Environment-Induced Effects on Quantum Chaos: Decoherence, Delocalization and Irreversibility *

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Abstract

Decoherence in quantum systems which are classically chaotic is studied. It is well-known that a classically chaotic system when quantized loses many prominent chaotic traits. We show that interaction of the quantum system with an environment can under general circumstances quickly diminish quantum coherence and reenact some characteristic classical chaotic behavior. We use the Feynman-Vernon influence functional formalism to study the effect of an ohmic environment at high temperature on two classically-chaotic systems: The linear Arnold cat map (QCM) and the nonlinear quantum kicked rotor (QKR). Features of quantum chaos such as recurrence in QCM and diffusion suppression leading to localization in QKR are destroyed in a short time due to environment-induced decoherence. Decoherence also undermines localization and induces an apparent transition from reversible to irreversible dynamics in quantum chaotic systems.

1 Decoherence and Disappearance of Recurrence in the Quantum Cat Map

Arnold’s cat map is a linear area-preserving map $T$ on a torus in phase space formed by identifying the boundaries of the interval $[0, 2\pi]$ in both the coordinate $Q$ and the momentum $P$ directions [1]. (Because of this the area of the torus is characterized by Planck’s constant which takes on the values $\hbar = 2\pi/\mathcal{N}$, where $\mathcal{N}$ is the number of sites in both the coordinate and the momentum directions in the phase space.) From time step $j$ to $j + 1$ it is given by

$$
\begin{pmatrix}
Q_{j+1} \\
P_{j+1}
\end{pmatrix}
= 
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
Q_j \\
P_j
\end{pmatrix}
= T \begin{pmatrix}
Q_j \\
P_j
\end{pmatrix}
$$

(1.1)
where $\det T = 1$ guarantees area preservation. The degree of chaos depends on the choice of $T$. The eigenvalues of $T$ are either both real or both imaginary. In the latter case, $T$ is elliptic, the motion becomes periodic and no sensitive dependence on the initial condition is observed. When $T$ is hyperbolic, the motion is chaotic.

Quantized cat map is studied in detail by Hannay and Berry [2]. The matrix has to assume a special form in order to yield nontrivial values of the propagator for the map. We choose

$$T_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

for the elliptic and the hyperbolic cases respectively.

For the special choice of the matrix elements $T_1, T_2$ made above the propagator takes on the simple forms,

$$U_1(j+1, j) = \sqrt{\frac{i}{N}} \exp[-\frac{i}{\hbar} Q_j Q_{j+1}], \quad U_2(j+1, j) = \sqrt{\frac{i}{N}} \exp[\frac{i}{\hbar}(Q_j^2 - Q_j Q_{j+1} + Q_{j+1}^2)].$$

Since each iteration describes a permutation among sites, each site belongs to a periodic orbit. Thus the quantum dynamics follows the classical way, resulting in the recurrence of the wave function (or equivalently, the Wigner function [2]).

We now couple the system linearly to a bath of $N$ harmonic oscillators with coordinates $q_\alpha$ and momentum $p_\alpha$ ($\alpha = 1, ..N$) described by the Hamiltonian $H_B$ and the interaction Hamiltonian $H_C$

$$H_B = \sum_{\alpha=1}^N \left( \frac{p_\alpha^2}{2} + \frac{\omega^2_\alpha q_\alpha^2}{2} \right), \quad H_C = \sum_{\alpha=1}^N C_\alpha Q q_\alpha.$$  (1.4)

where $Q$ is the coordinate of the system and $C_\alpha$ is the coupling constant of $Q$ to the $\alpha$th oscillator in the bath. By integrating out the bath variables, we get the reduced density matrix,

$$\rho_r(Q_j, Q_j', t) = \int \prod_{\alpha=1}^N dq_\alpha dq_\alpha' \exp \frac{i}{\hbar} \left[ S(Q) + S_C(Q, q_\alpha) + S_B(q_\alpha) - S(Q') - S_C(Q', q_\alpha') - S_B(q_\alpha') \right].$$

(1.5)

where $S$ is the classical action of the system which appears as the exponent of the propagator in (1.3). $S_B$, and $S_C$ are the actions for bath and interaction, respectively. The propagator $J_r$ for the reduced density matrix from time steps $j$ to $j+1$ is

$$J_r(Q_{j+1}, Q_{j+1}' | Q_j, Q_j', t) = \int DQ DQ' \exp \frac{i}{\hbar} [S(Q) - S(Q') + A(Q, Q')],$$

(1.6)

in a path-integral representation [3, 4, 5], where

$$\frac{i}{\hbar} A(Q, Q') = \frac{1}{\hbar^2} \int_0^t ds \int_0^s ds' r(s)[-i \mu(s - s') R(s') - \nu(s - s') r(s')]$$

(1.7)

is the influence action. Here $r \equiv \frac{Q - Q'}{2}$, $R \equiv \frac{Q + Q'}{2}$, and $\mu(s), \nu(s)$ are the dissipation and noise kernels respectively [3].
If we consider the simplest case of an ohmic bath at high temperature $kT > \hbar \Lambda >> \hbar \omega_\alpha$, and consider times shorter than the relaxation time, then we obtain a Gaussian form for the influence functional, with $\frac{i}{\hbar} A(Q, Q') = - \frac{2M\gamma kT}{\hbar^2} \Sigma_j r_j^2$, where the noise kernel becomes local $\nu(s) = 2M\gamma kT \delta(s)$ and $\gamma$ is the damping coefficient. The unit-time propagator becomes

$$ J_r(Q_{j+1}, Q'_{j+1} | Q_j, Q'_j) = \langle J_r(Q_{j+1}, Q'_{j+1} | Q_j, Q'_j, \xi) \rangle = \langle \exp \frac{i}{\hbar} [S(Q_{j+1}, Q_j) - S(Q'_{j+1}, Q'_j + \xi r_{j+1})] \rangle. $$

(1.8)

Here $\xi$ is a Gaussian white noise given by

$$ \langle \xi \rangle = 0, \quad \langle \exp \frac{i}{\hbar} \xi r \rangle = \exp[- \frac{2M\gamma kT}{\hbar^2} r^2] $$

(1.9)

where $\langle \rangle$ denotes statistical average over noise realization $\xi$. For the elliptic map, we get

$$ J_r(Q_{j+1}, Q'_{j+1} | Q_j, Q'_j, \xi) = \left( \frac{i}{N} \right)^{1/2} \exp \left[ \frac{i}{\hbar} (-r_j R_{j+1} - r_{j+1} R_j + \xi r_{j+1}) \right]. $$

(1.10)

and for the hyperbolic map,

$$ J_r(Q_{j+1}, Q'_{j+1} | Q_j, Q'_j, \xi) = \left( \frac{i}{N} \right)^{1/2} \exp \left[ \frac{i}{\hbar} (2r_j R_j + 2r_{j+1} R_{j+1} - r_j R_{j+1} - r_{j+1} R_j + \xi r_{j+1}) \right] $$

(1.11)

The Wigner function is defined as

$$ W(R, p) = \frac{1}{\pi \hbar} \int_0^{2\pi} \psi(R + r) \psi^*(R - r) \exp \left( \frac{2i}{\hbar} pr \right) dr. $$

(1.12)

where $p$ is the momentum conjugate to $r$. The propagator $K$ for the Wigner function is

$$ K(R_{j+1}, p_{j+1} | R_j, p_j, \xi) = \Sigma_{r_j} \Sigma_{r_{j+1}} J_r(Q_{j+1}, Q'_{j+1} | Q_j, Q'_j, \xi) \exp \left( \frac{2i}{\hbar} (p_j r_j - p_{j+1} r_{j+1}) \right). $$

(1.13)

This is reduced to the form of the classical cat map. For the elliptic case,

$$ R_j = -p_j + \xi, \quad p_j = R_{j+1}. $$

(1.14)

For the hyperbolic case,

$$ R_j = -p_j + 2R_{j+1} + \xi, \quad p_j = -3R_{j+1} + 2p_j - 2\xi. $$

(1.15)

Without noise, quantum evolution follows classical permutation \[4\] the phase space is divided by a finite number of different periodic orbits and the period is known to increase roughly proportional to $N$ with some irregular oscillation. When coupled to a bath, the cat map is exposed to a Gaussian noise in each time step. The discretized noise induces transitions between different periodic orbits in an irregular way. Interaction with an environment blurs the recurrence of physical quantities in the quantum map. Fig.1 shows $Tr \rho^2_r$, etc.
the linearized entropy (with the reversed sign) for various cases. If there is no interaction with the environment, the entropy is constant for both regular and chaotic cases. Quantum recurrence is evident even when the system is chaotic. When interaction sets in, \( Tr \rho^2 \) decays exponentially, showing that the system rapidly decoheres. The rate of decoherence is much faster in chaotic systems than in regular systems [6]. It suggests that recurrence would be less evident in a decohering chaotic system. In Fig. 2, we show the mean displacement of points in the phase space as a function of time steps. This is defined by \( l = \sqrt{\langle \Delta x^2 + \Delta p^2 \rangle} \), where \( \Delta x \) and \( \Delta p \) are the displacements from the initial phase space points, and \( \langle \rangle \) denotes averaging over noise distributions. In the chaotic case, we see that recurrence disappears with just a small amount of noise (Fig. 2a) whereas in the regular case, the same amount of noise does not alter the qualitative picture of recurrence (Fig. 2b). In both cases, the decohered quantum system behaves close to the classical picture in which the regular and chaotic dynamics are clearly distinguished. In spite of the discreteness of the points on the torus, the system behaves effectively classically due to the influence of the environment.

2 Decoherence and Delocalization in the Quantum Kicked Rotor

The kicked rotor is one of the most intensively studied models from both the quantum and classical point of view [7]. The Hamiltonian of the kicked rotor is given by

\[
H = \frac{p^2}{2m} + K \cos x \sum_{j=\infty}^{\infty} \delta(t - j) \tag{2.1}
\]

which describes a one-dimensional rotor subjected to a delta-functional periodic kick at \( t = j \). Here \( x \) is the angle of the rotor with period \( 2\pi \), \( m \) is the moment of inertia, \( p \) is the angular momentum, and \( K \) is the strength of the kick which measures the nonlinearity. When \( K > 1 \), the system becomes chaotic over the entire phase space.

The quantum dynamics of the kicked rotor is given by the corresponding Schrödinger equation

\[
\frac{i\hbar}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t) + K \cos x \sum_{j} \delta(t - j) \psi(x,t) \tag{2.2}
\]

where \( \psi \) is the wave function of the rotor.

Denoting \( \psi_j \) as the wave function \( \psi(x,t) \) at each discrete time \( t = j \), and integrating (2.2) from \( j \) to \( j + 1 \), we obtain

\[
\psi_{j+1}(x) = \exp[-i\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2}] \exp[-i\frac{K \cos x}{\hbar}] \psi_j(x) \tag{2.3}
\]

The quantum kicked rotor (QKR) is known to exhibit dynamical localization. After some relaxation time scale, the wave function becomes exponentially localized in the momentum space [8]. This may be interpreted as a particle moving in a lattice with a quasi-random
potential. This heuristic picture seems to justify the analogy between the quantum kicked rotor to the tight binding model with an exponentially decaying hopping parameter which is known to show Anderson localization \cite{9}. Dynamical localization in this context arises from the suppression of classical diffusive behavior by the quantum dynamics. However, as shown by Ott, et.al. \cite{10}, a small external noise can break the localization. Sufficient amount of noise would induce the quantum system to exhibit classical diffusive behavior. Dittrich and Graham studied this problem \cite{11} by coupling the system to a zero temperature harmonic oscillator bath and analysed solutions to the master equation. Cohen and Fishman presented the most detailed study of this problem for an ohmic bath \cite{12}. Here we want to approach these issues from an environment-induced decoherence point of view \cite{13}. We begin by calculating the density matrix for the kicked rotor coupled to an environment.

We introduce a linear coupling of the system momentum \( p \) with each oscillator coordinate \( q_\alpha (\alpha = 1,..N) \) in the bath in the form \( H_C = \Sigma_{\alpha=1}^{N} C_\alpha q_\alpha p \) (Here \( q,p \) without the subscript \( \alpha \) denote the system coordinate and momentum variables). As before, we assume an ohmic bath and examine the time period where dissipation is small. Under these assumptions, the unit time propagator for the wave function \( U_\xi (j+1, j) \) is given by

\[
U_\xi (j+1, j) = \exp\left[-i \frac{p^2}{2m} \right] \exp\left[-i \frac{K \cos x}{\hbar} \right] \exp\left[-i \frac{\xi p}{\hbar} \right] \tag{2.4}
\]

where, as before, the noise term \( \xi \) arises from using a Gaussian identity in the integral transform of the term involving the noise kernel in the influence functional. Summing over all noise realizations \( \langle \rangle \) gives the desired reduced density matrix,

\[
\rho_{rj}(p, p') = \langle \psi_{j+1, \xi}(p) \psi_{j-1, \xi}(p') \rangle \xi \tag{2.5}
\]

where

\[
\psi_{j+1, \xi}(p) = U_\xi (j+1, j) \psi_{j-1, \xi}(p) \tag{2.6}
\]

Loss of quantum coherence is measured by the density matrix becoming approximately diagonal. \( Tr\rho_r^2 \) can be expressed as

\[
Tr\rho_r^2 = \langle \Sigma_p \Sigma_{p'} \psi_\xi(p) \psi^*_\xi(p') \psi_{\xi'}(p') \psi^*_{\xi'}(p) \rangle_{\xi, \xi'} \tag{2.7}
\]

where \( \langle \rangle_{\xi, \xi'} \) denotes the statistical average of all possible noise histories of two independent noises \( \xi, \xi' \) defined at each time interval from \( j \) to \( j+1 \). At high temperatures \( \xi(\tau), \xi'(\tau) \) are reduced to two time-uncorrelated independent Gaussian white noises defined at each time step.

We see that there is a close relation between the breaking of dynamical localization and quantum decoherence. In Fig.3 we plot the linearized entropy \( Tr\rho_r^2 \) versus the energy \( \langle p^2 \rangle \). This shows that delocalization occurs as quantum coherence breaks down, suggesting that delocalization and decoherence occurs by the same mechanism. As the nonlinearity parameter \( K \) increases, the system decoheres more rapidly. At the same time, the amount of delocalization measured by the diffusion constant increases.
This may be explained in the following way: Because the coupling is through the momentum, the noise term does not involve any nonlinearity. The time scale for the system to lose coherence is given by \( t_D = (\lambda_{dB}/\delta p)^2/\gamma \), where \( \lambda_{dB} = \hbar/\sqrt{2\pi mkT} \) is the thermal de Broglie wavelength, and \( \delta p \) is the relevant momentum scale. After this time, noise will destroy the quantum coherence between such momentum separations. In the kicked rotor case, localization will occur due to the coherence around \( \delta p \sim \Delta \), where \( \Delta \sim \bar{\hbar} K \) is the localization length. Since \( l \sim K^2 \), this gives \( t_D \sim K^{-4} \). This shows that nonlinearity increases the rate of decoherence.

The relation between the diffusion constant \( D \) and the noise strength is given in [10, 12]. For our case, \( K/\bar{\hbar} \gg 1 \) and for weak noises, we can consider the particle as undergoing a random walk with hopping parameter \( 1/t_c \). Then \( D = \Delta^2/t_D = (\Delta^4/\lambda_{dB}^2)/\gamma \).

### 3 Decoherence and Irreversibility in Quantum Chaos

The Wigner function is often used to examine the quantum to classical transition. The Wigner function at time \( t = j \) is defined as

\[
W_j(X, p) = \frac{1}{4\pi\hbar} \int_{-\pi}^{\pi} dy \ e^{i\phi y} \rho_j(x + \frac{y}{2}, x - \frac{y}{2}),
\]

where \( X \equiv \frac{1}{2}(x + x'), y \equiv x - x' \). From (2.3), the unit-time propagator for the Wigner function of the QKR is found to be

\[
W_{j+1}(X, p) = e^{-\frac{K\sin x}{\hbar} \Delta_p} e^{-p\partial_x} W_j(X, p)
\]

where \( \Delta_p \equiv e^{\frac{K}{\hbar} \partial_x} - e^{-\frac{K}{\hbar} \partial_x} \) measures the effect of the kick. We can see the effects of quantum corrections is seen more clearly if we expand \( \Delta_p \) in orders of \( \hbar \):

\[
e^{-\frac{iK\sin x}{\hbar} \Delta_p} \approx e^{-K\sin x \partial_p} e^{\frac{K^2}{2\hbar} K\sin x \partial_p} \ldots
\]

The first exponential contains the classical propagator and the second contains quantum corrections of even orders of \( \hbar \). Thus we get

\[
W_{j+1}(X, p) \approx e^{\frac{K^2}{2\hbar} K\sin x \partial_p^2} W_j(X - (p + K\sin x), p - K\sin x)
\]

where the Wigner function with the new arguments depicts classical evolution. This map alone is the source of stretching and folding of volume in phase space which signify classical chaos.

For a linear system the Wigner function is known to show a smooth convergence to the classical Liouville distribution. But if the system Hamiltonian has a nonlinear term, quantum corrections associated with the higher derivatives of the potential pick up the rapid oscillations in the Wigner function and it no longer has a smooth classical limit [14]. However, upon interaction with an environment, a coarse-grained Wigner function can have a smooth classical limit [15] for nonlinear systems.
If the initial system wavefunction is described by a Gaussian wave packet with width \( \delta p(\gg h) \), we would expect to see a classical-like evolution of the packet at short times. When the width of the contracting wave packet gets so small as comparable to \( \bar{\hbar} \), the effect of quantum corrections from higher \( \bar{\hbar} \) order terms in (3.4) set in. By comparing the classical and quantum terms, we see that quantum corrections will become important when \( \delta p(t) \sim \bar{\hbar} \). Here \( \delta p(t) = \delta p(0) e^{-\lambda t} \), where the Lyapunov exponent \( \lambda \sim \ln(K/2) \). Thus we can deduce the Ehrenfest time for QKR to be \( t_E \sim \frac{1}{\lambda} \ln \frac{\delta p(0)}{\bar{\hbar}} \). Note that in the continuum case, this definition gives us a different time scale for each term in the expansion.

The major effect of the bath (at times short compared with the relaxation time) is the appearance of a diffusion term in (3.4),

\[
W_{j+1}(X, p) \approx e^{D \partial_x^2 + K \sin x \partial_p^4} W_j(X - (p - K \sin x), p - K \sin x)
\]  

(3.5)

Competition amongst the three terms with different physical origins is apparent: The first term in (3.5) is the quantum diffusion term, the second is the quantum correction term, and the third is purely classical evolution. As discussed by Zurek and Paz, if \( D \) is sufficiently large, the effect of quantum corrections becomes inconspicuous. In this case, the diffusion term traces out a small scale oscillating behavior before quantum corrections have a chance to change classical evolution. Then one may expect the time evolution of the Wigner function to be like that of classical evolution with noise. The role of quantum diffusion is to add some Gaussian averaging so that the contracting direction in phase space will be suppressed while it does not affect the stretching direction. As long as the width of the wave packet is large such that the first term is negligible, the evolution should be Liouvillian (time reversible if we assume infinite measurement precision). Furthermore, we expect that after the width of the packet along the contracting direction becomes comparable to the diffusion generated width (in the Gaussian wave packet), the dynamics will start showing irreversible behavior arising from coarse graining (as distinct from irreversibility from instability). Consequently, entropy should increase in this regime. In Fig. 4a, we plot the von Neumann entropy for the dynamics of (3.5). We can see three qualitatively different regimes: I. the Liouville regime: the entropy is constant and the dynamics is time reversible. II. the decohering regime: the entropy keeps increasing due to coarse graining. III. the finite size regime: due to the bounded nature of the phase space, the entropy shows saturation. Our result from quantitative analysis seems to confirm the qualitative description of Zurek and Paz who used the inverted harmonic oscillator potential as a generic source of instability. Since the phase space in their model is not bounded they do not see Regime III. Similar features appear in the quantum cap map (Fig. 4b). In this case, the full quantum dynamics can be calculated in a simple way. Resemblance with the result of a classical rotor with noise is obvious. However, in this case, the stable entropy is smaller than the maximum value which may be explained as a finite (phase space) size effect.

\(^2\)Ehrenfest time is customarily defined as the time when quantum dynamics can be adequately described by the classical equations, i.e., the time \( t < t_E \) when the Wigner function or the expectation value of any observable follow classical trajectories.
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Figure Captions

Figure 1 The linearized entropy (with reversed sign) $Tr\rho^2_r$ is plotted here as a function of time. If there is no environment, the entropy is constant for both hyperbolic and elliptic cases, indicating the purity of the state. For the hyperbolic map, even though classically this system is strongly chaotic, the corresponding quantum system does not show chaotic behavior. This situation changes drastically when the system interacts with a thermal bath: entropy keeps increasing due to coarse graining. Note that in the hyperbolic case (solid line) the rate of entropy increase is greater than in the elliptic case (dotted line). $N = 50$ is used here (also in Fig.2).

Figure 2 The mean phase space point displacement is shown. When there is no environment (dotted line), the system shows recurrence in both hyperbolic (a) and elliptic (b) cases. In the presence of an environment, the hyperbolic map loses the recurrence behavior (solid line) under a Gaussian noise with $\sigma = 0.08$ and maintains a near-constant value, indicating the ergodicity of the classical map. On the other hand, the elliptic map still shows recurrence with the same amount of noise, suggesting classical periodicity.

Figure 3 $Tr\rho^2_r$ (solid line, left scale) and $<p^2>$ (dashed line, right scale) are plotted against time for $K = 12$ and $\hbar = 1.52$. The upper solid line and the lower dashed line correspond to the case when there is noise, with $\sigma = 0.5$. As the noise strength increases to $\sigma = 1.5$, the decoherence time shortens, and $Tr\rho^2_r$ decays rapidly (the lower solid line). This accompanies the increase of diffusive behavior in $<p^2>$ (upper dashed line).

Figure 4 The von Neumann entropy is plotted versus time for (a) the quantum kicked rotor with an environment. Here, $\hbar = 1.52$, $\sigma = 0.08$ and $K = 1.2$. Entropy stays at zero (reversible dynamics) until a transition regime, after which the dynamics becomes irreversible. (b) the quantum cat map, with the same parameters and the same amount of noise. We see the same qualitative feature as in the QKR case.

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