CHARACTERS OF IRREDUCIBLE WHITTAKER MODULES FOR COMPLEX SEMISIMPLE LIE ALGEBRAS

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Abstract. Let \( g \) be a complex semisimple Lie algebra. We give a description of characters of irreducible Whittaker modules for \( g \) with any infinitesimal character, along with a Kazhdan-Lusztig algorithm for computing them. This generalizes Milicic-Soergel’s and Romanov’s results for integral infinitesimal characters.

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1. Introduction

Let \( \mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{n}, \mathfrak{h} \) be a complex semisimple Lie algebra, a Borel subalgebra, the nilpotent radical of the Borel, and a Cartan subalgebra. Let \( G \supset B \supset N, H \) denote a complex connected algebraic group with Lie algebra \( \mathfrak{g} \), and subgroups corresponding to \( \mathfrak{b}, \mathfrak{n}, \mathfrak{n} \) respectively. Let \( \Sigma \) be the root system of \( (\mathfrak{g}, \mathfrak{h}) \), and \( W \) the Weyl group of \( \Sigma \). Let \( U(\mathfrak{g}) \) be the universal enveloping algebra of \( \mathfrak{g} \), and let \( Z(\mathfrak{g}) \) be the center of \( U(\mathfrak{g}) \). By an infinitesimal character, we mean a \( \mathbb{C} \)-algebra homomorphism from \( \mathfrak{z}(\mathfrak{g}) \) to \( \mathbb{C} \). Via the Harish-Chandra homomorphism, any infinitesimal character corresponds to a unique \( W \)-orbit in \( \mathfrak{h}^* \).

The category of \textit{Whittaker modules}, denoted by \( \mathcal{N} \), is the full subcategory of all \( \mathfrak{g} \)-modules consisting of those that are finitely generated over \( \mathfrak{g} \), locally finite over \( \mathfrak{n} \), and locally finite over \( Z(\mathfrak{g}) \). Whittaker modules for real groups have been studied, for example in the work of Casselman-Hecht-Milicic [CHM00]. Over a complex semisimple Lie algebra, previous contributors include Kostant [Kos78] who studied the “non-degenerate” case, McDowell [McD85] who constructed and studied standard Whittaker modules, Milicic-Soergel who reproved Kostant’s results with geometry in [MiS14] and obtained a description of the composition factors of standard Whittaker modules and their multiplicities in the case of integral infinitesimal characters in [MiS97], and Romanov [Rom21] who established a Kazhdan-Lusztig algorithm for Whittaker modules with regular integral infinitesimal characters.

**Goal.** Generalize Milicic-Soergel’s and Romanov’s result to arbitrary infinitesimal character.

In the remaining part of this introduction, we give a brief presentation on the preliminaries on Whittaker modules and localization, explain the idea of proof, and state the main results precisely followed by an outline of the paper.

1.1. Preliminaries on Whittaker modules.** We present necessary facts on Whittaker modules without proof. References include [Kos78], [McD85], [MiS14], [Mil], and [Rom21].

If we insist that \( Z(\mathfrak{g}) \) acts on a Whittaker module via a character given by a Weyl group orbit \( \theta \), we obtain the full subcategory \( \mathcal{N}_\theta \) of \( \mathcal{N} \). If instead we require \( \xi - \eta(\xi) \) acts locally nilpotently for all \( \xi \in \mathfrak{n} \) for some character \( \eta : \mathfrak{n} \to \mathbb{C} \), we obtain \( \mathcal{N}_\eta \). Set \( \mathcal{N}_{\theta,\eta} = \mathcal{N}_\theta \cap \mathcal{N}_\eta \). Every object in \( \mathcal{N} \) has finite length, and each irreducible object is contained in one of the \( \mathcal{N}_{\theta,\eta} \)'s. Write \( \Pi \subset \Sigma^+ \subset \Sigma \) for the set of simple and positive roots defined by \( \mathfrak{b} \). Each \( \eta \) defines a subset \( \Theta \) of simple roots \( \Pi \subset \Sigma \) by

\[
\Theta = \{ \alpha \in \Pi \mid \eta|_{\mathfrak{g}_\alpha} \neq 0 \} \subseteq \Pi.
\]

Different \( \eta \)'s define different subcategories \( \mathcal{N}_{\theta,\eta} \)'s of \( \mathcal{N} \), but their categorical structures are similar whenever two \( \eta \)'s give the same \( \Theta \). If \( \Theta = \Pi \), we say that \( \eta \) is non-degenerate, in which case \( \mathcal{N}_{\theta,\eta} \) is semisimple with one irreducible object, and \( \mathcal{N}_\eta \) is equivalent to the category of finite dimensional \( Z(\mathfrak{g}) \)-modules. If \( \Theta = \emptyset \) (i.e. \( \eta = 0 \)), \( \mathcal{N}_{\theta,0} \) recovers a well-studied category containing Verma modules.

Our goal is to describe characters of irreducible objects in \( \mathcal{N}_{\theta,\eta} \) for any \( \theta \) and \( \eta \) in terms of characters of certain “standard modules” \( M(\lambda, \eta) \) which are defined by McDowell analogous to Verma modules. We will not need their precise definition. \( M(\lambda, \eta) \) admits an infinitesimal character given by the Weyl group orbit \( \theta \) of \( \lambda \). Also, \( M(\lambda_1, \eta) \cong M(\lambda_2, \eta) \) if and only if \( \lambda_1 \) and \( \lambda_2 \) are in the same \( W_\Theta \)-orbit, where \( W_\Theta \subset W \) is the parabolic subgroup defined by \( \Theta \). Therefore, if \( \lambda \in \theta \) is fixed (and will be chosen to be antidominant against \( \mathfrak{b} \)), standard modules in \( \mathcal{N}_{\theta,\eta} \) are parameterized by right \( W_\Theta \)-cosets in \( W \). For a right \( W_\Theta \)-coset \( C \), we will write \( \mathcal{w}^C \) for the unique longest element in \( C \) and write \( M(\mathcal{w}^C \lambda, \eta) \) for the corresponding standard module. McDowell showed that each \( M(\mathcal{w}^C \lambda, \eta) \) has a unique irreducible quotient denoted by \( L(\mathcal{w}^C \lambda, \eta) \), and any irreducible object arises in this way. Hence, \( L(\mathcal{w}^C \lambda, \eta) \) also has infinitesimal character given by \( \theta \) and lives in \( \mathcal{N}_{\theta,\eta} \).
Romanov developed in her thesis[Rom21, 2:2] a character theory on \( \mathcal{N}_{\vartheta,\eta} \). This is a map on objects of \( \mathcal{N}_{\vartheta,\eta} \) that factors through and is injective on the Grothendieck group \( K\mathcal{N}_{\vartheta,\eta} \). The characters of standard modules are computed explicitly in loc. cit.

1.2. Localization of Whittaker modules and twisted sheaves. The strategy of Milicic-Soergel’s, Romanov’s, and ours is to study \( \mathcal{D} \)-modules corresponding to Whittaker modules. In this subsection we introduce the localization framework related to Whittaker modules. References for facts below include [BeBe81], [BeBe93], [MiS14], [Mil], [Rom21].

Let \( X \) be the flag variety of \( \mathfrak{g} \), the variety of Borel subalgebras of \( \mathfrak{g} \). For a point \( x \in X \), the corresponding Borel subalgebra is denoted by \( \mathfrak{b}_x \). For each \( \lambda \in \mathfrak{h}^* \), Beilinson and Bernstein constructed in [BeBe81] a twisted sheaf of differential operators \( \mathcal{D}_\lambda \) on \( X \) whose global sections are equal to \( \mathcal{U}_\theta = \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})m_\theta \), where \( m_\theta \) is the maximal ideal of \( \mathcal{Z}(\mathfrak{g}) \) corresponding to the character determined by \( \theta \). Here, we use the parametrization as in [Mil], under which \( \mathcal{D}_{-\rho} = \mathcal{D}_X \). \( \lambda \in \mathfrak{h}^* \) is said to be antidual if \( \alpha^\vee(\lambda) \notin \mathbb{Z}_{>0} \) or all for all \( \alpha \in \Sigma^+ \); regular if \( \alpha^\vee(\lambda) \neq 0 \) for all \( \alpha \). If \( \lambda \) is antidominant and regular, Beilinson and Bernstein showed that taking global sections is an equivalence of categories

\[
\Gamma(X, -) : \text{Mod}_{qc}(\mathcal{D}_\lambda) \cong \text{Mod}(\mathcal{U}_\theta)
\]

between the category of quasi-coherent \( \mathcal{D}_\lambda \)-modules and the category of \( \mathcal{U}_\theta \)-modules, and a quasi-inverse is given by the localization functor \( \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} - \). If \( \lambda \) is only antidominant but not regular, \( \Gamma(X, -) \) is still exact, but some \( \mathcal{D}_\lambda \)-modules can have zero global section.

The subcategory \( \mathcal{N}_{\vartheta,\eta} \) of \( \text{Mod}(\mathcal{U}_\theta) \), under the above equivalence of categories, to the subcategory \( \text{Mod}_{\text{coh}}(\mathcal{D}_\lambda, N, \eta) \) consisting of \( \eta \)-twisted Harish-Chandra sheaves. This is the full subcategory of all coherent \( \mathcal{D}_\lambda \)-modules consisting of weakly \( N \)-equivariant \( \mathcal{D}_\lambda \)-modules such that the differential of \( N \)-action and the action of \( \eta \in \mathcal{D}_\lambda \) differ by \( \eta \). These modules are automatically holonomic.

In our notations for holonomic \( \mathcal{D} \)-modules, for a morphism \( f \) between smooth varieties, we have direct images \( f_+ \), \( f_! \) and inverse images \( f^+ \), \( f^! \). Here \( f_+ \) agrees with the one in [Bor+87, VI.5], and also agrees with the \( * \)-direct image in the usual six-functor formalism. \( f_! \) is the functor obtained by conjugating \( f_+ \) by holonomic duality. \( f^! \) agrees with the one defined in [Bor+87, VI.4]. When \( f \) is a closed immersion of a subvariety, \( H^0 f^! V \) consists of sections of \( V \) supported in the subvariety. \( f^+ \) is a shift of \( f^! \) by the relative dimension; forgetting the \( \mathcal{D} \)-module structures, \( f^+ \) agrees with the usual \( \mathcal{O} \)-module inverse image \( f^* \). \( \eta \)-twisted Harish-Chandra sheaves are functorial with respect to all these operations.

Let \( C(w) \), \( w \in W \) be the Schubert cells (i.e. \( N \)-orbits) on \( X \), with inclusion maps \( i_w : C(w) \to X \). There exist nonzero \( \eta \)-twisted Harish-Chandra sheaves on \( C(w) \) if and only if \( w \) is the longest element in the right \( W_\Theta \)-coset that contains it. If this is the case, the category \( \text{Mod}_{\text{coh}}(\mathcal{D}_{C(w)}, N, \eta) \) is semisimple, in which the unique irreducible object, denoted by \( \mathcal{O}_{C(w)}^\eta \), has \( \mathcal{O}_{C(w)} \) as the underlying structure of an \( N \)-equivariant \( \mathcal{O}_{C(w)} \)-module, but with an \( \eta \)-twisted \( \mathcal{D}_{C(w)} \)-action. We call the \( \mathcal{D} \)-module direct images

\[
\mathcal{I}(w, \lambda, \eta) = i_{w, +} \mathcal{O}_{C(w)}^\eta, \quad \mathcal{M}(w, \lambda, \eta) = i_{w, !} \mathcal{O}_{C(w)}^\eta
\]

the standard module and the costandard module, respectively. The standard module \( \mathcal{I}(w, \lambda, \eta) \) contains a unique irreducible submodule, denoted by \( \mathcal{L}(w, \lambda, \eta) \), and \( \mathcal{I}(w, \lambda, \eta) \) is the unique irreducible quotient of \( \mathcal{M}(w, \lambda, \eta) \). These exhaust all irreducible objects in \( \text{Mod}_{\text{coh}}(\mathcal{D}_\lambda, N, \eta) \). Romanov showed that if \( w \) is the longest element in a right \( W_\Theta \)-coset and \( \lambda \) is antidominant, Beilinson-Bernstein’s equivalence of categories sends \( \mathcal{M}(w, \lambda, \eta) \) to \( \mathcal{M}(w, \lambda, \eta) \) to either \( L(w, \lambda, \eta) \) or 0. If \( \lambda \) is furthermore regular, \( \mathcal{L}(w, \lambda, \eta) \) is always sent to \( L(w, \lambda, \eta) \). This allows us to study Whittaker modules using geometry on \( X \).

For certain reasons, \( \mathcal{I}(w, \lambda, \eta) \) is more convenient to work with than \( \mathcal{M}(w, \lambda, \eta) \). The holonomic duality \( \mathbb{D} \) sends \( \mathcal{I}(w, \lambda, \eta) \) and \( \mathcal{L}(w, \lambda, \eta) \) to \( \mathcal{M}(w, -\lambda, \eta) \) and \( \mathcal{L}(w, -\lambda, \eta) \), respectively. So we have
the following flowchart
\[
\begin{align*}
N_{\theta, \eta} & \xrightarrow{D_\lambda \otimes u_\theta} \text{Mod}_{\text{coh}}(D_\lambda, N, \eta) \xrightarrow{D} \text{Mod}_{\text{coh}}(D_{-\lambda}, N, \eta), \\
L(w^C, \lambda, \eta) & \mapsto L(w^C, \lambda, \eta) \mapsto L(w^C, -\lambda, \eta),
\end{align*}
\]
Modules in \(\text{Mod}_{\text{coh}}(D_{-\lambda}, N, \eta)\) are all holonomic and have finite length. Hence the set of irreducible objects form a basis for the Grothendieck group \(K\text{Mod}_{\text{coh}}(D_{-\lambda}, N, \eta)\). A standard argument using pullback-pushforward adjunctions shows that the set of standard modules also form a basis for \(K\text{Mod}_{\text{coh}}(D_\lambda, N, \eta)\). Therefore, our goal of finding coefficients of \(\text{ch}\, M(w^D, \lambda, \eta)\) in \(\text{ch}\, L(w^C, \lambda, \eta)\) is translated to finding the change of bases matrix. Of course, the special case of \(\eta = 0\) corresponds to Verma modules and has already been worked out.

In the rest of the paper (particularly in §3 and §4), we will use facts about \(D\)-modules without citing references, including the triangle for local cohomology, base change theorem, and Kashiwara’s equivalence of categories for closed immersions. These facts are contained in [Bor+87], IV.8.3, 8.4 and 7.11, respectively.

1.3. Idea of the algorithm. The character formula for irreducible Whittaker modules follows from a Kazhdan-Lusztig algorithm, the proof of which is in the same spirit as the algorithm for irreducible highest weight modules which we recall now. In the highest weight case (Verma case), the Grothendieck group of the category \(\text{Mod}_{\text{coh}}(D_\lambda, N)\) (with \(\eta = 0\)) has two natural bases given by standard modules and irreducible modules, respectively, both labeled by the Weyl group \(W\). The Hecke algebra \(H\) of \(W\) with \(\mathbb{Z}[q^{\pm 1}]\)-coefficients also have two bases: the defining basis \(\delta_w\)'s and the Kazhdan-Lusztig basis \(C_w\)'s. The Kazhdan-Lusztig basis is characterized by two conditions: 1) the expansion of \(C_w\) in terms of the \(\delta_v\)'s involve only those with \(v \leq w\) and the coefficient of \(\delta_w\) is 1, and 2) the product \(\delta_s C_w\), where \(s\) is a simple reflection so that \(\ell(ws) > \ell(w)\), is a \(\mathbb{Z}\)-linear combination of \(C_v\)'s, \(v \leq w\).s. These conditions inductively determine the Kazhdan-Lusztig basis and provides a recursive algorithm for computing it. To relate the geometric picture to \(H\), one constructs a comparison map from \(\text{Mod}_{\text{coh}}(D_\lambda, N, \eta)\) to \(H\) sending standard modules to the \(\delta_w\)'s and descends to a map on the Grothendieck group. The map is defined by sending \(F\) to a linear combination where the coefficient of \(q^p \delta_w\) is the \(p\)-th higher inverse image of \(F\) to the Schubert cell \(C(w)\). It is therefore to show that multiplication by \(\delta_s\) on \(H\) lifts to a geometric operation on the geometric side, and that the irreducible \(D\)-modules satisfy (1) and (2). (1) is automatically satisfied by construction of our comparison map. Multiplication of \(\delta_s\) is lifted to the “push-pull” operation along the natural map \(X \to X_s\) to the \(s\)-partial flag variety. This operation is a geometric version of Vogan’s \(U\)-functor in [Vog79]. The fact that (2) is satisfied on the geometric side follows from Decomposition Theorem of Beilinson, Bernstein and Deligne and is worked out by Milicic in his unpublished notes [Mil]. By an induction on \(\ell(w)\), our map sends the class of \(L(w, \lambda)\) to \(C_w\).

Details of this approach can be found in op. cit.

There are two main changes in the case of Whittaker modules with integral infinitesimal character. First, standard and irreducible \(D_\lambda\)-modules are now labeled by right \(W_\Theta\)-cosets. Based on the result of Milicic-Soergel, the correct replacement for the full Hecke algebra \(H\) is the parabolic Hecke algebra \(H_{\Theta\Theta}\), where a set of basis is now labeled by \(W_\Theta\backslash W\). The Kazhdan-Lusztig polynomials (the coefficients of \(\delta_v\) in \(C_w\)) are replaced by parabolic Kazhdan-Lusztig polynomials defined in [Soe97]. Second, the \(D_\lambda\)-modules at hand are no longer regular holonomic (merely holonomic). Therefore a decomposition theorem for general holonomic modules is needed and is proven by Mochizuki [Moc11]. Romanov adapted the strategy for Verma modules to the case of Whittaker modules in her thesis [Rom21] and obtained an algorithm.

The work of this paper generalizes Romanov’s algorithm to arbitrary infinitesimal characters. There are two extra complications compared to the case of integral infinitesimal characters. For one, although standard and irreducible modules are still parameterized by \(W_\Theta\backslash W\), our category
could be divided into smaller blocks, as is the case for Verma modules with non-integral infinitesimal characters. It was not clear what the replacement is for $\mathcal{H}_\Theta$ until it was found later by staring at examples: one should take a direct sum of various parabolic Hecke algebras $\mathcal{H}_{\Theta(u,\lambda)}$ of the integral Weyl group $W_\lambda$, where the sets of simple reflections $\Theta(u,\lambda)$ in $W_\lambda$ are determined by looking at the double cosets $W_\Theta \backslash W / W_\lambda$. For another, the “push-pull” operation along $X \rightarrow X_s$ does not exist when $\lambda$ is non-integral to $s$ – there is no sheaf of twisted differential operators on $X_s$ that pulls back to $D_\lambda$. As a result, induction on $\ell(w)$ cannot proceed as before. To remedy this, we use intertwining functor $I_s$ for non-integral $s$ in place of the $U$-functor. It is an equivalence of categories between $\mathcal{D}_\lambda$-modules and $\mathcal{D}_{s\lambda}$-modules. This allows us to increase $\ell(w)$ and at the same time translate induction hypothesis on the block of $\mathcal{D}_\lambda$-modules we are dealing with to a block of $\mathcal{D}_{s\lambda}$-modules without loosing any information. This idea of proof is suggested to me by Milicic.

Together with necessary modification and supplements on details, this proves the algorithm for Whittaker modules for regular infinitesimal characters. Extending to singular characters can be done without much difficulty.

1.4. The main results. To state the theorem, we need to introduce more notation. Recall that $\Pi \subset \Sigma^+ \subset \Sigma$ is the set of simple roots and the set of positive roots defined by $\mathfrak{b}$, $\Theta = \{ \alpha \in \Pi \mid \eta |_{\theta_\alpha} \neq 0 \}$, and $W_\Theta \subseteq W$ is the parabolic subgroup corresponding to $\Theta$. Let $\lambda \in \mathfrak{h}^*$ be an antidominant element in $\theta$. Let $W_\lambda = \{ w \in W \mid w\lambda - \lambda \in \mathbb{Z} \cdot \Sigma \}$, the integral Weyl group of $\lambda$, identified with the Weyl group of the integral root subsystem $\Sigma_\lambda = \{ \alpha \in \Sigma \mid \alpha(\lambda) \in \mathbb{Z} \} \subseteq \Sigma$. Finally, we take a set of positive roots in $\Sigma_\lambda$ by $\Sigma^+ = \Sigma_\lambda \cap \Sigma^+$, and let $\Pi_\lambda \subseteq \Sigma^+_\lambda$ be the corresponding set of simple roots. Note that roots in $\Pi_\lambda$ may not be simple in $\Sigma^+$. Recall also that $C$, $D$ denote right $W_\Theta$-cosets in $W$, and $w^C$, $w^D$ denote the unique longest elements in the cosets.

For $M(w^D\lambda,\eta)$ to appear in the character of $L(w^C\lambda,\eta)$, $D$ and $C$ must in the same $(W_\Theta, W_\lambda)$-double coset. Suppose $C$ is contained in a double coset $W_\Theta u W_\lambda$, where $u$ is the unique shortest element in this double coset (which exists by 2.3.3). Take the intersections of $uW_\lambda$ with various right $W_\Theta$-cosets in $W_\Theta u W_\lambda$. This is a partition of $uW_\lambda$. Left-translating back into $W_\lambda$, we obtain a partition of $W_\lambda$, which coincides with the partition given by right cosets of the parabolic subgroup $W_{\lambda,\Theta(u,\lambda)}$ of $W_\lambda$ corresponding to the subset of simple roots $\Theta(u,\lambda) = u^{-1} \Sigma_\Theta \cap \Pi_\lambda \subseteq \Pi_\lambda$ (2.4.2). This defines a map from the set of right $W_\Theta$-cosets in $W_\Theta u W_\lambda$ to the set of right $W_{\lambda,\Theta(u,\lambda)}$-cosets in $W_\lambda$, i.e. from $W_\Theta \backslash W_\Theta u W_\lambda$ to $W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$. We denote this map by $(-)|_\lambda$ (2.4.4). We say that $W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$ is the integral model for $W_\Theta \backslash W_\Theta u W_\lambda$ since, as will be apparent from the main theorem, $W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$ controls the characters of $L(w^C\lambda,\eta)$ for $C \in W_\Theta \backslash W_\Theta u W_\lambda$. Recall that there is a partial order $\leq$ on $W_\Theta$ inherited from the restriction of Bruhat order to the set of longest element in each coset. We denote the partial order on $W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$ by $\leq_{u,\lambda}$. Then $M(w^D\lambda,\eta)$ appears in the character of $L(w^C\lambda,\eta)$ only if $D$ is contained in $W_\Theta u W_\lambda$, and $D|_\lambda \leq_{u,\lambda} C|_\lambda$ (for which we will simply write $D \leq_{u,\lambda} C$).

The precise coefficient of $\text{ch} M(w^D\lambda,\eta)$ in $\text{ch} L(w^C\lambda,\eta)$ is described by parabolic Kazhdan-Lusztig polynomials for $(W_\lambda, \Pi_\lambda, \Theta(u,\lambda))$ (see §4.1 for the precise definition of these polynomials and a comparison with the version defined in [So97]). These are polynomials $P_{EF}^{u,\lambda}$ in a variable $q$ indexed by pairs of right $W_{\lambda,\Theta(u,\lambda)}$-cosets $(E, F)$ with $F \leq_{u,\lambda} E$. In particular, we have the polynomials $P_{CD}^{u,\lambda} = P_{C|_\lambda, D|_\lambda}^{u,\lambda}$. The character formula for regular infinitesimal characters is the following.

**Theorem** (Regular case, 5.1.2). Let $\theta$ be a regular Weyl group orbit in $\mathfrak{h}^*$, and let $\lambda \in \theta$ be antidominant. For any $C \in W_\Theta \backslash W$, let $W_\Theta u W_\lambda$ be the double $(W_\Theta, W_\lambda)$-coset containing $C$, where $u$ is the unique shortest element in this double coset. Then

$$\text{ch} L(w^C\lambda,\eta) = \sum_{D \leq_{u,\lambda} C} P_{CD}^{u,\lambda}(-1)\text{ch} M(w^D\lambda,\eta),$$

where $X^\vee$ is a variable.
where \( ch \) is the character map, and the \( P_{CD}^{u,\lambda} \)'s are parabolic Kazhdan-Lusztig polynomials for \((W_\lambda, \Pi_\lambda, \Theta(u, \lambda))\) as defined in 4.1.2.

The formula for singular infinitesimal characters (5.2.5) can be obtained with minor modifications. By standard arguments, this result can be viewed as coefficients of the change of bases matrix in the Grothendieck group \( KN_{\theta,\eta} \) of \( N_{\theta,\eta} \), which can be inverted. When \( \lambda \) is integral, we recover the description of multiplicities of irreducible modules in standard modules proven by Milicic-Soergel and Romanov.

As is mentioned earlier, the character formula follows from a recursive algorithm for computing Kazhdan-Lusztig polynomials for twisted Harish-Chandra sheaves and a comparison with the \( P_{CD}^{u,\lambda} \)'s. The precise algorithm can be found in §4.2.

1.5. Outline of the paper. The paper is organized as follows. §2 are devoted to studying the structure of left \( W_\lambda \)-cosets and double \((W_\Theta, W_\lambda)\)-cosets in the Weyl group. These results (some of which are probably known) reflect the block structure of \( \text{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta) \). In §3 we study the effect of non-integral intertwining functors on irreducible \( \mathcal{D} \)-modules. §4 contains the statement and the proof of the algorithm. The character formula is established in §5. Lastly, in §6, we provide an example on the \( A_3 \) root system.

1.6. Acknowledgements. I would like to thank Milicic for providing suggestions on the algorithm, as well as for his mentorship, guidance, and valuable discussions. I also want to thank Romanov for helpful conversations.

2. Double cosets in the Weyl group

In this section, we collect some known results on the integral root subsystem, and examine the structure of double \((W_\Theta, W_\lambda)\)-cosets in \( W \). Most results on here are either known or not hard. We include most of the proofs for completeness.

In §2.1 we define a cross-section of \( W/W_\lambda \) and examine the restriction of Bruhat order to each coset. §2.2 sets notations and collects some known facts on \( W_\Theta \backslash W_\lambda \). In §2.3, we construct a cross-section \( A_{\Theta,\lambda} \) of \( W_\Theta \backslash W/W_\lambda \) consisting of the unique shortest elements in each double coset (2.3.3). Next, we show in §2.4 that, if one looks at the partition of \( W_\Theta \backslash W \) given by double cosets \( W_\Theta \backslash W/W_\lambda \), then each block in this partition corresponds to a right coset in \( W_\lambda \) of a parabolic subgroup of \( W_\lambda \), called the “integral model” for this double coset. As mentioned in §1, the parabolic Kazhdan-Lusztig polynomials for \((W_\lambda, \Pi_\lambda)\) with respect to this parabolic subgroup describes the multiplicities of Whittaker modules indexed by right \( W_\Theta \)-cosets in this double coset. Lastly, in §2.5, we prove a lemma which enables a key induction step in §4.6.

Recall that \( \lambda \in h^* \), \( \Sigma_\lambda = \{ \alpha \in \Sigma \mid \alpha^\vee(\lambda) \in \mathbb{Z} \} \) is subsystem of integral roots, \( W_\lambda = \{ w \in W \mid w\lambda - \lambda \in \mathbb{Z} \cdot \Sigma \} \) is its Weyl group. \( \Sigma_\lambda^+ = \Sigma^+ \cap \Sigma_\lambda \) is the set of positive roots and \( \Pi_\lambda \subseteq \Sigma_\lambda^+ \) the set of simple roots. Write \( \leq_\lambda \) for the Bruhat order on \( W_\lambda \) determined by \( \Pi_\lambda \). \( \Theta \) is a subset of \( \Pi \), \( \Sigma_\Theta \subseteq \Sigma \) is the subsystem generated by \( \Theta \) and \( W_\Theta \subset W \) is the Weyl group of \( \Sigma_\Theta \).

2.1. Left \( W_\lambda \)-cosets and Bruhat order. For any \( u \in W \), define the set

\[
\Sigma_u^\lambda = \{ \alpha \in \Sigma^+ \mid u\alpha \in -\Sigma^+ \} = \Sigma^+ \cap (-u^{-1}\Sigma^+),
\]

i.e. the set of positive roots \( \alpha \) so that \( u\alpha \) is not positive. It is well known that

\[
A_\lambda = \{ u \in W \mid \Sigma_u^\lambda \cap \Sigma_\lambda = \emptyset \}
\]

is a cross-section of \( W/W_\lambda \). \( \Sigma_\lambda \), \( W_\lambda \) and \( A_\lambda \) satisfy the following elementary properties.

Lemma 2.1.1. Let \( \beta \) be a simple root and let \( u \in W \).

(a) \( u\Sigma_\lambda = \Sigma_{u\lambda} \);

(b) if \( u \in A_\lambda \), \( u\Sigma_\lambda^+ = \Sigma_{u\lambda}^+ \).
(c) if \( u \in A_\lambda \), \( u\Pi_\lambda = \Pi_{u\lambda} \);
(d) \( uW_\lambda u^{-1} = W_{u\lambda} \);
(e) if \( s_\beta \in A_\lambda \) and \( u \in A_\lambda \), \( us_\beta \in A_{s_\beta \lambda} \);
(f) \( s_\beta \in A_\lambda \) if and only if \( \beta \in \Pi - \Pi_\lambda \).

In particular, (c) and (d) implies that conjugation by \( u \in A_\lambda \) sends simple reflections in \( W_\lambda \) to simple reflections in \( W_{u\lambda} \). This implies:

**Corollary 2.1.2.** Let \( u \in A_\lambda \). Then conjugation by \( u \) is an isomorphism of posets

\[
(W_\lambda, \leq_\lambda) \simto (W_{u\lambda}, \leq_{u\lambda}).
\]

We want to show that \( A_\lambda \) consists of unique shortest elements in left cosets. We in fact have a stronger statement: left multiplication by an element in \( A_\lambda \) is a map from \( W_\lambda \) to \( W \) that preserves the Bruhat orders.

**Lemma 2.1.3.** Let \( w, s_\alpha \in W_\lambda \) with \( \alpha \in \Sigma^+_\lambda \), and let \( u \in A_\lambda \). Suppose \( s_\alpha w \leq_\lambda w \). Then \( us_\alpha w < uw \).

**Proof.** An inequality in \( W \) with respect to Bruhat order can be checked by a regular antidominant integral weight. That is, if \( \mu \) is such a weight in \( \mathfrak{h}^* \), then \( us_\alpha w < uw \) if and only if \( us_\alpha \mu < uw\mu \), where the second inequality means that \( uw\mu - us_\alpha w\mu \) is nonzero and is a non-negative sum of simple roots. Similarly, if \( \mu \) is the projection to span \( \Sigma_\lambda \) along the ker \( \alpha \)'s, \( \alpha \in \Sigma_\lambda \), then \( s_\alpha w \leq_\lambda w \) if and only if \( s_\alpha w\mu \leq_\lambda w\mu \).

Therefore, if we write \( \nu = \mu - \bar{\mu} \),

\[
s_\alpha w \leq_\lambda w \iff s_\alpha w\mu \leq_\lambda w\mu \iff s_\alpha w\mu + \sum_{\alpha_i \in \Pi_\lambda} a_i\alpha_i = w\mu \text{ for some } a_i \in \mathbb{Z}_{\geq 0} \text{ not all zero} \]

\[
\iff s_\alpha w\mu + \nu + \sum_{\alpha_i \in \Pi_\lambda} a_i\alpha_i = w\mu + \nu \text{ for some } a_i \in \mathbb{Z}_{\geq 0} \text{ not all zero} \]

\[
\iff s_\alpha w\mu + \sum_{\alpha_i \in \Pi_\lambda} a_i\alpha_i = w\mu \text{ for some } a_i \in \mathbb{Z}_{\geq 0} \text{ not all zero}
\]

where the last step is because \( \nu \) is annihilated by all coroots in \( \Sigma_\lambda^\vee \). Applying \( u \) to both sides we get

\[
us_\alpha w\mu + \sum_{\alpha_i \in \Pi_\lambda} a_i u\alpha_i = uw\mu \text{ for some } a_i \geq 0 \text{ not all zero}
\]

which implies \( us_\alpha w\mu < uw\mu \), by the fact that \( u\alpha_i \in u\Sigma_\lambda^+ \subseteq \Sigma^+ \). Thus \( us_\alpha w < uw \) as desired.  

**Corollary 2.1.4.** Let \( v, w \in W_\lambda \) and \( v \leq_\lambda w \). Then for any \( u \in A_\lambda \), \( uw \leq uw \).

**Proof.** If equality holds, then the statement is trivial. Otherwise, by the definition of Bruhat order, there exist \( \alpha_1, \ldots, \alpha_k \in \Sigma_\lambda^+ \) such that

\[
v = s_{\alpha_k} \cdots s_{\alpha_1} w \leq_\lambda \cdots \leq_\lambda s_{\alpha_1} w <_\lambda w.
\]

Apply 2.1.3 to each inequality, we see

\[
uw = us_{\alpha_k} \cdots s_{\alpha_1} w < \cdots < us_{\alpha_1} w < uw
\]

as desired.

**Corollary 2.1.5.** For any \( u \in A_\lambda \), \( u \) is the unique shortest element in \( uW_\lambda \) with respect to the restriction of Bruhat order to \( uA_\lambda \).

The next lemma is analogous to a similar statement for parabolic subgroups (2.2.3), which we will need in a few occasions. The proof is a standard argument using the lifting property [BjBr05, 2.2.7].
Lemma 2.1.6. Let $\alpha \in \Pi$, and $u \in A_\lambda$. Then either $s_\alpha u \in A_\lambda$, or $s_\alpha u \in u W_\lambda$.

Proof. Suppose $s_\alpha u \notin u W_\lambda$. Then $s_\alpha u$ is in a different left $W_\lambda$-coset, i.e. $s_\alpha u = rv \in r W_\lambda$ for some $v \in W_\lambda$ and $r \in A_\lambda$ with $r \neq u$. So there exists some $v \in W_\lambda$ such that $s_\alpha u = rv$. We need to show that $v = 1$.

Write $w_1 \prec w_2$ when $w_1 < w_2$ and $l(w_1) = l(w_2) - 1$. From the relation $s_\alpha u = rv$, either $rv \prec u$ or $rv \succ u$. Also $s_\alpha u v^{-1} = r$, so either $r \prec u v^{-1}$ or $r \succ u v^{-1}$. From 2.1.5, we also know $r \preceq rv$ and $u \preceq u v^{-1}$. We have the following four possibilities.

(a) \[ r \succ u v^{-1} \]\[
\begin{array}{c}
\preceq \\
\prec \\
\end{array}
\]
\[ rv \prec u \]
is impossible since it implies $u > u$.

(b) \[ r \succ u v^{-1} \]\[
\begin{array}{c}
\preceq \\
\prec \\
\end{array}
\]
\[ rv \succ u \]
violates $rv \succ u$. Therefore we must have $rv = r$ and hence $v = 1$.

(c) \[ r \preceq u v^{-1} \]\[
\begin{array}{c}
\preceq \\
\prec \\
\end{array}
\]
\[ rv \prec u \]
Same argument as in (b) shows that $v = 1$.

(d) \[ r \preceq u v^{-1} \]\[
\begin{array}{c}
\preceq \\
\prec \\
\end{array}
\]
\[ rv \succ u \]
Let $k = l(u v^{-1}) - l(u)$. Then
\[
l(rv) \geq l(r) = l(u v^{-1}) - 1 \\
= l(u) + k - 1 \\
= l(rv) - 1 + k - 1 \\
= l(rv) + k - 2
\]
and $0 \leq k \leq 2$. If $k = 2$, then $l(r) = l(rv)$ and $v = 1$. If $k = 0$, then $l(u) = l(u v^{-1})$ and $v = 1$. Suppose $k = 1$. Applying the lifting property twice, we see $r \preceq u$ and $u \preceq r$. So $r = u$, contradicting our assumption for $r$. Therefore we must have $v = 1$.

Thus $v = 1$ in all cases, as desired.

2.2. Notations and preliminaries on $W_\Theta \backslash W$. We recall some well-known facts of right $W_\Theta$-cosets and partial orders without proof.

Write $w_\Theta \in W_\Theta$ for the longest element. The set
\[
\Theta W = \{ w \in W \mid w^{-1} \Theta \subseteq -\Sigma^+ \}
\]
is a cross-section of $W_\Theta \backslash W$ consisting of the longest elements in each coset, and
\[
w_\Theta \Theta W = \{ w \in W \mid w^{-1} \Theta \subseteq \Sigma^+ \}
\]
is a cross-section consisting of the shortest elements in each coset. For a right $W_\Theta$-coset $C$, we write $w^C$ for the corresponding element in $\Theta W$. The restriction of Bruhat order on the set $\Theta W$ defines a partial order $\preceq$ on $W_\Theta \backslash W$. We will use the phrase “the length of $C$” to refer to the length of the element $w^C$. If $\Theta(c, \lambda)$ is a subset of $\Pi_\lambda$ defining the parabolic subgroup $W_{\lambda, \Theta(c, \lambda)} \subseteq W_\lambda$, we write “$\preceq_{a, \lambda}$” for the partial order on $W_{\lambda, \Theta(c, \lambda)} \backslash W_\lambda$.

The following facts will be used throughout this section.

Lemma 2.2.2. Any element in $W_\Theta$ permutes positive roots outside $\Sigma_\Theta^+$, that is, it permutes the set $\Sigma^+ - \Sigma_\Theta^+$.

Lemma 2.2.3. Let $C$ be a right $W_\Theta$-coset and $\alpha \in \Pi$. Then exactly one of the following happens.
2.3.1. Let \( C \) be the largest right \( \Theta \)-coset. In this case \( w^C s_\alpha = w^C s_\alpha \), and for any \( w \in C \), \( w s_\alpha = w \).

(b) \( C s_\alpha = C \).

(c) \( C s_\alpha < C \). In this case \( w^C s_\alpha = w^C s_\alpha \), and for any \( w \in C \), \( w s_\alpha < w \).

Moreover, the identity coset \( W_\Theta \) is the only right \( W_\Theta \)-coset \( C \) such that \( C s_\alpha \geq C \) for all \( \alpha \in I \).

**Corollary 2.2.4.** Let \( C, D \) be two right \( W_\Theta \)-cosets. Let \( v \in D, w \in C \). If \( v \leq w \), then \( D \leq C \).

### 2.3. A cross-section of \( W_\Theta \backslash W/W_\lambda \)

Define the set \( A_{\Theta, \lambda} = A_\lambda \cap (w_\Theta^\Theta W) \). We will show (in 2.3.3) that this is a cross-section of \( W_\Theta \backslash W/W_\lambda \) consisting of the unique shortest elements in each double coset. Later results, as well as the main theorem of the paper, will often be formulated using this set.

**Lemma 2.3.1.** Let \( u, r \in A_\lambda \). Suppose \( u \) and \( r \) are in the same \( (W_\Theta, W_\lambda) \)-coset. Then \( u \) and \( r \) are contained in the same right \( W_\Theta \)-coset.

**Proof.** The case \( u = r \) is trivial. Assume \( u \neq r \). By assumption, \( r = w uv^{-1} \) for some \( w \in W_\Theta \) and \( v \in W_\lambda \). We will do induction on \( \ell(w) \).

Consider the case \( \ell(w) = 1 \). Then \( w = s_\alpha \) for some \( \alpha \in \Theta \), and \( s_\alpha u = rv \). By 2.1.6 \( v = 1 \). Hence \( r = s_\alpha u \in W_\Theta u \) which is in the same right \( W_\Theta \)-coset as \( u \).

Consider \( \ell(w) = k > 1 \). Write \( w = s_\alpha w' \) for some \( \alpha \in \Theta \) and some \( w' \in W_\Theta \) so that \( \ell(w) = \ell(w') + 1 \). Then \( w'u' = (s_\alpha u)v \). We have two possibilities.

(a) \( s_\alpha r \in rW_\lambda \). Then there exists \( u' \in W_\lambda \) such that \( s_\alpha r = u' v \), so \( w'u(u'v)^{-1} = r \). Since \( \ell(w') \leq k - 1 \), by the induction assumption, \( u \) and \( r \) are in the same right \( W_\Theta \)-coset.

(b) \( s_\alpha r \notin rW_\lambda \). Then by 2.1.6, \( s_\alpha r \in A_\lambda \). From the equation \( w'u(u'v)^{-1} = s_\alpha r \), \( \ell(w') \leq k - 1 \) and the induction assumption, we see that \( u \) and \( s_\alpha r \) are in the same right \( W_\Theta \)-coset. Since \( s_\alpha r \) and \( r \) are in the same right \( W_\Theta \)-coset, so are \( u \) and \( r \).

**Proposition 2.3.2.** Consider any double coset \( W_\Theta wW_\lambda \) in \( W \).

(a) \( W_\Theta wW_\lambda \) contains a unique smallest right \( W_\Theta \)-coset \( C \), in the sense that \( C \leq C' \) for any \( C' \in W_\Theta \backslash W_\Theta wW_\lambda \).

(b) \( A_\lambda \cap (W_\Theta wW_\lambda) \subseteq C \).

(c) The unique shortest element in \( C \) is in \( A_\lambda \).

**Proof.** By 2.3.1, there exists a right \( W_\Theta \)-coset \( C \), contained in \( W_\Theta wW_\lambda \), such that \( A_\lambda \cap (W_\Theta wW_\lambda) \subseteq C \). Let \( y \) be the unique shortest element in \( C \). Say \( y \in wW_\lambda \) for some \( w \in A_\lambda \). Then \( u \leq y \) by 2.1.5. If \( y \neq u \), we will have \( u < y \), and hence by minimality of \( y \), \( u \) is in a different right \( W_\Theta \)-coset than \( y \), contradicting the construction of \( C \). Hence we must have \( y = u \), i.e., the unique shortest element in \( C \) is in \( A_\lambda \). Lastly, for any other right \( W_\Theta \)-coset \( C' \) in \( W_\Theta wW_\lambda \), let \( y' \) be its unique shortest element. \( y' \) is contained in one of the left \( W_\lambda \)-cosets, say \( y' \in u'W_\lambda \) for some \( u' \in A_\lambda \). Then \( u' \leq y' \) by 2.1.5. Also \( u' \neq y' \) (otherwise \( C' \ni y' = u' \in C \) which would imply \( C = C' \)). Hence \( u' < y' \). Therefore \( C < C' \) by 2.2.4. Thus \( C \) is the unique smallest right \( W_\Theta \)-coset in \( W_\Theta wW_\lambda \).

**Corollary 2.3.3.** \( A_{\Theta, \lambda} = A_\lambda \cap (w_\Theta^\Theta W) \) is a cross-section of \( W_\Theta \backslash W/W_\lambda \) consisting of the unique shortest elements in each double coset. For each \( u \in A_{\Theta, \lambda} \), \( W_\Theta u \) is the unique smallest right \( W_\Theta \)-coset in \( W_\Theta wW_\lambda \).

**Proof.** Take any double coset \( W_\Theta wW_\lambda \). By 2.3.2(c), if we take the shortest element \( u \) in the smallest right \( W_\Theta \)-coset in this double coset, then \( u \in A_\lambda \). Hence \( u \in A_\lambda \cap (w_\Theta^\Theta W) \). On the other hand, by 2.3.2(b), any other right \( W_\Theta \)-coset in \( W_\Theta wW_\lambda \) has empty intersection with \( A_\lambda \). Therefore \( A_\lambda \cap (w_\Theta^\Theta W) \cap W_\Theta wW_\lambda = \{u\} \). This shows that \( A_\lambda \cap (w_\Theta^\Theta W) \) is a cross-section.

### 2.4. Integral models

By results of the previous subsection, for each double coset \( W_\Theta uW_\lambda \) one can choose \( u \) to be in \( A_{\Theta, \lambda} \). Then \( uW_\lambda \) is contained in \( W_\Theta uW_\lambda \) that intersects with different right \( W_\Theta \)-cosets. It will turn out that these intersections produce a parabolic subgroup in \( W_\lambda \), and
the parabolic Kazhdan-Lusztig polynomials that arise determine the coefficients in the character formula.

**Lemma 2.4.1.** Let \( u \in A_{\Theta, \lambda} \). Then \( \Sigma_\Theta \cap \Pi_{u, \lambda} \) is a set of simple roots for the root system \( \Sigma_\Theta \cap \Sigma_{u, \lambda} \).

**Proof.** Let \( \beta \in \Sigma_\Theta \cap \Sigma_{u, \lambda} \). Write \( \beta \) as a \( \mathbb{Z}_{\geq 0} \)-linear combination in terms of reflections of roots in \( \Pi_{u, \lambda} \). If one of the summands is from \( \Pi_{u, \lambda} - \Sigma_\Theta \), then writing \( \beta \) as a sum of reflections of roots in \( \Pi \), there is a summand that comes from \( \Pi - \Theta \). This implies \( \beta \notin \Sigma_\Theta \), a contradiction. Hence \( \beta \) is a sum of reflections of roots from \( \Sigma_\Theta \cap \Pi_{u, \lambda} \). Therefore \( \Sigma_\Theta \cap \Pi_{u, \lambda} \) spans \( \Sigma_\Theta \cap \Sigma_{u, \lambda} \). Since \( \Sigma_\Theta \cap \Pi_{u, \lambda} \) is a subset of simple roots in \( \Sigma_{u, \lambda} \), roots contained in \( \Sigma_\Theta \cap \Pi_{u, \lambda} \) remain simple in \( \Sigma_\Theta \cap \Sigma_{u, \lambda} \). Thus \( \Sigma_\Theta \cap \Pi_{u, \lambda} \) is a set of simple roots for \( \Sigma_\Theta \cap \Sigma_{u, \lambda} \). 

Write \( W_{u, \lambda, \Sigma_\Theta \cap \Pi_{u, \lambda}} \) for the parabolic subgroup of \( W_{u, \lambda} \) corresponding to \( \Sigma_\Theta \cap \Pi_{u, \lambda} \). Then \( W_{u, \lambda, \Sigma_\Theta \cap \Pi_{u, \lambda}} \) is the Weyl group of \( \Sigma_\Theta \cap \Sigma_{u, \lambda} \) and is a subgroup of \( W_{\Theta} \cap W_{u, \lambda} \).

**Proposition 2.4.2.** For any \( u \in A_{\Theta, \lambda} \), \( W_{\Theta} \cap W_{u, \lambda} = W_{u, \lambda, \Sigma_\Theta \cap \Pi_{u, \lambda}} \). In particular, \( W_{\Theta} \cap W_{u, \lambda} \) is a parabolic subgroup of \( W_{u, \lambda} \).

**Proof.** \( W_{u, \lambda, \Sigma_\Theta \cap \Pi_{u, \lambda}} \) is certainly contained in \( W_{\Theta} \cap W_{u, \lambda} \). Let \( w \in W_{\Theta} \cap W_{u, \lambda} \). Being in \( W_{\Theta} \), \( w \) permutes roots in \( \Sigma_\Theta \); being in \( W_{u, \lambda} \), \( w \) permutes roots in \( \Sigma_{u, \lambda} \). Hence \( w \) permutes roots in \( \Sigma_\Theta \cap \Sigma_{u, \lambda} \), and it sends the set \( \Sigma^+ \cap (\Sigma_\Theta \cap \Sigma_{u, \lambda}) \) of positive roots in \( \Sigma_\Theta \cap \Sigma_{u, \lambda} \) to another set of positive roots \( w\Sigma^+ \cap (\Sigma_\Theta \cap \Sigma_{u, \lambda}) \). Since \( W_{u, \lambda, \Sigma_\Theta \cap \Pi_{u, \lambda}} \) is the Weyl group of \( \Sigma_\Theta \cap \Sigma_{u, \lambda} \), there exists a unique element \( v \in W_{u, \lambda, \Sigma_\Theta \cap \Pi_{u, \lambda}} \) that sends \( w\Sigma^+ \cap (\Sigma_\Theta \cap \Sigma_{u, \lambda}) \) back to \( \Sigma^+ \cap (\Sigma_\Theta \cap \Sigma_{u, \lambda}) \). Hence \( vw \) permutes \( \Sigma^+ \cap (\Sigma_\Theta \cap \Sigma_{u, \lambda}) = \Sigma^+_{u, \lambda} \cap \Sigma^+_\Theta \). On the other hand, since \( vw \in W_{\Theta} \), by 2.2.2 it permutes \( \Sigma^+ - \Sigma^+_\Theta \); thus it is also in \( W_{u, \lambda} \). Hence, it permutes \( \Sigma^+_{u, \lambda} - \Sigma^+_\Theta \). As a result, \( vw \) permutes \( \left( \Sigma^+_{u, \lambda} \cap \Sigma^+_\Theta \right) \cup \left( \Sigma^+_{u, \lambda} - \Sigma^+_\Theta \right) \).

Since \( W_{u, \lambda} \) acts simply transitively on the set of sets of positive roots in \( \Sigma_{u, \lambda} \), we must have \( vw = 1 \). Therefore \( w = v^{-1} \in W_{u, \lambda, \Sigma_\Theta \cap \Pi_{u, \lambda}} \). Thus \( W_{\Theta} \cap W_{u, \lambda} = W_{u, \lambda, \Sigma_\Theta \cap \Pi_{u, \lambda}} \), as desired. 

For \( u \in A_{\Theta, \lambda} \), write

\[
\Theta(u, \lambda) = u^{-1}(\Sigma_\Theta \cap \Pi_{u, \lambda}) = u^{-1}\Sigma_\Theta \cap \Pi_{\lambda}.
\]

Since \( \Sigma_\Theta \cap \Pi_{u, \lambda} \) is a subset of simple roots in \( \Sigma_{u, \lambda} \), \( \Theta(u, \lambda) \) is a subset of simple roots in \( u^{-1}\Sigma_{u, \lambda} = \Sigma_{\lambda} \). Write \( W_{\lambda, \Theta(u, \lambda)} \) for the parabolic subgroup of \( W_{\lambda} \) corresponding to \( \Theta(u, \lambda) \).

**Proposition 2.4.3.** Let \( u \in A_{\Theta, \lambda} \). The left-multiplication-by-\( u \) map

\[
W_{\lambda} \rightrightarrows uW_{\lambda}
\]

sends right \( W_{\lambda, \Theta(u, \lambda)} \)-cosets in \( W_{\lambda} \) to the subsets \( C \cap uW_{\lambda} \) for right \( W_{\Theta} \)-cosets \( C \) in \( W_{\Theta}uW_{\lambda} \). All sets of the form \( C \cap uW_{\lambda} \) are obtained in this way.

Moreover, this map preserves the partial orders on cosets: if \( C', D' \in W_{\lambda, \Theta(u, \lambda)} \) \( \backslash W_{\lambda} \) are sent to \( C \cap uW_{\lambda} \) and \( D \cap uW_{\lambda} \), respectively, then \( D' \leq_{u, \lambda} C' \) implies \( D \leq C \).

**Proof.** Consider the smallest right \( W_{\Theta} \)-coset \( W_{\Theta}u \) of \( W_{\Theta}uW_{\lambda} \).

\[
W_{\Theta}u \cap uW_{\lambda} = (W_{\Theta} \cap uW_{\lambda}u^{-1})u
= (W_{\Theta} \cap W_{u, \lambda})u
= W_{u, \lambda, \Sigma_\Theta \cap \Pi_{u, \lambda}}u
= W_{u, \lambda, \Theta(u, \lambda)}u
= (uW_{\lambda, \Theta(u, \lambda)}u^{-1})u
= uW_{\lambda, \Theta(u, \lambda)}.
\]

Hence left multiplication by \( u \) sends \( W_{\lambda, \Theta(u, \lambda)} \) to \( W_{\Theta}u \cap uW_{\lambda} \). Since left multiplication by \( u \) commutes with right multiplication by elements of \( W_{\lambda} \), it sends right \( W_{\lambda, \Theta(u, \lambda)} \)-cosets in \( W_{\lambda} \) to
right $W_\lambda$-translates of $W_\Theta u \cap uW_\lambda$, which gives us $C \cap uW_\lambda$ for various right $W_\Theta$-cosets $C$ in $W_\Theta uW_\lambda$. Moreover, any right $W_\Theta$-coset $C$ in $W_\Theta uW_\lambda$ is obtained as a right $W_\Theta$-translation of $W_\Theta u$, hence the intersection $C \cap uW_\lambda$ is necessarily the image of a right $W_{\lambda,\Theta(u,\lambda)}$-coset.

To show that this map is order preserving, take two right $W_{\lambda,\Theta(u,\lambda)}$-cosets $C'$ and $D'$ such that $D' \leq_{u,\lambda} C'$. This means that the $\leq_{\lambda}$-longest elements $v^{D'}$, $v^{C'}$ of $D'$ and $C'$ satisfy $v^{D'} \leq_{\lambda} v^{C'}$. Since left multiplication by $u$ preserves Bruhat orders (2.1.4), $uv^{D'} \leq uv^{C'}$. Therefore $D \leq C$ by 2.2.4.

**Corollary 2.4.4.** As $u$ ranges over $A_{\Theta,\lambda}$, left multiplication by $W_\Theta u$ defines a bijection

$$\text{ind}_\lambda : \bigcup_{u \in A_{\Theta,\lambda}} W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda \xrightarrow{\sim} W_\Theta \backslash W$$

which is order-preserving when restricted to each $W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$ and commutes with right multiplication by $W_\lambda$. The image of $W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$ equals $W_\Theta \backslash W_\Theta uW_\lambda$.

**Notation 2.4.5.** We will write $(-)|_\lambda : W_\Theta \backslash W \rightarrow \bigcup_{u \in A_{\Theta,\lambda}} W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$ for the inverse map. If $C$ and $D$ are both sent to $W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$, we will write $C \leq_{u,\lambda} D$ for $C|_\lambda \leq_{u,\lambda} D|_\lambda$.

The map $(-)|_\lambda$ plays an important role towards our goal. As explained in the introduction, standard and irreducible Whittaker modules in $\mathcal{N}_{\theta,\eta}$ are parameterized by $W_\Theta \backslash W$, but compared to the integral case, $\mathcal{N}_{\theta,\eta}$ is divided into smaller blocks. The map $(-)|_\lambda$ reflects this division: on the level of standard/irreducible modules, modules that correspond to $C$'s in the same $(W_\Theta, W_{\lambda})$-coset are in the same block, and each block looks like an integral Whittaker category (at least on the level of standard and irreducible modules) modeled by $W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$.

We also need to understand how $(-)|_\lambda$ behaves under right multiplication by a non-integral simple reflection. This reflects the effect of non-integral intertwining functors which will be defined in §3 and will be used in the algorithm. Roughly speaking, right multiplication by a non-integral simple reflection translates $W_\Theta, W_\lambda$-coset structures to $(W_\Theta, W_{s\beta})$-coset structures, while conjugation by the same reflection translates right $W_{\lambda,\Theta(u,\lambda)}$-coset structures in $W_\lambda$ to $W_{s\beta,\lambda,\Theta(r,s\beta)}$-coset structures in $W_{s\beta\lambda}$.

**Lemma 2.4.6.** Let $u \in A_{\Theta,\lambda}$, $\beta \in \Pi - \Pi_\lambda$. Then $W_\Theta us\beta$ is the smallest right $W_\Theta$-coset in $W_\Theta us\beta W_{s\beta\lambda} = W_\Theta uW_{\lambda}s\beta$.

**Proof.** By 2.1.1(e)(f), $us\beta \in A_{s\beta\lambda}$. Hence the claim follows from 2.3.2. ■

**Corollary 2.4.7.** Let $u \in A_{\Theta,\lambda}$, $\beta \in \Pi - \Pi_\lambda$. If $r$ denotes the unique element in $A_{\Theta,s\beta\lambda} \cap W_\Theta us\beta W_{s\beta\lambda}$, then $W_\Theta r = W_\Theta us\beta$.

**Proposition 2.4.8.** Let $u \in A_{\Theta,\lambda}$, $\beta \in \Pi - \Pi_\lambda$. Let $r$ be the unique element in $A_{\Theta,s\beta\lambda} \cap W_\Theta us\beta W_{s\beta\lambda}$. Then conjugation by $s\beta$ is a bijection

$$s\beta(-)|s\beta : W_{s\beta,\lambda,\Theta(r,s\beta)} \backslash W_{s\beta\lambda} \xrightarrow{\sim} W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$$

that preserves the partial orders on right cosets. Moreover, the following diagram commutes

$$
\begin{array}{ccc}
W_{s\beta,\lambda,\Theta(r,s\beta)} \backslash W_{s\beta\lambda} & \xrightarrow{s\beta(-)|s\beta} & W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda \\
\text{ind}_{s\beta\lambda} & & \text{ind}_\lambda \\
W_\Theta \backslash W & \xrightarrow{(-)|s\beta} & W_\Theta \backslash W
\end{array}
$$

(2.4.9)

In particular, for any $C$, $D \in W_\Theta \backslash W_\Theta r W_{s\beta\lambda}$, $D \leq_{r,s\beta\lambda} C \iff D_{\beta} \leq_{u,\lambda} C\beta$. 

Proof. By 2.1.2, conjugation by $s_\beta$ is an isomorphism of groups and posets between $(W_{s_\beta\lambda}, \leq_{s_\beta\lambda})$ and $(W_\lambda, \leq_\lambda)$. By the preceding corollary, there exists $w \in W_\emptyset$ such that $wr = us_\beta$. Therefore
\[
s_\beta \Theta(u, \lambda) = s_\beta(u^{-1}\Sigma_\Theta \cap \Pi_\lambda) = (us_\beta)^{-1}\Sigma_\Theta \cap s_\beta \Pi_\lambda = (wr)^{-1}\Sigma_\Theta \cap \Pi_{s_\beta\lambda} = r^{-1}(w^{-1}\Sigma_\Theta) \cap \Pi_{s_\beta\lambda} = r^{-1}\Sigma_\Theta \cap \Pi_{s_\beta\lambda} = \Theta(r, s_\beta\lambda).
\]

Hence conjugation by $s_\beta$ sends $W_{s_\beta\lambda, \Theta(r, s_\beta\lambda)}$ to $W_\lambda, \Theta(u, \lambda)$ and therefore induces a bijection from $W_{s_\beta\lambda, \Theta(r, s_\beta\lambda)}/W_{s_\beta\lambda}$ to $W_\lambda, \Theta(u, \lambda)/W_\lambda$. Furthermore, since conjugation by $s_\beta$ preserves Bruhat orders, it also preserves the partial orders on right cosets.

To check that the diagram commutes, take any $D' \in W_{s_\beta\lambda, \Theta(r, s_\beta\lambda)}/W_{s_\beta\lambda}$. Along the top-right path, $D'$ is sent to
\[W_\emptyset u \cdot s_\beta D' s_\beta = W_\emptyset wrD' s_\beta = W_\emptyset rD' s_\beta,
\] which agrees with the image along the bottom-left path. 

2.5. A technical lemma. In the last part of this section, we prove a technical lemma that will be used in §4.6 in induction process.

Proposition 2.5.1. Let $u \in A_{\emptyset, \lambda}$ and $C \in W_\emptyset \backslash W_\emptyset u W_\lambda$. Suppose $C \neq W_\emptyset u$. Then there exist $\alpha \in \Pi_\lambda$, $s > 0$ and $\beta_1, \ldots, \beta_s \in \Pi$ such that, writing $z_0 = 1$, $z_i = s_{\beta_1} \cdots s_{\beta_i}$ and $z = z_s$, the following conditions hold:

(a) for any $0 \leq i \leq s - 1$, $\beta_{i+1}$ is non-integral to $z_i^{-1}\lambda$;
(b) $z_i^{-1}\alpha \in \Pi \cap \Pi_{z_i^{-1}\lambda}$;
(c) $C s_\alpha < u, \lambda C$;
(d) if $s > 0$, $C z < C$;
(e) $C s_\alpha z = C z s_{z^{-1}\alpha} < C z$.

This proposition is used in showing that the $q$-polynomials defined geometrically (by taking higher inverse images to Schubert cells; see §1.3 for an explanation of the idea) agree with the parabolic Kazhdan-Lusztig polynomials for the parabolic system $(W_\lambda, \Pi_\lambda, \Theta(u, \lambda))$. This is a proof by induction in the length of $C$. As mentioned in §1.3, the Kazhdan-Lusztig basis $C_w$ is partly characterized by properties of the product $\delta_s C_w$. If the simple reflection $s \in W_\lambda$ happens to be simple in $W$, then multiplication by $\delta_s$ on $C_w$ lifts to the geometric $U$-functor (push-pull along $X \to X_s$). However, if $s$ is not simple in $W$, no such $U$-functor exists. The strategy in this situation is to use non-integral intertwining functors to translate everything so that $s$ becomes simple in both the integral Weyl group and in $W$. On the $W_\lambda$ level, these non-integral intertwining functors correspond to applying conjugations $s_{\beta_1}(-)s_{\beta_i}$ by non-integral simple reflections so that $s \in W_\lambda$ is translated to $(s_{\beta_1} \cdots s_{\beta_i})^{-1}s_{\beta_1} \cdots s_{\beta_i}$, which is simple in $W_{s_{\beta_1} \cdots s_{\beta_i}\lambda}$. On the $W$ level, they correspond to right multiplication on $C$ by $s_{\beta_1} \cdots s_{\beta_s}$. Also, one needs to ensure that the length of $C$ decreases after these non-integral reflections in order to apply the induction hypothesis on $C$. The existence of such a chain of non-integral reflections is guaranteed by the proposition.

Proof. Since $C \neq W_\emptyset u$, in particular $C \neq W_\emptyset$, there exists a simple reflection $s_\gamma$ such that $C s_\gamma < C$.

If there exists $\alpha \in \Pi \cap \Pi_\lambda$ such that $C s_\alpha < C$, then this $\alpha$ together with $s = 0$ satisfies the requirement: (a) and (d) are void, while (b) and (c) are true by construction. We need to verify (c). Since $s_\alpha$ is simple in $(W_\lambda, \Pi_\lambda)$, we have three mutually exclusive possibilities: $C s_\alpha < u, \lambda C$, $C s_\alpha = C$, or $C s_\alpha > u, \lambda C$. Since the map $\text{ind}_\lambda$ preserves the partial order, they imply $C s_\alpha < C$,
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2.4.6

Cs_α = C and Cs_α > C, respectively. By our choice of α, the last two possibilities cannot happen. Hence we must have Cs_α <_{w,λ} C and (c) holds.

Suppose such α does not exist. Then any simple reflection that decreases the length of C via right multiplication must be non-integral to λ. Let s_β₁, β₁ ∈ Π − Π_λ, be one of those. If there exists α' ∈ Π ∩ Π_β₁ with Cs_β₁s_α' < Cs_β₁, we claim that α := s_β₁α' ∈ s_β₁Π_α = Π_λ, s = 1 and β₁ satisfy our requirements. (a) and (d) follows by our choice of s_β₁, (e) follows from the conditions on α'. For (b),

\[ z^{-1}α = s_β₁s_β₁α' = α' ∈ Π ∩ Π_β₁ \]

by definition of z and α'. For (c), arguing in the same way, we only need to rule out Cs_α ≥ C, which would imply ℓ(C) − ℓ(Cs_αs_β₁) ∈ {−2, −1, 0, 1}. On the other hand,

\[ C > Cs_β₁s_α' = Cs_β₁s(s_β₁α) = Cs_β₁s_αs_β₁ = Cs_αs_β₁. \]

So ℓ(C) − ℓ(Cs_αs_β₁) ≥ 2 and (c) holds.

If such α' does not exist, then we can find β_2, ..., β_s ∈ Π such that Cz_{i+1} < Cz_i for all 1 ≤ i ≤ s − 1 until we get to a point where there exists α'' ∈ Π ∩ Π_{z−1} with Cz_{α''} < Cz (termination of this process is proven in the next paragraph). We claim that α := za'' ∈ Π_{z−1} = Π_λ, s and β₁, ..., β_s satisfy our requirements. The verification is essentially the same as in the previous case. (a), (b), (d) and (e) are satisfied by our choice of β's and α''. For (c), we have an inequality

\[ Cz > Czs_{α''} = Czs_{z−1} = Cz^{-1}s_αz = Cs_αz \quad (2.5.2) \]

where ℓ(w_{Cz}) = ℓ(w_Cz) = ℓ(w_C) − s. Also w_Czs_{α''} = w_Cz_{α''} = w_Cs_{α''}. Hence

\[ ℓ(w_Cs_α) = ℓ(w_Cs_{α''}z^{-1}) = ℓ(w_{Cα''}z^{-1}) ≤ ℓ(w_{Cα''}) + s = ℓ(w_{Cz}) − 1 + s = ℓ(w_C) − 1 < ℓ(w_C). \]

This rules out Cs_α ≥ C and (c) is thus verified.

Lastly, let us show that this process of finding α'' must terminate no later than when we get to ℓ(w_{Cz}) = ℓ(w_{θ}) + 1. That is, we show that when ℓ(w_{Cz}) = ℓ(w_{θ}) + 1, such an α'' must exist.

The condition ℓ(w_{Cz}) = ℓ(w_{θ}) + 1 implies Cz = W_{θ}s_γ > W_{θ} for some simple reflection s_γ. If γ ∈ Π ∩ Π_{z−1}, then s_γ ∈ A_{z−1}. Also, since W_{θ}s_γ > W_{θ}, any element of W_{θ}s_γ must have length ≥ 1. Hence s_γ is the shortest element of W_{θ}s_γ, i.e., s_γ ∈ w_{θ}W_{θ}. Therefore s_γ ∈ A_{z−1} ∩ (w_{θ}W_{θ}) = A_{θ,z−1}. Since C = W_{θ}s_{z−1}, by (repeatedly applying) 2.4.6, we see that C is the smallest right W_{θ}-coset in the (W_{θ}, W_{λ})-coset containing it, that is, C = W_{θ}u. This contradicts our assumption on C. Therefore γ ∈ Π ∩ Π_{λ}, and α'' = γ satisfies our requirement for α''. Thus the process terminates.

3. Non-integral Intertwining Functors

In this section, we given definitions of non-integral intertwining functors and show that they translate the Kazhdan-Lusztig polynomials for our Whittaker modules. See §1.2 the geometric setup and related notations.

For any w ∈ W, let Z_w denote the subset of X × X consists of pairs (x, y) such that b_x and b_y are in relative position w. This means that for any common Cartan subalgebra c and any representative of w in N_G(c) (also denoted by w), b_x = Ad(w(b_y)). If w is fixed, we write

\[ X \xleftarrow{p_1} Z_w \xrightarrow{p_2} X \]

for the two projections. For an integral weight μ ∈ h*, write O_X(μ) for the G-equivariant line bundle on X where the b-action on the geometric fiber at x_b ∈ X (the point on X that corresponds to b) is given by μ.

Definition 3.0.1. For w ∈ W and λ ∈ h*, the intertwining functor LI_w is defined to be

\[ LI_w : D^b(D_λ) → D^b(D_{wλ}). \]
\[ \mathcal{F}^* \to p_{1+}(p_1^+\mathcal{O}_X(\rho - wp) \otimes p_2^+\mathcal{F}^*) \]
\[ \cong \mathcal{O}_X(\rho - wp) \otimes p_{1+}p_2^+\mathcal{F}^*. \]

Write \( I_w \) for \( H^0LI_w \). It is shown in [Mil, L.3] that \( LI_w \) is the left derived functor of \( I_w \).

For properties of intertwining functors including 3.0.2 below, readers can refer to loc. cit. The main property we will use is

**Theorem 3.0.2.** If \( \beta \in \Pi - \Pi_\lambda \), then \( I_{s_\beta} \) is an equivalence of categories

\[ I_{s_\beta} : \text{Mod}_{qc}(D_\lambda) \cong \text{Mod}_{qc}(\mathcal{D}_{s_\beta \lambda}) \]

whose quasi-inverse is \( I_{s_\beta} \).

To use these functors for our purpose, we need to compute the action of intertwining functors on standard and irreducible modules. Romanov computed the following result for \( Cs_\beta > C \). The main ingredients of the proof there are base change formula and projection formula for \( D \)-modules.

**Proposition 3.0.3** ([Rom21, 3.15]). Let \( \beta \in \Pi \) and \( C \in \mathcal{W}_1 \setminus W \) such that \( Cs_\beta > C \). Then for any \( \lambda \in \mathfrak{h}^* \),

\[ LI_{s_\beta} \mathcal{I}(w^C, \lambda, \eta) = \mathcal{I}(w^C s_\beta, s_\beta \lambda, \eta). \]

Combined with 3.0.2 we get

**Corollary 3.0.4.** Let \( \beta \in \Pi - \Pi_\lambda \) and \( C \in \mathcal{W}_1 \setminus W \) such that \( Cs_\beta \neq C \). Then

\[ I_{s_\beta} \mathcal{I}(w^C, \lambda, \eta) = \mathcal{I}(w^C s_\beta, s_\beta \lambda, \eta). \]

**Proof.** Suppose \( Cs_\beta > C \), then the statement follows from 3.0.3. But since \( I_{s_\beta} \) is an equivalence of categories with inverse \( I_{s_\beta} \),

\[ \mathcal{I}(w^C, \lambda, \eta) = I_{s_\beta} \mathcal{I}(w^C s_\beta, s_\beta \lambda, \eta). \]

It remains to consider the case \( Cs_\beta = C \).

For a simple root \( \beta \), write \( X_\beta \) for the partial flag variety of type \( \beta \), and write \( p_\beta : X \to X_\beta \) for the natural projection. This is a Zariski-local \( \mathbb{A}^1 \)-fibration. \( x \) and \( y \) are contained in the same \( p_\beta \)-fiber (i.e. \( p_\beta(x) = p_\beta(y) \)) if and only if \( b_x \) and \( b_y \) are in relative position 1 or \( s_\beta \).

**Lemma 3.0.5.** Let \( C \in \mathcal{W}_1 \setminus W \) and \( \beta \in \Pi \) such that \( Cs_\beta = C \). Set

\[ S = \{(x, y) \in C(w^C) \times C(w^C) \mid b_x \text{ and } b_y \text{ are in relative position } s_\beta \} \subset Z_{s_\beta}. \]

Write \( C(w^C) \xrightarrow{p_1|_S} S \xrightarrow{p_2|_S} C(w^C) \) for the projections. Then

\[ (p_1|_S)^* + (p_2|_S)^* \mathcal{O}_{C(w^C)} = \mathcal{O}_{C(w^C)}. \]

**Proof.** For convenience, write \( w = w^C \), \( p_1 = p_1|_S \) and \( p_2 = p_2|_S \). Set

\[ S' = C(w) \times p_\beta(C(w)) C(w) = \{(x, y) \in C(w) \times C(w) \mid p_\beta(x) = p_\beta(y)\}. \]

Then \( S \subset S \cup \Delta_{C(w)} = S' \subset Z_{s_\beta} \), where \( \Delta_{C(w)} \) denotes the diagonal. Write \( C(w) \xrightarrow{\eta_1} S' \xrightarrow{\eta_2} C(w) \) for the projections, and \( \Delta_{C(w)} \xrightarrow{i_\Delta} S' \xrightarrow{i_S} S \) for the inclusions. Then \( i_\Delta \) is a closed immersion with
relative dimension 1, and \( i_S \) is open. We have the following diagram

\[
\begin{array}{ccc}
S & \xrightarrow{i_S} & S' \\
\downarrow{p_1} & & \downarrow{q_2} \\
C(w) & \xrightarrow{q_1} & C(w) \\
& \downarrow{p_\beta} & \downarrow{p_\beta} \\
& C(w) & \xrightarrow{p_\beta} p_\beta(C(w))
\end{array}
\] (3.0.6)

where the bottom-right square is Cartesian.

Applying the triangle for local cohomology to \( q_2^+ \mathcal{O}^\eta_{C(w)} \), we get

\[
i_{\Delta} + i_{\Delta}^! q_2^+ \mathcal{O}^\eta_{C(w)} \longrightarrow q_2^+ \mathcal{O}^\eta_{C(w)}.
\]

Applying \( q_1 \), we get

\[
q_1 + i_{\Delta} + i_{\Delta}^! q_2^+ \mathcal{O}^\eta_{C(w)} \longrightarrow q_1 + q_2^+ \mathcal{O}^\eta_{C(w)}.
\]

Applying base change to the bottom-right square in (3.0.6), \( q_1 + q_2^+ \mathcal{O}^\eta_{C(w)} \cong p_\beta^+ p_\beta^+ \mathcal{O}^\eta_{C(w)} \). Here \( p_\beta^+ \mathcal{O}^\eta_{C(w)} \) is an \( \eta \)-twisted Harish-Chandra sheaf on \( p_\beta(C(w)) \). But \( p_\beta(C(w)) \) is isomorphic to \( C(ws_\beta) \) as an \( N \)-variety via \( p_\beta \), and since \( ws_\beta \) is not the longest element in \( W_\Theta ws_\beta = W_\Theta w \), we know there is no \( \eta \)-twisted Harish-Chandra sheaf on \( C(ws_\beta) \) except 0. Hence \( p_\beta^+ \mathcal{O}^\eta_{C(w)} = 0 \) and thus \( q_1 + q_2^+ \mathcal{O}^\eta_{C(w)} = 0 \). As a result,

\[
q_1 + i_S + i_S^! q_2^+ \mathcal{O}^\eta_{C(w)} = q_1 + i_{\Delta} + i_{\Delta}^! q_2^+ \mathcal{O}^\eta_{C(w)}[1].
\]

The left side simplifies to \( p_1 p_2^+ \mathcal{O}^\eta_{C(w)} \). For the right side, \( q_1 + i_{\Delta} = (q_1 \circ i_{\Delta})_+ \) and \( q_1 \circ i_{\Delta} \) is the projection \( \Delta_{C(w)} \to C(w) \) along the first coordinate which is an \( N \)-equivariant isomorphism. Moreover,

\[
i_{\Delta}^! q_2^+ \mathcal{O}^\eta_{C(w)}[1] = i_{\Delta}^! q_2^+ \mathcal{O}^\eta_{C(w)}[1][[-1] = (q_2 \circ i_{\Delta})^+ \mathcal{O}^\eta_{C(w)};
\]

and \( q_2 \circ i_{\Delta} \) is the projection \( \Delta_{C(w)} \to C(w) \) along the second coordinate, also an \( N \)-equivariant isomorphism. Thus

\[
p_1 p_2^+ \mathcal{O}^\eta_{C(w)} = (q_1 \circ i_{\Delta})_+ (q_2 \circ i_{\Delta})^+ \mathcal{O}^\eta_{C(w)} = \mathcal{O}^\eta_{C(w)};
\]

Lemma 3.0.7. Let \( s_\beta \in \Pi \) and \( C \in W_\Theta \backslash W \) such that \( C s_\beta = C \). Write \( \iota : C(w^C) \to C(w^C) \cup C(w^C s_\beta) \) for the inclusion. Then for any \( \mathcal{F} \in \text{Mod}_{coh}(\mathcal{D}_{C(w^C) \cup C(w^C s_\beta)}, N, \eta) \),

\[
\mathcal{F} = \iota_+ \iota^! \mathcal{F} = (\iota_+ \mathcal{O}^\eta_{C(w^C)}) \otimes \text{rank} \iota^! \mathcal{F}
\]

where \( \text{rank} \) stands for the rank as a free \( \mathcal{O} \)-module.

Proof. Write \( w = w^C \). The assumption implies that \( ws_\beta \in C, ws_\beta < w \), and that \( C(w) \) and \( C(ws_\beta) \) are open and closed in \( C(w) \cup C(ws_\beta) \), respectively.
Since the category of $\eta$-twisted Harish-Chandra sheaves on $C(w)$ is semisimple, $i^!F$ is a direct sum of copies of $\mathcal{O}^{\eta}_{C(w)}$. This implies the second equality. For the first equality, adjunction gives a map

$$\mathcal{F} \to i_{+1}i^!F$$

(3.0.8)

whose kernel and cokernel are supported on $C(ws_{\beta})$, which are equal to direct images of $\eta$-twisted Harish-Chandra sheaves on $C(ws_{\beta})$ by Kashiwara’s equivalence. But $ws_{\beta}$ is not the longest element in $C$, so there is no such module on $C(ws_{\beta})$ except zero. Hence (3.0.8) is an isomorphism, which establishes the first equality.

**Proposition 3.0.9.** Let $C \in W_{\Theta}/W$, $\beta \in \Pi$ such that $Cs_{\beta} = C$. Then

$$LI_{s_{\beta}} \mathcal{I}(w^C, \lambda, \eta) = \mathcal{I}(w^C, s_{\beta}, \lambda, \eta).$$

**Proof.** Write $w = w^C$. Let

$$F = Z_{s_{\beta}} \times_{p_2, X, i_w} C(w) = \{(x, y) \in X \times C(w) \mid b_x \text{ and } b_y \text{ are in relative position } s_{\beta}\}$$

and let $S$ be as in 3.0.5. Then $S$ is a subvariety of $F$. It’s easy to see that $p_1(F) = \{x \in X \mid \exists y \in C(w) \text{ such that } b_x \text{ and } b_y \text{ are in relative position } s_{\beta}\} = C(w) \cup C(ws_{\beta})$. So we have the following diagram

![Diagram](3.0.10)

The right-most square is Cartesian by definition of $F$. The top-left square is also Cartesian, i.e. $S$ is the preimage of $C(w)$ along $p_1: F \to C(w) \cup C(ws_{\beta})$. By definition of intertwining functors and base change,

$$LI_{s_{\beta}} \mathcal{I}(w, \lambda, \eta) = \mathcal{O}_X(\rho - s_{\beta} \rho) \otimes_{\mathcal{O}_X} p_1 + p_2^!i_w + \mathcal{O}^\eta_{C(w)}$$

$$= \mathcal{O}_X(\rho - s_{\beta} \rho) \otimes_{\mathcal{O}_X} p_1 + i_{F} + (p_2 | F)^{+} \mathcal{O}^\eta_{C(w)}$$

$$= \mathcal{O}_X(\rho - s_{\beta} \rho) \otimes_{\mathcal{O}_X} j_{w} + (p_1 | F) + (p_2 | F)^{+} \mathcal{O}^\eta_{C(w)}. $$

(3.0.11)

We claim that $(p_1 | F)^{+} + (p_2 | F)^{+} \mathcal{O}^\eta_{C(w)} = t_w + \mathcal{O}^\eta_{C(w)}$. By 3.0.7,

$$(p_1 | F)^{+} + (p_2 | F)^{+} \mathcal{O}^\eta_{C(w)} = t_w + t_w(i_{w}(p_1 | F) + (p_2 | F)^{+} \mathcal{O}^\eta_{C(w)}).$$

Apply base change using the top-left square in (3.0.10),

$$t_w(i_{w}(p_1 | F) + (p_2 | F)^{+} \mathcal{O}^\eta_{C(w)}) = (p_1 | S) + a_{S}(p_2 | F)^{+} \mathcal{O}^\eta_{C(w)} = (p_1 | S) + a_{S}(p_2 | F)^{+} \mathcal{O}^\eta_{C(w)}$$

Note that $p_2 | F \circ a_S = p_2 | S$. Hence, by 3.0.5, the sheaf in the above equation equals $\mathcal{O}^\eta_{C(w)}$. Therefore

$$(p_1 | F)^{+} + (p_2 | F)^{+} \mathcal{O}^\eta_{C(w)} = t_w + t_w(i_{w}(p_1 | F) + (p_2 | F)^{+} \mathcal{O}^\eta_{C(w)} = t_w + \mathcal{O}^\eta_{C(w)}$$

as claimed. As a result,

$$(3.0.11) = \mathcal{O}_X(\rho - s_{\beta} \rho) \otimes_{\mathcal{O}_X} j_{w} + t_w + \mathcal{O}^\eta_{C(w)}$$
\[ = \mathcal{O}_X(\rho - s_\beta \rho) \otimes_{\mathcal{O}_X} i_{w+} \mathcal{O}^n_{C(w)} \]
\[ = \mathcal{I}(w, s_\beta \lambda, \eta) \]
which proves the proposition.

**Corollary 3.0.12.** Let \( \beta \in \Pi - \Pi_\lambda \). Let \( C \in \mathcal{W}_\Theta \backslash W \). Then
\[ I_{s_\beta, \mathcal{I}}(w^C, \lambda, \eta) = I_{w^C, s_\beta \lambda, \eta} \]
\[ I_{s_\beta, \mathcal{L}}(w^C, \lambda, \eta) = L(w^C_{s_\beta}, s_\beta \lambda, \eta) \]
(note that we have \( w^C_{s_\beta} \) instead of \( w^C s_\beta \) on the right hand sides).

**Proof.** The statement about standard modules is the combination of 3.0.4 and 3.0.9. Since \( I_{s_\beta} \) is an equivalence of categories, it must send the unique irreducible submodule of \( \mathcal{I}(w^C, \lambda, \eta) \) to the unique irreducible submodule of \( \mathcal{I}(w^C_{s_\beta}, s_\beta \lambda, \eta) \), i.e. it must send \( L(w^C, \lambda, \eta) \) to \( L(w^C_{s_\beta}, s_\beta \lambda, \eta) \). 

Next, we show that non-integral intertwining functors also preserves pullback of irreducible modules to strata.

**Proposition 3.0.13.** Let \( \beta \in \Pi - \Pi_\lambda, C, D \in \mathcal{W}_\Theta \backslash W \) and \( p \in \mathbb{Z} \). Then
\[ \text{rank } HP_{w^D, s_\beta}^p L(w^C, \lambda, \eta) = \text{rank } HP_{w^C, s_\beta}^p L(w^C_{s_\beta}, s_\beta \lambda, \eta). \]

The proof we give below uses the same tools as in the previous proposition. There is an alternative proof which we briefly mention. One shows that rank \( HP_{w^D} L(w^C, \lambda, \eta) \) equals the dimension of the \( p \)-th \( D_\lambda \)-module Ext group of \( M(w^C, \lambda, \eta) \) and \( L(w^C, \lambda, \eta) \) using facts on derived categories of highest weight categories. The proposition follows from the fact that \( I_{s_\beta} \) is an equivalence of categories and induces an isomorphism on Ext-groups.

**Proof.** Write \( w = w^D \).

There are two cases, \( D s_\beta \neq D \) or \( D s_\beta = D \). Consider the first case. Assume \( D s_\beta < D \). Then \( w^{D s_\beta} = w^D s_\beta = w s_\beta \). Let
\[ F = C(w s_\beta) \times_{i_{w s_\beta}, X \times^p} Z s_\beta = \{ (x, y) \in C(w s_\beta) \times X : b_x \text{ and } b_y \text{ are in relative position } s_\beta \}. \]

Then the second projection \( p_2|_F : F \rightarrow X \) induces an isomorphism of \( F \) onto \( C(w) \), and we have the following commuting diagram

\[ \begin{array}{ccc}
F & \xrightarrow{i_F} & F \\
\downarrow{p_1|_F} & & \downarrow{p_2|_F} \\
C(w s_\beta) & \xrightarrow{Z} & C(w) \\
i_{w s_\beta} & & i_w \\
\end{array} \]

where the left square is Cartesian. Using base change,
\[ \text{rank } HP_{w s_\beta}^p L(w^C_{s_\beta}, s_\beta \lambda, \eta) = \text{rank } HP_{w^C, s_\beta}^p L(w^C, \lambda, \eta) \]
\[ = \text{rank } HP_{w s_\beta}^p I_{s_\beta} L(w^C, \lambda, \eta) \]
\[ = \text{rank } HP(p_1|_F)^+ (p_2|_F)^+ i_{w}^p L(w^C, \lambda, \eta)[-1]. \quad \text{(3.0.14)} \]

Here in (3.0.14) we did not write the twist by the line bundle \( \mathcal{O}_X(\rho - w \rho) \) because there is no twist on \( C(w s_\beta) \). Since \( \text{Mod}_{\text{coh}}(D_{C(w)}, N, \eta) \) is semisimple, \( i_{w}^p L(w^C, \lambda, \eta) \) is a direct sum of \( \mathcal{O}^n_{C(w)} \)'s at different degrees. So \( (p_2|_F)^+ i_{w}^p L(w^C, \lambda, \eta) \) is a direct sum of \( \mathcal{O}^n_{C(w)} \)'s at different degrees by the fact that \( p_2|_F \) is an isomorphism, and the rank at degree \( p \) being \( \text{rank } HP_{w^C, s_\beta}^p \mathcal{L}(w^C, \lambda, \eta) \). Hence it is enough
to compute \((p_1|_F)_+ \mathcal{O}_F^\eta\), for which we use the fact that a map of homogeneous spaces of a unipotent group is isomorphic to a coordinate projection of affine spaces, that is, we have the following commutative diagram where all maps are \(N\)-equivariant, for some \(N\)-actions on \(\mathbb{A}^1 \times \mathbb{A}^{\ell(w_\beta)}\) and \(\mathbb{A}^{\ell(w_\beta)}\):

\[
\begin{array}{ccc}
F & \xrightarrow{p_1|_F} & C(w_\beta) \\
\cong & & \cong \\
\mathbb{A}^1 \times \mathbb{A}^{\ell(w_\beta)} & \xrightarrow{pr_1} & \mathbb{A}^{\ell(w_\beta)}
\end{array}
\]

So it suffices to compute \(pr_1+\mathcal{O}_{\mathbb{A}^1 \times \mathbb{A}^{\ell(w_\beta)}}^\eta\). Since \(pr_1\) is a coordinate projection, \(pr_1+\mathcal{O}_{\mathbb{A}^{\ell(w_\beta)}}^\eta = \mathcal{O}_{\mathbb{A}^{\ell(w_\beta)}}\) (we remark that, without the assumption of \(D_{s_\beta} \neq D, w_\beta \) and \(w\) can be in the same right \(W_\Theta\)-coset, in which case \(\mathcal{O}_{\mathbb{A}^{\ell(w_\beta)}}^\eta\) does not exist). On the other hand, by functoriality of \(\eta\)-twisted Harish-Chandra sheaves, we must have \(pr_1+\mathcal{O}_{\mathbb{A}^{\ell(w_\beta)}}^\eta = \mathcal{O}_{\mathbb{A}^1 \times \mathbb{A}^{\ell(w_\beta)}}^\eta\). We conclude that

\[
\mathcal{O}_{\mathbb{A}^1 \times \mathbb{A}^{\ell(w_\beta)}}^\eta = \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_{\mathbb{A}^{\ell(w_\beta)}}^\eta.
\]

As a result, writing \(p : \mathbb{A}^1 \rightarrow \{\ast\}\) for the unique morphism to a point,

\[
pr_1+\mathcal{O}_{\mathbb{A}^1 \times \mathbb{A}^{\ell(w_\beta)}}^\eta = (p+\mathcal{O}_{\mathbb{A}^1}) \boxtimes ((Id_{\mathbb{A}^{\ell(w_\beta)}})+\mathcal{O}_{\mathbb{A}^{\ell(w_\beta)}}^\eta)
\]

\[
= C[1] \boxtimes \mathcal{O}_{\mathbb{A}^{\ell(w_\beta)}}^\eta
\]

\[
= \mathcal{O}_{\mathbb{A}^{\ell(w_\beta)}}^\eta[1].
\]

Therefore \((p_1|_F)_+ \mathcal{O}_F^\eta = \mathcal{O}_{C(w_\beta)}[1]\) and hence

\[
\text{rank } H^p_{w_{s_\beta}} L^C_{C_{s_\beta}, s_\beta \lambda, \eta} = (3.0.15) = \text{rank } H^p_{w} L^C_{C, \lambda, \eta}.
\]

Now consider the case \(D_{s_\beta} = D\). In this case \(w_\beta \) and \(w\) are in relative position \(s_\beta\) and set \(S\) as in 3.0.5, viewed as a subvariety of \(F\). Then the following diagram commutes

\[
\begin{array}{ccc}
F & \xrightarrow{i_F} & C(w) \\
\downarrow p_1|_F & & \downarrow p_2|_F \\
C(w) & \xrightarrow{i_{s_\beta}} & Z_{s_\beta} \\
\downarrow i_w & & \downarrow j_w \\
X & \xrightarrow{\iota_w} & X \\
\end{array}
\]

where the left-most square and the top-right square are Cartesian. Using base change,

\[
i^1_{w{s_\beta}} L^C_{C_{s_\beta}, s_\beta \lambda, \eta} = i^1_{w} L^C_{s_\beta \lambda, \eta} [1].
\]

By 3.0.7, \(j^1_{w} L^C_{C, \lambda, \eta} = \iota_{w} j^1_{w} L^C_{C, \lambda, \eta}\). Hence

\[
(3.0.16) = (p_1|_F)_+ (p_2|_F)_+ i^1_{w} L^C_{C, \lambda, \eta}[1]
\]

\[
= (p_1|_F)_+ b_{S} (p_2|_S)_+ i^1_{w} L^C_{C, \lambda, \eta}.
\]
Here \( p_1|_F \circ b_S = p_1|_S \). Also \( \mathcal{L}(w^C, \lambda, \eta) \) is a direct sum of \( \mathcal{O}_{C(w)}^\mu \) in various degrees. Hence by 3.0.5,
\[
\text{rank } H^p_i \mathcal{L}(w^C, \lambda, \eta) = \text{rank } H^p(3.0.17) = \text{rank } H^p_i \mathcal{L}(w^C, \lambda, \eta).
\]

4. Main algorithm

In this section, we formulate an algorithm for computing a set of polynomials in \( q \) indexed by pairs of right \( W_\Theta \)-cosets whose evaluation at \( q = -1 \) leads to the character formula for irreducible modules. This is in the same spirit as ordinary Kazhdan-Lusztig polynomials for category \( \mathcal{O} \). The algorithm we will prove is suggested by Milicic.

In §4.1, we define the parabolic Kazhdan-Lusztig polynomials, the module \( \mathcal{H}_\Theta \), and related notations. The statement of the algorithm is contained in §4.2. Proof of the algorithm is divided into subsections that follow.

4.1. The parabolic Hecke algebra and parabolic Kazhdan-Lusztig polynomials. Recall the sets \( A_{\Theta, \lambda} = A_{\lambda} \cap (w_\Theta \Theta W) \) and \( \Theta(u, \lambda) \subseteq \Pi_\lambda \) defined in §2.3 and §2.4. Recall also that we have a partial order on \( W_\Theta \) inherited from the Bruhat order on \( \Theta W \), denoted by \( \leq \). Similarly, we have a partial order on \( W_{\lambda, \Theta(u, \lambda)} \) which we denote by \( \leq_{u, \lambda} \).

Let \( \mathcal{H}_\Theta \) be the free \( \mathbb{Z}[q, q^{-1}] \)-modules with basis \( \delta_C, C \in W_\Theta \). For any \( \alpha \in \Pi_\lambda \), define a \( \mathbb{Z}[q, q^{-1}] \)-linear operator on \( \mathcal{H}_\Theta \) by
\[
T_\alpha(\delta_C) = \begin{cases} q\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha > C; \\ 0 & \text{if } Cs_\alpha = C; \\ q^{-1}\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha < C. \end{cases}
\]

The module \( \mathcal{H}_\Theta \) is supposed to be the underlying module of the parabolic Hecke algebra, and \( T_\alpha \) is supposed to encode information of the parabolic Kazhdan-Lusztig basis. However, we will not explicitly use the multiplicative structure on \( \mathcal{H}_\Theta \).

If \( u \) is an element in \( A_{\Theta, \lambda} \), let \( \mathcal{H}_{\Theta(u, \lambda)} \) be the free \( \mathbb{Z}[q, q^{-1}] \)-module with basis \( \delta_E, E \in W_{\lambda, \Theta(u, \lambda)} \). Define the operator \( T^\alpha_{u, \lambda} \) in the same way for any \( \alpha \in \Pi_\lambda \), replacing \( C \) by \( E \) and \( >_{u, \lambda} \) by \( >, < \), respectively.

We will use a left action of \( W \) on \( \mathcal{H}_\Theta \) defined by \( w \cdot \delta_C = \delta_{wC} \). Similarly, a right action of \( W \) on \( \mathcal{H}_\Theta \) is defined by \( \delta_C \cdot w = \delta_{Cw} \). We will simply write \( wC \), \( Cw \) for the actions, omitting the dots. \( w(-1)w^{-1} \) then denotes the simultaneous action of \( w \) on the left and \( w^{-1} \) on the right. By 2.4.8, \( s_\beta(-)s_\beta \) defines a bijection
\[
s_\beta(-)s_\beta : W_{\lambda, \Theta(u, \lambda)} \rightarrow W_{\lambda, \Theta(r, s_\beta \lambda)} \lambda \quad \text{where } r \text{ is the unique element in } A_{\Theta, s_\beta \lambda} \cap W_{\Theta(u, \lambda)} s_\beta W_{\beta \lambda}.
\]

Recall that we have a bijection
\[
(-)|_\lambda : W_\Theta \rightarrow \bigcup_{u \in A_{\Theta, \lambda}} W_{\lambda, \Theta(u, \lambda)} \lambda
\]
defined in 2.4.5. We extend \( (-)|_\lambda \) to a map
\[
(-)|_\lambda : \mathcal{H}_{\Theta(u, \lambda)} \rightarrow \bigoplus_{u \in A_{\Theta, \lambda}} \mathcal{H}_{\Theta(u, \lambda)} \lambda, \quad \delta_C \mapsto \delta_C|_\lambda.
\]

The following theorem, proven in \([\text{Rom21, 5.1}]\), defines a set of polynomials indexed by pairs of right cosets. It is verified in \( \text{op. cit.} \). Remark 6.4 that these polynomials agree with the parabolic Kazhdan-Lusztig polynomials. More precisely, the polynomials \( P_{CD} \) are equal to the polynomials \( n_{y,x} \) in \([\text{Soe97, 3.2.1}]\) for \( x = w_\Theta w^C, y = w_\Theta w^D \).
For a right coset $E \in W_{\Theta} \backslash W$, we write $(W_{\Theta} \backslash W)_{\leq E}$ for the set of those cosets $F$ such that $F \leq G$.

**Parabolic Kazhdan-Lusztig polynomials for $(W, \Pi, \Theta)$ 4.1.1.** For any $E \in W_{\Theta} \backslash W$, there exists a unique set of polynomials $\{P_{CD}\} \subset q\mathbb{Z}[q]$ indexed by

$$\{(C, D) \mid C, D \in (W_{\Theta} \backslash W)_{\leq E}; D < C\}$$

such that the function

$$\psi : (W_{\Theta} \backslash W)_{\leq E} \rightarrow \mathcal{H}_\Theta, \quad C \mapsto \delta_C + \sum_{D < C} P_{CD} \delta_D$$

satisfies the following property: for any $C \in W_{\Theta} \backslash W$ with $C \neq W_{\Theta}$, there exist $\alpha \in \Pi$ and $c_D \in \mathbb{Z}$ such that $C s_\alpha < C$ and

$$T_\alpha(\psi(C s_\alpha)) = \sum_{D \in C} c_D \psi(D).$$

Moreover, the polynomials $P_{CD}$ do not depend on the choice of $E$. \[1\]

We apply the same definition to $(W_{\lambda}, \Pi_{\lambda}, \Theta(u, \lambda))$:

**Parabolic Kazhdan-Lusztig polynomials for $(W_{\lambda}, \Pi_{\lambda}, \Theta(u, \lambda))$ 4.1.2.** For any $E \in W_{\lambda, \Theta(u, \lambda)} \backslash W_{\lambda}$, there exists a unique set of polynomials $\{P_{FG}^{u, \lambda}\} \subset q\mathbb{Z}[q]$ indexed by

$$\{(F, G) \mid F, G \in (W_{\lambda, \Theta(u, \lambda)} \backslash W_{\lambda})_{\leq u, \lambda} E; G <_{u, \lambda} F\}$$

such that the function

$$\psi_{u, \lambda} : (W_{\lambda, \Theta(u, \lambda)} \backslash W_{\lambda})_{\leq u, \lambda} E \rightarrow \mathcal{H}_{\Theta(u, \lambda)}, \quad F \mapsto \delta_F + \sum_{G <_{u, \lambda} F} P_{FG}^{u, \lambda} \delta_G$$

satisfies the following property: for any $F \in W_{\lambda, \Theta(u, \lambda)} \backslash W_{\lambda}$ with $F \neq W_{\lambda, \Theta(u, \lambda)}$, there exist $\alpha \in \Pi_{\lambda}$ and $c_G \in \mathbb{Z}$ such that $F s_\alpha <_{u, \lambda} F$ and

$$T_\alpha^{u, \lambda}(\psi_{u, \lambda}(F s_\alpha)) = \sum_{G <_{u, \lambda} F} c_G \psi_{u, \lambda}(G). \tag{4.1.3}$$

Moreover, the polynomials $P_{FG}^{u, \lambda}$ do not depend on the choice of $E$.

We will write $P_{CD}^{u, \lambda}$ instead of $P_{C_{(\lambda)}, D_{(\lambda)}^{(\lambda)}}^{u, \lambda}$ for convenience. Set $P_{EE}^{u, \lambda} = 1$ for all $E \in W_{\lambda, \Theta(u, \lambda)} \backslash W_{\lambda}$.

4.2. Main algorithm. Define the map

$$\nu : \text{Obj Mod}_{coh}(D_{(\lambda)}, N, \eta) \rightarrow \mathcal{H}_{\Theta}, \quad F \mapsto \sum_{D \in W_{\Theta} \backslash W} \sum_{p \in \mathbb{Z}} \left( \text{rank} H^p i_{wD}^! F \right) q^p \delta_D.$$

Clearly, this map can be defined for any complexes of $D_{(\lambda)}$-modules with $\eta$-twisted $N$-equivariant cohomologies. The following easy property of $\nu$ is immediate:

**Lemma 4.2.1.**

$$\nu(\mathcal{I}(w^C, \lambda, \eta)) = \delta_C.$$

**Proof.** Let $D \in W_{\Theta} \backslash W$. Then $i_{wD}^! \mathcal{I}(w^C, \lambda, \eta) = i_{wD}^! i_{wC}^! + O_{C(w^C)}^{(\eta)}$. If $C = D$, this is $O_{C(w^C)}^{(\eta)}$ by Kashiwara’s theorem. Otherwise, this is 0 by base change. Hence the claim follows by the definition of $\nu$. \hfill \blacksquare

\[1\] Romanov actually denotes the map by $\varphi$. We reserve the notation $\varphi$ to be used in the main algorithm 4.2.2.
Theorem 4.2.2 (Kazhdan-Lusztig Algorithm for Whittaker modules). Fix a character $\eta : \mathfrak{n} \to \mathbb{C}$. For any $\lambda \in \mathfrak{h}^*$, there exists a unique map

$$\varphi_\lambda : W_\Theta \setminus W \to \mathcal{H}_\Theta$$

such that for any $C \in W_\Theta \setminus W$, if we write $u$ for the unique element in $A_{\Theta, \lambda}$ such that $C$ is contained in $W_u W_\lambda$, the following conditions hold:

1. for some $P_{\nu, \lambda}^{\mu, \lambda} \in q \mathbb{Z}[q]$,

$$\varphi_\lambda(C) = \delta_C + \sum_{D \in W_\Theta \setminus W_u W_\lambda, \ D \leq u, \alpha \in \Pi, \lambda} P_{\nu, \lambda}^{\mu, \lambda} \delta_D.$$

2. for any $\alpha \in \Pi \cap \Pi_\lambda$ with $C_{S_{\alpha}} \subset C$, there exist $c_D \in \mathbb{Z}$ such that

$$T_\alpha(\varphi_\lambda(C_{S_{\alpha}})) = \sum_{D \in W_\Theta \setminus W_u W_\lambda, \ D \leq u, \alpha \in \Pi, \lambda} c_D \varphi_\lambda(D).$$

3. for any $\beta \in \Pi \setminus \Pi_\lambda$ such that $C_{S_{\beta}} \subset C$,

$$\varphi_{s_{\beta} \lambda}(C_{S_{\beta}}) = \varphi_\lambda(C) s_{\beta}$$

(recall that the action $\mathcal{H}_\Theta \lhd W$ is given by $\delta_C \cdot w = \delta_{Cw}$).

4. The polynomials $P_{\nu, \lambda}^{\mu, \lambda}$ are parabolic Kazhdan-Lusztig polynomials for $(W_\lambda, \Pi_\lambda, \Theta(u, \lambda))$. Moreover, the map $\varphi_\lambda$ is given by

$$\varphi_\lambda(C) = \nu(\mathcal{L}(w^C, \lambda, \eta)).$$

It is necessary that the theorem is stated for any $\lambda$ as we will often translate between different $\lambda$’s in the proof.

There are two aspects to this theorem. First, in view of the geometric picture §1.3, part (4) of the theorem says that the composition

$$\text{Obj} \text{Mod}_{\text{coh}}(D_\lambda, N, \eta) \xrightarrow{\nu} \mathcal{H}_\Theta \xrightarrow{(-)_{\lambda}} \bigoplus_{u \in A_{\Theta, \lambda}} \mathcal{H}_{\Theta(u, \lambda)}$$

sends irreducible modules to parabolic Kazhdan-Lusztig basis elements and standard modules to the standard basis. Thus, characters of irreducible modules can be obtained by looking at the Hecke algebra side. Second, parts (1) through (3) of the theorem provides an algorithm for computing the polynomials $P_{\nu, \lambda}^{\mu, \lambda}$’s.

The reader should think of $\varphi_\lambda(C)$ as a record of the expansion of $\mathcal{L}(w^C, \lambda, \eta)$ in the Grothendieck group in terms of the standard modules. (1) comes from the fact that $\mathcal{L}(w^C, \lambda, \eta)$ is supported on the closure of $C(w^C)$, and a standard module can appear in the character of $\mathcal{L}(w^C, \lambda, \eta)$ only if it is supported inside $\overline{C(w^C)}$ and if it is in the same block as $\mathcal{L}(w^C, \lambda, \eta)$. (2) reflects the action of the $U$-functor on irreducible Whittaker modules. (3) reflects the fact that non-integral intertwining functor is an equivalence of categories. So information of expansion in one category is translated fully to another category. The operations of (2) and (3) allows one to start with the base case $\varphi_\lambda(W_\Theta)$ and inductively compute any $\varphi_\lambda(C)$. For (4), the reader should refer to §2.5 for the idea behind the proof.

Let us begin the proof of the theorem. Uniqueness is determined by (1), (4) and the uniqueness of parabolic Kazhdan-Lusztig polynomials. For existence, we will show that $\varphi_\lambda(C) = \nu(\mathcal{L}(w^C, \lambda, \eta))$ satisfies the requirements (1)-(4) by induction on $\ell(w^C)$.

Consider the base case $\ell(w^C) = \ell(w_\Theta)$, that is, $C = W_\Theta$, $w^C = w_\Theta$. The argument for this case in the same as in [Rom21]. We include the details because it is short. Recall that $B \supseteq H$ are the Borel subgroup and the maximal torus corresponding to $\mathfrak{b}$ and $\mathfrak{h}$, respectively. Write $P_\Theta \supseteq B$ for the parabolic subgroup of $G$ of type $\Theta$. Let $O$ be the unique closed $P_\Theta$-orbit in $X$ with inclusion
maps \( i_{w_0, O} : C(w_\Theta) \hookrightarrow O \) and \( i_O : O \hookrightarrow X \). Then \( O \) is isomorphic to the flag variety of the largest semisimple quotient of \( p_\Theta \) in which \( C(w_\Theta) \) is the open Schubert cell, and \( O \) is a closed subvariety of \( X \). Write

\[
\mathcal{I}(w_\Theta, \lambda, \eta) = i_{w_0} + \mathcal{O}_C(w_\Theta) = i_O + i_{w_0, O} + \mathcal{O}_C(w_\Theta).
\]

By [MiS14, 5], \( i_{w_0, O} + \mathcal{O}_C(w_\Theta) \) is an irreducible \( D \)-module on \( O \) (this is the non-degenerate case: the restriction \( \eta \) to the nilpotent radical of a Borel in \( s_\Theta \) does not vanish on any simple root spaces). Hence \( \mathcal{I}(w_\Theta, \lambda, \eta) \) is irreducible by Kashiwara, and \( \mathcal{I}(w_\Theta, \lambda, \eta) = \mathcal{L}(w_\Theta, \lambda, \eta) \). As a result

\[
\nu(\mathcal{L}(w_\Theta, \lambda, \eta)) = \nu(\mathcal{I}(w_\Theta, \lambda, \eta)) = \delta_{W_\Theta}
\]

by 4.2.1. Therefore, the function \( \varphi_\lambda(C) \) satisfies (1) for \( C = W_\Theta \). The conditions (2)-(4) are void. This completes the base case.

Now consider the case \( \ell(w^C) = k > \ell(w_\Theta) \). The verification of (1)-(4) for \( C \) is divided into subsections.

4.3. **Verification of 4.2.2(3) for \( \ell(w^C) = k \).** Assume \( \beta \in \Pi - \Pi_\lambda \) is such that \( C s_\beta < C \). By definition,

\[
\varphi_\lambda(C) s_\beta = \left( \sum_{D \in W_\Theta} \sum_{p \in \mathbb{Z}} \left( \text{rank } H^p i_{w^D}^1 \mathcal{L}(w^C, \lambda, \eta) \right) q^p \delta_D \right) s_\beta
\]

and

\[
\varphi_{s_\beta \lambda}(C s_\beta) = \sum_{D \in W_\Theta} \sum_{p \in \mathbb{Z}} \left( \text{rank } H^p i_{w^D}^1 \mathcal{L}(w^{C s_\beta}, s_\beta \lambda, \eta) \right) q^p \delta_D s_\beta
\]

where in the last equality we rearranged the sum by the bijection \( W_\Theta \backslash W \xrightarrow{\sim} W_\Theta \backslash W, D \mapsto D s_\beta \). Hence it suffices to show that, for any \( D \in W_\Theta \backslash W \) and any \( p \in \mathbb{Z} \),

\[
\text{rank } H^p i_{w^D}^1 \mathcal{L}(w^C, \lambda, \eta) = \text{rank } H^p i_{w^{D s_\beta}}^1 \mathcal{L}(w^{C s_\beta}, s_\beta \lambda, \eta).
\]

This follows by 3.0.13.

4.4. **Verification of 4.2.2(2) for \( \ell(w^C) = k \).** This part of the argument is modified from [Mil, V.2] and is almost identical to [Rom21, §5] with slight modifications. Instead of reproving all the details, we briefly review Romanov’s argument and point out the main change for our situation.

Let \( \alpha \in \Pi \cap \Pi_\lambda \) be such that \( C s_\alpha < C \). Let \( X_\alpha \) be the partial flag variety with natural projection \( p_\alpha : X \to X_\alpha \). For any \( D \in W_\Theta \backslash W \) such that \( w^D s_\alpha < w^D \), write \( Z_D = C(w^D) \cup C(w^D s_\alpha) \) with inclusion map \( j_D : Z_D \hookrightarrow X \) (not to be confused with the variety \( Z_w \subseteq X \times X \) in the definition of intertwining functors). The main tools in \textit{loc. cit.} are the functor

\[
U_\alpha := p_\alpha^! p_{\alpha^+}
\]

and the triangle for local cohomology for the inclusion \( C(w^D s_\alpha) \hookrightarrow Z_D \). The first goal is to show that \( \nu \circ U_\alpha \) and \( T_\alpha \circ \nu \) agree on \( \mathcal{L}(w^C s_\alpha, \lambda, \eta) \). One applies the triangle and \( U_\alpha \) to \( j_D^! \mathcal{L}(w^C s_\alpha, \lambda, \eta) \). One of the vertices of the resulting triangle is given by restriction of \( U_\alpha \mathcal{L}(w^C s_\alpha, \lambda, \eta) \) to \( Z_D \). The other two vertices have locally free cohomologies at all degrees, and the ranks are given by the higher pullbacks of \( \mathcal{L}(w^C s_\alpha, \lambda, \eta) \) to the cells in \( Z_D \). Using the parity degree vanishing (\textit{op. cit.} 5.3) of parabolic Kazhdan-Lusztig polynomials, Romanov was able to split the long exact sequence of
The main change happens at the last step: compared with (4.4.2), the right hand side of our equation 4.2.2(2) is only summing over right $W_\phi$-cosets $D$’s such that $D$ is in the same double $(W_\phi, W_\lambda)$-coset as $C$ with $D \leq u, \lambda \ C$. To get this restricted sum, we apply the induction assumption to $Cs_\alpha$. In more detail, 4.2.2(1) for $Cs_\alpha$ compared with the definition of $\varphi_\lambda(Cs_\alpha) = \nu(L(w^C s_\alpha, \lambda, \eta))$ implies that

$$i^j_{wD} L(w^C s_\alpha, \lambda, \eta) = 0$$

whenever $D$ is either not in the same double $(W_\phi, W_\lambda)$-coset as $C$ and whenever $D$ is in the same double coset as $C$ but $D \not\leq u, \lambda \ C$. Combined with triangle mentioned above, $j^j_{D} U_\alpha L(w^C s_\alpha, \lambda, \eta) = 0$ for these $D$’s, and so are any of its direct summands. Since $i^j_{wD} L(w^D, \lambda, \eta) = O^\eta_{C(w^D)} \neq 0$, in particular $j^j_{D} L(w^D, \lambda, \eta) \neq 0$, we conclude that $L(w^D, \lambda, \eta)$ is not a direct summand of $U_\alpha L(w^C s_\alpha, \lambda, \eta)$. We record this separately as a lemma for later use.

**Lemma 4.4.3.** Let $\alpha \in \Pi \cap \Pi_\lambda$ and $C \in W_\phi \backslash W$ be such that $Cs_\alpha < C$. If $D \in W_\phi \backslash W$ is either not in the same double $(W_\phi, W_\lambda)$-coset as $C$, or if $D$ is in the same double coset as $C$ but $D \not\leq u, \lambda \ C$, then $j^j_{D} U_\alpha L(w^C s_\alpha, \lambda, \eta) = 0$. In particular,

$$U_\alpha L(w^C s_\alpha, \lambda, \eta) = \bigoplus_{D \in W_\phi \backslash W_\phi \cap W_\lambda \atop D \leq u, \lambda \ C} L(w^D, \lambda, \eta)^{\otimes c_D}$$

for some $c_D \in \mathbb{Z}$.

Therefore

$$T_\alpha(\varphi_\lambda(Cs_\alpha)) = \nu(U_\alpha L(w^C s_\alpha, \lambda, \eta))$$

$$\hspace{1cm} = \nu \left( \bigoplus_{D \in W_\phi \backslash W_\phi \cap W_\lambda \atop D \leq u, \lambda \ C} L(w^D, \lambda, \eta)^{\otimes c_D} \right)$$

$$\hspace{1cm} = \sum_{D \in W_\phi \backslash W_\phi \cap W_\lambda \atop D \leq u, \lambda \ C} c_D \nu(L(w^D, \lambda, \eta))$$

$$\hspace{1cm} = \sum_{D \in W_\phi \backslash W_\phi \cap W_\lambda \atop D \leq u, \lambda \ C} c_D \varphi_\lambda(D).$$

and 4.2.2(3) is verified for $C$. 

cohomologies of the triangle into short exact sequences, producing a description of $U_\alpha L(w^C s_\alpha, \lambda, \eta)$ in terms of ranks of its restrictions to cells. One obtains the desired equality

$$\nu(U_\alpha L(w^C s_\alpha, \lambda, \eta)) = T_\alpha(\nu(L(w^C s_\alpha, \lambda, \eta)))$$

by plugging in these rank descriptions to the definition of $\nu(U_\alpha L(w^C s_\alpha, \lambda, \eta))$. 

Therefore the decomposition theorem of holonomic $D$-modules [Moc11] and semisimplicity of $U_\alpha$ [Mil, V.2.7], there exist some $c_D \in \mathbb{Z}$ so that

$$T_\alpha(\varphi_\lambda(Cs_\alpha)) = \sum_{D \in \mathcal{C}} c_D \varphi_\lambda(D)$$

for any integral $\lambda$.

Roughly the same argument applies to our situation and produces the same comparison (4.4.1).
4.2. Verification of 4.2.2 (1) for \( \ell(w^C) = k \). Suppose there exists \( \beta \in \Pi - \Pi_\lambda \) such that \( Cs_\beta < C \). Then the non-integral intertwining functor \( I_{s_\beta} \) sends \( \mathcal{L}(w^C, \lambda, \eta) \) to \( \mathcal{L}(w^{C_\beta}, s_\beta \lambda, \eta) \), allowing us to translate induction assumption for the latter module to the former. In more detail, the induction assumption applies to \( C_\beta \). Hence, from 4.2.2(1) for \( C_{s_\beta} \) and \( s_\beta \lambda \), we obtain

\[
\varphi_{s_\beta \lambda}(C_{s_\beta}) = \delta_{C_{s_\beta}} + \sum_{D \in W_{\Theta, r}W_{s_\beta}} Q_D \delta_D,
\]

for some polynomials \( Q_D \in q\mathbb{Z}[q] \), where \( r \) is the unique element in \( A_{\Theta, s_\beta \lambda} \) such that \( C_{s_\beta} \in W_{\Theta} \setminus W_{\Theta, r}W_{s_\beta} \). Applying 4.2.2(3) for \( C \),

\[
\varphi_{\lambda}(C) = \varphi_{s_\beta \lambda}(C_{s_\beta}) \delta_{C} + \sum_{D \in W_{\Theta, r}W_{s_\beta}} Q_D \delta_{D_{s_\beta}}.
\]

By 2.4.6 and its corollary, there exists \( w \in W_{\Theta} \) with \( wr = us_\beta \). Hence

\[
W_{\Theta}W_{s_\beta} = W_{\Theta}W_{rs_\beta}W_{s_\beta} = W_{\Theta}us_\beta W_{s_\beta} = W_{\Theta}uW_{s_\beta},
\]

and we see that \( D \in W_{\Theta} \setminus W_{\Theta, r}W_{s_\beta} \) if and only if \( D_{s_\beta} \in W_{\Theta} \setminus W_{\Theta, u}W_{s_\beta} \). By 2.4.8,

\[
D_{r, s_\beta} \in C_{s_\beta} \iff D_{s_\beta} \in C.
\]

Hence

\[
\varphi_{\lambda}(C) = \delta_{C} + \sum_{D_{s_\beta} \in W_{\Theta} \setminus W_{\Theta, u}W_{s_\beta}} Q_D \delta_{D_{s_\beta}}
\]

for some \( Q_D \in q\mathbb{Z}[q] \), and 4.2.2(1) holds for \( C \) in this case.

If such \( \beta \) does not exist, then there exists a simple integral root \( \alpha \) with \( C_{s_\alpha} < C \). By induction assumption, we understand \( \mathcal{L}(w^{C_{s_\alpha}}, \lambda, \eta) \). So we hope a close analysis of \( U_\alpha \) will tell us about the information of \( \mathcal{L}(w^C, \lambda, \eta) \).

From the discussion of §4.4 we know \( \mathcal{L}(w^C, \lambda, \eta) \) is a direct summand of \( U_\alpha \mathcal{L}(C_{s_\alpha}, \lambda, \eta) \). The same holds if we apply \( j_D^1 \) or \( i^1_{w, D} \) to these two modules. However, we have seen from 4.4.3 that \( j_D^1 U_\alpha \mathcal{L}(w^C s_\alpha, \lambda, \eta) = 0 \) whenever \( D \) does not satisfy the conditions \( D \in W_{\Theta} \setminus W_{\Theta, u}W_{s_\beta} \) and \( D \leq_{u, \lambda} C \). Consequently \( i^1_{w, D} \mathcal{L}(w^C, \lambda, \eta) = i^1_{w, D} U_\alpha \mathcal{L}(w^C s_\alpha, \lambda, \eta) = 0 \) for such \( D \)’s, and hence

\[
\varphi_{\lambda}(C) = \sum_{D \in W_{\Theta} \setminus W_{p_\mathbb{Z}}} \sum_{D_{s_\beta} \leq_{u, \lambda} C} \left( \text{rank } H^p i^1_{w, D} \mathcal{L}(w^C, \lambda, \eta) \right) q^p \delta_D
\]

It suffices to compute the remaining coefficients. Consider \( D = C \). Since \( \mathcal{I}(w^C, \lambda, \eta) \) is a direct image, it contains no section supported in \( \partial C(w^C) \) except 0. The same holds for \( \mathcal{L}(w^C, \lambda, \eta) \) being a submodule of \( \mathcal{I}(w^C, \lambda, \eta) \). Hence \( \mathcal{L}(w^C, \lambda, \eta)|_{X - \partial C(w^C)} \) is a nonzero submodule of \( \mathcal{I}(w^C, \lambda, \eta)|_{X - \partial C(w^C)} \). But \( \mathcal{I}(w^C, \lambda, \eta)|_{X - \partial C(w^C)} \) is irreducible by Kashiwara’s equivalence of categories for the closed immersion \( C(w^C) \rightarrow X - \partial C(w^C) \), so \( \mathcal{L}(w^C, \lambda, \eta)|_{X - \partial C(w^C)} = \)
4.2.2. \( I(w^C, \lambda, \eta)|_{x - \partial C(w^C)} \), and their further pullback to \( C(w^C) \) is \( O_{C(w^C)}^p \). Hence the coefficient of \( \delta_C \) is
\[
\sum_{p \in \mathbb{Z}} (\text{rank } H^p I_{w^D}^C \mathcal{L}(w^C, \lambda, \eta)) q^p = 1.
\]
When \( D < C \), we have \( H^0 I_{w^D}^C \mathcal{L}(w^C, \lambda, \eta) = 0 \) since \( H^0 I_{w^D}^C \) takes sections supported in \( C(w^D) \) and \( C(w^D) \subseteq \partial C(w^C) \). Hence the coefficient of \( \delta_D \) is
\[
\sum_{p \in \mathbb{Z}} (\text{rank } H^p I_{w^D}^C \mathcal{L}(w^C, \lambda, \eta)) q^p \in q\mathbb{Z}[q].
\]
Thus 4.2.2(1) holds for \( C \).

4.6. Verification of 4.2.2(4) for \( \ell(w^C) = k \). Based on our definition of parabolic Kazhdan-Lusztig polynomials 4.1.2, we need to find \( \alpha \in \Pi \lambda \) such that \( C s_\alpha < u, \lambda \) \( C \) and equation (4.1.3) holds for the function
\[
\psi_{u, \lambda}(C|\lambda) := \varphi_\lambda(C)|\lambda.
\]
See §2.5 for an explanation of the geometric idea behind this proof.

If \( \alpha \) can be chosen to be in \( \Pi \cap \Pi \lambda \), then by the following lemma, (4.1.3) follows from 4.2.2(2) for \( C \).

**Lemma 4.6.1.** Let \( \alpha \in \Pi \cap \Pi \lambda \). Then for each \( u \in A_{\Theta, \lambda} \)
\[
(-)|\lambda \circ T_\alpha = T_{\alpha}^u |\lambda \circ (-)|\lambda
\]
as maps from \( \text{ind}_\lambda \mathcal{H}_{\Theta(u, \lambda)} \subseteq \mathcal{H}_{\Theta} \) to \( \mathcal{H}_{\Theta(u, \lambda)} \). In other words, the following diagram commutes
\[
\begin{array}{ccc}
\mathcal{H}_{\Theta} & \xrightarrow{T_{\alpha}} & \mathcal{H}_{\Theta} \\
(-)|\lambda \downarrow & & \downarrow (-)|\lambda \\
\bigoplus_{u \in A_{\theta, \lambda}} \mathcal{H}_{\Theta(u, \lambda)} & \xrightarrow{\bigoplus_{u} T_{\alpha}^u} & \bigoplus_{u \in A_{\theta, \lambda}} \mathcal{H}_{\Theta(u, \lambda)}.
\end{array}
\]

The proof is left as an exercise. It consist of unwrapping definitions and using the facts that \( \text{ind}_\lambda \) preserves partial orders on right cosets 2.4.3.

If such \( \alpha \) cannot be found, we will need to use non-integral intertwining functors to move \( \alpha \) to some simple root \( z^{-1}_1 \alpha \) and move \( \mathcal{L}(w^C, \lambda, \eta) \) to some irreducible module supported on a smaller orbit, then translate the induction assumption there back. The translation requires the following lemma. The proof is similar to the previous one, using 2.4.8 instead.

**Lemma 4.6.2.** Let \( \alpha \in \Pi \cap \Pi \lambda \), \( \beta \in \Pi - \Pi \lambda \). For any \( u \in A_{\Theta, \lambda} \), let \( r \in A_{\Theta, s_{\beta} \lambda} \) be the unique element such that \( W_{\Theta} u s_{\beta} W_{s_{\beta} \lambda} = W_{\Theta} r W_{s_{\beta} \lambda} \). Then
\[
(s_{\beta}(-) s_{\beta}) \circ T_{\alpha}^u = T_{s_{\beta} \alpha} r_{s_{\beta} \lambda} \circ (s_{\beta}(-) s_{\beta})
\]
as maps from \( \mathcal{H}_{\Theta(u, \lambda)} \) to \( \mathcal{H}_{\Theta(r, s_{\beta} \lambda)} \), where \( s_{\beta}(-) s_{\beta} \) denotes conjugation by \( s_{\beta} \). In other words, the following diagram commutes
\[
\begin{array}{ccc}
\mathcal{H}_{\Theta(u, \lambda)} & \xrightarrow{T_{\alpha}^u} & \mathcal{H}_{\Theta(u, \lambda)} \\
 s_{\beta}(-) s_{\beta} \downarrow & & \downarrow s_{\beta}(-) s_{\beta} \\
\mathcal{H}_{\Theta(r, s_{\beta} \lambda)} & \xrightarrow{T_{s_{\beta} \alpha}} & \mathcal{H}_{\Theta(r, s_{\beta} \lambda)}.
\end{array}
\]

Choose \( \alpha \in \Pi \lambda \), \( s \geq 0 \) and \( \beta_1, \ldots, \beta_s \in \Pi \) such that if we write \( z_0 = 1, z_i = s_{\beta_1} \cdots s_{\beta_i} \) and \( z = z_s \), the following conditions hold:

(a) for any \( 0 \leq i \leq s - 1, \beta_{i+1} \) is non-integral to \( z_i^{-1} \).
(b) $z^{-1} \alpha \in \Pi \cap \Pi z^{-1} \lambda$;
(c) $Cs_\alpha \prec_{u, \lambda} C$;
(d) if $s > 0$, $C z < C$;
(e) $C s_\alpha z = C z s^{-1}_\alpha < C z$.

Such a choice exists by 2.5.1. Combining the lemmas with the diagram (2.4.9), we obtain a commutative diagram

$$
\begin{array}{c}
\mathcal{H}_\Theta \\
\downarrow \frac{\text{(-)}z^{-1}}{\text{(-)}|z^{-1}_\lambda} \\
\mathcal{H}_\Theta (r, z^{-1}_\lambda) \\
\bigoplus \mathcal{H}_\Theta (u, z^{-1}_\lambda) \\
\downarrow r \in A_{\Theta z^{-1}_\lambda} \\
\bigoplus \mathcal{H}_\Theta (u, z^{-1}_\lambda) \\
\downarrow u \in A_{\Theta z^{-1}_\lambda} \\
\mathcal{H}_\Theta \\
\end{array}
\xrightarrow{T_z^{-1}_\alpha} \xrightarrow{\bigoplus \mathcal{H}_\Theta (r, z^{-1}_\lambda) \oplus \bigoplus \mathcal{H}_\Theta (u, z^{-1}_\lambda) \oplus \mathcal{H}_\Theta (r, z^{-1}_\lambda)} \xrightarrow{T_\Theta} \mathcal{H}_\Theta
$$

(4.6.3)

Since $C z < C$, the induction assumption applies to $C z$ for $z^{-1}_\lambda$. In particular, if we apply 4.2.2(2) to $C z s^{-1}_\alpha < C z$ and $z^{-1}_\lambda$, we obtain the equation

$$
T_z^{-1}_\alpha (\varphi_{z^{-1}_\lambda}(C z s^{-1}_\alpha)) = \sum_{D \in W_\Theta \backslash W_\Theta r W_{z^{-1}_\lambda}} c_\lambda \varphi_{z^{-1}_\lambda}(D)
$$

(4.6.4)

where $r$ is the unique element in $A_{\Theta z^{-1}_\lambda}$ such that $C z \in W_\Theta \backslash W_\Theta r W_{z^{-1}_\lambda}$. We apply to both sides $(-)|z^{-1}_\lambda$ followed by $z(-)z^{-1}$.

If we view $\varphi_{z^{-1}_\lambda}(C z s^{-1}_\alpha)$ as an element in the middle $\mathcal{H}_\Theta$ in the diagram, the left side of (4.6.4) lands in $\mathcal{H}_\Theta (u, \lambda)$ in the bottom middle of the diagram through the rightmost path after applying $(-)|z^{-1}_\lambda$ and $z(-)z^{-1}$. Going through the leftmost path instead, this element in $\mathcal{H}_\Theta (u, \lambda)$ becomes

$$
T_\alpha^\lambda (\varphi_{z^{-1}_\lambda}(C z s^{-1}_\alpha)z^{-1}_\lambda).
$$

Rewrite $C z s^{-1}_\alpha = C s_\alpha z$ and use 4.2.2(3) for $C s_\alpha$, the above quantity becomes

$$
T_\alpha^\lambda (\varphi_\lambda (C s_\alpha)|\lambda) = T_\alpha^\lambda (\psi_\lambda (C s_\alpha)|\lambda).
$$

Viewing the right side of (4.6.4) as an element in the middle $\mathcal{H}_\Theta$ in the diagram, $(-)|z^{-1}_\lambda$ and $z(-)z^{-1}$ sends it to $\mathcal{H}_\Theta (u, \lambda)$ at the bottom-left along the middle path. Going through the leftmost path instead, this element becomes

$$
\sum_{D \in W_\Theta \backslash W_\Theta r W_{z^{-1}_\lambda}} c_\lambda \varphi_\lambda (Dz^{-1})|\lambda = \sum_{D \in W_\Theta \backslash W_\Theta r W_{z^{-1}_\lambda}} c_\lambda \psi_\lambda ((Dz^{-1})|\lambda).
$$

As in the first part of §4.5, we can rewrite the subscript of the sum. There is an element $w \in W_\Theta$ such that $wr = uz$ by 2.4.6. Hence

$$
W_\Theta r W_{z^{-1}_\lambda} = W_\Theta wr z^{-1}_\lambda W_\lambda z
\quad = W_\Theta uz z^{-1}_\lambda W_\lambda z
\quad = W_\Theta uz W_\lambda z,
$$

and $D \in W_\Theta \backslash W_\Theta r W_{z^{-1}_\lambda}$ if and only if $Dz^{-1} \in W_\Theta \backslash W_\Theta u W_\lambda$. Moreover, by 2.4.8,

$$
D \leq_{r, z^{-1}_\lambda} C z \iff Dz^{-1} \leq_{u, \lambda} C.
$$
Hence (4.6.4) becomes
\[ T_{\alpha}^{u,\lambda}(\psi_{u,\lambda}(C_{\lambda})) = \sum_{Dz^{-1} e W_{\Theta} \cap W_{\Theta} u W_{\lambda}} c_{D}\psi_{u,\lambda}((Dz^{-1})|_{\lambda}). \]

Therefore, \( \alpha \in \Pi_{\lambda} \) is such that \( C_{\lambda} < u,\lambda C \) and equation (4.1.3) holds for \( C_{\lambda} \alpha \). By 4.1.2, the polynomials \( P_{CD}^{u,\lambda} \) are parabolic Kazhdan-Lusztig polynomials for \( (W_{\lambda}, \Pi_{\lambda}, \Theta(u, \lambda)) \). Thus 4.2.2(4) holds for \( C \).

This completes the proof of 4.2.2.

5. Character formula for irreducible modules

5.1. Regular case. By standard arguments, the algorithm 4.2.2 leads to a character formula for irreducible Whittaker modules for regular infinitesimal characters.

Let \( \lambda \in \mathfrak{h}^* \) be antidominant regular. As explained in §1.2, Beilinson-Bernstein’s localization and holonomic duality are equivalences of categories which send Whittaker modules to \( D \)-modules. Combined with the map \( \nu \), we obtain the composition

\[
\begin{align*}
N_{\theta, \eta} & \xrightarrow{\nu_{\lambda}^{-1}} \text{Mod}_{\text{coh}}(D_{\lambda}, N, \eta) \xrightarrow{\nu} \text{Mod}_{\text{coh}}(D_{-\lambda}, N, \eta) \xrightarrow{(\cdot)|_{-\lambda}} \bigoplus_{u \in A_{\theta, -\lambda}} \text{Mod}_{\Theta(u, -\lambda)}, \\
L(w^C, \lambda, \eta) & \xleftarrow{\nu_{\lambda}^{-1}} \mathcal{L}(w^C, \lambda, \eta) \xrightarrow{\nu_{\lambda}^{-1}} \mathcal{L}(w^C, -\lambda, \eta) \xrightarrow{\varphi_{-\lambda}(C)} \varphi_{\lambda}(C)|_{-\lambda} \xrightarrow{\delta_{-\lambda}} \delta_{C_{-\lambda}}.
\end{align*}
\]

Since the coefficients \( \sum_{p \in \mathbb{Z}} (-1)^p \text{rank} \, H_{t_\lambda}^p \mathcal{F} \) in the definition of \( \nu \) is additive with respect to short exact sequences, \( \nu \) factors through the Grothendieck group

\[ \nu|_{q=-1} : K \text{Mod}_{\text{coh}}(D_{\lambda}, N, \eta) \longrightarrow \text{Mod}_{\Theta} |_{q=-1} \]

which is an isomorphism by 4.2.1. Therefore we have an isomorphism

\[ K\mathcal{N}_{\theta, \eta} \cong \bigoplus_{u \in A_{\theta, -\lambda}} \text{Mod}_{\Theta(u, -\lambda)} |_{q=-1} \]

\[ [L(w^C, \lambda, \eta)] \mapsto \varphi_{\lambda}(C)|_{-\lambda}|_{q=-1} \]

\[ [M(w^C, \lambda, \eta)] \mapsto \delta_{C_{-\lambda}}|_{q=-1}. \]

Hence 4.2.2(1) and (4) imply

\[ [L(w^C, \lambda, \eta)] = \sum_{D \in W_{\Theta} \setminus W_{\Theta} u W_{\lambda}, D \leq \lambda} P_{CD}^{u,\lambda}(-1)[M(w^D, \lambda, \eta)] \]

in \( K\mathcal{N}_{\theta, \eta} \). Note that \( \Sigma_{\lambda} = \Sigma_{-\lambda} \) as subsets of \( \Sigma \) and \( W_{\lambda} = W_{-\lambda} \) as subgroups of \( W \). Hence all the combinatorial structures defined based on \( \lambda \) and \( -\lambda \) are canonically identified. Further applying the character map, we thus obtain

**Theorem 5.1.2** (Character formula for irreducible Whittaker modules: Regular case). Let \( \lambda \in \mathfrak{h}^* \) be antidominant and regular. Let \( \eta : n \to C \) be any character. For any \( C \in W_{\Theta} \setminus W_{\Theta} u W_{\lambda} \), let \( u \in A_{\theta, \lambda} \) be the unique element such that \( C \subseteq W_{\Theta} u W_{\lambda} \). Then

\[ \text{ch} \, L(w^C, \lambda, \eta) = \sum_{D \in W_{\Theta} \setminus W_{\Theta} u W_{\lambda}, D \leq \lambda} P_{CD}^{u,\lambda}(-1) \text{ch} \, M(w^D, \lambda, \eta), \]

where the \( P_{CD}^{u,\lambda} \)'s are parabolic Kazhdan-Lusztig polynomials for \( (W_{\lambda}, \Pi_{\lambda}, \Theta(u, \lambda)) \) as defined in 4.1.2.

When \( \lambda \) is integral, we have a simpler description, which we state separately.
Corollary 5.1.4 (Character formula for irreducible Whittaker modules: Regular integral case). Let $\lambda \in \mathfrak{h}^*$ be antidominant, regular and integral. Let $\eta : \mathfrak{n} \to \mathbb{C}$ be any character. For any $C \in W_\Theta \backslash W$, 

$$\text{ch } L(w^C \lambda, \eta) = \sum_{D \in W_\Theta \backslash W} P_{CD}(-1) \text{ch } M(w^D \lambda, \eta),$$

where the $P_{CD}$'s are parabolic Kazhdan-Lusztig polynomials for $(W, \Pi, \Theta)$ as defined in 4.1.1.

Inverting the matrix $(P_{CD}(-1))_{C,D}$, we recover the description in [Mi97] and [Rom21] of multiplicities of irreducible Whittaker modules in standard Whittaker modules with antidominant regular integral infinitesimal characters.

At another extreme, when $\eta = 0$ (i.e. $\Theta = \emptyset$), we recover the well-known non-integral Kazhdan-Lusztig conjecture for Bernstein-Gelfand-Gelfand’s category $\mathcal{O}$.

Corollary 5.1.5 (Kazhdan-Lusztig conjecture for category $\mathcal{O}$). Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. For any $w \in W$, 

$$\text{ch } L(w^\lambda) = \sum_{w \in W^\lambda} P_{ww}^\lambda(-1) \text{ch } M(v^\lambda),$$

where the $P_{ww}^\lambda$'s are Kazhdan-Lusztig polynomials for $(W^\lambda, \Pi^\lambda, \emptyset)$ as defined in 4.1.2, $M(v^\lambda)$ is the Verma module of highest weight $v^\lambda - \rho$, and $L(w^\lambda)$ is the unique irreducible quotient of $M(w^\lambda)$.

5.2. Singular case. The singular case can be deduced from the regular case easily.

Let $\lambda \in \mathfrak{h}^*$ be antidominant and singular. We still have the maps (5.1.1), but the exact functor $\Gamma(X, -)$ is no longer an equivalence of categories and only descends to a surjection $\text{KMod}_{\text{coh}}(D_{\lambda, \eta}) \to KN_{\Theta, \eta}$ on Grothendieck groups. However, the identification $\Gamma(X, M(w^D, \lambda, \eta)) = M(w^D \lambda, \eta)$ still holds. Therefore, the argument for regular case produces the equality

$$\text{ch } \Gamma(X, L(w^C, \lambda, \eta)) = \sum_{D \in W_\Theta \backslash W} P_{CD}^w(-1) \text{ch } M(w^D \lambda, \eta). \tag{5.2.1}$$

However, $\Gamma(X, L(w^C, \lambda, \eta))$ could be zero, and the $M(w^D \lambda, \eta)$’s could coincide for different $D$’s. Therefore, it suffices to describe which $M(w^D \lambda, \eta)$’s coincide and which $\Gamma(X, L(w^C, \lambda, \eta))$’s are zero.

The first question has an easy answer. Recall that for $C, D \in W_\Theta \backslash W$, $M(w^D \lambda, \eta) = M(w^C \lambda, \eta)$ if and only if $W_\Theta w^D \lambda = W_\Theta w^C \lambda$. Let $W^\lambda$ be the stabilizer of $\lambda$ in $W$. Then the above condition is equivalent to $W_\Theta w^D W^\lambda = W_\Theta w^C W^\lambda$, i.e. that $C$ and $D$ are in the same double $(W_\Theta, W^\lambda)$-coset.

Lemma 5.2.2. Let $\lambda \in \mathfrak{h}^*$ be antidominant and let $\eta : \mathfrak{n} \to \mathbb{C}$ be a character. The following are equivalent:

(a) $M(w^C \lambda, \eta) = M(w^D \lambda, \eta)$;
(b) $\Gamma(X, M(w^C, \lambda, \eta)) = \Gamma(X, M(w^D, \lambda, \eta))$;
(c) $C$ and $D$ are in the same double $(W_\Theta, W^\lambda)$-coset.

Therefore, for a fixed standard Whittaker module $M$, there is a unique double coset $W_\Theta w W^\lambda$ such that $\Gamma(X, M(w^D, \lambda, \eta)) = M$ for all $D \in W_\Theta \backslash W_\Theta w W^\lambda$.

The following proposition answers the second question.

Proposition 5.2.3. Let $\lambda \in \mathfrak{h}^*$ be antidominant and let $\eta : \mathfrak{n} \to \mathbb{C}$ be a character. Let $v \in W$. Then the set $W_\Theta \backslash W_\Theta v W^\lambda$ of right $W_\Theta$-cosets contains a unique smallest element $C$. Furthermore,

(a) $\Gamma(X, L(w^C, \lambda, \eta)) = L(w^C \lambda, \eta) \neq 0$; and
(b) $\Gamma(X, L(w^D, \lambda, \eta)) = 0$ for any $D \in W_\Theta \backslash W_\Theta v W^\lambda$ not equal to $C$. 

In other words, for a fixed standard Whittaker module $M$, among all the costandard $D_{\lambda}$-modules that realizes $M$, the irreducible quotient of the one with the smallest support realizes the unique irreducible submodule of $M$.

**Proof.** Write $M = M(v,\lambda,\eta)$ and $L = L(v,\lambda,\eta)$.

First, there is one and at most one $D$ in $W_{\Theta}\backslash W$ with $\Gamma(X, L(w^D,\lambda,\eta)) = L$. This is because, by the theory of localization, there is a unique irreducible $D_{\lambda}$-module $V$ with $\Gamma(X, V) = L$ (since the irreducible quotient of $D_{\lambda} \otimes_a L$). By the classification of irreducible twisted Harish-Chandra sheaves, $V$ equals to $L(w^D,\lambda,\eta)$ for a single $D \in W_{\Theta}\backslash W$.

Since $L(w^D,\lambda,\eta)$ is the unique irreducible quotient of $\mathcal{M}(w^D,\lambda,\eta)$ and $\Gamma(X,-)$ is exact on $D_{\lambda}$-modules, $\Gamma(X, L(w^D,\lambda,\eta))$ equals the unique irreducible quotient $L(w^D\lambda,\eta)$ of $M(w^D\lambda,\eta)$. Therefore, the equality $L(v,\lambda,\eta) = \Gamma(X, V) = \Gamma(X, L(w^D,\lambda,\eta)) = L(w^D\lambda,\eta)$ implies $M(w^C\lambda,\eta) = M(w^D\lambda,\eta)$ which forces our $D$ to be in the double coset $W_{\Theta}vW^\lambda$.

It remains to show that such a $D$ is minimum in $W_{\Theta}\backslash W_{\Theta}vW^\lambda$. Let $C$ be a minimal element in $W_{\Theta}\backslash W_{\Theta}vW^\lambda$. The composition factors of $M(w^C,\lambda,\eta)$ consist of some $L(w^E,\lambda,\eta)$’s with $E \leq C$. Taking global sections, we see that the composition factors of $M = \Gamma(X, M(w^C,\lambda,\eta))$ consist of some $\Gamma(X, L(w^E,\lambda,\eta))$’s that are nonzero and with $E \leq C$. On the other hand, $L = \Gamma(X, L(w^D,\lambda,\eta))$ is a composition factor of $L$. Hence $\Gamma(X, L(w^D,\lambda,\eta)) = \Gamma(X, L(w^E,\lambda,\eta))$ for some $E \leq C$. By the same uniqueness statement appeared in the preceding paragraph, $L(w^D,\lambda,\eta) = L(w^E,\lambda,\eta)$ and hence $D = E \leq C$. By the minimality of $C$, $D = C$. Thus $C = D$ is the minimum element in $W_{\Theta}\backslash W_{\Theta}vW^\lambda$ and $\Gamma(X, L(w^C,\lambda,\eta)) = L$.

There exists a number $c \in \mathbb{C}$ so that $W^\lambda = W_{c\lambda}$. Hence by 2.3.3, the set

$$A^n_{\Theta} : = A_{c\lambda} \cap (w_{\Theta}^\Theta W)$$

is a cross-section of $W_{\Theta}\backslash W/W^\lambda$ consisting of the unique shortest elements in each double coset.

**Corollary 5.2.4.** Let $\lambda \in \mathfrak{h}^*$ be antidominant and let $\eta : \mathfrak{n} \to \mathbb{C}$ be a character. Let $C \in W_{\Theta}\backslash W$. The following are equivalent:

(a) $C = W_{\Theta}v$ for some $v \in A^n_{\Theta}$;
(b) $\Gamma(X, L(w^C,\lambda,\eta)) \neq 0$;
(c) $\Gamma(X, L(w^C,\lambda,\eta)) = L(w^C\lambda,\eta)$.

Using these observations, we can write down a character formula for general infinitesimal characters.

**Theorem 5.2.5** (Character formula for irreducible Whittaker modules: General case). Let $\lambda \in \mathfrak{h}^*$ be antidominant. Let $\eta : \mathfrak{n} \to \mathbb{C}$ be any character. For any $v \in A^n_{\Theta}$, let $C = W_{\Theta}v$, and let $u \in A_{\Theta,\lambda}$ be the unique element such that $C \subseteq W_{\Theta}uW^\lambda$. Then

$$\text{ch } L(v,\lambda,\eta) = \text{ch } L(w^C\lambda,\eta) = \sum_{z \in A^n_{\Theta} \cap (W_{\Theta}uW^\lambda)} \left( \sum_{D \in W_{\Theta}\backslash W_{\Theta}vW^\lambda} \sum_{D \in u,C} P_{CD}^{u,\lambda}(-1) \right) \text{ch } M(z,\lambda,\eta),$$

(5.2.6)

where the $P_{CD}^{u,\lambda}$’s are parabolic Kazhdan-Lusztig polynomials for $(W_\lambda, \Pi_\lambda, \Theta(u,\lambda))$ as defined in 4.1.2. As $v$ ranges over $A^n_{\Theta}$, $L(v,\lambda,\eta)$ exhausts all irreducible objects in $\mathcal{N}_{\Theta,\eta}$.

**Proof.** The right hand side is obtained by rearranging the right side of (5.2.1) based on 5.2.2. The left hand side and the last statement (that those $L(v,\lambda,\eta)$’s exhaust all irreducibles) follows from 5.2.4 and Beilinson-Bernstein’s equivalence of categories in the singular case. 

$\blacksquare$
6. An example in $A_3$

The $A_3$ root system (pictured below) is the smallest example in which all nontrivial phenomena appear. To make the picture more readable, only the positive roots are connected to the origin. Here $\lambda$ can be chosen to be $\lambda = -m\rho + c(-\alpha + 2\beta + \gamma)$ for any nonzero number $c$ transcendental over $\mathbb{Q}$ and any large enough integer $m$ so that $\lambda$ is antidominant regular ($-\alpha + 2\beta + \gamma$ is a vector perpendicular to the plane spanned by $\alpha + \beta$ and $\gamma$).

\[ \beta + \gamma \]

In the above diagram, $\{\alpha, \beta, \gamma\}$ are simple roots. $\Theta = \{\beta, \gamma\}$ which are indicated by double lines in the picture. Roots in $\Sigma_\lambda$ are marked by $\bigcirc$, and those not in $\Sigma_\lambda$ are marked by $\bullet$.

Below is a diagram of the Weyl group, arranged in a way so that elements in the same right $W_\Theta$-coset are grouped together and are connected by double lines. Elements surrounded by shapes are the longest elements in right $W_\Theta$-cosets. Elements that are crossed out are those in $W_\lambda$. There are two double $(W_\Theta, W_\lambda)$-cosets: elements in $W_\Theta s_\gamma s_\beta W_\lambda$ are underlined; elements in $W_\Theta W_\lambda$ are those that are not underlined.

Let’s first look at the double coset $W_\Theta s_\gamma s_\beta W_\lambda$, with $u = s_\gamma s_\beta \in A_{\Theta, \lambda}$. It equals a single left $W_\lambda$-coset $s_\gamma s_\beta W_\lambda$ and a single right $W_\Theta$-coset $W_\Theta s_\gamma s_\beta$. Hence

$\Theta(s_\gamma s_\beta, \lambda) = \Pi_\lambda$, $W_{\lambda, \Theta(s_\gamma s_\beta, \lambda)} = W_\lambda$,
and \( (W_\Theta s_\alpha s_\beta) \mid_\lambda = W_\lambda 1 \), the unique right \( W_\lambda \)-coset in \( W_\lambda \). Therefore
\[
\varphi_\lambda(W_\Theta s_\gamma s_\beta) = \delta_{W_\Theta s_\gamma s_\beta}, \\
\text{ch } L(s_\gamma s_\beta \lambda, \eta) = \text{ch } M(s_\gamma s_\beta \lambda, \eta).
\]

Now let’s look at the other double coset \( W_\Theta W_\lambda \), with \( u = 1 \) and
\[
\Theta(1, \lambda) = \{ \alpha + \beta \}, \quad W_{\lambda, \Theta(1, \lambda)} = \{ 1, s_\alpha + \beta \}.
\]

For convenience, we write
\[
W_\bullet := W_{\lambda, \Theta(1, \lambda)}.
\]

The root system \( \Sigma_\lambda \) and a diagram for \( (W_\lambda, \Pi_\lambda, \Theta_\lambda^1) \) are

\[
\begin{array}{ccc}
\alpha + \beta & \alpha + \beta + \gamma \\
\gamma & & \gamma \\
\end{array}
\]

The map \((-)\mid_\lambda \) restricted to \( W_\Theta \setminus W_\Theta W_\lambda \) can be visualized as

\[
\begin{array}{ccc}
s_{\alpha + \beta + \gamma} & s_{\gamma} s_{\alpha + \beta} \\
s_{\alpha + \beta} s_\gamma & & s_\gamma \\
[s_{\alpha + \beta}] & & 1 \\
\end{array}
\]

where a coset on the right hand side is sent to the coset on the left with the same shape. The parabolic Kazhdan-Lusztig polynomials for \( (W_\lambda, \Pi_\lambda, \Theta_\lambda^1) \) are

| \( P_{EF}^{1, \lambda} \) | \( E \) | \( F \) | \( W_\bullet s_{\alpha + \beta} \) | \( W_\bullet s_{\alpha + \beta} s_\gamma \) | \( W_\bullet s_{\alpha + \beta + \gamma} \) |
|-----------------|-------|---------|-----------------|-----------------|-----------------|
| \( W_\bullet s_{\alpha + \beta} \) | \( q \) | \( q \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( W_\bullet s_{\alpha + \beta} s_\gamma \) | \( q \) | \( 1 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( W_\bullet s_{\alpha + \beta + \gamma} \) | \( 0 \) | \( q \) | \( 1 \) | \( 1 \) | \( 1 \) |

Hence
\[
\varphi_\lambda(W_\Theta s_\alpha s_\beta s_\alpha) = P_{(W_\bullet s_{\alpha + \beta}),(W_\bullet s_{\alpha + \beta})}^{1, \lambda} \delta_{W_\Theta s_\alpha s_\beta s_\alpha} + P_{(W_\bullet s_{\alpha + \beta}),(W_\bullet s_{\alpha + \beta} s_\gamma)}^{1, \lambda} \delta_{W_\Theta s_\alpha s_\beta s_\gamma} + P_{(W_\bullet s_{\alpha + \beta}),(W_\bullet s_{\alpha + \beta + \gamma})}^{1, \lambda} \delta_{W_\Theta w_0} = \delta_{W_\Theta s_\alpha s_\beta s_\alpha},
\]

\[
\varphi_\lambda(W_\Theta s_\alpha s_\beta s_\gamma) = P_{(W_\bullet s_{\alpha + \beta}),(W_\bullet s_{\alpha + \beta} s_\gamma)}^{1, \lambda} \delta_{W_\Theta s_\alpha s_\beta s_\gamma} + P_{(W_\bullet s_{\alpha + \beta}),(W_\bullet s_{\alpha + \beta + \gamma})}^{1, \lambda} \delta_{W_\Theta s_\alpha s_\beta s_\gamma}.
\]
Specializing to $q = -1$, we get
\[
\begin{align*}
\text{ch } L(s_{\alpha}s_{\beta}s_{\alpha}\lambda, \eta) &= \text{ch } M(s_{\alpha}s_{\beta}s_{\alpha}\lambda, \eta), \\
\text{ch } L(s_{\alpha}s_{\beta}s_{\alpha}s_{\gamma}\lambda, \eta) &= -\text{ch } M(s_{\alpha}s_{\beta}s_{\alpha}\lambda, \eta) + \text{ch } M(s_{\alpha}s_{\beta}s_{\alpha}s_{\gamma}\lambda, \eta), \\
\text{ch } L(w_0\lambda, \eta) &= -\text{ch } M(s_{\alpha}s_{\beta}s_{\alpha}s_{\gamma}\lambda, \eta) + \text{ch } M(w_0\lambda, \eta).
\end{align*}
\]

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