SHAPLEY VALUE FOR DIFFERENTIAL NETWORK GAMES: 
THEORY AND APPLICATION

LEON PETROSYAN*
St. Petersburg State University
Univereitetskaya Nab. 7/9 Saint-Petersburg, Russia

DAVID YEUNG
SRS Consortium for Advanced Study in Cooperative Dynamic Games
Shue Yan University
10 Wai Tsui Cres, North Point, Hong Kong

Abstract. This paper presents a time-consistent dynamic Shapley value imputation for a class of differential network games. A novel form for measuring the worth of coalitions – named as cooperative-trajectory characteristic function – is developed for the Shapley value imputation. This new class of characteristic functions is evaluated along the cooperative trajectory. It measures the worth of coalitions under the process of cooperation instead of under minimax confrontation or the Nash non-cooperative stance. The resultant dynamic Shapley value imputation yields a new cooperative solution in differential network games.

1. Introduction. Theory and applications of network games or games on networks have been growing in recent research. Mazalov and Chirkova (2019) [7] provided a comprehensive disquisition on the topic. Given that most practical game situations are dynamic (intertemporal) rather than static, differential network games have become a field that attracts theoretical and technical developments. The impacts of the classic books by Isaacs (1965)[5] and by Krasovskii (1985) [6] on the fundamentals of differential games are profound. Applications in dynamic network games have covered a considerably wide scope. Wie (1993 and 1995) [16, 17] developed differential game models for studying traffic network. Zhang et al. (2018) [22] presented a differential game of network defense. Pai (2010) [9] provided a differential game formulation of a controlled network. Meza and Lopez-Barrientos (2016) [8] examined a differential game of a duopoly with network externalities. Cao et al. (2008) [1] studied the minimax equilibrium of differential network games. Chirkova (2017) [2] considered optimal arrivals in a two-server random access loss system. Petrosyan and Sedakov (2009) [13] examined multistage networking games with full information. Petrosyan (2010) [12] developed cooperative differential games on networks. Gao and Pankratova (2017) [3] presented a review of papers in dynamic network games.

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* Corresponding author: Leon Petrosyan.
Coordinating players in a network to maximize their joint gain and distribute the cooperative gains in a dynamically stable solution is a topic of ongoing research. The Shapley (1953) [15] value is credited to be one of the best solutions in attributing a fair gain to each player in a complex situation like a network. However, the determination of the worth of the subsets of players (characteristic function) in the Shapley value is not indisputably unique. In addition, in a differential game, the worth of the coalitions of players changes as the game evolves. In this paper, we present a novel characteristic function – named as cooperative-trajectory characteristic function – to generate a time consistent Shapley value solution in a class of network differential games. In computing the values of characteristic function for coalitions, we maintain the cooperative strategies for all players and evaluate the worth of the coalitions along the cooperative trajectory. Thus, the worth of coalition $S$ is its cooperative payoff with the exclusion of the gains through network connection from players outside the coalition. The rationale for such formulation is to attribute the contributions of the players in the process of cooperation. Worth-noting is that this new set of characteristic functions is time consistent. The time consistency property of the characteristic function has not been shared by existing characteristic functions in differential games (see Gromova (2016) [4], Petrosyan (1995) [11], Petrosyan and Zaccour (2003) [14], Yeung (2010) [18], Yeung and Petrosyan (2016 and 2018) [20, 21]). It is the first time that the worth of coalitions is measured under the process of cooperation instead of under min-max confrontation or Nash non-cooperative stance. There are different solution concepts (optimality principles) in cooperative differential game theory taken from classical static approach. The most frequently used solutions are the core and the Shapley value. The core is a set-valued solution concept and in many cases can be void (may not exist). The Shapley value exist always and is a single valued solution. The most preferable situation is when the core is not void and the Shapley value belongs to the core. Exactly this happens when the solution is constructed on the bases of cooperative trajectory characteristic function since this characteristic function is convex (proposition 2.1). This property with, time-consistency of the constructed Shapley value and computational simplicity of the characteristic function is an important background of what follows. We then develop a dynamic Shapley value solution along the optimal trajectory using cooperative-trajectory characteristic function. An imputation distribution procedure is formulated so that the dynamic Shapley value solution can be realized. An application in cooperative regional development in a network of regions is provided.

2. Game formulation and characteristic functions. In this section, we present a class of differential network games and develop a new system of characteristic functions which is derived along the optimal cooperative trajectory.

2.1. Formulation of a class of differential network games. Consider a class of $n$-person differential games on network with game horizon $[t_0, T]$. The players are connected in a network system. We use $N = \{1, 2, \cdots, n\}$ to denote the set of players in the network. The nodes of the network are used to represent the players from the set $N$. We also denote the set of nodes by $N$ and denote the set of all arcs in network $N$ by $L$. The arcs in $L$ are the arc $(i, j) \in L$ for players $i, j \in N$. For notational convenience, we denote the set of players connected to player $i$ as $K(i) = \{j : arc(i, j) \in L\}$, and the set $K(i) = K(i) \cup \{i\}$, for $i \in N$. 
Let $x^i(\tau) \in R^m$ be the state variable of player $i \in N$ at time $\tau$, and $u^i(\tau) \in U^i \subset R^k$ the control variable of player $i \in N$.

Every player $i \in N$ can cut the connection with any other players from the set $\tilde{K}(i)$ at any instant of time.

The state dynamics of the game is

$$\dot{x}^i(\tau) = f^i(x^i(\tau), u^i(\tau)), \quad x^i(t_0) = x^i_0, \quad \text{for } \tau \in [t_0, T] \text{ and } i \in N.$$  \hspace{1cm} (1)

The function $f^i(x^i, u^i)$ is continuously differentiable in $x^i$ and $u^i$.

The payoff function of player $i$ depends upon his state variable, his own control variable and the state variables of players from the set $\tilde{K}(i)$.

In particular, the payoff of player $i$ is given as

$$H_i(x^i_0, x^i_0, \ldots, u^n) = \sum_{j \in K(i)} \int_{t_0}^{T} h^i_j(x^i(\tau), x^j(\tau), u^i(\tau)) d\tau + q_i(x^i(T)).$$  \hspace{1cm} (2)

The term $h^i_j(x^i(\tau), x^j(\tau), u^i(\tau))$ is the instantaneous gain that player $i$ can obtain through network link with player $j \in \tilde{K}(i)$ and $h^i_j(x^i(\tau), x^j(\tau), u^i(\tau))$ is the instantaneous gain that player $i$ can obtain by itself. The terminal payoff that player $i$ received at time $T$ is $q_i(x^i(T))$. The functions $h^i_j(x^i(\tau), x^j(\tau), u^i(\tau))$, for $j \in \tilde{K}(i)$ are non-negative. For notational convenience, we use $x(t)$ to denote the vector $(x^1(t), x^2(t), \ldots, x^n(t))$.

A feedback Nash equilibrium solution under non-cooperation can be characterized as

**Theorem 2.1.** A set of strategies $u^i = \phi^i(t,x)$, $i \in N$ and $t \in [t_0, T]$, constitutes a feedback Nash equilibrium solution to the differential network game (1)-(2) if there exist functions $V^i(t, x)$ such that the following recursive relations are satisfied:

$$V^i(t, x) = q_i(x^i),$$  \hspace{1cm} (3)

$$-V^i(t, x) = \max_{u^i} \left\{ \sum_{j \in K(i)} h^i_j(x^i, x^j, u^i) + V^j_{x^i}(t, x) f^j(x^i, u^i) + \sum_{\ell \in N \setminus \{i\}} V^j_{x^\ell}(t, x) f^j(x^\ell, \phi^\ell(t, x)) \right\} =$$

$$= \sum_{j \in K(i)} h^i_j(x^i, x^j, \phi^j(t, x)) + V^j_{x^i}(t, x) f^j(x^i, \phi^j(t, x)) + \sum_{\ell \in N \setminus \{i\}} V^i_{x^\ell}(t, x) f^\ell(x^\ell, \phi^\ell(t, x)),$$  \hspace{1cm} (4)

for $t \in [t_0, T]$ and $i \in N$.

**Proof.** Invoking the technique of dynamic programming technique, $V^i(t, x)$ is the maximized payoff of player $i$ for given Nash equilibrium strategies $\{\phi^\ell(t, x)\}$, for $\ell \in N$ and $\ell \neq i$ of the other $n-1$ players. Hence a Nash equilibrium appears.

Performing the maximization operator in Theorem 2.1., we obtain

$$\frac{\partial}{\partial u^i} \sum_{j \in K(i)} h^i_j(x^i, x^j, u^i) + V^j_{x^i}(t, x) \frac{\partial}{\partial u^i} f^j(x^i, u^i) = 0.$$  \hspace{1cm} (5)
Under non-cooperation, the potential externalities (positive or negative) of a player’s state on the payoffs of connected players are not considered in the player’s optimization process. Hence, the non-cooperative equilibrium is, in general, not a Pareto optimal solution.

2.2. Cooperation and characteristic functions. Now we consider the case when all the players want to cooperate and enhance their payoffs under cooperation. Two important factors for a successful cooperation scheme are (i) group optimality to be maintained so that all potential gains are captured, and (ii) any player deviating from the cooperative scheme would receive a smaller payoff. Moreover, to create a cooperative solution in which every player would commit from the beginning to the end, the proposed optimality principles must remain effective throughout the cooperation period. Specifically, the agreed-upon optimality principle of sharing be maintained in the entire game. This is the ‘classic’ game-theoretic problem of time consistency (see Yeung and Petrosyan (2004) and (2016) [19, 20]). To achieve group optimality, the players maximize their joint payoff

\[
\sum_{i \in N} \left( \sum_{j \in K(i)} \int_{t_0}^{T} h^i_j (x^i(\tau), x^j(\tau), u^i(\tau)) d\tau + q_i(x^i(T)) \right)
\]

subject to dynamics (1).

We use \( \bar{x}(t) = (\bar{x}^1(t), \bar{x}^2(t), \ldots, \bar{x}^n(t)) \) to denote the optimal cooperative trajectory of problem of maximizing (6) subject to (1). We let the corresponding optimal cooperative strategies of player \( i \) be denoted by \( \bar{u}^i(t) \), for \( t \in [t_0, T] \) and \( i \in N \). The maximized joint cooperative payoff involving all players can then be expressed as

\[
\max_{u^1, u^2, \ldots, u^n} \left\{ \sum_{i \in N} \left( \sum_{j \in K(i)} \int_{t_0}^{T} h^i_j (x^i(\tau), x^j(\tau), u^i(\tau)) d\tau + q_i(x^i(T)) \right) \right\}
\]

subject to dynamics (1).

Next, we consider distributing the cooperative payoffs to the participating players under an agreeable scheme. Given that the contributions of an individual player to the joint payoff through linked players can be diverse, the Shapley (1953) [15] value provides one of the best solutions in attributing a fair gain to each player in a complex network. One of the contentious issues in using the Shapley value is the determination of the worth of subsets of players (characteristic function). In cooperative differential games, the characterization of the worth of a coalition of players \( S \) is not indisputably unique except for the case of zero-sum games in which coalition \( S \) seeks to maximize its payoff while coalition \( N \setminus S \) seeks to minimize the payoff. In this section, we present a new formulation of the worth of coalition \( S \subset N \). In computing the values of characteristic function for coalitions, we evaluate contributions of the players in the process of cooperation and maintain the cooperative strategies for all players along the cooperative trajectory. In particular, we evaluate the worth of the coalitions along the cooperative trajectory as

\[
V(S; x_0, T - t_0) = \sum_{i \in S} \left( \sum_{j \in K(i) \cap S} \int_{t_0}^{T} h^i_j (\bar{x}^i(\tau), \bar{x}^j(\tau), \bar{u}^i(\tau)) d\tau + q_i(\bar{x}^i(T)) \right).
\]
Note that the worth of coalition $S$ is measured by the sum of the payoffs of the players in the coalition in the cooperation process with the exclusion of the gains from players outside coalition $S$. Thus, the characteristic function reflecting the worth of coalition $S$ in (8) is formulated along the cooperative trajectory $\vec{x}(t)$. This is a novel feature and we name it as cooperative-trajectory characteristic function.

Similarly, the cooperative-trajectory characteristic function at time $t \in [t_0, T]$ can be evaluated as

$$V(S; \vec{x}(t), T - t) = \sum_{i \in S} \left( \sum_{j \in K(i) \cap S} \int_t^T h^i_j(\vec{x}^i(\tau), \vec{x}^j(\tau), \vec{u}^i(\tau)) d\tau + q_i(\vec{x}^i(T)) \right).$$

(9)

For simplicity in notation, we denote the gain that player $i$ can obtain through the network link with player $j \in K(i)$ as

$$\alpha_{ij}(\vec{x}(t), T - t) = \int_t^T h^i_j(\vec{x}^i(\tau), \vec{x}^j(\tau), \vec{u}^i(\tau)) d\tau,$$

(10)

for $t \in [t_0, T]$.

Using the notations in (10), we can express the worth of coalition $S$ in (8) in the start of the cooperation scheme as

$$V(S; x_0, T - t_0) = \sum_{i \in S} \left( \sum_{j \in K(i) \cap S} \alpha_{ij}(x_0, T - t_0) + q_i(\vec{x}^i(T)) \right),$$

(11)

and the worth of coalition $S$ along the cooperative trajectory $\vec{x}(t)$ as

$$V(S; \vec{x}(t), T - t) = \sum_{i \in S} \left( \sum_{j \in K(i) \cap S} \alpha_{ij}(\vec{x}(t), T - t) + q_i(\vec{x}^i(T)) \right)$$

(12)

for $t \in [t_0, T]$.

An important property of the cooperative-trajectory characteristic function as a measure of the worth of coalition in the Shapley value is given below.

**Proposition 1.** The following inequalities holds for cooperative-trajectory characteristic function:

$$V(S_1 \cup S_2; x_0, T - t_0) \geq V(S_1; x_0, T - t_0) + V(S_2; x_0, T - t_0) - V(S_1 \cap S_2; x_0, T - t_0).$$

(13)

Similarly, along the cooperative trajectory $\vec{x}(t)$, the following inequalities holds

$$V(S_1 \cup S_2; \vec{x}(t), T - t) \geq V(S_1; \vec{x}(t), T - t) + V(S_2; \vec{x}(t), T - t) - V(S_1 \cap S_2; \vec{x}(t), T - t), \text{ for } t \in [t_0, T].$$

(14)

**Proof.** See Appendix A.

The inequalities in Proposition 1 implies that the game is convex and so are the subgames along the cooperative trajectory. This also means that the core of the game is not void and the Shapley value belongs to the core.

**Proposition 2.** The cooperative-trajectory characteristic function is time consistent.
Proof. Using (8)-(9), the cooperative-trajectory characteristic function can be expressed as

\[
V(S; x_0, T - t_0) = \sum_{i \in S} \left( \sum_{j \in K(i) \cap S} \int_{t_0}^{T} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau), \bar{u}^i(\tau))d\tau + q_i(\bar{x}^i(T)) \right) 
\]

\[
= \sum_{i \in S} \sum_{j \in K(i) \cap S} \int_{t_0}^{t} h_i^j(\bar{x}^i(\tau), \bar{x}^j(\tau), \bar{u}^i(\tau))d\tau + V(S; \bar{x}(t), T - t), \text{ for } S \subset N. \quad (15)
\]

Hence, the cooperative-trajectory characteristic function \( V(S; x_0, T - t_0) \) is time consistent.

The time consistency property of the characteristic function has not been shared by existing characteristic functions in differential games (see Gromova (2016) [4], Petrosyan (1995) [11], Petrosyan and Zaccour (2003) [14], Yeung (2010) [18], Yeung and Petrosyan (2016 and 2018) [20, 21]). It is the first time that the worth of coalitions is measured under the process of cooperation instead of under min-max confrontation or Nash non-cooperative stance. Finally, any individual player attempting to act independently will have the links to other players in the network being cut off.

3. Dynamic shapley value and imputation distribution. In this section, we develop a dynamic Shapley value imputation using the cooperative-trajectory characteristic function and formulate an imputation distribution procedure (IDP) such that the Shapley value imputation can be realized.

3.1. Optimal trajectory dynamic shapley value. Now, we consider allocating the grand coalition cooperative network gain \( V(N; x_0, T - t_0) \) to individual players according to the Shapley value imputation with cooperative-trajectory characteristic functions. Player \( i \)'s payoff under cooperation would become

\[
Sh_i(x_0, T-t_0) = \sum_{S \subset N, S \ni i} \frac{(|S| - 1)! (n - |S|)!}{n!} \left[ V(S; x_0, T-t_0) - V(S \setminus \{i\}; x_0, T-t_0) \right],
\]

for \( i \in N \).

Invoking (11), the difference \( V(S; x_0, T-t_0) - V(S \setminus \{i\}; x_0, T-t_0) \) can be expressed as

\[
= \sum_{i \in S} \left( \sum_{j \in K(i) \cap S} \alpha_{ij}(x_0, T-t_0) + q_i(\bar{x}^i(T)) \right) - \\
\sum_{i \in S \setminus \{i\}} \left( \sum_{j \in K(i) \cap S \setminus \{i\}} \alpha_{ij}(x_0, T-t_0) + q_i(\bar{x}^i(T)) \right)
\]

\[
= \sum_{j \in K(i) \cap S} \alpha_{ij}(x_0, T-t_0) + \sum_{j \in K(i) \cap S \setminus \{i\}} \alpha_{ij}(x_0, T-t_0) + q_i(\bar{x}^i(T)). \quad (17)
\]
Therefore, we can obtain the cooperative payoff of player $i$ under the Shapley value as

$$Sh_i(x_0, T - t_0) = \sum_{S \subseteq N \atop S \ni i} \frac{(|S| - 1)! (n - |S|)!}{n!} \left[ \sum_{j \in K(i) \cap S} \alpha_{ij}(x_0, T - t_0) \right]$$

$$+ \sum_{j \in K(i) \cap S} \alpha_{ji}(x_0, T - t_0) + q_i(\bar{x}^i(T)) \right]. \quad (18)$$

However, in a dynamic framework, the agreed upon optimality principle for sharing the gain has to be maintained throughout the cooperation duration (see Yeung and Petrosyan (2004 and 2016) [19, 20]) for a dynamically consistent solution. Applying the Shapley value imputation in (18) to any time instance $t \in [t_0, T]$, we obtain:

$$Sh_i(\bar{x}(t), T - t) = \sum_{S \subseteq N \atop S \ni i} \frac{(|S| - 1)! (n - |S|)!}{n!} \left[ \sum_{j \in K(i) \cap S} \alpha_{ij}(\bar{x}(t), T - t) + \alpha_{ji}(\bar{x}(t), T - t) + q_i(\bar{x}^i(T)) \right] \times \left( \sum_{j \in K(i) \cap S} \alpha_{ij}(\bar{x}(t), T - t) + \sum_{j \in K(i) \cap S} \alpha_{ji}(\bar{x}(t), T - t) + q_i(\bar{x}^i(T)) \right), \quad (19)$$

for $t \in [t_0, T]$.

The Shapley value imputation in (18)-(19) is based on characteristic function evaluates along the optimal cooperative trajectory and it attributes the contributions of the players under the optimal cooperation process. Indeed, it can be regarded as optimal trajectory dynamic Shapley value. In addition, this Shapley value imputation (18)-(19) fulfils the property of time consistency.

**Proposition 3.** The Shapley value imputation in (18)-(19) satisfies the time consistency property.

**Proof.** Substituting the values of $\alpha_{ij}(x_0, T - t_0)$ into (18), we obtain

$$Sh_i(x_0, T - t_0) = \sum_{S \subseteq N \atop S \ni i} \frac{(|S| - 1)! (n - |S|)!}{n!} \left[ \sum_{j \in K(i) \cap S} \int_{t_0}^{T} [h_i^j(\bar{x}^i(\tau), \bar{x}^i(\tau)), (\bar{u}^i(\tau) + h_j^j(\bar{x}^i(\tau), \bar{x}^i(\tau), \bar{u}^i(\tau))]d\tau + q_i(\bar{x}^i(T))] \right]$$

$$\times \sum_{j \in K(i) \cap S} \int_{t_0}^{T} [h_i^j(\bar{x}^i(\tau), \bar{x}^i(\tau), \bar{u}^i(\tau)) + h_j^j(\bar{x}^i(\tau), \bar{x}^i(\tau), \bar{u}^i(\tau))]d\tau$$

$$+ \sum_{S \subseteq N \atop S \ni i} \frac{(|S| - 1)! (n - |S|)!}{n!} \left[ \sum_{j \in K(i) \cap S} \int_{t}^{T} [h_i^j(\bar{x}^i(\tau), \bar{x}^i(\tau), \bar{u}^i(\tau))]d\tau \right]$$
Invoking (19), we have

$$\text{Sh}_i(x_0, T - t_0) = \sum_{S \subset N} \frac{(|S| - 1)!(|n - |S||)!}{n!} \times$$

$$\times \sum_{j \in K(i) \cap S} \int_{t_0}^{t} [h^j_i(\bar{x}^i(\tau), \bar{x}^j(\tau), \bar{u}^i(\tau)) + h^j_i(\bar{x}^j(\tau), \bar{x}^i(\tau), \bar{u}^j(\tau))]d\tau$$

$$+ \text{Sh}_i(\bar{x}(t), T - t), \; i \in N$$

which exhibits the time consistency property of the Shapley value imputation $\text{Sh}_i(\bar{x}(T), T - t)$, for $t \in [t_0, T]$. \hfill \Box

This is the first time that a Shapley value measure itself in a dynamic framework fulfils the property of time consistency (see existing dynamic Shapley value measures which do not share this property in Gromova (2016) [4], Petrosyan (1995)[11], Petrosyan and Zaccour (2003) [14], Yeung (2010) [18], Yeung and Petrosyan (2016 and 2018)) [20, 21]. Using this Shapley value formulation, the cooperative game solution would automatically satisfy the condition of cooperative time consistency (see Yeung and Petrosyan 2004 and 2016 [19, 20]). Crucial to the analysis is the design of an IDP such that the Shapley imputation (18)-(19) can be realized. This will be done in the next section.

3.2. Imputation distribution procedure. To derive an imputation procedure that leads to the realization of the optimal trajectory dynamic Shapley value in (18)-(19), we first introduce the “instantaneous characteristic function” introduced by Petrosian et al. (2016). Differentiating the characteristic function (9) with respect to $t$ yields

$$\sum_{i \in S} \sum_{j \in K(i) \cap S} h^j_i(\bar{x}^i(t), \bar{x}^j(t), \bar{u}^i(t)) = -\frac{d}{dt} V(S; \bar{x}(t), T - t).$$

(22)

Following Petrosian et al. (2016) [11] we define

$$W(S; \bar{x}(t), T - t) = \sum_{i \in S} \sum_{j \in K(i) \cap S} h^j_i(\bar{x}^i(t), \bar{x}^j(t), \bar{u}^i(t)), \; \text{for } S \subset N,$$  

(23)

as the “instantaneous characteristic function” of the game at time $t \in [t_0, T]$.

Following the proof of Proposition 1, we can obtain the inequality

$$W(S_1 \cup S_2; \bar{x}(t), T - t) \geq W(S_1; \bar{x}(t), T - t) +$$

$$+ W(S_2; \bar{x}(t), T - t) - W(S_1 \cap S_2; \bar{x}(t), T - t), \; \text{for } t \in [t_0, T].$$

(24)

We introduce the instantaneous core $C(\bar{x}(t), T - t)$ as the set of imputations

$$\xi(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_n(t))$$
such that
\[
\sum_{i=1}^{n} \xi_i(t) = W(N; \tilde{x}(t), T - t) \quad \text{and} \quad \sum_{i \in S} \xi_i(t) \geq W(S; \tilde{x}(t), T - t), \ i \in N \quad (25)
\]
for \( S \subset N, S \neq N \) and \( t \in [t_0, T] \).

Given the properties of \( W(S; \tilde{x}(t), T - t) \) in (24), the set \( C(\tilde{x}(t), T - t) \) is non-empty. An imputation distribution procedure (IDP) leading to the realization of the optimal trajectory Shapley value imputation (18)-(19) has to satisfy

\[
\int_{t_0}^{T} \xi_i(\tau) d\tau + q_i(x^i(T)) = Sh_i(x_0, T - t_0)
\]

and

\[
\int_{t}^{T} \xi_i(\tau) d\tau + q_i(x^i(T)) = Sh_i(\tilde{x}(t), T - t), \ \text{for} \ t \in [t_0, T].
\]

In the theorem below, we derive an IDP which leads to the realization of the optimal trajectory dynamic Shapley value imputation (18)-(19).

**Theorem 3.1.** An imputation distribution procedure (IDP) giving player \( i \in N \) at time \( t \in [t_0, T] \) an allotment

\[
\beta_i(t) = \sum_{\substack{S \subset N \ \text{with} \ S \ni i \ \text{and} \ S \supset i}} \frac{(|S| - 1)! (n - |S|)!}{n!} \times \left( \sum_{j \in K(i) \cap S} \int_{t_0}^{T} h^j_i([\bar{x}(\tau), \bar{x}^i(\tau), \bar{u}^i(\tau)] + h^j_i(\bar{x}(\tau), \bar{x}^i(\tau), \bar{u}^i(\tau))] d\tau + q_i(x^i(T)) \right)
\]

would lead to the realization of the Shapley value imputation in (18)-(19).

**Proof.** Using the IDP in (26), we obtain

\[
\int_{t_0}^{T} \beta_i(\tau) d\tau + q_i(x^i(T)) = \sum_{\substack{S \subset N \ \text{with} \ S \ni i \ \text{and} \ S \supset i}} \frac{(|S| - 1)! (n - |S|)!}{n!} \times \left( \sum_{j \in K(i) \cap S} \int_{t_0}^{T} h^j_i([\bar{x}(\tau), \bar{x}^i(\tau), \bar{u}^i(\tau)] + h^j_i(\bar{x}(\tau), \bar{x}^i(\tau), \bar{u}^i(\tau))] d\tau + q_i(x^i(T)) \right)
\]

\[
= Sh_i(x_0, T - t_0), \ i \in N
\]

and

\[
\int_{t}^{T} \beta_i(\tau) d\tau + q_i(x^i(T))
\]

\[
\sum_{\substack{S \subset N \ \text{with} \ S \ni i \ \text{and} \ S \supset i}} \frac{(|S| - 1)! (n - |S|)!}{n!} \left( \sum_{j \in K(i) \cap S} \int_{t_0}^{T} h^j_i([\bar{x}(\tau), \bar{x}^i(\tau), \bar{u}^i(\tau)] + h^j_i(\bar{x}(\tau), \bar{x}^i(\tau), \bar{u}^i(\tau))] d\tau + q_i(x^i(T)) \right) = Sh_i(\tilde{x}(t), T - t), \ i \in N
\]

(27)
and

\[ S_{0i}(x_0, T - t_0) = \int_{t_0}^{t} \beta_i(\tau) d\tau + S_{ti}(\bar{x}(t), T - t), \quad \text{for } t \in [t_0, T], \quad i \in N. \quad (29) \]

Hence, Theorem 3.1 follows. \( \square \)

One can readily verify that the imputation distribution procedure
\[ \beta(t) = (\beta_1(t), \beta_2(t), \ldots, \beta_n(t)) \]
in (26) satisfies

\[ \sum_{i=1}^{n} \beta_i(t) = W(N; \bar{x}(t), T - t) \quad \text{and} \quad \sum_{i \in S} \beta_i(t) \geq W(S; \bar{x}(t), T - t), \quad i \in N, \quad (30) \]

for \( S \subset N \), \( S \neq N \) and \( t \in [t_0, T] \). Hence, \( \beta(t) \) belongs to the instantaneous core \( C(\bar{x}(t), T - t) \).

4. An application in regional cooperation. We consider an application of the above analysis to cooperation in regional development in a network of \( n \) regions. Let \( x^i(\tau) \) denote the capital resources of region \( i \), for \( i \in N \). The payoff of region \( i \) is positively related to its capital resources and the capital resources of regions linked in the region network. In particular, the payoff of region \( i \) is

\[ H_i(x^i_{0}, x^{K(i)}_0, u^i) = \sum_{j \in K(i)} \int_{t_0}^{T} [a^{i(j)}x^{j}(\tau) + b^{i(j)}x^{j}(\tau) - w^{i(j)}c(u^i(\tau))^2] d\tau + q_ix^i(T), \quad (31) \]

for \( i \in N \), where \( u^i(\tau) \) is the investment of capital resources by region \( i \), \( a^{i(j)}x^{j}(\tau) + b^{i(j)}x^{j}(\tau) \) is the gain region \( i \) obtains through interaction with region \( j \), \( c(u^i(\tau))^2 \) is the cost of capital investment attributed to the interaction between region \( i \) and region \( j \), the terminal payoff of region \( i \) is \( q_ix^i(T) \), and \( c, a^{i(j)}, b^{i(j)}, w^{i(j)}, q_i \geq 0 \).

In particular, \( w^{i(i)} = 1 \) and \( w^{i(j)} = 0 \), for \( j \in K(i) \). This specification shows that the gain region \( i \) obtains by itself without interaction with other regions would bear the full cost of capital investment. The capital resources accumulation dynamics of region \( i \) is

\[ \dot{x}^i(\tau) = u^i(\tau) - \delta^i x^i(\tau), \quad x^i(t_0) = x^i_0, \quad \text{for } i \in N \quad \text{and} \quad \tau \in [t_0, T]. \quad (32) \]

We use \( x(t) \) to denote the vector \( (x^1(t), x^2(t), \ldots, x^n(t)) \) and \( V(N; x, T - t) \) to denote the maximal value of the grand coalition payoff at time \( t \in [t_0, T] \) that solves the problem

\[
\max_{u^1, u^2, \ldots, u^n} \left\{ \sum_{i \in N} \left( \sum_{j \in K(i)} \int_{t}^{T} [a^{i(j)}x^{j}(\tau) + b^{i(j)}x^{j}(\tau) - w^{i(j)}c(u^i(\tau))^2] d\tau + q_i x^i(T) \right) \right\}
\]

subject to capital accumulation dynamics (32).

The solution to the dynamic optimization problem (32)-(33) can be solved by optimal control or by Bellman’s dynamic programming techniques. To derive the control strategies explicitly in a relatively more direct way, we adopt the dynamic
programming method. The payoff of the grand coalition \( V(N; x, T - t) \) can be characterized as follows:

\[
V(N; x, 0) = \sum_{i=1}^{n} q_i x^i
\]

\[-V_t(N; x, T - t) = \max_{u^1, u^2, \ldots, u^n} \left\{ \sum_{i \in N} \sum_{j \in K(i)} \left[ a^{(i)} x^i + b^{(i)} x^j - w^{(i)} c(u^i)^2 \right] + \sum_{i \in N} V_{x^i}(N; x, T - t)(u^i - \delta^i x^i) \right\}, \text{ for } t \in [t_0, T]. \tag{35}\]

Performing the maximization operator in (35) yields:

\[
u^i = \frac{V_{x^i}(N; x, T - t)}{2c}, \text{ for } i \in N. \tag{36}\]

**Proposition 4.** The value function representing the cooperative joint payoff of all the \( n \) regions can be obtained as

\[
V(N; x, T - t) = \sum_{i \in N} A_i(t) x^i + C(t) \text{ for } t \in [t_0, T] \tag{37}\]

where

\[
A_i(t) = \frac{1}{\delta^i} \left[ \delta^i q_i - \sum_{j \in K(i)} a^{(j)} - \sum_{j \in K(i)} b^{(j)} \right] \exp[\delta^i(t - T)] + \frac{1}{\delta^i} \left[ \sum_{j \in K(i)} a^{(j)} + \sum_{j \in K(i)} b^{(j)} \right]
\]

and

\[
C(t) = \int_{t}^{T} \sum_{i \in N} \left( \frac{A_i(\tau)}{4c} \right)^2 d\tau, \text{ for } i \in N.
\]

**Proof.** See Appendix B. \( \square \)

Using Proposition 5, the optimal cooperative strategies in (36) can be expressed as

\[
\bar{u}^i(t) = \frac{A_i(t)}{2c}, \text{ for } i \in N \text{ and } t \in [t_0, T]. \tag{38}\]

Substituting (38) into (32), one can obtain the dynamics of the cooperative trajectories as

\[
x^i(t) = \frac{A_i(t)}{2c} - \delta^i x^i(t), x^i(t_0) = x^i_0, \text{ for } i \in N \text{ and } t \in [t_0, T]. \tag{39}\]

System (39) is a system of independent first order linear equations which solution can be obtained by standard techniques. We use \( \{\bar{x}(t)\}_{t=t_0}^{T} \) to denote the solution of (39). The cooperative-trajectory characteristic function reflecting the worth of coalition \( S \) becomes

\[
V(S; \bar{x}(t), T - t) = \sum_{i \in S} \left| \sum_{j \in K(i) \cap S} \int_{t}^{T} \left( a^{(j)} \bar{x}^j(\tau) + b^{(j)} \bar{x}^j(\tau) - w^{(j)} \frac{[A_i(\tau)]^2}{4c} \right) d\tau + q_i \bar{x}^i(T) \right|
\]
Similarly, for subgames along the cooperative trajectory we can obtain the Shapley value imputation to player $i$ as

\[ Sh_i(\bar{x}(t), T - t) = \sum_{S \subset N} \frac{(|S| - 1)! (n - |S|)!}{n!} \times \]

\[ \int_t^T \left( a^{i(j)}(\tau) + b^{i(j)}(\bar{x}(\tau) - w^{i(j)} \frac{[A_i(\tau)]^2}{4c} \right) d\tau + q_i \bar{x}(T) \right), \quad i \in N. \]
+ \int_t^T \left( a^{(i)} \tilde{x}^i(\tau) + b^{(i)} \bar{x}^j(\tau) - w^{(i)} \frac{[A_i(\tau)]^2}{4c} \right) d\tau + q_i \bar{x}^i(T) \right), \ i \in N. \quad (44)

The time consistency property of the Shapley value is clearly demonstrated by

\[ Sh_i(x_0, T - t_0) = \sum_{S \subset N, S \ni i} \frac{(|S| - 1)!(n - |S|)!}{n!} \times \]

\[ \times \sum_{j \in K(i) \cap S} \int_{t_0}^T \left( a^{(j)} \tilde{x}^i(\tau) + b^{(j)} \bar{x}^j(\tau) - w^{(j)} \frac{[A_i(\tau)]^2}{4c} \right) d\tau \]

\[ + Sh_i(\bar{x}(t), T - t), \ i \in N. \quad (45) \]

Differentiating the cooperative-trajectory characteristic function in (40) with respect to \( t \), we obtain

\[ \sum_{i \in S} \sum_{j \in K(i) \cap S} \left( a^{(j)} \tilde{x}^i(t) + b^{(j)} \bar{x}^j(t) - w^{(j)} \frac{[A_i(t)]^2}{4c} \right) = - \frac{d}{dt} V(S; \bar{x}(t), T - t). \quad (46) \]

Then, we obtain the instantaneous characteristic function

\[ W(S; \bar{x}(t), T - t) = \sum_{i \in S} \sum_{j \in K(i) \cap S} \left( a^{(j)} \tilde{x}^i(t) + b^{(j)} \bar{x}^j(t) - w^{(j)} \frac{[A_i(t)]^2}{4c} \right). \quad (47) \]

Invoking Theorem 3.1, if we adopt an IDP

\[ \beta_i(t) = \]

\[ \sum_{S \subset N, S \ni i} \frac{(|S| - 1)!(n - |S|)!}{n!} \sum_{i \in S} \sum_{j \in K(i) \cap S} \left( a^{(j)} \tilde{x}^i(t) + b^{(j)} \bar{x}^j(t) - w^{(j)} \frac{[A_i(t)]^2}{4c} \right), \]

\[ \quad (48) \]

for \( \beta_i(t) \geq 0 \) and \( t \in [t_0, T] \), we obtain the Shapley value

\[ Sh_i(x_0, T - t_0) = \int_{t_0}^T \beta_i(\tau)d\tau + q_i \bar{x}^i(T) = \int_{t_0}^T \beta_i(\tau)d\tau + Sh_i(\bar{x}(t), T - t), \quad (49) \]

which is time-consistent.

Finally, one can readily demonstrate

\[ \sum_{i=1}^n \beta_i(t) = W(N; \bar{x}(t), T - t) \text{ and } \sum_{i \in S} \beta_i(t) \geq W(S; \bar{x}(t), T - t). \quad (50) \]

5. Conclusions. This paper presents a time-consistent dynamic Shapley value imputation for a class of differential network games. A novel form for measuring the worth of coalitions – named as cooperative-trajectory characteristic function – for the Shapley value imputation is developed. In computing this type of characteristic function, we evaluate contributions of the players in the process of cooperation and maintain the cooperative strategies for all players along the cooperative trajectory. The new features of the cooperative-trajectory characteristic function include (i) the worth of coalitions is derived in the process of cooperation along the cooperative trajectory, (ii) the marginal contributions of individual players are evaluated...
based on their cooperative actions/strategies, (iii) the cooperative-trajectory character-
stic function itself is time consistent. Using this characteristic function, an
optimal trajectory dynamic Shapley value solution is derived. Worth-noting is the
resultant Shapley value itself fulfills the property of time consistency. This is the
first time that cooperative-trajectory characteristic function is used to derive the
Shapley value in network differential games, further fruitful research along this line
is expected.

Another important point is the applicability of proposed solutions based on newly
deﬁned characteristic function to wide class of dynamic real-life game-theoretic
problems. This is because of simple method used for computation of character-
stic function and the Shapley value. It is sufﬁcient to make one maximization
operation for computation of maximal joint payoff of players and corresponding co-
operative trajectory. The only limitation is connected with the level of complexity
of the maximization problem (7) but this type of problems are classical in the mod-
ern control theory and there is a wide variety of effective methods to solve them
dynamic programming, Pontryagin maximum principle, variational methods).

Appendix A: Proof of Proposition 1.
Using (11), we have
\[
V(S_1 \cup S_2; x_0, T - t_0) = \sum_{i \in S_1} \sum_{j \in K(i) \cap (S_1 \cup S_2)} a_{ij}(x_0, T - t_0) = \\
= \sum_{i \in S_1} \sum_{j \in K(i) \cap S_1} a_{ij}(x_0, T - t_0) + \sum_{i \in S_2} \sum_{j \in K(i) \cap S_2} a_{ij}(x_0, T - t_0) - \\
- \sum_{i \in S_1} \sum_{j \in K(i) \cap S_1} a_{ij}(x_0, T - t_0) \\
+ \sum_{i \in S_2} \sum_{j \in K(i) \cap S_2} a_{ij}(x_0, T - t_0) + \sum_{i \in S_1} \sum_{j \in K(i) \cap S_1} a_{ij}(x_0, T - t_0) \\
\geq \sum_{i \in S_1} \sum_{j \in K(i) \cap S_1} a_{ij}(x_0, T - t_0) + \sum_{i \in S_2} \sum_{j \in K(i) \cap S_2} a_{ij}(x_0, T - t_0) - \\
- \sum_{i \in S_1} \sum_{j \in K(i) \cap S_2} a_{ij}(x_0, T - t_0) \\
= V(S_1; x_0, T - t_0) + V(S_2; x_0, T - t_0) - V(S_1 \cap S_2; x_0, T - t_0). \quad (A.1)
\]
Following the above analysis, one can show that
\[
V(S_1 \cup S_2; \bar{x}(t), T - t) \geq V(S_1; \bar{x}(t), T - t) + V(S_2; \bar{x}(t), T - t) \\
- V(S_1 \cap S_2; \bar{x}(t), T - t), \text{ for } t \in [t_0, T]. \quad (A.2)
\]
Hence Proposition 1 follows.

Appendix B: Proof of Proposition 5 Using (37) and substituting (36) into (35)
yields
\[
- \sum_{i \in N} \dot{A}_i(t)x^i + \dot{C}(t) = \left\{ \sum_{i \in N} \sum_{j \in K(i)} \left[ a^{ij}x^i + b^{ij}x^j - c^{ij} \left( \frac{A_i(t)}{2c} \right)^2 \right] \\
+ \sum_{i \in N} A_i(t) \left( \frac{A_i(t)}{2c} - \delta^ix^i \right) \right\}, \text{ for } t \in [t_0, T]. \quad (B.1)
\]
Note that both the right-hand-side and the left-hand-side of (B.1) are linear functions in \(x^1, x^2, \cdots, x^n\). Grouping terms, we have

\[
\dot{A}_i(t)x^i = -\sum_{j \in K(i)} a^{i(j)}x^j - \sum_{j \in \tilde{K}(i)} b^{i(j)}x^j + A_i(t)\delta^i, \text{ for } i \in N, \quad (B.2)
\]

and

\[
\dot{C}(t) = \sum_{i \in N} \sum_{j \in K(i)} w^{i(j)}c \left( \frac{A_i(t)}{2c} \right)^2 - \sum_{i \in N} \frac{[A_i(t)]^2}{2c} = -\sum_{i \in N} \frac{[A_i(t)]^2}{4c}. \quad (B.3)
\]

Using (34) and (37), we obtain

\[
\sum_{i \in N} A_i(T)x^i + C(T) = \sum_{i=1}^n q_i x^i(T). \quad (B.4)
\]

The condition in (B.4) implies

\[
A_i(T) = q_i \text{ and } C(T) = 0. \quad (B.5)
\]

For (B.2) to hold, it is required that

\[
\dot{A}_i(t) = A_i(t)\delta^i - \sum_{j \in K(i)} a^{i(j)} - \sum_{j \in \tilde{K}(i)} b^{i(j)}, A_i(T) = q_i, \text{ for } i \in N. \quad (B.6)
\]

Equation (B.6) is a system of \(n\) independent first order linear differential equations which solution can be obtained readily as:

\[
A_i(t) = \frac{1}{\delta^i} \left[ \delta^i q_i - \sum_{j \in K(i)} a^{i(j)} - \sum_{j \in \tilde{K}(i)} b^{i(j)} \right] \exp[\delta^i(t - T)] + \frac{1}{\delta^i} \left[ \sum_{j \in K(i)} a^{i(j)} + \sum_{j \in \tilde{K}(i)} b^{i(j)} \right]. \quad (B.7)
\]

With the boundary conditions being \(C(T) = 0\) and \(A_i(t)\) from (B.7), the solution to the differential equation system (B.3) can be expressed as

\[
C(t) = \int_t^T \sum_{i \in N} \frac{[A_i(\tau)]^2}{4c} d\tau, \text{ for } i \in N. \quad (B.8)
\]

Hence Proposition 5.

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E-mail address: l.petrosyan@spbu.ru

E-mail address: dwkyeung@hksyu.edu