This paper is devoted to the study of the higher index theory of codimension 2 submanifolds originated by Gromov–Lawson and Hanke–Pape–Schick. The first main result is to construct the ‘codimension 2 transfer’ map from the Higson–Roe analytic surgery exact sequence of a manifold $M$ to that of its codimension 2 submanifold $N$ under some assumptions on homotopy groups. This map sends the primary and secondary higher index invariants of $M$ to those of $N$. The second is to establish that the codimension 2 transfer map is adjoint to the co-transfer map in cyclic cohomology, defined by the cup product with a group cocycle. This relates the Connes–Moscovici higher index pairing and Lott’s higher $\rho$-number of $M$ with those of $N$.
assumptions on homotopy groups (listed in Proposition 2.8), and if the Rosenberg index $\alpha_\pi(N)$ vanishes, then $M$ does not admit any psc metric. The corresponding result in higher signature is established by Higson–Schick–Xie [HSX18].

In recent researches it has turned out that this new obstruction does not go beyond the standard one, the Rosenberg index of $M$ itself. Namely, the non-vanishing of $\alpha_\pi(N)$ implies the non-vanishing of $\alpha_\pi(M)$. In [Ni18,NSZ19], Nitsche, Schick and Zeidler suppose a framework for proving this; the codimension 2 transfer map. They have constructed a homomorphism between K-homology groups of the classifying spaces

$$\tau_\sigma: K_\ast(B\Gamma) \to K\oplus_2(B\pi),$$

which relates the K-homology classes of the Dirac operator of $M$ with that of $N$. Following this, in [KS20], Schick and the author construct a homomorphism between group C*-algebra K-theory

$$\tau_\sigma: K_\ast(C^*\Gamma) \to K\oplus_2(C^*\pi),$$

based on the index pairing with a flat bundle of Calkin algebras. It is proved in [KS20, Theorems 1.2, 1.3] that $\tau_\sigma$ sends the Rosenberg index of $\bar{M}$, the boundary map in $K$-theory associated to the short exact sequence of coarse C*-algebras $0 \to C^*(\bar{M})^\Gamma \to D^*(\bar{M})^\Gamma \to Q^*(\bar{M})^\Gamma \to 0$ (where $\bar{M}$ denotes the universal covering of $M$), to that of $N$. Moreover, it relates the primary and the secondary higher index invariants of $M$ with those of $N$ in the following sense.

**Theorem 1.3.** There is a homomorphism of long exact sequences

$$\cdots \longrightarrow K_\ast(C^*(\bar{M})^\Gamma) \longrightarrow K_\ast(D^*(\bar{M})^\Gamma) \longrightarrow K_\ast(Q^*(\bar{M})^\Gamma) \longrightarrow K\oplus_1(C^*(\bar{M})^\Gamma) \longrightarrow \cdots$$

$$\downarrow \tau_\sigma \quad \quad \downarrow \tau_\sigma \quad \quad \downarrow \tau_\sigma\quad \quad \downarrow \tau_\sigma$$

$$\cdots \longrightarrow K\oplus_2(C^*(\bar{N})^\pi) \longrightarrow K\oplus_2(D^*(\bar{N})^\pi) \longrightarrow K\oplus_2(Q^*(\bar{N})^\pi) \longrightarrow K\oplus_3(C^*(\bar{N})^\pi) \longrightarrow \cdots.$$

Moreover, the following hold:

1. The map $\tau_\sigma: K_\ast(C^*(\bar{M})^\Gamma) \to K\oplus_2(C^*(\bar{N})^\pi)$ coincides with (1.2).
2. The map $\tau_\sigma: K_\ast(Q^*(\bar{M})^\Gamma) \to K\oplus_2(Q^*(\bar{N})^\pi)$ coincides with (1.1).
3. The equalities $\tau_\sigma(\rho(g_M)) = \rho(g_N)$ and $\tau_\sigma(\rho_{\text{sgn}}(f_M)) = 2\rho_{\text{sgn}}(f_N)$ hold.

Here, in (3), $g_M$ is a psc metric on $M$ which is of the form $g_{D^2} + g_N$ on a tubular neighborhood $U \cong N \times D^2$ of $N$, and $f_M: M' \to M$ is an oriented homotopy equivalence whose restriction to $f^{-1}(N)$, denoted by $f_N$, is also an oriented homotopy equivalence. They associate the higher $\rho$-invariants $\rho(g_M)$, $\rho(g_N)$, $\rho_{\text{sgn}}(f_M)$ and $\rho_{\text{sgn}}(f_N)$ (a detailed definition is reviewed in [4.8] and [4.10]). We remark that (1), (2) and the commutativity of the diagram reproves [KS20, Theorems 1.2, 1.3].

The proof, given in Theorems 4.5 and 4.12 consists of two steps. First, we construct a $\ast$-homomorphism lifting $\Gamma$-invariant operators on $\bar{M}$ onto the $\Gamma$-covering $\bar{M}$ of $M := (\bar{M}/\pi) \setminus (N \times D^2)$. This covering is constructed in [HPS15, Theorem 4.3] and plays a key role for proving their main result. The lifting of operators to a covering space does not form a $\ast$-homomorphism of coarse C*-algebras in general, but in our setting it makes sense modulo the boundary, i.e., as a $\ast$-homomorphism to the quotient of coarse C*-algebras $C^*(\bar{M})^\Gamma/C^*(\bar{N} \subset \bar{M})^\Gamma$. Second, we apply the ‘boundary of Dirac is Dirac’ and ‘boundary of signature is 2 times signature’ principles to the manifold with boundary $\bar{M}$. Since $\bar{N}$ is diffeomorphic to $\bar{N} \times \mathbb{R}$, the boundary map in $K$-theory sends the Dirac operator on $\bar{M}$ to that of $\bar{N} \times \mathbb{R}$, which is identified with the Dirac operator on $\bar{N}$ by the partitioned manifold index theorem [Roe88].

The second main theorem relates the codimension 2 transfer map and the pairing of higher index invariants with cyclic cocycles of the group algebra. The cyclic cohomology group of $\mathbb{C}^\ast[\Gamma]$ is described in terms of the group cohomology by Burghelea [Bur83]. More specifically, $HC^\ast(\mathbb{C}^\ast[\Gamma])$ decomposes
into the direct product of the ‘localized’ and the ‘delocalized’ parts, where the former is isomorphic to the cohomology of \( \Gamma \) itself and the latter is isomorphic to the product of cohomology of normalizer subgroups of \( \Gamma \). If the group \( \Gamma \) is hyperbolic, then a cyclic cocycle on \( C^* \Gamma \) induces a homomorphism \( K_* (C^* \Gamma) \to C \), \( \{ \text{[Jol89, Pus10]} \} \). It plays a key role in the proof of the Novikov conjecture for hyperbolic groups by Connes-Moscovici \( \{ \text{CM90]} \). There is also a secondary analog of this pairing, Lott’s higher \( \rho \)-number or the delocalized \( \eta \)-invariant \( \{ \text{[Lot92b]} \}, \) in which there has been a growing interest in recent researches such as Chen–Wang–Xie–Yu \( \{ \text{CXY19]} \) and Piazza–Schick–Zenobi \( \{ \text{PSZ19]} \).

The codimension 2 co-transfer map of cohomology groups is defined in a dual way as \( (1.1) \). With the language of group cohomology, this is realized by the cup product with a second cohomology class \( \sigma \in H^2(\Gamma; \mathbb{Z}[\Gamma/\pi]) \) (by this reason it is written as \( \sigma \cdot \phi \)). This \( \sigma \) is an essential ingredient of the construction of codimension 2 transfer maps in homology, studied in Section \( 2 \). We show that this co-transfer map is adjoint to \( \tau_\sigma \) up to the constant \( 2\pi i \) with respect to the pairing of \( K \)-theory and cyclic cohomology. For the definition of numerical higher indices \( \alpha_\phi(M) \), \( \text{Sgn}_\phi(M) \) and higher \( \rho \)-numbers \( \vartheta_\phi(M) \), \( \vartheta_{\text{sgn}}(f_M) \), see Definition \( 5.20 \).

**Theorem 1.4.** Let \( \Gamma \) be a hyperbolic group and let \( \pi \) be its hyperbolic subgroup. Let \( \phi \in H_2^\text{del,loc}(\Gamma) \) and \( \psi \in H_2^\text{del,loc}(\Gamma) \). Let \( M \) be a closed manifold and let \( N \) be a codimension 2 submanifold of \( M \) satisfying (1), (2), (3) of \( 2.16 \). The following equalities hold:

1. \( 2\pi i \alpha_{\phi,\sigma}(M) = \alpha_\phi(N) \) and \( \pi i \text{Sgn}_{\phi,\sigma}(M) = \text{Sgn}_\phi(N) \).
2. \( 2\pi i \vartheta_{\phi,\sigma}(g_M) = \vartheta_\phi(g_N) \) and \( \pi i \vartheta_{\text{sgn}}(f_M) = \vartheta_{\text{sgn}}(f_N) \).

The proof is given in Theorem \( 5.23 \). There are two ingredients of the proof. The first is to identify the codimension 2 co-transfer map of cyclic cohomology groups with the cup product with \( \sigma \). This is performed on the basis of the identification of \( \sigma \) with the Dixmier–Douady class of a twist (central extension) of the action groupoid \( B \rtimes \Gamma \), where \( B \) is a bouquet of circles studied in Section \( 3 \). The second is to define the codimension 2 transfer map in terms of unconditional Banach algebras. This enables us to make the codimension 2 transfer map compatible with the ‘mapping analytic surgery to homology’ formalism of the higher index pairing developed by Piazza–Schick–Zenobi \( \{ \text{PSZ19]} \). We remark that this theorem produces a class of non-trivial computations of the higher \( \rho \)-numbers.

This paper is organized as follows. In Section \( 2 \) we review the construction of the codimension 2 transfer map in general homology theory and discuss its relation with an extension of groups. In Section \( 3 \) we revisit the construction of the \( C^* \)-algebraic transfer map in \( \{ \text{KS20]} \) from the viewpoint of groupoid cocycles. Moreover, we also give a coarse geometric view of this construction. In Section \( 4 \) we prove our first main theorem, Theorem \( 1.3 \). We first construct a map between Higson–Roe analytic surgery sequences, and next show that these maps relate the higher index invariants of \( M \) to those of \( N \). In Section \( 5 \) we prove our second main theorem, Theorem \( 1.4 \). We first study the codimension 2 co-transfer of cyclic cohomology groups of \( C(\Gamma) \), and then extend it to the unconditional Banach algebra \( \mathcal{A}\Gamma \).

### 2. Codimension 2 transfer map via a second cohomology class

In this section, we review the (codimension 2) submanifold transfer map of general homology theory introduced by Nitsche, Schick, and Zeidler \( \{ \text{Nitsche, Schick, Zeidler 2018, NSZ19]} \). We slightly rearrange the exposition on the basis of a second cohomology class. This provides us a systematic construction of codimension 2 inclusions of manifolds to which the theory is applied. We also discuss a realization of this second cohomology class by an extension of fundamental groups.

#### 2.1. Setting

Let \( \Gamma \) be a finitely presented discrete group and let \( \pi \) be its finitely presented subgroup.

**Definition 2.1.** We define the ‘compactly supported’ cohomology group of the topological space \( E\Gamma/\pi \) as the reduced cohomology group of the pointed space

\[
(ET/\pi)^\dagger := (ET \times \Gamma (\Gamma/\pi)^\dagger)/(ET \times \Gamma \{\ast\}),
\]

where \( (\Gamma/\pi)^\dagger \) denotes the 1-point compactification \( ET/\pi \sqcup \{\ast\} \). In the same way, we also define the group \( h^n_\pi(ET/\pi) := h^n((ET/\pi)^\dagger) \) for a general cohomology theory \( h^* \).
This coincides with the compactly supported cohomology in the usual sense if and only if $B\Gamma$ is compact. To be more precise, this group should be called the cohomology with fiberwisely compact support over $B\Gamma$, but in this paper we just call it, in short, the compactly supported cohomology.

We represent a second cohomology class $\sigma \in H^2_c(ET/\pi; \mathbb{Z})$ in several ways. First, it is represented by a continuous map

$$F_{\sigma} : (ET/\pi)^\dagger \to K(\mathbb{Z}, 2).$$

Second, it is represented by a compactly supported complex line bundle $L$ over $ET/\pi$, i.e., a line bundle over $(ET/\pi)^\dagger$. This bundle is related with $F_{\sigma}$ as $L := F_{\sigma}^*(\tilde{L})$, where $\tilde{L} \to BU(1) = K(\mathbb{Z}, 2)$ is the universal line bundle. If $B\Gamma$ is modeled by a closed aspherical manifold, then the zero locus of a generic section $s : ET/\pi \to L$ is a closed codimension 2 submanifold $N \subset ET/\pi$, which represents $\sigma$ through the Poincaré duality. Third, this cohomology class is represented by an extension of groups, which is discussed later in Subsection 2.4.

We impose several assumptions on this cohomology class $\sigma$. Let $q : ET/\pi \to B\Gamma$ denote the projection and let $j : ET/\pi \to (ET/\pi)^\dagger$ denote the inclusion, which induces the homomorphism

$$j^* : H^2_c(ET/\pi; \mathbb{Z}) \to H^2_c(ET/\pi; \mathbb{Z}) \cong H^2(B\pi; \mathbb{Z}).$$

**Assumption 2.2.** We consider the following assumptions:

(A1) There is an open subset $U_{\sigma} \subset ET/\pi$ such that $q|_{U_{\sigma}}$ is injective and $\sigma$ is contained in the image of the map $H^2_c(U_{\sigma}; \mathbb{Z}) \to H^2_c(ET/\pi; \mathbb{Z})$ induced from the inclusion.

(A2) The equality $j^*(\sigma) = 0$ holds.

By abuse of notation, when $\sigma$ satisfies (A1), we use the same letter $\sigma$ for a choice of its preimage in $H^2_c(U_{\sigma}; \mathbb{Z})$.

There is an immediate implication of (A2). Let $\mu : BU(1) \to B\text{SU}(2)$ denote the continuous map induced from the group homomorphism $U(1) \to SU(2)$ given by $z \mapsto \text{diag}(z, \bar{z})$.

**Lemma 2.3.** Let $\sigma$ satisfy (A2) of Assumption 2.2 Then the following holds;

(A2') the composition $\mu \circ F_{\sigma} : (ET/\pi)^\dagger \to B\text{SU}(2)$ is null-homotopic.

Moreover, if $\sigma$ also satisfy (A1) of Assumption 2.2 then $\mu \circ F_{\sigma}$ is null-homotopic as a map from the 1-point compactification $U_{\sigma}^\dagger$.

**Proof.** We may assume that $B\Gamma$ is modeled by a locally compact Hausdorff space. Let $L$ be the line bundle over $(ET/\pi)^\dagger$ representing $\sigma$ as above. Let $s : ET/\pi \to L$ be a continuous section such that $V := \{x \in (ET/\pi)^\dagger \mid |s(x)| < 1\}$ is relatively compact. If we additionally assume (A1), we choose $s$ in the way that $|s(x)| \geq 1$ for any $x \in U_{\sigma}^\dagger$.

Since $\sigma|_V = j^*(\sigma)|_V$ is trivial, there is a non-vanishing section $t : V \to L^*|_V$. Set $\tilde{s} := \chi \cdot (s + 0) + (1 - \chi) \cdot (0 \oplus t)$, where $\chi : ET/\pi \to [0, 1]$ is a bump function satisfying $\text{supp}(\chi) \subset V$ and $\chi \equiv 1$ on $\{x \in ET/\pi \mid |s(x)| < 1/2\}$. This is a non-vanishing section of the $SU(2)$-bundle $L \oplus L^*$. Therefore, the map $\mu \circ F_{\sigma}$ is null-homotopic. Moreover, if we additionally assume (A1), the section $(s, t)$ is an extension of $(s, 0)$ on $U_{\sigma}^\dagger$, and hence a null-homotopy of $\mu \circ F_{\sigma}$ is taken in $\text{Map}(U_{\sigma}^\dagger, B\text{SU}(2))$. $\square$

**Example 2.4.** Let $\Sigma_{g,1}$ denote the closed oriented surface of genus $g \geq 1$ and a single boundary. Let $\text{Diff}(\Sigma_{g,1}, \partial)$ denote the group of diffeomorphisms on $\Sigma_{g,1}$ fixing the boundary, which is regarded as a subgroup of $\text{Diff}(\Sigma_{g})$. Let $\pi$ be a finitely presented group with a homomorphism $\pi \to \pi_0(\text{Diff}(\Sigma_{g,1}, \partial))$. This induces an action $\pi \curvearrowright \pi_1(\Sigma_{g})$. Set $\Gamma := \pi_1(\Sigma_{g}) \rtimes \pi$. The homomorphism $\pi \to \pi_0(\text{Diff}(\Sigma_{g,1}, \partial))$ also induces a fiber bundle $E \to B\pi$ with the fiber $\Sigma_{g}$ since the identity component of $\text{Diff}(\Sigma_{g,1}, \partial)$ is contractible. Note that the total space $E$ is a model of $B\Gamma$. Moreover, by the construction, this bundle has a section $B\pi \to E$ such that the neighborhood of its image is isomorphic to $B\pi \times \mathbb{D}^2$.

Now, the space $ET/\pi$ is a $\Sigma_{g}$-bundle over $B\pi$ including a copy of $B\pi \times \mathbb{D}^2$. The Poincaré dual of $B\pi \subset B\pi \times \mathbb{D}^2 \subset ET/\pi$ determines a second cohomology class $\sigma \in H^2_c(ET/\pi; \mathbb{Z})$. This $\sigma$ satisfies Assumption 2.2.
2.2. Codimension 2 transfer map. The codimension 2 transfer map of general homology theories, constructed by Nitsche, Schick and Zeidler [Ni18,NSZ19], is a homomorphism

$$\tau_{\sigma}: h_*(B\Gamma) \to h_{*\cdot 2}(B\pi).$$

Here $h_*$ is a general homology theory satisfying the following nice property. We say that a generalized homology theory $h_*$ has lf-lifting if, for any covering $Y \to Y$ and any open subspace $U \subset Y$ such that the restriction of the projection $U \to Y$ is a proper map, there is a homomorphism

$$\nu_{Y,U}: h_*(Y) \to h_*(Y, Y \setminus U)$$

such that, if there is another covering $\tilde{Y} \to Y$, open subspace $V \subset \tilde{Y}$ and a covering map $F: \tilde{Y} \to Y$ such that $F|U$ is injective and $F(U) \supset V$, then $F_* \circ \nu_{Y,U} = \nu_{\tilde{Y},V}$ holds. This is a slight modification of the notion of locally-finite restriction introduced by Nitsche (see e.g. [NSZ19, Definition 4.2]), which works even if $h_*$ does not have equivariant theory but the target is restricted to Galois covering spaces (i.e., free and proper actions). For example, ordinary homology theory, cobordism theories, real and complex K-theories have lf-lifting.

Let $h^*$ be a multiplicative generalized cohomology theory which is complex oriented. Let $c^h_1 \in h^2(K(\mathbb{Z},2))$ denote the complex orientation. By abuse of notation, we use the same letter $\sigma$ for the second $h$-cohomology class $F^\sigma_*(\xi) \in h^2(ET/\pi)$. We also define $\sigma \in h^2(ET/\pi)$ for a generalized homology theory $h^*$ which is not necessarily complex oriented, like spin cobordism theory or KO-theory. For this sake, additionally we assume that $\sigma$ satisfies (A2') in Lemma 2.3

**Lemma 2.5.** Assume $\sigma$ satisfies (A2') in Lemma 2.3 Then there is a lifting of $F_\sigma$ as

$$\xymatrix{ \overline{F_\sigma} \ar[rrd] & & S^2 \\ & (ET/\pi)^{\一度} & \ar[rr]^<<<<<<<<<<<{\sim} & & BU(1). }$$

**Proof.** This is obvious since $S^2 \cong SU(2)/U(1) \to BU(1) \to BSU(2)$ is a fibration. \hfill $\square$

Let $1_h \in h^2(S^2,*) \cong h^0$ denote the ring unit. In the same way as above, we use the same letter $\sigma$ for the second $h$-cohomology class $F^\sigma_*(1_h) \in h^2(ET/\pi)$.

Now we are ready to state the definition of the codimension 2 transfer map. We mention that, by replacing $F_\sigma$ if necessary, we may assume that the restriction of the projection $ET/\pi \to B\Gamma$ onto the support of $F_\sigma$, i.e., the open subset $V := F^{-1}_\sigma(BU(1) \setminus \{\ast\}) \subset ET/\pi$, is a relatively proper map. Hence $\sigma \in h^2(ET/\pi)$ is the image of an $h$-cohomology class $\sigma_V \in h^2(ET/\pi, (ET/\pi) \setminus V)$.

**Definition 2.6.** Let $h_*$ be a multiplicative general homology theory having lf-lifting and let $\sigma \in H^2_\Gamma(ET/\pi;\mathbb{Z})$. Assume that either $h_*$ is complex oriented, or $\sigma$ satisfies (A2') in Lemma 2.3. Then the homomorphism $\tau_\sigma: h_*(B\Gamma) \to h_*(B\pi)$ is defined as the composition

$$\tau_\sigma: h_*(B\Gamma) \xrightarrow{\nu_{ET/\pi,V}} h_*(ET/\pi, (ET/\pi) \setminus V) \xrightarrow{\sigma_V \circ \cdot} h_{*\cdot 2}(ET/\pi) \cong h_{*\cdot 2}(B\pi),$$

where $V \subset ET/\pi$ is the open subset as above.

This definition is independent of the choice of $V$ and $\sigma_V$ because, if there is another choice $V'$ and $\sigma_{V'}$, there is $V'' \subset ET/\pi$ which includes $V \cup V'$ and the image of $\sigma_V$ and $\sigma_{V'}$ coincides in $H^2_\Gamma(V'',\mathbb{Z})$.

2.7. The map $\tau_\sigma$ is understood in the following way. Let $M$ be a closed manifold with $\pi_1(M) \cong \Gamma$ and let $\xi_M: M \to B\Gamma$ denote the classifying map of the universal covering $\tilde{M}$. Then $\xi_M^*(\sigma) \in H^2_\Gamma(M/\pi;\mathbb{Z})$ is represented by a closed submanifold $N \subset \tilde{M}/\pi$ of codimension 2. Let $U \subset \tilde{M}$ denote the tubular neighborhood of $N$. In the same way as Definition 2.6 we also define

$$\tau_\sigma^{M,N}: h_*(M) \to h_*(\tilde{M}/\pi, \tilde{M}/\pi \setminus U) \xrightarrow{\xi_M^*(\sigma) \circ \cdot} h_{*\cdot 2}(U) \cong h_{*\cdot 2}(N).$$
By definition the diagram

\[
\begin{array}{ccc}
  h_*(M) & \xrightarrow{\tau^{M,N}_*} & h_{*-2}(N) \\
  (\xi_M)_* & \downarrow & (\xi_N)_* \\
  h_*(B\Gamma) & \xrightarrow{\tau_\sigma} & h_{*-2}(B\pi)
\end{array}
\]

commutes, where \(\xi_N\) denotes the classifying map of the universal covering of \(N\).

The map \(\tau^{M,N}_\sigma\) is the intersection with \(N\), taken in the covering space \(\wt{M}/\pi\) instead of \(M\). In particular, if \(M\) is \(h_\ast\)-oriented, the fundamental class \([M]\) is sent to the fundamental class \([N]\). If we assume (A1) of Assumption 2.2, then we may choose \(N\) as a submanifold of \(\xi^{-1}_M(U_\pi)\), and hence there is a submanifold \(N \subset M\) which lifts to its copy in \(\wt{M}/\pi\). In this case \(\tau^{M,N}_\sigma\) is the intersection with \([N] \in H^2(M;\Z)\).

Let \(\iota : N \to M\) denote the inclusion. Then \(\iota^\ast(\sigma) = c_1(L|_N) = c_1(\nu N)\). Now, the additional assumption (A2) corresponds to the triviality of the normal bundle \(\nu N\), since we have \(\iota^\ast(\sigma) = \iota^\ast(j^\ast(\sigma)) = 0\).

2.3. Transfer map via codimension 2 submanifold. Here we relate the transfer map \(\tau_\sigma\) with the topology of codimension 2 submanifolds studied in \cite{GL83, HPS15}. Let \(M\) be a closed manifold and let \(N\) be a codimension 2 submanifold of \(M\). Set \(\Gamma = \pi_1(M)\) and \(\pi := \pi_1(N)\). Assume that the homomorphism \(\pi \to \Gamma\) induced from the inclusion \(N \subset M\) is injective. Let \(p : M/\pi \to M\) denote the covering map, where \(\wt{M}\) denotes the universal covering space. The connected component of the inverse image \(p^{-1}(N)\) corresponds in one-to-one to the double coset space \(\pi \backslash \Gamma / \pi\), and the unit \(\pi \epsilon \pi\) corresponds to a copy of \(N\) in \(\wt{M}/\pi\). Therefore, the Poincaré dual \(PD[N]\) of \(N\) determines second cohomology classes in both \(H^2(M;\Z)\) and \(H^2_\pi(\wt{M};\Z)\).

The following proposition is a rephrasing of a part of \cite{Nit18} Theorem 5.3.6 and \cite{NSZ19} Theorem 5.1.

**Proposition 2.8.** Let \(M\) be a closed manifold and let \(N \subset M\) be a codimension 2 closed submanifold. Let \(\Gamma := \pi_1(M), \pi := \pi_1(N)\). Assume that

1. \(\pi \to \Gamma\) is injective,
2. \(\pi_2(N) \to \pi_2(M)\) is surjective, and
3. the normal bundle \(\nu N\) is trivial.

Then there is a cohomology class \(\sigma = \sigma_{M,N} \in H^2_\pi(ET/\pi;\Z)\) which satisfies Assumption 2.2 and the pull–back \(\xi^*_M(\sigma)\) by the classifying map \(\xi_M : M \to B\Gamma\) coincides with the Poincaré dual \(PD[N]\) in \(H^2_\pi(M/\pi;\Z)\).

The proof of this proposition relies on the following key observation by Hanke–Pape–Schick, which is proved in the proof of \cite{HPS15} Theorem 4.3. This claim is an essential ‘geometric’ ingredient of the codimension 2 obstruction theory.

**Lemma 2.9** (\cite{HPS15}). Let \(M\) and \(N\) satisfy (1), (2), (3) of Proposition 2.8. Let \(U\) denote the tubular neighborhood of \(N\), which is homeomorphic to \(N \times \mathbb{D}^2\). Then the homomorphism \(\pi_1(DU) \to \pi_1((\wt{M}/\pi) \setminus U)\) is split injective.

A model of \(B\Gamma\) is obtained by attaching \(k\)-cells \(\{D_k\}_{k \in \mathbb{N}}\), for \(k \geq 3\), to \(M\). Corresponding to this, we obtain a model of \(ET/\pi\) by attaching \(\Gamma/\pi\) copies of cells \(\{D_{i,g}\pi\}_{i \in \mathbb{N}, g \pi \in \Gamma/\pi}\) to \(\wt{M}/\pi\). Let \(X_0 := \wt{M}/\pi\) and we inductively define the CW-complex \(X_i\) to be the one obtained by attaching \(\{D_{i,g}\pi\}_{g \pi \in \Gamma/\pi}\) to \(X_{i-1}\). Let \(K_0 := U\) and let \(K_i\) be the subcomplex obtained by attaching to \(K_{i-1}\) the (finite number of) \(D_{i,g}\pi\)’s whose boundary intersects with \(K_{i-1}\). We extend the maps \(F_i : (X_i, X_i \setminus K_i) \to (\mathbb{D}^2, \partial \mathbb{D}^2)\)
inductively, and define \( F : (ET / \pi, ET / \pi \setminus K) \to (D^2, \partial D^2) \) as their limit. Now the pull-back of the generator of \( H^2(D^2, \partial D^2; \mathbb{Z}) \) is the desired cohomology class.

When extending \( F_{i-1} \) to \( F_i \), it is chosen such that \( q|_{V_i} \) is injective, where \( V_i \) denotes the inverse image of the closure of \( B_\varepsilon(0) \subset D^2 \). By setting \( U_\varepsilon := F^{-1}(B_\varepsilon(0)) \), this shows (A1). Indeed, since \( p(V_{i-1}) \cap \partial D_i \) is identified with the disjoint union of \( \partial D_{i,g} \cap p(V_{i-1}) \), there is an open neighborhood \( U_{i,g} \) of each \( \partial D_{i,g} \cap p(V_{i-1}) \) in \( D_i \) which are mutually disjoint. We can choose an extension \( F_i \) on \( D_{i,g} \) in the way that \( F_i(U_{i,g}) \subset B_\varepsilon(0) \), which implies the injectivity of \( p|_{V_i} \) as desired.

Let \( N \) denote the classifying map. The assumption (3) implies \( c_1(\nu N) = \xi^*_N(\sigma) = 0 \). Now (A2) follows from the injectivity of \( \xi^*_N \) in the second cohomology groups, which follows from 2-connectedness of \( \xi_N \).

**Notation 2.10.** Let \( M \) be a closed manifold and let \( N \subset M \) be a codimension 2 submanifold of \( M \) satisfying (1), (2), (3) of Proposition 2.8. Let \( U \) be a tubular neighborhood of \( N \). Let \( \tilde{M} \) denote the universal covering of \( M \). By the assumption of fundamental groups, there is a copy of \( U \) in the \( \Gamma / \pi \)-covering \( \tilde{M} / \pi \) over \( M \). Let \( \mathring{M} \) denote the complement of \( U \). Let \( \mathring{M} \) denote its \( \pi \)-covering, which is a manifold with the boundary and is a closed subset of \( \tilde{M} \). We write \( \overline{M} \) for the \( Z \)-covering of \( \mathring{M} \) associated to the splitting given in Lemma 2.9.

\[
\begin{align*}
\tilde{M} & \xrightarrow{\pi} \mathring{M} / \pi \xrightarrow{\Gamma / \pi} M \\
\mathring{M} & \xrightarrow{\pi} \mathring{M} / \pi \xrightarrow{\Gamma / \pi} M.
\end{align*}
\]

We write \( (N), \mathring{N}, \overline{N} \) for the boundary of \( M \), \( \tilde{M} \), \( \overline{M} \) respectively. Note that \( N \cong N \times S^1, \mathring{N} \cong \mathring{N} \times S^1 \) and \( \overline{N} \cong \overline{N} \times \mathbb{R} \). We also write \( M^o \) for the complement of \( U \).

2.4. **Extension of fundamental groups.** The cohomology group \( H^*_c(ET / \pi; \mathbb{Z}) \) is identified with the group cohomology \( H^*(\Gamma; \mathbb{Z}[\Gamma / \pi]) \). Here, \( \mathbb{Z}[\Gamma / \pi] \) denote the free abelian group of finitely supported \( \mathbb{Z} \)-valued functions on \( \Gamma / \pi \), in other words, the direct sum of \( \Gamma / \pi \) copies of \( \mathbb{Z} \), whose \( \Gamma \)-module structure is induced from the regular \( \Gamma \)-action on \( \Gamma / \pi \). To be precise, this identification is given in the following way.

**Lemma 2.11.** The groups \( H^*_c(ET / \pi; \mathbb{Z}) \) and \( H^*(\Gamma; \mathbb{Z}[\Gamma / \pi]) \) are isomorphic.

**Proof.** We consider the relative version of the Serre spectral sequence for the pair of Serre fibrations \( (ET \times_\Gamma (\Gamma / \pi)^+, ET \times_\Gamma \{\ast\}) \) over \( B\Gamma \) (cf. [McC85] Exercise 5.6). It is a spectral sequence converging to \( H^*_c(ET / \pi; \mathbb{Z}) \) whose \( E_2 \)-page is

\[
E_2^{p,q} \cong \begin{cases} 
H^p(B\Gamma; \mathbb{Z}[\Gamma / \pi]) & \text{if } q = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

This shows the lemma. \( \square \)

Through this isomorphism, the cohomology class \( \sigma \in H^2(\Gamma; \mathbb{Z}[\Gamma / \pi]) \) corresponds to a group extension

\[
1 \to \mathbb{Z}[\Gamma / \pi] \to G \to \Gamma \to 1.
\]

We relate the middle group \( G \) with a fundamental group of manifolds. Let \( M \) be a closed manifold with \( \pi_1(M) \cong \Gamma \) and let \( N \subset M \) be a codimension 2 submanifold representing \( f^* \sigma \) as in 2.7. In the same way as Notation 2.10, we write as \( M^o := M \setminus U \) and \( \mathring{M} := (M / \pi) \setminus U \), where \( U \) is a tubular neighborhood of \( N \).

**Lemma 2.13.** We assume that \( \sigma \) satisfies (A1) of Assumption 2.2. Then the homomorphism \( i_* : \pi_1(M^o) \to \pi_1(M) \cong \Gamma \) factors through \( G \), that is, there is a group homomorphism \( \phi : \pi_1(M^o) \to G \) such that the
splits, i.e., there is a homomorphism

\[ \xi_M : M \to \Gamma \]

and hence is homotopic to \((\xi_M \circ \iota)\). Hence, through this isomorphism, the map \(H^2(\Gamma; \mathbb{Z}/\pi)\) is a representative of \(\iota\) as desired.

\[ \boxed{\text{Proof.}} \]

**Remark 2.15** Let \(\sigma \in H^2(\Gamma; \mathbb{Z}/\pi)\) be a link of \(N \subset \tilde{M}/\pi\). Its fundamental group \(\pi_1(N^1) \cong \mathbb{Z}\) is trivial. Therefore the pull-back extension

\[ 1 \to \mathbb{Z}[\Gamma/\pi] \to \mathcal{G} \times \Gamma \to \mathcal{G}^\prime \to 1 \]

splits, i.e., there is a homomorphism

\[ 1 \longrightarrow \mathbb{Z}[\Gamma/\pi] \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}^\prime \longrightarrow 1 \]

as desired.

We rephrase the condition (A2) of Assumption 2.2 in terms of group extension. Let \(\mathbb{Z}[\Gamma/\pi]^\prime\) denote the completion of \(\mathbb{Z}[\Gamma/\pi]\), i.e., the direct product \(\prod_{\Gamma/\pi} \mathbb{Z}\). Then \(\mathbb{Z}[\Gamma/\pi]^\prime\) is isomorphic to the cohomology group \(H^4(\mathbb{E}\Gamma/\pi; \mathbb{Z})\). Moreover, through this isomorphism, the map \(\mathbb{Z}[\Gamma/\pi] \to \mathbb{Z}[\Gamma/\pi]^\prime\) induced from the inclusion of coefficients \(\mathbb{Z}[\Gamma/\pi] \to \mathbb{Z}[\Gamma/\pi]^\prime\) is identified with \(j^*\). Let \(S^1 \subset \mathcal{M}\) be a link of \(N \subset \tilde{M}/\pi\). Its fundamental group \(\pi_1(S^1) \cong \mathbb{Z}\) is sent to \(\mathbb{Z} \cdot \delta_{\pi} \subset \mathbb{Z}[\Gamma/\pi] \subset G\) by the map \(\varphi\) in Lemma 2.13.

**Lemma 2.14.** Assume that \(\sigma \in H^2(\mathbb{E}\Gamma/\pi; \mathbb{Z})\) satisfies Assumption 2.2. Then the homomorphism \(\pi_1(S^1) \cong \mathbb{Z} \to \pi_1(\tilde{M}/\pi)\) is split injective.

**Proof.** Let \(H\) and \(\hat{H}\) denote the preimage of \(\pi\) by the quotient \(G \to \Gamma\) and \(\hat{G} \to \Gamma\) respectively. Since (A2) is equivalent to \(j^*(\sigma) = 0\), we have that the extension

\[ 1 \to \mathbb{Z}[\Gamma/\pi]^\prime \to \hat{G} \to \Gamma \to 1 \]

is trivial. Therefore the pull-back extension \(1 \to \mathbb{Z}[\Gamma/\pi]^\prime \to \hat{H} \to \pi \to 1\), and hence its quotient \(1 \to \mathbb{Z} \to \hat{H}/\hat{Z}^\perp \to \pi \to 1\)

by the normal subgroup \(\hat{Z}^\perp := \prod_{(\Gamma/\pi) \setminus \{e\}} \mathbb{Z}\) of \(\hat{H}\), are also trivial. That is, \(\hat{H}/\hat{Z}^\perp \cong \pi \times \mathbb{Z}\). Now the composition

\[ \pi_1(\tilde{M}/\pi) \cong H \to \hat{H} \to \hat{H}/\hat{Z}^\perp \cong \pi \times \mathbb{Z} \to \mathbb{Z} \]

is the desired splitting of \(\mathbb{Z} \to H\).

**Remark 2.15.** Let \(s : \Gamma \to G\) be a set-theoretical section. Then the function \((g, h) \mapsto s(g)s(h)s(gh)^{-1} \in \mathbb{Z}[\Gamma/\pi]\) satisfies the 2-cocycle relation and is a representative of \(\sigma \in H^2(\Gamma; \mathbb{Z}[\Gamma/\pi])\). Hereafter we also use the letter \(\sigma(g, h)\) for this 2-cocycle. The codimension 2 transfer map in group homology is given by the cap product as

\[ H_*^{\mathbb{Z}}(\Gamma; \mathbb{Z}) \to H_*^{\mathbb{Z}}(\Gamma; \mathbb{Z}[\Gamma/\pi]) \xrightarrow{\sigma_*} H_{*-2}^{\mathbb{Z}}(\Gamma; \mathbb{Z}[\Gamma/\pi]) \to H_{*-2}(\pi; \mathbb{Z}). \]
Conversely, the codimension 2 co-transfer map in group cohomologies is defined by the cup product as
\[ H^{i-2}(\pi; \mathbb{Z}) \cong H^{i-2}(\Gamma; \mathbb{Z}/\pi) \xrightarrow{\sigma \cup} H^i(\Gamma; \mathbb{Z}/\pi) \rightarrow H^i(\Gamma; \mathbb{Z}), \]
where the last morphism is induced from the map of coefficients \( \mathbb{Z}[\Gamma/\pi] \to \mathbb{Z} \) sending \( \sum n_{g\pi} \delta_{g\pi} \in \mathbb{Z}[\Gamma/\pi] \) to \( \sum n_{g\pi} \in \mathbb{Z} \).

We discuss a converse of Proposition 2.8 getting a codimension 2 submanifold \( N \subset M \) from the cohomology class \( \sigma \).

2.16. Let \( \Gamma \) be a finitely presented group, let \( \pi \) be a subgroup and let \( M \) be a closed manifold with \( \pi_1(M) \cong \Gamma \). Let the cohomology class \( \sigma \in H^2_\text{c}(E\Gamma/\pi; \mathbb{Z}) \) satisfy Assumption 2.2, and let \( N \subset \xi_{x_1}^{-1}(U) \subset M \) be a codimension 2 submanifold of \( M \) as in 2.7. Then the pair \((M, N)\) satisfies the following.

1. There is a homomorphism \( \pi_1(N) \to \pi \).
2. The normal bundle \( \nu N \) is trivial.
3. The homomorphism \( \pi_1(S^1) \to \pi_1(M) \) splits.

Indeed, (1) follows from (A1) since \( N \) is identified with a submanifold of \( \tilde{M}/\pi \), (2) is checked in the same way as the last paragraph of 2.7, and (3) follows from Lemma 2.14. These three conditions are enough for the construction of codimension 2 transfer discussed in the coming sections.

3. C*-algebraic codimension 2 transfer revisited

A C*-algebraic codimension 2 transfer map (1.2) is constructed in [KS20] by using a representation of \( \Gamma \) onto the Calkin algebra of a Hilbert C*-module. In this section, we revisit this construction from the viewpoint of the second cohomology class \( \sigma \in H^2_\text{c}(E\Gamma/\pi; \mathbb{Z}) \) introduced in the previous section. We associate to \( \sigma \) a twist, i.e. a \( \mathbb{T} \)-valued 2-cocycle, of the action groupoid \( B \times \Gamma \), where \( B \) is a bouquet of circles. The associated twisted crossed product C*-algebra \( C(B) \rtimes_\sigma \Gamma \) is related to the Calkin algebra used in [KS20].

Throughout the paper, we consider the maximal completions for group C*-algebras or crossed products unless otherwise noted.

Remark 3.1. This and the next sections are based on complex K-theory, but written in such a way that all statements and discussions are immediately extended to Real K-theory (for example, we treat the degree 2 shift of K-theory faithfully). For this sake, we mix some notations specific to Real K-theory into our complex-based discussion. Let \( \mathbb{R}^{0,1} \) denote the Real space \( \mathbb{R} \) equipped with the involution \( t \to -t \) and let \( S^{0,1} := C_0(\mathbb{R}^{0,1}) \), the Real C*-algebra of continuous functions on \( \mathbb{R}^{0,1} \) vanishing at infinity. Let \( \mathbb{T}^{0,1} \) denote the 1-point compactification of \( \mathbb{R}^{0,1} \), which is a circle with 2 fixed points of the involution.

3.1. C*-algebraic codimension 2 transfer and twisted crossed product. Let \( \Pi \) denote the direct product group \( \pi \times \mathbb{Z} \) and let \( \mathbb{B}C^*\Pi \) and \( \mathbb{E}C^*\Pi \) denote the C*-algebra of bounded adjointable operators and compact operators on the Hilbert C*-\( \Pi \)-module \( \ell^2(\Gamma/\pi) \otimes C^*\Pi \) respectively. In [KS20], a \( * \)-homomorphism
\[ \phi: C^*\Gamma \to \mathbb{Q}C^*\Pi \]
is constructed, where \( \mathbb{Q}C^*\Pi \) is the Calkin algebra \( \mathbb{B}C^*\Pi / \mathbb{E}C^*\Pi \). The C*-algebraic codimension 2 transfer map (1.2) is defined by the composition
\[ (3.2) \tau_\sigma: K_s(C^*\Gamma) \xrightarrow{\phi} K_s(\mathbb{Q}C^*\Pi) \xrightarrow{\beta} K_{s-1}(\mathbb{E}C^*\Pi) \xrightarrow{\beta} K_{s-1}(C^*\pi), \]
where \( \beta \) denotes the boundary map of associated to the extension \( 0 \to \mathbb{E}C^*\Pi \to \mathbb{B}C^*\Pi \to \mathbb{Q}C^*\Pi \to 0 \) and \( \beta \) denotes the projection onto the second direct summand of
\[ K_{s-1}(\mathbb{E}C^*\Pi) \cong K_{s-1}(C^*\pi) \cong K_{s-1}(C^* \cdot C(\mathbb{T}^{0,1})) \cong K_{s-1}(C^*\pi) \oplus K_{s-2}(C^*\pi), \]
which is actually given by the Kasparov product with an element \( \beta \in K_{s-1}(C(\mathbb{T}^{0,1})) \).

We shortly recall the construction of \( \phi \). Let \( V \) denote the Mishchenko bundle, i.e., the C*-\( \Pi \)-module bundle
\[ V := \overline{M^0} \times \Pi \ C^*\Pi \to M^0 \]
over $\mathcal{M}^0 := \widetilde{\mathcal{M}}^0 / \pi \subset \mathcal{M}$. Let $\tilde{p}_! V$ denote its push-forward onto $M^0$ with respect to the projection $\tilde{p}$. This is a Hilbert $C^*\Pi$-module bundle whose fiber is

$$(\tilde{p}_! V)_x \cong \ell^2(\Gamma / \pi) \otimes C^*\Pi.$$ 

By Lemma 2.13, the group $G$ defined as in (2.12) acts on $(\tilde{p}_! V)_x$ by the monodromy representation, which gives rise to a $*$-homomorphism $\tilde{\phi}: G \to \mathcal{U}(\mathbb{B}_{C^*\Pi})$. Moreover, the generator $t := \delta_{e\pi} \in \mathbb{Z}[\Gamma / \pi] \subset G$ acts on the fiber $(\tilde{p}_! V)_x$ by a compact operator. Therefore, the $G$-action reduces to a homomorphism from $\Gamma = G / \langle t \rangle$ to the unitary group of $\mathbb{Q}_{C^*\Pi}$, which induces the desired $*$-homomorphism $\phi$.

This $\phi$ associates a $C^*$-algebra extension

$$0 \to \mathbb{K}_{C^*\Pi} \to \mathbb{B}_{C^*\Pi} \oplus \mathbb{Q}_{C^*\Pi} C^*\Gamma \to C^*\Gamma \to 0,$$

where the middle $C^*$-algebra is the fiber product, i.e., the subalgebra of the direct sum $\mathbb{B}_{C^*\Pi} \oplus C^*\Gamma$ consisting of pairs $(x, y)$ satisfying $q(x) = \phi(y) \in \mathbb{Q}_{C^*\Pi}$ (here $q: \mathbb{B}_{C^*\Pi} \to \mathbb{Q}_{C^*\Pi}$ denotes the quotient).

By the maximality of the norm on the group $C^*$-algebra $C^\ast(M)$ consisting of pairs $g \in \mathcal{M}$ imposed to the group $C^*$-algebra $C^\ast(M)$, this $\phi$ is identified with the $2$-cocycle

$$\sigma_{\tilde{\phi}}: \Gamma \times \Gamma \to \mathbb{C}, \quad \sigma_{\tilde{\phi}}(g, h) := \chi(g h),$$

of the action groupoid $\Gamma \times \Gamma$.

**Remark 3.3.** The $2$-cocycle $\sigma_{\tilde{\phi}} \in Z^2(\Gamma, \mathbb{C})$ is complex-conjugation invariant in the following sense. The group $\mathbb{Z}_2$ acts on the Real space $\mathbb{T}$ by the involution and on the sheaf $\mathbb{T}$ by complex conjugation. Then $\sigma_{\tilde{\phi}}$ is $\mathbb{Z}_2$-invariant in $Z^2(\Gamma, \mathbb{T})$. This enables us to impose the canonical Real structure on the twisted crossed product $C(\Gamma \times \mathbb{C}) \cong C^*G$ determined by

$$\overline{\overline{f}} \cdot u_g = \overline{\overline{f}} \cdot u_g, \quad \text{where } \overline{\overline{f}}(\chi) = f(\overline{\chi}) \text{ for } \chi \in T.$$ 

Note that this Real structure is the same with the standard Real structure imposed to the group $C^*$-algebra $C^*G$.

For $g \pi \in \Gamma / \pi$, let $X_{g\pi}$ denote the submodule $\ell^2(\{g\pi\}) \otimes C^*\Pi$ of $\ell^2(\Gamma / \pi) \otimes C^*\Pi$. Since the $\mathbb{Z}$-covering $\widetilde{\mathcal{M}}^0 \to \mathcal{M}^0$ extends to $\widetilde{\mathcal{M}} \to \mathcal{M}$, the monodromy of the element $\delta_{g\pi} \in \mathbb{Z}[\Gamma / \pi] \subset G$ is the diagonal unitary given by

$$(3.4) \quad u_{\delta_{g\pi}}|_{X_{g\pi}} = \begin{cases} u_t & \text{if } g = h, \\ 1 & \text{otherwise}, \end{cases}$$

where $u_t \in C^*\mathbb{Z} \subset C^*\Pi$ denotes the generator. In particular, each $u_{\delta_{g\pi}}$ is a compact operator on $\ell^2(\Gamma / \pi) \otimes C^*\Pi$. This shows that the $*$-homomorphism

$$\phi: C^\ast(\mathbb{Z}[\Gamma / \pi]) \cong C(T) \to \mathbb{B}_{C^*\Pi}$$

factors through $C(B)$, where

$$B := (\mathbb{R}^{0,1} \times \Gamma / \pi)^+ \subset T$$

denotes the bouquet of $\Gamma / \pi$ copies of circles.
Let $\hat{\sigma}_B$ denote the restriction of $\hat{\sigma}_T$ to the subgroupoid $B \rtimes \Gamma$. By abuse of notation, we simply write $\sigma$ for this cocycle. Then the above discussion is summarized to be the existence of a factorization indicated as the dotted arrow:

$$
\begin{array}{ccc}
C^*G & \hookrightarrow & C(B) \rtimes_\sigma \Gamma \\
\downarrow & & \downarrow \\
B_{C^*\Gamma} & \rightarrow & Q_{C^*\Gamma}.
\end{array}
$$

**Lemma 3.5.** Assume that $\sigma$ satisfies (A2) of Assumption 2.2 Then the twisted crossed product $C_0(B_0) \rtimes_\sigma \Gamma$ is isomorphic to $S^{0,1}C^*\pi \otimes \mathbb{K}(\ell^2(\Gamma/\pi))$.

**Proof.** The lemma follows from the fact that the restriction of $\sigma$ onto $C_0(B_0)$ is a coboundary in the multiplier $C^*$-algebra $C_b(B_0)$, which follows from (A2). In more detail, by (A2) we have a 1-cocycle $b: \Gamma \rightarrow \mathbb{Z}[\Gamma/\pi]^*$ such that

$$\sigma(g, h) = b(g)\gamma_g(b(h))b(gh)^{-1}$$

holds. Since $b(g) \in \mathbb{Z}[\Gamma/\pi]^*$ determines a bounded continuous function $b(g)|_B \in C_b(B_0)$, we get a $*$-homomorphism $\varphi: C_0(B_0) \rtimes_{\sigma, \text{alg}} \Gamma \rightarrow C_0(B_0) \rtimes_{\text{alg}} \Gamma$ determined by $\varphi(f \cdot u_g) = (f \cdot b(g)) \cdot u_g$ for any $f \in C_0(B_0)$ and $g \in \Gamma$. This extends to a $*$-isomorphism of maximal $C^*$-completions.

Consequently, we obtain the following commutative diagram of exact sequences;

$$
\begin{array}{cccccc}
0 & \rightarrow & S^{0,1}C^*\pi \otimes \mathbb{K} & \rightarrow & C(B) \rtimes_\sigma \Gamma & \rightarrow & C^*\Gamma & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & B_{C^*\Pi} & \rightarrow & B_{C^*\Pi} \oplus Q_{C^*\Pi} & \rightarrow & C^*\Gamma & \rightarrow & 0.
\end{array}
$$

Therefore, the $C^*$-algebraic codimension 2 transfer map (3.2) is rewritten as

$$
(3.7) \quad \tau_\sigma: K_s(C^*\Gamma) \xrightarrow{\partial} K_{s-1}(S^{0,1}C^*\pi \otimes \mathbb{K}) \cong K_{s-2}(C^*\pi),
$$

where $\partial$ denotes the boundary map of the upper exact sequence of (3.6).

**Remark 3.8.** This construction does not produce the reduced version of codimension 2 transfer map, i.e., the map $K_s(C^*\pi) \rightarrow K_{s-2}(C^*\pi)$. This is due to the fact that the reduced crossed product functor is not exact at the middle if $\Gamma$ is not an exact group (for this topic, we refer the reader to [BO08 Subsection 5.1]).

### 3.2. Dixmier–Douady description of the second cohomology class $\sigma$

The 2-cocycle $\sigma$ lies in the second cohomology group $H^2_\Gamma(\mathbb{E}\Gamma/\pi; \mathbb{Z})$, which is the same thing as the compactly supported equivariant cohomology group $H^2_\Gamma(\mathbb{C}^\times, \mathbb{C}^\times; \mathbb{Z})$. On the other hand, the groupoid twist $\sigma$ determines an element of the second reduced equivariant cohomology group $H^2_\Gamma(B, \ast; \mathbb{T})$. Since $B$ is the suspension of the based space $(\Gamma/\pi)^+$, there is an isomorphism

$$
(3.9) \quad H^2_\Gamma(\Gamma; \mathbb{Z}[\Gamma/\pi]) \cong H^2_\Gamma(B, \ast; \mathbb{T}) \cong H^2_\Gamma(B, \ast; \mathbb{Z}).
$$

Indeed, the cohomology classes determined by $\sigma$ on both sides of the above isomorphism are identified in the following way.

**Proposition 3.10.** The group 2-cocycle $\sigma \in Z^2(\Gamma; \mathbb{Z}[\Gamma/\pi])$ and the groupoid 2-cocycle $\hat{\sigma}_B \in Z^2(\mathbb{C}^\times; \mathbb{T})$ are in the same cohomology class through the isomorphism (3.9).

This justifies our simplified notation using $\sigma$ instead of $\hat{\sigma}_B$.

**Proof.** The identity map $z: S^1 \rightarrow \mathbb{T}$ is a Čech 0-cocycle of the sheaf $\mathbb{T}$ of $\mathbb{T}$-valued functions on $S^1$. The isomorphism

$$
H^2_\Gamma((\Gamma/\pi)^+, \ast; \mathbb{Z}) \cong H^2_\Gamma(B, \ast; \mathbb{T})
$$
is induced from the cup product
\[ \tilde{C}_0^2((\Gamma/\pi)^+, \ast; \mathbb{Z}) \otimes \tilde{C}_0^0(S^1, \ast; \mathbb{T}) \to \tilde{C}_0^2((\Gamma/\pi)^+ \wedge S^1, \ast; \mathbb{T}) \]
with \( z \in \tilde{C}_0^0(S^1, \ast; \mathbb{T}) \). By definition we have \( \sigma \cup z = \tilde{\sigma} |_B \).

3.3. From groupoid \( C^* \)-algebra to Roe algebra. In this subsection, we relate the twisted crossed product \( C^* \)-algebra \( C(B) \times_\sigma \Gamma \) with invariant Roe algebras. Let \( \Gamma, \pi, \sigma, M \) and \( N \) be as in 2.16 and let \( \tilde{M}, \tilde{\Gamma} \) be the covering spaces as in 2.7.

3.11. We fix a Riemannian metric on \( M \). Let \( C^*(\tilde{M})^\Gamma \) denote the \( \Gamma \)-invariant (maximal) Roe algebra of \( \tilde{M} \), i.e., the completion of the \( * \)-algebra \( \mathbb{C}[\tilde{M}]^\Gamma \) of bounded operators on \( L^2(\tilde{M}) \) such that
1. the propagation of \( T \) is finite,
2. \( T \) is locally compact, i.e., \( fT, Tf \in \mathbb{K}(L^2(\tilde{M})) \), and
3. \( T \) is \( \Gamma \)-invariant.

Here, the propagation of \( T \) is defined to be the supremum of the distance \( d(x, y) \) of \( x, y \in \tilde{M} \) such that \( f_2Tf_1 \neq 0 \) for any \( f_1, f_2 \in C_c(\tilde{M}) \) such that \( f_1(x) \neq 0, f_2(y) \neq 0 \). The norm on \( \mathbb{C}[\tilde{M}]^\Gamma \) is chosen as the maximal one among all possible \( C^* \)-norms on it (the well-definedness is proved in [GMY08, 3.5]). We also define the \( \Pi \)-invariant Roe algebra \( C^*(\tilde{M})^\Pi \) in the same way.

As is constructed in [CWX20] Lemma 2.12, there is a \(*\)-homomorphism
\[ (3.12) \quad \epsilon_\mathbb{Z}: C^*(\tilde{M})^\Pi \to C^*(\tilde{M}), \]
which sends the operator \( T \in \mathbb{C}[\tilde{M}]^\Pi \) represented by a kernel function \( t(x, y) \in \tilde{M} \times \tilde{M} \to \mathbb{C} \) to the operator on \( \tilde{M} \) which is represented by the kernel function
\[ \epsilon_\mathbb{Z}(t)(x, y) = \sum_{\pi(y)=y} t(\tilde{x}, y), \]
which is well-defined independent of the choice of \( \tilde{x} \in \tilde{M} \) such that \( \pi(\tilde{x}) = x \). The \(*\)-homomorphism of this kind does exist for any Galois covering space and a surjection of the group. More generally, for a free proper \( G \)-space \( X \) and a normal subgroup \( N \trianglelefteq G \), a \(*\)-homomorphism \( \epsilon_N: C^*(X)^G \to C^*(X/N)^G/N \) is defined in the same way. We simply write this \(*\)-homomorphism as \( \epsilon \) if \( N \) is clear from the context.

Let \( C^*(\tilde{M})^\Gamma \) denote the kernel of \( \epsilon_\mathbb{Z} \).

Set \( M^* := M \setminus N \) and let \( \tilde{M}^*, \tilde{M}^\ast \) and \( \tilde{M}^\Pi \) denote its covering space whose fiber is \( \Gamma, G/Z \) and \( G \) respectively, where \( Z^\perp := \mathbb{Z}((\Gamma/\pi) \setminus \{ \pi \}) \). Since \( \tilde{M}^\ast \) is a dense open subspace of \( \tilde{M} \), the identification \( L^2(\tilde{M}) \cong L^2(\tilde{M}^\ast) \) induces \(*\)-isomorphisms \( C^*(\tilde{M})^\ast \cong C^*(\tilde{M}^\ast)^\ast \) and \( C^*(\tilde{M})^\Pi \cong C^*(\tilde{M}^\Pi)^\Pi \). We choose a tubular neighborhood \( V \) of \( N \) including the closure of \( U \) and a diffeomorphism between \( V \setminus N \) and \( V \setminus U \) which is the identity at the boundary. This gives rise to a \(*\)-isomorphism of Roe algebras \( \chi: C^*(\tilde{M}^\ast)^\Pi \to C^*(\tilde{M})^\ast \).

We write the restriction map as the inclusion of invariant Roe algebras
\[ \text{res}_\mathbb{R}: C^*(\tilde{M})^\Gamma \to C^*(\tilde{M})^\ast \].

Also, we identify the group \( C^* \)-algebra \( C^*(\Gamma) \otimes \mathbb{K} \) with the invariant Roe algebra. Let \( U \) be a fundamental domain of the \( \Gamma \)-space \( \tilde{M} \), i.e., a 1-connected dense open subset of \( M \). We choose \( U \) in the way that \( U \cap N = \emptyset \). Then the isomorphism \( L^2(\tilde{M}) \cong \ell^2(\Gamma) \otimes L^2(U) \) gives rise to \( C^*(\tilde{M})^\Gamma \cong C^*(\Gamma) \otimes \mathbb{K}(L^2(U)) \). In summary, we get a \(*\)-homomorphism
\[ (3.13) \quad \phi: C^*(\Gamma) \otimes \mathbb{K} \cong C^*(\tilde{M})^\Gamma \xrightarrow{\text{res}_\mathbb{R}} C^*(\tilde{M}^\ast)^\ast \xrightarrow{\chi} C^*(\tilde{M})^\ast. \]

Lemma 3.14. There is a homomorphism of \( C^* \)-algebra extensions
\[ \begin{array}{cccccc}
0 & \to & S^0 \otimes C^*(\pi) & \to & C^*(\tilde{M}) & \to & 0 \\
\phi & & \downarrow & & \downarrow & & \phi \\
0 & \to & C^*(\tilde{M})^\Pi & \to & C^*(\tilde{M})^\ast & \to & 0.
\end{array} \]
such that the right vertical map \( \phi \) is the same as (3.13). Moreover, the image of the left vertical map \( \phi \) coincides with \( C^*(\mathcal{N} \subset \mathcal{M})^\Pi \cap C^*(\mathcal{M})^\Pi \).

**Proof.** Let \( Z := \mathbb{Z}[(\Gamma/\pi)] \) and \( Z^\perp := \mathbb{Z}[(\Gamma/\pi) \setminus \{ e\pi \}] \) as above. By definition of the \( * \)-homomorphism \( \epsilon_Z \) as in (3.12), the diagram

\[
\begin{array}{c}
C^*G \otimes \mathbb{K} \\
\downarrow \psi \\
C^*(\mathcal{M}^\Pi) \\
\end{array}
\begin{array}{c}
\cong \\
\cong \\
\cong \\
\end{array}
\begin{array}{c}
C^*(\mathcal{M}^\Pi)^G \\
\xrightarrow{\text{res}} C^*G \\
\xrightarrow{\epsilon_Z} C^*(\mathcal{M}^\Pi)^\Pi \\
\end{array}
\begin{array}{c}
\text{def.} \\
\text{def.} \\
\text{def.} \\
\end{array}
\begin{array}{c}
\xrightarrow{\text{res}_\Gamma} C^*(\mathcal{M}^\Pi)^\Gamma \\
\xrightarrow{\epsilon_Z} C^*(\mathcal{M}^\Pi)^\Gamma \\
\xrightarrow{\epsilon_Z} C^*(\mathcal{M}^\Pi)^\Gamma \\
\end{array}
\begin{array}{c}
\xrightarrow{X} C^*(\mathcal{M})^\Pi \\
\xrightarrow{\epsilon_Z} C^*(\mathcal{N} \subset \mathcal{M})^\Pi \\
\xrightarrow{\epsilon_Z} C^*(\mathcal{N} \subset \mathcal{M})^\Pi \\
\end{array}
\]

commutes.

Through the unitary isomorphism

\[
L^2(\mathcal{M}^\ast) \cong L^2(U) \otimes \ell^2(G/Z^\perp) \cong L^2(U) \otimes \ell^2\Gamma \otimes \ell^2\mathbb{Z},
\]

the representation \( \phi \) in the above paragraph is identified with the monodromy representation \( \tilde{\phi} \) introduced in Subsection 3.1 and hence factors through \( C(B) \rtimes_\sigma \Gamma \). Moreover, by (3.4), the image of \( C_0(B_0) \rtimes_\sigma \Gamma \) acts on the Hilbert \( C^*\Pi \)-module \( \ell^2(\Gamma/\pi) \otimes C^*\Pi \) by a compact operator. This shows that \( \phi \) sends \( (C_0(B_0) \rtimes_\sigma \Gamma) \otimes \mathbb{K} \) to \( C^*(\mathcal{N} \subset \mathcal{M})^\Pi \cap C^*(\mathcal{M})^\Pi \). \( \square \)

**Remark 3.16.** Lemma 3.14 is also proved more algebraically with the language of twisted crossed products. Through the unitary (3.15), the Roe algebras are identified with crossed products as

\[
C^*(\mathcal{M})^\Pi \cong (c_b(\Gamma/\pi) \otimes C^*\mathbb{Z}) \rtimes_\sigma \Gamma,
\]

\[
C^*(\mathcal{M})^\Pi_0 \cong (c_b(\Gamma/\pi) \otimes S^0) \rtimes_\sigma \Gamma.
\]

Hence the map of exact sequences in the statement of Lemma 3.14 is identified with

\[
0 \rightarrow S^{0,1}c_0(\Gamma/\pi) \rtimes_\sigma \Gamma \rightarrow C(B) \rtimes_\sigma \Gamma \rightarrow C^*\Gamma \rightarrow 0
\]

and

\[
0 \rightarrow S^{0,1}c_0(\Gamma/\pi) \rtimes_\sigma \Gamma \rightarrow (c_b(\Gamma/\pi) \otimes C^*\mathbb{Z}) \rtimes_\sigma \Gamma \rightarrow c_b(\Gamma/\pi) \rtimes \Gamma \rightarrow 0,
\]

after taking tensor product with \( \mathbb{K}(L^2(U)) \).

Let \( s_0 : C^*\Gamma \rightarrow C(B) \rtimes_\sigma \Gamma \) be a set-theoretical section. Then Lemma 3.14 shows that the composition

\[
s := q \circ \phi \circ s_0 : C^*(\mathcal{M})^\Gamma \rightarrow C^*(\mathcal{N} \subset \mathcal{M})^\Pi
\]

is a \( * \)-homomorphism such that the diagram

\[
\begin{array}{c}
C^*(\mathcal{M})^\Gamma \\
\downarrow \text{res}_\Gamma \\
C^*(\mathcal{N} \subset \mathcal{M})^\Pi
\end{array}
\begin{array}{c}
\cong \\
\cong \\
\cong \\
\end{array}
\begin{array}{c}
C^*(\mathcal{N} \subset \mathcal{M})^\Pi/\mathcal{N} \subset \mathcal{M}^\Pi \\
\xrightarrow{q} C^*(\mathcal{N} \subset \mathcal{M})^\Pi \\
\end{array}
\begin{array}{c}
\xrightarrow{\epsilon} C^*(\mathcal{M})^\Pi \\
\xrightarrow{\epsilon} C^*(\mathcal{M})^\Pi \\
\end{array}
\]

commutes.

For the latter use, we give an explicit and intuitive description of this \( * \)-homomorphism \( s \). Let \( R \) denote the injectivity radius of \( M \) with respect to a fixed Riemannian metric. We use the neighborhood \( B_{R/4}(N) := \{ x \in \mathcal{M} \mid d(x, N) < R/4 \} \) of \( N \) as \( U \). Then, as is constructed in [C-W-Y 20, Proposition 2.8], an operator \( T \in B(L^2(\mathcal{M})) \) with \( \text{Prop} T < R/4 \) lifts to \( \tilde{T} \in B(L^2(\mathcal{M})) \) which is uniquely characterized by the property \( \text{Prop} \tilde{T} = \text{Prop} T < R/4 \) (we remark that any two points \( x, y \in \tilde{M} \) with \( d(x, y) < R/4 \) is connected by a unique geodesic in \( \mathcal{M}^\ast \) with the length less that \( R/4 \)). Let \( P \) denote the projection onto the subspace \( L^2(\mathcal{M}) \subset L^2(\tilde{M}) \).

**Proposition 3.18.** Assume that \( T \in C^*(\mathcal{M} \subset \mathcal{M})^\Gamma \) satisfies \( \text{Prop} T < R/4 \). Then the equality \( s(T) = q(\tilde{T}P\tilde{T}) \) holds, where \( q \) denotes the quotient map.
Proof. We use the neighborhood $B_{R/2}(N)$ for $V$ in the proof of Lemma 3.14. Let $\hat{U}$, $\hat{U}'$ and $\hat{U}'$ denote its inverse image in the covering spaces $\hat{M}^\ast$, $\hat{M}'$ and $\hat{M}'$ respectively. Let $\chi$ be a smooth bump function on $\hat{M}$ such that $\chi \equiv 0$ on $V$ and $\chi \equiv 1$ on the complement of $B_{3R/4}(N)$. We use the same letter $\chi$ for its lift to $\hat{M}$.

For $T \in C^\ast(\hat{M})^\Gamma$ with $\operatorname{Prop} T < R/4$, set $T_0 := T \chi \in C^\ast(\hat{M})^\Gamma$ and $T_1 := T(1 - \chi)$. Then the decomposition $T = T_0 + T_1$ satisfies that $\operatorname{supp} T_0 \subset \hat{M}'$ and $\operatorname{supp} T_1 \subset B_R(N)$. We apply \cite{CWY20} Proposition 2.8 to get a lift $\hat{T}_0 \in C^\ast(\hat{M}^\ast)^G$ with propagation less than $R/4$. By the proof of Lemma 3.14 we obtain that

$$s(T_0) = e(T_0) = T_0 \text{ modulo } C^\ast(\hat{N} \subset \hat{M})^\Pi.$$

Here the equality $e(T_0) = T_0$ follows from the fact $\operatorname{Prop}(e(T_0)) < R/4$, which follows from the definition of the map $\epsilon$.

The remaining task is to show that $s(T_1) = \overline{PT_1P}$ modulo the boundary. Set $W := B_R(N) \setminus N$ and we write its inverse images in $\hat{M}^\ast$, $\hat{M}'$ and $\hat{M}^\ast$ as $\hat{W}$, $\hat{W}'$ and $\hat{W}^\ast$ respectively. Then $\hat{W}$ is a disjoint union of connected spaces $\hat{W}_{g\pi}$ parametrized by $\Gamma/\pi$. Each $\hat{W}_{g\pi}$ has a $\Gamma$-covering $\hat{W}_{g\pi}$ such that

$$\hat{W} = \bigsqcup_{g\pi \in \Gamma/\pi} \hat{W}_{g\pi} \times \mathbb{Z}[X_g],$$

$$\hat{W} = \hat{W}_{\pi} \sqcup \bigsqcup_{g\pi \in \Gamma/\pi \setminus \{\pi\}} \hat{W}_{g\pi} \times \mathbb{Z},$$

where $X_g := (\Gamma/\pi) \setminus \{g\pi\}$. Since $T_1$ is supported in $\hat{W}$, it is decomposed into an infinite sum $\sum_{g\pi \in \Gamma/\pi} T_{1,g\pi}$. Let $S_{g\pi}$ be an arbitrary lift of $T_{1,g\pi}$ to $C^\ast(\hat{W}_{g\pi})^\mathbb{Z}$, i.e., $\epsilon(\mathbb{Z}S_{g\pi}) = T_{1,g\pi} \in C^\ast(\hat{W}_{g\pi})$. Then

$$\sum_{g\pi \in \Gamma/\pi} S_{g\pi} \otimes 1 \in \prod_{g\pi \in \Gamma/\pi} C^\ast(\hat{W}_{g\pi})^\mathbb{Z} \otimes C^\ast(\mathbb{Z}[X_g])^\mathbb{Z}[X_g]$$

is a lift of $T_1$ to $\hat{W}$, and hence $s(T_1)$ is

$$\epsilon(\mathbb{Z} \left( \sum_{g\pi \in \Gamma/\pi} S_{g\pi} \otimes 1 \right)) = S_{\pi} \otimes 1 \otimes \mathbb{Z} \otimes 1 \in C^\ast(\hat{W}_{\pi})^\mathbb{Z} \otimes \prod_{g\pi \neq \pi} C^\ast(\hat{W}_{g\pi} \times \mathbb{Z}) \otimes \mathbb{Z},$$

which coincides with

$$0 \oplus \sum_{g\pi \neq \pi} T_{g\pi} \otimes 1 = \overline{PT_1P} \text{ modulo } C^\ast(\hat{N} \subset \hat{M})^\Pi.$$  

This finishes the proof. \qed

This proposition shows that the map $\tau_\sigma$ constructed in (3.7) is rephrased as the composition

$$\tau_\sigma = \partial \circ s_\ast : K_\ast(\hat{M}^\ast)^\Gamma \rightarrow K_\ast \left( \frac{C^\ast(\hat{M}^\ast)^\Pi}{C^\ast(\hat{N} \subset \hat{M})^\Pi} \right) \rightarrow K_{\ast-1}(\hat{N}^\ast(\hat{N} \subset \hat{M})^\Pi).$$

Indeed, the range of this map is isomorphic to $K_{\ast-2}(\hat{N}^\ast) \cong K_{\ast-2}(\hat{N}^\ast)$, as is discussed in next section (Remark 3.2).

4. Codimension 2 transfer of the secondary index invariants

In this section we give a proof of Theorem 4.3. We extend the codimension 2 transfer map to the Higson–Roe analytic surgery sequence, i.e., the long exact sequence associated to the short exact sequence of coarse $C^\ast$-algebras

$$0 \rightarrow C^\ast(\hat{M})^\Gamma \rightarrow D^\ast(\hat{M})^\Gamma \rightarrow Q^\ast(\hat{M})^\Gamma \rightarrow 0.$$

This is obtained by extending the $\ast$-homomorphism (3.17) to $D^\ast(\hat{M})^\Gamma$. Combined with the ‘boundary of Dirac is Dirac’ and ‘boundary of signature is 2’ times signature’ principles, we also show that the resulting homomorphism in $K$-theory relates the primary and the secondary higher index invariants of $M$ and those of $N$.
4.1. Codimension 2 transfer of the Higson–Roe analytic surgery sequence. We start with a short review of the pseudo-local coarse C*-algebra, in particular the maximal C*-completion. For a more detail on the definitions, see for example [Roe96] and [GWY08, OY09].

4.1. Let $D^*_{alg}(\widetilde{M})^\Gamma$ denote the set of operators on $T \in \mathbb{B}(L^2(\widetilde{M}))$ such that

1. $T$ is $\Gamma$-invariant,
2. $T$ is pseudo-local, i.e., $[T, f] \in \mathbb{K}(L^2(\widetilde{M}))$ for any $f \in C_c(\widetilde{M})$,
3. the propagation of $T$ is finite.

An operator $T \in D^*_{alg}(\widetilde{M})^\Gamma$ satisfies $TS, ST \in \mathbb{C}[\widetilde{M}]^\Gamma$ for any $S \in \mathbb{C}[\widetilde{M}]^\Gamma$, and moreover, the multiplier norm

$$
\|T\|_{\text{max}} := \sup_{S \in \mathbb{C}[\widetilde{M}]^\Gamma \setminus \{0\}} \|TS\|_{\text{max}} / \|S\|_{\text{max}}
$$

is finite ([OY09 Lemma 2.16]). Hence $D^*_{alg}(\widetilde{M})^\Gamma$ is regarded as a *-subalgebra of the multiplier C*-algebra $\mathcal{M}(C^*(\widetilde{M})^\Gamma)$. Let $D^*(\widetilde{M})^\Gamma$ denote its closure.

In the same way, we also define the C*-subalgebra $D^*(\overline{\mathcal{N} \subset \overline{M}})^H$ of $\mathcal{M}(C^*(\overline{\mathcal{N} \subset \overline{M}})^H)$. The set $D^*_{alg}(\overline{\mathcal{N} \subset \overline{M}})^H$ of bounded operators on $L^2(\overline{M})$ satisfying the conditions (1), (2), (3) above and

4. $T$ is supported on a $R$-neighborhood of $\overline{N}$, and
5. $Tf, ft \in \mathbb{K}(L^2(\overline{M}))$ for any $f \in C_c(\overline{M} \setminus \overline{N})$,

forms a *-ideal of $D^*_{alg}(\overline{M})^\Gamma$. Hence its closure, denoted by $D^*(\overline{\mathcal{N} \subset \overline{M}})^H$, is a *-ideal of $D^*_{alg}(\overline{M})^\Gamma$.

The quotient C*-algebras are written as

$$
\mathcal{Q}(\overline{M})^H := D^*(\overline{M})^H / C^*(\overline{M})^H,
\mathcal{Q}^*(\overline{\mathcal{N} \subset \overline{M}})^H := D^*(\overline{\mathcal{N} \subset \overline{M}})^H / C^*(\overline{\mathcal{N} \subset \overline{M}})^H.
$$

Then $\mathcal{Q}^*(\overline{\mathcal{N} \subset \overline{M}})^H$ is a *-ideal of $\mathcal{Q}^*(\overline{M})^H$. Moreover, there are isomorphisms

$$
K^H_{1-\eta}(\overline{M}) \cong K_\eta(\mathcal{Q}^*(\overline{M})^H),
K^H_{1-\eta}(\overline{\mathcal{N}}, \overline{M}) \cong K_\eta(\mathcal{Q}^*(\overline{M})^H / \mathcal{Q}^*(\overline{\mathcal{N} \subset \overline{M}})^H),
$$

where the groups in the left hand side are equivariant K-homology groups.

Remark 4.2. The K-group $K_\eta(C^*(\overline{\mathcal{N} \subset \overline{M}})^H)$ is isomorphic to $K_\eta(C^*(\overline{M})^H)$, and the same is true for $D^*$ and $\mathcal{Q}^*$ cases [Sie12 Proposition 4.3.34]. These K-groups are isomorphic to the coarse K-groups of $\overline{N}$ in the following way.

First, in $C^*$-case, the map

$$
\varepsilon_\eta \oplus (\partial_{MV} \circ \text{res}_{\mathcal{H}}^\eta) : K_\eta(C^*(\overline{M})^H) \to K_\eta(C^*(\overline{N})^\pi) \oplus K_{\eta-1}(C^*(\overline{N})^\pi)
$$

is an isomorphism. Here, $\varepsilon$ is the map (5.12), $\text{res}_{\mathcal{H}}^\eta$ is the inclusion $C^*(\overline{M})^H \to C^*(\overline{N})^\pi$, and $\partial_{MV}$ denotes the coarse Mayer–Vietoris boundary map [HRY93]. This is due to the Künneth theorem [RS86] since $C^*(\overline{N})^\pi \cong C^*(\overline{N})^\pi \otimes C^*Z$.

The homomorphisms $\varepsilon_\eta \oplus (\partial_{MV} \circ \text{res}_{\mathcal{H}}^\eta)$ are also defined for $D^*$ and $\mathcal{Q}^*$ coarse C*-algebras, and are isomorphic. Indeed, the *-homomorphism (5.12) extends to $D^*$ and $\mathcal{Q}^*$ algebras by the same construction. Also, the coarse Mayer–Vietoris boundary map is defined for [Sie12]. In $\mathcal{Q}^*$-case, this follows from the Künneth theorem of K-homology group $K^H_{1}(\mathcal{N})$. The $D^*$-case is proved by the five lemma.

Proposition 4.3. The *-homomorphism

$$
s : C^*(\overline{M})^\Gamma \to C^*(\overline{M})^\Gamma / C^*(\overline{\mathcal{N} \subset \overline{M}})^H
$$

constructed in (5.17) lifts to

$$
s : D^*(\overline{M})^\Gamma \to D^*(\overline{M})^\Gamma / D^*(\overline{\mathcal{N} \subset \overline{M}})^H.
$$
Proof. Let $\varepsilon > 0$ be the positive number in Proposition [3.18]. For $T \in D^s(M)^F$, there is a decomposition $T = T_0 + T_1$, where $T_0 \in D^s(M)^F$ satisfies $\text{Prop}(T_0) < R/4$ and $T_1 \in C^s(M)^F$ (cf. [Roe96] Lemma 5.8). We define the $*$-homomorphism $s$ as

$$s(T) := \overline{T}_0F + s(T_1).$$

Here $\overline{T}_0$ and $s(T_1)$ are as in Proposition [3.18]. This is well-defined independent of the choice of the decomposition $T = T_0 + T_1$. Indeed, for another decomposition $T = T'_0 + T'_1$ we have

$$(\overline{T}_0 + s(T_1)) - (\overline{T}'_0 + s(T'_1)) = \overline{S} - s(S),$$

where $S := T_0 - T'_0 = T'_1 - T_1$ is an operator with $\text{Prop}(S) < R/4$ and $S \in C^s(M)^F$. By Proposition [3.18] the difference $\overline{S} - s(S)$ lies in $C^s(\overline{N} \subset \overline{M})^F \subset D^s(\overline{N} \subset \overline{M})^F$.

We show that this $s$ is multiplicative. For $T, S \in D^s(M)^F$, we choose decompositions $T = T_0 + T_1$ and $S = S_0 + S_1$ such that the propagation of $T_0$ and $S_0$ are less than $R/8$. Then, since $\text{Prop}(T_0S_0) < R/4$ and $ST - T_0S_0 \in C^s(M)^F$, we have $s(TS) = \overline{TT_0S_0F} + s(TS - T_0S_0)$, and hence

$$s(T)s(S) - s(TS) = (\overline{TT_0F} \cdot \overline{SP} - \overline{TT_0S_0F}) + (\overline{TT_0F} \cdot s(S_1) - s(T_0S_1)) + (s(T_1) \cdot \overline{PS_0F} - s(T_0S_1)) + (s(T_1)s(S) - s(T_1S_1)).$$

It is straightforward to see that the first term $\overline{TT_0F} \cdot \overline{SP} - \overline{TT_0S_0F} = \overline{T[T_0, P]SP}$ is contained in $D^s(\overline{N} \subset \overline{M})^F$ and the second, third and fourth terms are in $C^s(\overline{N} \subset \overline{M})^F$.

We write $\partial_D$ and $\partial_Q$ for the boundary homomorphism in K-theory associated to the exact sequences

$$0 \rightarrow D^s(\overline{N} \subset \overline{M})^F \rightarrow D^s(\overline{M})^F \rightarrow D^s(\overline{N} \subset \overline{M})^F \rightarrow 0,$$

$$0 \rightarrow Q^s(\overline{N} \subset \overline{M})^F \rightarrow Q^s(\overline{M})^F \rightarrow Q^s(\overline{N} \subset \overline{M})^F \rightarrow 0,$$

respectively.

**Definition 4.4.** The codimension 2 transfer maps are defined as

$$\tau_\sigma := \partial_D \circ \text{res}_\sigma \circ \partial_D \circ s_\sigma : K_\sigma(D^s(M)^F) \rightarrow K_{\sigma - 2}(D^s(\overline{N})^F),$$

$$\tau_\sigma := \partial_D \circ \text{res}_\sigma \circ \partial_Q \circ s_\sigma : K_\sigma(Q^s(M)^F) \rightarrow K_{\sigma - 2}(Q^s(\overline{N})^F).$$

**Theorem 4.5.** The following diagram of long exact sequences commutes:

$$\cdots \rightarrow K_\sigma(C^s(M)^F) \rightarrow K_\sigma(D^s(M)^F) \rightarrow K_\sigma(Q^s(M)^F) \rightarrow K_{\sigma - 1}(C^s(M)^F) \rightarrow \cdots$$

$$\downarrow \tau_\sigma \downarrow \tau_\sigma \downarrow \tau_\sigma$$

$$\cdots \rightarrow K_{\sigma - 2}(C^s(\overline{N})^F) \rightarrow K_{\sigma - 2}(D^s(\overline{N})^F) \rightarrow K_{\sigma - 2}(Q^s(\overline{N})^F) \rightarrow K_{\sigma - 3}(C^s(\overline{N})^F) \rightarrow \cdots.$$

**Proof.** This follows from the commutativity of the diagrams

$$0 \rightarrow C^s(M)^F \rightarrow D^s(M)^F \rightarrow Q^s(M)^F \rightarrow 0$$

and

$$0 \rightarrow C^s(\overline{N} \subset \overline{M})^F \rightarrow C^s(M)^F \rightarrow C^s(M)^F / C^s(\overline{N} \subset \overline{M})^F \rightarrow 0$$

and

$$0 \rightarrow D^s(\overline{N} \subset \overline{M})^F \rightarrow D^s(M)^F \rightarrow D^s(M)^F / D^s(\overline{N} \subset \overline{M})^F \rightarrow 0$$

and

$$0 \rightarrow Q^s(\overline{N} \subset \overline{M})^F \rightarrow Q^s(M)^F \rightarrow Q^s(M)^F / Q^s(\overline{N} \subset \overline{M})^F \rightarrow 0,$$
both of which are obvious from the definitions. □

4.2. **Secondary higher index invariants.** Next, we study the behavior of K-theory classes with the geometric origin, the primary and secondary higher index invariants of manifolds, under the codimension 2 transfer map.

4.6. Let \( R > 0 \) be injectivity radius of \( M \). Let \( D \) be a \( \Gamma \)-invariant formally self-adjoint elliptic differential operator of first order on a \( \Gamma \)-vector bundle on \( \tilde{M} \). Assume that the norm of the principal symbol \( \sigma(D) \) restricted to the unit cosphere bundle \( S(T^*\tilde{M}) \) is bounded by 1. Let \( \tilde{D}_\infty \) be an elliptic operator on \( \tilde{M}_\infty := \tilde{M} \cup_{\tilde{\gamma}} \mathbb{N} \times \mathbb{R}_{\geq 0} \), equipped with a complete metric extending that of \( \tilde{M} \), whose restriction to \( \tilde{M} \) coincides with the lift of \( D \).

Let \( \chi \) be a real-valued odd function on \( \mathbb{R} \) whose Fourier transform \( \hat{\chi} \) is smooth on \( \mathbb{R} \setminus \{0\} \) and satisfies \( \text{supp}(\hat{\chi}) \subseteq [-R/4, R/4] \) and \( \hat{\chi}|_{[-R/8, R/8]} \equiv x^{-1} \). The both functional calculi \( \chi(D) \) and \( \chi(\tilde{D}_\infty) \) are pseudo-local bounded operators with finite propagation and moreover, \( \chi(D)^2 - 1 \) and \( \chi(\tilde{D}_\infty)^2 - 1 \) lie in the Roe algebra. By [Roe88, Proposition 2.3], the propagation of \( \chi(D) \) is less than \( R/4 \) and \( \chi(D)\xi \) depends only on geometric data of the \( R/2 \)-neighborhood of the support of \( \xi \in L^2(\tilde{M}) \) (the same is also true for \( \chi(\tilde{D}_\infty) \)). This and Proposition 3.18 show that

\[
\rho_{s}(\chi(D)) = P\chi(\tilde{D}_\infty)P \mod {D^*((\mathbb{N} \subset \tilde{M}))^\Gamma}.
\]

We review the construction of K-theory classes associated with elliptic operators with the geometric origin.

4.7. Let \( M \) be an \( n \)-dimensional closed spin manifold. Then the Dirac operator \( \psi_M : L^2(\tilde{M}, S) \to L^2(\tilde{M}, S) \) is odd and self-adjoint. The functional calculus \( \psi(\psi_M) \) is bounded, odd, self-adjoint, pseudo-local and finite propagation, and its image in \( Q^*(\tilde{M})^\Gamma \) is a unitary. Hence it determines the Dirac fundamental class

\[
[M] := [\chi(\psi_M)] \in K_{n+1}(Q(\tilde{M})^\Gamma).
\]

More precisely, we consider the \( C_{0,q} \)-Dirac operator when \( n \equiv -q \) modulo 8, which determines an element of the \( K_{n+1} \)-group, as is discussed in Remark A.1.

In the same way the Dirac operator \( \psi_M \) on \( \tilde{M}_\infty \) determines

\[
[\tilde{M}, \mathbb{N}] := [P\chi(\psi_M)P] \in K_{n+1}(Q^*(\tilde{M})^\Pi/Q^*(\mathbb{N} \subset \tilde{M})^\Pi),
\]

where \( P \) is the projection onto \( L^2(\tilde{M}) \). By 4.6, we have

\[
\rho_{s}([\psi_M]) = P\chi(\psi_M)P \in Q^*(\tilde{M})^\Pi/Q^*(\mathbb{N} \subset \tilde{M})^\Pi,
\]

and hence

\[
\rho_{s}([\tilde{M}, \mathbb{N}]) \in K_{n+1}(Q^*(\tilde{M})^\Pi/Q^*(\mathbb{N} \subset \tilde{M})^\Pi) \cong K^{\Pi}_{n+1}(\tilde{M}, \mathbb{N}).
\]

4.8. Let us assume that an \( n \)-dimensional closed spin manifold \( M \) is imposed a positive scalar curvature metric \( g_M \). Then, as is proved in [GXY20] Theorem 1.1], the operator \( \chi(\psi_M) \) is invertible in \( D^*(\tilde{M})^\Gamma \). Hence it determines the K-theory class

\[
\rho(g_M) := [\chi(\psi_M)] \in K_{n+1}(D^*(\tilde{M})^\Gamma).
\]

called the higher \( \rho \)-invariant.

We additionally assume that the psc metric \( g \) is of the form \( g_{D^2} + g_N \) on a tubular neighborhood \( U = N \times \mathbb{D}^2 \) of \( N \), where \( g_{D^2} \) denotes the standard flat metric on \( \mathbb{D}^2 \). We extend the metric on \( M \) to a Riemannian metric \( g_M \) on \( \tilde{M}_\infty \), which has uniformly positive scalar curvature and of the form \( ds^2 + g_N \) on \( N \times \mathbb{R}_{\geq 0} \), where \( ds^2 \) is the metric on \( \mathbb{R}_{\geq 0} \). The Dirac operator \( \psi_M \) on \( \tilde{M}_\infty \) with respect to the pull-back Riemannian metric on \( \tilde{M}_\infty \) determines an invertible element \( \chi(\psi_M) \in D^*(\tilde{M}_\infty)^\Gamma \). Through the isomorphism

\[
D^*(\tilde{M}_\infty)^\Pi/D^*(\mathbb{N} \subset \tilde{M}_\infty) \cong D^*(\tilde{M})^\Pi/D^*(\mathbb{N} \subset \tilde{M})^\Pi,
\]

it determines the higher \( \rho \)-invariant

\[
\rho(g_M) := [P\chi(\psi_M)P] \in K_{n+1}(D^*(\tilde{M})^\Pi/D^*(\mathbb{N} \subset \tilde{M})^\Pi).
\]
By 4.6 we have $s(\chi(\mathcal{D}_M)) = P \chi(\mathcal{D}_M) P$, which implies

$$s_*(\rho(g_M)) = \rho(g_M).$$

**4.9.** Let $M$ be an $n$-dimensional closed oriented manifold. If $n$ is even, the signature operator $D^\text{sgn}_M := d + d^*$ acting on $L^2(\widetilde{M}, \Lambda^\bullet_c TM)$ is an odd self-adjoint elliptic operator, and hence determines a $K$-theory class

$$[M]_{\text{sgn}} := [\chi(D^\text{sgn}_M)] \in K_{n+1}(Q^*(\widetilde{M})\Gamma).$$

A similar construction also works when $n$ is odd (see Remark [A.1] for a more detail). In the same way, the signature operator $D^\text{sgn}_M$ on $\mathcal{M}_\infty$ also determines

$$[\mathcal{M}, \mathcal{N}]_{\text{sgn}} := [P \chi(D^\text{sgn}_M)P] \in K_{n+1}(Q(\mathcal{M})\Pi / Q(\mathcal{N} \subset \mathcal{M})\Pi).$$

By 4.6 we have

$$s(\chi(D^\text{sgn}_M)) = P \chi(D^\text{sgn}_M) P \in Q^*(\mathcal{M})\Pi / Q^*(\mathcal{N} \subset \mathcal{M})\Pi,$$

and hence

$$s_*[M]_{\text{sgn}} = [\mathcal{M}, \mathcal{N}]_{\text{sgn}} \in K_{n+1}(Q(\mathcal{M})\Pi / Q(\mathcal{N} \subset \mathcal{M})\Pi) \cong K_n(\mathcal{M}, \mathcal{N}).$$

**4.10.** Let $f: M' \to M$ be an oriented homotopy equivalence of $n$-dimensional manifolds. We construct an invertible perturbation of the signature operator following the construction of [HS92] Section 3.2 (we also refer the reader to [LS07]). We choose a submersion $F: \mathbb{D}^k \times M' \to M$ such that $f|_{\{0\} \times M'} = f$ and a differential form $v \in \Omega^n_c(\mathbb{D}^k)$ with $\int_{\mathbb{D}^k} v = 1$. We write $p: M' \times \mathbb{D}^k \to M'$ for the projection. We define the linear map $T := p_1 \circ e(v) \circ F^*$, where $F^*$ denotes the pull-back, $e(v)$ denotes the exterior product with $v$, and $p_1$ denotes thefiberwise integral along $p$. Namely

$$T(\omega) := \int_{\mathbb{D}^k} v \wedge F^*(\omega).$$

As is shown in [HS92] Lemma 3.2, there are operators $y: \Omega^*(\widetilde{M}) \to \Omega^*(\widetilde{M})$ and $z \in \Omega^*(\widetilde{M}') \to \Omega^*(\widetilde{M}')$ such that $1 - T'^T T = yd_M + zd_M'y$ and $1 - TT'^T = zd_M' + d_M'z$.

Let $\gamma$ denote the involution acting on $\Omega^p$ by $(-1)^p$, and let $\tau$ denote the Hodge $*$-operator (we use the same letter for the operators on both $M$ and $M'$). Let $a^\dagger$ stand for $\tau a^* \tau$. Following to [HS92] Lemma 2.1, we define the operators

$$d_{f_M} = \begin{pmatrix} d_M & \alpha T^T \\ 0 & -d_M' \end{pmatrix}, \quad L := \begin{pmatrix} 1 & T^T(\tau L)^{-1} \\ T(i\gamma + \alpha y) & 0 \end{pmatrix},$$

where $\alpha > 0$ is chosen in the way that $L$ is invertible (the existence of such $\alpha > 0$ is shown in the proof of [HS92] Lemma 2.1). Set $S := \tau L \cdot |\tau L|^{-1}$.

Following the idea of [GXY20] Section 3.1, we regard the operators as regular unbounded multipliers on the maximal uniform Roe algebra $C^*_{\text{max}}(\mathcal{M}, \mathcal{H}_f)\Gamma$ with respect to the $\Gamma$-equivariant $C_0(\mathcal{M})$-module

$$\mathcal{H}_f := L^2(\mathcal{M}, \Lambda^\bullet_c TM) \oplus L^2(\mathcal{M}', \Lambda^\bullet_c TM').$$

Then the relations $1 - T'^T T = yd_M + d_M'y$ and $1 - TT'^T = zd_M' + d_M'z$ show that the differential $d_{f_M}$ is acyclic as operators on this Hilbert module, and hence, the operator

$$D^\text{sgn}_{f_M} := \begin{cases} (d_{f_M} - Sd_{f_M}S) \circ |\tau L|^{-1} & \text{if } n \text{ is even,} \\ -i(d_{f_M}S + Sd_{f_M}) \circ |\tau L|^{-1} & \text{if } n \text{ is odd}, \end{cases}$$

is self-adjoint and invertible. Moreover it anticommutes with the $\mathbb{Z}_2$-grading $S$ if $n$ is even. We also remark that this operator satisfies the assumption of [4.6] since both $T$ and $y$ have zero propagation. We define the signature higher $\rho$-invariant as

$$\rho_{\text{sgn}}(f_M) := [\chi(D^\text{sgn}_{f_M})] \in K_{n+1}(D^*(\widetilde{\mathcal{M}})\Gamma).$$

We additionally assume that $f_M$ restricts to a homotopy equivalence $f_N: N' \to N$ of codimension 2 submanifolds. We choose the submersion $F$ as $F|_{N'}$ is also a submersion. Then $f_M$ and $F$ extends to a homotopy equivalence $f_M: \mathcal{M}'_\infty \to \mathcal{M}_\infty$ and a submersion $F_M: \mathcal{M}'_\infty \times \mathbb{D}^k \to \mathcal{M}_\infty$ respectively.
By the same construction for these maps, we obtain the operator $D_{fM}^{sgn}$, which is locally a lift of $D_{fM}^{sign}$. It determines the higher $\rho$-invariant

$$\rho_{sgn}(fM) := [P\chi(D_{fM}^{sgn})P] \in K_{n+1}(D^*(\overline{M})^\Pi / D^*(\overline{N} \subset \overline{M})^\Pi).$$

By 4.6 we have

$$s(\chi(D_{fM}^{sgn})) = P\chi(D_{fM}^{sgn})P \in D^*(\overline{M})^\Pi / D^*(\overline{N} \subset \overline{M})^\Pi,$$

and hence

$$s_\ast \rho_{sgn}(fM) = \rho_{sgn}(fM) \in K_{n+1}(D^*(\overline{M})^\Pi / D^*(\overline{N} \subset \overline{M})^\Pi).$$

The ‘boundary of Dirac is Dirac’ and ‘boundary of signature is $2^\epsilon$ times signature’ principles claims the following equalities.

**Lemma 4.11.** The following equalities hold;

1. $\partial Q([\overline{M}, \overline{N}]) = [\overline{N}] = [N]$,
2. $\partial Q([\overline{M}, \overline{N}]_{sgn}) = 2^\epsilon [\overline{N}]_{sgn} = 2[N]_{sgn}$,
3. $\partial_D(\rho_{gMN}) = \rho(g_N) = \rho(g_N)$,
4. $\partial_D(\rho_{sgn}(f\overline{M})) = 2^\epsilon \rho_{sgn}(f\overline{N}) = 2\rho_{sgn}(f\overline{N}).$

where $\epsilon \in \{0, 1\}$ is determined as $\epsilon = 0$ if $\dim M$ is odd and $\epsilon = 1$ if $\dim M$ is even. Here, each of the second equalities are considered under the isomorphisms of $K$-groups given in Remark 4.2.

**Proof.** Each of the first equalities are the ‘boundary of Dirac is Dirac’ or the ‘boundary of signature is $2^\epsilon$ times signature’ principles, which is proved in [HR00b, Proposition 11.2.15], [HSX18, 2.13], [PS14, Theorem 1.22] (see also [Zei16, Theorem 5.15]) and [WYX20, Appendix D] respectively. Each of second equalities again follows from the ‘boundary of Dirac is Dirac’ or the ‘boundary of signature is $2^\epsilon$ times signature’ principles, applied for the manifold $N \cong N \times \mathbb{R}$. We remark that the proof of (4) requires a further discussion because our construction of the signature higher $\rho$-invariant is not the same as the one dealt with in [WYX20]. This is detailed in Appendix A. \qed

**Theorem 4.12.** Let $M$ be an $n$-dimensional closed manifold and let $N$ be a codimension 2 submanifold of $M$ satisfying (1), (2) and (3) of Proposition 2.8.

1. Let $M$ be a spin manifold. Then we have

$$\tau_\sigma([M]) = [N] \in K_{n-1}(Q^*(\tilde{N})^\pi) \cong K_{n-2}(N).$$

2. Let $M$ be oriented. Then we have

$$\tau_\sigma([M]_{sgn}) = [N]_{sgn} \in K_{n-1}(Q^*(\tilde{N})^\pi) \cong K_{n-2}(N).$$

3. Let $M$ be spin and equipped with a psc metric $g_M$ which is of product-like on $U \cong N \times \mathbb{D}^2$. Then we have

$$\tau_\sigma(\rho(g_M)) = \rho(g_N) \in K_{n-1}(D^*(\tilde{N})^\pi).$$

4. Let $f : M' \to M$ be a oriented homotopy equivalence such that the restriction $f|_N : f^{-1}(N) \to N$ is also a homotopy equivalence. Then we have

$$\tau_\sigma(\rho_{sgn}(fM)) = \rho_{sgn}(fN) \in K_{n-1}(D^*(\tilde{N})^\pi).$$

**Proof.** They are already proved in 4.7, 4.8, 4.9, 4.10 and Lemma 4.11. \qed

This, in combination with Theorem 4.5 reprovess the main theorem of [KS20].

**Corollary 4.13** ([KS20 Theorem 1.2, Theorem 1.3]). The codimension 2 transfer map $\tau_\sigma$ sends the Rosenberg index $\alpha_\Gamma(M)$ to $\alpha_\pi(N)$ and the higher signature $Sgn_\Gamma(M)$ to $2 Sgn_\pi(N)$.

**Example 4.14.** Let $\pi$ be a finite cyclic group equipped with $\pi \to \pi_0(\text{Diff}(\Sigma_{g,1}, \partial))$ as in Example 2.4.

Let $N$ be a closed manifold with $\pi_1(N) \cong \pi$ and let $M \to N$ be the surface bundle as in Example 2.4.
• Assume that $N$ is spin and is equipped with a psc metric $g_N$ such that $\rho(g_N) \neq 0$. This assumption is satisfied if there is a unitary representation $\nu \in R(\pi)$ such that the $\rho$-invariant $\rho_\psi(g_N)$ is non-zero, or alternatively, there is a conjugacy class $(h) \in \langle \pi \rangle$ such that the delocalized $\eta$-invariant [Lot99, Definition 7] is non-zero (this assumption implies $\rho(g_N) \neq 0$ [XY19, Theorem 1.1]). Moreover, we assume that the psc metric $ds^2 + g_N$ on $N \times S^1$ extends to a psc metric $g_M$ on $M$. Then we have $\tau_\sigma(\rho(g_M)) = \rho(g_N) \neq 0$, and hence $\rho(g_M) \neq 0$. This shows that $(M, g_M)$ is not psc null-cobordant.

• Let $f_N: N' \to N$ be an oriented homotopy equivalence with non-trivial higher $\rho$-invariant $\rho_{\text{sgn}}(f_N)$. In the same way as above, the non-vanishing of the $\rho$-invariants or delocalized $\eta$-invariants is sufficient for this. Then $M' := f_N(N')$ is also a surface bundle and $f_N$ extends to an oriented homotopy equivalence $f_M: M' \to M$. Then $\tau_\sigma(\rho_{\text{sgn}}(f_M)) = \rho_{\text{sgn}}(f_N) \neq 0$, and hence $\rho_{\text{sgn}}(f_M) \neq 0$. This shows that $f_M: M' \to M$ is not $h$-cobordant to the identity.

4.3. Geometric model of the codimension 2 transfer. We have given two different approaches for defining the codimension 2 transfer map of K-homology groups $\tau_\sigma: K_\ast(M) \to K_{\ast-2}(N)$; the cohomological construction in Definition 2.6 and 2.7 and the analytic (coarse geometric) one in Definition 4.4. Here we show that they are actually the same. We deal with the Baum–Douglas geometric model of the K-homology group [BD82, BHS07] as the domain and the range of the transfer map. We recall that a K-homology cycle is represented by a triple $(W, f, E)$, where $W$ is a spin$^c$ manifold, $f: W \to M$ and $E$ is a complex vector bundle over $W$.

Lemma 4.15. Let $(W, f, E)$ be a geometric K-homology cycle of $M$. Assume that $f$ is transverse to $N$ and set $X := f^{-1}(N)$. Then the map $\tau_\sigma: K_\ast(M) \to K_{\ast}(N)$ sends $[W, f, E]$ to $[X, f|_X, E|_X]$.

Proof. The definition of $\tau_\sigma$ given in Definition 2.6 and 2.7 is given by the cap product with $\sigma \in K^2(\tilde{M}/\pi, \tilde{M}/\pi \setminus U) \cong K^0(U) \cong K^2(M, M \setminus U)$. This $\sigma$ is by definition represented by $[L, \underline{\underline{s}}, s]$, where $L$ is the line bundle on $M$ representing $\sigma \in H^2(U; \mathbb{Z})$ and $s$ is a non-vanishing section of $L$ on $M \setminus U$. We have

$$[L, \underline{\underline{s}}, s] \cdot [W, f, E] = [W, f, E \otimes L] - [W, f, E] = [W \sqcup W^{\text{op}}, f \sqcup f, E \otimes L \sqcup E] = [\mathbb{S}((L \oplus \mathbb{R})|_X), f|_X \circ q, q^*(E \otimes L)|_X],$$

where $W^{\text{op}}$ is $W$ with the opposite orientation and $q: \mathbb{S}(L \oplus \mathbb{R}) \to X$ denotes the projection. Here the last equality comes from a cobordism of K-homology cycles. Now the right hand side is the vector bundle modification of $[X, f|_X, E|_X]$ by the trivial bundle $L \to X$.

The identification of geometric and analytic K-homology groups is given by the map $\Phi_M: K_\ast(M) \to K_{\ast+1}(Q^\ast(\tilde{M})^\Gamma)$ defined as

$$\Phi_M([W, f, E]) := (\text{Ad } V_f)_\ast([\chi(D_f^E)]) \in K_{\ast+1}(Q^\ast(\tilde{M})^\Gamma).$$

Here, $\chi$ is a function as in [4.7] $D_f^E$ is the spin$^c$ Dirac operator on $\tilde{W}$ twisted by the vector bundle $E$, and $V_f: L^2(\tilde{W}) \to L^2(\tilde{M})$ is a $\Gamma$-equivariant covering isometry of $f$. That is, $V_f$ is a $\Gamma$-equivariant isometry such that $V_f^* \varphi V_f - f^* \varphi \in \mathbb{H}(L^2(\tilde{W}))$ for any $\varphi \in C_c(\tilde{M})$ and there is $R > 0$ such that $\text{supp}(V_f)$ is included to the $R$-neighborhood of the graph of $f$ in $M \times W$. Note that such an isometry induces $*$-homomorphisms between $C^\ast$, $D^\ast$ and $Q^\ast$ coarse $C^\ast$-algebras.

Lemma 4.16. Let $M$ and $W$ be manifolds with fundamental group $\Gamma$ and let $f: W \to M$ be a smooth map inducing an isomorphism of fundamental groups. Let $N$ be a codimension 2 submanifold of $M$. Assume that $f$ is transverse to $N$ and set $X := f^{-1}(N)$.

Let $V_f: L^2(\tilde{W}) \to L^2(\tilde{M})$ and $V_f: L^2(\tilde{X}) \to L^2(\tilde{N})$ be equivariant covering isometries. Then the diagram

$$\begin{array}{ccc}
K_\ast(C^\ast(\tilde{W})^\Gamma) & \xrightarrow{(\text{Ad } V_f)_\ast} & K_\ast(C^\ast(\tilde{M})^\Gamma) \\
\downarrow{\tau_\sigma} & & \downarrow{\tau_\sigma} \\
K_\ast(C^\ast(\tilde{X})^\pi) & \xrightarrow{(\text{Ad } V_f)_\ast} & K_\ast(C^\ast(\tilde{N})^\pi)
\end{array}$$

20
commutes. The same commutativity also holds for both $D^*$ and $Q^*$ coarse C*-algebras.

**Proof.** Let $W := (\tilde{W}/\pi) \setminus X$, let $\overline{W}$ be its II-Galois covering, and let $\overline{X} := \partial\overline{W} \cong \tilde{X} \times \mathbb{R}$. Let $p: \overline{M} \to \tilde{M}$ denote the projection and let $\tilde{f}: \overline{W} \to \overline{M}$ be the lift of $f$. Then there is a covering isometry $V_f$ such that, for any open subset $U \subset \overline{M}$ such that $p|_U$ is injective, $V_f|_{L^2(f^{-1}(U))}$ is identified with $V_f|_{L^2(f^{-1}(\bar{p}(U)))}$ by $p$. Then the diagram

$$
\begin{array}{c}
C^*(\overline{W})^\Gamma \xrightarrow{\phi_M} C^*(\overline{W})^\Pi / C^*(\overline{X} \subset \overline{W})^\Pi \\
\downarrow \text{Ad } V_f \quad \quad \quad \downarrow \text{Ad } V_f \\
C^*(\tilde{M})^\Gamma \xrightarrow{\phi_N} C^*(\tilde{M})^\Pi / C^*(\tilde{N} \subset \tilde{M})^\Pi
\end{array}
$$

commutes. Since $(\text{Ad } V_f)_\ast$ is also compatible with the K-theory boundary map, this finishes the proof. The same proof also works for $D^*$ and $Q^*$ coarse C*-algebras. \hfill \Box

**Proposition 4.17.** The following diagram commutes:

$$
\begin{array}{c}
K_\ast(M) \xrightarrow{\Phi_M} K_{\ast+1}(Q^*(\tilde{M})^\Gamma) \\
\downarrow \tau_\sigma \quad \quad \downarrow \tau_\sigma \\
K_{\ast-2}(N) \xrightarrow{\Phi_N} K_{\ast-1}(Q(\tilde{N})^\pi).
\end{array}
$$

**Proof.** An element of $K_\ast(M)$ is represented by a triple $(W, f, E)$. Without loss of generality we may assume that $f$ is transverse of $N$. Then we have

$$(\tau_\sigma \circ \Phi_M)([W, f, E]) = (\tau_\sigma \circ (\text{Ad } V_f)_\ast)([\chi(D_E^W)]) = (([\text{Ad } V_f]_\ast \circ \tau_\sigma)([\chi(D_E^W)])) = ([\text{Ad } V_f]_\ast([\chi(D_E^W)]) = \Phi_N([X, f|_X, E|_X]) = \Phi_N(\text{ind } [X, f|_X, E|_X]).$$

Here, the second equality is due to Lemma 4.16. For the third equality, we use the fact that a twisted spin$^c$ Dirac operator is sent by $\tau_\sigma$ to its restriction to the codimension 2 submanifold, which is shown by the same argument as 4.7. \hfill \Box

We also study a geometric description of the codimension 2 transfer map between analytic structure groups. Inspired from the work of Deeley and Goffeng [DG17], in which a geometric model of the analytic structure group $K_\ast(D^*(\tilde{M})^\Gamma)$ is established, we study the map

$$\Phi_{B\Gamma, M}: K_\ast(B\Gamma, M) \to K_\ast(D^*(\tilde{M})^\Gamma)$$

defined by

$$\Phi_{B\Gamma, M}([W, f, E]) := (\text{Ad } V_f)_\ast([\chi(D_{B\Gamma}^W + A)]) - j_\ast \text{ind}_{APS}(D_{B\Gamma}^E, A),$$

where $A$ is an arbitrary choice of smoothing invertible perturbation of $D_{B\Gamma}^W$ and $\text{ind}_{APS}(D_{B\Gamma}^E, A)$ is the higher Atiyah–Patodi–Singer index with respect to the boundary condition $A$, in the sense of [PS16] Definition 2.27. Alternatively, this element is also defined by using coarse Mayer–Vietoris boundary map as

$$\Phi_{B\Gamma, M}([W, f, E]) := ((\text{Ad } V_f)_\ast \circ \partial_MV)([\chi(D_{B\Gamma}^E)])$$

where $B\Gamma W$ denotes the invertible double of $W$. Note that the operator $D_{B\Gamma W}^E$ is invertible by [XY14] Theorem 5.1.

**Proposition 4.20.** The maps (4.18) and (4.19) coincide.

**Proof.** Let $A_W$ be a smoothing perturbation such that $\text{ind}_{APS}(D_{B\Gamma}^E, A_W) = 0$ (the existence of such $A_W$ is proved in e.g. [Kub20] Lemma 4.6). Let $D_{W, \infty}^E + A_{W, \infty}$ be the perturbed Dirac operator on $\tilde{W}_\infty := \tilde{W} \sqcup_{\partial \tilde{W}} \tilde{W} \times [0, \infty)$ as in [PS16] Definition 2.20. By the triviality of the APS index, there is
a smoothing operator $B$ supported on $\hat{W}$ such that $D_{W,\infty}^E + A_{W,\infty} + B$ is invertible. Then we get the conclusion since

$$P_\chi(D_{W,\infty}^E + A_{W,\infty} + B)P = P_\chi(D_{\hat{W}}^E)P \in D^*(\hat{W})^\Gamma / D^*(\partial\hat{W} \subset \hat{W})^\Gamma,$$

where $P$ denotes the projection onto the $L^2$-section space on $\hat{W}$, and

$$\partial_{MV}([P_\chi(D_{W,\infty}^E + A_{W,\infty} + B)P]) = [\chi(D_{\hat{W}}^E + A_W)].$$

The definition $[4.18]$ of $\Phi_{B\Gamma, M}$ shows that the diagram

$$\begin{array}{ccccccc}
K_*(M) & \xrightarrow{\Phi_M} & K_*(BT) & \xrightarrow{\text{ind}} & K_*(BT, M) & \xrightarrow{\Phi_{B\Gamma, M}} & K_{*-1}(M) \\
\downarrow & & \downarrow & & \downarrow & & \\
K_{*-1}(Q^*(\hat{M})^\Gamma) & \xrightarrow{\Phi_M} & K_*(C^*(\hat{M})^\Gamma) & \xrightarrow{\Phi_{B\Gamma, M}} & K_{*-1}(D^*(\hat{M})^\Gamma) & \xrightarrow{\Phi_M} & K_{*-1}(Q^*(\hat{M})^\Gamma)
\end{array}$$

commutes.

We compare the analytic codimension 2 transfer map given in Definition $[4.4]$ with the relative cohomological transfer defined in the same way as Definition $[2.6]$ by the composition

$$\tau_\sigma : K_*(B\Gamma, M) \to K_*(E\Gamma / \pi, \hat{M} / \pi \cup U) \xrightarrow{\sigma \cap} K_*(U, U \cap \hat{M} / \pi) \to K_*(B\pi, N).$$

Recall that a relative K-homology cycle of the pair $(B\Gamma, M)$ is represented by a triple $(W, f, E)$, where $W$ is a compact spin$^c$-manifold with boundary, $f : (W, \partial W) \to (B\Gamma, M)$ and $E$ is a complex vector bundle on $W$.

**Lemma 4.21.** Let $(W, f, E)$ be a relative geometric K-homology cycle of the pair $(B\Gamma, M)$. Assume that $f|_{\partial W}$ is transverse to $N$ and set $\partial X := f^{-1}(N)$. Then there is a submanifold-with-boundary $(X, \partial X) \subset (W, \partial W)$ such that the map $\tau_\sigma : K_*(M) \to K_*(N)$ sends $[W, f, E]$ to $[X, f|_X, E|_X]$.

**Proof.** The existence of such $X$ follows from $[\partial X] = f^*\sigma \in H^2(\partial W; \mathbb{Z})$, which shows that there is a line bundle $L \to W$ such that $\partial X$ is the zero locus of a section $s : \partial W \to L|_{\partial W}$. The lemma is proved completely in the same way as Lemma $[4.15]$. □

**Theorem 4.22.** The following diagram commutes:

$$\begin{array}{cccccc}
K_*(B\Gamma, M) & \xrightarrow{\Phi_{B\Gamma, M}} & K_*(D^*(\hat{M})^\Gamma) \\
\downarrow{\tau_\sigma} & & \downarrow{\tau_\sigma} \\
K_*(B\pi, N) & \xrightarrow{\Phi_{B\pi, N}} & K_*(D(\hat{N})^\pi).
\end{array}$$

**Proof.** Let $\mathbb{D}W$ and $\mathbb{D}X$ denote the invertible double of $W$ and $X$ respectively. Let $\mathbb{D}\hat{W}$ denote the $\Gamma$-covering of $\mathbb{D}W$, let $\mathbb{D}\hat{W}$ denote the $\Pi$-covering of $(\mathbb{D}\hat{W} / \pi) \setminus \mathbb{D}X$, and let $\mathbb{D}\hat{X}$ denote the boundary of $\mathbb{D}\hat{W}$, which is $\Pi$-equivariantly diffeomorphic to $\mathbb{D}X \times \mathbb{R}$. Then the diagram

$$\begin{array}{cccccc}
K_*(D^*(\mathbb{D}\hat{W})^\Gamma) & \xrightarrow{\partial_{MV}} & K_*(D^*(\mathbb{D}\hat{X})^\Pi) & \xrightarrow{\partial_{MV}} & K_*(D^*(\mathbb{D}\hat{X})^\pi) \\
\downarrow \partial_{MV} & & \downarrow \partial_{MV} & & \downarrow \partial_{MV} \\
K_*(D^*(\hat{W})^\Gamma) & \xrightarrow{\partial_{MV}} & K_*(D^*(\hat{X})^\Pi) & \xrightarrow{\partial_{MV}} & K_*(D^*(\hat{X})^\pi)
\end{array}$$
commutes. This shows the commutativity of $\tau_\sigma$ and $\partial_{MV}$, and hence we get
\[
(\tau_\sigma \circ \Phi_{B;\Gamma,M})([W,f,E]) = (\tau_\sigma \circ (\Ad V_f)_* \circ \partial_{MV})([\chi(D_{BW}^E)]) \\
= ((\Ad V_f)_* \circ \tau_\sigma \circ \partial_{MV})([\chi(D_{BW}^E)]) \\
= ((\Ad V_f)_* \circ \partial_{MV} \circ \tau_\sigma)([\chi(D_{BW}^E)]) \\
= ((\Ad V_f)_* \circ \partial_{MV})([\chi(D_{BW}^E)]) \\
= \Phi_{B;\Gamma,N}([X,f,X], E[X]) \\
= (\Phi_{B;\Gamma,N} \circ \tau_\sigma)([W,f,E]).
\]

We remark that the same discussion also work in Real geometric K-homology only by replacing a triple $(W,f,E)$ with a representative of Real K-homology cycles, i.e., a spin manifold $W$, $f: W \to M$, and a real vector bundle $E$ on $W$ (cf. [RSV09, Section 2]).

5. Codimension 2 transfer and higher index pairing

In this section, we prove the compatibility of the codimension 2 transfer map studied in previous sections and the co-transfer map in cyclic cohomology groups. This reduces computations of the Connes–Moscovici higher index pairing and Lott’s higher $\rho$-numbers of $M$ to those of $N$.

5.1. Codimension 2 co-transfer in cyclic cohomology. For a general theory of cyclic homology and cohomology groups, we refer the readers to standard references [Con94, CST04, Lod98]. We define the $*$-subalgebra of smooth functions $C^\infty(B)$ on $B$, i.e.,
\[
C^\infty(B) := \bigoplus_{\Gamma/\pi} C_0^\infty(0, 1)^\Gamma \bigotimes_{\text{alg}} C_0^\infty(0, 1)^\Gamma.
\]
Here $C_0^\infty(0, 1)$ denotes the space of smooth functions on $\mathbb{T}$ which vanishes at $1 \in \mathbb{T}$. Then $C^\infty(B)$ contains the functions $\sigma(g,h)$ for any $g,h \in \Gamma$, and hence the algebraic twisted crossed product $C^\infty(B) \rtimes_{\sigma, \text{alg}} \Gamma$ makes sense. In the same way as [5.3], the cocycle $\sigma$ is untwisted when it is restricted to the ideal $C^\infty_0(B_0) := \bigoplus_{\Gamma/\pi} C_0^\infty(0, 1)$, we obtain a $*$-isomorphism
\[
C^\infty_0(B_0) \rtimes_{\sigma, \text{alg}} \Gamma \cong C_0^\infty_0(B_0) \rtimes_{\text{alg}} \Gamma \cong S\mathbb{C}[\pi] \otimes \mathbb{K}_{\text{alg}},
\]
where $S\mathbb{C}[\pi] := \mathbb{C}[\pi] \otimes C^\infty_0(0, 1)$, and hence an exact sequence
\[
0 \to S\mathbb{C}[\pi] \otimes \mathbb{K}_{\text{alg}} \to C^\infty(B) \rtimes_{\sigma, \text{alg}} \Gamma \to \mathbb{C}[\Gamma] \to 0.
\]

The periodic cyclic homology $H_{\text{pc}}^*(A)$ of a unital $\mathbb{C}$-algebra $A$ is the homology group of the periodic cyclic complex $(CC_*(A)[v^{\pm 1}], b + vB)$, where $CC_*(A) := A^+ \otimes A^{\otimes p}$, $v$ is a degree $-2$ formal variable and
\[
\begin{align*}
b(\tilde{a}_0 \otimes \cdots \otimes a_n) &= \sum_{j=0}^{n-1} (\tilde{a}_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n) + a_n \tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, \\
B(\tilde{a}_0 \otimes \cdots \otimes a_p) &= \sum_{j=0}^{p} (-1)^{pj} 1 \otimes a_j \otimes \cdots \otimes a_p \otimes a_0 \otimes \cdots \otimes a_{j-1}.
\end{align*}
\]
The cyclic homology $HC_*(A)$ is the homology group of the truncated complex $CC_*(A)[v^{\pm 1}]/CC_*(A)[v]$. Note that $H_{\text{pc}}^*(A)$ is obtained as the projective limit of $HC_*(A)$ by Connes’ $S$-map.

Dually, the periodic cyclic cohomology $HP^*(A)$ is the cohomology of the complex $(CC^*(A)[u^{\pm 1}], b + uB)$, where $CC^*(A) := \text{Hom}(CC_n(A), \mathbb{C})$, $b$ and $B$ denotes the adjoint of the corresponding maps defined above, and $u$ is a degree 2 formal variable.

The periodic cyclic homology have the excision property [Lod98, Theorem 2.2.17] implying the long exact sequence
\[
\cdots \to HP_n(S\mathbb{C}[\pi] \otimes \mathbb{K}_{\text{alg}}) \to HP_n(C^\infty(B) \rtimes_{\sigma, \text{alg}} \Gamma) \to HP_n(C[\Gamma]) \to \cdots.
\]
Since $S\mathbb{C}[\pi] \otimes \mathbb{K}_{\text{alg}}$ is Morita equivalent to $S\mathbb{C}[\pi]$, there is an isomorphism $HP_*(S\mathbb{C}[\pi] \otimes \mathbb{K}_{\text{alg}}) \cong HP_{*-1}(\mathbb{C}[\pi])$ (Lod98, Theorem 2.2.9).
Definition 5.2. We write $\tau_\sigma$ for the boundary map

$$\tau_\sigma := \partial : \text{HP}_*\langle \mathbb{C}[[\Gamma]] \rangle \to \text{HP}_{*-1}(\mathbb{S}\mathbb{C}[\pi] \otimes K_{\text{alg}}) \cong \text{HP}_{*-2}(\mathbb{C}[[\pi]])$$

of the cyclic homology associated to the exact sequence (5.1). Similarly, we define the co-transfer map $\tau_\sigma^*$ as the boundary map

$$\tau_\sigma^* := \delta : \text{HP}^{*-2}(\mathbb{C}[[\pi]]) \cong \text{HP}^{*-1}(\mathbb{S}\mathbb{C}[\pi] \otimes K_{\text{alg}}) \to \text{HP}^*(\mathbb{C}[[\Gamma]])$$

Due to the compatibility of the boundary map and the pairing of cyclic homology and cohomology groups, the equality

$$\langle \tau_\sigma(\xi), \phi \rangle_{\mathbb{C}[[\Gamma]]} = \langle \xi, \tau_\sigma^*(\phi) \rangle_{\mathbb{C}[[\Gamma]]}$$

holds for any $\xi \in \text{HP}_*\langle \mathbb{C}[[\Gamma]] \rangle$ and $\phi \in \text{HP}^{*-2}(\mathbb{C}[[\pi]])$.

Next, we observe that this codimension 2 transfer map respects the decomposition of the cyclic (co)homology group of the group algebra given by Burghelea [Bur85], which is generalized to the groups, the equality

$$\text{HP}(\text{a} \rtimes \text{alg} \Gamma) \cong \bigoplus_{\langle g \rangle \in \langle \Gamma \rangle} \text{HP}(\text{a} \rtimes \text{alg} \Gamma; \langle g \rangle)$$

where

$$\text{CC}_p(\text{a} \rtimes \text{alg} \Gamma; \langle g \rangle) := \text{span}\{(a_0 u_{g_0} \otimes a_1 u_{g_1} \otimes \cdots \otimes a_p u_{g_p} \mid g_0 g_1 \cdots g_q \in \langle g \rangle\}.$$
commutes, where \( j_1, j_2, j_3, j_4 \) are inclusions, considered by identifying \( \mathbb{C}[N_\pi(h)] \) with \( e(\{e\pi\}) \times_{\text{alg}} N_\pi(h) \) and \( \mathbb{C}[\pi] \) with \( e(\{e\pi\}) \times_{\text{alg}} N_\Gamma(h) \). By taking the sum over all \( \langle h \rangle \in \langle \pi; g \rangle \), we get

\[
\bigoplus_{\langle h \rangle} \text{HP}_*(\mathbb{C}[N_\pi(h)]; \langle h \rangle) \rightarrow \bigoplus_{\langle h \rangle} \text{HP}_*(c_c(X_h) \times_{\text{alg}} N_\Gamma(g); \langle h \rangle)
\]

(5.4)

Here we remove the homomorphism \( \text{Ad}(x^{-1}) \) since it induces the identity on the cyclic homology groups.

By [Nis90, Lemma 2.7], each \( (j_2)_* \) is an isomorphism for any \( \langle h \rangle \in \langle \pi; g \rangle \). Moreover, each \( (j_1)_* \) is also an isomorphism since the inclusion \( j_1 \) implements the Morita equivalence. By the same reason, \( \sum_{\langle j \rangle} (j_3)_* \) is also isomorphic. Indeed, the direct sum

\[
\bigoplus_{\langle g \rangle} \sum_{\langle h \rangle \in \langle \pi; g \rangle} (j_4)_* : \bigoplus_{\langle h \rangle \in \langle \pi \rangle} \text{HP}_*(\mathbb{C}[\pi]; \langle h \rangle) \rightarrow \bigoplus_{\langle g \rangle \in \langle \Gamma \rangle} \text{HP}_*(c_c(\Gamma/\pi) \times_{\text{alg}} \Gamma; \langle g \rangle)
\]

coincides with the map \( \text{HP}_*(\mathbb{C}[\pi]) \rightarrow \text{HP}_*(c_c(\Gamma/\pi) \times_{\text{alg}} \Gamma) \), and hence is isomorphic. This shows that \( \sum_{\langle j \rangle} (j_3)_* \) is also isomorphic, and in particular, \( \text{HP}_*(c_c(Y) \times_{\text{alg}} \Gamma; \langle g \rangle) = 0 \).

**Lemma 5.5.** The map \( \tau_\sigma \) respects the decomposition (5.3), i.e., \( \tau_\sigma \) is the direct sum of homomorphisms

\[
\tau^{(g)}_\sigma : \text{HP}_*(\mathbb{C}[\Gamma]; \langle g \rangle) \rightarrow \text{HP}_{*-2}(c_c(\Gamma/\pi) \times_{\text{alg}} \Gamma; \langle g \rangle) \cong \bigoplus_{\langle h \rangle \in \langle \pi; g \rangle} \text{HP}_{*-2}(\mathbb{C}[\pi]; \langle h \rangle).
\]

**Proof.** In general, if one has an extension \( 0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0 \) of \( \Gamma \)-algebras, then the boundary map \( \partial : \text{HP}_n(A/I \times \Gamma) \rightarrow \text{HP}_{n-1}(I \times \Gamma) \) respects the decomposition (5.3) since the decomposition is given in chain level. This proves the lemma since the twisted action \( C^\infty(B, \Gamma) \) is Morita equivalent to a genuine action onto \( C^\infty(B, K_\sigma) \) where \( K_\sigma \) is a \( \Gamma \)-equivariant bundle of compact operator algebras (see Remark [B.4]). \( \square \)

In [Bur83], each group \( \text{HP}_*(\mathbb{C}[\Gamma]; \langle g \rangle) \) is related to the group homology. Indeed, there is an isomorphism

\[
\Phi_\Gamma : H_*(\text{H}_s(\Gamma; \mathbb{C}) \rightarrow \text{HP}_*(\mathbb{C}[\Gamma]; e),
\]

where \( H_*(\cdot) \) denotes the product of homology group \( \prod_{k \in \mathbb{Z}} H_*^{4k}(\cdot) \). We compare the codimension 2 transfer maps in group homology, given in Remark [2.15] and cyclic homology groups.

**Lemma 5.6.** The diagram

\[
\begin{array}{ccc}
H_*(\text{H}_s(\Gamma; \mathbb{C}) & \xrightarrow{2\pi i\sigma} & H_*(\pi; \mathbb{C}) \\
\Phi_\Gamma \downarrow & & \Phi_\sigma \\
\text{HP}_*(\mathbb{C}[\Gamma]; e) & \xrightarrow{\tau_\sigma} & \text{HP}_*(\mathbb{C}[\pi]; e).
\end{array}
\]

commutes.

**Proof.** First, we study the map \( \tau_\sigma \) in twisted equivariant (co)homology, instead of cyclic homology. Let \( (\Omega^*(B), d) \) denote the de Rham complex on \( B \), defined precisely as \( \Omega^*(B) = \mathbb{C} \cdot 1 + \Omega^0(B) \), where \( (\Omega^0(B), d) \) is the complex of \( \Omega^0(B) \) on the manifold \( B \) vanishing at infinity and \( d(1) = 0 \). We regard \( \Omega^*(B) \) as a complex of \( \Gamma \)-modules and define the chian and cochain double complexes \( C_*(\Gamma, \Omega^*(B)) \) and \( C^*(\Gamma, \Omega^*(B)) \) with coefficients in \( \Omega^*(B) \). The cohomology of the total complex of \( C^*(\Gamma, \Omega^*(B)) \) is isomorphic to the equivariant cohomology \( H_1^*(B; \mathbb{C}) \).

Let us define the \((2, 1)\)-cocycle \( \Theta \in C^2(\Gamma; \Omega^1(B)) \) as

\[
\Theta(g, h) := \sigma(g, h)^{-1} d\sigma(g, h) \in \Omega^1(B).
\]

Since \( \int z^{-1} dz = 2\pi i \), this \( \Theta \) represents the cohomology class \( 2\pi i \sigma \in H^2_\text{eff}(B; \mathbb{C}) \) (cf. Proposition [3.10]).
The twisted de Rham differential $\delta_\Theta := \delta + \nu d_{\text{dR}} + \Theta$ on the double complex $C_\bullet(\Gamma, \Omega^\bullet(B))[[v^{\pm 1}]]$, where $\nu$ is a formal symbol of degree $-2$, gives rise to an exact sequence

\[(5.7) \quad 0 \to C_\bullet(\Gamma, \Omega^\bullet(B))[[v^{\pm 1}]] \to C_\bullet(\Gamma, \Omega^\bullet(B))[[v^{\pm 1}]] \to C_\bullet(\Gamma, \mathbb{C})[[v^{\pm 1}]] \to 0.\]

Since the differential $\delta_\Theta$ is of the form $\left( \delta^{\Theta} \right)^0$ under the direct sum decomposition $C_\bullet(\Gamma, \Omega^\bullet(B)) \cong C_\bullet(\Gamma, \mathbb{C}) \oplus C_\bullet(\Gamma, \Omega^\bullet(B))$, the boundary map of the associated long exact sequence is the cap product with $\Theta$. By Lemma 3.3, the group $H_\bullet(\Gamma, \Omega^\bullet(B))[[v^{\pm 1}]]$, $\delta_\Theta$ is isomorphic to the equivariant homology $H^\Gamma_{[-1]}(\Gamma/\pi; \mathbb{C})$ with degree shift by 1. Under this isomorphism, the cap product with $[\Theta]$ is identified with the cap product in equivariant cohomology with $2 \pi i \sigma \in H^2_{\text{dR}}(\Gamma/\pi; \mathbb{C})$.

The remaining task is to relate the exact sequence of group homologies in (5.7) with that of cyclic chain complexes. This is essentially due to the work of Angel [Ang13], but we need slightly modified version of it. The detailed discussion of this part is given in Appendix B. 

We also describe the delocalized part of the cyclic homology in terms of group homology. For $h \in \langle \pi \rangle$, let

$$\sigma^{(h)} := \psi_{\langle h \rangle} \langle \pi \rangle \sigma \in H^2(N_\Gamma(h); \mathbb{Z}[X_h]),$$

where $\psi: N_\Gamma(h) \to \Gamma$ denote the inclusion and $\psi_{\langle h \rangle}: \mathbb{Z}[\Gamma/\pi] \to \mathbb{Z}[X_h]$ denote the quotient. It corresponds to an extension

$$0 \to \mathbb{Z}[X_h] \to \mathbb{Z}[X_h] \rtimes_{\pi} N_\Gamma(h) \to N_\Gamma(h) \to 0.$$ 

Set $N_{\Gamma,g} := N_\Gamma(g)/\langle g \rangle$ and $N_{\pi,h} := N_{\pi}(h)/\langle h \rangle$. If $g \in \Gamma$ is a torsion element, then the homomorphism

$$HC_s(c_c(X_h) \rtimes_{\text{alg}} N_\Gamma(h); \langle h \rangle) \to HC_s(c_c(X_h) \rtimes N_{\Gamma,h}; e)$$

is isomorphic since $h$ acts on $X_h$ trivially [Nis90, 2.7].

**Proposition 5.8.** If $g \in \Gamma$ is a torsion element, the diagram

$$\begin{align*}
H_\bullet(\Gamma, \mathbb{C}) & \xrightarrow{\sum \sigma^{(h)}_{\langle h \rangle} \oplus_{(h) \in \langle \pi \rangle, g} H_\bullet(N_{\pi,h}; \mathbb{C})} \oplus_{(h) \in \langle \pi \rangle, g} H_\bullet(N_{\pi,h}; \mathbb{C}) \\
HP_\bullet(\mathbb{C}[\Gamma]; \langle g \rangle) & \xrightarrow{\tau^{(h)}_{\langle g \rangle} \oplus_{(h) \in \langle \pi \rangle, g} HP_\bullet(\mathbb{C}[\pi]; \langle h \rangle)} 
\end{align*}$$

commutes.

**Proof.** By the above arguments, the diagram

$$\begin{align*}
HC_s(\mathbb{C}[N_{\Gamma,g}]; e) & \xrightarrow{\partial} \oplus_{(h)} HC_s(\mathbb{C}[X_h] \rtimes N_{\Gamma,h}; e) \cong \oplus_{(h)} HC_s(\mathbb{C}[N_{\pi,h}]; e) \\
\cong & \oplus_{(h)} HC_s(\mathbb{C}[\Gamma]; \langle g \rangle) \xrightarrow{\partial} HC_s(c_c(X_h) \rtimes \Gamma; \langle g \rangle) \cong \oplus_{(h)} HC_s(\mathbb{C}[\pi]; \langle h \rangle)
\end{align*}$$

commutes. After taking the projective limit by Connes’ $S$-map, the upper horizontal maps is identified with $2 \pi i \sigma^{(h)}$ by Lemma 5.6. This finishes the proof. 

**Remark 5.9.** As is shown in [Bur85], if $g$ is an infinite order element, then

$$HP_\bullet(\mathbb{C}[\Gamma]; \langle g \rangle) \cong T^g_\bullet(\mathbb{C}; \mathbb{C}) := \lim_{\longleftarrow} H_\bullet(N_{\Gamma,g}; \mathbb{C}, S),$$

where $S$ is the Gysin map of the fibration $S^1 \to BN_\Gamma(g) \to BN_{\Gamma,g}$. The same proof as Proposition 5.8 identifies $\tau^{(h)}$ with the sum of the cap product with $\sigma^{(h)}$ defined on $T^g_\bullet(\mathbb{C}; \mathbb{C})$.

The proof of Proposition 5.8 also shows the dual statement in cohomology theories. Note that the cyclic cohomology group $HC^*(\mathbb{C}[\Gamma])$ decomposes into the direct product of groups $HC^*(\mathbb{C}[\Gamma]; \langle g \rangle)$. Let $H^{[\pi]}(\cdot)$ stands for the direct sum $\bigoplus_{k \in \mathbb{Z}} H^{*[+2k]}(\cdot)$.
Corollary 5.10. If $g \in \Gamma$ is a torsion element, the diagram

$$
\begin{align*}
H^\bullet(N_{\pi}; \mathbb{C}) & \xrightarrow{\sum 2\pi\sigma(h)} \bigoplus_{(h) \in \langle \pi; g \rangle} H^\bullet(N_{\pi,h}; \mathbb{C}) \\
HP^\bullet(\mathbb{C}[\Gamma]; \langle g \rangle) & \xrightarrow{\tau_{\delta}(h)} \bigoplus_{(h) \in \langle \pi; g \rangle} HP^\bullet(\mathbb{C}[\pi]; \langle h \rangle)
\end{align*}
$$

commutes.

5.2. Mapping codimension 2 transfer of analytic surgery to that of homology. Next we relate the codimension 2 transfer map with the ‘mapping analytic surgery to homology’ formalism developed by [PSZ19]. For this sake, we firstly rebuild the codimension 2 transfer maps in the unconditional crossed product Banach algebras. In contrast to Sections 3 and 4 hereafter $C^*(\tilde{M})^\Gamma$ stands for the $C^*$-algebra completion of $C[\tilde{M}]^\Gamma$ with respect to the reduced norm, i.e., the operator norm on $L^2(\tilde{M})$. Other coarse $C^*$-algebras $D^*(\tilde{M})^\Gamma$ and $Q^*(\tilde{M})^\Gamma$ are also defined in a consistent way.

Let $\| \cdot \|_{A\Gamma}$ be a submultiplicative norm on the group algebra $\mathbb{C}[\Gamma]$. Assume this is unconditional, i.e., if $\sum a_g u_g, \sum b_g u_g \in \mathbb{C}[\Gamma]$ satisfies $|a_g| \leq |b_g|$ for any $g \in \Gamma$, then $\| \sum a_g u_g \|_{A\Gamma} \leq \| \sum b_g u_g \|_{A\Gamma}$ holds. We write $A\Gamma$ for the completion of $\mathbb{C}[\Gamma]$ with respect to this norm. We also assume that $A\Gamma$ is closed under holomorphic functional calculus. Let $\| \cdot \|_{A\pi}$ the restriction of $\| \cdot \|_{A\Gamma} \to \mathbb{C}[\pi]$, which is also a submultiplicative unconditional norm on $\mathbb{C}[\pi]$ and the completion $A\pi := \mathbb{C}[\pi]$ is closed under holomorphic functional calculus.

For a $\Gamma$-$C^*$-algebra $B$, we define the norm $\| \cdot \|_{A(B,\Gamma)}$ on the algebraic crossed product $B \rtimes_{\text{alg}} \Gamma$ as

$$\| \sum g f_g u_g \|_{A(B,\Gamma)} := \| \sum g \| f_g \| u_g \|_{A\Gamma}.$$  (5.11)

We write $A(B, \Gamma)$ for the completion of $B \rtimes_{\text{alg}} \Gamma$ with respect to this norm.

Lemma 5.12. Let $0 \to I \to B \xrightarrow{\varphi} B/I \to 0$ be an extension of $\Gamma$-$C^*$-algebras such that there is a (possibly non-equivariant) contractive section $\nu: B/I \to B$. Then

$$0 \to A(I, \Gamma) \to A(B, \Gamma) \to A(B/I, \Gamma) \to 0$$

is an exact sequence of Banach algebras.

Proof. The exactness at the left and the right are obvious. Hence it suffices to show that the map $\tilde{q}: A(B, \Gamma)/(A(I, \Gamma) \to A(B/I, \Gamma)$ induced from $\varphi$ is an isomorphism. For any $x \in A(B, \Gamma)$ and an arbitrary $\varepsilon > 0$, we take a finite sum $x' := \sum_g b_g u_g \in B \rtimes_{\text{alg}} \Gamma$ such that $\| x - x' \|_{A(B,\Gamma)} < \varepsilon$. Let $y' := \sum_g (\nu \circ \varphi)(b_g) \cdot u_g \in I \rtimes_{\text{alg}} \Gamma$. Then

$$x' - y' = \sum_g (b_g - (\nu \circ \varphi)(b_g)) \cdot u_g \in I \rtimes_{\text{alg}} \Gamma$$

and hence $\tilde{q}(x') = \tilde{q}(y')$. Therefore, we obtain

$$\| \tilde{q}(x) \| \leq \| \tilde{q}(x') \| + \varepsilon = \inf_{\tilde{q}(x') = \tilde{q}(y')} \| y \|_{A(B,\Gamma)} + \varepsilon \leq \| y' \|_{A(B,\Gamma)} + \varepsilon$$

$$= \| \sum g (\nu \circ \varphi)(b_g) \|_{A\Gamma} + \varepsilon$$

$$\leq \sum g \| (\nu \circ \varphi)(b_g) \|_{B/I \rtimes \Gamma} + \varepsilon = \| q(y') \|_{A(B/I,\Gamma)} + \varepsilon.$$

Here, for the last inequality we use the unconditionality condition of the norm $\| \cdot \|_{A\Gamma}$. This shows the lemma. □

Let $K_{\sigma}$ be a $\Gamma$-equivariant compact operator algebra bundle over $B$ whose Dixmier-Douady class is $\sigma \in H^2(B; T)$ (a construction is given in Remark 3.4). Let $C(B, K_{\sigma})$ denote the $C^*$-algebra of continuous section of $K_{\sigma}$. Then the crossed product $C(B, K_{\sigma}) \rtimes \Gamma$ is Morita equivalent to $C(B) \rtimes_{\sigma} \Gamma$. 27
Apply Lemma 5.12 to the extension $0 \to C_0(B_0, \mathcal{K}_\sigma) \to C(B, \mathcal{K}_\sigma) \to \mathbb{C} \to 0$, we obtain an exact sequence

$$0 \to A(C_0(B_0, \mathcal{K}_\sigma), \Gamma) \to A(C(B, \mathcal{K}_\sigma), \Gamma) \to A(\mathbb{K}, \Gamma) \to 0$$

of Banach algebras. This induces the boundary map

$$\tau_\sigma := \partial : \text{HP}_*(A\Gamma) \to \text{HP}_{*-1}(A(C_0(B_0, \mathcal{K}_\sigma), \Gamma)) \cong \text{HP}_{*-2}(A\pi),$$

which is compatible with Definition 5.2. The last isomorphism follows from our assumption on the unconditional norm $\| \cdot \|_{A\pi}$, which implies that $SA\pi$ is identified with a full corner subalgebra of $A(C_0(B_0, \mathcal{K}), \Gamma) = A(SC, (\Gamma/\pi) \otimes \mathbb{K}, \Gamma)$.

The following lemma shows that the codimension 2 transfer maps constructed in Banach algebraic and C*-algebraic settings are compatible.

**Lemma 5.13.** Assume that $\Gamma$ is exact (cf. Remark 3.8). Then there is an inclusion of exact sequences

$$0 \longrightarrow A(C_0(B_0, \mathcal{K}_\sigma), \Gamma) \longrightarrow A(C(B, \mathcal{K}_\sigma), \Gamma) \longrightarrow A(\mathbb{K}, \Gamma) \longrightarrow 0$$

$$0 \longrightarrow C_0(B_0, \mathcal{K}_\sigma) \rtimes_{\text{red}} \Gamma \longrightarrow C(B, \mathcal{K}_\sigma) \rtimes_{\text{red}} \Gamma \longrightarrow \mathbb{K} \rtimes_{\text{red}} \Gamma \longrightarrow 0.$$

**Proof.** By assumption of the norm $\| \cdot \|_{A\Gamma}$, the right vertical map is contractive. The left vertical map is also contractive since the algebras $A(C_0(B_0, \mathcal{K}_\sigma), \Gamma)$ and $C_0(B_0, \mathcal{K}_\sigma) \rtimes_{\text{red}} \Gamma$ are Morita equivalent to $SA\pi$ and $SC_{\text{red}}^{\ast}\pi$ respectively, and moreover $A\pi$ is a subalgebra of $C_{\text{red}}^{\ast}\pi$ (note that the closure of $C[\pi]$ in $C_{\text{red}}^{\ast}\pi$ coincides with $C_{\text{red}}^{\ast}\Gamma$).

To see that the middle map is bounded, let $x = \sum g_b u_y \in C(B, \mathcal{K}, \sigma) \rtimes_{\text{alg}} \Gamma$. For simplicity, we write the unconditional norms as $\| \cdot \|_{A\Gamma}$ and reduced crossed product norms as $\| \cdot \|_{C\ast\Gamma}$. In the same way as Lemma 5.12 let $y := \sum (\nu \circ q)(g_b) u_y$. Then the assumption of unconditionality on $A\Gamma$ shows that

- $\|y\|_{A\Gamma} \leq \|x\|_{A\Gamma}$,
- $\|x - y\|_{C\ast\Gamma} \leq \|x - y\|_{A\Gamma} \leq 2\|x\|_{A\Gamma}$,
- $\|y\|_{A\Gamma} \leq \|q(y)\|_{A\Gamma} = \|q(x)\|_{A\Gamma} \leq \|x\|_{A\Gamma}$.

which implies that $\|x\|_{C\ast\Gamma} \leq 3\|x\|_{A\Gamma}$. This shows that the middle vertical map is bounded on a dense subalgebra of $A(C(B, \mathcal{K}_\sigma), \Gamma)$, and hence extends to a bounded algebra homomorphism. \qed

The decomposition of periodic cyclic homology group $\text{HP}(\mathbb{C}[\Gamma])$ into the localized and delocalized parts extends to a long exact sequence

$$\cdots \to \text{HP}^*_p(A\Gamma) \to \text{HP}_* (A\Gamma) \to \text{HP}^{\delta_\ast}_*(A\Gamma) \overset{\delta_p}{\to} \text{HP}_*(A\Gamma) \to \cdots.$$ 

In [PSZ19] Definition 6.36, Piazza–Schick–Zenobi has constructed the Chern character maps taking value in the non-commutative de Rham homology groups

$$\text{Ch}_p : K_* (\widetilde{M}^\delta) \to \text{HP}^p_{[\delta]} (A\Gamma),$$

$$\text{Ch}_p^{\delta_*} : K_* (D^\ast (\widetilde{M}^\delta)) \to \text{HP}^{\delta_*}_{[\delta]} (A\Gamma),$$

$$\text{Ch}_p^{\ast} : K_* (Q^\ast (\widetilde{M}^\delta)) \to \text{HP}^{\ast}_{[\delta]} (A\Gamma),$$

by using the pseudodifferential operator algebras (whose K-theory is identified with those of coarse C*-algebras). Here, $\text{HP}^p_{[\delta]} (A\Gamma)$ denotes the periodic reduced noncommutative de Rham homology group [Kar87], which embeds into the reduced periodic cyclic homology $\overline{\text{HC}}_*(A\Gamma) := \text{coker}(\text{HP}_*(\mathbb{C}) \to \text{HP}_*(A\Gamma)).$

**Remark 5.15.** In [PSZ19], the map $\text{Ch}_p^{\ast}$ is defined to be the composition of the Chern character of noncommutative de Rham homology and Lott’s trace map [Lot92a]. This map coincides with the Chern–Connes character in cyclic homology theory. Indeed, this is checked by comparing [Kar87] 1.17, Example 1.15 and [Lod98] Theorem 8.3.2, Corollary 8.3.5, noting that the inclusion $\text{HP}^p_{[\delta]} (A\Gamma) \subset \overline{\text{HC}}_{\ast} (A\Gamma)$ is induced from the isomorphism $\text{HC}_{\ast} (A\Gamma)/1.1 b \cong \overline{\text{HC}}_{\ast} (A\Gamma)/1.1 d$ (see [Lod98] Theorem 2.6.8)).
In this paper we deal with a minor modification of (5.14). Let $V$ be a vector bundle over $M$ and let $X$ denote the sphere bundle $S(V \oplus \mathbb{R})$, which includes $M$ as a submanifold. We use the letter $\mathcal{V}$ for the tubular neighborhood of $M$ in $X$, on which a Riemannian metric is imposed in the way that the projection $\mathcal{V} \to M$ is a coarse equivalence. Let $\tilde{X}$ denote the $\Gamma$-Galois covering of $X$. In [PSZ19 Definition 5.14], the Banach algebra completions $\Psi_{\text{AF}}^0(\tilde{X})$ and $\Psi_{\text{AF}}^\infty(\tilde{X})$ of the algebras of $\Gamma$-invariant pseudodifferential operators of order 0 and $-\infty$ are defined. We write $\Psi_{\text{AF}}^k(\mathcal{V})$ ($k = 0, -\infty$) for the closed subalgebras of $\Psi_{\text{AF}}^k(\tilde{X})$ consisting of operators supported on $\mathcal{V}$. Then $\Psi_{\text{AF}}^\infty(\mathcal{V})$ is a dense subalgebra of $C^\infty(\mathcal{V}) = C^\infty_\Gamma \otimes \mathbb{K}$ and the quotient $\Psi_{\text{AF}}^0(\mathcal{V})/\Psi_{\text{AF}}^\infty(\mathcal{V})$ is isomorphic to the algebra of principal symbols, i.e., the continuous functional algebra $C_0(S\mathcal{V})$ on the sphere bundle of $T\mathcal{V}$. Moreover, $\Psi_{\text{AF}}^0(\mathcal{V})$ is a subalgebra of $D^*(\mathcal{V})$. As is shown in [Zen19 Theorem 3.5], there is an isomorphism of $K$-groups of mapping cones

$$K_*\left(\mathcal{V} \to \Psi_{\text{AF}}^0(\mathcal{V})\right) \cong K_*\left(\mathcal{V} \to D^*(\mathcal{V})\right)$$

such that the composition

$$K_*\left(\mathcal{V} \to \Psi_{\text{AF}}^0(\mathcal{V})\right) \to K_*\left(\mathcal{V} \to \Psi_{\text{AF}}^0(\mathcal{V})\right) \cong K_*\left(\mathcal{V} \to D^*(\mathcal{V})\right)$$

is the same map as the one induced from the inclusion $\Psi_{\text{AF}}^0(\mathcal{V}) \subset D^*(\mathcal{V})$.

Note that a five lemma argument and the Thom isomorphism $K_*\mathcal{V}) \cong K_* (M)$ shows that $D^*(\tilde{M})$ and $D^*(\mathcal{V})$ have the same $K$-theory. We write $\text{Ch}_\Gamma, \text{Ch}_\Gamma^\infty$ and $\text{Ch}_\Gamma^\text{del}$ for the compositions

$$\text{Ch}_\Gamma: \text{Ch}_\Gamma^\text{del}: \text{Ch}_\Gamma^\infty,$$

where the last maps $\text{Ch}_\Gamma, \text{Ch}_\Gamma^\infty, \text{Ch}_\Gamma^\text{del}$ are the Chern character maps (5.14) of Piazza–Schick–Zenobi (more precisely, the composition of Chern character maps on $\tilde{X}$ and the inclusions induced from $\Psi_{\text{AF}}^k(\tilde{X}) \to \Psi_{\text{AF}}^k(\tilde{X})$). Here we remark that one gets the same map if they use another open embedding $\mathcal{V} \subset Y$ inducing isomorphism of $\pi_1$ instead of $\mathcal{V} \subset X$. It directly follows from the definition of Chern character maps. By [PSZ19 Theorem 6.38], the diagram

$$\cdots \longrightarrow K\left(\mathcal{V} \to \Psi_{\text{AF}}^0(\mathcal{V})\right) \longrightarrow K\left(\mathcal{V} \to D^*(\mathcal{V})\right) \longrightarrow \cdots$$

commutes.

**Lemma 5.16.** Let $W$ be a compact spin$^c$ manifold with boundary $\partial W$. Assume that the Euler number of $\partial W$ is zero. Let $f: (W, \partial W) \to (B\Gamma, \mathcal{V})$ be a continuous map of pairs such that $f|_{\partial W}$ is a smooth embedding with trivial normal bundle. Let $E$ be a vector bundle over $W$. Then there is $x_{\partial W, f, E} \in K_*\left(\mathcal{V} \to \Psi_{\text{AF}}^0(\mathcal{V})\right)$ with the following properties;

1. the class $x_{\partial W, f, E}$ is sent to $\Phi_{\text{AF}}([W, f, E]) \in K_*\left(\mathcal{V} \to D^*(\mathcal{V})\right)$,
2. $\text{Ch}_\Gamma^\text{del}(x_{\partial W, f, E}) = 0$.

**Proof.** Firstly we remark that any pseudosymmetric operator acting on sections of a vector bundle $E$ over a manifold $M$ is viewed as an element of $\mathbb{M}(\Psi_{\text{AF}}^0(\mathcal{V}))$ for some $N$ by identifying $L^2(\mathcal{M}, E)$ with $PL^2(\mathcal{M}, \mathbb{C}^N)$, where $P \in C(M, \mathbb{M}_{\mathbb{R}})$ is the support projection of $E$ embedded into a trivial bundle $\mathbb{C}^N$.

Here we focus on the case that $\dim W$ is even, and just remark that the same proof also works for odd-dimensional case by considering Dirac operators with $Cl_1$-symmetry. Let $D_{\partial W}$ and $D_W$ denote the spin$^c$-Dirac operator on $\partial W$ and $W$ twisted by $E$ respectively. Let $A_W$ be a smoothing operator as
Proposition 4.20, i.e., we have \((D_{\partial W} + A_W)^2 \geq \varepsilon^2\) and \(\text{ind}_{APS}(D_W, A_W) = 0\). By (4.18), we have

\[
\chi(D_{\partial W} + A_W) = \Phi_{BT,M}([W, f, E]) \in K_0(D^*(\partial \tilde{W})^\Gamma).
\]

Let \(D^m_r\) denote the \(m\)-dimensional open disk of radius \(r > 0\). We define operators \(C\) and \(D\) on \(L^2(D^m_r, \Delta_{m,m})\), where \(\Delta_{m,m}\) is the unique irreducible representation of the Clifford algebra \(C\ell_{m,m}\), as

\[
D := \sum c(f_i)\partial_{x_i}, \quad C := \frac{1}{r} \sum c(e_i) x_i,
\]

where \(c(e_i)\) and \(c(f_i)\) are self-adjoint generators of \(C\ell_{m,m}\) with \(c(e_i)^2 = 1\) and \(c(f_i)^2 = -1\). Then \(C\) is the Clifford multiplication with the radial vector field on \(D^m_r\), and in particular \(C^2 - 1 \in C_0(D^m_r)\). Set \(S^E_{\partial W,m} := E \otimes S_{\partial W} \otimes \Delta_{m,m}\). By the assumption on the vanishing of the Euler number of \(\partial W\), there is a non-vanishing vector field \(\xi\) on \(\partial W \times D^m_r\) which coincides with the radial vector field outside a compact subset, and hence its Clifford multiplication acting on the bundle \(S^E_{\partial W,m}\) satisfies

\[
C - c(\xi) \in C_0(W \times D^m_r, \text{End}(S^E_{\partial W,m})).
\]

Now, the bounded self-adjoint operator

\[
B := (1 - C^2)^{1/4} \chi(D_{\partial W} + A_W + D)(1 - C^2)^{1/4} + C
\]

on \(\partial W \times D^m_r\) is a 0-th order pseudo-differential operator whose principal symbol satisfies \(\sigma(B) - c(\xi) \in C_0(SV, \text{End}(S^E_{\partial W,m}))\). Moreover, it is invertible if \(r\) is sufficiently large so that

\[
\|[(1 - C^2)^{1/4}, \chi(D_{\partial W} + A_W + D)]\| < \varepsilon/4.
\]

Without loss of generality, we may assume that the Riemannian metric on \(r\)-neighborhood of \(\partial W\) is the product metric \(g_{\partial W} + g_{\mathbb{D}^m_r}\). We define \(x_{W,f,E}\) as the difference class

\[
x_{W,f,E} := \iota_*(\kappa(B, c(\xi))) \in K_0(\Psi^0_{\text{AT}}(\tilde{V})),
\]

where \(\kappa: \Psi^0_{\text{AT}}(\partial \tilde{W} \times D^m_r) \to \Psi^0_{\text{AT}}(\tilde{V})\) is induced from the isometric open embedding. The inclusion \(\Psi^0_{\text{AT}}(\tilde{V}) \subset D^*(\tilde{V})^\Gamma\) sends this \(x_{W,f,E}\) to

\[
[\chi(D_{\partial W} + A_W)] \otimes \beta \in K_{\ast}(D^*(\tilde{V})^\Gamma),
\]

where \(\otimes\) denotes the secondary exterior product in the sense of Appendix A. Hence it is identified with \(\Phi_{BT,M}([W, f, E])\) under the isomorphism \(K_{\ast}(D^*(\tilde{M})^\Gamma) \cong K_{\ast}(D^*(\tilde{V})^\Gamma)\).

Finally we show that \(\text{Ch}^\text{del}_{\tilde{V}}(x_{W,f,E}) = 0\). This follows from [PSZ19] Proposition 6.71, in which the delocalized \(\rho\)-invariant is identified with the higher APS index. Set \(W := W \times S^m\). Note that both \(B\) and \(c(\xi)\) extends to a pseudo-differential operator on \(\partial W\) twisted by \(S_{\partial W} \otimes E \otimes \mathcal{E}\), where \(\mathcal{E}\) is the \(S^2\)-graded vector bundle on \(S^m\) obtained by clutching trivial bundles \(D^m_r \times \Delta_{m,m}\) and \(D^m_r \times (\Delta_{m,m})^\oplus 2\) by \(\text{id} \oplus C_{\partial \mathbb{D}^m_r}\). Therefore, the image of \(x_{W,f,E}\) in the \(K\)-group of \(\Psi^0_{\text{AT}}(\tilde{W})\) is \([B] - [c(\xi)]\).

Since \(\text{Ch}^\text{del}\) factors through the mapping cone \(C(C(W) \to \Psi^0_{\text{AT}}(\tilde{W}))\), we have \(\text{Ch}^\text{del}_{\tilde{V}}([c(\xi)]) = 0\). Moreover, since \(B\) is homotopic to \(\chi(D_{\partial W} + A_W + D_{\tilde{S}_m})\) in \(\Psi^0_{\text{AT}}(\tilde{W})\), we have

\[
\text{Ch}^\text{del}_{\tilde{V}}(x_{W,f,E}) = \text{Ch}^\text{del}_{\tilde{V}}([B]) = \text{Ch}^\text{del}_{\tilde{V}}([\chi(D_{\partial W} + A_W + D_{\tilde{S}_m})]) = \text{ind}_{APS}(D_W + D_{\tilde{S}_m}, A_W \oplus 1) \quad \text{ind}_{APS}(D_W, A_W) \cdot \text{ind}(D_{\tilde{S}_m}) = 0.
\]

Here the third equality is [PSZ19] Proposition 6.71 and the forth equality is the multiplicativity of the higher APS index.

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Since (1), (2), (3) of Theorem 4.22 we have the subgroup $\Gamma$. Assume that it in the way that there are mutually disjoint isometric open embeddings of tangent bundles $T\Gamma$. Generates $\Gamma$. The commutativity of the left diagram is due to the functoriality, more precisely the compatibility of a diagram chasing argument similar to the above proof of Theorem 5.17 shows that $\tau_\sigma$ with this problem. We only remark that $\tau_\sigma$ commutative diagram. In the same way, a diagram chasing argument shows the commutativity of the middle diagram on the 'complement' of $\mathcal{X}$. We show the commutativity of the right diagram. Recall that $\tau_\sigma$ is injective. This finishes the proof since $\text{H}^*_p(C[\Gamma]) \rightarrow \text{H}^*_p(C[\Gamma])$ is injective.

In the same way, a diagram chasing argument shows the commutativity of the middle diagram on the subgroup $\mathcal{X} := \text{Im}(K_*(\tilde{N})^\Gamma) \rightarrow K_*(D^*(\tilde{M})^\Gamma)$ as

\[
\begin{array}{c}
\text{Ch}_{\Gamma,V} \\
\text{K}_*(C_0(T^*M)) \longrightarrow K_*(\Psi_{C^\infty}^\Gamma(\tilde{M})) \longrightarrow K_*(\Psi_{C^\infty}^\Gamma(N)) \longrightarrow K_*(C_0(T^*N))
\end{array}
\]

Here we write the mapping cone $C(C(M) \rightarrow C(SM))$ as $C_0(T^*M)$ in short. This finishes the proof. The remaining task is to show the commutativity of the diagram on the 'complement' of $\mathcal{X}$. By the assumption of the exactness, $\Gamma$ has the strong Novikov property [Oza00][HR00a][Yu00], and hence we have

\[
\begin{align*}
\text{K}_*(C^*(\tilde{M})^\Gamma) &\cong K^\Gamma_*(E\Gamma, ET) \oplus K^\Gamma_*(ET) \subset \text{K}_*(D^*(\tilde{M})^\Gamma) \\
\text{K}_*(D^*(\tilde{M})^\Gamma) &\cong K^\Gamma_*(E\Gamma, ET) \oplus K^\Gamma_*(ET, \tilde{M})
\end{align*}
\]

Let $x_1, \ldots, x_k \in K_*(BG, M)$ be a finite number of elements such that their images generate the subgroup $\text{Im}(\partial: K_*(BG, M) \rightarrow K_{*+1}(M))$. The, by the above isomorphism, $x_1, \ldots, x_k$ and $\mathcal{X}$ generates $K_*(D^*(\tilde{M})^\Gamma)$. Let $[W_i, f_i, E_i]$ be geometric cycles representing $x_i$. We may assume that the tangent bundles $T(\partial W_i)$ are trivial. We choose a vector bundle $V \rightarrow M$ and a Riemannian metric on it in the way that there are mutually disjoint isometric open embeddings $\partial W_i \times \mathbb{R}^m \subset V$. Then, by Theorem 4.22 we have $\text{Ch}_{\Gamma,V}^\text{del}(x_i) = \text{Ch}_{\Gamma,V}(x_i) = 0$. Moreover, by Lemma 5.16 we can apply the same argument to $\tau_\sigma(x_i) = [W_i \cap N, f_i|_{W_i \cap N}, E_i|_{W_i \cap N}]$, and get $\text{Ch}_{\Gamma,V}^\text{del}(\tau_\sigma(x_i)) = 0$. This finishes the proof.

\[
\text{Remark 5.18.} \quad \text{It is natural to ask whether } \text{Ch}_{\Gamma,V}^\text{del} \text{ coincides with } \text{Ch}_{\Gamma,V}^\text{del} \text{ in general, but here we do not deal with this problem. We only remark that } \text{Ch}_{\Gamma} = \text{Ch}_{\Gamma,V} \text{ is easily verified from the definition, and hence a diagram chasing argument similar to the above proof of Theorem 5.17 shows that } \text{Ch}_{\Gamma}^\text{del} = \text{Ch}_{\Gamma,V}^\text{del} \text{ on the subgroup } \mathcal{X}.
\]

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5.3. The higher index pairing. Finally, we relate the primary and secondary higher index pairings with the codimension 2 transfer map.

Let $\Gamma$ be a finitely presented hyperbolic group and let $\pi$ be its hyperbolic subgroup (note that any hyperbolic group is exact [Ada94]). Let $S$ be a generating set of $\Gamma$ such that $S \cap \pi$ generates $\pi$. Let $\ell$ and $\ell_\pi$ denote the associated word-length function on $\Gamma$ and $\pi$ respectively. Note that $\ell(h) \leq \ell_\pi(h)$ holds for any $h \in \pi$. Joissaint [JoI89] and Lafforgue [Laf00,Laf02], there is an increasing sequence of unconditional norms

$$\| \sum g a_g u_g \|_{A^k_n,\Gamma}^2 := \sum_g |a_g|^2 (1 + \ell(g))^{2n}$$

of the group algebra $C[\Gamma]$. It is shown in [Laf00] Proposition 1.2 that this norm is submultiplicative for sufficiently large $n$. In [Pus10], Puschmann has introduced a new series of unconditional norms $\| \cdot \|_{A^k_n,\Gamma}$ constructed from $\| \cdot \|_{A^k_n,\pi_i}$ for $k \in \mathbb{N}$. We write $A^k_n,\Gamma$ for the Banach $*$-algebra completion of $C[\Gamma]$ with respect to $\| \cdot \|_{A^k_n,\Gamma}$.

5.19. The family $\{\| \cdot \|_{A^k_n,\Gamma}\}_{n,k \in \mathbb{N}}$ is an increasing sequence of submultiplicative norms on $B \rtimes_{\text{red}} \Gamma$. They have the following properties:

1. Each $A^k_n,\Gamma$ is closed under holomorphic functional calculus.
2. There are inclusions $C[\Gamma] \subset A^k_n,\Gamma \subset A^k_m,\Gamma \subset C^*_{\text{red}},\Gamma$ for $n \geq m$ and $k \geq l$.
3. There is a splitting $\kappa_{\Gamma} : \text{HP}^*(C[\Gamma]) \to \text{HP}^*(A^\Gamma)$ of the map $\text{HP}^*(A^\Gamma) \to \text{HP}^*(C[\Gamma])$ induced from the inclusion.
4. Let $\text{HP}^*(A^\Gamma; (g))$ denote the cohomology group of the closure of the subcomplex $C^*_{\text{cocomplex}}(C[\Gamma]; (g))$ in the cyclic cocomplex $C^*_{\text{cocomplex}}(A^\Gamma)$. There is a splitting surjection $c_{(g)} : \text{HP}^*(A^\Gamma) \to \text{HP}^*(A^\Gamma; (g))$ of the map induced from the inclusion.

In the same way, for a $\Gamma$-$C^*$-algebra $B$, we write the projective limit $A^\Gamma(B)$, we write the projective limit $B \rtimes_{\text{red}} \Gamma$. By Lemma 5.14 we have $A^\Gamma(B,g) \subset B \rtimes_{\text{red}} \Gamma$ when $B$ is one of the $C^*$-algebras $C_0(B_0,K_\sigma)$, $C(B,K_\sigma)$ or $K$.

By [Laf02](3), the pairing of $\text{HP}^*(A^\Gamma)$ and $\text{HP}^*(C[\Gamma]; (g))$ makes sense. The invariants treated in this section are defined by using this pairing, as are introduced in [CM90,Laf92b]. We also refer the reader to [PSZ19] Section 6).

Definition 5.20. Let $\Gamma$ be a (subgroup of a) hyperbolic group, let $M$ be a closed manifold with $\pi_1(M) \cong \Gamma$ and let $\pi$ be its codimension 2 submanifold satisfying (1), (2), (3) of 2.16. Let $\phi \in \text{HC}^n_{\text{red}}(C[\Gamma])$ and let $\psi \in \text{HC}^m_{\text{red}}(C[\Gamma])$ be fixed vector bundle $V$ on $M$ as in Theorem 5.17.

1. If $M$ is spin, the higher index is defined as $\alpha_{\phi}(M) := \langle \text{Ch}_{\Gamma,V}(\xi_M^\phi), \phi \rangle$.
2. If $M$ is spin and equips a psc metric $g_M$, the higher $\rho$-number is defined as $\theta_{\psi}(g_M) := \langle \text{Ch}_{\Gamma,V}(\xi_M^\psi), \phi \rangle$.
3. The higher signature is defined as $\text{Sgn}_{\phi}(M) := \langle \text{Ch}_{\Gamma,V}(\text{Sgn}_M(M)), \phi \rangle$.
4. If $M$ equips an oriented homotopy equivalence $f_M : M' \to M$, the higher $\rho$-number is defined as $\theta_{\psi}(f_M) := \langle \text{Ch}_{\Gamma,V}(\text{Sgn}(f_M)), \phi \rangle$.

The Connes–Moscovici higher index theorem [CM90] Theorem 5.4] states that, for any $\phi \in H^{2\theta}(\Gamma;C)$, the equalities

$$\alpha_{\phi}(M) = (\frac{-1}{2\pi i})^{n} \langle \hat{\Lambda}(M)\xi_M^\phi, [M] \rangle, \quad (5.21)$$

$$\text{Sgn}_{\phi}(M) = (\frac{-1}{2\pi i})^{n} \langle L(M)\xi_M^\phi, [M] \rangle$$

hold, where $n := \dim M$. Note that our convention of the pairing is different from the one in [CM90] by the normalization factor $q!/(2q)!$. 

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Lemma 5.22. The following diagram commutes:

\[
\begin{array}{ccc}
  \text{HP}^*(\mathbb{C}[\pi]) & \xrightarrow{\kappa_{\pi}} & \text{HP}^*(A_{\pi}) \\
  \phi^* & & \phi^* \\
  \text{HP}^*(\mathbb{C}[\Gamma]) & \xrightarrow{\kappa_{\Gamma}} & \text{HP}^*(A_{\Gamma}). \\
\end{array}
\]

Proof. Let \( \text{CC}^*_{\text{pol}}(\mathbb{C}[\Gamma]) \) denote the subcomplex of \( \text{CC}^*_{\text{pol}}(\mathbb{C}[\Gamma]) \) consisting of cyclic cochains of polynomial growth. Since \( \Gamma \) is hyperbolic, any polynomial growth cyclic cochain extends to \( A_{\Gamma} \) and \( \text{CC}^*_{\text{pol}}(\mathbb{C}[\Gamma]) \rightarrow \text{CC}^*(\mathbb{C}[\Gamma]) \) is a quasi-isomorphism. The splitting \( \kappa_{\Gamma} \) is induced from \( \text{CC}^*_{\text{pol}}(\mathbb{C}[\Gamma]) \rightarrow \text{CC}^*(A_{\Gamma}) \).

We generalize the notion of polynomial growth for cyclic cohomology groups of crossed products. For a cyclic cocycle \( \phi \in \text{CC}^*(B \rtimes_{\text{alg}} \Gamma; \langle g \rangle) \), we define \( \phi_{g_0,g_1,\ldots,g_n} \in \text{Hom}(B^\otimes n+1, \mathbb{C}) \) and \( \phi_{1,g_1,\ldots,g_n} \in \text{Hom}(B^\otimes n, \mathbb{C}) \) as

\[
\phi_{g_0,g_1,\ldots,g_n}(b_0, b_1, \ldots, b_n) := \phi(b_0 u_{g_0}, b_1 u_{g_1}, \ldots, b_n u_{g_n}), \\
\phi_{1,g_1,\ldots,g_n}(b_1, \ldots, b_n) := \phi(1, b_1 u_{g_1}, \ldots, b_n u_{g_n}).
\]

We say that \( \phi \) is of polynomial growth if \( \|\phi_{g_0,\ldots,g_n}\| \leq C(1 + \ell(g_0) \cdots \ell(g_n))^N \) and \( \|\phi_{1,g_1,\ldots,g_n}\| \leq C(1 + \ell(g_1) \cdots \ell(g_n))^N \) hold for some \( C > 1 \) and \( N \in \mathbb{Z}_{>0} \). Then, for any \( x_0 = \sum b_{0,g} u_g, \ldots, x_n = \sum b_{n,g} u_g \in B \rtimes_{\text{alg}} \Gamma \), we have a bound

\[
\|\phi(x_0, \ldots, x_n)\| \leq \sum_{g_0,\ldots,g_n \in \langle g \rangle} \|\phi_{g_0,\ldots,g_n}\| \cdot \|b_{0,g_0}\| \cdots \|b_{n,g_n}\|
\]

Lemma 5.22. This finishes the proof of the lemma.

\[\square\]

Theorem 5.23. Let \( \Gamma \) be a hyperbolic group and let \( \pi \) be its finitely presented subgroup. Let \( \phi \in \text{HC}_{q-2}^\pi(\mathbb{C}[\pi]) \) and \( \psi \in \text{HC}_{q-2}^{\text{del}}(\mathbb{C}[\pi]) \). Then, for \( M, N, g_M, g_N, \) and \( f_N \) as in the statement of Theorem 5.12, the following equalities hold:

1. \( 2\pi \alpha_{\phi,\psi}(M) = \alpha_{\phi}(N) \).
(2) \(2\pi i \theta_{\sigma \phi}(g_M) = \theta_\psi(g_N)\).
(3) \(\pi i \text{Sgn}_{\sigma, \phi}(M) = \text{Sgn}_\phi(N)\).
(4) \(\pi i \theta_{\sigma \phi}^g(f_M) = \theta_{\sigma \phi}^g(f_N)\).

**Proof.** This follows from Corollary 5.10, Theorem 5.17, and Lemma 5.22. \(\square\)

**Remark 5.24.** Theorem 5.23 (1), (3) are also proved by the Connes–Moscovici higher index theorem (5.21) as

\[
\alpha_{\sigma \phi}(M) = \frac{(-1)^n}{(2\pi i)^g} \langle f^*(\sigma \cdot \phi) \hat{A}(M), [M] \rangle = \frac{(-1)^n}{(2\pi i)^g} \langle \{PD[N] \cdot f^*(\phi) \cdot \hat{A}(M), [M] \rangle
\]

\[
= \frac{1}{2\pi i} \cdot \frac{(1)^{n-2}}{(2\pi i)^{n-1}} \langle f^*(\phi) \cdot \hat{A}(N), [N] \rangle = \frac{1}{2\pi i} \alpha_\phi(N).
\]

The same is also true for the higher signature.

**Example 5.25.** Let \(M, N, g_M, g_N, f_M\) and \(f_N\) be as in Example 4.14. Assume that the delocalized \(\eta\)-invariants \(\eta(h_i)(g_M)\) and \(\eta(h_i)(f_N)\) do not vanish (recall that they are the same thing as the higher \(\rho\)-number of the cyclic cocycle \(\text{tr}_{h_i} \in \text{HC}(\mathbb{C}[\pi])\) corresponding to \(\langle h_i \rangle \in (\pi)\) as is shown in [XY19] Theorem 1.1). Now the higher \(\rho\)-number of \(g_M\) with respect to the delocalized cyclic cocycle \(\sigma \cdot \text{tr}_h (h)\) is calculated as

\[
\theta_{\sigma \text{tr}_h}(g_M) = (2\pi i)^{-1} \theta_{\text{tr}_h}(g_N) = (2\pi i)^{-1} \eta(h_i)(g_N),
\]

\[
\theta_{\sigma \text{tr}_h}(f_M) = (\pi i)^{-1} \theta_{\text{tr}_h}(f_N) = (\pi i)^{-1} \eta(h_i)(f_N).
\]

They are non-trivial examples in which the higher \(\rho\)-number of higher degree is determined.

**Appendix A. Secondary external product via pseudo-local \(C^*\)-algebra**

In this appendix, we define the secondary external product in coarse \(C^*\)-algebra K-theory, the product of the analytic structure set and the K-homology group, in terms of the pseudo-local coarse \(C^*\)-algebra \(D^* (\hat{M})^\Gamma\), instead of Yu’s localization algebra studied in [Zei16]. The construction is inspired from the definition of the Kasparov product [Kas80].

**Remark A.1.** We use the K-theory of \(\mathbb{Z}_2\)-graded \(C^*\)-algebras by Van Daele [VD83] for coarse \(C^*\)-algebras with Clifford algebra symmetry. In general, for a \(\mathbb{Z}_2\)-graded (Real) \(C^*\)-algebra \(A\), its (Real) K-theory \(K_1(A)\) is defined to be the set of homotopy classes of (Real) odd self-adjoint unitaries on \(A\). We define the coarse \(C^*\)-algebras \(C^*_{p,q}(\hat{M})^\Gamma\), \(D^*_{p,q}(\hat{M})^\Gamma\), \(Q^*_{p,q}(\hat{M})^\Gamma\) consisting of operators which is graded-commutative with the action of Clifford algebras. Then \(K_1(C^*_{p,q}(\hat{M})^\Gamma) \cong K_1(\mathbb{Z}_2)\) holds, and the same is also true for \(D^*\) and \(Q^*\).

For a spin manifold \(M\) with \(n := \dim M = 8m - q\) (where \(q = 0, \cdots, 7\)), the Dirac operator on the spinor bundle of \(\text{Spin}(M) \times_{\text{Spin}_n} \Delta_{8m}\) is equipped with an additional symmetry of \(Cl_{0,q}\), and hence determines a K-theory class \(\chi(\mathbb{D}_M) \in K_1(Q^*_{0,q}(\hat{M})^\Gamma)\) (cf. [4.7]). Similarly, the signature operator on an odd-dimensional manifold determines a complex K-theory class \(\chi(D^*_{\text{sgn}}(M^\Gamma)) \in K_1(Q^*_{0,1}(\hat{M})^\Gamma)\) (we refer to [RW06] Definition and Notation 1.1).

**Definition A.2.** The external product

\[
\hat{\otimes} : K_1(D^*_{p_1,q_1}(M_1)^{\Gamma_1}) \otimes K_1(Q^*_{p_2,q_2}(\hat{M}_2)^{\Gamma_2}) \to K_1(D^*_{p,q}(\hat{M}_1 \times \hat{M}_2)^{\Gamma_1 \times \Gamma_2}),
\]

where \(p = p_1 + p_2\) and \(q = q_1 + q_2\), is defined as

\[
[F_1] \hat{\otimes} [F_2] := [F_1 \otimes 1 + (1 - F_1^2) \otimes F_2],
\]

where \(f(x) := x/|x|\).

This definition makes sense because the operator \(F_1 \otimes 1 + (1 - F_1^2) \otimes F_2\) is invertible in \(D^*_{p+q}(\hat{M}_1 \times \hat{M}_2)^{\Gamma_1 \times \Gamma_2}\). Indeed, this is seen as

\[
(F_1 \otimes 1 + (1 - F_1^2) \otimes F_2)^2 = F_1^2 \otimes 1 + (1 - F_1^2) \otimes F_2^2 \geq F_1^2 \otimes 1 > 0.
\]

This also shows that \([F_1] \hat{\otimes} [F_2]\) is well-defined independent of the choice of representatives \(F_1\) and \(F_2\).
Lemma A.3. For $i = 1, 2$, let $D_i$ be a $\Gamma_i$-invariant $\mathbb{Z}_2$-graded elliptic first-order differential operator on $M_i$ anticommuting with a $\mathbb{Z}_2$-graded representation of $\mathfrak{gl}_{p_i,q_i}$. Moreover, we assume that $D_1$ is invertible. Then we have
\[
\chi(D_1) \otimes \chi(D_2) = \chi(D_1 \otimes 1 + 1 \otimes D_2).
\]

Proof. This is a standard argument in Kasparov theory. We just refer the reader to [BJ83] or [Bla98 Proposition 18.10.1].

Lemma A.3 shows that
\[
\chi(D_{\mathfrak{M}^i_1}) \otimes \chi(D_{\mathfrak{M}^i_2}) = \chi(D_{\mathfrak{M}^i_1 \times \mathfrak{M}^i_2}),
\]
\[
\chi(D_{\mathfrak{M}^i_1 \times \mathfrak{M}^i_2}) = 2^\epsilon \chi(D_{\mathfrak{M}^i_1 \times \mathfrak{M}^i_2}),
\]
where $\epsilon = 1$ if both $\dim M_1$ and $\dim M_2$ are odd, and otherwise $\epsilon = 0$. In particular, when $M_2 = \mathbb{R}$, these equalities means that
\[
\rho(g_{\mathbb{R} \times \mathbb{R}}) = \rho(g_{\mathbb{R}}) \otimes [\mathbb{R}],
\]
\[
\rho_{\text{sgn}}(f_{\mathbb{R} \times \mathbb{R}}) = 2^\epsilon \rho_{\text{sgn}}(f_{\mathbb{R}}) \otimes [\mathbb{R}]_{\text{sgn}},
\]
where $\epsilon$ is 0 or 1 if $\dim N$ is even or odd. Therefore, the coarse Mayer–Vietoris boundary map
\[
\partial_{\text{MV}} : K_*(D_{p,q}(\tilde{N} \times \mathbb{R}))^n \to K_*(D_{p,q}(\tilde{N} \subset \tilde{N} \times \mathbb{R}))^n
\]
sends the higher $\rho$-invariants of $N \times \mathbb{R}$ as
\[
\partial_{\text{MV}}(\rho(g_{\mathbb{R} \times \mathbb{R}})) = \partial_{\text{MV}}(\rho(g_{\mathbb{R}})) \otimes [\mathbb{R}] = \rho(g_{\mathbb{R}}) \otimes \partial_{\text{MV}}[\mathbb{R}] = \rho(g_{\mathbb{R}}),
\]
\[
\partial_{\text{MV}}(\rho_{\text{sgn}}(f_{\mathbb{R} \times \mathbb{R}})) = \partial_{\text{MV}}(2^\epsilon \rho(f_{\mathbb{R}}) \otimes [\mathbb{R}]) = 2^\epsilon \rho(f_{\mathbb{R}}) \otimes \partial_{\text{MV}}[\mathbb{R}] = 2^\epsilon \rho(g_{\mathbb{R}}).
\]

This completes the proof of Lemma 4.11 (3), (4).

Appendix B. Cyclic homology of a crossed product and group homology

In this appendix, we give a more detailed discussion on the proof of Lemma 5.6. We show that the exact sequence of cyclic chain complexes
\[
0 \to \ker \theta \to CC_*(C(B, \mathcal{K}_\sigma) \rtimes_{\text{alg}} \Gamma) \otimes [u^\pm 1] \to CC_*(C[\Gamma]) \otimes [v^\pm 1] \to 0
\]
is quasi-isomorphic to the exact sequence of chain complexes
\[
0 \to C_*(\Gamma, \Omega^*(B_0)) \otimes [v^\pm 1] \to C_*((\Gamma, \Omega^*(B))) \otimes [v^\pm 1] \to C_*((\Gamma, \Gamma)) \otimes [v^\pm 1] \to 0
\]
given in (5.7) (note that the excision property of cyclic homology states that the inclusion $CC_*(C_0(B_0, \mathcal{K}_\sigma) \rtimes_{\text{alg}} \Gamma)) \otimes [u^\pm 1] \to \ker \theta$ is quasi-isomorphic). To this end, we construct maps from (B.1) and (B.2) to another exact sequence of complexes
\[
0 \to \Omega_0(B_0 \rtimes \Gamma) \otimes [u^\pm 1]' \to \Omega(B \rtimes \Gamma) \otimes [u^\pm 1]' \to \Omega(\text{pt} \rtimes \Gamma) \otimes [u^\pm 1]' \to 0.
\]

Here, for a vector space $V$, $V'$ stands for the algebraic dual vector space $\text{Hom}(V, \mathbb{C})$.

We start with the definition of (B.3). Following [Ang13 Subsection 2.3], let $\Omega^*(B)$ denotes the dual cocomplex of $\Omega^*(B)$, which is equipped with the degree $-1$ differential $d_{\text{dR}}$. We define the $\mathbb{Z}$-graded vector space $\Omega^*(B \rtimes \Gamma) \otimes [u^\pm 1]'$, where $u$ is a degree 2 formal symbol, as
\[
\Omega^*(B \rtimes \Gamma) \otimes [u^\pm 1]' := \left\{ (\omega_{(n)}) \in \prod_{n \geq 0} \Omega^*(B \rtimes \Gamma^n) \otimes \Omega^*(\Delta^n) \mid (\text{id} \times \delta^i)^* \omega_{(n)} = (\delta_i \times \text{id})^* \omega_{(n-i)} \right\},
\]
where $\delta_i : B \rtimes \Gamma^n \to B \rtimes \Gamma^{n-1}$ and $\delta^i : \Delta^{n-1} \to \Delta^n$ denote the $i$-th face maps. We write $\omega(g_1, \cdots, g_n)$ for the restriction of $\omega$ to $B \times \{g_1, \cdots, g_n\} \times \Delta^n$. The twisted simplicial de Rham differential is defined as $d_{\Theta} := ud_{\text{dR}} + d_{\Delta} + \Theta$, where $d_{\Delta}$ is the de Rham differential on $\Omega^*(\Delta^n)$ and $\Theta \in \Omega^{1,2}(B \rtimes \Gamma)$ is the differential form
\[
\Theta(g_1, \cdots, g_n) := \sum_{1 \leq i \leq j \leq k} 2\alpha(g_1 \cdots g_i, g_{i+1} \cdots g_j)dt_idt_j,
\]
where \( \alpha(g, h) = \sigma(g, h)^{-1} \sigma(g, h) \in \Omega^1(B) \). The dual of the short exact sequence \( 0 \to \Omega^\bullet_0(B_0) \to \Omega^\bullet(B) \to \mathcal{C} \to 0 \) gives rise to
\[
0 \to \Omega^\bullet(pt \times \Gamma)[u^{\pm 1}] \to \Omega(B \times \Gamma)[u^{\pm 1}] \to \Omega^\bullet_0(B_0 \times \Gamma)[u^{\pm 1}] \to 0,
\]
which is dual to (B.3).

**Remark B.4.** We review the Packer–Raeburn construction [PR89] Theorem 3.4. Let \( \mathcal{H}_\sigma \) be the Hilbert bundle \( B \times \ell^2(\Gamma) \), on which \( \Gamma \) acts as
\[ u_g \xi := ((\gamma_g \otimes \lambda_g) \circ v(g)) \xi \]
for any \( \xi \in C(B, \mathcal{H}_\sigma) \), where
\[ v(g) := \text{diag}(\sigma(g, h)^\ast)_{h \in \Gamma} \in C(B, \mathcal{B}(\ell^2\Gamma)). \]

Let \( \mathcal{K}_\sigma \) denote the associated compact operator algebra bundle on \( B \). Then, the relation
\[ u_g u_h = \sigma(g, h, x)^\ast u_{gh} \]
holds, i.e., \{\( u_g \)\} is a \( \sigma^\ast \)-twisted unitary representation of the groupoid \( B \times \Gamma \). This implements a twisted \( \Gamma \)-equivariant Morita equivalence between \( (C(B), \sigma) \) and the untwisted \( \Gamma \)-\( C^\ast \)-algebra \( C(B, \mathcal{K}_\sigma) \). In particular, \( C(B) \times_\sigma \Gamma \) is Morita equivalent to \( C(B, \mathcal{K}_\sigma) \times \Gamma \).

**Lemma B.5.** There are homomorphisms from the exact sequences (B.1) and (B.2) to (B.3), which induces isomorphism of homology.

**Proof.** The pairing \( \langle \cdot, \cdot \rangle \) of \( C^\ast_\bullet(\Gamma, \Omega^\bullet(B)) \) is defined by
\[
\langle \sum_{g_1, \cdots, g_m} \xi_{g_1, \cdots, g_m} v^k (\omega(n)) u^l \rangle := \delta_{k,l} \sum_{g_1, \cdots, g_m} \int_{\triangle^n} \xi_{g_1, \cdots, g_m} \wedge \omega_{g_1, \cdots, g_m}
\]
satisfies \( \langle \xi u, d_{\text{IR}} \omega \rangle = \langle d_{\text{IR}} \xi, \omega \rangle \), \( \langle \xi, d_{\Delta} \omega \rangle = \langle d_{\Gamma} \xi, \omega \rangle \) and \( \langle \xi, \Theta \omega \rangle = \langle \Theta \xi, \omega \rangle \). Moreover, this pairing gives rise to a commutative diagram
\[
\begin{array}{ccc}
C_\ast(\Gamma, \Omega^\bullet_0(B_0))[u^{\pm 1}] & \longrightarrow & C_\ast(\Gamma, \Omega^\bullet(B))[u^{\pm 1}] \\
\downarrow & & \downarrow \\
\Omega^\bullet_0(B_0 \times \Gamma)[u^{\pm 1}] & \longrightarrow & \Omega(B \times \Gamma)[u^{\pm 1}] \\
\end{array}
\]
Moreover, the left and the right vertical maps induce the isomorphism of homology. Indeed, the homology groups of two complexes at the left (resp. the right) are both isomorphic to the group homology \( H_{[\ast - 1]}(\pi; \mathcal{C}) \) (resp. \( H_{[\ast]}(\Gamma; \mathcal{C}) \)).

In [Ang13], Angel constructed a pairing of (B.1) and (B.3) as a variation of the JLO character. Although the space \( B \) is not a manifold, the same definition also works in our setting. Moreover, due to the 1-dimensionality of \( B \), the construction is partly simplified.

The simplicial connection and curvature forms of \( \mathcal{H}_\sigma \) is defined by gluing the connections and curvature forms
\[
\nabla^k_u(g_1, \cdots, g_k) := d_{\text{IR}} + u^{-1} d_{\Delta} + t_1 A(g_1) + t_2 A(g_1 g_2) + \cdots + t_k A(g_1 \cdots g_k),
\]
\[
\partial^{(k)}_l(g_1, \cdots, g_k) := \sum_{1 \leq i \leq j \leq k} \alpha(g_1 \cdots g_i g_{i+1} \cdots g_j) (t_i dt_j - t_j dt_i).
\]

The JLO homomorphism
\[ T_{\text{JLO}} : C_\ast(\Gamma, CC_\bullet(C^\infty(B, \mathcal{K}_\sigma))) \to \Omega^\bullet(B \times \Gamma)[u^{\pm 1}], \]
is defined as
\[
\langle (\omega(n)) u^k, T_{\text{JLO}}(((\tilde{a}_0 \otimes \cdots \otimes a_p), (g_1, \cdots, g_q)) v^l) \rangle
\]
\[= \delta_{k,l} \int_{\triangle^n} \omega_{g_1, \cdots, g_q} \wedge \int_{\triangle^n} \text{Tr}(\tilde{a}_0 e^{s \delta_{\theta}(\omega)} (\nabla^0_u a_1) e^{s \delta_{\theta}(\omega)} \cdots e^{s \delta_{\theta}(\omega)} (\nabla^0_u a_q)) ds_1 \cdots ds_q.
\]
It is shown in the same way as [Ang13] Theorem 5.5 that it is a chain map.

Moreover, in [Ang13] Subsection 5.3 (we also refer to [Nis90] 2.5), a quasi-isomorphism
\[ \Psi_A : C_\ast(\Gamma, CC_\bullet(A)) \to (CC_\bullet(A \times_\text{alg} \Gamma))[u^{\pm 1}], b + uB \]
\[\text{[36]}\]
is constructed for any $\Gamma$-algebra $A$. By the construction, this $\Psi_A$ is functorial. Therefore, $\Psi_A \circ T_{JLO}$ gives rise to a commutative diagram

\[
\begin{array}{c}
\ker \theta \\
\downarrow \\
CC_\ast(C^\infty(\mathcal{B}, K_\sigma) \ast_{\text{alg}} \Gamma, e)[u^{\pm 1}] \\
\downarrow \\
\Omega(\mathcal{B}_0 \times \Gamma)[u^{\pm 1}]' \\
\downarrow \\
\Omega(\mathcal{B} \times \Gamma)[u^{\pm 1}]' \\
\downarrow \\
\Omega(\mathcal{B} \times \Gamma)[u^{\pm 1}]'.
\end{array}
\]

Since the left and the right vertical maps are both isomorphic, this finishes the proof of the lemma. □

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