ODE Analysis of Stochastic Gradient Methods with Optimism and Anchoring for Minimax Problems and GANs

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Abstract

Despite remarkable empirical success, the training dynamics of generative adversarial networks (GAN), which involves solving a minimax game using stochastic gradients, is still poorly understood. In this work, we analyze last-iterate convergence of simultaneous gradient descent (simGD) and its variants under the assumption of convex-concavity, guided by a continuous-time analysis with differential equations. First, we show that simGD, as is, converges with stochastic sub-gradients under strict convexity in the primal variable. Second, we generalize optimistic simGD to accommodate an optimism rate separate from the learning rate and show its convergence with full gradients. Finally, we present anchored simGD, a new method, and show convergence with stochastic subgradients.

1 Introduction

Training of generative adversarial networks (GAN) [19], solving a minimax game using stochastic gradients, is known to be difficult. Despite the remarkable empirical success of GANs, further understanding the global training dynamics empirically and theoretically is considered a major open problem [18, 54, 39, 37, 48].

The local training dynamics of GANs is understood reasonably well. Several works have analyzed convergence assuming the loss functions have linear gradients and the training uses full (deterministic) gradients. Although the linear gradient assumption is reasonable for local analysis (even though the loss functions may not be continuously differentiable due to ReLU activation functions) such results say very little about global convergence. Although the full gradient assumption is reasonable when the learning rate is small, such results say very little about how the randomness affects the training.

This work investigates global convergence of simultaneous gradient descent (simGD) and its variants for zero-sum games with a convex-concave cost using using stochastic subgradients. We specifically study convergence of the last iterates rather than the averaged iterates.

Section 2 presents convergence of simGD with stochastic subgradients under strict convexity in the primal variable. The goal is to establish a minimal sufficient condition of global convergence for simGD without modifications. Section 3 presents a generalization of optimistic simGD [8], which allows an optimism rate separate from the learning rate. We prove the generalized optimistic simGD using full gradients converges, and experimentally demonstrate that the optimism rate must be tuned separately from the learning rate when using stochastic gradients. However, it is unclear whether optimistic simGD is theoretically compatible with stochastic gradients. Section 4 presents anchored simGD, a new method, and presents its convergence with stochastic subgradients. The presentation and analyses of Sections 2, 3, and 4 are guided by continuous-time first-order ordinary differential equations.

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equations (ODE). In particular, we interpret optimism and anchoring as discretizations of certain regularized dynamics. Section 5 experimentally demonstrates the benefit of optimism and anchoring for training GANs in some setups.

**Prior work.** There are several independent directions for improving the training of GANs such as designing better architectures, choosing good loss functions, or adding appropriate regularizers \([54, 2, 60, 11, 20, 64, 59, 37, 38, 41]\). In this work, we accept these factors as a given and focus on how to train (optimize) the model effectively.

Optimism is a simple modification to remedy the cycling behavior of simGD, which can occur even under the bilinear convex-concave setup \([8, 9, 10, 56, 17, 28, 42, 50]\). These prior work assume the gradients are linear and use full gradients. Although the recent name ‘optimism’ originates from its use in online optimization \([6, 55, 56, 62]\), the idea dates back to Popov’s work in the 1980s \([53]\) and has been studied independently in the mathematical programming community \([35, 32, 34, 33, 7]\).

Classical literature analyze convergence of the Polyak-averaged iterates (which assigns less weight to older iterates) when solving convex-concave saddle point problems using stochastic subgradients \([47, 46, 24, 17]\). For GANs, however, last iterates or exponentially averaged iterates \([66]\) (which assigns more weight to newer iterates) are used in practice. Therefore, the classical work using Polyak averaging do not fully explain the empirical success of GANs.

The classical techniques used for the analyses of this work, the stochastic approximation technique \([12, 22]\), ideas from control theory \([22, 45]\), ideas from variational inequalities and monotone operator theory \([16, 17]\), and continuous-time ODE analysis \([22, 7]\), have been utilized for analyzing GANs.

Finally, we point out that the results of this work are broadly applicable beyond GANs since minimax game formulations are also used in other areas of machine learning such as actor-critic models \([51]\) and domain adversarial networks \([14, 13, 15, 31]\).

## 2 Stochastic simultaneous subgradient descent

Consider the cost function \(L : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}\) and the minimax game \(\min_x \max_u L(x, u)\). We say \((x_*, u_*) \in \mathbb{R}^m \times \mathbb{R}^n\) is a solution to the minimax game or a *saddle point* of \(L\) if

\[
L(x_*, u) \leq L(x_*, u_*) \leq L(x, u_*) , \quad \forall x \in \mathbb{R}^m, u \in \mathbb{R}^n.
\]

We assume \(L\) is convex-concave and has a saddle point. (A0)

By convex-concave, we mean \(L(x, u)\) is a convex function in \(x\) for fixed \(u\) and a concave function in \(u\) for fixed \(x\). Define

\[
G(x, u) = \begin{bmatrix}
\partial_x L(x, u) \\
\partial_u(-L(x, u))
\end{bmatrix},
\]

where \(\partial_x\) and \(\partial_u\) respectively denote the convex subdifferential with respect to \(x\) and \(u\). For simplicity, write \(z = (x, u) \in \mathbb{R}^{m+n}\) and \(G(z) = G(x, u)\). Note that 0 \(\in G(z)\) if and only if \(z\) is a saddle point. Since \(L\) is convex-concave, the operator \(G\) is monotone \([58]\):

\[
(g_1 - g_2)^T(z_1 - z_2) \geq 0 \quad \forall g_1 \in G(z_1), g_2 \in G(z_2), z_1, z_2 \in \mathbb{R}^{m+n}.
\]

Let \(g(z; \omega)\) be a stochastic subgradient oracle, i.e., \(\mathbb{E}_{\omega}g(z; \omega) \in G(z)\) for all \(z \in \mathbb{R}^{m+n}\), where \(\omega\) is a random variable. Consider *Simultaneous Stochastic Sub-Gradient Descent* (SSSGD)

\[
z_{k+1} = z_k - \alpha_k g(z_k; \omega_k)
\]

for \(k = 0, 1, \ldots\), where \(z_0 \in \mathbb{R}^{m+n}\) is a starting point, \(\alpha_0, \alpha_1, \ldots\) are positive learning rates, and \(\omega_0, \omega_1, \ldots\) are IID random variables. (We read SSSGD as “triple-SGD”.) In this section, we provide convergence of SSSGD when \(L(x, u)\) is strictly convex in \(x\).

### 2.1 Continuous-time illustration

To understand the asymptotic dynamics of the stochastic discrete-time system, we consider a corresponding deterministic continuous-time system. For simplicity, assume \(G\) is single-valued and smooth. Consider

\[
\dot{z}(t) = -g(t), \quad g(t) = G(z(t))
\]
We can alternatively assume \( L(\omega) \) where \( x \) and \( z \) where we used (1). However, there is no mechanism forcing \( z(t) \) to converge to a solution. Consider the two examples \( L_0(x, u) = xu \) and \( L_\rho(x, u) = (\rho/2)x^2 + xu \) with
\[
G_0(x, u) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad G_\rho(x, u) = \begin{bmatrix} \rho & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}
\]
(2)
where \( x \in \mathbb{R} \) and \( u \in \mathbb{R} \) and \( \rho > 0 \). Note that \( L_0(x, u) = xu \) is the canonical counter example that also arises as the Dirac-GAN [37]. See Figure [1].

The classical LaSalle–Krasnovskii invariance principle [26][27] states that (paraphrased) if \( z_\infty \) is a limit point of \( z(t) \), then the dynamics starting at \( z_\infty \) will have a constant distance to \( z_\star \). On the left of Figure [1] we can see that \( \|z(t) - z_\star\|^2 \) is constant as \( d \frac{1}{2} \|z(t) - z_\star\|^2 = 0 \) for all \( t \). On the right of Figure [1] we can see that although \( d \frac{1}{2} \|z(t) - z_\star\|^2 = 0 \) when \( z(t) = (0, u) \) for \( u \neq 0 \) (the dotted line) this 0 derivative is temporary as \( z(t) \) will soon move past the dotted line. Therefore, \( z(t) \) can maintain a constant constant distance to \( z_\star \) only if it starts at 0, and 0 is the only limit point of \( z(t) \).

2.2 Discrete-time convergence analysis

Consider the further assumptions
\[
\sum_{k=0}^{\infty} \alpha_k = \infty, \quad \sum_{k=0}^{\infty} \alpha_k^2 < \infty \quad (A1)
\]
\[
\mathbb{E}_{\omega_1, \omega_2} \|g(z_1; \omega_1) - g(z_1; \omega_2)\|^2 \leq R_1^2 \|z_1 - z_2\|^2 + R_2^2 \quad \forall z_1, z_2 \in \mathbb{R}^{m+n}, \quad (A2)
\]

where \( \omega_1 \) and \( \omega_2 \) are independent random variables and \( R_1 \geq 0 \) and \( R_2 \geq 0 \). These assumptions are standard in the sense that analogous assumptions are used in convex minimization to establish almost sure convergence of stochastic gradient descent.

Theorem 1. Assume (A0), (A1), and (A2). Furthermore, assume \( L(x, u) \) is strictly convex in \( x \) for all \( u \). Then SSSGD converges in the sense of \( z_k \overset{a.s.}{\to} z_\star \) where \( z_\star \) is a saddle point of \( L \).

We can alternatively assume \( L(x, u) \) is strictly concave in \( u \) for all \( x \) and obtain the same result.

The proof uses the stochastic approximation technique of [12]. We show that the discrete-time process converges (in an appropriate topology) to continuous-time trajectories satisfying a differential inclusion and use the LaSalle–Krasnovskii invariance principle to argue that limit points are solutions.

Related prior work. Theorem 3.1 of [36] proves a similar convergence result under the stronger assumption of strict convex-concavity in both \( x \) and \( u \) for the more general mirror descent setup.
We reparameterize the dynamics
\[
\dot{\zeta}(t) = -\alpha G_\beta(\zeta(t)),
\]
where \(G_\beta\) is the Moreau–Yosida regularization of \(G\). With a change of variables we get
\[
\dot{z}(t) = -\alpha g(t) - \beta g(t), \quad g(t) = G(z(t)),
\]
and the discretization \(\dot{z}(t) \approx z_{k+1} - z_k\) and \(\hat{g}(t) \approx G(z_k) - G(z_{k-1})\) yields SimGD-O.

We further explain. The Moreau–Yosida [43, 67] regularization of \(G\) with parameter \(\beta > 0\) is
\[
G_\beta = \beta^{-1}(I - (I + \beta G)^{-1}).
\]
To clarify, \(I : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}\) is the identity mapping and \((I + \beta G)^{-1}\) is the inverse (as a function) of \(I + \beta G\), which is well-defined by Minty’s theorem [40]. It is straightforward to verify that
\(G_\beta(z) = 0\) if and only if \(G(z) = 0\), i.e., \(G_\beta\) and \(G\) share the same equilibrium points. For small \(\beta\), we can think of \(G_\beta\) as an approximation \(G\) that is better-behaved. Specifically, \(G\) is merely monotone (satisfies (1)), but \(G_\beta\) is furthermore \(\beta\)-cocoercive, i.e.,
\[
(G_\beta(z_1) - G_\beta(z_2))^T (z_1 - z_2) \geq \beta \|G_\beta(z_1) - G_\beta(z_2)\|^2 \quad \forall z_1, z_2 \in \mathbb{R}^{m+n}. \tag{3}
\]
We reparameterize the dynamics \(\dot{\zeta}(t) = -\alpha G_\beta(\zeta(t))\) with \(z(t) = (I + \beta G)^{-1}(\zeta(t))\) and \(g(t) = G(z(t))\) to get \(\dot{z}(t) = \beta g(t)\) and
\[
\dot{z}(t) + \beta \dot{g}(t) = \dot{\zeta}(t) = -\frac{\alpha}{\beta} (\zeta(t) - z(t)) = -\alpha g(t).
\]
This gives us \( \dot{z}(t) = -\alpha g(t) - \beta \dot{g}(t) \).

We now investigate convergence. Let \( z_* \) satisfy \( G(z_*) = 0 \) (and therefore \( G_\beta(z_*) = 0 \)). Then
\[
\frac{d}{dt} \frac{1}{2} \|z(t) - z_*\|^2 = (\zeta(t) - z_*)^T \dot{\zeta}(t) = -\alpha (\zeta(t) - z_*)^T G_\beta(\zeta(t)) \leq -\alpha \beta \|G(\zeta(t))\|^2,
\]
where we use cocoercivity. This translates to
\[
\frac{d}{dt} \frac{1}{2} \|z(t) + \beta g(t) - z_*\|^2 \leq -\alpha \beta \|g(t)\|^2. \tag{4}
\]

The quantity \( \|g(t)\|^2 \) is nonincreasing since
\[
\frac{d}{dt} \frac{1}{2} \|g(t)\|^2 = -\frac{1}{\alpha} \dot{\zeta}(t)^T \dot{g}(t) = -\frac{1}{\alpha} \lim_{h \to 0} \frac{1}{h^2} (\zeta(t + h) - \zeta(t))^T (G_\beta(\zeta(t + h)) - G_\beta(\zeta(t))) \leq -\frac{\beta}{\alpha} \lim_{h \to 0} \frac{1}{h^2} \|G_\beta(\zeta(t + h)) - G_\beta(\zeta(t))\| = -\frac{\beta}{\alpha} \|\dot{g}(t)\|^2 \leq 0,
\]
where we use cocoercivity. Finally, integrating (4) on both sides gives us
\[
\frac{1}{2} \|z(t) + \beta g(t) - z_*\|^2 - \frac{1}{2} \|z(0) + \beta g(0) - z_*\|^2 \leq -\alpha \beta \int_0^t \|g(s)\|^2 \, ds \leq -\alpha \beta t \|g(t)\|^2
\]
\[
\|g(t)\|^2 \leq \frac{1}{2\alpha \beta t} \|z(0) + \beta g(0) - z_*\|^2.
\]

This analysis was inspired by [5, 7]: Attouch et al. studied continuous-time dynamics with Moreau–Yosida regularization and Csetnek et al. interpreted a forward-backward-forward-type method as a discretization of continuous-time dynamics with the Douglas–Rachford operator.

**Other interpretations of optimism.** Daskalakis et al. interprets optimism as augmenting “follow the regularized leader” with the (optimistic) prediction that the next gradient will be the same as the current gradient in online learning setup [8]. Peng et al. interprets optimism as “centripetal acceleration” [50] but does not provide a formal analysis with differential equations.

### 3.2 Discrete-time convergence analysis

The discrete-time method SimGD-O converges under the assumption
\[
L \text{ is differentiable and } \nabla L \text{ is } R\text{-Lipschitz continuous.} \tag{A3}
\]

**Theorem 2.** Assume (A0) and (A3). If \( 0 < \alpha < (2\beta - 1)/(4\beta^2 R) \) and \( 1/2 < \beta \), then SimGD-O converges in the sense of
\[
\min_{i=0, \ldots, k} \|G(z_k)\|^2 \leq \frac{1 + \beta^2 \alpha^2 R^2}{\alpha^2 (\beta - 1/2 - 2\beta^2 R) k} \|z_0 + \beta G(z_0) - z_*\|^2.
\]

Furthermore, \( z_k \to z_* \), where \( z_* \) is a saddle point of \( L \).

The proof can be considered a discretization of the continuous-time analysis.

**Related prior work.** Peng et al. [50] show convergence of convergence of simGD-O for \( \alpha \neq \beta \) and bilinear \( L \). Malitsky et al. [34, 7] show convergence of simGD-O when \( \alpha = \beta \) and convex-concave \( L \). Theorem 2 establishes convergence \( \alpha \neq \beta \) and \( L \) is convex-concave.

### 3.3 Difficulty with stochastic gradients

Training in machine learning usually relies on stochastic gradients, rather than full gradients. We can consider a stochastic variation of SimGD-O:
\[
z_{k+1} = z_k - \alpha_k g(z_k; \omega_k) - \beta_k (g(z_k; \omega_k) - g(z_{k-1}; \omega_{k-1})) \tag{SimGD-OS}
\]
with learning rate \( \alpha_k \) and optimism rate \( \beta_k \).
Figure 2 presents experiments of SimGD-OS on a simple bilinear problem. The choice $\beta_k = \alpha_k$ where $\alpha_k \to 0$ does not lead to convergence. Discretizing $\dot{z}(t) = -\alpha g(t) - \beta g(t)$ with a diminishing step $h^k$ leads to the choice $\alpha_k = \alpha h^k$ and $\beta_k = \beta$, but this choice as well does not converge. Rather, both $\alpha_k$ and $\beta_k$ must be diminishing and $\alpha_k = o(\beta_k)$, i.e., $\alpha_k$ must diminish faster than $\beta_k$ for convergence. Rather, it is necessary to tune $\alpha_k$ and $\beta_k$ separately as in Theorem 2 to obtain convergence and dynamics appear to be sensitive to the choice of $\alpha_k$ and $\beta_k$. One explanation of this difficulty is that the finite difference approximation $\alpha_k^{-1} (g(z_k; \omega_k) - g(z_{k-1}; \omega_{k-1})) \approx g(t)$ is unreliable when using stochastic gradients.

Whether the observed convergence holds generally in the nonlinear convex-concave setup and whether optimism is compatible with subgradients is unclear. This motivates anchoring of the following section which is provably compatible with stochastic subgradients.

**Related prior work.** Gidel et al. [17] show averaged iterates of SimGD-OS converges if iterates are projected onto a compact set. Mertikopoulos et al. [36] show almost sure convergence of SimGD-OS under *strict* convex-concavity. However, such analyses do not provide a compelling reason to use optimism since SimGD without optimism already converges under these setups.

### 4 Simultaneous GD with anchoring

Consider setup of Section 3. We propose Anchored Simultaneous Gradient Descent

$$z_{k+1} = z_k - \frac{1 - p}{(k + 1)^p} G(z_k) + \frac{(1 - p)\gamma}{k + 1} (z_0 - z_k) \quad \text{(SimGD-A)}$$

for $k \geq 0$, where $z_0 \in \mathbb{R}^{m+n}$ is a starting point, $p \in (1/2, 1)$, and $\gamma > 0$ is the *anchor rate*. The last term, the anchoring term, was inspired by Halpern’s method [21, 65, 29] and James–Stein estimator [61, 23]. In this section, we provide a continuous-time illustration of SimGD-A and provide convergence for both the deterministic and stochastic setups.

#### 4.1 Continuous-time illustration

Consider the continuous-time dynamics

$$\dot{z}(t) = -g(t) + \frac{\gamma}{t} (z_0 - z(t)),$$

for $t \geq 0$, where $\gamma \geq 1$ and $z(0) = z_0$. We obtain SimGD-A by discretizing this ODE with diminishing steps $(1 - p)/(k + 1)^p$.

Define $g(t) = G(z(t))$. Then

$$0 \leq \frac{1}{h^2} \langle z(t + h) - z(t), g(t + h) - g(t) \rangle \to \langle \dot{z}(t), \dot{g}(t) \rangle \quad \text{as } h \to 0.$$

Using this, we have

$$\frac{d}{dt} \frac{1}{2} \|\dot{z}(t)\|^2 = -\langle \dot{z}(t), \dot{g}(t) \rangle - \left(\frac{\gamma}{t} \|\dot{z}(t)\|^2 + \frac{\gamma}{t^2} (z(t) - z_0, \dot{z}) \right),$$

$$\leq -\frac{\gamma}{t} \|\dot{z}(t)\|^2 + \frac{\gamma}{t^2} (z(t) - z_0, \dot{z}).$$

Using $\gamma \geq 1$, we have

$$\frac{d}{dt} \frac{1}{2} \|\dot{z}(t)\|^2 + \frac{1}{t} \|\dot{z}(t)\|^2 \leq \frac{\gamma}{t^2} (z(t) - z_0, \dot{z}).$$

Multiplying by $t^2$ and integrating both sides gives us

$$\frac{t^2}{2} \|\dot{z}(t)\|^2 \leq \frac{\gamma}{2} \|z(t) - z_0\|^2.$$
Reorganizing, we get

\[ t^2 \frac{\|g(t)\|^2}{2} - \frac{\gamma}{t} \langle g(t), z_0 - z(t) \rangle + \frac{\gamma^2}{2} \|z(t) - z_0\|^2 \leq \frac{\gamma}{2} \|z(t) - z_0\|^2 \]

Using \( \gamma \geq 1 \), the monotonicity inequality, and Young’s inequality, we get

\[ \|g(t)\|^2 \leq \frac{2\gamma}{t} \langle g(t), z_0 - z(t) \rangle \leq \frac{2\gamma}{t} \|g(t)\|^2 + \frac{2\gamma^2}{t^2} \|z_0 - z_*\|^2 \]

and conclude

\[ \|g(t)\|^2 \leq \frac{4\gamma^2}{t^2} \|z_0 - z_*\|^2. \]

Interestingly, anchoring leads to a faster rate \( O(1/t^2) \) compared to the rate \( O(1/t) \) of optimism in continuous time. The discretized method, however, is not faster than \( O(1/k) \).

### 4.2 Discrete-time convergence analysis and compatibility with stochastic subgradients

**Theorem 3.** Assume (A0) and (A3). If \( p \in (1/2, 1) \) and \( \gamma \geq 2 \), then SimGD-A converges in the sense of

\[ \|G(z_k)\|^2 \leq \frac{C}{k^{2-2p}} + O\left( \frac{1}{k} \right) \]

for \( k \geq 1 \) for some \( C > 0 \).

The constant \( C \) is computable, although it is complicated. The proof can be considered a discretization of the continuous-time analysis.

Consider the setup of Section 2. We propose Anchored Simultaneous Stochastic SubGradient Descent

\[ z_{k+1} = z_k - \frac{1 - p}{(k + 1)^p} g(z_k; \omega_k) + \frac{(1 - p)\gamma}{(k + 1)^{1-\varepsilon}} (z_0 - z_k) \quad \text{(SSSGD-A)} \]

**Theorem 4.** Assume (A0) and (A2). If \( p \in (1/2, 1) \), \( \varepsilon \in (0, 1/2) \), and \( \gamma > 0 \), then SSSGD-A converges in the sense of \( z_k \xrightarrow{L^2} z_* \), where \( z_* \) is a saddle point.

To the best of our knowledge, Theorem 4 is the first result establishing last-iterate convergence for convex-concave cost functions using stochastic subgradients.

### 5 Experiments

In this section, we experimentally demonstrate the effectiveness of optimism and anchoring for training GANs. We train Wasserstein-GANs with gradient penalty on the MNIST and CIFAR-10 dataset and plot the Fréchet Inception Distance (FID). The experiments were
implemented in PyTorch [49]. We combine Adam with optimism and anchoring (described precisely in Appendix F) and compare it against the baseline Adam optimizer [25]. The generator and discriminator architectures and the hyperparameters are described in Appendix F. For optimistic and anchored Adam, we roughly tune the optimism and anchor rates and show the curve corresponding to the best parameter choice. Figure 3 shows an ensemble of samples generated at the end of the training period.

Figure 4 shows that the MNIST setup benefits from anchoring but not from optimism, while the CIFAR-10 setup benefits from optimism but not from anchoring. We leave comparing the effects of optimism and anchoring in practical GAN training (where the cost function is not convex-concave) as a topic of future work.

6 Conclusion

In this work, we analyzed the convergence of SSSGD, Optimistic simGD, and Anchored SSSGD. Under the assumption that the cost $L$ is convex-concave, Anchored SSSGD provably converges under the most general setup. Through experiments, we showed that the practical GAN training benefits from optimism and anchoring in some (but not all) setups.

Generalizing these results to accommodate projections and proximal operators, analogous to projected and proximal gradient methods, is an interesting direction of future work. Weight clipping [2] and spectral normalization [41] are instances where projections are used in training GANs.

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A Notation and preliminaries

Write $\mathbb{R}_+$ to denote the set of nonnegative real numbers and $\langle \cdot, \cdot \rangle$ to denote inner product, i.e., $\langle u, v \rangle = u^T v$ for $u, v \in \mathbb{R}^{m+n}$.

We say $A$ is a point-to-set mapping on $\mathbb{R}^d$ if $A$ maps points of $\mathbb{R}^d$ to subsets of $\mathbb{R}^d$. For notational simplicity, we write

$$\langle A(x) - A(y), x - y \rangle = \{ (u - v, x - y) \mid u \in A(x), v \in A(y) \}.$$ 

Using this notation, we define monotonicity of $A$ with

$$\langle A(x) - A(y), x - y \rangle \geq 0 \quad \forall x, y \in \mathbb{R}^d,$$

where the inequality requires every member of the set to be nonnegative. We say a monotone operator $A$ is maximal if there is no other monotone operator $B$ such that the containment

$$\{(x, u) \mid u \in A(x)\} \subset \{(x, u) \mid u \in B(x)\}$$

is proper. If $L : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is convex-concave, then the subdifferential operator

$$G(x, u) = \left[ \begin{array}{c} \partial_x L(x, u) \\ \partial_u (-L)(x, u) \end{array} \right]$$

is maximal monotone. By Proposition 20.36, $G(z)$ is closed-convex for any $z \in \mathbb{R}^{m+n}$. By Proposition 20.38(iii) maximal monotone operators are upper semicontinuous in the sense that if $G$ is maximal monotone, then $g_k \to G(z_k)$ for $k = 0, 1, \ldots$ and $(z_k, g_k) \to (z_\infty, g_\infty)$ imply $g_\infty \in G(z_\infty)$. (In other words, the graph of $G$ is closed.) Define $Zer(G) = \{z \in \mathbb{R}^d \mid 0 \in G(z)\}$, which is the set of saddle-points or equilibrium points. When $G$ is maximal monotone, $Zer(G)$ is a closed convex set. Write

$$P_{Zer(G)}(z_0) = \arg \min_{z \in Zer(G)} \| z - z_0 \|$$

for the projection onto $Zer(G)$.

Write $C(\mathbb{R}_+, \mathbb{R}^d)$ for the space of $\mathbb{R}^d$-valued continuous functions on $\mathbb{R}_+$. For $f_n : \mathbb{R}_+ \to \mathbb{R}^{m+n}$, we say $f_n \to f$ in $C(\mathbb{R}_+, \mathbb{R}^d)$ if $f_n \to f$ uniformly on bounded intervals, i.e., for all $T < \infty$, we have

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \| f_n(t) - f(t) \| = 0.$$

In other words, we consider the topology of uniform convergence on compact sets.

We rely on the following inequalities, which hold for any $a, b \in \mathbb{R}^{m+n}$ any $\varepsilon > 0$.

$$\| a + b \| \leq 2\| a \| + 2\| b \| \quad (5)$$

$$\langle a, b \rangle \leq \frac{1}{2\varepsilon} \| a \|^2 + \frac{\varepsilon}{2} \| b \|^2 \quad (6)$$

In particular, (6) is called Young’s inequality.

**Lemma 1** (Theorem 5.3.33 (1)). Let $(m_k, F_k)$ be a martingale such that

$$\mathbb{E}[\| m_k \|^2] < \infty$$

for all $k \geq 0$ and

$$\sum_{k=0}^{\infty} \mathbb{E} \left[ \| m_{k+1} - m_k \|^2 \mid F_k \right] < \infty$$

then $m_k$ converges almost surely to a limit.

**Lemma 2** (Robbins–Siegmund (57)). Let $\{V_k\}_{k \in \mathbb{N}_+}$, $\{S_k\}_{k \in \mathbb{N}_+}$, $\{U_k\}_{k \in \mathbb{N}_+}$, and $\{\beta_k\}_{k \in \mathbb{N}_+}$ be nonnegative $F_k$-measurable random sequences satisfying

$$\mathbb{E}_k V_{k+1} \leq (1 + \beta_k) V_k - S_k + U_k.$$ 

If, furthermore,

$$\sum_{k=1}^{\infty} \beta_k < \infty, \quad \sum_{k=1}^{\infty} U_k < \infty$$

holds almost surely, then

$$V_k \to V_\infty, \quad S_k \to 0$$

almost surely, where $V_\infty$ is a random limit.
Define
\[ \tilde{G}(z) = \mathbb{E}_\omega g(z; \omega) \in G(z). \]
Note that \( 0 \neq \tilde{G}(z_*) \) is possible even if \( 0 \in G(z_*) \) when \( L \) is not continuously differentiable.

**Lemma 3.** Under Assumptions [A0] and [A2], we have
\[ \mathbb{E}_\omega \|g(z; \omega)\|^2 \leq R_3^2 \|z - z_*\|^2 + R_4^2 \]
for some \( R_3 > 0 \) and \( R_4 > 0 \).

**Proof.** Let \( z_* \) be a saddle point, which exists by Assumption [A0]. Let \( \omega \) and \( \omega' \) be independent and identically distributed. Then
\[
\mathbb{E}_\omega \|g(z; \omega)\|^2 \leq \mathbb{E}_\omega \|g(z; \omega)\|^2 + \mathbb{E}_\omega \|g(z_*; \omega') - \tilde{G}(z_*)\|^2 \\
= \mathbb{E}_\omega,\omega' \|g(z; \omega) - g(z_*; \omega') + \tilde{G}(z_*)\|^2 \\
\leq \mathbb{E}_\omega,\omega' \|g(z; \omega) - g(z_*; \omega')\|^2 + 2 \|\tilde{G}(z_*)\|^2 \\
\leq 2R_3^2 \|z - z_*\|^2 + 2R_4^2 + 2\|\tilde{G}(z_*)\|^2
\]
where we use the fact that \( g(z_*; \omega') - \tilde{G}(z_*) \) is a zero-mean random variable, Assumption [A2], and (5). The stated result holds with \( R_3^2 = 2R_1^2 \) and \( R_4^2 = 2R_2^2 + 2\|\tilde{G}(z_*)\|^2 \).

### B Proof of Theorem [1]

Consider the differential inclusion
\[ \dot{z}(t) \in -G(z(t)) \] (7)
with the initial condition \( z(0) = z_0 \). We say \( z : [0, \infty) \to \mathbb{R}^{m+n} \) satisfies (7) if there is a Lebesgue integrable \( \zeta : [0, \infty) \to \mathbb{R}^{m+n} \) such that
\[ z(t) = z_0 + \int_0^t \zeta(s) \, ds, \quad \zeta(t) \in -G(z(t)), \quad \forall t \geq 0 \]
(8)

**Lemma 4 (Theorem 5.2.1 [4]).** If \( G \) is maximal monotone, the solution to (7) exists and is unique. Furthermore, \( \phi_t : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n} \) is \( 1 \)-Lipschitz continuous for all \( t \geq 0 \).

Write \( z(t) = \phi_t(z_0) \) and call \( \phi_t : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n} \) the time evolution operator. In other words, \( \phi_t \) maps the initial condition of the differential inclusion to the point at time \( t \).

**Lemma 5 (LaSalle–Krasnovskii).** If \( z(\cdot) \) satisfies (7), then \( z(t) \to z_\infty \) as \( t \to \infty \) and \( z_\infty \in \text{Zer}(G) \).

This proof can be considered an adaptation of the LaSalle–Krasnovskii invariance principle [26, 27] to the setup of differential inclusions. The standard result applies to differential equations.

**Proof.** Consider any \( z_* \in \text{Zer}(G) \), which exists by Assumption [A0]. Since \( z(t) \) is absolutely continuous, so is \( \|z(t) - z_*\|^2 \), and we have
\[ \frac{d}{dt} \|z(t) - z_*\|^2 = \langle \zeta(t), z(t) - z_* \rangle \leq 0 \]
for almost all \( t > 0 \), where \( \zeta(\cdot) \) is as defined in (8) and the inequality follows from (1), monotonicity of \( G \). Therefore, \( \|z(t) - z_*\|^2 \) is a nonincreasing function of \( t \). Therefore \( z(t) \) is bounded and
\[ \lim_{t \to \infty} \|z(t) - z_*\| = \chi \]
for some limit \( \chi \geq 0 \) since nonincreasing lower-bounded sequences have limits.

Let \( t_k \to \infty \) such that \( z(t_k) \to z_\infty \), i.e., \( z_\infty \) is a limit point of \( z(\cdot) \). Then, \( \|z_\infty - z_*\|^2 = \chi \). Since \( \phi_t(\cdot) \) (with fixed \( t \)) is continuous by Lemma 4, we have
\[ \lim_{k \to \infty} \phi_{s+t_k}(z_0) = \phi_s(\phi_{t_k}(z(0))) \to \phi_s(z_\infty) \]
for all \( s \geq 0 \). This means \( \phi_s(z_\infty) \) is also a limit point of \( z(\cdot) \) and
\[ \|\phi_s(z_\infty) - z_*\| = \chi \]
Then for any sequence \( \{z_k\} \), we verify the conditions of Lemma 6 and make the argument that the noisy discrete time process is close to the noiseless continuous time process and the two processes converge to the same limit.

\[
z_\infty = (x_\infty, u_\infty) \quad \text{and let} \quad z_* = (x_*, u_*) \in \text{Zer}(G). \text{ If } x_\infty \neq x_* \quad \langle G(x_\infty, u_\infty), (x_\infty, u_\infty) - (x_*, u_*) \rangle > 0
\]

by strict convexity. In light of (9), we conclude \( x_\infty = x_* \).

\[
\text{Write } (\phi^*_s(z_*), \phi^u_s(z_*)) = (\phi_s(z_*)). \text{ Then}
\]

\[
0 \in \langle G(\phi_s(z_\infty)), \phi_s(z_\infty) - z_* \rangle
\]

\[
\langle \partial u(-L)(x_*, \phi^u_s(z_\infty)) - \partial u(-L)(x_*, \phi_s(z_*)), \phi^u_s(z_\infty) - u_* \rangle
\]

\[
\geq -L(x_*, \phi^u_s(z_\infty)) + L(x_*, u_*) \geq 0,
\]

where the first inequality follows from concavity of \( L(x, u) \) in \( u \) and the second inequality follows from the fact that \( u_* \) is a maximizer when \( x_* \) is fixed. Therefore, we have equality throughout, and \( L(x_*, \phi^u_s(z_\infty)) = L(x_*, u_*) \), i.e., \( \phi^u_s(z_\infty) \) also maximizes \( L(x_*, \cdot) \) and \( u_* \) is a solution.

Finally, since \( z_\infty \) is a solution, \( \|z(t) - z_\infty\| \) converges to a limit as \( t \to \infty \). Since \( \|z(t_k) - z_\infty\| \to 0 \), we conclude that \( \|z(t) - z_\infty\| \to 0 \) as \( t \to \infty \).

**Lemma 6 (Theorem 3.7 of [12])**. Consider the update

\[
z_{k+1} = z_k - \alpha_k(\zeta_k + \xi_k), \quad \zeta_k \in G(z_k).
\]

Define \( t_k = \sum_{i=1}^k \alpha_i \) and

\[
x(t) = x_k + \frac{t - t_k}{t_{k+1} - t_k} (x_{k+1} - x_k), \quad t \in [t_k, t_{k+1}).
\]

Define the time-shifted process

\[
x^\tau(\cdot) = x(t + \cdot).
\]

Let the following conditions hold:

(i) The iterates are bounded, i.e., \( \sup_k \|z_k\| < \infty \) and \( \sup_k \|\zeta_k\| < \infty \).

(ii) The stepsizes \( \alpha_k \) satisfy Assumption (A1).

(iii) The weighted noise sequence converges: \( \sum_{k=0}^\infty \alpha_k \xi_k = v \) for some \( v \in \mathbb{R}^d \).

(iv) For any increasing sequence \( n_k \) such that \( z_{n_k} \to z_\infty \), we have

\[
\lim_{n \to \infty} \text{dist} \left( \frac{1}{m} \sum_{k=1}^m \zeta_{n_k}, G(z_\infty) \right) = 0.
\]

Then for any sequence \( \{\tau_k\}_{k=1}^\infty \subset \mathbb{R}_+ \), the sequence of functions \( \{z^{\tau_k}(\cdot)\} \) is relatively compact in \( C(\mathbb{R}_+, \mathbb{R}^d) \). If \( \tau_k \to \infty \), all limit points of \( \{z^{\tau_k}(\cdot)\} \) satisfy the differential inclusion.

We verify the conditions of Lemma 6 and make the argument that the noisy discrete time process is close to the noiseless continuous time process and the two processes converge to the same limit.

**Verifying conditions of Lemma 6**

Condition (i). Let \( z_* \in \text{Zer}(G) \). Write \( F_k \) for the \( \sigma \)-field generated by \( \omega_0, \ldots, \omega_{k-1} \). Write \( \bar{G}(z) = \text{Eg}(z; \omega) \in G(z) \). Then

\[
\|z_{k+1} - z_*\|^2 = \|z_k - z_*\|^2 - 2\alpha_k(z_k - z_*, g(z_k; \omega_k)) + \alpha^2_k\|g(z_k; \omega_k)\|^2
\]

\[
\text{E} \left[ \|z_{k+1} - z_*\|^2 \mid F_k \right] \leq \|z_k - z_*\|^2 - 2\alpha_k(z_k - z_*, \bar{G}(z_k)) + \alpha^2_k (R^2 \|z_k - z_*\|^2 + R^2)
\]

\[
= (1 + \alpha^2_k R^2)\|z_k - z_*\|^2 - 2\alpha_k(z_k - z_*, \bar{G}(z_k)) + \alpha^2_k R^2.
\]
where we used Assumption (A2) and Lemma 3. Since \( \sum_{k=0}^{\infty} \alpha_k^2 < \infty \) by Assumption (A1), this inequality and Lemma 2 tells us
\[
\|z_k - z_*\|^2 \to \text{limit}
\]
for some limit, which implies \( z_k \) is a bounded sequence. Since \( z_k \) is bounded, so it \( \hat{G}(z_k) \) since
\[
\|\hat{G}(z_k)\|^2 \leq E_\omega \|g(z_k; \omega)\|^2 \\
\leq R_4^2 \sup_k \|z_k - z_*\|^2 + R_4^2
\]
where we used Lemma 3.

Condition (ii). This condition is assumed.

Condition (iii). Define
\[
m_k = \sum_{i=0}^{k} \alpha_i \xi_i,
\]
then \((m_k, F_k)\) is a martingale and
\[
\sum_{k=0}^{\infty} \mathbb{E} \left[ \|m_{k+1} - m_k\|^2 \middle| F_k \right] = \sum_{k=0}^{\infty} \alpha_k^2 \mathbb{E} \left[ \|\xi_k\|^2 \middle| F_k \right] \\
\leq \sum_{k=0}^{\infty} \alpha_k^2 \mathbb{E}_\omega \left[ \|g(z_k; \omega)\|^2 \middle| F_k \right] \\
\leq \sum_{k=0}^{\infty} \alpha_k^2 \left( R_4^2 \|z_k - z_*\|^2 + R_4^2 \right) \\
\leq \sum_{k=0}^{\infty} \alpha_k^2 \left( \sup_k 2R_4^2 \|z_k\| + 2R_4^2 \|z_*\|^2 + R_4^2 \right) < \infty
\]
almost surely, where the first inequality is the second moment upper bounding the variance, the second inequality is Lemma 3, and the third inequality is (5) and condition (i). Finally, we have (iii) by Lemma 1.

Condition (iv). As discussed in Section A, \( G \) is maximal monotone, which implies \( G \) is upper semicontinuous, i.e., \((z_{n_k}, g_{n_k}) \to (z_\infty, g_\infty)\) implies \( g_\infty \in G(z_\infty) \), and \( G(z_\infty) \) is a closed convex set. Therefore, \( \text{dist}(\zeta_{n_k}, G(z_\infty)) \to 0 \) as otherwise we can find a further subsequence such that converging to \( \zeta_\infty \) such that \( \text{dist}(\zeta_\infty, G(z_\infty)) \) > 0. (Here we use the fact that \( \zeta_k \) is bounded due to condition (i)). Since \( G(z_\infty) \) is a convex set,
\[
\text{dist}(\zeta_{n_k}, G(z_\infty)) \to 0 \implies \frac{1}{m} \sum_{k=1}^{m} \text{dist}(\zeta_{n_k}, G(z_\infty)) \to 0 \implies \text{dist} \left( \frac{1}{m} \sum_{k=1}^{m} \zeta_{n_k}, G(z_\infty) \right) \to 0.
\]

Final proof. Let \( z_{n_k} \to z_\infty \) be a limit point. By Lemma 5 there is a \( T \geq 0 \) such that \( \|\phi_t z_\infty - \phi_\infty z_\infty\| < \varepsilon \) for all \( t \geq T \), where \( \phi_\infty z_\infty \to \phi_\infty z_\infty \) as \( s \to \infty \) and \( \phi_\infty z_\infty \) is a saddle point by Lemma 5. By Lemma 6, \( \lim_{n_k \to \infty} \|z_{n_k}(T) - \phi_\infty z_\infty\| < \varepsilon \). Since this holds for all \( \varepsilon > 0 \), we conclude that there is a further subsequence of \( z_{n_k} \) converging to \( \phi_\infty z_\infty \). Since \( \|z_k - \phi_\infty z_\infty\| \) converges to a limit and converges to 0 on this further subsequence, we conclude \( \|z_k - \phi_\infty z_\infty\| \to 0 \) almost surely.

C Proof of Theorem 2

Throughout this section, write \( g_k = G(z_k) \) for \( k \geq 1 \). Without loss of generality, assume \( \alpha = 1 \). Then
\[
\|z_{k+1} + \beta g_k - z_*\|^2 = \|z_k + \beta g_{k-1} - z_*\|^2 - 2\langle g_k, z_k - z_* \rangle - \langle g_k, 2\beta g_{k-1} - g_k \rangle \\
\leq \|z_k + \beta g_{k-1} - z_*\|^2 - \langle g_k, 2\beta g_{k-1} - g_k \rangle,
\]
where the inequality follows from (11), monotonicity of $G$, and 

$$-\langle g_k, 2\beta g_{k-1} - g_k \rangle = 4\beta^2 (g_k - g_{k-1}, z_k - z_{k-1}) - (2\beta - 1)\|z_{k+1} - z_k\|^2 - \beta^2 (1 + 2\beta)\|g_k - g_{k-1}\|^2 \leq 4\beta^2 (g_k - g_{k-1}, z_k - z_{k+1}) - (2\beta - 1)\|z_{k+1} - z_k\|^2.$$ 

We can bound

$$4\beta^2 (g_k - g_{k-1}, z_k - z_{k+1}) \leq \frac{2\beta^2}{R}\|g_k - g_{k-1}\|^2 + 2\beta^2 R\|z_{k+1} - z_k\|^2 \leq 2\beta^2 R\|z_k - z_{k-1}\|^2 + 2\beta^2 R\|z_{k+1} - z_k\|^2,$$

where the first inequality follows from (6). Young’s inequality, with $\varepsilon = R$ and the second inequality follows from Assumption (A3), R-Lipschitz continuity of $G$. Putting these together we get

$$\|z_{k+1} + \beta g_k - z_*\|^2 \leq \|z_k + \beta g_{k-1} - z_*\|^2 + 2\beta^2 R\|z_k - z_{k-1}\|^2 - (2\beta - 1 - 2\beta^2)\|z_{k+1} - z_k\|^2. \quad (10)$$

Since $\beta > 1/2$ and $R < (2\beta - 1)(4\beta^2)$ is assumed for Theorem 2, we have

$$2\beta^2 R < (2\beta - 1 - 2\beta^2)R.$$

By summing (10), we have

$$(2\beta - 1 - 2\beta^2 R) \sum_{i=0}^{k} \|z_{i+1} - z_i\|^2 - 2\beta^2 R \sum_{i=0}^{k} \|z_i - z_{i-1}\|^2 \leq \|z_0 + \beta g_{-1} - z_*\|^2$$

$$(2\beta - 1 - 4\beta^2 L) \sum_{i=0}^{k} \|z_{i+1} - z_i\|^2 \leq \|z_0 + \beta g_{-1} - z_*\|^2,$$ 

(11)

where we use $z_0 = z_{-1}$.

Next,

$$\|g_k\|^2 = \|z_{k+1} - z_k + \beta (g_k - g_{k-1})\|^2 \leq 2\|z_{k+1} - z_k\|^2 + 2\beta^2 \|g_k - g_{k-1}\|^2 \leq 2\|z_{k+1} - z_k\|^2 + 2\beta^2 R^2 \|z_k - z_{k-1}\|^2$$

where we use (5). Using (11), we get

$$\sum_{i=1}^{k} (2\|z_{i+1} - z_i\|^2 + 2\beta^2 R^2 \|z_i - z_{i-1}\|^2) \leq \frac{2 + 2\beta^2 R^2}{2\beta - 1 - 4\beta^2 R} \|z_0 + \beta g_{-1} - z_*\|^2.$$

Therefore, $2\|z_{k+1} - z_k\|^2 + 2\beta^2 R^2 \|z_k - z_{k-1}\|^2 \rightarrow 0$ and $\|g_k\|^2 \rightarrow 0$. Moreover, we have

$$\min_{i=0,\ldots,k} \|g_i\|^2 \leq \frac{2 + 2\beta^2 R^2}{(2\beta - 1 - 4\beta^2 R)k} \|z_0 + \beta g_{-1} - z_*\|^2.$$

By scaling $G$ by $\alpha$, we get the first stated result.

By summing (10), we have

$$\|z_{k+1} + \beta g_{k-1} - z_*\|^2 \leq \|z_0 + \beta g_{-1} - z_*\|^2,$$

and using the triangle inequality we get

$$\|z_k - z_*\| \leq \|z_0 + \beta g_{-1} - z_*\| + \|g_k - g_{k-1}\| \rightarrow \|z_0 + \beta g_{-1} - z_*\|$$

as $k \rightarrow \infty$. (Remember $g_k \rightarrow 0$.) So $z_k$ is a bounded sequence, and let $z_\infty$ be the limit of a convergent subsequence $z_{n_k}$. Since $G$ is a continuous mapping with $g_{n_k} = G(z_{n_k})$, $z_{n_k} \rightarrow z_\infty$, and $g_{n_k} \rightarrow 0$, we have $G(z_\infty) = 0$.

Finally, we show that the entire sequence $z_k$ converges to $z_\infty$. Reorganizing (10), we get

$$\|z_{k+1} + \beta g_k - z_*\|^2 + 2\beta^2 R\|z_{k+1} - z_k\|^2 \leq \|z_{k+1} + \beta g_{k-1} - z_*\|^2 + 2\beta^2 R\|z_k - z_{k-1}\|^2 - (2\beta - 1 - 4\beta^2 R)\|z_{k+1} - z_k\|^2.$$

So $\|z_{k+1} + \beta g_k - z_*\|^2 + 2\beta^2 R\|z_{k+1} - z_k\|^2$ is a nonincreasing sequence, and the following limit exists

$$\lim_{k \rightarrow \infty} \|z_{k+1} + \beta g_{k-1} - z_*\|^2 + 2\beta^2 R\|z_k - z_{k-1}\|^2 = \|z_\infty - z_*\|^2.$$

Since $z_*$ can be any equilibrium point, we let $z_* = z_\infty$. This proves $\|z_k - z_\infty\|^2 \rightarrow 0$, i.e., $z_k \rightarrow z_\infty$. \qed
D Proof of Theorem 3

We quickly state a few identities and inequalities we later use. As the verification of these results are elementary, we only provide a short summary of their proofs.

Lemma 7. For \( p \in (0, 1) \) and \( k \geq 1 \),

\[
\frac{p}{k} - \frac{p(1-p)}{2k^2} < \frac{(k+1)^p - kp}{kp} < p.
\]

The proof follows from a basic application of the inequality

\[
1 + px - \frac{p(1-p)}{2}x^2 \leq (1 + x)^p \leq 1 + px
\]

for \( x \in [0, 1] \) and \( p \in (0, 1) \).

Lemma 8. For \( p \in (0, 1) \) and \( k \geq 1 \),

\[
\frac{p}{k + 1} < \frac{(k+1)^p - kp}{kp}.
\]

The proof follows from integrating the decreasing function \( p/x^{1-p} \) from \( k \) to \( k + 1 \).

Lemma 9. For \( p \in (0, 1) \) and \( k \geq 1 \),

\[
0 \leq \frac{p}{k(k + 1)} - \frac{(k+1)^p - kp}{kp(k + 1)} \leq \frac{p(1-p)}{2k^3}.
\]

The proof follows from Lemma 7.

Lemma 10. Given any \( V_0, V_1, \ldots \in \mathbb{R} \), we have

\[
\sum_{j=1}^{k} \left( \frac{j(j+1)}{2} (V_j - V_{j-1}) + jV_{j-1} \right) = \frac{k(k+1)}{2} V_k
\]

The proof follows from basic calculations.

Lemma 11. Let \( z_0, z_1, \ldots \in \mathbb{R}^{m+n} \) be an arbitrary sequence. Then for any \( k = 0, 1, \ldots \),

\[
\frac{1}{2}||z_{k+1} - z_0||^2 - \frac{1}{2}||z_k - z_0||^2 = \left< z_{k+1} - z_k, \frac{1}{2}(z_{k+1} + z_k) - z_0 \right>.
\]

The proof follows from basic calculations.

D.1 Main analysis

Throughout this section, write \( g_k = G(z_k) \) for \( k \geq -1 \).

Lemma 12. Let \( \{V_k\}_{k \in \mathbb{N}_+} \) and \( \{U_k\}_{k \in \mathbb{N}_+} \) be nonnegative (deterministic) sequences satisfying

\[
V_{k+1} \leq \left( 1 - \frac{C_1}{k^{1-\varepsilon}} + f(k) \right) V_k + \frac{C_2}{k^{1-\varepsilon}} \sqrt{V_k} + U_k
\]

where \( C_1 > 0, C_2 > 0, f(k) = o(1/k^{1-\varepsilon}) \) with \( \varepsilon \in [0, 1) \), and

\[
\sum_{k=1}^{\infty} U_k < \infty.
\]

Then \( \limsup_{k \to \infty} V_k \leq C_2^2/C_1^2 \).

Proof. For any \( \delta \in (0, C_1) \), there is a large enough \( K \geq 0 \) such that for all \( k \geq K \),

\[
\frac{C_1}{k^{1-\varepsilon}} - f(k) \geq \frac{C_1 - \delta/2}{k^{1-\varepsilon}}.
\]
Define 

$$\nu = \frac{C_2^2}{(C_1 - \delta)^2}$$

for \( k \geq 0 \). Then

$$V_{k+1} \leq \left( 1 - \frac{C_1 - \delta/2}{k^{1-\varepsilon}} \right) V_k + \frac{C_2^2}{(C_1 - \delta)k^{1-\varepsilon}} \max \left\{ \frac{\sqrt{V_k}}{\nu}, V_k \right\} + U_k$$

$$V_{k+1} - \nu \leq \left( 1 - \frac{C_1 - \delta/2}{k^{1-\varepsilon}} \right) (V_k - \nu) - \frac{C_2^2 \delta}{2k^{1-\varepsilon}(C_1 - \delta)^2} + \frac{C_2^2}{(C_1 - \delta)k^{1-\varepsilon}} \max \left\{ \frac{\sqrt{V_k}}{\nu} - 1, \frac{V_k}{\nu} - 1 \right\} + U_k$$

Note that \( \max \{ \sqrt{x} - 1, x - 1 \} \leq \max \{0, x - 1\} \) for all \( x \geq 0 \). So

$$V_{k+1} - \nu \leq \left( 1 - \frac{C_1 - \delta/2}{k^{1-\varepsilon}} \right) (V_k - \nu) + \frac{C_2^2}{(C_1 - \delta)k^{1-\varepsilon}} \max \left\{ 0, \frac{V_k}{\nu} - 1 \right\} + U_k$$

$$= \left( 1 - \frac{C_1 - \delta/2}{k^{1-\varepsilon}} \right) (V_k - \nu) + \frac{C_1 - \delta}{k^{1-\varepsilon}} \max \{0, V_k - \nu\} + U_k$$

$$\leq \left( 1 - \frac{C_1 - \delta}{k^{1-\varepsilon}} \right) \max \{0, V_k - \nu\} + \frac{C_1 - \delta}{k^{1-\varepsilon}} \max \{0, V_k - \nu\} + U_k$$

$$= \left( 1 - \frac{\delta}{2k^{1-\varepsilon}} \right) \max \{0, V_k - \nu\} + U_k$$

for large enough \( k \). Since

$$0 \leq \left( 1 - \frac{\delta}{2k^{1-\varepsilon}} \right) \max \{0, V_k - \nu\} + U_k$$

for large enough \( k \), we have

$$\max \{0, V_{k+1} - \nu\} \leq \left( 1 - \frac{\delta}{2k^{1-\varepsilon}} \right) \max \{0, V_k - \nu\} + U_k$$

With a standard recursion argument (e.g. Lemma 3 of [52]) we conclude \( \max \{0, V_k - \nu\} \to 0 \).

Since this holds for any \( \delta > 0 \), we conclude \( \limsup_{k \to \infty} V_k \leq C_2^2/C_1^2 \).

\( \square \)

**Lemma 13.**

$$\|z_k - z_*\|^2 \leq C$$

for all \( k \geq 0 \) for some \( C > 0 \). (This result depends on assumption \( p > 1/2 \).)

**Proof.**

$$\|z_{k+1} - z_*\|^2 = \|z_k - z_*\|^2 - \frac{2(1 - p)}{(k + 1)^p} \langle g_k, z_k - z_* \rangle + \frac{2\gamma(1 - p)}{k + 1} \langle z_0 - z_k, z_k - z_* \rangle$$

$$+ \left\| 1 - \frac{p}{(k + 1)^p} g_k + \frac{\gamma(1 - p)}{k + 1} (z_0 - z_k) \right\|^2$$

$$\leq \left( 1 - \frac{2\gamma(1 - p)}{k + 1} \right) \|z_k - z_*\|^2 + \frac{2\gamma(1 - p)}{k + 1} \langle z_0 - z_* , z_k - z_* \rangle$$

$$+ \frac{2(1 - p)^2}{(k + 1)^{2p}} \|g_k\|^2 + \frac{4\gamma^2(1 - p)^2}{(k + 1)^2} \|z_0 - z_k\|^2$$

$$\leq \left( 1 - \frac{2\gamma(1 - p)}{k + 1} + \frac{4\gamma^2(1 - p)^2}{(k + 1)^2} \right) \|z_k - z_*\|^2 + \frac{2\gamma(1 - p)}{k + 1} \|z_0 - z_*\| \|z_k - z_*\|$$

$$+ \frac{2(1 - p)^2}{(k + 1)^{2p}} \|z_k - z_0\|^2 + \frac{4\gamma^2(1 - p)^2}{(k + 1)^2} \|z_0 - z_*\|^2$$

$$= \left( 1 - \frac{2\gamma(1 - p)}{k + 1} + \frac{4\gamma^2(1 - p)^2}{(k + 1)^2} + R_1^2 \frac{4(1 - p)^2}{(k + 1)^{2p}} \right) \|z_k - z_*\|^2$$

$$+ \frac{2\gamma(1 - p)}{k + 1} \|z_0 - z_*\| \|z_k - z_*\| + \frac{4\gamma^2(1 - p)^2}{(k + 1)^2} \|z_0 - z_*\|^2$$
where the first inequality follows from (1), the monotonicity inequality, and (5) and the second inequality follows from Assumption A3. We conclude the statement with Lemma 12.

Lemma 14.

\[ \| z_{k+1} - 2z_k + z_{k-1} \|^2 \leq 4(1-p)^2 \left( \frac{\gamma^2}{k^2} + \frac{R^2}{k^{2p}} \right) \| z_k - z_{k-1} \|^2 + 4(1-p)^2 \left( \frac{p^2 R^2}{k^{2+2p} + \gamma^2} \right) \| z_0 - z_k \|^2 \]

Proof.

\[
\| z_{k+1} - 2z_k + z_{k-1} \|^2 \\
= \left\| \frac{1-p}{(k+1)^p} g_k - \frac{1-p}{k^p} g_{k-1} - \frac{(1-p)\gamma}{k+1} (z_0 - z_k) + \frac{(1-p)\gamma}{k} (z_0 - z_{k-1}) \right\|^2 \\
\leq 2 \left\| \frac{1-p}{(k+1)^p} g_k - \frac{1-p}{k^p} g_{k-1} \right\|^2 + 2 \left\| \frac{(1-p)\gamma}{k+1} (z_0 - z_k) - \frac{(1-p)\gamma}{k} (z_0 - z_{k-1}) \right\|^2 \\
\leq \frac{4(1-p)^2}{k^{2p}} \| g_k - g_{k-1} \|^2 + 4 \left( \frac{1-p}{(k+1)^p} - \frac{1-p}{k^p} \right)^2 \| g_k \|^2 \\
+ \frac{4\gamma^2(1-p)^2}{k^2} \| z_k - z_{k-1} \|^2 + 4 \left( \frac{(1-p)\gamma}{k+1} - \frac{(1-p)\gamma}{k} \right)^2 \| z_0 - z_k \|^2 \\
\leq 4(1-p)^2 \left( \frac{\gamma^2}{k^2} + \frac{R^2}{k^{2p}} \right) \| z_k - z_{k-1} \|^2 + 4(1-p)^2 \left( \frac{p^2 R^2}{k^{2+2p} + \gamma^2} \right) \| z_0 - z_k \|^2
\]

where the first and second inequalities follow from (5) and the third inequality follows from Assumptions A3 and Lemma 7. □
Main proof. The key idea is to mimic the continuous-time analysis for the discrete-time setup by bounding the higher-order terms. We have

\[
\frac{1}{2} \|z_{k+1} - z_k\|^2 - \frac{1}{2} \|z_k - z_{k-1}\|^2
= \left\langle \frac{1}{2} (z_{k+1} - z_{k-1}), z_{k+1} - 2z_k + z_{k-1} \right\rangle
= -\frac{1-p}{k^p} \langle z_{k} - z_{k-1} - g_k, z_k \rangle + (1-p) \frac{(k+1)^p - k^p}{k^p (k+1)^p} \langle z_{k} - z_{k-1}, g_k \rangle - \frac{\gamma(1-p)}{k} \|z_k - z_{k-1}\|^2
- \frac{\gamma(1-p)}{k (k+1)} \langle z_{k} - z_{k-1}, z_0 - z_k \rangle + \frac{1}{2} \|z_{k+1} - 2z_k + z_{k-1}\|^2
\leq (1-p) \frac{(k+1)^p - k^p}{k^p (k+1)^p} \langle z_{k} - z_{k-1}, g_k \rangle - \frac{\gamma(1-p)}{k} \|z_k - z_{k-1}\|^2
- \frac{\gamma(1-p)}{k (k+1)} \langle z_{k} - z_{k-1}, z_0 - z_k \rangle + \frac{1}{2} \|z_{k+1} - 2z_k + z_{k-1}\|^2
= -\left\langle \frac{\gamma(1-p)}{k} + \frac{(k+1)^p - k^p}{k^p} - \frac{\gamma(1-p)}{2k (k+1)^2} \right\rangle \|z_k - z_{k-1}\|^2
- \frac{(k+1)^p - k^p}{k^p} \langle z_{k} - z_{k-1} - z_{k+1} + 2z_k + z_{k-1} \rangle + \frac{1}{2} \|z_{k+1} - 2z_k + z_{k-1}\|^2
- \gamma(1-p) \left\langle \frac{1}{k (k+1)} - \frac{(k+1)^p - k^p}{k^p (k+1)^p} \right\rangle \langle z_{k} - z_{k-1}, z_0 - \frac{1}{2} (z_{k} + z_{k-1}) \rangle
\leq -\left\langle \frac{\gamma(1-p)}{k} + \frac{p(1-p)}{2k^2} - \frac{\gamma(1-p)}{2k (k+1)^2} \right\rangle \|z_k - z_{k-1}\|^2
- \frac{(k+1)^p - k^p}{k^p} \langle z_{k} - z_{k-1} - z_{k+1} + 2z_k + z_{k-1} \rangle + \frac{1}{2} \|z_{k+1} - 2z_k + z_{k-1}\|^2
- \gamma(1-p) \left\langle \frac{1}{k (k+1)} - \frac{(k+1)^p - k^p}{k^p (k+1)^p} \right\rangle \langle z_{k} - z_{k-1}, z_0 - \frac{1}{2} (z_{k} + z_{k-1}) \rangle
\leq -\left(\frac{\gamma(1-p)}{k} + \frac{p(1-p)}{2k^2} - \frac{\gamma(1-p)}{2k (k+1)^2} \right) \|z_k - z_{k-1}\|^2
+ \frac{p}{2k^2} \|z_{k+1} - 2z_k + z_{k-1}\|^2
+ \frac{1}{2} \|z_{k+1} - 2z_k + z_{k-1}\|^2
- \gamma(1-p) \left\langle \frac{1}{k (k+1)} - \frac{(k+1)^p - k^p}{k^p (k+1)^p} \right\rangle \langle z_{k} - z_{k-1}, z_0 - \frac{1}{2} (z_{k} + z_{k-1}) \rangle
= -\frac{1+p}{2k} \|z_{k+1} - 2z_k + z_{k-1}\|^2
- \frac{\gamma(1-p)^2}{k (k+1)} \langle z_{k} - z_{k-1}, z_0 - \frac{1}{2} (z_{k} + z_{k-1}) \rangle
- \gamma(1-p) \left\langle \frac{1}{k (k+1)} - \frac{(k+1)^p - k^p}{k^p (k+1)^p} - \frac{1-p}{k (k+1)} \right\rangle \langle z_{k} - z_{k-1}, z_0 - \frac{1}{2} (z_{k} + z_{k-1}) \rangle
= C_1(k,p)
\]

where the first inequality follows from (1), the monotonicity inequality, the second inequality follows from Lemma 7 and (5), and the third inequality follows from Lemma 7 and (6). Young’s inequality, with \( \varepsilon = k. \)
By Lemma 9, \( |C_1(k, p)| \leq \frac{p(1-p)}{2k^2} \). Using (6), Young’s inequality, with \( \varepsilon = 1/k \) and (5) we get

\[
-\gamma (1-p) \left( 1 - \frac{p + k^p}{k(k+1)} \right) \left( \frac{p}{k(k+1)} \right)^2 \left( \frac{1}{k(k+1)} \right) \left( z_k - z_{k-1} \right) - \frac{1}{2} \left( z_k + z_{k+1} \right)
\]

\[
\leq \frac{\gamma p(1-p)^2}{4k^2} \|z_k - z_{k-1}\|^2 + \frac{\gamma p(1-p)^2}{4k^2} \|z_0 - \frac{1}{2}(z_k + z_{k+1})\|^2
\]

\[
\leq \frac{\gamma p(1-p)^2}{4k^2} \|z_k - z_{k-1}\|^2 + \frac{\gamma p(1-p)^2}{8k^4} (\|z_0 - z_k\|^2 + \|z_0 - z_{k-1}\|^2).
\]

Putting these together we get

\[
\frac{1}{2} \|z_{k+1} - z_k\|^2 - \frac{1}{2} \|z_k - z_{k-1}\|^2 + \frac{1}{k+1} \|z_k - z_{k-1}\|^2 - \frac{\gamma (1-p)^2}{k(k+1)} \left( z_k - z_{k-1} \right) - \frac{1}{2} \left( z_k + z_{k+1} \right)
\]

\[
\leq - \left( \frac{(\gamma - 1)(1-p)}{k} + \frac{\gamma p(1-p)^2}{2k(k+1)^2} - \frac{p(1-p)}{2k^2} - \frac{\gamma (1-p)}{2k(k+1)} - \frac{1}{2k^2} \right) \|z_k - z_{k-1}\|^2
\]

\[
+ \frac{1+p}{2} \|z_{k+1} - 2z_k + z_{k-1}\|^2 + \frac{\gamma p(1-p)^2}{8k^4} (\|z_0 - z_k\|^2 + \|z_0 - z_{k-1}\|^2)
\]

With Lemma 13 and Lemma 14 we get

\[
\frac{1}{2} \|z_{k+1} - z_k\|^2 - \frac{1}{2} \|z_k - z_{k-1}\|^2 + \frac{1}{k+1} \|z_k - z_{k-1}\|^2 - \frac{\gamma (1-p)^2}{k(k+1)} \left( z_k - z_{k-1} \right) - \frac{1}{2} \left( z_k + z_{k+1} \right)
\]

\[
\leq - \left( \frac{(\gamma - 1)(1-p)}{k} + \frac{\gamma p(1-p)^2}{2k(k+1)^2} - \frac{p(1-p)}{2k^2} - \frac{\gamma (1-p)}{2k(k+1)} - \frac{1}{2k^2} \right) \|z_k - z_{k-1}\|^2
\]

\[
+ \left( 2(1+p)(1-p)^2 \left( \frac{p^2 R^2}{k^2+2p} + \frac{\gamma^2}{k^4} \right) \right) C_2
\]

\[
\leq - \left( \frac{(\gamma - 1)(1-p)}{k} + \mathcal{O} \left( \frac{1}{k^2p^2} \right) \right) \|z_k - z_{k-1}\|^2 + \mathcal{O} \left( \frac{1}{k^2p^2} \right)
\]

Note that there is a \( K \in \mathbb{N} \) such that \( C_3(k, \gamma, p, R) \leq 0 \) for all \( k \geq K \) (with \( \gamma, p, \) and \( R \) fixed). We multiply both sides with \( k(k+1) \) and sum both sides from \( k = 1 \) to \( k = k \), and apply Lemma 10 and Lemma 11 to get

\[
\frac{k(k+1)}{2} \|z_{k+1} - z_k\|^2 \leq \frac{\gamma (1-p)^2}{2} \|z_k - z_0\|^2 + C_4 + \mathcal{O} \left( \frac{1}{k^{2p-1}} \right)
\]

where \( C_4 < \infty \) since \( C_3(k, \gamma, p, R) > 0 \) for only finitely many \( k \). Reorganizing we get

\[
\frac{k(k+1)(1-p)^2}{2(k+1)^{2p}} \|g_k\|^2 + \frac{k(1-p)^2}{2(k+1)} \|z_0 - z_k\|^2 - \frac{k(1-p)^2}{(k+1)^{p}} (g_k, z_0 - z_k)
\]

\[
\leq \frac{\gamma (1-p)^2}{2} \|z_k - z_0\|^2 + C_4 + \mathcal{O} \left( \frac{1}{k^{2p-1}} \right)
\]

Reorganizing yet again we get

\[
\frac{k(k+1)(1-p)^2}{2(k+1)^{2p}} \|g_k\|^2 - \frac{k(1-p)^2}{(k+1)^{p}} (g_k, z_0 - z_k)
\]

\[
\leq \frac{\gamma (1-p)^2}{2} \left( 1 - \frac{\gamma k}{k+1} \right) \|z_k - z_0\|^2 + C_4 + \mathcal{O} \left( \frac{1}{k^{2p-1}} \right)
\]

\[
\leq C_4 + \mathcal{O} \left( \frac{1}{k^{2p-1}} \right),
\]

where we use the assumption that \( \gamma \geq 2 \). Reorganizing again, we get

\[
\|g_k\|^2 \leq \frac{2\gamma}{(k+1)^{1-p}} (g_k, z_0 - z_k) + \frac{2C_4}{(1-p)^{2}(k+1)^{1-2p}} + \mathcal{O} \left( \frac{1}{k} \right)
\]

\[
\leq \frac{2\gamma}{(k+1)^{1-p}} (g_k, z_0 - z_k) + \frac{4C_4}{(1-p)^{2}(k+1)^{2-2p}} + \mathcal{O} \left( \frac{1}{k} \right)
\]

\[
\leq \frac{1}{2} \|g_k\|^2 + \frac{2\gamma^2}{(k+1)^{2-2p}} \|z_0 - z_k\|^2 + \frac{4C_4}{(1-p)^{2}(k+1)^{2-2p}} + \mathcal{O} \left( \frac{1}{k} \right)
\]

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for $k \geq 1$, where the second inequality follows from (1), the monotonicity inequality, and the third inequality follows from (6), Young’s inequality, with $\varepsilon = \gamma/(k+1)^{1-p}$. Finally, we have

$$\|g_k\|^2 \leq \frac{C}{k^{2-2p}} + O\left(\frac{1}{k}\right)$$

with $C = 4\gamma^2 + 8C_4/(1-p)^2$.

### Proof of Theorem 4

**Lemma 15** (Proposition 23.31 and Theorem 23.44 of [5]). Let $G$ be a maximal monotone operator such that $\text{Zer}(G) \neq \emptyset$. Then $(I + \tau G)^{-1}(z_0) \to P_{\text{Zer}(G)}(z_0)$ and

$$\|(I + (\tau + s)G)^{-1}(z_0) - (I + \tau G)^{-1}(z_0)\| \leq O\left(\frac{s}{\tau}\right)$$

for any $s \geq 0$ as $\tau \to \infty$.

**Lemma 16.** Let $\varepsilon \in (0, 1)$. Let $\{V_k\}_{k \in \mathbb{N}_+}$ and $\{U_k\}_{k \in \mathbb{N}_+}$ be nonnegative (deterministic) sequences satisfying

$$V_{k+1} \leq \left(1 - \frac{C}{k^{1-\varepsilon}} + f(k)\right)V_k + g(k)\sqrt{V_k} + U_k$$

where $C > 0$, $f(k) = o(1/k^{1-\varepsilon})$, $g(k) = O(1/k)$, and

$$\sum_{k=1}^{\infty} U_k < \infty.$$

Then $V_k \to 0$.

**Proof.** For any $\delta > 0$, there is a large enough $K \geq 0$ such that

$$V_{k+1} \leq \left(1 - \frac{C - \delta}{k^{1-\varepsilon}}\right)V_k + \frac{\delta}{k^{1-\varepsilon}}\sqrt{V_k} + U_k$$

for all $k \geq K$. By Lemma 12, we conclude $\limsup_{k \to \infty} V_k \leq \delta^2/(C - \delta)^2$. Since this holds for all $\delta > 0$, we conclude $V_k \to 0$.

### Main analysis.

Define

$$\zeta_{k+1} = (I + (k+1)^{1-p-\varepsilon}/\gamma G)^{-1}(z_0).$$

Then

$$0 = -\frac{1-p}{(k+1)^p}G(\zeta_{k+1}) + \frac{(1-p)\gamma}{(k+1)^{1-\varepsilon}}(z_0 - \zeta_{k+1})$$

Since $p < 1$, Lemma 15 gives us

$$\zeta_k \to P_{\text{Zer}(G)}(z_0).$$
where the first inequality follows from (1), the monotonicity inequality, Cauchy-Schwartz inequality, variation of the Optimistic Adam of [8], which uses element-wise square operation.

In this section, we provide further details of the experiments of Section 5. Our Optimistic Adam is a

\[ F \text{ Experiment details} \]

Since \( \zeta \rightarrow P_{\text{Zer}(G)}(z_0) \), we conclude \( z_k \overset{L^2}{\rightarrow} P_{\text{Zer}(G)}(z_0) \).

\[ \square \]

F Experiment details

In this section, we provide further details of the experiments of Section 5. Our Optimistic Adam is a variation of the Optimistic Adam of [8], which uses \( \beta = 1 \) while we allow for a general optimism rate \( \beta > 0 \). For Anchored Adam, we do not diminish the strength of the anchor proportional to \( \frac{1}{k^1} \) since Adam does not diminish the learning rate. Rather, we maintain a constant anchor strength \( \gamma \) but refresh the anchor point every \( T \) iterations. The notation \( \nabla^2 \) in algorithm tables denote the element-wise square operation.
Generator

- latent space 100 (Gaussian noise)
- dense 128 ReLU
- dense 256 ReLU
- dense 512 ReLU
- dense 1024 ReLU
- dense 1024 tanh

Discriminator

- Resize the input image 28 × 28 to 32 × 32
- dense 512 ReLU
- dense 256 ReLU
- dense 1 ReLU

Table 1: Generator and discriminator architectures for the MNIST experiment

| Batch size = 64 |
|-----------------|
| Adam learning rate = 0.0002 |
| Adam $\beta_1 = 0.5$ |
| Adam $\beta_2 = 0.999$ |
| max iteration = 200000 |
| GAN objective = “WGAN-GP” |
| Gradient penalty parameter $\lambda = 10$ |
| $n_{\text{dis}} = 5$ |
| Optimizer = “Adam”, “Optimistic Adam”, or “Anchored Adam” |
| Optimism rate $\rho = 1$ |
| Anchor rate $\gamma = 1$ |
| Anchor refresh period $T = 10000$ |

Table 2: Hyperparameters for the MNIST experiment

Optimistic Adam

**Parameters:** learning rate $\eta$, exponential decay rates for moment estimates $\beta_1, \beta_2 \in [0, 1)$, optimism rate $\rho > 0$, and initial parameters $z_0$

**Repeat** $k = 0, 1, 2, \ldots, K$ (iteration):

1. Compute stochastic gradient $\nabla_{z,k} = G(z_k)$
2. Update biased estimate of first moment: $m_k = \beta_1 m_{k-1} + (1 - \beta_1) \nabla_{z,k}$
3. Update biased estimate of second moment: $v_k = \beta_2 v_{k-1} + (1 - \beta_2) \nabla^2_{z,k}$
4. Scale the step-size: $\hat{\eta}_k = \eta \sqrt{1 - \beta_2^k} / (1 - \beta_1^k)$
5. Perform optimistic gradient step: $z_k = z_{k-1} - \hat{\eta}_k (1 + \rho) \frac{m_k}{\sqrt{v_k + \epsilon}} + \hat{\eta}_{k-1} \rho \frac{m_{k-1}}{\sqrt{v_{k-1} + \epsilon}}$

Return $z_K$

Anchored Adam

**Parameters:** learning rate $\eta$, exponential decay rates for moment estimates $\beta_1, \beta_2 \in [0, 1)$, anchor rate $\gamma > 0$, anchor update period $T$, and initial parameters $z_0$

**Repeat** $k = 0, 1, 2, \ldots, K$ (iteration):

1. set anchor $a_k = z_k$ if mod$(k, T) = 0$ else $a_k = a_{k-1}$
2. Compute stochastic gradient $\nabla_{z,k} = G(z_k)$
3. Update biased estimate of first moment: $m_k = \beta_1 m_{k-1} + (1 - \beta_1) \nabla_{z,k}$
4. Update biased estimate of second moment: $v_k = \beta_2 v_{k-1} + (1 - \beta_2) \nabla^2_{z,k}$
5. Scale the step-size: $\hat{\eta}_k = \eta \sqrt{1 - \beta_2^k} / (1 - \beta_1^k)$
6. Perform anchored gradient step: $z_k = z_{k-1} - \hat{\eta}_k \frac{m_k}{\sqrt{v_k + \epsilon}} + \gamma (a_k - z_{k-1})$

Return $z_K$
### Generator
- latent space $128$ (Gaussian noise)
- dense $4 \times 4 \times 512$ batchnorm ReLU
- $4 \times 4$ conv.T stride=2 $256$ batchnorm ReLU
- $4 \times 4$ conv.T stride=2 $128$ batchnorm ReLU
- $4 \times 4$ conv.T stride=2 $64$ batchnorm ReLU
- $4 \times 4$ conv.T stride=1 $3$ weightnorm tanh

### Discriminator
- Input Image $32 \times 32 \times 3$
- $3 \times 3$ conv. stride=1 $64$ lReLU
- $3 \times 3$ conv. stride=2 $128$ lReLU
- $3$ conv. stride=1 $128$ lReLU
- $3$ conv. stride=2 $256$ lReLU
- $3$ conv. stride=1 $256$ lReLU
- $3$ conv. stride=2 $512$ lReLU
- $3$ conv. stride=1 $512$ lReLU
- dense $1$

Table 3: Generator and discriminator architectures for the CIFAR-10 experiment

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### Hyperparameters
- batch size = 64
- Adam learning rate = 0.0001
- Adam $\beta_1 = 0.0$
- Adam $\beta_2 = 0.9$
- max iteration = 100000
- GAN objective = “WGAN-GP”
- Gradient penalty parameter $\lambda = 1$
- $n_{\text{dis}} = 1$
- Optimizer = “Adam”, “Optimistic Adam”, or “Anchored Adam”
- Optimism rate $\rho = 1$
- Anchor rate $\gamma = 1$
- Anchor refresh period $T = 10000$

Table 4: Hyperparameters for the CIFAR-10 experiment