if all compositions of maps in the chain are degenerate.

**Theorem 4.2.** Let $L$ be a sequence of $N$–generated limit groups. Then for all $K$ there exist $M = M(N,K)$ such that if $\|L\|_{pl} > M$ then there exists a maximal surjective $H/\Box L$ such that $\|H\|_{pl} > K$.

To prove Theorem 4.2 we introduce sequences of strict resolutions as a formal device.

**Definition 4.3.** A sequence of strict resolutions is a sequence $\mathcal{S} = (S_i)_{i=1...n}$ of proper partial strict resolutions

\[
S_i := (S_i(k_i) \to \cdots \to S_i(1))
\]

equipped with homomorphisms

\[
\psi_i: S_i(1) \to S_{i+1}(k_{i+1})
\]

Notice that we chose to index the partial strict resolutions by decreasing, rather than increasing, indices.

The height of a sequence of strict resolutions is the length of the shortest partial strict resolution appearing in the sequence: $h(\mathcal{S}) = \min \{k_i \mid k_i = \|S_i\|\}$.

A refinement of a sequence of strict resolutions $\mathcal{S}$ is a sequence of strict resolutions $\mathcal{S}'$ such that if $S_i$ is the $i$-th strict resolution in $\mathcal{S}$ and $S_i'$ is the $i$-th strict resolution in $\mathcal{S}'$ then the resolutions $S_i$ and $(S_i'(j))_{j \leq k_i}$ coincide and the composition of maps $S_{i-1}'(1) \to S_i'(k_i)$ agrees with $\psi_i$. A refinement is proper if $k_i' > k_i$ for some $i$.

A subsequence of a sequence of strict resolutions $\mathcal{S}$ is a sequence of strict resolutions $\mathcal{S}'$ such that $\mathcal{S}'$ is obtained from $\mathcal{S}$ by deleting some of the strict resolutions appearing in $\mathcal{S}$ and composing maps.
The length of a sequence of strict resolutions $\mathcal{S}$ is the number of strict resolutions appearing, and is denoted by $\|\mathcal{S}\|$. Subsequences and refinements of sequences of strict resolutions are illustrated in Figure 1, as is the relationship between sequences of strict resolutions and resolutions of sequences. If $\mathcal{S}$ is a sequence of strict resolutions such that $(S_i(1))$ appears as a subsequence of subgroups of a sequence $\mathcal{L}$, that is, there is an inclusion $(S_i(1)) \hookrightarrow \mathcal{L}$, then the sequence $(S_i(k_i))$ is a resolution of $\mathcal{L}$.

$$L_{i,j} \quad \downarrow \quad \downarrow \quad S_j(k_j) \quad \downarrow \quad \downarrow \quad S_l(k_l)$$

$S_i(1) \quad \rightarrow \rightarrow \rightarrow \rightarrow S_j(1) \rightarrow \rightarrow \rightarrow S_l(1)$

**Figure 1.** Raising the height.

**Lemma 4.4.** Fix $N$. For all $K$ there exists $M = M(K, N)$ such that if $\mathcal{S} = (S_i)$ is a sequence of strict resolutions and $\text{rk}(S_i(k_1)) = N$ then if $\|\mathcal{S}\| > M$ then there is a refinement $\mathcal{S}'$ of a subsequence of $\mathcal{S}$ such that $\|\mathcal{S}'\| \geq K$ and no subsequence of $\mathcal{S}'$ admits a proper refinement.

We now prove Theorem 4.2, assuming Lemma 4.4.

**Proof of Theorem 4.2.** Choose $K > 0$, and let $\mathcal{L}$ be a sequence of length $M(K + 1, N)$. Let $S_i$ be the trivial partial strict resolution $(\mathcal{L}(i))$ consisting of a single group. Set $\mathcal{S} = (S_i)$. By Lemma 4.4 there is a refinement $\mathcal{S}'$ of a subsequence of $\mathcal{S}$ with $\|\mathcal{S}'\| \geq K + 1$ with the property that no subsequence of $\mathcal{S}'$ admits a proper refinement. In particular, if $\mathcal{S}' = \{S'_i\}_{i=1,K+1}$ then every map $S'_i(1) \rightarrow S_j(k_j')$ is degenerate. If $i > 1$ then $S'_i(k_i) \rightarrow S_j(k_j')$ is degenerate, otherwise a subsequence of $\mathcal{S}'$ admits a proper refinement since $S'_i(1) \rightarrow S'_j(k_j')$ factors through $S'_i(k_i') \rightarrow S'_j(k_j')$ for $i' < i < j$. By removing the first element of $\mathcal{S}'$ we obtain the desired sequence: set $\mathcal{H}(i) = S'_i(k_i')$. □

We now prove Lemma 4.4. The proof relies on the Ramsey theorem.²

**Theorem 4.5** (Ramsey Theorem (see [GR90])). Let $K_n$ be the complete graph on $n$ vertices, and let $M > 0$. Then there exists $R(M)$ so that if $n > R(M)$ and the

²What is actually needed lies somewhere between the Ramsey theorem and the pigeonhole principle.
edges of \( K_n \) are bicolored, then there exists a complete monochromatic subgraph \( K_M \subset K_n \).

**Proof of Lemma 4.4.** The proof is by induction on the height of a sequence of strict resolutions. If \( \mathcal{S} \) is a sequence of strict resolutions of \( N \)-generated limit groups then \( h(\mathcal{S}) \leq 3N \). Let \( G(\mathcal{S}) \) be the complete graph whose vertex set is the index set for \( \mathcal{S} \). Color an edge \((i, j), \) \( i < j \) white if \( S_i(1) \to S_j(k_j) \) factors through a proper strict homomorphism \( L_{i,j} \to S_j(k_j) \), and black otherwise. If \( \| \mathcal{S} \| > R(K + 1) \) then there exists a complete monochromatic subgraph \( G_m \) of \( G(\mathcal{S}) \) with \( K + 1 \) vertices. If \( G_m \) is colored black, then set \( \mathcal{S}' \) to be the subsequence of \( \mathcal{S} \) indexed by the vertices of \( G_m \) sans first vertex. Then \( \mathcal{S}' \) has length \( K \) and admits no proper refinement.

If \( G_m \) is colored white, then let \( \mathcal{S}'' \) be the refinement of the subsequence of \( \mathcal{S} \) indexed by the vertices of \( G_m \), and whose strict resolutions are constructed as follows: If \((i, j) \in G_m(0), i < j \), such that if \( l \in G_m(0) \) then \( l \leq i \) or \( l \geq j \), then prepend \( L_{i,j} \) to \( S_j \) to build a proper strict resolution

\[
L_{i,j} \to S_j(k_j) \to \cdots \to S_j(1)
\]

Remove the first strict resolution from \( \mathcal{S}'' \) and call the resulting sequence of strict resolutions \( \mathcal{S}' \). Since \( L_{i,j} \to S_j(k_j) \) is strict and proper for all \( j > 1, h(\mathcal{S}'') > h(\mathcal{S}). \)

If \( h(\mathcal{S}) = 3N \) then \( \mathcal{S} \) satisfies the theorem. Set \( R_1(K) = R(K + 1) \). Then if \( \mathcal{S} \) has length at least \( M(N, K) = (R_1)^{3N}(K) \) it has a subsequence of length \( (R_1)^{3N-1}(K) \) admitting a refinement of height at least \( h(\mathcal{S}) + 1 \). Inducting on the height (which takes at most \( 3N \) steps by the bound on the length of a strict resolution) we see that \( \mathcal{S} \) has a subsequence of length \( K \) which has a refinement which admits no proper refinements.

\[\square\]

## 5. Constructing Strict Homomorphisms

So far we have only used strictness in a purely formal way, ignoring its geometric content. In this section we put Theorem 2.9 to work, and show explicitly how to factor a homomorphism \( G \to H \) of limit groups through a strict \( \Phi(G) \to H \). In section 6 we construct a complexity “\( sc \),” modeled on the Scott complexity, which is nondecreasing under degenerate maps and takes boundedly many values for limit groups with a given first betti number. We then prove a theorem which says that if equality of \( sc \) holds under a degenerate map then the JSJ decompositions of the groups in question strongly resemble one another, or are “aligned”. Combining this with Theorem 4.2 we have a method for aligning JSJ decompositions. Before we begin, we give an example which should serve to motivate the constructions of subsections 5.1 and 5.2.

**Example 5.1.** Let \( \varphi : G \to H \) be a homomorphism from a finitely generated group with one edge abelian splitting \( G = G_1 * E \) to a limit group. Let \( e \in E \) and let \( \tau \) be the generalized Dehn twist in \( E \) by \( e \): \( \tau(g) = g \) if \( g \in G_1 \) and \( \tau(g) = ege^{-1} \) if \( g \in G_2 \). Consider a sequence of homomorphisms \( f_n : H \to \mathbb{R} \) which converges to \( H \). Let \( g_n \) be the sequence \( f_n \circ \varphi \circ \tau^{m(n)} \). We choose \( m(n) \) later.
Pass to a convergent subsequence of \( g_n \), let \( \Phi_s G \) be the quotient of \( G \) by the stable kernel, and let \( \eta: G \to \Phi_s G \) be the quotient map. Suppose \( E \) has trivial image in \( H \). Then we define \( \Phi_s G \) differently, and declare it to be \( \varphi(G_1/E) \star \varphi(G_2/E) \). Note also that \( \tau \) pushes forward to an automorphism \( \tau' \) of \( \Phi_s G \). If one of \( G_1 \) or \( G_2 \) has abelian image in \( H \) then \( \tau \) also pushes forward to \( \Phi_s G \).

Suppose that \( \tau \) pushes forward to an automorphism \( \tau' \) of \( \Phi_s G \) and that neither \( G_1 \) nor \( G_2 \) has abelian image in \( H \). Then \( \varphi \circ f_n = g_n \circ (\tau')^{-m(n)} \circ \eta \). Then \( \text{ker}(\eta) < \text{ker}(\varphi) \) and there is an induced strict homomorphism \( \Phi_s \varphi: \Phi_s G \to H \) whose composition with \( \eta \) is \( \varphi \). If \( m(n) \) is sufficiently large the limiting action of \( \Phi_s G \) on the \( \mathbb{R} \)-tree for the sequence \( g_n \) is simplicial, induces a graph of groups decomposition \( \Delta \) of \( \Phi_s G \) with one edge, and \( G_1 \) and \( G_2 \) both have “elliptic” images in \( \Delta \): the graph of groups decomposition has the form \( \overline{G}_1 \ast_{E'} \overline{G}_2 \), \( G_1 \) maps to the envelope of \( \overline{G}_1 \), \( G_2 \) maps to the envelope of \( \overline{G}_2 \), \( E \) maps to the centralizer of \( E' \) and \( \eta \) respects incidence and conjugacy data of graphs of groups, that is, \( \tau \) pushes forward to \( \tau' \).

This is a motivating example for our alignment of JSJ decompositions approach to Theorem 1.2: If the homomorphism from the example above is degenerate, a condition which can be created given sufficiently long sequences of epimorphisms of limit groups by Theorem 4.2, we see roughly that one of three things can happen to a one-edged splitting of the domain: either the homomorphism factors through a free product seen by the edge, a vertex group has abelian image, or the target group splits over the centralizer of the image of the edge group. The above example uses limiting actions to suggest ways in which the range of a degenerate map inherits splittings from the domain. Rather than take this approach throughout, the kind of information which must be recorded requires that we manually construct the group \( \Phi_s G \) (hence \( H \)) from the data \( G \) and \( \varphi \).

5.1. **Freely decomposable groups.** In this subsection we assign to a homomorphism \( \varphi: G \to H \) a strict homomorphism \( \Phi_s \varphi: \Phi_s G \to H \) and a complexity \( \text{sc} \) which is nondecreasing for such \( G \to \Phi_s G \).

**Definition 5.2** ([Sco73]). Let \( G_1 \ast \cdots \ast G_p \ast F_q \) be a Grushko factorization of a finitely generated group \( G \). The **Scott Complexity** of \( G \) is the lexicographically ordered pair \( \text{sc}(G) = (p + q, q) \).

Let \( \varphi: G \to H \) be a homomorphism of finitely generated groups \( G \) and \( H \). Then

\[
\text{sc}(\varphi) := \max \{ \text{sc}(L/K) \mid \varphi \text{ factors through } L/K \}
\]

Let \( G = G_1 \ast \cdots \ast G_{\text{pc}} \ast F_{\text{qG}} \) and \( H = H_1 \ast \cdots \ast H_{\text{ph}} \ast F_{\text{qh}} \). Given a homomorphism \( \varphi: G \to H \) we define a quotient \( \eta: G \to \Phi_s G \), and an induced map \( \Phi_s \varphi: \Phi_s G \to H \) such that \( \Phi_s \varphi \circ \eta = \varphi \). For each freely indecomposable free factor \( G_i \) of \( G \), let \( L^i \) be a group with highest Scott complexity that \( \varphi|_{G_i}: G_i \to \varphi(G_i) \) factors through. Set

\[
\Phi_s G := L^1 \ast \cdots \ast L^{\text{pc}} \ast F_{\text{qG}}
\]
Each $L^i_j$ has a Grushko decomposition $L^i_1 \cdots L^i_{p_i} \ast \mathbb{F}_{q_i}$; replace each $L^i_j$ by its image in $H$ and call the resulting group $\Phi_{s} G$. There is an induced map $\Phi_{s} \varphi : \Phi_{s} G \to H$.

Before we begin show that the Scott complexity behaves well under degenerate maps, we need a lemma to show that the homomorphism constructed above is strict.

**Lemma 5.3.** Let $\pi : G \to H$ have nonabelian image, $H$ a limit group. Suppose $\pi$ is injective on freely indecomposable free factors of $G$. Then $\pi$ is strict. In particular, $G$ is a limit group.

The proof of Lemma 5.3 relies on the following bit of folklore: Let $g_i$, $i = 0, \ldots, m$, and $t$ be nontrivial words in a free group. If $[g_i,t] \neq 1$ for all $i$, then for sufficiently large $n$, the word

$$g_{0}t^{n}g_{1} \cdots g_{m-1}t^{n}g_{m}$$

is not trivial.

**Proof of Lemma 5.3.** The strategy is to find automorphisms $\phi_n$ of $G$ and a sequence of homomorphisms $f_n : H \to \mathbb{F}$ converging to $H$ such that $\text{Ker}(f_n \circ \pi \circ \phi_n) = \{1\}$. Express $G$ as a most refined free product

$$G_1 \ast \cdots \ast G_{k-1} \ast G_k \ast \cdots \ast G_{l-1} \ast G_l \ast \cdots \ast G_{p+q}$$

with $G_i$ nonabelian for $i \leq k - 1$, noncyclic free abelian for $l - 1 \geq i \geq k$, and $G_i \cong \mathbb{Z}$ for $i \geq l$. Suppose a basis element $x$ generating some $G_i \ i \geq l$ has trivial image in $H$. Let $\{g_{i,j}\}$ be a generating set for $G_i$. Precompose $\pi$ by the automorphism which maps $x$ to $xg$ for some $g \in G_1$ and is the identity on the rest of $G$. In this way, arrange that no element of a fixed basis for the free part of a Grushko decomposition of $G$ has trivial image in $H$. Under the new map $\pi$, every free factor in some fixed Grushko free factorization embeds in $H$.

Let $B_{i,n}$ be the ball of radius $n$ in the Cayley graph of $G_i$ with respect to $\{g_{i,j}\}$. For each $n$, choose a homomorphism $f_n : H \to \mathbb{F}$ such that $f_n$ has nonabelian image and embeds $\pi(B_{i,n})$ for each $i$.

Suppose that $G_1$ is freely indecomposable nonabelian and that $f_n$ has nonabelian image when restricted to $\pi(G_1)$. Since $f_n(\pi(G_1))$ is nonabelian there is an element $c_n$ in $G_1$ such that $[f_n(\pi(c_n)), f_n(\pi(g))] \neq 1$ for all $g \in \bigcup B_{i,n}$. Fix $m$ and choose integers $m_i$, $i > 1$, such that $|m_i - m_j| > m$ for $i \neq j$. Let $h = h_1 \cdots h_t$ be a word in $\bigcup B_{i,n}$ such that for all $l$, $h_l$ and $h_{l+1}$ are contained in distinct $B_{i,n}$’s, let $\Omega_{n,s}$ be the collection of all such words with length at most $s$, and let $i(h_j)$ be the index $i$ such that $h_j \in B_{i(h_j),n}$. Let $\varphi_m$ be the automorphism of $G$ which is the identity on $G_1$ and which maps $G_i$ to $c_n^{-m}G_i c_n^{m}$. The image of $h$ in $\mathbb{F}$ is

$$f_n \varphi_m(c_n)^{m(h_1)} f_n \varphi(h_1) f_n \varphi(c_n)^{-m(h_2)} f_n \varphi(h_1)^{m(h_2)} f_n \varphi(h_2) \cdots$$

Since $h_i$ and $h_{i+1}$ are contained in distinct factors of $G$, the terms

$$f_n \varphi(c_n)^{-m(h_j)} f_n \varphi(c_n)^{m(h_{j+1})}$$

are.
are at least $m$--th powers of the image of $c_n$. By the folklore mentioned prior to this proof, for sufficiently large $m$, $f_n \circ \pi \circ \varphi_m(h) \neq 1$. Since $\Omega_{n,s}$ has finitely many elements, choose $m$ large enough so that $f_n \circ \pi \circ \varphi_m$ embeds $\Omega_{n,s}$. Thus we may treat $m$ as a function of $n$ and $s$. The family $\Omega_{n,s}$ exhausts $G$, and since $f_n \circ \pi \circ \varphi_m(n,s)$ embeds $\Omega_{n,s}$,

$$\text{Ker}_{n,s \to \infty}(f_n \circ \pi \circ \varphi_m(n,s)) = \{1\}$$

Therefore $\pi$ is strict by definition.

Now suppose all indecomposable factors of $G$ are free abelian. Then the image of, without loss of generality, $G_1 * G_2$ is nonabelian. Simply repeat the argument using the factor $G_1 * G_2$ rather than $G_1$.

**Theorem 5.4** (Scott complexity is monotone ([Swa04])). If $G \to H$ is a degenerate map of nonabelian limit groups, then $\text{rk}(G) \geq \text{rk}(H)$. If $\text{rk}(G) = \text{rk}(H)$ then $\text{sc}(G) \leq \text{sc}(H)$. If $\text{sc}(G) = \text{sc}(H)$ then $p_i = 1, q_i = 0$ (Definition 5.2), and no map $G_i \to \text{Im}(G_i)$ is trivial or factors through a free product.

**Proof.** Clearly $\text{rk}(G) \geq \text{rk}(H)$. If their ranks are equal then no $p_i$ is zero. Let $r_i = \text{rk}(G^i)$. If $p_i = 0$ then the induced map $\mathbb{F}_{r_i} \to \mathbb{F}_{q_i}$ cannot be an isomorphism since free groups are Hopfian. Thus if some $p_i = 0$ then $\text{rk}(G) > \text{rk}(H)$. If some $L_j^i$ has trivial image in $H$ then $\text{rk}(G) > \text{rk}(H)$, thus all $L_j^i$ have nontrivial image, and a fortiori, all $G_i$. Since $\text{sc}(L_j^i)$ is maximal out of all groups $\varphi|G_i$ factors through, every $L_j^i$ has freely indecomposable image in $H$.

Since $\Phi_s \varphi \to H$ is injective on freely indecomposable free factors and has nonabelian image, by Lemma 5.3, it is strict, and since $\varphi$ is degenerate, $\Phi_s \varphi$ is an isomorphism. Thus $p_H = \sum_i p_i$ and $q_H = q_G + \sum_i q_i$.

Computing that the Scott complexity of $H$ is at most that of $G$ is now easy:

$$p_H + q_H = \sum_i p_i + q_G + \sum_i q_i$$

Since each $p_i \geq 1$, $p_H \geq p_G$ and $q_H \geq q_G$. Thus $p_H + q_H \geq p_G + q_G$. If there is equality in the first coordinate then $p_i$ must be 1 for all $i$, and if there is equality in the second coordinate then $q_i = 0$ for all $i$. Thus $\text{sc}(G) \leq \text{sc}(H)$, with equality only if $\text{sc}(\varphi|G_i) = (1, 0)$, that is, no restriction $G_i \to \text{Im}(G_i)$ factors through a nontrivial free product.

In light of this, we set $c_{fd}(G) = (\text{rk}(G), b_1(G), -\text{sc}(G))$. If $G \to H$ is degenerate and $c_{fd}(G) = c_{fd}(H)$ then for each $i$ there is a unique $j(i)$ such that $G_j(i) \to H_i$ (up to conjugacy), and $H \cong \ast_i \text{Im}(G_i) \ast \mathbb{F}_{q_i}$. By Theorem 5.4, $c_{fd}$ is a nonincreasing function under degenerate maps. If $G_j$ is abelian and $\varphi(G_j) = H_i$, then $H_i$ is abelian. If $c_{fd}(G) = c_{fd}(H)$ then for all abelian freely indecomposable free factors $G_{j(i)}$, $\varphi|G_j(i) : G_j(i) \to H_i$ is an isomorphism, and for nonabelian freely indecomposable free factors, is degenerate.

A chain $\mathcal{L}$ of limit groups is **indecomposable** if no map $\mathcal{L}(i) \to \mathcal{L}(j)$ factors through a free product. Not only are the groups in an indecomposable sequence indecomposable, all compositions of maps are too. By Theorem 4.2 and the fact
that for limit groups of a fixed rank there are only finitely many values the Scott complexity can take, we have the following “alignment theorem.”

**Theorem 5.5** (Reduction to indecomposable sequences). Let \( \mathcal{L} \) be a rank \( N \) sequence of epimorphisms of limit groups. For all \( K \) there exists \( M = M(K, N) \) such that if \( \|\mathcal{L}\|_{pl} \geq M \) then there exists a maximal resolution \( \tilde{\mathcal{L}} \sqcup \mathcal{L} \) such that \( \|\tilde{\mathcal{L}}\|_{pl} > K \) and \( c_{fd} \) is constant along \( \tilde{\mathcal{L}} \).

In particular, \( \tilde{\mathcal{L}} \) splits as a graded free product of sequences

\[
\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_1 * \cdots * \tilde{\mathcal{L}}_p * \mathcal{F}
\]

where \( \mathcal{F} \) is the constant sequence \( (\mathbb{F}_q) \) for some \( q \). The sequences \( \tilde{\mathcal{L}}_i \sqcup \mathcal{L} \) are indecomposable maximal resolutions of their images. □

5.2. **Freely indecomposable groups.** Our goal is to understand degenerate maps of limit groups. In the previous subsection we saw that for freely decomposable groups, a degenerate map either decreases the rank, decreases the first betti number, raises the Scott complexity or respects Grushko factorizations. Scott complexity is blind to the behavior of restrictions of degenerate maps to the freely indecomposable factors in the Grushko decompositions of limit groups. In this subsection we construct a natural generalization of Scott complexity to JSJ decompositions. In this subsection \( \varphi: G \to H \) always has Scott complexity \((1,0)\), that is, \( G \) is freely indecomposable and \( \varphi \) doesn’t factor through a free product. Such a map is indecomposable.

In this subsection we mimic the construction of \( \Phi_s G \) for indecomposable homomorphisms: given an indecomposable \( \varphi: G \to H \), we build a quotient group \( \Phi_s G \) of \( G \) and a strict homomorphism \( \Phi_s G \to H \) which \( \varphi \) factors through. In particular, if \( \varphi \) is degenerate, then \( \Phi_s G \to H \) is an isomorphism. The construction of \( \Phi_s \) is very explicit, and will be used in the next section to show that the JSJ decomposition of \( \Phi_s G \) strongly resembles the JSJ decomposition of \( G \).

Our main tool is a variation on the Bestvina-Feighn folding machinery [BF91], which is roughly as follows. If \( T \) and \( S \) are faithful simplicial \( G \)-trees and \( \phi: T \to S \) is a \( G \)-equivariant morphism then \( \phi \) can be realized as a composition of elementary folds à la Stallings. As in the case of a free group, one may ignore the equivariance by studying the situation in the quotient graphs \( T/G \) and \( S/G \). The catch for non-free actions is that vertex and edge stabilizers are not trivial and we must consider a wider variety of morphisms of graphs of groups. The elementary folds one needs to consider are listed in in Figure 2.

A type I fold is the decorated version of a Stallings fold. Type II folds have no effect on the quotient graph \( T/G \), but if \( \bar{v} \) is a lift of \( v \) then the set of lifts of \( e \) (as an oriented edge) adjacent to \( v \) are in one to one correspondence with the cosets of \( E \) in \( V \). If \( g \) is pulled across the edge then the cosets of \( E \) fall into cosets of \( \langle E, g \rangle \), and \( T \) is folded accordingly. A type III fold is a composition of subdivision followed by type I and type IV folds.

Slightly more general than a type IV fold is the following move, which we call a type IVB fold: Let \( \Delta \) be a graph of groups decomposition of \( G \) and \( \Gamma \) a connected subgraph of \( \Delta \), and let \( G_{\Gamma} \) be the subgroup of \( G \) carried by \( \Gamma \). Collapse the graph \( \Delta \)
to form a graph $\Delta/\Gamma$ with distinguished vertex $\gamma$ which is the image of $\Gamma$. Assign the group $G_\Gamma$ to $\gamma$, retaining the labels on all edges and vertices not contained in $\Gamma$, to form a graph of groups decomposition $\Delta/\Gamma$ of $G$. Then $\Delta \rightarrow \Delta/\Gamma$ is a composition of collapses and type IV folds. If $\Gamma$ is disconnected, it is understood that the move is carried out component by component.

For general $T \rightarrow S$, Dunwoody adds vertex morphisms to this list [Dun98], wherein one is allowed to pass to a quotient of a vertex group. We will confine ourselves to a particular type of vertex morphism, the strict vertex morphisms, tailor-made for producing desirable quotients of limit groups.

5.2.1. Almost-strict homomorphisms. In this sub-subsection we show that if $\varphi: G \rightarrow H$ is indecomposable and $G$ is equipped with a GAD $\Delta$, then $\varphi$ factors through an almost-strict (Definition 5.8) homomorphism $\Phi_{\text{as}} G \rightarrow H$ such that $\Phi_{\text{as}} G$ has a GAD which resembles a degenerated and blown up $\Delta$. To begin we show that if an indecomposable homomorphism of limit groups doesn’t factor through a free product then edge groups don’t have trivial image.

**Lemma 5.6.** Suppose $G$ and $H$ are limit groups, $\varphi: G \rightarrow H$ indecomposable. If $G$ splits nontrivially over an abelian subgroup $E$ then $\varphi(E) \neq \{1\}$.

**Proof.** If the edge associated to $E$ is nonseparating then $\varphi$ factors through a group of the form $G' \ast \mathbb{Z}$, and if the edge is separating, then one of two possibilities occurs: Let $G_1$ and $G_2$ be the vertex groups of the one edged splitting. If $E$ has trivial image then at least one $G_i$ (say 1) has trivial image in $H$. If this is the case then $\varphi|_{G_1}$ factors through abelianization of $G_1$. If $G_1$ is nonabelian, then since limit groups have nontrivial homology relative to abelian subgroups, $\varphi$ factors through a nontrivial free product $G_1^{ab}/E \ast G_2/E$. If $G_1$ is abelian, then $E$ must have index
1 in \( G_1 \), otherwise \( G_1/E \) is nontrivial, again contradicting indecomposability of \( \varphi \).

Before we begin the construction of \( \Phi_s \) in earnest, we give a construction which vaguely resembles factoring through a group with maximal Scott complexity (Subsection 5.1, particularly the construction in Definition 5.2).

**Definition 5.7.** Let \( G \) be a finitely generated group with a graph of groups decomposition \( \Delta, G \cong G' \ast_{A_i} B_i \), \( B_i \) abelian. The complexity \( c_a(G, \Delta) \) is the sum \( \sum_i \text{rk}(B_i/A_i) \). Let \( E \) be a collection of abelian subgroups of \( G \). Then

\[
    c_a(G; E) := \max_{\Delta} \{ c_a(G, \Delta) \mid E \in \mathcal{E} \Rightarrow E \text{ conjugate into some } B_i \text{ of } \Delta \}
\]

If \( \varphi: G \to H \), \( E \) a family of abelian subgroups of \( G \), let \( \varphi(E) = \{ \varphi(E) \mid E \in \mathcal{E} \} \). Then set

\[
    c_a(\varphi, \mathcal{E}) := \max_{G'} \{ c_a(G', \varphi'(\mathcal{E})) \mid \varphi \text{ factors through a map } \varphi': G \twoheadrightarrow G' \}
\]

If \( \varphi: G \to H \) then we say that \( G' \) realizes \( c_a(\varphi) \) if \( \varphi \) factors through \( \varphi': G \to G' \) and \( c_a(G', \varphi'(\mathcal{E})) = c_a(\varphi, \mathcal{E}) \)

As an intermediate step between \( G \) and \( \Phi_sG \) we construct an intermediate quotient \( \Phi_{as}G \) and a homomorphism \( \Phi_{as}: G \twoheadrightarrow H \) which is *almost-strict*.

**Definition 5.8** *(Almost-strict).* Let \( \varphi: G \to H \) be indecomposable, \( H \) a limit group, and \( \Delta \) a GAD of \( G \). If \( \varphi \) satisfies the following axioms it is *almost-strict* (with respect to \( \Delta \)).

- **AS1:** \( \varphi \) embeds rigid vertex groups of \( \Delta \) in \( H \).
- **AS2:** \( \varphi \) embeds edge groups of \( \Delta \) in \( H \).
- **AS3:** [QH subgroups] are nonabelian.
- **AS4:** No rigid vertex group of \( \Delta \) has a nontrivial splitting in which all incident edge groups are elliptic.
- **AS5:** Every edge of \( \Delta \) is incident to an abelian vertex group.

To represent an almost-strict homomorphism we use the notation \( AS(\varphi, G, \Delta, H) \).

The reader should note that this list is missing the third and fourth conditions from Theorem 2.9. The last axiom is for technical reasons which appear in the proofs of Theorem 5.19, Lemma 6.16, and in Definition 6.14.

We say that a map of a pair \((G, \mathcal{E})\) to a group \( H \) is *indecomposable relative to the collection \( \mathcal{E} \), or \( \mathcal{E} \)-indecomposable*, if all images of elements of \( \mathcal{E} \) are nontrivial and it doesn’t factor through a free product in which all images of elements of \( \mathcal{E} \) are elliptic.

We now define two types of *strict relative quotients* of groups.

**Definition 5.9** *(Type I strict relative quotient).* Let \((R, \mathcal{E})\) be finitely generated group with a finite collection of abelian subgroups \( \mathcal{E} \), and suppose that \( \varphi: R \to H \) is \( \mathcal{E} \)-indecomposable and that \( H \) is a limit group. Let \( \mathcal{R} \) be the collection of quotients of \( R \) such that for all \( R' \in \mathcal{R}, \pi: R \twoheadrightarrow R', \varphi \) factors through a \( \text{Mod}(R', \pi(\mathcal{E}))-\text{strict } \varphi': R' \to H \).
Define a partial order on $\mathcal{R}$ as follows. If $R_1$ and $R_2 \in \mathcal{R}$ and the map $R_1 \to R$ factors through $R_1 \to R_2$ then $R_1 \geq R_2$, with equality if and only if $R_1 \to R_2$ is an isomorphism. By Theorem 2.10 $\mathcal{R}$ has maximal elements. Choose one such maximal element and call it $SR$. The quotient group $SR$ is a type I strict relative quotient of $R$.

Let $\varphi: G \to H$ be an indecomposable homomorphism of limit groups, $R$ a vertex group of a GAD $\Delta$ of $G$. The restrictions $\varphi|_R$ are $\mathcal{E}(R)$–indecomposable.

**Definition 5.10** (Type II strict relative quotient). Let $R$ and $H$ be limit groups and suppose $\varphi: (R, \mathcal{E}) \to H$ is $\mathcal{E}$–indecomposable. Choose a group $\overline{R} = S \ast_{A_i} B_i$ realizing $c_a(\varphi, \mathcal{E}(R))$. Without loss, we may assume that for all $E \in \mathcal{E}(R)$, by introducing a new edge and valence one vertex, both with the group $E$ attached, that there is some $B_i$ into which the image of $E$ is conjugate. Conjugate the collection $\mathcal{E}$ so that each element has image contained in some $B_i$, rather than just being conjugate into one. This operation has no effect on $c_a$. For each $E_i \in \mathcal{E}$ let $F_i$ be the kernel of the map $\text{Im}(E_i) \to H$. Suppose that the image of $E_i$ is contained in $B_j(i)$. Let $F_i = F_i \cap A_j(i)$ and let $I(j)$ be the collection of indices $i$ such that $E_i < B_j$. Now build the quotient groups

$$\overline{R}' = S' \ast_{A_j'} B_j'$$

$S' := S/\langle F_i \rangle$, $A_j' := A_j/\langle F_i \rangle_{i \in I(j)}$, $B_j' := B_j/\langle F_i \rangle_{i \in I(j)}$

and

$$\overline{R} = S'' \ast_{A_j''} B_j'$$

$S'' := S/\langle F_i \rangle$, $A_j'' := A_j/\langle F_i \rangle_{i \in I(j)}$, $B_j' := B_j/\langle F_i \rangle_{i \in I(j)}$

The map $\overline{R} \to H$ factors through the obvious maps $\overline{R} \to \overline{R}' \to \overline{R}'' \to H$. Now if $c_a(S', \{A_j'\}) > 0$ then $c_a(\overline{R}', \text{Im}(\mathcal{E})) > c_a(\overline{R}, \text{Im}(\mathcal{E}))$. By construction, if $E \in \mathcal{E}$, then the image of $E$ in $\overline{R}''$ embeds in $H$.

Now pass to the type I strict relative quotient of $S'$ with respect to the collection $A_j'$. Rather than call the quotient $SS'$, as above, since $S'$ depends only on $R$ and the map to $H$, call the quotient $S_{II}(R)$. Call the image of $A_j'$ in $S_{II}(R) A_j''$, and let $F_j'' := \text{ker}(A_j'' \to H)$. Now build the following quotient of $R$:

$$(ASR =) AS(R, \mathcal{E}) := S_{II}(R) \ast_{A_j''} \langle B_j''/F_j'' \rangle$$

The quotient $ASR$ is a type II strict relative quotient of $R$. By the same reasoning as above, $c_a(SS', \{A_j''\}) = 0$. The groups $A_j''$ embed in $H$. The reader must be warned that in general $ASR$ is not a limit group.

**Remark 5.11.** If $c_a(\varphi, \mathcal{E}(R)) = 0$ then $A_i$ is finite index in $B_i$ for all $i$ in the decomposition $\overline{R} = S \ast_{A_i} B_i$. If this is the case then $S(R, \mathcal{E})$ and $AS(R, \mathcal{E})$ agree. In light of this, in the event that $c_a = 0$, we freely ignore the distinction between type I and type II strict relative quotients whenever convenient.

Let $(L, \mathcal{E})$ be a limit group and let JSJ$(L, \mathcal{E})$ be its relative abelian JSJ. We make the following normalizations on JSJ$(L, \mathcal{E})$:

- If $E, E' \in \mathcal{E}$ and $Z_L(E)$ is conjugate to $Z_L(E')$ then $[E, E'] = \{1\}$. 

The centralizer of $E$ is always an abelian vertex group of JSJ$(L, E)$. This can be accomplished by subdivision or introduction of valence one vertex groups, as necessary.

We are now ready to construct, given an indecomposable map $G \to H$, $\Delta$ a GAD of $G$, a quotient $\Phi_{\text{as}} G$ of $G$, equipped with a GAD $\Phi_{\text{as}} \Delta$ induced by $\Delta$, and a $\Phi_{\text{as}} \Delta$–almost strict $\Phi_{\text{as}} G \to H$, such that the following diagram commutes.

\[
\begin{array}{ccc}
\Phi_{\text{as}} G & \longrightarrow & H \\
\downarrow & & \\
G & \longrightarrow & H
\end{array}
\]

Let $G$ be a group with an abelian decomposition $\Delta$ and an indecomposable map $\pi : G \to H$, $H$ a limit group. Build a group $\Phi_{\text{as}} G$ with a splitting $\Phi_{\text{as}} \Delta$ by taking the following quotients of vertex groups of $G$. For each edge group of $\Delta$, pass to the image in $H$. For each rigid vertex group of $\Delta$ pass to $\mathcal{A}S(R, E(R))$ if $R$ has nonabelian image in $H$, otherwise pass to $S(R, E(R))$. For all other cases pass to $S(V, E(V))$. The gluing data from $\Delta$ descends to gluing data on this new collection of groups. Call the resulting group $\Phi_{\text{as}} G$. For each rigid vertex group $R_i$ such that $\varphi(R_i)$ is nonabelian, $\mathcal{A}S(R_i, E(R_i))$ has the form $S_i \ast_{A_{i,j}} B_{i,j}$, and for each element $E \in E(R_i)$, the image of $E$ maps to some $B_{i,j}$. Replace $\mathcal{A}S R_i$ in $\Delta$ by $S_i \ast_{A_{i,j}} B_{i,j}$, and for each $i$ refine the splitting on $\Delta$ with the normalized relative JSJ decomposition JSJ$(S_i, \{ A_{i,j} \})$. The resulting decomposition of $\Phi_{\text{as}} G$ is called $\Phi_{\text{as}} \Delta$. Note that every nonabelian vertex group of $\Phi_{\text{as}} \Delta$ is either a nonabelian vertex group from some JSJ$(S_i, \{ A_{i,j} \})$ or is a QH vertex group inherited from $\Delta$. Call the induced homomorphism of $\Phi_{\text{as}} G \to H$ $\Phi_{\text{as}} \pi$.

**Definition 5.12** (Almost-strict factorization). The group $\Phi_{\text{as}} G$ with GAD $\Phi_{\text{as}} \Delta$ and homomorphism $\Phi_{\text{as}} \pi$ from above is an almost-strict factorization of $\pi$.

The induced homomorphism $\Phi_{\text{as}} \pi : \Phi_{\text{as}} G \to H$, may not be strict, but it is reasonably close: Since the automorphisms in $\text{Mod}(SS, E(S))$ fix the incident edge groups, they extend to automorphisms of $\Phi_{\text{as}} G$. Hence, for every sequence $f_n : H \to \mathbb{F}$ converging to $H$, there exists a sequence of automorphisms $\phi_n \in \text{Mod}(SR, E(SR))$, fixing vertex groups up to conjugacy, such that $f_n \circ \Phi_{\text{as}} \pi \circ \phi_n$ is stably trivial on every vertex group of $\Phi_{\text{as}} \Delta$. Thus if $\Phi_{\text{as}} \pi$ isn’t strict, the failure must lie elsewhere in the bullets of Theorem 2.9.

The vertices of $\Delta$ came with labels Q, R, and A, corresponding to quadratically hanging, rigid, and abelian vertex groups. In the group $\Phi_{\text{as}} G$ we label the vertices of $\Phi_{\text{as}} \Delta$ as follows.

- Label abelian vertex groups ‘A’.
- Label QH vertex groups ‘Q’.
- Label rigid vertex groups coming from the relative decompositions JSJ$(SS)$ ‘R’.

That $\Phi_{\text{as}} \pi : \Phi_{\text{as}} G \to H$ is almost-strict follows immediately from the definitions. The JSJ decomposition of $G$, in the event that $G$ has one and $\Delta = \text{JSJ}(G)$, gives some information about the JSJ decomposition of $H$. 
Definition 5.13. If \( G \) is a group then \( G^{ab} \) is the abelianization of \( G \) modulo torsion. If \( E_i \) is a collection of subgroups of \( G \) then the smallest subgroup of \( G^{ab} \) which is closed under taking roots and contains the images of \( E_i \) is the peripheral subgroup of \( G^{ab} \) and is denoted by \( P(G^{ab}) \). If \( A \) is an abelian vertex group of a GAD then this definition of the peripheral subgroup agrees with the original definition.

Lemma 5.14. Let \( \pi : G \rightarrow H \) be indecomposable, \( G \) a limit group, \( \Delta \) a GAD of \( G \), and \( G \rightarrow \Phi_{ab} G \), a \( \Phi_{ab} \) \( \Delta \)-almost strict factorization of \( \pi \) as above.

If \( V \) is a vertex of \( \Delta \) which has abelian image then \( SV \cong V^{ab} / \ker(P(V^{ab}) \rightarrow H) \). By indecomposability of \( \pi \), if \( V \) is a QH subgroup then \( V \) is a punctured sphere or projective plane.

If \( V \) is a QH vertex group which has nonabelian image in \( H \) then \( S(V, \partial V) \cong \langle V, \partial V \rangle \).

Suppose that \( V \) has a graph of groups decomposition \( V \cong \Gamma(A_i, F_j) \) over torsion free abelian vertex groups \( A_i \) and nontrivial edge groups \( F_j \). Suppose further that each edge group of \( \Delta \) incident to \( V \) is elliptic in \( \Gamma \). Let \( P(A_i) \) be the subgroup of \( A_i \) generated by incident edge groups (in \( \Gamma \)) and those elements of \( \mathcal{E}(V) \) conjugate into \( A_i \). Call the underlying graph of \( \Gamma(A_i, F_j) \) \( \Gamma \) as well. Then

\[
\sum_i \rk(A_i/P(A_i)) + b_1(\Gamma) \leq \rk(SV/\langle \im(\mathcal{E}(V)) \rangle)
\]

Proof. If \( V \) is QH and has any genus then any simple closed curve cutting off a handle has trivial image. Thus \( \pi \) factors through a free product \( G \ast \mathbb{Z}^2 \). Contradiction.

The statement that QH subgroups with nonabelian image are isomorphic to their type I strict relative quotients is [Sel01, Lemma 5.13].

Suppose \( V \) is as in the last paragraph of the lemma. By indecomposability of \( \pi \) and Lemma 5.6, no \( F_j \) has trivial image in \( H \). Construct an abelian quotient \( \overline{V} \) of \( V \) as follows: Let \( T \) be a maximal tree in \( \Gamma \) and let \( F_{i_1}, \ldots, F_{i_m} \) be the edge groups of \( \Gamma \) not carried by edges in \( T \). Let \( V_T \) be the subgraph of groups of \( V \) obtained by restriction to \( T \), and let \( \mathcal{E}(V_T) \) be the collection \( \mathcal{E}(V) \) along with the \( F_{i_j} \). The inequality

\[
\sum_i \rk(A_i/P(A_i)) \leq \rk(SV_T/\langle \im(\mathcal{E}(V) \cup \{ F_{i_j} \}) \rangle)
\]

holds. Let \( t_j \) be the stable letter associated to \( F_{i_j} \). Then the image of \( t_j \) in \( H \) conjugates the image of \( F_{i_j} \) to another subgroup of the image of \( (PSV_T) \). Since abelian subgroups of limit groups are malnormal, the two inclusions \( F_{i_j} \rightarrow SV_T \) must agree with one another. Since limit groups are commutative transitive, the map \( V_T * F_{i_j} \rightarrow H \) factors through \( V_T \oplus \langle t_j \rangle \), a group which satisfies the inequality of the lemma. Repeating over all edge groups \( F_{i_j} \) we find an abelian quotient \( V' \) of \( V \), a \( \text{Mod}(V', \im(\mathcal{E}(V))) \)-strict \( V' \rightarrow H \) satisfying the lemma. Since \( V' \rightarrow H \) factors through \( SV \rightarrow H \), \( V' \cong SV \) by maximality.

5.2.2. From almost-strict to strict. Fix \( AS(\varphi, G, \Delta, H) \) for the remainder of this section. We build an infinite sequence of groups \( G \rightarrow G_1 \rightarrow \cdots G_i \cdots \rightarrow H \) such
that each induced map \( G_i \to H \) is almost strict, closer to strict than the previous homomorphism, and the direct limit homomorphism \( \lim_{\to} G_i \to H \) is strict, and that for all but finitely many \( i, G_i \to G_{i+1} \) is an isomorphism.

We now define two maps which takes as input almost strict \( AS(\varphi, G, \Delta, H) \) and output almost strict \( AS(\Phi^*\varphi, \Phi^*G, \Phi^*\Delta, H) \) such that the induced homomorphisms \( \Phi^*\varphi: \Phi^*G \to H \) are closer to satisfying the bullets from Theorem 2.9.

**Defining \( \Phi_a \)**

First, we take an almost strict \( AS(\varphi, G, \Delta, H) \) and adjust the GAD \( \Delta \). Let \( \sim_a \) be the equivalence relation generated by adjacency of abelian vertex groups. Let \( [A] \) be an \( \sim_a \) equivalence class and let \( \Gamma_{[A]} \) be the subgraph of \( \Delta \) with vertices from \( [A] \) and edges connecting members of \( [A] \). Now perform a sequence of type IVB folds to collapse the subgraphs \( \Gamma_{[A]} \), as \( [A] \) varies over all \( \sim_a \) equivalence classes. Call the vertex associated to \( [A] \) \( v_{[A]} \).

Let \( \Phi_a G \) be the group obtained from \( G \) by passing from \( G \) to \( S_{G \cdot v_{[A]}} \) for each \( \sim_a \) equivalence class \( A \). The abelian splitting \( \Phi_a \Delta \) of \( \Phi_a G \) is the one it inherits from the collapsed \( \Delta \). The induced map \( \Phi_a G \to H \) is denoted by \( \Phi_a \varphi \).

The induced map of the quotient group \( \Phi_a G \) with the GAD it inherits from \( G \) is almost-strict.

**Defining \( \Phi_r \)**

We now define a third homomorphism \( \Phi_r \), in addition to \( \Phi_{as} \) and \( \Phi_a \), which brings us closer to satisfying the third bullet of Theorem 2.9.

**Definition 5.15.** Let \( R_1 \) be a limit group, \( E \) a family of nonconjugate abelian subgroups of \( R_1 \) such that the relative JSJ decomposition \( JSJ(R_1; E) \) is trivial. An envelope of \( R_1 \) is a limit group \( R_2 \) such that

- There is a collection \( \{P_i\}_{i \in I} \) of free abelian groups such that \( R_2 \) is a limit quotient of \( R_1 \star_{E_i} P_i \),
- There is a map from \( R_1 \star_{E_i} P_i \) to a fixed limit group \( H \) called the target.
- \( R_1 \) \to \( R_2 \) is injective and the map \( R_1 \star_{E_i} P_i \to H \) factors through the map to \( R_2 \).
- The map from \( R_2 \) to \( H \) is \( \text{Mod}(R_2; \{\text{Im}(P_i)\}) \)–strict.

We call the group \( R_1 \star_{E_i} P_i \) a pre-envelope. We call \( R_1 \) the core of the pre-envelope. The complexity of a pre-envelope is the ordered triple

\[
\left( \sum_i \text{rk}(P_i/E_i), |I|, m \right)
\]

Where \( m \) is the number of indices \( i \) such that \( E_i \) is not maximal abelian in \( R_1 \).

Our main lemma regarding envelopes is that they can be written nicely as the output of the process of iteratively adjoining roots (Definition 5.16) and extension of centralizers.

**Definition 5.16 (Iteratively adjoining roots).** Let \( L \) be a limit group. Then \( L' \) is obtained from \( L \) by iteratively adjoining roots if there is a finite sequence of limit groups \( L_i, L_0 = L, L_n = L' \) such that
Lemma 5.17. Let $R_2$ be an envelope of $R_1$, with $\text{Mod}(R_2; \{\text{Im}(P_i)\})$ strict homomorphism $\varphi: R_2 \to H$ which embeds $R_1$. Then $R_2$ can be decomposed as an iterated adjunction of roots and an extension of centralizers, and can be realized as a quotient pre-envelope of the pre-envelope $R_1 \ast_{E_i} P_i$ of lower complexity.

Let $\{F_j\}$ be the set of equivalence classes of $E_i$ in $R_2$ such that $E_i, E'_i \in F_j$ if and only if $E_i$ and $E'_i$ have conjugate centralizers. For each class $F_j$ choose a single representative element $F_j$, and for $j$ let $I_j$ be the set of indices $i$ such that $E_i \in F_j$. Then there are direct sum decompositions $P_i \cong P_i^j \oplus C_i$ such that $E_i < C_i$, quotients $D_j$ of $\oplus_{i \in I_j} P_i^j$, and $R_2$ can be written as

$$R_1 := R_1 \left[\sqrt{E_j}\right], \quad R_2 \cong R_1 \ast_{P_i^j} (Z_{P_i}^j(F_j) \oplus D_j)$$

Moreover, the group $R_1$ is a quotient of $R_1 \ast_{E_i} C_i$, has trivial JSJ relative to the centralizers of the images of the $E_i$, and $\varphi$ is an embedding.

First we need a basic lemma.

Lemma 5.18. Let $H$ be a limit group, $L$ a pre-envelope $G \ast_{A_i} B_i$, $A_i$ maximal abelian in $G$, closed under taking roots in $B_i$, and $A_i$ not conjugate to $A_j$ for $i \neq j$ in $G$. Let $\phi: L \to H$ be a homomorphism such that all restrictions $\phi|_{B_i}$ and $\phi|_{A_i}$ are injective. Then $\phi$ is injective.

The argument is standard, and follows from normal forms, induction on the height of the analysis lattice, and the fact that if two elements of a group are non-conjugate then they remain nonconjugate after extension of centralizers.

Proof of Lemma 5.17. We define three homomorphisms of pre-envelopes. Each homomorphism yields a pre-envelope of lower complexity, or the pre-envelope embeds in the target.

Suppose $E_i$ is not maximal abelian in $R_1$. Let $C_i$ be the smallest direct summand of $P_i$ containing $E_i$, and let $D_i$ be a complimentary direct summand. By commutative transitivity the map $R_1 \ast_{E_i} P_i \to R_2$ factors through

$$P_{i \neq j} \ast_{E_j} R_1 \ast_{Z_{R_1}(E_i)} ((Z_{R_1}(E_i) \ast_{E_i} C_i)^{ab} \oplus D_i)$$

The last coordinate of the complexity decreases.

If, say, $E_1$ and $E_2$ have conjugate centralizers, then the map $R_1 \ast_{E_i} P_i \to H$ factors through

$$R_1' := P_{i > 2} \ast_{E_j > 2} R_1 \ast_{Z_{R_1}(E_1)} (Z_{R_1}(E_1) \oplus D_1 \oplus D_2)$$

In this case the pre-envelope $R_1'$ has lower complexity than $R_1$, as the second coordinate of the complexity strictly decreases.

We now define the third quotient of a pre-envelope. This procedure is only to be applied if the previous two cannot.

Let $C_i$ be the largest direct summand of $P_i$ such that the image of $E_i$ has finite index image in the image of $C_i$, let $D_i$ be a complimentary direct summand, and
let $\overline{C}_i$ be the image of $C_i$ in $R_2$. Let $R'_1$ be the image of $R_1 \ast_{E_i} \overline{C}_i$ in $R_2$. Since $E_i$ is finite index in $\overline{C}_i$, the relative JSJ decomposition of $R'_1$ is also trivial and $R'_1$ must embed in $R_2$ and the target $H$.

Then $R_2$ is a limit quotient of a new pre-envelope

$$R'_2 := R'_1 \ast_{\overline{C}_i} (\overline{C}_i \oplus D_i)$$

Since $C_i$ contains $E_i$, the complexity of $R'_2$ is at most that of $R'_1 \ast_{E_i} P_i$, and the core $R'_1$ is obtained from $R_1$ by adjoining roots. If the complexity doesn’t decrease, then each $P_i$ must embed in $R_2$, each $E_i$ must be maximal abelian in $R_1$, and by Lemma 5.18, $R_1 \ast_{E_i} P_i$ embeds in the target $H$, $R_2 = R_1 \ast_{E_i} P_i$, and $R_2$ trivially has the structure from the lemma. □

Let $AS(\varphi, G, \Delta, H)$ be almost strict. The homomorphism $\varphi$ may not be strict because it may not embed envelopes of rigid vertex groups. We first adjust $\Delta$. Let $e$ be an edge incident to a rigid vertex group $R$, $\tau(e) = r$, and let $A$ be the abelian vertex group attached to $\iota(e)$ and provided by axiom AS5. Subdivide $e$ and pull $P(A)$ across the newly introduced edge adjacent to $A$. Repeat for all edges incident to rigid vertex groups, and call the new abelian decomposition of $G \Delta'$. Let $A(E)$ be the abelian vertex group adjacent to and centralizing $E$. For each rigid vertex group of $\Delta'$, the star of $R$ has the following form

$$St(R) \cong R \ast_{E \in \varepsilon(R)} P(A(E))$$

Now replace $St(R)$ by its type I strict relative quotient $S(St R, \{P(A_i)\})$. By Lemma 5.17

$$S(St R, \{P(A_i)\}) \cong \overline{R} \ast_{Z(F_j)} (C(F_j) \oplus D_j)$$

and each $P(A_i)$ has image contained in some $\overline{C}_i \ast_{Z(F_j)} (C(F_j) \oplus D_j)$. Since $P(A_i)$ embeds in $H$ it embeds in $C(F_j) \oplus D_j$. Repeat this process for all rigid vertex groups of $\Delta'$ and call the quotient group $\Phi_i G$, the induced decomposition $\Phi_i \Delta$, and the induced homomorphism $\Phi_i \varphi$. That the tuple $(\Phi_i \varphi, \Phi_i G, \Phi_i \Delta, H)$ is almost strict follows immediately from the definitions.

We can now define the direct limit promised at the beginning of this sub-subsection. Let $G \to H$ be a homomorphism of limit groups which doesn’t factor through a free product. Let $G_i$ be the group

$$G_0 = \Phi_{as} G, \quad G_{2n+1} = \Phi_{a} G_{2n}, \quad G_{2n} = \Phi_{r} G_{2n-1}$$

Similarly define $\Delta_{2n+1}$ and $\Delta_{2n+2}$. Let $G_\infty$ be the direct limit of the sequence $(G_n)$. The homomorphism $G \to H$ factors through $G_\infty$.

**Theorem 5.19.** The direct limit $G_\infty$ is a limit group, $G_n \to G_{n+1}$ is an isomorphism for all but finitely many $n$, and the rigid vertex groups in the decomposition of $G_\infty$ induced by the JSJ decomposition of $G$ are obtained from the vertex groups of $\Phi_{as} G$ by iteratively adjoining roots.

**Proof of Theorem 5.19.** By Lemma 5.14 $c_\alpha(G_n, \Delta_n)$ (Definition 5.7) is nondecreasing. Since $b_1(G) \leq b_1(G_n)$ we may drop finitely many terms from the beginning of the sequence and assume that the sequence $c_\alpha(G_n, \Delta_n)$ is constant. If this is the case then the underlying graphs of $\Delta_i$ have constant betti number. Let
\[ \{ R^i_j \}, \text{be the rigid vertex groups of } \Delta_i. \] By Lemma 5.17 \( R^{i+1}_j \) is obtained from \( R^i_j \) by iteratively adjoining roots. Let \( v^i_j \) be the number of edges incident to \( R^i_j \). The sequence \( (v^i_j)_{i=1,\infty} \) is nonincreasing. Again, by dropping finitely many terms from the beginning of the sequence, we may assume that all the sequences \( v^i_j \) constant. Once this is the case, the sequence of numbers of conjugacy classes of abelian vertex groups of \( G_{2n+1} \) is constant.

Let \( A^{2n+1}_k \) be the collection of sequences of abelian vertex groups, \( A^{2n+1}_k < G_{2n+1} \) such that \( A^{2n+1}_k \to A^{2n+3}_k \) for all \( n \). Inside \( \Delta_{2n+1} \) there are subgraphs of groups of the form

\[
\begin{align*}
R^{2n+1}_{i(l)} &\to E^{2n+1}_{i(l)} & A^{2n+1}_{k(l)}
\end{align*}
\]

In passing to \( G_{2n+2} \), by subdividing and pulling, subgraphs of this form are transformed to subgraphs of groups of the form

\[
\begin{align*}
R^{2n+2}_{i(l)} &\to E^{2n+1}_{i(l)} & P(A^{2n+1}_{k(l)}) &\to P(A^{2n+3}_{k(l)}) & A^{2n+1}_{k(l)}
\end{align*}
\]

Which, after an application of Lemma 5.17, become the subgroups

\[
\begin{align*}
R^{2n+2}_{i(l)} &\to \mathcal{Z}(E^{2n+1}_{i(l)}) & \mathcal{Z}(E^{2n+1}_{i(l)}) &\to D^{2n+2}_{i(l)} & A^{2n+1}_{k(l)}
\end{align*}
\]

contained in \( G_{2n+2} \). The centralizer \( \mathcal{Z}(E^{2n+1}_{i(l)}) \) should be taken in \( R^{2n+2}_{i(l)} \). Now to pass to \( G_{2n+3} \), the edges labeled \( P(A^{2n+1}_{k(l)}) \) are crushed when collapsing the subtrees (they are trees since \( c_a \) is constant) \( \Gamma[A] \subset \Delta_{2n+1} \). Since \( E^{2n+1}_{i(l)} \to P(A^{2n+1}_{k(l)}) \) and \( P(A^{2n+1}_{k(l)}) \) is generated by incident edge groups, and since \( E^{2n+1}_{i(l)} < \mathcal{Z}R^{2n+2}_{i(l)}(E^{2n+1}_{i(l)}) \equiv E^{2n+2}_{i(l)}, P(A^{2n+1}_{k(l)}) < P(A^{2n+3}_{k(l)}) \). Since \( P(A^{2n+1}_{k(l)}) \) embeds in \( H \) by construction, the sequences \( P(A^{2n+1}_{k(l)}) < P(A^{2n+3}_{k(l)}) \) map to sequences of subgroups of a finitely generated free abelian subgroup of \( H \). Thus, for sufficiently large \( n \), \( P(A^{2n+1}_{k(l)}) = P(A^{2n+3}_{k(l)}) \). Likewise, the sequences \( \mathcal{Z}R^{2n+2}_{i(l)}(E^{2n+1}_{i(l)}) \) are stable for sufficiently large \( n \), hence \( R^{2n+2}_{i(l)} \to R^{2n+4}_{i(l)} \) is an isomorphism for all \( i \) and sufficiently large \( n \).

Let \( n \) be large enough to satisfy the above. If \( E \) is adjacent to \( R \) and \( E \) doesn’t have maximal abelian image in \( R \) then \( \Phi \circ \Phi \) strictly increases the rank of some peripheral subgroup, contradicting the stability of ranks of peripheral subgroups (Recall the normalization that every edge group be adjacent to a maximal abelian vertex group.). If the envelope of a rigid vertex group doesn’t embed, then the either the rank of an edge group must increase under \( \Phi \), or a peripheral subgroup must fail to embed, neither of which is possible. By Lemma 5.18 the envelopes must embed. By Theorem 2.9, \( G_n \to H \) is strict for sufficiently large \( n \).

That the rigid vertex groups are obtained by iteratively adjoining roots follows immediately from the construction of \( \Phi \) and Lemma 5.17. That \( G_n \to G_{n+1} \) is an isomorphism for sufficiently large \( n \) follows from the fact that stability of edge and peripheral subgroups implies that the quotients \( \Phi \) and \( \Phi \) are isomorphisms. \( \Box \)
Set $\Phi_s G := G_\infty$, likewise for $\Phi_s \Delta, \Phi_s \varphi$. We say that $\Phi_s \Delta$ is the \textit{push-forward} of $\Delta$.

6. DEGENERATIONS OF JSJ DECOMPOSITIONS

We saw in the previous section that given an indecomposable homomorphism $\pi: G \to H$, one can construct a quotient $\Phi_s(G)$ of $G$ and a strict homomorphism $\Psi_s(G) \to H$ such that the composition of the quotient map and the strict homomorphism is $\pi$. Our approach to Krull dimension for limit groups is to use Theorems 5.4 and 4.2 to reduce the problem of existence of arbitrarily long chains of epimorphisms of limit groups to an analysis of degenerate chains. Let $L$ be a chain of freely indecomposable limit groups such that all maps $L(i) \to L(j), i < j$, are indecomposable. Such a chain is \textit{indecomposable}. The next step in our analysis is to control the degenerations of the sequence of JSJ decompositions $\text{JSJ}(L(i))$.

The strict homomorphism $\Phi_s(G) \to H$ is a (finite) direct limit of quotients of $G$ obtained by iterating the constructions $\Phi_a$ and $\Phi_r$. These homomorphisms were designed to ensure that in the direct limit the bullets from Theorem 2.9 were satisfied. The possible degenerations of the abelian JSJ of $G$ under $\Phi_s$ were not completely characterized since we were only interested in constructing a strict factorization.

The chains of freely indecomposable free factors produced by Theorem 5.4 are necessarily degenerate.

In this section we construct a complexity, computed from the JSJ decomposition and modeled on the Scott complexity, which is nondecreasing on degenerate indecomposable chains and which takes boundedly many values for limit groups with a given first betti number. It will be constructed out of three kinds of data from the JSJ decomposition: A complexity which manages $b_1$, the complexity of QH vertex groups not reflected in the first betti number, and some combinatorics of the underlying graphs of JSJ decompositions.

We start by measuring as much of the first betti number of a limit group which is immediately detectable in its JSJ decomposition. Define the following quantities:

- The sum of relative ranks of abelian vertex groups:
  \[ c_a(G) := \sum_A b_1(A/P(A)) \]

- The complexity of surface vertex groups: If $\Sigma$ is a surface with boundary then
  \[ c_g(G) := \sum_{\Sigma(Q)} b_1(\Sigma, \partial \Sigma) \]
  with the sum over all surfaces $\Sigma(Q)$ representing QH vertex groups $Q$ of $G$. If the computation is done in a GAD $\Delta$ we write this as $c_g(G, \Delta)$, otherwise the abelian JSJ is implied.

- Let $\Gamma$ be the underlying graph of the JSJ decomposition. The betti number of the underlying graph of the abelian JSJ decomposition:
  \[ c_b(G) := b_1(\Gamma) \]
Our first approximation to the Scott complexity of a limit group is the quantity
\[ \text{sc}_1(G) := (c_a(G), c_g(G), c_b(G)) \]
We have already shown, in Lemma 5.14, that the first coordinate of \( \text{sc}_1 \) is nondecreasing under indecomposable degenerate maps of limit groups. Lemma 6.2 is the analog of Lemma 5.14 for surfaces with boundary. First, a simple lemma.

**Lemma 6.1.** Let \( \Sigma \) be a surface with boundary, written as a graph of surfaces with boundary \( \Gamma(\Sigma_i) \). Then
\[ b_1(\Sigma, \partial \Sigma) \geq b_1(\Gamma) + \sum_i b_1(\Sigma_i, \partial \Sigma_i) \]

**Proof.** There is a map \( \Sigma \rightarrow \mathbb{Z}^{b_1(\Gamma)} \) which kills \( \partial \Sigma \) and all \( \Sigma_i \). \( \square \)

**Lemma 6.2.** Suppose \( c_a(G) = c_a(H) \). Then \( c_g(G) \leq c_g(H) \). If equality holds then \( c_b(G) \leq c_b(H) \).

**Proof.** Let \( \Delta = \Phi_s(\text{JSJ}(G)) \) be the push forward of the abelian JSJ of \( G \) under \( \Phi_s \). Since \( G \rightarrow H \) is degenerate, the map \( \Phi_s(G) \rightarrow H \) is an isomorphism. Thus, we only need to deduce how the JSJ of \( H \) is obtained from \( \Delta \).

The main difficulty is that there may be splittings of \( H \) which are hyperbolic-hyperbolic with respect to one-edged splittings corresponding to edge groups incident to rigid vertex groups of \( \Delta \). We claim the following: If \( R \) is a rigid vertex group of \( \Delta \) then either \( R \) is a rigid vertex group in the JSJ of \( H \) or \( R \) is represented by a non-QH subsurface of a QH vertex group of \( H \). Such a subsurface group must be either a twice punctured projective plane or a thrice punctured sphere.

It is a well known fact from the JSJ theory that if \( \Delta \) is a GAD of \( H \) and \( H \) has an abelian JSJ decomposition, then \( \Delta \) can be obtained (essentially) from the JSJ by choosing a family of simple closed curves \( c_i \) on QH vertex groups and regarding the resulting subsurfaces as vertex groups and the curves \( c_i \) as edge groups. After slicing the surface vertex groups like this, a sequence of folds produces \( \Delta \).

Thus every rigid vertex group \( R \) of \( \Delta \) is generated by rigid, abelian, subsurface groups and stable letters of the JSJ of \( H \). If it consists of more than two, then \( R \) has a splitting relative to its incident edge groups, contrary to rigidity of \( R \). Thus \( R \) is either a subsurface or a rigid vertex group from the JSJ of \( H \). If \( R \) is a subsurface group then, since \( R \) isn’t a QH vertex group of the relative abelian JSJ of \( R \), it must be one of the two non-QH surface groups above. We recover the abelian JSJ of \( H \) by gluing the QH vertex groups of \( \Delta \) and the rigid subsurface groups along their boundary components.

Define, in analogy with \( \sim_a \), an equivalence relation \( \sim_q \) on subsurface groups \( R \), QH vertex groups \( Q \), and cyclic abelian vertex groups \( Z \): If \( V \) is a cyclic abelian vertex group of valence two, regard it as the fundamental group of an annulus, and of valence one, of a Möbius band. If two such groups \( V_1 \) and \( V_2 \) are adjacent, being the endpoints of an edge \( e \), then \( V_1 \sim_q V_2 \) if the inclusions \( E \hookrightarrow V_i \) are isomorphisms with boundary components. As for \( \sim_a \) equivalence classes \( [A] \), let \( \Gamma_Q \) be the subgraph spanned by edges connecting vertices of a \( \sim_q \) equivalence class \( Q \).
If a cycle \( C \) of such vertex groups appears in \( \Gamma_G \), then by Lemma 6.1
\[
c_g(H, \Delta/C) + c_b(H, \Delta/C) \geq c_g(H, \Delta) + c_b(H, \Delta)
\]
Collapsing all such cycles through a sequence of collapses and type IV folds, we see that
\[
c_g(G) + c_b(G) \leq c_g(H, \Delta) + c_b(H, \Delta) \leq c_g(H) + c_b(H).
\]
If equality holds then no cycles are collapsed, and if a cycle is collapsed then the inequality is strict.

With this the proposition that \( \text{sc}_1 \) is nondecreasing under degenerate maps of freely indecomposable limit groups is established. We now turn to the problem of adding terms to the complexity \( \text{sc}_1 \) which handle the possibility that nonabelian vertex groups may have abelian image under degenerate maps. For the remainder of the section, \( G \to H \) is a degenerate map such that \( \text{sc}_1(G) = \text{sc}_1(H) \). Note that if \( R \) is a valence one vertex group then, since the first betti number relative to an abelian subgroup is greater than 0, \( R \) cannot have abelian image in \( H \), otherwise \( c_a(H) > c_a(G) \). Likewise, the underlying graph of the JSJ of any \( SR \), for a rigid vertex group \( R \) of \( G \), cannot contain a loop (which makes a contribution to \( c_b) \) or, through abelian vertex groups in their relative JSJ’s, make any contributions to \( c_a \), in other words, they must be trees with abelian subgroups equal to their peripheral subgroups (modulo the images of incident edge groups). Let \( \Gamma_A \) be a subgraph of the underlying graph of the JSJ of \( G \) such that \( \Gamma \) has abelian image in \( H \). If \( c_a(G) = c_a(H) \) then \( \Gamma_A \) must be a tree. If \( A \ast_{E_i} G' \) is the splitting of \( G \) obtained by collapsing all edges of \( G \) except those adjacent to \( \Gamma_A \) but not contained in \( \Gamma_A \), then no two images \( \text{Im}(E_i), \text{Im}(E_j) \) can have conjugate centralizers in \( \text{Im}(G') \), otherwise \( c_a(H) > c_a(G) \).

**Lemma 6.3.** Suppose \( G \to H \) is degenerate and \( \text{sc}_1(G) = \text{sc}_1(H) \). Suppose \( R \) is a nonabelian vertex group of \( G \) (either rigid or QH) with abelian image in \( H \). Let \( e_1, \ldots, e_n \) be the edges of \( G \) such that \( e_i \) doesn’t connect \( r \), the vertex carrying \( R \), to a valence one abelian vertex group of \( G \). Then the collection of images of \( E_i, \langle E_i \rangle < R \), generate \( H_1(R) \), and if an incident edge is separating, then neither vertex group in the corresponding one-edged splitting has abelian image in \( H \). In particular, if \( R \) is QH, then at most one edge incident to \( R \) connects it to a valence one abelian vertex group. In either case, the abelian vertex group of \( H \) containing the image of \( R \) has valence at least two.

**Proof.** The statement about incident edges generating \( R \) homologically follows from the observation that if not, then there is a map \( R \to \mathbb{Z} \) which kills every incident edge. Let \( A_1 \) and \( A_2 \) be valence one abelian vertex groups adjacent to \( Q \), a QH vertex group of \( G \). Let \( P \) be a pair of pants with boundary components \( l_1, l_2 \) and \( w \), such that \( l_i \) is the leg of \( P \) attached to \( A_i \). Then the group \( A = A_1 \ast_{l_1} P \ast_{l_2} A_2 \) satisfies \( b_1(\mathcal{SA}, w) > b_1(A_1, l_1) + b_1(A_2, l_2) \), leading to a strict increase in \( c_a \).

Since the first betti number of a nonabelian limit group relative to an abelian subgroup is at least one, \( n \) is at least two. Suppose that some edge \( e_i \) is separating, \( G = G_1 \ast_{E_i} G_2 \), and \( R \) (or \( Q \)) is contained in \( G_1 \). If \( G_2 \) has abelian image in \( H \) then the relative JSJ decomposition \( \text{JSJ}(G_2, E_i) \) is a tree. If \( G_2 \) contains a rigid vertex group \( R' \) of \( G \) then \( R' \) isn’t a valence one vertex group, is a cut-point in
the underlying graph of JSJ$(G)$, and has as many complimentary components as it does incident edge groups. Likewise, if $Q'$ is a QH subgroup of $G$ contained in $G_2$, then the vertex associated to $Q'$ must be a cut-point in the underlying graph of the JSJ and in fact must have as many complimentary components as it does boundary components.

Thus, if $G_2$ contains a nonabelian vertex group of $G$, then it must contain one for which all incident edges but one are attached to valence one abelian vertex groups of $G$. This is of course an impossibility, as $c_a(G)$ is then strictly larger than $c_a(H)$. □

If $Q$ is a QH vertex group of $G$ which has abelian image in $H$ then, since $G \to H$ doesn’t factor through a free product, $Q$ can have no genus and is either a punctured sphere or projective plane. Since $Q$ isn’t a punctured torus, $\chi(Q) \leq -2$ and $Q$ has at least three punctures. Let $\Delta$ be the splitting obtained by collapsing all edges of JSJ$(G)$ not adjacent to $Q$. If $c$ and $d$ are boundary components of $Q$ which are attached to the same vertex group in $\Delta$, then their images have nonconjugate centralizers, otherwise $c_a$ strictly increases.

The vertex group in $\Phi_0^\ast \Phi_a^G$ which is the image of $Q$ is abelian, and, since $c_a(G) = c_a(H)$, the remark about one edged splittings and the fact that no $\Gamma_A$ can contain a loop the vertex group containing $\text{Im}(Q)$ has valence at least three if $Q$ is not adjacent to a valence one abelian vertex group, has valence at least two if $Q$ is adjacent to a single valence one abelian vertex group, and if $Q$ is adjacent to two valence one abelian vertex groups $A_1$ and $A_2$, $c_a(H) > c_a(G)$. The last part can be seen by finding a pair of pants $P$ in $Q$ with one cuff attached to each abelian vertex group. Let $w$ be the waist of the pair of pants. Passing to the abelianization of $A = A_1 \ast_{\mathbb{Z}} P \ast_{\mathbb{Z}} A_2$, an elementary computation shows that the relative betti number satisfies $b_1(SA, w) > b_1(A_1, l_1) + b_1(A_2, l_2)$, leading to an increase of $c_a$.

The next step in our analysis is to show that a certain quantity can be appended to $sc_1$ such that under degenerate maps which don’t raise $sc_1$, the quantity appended is nondecreasing, and if equality holds in the new coordinates, then only valence two nonabelian rigid vertex groups can have abelian image (and ruling out the possibility that QH subgroups have abelian image). There is also the possibility that QH subgroups can grow, in the sense that their Euler characteristics can decrease, and that rigid vertex groups can engender multiple child rigid vertex groups. We also control these phenomena.

For the remainder of the section, $G \to H$ is degenerate and $sc_1(G) = sc_1(H)$.

**Definition 6.4.** If $\Gamma$ is a finite graph, $v$ a vertex of $\Gamma$, then $\kappa(v)$ is

$$1 - \frac{1}{2} \text{valence}(v)$$

by definition $\chi(\Gamma) = \sum_{v \in V^0} \kappa(v)$. Let $\Gamma$ be the underlying graph of an abelian decomposition $\Delta$ of a limit group $G$. Then define

$$\kappa_N(\Delta) := \sum_{G_v \text{ nonabelian}} \kappa(v)$$
and
\[ \kappa_A(\Delta) := \sum_{G_v \text{ abelian}} \kappa(v) \]
By definition, \( \kappa_A(\Delta) + \kappa_N(\Delta) = \chi(\Gamma) \).

Suppose \( v \) is a valence one vertex of \( \Delta \). Then, regardless of whether or not \( G_v \)

is abelian, \( G_v \) contributes at least 1 to \( b_1(G) \). Let \( \kappa^+_N|A \) be the total contribution

of nonabelian|abelian valence one vertex groups to \( \kappa^+_N|A \), and let \( \kappa^-_N|A \) be the total

contribution of nonabelian|abelian vertex groups with valence at least three. All

other vertex groups have valence two and make no contribution to either \( \kappa_N \) or \( \kappa_A \).

We now show that \( \kappa_N(\Delta) \) takes only boundedly many values for abelian decom-

positions of limit groups with a given first betti number.

Lemma 6.5 (Bounding \( \kappa_N(\Delta) \)). Let \( G \) be a limit group. Then
\[
\frac{1}{2} b_1(G) \geq \kappa_N(\Delta) \geq 1 - \frac{3}{2} b_1(G)
\]
for all abelian decompositions \( \Delta \) of \( G \).

Proof. Since each valence one vertex group of \( \Delta \) contributes at least 1 to \( b_1(G) \),
we have that \( \kappa^+_N|A \leq \frac{1}{2} b_1(G) \) since \( \kappa^+_N|A \leq \kappa^+_N|A \).

Since \( \kappa_N + \kappa_A = 1 - b_1(\Delta) \geq 1 - b_1(G) \) we have that
\[
\kappa_N + \frac{1}{2} b_1(G) \geq 1 - b_1(G)
\]
Thus \( \kappa_N \geq 1 - \frac{3}{2} b_1(G) \). The same also holds for \( \kappa_A \). \( \Box \)

Definition 6.6 (Very weakly JSJ respecting). Suppose \( G \rightarrow H \) is degenerate and
\( sc_1(G) = sc_1(H) \). Suppose that for all rigid vertex groups \( R \) of \( G \), \( R \) has non-
abelian image in \( H \) unless it has valence two, the relative JSJ decompositions
\( \Delta_{ASR} \) are trees, edge groups incident to \( R \) have images in valence one abelian
vertex groups of \( \Delta_{ASR} \), and no two incident edge groups have conjugate central-
izer in \( ASR \). Moreover, in the process of taking the direct limit \( \varprojlim_i G_i \rightarrow H \), at no
point do do edges incident to a rigid vertex group of \( G_i \) have conjugate centralizers,
after application of Lemma 5.17, in the associated rigid vertex group of \( G_i+1 \). Such
a homomorphism is very weakly JSJ respecting.

Lemma 6.7. Suppose \( \varphi: G \rightarrow H \) is degenerate and \( sc_1(G) = sc_1(H) \). Then
\( \kappa_N(G) \leq \kappa_N(H) \), with equality only if \( \varphi \) is very weakly JSJ respecting.

Proof. Let \( R \) be a rigid vertex. By the assumption on \( sc_1 \), if \( R \) has valence one
then \( R \) doesn’t have abelian image. Thus we only need to consider rigid vertices
with valence at least three: those with valence two are free to have abelian image
without disturbing \( \kappa_N \). Let \( R \) have valence at least three. If \( R \) has abelian image in \( H \)
then \( \kappa_N \) must increase by at least \( \frac{1}{2} \). We only need analyze rigid vertex groups
with nonabelian image. Let \( \Delta_R \) be the relative JSJ decomposition of \( SR \).

By our standing assumption that JSJ decompositions be bipartite, with one class
the abelian vertex groups, the other the nonabelian vertex groups, and if two edge
groups are incident to a rigid vertex group \( R \), then they have nonconjugate centralizer, we may assume that \( \Delta_R \) has a form such that all images of edge groups incident to \( R \) map to abelian vertex groups of \( \Delta_R \) (If an incident edge group has image in a rigid vertex group and its centralizer isn’t conjugate to the centralizer of an incident edge group in \( \Delta_R \), simply introduce a new edge and pull the centralizer). Moreover, in \( \Phi_{as}(G) \), since \( c_a(G) = c_a(H) \), all abelian vertex groups of \( \Phi_{as}(G) \) contributed by \( SR \) are equal to their peripheral subgroups. In particular, there are no valence one abelian vertex groups which don’t contain the image of some edge group incident to \( R \).

Let \( v_1, \ldots, v_n \) be the vertices of \( \Delta_R \) corresponding to abelian vertex groups, and let \( T \) be the subtree of \( \Delta_R \) spanned by the collection \( \{v_i\} \), and let \( T_j \) be the collection of closures of complimentary components of \( T \) in \( \Delta_F \). Note that if \( T_j \cap T_j' \neq \emptyset \) then the intersection is a single point and coincides with the intersections \( T \cap T_j \) and \( T \cap T_j' \).

We now compute the contribution of \( T \) to \( \kappa_N \). First, we claim that if two images of edge groups incident to \( R \) have conjugate centralizing elements in \( ASR \), or if some incident edge group maps to a non valence one vertex of \( T \) then

\[
\sum_{v \in T, ASRv \text{ nonabelian}} \kappa(v) > \kappa_N(R)
\]

Equality holds if we pretend that every vertex group from \( T \) is nonabelian and that the centralizers of images of incident edge groups are nonconjugate. If any vertex of valence greater than two is abelian, the inequality above must hold. If all vertices of valence at least three are nonabelian, then no two images of incident edge groups can have conjugate centralizers, since these correspond to new abelian vertex groups (of \( \Phi_{as} \Delta \)) with valence at least three. If an incident edge group has image in a non-valence one abelian vertex group of \( T \) then the corresponding vertex of \( \Phi_{as} \) has valence at least three, and the corresponding contribution to the computation of \( \kappa_N \) is strictly positive.

Now we compute the contribution of the trees \( T_j \) to \( \kappa_N \). As above, if we pretend that each tree consists of only nonabelian vertex groups, then the contribution each tree makes to \( \kappa_N \) over that of \( R \) is at least \( \frac{1}{2} \). If any vertex groups with valence at least three are abelian, then the contribution of \( T_j \) to \( \kappa_N \) can only increase since no valence one vertex groups not centralizing images of edge groups incident to \( R \) are abelian, by the hypothesis that \( s_{c1}(G) = s_{c1}(H) \).

To verify that \( \kappa_N(G) \leq \kappa_N(H) \) we only need to check that iterated application of \( \Phi_{as} \circ \Phi_r \) to \( \Phi_{as}(G) \) cannot decrease \( \kappa_N \). This follows immediately from the fact that no rigid vertex group of \( \Phi_{as}(G) \) splits after an application of \( \Phi_r \) (lemma 5.17) and the observation that folding edges of nonabelian vertex groups with conjugate centralizers can only increase \( \kappa_N \). If equality of \( \kappa_N \) holds then no such folding can ever occur.

The last step is to observe that when crushing trees of \( \text{QH} \) and subsurface groups of the abelian decomposition \( H \) inherits from \( \Phi_{as}(G) \) to build the \( \text{QH} \) subgroups of \( H \), as in the proof of Lemma 6.2, \( \kappa_N \) doesn’t change. \( \Box \)
The final step in this coarse analysis of the degenerations of JSJ decompositions is the following observation. The quantity

$$c_q(G) := \sum_{Q \in \text{JSJ}(G)} |\chi(Q)|$$

is the total complexity of QH subgroups.

**Lemma 6.8.** If $sc_1(G) = sc_1(H)$ and $\kappa_N(G) = \kappa_N(H)$ then

$$c_q(G) \leq c_q(H)$$

If equality holds then no vertex groups of $SR$ correspond to sub-surface groups of $\text{QH}$ vertex groups of $H$.

**Proof.** Since $\kappa_N(G) = \kappa_N(H)$, no nonabelian vertex group with valence one or at least three has abelian image. If $SR$ contributes a subsurface group to a QH subgroup of $H$ then the total Euler characteristic must decrease since the subsurface group must be either a multiply punctured sphere or projective plane with Euler characteristic at most $-1$. $\square$

**Definition 6.9 (Essential, Vulnerable).** We say that a vertex of the abelian JSJ decomposition of a limit group is **essential** if it satisfies at least one of the following:

- It is QH.
- It isn’t valence two.
- It is abelian and isn’t equal to its peripheral subgroup.

The number of essential vertices of $G$ is denoted by

$$v_e(G)$$

Let $\beta(G)$ be the set of essential vertices in $\text{JSJ}(G)$, and let $\alpha(G)$ be the set of unoriented reduced edge paths $\alpha: [0, 1] \to \text{JSJ}(G)$ such that $\alpha^{-1}(\beta(G)) = \{0, 1\}$. The elements of $\alpha(G)$ fall into six classes, corresponding to the types of endpoints: for example, $AA(G)$ is the set of elements of $\alpha(G)$ whose endpoints are both essential abelian vertices. A path in $AA(G)$, $AQ(G)$, or $QQ(G)$, is **vulnerable** if it crosses a rigid vertex group. Let $c_v(G)$ be the number of vulnerable paths in $AQ(G) \sqcup QQ(G)$.

Let $c_{\#r}(G)$ be the number of essential rigid vertex groups of $G$.

We can now define the Scott complexity.

**Definition 6.10 (Scott complexity).** The **Scott complexity** of a freely indecomposable limit group is the following lexicographically ordered quantity:

$$sc(G) := (c_a(G), c_g(G), c_b(G), \kappa_N(G), c_q(G), c_{\#r}(G), -v_e(G), -c_v(G))$$

Note that $sc$ takes only boundedly many values for a given first betti number. That the last four take only boundedly many values follows from the part of the proof of Lemma 2.7 bounding the complexity of the JSJ decomposition in terms of $b_1$ ($c_q$ and $c_{\#r}$) and Lemma 6.5 ($\kappa_N$).

Fix a degenerate $\varphi: G \to H$ such that $sc_1$ and $\kappa_N$ are constant. By Lemma 6.7, $\varphi$ is very weakly JSJ respecting. If, additionally, $c_q(G) = c_q(H)$, then for all rigid
vertex groups $R$ of $G$, $\text{ASSR} = SS \ast_{F_i} B_{i_1}$, no vertex group of JSJ($SS, E(SS)$) is a subsurface group of a QH subgroup of $H$, and we conclude by Lemma 5.17 that all rigid vertex groups of $H$ are obtained by iteratively adding roots to rigid vertex groups from the collection $\{S_1, R_i | R_i$ rigid in $G\}$. If the relative JSJ of some $S_1, R_i$ has more than one non-valence two vertex then $c_{\#r}(G) < c_{\#r}(H)$. Thus, if $\text{sc}(G) = \text{sc}(H)$ then, for all rigid vertex groups $R_i$, the underlying graph of $E S_1, R_i$ is a tree, has only one vertex $v$ of valence at least three, the valence of $v$ is precisely the valence of $R$, and every valence one vertex contains the image of exactly one incident edge group.

Let

$$\text{sc}_2(G) = (c_a(G), c_q(G), c_b(G), c_\kappa(G), c_q(G), c_{\#r}(G))$$

and suppose that $\text{sc}_2(G) = \text{sc}_2(H)$ and $-\nu(e)(G) < -\nu(e)(H)$. There must be a path $p$ in $AA(G)$ or $QQ(G)$ such that every rigid vertex group crossed has abelian image in $H$. (No path in $AQ(G)$ contains an essential vertex, so we need not consider this case.) Abelianizing these groups leaves $\kappa_N$ and $c_q$ unchanged. If $p \in AA(G)$ then the number of essential abelian vertices of $H$ is less than that of $G$. Now we consider the quantity $c_v$, under the assumption that $-\nu(e)$ is constant. The number of vulnerable paths cannot increase, as the topology and labeling of essential vertices of the underlying graphs of JSJ’s is unchanged under $G \rightarrow H$.

**Lemma 6.11.** If $\text{sc}(G) = \text{sc}(H)$ then at least one rigid vertex group of $G$ crossed by a vulnerable path in $AQ(G) \sqcup QQ(G) \sqcup AA(G)$ has nonabelian image in $H$.

The lemma follows immediately from the definitions.

**Definition 6.12 (Weakly JSJ respecting).** If $\varphi: G \rightarrow H$ is degenerate and $\text{sc}(G) = \text{sc}(H)$, then $\varphi$ is **weakly JSJ respecting**.

In conclusion, we have the following. Let $C_n$ be the number of values $\text{sc}$ takes for limit groups with first betti number $n$.

**Lemma 6.13.** Let $\mathcal{L}$ be a degenerate chain of limit groups. If $\|\mathcal{L}\| > K \cdot C_n$ and $b_1(\mathcal{L}) = n$, then there is a weakly JSJ respecting subchain of $\mathcal{L}$ of length $K$.

We can now define the GAD’s which are respected under maps of constant Scott complexity.

**Definition 6.14 (\(\Delta\)-admissible; \(\Delta\)-stable; almost-JSJ respecting).** A virtual JSJ decomposition of a limit group $L$ is an abelian decomposition $\Delta$ such that the Scott complexity of $L$ measured with respect to $\Delta$ is the same as the Scott complexity of $L$ measured with respect to JSJ($L$).

Let $\varphi: G \rightarrow H$ be degenerate. If $\Delta$ is a virtual JSJ decomposition of $G$ then $\varphi$ is **\(\Delta\)-admissible** if $\text{sc}(G) = \text{sc}(H)$ and every nonabelian rigid vertex group of $\Delta$ has nonabelian image in $H$. If $\varphi$ is $\Delta$-admissible then there is a well defined push forward of $\Delta$ to $H$ which is also a virtual JSJ decomposition.

Let $\mathcal{L}$ be a weakly JSJ respecting chain, and let $\Delta$ be a virtual JSJ decomposition of $\mathcal{L}(1)$. We say that $\mathcal{L}$ is **\(\Delta\)-admissible** if and all $\varphi_{1, j}$ are $\Delta$-admissible.

Suppose $\varphi: G \rightarrow H$ is $\Delta$-admissible. Then $\varphi$ is **\(\Delta\)-stable** if, for every rigid vertex group $R$ of $\Delta$, $c_a(\varphi|R, E(R)) = 0$ (See Definition 5.7).
Let $\mathcal{L}$ be a weakly JSJ respecting chain such that $\varphi_{i,j}$ is JSJ($\mathcal{L}(i)$)$\,$–stable for all $i$ and $j$. If $\mathcal{L}(i)$ and $\mathcal{L}(j)$ have the same number of rigid vertex groups for all $i$ and $j$, and the betti numbers of rigid vertex groups of JSJ decompositions are constant then $\mathcal{L}$ is almost-JSJ respecting.

The following lemma follows easily from the definitions and conventions on JSJ decompositions.

**Lemma 6.15.** A virtual JSJ decomposition is obtained by collapsing a forest $F$ in JSJ($G$) of the following form, and all such forest collapses yield virtual JSJ decompositions:

- All valence one vertices of $F$ are rigid, all valence two vertices are rigid or abelian.
- Each component of $F$ contains at most one essential vertex. If $F$ contains an essential vertex then that vertex is rigid. In fact, every component of $F$ is star-shaped, and all non-valence-two vertices are correspond to rigid vertex groups of the JSJ of $G$.

If $G \rightarrow H$ is degenerate, indecomposable, and $sc(G) = sc(H)$ then the JSJ decompositions of $G$ and $H$ resemble one another quite strongly. The next step is to show that under weakly JSJ respecting chains, one can choose, uniformly in $b_1$, subchains such that no nonabelian rigid vertex groups have abelian image and such that rigid vertex groups are obtained from images of previously occurring rigid vertex groups by iteratively adjoining roots and passing to limit quotients.

**Lemma 6.16.** Fix $n$. For all $K$ there exists $M = M(K,n)$ such that if $\mathcal{L}$ is $\Delta$–admissible degenerate chain with $b_1(\mathcal{L}) = n$, $\|\mathcal{L}\|_{pl} \geq M$, then there is a subchain $\mathcal{L}'$ of $\mathcal{L}$ such that

- $\|\mathcal{L}'\|_{pl} \geq K$
- If $\Delta$ is the push-forward of $\Delta$ to $\mathcal{L}'(1)$ then $\mathcal{L}'$ is $\Delta$–stable.
- Let $k$ be an integer and let $\mathcal{L}'_H$ (head?) and $\mathcal{L}'_T$ (tail?) be the two subchains of $\mathcal{L}$ obtained by restricting to the first $k$ and last $\|\mathcal{L}'\| - k$ indices, respectively, ($k$ may be 0 or $\|\mathcal{L}'\|$) such that $k$ is the last index for which the push forward of $\Delta$ to $\mathcal{L}(k)$ is $\mathcal{L}(k)$'s abelian JSJ decomposition. Then there is a virtual JSJ decomposition $\Delta'$ of $\mathcal{L}(k + 1)$ for which $\Delta'_T$ is $\Delta'$–admissible and the number of vertex groups of $\Delta'$ is strictly greater than the number of vertex groups of $\Delta$.

**Proof.** Let $\Delta$ be the virtual JSJ decomposition of $\mathcal{L}(1)$. We construct virtual abelian decompositions $\Delta_i$ for $\mathcal{L}(i)$ inductively. Rather than constructing $\mathcal{L}(i + 1)$ from $\mathcal{L}(i)$, first by applying strict vertex morphisms to rigid vertex groups from the JSJ of $\mathcal{L}(i)$, we apply them only to the vertex groups of the virtual decompositions $\Delta_i$. Let $R$ be a rigid vertex group of $\Delta_i$, and construct $AS(R, E(R)) \cong S_{II}(R) *_{F_j} B_j$ as the construction of $\Phi_{as}((\mathcal{L}(i))$ with respect to $\Delta$ demands. If $c_a(\varphi_{i,i+1}, R, E) > 0$, i.e., if $\varphi_{i,i+1}$ is not $\Delta$–stable, $b_1(S_{II}(R)) < b_1(B_j)$.

Now consider the relative JSJ decomposition of $S_{II}R$. We claim that since $\Delta$ is a virtual JSJ decomposition of $\mathcal{L}(i)$, the relative JSJ of $S_{II}R$ has at most one
essential vertex and that vertex has the same valence as \( R \). If \( S_{ij} R \) contains more than one essential vertex then, since \( \text{sc}(\mathcal{L}(i)) = \text{sc}(\mathcal{L}(i+1)) \), all but one of them is a valence one abelian vertex.

Consider the direct limit \( \varprojlim \mathcal{L}(i)_n = \mathcal{L}(i+1) \). Repeated application of Lemma 5.17 shows that the rigid vertex groups of \( \mathcal{L}(i+1) \) are obtained by iteratively adjoining roots to the vertex groups of \( S_{ij} R \), as \( R \) varies over all vertex groups of \( \Delta_i \). Thus every homomorphism induced by \( R \) showing that the rigid vertex groups of \( \mathcal{L}(i+1) \) can give rise to an essential vertex of \( \mathcal{L}(i+1) \). For each rigid vertex group of \( \Delta_i \), declare the subtree of groups of the JSJ decomposition of \( \text{JSJ}(\mathcal{L}(i+1)) \) induced by \( S_{ii} R \) a vertex group of an abelian decomposition \( \Delta_{i+1} \) of \( \mathcal{L}(i+1) \). Since no new essential vertices are created, the decomposition \( \Delta_{i+1} \) is a virtual JSJ decomposition. Call the vertex of \( \Delta_{i+1} \) associated to \( R \), the push forward of \( R \), \( \varphi_{i,i+1}(R) \). Suppose \( c_\delta(R, \mathcal{E}(R)) > 0 \), then \( b_1(SR) < b_1(R) \). Since \( \varphi_{i,i+1}(R) \) is obtained by iteratively adjoining roots to \( SR \), the betti number cannot increase and \( b_1(\varphi_{i,i+1}(R)) < b_1(R) \).

Fix \( n \), let \( b_n \) be the largest number such that a vertex group \( R \) in a virtual JSJ decomposition of a limit group \( L \) with \( b_1(L) = n \) can have \( b_1(R) = b_n \), and let \( r_n \) be the maximum number of nonabelian rigid vertex groups in a virtual JSJ decomposition of a limit group \( L \) with \( b_1(L) = n \). Let \( M = K \cdot b_n^r \). If \( \mathcal{L} \) has length \( M \) then it has a subchain \( \mathcal{L}' \), of length \( K \), such that for every rigid vertex group \( R_{i,j} \) of every decomposition \( \Delta_i \), \( c_\delta(\varphi_{i,i+1}|_{R_{i,j}}) = 0 \) (incident edge groups are implied). Thus every homomorphism \( \mathcal{L}'(i) \rightarrow \mathcal{L}'(j) \) is \( \Delta \)-stable, for \( \Delta \) inherited from \( \mathcal{L}(1) \).

Now examine \( \mathcal{L}'(\mathcal{T} \cdot (1)) \), and suppose that the relative JSJ of \( R \) is nontrivial. Then \( R \) must have a splitting \( R = R' \ast E \ast E'' \ast R'' \ast R' \) and \( R'' \) nonabelian, \( E = (E, E') \) and maximal abelian (recall that \( A \) is the centralizer of its incident edge groups by our choice of normalization of the JSJ), and with all edge groups incident to \( R \) elliptic and not centralized by \( A \). Then neither \( R' \) nor \( R'' \) has abelian image in any \( \mathcal{L}_{\mathcal{T}}(k) \): if this is the case then, since limit groups have nontrivial homology relative to any abelian subgroup, the map \( R \rightarrow \mathcal{L}_{\mathcal{T}}(k) \) would factor through (say) \( R' \ast E \ast E' \ast (E'')^{ab} \), which has nonzero \( c_\delta \) since the edges incident to \( R \) are elliptic. Since \( \Delta_{i+1} \) is the outcome of a forest collapse, removal of the edges labeled \( E \) and \( E'' \) from the forest gives a new forest which yields a new virtual JSJ decomposition \( \Delta' \) of \( \mathcal{L}_{\mathcal{T}}(1) \). The new decomposition has more vertices than \( \Delta \), and, since neither \( R' \rightarrow \mathcal{L}_{\mathcal{T}}(k) \) nor \( R'' \rightarrow \mathcal{L}_{\mathcal{T}}(k) \) has abelian image for any \( k \), the chain \( \mathcal{L}_{\mathcal{T}} \) is \( \Delta' \)-admissible.

We use Lemma 6.16 to control those degenerations of JSJ decompositions which are invisible to the Scott complexity. Repeated application of the lemma uniformly many times in the first betti number gives us the following theorem.

**Theorem 6.17.** Let \( \mathcal{L} \) be a degenerate chain of limit groups with \( \text{sc}(\mathcal{L}(i)) = \text{sc}(\mathcal{L}(i+1)) \) for all \( i \). Then for all \( K \) there exists \( M = M(K, b_1(\mathcal{L})) \) such that if \( ||\mathcal{L}|| > M \) then there is an almost-JSJ respecting subchain \( \mathcal{L}' \) of \( \mathcal{L} \) with length at least \( K \).
Proof: We only need to show that such chains are \(\Delta\)-admissible for some \(\Delta\) and that Lemma 6.16 only needs to be used boundedly many times, depending only on \(n\).

By Lemma 6.11 no rigid vertex groups crossed by vulnerable paths in \(AQ(\mathcal{L}(1))\) or \(QQ(\mathcal{L}(1))\) might have abelian image in some \(\mathcal{L}(k)\). We construct a forest in \(\text{JSJ}(\mathcal{L}(1))\) as follows. Let \(p\) be a vulnerable path, and let \(p'\) be the longest proper sub-path of \(p\) with endpoints which are rigid vertex groups of \(G\). The sub-graph of groups spanned by the image of \(p'\) doesn’t have abelian image in any \(\mathcal{L}(k)\), by Lemma 6.11.

For \(p \in RR(\mathcal{L}(1)) \sqcup RA(\mathcal{L}(1)) \sqcup RQ(\mathcal{L}(1))\) choose an orientation of \(p\) let \(p'\) be the subpath which begins at \(p(1)\) and ends at the last inessential rigid vertex group crossed by \(p\).

The forest which is the union of all images of \(p'\)'s constructed gives a virtual JSJ decomposition \(\Delta\) of \(\mathcal{L}(1)\) by Lemma 6.15, and the chain \(\mathcal{L}\) is \(\Delta\)-admissible by the previous paragraphs.

Let \(M(K,n) = K \cdot b_n^a\) be the constant from Lemma 6.16. If \(\|\mathcal{L}\| > K \cdot b_n^a\) then \(\mathcal{L}\) is long enough to apply the lemma \(r_n\) times, yielding a subchain of length \(K\) which is both JSJ–admissible and JSJ–stable. If \(\mathcal{L}'\) is JSJ–admissible and JSJ–stable and for some rigid vertex group \(\mathcal{R}\) of \(\mathcal{L}'(k)\), the decomposition JSJ(\(S_{II}(\mathcal{R},E)\)) isn’t trivial, then the JSJ decomposition of \(\mathcal{L}'(k+1)\) has strictly more rigid vertex groups than \(\mathcal{L}'(k)\). Since \(c_a(\varphi_{n,n+1}) = 0\) for all \(n\), in the group \(AS(\mathcal{R}) = S_{II}(\mathcal{R}) \ast A_j B_j, B_j = A_j\) and \(S_{II}(\mathcal{R}) = AS(\mathcal{R})\), hence it’s safe to call \(AS(\mathcal{R})\) \(S(\mathcal{R})\). If \(\mathcal{L}'\) has length \(K \cdot r_n\) then it contains a subchain of length \(K\) such that all relative decompositions of \(S(\mathcal{R})\) are trivial, as the theorem asks. \(\square\)

In the remainder of the section we define some technical refinements of the notion of almost-JSJ respecting which are used in Section 9.

**Definition 6.18.** A GAD \(\Delta\) of a limit group \(L\) misses \(A\) if \(A\) is an abelian vertex group of \(\text{JSJ}(L)\) and no splitting of \(L\) over a subgroup of \(A\) is visible in \(\Delta\).

A GAD which misses \(A\) is obtained from the abelian JSJ by collapsing the star of the vertex which carries \(A\) and possibly collapsing further edges. If \(\mathcal{A}\) is a collection of abelian vertex groups of \(L\) then the GAD of \(L\) obtained by collapsing all stars of elements of \(L\) is denoted by \(\text{JSJ}_{\mathcal{A}}(L)\). It follows from the construction of \(\Phi_s\) that if \(\varphi: G \to H\) is degenerate and almost-JSJ respecting then the vertex groups of \(\text{JSJ}_{\Phi_s(\mathcal{A})}(H)\) are obtained from the images of the vertex groups of \(\text{JSJ}_{\mathcal{A}}(G)\) by iteratively adjoining roots.

We leave the following lemmas as an exercise.

**Lemma 6.19.** Suppose \(\varphi: G \to H\) is almost-JSJ respecting, and that \(R\) is a nonabelian non-QH vertex group of \(\text{JSJ}_{\mathcal{A}}(G)\). Then \(b_1(\rho) \geq b_1(\varphi_{\#}(\rho))\). If \(c_a(\varphi, E(\rho)) > 0\) then the inequality is strict.

Suppose \(\varphi: G \to H\) is almost-JSJ respecting. If equality of betti numbers holds in the above lemma for all vertex groups \(\rho\) of \(\text{JSJ}_{\mathcal{A}}(G)\), as \(\mathcal{A}\) varies over all collections of abelian vertex groups of \(G\), then we call \(\varphi\) \(b_1\)-respecting.
Theorem 6.20. For all $K$ there exists $M = M(K, b_1(L))$, such that if $L$ is a JSJ stable degenerate chain and $|L| \geq M$, then there is a subchain $L'$ of $L$ of length at least $K$ such that all maps from $L'$ are $b_1$-respecting.

We call chain satisfying Theorem 6.20 JSJ respecting.

7. QCJSJ respecting

We start by proving a lemma about ranks of abelian subgroups of limit groups. As a consequence, we can assume that degenerate JSJ respecting chains have subsequences whose subsequences of abelian subgroups are well-behaved.

Lemma 7.1. Let $G$ be a limit group with $b_1(G) = n$. Then all abelian subgroups of $G$ have rank at most $n$. If $G \twoheadrightarrow H$, $A < G$ abelian, if $\mathbb{Z}^2 < \ker(A \to H)$ then $b_1(G) > b_1(H)$.

Proof. Let $G = L_0 \twoheadrightarrow L_1 \twoheadrightarrow \cdots$ be a strict resolution of $G$. Let $A < G$ be an abelian subgroup. If $A$ is cyclic then the lemma holds. Suppose $A$ has rank greater than two. Since $A$ isn’t infinite cyclic it acts elliptically in the abelian JSJ decomposition of $G$, and is thus contained in an abelian vertex group or a rigid vertex group. If $A$ is contained in rigid vertex groups all the way down the resolution $L_0 \twoheadrightarrow \cdots$ then, since the last group in the strict resolution is free, $A$ must be cyclic. Let $i_0$ be the first index such that $A$ is contained in an abelian vertex group but not completely contained in the peripheralsubgroup of an abelian subgroup $B$ of $L_{i_0}$. If $L_{i_0}$ is freely decomposable then we are done by induction since $A$ is contained in a free factor of $L_{i_0}$. By linear algebra

$$\text{rk}(A \cap P(B)) + \text{rk}(B/P(B)) \geq \text{rk}(A)$$

Let $C$ be a complementary direct summand such that $A \cong (A \cap P(B)) \oplus C$. The summand $C$ has rank at most $\text{rk}(B/P(B))$ and every element of $B/P(B)$ represents a nontrivial element of $H^1(L_{i_0}; L_{i_0,P})$. Continue this process on $L_{i_0,P}$ with respect to $A \cap P(B)$, peeling off direct summands until $A \cap P(B)$ has rank 1.

By induction applied to $A \cap P(B) < L_{i_0,P}$, which has lower betti number than $L_{i_0}$, $\text{rk}(A \cap P(B)) \leq n - \text{rk}(B/P(B))$. Combining this with the above inequality, $\text{rk}(A) \leq n$. At the last step in the induction, when $A \cap P(B)$ has rank 1 or 0, the group chosen in the resolution has nontrivial homology relative to $A$.

The above argument establishes that $\ker(A \to H_1(G))$ has rank at most one.

Suppose that $L$ is JSJ respecting. Let $\{A^n_s\}$ be the set of abelian vertex groups, the set of conjugacy classes of centralizers of edge groups of $L(n)$, and let $A_s$ be the sequence

$$A^n_s \to A^{n+1}_s$$

The homomorphism $\varphi_{n,m}$ may not be injective on $A^n_s$, but since $\varphi_{n,m}$ doesn’t factor through a free product, it’s image is not trivial. Define in a similar manner $E^n_l$ and $E_l$ for the edge groups $E^n_l$ of $L(n)$ and sequences of edge groups of $L$. If
\[ \text{rk}(A^m_n) < \text{rk}(A^m_n) - 2 \text{ or } \text{rk}(E^m_n) < \text{rk}(E^m_n) \text{ then } b_1(L_i(m)) < b_1(L_i(n)). \] Thus if \( \text{rk}(E^m_n) > 2 \) and the sequence \( b_1(L_i) \) is constant, then \( A^m_s \) has rank at least two, as does \( A^m_s \) for all \( m' > m \).

**Definition 7.2.** A degenerate JSJ respecting chain \( \mathcal{L} \) is QCJSJ respecting if, for each sequence \( S \in \{A_s, E_t\} \), exactly one of the following holds:

1. \( \text{rk}(S^m_n) = \text{rk}(S^m_n) > 2 \) and \( S^m_n \xrightarrow{} S^m_n \) for all \( m \) and \( n \)
2. \( \text{rk}(S^m_n) = \text{rk}(S^m_n) > 2 \) and \( S^m_n \rightarrow S^m_n \) has infinite cyclic kernel for all \( n \) and \( m \).
3. \( \text{rk}(S^m_n) = \text{rk}(S^m_n) = 2 \) and \( S^m_n \xrightarrow{} S^m_n \) for all \( m \) and \( n \)
4. \( \text{rk}(S^m_n) = 2 \) and \( S^m_n \rightarrow S^{m+1} \) has infinite cyclic image
5. \( S^m_n \cong \mathbb{Z} \) for all \( n \).

A sequence of edge groups satisfying one of the first three bullets is **big**, otherwise it is **small**. A sequence of edge groups satisfying either of the second or fourth bullets is **flexible**, otherwise it is **rigid**.

The main application of Lemma 7.1 is that QCJSJ respecting sequences can be derived from JSJ respecting sequences.

**Lemma 7.3.** For all \( K \) there exists \( M = M(K, b) \) such that if \( \mathcal{L} \) is a degenerate JSJ respecting sequence and \( \|\mathcal{L}\| > M(K, b_1(\mathcal{L})) \), then \( \mathcal{L} \) has a QCJSJ respecting subsequence with length at least \( K \).

**Proof.** Follows immediately from the discussion prior to the lemma, the bound on the number of edge groups depending only on \( b_1 \), and Lemma 7.1. \( \square \)

**Definition 7.4.** Degenerate QCJSJ respecting chains are cooked up so that certain important sequences of subgroups are respected under all maps. Let \( \mathcal{L} \) be QCJSJ respecting. For each JSJ decomposition \( \text{JSJ}(\mathcal{L}(i)) \), form a new decomposition \( \text{JSJ}_B(\mathcal{L}(i)) \) by folding together all big incident edges incident to abelian vertex groups, as in Figure 3. For each abelian vertex group \( A \) of \( \mathcal{L}(k) \), let \( P_B(A) \) be the subgroup of \( A \) generated by big incident edges.

![Figure 3. Folding big edges together.](image)

Let \( \mathcal{L} \) be a QCJSJ respecting sequence, and let \( b^k_1, \ldots, b^k_{m} \) be the edges of \( \text{JSJ}_B(\mathcal{L}(k)) \) which carry big edge groups (the edges labeled \( P_B(A) \), for some abelian vertex group \( A \) from the JSJ), and let \( \Gamma^k_1, \ldots, \Gamma^k_m \) be the connected components
of the union of the $b_i^k$. The subgroups carried by the $\Gamma_j^k$, are the rigid vertex groups of the QCJSJ decomposition, and the graph of groups obtained by collapsing the subgraphs $\Gamma_j^k$ of JSJ$_B(\mathcal{L}(k))$ is the quasicyclic JSJ decomposition, or QCJSJ, for short.

For each abelian vertex group $A^n$ of $\mathcal{L}(n)$ we have isolated $P_B(A^n)$, the subgroup generated by big incident edges. There is another special subgroup, $P_S(A^n)$, the subgroup generated by small incident edges. Since big|small edges of $\mathcal{L}(n)$ map to big|small edges of $\mathcal{L}(n+1)$, there is a map

$$P_B|S(A^n) \to P_B|S(A^{n+1})$$

Let $\{R_i^k\}$ be the collection of vertex groups in QCJSJ($\mathcal{L}(k)$). Since $\varphi_{k,l}$ sets up a one to one correspondence between vertex groups and big|small edges of JSJ($\mathcal{L}(k)$) and JSJ($\mathcal{L}(l)$), and the induced maps $\mathcal{L}(k) \to \mathcal{L}(l)$ respect vertex groups and big|small edges and the incidence conditions of the JSJ decompositions, there is a one-to-one correspondence between vertex groups of QCJSJ($\mathcal{L}(k)$) and QCJSJ($\mathcal{L}(l)$), i.e., there is a unique vertex group $R_i^k$ in QCJSJ($\mathcal{L}(l)$) such that $\varphi_{k,l}(R_i^k) < R_i^l$.

Recall our normalization that every edge group be centralized by an abelian vertex group and that every edge group is primitive. Since $\mathcal{L}$ is QCJSJ respecting, the kernels of the induced maps on $P_B(A^n)$ and $P_S(A^n)$ are at most infinite cyclic, and if both of them have kernel, the kernel is contained in the intersection $P_B(A^n) \cap P_S(A^n)$.

Define the following sequences of subgroups of $\mathcal{L}$:

$$\mathcal{R}_i^n(m) := \varphi_{n,m}(R_i^n)$$

There are obvious inclusions $\mathcal{R}_i^n \hookrightarrow \mathcal{L}$.

**Lemma 7.5.**

- $\mathcal{R}_i(n+1) = \mathcal{R}_i^{n+1}(n+1)$ is obtained by iteratively adjoining roots to $\mathcal{R}_i^n(n+1)$.
- $b_1(R_i^n) \leq b_1(\mathcal{L}(n))$
- If $b_1(\mathcal{R}_i) = b_1(\mathcal{L})$ for some $i$ then there is only one nonabelian vertex in each QCJSJ decomposition.

This essentially follows from the definitions.

**Proof:** The first bullet follows from the fact that the vertex groups of QCJSJ($\mathcal{L}(n)$) are generated by vertex groups from JSJ($\mathcal{L}(n)$) and stable letters from JSJ($\mathcal{L}(n)$), and that the vertex groups of JSJ($\mathcal{L}(n+1)$) are obtained from the images of vertex groups of JSJ($\mathcal{L}(n)$) by iteratively adjoining roots, by Lemma 5.17.

Let $A$ be an abelian vertex group of $\mathcal{L}(n)$ such that $\text{Ker}(A \to \mathcal{L}(n+1)) \cong \mathbb{Z}$. Then at least one edge incident to $A$ is flexible. The only case which needs consideration is when none of the big edges incident to $A$ is flexible. In this case the rank of $P_B(A)$ is strictly less than the rank of $A$, and the image of any small flexible edge adjacent to $A$ has at most rank one intersection with $P_B(A)$.
The inequality of betti numbers follows from the fact that if \( E \) is a small flexible edge, then either \( E \) is incident to an abelian vertex group \( A \) with no big flexible edges or with at least one big flexible edge. In the former case, the vertex group has betti number at least one less than that of the subgroup obtained by attaching \( A \) to the rigid vertex group along \( P_B(A) \), which has betti number at most one greater than the ambient group. In the latter case, both inclusions of \( E \) into vertex groups of the one edged splitting induced by \( E \) have, homologically, one dimensional images.

The third bullet follows from the same argument. □

The following follow from Theorem 6.20, Lemma 7.3, and Lemma 2.7.

**Theorem 7.6** (Alignment Theorem). Let \( \mathcal{L} \) be an indecomposable sequence of limit groups, \( \text{rk}(\mathcal{L}) = N \). For all \( K \) there exists \( M \) such that if \( \|\mathcal{L}\| > M \) then there is a maximal QCJSJ respecting resolution \( \tilde{\mathcal{L}} \boxslash \mathcal{L} \) with \( \|\tilde{\mathcal{L}}\| \geq K \).

**Corollary 7.7** (Alignment Corollary). Let \( (\iota: \mathcal{G} \hookrightarrow \mathcal{L}) \in \text{Seq}(\mathcal{L}, b, d) \) be indecomposable. For all \( K \) there exists \( M = M(\text{Comp}(\iota)) \) such that if \( \|\mathcal{G}\| > M \) then there is a maximal QCJSJ respecting resolution \( \tilde{\mathcal{G}} \boxslash \mathcal{G} \) with \( \|\tilde{\mathcal{G}}\| \geq K \).

8. LIFTING DIMENSION BOUNDS

Suppose \( \mathcal{L} \) is a sequence of epimorphisms of \( N \)-generated limit groups. There is a trivial resolution \( \mathcal{L} \boxslash \text{id} \mathcal{L} \) which is simply the identity resolution. This resolution has complexity \( \text{Comp}(\mathcal{L} \boxslash \mathcal{L}) \leq (N, 6N) \).

As a consequence of Theorem 5.5 we have the following important corollary.

**Corollary 8.1** (Reduction to indecomposable sequences). Let \( \iota: \mathcal{G} \hookrightarrow \mathcal{L} \) be an inclusion of sequences. For all \( K \) there exists \( M = M(\text{Comp}(\iota)) \) such that if \( \|\mathcal{G}\| > K \) there exists a maximal resolution \( \tilde{\mathcal{G}} \boxslash \mathcal{G} \hookrightarrow \mathcal{L} \) of \( \mathcal{G} \) such that \( \|\tilde{\mathcal{G}}\| > K \) such that \( c_{fd} \) is constant along \( \tilde{\mathcal{G}} \).

In particular, \( \tilde{\mathcal{G}} \) splits as a graded free product of sequences

\[
\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_1 \ast \cdots \ast \tilde{\mathcal{G}}_p \ast \mathcal{F}
\]

where \( \mathcal{F} \) is the constant sequence \((\mathbb{F}_q)\) for some \( q \). The sequences \( \tilde{\mathcal{G}}_i \boxslash \mathcal{G} \) are indecomposable maximal resolutions of their images. If \( \text{sc}(\tilde{\mathcal{G}}) > (1, 0) \) then

\[
\text{Comp}(\tilde{\mathcal{G}}_i \boxslash \mathcal{L}) < \text{Comp}(\mathcal{G} \hookrightarrow \mathcal{L})
\]

**Proof of Corollary 8.1.** By Lemma 2.7, the rank of \( \mathcal{G} \) is bounded above by some function of \( \text{Comp}(\mathcal{G}) \). Now apply Theorem 5.5. □

**Remark 8.2.** Let \((b_0, d_0)\) be a minimal complexity for which Theorem 3.5 fails, should there be one. Let \( \tilde{\mathcal{G}} \boxslash \mathcal{L} \) be a maximal resolution provided by the corollary.

Suppose that \( \tilde{\mathcal{G}} \) consists of freely decomposable groups and decomposes as a nontrivial free product of resolutions

\[
(\tilde{\mathcal{G}}_1 \ast \cdots \ast \tilde{\mathcal{G}}_p \ast \mathcal{F}_q) \boxslash \mathcal{L}
\]
Each of the free factors gives rise to a maximal resolution
\[ \tilde{G}_i \sqsubset \mathcal{L}, \quad \mathcal{F}_q \sqsubset \mathcal{L} \]
of complexity less than \( \text{Comp}(\mathcal{G} \sqsubset \mathcal{L}) \), and with equality only if \( (p, q) = (1, 0) \).

Suppose we are in the case \( p = 1, q > 0 \) or \( p \geq 2 \). In this case there are at most \( N \) (this is an overestimation) free factors of \( \mathcal{L} \), each of which has complexity strictly less than that of \( \mathcal{L} \). Let \( B \) be the maximal proper length of a sequence with complexity less than that of \( \mathcal{G} \). Then if \( M > 3BN \) there exist three consecutive indices \( i, i+1, i+2 \) such that the maps
\[ \tilde{G}_j(i) \to \tilde{G}_j(i+1) \to \tilde{G}_j(i+2) \]
are isomorphisms for all \( j \). The same is true for the free part of \( \tilde{G} \). It follows immediately that \( \mathcal{G}(k_i) \to \mathcal{G}(k_{i+1}) \) must be an isomorphism.

Thus if Theorem 3.5 fails, by the corollary and the above, if \( \mathcal{G}_i \hookrightarrow \mathcal{L} \in \text{Seq}(\mathcal{L}, b, d) \) is a sequence such that \( \| \mathcal{G}_i \|_d > i \) then the maximal resolutions \( \tilde{G}_i \sqsubset \mathcal{G}_i \) of \( \mathcal{G}_i \hookrightarrow \mathcal{L} \) must have \( \text{sc}(\tilde{G}) = (1, 0) \).

Suppose now that we are in the case \( (p, q) = (1, 0) \). As before, if \( \mathcal{G} \) has length \( M = M(K, N) \), the sequence \( \tilde{G} \) has length at least \( K \) and consists of indecomposable maps. Since \( \tilde{G} \) is QCJSJ respecting, the sequence \( c_a(\tilde{G}) \) is constant. We now show how to lift a dimension bound for sequences simpler than \( \mathcal{G} \) in the event that \( c_a(\tilde{G}) > 0 \).

Since \( c_a(\tilde{G}) \) is constant along \( \tilde{G} \), the maps \( \tilde{G}(i) \to \tilde{G}(j) \) map \( \tilde{G}(i)_P \) onto \( \tilde{G}(j)_P \). If \( \tilde{G}(i) \to \tilde{G}(j) \) is JSJ respecting, every automorphism in \( \text{Mod}(\tilde{G}(i)_P, \tilde{G}(i)_P) \) pushes forward to an element of \( \text{Mod}(\tilde{G}(j)_P, \tilde{G}(i)_P) \), hence if the map \( \tilde{G}(i)_P \to \tilde{G}(j)_P \) is an isomorphism then \( \tilde{G}(i) \to \tilde{G}(j) \) is \( \text{Mod}(\tilde{G}(i)_P, \tilde{G}(i)_P) \)-strict. Since every automorphism pushes forward, this map is an isomorphism. Now if \( c_a(\tilde{G}) > 0 \) then \( b_1(\tilde{G}_P) < b_1(\tilde{G}) \) and
\[ \text{Comp}(\tilde{G}_P \sqsubset \mathcal{G}) < \text{Comp}(\tilde{G} \sqsubset \mathcal{G}) \]

The same sort of analysis can be carried out if \( \tilde{G} \) has a sequence of QH subgroups: Let \( L \) be a limit group and let \( \text{JSJ}_Q(L) \) be the graph of groups decomposition of \( L \) obtained by collapsing all edges not adjacent to QH subgroups. Then \( \tilde{G} \) respects the decompositions \( \text{JSJ}_Q(L) \), and maps the vertex groups of \( \text{JSJ}_Q(\tilde{G}(i)) \) onto the associated vertex groups of \( \text{JSJ}_Q(\tilde{G}(i+1)) \). Thus we derive boundedly many \( (b_1(\tilde{G})) \) sequences \( \mathcal{V}_i \hookrightarrow \tilde{G} \) of vertex groups. The resolutions \( \mathcal{V}_i \sqsubset \mathcal{L} \) have lower complexity than \( \tilde{G} \sqsubset \mathcal{L} \). Thus, given \( K \) there exists \( M(b_1(\tilde{G}), K) \) such that of \( \| \tilde{G} \| > M \) then it has a subsequence of length \( K \) such that all maps on vertex groups are injective. As above, modular automorphisms from \( \text{Mod}(\tilde{G}, \text{JSJ}_Q(\tilde{G})) \) push forward and we discover that some \( \tilde{G}(i) \to \tilde{G}(i+1) \) must be an isomorphism. Thus we may assume that the QCJSJ respecting sequences derived have no QH subgroups.

The above remark essentially contains a proof of Theorem 3.5. If a QCJSJ \( \tilde{G} \sqsubset \mathcal{G} \) respecting resolution has no QH subgroups and all abelian vertex groups
are equal to their peripheral subgroups we must work a little harder and use an
analysis of sequences of images of vertex groups of QCJSJ(\(\tilde{G}\)) to conclude that if
\(\tilde{G}\) is sufficiently long then it contains an isomorphism. In the next section we show
how to apply the construction of QCJSJ respecting resolutions multiple times\(^3\) in
order to find sequences of strictly lower complexity.

Our application of Theorem 1.4 is a way to produce this “lift” of a dimension
bound from sequences of vertex groups to a dimension bound for the ambient
chains. Theorem 7.6 is used to express QCJSJ respecting sequences as graphs
of sequences of groups obtained by passing to quotients and iteratively adjoining
roots. Theorem 1.4, on the other hand, is only stated (and possibly only true for)
sequences obtained by adjoining roots a single time along a single fixed collection
of elements. To cope with this deficiency we construct a collection of subsequences
of subgroups to which Theorem 1.4 can be applied.

Let \(\mathcal{R}_i\) be a sequence of vertex groups of QCJSJ(\(\tilde{G}\)). Let \(E_1, \cdots, E_m\) be the
edge groups incident to some vertex group \(\mathcal{R}_{i}^{n-1}(n-1) < \tilde{G}(n-1)\), and let
\(F_1, \cdots, F_m\) be the corresponding edges of \(\tilde{G}(n)\). Let \(C_j\) be the closure of the
image of \(E_j\) in \(F_j\), the subgroup of \(F_j\) consisting of all elements which have powers
lying in the image of \(E_j\). Now let \(S_i(n) = \langle \mathcal{R}_i^{n-1}(n) *_{\text{Im}(E_j)} C_j \rangle < \mathcal{R}_i^n(n)\) and
\(S_i^0(n) = \varphi_{n,m}(S_i(n))\). We leave it as an exercise for the reader to show that
\(S_i(n)\) is obtained from \(S_i^{n-1}(n)\) by adjoining roots to the collection
\[
\mathcal{E}_i^n = \{ S_i^{n-1}(n) \cap C_j \}_{j=1\ldots m}
\]
A priori, it is only obtained from \(\mathcal{R}_i^{n-1}(n)\) by adjoining roots. Note that \(\|\mathcal{E}_i^n\| \leq 2 b_1(\tilde{G})\).

**Theorem 8.3** (Krull assuming short sequences of vertex groups). Let \(\mathcal{L}\) be a freely
indecomposable QCJSJ respecting JSJ respecting chain of limit groups, and sup-
pose
\[
\mathcal{L} \varnothing \mathcal{H}, \quad \text{Comp}(\mathcal{L} \varnothing \mathcal{H}) = (b,d)
\]
Furthermore, assume that, for all sequences \(\mathcal{R}_i^n < \mathcal{L}\) of images of vertex groups of
quasi-cyclic JSJ decompositions, \(\|\mathcal{R}_i^n\|_{pl} < D\).

Then there is a constant \(D' = D'(D,b,d)\) such that \(\|\mathcal{L}\|_{pl} < D'\).

Note that the presence of \(\mathcal{H}\) is not strictly necessary. We only include it so the
statement of the theorem meshes more smoothly with the way it is used.

**Proof of Theorem 8.3.** Suppose there exist such \(\mathcal{L}\) of arbitrary proper length. First,
suppose that the sequences \(\mathcal{R}_i^n\) are in fact constant. Then there are no flexible
edges and the quasi cyclic JSJ decomposition agrees with the cyclic JSJ decom-
position and all peripheral subgroups of abelian vertex groups embed in \(\mathcal{L}(n+1)\). By
Lemma 5.18 the envelopes of all rigid vertex groups of \(\mathcal{L}(n)\) embed in \(\mathcal{L}(n+1)\),
therefore \(\mathcal{L}(n) \rightarrow \mathcal{L}(n+1)\) is strict.

\(^3\)Twice.
Observe now that every element of the modular group of \(L(n)\) pushes forward to \(\text{Mod}(L(n+1))\). It is then an easy exercise to show that such a strict epimorphism is an isomorphism.

Since \(S^n_i\) is a sequence of subgroups of \(R^n_i\), they have proper length bounded by \(D\) as well.

The number of sequences of vertex groups of QCJSJ\((L)\) depends only on \((b, d)\) by the proof of Lemma 2.7 (Or acylindrical accessibility. See [Sel97] or [Wei02]). Call the bound \(B = B(b, d)\). Then if \(\|L\| > D^{2K}\) then there is a subsequence \(L'\) of length \(K\) such that for each \(n\) and \(i\) the sequences

\[
S^n_i(n + 1) \rightarrow \cdots
\]

are constant. By Theorem 1.4 applied to the pair of sequences (indexed by \(n\))

\[
(S^{n-1}_i(K), (S^n_i(n), E_i)), \quad n < K
\]

for all but \(C = C(b, d, |E(\mathcal{R}^n)|)\) indices, the maps \(S^{n-1}_i(n - 1) \rightarrow S^{n-1}_i(K) \cong S^{n-1}_i(n)\) are isomorphisms. Since the number of edge groups incident to a rigid vertex group only depends on \(b, C\) is independent of \(L\).

By construction, \(S^n_i(n)\) intersects each small flexible edge incident to \(R^n_i(n)\) in a finite index subgroup.

For each index \(n\) such that \(S^{n-1}_i(n - 1) \rightarrow S^{n-1}_i(n)\) is an isomorphism, we then have that a small flexible edge incident to \(R^n_i(n)\) must embed in \(R^{n+1}_i(n + 1)\), contradicting the construction of the QCJSJ decompositions. We conclude that the QCJSJ decompositions coincide with the cyclic JSJ decompositions, and that \(R^n_i = S^n_i\).

Since the rank of \(R^n_i\) is bounded in terms of the rank of the resolution \(L \boxtimes H\), \(C\) depends only on \((b, d)\). Since the number of rigid vertex groups is bounded by \(B\), if \(\|L\| > C^B\) and \(L\) satisfies (\(*\)), then \(L\) contains an isomorphism by the first two paragraphs of the proof. For general \(L \boxtimes H\) satisfying the hypotheses of the theorem, if \(\|L\| > D' = D^{B(n)C(N)B(N)}\) then \(L\) contains an isomorphism.

\section{Decreasing the Complexity}

We prove a useful lemma before we begin.

\textbf{Lemma 9.1.} Suppose \(\varphi: G \rightarrow H\) is degenerate and indecomposable. Let \(\Delta\) be a cyclic decomposition of \(G\), and suppose that a noncyclic vertex group of \(\Delta\) with first betti number at least two has cyclic image in \(H\). Then the vertex groups of the cyclic JSJ decomposition, and therefore of the QCJSJ as well, of \(H\) satisfy \(b_1(H_{\varphi}) < b_1(H)\).

\textbf{Proof:} This is proven by following the construction of \(\Phi_\ast H\). Build \(\Phi_{\ast G}(\varphi): \Phi_{\ast G}(G) \rightarrow H\) with respect to the decomposition \(\Delta'\) obtained by collapsing all edges not adjacent to the distinguished vertex \(v\). Since the theorem is trivially true if \(H\) has any QH vertex groups or principle cyclic splittings of the form \(A *_{\mathbb{Z}} G'\), we may assume that \(\Phi_{\ast G}(\Delta')\), the decomposition obtained by pushing forward \(\Delta'\) and blowing up vertex groups into their relative JSJ decompositions, has no QH vertex groups. There is a bijection between nonabelian vertex groups of \(\Phi_{\ast G}(\Delta')\) and nonabelian
vertex groups of $H$. Let $W$ be a nonabelian vertex group of $\Phi_{\text{as}}(G)$. The associated vertex group of $H$ is obtained from $W$ by iteratively adjoining roots. Since $\Phi_{\text{as}}H$ is almost strict, it embeds $W$, hence if $E$ is an edge group adjacent to $W$ and if the centralizer of $E$ in $W$ is noncyclic, the centralizer of the associated edge of $H$ has noncyclic centralizer. Since every edge of $H$ centralizes the image of some edge of $\Phi_{\text{as}}(\Delta')$, and since every limit group has a principle cyclic splitting, there is some $\sim_a$ equivalence class $A$ (recall that $\sim_a$ is the equivalence relation on abelian vertex groups and edge groups generated by adjacency) which contains only infinite cyclic edge and vertex groups, and such that if an edge $e$ of $A$ is adjacent to a nonabelian vertex group $W$ of $\Phi_{\text{as}}(\Delta')$, then the centralizer of $e$ in $W$ is infinite cyclic. Moreover, at least two (oriented) edges from such a class must be adjacent to nonabelian vertex groups.

Since $\varphi$ is degenerate, we may write $H$ as a direct limit

$$H = \lim_{\to} G_n, \quad G_n := (\Phi_1 \circ \Phi_a)^n \Phi_{\text{as}}(G)$$

Let $A_{n,i}, i = 1..m_n$ be the collection of $\sim_a$ equivalence classes for $G_n$. By the discussion above, every equivalence class for $G_n$ contains the image of some equivalence class of $G_{n-1}$ and there is a (possibly one-to-many) map $\{A_{n,i}\}_{i=1..m_n} \rightarrow \{A_{n-1,j}\}_{j=1..m_{n-1}}$.

Since the direct limit has finite length, $H = G_m$ for some $m$. Choose $\sim_a$ equivalence classes $A_{j,i(j)}, j = 0..m$ such that the image of $\Gamma_{A_{j,i(j)}}$ is a vertex of $A_{j+1,i(j+1)}$. If the distinguished vertex $v$ is an element of $A_{0,i(0)}$ then the subgraphs of groups carried by the complimentary components of $v$ have lower first betti number than $\Phi_{\text{as}}(G)$. Otherwise, an easy homological argument shows that the complimentary components of $A_{0,i(0)}$ have lower first betti number than $\Phi_{\text{as}}(G)$. This state of affairs is unchanged by an application of $\Phi_a$, and remains unchanged after an application of $\Phi_1$. We see in the limit that to $A_{m,j(m)}$ there is an associated principle cyclic splitting (there may be more than one), and that the vertex group of this splitting has strictly lower first betti number than the ambient group $H$. $\square$

To prove Theorem 3.5, of which Theorem 1.2 is a consequence, we mimic Sela’s construction of the cyclic analysis lattice, but for sequences of limit groups. We don’t build an entire analysis lattice, only the few branches necessary for the inductive proof of Theorem 3.5.

At each level of the cyclic analysis lattice of a limit group, we pass either to the freely indecomposable free factors of a limit group, or, if freely indecomposable, we pass to the vertex groups of the cyclic JSJ decomposition. The construction of QCJSJ decompositions and maximal Grushko–respecting resolutions give us the ability to mimic this construction for sequences of limit groups.

Let $V$ be a vertex group in the cyclic JSJ of a limit group $L$. The neighborhood of $V$, as opposed to the envelope, is the subgroup of $L$ generated by $V$ and the centralizers of incident edge groups. The neighborhood of such a $V$ always has the form $V \ast E_i A_i$, where $E_i$ are the edge groups incident to $V$ such that the centralizer
of $E_i$ isn’t contained in $V$ (it can be contained in a conjugate of $V$), and the $A_i$ are the centralizers of such $E_i$.

If $G < L$ is a subgroup of $L$ which is freely indecomposable relative to a collection of JSJ$_C(L)$–elliptic subgroups then every abelian vertex group of the decomposition of $G$ inherited by its action on the minimal $G$–invariant subtree of the Bass-Serre tree for the cyclic JSJ of $L$ is adjacent to a nonabelian vertex group.

**Lemma 9.2.** Let $\mathcal{G} \varsubsetneq \text{Im}(\mathcal{G}) \hookrightarrow \mathcal{L}$ be a maximal QCJSJ respecting resolution of a subsequence of $\mathcal{L}$. Let $\mathcal{R}$ be a sequence of vertex groups of QCJSJ($\mathcal{G}$), and suppose that $\mathcal{P}$ is an indecomposable maximal QCJSJ respecting resolution of the image $\text{Im}(\mathcal{R}) \hookrightarrow \mathcal{L}$ and $c_\alpha(\mathcal{P}(n)) = 0$ for all $i$. Suppose further that $\mathcal{P}$ has no QH sequences of vertex groups, only one sequence $\mathcal{M}$ of vertex groups in its QCJSJ decomposition, and that $b_1(\mathcal{M}) = b_1(\mathcal{G})$. Let $\{1, \ldots, m\}$ be the index set for $\mathcal{R}$. Then the vertex groups of $\mathcal{M}(n)$ map to a neighborhood of a vertex group of JSJ$_C(\text{Im}(\mathcal{G})(n))$ for $m - 2 \geq n \geq 2$.

**Proof:** Let $A_i$ be the set of sequences of big (Definition 7.2) conjugacy classes of abelian vertex groups of $\mathcal{P}$. Let $\Delta_n = \Delta_n(V^n_i, A^n_j, E^n_k)$ be the decomposition of $\mathcal{P}(n)$ obtained by collapsing all edges not adjacent to some $A^n_j(n)$. This decomposition has the property that every vertex group $A^n_j$ has elliptic image in $\text{Im}(\mathcal{G})(n)$. Moreover, since $\mathcal{P}$ is QCJSJ respecting, $c_\alpha(\varphi_{n,n+1}|V^n_i) = 0$. By definition $\mathcal{P}$ is $\Delta$–stable.

The first thing to show is that the image of each $V^n_i$ in $\mathcal{G}(n + 1)$ is contained in a vertex group of JSJ($\mathcal{G}(n + 1)$). This fact will be used in the second half of the proof to show that if the vertex groups of a special decomposition of $\mathcal{P}$ don’t map to neighborhoods of vertex groups of $\mathcal{G}$, then there is an intermediate group between $\mathcal{P}(n)$ and $\mathcal{P}(n + 2)$ which has more noncyclic abelian vertex groups than $\mathcal{P}(n)$, which we use to derive a contradiction to the assumption that $\mathcal{P}$ is JSJ respecting. Consider the commutative diagram in Figure 4.

\[
\begin{array}{ccc}
\mathcal{P}(n - 2) & \xrightarrow{\varphi_{n-2,n-1}} & \mathcal{P}(n - 1) & \xrightarrow{\varphi_{n-1,n}} & \mathcal{P}(n) \\
\pi_{n-2} & & & & \\
\text{Im}(\mathcal{R})(n - 2) & \xrightarrow{\psi_{n-2}} & \text{Im}(\mathcal{R})(n - 1) \quad & & \\
& & & & \uparrow \pi_{n-1} \\
& & & & \mathcal{R}(n - 1) < \mathcal{G}(n - 1) \\
\end{array}
\]

**Figure 4.** Deducing that the abelian JSJ of $\mathcal{P}$ resembles the induced decomposition of $\mathcal{R}$.

Let $\Delta$ be the decomposition of $\mathcal{R}(n - 1)$ induced by JSJ$_B(\mathcal{G}(n - 1))$. We can write $\Delta = \Delta(W_i, B_j, F_k)$, where $W_i$ are rigid vertices from JSJ($\mathcal{G}(n - 1)$), the $B_j$ are subgroups $P_B(A_j)$ of abelian vertex groups of $\mathcal{G}(n - 1)$, and the $F_k$ are big edge groups connecting them. To be included in $\Delta$, a vertex group $P_B(A)$ of
\(\text{JSJ}_B(\mathcal{G}(n - 1))\) must have at least two big incident edges, otherwise the associated one-edged splitting is trivial. Since \(\mathcal{P}\) is degenerate, the composition \(\psi_{n-1} \circ \pi'_{n-1}\) is degenerate as well, since

\[
\psi_{n-1} \circ \psi_{n-2} = \psi_{n-1} \circ \pi'_{n-1} \circ \psi'_{n-2} \circ \pi_{n-2}
\]

Now construct \(\Phi_\delta(\psi_{n-1} \circ \pi'_{n-1}): \Phi_\delta(\mathcal{R}(n - 1)) \to \mathcal{P}(n)\), starting with the decomposition \(\Delta\). We start by building an almost-strict \(\Phi_{\text{as}}(\mathcal{R}(n - 1)) \to \mathcal{P}(n)\). For each vertex group \(W_i\) we build first \(A\mathcal{S}(W_i) = S_{\mathcal{H}}(W_i) \ast E_{i,j} B_{i,j}\). If \(F \in \mathcal{E}(W_i)\) then the image of \(F\) is contained in some \(B_i\). Since the edge groups incident to \(W_i\) are nonconjugate, and remain nonconjugate in \(\mathcal{G}(n)\), each \(B_{i,j}\) contains a unique such \(F\). Let \(W_{i,1}, \ldots, W_{i,n_i}\) be the vertex groups of the relative JSJ of \(S_{\mathcal{H}}(W_i)\).

The image of \(\Phi_{\text{as}}(\mathcal{R}(n - 1)) \to \mathcal{P}(n)\) is contained in a vertex group of the JSJ, and since the nonabelian vertex groups of \(\mathcal{P}(n)\) are obtained from the vertex groups \(W_{i,j}\) by iteratively adjoining roots, the vertex groups of \(\mathcal{P}(n)\) have images contained vertex groups of \(\mathcal{G}(n)\). Moreover, for \(i \neq j\), \(W_{i,j}\) and \(W_{i',j'}\) have image contained in district vertex groups of \(\mathcal{G}(n + 1)\). In order to check the condition that the vertex groups of \(\Delta_n\) have images contained in vertex groups of \(\mathcal{G}(n + 1)\) all we need to check is that if some \(W_{i,j}\) and \(W_{i',j'}\) have incident edges which are in the same \(\sim_a\) equivalence class \([A]\), then the vertex group of \(\mathcal{P}(n)\) which centralizes the image of the subgroup carried by \([A]\) isn’t infinite cyclic. Since \(c_a(\mathcal{P}) = 0\) the topology of the underlying graph of the decomposition of \(\Phi_{\text{as}}(\mathcal{R}(n - 1))\) is identical to the topology of the underlying graph of the abelian JSJ of \(\mathcal{P}(n)\), hence if there is such a pair of groups then there is a path \(p\) between \(W_{i,j}\) and \(W_{i',j'}\) in the graphs of groups decomposition of \(\Phi_{\text{as}}(\mathcal{R}(n - 1))\) which corresponds to the path between their image vertex groups which crosses the centralizer of their intersection. If \(i \neq i'\) then \(p\) passes through some vertex \(A\mathcal{S}(B_k)\) for some \(k\). Since \(B_k\) is big, the image of \(B_k\) in \(\mathcal{P}(n)\) doesn’t have cyclic image in any further element of \(\mathcal{G}\). Furthermore, if we apply \(\Phi_n\), altering \(p\) appropriately, this state of affairs is unchanged, and a further application of \(\Phi\), also altering \(p\) appropriately, also leaves this state of affairs unchanged, thus any path from the vertex group of \(\mathcal{P}(n)\) associated to \(W_{i,j}\) to the vertex group associated to \(W_{i',j'}\) must pass through a big abelian vertex group.

Now consider the commutative diagram in Figure 5. Define

\[
\mathcal{P}(n)' := \Delta_n(\pi_n(V_i^n), \pi_n(A_i^n), \pi_n(E_i^n))
\]

Let \(V_i^n = \pi_n(V_i^n)\), and let \(\Delta_{n,i}\) be the decomposition over cyclic groups that \(V_i^n\) inherits from its action on the Bass-Serre tree associated to the cyclic JSJ decomposition of \(\text{Im}(\mathcal{G})(n)\). Since \(A_i^n\) is big, it has elliptic image in \(\text{Im}(\mathcal{G})(n)\), therefore the edges \(E_i^n\) have elliptic images in \(\text{Im}(\mathcal{G})(n)\), and the decompositions \(\Delta_{n,i}\) can be used to refine the decomposition \(\Delta_n\) of \(\mathcal{P}(n)'\) to a decomposition \(\Delta\).

Let \(\Gamma_{n,i}\) be the underlying graph of \(\Delta_{n,i}\), and let \(\Gamma_n\) be the underlying graph of the cyclic JSJ of \(\text{Im}(\mathcal{G})(n)\). The natural map of graphs of groups \(\Delta_{n,i}\) to the cyclic JSJ of \(\text{Im}(\mathcal{G})(n)\) induces a combinatorial map of underlying graphs \(\tau_{n,i}: \Gamma_{n,i} \to \Gamma_n\). Since \(\text{Im}(\mathcal{G})(n) \to \mathcal{G}(n + 1)\) is degenerate and the image of \(V_i\) in \(\mathcal{G}(n + 1)\) is contained in a vertex group of the abelian JSJ of \(\mathcal{G}(n + 1)\), \(\tau_{n,i}\) is homotopically
trivial, therefore $\iota_{n,i}$ factors through a map $\tilde{\iota}_{n,i} : \Gamma_{n,i} \to \tilde{\Gamma}_n$, the universal cover of $\Gamma_n$. Furthermore, for at most one set of preimages $I = \tilde{\iota}_{n,i}^{-1}(w)$, $w \in \tilde{\Gamma}_n$, can the vertex groups of $\Delta_{n,i}$ have nonabelian image in $G(n+1)$. Let $I_e$ be this set of vertices, and let $e$ be the associated vertex of $\tilde{\Gamma}_n$. See Figure 6.

Choose $F \in \mathcal{E}(V^n_i)$. Since $F$ is adjacent to some $A^\beta_j$, and since the $A^\beta_j$ have elliptic images in $G$, it is elliptic in $\Delta_{n,i}$. Since $c_\alpha(P) = 0$, if a valence one vertex group $K$ of some $\Delta_{n,i}$ is abelian (and is necessarily noncyclic) or has abelian image in $G(n+1)$, then some $F_i$ must have image in $K$, a contradiction to the fact that $P$ is JSJ respecting.

Suppose an abelian vertex group of $\Delta_{n,i}$ with valence at least two contains the image of $F$. If the vertex is nonseparating then $\Delta_{n+1}$ has strictly more edges than $\Delta_n$ since $c_\alpha(P) = 0$, a contradiction. If it is separating then either $\Delta_{n+1}$ has more edges than $\Delta_n$, an impossibility, or one of the complementary components has abelian image in $G(n+1)$, then no other edge from $\Delta_n$ has image in the component with abelian image and we conclude again that $c_\alpha(P) > 0$, another contradiction. Thus $F$ has image in a nonabelian vertex group of $\Delta_{n,i}$ and this vertex group doesn’t have abelian image in $G(n+1)$. If any vertex group $K$ (abelian or not) of $\Delta_{n,i}$ which does not contain the image of any $F_i$ has abelian image in $G(n+1)$ then, since $\Delta_{n+1}$ has the same number of abelian vertex groups as $\Delta_n$, $K$ must have cyclic image in $P(n+2)$. By Lemma 9.1 applied to the composition $\mathcal{P}(n) \to \mathcal{P}(n+2)$, the vertex group of QCJSJ($\mathcal{P}(n+2)$) has strictly lower betti
number than \( \mathcal{P}(n + 2) \), contradictory to hypothesis. We conclude that all vertex groups carried by vertices from \( \Gamma_{n,i}^{(0)} \setminus I_e \) are infinite cyclic.

We must then have that all noncyclic vertices of \( \Delta_{n,i} \) are in \( I_e \). Since all noncyclic vertices are in \( I_e \), every cyclic vertex must be at most distance one from \( I \), and we conclude that \( V_i \) is contained in a neighborhood of a vertex group of the cyclic JSJ of \( \text{Im}(\mathcal{G})(n) \).

Let \( \mathcal{M}(n) \) be the vertex group of the QCJSJ of \( \mathcal{P}(n) \). Every vertex group of \( \text{JSJ}_B(\mathcal{P}(n)) \) is contained in some \( V_i \), hence every vertex group of \( \mathcal{M}(n) \) has image in a neighborhood of a vertex group of the cyclic JSJ of \( \text{Im}(\mathcal{G})(n) \). Suppose \( M_1 \) and \( M_2 \) are vertex groups of \( \text{JSJ}_B(\mathcal{P}(n)) \) which are adjacent in \( \mathcal{M}(n) \). Then \( M_1 \) and \( M_2 \) are connected by a big edge \( E \). If \( M_1 \star_E M_2 \) didn’t map to a neighborhood of a vertex group of the cyclic JSJ of \( \text{Im}(\mathcal{G})(n) \) then, since all edges incident to \( M_1 \) are big, \( M_1 \) is elliptic, and the edges of \( \text{JSJ}_e(\text{Im}(\mathcal{G})(n)) \) are infinite cyclic, \( c_a(\varphi_{n,n+1}M_1, E(M_1)) > 0 \), contradicting the assumption that \( \mathcal{P} \) is JSJ respecting.

**Lemma 9.3.** Let \( L \) be a limit group and \( N \) be a neighborhood of a vertex group \( V \) of the cyclic JSJ. Let \( A_1, \ldots, A_k \) be the centralizers of edge groups incident to \( V \) which aren’t contained in \( V \). A freely indecomposable subgroup \( G \) of \( N \) has the property that its vertex groups are contained in conjugates of \( V \) or it has a principle cyclic splitting of the form \( G = G' \star_{\mathbb{Z}} A' \), for some \( A' < A_1 \), for some \( j \).

In the previous section we showed how to lift a dimension bound for sequences of vertex groups of QCJSJ decompositions to a dimension bound for QCJSJ respecting sequences of limit groups. The next theorem allows us to apply the construction of maximal resolutions of sequences of images of vertex groups twice, in the event that no “obvious” reductions in complexity are possible (nontrivial \( c_a \), freely decomposable maximal resolutions) to arrive at resolutions of lower complexity.

**Theorem 9.4 (Decrease the complexity).** Let \( \mathcal{L}, \mathcal{G}, \mathcal{R}, \mathcal{P}, \mathcal{M} \) be as in Lemma 9.2. Let \( \mathcal{H} \varnothing \text{Im}(\mathcal{M}) \) be a QCJSJ respecting maximal resolution of the image of \( \mathcal{M} \).

Let \( \mathcal{S} \) be a sequence of vertex groups of QCJSJ(\( \mathcal{H} \)), and suppose that \( Q \) is an indecomposable maximal QCJSJ respecting resolution of the image \( \text{Im}(\mathcal{S}) \varnothing \mathcal{L} \) and \( c_a(Q(n)) = 0 \) for all \( i \). Suppose further that \( Q \) has no QH sequences of vertex groups, only one sequence \( N \) of vertex groups in its QCJSJ decomposition, and that \( b_1(N) = b_1(Q) \). Let \( \{1, \ldots, m\} \) be the index set for \( S \). Then the vertex groups \( N(n) \) map to a vertex group of JSJ of \( \text{Im}(\mathcal{G})(n) \) for \( m - 4 \leq n \geq 4 \).

In particular,\(^4\)

\[
\text{Comp}(N_{\{4,\ldots,m-4\}} \varnothing \mathcal{L}) < \text{Comp}(\mathcal{G} \varnothing \mathcal{L})
\]

**Proof.** Consider the diagram in Figure 7. Each block, separated by the long equals sign, represents a use of Lemma 9.2.

By Lemma 9.3, since \( c_a(Q) = 0 \), the neighborhoods of vertex groups of the images \( \text{Im}(\mathcal{M})(n) \) are completely contained in vertex groups of the cyclic JSJ of

\(^4\)These are the same hypotheses as Lemma 9.2.
Im(\mathcal{G}). By Lemma 9.2 applied to the tuple (\mathcal{H}, \mathcal{S}, \mathcal{Q}, \mathcal{N}), Im(\mathcal{N})(n) is contained in a vertex group of the cyclic JSJ of Im(\mathcal{G})(n), hence \text{Comp}(\mathcal{N}((4, \ldots, m-4) \boxslash \mathcal{L})) < \text{Comp}(\mathcal{G} \boxslash \mathcal{L}) \) \quad \square

Now we can continue the analysis using the sequences Im(\mathcal{N})((4, \ldots, m-4)) which have strictly lower depth than \mathcal{L}.

To finish the proof of Theorem 3.5 we combine the work from previous sections. Corollary 7.7, Corollary 8.1, Remark 8.2, Theorem 8.3, Theorem 9.4, and the uniform bound on depths of rank \( n \) limit groups provided by Theorem 2.11 formally imply Theorems 3.5 and 1.2.

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