Thermal and mechanical equilibrium among weakly interacting systems in generalized thermostatistics framework

A.M. Scarfone

Istituto Nazionale di Fisica della Materia (CNR-INFM) and Physics Department
Unità del Politecnico di Torino, Corso Duca degli Abruzzi 24,
I-10129 Torino, Italy

Abstract

We consider two statistically independent systems described by the same entropy belonging to the two-parameter family of Sharma-Mittal. Assuming a weak interaction among the systems, allowing in this way an exchange of heat and work, we analyze, both in the entropy representation and in the energy representation, the evolution toward the equilibrium. The thermodynamics evolution is controlled by two scalar quantities identified with the temperature and the pressure of the system. The thermodynamical stability conditions of the equilibrium state are analyzed in both representations. Their relationship with the concavity conditions for the entropy and with the convexity conditions for the energy are spotlighted.

Key words: Sharma-Mittal entropy, thermodynamical equilibrium, thermodynamical stability.
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1 Introduction

When two different systems, posed in thermodynamical contact, exchange heat and work, they evolve toward an equilibrium configuration. In the entropy representation, the evolution toward equilibrium is controlled by the increase of entropy, which reaches its maximum value according to the maximum entropy principle. Differently, in the energy representation, the evolution toward equilibrium is ruled by the decrease of energy, which reaches its minimum value according to the minimum energy principle.

As known, the formal development of the thermodynamical theory can be equivalently carried on in both these formalisms [1] and many physical implications can be obtained by applying the extremal principles. For instance, one
can derive a definition of temperature and pressure as the variables controlling the exchange of heat and work [2] and obtain the thermodynamical stability conditions (TSCs) of the equilibrium state.

In this paper we will be particularly concerned with some questions related to the approach toward the equilibrium among two weakly interacting systems described by the same entropy belonging to the two-parameter family of Sharma-Mittal (SM) [3]. Many one-parameter entropies introduced in literature, in the framework of the generalized statistical mechanics, belong to the SM family and can be thus considered in a unifying scheme. Among them, we recall the Rényi entropy [4], the Tsallis entropy [5], the Landsberg-Vedral entropy [6], and others [7]. Remarkably, these entropies admit a probability distribution function with an asymptotic power law behavior which differs from the exponential behavior showed by the Gibbs distribution.

The SM entropy, introduced initially in the information theory, has been recently reconsidered in the framework of the generalized thermostatistics [7]. In [8] a kinetic approach based on a nonlinear Fokker-Planck equation related to the SM entropy has been discussed. Physical applications in the study of a weakly interacting gas [9] and in the context of the specific heat in the non extensive statistical picture [10] have been reported, whilst in [11], it has been rediscovered on the Kolmogorov-Nagumo average framework [12].

The purpose of this paper is twofold. Firstly, we study, both in the entropy and in the energy representation, the approach toward the equilibrium of two systems weakly interacting described by the same entropy. It is shown that the evolution toward the equilibrium is controlled by two scalar quantities, which can be identified with the temperature and the pressure of the system. Alternative definitions of temperature and pressure, in presence of a generalized entropy, have been previously advanced in literature [13,14,15,16]. They are based on a generalization of the thermodynamical zero law, which is substantially different from the dynamical approach discussed in this work.

Successively, we explore the TSCs for the equilibrium state. In the Boltzmann-Gibbs theory, TSCs are equivalent to the concavity conditions for the entropy or to the convexity conditions for the energy. Since the SM entropy fulfil a not linear “composability” rule, it is show, in accordance with the existing literature [15,16,17], that in this case TSCs are not equivalent to the concavity conditions for the entropy. Differently, by assuming for the energy a linear composition, in the energy representation TSCs are merely consequences of the convexity conditions for the energy. In this sense, the “composability” rule of the relevant physical quantities play a rôle in the derivation of the TSCs.

The plan of the paper is the following. In Section 2, we revisit the SM entropy recalling some useful proprieties. In Section 3, we study the approach to the equilibrium according to the maximal entropy principle, whilst, in Section 4, we derive the TSCs in the entropy representation. In Section 5, equilibrium and its stability are reconsidered in the energy representation. The conclusions are reported in Section 6.
2 The Sharma-Mittal entropy

Let us introduce the SM entropy in the form

$$S_{q,2-r}(p) = \ln_q \left( \sum_{i=1}^{W} p_i^{2-r} \right)^{1/(r-1)},$$

(throughout this paper we take $k_B = 1$), where

$$\ln_q(x) = \frac{x^{1-q} - 1}{1 - q},$$

is the $q$-deformed logarithm and, $q > 0$ and $r < 2$, are two real parameters.

In Eq. (2.1) $p \equiv \{p_i\}_{i=1}^{W}$ is a discrete distribution function and $W$ denotes the number of microstates accessible by the system.

For our convenience, Eq. (2.1) differs from the definition given in Ref. [7] and is related to this one by $r \to 2 - r$.

Equation (2.1) includes some one-parameter entropies already known in literature: the Rényi entropy $S_{2-r}^R(p) = \ln(\sum_i p_i^{2-r})/(r-1)$ [4] for $q = 1$, the Tsallis entropy $S_q^T(p) = (\sum_i p_i^q - 1)/(1 - q)$ [5] for $r = 2 - q$, the Landsberg-Vedral entropy [6] $S_{2-q}^{LV} = [(\sum_i p_i^{2-q})^{-1} - 1]/(1 - q)$ for $r = q$, the Gaussian entropy [7] $S_q^G = \ln_q[\exp(-\sum_i p_i \ln(p_i))]$ for $r = 1$, the escort entropy [18] $S_q^E = [(\sum_i p_i^{1/q})^{-q} - 1]/(1 - q)$ for $r = 2 - 1/q$ and, last but not least, the Boltzmann-Gibbs entropy $S_{BG}^G(p) = -\sum_i p_i \ln(p_i)$, recovered in the $(q, r) \to (1, 1)$ limit.

In the entropy representation the canonical distribution at equilibrium can be obtained from the following variational problem

$$\frac{\delta}{\delta p_j} \left( S_{(m)}(p) - \gamma \sum_{i=1}^{W} p_i - \beta \sum_{i=1}^{W} p_i E_i \right) = 0,$$

(2.3)

[for sake of simplicity hereinafter we introduct the notation $\{m\} \equiv (q, 2-r)$]

where the constraints on the normalization $\sum_i p_i = 1$ and on the linear mean energy $U = \sum_i p_i E_i$ are taken into account through the Lagrange multipliers $\gamma$ and $\beta$, respectively. We observe that in Refs. [7,9] the mean energy is defined by means of “escort” probability distribution. Remarkably, these two different approaches are related according to the “$q \to 1/q$” symmetry [7].

In a similar way, in the energy representation, the distribution at equilibrium can be obtained from the following variational problem

$$\frac{\delta}{\delta p_j} \left( U(p) - \gamma' \sum_{i=1}^{W} p_i - \beta' \ln_q \left( \sum_{i=1}^{W} p_i^{2-r} \right)^{1/(r-1)} \right) = 0,$$

(2.4)
where now the constraints, given by the normalization and the entropy (2.1), are taken into account by the Lagrange multipliers $\gamma'$ and $\beta'$. Equations (2.3) and (2.4) differ only for a redefinition of the Lagrange multipliers according to $\beta = 1/\beta'$ and $\gamma = -\gamma'/\beta'$. As a consequence, both the variational problems give the same distribution

$$p_j = \frac{1}{Z_{(m)}} \exp_r \left( -\frac{\bar{\beta}}{2 - r} (E_j - U) \right),$$

(2.5)

with the $q$-deformed exponential, the inverse function of the $q$-deformed logarithm, given by

$$\exp_q(x) = [1 + (1 - q) x]^\frac{1}{1-q},$$

(2.6)

and $[x]_+ = \max(x, 0)$ defines a cut-off condition for $r < 1$, whereas the distribution shows an asymptotic power law behavior $p(E) \sim E^{-1/(r-1)}$ for $r > 1$. The quantity $\bar{\beta}$ is given by

$$\bar{\beta} = \frac{\beta}{(Z_{(m)})^{1-q}} = \frac{\beta}{1 + (1 - q) S_{(m)}},$$

(2.7)

and the normalization function $Z_{(m)}$, defined in

$$Z_{(m)} = \left( \sum_{i=1}^{W} p_i^{2-r} \right)^{1/(r-1)},$$

(2.8)

is a function of the Lagrange multipliers $\gamma$ and $\beta$

$$\left( Z_{(m)} \right)^{1-q} = \frac{r - 1}{2 - r} (\gamma + \beta U).$$

(2.9)

The function $Z_{(m)}$ is related to the canonical partition function $Z_{(m)}$ in

$$\ln_q \left( Z_{(m)} \right) = \ln_q \left( Z_{(m)} \right) - \beta U,$$

(2.10)

so that, from the definition (2.1) we obtain

$$S_{(m)} = \ln_q \left( Z_{(m)} \right) + \beta U.$$

(2.11)

This relationship between the entropy and the partition function, through the introduction of a suitable deformed logarithm, is recurrent in different generalized formulations of the statistical mechanics (in addition to the classical
Boltzmann-Gibbs theory) [19,20,21].
Because \( \exp_q(x) \) is a monotonic and increasing function, from Eq. (2.5) it could appear that the most probable state corresponds to the fundamental energy level. On the other hand, let us introduce the multiplicity \( \Omega(E_j, V) \) of a macrostate with energy \( E_j \). It depends on the volume \( V \) of the system and represents the number of possible microstates with the same energy \( E_j \). By taking into account that all the probabilities \( p_i \) of the microstates belonging to the same macrostate, labeled by the energy \( E_j \), have the same value, we can introduce the relevant probability \( P(E_j, V) \) of a macrostate as

\[
P(E_j, V) = \frac{\Omega(E_j, V)}{Z_{(m)}} \exp_r \left( -\frac{\beta}{2 - r} (E_j - U) \right). \tag{2.12}
\]

Therefore, the most probable state, which maximize the relevant probability \( P(E_j, V) \), is given by the competition among \( p_j \), which is a monotonic decreasing function with respect to the energy and \( \Omega(E_j, V) \) which is typically a monotonic increasing function.

In the microcanonical picture, since all the microstates have the same energy \( U \), the relevant probability is given by \( P(U, V) = 1 \) because \( P(E_j, V) = 0 \) for \( E_j \neq U \). In this case \( \Omega(U, V) = W \), Eq. (2.5) reduces to the uniform distribution with \( p_i = 1/W \) and entropy (2.1) assumes the expression

\[
S_{(m)}(U, V) = \ln_q \left( W \right). \tag{2.13}
\]

Equation (2.13) mimics the Boltzmann formula for the entropy, recovered in the \( q \to 1 \) limit.

In the following we derive some proprieties of entropy \( S_{(m)}(U, V) \) useful in the next sections.

Firstly, from Eq. (2.3), using Eq. (2.8), it follows

\[
\frac{2 - r}{r - 1} \left( \frac{Z_{(m)}}{Z_{(m)}} \right)^{r+q-2} p_j^{1-r} = \gamma + \beta E_j , \tag{2.14}
\]

and, after deriving Eq. (2.1), we obtain the relation

\[
dS_{(m)} = \frac{2 - r}{r - 1} \left( \frac{Z_{(m)}}{Z_{(m)}} \right)^{r+q-2} \sum_{j=1}^{W} p_j^{1-r} dp_j = \sum_{j=1}^{W} (\gamma + \beta E_j) dp_i . \tag{2.15}
\]

By taking into account that \( \sum_i dp_i = 0 \), as it follows from the normalization on \( p_i \), and recalling that \( dU = \sum_i dE_i p_i + \sum_i E_i dp_i \equiv \delta \mathcal{L} + \delta \mathcal{Q} \) (first law), under the “no work” condition \( \delta \mathcal{L} \equiv \sum_i dE_i p_i = 0 \), we obtain the fundamental
thermodynamics relation
\[
\left( \frac{\partial S^{(m)}}{\partial U} \right)_V = \beta .
\] (2.16)

Moreover, from the definition (2.7) we have
\[
\left( \frac{\partial \tilde{\beta}}{\partial U} \right)_V = \frac{1}{1 + (1 - q) S^{(m)}} \left[ \frac{\partial^2 S^{(m)}}{\partial U^2} - \frac{1 - q}{1 + (1 - q) S^{(m)}} \left( \frac{\partial S^{(m)}}{\partial U} \right)^2 \right]_V ,
\] (2.17)

which, for a stable equilibrium configuration, implies
\[
\left( \frac{\partial \tilde{\beta}}{\partial U} \right)_V < 0 ,
\] (2.18)

(a sketch of this statement will be given in Section 4 [cfr. Eq. (4.5)]).

Secondly, we recall the “composability” rule of entropy \( S^{(m)}(U, V) \) for two statistically independent systems A and B, in the sense of \( p_{ij}^{AUB} = p_i^A \cdot p_j^B \).

From the definition (2.1) it follows that
\[
S^{AUB}^{(m)} = S^{A}_{}^{(m)} + S^{B}_{}^{(m)} + (1 - q) S^{A}_{}^{(m)} S^{B}_{}^{(m)} ,
\] (2.19)

where the “super-additivity” \((q < 1)\) and the “sub-additivity” \((q > 1)\) behaviors are controlled only by the parameter \(q\). Linear composability is recovered for \(q = 1\), i.e. for the Rényi family.

Finally, we discuss the concavity conditions of entropy and the convexity conditions of energy. As it is well known, the concavity conditions for the given problem follow from the analysis of the sign of the eigenvalues of the Hessian matrix associated to the function \( S^{(m)}(U, V) \). In particular, by requiring that the following quadratic form
\[
\phi(y) = \frac{\partial^2 S^{(m)}}{\partial U^2} y_U^2 + 2 \frac{\partial^2 S^{(m)}}{\partial U \partial V} y_U y_V + \frac{\partial^2 S^{(m)}}{\partial V^2} y_V^2
\] (2.20)

be negative definite for any arbitrary vector \( y \equiv (y_U, y_V) \), we obtain the relations
\[
\frac{\partial^2 S^{(m)}}{\partial U^2} < 0 , \quad \frac{\partial^2 S^{(m)}}{\partial U^2} \frac{\partial^2 S^{(m)}}{\partial V^2} - \left( \frac{\partial^2 S^{(m)}}{\partial U \partial V} \right)^2 > 0 ,
\] (2.21)

stating the concavity conditions of the SM entropy (remark that Eqs. (2.21) imply the further relation \(\frac{\partial^2 S^{(m)}}{\partial V^2} < 0\)).
In a similar way, the energy $U(S_{(m)}, V)$ is a convex function with respect to $S_{(m)}$ and $V$ if the following quadratic form

$$
\psi(y) = \frac{\partial^2 U}{\partial S_{(m)}^2} y_S^2 + 2 \frac{\partial^2 U}{\partial S_{(m)} \partial V} y_S y_V + \frac{\partial^2 U}{\partial V^2} y_V^2 \tag{2.22}
$$

is positive definite for any arbitrary vector $y \equiv (y_S, y_V)$. Easily, we obtain the relations

$$
\frac{\partial^2 U}{\partial S_{(m)}^2} > 0, \quad \frac{\partial^2 U}{\partial S_{(m)}^2} \frac{\partial^2 U}{\partial V^2} - \left( \frac{\partial^2 U}{\partial S_{(m)} \partial V} \right)^2 > 0, \tag{2.23}
$$

which state the convexity conditions for the energy.

3 Thermal and mechanical equilibrium

We consider an initial situation where two isolated systems A and B, with mean energies $U^A$ and $U^B$ and volumes $V^A$ and $V^B$, respectively, are described by the same entropy. Exchange of heat (energy) and work (volume), which may take place between the systems, are initially prohibited.

Latter, some constraints are relaxed allowing, in this way, to establish weak interactions among them. The whole system $A \cup B$ is now subjected to new constraints given by the total energy $U^{A\cup B} = U^A + U^B$ and the total volume $V^{A\cup B} = V^A + V^B$ which we assume to be conserved in time. This means that we are neglecting the interaction among the two systems. In fact, in the limit of zero interaction the energies and volumes are strictly additive, however, a small interaction between the parts is required to enable some exchange of heat and work among them. In the same way, we can pose

$$
p^{A\cup B}_{ij} = p^A_i p^B_j (1 + \delta p_{ij}), \tag{3.1}
$$

where $\delta p_{ij}$ takes into account the correlations between the systems. Under the hypothesis of very weak interaction it is reasonable to neglect this term, i.e., we assume that the statistical independence among the systems is preserved in time.

When the systems are posed in thermodynamical contact, the entropy is not at its maximum value due to the new constraints. The system will evolve toward a new equilibrium increasing its entropy, $\delta S^{A\cup B}_{(m)} > 0$, until reaches its extreme limit.
By evaluating the variation of $S_{\{m\}}^{A\cup B}$, up to the first order in $\delta U$ and $\delta V$, from Eq. (2.19) we obtain

$$
\delta S_{\{m\}}^{A\cup B} = \left[ 1 + (1 - q) S_{\{m\}}^B \right] \left[ \frac{\partial S_{\{m\}}^A}{\partial U} \delta U + \frac{\partial S_{\{m\}}^A}{\partial V} \delta V \right] 
- \left[ 1 + (1 - q) S_{\{m\}}^A \right] \left[ \frac{\partial S_{\{m\}}^B}{\partial U} \delta U + \frac{\partial S_{\{m\}}^B}{\partial V} \delta V \right] 
= \left[ 1 + (1 - q) S_{\{m\}}^A \right] \left[ 1 + (q - 1) S_{\{m\}}^B \right]
\times \left\{ \left[ \frac{1}{1 + (1 - q) S_{\{m\}}^A} \left( \frac{\partial S_{\{m\}}^A}{\partial U} \right) \right] - \frac{1}{1 + (1 - q) S_{\{m\}}^B} \left( \frac{\partial S_{\{m\}}^B}{\partial U} \right), \delta U \right\} > 0,
$$

(3.2)

where, we pose $\delta U^A = -\delta U^B \equiv \delta U$ and $\delta V^A = -\delta V^B \equiv \delta V$, according to the conservation of $U^{A\cup B}$ and $V^{A\cup B}$.

Assuming firstly $\delta V = 0$, from Eq. (3.2) it follows

$$
\beta^A \left[ 1 + (1 - q) S_{\{m\}}^A \right] - \beta^B \left[ 1 + (1 - q) S_{\{m\}}^B \right] \delta U > 0,
$$

(3.3)

since $1 + (1 - q) S_{\{m\}} > 0$. Equation (3.3) can be written in

$$
(\beta^A - \beta^B) \delta U > 0,
$$

(3.4)

which implies that $\text{sgn}(\delta U) = \text{sgn}(\beta^A - \beta^B)^1$. This means that energy flows always from the system with smaller $\beta$ to the system with larger $\beta$. Such a process goes on until the equilibrium, stated by the equality $\beta^A = \beta^B$, is reached.

The main facts of $\bar{\beta}$ reflect the same physical proprieties of $\beta = 1/T$ of the standard thermodynamics [2] and can be summarized in the following points:

a) Two systems which cannot exchange energy have in general different $\bar{\beta}$’s.

b) When two weakly interacting systems exchange energy, their respective values of $\bar{\beta}$ become equal when equilibrium is reached.

c) Between two weakly interacting systems the energy flows always from the system with the smaller $\bar{\beta}$ to the system with the larger $\bar{\beta}$.

1 The sign function $\text{sgn}(x)$ is defined in $\text{sgn}(x) = +1$ for $x > 0$ and $\text{sgn}(x) = -1$ for $x < 0$. 
d) The mean energy of a system, in a stable equilibrium configuration, increases monotonically as $\tilde{\beta}$ decreases, according to Eq. (2.18). The parameter $\tilde{\beta}$ is a variable controlling the exchange of energy among the systems and can be identified with the temperature according to

$$T = \frac{1}{\tilde{\beta}}. \quad (3.5)$$

Equations (2.7), (2.16) and (3.5) establish a relationships between temperature and entropy which differs from the standard one $\beta = 1/T = (\partial S/\partial U)_V$. In other words, the inverse of the temperature differs form the Lagrange multiplier $\beta$ associated to the mean energy $U$. The standard relationship is recovered only in the $q \to 1$ limit, where the SM entropy reduces to the Rényi entropy. For the Tsallis’ case, with $r = 2 - q$, Eq. (3.5) coincides with the definition of temperature obtained in Ref. [13] and derived by means of separability constant between the thermal equilibrium of two systems.

In the microcanonical picture Eq. (3.5) coincides with to the standard definition of the Boltzmann temperature. In fact, in this case Eq. (2.5) collapses to the uniform distribution $p_i = 1/W$ and, accounting for Eq. (2.13), we obtain

$$\frac{1}{T} = \left( \frac{\partial}{\partial U} \ln W \right)_V. \quad (3.6)$$

This fact agrees with already known results obtained by using the Tsallis’ entropy [22] and the Sharma-Taneja-Mittal entropy [16,23]. Similar arguments can be applied to obtain a definition of pressure. In fact, by posing $\delta U = 0$, from Eq. (3.2) we obtain

$$\left[ \frac{1}{1 + (1 - q) S_{\{m\}}^A (\partial S_{\{m\}}^A/\partial V)^U} - \frac{1}{1 + (1 - q) S_{\{m\}}^B (\partial S_{\{m\}}^B/\partial V)^U} \right] \delta V > 0, \quad (3.7)$$

which advances the following definition

$$P = \frac{T}{1 + (1 - q) S_{\{m\}}^A} \left( \frac{\partial S_{\{m\}}^A}{\partial V} \right)_U. \quad (3.8)$$

Recalling that $\delta U = 0$ implies thermal equilibrium, i.e. $\tilde{\beta}^A = \tilde{\beta}^B \equiv \beta > 0$, Eq. (3.7) can be rewritten in

$$\left( P^A - P^B \right) \delta V \geq 0, \quad (3.9)$$

so that $\text{sgn}(\delta V) = \text{sgn}(P^A - P^B)$, which means that the system with greater pressure increases its volume, whilst the system with lowest pressure reduces
its volume. It is worth to observe that, by taking into account Eq. (2.16) and the relation \((\partial U/\partial S)_{V}(\partial S_{(m)}/\partial V)_{U} = -(\partial U/\partial V)_{S}\), Eq. (3.8) can be written in

\[
P = -\left(\frac{\partial U}{\partial V}\right)_{S},
\]

which coincides with the definition of pressure given in the standard thermodynamics. What is different, as stated by Eq. (3.8), is the relationships between the pressure \(P\) and the entropy \(S_{(m)}(U, V)\).

### 4 Thermodynamical stability

The thermodynamical stability conditions for the entropy \(S_{(m)}(U, V)\) can be obtained by analyzing the sign of the entropy changes produced by perturbing the system away from the equilibrium. To begin with, we expand the variation of the entropy \(\delta S_{(m)}\) up to the second order in \(\delta U\) and \(\delta V\). By recalling that at the equilibrium the first order terms vanish, we obtain

\[
\begin{align*}
\frac{1}{2} \left[ & 1 + (1 - q) S_{(m)}^{B} \right] \left[ \frac{\partial^{2} S_{(m)}^{A}}{(\partial U^{A})^{2}} - \frac{1 - q}{1 + (1 - q) S_{(m)}^{A}} \left( \frac{\partial S_{(m)}^{A}}{\partial U^{A}} \right)^{2} \right] \\
& + \left[ 1 + (1 - q) S_{(m)}^{A} \right] \left[ \frac{\partial^{2} S_{(m)}^{B}}{(\partial U^{B})^{2}} - \frac{1 - q}{1 + (1 - q) S_{(m)}^{B}} \left( \frac{\partial S_{(m)}^{B}}{\partial U^{B}} \right)^{2} \right] \right) (\delta U)^{2} \\
& + \frac{1}{2} \left[ 1 + (1 - q) S_{(m)}^{B} \right] \left[ \frac{\partial^{2} S_{(m)}^{A}}{(\partial V^{A})^{2}} - \frac{1 - q}{1 + (1 - q) S_{(m)}^{A}} \left( \frac{\partial S_{(m)}^{A}}{\partial V^{A}} \right)^{2} \right] \\
& + \left[ 1 + (1 - q) S_{(m)}^{A} \right] \left[ \frac{\partial^{2} S_{(m)}^{B}}{(\partial V^{B})^{2}} - \frac{1 - q}{1 + (1 - q) S_{(m)}^{B}} \left( \frac{\partial S_{(m)}^{B}}{\partial V^{B}} \right)^{2} \right] \right) (\delta V)^{2} \\
& + \left[ 1 + (1 - q) S_{(m)}^{B} \right] \left[ \frac{\partial^{2} S_{(m)}^{A}}{\partial U^{A} \partial V^{A}} - \frac{1 - q}{1 + (1 - q) S_{(m)}^{A}} \frac{\partial S_{(m)}^{A}}{\partial U^{A}} \frac{\partial S_{(m)}^{A}}{\partial V^{A}} \right] \\
& + \left[ 1 + (1 - q) S_{(m)}^{A} \right] \left[ \frac{\partial^{2} S_{(m)}^{B}}{\partial U^{B} \partial V^{B}} - \frac{1 - q}{1 + (1 - q) S_{(m)}^{B}} \frac{\partial S_{(m)}^{B}}{\partial U^{B}} \frac{\partial S_{(m)}^{B}}{\partial V^{B}} \right] \right) \delta U \delta V < 0,
\end{align*}
\]

(4.1)
and introducing the notation

$$S_{XY} = \frac{\partial^2 S_{(m)}}{\partial X \partial Y} - \frac{1 - q}{1 + (1 - q) S_{(m)}} \frac{\partial S_{(m)}}{\partial X} \frac{\partial S_{(m)}}{\partial Y},$$ \hspace{1cm} (4.2)

where $X$ and $Y$ stand for $U$ or $V$, Eq. (4.1) can be written as

$$\frac{1}{2} \left[ 1 + (1 - q) S^B_{(m)} \right] \left[ S^A_{UU} (\delta U)^2 + S^A_{UV} \delta U \delta V + S^A_{VV} (\delta V)^2 \right] + \frac{1}{2} \left[ 1 + (1 - q) S^A_{(m)} \right] \left[ S^B_{UU} (\delta U)^2 + S^B_{UV} \delta U \delta V + S^B_{VV} (\delta V)^2 \right] < 0 \hspace{1cm} (4.3)$$

This equation is fulfilled if the following inequalities

$$S_{UU} < 0, \quad S_{UU} S_{VV} - (S_{UV})^2 > 0,$$ \hspace{1cm} (4.4)

are separately satisfied by both systems. [A further relation $S_{VV} < 0$ follows from Eqs. (4.4)].

Explicitly, we have

$$\frac{\partial^2 S_{(m)}}{\partial U^2} < \frac{1 - q}{1 + (1 - q) S_{(m)}} \left( \frac{\partial S_{(m)}}{\partial U} \right)^2,$$ \hspace{1cm} (4.5)

$$\frac{\partial^2 S_{(m)}}{\partial U^2} \frac{\partial^2 S_{(m)}}{\partial V^2} - \left( \frac{\partial^2 S_{(m)}}{\partial U \partial V} \right)^2 > \frac{1 - q}{1 + (1 - q) S_{(m)}} B_{(m)},$$ \hspace{1cm} (4.6)

where

$$B_{(m)} = \left( \frac{\partial^2 S_{(m)}}{\partial U^2} \right)^{-1} \left\{ \left( \frac{\partial^2 S_{(m)}}{\partial U^2} \frac{\partial S_{(m)}}{\partial V} - \frac{\partial^2 S_{(m)}}{\partial U \partial V} \frac{\partial S_{(m)}}{\partial U} \right)^2 \right. \\
+ \left. \left( \frac{\partial S_{(m)}}{\partial U} \right)^2 \left[ \frac{\partial^2 S_{(m)}}{\partial U^2} \frac{\partial^2 S_{(m)}}{\partial V^2} - \left( \frac{\partial^2 S_{(m)}}{\partial U \partial V} \right)^2 \right] \right\}, \hspace{1cm} (4.7)$$

is a negative definite quantity for a concave entropy.

Equations (4.5)-(4.6) are the thermodynamical stability conditions for the entropies belonging to the SM family. They reduce to the concavity conditions (2.21) in the $q \to 1$ limit.

We observe that for “super-additive” and “additive” systems, with $q \leq 1$, Eqs. (4.5)-(4.6) are satisfied if the concavity conditions (2.21) holds. Moreover, when $q < 1$ the equilibrium configuration is stable also if the entropy shows a small convexity. In this sense the “super-additive” systems exhibit a kind of “super-stability”. We observe that Eq. (4.5) implies Eq. (2.18) so that, if the
TSCs are fulfilled, among two bodies in thermal contact heat always flows from hot body to cold body, freely from the concavity arguments on the entropy. Differently, for “sub-additive” systems with $q > 1$, the concavity conditions are not enough to guarantee the thermodynamical stability of the equilibrium configuration. In this case, entropies with a not very pronounced concavity, can still violate Eqs. (4.5)-(4.6) as well as Eq. (2.18). Thus, we can state a kind of “sub-stability” for “sub-additive” systems according to Eqs. (4.5)-(4.6).

5 Equilibrium and stability in the energy representation

The study of the approach in the direction of the equilibrium and the analysis of its stability in energy representation require only a straightforward transcription of language.

Let us consider two isolated systems A and B initially at equilibrium, constrained by their respective entropies $S_A^{(m)}$, $S_B^{(m)}$ and volumes $V_A$, $V_B$. When certain constraints are removed, a weak interaction among the systems starts on, giving origin to an exchange of heat (entropy) and work (volume). We assume the total entropy $S_{A∪B}^{(m)} = S_A^{(m)} + S_B^{(m)} + (1 - q) S_A^{(m)} S_B^{(m)}$ and the total volume $V_{A∪B} = V_A + V_B$ constants in time.

According to the minimum energy principle the system will evolve toward a new equilibrium with lower energy. By evaluating, up the first order in $\delta S_{(m)}$ and $\delta V$, the changing in the energy $U_{A∪B}$, we obtain

$$
\delta U_{A∪B} = \left[ \left( \frac{\partial U_A}{\partial S_A^{(m)}} \right)_V \delta S_A^{(m)} + \left( \frac{\partial U_B}{\partial S_B^{(m)}} \right)_V \delta S_B^{(m)} \right] + \left[ \left( \frac{\partial U_A}{\partial V_A} \right)_S \delta V_A + \left( \frac{\partial U_B}{\partial V_B} \right)_S \delta V_B \right] < 0 ,
$$

and by posing $\delta S_A^{(m)}/[1 + (1 - q) S_A^{(m)}] = -\delta S_B^{(m)}/[1 + (1 - q) S_B^{(m)}] \equiv \delta \Sigma$ and $\delta V_A = -\delta V_B \equiv \delta V$, Eq. (5.1) becomes

$$
\delta U_{A∪B} = \left[ 1 + (1 - q) S_A^{(m)} \right] \left( \frac{\partial U_A}{\partial S_A^{(m)}} \right)_V - \left[ 1 + (1 - q) S_A^{(m)} \right] \left( \frac{\partial U_B}{\partial S_B^{(m)}} \right)_V \delta \Sigma + \left[ \left( \frac{\partial U_A}{\partial V_A} \right)_S - \left( \frac{\partial U_B}{\partial V_B} \right)_S \right] \delta V .
$$
Assuming firstly $\delta V = 0$ and by taking into account Eqs. (2.7) and (2.16), from Eq. (5.2) we obtain

$$
\left( \frac{1}{\beta^A} - \frac{1}{\beta^B} \right) \delta \Sigma \leq 0 ,
$$

(5.3)

which is equivalent to Eq. (3.4). We remark only that Eq. (5.3) is a relation in $S_{(m)}$ and $V$ which is more conveniently assumed when the total energy $U^{A∪B}$ is known, whereas Eq. (3.4) is a relation in $U$ and $V$ which is more conveniently assumed when the total entropy $S^{A∪B}_{(m)}$ is known.

In a similar way, by posing $\delta \Sigma = 0$, from Eq. (5.2) we obtain

$$
\left[ \left( \frac{\partial U^A}{\partial V^A} \right)_S - \left( \frac{\partial U^B}{\partial V^B} \right)_S \right] \delta V \leq 0 ,
$$

(5.4)

which coincides with Eq. (3.9) according to the definition (3.10).

Finally, in order to obtain the thermodynamical stability conditions in the energy representation we proceed by expanding the variation of the energy $\delta U^{A∪B}$ around the equilibrium, up to the second order in $\delta \Sigma$ and $\delta V$. We obtain

$$
\begin{align*}
&\frac{1}{2} \left\{ \frac{\partial^2 U^A}{(\partial S^A_{(m)})^2} \left[ 1 + (1 - q) S^A_{(m)} \right]^2 + \frac{\partial^2 U^B}{(\partial S^B_{(m)})^2} \left[ 1 + (1 - q) S^B_{(m)} \right]^2 \right\} (\delta \Sigma)^2 \\
&+ \frac{1}{2} \left[ \frac{\partial^2 U^A}{\partial S^A_{(m)} \partial V^A} \left[ 1 + (1 - q) S^A_{(m)} \right] + \frac{\partial^2 U^B}{\partial S^B_{(m)} \partial V^B} \left[ 1 + (1 - q) S^B_{(m)} \right] \right] \delta \Sigma \delta V \\
&+ \frac{1}{2} \left[ \frac{\partial^2 U^A}{(V^A)^2} + \frac{\partial^2 U^A}{(V^A)^2} \right] (\delta V)^2 > 0 ,
\end{align*}
$$

(5.5)

and recalling that $1 + (1 - q) S_{(m)} > 0$, from Eq. (5.5) it follows

$$
\frac{\partial^2 U}{\partial S^2_{(m)}} > 0 , \quad \frac{\partial^2 U}{\partial S^2_{(m)}} \frac{\partial^2 U}{\partial V^2} - \left( \frac{\partial^2 U}{\partial S_{(m)} \partial V} \right)^2 > 0 ,
$$

(5.6)

that coincides with the convexity conditions for the energy (2.23). This result was expected and shows us that when the “composability rule” of a thermodynamical quantity is linear, like the energy in the present case, the structures of the TSCs are equivalent to concavity (convexity) arguments of the same quantity, freely from the “composability” proprieties of the other thermodynamical quantities (for a discussion of a thermostatistics theory based on non linear additive energies see, for instance, Ref. [24]).
6 Conclusions

In this work we have studied the thermodynamical equilibrium and its stability among two systems weakly interacting and described by the same entropy belonging to the family of Sharma-Mittal. We have derived a definition of temperature and pressure, controlling the exchange of heat and work between the two systems. It has been shown that temperature and pressure, obtained from dynamical arguments, coincides with the ones already known in literature and derived from statical considerations. We have inquired on the TSCs both in the entropy and in the energy representation. It is shown that, due to the nonlinear “composability” rule of entropy, the concavity conditions alone are not necessary nor sufficient conditions for the stability of the equilibrium. In particular, when the system is sub-additive the concavity conditions do not imply the stability whereas, when the system is super-additive, the concavity conditions imply the stability of the equilibrium configuration. A different situation is obtained in the energy representation where, by assuming a linear “composability”, the TSCs imply the convexity conditions for the energy. This shows that, although the two representations are equivalent, the analysis of some thermodynamical proprieties, like for instance the TSCs, could be performed more easily in one than in the other representation.

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