Complex Dynamics of Beddington–DeAngelis-Type Predator-Prey Model with Nonlinear Impulsive Control

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1. Introduction

Since pest outbreak can cause serious economic loss, pest control has been becoming an increasing concern to entomologists and society all over the world. Several pest control strategies can be used for farmers. As well known, chemical pesticide can directly and rapidly kill large proportion of pest, and it is the only way to prevent economic losses in many cases. Biological control is the practice of releasing of natural enemies to control pests [1, 2], sometimes which has a highly efficacious and more active role in some pest situations. However, in order to avoid the resistance development of pests to the control tactic and to protect the environment quality, different pest control techniques should be combined together rather than against overuse of a single control strategy. In particular, integrated pest management(IPM) incorporates a variety of cultural, biological, and chemical methods to high efficiency control of the pest populations, which has been proved that it is more efficient long-term strategy for pest control than the classical one (such as biological control or chemical control) [3–5].

It is reasonable and accurate that impulsive differential equations mathematically simulate the evolution of biological behaviors and complex biological phenomena, which provide conditions for people to assist in the design of IPM strategies and understand the biological phenomena from a mathematical point of view [1, 3, 6–9]. In recent years, the impulsive differential systems with integrated pest management have been systematically studied and developed rapidly [10, 11], which enriched its basic theory and analytical techniques of impulsive differential system [12–20]. However, one of the main assumptions in previous literature is that a certain proportion of pest population is killed when the pesticide is applied. Meanwhile, a constant natural enemy is released [21–28], which means that the agricultural resources have almost no effect on IPM.

In reality, the release methods and ratios of numbers of natural enemies will inevitably be affected by the limitation of agricultural resources because of the unbalance development of agricultural, such as agricultural, capital, labour forces, biological resources, and pesticides. Therefore, the release ratios of numbers of natural enemies according to current density in the field could significantly affect the effect...
of pest control strategy. In order to take the resource limitation into the IPM strategy, several predator-prey models with nonlinear impulse have been proposed [29–32] and mainly focused on establishing the global stability conditions. However, the nonlinear impulse function mentioned above is only related to the density of the natural enemy population in the field. Based on resource limitation, the densities of the pest and natural enemy should be carefully monitored before IPM measures are applied. A more realistic case is that the methods for the instantaneous releasing numbers of natural enemies should be based on the dynamic changes of pest and natural enemy densities. In other words, the higher the number of pest population or the lower the number of predator population in the field, the higher the number of predator population should be released and vice versa, which has not been studied until now.

Therefore, in order to take the resource limitation into account and to understand how the nonlinear density regulatory factor for the natural enemies affects the dynamics of predator-prey model, we propose the following predator-prey model with Beddington–DeAngelis functional response and nonlinear impulse control:

\[
\begin{align*}
\frac{dx(t)}{dt} &= r x(t) \left( 1 - \frac{x(t)}{K} \right) - \frac{a x(t) y(t)}{c + x(t) + by(t)}, \quad t \neq nT, \\
\frac{dy(t)}{dt} &= \frac{b x(t) y(t)}{c + x(t) + by(t)} - \delta y(t), \\
x(t^+) &= q_1 x(t), \\
y(t^+) &= q_2 y(t) + \frac{\lambda_1 x(t)}{1 + \theta_1 x(t)} + \frac{\lambda_2}{1 + \theta_2 y(t)}, \quad t = nT.
\end{align*}
\]

(1)

where \(x(t)\) and \(y(t)\) are the densities of prey and predator populations, respectively, and all parameters are positive constants. IPM strategy (the nonlinear impulse) is applied at each discrete time point \(nT\), \(0 \leq q_1, q_2 \leq 1\) present survival rate of prey and predator after harvesting or pesticides; \(q_2 \geq 1\) means that the pesticides only affect the pest and an impulsive increase of the predator population density is induced by release of predators. Moreover, we choose the nonlinear saturation functions or density-dependent functions as follows:

\[
y(t^+) = q_2 y(t) + \frac{\lambda_1 x(t)}{1 + \theta_1 x(t)} + \frac{\lambda_2}{1 + \theta_2 y(t)}, \quad t = nT.
\]

(2)

\(\lambda_1, \lambda_2 \geq 0\) is the maximal release amount of the predator according to the densities of prey and predator populations respectively, and \(\theta_1, \theta_2 \geq 0\) represent the shape parameter. In particular, the system with \(\lambda_1 = 0, \theta_2 = 0\) (i.e., linear impulsive perturbations) has been investigated in [27, 28]. We assume that the densities of the natural enemy populations are updated to \(y(t^+) = q_2 y(t) + (\lambda_1 x(t)/1 + \theta_1 x(t)) + (\lambda_2/1 + \theta_2 y(t))\) at each time point \(nT\), which is a more reasonable control strategy than previous literature [31, 32].

The purpose of this paper proposes a Beddington–DeAngelis interference model with nonlinear impulsive control to address how the nonlinear impulse control actions affect the successful pest control strategies. By using the Floquet theorem and small-amplitude perturbation skills, we obtain that the pest-eradication periodic solution is globally stable if the period of impulse \(T\) is less than a critical value, and a sufficient condition for the permanence of the system is obtained. Moreover, when the trivial periodic solution loses its stability, we obtain that a nontrivial periodic solution appears via a supercritical bifurcation by employing a bifurcation theorem. By bifurcation diagrams, we show that the model presents more rich and interesting dynamic behavior including periodic doubling bifurcation, period-halving bifurcations, chaotic solutions, and multi-stability. Finally, we give some related biological implications.

2. Global Stability of the Pest-Eradication Periodic Solution

As we know, eradicating the pest population is an important purpose of IPM strategy, so the existence and global stability of the pest-eradication periodic solution play a crucial role in studying the dynamical behavior. For this, we firstly study the properties of the subsystem

\[
\begin{align*}
\frac{dy(t)}{dt} &= -\delta y(t), \quad t \neq nT, \\
y(t^+) &= q_2 y(t) + \frac{\lambda_2}{1 + \theta_2 y(t)}, \quad t = nT.
\end{align*}
\]

(3)

Subsystem (3) is a nonlinear growth model, by using the same methods as those in reference [32], and we can have the following result.

Lemma 1. When \(1 - q_2 \exp(-\delta T) > 0\), model (3) has a globally stable periodic solution:

\[
y_p(t) y^* \exp(-\delta (t - nT)), \quad t \in (nT, (n + 1)T),
\]

where \(y^* = ((\sqrt{A} + 1)/((2A \exp(-\delta T)))\) with \(A = ((4A_1A_2 \exp(-\delta T))/(1 - q_2 \exp(-\delta T)))\) is a positive constant.

Therefore, when \(q_2 \exp(-\delta T) < 1\), we have that model (1) has complete expression for pest-eradication periodic solution \((x_p(t), y_p(t)) = (0, y^* \exp(-\delta (t - nT)))\).

Next, we will present a condition which guarantees the local and global asymptotic stability of pest-eradication periodic solution \((x_p(t), y_p(t))\) of model (1).

Theorem 1. The pest-eradication periodic solution \((x_p(t), y_p(t))\) is locally asymptotically stable provided that

\[
T < \frac{1}{\alpha} \frac{1}{q_1} \frac{1}{rb^2} \ln \left( \frac{c + by^* \exp(-\delta T)}{c + by^*} \right).
\]

(5)
Furthermore, \((x_p(t), y_p(t))\) is globally asymptotically stable provided
\[
T < \frac{1}{r} \ln \frac{1}{q_1} - \frac{\alpha}{rbd} \ln \left( \frac{c + K + by^* \exp(-\Delta T)}{c + K + by^*} \right).
\] (6)

Proof. The local stability of the pest-eradication solution may be determined by the behavior of small-amplitude perturbations of the solution. Defining \(u(t) = x(t) - x_p(t), v(t) = y(t) - y_p(t)\), then the fundamental matrix \(\Phi(t)\) of model (1) satisfies
\[
\frac{d\Phi(t)}{dt} = \begin{pmatrix}
-ay_p(t) & 0 \\
\beta y_p(t) & -\delta
\end{pmatrix} \Phi(t),
\] (7)

has absolute value less than one, then the solution \((x_p(t), y_p(t))\) of model (1) is locally stable, and * is not needed to calculate the exact form. Note that all multipliers are
\[
\mu_1 = q_1 \exp \left( \int_0^T \left( r - \frac{ay_p(t)}{c + by_p(t)} \right) dt \right),
\]
\[
\mu_2 = \left( q_2 - \frac{\lambda_2 \theta_2}{(1 + \theta_2 y_p(T))^2} \right) \exp(-\Delta T).
\] (10)

It is easy to see that \(\mu_2 \leq q_2 \exp(-\Delta T) < 1\). Furthermore,
\[
\frac{\lambda_2 \theta_2 \exp(-\Delta T)}{(1 + \theta_2 y_p(T))^2} \leq \frac{4\lambda_2 \theta_2 \exp(-\Delta T)}{(1 + \theta_2 y_p(T))^2 + 1} < 1.
\] (11)

So, we obtain \(\mu_2 \geq q_2 \exp(-\Delta T) - 1 > -1\), which means \(\mu_2 < 1\). Since \(\int_0^T (r - (ay_p(t)/c + by_p(t)))dt = (1/b\delta), \ln ((c + by^* \exp(-\Delta T))/(c + by^*))\), according to the Floquet theory of impulsive differential equation, the pest-eradication periodic solution \((x_p(t), y_p(t))\) is locally asymptotically stable if
\[
T < \frac{1}{r} \ln \frac{1}{q_1} - \frac{\alpha}{rbd} \ln \left( \frac{c + by^* \exp(-\Delta T)}{c + by^*} \right).
\] (12)

and \(\Phi(0) = I\) is the identity matrix. From the third and fourth equations of (1), one has that
\[
\begin{pmatrix}
u(nT^+) \\
n(nT^+)
\end{pmatrix} = \begin{pmatrix}q_1 & 0 \\
\lambda_1 q_2 - \frac{\lambda_2 \theta_2}{(1 + \theta_2 y_p(T))^2} & 1\end{pmatrix} \begin{pmatrix}u(nT) \\
v(nT)
\end{pmatrix} = B(T) \begin{pmatrix}u(nT) \\
v(nT)
\end{pmatrix}.
\] (8)

Therefore, if each eigenvalues of the following matrix
\[
M = B(T)\Phi(T)
\]

\[
= \begin{pmatrix}
q_1 \exp \left( \int_0^T \left( r - \frac{ay_p(t)}{c + by_p(t)} \right) dt \right) \\
q_2 - \frac{\lambda_2 \theta_2}{(1 + \theta_2 y_p(T))^2} \exp(-\Delta T)
\end{pmatrix}
\]

Next, we will show the global attractivity provided condition (6) is satisfied. From the comparison theorem of impulsive equation, we obtain \(y(t) \geq y_p(t) - \epsilon\) for all \(t\) large enough. Also, it is easy to see that \(x(t) < K + \epsilon\) for all \(t\) large enough.

For simplicity, we may assume that \(y(t) \geq y_p(t) - \epsilon\) and \(x(t) < K + \epsilon\) for all \(t \geq 0\). If condition (6) holds true, then we choose an \(\epsilon > 0\) such that
\[
\eta = q_1 \exp \left( \int_0^T \left( r - \frac{a(y_p(t) - \epsilon)}{c + K + \epsilon + b(y_p(t) - \epsilon)} \right) dt \right) < 1.
\] (13)

From model (1), we obtain
\[
\begin{cases}
dx(t)/dt \leq x(t) \left( r - \frac{a(y_p(t) - \epsilon)}{c + K + \epsilon + b(y_p(t) - \epsilon)} \right), & t \neq nT, \\
x(t) = q_1 x(t), & t = nT.
\end{cases}
\] (14)

Integrating on \((nT, (n+1)T]\), one obtains
\[ x((n+1)T) = q_1 x(nT) \exp \left[ \int_{nT}^{(n+1)T} \left( r - \frac{a y_p(t) - \epsilon}{c + K + \epsilon + b(y_p(t) - \epsilon)} \right) \, dt \right] \]

\[ = x(nT) \eta. \]  

(15)

Thus, \( x(nT) = x(0^+) \eta^n \) and consequently \( x(nT) \to 0 \) as \( n \to \infty \). Therefore, \( x(t) \to 0 \) as \( n \to \infty \), since \( 0 < x(t) \leq q_1 x(nT) \exp(rT) \) for \( t \in (nT, (n+1)T) \).

Following, we only need to prove \( y(t) \to y_p(t) \) as \( t \to \infty \). There must exist a \( 0 < \epsilon_1 < (\delta c/\beta) \) and \( T_1 > 0 \) such that \( 0 < x(t) < \epsilon_1 \) for \( t > T_1 \). Again, for simplicity, it is assumed that \( 0 < x(t) < \epsilon_1 \) holds true for \( t \geq 0 \). Then, we deduce that \( (\lambda_1 x(t)/1 + \theta_1 x(t)) < \epsilon_1 \lambda_1 \), and

\[-\delta y(t) \leq \frac{dy}{dt} \leq y(t) \left( \frac{\beta \epsilon_1}{c - \delta} \right), \]

from which we can have the following equation:

\[
\begin{align*}
\frac{dz(t)}{dt} &= \left( \frac{\beta \epsilon_1}{c - \delta} \right) z(t) = -\delta z(t), \quad t \neq nT, \\
z(t^+) &= q_2 z(t) + \epsilon_1 \lambda_1 + \frac{\lambda_2}{\lambda_2^2 + \theta_2^2 z(t)}, \quad t = nT.
\end{align*}
\]

(17)

By Lemma 1, model (17) has a globally asymptotically stable periodic solution \( z_p(t) = z^* \exp(-\delta t) \), where

\[ z^* = ((-A_2 + \sqrt{A_2^2 + 4B_1 (\xi_1 \lambda_3 + \lambda_2)})/(2B_1)) \]

with \( A_1 = 1 - (q_2 + \epsilon_1 \lambda_1) \exp(-\delta T), B_1 = \theta_2 (1 - q_2 \exp(-\delta T)) \exp(-\delta T) \) and \( t \in \mathbb{R} \).

According to the comparison theorem, we get

\[ y_p(t) \leq y(t) \]

and

\[ z(t) \to z_p(t), \quad z(t) \to y_p(t) \]

as \( t \to \infty \). Hence, for any \( \epsilon_1 > 0 \), we have

\[ y_p(t) - \epsilon_1 < y(t) < z_p(t) + \epsilon_1, \]

for \( T_2 \geq T_1 > 0 \). Furthermore, let \( \epsilon \to 0 \), we get \( y_p(t) - \epsilon_1 < y(t) < y_p(t) + \epsilon_1 \) for \( t \) large enough. In other words, \( y(t) \to y_p(t) \) as \( t \to \infty \) for \( t \) large enough. The proof is completed.

3. Permanence

Persistence is an important property of dynamical systems for addressing the long-term survival of all components of a system. Now, we investigate the sufficient condition for the permanence of model (1).

**Theorem 2.** Model (1) is permanent if \( T > (1/r) \ln(1/q_1) - (a/\beta b\delta) \ln(c + by^* \exp(-\delta T) + c + by^*) \) holds true.

**Proof.** Suppose that \( (x(t), y(t)) \) is a solution of (1) with \( x(0) > 0, y(0) > 0 \). It is easy to know that \( x(t) < M, y(t) < M \) for all \( t > 0, M > (r/c) \). Define \( m_1 = y^* \exp(-\delta T) - \epsilon \). From Theorem 1, it is easy to see that \( y(t) \geq m_1 \) for \( t \) large enough.

Then, we shall find an \( m_1 \) such that \( x(t) \geq m_1 \) for all \( t \) that are large enough. We will do it in the following two steps.

**Step 1.** Let \( m_1 > 0, \epsilon_1 > 0 \) be small enough such that \( \delta_2 = \delta - (\beta m_1/c + \epsilon_1) > 0 \) and \( \eta = q_1 \exp(rT - (rm_1/T)/K) - (a/b\delta_2) \ln(c + bu^* \exp(-\delta_2 T)/c + b - (\alpha \epsilon_1^* T/c)) > 1 \), where \( u^* = ((-A_1 + \sqrt{A_1^2 + 4B_1 (m_1 \lambda_1 + \lambda_2)})/(2B_1)) \) with \( A_1 = 1 - (q_2 + m_1 \lambda_1) \exp(-\delta_2 T), B_2 = \theta_2 (1 - q_2 \exp(-\delta_2 T)) \exp(-\delta_2 T) \).

By Theorem 1, we then obtain \( y(t) \leq u(t) \) and \( y(t) \to u_p(t), t \to \infty \), where \( u(t) \) is the solution of

\[
\begin{align*}
\frac{du(t)}{dt} &= -\delta_2 u(t), \quad t \neq nT, \\
\left\{ \begin{array}{ll}
\frac{du(t)}{dt} &= -\delta_2 u(t), \quad t \neq nT, \\
\frac{u(t)}{dt} &\leq q_2 u(t) + \lambda_1 m_3 + \frac{\lambda_2}{1 + \theta_2 u(t)}, \quad t = nT, \\
\end{array} \right.
\end{align*}
\]

(20)

and \( u_p(t) = u^* \exp(-\delta_2 (t - nT)), t \in (nT, (n+1)T] \) with \( u^* = ((-A_1 + \sqrt{A_1^2 + 4B_1 (m_1 \lambda_1 + \lambda_2)})/(2B_1)) \).

Consequently, there exists a \( T > 0 \) such that

\[
\begin{align*}
\frac{dx(t)}{dt} &\geq x(t) \left( r - \frac{rm_3}{K} - \frac{ay(t)}{c + by(t)} \right), \\
&\geq x(t) \left( r - \frac{rm_3}{K} - \frac{au_p(t) + a\epsilon_1}{c + bu_p(t) + b\epsilon_1} \right), \\
&\geq x(t) \left( r - \frac{rm_3}{K} - \frac{au_p(t) - a\epsilon_1}{c + bu_p(t) - a\epsilon_1} \right),
\end{align*}
\]

(21)

for \( t > T \). Furthermore, we get

\[
\begin{align*}
\frac{dx(t)}{dt} &\geq x(t) \left( r - \frac{rm_3}{K} - \frac{au_p(t) - a\epsilon_1}{c + bu_p(t) - a\epsilon_1} \right), \quad t \neq nT, \\
x(t^+) &= q_1 x(t), \quad t = nT, \\
\end{align*}
\]

(22)

for \( t > T \). Integrating (22) on \( (nT, (n+1)T] \), where \( nT > T \), we obtain

\[
x((n+1)T) \geq x(nT) \exp \left[ \int_{nT}^{(n+1)T} \left( r - \frac{rm_3}{K} - \frac{au_p(t) - a\epsilon_1}{c + bu_p(t) - a\epsilon_1} \right) \, dt \right]
\]

\[ = x(nT) \eta. \]

(23)
Then, \( x((n+k)T) > x(nT)\eta^n \to \infty \) as \( k \to \infty \), which is a contradiction. Therefore, there exists a \( t_1 > 0 \) such that \( x(t_1) \geq m_3 \).

**Step 2.** If \( x(t) \geq m_3 \) for all \( t \geq t_1 \), then model (1) is permanent. If not, we can define \( t^* = \inf_{x \geq m_3} \), then \( x(t) \geq m_3 \) for \( t \in [t_1, t^*] \) since the continuity of \( x(t) \) and \( x(t^*) = m_3 \). We only need to consider two possible cases.

**Case (1):** \( t^* = n_1 T \). For some \( n_1 \in \mathbb{Z}_+ \), then we have \( q_1 m_3 \leq x(t_1) = q_1 x(t^*) < m_3 \). Select \( n_2, n_3 \in \mathbb{Z}_+ \) such that

\[
q_1 \eta^n \exp((n_2 + 1)\eta n_1 T) > 1,
\]

(24)

where \( \rho = r - (m_3 r / K) - (a/c)M < 0 \). Let \( T = n_2 T + n_3 T \), we will show that there exists a \( t^* < t \leq T \) such that \( x(t) \geq m_3 \). If not, by (20) with \( u(t^*) = y(t) \), we can see that \( y(t) \leq u(t) = u(t_1) + \epsilon_1 \) for \( t \in [t^* + n_2 T, t^* + T] \). An argument similar to Step 1 yields

\[
x(t^* + T) \geq x(t^* + n_2 T)\eta^n.
\]

(25)

Since \( y(t) \leq M \), when \( t \in [t^*, t^* + n_2 T] \), we get

\[
\begin{align*}
\frac{dx(t)}{dt} &\geq x(t)\left(r - \frac{m_3 r}{K} - \frac{a}{c} M\right), \quad t \neq n_3 T, \\
x(t^*) &= q_1 x(t), \quad t = n_3 T.
\end{align*}
\]

(26)

Furthermore, we integrate equation (26) on \([t^*, t^* + n_2 T] \), then we know

\[
x(t^* + n_2 T) \geq m_3 q_1^{n_2 - 1} \exp(n_2 \rho T).
\]

(27)

From above, we obtain \( x(t^* + T) \geq m_3 q_1^{n_2 - 1} \exp(n_2 \rho T) \eta^n > m_3 \) which leads to a contradiction.

Now, let \( \bar{T} = \inf_{x \geq m_3} \), then \( x(t) \leq m_3 \) for \( t \in (t^*, \bar{T}) \) and \( x(\bar{T}) = m_3 \). For any \( t^* < t < \bar{T} \), let us assume that there exists \( k \in \mathbb{Z}_+ \) such that \( t \in (t^* + (k - 1)T, t^* + kT) \) and \( k \leq n_2 + n_3 \), so, from (26), we obtain

\[
x(t) \geq x(t^*)q_1^{k-1} \exp((k - 1)\rho T)\exp(\rho((t - t^*) + (k - 1)T))
\]

\[
\geq m_3 q_1^k \exp(k\rho T)
\]

\[
\geq m_3 q_1^{n_2 + n_3} \exp(\rho(T + n_2 + n_3))
\]

(28)

Let \( m_1 = m_3 q_1^{n_2 + n_3} \exp((n_2 + n_3)\eta n_1 T) \), thus, for \( t \in (T, \bar{T}) \), we get \( x(t) \geq m_1 \). For \( t > \bar{T} \), we can continue the same arguments since \( x(\bar{T}) \geq m_3 \).

**Case (2):** \( t^* \neq n_1 T, n_1 \in \mathbb{Z}_+ \). We have \( x(t^*) = m_3 \) since \( x(t) \) is continuous. Suppose \( t^* \in (n_1 T, (n_1 + 1)T) \), \( n_1 \in \mathbb{Z}_+ \), we consider the following two cases for \( t \in (t^*, (n_1 + 1)T) \).

**Case (2a):** \( x(t) < m_3 \) for \( t \in (t^*, (n_1 + 1)T) \). In this case, we continue this process by using step case (1), we can prove that there exists an \( \bar{T} = \inf_{x \geq m_3} \), such that \( x(t) \leq m_3 \), \( t \in (t^*, \bar{T}) \), and \( x(\bar{T}) = m_3 \). For any \( t \in (t^*, \bar{T}) \), suppose \( t \in [n_1 T + (l - 1)T, n_1 T + lT] \), \( l \leq 1 + n_2 + n_3 \), we obtain

\[
x(t) \geq m_3 q_1^{l-1} \exp(l\eta n_1 T)
\]

\[
\geq m_3 q_1^{n_2 + n_3} \exp((n_2 + n_3 + 1)\rho T) \leq m_1.
\]

(29)

Since \( m_1 < m_1 \), thus, for \( t \in (t^*, \bar{T}) \), we have \( x(t) \geq m_1 \).

**Case (2b):** there is a \( t^* \in (t^*, (n_1 + 1)T) \) such that \( x(t^*) \geq m_3 \). In this case, a similar argument as above, there is \( \bar{T} = \inf_{x \geq m_3} \) such that \( x(t) \leq m_3 \), \( t \in (t^*, \bar{T}) \). Therefore, integrating equation (26) on \([t^*, t] (t \leq \bar{T}) \), we can get that \( x(t) \geq x(t^*)\exp(\rho(t - t^*)) \geq m_1 \). Thus, the similar argument can be continued for both cases since \( x(t) \geq m_1 \) for some \( t > t_1 \). This completes the proof.

**Remark 1.** Define \( T^* = (1/r)\ln(1/q_1) - (a/rb\delta)\ln ((x + by)^* \exp(-\delta T)/(x + by)^*)) \), the pest-eradication periodic solution loses its stability if \( T > T^* \). Therefore, \( T^* \) is the critical threshold value to discriminate between stability and permanence.

**4. Bifurcation**

Now, we will deal with the existence of nontrivial solution near the pest-eradication solution. We use the bifurcation theory in earlier publications [33].
Theorem 3. Model (1) has a positive nontrivial periodic solution when $T = T^*$, which is supercritical if $q_2 > \theta_2\lambda_2$ and $Ka < 4rbc$.

Proof. It is convenient for the computation to exchange $x$ and $y$ and change the period $T$ to $\tau$.

\[
\begin{align*}
\frac{dx(t)}{dt} &= \frac{\beta x(t)y(t)}{c + y(t) + bx(t)} - \delta x, \\
\frac{dy(t)}{dt} &= ry(t) - \frac{ax(t)y(t)}{c + y(t) + bx(t)}, \\
x(t') &= q_2x(t) + \frac{\lambda_1y(t)}{1 + \theta_1y(t)} + \frac{\lambda_2}{1 + \theta_2x(t)}, \\
y(t') &= q_1y(t),
\end{align*}
\]  

(30)

Let $\Phi$ be the solution of the pulse-free system associated with system (30). Also, we get $X(t) = \Phi(t, X_0)$ with $X_0 = X(0)$. We define the mapping $\Theta_1, \Theta_2: R^2 \rightarrow R^2$ by

\[
\Theta_1(x, y) = q_2x + \frac{\lambda_1y}{1 + \theta_1y} + \frac{\lambda_2}{1 + \theta_2x},
\]

(31)

and the map $F_1, F_2: R^2 \rightarrow R^2$ by

\[
\begin{align*}
F_1(x, y) &= \frac{\beta x(t)y(t)}{c + y(t) + bx(t)} - \delta x, \\
F_2(x, y) &= ry(t) - \frac{ax(t)y(t)}{c + y(t) + bx(t)}.
\end{align*}
\]

(32)

Furthermore, to establish the nontrivial periodic solution of system (30), we employ the techniques in [33]. Then, we have

\[
\begin{align*}
\frac{\partial \Phi_1(t, X_0)}{\partial x} &= \exp\left(\int_0^t \frac{\partial F_1(\xi(t))}{\partial x} dt\right), \\
\frac{\partial \Phi_1(t, X_0)}{\partial y} &= \int_0^t \exp\left(\int_0^s \frac{\partial F_1(\xi(s))}{\partial y} ds\right) \frac{\partial F_1(\xi(s))}{\partial x} \exp\left(\int_0^s \frac{\partial F_2(\xi(s))}{\partial y} ds\right) dv, \\
\frac{\partial \Phi_2(t, X_0)}{\partial x} &= 0, \\
\frac{\partial \Phi_2(t, X_0)}{\partial y} &= \exp\left(\int_0^t \frac{\partial F_2(\xi(t))}{\partial y} dt\right) > 0.
\end{align*}
\]

Thus, by simple calculations, we obtain

\[
d_0' = 1 - \frac{\partial \Theta_2}{\partial y} \cdot \frac{\partial \Phi_2}{\partial y} (t_0, X_0)
\]

(34)

\[
= 1 - q_1 \exp\left(\int_0^{t_0} \left(\frac{a}{c + by_p(t)} \cdot \frac{y_p(t)}{c + by_p(t)}\right) dt\right).
\]

where $t_0$ is the root of $d_0' = 0$. Actually, it is easy to see that $d_0' = 0$ is equivalent to $t_0 = T^*$.

Also, we obtain that

\[
a_0' = 1 - \left(\frac{\partial \Theta_1}{\partial x} \cdot \frac{\partial \Phi_1}{\partial x} (t_0, X_0) + \exp(-DT)\right) > 0,
\]

\[
b_0' = \left(\frac{\partial \Theta_1}{\partial y} \cdot \frac{\partial \Phi_1}{\partial y} (t_0, X_0) + \frac{\partial \Theta_2}{\partial y} \cdot \frac{\partial \Phi_2}{\partial y} (t_0, X_0)\right)
\]

(35)

Note that $b_0' > 0$ if $q_2 > \theta_2\lambda_2$.

Following, we should calculate the second-order partial derivatives:
Figure 1: Bifurcation diagrams of model (1) with respect to $T$. The parameter values are as follows. $r = 2.72$, $K = 10$, $c = 1$, $b = 0.001$, $\alpha = 1.2$, $\beta = 0.48$, $\delta = 0.59$, $\theta_1 = 12$, $\theta_2 = 1$, $q_1 = 0.62$, $q_2 = 0.98$, $\lambda_1 = 4.4$, and $\lambda_2 = 10.2$.

Figure 2: Two coexisting attractors of model (1) with $T = 7$. The other parameter values are identical to those in Figure 1. The initial conditions are as follows: $(A - B)(x_0, y_0) = (1.4, 2.1)$; $(C - D)(x_0, y_0) = (1.8, 2.6)$. 
\[
\frac{\partial^2 \Phi^2(\tau_0, X_0)}{\partial x \partial y} = \int_0^{\tau_0} \exp \left( \int_s^{\tau_0} \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) \frac{\partial^2 F_2(\zeta(t))}{\partial x \partial y} \exp \left( \int_0^s \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) ds
\]
\[
= -\int_0^{\tau_0} \exp \left( \int_s^{\tau_0} \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) \frac{b_2 a y_F(s)}{c + b y_F(s)} \exp \left( \int_0^s \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) ds
\]
\[
< 0.
\]

Figure 3: Bifurcation diagrams of model (1) with respect to \( \lambda_1 \). The other parameters are identical to those in Figure 1 and \( K = 10.5, T = 10 \).

Figure 4: Two coexisting attractors of model (1) with \( \lambda_1 = 9.4 \). The other parameter values identical to those in Figure 3. The initial conditions are as follows: \( (A - B)(x_0, y_0) = (1.4, 2.1) \); \( (C - D)(x_0, y_0) = (1.8, 2.6) \).
By the same methods as shown above, we have

\[
\frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial y^2} = \int_0^{\tau_0} \exp\left(\int_s^{\tau_0} \frac{\partial F_1(\zeta(t))}{\partial y} \, dt\right) \frac{\partial^2 F_2(\zeta(s))}{\partial y \, \partial x} \exp\left(\int_0^s \frac{\partial F_1(\zeta(t))}{\partial x} \, dt\right) \frac{\partial F_1(\zeta(s))}{\partial y} \, ds \\
+ \int_0^{\tau_0} \exp\left(\int_s^{\tau_0} \frac{\partial F_1(\zeta(t))}{\partial y} \, dt\right) \frac{\partial^2 F_2(\zeta(s))}{\partial y \, \partial x} \exp\left(\int_0^s \frac{\partial F_1(\zeta(t))}{\partial x} \, dt\right) \frac{\partial F_1(\zeta(s))}{\partial y} \, ds \\
- \int_0^{\tau_0} \exp\left(\int_s^{\tau_0} \frac{\partial F_1(\zeta(t))}{\partial y} \, dt\right) \frac{b a y_p(s)}{c + b y_p(s)} \exp\left(\int_0^s \frac{\partial F_2(\zeta(t))}{\partial y} \, dt\right) c + b y_p(s) \exp\left(\int_0^s \frac{\partial F_1(\zeta(t))}{\partial y} \, dt\right) \, ds,
\]

\[
\frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial \tau \partial y} = \frac{\partial F_1(\zeta(\tau_0))}{\partial y} \exp\left(\int_0^{\tau_0} \frac{\partial F_2(\zeta(t))}{\partial y} \, dt\right) \\
= \left( r - \frac{\alpha y_p(\tau_0)}{c + b y_p(\tau_0)} \right) \exp\left(\int_0^{\tau_0} \left( r - \frac{\alpha y_p(t)}{c + b y_p(t)} \right) \, dt\right).
\]

Therefore, we can deduce that
Now, we determine the sign of $B$. For this, let $g(t) = r - ((ay_p(t))/(c + by_p(t)))$, then $g'(t) = (abd + \exp(-\delta t))/(c + by_p(t))^2 > 0$, so $g(t)$ is strictly increasing. Since $\int_0^t g(t)dt = \ln(1/q_1) > 0$, which indicates that $g(t_0) > 0$, i.e. $(\partial^2 \Phi_2(t_0, X_0)/\partial y \partial \tau) > 0$. Therefore, $B < 0$. Defining $\varphi(t) = -(r/K) + (at + (c + bt)^2)$, it is easy to know that $-(r/K) + (at + (c + bt)^2) < 0$ if $K\alpha < 4rb$, which means $\varphi(y_p(t)) = -(r/K) + (ay_p(t))/(c + by_p(t)) < 0$ for all $y_p(t) > 0$. So, we have $(\partial^2 \Phi_2(t_0, X_0)/\partial y^2) < 0$. From above, we have $C > 0$.

Hence, $BC < 0$; according to Theorem 2 of [33], a supercritical bifurcation occurs at $T = T^*$.

5. Numerical Simulation

To confirm our theoretical results and facilitate their interpretation, we will focus on the complex dynamics by bifurcation analysis numerically, which can obtain the properties of a dynamics system.

Firstly, we investigate the effect of pulse period $T$ on dynamical system. Figure 1 shows that model (1) could exist with complex and interesting dynamic behavior with increase of parameter $T$, such as period-doubling bifurcation, period-halving bifurcations, chaos band, and non-unique dynamics, i.e., several attractors may coexist with the
same $T$. For example, Figure 2 indicates that two different attractors can coexist with each other with the same $T = 7$. If we choose different initial value $(x_0, y_0) = (1.4, 2.1)$ and $(x_0, y_0) = (1.8, 2.1)$, a $3T$-periodic solution coexists with $T$-periodic solution, which indicates that the final stable states of pest and natural enemy population depend on their initial densities. All these results confirm that varying impulsive period $T$ could dramatically change the dynamics of model (1).

It follows from Figure 3 that the nonlinear impulsive parameter $\lambda_1$ affects the dynamics of model (1). As parameter $\lambda_1$ increases, system (1) experiences period-doubling bifurcation, chaotic, period-halving bifurcations, and multiple stability. When $\lambda_1 = 9.4$, two attractors with different amplitudes appear, i.e., a $T$-periodic solution and $2T$-periodic solution coexist (see Figure 4). Therefore, the initial values of both the pest and predator populations are crucial. The above results reveal that the parameter $\lambda_1$ can dramatically change dynamics of system (1).

From the bifurcation diagrams Figure 5, we observe that a positive periodic solution appears when the pest-eradication solution loses its stability. The behavior of positive periodic solution is kept until $q_2 \approx 3.9$, and then a period-doubling bifurcation occurs, which means that a $T$-periodic solution disappears suddenly at this point and $2T$-periodic solution appears. With the increasing of $q_2$, a series of period-doubling bifurcations lead model (1) from periodicity to chaos. When $q_2 \geq 5.16$ and nearby 5.16, the chaos disappears and a $3T$-periodic solution appears. As $q_2$ increases further, the evidence for $3T$-periodic solution leading to chaos can be seen. Following these, the system displays a series of period-halving bifurcations.

Similarly, we investigate the effect of parameter $q_1$ on dynamic of system. The bifurcation diagrams with respect to parameter $q_1$ in the range $[0, 0.8]$ are shown in Figure 6. We can observe that model (1) also displays very complex dynamical behaviors with $q_1$ increasing.

6. Conclusion

Based on resource limitation, the optimal pest control strategy is that the instantaneous releasing numbers of natural enemies should be adjusted according to the densities of both pest and natural enemy in the field. A more natural understanding is that when the higher the number of pest population or the lower the number of predator population in the field, the higher the number of predator population should be released and the converse is also true. For this, we have investigated effects of nonlinear impulsive perturbations on a predator-prey model with Beddington–DeAngelis functional response. We have proven that there is a global stability of pest-eradication periodic solution if the impulsive period $T < T^*$ by using the Floquet theorem and small amplitude perturbation skills, and model (1) is permanent when the period $T > T^*$. Hence, $T = T^*$ plays a bifurcation threshold, and the system bifurcates to a positive periodic solution via supercritical bifurcation once a threshold condition is reached. By bifurcation diagrams, we can show that the system contains very rich dynamical behavior, including period-doubling bifurcation, period-halving bifurcations, chaos, and nonunique attractors, i.e., the system could exist with two stable positive periodic solutions and even more complex dynamics (see Figures 2 and 4). Also, bifurcation analyses reveal that the final dynamics of the system depends on the initial densities, and the nonlinear impulsive may result in complexity of pest control. All those results confirm that the pest control strategy should be carefully designed once the nonlinear impulsive control measures have been taken into account.

Based on the present study, we found that the system with nonlinear impulsive control actions provides more rich results and more realistic than the previous systems with linear impulsive control, and thus nonlinear impulsive control should be taken into account when implementing integrated pest management. However, the aim of IPM should reduce pest populations to below the economic threshold rather than eradication, which can be naturally and accurately described by the state-dependent impulsive differential equations and result in more difficulty for analyzing the global dynamics. We leave for future research.

Data Availability

There were no data used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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