Stability of the Nonlinear Milne Problem for Radiative Heat Transfer System

Mohamed Ghattassi, Xiaokai Huo & Nader Masmoudi

Communicated by A. Figalli

Abstract

This paper focuses on the nonlinear Milne problem of the radiative heat transfer system on the half-space. The nonlinear model is described by a second order ODE for temperature coupled to transport equation for radiative intensity. The nonlinearity of the fourth power Stefan–Boltzmann law of black body radiation, brings additional difficulty in mathematical analysis, compared to the well-developed theory for the Milne problem of the linear transport equation. To overcome this difficulty, the monotonicity properties of the second order ODE are used, together with the uniform estimate and compactness method, to prove the existence of the nonlinear Milne problem and to show the exponential decay of solutions. Moreover, the linear stability of the problem is established under a spectral assumption on its solutions, and the uniqueness of the nonlinear Milne problem is established in a neighborhood of solutions satisfying a spectral assumption or when the boundary conditions are close to the well-prepared case. The current work extends the study of Milne problem for linear transport equations and provides a comprehensive study on the nonlinear Milne problem of radiative heat transfer systems.

Contents

1. Introduction .................................... 2
   1.1. Main Results ................................ 5
   1.2. Organization ............................... 7
2. The Nonlinear Milne Problem on the Bounded Interval .............. 8
   2.1. Monotonicity of the Second Order ODE ................... 9
   2.2. Existence and Uniqueness ........................... 11
   2.3. Weighted Estimate .............................. 15
3. The Nonlinear Milne Problem on the Half Line .................. 18
   3.1. Existence ................................... 18
   3.2. Weighted Estimate .............................. 21
   3.3. Exponential Decay .............................. 23
1. Introduction

We consider the system
\[
\begin{align*}
\partial^2_x T &= -\langle \psi - T^4 \rangle, \\
\mu \partial_x \psi &= -(\psi - T^4)
\end{align*}
\] (1.1)
\[
\begin{align*}
\mu \partial_x \psi &= -(\psi - T^4)
\end{align*}
\] (1.2)
in the space \((x, \mu) \in \mathbb{R}_+ \times [-1, 1]\), where \(T = T(x)\) and \(\psi = \psi(x, \mu)\) are the temperature and radiative intensity, respectively. The bracket \(\langle \cdot \rangle\) denotes the integral over \(\mu \in [-1, 1]\), that is \(\langle f \rangle := \int_{-1}^{1} f(\mu) \, d\mu\). The boundary conditions are taken to be the non-homogeneous Dirichlet conditions
\[
\begin{align*}
T(0) &= T_b, \\
\psi(0, \mu) &= \psi_b(\mu), \quad \text{for any } \mu \in (0, 1],
\end{align*}
\] (1.3)
\[
\begin{align*}
T(0) &= T_b, \\
\psi(0, \mu) &= \psi_b(\mu), \quad \text{for any } \mu \in (0, 1],
\end{align*}
\] (1.4)
where \(T_b \geq 0\) is a given constant and \(\psi_b = \psi_b(\mu) \geq 0\) is a given function of \(\mu \in (0, 1]\). Here \(\psi_b\) describes the radiative intensity transmitted into the medium from outside. In this paper, the above problem is called the nonlinear Milne problem of radiative heat transfer. The main objective of our work is to study the existence, uniqueness, behavior of solutions at infinity and stability of this nonlinear Milne problem.

System (1.1)–(1.2) was derived in the boundary layer problem for the diffusive limit of the radiative heat transfer system
\[
\begin{align*}
\partial_t T^\varepsilon - \Delta T^\varepsilon &= \frac{1}{\varepsilon^2} \langle \psi^\varepsilon - (T^\varepsilon)^4 \rangle, \\
\partial_t \psi^\varepsilon + \frac{1}{\varepsilon} \omega \cdot \nabla \psi^\varepsilon &= -\frac{1}{\varepsilon^2} (\psi^\varepsilon - (T^\varepsilon)^4),
\end{align*}
\] where \(T^\varepsilon = T^\varepsilon(t, X), \psi^\varepsilon = \psi(t, X, \omega)\) with \(X \in \mathbb{R}^3, \omega \in \mathbb{S}^2\). System (1.1)–(1.2) is the boundary layer problem of the corresponding steady state problem of the above radiative heat transfer system, for more details we refer the reader to [20] and references therein. When the boundary data is well-prepared such that \(\psi_b = T^4_b\), system (1.1)–(1.2) has the constant solution \((T_b, \psi_b I_{\mu \in [0, 1]})\), where \(I_{\mu \in [0, 1]}\) denotes the characteristic function on \([0, 1]\), and no boundary layer exists. In this case the diffusive limit for the radiative transfer system was rigorously
justified in our previous work [12] based on the weak convergence method and the relative entropy method. In general, when \( \psi_b \neq T_b^4 \), it is crucial to study system (1.1)–(1.2) because the boundary layer effects make the interior solution obtained by asymptotic analysis invalid near the boundary. The solution to (1.1)–(1.2) provides a better description of the radiative transfer system near the boundary. Moreover, the boundary condition for the interior solutions needs to be determined by the limit of the solutions to the nonlinear Milne problem (1.1)–(1.2) as \( x \to \infty \). Based on this work, we are able to provide a rigorous justification of the diffusive limit [11,13]. Hence it is important to study the nonlinear Milne problem and show the behavior of its solutions at infinity.

There has been a well-developed theory of the Milne problem for the linear transport equations. When the equation for temperature is not considered, system (1.1)–(1.2) becomes

\[
\mu \partial_x \psi = \frac{1}{2} \langle \psi \rangle - \psi. \tag{1.5}
\]

This problem, referred to as the Milne problem for the linear transport equation, dates back to the work [4,5]. In [5], the above problem was derived from the linear transport equation and the equation was shown to decay exponentially to a constant by using the theory of stochastic processes. Later, this was also proved by using compactness method and weighted estimates in [4]. Since 0 belongs to the spectral of the linear operator \( L \psi := -\mu \partial_x \psi - \psi + \frac{1}{2} \langle \psi \rangle \), one cannot use semigroup theory to show the existence of solutions for the above problem. Instead, the maximum principle is used to show the existence of solutions in \( L^\infty \). Moreover, using a weighted estimate, the exponential convergence and the uniqueness of solutions is shown in [4]. Recently, a geometric correction to the above Milne problem is proposed in [27] to better describe the behavior of solutions near the boundary when the boundary is not flat. A weighted estimate accounting for the geometric corrections was provided therein to show the exponential convergence of solutions.

Compared to the Milne problem for linear transport equations, there are only few works on the nonlinear Milne problem. For example, in [20], the existence of weak solutions in \( L^\infty_{\text{loc}}(\mathbb{R}_+^+) \times L^\infty_{\text{loc}}(\mathbb{R}_+^+ \times [-1, 1]) \) was shown under the assumption that \( \gamma_1 \leq T_b \leq \gamma_2 \) for some constants \( \gamma_1 > 0, \gamma_2 \geq 0 \) by using the Leray–Schauder fixed point theorem. However, in [20] the analysis is restricted to the case where no genuine boundary layer appears at variance with the present contribution, namely, there is no information about the uniqueness or the behaviour of solutions at infinity. In [19], existence and uniqueness are shown for system (1.1)–(1.2) with Dirichlet boundary data in a bounded interval by using the maximum principle. For the half-space nonlinear Milne problem, the boundary condition at infinity is not known and such method cannot be directly adapted to our problem.

The major difficulty of the study of the nonlinear Milne problem (1.1)–(1.4) lies in two aspects. First, the second order ODE (1.1) and the transport equation (1.2) are coupled and need to be studied together. Maximum principle holds for equation (1.1) and (1.2) separately, but is not known to hold for the system. Second, the \( T^4 \) term is nonlinear, which is due to the Stefan–Boltzmann law of black body radiation. This physical term, makes the Milne problem nonlinear and the theory of
linear operators not applicable. The lack of maximum principle and the nonlinear nature of the problem, make the study of our problem challenging. To overcome the above difficulty, several techniques are used. First, the maximum principle for the second order ODE (1.1) and for the transport equation (1.2) implies the monotonicity of solutions with respect to source term and boundary data. This monotonicity property could help us to construct an approximate sequence converging to the solutions of (1.1)–(1.4).

This technique was used in [9] to study the Milne problem for a similar system but without the second order derivative $\partial_x^2 T$. The recent paper of Golse and Pironneau [16] considers a set of radiative heat transfer system that describes a fluid coupled with radiation including numerical algorithms to approximate the system. The proof is based on an argument given by [24] where the maximum principle is also shown. Here we need to use the monotonicity property of the second order ODE to construct a sequence of functions that converge to a solution of the nonlinear Milne problem. Second, to prove the exponential decay property of the solution, we cannot directly follow [4] since the property that $\partial_x \psi$ satisfies the same equation as $\psi$ is used to show the exponential decay of solution for the Milne problem (1.5). However, in our case, due to nonlinearity, $(\partial_x T, \partial_x \psi)$ satisfies a different system. Fortunately, we are able to derive a weighted estimate in a similar way as [4] that leads to the exponential decay of $T$ as well as $\partial_x T$, which also implies the decay property of $\psi$ by directly using the formulas for linear transport equation. Third, unlike the linear Milne problem (1.5), here the uniqueness of nonlinear Milne problem (1.1)–(1.4) does not follow from the weighted estimates and decay property of solutions due to the nonlinearity. We solve this issue by using linear stability analysis and propose a spectral assumption that leads to weighted estimates and the uniqueness of the linearized system. Last, a sufficient condition is proposed for the spectral assumption by using the generalized Hardy’s inequality. This allows us to prove the uniqueness of the nonlinear Milne problem in the vicinity of solutions satisfying the sufficient condition or near well-prepared boundary data where $\psi_b - T^4_b$ is small.

The system (1.1)–(1.2) could be generalized by considering the dependence on frequency of the radiative density. The corresponding frequency dependent model reads as

\[
\partial_x^2 T + \int_0^\infty \int_{-1}^1 (\psi - B_v(T)) \, d\mu \, dv = 0,\\
\mu \partial_x \psi + \psi - B_v(T) = 0, 
\]

where $T = T(x)$, $\psi = \psi(x, \mu, v)$ and $v \in (0, \infty)$ is the frequency. The nonlinear function $B_v(T)$ is usually taken to be the Planck function

\[
B_v(T) = \frac{2h
u^3}{c^2} \frac{1}{e^{\frac{hv}{kT}} - 1},
\]

where $c$ is the speed of light, $k$ is the Boltzmann constant and $h$ is the Planck constant. When the diffusive operator $\partial_x^2 T$ is neglected, the corresponding Milne problem is studied in [9,21,25]. The existence of the above frequency dependent half-space
problem was proved in [25], where later a simplified demonstration was provided in [9]. However, the existence and uniqueness for the nonlinear Milne problem (1.1)–(1.2) and the above frequency dependent problem has not been proved in the literature.

Our work could be treated as a development of the method of weighted estimates first developed in [4] to the nonlinear coupled Milne problem. The work [4] paves the way to use rigorous mathematical analysis to study the boundary layer problem of kinetic PDEs. The same technique was used in [27] to study the Milne problem of linear transport equation with geometric corrections. Such techniques have been also used and developed for the other kinetic equations, in particular Boltzmann equations. For example, existence, uniqueness and asymptotic behavior of the linearized Boltzmann equation for a gas of hard spheres is studied by using more complicated weighted estimates in [2]. The analysis is extended to gas mixtures in [1] and to the Vlasov–Boltzmann–Poisson equation in [7]. The nonlinear Milne problem for Boltzmann equation was considered in [15], where the existence and exponential decay property of solutions are proved for the Milne problem for the nonlinear Boltzmann equation in one-dimension with a slightly perturbed reflection boundary condition using weighted estimates and the Banach fixed-point theorem. The half space boundary layer problem for the steady Boltzmann equation with slab symmetry for a monatomic, hard sphere gas was treated in [6], where the gas is assumed to be in contact with its condensed phase. The existence and uniqueness of a decaying type solution is proved in the neighborhood of the Maxwellian of zero bulk velocity and same-as the condensed phase temperature.

Besides these works, there are also some other ways to study the Milne problems of Boltzmann equations, for example [1,7,17,22,26]. Among these, in [22], the authors use the theory of invariant manifolds and use Green functions to construct linear invariant manifolds for the half-space Milne problem of Boltzmann equation with Dirichlet boundary conditions. Based on the macro-micro or hydrodynamics-kinetic decomposition of solutions, existence of solutions to the same nonlinear problem is proved in [26] and it is found that the Mach number of Maxwellian affects the existence. For the case of reflective boundary conditions, the existence is shown recently in [17], where uniqueness is also established under some constraint conditions. For a review of the boundary layer problem for Boltzmann equation, we refer the reader to [3]. Since techniques and methods for the Milne problem of linear transport equations have been extended to study the Milne problem for the Boltzmann equation, for example from [4] to [2] and [6], the study of the nonlinear Milne problem (1.1)–(1.4) could provide tools and insights for other linear and nonlinear Milne problem for system of kinetic equations.

### 1.1. Main Results

Our first result is the existence and exponential decay properties of solutions to the nonlinear Milne problem (1.1)–(1.4), stated in the following theorem:

**Theorem 1.** (Existence of the nonlinear Milne problem) Given $T_b \geq 0$, $\psi_b = \psi_b(\mu) \geq 0$ for any $\mu \in (0, 1]$, there exists a bounded non-negative weak solution
\((T, \psi) \in L^2_{\text{loc}}(\mathbb{R}_+) \times L^2_{\text{loc}}(\mathbb{R}_+ \times [-1, 1])\) to system (1.1)–(1.2). Moreover, there exists a constant \(T_\infty \geq 0\) such that for any \(0 \leq \alpha < 1\),

\[
|T(x) - T_\infty| \leq Ce^{-\alpha x},
\]

\[
|\psi(x, \mu) - T_4^4| \leq Ce^{-\alpha x},
\]

for any \(\mu \in [-1, 1]\), \(x \in \mathbb{R}_+\), where \(C > 0\) is a constant depending on \(\alpha\) and \(|\psi_b - T_4^4|\).

The above theorem gives the existence of bounded solutions for system (1.1)–(1.2) and shows that \(T\) converges exponentially to some nonnegative constant \(T_\infty\) as \(x \to \infty\).

The existence of the nonlinear Milne problem has been proved in [20]. However, their results only give the existence and do not provide the asymptotic behavior of the solutions at infinity. This is due to the Schauder fixed point theorem which is used in [20] to obtain existence. A more refined analysis using the monotonicity of solutions for the second order ODE and for the transport equation, beyond the maximum principle, enables us to show the exponential decay behavior of solutions.

In order to study the uniqueness of the nonlinear Milne problem, we consider the linearized system

\[
\begin{align*}
\partial_x^2 g &= -\langle \phi - 4T^3 g \rangle + S_1, \\
\mu \partial_x \phi &= -\langle \phi - 4T^3 g \rangle + S_2
\end{align*}
\]

This is supplemented with the boundary conditions

\[
\begin{align*}
g(0) &= g_b, \\
\phi(0, \mu) &= \phi_b(\mu), \text{ for any } \mu \in (0, 1).
\end{align*}
\]

Here \(S_1 = S_1(x)\) and \(S_2 = S_2(x, \mu)\) are given functions and \(T = T(x)\) is the solution to system (1.1)–(1.2).

Our second result is the existence and uniqueness of the above linearized system. Before we state the result, we propose the following spectral assumption:

**Spectral Assumption:**

(A) We say that the function \(T \in C^1(\mathbb{R}_+)\) satisfies the spectral assumption if there exists constants \(\beta_0 > 0\) and \(0 < M < 1\), such that

\[
M \int_0^\infty e^{2\beta_0 x} (2T^3)^2 |\partial_x f|^2 \, dx \geq 4 \int_0^\infty e^{2\beta_0 x} |\partial_x (2T^3)|^2 f^2 \, dx
\]

for all measurable function \(f \in C^1(\mathbb{R}_+)\) with \(f(0) = 0\).

All constant functions satisfy the spectral assumption since the right side of (1.11) would be equal to zero. It will be proved in Lemma 5 that solutions to the problem (1.1)–(1.2) satisfy the spectral assumption when the boundary data is close to the well-prepared case.

We now present our second result.
Theorem 2. Suppose $T$ satisfies the spectral assumption (A) for some constants $0 < \beta < 1$ and $M > 0$. Assume $S_1, S_2$ decay to zero exponentially such that
text{\[
\int_0^{\infty} e^{2\beta x} |S_1|^2 \, dx < \infty, \quad \int_0^{1} e^{2\beta x} |S_2|^2 \, d\mu \, dx.
\]}

Then, there exists a unique bounded solution $(g, \phi) \in L^2_{\text{loc}}(\mathbb{R}^+) \times L^2_{\text{loc}}(\mathbb{R}^+ \times [-1, 1])$ to system (1.7)–(1.8) with boundary conditions (1.9)–(1.10). Moreover, there exists a constant $g_\infty$ such that
text{\begin{align}
|g(x) - g_\infty| &\leq C e^{-\beta x}, \\
|\phi(x, \mu) - 4T^3_{\infty}g_\infty| &\leq C e^{-\beta x},
\end{align}}

for any $x \in \mathbb{R}^+, \mu \in [-1, 1]$, where $C > 0$ is a constant depending on $\beta$ and $\phi_b - 4T^3_b g_b$.

Our third result concerns the uniqueness for the nonlinear Milne problem. Due
to the nonlinearity of the spectral assumption, its uniqueness cannot be obtained
directly from Theorem 2. However, by using the above theorem, we are able to prove uniqueness for the nonlinear Milne problem in the neighborhood of functions satisfying the spectral assumption (A) or for the case when $\psi_b - T^4_b$ is small (close to well-prepared case).

Theorem 3. Assume the hypotheses of Theorem 1 hold. Suppose $(T, \psi)$ is a solution to system (1.1)–(1.2). Then if $T$ satisfies the spectral assumption (A), the solution to system (1.1)–(1.4) is unique in the set
text{\[
\mathcal{V}_\varepsilon(T) = \{(G, \phi) \in L^2_{\text{loc}}(\mathbb{R}^+) \times L^2_{\text{loc}}(\mathbb{R}^+ \times [-1, 1]) : \|G - T\|_{L^2(\mathbb{R}^+)} \leq \varepsilon\},
\]}

for $\varepsilon > 0$ sufficiently small.

Moreover, if the spectral assumption (A) is replaced by the condition
text{\[
\frac{1}{2} \int_0^{1} \mu(\psi_b - T^4_b)^2 \, d\mu \leq C_b
\]}

for some constant $C_b$ (given in (5.14)), then the solution to system (1.1)–(1.4) is unique.

1.2. Organization

The paper is organized as follows: in Sect. 2, the existence of solutions to system
(1.1)–(1.2) is proved in a bounded interval, in Theorem 4. To prove this theorem, the
monotonicity for the second order ODE is shown in (1.1) in Sect. 2.1 and Theorem
2 is proved in Sect. 2.2 for existence and uniqueness, and Sect. 2.3, for the weighted
estimate. Section 3 is devoted to the proof of Theorem 1. The existence of solutions
and weighted estimates are shown in Sect. 3.1 and Sect. 3.2, respectively, both by
passing to the limit $B \to \infty$ in the solutions on the bounded interval $[0, B]$. The
exponential decay property (1.6) of solutions is shown in Sect. 3.3. Section 4 is
devoted to the study of the linearized Milne problem. Followed by a discussion
on the spectral assumption (A) in Sect. 4.1, the existence of weak solutions to
the linearized Milne problem on a bounded interval \([0, B]\) is shown in Sect. 4.2. The solution is then extended to the half-space, with existence proved in Sect. 4.3, weighted estimate and decay proved in Sect. 4.4, uniqueness in 4.5, thus proving Theorem 2. Finally, we prove Theorem 3 in Sect. 5, where the sufficient condition (5.1) is shown to imply the spectral assumption (A) in Sect. 5.1 and the uniqueness of solutions for system (1.1)–(1.2) is established for the case (1.14) in Sect. 5.2 and \(T \in \mathcal{V}_\epsilon(T)\) in Sect. 5.3.

2. The Nonlinear Milne Problem on the Bounded Interval

In this section, we consider system (1.1)–(1.2) on a bounded interval \(x \in [0, B]\),

\[
\partial_x^2 T^B = -\langle \psi^B - (T^B)^4 \rangle, \quad (2.1)
\]

\[
\mu \partial_x \psi^B = -\left( \psi^B - (T^B)^4 \right), \quad (2.2)
\]

associated to the following boundary conditions:

\[
T^B(0) = T_b, \quad \partial_x T^B(B) = 0, \quad (2.3)
\]

\[
\psi^B(0, \mu) = \psi_b(\mu), \quad \psi^B(B, \mu) = \psi^B(B, -\mu), \quad \text{for any } \mu \in (0, 1]. \quad (2.4)
\]

The boundary conditions at \(x = B\) are motivated by the fact that solutions to the nonlinear Milne problem (1.1)–(1.2) are expected to converge to some constants at infinity. If the right boundary conditions at \(x = B\) are replaced by Dirichlet boundary conditions, the existence for the above system was proven in [19] by using the fixed point theorem together with the maximum principle. However, here we do not know the value of \(T^B, \psi^B\) on the boundary \(x = B\). The method of [19] cannot be used. A different approach based on the theory of subsolutions and fixed point theorems is used here to prove the existence of the solution for system (2.1)–(2.2). The theory of subsolutions is used in [9] to show the existence of solutions to system (1.1)–(1.2) without the diffusion operator \(\partial_x^2 T\). Here we will study the property of solutions for (2.1) and show the existence for system (2.1)–(2.2). The existence theorem reads as follows:

**Theorem 4.** Assume \(0 \leq T_b \leq \gamma\), \(0 \leq \psi_b = \psi_b(\mu) \leq \gamma^4\) for any \(\mu \in (0, 1]\) and for some constant \(\gamma > 0\). Then there exists a unique solution \((T^B, \psi^B) \in C^2([0, B]) \times C^1([0, B] \times [-1, 1])\) to system (2.1)–(2.2) with boundary conditions (2.3)–(2.4), and the solution satisfies

\[
0 \leq T^B(x) \leq \gamma, \quad 0 \leq \psi^B(x, \mu) \leq \gamma^4, \quad \text{for any } x \in [0, B], \mu \in [-1, 1].
\]

Furthermore, the estimate

\[
\int_0^B e^{2\alpha x} 4(T^B)^3 |\partial_x T^B|^2 \, dx + (1 - \alpha) \int_0^B \int_{-1}^1 e^{2\alpha x} (\psi^B - (T^B)^4)^2 \, d\mu \, dx
\]

\[
+ \frac{1}{2} \int_{-1}^1 |\mu|(\psi^B(0, \cdot) - T_b^4)^2 \, d\mu \leq \frac{1}{2} \int_0^1 \mu(\psi_b - T_b^4)^2 \, d\mu \quad (2.5)
\]

holds for any \(\alpha \in [0, 1]\).
The above existence theorem is proved by constructing a monotonic sequence of functions that converges to the solution of system (2.1)–(2.2). Before we construct the sequence, we first show the monotonicity property of equation (2.1).

2.1. Monotonicity of the Second Order ODE

The monotonicity property of the second order ODE (2.1) is given in the following lemma:

**Lemma 1.** Given $0 \leq T_b \leq \gamma$ for some constant $\gamma > 0$. Let $\phi$ be a strictly increasing bounded function on $\mathbb{R}$ and $g \in C([0, B])$ be a continuous bounded function satisfying $0 \leq g \leq \phi(\gamma)$. Then there exists a unique bounded solution $T \in C^2([0, B])$ to the equation

$$-\partial_x^2 T(x) + \phi(T(x)) = g(x), \quad \text{for any } x \in [0, B],$$

supplemented by the boundary conditions

$$T(0) = T_b, \quad \partial_x T(B) = 0,$$

and the solution satisfies $0 \leq T(x) \leq \gamma$ for any $x \in [0, B]$. Moreover, let $T_1, T_2$ be two solutions to the above equation with source terms $g_1, g_2$ and boundary data $T_{b_1}, T_{b_2}$, respectively. If $0 \leq g_1(x) \leq g_2(x) \leq \phi(\gamma)$ for all $x \in [0, B]$ and $0 \leq T_{b_1} \leq T_{b_2} \leq \gamma$, then $0 \leq T_1(x) \leq T_2(x) \leq \gamma$ for all $x \in [0, B]$.

**Proof. Existence.** Letting $h \in C([0, B])$, the solution to the problem

$$\begin{cases}
-\partial_x^2 f = h, & \text{on } [0, B], \\
f(0) = 0, & \partial_x f(B) = 0
\end{cases}$$

can be explicitly written as

$$f(x) = \int_0^x \int_t^B h(s) \, ds \, dt.$$ \hspace{1cm} (2.7)

Denote the above mapping by $f = G h$. Then $G$ is a linear operator from $C([0, B])$ to $C^2([0, B])$. We next show the mapping $\mathcal{N}$ defined by

$$\mathcal{N} T := G(g - \phi(T)) + T_b$$

has a fixed point. To show this, we recall the statement of the Leray–Schauder Theorem in Appendix E, [14, Theorem 11.3].

- We consider the Banach space $D = C([0, B])$ equipped with the supremum norm

$$\| f \|_D := \sup_{x \in [0, B]} |f(x)|.$$  \hspace{1cm}

Since $\mathcal{N} : C([0, B]) \to C^2([0, B])$ and $C^2([0, B])$ is compactly embedded in $C([0, B])$, where $C^2([0, B])$ is equipped with the norm

$$\| f \|_{C^2([0, B])} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^2} + \| f \|_{C([0, B])},$$

$\mathcal{N}$ is a compact map from $C([0, B])$ to itself.
We now need to show that the set
\[ A := \left\{ T \in C([0, B]) : T = \sigma N T, \text{ for some } \sigma \in [0, 1] \right\}, \]
is bounded. To show this, suppose \( T = \sigma N T \), then
\[ T = \sigma (G(g - \phi(T)) + T_b), \quad (2.8) \]
which satisfies
\[ -\partial_x^2 T + \sigma \phi(T(x)) = \sigma g(x), \quad x \in [0, B] \quad \text{and} \quad T(0) = \sigma T_b, \quad \partial_x T(B) = 0. \quad (2.9) \]

We first show \( T \geq 0 \). We use a contradiction argument. Suppose the minimum of \( T \) is less than zero. Then let \( x_m \in [0, B] \) be the minimum point, \( T(x_m) < 0 \). By the monotonicity of \( \phi \), \( \phi(T(x_m)) < \phi(0) < 0 \). Hence
\[ -\partial_x^2 T(x_m) = \sigma (g(x_m) - \phi(x_m)) > 0, \]
that is around \( x_m \), \( T(x) \) is concave, which implies \( x_m \) cannot be a local minimum point. Hence \( x_m \notin (0, B) \). If \( x_m = B \), due to \( -\partial_x^2 T(B) > 0 \) and \( \partial_x T(B) = 0 \), \( x_m = B \) should be a maximum point rather than a minimum point. Therefore, \( x_m \neq B \). Otherwise if \( x_m = 0 \), \( T(0) = \sigma T_b \geq 0 \), which contradicts to the assumption \( T(x_m) < 0 \), so \( x_m \neq 0 \). Therefore, \( x_m \notin [0, B] \) and thus the set of minimum points with negative value is empty. Therefore, we conclude that \( T(x) \geq 0 \) for all \( x \in [0, B] \).

We can follow a similar contradiction argument to show that \( T \leq \gamma \). Suppose \( T \) attains its maximum at \( x = x_M \) with \( T(x_M) = \gamma \). Due to monotonicity of \( \phi \), \( \phi(T(x_M)) > \phi(\gamma) \) and thus
\[ -\partial_x^2 T(x_M) = \sigma (g(x_M) - \phi(T(x_M))) < \sigma (\phi(\gamma) - \phi(\gamma)) < 0, \]
which implies that \( x_M \) is a local minimum. Hence \( x_M \notin (0, B) \). If \( x_M = 0 \), then \( T(x_M) = \sigma T_b \leq \gamma \), so \( x_M \neq 0 \). If \( x_M = B \), due to \( \partial_x T(B) = 0 \), \( \partial_x^2 T(B) \geq 0 \) also implies that \( B \) is a minimum point, which contradicts the assumption that \( x_M = B \) is a maximum point. Therefore, we conclude that the set of maximum point with values bigger than \( \gamma \) is empty. Therefore, \( T(x_M) \leq \gamma \) and so \( T(x) \leq \gamma \) for all \( x \in [0, B] \). Finally, from \( 0 \leq T \leq \gamma \), we conclude that
\[ 0 \leq \|T\|_{C([0, B])} \leq \gamma \quad \text{for any} \quad \sigma \in [0, 1]. \]

We can thus conclude that the set \( A \) is bounded for any \( \sigma \in [0, 1] \).

Consequently, the hypotheses of Leray–Schauder fixed point theorem are verified, which implies that \( N \) has a fixed point in \( C([0, B]) \). Therefore, there exists a solution in \( C([0, B]) \) for equation (2.6). Moreover, from (2.8) and the formula (2.7), \( T \in C^2([0, B]) \).

**Uniqueness.** Suppose \( T_1 \) and \( T_2 \) are two distinct solutions to (2.6), then \( T_1 - T_2 \) satisfies
\[ \partial_x^2 (T_1 - T_2) - (\phi(T_1) - \phi(T_2)) = 0, \]
with
\[(T_1 - T_2)(0) = 0, \quad \partial_x(T_1 - T_2)(B) = 0.\]

We first show any maximum point \(x_M \in [0, B]\) of \(T_1(x) - T_2(x)\) cannot be positive. Otherwise, \(T_1(x_M) - T_2(x_M) > 0\) and then by the monotonicity of \(\phi, \phi(T_1(x_M)) > \phi(T_2(x_M)) > 0\), hence \(\partial^2_x(T_1 - T_2)(x_M) = \phi(T_1(x_M)) - \phi(T_2(x_M)) > 0\), which implies \(T_1 - T_2\) is convex near \(x_M\). Thus \(x_M\) can only be a local minimum or on the boundary. By the assumption, \(x_M\) is a maximum point, \(x_M \notin (0, B)\). If \(x_M = 0, T_1(0) - T_2(0) = 0\) contradicts the assumption \((T_1 - T_2)(x_M) > 0\), so \(x_M \neq 0\). If \(x_M = B\), then \(\partial_x(T_1 - T_2)(B) = 0\) and \(\partial^2_x(T_1 - T_2)(B) = 0\) implies \(B\) is a local minimum and thus cannot be a maximum point. Hence the set of \(\{x : (T_1 - T_2)(x) > 0\}\) is empty.

Similarly, we can show any minimum point \(x_m \in [0, B]\) of \(T_1(x) - T_2(x)\) cannot have values \(T_1(x_m) - T_2(x_m) < 0\). Otherwise, \(T_1(x_m) - T_2(x_m) < 0\) and \(\partial^2_x(T_1 - T_2)(x_m) = \phi(T_1(x_m)) - \phi(T_2(x_m)) < 0\) and \(T_1 - T_2\) is locally concave at \(x_m\). So \(x_m\) cannot be a local minimum, \(x_m \notin (0, B)\). The fact that \((T_1 - T_2)(0) = 0\) implies \(x_m \neq 0\). The condition \(\partial_x(T_1 - T_2)(B) = 0\) and \(\partial^2_x(T_1 - T_2)(B) < 0\) implies that \(B\) is a local maximum point and thus cannot be a minimum point, that is \(x_M \notin B\). Therefore, we conclude that the set of \(\{x : (T_1 - T_2)(x) < 0\}\) is empty.

Combining the above arguments, the set \(\{x : (T_1 - T_2)(x) \neq 0\}\) is empty. That is, \(T_1 = T_2\) on \([0, B]\) and the solution is unique.

**Monotonicity.** Assume \(g_1(x) \leq g_2(x)\) on \(x \in [0, B]\) and \(T_{b1} \leq T_{b2}\), and \(T_1, T_2\) are two solutions with source terms \(g_1, g_2\) and boundary data \(T_{b1}, T_{b2}\), respectively. Suppose the maximum of \(T_1(x) - T_2(x)\) occurs at \(x = x_M \in (0, B)\) with \(T_1(x) - T_2(x) > 0\). Then \(\partial^2_x(T_1 - T_2)(x_M) = \phi(T_1(x_M)) - \phi(T_2(x_M)) > 0\) and so \(x = x_M\) cannot be a local maximum, \(x_M \notin (0, B)\). If the maximum occurs at \(x = B\) and \((T_1 - T_2)(B) > 0\), then \(\partial^2_x(T_1 - T_2)(B) > 0\) and so \(x = B\) is a minimum point due to \(\partial_x(T_1(B) = \partial_x(T_2(B), which contradicts the assumption that \(x = B\) is a maximum point. If the maximum occurs at \(0\), we have \(T_1(0) - T_2(0) = T_{b1} - T_{b2} \leq 0\), which contradicts the assumption \(T_1(x_M) - T_2(x_M) > 0\). Therefore, we have \(T_1 \leq T_2\) on \([0, B]\). \(\square\)

### 2.2. Existence and Uniqueness

With the monotonicity of solutions to the second order ODE in Lemma 1 and the monotonicity of solutions for the linear transport equation in Lemma 7 in Appendix A, we are prepared to prove the existence for system (2.1)–(2.2). Before we give the proof, we first introduce the concept of subsolutions.

**Definition 1.** We call \((T, \psi) \in C^2([0, B]) \times C^1([0, B] \times \mathbb{R})\) a subsolution of (2.1)–(2.2) if the following inequalities hold:

\[-\partial^2_xT - \left< \psi - T^4 \right> \leq 0,\]
\[\mu \partial_x \psi + (\psi - T^4) \leq 0,\]

and on the boundary they satisfy \(0 \leq T(0) \leq T_{b}, 0 \leq \psi(0, \mu) \leq \psi_{b}(\mu)\) for \(\mu \in (0, 1)\).
**Proof of Theorem 4, existence and uniqueness.** The proof is divided into four steps. An approximate sequence is constructed in the first step. Then, we prove the existence of solutions for system (2.1)–(2.2). After that, the uniqueness of solutions is shown using a contradiction argument. Finally, a weighted energy estimate is derived.

**Step 1: Construction of an approximate sequence.** Let \(\phi(T)\) be a strictly increasing function defined by

\[
\phi(T) = \begin{cases} 
\frac{T}{1-T} & \text{if } T < 0, \\
T^4 & \text{if } 0 \leq T \leq \gamma, \\
\gamma^4 + \frac{T-\gamma}{1+T-\gamma} & \text{if } T > \gamma.
\end{cases}
\]

The above function satisfies \(\phi(T) < 0\) if \(T < 0\) and \(\phi(T) > \gamma^4\) if \(T > \gamma\).

We construct a sequence of functions \(\{T_k = T_k(x), \psi_k = \psi_k(x, \mu)\}_{k=0}^{\infty}\) starting with \(T_0, \psi_0\) being a subsolution given by Definition 1.1 that is

\[
-\partial_x^2 T^0 + 2\phi(T^0) \leq (\psi^0), \quad (2.12)
\]

\[
\mu \partial_x \psi^0 + \psi^0 \leq (T^0)^4, \quad (2.13)
\]

with

\[
0 \leq T^0(0) \leq T_b, \quad \partial_x T^0(B) = 0,
\]

\[
0 \leq \psi^0(0, \mu) \leq \psi_b(\mu), \quad \psi(B, \mu) = \psi(B, -\mu), \quad \text{for any } \mu \in (0, 1].
\]

Given \((T^k, \psi^k)\), \((T^{k+1}, \psi^{k+1})\) is obtained by solving

\[
-\partial_x^2 T^{k+1} + 2\phi(T^{k+1}) = (\psi^k), \quad (2.14)
\]

\[
\mu \partial_x \psi^{k+1} + \psi^{k+1} = (T^k)^4, \quad (2.15)
\]

with boundary conditions

\[
T^{k+1}(0) = T_b, \quad \partial_x T^{k+1}(B) = 0,
\]

\[
\psi^{k+1}(0, \mu) = \psi_b, \quad \psi^{k+1}(B, \mu) = \psi^{k+1}(B, -\mu), \quad \text{for any } \mu > 0.
\]

**Step 2: Existence.** Assume \((T^k, \psi^k) \in C^2([0, B]) \times C^1([0, B] \times [-1, 1])\) satisfy \(0 \leq T^k \leq \gamma\) and \(0 \leq \psi^k \leq \gamma^4\), then Lemma 1 and Lemma 7 imply that there exists a unique solution \((T^{k+1}, \psi^{k+1}) \in C^2([0, B]) \times C^1([0, B] \times [-1, 1])\) to system (2.14)–(2.15) and the solution satisfies \(0 \leq T^{k+1} \leq \gamma\) and \(0 \leq \psi^{k+1} \leq \gamma^4\).

By induction, we conclude that for any \(k \geq 1\) and \(k \in \mathbb{N}\),

\[
0 \leq T^k(x) \leq \gamma, \quad 0 \leq \psi^k(x, \mu) \leq \gamma^4, \quad \text{for any } x \in [0, B], \mu \in [-1, 1]. \quad (2.16)
\]

\[1\] We may take \(T^0 = 0\) and \(\psi^0 = 0\), which is a subsolution.
Next we prove by induction that $T^{k+1}(x) \geq T^k(x)$, $\psi^{k+1}(x, \mu) \geq \psi^k(x, \mu)$ for any $x \in [0, B]$ and $\mu \in [-1, 1]$. First $T^1$, $\psi^1$ solves

$$-\partial_x^2 T^1 + 2\phi(T^1) = (\psi^0), \quad \mu \partial_x \psi^1 + \psi^1 = (T^0)^4.$$  

Comparing this with equations (2.12)-(2.13) for $(T^0, \psi^0)$, we get from Lemma 1 and Lemma 7 that $T^1(x) \geq T^0(x)$ and $\psi^1(x, \mu) \geq \psi^0(x, \mu)$ for $x \in [0, B]$ and $\mu \in [-1, 1].$

Suppose $T^k(x) \geq T^{k-1}(x)$ and $\psi^k(x, \mu) \geq \psi^{k-1}(x, \mu)$, then

$$-\partial_x^2 T^{k+1} + 2\phi(T^{k+1}) = (\psi^k),$$  

$$-\partial_x^2 T^k + 2\phi(T^k) = (\psi^{k-1}),$$  

and by Lemma 1, $T^{k+1}(x) \geq T^k(x)$ for any $x \in [0, B]$. Similarly, we have

$$\mu \partial_x \psi^{k+1} + \psi^{k+1} = (T^k)^4,$$  

$$\mu \partial_x \psi^k + \psi^k = (T^{k-1})^4,$$  

and Lemma 7 implies that $\psi^{k+1}(x, \mu) \geq \psi^k(x, \mu)$ for any $x \in [0, B]$ and $\mu \in [-1, 1]$. Therefore, we conclude that for any $k \geq 1$ and $k \in \mathbb{N},$

$$T^{k+1}(x) \geq T^k(x), \quad \psi^{k+1} \geq \psi^k(x), \quad \text{for any } x \in [0, B], \mu \in [-1, 1].$$  

Combining this with (2.16), we conclude that $\{T^k\}_{k=0}^{\infty}, \{\psi^k\}_{k=0}^{\infty}$ are bounded increasing sequences. Hence there exists a converging subsequence such that

$$\lim_{k \to \infty} T^k(x) = T^B(x), \quad \lim_{k \to \infty} \psi^k(x, \mu) = \psi^B(x, \mu),$$

for any $x \in [0, B], \mu \in [-1, 1]$. Moreover, by the Beppo Levi’s lemma, it holds that

$$\lim_{k \to \infty} \int_0^x T_k(y)dy = \int_0^x T^B(y)dy, \quad \lim_{k \to \infty} \int_0^x \psi_k(y, \mu)dy = \int_0^x \psi^B(y, \mu)dy.$$  

Hence, we can pass to the limit $k \to \infty$ in

$$T^{k+1} = \mathcal{G} \left( (\psi^k) - \phi(T^{k+1}) \right) + T_b,$$

and get

$$T^B = \lim_{k \to \infty} \mathcal{G} \left( (\psi^k) - \phi(T^k) \right) + T_b = \mathcal{G} \left( (\psi) - \phi(T) \right) + T_b$$

and by the continuity of the operator $\mathcal{G}$, $T^B \in C^2([0, B])$. In a similar way we can also get $\psi \in C^1([0, B] \times [-1, 1])$. Hence, we can apply the Dini’s theorem and conclude that the convergence of $(T_k, \psi_k)$ to $(T^B, \psi^B)$ is uniform in $C^2([0, B]) \times C^1([0, B] \times [-1, 1]).$
Moreover, since $0 \leq T_k, \psi_k \leq \gamma$, $(T^B_k, \psi^B_k)$ satisfies $0 \leq T^B \leq \gamma$, $0 \leq \psi^B \leq \gamma^4$. We can pass to the limit in (2.14)–(2.15) and get that $(T^B, \psi^B)$ satisfies the system

$$-\partial_x^2 T^B + 2\phi(T^B) = \langle \psi^B \rangle,$$

$$\mu \partial_x \psi^B + \psi^B = (T^B)^4,$$

with

$$T^B(0) = T_b, \quad \partial_x T^B(B) = 0,$$

$$\psi^B(0, \mu) = \psi_b, \quad \psi^B(B, \mu) = \psi(B, -\mu), \quad \text{for any } \mu > 0.$$

Since $0 \leq T^B \leq \gamma$, $\phi^B(T^B) = (T^B)^4$ and the above system is equivalent to (2.1)–(2.2). Therefore, there exists a solution $(T^B, \psi^B) \in C^2([0, B]) \times C^1([0, B] \times [-1, 1])$ to system (2.1)–(2.2) with boundary conditions (2.3)–(2.4).

**Step 3: Uniqueness.** Suppose $(T^B_1, \psi^B_1), (T^B_2, \psi^B_2)$ are two solutions to (2.1)–(2.2) with boundary conditions (2.3)–(2.4). Define

$$T^0 = \max(T^B_1, T^B_2), \quad \psi^0 = \max(\psi^B_1, \psi^B_2),$$

which satisfies (2.12)–(2.13) and is thus a subsolution. We construct a sequence of functions $(\{T^k(x), \psi^k(x, \mu)\})_{k=0}^\infty$ by solving iteratively

$$-\partial_x^2 T^{k+1} + 2\phi(T^{k+1}) = \langle \psi^k \rangle,$$

$$\mu \partial_x \psi^{k+1} + \psi^{k+1} = (T^k)^4.$$

Then Lemma 7 implies that there exists a subsequence such that $(T^k, \psi^k)$ converges to a solution $(T^B, \psi^B)$ of (1.1)–(1.2) as $k \to \infty$ and $T^B \geq T^0, \psi^B \geq \psi^0$. Hence $g^B = T^B - T^B_1, \varphi^B = \psi^B - \psi^B_1$ is a nonnegative solution to the following system

$$-\partial_x^2 g^B + 2((T^B)^4 - (T^B_1)^4) = \langle \varphi \rangle,$$

$$\mu \partial_x \varphi^B + \varphi^B = ((T^B)^4 - (T^B_1)^4),$$

with

$$g^B(0) = 0, \quad \partial_x g^B(B) = 0,$$

$$\varphi^B(0, \mu) = 0, \quad \varphi^B(B, \mu) = \varphi^B(B, -\mu), \quad \text{for any } \mu > 0.$$

Comparing (2.17) to (2.18) gives

$$\partial_x (\partial_x g^B - \langle \mu \varphi^B \rangle) = 0.$$

Due to the boundary conditions $\partial_x g^B(B) = 0$ and $\langle \mu \varphi^B(B, \cdot) \rangle = 0$,

$$\partial_x g^B(x) = \langle \mu \varphi^B(x, \cdot) \rangle, \quad \text{for any } x \in [0, B].$$

Using (B.4), we have
\[ \varphi^B(0, \mu) = - \int_0^B \frac{1}{\mu} e^{\frac{x-s}{\mu}} (T^B)^4 - (T^B_1)^4(s) \, ds \]

If \( T^B(x) > T^B_1(x) \) on some interval \((a, b) \subset [0, B]\), then the above equation implies that \( \varphi^B(0, \mu) > 0 \) for \( \mu < 0 \). This, together with the boundary condition \( \varphi^B(0, \mu) = 0 \) for \( \mu > 0 \), implies

\[ \partial_x g^B(0) = \langle \mu \varphi^B(0, \cdot) \rangle = \int_{-1}^0 \mu \varphi^B(0, \mu) \, d\mu < 0. \]

This contradicts the fact that \( g^B(x) \geq 0 \) for any \( x \in [0, B] \) and \( g^B(0) = 0 \). Hence \( T^B(x) = T^B_1(x) \) for \( x \in [0, B] \) almost everywhere.

Due to the fact that \( T^B = T^B_1 \), equation (2.18) becomes \( \mu \partial_x \varphi^B + \varphi^B = 0 \), with boundary conditions \( \varphi^B(0, \mu) = 0 \), \( \varphi(B, \mu) = \varphi(B, -\mu) \) for \( \mu > 0 \). Therefore, \( \psi^B(x, \mu) = \psi^B_1(x, \mu) \) for \( x \in [0, B], \mu \in [-1, 1] \). By the same argument we also obtain \( T^B = T^B_2, \psi^B = \psi^B_2 \). Hence \( T^B_1 = T^B_2 \) and \( \psi^B_1 = \psi^B_2 \), that is the solution to system (2.1)–(2.4) is unique. \( \Box \)

### 2.3. Weighted Estimate

Next we derive the weighted estimate (2.5).

**Proof of Theorem 4, weighted estimate.** We multiply (2.1) by \((T^B)^4\), integrate over \([0, x]\) and multiply (2.2) by \( \psi^B \), integrate over \((z, \mu) \in [0, x] \times [-1, 1]\), and combine the results to get

\[ -\int_0^x (T^B)^4 \partial_z^2 T^B \, dz + \frac{1}{2} \int_0^x \int_{-1}^1 \mu \partial_z (\psi^B)^2 \, d\mu \, dz \]

\[ + \int_0^x \int_{-1}^1 (\psi^B - (T^B)^4)^2 \, d\mu \, dz = 0. \quad (2.19) \]

Using integration-by-parts, we obtain

\[ -\int_0^x (T^B)^4 \partial_z^2 T^B \, dz = \int_0^x 4(T^B)^3 |\partial_z T^B|^2 \, dz - (T^B)^4(z) \partial_z T^B(z) \bigg|_z=0. \quad (2.20) \]

and

\[ \int_0^x \int_{-1}^1 \mu \partial_z (\psi^B)^2 \, d\mu \, dz = \int_{-1}^1 \mu (\psi^B)^2(z, \cdot) \, d\mu \bigg|_{z=0}. \quad (2.21) \]

Comparing equation (2.1) to (2.2) gives

\[ \partial_z^2 T^B = \partial_x \langle \mu \psi^B \rangle, \quad (2.22) \]
this together with $\partial_z T^B(B) = \langle \mu \psi^B(B, \cdot) \rangle = 0$ implies
\[ \partial_z T^B(x) = \langle \mu \psi^B(x, \cdot) \rangle, \quad \text{for any } x \in [0, B]. \quad (2.23) \]

Using this and taking (2.20) and (2.21) into (2.19), we get
\[ \int_0^x 4(T^B)^3 |\partial_z T^B|^2 \, dx + \int_0^x \int_{-1}^1 (\psi^B - (T^B)^4)^2 \, d\mu \, dz + \frac{1}{2} \int_{-1}^1 \mu(\psi^B(x, \cdot) - (T^B(x))^4)^2 \, d\mu - \frac{1}{2} \int_{-1}^1 (\mu(\psi^B(0, \cdot) - T^B_0)^2) \, d\mu = 0. \]

The boundary condition (2.4) implies
\[ \int_{-1}^1 \mu(\psi^B(0, \cdot) - T^B_0)^2 \, d\mu = \int_{-1}^0 \mu(\psi^B(0, \cdot) - T^B_0)^2 \, d\mu + \int_0^1 \mu(\psi_b - T^B_0)^2 \, d\mu. \quad (2.24) \]

Taking this into the previous equation leads to the estimate (2.5) with $\alpha = 0$:
\[ \int_0^x 4(T^B)^3 |\partial_z T^B|^2 \, dx + \int_0^x \int_{-1}^1 (\psi^B - (T^B)^4)^2 \, d\mu \, dz + \frac{1}{2} \int_{-1}^1 \mu(\psi^B(x, \cdot) - (T^B(x))^4)^2 \, d\mu + \frac{1}{2} \int_{-1}^0 |\mu|(\psi^B(0, \cdot) - T^B_0)^2 \, d\mu = \frac{1}{2} \int_0^1 \mu(\psi_b - T^B_0)^2 \, d\mu. \quad (2.25) \]

Note that here we have that
\[ \int_{-1}^1 \mu(\psi^B(x, \cdot) - (T^B(x))^4)^2 \, d\mu \geq 0 \]
for any $x \in [0, B]$, which is due to the fact that $\int_{-1}^1 \mu(\psi^B(x, \cdot) - (T^B(x))^4)^2 \, d\mu$ is a non-increasing function of $x$ and
\[ \int_{-1}^1 \mu(\psi^B(B, \cdot) - (T^B(B))^4)^2 \, d\mu = 0, \quad (2.26) \]

which is due to the boundary condition $\psi^B(B, \mu) = \psi^B(B, -\mu)$. The non-increasing fact of $\int_{-1}^1 \mu(\psi^B(x, \cdot) - (T^B(x))^4)^2 \, d\mu$ can be seen if we integrate over $[x_1, x_2]$ with $0 \leq x_1 < x_2 \leq B$ in the derivation above, the inequality
\[ \frac{1}{2} \int_{-1}^1 \mu(\psi^B(x, \cdot) - (T^B(x))^4)^2 \, d\mu \bigg|_{x_1}^{x_2} = -\int_{x_1}^{x_2} 4(T^B)^3 |\partial_z T^B|^2 \, dz - \int_{x_1}^{x_2} \int_{-1}^1 (\psi^B - (T^B)^4)^2 \, d\mu \, dz \leq 0 \]
is satisfied.

Next, let $\alpha \in (0, 1)$ be a constant. We multiply (2.1) by $e^{2\alpha x}(T^B)^4$, and (2.2) by $e^{2\alpha x} \psi^B$, and integrate to get
\[-\int_0^x e^{2\alpha z}(T^B)^4 \partial_z^2 T^B \, dz + \frac{1}{2} \int_0^x \int_{-1}^1 \mu e^{2\alpha z} \partial_z^2 (\psi^B)^2 \, d\mu \, dz \]
\[+ \int_0^x \int_{-1}^1 e^{2\alpha z} (\psi^B - (T^B)^4)^2 \, d\mu \, dz = 0. \tag{2.27}\]

Using integration-by-parts, we get
\[-\int_0^x e^{2\alpha z}(T^B)^4 \partial_z^2 T^B \, dz = \int_0^x e^{2\alpha z} (T^B)^3 |\partial_z T^B|^2 \, dz + 2\alpha \int_0^x e^{2\alpha z} (T^B)^4 \partial_z T^B \, dz \]
\[= e^{2\alpha z} (T^B)^4 (z) \partial_z T^B (z) \bigg|_{z=0}^x, \tag{2.28}\]

and
\[\int_0^x \int_{-1}^1 \mu e^{2\alpha z} \partial_z (\psi^B)^2 \, d\mu \, dz \]
\[= -2\alpha \int_0^x \int_{-1}^2 \mu e^{2\alpha z} (\psi^B)^2 \, d\mu \, dz + \int_{-1}^1 \mu e^{2\alpha z} (\psi^B)^2 (z, \cdot) \, d\mu \bigg|_{z=0}^x.\]

Taking the above equations into (2.27) and using the relation (2.23) gives
\[\int_0^x e^{2\alpha z} 4(T^B)^3 |\partial_z T^B|^2 \, dz + \int_0^x \int_{-1}^1 e^{2\alpha z} (\psi^B - (T^B)^4)^2 \, d\mu \, dz \]
\[= \alpha \int_0^x \int_{-1}^1 \mu e^{2\alpha z} (\psi^B - (T^B)^4)^2 \, d\mu \, dz \]
\[= -\frac{1}{2} \int_{-1}^1 \mu e^{2\alpha z} (\psi^B (z, \cdot) - (T^B (z))^4)^2 \, d\mu \bigg|_{0}^x. \]

Using the relation (2.24) and \(1 - \alpha \mu \geq 1 - \alpha\) for \(0 \leq \alpha < 1\) in the above equation leads to
\[\int_0^x e^{2\alpha z} 4(T^B)^3 |\partial_z T^B|^2 \, dz + (1 - \alpha) \int_0^x \int_{-1}^1 e^{2\alpha z} (\psi^B - (T^B)^4)^2 \, d\mu \, dz \]
\[+ \frac{1}{2} \mu \int_{-1}^1 e^{2\alpha x} (\psi^B (x, \cdot) - (T^B (x))^4)^2 \, d\mu + \frac{1}{2} \int_{-1}^0 |\mu|((\psi^B (0, \cdot) - T_b)^4)^2 \, d\mu \]
\[\leq \frac{1}{2} \int_{-1}^1 \mu (\psi_b - T_b^4)^2 \, d\mu, \tag{2.29}\]

and finishes the proof of Theorem 4. \(\square\)

**Remark 1.** The existence results for system (2.1)–(2.4) has been proved in [20, Theorem 3.3]. In addition, consider system (2.14)–(2.15), the maximum principle stated in Lemma 1 and Lemma 7 implies that (2.16) holds for \((k + 1)\) if it holds for \((k)\). Therefore, (2.14)–(2.15) defines an operator mapping \(L^\infty([0, B]) \times \)
$W^{2, \infty}([0, B])$ to itself. A direct application of Schauder’s fixed point theorem leads to the existence of a fixed point, which is a solution to system (2.1)–(2.4). The existence on the half line can be obtained since the $L^\infty$ bounds does not depend on $B$. However, Schauder’s fixed point theorem does not provide uniqueness or behaviour of solutions at infinity. Here by using the monotonicity of the elliptic and transport equation, we are able to prove uniqueness of solutions and obtain the weighted estimate. This weighted estimate leads to the exponential decay of solutions for the half-space problem at infinity.

3. The Nonlinear Milne Problem on the Half Line

In this section, we consider the existence of the nonlinear Milne problem and prove Theorem 1. We first pass to the limit $B \to \infty$ in system (2.1)–(2.2) and show the existence of weak solutions to system (1.1)–(1.2). Then we derive a weighted estimate and show the exponential decay behavior of solutions.

3.1. Existence

We first pass to the limit $B \to \infty$ on system (2.1)–(2.2) and show the existence of weak solutions for system (1.1)–(1.2).

First we show the convergence of $T_B(x)$ as $B \to \infty$. To show this, we first prove that there exists a subsequence $\{T_{B_k}(B_k)\}$ converging to some constant $T_\infty$. From the weighted estimate (2.5) (taking $x = B$ in (2.5)), we can get

$$
(T_B)^{5/2}(B) - T_b^{5/2} = \int_0^B \partial_x(T_B)^{5/2} \, dx = \int_0^B e^{-\alpha x} e^{\alpha x} \partial_x(T_B)^{5/2} \, dx \leq 1\frac{1}{(2\alpha)^{1/2}} \left(1 - e^{-2\alpha B}\right)^{1/2} \left(\int_0^B e^{2\alpha x} 4(T_B)^3|\partial_x(T_B)|^2 \, dx\right)^{1/2} \left(\int_0^B e^{-2\alpha x} \, dx\right)^{1/2} \left(\frac{25}{16}\int_0^B e^{2\alpha x} 4(T_B)^3|\partial_x(T_B)|^2 \, dx\right)^{1/2} \leq \frac{5}{4(2\alpha)^{1/2}} \left(\frac{1}{2} \int_0^1 \mu(\psi_b - T_b^4)^2 \, d\mu\right)^{1/2}.
$$

Hence $(T_B)^{5/2}(B)$ is bounded for any $B > 0$. By the Bolzano-Weierstrass theorem, there exists a subsequence $\{B_k\}$ such that

$$
T_{B_k}(B_k) \to T_\infty, \quad k \to \infty,
$$

where $T_\infty$ is some constant. Since $0 \leq T^k(B_k)(B_k) \leq \gamma$ according to Theorem 4, $0 \leq T_\infty \leq \gamma$. 
Next we show $T^B(x) - T^B(B)$ is uniformly bounded in $L^2([0, B])$. From estimate (2.5), we have the inequality

$$(1 - \alpha) \int_0^B \int_{-1}^1 e^{2\alpha x} (\psi B - (T^B)^4)^2 \ dx \ d\mu \leq \frac{1}{2} \int_0^1 \mu(\psi B - T^B_0)^2 \ d\mu.$$ 

Using this estimate and the relation

$$\left( \int_{-1}^1 \mu \psi B \ d\mu \right)^2 = \left( \int_{-1}^1 \mu(\psi B - (T^B)^4) \ d\mu \right)^2 \leq \int_{-1}^1 \mu^2 \ d\mu \int_{-1}^1 (\psi B - (T^B)^4)^2 \ d\mu$$

we can get

$$\int_0^B e^{2\alpha x} |(\mu \psi B)|^2 \ dx \leq \frac{2}{3} \int_0^B \int_{-1}^1 e^{2\alpha x} (\psi B - (T^B)^4)^2 \ dx \ d\mu \leq \frac{1}{3} \frac{1}{1 - \alpha} \int_0^1 \mu(\psi B - T^B_0)^2 \ d\mu.$$ 

Due to (2.23), the above inequality implies

$$\int_0^B e^{2\alpha x} |\partial_x T^B|^2 \ dx \leq \frac{1}{3} \frac{1}{1 - \alpha} \int_0^1 \mu(\psi B - T^B_0)^2 \ d\mu. \tag{3.1}$$

Therefore,

$$\int_0^B (T^B(x) - T^B(B))^2 \ dx = \int_0^B \left( \int_x^B \partial_z T^B(z) \ dz \right)^2 \ dx$$

$$= \int_0^B \left( \int_x^B e^{-\alpha z} e^{\alpha z} \partial_z T^B(z) \ dz \right)^2 \ dx$$

$$\leq \frac{1}{2\alpha} \int_0^B e^{2\alpha z} |\partial_z T^B|^2 \ dx \cdot \int_x^B e^{-2\alpha z} \ dz$$

$$\leq \frac{1}{6} \frac{1}{1 - \alpha} \int_0^1 \mu(\psi B - T^B_0)^2 \ d\mu$$

$$\leq \frac{1}{6} \frac{1}{2^{2\alpha^2} (1 - \alpha)} \int_0^1 \mu(\psi B - T^B_0)^2 \ d\mu,$$

where $(e^{2\alpha B} - 2\alpha B - 1)e^{-2\alpha B} = 1 - 2\alpha B e^{-2\alpha B} - e^{-2\alpha B} \leq 1$. Hence $T^B(x) - T^B(B) \in L^2_{loc}(\mathbb{R}_+)$ is uniformly bounded. Moreover, due to the boundary condition (2.3), $\partial_x T^B(B) = 0$ and so the estimate (3.1) implies

$$\int_0^B |\partial_x T^B(x) - \partial_x T^B(B)|^2 \ dx \leq \frac{1}{3(1 - \alpha)} \int_0^1 \mu(\psi B - T^B_0)^2 \ d\mu,$$
and thus \( T^B(x) - T^B(B) \) is uniformly bounded in \( H^1_{\text{loc}}(\mathbb{R}^+) \). Moreover, \( \|\partial^2_x T^B\|_{L^2([0,B])} = \|\langle \psi^B - (T^B)^4\rangle\|_{L^2([0,B])} \leq \sqrt{2}\|\psi^B - (T^B)^4\|_{L^2([0,B] \times [-1,1])} \) is uniformly bounded. Therefore, there exists a subsequence \( B_k \) such that

\[
T^{B_k}(x) - T^{B_k}(B_k) \rightharpoonup T(x) - T_\infty, \quad \text{weakly in} \ H^2_{\text{loc}}(\mathbb{R}^+).
\]

By the continuous embedding \( C^1_{\text{loc}} \subset H^2_{\text{loc}}(\mathbb{R}^+) \), we get from the above convergence

\[
T^{B_k}(x) \rightarrow T(x), \quad \text{strongly in} \ C^1_{\text{loc}}(\mathbb{R}^+).
\]  

Next we show the convergence of \( \psi^B(x) \). Due to the uniform estimate (2.25), \( \|\psi^B - (T^B)^4\|_{L^2([0,B] \times [-1,1])} \) is uniformly bounded in \( B \). This, together with (3.3) implies that there exists a subsequence \( \{\psi^{B_k}\} \) such that as \( k \rightarrow \infty \),

\[
\psi^{B_k}(x, \mu) - (T^{B_k})^4(x) \rightharpoonup \psi(x, \mu) - T^4, \quad \text{weakly in} \ L^2_{\text{loc}}(\mathbb{R}^+ \times [-1,1]).
\]

Next we show the limit \( T, \psi \) solves system (1.1)–(1.2) with boundary conditions (1.3)–(1.4). First we show the limit \( T(x), \psi(x) \) satisfies the boundary condition (1.3)–(1.4). Due to the trace operator \( u \mapsto u(0) \) is continuous (see [8, Ex 8.18]) for any \( u \in H^1_{\text{loc}}(\mathbb{R}^+) \), we can get from (3.2) that \( T(0) = \lim_{k \rightarrow \infty} T^{B_k}(0) = T_0 \). One can also check that \( \varphi \rightarrow \varphi(0, \cdot) \) is also continuous for any \( \varphi \in L^2_{\text{loc}}(\mathbb{R}^+ \times [-1,1]) \cap \{\varphi : \mu \partial_x \varphi + \varphi \in C_{\text{loc}}(\mathbb{R}^+)\} \) (one can use the formulas (B.4)–(B.4) to show this). Hence, \( \psi^{B_k}(0, \mu) \rightarrow \psi^k(0, \mu) \) as \( B_k \rightarrow \infty \). To show \( (T, \psi) \) satisfies (1.1)–(1.2), we apply a test function \( h = h(x) \in C^1([0,x]), \varphi \in C^1([0,x] \times [-1,1]) \) to (2.1) and (2.2), integrate by parts, and obtain

\[
\int_0^x \partial_x T^B \partial_z h \, dz - \partial_z T^B(z) h(z) \bigg|_0^x + \int_0^x \int_{-1}^1 (\psi^B - (T^B)^4) h \, d\mu \, dz = 0,
\]

\[
\int_0^x \int_{-1}^1 \mu \psi^B \partial_z \varphi \, d\mu \, dz - \int_{-1}^1 \mu \varphi(z, \cdot) \psi^B(z, \cdot) \, d\mu \bigg|_0^x + \int_0^x \int_{-1}^1 (\psi^B - (T^B)^4) \varphi \, d\mu \, dz = 0,
\]

Due to (3.3), we can pass to the limit

\[
\int_0^x \partial_z T^B \partial_z h \, dz \rightarrow \int_0^x \partial_z T \partial_z h \, dz, \quad \text{as} \ B \rightarrow \infty.
\]

Due to (3.3)–(3.4), we can pass to the limit

\[
\int_0^x \int_{-1}^1 \mu \psi^B \partial_z \varphi \, d\mu \, dz \rightarrow \int_0^x \int_{-1}^1 \mu \psi \partial_z \varphi \, d\mu \, dz, \quad \text{as} \ B \rightarrow \infty,
\]

Using (3.4), we can pass to the limit as \( B \rightarrow \infty \), and

\[
\int_0^x \int_{-1}^1 (\psi^B - (T^B)^4) h \, d\mu \, dz \rightarrow \int_0^x \int_{-1}^1 (\psi - T^4) h \, d\mu \, dz,
\]

\[
\int_0^x \int_{-1}^1 (\psi^B - T^4) \varphi \, d\mu \, dz \rightarrow \int_0^x \int_{-1}^1 (\psi - T^4) \varphi \, d\mu \, dz.
\]
By the continuity of the trace operator, we get from the above facts,
\[ \partial_x T^B(x) \rightarrow \partial_x T(x), \quad \text{almost everywhere.} \]

Due to \( \mu \partial_x \psi + \psi \in C_{\text{loc}}(\mathbb{R}_+), \psi^B(x, \mu) \rightarrow \psi(x, \mu) \) in \( L^2_{\mu}([-1, 1]) \) almost everywhere,
\[ \int_{-1}^{1} \mu \varphi \psi^B \, d\mu \rightarrow \int_{-1}^{1} \mu \varphi \psi \, d\mu. \]

Combining the above limits, we can pass to the limit \( B \rightarrow \infty \) in (3.5)–(3.6) and
\begin{align*}
\int_{0}^{x} \partial_z T \partial_z h - \partial_z T(z) h(z) \bigg|_0^x + \int_{0}^{x} \int_{-1}^{1} (\psi - (T^4) h \, d\mu \, dz = 0, \\
\int_{0}^{x} \int_{-1}^{1} \mu \psi \partial_z \varphi \, d\mu \, dz - \int_{-1}^{1} \int_{0}^{1} \mu \varphi(z, \cdot) \psi(z, \cdot) \, d\mu \bigg|_0^x \\
+ \int_{0}^{x} \int_{-1}^{1} (\psi - T^4) \varphi \, d\mu \, dz = 0
\end{align*}
holds. Therefore, we have proved that there exists a weak solution \((T, \psi) \in C_{\text{loc}}(\mathbb{R}_+) \times L^2_{\text{loc}}(\mathbb{R}_+ \times [-1, 1])\) to system (1.1)–(1.2).

3.2. Weighted Estimate

Due to the lower semi-continuity of the norm \( \| \cdot \|_{L^2([0,x])} \), the estimate (2.25) and the strong convergence (3.3) of \( T \) and the weak convergence (3.2) of \( \partial_x T^B \) imply
\[ \int_{0}^{x} 4T^3 |\partial_z T|^2 \, dz \leq \liminf_{B \rightarrow \infty} \int_{0}^{x} 4(T^B)^3 |\partial_z T^B|^2 \, dz. \]

Due to the weak convergence (3.4) of \( \psi^B - (T^B)^4 \),
\[ \int_{0}^{x} \int_{-1}^{1} (\psi - T^4)^2 \, d\mu \, dz \leq \liminf_{B \rightarrow \infty} \int_{0}^{x} \int_{-1}^{1} (\psi^B - (T^B)^4)^2 \, d\mu \, dz. \]

From (2.25), \( \langle \mu(\psi^B - (T^B)^2) \rangle \) is non-negative and uniformly bounded in \( L^\infty_{\text{loc}}(\mathbb{R}_+) \). Hence
\[ \langle \mu(\psi^B - (T^B)^2) \rangle \rightharpoonup^* \langle \mu(\psi - T^4)^2 \rangle, \quad \text{weakly-* in } L^\infty_{\text{loc}}(\mathbb{R}_+). \]

Therefore,
\[ \limsup_{B \rightarrow \infty} \langle \mu(\psi^B - (T^B)^4) \rangle \geq \langle \mu(\psi - T^4)^2 \rangle \geq 0. \]

Combining the above inequality with (3.9)–(3.10), we can take \( \limsup \) on (2.25) and using \( \limsup_{n \rightarrow \infty} (a_n + b_n) \geq \limsup_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n \) to get
\[
\int_0^x 4T^3 |\partial_x T|^2 \, dz + \int_0^x \int_{-1}^1 (\psi - T^4)^2 \, d\mu \, dz + \frac{1}{2} \langle \mu(\psi(x, \cdot) - T^4(x))^2 \rangle \\
+ \frac{1}{2} \int_{-1}^0 |\mu| (\psi(0, \cdot) - T_b^4)^2 \, d\mu \leq \frac{1}{2} \int_0^1 \mu(\psi_b - T_b^4)^2 \, d\mu \tag{3.11}
\]
holds for any \( x \in \mathbb{R}_+ \). Note in the above \( \langle \mu(\psi(x, \cdot) - T^4(x))^2 \rangle \geq 0 \) for any \( x \in \mathbb{R}_+ \).

Similarly, we can pass to the limit \( B \to \infty \) in (2.5) and use the weak convergence up to a subsequence implied by (2.5):
\[
e^{ax} |\partial_x (T^B)^{2/3}| \to e^{ax} |\partial_x T^2|, \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}_+),
\]
\[
e^{ax} (\psi - (T^B)^4) \to e^{ax} (\psi - T^4), \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}_+ \times [-1, 1]),
\]
\[
e^{2ax} \langle \mu(\psi - (T^B)^4)^2 \rangle \to * e^{2ax} \langle \mu(\psi - T^4)^2 \rangle, \text{ weakly-* in } L^\infty_{\text{loc}}(\mathbb{R}_+),
\]
\[
\int_{-1}^0 |\mu| (\psi^B(0, \cdot) - T_b^4)^2 \, d\mu \to \int_{-1}^0 |\mu| (\psi(0, \cdot) - T_b^4)^2 \, d\mu, \text{ weakly-* in } L^\infty_{\text{loc}}(\mathbb{R}_+).
\]
Consequently, we obtain
\[
\int_0^x e^{2ax} 4T^3 |\partial_x T|^2 \, dz \leq \liminf_{B \to \infty} \int_0^x e^{2ax} 4(T^B)^3 |\partial_x T^B|^2 \, dz,
\]
\[
\int_0^x \int_{-1}^1 e^{2ax} (\psi - T^4)^2 \, d\mu \, dz \leq \liminf_{B \to \infty} \int_{-1}^x \int_{-1}^1 e^{2ax} (\psi^B - (T^B)^4)^2 \, d\mu \, dz,
\]
\[
\limsup_{B \to \infty} \mu(\psi^B - (T^B)^4)^2 \geq \langle \mu(\psi - T^4)^2 \rangle \geq 0,
\]
\[
\lim_{B \to \infty} \int_{-1}^0 |\mu| (\psi^B(0, \cdot) - T_b^4)^2 \, d\mu = \int_{-1}^0 |\mu| (\psi(0, \cdot) - T_b^4)^2 \, d\mu.
\]
Therefore, we take \( \lim sup_{B \to \infty} \) in (2.5) and use the above inequalities to get
\[
\int_0^x e^{2ax} 4T^3 |\partial_x T|^2 \, dz + (1 - \alpha) \int_0^x \int_{-1}^1 e^{2ax} (\psi - T^4)^2 \, d\mu \, dz \\
+ \frac{1}{2} \int_{-1}^0 e^{2ax} \langle \mu(\psi(x, \cdot) - T^4(x))^2 \rangle + \frac{1}{2} \int_{-1}^0 |\mu| (\psi(0, \cdot) - T_b^4)^2 \, d\mu \\
\leq \frac{1}{2} \int_0^1 \mu(\psi_b - T_b^4)^2 \, d\mu, \text{ for any } x \in \mathbb{R}_+.
\]
From the above estimate, \( e^{2ax} \langle \mu(\psi - T^4)^2 \rangle \) is bounded, hence \( \langle \mu(\psi - T^4)^2 \rangle \) vanishes as \( x \to \infty \). Since \( \alpha < 1 \) is arbitrary, we also have \( e^{2ax} \langle \mu(\psi - T^4)^2 \rangle \to 0 \) as \( x \to \infty \). We thus let \( x \to \infty \) in the above inequality and get the following weighted estimate
\[
\int_0^\infty e^{2ax} 4T^3 |\partial_x T|^2 \, dx + (1 - \alpha) \int_0^\infty \int_{-1}^1 e^{2ax} (\psi - T^4)^2 \, d\mu \, dx \\
+ \frac{1}{2} \int_{-1}^0 |\mu| (\psi(0, \mu) - T_b^4)^2 \, d\mu \leq \frac{1}{2} \int_0^1 \mu(\psi_b - T_b^4)^2 \, d\mu, \tag{3.12}
\]
for any \( \alpha \in (0, 1) \).
3.3. Exponential Decay

Now, we will establish the exponential decay property of solutions to system (1.1)–(1.2) using the estimate (3.12). Let \( x_2 > x_1 \geq 0 \), we have from (3.12),

\[
|T^\frac{5}{2}(x_1) - T^\frac{5}{2}(x_2)| = \left| \int_{x_1}^{x_2} \partial_z T^\frac{5}{2} \, dz \right| = \left| \int_{x_1}^{x_2} e^{-\alpha z} e^{\alpha z} \partial_z T^\frac{5}{2} \, dz \right|
\]

\[
\leq \left( \int_{x_1}^{x_2} e^{-2\alpha z} \, dz \right)^{\frac{1}{2}} \left( \frac{25}{16} \int_{x_1}^{x_2} e^{2\alpha z} 4T^3 |\partial_z T|^2 \, dz \right)^{\frac{1}{2}}
\]

\[
\leq \frac{5}{4} \frac{1}{\sqrt{2\alpha}} \left( e^{-2\alpha x_1} - e^{-2\alpha x_2} \right)^{\frac{1}{2}} \left( \frac{1}{2} \int_0^1 \mu(\psi_b - T^4_b)^2 \, d\mu \right)^{\frac{1}{2}}.
\]

Taking \( x_1, x_2 \geq -(2 \log \varepsilon)/2\alpha \) with \( M > 0 \) large enough, we can get

\[
(e^{-\alpha x_1} - e^{-\alpha x_2}) \leq e^{-\alpha x_1} + e^{-\alpha x_2} \leq 2\varepsilon^2,
\]

and so

\[
|T^\frac{5}{2}(x_1) - T^\frac{5}{2}(x_2)| \leq \varepsilon \frac{5}{4} \frac{1}{\sqrt{2\alpha}} \left( \int_0^1 \mu(\psi_b - T^4_b)^2 \, d\mu \right)^{\frac{1}{2}},
\]

and hence

\[
\lim_{x \to \infty} T(x) = T_\infty \text{ exists and } 0 \leq T_\infty \leq \gamma \text{ is finite.}
\]

Next we show \(|T(x) - T_\infty|\) decays to zero exponentially as \( x \) goes to infinity. We first show that \( \partial_x T(x) = \langle \mu \psi(x, \cdot) \rangle \) holds for any \( x \in \mathbb{R}^+ \). To see this, we follow the work [20] and compare (1.1) with (1.2) to get

\[
\partial_x (\partial_x T - \langle \mu \psi \rangle) = 0.
\]

Let \( j := \partial_x T - \langle \mu \psi \rangle \). Multiplying (1.2) by \( \mu \) and integrating over \([-1, 1]\) gives

\[
\partial_x \langle \mu^2 \psi \rangle = \langle \mu \psi \rangle.
\]  

We thus get \( j = \partial_x (T + \langle \mu^2 \psi \rangle) \). By the boundness of \( T, \psi, j = 0 \) and \( T(x) + \langle \mu^2 \psi(x, \cdot) \rangle \) is a constant, that is

\[
\partial_x T(x) = \langle \mu \psi(x, \cdot) \rangle, \quad \text{for all } \forall x \in \mathbb{R}^+.
\]

From the estimate (3.12), we obtain

\[
(1 - \alpha) \int_0^x \int_{-1}^1 e^{2\alpha z}(\psi - T^4) \, d\mu \, dz \leq \frac{1}{2} \int_0^1 \mu(\psi_b - T^4_b)^2 \, d\mu,
\]

Therefore,
Due to the relation (3.14), the above inequality is equivalent to
\[
\int_0^x e^{2\alpha z} |\partial_z T|^2 \, dz \geq \frac{1}{3(1-\alpha)} \int_0^1 \mu(\psi_b - T_b^4)^2 \, d\mu.
\] (3.15)

We get from the above inequality that
\[
\int_0^x e^{2\alpha z} |\partial_z T|^2 \, dz \geq \frac{1}{3(1-\alpha)} \int_0^1 \mu(\psi_b - T_b^4)^2 \, d\mu.
\] (3.16)

Moreover, using (1.1) and the estimate (3.12), we have
\[
\int_0^x e^{2\alpha z} |\partial_z^2 T|^2 \, dz = \int_0^x e^{2\alpha z} |\partial z \, T|^2 \, dz \leq \int_0^x \int_{-1}^1 e^{2\alpha z} (\psi - T^4)^2 \, d\mu \, dz \leq \frac{1}{2} \int_0^1 \mu(\psi_b - T_b^4)^2 \, d\mu.
\]

By (3.16) and the above inequality, Barbalat’s lemma ([10]) implies
\[
\partial_z T(x) \to 0, \quad \text{as} \quad x \to \infty.
\]
Finally, we show the decay of $\psi$. We can pass to the limit $B \to \infty$ in the formula (B.4)–(B.4) and show $\psi$ satisfies the formula (C.2)–(C.2). Therefore, for $\mu > 0$,  

$$|\psi(x, \mu) - T^4_\infty|$$  

and for $\mu < 0$,  

$$|\psi(x, \mu) - T^4_\infty|$$  

\begin{align*}
\leq |\psi_b - T^4_\infty| e^{-\frac{x}{\mu}} + |\psi_b - T^4_\infty| e^{-\frac{x}{\mu}} + \int_0^x |T(s) - T_\infty| \cdot (T^2(s) + T^2_\infty)(T(s) + T_\infty) ds \\
\leq |\psi_b - T^4_\infty| e^{-\frac{x}{\mu}} + 4(T_b + M_\alpha)^3 M_\alpha e^{-\frac{x}{\mu}} + \int_0^x \left[ \frac{1}{\mu} e^{-\frac{x-s}{\mu}} M_\alpha e^{-\alpha x} 4(T_b + 2M_\alpha)^3 \right] ds \\
= |\psi_b - T^4_\infty| e^{-\frac{x}{\mu}} + 4(T_b + M_\alpha)^3 M_\alpha e^{-\frac{x}{\mu}} + 4(T_b + 2M_\alpha)^3 M_\alpha \frac{1}{1 - \mu \alpha} (e^{-\alpha x} - e^{-\frac{x}{\mu}}) \\
\leq |\psi_b - T^4_\infty| e^{-\frac{x}{\mu}} + 4(T_b + 2M_\alpha)^3 M_\alpha \frac{1}{1 - \mu \alpha} e^{-\alpha x},
\end{align*}

(3.20)

The above two inequalities give the second inequality of (1.6) and finish the proof of Theorem 1.

Remark 2. We can show that $T(x) > 0$ for any $x \in (0, \infty)$ provided $T_b, \psi_b$ are not all zero. For the case $T_b = 0, \psi_b = 0$, it is obvious that zero is the solution to system (1.1)–(1.2). For example, if we take $T_b > 0, \psi_b = 0$, and for the case
$T_b > 0$, $\psi_b = 0$, we assume $T(x) = 0$ on $(a, b) \subset \mathbb{R}_+$. Then the formula (see (B.4)–(B.4))
\[
\langle \psi(x, \cdot) \rangle = \int_0^1 \psi_b(\mu) e^{-\frac{x}{\mu}} d\mu + \int_0^1 \int_0^\infty \frac{1}{\mu} e^{-\frac{|x-s|}{\mu}} T^4(s) ds d\mu
\]
implies $\langle \psi(x, \cdot) \rangle > 0$ for any $x \in (a, b)$. From (1.1), $\partial_x^2 T(x) = -\langle \psi(x, \cdot) - T^4(x) \rangle = -\langle \psi(x, \cdot) \rangle < 0$ and thus $T(x)$ is concave on the interval $(a, b)$, which contradicts the assumption $T(x) = 0$ on this interval. Therefore, $T(x) > 0$ for any $x \in \mathbb{R}_+$. For the case $T_b = 0$ and $\psi_b(0, \mu) \neq 0$ on the interval $\mu \in (c, d) \subset (0, 1)$. Then $\int_0^1 \psi_b(\mu) d\mu > 0$ and by the above formula, $\langle \psi(x, \cdot) \rangle > 0$. Hence by the same contradiction argument, we have $T(x) > 0$ for $x \in \mathbb{R}_+$. Finally for the case $T_b, \psi_b$ both are not zero, $\int_0^1 \psi_b d\mu > 0$ and so $\langle \psi(x, \cdot) \rangle > 0$. Therefore, following the same contradiction argument, $T(x) > 0$ for all $x \in \mathbb{R}_+$. We conclude that in all cases $T(x) > 0$ for all $x \in \mathbb{R}_+$ if $T_b, \psi_b$ are not both zero. However, there is no information about $\lim_{x \to \infty} T(x)$.

The above remark is used in the next section to define a weighted $L^2$ space (see the proof of Lemma 3 for details).

4. The Linearized Problem

This section focuses on the study of the linearized system (1.7)–(1.8) of (1.1)–(1.2). We will present the spectral assumption and prove Theorem 2. Let $T$ be a non-trivial solution to system (1.1)–(1.2). We consider system (1.7)–(1.8),
\[
\begin{align*}
\partial_x^2 g &= -\langle \phi - 4T^3 g \rangle + \langle S_1 \rangle, \\
\mu \partial_x \phi &= -\langle \phi - 4T^3 g \rangle + S_2,
\end{align*}
\]
with boundary conditions
\[
\begin{align*}
g(0) &= 0, \\
\phi(0, \mu) &= \phi_b(\mu), \text{ for any } \mu \in (0, 1],
\end{align*}
\]
where the source terms $S_1, S_2$ are functions that decay to zero at infinity. We will prove Theorem 2 in this section. Throughout this section, in order to simplify the notation for the reader, we assume $S_2 = S_1$. For $S_2 \neq S_1$, one can use superposition principle to construct solutions and reduce the problem to the above case (see Appendix A for the proof of Theorem 2 in the general case.)

Since $S_1$ does not have a definite sign, the solutions to the above system are not necessarily positive. Moreover, $T$ may not be monotonic.

Therefore, the monotonicity technique used in section 2 to show the existence of solutions for the nonlinear system (1.1)–(1.2) is not applicable here for the linearized model. Instead, we use the Banach fixed point theorem to show the existence of the linearized model on a bounded interval $[0, B]$. 
Then, a uniform weighted estimate is derived, allowing us to pass to the limit \( B \rightarrow \infty \) and show the existence for system (4.1)–(4.2). Finally, a contradiction argument is used for showing the uniqueness of solutions.

To establish the stability estimates and show the existence and uniqueness for system (4.1)–(4.2), a spectral assumption is proposed. With the help of the spectral assumption, we derive the weighted estimate on solutions of system (4.1)–(4.2). Some inequalities are derived from the spectral assumption in section 4.1 and the assumption will be shown later in the last section to hold when boundary conditions are close to the well-prepared case.

### 4.1. The Spectral Assumption

Now, we derive some inequalities from the spectral assumption (A).

**Lemma 2.** Assume \( T \in C^1_{\text{loc}}(\mathbb{R}_+) \) satisfies the spectral assumption (A). Then for any function \( f \in C^1_{\text{loc}}(\mathbb{R}_+) \) with \( f(0) = 0 \), the following inequalities hold:

\[
M \int_0^\infty (2T^2) |\partial_x f|^2 \, dx \geq 4 \int_0^\infty |\partial_x (2T^2)|^2 f^2 \, dx, \tag{4.5}
\]

\[
\int_0^\infty 4T^2 |\partial_x f|^2 \, dx + \int_0^\infty \partial_x (4T^3) f \partial_x f \, dx \geq 0, \tag{4.6}
\]

and

\[
\int_0^\infty e^{2\beta x} 4T^3 |\partial_x f|^2 \, dx + \int_0^\infty e^{2\beta x} \partial_x (4T^3) f \partial_x f \, dx \geq 0, \tag{4.7}
\]

where \( M \) is the constant in the spectral assumption (A), and \( \beta \) is any constant satisfying \( \beta \in [0, \beta_0] \) where \( \beta_0 \) is given in the spectral assumption (A).

**Proof.** We first prove (4.5). Let \( F \in C^1_{\text{loc}}(\mathbb{R}_+) \) be any function defined on the half-line and \( F(x) \geq 0 \) for any \( x \in \mathbb{R}_+ \). Taking \( f = \int_0^x F(t) \, dt \) in (1.11) leads to

\[
M \int_0^\infty e^{2\beta x} (2T^2)^2 F^2(x) \, dx \geq 4 \int_0^\infty e^{2\beta x} |\partial_x (2T^2)|^2 \left( \int_0^x F(t) \, dt \right)^2 \, dx,
\]

for the same constant \( M < 1 \) as in the spectral assumption (A). Let \( G(x) = e^{\beta x} F(x) \geq 0 \), the above inequality implies

\[
M \int_0^\infty (2T^2)^2 G^2(x) \, dx \geq 4 \int_0^\infty e^{2\beta x} |\partial_x (2T^2)|^2 \left( \int_0^x e^{-\beta t} G(t) \, dt \right)^2 \, dx
\]

\[
= 4 \int_0^\infty |\partial_x (2T^2)|^2 \left( \int_0^x e^{\beta(x-t)} G(t) \, dt \right)^2 \, dx
\]

\[
\geq 4 \int_0^\infty |\partial_x (2T^2)|^2 \left( \int_0^x G(t) \, dt \right)^2 \, dx,
\]

since \( e^{\beta(x-t)} \geq 1 \) for any \( t \leq x \). Due to \( \int_0^x |G(t)| \, dt \geq \int_0^x G(t) \, dt \), the above inequality also holds for all \( G(x) \in C^1_{\text{loc}}(\mathbb{R}_+) \). Let \( g(x) = \int_0^x G(t) \, dt \), then (4.5) holds for any \( g \in C^1_{\text{loc}}(\mathbb{R}_+) \) satisfying \( g(0) = 0 \).
Next we show (4.6). The spectral assumption (A) implies
\[ \int_0^\infty 4T^3 |\partial_x f|^2 \, dx \geq \int_0^\infty 36T |\partial_x T|^2 f^2 \, dx. \]

By Young’s inequality, we deduce that
\[ \left| \int_0^\infty \partial_x (4T^3 f) \partial_x f \, dx \right| \leq \frac{1}{2} \int_0^\infty \frac{1}{4T^3} |\partial_x (4T^3)|^2 f^2 \, dx + \frac{1}{2} \int_0^\infty 4T^3 |\partial_x f|^2 \, dx \]
\[ = \frac{1}{2} \int_0^\infty 36T |\partial_x T|^2 f^2 \, dx + \frac{1}{2} \int_0^\infty 4T^3 |\partial_x f|^2 \, dx. \]

Combining this with the previous inequality, we obtain
\[ \int_0^\infty 4T^3 |\partial_x f|^2 \, dx + \int_0^\infty \partial_x (4T^3 f) \partial_x f \, dx \]
\[ \geq \int_0^\infty 4T^3 |\partial_x f|^2 \, dx - \frac{1}{2} \int_0^\infty 36T |\partial_x T|^2 f^2 \, dx - \frac{1}{2} \int_0^\infty 4T^3 |\partial_x f|^2 \, dx \]
\[ = \frac{1}{2} \int_0^\infty 4T^3 |\partial_x f|^2 \, dx - \frac{1}{2} \int_0^\infty 36T |\partial_x T|^2 f^2 \, dx \geq 0, \]
and thus (4.6) holds.

Similarly, the spectral assumption (A) also implies for \( \beta \in [0, \beta_0] \),
\[ \int_0^\infty e^{2\beta x} 4T^3 |\partial_x f|^2 \, dx \geq \int_0^\infty e^{2\beta x} 36T |\partial_x T|^2 f^2 \, dx. \]

This implies
\[ \int_0^\infty e^{2\beta x} 4T^3 |\partial_x f|^2 \, dx + \int_0^\infty e^{2\beta x} \partial_x (4T^3 f) \partial_x f \, dx \]
\[ \geq \int_0^\infty e^{2\beta x} 4T^3 |\partial_x f|^2 \, dx - \frac{1}{2} \int_0^\infty e^{2\beta x} 4T^3 |\partial_x f|^2 \, dx - \frac{1}{2} \int_0^\infty e^{2\beta x} 36T |\partial_x T|^2 f^2 \, dx \]
\[ = \frac{1}{2} \int_0^\infty e^{2\beta x} 4T^3 |\partial_x f|^2 \, dx - \frac{1}{2} \int_0^\infty e^{2\beta x} 36T |\partial_x T|^2 f^2 \, dx \geq 0, \]
which is (4.7) and the proof is finished. \( \square \)

### 4.2. Existence on the Bounded Interval

We now consider system (4.1)–(4.2) on the bounded interval \([0, B]\),
\[
\partial_x^2 g^B = -\langle \phi^B - 4T^3 g^B \rangle + \langle S_1 \rangle, \quad (4.8) \\
\mu \partial_x \phi^B = -\langle \phi^B - 4T^3 g^B \rangle + S_1, \quad (4.9)
\]
with boundary conditions
\[
g^B(0) = 0, \quad \partial_x g^B(B) = 0, \quad (4.10) \\
\phi^B(0, \mu) = \phi_0(\mu), \quad \phi^B(B, \mu) = \phi^B(B, -\mu), \quad \text{for any } \mu > 0, \quad (4.11)
\]
where \( T^B \) is the solution to the system (2.1)–(2.2).

Define the weighted space

\[
L^2_m([0, B]) := \left\{ f : \int_0^B m(x) f^2(x) \, dx < \infty \right\},
\]

where \( m(x) \geq 0 \) is a given nonnegative valued function. We now prove the following existence lemma:

**Lemma 3.** Let \( S_1 : [0, B] \mapsto \mathbb{R} \) be a continuous function on \([0, B] \). Then there exists a unique solution \( (g^B, \phi^B) \in C^1([0, B]) \times C([0, B] \times [-1, 1]) \) to system (4.8)–(4.9) with boundary conditions (4.10)–(4.11), and the solution satisfies

\[
\int_0^B \int_{-1}^1 e^{2\beta x} (\phi^B - 4T^3 g^B)^2 \, d\mu \, dx + \frac{1}{1 - \beta} \int_{-1}^0 |\mu|(\phi^B)^2(0, \cdot) \, d\mu \
\leq \frac{1}{1 - \beta} \int_0^1 \mu \phi^2_B \, d\mu + \frac{2}{(1 - \beta)^2} \int_0^B e^{2\beta x} |S_1|^2 \, dx.
\]

(4.12)

**Proof.** The proof is divided into two steps. In the first step, we prove existence. We then derive the uniform estimate in the second step.

**Step 1: Existence.** For a given \( g^B \), the solution to (4.9), according to (B.4)–(B.4), can be expressed as

\[
\phi^B = \begin{cases}
\phi_b(\mu) e^{-\frac{x}{\mu}} + \int_0^x \frac{1}{\mu} e^{-\frac{x-s}{\mu}} 4T^3(s) g^B(s) \, ds, & \text{for } \mu > 0, \\
\phi_b(-\mu) e^{\frac{2B-x}{\mu}} - \int_0^B \frac{1}{\mu} e^{\frac{2B-x-s}{\mu}} 4T^3(s) g^B(s) \, ds & \text{for } \mu < 0,
\end{cases}
\]

(4.13)

We denote the above relation by the mapping \( \phi^B = \Phi_{\phi_b}(g^B) \). Then we have

\[
\langle \phi^B(x, \cdot) \rangle = \int_0^1 \phi^B(x, \mu) \, d\mu + \int_{-1}^0 \phi^B(x, \mu) \, d\mu \\
= \int_0^1 \phi_b(\mu) e^{-\frac{x}{\mu}} \, d\mu + \int_0^1 \phi_b(\mu) e^{-\frac{2B-x}{\mu}} \, d\mu \\
+ \int_0^B \int_0^1 \frac{1}{\mu} e^{-\frac{x-s}{\mu}} 4(T^B)^3(s) g^B(s) \, d\mu \, ds \\
+ \int_0^B \int_0^1 \frac{1}{\mu} e^{\frac{2B-x-s}{\mu}} 4(T^B)^3(s) g^B(s) \, d\mu \, ds.
\]

(4.14)

System (4.8)–(4.9) is equivalent to the equation

\[
-\partial^2_x g^B + 8T^3 g^B - \langle \Phi_{\phi_b}(g^B) \rangle = 2S_1,
\]

(4.15)

with

\[
g^B(0) = 0, \quad \partial_x g^B(B) = 0.
\]
To prove the existence for the above equation, we construct a sequence \( \{g_k\}_{k=0}^{\infty} \) with \( g^0(x) = g_b \) and \( g^k \) is solved via
\[
-\partial_x^2 g_k + 8T^3 g_k - \langle \Phi_{\phi_b}(g_{k-1}) \rangle = 2S_1,
\]
with
\[
g_k(0) = 0, \quad \partial_x g_k(B) = 0.
\]
Denote the above mapping by \( \mathcal{G} \) with \( g_k = \mathcal{G}g_{k-1} \). Then according to [18, Chapter 2.2.3], \( \mathcal{G} \) is invertible on the space \( C([0, B]) \) and thus the above equation has a solution \( g_k \in C([0, B]) \). Moreover, by (4.16), \( g_k \in C^2([0, B]) \). Next we show \( \mathcal{G} \) is a contraction mapping in the weighted space \( L^2_{(16T^6)}([0, B]) \).

We first show \( \mathcal{G} \) maps the space \( L^2_{(16T^6)}([0, B]) \) onto itself. We multiply (4.16) by \( 4(T)^3 g_k \) and integrate over \([0, B]\) to get
\[
-\int_0^B 4T^3 g_k \partial_x^2 g_k \, dx + \int_0^B 32T^6 g_k^2 \, dx = \int_0^B 4T^3 g_k \langle \Phi_{\phi_b}(g_{k-1}) \rangle \, dx. \tag{17}
\]
Since \( T \) satisfies the spectral assumption (A), by Lemma 2, inequality (4.6) holds, which implies after taking \( f = g_k 1_{0 \leq x \leq B} \),
\[
-\int_0^B 4T^3 g_k \partial_x^2 g_k \, dx = \int_0^B 4T^3 |\partial_x g_k|^2 \, dx + \int_0^B \partial_x (4T^3) g_k \partial_x g_k \, dx \geq 0. \tag{18}
\]
Then, Young’s inequality implies
\[
\int_0^B 4T^3 g_k \langle \Phi_{\phi_b}(g_{k-1}) \rangle \, dx \leq \int_0^B 16T^6 g_k^2 \, dx + \frac{1}{4} \int_0^B |\langle \Phi_{\phi_b}(g_{k-1}) \rangle|^2 \, dx. \tag{19}
\]
Using the formula (4.14) and Young’s convolution inequality, we obtain
\[
\left\| \int_0^x f(s) \int_0^1 \frac{1}{\mu} e^{-\frac{x-s}{\mu}} \, d\mu \, ds \right\|_{L^2([0, B])}^2 \leq \left\| \int_0^1 \frac{1}{\mu} e^{-\frac{x}{\mu}} \, d\mu \right\|_{L^1([0, B])}^2 \| f \|_{L^2([0, B])}^2 \leq C(B) \| f \|_{L^2([0, B])}^2,
\]
and
\[
\left\| \int_x^B f(s) \int_0^1 \frac{1}{\mu} e^{\frac{s-x}{\mu}} \, d\mu \, ds \right\|_{L^2([0, B])}^2 \leq \left\| \int_0^1 \frac{1}{\mu} e^{\frac{s}{\mu}} \, d\mu \right\|_{L^1([0, B])}^2 \| f \|_{L^2([0, B])}^2 \leq C(B) \| f \|_{L^2([0, B])}^2,
\]
where \( C(B) > 0 \) is a constant depending only on \( B \). Similarly,
\[
\left\| \int_0^B \int_0^1 \frac{1}{\mu} e^{\frac{2B-x-s}{\mu}} f(s, \mu) \, d\mu \, ds \right\|_{L^2([0, B])} \leq C(B) \| f \|_{L^2([0, B] \times [-1, 1])}.
\]
By the above inequalities and (4.14), we deduce
\[
\|\langle \Phi_{\phi_b}(g_{k-1}) \rangle \|_{L^2([0,B])}^2 \leq C(B, \phi_b) + C(B) (\|4T^3 g_{k-1}\|_{L^2([0,B])})^2.
\]

Taking this into (4.19) leads to
\[
\int_0^B 4T^3 g_k \langle \Phi_{\phi_b}(g_{k-1}) \rangle \, dx \leq \int_0^B 16(T^B)^6 g_k^2 \, dx + C(B) \int_0^B 16T^6 g_{k-1}^2 \, dx.
\]

Combining the above inequality with (4.18), (4.17) implies
\[
\int_0^B 16T^6 g_k^2 \, dx \leq C(B) \int_0^B 16T^6 g_{k-1}^2 \, dx.
\]

Hence \(G\) maps from \(L^2_{16T^6}([0, B])\) into itself.

We next show that \(G\) is a contraction mapping. Let \(h_k = g_k - g_{k-1}\), we have
\[
\partial_x^2 h_k = 8T^3 h_k - \langle \Phi_0(h_{k-1}) \rangle, \quad \text{(4.20)}
\]
with
\[
h_k = 0, \quad \partial_x h_k(B) = 0.
\]

We multiply (4.20) by \(4(T^B)^3 h_k\) and integrate over \([0, B]\) to get
\[
- \int_0^B 4T^3 h_k \partial_x^2 h_k \, dz + \int_0^B 32T^6 h_k^2 \, dx - \int_0^B \langle \Phi_0(h_{k-1}) \rangle 4T^3 h_k \, dx = 0.
\]

The inequality (4.6) implied by the spectral assumption on \(T^B\) leads to
\[
- \int_0^B 4T^3 h_k \partial_x^2 h_k \, dx = \int_0^B 4T^3 |\partial_x h_k|^2 \, dz + \int_0^B \partial_x (4T^3 h_k) \partial_x h_k \, dz \geq 0.
\]

Recalling (4.14), we have
\[
\int_0^B 4T^3 h_k \langle \Phi_{\phi_b=0}(h_{k-1}) \rangle \, dx \quad \text{(4.23)}
\]
\[
= \int_0^B \int_0^B \int_0^1 \int_0^1 \frac{1}{\mu} e^{-\frac{|x-s|}{\mu}} 16T^3(s)h_{k-1}(s)T^3(x)h_k(x) \, d\mu \, ds \, dx
\]
\[
+ \int_0^B \int_0^1 \int_0^1 \int_0^1 \frac{1}{\mu} e^{-\frac{2B-x-s}{\mu}} 16T^3(s)h_{k-1}(s)T^3(x)h_k(x) \, d\mu \, ds \, dx
\]
\[
= - \frac{1}{2} \int_0^B \int_0^B \int_0^1 \int_0^1 \frac{1}{\mu} e^{-\frac{|x-s|}{\mu}} (4T^3(s)h_{k-1}(s) - 4T^3(x)h_k(x))^2 \, d\mu \, ds \, dx
\]
\[
- \frac{1}{2} \int_0^B \int_0^B \int_0^1 \int_0^1 \frac{1}{\mu} e^{-\frac{2B-x-s}{\mu}} (4T^3(s)h_{k-1}(s) - 4T^3(x)h_k(x))^2 \, d\mu \, ds \, dx
\]
\[
+ \frac{1}{2} \int_0^B \int_0^B \int_0^1 e^{-\frac{|x-s|}{\mu}} (16T^6(s)h_{k-1}^2(s) + 16T^3(x)h_k^2(x)) \, d\mu \, ds \, dx
\]
+ \frac{1}{2} \int_0^B \int_0^B \int_0^1 \frac{1}{\mu} e^{-2B-x-s} \left(16T^6(s)h_{k-1}^2(s) + 16T^3(x)h_k^2(x)\right) d\mu \, ds \, dx

= \frac{1}{2} \int_0^B \int_0^B \int_0^1 \frac{1}{\mu} e^{-\frac{2B-x-s}{\mu}} \left(4T^3(s)h_{k-1}(s) - 4T^3(x)h_k(x)\right)^2 d\mu \, ds \, dx

- \frac{1}{2} \int_0^B \int_0^1 \frac{1}{\mu} e^{-\frac{2B-x-s}{\mu}} \left(4T^3(s)h_{k-1}(s) - 4T^3(x)h_k(x)\right)^2 d\mu \, ds \, dx

+ \frac{1}{2} \int_0^1 \left(1 - e^{-\frac{B-s}{\mu}} + e^{-\frac{B-s}{\mu}} - e^{-\frac{2B-s}{\mu}}\right)

\cdot 16T^6(x)(h_k^2(x) + h_{k-1}^2(x)) d\mu \, dx

\leq \frac{1}{2} \int_0^B \int_0^1 16(2 - e^{-\frac{s}{\mu}} - e^{-\frac{2B-s}{\mu}})T^6(x)(h_k^2(x) + h_{k-1}^2(x)) d\mu \, dx.

(4.24)

Taking (4.22) and (4.23) into (4.21) gives

\frac{1}{2} \int_0^B \int_0^1 16(2 + e^{-\frac{s}{\mu}} + e^{-\frac{2B-s}{\mu}})T^6(x)h_k^2(x) d\mu \, dx

- \frac{1}{2} \int_0^B \int_0^1 16(2 - e^{-\frac{s}{\mu}} - e^{-\frac{2B-s}{\mu}})T^6(x)h_{k-1}^2(x) d\mu \, dx \leq 0.

Due to

\min_{x \in [0,B]} \left(e^{-\frac{s}{\mu}} + e^{-\frac{2B-s}{\mu}}\right) = e^{-\frac{B}{\mu}} + e^{-\frac{B}{\mu}} = 2e^{-\frac{B}{\mu}},

the previous inequality implies

\int_0^B 16T^6(x)h_k^2(x) \, dx \leq \delta \int_0^B 16T^6(x)h_{k-1}^2(x) \, dx,

(4.25)

with

\delta = \frac{\int_0^1 (1 - e^{-\frac{B}{\mu}}) \, d\mu}{\int_0^1 (1 + e^{-\frac{B}{\mu}}) \, d\mu} < 1.

Then (4.25) can be written as

\|h_k\|_{L^2_{16T^6}([0,B])}^2 \leq \delta \|h_{k-1}\|_{L^2_{16T^6}([0,B])}^2,

which is

\|g_k - g_{k-1}\|_{L^2_{16T^6}([0,B])}^2 \leq \delta \|g_{k-1} - g_{k-2}\|_{L^2_{16T^6}([0,B])}^2,

with \delta < 1. Thus the map \mathcal{G} : L^2_{16T^6}([0,B]) \mapsto L^2_{16T^6}([0,B]) is a contraction mapping. By the Banach fixed point theorem, there exists a unique solution to the equation (4.15) in \Lk16T^6([0,B]), that is there exists a unique solution

g^B \in L^2_{16T^6}([0,B])

to the equation (4.15). By equation (4.15), \partial_x g^B \in L^2([0,B])
and thus \( g^B \in C^1([0, B]) \). Since \( g^B \in L^2_{16/7}(\{0, B\}) \) is equivalent to \( 4T^3g^B \in L^2([0, B]) \). By the formula (4.13), \( \phi^B \in C([0, B] \times [-1, 1]) \).

**Step 2. The weighted estimates.** The weighted estimate is derived in a similar manner as in section 2.3 for the nonlinear case. The major difference is that we need the spectral assumption to estimate the term involving \( T \). First, (2.23) still holds for \( g^B, \phi^B \), that is

\[
\partial_x g^B(x) = \langle \mu \phi^B(x, \cdot) \rangle, \quad \text{for any } x \in [0, B]. \tag{4.26}
\]

This can be seen from equating (4.8) with (4.9) and use the boundary conditions (4.10)–(4.11), similarly as the derivation of (2.23). To obtain the weighted estimate, we multiply (4.8) by \( e^{2\beta x}4T^3 g^B \) and (4.9) by \( e^{2\beta x} \phi^B \) and integrating over \([0, B]\) and \([-1, 1]\) to obtain

\[
- \int_0^B e^{2\beta x}4T^3 g^B \partial_x^2 g^B \, dx + \int_0^B \int_{-1}^1 \mu e^{2\beta x} \partial_x (\phi^B)^2 \, d\mu \, dx
+ \int_0^B \int_{-1}^1 e^{2\beta x}(\phi^B - 4T^3 g^B)^2 \, d\mu \, dx = \int_0^B \int_{-1}^1 e^{2\beta x}(\phi^B - 4T^3 g^B)S_1 \, dx. \tag{4.27}
\]

The second term on the left of the above equation can be treated similarly as (2.29) using integration-by-parts:

\[
\int_0^B \int_{-1}^1 \mu e^{2\beta x} \partial_x (\phi^B)^2 \, d\mu \, dx
= -\beta \int_0^B \int_{-1}^1 e^{2\beta x} \mu (\phi^B)^2 \, d\mu \, dx + \frac{1}{2} \int_{-1}^1 e^{2\beta x} \mu (\phi^B)^2 \, d\mu \bigg|_0^B
= -\beta \int_0^B \int_{-1}^1 e^{2\beta x} \mu (\phi^B)^2 \, d\mu \, dx - \frac{1}{2} \int_{-1}^1 \mu (\phi^B)^2 (0, \cdot) \, d\mu. \tag{4.28}
\]

The first term on the left of equation (4.27) can be treated the same as (2.28) using the integration-by-parts:

\[
- \int_0^B e^{2\beta x}4T^3 g^B \partial_x^2 g^B \, dx
= \int_0^B e^{2\beta x}4T^3 |\partial_x g^B|^2 \, dx + \int_0^B e^{2\beta x} \partial_x (2T^3) \partial_x (g^B)^2 \, dx
+ 2\beta \int_0^B e^{2\beta x}4T^3 g^B \partial_x g^B \, dx - e^{2\beta x}4T^3 g^B \partial_x g^B \bigg|_0^B.
\]

However, compared to (2.28), the non-constant function \( T \) leads to the additional term, which is the second term on the right of the above equation. In order to deal with this difficulty, we use the spectral assumption (A) and the inequality (4.7) from Lemma 2 to obtain

\[
- \int_0^B e^{2\beta x}4T^3 g^B \partial_x^2 g^B \, dx \geq 2\beta \int_0^B e^{2\beta x}4T^3 g^B \partial_x g^B \, dx. \tag{4.29}
\]
Note that here we can take function $f(x) = g_B(x)$ for $x \in [0, B]$ and $f(x) = 0$ for $x > B$ in (1.11) of the spectral assumption, so that the above inequality holds.

We can combine the term on the right of the above inequality with the first right term of equation (4.28) and by using the relation (4.26) to get:

$$2\beta \int_0^B e^{2\beta x} 4T^3 g_B \partial_x g_B \, dx - \beta \int_0^B \int_{-1}^1 e^{2\beta x} \mu(\phi_B^2) \, d\mu \, dx$$

$$= -\beta \int_0^B \int_{-1}^1 \mu(\phi_B - 4T^3 g_B)^2 \, d\mu \, dx. \quad (4.30)$$

Moreover, since here we have the source term $S_1$, we need to estimate the corresponding term as

$$\int_0^B \int_{-1}^1 e^{2\beta x} (\phi_B - 4T^3 g_B)^2 \, d\mu \, dx$$

$$\leq \xi \int_0^B \int_{-1}^1 e^{2\beta x} (\phi_B - 4T^3 g_B)^2 \, d\mu \, dx + \frac{1}{2\xi} \int_0^B e^{2\beta x} S_1^2 \, dx.$$

Combing the above inequality with (4.29), (4.28) and (4.30), the equality (4.27) becomes

$$\int_0^B \int_{-1}^1 (1 - \xi - \mu \beta) e^{2\beta x} (\phi_B - 4T^3 g_B)^2 \, d\mu \, dx$$

$$+ \frac{1}{2} \int_{-1}^0 |\mu|(\phi_B^2)(0, \cdot) \, d\mu \leq \frac{1}{2} \int_{-1}^1 \mu \phi_B^2 \, d\mu + \frac{1}{2\xi} \int_0^B e^{2\beta x} S_1^2 \, dx.$$

Taking $\beta < 1$, then $1 - \xi - \mu \beta \geq 1 - \xi - \beta$. Taking $\xi = (1 - \beta)/2$, the above inequality implies (4.12) and finishes the proof of Lemma 3. □

### 4.3. Existence on the Half-Space

Next we pass to the limit $B \to \infty$ in system (4.8)–(4.9) and show the existence for system (4.1)–(4.2).

**Proof of Theorem 2, existence.** By the relation (4.26), we have

$$\int_0^B e^{2\beta x} |\partial_x g_B|^2 \, dx = \int_0^B e^{2\beta x} |\mu \phi_B|^2 \, dx.$$

Using the fact $\langle \mu 4T^3 g_B \rangle = 0$ and Hölder’s inequality, the above equation equals

$$\int_0^B e^{2\beta x} |\partial_x g_B|^2 \, dx = \int_0^B e^{2\beta x} \left( \int_{-1}^1 \mu(\phi_B - 4T^3 g_B) \, d\mu \right)^2 \, dx$$

$$\leq \int_0^B \left( \int_{-1}^1 \mu \, d\mu \right) \left( \int_{-1}^1 (\phi_B - 4T^3 g_B)^2 \, d\mu \right) \, dx$$

$$\leq \frac{2}{3} \int_0^B \int_{-1}^1 e^{2\beta x} (\phi_B - 4T^3 g_B)^2 \, d\mu \, dx$$
\[
\leq \frac{2}{3} \frac{1}{1 - \beta} \int_0^1 \mu \phi_B^2 \, d\mu + \frac{4}{3(1 - \beta)^2} \int_0^\infty e^{2\beta x} |S_1|^2 \, dx,
\]

which is uniformly bounded. Hence

\[
|g_B(x)| = |g_B(x) - g_B(0)| = \left| \int_0^x \partial_z g_B(z) \, dz \right|
\]

\[
= \left| \int_0^x e^{-\beta z} e^{\beta z} \partial_z g_B(z) \, dz \right|
\]

\[
\leq \left( \int_0^x e^{2\beta z} |\partial_z g|^2 \, dz \right)^{\frac{1}{2}} \left( \int_0^x e^{-2\beta z} \, dz \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{\sqrt{2\beta}} e^{-\beta x} \left( \frac{2}{3} \frac{1}{1 - \beta} \int_0^1 \mu \phi_B^2 \, d\mu + \frac{4}{3(1 - \beta)^2} \int_0^\infty e^{2\beta x} |S_1|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
= N_\beta e^{-\beta x}.
\]

Hence \( g_B \in L^2_{\text{loc}}(\mathbb{R}_+) \). By the equation (4.8),

\[
\int_0^B e^{2\beta x} \partial_x^2 g_B^2 \, dx \leq \frac{2}{3} \int_0^B \int_{-1}^1 (\phi_B - 4T^3 g_B)^2 \, d\mu \, dx + 2 \int_0^B e^{2\beta x} |S_1|^2 \, dx
\]

and is also uniformly bounded. Hence \( g_B \in H^2_{\text{loc}}(\mathbb{R}_+) \) is uniformly bounded and we can find a subsequence that

\[
g_B \rightharpoonup g, \quad \text{weakly in } H^2_{\text{loc}}(\mathbb{R}_+),
\]

\[
g_B \to g, \quad \text{strongly in } C^1_{\text{loc}}(\mathbb{R}_+).
\]

(4.32)

Moreover, by the continuity of the trace operator, we can pass to the limit \( B \to \infty \) in (4.10) and get that \( g(0) = 0 \).

By the above convergence result and the uniform boundness of \( \int_0^B \int_{-1}^1 (\phi_B - 4T^3 g_B)^2 \, d\mu \, dx \), there exists subsequence such that

\[
\phi_B - 4T^3 g_B \rightharpoonup \phi - 4T^3 g, \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}_+ \times [-1, 1]).
\]

Hence

\[
\phi_B \rightharpoonup \phi, \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}_+ \times [-1, 1]).
\]

(4.33)

Moreover, by equation (4.9),

\[
\int_0^B \int_{-1}^1 |\partial_x (\mu \psi)|^2 \, d\mu \, dx \leq \int_0^B \int_{-1}^1 (\phi_B - 4T^3 g_B)^2 \, d\mu \, dx + 2 \int_0^B |S_1|^2 \, dx,
\]

is uniformly bounded, hence \( \partial_x (\mu \psi) \in L^2_{\text{loc}}(\mathbb{R}_+ \times [-1, 1]) \). Thus we can use the trace theorem and pass to the limit in (4.11) to get that \( \phi(0, \mu) = \phi_b(\mu) \) for any \( \mu > 0 \). Moreover, we can pass the weak limit in (4.8)–(4.9) and use (4.32)
and (4.33) and obtain that \((g, \phi)\) satisfies the system (4.1)–(4.2) with boundary conditions (4.3)–(4.4).

The proof is divided into four steps. In the first step, we pass to the limit \(B \to \infty\) in (4.10)–(4.11) and prove the limit satisfies system (4.1)–(4.2) in the weak sense. Then we show the uniform estimate and weighted decay property (1.12).

### 4.4. Exponential Decay

Next we derive the uniform estimate for system (4.1)–(4.2) on the half-space and show the exponential decay properties.

**Proof of Theorem 2, exponential decay.** The weak convergence of \(\phi^B - 4(T^B)^3 g^B\) in \(L^2_{\text{loc}}(\mathbb{R}_+)\) and the weak lower semi-continuity of the norm \(\| \cdot \|_{L^2}\) implies that

\[
\|e^{\beta x} (\phi - 4T^3 g)\|_{L^2(\mathbb{R}_+ \times [-1, 1])} \leq \liminf_{B \to \infty} \|e^{\beta x} (\phi^B - 4(T^B)^3 g^B)\|_{L^2([0, B] \times [-1, 1])}.
\]

Moreover, by the trace theorem, we have

\[
\lim_{B \to \infty} \int_{-1}^{0} |\mu| (\phi^B)^2(0, \cdot) \, d\mu = \int_{-1}^{0} |\mu| \phi^2(0, \cdot) \, d\mu.
\]

Therefore, we can take the \(\liminf_{B \to \infty}\) in (4.12) and obtain the following weighted estimate:

\[
\int_{0}^{\infty} \int_{-1}^{1} e^{2\beta x} (\phi^B - 4(T^B)^3 g^B)^2 \, d\mu \, dx + \frac{1}{1 - \beta} \int_{-1}^{0} |\mu| \phi^2(0, \cdot) \, d\mu \\
\leq \frac{1}{1 - \beta} \int_{0}^{1} \mu \phi_0^2 \, d\mu + \frac{2}{(1 - \beta)^2} \int_{0}^{\infty} e^{2\beta x} |S_1|^2 \, dx.
\]  

(4.34)

To show the exponential decay property, we first note that the condition (4.26) still holds for \(g, \phi\) on the half-line. We can pass to the limit in (4.26) due to (4.32) and (4.33), and

\[
\partial_x g(x) = \langle \mu \phi(x, \cdot) \rangle, \quad \text{for } x \in \mathbb{R}_+ \text{ almost everywhere.}
\]

(4.35)

In order to show the above relation still holds at infinity, that is \(\lim_{x \to \infty} \partial_x g(x) = \lim_{x \to \infty} \langle \mu \phi(x, \cdot) \rangle\), we need to show that both terms of the above equation are bounded. To show this, first comparing the equation (4.1) to ((4.2)) gives

\[
\partial_x (\partial_x g - \langle \mu \phi \rangle) = 0.
\]

By equation (4.2) and the estimate (4.34), similarly as (4.31),

\[
\int_{0}^{\infty} e^{2\beta x} |\langle \mu \phi \rangle|^2 \, dx \leq \frac{2}{3} \int_{0}^{\infty} \int_{-1}^{1} (\phi - 4T^3 g)^2 \, d\mu \, dx \\
\leq \frac{2}{3(1 - \beta)} \int_{0}^{1} \mu \phi_0^2 \, d\mu + \frac{4}{3(1 - \beta)^2} \int_{0}^{\infty} e^{2\beta x} |S_1|^2 \, dx.
\]  

(4.36)
Moreover, we have
\[ \partial_x \langle \mu \phi \rangle = -\langle \phi - 4T^3g \rangle + \langle S_1 \rangle, \]
and so
\[ \int_0^\infty e^{2\beta x} |\partial_x \langle \mu \phi \rangle|^2 \, dx \]
\[ \leq \int_0^\infty e^{2\beta x} |\langle \phi - 4T^3g \rangle|^2 \, dx + \int_0^\infty e^{2\beta x} |\langle S_1 \rangle|^2 \, dx \]
\[ \leq 2 \int_0^\infty \int_{-1}^1 e^{2\beta x} (\phi - 4T^3g)^2 \, dx + 2 \int_0^\infty e^{2\beta x} |S_1|^2 \, dx \]
\[ \leq 2 \int_0^\infty e^{2\beta x} |S_1|^2 \, dx + \frac{2}{2y} \int_0^1 \mu \phi^2 \, d\mu + \frac{4}{(1-\beta)^2} \int_0^\infty e^{2\beta x} |S_1|^2 \, dx \]
(4.37)

From (4.36) and (4.37), we can apply the Barbalat’s lemma [10] and deduce
\[ \langle \mu \phi(x, \cdot) \rangle \to 0, \quad \text{as} \quad x \to \infty \]
Hence
\[ |\langle \mu \phi(x, \cdot) \rangle| = \left| \int_x^\infty e^{-\beta z} e^{2\beta z} \partial_z \langle \mu \phi \rangle \, dz \right| \leq \frac{1}{\sqrt{2\beta}} e^{-\beta x} \left( \int_0^\infty e^{2\beta z} |\partial_z \langle \mu \phi \rangle|^2 \, dz \right)^{\frac{1}{2}}, \]
and is uniformly bounded. Thus (4.35) holds for all \( x \in [0, \infty) \) and also at infinity \( \lim_{x \to \infty} \partial_x g(x) = \lim_{x \to \infty} \langle \mu \phi(x, \cdot) \rangle \). Using the relation (4.35) and (4.36), we obtain
\[ \int_0^\infty e^{2\beta x} |\partial_x g|^2 \, dx \leq \frac{2}{3} \frac{1}{1-\beta} \int_0^1 \mu \phi^2 \, d\mu + \frac{4}{3(1-\beta)^2} \int_0^\infty e^{2\beta x} |S_1|^2 \, dx = 2\beta N_{\beta}^2. \]
Hence
\[ |g(x)| = \left| \int_0^x \partial_z g(z) \, dz \right| = \left| \int_0^x e^{-\beta z} e^{2\beta z} \partial_z g \, dz \right| \]
\[ \leq \frac{1}{\sqrt{2\beta}} \left( \int_0^\infty e^{2\beta z} |\partial_z g|^2 \, dz \right)^{\frac{1}{2}} = N_{\beta}. \]
We can let \( x \to \infty \) and obtain \( g_\infty := \lim_{x \to \infty} g(x) \) exists and
\[ |g_\infty| \leq N_{\beta}. \]
We also have
\[ |g(x) - g_\infty| = \left| \int_x^\infty e^{-\beta z} e^{2\beta z} \partial_x g \, dz \right| \]
\[ \leq \frac{1}{\sqrt{2\beta}} e^{-\beta x} \left( \int_0^\infty e^{2\beta z} |\partial_x g|^2 \, dx \right)^{\frac{1}{2}} = N_{\beta} e^{-\beta x}. \]
which gives the first inequality in (1.12). The other two inequalities in (1.12) can be derived by using the formula (C.2)–(C.2) in a similar manner as (3.19)–(3.21). For \( \mu > 0 \),

\[
\begin{align*}
|\psi(x, \mu) - 4T^3_{\infty}g_{\infty} - \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} S_1(s) \, ds| & \quad (4.38) \\
= |\psi_b e^{-\frac{x}{\mu}} + \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} (4T^3(s)g(s) + S_1(s)) \, ds - 4T^3_{\infty}g_{\infty}| \\
= |\psi_b e^{-\frac{x}{\mu}} + \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} (4T^3(s)g(s) - T^4_{\infty}) \, ds - 4T^3_{\infty}g_{\infty}e^{-\frac{x}{\mu}} \, ds| \\
\leq |\psi_b - 4T^3_{\infty}g_{\infty}| e^{-\frac{x}{\mu}} + \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} |4T^3(s)g(s) - 4T^3_{\infty}g_{\infty}| \, ds \\
\leq |\psi_b - 4T^3_{\infty}g_{\infty}| e^{-\frac{x}{\mu}} + \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} 4|T^3(s) - T^3_{\infty}|g(s) \, ds \\
& \quad + \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} 4T^3_{\infty}|g(s) - g_{\infty}| \, ds \\
= |\psi_b - 4T^3_{\infty}g_{\infty}| e^{-\frac{x}{\mu}} + 12(T_b + 2M_\alpha)^2 M_\alpha e^{-\alpha x} N_\beta \frac{1}{1 - \mu \alpha} (e^{-\beta x} - e^{-\frac{x}{\mu}}) \\
& \quad + 4(T_b + M_\alpha)^3 N_\beta \frac{1}{1 - \mu \beta} (e^{-\beta x} - e^{-\frac{x}{\mu}}) \\
\leq |\psi_b - 4T^3_{\infty}g_{\infty}| e^{-\frac{x}{\mu}} + 4(T_b + 2M_\alpha)^2 (4M_\alpha + T_b) N_\beta e^{-\beta x}, \quad (4.39)
\end{align*}
\]

where in the last inequality we take \( \alpha = \beta \). For \( \mu < 0 \),

\[
\begin{align*}
|\psi(x, \mu) - 4T^3_{\infty}g_{\infty} + \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} S_1(s) \, ds| \\
= - \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} 4T^3(s)g(s) \, ds - 4T^3_{\infty}g_{\infty} \\
= - \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} 4T^3(s)g(s) \, ds + \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} 4T^3_{\infty}g_{\infty} \, ds \\
\leq - \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} |4T^3(s)g(s) - 4T^3_{\infty}g_{\infty}| \, ds \\
\leq - \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} 4|T^3(s) - T^3_{\infty}|g(s) \, ds - \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} 4T^3_{\infty}|g(s) - g_{\infty}| \, ds \\
\leq - \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} 4\cdot 3(T_b + 2M_\alpha)^2 M_\alpha e^{-\alpha x} N_\beta \, ds \\
& \quad - \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} 4(T_b + M_\alpha)^3 N_\beta e^{-\beta x} \, dx
\end{align*}
\]
\[ \begin{align*}
&= 12(T_b + 2M_\alpha)^2 M_\alpha N_\beta \frac{1}{1 - \mu \alpha} e^{-\alpha x} + 4(T_b + M_\alpha)^3 N_\beta \frac{1}{1 - \mu \beta} e^{-\beta x} \\
&= -4(T_b + 2M_\alpha)^3 M_\alpha \frac{1}{1 + \mu \alpha} e^{-\alpha x} \\
&\leq 4(T_b + 2M_\alpha)^2 (4M_\alpha + T_b) N_\beta e^{-\beta x},
\end{align*} \]

where in the last inequality we also take \( \alpha = \beta \). Since for \( \mu > 0 \)
\[ \left| \int_0^x \frac{1}{\mu} e^{-\frac{x-s}{\mu}} S_1(s) \, ds \right| = \left| \int_0^x \frac{1}{\mu} e^{-\frac{x-s}{\mu}} e^{-\beta s} e^{\beta s} S_1(s) \, ds \right| \]
\[ \leq \left( \int_0^x \frac{1}{\mu^2} e^{-\frac{2(x-s)}{\mu}} e^{-2\beta s} \, ds \right)^{\frac{1}{2}} \left( \int_0^x e^{2\beta s} |S_1(s)|^2 \, ds \right)^{\frac{1}{2}} \]
\[ \leq \frac{1}{1 - 2\mu \beta} (e^{-2\beta x} - e^{-\frac{2 x}{\mu}})^{\frac{1}{2}} \|e^{\beta x} S_1\|_{L^2(\mathbb{R}^+)} \]
\[ \leq \frac{1}{1 - 2\mu \beta} \|e^{\beta x} S_1\|_{L^2(\mathbb{R}^+)} e^{-\beta x}, \]

and similarly for \( \mu < 0 \),
\[ \left| \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} S_1(s) \, ds \right| = \left| \int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} e^{-\beta s} e^{\beta s} S_1(s) \, ds \right| \]
\[ \leq \left( \int_x^\infty \frac{1}{\mu^2} e^{-\frac{2(x-s)}{\mu}} e^{-2\beta s} \, ds \right)^{\frac{1}{2}} \left( \int_x^\infty e^{2\beta s} |S_1(s)|^2 \, ds \right)^{\frac{1}{2}} \]
\[ \leq \frac{1}{1 - 2\mu \beta} \|e^{-\beta x} S_1\|_{L^2(\mathbb{R}^+)} \]

Combining the above inequalities with (4.38) and (4.40), we obtain the last inequality in (1.12). \( \square \)

4.5. Uniqueness

Suppose \((g_1, \phi_1)\) and \((g_2, \phi_2)\) are two solutions of the system (4.1)–(4.2) with boundary conditions (4.3)–(4.4). Then \( h = g_1 - g_2 \) and \( \varphi = \phi_1 - \phi_2 \) satisfy the system
\[ \partial_x^2 h = -\langle \varphi - 4T^3 h \rangle, \]
\[ \mu \partial_x \varphi = -\langle \varphi - 4T^3 h \rangle, \]

and the boundary conditions
\[ h(0) = 0, \quad \varphi(0, \mu) = 0, \text{ for } \mu > 0. \]

From the previous section, we have that \( h, \varphi \) also verify that
\[ \lim_{x \to \infty} \partial_x h = 0, \quad \lim_{x \to \infty} \langle \mu \varphi(x, \cdot) \rangle = 0, \]
and

\[ \partial_\chi h(x) = \langle \mu \varphi(x, \cdot) \rangle, \quad \text{for any } x \in \mathbb{R}_+. \tag{4.45} \]

We now multiply (4.41) by \(4T^3 h\) and (4.42) by \(\varphi\) and integrate to get

\[ - \int_0^\infty 4T^3 h \partial_\chi^2 h \, dx + \int_0^\infty \mu \partial_\chi \frac{\varphi^2}{2} \, d\mu \, dx + \int_0^\infty \int_{-1}^1 (\varphi - 4T^3 h)^2 \, d\mu \, dx = 0. \]

We integrate by parts and by using (4.43)–(4.44) and the inequality (4.7) due to the spectral assumption (A), we obtain

\[ - \int_0^\infty 4T^3 h \partial_\chi^2 h \, dx + \int_0^\infty \mu \partial_\chi \frac{\varphi^2}{2} \, d\mu \, dx = - \Bigg| \int_0^\infty \frac{1}{2} \langle \mu \varphi(0, \cdot) \rangle^2 d\mu \Bigg|. \]

The relation (4.45) is used in the second equality. Taking the above inequality into the previous equation leads to

\[ \frac{1}{2} \int_{-1}^0 |\mu| \langle \varphi(0, \cdot) \rangle^2 d\mu + \int_0^\infty \int_{-1}^1 (\varphi - 4T^3 h)^2 \, d\mu \, dx \leq 0. \]

This implies that

\[ \varphi - 4T^3 h = 0, \, \varphi(0, \mu) = 0, \quad \text{for any } \mu \in [-1, 1], \]

and so

\[ \partial_\chi^2 h = 0, \, \partial_\chi \langle \mu \varphi \rangle = 0, \, \langle \mu \varphi(0, \cdot) \rangle = 0. \]

Therefore, we obtain

\[ \langle \mu \varphi(x, \cdot) \rangle = 0, \quad \text{for any } x \geq 0 \]

and

\[ \partial_\chi h(x) = \langle \mu \varphi(x, \cdot) \rangle = 0, \quad \text{for any } x \geq 0. \]

Consequently, we have \(h \equiv 0\) and \(\varphi \equiv 4T^3 h \equiv 0\). Finally, system (4.41)–(4.42) has only zero solutions. Therefore, the solution to system (4.1)–(4.2) is unique. \(\Box\)

5. Uniqueness of the Nonlinear Milne Problem

This section is devoted to the proof of Theorem 3. We first show the uniqueness for small boundary data and then for small perturbations in a neighborhood of functions satisfying the spectral assumptions.
5.1. Sufficient Condition for the Spectral Assumption

We next give some sufficient and necessary conditions for the spectral assumption \((A)\) using Hardy’s inequality.

**Lemma 4.** The spectral assumption \((A)\) holds for function \(T \in C^1(\mathbb{R}_+)\) if

\[
A_0 := \sup_{r \in (0, \infty)} \left( \int_r^\infty e^{2\beta x} 36T|\partial_x T|^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^r \frac{1}{e^{2\beta x} 4T^3} \, dx \right)^{\frac{1}{2}} < \frac{1}{2},
\]

(5.1)

for some constant \(\beta > 0\). Moreover, if the spectral assumption \((A)\) holds, then \(A_0 < 1\). Furthermore, if \(T\) satisfies \(T \leq |T| \leq C, |\partial_x T| \leq Ce^{-\alpha x}\) for some constants \(T_0 > 0, C > 0\) and \(\alpha > 0\). Then the spectral assumption \((A)\) holds if

\[
A_1 := \sup_{r \in (0, \infty)} \left( \int_r^\infty 36T|\partial_x T|^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^r \frac{1}{4T^3} \, dx \right)^{\frac{1}{2}} < \frac{1}{2}.
\]

(5.2)

**Proof.** This lemma is a direct consequence of the generalized Hardy’s inequality (Lemma 8 in Appendix D). We take \(u(x) = e^{2\beta x} 36T(x)|\partial_x T(x)|^2, v(x) = e^{2\beta x} 4T^3(x), F(x) = \partial_x f(x)\) and \(b = \infty, p = p' = 2\) in (D.1) and get that

\[
\int_0^\infty f^2 e^{2\beta x} 36T|\partial_x T|^2 \, dx \leq M_0 \int_0^\infty |\partial_x f|^2 e^{2\beta x} 4T^3 \, dx,
\]

(5.3)

holds for any \(f \in C^1(\mathbb{R}_+)\) with \(f(0) = 0\), with the best constant \(M_0\) satisfying \(A_0^2 \leq M_0 \leq 4A_0^2\), where \(A_0\) is given by (5.1).

On the other hand, if the spectral assumption \((A)\) holds, then the best constant \(M_0\) in (5.3) satisfies \(M_0 < 1\). Since \(A_0^2 \leq M_0 \leq 4A_0^2\), where \(A_0\) is given by (5.1),

Finally, if \(|T(x)| \leq C, |\partial_x T(x)| \leq Ce^{-\alpha x}\), then

\[
T(x)|\partial_x T(x)|^2 \leq Ce^{-\alpha x}.
\]

(5.4)

This implies

\[
\int_r^\infty (e^{2\beta x} - 1)|\partial_x T|^2 \, dx \leq C \int_r^\infty (e^{2\beta x} - 1)e^{-\alpha x} \, dx
\]

\[
= Ce^{-\alpha r} \frac{2\beta - \alpha(1 - e^{2\beta r})}{\alpha(\alpha - 2\beta)}
\]

\[
\leq C \frac{1}{(\alpha - 2\beta)} e^{-(\alpha - 2\beta)r} - C \frac{1}{\alpha} e^{-\alpha r}
\]

(5.5)

for \(\beta < \alpha/2\).

Assume \(T \geq T_m > 0\) for some constant \(T_m > 0\). Then

\[
\int_0^r \frac{1}{4T^3} \, dx \leq \frac{1}{4T_m^3} r
\]

(5.6)

Therefore,
\begin{equation}
\int_{r}^{\infty} e^{2\beta x} 36T |\partial_x T|^2 \, dx \int_{0}^{r} \frac{1}{e^{2\beta x}4T^3} \, dx
\end{equation}
(5.7)

for \( \beta \) sufficiently small, the first term on the last line of the above inequality can be sufficiently small and so
\[
A_0 < A_1 + \varepsilon < \frac{1}{2}.
\] (5.9)

Hence by the previous argument, the spectral assumption (A) holds.

### 5.2. Uniqueness for Almost Well-Prepared Boundary Data

We first show that the spectral assumption is not empty and its holds when \( \int_{0}^{1} (\psi - T_4)^2 \, d\mu \) is small using the Hardy’s inequality, [Appendix D]. We then give the uniqueness of the solution for system (1.1)–(1.2).

**Lemma 5.** Assume the boundary data \((T_b, \psi_b)\) satisfies (1.14) for some constant \(C_b\) satisfying (5.14). Let \((T, \psi)\) be the corresponding solution obtained in Theorem 1. Then \(T\) satisfies the inequality (5.1). As a consequence, \(T\) also satisfies the spectral assumption (A).

**Proof.** By Lemma 4, the spectral assumption (A) holds if the condition (5.1) is fulfilled. We next show (5.1) holds when \( \int_{0}^{1} \mu (\psi - T_4)^2 \, d\mu \) is sufficiently small. Define \(C_b := \frac{1}{2} \int_{0}^{1} \mu (\psi - T_4)^2 \, d\mu\),
\[
|\partial_x T(x)| \leq \frac{1}{\sqrt{2\alpha}} e^{-\alpha x} C_b^\frac{1}{2}.
\] (5.10)

Theorem 1 implies that \((T, \psi)\) are uniformly bounded:
\[
0 \leq T(x) \leq \gamma, \quad 0 \leq \psi(x) \leq \gamma^4,
\]
for any \(x \geq 0\). Hence for \(\beta < \alpha\),
\[
\int_{r}^{\infty} e^{2\beta x} 36T |\partial_x T|^2 \, dx \leq \frac{36}{2(\alpha - \beta)\sqrt{2\alpha}} \gamma C_b e^{-2(\alpha - \beta)r}.
\] (5.11)

On the other hand, due to (3.16), we get
\[
|T(x) - T_b| = \left| \int_{0}^{x} \partial_z T \, dz \right| = \left| \int_{0}^{x} e^{-\alpha z} e^{\alpha z} \partial_z T \, dz \right|
\]
\[
\int_0^x e^{-2\alpha z} \, dz \leq \left( \int_0^x e^{2\alpha z} \, dz \right)^{\frac{1}{2}} \left( \int_0^x |\partial_z T|^2 \, dz \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2\alpha}} \sqrt{1 - e^{-2\alpha x}} \sqrt{\frac{2}{3(1 - \alpha)}} C_b \leq \sqrt{\frac{1 - e^{-2\alpha x}}{3\alpha(1 - \alpha)}} C_b,
\]

and so

\[ T(x) \geq T_b - \sqrt{\frac{1 - e^{-2\alpha x}}{3\alpha(1 - \alpha)}} C_b, \quad \text{for any } 0 \leq x \leq r, \]

if \( C_b \) is sufficiently small. For example, we can take \( C_b = \frac{1}{2} T_b \sqrt{3\alpha(1 - \alpha)} \) and the above inequality implies

\[ T(x) \geq \frac{1}{2} T_b. \]  

(5.12)

Hence

\[ \int_0^r \frac{1}{e^{2\beta x} 4T^3} \, dx \leq \frac{1}{\beta T_b^3}. \]  

(5.13)

Combining this inequality with (5.11) gives

\[ \int_r^{\infty} e^{2\beta x} 36T |\partial_x T|^2 \, dx \cdot \int_0^r \frac{1}{e^{2\beta x} 4T^3} \, dx \leq \frac{36}{2(\alpha - \beta) \sqrt{2\alpha}} \gamma C_b \frac{1}{\beta T_b^3} e^{-2(\alpha - \beta)r}. \]

Since \( e^{-2(\alpha - \beta)r} \leq 1 \), inequality (5.1) is realized if

\[ \frac{36}{2(\alpha - \beta) \sqrt{2\alpha}} \gamma C_b \frac{1}{\beta T_b^3} < \frac{1}{4}. \]

Therefore, inequality (5.1) is fulfilled if we take

\[ C_b = \min \left\{ \frac{1}{2} T_b \sqrt{3\alpha(1 - \alpha)}, \frac{1}{72} \gamma^{-1} T_b^3 \beta (\alpha - \beta) \sqrt{\alpha} \right\} \]  

(5.14)

in (1.14). By Lemma 4, the spectral assumption (A) also holds. \( \square \)

With this lemma, we can prove Theorem 3 for the case when boundary data is almost well-prepared.

**Proof of Theorem 3, near well-prepared boundary data.** Let \((T, \psi)\) and \((T_1, \psi_1)\) be two solutions to system (1.1)–(1.2) with boundary conditions (1.3)–(1.4) satisfying the assumption of Theorem 3. Then \( h := T_1 - T, \phi = \psi_1 - \psi \) satisfies

\[ \partial^2_x h = -\langle \phi - wh \rangle, \]  

(5.15)

\[ \mu \partial_x \phi = - (\phi - wh), \]  

(5.16)
with boundary conditions

\[ h(0) = 0, \quad \phi(0, \mu) = 0, \quad \text{for } \mu > 0, \]

where \( w = (T_1^4 - T^4)/(T_1 - T) \). By Lemma 5, \( T \) and \( T_1 \) both satisfy the spectral assumption (A) and thus inequality (4.7) is satisfied for both \( T \) and \( T_1 \). Moreover, one can show in the same way that \( T_1^2 T \) (or \( T_1^2 T_1 \)) also satisfies the inequality

\[
\int_0^\infty e^{2\beta x} 4T^2 T_1 |\partial_x f|^2 \, dx \geq \int_0^\infty e^{2\beta x} \frac{1}{4T^2 T_1} |\partial_x (4T^2 T_1)|^2 f^2 \, dx \quad (5.17)
\]

which holds if

\[
\int_0^\infty e^{2\beta x} \left( 32T_1 |\partial_x T|^2 + 8 \frac{T_1^2}{T_1} |\partial_x T_1|^2 \right) f^2 \, dx \leq \int_0^\infty e^{2\beta x} 4T^2 T_1 |\partial_x T_1|^2 \, dx.
\]

According to Hardy’s inequality (D.1), a sufficient condition for the above inequality to hold is (see Lemma 4)

\[
\sup_{r \in (0, \infty)} \left( \int_r^\infty e^{2\beta x} \left( 32T_1 |\partial_x T|^2 + 8 \frac{T_1^2}{T_1} |\partial_x T_1|^2 \right) \, dx \cdot \int_0^r e^{2\beta x} \frac{1}{4T_1^2} \, dx \right) \leq \frac{1}{4}. \quad (5.18)
\]

Taking \( C_b = \frac{1}{2} T_b \sqrt{3\alpha(1 - \alpha)} \), (5.12) implies \( T(x), T_1(x) \geq \frac{1}{2} T_b \). Due to the fact that \( T, T_1 \leq \gamma \), we get

\[
\int_r^\infty e^{2\beta x} \left( 32T_1 |\partial_x T|^2 + 8 \frac{T_1^2}{T_1} |\partial_x T_1|^2 \right) \, dx \leq \frac{32}{2(\alpha - \beta) \sqrt{2\alpha}} C_b e^{-2(\alpha - \beta) r} + \frac{8}{(\alpha - \beta) \sqrt{2\alpha} T_b} \gamma^2 C_b e^{-2(\alpha - \beta) r}.
\]

On the other hand, \( T(x), T_1(x) \geq \frac{1}{2} T_b \), on \( (0, \infty) \) implies

\[
\int_0^r \frac{1}{4T_1^2} e^{2\beta x} \, dx \leq \frac{1}{\beta T_b^3}.
\]

Hence (5.13) is also realized. Then

\[
\int_r^\infty e^{2\beta x} \left( 32T_1 |\partial_x T|^2 + 8 \frac{T_1^2}{T_1} |\partial_x T_1|^2 \right) \, dx \cdot \int_0^r \frac{1}{4T_1^2} e^{2\beta x} \, dx \leq \left( \frac{32}{2(\alpha - \beta) \sqrt{2\alpha}} \gamma + \frac{8}{(\alpha - \beta) \sqrt{2\alpha} T_b} \gamma^2 \right) \frac{C_b}{\beta T_b^3} e^{-2(\alpha - \beta) r}.
\]

Hence (5.18) is fulfilled if
Moreover, by the proof of Lemma 4, the solution to system (1.1)–(1.2) is unique. Consequently, we have expected to the smaller.

Let \( \psi(0) \) be given by (5.1). Let \( V \) defined in (5.1)

Then (5.17) holds and \( w = (T_1^4 - T^4)/(T_1 - T) \) satisfies the spectral assumption (A).

Moreover, by the proof of Lemma 4, \( M \leq 4 \left( \frac{32}{2(\alpha-\beta)\sqrt{2\alpha}} \gamma + \frac{8}{(\alpha-\beta)\sqrt{2\alpha}T_b} \gamma^2 \right) \frac{C_b}{\beta T_b} \).

With a smaller \( C_b \), the constant \( M \) in (1.11) of the spectral assumption (A) is expected to the smaller.

Therefore, by Theorem 2, the solution to the linear problem (5.15)–(5.16) is unique. Consequently, we have \( h = 0, \psi = 0 \), which is \( T_1 = T, \psi_1 = \psi \), that is the solution to system (1.1)–(1.2) is unique. \( \square \)

5.3. Uniqueness under Small Perturbations

First we show that the spectral assumption holds under small perturbations.

**Lemma 6.** Assume \( T \) satisfies the inequality (5.1). Let \( V_e(T) \) be the function space defined in (1.13). Then there exists a small constant \( \varepsilon > 0 \) such that the inequality (5.1) holds for any \( T_1 \in V_e(T) \). As a consequence, the spectral assumption (A) is realized for any \( T_1 \in V_e(T) \).

**Proof.** Let \( A_0 \) be given by (5.1). Let \( h = (T_1 - T)/\varepsilon \). We then calculate

\[
\int_r^\infty e^{2\beta x} 36T_1 |\partial_x T_1|^2 \, dx \cdot \int_0^r \frac{1}{4T_1^3 e^{2\beta x}} \, dx
\]

\[
= \int_r^\infty e^{2\beta x} 36(T + \varepsilon h)|\partial_x T + \varepsilon \partial_x h|^2 \, dx \cdot \int_0^r \frac{1}{4(T + \varepsilon h)^3 e^{2\beta x}} \, dx
\]

\[
= \int_r^\infty e^{2\beta x} 36T |\partial_x T|^2 \, dx \cdot \int_0^r \frac{1}{4T^3 e^{2\beta x}} \, dx
\]

\[
\quad + \varepsilon \int_r^\infty e^{2\beta x} 36h |\partial_x T + \varepsilon \partial_x h|^2 \, dx \cdot \int_0^r \frac{1}{4(T + \varepsilon h)^3 e^{2\beta x}} \, dx
\]

\[
\quad + \varepsilon \int_r^\infty e^{2\beta x} 36T |\partial_x T|^2 \, dx \cdot \int_0^r \frac{-3T^2 h - 3\varepsilon Th^2 - \varepsilon^2 h^3}{4(T + \varepsilon h)^3 T^3 e^{2\beta x}} \, dx
\]

\[
\quad + \varepsilon \int_r^\infty e^{2\beta x} 36(2\partial_x T \partial_x h + \varepsilon |\partial_x h|^2) \, dx \cdot \int_0^r \frac{1}{4(T + \varepsilon h)^3 e^{2\beta x}} \, dx.
\]

Consequently,

\[
(A')^2 := \sup_{r \in (0, \infty)} \left( \int_r^\infty e^{2\beta x} 36(T + \varepsilon h)|\partial_x T + \varepsilon \partial_x h|^2 \, dx \cdot \int_0^r \frac{1}{4(T + \varepsilon h)^3 e^{2\beta x}} \, dx \right)
\]

\[
\leq A_0^2 + C_1 \varepsilon.
\]
for some constant $C_1 > 0$. By Hardy’s inequality, we have

$$C_{hd}' \int_0^\infty e^{2\beta x} 4(T + \varepsilon h)^3 |\partial_x f|^2 \, dx \geq \int_0^\infty e^{2\beta x} 36(T + \varepsilon h)|\partial_x(T + \varepsilon h)|^2 f^2 \, dx$$

where $C_{hd}' \leq 4(A')^2 \leq 4A_0^2 + 4C_1\varepsilon$. Therefore if $A_0^2 < 1/4$, then $(A')^2 < 1/4$ for $\varepsilon$ sufficiently small, and the inequality 5.1 holds for $T + \varepsilon h$ if $T$ satisfies (5.1) with $A_0 < 1/2$. Therefore, we conclude that the inequality (5.1) holds for any $T_1 \in C^1(\mathbb{R}_+)$ such that $\|T_1 - T\|_{H^1(\mathbb{R}_+)} \leq C_2\varepsilon$ for some constant $C_2 > 0$, where $T$ satisfies (5.1) with $A_0 < 1/2$.

Moreover, for a smaller $A_0$, $\varepsilon$ could be larger in order for $4A_0^2 + 4C_1\varepsilon < 1$ to hold. A smaller $A_0$ implies a smaller $M$ in (1.11). Therefore, it is expected that a smaller $M$ in the spectral assumption is more stable with respect to perturbations.

We can also show when $\varepsilon$ is large, the spectral assumption does not hold. Assume $\varepsilon$ is large, then the dominant term is

$$\int_r^\infty e^{2\beta x} 36h|\partial_x h|^2 \, dx \cdot \int_0^r \frac{1}{4h^3e^{2\beta x}} \, dx,$$

which can become large for $h \in L^2_{\text{loc}}(\mathbb{R}_+)$. For example, we may take $\beta = 0$, $h(x) = xe^{-x}$, the integral $\int_r^\infty 36h|\partial_x h|^2 \, dx$ is finite positive while $\int_0^r \frac{1}{4x^3} \, dx = \int_0^r \frac{3x}{4x^3} \, dx$ diverges. Therefore, when $\varepsilon \to \infty$, $(A')^2 \to \infty$ and $C_{hd}' \geq (A')^2$ also goes to $\infty$. Hence Lemma 6 only holds for $\varepsilon$ sufficiently small. □

We proceed to prove Theorem 3.

Proof of Theorem 3: The case of small perturbations. Suppose $T$, $T_1$ are two functions satisfying the assumptions of Theorem 3 and $T_1 \in \mathcal{V}_\varepsilon(T)$ with $\mathcal{V}_\varepsilon(T)$ given in (1.13). Setting $h = (T_1 - T)/\varepsilon$, then $w := (T_1 - T^4)/(T_1 - T)$ is

$$w = \frac{(T + \varepsilon h)^4 - T^4}{\varepsilon h} = 4T^3 + 6\varepsilon T^2 h + 4\varepsilon^2 h^2 T + \varepsilon^3 h^3$$

is a small perturbation of $4T^3$, thus by the proof of Lemma 6 and Lemma 2, (4.7) still holds with $4T^3$ replaced by $w$. Thus by Theorem 2, $T_1 = T$ and the solution is unique in $\mathcal{V}_\varepsilon(T)$. This finishes the proof of Theorem 3. □

Acknowledgements. The work of N.M. is supported by NSF Grant DMS-1716466 and by Tamkeen under the NYU Abu Dhabi Research Institute grant of the center SITE. The work of M.G. is supported by Tamkeen under the NYU Abu Dhabi Research Institute grant of the center SITE.

Declarations

Conflict of interest The authors declare no potential conflict of interest.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other right-
Appendix A: Proof of Theorem 2 for the General Case

Consider

\[ \partial^2_x g = -\langle \phi - 4T^3g \rangle + \langle S_1 \rangle, \quad (A.1) \]
\[ \mu \partial_x \phi = -(\phi - 4T^3g) + S_2, \quad (A.2) \]

supplemented with the following boundary conditions

\[ g(0) = g_b, \quad (A.3) \]
\[ \phi(0, \mu) = \phi_b(\mu), \text{ for any } \mu \in (0, 1]. \quad (A.4) \]

Then the following theorem holds:

**Theorem 5.** Suppose $T$ satisfies the spectral assumption (A) for some constants $0 < \beta < 1$ and $M > 0$. Assume $S_1 = S_1(x, \beta)$, $S_2 = S_2(x, \beta)$ decays to zero exponentially such that $\|e^{\beta x}S_1\|_{L^2(\mathbb{R}_+ \times [-1, 1])}$, $\|e^{\beta x}S_1\|_{L^2(\mathbb{R}_+ \times [-1, 1])} < \infty$. Then there exists a unique bounded solution $(g, \phi) \in L^2_{\text{loc}}(\mathbb{R}_+) \times L^2_{\text{loc}}(\mathbb{R}_+ \times [-1, 1])$ to system (A.1)–(A.2) with boundary conditions (A.3)–(A.4), and the solution satisfies

\[ |g(x) - g_\infty| \leq Ce^{-\beta x}, \quad |\phi(x, \mu) - 4T^3g_\infty| \leq Ce^{-\beta x}, \]

for some constant $C > 0$ and $g_\infty \in \mathbb{R}$, where $T_\infty = \lim_{x \to \infty} T(x)$.

**Proof.** First we can take $h = g - g_b e^{-x}$ and system (A.1)–(A.2) becomes

\[ \partial^2_x h + \langle \phi - 4T^3g \rangle = \langle S_1 \rangle + \langle -\frac{1}{2}g_b e^{-x} + 4T^3g_b e^{-x} \rangle =: \langle \tilde{S}_1 \rangle, \]
\[ \mu \partial_x \phi + \phi - 4T^3g = S_2 + 4T^3g_b e^{-x} := \tilde{S}_2, \]

with boundary conditions

\[ g(0) = 0, \quad \phi(0, \mu) = \phi_b(\mu), \text{ for any } \mu \in (0, 1]. \]

In order to solve the above problem, we first construct $(g^1, \phi^1)$ by solving the following system

\[ \partial^2_x g^1 + \langle \phi^1 - 4T^3g^1 \rangle = \langle \tilde{S}_1 \rangle, \]
\[ \mu \partial_x \phi^1 + \phi^1 - 4T^3g^1 = \tilde{S}_2 - \frac{1}{2}\langle \tilde{S}_2 - \tilde{S}_1 \rangle, \quad (A.5) \]

with boundary conditions

\[ g^1(0) = 0, \quad \phi^1(0, \mu) = \phi_b(\mu), \text{ for any } \mu \in (0, 1]. \]

Next let $\varphi = \mu G$ with $G$ being the solution of

\[ G(x) = -\frac{3}{2} \int_x^\infty \langle \tilde{S}_2 - \tilde{S}_1 \rangle \, dz. \]
Then $\varphi$ satisfies

$$
\langle \mu \partial_x \varphi + \mu \varphi \rangle = \langle \mu^2 \partial_x G \rangle = \frac{3}{2} \partial_x G = \langle \tilde{S}_2 - \tilde{S}_1 \rangle.
$$

Next we construct $(g^2, \phi^2)$ by solving the system

$$
\begin{align*}
\partial^2_x g^2 + \langle \phi^2 - 4T^3 g^2 \rangle &= 0, \\
\mu \partial_x \phi^2 + \phi^2 - 4T^3 g^2 &= -(\mu \partial_x \varphi + \mu \varphi) + \frac{1}{2} \langle \tilde{S}_2 - \tilde{S}_1 \rangle,
\end{align*}
$$

(A.6)

with boundary conditions

$$
g(0) = 0, \quad \varphi(0, \mu) = -\mu G(0), \quad \text{for any } \mu \in (0, 1].
$$

Then one can show directly that $(g = h_b e^{-x} + g^1 + g^2, \phi = \mu \varphi + \phi^1 + \phi^2)$ satisfies system (A.1)–(A.2) with boundary conditions (A.3)–(A.4).

We can also obtain the decay estimate using Theorem 2. First, since $\langle \tilde{S}_1 \rangle = \langle \tilde{S}_2 - \frac{1}{2} (\tilde{S}_2 - \tilde{S}_1) \rangle$, we can apply Theorem 2 to get the existence and uniqueness of solutions to system (A.5). Moreover, we can get from estimate (1.12)

$$
|g^1(x) - s^1_\infty| \leq N^1_\beta e^{-\beta x},
$$

$$
|\phi^1(x, \mu) - 4T^3 g^1_\infty| \leq C_1(\beta, \psi_b, T_b, M_\alpha, N^1_\beta) e^{-\beta x}, \quad \text{for any } \mu \in [-1, 1],
$$

with

$$
N^1_\beta = \frac{1}{\sqrt{2\beta}} \left( \frac{2}{3(1-\beta)} \int_0^1 \mu \phi^2_b d\mu + \frac{2}{3(1-\beta)^2} \| e^{\beta x} (\tilde{S}_2) \right. \\
- \left. \frac{1}{2} \langle \tilde{S}_2 - \tilde{S}_1 \rangle \|_{L^2(\mathbb{R}_+ \times [-1, 1])} \right)^{\frac{1}{2}}.
$$

Moreover,

$$
|g^1(x)| \leq 2N^1_\beta, \quad |\phi^1(x, \mu)| \leq C(\beta, \psi_b, T_b, M_\alpha, N^1_\beta).
$$

Similarly, we can also get the existence and uniqueness of (A.6) since

$$
\langle -(\mu \partial_x \varphi + \mu \varphi) + \frac{1}{2} \langle \tilde{S}_2 - \tilde{S}_1 \rangle \rangle = 0.
$$

Moreover, we can also get the estimate

$$
|g^2(x) - g^2_\infty| \leq N^2_\beta e^{-\beta x},
$$

$$
|\phi^2(x, \mu) - 4T^3 g^2_\infty| \leq C_1(\beta, \psi_b, T_b, M_\alpha, N^2_\beta) e^{-\beta x}, \quad \text{for any } \mu \in [-1, 1],
$$

with
\[ N_\beta^2 = \frac{1}{\sqrt{2\beta}} \left( \frac{2}{3(1-\beta)} \int_0^1 \mu G^2(0) \, d\mu + \frac{2}{3(1-\beta)^2} \| e^{\beta x} ((-\mu \partial_x \varphi + \mu \varphi) + \frac{1}{2} (\tilde{S}_2 - \tilde{S}_1)) \|_{L^2(\mathbb{R} \times [-1,1])}^2 \right)^{\frac{1}{2}}. \]

We can combine the above estimates and obtain

\[ |g(x) - g_\infty| \leq Ce^{-\beta x}, \quad |\phi(x, \mu) - 4T_\infty^3 g_\infty| \leq Ce^{-\beta x}, \]

with \( C \) being a constant depending on \( g_b, \phi_b, \beta, \psi_b, T_b, M_\alpha \). \( \square \)

**Appendix B: Monotonicity of Linear Transport Equation**

**Lemma 7.** Given \( \psi_b : (0, 1) \mapsto \mathbb{R} \), satisfying \( 0 \leq \psi_b \leq \gamma \) for some constant \( \gamma > 0 \) for any \( \mu \in (0, 1) \). Let \( h \in C([0, B]) \) satisfy \( 0 \leq h(x) \leq \gamma \) for any \( x \in [0, B] \). Then there exists a unique solution \( \psi \in C^1([0, B] \times [-1, 1]) \) to the equation

\[
\begin{align*}
\mu \partial_x \psi + \psi &= h, \quad x \in (0, B), \quad \mu \in [-1, 1], \\
\psi(0, \mu) &= \psi_b(\mu), \quad \psi(B, \mu) = \psi(B, -\mu), \quad \text{for any} \ \mu \in (0, 1),
\end{align*}
\]

and the solution verifies \( 0 \leq \psi(x, \mu) \leq \gamma \) for \( x \in [0, B], \ \mu \in [-1, 1] \).

Furthermore, let \( \psi_1, \psi_2 \) be two solutions corresponding to the source data \( h_1, h_2 \) and boundary data \( \psi_{b1}, \psi_{b2} \), respectively. If \( 0 \leq h_1(x) \leq h_2(x) \leq \gamma \) for any \( x \in [0, B] \) and \( 0 \leq \psi_{b1}(\mu) \leq \psi_{b2}(\mu) \leq \gamma \) for \( \mu \in [0, 1] \), then \( 0 \leq \psi_1(x, \mu) \leq \psi_2(x, \mu) \leq \gamma \) for any \( x \in [0, B] \) and \( \mu \in [-1, 1] \).

**Proof.** The existence and uniqueness of (B.1) can be shown by the method of characteristic lines. Actually, we can compute the solution to (B.1) directly. For \( \mu > 0 \), we integrate from \([0, x]\) and get

\[
\psi(x, \mu) = \psi_b(\mu) e^{-\frac{x}{\mu}} + \int_0^x \frac{1}{\mu} h(s) e^{-\frac{x-s}{\mu}} \, ds. \tag{B.3}
\]

For \( \mu < 0 \) we integrate (B.1) over \([x, B]\) and use the boundary condition \( \psi(B, \mu) = \psi(B, -\mu) \) and get

\[
\psi(x, \mu) = \psi_b(-\mu) e^{\frac{2B-x}{\mu}} - \int_0^B \frac{1}{\mu} e^{\frac{2B-x-s}{\mu}} h(s) \, ds - \int_x^B \frac{1}{\mu} e^{\frac{x-s}{\mu}} h(s) \, ds, \tag{B.4}
\]

The boundness of \( \psi \) can be obtained directly by for \( \mu > 0 \) from (B.4),

\[
\psi(x, \mu) \leq \gamma e^{-\frac{x}{\mu}} + \int_0^x \frac{\gamma}{\mu} e^{-\frac{x-s}{\mu}} \, ds = \gamma,
\]

and for \( \mu < 0 \) from (B.4),

\[
\psi(x, \mu) \leq \gamma e^{\frac{2B-x}{\mu}} - \int_0^B \frac{\gamma}{\mu} e^{\frac{2B-x-s}{\mu}} \, ds - \int_x^B \frac{\gamma}{\mu} e^{\frac{x-s}{\mu}} \, ds = \gamma.
\]

The monotonic property follows directly from the expressions (B.4) and (B.4). We can see from these expressions that \( h_1(x) \leq h_2(x) \) for all \( x \in \mathbb{R}_+ \) implies \( \psi_1(x) \leq \psi_2(x) \) for all \( x \in \mathbb{R}_+ \). \( \square \)
Appendix C: Formula for the Transport Equation

Suppose \( h \in L^\infty(\mathbb{R}_+) \), the solution to the linear transport equation

\[
\mu \partial_x \psi + \psi = h, \quad \text{for } x \in [0, \infty], \; \mu \in [-1, 1],
\]

\[
\psi(0, \mu) = \psi_b, \quad \text{for } \mu \in (0, 1]
\]
is given by

\[
\psi(x, \mu) = \psi_b(\mu) e^{-\frac{x}{\mu}} + \int_0^x \frac{1}{\mu} h(s) e^{-\frac{x-s}{\mu}} \, ds. \tag{C.1}
\]

for \( \mu > 0 \) and

\[
\psi(x, \mu) = -\int_x^\infty \frac{1}{\mu} e^{-\frac{x-s}{\mu}} h(s) \, ds, \quad \text{for } \mu < 0. \tag{C.2}
\]

The formula (C.2) is the same as (B.4) and can be shown the same way. The formula (C.2) is obtained by sending \( B \to \infty \) in (B.4). The condition \( h \in L^\infty \) is used, since, for example,

\[
0 \leq \int_0^B \frac{1}{\mu} e^{-\frac{2B-x-s}{\mu}} h(s) \, ds = e^{-\frac{2B-\mu}{\mu}} \frac{1}{\mu} \int_0^B e^{-\frac{s}{\mu}} h(s) \, ds \leq e^{-\frac{2B-\mu}{\mu}} \| h \|_{L^\infty([0, \infty))} (1 - e^{-\frac{B}{\mu}})
\]

By passing to the limit \( B \to \infty \) the above inequality becomes

\[
0 \leq \lim_{B \to \infty} \int_0^B \frac{1}{\mu} e^{-\frac{2B-x-s}{\mu}} h(s) \, ds \leq 0,
\]

and hence \( \lim_{B \to \infty} \int_0^B \frac{1}{\mu} e^{-\frac{2B-x-s}{\mu}} h(s) \, ds = 0. \)

Appendix D: Generalized Hardy’s Inequality

First we recall the following generalized Hardy’s inequality:

**Lemma 8.** (Generalized Hardy’s inequality [23]) Let \( p > 1 \) and \( 0 < b \leq \infty \). The inequality

\[
\int_0^b \left( \int_0^x F(t) \, dt \right)^p u(x) \, dx \leq C_{hd} \int_0^b F(x)^p v(x) \, dx \tag{D.1}
\]

holds for all measurable function \( F(x) \geq 0 \) on \((0, b)\) if and only if

\[
A = \sup_{r \in (0, b)} \left( \int_0^b u(x) \, dx \right)^{\frac{1}{p}} \left( \int_0^r (v(x))^{1-p'} \, dx \right)^{\frac{1}{p'}} < \infty,
\]

and the best constant of \( C_{hd} \) satisfies \( A \leq C_{hd}^{1/p} \leq p^{1/p} (p')^{1/p'} A \) where \( \frac{1}{p'} + \frac{1}{p} = 1. \)
Appendix E: Leray–Schauder Theorem

**Theorem 6.** [14, Theorem 11.3] Let $\mathcal{N}$ be a compact mapping of a Banach space $D$ into itself, and suppose there exists a constant $K$ such that

$$\|x\|_D < K$$

for all $x \in D$ and $\sigma \in [0, 1]$ satisfying $x = \sigma \mathcal{N}x$. Then $\mathcal{N}$ has a fixed point.

**References**

1. **Aoki, K., Bardos, C., Takata, S.:** Knudsen layer for gas mixtures. *J. Stat. Phys.* **112**, 629–655, 2003
2. **Bardos, C., Caflisch, R.E., Nicolaenko, B.:** The Milne and Kramers problems for the Boltzmann equation of a hard sphere gas. *Commun. Pure Appl. Math.* **39**, 323–352, 1986
3. **Bardos, C., Golse, F., Sone, Y.:** Half-space problems for the Boltzmann equation: a survey. *J. Stat. Phys.* **124**, 275–300, 2006
4. **Bardos, C., Santos, R., Sentis, R.:** Diffusion approximation and computation of the critical size. *Trans. Am. Math. Soc.* **284**, 617–649, 1984
5. **Bensoussan, A., Lions, J.-L., Papanicolaou, G.C.:** Boundary layers and homogenization of transport processes. *Publ. Res. Inst. Math. Sci.* **15**, 53–157, 1979
6. **Bernhoff, N., Golse, F.:** On the boundary layer equations with phase transition in the kinetic theory of gases. *Arch. Ration. Mech. Anal.* **240**, 51–98, 2021
7. **Bostan, M., Gamba, I.M., Goudon, T., Vasseur, A.:** Boundary value problems for the stationary Vlasov–Boltzmann–Poisson equation. *Indiana Univ. Math. J.* **2010**, 1629–1660, 2010
8. **Brezis, H.:** *Functional Analysis. Sobolev Spaces and Partial Differential Equations*. Springer, Berlin, 2010
9. **Clouët, J., Sentis, R.:** Milne problem for non-grey radiative transfer. *Kinetic Relat. Models* **2**, 345, 2009
10. **Farkas, B., Wegner, S.-A.:** Variations on Barbálat’s lemma. *Am. Math. Mon.* **123**, 825–830, 2016
11. **Ghattassi, M., Huo, X., Masmoudi, N.:** Diffusive limits of the steady state radiative heat transfer system: curvature effects, 2022
12. **Ghattassi, M., Huo, X., Masmoudi, N.:** On the diffusive limits of radiative heat transfer system i: Well-prepared initial and boundary conditions. *SIAM J. Math. Anal.* **54**, 5335–5387, 2022
13. **Ghattassi, M., Huo, X., Masmoudi, N.:** Diffusive limits of the steady state radiative heat transfer system: boundary layers. *J. Math. Pures Appl.* **175**, 181–215, 2023
14. **Gilbarg, D., Trudinger, N.S.:** *Elliptic Partial Differential Equations of Second Order*, vol. 224. Springer, Berlin, 2015
15. **Golse, F., Perthame, B., Sulem, C.:** On a boundary layer problem for the nonlinear Boltzmann equation. *Arch. Ration. Mech. Anal.* **103**, 81–96, 1988
16. **Golse, F., Pironneau, O.:** Radiative transfer in a fluid. *Revista de la Real Academia de Ciencias Exactas. Físicas y Naturales. Serie A. Matemáticas* **117**, 37, 2023
17. **Huang, F., Wang, Y.:** Boundary layer solution of the Boltzmann equation for diffusive reflection boundary conditions in half-space. *SIAM J. Math. Anal.* **54**, 3480–3534, 2022
18. **Kato, T.:** *Perturbation Theory for Linear Operators*, vol. 132. Springer, Berlin, 2013
19. **Kelley, C.T.:** Existence and uniqueness of solutions of nonlinear systems of conductive-radiative heat transfer equations. *Transp. Theory Stat. Phys.* **25**, 249–260, 1996
20. **Klar, A., Schmeiser, C.:** Numerical passage from radiative heat transfer to nonlinear diffusion models. *Math. Models Methods Appl. Sci.* **11**, 749–767, 2001
21. Larsen, E., Pomraning, G., Badham, V.: Asymptotic analysis of radiative transfer problems. *J. Quant. Spectrosc. Radiat. Transf.* **29**, 285–310, 1983

22. Liu, T.-P., Yu, S.-H.: Invariant manifolds for steady Boltzmann flows and applications. *Arch. Ration. Mech. Anal.* **209**, 869–997, 2013

23. Masmoudi, N.: About the hardy inequality. *An Invitation to Mathematics*. Springer, pp. 165–180, 2011

24. Mercier, B.: Application of accretive operators theory to the radiative transfer equations. *SIAM J. Math. Anal.* **18**, 393–408, 1987

25. Sentis, R.: Half space problems for frequency dependent transport equations. Application to the Rosseland approximation of the radiative transfer equations. *Transp. Theory Stat. Phys.* **16**, 653–697, 1987

26. Ukai, S., Yang, T., Yu, S.-H.: Nonlinear boundary layers of the Boltzmann equation: I. Existence. *Commun. Math. Phys.* **236**, 373–393, 2003

27. Wu, L., Guo, Y.: Geometric correction for diffusive expansion of steady neutron transport equation. *Commun. Math. Phys.* **336**, 1473–1553, 2015

Mohamed Ghattassi, Nader Masmoudi
Department of Mathematics,
New York University in Abu Dhabi, Saadiyat Island,
P.O. Box 129188, Abu Dhabi
United Arab Emirates.
e-mail: mg6888@nyu.edu

and

Mohamed Ghattassi
Department of Mathematics, Faculty of Science of Gabès,
University of Gabès,
Gabès
Tunisia.

and

Xiaokai Huo
Department of Mathematics,
Iowa State University,
Ames
IA
50011 USA.
e-mail: xhuo@iastate.edu

and

Nader Masmoudi
Courant Institute of Mathematical Sciences,
New York University,
251 Mercer Street,
New York
NY
10012 USA.
e-mail: nm30@nyu.edu

(Received July 22, 2022 / Accepted July 25, 2023)
Published online September 27, 2023
© The Author(s), under exclusive licence to Springer-Verlag GmbH, DE, part of Springer Nature (2023)