Calculation of 1–loop Hexagon Amplitudes in the Yukawa Model

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Abstract

We calculate a class of one–loop six–point amplitudes in the Yukawa model. The construction of multi–particle amplitudes is done in the string inspired formalism and compared to the Feynman diagrammatic approach. We show that there exists a surprisingly efficient way of calculating such amplitudes by using cyclic identities of kinematic coefficients and discuss in detail cancellation mechanisms of spurious terms. A collection of formulas which are useful for the calculation of massless hexagon amplitudes is given.
1 Introduction

Present and future collider experiments will provide us with more and more experimental data containing information on multi–particle final states. With increasing precision the quest for the inclusion of quantum corrections arises. This is especially true for QCD observables, as calculations on the Born level are typically plagued by large scale uncertainties and therefore are hardly predictive.

Whereas next-to-leading order calculations for $2 \rightarrow 3$ processes have become available in the last years [1, 2, 3, 4], the step to $2 \rightarrow 4$ processes, or even higher, has not been made yet. The reason lies in the fact that the computation of the corresponding amplitudes is highly nontrivial. Although iterative reduction methods allowing for a brute force approach are understood for such amplitudes [5], it turns out that it is necessary to understand better recombination and simplification mechanisms at intermediate steps of the calculation in order to avoid intractably large expressions.

In the last few years methods either directly based on string perturbation theory [6, 7] or on a string-like rewriting of field theory amplitudes [8] have been used to derive a number of “master formulas” for one-loop $N$–point amplitudes. Those are generating functionals which upon expansion yield, for any $N$, a closed parameter integral expression for the amplitude. At the one-loop level, master formulas have been derived for the QCD gluon amplitudes on–shell [9] as well as off–shell [9], the scalar/spinor QED photon amplitudes in vacuum [3] as well as in a constant field [10], the graviton amplitudes in quantum gravity [6], and for amplitudes involving a fermion loop and either vectors and axialvectors [11] or scalars and pseudoscalars [12]. A multi-loop generalisation exists for the case of the QED photon amplitudes [6]. The resulting integral representations are related to standard Feynman parameter integrals in a well-understood way [14]. Nevertheless, due to their superior organisation they often allow one to exploit at the integral level properties of an amplitude which normally would be seen only at later stages in a Feynman graph calculation [12].

Although the string inspired formalism allows for an elegant formulation of amplitudes in terms of a manifest Lorentz structure one is, except in certain particularly favourable cases [6, 12], not at all dispensed from doing cumbersome algebraic work. The complexity of doing tensor reduction in momentum space translates into the need to reduce Feynman parameter integrals with nontrivial numerators to genuine $N$–point scalar integrals. Finally these have to be reduced further down to known scalar integrals. Substantial cancellations are typical in all these steps and progress in finding efficient calculation methods relies on a better understanding of these mechanisms.

The calculation presented here was triggered by the observation that the string inspired master formulas derived in [12] for the one-loop $N$–point functions in the Yukawa model allow one to directly express these Green’s functions in terms of scalar parameter integrals, without the appearance of tensor integrals which one would normally expect to encounter in a parameter integral computation of a fermion loop amplitude. Their computation should therefore be considerably simpler than the one of the $N$–photon or gluon amplitudes, so that their study can serve as an intermediate step towards the computation of gauge theory amplitudes.

In section 2 we will present the construction of the $N$–point scalar amplitude coming from the Yukawa couplings in the string inspired formalism. In section 3 we will formulate the same amplitude in terms of Feynman diagrams and show the equivalence of the expressions. Then we will derive a compact expression for the amplitude with special emphasis on the cancellation mechanisms at work. Section 4 contains our conclusions.

2 Constructing multi–particle amplitudes in the string inspired formalism

The minimal setting for the amplitudes in question is a Yukawa Model with both a scalar $\phi$ and a pseudoscalar $\phi_5$,

$$L_{\text{yuk}} = \bar{\psi} \left[ i\partial - m - g\phi - ig_5\gamma^5\phi_5 \right] \psi + L_{\phi,\phi_5}$$ \hspace{1cm} (1)
We did not write out the scalar/pseudoscalar part of the Lagrangian, \( L_{\phi, \phi_5} \), as it is not used in the following. Still we want to note that the quartic interactions have to be included if the model is to be renormalizable, since the (pseudo)scalar four-point functions are divergent.

Based on earlier work on the worldline representation of Yukawa couplings (refs. [13, 16]), in the following master formulas for the fermion loop contributions to the one-particle irreducible one-loop amplitudes with an even number of legs \( N \) have been derived: For the \( N - \) scalar case this formula reads4

\[
\Gamma_{\phi \phi \phi \phi^{\prime}}^{[p_1, \ldots, p_N]} = -2(i g)^N \int_0^\infty \frac{dT}{T} e^{-m^2T} (4\pi T)^{-\frac{D}{2}} \prod_{i=1}^N \int_0^T d\tau_i \int d\theta_1 \cdots \int d\theta_N \int d\varepsilon_1 \cdots \int d\varepsilon_N \times \exp \left\{ -\sum_{i,j=1}^N \left[ \frac{1}{2} (G_{Bij} + \theta_i \theta_j G_{Fij}) p_i \cdot p_j + \frac{1}{2} (G_{Fij} + \theta_i \theta_j 2\delta_{ij}) \varepsilon_i \varepsilon_j \right] + 2 i m \sum_{j=1}^N \varepsilon_j \theta_j \right\}
\]

Here \( T \) is the global proper-time variable for the loop fermion, and \( G_{Bij}, G_{Fij}, \delta_{ij} \) denote the basic worldline Green’s functions13

\[
G_{Bij} = | \tau_i - \tau_j | - \frac{(\tau_i - \tau_j)^2}{T}
\]

\[
G_{Fij} = \text{sign}(\tau_i - \tau_j)
\]

\[
\delta_{ij} = \delta(\tau_i - \tau_j)
\]

The Grassmann variables \( \theta_1, \ldots, \theta_N \) and “polarisation scalars” \( \varepsilon_1, \ldots, \varepsilon_N \) are all anticommuting with each other, as well as with \( d\theta_1, \ldots, d\theta_N \) and \( d\varepsilon_1, \ldots, d\varepsilon_N \). The Grassmann integration rules are \( \int d\theta_i \theta_i = \int d\varepsilon_i \varepsilon_i = 1 \).

The corresponding formula for the pseudoscalar case is somewhat simpler,

\[
\Gamma_{\phi \phi \phi \phi^{\prime}}^{[p_1, \ldots, p_N]} = -2(i g_5)^N \int_0^\infty \frac{dT}{T} e^{-m^2T} (4\pi T)^{-\frac{D}{2}} \prod_{i=1}^N \int_0^T d\tau_i \int d\theta_1 \cdots \int d\theta_N \int d\varepsilon_1 \cdots \int d\varepsilon_N \times \exp \left\{ -\sum_{i,j=1}^N \left[ \frac{1}{2} (G_{Bij} + \theta_i \theta_j G_{Fij}) p_i \cdot p_j + \frac{1}{2} (G_{Fij} + \theta_i \theta_j 2\delta_{ij}) \varepsilon_i \varepsilon_j \right] \right\}
\]

Note that these formulas are also valid off-shell. In the massless case, which is the one which we are going to exploit in the present paper, both formulas coincide, as they should since here the chiral symmetry is unbroken. In contrast to, e.g., the master formula for the \( N - \) photon amplitude, for the case at hand the result of the Grassmann integrations is easy to write down in closed form. It is a matter of simple combinatorics to see that

\[
\int d\theta_1 \cdots d\theta_N \int d\varepsilon_1 \cdots d\varepsilon_1 \exp \left\{ -\sum_{i,j=1}^N \left[ \frac{1}{2} (G_{Bij} + \theta_i \theta_j G_{Fij}) p_i \cdot p_j + \frac{1}{2} (G_{Fij} + \theta_i \theta_j 2\delta_{ij}) \varepsilon_i \varepsilon_j \right] \right\} = \sum_{j=0}^{N/2} Y_{N,N-j} \exp \left\{ -\sum_{j<l=1}^N G_{Bjl} p_j \cdot p_l \right\}
\]

\(^3\)Note that in contrast to [14] we use here the metric signature \((+, -, -, -)\)

\(^4\)Those appear here because the derivation of these formulas uses dimensional reduction from six-dimensional gauge theory. For details see [12, 13].
where

$$Y_{N,N-j} = \frac{(-1)^{N/2-j}}{j!(N-2j)!} \sum_{\pi \in S_N} \delta_{\pi_1 \pi_2} \delta_{\pi_3 \pi_4} \cdots \delta_{\pi_{2j-1} \pi_{2j}} \text{Alt}(\sigma_{\pi_{2j+1} \pi_{2j+2} \cdots \pi_{2j+N}}, \delta_{\pi_j \pi_{j+1}} \cdots \delta_{\pi_{N-1} \pi_N})$$

$$\times \text{Alt}(\phi_{\pi_{2j+1} \pi_{2j+2} \cdots \pi_{2j+N}}, \phi_{\pi_j \pi_{j+1}} \cdots \phi_{\pi_{N-1} \pi_N})$$

(6)

Here we used the abbreviations

$$\sigma_{ij} = \text{sign}(\tau_i - \tau_j)$$

$$\phi_{ij} = \sigma_{ij} p_i \cdot p_j$$

$$\text{Alt}(T_{j_1 j_2}, T_{j_3 j_4}, \cdots, T_{j_{2k-1} j_{2k}}) = \frac{1}{k!2^k} \sum_{\pi \in S_{2k}} \text{sign}(\pi) T_{\pi_{j_1}, \pi_{j_2}} T_{\pi_{j_3}, \pi_{j_4}} \cdots T_{\pi_{j_{2k-1}}, \pi_{j_{2k}}}$$

(7)

Note that the variable $j$ counts the numbers of “pinches” in a term, so that $(N-j)$ is the number of nontrivial integrations. This integral represents the whole amplitude; the usual summation over “crossed” diagrams here is replaced by the integration over the various ordered sectors of the $N$-fold integral. For a fixed ordering any given term in $Y_{N,N-j}$ produces just a factor times the standard $(N-j)$-point scalar integral $I_{N-j}^N$ in $n$ dimensions. For the standard ordering $\tau_1 > \cdots > \tau_N$ one has

$$\text{Alt}(\sigma_{\pi_1 \pi_2} \cdots \sigma_{\pi_{N-1} \pi_N}) = \text{sign}(\pi)$$

(8)

$$\text{Alt}(\phi_{\pi_1 \pi_2} \cdots \phi_{\pi_{N-1} \pi_N}) = \text{sign}(\pi) \frac{1}{4} \text{tr}(p_1, \ldots, p_n)$$

(9)

Inside the traces contraction of the momenta with Dirac matrices is understood throughout this paper. Thus in the massless case we can write the contribution from this standard sector as

$$\Gamma_{\text{yuk}}^{1 \ldots N}[p_1, \ldots, p_N] = - \frac{g_5^N}{(4\pi)^{n/2}} \frac{1}{N} \mathcal{A}(p_1, \ldots, p_N) = - \frac{g_5^N}{(4\pi)^{n/2}} \frac{1}{N} \sum_{j=0}^{N/2} \mathcal{A}_{N,N-j}(p_1, \ldots, p_N)$$

$$2 \mathcal{A}_{N,N-j}(p_1, \ldots, p_N) = \frac{(-1)^{N}}{j!(N-2j)!} \sum_{\pi \in S_N} \text{tr}(p_{\pi_{j_1+1}}, \ldots, p_{\pi_N}) \int_0^\infty \frac{dT}{T^{n/2+1}}$$

$$\times \int_0^T d \tau_1 \int_0^{\tau_1} d \tau_2 \cdots \int_0^{\tau_{N-1}} d \tau_N \delta_{\pi_1 \pi_2} \cdots \delta_{\pi_{j-1} \pi_j} \exp \left[ - \sum_{j < l = 1}^N G_{Bj} p_j \cdot p_l \right]$$

$$= \sum_{j \text{ pairs}} \text{tr}(p_1, \ldots, p_{r_1-1}, p_{r_1+1}, p_{r_1+2}, \ldots, p_{r_j-1}, p_{r_j+1}, \ldots, p_{N})$$

$$\times I_{N-j}^{N-1}(p_1, \ldots, p_{r_1-1}, p_{r_1} + p_{r_1+1}, p_{r_1+2}, \ldots, p_{r_j} + p_{r_j+1}, \ldots, p_{N})$$

(10)

Here in the last expression the sum runs over all possible ways of deleting $j$ pairs of adjacent $p_j$’s from the trace $\text{tr}(p_1, \ldots, p_N)$ (including the pair $p_N, p_1$). Note that in the standard sector only $\delta$–functions with adjacent indices contribute, and for those we have to take a factor of $1/2$ into account since each contribution is shared between two adjacent sectors.

The whole amplitude is obtained by summing over all permutations of $\Gamma_{\text{yuk}}^{1 \ldots N}[p_1, \ldots, p_N]$ in $p_1, \ldots, p_N$. Due to cyclic and parity invariance this sum contains only $(N-1)!/2$ different terms.

Thus there master formula allows us to directly express this $N$–point amplitude in terms of scalar integrals. Note that from the master formulas it is clear that, in the pseudoscalar case, we can
generalise this result to the massive case simply by replacing all the scalar integrals by their massive counterparts. In the scalar case, to the contrary, a number of additional integrals would appear.

In the following we will focus on the massless case with $N = 6$. All amplitudes with six scalars/pseudoscalars are related by chiral invariance. Setting $g = g_5$ one has

$$
\Gamma_{yuk}^{\phi}[p_1, p_2, p_3, p_4, p_5, p_6] = \Gamma_{yuk}^{\phi_{2 \phi \phi}} = \Gamma_{yuk}^{\phi_2 \phi_2 \phi} = \Gamma_{yuk}^{\phi_3}
$$

(11)

Hence it is enough to compute one of these amplitudes only.

### 3 Calculation of the hexagon amplitude $\Gamma_{yuk}^{\phi}$

Now we turn to the calculation of the hexagon amplitude $\Gamma_{yuk}^{\phi}[p_1, p_2, p_3, p_4, p_5, p_6]$. First we rederive the amplitude in the Feynman diagrammatic approach. The amplitude can be written as a sum over $6!$ permutations of the external momentum vectors $p_1, \ldots, p_6$

$$
\Gamma_{yuk}^{\phi}[p_1, p_2, p_3, p_4, p_5, p_6] = -\frac{g^6}{(4\pi)^{n/2}} \frac{1}{6} \sum_{\pi \in S_6} A(p_{\pi_1}, p_{\pi_2}, p_{\pi_3}, p_{\pi_4}, p_{\pi_5}, p_{\pi_6})
$$

(12)

Each permutation corresponds to a single Feynman diagram. The amplitude for the trivial permutation is given by

$$
A(p_1, p_2, p_3, p_4, p_5, p_6) = \int \frac{d^n k}{(2\pi)^n} \frac{\text{tr}(q_1, q_2, q_3, q_4, q_5, q_6)}{q_1^2 q_2^2 q_3^2 q_4^2 q_5^2 q_6^2}
$$

(13)

where $q_j = k - r_j = k - p_1 - \ldots - p_j$. The trivial permutation corresponds to the standard ordering of the worldline integral $[1]$. From translation invariance it follows that

$$
A(p_j, p_{j+1}, p_{j+2}, p_{j+3}, p_{j+4}, p_{j+5}) = A(p_{j+1}, p_{j+2}, p_{j+3}, p_{j+4}, p_{j+5}, p_j)
$$

(14)

Hence it is enough to sum in (12) only over non cyclic permutations, i.e. $\pi \in S_6/Z_6$, and to multiply by a factor 6. Note that all indices labelling momenta are understood to be mod 6 throughout this paper. Working out the trace gives a sum of products of terms $q_k \cdot q_j$ which can be written as $(j > k)$

$$
2q_k \cdot q_j = -(q_k - q_j)^2 + q_k^2 + q_j^2 = -(p_j + p_{j-1} + \ldots + p_{k+1})^2 + q_k^2 + q_j^2
$$

(15)

It is now immediately clear that the whole loop momentum dependence in the numerators is only through inverse propagators which cancel directly. This means that each graph can simply be represented as a linear combination of scalar integrals. For the trivial permutation we find

$$
2A(p_1, p_2, p_3, p_4, p_5, p_6) = \text{tr}(1) I_3^\alpha(p_{12}, p_{34}, p_{56}) + \text{tr}(1) I_3^\alpha(p_{23}, p_{45}, p_{61}) + \text{tr}(p_1, p_2) I_3^\alpha(p_{12}, p_{34}, p_{56}) + \text{tr}(p_2, p_3) I_3^\alpha(p_{23}, p_{45}, p_{61}) + \text{tr}(p_3, p_4) I_3^\alpha(p_{34}, p_{56}, p_{12}) + \text{tr}(p_4, p_5) I_3^\alpha(p_{45}, p_{56}, p_{12}) + \text{tr}(p_5, p_6) I_3^\alpha(p_{56}, p_{12}, p_{34}) + \text{tr}(p_6, p_1) I_3^\alpha(p_{56}, p_{12}, p_{34}) + \text{tr}(p_1, p_4) I_3^\alpha(p_{12}, p_{34}, p_{56}) + \text{tr}(p_2, p_5) I_3^\alpha(p_{23}, p_{45}, p_{61}) + \text{tr}(p_3, p_6) I_3^\alpha(p_{34}, p_{56}, p_{12}) + \text{tr}(p_1, p_2, p_3, p_4) I_3^\alpha(p_{56}, p_{12}, p_{34}, p_{56}) + \text{tr}(p_2, p_3, p_4, p_5) I_3^\alpha(p_{61}, p_{23}, p_{45}, p_{61}) + \text{tr}(p_3, p_4, p_5, p_6) I_3^\alpha(p_{56}, p_{12}, p_{34}, p_{56}) + \text{tr}(p_4, p_5, p_6, p_1) I_3^\alpha(p_{23}, p_{45}, p_{61}) + \text{tr}(p_5, p_6, p_1, p_2) I_3^\alpha(p_{56}, p_{12}, p_{34}, p_{56}) + \text{tr}(p_6, p_1, p_2) I_3^\alpha(p_{23}, p_{45}, p_{61}) + \text{tr}(p_1, p_2, p_3, p_4, p_5, p_6) I_3^\alpha(p_{12}, p_{34}, p_{56}) + \text{tr}(p_2, p_3, p_4, p_5, p_6, p_1) I_3^\alpha(p_{23}, p_{45}, p_{61}) + \text{tr}(p_3, p_4, p_5, p_6, p_1) I_3^\alpha(p_{34}, p_{56}, p_{12}) + \text{tr}(p_4, p_5, p_6, p_1) I_3^\alpha(p_{45}, p_{56}, p_{12}) + \text{tr}(p_5, p_6, p_1) I_3^\alpha(p_{56}, p_{12}, p_{34}) + \text{tr}(p_6, p_1) I_3^\alpha(p_{56}, p_{12}, p_{34})
$$

(16)
in agreement with formula \[10\] derived in the string inspired formalism. The arguments of the \(N\)-point scalar integrals are the momenta of the external legs. We use the abbreviation \(p_{ijk\ldots} = p_i + p_j + p_k + \ldots\). The spinor traces can be expressed by Mandelstam variables defined by the 9 cuts of the hexagon graph, but the form given above is not only most compact but also most convenient to proceed.

\[
\begin{align*}
\text{tr}(1) &= 4 \\
\text{tr}(p_i,p_j) &= 2s_{ij} \\
\text{tr}(p_1,p_4) &= 2s_{14} = 2(s_{23} + s_{56} - s_{123} - s_{234}) \\
\text{tr}(p_1,p_2,p_3,p_4) &= s_{12}(s_{234} - s_{23}) + s_{23}(s_{56} - s_{34}) + s_{123}(s_{34} - s_{234}) \\
\text{tr}(p_1,p_2,p_3,p_4,p_5,p_6) &= s_{123}s_{234}s_{345} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345} - s_{34}s_{123}s_{61} 
\end{align*}
\]

with \(s_{l,l+1} = (p_l + p_{l+1})^2\), \(s_{l-1,l,l+1} = (p_{l-1} + p_l + p_{l+1})^2\). The remaining traces are defined by cyclic relabelling. In the following the momenta inside the traces will be represented by their indices only.

Before turning to the computation of the amplitude, we first note that the amplitude is free of infrared poles. This can be seen by power counting for the soft and collinear poles separately. To investigate the soft limit we replace in \(13\) \(k^\mu\) by \(\lambda k^\mu\) and let \(\lambda \to 0\). As the integrand times the measure behave as \(\lambda^d d\lambda/\lambda^4\), no poles related with the soft limit \(\lambda \to 0\) are present. To see if collinear limits lead to a divergence it is enough to study the limit \(k||p_1\), i.e. \((k - p_1)^2 \to 0\) with \(|k^\mu|\) nonzero for at least one component. To do so we parametrise the loop momentum as

\[
k^\mu = zp_1^\mu + \frac{k^2 + k_{\perp}^2}{2 z n \cdot p_1} n^\mu + k_T^\mu
\]

Here \(n^\mu\) is an arbitrary light-like four vector not collinear to \(p_1\) with \(\vec{n}_\perp T = 0\). The only dangerous propagator in this collinear limit is \((k - p_1)^2 = -(k^2(1 - x) + k_T^2)/x\). It is easy to see that the numerator is proportional to \(k_T\) and thus the \(k_T\) integration does not lead to a pole in the collinear limit \(k \to xp_1\), since the integral behaves as

\[
\int_0^1 dk_T^2 (k_T^2)^{-1/2 + \epsilon} (\text{const.} + \mathcal{O}(k_T^2))
\]

Physically speaking the collinear splitting of a massless spin 0 particle into a massless fermion antifermion pair is infrared safe.

We turn now to the explicit calculation of the hexagon amplitude. We will draw special attention to the cancellation mechanisms of the spurious poles and to spurious finite terms. First we will reduce hexagon and pentagon integrals to box integrals. Then the explicit expressions for the box integrals are inserted. Finally the coefficients of different terms are combined and simplified by using linear relations for the reduction coefficients. We note already that in none of these steps the size of the expression will blow up.

To deal with the \(N\)-point scalar integrals one has to use reduction formulas \([1, 5]\). Pentagon integrals can always be represented in terms of box integrals plus a term which is of order \(\epsilon\), while hexagon integrals decay into pentagon integrals. Following \([5]\), the reduction formula for the hexagon integral reads:

\[\text{By Lorentz invariance one can choose } p_1 = p_1^0(1, \vec{0}_T, 1).\]
The coefficients $b_j$ are defined by the linear equation

$$(\hat{S} \cdot b)_j = 1 \iff b_j = \sum_{k=1}^{6} \hat{S}_{kj}^{-1} \text{ where } \hat{S}_{kj} = (r_k - r_j)^2$$  \hspace{1cm} (21)$$

The Gram matrix $G_{kl} = 2 r_l \cdot r_k$ is related to $\hat{S}$ by $\hat{S}_{kl} = -G_{kl} + r_k^2 + r_l^2$. For $N \geq 6$ and 4-dimensional external momenta one has $\det(G) = 0$, which leads to a non-linear constraint between the Mandelstam variables. We note that this constraint is represented linearly in terms of the coefficients $b_j$. One has

$$\det(G) = 0 \iff \sum_{j=1}^{6} b_j = 0$$  \hspace{1cm} (22)$$

By solving eq. (21) with Cramer’s rule one sees that the constraint (22) relates sums of determinants of 5 by 5 matrices. Expressing it in terms of Mandelstam variables leads to a huge expression just representing zero. The guideline to keep the sizes of expressions under control in calculations of multi-particle processes is thus to use representations of amplitudes where the $b_j$ are kept manifestly and to use relations (21) and (22) to perform cancellations as far as possible.

Applying the reduction formula (20) above to reduce the hexagon, we observe that the coefficients of the hexagon and pentagon integrals in the amplitude combine in a nice way to form a resulting coefficient for a given pentagon which is again proportional to $b_j$. The resulting coefficient of
\( I^n_r (p_{12}, p_3, p_4, p_5, p_6) \) in (41) is

\[ \text{tr}(3456) + \text{tr}(123456) b_1 = -2s_{34}s_{45}s_{56} b_4, \]

(23)

analogous for all cyclic permutations. Not only the size of the coefficients did not increase but also one can still use linear relations. This will turn out to be of major importance in what follows. Now we reduce the pentagons to boxes using the reduction formula (33) given in the appendix. We obtain

\[
A(p_1, p_2, p_3, p_4, p_5, p_6) = \frac{2}{3} I^n_3 (p_{12}, p_{34}, p_{56}) + \frac{\text{tr}(14)}{4} I^n_4 (p_1, p_{23}, p_4, p_{56}) + \frac{b_2}{2E_2} \left( s_{23}s_{34} \left[ \text{tr}(1234) - 2s_{12} (s_{234} - s_{23}) \right] I^n_4 (p_2, p_3, p_4, p_{56}) + s_{12}s_{23} \left[ \text{tr}(1234) - 2s_{34} (s_{123} - s_{23}) \right] I^n_4 (p_1, p_{23}, p_4, p_{56}) + \text{tr}(1234) E_2 I^n_4 (p_1, p_{23}, p_4, p_{56}) \right) + 5 \text{cyclic permutations} \quad (24)
\]

As a shorthand notation we use \( E_1 = s_{123}s_{345} - s_{12}s_{45} \). The \( E_j \) for \( j > 1 \) are defined by cyclic permutation. Note that \( E_j = E_{j+3} \).

The amplitude is now expressed in terms of four functions: The triangle with all three legs off-shell, box integrals with two off-shell legs at adjacent corners \( I^n_4 (p_1, p_{23}, p_4, p_{56}) \) and 5 permutations), box integrals with two off-shell legs at opposite corners \( I^n_4 (p_1, p_{23}, p_4, p_{56}) \) and 2 permutations, and box integrals with one off-shell leg \( I^n_4 (p_1, p_2, p_3, p_{56}) \) and 5 permutations). We now collect and combine the coefficients of particular terms in the cyclic sum. Already at this stage nontrivial cancellations happen. First consider the coefficient \( C^{op} \) of the "opposite" box \( I^n_4 (p_1, p_{23}, p_4, p_{56}) = I^n_4 (p_4, p_{56}, p_1, p_{23}) \) in (24). It is given by

\[
C^{op} = \frac{1}{2} \left\{ \text{tr}(14) + b_2 \text{tr}(1234) + b_5 \text{tr}(4561) \right\} = \left( s_{56} + s_{23} - s_{234} - s_{123} \right) - b_2 E_2 - s_{12} (s_{234} - s_{23}) - s_{34} (s_{123} - s_{23}) / 2 \]

\( -b_5 E_2 - s_{45} (s_{234} - s_{56}) - s_{61} (s_{123} - s_{56}) / 2 \) \quad (25)

Using \( \hat{S} \cdot b = 1 \) to replace \( b_2 s_{12}, b_2 s_{34}, b_5 s_{45}, b_5 s_{61} \), one finds the useful relation

\[
2 C^{op} = \text{tr}(14) + b_2 \text{tr}(1234) + b_5 \text{tr}(4561) = -E_2 \sum_{j=1}^6 b_j = 0 \quad (26)
\]

Hence the coefficients of the box integrals with two off-shell legs at opposite corners are identically zero! One can combine the coefficients of the adjacent boxes as a linear combination of \( b_j \)'s in a similar way. To investigate further cancellations we insert the expressions for the box integrals given in the appendix, (11) and (12), into (24) to obtain
Expression (27) contains spurious double and single poles. To see the cancellation of the pole terms it is enough to look at one double pole term, e.g. \((-s_{12})^{-\epsilon}/\epsilon^2\), and one single pole term, e.g. \([(-s_{12})^{-\epsilon} - (-s_{123})^{-\epsilon}]/\epsilon^2\), separately. The cancellation of the others then follows by cyclic symmetry. The coefficient of \((-s_{12})^{-\epsilon}/\epsilon^2\) in the cyclic sum in (27) is given by

\[
 b_1 + \frac{b_1}{2E_1} \{ 2 \text{tr}(1234) - 2s_{61} (s_{123} - s_{12}) - 2s_{23} (s_{345} - s_{12}) \} = b_1 + \frac{b_1}{2E_1} \{ -2E_1 \} = 0 \quad (28)
\]

The coefficient of \([(-s_{12})^{-\epsilon} - (-s_{123})^{-\epsilon}]/\epsilon^2\) in the cyclic sum (27) is given by

\[
 \frac{b_1}{2E_1} \{ \text{tr}(6123) - 2s_{61} (s_{123} - s_{12}) \} + \frac{b_2}{2E_2} \{ 2 \text{tr}(1234) - 2s_{34} (s_{123} - s_{23}) - 2s_{12} (s_{234} - s_{23}) \} + \frac{b_4}{2E_4} \{ \text{tr}(3456) - 2s_{56} (s_{345} - s_{45}) \} - b_3 = -b_3 + \frac{b_2}{E_2} (-E_2) - \frac{b_2 + b_3}{E_1} (-E_1) = 0 \quad (29)
\]

where again \(\hat{S} \cdot b = 1\) and (26) have been used. The remaining structures are now \(I_3^\alpha\), \(F_{2\alpha}\) and \(F_1\). The latter two contain dilogarithms with single ratios of Mandelstam variables, products of logarithms and \(\pi^2\) terms. The fact that the single poles stemming from the differences \([(-s_{12})^{-\epsilon} - (-s_{123})^{-\epsilon}]/\epsilon^2\) (which in turn are related to the triangles with two off-shell external legs occurring in the reduction of box integrals) cancel independently from those stemming from the double pole terms (like \((-s_{12})^{-\epsilon}/\epsilon^2\), which are related to the triangles with one off-shell external leg) has an important consequence for the finite part of the amplitude: By examination of expressions (11) to (13) for the box integrals one observes that a term \([(-a)^{-\epsilon} - (-b)^{-\epsilon}]/\epsilon^2\) always is associated with a dilogarithm \(Li_2(1 - a/b)\) in the finite part of the same box integral. Therefore the cancellation of the terms \([(-a)^{-\epsilon} - (-b)^{-\epsilon}]/\epsilon^2\) in the amplitude immediately leads to the cancellation of the dilogarithms. The \(\pi^2\) terms present in the box with one off-shell leg also cancel in (27) due to relations (21) and (20). Hence the only terms which survive are the triangle graphs and some logarithmic terms stemming from the finite parts of the box integrals, such that we finally obtain
\[ A(p_1, p_2, p_3, p_4, p_5, p_6) = G(p_1, p_2, p_3, p_4, p_5, p_6) + 5 \text{ cyclic permutations} \] (30)

with

\[
G(p_1, p_2, p_3, p_4, p_5, p_6) = \frac{2}{3} I_3^N(p_{12}, p_{34}, p_{56})
\]

\[
+ \left\{ \frac{b_1}{E_1} [\text{tr}(6123) - 2s_{61}(s_{123} - s_{12})] + \frac{b_2}{E_2} [\text{tr}(1234) - 2s_{34}(s_{123} - s_{23})] \right\}
\]

\[ \times \log \left( \frac{s_{12}}{s_{123}} \right) \log \left( \frac{s_{23}}{s_{123}} \right) \]

\[ + \left\{ -b_1 + \frac{b_2}{2E_2} [\text{tr}(1234) - 2s_{34}(s_{123} - s_{23})] + \frac{b_6}{2E_6} [\text{tr}(5612) - 2s_{56}(s_{345} - s_{61})] \right\} \]

\[ \times \left[ \log \left( \frac{s_{12}}{s_{234}} \right) \log \left( \frac{s_{56}}{s_{234}} \right) + \log \left( \frac{s_{34}}{s_{234}} \right) \log \left( \frac{s_{12}}{s_{56}} \right) \right] \] (31)

Note that \( G(p_1, p_2, p_3, p_4, p_5, p_6) \) has no spurious singularities. We checked that the numerator of expression (31) vanishes in the limits where its denominator vanishes.

Finally, the full amplitude is given by the sum over permutations of the function \( G \)

\[ \Gamma_y[p_1, p_2, p_3, p_4, p_5, p_6] = -\frac{g^6}{(4\pi)^2} \sum_{\pi \in S_6} G(p_{\pi_1}, p_{\pi_2}, p_{\pi_3}, p_{\pi_4}, p_{\pi_5}, p_{\pi_6}) \] (32)

From (13) it is clear that half of the 6! permutations simply correspond to a change in orientation of the fermion line which does not change the value of the integral. It is thus enough to sum in (32) over orientation conserving permutations, i.e. \( \pi \in S_6/Z_2 \), and to multiply by a factor two.

4 Conclusion

We calculated a certain class of hexagon amplitudes in the Yukawa model. First, \( N \)-point amplitudes with scalars/pseudoscalars as external particles attached to a fermion loop were constructed using string inspired methods. The amplitudes turned out to be represented in terms of scalar integrals only. Thus the Yukawa model is an adequate testing ground to study nontrivial cancellations appearing in scalar integral reductions in isolation from additional complications due to a nontrivial tensor structure in more realistic situations such as gauge theory amplitudes. Focusing on the massless case and \( N = 6 \) for a representative amplitude we first demonstrated the equivalence of the string inspired to the Feynman diagrammatic approach, then we explicitly calculated the amplitude. This was done by using reduction formulas for scalar \( N \)-point integrals. It was shown in detail how cancellations can be made manifest at each step of the calculation by using linear relations between reduction coefficients. This saved us from dealing with large expressions at any stage of the calculation. With the present method there is no explosion of terms typical for multi-leg calculations. The final answer is surprisingly compact and contains — apart from 3-point functions with 3 off-shell legs — only some products of logarithms. A reason for that lies certainly in the fact that the amplitudes under consideration are infrared finite.

In the case of off-shell amplitudes the increasing number of kinematic invariants will lead to larger expressions. Still, reduction coefficients will obey linear relations similar to the ones used in deriving the on-shell amplitudes. The same is true in the case of massive particles. Thus one can expect analogous cancellation mechanisms in both cases. This deserves further study.
As a next step more realistic examples have to be considered including gauge bosons and a nontrivial infrared structure. Again, it is justified to speculate that the recombination of scalar integrals will work similarly. Hopefully this work is a step towards efficient algorithms to calculate multi–particle amplitudes at one loop.

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Appendix

Explicit reduction formulas for scalar integrals

We collect here reduction formulas relevant for the computation of massless hexagon amplitudes. The given formulas are sufficient to deal with any scalar integral arising in the calculation of massless hexagon amplitudes.

The pentagon integrals with one external leg off-shell are reduced by the formula

\[
I_5^n(p_{12}, p_3, p_4, p_5, p_6) = \frac{1}{2E_1} \left\{ \frac{1}{s_{34}} \left[ -\text{tr}(3456) + 2s_{34}(s_{123} - s_{45}) \right] I_4^p(p_4, p_5, p_6, p_{123}) + \frac{1}{s_{56}} \left[ -\text{tr}(3456) + 2s_{56}(s_{345} - s_{45}) \right] I_4^p(p_3, p_4, p_5, p_{612}) + \frac{1}{s_{34}s_{45}s_{56}} \left[ -\text{tr}(3456) \right] E_1 \right\} I_4^n(p_6, p_{12}, p_3, p_{45})
\]

\[
+ \frac{1}{s_{34}s_{45}} \left[ s_{34} \text{tr}(3456) + 2s_{34}s_{45}(s_{345} - s_{12}) \right] I_4^n(p_5, p_6, p_{12}, p_{34}) + \frac{1}{s_{45}s_{56}} \left[ s_{123} \text{tr}(3456) + 2s_{45}s_{56}(s_{123} - s_{12}) \right] I_4^n(p_5, p_6, p_{12}, p_{34}) \]

\[
+ \frac{1}{s_{123}} \left\{ \frac{1}{s_{34}s_{45}s_{56}} \left[ s_{123} \text{tr}(3456) + 2s_{45}s_{56}(s_{123} - s_{12}) \right] I_4^n(p_3, p_4, p_{56}, p_{12}) \right\} \tag{33}
\]

where $E_1 = s_{123}s_{345} - s_{12}s_{45}$. For the boxes three cases have to be distinguished: 2 off–shell legs at opposite corners, 2 off–shell legs at adjacent corners, and 1 off–shell leg. Note that the infrared poles of the boxes are contained in triangle graphs with one and/or two legs off–shell. The off–shell momenta are sums of light–like vectors. For the ”adjacent” case we find

\[
I_3^n(p_1, p_2, p_{34}, p_{56}) = \frac{2s_{34}s_{56} + s_{234}(s_{12} - s_{56} - s_{34})}{s_{234}s_{12}} I_3^n(p_{12}, p_{34}, p_{56}) + \frac{s_{234} - s_{56}}{s_{234}s_{12}} I_3^n(p_1, p_{234}, p_{56}) + \frac{1}{s_{234}} I_3^n(p_1, p_2, p_{3456}) + \frac{s_{234} - s_{34}}{s_{234}s_{12}} I_3^n(p_2, p_{34}, p_{561})
\]

\[
+ 2(n - 3) \frac{s_{34}s_{56} - s_{234}(s_{34} + s_{56} - s_{12} - s_{234})}{s_{12}s_{234}} I_4^{n+2}(p_1, p_2, p_{34}, p_{56}) \tag{34}
\]
The "opposite" case gives

\[
I^4_4(p_1, p_2, p_3, p_{456}) = \frac{s_{123} - s_{56}}{s_{123} s_{34} - s_{23} s_{56}} I^4_3(p_4, p_{56}, p_{123}) + \frac{s_{234} - s_{56}}{s_{123} s_{34} - s_{23} s_{56}} I^4_3(p_1, p_{234}, p_{56})

+ \frac{s_{123} - s_{23}}{s_{123} s_{34} - s_{23} s_{56}} I^4_3(p_1, p_{23}, p_{456}) + \frac{s_{234} - s_{23}}{s_{123} s_{34} - s_{23} s_{56}} I^4_3(p_4, p_{561}, p_{23})

+ 2(n - 3) \left[ \frac{s_{234} + s_{123} - s_{23} - s_{56}}{s_{23} s_{56}} I^{n+2}_4(p_1, p_{23}, p_4, p_{56}) \right]
\]

(35)

and finally the case with one leg off-shell

\[
I^4_n(p_1, p_2, p_3, p_{456}) = \frac{s_{12} - s_{123}}{s_{12} s_{23}} I^3_3(p_3, p_{456}, p_{12}) + \frac{s_{23} - s_{123}}{s_{12} s_{23}} I^3_3(p_1, p_{23}, p_{456})

+ \frac{1}{s_{23}} I^3_3(p_1, p_2, p_{3456}) + \frac{1}{s_{12}} I^3_3(p_2, p_3, p_{4561})

+ 2(n - 3) \frac{s_{12} + s_{23} - s_{123}}{s_{12} s_{23}} I^{n+2}_4(p_1, p_2, p_3, p_{456})
\]

(36)

All dilogarithms are collected in the terms \(I_4^{n+2}\) and the triangles with 3 legs off-shell. In the case of the box with two adjacent legs off–shell, \(I_3^3\) and \(I_4^{n+2}\) combine to a much simpler expression than the single expressions individually. This indicates that the splitting into triangles and remainder terms is only useful, if infrared singularities are present. Explicit formulas for the scalar integrals are given below.

**List of scalar integrals**

The triangles with one and two on–shell legs are given by

\[
I_3^3(p_1, p_2, p_{3456}) = \frac{r_G}{\epsilon^2} \left[ \frac{(-s_{12})^{-\epsilon}}{s_{12}} \right]
\]

\[
I_3^3(p_1, p_{23}, p_{456}) = \frac{r_G}{\epsilon^2} \left[ \frac{(-s_{23})^{-\epsilon} - (-s_{123})^{-\epsilon}}{s_{23} - s_{123}} \right]
\]

\[
r_G = \frac{\Gamma(1 + \epsilon) \Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)}
\]

For the finite triangle with all legs off–shell we quote the four dimensional representation \(\Gamma\)

\[
I_3(p_{12}, p_{34}, p_{56}) = -\frac{1}{\sqrt{\lambda}} \left\{ 2 Li_2 \left[ 1 - \frac{1}{y_2} \right] + 2 Li_2 \left[ 1 - \frac{1}{x_2} \right] + \frac{\pi^2}{3} \right.

+ \frac{1}{2} \left[ \log^2 \left( \frac{x_1}{y_1} \right) + \log^2 \left( \frac{x_2}{y_2} \right) - \log^2 \left( \frac{x_1}{y_1} \right) + \log^2 \left( \frac{x_1}{y_2} \right) \right] \} \]

(39)

\[x_{1,2} = \frac{s_{12} + s_{34} - s_{56} + \sqrt{\lambda}}{2s_{12}}\]

\[y_{1,2} = \frac{s_{12} - s_{34} + s_{56} + \sqrt{\lambda}}{2s_{12}}\]

\[\lambda = s_{12}^2 + s_{34}^2 + s_{56}^2 - 2s_{12}s_{34} - 2s_{34}s_{56} - 2s_{56}s_{12} - i \text{sign}(s_{12})\delta\]
The infinitesimal imaginary assures that the formula is valid in all kinematic regions by using

$$\sqrt{\lambda + i\delta} = \begin{cases} \sqrt{\lambda} + i\delta & , \lambda \geq 0 \\ i\sqrt{-\lambda} & , \lambda < 0 \end{cases}$$

(40)

In the splitting of the box integrals into divergent and finite pieces the grouping of the \((-s_{ij})^{-\epsilon}\) terms is induced by the triangle graphs. We keep this form in which the separation of single and double poles is manifest.

$$I_4^n(p_1, p_2, p_3, p_{456}) = \frac{1}{s_{12}s_{23}} \left\{ \frac{r_T}{\epsilon^2} \left[ (-s_{12})^{-\epsilon} + (-s_{23})^{-\epsilon} + \left((-s_{12})^{-\epsilon} - (-s_{123})^{-\epsilon}\right) \right] - 2 F_1(s_{12}, s_{23}, s_{123}) \right\}$$

(41)

$$I_4^n(p_1, p_2, p_{34}, p_{56}) = \frac{1}{s_{12}s_{23}s_{234}} \left\{ \frac{r_T}{\epsilon^2} \left[ (-s_{12})^{-\epsilon} + \left((-s_{234})^{-\epsilon} - (-s_{34})^{-\epsilon}\right) \right] + \left((-s_{234})^{-\epsilon} - (-s_{34})^{-\epsilon}\right) \right\} - 2 F_{2A}(s_{12}, s_{234}, s_{34}, s_{56})$$

(42)

$$I_4^n(p_1, p_{23}, p_4, p_{56}) = \frac{1}{s_{123}s_{234} - s_{23}s_{56}} \left\{ \frac{r_T}{\epsilon^2} \left[ (-s_{123})^{-\epsilon} - (-s_{23})^{-\epsilon} \right] + \left((-s_{123})^{-\epsilon} - (-s_{56})^{-\epsilon}\right) \right\} - 2 F_{2B}(s_{123}, s_{234}, s_{23}, s_{56})$$

(43)

The finite terms are given by logarithms and dilogarithms.

$$F_1(s_{12}, s_{23}, s_{123}) = - Li_2 \left( 1 - \frac{s_{12}}{s_{123}} \right) - Li_2 \left( 1 - \frac{s_{23}}{s_{123}} \right) - \log \left( \frac{s_{12}}{s_{123}} \right) \log \left( \frac{s_{23}}{s_{123}} \right) + \frac{\pi^2}{6}$$

(44)

$$F_{2A}(s_{12}, s_{234}, s_{34}, s_{56}) = Li_2 \left( 1 - \frac{s_{34}}{s_{234}} \right) + Li_2 \left( 1 - \frac{s_{56}}{s_{234}} \right) + \frac{1}{2} \log \left( \frac{s_{12}}{s_{234}} \right) \log \left( \frac{s_{56}}{s_{234}} \right) + \frac{1}{2} \log \left( \frac{s_{34}}{s_{234}} \right) \log \left( \frac{s_{12}}{s_{56}} \right)$$

(45)

$$F_{2B}(s_{123}, s_{234}, s_{23}, s_{56}) = - Li_2 \left( 1 - \frac{s_{23}56}{s_{123}234} \right) + Li_2 \left( 1 - \frac{s_{23}}{s_{123}} \right) + Li_2 \left( 1 - \frac{s_{56}}{s_{123}} \right) + Li_2 \left( 1 - \frac{s_{56}}{s_{234}} \right) + \frac{1}{2} \log^2 \left( \frac{s_{123}}{s_{234}} \right)$$

(46)
The pentagon integral is now expressible as a pole part and a remainder as follows:

\[
I_5^p(p_{12}, p_{23}, p_{45}, p_6) = I_5^p(p_{12}, p_{23}, p_{45}, p_6)|_{\text{pole part}}
- \frac{1}{s_{34} s_{45} s_{56}} F_{\text{Penta}}^p(s_{123}, s_{34}, s_{45}, s_{56}, s_{45}, s_{12})
\]

(47)

\[
I_5^p(p_{12}, p_{23}, p_{45}, p_6)|_{\text{pole part}} =
\frac{r_p}{\epsilon^2} \left\{ \frac{(-s_{34})^{-\epsilon} + (-s_{45})^{-\epsilon} + (-s_{56})^{-\epsilon}}{s_{123} s_{34} s_{45} s_{56}} \right. \\
+ \frac{(s_{56} - s_{123})[(-s_{123})^{-\epsilon} - (-s_{56})^{-\epsilon}]}{s_{34} s_{45} s_{56} s_{123}} \\
+ \frac{(s_{123} - s_{12})[(-s_{123})^{-\epsilon} - (-s_{12})^{-\epsilon}]}{s_{34} s_{45} s_{56} s_{123}} \\
+ \frac{(s_{123} - s_{45})[(-s_{45})^{-\epsilon} - (-s_{123})^{-\epsilon}]}{s_{45} s_{56} (s_{123} s_{345} - s_{12} s_{45})} \\
\left. + \frac{(s_{45} - s_{12})[(-s_{45})^{-\epsilon} - (-s_{12})^{-\epsilon}]}{s_{45} s_{56} (s_{123} s_{345} - s_{12} s_{45})} \right\}
\]

(48)

\[
F_{\text{Penta}}^p(s_{123}, s_{34}, s_{45}, s_{56}, s_{45}, s_{12}) =
\frac{1}{E_1} \left\{ \left[ E_1 + s_{34} (s_{123} - s_{45}) - s_{56} (s_{45} - s_{45}) \right] F_1(s_{45}, s_{56}, s_{123}) \\
+ \left[ \frac{s_{45} s_{12}}{s_{45}} (s_{45} - s_{12}) - \frac{s_{34} (s_{45} - s_{45})}{s_{34}} E_1 + s_{56} (s_{45} - s_{45}) \right] F_2A(s_{56}, s_{34}, s_{12}, s_{34}) \\
+ \left[ E_1 - s_{34} (s_{123} - s_{45}) - s_{56} (s_{45} - s_{45}) \right] F_2B(s_{45}, s_{123}, s_{12}, s_{45}) \\
+ \left[ \frac{s_{56} s_{12}}{s_{123}} (s_{123} - s_{12}) - \frac{s_{12} (s_{123} - s_{12})}{s_{123}} E_1 + s_{34} (s_{123} - s_{45}) \right] F_2A(s_{34}, s_{123}, s_{56}, s_{12}) \\
+ \left[ E_1 - s_{34} (s_{123} - s_{45}) + s_{56} (s_{45} - s_{45}) \right] F_1(s_{34}, s_{45}, s_{345}) \right\}
\]

(49)
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