POINTWISE CONVERGENCE OF MULTIPLE ERGODIC AVERAGES AND STRICTLY ERGODIC MODELS

By

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Abstract. By building some suitable strictly ergodic models, we prove that for an ergodic system $(X, X, \mu, T)$, $d \in \mathbb{N}$, $f_1, \ldots, f_d \in L^\infty(\mu)$, the averages
\[
\frac{1}{N^2} \sum_{(n,m) \in [0,N-1]^2} f_1(T^n x) f_2(T^{n+m} x) \cdots f_d(T^{n+(d-1)m} x)
\]
converge to a constant $\mu$ a.e.

Deriving some results from the construction, for distal systems we answer positively the question if the multiple ergodic averages converge a.e. That is, we show that if $(X, X, \mu, T)$ is an ergodic distal system, and $f_1, \ldots, f_d \in L^\infty(\mu)$, then the multiple ergodic averages
\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \cdots f_d(T^{dn} x)
\]
converge $\mu$ a.e.

1 Introduction

1.1 Main results. Throughout this paper, by a topological dynamical system (t.d.s. for short) we mean a pair $(X, T)$, where $X$ is a compact metric space and $T$ is a homeomorphism from $X$ to itself. A measurable system (m.p.t. for short) is a quadruple $(X, X, \mu, T)$, where $(X, X, \mu)$ is a Lebesgue probability space and $T : X \to X$ is an invertible measure preserving transformation.

Let $(X, X, \mu, T)$ be an ergodic m.p.t. We say that $(\hat{X}, \hat{T})$ is a topological model (or just a model) for $(X, X, \mu, T)$ if $(\hat{X}, \hat{T})$ is a t.d.s. and there exists an invariant probability measure $\hat{\mu}$ on the Borel $\sigma$-algebra $\mathcal{B}(\hat{X})$ such that the systems $(X, X, \mu, T)$ and $(\hat{X}, \mathcal{B}(\hat{X}), \hat{\mu}, \hat{T})$ are measure theoretically isomorphic.

The well-known Jewett–Krieger theorem [29, 30] states that every ergodic system has a strictly ergodic model. We note that one can add some additional

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properties to the topological model. For example, in [31] Lehrer showed that the strictly ergodic model can be required to be a topological (strongly) mixing system in addition.

Now let $\tau_d(\hat{T}) = \hat{T} \times \cdots \times \hat{T}$ (d times) and $\sigma_d(\hat{T}) = \hat{T} \times \hat{T}^2 \times \cdots \times \hat{T}^d$. The group generated by $\tau_d(\hat{T})$ and $\sigma_d(\hat{T})$ is denoted $\langle \tau_d(\hat{T}), \sigma_d(\hat{T}) \rangle$. For any $x \in \hat{X}$, let $N_d(\hat{X}, x) = \Omega((x, \ldots, x), \langle \tau_d(\hat{T}), \sigma_d(\hat{T}) \rangle)$, the orbit closure of $(x, \ldots, x)$ (d times) under the action of the group $\langle \tau_d(\hat{T}), \sigma_d(\hat{T}) \rangle$. We remark that if $(\hat{X}, \hat{T})$ is minimal, then all $N_d(\hat{X}, x)$ coincide, which will be denoted by $N_d(\hat{X})$. It was shown by Glasner [19] that if $(\hat{X}, \hat{T})$ is minimal, then $(N_d(\hat{X}), \langle \tau_d(\hat{T}), \sigma_d(\hat{T}) \rangle)$ is minimal.

In this paper, first we will show the following theorem.

**Theorem A.** Let $(X, \mathcal{X}, \mu, T)$ be an ergodic m.p.t. and $d \in \mathbb{N}$. Then it has a strictly ergodic model $(\hat{X}, \hat{T})$ such that $(N_d(\hat{X}), \langle \tau_d(\hat{T}), \sigma_d(\hat{T}) \rangle)$ is strictly ergodic.

As a consequence, we have

**Theorem B.** Let $(X, \mathcal{X}, \mu, T)$ be an ergodic m.p.t. and $d \in \mathbb{N}$. Then for $f_1, \ldots, f_d \in L^\infty(\mu)$ the averages

$$\frac{1}{N^2} \sum_{(n,m) \in [0,N-1]^2} f_1(T^nx)f_2(T^{n+m}x) \cdots f_d(T^{n+(d-1)m}x)$$

converge to a constant $\mu$ a.e.

We remark that theorems similar to Theorems A and B can be established for cubes (see [28]). Moreover, the convergence in Theorem B can be stated for any tempered Følner sequence $\{F_N\}_{N \geq 1}$ of $\mathbb{Z}^2$ instead of $[0, N-1]^2$.

It is a long open question if the multiple ergodic averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^nx) \cdots f_d(T^{dn}x)$$

converge a.e. Using some results developed when proving Theorem A, we answer the question positively for distal systems. Namely, we have

**Theorem C.** Let $(X, \mathcal{X}, \mu, T)$ be an ergodic distal system, and $d \in \mathbb{N}$. Then for all $f_1, \ldots, f_d \in L^\infty(\mu)$

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^nx) \cdots f_d(T^{dn}x)$$

converge $\mu$ a.e.
Note that Furstenberg’s structure theorem [18] states that each ergodic system is a weakly mixing extension of an ergodic distal system. Thus, by Theorem C the open question is reduced to dealing with the weakly mixing extensions. To prove Theorem C, we show the following result, which is of independent interest.

**Theorem D.** Let \((X, \mathcal{X}, \mu, T)\) be an ergodic system and \(d \in \mathbb{N}\). Then there exists a family \(\{\mu^{(d)}_x\}_{x \in \mathcal{X}}\) of probability measures on \(X^d\) such that

1. for \(\mu\) a.e. \(x \in X\), \(\mu^{(d)}_x\) is ergodic under \(T \times T^2 \times \cdots \times T^d\),
2. for all \(f_1, \ldots, f_d \in L^\infty(\mu)\),
   \[
   \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^nx)f_2(T^{2n}x)\cdots f_d(T^{dn}x) \rightarrow \int_{X^d} f_1(x_1)f_2(x_2)\cdots f_d(x_d) \, d\mu^{(d)}_x(x_1, x_2, \ldots, x_d)
   
   \text{as } N \to \infty, \text{ where convergence is in } L^2(\mu),
   
   (1.2)
   
3. for \(\mu\) a.e. \(x \in X\), \((p_j)_*\mu^{(d)}_x \ll \mu\) for \(1 \leq j \leq d\), where \(p_j : X^d \to X\) is the projection to the \(j\)-th coordinate.

We note that the idea which we used in this paper to show pointwise convergence theorems can be applied to other situations (together with other tools); see, for example, [14]. Moreover, we have the following conjecture.

**Conjecture.** Let \((X, \mathcal{X}, \mu, T)\) be an ergodic system. Then it has a topological model \((\hat{X}, \hat{T})\) such that for a.e. \(x \in \hat{X}\), \((x, \ldots, x)\) is a generic point of some ergodic measures \(\mu^{(d)}_x\) invariant under \(\hat{T} \times \cdots \times \hat{T}^d\).

We also conjecture that the measures \(\mu^{(d)}_x\) are the ones defined in Theorem D. Once the conjecture is proven, then the multiple ergodic averages converge a.e. by a similar argument that we used to prove Theorem B.

**1.2 Backgrounds.** In this subsection we will give some background for our research.

**1.2.1 Ergodic averages.** In this subsection we recall some results related to pointwise ergodic averages.

The first pointwise ergodic theorem was proved by Birkhoff in 1931. Following Furstenberg’s beautiful work on the dynamical proof of Szemerédi’s theorem in 1977, problems concerning the convergence of multiple ergodic averages (in \(L^2\) or pointwisely) have attracted a lot of attention.
The convergence of the averages
\[ \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{nx}) \cdots f_d(T^{dn}x) \]
in $L^2$ norm was established by Host and Kra [26, Theorem 1.1] (see also Ziegler [46]). We note that in their proofs, the characteristic factors play a great role. The convergence of the multiple ergodic average for commuting transformations was obtained by Tao [38] using the finitary ergodic method; see [3, 25] for more traditional ergodic proofs by Austin and Host respectively. Recently, the convergence of multiple ergodic averages for nilpotent group actions was proved by Walsh [40].

The first breakthrough on pointwise convergence of (1.3) for $d > 1$ is due to Bourgain, who showed in [8] that for $d = 2$, for $p, q \in \mathbb{N}$ and for all $f_1, f_2 \in L^\infty$, the limit of $\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{pn}x)f_2(T^{qn}x)$ exists a.e. Before Bourgain’s work, Lesigne showed this convergence holds if the system is distal, with $T^p, T^q$ and $T^{p-q}$ ergodic [33]. Also in [12, 1], it was shown that the problem of the almost everywhere convergence of (1.3) can be deduced from the case when the m.p.t. has zero entropy. One can also find some results dealing with weakly mixing transformations in [1].

Recently there are some results on the limiting behavior of the averages along cubes, and we refer to [6, 26, 2, 9] for details. Also in [9], Chu and Frantzikinakis obtained the following result. For $i = 1, 2, \ldots, d$, let $T_i : X \to X$ be m.p.t., $f_i \in L^\infty(\mu)$ be functions, $p_i \in \mathbb{Z}[t]$ be non-constant polynomials such that $p_i - p_j$ is non-constant for $i \neq j$, and $b : \mathbb{N} \to \mathbb{N}$ be a sequence such that $b(N) \to \infty$ and $b(N)/N^{1/h} \to 0$ as $N \to \infty$, where $h$ is the maximum degree of the polynomials $p_i$. Then the averages
\[ \frac{1}{Nb(N)} \sum_{1 \leq m \leq N, 1 \leq n \leq b(N)} f_1(T_1^{m+p_1(n)}x) \cdots f_d(T_d^{m+p_d(n)}x) \]
converge pointwise as $N \to \infty$.

1.2.2 Topological models. The pioneering work on topological models was done by Jewett in [29]. He proved the theorem under the additional assumption that $T$ is weakly mixing, and the general case was proved by Krieger in [30] soon after. The papers of Hansel and Raoult [23], Bellow and Furstenberg [4], and Denker [10] gave different proofs of the theorem in the general ergodic case (see also [11]). One can add some additional properties to the topological model. For example, in [31] Lehrer showed that the strictly ergodic model can be required as a topological (strongly) mixing system in addition. Our Theorem A strengthens the
Jewett–Krieger Theorem in another direction, i.e., we can require the model to be strictly ergodic under some group actions on some subsets of the product space.

It is well known that each m.p.t. has a topological model [18]. There are universal models, models for some group actions and models for some special classes. Weiss [42] showed the following nice result: there exists a minimal t.d.s. \((X, T)\) with the property that for every aperiodic ergodic m.p.t. \((Y, \mathfrak{Y}, \nu, S)\) there exists a \(T\)-invariant Borel probability measure \(\mu\) on \(X\) such that the systems \((Y, \mathfrak{Y}, \nu, S)\) and \((X, \mathcal{B}(X), \mu, T)\) are measure theoretically isomorphic. Weiss [41] (see also [44, 20, 22]) showed that the Jewett–Krieger Theorem can be generalized from \(\mathbb{Z}\)-actions to commutative group actions. An ergodic system has a doubly minimal model if and only if it has zero entropy [43] (other topological models for zero entropy systems can be found in [24, 15]); and an ergodic system has a strictly ergodic, UPE (uniform positive entropy) model if and only if it has positive entropy [21].

Note that not any dynamical properties can be added in the uniquely ergodic models. For example, Lindernstrauss showed that every ergodic measure distal system \((X, \mathfrak{X}, \mu, T)\) has a minimal topologically distal model [34]. This topological model need not, in general, be uniquely ergodic. In other words, there are measurable distal systems for which no uniquely ergodic topologically distal models exist [34]. We refer to [22] for more information on the topic.

We say that \(\hat{\pi} : \hat{X} \to \hat{Y}\) is a **topological model** for a factor map \(\pi : (X, \mathfrak{X}, \mu, T) \to (Y, \mathfrak{Y}, \nu, S)\) if \(\hat{\pi}\) is a topological factor map and there exist measure theoretical isomorphisms \(\phi\) and \(\psi\) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & \hat{X} \\
\downarrow{\pi} & & \downarrow{\hat{\pi}} \\
Y & \xrightarrow{\psi} & \hat{Y}
\end{array}
\]

is commutative, i.e., \(\hat{\pi}\phi = \psi\pi\). Weiss [41] generalized the Jewett–Krieger Theorem to the relative case. Namely, he proved that if \(\pi : (X, \mathfrak{X}, \mu, T) \to (Y, \mathfrak{Y}, \nu, S)\) is a factor map with \((X, \mathfrak{X}, \mu, T)\) ergodic and \((\hat{Y}, \hat{\mathfrak{Y}}, \hat{\nu}, \hat{S})\) is a uniquely ergodic model for \((Y, \mathfrak{Y}, \nu, T)\), then there is a uniquely ergodic model \((\hat{X}, \hat{\mathfrak{X}}, \hat{\mu}, \hat{T})\) for \((X, \mathfrak{X}, \mu, T)\) and a factor map \(\hat{\pi} : \hat{X} \to \hat{Y}\) which is a model for \(\pi : X \to Y\). We will refer to this theorem as **Weiss’s Theorem**. We note that in [41] Weiss pointed out that the relative case holds for commutative group actions.
1.3 Main ideas of the proofs. Now we describe the main ideas and ingredients in the proofs.

To prove Theorem A the first fact we face is that for an ergodic m.p.t. 
\((X, \hat{X}, \mu, T)\), not every strictly ergodic model is the one we need in Theorem A.\(^1\) This indicates that to obtain Theorem A, the Jewett–Krieger Theorem is not enough for our purpose. Fortunately, we find that Weiss’s Theorem is the right tool.

Precisely, for \(d \geq 3\) let \(\pi_{d-2} : X \to Z_{d-2}\) be the factor map from \(X\) to its \(d-2\)-step nilfactor \(Z_{d-2}\). By the results of Host–Kra–Maass in \([27]\), \(Z_{d-2}\) may be regarded as a topological system in the natural way. Using Weiss’s Theorem there is a uniquely ergodic model \((\hat{X}, \hat{X}, \hat{\mu}, \hat{T})\) for \((X, X, \mu, T)\) and a factor map \(\hat{\pi}_{d-2} : \hat{X} \to Z_{d-2}\) which is a model for \(\pi_{d-2} : X \to Z_{d-2}\).

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & \hat{X} \\
\pi_{d-2} \downarrow & & \downarrow \hat{\pi}_{d-2} \\
Z_{d-2} & \longrightarrow & Z_{d-2}
\end{array}
\]

We then show that \((\hat{X}, \hat{T})\) is what we need. To this aim, we need to understand well the ergodic decomposition of \(d\)-fold self-joinings of \(X\). We first study the \(\sigma\)-algebra of \(\sigma_d\)-invariant sets under \(\sigma_d\), and show that we can always deduce this \(\sigma\)-algebra from the one on its nilfactors. Then via studying nilsystems, we get the ergodic decomposition of Furstenberg self-joinings under the action \(\sigma_d\). This is the main tool we develop to prove Theorem A.

Once Theorem A is proven, Theorem B will follow by an argument using some well-known theorems related to pointwise convergence for \(\mathbb{Z}^d\) actions (see, for example, \([35]\) by Lindenstrauss) and for uniquely ergodic systems.

Let \((X, \hat{X}, \mu, T)\) be an ergodic distal system. Then \(\pi_{d-1} : X \to Z_{d-1}\) is a distal extension. By Furstenberg’s Structure Theorem, \(\pi_{d-1}\) is decomposed into isometric extensions and inverse limit. We show that the property of almost everywhere convergence of the multiple ergodic averages \((1.3)\) is preserved by these isometric extensions, and then we conclude Theorem C. This argument is inspired by Lesigne’s work in \([33]\).

1.4 Organization of the paper. We organize the paper as follows. In Section 2 we introduce some basic notions and results needed in the paper. In Section 3 we study the ergodic decomposition of self-joinings under \(T \times T^2 \times \cdots \times T^d\). Then in Sections 4 and 5 we prove Theorems A, B, C and D respectively.

\(^1\)Take any weakly mixing strictly ergodic model \((X, T, \mu)\) of an m.d.s. with discrete spectrum. Then \((N_d(X), (\tau_d, \sigma_d))\) is not strictly ergodic under \((\tau_d, \sigma_d)\) when \(d \geq 3\); in fact in this case \(N_d(X) = X^d\) and the invariant measures \(\mu \times \cdots \times \mu \neq \mu^d\), where \(\mu^d\) is defined in \([17]\).
Acknowledgments. We thank the referee for the very careful reading and many useful comments, which helped us to improve the writing of the paper and simplify some proofs. In particular, the comments helped us to rewrite Proposition 3.10 and obtain Corollary 3.3 and Corollary 4.3 which simplify the proof of Theorem A.

2 Preliminaries

In this section we introduce some basic notions in ergodic theory and topological dynamics. In this paper, instead of just considering a single transformation $T$, we will consider commuting transformations $T_1, \ldots, T_k$ of $X$. We only recall some basic definitions and properties of systems for one transformation. Extensions to the general case are straightforward.

2.1 Ergodic theory and topological dynamics.

2.1.1 Measurable systems. For a m.p.t. $(X, \mathcal{X}, \mu, T)$ we write $\mathcal{I}(X, \mathcal{X}, \mu, T)$ for the $\sigma$-algebra $\{A \in \mathcal{X} : T^{-1}A = A\}$ of invariant sets. Sometimes we will use $\mathcal{I}$ or $\mathcal{I}(T)$ for short. A m.p.t. is ergodic if all the $T$-invariant sets have measure either 0 or 1; $(X, \mathcal{X}, \mu, T)$ is weakly mixing if the product system $(X \times X, \mathcal{X} \times \mathcal{X}, \mu \times \mu, T \times T)$ is ergodic.

A homomorphism from m.p.t. $(X, \mathcal{X}, \mu, T)$ to $(Y, \mathcal{Y}, \nu, S)$ is a measurable map $\pi : X_0 \to Y_0$, where $X_0$ is a $T$-invariant subset of $X$ and $Y_0$ is an $S$-invariant subset of $Y$, both of full measure, such that $\pi_*\mu = \mu \circ \pi^{-1} = \nu$ and $S \circ \pi(x) = \pi \circ T(x)$ for $x \in X_0$. When we have such a homomorphism we say that $(Y, \mathcal{Y}, \nu, S)$ is a factor of $(X, \mathcal{X}, \mu, T)$. If the factor map $\pi : X_0 \to Y_0$ can be chosen to be bijective, then we say that $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ are (measure theoretically) isomorphic (bijective maps on Lebesgue spaces have measurable inverses). A factor can be characterized (modulo isomorphism) by $\pi^{-1}(\mathcal{Y})$, which is a $T$-invariant sub-$\sigma$-algebra of $\mathcal{X}$, and conversely any $T$-invariant sub-$\sigma$-algebra of $\mathcal{X}$ defines a factor. By a classical result and abuse of terminology we denote by the same letter the $\sigma$-algebra $\mathcal{Y}$ and its inverse image by $\pi$. In other words, if $(Y, \mathcal{Y}, \nu, S)$ is a factor of $(X, \mathcal{X}, \mu, T)$, we think of $\mathcal{Y}$ as a sub-$\sigma$-algebra of $\mathcal{X}$.

We say that $(X, \mathcal{X}, \mu, T)$ is an inverse limit of a sequence of factors $(X, \mathcal{X}_j, \mu, T)$ if $(\mathcal{X}_j)_{j \in \mathbb{N}}$ is an increasing sequence of $T$-invariant sub-$\sigma$-algebras such that $\bigvee_{j \in \mathbb{N}} \mathcal{X}_j = \mathcal{X}$ up to sets of measure zero.
2.1.2 Topological dynamical systems. A t.d.s. \((X, T)\) is transitive if there exists some point \(x \in X\) whose orbit \(O(x, T) = \{ T^n x : n \in \mathbb{Z} \}\) is dense in \(X\) and we call such a point a transitive point. The system is minimal if the orbit of any point is dense in \(X\). This property is equivalent to saying that \(X\) and the empty set are the only closed invariant sets in \(X\).

A factor of a t.d.s. \((X, T)\) is another t.d.s. \((Y, S)\) such that there exists a continuous and onto map \(\phi : X \to Y\) satisfying \(S \circ \phi = \phi \circ T\). In this case, \((X, T)\) is called an extension of \((Y, S)\). The map \(\phi\) is called a factor map.

2.1.3 \(M(X)\) and \(M_T(X)\). For a t.d.s. \((X, T)\), denote by \(M(X)\) the set of all probability measures on \(X\). Let \(M_T(X) = \{ \mu \in M(X) : T_* \mu = \mu \circ T^{-1} = \mu \}\) be the set of all \(T\)-invariant Borel probability measures of \(X\) and \(M_T^e(X)\) be the set of ergodic elements of \(M_T(X)\). It is well known that \(M_T^e(X) \neq \emptyset\).

A t.d.s. \((X, T)\) is called uniquely ergodic if there is a unique \(T\)-invariant probability measure on \(X\). It is called strictly ergodic if it is uniquely ergodic and minimal.

2.1.4 Topological distal systems. A t.d.s. \((X, T)\) (with metric \(\rho\)) is called topologically distal if \(\inf_{n \in \mathbb{Z}} \rho(T^n x, T^n x') > 0\) whenever \(x, x' \in X\) are distinct.

2.2 Conditional expectation. If \(\mathcal{Y}\) is a \(T\)-invariant sub-\(\sigma\)-algebra of \(\mathcal{X}\) and \(f \in L^1(\mu)\), we write \(\mathbb{E}(f|\mathcal{Y})\), or \(\mathbb{E}_\mu(f|\mathcal{Y})\) if needed, for the conditional expectation of \(f\) with respect to \(\mathcal{Y}\). The conditional expectation \(\mathbb{E}(f|\mathcal{Y})\) is characterized as the unique \(\mathcal{Y}\)-measurable function in \(L^2(\mathcal{Y}, \mathcal{Y}, \nu)\) such that

\[
\int_Y g \mathbb{E}(f|\mathcal{Y}) d\nu = \int_X g \circ \pi f d\mu
\]

for all \(g \in L^2(\mathcal{Y}, \mathcal{Y}, \nu)\). We will frequently make use of the identities

\[
\int \mathbb{E}(f|\mathcal{Y}) d\mu = \int f d\mu \quad \text{and} \quad T \mathbb{E}(f|\mathcal{Y}) = \mathbb{E}(Tf|\mathcal{Y}).
\]

We say that a function \(f\) is orthogonal to \(\mathcal{Y}\), written as \(f \perp \mathcal{Y}\), when it has a zero conditional expectation on \(\mathcal{Y}\). If a function \(f \in L^1(\mu)\) is measurable with respect to the factor \(\mathcal{Y}\), we write \(f \in L^1(\mathcal{Y}, \mathcal{Y}, \nu)\).

The disintegration of \(\mu\) over \(\nu\), written as \(\mu = \int \mu_y d\nu(y)\), is given by a measurable map \(y \mapsto \mu_y\) from \(Y\) to the space of probability measures on \(X\) such that

\[
\mathbb{E}(f|\mathcal{Y})(y) = \int_X f d\mu_y
\]

\(\nu\)-almost everywhere.
2.3 Joining.

2.3.1 Joining and conditional product measure. The notions of joining and conditional product measure are introduced by Furstenberg in [17]. Let \((X_i, \mu_i, T_i), i = 1, \ldots, k\), be m.p.t., and let \((Y_i, \nu_i, S_i)\) be corresponding factors and \(\pi_i : X_i \to Y_i\) the factor maps. A measure \(\nu\) on \(Y = \prod_i Y_i\) defines a joining of the measures on \(Y_i\) if it is invariant under \(S_1 \times \cdots \times S_k\) and maps onto \(\nu_j\) under the natural map \(\prod_i Y_i \to Y_j\). When \(S_1 = \cdots = S_k\), we then say that \(\nu\) is a \(k\)-fold self-joining.

Let \(\nu\) be a joining of the measures on \(Y_i, i = 1, \ldots, k\), and let \(\mu_i = \int \mu_{X_i, y_i} d\nu_i(y_i)\) represent the disintegration of \(\mu_i\) with respect to \(\nu_i\). Let \(\mu\) be a measure on \(X = \prod_i X_i\) defined by

\[
\mu = \int_Y \mu_{X_1, y_1} \times \mu_{X_2, y_2} \times \cdots \times \mu_{X_k, y_k} d\nu(y_1, y_2, \ldots, y_k).
\]

Then \(\mu\) is called the conditional product measure with respect to \(\nu\).

Equivalently, \(\mu\) is conditional product measure relative to \(\nu\) if and only if for all \(k\)-tuple \(f_i \in L^\infty(X_i, \mu_i), i = 1, \ldots, k\)

\[
\int_X f_1(x_1)f_2(x_2) \cdots f_k(x_k) d\mu(x_1, x_2, \ldots, x_k) = \int_Y E(f_1|y_1)(y_1)E(f_2|y_2)(y_2) \cdots E(f_k|y_k)(y_k) d\nu(y_1, y_2, \ldots, y_k).
\]

2.3.2 Relatively independent joining. Let \((X_1, X_1, \mu_1, T_1)\) and \((X_2, X_2, \mu_2, T_1)\) be two systems and let \((Y, \nu, S)\) be a common factor with \(\pi_i : X_i \to Y\) for \(i = 1, 2\) the factor maps. Let \(\mu_i = \int \mu_{X_i, y_i} d\nu(y)\) represent the disintegration of \(\mu_i\) with respect to \(Y\). Let \(\mu_1 \times_\nu \mu_2\) denote the measure defined by

\[
\mu_1 \times_\nu \mu_2(A) = \int_Y \mu_{X_1, y} \times \mu_{X_2, y} d\nu(y),
\]

for all \(A \in X_1 \times X_2\). The system \((X_1 \times X_2, X_1 \times X_2, \mu_1 \times_\nu \mu_2, T_1 \times T_2)\) is called the relative product of \(X_1\) and \(X_2\) with respect to \(Y\) and is denoted \(X_1 \times_\nu X_2\). \(\mu_1 \times_\nu \mu_2\), also called relatively independent joining of \(X_1\) and \(X_2\) over \(Y\).

2.4 HK-seminorms. When \(f_i, i \in I\), are functions on the set \(X\), we define a function \(\bigotimes_{i \in I} f_i\) on \(X^I\) by

\[
\bigotimes_{i \in I} f_i(x) = \prod_{i \in I} f_i(x_i),
\]

where \(x = (x_i) \in X^I\).
2.4.1 Let \((X, \mathcal{X}, \mu, T)\) be an ergodic system and \(k \in \mathbb{N}\). We define a measure \(\mu^{[k]}\) on \(X^{2^k}\) invariant under \(T^{[k]} = T \times T \times \cdots \times T\) (\(2^k\) times) by

\[
\mu^{[1]} = \mu \times \cdots \times \mu = \mu \times \mu;
\]
for \(k \geq 1,
\mu^{[k+1]} = \mu^{[k]} \times \cdots \times \mu^{[k]}.
\]

Write \(x = (x_0, x_1, \ldots, x_{2^k-1})\) for a point of \(X^{2^k}\). We define a seminorm \(\| \cdot \|_k\) on \(L^\infty(\mu)\) by

\[
\| f \|_k = \left( \int_{X^{2^k}} \left( \bigotimes_{i=0}^{2^k-1} f(x_i) d\mu^{[k]}(x) \right)^{1/2^k} \right)^{1/2^k} = \left( \int_{X^{2^k}} \prod_{i=0}^{2^k-1} f(x_i) d\mu^{[k]}(x) \right)^{1/2^k}.
\]

That \(\| \cdot \|_k\) is a seminorm\(^2\) can be proved as in [26], and we call it the **Host–Kra seminorm** (HK seminorm for short).

As \(X\) is assumed to be ergodic, the \(\sigma\)-algebra \(\mathcal{F}^{[0]}\) is trivial and \(\mu^{[1]} = \mu \times \mu\). We therefore have

\[
\| f \|_1 = \left( \int_{X^2} f(x_0) f(x_1) d\mu \times \mu(x_0, x_1) \right)^{1/2} = \left| \int f d\mu \right|.
\]

It is shown in [26] that for all \(f_i \in L^\infty(\mu), i \in \{0, 1, \ldots, 2^k - 1\},\)

\[
\left| \int \bigotimes_{i=0}^{2^k-1} f_i d\mu^{[k]} \right| \leq \prod_{i=0}^{2^k-1} \| f_i \|_k.
\]

The following lemma follows immediately from the definition of the measures and the Ergodic Theorem.

**Lemma 2.1.** For every integer \(k \geq 0\) and every \(f \in L^\infty(\mu)\), one has

\[
\| f \|_{k+1} = \left( \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \| f \circ T^n \|_k \right)^{1/2^{k+1}}.
\]

Note that (2.6) can be considered as an alternate definition of the seminorms.

2.4.2 A factor \((Z, \mathcal{Z})\) of \(X\) is **characteristic** for averages

\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \cdots f_d(T^{dn} x)
\]

\(^2\)Here for simplicity we give the formula for real functions, and one can give the formula for complex functions similarly.
if the limiting behavior of \((2.7)\) only depends on the conditional expectation of \(f_i\) with respect to \(Z\):

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (T^n f_1 T^{2n} f_2 \cdots T^{dn} f_d - T^n \mathbb{E}(f_1 | Z) T^{2n} \mathbb{E}(f_2 | Z) \cdots T^{dn} \mathbb{E}(f_d | Z)) \to 0
\]

for any \(f_1, \ldots, f_d \in L^\infty(X, \mathcal{X}, \mu)\). The minimal characteristic factor of \((2.7)\) always exists \([26, 46]\), and it is denoted by \((Z_d^{-1}, \ldots, Z_d^{-1}, \mu_d^{-1})\). An important property is

**Proposition 2.2** ([26, Lemma 4.3]). For a \(f \in L^\infty(\mu)\), \(\|f\|_k = 0\) if and only if \(\mathbb{E}(f | Z_{k-1}) = 0\).

### 2.5 Nilsystems.

Let \(G\) be a group. For \(g, h \in G\), we write

\([g, h] = ghg^{-1}h^{-1}\)

for the commutator of \(g\) and \(h\) and we write \([A, B]\) for the subgroup spanned by \(\{[a, b] : a \in A, b \in B\}\). The commutator subgroups \(G_j, j \geq 1\), are defined inductively by setting \(G_1 = G\) and \(G_{j+1} = [G_j, G]\). Let \(k \geq 1\) be an integer. We say that \(G\) is \(k\)-step nilpotent if \(G_{k+1}\) is the trivial subgroup.

Let \(G\) be a \(k\)-step nilpotent Lie group and \(\Gamma\) a discrete cocompact subgroup of \(G\). The compact manifold \(X = G/\Gamma\) is called a \(k\)-step nilmanifold. The group \(G\) acts on \(X\) by left translations and we write this action as \((g, x) \mapsto gx\). The Haar measure \(\mu\) of \(X\) is the unique probability measure on \(X\) invariant under this action. Let \(\tau \in G\) and \(T\) be the transformation \(x \mapsto \tau x\) of \(X\). Then \((X, \mu, T)\) is called a \(k\)-step nilsystem.

Here are some basic properties of nilsystems.

**Theorem 2.3** ([36, 32]). Let \((X = G/\Gamma, \mu, T)\) be a \(k\)-step nilsystem with \(T\) the translation by the element \(t \in G\). Then:

1. \((X, T)\) is uniquely ergodic if and only if \((X, \mu, T)\) is ergodic if and only if \((X, T)\) is minimal if and only if \((X, T)\) is transitive.

2. Let \(Y\) be the closed orbit of some point \(x \in X\). Then \(Y\) can be given the structure of a nilmanifold, \(Y = H/\Lambda\), where \(H\) is a closed subgroup of \(G\) containing \(t\) and \(\Lambda\) is a closed cocompact subgroup of \(H\).

One can generalize the above results to the action of several translations. For example, let \(X = G/\Gamma\) be a nilmanifold with Haar measure \(\mu\) and let \(t_1, \ldots, t_k\) be commuting elements of \(G\). If the group spanned by the translations \(t_1, \ldots, t_k\) acts ergodically on \((X, \mu)\), then \(X\) is uniquely ergodic for this group. For more details, refer to \([32]\).
2.6 System of order $d - 1$ and topological system of order $(d - 1)$. In [26], it is shown that $(\mathbb{Z}_{d-1}, \mathbb{Z}_{d-1}, \mu_{d-1}, T)$ has a very nice structure.

**Theorem 2.4 ([26]).** Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $d \in \mathbb{N}$. Then the system $(\mathbb{Z}_{d-1}, \mathbb{Z}_{d-1}, \mu_{d-1}, T)$ is a (measure theoretic) inverse limit of $d - 1$-step nilsystems. $(\mathbb{Z}_{d-1}, \mathbb{Z}_{d-1}, \mu_{d-1}, T)$ is called a system of order $d - 1$.

One also has the topological version of this notion, i.e., the topological inverse limit of nilsystems. First recall the definition of an inverse limit of t.d.s. If $(X_i, T_i)_{i \in \mathbb{N}}$ are t.d.s. with $\text{diam}(X_i) \leq 1$ and $\pi_i : X_{i+1} \to X_i$ are factor maps, the inverse limit of the systems is defined to be the compact subset of $\prod_{i \in \mathbb{N}} X_i$ given by

$$\{ (x_i)_{i \in \mathbb{N}} : \pi_i(x_{i+1}) = x_i, i \in \mathbb{N} \},$$

and we denote it by $\lim_{\leftarrow}(X_i, T_i)_{i \in \mathbb{N}}$. It is a compact metric space endowed with the distance $\rho((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) = \sum_{i \in \mathbb{N}} 1/2^i \rho_i(x_i, y_i)$, where $\rho_i$ is the metric in $X_i$. We note that the maps $T_i$ induce naturally a transformation $T$ on the inverse limit.

**Definition 2.5 ([27]).** An inverse limit of $(d - 1)$-step minimal nilsystems is called a topological system of order $(d - 1)$.

By Theorem 2.3, a topological system of order $(d - 1)$ is uniquely ergodic for each $d \in \mathbb{N}$.

If $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\epsilon \in \{0, 1\}^d$, we define

$$n \cdot \epsilon = \sum_{i=1}^{d} n_i \epsilon_i.$$

**Definition 2.6.** Let $(X, T)$ be a t.d.s. and let $d \in \mathbb{N}$. The points $x, y \in X$ are said to be regionally proximal of order $d$ if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that $\rho(x, x') < \delta$, $\rho(y, y') < \delta$, and

$$\rho(T^{n \cdot \epsilon}x', T^{n \cdot \epsilon}y') < \delta \quad \text{for any } \epsilon \in \{0, 1\}^d \setminus \{(0, 0, \ldots, 0)\}.$$

The set of regionally proximal pairs of order $d$ is denoted by $\text{RP}^{[d]}$ (or by $\text{RP}^{[d]}(X, T)$ in case of ambiguity), and is called the regionally proximal relation of order $d$.

The above definition was introduced in [27] by Host–Kra–Maass and it was proved that for a minimal distal system, $\text{RP}^{[d]}$ is an equivalence relation and $X/\text{RP}^{[d]}$ is a topological system of order $d$. Later it was shown that it is an equivalence relation for any minimal systems by Shao–Ye in [37]. We will use the following theorems in the paper.
Theorem 2.7 ([27, Theorem 1.2]). Let \((X, T)\) be a minimal topologically distal system and let \(d \in \mathbb{N}\). Then \((X, T)\) is a topological system of order \(d\) if and only if \(\text{RP}^d = \Delta_X\).

Theorem 2.8 ([27, Subsection 5.1]). Any system of order \(d\) is isomorphic in the measure theoretic sense to a topological system of order \(d\).

Lemma 2.9 ([13, Lemma A.3]). Let \((X, T)\) be a system of order \(d\). Then the maximal measurable and topological factors of order \(j\) coincide, where \(j \leq d\).

3 Ergodic decomposition of self-joinings under \(T \times T^2 \times \cdots \times T^d\)

In this section we study ergodic decomposition of self-joinings under \(T \times T^2 \times \cdots \times T^d\).

The theorems in this section are important for our proofs, and also they have their own interest.

3.1 Furstenberg self-joining. Let \(T : X \to X\) be a map and \(d \in \mathbb{N}\). Set
\[
\tau_d = \tau_d(T) = T \times \cdots \times T \text{ (d times)},
\]
\[
\sigma_d = \sigma_d(T) = T \times T^2 \times \cdots \times T^d
\]
and
\[
\sigma_d' = \sigma_d'(T) = \text{id} \times T \times \cdots \times T^{d-1} = \text{id} \times \sigma_{d-1}.
\]
Note that \(\langle \tau_d, \sigma_d \rangle = \langle \tau_d, \sigma_d' \rangle\). For any \(x \in X\), let \(N_d(X, x) = \overline{O((x, \ldots, x), \langle \tau_d, \sigma_d \rangle)}\), the orbit closure of \((x, \ldots, x)\) (\(d\) times) under the action of the group \(\langle \tau_d, \sigma_d \rangle\).

We remark that if \((X, T)\) is minimal, then all \(N_d(X, x)\) coincide, which will be denoted by \(N_d(X)\). It was shown by Glasner [19] that if \((X, T)\) is minimal, then \((N_d(X), \langle \tau_d, \sigma_d \rangle)\) is minimal. Hence if \((N_d(X), \langle \tau_d, \sigma_d \rangle)\) is uniquely ergodic, then it is strictly ergodic.

Definition 3.1. Let \((X, T)\) be a t.d.s. with \(\mu \in M_T(X)\). For \(d \geq 1\) let \(\mu^{(d)}\) be the measure on \(X^d\) defined by
\[
\int_{X^d} \bigotimes_{j=1}^{d} f_j \, d\mu^{(d)}(x) = \lim_{N \to +\infty} \frac{1}{N} \int_X \prod_{j=1}^{d} f_j(T^{jd}x) \, d\mu(x)
\]
for \(f_j \in L^\infty(X, \mu), 1 \leq j \leq d\), where the limit exists by [26, Theorem 1.1].

We call \(\mu^{(d)}\) the Furstenberg self-joining. Clearly, it is invariant under \(\tau_d\) and \(\sigma_d\).
For a t.d.s. \((X, T), \mu \in M_T(X)\) and \(d \in \mathbb{N}\), it is easy to see that
\[
\frac{1}{N} \sum_{n=0}^{N-1} \sigma_n^d \mu^d \rightarrow \mu^{(d)}, \quad N \to \infty, \quad \text{weak}^* \text{ in } M(X^d),
\]
where \(\mu^d\) is the diagonal measure on \(X^d\) as defined in [17], i.e., it is defined on \(X^d\) as follows:
\[
\int_{X^d} f_1(x_1) \cdots f_d(x_d) \, d\mu^d(x_1, \ldots, x_d) = \int_X f_1(x) \cdots f_d(x) \, d\mu(x),
\]
where \(f_1, \ldots, f_d \in C(X)\).

3.2 The \(\sigma\)-algebra of invariant sets under \(\sigma_d = T \times T^2 \times \cdots \times T^d\). In this subsection we study the \(\sigma\)-algebra of invariant sets under \(\sigma_d = T \times T^2 \times \cdots \times T^d\). We will show that we can always deduce this \(\sigma\)-algebra from the one on its nilfactors.

For a m.p.t. \((X, \mathcal{X}, \mu, T)\) and \(d \in \mathbb{N}\), recall that a measure \(\lambda\) on \(X^d\) is called \(d\)-fold self-joining of \(X\), if it is \(\tau_d\)-invariant and maps onto \(\mu\) under the nature \(j\)-th coordinate projection \(X^d \to X, 1 \leq j \leq d\). The proof of the following lemma is similar to the proof of Theorem 12.1 in [26].

**Lemma 3.2.** Let \((X, \mathcal{X}, \mu, T)\) be an ergodic system and \(d \geq 1\) be an integer. Suppose that \(\lambda\) is a \(d\)-fold self-joining of \(X\). Assume that \(f_1, \ldots, f_d \in L^\infty(X, \mu)\) with \(\|f_j\|_{\infty} \leq 1\) for \(j = 1, \ldots, d\). Then
\[
\limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x_1) f_2(T^{2n} x_2) \cdots f_d(T^{dn} x_d) \right\|_{L^2(X^d, \lambda)} \leq \min_{1 \leq l \leq d} \{ l \cdot \|f_l\|_1 \}.
\]

**Proof.** We proceed by induction. For \(d = 1\), by the Ergodic Theorem,

\[
\left\| \frac{1}{N} \sum_{n=0}^{N-1} T^n f_1 \right\|_{L^2(\mu)} \to \left\| \int f_1 \, d\mu \right\| = \|f_1\|_1.
\]

Let \(d \geq 1\) and assume that (3.1) holds for \(d\) and any \(d\)-fold self-joining of \(X\). Let \(f_1, \ldots, f_{d+1} \in L^\infty(\mu)\) with \(\|f_j\|_{\infty} \leq 1\) for \(j = 1, \ldots, d+1\). Let \(\lambda\) be any \(d+1\)-fold self-joining of \(X\). Choose \(l \in \{ 2, 3, \ldots, d+1 \}\). (The case \(l = 1\) is similar.) Write
\[
\bar{\xi}_n = \bigotimes_{j=1}^{d+1} T^j f_j = f_1(T^n x_1) f_2(T^{2n} x_2) \cdots f_{d+1}(T^{(d+1)n} x_{d+1}).
\]

By the van der Corput lemma [5],
\[
\limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} \bar{\xi}_n \right\|_{L^2(\lambda)}^2 \leq \limsup_{H \to \infty} \frac{1}{H} \sum_{h=0}^{H-1} \limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} \int_{\mathbb{Z}} \bar{\xi}_{n+h} : \xi_n \, d\lambda \right\|.
\]
Letting $M$ denote the last lim sup, we need to show that $M \leq l^2 \| f_l \|^2_{d+1}$. For any $h \geq 1$,
\[
\left| \frac{1}{N} \sum_{n=0}^{N-1} \int \xi_{n+h} \cdot \xi_n \, d\lambda \right|
\]
\[
= \left| \int (f_1 \cdot T^{hF_1}) \otimes \frac{1}{N} \sum_{n=0}^{N-1} (\sigma_d)^n \otimes f_j \cdot T^{jhF_j} \, d\lambda (x_1, \ldots, x_{d+1}) \right|
\]
\[
\leq \| f_1 \cdot T^{hF_1} \|_{L^2(\lambda)} \cdot \left| \frac{1}{N} \sum_{n=0}^{N-1} (\sigma_d)^n \otimes f_j \cdot T^{jhF_j} \right|_{L^2(\lambda)}
\]
\[
= \| f_1 \cdot T^{hF_1} \|_{L^2(\mu)} \cdot \left| \frac{1}{N} \sum_{n=0}^{N-1} (\sigma_d)^n \otimes f_j \cdot T^{jhF_j} \right|_{L^2(\lambda')}
\]
where $\lambda'$ is the image of $\lambda$ to the last $d$ coordinates. It is clear $\lambda'$ is a $d$-fold self-joining of $X$, and by the inductive assumption,
\[
\left| \frac{1}{N} \sum_{n=0}^{N-1} \int \xi_{n+h} \cdot \xi_n \, d\lambda \right| \leq l^2 \| f_1 \cdot T^{lHF_l} \|_{d}.
\]

We get
\[
M \leq l \cdot \limsup_{H \to \infty} \frac{1}{H} \sum_{h=0}^{H-1} \| f_1 \cdot T^{hF_1} \|_{d} \leq l^2 \cdot \limsup_{H \to \infty} \frac{1}{H} \sum_{h=0}^{H-1} \| f_1 \cdot T^{hF_1} \|_{d}
\]
\[
\leq l^2 \cdot \limsup_{H \to \infty} \left( \frac{1}{H} \sum_{h=0}^{H-1} \| f_1 \cdot T^{hF_1} \|_{d}^2 \right)^{1/2^d}
\]
\[
= l^2 \cdot \| f_1 \|_{d+1}^2.
\]
The last equation follows from Lemma 2.1. The proof is completed. \qed

**Corollary 3.3.** Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $d \geq 2$ be an integer. Suppose that $\lambda$ is a $d$-fold self-joining of $X$ and it is invariant under $\sigma_d$. Assume that $f_1, \ldots, f_d \in L^\infty(X, \mu)$ with $\| f_j \|_{\infty} \leq 1$ for $j = 1, \ldots, d$. Then
\[
(3.2) \quad \left| \int f_1(x_1)f_2(x_2) \cdots f_d(x_d) d\lambda(x_1, \ldots, x_d) \right| \leq d \min_{1 \leq l \leq d} \{ \| f_l \|_{d-1} \}.
\]

**Lemma 3.4.** Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $d \in \mathbb{N}$. Suppose that $\lambda$ is a $d$-fold self-joining of $X$ and it is $\sigma_d$-invariant. Assume that $f_1, \ldots, f_d \in L^\infty(X, \mu)$. Then
\[
(3.3) \quad \mathbb{E}\left( \bigotimes_{j=1}^{d} f_j \mid \mathcal{J}(X^d, \mathcal{X}^d, \lambda, \sigma_d) \right) = \mathbb{E}\left( \bigotimes_{j=1}^{d} \mathbb{E}(f_j \mid \mathcal{Z}_{d-1}) \mid \mathcal{J}(X^d, \mathcal{X}^d, \lambda, \sigma_d) \right).
\]
Proof. By telescoping, it suffices to show that

\[
\mathbb{E}\left( \bigotimes_{j=1}^{d} f_j \middle| \mathcal{J}(X^d, X^d, \lambda, \sigma_d) \right) = 0
\]

whenever \(\mathbb{E}(f_k|Z_{d-1}) = 0\) for some \(k \in \{1, 2, \ldots, d\}\). This condition implies that \(\|f_k\|_d = 0\) by Proposition 2.2. By the Ergodic Theorem and Lemma 3.2, we have

\[
\left\| \mathbb{E}\left( \bigotimes_{j=1}^{d} f_j \middle| \mathcal{J}(X^d, X^d, \lambda, \sigma_d) \right) \right\|_{L^2(\lambda)} = \lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x_1) f_2(T^{2n} x_2) \cdots f_d(T^{dn} x_d) \right\|_{L^2(\lambda)} \leq k \cdot \|f_k\|_d = 0.
\]

So the lemma follows. \(\square\)

**Proposition 3.5.** Let \((X, \mathcal{X}, \mu, T)\) be ergodic and \(d \in \mathbb{N}\). Suppose that \(\lambda\) is a \(d\)-fold self-joining of \(X\) and it is \(\sigma_d\)-invariant. Then the \(\sigma\)-algebra \(\mathcal{J}(X^d, \mathcal{X}^d, \lambda, \sigma_d)\) is measurable with respect to \(Z_{d-1}^d\).

Proof. Every bounded function on \(X^d\) which is measurable with respect to \(\mathcal{J}(X^d, \mathcal{X}^d, \lambda, \sigma_d)\) can be approximated in \(L^2(\lambda)\) by finite sums of functions of the form \(\mathbb{E}(\bigotimes_{j=1}^{d} f_j |\mathcal{J}(X^d, \mathcal{X}^d, \lambda, \sigma_d))\) where \(f_1, \ldots, f_d\) are bounded functions on \(X\). By Lemma 3.4 one can assume that these functions are measurable with respect to \(Z_{d-1}\). In this case \(\bigotimes_{j=1}^{d} f_j\) is measurable with respect to \(Z_{d-1}^d\). Since this \(\sigma\)-algebra \(Z_{d-1}^d\) is invariant under \(\sigma_d\),

\[
\mathbb{E}\left( \bigotimes_{j=1}^{d} f_j \middle| \mathcal{J}(X^d, \mathcal{X}^d, \lambda, \sigma_d) \right) = \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left( \bigotimes_{j=1}^{d} f_j \right) \circ \sigma^n_d
\]

is also measurable with respect to \(Z_{d-1}^d\). Therefore \(\mathcal{J}(X^d, \mathcal{X}^d, \lambda, \sigma_d)\) is measurable with respect to \(Z_{d-1}^d\). \(\square\)

Let \(\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)\) be a homomorphism; \(\pi\) is ergodic or \((X, \mathcal{X}, \mu, T)\) is an ergodic extension of \((Y, \mathcal{Y}, \nu, S)\) if \(T\)-invariant sets of \(X\) are contained in \(Y\), i.e., \(\mathcal{J}(T) \subseteq \mathcal{Y}\).

**Corollary 3.6.** Let \((X, \mathcal{X}, \mu, T)\) be an ergodic system and \(d \in \mathbb{N}\). Suppose that \(\lambda\) is a \(d\)-fold self-joining of \(X\) and it is \(\sigma_d\)-invariant. Then the factor map \(\pi_{d-1}^d : (X^d, \mathcal{X}^d, \lambda, \sigma_d) \to (Z_{d-1}^d, \mathcal{Z}_{d-1}^d, \tilde{\lambda}, \sigma_d)\) is ergodic, where \(\tilde{\lambda}\) is the image of \(\lambda\).

In particular, one has that \(\mathcal{J}(X^d, \lambda, \sigma_d)\) is isomorphic to \(\mathcal{J}(Z_{d-1}^d, \tilde{\lambda}, \sigma_d)\).
3.3 Ergodic decomposition of Furstenberg self-joining of nilsystems under action $\sigma_d$. In the previous subsection we showed that to study the $\sigma$-algebra of invariant sets under $\sigma_d = T \times T^2 \times \cdots \times T^d$, we only need to study the one on its nilfactors. Hence in this subsection we study the ergodic decomposition of Furstenberg self-joinings of nilsystems under the action $\sigma_d$.

In this subsection $d \geq 2$ is an integer, and $(X = Z_{d-1}, \mathbb{Z}_{d-1}, \mu_{d-1}, T)$ is a topological system of order $d - 1$. Recall $N_\ell = N_\ell(X) = \overline{\mathcal{O}(\Delta_\ell(X), \sigma_\ell)} = \overline{\mathcal{O}(x, \ldots, x, \tau_\ell, \sigma_\ell)} \subset X_\ell$
and
$$N_\ell[x] := \overline{\mathcal{O}((x, \ldots, x), \sigma_\ell^\prime)} = \{x\} \times \overline{\mathcal{O}((x, \ldots, x), \sigma_{\ell-1})},$$
where $\ell \geq 2$ and $x \in X$.

3.3.1 Basic properties. First we recall some basic properties.

Theorem 3.7 ([7, 45]). With the above notations, we have:

1. (Theorem 2.3) The $(N_\ell, (\tau_\ell, \sigma_\ell))$ is ergodic (and thus uniquely ergodic) with the Furstenberg self-joining $\mu_{d-1}^{(d)}$.

2. (Theorem 2.3) For each $x \in X$, the system $(N_\ell[x], \sigma_\ell^\prime)$ is uniquely ergodic with some measure $\delta_x \times \mu_{d-1}^{(d)}(x)$.

3. ([7, Lemma 5.3]) $\mu_{d-1}^{(d)} = \int_X \delta_x \times \mu_{d-1,x}^{(d)} d\mu_{d-1}(x)$.

4. (Ziegler) Let $f_1, f_2, \ldots, f_{d-1}$ be continuous functions on $X$ and let $\{M_i\}$ and $\{N_i\}$ be two sequences of integers such that $N_i \to \infty$. For $\mu_{d-1}$-almost every $x \in X$,

$$\frac{1}{N_i} \sum_{n=M_i}^{N_i+M_i-1} f_1(T^n x)f_2(T^{2n} x) \cdots f_{d-1}(T^{(d-1)n} x) \to \int f_1(x_1)f_2(x_2) \cdots f_{d-1}(x_{d-1}) d\mu_{d-1,x}^{(d)}(x_1, x_2, \ldots, x_{d-1})$$
as $i \to \infty$.

Remark 3.8. In fact, in [7, 45] Theorem 3.7 is for nilsystems. Via an inverse limit argument, it is easy to see that Theorem 3.7 holds for topological systems of order $d$.

3.3.2 The ergodic decomposition of $\mu_{d-1}^{(d)}$ under $\sigma_d$. Now we study the ergodic decomposition of $\mu_{d-1}^{(d)}$ under $\sigma_d$. By Theorem 2.3 for each $x \in X$, let $\nu_{d-1,x}^{(d)}$ be the unique $\sigma_d$-invariant measure on $\overline{\mathcal{O}(x^d, \sigma_d)}$, where $x^d = (x, x, \ldots, x) \in X^d$. Then

$$\varphi : X \longrightarrow M(N_d); \quad x \mapsto \nu_{d-1,x}^{(d)}$$
is a Borel map and \( \varphi(X) \subseteq M^e_d(N_d) \). This fact follows from the fact that 
\[ x \mapsto \frac{1}{N} \sum_{n < N} \delta_{\varphi^n x} \] is continuous and \( \frac{1}{N} \sum_{n < N} \delta_{\varphi^n x} \) converges to \( \nu^{(d)}_{d-1,x} \) weakly.

It is easy to check that \( \int_X \nu^{(d)}_{d-1,x} \, d\mu_{d-1}(x) \) is \( (\tau_d, \sigma_d) \)-invariant and hence it is equal to \( \mu^{(d)}_{d-1} \) by the uniqueness. Hence we have

\[(3.6) \quad \mu^{(d)}_{d-1} = \int_X \nu^{(d)}_{d-1,x} \, d\mu_{d-1}(x).\]

Now we will prove the following result:

**Theorem 3.9.** \( \mu^{(d)}_{d-1} = \int_X \nu^{(d)}_{d-1,x} \, d\mu_{d-1}(x) \) is the ergodic decomposition of \( \mu^{(d)}_{d-1} \) under \( \sigma_d \).

First we have the following claim:

**Claim.** There exists a continuous map \( \psi : N_d \to X \) such that

\[ x_1 = \psi(x_2, x_3, \ldots, x_{d+1}) \]

for every \( (x_1, x_2, x_3, \ldots, x_{d+1}) \in N_{d+1} \).

**Proof of Claim.** The claim follows from the following fact: the projection

\[ p_2 : N_{d+1}(X) \to N_d(X); \quad (x_1, x_2, \ldots, x_{d+1}) \mapsto (x_2, \ldots, x_{d+1}) \]

is a bijection.

By definition, it is clear that \( p_2 \) is onto. Now we show that \( p_2 \) is also injective. Let \( (x_1, x_2, \ldots, x_{d+1}), (y_1, x_2, \ldots, x_{d+1}) \in N_{d+1} \). We will show that \( x_1 = y_1 \). First, by the definition of \( N_{d+1}(X) \), there exists \( (x^*, y^*) \in \bar{O}((x_1, y_1), T \times T) \) and \( x \in X \) such that

\[ (x^*, x, \ldots, x), (y^*, x, \ldots, x) \in N_{d+1}. \]

Thus for any \( \delta > 0 \), there is some \( n, m \in \mathbb{Z} \) such that \( \rho(T^{m+n}x, x) < \delta / 2 \) for all \( j = 1, 2, \ldots, d \) and \( \rho(T^{m}x, x^*) < \delta / 2 \). Let \( x' = T^nx, y' = T^{n+m}x', \) and let \( n = (n, n, \ldots, n) \in \mathbb{Z}^{d-1} \). Then \( \rho(x', x^*) < \delta / 2, \rho(y', x) < \delta / 2 \) and

\[ \{ n \cdot \epsilon : \epsilon \in \{0, 1\}^{d-1} \setminus \{(0, 0, \ldots, 0)\} \} = \{ n, 2n, \ldots, (d - 1)n \}. \]

So we have that

\[
\rho(T^{n+\epsilon}x', T^{n+\epsilon}y') = \rho(T^{n+\epsilon-m}x, T^{n+\epsilon-m+n}x) \leq \rho(T^{n+\epsilon-m}x, x) + \rho(x, T^{n+\epsilon-m+n}x) \\
\leq 2 \max_{1 \leq j \leq d} \rho(T^{jn+n}x, x) = \delta.
\]

By the definition of \( \text{RP}^{[d-1]}(X) \) one has that \( (x^*, x) \in \text{RP}^{[d-1]}(X) \). Hence \( x^* = x \) by Theorem \([2,7] \). Similarly, one has that \( y^* = x \). Thus \( x^* = y^* \) and so \( x_1 = y_1 \) since \( (X, T) \) is distal. This shows that \( p_2 \) is injective. The proof of the Claim is completed.
Now we show that
\[ \varphi : X \rightarrow M(N_d); \]
\[ x \mapsto v_{d-1,x}^{(d)} \]
is one-to-one. Since \( \psi^{-1}(x) \supseteq \mathcal{O}((x^d, \sigma_d)) \) for any \( x \in X \), one has that
\[ \mathcal{O}((x^d, \sigma_d)) \cap \mathcal{O}((y^d, \sigma_d)) = \emptyset \]
whenever \( x \neq y \). Thus \( v_{d-1,x}^{(d)} \neq v_{d-1,y}^{(d)} \) whenever \( x \neq y \).

By the above discussion, one has that \( \varphi \) is a one-to-one Borel map. Hence by the Souslin Theorem (see, e.g., [20, Theorem 2.8 (2)]), \( \varphi(X) \) is a Borel subset of \( M(N_d) \) and \( \varphi \) is a Borel isomorphism from \( X \) to \( \varphi(X) \). Let \( \kappa = \varphi_*(\mu_{d-1}) \). Then \( \kappa \) is a Borel probability measure on the \( G_\delta \) subset \( M_\sigma^e(N_d) \) of \( M(N_d) \), \( \kappa(\varphi(X)) = 1 \) and
\[
\mu_{d-1}^{(d)} \int_X \varphi(x) \, d\mu_{d-1}(x) = \int_{M_\sigma^e(N_d)} \theta d\kappa(\theta)
\]
is the ergodic decomposition of \( \mu_{d-1}^{(d)} \) under \( \sigma_d \).

Let \( N_d \) be the Borel \( \sigma \)-algebra of \( N_d \). By the ergodic decomposition theorem (see, e.g., [39, Theorem 4.2]), there exists a Borel map \( \xi : N_d \rightarrow M_\sigma^e(N_d) \) such that
\begin{enumerate}
  \item \( \xi(\sigma_d x) = \xi(x) \) for any \( x \in N_d \),
  \item for any \( \theta \in M_\sigma^e(N_d) \), \( \theta(\xi^{-1}(\theta)) = 1 \),
  \item for any \( \eta \in M_\sigma(N_d) \),
\end{enumerate}
\[
\eta(A) = \int_{N_d} \xi(x)(A) \, d\eta(x)
\]
for any \( A \in N_d \).

Let \( M_\sigma^e(N_d) \) be the Borel \( \sigma \)-algebra of \( M_\sigma^e(N_d) \) and \( \nu = \xi_*(\mu_{d-1}^{(d)}) \). Then
\[
\xi : (N_d, N_d, \mu_{d-1}^{(d)}) \rightarrow (M_\sigma^e(N_d), M_\sigma^e(N_d), \nu)
\]
is a measure-preserving map. By [39, Lemma 4.2],
\[
\xi^{-1}(M_\sigma^e(N_d)) = \mathcal{J}(N_d, N_d, \mu_{d-1}^{(d)}, \sigma_d) \mod \mu_{d-1}^{(d)}
\]
and
\[
\mu_{d-1}^{(d)} = \int_{N_d} \xi(x) \, d\mu_{d-1}^{(d)}(x) = \int_{M_\sigma^e(N_d)} \theta d\nu(\theta)
\]
is the disintegration of \( \mu_{d-1}^{(d)} \) over \( \nu \) by \( \xi \).
Hence the uniqueness of the representation in Choquet’s theorem implies \( \nu = \kappa \) so that \( \nu(\varphi(X)) = 1 \). Now

\[
\varphi^{-1} : (M_{\sigma_d}^e(N_d), M_{\sigma_d}^e(N_d), \nu) \rightarrow (X, \mathcal{X}, \mu_{d-1})
\]

is an isomorphism.

Let

\[
E = \{(x_2, x_3, \ldots, x_{d+1}) \in N_d : \psi(x_2, x_3, \ldots, x_{d+1}) = \varphi^{-1} \circ \zeta(x_2, x_3, \ldots, x_{d+1})\}. 
\]

Then \( E \) is a Borel subset of \( N_d \). Now for any \( x \in X \), it is not hard to see that

\[
\varphi(x)(E) \supseteq \varphi(x)(\overline{O((x^d, \sigma_d) \cap \zeta^{-1}(\varphi(x)))}) = \varphi(x)(\zeta^{-1}(\varphi(x))) \quad (iii)
\]

and so

\[
\mu_{d-1}(E) = \int_X \varphi(x)(E) \, d\mu_{d-1}(x) = 1.
\]

This implies that \( \psi = \varphi^{-1} \circ \zeta \) for \( \mu_{d-1} \)-a.e. \( x \in X \).

To sum up, we have

**Proposition 3.10.** Let \( \psi : N_d \rightarrow X \) be the continuous map such that \( x_1 = \psi(x_2, x_3, \cdots, x_{d+1}) \) for every \( (x_1, x_2, x_3, \ldots, x_{d+1}) \in N_{d+1} \). Then

\[
\psi : (N_d, N_d, \mu_{d-1}^{(d)}) \rightarrow (X, \mathcal{X}, \mu_{d-1})
\]

is a measure preserving map such that

1. \( \psi^{-1}(X) = \{(N_d, N_d, \mu_{d-1}^{(d)}, \sigma_d) \mod \mu_{d-1}^{(d)}\} \),
2. the disintegration \( \mu_{d-1}^{(d)} = \int_X (\mu_{d-1}^{(d)})_x d\mu_{d-1}(x) \) of \( \mu_{d-1}^{(d)} \) over \( \mu_{d-1}^{(d)} \) by \( \psi \) is the ergodic decomposition of \( \mu_{d-1}^{(d)} \) under \( \sigma_d \),
3. \( (\mu_{d-1}^{(d)})_x = \psi_{d-1,x}^{(d)} \) for \( \mu_{d-1}^{(d)} \)-a.e. \( x \in X \).

**Remark 3.11.** Note that \( N_{d+1}[x] = \{x\} \times \overline{O(x^d, \sigma_d)} \). It follows that for all \( x \)

\[
\mu_{d-1,x}^{(d+1)} = \psi_{d-1,x}^{(d)}.
\]

Hence

\[
(\mu_{d-1}^{(d)})_x = \psi_{d-1,x}^{(d)} = \mu_{d-1,x}^{(d+1)}
\]

for \( \mu_{d-1}^{(d)} \)-a.e. \( x \in X \). Thus usually \( (\mu_{d-1}^{(d)})_x \) is different from \( \mu_{d-1,x}^{(d+1)} \).

It is easy to see that Theorem 3.9 follows from Proposition 3.10.
3.4 Ergodic decomposition of Furstenberg self-joining under the action $\sigma_d$. Let $(X, T)$ be a minimal t.d.s. with measure $\mu$, and let

$$\pi_{d-1} : (X, T) \rightarrow (Z_{d-1}, T)$$

be the topological factor map, where $Z_{d-1}$ is both a topological system of order $d-1$ and a system of order $d-1$ with measure $\mu_{d-1}$. Notice that in the next section we will show that for each ergodic system, one can always find such a minimal topological model.

Recall that for a t.d.s. $(X, T)$ and $d \in \mathbb{N}$, we define $\mu^{(d)}$ as the weak* limit points of sequence $\{ \frac{1}{N} \sum_{n=0}^{N-1} \sigma^n \mu^d \}$ in $M(X^d)$. Then $\mu^{(d)}$ is a $d$-fold self-joining of $X$ and it is $(\tau_d, \sigma_d)$-invariant. By definition, the image of $\mu^{(d)}$ under $\pi_{d-1}^d$ is $\mu_{d-1}^{(d)}$.

By Corollary 3.6 the factor map

$$\pi_{d-1}^d : (X^d, \mathbb{X}^d, \mu^{(d)}, \sigma_d) \rightarrow (Z_{d-1}^d, \mathbb{Z}_{d-1}^d \mu_{d-1}^{(d)}, \sigma_d)$$

is ergodic. Hence $\mathcal{I}(X^d, \mathbb{X}^d, \mu^{(d)}, \sigma_d) = \mathcal{I}(Z_{d-1}^d, \mathbb{Z}_{d-1}^d, \mu_{d-1}^{(d)}, \sigma_d)$. By (3.6),

$$\mu_{d-1}^{(d)} = \int_{Z_{d-1}} \nu_{d-1, x} \, d\mu_{d-1}(x)$$

is the ergodic decomposition of $\mu_{d-1}^{(d)}$ under $\sigma_d$.

Let $\phi = \pi_{d-1}^d |_{N_d(X)}$ and

$$\psi : (N_d(Z_{d-1}), N_d(Z_{d-1}), \mu_{d-1}^{(d)}) \rightarrow (Z_{d-1}, \mathbb{Z}_{d-1}, \mu_{d-1})$$

be the measure-preserving map defined in Proposition 3.10. Then by Corollary 3.6 and the fact that $\mu^{(d)}(N_d(X)) = 1$ and $\mu_{d-1}^{(d)}(N_d(Z_{d-1})) = 1$, one has that

$$\phi : (N_d(X), N_d(X), \mu^{(d)}, \sigma_d) \rightarrow (N_d(Z_{d-1}), N_d(Z_{d-1}), \mu_{d-1}^{(d)}, \sigma_d)$$

is a factor map with

$$\mathcal{I}(N_d(X), N_d(X), \mu^{(d)}, \sigma_d) = \phi^{-1} \mathcal{I}(N_d(Z_{d-1}), N_d(Z_{d-1}), \mu_{d-1}^{(d)}, \sigma_d). \quad (3.8)$$

Combining this with Proposition 3.10 (1), we have

$$\phi : (N_d(X), N_d(X), \mu^{(d)}) \rightarrow (N_d(Z_{d-1}), N_d(Z_{d-1}), \mu_{d-1}^{(d)}) \rightarrow (Z_{d-1}, \mathbb{Z}_{d-1}, \mu_{d-1})$$

and $\phi^{-1}(\psi^{-1}(\mathbb{Z}_{d-1})) = \mathcal{I}(N_d(X), \mu^{(d)}, \sigma_d)$. From this, let

$$\mu^{(d)} = \int_{Z_{d-1}} \psi^{(d)}_s \, d\mu_{d-1}(s) \quad (3.10)$$
be the disintegration of $\mu^{(d)}$ over $\mu_{d-1}$ by $\psi \circ \phi$. Since
$$\phi^{-1}(\psi^{-1}(\mathbb{Z}_{d-1})) = \mathcal{N}_d(\mathcal{X}), \mu^{(d)}, \sigma_d,$$
(3.10) is the ergodic decomposition of $\mu^{(d)}$ under $\sigma_d$ (see, e.g., [20, Theorem 8.7]). Moreover, for $\mu_{d-1}$-a.e. $s \in \mathbb{Z}_{d-1}$,
$$\phi_s(\nu_s^{(d)}) = (\mu_{d-1})_s = \nu_{d-1, s}^{(d)},$$
by [16, Corollary 5.24], (3.8) and Proposition 3.10, where
$$\mu_{d-1} = \int_{\mathbb{Z}_{d-1}} (\mu_{d-1})_s d\mu_{d-1}(s)$$
is the disintegration of $\mu_{d-1}$ over $\mu_{d-1}$ by $\psi$.

To sum up, we have the following result:

**Theorem 3.12.** $\mu^{(d)} = \int_{\mathbb{Z}_{d-1}} v_s^{(d)} d\mu_{d-1}(s)$ is the ergodic decomposition of $\mu^{(d)}$ under $\sigma_d$.

## 4 Proof of Theorems A and B

In this section we prove Theorems A and B. First we give the proof of Theorem A by using the tools developed in Section 3.

### 4.1 Another form of Theorem A.

**Definition 4.1.** Let $(\mathcal{X}, \mathcal{X}, \mu, T)$ be an ergodic m.p.t. and $(\hat{\mathcal{X}}, \hat{T})$ be its model. For $d \in \mathbb{N}$, $(\hat{\mathcal{X}}, \hat{T})$ is called a $\langle \tau_d, \sigma_d \rangle$-strictly ergodic model for $(\mathcal{X}, \mathcal{X}, \mu, T)$ if $(\hat{\mathcal{X}}, \hat{T})$ is a strictly ergodic model and $(\mathcal{N}_d(\hat{\mathcal{X}}), \langle \tau_d(\hat{T}), \sigma_d(\hat{T}) \rangle)$ is strictly ergodic.

From the statement of Theorem A one does not know what the model looks like. The following statement avoids this weakness and is suitable for the induction. Note that we let $Z_0 = \{pt\}$ be the trivial system.

**Theorem 4.2.** Let $(\mathcal{X}, \mathcal{X}, \mu, T)$ be an ergodic m.p.t., $d \geq 2$ and
$$\pi_{d-2} : X \longrightarrow Z_{d-2}$$
be the factor map to the $Z_{d-2}$. Assume that $Z_{d-2}$ is isomorphic to a topological system of order $d-2$ (see Theorem 2.8 still denote it by $Z_{d-2}$). Then any strictly ergodic system $\hat{\mathcal{X}}$ obtained from Weiss’s theorem is a $\langle \tau_d, \sigma_d \rangle$-strictly ergodic model.

$$\begin{array}{ccc}
X & \longrightarrow & \hat{\mathcal{X}} \\
\pi_{d-2} \downarrow & & \hat{\pi}_{d-2} \\
Z_{d-2} & \longrightarrow & Z_{d-2}
\end{array}$$

Theorem 4.2 follows immediately from the following corollary, which follows from Corollary 3.3.
Corollary 4.3. Let \((X, \mathcal{X}, \mu, T)\) be an ergodic system and \(d \geq 2\) be an integer. Let \(\pi_{d-2} : (X, \mathcal{X}, \mu, T) \to (Z_{d-2}, \mathcal{Z}_{d-2}, \mu_{d-2}, T)\) be its factor of order \(d - 2\). Suppose that \(\lambda\) is a \(d\)-fold self-joining of \(X\) and it is \(\sigma_d\)-invariant. If \(\mu^{(d)}_{d-2}\) is the image of \(\lambda\) under \(\pi^d_{d-2}\), then \(\lambda\) is the conditionally independent measure with respect to \(\mu^{(d)}_{d-2}\).

The following result is a direct consequence of Theorem 4.2.

Corollary 4.4. Let \((X, T)\) be a uniquely ergodic t.d.s with invariant measure \(\mu\) and \(d \geq 2\). Assume that the measure theoretic factor map \(\pi^d_{d-2} : X \to Z^d_{d-2}\) is (equal \(\mu\)-a.e. to) a continuous factor map. Then \((N_d(X), \langle \tau_d, \sigma_d \rangle)\) is unique ergodic, and the unique invariant measure is the Furstenberg self-joining \(\mu^{(d)}\).

4.2 Another proof of Theorem 4.2 and its consequence. In this subsection, we give another proof of Theorem 4.2. By this proof, we will get the following result, which is key to the proof of Theorem D.

Theorem 4.5. Let \((X, T)\) be a uniquely ergodic t.d.s with invariant measure \(\mu\) and \(d \geq 1\). Assume that the measure theoretic factor map \(\pi_{d-1} : X \to Z_{d-1}\) is (equal \(\mu\)-a.e. to) a continuous factor map. Let \(\mu^{(d)} = \int_{Z_{d-1}} \nu_s^{(d)} d\mu_{d-1}(s)\) be the ergodic decompositions of \(\mu^{(d)}\) under \(\sigma_d\) as in Theorem 3.12 and let \(\mu = \int_{Z_{d-1}} \psi_s d\mu_{d-1}(s)\) be the disintegration of \(\mu\) over \(\mu_{d-1}\). Then

\[
\mu^{(d+1)} = \int_{Z_{d-1}} \theta_s \times \nu_s^{(d)} d\mu_{d-1}(s).
\]  

Another proof of Theorem 4.2. Let \((X, T)\) be a strictly ergodic system and let \(\mu\) be its unique \(T\)-invariant measure.

When \(d = 2\). Note that \(X^2 = X \times X\), \(\tau_2 = T \times T\), \(\sigma_2 = T \times T^2\) and \(\sigma'_2 = \text{id} \times T\).

It is easy to see that \(N_2(X) = X \times X\) and \(\mu \times \mu\) is unique \(\langle \tau_2, \sigma_2 \rangle\)-invariant measure on \(X \times X\).

Next assume that Theorem 4.2 holds for \(d \geq 2\). We show it also holds for \(d + 1\). Let \(\pi_{d-1} : X \to Z_{d-1}\) be the factor map from \(X\) to \(Z_{d-1}\), the system of order \(d - 1\). We build \(\hat{X}\) in the following way by Weiss’s Theorem and Theorem 2.8

\[
\begin{align*}
X & \longrightarrow \hat{X} \\
\pi_{d-1} & \downarrow \hat{\pi}_{d-1} \\
Z_{d-1} & \longrightarrow Z_{d-1}
\end{align*}
\]

Without loss of generality we assume that \(X = \hat{X}\).

Now we show that \((N_{d+1}(X), \langle \tau_{d+1}, \sigma_{d+1} \rangle)\) is uniquely ergodic.
Let \( \zeta : Z_{d-1} \rightarrow Z_{d-2} \) be the factor map to the maximal topological factor of order \( d - 2 \). By Theorem 2.9, \( \zeta \) is also the factor map to the maximal factor of order \( d - 2 \). By the inductive assumption, \((N_d(X), \langle \tau_d, \sigma_d \rangle)\) is uniquely ergodic, and we denote its unique measure by \( \mu^{(d)} \).

By Theorem 3.12

\[
\mu^{(d)} = \int_{Z_{d-1}} \nu_s^{(d)} \, d\mu_{d-1}(s)
\]

is the ergodic decomposition of \( \mu^{(d)} \) under \( \sigma_d \).

Let \( \lambda \) be a \( \langle \tau_{d+1}, \sigma_{d+1} \rangle \)-invariant measure of \( N_{d+1}(X) \) and \( \mu = \int_{Z_{d-1}} \theta_s \, d\mu_{d-1}(s) \) be the disintegration of \( \mu \) over \( \mu_{d-1} \). We will show that

\[
\lambda = \int_{Z_{d-1}} \theta_s \times \nu_s^{(d)} \, d\mu_{d-1}(s)
\]

which implies that \( \lambda \) is unique.

To do this let

\[
p_1 : (N_{d+1}(X), \langle \tau_{d+1}, \sigma_{d+1} \rangle) \rightarrow (X, T); \quad (x_1, x) \mapsto x_1
\]

\[
p_2 : (N_{d+1}(X), \langle \tau_{d+1}, \sigma_{d+1} \rangle) \rightarrow (N_d(X), \langle \tau_d, \sigma_d \rangle); \quad (x_1, x) \mapsto x
\]

be the projections (here we use the fact that \( \langle \tau_d, T^2 \times \cdots \times T^{d+1} \rangle = \langle \tau_d, \sigma_d \rangle \)). Then \( (p_2)_*(\lambda) \) is a \( \langle \tau_d, \sigma_d \rangle \)-invariant measure of \( N_d(X) \). By the assumption on \( d \), \( (p_2)_*(\lambda) = \mu^{(d)} \). Hence we may assume that

\[
\lambda = \int_{X^d} \lambda_x \times \delta_x \, d\mu^{(d)}(x)
\]

is the disintegration of \( \lambda \) over \( \mu^{(d)} \). Since \( \lambda \) is \( \sigma_{d+1} = \text{id} \times \sigma_d \)-invariant, we have

\[
\lambda = \text{id} \times \sigma_d \lambda = \int_{X^d} \lambda_x \times \sigma_d \delta_x \, d\mu^{(d)}(x) = \int_{X^d} \lambda_x \times \delta_{\sigma_d(x)} \, d\mu^{(d)}(x)
\]

\[
= \int_{X^d} \lambda_{(\sigma_d)^{-1}(x)} \times \delta_x \, d\mu^{(d)}(x).
\]

The uniqueness of disintegration implies that

\[
\lambda_{(\sigma_d)^{-1}(x)} = \lambda_x, \quad \mu^{(d)} \text{ a.e.}
\]

Define

\[
F : (X^d, \mu^{(d)}, \sigma_d) \rightarrow M(X); \quad x \mapsto \lambda_x.
\]

By (4.5), \( F \) is a \( \sigma_d \)-invariant \( M(X) \)-valued function. Hence \( F \) is \( \mathcal{I}(X^d, X^d, \mu^{(d)}, \sigma_d) \)-measurable, and this implies \( \lambda_x = \lambda_{\psi(\phi(x))} = \lambda_s, \quad \mu^{(d)} \text{ a.e.} \), where \( s, \psi \) and \( \phi \) are defined in (3.9).
Thus by (4.4) one has that
\[
\lambda = \int_{X^d} \lambda_x \times \delta_x \, d\mu^{(d)}(x) = \int_{X^d} \lambda_x \psi(\phi(x)) \times \delta_x \, d\mu^{(d)}(x)
\]
\[
(4.6)
= \int_{Z_{d-1}} \int_{X^d} \lambda_s \times \delta_x \, dv^{(d)}_s(x) \, d\mu_{d-1}(s)
\]
\[
= \int_{Z_{d-1}} \lambda_s \times \left( \int_{X^d} \delta_x \, dv^{(d)}_s(x) \right) \, d\mu_{d-1}(s) = \int_{Z_{d-1}} \lambda_s \times v^{(d)}_s \, d\mu_{d-1}(s).
\]

In the sequel we will show that \( \lambda_s = \theta_s \) for \( \mu_{d-1} \)-a.e. \( s \in Z_{d-1} \) and it is clear that (4.3) follows from this fact and (4.6) immediately.

Let \( \pi_{d-1}^{d+1} : (N_{d+1}(X), (\tau_{d+1}, \sigma_{d+1})) \to (N_{d+1}(Z_{d-1}), (\tau_{d+1}, \sigma_{d+1})) \) be the natural factor map. By Theorem 3.7 (\( N_{d+1}(Z_{d-1}), (\tau_{d+1}, \sigma_{d+1}), \mu_{d-1}^{(d+1)} \)) is uniquely ergodic.

Hence
\[
\int_{Z_{d-1}} (\pi_{d-1}^{d+1})_s(\lambda_s \times v^{(d)}_s) \, d\mu_{d-1}(s) = (\pi_{d-1}^{d+1})_s(\lambda) = \mu_{d-1}^{(d+1)} = \int_{Z_{d-1}} \delta_s \times \mu_{d-1,s}^{(d+1)} \, d\mu_{d-1}(s).
\]

The last equality follows from Theorem 3.7(3), since for \( \mu_{d-1} \)-a.e. \( s \in Z_{d-1} \), the system \( (\overline{O}(\cdot, \ldots, s), \sigma_{d+1}) \) is uniquely ergodic with some measure \( \delta_s \times \mu_{d-1,s}^{(d+1)} \). Hence \( \mu_{d-1,s}^{(d+1)} \) is the unique ergodic measure of \( (\overline{O}(\cdot, \ldots, s), \sigma_d, \sigma_d) \), i.e., \( \mu_{d-1,s}^{(d+1)} = v_{d-1,s}^{(d)} \).

Note that
\[
(\pi_{d-1}^{d+1})_s(v^{(d)}_s) = \phi_s(v^{(d)}_s) = (\mu_{d-1,s}^{(d)})_s = v^{(d)}_{d-1,s} = \mu_{d-1,s}^{(d+1)}
\]
and \( (\mu_{d-1,s}^{(d)})(\psi^{-1}(s)) = 1 \) for \( \mu_{d-1} \)-a.e. \( s \in Z_{d-1} \). We claim that
\[
(\pi_{d-1}^{d+1})_s(\lambda_s \times v^{(d)}_s) = \delta_s \times \mu_{d-1,s}^{(d+1)}
\]
for \( \mu_{d-1} \)-a.e. \( s \in Z_{d-1} \). We postpone the verification of (4.7) to the next subsection.

It is clear that (4.7) implies
\[
(\pi_{d-1}^{d+1})_s(\lambda_s) = \delta_s
\]
for \( \mu_{d-1} \)-a.e. \( s \in Z_{d-1} \). Since \( (p_1)_s(\lambda) = \mu \), it follows from (4.6) that
\[
\mu = \int_{Z_{d-1}} \lambda_s \, d\mu_{d-1}(s).
\]

Now, (4.8) and (4.9) imply that \( \mu = \int_{Z_{d-1}} \lambda_s \, d\mu_{d-1}(s) \) is also the disintegration of \( \mu \) over \( \mu_{d-1} \). So we conclude that
\[
\lambda_s = \theta_s \quad \text{for } \mu_{d-1} \text{-a.e. } s \in Z_{d-1}
\]
by the uniqueness of the disintegration. The proof is complete. \( \square \)

4.2.1 Proof of (4.7). Assume to the contrary that (4.7) does not hold. Then
\[
\mu_{d-1} \{ s \in Z_{d-1} : (\pi_{d-1}^{d+1})_s(\lambda_s \times v^{(d)}_s) \neq \delta_s \times \mu_{d-1,s}^{(d+1)} \} > 0.
\]
So there is some function \( f \in C(N_{d+1}(Z_{d-1})) \) such that \( \mu_{d-1}(C) > 0 \), where

\[
C = \{ s \in Z_{d-1} : (\pi_{d+1}^d)_*(\lambda_s \times \nu_s^d)(f) > \delta_s \times \mu_{d-1,s}^{(d+1)}(f) \}.
\]

Let \( B = \psi^{-1}(C) \) and \( A = p_{d-1,2}^{-1}(B) \), where

\[
p_{d-1,2} : (N_{d+1}(Z_{d-1}), \langle \tau_{d+1}, \sigma_{d+1} \rangle) \rightarrow (N_d(Z_{d-1}), \langle \tau_d, \sigma_d \rangle); \ (s_1, s) \mapsto s
\]

is the projection. Since \((\pi_{d-1}^d)_*(\nu_s^d) = \mu_{d-1,s}^{(d+1)} = (\mu_{d-1})_s \) and \((\mu_{d-1})_s(\psi^{-1}(s)) = 1\) for \( \mu_{d-1}\)-a.e. \( s \in Z_{d-1} \), one has that for \( \mu_{d-1}\)-a.e. \( s \in Z_{d-1} \),

\[
\mu_{d-1,s}^{(d+1)}(B) = \begin{cases} 
1 & \text{if } s \in C, \\
0 & \text{if } s \not\in C,
\end{cases}
\]

and

\[
(\pi_{d-1}^d)_*(\lambda_s \times \nu_s^d)(A) = (\pi_{d-1}^d)_*(\nu_s^d)(B) = \begin{cases} 
1 & \text{if } s \in C, \\
0 & \text{if } s \not\in C,
\end{cases}
\]

Moreover, for \( \mu_{d-1}\)-a.e. \( s \in Z_{d-1} \), one has that for \( \mu_{d-1}\)-a.e. \( s \in Z_{d-1} \),

\[
\delta_s \times \mu_{d-1,s}^{(d+1)}(A) = \mu_{d-1,s}^{(d+1)}(B) = \begin{cases} 
1 & \text{if } s \in C, \\
0 & \text{if } s \not\in C.
\end{cases}
\]

Thus

\[
\mu_{d-1}^{(d+1)}(f \cdot 1_A) = \int_{Z_{d-1}} f \cdot 1_A \ d\mu_{d-1}^{(d+1)}
\]

\[
= \int_{N_{d+1}(Z_{d-1})} \left( \int_{Z_{d-1}} f \cdot 1_A \ d\delta_s \times \mu_{d-1,s}^{(d+1)} \right) d\mu_{d-1}(s)
\]

\[
= \int_{C} \delta_s \times \mu_{d-1,s}^{(d+1)}(f) d\mu_{d-1}(s) < \int_{C} (\pi_{d-1}^d)_*(\lambda_s \times \nu_s^d)(f) \ d\mu_{d-1}(s)
\]

\[
= \int_{Z_{d-1}} \left( \int_{N_{d+1}(Z_{d-1})} f \cdot 1_A \ d(\pi_{d-1}^d)_*(\lambda_s \times \nu_s^d) \right) d\mu_{d-1}(s)
\]

\[
= \mu_{d-1}^{(d+1)}(f \cdot 1_A),
\]

a contradiction. Hence \((4.7)\) holds.

4.3 Proof of Theorem B. In this subsection we show how to obtain Theorem B from Theorem A. We need the following formula which is easily verified.

Lemma 4.6. Let \( \{a_i\}, \{b_i\} \subseteq \mathbb{C} \). Then

\[
\prod_{i=1}^k a_i - \prod_{i=1}^k b_i = (a_1 - b_1)b_2 \cdots b_k + a_1(a_2 - b_2)b_3 \cdots b_k + \cdots + a_1 \cdots a_{k-1}(a_k - b_k).
\]
The proof of Theorem B. Since \((X, \mathcal{X}, \mu, T)\) has a \(\langle \tau_d, \sigma_d \rangle\)-strictly ergodic model, we may assume that \((X, T)\) itself is a minimal t.d.s. and \(\mu\) is its unique measure such that \((N_d(X), \langle \tau_d, \sigma_d \rangle)\) is uniquely ergodic with the unique measure \(\mu^{(d)}\).

Fix \(f_1, \ldots, f_d \in L^\infty\) and let \(\epsilon > 0\). Without loss of generality, we assume that for all \(1 \leq j \leq d\), \(\|f_j\|_\infty \leq 1\). Choose continuous functions \(g_j\) such that \(\|g_j\|_\infty \leq 1\) and \(\|f_j - g_j\|_1 < \epsilon/d\) for all \(1 \leq j \leq d\). We have

\[
\left| \frac{1}{N^2} \sum_{n \in [0,N-1]} \prod_{j=1}^d f_j(T^{n+(j-1)m}x) \right| - \int_{N_d(X)} \prod_{j=1}^d f_j \mu^{(d)}(dx)
\]

\[
\leq \left| \frac{1}{N^2} \sum_{n \in [0,N-1]} \prod_{j=1}^d f_j(T^{n+(j-1)m}x) \right| - \frac{1}{N^2} \sum_{n \in [0,N-1]} \prod_{j=1}^d g_j(T^{n+(j-1)m}x)
\]

\[
+ \left| \frac{1}{N^2} \sum_{n \in [0,N-1]} \prod_{j=1}^d g_j(T^{n+(j-1)m}x) \right| - \int_{N_d(X)} \prod_{j=1}^d g_j \mu^{(d)}(dx)
\]

\[
+ \left| \int_{N_d(X)} \prod_{j=1}^d g_j \mu^{(d)}(dx) - \int_{N_d(X)} \prod_{j=1}^d f_j \mu^{(d)}(dx) \right|.
\]

Now by the Pointwise Ergodic Theorem for \(\mathbb{Z}^2\) applied to \((n, m) \mapsto T^{n+(j-1)m}\) (see, for example, [35]) we have that for all \(1 \leq j \leq d\)

\[
\frac{1}{N^2} \sum_{n \in [0,N-1]} \sum_{m \in [0,N-1]} |f_j(T^{n+(j-1)m}x) - g_j(T^{n+(j-1)m}x)| \rightarrow \|f_j - g_j\|_1, \quad N \rightarrow \infty.
\]

for \(\mu\) a.e. Hence by Lemma 4.6

\[
\limsup_{N \rightarrow \infty} \left| \frac{1}{N^2} \sum_{n \in [0,N-1]} \prod_{j=1}^d f_j(T^{n+(j-1)m}x) \right| - \frac{1}{N^2} \sum_{n \in [0,N-1]} \prod_{j=1}^d g_j(T^{n+(j-1)m}x)
\]

\[
\leq \sum_{j=1}^d \left[ \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{n \in [0,N-1]} \prod_{j=1}^d |f_j(T^{n+(j-1)m}x) - g_j(T^{n+(j-1)m}x)| \right]
\]

\[
= \sum_{j=1}^d \|f_j - g_j\|_1 \leq \epsilon, \quad a.e.
\]

Since \(g_1 \otimes \cdots \otimes g_d : X^d \rightarrow \mathbb{R}\) is continuous and \((N_d(X), \langle \tau_d, \sigma_d \rangle, \mu^{(d)})\) is uniquely ergodic, we have

\[
\lim_{N \rightarrow \infty} \left| \frac{1}{N^2} \sum_{n \in [0,N-1]} \prod_{j=1}^d g_j(T^{n+(j-1)m}x) - \int_{N_d(X)} \prod_{j=1}^d g_j \mu^{(d)}(dx) \right| = 0.
\]
Since the $j$-th marginal of $\mu^{(d)}$ is equal to $\mu$, by Lemma 4.6 we have

\begin{equation}
\left| \int_{N_d(X)} g_j d\mu^{(d)} - \int_{N_d(X)} f_j d\mu^{(d)} \right| \leq \sum_{j=1}^d \int_{X} |g_j - f_j| d\mu \leq \epsilon.
\end{equation}

So combining (4.10)–(4.14), we have

\[
\limsup_{N \to \infty} \left| \frac{1}{N^2} \sum_{m \in [0, N-1]} \prod_{j=1}^d f_j(T^{n+j-1}m_X) - \int_{N_d(X)} \prod_{j=1}^d f_j d\mu^{(d)} \right| \leq 2\epsilon, \quad \text{a.e.}
\]

Since $\epsilon$ is arbitrary, the proof is completed. \hfill \Box

4.4 Proof of Theorem D.

Proof. First we assume that $(X, T, \mu)$ is the strictly ergodic system obtained in Theorem 4.2 by setting $X = \hat{X}$ with the unique measure $\mu^{(d+1)}$.

Let $\pi_{d-1} : X \to Z_{d-1}$ be the factor map and $\mu = \int_{Z_{d-1}} \theta_s d\mu_{d-1}(s)$ be the disintegration of $\mu$ over $\mu_{d-1}$. Then by Theorem 4.5 we have

\begin{equation}
\mu^{(d+1)} = \int_{Z_{d-1}} \theta_s \times \nu^{(d)}_{\pi_{d-1}(x)} d\mu_{d-1}(s).
\end{equation}

For $x \in X$ let

\begin{equation}
\mu^{(d)}_{x} = \nu^{(d)}_{\pi_{d-1}(x)}.
\end{equation}

By definition, for $\mu$ a.e. $x \in X$, $\mu^{(d)}_{x}$ is ergodic under $T \times T^2 \times \cdots \times T^d$.

Now we verify that $\{ \mu^{(d)}_{x} \}_{x \in X}$ satisfies (1.2). First, together with

\[
\mu = \int_{Z_{d-1}} \theta_s d\mu_{d-1}(s),
\]

we can rewrite (4.15) as

\begin{equation}
\mu^{(d+1)} = \int_{X} \delta_x \times \mu^{(d)}_{x} d\mu(x).
\end{equation}

In fact,

\[
\int_{X} \delta_x \times \mu^{(d)}_{x} d\mu(x) = \int_{Z_{d-1}} \left( \int_{X} \delta_x \times \nu^{(d)}_{\pi_{d-1}(x)} d\theta_s(x) \right) d\mu_{d-1}(s)
\]

\[
= \int_{Z_{d-1}} \left( \int_{X} \delta_x d\theta_s(x) \right) \times \nu^{(d)}_{\pi_{d-1}(x)} d\mu_{d-1}(s)
\]

\[
= \int_{Z_{d-1}} \theta_s \times \nu^{(d)}_{\pi_{d-1}(x)} d\mu_{d-1}(s) = \mu^{(d+1)}.
\]
Now we show (1.2). By Theorem 1.1 in [26], let the left side of (1.2) converge (in \( L^2 \)) to some function \( g \). Now we show \( g \) is equal to the right side of (1.2). Let \( f \in L^\infty(X) \); we have
\[
\int f(x)g(x) \, d\mu(x)
= \lim_{N \to \infty} \int_X \frac{1}{N} \sum_{n=0}^{N-1} f(x)f_1(T^nx)f_2(T^{2n}x) \cdots f_d(T^{dn}x) \, d\mu(x)
= \lim_{N \to \infty} \int_X \frac{1}{N} \sum_{n=0}^{N-1} f(T^nx)f_1(T^{2n}x)f_2(T^{3n}x) \cdots f_d(T^{(d+1)n}x) \, d\mu(x)
= \int_{\mathcal{X}^d} f(x_0)f_1(x_1) \cdots f_d(x_d) \, d\mu^{(d)}(x_0, x_1, \ldots, x_{d-1}) \quad \text{(by Definition 3.1)}
= \int_X f(x) \left( \int_{\mathcal{X}^d} f_1(x_1)f_2(x_2) \cdots f_d(x_d) \, d\mu_x^{(d)}(x_1, x_2, \ldots, x_d) \right) \, d\mu(x).
\]
Thus \( g(x) = \int_{\mathcal{X}^d} f_1(x_1)f_2(x_2) \cdots f_d(x_d) \, d\mu_x^{(d)}(x_1, x_2, \ldots, x_d) \), \( \mu \) a.e. \( x \in X \). Note that (4.17) is used in the last equality.

For \( j \in \{1, 2, \ldots, d\} \), \((p_j)_*(\mu_x^{(d)})\) is a \( T^j \)-invariant measure of \( X \). Let \( v_j = (p_j)_*(\mu_x^{(d)}) \). Since \((X, T, \mu)\) is uniquely ergodic, it is easy to see that \( \mu = [v_j + T_*v_j + \cdots + (T^{j-1})*v_j]/j \). It follows that \((p_j)_*(\mu_x^{(d)}) = v_j \ll \mu \). Hence the theorem holds for the system \((X, \mu, T)\).

Now we prove the result for any ergodic system \((X, \mathcal{X}, \mu, T)\). By the proof of Theorem A, \((X, \mathcal{X}, \mu, T)\) has a strictly ergodic model \((\hat{X}, \hat{\mathcal{T}}, \hat{\mu})\). Let \( \phi : X \to \hat{X} \) be the isomorphism. It is clear that \( \phi^d : \mathcal{X}^d \to \hat{\mathcal{X}}^d \) is also an isomorphism.

By the proof above, we have shown that for \((\hat{X}, \hat{\mathcal{T}}, \hat{\mu})\), there exists a family \( \{\hat{\mu}_x^{(d)}\}_{x \in \hat{X}} \) of probability measures on \( \hat{\mathcal{X}}^d \) such that it satisfies conditions (1)–(3) listed in the theorem. Define \( \mu_x^{(d)} = \hat{\mu}_x^{(d)} \circ \phi^d \). By (3), for \( \hat{\mu} \) a.e. \( \hat{x} \in \hat{X} \), \((p_j)_*(\hat{\mu}_x^{(d)}) \ll \hat{\mu} \) for \( 1 \leq j \leq d \), where \( p_j : \hat{\mathcal{X}}^d \to \hat{X} \) is the projection to the \( j \)-th coordinate. It follows that \( \mu_x^{(d)} \) is well-defined. Then it is not hard to check that \( \{\mu_x^{(d)}\}_{x \in X} \) also satisfies (1)–(3). The proof is completed. \( \square \)

5 Proof of Theorem C

In this section we will prove Theorem C. To do this, first we derive some properties from the result proved in the previous sections. Then using the properties and a lemma we show that the pointwise convergence can be lifted from a distal system to its isometric extension under some conditions. Finally, we conclude Theorem C by the structure theorem for distal systems.
5.1 Isometric extensions. Isometric extensions and weakly mixing extensions are two basic extensions in the Furstenberg structure theorem for a m.p.t. Let \( \pi : (X, X, \mu, T) \to (Y, Y, \nu, S) \) be a factor map. The \( L^2(X, X, \mu) \) norm is denoted by \( \| \cdot \| \) and the \( L^2(X, X, \mu, \nu) \) norm by \( \| \cdot \|_\nu \) for \( \nu \)-almost every \( y \in Y \).

Recall \( \{ \mu_y \}_{y \in Y} \) is the disintegration of \( \mu \) relative to \( \nu \). A function \( f \in L^2(X, X, \mu) \) is almost periodic over \( Y \) if for every \( \epsilon > 0 \) there exist \( g_1, \ldots, g_l \in L^2(X, X, \mu) \) such that for all \( n \in \mathbb{Z} \)

\[
\min_{1 \leq j \leq l} \| T^n f - g_j \|_\nu < \epsilon
\]

for \( \nu \) almost every \( y \in Y \). One writes \( f \in AP(Y) \). Let \( K(X|Y, T) \) be the closed subspace of \( L^2(X) \) spanned by the almost periodic functions over \( Y \). When \( Y \) is trivial, \( K(X, T) = K(X|Y, T) \) is the closed subspace spanned by eigenfunctions of \( T \).

Now \( X \) is an isometric extension of \( Y \) if \( K(X|Y, T) = L^2(X) \) and it is a (relatively) weak mixing extension of \( Y \) if \( K(X|Y, T) = L^2(Y) \).

It can be shown that if \( X \) is an isometric extension of an m.p.t. \((Y, \nu, S)\), then \( X \) is isomorphic to a skew product \( X' = Y \times M \), where \( M = G/H \) is a homogeneous compact metric space; \( \mu' = \nu \times m_M \) with \( m_M \) is the unique probability measure invariant under the transitive group of isometries \( G \). Moreover, the action of \( T' \) on \( X' \) is given by

\[
T'(y, gH) = (Sy, \rho(y)gH),
\]

where \( \rho : Y \to G \) is a cocycle. We denote \( X' \) by \( Y \times_\rho G/H \) and \( T' \) by \( T_\rho \). When \( H \) is trivial, we say \( Y \times_\rho G \) is a group extension of \( Y \). We refer to [20] for the details.

**Lemma 5.1.** Let \( \pi : (X, X, \mu, T) \to (Y, Y, \nu, S) \) be a factor map between ergodic systems with \( Z_{d-1}(X) = Z_{d-1}(Y) \), and \( d \in \mathbb{N} \). Assume that \( \{ \mu_x^{(d)} \}_{x \in X} \) and \( \{ v_y^{(d)} \}_{y \in Y} \) are the families of measures defined in Theorem D respectively. Then for given \( f_1, \ldots, f_d \in L^\infty(\mu) \), one has that for \( \mu \) a.e. \( x \in X \)

\[
\int_{X^d} f_1(x_1) \cdots f_d(x_d) \, d\mu_x^{(d)}(x_1, \ldots, x_d) = \int_{Y^d} \mathbb{E}(f_1|Y)(y_1) \mathbb{E}(f_2|Y)(y_2) \cdots \mathbb{E}(f_d|Y)(y_d) \, dv_y^{(d)}(y_1, y_2, \ldots, y_d).
\]

**Proof.** Since \( Z_{d-1}(X) = Z_{d-1}(Y) \), by Theorem 12.1 in [26], \( Y \) is also a
characteristic factor of $X$. That is
\[
\left\| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_d(T^{dn} x) \right\|_{L^2} - \frac{1}{N} \sum_{n=0}^{N-1} E(f_1|Y)(T^n x) E(f_2|Y)(T^{2n} x) \cdots E(f_d|Y)(T^{dn} x) \nrightarrow 0
\]
as $N \to \infty$. Moreover, we obtain (5.1) by applying Theorem D (2) to $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$. □

In the proof of Proposition 5.3 we need the following Proposition 5.2. Recall that for a compact metric space $X$, $M(X)$ is the set of all Borel probability measures on $X$ with weak* topology. If $T$ is a continuous map from $X$ to itself, then it is well known that for all $x \in X$ each limit point of \( \{ \frac{1}{N} \sum_{n=0}^{N-1} T^n \delta_x \}_{N \in \mathbb{N}} \) is $T$-invariant. If $T$ is measurable instead of continuity, then more will be involved. Proposition 5.2 will deal with the similar situation in $X^d$ for our purpose.

We remark that when $d = 2$ and all transformations are ergodic, this proposition was proved in [33, Proposition 3]. We leave the proof of this result to the appendix, which is similar to the one in [33] but much more involved.

**Proposition 5.2.** Let $(X, \mathcal{X}, \mu)$ be a probability space with $X$ compact metric space and $d \in \mathbb{N}$. For $1 \leq i \leq d$, let $T_i : X \to X$ be measure preserving transformations with finitely many ergodic components. Then there is a measurable set $X_\ast$ with $\mu(X_\ast) = 1$ such that for $x \in X_\ast$ each weak* limit point $\lambda$ of the sequence
\[
\left\{ \frac{1}{N} \sum_{n=0}^{N-1} (T_1 \times T_2 \times \cdots \times T_d)^n \delta_{(x,x,\ldots,x)} \right\}_N
\]
is in $M(X^d)$, and $\lambda$ is $T_1 \times \cdots \times T_d$-invariant.

The following proposition is crucial for our proof.

**Proposition 5.3.** Let $\pi : (X = Y \times_{\rho} G/H, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$ be an isometric extension between two ergodic systems with $Z_{d-1}(X) = Z_{d-1}(Y)$, and $d \in \mathbb{N}$. If
\[
\frac{1}{N} \sum_{n=0}^{N-1} f'_1(T^n y) f'_2(T^{2n} y) \cdots f'_d(T^{dn} y)
\]
converge $\nu$ a.e. for any given $f'_1, \ldots, f'_d \in L^\infty(\nu)$, then
\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_d(T^{dn} x)
\]
converge $\mu$ a.e. for any given $f_1, \ldots, f_d \in L^\infty(\mu)$. 
Proof. We may assume that $Y$ is a compact metric space. By the assumption of the theorem, there is some measurable set $Y_0 \in \mathcal{Y}$ with $\nu(Y_0) = 1$ such that for $y \in Y_0$ and for all $f'_1, \ldots, f'_d \in C(Y)$,

$$\frac{1}{N} \sum_{n=0}^{N-1} f'_1(T^ny) f'_2(T^{2n}y) \cdots f'_d(T^{dn}y)$$

converge since $C(Y)$ is separable. By Theorem D, we may assume that for all $y \in Y_0$ and for all $f'_1, \ldots, f'_d \in C(Y)$,

$$\frac{1}{N} \sum_{n=0}^{N-1} f'_1(T^ny) f'_2(T^{2n}y) \cdots f'_d(T^{dn}y) \rightarrow \int_{Y^d} f'_1(y_1)f'_2(y_2) \cdots f'_d(y_d) \, d\nu_y(y_1, y_2, \ldots, y_d)$$

as $N \to \infty$, since the almost everywhere limit coincides with the limit in $L^2$.

Let $X_0 = \{(y, gH) : y \in Y_0, g \in G\}$. Then $\mu(X_0) = \mu(Y_0 \times G/H) = 1$. Since $C(X)$ is a separable space, by Lemma 5.1, there is measurable set $X_1$ such that $\mu(X_1) = 1$ and (5.1) holds for all continuous functions. In Proposition 5.2 we take $T_1 = T, \ldots, T_d = T^d$ and let $X_\ast$ be the set defined there.

Now fix $x = (y, gH) \in X_0 \cap X_1 \cap X_\ast$. Let $\lambda$ be a weak$^*$ limit point of the sequence

$$\left\{ \frac{1}{N} \sum_{n=0}^{N-1} (T \times T^2 \times \cdots \times T^d)_x^n \delta_{(x, x, \ldots, x)} \right\}_N$$

in $M(X^d)$. By Proposition 5.2, $\lambda$ is $T \times \cdots \times T^d$-invariant. We are going to show that $\lambda = \mu_x^{(d)}$.

Let $\pi^d : X^d \to Y^d$, $(x_1, \ldots, x_d) \mapsto (\pi(x_1), \ldots, \pi(x_d))$. Then $\pi_\ast^d \lambda$ is a weak$^*$ limit point of the sequence

$$\left\{ \frac{1}{N} \sum_{n=0}^{N-1} (S \times S^2 \times \cdots \times S^d)_y^n \delta_{(y, y, \ldots, y)} \right\}_N$$

in $M(Y^d)$. By the assumption, we know that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} (S \times S^2 \times \cdots \times S^d)_y^n \delta_{(y, y, \ldots, y)} = \nu^{(d)}_y.$$ 

Thus $\pi_\ast^d \lambda = \nu^{(d)}_y$. 

Let $\psi_1, \ldots, \psi_d \in C(G)$ such that $\psi_i \geq 0$ and $\int_G \psi_i \, dm = 1$ for all $i \in \{1, \ldots, d\}$. Now define a new measure $\lambda_{\psi_1, \ldots, \psi_d}$ as follows:

$$\lambda_{\psi_1, \ldots, \psi_d}(f_1 \otimes \cdots \otimes f_d) = \int_{G^d \times X^d} f_1(y_1, h_1 g_1 H) \cdots f_d(y_d, h_d g_d H) \times \psi_1(h_1) \cdots \psi_d(h_d) \, dh_1 \cdots dh_d \, d\lambda(x_1, \ldots, x_d),$$

(5.2)

where $x_i = (y_i, g_i H)$ and $f_i \in C(X)$ for all $i \in \{1, 2, \ldots, d\}$.

Then by $\pi^{(d)}_* \lambda = \nu^{(d)}$, (5.1), and $E(|f_i| Y)(y_i) = \int_G |f_j(y_j, h_i H)| \, dh_i$, $1 \leq j \leq d$ we have

$$|\lambda_{\psi_1, \ldots, \psi_d}(f_1 \otimes \cdots \otimes f_d)| \leq \prod_{i=1}^d \sup_{g \in G} |\psi_i(g)| \times \int_{G^d \times X^d} |f_1(y_1, h_1 g_1 H) \cdots f_d(y_d, h_d g_d H)| \, dh_1 \cdots dh_d \, d\lambda(x_1, \ldots, x_d)$$

$$= \prod_{i=1}^d \sup_{g \in G} |\psi_i(g)| \times \int_{G^d \times Y^d} |f_1(y_1, h_1 H) \cdots f_d(y_d, h_d H)| \, dh_1 \cdots dh_d \, dv^{(d)}(y_1, \ldots, y_d)$$

$$= \prod_{i=1}^d \sup_{g \in G} |\psi_i(g)| \int_{Y^d} E(|f_1| Y)(y_1) E(|f_2| Y)(y_2) \cdots E(|f_d| Y)(y_d) \, dv^{(d)}(y_1, y_2, \ldots, y_d)$$

$$= \prod_{i=1}^d \sup_{g \in G} |\psi_i(g)| \int_{X^d} |f_1(x_1) \cdots f_d(x_d)| \, d\mu^{(d)}_x(x_1, \ldots, x_d).$$

Thus we have

$$\lambda_{\psi_1, \ldots, \psi_d} \ll \mu^{(d)}_x.$$

(5.3)

Note that $\rho : Y \to G$ is a cocycle. For each $n \in \mathbb{N}$, let

$$\rho^{(n)}(y) = \rho(S^{n-1}y) \rho(S^{n-2}y) \cdots \rho(y).$$

Then we have

$$T^n(y, gH) = (S^n, \rho^{(n)}(y)gH).$$
Now, in addition, we assume that \( \psi \) satisfy \( \psi_i(h^{-1}gh) = \psi_i(g) \) for all \( 1 \leq i \leq d \). Then for \( f_1, \ldots, f_d \in C(X) \), we have

\[
\begin{align*}
\lambda \psi_1, \ldots, \psi_d((f_1 \otimes \cdots \otimes f_d) \circ T \times \cdots \times T^d) & = \int_{G^d \times X^d} f_1(Sy_1, \rho(y_1)h_1g_1H) \cdots f_d(S^d_y d, \rho^{(d)}(y_d)g_dH) \times \psi_1(h_1) \cdots \psi_d(h_d) dh_1 \cdots dh_d \lambda(x_1, \ldots, x_d) \\
& = \int_{G^d \times X^d} f_1(Sy_1, h_1 \rho(y_1)g_1H) \cdots f_d(S^d_y d, h_d \rho^{(d)}(y_d)g_dH) \times \psi_1(h_1) \cdots \psi_d(h_d) dh_1 \cdots dh_d \lambda(x_1, \ldots, x_d) \\
& = \frac{\lambda \psi_1, \ldots, \psi_d(f_1 \otimes \cdots \otimes f_d)}{\lambda \psi_1, \ldots, \psi_d((f_1 \otimes \cdots \otimes f_d) \circ T \times \cdots \times T^d)}.
\end{align*}
\]

That is, \( \lambda \psi_1, \ldots, \psi_d \) is \( T \times \cdots \times T^d \)-invariant. Since \( \mu_{\otimes}^{(d)} \) is ergodic, we have that

\[
\lambda \psi_1, \ldots, \psi_d = \mu_{\otimes}^{(d)}.
\]

Now we will define a sequence \( \{ \phi_n \} \) such that all \( \phi_n \) satisfy the properties which \( \psi_i \) hold above, and

\[
\lambda \phi_n \rightarrow \lambda, \quad n \rightarrow \infty.
\]

Then we get that \( \lambda = \mu_{\otimes}^{(d)} \).

Since \( G \) is a compact metric group, there is an invariant metric \( \varrho \). For all \( n \in \mathbb{N} \), let

\[
\varphi_n(g) = 1/n - \inf \{ 1/n, \varrho(e, g) \}.
\]

Then let

\[
\phi_n = \frac{\varphi_n}{\int_G \varphi_n dm}.
\]

Note that \( \phi_n \) is supported on \( A_n = \{ g \in G : \varrho(e, g) < \frac{1}{n} \} \). It follows that for given \( y_1, \ldots, y_d, f_1(y_1, h_1g_1H) \cdots f_d(y_d, h_dg_dH) \) is close to \( f_1(y_1, g_1H) \cdots f_d(y_d, g_dH) \) uniformly on \( A_n \). Then using the fact that \( \int_G \phi_n dm = 1 \) we deduce that \( \phi_n \) is what we need.

To sum up, we have proved that for all \( x \in X_0 \cap X_1 \cap X_* \), \( \mu_{\otimes}^{(d)} \) is the unique weak* limit point of sequence

\[
\left\{ \frac{1}{N} \sum_{n=0}^{N-1} (T \times T^2 \times \cdots \times T^d)^n \delta_{(x, x, \ldots, x)} \right\}_N
\]

in $M(X^d)$. Hence for all $x \in X_0 \cap X_1 \cap X_\ast$, we have that for all $f_1, \ldots, f_d \in C(X)$
\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_d(T^{dn} x)
\rightarrow \int_{X^d} f_1(x_1) f_2(x_2) \cdots f_d(x_d) \, d\mu_x^{(d)}(x_1, x_2, \ldots, x_d)
\]  
(5.4)
as $N \to \infty$. Note that $\mu(X_0 \cap X_1 \cap X_\ast) = 1$.

Now by the same approximation argument as in the proof of Theorem B, we have that for all $f_1, \ldots, f_d \in L^\infty(\mu)$,
\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_d(T^{dn} x)
\]converges $\mu$ a.e. The proof is completed. \hfill \Box

5.2 Proof of Theorem C. In this final subsection we will prove Theorem C. We start with the definition of a distal system.

**Definition 5.4.** Let $\pi : (X, X, \mu, T) \to (Y, Y, \nu, T)$ be a factor between two ergodic systems. We call the extension $\pi$ a **distal extension** if there exists a countable ordinal $\eta$ and a directed family of factors $(X_\theta, X_\theta, \mu_\theta, T)$, $\theta \leq \eta$ such that:

1. $X_0 = Y$ and $X_\eta = X$.
2. For $\theta < \eta$, the extension $\pi_\theta : X_{\theta+1} \to X_\theta$ is isometric and non-trivial (i.e., not an isomorphism).
3. For a limit ordinal $\lambda \leq \eta$, $X_\lambda = \lim_{\theta < \lambda} X_\theta$ (i.e., $X_\lambda = \bigvee X_\theta$).

If $X$ is a distal extension of the trivial system, then $(X, X, \mu, T)$ is called a **distal system**.

**The proof of Theorem C.** We say a system $(X, X, \mu, T)$ satisfies $(\ast)$, if for all $f_1, \ldots, f_d \in L^\infty(\mu)$
\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \cdots f_d(T^{dn} x)
\]converge $\mu$ a.e. The aim is to prove each distal system satisfies $(\ast)$. We will use the structure of distal systems and Proposition[5.3] to complete the proof.

Let $\pi_{d-1} : X \to Z_{d-1}$ be the factor map. Then $\pi_{d-1}$ is distal since $X$ is distal. By the definition of a distal extension, there exists a countable ordinal $\eta$ and a directed family of factors $(X_\theta, X_\theta, \mu_\theta, T)$, $\theta \leq \eta$ such that:
(1) \( X_0 = Z_{d-1} \) and \( X_\eta = X \).

(2) For \( \theta < \eta \), the extension \( \pi_\theta : X_{\theta+1} \rightarrow X_\theta \) is non-trivial isometric.

(3) For a limit ordinal \( \lambda \leq \eta \), \( X_\lambda = \lim_{\leftarrow \theta < \lambda} X_\theta \).

Then we have:

(i) By Theorem 3.7, \( X_0 = Z_{d-1} \) satisfies (\( \ast \)).

(ii) For \( \theta < \eta \) the extension \( \pi_\theta : X_{\theta+1} \rightarrow X_\theta \) is non-trivial isometric. If \( X_\theta \) satisfies (\( \ast \)), then by Proposition 5.3, \( X_{\theta+1} \) also satisfies (\( \ast \)).

(iii) For a limit ordinal \( \lambda \leq \eta \), if all \( \theta < \lambda \), \( X_\theta \) satisfies (\( \ast \)), then it is easy to verify that the inverse limit \( X_\lambda = \lim_{\leftarrow \theta < \lambda} X_\theta \) also satisfies (\( \ast \)).

(iv) By (i)–(iii), \( X_\eta = X \) satisfies (\( \ast \)).

The proof is complete. □

Appendix A. Proof of Proposition 5.2

Proof of Proposition 5.2. Notice that when all transformations \( T_i \) are continuous, the result follows by the standard argument and in this case \( X_\ast = X \).

Now we deal with the general case. Since \( X \) is a compact metric space, \( C(X) \) is separable. Let \( C_1 \) be a countable dense subset of \( C(X) \). Let

\[
C_2 = \{ \| g_1 \circ T_i - g_2 \|_d : 1 \leq i \leq d, \; g_1, g_2 \in C_1 \}.
\]

It is obvious that \( C_2 \) is countable. Since \( T_i \) has only finitely many ergodic components, it follows that \( J_{T_i} \) is a finite \( \sigma \)-algebra and hence is generated by a finite measurable partition \( \beta_i \) for each \( 1 \leq i \leq d \). For a given \( f \in C_2 \), there is some \( X_f \in \mathcal{X} \) with \( \mu(X_f) = 1 \) such that for all \( x \in X_f \), one has \( \mu(\beta_i(x)) > 0 \) and

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n_{\beta_i(x)}x) = \mathbb{E}(f|J_{T_i})(x) = \frac{\int_{\beta_i(x)} f(y)d\mu(y)}{\mu(\beta_i(x))}, \quad \forall \; 1 \leq i \leq d,
\]

where \( \beta_i(x) \) is the atom of \( \beta_i \) containing \( x \). Let

\[
X_\ast = \bigcap_{f \in C_2} X_f.
\]

Since \( C_2 \) is countable, we conclude that \( X_\ast \in \mathcal{X} \) and \( \mu(X_\ast) = 1 \).

Recall that the topology of \( C(X) \) is the uniform convergence topology. Let

\[
C_3 = \{ \| g_1 \circ T_i - g_2 \|_d : 1 \leq i \leq d, \; g_1, g_2 \in C(X) \}.
\]

Then each element of \( C_3 \) is the uniform limit of elements of \( C_2 \). It is easy to show that (A.1) holds for all \( x \in X_\ast \) and for all \( f \in C_3 \).
Let \( x_0 \in X_* \) and let \( \lambda \) be a weak limit point of the sequence
\[
\left\{ \frac{1}{N} \sum_{n=0}^{N-1} (T_1 \times T_2 \times \cdots \times T_d)^n \delta(x_0, x_0, \ldots, x_0) \right\}_{N}.
\]
Now we show \( \lambda \) is \( T_1 \times \cdots \times T_d \)-invariant.

For all \( f_1, \ldots, f_d \in C(X) \) we have that
\[
\left| \frac{1}{N} \sum_{n=0}^{N-1} d \prod_{i=1}^{d} f_i(T^n_i x_0) \right| \leq \prod_{i=1}^{d} \left( \frac{1}{N} \sum_{n=0}^{N-1} |f_i| d(T^n_i x_0) \right)^{1/d}.
\]
Since for each \( 1 \leq i \leq d, \) \( |f_i|^d \in C_3, \) by (A.1) we have
\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} d \prod_{i=1}^{d} f_i(T^n_i x_0) \right| \leq \prod_{i=1}^{d} (\mathbb{E}(|f_i|^d | \mathcal{J}_T)(x_0))^{1/d}.
\]
In particular, we deduce that
\[(A.2) \quad \lambda(f_1 \otimes \cdots \otimes f_d) \leq \prod_{i=1}^{d} (\mathbb{E}(|f_i|^d | \mathcal{J}_T)(x_0))^{1/d}.
\]
Now we show that \( \lambda(f_1 \circ T_1 \otimes \cdots \otimes f_d \circ T_d) = \lambda(f_1 \otimes \cdots \otimes f_d) \) for all \( f_1, \ldots, f_d \in C(X). \) Let \( M > 0 \) such that \( \|f_i\| \leq M, \) \( 1 \leq i \leq d. \) Let \( \delta = \min_{1 \leq i \leq d} \{ \mu(\beta_i(x_0)) \}. \) Then \( \delta > 0, \) and hence for any \( \epsilon > 0 \) one can choose functions \( g_i \in C(X), \) \( 1 \leq i \leq d \) such that
\[
\|f_i \circ T_i - g_i\|_{L^d(\mu)} < \epsilon \delta^{1/d} \quad \text{and} \quad \|g_i\| \leq M, \quad 1 \leq i \leq d.
\]
Thus
\[(A.3) \quad (\mathbb{E}(|f_i \circ T_i - g_i|^d | \mathcal{J}_T)(x_0))^{1/d} = \left( \frac{\int_{\beta_i(x_0)} |f_i \circ T_i - g_i|^d(y) d\mu(y)}{\mu(\beta_i(x_0))} \right)^{1/d} < \epsilon
\]
for \( 1 \leq i \leq d. \) By Lemma 4.6 and (A.2),
\[
|\lambda(f_1 \circ T_1 \otimes \cdots \otimes f_d \circ T_d) - \lambda(g_1 \otimes \cdots \otimes g_d)|
\leq \sum_{i=1}^{d} \left| \lambda \left( \bigotimes_{j=1}^{i-1} f_j \circ T_j \otimes (f_i \circ T_i - g_i) \otimes \bigotimes_{k=i+1}^{d} g_k \right) \right|
\leq \sum_{i=1}^{d} M^{d-1} (\mathbb{E}(|f_i \circ T_i - g_i|^d | \mathcal{J}_T)(x_0))^{1/d} \leq dM^{d-1} \epsilon.
\]
Also we have
\[
\frac{1}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{d} f_i(T^n_1 x_0) - \frac{1}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{d} g_i(T^n_1 x_0) \leq \frac{1}{N} \prod_{i=1}^{d} f_i(x_0) - \prod_{i=1}^{d} f_i(T_1^N x_0) + \frac{1}{N} \sum_{n=0}^{N-1} \left[ \prod_{i=1}^{d} f_i(T^n_1 x_0) - \prod_{i=1}^{d} g_i(T^n_1 x_0) \right]
\]
\[
(A.5) \leq \frac{2M^d}{N} + \sum_{i=1}^{d} \left[ \frac{1}{N} \sum_{n=0}^{N-1} \left( \prod_{j=1}^{i-1} f_j(T^{n+1}_j x_0) \right) \times (f_i(T^n_1 x_0) - g_i(T^n_1 x_0)) \left( \prod_{k=i+1}^{d} g_k(T^n_k x_0) \right) \right]^{1/d} \leq \frac{2M^d}{N} + \sum_{i=1}^{d} M^{d-1} \left( \frac{1}{N} \sum_{n=0}^{N-1} |f_i \circ T_i - g_i|^d(T^n_1 x_0) \right)^{1/d}.
\]

By (A.1) and (A.3), it follows that
\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{d} f_i(T^n_1 x_0) - \frac{1}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{d} g_i(T^n_1 x_0) \right|
\]
\[
(A.6) \leq \sum_{i=1}^{d} M^{d-1} \left( \mathbb{E}(|f_i \circ T_i - g_i|^d |\mathcal{T}_N(x_0)) \right)^{1/d} \leq dM^{d-1} \epsilon.
\]

In particular,
\[
(A.7) \quad |\hat{\lambda}(f_1 \otimes \cdots \otimes f_d) - \hat{\lambda}(g_1 \otimes \cdots \otimes g_d)| \leq dM^{d-1} \epsilon.
\]

By (A.4) and (A.7), we conclude that
\[
(A.8) \quad |\hat{\lambda}(f_1 \circ T_1 \otimes \cdots \otimes f_d \circ T_d) - \hat{\lambda}(f_1 \otimes \cdots \otimes f_d)| \leq 2dM^{d-1} \epsilon.
\]

Since \( \epsilon \) is arbitrary, the above inequality implies that
\[
\hat{\lambda}(f_1 \circ T_1 \otimes \cdots \otimes f_d \circ T_d) = \hat{\lambda}(f_1 \otimes \cdots \otimes f_d)
\]
for all \( f_1, \ldots, f_d \in C(X) \). Since the linear span of \( C(X)^d \) is dense in \( C(X^d) \), we have
\[
\hat{\lambda}(F \circ (T_1 \times \cdots \times T_d)) = \hat{\lambda}(F)
\]
for all \( F \in C(X^d) \). That is, \( \hat{\lambda} \) is \( T_1 \times \cdots \times T_d \)-invariant. \( \square \)
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