A Note on the Inversion Complexity of Boolean Functions in Boolean Formulas

Hiroki Morizumi
Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan
morizumi@ecei.tohoku.ac.jp

Abstract

In this note, we consider the minimum number of NOT operators in a Boolean formula representing a Boolean function. In circuit complexity theory, the minimum number of NOT gates in a Boolean circuit computing a Boolean function \( f \) is called the inversion complexity of \( f \). In 1958, Markov determined the inversion complexity of every Boolean function and particularly proved that \( \lceil \log_2(n+1) \rceil \) NOT gates are sufficient to compute any Boolean function on \( n \) variables. As far as we know, no result is known for inversion complexity in Boolean formulas, i.e., the minimum number of NOT operators in a Boolean formula representing a Boolean function. The aim of this note is showing that we can determine the inversion complexity of every Boolean function in Boolean formulas by arguments based on the study of circuit complexity.

1 Introduction

When we consider Boolean circuits with a limited number of NOT gates, there is a basic question: Can a given Boolean function be computed by a circuit with a limited number of NOT gates? This question has been answered by Markov [2] in 1958. The inversion complexity of a Boolean function \( f \) is the minimum number of NOT gates required to construct a Boolean circuit computing \( f \), and Markov completely determined the inversion complexity of every Boolean function \( f \). In particular, it has been shown that \( \lceil \log_2(n+1) \rceil \) NOT gates are sufficient to compute any Boolean function.

After more than 30 years from the result of Markov, Santha and Wilson [4] investigated the inversion complexity in constant depth circuits and showed that on the restriction \( \lceil \log_2(n+1) \rceil \) NOT gates are not sufficient to compute a Boolean function. The result has been extended to bounded depth circuits by Sung and Tanaka [5]. Recently we completely determined the inversion complexity of every Boolean function in non-deterministic circuits, and particularly proved that one NOT gate is sufficient to compute any Boolean function if we can use an arbitrary number of guess inputs [3].

A Boolean circuit whose gates have fan-out one is called a formula. Formulas are one of well-studied circuit models in circuit complexity theory. Note that a Boolean circuit whose gates have fan-out one corresponds to a Boolean formula. In this note, we investigate the inversion complexity in formulas, which corresponds to the minimum number of NOT operators in a Boolean formula representing a Boolean function. As far as we know,
there is no result for the inversion complexity in formulas. We completely determine the inversion complexity of every Boolean function in formulas.

2 Preliminaries

A circuit is an acyclic Boolean circuit which consists of AND gates of fan-in two, OR gates of fan-in two and NOT gates. A formula is a circuit whose gates have fan-out one. We denote the number of NOT gates in a formula $C$ by $\text{not}(C)$.

Let $x$ and $x'$ be Boolean vectors in $\{0, 1\}^n$. $x \leq x'$ means $x_i \leq x'_i$ for all $1 \leq i \leq n$. $x < x'$ means $x \leq x'$ and $x_i < x'_i$ for some $i$. A Boolean function $f$ is called monotone if $f(x) \leq f(x')$ whenever $x \leq x'$.

A chain is an increasing sequence $x^1 < x^2 < \cdots < x^k$ of Boolean vectors in $\{0, 1\}^n$. The decrease $d_X(f)$ of a Boolean function $f$ on a chain $X$ is the number of indices $i$ such that $f(x^i) \nleq f(x^{i+1})$ for $1 \leq i \leq k - 1$. The decrease $d(f)$ of $f$ is the maximum of $d_X(f)$ over all increasing sequences $X$. We denote the inversion complexity of a Boolean function $f$ in circuits by $I(f)$. Markov gave the tight bound of $I(f)$ for every Boolean function $f$.

**Proposition 1** (Markov[2]). For every Boolean function $f$,

$$I(f) = \lceil \log_2(d(f) + 1) \rceil.$$  

3 Inversion Complexity in formulas

3.1 Result

We denote by $I_{\forall}(f)$ the inversion complexity of a Boolean function $f$ in formulas. We consider only single-output Boolean functions. The result of this note is the following one.

**Theorem 1.** For every Boolean function $f$,

$$I_{\forall}(f) = d(f).$$

In the rest, we prove Theorem 1.

3.2 Upper bound

We prove $I_{\forall}(f) \leq d(f)$. We use a similar argument to one which is used to prove Proposition 1.

**Proof (the upper bound of $I_{\forall}(f)$).** We use induction on $d(f)$.

*Base:* $d(f) = 0$. Then $f$ is monotone and $I_{\forall}(f) = 0$.

*Induction Step:* Suppose

$$I_{\forall}(f') \leq d(f')$$

for every Boolean function $f'$ such that $d(f') \leq d(f) - 1$. 



2
First we separate $f$ to two functions $f_1$ and $f_2$ as follows. See Fig. 1. Let $S$ be the set of all vectors $x \in \{0, 1\}^n$ such that for every chain $X$ starting with $x$, $d_X(f) = 0$. We define $f_1$ and $f_2$ as follows:

$$f_1(x) = \begin{cases} f(x) & \text{if } x \in S; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_2(x) = \begin{cases} 1 & \text{if } x \in S; \\ f(x) & \text{otherwise.} \end{cases}$$

We define $f_t$ as follows:

$$f_t(x) = \begin{cases} 1 & \text{if } x \in S; \\ 0 & \text{otherwise.} \end{cases}$$

By the definitions of $f_1$ and $S$,

$$d(f_1) = 0. \quad (1)$$

Next we show that $d(f_t) = 0$. Let $x$ and $x'$ be Boolean vectors in $\{0, 1\}^n$ such that $x \leq x'$. Suppose that $f_t(x') = 0$, i.e., $x' \notin S$. Since $x' \notin S$, there is a chain $X'$ starting with $x'$ and such that $d_{X'}(f) \geq 1$. Then for a chain $X$ which starts with $x$ and includes $X'$, $d_X(f) \geq 1$. Therefore $x \notin S$. Thus if $f_t(x') = 0$, then $f_t(x) = 0$, which means

$$d(f_t) = 0. \quad (2)$$

Finally we show that

$$d(f_2) \leq d(f) - 1. \quad (3)$$

We assume that $d(f_2) > d(f) - 1$. Since $f_2(x) = 1$ for $x \in S$, there is a chain $X_1$ ending in a vector $x' \notin S$ and such that $d_{X_1}(f) = d_{X_1}(f_2) > d(f) - 1$. Since the $x'$ is not in $S$, there is a chain $X_2$ starting with $x'$ and such that $d_{X_2}(f) \geq 1$. Let $X'$ be the chain which is obtained by connecting $X_1$ and $X_2$. Then,

$$d_X(f) = d_{X_1}(f) + d_{X_2}(f) > (d(f) - 1) + 1 = d(f).$$
Thus a contradiction happens.

By the supposition and Eq. (1) to (3), there are a formula $C_2$ computing $f_2$ such that $not(C_2) \leq d(f_2)$ and formulas $C_1$ and $C_t$ computing $f_1$ and $f_t$ respectively such that $not(C_1) = not(C_t) = 0$. We construct a formula $C$ computing $f$ from $C_1, C_2$ and $C_t$ as $C$ computes the following:

$$f_1 \lor (f_2 \land \neg f_t).$$

The number of NOT gates in $C$ is

$$not(C) = not(C_1) + not(C_2) + not(C_t) + 1 \leq d(f_2) + 1 \leq d(f)$$

We show that $C$ computes $f$ for each of the following two cases.

Case 1: The input $x$ is in $S$.
Then $f_1(x) = f(x)$ and $f_t(x) = 1$. Therefore

$$f_1 \lor (f_2 \land \neg f_t) = f_1 = f.$$ 

Case 2: The input $x$ is not in $S$.
Then $f_1(x) = 0$, $f_2(x) = f(x)$ and $f_t(x) = 0$. Therefore

$$f_1 \lor (f_2 \land \neg f_t) = f_2 = f.$$ 

Thus the formula $C$ computes $f$ and has at most $d(f)$ NOT gates. Therefore $I_{for}(f) \leq d(f).$

\[\square\]

### 3.3 Lower bound

We prove $I_{for}(f) \geq d(f)$. If the input of a NOT gate $N$ is 0 and the output is 1, then we call the state of $N$ up. If otherwise, we call the state down. We denote by $not_d(C, x)$ the number of NOT gates whose states are down in a formula $C$ given $x$ as the input of $C$.

**Proof (the lower bound of $I_{for}(f)$).** Let $C$ be a formula computing $f$. Let $X$ be an increasing sequence $x^1 < x^2 < \cdots < x^k$ of Boolean vectors in $\{0, 1\}^n$ such that $d_X(f) = d(f)$.

**Lemma 1.** Let $x$ and $x'$ be Boolean vectors in $\{0, 1\}^n$ such that $x < x'$, $f(x) = 1$ and $f(x') = 0$. Then,

$$not_d(C, x') - not_d(C, x) \geq 1.$$ 

**Proof.** We change the input of $C$ from $x$ to $x'$. Let $N_1, N_2, \ldots, N_m$ be all NOT gates which change from down state to up state at the time. Since $x < x'$, each $N_i$ for $1 \leq i \leq m$ is connected from $N_i'$ which changes from up state to down state by a path including no NOT gate. Since the output of $C$ changes from 1 to 0, the output of $C$ is also connected from $N_0'$ which changes from up state to down state by a path including no NOT gate. $N_1', N_2', \ldots, N_m'$ and $N_0'$ are distinguished from each other, since $C$ is a formula. Thus the number of NOT gates whose states are down increases by at least one. \[\square\]
Lemma 2. Let $x$ and $x'$ be Boolean vectors in $\{0,1\}^n$ such that $x < x'$. Then,
\[ \text{not}_d(C, x') - \text{not}_d(C, x) \geq 0. \]

Proof. We can use a similar argument to one of Lemma 1. In this case, we do not consider $N'_o$.

Since on $X$ the number of indices $i$ such that $f(x^i) = 1$ and $f(x^{i+1}) = 0$ is at least $d(f)$, by Lemma 1 and 2,
\[ \text{not}_d(C, x^k) - \text{not}_d(C, x^1) \geq d(f). \]
Thus $C$ includes at least $d(f)$ NOT gates.

References

[1] M. Fischer, Lectures on network complexity, Technical Report 1104, CS Department, Yale University, 1974 (revised 1996).

[2] A. A. Markov, On the inversion complexity of a system of functions, J. ACM 5(4), pp. 331–334, 1958.

[3] H. Morizumi, Limiting negations in non-deterministic circuits, manuscript, 2008.

[4] M. Santha and C. Wilson, Limiting negations in constant depth circuits, SIAM J. Comput. 22(2), pp. 294–302, 1993.

[5] S. Sung and K. Tanaka, Limiting negations in bounded-depth circuits: an extension of Markov’s theorem, Proc. of 14th ISAAC, LNCS vol. 2906, pp. 108–116, 2003.