Exact bosonization in arbitrary dimensions

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We extend the previous results of exact bosonization, mapping from fermionic operators to Pauli matrices, in 2D and 3D to arbitrary dimensions. This bosonization map gives a duality between any fermionic system in arbitrary \( n \) spatial dimensions and a class of \((n-1)\)-form \( \mathbb{Z}_2 \) gauge theories in \( n \) dimensions with a modified Gauss’s law. This map preserves locality and has an explicit dependence on the second Stiefel-Whitney class and a choice of spin structure on the spatial manifold. A formula for Stiefel-Whitney homology classes on lattices is derived. In the Euclidean path integral, this exact bosonization map is equivalent to introducing a topological Steenrod square term to the space-time action.

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I. INTRODUCTION AND SUMMARY

It is well known that every fermionic lattice system in 1D is dual to a lattice system of spins with a \( \mathbb{Z}_2 \) global symmetry (and vice versa). The duality is kinematic (independent of a particular Hamiltonian) and arises from the Jordan-Wigner transformation. Recently, it has been shown that any fermionic lattice system in 2D is dual to a \( \mathbb{Z}_2 \) gauge theory with an unusual Gauss’s law [1]. The fermion can be identified with the flux excitation of the gauge theory, which is described by the Chern-Simons-like term \( i \pi \int \frac{1}{2} A \wedge \delta A \) in the space-time action. The 2D duality is also kinematic. This approach has been generalized to 3D [2]. Every fermionic lattice system in 3D is dual to a \( \mathbb{Z}_2 \) 2-form gauge theory with an unusual Gauss’s law. Here, 2-form gauge theory means that the \( \mathbb{Z}_2 \) variables live on faces (2-simplices), while the parameters of the gauge symmetry live on edges (1-simplices). 2-form gauge theories in 3+1D have local flux excitations, and the unusual Gauss’s law ensures that these excitations are fermions. This Gauss’s law can be described by the Steenrod square topological action \( i \pi \int B \cup B + B \cup \delta B \). The form of the modified Gauss’s law was first observed in Ref. [3]: A bosonization of fermionic systems in \( n \) dimensions must have a global \((n-1)\)-form \( \mathbb{Z}_2 \) symmetry with a particular ’t Hooft anomaly. The standard Gauss’s law leads to a trivial ’t Hooft anomaly, so bosonization requires us to modify it in a particular way.

In this paper, we extend these results to arbitrary \( n \) dimensions. We show that every fermionic lattice system in \( n \) dimension is dual to a \( \mathbb{Z}_2 \) \((n-1)\)-form gauge theory with a modified Gauss’s law. Our bosonization map is kinematic and local in the same sense as the Jordan-Wigner map.\(^1\)

\( \text{local observable on the fermionic side, including the Hamiltonian density, is mapped to a local gauge-invariant observable on the} \mathbb{Z}_2 \text{ gauge theory side. In the Euclidean picture, we show explicitly that our bosonization map is equivalent to introducing the topological term in the action,} \)

\begin{equation}
S_{\text{top}} = i \pi \int Y \left( A_{n-1} \cup A_{n-2} + A_{n-1} \cup \delta A_{n-1} \right),
\end{equation}

where \( A_{n-1} \) is a \((n-1)\)-form gauge field, a \((n-1)\)-cochain \( A_{n-1} \in C^{n-1}(Y, \mathbb{Z}_2) \), and \( Y \) is \((n+1)\)-dimensional space-time manifold. When \( A_{n-1} \) is closed, i.e., \( \delta A = 0 \), this term reduces to the Steenrod square operator [4]. This Steenrod square term appears in the Lagrangian of fermionic symmetry-protected topological (SPT) phases [5] and it is indirectly argued that this term plays the role of statistical transmutation, which makes the theory fermionic [6,7]. Our approach provides an explicit Hamiltonian picture and the bosonization or fermionization procedure is exact, which gives the direct construction for supercohomology fermionic SPT phases. The quantum circuit for the supercohomology SPT ground state and its commuting projector Hamiltonian are derived explicitly in Ref. [8]. All supercohomology fermionic SPT phases in arbitrary dimensions can be constructed from the bosonization map presented in this paper.

There are already several proposals for an analog of the Jordan-Wigner map in arbitrary dimensions [9–12]. Our construction is most similar to that of Bravyi and Kitaev [9]. One advantage of our construction is that we can clearly identify the kind of \( n \)-dimensional bosonic systems that are dual to fermionic systems: They possess global \((n-1)\)-form \( \mathbb{Z}_2 \) symmetry with a specific ’t Hooft anomaly, as proposed in Ref. [3]. It is also manifest in our approach that the bosonization map depends on a choice of spin structure.

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\(^1\)We only consider the locality preserving map here. Although Jordan-Wigner transformation can map a single fermionic operator into spins, it contains a string operator, which is highly nonlocal. Our bosonization map and Jordan-Wigner transformation both preserve the locality of observables in fermionic systems.

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II. CHAINS, COCHAINS, AND HIGHER CUP PRODUCTS

In this section, we introduce the mathematical tools used in this paper. Our notations and conventions are also described. We will always work with an arbitrary triangulation of a closed simply connected $n$-dimensional manifold $M_n$ equipped with a branching structure (orientations on edges without forming a loop in any triangle). The vertices, edges, faces, and tetrahedra are denoted $v$, $e$, $f$, $t$, respectively. The general $d$ simplex is denoted as $\Delta_d$. We can label the vertices of $\Delta_d$ as $0, 1, 2, \ldots, d$ such that the directions of edges are from the small number to the larger number. We denote this $d$ simplex as $\Delta_d = (01 \ldots d)$. Its boundaries are $(d-1)$-simplices $(0, \ldots, \hat{i}, \ldots, d)$ for $i = 0, \ldots, d$, where $\hat{i}$ means $i$ is omitted. A formal sum of $d$-simplices modulo 2 forms an element of the chain $C_d(M_n, Z_2)$.

For every $v$, we define its dual 0-cochain $\nu$, which takes value 1 on $v$, and 0 otherwise, i.e., $\nu(v') = \delta_{v,v'}$. Similarly, $\varepsilon$ is an 1-cochain $\varepsilon(e') = \delta_{e,e'}$, and so forth, i.e., $\Delta_d$ being a $d$-cochain $\Delta_d(\Delta_d') = \delta_{\Delta_d,\Delta_d'}$. All dual cochains will be denoted in bold. A $d$-cochain $c_d \in C_d(M_n, Z_2)$ can be identified as a $Z_2$ field living on each $d$-simplex $\Delta_d$, with the value $c_d(\Delta_d)$. An evaluation of a cochain $c$ on a chain $\varepsilon'$ is the sum of $c$ evaluated on simplices in $\varepsilon'$, which is denoted $\int_{\varepsilon'} c = \sum_{\Delta \in \varepsilon'} c(\Delta)$. When the integration range is not written, $c$ is assumed to be the top dimension and $\int_{\varepsilon'} c \equiv \int_{\Delta} c$.

The boundary operator is denoted by $\partial$. For an $n$-simplex $\Delta_n$, $\partial \Delta_n$ consists of all boundary $(n-1)$-simplices of $\Delta_n$:

$$\partial((0, 1, 2, \ldots, d)) = \sum_{i=0}^{d} (0, \ldots, \hat{i}, \ldots, d).$$ (2)

The coboundary operator is denoted by $\delta$ (not to be confused with the Kronecker delta previously). On a 0-cochain $\nu$, $\partial \nu$ is an 1-cochain acting on edges, and is 1 if $\partial \nu$ contains $v$ and 0 otherwise:

$$\delta v(e) = \nu(\partial e) = \delta_{v,e}. (3)$$

It is similar for simplices in any dimension. For any $d$-cochain $c \in C^d(M_n, Z_2)$, its coboundary $\delta c \in C^{d+1}(M_n, Z_2)$ acting on a $(d+1)$-simplex $\Delta_{d+1} = (0, 1, \ldots, d+1)$ is defined by

$$\delta c(\Delta_{d+1}) \equiv c(\partial \Delta_{d+1}) = \sum_{i=0}^{d+1} c((0, \ldots, \hat{i}, \ldots, d+1)).$$ (4)

The cup product $\cup$ of a $p$-cochain $\alpha_p$ and a $q$-cochain $\beta_q$ is a $(p+q)$-cochain defined as

$$[\alpha_p \cup \beta_q](0, 1, \ldots, p+q)$$

$$= \alpha_p((01 \ldots p)) \beta_q((p, p+1, \ldots, p+q))$$

$$= \alpha_p((0 \sim p)) \beta_q((p \sim p+q)).$$ (5)

A direct construction of branching structure is to arbitrarily assign different real numbers on all vertices. For each edge, the arrow is pointed from the smaller number to the larger number.
generalized to higher dimensions: the $\cup_{n-2}$ of two $(n-1)$-cochains acting on an $n$-simplex is the sum of the product of $(n-1)$-cochains acting on its boundary $(n-1)$-simplices with the same orientation. This geometry interpretation of higher cup products is crucial since it is further shown that this property coincides with the commutation relations of fermionic hopping operators. The fermionic statistic is captured by higher cup products and this makes it convenient to derive the topological action for fermionic theories. Although not directly used in this paper, a higher order product of arbitrary cochains has a nice geometrical interpretation [13]: The higher cup product measures the intersection between dual cells and thickened, shifted version of other dual cells, where the thickening and shifting are determined by the vector frame field. For example, the simplest cup product formula

$$\alpha_1 \cup \beta_1((012)) = \alpha_1((01))\beta_1((12)), \quad (11)$$

can be viewed as the intersection point in Fig. 2.

It should be emphasized that the cup products satisfy the recursive property

$$\alpha \cup_a \beta_+ \cup_a \alpha = \alpha \cup_{a+1} \delta \beta + \delta \alpha \cup_{a+1} \beta + \delta(\alpha \cup_{a+1} \beta), \quad (12)$$

which can be interpreted as that the noncommutative property of the $\cup_a$ product is equal to the failure of the product rule of the coboundary operation $\delta$ on the $\cup_{a+1}$ product.

Finally, $\Delta^1_n \supset \Delta^2_n$ or $\Delta^2_n \subset \Delta^1_n$ means that the simplex $\Delta^1_n$ contains $\Delta^2_n$ as a subsimplex. A general rule of thumb is that the subset symbol always points to one higher dimension.

III. REVIEW OF BOSON-FERMION DUALITY IN (2+1)D AND (3+1)D

We begin by reviewing the duality between fermions and $\mathbb{Z}_2$ lattice gauge theory in both two spatial dimensions [1] and three spatial dimensions [2]. On each face $f$ of the 2-manifold $M_2$, we place a single pair of fermionic creation-annihilation operators $c_f, c^\dagger_f$, or equivalently a pair of Majorana fermions:

$$\gamma_f = c_f + c^\dagger_f, \quad \gamma_f' = \frac{c_f - c^\dagger_f}{i}. \quad (13)$$

The algebra of Majorana fermions is

$$\{\gamma_f, \gamma_{f'}\} = \{\gamma_f', \gamma_{f''}\} = 2\delta_{f,f'}, \quad \{\gamma_f, \gamma_{f'}\} = 0, \quad (14)$$

where $\{A, B\} = AB - BA$ is the anticommutator. The even fermionic algebra consists of local observables with a trivial fermionic parity (i.e., $P_F = \prod_f (-1)^{\epsilon_f c_f} = 1$). It is generated by the on-site fermion parity,

$$P_f = -i\gamma_f\gamma_f', \quad (15)$$

and the fermionic hopping operator on every edge $e$,

$$S_e = i\gamma_{L(e)}\gamma_{R(e)}, \quad (15)$$

where the sign from the commutation occurs only when the arrows on the two edges follow head to tail and are on the same triangle, i.e., edges $(e, e')$ being $(01), (12)$ of a triangle $(012)$. In general, for any 1-cochains $\lambda$ and $\lambda'$,

$$S_{\lambda + \lambda'} \equiv (-1)^{\int\lambda \wedge \lambda'} S_{\lambda}S_{\lambda'}, \quad (16)$$

In other words, $S_e$ is the product of $S_o$ over $\{e|\lambda(e) = 1\}$ and the sign in front is consistent with the commutation relations. If we consider the product of fermionic hopping operators on edges around a vertex $v$, the Majorana operators cancel out up to some $P_f$ terms. The two generators $P_f$ and $S_f$ satisfy the following constraint at each vertex $v$ [1]:

$$(-1)^{\int\varepsilon v S_{0e}} \prod_f P_f^{\varepsilon_{f}w_{f}+f_{f}w_{v}} = 1, \quad (17)$$

where $w_2 \in C_0(M_2, \mathbb{Z}_2)$ is the 0-chain which is Poincaré dual to the second Stiefel-Whitney cohomology class $w_2(M_2)$. The explicit expression of $w_2$ is given in the Appendix. We require $M_2$ to be a spin manifold, i.e., the second Stiefel-Whitney class is exact: $w_2 = \partial E$ for some $E \in C_1(M_2, \mathbb{Z}_2)$. The 1-chain $E$ is a choice of the spin structure. The nonaxcessness of the second Stiefel-Whitney class is the obstruction to determine this 1-chain $E$, which prevents us from defining a self-consistent bosonization map, which dualizes the even sector of fermionic Hilbert space to a $\mathbb{Z}_2$ gauge theory.

The bosonic dual of this system involves $\mathbb{Z}_2$-valued spins on the edges of the triangulation. The bosonic algebra are generated by Pauli matrix on edges:

$$X_e = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y_e = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z_e = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (18)$$

For every face $f$, we define the flux operator,

$$W_f = \prod_{e \subset f} Z_e, \quad (19)$$
and for every edge $e$ we define a unitary operator $U_e$ which squares to 1:

$$U_e = X_e \left( \prod_{e'} Z_{e'}^{\epsilon_{e'e}} \right),$$

where $X_e, Z_e$ are Pauli matrices acting on a spin at the edge $e$. It has been shown in Ref. [1] that the sets $\{U_e, W_f\}$ and $\{S_\epsilon, P_f\}$ satisfy the same commutation relations. The boson-fermion duality map defined on the manifold $M_2$ is

$$W_f = \prod_{e \in f} Z_e \leftrightarrow P_f = -i\gamma_f \gamma_f', \quad U_e = X_e \left( \prod_{e'} Z_{e'}^{\epsilon_{e'e}} \right) \leftrightarrow (-1)^{\epsilon_{ef}} S_e$$

where the 1-chain $w_2 \in C_0(M_2, \mathbb{Z}_2)$ is the chain representation of the second Stiefel-Whitney class and the 1-chain $E \in C_1(M_2, \mathbb{Z}_2)$ denotes a choice of spin structure ($\partial E = w_2$).

IV. EXACT BOSONIZATION IN $n$ DIMENSIONS

From the 2D and 3D formulas Eqs. (21) and (22), it is very natural to conjecture the $n$-dimensional boson-fermion duality. Consider a spin manifold $M_n$ in spatial $n$ dimensions. The fermions live at the center $n$-simplices, i.e., $\gamma_{\Delta_n-1}$ and $Z_{\Delta_n-1}$, for each $\Delta_n$. The Pauli matrices live on $(n - 1)$-simplices, i.e., $X_{\Delta_n-1}$ and $Z_{\Delta_n-1}$, for each $\Delta_n$. The $n$-dimensional boson-fermion duality should be

$$W_{\Delta_n} = \prod_{\Delta_{n-1} \subset \Delta_n} Z_{\Delta_{n-1}} \leftrightarrow P_t = -i\gamma_{\Delta_n} \gamma_{\Delta_n},$$

$$U_{\Delta_{n-1}} \equiv X_{\Delta_{n-1}} \left( \prod_{\Delta_{n-1}} Z_{\Delta_{n-1}}^{\epsilon_{\Delta_{n-1} \Delta_{n-1}'} S_{\Delta_{n-1}}} \right) \leftrightarrow (-1)^{\epsilon_{\Delta_{n-1}}} S_{\Delta_{n-1}}$$

where $w_2 \in C_{n-2}(M_{n-1}, \mathbb{Z}_2)$ is the chain representation of the second Stiefel-Whitney class, $E \in C_{n-1}(M_{n-1}, \mathbb{Z}_2)$ denotes a choice of spin structure ($\partial E = w_2$), and for general $(n - 1)$-cochain $\lambda_{n-1}$ and $\lambda'_{n-1}$, the product of $S$ operators is defined as

$$S_{\lambda_{n-1} + \lambda'_{n-1}} = (-1)^{\int \lambda_{n-1} \lambda'_{n-1}} S_{\lambda_{n-1}} S_{\lambda'_{n-1}}.$$ (24)

This $n$-dimensional boson-fermion duality Eq. (23) is the main theorem of this paper, which will be proved by the end of this section.

A. Commutation relations

Consider an $n$-simplex $\Delta_n = \{012\ldots n\}$. Its boundary contains all $(n - 1)$-simplex $\{\partial \Delta_n\} = \{0 \ldots \hat{i} \ldots n\}$ where $\hat{i}$ means the vertex $i$ is omitted. We define the orientation of $\partial \Delta_n$ as $O(\partial \Delta_n) = (-1)^i$. For $+$-oriented $\Delta_n$, if $O(\partial \Delta_n) = 1$, the boundary $\partial \Delta_n$ is outward, and if $O(\partial \Delta_n) = -1$, the boundary $\partial \Delta_n$ is inward. For $-$-oriented $\Delta_n$, the inward and outward boundaries are opposite. $S_{\Delta_n}, S_{\Delta_n}'$ anticommute only when $\Delta_n$ and $\Delta_n'$ are both inward or both outward boundaries of some $n$-simplex, i.e., $\Delta_n, \Delta_n' \in \partial \Delta_n$. We are going to prove that this is equivalent to

$$S_{\Delta_n} S_{\Delta_n'} = (-1)^{\int \Delta_n \Delta_n'} \Delta_n S_{\Delta_n} S_{\Delta_n'}.$$ (25)
From the definition of the higher cup product Eq. (8), we have

\[ \{ \Delta_{n-1} \cup_{n-2} \Delta'_{n-1} \}(0, 1, \ldots, n) = \sum_{0 \leq i_0 < i_1 < \cdots < i_{n-1} \leq n} \Delta_{n-1}(0 \sim i_0, i_1 \sim i_2, i_3 \sim i_4, \ldots) \Delta'_{n-1}(i_0 \sim i_1, i_2 \sim i_3, \ldots) \]

\[ = \sum_{0 \leq j_1 < j_2 < \cdots \leq n, j_1, j_2 \text{ even}} \Delta_{n-1}(0 \ldots \hat{j}_2 \ldots n) \Delta'_{n-1}(0 \ldots \hat{j}_1 \ldots n) \]

\[ + \sum_{0 \leq k_1 < k_2 < \cdots \leq n, k_1, k_2 \text{ odd}} \Delta_{n-1}(0 \ldots \hat{k}_1 \ldots n) \Delta'_{n-1}(0 \ldots \hat{k}_2 \ldots n). \]

(26)

The \( \cup_{n-2} \) only contains the product of boundaries \( \Delta^i_{n-1} \) with the same orientation (inward or outward) and each pair of \( \Delta^i_{n-1}, \Delta'_{n-1} \) with the same orientation appears exactly once. Therefore, the \( \cup_{n-2} \) expression in Eq. (25) captures the commutation relations of fermionic hopping operators \( S_{\Delta_n} \). It is easy to check that bosonic operators \( U_{\Delta_n} \) satisfy the same commutation relations:

\[ U_{\Delta_{n-1}} U'_{\Delta_{n-1}} = (-1)^{j_1} \Delta_{n-1}^{i_1} \Delta_{n-1}^{i_2} A_{n-1}^{i_1} A_{n-1}^{i_2} U_{\Delta_{n-1}} U'_{\Delta_{n-1}}. \]

(27)

Therefore, \( \{ S_{\Delta_{n-1}}, P_{\Delta_n} \} \) and \( \{ U_{\Delta_{n-1}}, W_{\Delta_n} \} \) in Eq. (23) have the same commutation relations.

**B. Gauge constraints**

In this section, we will derive the constraints on fermionic operators:

\[ (-1)^{j_1} \int_{\Delta_{n-2}} \delta_{\Delta_{n-1}} \prod_{\Delta_n} p^{\Delta_{n-2}, \Delta_n, +} A_{\Delta_n, \Delta_n, -} A_{\Delta_n, \Delta_n, -} = 1. \]

(28)

This follows directly from the following two lemmas.

**Lemma 1.** The Majorana operators in \( S_{\Delta_{n-2}} \) cancel out with Majorana operators in \( \prod_{\Delta_n} p^{\Delta_{n-2}, \Delta_n, +} A_{\Delta_n, \Delta_n, -} A_{\Delta_n, \Delta_n, -} \).

**Lemma 2.** The sign of \( S_{\Delta_{n-2}} \) and the product of on-site fermion parities \( \prod_{\Delta_n} p^{\Delta_n} \) is \( (-1)^{\sum_{i=1}^{n} i} \). Where we order \((n - 1)\)-simplices \( \{ \Delta_{n-1} \} \) clockwise as \( \Delta^1_{n-1}, \Delta^2_{n-1}, \cdots, \Delta^n_{n-1} \) and \( \Delta^0_{n-1} \equiv \Delta^0_{n-1} \), as shown in Fig. 4. This sign is a chain representative of the second Stiefel-Whitney class:

\[ (-1)^{\sum_{i=1}^{n} i}. \]

(29)

**Proof of Lemma 1.** Let us denote \( \Delta_n = (01 \ldots n) \) formed by \( \Delta_{n-2} \) and two \((n - 1)\)-simplices \( \Delta^0_{n-1} \) and \( \Delta^1_{n-1} \), shown in Fig. 3(a). We know that \( S_{\Delta_{n-2}} \) contains \( \gamma_{\Delta_n} \gamma'_{\Delta_n} \) and if only if \( \Delta^0_{n-1}, \Delta^1_{n-1} \) are one inward boundary and one outward boundary of \( n \)-simplex \( \Delta_n \), as indicated in Figs. 3(b) and 3(c). For the product of \( P_{\Delta_n} \), we simplify the integral as

\[ \int_{\Delta_{n-2}} \Delta_n \cup_{n-2} \Delta_n \cup_{n-2} \Delta_n = \int \delta_{\Delta_{n-2}} \cup_{n-1} \Delta_n. \]

(30)

where we have used the property \( \delta(\alpha \cup_{n-1} \beta) = \delta \alpha \cup_{n-1} \beta + \alpha \cup_{n-2} \beta + \beta \cup_{n-2} \alpha \) and \( \delta_{\Delta_n} = 0 \) (since \( n \) is the top dimension). The integral Eq. (30) has only the
contribution from $\Delta_n = (01 \ldots n)$:

$$\int_{\Delta_{n-2} \cup \Delta_{n-2} \cup \Delta_n \cup \Delta_{n-2} \Delta_{n-2}} = \left[(\Delta_{n-1}^L + \Delta_{n-1}^R) \cup \Delta_{n-1} \Delta_n\right](01 \ldots n)$$

$$= \sum_{0 \leq i < j \leq \ldots < k \leq \ldots \leq n} (\Delta_{i-1}^L + \Delta_{i-1}^R)(0 \sim i_0, i_1 \sim i_2, i_3 \sim i_4, \ldots) \Delta_n(i_0 \sim i_1, i_2 \sim i_3, \ldots)$$

$$= \sum_{0 \leq j < n} (\Delta_{j-1}^L + \Delta_{j-1}^R)((0 \ldots j \ldots n)) \Delta_n((0 \ldots n))$$

$$= \sum_{0 \leq j < n} (\Delta_{j-1}^L + \Delta_{j-1}^R)((0 \ldots j \ldots n)),$$

(31)

which is 1 if and only $\Delta_{n-1}^L, \Delta_{n-1}^R$ are one inward boundary and one outward boundary of the $n$-simplex $\Delta_n$. This shows that product of $P_{\Delta_n}$ contain $P_{\Delta_n} \sim \gamma_{\Delta_n} \gamma_{\Delta_n}$ if and only if $\Delta_{n-1}^L, \Delta_{n-1}^R$ are one inward boundary and one outward boundary of the $n$-simplex $\Delta_n$. This cancels out with $S_{\Delta_{n-2}}$ exactly.

**Proof of Lemma 2.** We compare the signs between

$$S_{\Delta_{n-2}} = (-1)^{\sum_{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}}} \times \prod_{\Delta_{n-1} \cup \Delta_{n-1}} S_{\Delta_{n-1}}$$

and

$$\prod_{\Delta_{n-1}} P_{\Delta_n}^{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}},$$

(32)

(33)

where we have used the definition of $S_{\Delta_{n-1}}$ in Eq. (24). As shown in Fig. 4,

$$S_{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}} = S_{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}} \times \prod_{\Delta_{n-1} \cup \Delta_{n-1}} S_{\Delta_{n-1}}$$

(34)

We can check that

$$S_{\Delta_{n-1} \cup \Delta_{n-1}} = (-1)^{\sum_{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}}} \times \prod_{\Delta_{n-1} \cup \Delta_{n-1}} p_{\Delta_n}^{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}}$$

(35)

since $S_{\Delta_{n-1} \cup \Delta_{n-1}}$ is 1 (or $-P_{\Delta_n} P_{\Delta_n}$) if $\Delta_{n-1}^L, \Delta_{n-1}^R$ are both inward or outward (or one inward and one outward) in $\Delta_n = a$. Therefore,

$$S_{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}} = (-1)^{\sum_{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}}} \times \prod_{\Delta_{n-1} \cup \Delta_{n-1}} p_{\Delta_n}^{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}}$$

(36)

Together with Eq. (32), we have

$$S_{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}} = (-1)^{\sum_{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}}} \times \prod_{\Delta_{n-1} \cup \Delta_{n-1}} p_{\Delta_n}^{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}}$$

(37)

From the definition of $U_{n-2}$ product Eq. (26),

$$\sum_{i=1}^{d} \int_{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}}$$

(38)

(39)

where the relation between $\Delta_{n-2}, \Delta_{n-1}^L, \Delta_{n-1}^R$ and $\Delta_n$ is shown in Fig. 4. The distinct orientations of $-\Delta_n$ and $+\Delta_n$ in the summation come from the fact that $j_1, j_2$ and $k_1, k_2$ in Eq. (26) have opposite orders. Equation (38) is related to $w_2$ by the following Lemma 3, which is proved in the Appendix. Therefore, we derive

$$\sum_{\Delta_{n-1}} (-1)^{\sum_{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}}} \times \prod_{\Delta_{n-1} \cup \Delta_{n-1}} p_{\Delta_n}^{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}}$$

FIG. 4. By the operations defined in Fig. 3, we can simplify the product $S_{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}} = S_{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}} \times \prod_{\Delta_{n-1} \cup \Delta_{n-1}} p_{\Delta_n}^{\Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1} \cup \Delta_{n-1}}$. 

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where
\[ c(\Delta_{n-2}) = 1 + \sum_{\text{oriented } \Delta_n = (0 \ldots n) \text{ even}} \sum_{j_1 < j_2} \sum_{j_1, j_2 \in \text{even}} \Delta_{n-2}(0 \ldots \hat{j_1} \ldots \hat{j_2} \ldots n) \]
\[ + \sum_{\text{oriented } \Delta_n = (0 \ldots n) \text{ odd}} \sum_{j_1 < j_2} \sum_{j_1, j_2 \in \text{odd}} \Delta_{n-2}(0 \ldots \hat{j_1} \ldots \hat{j_2} \ldots n). \]
(41)

We can modify the sign of \( S_{\Delta_{n-1}} \) as
\[ S_{\Delta_{n-1}}^E \equiv (-1)^{\frac{1}{2} \lambda_n} S_{\Delta_{n-1}}, \]
(42)
where \( E \in C_{n-1}(M_n, \mathbb{Z}_2) \) is a choice of spin structure satisfying \( \delta E = w_2 \). In these modified operators, the constraint on the fermionic operator becomes
\[ S_{\Delta_{n-2}}^E \prod_{\Delta_n} f_{\Delta_n} \sum_{\lambda_n}^1 \Delta_{n-2}^0 + \Delta_{n-2}^1 = 1, \]
(43)
which is mapped to
\[ G_{\Delta_{n-2}} = U_{\lambda_n} \prod_{\Delta_n} \left( \sum_{\lambda_n}^1 \Delta_{n-2}^0 + \Delta_{n-2}^1 \right), \]
(44)
where \( U_{\lambda_n} \) is defined by
\[ U_{\lambda_n} \equiv (-1)^{\lambda_n} U_{\lambda_n+1}^1 U_{\lambda_n}^0, \]
(45)
and it can be calculated directly from \( U_{\lambda_n} \) defined in Eq. (23):
\[ U_{\lambda_n} \equiv \prod_{\lambda_n \in \Lambda_{n-1}}^1 \Delta_{n-2}(0 \ldots \hat{j_1} \ldots \hat{j_2} \ldots n). \]
(46)
We need to impose this gauge constraint \( G_{\Delta_{n-2}} = 1 \) on bosonic operators for every \((n - 2)\)-simplex \( \Delta_{n-2} \).
We also need to impose the even total parity constraint for fermions,
\[ \prod_{\Delta_n} P_{\Delta_n} = 1, \]
(47)
since it is mapped to the bosonic operator \( \prod_{\Delta_n} W_{\Delta_n} = 1 \). After imposing the gauge constraints, the \( n \)-dimensional boson-fermion duality Eq. (23) is completed.

V. MODIFIED GAUSS’S LAW AND EUCLIDEAN ACTION

A. Gauss’s law as boundary anomaly
First, we consider the standard \((n - 1)\)-form \( \mathbb{Z}_2 \) lattice gauge theory on the \( n \)-dimensional manifold \( M_n \),
\[ H^0 = -J_1 \sum_{\Delta_n} X_{\Delta_n-1} - J_2 \sum_{\Delta_n} W_{\Delta_n}, \]
(48)
with the gauge constraint (Gauss’s law):
\[ G_{\Delta_{n-2}} = \prod_{\Delta_n \in \Lambda_{n-2}} X_{\Delta_{n-2}} = 1. \]
(49)
It is well known that its Euclidean theory is \((n + 1)\)-dimensional Ising model (with a certain choice of \( J_1 \) and \( J_2 \)) [14],
\[ S_{\text{Ising}}(\Delta_{n-1}) = -J \sum_{\Delta_n \in \mathcal{C}} |\delta \Delta_{n-1}(\Delta_n)|, \]
(50)
where \( A \in C_{n-1}^1(Y, \mathbb{Z}_2) \) is a \((n - 1)\)-cochain on the space-time manifold \( Y \), \( |\delta A| = 0, 1 \) gives \( \delta A \) (mod 2), and \( J \) depends on \( J_1 \) and \( J_2 \). In this case, \( S_{\text{Ising}} \) is invariant under the gauge transformation \( \Delta_{n-1} \rightarrow \Delta_{n-1} + \delta \Delta_{n-2} \) for arbitrary \((n - 2)\)-cochain \( \Delta_{n-2} \in C_{n-2}^1(Y, \mathbb{Z}_2) \). Therefore, \( S_{\text{Ising}} \) has no boundary anomaly under the standard Gauss’s law.
Now, we propose a class of \( \mathbb{Z}_2 \) lattice gauge theory,
\[ H = -J_1 \sum_{\Delta_n} U_{\Delta_n-1} - J_2 \sum_{\Delta_n} W_{\Delta_n}, \]
(51)
with the modified Gauss’s law (gauge constraints) at \((n - 2)\)-simplices:
\[ G_{\Delta_{n-2}} = \prod_{\Delta_n \in \Lambda_{n-2}} X_{\Delta_{n-2}}(\prod_{\Delta_n \in \Lambda_{n-2}} Z_{\Delta_{n-2}}) = 1. \]
(52)
This model describes a free fermion system, since it is dual to
\[ H_f = -J_1 \sum_{\Delta_n} (-1)^{\frac{1}{2} \lambda_n} U_{\lambda_n} Y_{\lambda_n}^1 Y_{\lambda_n}^0 \]
(53)
The modified Gauss’s law Eq. (52) on a \((n - 2)\)-simplex \( \Delta_{n-2} \), or equivalently on the dual \((n - 2)\)-cochain \( \lambda_{n-2} \), can be generalized to an arbitrary \((n - 2)\)-cochain \( \lambda_{n-2} = \sum \lambda_{n-2} \); the Gauss’s law is
\[ 1 = G_{\lambda_{n-2}} = \prod_i G_{\lambda_{n-2}^i}. \]
(54)
where the sign comes from anticommutation of \( X \) and \( Z \) on the same simplex. This can be proved by induction.
(1) We first check for \( \lambda_{n-2} = \Delta_{n-2} \), where \( \lambda_{n-2} \) contains a single \((n - 2)\)-simplex. We have \( \Delta_{n-2} \cup_{n-4} \Delta_{n-2}^1 + \Delta_{n-2} \cup_{n-4} \Delta_{n-2}^1 \delta \Delta_{n-2} = 0 \) by the definition of higher cup products since the vertices in \( \delta \) cannot match. For example, \( \Delta_{n-2} \) acts only nontrivial on a \((n - 2)\)-simplex with \((n - 1)\) vertices, while \( \Delta_{n-2} \cup_{n-4} \Delta_{n-2}^1 \) has the input of \((n + 1)\) vertices, which has 2 extra vertices at least, \( \lambda_{n-2} \) vanishes when it acts on any simplex with the extra vertices. The gauge constraint reduces the original form Eq. (52).
(2) It is straightforward to check \( G_{\lambda_{n-2}^1} G_{\lambda_{n-2}^1} = G_{\lambda_{n-2}^1 \lambda_{n-2}^1} \), using the recursive property of cup products:
\[ \alpha \cup_{n+1} \beta + \beta \cup_{n+1} \alpha = \alpha \cup_{n+1} \delta \beta + \delta \alpha \cup_{n+1} \beta + \delta(\alpha \cup_{n+1} \beta). \]
(55)
Consider now the following \((n - 1)\)-form gauge theory defined on a general triangulated \((n + 1)\)-dimensional manifold \(Y\):

\[
S(A_{n-1}) = - \sum_{\Delta_n \subset Y} |\delta A_{n-1}(\Delta_n)| + i \pi \int_Y (A_{n-1} \cup_{n-3} A_{n-1}) \\
+ A_{n-1} \cup_{n-2} \delta A_{n-1},
\]

(56)

where \(A_{n-1} \in C^{n-1}(Y, \mathbb{Z}_2)\) and the gauge symmetry acts by \(A_{n-1} \to A_{n-1} + \delta \Lambda_{n-2} \) for \(\Lambda_{n-2} \in C^{n-2}(Y, \mathbb{Z}_2)\). The second term is the generalized Steenrod square defined in Ref. [5]. The action is gauge invariant up to a boundary term

\[
S(A_{n-1} + \delta \Lambda_{n-2}) - S(A_{n-1}) \\
= i \pi \int_{\partial Y} (\Lambda_{n-2} \cup_{n-4} A_{n-2} + A_{n-2} \cup_{n-3} \delta \Lambda_{n-2} \\
+ \delta \Lambda_{n-2} \cup_{n-2} A_{n-1}) \\
= i \pi \int_{\partial Y} (\Lambda \cup_{n-4} A + A \cup_{n-3} \delta A + \delta \Lambda \cup_{n-2} A),
\]

(57)

where we have omitted the subscript of \(A_{n-1}\) and \(\Lambda_{n-2}\) for simplicity. This boundary term determines the Gauss’s law for the wave-function \(\Psi(A)\) on the spatial slice \(M = \partial Y\),

\[
\Psi(A + \delta \Lambda) = (-1)^{\omega(A, \Lambda)} \Psi(A),
\]

(58)

where \(\omega(A, \Lambda) = \int_M (\Lambda \cup_{n-4} A + A \cup_{n-3} \delta A + \delta \Lambda \cup_{n-2} A)\). The Gauss’s law is the same as the gauge constraint Eq. (54) if we identify \(Z_{\Delta_{n-1}}\) as \((-1)^{\omega_{n-1}(\Delta_{n-1})}\) and \(X_{\Delta_{n-1}}\) acts as the transformation \(\Delta_{n-1} \to \Delta_{n-1} + \Delta_{n-1}\). The modified Gauss’s law Eq. (52) represents the boundary anomaly of topological action Eq. (56) as we claimed.

In the following subsection, we derive the Euclidean action of the modified \(\mathbb{Z}_2\) lattice gauge theory Eq. (51) explicitly, which is analogous to Eq. (56).

### B. Euclidean path integral of lattice gauge theories

Start with the Hamiltonian of modified \(\mathbb{Z}_2\) lattice gauge theory,

\[
H = -J_1 \sum_{\Delta_{n-1}} U_{\Delta_{n-1}} - J_2 \sum_{A_n} W_{A_n} \\
= -J_1 \sum_{\Delta_{n-1}} X_{\Delta_{n-1}} \left( \prod_{\Delta_{n-1}' \supset \Delta_{n-1}} Z_{\Delta_{n-1}'} \right) \\
- J_2 \sum_{\Delta_{n-1} \subset \Delta_{n-1}'} \prod_{\Delta_{n-1} \subset \Delta_{n-1}'} Z_{\Delta_{n-1}'}.
\]

(59)

with gauge constraints

\[
G_{\Delta_{n-2}} = \prod_{\Delta_{n-1} \subset \Delta_{n-2}} X_{\Delta_{n-1}} \left( \prod_{\Delta_{n-1}' \supset \Delta_{n-1}} Z_{\Delta_{n-1}'} \right) = 1.
\]

(60)

The partition function is

\[
Z = \text{Tr} \, e^{-\beta H} = \text{Tr} \, T^M,
\]

(61)

where we use Trotter-Suzuki decomposition in the imaginary time direction and \(T\) is the transfer matrix defined as

\[
T = \left( \prod_{\Delta_{n-2}} G_{\Delta_{n-2}} \right) e^{-\beta H}.
\]

(62)

The first factor arises from the gauge constraints on the Hilbert space. The space-time manifold consists of many time slices labeled by layers \(\{i\}\). In the \(i\)th layer, we insert a complete basis (in Pauli matrix \(Z_{\Delta_{n-1}}\)): \(b_i^{n-2} \in C^{n-1}(M_n, \mathbb{Z}_2)\) (a \(\mathbb{Z}_2\) field on each \(\Delta_{n-1}\) of the spatial manifold \(M_n\) such that \(Z_{\Delta_{n-1}} = (-1)^{\bar{\psi}_n(\Delta_{n-1})}\)). The transfer matrix \(T\) between the \(i\)th layer and the \((i + 1)\)th layer contains gauge constraints on every spatial \((n - 2)\)-simplex \(\Delta_{n-2}\):

\[
\delta G_{\Delta_{n-2}} = \frac{1}{2} \left[ \prod_{\Delta_{n-2}} \left( \prod_{\Delta_{n-1} \subset \Delta_{n-2}} G_{\Delta_{n-1}} \right)^{1/2} \right],
\]

(63)

where we introduce the Lagrangian multiplier \(a^{i+1/2}_{n-2} \in C^{n-2}(M_n, \mathbb{Z}_2)\) (a \(\mathbb{Z}_2\) field living on each \(\Delta_{n-2}\) of the spatial manifold \(M_n\)). Notice that \(a^{i+1/2}_{n-2}\) defined between two time slices lives on the spatial \((n - 2)\)-simplex \(\Delta_{n-2}\), which can be interpreted as the space-time \((n - 1)\)-simplex between the two layers. From the similar calculation in Ref. [2], we have

\[
Z = \sum_{\{a^{i+1/2}_{n-2}, b_i^{n-1}\}} \exp \{ i \sum_i S_{\text{Ising}} + S_{\text{Top}} \{ a^{i+1/2}_{n-2}, b_i^{n-1} \} \},
\]

(64)

where

\[
S_{\text{Ising}} \{ a^{i+1/2}_{n-2}, b_i^{n-1} \} = \sum_i \left( -J_1 \sum_{\Delta_{n-1}} \delta b_i^{n-1}(\Delta_{n-1}) - J_2 \sum_{\Delta_{n-1}} \left[ b_i^{n-1} + b_i^{n+1} + \delta a^{i+1/2}_{n-2}(\Delta_{n-1}) \right] \right)
\]

(65)

and

\[
S_{\text{Top}} \{ a^{i+1/2}_{n-2}, b_i^{n+1} \} = i \pi \sum_i \int_{M_n} a^{i+1/2}_{n-2} \cup_{n-4} a^{i+1/2}_{n-2} + a^{i+1/2}_{n-2} \cup_{n-3} \delta a^{i+1/2}_{n-2} + \delta a^{i+1/2}_{n-2} \cup_{n-2} b_i^{n+1} \\
+ b_i^{n+1} \cup_{n-2} \left( b_i^{n-1} + b_i^{n+1} + \delta a^{i+1/2}_{n-2} \right).
\]

(66)
Here \( J_1, J_2 \) are constants depending on \( J_1, J_2, \delta \tau \) in the original Hamiltonian and we assume \( J_1 = J_2 = J \) for simplicity. \( \lfloor \ldots \rfloor \) gives the argument’s parity 0 or 1. The gauge transformations act as
\[
\begin{align*}
\delta b_{n-1}^i &\to b_{n-1}^i + \delta \lambda^i, \\
\delta a_{n-2}^{i+1/2} &\to a_{n-2}^{i+1/2} + \delta \mu^i + \lambda^i
\end{align*}
\]
where \( \lambda^i \) are arbitrary \((n-2)\)-cochains and \( \mu^i \) are arbitrary \((n-3)\)-cochains.

If we interpret \( a_{n-2}^{i+1/2} \) as space-time \((n-1)\)-cochains, we can rewrite
\[
\left\{ a_{n-2}^{i+1/2}, b_{n-1}^i \right\} \to A_{n-1}^i \in C^{n-1}(Y, \mathbb{Z}_2),
\]
which is a \( \mathbb{Z}_2 \) field living on \((n-1)\)-simplicies in space-time manifold \( Y \). It is natural to write \( S_{\text{ising}} \) in Eq. (65) as
\[
S_{\text{ising}} = - \sum_{\Delta_2 \subset Y} [\delta A_{n-1}(\Delta_2)].
\]
(69)
The space-time manifold \( Y = M_0 \times [-\infty, 0] \) (spatial and temporal parts) is not a triangulation, since we only triangulate the spatial manifold \( M_0 \) under the discretized time. The (higher) cup products are not well-defined in \( Y \). However, we can still write an expression
\[
S_{\text{top}} = \delta \tau \int_{Y'} \left( A_{n-1} \cup_{n-2} A_{n-1} + A_{n-1} \cup_{n-2} \delta A_{n-1} \right)
\]
in \((n+1)\)-dimensional triangulation \( Y' \) such that \( Y' \) is a refinement of \( Y \). We can check that Eqs. (66) and (70) produce the same boundary term under gauge transformations.

**VI. CONCLUSIONS**

We have extended the the exact bosonization Eq. (21) in 2D and Eq. (22) in 3D to arbitrary dimensions. The dictionary for \( n \)-dimensional boson-fermion duality is given in Eq. (23). This bosonization is a duality between any fermionic system in arbitrary \( n \) spatial dimensions and \((n-1)\)-form \( \mathbb{Z}_2 \) gauge theories in \( n \) dimensions with gauge constraints (the modified Gauss’s law). This map preserves locality: Every local even fermionic observable is mapped to a local gauge-invariant bosonic operator. The formula has an explicit dependence on the second Stiefel-Whitney class of the manifold and a choice of spin structure is needed. As a side product, we discover formula Eq. (29) for Stiefel-Whitney homology classes on lattices. In the Euclidean picture, we have shown that the Euclidean path integral of the \( n \)-dimensional \( \mathbb{Z}_2 \) gauge theory with modified Gauss’s law is the \((n+1)\)-dimensional Ising model with an additional topological Steenrod square Eq. (56) term.

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**APPENDIX: A FORMULA FOR STIEFEL-WHITNEY HOMOLOGY CLASSES**

In this section, we prove Lemma 3, Eq. (41). First, let us recall the theorem proved in Ref. [15]. Let \( s \) be a \( p \)-simplex, say \( s = (v_0, v_1, \ldots, v_p) \). Let \( k \) be another simplex which has \( s \) as a face; i.e., \( s \subset k \) (\( s \) may be equal to \( k \)). Let
\[
\begin{align*}
B_{-1} &\equiv \text{set of vertices of } k \text{ less than } v_0, \\
B_0 &\equiv \text{set of vertices of } k \text{ between } v_0 \text{ and } v_1, \\
B_m &\equiv \text{set of vertices of } k \text{ between } v_m \text{ and } v_{m+1}, \\
B_p &\equiv \text{set of vertices of } k \text{ greater than } v_p.
\end{align*}
\]
(67)
We say that \( s \) is regular in \( k \), if \( \#(B_m) = 0 \) for every odd \( m \). Let \( \partial_s(k) \) denote the mod 2 chain which consists of all \( p \)-dimensional simplices \( s \) in \( k \) so \( s \) is regular in \( k \). For example, \((012)\) and \((023)\) are regular in \((0123)\) and therefore \( \partial_2((0123)) = (012) + (023) \). The theorem is [15]

**Theorem 1.** \( \sum_{k: \dim k \geq (n-2)} \partial_{n-2}(k) \) is a \((n-2)\)-chain which represents \( u_{n-2} \).

In particular, for any \( n \)-simplex \( \Delta_n = \{0 \ldots n\} \), all \((n-1)\)-simplices regular in \( \Delta_n \) are
\[
\{0 \ldots i \ldots n\} \forall i \in \text{odd}
\]
(68)
and all \((n-2)\)-simplices regular in \( \Delta_n \) are
\[
\{0 \ldots i \ldots j \ldots n\} \forall i \in \text{odd}, j \in \text{even}, i < j.
\]
(69)
We now use this theorem to prove Lemma 3.

**Proof of Lemma 3.** For every \((n-2)\)-simplex \( \Delta_{n-2} \), it is regular in itself. This contributes the 1 in the coefficient of \( c(\Delta_{n-2}) \) in Eq. (41).

For every \((n-1)\)-simplex \( \Delta_{n-1} \), it is a boundary of two \( n \)-simplices \( \Delta_n^L \) and \( \Delta_n^R \), with \( \Delta_{n-1} \) being an outward boundary of \( \Delta_n^L \) and an inward boundary of \( \Delta_n^R \). We define that \( \Delta_{n-1} \) belongs to \( \Delta_n^R \) and the summation of \( \dim k = n-1, n \) in Theorem 1 can be written as
\[
\sum_{\Delta_{n-1}} \partial_{n-2}(\Delta_{n-1}) + \sum_{\Delta_n} \partial_{n-2}(\Delta_n)
\]
\[
= \sum_{\Delta_n} \left[ \partial_{n-2}(\Delta_n) + \sum_{\Delta_{n-1} \in \Delta_n | \Delta_{n-1} \text{ is inward}} \partial_{n-2}(\Delta_{n-1}) \right].
\]
(70)
If \( \Delta_n = \{0 \ldots n\} \), is \( + \) oriented, the terms in the summation are
\[
\partial_{n-2}(\{0 \ldots n\}) + \sum_{0 \leq i < j \leq n, i \text{ odd}} \partial_{n-2}(\{0 \ldots \hat{i} \ldots j \ldots n\})
\]
\[
+ \sum_{0 \leq i < j \leq n, i \text{ odd}} \partial_{n-2}(\{0 \ldots j \ldots \hat{i} \ldots n\})
\]
\[
+ \sum_{j > i \leq n, j \text{ even}} \partial_{n-2}(\{0 \ldots \hat{i} \ldots \hat{j} \ldots n\})
\]
\[
= \sum_{i, j \leq n, i, j \text{ odd}} \{0 \ldots \hat{i} \ldots \hat{j} \ldots n\},
\]
(71)
where we have used the definition of regular simplex defined above. Similarly, we can derive that if $\Delta_n = \langle 0 \ldots n \rangle$ is – oriented, the term is

$$
\sum_{i, j < j, i \in \text{even}, j \in \text{even}} \langle 0 \ldots \hat{i} \ldots \hat{j} \ldots n \rangle.
$$

(A6)

Combining Eqs. (A5) and (A6) with the 1 from $\dim k = n - 2$ in Theorem 1, we have

$$
w_2 = \sum_{\Delta_{n-2}} c(\Delta_{n-2}) \Delta_{n-2},
$$

(A7)

where

$$
c(\Delta_{n-2}) = 1 + \sum_{- \text{oriented } \Delta_n = \langle 0 \ldots n \rangle \ j_1 < j_2 | j_1, j_2 \in \text{even}} \Delta_{n-2}(\langle 0 \ldots j_1 \ldots j_2 \ldots n \rangle)
+ \sum_{+ \text{oriented } \Delta_n = \langle 0 \ldots n \rangle \ k_1 < k_2 | k_1, k_2 \in \text{odd}} \Delta_{n-2}(\langle 0 \ldots k_1 \ldots k_2 \ldots n \rangle).
$$

(A8)

\[\square\]

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