Some properties for superprocess under a stochastic flow

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Abstract

For a superprocess under a stochastic flow, we prove that it has a density with respect to the Lebesgue measure for \( d = 1 \) and is singular for \( d > 1 \). For \( d = 1 \), a stochastic partial differential equation is derived for the density. The regularity of the solution is then proved by using Krylov’s \( L_p \)-theory for linear SPDE. A snake representation for this superprocess is established. As applications of this representation, we prove the compact support property for general \( d \) and singularity of the process when \( d > 1 \).

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1 Introduction

Superprocesses under stochastic flows have been studied by many authors since the work of Wang (\cite{W1},\cite{W2}) and Skoulakis and Adler \cite{SA}. At an early stage, this problem was studied as the high-density limit of a branching particle system while the motion of

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each particle is governed by an independent Brownian motion as well as by a common
Brownian motion which determines the stochastic flow. The limit is characterized by
a martingale problem whose uniqueness is established by a moment duality. Before we
go any further, let us introduce the model in more detail.

Let \( b : \mathbb{R}^d \rightarrow \mathbb{R}^d, \sigma_1, \sigma_2 : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d} \) be measurable functions. Let \( W, B_1, B_2, \cdots \) be independent \( d \)-dimensional Brownian motions. Consider a branching particle system performing independent binary branching. Between branching times, the motion of the \( i \)th particle is governed by the following stochastic differential equation (SDE):

\[
d\eta_i(t) = b(\eta_i(t))dt + \sigma_1(\eta_i(t))dW(t) + \sigma_2(\eta_i(t))dB_i(t). \tag{1.1}
\]

It is proved by Skoulakis and Adler [9] that the high-density limit \( X_t \) is the unique solution to the following martingale problem (MP): \( X_0 = \mu \in \mathcal{M}_F(\mathbb{R}^d) \), where \( \mathcal{M}_F(\mathbb{R}^d) \) denotes the space of finite nonnegative measures on \( \mathbb{R}^d \) and for any \( \phi \in C^2_0(\mathbb{R}^d) \),

\[
M_t(\phi) \equiv \langle X_t, \phi \rangle - \langle \mu, \phi \rangle - \int_0^t \langle X_s, L\phi \rangle \, ds \tag{1.2}
\]

is a continuous martingale with quadratic variation process

\[
\langle M(\phi) \rangle_t = \int_0^t \left( \langle X_s, \phi^2 \rangle + \left| \langle X_s, \sigma_1^T \nabla \phi \rangle \right|^2 \right) \, ds \tag{1.3}
\]

where

\[
L\phi = \sum_{i=1}^d b^i \partial_i \phi + \frac{1}{2} \sum_{i,j=1}^d a^{ij} \partial^2_{ij} \phi,
\]

\[
a^{ij} = \sum_{k=1}^d \sum_{\ell=1}^2 \sigma_{k \ell}^i \sigma_{k \ell}^j, \quad \partial_i \text{ means the partial derivative with respect to the } i\text{th component of } x \in \mathbb{R}^d, \quad \sigma_1^T \text{ is the transpose of the matrix } \sigma_1, \quad \nabla = (\partial_1, \cdots, \partial_d)^T \text{ is the gradient operator and } \langle \mu, f \rangle \text{ represents the integral of the function } f \text{ with respect to the measure } \mu. \]

It was conjectured in [9] that the conditional log-Laplace transform of \( X_t \) should be the unique solution to a nonlinear stochastic partial differential equation (SPDE). Namely

\[
\mathbb{E}_\mu \left( e^{-\langle X_t, f \rangle} \bigg| W \right) = e^{-\langle \mu, y_{0,1} \rangle} \tag{1.4}
\]
and
\[ y_{s,t}(x) = f(x) + \int_s^t (Ly_{r,t}(x) - y_{r,t}(x)^2) \, dr + \int_s^t \nabla^T y_{r,t}(x) \sigma_1(x) \, \hat{d}W(r) \] (1.5)

where \( \hat{d}W(r) \) represents the backward Itô integral:
\[
\int_s^t g(r) \hat{d}W(r) = \lim_{|\Delta| \to 0} \sum_{i=1}^n g(r_i) (W(r_i) - W(r_{i-1}))
\]
where \( \Delta = \{r_0, r_1, \ldots, r_n\} \) is a partition of \([s, t]\) and \( |\Delta| \) is the maximum length of the subintervals.

This conjecture was confirmed by Xiong [13] under the following conditions (BC) which will be assumed throughout this paper: \( f \geq 0, \ b, \ \sigma_1, \ \sigma_2 \) are bounded with bounded first and second derivatives. \( \sigma_2^T \sigma_2 \) is uniformly positive definite, \( \sigma_1 \) has third continuous bounded derivatives. \( f \) is of compact support.

Making use of the conditional log-Laplace functional, the long-term behavior of this process is studied in [14]. Also, the model has been extended in that paper to allow infinite measures \( \mu \in \mathcal{M}_{tem}(\mathbb{R}^d) \), namely, \( \int_{\mathbb{R}^d} e^{-\lambda |x|} \mu(dx) < \infty \) for some \( \lambda > 0 \). We shall assume \( \mu \in \mathcal{M}_{tem}(\mathbb{R}^d) \) throughout this paper. A similar model has been investigated by Wang [12] and Dawson et al [1] when the spatial dimension is 1. Further, in that case, it is proved by Dawson et al [2] that their process is density-valued and solves a SPDE. The regularity of the solution was left open in that article.

This paper is organized as follows: In Section 2, we establish a snake representation for \( X_t \). As immediate consequences to this representation, we get the compact support property of \( X_t \) (for all \( d \)) and for \( d > 1 \), \( X_t \) takes values in the set of singular measures. Then, for \( d = 1 \), we prove in Section 3 that \( X_t \) is absolutely continuous with respect to Lebesgue measure and show that the density \( X(t, x) \) satisfies the following SPDE
\[
\partial_t X = L^*X - \partial_x (\sigma_1 X) \dot{W}_t + \sqrt{X} \dot{B}_{tx} \] (1.6)
where $B$ is a Brownian sheet and $L^*$ is the adjoint operator of $L$. The main result of this paper is to show the Hölder continuity of $X(t, x)$.

Here is the main result. First recall that for $n \in \mathbb{R}$ and $p \in [2, \infty)$, $H^n_p$ is the space of Bessel potentials with norm
\[
\|u\|_{n,p} = \|(I - \Delta)^{n/2}u\|_p.
\]

**Theorem 1.1** Suppose that Condition (BC) is satisfied. Then

i) If $d > 1$, then $X_t$ is singular a.s.

ii) If $d = 1$, then $X_t$ is absolutely continuous with respect to Lebesgue measure and the density satisfies the SPDE (1.6).

iii) If in addition, $\mu$ satisfies $\mu \in H^{1/2 - \epsilon - 2/p}_p$ with $\epsilon \in (0, 1/8)$ and $p > 1/\epsilon$, and also satisfies
\[
\sup_{t,x} \langle \mu, \varphi_t(x - \cdot) \rangle < \infty,
\]
then the density $X(t, x)$ is Hölder continuous in $x$ with index $1/2 - 2\epsilon$ for (a.e.) $t$ a.s., where $\varphi_t(x)$ is the density of a normal random variable with mean 0 and variance $t$.

Note that (1.7) is satisfied if $\mu$ has bounded density with respect to Lebesgue measure.

Suppose that we apply the usual integral equation as in [10], Chapter 3, for (1.6) in order to prove the Hölder continuity. Then formally we have
\[
X(t, x) = \int p_0(t, x, y)X(0, y)dy + \int_0^t \int \sigma_1(y)X(s, y)\partial_y p_0(t - s, x, y)dydW(s)
+ \int_0^t \int \sqrt{X(s, y)}p_0(t - s, x, y)B(dsdy)
\]
where $p_0$ is the transition function of the Markov process with generator $L$. However, the second term on the right hand side of the above equation is about
\[
\int_0^t (t - s)^{-1/2}dW(s)
\]
which is not convergent. Therefore, the convolution argument used by Konno and Shiga [5] does not apply to our model. In Section 4, we freeze the nonlinear term in (1.6) and apply Krylov’s $L_p$-theory for linear SPDE to get the Hölder continuity with index slightly less than $\frac{1}{2}$ for $X$.

Note that the SPDE in [2] is (1.6) in current paper with $\dot{W}_t$ replaced by a space-time noise which is colored in space and white in time. The method of this paper can be applied to that equation to prove the regularity for its solution.

2 Snake representation

In this section, we construct a path-valued process $Y_t$ such that the process $X_t$ can be represented according to this process. Then, as an easy application of this representation, we derive the properties for $X_t$.

For the convenience of the reader, we recall some basic definitions and facts taken from Le Gall [8]. Let $\zeta \geq 0$ and let $f$ be a continuous function from $\mathbb{R}_+$ to $\mathbb{R}^d$ such that $f(s) = f(\zeta)$, $\forall$ $s \geq \zeta$. We call such pair $(f, \zeta)$ a stopped path with $\zeta$ being the lifetime of the path. We denote the collection of all stopped paths by $\mathcal{W}$. For $(f, \zeta), (f', \zeta') \in \mathcal{W}$, define a distance

$$\delta((f, \zeta), (f', \zeta')) = \sup_{s \geq 0} |f(s) - f'(s)| + |\zeta - \zeta'|.$$ 

Then $(\mathcal{W}, \delta)$ is a Polish space. In [8], Le Gall constructed a continuous time-homogeneous strong Markov process $(Z_t, \zeta)$ taking values on $\mathcal{W}$. $\zeta_t$ is a one-dimensional reflecting Brownian motion. Given $\zeta$, the process $Z$ has the following property: for all $r < t$, and for all $s \leq m_{r,t} := \inf_{r \leq u \leq t} \zeta_u$ we have $Z_r(s) = Z_t(s)$. Furthermore, given $m_{r,t}$ and $Z_r(m_{r,t})$, the processes $Z_r(s) : s \geq m_{r,t}$ and $Z_t(s) : s \geq m_{r,t}$ are conditionally independent Brownian motions with lifetimes $\zeta_r$ and $\zeta_t$ respectively.
Denote the strong solution to the SDE

$$d\eta(t) = b(\eta(t))dt + \sigma_1(\eta(t))dW(t) + \sigma_2(\eta(t))dB(t)$$

by $\eta(t) = F(t, W, B)$. Define the following path-valued process

$$\mathcal{Y}_t(s) = F(s, W, Z_t)$$

with the life-time process $\zeta_t$.

**Lemma 2.1** $(\mathcal{Y}_t, \zeta_t)$ is a continuous $\mathcal{W}$-valued process.

Proof: Note that for all $r < t$ and for all $s < m_{r,t}$, we have $\mathcal{Y}_r(s) = \mathcal{Y}_t(s)$. Furthermore, for given $\mathcal{Y}_r(m_{r,t})$, the processes $\mathcal{Y}_r(s) : s \geq m_{r,t}$ and $\mathcal{Y}_t(s) : s \geq m_{r,t}$ are the motions of two particles (say, $\eta_1$ and $\eta_2$) given as in the introduction with lifetimes $\zeta_r$ and $\zeta_t$ starting from the same position $\mathcal{Y}_r(m_{r,t})$. A simple application of Burkholder’s inequality gives

$$E \left[ \sup_{m \leq s \leq M} |\eta_1(s) - \eta_2(s)|^k \right] \leq K |M - m|^{k/2},$$

where $m = m_{r,t}$ and $M = \zeta_r \lor \zeta_t$. Denote by $E^\zeta$ the conditional expectation given $\zeta$.

Then

$$E \left[ \sup_{s \geq 0} |\mathcal{Y}_r(s) - \mathcal{Y}_t(s)|^k \right] = E \left[ E^\zeta \left\{ \sup_{s \geq m_{r,t}} |\mathcal{Y}_r(s) - \mathcal{Y}_t(s)|^k \right\} \right]$$

$$\leq E \left[ K |\zeta_r + \zeta_t - 2m_{r,t}|^{k/2} \right]$$

$$\leq K E \left[ \sup_{s \in [r,t]} |\zeta_s - \zeta_r|^{k/2} \right]$$

$$\leq K |t - r|^{k/4}.$$ 

The conclusion follows from Kolmogorov’s criteria by taking $k > 4$; see \[10\] for Kolmogorov’s criteria.
Theorem 2.2

\[ X_t(f) = \int_t^\tau f(\mathcal{Y}_s(\zeta_s))d\ell_s^t \]  \hspace{1cm} (2.1)

where \( \ell^t \) is the local time process of \( \zeta \) at level \( t \) and

\[ \tau = \inf\{s : \ell_s^0 \geq 1\}. \]

Proof: Fix a parameter \( h > 0 \). For every \( t \geq 0 \), denote by \([a^1_t, b^1_t], [a^2_t, b^2_t], \ldots, [a^{N_t}_t, b^{N_t}_t]\) the excursion intervals of \((\zeta_s)_{0 \leq s \leq \tau}\) above level \( t \), corresponding to excursions of height greater than \( h \). Set

\[ X^h_t = 2h \sum_{i=1}^{N_t} \delta_{a^i_t(t)}. \]

Then \( X^h_t \) is the measure-valued process corresponding to the branching particle system described as follows: At time \( t = 0 \), we have \( N_0 \) particles in \( \mathbb{R}^d \) with Poisson random measure with intensity measure \( h^{-1}\mu \). The particles then move according to (1.1) with common \( W \) and independent \( B_i \)'s. Each of them has a finite lifetime (independent of others) which is exponential with mean \( h \). When a particle dies, it gives rise to either 0 or 2 new particles with probability \( \frac{1}{2} \). The new particles start from the position of the their father. As in the proof of Theorem 2.1 in [8], by the well-known approximation of Brownian local time by upcrossing numbers, we have that \( X^h_t \) converges weakly to \( X_t \), where \( X_t \) is given by the right hand side of (2.1). \[ \square \]

As an application of the snake representation, we have the following immediate consequence.

Corollary 2.3 If \( \mu \) is a finite measure, then for any \( t > 0 \), \( X_t \) has compact support a.s.

Proof: By the snake representation, there exists a finite set \( I \) such that

\[ \langle X_t, f \rangle = \sum_{i \in I} \int_0^{\tau_i} f(\mathcal{Y}^i_s) d\ell^i_s(\zeta^i) \]
where \( \hat{Y}^i_s \) is the tip of the \( i \)th snake. It is not hard to show that \( \hat{Y}^i_s \) is continuous and hence, for any \( t_0 > 0 \),

\[
\bigcup_{t \geq t_0} \text{supp}(X_t) \subset \bigcup_{i \in I} \text{Range} \left( \hat{Y}^i \right) = \bigcup_{i \in I} \{ \hat{Y}^i_s : 0 \leq s \leq \tau_i \}
\]  

(2.2)
is compact.

To consider the case for \( \mu \) being \( \sigma \)-finite, the following conditional martingale problem (CMP) is useful. The following lemma was proved in [1]...

**Lemma 2.4** i) If \( X_t \) is the solution to MP, then there exists a Brownian motion \( W_t \) such that for any \( \phi \in C_0^2(\mathbb{R}^d) \),

\[
N_t(\phi) \equiv \langle X_t, \phi \rangle - \langle \mu, \phi \rangle - \int_0^t \langle X_s, L\phi \rangle \, ds - \int_0^t \langle X_s, \sigma_1^T \nabla \phi \rangle \, dW_s
\]  

(2.3)
is a continuous \((\mathbb{P}, \mathcal{G}_t)\)-martingale with quadratic variation process

\[
\langle N(\phi) \rangle_t = \int_0^t \langle X_s, \phi^2 \rangle \, ds
\]  

(2.4)

where \( \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_W^\infty \).

ii) If \( X_t \) is a solution to CMP, then it is a solution to MP.

As another application of the snake representation, we have

**Corollary 2.5** If \( d \geq 2 \), then \( X_t \) is singular.

Proof: If \( \mu \) is finite and \( d > 1 \), it follows from (2.2) the support is of Lebesgue measure 0 since \( \{ \hat{Y}^i_s : 0 \leq s \leq \tau_i \} \) is a continuous (one-dimensional) curve in \( \mathbb{R}^d \). If \( \mu \) is \( \sigma \)-finite, we can take \( \mu = \sum_{n=1}^\infty \mu^n \) with \( \mu^n \) finite. Construct the solution \( X^n_t \) to CMP with the same \( W \) and with initial \( \mu^n, n = 1, 2, \ldots \). Then

\[
X_t = \sum_{n=1}^\infty X^n_t
\]
is the solution to CMP with initial \( \mu \). Then \( \text{supp}(X^n_t) \) has Lebesgue measure 0 and hence, so does the support of \( X_t \). This implies that \( X_t \) is a singular measure a.s.  
\[\blacksquare\]
3 SPDE for $d = 1$

In this section, we prove that $X_t$ has a density which satisfies the SPDE (1.6) whose mild form is

$$
\langle X_t, f \rangle = \langle \mu, f \rangle + \int_0^t \langle X_s, Lf \rangle \, ds + \int_0^t \langle X_s, \sigma_1 f' \rangle \, dW_r \\
+ \int_0^t \int \sqrt{X_s(x)} f(x) B(dsdx). 
$$

(3.1)

Let $p_0(t, x, y)$ and $q_0(t, (x_1, x_2), (y_1, y_2))$ be the transition density functions of the Markov processes $\eta_1(t)$ and $(\eta_1(t), \eta_2(t))$ respectively. By Theorem 1.5 of [13], we have

$$
\mathbb{E} \left[ \langle X_t, f \rangle \right] = \int_{\mathbb{R}^2} f(y)p_0(t, x, y)dy\mu(dx) 
$$

(3.2)

and

$$
\mathbb{E} \left[ \langle X_t, f \rangle \langle X_t, g \rangle \right] 
= \int_{\mathbb{R}^4} f(y_1)g(y_2)q_0(t, (x_1, x_2), (y_1, y_2))dy_1dy_2\mu(dx_1)\mu(dx_2) \\
+ 2\int_0^t \int_{\mathbb{R}^4} p_0(t-s, z, y)f(z_1)g(z_2)q_0(s, (y, y), (z_1, z_2))dz_1dz_2dy\mu(dz). 
$$

(3.3)

**Theorem 3.1** If $\mu(\mathbb{R}) < \infty$, then $X_t \in H_0 \equiv L^2(\mathbb{R})$ a.s.

Proof: Take $f = p_0(\epsilon, x, \cdot)$ and $g = p_0(\epsilon', x, \cdot)$ in (3.3). Note that as $\epsilon, \epsilon' \rightarrow 0$,

$$
\int_{\mathbb{R}^4} p_0(\epsilon, x, z_1)p_0(\epsilon', x, z_2)p_0(t-s, z, y)q_0(t, y, y), (z_1, z_2))dz_1dz_2 \\
\rightarrow p_0(t-s, z, y)q_0(t, (y, y), (x, x)).
$$

Note that by Theorem 6.4.5 in Friedman [3], we have

$$
p_0(\epsilon, x, y) \leq c\varphi_{\epsilon'}(x-y),
$$

$$
q_0(s, (y, y), (z_1, z_2)) \leq c\varphi_{\epsilon'}(y-z_1)\varphi_{\epsilon'}(y-z_2)
$$
where $\varphi_t(x)$ is the normal density with mean 0 and variance $t$ (introduced earlier).

Note that $c'$ is a constant which is usually greater than 1. Since it does not play an essential role, to simplify the notations, we assume $c' = 1$ throughout the rest of this paper. Hence,

$$
\int_{\mathbb{R}^2} p_0(\epsilon, x, z_1)p_0(\epsilon', x, z_2)p_0(t - s, z, y)q_0(s, (y, y), (z_1, z_2))dz_1dz_2 \\
\leq c \int_{\mathbb{R}^2} \varphi_\epsilon(x - z_1)\varphi_{\epsilon'}(x - z_2)\varphi_{t-s}(z - y)\varphi_s(y - z_1)\varphi_s(y - z_2)dz_1dz_2 \\
= c\varphi_{s+\epsilon}(x - y)\varphi_{s+\epsilon'}(x - y)\varphi_{t-s}(z - y).
$$

As

$$
\lim_{\epsilon, \epsilon' \to 0} \int_0^T dt \int dx \int_0^t ds \int_{\mathbb{R}^2} \varphi_{s+\epsilon}(x - y)\varphi_{s+\epsilon'}(x - y)\varphi_{t-s}(z - y)dy\mu(dz) \\
= \lim_{\epsilon, \epsilon' \to 0} \int_0^T dt \int dx \int_0^t ds \varphi_{2s+\epsilon+\epsilon'}(0)\mu(\mathbb{R}) \\
= \int_0^T dt \int dx \int_0^t ds \varphi_{2s}(0)\mu(\mathbb{R}) \\
= \int_0^T dt \int dx \int_0^t ds \int_{\mathbb{R}^2} \varphi_{t-s}(z - y)\varphi_s(x - y)\varphi_s(x - y)dy\mu(dz),
$$

by the dominated convergence theorem, we see that as $\epsilon, \epsilon' \to 0$,

$$
\int_0^T dt \int dx \int_0^t ds \int_{\mathbb{R}^2} p_0(t - s, z, y)p_0(\epsilon, x, z_1)p_0(\epsilon', x, z_2)q_0(s, (y, y), (z_1, z_2))dz_1dz_2dy\mu(dz) \\
\to \int_0^T dt \int dx \int_0^t ds \int_{\mathbb{R}^2} p_0(t - s, z, y)q_0(t, (y, y), (x, x))dy\mu(dz).
$$

Similarly, we have

$$
\int_0^T dt \int dx \int_{\mathbb{R}^4} p_0(\epsilon, x, y_1)p_0(\epsilon', x, y_2)q_0(t, (x_1, x_2), (y_1, y_2))dy_1dy_2\mu(dx_1)\mu(dx_2) \\
\to \int_0^T dt \int dx \int_{\mathbb{R}^2} q_0(t, (x_1, x_2), (x, x))\mu(dx_1)\mu(dx_2).
$$

Hence

$$
\int_0^T dt \int dx \left( \langle X_t, p(\epsilon, x, \cdot) \rangle \langle X_t, p(\epsilon', x, \cdot) \rangle \right)
$$
\[
\int_0^T dt \int dx \int_{\mathbb{R}^2} q_0(t,(x_1,x_2),(x,x)) \mu(dx_1)\mu(dx_2)
+ \int_0^T dt \int dx \int_0^t ds \int_{\mathbb{R}^2} p_0(t-s,x,y)q_0(t,(y,y),(x,x)) dy \mu(dx).
\]

From this, we can show that \(\{\langle X_t, p_0(\epsilon,x,\cdot) \rangle : \epsilon > 0\}\) is a Cauchy sequence in \(L^2(\Omega \times [0,T] \times \mathbb{R})\). This implies the existence of the density \(X_t(x)\) of \(X_t\) in \(L^2(\Omega \times [0,T] \times \mathbb{R})\).

Next theorem considers infinite measure.

**Theorem 3.2** If \(\mu \in \mathcal{M}_{\text{tem}}(\mathbb{R}^d)\), then \(X_t\) has a density \(X_t(x)\).

Proof: If \(\mu\) is \(\sigma\)-finite, we can construct \(X^n\) with \(X_0^n = \mu^n\) being finite as those in the proof of Corollary 2.5. Then

\[
X_t = \sum_{n=1}^{\infty} X^n_t
\]

is the solution to CMP with initial \(\mu\). Let

\[
X_t(x) = \sum_{n=1}^{\infty} X^n_t(x). \tag{3.4}
\]

By (3.2), we have

\[
\mathbb{E} X^n_t(x) = \int_{\mathbb{R}} p_0(t,y,x)\mu^n(dy).
\]

As

\[
p_0(t,x,y) \leq c\varphi_t(x-y) \leq c(t,\lambda,x)e^{-\lambda|y|},
\]

for any \(\lambda > 0\), we have

\[
\mathbb{E} \sum_{n=1}^{\infty} X^n_t(x) = \sum_{n=1}^{\infty} \int_{\mathbb{R}} p_0(t,y,x)\mu^n(dy)
= \int_{\mathbb{R}} p_0(t,y,x)\mu(dy) < \infty.
\]

Hence, \(X_t(x)\) is well-defined by (3.2). It is then easy to show that \(X_t(x)dx = X_t(dx)\).
Finally, we derive the SPDE satisfied by the density.

**Theorem 3.3** If \( d = 1 \), then \( X_t \) is the (weak) unique solution to the SPDE (3.1).

Proof: Note that \( N_t(\phi) \) is a continuous \((\mathbb{P}, \mathcal{G}_t)\)-martingale with quadratic variation process

\[
\langle N(\phi) \rangle_t = \int_0^t \int_{\mathbb{R}} \left( \sqrt{X_s(x)\phi(x)} \right)^2 \, dx \, ds.
\]

By the martingale representation theorem ([4], Theorem 3.3.5), there exists an \( L^2(\mathbb{R}) \)-cylindrical Brownian motion \( \tilde{B} \) on an extension of \((\Omega, \mathcal{F}, \mathcal{G}_t, \mathbb{P})\) such that

\[
N_t = \int_0^t \left\langle \sqrt{X_s}, \, d\tilde{B}_s \right\rangle_{L^2(\mathbb{R})}.
\]

There exists a standard Brownian sheet \( B \) such that

\[
\tilde{B}_t(h) = \int_0^t \int_{\mathbb{R}} h(x) B(dsdx), \quad \forall h \in L^2(\mathbb{R}).
\]

Therefore,

\[
N_t(\phi) = \int_0^t \int_{\mathbb{R}} \sqrt{X_s(x)\phi(x)} B(dsdx).
\]

As \( B \) is a Brownian sheet on an extension of \( \mathcal{G}_t \), it is easy to show that \( B \) is independent of \( W \).

\[\blacksquare\]

### 4 Hölder Continuity

This section is devoted to the proof of the main result: Theorem 1.1 (iii). Namely, in this section, we consider the regularity of the solution to the nonlinear SPDE (1.6). We use the linearization and Krylov’s \( L_p \)-theory for linear SPDE.

We will paraphrase the condition (BC) to find some reasonable assumptions for \( \sigma_1, \sigma_2, b \) to make our regularity argument easy. Note that these functions are scalar functions since we are dealing with the situation \( d = 1 \). Therefore, we have \( L = \frac{1}{2} a \partial_{xx} + b \partial_x \) and \( L^* = \frac{1}{2} a' \partial_{xx} + (a' - b) \partial_x + (\frac{1}{2} a'' - b'). \)
We start by defining some basic spaces. We denote

\[ [f]_0 = \sup_{x \in \mathbb{R}} |f(x)|, \quad [f]_\gamma = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma} \]

for \( \gamma \in (0, 1] \). Using this notation, we define

\[ \|f\|_{C^0,\gamma} = [f]_0 + [f]_\gamma, \quad \|f\|_{C^1,\gamma} = [f]_0 + [f']_0 + [f']_\gamma \]

\[ \|f\|_{C^1} = [f]_0 + [f']_0, \quad \|f\|_{C^2} = [f]_0 + [f']_0 + [f'']_0 \]

assuming that \( f' \) or \( f'' \) exist if they appear in the corresponding definition. Then we define the Banach spaces:

\[ C^{0,\gamma} = \{ f : \|f\|_{C^{0,\gamma}} < \infty \}, \quad C^{1,\gamma} = \{ f : \|f\|_{C^{1,\gamma}} < \infty \} \]

\[ C^1 = \{ f : \|f\|_{C^1} < \infty \}, \quad C^2 = \{ f : \|f\|_{C^2} < \infty \}. \]

**Remark 4.1** Zygmund spaces \( C^{0,\gamma}, C^{1,\gamma} \) are the usual Hölder spaces if \( \gamma \in (0, 1) \). It is easy to see that we have \( \|f\|_{C^{0,\gamma}} \leq 2\|f\|_{C^{0,1}}, \|f\|_{C^{1,\gamma}} \leq 2\|f\|_{C^{1,1}} \) and \( \|f\|_{C^{0,1}} \leq \|f\|_{C^1}, \|f\|_{C^{1,1}} \leq \|f\|_{C^2} \) when \( f' \) or \( f'' \) exists.

Now, we state assumptions on \( \sigma_1, \sigma_2, b \). First, our condition (BC) gives us the following assumption:

\[ \sigma_1, \sigma_2 \in C^2, \quad b \in C^1 \quad (4.1) \]

which, in particular, implies \( a = \sigma_1^2 + \sigma_2^2 \in C^2 \). We also assume that

\[ \delta \leq \frac{1}{2} a, \quad \frac{1}{2} \sigma_2^2 \leq K, \quad \|\sigma_1\|_{C^2}, \|\sigma_2\|_{C^2}, \|b\|_{C^1} \leq K \quad (4.2) \]

for some positive constants \( \delta, K \).

Next, we recall the basic definitions of some function spaces defined in [7]. In addition to the definition about space of Bessel potentials in the Theorem 1.1, we also
define the following: For $n \in \mathbb{R}$ and $p \in [2, \infty)$ let $H^n_p(l_2)$ be the space with norm

$$\|g\|_{n,p} = \left\| (I - \Delta)^{n/2} g \right\|_{l_2}$$

for $l_2$-valued functions $g = \{g^k\}$. Then we define

$$\mathbb{H}^n_p(T) = \mathbb{L}_p(\Omega \times [0, T], \mathcal{P}, H^n_p(l_2)) \quad \text{and} \quad \mathbb{H}^n_p(T, l_2) = \mathbb{L}_p(\Omega \times [0, T], \mathcal{P}, H^n_p(l_2))$$

where $\mathcal{P}$ is the predictable $\sigma$-field. We denote $\mathbb{L}_p(T) = \mathbb{H}^0_p(T)$. Let \{w^k_t : k = 1, 2, \ldots\} be a family of independent one-dimensional Brownian motions.

We say $u \in \mathcal{H}^n_p(T)$ if $\partial_{xx} u \in \mathcal{H}^{n-2}_p(T)$ and $u(0, \cdot) \in \mathbb{L}_p(\Omega, H^{n-2/p}_p)$ and there exists $(f, g) \in \mathcal{H}^{n-2}_p(T) \times \mathcal{H}^{n-1}_p(T, l_2)$ such that $\forall \phi \in C^\infty_0(\mathbb{R})$, (a.s.)

$$\langle u_t, \phi \rangle = \langle u_0, \phi \rangle + \int_0^t \langle f_s, \phi \rangle \, ds + \sum_{k=0}^\infty \int_0^t \langle g^k_s, \phi \rangle \, dw^k_s$$

holds for all $t \leq T$. We denote

$$\|u\|_{\mathcal{H}^n_p(T)} = \|\partial_{xx} u\|_{\mathcal{H}^{n-2}_p(T)} + \|f\|_{\mathcal{H}^{n-2}_p(T)} + \|g\|_{\mathcal{H}^{n-1}_p(T, l_2)} + \left( E \|u_0\|_{\mathcal{H}^{n-2/p}_p} \right)^{1/p}$$

Reader can find motivation of this definition and detailed remarks in [7].

Now, we fix $\epsilon \in (0, \frac{1}{4})$ and proceed to the Proof of Theorem 1.1 (iii): First, we freeze the nonlinear term of SPDE (1.6) and consider the following auxiliary linear SPDE for $Y_t(x)$:

$$\left\{ \begin{array}{ll}
\partial_t Y &= L^* Y + \sqrt{X} \dot{B}_t \, x \\
Y_0 &= \mu
\end{array} \right. \quad \text{(4.3)}$$

where we assume $\mu \in H^{4-n-2/p}_p$.

Then $Z = X - Y$ satisfies

$$\left\{ \begin{array}{ll}
\partial_t Z &= L^* Z - (\partial_x (\sigma_1 Z) + \partial_x (\sigma_1 Y)) \dot{W}_t \\
Z_0 &= 0.
\end{array} \right. \quad \text{(4.4)}$$

We apply Theorem 8.5 of [7] to (4.3). To do this we need the coefficients of $L^*$ and $\sqrt{X}$ to satisfy

$$\|a\|_{C^{1.1}} < \infty, \quad \|a' - b\|_{C^{0.1}} < \infty, \quad \left[ \frac{1}{2} a'' - b' \right]_0 < \infty, \quad \|\sqrt{X}\|_{L_p(T)} < \infty.$$
In fact, we have
\[ \|a\|_{C^{1,1}} \leq K, \quad \|a' - b\|_{C^{0,1}} \leq 2K, \quad \left[ \frac{1}{2}a'' - b' \right]_0 \leq 2K \]
by our assumptions (4.1) and (4.2) and Remark 4.1. We will prove \( \|\sqrt{X}\|_{L^p(T)} < \infty \) later and take this for granted in this proof.

Now, by Theorem 8.5 of [7] to (4.3) and the first assertion of Lemma 8.4 and the fact that \( \mu \) is nonrandom, we have a unique solution \( Y \) in \( H^{\frac{1}{2} - \epsilon}(T) \) with estimate
\[ \|Y\|_{H^{\frac{1}{2} - \epsilon}(T)} \leq N(\|\sqrt{X}\|_{L^p(T)} + \|\mu\|_{\frac{1}{2} - \epsilon - 2/p,p}) \]  
(4.5)
where \( N \) depends only on \( \epsilon, p, \delta, K, T \).

Now we use Theorem 5.1 in [7] for equation (4.4) above with \( n = -\frac{3}{2} - \epsilon \in (-2, -\frac{3}{2}) \).
Note \( \partial_x(\sigma_1 Z) = \sigma_1 \partial_x Z + \partial_x \sigma_1 Z \). If we read [7] carefully, we can see that the following conditions are required:

(i) \[ \delta \leq \frac{1}{2}a - \frac{1}{2}\sigma_1^2 (= \frac{1}{2}\sigma_2^2) \leq K_1 \]
for some positive \( \delta, K_1 \).

(ii) \( a, \sigma_1 \) are Lipschitz continuous with Lipschitz constant \( K_1 \).

(iii) \( a \in C^{1,\gamma_1}, \sigma_1 \in C^{0,\gamma_2} \) for some \( \gamma_1, \gamma_2 \in (0,1) \) and \( \|a\|_{C^{1,\gamma_1}} + \|\sigma\|_{C^{0,\gamma_2}} \leq K_1 \)

(iv) \( \|a' - b\|_{C^{0,\gamma_3}} + [\frac{1}{2}(a'' - b')]_0 + [\partial_x \sigma_1]_0 \leq K_1 \) for some \( \gamma_3 \in (0,1) \).

(v) \( \partial_x(\sigma_1 Y) \in H^{n+1}_p(T) (= \mathbb{H}^{\frac{1}{2} - \epsilon}(T)) \).

But, conditions (i) through (iv) are satisfied under (4.1) and (4.2) and Remark 4.1. Note that we can take some constant multiple of \( K^2 \) as \( K_1 \). On the other hand, (v) is
also satisfied. For

\[ \| \partial_x (\sigma_1 Y) \|_{H_p^{\frac{1}{2} - \epsilon}(T)} \leq N \| \sigma_1 Y \|_{H_p^{\frac{1}{2} - \epsilon}(T)} \]  

(4.6)

\[ \leq N \| \sigma \|_{C^0} \| Y \|_{H_p^{\frac{1}{2} - \epsilon}(T)} \]  

(4.7)

\[ \leq N \| \sigma \|_{C^1} \| Y \|_{H_p^{\frac{1}{2} - \epsilon}(T)} \]  

(4.8)

\[ \leq N \| Y \|_{H_p^{\frac{1}{2} - \epsilon}(T)} \]  

(4.9)

\[ \leq N \| \sqrt{X} \|_{L_p(T)} + N \| \mu \|_{\frac{1}{2} - \epsilon/p, p} < \infty. \]  

(4.10)

(4.6) follows the observation \( \partial_x = \partial_x (I - \Delta)^{-1/2} (I - \Delta)^{1/2} \) and the boundness of the operator \( \partial_x (I - \Delta)^{-1/2} \). (4.7) follows Lemma 5.1 (i) in [7]. Up to this step, \( N \) only depends on \( \epsilon, p \). Note that \( \frac{1}{2} - \epsilon + \frac{1}{4} \) is still in \((0, 1)\) since \( \frac{1}{2} - \epsilon \in (\frac{1}{4}, \frac{1}{2}) \). Hence, we have (4.8) by (4.2) and Remark 4.1. (4.9) follows Theorem 3.7 in [7] and \( N \) depends only on \( \epsilon, p, K, T \) now. Finally, (4.5) gives us (4.11) with \( N = N(\epsilon, p, \delta, K, T) \).

Therefore, we have a unique solution \( Z \) in \( H_p^{\frac{1}{2} - \epsilon}(T) \) with

\[ \| Z \|_{H_p^{\frac{1}{2} - \epsilon}(T)} \leq N \| \partial_x (\sigma_1 Y) \|_{H_p^{\frac{1}{2} - \epsilon}(T)} \leq N \| \sqrt{X} \|_{L_p(T)} + N \| \mu \|_{\frac{1}{2} - \epsilon/2, p} \]  

(4.11)

where \( N = N(\epsilon, p, \delta, K, T) \).

Thus, combining (4.5) and (4.11), we have \( X = Y + Z \in H_p^{\frac{1}{2} - \epsilon}(T) \) with estimate

\[ \| X \|_{H_p^{\frac{1}{2} - \epsilon}(T)} \leq N \| \sqrt{X} \|_{L_p(T)} + N \| \mu \|_{\frac{1}{2} - \epsilon/2, p}. \]  

(4.12)

By the embedding Theorem 7.1 in [7], this implies

\[ \left( E \int_0^T \| X_t \|_{C^{\frac{1}{2} - \epsilon/p}}^p dt \right)^{1/p} \leq N \| \sqrt{X} \|_{L_p(T)} + N \| \mu \|_{\frac{1}{2} - \epsilon/2, p}. \]  

So, for large \( p > \frac{1}{\epsilon} \), we have

\[ \| X_t \|_{C^{\frac{1}{2} - \epsilon}} < \infty \]

for (a.e.) \( t \in [0, T] \) a.s., we are done with the proof.  

Finally, we use the moment dual to prove that

\[ \mathbb{E} \int_0^T \int_{\mathbb{R}} X(t, x)^n dx dt < \infty \]  

(4.13)

for all \( n \in \mathbb{N} \).

Let \( n_t \) be a pure-death Markov chain with \( n_0 = 0 \) and, at a rate \( \frac{1}{2} n(n - 1) \), jumps from \( n \) to \( n - 1 \). Let \( 0 = \tau_0 < \tau_1 < \cdots < \tau_{n-1} \) be the jump times. Let \( f_0 = \delta_{y}^{\otimes n} \) and for \( t < \tau_1 \), \( f_t(y) = p^n_\tau(t, (x, \cdots, x), y) \), \( \forall y \in \mathbb{R}^n \) where \( p^n_\tau \) is the transition function of the \( n \)-dimensional diffusion \((\eta_1(t), \cdots, \eta_n(t))\). For \( f \in C(\mathbb{R}^n) \), let \( G_{ij} f \in C(\mathbb{R}^{n-1}) \) be given by

\[ G_{ij} f(y_1, \cdots, y_{n-2}, y_{n-1}) = f(y_1, \cdots, y_{n-1}, \cdots, y_{n-1}, \cdots, y_{n-2}) \]

where \( y_{n-1} \) is at \( i \)th and \( j \)th position. Let

\[ f_{\tau_1} = \Gamma_1 f_{\tau_1} \]

where \( \Gamma_1 \) is a random element taking values in \( \{ G_{ij} : 1 \leq i < j \leq n \} \) uniformly. We continue this procedure to get the process \( f_t \). Replace \( f_0 \) by a smooth function \( f^k_0 \geq 0 \) approximating \( f_0 \). Denote the process constructed above with \( f^k_0 \) in place of \( f_0 \) by \( f^k_t \).

Similar to Theorem 11 in Xiong and Zhou [15], we have

\[ \mathbb{E} \langle X_t^{\otimes n}, f^k_0 \rangle = \mathbb{E} \left( \langle \mu^{\otimes n}, f_t^{k} \rangle \exp \left( \frac{1}{2} \int_0^t n_s(n_s - 1) ds \right) \right) . \]

Taking limits and using Fatou’s lemma, we have

\[ \mathbb{E} X(t, x)^n \leq \mathbb{E} \left( \langle \mu^{\otimes n}, f_t \rangle \exp \left( \frac{1}{2} \int_0^t n_s(n_s - 1) ds \right) \right) \]

\[ \leq \exp \left( \frac{1}{2} n(n - 1)t \right) \sum_{i=1}^{n} \mathbb{E} \left( \langle \mu^{\otimes n}, f_t \rangle 1_{\tau_{i-1} \leq t < \tau_i} \right) . \]

Let \( i = 3 \). Then

\[ f_t(x_1, \cdots, x_{n-2}) \leq c \int_{\mathbb{R}^{n-2}} \Pi_{i=1}^{n-2} \varphi_{t-\tau_2}(x_i - y_i) \Gamma_2 f_{\tau_2}(y) dy \]
\[
\begin{align*}
\leq c \int_{\mathbb{R}^{n-2}} & \Pi_{i=1}^{n-2} \varphi_{t-\tau_2}(x_i - y_i) \sum_{1 \leq k < \ell} \frac{2}{(n-2)(n-3)} \\
& f_{\tau_2}(y_1, \cdots, y_{n-2}, \cdots, y_{n-3})dy \\
\leq c \int_{\mathbb{R}^{n-2}} & \Pi_{i=1}^{n-2} \varphi_{t-\tau_2}(x_i - y_i) \sum_{1 \leq k < \ell} \frac{2}{(n-2)(n-3)} \\
& \int_{\mathbb{R}^{n-1}} \Pi_{j=1}^{n-1} \varphi_{\tau_2-\tau_1}(y_j - z_j) \varphi_{\tau_1}(z_1 - x) \cdots \\
& \cdots \varphi_{\tau_1}(z_{n-2} - x) \varphi_{\tau_1}(z_{n-1} - x)^2dz.
\end{align*}
\]

Thus
\[
\langle \mu^\otimes_{n-2}, f_t \rangle \leq c \int_{\mathbb{R}} \varphi_{t-\tau_2}(x_{n-2} - y_{n-2}) \mu(dx_{n-2}) \int_{\mathbb{R}} dy_{n-2} \sum_{1 \leq k < \ell} \frac{2}{(n-2)(n-3)} \varphi_{\tau_2-\tau_1}(y_{n-2} - z_k) \varphi_{\tau_2-\tau_1}(y_{n-2} - z_\ell) \\
\leq c \int_{\mathbb{R}} \varphi_{t-\tau_2}(x_{n-2} - y_{n-2}) \mu(dx_{n-2}) \int_{\mathbb{R}} dy_{n-2} \frac{1}{\sqrt{\tau_1(\tau_2 - \tau_1)}} \varphi_{\tau_2}(y_{n-2} - x).
\]

Therefore
\[
\int_{\mathbb{R}} \langle \mu^\otimes_n, f_t \rangle 1_{\tau_2 \leq t < \tau_3} dx \leq c\mu(\mathbb{R})E \frac{1}{\sqrt{\tau_1(\tau_2 - \tau_1)}} < \infty.
\]

The other terms can be proved similarly. This finishes the proof of Theorem 1.1.

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