Robustness of Uncertain Switching Nonlinear Feedback Systems against Large Time-Variation

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Abstract—For a non-linear MIMO feedback system, the robustness against uncertain time-variations in the feedback loop is investigated in an input-output framework. A general sufficient condition in terms of a bound on average rates of time-variation for the system to be stable is derived. The condition gives a tolerable limit on infrequent large variations or slow time-variation rate of a non-linear MIMO adaptive switching system.

I. INTRODUCTION

Typically, a feedback system having a time-varying non-linear loop function is equivalent to a feedback system with its loop function that switches among a family of time-invariant functions. These time-invariant functions are referred as frozen-time loop functions where each of them represents the loop function frozen at one certain time instant during the switching sequence. If the distances between all the frozen-time loop functions and the nominal loop function model, if exists, are bounded by a constant, then one can use classical small-gain theorem [1] to determine the stability of the feedback system. However, in a more general case where the loop function is persistently time-varying and such a nominal loop function model does not exist, small-gain theorem does not conclude the stability of the feedback system. On the other hand, sufficient conditions for preserving stability of time-varying feedback systems in such general cases have been derived in the past in terms of maximum differences between consecutive frozen-time loop functions with various assumptions. Desoer [2] considered the discrete-time case where frozen-time functions are Hurwitz matrices which are linear and memoryless. In [3], Solo relaxed one of Desoer’s assumptions for continuous-time cases where frozen-time functions are matrices which are not all Hurwitz. Zames and Wang [4] improved Desoer’s result by considering the slightly more general linear case in which frozen-time loop functions are assumed to be bounded, exponentially stabilizing, and time-invariant convolution operators.

The results in [2], [3], and [4] are relevant for stability analysis of adaptive control when plants have large time-variations. An adaptive control logic tries to preserve stability and meet additional performance specifications simultaneously. In this context, a multiple model adaptive control [5], [6] is developed to return a model close to the perturbed plant and corresponding designed controller. A system with Hysteresis Logic (HL) that switches controllers based on their real-time data-driven performance evaluations is developed by Morse, Mayne, and Goodwin in [7], while a model-based switched system with HL that takes into account plant uncertainties or noise is investigated in [8], [9], and [10]. A HL based switched system with an additional reset feature which safely discards past evaluated controller performances is developed by Battistelli, Hesperana, Mosca, and Tesi in [11]. The results of [11] are generalized in [12] by (i) considering non-linear plants and controllers and (ii) adding a bumpless switching feature. A switching system that considers real-time and data-driven controller performance based on loop-shape specifications is developed in [13].

In a feedback system having an adaptive switching controller and a time-varying plant to be compensated, switching among controllers essentially leads to switching among loop functions. Therefore, the results in [2], [3], and [4] can be used to analyze stability of adaptive switching systems. However, to investigate the impact of plant variations on stability of an adaptive switching control system, it is impractical to assume that either the adaptive loop function is linear or all the frozen-time loop functions are stabilizing. Adaptive loop functions are inherently non-linear. For example, in the HL switching algorithm [7], the non-linear maximum operator causes adaptive loop functions to be non-linear. Also, switching algorithms may momentarily insert a destabilizing controller in the loop causing the resultant frozen-time loop function to be destabilizing [14]. Moreover, the results in [2] and [4] are compatible to slowly time-varying loop functions as they consider maximum difference between consecutive frozen-time loop functions, but they are not compatible to the adaptive loop functions having infrequent and large time-variations. The recent works [15] and [16] too investigated stability of feedback systems with time-varying loop functions considering at least one of the assumptions on frozen-time loop functions that they are (i) stabilizing all the time, (ii) linear, and (iii) slightly different from adjacent frozen-time loop functions.

Therefore, we aim to solve a problem of determining under what condition in terms of average time-variation rate...
of time-varying non-linear loop function, stability of non-linear feedback system can be preserved without the three aforementioned assumptions on frozen-time loop functions.

The organization of the present paper is as follows. The preliminary facts are given in Section II. The problem formulation is described in Section III, followed by the main results in Section IV. A comparison of the results in the present paper with those in [4] is discussed in Section V, followed by an application for adaptive control in Section VI. Two simulation examples are presented in Section VII, followed by conclusions in Section VIII.

II. PRELIMINARIES

In the present paper, we consider discrete-time signals and systems. The sets of integers and real numbers are denoted by \( \mathbb{Z} \) and \( \mathbb{R} \) respectively. The transpose and the Euclidean norm are denoted by \( [\cdot]' \) and \( [\cdot] \) respectively. We denote the spectral radius of a matrix \( A \) as \( \lambda_{max}(A) \).

**Definition 1:** (Signal): A real-valued function \( x(t) \) of time \( t \) is said to be a signal mapping \( t \in \mathbb{Z} \) to \( x \in \mathbb{R}^n \), where \( n \in \mathbb{Z}_+ \setminus \{0\} \).

**Definition 2:** (Signal Norm): Given \( \sigma \geq 1 \) and \( p \in [1, \infty] \), for any signal \( x \) and for all \( t_1, t_2 \in \mathbb{Z} \), where \( t_1 < t_2 \), the moving-window fading-memory \( \ell_{\sigma p} \)-semi norm \([11, [17],[18]]\) is defined as

\[
\|x\|_{\sigma p, [t_1, t_2]} = \left\{ \sum_{\tau = t_1}^{t_2} \sigma^{-p(t_2-\tau)}|x(\tau)|^p \right\}^{\frac{1}{p}}, \quad \text{if } p \in [1, \infty),
\]

\[
\sup_{\tau \in [t_1, t_2]} \sigma^{-(t_2-\tau)}|x(\tau)|, \quad \text{if } p = \infty,
\]

where \( [\cdot] \) denotes the Euclidean norm. For brevity, the notation \( \|x\|_{\sigma p, [t_1, t_2]} \) is simplified as (i) \( \|x\|_{\sigma p, t} \) if \( t_1 = -\infty \) and \( t_2 = t \) and (ii) \( \|x\|_{\sigma p, [t_1, t_2]} \) if \( \sigma = 1 \). The extended space \( \ell_{\sigma p}^0 \) is defined as \( \ell_{\sigma p}^0 = \{ \alpha: \alpha(t) \in \mathbb{R}^n, n \in \mathbb{Z}_+ \setminus \{0\} \} \).

**Lemma 1:** For any signal \( x \), it holds \( \forall \tau \in [1, \infty), \forall \sigma \geq 1, \forall \tau \in \mathbb{Z} \), and \( \forall t \leq \tau \) that \( |x|_{\sigma p, t} \leq \sigma^{t-\tau} |x|_{\sigma p, \tau} \).

**Proof:** Refer [11] and [12].

**Definition 3:** (System): Given \( \sigma \geq 1 \) and \( p \in [1, \infty] \), a system or operator \( H \) with input \( u \) and output \( y \) is a mapping \( u^n \in \ell_{\sigma p} \) to \( y \in \ell_{\sigma p}^m \), where \( n, m \in \mathbb{Z}_+ \setminus \{0\} \).

**Definition 4:** (System Norm): Given \( \sigma \geq 1 \) and \( p \in [1, \infty] \), the moving-window fading-memory \( \ell_{\sigma p} \)-semi norm of a system \( H \) with input \( u \) is defined as \( \|H\|_{\sigma p} = \sup_{\tau \in \mathbb{Z}} \|H\|_{\sigma p, \tau} \) if the supremum exists, else \( \|H\|_{\sigma p} = \infty \), where

\[
\|H\|_{\sigma p, \tau} = \sup_{\|u\|_{\sigma p, \tau} > 0} \|H u\|_{\sigma p, \tau}
\]

is the norm of system \( H \) at time \( \tau \in \mathbb{Z} \). For simplicity, \( \|H\|_{\sigma p} = \|H\|_{\sigma p, \tau} \) when \( \sigma = 1 \).

**Definition 5:** (Stability and Degree of Stability): Given \( \sigma \geq 1 \), \( p \in [1, \infty] \), a system \( H \) is said to be weakly \( \ell_{\sigma p} \)-stable if there exist a constant \( c \in \mathbb{R}_+ \) and an infinite time sequence \( \{t_i: t_{i-1} < t_i, i \in \mathbb{Z} \} \) with \( t_i \to \infty \) as \( i \to \infty \) such that

\[
\|H\|_{\sigma p, t_i} \leq c, \quad \forall i \in \mathbb{Z}
\]

If \( \{t_i\} = \mathbb{Z} \) then \( H \) is said to be \( \ell_{\sigma p} \)-stable which we denote as \( \ell_{\infty} \)-stability for \( \sigma = 1 \) and \( p = \infty \). Given a system \( H \), the supremum of the set of \( \sigma \) for which \([11]\) holds is called the degree \( \sigma_{0} \) of stability of \( H \).

**Remark 1:** By \([11]\) and \([12]\), if a linear and time-invariant system \( H \) has finite \( \ell_{\sigma p} \)-semi norm with degree \( \sigma \geq 1 \), then \( H \) has all its poles within the circle of radius \( \frac{1}{\sqrt{\sigma}} \).

**Remark 2:** If \( \sigma_1 > \sigma_2 \), where \( \sigma_1 \triangleq \text{arg sup}_{\tau \geq 1} \|H_1\|_{\sigma p} < \infty \) and \( \sigma_2 \triangleq \text{arg sup}_{\tau \geq 1} \|H_2\|_{\sigma p} < \infty \), then system \( H_1 \) is comparatively more stable than system \( H_2 \) by \([12]\).

**Remark 3:** The adaptive switching control with reset mechanism, proposed in \([11]\) and \([12]\) adaptively generates an infinite time sequence \( \{t_k: t_{k-1} < t_k, k \in \mathbb{Z} \} \) with \( t_k \to \infty \) as \( k \to \infty \). By assuming finite-order plant and linear time-invariant controllers, it is proved in \([11]\) Theorem 1 that the adaptive switching control \([11]\) preserves \( \ell_{\infty} \)-stability. On the other hand, by relaxing these assumptions, it is proved in \([12]\) Theorem 3 that the adaptive switching control \([12]\) too preserves \( \ell_{\infty} \)-stability.

**Definition 6:** (Backward Shift and Truncation Operators): The operator \( T \) is defined as the backward shift operator by

\[
(T^\theta x)(t) = x(t - \theta)
\]

for all \( x \in \ell_{\sigma p}^0, t \in \mathbb{Z}, n \in \mathbb{Z}_+ \setminus \{0\} \) and \( \theta \in \mathbb{Z} \). The operator \( P_{\tau} \) is defined as the truncation operator by

\[
(P_{\tau} x)(t) = \begin{cases} x(t), & \forall t \leq \tau, \\ 0, & \text{otherwise.} \end{cases}
\]

for all \( \tau \in \mathbb{Z} \).
Lemma 3: Given a non-linear system $H$ with input $u$ and a pair of times $t, \tau \in \mathbb{Z}$, where $t \leq \tau$, then $\forall u \in \ell^p_{\sigma pe}$ with $n \in \mathbb{Z}^+ \setminus \{0\}$, we have

$$\left( \nabla H_{\tau} u \right) (t) = \begin{cases} \left( \sum_{i=t+1}^{\tau} \nabla h_i \right) + t - t, & \text{if } t < \tau, \\ 0, & \text{if } t = \tau. \end{cases}$$

Proof: The lemma is an immediate consequence of Definition 8.

Remark 5: For a system $H$ with input $u \in \ell^p_{\sigma pe}$, where $n \in \mathbb{Z}^+ \setminus \{0\}$, the reference 4 defines the term $d_\sigma(H)$ by $d_\sigma(H) \triangleq \sup_{t \in \mathbb{Z}, u \in \ell^p_{\sigma pe}} ||T(H u)(t) - (HT u)(t)||_{\sigma,\infty}$, which is equal to $\sup_{t \in \mathbb{Z}} \left\{ ||\nabla h_t||_{\sigma,\infty} \right\}$ according to our Definition 8.

A system $H$ is said to be slowly time-varying when $||\nabla h_t||_{\sigma,\infty}$ is small for all $t \in \mathbb{Z}$ and it is said to be infrequently varying over the interval $L$ when $||\nabla h_t||_{\sigma,\infty}$ has small average over the interval $L$.

The $N$-width average variation rate of a time-varying non-linear system is defined as follows.

Definition 9: (N-Width Average Variation Rate): Given a causal non-linear system $H$ and an integer $N \in \mathbb{Z}^+ \setminus \{0\}$, the $N$-width average variation rate of $H$ is defined as

$$d_{\sigma,N}(H)(t) \triangleq \frac{1}{N} \sum_{i=t-N+1}^{t} ||\nabla h_i||_{\sigma,\infty}. \quad (2)$$

We define $\bar{d}_{\sigma,N}(H)$ as the least upper bound on $d_{\sigma,N}(H)(t)$ for all $t \in \mathbb{Z}$, i.e.,

$$\bar{d}_{\sigma,N}(H) \triangleq \sup_{t \in \mathbb{Z}} d_{\sigma,N}(H)(t).$$

Remark 6: A special case of Definition 9 having $N = 1$ is discussed in 4.

Lemma 4: Consider constants $\sigma > 1$ and $N \in \mathbb{Z}^+ \setminus \{0\}$. Consider a system $H = G K$ where $K$ is a time-invariant non-linear system with finite $||K||_{\sigma,\infty}$ and $G$ is a time-varying non-linear system. Let the frozen-time snapshots of $G$ and $H$ at time $t \in \mathbb{Z}$ be denoted by $g_t$ and $h_t$ respectively. Then we have

$$||\nabla h_t||_{\sigma,\infty} \leq ||\nabla g_t||_{\sigma,\infty} ||K||_{\sigma,\infty} \quad (3)$$

and

$$\bar{d}_{\sigma,N}(H) \leq ||K||_{\sigma,\infty} \bar{d}_{\sigma,N}(G). \quad (4)$$

Remark 7: A solution to problem \ problem implies the system $\Sigma$ is weakly $\ell_{\infty}$-stable with respect to a given time sequence $\{t_i\}$. In case $\{t_i\} = \mathbb{Z}$, the system $\Sigma$ is $\ell_{\infty}$-stable as well.

III. PROBLEM FORMULATION

We consider the general feedback system $\Sigma$ in Fig. 1 which can be described as $\Sigma = (I - G T)^{-1} F$ shown in Fig. 2 where $F$ and $G$ are causal non-linear operators. The main problem is formulated as follows.

**Problem 1:** Consider $\sigma \geq 1$. Consider the non-linear feedback system $\Sigma$ in Fig. 2 where $G : \ell^m_{\sigma,\infty} \mapsto \ell^m_{\sigma,\infty}$ and $F : \ell^m_{\sigma,\infty} \mapsto \ell^m_{\sigma,\infty}$ with $m, n \in \mathbb{Z}^+ \setminus \{0\}$ and $||F||_{\infty} < \infty$. Given a time sequence $\{t_i : t_{i-1} \leq t_i, t_i \in \mathbb{Z}\}$, find a sufficient condition such that, for all $i \in \mathbb{Z}$, the inequality $||\Sigma||_{\infty,\tau, i} \leq c$ holds for some constant $c > 0$.

**Remark 7:** A solution to problem \ problem implies the system $\Sigma$ is weakly $\ell_{\infty}$-stable with respect to a given time sequence $\{t_i\}$. In case $\{t_i\} = \mathbb{Z}$, the system $\Sigma$ is $\ell_{\infty}$-stable as well.

IV. MAIN RESULTS

We derive a solution to the Problem 1 without posing any assumptions on the time-varying MIMO feedback system in Fig. 1. In the later part, we consider some special cases of the derived results as well as compare them with the results from Zames and Wang’s paper 4.

Lemma 5: Consider $\sigma, \sigma_0 \in \mathbb{R}^+$ where $1 \leq \sigma \leq \sigma_0$. Consider two causal non-linear systems $H$ and $G$. Define

$$c_{\sigma,\sigma_0}(G, t) \triangleq \sup_{i \geq 1} \left( \frac{\sigma}{\sigma_0} \right)^i \sum_{q=t-i+1}^{t} ||\nabla g_q||_{\sigma,\infty} \quad (5)$$

Then $\forall t \in \mathbb{Z}$, we have

$$||h_t \nabla G_t||_{\sigma,\infty, t} \leq ||h_t||_{\sigma,\infty, t} c_{\sigma,\sigma_0}(G, t). \quad (6)$$

Proof: Refer Appendix A.

The following lemma is derived for systems with bounded variation rate defined in Definition 9

Lemma 6: Consider $\sigma, \sigma_0 \in \mathbb{R}^+$ where $1 \leq \sigma \leq \sigma_0$. Consider a causal non-linear system $G$ having $d_{\sigma,N}(G) \in \mathbb{R}^+$ for some $N \in \mathbb{Z}^+$. Define

$$c_{\sigma,N}(G) \triangleq \left( e \ln \left( \frac{\sigma_0}{\sigma} \right) \right)^{-1} \left( \frac{\sigma}{\sigma_0} \right)^{N-1} \bar{d}_{\sigma,N}(G). \quad (7)$$

Proof: Let the input to the system $H$ be denoted by $x$. Then by Definitions 4 and 8 for all $t \in \mathbb{Z}$, we have

$$||\nabla h_t x||_{\sigma,\infty,t} = \left( g_{t_{i-1}}(K x) - g_t(K x) \right)_{\sigma,\infty,t} \leq ||\nabla g_t||_{\sigma,\infty} ||K||_{\sigma,\infty} ||x||_{\sigma,\infty,t}.$$
Then for all \( t \in \mathbb{Z} \), we have
\[
c_{\sigma,\sigma_0}(G, t) \leq c_{\sigma,N}(G).
\]

Proof: Refer Appendix B. \( \square \)

**Lemma 7:** Consider \( \sigma, \sigma_0 \in \mathbb{R}_+ \) where \( 1 \leq \sigma < \sigma_0 \). Consider two causal non-linear systems \( H \) and \( G \) having \( \overline{d}_{\sigma,N}(G) \in \mathbb{R}_+ \) for some \( N \in \mathbb{Z}_+ \setminus \{0\} \). Then \( \forall t \in \mathbb{Z} \), we have
\[
\| h_t \nabla G \|_{\sigma,\infty,t} \leq \| h_t \|_{\sigma_0,\infty,t} c_{\sigma,N}(G).
\]

Proof: The lemma is a consequence of Lemma 5 and Lemma 6. \( \square \)

**Remark 8:** In [4], it is derived that \( \| h_t \nabla K \|_{\sigma,\infty} \leq \| h_t \|_{\sigma,\infty} c^{-1}(e \ln \left( \frac{\sigma}{\sigma_0} \right))^{-1} \| d_\sigma \( K \). \) By Remark 5 and Definition 8, \( d_\sigma \( K \) = \sup_{\tau \in \mathbb{Z}} \| \nabla k_\tau \|_{\sigma,\infty} \). Therefore Lemma 7 generalizes the result in [4] by considering \( N \geq 1 \).

A solution for Problem 1 is proposed as follows.

**Theorem 1:** Consider \( \sigma, \sigma_0 \in \mathbb{R}_+ \) where \( 1 \leq \sigma < \sigma_0 \), \( \rho \in \mathbb{R}_+ \), and \( \{t_i : t_{i-1} \leq t_i \leq t_{i+1} \} \subset \mathbb{Z} \). Consider the non-linear feedback system \( \Sigma \) in Fig. 8 where \( F : \ell^m_{\sigma,\infty} \rightarrow \ell^m_{\sigma,\infty} \) having \( \| F \|_{\infty} < \infty \) and \( G : \ell^m_{\sigma,\infty} \rightarrow \ell^m_{\sigma,\infty} \) with \( n, m \in \mathbb{Z}_+ \setminus \{0\} \). Let \( s_t \) and \( l_t \) be the TI frozen-time snapshots of \( (I - G_t T)^{-1} \) and \( (I - G_t T)^{-1} G_t T \) respectively. Define
\[
\psi(\hat{t}) = \max \left\{ \min \left\{ \| l_t \|_{\sigma_0,\infty,t}, c_{\sigma,\sigma_0}(G, t), \| g_t \|_{\sigma,\infty,t} \right\}, \sigma^{-1} \right\},
\]
\[
\hat{c} \triangleq \sigma^{-1} \left( \frac{\sigma}{1 - \sigma} + 1 \right), \quad \hat{\beta} \triangleq \max \left\{ \sigma^{-1}, \| l_t \|_{\sigma_0,\infty,t}, c_{\sigma,\sigma_0}(G, t), \| g_t \|_{\sigma,\infty,t} \right\},
\]
and \( c_{\sigma,\sigma_0}(G, t) \) is defined in [5]. If
\[
\rho^{\hat{t} - t} \geq \prod_{j=t+1}^{t_i} \psi(\hat{t}), \quad \forall t \in [t_{i-1}, t_i - 1], \forall i \in \mathbb{Z},
\]
then for all \( i \in \mathbb{Z} \) we have
\[
\| x \|_{\sigma,\infty,t_i} \leq \rho^{\hat{t} - t_i - 1} \| x \|_{\sigma,\infty,t_{i-1}} + \beta \| u \|_{\sigma,\infty,t_i}.
\]
Furthermore, if \( \rho \in (\sigma^{-1}, 1) \), then for all \( i \in \mathbb{Z} \) we have \( \| x \|_{\infty,\infty} \leq \hat{c} \). \( \square \)

Proof: Refer Appendix C. \( \square \)

**Remark 9:** To hold condition (12), it is not necessary for the frozen-time snapshots \( l_t \) to be stable for all time. It can be unstable at some times other than \( \{t_i\} \) provided \( c_{\sigma,\sigma_0}(G, t) \) is small enough for all \( t \in [t_{i-1} + 1, t_i] \) and for all \( i \in \mathbb{Z} \). \( \square \)

Like Zames and Wang’s sufficient condition [4] inequality (2.22) for system \( \Sigma \) to be\( \ell_{\infty}\)-stable, Theorem 1 considers the frozen-time snapshots \( l_t \), but it does not consider assumptions that \( F \) and \( G \) are linear, \( G \) is stabilizing, and \( \{t_i\} \} = \mathbb{Z} \). Therefore, Theorem 1 is a generalization of [4] inequality (2.22).

**Corollary 1:** Let \( \{t_i : t_{i-1} \leq t_i \subset \mathbb{Z} \} \) be a time sequence and let \( \rho \in (0, 1) \) be a constant. Define
\[
\psi_N(t) \triangleq \min \left\{ \| l_t \|_{\sigma_0,\infty,t} c_{\sigma,N}(G), \| g_t \|_{\sigma,\infty,t} \right\}, \sigma^{-1},
\]
where \( c \) is defined as in [9] and \( c_{\sigma,N}(G) \) is defined in [7]. If
\[
\rho^{\hat{t} - t} \geq \prod_{j=t+1}^{t_i} \psi_N(j), \quad \forall t \in [t_{i-1}, t_i - 1], \forall i \in \mathbb{Z},
\]
then
\[
\| x \|_{\infty,\infty} \leq \hat{c}, \quad \forall t \in \mathbb{Z}.
\]

Proof: By Lemma 7 \( c_{\sigma,\sigma_0}(G, t) \leq c_{\sigma,N}(G) \) for all \( t \in \mathbb{Z} \), which implies \( \psi(t) \leq \psi_N(t) \) for all \( t \in \mathbb{Z} \). Therefore if (15) holds then (12) holds. Hence, the corollary is proved.

The following corollary gives a sufficient condition for the system \( \Sigma \) to be \( \ell_{\infty}\)-stable for all time.

**Corollary 2:** Let \( \{t_i : t_{i-1} \leq t_i \subset \mathbb{Z} \} \) be a time sequence and let \( \rho \in (0, 1) \) be a constant. Define
\[
\hat{\gamma}(t) \triangleq \left\{ \begin{array}{ll}
\max \left\{ \sigma^{-1}, \| l_t \|_{\sigma_0,\infty,t}, c_{\sigma,\sigma_0}(G, t), \| g_t \|_{\sigma,\infty,t} \right\}, & \text{if } G_t \text{ is stabilizing,} \\
\max \left\{ \sigma^{-1}, \| g_t \|_{\sigma,\infty,t} \right\}, & \text{if } G_t \text{ is destabilizing,}
\end{array} \right.
\]
and \( \hat{t} \) is defined as in (10). If
\[
\rho^{\hat{t} - t} \geq \prod_{j=t+1}^{t_i} \hat{\gamma}(\hat{t}), \quad \forall t \in [t_{i-1}, t_i - 1], \forall i \in \mathbb{Z},
\]
then
\[
\| x \|_{\infty,\infty} \leq \hat{c}, \quad \forall t \in \mathbb{Z}.
\]

Proof: Refer Appendix D. \( \square \)

**Remark 10:** By (19), \( \hat{\gamma}(t) \) is bounded for all \( t \in \mathbb{Z} \) because \( \| F \|_{\infty} < \infty \) and \( \| s_t \|_{\sigma,\infty,t} < \infty \) when \( G_t \) is stabilizing. Therefore, by (16), (18), and (19), \( \hat{c} \) is bounded provided \( t \) is finite.

The following lemma gives a sufficient condition for the system \( \Sigma \) to be \( \ell_{\infty}\)-stable for all time given the upper bound \( d_{\sigma,N}(G) \), on average variation rate of loop function \( G \).

**Lemma 8:** Let \( \{t_i : t_{i-1} \leq t_i \subset \mathbb{Z} \} \) be a time sequence and let \( \rho \in (0, 1) \) be a constant. Consider \( \hat{c} \) defined in (16). Define
\[
\hat{\psi}_N(t) \triangleq \left\{ \begin{array}{ll}
\max \left\{ \sigma^{-1}, \| l_t \|_{\sigma_0,\infty,t}, c_{\sigma,N}(G) \right\}, & \text{if } G_t \text{ is stabilizing,} \\
\max \left\{ \sigma^{-1}, \| g_t \|_{\sigma,\infty,t} \right\}, & \text{if } G_t \text{ is destabilizing,}
\end{array} \right.
\]
If
\[
\rho^{\hat{t} - t} \geq \prod_{j=t+1}^{t_i} \hat{\psi}_N(j), \quad \forall t \in [t_{i-1}, t_i - 1], \forall i \in \mathbb{Z},
\]
then

$$\|\Sigma\|_{\infty,t} \leq \tilde{c}, \quad \forall t \in \mathbb{Z}. \tag{24}$$

**Proof:** By Lemma 8, \(c_{\sigma,\sigma_0}(G,t) \leq c_{\sigma,N}(G)\) for all \(t \in \mathbb{Z}\), which implies \(\psi(t) \leq \psi_N(t)\) for all \(t \in \mathbb{Z}\). Therefore if (23) holds then (21) holds. Hence, the lemma is proved. ■

**Lemma 9:** Consider \(\sigma,\sigma_0 \in \mathbb{R}_+,\) where \(1 \leq \sigma < \sigma_0\), and \(\rho \in (\sigma^{-1},1)\). Consider the non-linear feedback system \(\Sigma\) in Fig. 2 where \(F : l^n_{\infty} \to l^m_{\infty}\) having \(\|F\|_{\infty} < \infty\) and \(G : l^m_{\infty} \to l^n_{\infty}\) with \(n,m \in \mathbb{Z}_+\{0\}\). Let \(s_t\) and \(l_t\) be the TI frozen-time snapshots of \((I - G_tT)^{-1}\) and \((I - G_tT)^{-1}G_tT\) respectively such that \(\sup_{t \in \mathbb{Z}} \|s_t\|_{\infty}, \|l_t\|_{\infty} < \infty\). Define

$$\tilde{c} \triangleq \frac{\|F\|_{\infty} \sup_{t \in \mathbb{Z}} \|s_t\|_{\infty}}{1 - \rho}. \tag{25}$$

Then for all \(t \in \mathbb{Z}\), we have \(\|\Sigma\|_{\infty,t} \leq \tilde{c}\).

**Proof:** If (25) holds then (12) holds too for the special case \(t = t_{i-1} - 1\) by (5) and (8). If \(t = 1\) by (4), (10) and (11), then we get \(\beta = \frac{\beta}{\|F\|_{\infty} \sup_{t \in \mathbb{Z}} \|s_t\|_{\infty}} \sigma\). Therefore, by (24) ■

The following corollary is derived from Lemma 9 given the upper bound \(\tilde{d}_{\sigma,N}(G)\), on average variation rate of loop function \(G\).

**Corollary 3:** Define

$$\tilde{d}_{\sigma,N}(G) \triangleq \left(\frac{\sigma}{\sigma_0}\right)^{-N} \left(e^{\ln\left(\frac{\sigma}{\sigma_0}\right)} \left(\sup_{t \in \mathbb{Z}} \|l_t\|_{\infty}\right)^{-1}\right) \rho, \tag{26}$$

Then for all \(t \in \mathbb{Z}\), we have \(\|\Sigma\|_{\infty,t} \leq \tilde{c}\).

**Proof:** Let \(\tilde{d}_{\sigma,N}(G)\) defined in Definition 9 and \(\tilde{c}\) defined in (24). If

$$\tilde{d}_{\sigma,N}(G) \leq \tilde{d}_{\sigma,N}(G), \tag{27}$$

then for all \(t \in \mathbb{Z}\), we have \(\|\Sigma\|_{\infty,t} \leq \tilde{c}\).

**Remark 11:** Corollary 3 gives an upper bound on average variation rate of loop function \(G\) for closed-loop system \(\Sigma\) to be stable with degree 1 given the frozen-time extension \(G_t\) is stabilizing for all \(t \in \mathbb{Z}\), i.e., \(\sup_{t \in \mathbb{Z}} \|l_t\|_{\infty} < \infty\) for \(t \in \mathbb{Z}\).

**Remark 12:** For the system \(\Sigma\) to be \(\ell_{\infty}\)-stable, Zames and Wang’s sufficient condition [4] inequality (22.22) is

$$\|\nabla g_t\|_{\infty} \leq \tilde{d}_{\sigma,1}(G), \forall t \in \mathbb{Z}. \tag{28}$$

where

$$\tilde{d}_{\sigma,1}(G) \triangleq \left(e^{\ln\left(\frac{\sigma}{\sigma_0}\right)} \left(\sup_{t \in \mathbb{Z}} \|l_t\|_{\infty}\right)^{-1}\right) \rho, \tag{29}$$

with \(\rho \in (0,1)\). In [4], Zames and Wang considered the sensitivity function \((I - GT)^{-1}\) and proved \(\|\|I - GT\|^{-1}\|_{\infty} \leq \sup_{t \in \mathbb{Z}} \|s_t\|_{\infty}\|s_t\|_{\infty}^{-1}\) if (28) holds. On the other hand, we considered the system \(\Sigma = (I - GT)^{-1}F\), and proved in Lemma 9 that \(\|I - GT\|^{-1}F\|_{\infty} \leq \tilde{c} = \frac{\|F\|_{\infty} \sup_{t \in \mathbb{Z}} \|s_t\|_{\infty}}{1 - \rho}\) if the sufficient condition (25) holds.

In the following lemma, the relation between our sufficient condition (12) and Zames and Wang’s sufficient condition (28) is discussed.

**Lemma 10:** Consider \(\sigma,\sigma_0 \in \mathbb{R}_+,\) where \(1 \leq \sigma < \sigma_0\), and \(\rho \in (\sigma^{-1},1)\). Consider the non-linear feedback system \(\Sigma\) in Fig. 2 where \(F : l^n_{\infty} \to l^m_{\infty}\) having \(\|F\|_{\infty} < \infty\) and \(G : l^m_{\infty} \to l^n_{\infty}\) with \(n,m \in \mathbb{Z}_+\{0\}\). Let \(s_t\) and \(l_t\) be the TI frozen-time snapshots of \((I - G_tT)^{-1}\) and \((I - G_tT)^{-1}G_tT\) respectively. Then the sufficient condition (12) and (20) hold whenever Zames and Wang’s sufficient condition (28) holds, and there exist cases where Zames and Wang’s sufficient condition (28) does not hold when the sufficient condition (12) and (20) hold.

**Proof:** Refer Appendix E.

**Remark 13:** By Lemma 10, the system \(\Sigma\) is \(\ell_{\infty}\)-stable when \(G\) varies with periodic large-variation such that for all \(q \in \mathbb{Z}\), \(\|\nabla g_t\|_{\infty} \leq \tilde{d}_{\sigma,1}(N\tilde{d}_{\sigma,N}), \forall t = qN,\) and \(\|\nabla g_t\|_{\infty} = 0, \forall t \neq qN\).

**V. COMPARISON WITH [4]**

Zames and Wang [4] proved that

$$\|H\|_{\sigma} \leq \sigma^{-1} \left(e^{\ln\left(\frac{\sigma}{\sigma_0}\right)} \left(\sup_{t \in \mathbb{Z}} \|l_t\|_{\infty}\right)^{-1}\right) \|H\|_{\sigma_0} \|d_{\sigma}(G)\|, \tag{30}$$

where both \(H\) and \(G\) are causal and linear with bounded \(\|H\|_{\sigma_0}\) and \(\|G\|_{\sigma}\). By (4), \(\|H\|_{\sigma} = \sup_{t \in \mathbb{Z}} \|h_t\|_{\sigma_0}\). By Remark 5, \(d_{\sigma}(G) = \sigma d_{\sigma,N}(G)\) when \(N = 1\). By (4) and Definition 8, \(\|H\|_{\sigma} = \sup_{t \in \mathbb{Z}} \|h_t\|_{\sigma_0}\). Therefore (30) is a special case of our Lemma 7 when the worst-case variation rate of \(G\) is bounded, and \(H\) and \(G\) are causal, stable, and linear. Therefore, Lemma 7 generalizes (30) by considering non-linear and unstable \(H\) and \(G\) and by relaxing the assumption that the worst-case variation rate of \(G\) is bounded.

Theorem 1 generalizes Zames and Wang’s sufficient condition (25) by relaxing assumptions that \(F\) and \(G\) in Fig. 1 are linear and the worst-case variation rate of \(G\) is bounded. Theorem 1 allows unstable \((I - GT)^{-1}G\) such that its frozen-time snapshot is not necessarily \(\ell_{\infty}\)-stable for all time. Furthermore, Theorem 1 considers weakly \(\ell_{\infty}\)-stability of the system \(\Sigma\) at given time sequence \(\{t_i\}\), which generalizes Zames and Wang’s sufficient condition (25) where \(\ell_{\infty}\)-stability of the system \(\Sigma\) is considered for all time. By Lemma 10, Zames and Wang’s sufficient condition (25) is a special case of Theorem 1.
VI. BOUND ON PLANT TIME-VARIATION RATE FOR ADAPTIVE CONTROL

An interesting question in adaptive control is how much plant time-variation rate can be tolerated. In this section, we answer this question with the help of Corollary 3 to derive an upper bound on allowed average plant time-variation rates in adaptive control framework. We consider the adaptive switching system developed in [12] as follows. For an unknown slowly time-varying nonlinear plant $P$, the paper [12] proposes an algorithm that returns a stabilizing adaptive switching controller $K$ of the form shown in Fig. 4 that the resultant closed-loop adaptive system $\Sigma_1(K, P)$ of the form shown in Fig. 3 with resetting is exponentially stable and has bounded $\ell_\infty$-norm subject to the assumption that the adaptive control problem is feasible in the sense there always exist at least one candidate controller capable of stabilizing the slowly time-varying plant $P$, where $\mathcal{M} = \{1, 2, \ldots, m\}$ is the set of candidate controllers’ indices. The importance of our Corollary 3 is that it not only confirms that the system will remain stable in the presence of slow and/or infrequent large plant time-variation but also gives a quantitative bound on the amount of tolerable average rate of plant time-variation, provided that the frozen-time adaptive problem for the frozen time plants $P_t$ are feasible for all $t \in \mathbb{Z}_+$ and the average variation rate of the frozen-time snapshots of the open-loop system is small enough.

The adaptive switching system $\Sigma_1$ can be converted to the generic feedback system in Fig. 1 by letting $F = [I \ P]^T N_t^r$, $x = \zeta$, and $G T = [I \ P]^T [1 - D_t^u - N_t^y]$. Let $l_t$ be the frozen-time snapshot of $(I - G_t T)^{-1} G_t T$ for all $t \in \mathbb{Z}$. The paper [12] has proved that the proposed nonlinear adaptive controller $K$ achieves $\ell_\infty$-stability for all $P_t$, $t \in \mathbb{Z}$ with some degree $\lambda > 1$ which is subject to the feasibility assumption [12] Assumption A2, and thus we have $\|l_t\|_{\lambda^\infty}$ bounded for all $t \in \mathbb{Z}$. We assume that the $N$-width average time-variation of plant $P$ is bounded by $\tilde{d}_{\sigma,N}(P)$ for some $N \in \mathbb{Z}_+ \setminus \{0\}$ and $\sigma \in (1, \lambda)$. Then by Lemma 4 we have

$$\tilde{d}_{\sigma,N}(G) \leq \max_{i \in \mathcal{M}} \left\| [I - D_t^u - N_t^y] \right\|_{\sigma^\infty} \tilde{d}_{\sigma,N}(P) \quad (31)$$

where $\max_{i \in \mathcal{M}} \left\| [I - D_t^u - N_t^y] \right\|_{\sigma^\infty}$ is finite according to controller realization in [12]. According to Corollary 3 and [31], if for some $\rho \in (\sigma^{-1}, 1)$ the term $\tilde{d}_{\sigma,N}(P)$ satisfies

$$\tilde{d}_{\sigma,N}(P) \leq \left( \max_{i \in \mathcal{M}} \left\| [I - D_t^u - N_t^y] \right\|_{\sigma^\infty} \right)^{\lambda - \frac{N}{\lambda}} \left( \frac{\lambda}{\sigma} \right)^{\frac{N}{\lambda}} (\lambda - \rho)^{\frac{N}{\lambda}} \quad (32)$$

then $\ell_\infty$-stability of the system $\Sigma_1$ is preserved. Therefore, as long as the $N$-width average variation rate of plant $P$ does not violate the inequality (32), the nonlinear adaptive controller $K$ developed in [12] preserves $\ell_\infty$-stability of the adaptive switching system $\Sigma_1$.

VII. SIMULATION

In this section, MATLAB simulations are presented. The Examples 1 is demonstrated to support the Corollary [2] and the Example 2 shows a case where Zames and Wang’s condition [28] does not hold while our condition holds and concludes $\ell_\infty$-stability of a system.

Example 1: Consider the system $\Sigma$ in Fig. 1. Let $F$ be an identity matrix. Let the persistently destabilizing loop function $G$ be equal to a time-varying non-linear system $\Phi H_t$ such that $(G x(t)) = \Phi H_t x(t)$ for all $t \in \mathbb{Z}$ where $x \in \ell^2_\infty$.

The system $\Phi$ with input $[v_1 \ v_2]'$ and output $[w_1 \ w_2]'$ is a dead-zone operator such that for all $i \in \{1, 2\}$,

$$w_i = \begin{cases} v_1 - 0.5, & \text{if } v_i \geq 0.5, \\ 0, & \text{if } v_i \in (-0.5, 0.5), \\ v_1 + 0.5, & \text{if } v_i \leq 0.5, \end{cases} \quad (33)$$

and the system $H_t$ is time-varying such that (i) $(H_t x) = A_t x + B_t (x(t-1))$ where $\lambda_{\max}(B_t) < 1$ for all $t \in [0, 995]$, (ii) $\lambda_{\max}(A_t) < 1$ whenever the function $I_a(t) = 1$, and $\lambda_{\max}(A_t) > 1$ whenever $I_a(t) = 0$ for all $t \in [0, 995]$ as shown in Fig. 5(a), (iii) $H_t$ is destabilizing whenever $I_a(t) = 1$ and $H_t$ is stabilizing whenever $I_a(t) = 0$, and (iv) $H_t \neq H_j$, for all $t \in [0, 995]$ and $i \neq j$.

We simulated the above system $\Sigma$ in MATLAB with zero initial conditions and $u(t) = 2 \exp \left( \frac{t}{20} \right) \cos \left( \frac{\pi}{2} \right) [1 \ 1]'$ for $t = 0 : 995$. We considered $\sigma = 1.2, \sigma_0 = 1.4$, and $\rho = 0.94$. The simulation results are shown in Fig. 5(b).

Persistent and abrupt time-variations in the loop function $G$ are shown in Fig. 6(a). First we computed the terms
Therefore, Zames and Wang’s sufficient condition \( \text{(28)} \) does not hold for all \( t \in [0, 0.995] \) as shown in Fig. 6(a) and so it does not conclude that the system \( \Sigma \) is \( \ell_{\infty, \infty} \)-stable. This proves for this example that our sufficient condition \( \text{(20)} \) is less conservative than Zames and Wang’s sufficient condition \( \text{(28)} \), i.e., the condition \( \text{(20)} \) holds while the condition \( \text{(28)} \) does not hold.

**Example 2:** Consider the system \( \Sigma \) in Fig. 1. Let \( F \) be an identity matrix. Let the loop function \( G \) be equal to the system \( H_t \) such that \( (Gx)(t) = H_t, x(t) \) for all \( t \in \mathbb{Z} \), where \( x \in l^2_{\infty, \infty} \). And the system \( H_t \) is a time-varying \( 2 \times 2 \) real matrix such that \( (i)|\max(H_t)| < 1 \) for all \( t \in \mathbb{Z}_+ \) and (ii) \( H_t \neq H_j, \forall i, j \in \mathbb{Z}_+ \) and \( i \neq j \). The frozen-time snapshot \( t_i \) of \( (I - G_tT)^{-1} G_tT \) has \( \sup_{t \in \mathbb{Z}_+} \|l_i\|_{\sigma, \infty} = 4.8839 \) with considered \( G \).

We simulated the above system \( \Sigma \) in MATLAB with zero initial conditions and \( u = [\cos(t/2), \cos(t/2)]' \) for \( t = 0 : 982 \). We considered \( \sigma = 1.2, \sigma_0 = 1.44 \), and \( \rho = 0.9 \). The simulation results are shown in Fig. 8.

Fig. 8(a) shows persistent and abrupt time-variations in \( G \). First we computed the terms \( c_{\sigma, \sigma_0}(G)(t) \) and \( \hat{\psi}(t) \) for all \( t \in [0, 0.995] \) by \( \text{(5)} \) and \( \text{(17)} \) respectively. We choose a time sequence \( \{t_i\} \) shown in Fig. 6(b). Then by the sufficient condition \( \text{(20)} \), the \( \ell_{\infty, \infty} \)-stability of the system \( \Sigma \) is preserved since \( (i) \|x\|_{\infty, 0} = 0 \) because of the zero initial condition and (ii) condition \( \text{(20)} \) holds for all \( t_i \in \{t_i\} \). For example, Fig. 7(a) and (b) show the condition \( \text{(20)} \) holds for \( (t_{i-1}, t_i) = (313, 330) \) and \( (t_{i-1}, t_i) = (643, 654) \) respectively. By \( \text{(16)} \), we compute \( \hat{c} = 79129 \). Therefore, by Corollary \( \text{(2)} \), \( \|\Sigma\|_{\infty, t} \leq \hat{c} \) which can be verified in Fig. 5(b) where \( \|x\|_{\infty, t} \leq \hat{c} \) for all \( t \in [0, 0.995] \).

On the other hand, since \( H(t) \) is destabilizing whenever \( \mathcal{I}_q(t) = 1 \) for all \( t \in [0, 0.995] \), the frozen-time snapshot \( t_i \) of \( (I - G_tT)^{-1} G_tT \) is unstable, and \( \sup_{t \in \mathbb{Z}_+} \|l_i\|_{\sigma, \infty} = \infty \) according to Definition \( \text{(4)} \). Zames and Wang’s sufficient condition \( \text{(28)} \) for the system \( \Sigma \) to be \( \ell_{\infty, \infty} \)-stable is \( \|g_t\|_{\sigma, \infty} \leq d_{\sigma, 1}(G) \) for all \( t \in \mathbb{Z} \). By \( \text{(29)} \), we computed \( d_{\sigma, 1}(G) = 0 \). But, \( \|g_t\|_{\sigma, \infty} > 0 \) for all \( t \in [0, 0.995] \) as shown in Fig. 6(a).
condition (20) is less conservative than Zames and Wang’s sufficient condition (28), i.e. the condition (20) holds while the condition (28) does not hold.

VIII. CONCLUSION

In this article, the input-output stability of a general time-varying MIMO non-linear feedback system has been investigated by a generalizing the results in [4]. A general sufficient condition to preserve stability of the feedback system has been derived by relaxing three assumptions [4] on the adaptive feedback loop function that (i) it is linear, (ii) its frozen-time snapshot is stabilizing all the time, and (iii) variation between its adjacent frozen-time snapshots is bounded. The sufficient condition gives a tolerable limit on average time-variation rate of the adaptive feedback loop function of a MIMO non-linear adaptive switching system to preserve its \( \ell_\infty \)-stability.

Our sufficient condition is less conservative compared to the sufficient condition in [4]. Whenever the condition [4] holds, our condition holds as well. In case when the adaptive feedback loop function has infrequent large time-variations, our condition holds but the condition [4] does not hold. Therefore, our condition is more practical to conclude stability of adaptive switching systems that are inherently non-linear and subject to infrequent large variations possibly due to unexpected component failures.

Appendix A
Proof of Lemma [5]

Consider the system \( h_t \nabla G_t \) with input \( u \in \ell^\infty_{n,\infty} \) and output \( y \in \mathbb{R}^m \) where \( n, m \in \mathbb{Z}_+ \). Then by Lemma [3] \( \forall t \in \mathbb{Z} \) we have

\[
y(t) = h_t \nabla G_t u = h_t \begin{bmatrix} \vdots \\
\sum_{i=t-1}^t \nabla g_i \\
\nabla g_i \mathcal{T} u \\
0 \end{bmatrix} \mathcal{T}^2 u.
\]

Next,

\[
|y(t)| \leq \|h_t\|_{\sigma_0,\infty,t} \begin{bmatrix} \vdots \\
\sum_{i=t-1}^t \nabla g_i \\
\nabla g_i \mathcal{T} u \\
0 \end{bmatrix} \mathcal{T}^2 u \\
\|u\|_{\sigma_0,\infty} \\
\|u\|_{\sigma_0,\infty,t} \\
\|u\|_{\sigma_0,\infty,t} \\
\|u\|_{\sigma_0,\infty,t} \leq \|h_t\|_{\sigma_0,\infty,t} \sup_{i \geq 1} \left[ \sigma \sum_{q=t-i+1}^t \|\nabla g_q\|_{\sigma_\infty} \right] \|u\|_{\sigma_0,\infty,t} \\
\end{bmatrix}
\]

(34)

(35)

where (34) is by Lemma [1] and (35) is by the definition of \( \ell_\infty \)-norm. Hence, the claim is proved by (35) and Definition [9].

Appendix B
Proof of Lemma [6]

Consider \( i \in \mathbb{Z}_+ \setminus \{0\} \). Since \( \forall t \in \mathbb{Z}, \exists j \in \mathbb{Z}_+ \setminus \{0\} \) such that \( t - i + 1 \in [t - jN + 1, t - (j - 1)N] \) and \( i \in [(j - 1)N + 1, jN] \), thus by Definition [9].
where (36) is due to sup}_{x \geq 0} x y^{-x} \leq (e \ln(y))^{-1}, \forall y > 1 [4]. Hence, the claim is proved by (38) and (39).

Appendix C
Proof of Theorem 1

Additionally, by Fig. 11 \|s_t\|_{\infty,t} \geq 1 for t \in Z and Definition 8 we have

\[ x(t) = f_t u + g_t T x, \]

\[ \|x(t)\| \leq \|F\|_{\infty} \|u\|_{\infty,t} + \|g_t\|_{\infty,t} \|T x\|_{\infty,t} \] (by Lemma 2)

\[ \|x(t)\| \leq \|s_t\|_{\infty,t} \|F\|_{\infty} \|u\|_{\infty,t} + \|g_t\|_{\infty,t} \|T x\|_{\infty,t} \]

\[ \|x(t)\| \leq \|s_t\|_{\infty,t} \|F\|_{\infty} \|u\|_{\infty,t} + \|g_t\|_{\infty,t} \|x\|_{\infty,t-1} \] (39)

Next, the property of \( \ell_\infty \)-semi norm \( \|x\|_{\infty,t} \leq \max \{\sigma^{-1} \|x\|_{\infty,t-1}, |x(t)|\} \) along with (38) and (39) are used to get

\[ \|x\|_{\infty,t} \leq \max \{\sigma^{-1} \|x\|_{\infty,t-1}, \min \{|s_t\|_{\infty,t}, \|F\|_{\infty} \|u\|_{\infty,t}, + |l_t|_{\infty}\sigma_0(G, t)\} \}

\|x\|_{\infty,t} \leq \|s_t\|_{\infty,t} \|F\|_{\infty} \|u\|_{\infty,t} + \psi(t) \|x\|_{\infty,t-1}, \]

\[ \|x\|_{\infty,t} \leq \max \{\sigma^{-1} \|x\|_{\infty,t-1}, \min \{|s_t\|_{\infty,t} \|F\|_{\infty} \|u\|_{\infty,t}, + |l_t|_{\infty}\sigma_0(G, t)\} \}

\|x\|_{\infty,t} \leq \|s_t\|_{\infty,t} \|F\|_{\infty} \|u\|_{\infty,t} + \psi(t) \|x\|_{\infty,t-1}, \]

(40)

where \( \psi(t) \) is defined in [3] and [10] is by [8]. Next, by applying the Grönwall-Bellman Lemma 19 on (40), it is true that

\[ \|x\|_{\infty,t} \leq \max \{\sigma^{-1} \|x\|_{\infty,t-1}, \min \{|s_t\|_{\infty,t}, \|F\|_{\infty} \|u\|_{\infty,t}, + |l_t|_{\infty}\sigma_0(G, t)\} \}

\|x\|_{\infty,t} \leq \|s_t\|_{\infty,t} \|F\|_{\infty} \|u\|_{\infty,t} + \psi(t) \|x\|_{\infty,t-1}, \]

(41)

\[ \|x\|_{\infty,t} \leq \max \{\sigma^{-1} \|x\|_{\infty,t-1}, \min \{|s_t\|_{\infty,t}, \|F\|_{\infty} \|u\|_{\infty,t}, + |l_t|_{\infty}\sigma_0(G, t)\} \}

\|x\|_{\infty,t} \leq \|s_t\|_{\infty,t} \|F\|_{\infty} \|u\|_{\infty,t} + \psi(t) \|x\|_{\infty,t-1}, \]

(42)

where (41) is by [12] and by \( \|u\|_{\infty,t} \leq \|u\|_{\infty,t}, \forall t \leq t_i, \) and (42) is by [10] and [11]. By considering \( \rho \in (\sigma^{-1}, 1) \) and applying the Grönwall-Bellman Lemma 19 on (42), \( \forall \alpha < j \) it is true that

\[ \|x\|_{\infty,t_j} \leq \rho^{t_f-t} \|x\|_{\infty,t_j} + \sum_{k=0}^{j} \rho^{j-k} \beta \|u\|_{\infty,t_k}, \]

(43)

\[ \|x\|_{\infty,t_j} \leq \rho^{t_f-t} \|x\|_{\infty,t_j} + \frac{\beta}{1-\rho} \|u\|_{\infty,t_k}, \]

(44)

1 According to Definitions 3 and 6, \( F, G_t, \) and \( \nabla G_t \) are not necessarily real matrices, and hence \( \Sigma \) is not necessarily a system in the state-space representation. In a special case where \( F, G_t, \) and \( \nabla G_t \) are memory-less systems and thus can be represented as real matrices, \( \Sigma \) is a system expressed in the state-space representation.
where (43) is by $\rho_j^{i-x_k} \leq \rho$ and (44) is by $\|u\|_{\infty,t_k} \leq \|u\|_{\infty,t_i}$, respectively. Next, by choosing $a$ such that $\|x\|_{\sigma=t_a} = 0$, the following inequality holds:

$$\|x\|_{\sigma=t_j} \leq \frac{\beta}{1-\rho} \|u\|_{\infty,t_i}. \quad (45)$$

Therefore for all $t \in [t_j-1 + 1, t_j]$,

$$\sigma(t_j-t) \|x(t)\| \leq \frac{\beta}{1-\rho} \|u\|_{\infty,t_i}, \quad (46)$$

and

$$\|x\|_{\sigma=t_j} \leq \frac{\beta}{1-\rho} \|u\|_{\infty,t_i}, \quad (47)$$

$$\|x\|_{\infty,[t_j-1+1,t_j]} \leq \|x\|_{\infty,[t_j-1+1,t_j]} \leq \|x\|_{\infty,t_i}, \quad (48)$$

and

$$\|x\|_{\infty,[t_j-1+1,t_j]} \leq c \|u\|_{\infty,t_i}, \quad (49)$$

where (46) is by (45) and Definition 2, (47) is by (48) and Definition 2, (48) is by (49), and (49) is by $\|x\|_{\infty,t_i} \leq \sup_{t \leq t_i} \|x\|_{\infty,[t_j-1+1,t_j]}$ respectively. Hence, the claim follows by noting $x = \Sigma u$.

**Appendix D**

**Proof of Corollary 2**

Let $t \in [t_i-1 + 1, t_i]$ for some $i \in \mathbb{Z}$. By Fig. 1 we have

$$x(t) = f_i u + g_i T x, \quad |x(t)| \leq \|F\|_{\infty} \|u\|_{\infty,t} + \|g_i\|_{\sigma=t_i} \|T x\|_{\sigma=t_i} \quad \text{(by Lemma 2)}$$

$$|x(t)| \leq \|F\|_{\infty} \|u\|_{\infty,t} + \|g_i\|_{\sigma=t_i} \|x\|_{\sigma,t_i-1} \quad \text{(50)}$$

Next, by the property of $\ell_{\sigma,t}$-semi norm $\|x\|_{\sigma,t} \leq \max \left\{ \sigma^{-1} \|x\|_{\sigma,t-1} \mid |x(t)| \right\}$, and by (38) and (50), we have

$$\|x\|_{\sigma,t} \leq \max \left\{ \sigma^{-1} \|x\|_{\sigma,t-1} \right\}, \quad \text{and} \quad \|x\|_{\sigma,t} \leq \max \left\{ \sigma^{-1} \|x\|_{\sigma,t-1} \right\}$$

and

$$\|x\|_{\sigma,t} \leq \max \left\{ \sigma^{-1} \|x\|_{\sigma,t-1} \right\}, \quad \|x\|_{\sigma,t} \leq \max \left\{ \sigma^{-1} \|x\|_{\sigma,t-1} \right\}$$

By (17), (19), (51) and (52), we have

$$\|x\|_{\sigma,t} \leq \gamma(t) \|u\|_{\infty,t} + \tilde{\psi}(t) \|x\|_{\sigma,t-1}. \quad (53)$$

Next, by applying the Grönwall-Bellman Lemma 19 on (53), it is true that for $t \in [t_i-1 + 1, t_i]$, we have

$$\|x\|_{\infty,t_i} \leq \left( \prod_{\tau=t_i-1+1}^{t} \tilde{\psi}(\tau) \right) \|x\|_{\infty,t_{i-1}} + \gamma(t) \|u\|_{\infty,t} \quad \text{(54)}$$

$$\leq \left( \sum_{\tau=t_i-1+1}^{t} \tilde{\psi}(\tau) \right) \|x\|_{\infty,t_{i-1}} + \gamma(t) \|u\|_{\infty,t} \quad \text{(55)}$$

where (54) is by $\tilde{\psi}(t) \geq \gamma^{-1}$ for all $t \in \mathbb{Z}$ and (20), (55) is by (20) and (18). By applying the Grönwall-Bellman Lemma 19 on (55), we have

$$\|x\|_{\sigma,t} \leq \rho^{i-t} \|x\|_{\sigma,t_0} + \sum_{k=t_i}^{j} \rho^{i-t} \|u\|_{\infty,t_k} \quad \text{(56)}$$

$$\leq \rho^{i-t} \|x\|_{\sigma,t_0} + \frac{\beta}{1-\rho} \|u\|_{\infty,t_k} \quad \text{(57)}$$

where (56) is by $\rho_j^{i-t} \leq \rho$ and (57) is by $\|u\|_{\infty,t_0} \leq \|u\|_{\infty,t_k}$ respectively. Next, by choosing $a$ such that $\|x\|_{\sigma,t_i} = 0$, the following inequality holds:

$$\|x\|_{\sigma,t_{i-1}} \leq \frac{\beta}{1-\rho} \|u\|_{\infty,t_{i-1}} \quad \text{(58)}$$

Therefore by (55) and (58), for all $t \in [t_{i-1} + 1, t_i]$, we have

$$\|x\|_{\sigma,t} \leq (\sigma \rho_j^t) \|x\|_{\sigma,t_{i-1}} + \frac{\beta}{1-\rho} \|u\|_{\infty,t_{i-1}} \quad \text{(59)}$$

where (59) and $\cup_{i \in \mathbb{Z}} [t_{i-1} + 1, t_i] = \mathbb{Z}$, we have $\|x\|_{\infty,t} \leq \tilde{\psi}(t) \|u\|_{\infty,t}$ and thus $\|x\|_{\sigma,t} \leq \tilde{\psi}(t) \|u\|_{\infty,t}$.

**Appendix E**

**Proof of Lemma 10**

Since the sufficient condition (27) is a special case of the sufficient conditions (12) and (20) with $\sup_{i \in \mathbb{Z}} s_i \|u\|_{\infty} < \infty, \|x\|_{\infty,\sigma} < \infty, \{t_i \} = \mathbb{Z}$, and the $N$-width average variation rate of $G$ is bounded, it is true that (27) $\Rightarrow$ (12) and (27) $\Rightarrow$ (20). Therefore, to prove that (12) and (20) hold whenever (28) holds and there exist cases where (12) and
hold while (28) does not hold, it suffices to prove (i) (28) ⇒ (27), and (ii) (27) ⇒ (28).

(i) Let \( N = 1 \), and thus \( \hat{d}_{\sigma,N} = \hat{d}_{\sigma,1} \) by (26) and (20). Since \( d_{\sigma,N}(G) \triangleq \sup_{t \in \mathbb{Z}} d_{\sigma,N}(G)(t) \), we have \( (28) \Leftrightarrow (27) \) by \( d_{\sigma,N}(G) = \hat{d}_{\sigma,1}(G) \) and Definitions 8 and 9. Therefore it is true that \( (28) \Rightarrow (27) \).

(ii) Consider a case where \( N > 1, N \left( \frac{a_0}{\sigma} \right)^{-1-N} > 1 \), and a time \( \tau \in \mathbb{Z} \) such that

\[
\| \nabla g_{\tau} \|_{\infty} \in \left( \hat{d}_{\sigma,1}(G), N \hat{d}_{\sigma,N}(G) \right)
\]

and

\[
\sum_{i=t-N+1}^{t} \| \nabla g_{i} \|_{\infty} \leq N \hat{d}_{\sigma,N}, \forall t \in \mathbb{Z}.
\]

Since \( N \left( \frac{a_0}{\sigma} \right)^{-1-N} > 1 \), it is true that \( N \hat{d}_{\sigma,N} > \hat{d}_{\sigma,1} \) and \( \hat{d}_{\sigma,N} \), \( N \hat{d}_{\sigma,N} \) is not empty. Next, by \( \| \nabla g_{\tau} \|_{\infty} > \hat{d}_{\sigma,1} \) (28) does not hold. On the other hand, by \( \hat{d}_{\sigma,N}(G) \triangleq \sup_{t \in \mathbb{Z}} d_{\sigma,N}(G)(t) \) and by \( \| \nabla g_{i} \|_{\infty} \leq \hat{d}_{\sigma,N}(G) \), inequality (27) holds.

Then it is true that \( (27) \Rightarrow (28) \).

By (i), (ii), \( (27) \Rightarrow (28) \) and \( (27) \Rightarrow (20) \), it is true that \( (28) \Rightarrow (20) \), and \( (20) \Rightarrow (28) \).

Therefore the lemma is proved. □

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