ONE MORE RECURSIVE-THEORETIC CHARACTERIZATION OF THE TOPOLOGICAL VAUGHT CONJECTURE

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Abstract. We prove in ZF a recursive-theoretic characterization of the Topological Vaught Conjecture by revisiting the fact that orbits in Polish G-spaces are Borel sets.

1. Introduction

The question of characterizing the (Topological) Vaught Conjecture in terms of recursion theory has been investigated by Montalban [10, 11], where he provides the answer under some determinacy hypothesis. In this note we provide one more characterization of the topological version of the conjecture in a recursive-theoretic language, which is actually provable in the Zermelo-Fraenkel theory ZF. Another distinctive aspect of our characterization is that, unlike [11], it refers to the Polish group actions rather to the more general case of analytic equivalence relations, although one of the directions holds in the latter case. We do not know if the other direction is also true in the general case.

We will make substantial use of many results from effective descriptive set theory, and we make no attempt in presenting any form of introduction to the latter. The standard textbook on the subject is [12]. We recall however some central notions.

The natural numbers are identified with the first infinite ordinal \( \omega \) and the Baire space \( \omega^\omega \) is denoted by \( N \). The Cantor space is \( 2^\omega \), i.e., the set of all finite sequences of elements of \( \omega \), including the empty one. The length \( \text{lh}(s) \) of a given \( s \in \omega^\omega \) is the unique number \( n \) for which \( s = (s_0, \ldots, s_{n-1}) \); as usual \( \omega^n \) the set of all \( s \in \omega^\omega \) with length \( n \). We also fix a recursive injection \( \langle \cdot \rangle : \omega^\omega \to \omega \). If \( t = \langle s_0, \ldots, s_{n-1} \rangle \) and \( i < n \) we denote by \( (t)_i \) the number \( s_i \). If \( t \) does not have the preceding form or \( i \geq n \) we let \( (t)_i \) be 0. We fix once and for all the enumeration \( (q_s)_{s \in \omega} \) of all non-negative rational numbers, \( q_s = (s)_0 \cdot ((s)_1 + 1)^{-1} \). Given \( \alpha \in N \) and \( n \in \omega \) we denote by \( \alpha \upharpoonright n \) the finite sequence \( (\alpha(0), \ldots, \alpha(n-1)) \).

We say that a Polish space \( \mathcal{X} \) is recursive if there is a compatible metric \( d \) on \( \mathcal{X} \) such that \( (\mathcal{X}, d) \) is recursively presented cf. [12, 3B].

A Polish G-space is a triple \( (\mathcal{X}, G, \cdot) \) such that \( \mathcal{X} \) is a Polish space, \( G \) is a Polish group, and \( \cdot : G \times \mathcal{X} \to \mathcal{X} \) is a continuous action on \( \mathcal{X} \). By \( E_G \) we always mean...
the induced orbit equivalence relation:
\[ x E_G y \iff (\exists g \in G)[x = g \cdot y], \]
where \( x, y \in X \). By \( G \cdot x \) we mean the equivalence class or else the orbit of \( x \).

An equivalence relation \( E \) on some Polish space \( X \) has perfectly many classes if there is a non-empty perfect set \( P \subseteq X \) such that for all \( x, y \in P \) with \( x \neq y \) we have that \( (x, y) \notin E \).

The famous *Topological Vaught Conjecture* states that for every Polish \( G \)-space \((X, G, \cdot)\) the orbit equivalence relation \( E_G \) has either countably many or perfectly many classes.

A **recursive Polish \( G \)-space** is a triple \((X, G, \cdot)\) with the following properties:

1. The sets \( X \) and \( G \) are recursive Polish spaces.
2. The set \( G \) is a group and the function \((x, y) \mapsto xy^{-1}\) is recursive.
3. The function \( \cdot : G \times X \to X \) is a group action, and is recursive; cf. [2].

The preceding notions relativize with respect to some parameter \( \varepsilon \in \mathcal{N} \), e.g., we can talk about \( \varepsilon \)-recursive (or else recursive in \( \varepsilon \)) Polish \( G \)-spaces. In fact every Polish \( G \)-space is \( \varepsilon \)-recursive for some suitable \( \varepsilon \in 2^\omega \).

The notion of a partial recursive function extends to recursive Polish spaces in a natural way cf. [12, 7B]. We denote by \( \{e\}^x \) the \( e \)-th partial \( x \)-recursive function on \( \omega \) to \( \omega \), where \( x \) belongs to some recursive Polish space \( X \). (The latter space should be clear from the context.)

By \( \omega^1_{\text{CK}} \) we mean the least non-recursive ordinal and by \( \omega^1_x \) the least non-\( x \)-recursive one. Given points \( x, y \) in a recursive Polish space we write \( x \leq_h y \) for \( x \in \Delta^1_1(y) \). The symbol \( \leq_T \) stands for Turing reducibility between members of \( 2^\omega \).

## 2. The characterization

We can now state our recursive-theoretic characterization of the Topological Vaught Conjecture. For simplicity we state the result for recursive Polish \( G \)-spaces, but of course the analogous result holds also in the relativized case.

**Theorem 1.** For every recursive Polish \( G \)-space \((X, G, \cdot)\) the following are equivalent.

1. The induced orbit equivalence relation \( E_G \) does not have perfectly many classes.
2. For all \( \alpha \) the set of orbits \( \{G \cdot x \mid \omega^1_1(\alpha, x) = \omega^1_1\} \) is countable \(^1\)

In fact the implication \( (2) \implies (1) \) holds for arbitrary \( \Sigma^1_1 \) equivalence relations \( E \) in recursive Polish spaces, i.e.,

if \( X \) is a recursive Polish space and \( E \) is a \( \Sigma^1_1 \) equivalence relation on \( X \), for which the \( \{x \in E \mid \omega^1_1(\alpha, x) = \omega^1_1\} \) is countable for all \( \alpha \in \mathcal{N} \), then \( E \) does not have perfectly many classes.

\(^1\)The proof of the Silver Dichotomy Theorem is a standard application of the Gandy-Harrington topology in order to produce perfectly many classes. In order to do so one utilizes the fact that the latter topology is Polish on all sets of the form \( \{x \mid \omega^1_1(\beta, x) = \omega^1_\beta\} \) cf. [3]. Theorem [4] suggests that, in the case of Polish group actions, the preceding technique may fail to produce perfectly many classes, because the sets, on which the Gandy-Harrington topology is most useful, induce only countably many classes.
Before proceeding to the proof we find it useful to discuss some well-known facts. The countable linear orderings are encoded by members of the Baire space in a natural way. For every $\alpha \in \mathcal{N}$ we define $\leq_\alpha \subseteq \omega \times \omega$ as follows

$$\text{Field}(\alpha) = \{n \in \omega \mid \alpha(\langle n, n \rangle) = 1\}$$

$$n \leq_\alpha m \iff n, m \in \text{Field}(\alpha) \& \alpha(\langle n, m \rangle) = 1.$$ 

The set LO of codes of countable linear orderings is defined as follows

$$\text{LO} = \{\alpha \in \mathcal{N} \mid \leq_\alpha \text{ is a linear ordering on Field}(\alpha)\}.$$  

The set of codes of well-orderings is

$$\text{WO} = \{\alpha \in \text{LO} \mid \leq_\alpha \text{ is a well-ordering on Field}(\alpha)\}.$$ 

We also put $\text{MY}(x) \equiv M(x) = \{y \in Y \mid \omega(x, y) = 1\}$, where $X, Y$ are recursive Polish spaces. It is well-known that $\text{MY}(x)$ is a Borel and $\Sigma^1_1$ subset of $Y$, see [4]. As it was proved by Spector cf. [15] the set $\text{MY}(x)$ contains all points in $\Delta^1_1(x)$, and so from the Thomasson-Hinnman Theorem cf. [16, 5] the set $\text{MY}(x)$ is comeager.

The hyperjump $W^x$ of $x \in \mathcal{X}$ is defined by

$$W^x = \{e \in \omega \mid \{e\}^x \text{ is total and in WO}\}.$$ 

It is a well-known result of Spector that $\omega(x, y) = 1$ if and only if $W^x \not\leq h y$. So the second assertion of Theorem 1 essentially says that for all $\alpha$ the set of orbits in the hypercone with basis $W^\alpha$ is co-countable.

**Remark 2.** We do not know if the direct implication of Theorem 1 extends to all analytic equivalence relations. The standard example of an analytic equivalence relation with uncountably many but not perfectly many classes does satisfy the second assertion of Theorem 1:

Given $x, y \in \text{LO}$ we define

$$xEy \iff [x, y \not\in \text{WO}] \text{ or } [\leq_x, \leq_y \text{ are isomorphic}].$$

Clearly the preceding $E$ is a $\Sigma^1_1$ equivalence relation on LO with uncountably many classes. Moreover it is not hard to verify that it does not have perfectly many classes, since for any non-empty perfect $P \subseteq \text{LO}$ as in the definition of “perfectly many” we would be able to find some non-empty perfect $P' \subseteq P$ with $P' \subseteq \text{WO}$. This would imply:

$$y \in \text{WO} \iff y \in \text{LO} \& (\exists x \in P')[\leq_y \text{ embeds in } \leq_x].$$ 

The latter would imply that $\text{WO}$ is a $\Sigma^1_1$ set, a contradiction.

Now given $\alpha \in \mathcal{N}$ the set $\omega_\alpha$ is countable, so there is a sequence $(y_n^\alpha)_{n \in \omega}$ of $\alpha$-recursive well-orderings such that for each $\xi < \omega_\alpha^\alpha$ it holds $\xi = |y_n^\alpha|$ for some $n \in \omega$. It follows easily that

$$\{[x]_E \mid x \in \text{WO} \& \omega_1^{(\alpha, x)} = \omega_1^\alpha\} = \{[y_n^\alpha]_E \mid n \in \omega\}.$$ 

Since there is only one class $[z]_E$ for $z \not\in \text{WO}$ it follows that the set $\{[x]_E \mid \omega_1^{(\alpha, x)} = \omega_1^\alpha\}$ is countable for all $\alpha \in \mathcal{N}$, and hence the equivalence relation $E$ satisfies the second assertion of Theorem 1 too.
One more example of a $\Sigma^1_2$ equivalence relation with uncountably many but not perfectly many classes is

$$xFy \iff \omega^x_1 = \omega^y_1,$$

where $x, y \in 2^\omega$.

One way to see the latter is by applying the main result of [11], see Remark 4 below. Given $\alpha \in \mathcal{N}$, using again that $\omega^\alpha_1$ is countable, we choose a sequence $(x_n)_{n \in \omega}$ in $2^\omega$ such that

$$\{\omega^x_n \mid x \in 2^\omega \} = \{\omega^y_n \mid n \in \omega \}.$$

Then for every $x \in 2^\omega$ with $\omega^x_1(\alpha, x) = \omega^\alpha_1$, we have in particular that $\omega^x_1 \leq \omega^\alpha_1$ and so there is some $n$ such that $\omega^x_1 = \omega^\alpha_1$, i.e., $xFx_n$. Thus the set of classes

$$\{[x]_F \mid \omega^x_1(\alpha, x) = \omega^\alpha_1 \} \equiv \{(x_n)_{n \in \omega} \}.$$

Hence the relation $F$ satisfies the second assertion of Theorem 11 as well.

As it is well-known cf. [2, Theorem 7.3.1] in a Polish $G$-space one can decompose the domain $\mathcal{X}$ into an $\omega_1$-sequence $(A_\xi)_{\xi < \omega_1}$ of Borel sets such that each restriction $E_G \cap (A_\xi \times A_\xi)$ is a Borel set. Moreover the $A_\xi$‘s can be chosen to be $E_G$-invariant. We provide the following related result.

**Proposition 3.** For every recursive Polish $G$-space $(\mathcal{X}, G, \cdot)$ and for all $\alpha \in \mathcal{N}$ the sets $E_G \cap (M(\alpha) \times \mathcal{X})$ and $E_G \cap (\mathcal{X} \times M^\mathcal{X}(\alpha))$ are Borel.

The proof of the proposition above will be given in the sequel. This result is also related to a result of Sami. To explain this better, we set first

$$\omega_{G^{x, \varepsilon}} = \min\{\omega_{G^{x, \varepsilon}(\alpha, x, \varepsilon)} \mid g \in G\}$$

where $(\mathcal{X}, G, \cdot)$ is $\varepsilon$-recursive. Sami proved that in every $\varepsilon$-recursive Polish $G$-space $(\mathcal{X}, G, \cdot)$ it holds: (i) every orbit $G \cdot x$ is a $\Pi^0_{G^{x, \varepsilon}} \times 2$ set; (ii) if there exists some $\xi < \omega_1$ such that every orbit is $G \cdot x$ is a $\Pi^0_\xi$ set, then $E_G$ is Borel; and therefore (iii) if $\omega_{G^{x, \varepsilon}} = \omega_{G^{x, \varepsilon}}$ for all $x \in \mathcal{X}$ then $E_G$ is a Borel equivalence relation.

Proposition 8 gives another proof of the preceding statement (iii). To see this assume that for all $x \in \mathcal{X}$ it holds $\omega_{G^{x, \varepsilon}} = \omega_{G^{x, \varepsilon}}$, i.e., there is some $z \in M(\varepsilon)$ such that $zE_Gx$. Then for all $x, y \in \mathcal{X}$ we have

$$xE_Gy \iff (\forall z \in M(\varepsilon))[xE_Gz \iff zE_Gy]$$

$$\iff (\forall z)[z \notin M(\varepsilon) \lor (x, z) \notin E_G \lor (z, y) \notin E_G \cap (M(\varepsilon) \times \mathcal{X})].$$

Using Proposition 8 it follows that $E_G$ is coanalytic and therefore it is moreover a Borel subset of $\mathcal{X} \times \mathcal{X}$.

Our characterization is proved with the help of the preceding proposition.

**Proof of Theorem 7.** For the left-to-right-hand direction, given $\alpha \in \mathcal{N}$ we consider the restriction $F := E_G \cap (M^\mathcal{X}(\alpha) \times M^\mathcal{X}(\alpha))$. Then $F$ is a Borel equivalence relation on the Borel set $M^\mathcal{X}(\alpha)$.

If the conclusion were not true then $F$ would have uncountably many equivalence classes and so from Silver’s Dichotomy [14] there would be some non-empty perfect set $P \subseteq M^\mathcal{X}(\alpha)$ such that for all $x, y \in P$ with $x \neq y$ it holds $(x, y) \notin E_G$. In particular $E_G$ would have perfectly many classes, a contradiction.
For the converse direction, consider some $\alpha$-recursive injection $\pi : 2^\omega \rightarrow \mathcal{X}$. It is enough to show that for some $z \neq w$ in $2^\omega$ we have that $\pi(z)_{E_G} \pi(w)$. The set

$$A = \{ z \in 2^\omega \mid \omega_1^{(\alpha, \pi(z))} = \omega_1^\alpha \}$$

is easily a $\Sigma^1_1(\alpha)$ subset of $2^\omega$. Moreover it contains all points in $\Delta^1_1(\alpha)$ and so $A$ is comeager. In particular $A$ is an uncountable set. From our hypothesis the set of classes $B = \{ [x]_{E_G} \mid \omega_1^{(\alpha, x)} = \omega_1^\alpha \}$ is countable. Clearly the function $z \mapsto [\pi(z)]_{E_G}$ carries $A$ inside $B$. Since $B$ is a countable set and $A$ is an uncountable one, it follows that the latter function cannot be one-to-one on $A$, i.e., there are $z \neq w$ in $A$ such that $[\pi(z)]_{E_G} = [\pi(w)]_{E_G}$. In other words $\pi(z)_{E_G} \pi(w)$.

The proof of the latter direction when we have an arbitrary $\Sigma^1_1(\varepsilon)$ equivalence relation $E$ in an $\varepsilon$-recursive Polish space $\mathcal{X}$ is exactly the same. \(\square\)

**Remark 4.** (a) Montalban [11] proved that under the axiom of $\Sigma^1_1$-determinacy an analytic equivalence relation $E$ on $2^\omega$ does not have perfectly many classes exactly when there is some $\varepsilon \in \mathcal{N}$ such that for all $\alpha \geq_T \varepsilon$ every $x \leq_h \alpha$ is $E$-equivalent to some $y \leq_T \alpha$ (the latter property is called HYP-is-recursive on a cone). Moreover he showed that the converse direction is in fact provable in $ZF$.

The argument that we used to prove the converse direction of Theorem 1 provides also a somewhat shorter (although with less information) proof of the converse direction of Montalban’s preceding result. To see this assume that $E$ satisfies that HYP-is-recursive on the cone with basis $\varepsilon$ and let $\pi : 2^\omega \rightarrow 2^\omega$ be an $\alpha$-recursive injection with $\alpha \geq_T \varepsilon$ and $E$ is $\Sigma^1_1(\alpha)$. Consider the set

$$A = \{ z \in 2^\omega \mid (\exists y \leq_T \alpha)[(\pi(z), y) \in E] \}.$$

Clearly $A$ is a $\Sigma^1_1(\alpha)$ subset of $2^\omega$, and from our hypothesis it contains all points $z \in 2^\omega$ with $z \leq_h \alpha$. Hence $A$ is comeager. On the other hand $A = \cup_e A_e$, where

$$A_e = \{ z \in 2^\omega \mid \{ e \}^\alpha \text{ is total and } (\pi(z), \{ e \}^\alpha) \in E \}.$$

Hence for some $e$ the set $A_e$ is non-meager and in particular it contains two distinct points $z \neq w$. We then have $\pi(z) E \{ e \}^\alpha E \pi(w)$. Hence $E$ cannot have perfectly many classes.

(b) It is clear from Montalban’s characterization and Theorem 1 that for orbit equivalence relations the condition “HYP-is-recursive on a cone” implies (in $ZF$) condition (b) of the latter theorem. It would be interesting to see if there is a direct proof of this fact, which may also work for arbitrary analytic equivalence relations.

**Orbits are Borel sets.** It is a known result of D. E. Miller [9] Theorem 2’ that orbits of Borel actions of Polish groups are Borel sets. Sami’s result (i) that we mentioned above is a refinement of the latter fact. Another such refinement is given by Becker [1], from which it follows that every orbit $G \cdot x$ in a recursive Polish $G$-space is a $\Delta^1_1(W^x)$ set. Although not explicitly mentioned by Becker, the following fact is immediate from his proof.

**Proposition 5** (cf. [1]). For every recursive Polish $G$-space $(\mathcal{X}, G, \cdot)$ we have for all $x, y \in \mathcal{X}$ that

$$x E_G y \iff (\exists g \in \Delta^1_1(W^x, y))[y = g \cdot x].$$

In particular every orbit $G \cdot x$ is a $\Delta^1_1(W^x)$ set.\(^2\)

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\(^2\)Notice that from the Kleene Basis Theorem we can always find such a $g$ in $\Delta^1_1(W^{(x, y)})$. The merit of this result is that we can relax the hyperjump on one of the variables. Moreover
To see how the latter proposition follows from Becker’s arguments, we go to the proof of [1] Lemma 3.5] and we notice that the unique $g \in G$, which satisfies that $g \in K$ and $g \cdot y = z$ is a $\Delta_1^1(T_H, y, z)$ point. This is because the set $K$ is $\Delta_1^1(T_H)$ (as Becker remarks in order to check this one has to review an earlier result of Dixmier [3]) and therefore the preceding $g$ is a member of a $\Delta_1^1(T_H, y, z)$ singleton. Since $T_H$ is a $\Sigma_1^1(y)$ subset of the naturals we have that $T_H \leq_T W^y$. Hence $g \in \Delta_1^1(W^y, z)$.

We find it useful to provide a more detailed sketch of the proof of Proposition 5 where the effective arguments are somewhat easier to follow. But before we do this we show that Proposition 3, which we used to prove our characterization, follows from Proposition 5.

Spector cf. [15] proved that for all $x, y \in 2^\omega$ the following hold: (a) $W^x \leq_h y$ implies $\omega_1^x < \omega_1^y$; (b) $\omega_1^x < \omega_1^y$ and $x \leq_h y$ implies $W^x \leq_h y$. It is then an easy corollary that if $\omega_1^{(x,y)} = \omega_1^x$ then $W^{(x,y)} \leq_h (W^x, y)$, for all $x, y \in 2^\omega$. In particular if $\omega_1^x = \omega_1^W$ then $W^x \leq_h (W, x)$ for all $x \in 2^\omega$. To see this assume that $\omega_1^{(x,y)} = \omega_1^x$ and apply (b) of the preceding result of Spector with $x' = x \oplus y := (x(0), y(0), x(1), y(1), \ldots)$ and $y' = W^x \oplus y$.

It is well-known that if $T$ is a perfect tree which is generic in the sense of Sacks forcing then $W^T \leq_h (W, T)$. In particular the latter relation holds for almost all $T$. Summing up we have the following.

**Corollary 6** (see also 3.13 in [13]). For every recursive Polish space $X$, all $\alpha \in \mathcal{N}$, and all $x \in X$ with $\omega_1^{(x, \alpha)} = \omega_1^x$ (in particular for almost all $x \in X$) we have that $W^{(\alpha, x)} \leq_h (W^x, x)$.

**Remark 7.** These results give a short effective proof that analytic sets have the Baire property. Let us see how. Suppose that $P \subseteq X$ is (without loss of generality) $\Sigma_1^1$ and that $F \subseteq 2^\omega \times \mathbb{N}$ is $\Pi^0_1$ such that

$$P(x) \iff (\exists \beta \in \mathbb{N}) F(x, \beta).$$

Using the Kleene Basis Theorem cf. [7] (see also [12], 4E.8]) and Corollary 6 we have that for all $x \in M \equiv M(\emptyset)$,

$$P(x) \iff (\exists \beta \in \Delta_1^1(W^x)) F(x, \beta) \iff (\exists \beta \in \Delta_1^1(W, x)) F(x, \beta).$$

The latter shows that the set $P$ is computed by a $\Pi^0_1(W)$ relation on $M$. Since $M$ is Borel it follows that the set $P \cap M$ is both analytic and coanalytic, and so from Souslin’s Theorem it is also Borel. Moreover the set $P \setminus (P \cap M)$ is meager because $M$ is comeager.

We can now explain how to derive Proposition 3 from Proposition 5. Given a recursive $(X, G, \cdot)$, by applying the Corollary 6 as above, we can see round-robin by combining these two type of refinements (Sami’ s result and Proposition 5) with Louveau Separation it follows that every orbit $G \cdot x$ in a recursive Polish space is lightface $\Pi^0_1(\omega_1^{G \cdot x + \omega_1^{(\beta, x)}})$ set, for some $\beta \leq_h W^x$.

Although these results were initially given for members of the Cantor space, they hold also in recursive Polish spaces, with very mild modifications. For example we exchange $x \oplus y$ with the pair $(x, y)$ and we consider the Polish space $X \times X$. 


style that for all $\alpha$ and all $(x, y) \in \mathcal{M}(\alpha) \times \mathcal{X}$,
\[ x \in E_G y \iff (\exists g \in \Delta^1_1(W^*, y))[y = g \cdot x] \]
\[ \iff (\exists g \in \Delta^1_1(W^{(\alpha, x)}, y))[y = g \cdot x] \]
\[ \iff (\exists g \in \Delta^1_1(W^\alpha, x, y))[y = g \cdot x]. \]

Hence the set $E_G \cap (\mathcal{M}(\alpha) \times \mathcal{X})$ is defined by a $\Pi^1_1(W^\alpha)$-formula, and is in particular a coanalytic subset of $\mathcal{X} \times \mathcal{X}$. Since it is evidently an analytic set as well, it follows from the Souslin Theorem that the set $E_G \cap (\mathcal{M}(\alpha) \times \mathcal{X})$ is Borel. The result for $E_G \cap (\mathcal{X} \times \mathcal{M}(\alpha))$ is proved similarly using the fact that $E_G$ is symmetric.

We conclude with a more detailed sketch of the proof of Proposition 8. We fix a recursive Polish $G$-space $(\mathcal{X}, G, \cdot)$. Recall that the stabilizer $G_x$ of $x \in \mathcal{X}$ is the set \{ $g \in G \mid g \cdot x = x$ \}. Given a Polish space $\mathcal{Y}$, by $F(\mathcal{Y})$ we mean the set of all closed subsets of $\mathcal{Y}$ with the Effros-Borel structure cf. [12, C]. A function $\delta : F(\mathcal{Y}) \to \mathcal{Y}$ is a choice function if for all $\emptyset \neq F \in F(\mathcal{Y})$ we have that $\delta(F) \in F$.

**Step 1.** By easy calculations one can check that for every choice function $\delta : F(G) \to G$ it holds
\[ y \in G \cdot x \iff (\exists g)[\delta(gG_x) = g \& y = g \cdot x] \]
\[ \iff (\exists g)[\delta(gG_x) = g \& y = g \cdot x], \]
where $\exists!$ stands for “there exists unique”.

**Step 2.** A Souslin scheme on a Polish space $\mathcal{Y}$ is any family $(U_s)_{s<\omega}$ of subsets of $\mathcal{Y}$. We say that a given Souslin scheme $(U_s)_{s<\omega}$ is good if $U_0 = \mathcal{Y}$, $U_s \cdot i \subseteq U_s$, $U_s = \bigcup_i U_s \cdot i$ and $\text{diam}(U_s) \leq 2^{-\text{lh}(s)}$ if $s \neq 0$, where $s \cdot i = (s_0, \ldots, s_{n-1}, i)$ and $n$ is the length of $s$.

The associated function of a good Souslin scheme $U := (U_s)_s$ on $\mathcal{Y}$ is $f : \mathcal{N} \to \mathcal{Y} : \{ f(\alpha) \} = \cap_n U(\alpha(0), \ldots, \alpha(n-1))$. It is easy to verify that the associated function of a good Souslin scheme is continuous, and if moreover the good Souslin scheme consists of open sets, then the associated function is also open.

**Proposition 8 (Folklore?).** Every recursive Polish space $\mathcal{Y}$ admits a good Souslin scheme, which has a recursive associated function.

**Proof.** Let $d$ be a compatible metric for $\mathcal{Y}$ and $\overline{r} = (r_j)_{j<\omega}$ a compatible recursive presentation. We denote by $N(k)$ be the open ball $B(r_{(k_0)}, q(k_1))$ with center $r_{(k_0)}$ and $d$-radius $q(k_1)$, where $(q_i)_{i<\omega}$ is the enumeration of all non-negative rational numbers that we fixed in the introduction.

By dividing $d$ with $1 + d$ we may assume that $d \leq 1$. We fix some $k_0 \in \omega$ such that $N(k_0) = \mathcal{Y}$.

The idea is to write each non-empty recursive union of some basic neighborhoods $N(m)$, $m \in I_k$, with $\overline{N(m)} \subseteq N(k)$ and radius($N(m)$) $\leq 2^{-1} \cdot \text{radius}(N(k))$. Here “recursive” means that the set $I(k, n) \iff n \in I_k$ is recursive.

Given $r_j \in N(k)$ then for any $t \in \omega$ with $0 < q_t < q(k_1) - d(r_j, r_{(k_0)})$ we have that $\overline{N((j, t))} \subseteq N(k)$. We define
\[ I(k, m) \iff d(r(m_0), r_{(k_0)}) < q(k_1) \& 0 < q(m_1) < q(k_1) - d(r_j, r_{(k_0)}) \]
\[ \& q(m_1) \leq 2^{-1} \cdot q(k_1), \]
so that $\overline{N(m)} \subseteq N(k)$ and \(\text{radius}(N(m)) \leq 2^{-1}\cdot \text{radius}(N(k))\), when \(I(k, m)\) holds. Clearly \(I\) is a recursive set and each \(k\)-section \(I_k\) of \(I\) is non-empty, provided that \(q(k)_1 > 0\) (and thus \(N(k) \neq \emptyset\)).

We prove round-robin style that
\[
\cup_{m \in I_k} N(m) = \cup_{m \in I_k} \overline{N(m)} = N(k)
\]
for all \(k \in \omega\).

The left-to-right inclusions are clear. Now suppose that \(x \in N(k)\) and choose some \(t \in \omega\) such that \(0 < 2 \cdot q_t < q(k)_1 - d(x, r(k)_0)\). We consider some \(r_j \in B(x, q_t)\). We then have
\[
d(r_j, r(k)_0) < d(r_j, r(k)_0) + q_t \leq d(r_j, x) + d(x, r(k)_0) + q_t < 2 \cdot q_t + d(x, r(k)_0) < q(k)_1.\]
This shows that \(d(r_j, r(k)_0) < q(k)_1\) and also that \(q_t < q(k)_1 - d(r_j, r(k)_0)\). Moreover from the inequality \(2 \cdot q_t + d(x, r(k)_0) < q(k)_1\) we obtain that \(q_t < 2^{-1} \cdot q(k)_1\). Hence \(I(k, (j, t))\) holds. Moreover \(d(r_j, x) < q_t\), i.e., \(x \in N(j, t)\). This settles the inclusion \(N(k) \subseteq \cup_{m \in I_k} N(m)\).

Finally we define recursively on \(lh(s)\) the Souslin scheme \((U_s)_s\) and the auxiliary function \(\tau : \omega^{<\omega} \to \omega\) as follows:
\[
(U_0, \tau(\emptyset)) = (N(k_0), k_0) = (\emptyset, k_0)
\]
\[
(U_s \cdot m, \tau(s \cdot m)) = \begin{cases} (N(m), m), & \text{if } I(\tau(s), m), \\ (N(m_s), m_s), & \text{if } \neg I(\tau(s), m), \end{cases}\]

By an easy induction on the length of \(s\) one can see that \(U_s = N(\tau(s)) \neq \emptyset\) for all \(s\). It is also easy to verify that \(U_s = \cup_{m \in \omega} U_s \cdot m = \cup_{m \in \omega} U_s \cdot m\), and that \(\text{radius}(U_s) \leq 2^{-lh(s)}\) for all \(s\). Hence \((U_s)_s\) is a good Souslin scheme.

Finally we show that the associated function \(f : \mathcal{N} \to \mathcal{Y}\) is recursive. Since \(f(\alpha) \in U_{\alpha \upharpoonright n} = N(\tau(\alpha \upharpoonright n))\) and the radius of the latter set is at most \(2^{-n}\) we have that \(d(f(\alpha), r(\alpha \upharpoonright n)_0) \leq 2^{-n}\) for all \(n\). It is then easy to verify that
\[
f(\alpha) \in N(m) \iff d(f(\alpha), r(m)_0) < q(m)_1,
\]
\[
\iff (\exists n)[2^{-n} < q(m)_1 - d(r(m)_0, r(\alpha \upharpoonright n)_0)],
\]
for all \(\alpha, m\). Hence \(f\) is recursive. \(\square\)

Now we fix a recursive Polish \(G\)-space \((\mathcal{X}, G, \cdot)\) and a good Souslin scheme \((U_s)_s\) for \(G\) with a recursive associated function \(f : \mathcal{N} \to G\).

For all non-empty \(F \in F(G)\) we define the pruned tree
\[
T_F = \{s \in \omega^{<\omega} \mid F \cap U_s \neq \emptyset\}
\]
and we let \(\alpha_F\) be the leftmost infinite branch of \(T_F\). We also consider the choice function
\[
\delta : F(G) \to G : F \mapsto f(\alpha_F)\]

**Step 3.** There exists an arithmetical relation \(A \subseteq \mathcal{N} \times \mathcal{X} \times G \times \omega^{<\omega}\) such that
\[
g \cdot G x \cap U_s \neq \emptyset \iff A(W^x, x, g, s)
\]

\footnote{Formally we define partial functions, as our definition does not exclude a priori the possibility that \(I_{\tau(s)}\) is the empty set and so \(m_\alpha\) is not defined. Of course, since we always have positive radii, the latter case never occurs, and therefore our functions are in fact total.}

\footnote{Notice that \(F(G)\) may not be a recursive Polish space. However this is not an obstacle, since in the computations we can easily bypass any direct reference to \(F(G)\).}
for all $x, g, s$.

In order to prove this we check first that for all $x, g, s$ it holds

$$(\exists h)[hx = x & g \cdot h \in U_s] \iff (\exists i, k)[(\forall j)[r_j \in N(G, k) \implies g \cdot r_j \in U_s \cdot i]]$$

and using the arithmetical relation $A$ we conclude that

where $(r_j)_{j \in \omega}$ is the recursive presentation of $G$ and $N(G, k)$ is the $k$-th basic neighborhood of $G$ which comes from $(r_j)_{j \in \omega}$.

The right-hand side of the preceding equivalence essentially says that there is a basic neighborhood $N(G, k)$ of $G$, which contains a member of the stabilizer of $x$, and is contained in a set of the form $g \cdot U_s \cdot i \subseteq g \cdot U_s$. The latter equivalence is proved using the property of the Souslin scheme being good, the density of $(r_j)_{j \in \omega}$ and the continuity of the group action.

Having established (2) we observe that, using the Kleene Basis Theorem, the $h$ on the right-hand side of the latter equivalence can be chosen to be recursive in $W^x$. Hence by taking the arithmetical set

$$C(\alpha, x, k) \iff (\exists h \leq_T \alpha)[h \cdot x = x & h \in N(G, k)]$$

we conclude that

$g \cdot G_x \cap U_s \neq \emptyset \iff (\exists h)[hx = x & g \cdot h \in U_s] \iff (\exists i, k)[D(g, k, s, i) & C(W^x, x, k)],$

where $D$ is defined according to (2). We then take

$$A(\alpha, x, g, s) \iff (\exists i, k)[D(g, k, s, i) & C(\alpha, x, k)].$$

**Step 4.** There exists an arithmetical relation $Q \subseteq \mathcal{N} \times G \times \omega$ such that

$$\delta(g \cdot G_x) \in N(G, k) \iff Q(W^x, x, g, k)$$

for all $x, g, k$.

To see this, let $N(\mathcal{N}, s)$ for $s \in \omega^{<\omega}$ be the usual $s$-th basic neighborhood of $\mathcal{N}$. Since $f : \mathcal{N} \to G$ is recursive we have that

$$f(\alpha) \in N(G, k) \iff (\exists s)[\alpha \in N(\mathcal{N}, s) & R^s(s, k)]$$

for some recursive $R^s \subseteq \omega^{<\omega} \times \omega$. Then we can easily see that

$$\delta(g \cdot G_x) \in N(G, k) \iff f(\alpha_g G_x) \in N(G, k) \iff (\exists s)[\alpha_g G_x \in N(\mathcal{N}, s) & R^s(s, k)],$$

where $\alpha_g G_x$ is as above the left-most infinite branch of the tree $T_F$ for $F = g \cdot G_x \in F(G)$.

Now we observe that

$$\alpha_g G_x \in N(\mathcal{N}, s) \iff g \cdot G_x \cap U_s \neq \emptyset & (\forall t \in \omega^{\text{lh}(s)})[t \leq_{\text{lex}} s \implies g \cdot G_x \cap U_t = \emptyset]$$

and using the arithmetical relation $A$ in the preceding step, it follows that there exists an arithmetical relation $Q \subseteq \mathcal{N} \times G \times \omega^{<\omega}$ such that

$$\alpha_g G_x \in N(\mathcal{N}, s) \iff Q(W^x, x, g, s)$$

for all $x, g, i, j, s$.

**Step 5.** For all $x, y \in \mathcal{X}$ the (possibly empty) set

$$A_{x,y} := \{ g \in G \mid \delta(g \cdot G_x) = g \& y = g \cdot x \}$$
is arithmetical in \((W^x, y)\). This is immediate from the key property of the set \(Q\) in Step 4 and the fact that \(\Sigma^0_n(W^x, x, y) = \Sigma^0_n(W^x, y)\) for every \(n \geq 1\). From Step 1, it follows that \(A_{x,y}\) is at most a singleton, and therefore when it is non-empty its unique point is \(\Delta^1_1(W^x, y)\). Hence
\[
y \in G \cdot x \iff A_{x,y} \neq \emptyset
\]
\[
\iff (\exists g \in \Delta^1_1(W^x, y))[\delta(g \cdot G_x) = g \iff y = g \cdot x]
\]
\[
\iff (\exists g \in \Delta^1_1(W^x, y))[y = g \cdot x].
\]
This finished the sketch of the proof.

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