Negative Dimension in General and
Asymptotic Topology

V.P. Maslov*

Abstract

We introduce the notion of negative topological dimension and the notion of weight for the asymptotic topological dimension. Quantizing of spaces of negative dimension is applied to linguistic statistics.

Recently, Yu. I. Manin has presented his considerations for the density of spaces of negative dimension [1].

1. Let us consider the simplest examples of (Haar) measures in the general case for the \(n\)-dimensional space. Let \(S_n\) be an \(n\)-dimensional ball of radius \(r\). In the spherical coordinates, the volume \(\mu(S_n)\) of the ball is equal to \(\text{const} \int_0^1 r^{n-1} dr = \text{const} \, r^n\). Here \(r^{n-1}\) stands for the density.

In the sense of the Fourier transform, the multiplication by a coordinate is dual to the corresponding derivation. Therefore, we can speak of dual \(n\) times differentiable functions in the Sobolev space \(W^2_2\). Dirac distinguished between the left and right components "bra" and "ket" in the "bracket" inner product. The "dual" space of this space according to Dirac is the space \(W^{-2}_2\) of Sobolev distributions (generalized functions).

In the same way we can define the functions in \(W^s_2\) by the "inner product," where \(s\) is a positive noninteger number, and the space \(W^{-s}_2\) as the "inner product" conjugate to \(W^s_2\).

One can similarly proceed with the density (or the weight) \(r^s\) and \(r^{-s}\), by using, for instance, the Riesz kernel or the Bessel potential to represent functions in \(W^s_2\).

Let us present an example of a space (of noninteger positive dimension) equipped with the Haar measure \(r^\sigma\), where \(0 \leq \sigma \leq 1\).

On the closed interval \(0 \leq x \leq 1\) there is a scale \(0 \leq \sigma \leq 1\) of Cantor dust with the Haar measure equal to \(x^\sigma\) for any interval \((0, x)\) similar to the entire given set of the Cantor dust. The direct product of this scale by the Euclidean cube of dimension \(k - 1\) gives the entire scale \(k + \sigma\), where \(k \in \mathbb{Z}\) and \(\sigma \in (0, 1)\).

General definition of spaces of negative dimension. Let \(M_{t_0}\) be a compactum, of Hausdorff dimension \(t_0\), which is an element of a \(t\)-parameter scale of mutually embedded compacta, \(0 < t < \infty\). Two scales of this kind are said to be equivalent with respect to the compactum \(M_{t_0}\) if all compacta in these scales coincide for any \(t \geq t_0\). We say that the compactum \(M_{t_0}\) is a hole in this equivalent set of scales and the number \(-t_0\) is the negative dimension of this equivalence class.

We consider the space of negative dimension \(-D = -k - \sigma\) with respect to the given above scale.

---

*Moscow State University, Physics Department, v.p.maslov@mail.ru
2. As in [2], the values of the random variable $x_1, \ldots, x_s$ are ordered in absolute value. Some of the numbers $x_1, \ldots, x_s$ may coincide. Then these numbers are combined adding the corresponding "probabilities", i.e., the ratio of the number of "hits" at $x_i$ to the general number of trials. The number of equal $x_i : x_i = x_{i+1} = \cdots = x_{i+k}$ will be called the multiplicity $q_i$ of the value $x_i$. In our consideration, both the number of trials $N$ and $s$ tend to infinity.

Let $N_i$ be the number of "appearances" of the value $x_i : x_i < x_{i+1}$, then

$$\sum_{i=1}^{s} \frac{N_i}{N} x_i = M,$$  \hfill (1)

where $M$ is the mathematical expectation.

The cumulative probability $P_k$ is the sum of the first $k$ probabilities in the sequence $x_i$: $P_k = \frac{1}{s} \sum_{i=1}^{k} N_i$, where $k < s$. We denote $NP_k = B_k$.

If all the variants for which

$$\sum_{i=1}^{s} N_i = N$$ \hfill (2)

and

$$\sum_{i=1}^{s} N_i x_i \leq E, \quad E = MN \leq N\overline{q},$$ \hfill (3)

where $\overline{q} = \frac{\sum_{i=1}^{s} q_i x_i}{Q}, \quad Q = \sum_{i=1}^{s} q_i$, are equivalent (equiprobable), then [3] [4] [5] the majority of the variants will accumulate near the following dependence of the "cumulative probability" $B_l\{N_i\} = \sum_{i=1}^{l} N_i$,

$$\sum_{i=1}^{l} N_i = \sum_{i=1}^{l} \frac{q_i}{e^{\beta x_i - \nu} - 1},$$ \hfill (4)

where $\beta'$ and $\nu'$ are determined by the conditions

$$B_s = N,$$ \hfill (5)

$$\sum_{i=1}^{s} \frac{q_i x_i}{e^{\beta x_i - \nu} - 1} = E,$$ \hfill (6)

as $N \to \infty$ and $s \to \infty$.

We introduce the notation: $\mathcal{M}$ is the set of all sets $\{N_i\}$ satisfying conditions (2) and (3); $N\{\mathcal{M}\}$ is the number of elements of the set $\mathcal{M}$.

**Theorem 1** Suppose that all the variants of sets $\{N_i\}$ satisfying the conditions (2) and (3) are equiprobable. Then the number of variants $N$ of sets $\{N_i\}$ satisfying conditions (2) and (3) and the additional relation

$$\left| \sum_{i=1}^{l} N_i - \sum_{i=1}^{l} \frac{q_i}{e^{\beta x_i - \nu} - 1} \right| \geq N^{(3/4 + \varepsilon)}$$ \hfill (7)

is less than $\frac{c_1 N\{\mathcal{M}\}}{N^m}$ (where $c_1$ and $m$ are any arbitrary numbers, $\sum_{i=1}^{l} q_i \geq \varepsilon Q$, and $\varepsilon$ is arbitrarily small).
Proof of Theorem 1.

Let \( A \) be a subset of \( M \) satisfying the condition

\[
| \sum_{i=1}^l N_i - \sum_{i=1}^l e^{\beta x_i - \nu} | \leq \Delta,
\]

where \( \Delta, \beta, \nu \) are some real numbers independent of \( l \).

We denote

\[
| \sum_{i=l+1}^s N_i - \sum_{i=l+1}^s e^{\beta x_i - \nu} | = S_{s-l};
\]

\[
| \sum_{i=1}^l N_i - \sum_{i=1}^l e^{\beta x_i - \nu} | = S_l.
\]

Obviously, if \( \{N_i\} \) is the set of all sets of integers on the whole, then

\[
\mathcal{N}\{M \setminus A\} = \sum_{\{N_i\}} (\Theta(E - \sum_{i=1}^s N_i x_i) \delta_{(\sum_{i=1}^s N_i)x_i} \Theta(S_l - \Delta) \Theta(S_{s-l} - \Delta)),
\]

where \( \sum N_i = N \).

Here the sum is taken over all integers \( N_i \), \( \Theta(x) \) is the Heaviside function, and \( \delta_{k_1,k_2} \) is the Kronecker symbol.

We use the integral representations

\[
\delta_{N,N'} = \frac{e^{-\nu N}}{2\pi} \int_{-\pi}^{\pi} d\varphi e^{-iN\varphi} e^{\nu N' e^{i\varphi}},
\]

(9)

\[
\Theta(y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \frac{1}{x - i} e^{\beta y(1+i\alpha)}.
\]

(10)

Now we perform the standard regularization. We replace the first Heaviside function \( \Theta \) in (8) by the continuous function

\[
\Theta_\alpha(y) = \begin{cases} 
0 & \text{for } \alpha > 1, \ y < 0 \\
1 - e^{\beta y(1-\alpha)} & \text{for } \alpha > 1, \ y \geq 0,
\end{cases}
\]

\[
\Theta_\alpha(y) = \begin{cases} 
e^{\beta y(1-\alpha)} & \text{for } \alpha < 0, \ y < 0 \\
1 & \text{for } \alpha < 0, \ y \geq 0,
\end{cases}
\]

where \( \alpha \in (-\infty, 0) \cup (1, \infty) \) is a parameter, and obtain

\[
\Theta_\alpha(y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\beta y(1+i\alpha)} \left( \frac{1}{x - i} - \frac{1}{x - \alpha i} \right) dx.
\]

(11)

If \( \alpha > 1 \), then \( \Theta(y) \leq \Theta_\alpha(y) \).

Let \( \nu < 0 \). We substitute (9) and (10) into (8), interchange the integration and summation, then pass to the limit as \( \alpha \to \infty \) and obtain the estimate

\[
\mathcal{N}\{M \setminus A\} \leq \left| \frac{e^{-\nu N + \beta E}}{i(2\pi)^2} \int_{-\pi}^{\pi} \left[ \exp(-iN\varphi) \sum_{\{N_j\}} \exp\{-\beta \sum_{j=1}^s N_j x_j + (i\varphi + \nu) \sum_{j=1}^s N_j \} \right] d\varphi \times \Theta(S_l - \Delta) \Theta(S_{s-l} - \Delta) \right|,
\]

(12)
where \( \beta \) and \( \nu \) are real parameters such that the series converges for them.

To estimate the expression in the right-hand side, we bring the absolute value sign inside the integral sign and then inside the sum sign, integrate over \( \varphi \), and obtain

\[
\mathcal{N}\{\mathcal{M} \setminus \mathcal{A}\} \leq \frac{e^{-\nu N + \beta E}}{2\pi} \sum_{\{N_i\}} \exp\{-\beta \sum_{i=1}^{s} N_i x_i + \nu \sum_{i=1}^{s} N_i\} \times \\
\times \Theta(S_i - \Delta) \Theta(S_{s-l} - \Delta).
\]

We denote

\[
Z(\beta, N) = \sum_{\{N_i\}} e^{-\beta \sum_{i=1}^{s} N_i x_i},
\]

where the sum is taken over all \( N_i \) such that \( \sum_{i=1}^{s} N_i = N \),

\[
\zeta_l(\nu, \beta) = \prod_{i=1}^{l} \xi_i(\nu, \beta); \zeta_{s-l}(\nu, \beta) = \prod_{i=l+1}^{s} \xi_i(\nu, \beta);
\]

\[
\xi_i(\nu, \beta) = \frac{1}{1 - e^{\nu - \beta x_i}} q_i, \quad i = 1, \ldots, l.
\]

It follows from the inequality for the hyperbolic cosine \( \cosh(x) = (e^x + e^{-x})/2 \) for \( |x_1| \geq \delta; |x_2| \geq \delta \):

\[
\cosh(x_1) \cosh(x_2) = \cosh(x_1 + x_2) + \cosh(x_1 - x_2) > \frac{e^\delta}{2}
\]

that the inequality

\[
\Theta(S_{s-l} - \Delta) \Theta(S_l - \Delta) \leq e^{-c\Delta} \cosh\left(c \sum_{i=1}^{l} N_i - c\phi_l\right) \cosh\left(c \sum_{i=l+1}^{s} N_i - c\phi_{s-l}\right),
\]

where

\[
\phi_l = \sum_{i=1}^{l} \frac{q_i}{e^{\beta x_i - \nu'} - 1}; \quad \phi_{s-l} = \sum_{i=l+1}^{s} \frac{q_i}{e^{\beta x_i - \nu'} - 1},
\]

holds for all positive \( c \) and \( \Delta \).

We obtain

\[
\mathcal{N}\{\mathcal{M} \setminus \mathcal{A}\} \leq e^{-c\Delta} \exp(\beta E - \nu N) \times \\
\times \sum_{\{N_i\}} \exp\{-\beta \sum_{i=1}^{l} N_i x_i + \nu \sum_{i=1}^{l} N_i\} \cosh\left(\sum_{i=1}^{l} cN_i - c\phi\right) \times \\
\times \exp\{-\beta \sum_{i=l+1}^{s} N_i x_i + \nu \sum_{i=l+1}^{s} N_i\} \cosh\left(\sum_{i=l+1}^{s} cN_i - c\phi\right) = \\
= e^{\beta E} e^{-c\Delta} \times \\
\times (\zeta_l(\nu - c, \beta) \exp(-c\phi_l) + \zeta_l(\nu + c, \beta) \exp(c\phi_l)) \times \\
\times (\zeta_{s-l}(\nu - c, \beta) \exp(-c\phi_{s-l}) + \zeta_{s-l}(\nu + c, \beta) \exp(c\phi_{s-l})).
\]

Now we use the relations

\[
\frac{\partial}{\partial \nu} \ln \zeta_l|_{\beta=\beta', \nu=\nu'} \equiv \phi_l; \quad \frac{\partial}{\partial \nu} \ln \zeta_{s-l}|_{\beta=\beta', \nu=\nu'} \equiv \phi_{s-l}
\]
and the expansion $\zeta_l(\nu \pm c, \beta)$ by the Taylor formula. There exists a $\gamma < 1$ such that

$$\ln(\zeta_l(\nu \pm c, \beta)) = \ln \zeta_l(\nu, \beta) \pm c(\ln \zeta_l')(\nu, \beta) + \frac{c^2}{2}(\ln \zeta_l'')(\nu \pm \gamma c, \beta).$$

We substitute this expansion, use formula (18), and see that $\phi_{\nu, \beta}$ is cancelled.

Another representation of the Taylor formula implies

$$\ln(\zeta_l(\nu + c, \beta)) = \ln(\zeta_l(\beta, \nu)) + \frac{c}{\beta} \partial \ln(\zeta_l(\beta, \nu)) +$$

$$+ \int_{\nu}^{\nu+c/\beta} d\nu' (\nu + c/\beta - \nu') \frac{\partial^2}{\partial \nu'^2} \ln(\zeta_l(\beta, \nu')).$$

(19)

A similar expression holds for $\zeta_{s-l}$.

From the explicit form of the function $\zeta_l(\beta, \nu)$, we obtain

$$\frac{\partial^2}{\partial \nu^2} \ln(\zeta_l(\beta, \nu)) = \beta^2 \sum_{i=1}^l g_i \frac{\exp(-\beta(x_i + \nu))}{(\exp(-\beta(x_i + \nu)) - 1)^2} \leq \beta^2 Qd,$$

(20)

where $d$ is given by the formula

$$d = \frac{\exp(-\beta(x_1 + \nu))}{(\exp(-\beta(x_1 + \nu)) - 1)^2}.$$

The same estimate holds for $\zeta_{s-l}$.

Taking into account the fact that $\zeta_l \zeta_{s-l} = \zeta_s$, we obtain the following estimate for $\beta = \beta'$ and $\nu = \nu'$:

$$N \{\mathcal{M} \setminus A\} \leq \zeta_s(\beta', \nu') \exp(-c\Delta + \frac{c^2}{2} \beta^2 Qd) \exp(E\beta' - \nu' N).$$

(21)

Now we express $\zeta_s(\nu', \beta')$ in terms $Z(\beta, N)$. To do this, we prove the following lemma.

**Lemma 1** Under the above assumptions, the asymptotics of the integral

$$Z(\beta, N) = \frac{e^{-\nu N}}{2\pi} \int_{-\pi}^{\pi} d\alpha e^{-iN\alpha} \zeta_s(\beta, \nu + i\alpha)$$

(22)

has the form

$$Z(\beta, N) = Ce^{-\nu N} \frac{\zeta_s(\beta, \nu)}{[(\partial^2 \ln \zeta_s(\beta, \nu))/(\partial^2 \nu)]} (1 + O(\frac{1}{N})),$$

(23)

where $C$ is a constant.

We have

$$Z(\beta, N) = \frac{e^{-\nu N}}{2\pi} \int_{-\pi}^{\pi} e^{-iN\alpha} \zeta_s(\beta, \nu + i\alpha) d\alpha = \frac{e^{-\nu N}}{2\pi} \int_{-\pi}^{\pi} e^{NS(\alpha, N)} d\alpha,$$

(24)

where

$$S(\alpha, N) = -i\alpha + \ln \zeta_s(\beta, \nu + i\alpha) = -i\alpha - \sum_{i=1}^s q_i \ln[1 - e^{\nu+i\alpha - \beta x_i}].$$

(25)
Here $S$ depends on $N$, because $s$, $x_i$, and $\nu$ also depend on $N$; the latter is chosen so that the point $\alpha = 0$ be a stationary point of the phase $S$, i.e., from the condition

$$N = \sum_{i=1}^{s} \frac{q_i}{e^{\beta x_i - \nu} - 1}.$$  \hspace{1cm} (26)

We assume that $a_1 N \leq s \leq a_2 N$, $a_1, a_2 = \text{const}$, and, in addition, $0 \leq x_i \leq B$ and $B = \text{const}$, $i = 1, \ldots, s$. If these conditions are satisfied in some interval $\beta \in [0, \beta_0]$ of the values of the inverse temperature, then all the derivatives of the phase are bounded, the stationary point is nondegenerate, and the real part of the phase outside a neighborhood of zero is strictly less than its value at zero minus some positive number. Therefore, calculating the asymptotics of the integral, we can replace the interval of integration $[-\pi, \pi]$ by the interval $[-\varepsilon, \varepsilon]$. In this integral, we perform the change of variable

$$z = \sqrt{S(0, N) - S(\alpha, N)}. \hspace{1cm} (27)$$

This function is holomorphic in the disk $|\alpha| \leq \varepsilon$ in the complex $\alpha$-plane and has a holomorphic inverse for a sufficiently small $\varepsilon$. As a result, we obtain

$$\int_{-\varepsilon}^{\varepsilon} e^{NS(\alpha, N)} d\alpha = e^{NS(0, N)} \int_{\gamma} e^{-Nz^2} f(z) dz,$$  \hspace{1cm} (28)

where the path $\gamma$ in the complex $z$-plane is obtained from the interval $[-\varepsilon, \varepsilon]$ by the change (27) and

$$f(z) = \left( \frac{\partial \sqrt{S(0, N) - S(\alpha, N)}}{\partial \alpha} \right)^{-1} \bigg|_{\alpha = \alpha(z)}. \hspace{1cm} (29)$$

For a small $\varepsilon$ the path $\gamma$ lies completely inside the double sector $\text{re}(z^2) > c(\text{re} z)^2$ for some $c > 0$; hence it can be “shifted” to the real axis so that the integral does not change up to terms that are exponentially small in $N$. Thus, with the above accuracy, we have

$$Z(\beta, N) = \frac{e^{-\nu N}}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{-Nz^2} f(z) dz. \hspace{1cm} (30)$$

Since the variable $z$ is now real, we can assume that the function $f(z)$ is finite (changing it outside the interval of integration), extend the integral to the entire axis (which again gives an exponentially small error), and then calculate the asymptotic expansion of the integral expanding the integrand in the Taylor series in $z$ with a remainder. This justifies that the saddle-point method can be applied to the above integral in our case.

**Lemma 2** The quantity

$$\frac{1}{N(M)} \sum_{\{N_i\}} e^{-\beta \sum_{i=1}^{s} N_i x_i}, \hspace{1cm} (31)$$

where $\sum N_i = N$ and $x_i N_i \leq E - N^{1/2 + \varepsilon}$, tends to zero faster than $N^{-k}$ for any $k, \varepsilon > 0$.

We consider the point of minimum in $\beta$ of the right-hand side of (17) with $\nu(\beta, N)$ satisfying the condition

$$\sum_{i} \frac{q_i}{e^{\beta x_i - \nu(\beta, N)} - 1} = N.$$.  \hspace{1cm} (32)
It is easy to see that it satisfies condition (5). Now we assume that the assumption of the lemma is not satisfied.

Then for \( \sum N_i = N, \sum x_i N_i \geq E - N^{1/2+\varepsilon} \), we have

\[
e^\beta E \sum_{\{N_i\}} e^{-\beta \sum_{i=1}^{N} x_i N_i} \geq e^{(N^{1/2+\varepsilon})\beta}.
\]

Obviously, \( \beta \ll \sqrt{N} \) provides a minimum of (17) if the assumptions of Lemma 1 are satisfied, which contradicts the assumption that the minimum in \( \beta \) of the right-hand side of (17) is equal to \( \beta' \).

We set \( c = \frac{\Delta N_1}{N^{1+\varepsilon}} \) in formula (21) after the substitution (23); then it is easy to see that the ratio

\[
\frac{\mathcal{N}(\mathcal{M} \setminus \mathcal{A})}{\mathcal{N}(\mathcal{M})} \approx \frac{1}{N^m},
\]

where \( m \) is an arbitrary integer, holds for \( \Delta = N^{3/4+\varepsilon} \). The proof of the theorem is complete.

We will prove a cumulative formula in which the densities coincide in shape with the Bose–Einstein distribution. The difference consists only in that, instead of the set \( \lambda_n \) of random variables or eigenvalues of the Hamiltonian operator, the Bose–Einstein formula contains some of their averages over the cells [6]. In view of the theorem given below, one can prove that the \( \varepsilon_i \), which are averages of the energy \( \lambda_k \) at the \( i \)th cell, are nonlinear averages in the sense of Kolmogorov [4].

3. Now we consider the notion of the lattice dimension.

We consider a straight line, a plane, and a three-dimensional space. We separate points \( i = 0, 1, 2, \ldots \) on the line and points \( x = i = 0, 1, 2, \ldots, y = j = 0, 1, 2, \ldots \) on the coordinate axes \( x, y \) on the plane. We associate this set of points \((i, j)\) with the points on the straight line (with the positive integers \( l = 1, 2, \ldots \)) up to the quantum constant \( \chi \) of the lattice.

According to M. Gromov’s definition [7], the asymptotic (topological) dimension of this lattice is equal to two.

We associate each point with a pair of points \( i \) and \( j \) according to the rule \( i + j = l \). The number of such points \( n_l \) is equal to \( l + 1 \). In addition, we assume that \( z = k = 0, 1, 2, \ldots \) on the axis, i.e., we set \( i + j + k = l \). In this case, the number of points \( q_l \) is equal to

\[
q_l = \frac{(l + 1)(l + 2)}{2}.
\]

If we set \( \lambda_i = l \) in formula (4), then, in the three-dimensional case, each \( i \) is associated with \( \frac{(i+1)(i+2)}{2} \) of mutually equal \( x_i = l \) (these are the multiplicities or the \( q_l \)-hold degeneracies of the spectrum of the oscillator). Formula (4) in this special case becomes

\[
N_i = \text{const} \sum_{i=0}^{l} \frac{(i+1)(i+2)}{2(e^{\beta_i - \nu} - 1)},
\]

\[
\Delta N_i = \text{const} \frac{(i+1)(i+2)}{2(e^{\beta_i - \nu} - 1)} \Delta_i, \quad \Delta_i = 1,
\]

\[
\Delta E_i = \text{const} \frac{i(i+1)(i+2)}{2(e^{\beta_i - \nu} - 1)} \Delta_i.
\]
for large $i$, $\Delta i \to 0$,

$$dE = \text{const} \frac{\omega^3 d\omega}{e^{\beta \omega} - 1}; \quad \beta = \frac{h}{T}$$

(cf. formula (60.4) in [8]).

Thus, we obtain a somewhat sharper version of the famous Planck formula for the radiation of a black body.

For the $D$-dimensional case, it is easy to verify that the sequence of weights (multiplicities) of the number of versions $i = \sum_{k=1}^{D} m_k$, where $m_k$ are arbitrary positive integers, has the form of the binomial coefficient

$$q_i(D) = \text{const} \frac{(i + D - 1)!}{i! D!},$$

where the constant depends on $D$.

Thus, for any $D$, formula (36) has the form

$$N_i = \text{const} \sum_{i=1}^{l} \frac{q_i(D)}{e^{\beta i} - 1}.$$  

(37)

For the positive integers, we have a sequence of weights $q_i$ (or, simply, a weight) of the form (36).

Our weight series can easily be continued to an arbitrary case by replacing the factorials with the $\Gamma$-functions; in this case, we assume that $D$ is negative.

This is the negative topological dimension (the hole dimension) of the quantized space (lattice).

If $D > 1$, then, as $i \to \infty$, a condensation of a sufficiently small perturbation occurs in the spectrum of the oscillator and the multiplicities split, i.e., the spectrum becomes denser as $i$ increases. The fact that $D$ is negative means that there is strong rarefaction in the spectrum as $i \to \infty$ (the constant in formula (37) must be sufficiently large).

For non-positive integer $D$, the terms $i = 0, 1, 2, 3, \ldots, -D$ become infinite. This means that they are very large in the experiment, which permits determining the lattice negative dimension corresponding to a given problem. We note that a new condensate occurs, which is possible for small $\beta$.

4. Now we consider an example of negative asymptotic dimension in linguistic statistics.

To each word a frequency dictionary assigns its occurrence frequency (i.e., the number of occurrences) of this word in the corresponding corpus of texts. There may be several words with same frequency.

There is an analogy between the Bose particles at the energy level of an oscillator $\lambda_i = i$ and the words with the occurrence frequency (occurrence number) $i$, namely, the words with the same occurrence frequency can be ordered in an arbitrary way, say, alphabetically, inversely alphabetically, or in any other order. The indexing of the ranks (the indices) of words within the family of a given occurrence frequency (occurrence number) is arbitrary. In this sense, the words are indistinguishable and are distributed according to the Bose statistic.

However, there is a difference between the approaches under consideration. In the frequency dictionary one evaluates the occurrence frequency of every word and then orders the words, beginning with the most frequently occurring words.

When there were no computers, it was difficult for a person to evaluate the number of words with equal occurrence frequency. By looking at a page as if it were a picture, a
person can determine a desired word on this page by its graphical form at every place of occurrence of the word. In this case, the person looks at a page of the text as if it were a photo, without going into the meaning. Similarly, if a person looks for a definite name in a long list of intrants who had entered a college, this person finds (or does not find) the desired name by eyes rather than reads all the names one after another.

An eye gets into the way of recognizing the desired image, and this ability intensifies as the viewed material increases: the more pages the eye scans, the less is the difficulty in finding the desired graphical form. Therefore, under a manual counting, it was simpler to recognize the desired word on a page without reading the text and to cross it out by a pencil, simultaneously counting the number of occurrences of the word. This procedure is repeated for any subsequent word, already using the text with the words crossed out (“holes”), which facilitates the search. In other words, the procedure is in the recognition of the image of the given word, similar to the recognition, say, of a desired mushroom in a forest without sorting out all the plants on the soil one after another. An ordinary computer solves problems of this kind by exhaustion, whereas a quantum computer (see [9]) makes this by recognizing the image.

However, for an ordinary computer, the number of operations needed to find the occurrence frequency of a word is less than the number of operations needed to find the number of words in the text with a given occurrence frequency.

One can say that the number of mushrooms we gathered (took away from the forest) is the number of holes we left in the forest. Similarly, the words we had “got out” from the text in the above way is an analog of holes rather than particles. Therefore, the linguists count the rank of words starting from the opposite end as compared to the starting end which would be used by physicists. The physicists would count the particles starting from the lowest level, whereas the holes, the absent electrons, would be counted from the highest level.

For this reason, the words in a frequency dictionary are associated with holes rather than particles. Correspondingly, the dimension in the distribution for frequency dictionaries is to be chosen as a “hole” dimension, which is negative.

The number of words encountered only once in the array of texts is approximately equal to 1/3 of the entire frequency dictionary which the number of words equal to \( N \). So as \( N \to \infty \) this is the condensate. It follows from the above that \( D = -1 \) for the dictionary. Hence, for \( \beta \ll 1 \) and \( \nu \sim 1 \), we have

\[
N_t = \text{const} \sum_{i=2}^{t} \frac{1}{i(i-1)(e^{\beta i-\nu} - 1)} \sim \text{const} \int_{\omega}^{\infty} \frac{d\omega}{\alpha \omega(\alpha \omega - 1)(e^{\beta \alpha \omega - \nu} - 1)},
\]

where \( \omega = l \) and \( \alpha \) is the scale constant. If \( \omega \) is finite and \( \beta \ll 1 \) the integral may be taken. Therefore one may set \( \beta = 0 \) for not too large \( \omega \).

Bellow we present some graphs of frequency dictionaries obtained from writings of A. N. Ostrovsky and N. Berdyaev.

References

[1] Yu. I. Manin. The notion of dimension in geometry and algebra. //ArXiv:math.AG/0502016 v1 1 Feb 2005.

[2] V. P. Maslov. Negative asymptotic topological dimension, a new condensate, and their connection with the quantized Zipf law. // Mat. Zametki [Math. Notes]. 2006, 80, No. 6, 856–863.
[3] V. P. Maslov. On a general theorem of set theory that leads to the Gibbs, Bose–Einstein, and Pareto distributions and to the Zipf–Mandelbrot law for the stock market. // Mat. Zametki [Math. Notes], 2005, 78, No. 6, 870–877.

[4] V. P. Maslov. The nonlinear average in economics. // Mat. Zametki [Math. Notes]. 2005, 78, No. 3, 377–395.

[5] V. P. Maslov. The lack-of-preference law and the corresponding distributions in frequency probability theory. // Mat. Zametki [Math. Notes]. 2006, 80, No. 2, 220–230.

[6] L. D. Landau, E. M. Lifshitz, Statistical Physics. Moscow: Nauka. 1964.

[7] M. Gromov. Asymptotic invariants of infinite groups. Geometric Group Theory, vol. 2. Cambridge University Press, 1993.

[8] L. D. Landau, E. M. Lifshitz, Quantum mechanics: non-relativistic theory. Course of Theoretical Physics, Vol. 3. Addison-Wesley Series in Advanced Physics. Pergamon Press Ltd., London–Paris; Addison-Wesley Publishing Co., Inc., Reading, Mass, 1958.

[9] V.P.Belavkin, V.P.Maslov. Design of the optimal dynamic analyzer: Mathematical Aspects of Sound and Visual Pattern Recognition. In: Mathematical aspects of computer engineering. Ed. by V.P.Maslov and K.A.Volosov. Moscow: Mir, 1988, p. 146-237.

[10] V. P. Maslov, M. V. Fedoryuk. Semi-Classical Approximation in Quantum Mechanics. 1981, D.Riedel Publ.Company. Dordrecht, Holland.

[11] V. P. Maslov. Methodes Operatorielles. Moscow:Mir, 1987.
Figure 1: A.N.Ostrovsky "Sin and Trouble Avoid Nobody." [Grekh da beda na kogo ne zhivet.]
Dependence of the inverted rank on the frequency. Practical and theoretical dependencies.
At the bottom, 2 frequencies are cut off: $\omega_{\text{min}} = 1$. At the top, 15 frequencies are cut off:
$\omega_{\text{max}} = 136$. The approximating curve is constructed with constants found from the following
two points: $\omega_1 = 2; \omega_2 = 43, \alpha = 0.5$. Mean quadratic error: $\sigma = 0.71589$. 
Figure 2: A.N.Ostrovsky "Lucrative Position." [Dokhodnoe mesto.] Dependence of the inverted rank on the frequency. Practical and theoretical dependencies. At the bottom, 2 frequencies are cut off: $\omega_{\text{min}} = 1$. At the top, 15 frequencies are cut off: $\omega_{\text{max}} = 149$. The approximating curve is constructed with constants found from the following two points: $\omega_1 = 2; \omega_2 = 42, \alpha = 1.4$. Mean quadratic error: $\sigma = 0.435321$.

Figure 3: A.N.Ostrovsky "Handsome Gentleman." [Krasavets muzhchina.] Dependence of the inverted rank on the frequency. Practical and theoretical dependencies. At the bottom, 2 frequencies are cut off: $\omega_{\text{min}} = 1$. At the top, 15 frequencies are cut off: $\omega_{\text{max}} = 136$. The approximating curve is constructed with constants found from the following two points: $\omega_1 = 2; \omega_2 = 30, \alpha = 0.5$. Mean quadratic error: $\sigma = 0.867221$. 
Figure 4: A.N.Ostrovsky "Good Friends Settle Things." [Svoi ljudi - sochtemsja.] Dependence of the inverted rank on the frequency. Practical and theoretical dependencies. At the bottom, 2 frequencies are cut off: $\omega_{\text{min}} = 1$. At the top, 15 frequencies are cut off: $\omega_{\text{max}} = 151$. The approximating curve is constructed with constants found from the following two points: $\omega_1 = 2; \omega_2 = 30; \alpha = 3.4$. Mean quadratic error: $\sigma = 0.456089$. 

Figure: Dependence of the inverted rank on the frequency.
Figure 5: A.N.Ostrovsky. Dictionary for all texts. Dependence of the inverted rank on the frequency. Practical and theoretical dependencies. At the bottom, 5 frequencies are cut off: \( \omega_{\text{min}} = 1 \). At the top, 15 frequencies are cut off: \( \omega_{\text{max}} = 497 \). The approximating curve is constructed with constants found from the following two points: \( \omega_1 = 5; \omega_2 = 86, \alpha = 0.5 \). Mean quadratic error: \( \sigma = 1.33802 \).

Figure 6: Nicolai Berdyaev. The Fate of Russia. Dependence of the inverted rank on the frequency. Practical and theoretical dependencies. At the bottom, 0 frequencies are cut off: \( \omega_{\text{min}} = 2 \). At the top, 7 frequencies are cut off: \( \omega_{\text{max}} = 850 \). The approximating curve is constructed with constants found from the following two points: \( \omega_1 = \text{List}; \omega_2 = 146, \alpha = 0.35 \). Mean quadratic error: \( \sigma = 17.365 \).
Figure 7: Nicolai Berdyaev. The Fate of Russia. Dependence of the variance $\sigma$ on the parameter $\alpha$. Range $[0.1,1.5]$ with step $\delta = 0.05$