Hamiltonian and Brownian systems with long-range interactions: III. The BBGKY hierarchy for spatially inhomogeneous systems

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Abstract

We study the growth of correlations in systems with weak long-range interactions. Starting from the BBGKY hierarchy, we determine the evolution of the two-body correlation function by using an expansion of the solutions of the hierarchy in powers of $1/N$ in a proper thermodynamic limit $N \to +\infty$. These correlations are responsible for the “collisional” evolution of the system beyond the Vlasov regime due to finite $N$ effects. We obtain a general kinetic equation that can be applied to spatially inhomogeneous systems and that takes into account memory effects. These peculiarities are specific to systems with unshielded long-range interactions. For spatially homogeneous systems with short memory time like plasmas, we recover the classical Landau (or Lenard-Balescu) equations. An interest of our approach is to develop a formalism that remains in physical space (instead of Fourier space) and that can deal with spatially inhomogeneous systems. This enlightens the basic physics and provides novel kinetic equations with a clear physical interpretation. However, unless we restrict ourselves to spatially homogeneous systems, closed kinetic equations can be obtained only if we ignore some collective effects between particles. General exact coupled equations taking into account collective effects are also given. We use this kinetic theory to discuss the processes of violent collisionless relaxation and slow collisional relaxation in systems with weak long-range interactions. In particular, we investigate the dependence of the relaxation time with the system size and provide a coherent discussion of all the numerical results obtained for these systems.

1 Introduction

Systems with long-range interactions are numerous in nature [1]. Some examples include self-gravitating systems, two-dimensional vortices, neutral and non-neutral plasmas, bacterial populations, defects in solids, etc... When the potential of interaction is attractive and unshielded, these systems can spontaneously organize into coherent structures accounting for the diversity of the objects observed in the universe. For example, self-gravitating systems organize into planets, stars, galaxies, clusters of galaxies... On the other hand, two-dimensional turbulent
flows organize into jets (like the gulf stream on the earth) or large-scale vortices (like Jupiter’s
great red spot in the jovian atmosphere). Biological populations (like bacteria, amoebae, en-
dotheilial cells,...) also interact via long-range signals through the phenomenon of chemotaxis.
Chemotactic aggregation leads to the spontaneous appearance of patterns like stripes and spots,
filaments, vasculature,... Although these astrophysical, hydrodynamical and biological systems
are physically different, they share a lot of analogies due to the long-range attractive nature of
the potential of interaction [2].

In view of the complexity of these systems, it is natural to try to understand their struc-
ture and organization in terms of statistical mechanics [1]. Since systems with long-range
interactions are generically spatially inhomogeneous, it is clear at first sights that the usual
thermodynamic limit $N \to +\infty$ with $N/V$ fixed is not valid. Therefore, the ordinary methods
of statistical mechanics and kinetic theory must be reformulated and adapted to these systems.
We shall assume, however, that the basic concepts are not altered so that the description of
these systems must be done in consistency with the foundations of statistical mechanics and
kinetic theory. In previous papers of this series [3, 4] (denoted Papers I and II), we have un-
dertaken a systematic study of the dynamics and thermodynamics of systems with long-range
interactions. In Paper I, we have considered the statistical equilibrium states and the static
correlation functions. We have shown that there exists a critical temperature $T_c$ (for Brownian
systems) or a critical energy $E_c$ (for Hamiltonian systems) above which the system is spatially
homogeneous and below which the homogeneous phase becomes unstable and is replaced by a
clustered phase. In Paper II, using an analogy with plasma physics, we have developed a kinetic
theory of systems with long-range interactions in the homogeneous phase. In the present paper
(Paper III), we propose new derivations of the kinetic equations that take into account non-
markovian effects and that can be applied to spatially inhomogeneous configurations. These
extensions are specific to systems with unshielded long-range interactions and they are novel
with respect to the much more studied case of neutral plasmas. They complete the results of
Paper II that were only valid for spatially homogeneous and markovian systems. However, a
limitation of the present approach is to neglect collective effects. These effects were taken into
account in Paper II for spatially homogeneous systems.

This paper is organized as follows. In Sec. 2, we derive a general kinetic equation for
Hamiltonian systems with weak long-range interactions from the BBGKY hierarchy. This
equation is valid at order $O(1/N)$ in an expansion of the solutions of the equations of the
hierarchy in powers of $1/N$ in the proper thermodynamic limit $N \to +\infty$ defined in Paper I.
For $N \to +\infty$, this kinetic equation reduces to the Vlasov equation. At order $O(1/N)$ it takes
into account the effect of “collisions” (more properly “correlations”) between particles due to
finite $N$ effects (graininess). It describes therefore the evolution of the system on a timescale
$N t_D$, where $t_D$ is the dynamical time. This general kinetic equation applies to systems that
can be spatially inhomogeneous and takes into account non-markovian effects. If we restrict
ourselves to spatially homogeneous systems and neglect memory terms, we recover the Landau
equation as a special case. In Secs. 3 and 4, we use this kinetic theory to discuss the processes
of violent collisionless relaxation and slow collisional relaxation in systems with weak long-range
interactions. We review several results obtained for self-gravitating systems, two-dimensional
vortices and the HMF model, emphasize their connections and try to explain them in the
light of the kinetic theory. In particular, we investigate the dependence of the relaxation time
with the system size. We also propose a scenario according to which, for a large class of
initial conditions, the transient states of the collisional relaxation of the HMF model could be
described by spatially homogeneous Tsallis distributions (polytropes) with a compact support
and with an index $q(t) \geq 1$ slowly decreasing with time until they become Vlasov unstable and
relax towards the Boltzmann distribution.
2 Kinetic equation from the BBGKY hierarchy

In this section, we derive a general kinetic equation (33) for Hamiltonian systems with weak long-range interactions. We start from the BBGKY hierarchy and use a systematic expansion of the solutions of the equations of this hierarchy in powers of $1/N$ in a proper thermodynamic limit $N \to +\infty$. The kinetic equation (33) is valid at order $O(1/N)$.

2.1 The $1/N$ expansion

We consider a system of $N$ particles with long-range interactions described by the Hamiltonian equations (I-1). Basically, the evolution of the $N$-body distribution function is governed by the Liouville equation (I-2). Introducing the reduced probability distributions (I-6), we can construct the complete BBGKY hierarchy (II-1). The first two equations of this hierarchy, governing the evolution of the one and two-body distributions, are given by Eqs. (II-2) and (II-3). We recall that $x$ stands for $(r, v)$. We now decompose the distributions functions in the form (I-14) and (I-15) where $P'_{2}(x_1, x_2, t)$ and $P'_{3}(x_1, x_2, x_3, t)$ are the two and three-body correlation functions (or cumulants). Substituting this decomposition in Eq. (II-2), we first obtain

$$\frac{\partial P_{1}}{\partial t} + v_{1} \frac{\partial P_{1}}{\partial r_{1}} + (N-1) \frac{\partial P_{1}}{\partial v_{1}} \int F(2 \to 1)P_{1}(x_2)dx_2 + (N-1) \frac{\partial P_{1}}{\partial v_{1}} \int F(2 \to 1)P'_{2}(x_1, x_2)dx_2 = 0, \tag{1}$$

where $F(j \to i)$ is the force by unit of mass created by particle $j$ on particle $i$. It is related to the potential of interaction $u_{ij} = u(|r_i - r_j|)$ by

$$F(j \to i) = -m \frac{\partial u_{ij}}{\partial r_{i}}. \tag{2}$$

Then, substituting the decompositions (I-14) and (I-15) in (II-3) and using Eq. (I) to simplify some terms, we get

$$\frac{\partial P'_{2}}{\partial t} + v_{1} \frac{\partial P'_{2}}{\partial r_{1}} + F(2 \to 1) \frac{\partial P'_{2}}{\partial v_{1}} + F(2 \to 1)P_{1}(x_2) \frac{\partial P_{1}}{\partial v_{1}}(x_1) - P_{1}(x_2) \frac{\partial}{\partial v_{1}} \int F(3 \to 1)P_{1}(x_1)P_{1}(x_3)dx_3 - \frac{\partial}{\partial v_{1}} \int F(3 \to 1)P'_{2}(x_1, x_3)P_{1}(x_2)dx_3 + (N-2) \frac{\partial}{\partial v_{1}} \int F(3 \to 1)P'_{2}(x_1, x_2)P_{1}(x_3)dx_3 + (N-2) \frac{\partial}{\partial v_{1}} \int F(3 \to 1)P'_{2}(x_2, x_3)P_{1}(x_1)dx_3 + (N-2) \frac{\partial}{\partial v_{1}} \int F(3 \to 1)P'_{3}(x_1, x_2, x_3)dx_3 + (1 \leftrightarrow 2) = 0. \tag{3}$$

Equations (I)-(3) are exact for all $N$ but the hierarchy is not closed. We shall now consider the thermodynamic limit defined in Paper I. It corresponds to $N \to +\infty$ in such a way that

\footnotesize{In Paper II, some terms were missing in Eq. (II-5) of the BBGKY hierarchy because we systematically took $N \to 1 \approx N$ and $N \to 2 \approx N$ which is not correct if we consider terms of order $O(1/N)$.}
the normalized temperature $\eta = \beta N m^2 u_*$ and the normalized energy $\epsilon = E / (u_* N^2 m^2)$ are fixed, where $u_*$ represents the typical strength of the potential of interaction. In general, the potential of interaction is written as $u(r_{ij}) = k \tilde{u}(r_{ij})$ where $k$ is the coupling constant (e.g., $G$ for self-gravitating systems or $k$ for the HMF model). By a suitable normalization of the parameters, this thermodynamic limit is such that the coupling constant behaves like $k \sim 1/N$ while the individual mass $m \sim 1$, the inverse temperature $\beta \sim 1$, the energy per particle $E/N \sim 1$ and the volume $V \sim 1$ are of order unity. This implies that $|x| \sim 1$ and $|F(j \rightarrow i)| \sim 1/N$. On the other hand, the dynamical time $t_D \sim R/v_{typ} \sim 1/\sqrt{kp} \sim 1$ is of order unity ($\rho \sim M/V$ is the average density and the typical velocity $v_{typ}$ has been obtained by equating the kinetic energy $\sim N m v^2$ and the potential energy $\sim N^2 m^2 k \tilde{u}$).

Since the normalized coupling constant $\beta m^2 u_*= \eta/N \sim 1/N$ goes to zero for $N \rightarrow +\infty$, we are studying systems with weak long-range interactions. It is argued in Papers I and II that there exists solutions of the whole BBGKY hierarchy such that the correlation functions $P_j$ scale like $1/N^{j-1}$. This implicitly assumes that the initial condition has no correlation, or that the initial correlations respect this scaling (if there are strong correlations in the initial state, the system will take a long time to erase them and the kinetic theory will be different from the one developed in the sequel). If this scaling is satisfied, we can consider an expansion of the solutions of the equations of the hierarchy in terms of the small parameter $1/N$. This is similar to the expansion in terms of the plasma parameter made in plasma physics. However, in plasma physics the systems are spatially homogeneous while, in the present case, we shall take into account spatial inhomogeneity. This brings additional terms in the kinetic equations that are absent in plasma physics. Therefore, strictly speaking, the hierarchy that we consider is different from the ordinary BBGKY hierarchy. Recalling that $P_1 \sim 1$, $P_2 \sim 1/N$ and $|F(j \rightarrow i)| \sim 1/N$, we obtain at order $1/N$:

$$
\frac{\partial P_1}{\partial t} + v_1 \frac{\partial P_1}{\partial r_1} + (N - 1) \frac{\partial P_1}{\partial v_1} \int F(2 \rightarrow 1) P_1(x_2) dx_2 + N \frac{\partial}{\partial v_1} \int F(2 \rightarrow 1) P_2'(x_1, x_2) dx_2 = 0,
$$

$$
\frac{\partial P_2'}{\partial t} + v_1 \frac{\partial P_2'}{\partial r_1} + \left[ F(2 \rightarrow 1) - \int F(3 \rightarrow 1) P_1(x_3) dx_3 \right] P_1(x_2) \frac{\partial P_1}{\partial v_1}(x_1)
+ N \frac{\partial P_2'}{\partial v_1} \int F(3 \rightarrow 1) P_1(x_3) dx_3 + N \frac{\partial}{\partial v_1} \int F(3 \rightarrow 1) P_2'(x_2, x_3) P_1(x_1) dx_3 + (1 \leftrightarrow 2) = 0.
$$

If we introduce the notations $\xi = N m P_1$ (distribution function) and $g = N^2 P_2'$ (two-body correlation function), we get

$$
\frac{\partial \xi}{\partial t} + v_1 \frac{\partial \xi}{\partial r_1} + N - 1 \langle F \rangle_1 \frac{\partial \xi}{\partial v_1} = -m \frac{\partial}{\partial v_1} \int F(2 \rightarrow 1) g(x_1, x_2) dx_2,
$$

$$
\frac{\partial g}{\partial t} + v_1 \frac{\partial g}{\partial r_1} + \langle F \rangle_1 \frac{\partial g}{\partial v_1} + \frac{1}{m^2} \int F(2 \rightarrow 1) f_2 \frac{\partial f_1}{\partial v_1}
+ \frac{\partial}{\partial v_1} \int F(3 \rightarrow 1) g(x_2, x_3, t) \frac{f_1}{m} dx_3 + (1 \leftrightarrow 2) = 0.
$$

$^2$Alternatively, we can assume that the mass of the particles scales like $m \sim 1/N$ while $k \sim u_* \sim 1$, $\beta \sim N$, $E \sim 1$ and $V \sim 1$. In this scaling, the total mass $M \sim N m$ is of order unity.
where we have introduced the abbreviations \( f_1 = f(r_1, v_1, t) \) and \( f_2 = f(r_2, v_2, t) \). We have also introduced the mean force (by unit of mass) created in \( r_1 \) by all the particles

\[
\langle F \rangle_1 = \int F(2 \to 1) \frac{f_2}{m} dr_2 dv_2,
\]

and the fluctuating force (by unit of mass) created by particle 2 on particle 1:

\[
\mathcal{F}(2 \to 1) = F(2 \to 1) - \frac{1}{N} \langle F \rangle_1.
\]

These equations are exact at the order \( O(1/N) \). They form therefore the right basis to develop a kinetic theory for Hamiltonian systems with weak long-range interactions. We note that these equations are similar to the BBGKY hierarchy of plasma physics but not identical. One difference is the \( (N-1)/N \) term in Eq. (6). The other difference is the presence of the fluctuating force \( \mathcal{F}(2 \to 1) \) instead of \( F(2 \to 1) \) due to the spatial inhomogeneity of the system. In plasma physics, the system is homogeneous over distances of the order of the Debye length so the mean force \( \langle F \rangle \) vanishes.

2.2 The Vlasov equation and beyond

Recalling that \( P_2' \sim 1/N \), we note that

\[
P_2(x_1, x_2, t) = P_1(x_1, t)P_1(x_2, t) + O(1/N).
\]

If we consider the limit \( N \to +\infty \) (for a fixed time \( t \)), we see that the correlations between particles can be neglected so that the two-body distribution function factorizes in two one-body distribution functions i.e. \( P_2(x_1, x_2, t) = P_1(x_1, t)P_1(x_2, t) \). Therefore the mean field approximation is exact in the limit \( N \to +\infty \). Substituting this result in Eq. (1), we obtain the Vlasov equation

\[
\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f}{\partial r_1} + \langle F \rangle_1 \frac{\partial f}{\partial v_1} = 0.
\]

This equation describes the collisionless evolution of the system up to a time at least of order \( N t_D \) (where \( t_D \) is the dynamical time). In practice, \( N \gg 1 \) so that the domain of validity of the Vlasov equation is huge (for example, in typical stellar systems \( N \sim 10^6 - 10^{12} \)). When the Vlasov equation is coupled to an attractive unshielded long-range potential of interaction, it can develop a process of violent relaxation towards a quasi stationary state (QSS). This process will be discussed specifically in Sec. 3.

If we want to describe the collisional evolution of the system, we need to consider finite \( N \) effects. Equations (6) and (7) describe the evolution of the system on a timescale of order \( N t_D \). The equation for the evolution of the smooth distribution function is of the form

\[
\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f}{\partial r_1} + \frac{N-1}{N} \langle F \rangle_1 \frac{\partial f}{\partial v_1} = C_N[f],
\]

where \( C_N \) is a “collision” term analogous to the one arising in the Boltzmann equation. In the present context, there are not real collisions between particles. The term on the right hand side of Eq. (12) is due to the development of correlations between particles as time goes on. It is related to the two-body correlation function \( g(x_1, x_2, t) \) which is itself related to the distribution function \( f(x_1, t) \) by Eq. (7). Our aim is to obtain an expression for the collision term \( C_N[f] \) at the order \( 1/N \). The difficulty with Eq. (7) for the two-body correlation function is that it is an
the integral in Eq. (7). Then, we get the coupled system due to a change of the correlation function. In this paper, we shall neglect the contribution of the integral in Eq. (7). Then, we get the coupled system

\[
\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f}{\partial r_1} + \frac{N-1}{N} \langle F \rangle_1 \frac{\partial f}{\partial v_1} = -m \frac{\partial}{\partial v_1} \int F(2 \rightarrow 1)g(x_1, x_2)dx_2, \tag{13}
\]

\[
\frac{\partial g}{\partial t} + \left[ v_1 \frac{\partial}{\partial r_1} + v_2 \frac{\partial}{\partial r_2} + \langle F \rangle_1 \frac{\partial}{\partial v_1} + \langle F \rangle_2 \frac{\partial}{\partial v_2} \right] g + \left[ F(2 \rightarrow 1) \frac{\partial}{\partial v_1} + F(1 \rightarrow 2) \frac{\partial}{\partial v_2} \right] \frac{f_1 f_2}{m} = 0. \tag{14}
\]

The integral that we have neglected contains “collective effects” that describe the polarization of the medium. In plasma physics, they are responsible for the Debye shielding, i.e. the fact that a charge is surrounded by a polarization cloud of opposite charges that diminish the interaction. These collective effects are taken into account in the Lenard-Balescu equation through the dielectric function (see Paper II). However, this equation is restricted to spatially homogeneous systems and based on a Markovian approximation. These assumptions are necessary to use Laplace-Fourier transforms in order to solve the integro-differential equation [7]. Here, we want to describe more general situations where the interaction is not shielded so that the system can be spatially inhomogeneous. If we neglect collective effects, we can obtain a general kinetic equation in a closed form [33] that is valid for systems that are not necessarily homogeneous and that can take into account memory effects. This equation has interest in its own right (despite its limitations) because its structure bears a lot of physical significance. Before deriving this general equation, we shall first consider the case of spatially homogeneous systems and make the link with the familiar Landau equation.

### 2.3 The Landau equation

For a spatially homogeneous system, the distribution function and the two-body correlation function can be written \( f = f(v_1, t) \) and \( g = g(v_1, v_2, r_1 - r_2, t) \). In that case, Eqs. (13)-(14) become

\[
\frac{\partial f_1}{\partial t} = m^2 \frac{\partial}{\partial v_1} \cdot \int \frac{\partial u}{\partial x} g(v_1, v_2, x, t)dx dv_2, \tag{15}
\]

\[
\frac{\partial g}{\partial t} + w \cdot \frac{\partial g}{\partial x} = \frac{\partial u}{\partial x} \cdot \left( \frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2} \right) f(v_1, t) \frac{f}{m}(v_2, t), \tag{16}
\]

where we have used the fact that \( F(1 \rightarrow 2) = -F(2 \rightarrow 1) \) and noted \( x = r_1 - r_2 \) and \( w = v_1 - v_2 \). Taking the Fourier transform of Eq. (16) and introducing the notations \( \partial = \partial/\partial v_1 - \partial/\partial v_2 \), \( f_1 = f(v_1, t) \) and \( f_2 = f(v_2, t) \), we obtain

\[
\frac{\partial \hat{g}}{\partial t} + ik \cdot w \hat{g} = i \frac{1}{m} \hat{u}(k) k \cdot \partial f_1 f_2. \tag{17}
\]

In terms of the Fourier transform of the correlation function, the kinetic equation (15) can be rewritten

\[
\frac{\partial f_1}{\partial t} = m^2 (2\pi)^d \frac{\partial}{\partial v_1} \cdot \int k \hat{u}(k) \text{Im} \hat{g}(v_1, v_2, k, t)dk dv_2. \tag{18}
\]
We shall assume that Im$\hat{g}(v_1, v_2, k, t)$ relaxes on a timescale that is much smaller than the timescale on which $f(v_1, t)$ changes. This is the equivalent of the Bogoliubov hypothesis in plasma physics. If we ignore memory effects, we can integrate the first order differential equation (17) by considering the last term as a constant. This yields
\[
\hat{g}(v_1, v_2, k, t) = \int_0^t d\tau \frac{i}{m} k \hat{u}(k) e^{-ik \cdot w \tau} \partial f_1(t) f_2(t),
\]
where we have assumed that no correlation is present initially: $g(t = 0) = 0$. Then, we can replace Im$\hat{g}(v_1, v_2, k, t)$ in Eq. (18) by its value obtained for $t \rightarrow +\infty$, which reads
\[
\text{Im} \hat{g}(k, v_1, v_2, +\infty) = \frac{\pi}{m} k \hat{u}(k) \delta(k \cdot w) \partial f_1(t) f_2(t). \tag{20}
\]
Substituting this relation in Eq. (18), we obtain the Landau equation in the form
\[
\frac{\partial f_1}{\partial t} = \pi (2\pi)^d m \frac{\partial}{\partial v_1^\mu} \int dv_2 dk k^\nu \hat{u}(k)^2 \delta(k \cdot w) \left( f_2 \frac{\partial f_1}{\partial v_1^\nu} - f_1 \frac{\partial f_2}{\partial v_2^\nu} \right). \tag{21}
\]
Other equivalent expressions of the Landau equation are given in Paper II. The Landau equation ignores collective effects. Collective effects can be taken into account by keeping the contribution of the last integral in Eq. (7). For spatially homogeneous systems, the calculations lead to the Lenard-Balescu equation discussed in Paper II (the Lenard-Balescu equation can be obtained from the Landau equation by replacing the potential $\hat{u}(k)$ by the “screened” potential $\hat{u}(k)/|\epsilon(k, k \cdot v_2)|$ including the dielectric function). The Landau and Lenard-Balescu equations conserve mass and energy (reducing to the kinetic energy for a spatially homogeneous system) and monotonically increase the Boltzmann entropy [6]. The collisional evolution is due to a condition of resonance between the particles orbits. For homogeneous systems, the condition of resonance encapsulated in the $\delta$-function appearing in the Landau and Lenard-Balescu equations corresponds to $k \cdot v_1 = k \cdot v_2$ with $v_1 \neq v_2$. For $d > 1$, the only stationary solution is the Maxwell distribution. Because of the $H$-theorem, the Landau and Lenard-Balescu equations relax towards the Maxwell distribution. Since the collision term in Eq. (21) is valid at order $O(1/N)$, the relaxation time scales like
\[
t_R \sim N t_D, \quad (d > 1) \tag{22}
\]
as can be seen directly from Eq. (21) by dimensional analysis (comparing the l.h.s. and the r.h.s., we have $1/t_R \sim u_2^2 N \sim 1/N$ while $t_D \sim R/v_{\text{typ}} \sim 1$ with the scalings introduced in Sec. 2.1). A more precise estimate of the relaxation time is given in [7]. For one-dimensional systems, like the HMF model, the situation is different. For $d = 1$, the kinetic equation (21) reduces to
\[
\frac{\partial f_1}{\partial t} = 2\pi^2 m \frac{\partial}{\partial v_1^\mu} \int dv_2 dk \frac{k^2}{|k|} \hat{u}(k)^2 \delta(v_1 - v_2) \left( f_2 \frac{\partial f_1}{\partial v_1^\nu} - f_1 \frac{\partial f_2}{\partial v_2^\nu} \right) = 0. \tag{23}
\]
Therefore, the collision term $C_N[f]$ vanishes at the order $1/N$ because there is no resonance. The kinetic equation reduces to $\partial f/\partial t = 0$ so that the distribution function does not evolve at all on a timescale $\sim N t_D$. This implies that, for one-dimensional homogeneous systems, the relaxation time to statistical equilibrium is larger than $N t_D$. Thus, we expect that
\[
t_R > N t_D, \quad (d = 1). \tag{24}
\]
The fact that the Lenard-Balescu collision term vanishes in 1D is known for a long time in plasma physics (see, e.g., the last paragraph in [8]) and has been rediscovered recently in the context of the HMF model [9] [4] [10].
2.4 The non Markovian kinetic equation

The above kinetic equations rely on the assumption that the correlation function relaxes much more rapidly than the distribution function. The Markovian approximation is expected to be a good approximation in the limit $N \to +\infty$ that we consider since the distribution function changes on a slow timescale of order $N t_D$ (where $t_D$ is the dynamical time) or even larger. However, for systems with long-range interactions, there are situations where the decorrelation time of the fluctuations can be very long so that the Markovian approximation may not be completely justified. This concerns in particular the case of self-gravitating systems for which the temporal correlation of the force decreases like $1/t$ (see [11] and Paper II). This is also the case for systems that are close to the critical point since the exponential relaxation time of the correlations diverges for $E \to E_c$ or $T \to T_c$ (see [12] [10] and Paper II). Therefore, it can be of interest to derive non-markovian kinetic equations that may be relevant to such systems. If we keep the time variation of $f(v, t)$ in Eq. (17), we obtain after integration
\[
\hat{g}(v_1, v_2, k, t) = \int_0^t d\tau \frac{i}{m} k \hat{u}(k) e^{-i k \cdot w \tau} \partial f_1(t - \tau) f_2(t - \tau). \tag{25}
\]
Inserting this relation in Eq. (15), we obtain a non Markovian kinetic equation
\[
\frac{\partial f_1}{\partial t} = (2\pi)^d m \frac{\partial}{\partial v_1^\mu} \int_0^t d\tau \int dv_2 dk^n k^{\nu} \hat{u}(k)^2 \cos(k \cdot w \tau) \left( \frac{\partial}{\partial v_1^{\nu}} - \frac{\partial}{\partial v_2^{\nu}} \right) f(v_1, t - \tau) f(v_2, t - \tau). \tag{26}
\]
In particular, for the HMF model, using the notations of Paper I, we get
\[
\frac{\partial f_1}{\partial t} = \frac{k^2}{4\pi} \frac{\partial}{\partial v_1^\mu} \int_0^t d\tau \int dv_2 \cos[(v_1 - v_2) \tau] \left( \frac{\partial}{\partial v_1^{\mu}} - \frac{\partial}{\partial v_2^{\mu}} \right) f(v_1, t - \tau) f(v_2, t - \tau). \tag{27}
\]
We note that, when memory terms are taken into account, the collision term does not vanish. However, if we make the Markovian approximation $f(v_1, t - \tau) \simeq f(v_1, t)$, $f(v_2, t - \tau) \simeq f(v_2, t)$ and extend the time integral to infinity\footnote{We could also consider an approximation where we make the Markovian approximation but keep the time integral going from 0 to $t$ (see Appendix A).}, we obtain
\[
\frac{\partial f_1}{\partial t} = \frac{k^2}{4} \frac{\partial}{\partial v_1^\mu} \int dv_2 \delta(v_1 - v_2) \left( \frac{\partial}{\partial v_1^{\mu}} - \frac{\partial}{\partial v_2^{\mu}} \right) f(v_1, t) f(v_2, t) = 0. \tag{28}
\]
When memory terms are neglected we recover the fact that the Landau collision term vanishes for a spatially homogeneous one-dimensional system. By working close to the critical point in the HMF model (where the exponential relaxation time of the correlations diverges), it may be possible to see non-markovian effects in numerical simulations of the $N$-body system. They should induce a small evolution of the homogeneous system on a timescale $N t_D$ as described by Eq. (27) or, more precisely, by its generalization taking into account collective effects (see Appendix A). This should not lead, however, to statistical equilibrium since Eq. (27) clearly does not tend to the Boltzmann distribution.

2.5 The kinetic equation for spatially inhomogeneous systems

Relaxing the assumption that the system is spatially homogeneous, the equation (14) for the correlation function can be written
\[
\frac{\partial g}{\partial t} + \mathcal{L} g = - \left[ \mathcal{F}(2 \to 1) \frac{\partial}{\partial v_1^\mu} + \mathcal{F}(1 \to 2) \frac{\partial}{\partial v_2^\mu} \right] \frac{f}{m} (x_1, t) \frac{f}{m} (x_2, t), \tag{29}
\]
where we have denoted the advective term by $\mathcal{L}$ (Liouvillean operator). Solving formally this equation with the Green function

$$G(t, t') = \exp \left\{ - \int_{t'}^t \mathcal{L}(\tau) d\tau, \right\} \tag{30}$$

we obtain

$$g(x_1, x_2, t) = - \int_0^t d\tau G(t, t - \tau) \left[ \mathcal{F}(2 \to 1) \frac{\partial}{\partial v_1} + \mathcal{F}(1 \to 2) \frac{\partial}{\partial v_2} \right] \frac{f}{m}(x_1, t - \tau) \frac{f}{m}(x_2, t - \tau). \tag{31}$$

The Green function constructed with the smooth field $\langle \mathbf{F} \rangle$ means that, in order to evaluate the time integral in Eq. (31), we must move the coordinates $r_i(t - \tau)$ and $v_i(t - \tau)$ of the particles with the mean field flow in phase space, adopting a Lagrangian point of view. Thus, in evaluating the time integral, the coordinates $r_i$ and $v_i$ placed after the Greenian must be viewed as $r_i(t - \tau)$ and $v_i(t - \tau)$ where

$$r_i(t - \tau) = r_i(t) - \int_0^\tau v_i(t - s) ds, \quad v_i(t - \tau) = v_i(t) - \int_0^\tau \langle \mathbf{F} \rangle(r_i(t - s), t - s) ds. \quad \tag{32}$$

Substituting Eq. (30) in Eq. (13), we get

$$\frac{\partial f}{\partial t} + v_1 \frac{\partial f}{\partial r_1} + \frac{N - 1}{N} \langle \mathbf{F} \rangle_1 \frac{\partial f}{\partial v_1} = \frac{\partial}{\partial v_1^\mu} \left[ \int_0^t d\tau \int d\mathbf{r}_2 d\mathbf{v}_2 \mathcal{F}^\mu(2 \to 1, t) G(t, t - \tau) \right] \times \left[ \mathcal{F}^\nu(2 \to 1) \frac{\partial}{\partial v_1^\nu} + \mathcal{F}^\nu(1 \to 2) \frac{\partial}{\partial v_2^\nu} \right] f(\mathbf{r}_1, \mathbf{v}_1, t - \tau) \frac{f}{m}(\mathbf{r}_2, \mathbf{v}_2, t - \tau). \quad \tag{33}$$

This returns the general kinetic equation obtained by Kandrup [13] with the projection operator formalism (note that we can replace $\mathcal{F}^\mu(2 \to 1, t)$ by $\mathcal{F}^\mu(2 \to 1, 0)$ in the first term of the r.h.s. of the equation since the fluctuations vanish in average). Equation (33) slightly differs from the equation obtained in [13] by a term $(N - 1)/N$ in the l.h.s. This new derivation of the kinetic equation (33) from a systematic expansion of the solutions of the BBGKY hierarchy in powers of $1/N$ is valuable because the present formalism is considerably simpler than the projection operator formalism and clearly shows which terms have been neglected in the derivation. It also clearly shows that the kinetic equation (33) is valid at order $1/N$ so that it describes the “collisional” evolution of the system on a timescale of order $N\tau_D$.

### 2.6 Summary of the different kinetic equations

Let us briefly summarize the different kinetic equations that appeared in our analysis. When collective terms are ignored, the kinetic equation describing the evolution of the system as a whole at order $1/N$ is

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial r} + \frac{N - 1}{N} \langle \mathbf{F} \rangle \frac{\partial f}{\partial v} = \frac{\partial}{\partial v^\mu} \left[ \int_0^t d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 \mathcal{F}^\mu(1 \to 0) G(t, t - \tau) \right] \times \left\{ \mathcal{F}^\nu(1 \to 0) \frac{\partial}{\partial v^\nu} + \mathcal{F}^\nu(0 \to 1) \frac{\partial}{\partial v_1^\nu} \right\} f(\mathbf{r}, \mathbf{v}, t - \tau) \frac{f}{m}(\mathbf{r}_1, \mathbf{v}_1, t - \tau). \quad \tag{34}$$

If we make a Markov approximation and extend the time integral to infinity, we obtain

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial r} + \frac{N - 1}{N} \langle \mathbf{F} \rangle \frac{\partial f}{\partial v} = \frac{\partial}{\partial v^\mu} \left[ \int_0^{+\infty} d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 \mathcal{F}^\mu(1 \to 0) G(t, t - \tau) \right] \times \left\{ \mathcal{F}^\nu(1 \to 0) \frac{\partial}{\partial v^\nu} + \mathcal{F}^\nu(0 \to 1) \frac{\partial}{\partial v_1^\nu} \right\} f(\mathbf{r}, \mathbf{v}, t) \frac{f}{m}(\mathbf{r}_1, \mathbf{v}_1, t). \quad \tag{35}$$
As we have indicated, the Markov approximation is justified for $N \rightarrow +\infty$ so that $\tau_{\text{corr}} \ll t_{\text{relax}} \sim N t_D$. We do not assume, however, that the decorrelation time is “extremely” short (i.e., $\tau_{\text{corr}} \rightarrow 0$). Therefore, in the time integral, the distribution functions must be evaluated at $(r(t - \tau), v(t - \tau))$ and $(r_1(t - \tau), v_1(t - \tau))$

\[
\mathbf{r}_i(t - \tau) = \mathbf{r}_i(t) - \int_0^\tau \mathbf{v}_i(t - s)ds, \quad \mathbf{v}_i(t - \tau) = \mathbf{v}_i(t) - \int_0^\tau \langle \mathbf{F} \rangle (\mathbf{r}_i(t - s), t)ds. \tag{36}
\]

Comparing Eq. (36) with Eq. (32), we have assumed that the mean force $\langle \mathbf{F} \rangle (\mathbf{r}, t)$ does not change substantially on the timescale $\tau_{\text{corr}}$ on which the time integral has essential contribution.

For a spatially homogeneous system, using the fact that Eq. (32) reduces to $\langle \mathbf{F} \rangle = 0$, Eq. (34) takes the form

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v^\mu} \int_0^t d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 \mathbf{F}^\mu(1 \rightarrow 0, t) \mathbf{F}^\nu(1 \rightarrow 0, t - \tau) \left( \frac{\partial}{\partial v^\nu} - \frac{\partial}{\partial v^\mu_1} \right) f(\mathbf{v}, t - \tau) \frac{f}{m}(\mathbf{v}_1, t - \tau). \tag{37}
\]

If we make the integration on $\mathbf{r}_1$, using the relation (A3) of Paper II, we obtain the non-markovian equation (26). If we make the Markovian approximation $f(\mathbf{v}, t - \tau) \simeq f(\mathbf{v}, t)$, $f(\mathbf{v}_1, t - \tau) \simeq f(\mathbf{v}_1, t)$ and extend the time integration to $+\infty$, we get

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v^\mu} \int_0^{+\infty} d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 \mathbf{F}^\mu(1 \rightarrow 0, t) \mathbf{F}^\nu(1 \rightarrow 0, t - \tau) \left( \frac{\partial}{\partial v^\nu} - \frac{\partial}{\partial v^\mu_1} \right) f(\mathbf{v}, t) \frac{f}{m}(\mathbf{v}_1, t). \tag{38}
\]

The integrals on $\tau$ and $\mathbf{r}_1$ can be performed as in Appendix A of Paper II and we finally obtain the Landau equation (21).

\section{Violent collisionless relaxation}

In this section, we physically discuss the process of violent collisionless relaxation in relation with the Vlasov equation (11) and point out several analogies between stellar systems, two-dimensional turbulence and the HMF model [1].

\subsection{Quasi Stationary States}

When the Vlasov equation is coupled to an attractive unshielded long-range potential of interaction it can develop a process of phase mixing and violent relaxation leading to the formation of a quasi-stationary state (QSS). This purely mean field process takes place on a very short time scale, of the order of a few dynamical times. This corresponds to the formation of galaxies in astrophysics, jets and vortices in geophysical and astrophysical flows and clusters in the HMF model. Lynden-Bell [14] has proposed to describe these QSS in terms of statistical mechanics, adapting the usual Boltzmann procedure so as to take into account the specificities of the Vlasov equation (in particular the conservation of the infinite class of Casimirs) [15]. This approach rests on the assumption that the collisionless mixing is efficient and that the ergodic hypothesis which sustains the statistical theory is fulfilled.

There are situations where the Lynden-Bell prediction works relatively well. However, there are other situations where the Lynden-Bell prediction fails. It has been understood since the beginning [14] that violent relaxation may be incomplete in certain cases so that the Lynden-Bell mixing entropy is not maximized in the whole available phase space. Incomplete relaxation [16] can lead to more or less severe deviations from the Lynden-Bell statistics. Physically, the system tries to reach the Lynden-Bell maximum entropy state during violent relaxation but, in some cases, it cannot attain it because the variations of the potential, that are the engine of the evolution, die
away before the relaxation process is complete (there may be other reasons for incomplete relaxation). Since the Vlasov equation admits an infinite number of stationary solutions, the coarse-grained distribution \( f(r, v, t) \) can be trapped in one of them \( f_{QSS}(r, v) \) and remain frozen in that quasi stationary state until collisional effects finally come into play (on longer timescales). This steady solution is not always the most mixed state (it can be only partially mixed) so it may differ from Lynden-Bell’s statistical prediction. Thus, for dynamical reasons, the system does not always explore the whole phase space ergodically. In general, the statistical theory of Lynden-Bell gives a relatively good first order prediction of the QSS without fitting parameter and is able to explain out-of-equilibrium phase transitions between different types of structures, depending on the values of the control parameters fixed by the initial condition. However, there are cases where the prediction does not work well (it can sometimes be very bad) because of incomplete relaxation. The difficulty is that we do not know a priori whether the prediction of Lynden-Bell will work or fail because this depends on the dynamics and it is difficult to know in advance if the system will mix well or not. Therefore, numerical simulations are necessary to determine how close to the Lynden-Bell distribution the system happens to be. Let us give some examples of complete and incomplete violent relaxation in stellar systems, 2D turbulence and for the HMF model.

3.2 Stellar systems

The concept of violent relaxation was first introduced by Lynden-Bell [14] to explain the apparent regularity of elliptical galaxies in astrophysics. However, for 3D stellar systems the prediction of Lynden-Bell leads to density profiles whose mass is infinite (the density decreases as \( r^{-2} \) at large distances). In other words, there is no maximum entropy state at fixed mass and energy in an unbounded domain. Furthermore, it is known that the distribution functions (DF) of galaxies do not only depend on the energy \( \epsilon = v^2/2 + \Phi(r) \) contrary to what is predicted by the Lynden-Bell statistical theory. This means that other ingredients are necessary to understand their structure [16]. However, the approach of Lynden-Bell is able to explain why elliptical galaxies have an almost isothermal core. Indeed, it is able to justify a Boltzmannian distribution \( f \sim e^{-\beta \epsilon} \) in the core without recourse to collisions which operate on a much longer timescale \( t_{\text{relax}} \sim (N/ \ln N)t_D \) [17]. By contrast, violent relaxation is incomplete in the halo. The concept of incomplete violent relaxation explains why galaxies are more confined than predicted by statistical mechanics (the density profile of elliptical galaxies decreases as \( r^{-4} \) instead of \( r^{-2} \) [18]). We note, however, that elliptical galaxies are not stellar polytropes so their DF cannot be fitted by the Tsallis distribution.

For one dimensional self-gravitating systems, the Lynden-Bell entropy has a global maximum at fixed mass and energy in an unbounded domain. Early simulations of the 1D Vlasov-Poisson system starting from a water-bag initial condition have shown a relatively good agreement with the Lynden-Bell prediction [19]. In other cases, the Vlasov equation (and the corresponding \( N \)-body system) can have a very complicated, non-ergodic, dynamics. For example, starting from an annulus in phase space, Mineau et al. [20] have observed the formation of phase-space holes which block the relaxation towards the Lynden-Bell distribution. In that case, the system does not even relax towards a stationary state of the Vlasov equation but develops everlasting oscillations.

For three dimensional self-gravitating systems confined within a box, Taruya & Sakagami [21] found numerically that the transient stages of the collisional relaxation of the \( N \)-stars system can be fitted by a sequence of polytropic (Tsallis) distributions with a time dependent \( q(t) \) index. Therefore, after the phase of violent relaxation, the system passes by a succession of quasi-stationary solutions of the Vlasov equation slowly evolving with time due to collisions (fi-
nite $N$ effects), until the gravothermal catastrophe associated with the Boltzmann distribution finally takes place.

### 3.3 Two-dimensional vortices

In the context of two-dimensional turbulence, Miller [22] and Robert & Sommeria [23] have developed a statistical mechanics of the 2D Euler equation which is similar to the Lynden-Bell theory (see [24] for a description of this analogy). This theory works relatively well to describe vortex merging [25] or the nonlinear development of the Kelvin-Helmholtz instability in a shear layer [26]. It can account for the numerous bifurcations observed between different types of vortices (monopoles, dipoles, tripoles,...) [27] and is able to reproduce the structure of geophysical and jovian vortices like Jupiter’s great red spot [28, 29].

However, some cases of incomplete relaxation have been reported. For example, in the plasma experiment of Huang & Driscoll [30], the MRS statistical theory gives a reasonable prediction of the QSS without fit but the agreement is not perfect [31]. The observed central density is larger than predicted by theory and the tail decreases more rapidly than predicted by theory, i.e. the vortex is more confined. This is related to the fact that mixing is not very efficient in the core and in the tail of the distribution (these features can be explained by developing a kinetic theory of violent relaxation [32]). As observed by Boghosian [33], the QSS can be fitted by a Tsallis distribution where the density drops to zero at a finite distance.

### 3.4 The HMF model

The nature of the quasi stationary states (QSS) observed in the HMF model has generated an intense (and lively) debate in the community of statistical physics.

Latora et al. [34] performed $N$-body numerical simulations starting from a water-bag initial condition with magnetization $M(0) = 1$ (unstationary). They observed the formation of QSS whose lifetime diverges with the system size $N$. The Boltzmann distribution of statistical equilibrium $f_B = Ae^{-\beta \epsilon}$ is reached on a timescale $t_{\text{relax}} \sim N$. In their Fig. 1(a), they compared the caloric curve $T(U)$ of these QSS (bullets) with the caloric curve corresponding to the Boltzmann distribution (full line). They found a range of energies $0.5 \lesssim U < U_c = 3/4$ where the caloric curve of the QSS disagrees with the caloric curve corresponding to the Boltzmann statistical equilibrium. We can interpret their results in another (complementary) manner. First of all, we note that the caloric curve of the QSS should be compared with the caloric curve predicted by the Lynden-Bell theory of violent relaxation since we are dealing with out-of-equilibrium structures (there is a priori no reason why the caloric curve of the QSS resulting from the violent collisionless relaxation should coincide with the caloric curve of the Boltzmann statistical equilibrium state resulting from the slow collisional relaxation). The question we now ask is: can the QSS be described by the Lynden-Bell theory? We note that, for a water-bag initial condition (two-levels), the distribution predicted by Lynden-Bell is similar to the Fermi-Dirac statistics $T_{\text{LB}} = \eta_0/(1 + e^{\beta \epsilon + \alpha})$ [15]. Furthermore, for the $M(0) = 1$ initial condition, we are in the dilute (non degenerate) limit of the Lynden-Bell statistical theory since the initial phase level $\eta_0 \to +\infty$ [35]. Therefore, for that particular initial condition, we remark that the distribution predicted by Lynden-Bell coincides with the Boltzmann statistics $T_{\text{LB}} = Ae^{-\beta \epsilon}$, although it applies to the out-of-equilibrium QSS. Because of this coincidence, we can use the caloric curve $T(U)$ reported in Fig. 1(a) of [34] to determine the domain of validity of the Lynden-Bell prediction. This curve shows that the Lynden-Bell prediction works well for $U > U_c = 3/4$ (i.e. in the region where the Lynden-Bell maximum entropy state is spatially homogeneous) and for $U \lesssim 0.5$ (i.e. in the region where the Lynden-Bell maximum entropy
state is strongly spatially inhomogeneous). However, for $0.5 \lesssim U < U_c$ (i.e., close to the transition energy), the Lynden-Bell prediction fails. In that case, Latora et al. [34] show that the distribution has the tendency to remain spatially homogeneous ($M_{\text{QSS}} \simeq 0$) and that the velocity distribution is non-gaussian. These results strongly differ from the Lynden-Bell prediction predicting a spatially inhomogeneous state with gaussian distribution $f_{\text{LB}} = Ae^{-\beta \epsilon}$, the same as the statistical equilibrium state $f_B = Ae^{-\beta \epsilon}$. Therefore, close to the critical energy $U_c$, violent relaxation is incomplete and leads to a non-ergodic behaviour. Since standard statistical mechanics breaks down (standard statistical mechanics in the present context refers, in our sense, to the Lynden-Bell theory), Latora et al. [34] propose to describe this regime in terms of Tsallis generalized thermodynamics. This is an interesting idea to explore since there are no many other alternatives when the evolution is non-ergodic (another alternative could be to develop a kinetic theory of violent relaxation as attempted in [36]).

Yamaguchi et al. [37] performed $N$-body numerical simulations starting from a water-bag initial condition with magnetization $M(0) = 0$. For $U = 0.69 > U^*_c = 7/12$, this initial condition is a stable steady state of the Vlasov equation. Furthermore, we remark that this spatially homogenous water-bag initial condition is a minimum of energy $E$ for a given mass $M$ and phase level value $\eta_0$ [35]. Therefore, this initial condition is the Lynden-Bell maximum entropy state. As a result, it does not evolve at all through the Vlasov equation. Yamaguchi et al. [37] show that it slowly evolves under the effect of collisions (finite $N$ effects) by passing through a series of stationary solutions of the Vlasov equation. This is similar to the results obtained by Taruya & Sakagami [21] for self-gravitating systems. Yamaguchi et al. [37] show that the Boltzmann distribution is reached on a timescale $t_{\text{relax}} \sim N^{1.7}t_D$ and that the velocity distribution of the transient states is given by the curve reported on their Fig. 12. Recently, Campa et al. [38] obtained similar results and showed that these transient states can be fitted by a semi-elliptical distribution. Interestingly, we remark that a semi-elliptical distribution is a Tsallis distribution $f_q(v) = [\mu - \beta(q - 1)v^2/2q]^{1/(q-1)}$ with an index $q = 3$ (if we note $1 - q_s$ instead of $q - 1$ this corresponds to $q_s = -1$ [4]). Therefore, we observe that the results of Yamaguchi et al. [37] and Campa et al. [38], like the results of Taruya & Sakagami [21] in astrophysics, show that Tsallis distributions may be useful to describe the transient states of a collisional relaxation. In their paper, Yamaguchi et al. [37] (see also [39]) reject the Tsallis distributions because of the absence of power law tails in their curves of Fig. 12. However, power law tails are obtained only for a subclass of Tsallis distributions corresponding to indices $q < 1$ (in our notations [10, 40]). For $q > 1$, the Tsallis distributions drop to zero at a finite value of the velocity, so they have a compact support. In particular, the distribution obtained by Yamaguchi et al. [37] and Campa et al. [38] appears to be well-fitted by a Tsallis distribution with $q = 3 > 1$ with a tail going to zero abruptly at a finite velocity $v_{\text{max}}$. In view of the lively debate and the controversy about the applicability of the Tsallis statistics to the HMF model [37, 39], it is amusing to realize that the distribution obtained numerically by Yamaguchi et al. [37] is in fact ... a Tsallis distribution! It has a compact support ($q > 1$) instead of power-law tails ($q < 1$).

Antoniazzi et al. [41] performed $N$-body numerical simulations starting from a water-bag...
For this value of energy, the Lynden-Bell theory predicts an out-of-equilibrium phase transition from a homogeneous state to an inhomogeneous state above a critical magnetization $M_{\text{crit}} = 0.897$ discovered in [35]. Numerical simulations [41] show that the Lynden-Bell prediction works relatively well for $M(0) < M_{\text{crit}}$. This is confirmed by Campa et al. [38] who find in addition that $t_{\text{relax}} \sim N^{1.7} t_D$ as for $M(0) = 0$. However, above the critical magnetization, the results of Latora et al. [34] and Campa et al. [38] indicate that the Lynden-Bell theory does not work since the observed QSS is homogeneous ($M_{\text{QSS}} \simeq 0$) with non-gaussian tails while the Lynden-Bell theory predicts an inhomogeneous state ($M_{\text{QSS}} \neq 0$) with gaussian tails. This discrepancy is particularly clear for the initial condition $M(0) = 1$ where the Lynden-Bell distribution coincides with the Boltzmann distribution $\mathcal{T}_{\text{LB}} \sim e^{-\beta \varepsilon}$ (non degenerate limit). Now, the early work of Latora et al. [34] indicates that this gaussian distribution is not observed and the recent work of Campa et al. [38] (for an isotropic water bag initial condition) shows that the QSS is well fitted by a semi-elliptical distribution. As we have seen, this is a particular Tsallis distribution with index $q = 3$ possessing a natural velocity cut-off. Such distributions, that rapidly drop to zero at a finite energy (here velocity) are typical products of incomplete relaxation. They are explained qualitatively by the fact that the high energy tail of the distribution in phase space does not mix well (a similar confinement is observed in the plasma experiment of Huang & Driscoll [30] discussed in Sec. 3.3). This confinement is consistent with a kinetic theory of incomplete violent relaxation [16, 32, 36]. Therefore, as proposed in [35], the out-of-equilibrium phase transition predicted by the Lynden-Bell theory could be associated with a change of regime in the dynamics. For $M(0) < M_{\text{crit}}$, the system mixes well, the evolution is ergodic and the violent relaxation is complete leading to the spatially homogeneous Lynden-Bell distribution. Here, usual thermodynamics (in the sense of Lynden-Bell) applies. By contrast, for $M(0) > M_{\text{crit}}$, violent relaxation seems to be incomplete. In that case, the system does not mix sufficiently well, the evolution is non ergodic and the observed QSS differs from the Lynden-Bell prediction. This is associated with the appearance of fractal-like phase space structures, aging, glassy behaviour, power-law decay of correlations and anomalous diffusion. Rapisarda & Pluchino [42] have proposed to describe these features in terms of Tsallis thermodynamics. On the other hand, in a recent paper, Pluchino et al. [43] have shown explicitly that, in this non-ergodic regime, time averages and ensemble averages differ. The time averages can be fitted by $q$-distributions. We propose that the ensemble averages could also be fitted by a $q$-distribution with an index $q > 1$ leading to a natural velocity cut-off. As we have seen, the index $q = 3$ corresponds to the semi-elliptical distribution observed by Campa et al. [38]. Summarizing the above discussion, it seems that the Lynden-Bell prediction works relatively well far from the transition line separating homogeneous and inhomogeneous states in the Lynden-Bell theory (see the phase diagrams reported in [35, 41]) but that it fails close to this transition line: for fixed $M(0) = 1$ this is around $U_c = 3/4$ (Fig. 1(a) of [34] shows a discrepancy with the Lynden-Bell theory in that region) and for fixed $U = 0.69$ this is around $M_{\text{crit}} = 0.897$ (the simulations of [34, 42, 43, 38] show a discrepancy with the Lynden-Bell theory for $M(0) \sim 1$). In that case, we have non-gaussian velocity distributions, phase space structures, glassy dynamics, aging, anomalous diffusion... We must however be very

5 The kinetic theory developed by Bouchet & Dauxois [9] and Chavanis [10] is valid when the distribution of the bath is spatially homogeneous. When the velocity distribution has gaussian tails, like the Lynden-Bell distributions obtained for $0 < M(0) < M_{\text{crit}}$ [11] (the case $M(0) = 0$ is special since the Lynden-Bell distribution coincides with the water-bag distribution with compact support), the velocity correlation function decays like $\langle v(0)v(t) \rangle \sim (\ln t)/t$ and the diffusion of angles is normal (with logarithmic corrections) [9]. If the velocity distribution of the bath is water-bag or semi-elliptic, like for $M(0) = 0$ [37, 38] or $M(0) = 1$ [34, 38], standard kinetic theory [9, 4, 10] predicts that the velocity correlation function has an exponential decay $\langle v(0)v(t) \rangle \sim e^{-t/\tau}$...
careful because these striking features, like phase space structures and anomalous diffusion, could be due to finite size effects \cite{46} and disappear for $N \rightarrow +\infty$ (note that $N = 256000$ in \cite{46}, $N = 10^5$ in \cite{41} while $N = 2000$ in \cite{42}). In the absence of phase space structures, we suggest that diffusion is normal because the bath distribution (semi-elliptical \cite{38}) has a compact support (see footnote 5).

Morita & Kaneko \cite{47} performed $N$-body numerical simulations starting from an initial condition with energy $U = 0.69$ and magnetization $M(0) = 1$ which is different from the water-bag. In that case, they find that the system does not relax to a QSS but exhibits oscillations whose duration diverges with $N$ (they find that the system relaxes towards the Boltzmann distribution on a timescale $t_{\text{relax}} \sim N$). Therefore, the Lynden-Bell prediction clearly fails. This long-lasting periodic or quasi periodic collective motion appears through Hopf bifurcation and is due to the presence of clumps (high density regions) in phase space. We remark that this behaviour is relatively similar to the one reported by Mineau et al. \cite{20} for self-gravitating systems, except that they observe phase space holes instead of phase space clumps.

4 Slow collisional relaxation

In this section, we discuss the process of slow collisional relaxation in relation with the kinetic equation (33).

4.1 About the $H$-theorem

When the system is spatially inhomogeneous, its collisional evolution can be very complicated and very little is known concerning kinetic equations of the form (33). For example, it is not straightforward to prove by a direct calculation that Eq. (33) conserves the energy. However, since Eq. (33) is exact at order $O(1/N)$, the energy must be conserved. Indeed, the integral constraints of the Hamiltonian system must be conserved at any order of the $1/N$ expansion (note that the neglect of collective effects in Eq. (33) may slightly alter the strict conservation of energy). On the other hand, we cannot establish the $H$-theorem for an equation of the form (33). It is only when additional approximations are implemented (markovian approximation and spatial homogeneity) that the $H$-theorem is obtained. To be more precise, let us compute the rate of change of the Boltzmann entropy $S_B = -\int \frac{k}{m} \ln \frac{k}{m} d\mathbf{r}_1 d\mathbf{v}_1$ with respect to the general kinetic equation (33). After straightforward manipulations obtained by interchanging the indices 1 and 2, it can be put in the form

$$
\dot{S}_B = \frac{1}{2m^2} \int d\mathbf{x}_1 d\mathbf{x}_2 \frac{1}{f_1 f_2} \int_0^t d\tau \left[ \mathcal{F}^{\mu}(2 \rightarrow 1) f_2 \frac{\partial f_1}{\partial v_1^\mu} + \mathcal{F}^{\mu}(1 \rightarrow 2) f_1 \frac{\partial f_2}{\partial v_2^\mu} \right]_t \times G(t, t - \tau) \left[ \mathcal{F}^{\nu}(2 \rightarrow 1) f_2 \frac{\partial f_1}{\partial v_1^\nu} + \mathcal{F}^{\nu}(1 \rightarrow 2) f_1 \frac{\partial f_2}{\partial v_2^\nu} \right]_{t - \tau}.
$$

(39)

We note that its sign is not necessarily positive. This depends on the importance of memory effects. In general, the Markovian approximation is justified for $N \rightarrow +\infty$ because the correlations decay on a timescale $\tau_{\text{corr}}$ that is much smaller than the relaxation time $t_{\text{relax}} \sim N t_D$ on leading to strictly normal diffusion of angles (this will be checked in a future contribution). However, if the system exhibits phase space structures like for $M(0) > M_{\text{crit}}$ and relatively small values of $N$ \cite{34, 42}, the approach of Bouchet & Dauxois \cite{9}, which assumes spatial homogeneity, is not valid anymore. It is precisely the presence of these phase space structures that induces anomalous diffusion as studied in \cite{42}. Therefore, there should not be any controversy since these authors \cite{9} and \cite{42} consider different situations as advocated in \cite{15, 35}.
which the distribution changes (as discussed in Sec. 2.4, this approximation is not completely obvious for self-gravitating systems and for systems that are close to the critical point). In that case, the entropy increases monotonically (see Sec. 2.3 for homogeneous systems). However, even if the energy is conserved and the entropy increases monotonically, it is not completely clear whether the general kinetic equation \[ \text{(33)} \] will relax towards the mean field Maxwell-Boltzmann distribution (1-24) of statistical equilibrium. It could be trapped in a steady state that is not the state of maximal entropy because there is no resonance anymore to drive the relaxation (this is the case for one dimensional homogeneous systems; see the discussion in Sec. 4.3). It could also undergo everlasting oscillations without reaching a steady state. The kinetic equation \[ \text{(33)} \] may have a rich variety of behaviors and its complete study is of great complexity.

### 4.2 The case of stellar systems

The case of stellar systems is special. These systems are spatially inhomogeneous but, in order to evaluate the collisional current in Eq. \[ \text{(33)} \], we can make a local approximation \[ \text{(18)} \] and work as if the system were homogeneous. This is justified by the divergence of the gravitational force \[ \mathbf{F}(2 \rightarrow 1) \] when two particles approach each other so that the fluctuations of the gravitational force are dominated by the contribution of the nearest neighbour \[ \mathbf{r}_2 \rightarrow \mathbf{r}_1 \]. The local approximation amounts to replacing \[ f(\mathbf{r}_2, \mathbf{v}_2, t - \tau) \] by \[ f(\mathbf{r}_1, \mathbf{v}_2, t - \tau) \] in Eq. \[ \text{(33)} \]. This approximation is justified by the fact that the diffusion coefficient diverges logarithmically when \[ \mathbf{r}_2 \rightarrow \mathbf{r}_1 \] (see below). Using the same argument, we can replace \[ \mathcal{F}^\mu(2 \rightarrow 1) \] by \[ F^\mu(2 \rightarrow 1) \] and \[ \mathcal{F}^\mu(1 \rightarrow 2) \] by \[ F^\mu(1 \rightarrow 2) = -F^\mu(2 \rightarrow 1) \]. We shall also make a markovian approximation \[ f(\mathbf{r}_1, \mathbf{v}_1, t - \tau) \approx f(\mathbf{r}_1, \mathbf{v}_1, t) \]

\[ f(\mathbf{r}_1, \mathbf{v}_2, t - \tau) \approx f(\mathbf{r}_1, \mathbf{v}_2, t) \] and extend the time integration to infinity. Then, Eq. \[ \text{(33)} \] becomes

\[
\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f}{\partial r_1} + \frac{N-1}{N} \left< \mathbf{F} \right>_1 \frac{\partial f}{\partial \mathbf{v}_1} = \frac{\partial}{\partial v_1^\mu} \int_0^{+\infty} \! \! dt \int d\mathbf{r}_2 d\mathbf{v}_2 F^\mu(2 \rightarrow 1, t) \mathcal{F}^\nu(2 \rightarrow 1, t - \tau) \times \left( \frac{\partial}{\partial v_1^\nu} - \frac{\partial}{\partial v_2^\nu} \right) f(\mathbf{r}_1, \mathbf{v}_1, t) f(\mathbf{r}_1, \mathbf{v}_2, t). \tag{40}
\]

Making a linear trajectory approximation \[ \mathbf{v}_i(t - \tau) = \mathbf{v}_i(t) \] and \[ \mathbf{r}_i(t - \tau) = \mathbf{r}_i - \mathbf{v}_i \tau \], we can perform the integrations on \[ \mathbf{r}_1 \] and \[ \tau \] like in Appendix A of Paper II. This yields the Vlasov-Landau equation

\[
\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f}{\partial r_1} + \left< \mathbf{F} \right>_1 \frac{\partial f}{\partial \mathbf{v}_1} = 2\pi m G^2 \ln \Lambda \frac{\partial}{\partial \mathbf{v}_1^\mu} \int d\mathbf{v}_2 \frac{\delta^{\mu\nu} w^2 - w^\mu w^\nu}{w^3} \left( \frac{\partial}{\partial v_1^\mu} - \frac{\partial}{\partial v_2^\mu} \right) f(\mathbf{r}_1, \mathbf{v}_1, t) f(\mathbf{r}_1, \mathbf{v}_2, t), \tag{41}
\]

where \[ \ln \Lambda = \int_0^{+\infty} dk/k \] is the Coulombian factor \[ \text{(18)} \]. It must be regularized at small and large scales by introducing appropriate cut-offs, writing \[ \ln \Lambda = \ln(L_{\text{max}}/L_{\text{min}}) \]. Note that the divergence at large scales does not occur in Eq. \[ \text{(33)} \]. It only arises if we assume that the system is spatially homogeneous and infinite (and if we make a Markovian approximation and extend the time integral to infinity). In their stochastic approach, Chandrasekhar & von Neumann \[ \text{(48)} \] argue that the Coulombian factor must be cut-off at the interparticle distance because the fluctuations of the gravitational force are described by the Holtzmark distribution (a particular Lévy law) that is dominated by the contribution of the nearest neighbour. However, Cohen et al. \[ \text{(49)} \], considering a Coulombian plasma, argue that the integral must be cut-off at the Debye length, which is larger than the interparticle distance. This is confirmed by the kinetic theory
of Lenard [50] and Balescu [51] which takes into account collective effects responsible for Debye shielding. In their kinetic theory, there is no divergence at large scales and the natural upper length scale appearing in the Coulombian factor is the Debye length. In plasmas, a charge is surrounded by a polarization cloud of opposite charges that diminishes the interaction. For gravitational systems, there is no shielding so we must stop the integration at $R$, the system size. Therefore, it is the finite spatial extent of the system that removes the Coulombian divergence. In a sense, the system size $R$ (or the Jeans length) plays the role of the Debye length in plasma physics. On the other hand, the divergence at small scales comes from the break up of the linear trajectory approximation when two stars approach each other. This divergence also occurs in Eq. (33) for the same reason: the unperturbed mean field motion becomes incorrect when two stars approach each other. In fact, to obtain Eq. (33), we have assumed that the correlation function $g$ is small with respect to $f$. This is true on average, but it is clear that correlations are important at small scales since two stars have the tendency to form a binary. Therefore, the expansion in powers of $1/N$ is not valid at any scale. One way to circumvent these difficulties is to use Eq. (33) or (41) without modification but introduce a cut-off at the Landau length corresponding to a deflection of 90° of the particles’ trajectory. Thus, we shall take $L_{\min} \sim Gm/v_{\text{typ}}^2$, where $v_{\text{typ}}$ is the typical velocity of a star. Therefore, the Coulombian factor is estimated by $\ln \Lambda = \ln(Rv_{\text{typ}}^2/Gm)$. Now, using a Virial type argument $v_{\text{typ}}^2 \sim \langle v^2 \rangle \sim GM/R$, we find that $\ln \Lambda \sim \ln N$. The relaxation time $t_R$ due to encounters can be estimated from the Vlasov-Landau equation (41) by comparing the scaling of the l.h.s. and r.h.s. This yields $1/t_R \sim mG^2 \ln \Lambda \rho/v_{\text{typ}}^3$. The dynamical time is $t_D \sim R/v_{\text{typ}} \sim 1/\sqrt{\rho G}$ where $\rho \sim M/R^3$ is the density. Comparing these two expressions, we get the scaling

$$t_R \sim \frac{N}{\ln N} t_D. \quad (42)$$

A more precise estimate of the relaxation time is given in [18, 7]. The Vlasov-Landau equation conserves the mass, the energy (kinetic + potential) and monotonically increases the Boltzmann entropy. The mean field Maxwell-Boltzmann distribution (I-24) is the only stationary solution of this equation (cancelling both the advective term and the collision term individually). Therefore, the system tends to reach this distribution on a timescale $(N/\ln N)t_D$. However, there are two reasons why it cannot attain it: (i) Evaporation: when coupled to the gravitational Poisson equation, the mean field Maxwell-Boltzmann distribution (I-24) yields a density profile with infinite mass so there is no physical distribution of the form (I-24) in an infinite domain [54, 55]. The system can increase the Boltzmann entropy indefinitely by evaporating. Therefore, the Vlasov-Landau equation (41) has no steady state with finite mass and the density profile tends to spread indefinitely. (ii) Gravothermal catastrophe: if the energy of the system is lower than the Antonov threshold $E_c = -0.335GM^2/R$ (where $R$ is the system size), it will undergo core collapse. This is called gravothermal catastrophe because the system can increase the Boltzmann entropy indefinitely by contracting and overheating. This process usually dominates over evaporation and leads to the formation of binary stars [18, 55].

### 4.3 One dimensional systems

One dimensional systems are also special. We have seen in Sec. 2.3 that one dimensional systems that are spatially homogeneous do not evolve at all on a timescale $\sim Nt_D$ or larger because of the absence of resonances. However, if the system is spatially inhomogeneous, new

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Note that the divergence at small scales does not occur in the binary encounter treatment of Chandrasekhar [52] and Rosenbluth et al. [53] which takes into account the exact two-body orbit of the particles instead of making a straight line approximation.
resonances can appear as described in [56] so that an evolution is possible on a timescale $N t_D$. Then, we can expect that one dimensional inhomogeneous systems will tend to approach the Boltzmann distribution on the timescale $N t_D$. To be more precise, let us consider the orbit-averaged-Fokker-Planck equation derived in [56]. Exploiting the timescale separation between the dynamical time and the relaxation time, we can average Eq. (33) over the orbits, assuming that at any stage of its evolution the system reaches a mechanical equilibrium on a short dynamical time. Therefore, the distribution function is a stationary solution of the Vlasov equation
\[ f \simeq f(\epsilon, t) \] [where $\epsilon = v^2/2 + \Phi$ is the individual energy] slowly evolving in time under the effect of “collisions” (= correlations due to finite $N$ effects). Introducing angle-action variables, we get an equation of the form [56]:
\[ \frac{\partial f}{\partial t} = \frac{1}{2} \sum_{m,m'} \int m A_{mm'}(J, J')^2 \delta(m\Omega(J) - m'\Omega(J')) \left\{ f(J') m \frac{\partial f}{\partial J} - f(J) m' \frac{\partial f}{\partial J'} \right\} dJ'. \] (43)

The important point to notice is that the evolution of the system is due to a condition of resonance between the pulsations $\Omega(J)$ of the particles’ orbits (this property probably extends to $d$ dimensions but is technically more complicated to show). Only particles whose pulsations satisfy $m\Omega(J) = m'\Omega(J')$ with $(m, J) \neq (m', J')$ participate to the diffusion current. This is similar to the collisional relaxation of two dimensional point vortices [57, 32]. It can be shown that Eq. (43) conserves mass and energy and monotonically increases entropy so that the system tends to approach the Boltzmann distribution of statistical equilibrium on a timescale $\sim N t_D$ [56]. However, it may happen that there is not enough resonances so that the system can be trapped in a quasi stationary state different from the Boltzmann distribution. This happens when the condition of resonance cannot be satisfied so that $m\Omega(J) \neq m'\Omega(J')$ for all $(m, J) \neq (m', J')$. In that case, the system is in a steady state of Eq. (43) which is not the Boltzmann distribution. This is what happens to point vortices in 2D hydrodynamics when the profile of angular velocity becomes monotonic [57]. In that case, the relaxation stops and the system will relax on a timescale larger than $N t_D$. We may wonder whether the same situation can happen to systems described by a kinetic equation of the form (33).

5 Conclusions and perspectives

In this paper, we have developed a kinetic theory for Hamiltonian systems with weak long-range interactions. A specificity of these systems is that they can be spatially inhomogeneous, which considerably complicates the kinetic theory. We have shown that the development of correlations between particles creates a current in the r.h.s. of Eq. (6) that is the counterpart of the collision term in the Boltzmann equation for neutral gases. Therefore, for Hamiltonian systems with weak long-range interactions, the evolution beyond the Vlasov regime is driven by “correlations” due to finite $N$ effects. We have obtained a kinetic equation (33) valid at order $O(1/N)$ that describes the evolution of the system on a timescale $\sim N t_D$. For homogeneous systems, this equation reduces to the Landau equation. For $d > 1$, the Landau equation relaxes towards the Boltzmann distribution. Therefore, for $d > 1$, the relaxation time scales like $t_{\text{relax}} \sim N t_D$. This scaling has been predicted and observed in a spatially homogeneous two-dimensional Coulombian plasma [4, 58, 7]. This scaling probably remains true for spatially inhomogeneous systems in $d > 1$ with the exception of self-gravitating systems that relax towards the mean-field Boltzmann distribution on a timescale $t_{\text{relax}} \sim (N/\ln N) t_D$, unless they experience evaporation or gravothermal catastrophe. For one dimensional systems, like the HMF model, the situation is more complicated. For Vlasov-stable homogeneous systems, the kinetic equation (33) reduces to $\partial f/\partial t = 0$. Therefore, there is no evolution on a timescale of...
the order $\sim Nt_D$. We conclude that the relaxation time is larger than $Nt_D$. We could imagine that the evolution is due to three-body, four-body,... correlations leading to a relaxation time of the order of $N^2t_D, N^3t_D,...$ However, Campa et al. [35], considering initial conditions with supercritical energy $U > U_c = 3/4$ for which the system is always spatially homogeneous, found that the relaxation time is extremely long scaling like $t_{\text{relax}} \sim e^N$. This suggests that the expansion of the BBGKY hierarchy in powers of $1/N$ may not be convergent in the homogeneous case and that another approach should be developed in that case. On the other hand, Morita & Kaneko [47] considering an initial condition with $U < U_c$ and $M(0) = 1$, found a relaxation time of the order $t_{\text{relax}} \sim Nt_D$. In their simulations, the system is always spatially inhomogeneous (the magnetization in the oscillatory regime is non-zero). As explained in Sec. 4.3, spatial inhomogeneities can create new resonances that drive the relaxation towards the Boltzmann equilibrium (BE) on a timescale $t_{\text{relax}} \sim Nt_D$ (predicted by the kinetic theory) that is much shorter than when the system remains spatially homogeneous. This could be an explanation (but not the only one) for the observed timescale in [47]. Yamaguchi et al. [37], considering a water-bag initial condition with $U < U_c$, and $M(0) = 0$, found a relaxation time $t_{\text{relax}} \sim N^{1.7}t_D$ with $\delta = 1.7$. In their simulations, the system is spatially homogeneous but it progressively becomes Vlasov unstable and undergoes a dynamical phase transition from the homogeneous QSS to the inhomogeneous BE. This instability considerably accelerates the relaxation towards the Boltzmann equilibrium with respect to the case $U > U_c$ [38] where the homogeneous distribution remains Vlasov stable until the end (see below). The same phase transition happens in the simulations of Latora et al. [34] who considered a water-bag initial condition with $U < U_c$ and $M(0) = 1$. Their system is roughly spatially homogeneous ($M_{\text{QSS}} \simeq 0$) but it also presents some phase space structures which may explain why they find a relaxation time of the order $t_{\text{relax}} \sim Nt_D$ shorter than $t_{\text{relax}} \sim N^{1.7}t_D$ (we have seen that spatial inhomogeneities can accelerate the relaxation by creating new resonances). More generally, for water-bag initial conditions, it would be interesting to determine how the exponent $\delta$ depends on the initial magnetization $M(0)$ and energy $U$. Considering the phase diagram in $(M(0), U)$ plane reported in [44], we suggest that $t_{\text{relax}} \sim e^N$ above the critical line $U_c = 3/4$ (no dynamical phase transition) and $t_{\text{relax}} \sim N^{1.7}t_D$ below the critical line $U_c = 3/4$ (dynamical phase transition). For given initial magnetization $M(0)$, we expect that $\delta$ diverges as we approach the critical energy $U_c = 3/4$ above which $t_{\text{relax}} \sim e^N$. Below the critical line $U_c = 3/4$ and above the transition line $U_{\text{crit}}(M(0))$ (the curve in Fig. 1 of [44]), observations [37] show that $\delta \sim 1.7$. Below the transition line $U_{\text{crit}}(M(0))$, the QSS is either spatially inhomogeneous (according to Lynden-Bell’s prediction if relaxation is complete) or spatially homogeneous with phase space structures in case of incomplete relaxation [34, 42]. In that case, observations [34] show that $\delta \sim 1$, which is consistent with the kinetic theory for inhomogeneous systems. Thus, for a given energy $U < U_c$, we expect that $\delta$ passes from $\delta > 1.7$ to $\delta = 1$ when $M(0)$ overcomes the critical magnetization $M_{\text{crit}}(U)$ [35, 44]. These ideas, that are consistent with the partial numerical information that we have at present, demand to be developed in more detail. In fact, this is a first attempt to connect the relaxation time to the phase diagram $(U, M(0))$ and the situation may be more complicated than that. In particular, the relaxation time seems to strongly depend on the detailed structure of the initial condition. For example, using an isotropic water bag initial condition with $M(0) = 1$, Campa et al. [38] find a relaxation time $t_{\text{relax}} \sim N^{1.7}t_D$ instead of the scaling $t_{\text{relax}} \sim Nt_D$ reported by Latora et al. [34]. This may be related to the lack of phase space structures in the simulations of [38].

Finally, we conclude by proposing the following scenario similar to the one proposed by Taruya & Sakagami [21] for self-gravitating systems. Analyzing the numerical results of [34, 37, 38], we argue that, for many initial conditions, the transient states of the collisional relaxation of the HMF model can be described by spatially homogeneous Tsallis distributions (polytropes)
with a time varying index $q(t) \geq 1$ (i.e. $n(t) \geq 1/2, 1 \leq \gamma(t) \leq 3$) corresponding to a compact support. More precisely, we parametrize these transient states by a distribution function of the form $f(v, t) = f(0, t) [1 - v^2/v_{max}(t)^2]^{n(t)-1/2}$. It is easy to show that $n(t)$ and $v_{max}(t)$ are related to each other by $n(t) = 2\pi v_{max}(t)^2/(kN\epsilon) - 1$ where $\epsilon = 4/(U - 1/2)$ is the conserved energy and $2\pi/(kN) = 1$ in usual notations. This simple relation follows from Eqs. (28) and (29) of [40] for homogeneous systems where $\Phi = 0, \rho = N/2\pi$ and $p = E/\pi$. It allows to determine $n(t)$ by simply measuring $v_{max}(t)$. These Tsallis distributions are quasi-stationary solutions of the Vlasov equation slowly evolving with time under the effect of collisions (finite $N$ effects). Initially, $n(t)$ is close to $n = 1$ (i.e. $\gamma = 2, q = 3$) corresponding to the semi-elliptical distribution observed by Campa et al. [38] in many circumstances. For $U = 0.69$ (i.e. $\epsilon = 0.76$) and $n = 1$ we get $v_{max} \simeq 1.23$. Progressively, $n(t)$ increases so as to attain the value $n \rightarrow +\infty$ (i.e. $\gamma = 1, q = 1$) corresponding to the Boltzmann equilibrium state for $t \rightarrow +\infty$. For $\epsilon < 1$ (i.e. $U < U_c = 3/4$), there exists a time $t_*$ at which $n(t) = n_{crit} = \epsilon/(1 - \epsilon)$ (corresponding to $\gamma(t) = \gamma_{crit} = 1/\epsilon, q(t) = q_{crit} = (\epsilon + 1)/(3\epsilon - 1)$) so that the Tsallis distribution becomes Vlasov unstable (see the criterion (156) of [10]) and the system rapidly relaxes towards the inhomogeneous Boltzmann distribution [1]. This accounts for the sudden dynamical phase transition observed in [37] from the homogeneous QSS ($M_{QSS} \simeq 0$) to the inhomogeneous BE ($M = M_{eq} \neq 0$). For $U = 0.69$, the transition corresponds to $n_{crit} \simeq 3.166$ leading to $v_{max} \simeq 1.78$ in qualitative agreement with [38]. In the supercritical case $\epsilon > 1$ (i.e. $U > U_c = 3/4$), the Tsallis distributions with $q(t) \geq 1$ are always Vlasov stable (since $q_{crit} = (\epsilon + 1)/(3\epsilon - 1) = (4U - 1)/(12U - 7) < 1$ or, alternatively, $\epsilon_{crit} = (q + 1)/(3q - 1) < 1$ or $U_{crit} = (7q - 1)/[4(3q - 1)] < U_c = 3/4$ which explains the long lifetime behaviour observed by Campa et al. [38]. This scenario suggests that Tsallis distributions can be attractors (or at least provide a good fit) for the transient states of the collisional relaxation, for a large class of initial conditions ($U = 0.69 < U_c$ with $M(0) = 0$ [37, 38]; $U = 0.69 < U_c$ with $M(0) = 1$ [34, 38]; and $U > U_c$ [38]). Note, however, that the previous scenario is not valid for all initial conditions so these attractors are not universal. For example, in the numerical simulations of Antoniazzi et al. [41], the system relaxes towards a Lynden-Bell distribution with gaussian tails for $0 < M < M_{crit} = 0.897$ and in the numerical simulations of Morita & Kaneko [17], the system develops everlasting oscillations. However, this picture now suggests to look in detail into the chaotic dynamics of the system in order to determine the basin of attraction of the Tsallis distributions [34, 37, 38], the Lynden-Bell distributions [41] and the oscillatory states [17]. We hope to develop these issues, and check the above scenario, in future communications.

A Some other kinetic equations

If we make the Markovian approximation $f(v_1, t - \tau) \simeq f(v_1, t)$ and $f(v_2, t - \tau) \simeq f(v_2, t)$ in Eq. (26) but do not extend the time integral to infinity, we obtain

$$\frac{\partial f_1}{\partial t} = \frac{\partial}{\partial v_1^i} \int dv_2 K^{i\nu}(w, t) \left( \frac{\partial}{\partial v_1^i} - \frac{\partial}{\partial v_2^\nu} \right) f(v_1, t)f(v_2, t),$$

(44)

with

$$K^{i\nu}(w, t) = (2\pi)^d m \int_0^t d\tau \int dk k^\mu k^\nu \hat{u}(k)^2 \cos(k \cdot w \tau).$$

(45)

Note that Taruya & Sakagami [21] interprete this transition as a generalized thermodynamical instability (in Tsallis sense) while we interprete it as a dynamical instability with respect to the Vlasov equation [40].
In $d = 3$, the components of this tensor can be calculated by introducing a spherical system of coordinates with the $z$-axis in the direction of $\mathbf{w}$. We find that

$$K^{\mu\nu}(\mathbf{w}, t) = A(w, t)\frac{w^2\delta^{\mu\nu} - w^\mu w^\nu}{w^2} + B(w, t)\frac{w^\mu w^\nu}{w^2}, \quad (46)$$

with

$$A(w, t) = \frac{8\pi^4 m}{w^2} \int_0^{+\infty} k^3 dk \hat{u}(k)^2 \int_0^{kw t} \left( 4 \frac{\sin \tau}{\tau^3} - 4 \frac{\cos \tau}{\tau^2} \right) d\tau, \quad (47)$$

$$B(w, t) = \frac{16\pi^4 m}{w^2} \int_0^{+\infty} k^3 dk \hat{u}(k)^2 \int_0^{kw t} \left( 2 \frac{\sin \tau}{\tau} + 4 \frac{\cos \tau}{\tau^2} - 4 \frac{\sin \tau}{\tau^3} \right) d\tau. \quad (48)$$

For $t \to +\infty$, the functions $A$ and $B$ reduce to

$$A(w) = \frac{8\pi^4 m}{w^2} \int_0^{+\infty} k^3 \hat{u}(k)^2 dk, \quad B(w) = 0, \quad (49)$$

and we recover the results (II-42) and (II-43) of Paper II. In $d = 2$, the components of the tensor (45) can be calculated by introducing a polar system of coordinates with the $x$-axis in the direction of $\mathbf{w}$. This leads to Eq. (46) with now

$$A(w, t) = \frac{8\pi^4 m}{w^2} \int_0^{+\infty} k^3 dk \hat{u}(k)^2 \int_0^{kw t} \frac{J_1(\tau)}{\tau} d\tau, \quad (50)$$

$$B(w, t) = \frac{8\pi^4 m}{w^2} \int_0^{+\infty} k^3 dk \hat{u}(k)^2 \int_0^{kw t} \left[ \frac{J_1(\tau)}{\tau} - J_2(\tau) \right] d\tau. \quad (51)$$

For $t \to +\infty$, the functions $A$ and $B$ reduce to

$$A(w) = \frac{8\pi^4 m}{w^2} \int_0^{+\infty} k^3 \hat{u}(k)^2 dk, \quad B(w) = 0, \quad (52)$$

and we recover the results (II-42) and (II-43) of Paper II. Finally, in $d = 1$, we obtain

$$K(w, t) = 4\pi m \int_0^{+\infty} k^2 \hat{u}(k) \frac{\sin(kw t)}{kw} dk. \quad (53)$$

For $t \to +\infty$, we find that

$$K(w) = 4\pi^2 m \delta(w) \int_0^{+\infty} k \hat{u}(k)^2 dk, \quad (54)$$

which returns Eq. (23). Finally, we recall that Eq. (20) ignores collective effects. As a simple generalization, we could replace in Eq. (20) the potential $\hat{u}(k)$ by the “screened” potential $\hat{u}(k)/|\epsilon(k, \mathbf{k} \cdot \mathbf{v}_2)|$ including the dielectric function (for the HMF model, this amounts to dividing the integrand of Eq. (27) by $|\epsilon(1, v_2)|^2$). This leads to

$$\frac{\partial f_1}{\partial t} = (2\pi)^d m \frac{\partial}{\partial v_1^i} \int_0^t d\tau \int d\mathbf{v}_2 d\mathbf{k} \delta^{\mu\nu} \frac{\hat{u}(k)^2}{|\epsilon(k, \mathbf{k} \cdot \mathbf{v}_2)|^2} \cos(\mathbf{k} \cdot \mathbf{w} \tau) \times \left( \frac{\partial}{\partial v_1^i} - \frac{\partial}{\partial v_2^i} \right) f(\mathbf{v}_1, t - \tau) f(\mathbf{v}_2, t - \tau). \quad (55)$$

Strictly speaking, this procedure is not rigorously justified for non Markovian systems since the dielectric function is obtained by assuming precisely that the distribution function does not change on the timescale of interest. Yet, this generalization could be performed heuristically in order to obtain a non Markovian kinetic equation taking into account some collective effects. If we make the Markovian approximation and extend the time integral to infinity, Eq. (55) returns the Lenard-Balescu equation.
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