SOME REMARKS ON $K_0$ OF NONCOMMUTATIVE TORI

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ABSTRACT

Using Rieffel’s construction of projective modules over higher dimensional noncommutative tori, we construct projective modules over some continuous field of C*-algebras whose fibers are noncommutative tori. Using a result of Echterhoff et al., our construction gives generators of $K_0$ of all noncommutative tori.

1 INTRODUCTION

The noncommutative 2-tori are one of the central objects in noncommutative geometry. They are amongst the most studied examples in the field. Their higher dimensional analogue on the other hand is not so well studied. Recall that the $n$-dimensional noncommutative torus $A_\theta$ is the universal C*-algebra generated by unitaries $U_1, U_2, U_3, \cdots, U_n$ subject to the relations

$$U_kU_j = e^{(2\pi i \theta_{jk})}U_jU_k$$

for $j, k = 1, 2, 3, \cdots, n$, where $\theta := (\theta_{jk})$ is a real skew symmetric $n \times n$ matrix. We call a skew-symmetric matrix totally irrational if the off-diagonal entries are rationally linearly independent and not rational.

Rieffel in [Rie88] had constructed projective modules over $n$-dimensional noncommutative tori while Elliott in [Ell84] computed the K-theory of these algebras. Elliott showed that the K-theory of $n$-dimensional noncommutative tori is independent of the parameter $\theta$ and also he computed the image of the canonical trace of $A_\theta$. It follows from Elliott’s computations that for totally irrational $\theta$ the canonical trace on $A_\theta$ is injective as a map from $K_0(A_\theta)$ to $\mathbb{R}$. So using the description of the image of the trace and Rieffel’s ([Rie88]) computations of traces of projective modules, we can compute a basis of $K_0$ for $A_\theta$ in the case where $\theta$ is totally irrational.

Based on the results of Rieffel and Elliott, Echterhoff, Lück, Phillips and Walters constructed in [Ech10] a continuous field of projective modules over the parameter space (certain space which consists of $2 \times 2$ skew symmetric matrices) of 2-dimensional noncommutative tori. Along with other results, the authors gave a basis of $K_0(A_\theta)$ using the range of the trace of 2-dimensional noncommutative tori. They also showed how this basis could

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be extended to provide elements of the basis of K-theory of some crossed products \( A_0 \times F \), where \( F \) is a finite cyclic group which is compatible with \( \theta \).

We take the similar approach of [Ech10] to provide bases for \( K_0(A_0) \) for any higher dimensional noncommutative tori.

This article is organised as follows:

In the second section of this article we recall some basics of groupoids, twisted groupoid C*-algebras and their K-theory. Though, in [Ech10], the authors didn’t work with groupoids, the groupoid approach turns out to be useful to understand the construction of their continuous field.

In the third section we construct the continuous field of projective modules and prove our main theorem.

In the fourth section we show how our theorem could be used to give explicit generators of the \( K_0 \) of the noncommutative tori: we work out the three and four dimensional cases in detail.

Notation: In the rest of the article \( e(x) \) denotes the number \( e^{2\pi ix} \).

2 TWISTED GROUPOID ALGEBRAS AND THEIR K-THEORY

We assume that the reader is familiar with basic notions of locally compact Hausdorff groupoids. We refer to the book of Renault [Ren80] for a basic course on groupoids and twisted representations of those. To introduce the notations we recall the definition of 2-cocycle on a groupoid.

**Definition 2.1.** Let \( G \) be a locally compact Hausdorff groupoid. A continuous map \( \omega : G^{(2)} \to \mathbb{T} \) is called a 2-cocycle if

\[
\omega(x,y)\omega(xy,z) = \omega(x,yz)\omega(y,z),
\]

whenever \((x,y),(y,z) \in G^{(2)}\) and

\[
\omega(x,d(x)) = 1 = \omega(t(x),x),
\]

for any \( x \in G \), where \( G^{(2)} \) denotes the composable pairs of \( G \) and \( d, t \) denote the domain and the range map respectively.

**Definition 2.2.** The C*-algebra \( C^*(G,\omega) \) is defined to be the enveloping C*-algebra of \( \omega \)-twisted left regular representation of the groupoid \( G \) (which is assumed to have Haar system).

**Example 2.3.** Let \( G \) be the group (hence groupoid) \( \mathbb{Z}^n \). For each \( n \times n \) real antisymmetric matrix \( \theta \), we can construct a 2-cocycle on this group by defining \( \omega_0(x,y) = e((x \cdot \theta y)/2) \). The corresponding group C*-algebra \( C^*(G,\omega_0) \) is easily seen to be isomorphic to the \( n \)-dimensional noncommutative torus as defined in the beginning of this article.

Two 2-cocycles \( \omega_1 \) and \( \omega_2 \) of a discrete group \( G \) are called cohomologus if there exists a map \( f : G \to \mathbb{T} \) such that

\[
\omega_1(x,y) = f(x)f(y)f(xy)\omega_2(x,y), \quad x,y \in G.
\]

This defines an equivalence relation on the set of 2-cocycles of \( G \). Let us denote the equivalence classes of 2-cocycles of \( G \) by \( H^2(G,\mathbb{T}) \). It is well known that \( H^2(\mathbb{Z}^n,\mathbb{T}) = \mathbb{T}^{n(n+1)/2} \).
Let $[a, b]$ be a closed interval. Let us consider the transformation groupoid $\mathbb{Z}^n \times [a, b]$ for trivial $\mathbb{Z}^n$ action on $[a, b]$. Suppose $\omega_r$ be a continuous family (with respect to $r \in [a, b]$) of 2-cocycles on the group $\mathbb{Z}^n$ (note that the continuity makes sense in this context). We define the following 2-cocycle $\omega$ on the groupoid $\mathbb{Z}^n \times [a, b]$: $\omega(x, y, r) = \omega_r(x, y)$, when $x, y$ belong to the $r$-fiber, and when $x, y$ do not belong to same $r$-fiber, $\omega(x, y, r)$ is defined to be zero. Then we have the following evaluation map

$$ev_r : C^*(\mathbb{Z}^n \times [a, b], \omega) \to C^*(\mathbb{Z}^n, \omega), \quad r \in [a, b].$$

The following theorem is due to Echterhoff et al. [Ech10].

**Theorem 2.4.** Let $[p_1], [p_2], \ldots, [p_m] \in K_0(C^*(\mathbb{Z}^n \times [a, b], \omega))$. Then the following are equivalent:

1. $[p_1], [p_2], \ldots, [p_m]$ form a basis of $K_0(C^*(\mathbb{Z}^n \times [a, b], \omega))$.
2. For some $r \in [a, b]$, the evaluated classes $[ev_r(p_1)], [ev_r(p_2)], \ldots, [ev_r(p_m)]$ form a basis of $K_0(C^*(\mathbb{Z}^n, \omega_r))$.
3. For every $r \in [a, b]$, the evaluated classes $[ev_r(p_1)], [ev_r(p_2)], \ldots, [ev_r(p_m)]$ form a basis of $K_0(C^*(\mathbb{Z}^n, \omega_r))$.

**Proof.** See remark 2.3 of [Ech10].

## 3 PROJECTIVE MODULES OVER BUNDLES OF NON-COMMUTATIVE TORI

As the pfaffian of an even dimensional skew symmetric matrix will play a central role in the construction of our continuous field, we recall the definition of the pfaffian.

**Definition 3.1.** The pfaffian of a $2p \times 2p$ skew symmetric matrix $A := (a_{ij})$ is a polynomial, denoted by $\text{pf}(A)$, in the entries $a_{ij}$ ($i < j$) such that $\text{pf}(A)^2 = \det A$ and $\text{pf}(I_0) = 1$, where $I_0$ is the block diagonal matrix constructed from $p$ identical $2 \times 2$ blocks

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It can be shown that $\text{pf}(A)$ always exists and is unique. To give some examples,

$$\text{pf} \begin{pmatrix} 0 & \theta_{12} \\ -\theta_{12} & 0 \end{pmatrix} = \theta_{12},$$

$$\text{pf} \begin{pmatrix} 0 & \theta_{12} & \theta_{13} & \theta_{14} \\ -\theta_{12} & 0 & \theta_{23} & \theta_{24} \\ -\theta_{13} & -\theta_{23} & 0 & \theta_{34} \\ -\theta_{14} & -\theta_{24} & -\theta_{34} & 0 \end{pmatrix} = \theta_{12}\theta_{34} - \theta_{13}\theta_{24} + \theta_{14}\theta_{23}.$$

Let us fix a number $n$. Also let $n = 2p + q$ for some $p$ and $q \in \mathbb{Z}_{\geq 0}$. Recall that a skew-symmetric matrix is totally irrational if the off diagonal entries are rationally linearly independent and not rational. Let us fix any totally irrational $n \times n$ skew symmetric matrix $\psi := \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$ with the top upper $2p \times 2p$ left corner $\psi_{11}$ having positive pfaffian. Also let
θ := \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} be any n × n skew-symmetric matrix such that it has similar properties as ψ, i.e. θ_{11}, the left 2p × 2p corner, has positive pfaffian.

Let I := [0, 1] and choose a path γ parametrised by I from ψ to θ in the set of n × n antisymmetric matrices, where ψ_{11} and θ_{11} are connected by a path γ_{11} in the space of 2p × 2p antisymmetric matrices with positive pfaffian. Since the latter is path connected (the space being isomorphic to GL_{2p}(\mathbb{R})/SP(2p, \mathbb{R})) the choice is always possible. The matrices ψ_{12} and θ_{12}, ψ_{21} and θ_{21}, ψ_{22} and θ_{22} are connected by straight line homotopies, which will be denoted by γ_{12}, γ_{21} and γ_{22}, respectively.

For r ∈ I, we have

\[ γ(r) = \begin{pmatrix} γ(r)_{11} & γ(r)_{12} \\ γ(r)_{21} & γ(r)_{22} \end{pmatrix} \]

(notice that γ(r)_{11} is the 2p × 2p block). Let Z^n × I be the transformation groupoid with the action of Z^n on I being trivial. We will construct a 2-cocycle on this groupoid. Fibre-wise, the C^*-algebra of this twisted groupoid algebra will be just the n-dimensional noncommutative tori with parameter γ. Define Ω(x, y, r) = e((x · γ(r)y)/2), when x, y are in r-fibre (r ∈ I), and if x, y belong to different fibres, then Ω(x, y, r) is defined to be zero. We will use the same approach of Rieffel [Rie88] to construct finitely generated projective C^*(Z^n × I, Ω,I) modules, which represent a suitable class (E_r) of projective module over C^*(Z^n, Ω_I) := C^*(Z^n, γ(r)), for each r ∈ I. To do this, we recall some constructions by Rieffel and Schwartz from [Lio33]. Define a new cocycle on the groupoid Ω^−1 by setting Ω^−1(x, y, r) = e((γ(r)'x · y)/2), where

\[ γ(r)' = \begin{pmatrix} γ(r)^{-1}_{11} & -γ(r)^{-1}_{12} \\ γ(r)_{21}γ(r)^{-1}_{11} & γ(r)_{22} - γ(r)_{21}γ(r)^{-1}_{11}γ(r)_{12} \end{pmatrix}, \]

when x, y are in r-fibre, otherwise we define Ω^−1(x, y, r) = 0. Set A = C^*(Z^n × I, Ω) and B = C^*(Z^n × I, Ω^−1). Then the fibre B_r of B, at r ∈ I, is the rotation algebra C^*(Z^n, −γ(r)'). Let M be the space \mathbb{R}P \times Z^n, G := M × \hat{M} and (·, ·) the natural pairing between M and \hat{M}. Consider the space E^∞ := \mathcal{S}(M, I) consisting of all complex functions on M × I which are smooth and rapidly decreasing in the first variable and continuous in the second variable in each derivative of the first variable. Denote the set of rapidly decreasing C(I)-valued functions on Z^n by A^∞ = \mathcal{S}(Z^n × I, Ω), viewed as a (dense) subalgebra of C^*(Z^n × I, Ω), and by B^∞ = \mathcal{S}(Z^n × I, Ω^−1), viewed as a (dense) subalgebra of C^*(Z^n × P, Ω^−1), which is constructed similarly.

Following Li [Lio33], we have the main theorem:

**Theorem 3.2.** E^∞ may be given an A^∞-B^∞ Morita equivalence bimodule structure, which can be extended to a strong Morita equivalence between A and B.

**Proof.** Following [Lio33], let

\[ T(r) = \begin{pmatrix} T(r)_{11} & 0 \\ 0 & I_q \\ T(r)_{31} & T(r)_{32} \end{pmatrix}, \]

where T(r)_{11} is a continuous family (with respect to r) of invertible matrices such that T(r)_{11} J_0 T(r)_{11} = γ(r)_{11}, J_0 := \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}, T(r)_{31} = γ(r)_{21} and T(r)_{32} is the matrix obtained from γ(r)_{22} by replacing the lower diagonal entries by zero.
We also define
\[
S(r) = \begin{pmatrix}
J_0(T(r)_{11})^{-1} & -J_0(T(r)_{11})^{-1}T(r)_{31} \\
0 & I_q \\
0 & T(r)_{32}
\end{pmatrix}.
\]

Let
\[
J = \begin{pmatrix}
J_0 & 0 & 0 \\
0 & 0 & I_q \\
0 & -I_q & 0
\end{pmatrix}
\]
and \(J'\) be the matrix obtained from \(J\) by replacing each negative entry of it by zero. Let us now denote the matrix \(T(r)\) by \(T_r\) and \(S(r)\) by \(S_r\). Note that \(T_r\) and \(S_r\) can be thought as maps \(\mathbb{R}^p \times \mathbb{R}^p \times \mathbb{Z}^n \to G\) (see the definition of embedding map in [Lio93, 2.1]). Let \(P'\) and \(P''\) be the canonical projections of \(G\) to \(M\) and \(\hat{M}\) respectively and \(T'_r\), \(T''_r\) be the maps \(P' \circ T_r\) and \(P'' \circ T_r\), respectively. Similarly we define \(S'_r\) and \(S''_r\) as \(P' \circ S_r\) and \(P'' \circ S_r\), respectively. Then the following formulas define an \(A^\infty-\mathcal{B}^\infty\) bimodule structure on \(E^\infty\):
\[
(fU^1_0)(x, r) = e(-T_r(1) \cdot J'T_r(1)/2) \langle x, T''_r(1) \rangle f(x - T'_r(1), r),
\]  
(2)
\[
\langle f, g \rangle_{A^\infty}(1) = e(-T_r(1) \cdot J'T_r(1)/2) \int_{\mathcal{G}} \langle x, -T''_r(1) \rangle g(x + T'_r(1), r) f(x, r) dx,
\]  
(3)
\[
(V_0^1f)(x, r) = e(-S_r(1) \cdot J'S_r(1)/2) \langle x, -S''_r(1) \rangle f(x + S'_r(1), r),
\]  
(4)
\[
\mathcal{B}^\infty \langle f, g \rangle(1) = e(S_r(1) \cdot J'S_r(1)/2) \int_{\mathcal{G}} \langle x, S''_r(1) \rangle g(x + S'_r(1), r) f(x, r) dx,
\]  
(5)

where \(l \in \mathbb{Z}^n\).

Using the proposition 2.2 of [Lio93] and the continuity of the families \(T_r\) and \(S_r\), the result follows. Completing the space \(E^\infty\) with respect to the defined inner products, we get an \(A-\mathcal{B}\) Morita equivalence bimodule. \(\square\)

If we denote the completion of \(E^\infty\) with respect to the inner product by \(\tilde{E}\), the fibre-wise Morita equivalence \(\tilde{E}_r\) is just the Morita equivalence between \(A_{\tilde{\gamma}(r)}\) and \(A_{-\tilde{\gamma}(r)}\) which Rieffel [Rie88] had considered. Since both \(\mathcal{B}\) and \(\mathcal{A}\) are unital, \(\tilde{E}\) is a finitely generated projective \(\mathcal{A}\)-module with respect to the given action of \(\mathcal{A}\) on \(\tilde{E}\) (see the argument before proposition 4.6 in [Ech10]). The trace of this module \(\tilde{E}_r\), which was originally computed by Rieffel [Rie88, proposition 4.3, page 289], can be shown to be exactly (up to a sign) the pfaffian of the upper left \(2p \times 2p\) corner of the matrix \(\gamma(r)\).

4 GENERATORS OF K_0 GROUPS OF NONCOMMUTATIVE TORI

From Elliott’s computation of image of traces for noncommutative tori and the fact that the trace \(\text{Tr} : K_0(A_0) \to \mathbb{R}\) is injective for totally irrational \(\theta\), we can use the main theorem and 2.4 to compute explicit generators of \(K_0(A_0)\) for all \(\theta\). We will explain the 3-dimensional and 4-dimensional cases.
in details and the \( n \geq 5 \) case will be just simple extrapolation of these two examples.

We recall the following facts which will play the key role. These facts are due to Elliott (taken from [EL08]):

**Lemma 4.1.** \( \text{Tr}(K_0(A_0)) \) is the subgroup of \( \mathbb{R} \) generated by 1 and the numbers \( \sum \pm (-1)^{[\xi]} \prod_{s=1}^{m} \theta_{i_s(i_{s-1})} \theta_{i_2(i_{s-2})} \) for \( 1 \leq j_1 < j_2 < \cdots < j_{2m} \leq n \), where the sum is taken over all elements \( \xi \) of the permutation group \( S_{2m} \) such that \( \xi(2s-1) < \xi(2s) \) for all \( 1 \leq s \leq m \) and \( \xi(1) < \xi(3) < \cdots < \xi(2m-1) \).

**Lemma 4.2.** \( \text{Tr} \) is injective for totally irrational \( \theta \).

Before going to explicit computations, we shall say some words about the pfaffian of an \( n \times n \) skew symmetric matrix \( A := (a_{ij}) \). Let \( 1 \leq l \leq \lfloor \frac{n}{2} \rfloor \).

**Definition 4.3.** A \( 2l \)-pfaffian minor (or just pfaffian minor) \( M^A_{2l} \) of a skew symmetric matrix \( A \) is the pfaffian of a submatrix of \( A \) consisting of rows and columns indexed by \( i_1, i_2, \ldots, i_{2l} \) for some \( i_1 < i_2 < \cdots < i_{2l} \).

Note that the number of \( 2l \)-pfaffian minors is \( \binom{n}{2l} \) and the number of all pfaffian minors is \( 2^{n-1} \).

Let us consider the \( n \times n \) antisymmetric matrix \( Z \) whose all entries above the diagonal are 1:

\[
Z = \begin{pmatrix}
0 & 1 & \cdots & \cdots & 1 \\
-1 & \ddots & \ddots & & \\
\vdots & \ddots & \ddots & \ddots & \\
\vdots & & \ddots & \ddots & 1 \\
-1 & \cdots & \cdots & -1 & 0
\end{pmatrix}
\]

**Proposition 4.4.** For any skew symmetric \( n \times n \) matrix \( A := (a_{ij}) \), there exists some positive integer \( t \), such that all pfaffian minors of \( A + tZ \) are positive.

**Proof.** For fixed \( l \) with \( 1 \leq l \leq \lfloor \frac{n}{2} \rfloor \), it is easily seen that \( M^{A+tZ}_{2l} \) is a polynomial in \( t \) and

\[
M^{A+tZ}_{2l} = t^l + t^{l-1}A_{l-1} + t^{l-2}A_{l-2} + \cdots + t^1A_1 + A_0
\]

for polynomials \( A_{l-1}, A_{l-2}, \ldots, A_0 \) in entries of \( A := (a_{ij}) \). Now we can choose such integer \( t \) such that \( t^l \) dominates the other entries of \( M^{A+tZ}_{2l} \).

Since we have only a finite number of pfaffian minors, we can also choose such integer \( t \) such that \( M^{A+tZ}_{2l} > 0 \) for all \( l \).

4.1 The 3-dimensional case

Let

\[
\theta = \begin{pmatrix}
0 & \theta_{12} & \theta_{13} \\
-\theta_{12} & 0 & \theta_{23} \\
-\theta_{13} & -\theta_{23} & 0
\end{pmatrix}
\]

Using the above proposition, assume that the 2-pfaffian minors of \( A_0, \text{pf}(M^\theta_{ij}) \), are positive (indeed, \( A_0 + tZ \) is isomorphic to \( A_0 \) for any integer \( t \)), where

\[
M^\theta_{ij} = \begin{pmatrix}
0 & \theta_{ij} \\
-\theta_{ij} & 0
\end{pmatrix}, \quad j > i \geq 1.
\]
From the lemma 4.1, one has

\[ \text{Tr}(K_0(A_0)) = \mathbb{Z} + \theta_{12} \mathbb{Z} + \theta_{13} \mathbb{Z} + \theta_{23} \mathbb{Z}. \]

When \( \theta \) is totally irrational (so that the trace is injective), we consider the projective \( A_0 \) module \( E^\theta_{12} := \mathcal{S}(\mathbb{R} \times \mathbb{Z}) \) constructed as in the main theorem at \( r = 0 \) fibre for the choice of \( M = \mathbb{R} \times \mathbb{Z} \). The trace of this module is \( \theta_{12} \) (see the remarks at the end of section 3).

Consider the following matrices

\[
\begin{pmatrix}
\theta_1 &=& \begin{pmatrix}
0 & \theta_{23} & -\theta_{12} \\
-\theta_{23} & 0 & -\theta_{13} \\
\theta_{12} & \theta_{13} & 0
\end{pmatrix}, \\
\theta_2 &=& \begin{pmatrix}
0 & \theta_{13} & \theta_{12} \\
-\theta_{13} & 0 & -\theta_{23} \\
-\theta_{12} & \theta_{23} & 0
\end{pmatrix}.
\]

Note that \( A_0, A_1 \), and \( A_2 \) are just rotations of each other and represent the same noncommutative tori. Let \( E \) be the projective modules over \( A_0 \) and \( A_1 \), respectively, as discussed above. Now, similarly to the previous case, we see that \( \text{Tr}(E^\theta_{13}) = \theta_{13} \) and \( \text{Tr}(E^\theta_{23}) = \theta_{23} \). Since \( \text{Tr} \) is injective for \( \theta \) (as it is totally irrational), we conclude that \( E^\theta_{12}, E^\theta_{13}, E^\theta_{23} \) along with trivial element generate \( K_0(A_0) \). Using our description of continuous field and 2.4, we conclude that \( E^\theta_{12}, E^\theta_{13}, E^\theta_{23} \) along with the trivial element generate \( K_0(A_0) \) for all \( \theta \).

### 4.2 The 4 dimensional case

Let

\[
\theta = \begin{pmatrix}
0 & \theta_{12} & \theta_{13} & \theta_{14} \\
-\theta_{12} & 0 & \theta_{23} & \theta_{24} \\
-\theta_{13} & -\theta_{23} & 0 & \theta_{34} \\
-\theta_{14} & -\theta_{24} & -\theta_{34} & 0
\end{pmatrix}.
\]

Without loss of generality (again using the proposition above), we can assume that the pfaffians \( \text{pf}(\theta) \) and \( \text{pf}(M^\theta_{ij}) \) are positive, where

\[
M^\theta_{ij} = \begin{pmatrix}
0 & \theta_{ij} \\
-\theta_{ij} & 0
\end{pmatrix}, \quad j > i \geq 1.
\]

Let \( \theta \) be totally irrational again. Then, similarly to the previous case, we get the six modules \( E^\theta_{12}, E^\theta_{13}, E^\theta_{14}, E^\theta_{23}, E^\theta_{24}, E^\theta_{34} \). These modules are completions of \( \mathcal{S}(\mathbb{R} \times \mathbb{Z}^2) \) for different actions of \( A_0 \). Since \( K_0(A_0) = \mathbb{Z}^6 \), we need to find another projective module which has trace \( \theta_{12} \theta_{34} - \theta_{13} \theta_{24} + \theta_{14} \theta_{23} \) (according to lemma 4.1). This module turns out to be the Bott class given by the completion of \( \mathcal{S}(\mathbb{R}^2) \) (as in the main theorem for \( M = \mathbb{R}^2 \)). Denote this module by \( E^\theta_{1234} \). It is very easy to check that \( \text{Tr}(E^\theta_{1234}) = \theta_{12} \theta_{34} - \theta_{13} \theta_{24} + \theta_{14} \theta_{23} \). So again using the main theorem and 2.4 we conclude that \( E^\theta_{12}, E^\theta_{13}, E^\theta_{14}, E^\theta_{23}, E^\theta_{24}, E^\theta_{34}, E^\theta_{1234} \) and \( E^\theta_{1234} \) along with the trivial element generate \( K_0(A_0) \) for each \( \theta \).

### Acknowledgements

The author wants to thank Siegfried Echterhoff for valuable discussions. This research was partially supported by the DFG through SFB 878.
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