Lorentzian Wormholes
in the Friedman-Robertson-Walker Universe

B. Mirza, M. Eshaghi, S. Dehdashti

Department of Physics, Isfahan University of Technology, Isfahan 84154, Iran

ABSTRACT

The metric of some Lorentzian wormholes in the background of the FRW universe is obtained. It is shown that for a de Sitter space-time the new solution is supported by Phantom Energy. The wave equation for a scalar field in such backgrounds is separable. The form of the potential for the Schrödinger type one dimensional wave equation is found.

PACS numbers: 04.20.-q, 04.20.Jb
Keywords: Lorentzian wormholes; weak energy condition.
1. Introduction

Since the pioneering article by Morris and Thorne [1], Lorentzian wormholes have attracted a lot of attention. A number of static and dynamic wormholes have been found both in general relativity and in alternative theories of gravitation. There are several reasons that support the interest in these solutions. One of them is the possibility of constructing time machines [2,3]. Another reason is related to the nature of the matter that generate these solutions. One major problem with the existence of traversable wormholes is that the matter required to support such a geometry essentially violates the energy conditions for general relativity [4,5,6,7]. To get around weak energy condition violations, there have been some works in non-standard gravity theories. Brans-Dicke gravity supports static wormholes both in vacuum [8] and with matter content that do not violate the weak energy condition by itself [9]. There are analysis in several others alternative gravitational theories, such as Einstein-Cartan models [10], Einstein-Gauss-Bonnet [11] and \( R + R^2 \) [12]. In all cases the violation of weak energy condition is a necessary condition for static wormhole to exist, although these changes in the gravitational action allows one to have normal matter while relagating the exoticity to non-standard fields. This leads to explore the issue of weak energy condition violation in non-static situations. Time dependence allow one to move the energy condition violations around in time [13,14,15,16,17,18], However for some comments see [19]. (Unfortunately radial null geodesics through the wormhole will still encounter energy condition violations, subject to suitable technical qualifications.)

In this paper we have obtained evolving worm holes in the FRW universe with a special choice for their shape function and with a Phantom Energy. For other ideas about growing wormholes by Phantom Energy accretion see [20, and refs]. The paper is organised as follows: In Section 2, the metric of a traversable wormhole in the FRW universe is obtained and then generalised to arbitrary dimensions. Finally the metric in an asymptotically de Sitte space is given and it is shown that the energy density is positive but the weak energy condition is violated. In the last section a one dimensional Schrödinger like equation is obtained which corresponds to the scalar wave equations in this kind of geometry.

2. A traversable wormhole in the FRW universe

In this part we are going to apply a new method [21] for obtaining a new solution for the evolving wormhole. We consider the following metric for a spherically symmetric and static wormhole

\[
ds^2 = -e^{-2\phi(r)} dt^2 + \frac{dr^2}{(1 - b(r)/r)} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)\]

(1)

where \( \phi(r) \) is the redshift functionand \( b(r) \) is the shape function. Here we shall restrict ourselves to the case \( \phi(r) = 0 \) and \( b(r) = b_0^m/r^m \), where \( b_0 \) is the radius of the throat of the wormhole and \( m \) is a real positive number. The following transformations can be use to rewrite the metric (1) in the isotropic spherical coordinates.

\[
s = 2^{-2/m} \ell, \quad t = 2^{-2/m} \nu, \quad \frac{b_0^m}{2} = b^m, \quad r = \frac{x}{2^{1/m}(1 + \frac{b^m}{x^m})^{2/m}}\]

(2)
So we get to
\[ d\ell^2 = -dv^2 + (1 + \frac{b^m}{x^m})^{4/m}(dx^2 + x^2 d\theta^2 + x^2 \sin^2 \theta d\phi^2) \] (3)

The metric for the FRW universe is given by
\[ d\ell^2 = -dv^2 + \frac{a^2(v)}{(1 + kx^2/4)^2}(dx^2 + x^2 d\theta^2 + x^2 \sin^2 \theta d\phi^2) \] (4)

where \( a(v) \) is the scale factor and \( k \) gives the curvature of the universe. By comparing (3) and (4) we set the metric for a wormhole embedded in the FRW universe as follows
\[ d\ell^2 = -A^2(v, x)dv^2 + B^2(v, x)(dx^2 + x^2 d\theta^2 + x^2 \sin^2 \theta d\phi^2) \] (5)

By considering \( G_{01} = 0 \) one has
\[ A(v, x) = f(v) \frac{\dot{B}}{2B} \] (6)

where dot denotes the derivative with respect to \( v \) and \( f(v) \) is an arbitrary function of \( v \). By comparing the \( g_{11} \) terms in (3) and (5), the possible form for the function \( B(x, v) \) is,
\[ B(v, x) = \left[w(v, x) + \frac{q(v)}{x^m}\right]^{2/m} \] (7)

By inserting eq. (7) in (6) we arrive at
\[ A(v, x) = \frac{f}{m} \frac{(\dot{w} + \frac{\dot{q}}{x^m})}{(w + \frac{q}{x^m})} \] (8)

In the case of \( v = \text{const} \) and the asymptotically flat conditions, \( A(v = \text{const}, x) \) should be reduced to the \( \sqrt{-g_{00}} \) term in eq.(3). We infer that the following identities should always hold
\[ f\dot{w} = mw, \quad f\dot{q} = mq \] (9)

We may define \( q \) as
\[ q(v) = b^m d^m(v) \quad \rightarrow \quad f = \frac{d}{d} \] (10)

where \( d(v) \) is an arbitrary function which is related to the scale factor of the universe. For the two limiting cases of small and large \( x \) values eq. (5) reduces to (3) and (4) respectively, and so we can identify the form of \( w(v, x) \) which is similar to the exact solution which has already been obtained for Black holes. [21]
\[ w(v, x) = \frac{d^m(v)}{(1 + kx^2/4)^{m/2}} \] (11)
The final form of the metric for a traversable wormhole in the background of FRW universe is given by

\[ d\ell^2 = -dv^2 + a^2(v)\left[\frac{1}{(1 + kx^2/4)^{m/2}} + \frac{b^m}{x^m}\right]^4/(4m)(dx^2 + x^2d^2\Omega) \] (12)

where \(a^2(v) = d^4(v)\) is the scale factor. This calculation can be extended to \((3+n)-\)dimensional wormholes. After a straightforward calculation for a \((3+n)-\)dimensional traversable wormhole in the background of FRW universe, we arrive at the following form for the metric

\[ d\ell^2 = -dv^2 + a^2(v)\left[\frac{1}{(1 + kx^2/4)^{mn/2}} + \frac{b^{mn}}{x^{mn}}\right]^4/(mn)(dx^2 + x^2d^2\Omega_{n+1}) \] (13)

Equation (13) suggests a kind of duality for wormholes in different dimensions. Consider \(p = mn\), then it is reasonable to expect that in a scattering process a \((3+n)\)-dimensional wormhole with an index \(m\) in (13) has a similar behaviour to another wormhole in \((3+1)\)-dimensions with index \(p\). There is also another kind of duality that is implied by (13), a \((3+n)\)-dimensional wormhole with an index \(m\) has a similar behaviour to a \((3+m)\)-dimensional wormhole with an index \(n\). In the following section we will consider a de Sitter space and show that for some special choices of the shape function the energy density could be positive.

3. Asymptotically de Sitter space-time

For \(a(t) = \exp(Ht), H = \text{const}\) and \(k = 0\), the metric (12) can be written as

\[ ds^2 = -dt^2 + e^{2Ht}[1 + \frac{b^m}{r^{mn}}]^{4/m}(dr^2 + r^2d^2\theta + r^2\sin^2\theta d\phi^2) \] (14)

For the large values of \(r\), (14) reduces to the de Sitter universe and for small values of \(x\) it describes a traversable wormhole. So we may suggest (14) as the metric for a traversable wormhole in the de Sitter universe. The Einstein equation is defined by

\[ G_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu} \] (15)

where we assume that in this case \(\Lambda = 3H^2\), Using an orthonormal reference frame with basis

\[ e_{\xi} = e_t \]

\[ e_{\tilde{r}} = \frac{e_r}{e^{Ht}(1 + \frac{b^m}{r^{mn}})^{2/m}} \]

\[ e_{\tilde{\theta}} = \frac{e_\theta}{e^{Ht}(1 + \frac{b^m}{r^{mn}})^{2/m}} \]
the energy momentum tensor of the perfect fluid is written as

\[ T_{\mu\nu} = diag(\rho, p, p, p) \]  
(17)

where \( \rho \) denotes the energy density and \( p \) is the pressure in the fluid. One may write the energy momentum tensor (17) as

\[ T_{\mu\nu} = diag(\rho, -\tau, p, p) \]  
(18)

where \( \tau \) denotes the stress (opposite to the pressure) in the \( e^\hat{r} \) direction. It can be shown that in the orthonormal frame (16), the metric (14) becomes

\[ g_{\mu\nu} = diag(-1, 1, 1, 1) \]  
(19)

After a straightforward calculation we arrive at

\[ \rho = \frac{e^{-2Ht}}{2\pi} \frac{b^m(1-m)}{r^{m+2}(1 + \frac{b^m}{r^m})^{\frac{1}{m}+2}} \]  
(20)

\[ -\tau = \frac{-e^{-2Ht}}{2\pi} \frac{b^m}{r^{m+2}(1 + \frac{b^m}{r^m})^{\frac{1}{m}+2}} \]  
(21)

The matter with the property, energy density \( \rho > 0 \) but pressure \( p < -\rho < 0 \) is known as Phantom Energy. So if \( 0 < m < 1 \) the energy source for the new solution (14) is Phantom Energy.

3. Scalar wave equation

The wave equation of the minimally coupled massless scalar field in a background which is defined by the metric in equation (12) is given by

\[ \nabla^\mu \nabla_\mu \Phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}g^{\mu\nu} \partial_\nu \Phi) \]  
(22)

In spherically symmetric space-time, the scalar field can be separated by variables

\[ \Phi_{lm} = Y_{lm}(\theta, \phi) \frac{u_l(r, t)}{r} \]  
(23)

where \( Y_{lm} \) is the spherical harmonics and \( l \) is the quantum angular momentum. If \( l = 0 \) and the scalar field \( \Phi(r) \) depends on \( r \), the wave equation becomes the following relation

\[ r^2 \left( \frac{1}{(1 + \frac{k r^2}{4})^{\frac{2}{3}}} + \frac{b^n}{r^m} \right) \frac{\partial}{\partial r} \Phi = const = A \]  
(24)
Thus the static scalar wave without propagation is easily found as the integral form of

$$\Phi = A \int r^{-2} \left( \frac{1}{(1 + kr^2)^{\frac{n}{2}}} + \frac{b^n}{r^n} \right) \frac{dr}{r} \quad (25)$$

If the scalar field depends on $r, t$; the wave equation after the separation of variables ($\vartheta, \varphi$) become

$$-\frac{\partial^2 u_l}{\partial t_*^2} + \frac{\partial^2 u_l}{\partial r_*^2} = V_l u_l \quad (26)$$

where the potential is

$$V(r) = \frac{1}{\left(1 + \frac{kr^2}{4}\right)^\frac{n}{2}} \left( \frac{L^2}{r^2} + \frac{2}{n} \frac{\partial}{\partial r} \ln \left( \frac{1}{(1 + kr^2)^{\frac{n}{2}}} + \frac{b^n}{r^n} \right) \right) \quad (27)$$

where $L^2 = l(l + 1)$ is the square of the angular momentum and the proper distance $r_*$ and time $t_*$ have the following relations to $r$ and $t$,

$$\frac{\partial}{\partial r_*} = \left( \frac{1}{1 + \frac{kr^2}{4}} \right)^\frac{n}{2} \frac{\partial}{\partial r} \quad (28)$$

$$\frac{\partial^2}{\partial t_*^2} = \frac{a}{\partial t} (a^3 \frac{\partial}{\partial t}) \quad (29)$$

The properties of the potential can be determined by the values $n, l, b, k$ and its shape. If the time dependence of the wave is harmonic as $u_l(r, t_*) = \hat{u}_l(r, w_*) e^{-i w_* t_*}$ the equation becomes

$$\left[ \frac{d^2}{dr_*^2} + w_*^2 - V_l(r) \right] \hat{u}_l(r, w_*) = 0 \quad (30)$$

It is just the Schrödinger equation with energy $w_*^2$ and potential $V_l(r)$. If $k = 0$, $V_l(r)$ approaches zero as $r \to \infty$, which means that the solution has the form of the plane wave $u_l \approx e^{\pm i w_*,r_*}$ asymptotically. The result shows that if a scalar wave passes through the wormhole the solution is changed from $e^{\pm i w_*,r_*}$ into $e^{\pm i w_*,r_*}$, which means that the potential affects the wave and experience the scattering. This part may be continued by using the method which is used in calculation of the quasinormal modes. (for a comprehensive review, see [22]).

3. Conclusion

In this work the metric for a traversable wormhole in the FRW universe is obtained. It is shown that for such wormholes the energy density may be positive although the weak energy condition is violated. The scalar wave equation in this space time is reduced to an effective one dimensional Schrödinger equation, which is helpful for investigating the quasi-normal modes.
References

[1] M. Morris and K. Thorne, Am. J. Phys. 56, 395 (1980).
[2] M. Morris, K. Thorne and U. Yurtserver, Phys. Rev. Lett. 61, 1446 (1988).
[3] V. Forlov and I. Novikov, Phys. Rev. D42, 1913 (1990).
[4] D. Hochberg and M. Visser, Phys Rev. D56, 4745 (1997).
[5] D. Hochberg and M. Visser, Phys Rev. D58, 044021 (1998).
[6] D. Ida and S.A. Heyward, Phys Lett A260, 175 (1999).
[7] J.L. Friedman, K. Schliech and D.M. Witt, Phys. Rev. Lett. 71, 1486 (1993).
[8] A. Agnese and M. La Camera, Phys. Rev. D51, 2011 (1995).
[9] L. A. Anchordoqui, et al. Phys. Rev. D55, 5226 (1997).
[10] L. A. Anchordoqui, Mod. Phys. Lett. A13, 1095 (1998).
[11] B. Bhawal and S. Kar, Phys. Rev. D46, 2464 (1992).
[12] D. Hochberg, Phys. Lett. B251, 349 (1990).
[13] T.A. Roman, Phys. Rev. D47, 1370 (1993). [gr-qc/9211012]
[14] D. Hochberg and T.W. Kephart, Phys. Rev. Lett. 70, 2665 (1993). [gr-qc/9211006]
[15] S. Kar, Phys Rev D49, 862 (1994).
[16] S. Kar and D. Sahdev, Phys. Rev. D53, 722 (1996).
[17] A. Wang and P. Letelier, Prog. Theor. Phys. 94, 137 (1995). [gr-qc/9506003]
[18] S. W. Kim, Phys. Rev. D53, 6889 (1996).
[19] D. Hochberg and M. Visser, Phys. Rev. Lett. 81, 746 (1998).
[20] R.F. Gonzalez-Diaz, [astro-ph/0507713].
[21] C. J. Gao and S. N. Zhang, Phys Lett B595, 28 (2004), [gr-qc/0407045].
[22] K. D. Kokkotas and B.G. Schmidt, ” Quasi-Normal modes of Stars and Black Holes”, Living Reviews in Relativity: [http://relativity.livingreviews.org/Articles/lrr-1999-2/index.html].