Critical behavior in topological ensembles

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We consider the relation between three physical problems: 2D directed lattice random walks in an external magnetic field, ensembles of torus knots and 5d Abelian SUSY gauge theory with massless hypermultiplet in $\Omega$ background. All these systems exhibit the critical behavior typical for the ”area+length” statistics of grand ensembles of 2D directed paths. In particular, using the combinatorial description, we have found the new critical behavior in the ensembles of the torus knots and in the instanton ensemble in 5d gauge theory. The relation with the integrable model is discussed.

Challenging questions appear often at the edges of traditional fields. As an example, the new branch of mathematical physics, the ”statistical topology” emerged recently by absorbing ideas from the statistical physics, theory of integrable systems, and algebraic topology (see, [1] for review). The scope of the statistical topology includes, on the one hand, mathematical problems involved in the construction of topological invariants of knots and links based on solvable models and, on the other hand, the physical and statistical problems related to summation over knot ensembles. In this work, we dwell predominantly to problems of the latter kind, demonstrating the emergence of a critical behavior in ensembles of torus knots.

Torus knots $T_{m,n}$ seem to be among the simplest objects in the knot theory. It is difficult to underestimate their role in different branches of mathematical physics. The topology of a torus knot is uniquely determined by the pair $(m,n)$, which fixes windings along two torus periods. In the Fig.1a an example of torus knot $T_{3,2}$ is given. Various explicit expressions for knot invariants are known, ranging from classical Jones-Kauffman polynomials [2, 3] to recently discovered superpolynomials [4]. New approaches to the torus knot invariants have been formulated recently in [5–7]. Less is known about properties of knot ensembles, where the particular topology of a knot diagram is considered as a topologically ”quenched” variable similar to the quenched disorder in statistical physics. The weighted summation over different torus knot types (i.e. different pairs $(m,n)$) means the consideration of the grand canonical ensemble (or, generating function) of torus knots.

In this Letter we consider the relation between three physical problems: i) two-dimensional directed lattice random walks in an external magnetic field, ii) ensembles of torus knots, and iii) the asymptotically free 5d Abelian SUSY gauge theory with a massless hypermultiplet. The reason for random walks to appear in the context of all these problems is as follows. The main tool for the evaluation of the torus knot invariants and Nekrasov partition function [20], is the so-called ”equivariant localization” which reduces the integral over the particular moduli space to the summation over the Young diagrams. The last problem can be reformulated as a weighted sums over the directed paths on a square lattice. On the other hand, we exploit the relation between the particular limits of a superpolynomial of $T_{n,n+1}$ torus knots, and equivariant integrals over the $n$-instanton moduli space for the 5d $U(1)$ SUSY gauge theory with one massless hypermultiplet. Collecting these different points of view, we interpret the generating function for ensembles of $T_{n,n+1}$ knots as the weighted sum in the 5d gauge theory over the instanton number with the fugacity, coinciding with the instanton action, and analyze the analytic structure of the corresponding generating function. Given an explicit expression for the free energy in the random walks problem, we use it to analyze the ensembles of knots and instantons. We show that at the ”gauge theory side” the derivative of the partition function with respect to the mass of the hypermultiplet, exhibits an unexpected critical behavior at some finite value of gauge coupling. In conclusion we speculate about some possible physical interpretation of established results.

In what follows we consider torus knot invariants expressed in terms of Dyck paths. Dyck path of length $2n$ on square lattice starts at the origin $(0,0)$, ends at point $(n,n)$ and consists of steps in upper and right direction, always staying above the diagonal of the square – see Fig.1b. The number of all the Dyck paths of length $2n$ is given by Catalan number, $C_n = \frac{1}{n+1} \binom{2n}{n}$. In what follows we sometimes refer to Dyck paths as to Brownian excursions (BE), having in mind an image of a charged particle on a lattice in external magnetic field, whose motion is subject to two restrictions: move only in upper and right directions and never intersect the diagonal. Calculating the action for such a particle, we can see that exponentiated external magnetic field is a fugacity for the area under the path, and the exponentiated mass is a fugacity for the length of the path. In this framework
it is natural to introduce a generating function for Dyck paths as a sum over \( n \).

To obtain torus knot invariants, we need two more statistics for Dyck paths, “corners” meaning number of points where path changes direction from up to right, and “bounces”, a definition for which can be found in [11, 12]. The polynomial invariant for \((n, n+1)\) torus knot is given by a partition function:

\[
Z_n(g, t, a) = \sum_{\pi_n \in \text{Dyck paths}} q^{\text{area}} t^{\text{corners}} a^{\text{bounces}}.
\]  

(1)

Figure 1: (a) Torus knot \(T_{2,3}\) (trefoil); (b) Sample of a Dyck path (Brownian excursion) with fixed area under a curve and fixed number of local peaks (up-corners). The partition function of such paths is counted by \( q \)-Narayana numbers.

Some limiting cases of (1) define known partition functions. Setting \( a = 1 \) in (1) we get the partition function of the “area+corner”–weighted Brownian excursions, known in enumerative combinatorics as \( q \)-Narayana numbers [9], \( N_n(q, t) \). They satisfy the functional recursion

\[
N_{n+1}(q, t) = (t-1)N_n(q, t) + \sum_{k=0}^{n} q^k N_n(q, t) N_{n-k}(q, t),
\]

(2)

which for \( N(s, q, t) = \sum_{n=0}^{\infty} s^n N_n(q, t) \), is turned into

\[
N(s, q, t) = 1 + (t-1)sN(s, q, t) + sN(s, q, t) N(sq, q, t).
\]

(3)

If the number of corners is not controlled (\( t = 1 \)), and only length and area are relevant, we arrive at the functional relation for \( q \)-Catalans [9, 10], \( C(s, q) = N(s, q, t = 1) \):

\[
C(s, q) = 1 + sC(s, q) C(qs, q), \quad C(s, q) = \frac{A_q(s)}{A_q(s/q)}.\]

(4)

where \( A_q(s) = \sum_{n=0}^{\infty} \frac{s^n (-s)^n}{(q^n)\Gamma(n)} \) is the \( q \)-Airy function and

\( (t; q)_n = \prod_{k=0}^{n-1} (1 - t q^k) \). The extension of \( q \)-Catalans to encompass “bounces” has been done in [11, 12], where it has been shown that the “bounce+area” statistic corresponds to the \((a, q)\)-Catalans. It should be pointed out that different authors use different notations for area–, corner–, and bounce–weighted statistics. It this paper we follow notations where \( q \) is attributed to the fugacity of the area, \( t \) - to the fugacity of corners, and \( a \) - to fugacity of bounces. Note the difference with respect to the notations in the papers [11, 12].

Let us describe how the critical behavior emerges at the BE side. In the works [13–15] it has been shown that in the double-scaling limit \( q \to 1^- \) and \( s \to \frac{1}{q}^- \) the function \( C(s, q) \) has the following form

\[
C(z) \sim C_{\text{reg}} + (1 - q)^{1/3} \frac{d}{dz} \ln \text{Ai}(4z); \quad z = \frac{\frac{3}{2} - s}{(1 - q)^{2/3}},
\]

(5)

where \( C_{\text{reg}} \) is the regular part at \( (q \to 1^- , s \to \frac{1}{q}^-) \) and

\[
\text{Ai}(z) = \frac{1}{\pi} \int_{0}^{\infty} \cos(\xi^3/3 + \xi z) d\xi \text{ is the Airy function.}
\]

The function \( C(s, 1) \) is the generating function for undeformed Catalan numbers:

\[
C(s, 1) = \frac{1 - \sqrt{1 - 4s}}{2s}.
\]

It makes sense in the area \( s < \frac{1}{4} \), and at the point \( s = \frac{1}{4} \) the first derivative of the generating function experiences a singularity which we interpret as critical behavior. The limit \( q = 1, s \to \frac{1}{q}^- \) can be read also from the asymptotic expression for \( C(s, q) \) (5):

\[
C(s, q) \bigg|_{q \to 1^-} \sim C_{\text{reg}} - 2 \sqrt{1 - 4s}.
\]

Note that the first non-singular term in (7) does not contain \( q \), so it is no matter in which order the limit in (5) is taken. However to define the double scaling behavior and derive the Airy-type asymptotic, the simultaneous scaling in \( s \) and \( q \) is required.

The same holds for Narayana numbers, counting Dyck paths with fixed fugacity of corners, \( t \): the square-root singularity of the generating function,

\[
N(s, t) = \frac{1 - (1-t)s - \sqrt{(1-s+st)^2 - 4st}}{2s},
\]

(8)

at \( t = 1 \) coincides with the one of Catalans (7). Whether the double scaling near the critical line \( (1-s+st)^2 - 4st = 0 \) for \( q \)-Narayanas with the Airy-type asymptotic (5) exists, is still an open question which deserves further investigation.

The discussion of the “area+length”–weighted scaling would be far from complete without mentioning that the asymptotic (5) describes the scaling of top line in a bunch of directed vicious walks. Proceeding as in [16], take the ensemble of \( N \) vicious walkers, define the averaged position of the top line and consider its fluctuations near the averaged position. In such a description
all vicious walkers lying below the top line play a role of a "mean field", which pushes the top line to some "atypical" equilibrium position, around which it fluctuates. It is naturally to suppose that the fluctuations of the top line in a mean-field approximation have the same scaling as the fluctuations of the "inflated" Brownian excursion with fixed area under the path. One actually can show that, following the line of reasoning of the work [17]. The solution of the inviscid Burgers equation \( \partial_t u_0(x, t) + u_0(x, t) \partial_x u_0(x, t) = 0 \) is \( u_0(z, t) = \frac{x}{\sqrt{2\pi N}} \) and gives the Wigner semicircle law centered at the point \( \frac{z}{\sqrt{N}} \). One can smear the function \( u_0(x, t) \) near the boundary value, \( x = x_c = \pm 2\sqrt{N} \), adding the fluctuations, i.e. passing to the Burgers equation with a weak diffusivity (0 < \( \nu < 1 \)): \( \partial_t u(x, t) + u(x, t) \partial_x u(x, t) = -\nu \partial_{xx} u(x, t) \).

Seeking for weakly fluctuating solutions of viscous Burgers equation near the top line, \( \langle t = N \rangle \), in the form [17] \( x = x_c + \nu^{\beta} y = 2t^{1/2} + \nu^{\beta} y, \) \( u(x, t) = \frac{x}{\sqrt{2}} + \nu^{\beta} w(s, t) = t^{-1/2} + \nu^{\beta} w(y, t) \) and substituting the ansatz for \( u(x, t) \) into the viscous Burgers equation, one gets the equation for \( w(y, t) \), which for \( \alpha = 2/3, \beta = 1/3 \) and appropriate boundary conditions is transformed in the limit \( \nu \to 0 \) into the dimensionless Ricatti equation [17]

\[
-y \partial_{yy} w - \nu \partial_y w = 0, \quad (9)
\]

having the solution (for \( t = N \))

\[
w(z) = 2 \frac{d}{dz} \ln \text{Ai}(2^{-1/3} z) \quad z = y N^{-1/2}.
\]  

In (10) one can recognize the singular part of the grand partition function of “area + length”-weighted Brownian excursion. To make this connection precise, define the partition function \( Z_n(A) \) of \( n \)-step directed 2d random walk in the upper half-plane of the square lattice (i.e. the Brownian excursion) with the fixed area, \( A \). Thus, one can straightforwardly identify \( Z(s, \nu) \) with \( u(y, N) = N^{-1/2} + \nu^{1/3} w(y, N) \) under the following redefinitions:

\[
\begin{align*}
1 - q & \leftrightarrow \nu, \\
\frac{1}{\nu} - s & \leftrightarrow -2^{-7/3} y N^{1/6}.
\end{align*}
\]

(11)

Turn now to torus knots \( T_{n,m} \) and introduce the super-polynomial [18], counting the Poincaré complex of the corresponding triple Khovanov homologies \( H_{ijk} \). The superpolynomial is a generalization of the HOMFLY polynomial of the knot and depends on three variables

\[
P_{n,m}(\tilde{\alpha}, \tilde{q}, \tilde{t}) = \sum_{ijk} \tilde{\alpha}^i \tilde{q}^j \tilde{t}^k \dim H_{ijk}.
\]

(12)

At \( \tilde{t} = -1 \) the superpolynomial \( P_{n,m}(\tilde{\alpha}, \tilde{q}, \tilde{t}) \) reduces to the HOMFLY polynomial. Besides, it can be interpreted as the generating function for the multiplicities of the particular sector of BPS states in the SUSY gauge theories [18]. We shall be interested in the critical behavior of the “area+length” type in the ensemble of torus knots and focus our attention at the particular case of \( T_{n,n+1} \) knots. Consider the generating function for superpolynomials introducing fugacity, \( s \), conjugated to the index \( n \), which controls the length of the Dyck path:

\[
Z(s, \tilde{\alpha}, \tilde{q}, \tilde{t}) = \sum_{n=0}^{\infty} P_{n,n+1}(\tilde{\alpha}, \tilde{q}, \tilde{t}) s^n. \quad (13)
\]

Hopefully there is an explicit expression for the superpolynomial of the \( T_{n,n+1} \) torus knots obtained in two different ways. The first way deals with the combinatorics of the Young diagrams and can be related to the BE approach [19]. The second approach has been found in [4] via localization on the fixed points of the torus action on the moduli space of \( n \) points in \( C^2 \). This approach will be used here to compare the superpolynomial generating function of torus knots with the instanton calculations in the particular 5d gauge theory.

The superpolynomial for \( (n, n + 1) \) knots expressed in terms of the Dyck paths reads as follows [19] (the definition of statistics and notations can be found there):

\[
P_{n,n+1}(\tilde{\alpha}, \tilde{q}, \tilde{t}) = \sum_{D} \tilde{q}^{\text{dinv}(D)} (\tilde{q}^{2} - |D|) \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} \sum_{\tilde{\alpha}} \sum_{\tilde{t}} (1 + \tilde{\alpha}^2 \tilde{q}^{2\beta} (P_{n})^2). \quad (14)
\]

where the area, bounce and corner statistics in the lattice paths are involved. Note however, that the fugacities usually used in the knot context and in the present work, are slightly different, hence we use here the “tilde” notation. To recover the known critical behavior for the generating function of superpolynomials, we have to consider the reduction of paths statistics to the “length+area” one. To this aim, let us first remind how the \( (q, t) \)-Catalans enter the game. Consider the lowest row in the expansion in a variable of the expression for the superpolynomial (14)

\[
P_{n,n+1}(\tilde{\alpha}, \tilde{q}, \tilde{t}) = \sum_{k} \tilde{\alpha}^k P_{n,n+1}(\tilde{q}, \tilde{t}). \quad (15)
\]

It turns out that the corresponding term at \( \tilde{\alpha} = 0 \) in the expansion coincides with the \( (q, t) \)-Catalan number:

\[
P_{n,n+1}^0(\tilde{q}, \tilde{t}) = C_n(\tilde{q}, \tilde{t}) = \sum_{D} \tilde{q}^{\text{dinv}(D)} \tilde{q}^{2} - |D|. \quad (16)
\]

Taking into account that \( C_n(\tilde{q}, \tilde{t}) = C_n(\tilde{t}, \tilde{q}) \), we can take \( \tilde{q} = 1 \) (or \( \tilde{t} = 1 \)) and obtain the sum over the paths controlled by the area only. The critical behavior of (16) occurs at \( \tilde{t} = 1, \tilde{q} \to 1^- \), \( s \to \frac{1}{\sqrt{2}} \) and should be compared with the scaling of the “area + length” counterpart in (5). Thus, we see that there is a critical behavior in the ensemble of torus knots of type \( T_{n,n+1} \) in the double scaling limit with respect to the generating parameters.

Let us make a short comment on the \( a \neq 0 \) case of a general superpolynomial. The nonvanishing \( \tilde{a} \) accounts for corners in the BE approach and some subtle
integrands in the localization approach [4]. It adds to the integrand over the moduli space the following sum \[ \sum_k (-\widetilde{a})^k \Lambda^k V \] which should be compared with the analogous sum in the instanton ensemble in 5d gauge theory.

Consider now the Nekrasov-like partition function [20] in the Abelian 5d SUSY gauge theory with the single massless hypermultiplet \( Q, \tilde{Q} \) in the \( \Omega \)-background. The coefficient in front of the Chern-Simons term is fixed \( k = 1 \) and the coupling constant in 5d theory is dimensionful, the fifth coordinate is compact. The Nekrasov partition function is trivial in this theory however we shall be interested in the vacuum matrix element \( \langle O \rangle \) of the particular operator \( O \). The calculation involves the weighted sum of the integrals over the instanton moduli where the parameters of the \( \Omega \)-background \( \epsilon_1, \epsilon_2 \) serve as the equivariant parameters for the two torus actions for the integration over the moduli space \( M_n \) of \( n \) point-like instantons located all at the origin. The instanton number is weighted with the gauge coupling \( s = e^{2\pi \tau} \), where \( \tau = 4\pi \beta g^{-2} \) and \( \beta \) is the radius of the compact fifth dimension. The partition function is evaluated as the sum over the Young diagrams, while the mass of the hypermultiplet provides the fugacity for the “corners”.

The desired operator \( O \) can be identified as follows [42]. First we have to relate deformed Catalan numbers \( C_n(q, t) \) with \( \langle O \rangle \) in the \( n \)-instanton sector. To this aim we use the important result [11]

\[
\chi^T(\text{Hilb}^n(\mathbb{C}^2, 0), V \otimes \Lambda^n V) = C_n(q, t),
\]

which interprets the \( (q, t) \)-deformed Catalans as equivariant integrals over the moduli space of \( n \)-Abelian instantons valued in the \( n \)-th power of the \( n \)-dimensional tautological bundle \( V \).

Looking at the representation of the \( (q, t) \)-Catalans in terms of the path on the Young tableau, the desired operator is identified in the \( n \)-instanton sector

\[
\langle O \rangle_n = \frac{1}{n!} \left\langle \hat{Q} \hat{Q} \left( 1 - (1 - q)(1 - t) \text{Tr} \Phi \right) \right\rangle_n,
\]

where \( \hat{Q}, \hat{Q} \) is the hypermultiplet, \( \text{Tr} \) substitutes the integral over the \( \mathbb{C}^2 \) in the \( \Omega \)-background and \( \Phi \) is the ”long scalar” in the \( \Omega \)-background. Hence the composite operator is the product of local and nonlocal 4-observables.

The parameter \( s \) counting the instanton numbers is identified with the bare gauge coupling since we consider the instanton sector only. Given the interpretation of the vacuum expectation value (vev) of the operator in the \( n \)-instanton sector we consider the weighted sum instantons to get the full nonperturbing contribution. It can be represented as a derivative with respect to the mass of the vev of ”generalized Wilson loop”:

\[
\frac{d}{dm} \langle (1 - (1 - q)(1 - t) \text{Tr} \Phi) \rangle \bigg|_{m=0} = \sum_n s^n C_n(q, t).
\]

The parameters \( (q, t) \) are related to the equivariant parameters of background as follows: \( q = e^{\epsilon_1}, \ t = e^{-\epsilon_2} \). Therefore in the Nekrasov-Shatashvili limit, the critical behavior for the \( q \)-Catalans corresponds to the following values of parameters in the 5d gauge theory

\[
\epsilon_1 = 0, \ \epsilon_2 \beta \to 0, \ \ln s = -8\pi^2 \beta g^{-2} = 4\pi \log 2.
\]

Keeping mass finite, we obtain the following sum in the integrand:

\[
\sum_k (-m)^{n-k+1} \Lambda^k V,
\]

which has to be compared with the similar sum in the superpolynomial.

This comparison implies that the mass of the hypermultiplet multiplied by \( \beta \) seems to be identical to the variable \( a \) in the superpolynomial and counts the ”corners” in the BE setup. This observation suggests the following conjecture relating the generating function for the superpolynomials and vev of nonlocal operator at arbitrary masses,

\[
Z(s, a, t, q) \propto \frac{d}{dm} \langle (1 - (1 - q)(1 - t) \text{Tr} \Phi) \rangle \langle \tau, m, \epsilon_1, \epsilon_2 \rangle,
\]

and the generating function for Narayana’s suggests an interesting critical behavior along the line in the (coupling constant, mass) plane at \( q = t = 1 \). We plan to discuss this point elsewhere.

Let us comment on the possible interpretation of the critical behavior. In the 5d gauge theory the partition function effectively counts the BPS instanton particles, hence one could question on the possible wall-crossing phenomena for the BPS multi-instanton bound states. If the wall-crossing interpretation is true, some stable bound composite state decays at this point and we have a wall-crossing phenomena. It should be emphasized that the ”wall-crossing” phenomena happens not for the partition function or spectrum, but for the derivative with respect to the flavor fugacity of the matrix element of nonlocal operator. That is, the ”wall-crossing” for the ”generalized Wilson loop” or bilinear condensate could take place. Note the possible relation with the discussion in [21] when the RG flow over the flavor fugacity implies the appearance of the additional surface operator. This situation also seems to have some similarities with the derivation of the chiral condensate in QCD via Casher-Banks relation [22].

The physics of the critical behavior has to be recognized at the knot side and the BE side as well. At the knot side large \( n \) knots dominate the generating function above phase transition and are suppressed below it. On the BE side we expect that the peculiar behavior of the random walks at the critical point yields the phenomena similar to the tricritical phase transition of Douglas-Kazakov type [23] characterized by the singularity (5). Note that the phase transition of Douglas-Kazakov type has been observed in [41].

At the end, let us mention that the scaling function \( C(z) \) which appeared many times throughout the text,
plays also a very important role, connecting BE to the integrable systems. It is known (see, for example, [24–26]) that \( w(z) \), defined in (10), is itself the generating function:

\[
w(z) \bigg|_{z = \infty} \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \Omega_k z^{-(3k-1)/2},
\]

where the coefficients \( \Omega_k \) have well-defined physical sense: representing \( \Omega_k \) in the form \( \Omega_k = \frac{2(3k-1)/2}{2k} \Gamma((3k-1)/2)(B^k) \), one can show [26, 27] that \( \langle B^k \rangle \) is the \( k \)-th moment of the area under the Brownian excursion on the unit interval. Defining \( K_k = \frac{\Omega_k}{2^{k+1}k!} \) one can show that \( K_k \) satisfy the recursion

\[
K_n = \frac{3n-4}{4} K_{n-1} + \sum_{k=1}^{n-1} K_k K_{n-k}; \quad K_0 = -\frac{1}{2}.
\]

On the other hand, the function \( w(x) \) appears it the theory of integrable systems as the solution of the rational Painlevé II equation (\( w(x) \equiv u(x) \)) at \( \alpha = 0 \):

\[
u''(x) = 2\nu^3(x) + 4\nu u(x) + 4 \left( \alpha + \frac{1}{2} \right). \tag{24}
\]

The connection of area-weighted generating function \( w(x) \) with the rational solutions of Painlevé II is not restricted exclusively by (24) and can be pushed for any \( \alpha = N + \frac{1}{2} \), where \( N = 0, 1, 2, \ldots \). Take into account that the rational solutions of (24) can be written (see [28, 29]) as \( u(x) = -\ln \frac{2\sigma_N(x)}{\sigma_N} \), where \( \sigma_N \equiv \sigma_N(x) \) is the \( \tau \)-function of the Toda system ([30]), written as a Hankel determinant

\[
\sigma_N = \det \left( \begin{array}{cccc}
a_0 & a_1 & \cdots & a_{N-1} \\
a_1 & a_2 & \cdots & a_N \\
\vdots & \vdots & \ddots & \vdots \\
a_{N-1} & a_N & \cdots & a_{2N-2} \end{array} \right). \tag{25}
\]

and the entries \( a_n \equiv a_n(x) \) satisfy the recursion [29]

\[
a_n = 2(n-2)a_{n-3} + \sum_{k=0}^{n-2} a_k a_{n-k-2}, \tag{26}
\]

with \( a_0 = x, \quad a_1 = 1, \quad a_2 = x^2 \). The associated generating function,

\[
G(x, s) = \sum_{j=0}^{\infty} a_j(x)(-2s)^{-j}, \tag{27}
\]

satisfies the Riccati equation [29] (compare to (9))

\[
-2G + G^2 + \partial_s G - (4s^2 + s^{-1})G + 4xs^2 = 0, \tag{28}
\]

whose solution is

\[
G(x, s) = 2s^2 + \frac{d}{ds} \ln Ai(s^2 - x). \tag{29}
\]

The equation (26) at large \( n \) resembles (though being different in details) the recursion (23) for the function \( K_n \). The connection between (23) and (26) can be set by comparing (22) and (27). Finally, we get

\[
a_j = (-2)^j \sum_{k=0}^{\infty} \frac{(-1)^k \Omega_k}{2k+1} \sum_{j=0}^{\infty} \left( \frac{1}{m} \right)^j (-x)^m \delta_{3k+2m+1,j}, \tag{30}
\]

where \( \delta_{ij} \) is the Kronecker \( \delta \)-function. Thus, we explicitly see how the linear combinations of moments of area-weighted Brownian excursions, \( \Omega_k \), are connected to the coefficients \( a_j \) in the expansion of the Toda \( \tau \)-function. Let us also point out the striking similarity of the recursion equation (23) for different momenta of area-weighted Dyck paths with the summation over genus \( g \) the partition function of the 6\( U(N)_k \times U(N)_{-k} \) Chern-Simons matter theory, also known as the ABJM theory [31]. We plan to discuss this question in details in the forthcoming paper [32].

In this Letter using the realization via the random walks we have found the critical behavior of the ensembles of the torus knots and instantons at the particular values of parameters. It is worth reminding that appearance of the singularity of type (5) is the manifestation of the third-order phase transition. In the seminal paper [33] it has been shown that the largest eigenvalue, \( \lambda_n \), of the Gaussian \( n \times n \) random matrix ensemble, converges at \( n \to \infty \) to \( \lambda_n \to 2\sqrt{n} + n^{1/6}\chi \), where the random variable \( \chi \) has a limiting \( n \)-independent distribution, \( \text{Prob}(\chi \leq x) = F_{\text{GUE}}(x) \), being the so-called Tracy-Widom distribution for GUE ensemble [34]. So, the normalized value \( \Lambda_n = \lambda_n/\sqrt{n} \) at large (but finite) \( n \) has an uncertainty (i.e. the width of the distribution) of order of \( n^{-1/3} \), typical for the 3rd order phase transitions. Above and below the critical value \( \Lambda_{\infty} = \lim \Lambda_n = 2 \), the tails of the distribution \( P(\Lambda) \) have different asymptotics, signifying existence of strong (for \( \Lambda < \Lambda_{\infty} \)) and weak (for \( \Lambda > \Lambda_{\infty} \)) couplings.

It would be interesting to extend the analysis to the whole ensemble of \( T_{n,m} \) torus knots and the whole set of fugacities. It would be important to recognize the counterparts of the "bounces" and "corners" in the particle path integral in continuum and obtain the interpretation of the external magnetic field in the BE approach as a kind of the Berry curvature from a 4d viewpoint. It would be also very interesting to extend our consideration and to relate the spectrum of Hofstadter model at BE side with the knot invariants and 4d instantons. We suppose to discuss this issue elsewhere [32]. It seems also very important to recognize the critical behavior observed in this Letter in the ensembles of branes [35], Hopfions [36] and "ensemble" of 3d theories classified by the torus knots [37].

Another interesting question concerns the direct relation between the limits of the superpolynomials and the correlator in the 5d theory discussed above. This seems to
be along the developments in [38]. However in our case the gauge coupling plays an essential role and it counts different torus knots. Let us emphasize that the knots live in the different internal space not in the space-time where the 5d gauge theory is defined. To some extend the situation reminds the classification of the phases in the solid state physics. The phases are classified by the invariants of the Berry connection in the momentum space like the Chern classes. Our situation to some extend generalizes this approach. The condensate which is the order parameter as well is interpreted in terms of the invariants of knots in the internal "momentum" space which presumably correspond to the Wilson loops of the Berry connection.

Let us remark that to obtain the critical behavior for the vev in 5d theory the Ω-background seems to be switched off and the pure 5d gauge theory can be considered. We hope to discuss the physics of the critical behavior taking into account the mass dependence else-

where. One more point deserving study is the following. Besides the usual instantons there are dyonic instantons in the nonabelian 5d gauge theory with two quantum numbers. In our Abelian case we expect the similar solutions due to the additional fundamental matter. The mapping of such dyonic instantons to the knot invariants seems to be a very interesting issue.

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[1] S. Nechaev Statistics of knots and entangled random walks (WSPC: Singapore, 1996), arXiv:cond-mat/ 9812205
[2] L.Kauffman and H. Saleur, Comm.Math.Phys. 141(1991)293
[3] M.Russo and V. Jones, Knot Theory, 2 (1993) 97
[4] E. Gorsky and A. Negut, Refined knot invariants and Hilbert schemes, arXiv:1304.3328 [math.RT]
[5] M. Aganagic and S. Shakirov, Refined Chern-Simons Theory and Topological String, [arXiv:1210.2733 [hep-th]]
[6] H. Ooguri and C. Vafa, “Knot invariants and topological strings,” Nucl. Phys. B 577, 419 (2000) [hep-th/ 9912123].
[7] A. Brini, B. Eynard, and M. Marino, Torus knots and mirror symmetry, Annales Henri Poincaré 13, 1873 (2012) [arXiv:1105.2012 [hep-th]]
[8] L. Carlitz and J. Riordan, Two element lattice permutation numbers and their q-generalization, Duke J. Math. 31 (1964), 371-388; J. Fürlinger and J. Hofbauer, q-Catalan numbers, J. Comb. Th. A 40 (1985), 248-264
[9] J. Cigler, q-Catalan numbers and q-Narayana polynomials, arXiv:math/0507225 [math.CO]
[10] A L Owczarek, and T. Prellberg, Pressure exerted by a vesicle on a surface, J. Phys. A: Math. Theor. 47 (2014) 215001; arXiv:1311.2174
[11] M. Haiman, t, q-Catalan numbers and the Hilbert scheme Discrete Math. 193 (1998), 201-224.
[12] J. Haglund, Conjectured statistics for the q,t-Catalan numbers, Adv. Math. 175 (2003) 319
[13] T. Prellberg, and R. Brak. Critical exponents from nonlinear functional equations for partially directed cluster models, J. Stat. Phys., 78 (1995) 701-730
[14] C. Richard, A. J. Guttmann, and I. Jensen, Scaling function and universal amplitude combinations for self-avoiding polygons, J.Phys. A: Math. Gen. 34 (2001) L495-L501
[15] C. Richard, Scaling Behaviour of Two-Dimensional Polygon Models, J. Stat. Phys., 108 (2002) 459-493
[16] P.L. Ferrari, M. Prahofer, and H. Spohn, Stochastic Growth in One Dimension and Gaussian Multi-Matrix Models, In proceedings of the 14th International Congress on Mathematical Physics (ICMP 2003), World Scientific (Ed. J.-C. Zambrini) (2006), 404-411, arXiv: math-ph/0310053
[17] J.-P. Blaizot, M.A. Nowak, Large N confinement, universal shocks and random matrices, Lectures given at the 49 Cracow School of Theoretical Physics, May 31 - June 10 (2009) Zakopane, Poland, arXiv:0911.3683
[18] N.M. Dunfield, S. Gukov and J. Rasmussen, The Superpolynomial for knot homologies, math/0505662 [math.GT]
[19] A.Oblomkov, J, Rasmussen and V. Shende with appendix of E. Gorsky, The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link, arxiv 1201.2115
[20] N.A. Nekrasov, Seiberg-Witten prepotential from instanton counting, Adv. Theor. Math. Phys. 7, 831 (2004) [hep-th/0206161].
[21] D. Gaiotto, L. Rastelli, and S.S. Razamat, Bootstrapping the superconformal index with surface defects, JHEP 1301, 022 (2013) [arXiv:1207.3577 [hep-th]]
[22] T. Banks and A. Casher, Chiral symmetry breaking in confining theories, Nuclear Physics B 169, (1980) 103-125
[23] M.R. Douglas and V.A. Kazakov, Large N phase transition in continuum QCD2, Physics Letters B, 319 (1993) 219-230
[24] P. Flajolet, and G. Louchard, Analytic variations on the Airy Distribution, Algorithmica, 31 (2001) 361-377
[25] C. Richard, Area distribution of the planar random loop boundary, J. Phys. A: Math. Gen., 37 (2004) 4493
[26] S. Janson, Brownian excursion area, Wright constants in graph enumeration, and other Brownian areas, Probability Surveys, 4 (2007) 80-145
[27] M.J. Kearney, and S.N. Majumdar, On the area under a continuous time Brownian motion till its first-passage
time, J. Phys. A: Math. Gen. 38 (2005) 4097
[28] K. Kajiwara, and Y. Ohta, Determinant structure of the rational solutions for the Painlevé II equation, J. Math. Phys., 37 (1996) 4693
[29] K. Iwasaki, K. Kajiwara, and T. Nakamura, Generating function associated with the rational solutions of the Painlevé II equation, J. Phys. A: Math. Gen., 35 (2002) L207-L211
[30] K. Kajiwara, M. Mazzocco, and Y. Ohta, A remark on the Hankel determinant formula for solutions of the Toda equation J. Phys. A: Math. Theor., 40 (2007) 12661
[31] H. Fujia, S. Hirano, and S. Moriyama, Summing Up All Genus Free Energy of ABJM Matrix Model, arXiv:1106.4631 [hep-th]
[32] K. Bulycheva, A. Gorsky, S. Nechaev, in preparation
[33] J. Baik, P. Deift, and K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations J. Amer. Math. Soc. 12, 1119 (1999)
[34] C.A. Tracy and H. Widom, Comm. Math. Phys. Level-spacing distributions and the Airy kernel 159, 151 (1994)
[35] K. Bulycheva and A. Gorsky, BPS states in the Omega-background and torus knots, JHEP 1404, 164 (2014) [arXiv:1310.7361 [hep-th]]
[36] M. Kobayashi and M. Nitta, Torus knots as Hopfions, Phys. Lett. B 728, 314 (2014) [arXiv:1304.6021 [hep-th]]
[37] H.-J. Chung, T. Dimofte, S. Gukov and P. Sulkowski, 3d − 3d Correspondence Revisited, [arXiv:1405.3663 [hep-th]]
[38] E. Witten, Fivebranes and Knots, arXiv:1101.3216 [hep-th]; D. Gaiotto and E. Witten, Knot Invariants from Four-Dimensional Gauge Theory, Adv. Theor. Math. Phys. 16, no. 3, 935 (2012) [arXiv:1106.4789 [hep-th]].
[39] E. Gorsky, arxiv 1013.0916, Zeta Functions in Algebra and Geometry, 213-232. Contemp. Math. 566
[40] N. Nekrasov and A. Okounkov, Seiberg-Witten theory and random partitions, hep-th/0306238; L. Baulieu, A. Losev, and N. Nekrasov, Chern-Simons and twisted supersymmetry in various dimensions, Nucl. Phys. B 522, 82 (1998) [hep-th/9707174]
[41] A. Marshakov and N. Nekrasov, “Extended Seiberg-Witten Theory and Integrable Hierarchy,” JHEP 0701, 104 (2007) [hep-th/0612019].
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