How is the Presence of Horizons and Localised Matter Encoded in the Entanglement Entropy?

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Motivated by the new theoretical paradigm that views spacetime geometry as emerging from the entanglement of a pre-geometric theory, we investigate the issue of the signature of the presence of horizons and localized matter on the entanglement entropy (EE) $S_E$ for the case of three-dimensional AdS (AdS$_3$) gravity. We use the holographically dual two-dimensional CFT on the torus and the related modular symmetry in order to treat bulk black holes and conical singularities (sourced by pointlike masses not shielded by horizons) on the same footing. In the regime where boundary tori can be approximated by cylinders we are able to give universal expressions for the EE of black holes and conical singularities. We argue that the presence of horizons/localized matter in the bulk is encoded in the EE in terms of (i) enhancement/reduction of the entanglement of the AdS$_3$ vacuum, (ii) scaling as area/volume of the leading term of the perturbative expansion of $S_E$, (iii) exponential/periodic behaviour of $S_E$, (iv) presence of unaccessible regions in the noncompact/compact dimension of the boundary cylinder. In particular, we show that the reduction effect of matter on the entanglement of the vacuum found by Verlinde for the de Sitter vacuum extends to the AdS$_3$ vacuum.

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I. INTRODUCTION

One of the most fruitful, recently proposed, concepts in the context of fundamental gravitational physics is that spacetime geometry and gravity emerge from quantum entanglement of a microscopic, pre-geometric theory [1–9]. It has been shown that the connectivity of the spacetime can be explained in terms of quantum entanglement [4]. It has been proved that entanglement entropy (EE) satisfies the same thermodynamical relations of the Bekenstein-Hawking entropy [5, 10]. The linearized Einstein equations have been derived from quite general principles involving quantum entanglement [6–8]. It has been also proposed that the accelerated expansion of the universe can be explained by a volume law contribution to the entanglement entropy producing a slow thermalization of the emergent spacetime [9]. In this framework the phenomenology commonly attributed to dark matter can be explained as a "dark force" originated by competition between area and volume contribution to EE [9].

This novel paradigm has been triggered by the advances in the understanding of the microscopic origin of the Bekenstein-Hawking entropy [11, 12], by the development of the AdS/CFT correspondence [13] and the related holographic interpretation of entanglement entropy [14, 15], by earlier work on the emergence of gravity [1] and, last but not the least, by the recent discussions about the black hole quantum information puzzle involving firewalls and complementarity [16–18].

A weak point of this new emergent gravity scenario, which in our opinion has not been adequately stressed, is that it requires for its formulation a background pre-existing geometric structure, like e.g. local Killing [1, 6–8] or cosmological [9] horizons. This is conceptually non consistent because one would like to have a background-independent formulation, in which gravity and spacetime emerges completely from a pre-geometric microscopic structure. This seems to support the point of view, according to which entanglement is not enough to explain all the quantum information puzzles and that some additional crucial feature is needed [18].

A promising way to try to shed light on this issue is to look at the way the information about the presence of background gravitational structures (horizons, vacuum spacetimes, spacetimes with singularities) is encoded in the EE. This is the main purpose of this paper. We investigate this issue in the context of AdS gravity in three spacetime dimensions (3D), in which pointlike matter generates conical singularities. Moreover in 3D, the AdS/CFT correspondence, the Ryu-Takayanagi (RT) formalism [14, 15, 19], and its extensions like complexified geodesic [20] and differential entropy [21] allows us to compute in closed form the EE of the dual two-dimensional (2D) Euclidean CFT living on the torus.

The modular symmetry of the torus will be used to exchange regions not accessible to measurement, to which an EE $S_E$ is associated, between different directions of Euclidean space and to treat bulk geometries with horizons and conical singularities on the same footing. After that, we work in the limit where the boundary tori can be approximated by cylinders. In this regime the modular symmetry will provide us with a web of dualities mapping the various boundary cylinders with unobservable regions. Moreover, in the cylinder approximation the EE of the Euclidean boundary CFT belongs to two universality classes corresponding to unobservable slices in the compact or noncompact dimension. This will allow us to assign a well-defined EE to the various gravitational bulk configurations and to discuss the signature corresponding to the presence of horizons or pointlike masses not shielded by an event horizon.

We find that the presence of an event horizons in the bulk is related to an enhancement of the entanglement of the AdS$_3$ vacuum, to an area law for the leading term of the perturbative expansion of $S_E$, to the presence of unobservable regions in the noncompact dimension of the cylinder and to exponential behaviour of the EE. Conversely, localized matter not shielded by an event horizon, i.e conical singularities in the bulk, are related to reduction of the entanglement of the AdS$_3$ vacuum, to a volume law for the leading term of the perturbative expansion of $S_E$, to the presence of unobservable regions in the compact dimension and to a periodic behaviour of the EE. In particular, this confirms the reduction effect of matter on the entanglement of the vacuum found in Ref. [9] for de Sitter spacetime.

We also briefly discuss EE in Minkowski spacetime, the concept of entwinement [21], complexified geodesic [20] and the concept of entanglement in time.

The structure of this paper is as follows. We start the paper by reviewing some aspects of AdS gravity in 3D and related bulk configuration (Sect. II). We discuss modular transformations in the torus in section III and their cylinder limit in Sect. IV. Then we move on to section V where we obtain entanglement entropy of the dual CFT for different locations of the slices in the cylinder approximation. In Sect. VI we compute the holographic entanglement entropy of various the 3D gravity bulk solutions. In section VII, we use gravitational tools to cross-check the results of Sect. VI. In Sect. VIII we derive the leading terms for the holographic EE in the large/small radius expansion associated to the various gravitational bulk configurations. In Sect. IX we perform a Wick rotation to derive the EE in Minkowski spacetime. In Sect. X we briefly discuss causal aspects of holographic EE. We conclude the paper by stating our conclusions (Sect. XI).
II. BULK GRAVITY CONFIGURATIONS

In this paper we will restrict our considerations to 3D AdS gravity. Classical, pure AdS$_3$ gravity is described by the action

$$A = \frac{1}{16\pi G_3} \int d^3x \left( R + \frac{2}{L^2} \right),$$

(2.1)

where $L$ is the de Sitter length and $G_3$ is 3D Newton constant.

The AdS$_3$/CFT$_2$ correspondence [13] dictates that in the large $\mathcal{N}$ limit, i.e in the regime where the central charges of the CFT$_2$ [22]

$$c = \bar{c} = \frac{3L}{2G_3} \gg 1,$$

(2.2)

3D AdS Gravity is holographically dual to 2D CFT defined on the $r \to \infty$ conformal boundary (torus).

A. The spectrum of AdS$_3$ gravity

Pure AdS gravity in 3D has no propagating degrees of freedom. Away from the sources the spacetime is always locally equivalent to AdS$_3$. Localised matter affects only the global properties of the spacetime. The solutions of AdS$_3$ classical gravity, which represent regular geometries, are usually classified in terms of three classes corresponding to the orbits (elliptic, hyperbolic, parabolic) of the SL$(2, R)$ group manifold [23–25]. However, a physical classification of the solutions can be obtained by starting from the BTZ black hole, i.e an asymptotically AdS solution with an event horizon, positive mass and finite temperature $T$. The BTZ black holes give the positive mass excitations. The vacuum of the theory is the zero mass, $T = 0$ solution and it is given by AdS$_3$ in Poincaré coordinates. In this paper we will consider this as the physical vacuum of the theory. The peculiarity of AdS$_3$ gravity is that in the theory we have an other vacuum solution, namely AdS$_3$ in global coordinates. This solution is separated from the AdS$_3$ Poincaré vacuum by solutions describing conical singularities. Although these geometries are not regular we will consider them as real physical solutions, because differently from gravity in four dimensions, they are sourced by a pointlike particle of positive mass.

In this paper we will therefore take in considerations all the solutions of AdS$_3$ classical gravity, i.e including also those representing conical singularities in space and time. As we will explain in the next section this is necessary in view of the modular symmetry of the CFT living in the boundary torus. For our purposes we will distinguish between four classes of solutions namely AdS$_3$ vacua, the BTZ Black hole and AdS$_3$ with conical defect along, respectively, the space and time direction.

- AdS$_3$ vacua

As we have mentioned above, we have two vacua: AdS$_3$ in global coordinates and AdS$_3$ in the Poincaré patch. Global AdS$_3$ can be represented by the metric

$$ds^2 = - \left( 1 + \frac{r^2}{L^2} \right) dt^2 + \left( 1 + \frac{r^2}{L^2} \right)^{-1} dr^2 + \frac{r^2}{L^2} d\phi^2.$$  

(2.3)

These global coordinates range over $t \in [-\infty, \infty], \phi \in [0, 2\pi L], r \in [0, \infty)$. By restricting $t \in [0, \beta]$ we get AdS$_3$ at finite temperature $T = 1/\beta$. This configuration is dual to a CFT with the same finite temperature and living on a torus $T(\beta, 2\pi L)$, where $\beta$ and $2\pi L$ represent, respectively, the cycles in the time and space direction. Also, one can take the Poincaré patch of AdS$_3$ which can be obtained by taking the limit $r \to \infty$ of (2.3) and is given by the metric:

$$ds^2 = - \left( \frac{r^2}{L^2} \right) dt^2 + \left( \frac{r^2}{L^2} \right)^{-1} dr^2 + \frac{r^2}{L^2} d\phi^2.$$  

(2.4)

Here, $\phi$ is no longer periodic, $\phi \in [-\infty, \infty], t \in [-\infty, \infty]$ and this solution is dual to a CFT living on a Plane.

In this paper we will consider AdS$_3$ in the Poincaré patch as the physical vacuum of the theory. The rational behind our choice is that this solution, differently from global AdS$_3$, is continuously connected both with the BTZ black hole part of the spectrum and with the part of the spectrum describing conical singularities, which are sourced by a pointlike mass. We can therefore consider both thermal excitation of the vacuum generated by a black hole with non vanishing mass and insertion of pointlike masses producing conical singularities.
- **BTZ black hole**
  It can be represented by the metric:
  \[
  ds^2 = -\frac{1}{L^2} \left( r^2 - r_+^2 \right) dt^2 + \frac{1}{L^2} \left( r^2 - r_+^2 \right)^{-1} L^2 dr^2 + \frac{r^2}{L^2} d\phi^2.
  \]  
  (2.5)
  The BTZ black hole has horizon radius \( r_+ \), inverse Hawking temperature \( \beta_H \), mass and thermal Bekenstein-Hawking entropy given by
  \[
  \beta_H = \frac{1}{T_H} = \frac{2\pi L^2}{r_+}, \quad M = \frac{r_+^2}{8G_3 L^2}, \quad S_{BH} = \frac{A}{4G_3} = \frac{\pi r_+}{2G_3}.
  \]  
  (2.6)
  The Poincaré patch of AdS\(_3\) can be considered as the ground state of BTZ black hole in which we have \( r_+ = 0, M = 0 \) and \( T_H = 0 \). The corresponding 3D Euclidean space has a contractible cycle along the \( t \) direction and for generic values of its periodicity \( \beta \) there is a conical singularity along the \( t \) direction. For \( \beta = \beta_H \), the conical singularity is removed and we have a CFT at a finite temperature living on a torus \( T(\beta_H, 2\pi L) \) with cycles \( \beta_H, 2\pi L \).

- **AdS\(_3\) with conical singularity in space**
  It can be obtained from Eq. (2.3) by rescaling the coordinates as
  \[
  t \to \Gamma t, \quad \phi \to \Gamma \phi \quad \text{and} \quad r \to r/\Gamma \quad \text{where} \quad \Gamma = \frac{r_+}{L}.
  \]  
  (2.7)
  With this rescaling we get the following metric:
  \[
  ds^2 = -\left( \Gamma^2 + \frac{r^2}{L^2} \right) dt^2 + \left( \Gamma^2 + \frac{r^2}{L^2} \right)^{-1} dr^2 + \frac{r^2}{L^2} d\phi^2,
  \]  
  (2.8)
  where the coordinates range is \( t \in [0, \infty], \phi \in [0, 2\pi L], r \in [0, \infty] \) and with the parameter \( 0 \leq \Gamma \leq 1 \) ranging from Poincaré AdS (\( \Gamma = 0 \)) to global AdS (\( \Gamma = 1 \)). For \( \Gamma \neq 1, 0 \) the corresponding 3D euclidean space has a contractible cycle along the spatial \( \phi \)-direction. If we restrict \( \phi \) as \( 0 \leq \phi \leq 2\pi L \) where \( \Gamma = r_+/L \) then we will have a space with a conical singularity owing to the deficit angle given as
  \[
  \delta \phi = 2\pi (1 - \Gamma) = 2\pi (1 - r_+/L) = 2\pi (1 - 2\pi L/\beta_{\text{con}}),
  \]  
  where in analogy with the inverse Hawking temperature we have defined,
  \[
  \beta_{\text{con}} \equiv \frac{2\pi L^2}{r_+} = \beta_H.
  \]  
  (2.9)
  By restricting the coordinates to \( t \in [0, \Gamma \beta_{\text{con}} = 2\pi L], \phi \in [0, 2\pi L], r \in [0, \infty] \), we have a spacetime with a conical singularity, whose \( r \to \infty \) conformal boundary is a torus \( \mathcal{T}(\Gamma \beta_{\text{con}}, 2\pi L) \). This AdS\(_3\) space with conical defect is dual to a CFT living on this torus. The conical singularity we have whenever \( \Gamma \neq 0, 1 \) represents the geometric distortion generated by a pointlike particle of mass \( m = (1 - \Gamma)/4G_3 \) [22, 26, 27]. In fact a stress tensor for a pointlike mass \( m \) provides a solution of Einstein equations in 3D given by Eq. (2.8). It is important to stress that what we are considering as the physical vacuum of the theory (AdS\(_3\) in the Poincaré patch) can be also obtained from the conical singularity spacetime by setting \( \Gamma = 0 \), which from Eq. (2.9) implies \( \beta_{\text{con}} = \infty \), and by decompactifying the \( t \) direction.

- **AdS\(_3\) with conical singularity in time**
  For generic values of the Euclidean time periodicity different from \( \beta_H \), the Euclidean version of the spacetime (2.5) describes a conical singularity in the time direction. We can always parametrize this periodicity in terms of \( r_+ \) and choose \( t \in [0, 2\pi r_+/L] \), where \( \Gamma \) is defined as above. In this way the conical singularity in time has the same characterisation of the conical singularity in space with time and space coordinates exchanged. The \( r \to \infty \) conformal boundary of the space is the torus \( \mathcal{T}(2\pi L, \Gamma \beta_{\text{con}}) \), which is the torus associated with conical singularities in space with exchanged cycles. This AdS\(_3\) space with conical defect in time is dual to a CFT living on this torus.
III. MODULAR TRANSFORMATIONS

One useful concept when dealing with a 2D CFT in the complex torus is that of modular invariance (see e.g. Ref. [28, 29]). The partition function of the theory must be invariant under modular transformations. The most important parameter for CFT on torus is the modular parameter \( \tau = \omega_1/\omega_2 \) where \( \omega_{1,2} \) are periods of the torus. With the help of this modular parameter \( \tau \), various modular transformations are defined which together form the modular group \( PSL(2, \mathbb{Z}) \): \( \tau \rightarrow (a\tau + b)/(c\tau + d) \) with \( ad - bc = 1 \). There are three types of modular transformations \([28, 29]\): \( T : \tau \rightarrow \tau + 1 \), \( S : \tau \rightarrow -1/\tau \), \( U : \tau \rightarrow \tau^{1/2} \). These transformations generate themselves by composition:

\[
\mathcal{T} = USU, \quad U = TST, \quad S = UT^{-1}UT, \quad (ST)^3 = S^2 = 1.
\]

Generally, lattice representation is used to look at modular transformations. In the lattice representation, torus is identified with vectors in the complex plane \( (\omega_{1,2}) \). Modular parameter is now given as \( \tau = \omega_2/\omega_1 = \tau_1 + i\tau_2 \). Considering a unit cell, modular transformations will act on this unit cell and for simplicity we can choose \( \omega_2 = \tau \) and \( \omega_1 = 1 \).

Of particular relevance for our investigations is the modular \( S \) transformation. It is realized in terms of the action on the complex coordinate on the torus \( z = t_E + ix_E \) (\( x_E, t_E \) are Euclidean space and time) and on the modular parameter \( \tau \) as

\[
z \rightarrow z' = \frac{z}{\tau}, \quad \tau \rightarrow \tau' = -\frac{1}{\tau}.
\]

In the context of the AdS\(_3\)/CFT\(_2\) correspondence \( S \) transformations have been used focusing on their action on the modular parameter \( \tau \) associated with the boundary tori dual to 3D bulk configurations. It has been shown that the modular parameter \( \tau_{\text{AdS}} \) of the CFT dual to AdS\(_3\) at finite temperature is related to the modular parameter \( \tau_{\text{BTZ}} \) of the CFT dual to the BTZ black hole by the modular modular \( S \) transformation \( \tau_{\text{AdS}} = -1/\tau_{\text{BTZ}} \) \([30, 31]\). Later, it has been shown that same relation holds for the modular parameter \( \tau_{\text{con}} \) of the CFT dual to AdS\(_3\) with space conical singularities: \( \tau_{\text{con}} = -1/\tau_{\text{BTZ}} \) \([32, 33]\).

Using the results of the previous section we can now also show that the boundary torus dual to AdS\(_3\) with conical singularity along the time direction belongs to the web generated by \( S \) transformations. In fact, we have seen that the boundary tori associated respectively to conical singularities in space and conical singularities in time are given respectively by \( T(\Gamma\beta_{\text{con}}, 2\pi\Gamma L) \) and \( T(2\pi\Gamma L, \Gamma\beta_{\text{con}}) \). They have their cycles exchanged and are therefore connected by a \( S \) duality transformation.

Having in mind the classification of the 3D bulk configuration given in Sect. II, we conclude that all the boundary tori dual to 3D AdS gravity bulk configurations are related one with the other by means of and \( S \) modular transformation. This means that compatibility of the AdS\(_3\)/CFT\(_2\) correspondence with the modular symmetry of the torus requires that with the exception of the vaca, the four types of bulk configurations listed in Sect. II (thermal AdS, BTZ black hole, conical singularities in space, conical singularities in time) have to be included in the physical spectrum of the theory. This is a quite non trivial result because conventional wisdom would require singular geometries to be excluded from the physical spectrum of pure AdS\(_3\) gravity.

As noted above, \( S \) modular transformations (3.1) are generally used only in terms of their action on the modular parameter \( \tau \) and not in terms of their action on the complex coordinate \( \tau \) of the torus. This is simply due to the obvious fact that the partition function for a CFT on the torus depends on \( \tau \) but not on \( z \). On the other hand this is not anymore true when one considers the EE of the CFT on the torus as we do in this paper. The EE of a QFT is defined as the von Neumann Entropy originated by tracing the density matrix over unobservable degrees of freedom localized in a codimension one region \( B \) of the manifold where the QFT lives. In the case of our 2D CFT this means that we have to assume that only a slices of say length \( l \) localized either in the space or time direction of the torus is accessible to measurement.

One can now easily show that the modular transformation (3.1) exchanges real and imaginary part of the complex coordinate \( z \), thus it can be used to exchange slices localized in the space and time directions. In fact, taking for simplicity \( \omega_1 = 1 \), \( \tau = \omega_2 = i\alpha \) (with \( \alpha \) real number) and a slice localized in the (Euclidean) time dimension, \( z = l \), using Eq. (3.1) we find

\[
z' = -\frac{id}{\alpha}, \quad \tau' = \frac{i}{\alpha}.
\]

This means moving from real to imaginary axis, hence using this aspect of \( S \) transformation we can exchange slices between (Euclidean) space and and time directions. In general one could consider the most general case in which slices have both non vanishing real and imaginary part. For simplicity, in this paper we will limit our considerations to slices localized completely either in the space or time dimension.
IV. CYLINDER APPROXIMATION AND MODULAR TRANSFORMATIONS ACTING ON BOUNDARY CYLINDERS

As we shall discuss in the next sections, the EE for 2D CFT in the torus has not an universal form but depends on the actual field content of the theory. Conversely for a 2D CFT in the cylinder the EE has an universal form. For this reason it is very useful to consider the cylinder approximation of the torus $T(\beta, \alpha)$ in the limit when one of the two compact dimensions decompactifies. This cylindric limit of the torus $T(\beta, \alpha)$ can be achieved in two different ways. (1), $\beta \gg \alpha$: in this case the time direction of the torus decompactifies and we have $T(\beta, \alpha) \rightarrow C(\alpha)$, where $C(\alpha)$ is defined as the cylinder with noncompact time direction and compact space direction of length $\alpha$; (2), $\alpha \gg \beta$: in this case the space direction of the torus that decompactifies and we have $T(\beta, \alpha) \rightarrow C(\beta)$, where $C(\beta)$ is defined as the cylinder with noncompact space direction and compact time direction of length $\beta$.

The modular transformations $S$ described in the previous section for the torus have a natural extension when we work in the cylinder approximation. Applying Eq. (3.1) to our torus $T(\beta, \alpha)$, we see that it is transformed into a torus with modular parameter $\tau' = i\beta/\alpha$, i.e into the torus $T(\alpha, \beta)$, where $\beta > \alpha$. Thus in the limit $\beta > \alpha$ the $S$ transformation acts as $C(\alpha) \rightarrow C(\alpha)$. Conversely, in the limit $\alpha > \beta$, $S$ acts as as $C(\beta) \rightarrow C(\beta)$. Hence, modular transformations act on the boundary cylinders by exchanging compact with non compact directions. By analogy with the usual thermal definition, where the periodicity of euclidean time, $\beta$ gives the scale of thermal correlation, i.e the inverse temperature $\beta = 1/T$, we can interpret the periodicity of the compact space dimension $\alpha$ as giving the scale of the quantum entanglement correlations. This can be used to define a sort of "quantum" temperature $\alpha = 1/TQ$.

It is important to stress that because the modular transformation $S$ exchanges the cycles of torus, it maps different regimes of the CFT. In the cylinder approximation this means that the modular map $C(\alpha) \rightarrow C(\alpha)$, relates the regime, where $T << T_Q$ (the regime where quantum correlations dominate) to the regime $T >> T_Q$, where thermal correlations dominate. Obviously, for the map $C(\beta) \rightarrow C(\beta)$ the opposite is true.

Using the results of section III, the previous discussion can be easily extended to the cylindric approximation of tori with unobservable regions. When unobservable regions are present the $S$ transformation not only maps the cylinder $C(\beta)$ into $C(\beta)$ but also moves simultaneously the unobservable region $B$ from the space to time direction of the cylinder and viceversa. Notice that the $S$ transformation cannot move regions from compact to noncompact direction of the cylinder. An unobservable region localized in the compact (non compact) direction already remains in the same direction.

On the other hand, slices can be moved from the compact to the non compact dimension of the cylinder first by using $S$ modular transformation in the torus then taking the infinite size limit of the appropriate direction. Notice that in this case the cylinders are not related by modular transformations. In the next section we will use this procedure to move slices from compact to non compact directions of the cylinder.

V. ENTANGLEMENT ENTROPY OF BOUNDARY CFT IN THE CYLINDER

In this section we will compute the EE of the boundary CFT for four different positions of slices and for the case in which the boundary CFT is defined on an infinite cylinder. The four different positions are obtained by putting slices in the Euclidean space/time direction and by considering cylinders with compact/noncompact space direction. We will consider infinite cylinders instead of tori because on cylinders the results for the EE are universal as compared to torus where the results depend on the details of underlying CFT theory. Later, on Sect. VI we will use these computations to associate a holographic EE to the four gravitational bulk configuration described in Sect. II.

In order to calculate the EE entropy for the CFT on the infinite cylinders we will use the following strategy. We will start from the term in the EE for a free Dirac fermion on the torus of Ref. [19], which gives in the infinite cylinder approximation the universal, leading contribution to the entropy:

$$S_E = -\text{Tr}(\rho_A \log \rho_A)$$

(5.1)

where $z = t_E + i x_E = i t_M + i x_M$ is the slice describing the subsystem $A$ and $\tau = \beta + i \alpha$ are, respectively,
the complex coordinate and the modular parameter of the torus, \((t_E, x_E)\) are euclidean coordinates, \((t_M, x_M)\) are Minkowskian coordinates\(^1\). Further, \(\beta\) is the periodicity for \(t_E\) and \(\alpha\) is the periodicity for \(x_E\), \(\theta\) is the theta function and \(\eta\) is the Dedekind eta function.

In the lattice representation of this torus and in the pictures that follow, we have the real axis along the vertical axis, whereas the imaginary axis is directed along the horizontal axis. Notice that we are using here the generalized form of the formula given in Ref. [19], where the formula has been used for spacelike slices only. Eq. (5.1) has a general validity and can be therefore used to compute the entanglement entropy for slices with both space and time components, just by giving \(z\) the desired value.

- **Slice along compact time dimension**
  We take the slice along time axis only, that is we consider a slice of the time coordinate of measure \(l_t\), \(z = l_t\) and \(\beta = 1\), \(\tau = i\alpha\). We now have \(q\) and \(y\) as required by the theta function,
  \[
  q = e^{2\pi i \tau} = e^{-2\pi \alpha}, \quad y = e^{2\pi i z} = e^{2\pi i l_t}.
  \]

Using the above values for \(q\) and \(y\), we have the form for \(S_1\) as:
  \[
  S_1 = \frac{c}{3} \log \frac{1}{\pi} \sin(\pi l_t) \prod_{m=1}^{\infty} \left( \frac{1 - y q^m (1 - y^{-1} q^m)}{(1 - q^m)^2} \right),
  \]

Pictorially this will look like:

\[
\begin{align*}
  l_t &\quad \equiv \quad \boxed{} \\
  (\beta) &\quad \equiv \quad \boxed{}
\end{align*}
\]

where the symbol \(\equiv\) stands for a slice.

\((\alpha)\)

Now, we take the Eq. (5.2) in the infinite spatial size limit \(\alpha >> 1\) which corresponds to \(T >> T_Q\) (where \(T_Q\) is the "quantum" temperature defined in the previous section. This gives us the entanglement entropy on the infinite cylinder \(C(\beta)\):
  \[
  S^{(tc)} = \frac{c}{3} \log \left| \frac{\beta a \pi}{\sin \left( \frac{\pi l_t}{\beta} \right)} \right|,
  \]

where we have reinstated the periodicity \(\beta\) of \(t_E\) and \(a\) is an UV cutoff. This equation has general validity and gives the EE entropy for slices localized along a compact time dimension.

- **Slice along noncompact space dimension**
  We now apply modular transformation \(S\) on \(z\) and \(\tau\) to get \(z'\) and \(\tau'\) which is:
  \[
  z' = \frac{z}{\tau} = -\frac{il_s}{\alpha}, \quad \tau' = -\frac{1}{\tau} = \frac{i}{\alpha},
  \]

where we have renamed \(l_t\) into \(l_s\) to stress its different character (measure of space slice instead of time slice). As a result, we now have \(q\) and \(y\) as:
  \[
  q = e^{2\pi i z} = e^{-2\pi / \alpha}, \quad y = e^{2\pi i z} = e^{2\pi i / \alpha}.
  \]

Modular transformation \(S\) has taken the slice from real to imaginary axis and pictorially this looks like:

\[
\begin{align*}
  (\alpha) &\quad \equiv \quad \boxed{} \\
  \frac{il_s}{(i \beta)} &\quad \equiv \quad \boxed{}
\end{align*}
\]

\(^1\) Notice that in this paper we are using a definition of the complex coordinate \(z\) of the torus different from that of Ref. [19], where it is defined as \(z = x_E + it_E\).
Using the modular transformation $S$, we now have the form for $S_1$ as:

$$S_1 = \frac{c}{3} \log \left| \frac{\alpha}{\pi} e^{\frac{\pi l_s}{\alpha}} \sinh \left( \frac{\pi l_s}{\alpha} \right) \prod_{m=1}^{\infty} \frac{(1 - y q^m)(1 - y^{-1} q^m)}{(1 - q^m)^2} \right|. \quad (5.6)$$

Now, we take Eq. (5.6) in the infinite space size, $\alpha \ll 1$ limit, implying $T \gg T_Q$, which gives us the entanglement entropy of the CFT on the infinite cylinder $C(\alpha)$ with slice localized in the spatial, noncompact, dimension:

$$S^{(snc)} = \frac{c}{3} \log \left| \frac{\alpha}{a \pi} \sinh \left( \frac{\pi l_s}{\alpha} \right) \right|. \quad (5.7)$$

• **Slice along compact space dimension**

Next, we apply the modular transformation $S$ on the cylinder described in section IV, to the cylinder $C(\beta)$ with slice in the compact time dimensions to obtain the cylinder $C(\beta)$ with slice in the compact space dimension. Pictorially this looks like:

![Diagram of a cylinder with slice in compact space dimension]

where, as usual, we have renamed $l_t$ into $l_s$. This cylinder approximation holds for $\alpha \gg \beta$, i.e for $T \ll T_Q$. Eq. (5.3) does not change under the modular transformation $C(\beta) \to C(\beta)$ and, correspondingly, we get the EE for a CFT on the $C(\beta)$ cylinder with slice $l_s$ along the compact space dimension.

$$S^{(sc)} = \frac{c}{3} \log \left| \frac{\beta}{a \pi} \sin \left( \frac{\pi l_s}{\beta} \right) \right|. \quad (5.8)$$

• **Slice along noncompact time dimension**

To get the EE for slice along non compact time dimensions we use the modular transformation $S$ on the cylinder described in section IV, to the cylinder $C(\alpha)$ with slice in the non compact space dimensions to obtain the cylinder $C(\alpha)$ with slice in the non compact time dimension. Pictorially, after renaming $l_s$ into $l_t$ this looks like:

![Diagram of a cylinder with slice in noncompact time dimension]

The modular transformation $C(\alpha) \to C(\alpha)$ leaves the EE (5.7) invariant. This allows us to write the EE of a CFT on the $C(\alpha)$ cylinder with slice $l_t$ along the noncompact time dimension:

$$S^{(tnc)} = \frac{c}{3} \log \left| \frac{\alpha}{a \pi} \sinh \left( \frac{\pi l_t}{\alpha} \right) \right|. \quad (5.9)$$

This cylinder approximation holds for $\beta \gg \alpha$, i.e for $T \ll T_Q$.

The four expressions (5.3),(5.8) and (5.7),(5.9) for the EE of a CFT on the cylinder we have derived in this section are universal and well-known [34–36]. However we want to stress here that their physical meaning and regime of validity is slightly different from what is usually considered. Eq. (5.3) and (5.7) give the contribution of, respectively, time and space entanglement at large temperature when thermal correlations dominate, whereas Eq. (5.8) and (5.9) give the same information but at small $T$, when quantum correlations dominate.
A. Planar approximation

The planar approximation describing a CFT in the plane can be obtained by considering in Eqs. (5.3),(5.8) the limit \( t_\perp, t_\parallel \ll \beta \) and in Eqs. (5.7),(5.9) the limit \( t_\perp, t_\parallel \ll \alpha \). We get the same expression for all four cases, giving the EE entropy for a CFT a zero temperature, non compact space dimension and with a space (time) slice of length \( l_\parallel \) (\( l_\perp \)):

\[
S_{\text{pl}}(5.10) = \frac{c}{3} \log \left| \frac{l_\perp}{a} \right|
\]

VI. HOLOGRAPHIC ENTANGLEMENT ENTROPY OF BULK GRAVITATIONAL CONFIGURATION

In this section we will show how, starting from the results of Sect. V, one can assign an holographic EE to the 3D bulk gravity configurations listed in Sect. II. Despite some progress, the problem of reducing the EE of bulk gravitational configuration to dual boundary CFT calculations is not solved [19, 32, 33, 37–43]. Our procedure is based on the AdS/CFT correspondence, which implies that in the large \( \epsilon \) limit we can use the dual CFT to describe 3D bulk configurations. The use of the dual CFT for calculating the EE has several advantages with respect to other non-holographic approaches, in which the EE entropy of gravity configurations, like black holes, is defined in terms of the EE of quantum states of matter fields in the given, classical, gravitational background [44, 45]. Typically, this bulk calculations are not universal, but depend on the number of matter fields and on the UV cutoff [44, 45]. Conversely, our approach does not suffer of these shortcomings. In a sense the number of matter fields is fixed by the central charge \( c \) of the CFT. Because correlations of the bulk theory are codified in a highly non local way in terms of bulk parameters characterising 3D bulk geometries.

Using arguments borrowed from the IR/UV relation in the AdS/CFT correspondence, in Ref. [32, 33] an identification \( \Lambda = 4\pi^2 L^2 / a \) was proposed for the case of the BTZ black hole. The proposal had a conjectural status. Let us now demonstrate it using geometric arguments. We also extend its validity to all the gravity solutions listed in Sect. II.

- Entanglement entropy of AdS\(_3\) with conical singularity in space

Let us start from the 3D bulk solution (2.8) describing AdS\(_3\) with conical singularities along the \( \phi \) direction. The dual boundary CFT lives in the boundary torus \( \mathcal{T}(2\pi L, 2\pi r_+) \). The key observation is that for \( r_+ < L \) we can see the conical singularity as originated by an observable slice of length \( 2\pi r_+ \) in the spatial cycle of total length \( 2\pi L \) of the boundary torus \( \mathcal{T}(2\pi r_+, 2\pi L) \). Notice that this torus is related to the original torus by a modular transformation (it has exchanged cycles). Moreover, putting \( r_+ = 0 \) the conical singularity in the bulk disappears. For \( r_+ > 0 \) we get the physical vacuum of the theory (in the bulk AdS\(_3\) in Poincaré patch, in the boundary a \( T = 0 \) CFT).

By rescaling the lengths by a factor \( L/r_+ \) we get the torus \( \mathcal{T}^{cs}(2\pi L, \beta_{\text{con}}) \) and for \( r_+ \ll L \) the cylinder \( C^{cs}(\beta_{\text{con}}) \) with slice length \( l_\perp = 2\pi \alpha \). The holographic EE for AdS\(_3\) with a conical singularity in space is therefore given by Eq. (5.8) with \( \beta = \beta_{\text{con}} = 2\pi L^2 / r_+ \) and \( l_\perp = 2\pi L \):

\[
S_{\text{cs}}(6.2) = \frac{c}{3} \log \left( \frac{\beta_{\text{con}}}{a\pi} \sin \left( \frac{2\pi L}{\beta_{\text{con}}} \right) \right)
\]
Notice that as expected this approximation holds for small temperatures, i.e for \( T = 1/\beta_{\text{con}} << T_Q = 1/(2\pi L) \).

\* Entanglement entropy of AdS\(_3\) with conical singularity in time

The boundary torus associated to AdS\(_3\) with conical singularity in time is obtained by applying the \( S \) transformation to \( T^{cs}(2\pi L, \beta_{\text{con}}) \) with slice length \( l_s = 2\pi L \). We get \( T^{et}(\beta_{\text{con}}, 2\pi L) \) with slice length \( l_t = 2\pi L \). Correspondingly, for \( L >> r_+ \), the above constructed boundary cylinder \( C^{et}(\beta_{\text{con}}) \) with spatial slice of length \( l_s = 2\pi L \), associated with AdS\(_3\) with conical singularity in space, is mapped by the \( S \) modular transformation in to the boundary cylinder \( C^{et}(\beta_{\text{con}}) \) with slice length \( l_t = 2\pi L \), associated with AdS\(_3\) with conical singularity in time. Thus, we get from Eq. (5.3) the EE of AdS\(_3\) with a conical singularity in time

\[
S^{(et)} = \frac{c}{3} \log \left| \frac{\beta_{\text{con}}}{a\pi} \sin \left( \frac{2\pi^2 L_t}{\beta_{\text{con}}} \right) \right|, \tag{6.3}
\]

where we have used \( L_t \equiv L \) to stress the fact that now \( L_t \) is the length of an euclidean time slice and the formula holds in the regime \( T >> T_Q \). Notice that although in Euclidean space \( L_t = L \) and the holographic EE for AdS\(_3\) with conical singularities in space and time are the same, the meaning of Eqs. (6.2) and (6.3) is rather different. Whereas Eq. (6.2) gives the holographic EE entropy of conical singularities in space in the regime where quantum correlations dominate, Eq. (6.3) gives the holographic EE entropy of conical singularities in time in the regime where thermal correlations dominate.

\* Entanglement entropy of the BTZ black hole

It is known that the boundary torus for AdS\(_3\) with a conical singularity in space \( T^{cs}(2\pi L, \beta_{\text{con}}) \) and the boundary torus \( T^{BTZ}(\beta_H, 2\pi L) \) associated with the BTZ black hole are related by a \( S \) modular transformation [32, 33]. Working in the, \( r_+ >> L \), cylinder approximation and using the results of Sect. V allows us to build the cylinder map \( C^{cs}(\beta_{\text{con}}) \rightarrow C^{BTZ}(\beta_H) \), where the two cylinders have slice of length \( 2\pi L \) respectively along the compact space and non-compact space directions. Correspondingly, Eq. (5.7) with \( \alpha = \beta_H \) and \( l_s = 2\pi L \) will give us the EE entropy of the BTZ black hole

\[
S^{(BTZ)} = \frac{c}{3} \log \left| \frac{\beta_H}{a\pi} \sinh \left( \frac{2\pi^2 L}{\beta_H} \right) \right|, \tag{6.4}
\]

which holds in the regime \( T >> T_Q \) and fully coincides with the result of [32, 33].

We can now close the web of modular transformations by mapping the cylinder \( C^{BTZ}(\beta_H) \) with slice in the noncompact space direction into the \( C^{BTZ}(\beta_H) \) with slice along the noncompact time dimension. Correspondingly Eq. (5.9) gives

\[
S^{(ET)} = \frac{c}{3} \log \left| \frac{\beta_H}{a\pi} \sinh \left( \frac{2\pi^2 L_t}{\beta_H} \right) \right|, \tag{6.5}
\]

giving the holographic EE entropy of the BTZ black hole in terms of Euclidean time correlations. Eq. (6.5) holds in the region \( T << T_Q \).

Notice that also here we have used \( L_t \equiv L \) to represent the length of an euclidean time slice. Considerations similar to those concerning the relation between Eqs. (6.2) and (6.3) apply also to Eqs. (6.4) and (6.5). Eq. (6.4) gives the holographic EE entropy of the BTZ black hole in the regime where thermal correlations dominate and in terms spatial correlations. On the other hand Eq. (6.5) gives the same quantity but in terms of time correlations and in the regime where quantum correlations dominate.

\* Entanglement entropy of the AdS\(_3\) vacuum

The leading term in the \( r_+ << L \) limit of Eqs. (6.2) and 6.5) give the entanglement entropy of AdS\(_3\) vacuum, which can be also expressed in terms of the IR cutoff \( \Lambda \) of Eq. (6.1),

\[
S^{\text{vac}} = \frac{c}{3} \log \left( \frac{2\pi L}{a} \right) = \frac{c}{3} \log \left( \frac{\Lambda}{L} \right). \tag{6.6}
\]

This equation can be obtained directly from the expression (5.10) giving the EE of the CFT in the plane setting \( l_{s,t} = 2\pi L \) and then using (6.1). Notice also that the EE of the AdS\(_3\) vacuum cannot be obtained from Eqs. (6.3) or (6.4) because they are large temperature \( (r_+ >> L) \) expansions.
Let us now briefly summarise the main results of this section. Pure 3D AdS gravity allows for two classes of solutions corresponding to singularities shielded by event horizons and naked (conical) singularities produced by a pointlike mass. In the cylinder approximation, i.e. in the large or small temperature limit, the holographic EE $S_E$ associated with these two classes of configurations falls also in two universality classes. Spacetimes with event horizons are always described by CFTs living in cylinders where regions of the non compact dimension are not accessible to observation and the corresponding EEs (6.4),(6.5) have an exponential behaviour. Spacetimes with conical singularities are always described by CFTs living in cylinders where regions of the compact dimension are not accessible to observation and the corresponding EEs (6.2),(6.3) have a periodic behaviour.

VII. HOLOGRAPHIC ENTANGLEMENT ENTROPY FROM GRAVITATIONAL TOOLS

In the framework of AdS/CFT correspondence, entanglement entropy of the boundary CFT can also be calculated using gravitational tools. In order to do so, one may use the holographic entanglement entropy formula as proposed by Ryu-Takayanagi [14, 15]. As per this formalism, considering a $d$-dimensional CFT, the entanglement entropy for a region $A$ having a $d-2$ dimensional boundary $\partial A$ is given by the minimal area of the $d-1$ dimensional surface $\gamma_A$ in AdS$_{d+1}$ whose boundary $\partial \gamma_A$ coincides with $\partial A$:

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_{d+1}}.$$  \hfill (7.1)

In this section we will show how the RT prescription and its extensions [20, 21] can be used to assign a holographic EE to the 3D bulk gravity configurations and to reproduce our results in Sect. VI.

- **BTZ black hole**

Entanglement Entropy for the 2D CFT dual to BTZ black hole in the bulk has been calculated, using the Ryu-Takayanagi formalism, in several papers (see e.g [19]). In this case the area $\text{Area}(\gamma_A)$ is equal to the minimal geodesic length between the two endpoints of 1-dim slice $A$ at constant time of length $l_s$ and we get back our equation (5.7) with $\beta = \beta_H$, where $\beta_H$ is the inverse Hawking temperature of the black hole.

As it has been observed in Ref. [19] the EE (7.1) gives also information about the holographic EE of the bulk configuration (the black hole) on which the Eq. (7.1) is evaluated. In particular, when $l_s$ becomes large the region $A$ covers most of the constant time-conformal boundary of the BTZ black hole. In this case, the minimal length geodesic consists of two disconnected pieces, one of finite length, wrapping around the black hole and an other, infinitely long of length $\propto c_3 \log(\epsilon/a)$ [19], giving essentially the EE of the AdS$_3$ vacuum. The finite term has to be identified with the holographic EE of the BTZ black hole. In the cylinder approximation $\beta \ll 2\pi L$ we have to consider $\phi$ living in the universal covering of the circle $S^1$. Thus, the configuration when the geodesic wraps around the black hole corresponds to $l_s = 2\pi L$, which inserted into Eq. (5.7) gives exactly our holographic EE for the BTZ black hole of Eq. (6.4).

- **Conical singularity in space**

Entanglement entropy for the 2D CFT dual to a spatial conical singularity in the bulk has been calculated in Ref. [21]:

$$S_{\text{con}} = \frac{c_1}{3} \log \left| \frac{R}{\pi (\Gamma \mu)} \sin(\Gamma \phi) \right|,$$ \hfill (7.2)

where in the above equation, $R$ is the total size of the system, $\Gamma \phi$ is the subsystem size in angular coordinates, $\mu$ is the UV cutoff.

Also in this case we expect Eq. (7.2), which gives the EE of the 2D CFT living in the boundary, to give the information about the holographic EE of the bulk configuration. This information can be easily extracted by recalling that, as pointed out in Sect. VI, for $r_+ < L$, we can see the conical singularity as originated by a slice of length $2\pi r_+$ in the spatial cycle of length $2\pi L$.

This means that in Eq. (7.2) we have to take $R = 2\pi L$ and $\Gamma \phi = \pi r_+ / L$. By appropriate identification of the UV cutoffs, $\Gamma \mu = a r_+ / L$ and using the definition (2.9) of $\beta_{\text{con}}$ we reproduce exactly our Eq. (6.2).

- **Conical singularity in time**
Since we are considering now timelike separated points on the boundary - and it is known that timelike geodesics in AdS do not reach the boundary - to calculate entanglement entropy for conical singularity in time, we use the complexified geodesic approach of Ref. [20]. We therefore use the complexified, renormalized, geodesic length for points separated by the interval $\Delta t_E$, as calculated in in [20] for the BTZ black hole background

$$L = 2 \ln \left[ \frac{2L}{r_+} \sin \left( \frac{r_+ \Delta t_E}{2L^2} \right) \right]$$

(7.3)

where $t_E$ is the Euclidean time with periodicity $2\pi$.

Formula (7.3) holds not only for a regular Euclidean manifold without conical singularities but after a rescaling of the $t_E$ also for a manifold with a conical singularity. Similarly to AdS$_3$ with conical singularities in the space direction, the case of conical singularity in the time direction can be described by considering it as described by time-slice of length $\Delta t_E = 2\pi L$. Using this relation and reinstating an appropriate UV cutoff a Eq. (7.3) gives our result of Eq. (6.3).

VIII. LEADING TERM IN THE LARGE/SMALL RADIUS EXPANSION OF THE ENTANGLEMENT ENTROPY AND AREA VERSUS VOLUME SCALING

In Sect. V we have shown that in the large/small radius limit the holographic EE of 3D gravity solutions falls in two classes. In this section we compute the leading terms of the large/small radius behaviour and, when possible, compare it with the thermal entropy of the system.

Let us first consider the BTZ black hole, i.e Eqs. (6.4) and (6.5). The cylinder approximation is valid for $r_+/L >> 1$. Expanding for $r_+/L >> 1$ both Eqs. (6.4) and (6.5) give

$$\Delta S = S_{BTZ} - S_{vac} = \frac{\pi r_+}{2G_3} - \frac{L}{2G_3} \ln \frac{\pi r_+}{L} + O(1) = S_{BH} - \frac{L}{2G_3} \ln S_{BH} + O(1).$$

(8.1)

where we have subtracted the contribution from the AdS vacuum $S_{vac}$ given by Eq. (6.6) and $S_{BH}$ is the thermal Bekenstein-Hawking entropy of the BTZ black hole.

Eq. (8.1) has a different interpretation when referred to Eq. (6.4) or to (6.5). In the first case the $r_+/L >> 1$ regime represents a large $T$ behaviour and Eq. (8.1) is what one naturally expects. In this regime thermal correlations dominate over quantum spatial correlations and the leading term in the EE is just a measure of thermal entropy. Also the subleading $\ln S_{BH}$ term is rather expected [45–47]. From the bulk point of view, the leading term in the large $T$ expansion of the EE is positive, scales as an area and reproduces correctly the area law for black holes. This means that, as expected, presence of an horizon in the bulk enhances the EE of the vacuum and has a holographic character.

On the other hand, from the point of view of the boundary CFT the leading term in Eq. (8.1) has the expected extensive, thermal, character. We can easily see from Eq. (5.7) that the regularized, large $T$ leading term of the EE for a CFT on the cylinder is given by

$$S_{snc} - S_{vac} = \frac{c}{3} \frac{\pi l_s}{\beta}.$$

(8.2)

With the identification $l_s = 2\pi L$, Eq. (8.2) agrees, as expected from the AdS/CFT correspondence, with the extensive Gibbs entropy $S_{Gibbs} = (2/3)c\pi^2 LT$ of a 2D CFT with a typical spatial scale $L$ and reproduces, after setting $\beta = \beta_H$, the Bekenstein-Hawking entropy of the BTZ black hole.

The interpretation of Eq. (8.1) when referred to Eq. (6.5), i.e to the case of time slices, is much more involved. It will be discussed in the next section.

Let us now consider conical singularities, i.e Eqs. (6.2) and (6.3). In this case the the cylinder approximation is valid for $r_+/L << 1$, i.e well below the Hawking-Page phase transition, where thermal AdS$_3$ is energetically preferred. However, in this paper we are not interested in stability questions and will just consider the contribution coming from conical singularities. Also in this case the discussion about time slices and the related Eq. (6.3) is more involved and it will be postponed to the next section.

Expanding Eqs. (6.2) we get at leading order in $r_+/L$

$$\Delta S = -\frac{c}{18} \frac{r_+^2}{L^2} = -\frac{\pi^2}{12G_3} \frac{r_+^2}{L}. $$

(8.3)
We see that from the 3D bulk point of view in the small $T$ limit the holographic EE of conical singularities is negative and scales as a volume. The $r_s/L << 1$ regime has the meaning of a small $T$ expansion, therefore in this regime spatial quantum correlations dominate over thermal correlations. From the point of view of the boundary CFT we have a super-extensive scaling behaviour. In fact expanding Eq. (5.8) we get at leading order

$$
\Delta S = -\frac{c}{18} \left( \frac{\pi l_s}{\beta} \right)^2 = -\frac{c}{18} \pi^2 l_s^2 T_Q^2,
$$

where $T_Q = 1/\beta$ is the "quantum" temperature introduced in Sect. IV. The negative sign in Eqs. (8.3) and (8.4) means physically that insertion of a point mass not shielded by an event horizon in the AdS$_3$ vacuum corresponds to a reduction of the EE of the vacuum and gives a volume contribution. This entanglement reduction due to matter has been also observed in a 4D de Sitter back ground in Ref [9]. Moreover, a volume-law scaling of the EE has been found in the subsystem-small-size regime of a class of non-local field theories [48].

IX. ENTANGLEMENT ENTROPY IN MINKOWSKI SPACE

In the previous sections we have calculated the holographic EE considering a spacetime with Euclidean signature. Now we want to discuss what happens when we consider it in Minkowski spacetime. Formally, the transition from Euclidean to Minkowski space can be accomplished just by considering the Wick rotation $t_E = it$. Obviously, the physical interpretation of the Wick rotation is far from being trivial.

As long as we take the subsystem $A$ as a spatial slice, the Wick rotation has no effect on our calculations of the EE. This means the Eqs. (5.7), (5.8) and correspondingly Eqs. (6.2), (6.4) do not change when passing to Minkowski space. Physically, this is a consequence of the fact that they are obtained from correlations in equilibrium states of a thermal QFT. On the other hand, this is not anymore true, when we consider the EE relative to time slices, i.e Eqs. (5.3), (5.9) and correspondingly Eqs. (6.3), (6.5). In the euclidean space, where time $t_E$ and space variables $x_E$ are treated on the same footing, these equations are simply related to (5.7), (5.8), (6.2), (6.4) by simply renaming the variables $t_E \leftrightarrow x_E$. The Wick rotation $t_l \rightarrow il_l$ (and correspondingly $L_l \rightarrow iL_l$) changes the nature of these equations. Periodic behaviour in the Euclidean becomes exponential in the Minkowskian. This can be applied to Eqs. (5.3) and (6.3), which hold in the high temperature regime. Physically, this means that after Wick rotation Eqs. (5.3) and (6.3) describe correlations in non equilibrium states of a thermal QFT in the regime dominated by thermal correlations. This interpretation of EE for timelike slices in terms of nonequilibrium correlations is very similar to the two-point functions calculations for timelike intervals of Ref. [20].

On the other hand, the Wick rotation applied to Eqs. (5.9), (6.5) transforms the exponential behaviour into periodic one. We do not have a clear interpretation of Eqs. (5.9), (6.5) in Minkowski space. This is because they hold in the regime $T << T_Q$ where quantum correlations dominate and a thermal interpretation is not appropriate. It is likely that the Minkowskian version the entanglement entropies (5.9), (6.5) should involve some notion of "entanglement in time", for instance like that proposed in Ref. [49, 50]. This point deserves further investigations.

X. CAUSALITY ASPECTS OF HOLOGRAPHIC ENTANGLEMENT ENTROPY

When entanglement entropy for a CFT is calculated using the replica trick, we are assured that locality and causality are preserved in the boundary theory. On the other hand, when one uses the Ryu-Takayanagi formalism, it is not completely clear how the minimal surfaces capture bulk information and express it in terms of boundary spatial entanglement. In [51] the authors have tried to answer to this question by constructing the causal holographic information. The construction, although similar to the Ryu-Takayanagi method, is indeed quite different from it. This causal holographic information is the proper area of the causal information surface, just like the entanglement entropy is given by the area of the minimal surface. Among the cases where the spatial entanglement entropy matches the causal holographic information are the Global AdS$_3$ and BTZ black hole giving support to the fact that at least in these cases the entanglement entropy given by Ryu-Takayanagi formula is causal.

Combining this causal holographic information associated with a hole in AdS, the authors in [52] have proposed the concept of differential entropy which has been used in [21] to calculate the entanglement entropy for conical singularity in space. Differential entropy also helps us to understand the causal facets of entanglement entropy related to conical singularity in space. This implies that our holographic derivation of the EE for the BTZ black and for 3D conical singularities in space, i.e Eqs. (6.2), (6.4) captures in a causal way bulk information.

These causality aspects become even more involved when we consider temporal entanglement entropy, i.e our Eqs. (6.3), (6.5). In Ref. [53] the authors do mention about the questionable aspect of causality related to the complexified
geodesics. These complexified geodesics do not guarantee any causal connection between the boundary and the bulk and presently can only be taken as a mathematical tool to calculate holographic temporal entanglement entropy.

Also one can use the image prescription given by [53] to calculate temporal holographic entanglement entropy but in this case also the image prescription is adjusted to take care of causality. In [21] the concept of entwining has been suggested to probe physics which is not captured by minimal geodesics. One possible application of it can be to probe black hole interior. It is likely that the temporal entanglement entropy can be considered as a manifestation of entwining where the internal degree of freedom in this case may come from the embedding space of AdS. This is also a point that deserves further investigation.

XI. CONCLUSION AND DISCUSSION

In this paper we have investigated the signature of the presence of horizons and localized matter in the holographic entanglement entropy in the framework of AdS3 gravity. Our main result is that bulk black hole excitations of the AdS3 Poincaré vacuum respect the area law and enhance the entanglement of the vacuum. Conversely, localized matter in the bulk generating conical singularities, gives a negative contribution to the entanglement of the vacuum and a contribution scaling as a bulk volume. This negative, bulk volume term corresponds for the boundary CFT to a super-extensive term.

Using the modular symmetries and working in the regime in which boundary tori, where the dual 2D CFT lives, can be approximated by cylinders we have been able to describe black holes and conical singularities in terms of boundary cylinders with unobservable regions in the Euclidean space or time directions. This allowed us to compute universal expressions for the holographic EE of the 3D bulk configurations. We have shown in this way that the presence of black hole excitations in the bulk are encoded in the holographic EE entropy in terms of presence of unobservable regions in the non compact direction of the cylinder and by the related exponential behaviour of the EE. Conversely, conical singularities are described by the presence of unobservable regions along the compact direction of the cylinder and by the related periodic behaviour of the EE.

There are several critical issues in our derivation. The first is the choice of the vacuum. We have chosen as vacuum AdS3 in Poincaré coordinates. This is the most natural choice in the context we are considering, because differently from AdS3 in global coordinates, the Poincaré vacuum is continuously connected both with the $T > 0$ black hole spectrum and with the part of the spectrum describing conical singularities. Nevertheless, one could also consider the possibility of choosing AdS3 in global coordinates as the vacuum.

A second, related, issue is represented by the Hawking-Page phase transition occurring when $r_+ \sim L$. Below this point the BTZ black hole becomes unstable and AdS3 at finite temperature becomes energetically preferred. Our cylinder approximations hold in the two limits $r_+ >> L$ or $r_+ << L$, i.e far away from the critical point. Whereas the regime $r_+ >> L$ is well understood - the path integral for AdS3 Euclidean gravity is dominated by the contribution coming from the BTZ black hole - what happens in the regime $r_+ << L$ in presence of localized matter is not completely clear. This point is also strongly related to the last issue, namely the fact that in principle singular geometries, like conical singularities, cannot be considered as allowed contributions to the Euclidean quantum gravity partition function. On the other hand, this regime is expected to be outside the domain of validity of the geometric spacetime description, i.e fully inside a would-be pre-geometric phase.

Our results rely heavily on the peculiarity of 3D AdS gravity, we therefore expect generalization to four or higher dimensions to be rather involved. For instance, in 4D gravity we have instead of conical singularities generated by a pointlike positive mass, naked curvature singularities generated by localized sources with negative mass.

[1] T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995) [gr-qc/9504004].
[2] T. Padmanabhan, “Thermodynamical Aspects of Gravity: New insights,” Rept. Prog. Phys. 73 (2010) 046901 [arXiv:0911.5004 [gr-qc]].
[3] E. P. Verlinde, “On the Origin of Gravity and the Laws of Newton,” JHEP 1104 (2011) 029 [arXiv:1001.0785 [hep-th]].
[4] M. Van Raamsdonk, “Building up spacetime with quantum entanglement,” Gen. Rel. Grav. 42 (2010) 2323 [Int. J. Mod. Phys. D 19 (2010) 2429] [arXiv:1005.3035 [hep-th]].
[5] H. Casini, M. Huerta and R. C. Myers, “Towards a derivation of holographic entanglement entropy,” JHEP 1105 (2011) 036 [arXiv:1102.0440 [hep-th]].
[6] T. Faulkner, M. Guica, T. Hartman, R. C. Myers and M. Van Raamsdonk, “Gravitation from Entanglement in Holographic CFTs,” JHEP 1403, 051 (2014) [arXiv:1312.7856 [hep-th]].
[7] B. Swingle and M. Van Raamsdonk, “Universality of Gravity from Entanglement,” [arXiv:1405.2933 [hep-th]].
[45] R. B. Mann and S. N. Solodukhin, “Quantum scalar field on three-dimensional (BTZ) black hole instanton: Heat kernel, effective action and thermodynamics,” Phys. Rev. D 55, 3622 (1997) [hep-th/9609085].
[46] D. V. Singh and S. Siwach, “Scalar Fields in BTZ Black Hole Spacetime and Entanglement Entropy,” Class. Quant. Grav. 30 (2013) 235034 [arXiv:1106.1005 [hep-th]].
[47] S. Sachan and D. V. Singh, “Logarithmic Corrections to the Entropy of Scalar Field in BTZ Black Hole Space-time,” Int. Journal of Mod. Phys. D26, (2017) 1750038 [arXiv:1412.7170 [hep-th]].
[48] N. Shiba and T. Takayanagi, JHEP 1402 (2014) 033 [arXiv:1311.1643 [hep-th]].
[49] S. J. Olson and T. C. Ralph, “Entanglement between the future and past in the quantum vacuum,” Phys. Rev. Lett. 106 (2011) 110404 [arXiv:1003.0720 [quant-ph]].
[50] S. J. Olson and T. C. Ralph, “Extraction of timelike entanglement from the quantum vacuum,” Phys. Rev. A 85, 012306 (2012) [arXiv:1101.2565 [quant-ph]].
[51] V. E. Hubeny and M. Rangamani, "Causal Holographic Information", JHEP 1206 (2012) 114, [arXiv:1204.1698 [hep-th]].
[52] V. Balasubramanian, B. D. Chowdhury, B. Czech, J. de Boer and M. P. Heller, "Bulk curves from boundary data in holography", Phys. Rev. D 89 (2014), 086004, [arXiv:1310.4204 [hep-th]].
[53] I. Y. Are’eva, M. A. Krantssov and M. D. Tikhanovskaya, "Improved image method for a holographic description of conical defects", Theor. Math. Phys. 189 (2016), 1660, [arXiv:1604.08905 [hep-th]].