Orbital and asymptotic stability for standing waves of a NLS equation with concentrated nonlinearity in dimension three

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Abstract

We begin to study in this paper orbital and asymptotic stability of standing waves for a model of Schrödinger equation with concentrated nonlinearity in dimension three. The nonlinearity is obtained considering a point (or contact) interaction with strength $\alpha$, which consists of a singular perturbation of the Laplacian described by a selfadjoint operator $H_\alpha$, and letting the strength $\alpha$ depend on the wavefunction: $i\dot{u} = H_\alpha u$, $\alpha = \alpha(u)$. It is well-known that the elements of the domain of such operator can be written as the sum of a regular function and a function that exhibits a singularity proportional to $|x - x_0|^{-1}$, where $x_0$ is the location of the point interaction. If $q$ is the so-called charge of the domain element $u$, i.e. the coefficient of its singular part, then, in order to introduce a nonlinearity, we let the strength $\alpha$ depend on $u$ according to the law $\alpha = -\nu|q|^\sigma$, with $\nu > 0$. This characterizes the model as a focusing NLS with concentrated nonlinearity of power type. For such a model we prove the existence of standing waves of the form $u(t) = e^{i\omega t}\Phi_\omega$, which are orbitally stable in the range $\sigma \in (0, 1)$, and orbitally unstable when $\sigma \geq 1$. Moreover, we show that for $\sigma \in (0, \frac{1}{\sqrt{2}})$ every standing wave is asymptotically stable in the following sense. Choosing initial data close to the stationary state in the energy norm, and belonging to a natural weighted $L^p$ space which allows dispersive estimates, the following resolution holds: $u(t) = e^{i\omega \infty t}\Phi_\omega \infty + U_\infty * \psi_\infty + r_\infty$, where $U$ is the free Schrödinger propagator, $\omega_\infty > 0$ and $\psi_\infty$, $r_\infty \in L^2(\mathbb{R}^3)$ with $\|r_\infty\|_{L^2} = O(t^{-5/4})$ as $t \to +\infty$. Notice that in the present model the admitted nonlinearity for which asymptotic stability of solitons is proved is subcritical, in the sense that it does not give rise to blow up, regardless of the chosen initial data.

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I. INTRODUCTION

In this paper we begin a systematic analysis of the stability of solitary waves for a nonlinear Schrödinger equation with a nonlinearity concentrated in space dimension three. In particular, we show that the standing waves of the model are asymptotically stable in the sense that the evolution of the system in a neighbourhood of a standing solution admits a soliton resolution expansion: at large times, the evolution decomposes as the sum of a standing wave (possibly with different parameters from those of the reference initial soliton), a free linear wave, and a small remainder with a spatial decay stronger than the linear dispersive one.

An analogous study concerning the NLS equation with a concentrated nonlinearity in dimension one was given in [4] and [22]. These papers have been a source of inspiration for the present work, in particular for what concerns the general scheme of analysis and for some proofs. However, the one and the three-dimensional models are different, in particular the latter is strongly singular and its energy space is not contained in $H^1(\mathbb{R}^3)$. This fact prevents us from following step by step the techniques and the results of the cited papers; in particular, no formal manipulations with delta distributions are possible, and the full definition of a delta interaction as a point perturbation of the Laplacian is needed in the analysis. We shall comment on that along the paper.

We start by giving a presentation of the model. According to [1], we construct a Schrödinger equation with concentrated nonlinearities in dimension three by starting from the standard three-dimensional linear Schrödinger operator with a so-called point or delta interaction ([3]). Point interactions are widely used in Quantum Mechanics as models of contact or zero-range interactions and they are intended to describe strongly concentrated potentials at a point. In order to rigorously define a delta interaction located at the origin of $\mathbb{R}^3$ we first consider the Laplacian restricted to the set $C^\infty_0(\mathbb{R}^3 \setminus \{0\})$ and obtain a symmetric non selfadjoint operator with deficiency indices $(1, 1)$. Second, by the classical Von Neumann-Krejn theory there exists a one-parameter family of selfadjoint extensions, which we denote by $H_\alpha$. The operator $H_\alpha$ is defined on the domain

$$D(H_\alpha) = \{ u \in L^2(\mathbb{R}^3) : u(x) = \phi(x) + qG_0(x) \text{ with } \phi \in L^2_{\text{loc}}(\mathbb{R}^3), \nabla \phi \in L^2(\mathbb{R}^3), \Delta \phi \in L^2(\mathbb{R}^3), q \in \mathbb{C}, \lim_{x \to 0} (u(x) - qG_0(x)) = \alpha q \},$$

where $G_0$ is the Green’s function of the Laplacian in three dimensions, i.e.

$$G_0(x) = \frac{1}{4\pi |x|^3}.$$  

(1)
and the action is given by $H_\alpha u(x) = -\Delta \phi(x)$, $x \in \mathbb{R}^3$. To summarize, any element of the domain decomposes in a regular part $\phi$ and a singular (Coulombian) part; the coefficient $q$ of the singular part is conventionally called charge, and the boundary condition imposes a relation between the charge and the value of the regular part at the origin depending on the so-called strength $\alpha$ of the point interaction, which is the parameter that fixes the selfadjoint extension.

An alternative equivalent and perhaps more direct construction, which better justifies the interpretation and the physical meaning of $H_\alpha$, can be given by defining $H_\alpha$ as a suitable scaling limit (in norm resolvent sense) of a family of Schrödinger operators of the form $-\Delta + V_\epsilon$, where $V_\epsilon$ is a short range potential that approximates a delta distribution as $\epsilon \to 0$. A closer analysis of the above scaling procedure shows that the point interaction cannot be interpreted as a kind of "laplacian plus delta distribution", differently from the one dimensional case; moreover the parameter $\alpha$ appearing in the above definition and characterizing the particular selfadjoint extension is related to zero energy resonances of the approximating operators. For details and further information see [3].

Whatever the definition given to the operator $H_\alpha$ is, we recall that, for $\alpha \geq 0$ (repulsive delta interaction), $H_\alpha$ is positive and its spectrum is purely absolutely continuous and coincides with $[0, +\infty)$, while for $\alpha < 0$ (attractive delta interaction) an isolated simple negative eigenvalue $\lambda = -(4\pi\alpha)^2$ appears, corresponding to a bound state. A second property relevant to the physical interpretation of the model and related to the value of $\alpha$ is that the scattering length of a delta interaction of strength $\alpha$ is given by $-(4\pi\alpha)^{-1}$. The closed and lower bounded quadratic form associated to $H_\alpha$ is

$$H_\alpha(u) = \int_{\mathbb{R}^3} |\nabla \phi(x)|^2 dx + \alpha |q|^2,$$

defined on the domain of finite energy states

$$V = \{ u \in L^2(\mathbb{R}^3) : u(x) = \phi(x) + qG_0(x), \text{ with } \phi \in L^2_{\text{loc}}(\mathbb{R}^3), \nabla \phi \in L^2(\mathbb{R}^3), q \in \mathbb{C} \},$$

which is a Hilbert space endowed with the norm

$$\|u\|^2_V = \|\nabla \phi\|_{L^2} + |q|^2.$$

Note that for a generic element $u$ of the form domain the charge $q$ and its regular part $\phi$ are independent of each other. Note also that the energy domain is strictly larger than $H^1(\mathbb{R}^3)$. So, the linear problem cannot be considered as a small perturbation of the standard free problem in the sense
of the quadratic forms (at variance with the one-dimensional case). An equivalent representation of
the energy space is obtained, fixed \( \lambda > 0 \), by
\[
V = \left\{ u = \phi_\lambda + qG_\lambda, \text{ with } \phi_\lambda \in H^1(\mathbb{R}^3), \ q \in \mathbb{C}, \ G_\lambda(x) = \frac{e^{-\lambda|x|}}{4\pi|x|} \right\},
\]
(6)
and one can define an equivalent energy norm by
\[
\|u\|_V^2 = \|\nabla \phi_\lambda\|_{L^2}^2 + |q|^2, \quad \forall u \in V.
\]
Notice that \( G_\lambda \in L^2(\mathbb{R}^3) \) and \( \phi_\lambda \in H^1(\mathbb{R}^3) \), while in the representation \( \text{(4)} \) the regular part belongs
to the homogeneous Sobolev space \( D^1(\mathbb{R}^3) \) and \( G_0 \notin L^2(\mathbb{R}^3) \).
Following \( \text{(1)} \), the nonlinear model can be defined by allowing the strength \( \alpha \) to depend on \( u \) as
\[
\alpha(u) = -\nu|q|^{2\sigma}, \quad \text{with } \nu > 0, \sigma > 0,
\]
so that
\[
D(H_{\alpha(u)}) = \{ u \in L^2(\mathbb{R}^3) : u(x) = \phi(x) + qG_0(x) \text{ with } \phi \in H^2_{\text{loc}}(\mathbb{R}^3), \ \Delta \phi \in L^2(\mathbb{R}^3), \ q \in \mathbb{C}, \lim_{x \to 0} (u(x) - qG_0(x)) = -\nu|q|^{2\sigma} q \},
\]
and \( H_{\alpha(u)}u = -\Delta \phi \). In the following sections, we often omit the notation \( H_{\alpha(u)} \) in favour of \( H_\alpha \) if no
risk of confusion exists between the linear and the nonlinear operator. We stress that the nonlinearity
we are considering is focusing. It can be interpreted as modeling the action of a defect in a medium
which exerts a nonlinear response to the propagation. We remark that a more general definition of
concentrated nonlinearities (with applications to the case of the wave equation) is given in \( \text{(11)} \).

We consider the evolution generated by the nonlinear operator \( H_{\alpha(u)} \), i.e.
\[
i \frac{du}{dt} = H_{\alpha(u)}u.
\]
(7)
In the present literature, there is some physical and numerical analysis of Schrödinger dynamics in
the presence of nonlinear defects, mainly focused on the milder one-dimensional case \( \text{(23, 27, 12)} \).
The more technical construction of the three-dimensional problem has hindered extended modelistic
study, numerical work as well as rigorous analysis. Moreover, a certain amount of literature is devoted
to NLS with nonhomogeneous (i.e. \( x \)-dependent and decaying) nonlinearities, yet with a relatively
low decay at infinity (see \( \text{[14, 18]} \) and references therein).

Local (for any \( \sigma > 0 \)) and global (for \( \sigma < 1 \)) well-posedness of the Cauchy problem associated to the
nonlinear Schrödinger equation \( \text{(7)} \) in the space \( V \) have been established in \( \text{(1)} \) and \( \text{(2)} \). In particular,
admits two conserved quantities called mass and energy, defined as

\[ M(u(t)) = \|u(t)\|_{L^2}^2, \quad E(u(t)) = \frac{1}{2}\|\nabla \phi(t)\|_{L^2}^2 - \frac{\nu}{2\sigma + 2}|q(t)|^{2\sigma + 2}. \]

In Section II we prove that equation (7) admits standing waves, i.e. solutions of the form \( u(x,t) = e^{i\omega t}\Phi_{\omega}(x) \), where the profile or amplitude \( \Phi_{\omega} \) up to a phase factor \( e^{i\theta} \) is given by

\[ \Phi_{\omega}(x) = \left( \frac{\sqrt{\omega}}{4\pi\nu} \right)^{\frac{1}{2\sigma}} e^{-\frac{\sqrt{\omega}|x|}{4\pi}}. \tag{8} \]

The set of standing waves is called the solitary manifold \( \mathcal{M} \), and the main concern of this paper consists in the study of the large-time evolution of initial data in the vicinity of \( \mathcal{M} \). A first result concerns stability and instability of standing waves. Stability has to be intended as orbital stability, i.e. Lyapunov stability up to symmetries of the equation, in this case up to gauge \((U(1))\) invariance.

The orbit of \( \Phi_{\omega} \) is then \( O(\Phi_{\omega}) = \{ e^{i\theta}\Phi_{\omega}(x), \theta \in \mathbb{R} \} \). Thus, by definition, the state \( \Phi_{\omega} \) is orbitally stable if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that

\[ d(\psi(0), O(\Phi_{\omega})) < \delta \quad \Rightarrow \quad d(\psi(t), O(\Phi_{\omega})) < \epsilon \quad \forall t > 0 \]

where \( d(\psi, O(\Phi_{\omega})) = \inf_{u \in O(\Phi_{\omega})} \| \psi - u \|_{\mathcal{V}} \). A stationary state is said to be unstable if it is not stable.

Then, we have the following result, proved in Section III:

**Theorem (Orbital Stability)** Let us consider (7). Then, for every \( \omega > 0 \),

(a) if \( 0 < \sigma < 1 \), then the state \( \Phi_{\omega} \) is orbitally stable

(b) if \( \sigma \geq 1 \), then \( \Phi_{\omega} \) is orbitally unstable.

The result directly follows from Weinstein [30] and Grillakis-Shatah-Strauss [20] theory for the case \( \sigma \neq 1 \), while for the case \( \sigma = 1 \) the pseudoconformal invariance of the equation gives the instability by blow-up.

The core of the paper is devoted to the study of the asymptotic stability of the family of stationary states. Asymptotic stability means, loosely speaking, that the solution \( u(t) \) corresponding to an initial datum \( u(0) \) close to the family of orbits, approaches some element of the family of orbits as \( t \to \infty \). The analysis makes use of the representation

\[ u(t,x) = e^{i\Theta(t)} \left( \Phi_{\omega(t)}(x) + \chi(t,x) \right), \tag{9} \]

where \( \Theta(t) = \int_0^t \omega(s)ds + \gamma(t) \), and \( \gamma(t) \) is a suitable phase. Namely, the solution is represented at every time as a modulated solitary wave, with time dependent parameters, up to a fluctuating
remainder $\chi$ which has to be controlled. Asymptotic stability of the family of standing waves means that the modulating parameters $\omega(t)$ and $\gamma(t)$ have a limit as $t \to \infty$, and the fluctuation $\chi$ is in some sense a small and decaying dispersive correction; the radiation damping through dispersion is responsible for the “dissipative” asymptotic behaviour of the solution $u$ around the family of relative equilibria $O(\Phi_\omega)$. Notice that, however, in general the solution does not converge to the solitary wave to which it was close initially.

The subject of asymptotic stability of solitary waves was pioneered by Soffer and Weinstein ([25], [26]), and Buslaev and Perelman ([5], [6]), who developed the main strategies and techniques, nowadays classical; a more recent presentation is contained in [7]. Many relevant later contributions refining and enlarging the hypotheses in the original papers, as well as concerning the kind of initial admitted data and nonlinearities, are contained in [8, 9, 16, 17, 21, 28, 29]. According to this consolidated analysis, one must preliminarily indagate the spectrum of the linearization of equation (7) around the solitary solution. Writing $u = e^{i\omega t}(\Phi_\omega + R)$ and identifying $R$ with the vector of its real and imaginary part, we obtain that it satisfies the canonical system

$$J \frac{dR}{dt} = \begin{bmatrix} H_{\alpha_1} + \omega & 0 \\ 0 & H_{\alpha_2} + \omega \end{bmatrix} R \equiv DR$$

where $H_{\alpha_j}$ are (linear) delta interaction hamiltonian operators with fixed strength $\alpha_j$ that depend on the stationary state $\Phi_\omega$ (through its charge) and on the parameters of the model $\nu, \sigma$ (see eq. (15)). So the dynamics of the linearization of the NLS around the standing wave $\Phi_\omega$ is controlled by the nonselfadjoint (Hamiltonian) matrix operator $L = JD$. The explicit characterization of the spectrum of the linearization $L$ is possible due to the detailed knowledge of the properties of operators $H_{\alpha_j}$. Such feature is unfrequent and allows to avoid further spectral assumptions. The complete result is given in Section IV, Theorem IV.2. Here it is sufficient to recall that in this paper we study asymptotic stability of standing waves in the range $\sigma \in (0, 1/\sqrt{2})$ only, which corresponds to $L$ having no eigenvalues different from zero and no resonances at the threshold of the essential spectrum. A forthcoming paper will treat the case $\sigma \in (1/\sqrt{2}, 1)$, where two simple eigenvalues $\pm i2\sigma\sqrt{1-\sigma^2}\omega$ appear.

Let us notice that the representation (9) amounts in fact to a change of coordinates from the original global $u$ to the new set $\{\omega, \gamma, \chi\}$, with a finite dimensional component given by $\{\omega, \gamma\}$, that describes the solitary manifold and an infinite dimensional one described by $\chi$. However, the representation is not unique, because any choice of $\omega, \gamma$ gives a corresponding choice of $\chi$ such that $u$ given by (9) is
a solution of (7); so one has to restrict in some way the behaviour of the new parameters \( \{\omega, \gamma, \chi\} \) of the solution. To this end, we exploit the fact that the solitary manifold can be naturally endowed with a symplectic structure (see Section II A) and it turns out that its tangent space \( T_{\Phi_\omega} \) coincides with the generalized kernel of the linearization \( L \). The generalized kernel is in turn non-trivial, so the propagator \( e^{-tL} \) has a component growing in time. A parametrization of the running approximate solitary wave in the neighborhood of the solitary manifold suitable for asymptotic analysis is hence obtained through a symplectic splitting in a component along the solitary manifold and a component transversal (symplectically orthogonal) to it. Requiring that the infinite dimensional component \( \chi \) is purely transversal, i.e. projects to zero on the directions of the discrete spectrum, here reduced to the generalized kernel of the linearization, provides the set of the so called modulation (coupled) equations for the parameters \( \omega(t) \) and \( \gamma(t) \), as well as a corresponding partial differential equation for \( \chi \) (see [15] for an enlightening description of the symplectic projection method). The goal is to establish the asymptotic behaviour of the solutions to the modulation equations with a simultaneous control of the decay of the nonlinear part \( \chi \), through the so-called majorant’s method (see [5–7]).

The main result of this paper is the following, and it is proven in Section VII.

**Theorem (Asymptotic stability)**

Assume \( \sigma \in (0, 1/\sqrt{2}) \). Let \( u \in C(\mathbb{R}^+, V) \) be a solution to equation (7) with \( u(0) = u_0 \in V \cap L^1_w \) and denote \( d = \|u_0 - e^{i\theta_0 \Phi_{\omega_0}}\|_{V \cap L^1_w} \), for some \( \omega_0 > 0 \) and \( \theta_0 \in \mathbb{R} \). Then, if \( d \) is sufficiently small, the solution \( u(t) \) can be decomposed as follows

\[
    u(t) = e^{i\omega_\infty t \Phi_{\omega_\infty}} + U \ast \psi_\infty + r_\infty(t),
\]

where \( \omega_\infty > 0 \) and \( \psi_\infty, r_\infty(t) \in L^2(\mathbb{R}^3) \), with \( \|r_\infty(t)\|_{L^2} = O(t^{-5/4}) \) as \( t \to +\infty \).

In the previous statement, \( L^1_w \) is defined in Section IV B and is a weighted space of integrable functions. The weight guarantees the validity of the dispersive estimates needed in order to control the decay of the transversal evolution, and it seems at present unavoidable in view of the singularity of finite energy states. Moreover, it imposes a certain localization on the the admitted initial data, which seems to be a technical requirement. The norm \( \| \cdot \|_{V \cap L^1_w} \) is defined as the maximum of the norms of the two Banach spaces \( V \) and \( L^1_w \).

Concerning the treatment of the modulation equations, one of the main additional difficulties with respect to standard models, and in particular with the case of concentrated nonlinearities in one dimension treated in [4] and [22], is that the equations controlling the evolution of the transversal part...
χ have domains that change with time. This fact forced us to make use of the variational formulation (i.e. in terms of quadratic forms) instead of the traditional strong formulation (i.e. in terms of operators and their domains). The same problem propagates to the proof of the asymptotics given in the above theorem. A last remark concerns the seemingly anomalous value of the nonlinearities where asymptotic stability is proven; this because in the typical situations, when standard NLS with or without potential is treated, it is difficult to have information about subcritical nonlinearities (but see the notably exception in [13]), and in particular pure power. On the other hand, the present model corresponds to an inhomogeneous (space dependent and strongly singular) nonlinearity; this seems to indicate that the analysis of specific models can give results not accessible to general theory, at least at present. The paper is organized as follows. In Section II we fix some notation, describe the set of standing waves for the system of interest, and deduce the linearized evolution around a standing solution. In Section III, using the method by Grillakis, Shatah and Strauss, we prove the orbital stability for any standing wave in the case of low nonlinearity power (i.e. σ < 1, and orbital instability in the case of large nonlinearity power (i.e. σ > 1); in the critical case (σ = 1) we directly show that any stationary state is affected by instability due to the vicinity of initial data that give rise to blow-up solutions. Section IV is devoted to the study of the linearized problem: first, the resolvent of the linearized generator of the evolution is explicitly constructed; then, it is used in order to derive dispersive estimates in suitable weighted spaces. In Section V we start the analysis of the asymptotic completeness by deducing the modulation equations of the system. In Section VI we prove the decay rate in time of the solutions to the modulation equations. Finally, in Section VII we prove the result about asymptotic stability. The paper ends by three appendices related to the content of Section IV. In the first appendix we construct the generalized kernel of the generator of the linearized evolution, in the second we prove a convenient expression for the resolvent of the same operator, while in the last appendix we give the explicit linearized dynamics along the generalized kernel.

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II. PRELIMINARIES

A. Hamiltonian structure

We consider $L^2(\mathbb{R}^3, \mathbb{C})$ as a real Hilbert space endowed with the scalar product

$$(u, v)_{L^2} = \Re \int_{\mathbb{R}^3} u \overline{v} \, dx = \int_{\mathbb{R}^3} (\Re v \Re u + \Im \overline{v} \overline{u}) \, dx.$$  \hspace{1cm} (11)

It is sometimes convenient to shift from the complex valued representation of $u$ to the vector real valued one through the identification $u = \Re u + i\Im u \mapsto (\Re u, \Im u) = (u_1, u_2)$. As a consequence, $H^s(\mathbb{R}^3, \mathbb{C}) \cong H^s(\mathbb{R}^3, \mathbb{R}^2)$, while multiplication by $i$ is equivalent to multiplication by the matrix $-J$, where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \hspace{1cm} (12)$$

The space $L^2(\mathbb{R}^3)$ is also a symplectic manifold when endowed with the symplectic form

$$\Omega(u, v) = \Im \int_{\mathbb{R}^3} u \overline{v} \, dx = \int_{\mathbb{R}^3} (\Re v \Im u - \Im v \Re u) \, dx = \int_{\mathbb{R}^3} (u_2 v_1 - u_1 v_2) \, dx.$$  \hspace{1cm} (13)

Along the paper we often shift between real and complex representation when no ambiguity occurs.

In our model the Hamiltonian functional coincides with the total energy and (exploiting the decomposition of an element of the form domain in regular and singular part) it is given by

$$E(u) = \frac{1}{2} \|\nabla \phi\|_{L^2}^2 - \frac{\nu}{2\sigma + 2} |q|^{2\sigma + 2}, \quad u = \phi + qG_0 \in V.$$  \hspace{1cm} (14)

Correspondingly, the NLS (7) takes the hamiltonian form

$$\frac{du}{dt} = J E'(u).$$  \hspace{1cm} (15)

where the prime denotes the differential of the considered functional at the point $u$.

B. Standing waves

Standing waves are solutions of the equation (7) of the form $u(x, t) = e^{i\omega t} \Phi_\omega(x)$. It immediately follows that if a standing wave exists, then the amplitude $\Phi_\omega$ satisfies the nonlinear equation

$$H_{\omega(\Phi_\omega)} \Phi_\omega + \omega \Phi_\omega = 0.$$  \hspace{1cm} (16)
Proposition II.1. Standing waves for equation (7) exist if and only if $\nu > 0$. In such a case the set of solitary waves is given by the two-dimensional manifold

$$\mathcal{M} = \{ e^{i\Theta} \Phi_\omega, \omega > 0, \Theta \in [0, 2\pi) \}$$

where the function

$$\Phi_\omega(x) = \left( \frac{\sqrt{\omega}}{4\pi \nu} \right)^{\frac{1}{2\nu}} e^{-\sqrt{\omega}|x|}$$

and the parameters $\omega$ and $\Theta$ play the role of local coordinates.

Proof. Recall that the function $G_0$ defined in (2) satisfies the equation $-\Delta G_0 = \delta$ where $\delta$ is the Dirac’s delta distribution centred at $x = 0$. Hence, for $x \neq 0$ equation (16) is equivalent to $-\Delta \Phi_\omega(x) + \omega \Phi_\omega(x) = 0$. Let us introduce, with a slight abuse, the function

$$f(r, \theta, \phi) = \Phi_\omega(x),$$

and consider the corresponding equation in spherical coordinates, namely

$$-\frac{\partial^2 f}{\partial r^2} - \frac{2}{r} \frac{\partial f}{\partial r} - \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} - \frac{\cos \phi}{r^2 \sin \phi} \frac{\partial f}{\partial \phi} - \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} + \omega f = 0,$$

and exploit the spherical harmonics expansion of the solution $f(r, \theta, \phi) = \sum_{l=0}^{+\infty} \sum_{j=-l}^{l} v_{l,j}(r)Y_{l,j}(\theta, \phi)$, where $Y_{l,j}$ denotes the set of spherical harmonics which is an orthonormal basis of $L^2([0, \pi] \times [0, 2\pi], \sin \theta d\theta d\phi)$. Since

$$\frac{\partial^2 Y_{l,j}}{\partial \phi^2} + \frac{\cos \phi}{\sin \phi} \frac{\partial Y_{l,j}}{\partial \phi} + \frac{1}{\sin^2 \phi} \frac{\partial^2 Y_{l,j}}{\partial \theta^2} = -\lambda Y_{l,j}, \quad \text{for some } \lambda \in \mathbb{C},$$

one has that $\lambda$ belongs to the set $\{\lambda_l := l(l+1), l \in \mathbb{N}\}$, and so the functions $v_{l,j}$ solve $-\nu v''_{l,j}(r) - \frac{2}{r} v'_{l,j}(r) + \left( \omega - \frac{\lambda_l}{r^2} \right) v_{l,j}(r) = 0$. Then, from formula 8.491.6 in [19],

$$v_{j,l}(r) = \frac{1}{\sqrt{r}} Z_{\lambda}(\sqrt{\omega} r),$$

where $Z_\nu$ is a Bessel’s function. By the asymptotic expansions 8.443 and 8.451.1 in [19] one immediately has that if $\lambda \neq 0$, then $v_{j,l}$ cannot belong to $L^2(\mathbb{R}^+, r^2 dr)$. Hence, we fix $\lambda = 0$ and denote $\Phi_\omega(x) = \frac{v(r)}{r}$, $r = |x|$. Thus $v$ has to be a square-integrable solution of $v''(r) - \omega v(r) = 0$, and finally

$$\Phi_\omega(x) = \frac{q e^{-\sqrt{\omega}|x|}}{4\pi |x|},$$

for some $q \in \mathbb{C}$ and $\omega > 0$. Furthermore, by the boundary condition in the definition of $D(H_{\alpha(u)})$, i.e.

$$\lim_{x \to 0}(\Phi_\omega(x) - q G_0(x)) = \alpha(\Phi_\omega)q,$$

one gets $-\frac{q}{4\pi \nu} \sqrt{\omega} = -\nu |q|^{2\sigma} q$, and supposing $\nu \neq 0$ one obtains $|q|^{2\sigma} = \frac{\sqrt{\omega}}{4\pi \nu}$. This requires $\nu > 0$, so

$$\Phi_\omega(x) = \left( \frac{\sqrt{\omega}}{4\pi \nu} \right)^{\frac{1}{2\nu}} e^{-\sqrt{\omega}|x|}.$$
which, up to a phase factor, gives the stated result. In the case \( \nu = 0 \), from boundary condition we get \( q = 0 \) or \( \omega = 0 \). If \( q = 0 \), then the function \( u \) vanishes. If \( \omega = 0 \), then one has \( u(x) = \frac{1}{4\pi|x|} \), which is the resonance function of the delta interaction with vanishing strength, but it is not an element of the operator domain, and it does not solve the stationary equation \([16]\). So for \( \nu = 0 \) standing waves do not exist.

In the following we denote \( q_\omega = \left( \frac{\sqrt{\omega}}{4\pi\nu} \right)^{\frac{1}{2\sigma}} \).

C. Linearization of \( H_{\alpha(u)} \) around \( \Phi_\omega \)

The linearization of equation \([7]\) around a stationary solution is not completely obvious, due to the fact that the nonlinearity is embodied in the domain of the operator \( H_{\alpha(u)} \) and not in the action of the operator itself. Nevertheless, we can consider the Hamiltonian associated to equation \([7]\) given by formula \([15]\) and notice that the nonlinearity no longer appears in the domain \( V \) but directly in the Hamiltonian functional. So we derive the linear operator which approximates \( H_{\alpha(u)} \) from the quadratic form which approximates \( E(\Phi_\omega) \) and obtain the following result.

**Proposition II.2.** The Hessian \( E''(\Phi_\omega) \) of the functional \( E \) can be represented as \( E''(\Phi_\omega)(h,k) = \langle H_{\alpha,\text{Lin}} h,k \rangle \), where \( H_{\alpha,\text{Lin}} \) is the linear operator given by

\[
H_{\alpha,\text{Lin}} = \begin{bmatrix}
H_{\alpha_1} & 0 \\
0 & H_{\alpha_2}
\end{bmatrix},
\]

where the operators \( H_{\alpha_1} \) and \( H_{\alpha_2} \) are the standard point interactions defined in the Introduction (see \([1]\)) and the fixed parameters \( \alpha_1 \) and \( \alpha_2 \) are given by

\[
\alpha_1 = -\nu(2\sigma + 1)|q_\omega|^{2\sigma} = -\frac{2\sigma + 1}{4\pi}\sqrt{\omega}, \quad \alpha_2 = -\nu|q_\omega|^{2\sigma} = -\frac{\sqrt{\omega}}{4\pi}.
\] (18)

\( H_{\alpha,\text{Lin}} \) is selfadjoint with respect to the real scalar product in \( L^2(\mathbb{R}^3, \mathbb{C}) \).

**Proof.** The first Gâteaux derivative of \( E(u) \) reads

\[
E'(u)[h] = \frac{d}{d\epsilon}\{E(u + \epsilon h)\}_{\epsilon=0} = \Re \int_{\mathbb{R}^3} \nabla \phi_u(x) \cdot \overline{\nabla \phi_h(x)} dx - \nu |q_u|^{2\sigma} \Re (q_u \overline{q_h}) \quad \forall u, h \in V,
\] (19)

while the second Gâteaux derivative at \( \Phi_\omega \) reads

\[
\frac{\partial^2}{\partial \epsilon \partial \lambda} \{E(\Phi_\omega + \epsilon h + \lambda k)\}_{\epsilon=0, \lambda=0} = \Re \int_{\mathbb{R}^3} \nabla \phi_h(x) \cdot \overline{\nabla \phi_k(x)} dx - \frac{\partial^2}{\partial \epsilon \partial \lambda} \left\{ \frac{\nu}{2\sigma + 2} |q_{\Phi_\omega + \epsilon h + \lambda k}|^{2\sigma + 2} \right\}_{\epsilon=0, \lambda=0}.
\]
The last term gives, after some calculation, the contribution (here $h = (h_1, h_2)$, $k = (k_1, k_2)$)
\[
\frac{\partial^2}{\partial \epsilon \partial \lambda} \left\{ -\frac{\nu}{2\sigma + 2}|q_{\Phi + \epsilon h + \lambda k}|^{2\sigma + 2} \right\}_{\epsilon = 0, \lambda = 0} = -\nu|q_\omega|^{2\sigma} \left[(2\sigma + 1)q_{h_1}q_{k_1} + q_{h_2}q_{k_2}\right].
\]
So $E''(\Phi_\omega)$ is given by the direct sum of two quadratic forms: one is acting on the real part of the functions $h$ and $k$, and the other on the imaginary part. The term related to the real part is a lower bounded quadratic form whose corresponding selfadjoint operator is $H_{\alpha_1}$, while the quadratic form related to the imaginary part corresponds to the operator $H_{\alpha_2}$ ($\alpha_1$ and $\alpha_2$ have been defined in (18)). Then, the operator $H_{\alpha, Lin}$ representing the entire quadratic form $E''(\Phi_\omega)$ is self-adjoint and the proof is complete.

Now, to get the linearized equation set $u(t) = e^{i\omega t}(\Phi_\omega + R)$ and obtain
\[
\frac{d}{dt}R = J(E'(\Phi_\omega) + \omega \Phi_\omega) + J(E''(\Phi_\omega) + \omega)R + \text{higher order terms} \simeq J(H_{\alpha, Lin} + \omega)R.
\]
Summing up, the linearized equation (7) becomes
\[
\frac{dR}{dt} = JDR,
\]
(20)
where $D = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}$, with
\[
L_j = H_{\alpha_j} + \omega, \quad j = 1, 2.
\]
(21)
Notice that the operator
\[
JD := L = \begin{bmatrix} 0 & L_2 \\ -L_1 & 0 \end{bmatrix},
\]
(22)
is not selfadjoint nor skew adjoint. Nevertheless, a standard application of Hille-Yosida theorem and a simple analysis of the resolvent of $L$ which takes into account the factorized structure $L = JD$ with $D$ s.a. shows that it generates a semigroup of linear operators with (at most) exponential growth in time. A more precise analysis of the resolvent of the operator $L$ will be given in Theorem IV.2 and in the appendix C we will prove that the semigroup has in fact a linear growth (see Theorem C.1) in the case here interesting, i.e. $\sigma \in (0, 1/\sqrt{2})$. 

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III. ORBITAL STABILITY

In order to prove the orbital stability of the stationary solutions to equation (7), we apply Grillakis-Shatah-Strauss theory, and in particular Theorem 2 in [20]. As a first step, we recall the following known fact proved in [3].

**Proposition III.1.** If $\alpha(u) = \alpha$ where $\alpha < 0$ is a constant, then

$$
\sigma(H_\alpha) \equiv \{-4(\pi \alpha)^2 \} \cup [0, +\infty).
$$

(23)

Thanks to the last proposition one can prove the following lemma which implies the spectral properties needed to verify Assumption 3 in [20].

**Lemma III.2.** The spectrum of the operator $D$ is

$$
\sigma(D) = \{-4\sigma(\sigma + 1)\omega, 0\} \cup [\omega, +\infty),
$$

and $\ker(D) = \text{span}\left\{\begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}\right\}$.

**Proof.** Since $D$ is the direct sum of the operators $L_1$ and $L_2$ acting on $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, its spectrum is given by the union of $\sigma(L_1)$ and $\sigma(L_2)$. From (23) follows

$$
\sigma(H_{\alpha_1}) = \{-2(\sigma + 1)^2\omega\} \cup [0, +\infty), \quad \sigma(H_{\alpha_2}) = \{-\omega\} \cup [0, +\infty).
$$

Then

$$
\sigma(L_1) = \sigma(H_{\alpha_1}) + \omega = \{-4\sigma(\sigma + 1)\omega\} \cup [\omega, +\infty), \quad \sigma(L_2) = \sigma(H_{\alpha_2}) + \omega = \{0\} \cup [\omega, +\infty).
$$

Hence, $\ker(L_1) = \{0\}$ and $\ker(L_2) = \text{span}\{\Phi_\omega\}$, which concludes the proof.

We can now prove the following

**Theorem III.3. (Orbital stability)** For each $\omega > 0$, if $0 < \sigma < 1$, then $\Phi_\omega$ is orbitally stable. If $\sigma > 1$, then $\Phi_\omega$ is orbitally unstable.

**Proof.** Well-posedness and existence of a branch of standing waves, i.e. Assumptions 1 and 2 in [20], are proved in [1] and [2] and in the previous section, while Assumption 3 is true thanks to Lemma
Hence, from Theorem 3 in [20] we have orbital stability if \( \frac{d}{d\omega} \| \Phi_\omega \|_{L^2(\mathbb{R}^3)}^2 > 0 \) and orbital instability if \( \frac{d}{d\omega} \| \Phi_\omega \|_{L^2(\mathbb{R}^3)}^2 < 0 \). In order to inspect the sign of \( \frac{d}{d\omega} \| \Phi_\omega \|_{L^2(\mathbb{R}^3)}^2 \), we compute

\[
\| \Phi_\omega \|_{L^2(\mathbb{R}^3)}^2 = \left( \frac{\sqrt{\omega}}{4\pi \nu} \right)^\frac{1}{4} \frac{1}{8\pi \sqrt{\omega}},
\]

hence \( \frac{d}{d\omega} \| \Phi_\omega \|_{L^2(\mathbb{R}^3)}^2 = \frac{1}{8\pi(4\pi \nu)^{1/4}} \frac{1}{2\sigma} \omega^{\frac{1-3\sigma}{2\sigma}} \), which concludes the proof.

A. The case \( \sigma = 1 \)

Since Theorem 3 in [20] does not give information about orbital stability of the stationary state \( e^{i\omega t} \Phi_\omega \) when \( \frac{d}{d\omega} \| \Phi_\omega \|_{L^2(\mathbb{R}^3)}^2 = 0 \), we need to inspect the case \( \sigma = 1 \) apart. In such case, equation (7) exhibits one additional symmetry (see [2]).

Remark III.4. Equation (7) is invariant under the pseudoconformal transformation

\[
\widehat{u}_{\nu,T}(t,x) = e^{-\frac{|x|^2}{(T-t)^{3/2}}} u\left( \frac{1}{T-t}, \frac{|x|}{T-t} \right).
\]

In [1] it is proved that equation (7) may have some non global solutions which blow up, in the following sense: the solution \( u(t) \) of equation (7) blows up (in the future) at time \( T < +\infty \) if

\[
\limsup_{t \to T^-} \| \nabla \phi_u \|_{L^2} = +\infty.
\]

Here \( \phi_u \) is the regular part of the function \( u \) and \( q_u \) is the corresponding charge according to the decomposition in (11). Due to energy conservation this condition is equivalent to \( \limsup_{t \to T^-} \| q_u(t) \| = +\infty \).

Thanks to the pseudoconformal invariance we prove that in any neighbourhood (in energy norm) of each standing wave there are initial data of a blow up solution.

Theorem III.5. Fix \( \sigma = 1 \) and \( \omega > 0 \). For any \( \delta > 0 \) there exists a blow up solution \( u(t) \in V \) such that \( \| u(0) - \Phi_\omega \|_V < \delta \).

Proof. Applying the pseudoconformal transformation to the solitary wave \( e^{i\omega t} \Phi_\omega \) one gets that for any \( T > 0 \), the function

\[
u_\nu,T(t,x) = e^{\frac{\omega}{4\pi \nu} T^{1/4}} e^{-\frac{\sqrt{\omega}}{4\pi \nu} \sqrt{T-t}|x|} e^{-i \frac{|x|^2}{4(T-t)}}
\]
is a solution to equation (7). Thus, for any $T > 0$, the initial datum $u_T(x) = e^{i\tilde{\omega} T} \frac{e^{-\sqrt{\omega}|x|}}{\sqrt{4\pi\nu 4\pi\sqrt{T}|x|}} e^{-i|x|^2/4T}$ gives rise to a solution that blows up at time $T$. Now, let $\tilde{\omega}$ depend on $T$ as $\tilde{\omega} = \omega T^2$, so that $u_T(x) = e^{-i|x|^2/4T} \Phi_{\omega}(x)$.

We prove the theorem by showing that $\| (e^{-i|x|^2/4T} - 1) \Phi_{\omega} \|_V \to 0$ as $T \to +\infty$. Indeed, noting that the function $(e^{-i|x|^2/4T} - 1) \Phi_{\omega}$ belongs to $H^1(\mathbb{R}^3)$,

$$\| (e^{-i|x|^2/4T} - 1) \Phi_{\omega} \|_V = \| \nabla ((e^{-i|x|^2/4T} - 1) \Phi_{\omega}) \|_{L^2} \leq \frac{1}{2T} \| \Phi_{\omega} \|_{L^2} + \frac{1}{4T} \| \cdot \|_{L^2}^2 \to 0, \quad T \to +\infty.$$ 

IV. SPECTRAL AND DISPERSIVE PROPERTIES OF LINEARIZATION $L$

Here we study the long time behaviour of equation (20), that is the linearization of (7) around the stationary solution $e^{i\omega t} \Phi_{\omega}$.

The generalized kernel of the operator $L$ (see (22)) is defined as $N_g(L) = \bigcup_{k \in \mathbb{N}} \ker(L^k)$.

In what follows let us denote

$$\varphi_{\omega}(x) = \frac{d\Phi_{\omega}}{d\omega}(x) = \frac{1}{4\sigma\omega} \left( \frac{\sqrt{\omega}}{4\pi\nu} \frac{\nu}{4\pi |x|} e^{-\sqrt{\omega}|x|} - \frac{1}{2\sqrt{\omega}} \left( \frac{\sqrt{\omega}}{4\pi\nu} \nu \frac{\sqrt{\omega}|x|}{4\pi} \right) \right),$$

$$g_{\omega}(x) = \frac{1}{4\sqrt{\pi\nu}} |x| \frac{e^{-\sqrt{\omega}|x|}}{4\pi |x|},$$

$$h_{\omega}(x) = \frac{1}{4\sqrt{\pi\nu}} \left( \frac{1}{4\omega} \frac{e^{-\sqrt{\omega}|x|}}{4\pi |x|} + \frac{1}{2\omega} e^{-\sqrt{\omega}|x|} + \frac{1}{2\sqrt{\omega}} |x| e^{-\sqrt{\omega}|x|} + \frac{1}{3} |x|^2 \frac{e^{-\sqrt{\omega}|x|}}{4\pi} \right).$$

In Appendix A we prove the following theorem.

**Theorem IV.1.** If the nonlinearity power $\sigma$ is different from 1, then $N_g(L) = \text{span} \left\{ \left( \begin{array}{c} 0 \\ \Phi_{\omega} \end{array} \right), \left( \begin{array}{c} \varphi_{\omega} \\ 0 \end{array} \right) \right\}$.

Moreover, if $\sigma = 1$, then $N_g(L) = \text{span} \left\{ \left( \begin{array}{c} 0 \\ \Phi_{\omega} \end{array} \right), \left( \begin{array}{c} \varphi_{\omega} \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ g_{\omega} \end{array} \right), \left( \begin{array}{c} 0 \\ h_{\omega} \end{array} \right) \right\}$.

In the following section we provide an explicit description of the spectrum of the non-selfadjoint operator $L$ and the dispersive estimates for the action of the propagator $e^{-Lt}$ upon the absolutely continuous subspace.
The purpose of this section is to prove an explicit formula for the resolvent of the linearized operator. For later convenience we denote

$$G_{\omega \pm i\lambda}(x) = \frac{e^{i\sqrt{-\omega \pm i\lambda}|x|}}{4\pi|x|}, \quad \omega > 0, \lambda \in \mathbb{C},$$

with the prescription $\Im \sqrt{-\omega \pm i\lambda} > 0.$

Furthermore, we make use of the notation $(g, h) := \int_{\mathbb{R}^3} g(x) h(x) \, dx.$

We prove the following

\textbf{Theorem IV.2.} The resolvent $R(\lambda) = (L - \lambda I)^{-1}$ of the operator $L$ defined in (22) is given by

$$R(\lambda) = \begin{bmatrix} -\lambda G_{\lambda^2} & -\Gamma_{\lambda^2} \\ \Gamma_{\lambda^2} & -\lambda G_{\lambda^2} \end{bmatrix} + \frac{4\pi}{W(\lambda^2)} i \begin{bmatrix} \Lambda_1 & i\Sigma_2 \\ -i\Sigma_1 & \Lambda_2 \end{bmatrix},$$

where

$$W(\lambda^2) = 32\pi^2 \alpha_1 \alpha_2 - 4i\pi (\alpha_1 + \alpha_2) \left( \sqrt{-\omega + i\lambda} + \sqrt{-\omega - i\lambda} \right) - 2\sqrt{-\omega + i\lambda} \sqrt{-\omega - i\lambda},$$

and formula (25) holds for all $\lambda \in \mathbb{C} \setminus \{ \lambda \in \mathbb{C} : W(\lambda^2) = 0, \text{ or } \Re(\lambda) = 0 \text{ and } |\Im(\lambda)| \geq \omega \}.$

Furthermore, the symbol $\ast$ in (25) denotes the convolution and

$$G_{\lambda^2}(x) = \frac{1}{2i\lambda} (G_{\omega - i\lambda}(x) - G_{\omega + i\lambda}(x)), \quad \Gamma_{\lambda^2}(x) = \frac{1}{2} (G_{\omega - i\lambda}(x) + G_{\omega + i\lambda}(x)).$$

Finally, the entries of the second matrix are finite rank operators whose action on $f \in L^2(\mathbb{R}^3)$ reads

$$\Lambda_1 f = [i\lambda(4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})(G_{\lambda^2}, f) - (4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})(\Gamma_{\lambda^2}, f)]G_{\omega + i\lambda} +$$

$$+ [i\lambda(4\pi\alpha_2 - i\sqrt{-\omega - i\lambda})(G_{\lambda^2}, f) + (4\pi\alpha_1 - i\sqrt{-\omega - i\lambda})(\Gamma_{\lambda^2}, f)]G_{\omega - i\lambda},$$

$$\Lambda_2 f = [i\lambda(4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})(G_{\lambda^2}, f) - (4\pi\alpha_2 - i\sqrt{-\omega + i\lambda})(\Gamma_{\lambda^2}, f)]G_{\omega + i\lambda} +$$

$$+ [i\lambda(4\pi\alpha_2 - i\sqrt{-\omega - i\lambda})(G_{\lambda^2}, f) + (4\pi\alpha_2 - i\sqrt{-\omega - i\lambda})(\Gamma_{\lambda^2}, f)]G_{\omega - i\lambda},$$

$$\Sigma_1 f = -[i\lambda(4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})(G_{\lambda^2}, f) - (4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})(\Gamma_{\lambda^2}, f)]G_{\omega + i\lambda} +$$

$$+ [i\lambda(4\pi\alpha_2 - i\sqrt{-\omega - i\lambda})(G_{\lambda^2}, f) + (4\pi\alpha_1 - i\sqrt{-\omega - i\lambda})(\Gamma_{\lambda^2}, f)]G_{\omega - i\lambda},$$

$$\Sigma_2 f = [i\lambda(4\pi\alpha_2 - i\sqrt{-\omega + i\lambda})(G_{\lambda^2}, f) - (4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})(\Gamma_{\lambda^2}, f)]G_{\omega + i\lambda} +$$

$$+ [i\lambda(4\pi\alpha_1 - i\sqrt{-\omega - i\lambda})(G_{\lambda^2}, f) + (4\pi\alpha_1 - i\sqrt{-\omega - i\lambda})(\Gamma_{\lambda^2}, f)]G_{\omega - i\lambda}.$$
\[ \Sigma_2 f = -i(4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})(G_{\lambda^2}, f) - (4\pi\alpha_2 - i\sqrt{-\omega + i\lambda})(\Gamma_{\lambda^2}, f)G_{\omega + i\lambda} + 
\]
\[ + i(4\pi\alpha_1 - i\sqrt{-\omega - i\lambda})(G_{\lambda^2}, f) + (4\pi\alpha_2 - i\sqrt{-\omega - i\lambda})(\Gamma_{\lambda^2}, f)G_{\omega - i\lambda}. \]

The spectrum of the operator \( L \) can be decomposed into an essential and a discrete part,

\[ \sigma(L) = \sigma_{\text{ess}}(L) \cup \sigma_d(L), \tag{27} \]

where the essential spectrum is

\[ \sigma_{\text{ess}}(L) = \mathbb{C}_+ \cup \mathbb{C}_- = \{ \lambda \in \mathbb{C} : \Re(\lambda) = 0 \text{ and } \Im(\lambda) \geq \omega \} \cup \{ \lambda \in \mathbb{C} : \Re(\lambda) = 0 \text{ and } \Im(\lambda) \leq -\omega \}, \]

and the discrete spectrum depends on the parameter \( \sigma \) as follows:

(a) if \( \sigma \in (0, 1/\sqrt{2}] \), then the only eigenvalue of \( L \) is 0 with algebraic multiplicity 2.

(b) if \( \sigma \in (1/\sqrt{2}, 1) \), then \( L \) has two simple eigenvalues \( \pm i\sigma\sqrt{1 - \sigma^2}\omega \) and the eigenvalue 0 with algebraic multiplicity 2.

(c) if \( \sigma = 1 \), then the only eigenvalue of \( L \) is 0 with algebraic multiplicity 4.

(d) if \( \sigma \in (1, +\infty) \), then \( L \) has two simple eigenvalues \( \pm 2\sigma\sqrt{\sigma^2 - 1}\omega \) and the eigenvalue 0 with algebraic multiplicity 2.

Before giving the proof, we need two preliminary lemmas.

**Lemma IV.3.** For any \( \mu \in \mathbb{C}, \omega > 0 \), the Green’s function \( G_\mu \) of the operator \( H_\mu \), defined by

\[ D(H_\mu) = H^4(\mathbb{R}^3), \quad H_\mu = \mu + (-\Delta + \omega)^2, \]

reads

\[ G_\mu(x) = \frac{1}{2i\sqrt{\mu}} \left( G_{\omega - i\sqrt{\mu}}(x) - G_{\omega + i\sqrt{\mu}}(x) \right). \tag{28} \]

**Proof.** By definition of Green’s function, \( G_\mu \) solves the equation \([\mu + (-\Delta + \omega)^2]G_\mu(x) = \delta(x)\). Taking the Fourier transform, one gets

\[ \tilde{G}_\mu(k) = \frac{1}{(2\pi)^{3/2}(\mu + (k^2 + \omega)^2)} = \frac{1}{2i\sqrt{\mu}} \left( G_{\omega - i\sqrt{\mu}}(k) - G_{\omega + i\sqrt{\mu}}(k) \right), \]

where the function \( G_{\omega \pm i\sqrt{\mu}} \) was defined in (21). The proof is complete. \( \square \)
Remark IV.4. The function $G_{\mu}$ is an element of $H^s(\mathbb{R}^3)$ for any $s < 7/2$.

Let us denote

$$H_{\mu}^{21} = \mu + L_2 L_1,$$

where $L_2$ and $L_1$ were defined in (21). Applying elementary rules on composition of operators, one can easily see that the domain of the operator $H_{\mu}^{21}$, which coincides with the domain of $L_2 L_1$, is given by

$$D(L_2 L_1) = \left\{ u \in L^2(\mathbb{R}^3) : u = \xi + pG_{\omega + i}\sqrt{\mu} + qG_{\omega - i}\sqrt{\mu}, \text{ with } \xi \in H^4(\mathbb{R}^3), p, q \in \mathbb{C}, \right\} \quad (29)$$

$$\xi(0) + ip\frac{\sqrt{-\omega - i\sqrt{\mu}}}{4\pi} + iq\frac{\sqrt{-\omega + i\sqrt{\mu}}}{4\pi} = \alpha_1(p + q),$$

$$(-\Delta + \omega)\xi(0) + \sqrt{\mu}p\frac{\sqrt{-\omega - i\sqrt{\mu}}}{4\pi} - \sqrt{\mu}q\frac{\sqrt{-\omega + i\sqrt{\mu}}}{4\pi} = \alpha_2\sqrt{\mu}(q - p).$$

In the following lemma the inverse operator of $H_{\mu}^{21}$ is constructed.

Lemma IV.5. For each $\mu \in \mathbb{C}$, the inverse of the operator $H_{\mu}^{21}$ is given by

$$(H_{\mu}^{21})^{-1} : L^2(\mathbb{R}^3) \rightarrow D(H_{\mu}^{21}) \quad f \mapsto G_{\mu} \ast f + p(f)G_{\omega + i}\sqrt{\mu} + q(f)G_{\omega - i}\sqrt{\mu}, \quad (30)$$

where the functionals $p,q : L^2(\mathbb{R}^3) \rightarrow \mathbb{C}$ act as

$$p(f) = \frac{4\pi}{i\sqrt{W(\mu)}}[i\sqrt{\mu}(4\pi\alpha_2 - i\sqrt{-\omega + i\sqrt{\mu}})(G_{\mu},f) - (4\pi\alpha_1 - i\sqrt{-\omega + i\sqrt{\mu}})(\Gamma_{\mu},f)],$$

$$q(f) = \frac{4\pi}{i\sqrt{W(\mu)}}[i\sqrt{\mu}(4\pi\alpha_2 - i\sqrt{-\omega - i\sqrt{\mu}})(G_{\mu},f) + (4\pi\alpha_1 - i\sqrt{-\omega - i\sqrt{\mu}})(\Gamma_{\mu},f)], \quad (31)$$

with

$$W(\mu) = 2(4\pi)2\alpha_1\alpha_2 - 4i\pi(\alpha_1 + \alpha_2)\left(\sqrt{-\omega + i\sqrt{\mu}} + \sqrt{-\omega - i\sqrt{\mu}}\right) - 2\sqrt{-\omega + i\sqrt{\mu}}\sqrt{-\omega - i\sqrt{\mu}}.$$

Proof. First we show that the definition of the functionals $p$ and $q$ ensures

$$G_{\mu} \ast f + p(f)G_{\omega + i}\sqrt{\mu} + q(f)G_{\omega - i}\sqrt{\mu} \in D(H_{\mu}^{21}) = D(L_2 L_1)$$

for all $f \in L^2(\mathbb{R}^3)$. Indeed, $p(f)$ and $q(f)$ solve the algebraic system given by the boundary condition in the definition of the domain (29), namely

$$\begin{cases}
\langle G_{\mu}, f \rangle + ip\frac{\sqrt{-\omega - i\sqrt{\mu}}}{4\pi} + iq\frac{\sqrt{-\omega + i\sqrt{\mu}}}{4\pi} = \alpha_1(p + q), \\
\langle \Gamma_{\mu}, f \rangle + \sqrt{\mu}p\frac{\sqrt{-\omega - i\sqrt{\mu}}}{4\pi} - \sqrt{\mu}q\frac{\sqrt{-\omega + i\sqrt{\mu}}}{4\pi} = \alpha_2\sqrt{\mu}(q - p).
\end{cases}$$
Now, denote by $\widehat{H}_0$ the operator that acts as the Laplacian on the subspace of the Schwartz functions in $\mathbb{R}^3$ that vanish in a neighbourhood of the origin. It is well-known (see [3]), that both selfadjoint operators $H_{\alpha_1}$ and $H_{\alpha_2}$ defined in Proposition 11.2 are restrictions of $\widehat{H}_0^*$ (i.e. the adjoint of $\widehat{H}_0$ as an operator in $L^2(\mathbb{R}^3)$), whose action on $G_{\omega \pm i\sqrt{\mu}}$ yields

$$[\mu + (\widehat{H}_0^* + \omega)^2]G_{\omega \pm i\sqrt{\mu}} = 0. \quad (32)$$

Recalling that $\mathcal{G}_\mu \in H^4(\mathbb{R}^3)$, it follows, for any $f \in L^2(\mathbb{R}^3)$,

$$\mathcal{H}^{21}_\mu (\mathcal{G}_\mu * f + p(f)G_{\omega + i\sqrt{\mu}} + q(f)G_{\omega - i\sqrt{\mu}}) = (\mu + (\widehat{H}_0^* + \omega)^2)(\mathcal{G}_\mu * f + p(f)G_{\omega + i\sqrt{\mu}} + q(f)G_{\omega - i\sqrt{\mu}}) =$$

$$= (\mu + (-\Delta + \omega)^2)(\mu + (-\Delta + \omega)^2)^{-1} f = f. \quad (32)$$

To conclude the proof one has to show

$$\mathcal{G}_\mu * (\mathcal{H}^{21}_\mu f) + p(\mathcal{H}^{21}_\mu f)G_{\omega + i\sqrt{\mu}} + q(\mathcal{H}^{21}_\mu f)G_{\omega - i\sqrt{\mu}} = f$$

for any $f \in D(\mathcal{H}^{21}_\mu)$. To this purpose let us set $f = \xi + aG_{\omega + i\sqrt{\mu}} + bG_{\omega - i\sqrt{\mu}}$ for some $\xi \in H^4(\mathbb{R}^3)$ and $a, b \in \mathbb{C}$ such that the boundary condition in (29) are satisfied, then, by (32)

$$\mathcal{H}^{21}_\mu f = [\mu + (-\Delta + \omega)^2] \xi$$

and, by system (31)

$$p(f) = a, \quad q(f) = b.$$

The proof is complete. \qed

**Remark IV.6.** The inverse of the operator $\mathcal{H}^{12}_\mu = \mu + L_1L_2$ is obtained exchanging $\alpha_1$ and $\alpha_2$ in the expression of $(\mathcal{H}^{12}_\mu)^{-1}$.

Now we can turn to the proof of Theorem IV.2

**Proof.** We preliminarily observe that

$$\Gamma_\mu(x) = (-\Delta + \omega)\mathcal{G}_\mu(x) = \frac{e^{i\sqrt{-\omega + i\sqrt{\mu}}|x|} + e^{i\sqrt{-\omega - i\sqrt{\mu}}|x|}}{8\pi|x|} = \frac{1}{2} \left( G_{\omega - i\sqrt{\mu}}(x) + G_{\omega + i\sqrt{\mu}}(x) \right).$$

As proven in Appendix [13] the following identity holds:

$$R(\lambda) = (L - \lambda I)^{-1} = \begin{bmatrix} -\lambda(\lambda^2 + L_2L_1)^{-1} & -L_2(\lambda^2 + L_1L_2)^{-1} \\ L_1(\lambda^2 + L_2L_1)^{-1} & -\lambda(\lambda^2 + L_1L_2)^{-1} \end{bmatrix} = \begin{bmatrix} -\lambda(\mathcal{H}^{21}_\lambda)^{-1} & -L_2(\mathcal{H}^{12}_\lambda)^{-1} \\ L_1(\mathcal{H}^{21}_\lambda)^{-1} & -\lambda(\mathcal{H}^{12}_\lambda)^{-1} \end{bmatrix},$$

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with \( \lambda \) in the resolvent set of \( L \), to be specified.

In order to find the explicit expressions for \( \Lambda_1 \) and \( \Lambda_2 \) given in (26), one sets \( \lambda = \sqrt{\mu} \) and then applies Lemma IV.5, Remark IV.6, and uses the definition of \( p \) and \( q \) given in (31). Besides, the operators \( \Sigma_1 \) and \( \Sigma_2 \) can be obtained applying \( L_1 \) and \( L_2 \) to \( (H^{21}_{\lambda_2})^{-1} \) and \( (H^{12}_{\lambda_2})^{-1} \), respectively, and using some elementary algebra.

The statement about the essential spectrum of \( L \) is a consequence of Weyl’s theorem (Theorem XIII.4 in [24]). On the other hand, the eigenvalues of \( L \) are given by the poles of the resolvent (25), or equivalently by the complex roots of the function \( W(\lambda) \); these can be computed through a lengthy but elementary calculation, here omitted.

**Remark IV.7.** As a by-product, the previous analysis of the complex roots of \( W(\lambda) \) reveals the presence of a resonance at the endpoints of essential spectrum for the case \( \sigma = \sqrt{2} \).

**B. Dispersive estimates for the linearized problem in the case \( \sigma \in (0, 1/\sqrt{2}) \)**

In this section we focus on the case \( \sigma \in (0, 1/\sqrt{2}) \) and study the behaviour for large \( t \) of the propagator \( e^{-Lt} \) restricted to the subspace associated to the essential spectrum of the operator \( L \).

In order to achieve an effective estimate, the following weighted \( L^p \) spaces are needed

\[
L^1_w(\mathbb{R}^3) = \left\{ f : \mathbb{R}^3 \to \mathbb{C} : \int_{\mathbb{R}^3} w(x)|f(x)|dx < +\infty \right\},
\]

and

\[
L^\infty_{w^{-1}}(\mathbb{R}^3) = \left\{ f : \mathbb{R}^3 \to \mathbb{C} : \text{esssup}_{x \in \mathbb{R}^3} (w(x))^{-1}|f(x)| < +\infty \right\},
\]

where \( w(x) = 1 + \frac{1}{|x|} \). The use of such spaces is due to the singularity of the elements of \( \Pi \). A similar choice was made in [10] for the sake of deriving dispersive estimates in the case of \( N \) delta interactions in \( \mathbb{R}^3 \).

**Theorem IV.8.** There exists a constant \( C > 0 \) such that

\[
\left| \frac{1}{2\pi i} \int_{\mathbb{R}^3} \int_{C_+ \cup C_-} (R(\lambda + 0) - R(\lambda - 0))(x)e^{-\lambda t}f(y) d\lambda dy \right| \leq C \left( \frac{1}{|x|} \right)^{3/2} \int_{\mathbb{R}^3} \left( 1 + \frac{1}{|y|} \right) |f(y)|dy
\]

for any \( f \in L^1_w(\mathbb{R}^3) \), where

\[
C_+ = \{ \lambda \in \mathbb{C} : \Re(\lambda) = 0 \text{ and } \Im(\lambda) \geq \omega \}, \quad C_- = \{ \lambda \in \mathbb{C} : \Re(\lambda) = 0 \text{ and } \Im(\lambda) \leq -\omega \}.
\]
Proof. One can compute the propagator $e^{-Lt}$ as the inverse Laplace transform of the resolvent of $L$.

In particular, by Theorem [V.2] and applying the residue theorem, it follows that for $t > 0$

$$e^{-Lt} = \frac{1}{2\pi i} \int_{iR+0} R(\lambda) e^{-\lambda t} d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=r} R(\lambda) e^{-\lambda t} d\lambda + \frac{1}{2\pi i} \int_{C_+ \cup C_-} (R(\lambda + 0) - R(\lambda - 0)) e^{-\lambda t} d\lambda,$$

with $r \in (0, \omega)$ and $R(\lambda \pm 0) = \lim_{\epsilon \to 0^\pm} R(\lambda \pm \epsilon)$.

We show the computations only for the component $R_{1,1}(\lambda)$ of the resolvent whose analytic expression is given in (23) and (26), since the other components can be handled in the same way.

Recalling the definition of $\alpha_1$ and $\alpha_2$ given in equation (18), $R_{1,1}(\lambda)$ can be written as an integral kernel, namely

$$R_{1,1}(\lambda; x, y) = i \frac{e^{i\sqrt{-\omega - i\lambda}|x - y|} - e^{i\sqrt{-\omega - i\lambda}|x - y|}}{8\pi |x - y|} +$$

$$+ \frac{-\sigma \sqrt{\omega} e^{i\sqrt{-\omega - i\lambda}|x|} e^{i\sqrt{-\omega - i\lambda}|y|} + |(\sigma + 1)\sqrt{\omega} + i\sqrt{-\omega + i\lambda}| e^{i\sqrt{-\omega - i\lambda}(|x| + |y|)}|}{8\pi |x||y|((2\sigma + 1)\omega + i(\sigma + 1)\sqrt{\omega} (\sqrt{-\omega - i\lambda + \sqrt{-\omega + i\lambda}}) - \sqrt{-\omega - i\lambda \sqrt{-\omega + i\lambda}})} +$$

$$- \frac{i}{8\pi |x||y|((2\sigma + 1)\omega + i(\sigma + 1)\sqrt{\omega} (\sqrt{-\omega - i\lambda + \sqrt{-\omega + i\lambda}}) - \sqrt{-\omega - i\lambda \sqrt{-\omega + i\lambda}}).}$$

Since from equation (33) it is clear that the computation of the integral on $C_+$ and on $C_-$ are analogous, we treat the cut $C_+$ only. On $C_+$, $\sqrt{-\omega + i\lambda}$ is continuous while, by the prescription $\Im(\sqrt{-\omega \pm i\lambda}) > 0$, considering $\epsilon$ as a real parameter, one has

$$\lim_{\epsilon \to 0^+} \sqrt{-\omega - i(\lambda + \epsilon)} = -\lim_{\epsilon \to 0^+} \sqrt{-\omega - i(\lambda - \epsilon)} = -\sqrt{-\omega - i\lambda}.$$

Performing the change of variable $k = \sqrt{-\omega - i\lambda}$, one can write

$$\int_{C_+} (R_{1,1}(\lambda + 0) - R_{1,1}(\lambda - 0)) e^{-\lambda t} d\lambda = i e^{-i\omega t} \int_{-\infty}^{+\infty} F(k) 2ke^{-ikt} dk,$$

where $F$ is the function $R(\lambda + 0) - R(\lambda - 0)$ expressed in the variable $k$.

The function $R_{1,1}$ defined in (33) is the sum of a convolution summand $R_{*,1,1}$ and a multiplication summand $R_{m,1,1}$, where

$$R_{*,1,1}(\lambda; x, y) = i \frac{e^{i\sqrt{-\omega + i\lambda}|x - y|} - e^{i\sqrt{-\omega - i\lambda}|x - y|}}{8\pi |x - y|}$$

and

$$R_{m,1,1}(\lambda; x, y) = i \frac{-\sigma \sqrt{\omega} e^{i\sqrt{-\omega - i\lambda}|x|} e^{i\sqrt{-\omega + i\lambda}|y|} + |(\sigma + 1)\sqrt{\omega} + i\sqrt{-\omega + i\lambda}| e^{i\sqrt{-\omega - i\lambda}(|x| + |y|)}|}{8\pi |x||y|((2\sigma + 1)\omega + i(\sigma + 1)\sqrt{\omega} (\sqrt{-\omega - i\lambda + \sqrt{-\omega + i\lambda}}) - \sqrt{-\omega - i\lambda \sqrt{-\omega + i\lambda}}).}$$

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\[-i \frac{[(\sigma + 1)\sqrt{\omega + i\sqrt{\omega + i\lambda}} - \sigma \sqrt{\omega e^{i\sqrt{\omega + i\lambda}}}}{8\pi|x||y| - i(2\sigma + 1)\omega + i(\sigma + 1)\sqrt{\omega (\sqrt{\omega - i\lambda + \sqrt{-\omega + i\lambda}) - \sqrt{-\omega - i\lambda\sqrt{-\omega + i\lambda}})}]
\]

Then we can define

\[F_\ast(k) = R_{\ast,1,1}(\lambda + 0) - R_{\ast,1,1}(\lambda - 0), \quad \text{and} \quad F_m(k) = R_{m,1,1}(\lambda + 0) - R_{m,1,1}(\lambda - 0).\]

One can easily compute \(F_\ast\) and gets \(F_\ast(k) = \frac{-\sin((x-y)k)}{4\pi|x-y|}\). Thus, by formula 3.851 in [19],

\[
\int_{-\infty}^{+\infty} F_\ast(k)2ke^{-ik^2}dk = \sin(|x-y|k)dk - \frac{1 + i}{16\sqrt{\pi}}t^{-\frac{3}{2}}e^{i\frac{|x-y|^2}{4}},
\]

for any \(t > 0\). Hence

\[
\left|\frac{1}{2\pi i} \int_{\mathbb{R}} \int_{-\infty}^{+\infty} F_\ast(k; y)dkf(y)dy\right| \leq \frac{1}{8\sqrt{\pi}}t^{-\frac{3}{2}}\int_{\mathbb{R}} |f(y)|dy. \quad (35)
\]

Let us estimate \(\int_{-\infty}^{+\infty} F_m(k)2ke^{-ik^2}dk\). One can notice that \(F_m(k)\) is the sum of terms of the form \(\frac{i}{8\pi|x|g(k)e^{\pm ik}}\), where \(g(k)\) is a rational function of \(k\) and \(\sqrt{-2\omega - k^2}\) possibly multiplied by \(e^{i\sqrt{-2\omega - k^2}}\), and \(s\) can be \(0, |x|, |y|\) or \(|x| + |y|\). Let us consider the term

\[g(k)e^{-ik(|x|+|y|)} = \frac{(\sigma + 1)\sqrt{\omega + i\sqrt{-2\omega - k^2}}}{(2\sigma + 1)\omega + i(\sigma + 1)\sqrt{\omega(\sqrt{-2\omega - k^2} - k) + k\sqrt{-2\omega - k^2}}}e^{-ik(|x|+|y|)},\]

which results from the second term in (33) referred to \(R_{m,1,1}(\lambda + 0)\).

Notice that \(g \in C^1(\mathbb{R}, \mathbb{C})\) and \(|g(k)| \sim \frac{1}{k}\) as \(k \to +\infty\), hence \(g \in L^2(\mathbb{R})\). Moreover,

\[
\frac{dg}{dk}(k) = \frac{-ik}{(2\sigma + 1)\omega + i(\sigma + 1)\sqrt{\omega(\sqrt{-2\omega - k^2} - k) + k\sqrt{-2\omega - k^2}}} + \frac{(\sigma + 1)\sqrt{\omega + i\sqrt{-2\omega - k^2}}}{(2\sigma + 1)\omega + i(\sigma + 1)\sqrt{\omega(\sqrt{-2\omega - k^2} - k) + k\sqrt{-2\omega - k^2}}}2
\]

\[
\cdot \left( -i(\sigma + 1)\sqrt{\omega}k + i(\sigma + 1)\sqrt{\omega + \sqrt{-2\omega - k^2} - k} - \frac{k^2}{\sqrt{-2\omega - k^2}} \right),
\]

which belongs to \(L^2(\mathbb{R})\) too, so \(g\) is an element of \(H^1(\mathbb{R})\), and as consequence \(\tilde{g} \in L^1(\mathbb{R})\), where \(\tilde{g}\) is the inverse Fourier transform of \(g\). Furthermore, one can compute the inverse Fourier transform of \(2ke^{-ik^2}\) as

\[
U_t(s) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} 2ke^{-ik^2}e^{-iks}dk = \frac{1}{(4\pi it)^{\frac{3}{2}}}e^{-\frac{s^2}{4it}}.
\]
From the last identity it follows
\[
\left| \frac{1}{2\pi i} \int_{\mathbb{R}^3} \int_{-\infty}^{+\infty} \frac{i}{8\pi|y|} g(k)e^{-ik(|x|+|y|)}2ke^{-ik2}dkf(y)dy \right| =
\]
\[
= \left| \int_{\mathbb{R}^3} \int_{-\infty}^{+\infty} \frac{1}{8\pi|y|} \tilde{g}(u)(u - |x| - |y|)duf(y)dy \right| \leq C \frac{1}{|x|} t^{-\frac{3}{2}} \int_{\mathbb{R}^3} |f(y)|dy,
\]
where the last inequality follows from Hölder inequality and \( C > 0 \). The other terms in \( F_m(k) \) are handled in an analogous way so we do not give details.

Summing up, let \( f \in L^1_w(\mathbb{R}^3) \). Then
\[
\left| \frac{1}{2\pi i} \int_{\mathbb{R}^3} \int_{C_+ \cup C_-} \left( R(\lambda + 0) - R(\lambda - 0) \right)e^{-\lambda t}d\lambda f(y)dy \right| \leq \frac{1}{2\pi} \left( \int_{\mathbb{R}^3} \left| \int_{C_+} \left( R(\lambda + 0) - R(\lambda - 0) \right)e^{-\lambda t}d\lambda f(y)dy \right| dy + \right.
\]
\[
+ \int_{\mathbb{R}^3} \left| \int_{C_-} \left( R(\lambda + 0) - R(\lambda - 0) \right)e^{-\lambda t}d\lambda f(y)dy \right| dy = \frac{1}{2\pi} (I + II).
\]

Let us estimate the integral \( I \). Thanks to the estimates (35) and (36) one has
\[
I = \int_{\mathbb{R}^3} \left| \int_{-\infty}^{+\infty} F(k)2ke^{-ik2}dkf(y)dy \right| \leq \int_{\mathbb{R}^3} \left| \int_{-\infty}^{+\infty} F_+(k)2ke^{-ik2}dkf(y)dy \right| + \int_{\mathbb{R}^3} \left| \int_{-\infty}^{+\infty} F_m(k)2ke^{-ik2}dk \right| dy \leq
\]
\[
\leq Ct^{-3/2} \left( \int_{\mathbb{R}^3} |f(y)|dy + \frac{1}{|x|} \int_{\mathbb{R}^3} \frac{|f(y)|}{|y|}dy \right) \leq C \left( 1 + \frac{1}{|x|} \right) t^{-3/2} \int_{\mathbb{R}^3} |f(y)| \left( 1 + \frac{1}{|y|} \right)dy.
\]
The integral \( II \) can be estimated in the same way, which completes the proof. \( \square \)

**Remark IV.9.** Evaluating the propagator \( e^{-Lt} \) at \( t = 0 \) one gets
\[
1 = \frac{1}{2\pi i} \int_{|\lambda| = \rho} R(\lambda)d\lambda + \frac{1}{2\pi i} \int_{C_+ \cup C_-} \left( R(\lambda + 0) - R(\lambda - 0) \right)d\lambda = P_0 + P_c.
\]

From Lemma [V.1] it will follow that the operators \( P_0 \) and \( P_c \) are symplectic projectors onto the subspaces associated to generalized kernel and to the continuous spectrum respectively. Finally, let us note that explicitly integrating the resolvent around its poles it turns out that the dynamics along the generalized kernel grows linearly in time. This fact is proved in Appendix C.
V. MODULATION EQUATIONS

In this section we restrict to the case $\sigma \in (0, 1/\sqrt{2})$, summarize the main technical steps and give some preliminary results towards the proof of asymptotic stability of standing waves. In particular, we write the so-called modulation equations that rule the evolution of a perturbed standing wave when splitted in a solitary component and a fluctuating one. We recall once more that the scalar product we adopt is the real scalar product on the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C})$ defined in (11). In order to make the reading easier, let us give a brief outline of the strategy to be employed. We follow the roadmap of the classical papers [25], [26], [5], [6], also adopted for the model with concentrated nonlinearity in dimension one in [4] and [22]. More specifically, we decompose the dynamics in the neighbourhood of the solitary manifold in a “longitudinal” and a “transversal” component with respect to the generalized kernel $N_g(L)$, given in Theorem IV.1 of the linearized operator $L$. In order to perform the required analysis, we exploit the symplectic structure introduced in Section II A. Let us begin by noticing that the solitary manifold $\mathcal{M}$ defined in (17) is a symplectic submanifold of $(L^2(\mathbb{R}^3, \mathbb{C}), \Omega)$, invariant under the flow of (7). Its tangent space at the standing wave $e^{i\theta}\Phi_\omega$ when $\theta = 0$ is two-dimensional and is generated by the vectors $\frac{d}{d\theta}(e^{i\theta}\Phi_\omega)_{\theta=0}$ and $\frac{d}{d\omega}(e^{i\theta}\Phi_\omega)_{\theta=0}$, in real representation given by

$$
\frac{d}{d\theta}(e^{i\theta}\Phi_\omega)_{\theta=0} \mapsto e_1 = \begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix} \quad \text{and} \quad \frac{d}{d\omega}(e^{i\theta}\Phi_\omega)_{\theta=0} \mapsto e_2 = \begin{pmatrix} \varphi_\omega \\ 0 \end{pmatrix},
$$

where $\varphi_\omega = \frac{d}{d\omega}\Phi_\omega$ was defined in Section IV. However, when no confusion arises we use the shorthand expressions $\Phi_\omega$ and $\varphi_\omega$ with the meaning of the corresponding real representative vectors (second component vanishing). As already remarked the couple of vectors $\{e_1, e_2\}$ is a basis for $N_g(L)$. It immediately seen that $\Omega(e_1, e_2) = \frac{1}{2} \frac{d}{d\omega}||\Psi_\omega||^2 \neq 0$, thanks to the condition $\sigma \in (0, 1/\sqrt{2})$ guaranteeing orbital stability. So the symplectic form is nondegenerate on the solitary manifold $\mathcal{M}$, which is a symplectic submanifold. By its very definition, $\mathcal{M}$ is invariant for the flow of (7).

The following lemma establishes the relation between the spectral projection $P_0$ introduced in Remark IV.9 and the symplectic projection onto the solitary manifold.

**Lemma V.1.** Let $\Delta = \frac{1}{2} \frac{d}{d\omega}||\Phi_\omega||^2_{L^2}$, then for any $f \in L^2(\mathbb{R}^3)$

$$
P_0 f = \frac{1}{\Delta} \Omega(f, \varphi_\omega) J\Phi_\omega - \frac{1}{\Delta} \Omega(f, J\Phi_\omega) \varphi_\omega ,
$$

where $\Omega(\cdot, \cdot)$ was defined in (13).
Proof. The explicit expression of the spectral projection $P_0 = \frac{1}{2\pi i} \int_{|\lambda|=r} R(\lambda) d\lambda$ can be recovered by Appendix C, and the equivalence with the r.h.s. follows by straightforward calculations.

Notice that the given representation of $P_0$ is well defined thanks to the fact that $\Delta > 0$, again as a consequence of the choice $\sigma \in (0, 1/\sqrt{2})$. Moreover, $P_0$ is a symplectically orthogonal projection, in the sense that given a couple $\{\zeta, f\}$ with $\zeta \in \text{Im} P_0$ and $f \in \text{Ker} P_0$, one has $\Omega(\zeta, f) = 0$. In particular it is useful to note that due to the definition of symplectic form $\Omega$, a state $f$ with vanishing component along the continuous spectrum of $L$ is orthogonal to the vectors $Je_1$ and $Je_2$, or in complex notation, to $\Phi_\omega$ and $i \frac{d}{d\omega} \Phi_\omega = i \varphi_\omega$.

After these preliminaries, as anticipated in formula (9), we write the solution to (7) as

$$u(t, x) = e^{i \Theta(t)} \left( \Phi_\omega(t)(x) + \chi(t, x) \right), \quad \Theta(t) = \int_0^t \omega(s) ds + \gamma(t),$$

with the final goal of proving that the solution decomposes in the sum of a solitary component and a dispersive one.

The local splitting of the invariant symplectic manifold $(L^2(\mathbb{R}^3, \mathbb{C}), \Omega)$ in two symplectically orthogonal manifolds, the finite dimensional solitary manifold $\mathcal{M}$ and the infinite dimensional range of the spectral projection on the continuous spectrum, suggests to symplectically project the flow according to this decomposition (see also Remark IV.9), in order to obtain the so called modulation equations. The projection along $\mathcal{M}$ ("longitudinal") gives rise to two ordinary differential equations for the frequency $\omega$ and the phase $\gamma$ of the solitary wave, depending parametrically on the fluctuating component $\chi$; while the projection on the continuous spectrum ("transversal") gives a partial differential equation for the remainder $\chi$ (with coefficients depending on $\gamma$ and $\omega$). The solution to the equation for the $\chi$ component will be shown to decay in time in suitable norms. As a consequence, one has the asymptotic behaviour of the solutions for the parameters $\omega$ and $\gamma$ of the solitary wave, to be shown in Section 6, and finally asymptotic stability, which will be the subject of Section 7.

To deduce the modulation equations it proves convenient to make use of the variational formulation of equation (7)

$$\left( i \frac{du}{dt}(t), v \right)_{L^2} = E'[u(t)](v) \quad \forall v \in V.$$  \hspace{1cm} (39)

To begin with, we replace in the previous equation the Ansatz (38).

By equation (16) and Proposition II.2, equation (39) can be rephrased as

$$\left( i \frac{d\chi}{dt}(t), v \right)_{L^2} = Q_{\alpha, \text{Lin}}(\chi(t), v) + \dot{\gamma}(t)(\Phi_\omega(t) + \chi(t), v)_{L^2} + \dot{\omega}(t) \left( -i \frac{d\Phi_\omega(t)}{d\omega}, v \right)_{L^2} + N(q_\chi(t), q_v(t))$$

(40)
for any $v \in V$.

Here $Q_{\alpha, Lin}$ is the quadratic form of the operator $D$ defined in (20) and acting as

$$Q_{\alpha, Lin}(\chi, v) = (\nabla \phi \chi, \nabla \phi v)_{L^2} - \frac{\sqrt{\omega}}{4\pi} \Re(q \chi \overline{q v}) - \frac{\sigma}{2\pi} \Re q \chi \Re q v + \omega(\chi, v)_{L^2},$$

and the nonlinear remainder $N(q \chi, q v)$ is given by

$$N(q \chi, q v) = -\nu |q \chi + q \omega|^2 \Re((q \chi + q \omega) \overline{q v}) + \nu (2\sigma + 1) |q \omega|^2 \Re q \chi \Re q v + \nu |q \omega|^2 \Im q \chi \Im q v + \nu |q \omega|^2 \Re(q \omega \overline{q v}).$$

In the previous equation, according to Section II B, $q \omega = \left(\frac{\sqrt{\omega}}{4\pi \nu}\right)^{\frac{1}{2\sigma}}$.

**Remark V.2.** The remainder $N(q \chi, q v)$ depends nonlinearly on $\chi$ (and $\omega$) and it is real linear in $v$; so, by Riesz representation theorem and with a slight abuse of notation, there exist a vector $N(q \chi)$ such that $N(q \chi, q v) = \Re(N(q \chi) \overline{q v})$. It is a peculiarity of this model that in fact it depends just on the charges of $\chi$ and $v$. Moreover, by its very definition, the remainder is the difference between the action of the complete vector field and its linear part at the solitary wave, and so it is quadratic in $q \chi$ near $\chi = 0$.

Corresponding expressions can be given with obvious modification in purely real form, which we omit for the sake of brevity. Since $\omega$, $\gamma$ and $\chi$ are all unknown the Ansatz makes the problem underdetermined, and a supplementary condition is needed to give a unique representation of the solution; a way to close the system for $\omega$, $\gamma$ and $\chi$ is to require that the $\chi$ component is decoupled from the discrete spectrum, i.e. $P_0 \chi = 0$, or equivalently to project equation (40) onto the symplectically orthogonal complement of the generalized kernel of $L$. The corresponding modulation equations take different forms according to the way one writes the projection and we give two of them for future reference. In the following we denote by $Q_L$ the bilinear form associated to the linear nonselfadjoint operator $L$.

**Theorem V.3. (Modulation equations I)** Let $\chi$ be a solution to equation (40) such that $P_0 \chi(t) = 0$ for all $t \geq 0$, and let the functions $\omega$ and $\gamma$ belong to $C^1(\mathbb{R})$; then $\omega$ and $\gamma$ solve the equations

$$\dot{\omega} = \Re \left( \frac{(JN(q \chi) q P_0^\ast (\Phi \omega + \chi))}{(\varphi \omega - \frac{dP_0^\ast}{d\omega} \chi, \Phi \omega + \chi)_{L^2}} \right),$$

and

$$\dot{\gamma} = \Re \left( \frac{(JN(q \chi) q J(\varphi \omega - \frac{dP_0}{d\omega} \chi))}{(\varphi \omega - \frac{dP_0}{d\omega} \chi, \Phi \omega + \chi)_{L^2}} \right).$$

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Proof. We adapt the reasoning in [7]. Equation (40) is equivalent to

\[
\left( \frac{d}{dt}(\Phi_\omega + \chi), v \right)_{L^2} = Q_L(\chi, v) + \dot{\gamma} \left( J(\Phi_\omega + \chi), v \right)_{L^2} + \Re(JN(q_\chi \overline{q} v)) \quad \forall v \in V.
\]  

(43)

Set \( v = P_0^* (\Phi_\omega + \chi) \); notice that differentiating in time \( P_0^* \chi = 0 \), one has

\[
P_0 \frac{d}{dt}(\Phi_\omega + \chi) = \dot{\omega} \left( \varphi - \frac{dP_0}{d\omega} \chi \right),
\]

where expressions such as \( \frac{dP_0}{d\omega} \chi \) are computed from the representation given in (37). Moreover, one immediately has the identities

\[
Q_L(\chi, P_0^* (\Phi_\omega + \chi)) = Q_L(P_0 \chi, (\Phi_\omega + \chi)) = 0
\]

and, using \( P_0 J = JP_0^* \),

\[
(J(\Phi_\omega + \chi), P_0^* (\Phi_\omega + \chi))_{L^2} = (J(\Phi_\omega + \chi), (P_0^*)^2(\Phi_\omega + \chi))_{L^2} = (JP_0^* (\Phi_\omega + \chi), P_0^* (\Phi_\omega + \chi))_{L^2} = 0.
\]

So one remains with

\[
\left( P_0 \frac{d}{dt}(\Phi_\omega + \chi), \Phi_\omega + \chi \right)_{L^2} = \Re(JN(q_\chi \overline{q} P_0^* (\Phi_\omega + \chi)))
\]

from which the equation for \( \dot{\omega} \) follows.

Now let us consider the test function \( JP_0 \frac{d}{dt}(\Phi_\omega + \chi) \), and notice the following facts, in which use is made of \( JP_0 = P_0^* J \).

\[
\left( \frac{d}{dt}(\Phi_\omega + \chi), JP_0 (\Phi_\omega + \chi) \right)_{L^2} = \left( \frac{d}{dt}(\Phi_\omega + \chi), JP_0^2 (\Phi_\omega + \chi) \right)_{L^2} = \left( P_0 \frac{d}{dt}(\Phi_\omega + \chi), JP_0 (\Phi_\omega + \chi) \right)_{L^2} = 0;
\]

\[
Q_L(\chi, JP_0 \frac{d}{dt}(\Phi_\omega + \chi)) = 0.
\]

It follows from the weak equation (43)

\[
\dot{\gamma} \left( \Phi_\omega + \chi, P_0 \frac{d}{dt}(\Phi_\omega + \chi) \right)_{L^2} = \Re(JN(q_\chi \overline{q} JP_0 \frac{d}{dt}(\Phi_\omega + \chi))
\]

and hence, after substituting the expression of \( P_0 \frac{d}{dt}(\Phi_\omega + \chi) \) determined above and cancelation of \( \dot{\omega} \) the equation for \( \dot{\gamma} \) follows. This ends the proof.

Two properties of the modulation equations which will be useful in the subsequent analysis are the following.
Corollary V.4. Under the hypotheses of Theorem V.3 and if it is known that \( \| \chi \|_{L^1_w} \) is sufficiently small, the right hand sides of (41) and (42) are smooth and there exists a continuous function \( R = R(\omega, \| \chi \|_{L^1_w}) \) such that, for any \( t \geq 0 \),
\[
|\dot{\omega}(t)| \leq R|q_\chi(t)|^2 \quad \text{and} \quad |\dot{\gamma}(t)| \leq R|q_\chi(t)|^2.
\]

The proof of the previous result is a consequence of two facts. In the first place \( (\varphi_\omega, \Psi_\omega)_{L^2} = \frac{1}{2\pi} \| \Phi_\omega \|_2^2 > 0 \) by condition \( \sigma \in (0, 1/\sqrt{2}) \) which gives orbital stability; secondarily, the nonlinear part in (40) actually depends only on the charges \( q_\chi \) and \( q_\nu \); provided that \( |q_\chi| \leq c \), there exists a positive constant \( C > 0 \) such that the denominators in (41) and (42) are strictly away from zero and
\[
|N(q_\chi)| \leq C|q_\chi|^2, \quad \forall \chi \in V.
\]

The second property concerns the compatibility of the orthogonality condition of the fluctuating part \( \chi \) with arbitrary choices of initial data. The following lemma assures in fact that the orthogonality condition \( P_0 \chi = 0 \) can be satisfied at the initial time in the neighbourhood of the solitary manifold without loss of generality.

Lemma V.5. Let \( u(t) \in C(\mathbb{R}^+, V) \) be a solution to equation (7) with \( u(0) = u_0 \in V \cap L^1_w \) and assume
\[
d = \| u_0 - e^{i\theta_0} \Phi_\omega \|_{V \cap L^1_w} \ll 1,
\]
for some \( \omega_0 > 0 \) and \( \theta_0 \in \mathbb{R} \).

Then, there exists a stationary wave \( e^{i\theta_0} \Phi_\omega \), and \( \chi_0(x) \) with \( P_0(\widetilde{\omega}_0)\chi_0 = 0 \) such that \( u_0(x) = e^{i\theta_0} (\Phi_\omega(x) + \chi_0(x)) \), and \( \| \chi_0 \|_{V \cap L^1_w} = O(d) \) as \( d \to 0 \).

The result is commonly stated as a preliminary step in the analysis of modulation equations (see for example [21], [13] and [4]). The proof is an application of the implicit function theorem making use again of the condition \( \frac{d}{dt} \| \Phi_\omega \|^2 \neq 0 \); we omit details and refer to the quoted references. As a consequence of the previous lemma, in all proofs in the rest of the paper we can assume \( P_0 \chi_0 = 0 \) where \( \chi_0 = \chi(0) \).

An equivalent form of the modulation equations for the soliton parameters \( \omega \) and \( \gamma \) can be obtained exploiting the characterization of the condition \( P_0 \chi = 0 \) through the (Hilbert) orthogonality \( (\chi, \Phi_\omega)_{L^2} = 0 = (\chi, i \varphi_\omega)_{L^2} \). In some respects they are more transparent and we give them making use of the complex writing.
Theorem V.6. (Modulation equations II) Let $\chi$ be a solution to equation (40) such that $P_0\chi(t) = 0$ for all $t \geq 0$, and let the functions $\omega$ and $\gamma$ belong to $C^1(\mathbb{R})$; then $\omega$ and $\gamma$ satisfy the equations

$$\dot{\omega} = \frac{((\chi, \varphi)_{L^2} + (\varphi, \Phi_{\omega})_{L^2})N(\chi, i\Phi_{\omega}) - (\chi, i\Phi_{\omega})_{L^2}N(\chi, \varphi)}{(\varphi, \Phi_{\omega})_{L^2}^2 - (\chi, \varphi)^2_{L^2}}$$  

$$\dot{\gamma} = \frac{((\chi, \varphi)_{L^2} - (\varphi, \Phi_{\omega})_{L^2})N(\chi, \varphi) + (\chi, \frac{d}{d\omega}\varphi_{\omega})_{L^2}N(\chi, i\Phi_{\omega})}{(\varphi, \Phi_{\omega})_{L^2}^2 - (\chi, \varphi)^2_{L^2}}$$

Proof. Differentiating in time the orthogonality conditions $(\chi, \Phi_{\omega})_{L^2} = 0 = (\chi, i\varphi)_{L^2}$, it easily follows that

$$(i\dot{\chi}, i\Phi_{\omega})_{L^2} = -\dot{\omega}(\chi, \varphi)_{L^2}, \quad (i\dot{\chi}, \varphi)_{L^2} = \dot{\omega}(\chi, i\frac{d}{d\omega}\varphi_{\omega})_{L^2}.$$ 

So testing the weak equation for $\chi$ with $i\Phi_{\omega}$ and $\varphi$ and taking into account properties of operators $L_1$ and $L_2$ and orthogonality conditions again, one obtains the system

$$\dot{\omega}((\chi, \varphi)_{L^2} - (\Phi_{\omega}, \varphi)_{L^2}) + \dot{\gamma}(\chi, i\Phi_{\omega})_{L^2} = -N(\chi, i\Phi_{\omega})$$

$$\dot{\omega}(\chi, \frac{d}{d\omega}\varphi_{\omega})_{L^2} - \dot{\gamma}((\Phi_{\omega}, \varphi)_{L^2} + (\chi, \varphi)_{L^2}) = N(\chi, \varphi).$$

The thesis follows solving for $\dot{\omega}$ and $\dot{\gamma}$.

Notice that to this second form of modulation equations apply similar remarks to the ones made for the first form. In particular, if a priori estimates on smallness of $\chi$ are known, the modulation equations are well defined thanks to the condition $\frac{d}{d\omega}\|\Phi_{\omega}\|^2 > 0$, and the analogous of Lemma [V.5] holds true.

VI. TIME DECAY OF WEAK SOLUTIONS

The goal of this section is to provide the time decay of the transversal component $\chi$ of the solution $u$ (see (31)) to equation (7); the result we achieve shows that $\chi$ is in fact not only a fluctuation, but also a decaying dispersive remainder and it paves the way to the proof of asymptotic stability of standing waves, that is given in the next section. To this end we follow the idea developed in [5],[6],[7] for the standard NLS and applied in [4] to the case of 1-d concentrated nonlinearities.
For any \( T > 0 \), define preliminarily the so-called majorant

\[ M(T) = \sup_{0 \leq t \leq T} \left[ (1 + t)^{3/2} \| \chi(t) \|_{L_w^{-1}} + (1 + t)^3 (|\dot{\gamma}(t)| + |\dot{\omega}(t)|) \right]. \]  

(46)

We aim at proving that the majorant is uniformly bounded in \( T \) by a constant \( \overline{M} = O(d) \), where \( d \) is the size of the dispersive component \( \chi \). The proof of such bound is the content of the following theorem.

**Theorem VI.1.** Let \( u \in C(\mathbb{R}^+, V) \) be a solution to equation (7) with \( u(0) = u_0 \in V \cap L_1^w \) and define \( d := \| u_0 - e^{i\theta_0} \Phi_{\omega_0} \|_{V \cap L_1^w} \), for some \( \omega_0 > 0 \) and \( \theta_0 \in \mathbb{R} \). Then, if \( d \) is sufficiently small, there are \( \omega, \gamma \in C^1(\mathbb{R}^+) \) which satisfy (41)-(42), and such that the solution \( u \) can be written as in (38).

Moreover, there is a positive constant \( \overline{M} > 0 \), depending only on the initial data, such that, for any \( T > 0 \), one has \( M(T) \leq \overline{M} \), and \( \overline{M} = O(d) \) as \( d \to 0 \). In particular

\[ \| \chi(t) \|_{L_w^{-1}} \leq \overline{M} (1 + t)^{-3/2} \quad \forall t > 0 \]  

(47)

\[ |\dot{\gamma}(t)| + |\dot{\omega}(t)| \leq \overline{M} (1 + t)^{-3} \quad \forall t > 0. \]  

(48)

The previous theorem is implied by the following proposition that is proven in Section VI C by using the results given in Sections VI A and VI B, and the dispersive properties of the linearization operator \( L \) given in Section IV B.

**Proposition VI.2.** Under the hypotheses of the previous theorem, assume that there exist some \( t_1 > 0 \) and \( \rho > 0 \) such that \( M(t_1) \leq \rho \). Then there are two positive numbers \( d_1 \) and \( \rho_1 \), independent of \( t_1 \), such that if \( d = \| \chi_0 \|_{V \cap L_1^w} < d_1 \) and \( \rho < \rho_1 \), then \( M(t_1) \leq \frac{\rho}{2} \).

Indeed, if Proposition VI.2 were true, then Theorem VI.1 would follow from the next argument: let \( \mathcal{I} \subset [0, +\infty) \) be defined as

\[ \mathcal{I} = \{ t_1 \geq 0 : \omega, \gamma \in C^1([0, t_1]), M(t_1) \leq \rho \}. \]

\( \mathcal{I} \) is obviously relatively closed in \([0, +\infty)\) with the topology induced by considering it as a subspace of \( \mathbb{R} \) with the standard Euclidean topology. On the other hand, the thesis of Proposition VI.2 and the estimates of Corollary V.4 imply that \( \mathcal{I} \) is also relatively open. Hence, the uniform estimate of Theorem VI.1 follows from the fact that \( \sup \mathcal{I} = +\infty \).
A. Frozen linearized problem

Note that the equation (40) is non autonomous. In order to make its study simpler, it is useful to exploit a further reparametrization of the solution $\chi(t)$. We fix a time $t_1 > 0$ and denote $\omega_1 = \omega(t_1)$ and $\gamma_1 = \gamma(t_1)$. Now define (in vector notation; we recall that $J$ corresponds to $-i$)

$$e^{-J\Theta(t)}\chi(t, x) = e^{-J\tilde{\Theta}(t)}\eta(t, x), \text{ where } \tilde{\Theta}(t) = \omega_1 t + \gamma_1. \quad (49)$$

The function $\eta$ satisfies the equation

$$\left( e^{J(\Theta - \tilde{\Theta})} \frac{d\eta}{dt}, v \right)_{L^2} = Q_L(e^{J(\Theta - \tilde{\Theta})}\eta, v) + (\omega_1 - \omega)(J\eta, v)_{L^2} + \dot{\gamma}(J\Phi, v)_{L^2}$$

$$- \dot{\omega} \left( \frac{d\Phi}{d\omega}, v \right)_{L^2} + JN(e^{J(\Theta - \tilde{\Theta})}\eta, q_v)_{L^2} \quad \forall v \in V. \quad (50)$$

We need a further manipulation which allows to rewrite the previous equation in a form which makes the role of reparametrization clear. To this end we need the following identities, which can be obtained from straightforward computations

- $Je^{J(\Theta - \tilde{\Theta})} = e^{J(\Theta - \tilde{\Theta})}J$;
- $Q_L(e^{J(\Theta - \tilde{\Theta})}u, v) - e^{J(\Theta - \tilde{\Theta})}Q_L(u, v) = \frac{(\sigma + 1)\sqrt{\omega}}{2\pi} \sin(\Theta - \tilde{\Theta})\sigma_3 q_u q_v$, for any $u, v \in V$, where
  $$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Making use of the previous identities, one rewrites the equation for $\eta$ as

$$\left( \frac{d\eta}{dt}, v \right)_{L^2} = (\omega_1 - \omega)(J\eta, v)_{L^2} + Q_L(\eta, v) + \left( e^{-J(\Theta - \tilde{\Theta})} \left( \dot{\gamma}J\Phi - \dot{\omega} \frac{d\Phi}{d\omega} \right), v \right)_{L^2} +$$

$$+ e^{-J(\Theta - \tilde{\Theta})} \frac{(\sigma + 1)\sqrt{\omega}}{2\pi} \sin(\Theta - \tilde{\Theta})\sigma_3 q_u q_v + e^{-J(\Theta - \tilde{\Theta})}JN(e^{J(\Theta - \tilde{\Theta})}q_v)_{L^2}, \quad \forall v \in V. \quad (51)$$

Let us define the linearization frozen at time $t_1$ as $L_I = L(\omega_1)$, and observe that for all $u, v \in V$

$$Q_L(u, v) - Q_{L_I}(u, v) = \frac{\sqrt{\omega} - \sqrt{\omega_1}}{4\pi} T q_u q_v - (\omega_1 - \omega)(Ju, v)_{L^2},$$

where $T = \begin{bmatrix} 0 & -1 \\ 2\sigma + 1 & 0 \end{bmatrix}$. Hence, equation (50) becomes

$$\left( \frac{d\eta}{dt}, v \right)_{L^2} = Q_{L_I}(\eta, v) + N_I(t, \omega, q_v)_{L^2} \quad \forall v \in V, \quad (51)$$

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where the time dependent nonlinear remainder (including now “dragging” terms due to reparametrization) is given for all $v \in V$ by

$$
N_I(t, \omega, q_\eta, q_v) = \left( e^{-J(\Theta - \tilde{\Theta})} \left( \gamma J \Phi_{\omega} - \omega \frac{d\Phi_{\omega}}{d\omega} \right), v \right)_{L^2} + \frac{\sqrt{\omega} - \sqrt{\omega_1}}{4\pi} T q_\eta q_v \\
+ e^{-J(\Theta - \tilde{\Theta})} \frac{(\sigma + 1)}{2\pi} \sin(\Theta - \tilde{\Theta}) \sigma_3 q_\eta q_v + e^{-J(\Theta - \tilde{\Theta})} JN(e^{J(\Theta - \tilde{\Theta})} q_\eta q_v).
$$

The gain in changing from original (40) for the dispersive component to equation (51) is that the latter is still non autonomous, but now the generator of the evolution is (in weak form) a sum of a fixed linear vector field (the frozen linearization $L_I$) and a nonlinear time dependent perturbation (see also [5]). This allows to use the known dispersive properties of linearization operator $L$ described in IV B.

B. Duhamel’s representation

In this subsection we write the equation (51) in Duhamel’s representation to better exploit the dispersive properties of the propagator $e^{L_I t}$. This is not a completely trivial task since our frozen equation is a variational equation and cannot be written in strong form. In order to reach our purpose, we consider (51) separating in the test function $v$ the regular and singular part accordingly to (6). So we begin by setting $v = \phi_\lambda^0 \in H^1(\mathbb{R}^3)$. We get

$$
\left( \frac{d\eta}{dt}(t), \phi_\lambda^0 \right)_{L^2} = (L_I \eta(t) + f_I(t), \phi_\lambda^0)_{L^2},
$$

where $f_I(t) = e^{-J(\Theta(t) - \tilde{\Theta}(t))} \left( \dot{\gamma}(t) J \Phi_{\omega(t)} - \dot{\omega}(t) \frac{d\Phi_{\omega(t)}}{d\omega} \right)$. Hence, by Duhamel’s principle one gets

$$
(\eta, \phi_\lambda^0)_{L^2} = \left( e^{L_I t} \eta_0 + \int_0^t e^{L_I (t-s)} f_I(s) ds, \phi_\lambda^0 \right)_{L^2}.
$$

If one considers the same equation with $v = q_v G_\lambda$ when $q_v \in \mathbb{C}$, one has

$$
\left( \frac{d\eta}{dt}(t), q_v G_\lambda \right)_{L^2} = (L_I \eta(t) + f_I(t) + g_I(t), q_v G_\lambda)_{L^2},
$$

where

$$
g_I(t) = e^{-J(\Theta(t) - \tilde{\Theta}(t))} \left( 4\sqrt{\lambda}(\sigma + 1) \sqrt{\omega(t)} \sin(\Theta(t) - \tilde{\Theta}(t)) \sigma_3 q_\eta(t) G_\lambda + \\
\right.
\left. + 8\pi \sqrt{\lambda} JN(e^{J(\Theta(t) - \tilde{\Theta}(t))} q_\eta(t)) G_\lambda \right) + 2\sqrt{\lambda}(\sqrt{\omega(t)} - \sqrt{\omega_1}) T q_\eta(t) G_\lambda,
$$

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where \(q_\eta\) is the charge of the function \(\eta\). Hence one has

\[
(\eta, q_\nu G_\lambda)_{L^2} = \left( e^{L_{I^*}} \eta_0 + \int_0^t e^{L_{I^*}(t-s)} (f_I(s) + g_I(s)) ds, q_\nu G_\lambda \right)_{L^2}.
\]

Summing up, for any \(v \in V\), the equation (51) can be rewritten as

\[
(\eta, v)_{L^2} = \left( e^{L_{I^*}} \eta_0 + \int_0^t e^{L_{I^*}(t-s)} f_I(s) ds, v \right)_{L^2} + \left( \int_0^t e^{L_{I^*}(t-s)} g_I(s) ds, q_\nu G_\lambda \right)_{L^2}.
\]

In what follows we will use the following estimate on the function \(g_I\).

**Lemma VI.3.** Under the hypotheses of Proposition VI.2 there exists a constant \(C > 0\) such that

\[
\|g_I(t)\|_{V \cap L^1_w} \leq C(\|q_\eta\|^2 + \rho \|q_\eta\|),
\]

for any \(t \leq t_1\).

**Proof.** First of all let us notice that it is possible to chose \(t_1\) in such a way that \(\omega(t) \geq c > 0\) for any \(0 \leq t \leq t_1\), then

\[
|\sqrt{\omega(t)} - \sqrt{\omega_1}| \leq C|\omega(t) - \omega_1| \leq C \int_t^{t_1} |\dot{\omega}(s)| ds \leq C \sup_{0 \leq t \leq t_1} [(1 + t)^3 |\dot{\omega}(t)|] \int_t^{t_1} (1 + s)^{-3} ds \leq C \rho,
\]

and

\[
|\Theta(t) - \tilde{\Theta}(t)| \leq \int_t^{t_1} |\dot{\omega}(t)| d\tau ds + \int_t^{t_1} |\dot{\gamma}(s)| ds \leq C \rho \int_t^{t_1} (1 + \tau)^{-3} d\tau ds + C \rho \int_t^{t_1} (1 + s)^{-3} ds \leq C \rho.
\]

The result follows since

\[
\|g_I(t)\|_{V \cap L^1_w} \leq C(\|\Theta(t) - \tilde{\Theta}(t)\|q_\eta(t)\| + |\sqrt{\omega(t)} - \sqrt{\omega_1}|\|q_\eta(t)\| + |q_\eta(t)|^2).
\]

We end the section with a technical result that allows to transfer dispersive estimates on the frozen fluctuating component \(P_c(L_I)\eta = P_c(\omega_1)\eta\) into estimates on \(\eta\). This is needed because \(\eta\) appears in the integral Duhamel’s equation where estimates have to be done, but the dispersive behaviour is at our disposal for \(P_c(\omega_1)\eta\). This is stated in the following lemma (see for analogous construction, for example, [16] and [4]).

**Lemma VI.4.** Let the hypotheses of Proposition VI.2 hold true and suppose that the quantity

\[
\sup_{0 \leq t \leq s \leq t_1} (|\omega(t) - \omega_1| + |\Theta(t) - \tilde{\Theta}(t)|) = \delta
\]

\[
\text{for any } t \leq t_1.
\]
is sufficiently small; then, for any $t \in [0,t_1]$ there is a bounded linear operator $\Pi(t) : P_c(\omega_1)(V \cap L^\infty_{w-1}) \rightarrow V \cap L^\infty_{w-1}$, and a positive constant $C = C(\delta, \omega_1) > 0$ such that $\eta(t) = \Pi(t)h(t)$, and
\[
C(\delta, \omega_1)^{-1}\|h\|_{V \cap L^\infty_{w-1}} \leq \|\eta\|_{V \cap L^\infty_{w-1}} \leq C(\delta, \omega_1)\|h\|_{V \cap L^\infty_{w-1}}.
\]

Proof. We give only a sketch of the standard proof, referring for details to the literature cited above. Set $\eta(t) = P_0(\omega_1)\eta + P_c(\omega_1)\eta = ik_1(t)\Phi_{\omega_1} + k_2(t)\frac{d}{dt}\Phi_{\omega_1} + h(t)$ . The condition $P_0\chi = 0$ makes time dependent functions $k_1$ and $k_2$ to satisfy a linear system with a source term depending on $h$; the coefficient matrix has an inverse uniformly bounded in $t$ and $t_1$ thanks to the conditions $(\Phi_{\omega_1}, \frac{d}{dt}\Phi_{\omega_1})_{L^2} > \text{const} > 0$ and $(\Phi_{\omega_1}, \frac{d}{dt}\Phi_{\omega_1})_{L^2} > \text{const} > 0$ valid for $|\omega - \omega_1|$ small enough. This gives a representation of $k_1$ and $k_2$ in terms of $h$ and as a consequence the required bound on the finite dimensional component. Now define $\Pi(t)h(t) = \eta(t) - ik_1\Phi_{\omega_1} - k_2\frac{d}{dt}\Phi_{\omega_1}$ and the complete bound follows.

\[\Box\]

C. Proof of Proposition VI.2

Estimate of $|\dot{\gamma}| + |\dot{\omega}|$.

Lemma VI.5. If $\eta \in V \cap L^\infty_{w-1}$ then its charge $q_\eta$ satisfies $|q_\eta| \leq 4\pi\|\eta\|_{L^\infty_{w-1}}$.

Proof. Since $\eta \in L^\infty_{w-1}(\mathbb{R}^3)$ then $\|\eta\|_{L^\infty_{w-1}} = \sup_{x \in \mathbb{R}^3} \left|\frac{|x|}{1+|x|}\phi_\eta(x) + \frac{q_\eta}{4\pi(1+|x|)}\right| \geq \frac{1}{4\pi}|q_\eta|$.

\[\Box\]

From the last lemma and Corollary V.3 one gets

$|\dot{\gamma}(t)| + |\dot{\omega}(t)| \leq c|q_\eta(t)|^2 \leq c_1\|q_\eta(t)\|^2_{L^\infty_{w-1}} \leq c_1(1 + t)^{-3}M(t)^2, \quad \forall t \in [0,t_1],$

with $c_1$ independent of $t_1$. Hence, one can choose $\rho_1^2 < \frac{1}{4c_1}$ and get $(1 + t)^3(|\dot{\gamma}(t)| + |\dot{\omega}(t)|) \leq c_1\rho_1^2 \leq \frac{\rho_1}{4}, \quad \forall t \in [0,t_1]$.

Estimate of $\|\eta\|_{L^\infty_{w-1}}$.

As explained in the previous section, for any $t \in [0,t_1]$ we have $\eta(t) = P_0(\omega_1)\eta(t) + P_c(\omega_1)\eta(t)$ (for the definitions of $P_0$ and $P_c$ see Remark [V.9]) and thanks to Lemma VI.3 we have $\eta(t) = \Pi h(t)$

where $\Pi(t) : P_c(\omega_1)(V \cap L^\infty_{w-1}) \rightarrow V \cap L^\infty_{w-1}$ is bounded.

In order to estimate $\|\eta\|_{L^\infty_{w-1}}$ we make use of the equation for $h$. For all $v \in V$, $h$ is a solution to
\[
\left(\frac{dh}{dt}, v\right)_{L^2} = Q_{L^1}(h, v) + (P_c(\omega_1)f_I, v)_{L^2} + (P_c(\omega_1)g_I, v)_{L^2} + (P_c(\omega_1)g_\lambda, v)_{L^2},
\]

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where \( f_I \) and \( g_I \) were defined at the beginning of Section VI B; hence, for any \( v \in V \), \( h \) satisfies

\[
(h, v)_{L^2} = \left( e^{L_I t} h_0 + \int_0^t e^{L_I(t-s)} P_c(\omega_1) f_I(s) ds, v \right)_{L^2} + \left( \int_0^t e^{L_I(t-s)} P_c(\omega_1) g_I(s) ds, q_0 G_\lambda \right)_{L^2}.
\]

In addition let us assume that \( v \in V \cap L^1_w \), hence by Hölder inequality

\[
(h, v)_{L^2} \leq \left( \| e^{L_I t} h_0 \|_{V \cap L^\infty_{w-1}} + \| \int_0^t e^{L_I(t-s)} P_c(\omega_1) f_I(s) ds \|_{V \cap L^\infty_{w-1}} \right) \| v \|_{L^1_{w}} + \]

\[
+ \left( \| \int_0^t e^{L_I(t-s)} P_c(\omega_1) g_I(s) ds \|_{V \cap L^\infty_{w-1}} \right) \| q_0 G_\lambda \|_{L^1_{w}}.
\]

Now we can apply the dispersive estimate proved in Theorem IV.8 and get

\[
\| e^{L_I t} h_0 \|_{V \cap L^\infty_{w-1}} \leq c(1 + t)^{-3/2} \| h_0 \|_{V \cap L^1_{w}} \leq c(1 + t)^{-3/2} d,
\]

where \( d \) was defined in the statement of the present proposition. Furthermore, again by Theorem IV.8,

\[
\left\| \int_0^t e^{L_I(t-s)} P_c(\omega_1) f_I(s) ds \right\|_{V \cap L^\infty_{w-1}} \leq c \int_0^t (1 + t - s)^{-3/2} \| f_I(s) \|_{V \cap L^1_{w}} ds \leq
\]

\[
\leq c \int_0^t (1 + t - s)^{-3/2} (|\dot{\gamma}(s)| + |\dot{\omega}(s)|) ds \leq c \int_0^t (1 + t - s)^{-3/2} \| \eta(s) \|_{L^\infty_{w-1}}^2 ds.
\]

Analogously, using Lemma VI.3 and Theorem IV.8,

\[
\left\| \int_0^t e^{L_I(t-s)} P_c(\omega_1) g_I(s) ds \right\|_{V \cap L^\infty_{w-1}} \leq c \int_0^t (1 + t - s)^{-3/2} \| g_I(s) \|_{V \cap L^1_{w}} ds \leq
\]

\[
\leq c \int_0^t (1 + t - s)^{-3/2} (\| \eta(s) \|_{L^\infty_{w-1}}^2 + \rho \| \eta(s) \|_{L^\infty_{w-1}}) ds.
\]

Let us define

\[
m(t) = \sup_{s \in [0, t]} (1 + s)^{3/2} \| \eta(s) \|_{L^\infty_{w-1}}.
\]

Now, using the above inequalities, Lemma 1.25, and exploiting the duality pairing defined by the inner product in \( L^2 \), it holds

\[
(1 + t)^{3/2} \| \eta(t) \|_{L^\infty_{w-1}} = (1 + t)^{3/2} \sup_{0 \neq v \in L^1_w} \left( \eta(t), v \right)_{L^2} \leq
\]

\[
\sup_{0 \neq v \in L^1_w} \| v \|_{L^1_{w}} \leq
\]
\[
\leq c \int_0^t (1 + t - s)^{-3/2} (\| \eta(s) \|_{L^\infty_{w^{-1}}}^2 + \rho \| \eta(s) \|_{L^\infty_{w^{-1}}}) ds
\]
\[
\leq c \left( d + m^2(t) \int_0^t (1 + t)^{3/2} (1 + s)^3 (1 + t - s)^{-3/2} ds + \rho m(t) \int_0^t (1 + t)^{3/2} (1 + s)^{-3/2} (1 + t - s)^{-3/2} ds \right).
\]
Observe that the constant \( c \) and both integrals appearing in the last inequality are bounded independently of \( t \), and this implies that for any \( t \in [0, t_1] \) we have
\[
m(t) \leq c(d + m^2(t_1) + \rho m(t)) \leq c(d + \rho^2) \leq c_2 d,
\]
provided \( d \) and \( \rho \) are small enough. Since the constant \( c_2 \) does not depend on \( t_1 \), we can choose \( d < \frac{\rho}{4c_2} \) and finally get
\[
m(t_1) \leq \frac{\rho}{4},
\]
concluding the proof of Proposition VI.2.

VII. ASYMPTOTIC STABILITY

Now we are in the position to prove the asymptotic stability result as stated in the next theorem. Before formulating the result, let us denote by \( U_t \) the integral kernel which defines the propagator of the free Laplacian in \( \mathbb{R}^3 \), namely
\[
U_t(x) = (4\pi it)^{-3/2} e^{i\frac{|x|^2}{4t}}.
\]

**Theorem VII.1.** Assume \( \sigma \in (0, 1/\sqrt{2}) \). Let \( u \in C(\mathbb{R}^+, V) \) be a solution to equation (7) with \( u(0) = u_0 \in V \cap L^1_w \) and denote \( d = \|u_0 - e^{i\theta_0} \Phi_{\omega_0}\|_{V \cap L^1_w} \), for some \( \omega_0 > 0 \) and \( \theta_0 \in \mathbb{R} \). Then, if \( d \) is sufficiently small, the solution \( u(t) \) can be decomposed as follows
\[
u(t) = e^{i\omega_\infty t} \Phi_{\omega_\infty} + U_t * \psi_\infty + r_\infty(t),
\]
where \( \omega_\infty > 0 \) and \( \psi_\infty, r_\infty(t) \in L^2(\mathbb{R}^3) \), with \( \|r_\infty(t)\|_{L^2} = O(t^{-5/4}) \) as \( t \to +\infty \).

**Proof.** Along the proof we assume that \( P_0(u_0 - e^{i\theta_0} \Phi_{\omega_0}) = 0 \), and we recall from Lemma VII.5 that there is no loss of generality in this choice. First of all let us notice that Theorem VII.1 implies \( \omega(t) \to \omega_\infty \), and \( \Theta(t) - \omega_\infty t \to 0 \), as \( t \to +\infty \). Next, let us define the modulated soliton as
\[
s(t, x) = e^{i\Theta(t)} \Phi_{\omega(t)}(x),
\]
and the function

\[ z(t, x) = u(t, x) - s(t, x). \]  

(55)

By equation (7) and (16) one has that, for any \( v \in V \), \( z(t) \) is also a solution to

\[ \left( \frac{dz}{dt}, v \right)_{L^2} = \Re \int_{\mathbb{R}^3} \nabla \phi_z \cdot \nabla \bar{\phi}_0 dx - \nu \Re \left( |q_u|^2 q_u - |q_s|^2 \bar{q}_s \right) + \left( \gamma s - i \omega \frac{ds}{d \omega}, v \right)_{L^2}. \]

As one can verify by direct differentiation, the solution of the last equation can be expressed as

\[ z(t, x) = U_t * z_0(x) + i \int_0^t U_{t-\tau}(x) q_z(\tau) d\tau - i \int_0^t U_{t-\tau} * f(s(\tau)) d\tau, \]

(56)

where we denoted \( f(s) = \gamma s - i \omega \frac{ds}{d \omega} \) and, according to (55), \( q_z(t) = q_u(t) - q_s(t) \). Let us consider the last integral in formula (56)

\[ \int_0^t U_{t-\tau} * f(s(\tau)) d\tau = U_t * \int_0^\infty U_{-\tau} * f(s(\tau)) d\tau - \int_t^\infty U_{t-\tau} * f(s(\tau)) d\tau, \]

and note that the regularity of \( s(t, x) \) implies \( \psi_1(x) = \int_0^\infty U_{-\tau} * f(s(\tau)) d\tau \in L^2(\mathbb{R}^3) \), and \( r_1(t, x) = -\int_t^\infty U_{t-\tau} * f(s(\tau)) d\tau \in L^2(\mathbb{R}^3) \). Moreover, from Theorem VI.1 and the unitarity of the evolution group of the free Laplacian we have \( \|r_1(t)\|_{L^2} = O(t^{-2}), t \to +\infty \).

To conclude the proof it is left to prove a similar asymptotic decomposition for the first integral in the formula (56). As before, one can write

\[ \int_0^t U_{t-\tau}(x) q_z(\tau) d\tau = U_t * \int_0^\infty U_{-\tau}(x) q_z(\tau) d\tau - \int_t^\infty U_{t-\tau}(x) q_z(\tau) d\tau. \]

First of all one needs to show that \( \psi_0(x) = \int_0^\infty U_{-\tau}(x) q_z(\tau) d\tau \) belongs to \( L^2(\mathbb{R}^3) \). To this aim, let us observe that \( \psi_0(x) = \frac{1}{(4\pi)^{3/2}} h \left( \frac{r^2}{4} \right) \), with \( h(y) = \int_0^\infty e^{-i y \tau} \tau^{-3/2} q_z(\tau) d\tau \), hence

\[ \|\psi_0\|_{L^2}^2 = \frac{1}{(4\pi)^2} \int_0^\infty \left( \frac{r^2}{4} \right)^{-1} dr \int_0^\infty |h(y)|^2 \sqrt{y} dy. \]

From the first and the last terms one gets \( \psi_0 \in L^2(\mathbb{R}^3) \) if and only if \( h \in L^2(\mathbb{R}^+, \sqrt{y} dy) \). On the other hand, one can perform the change of variable \( u = \frac{1}{r} \) in the integral function \( h \) and get

\[ h(y) = \int_0^\infty e^{-i y u} \frac{1}{\sqrt{u}} q_z \left( \frac{1}{u} \right) du = \int_0^\infty e^{-i y u} \left( \frac{1}{u} \right)^2 q_z \left( \frac{1}{u} \right) \sqrt{u} du, \]

where we set \( y = \frac{|x|^2}{4} \). Then \( \tilde{h}(u) = \frac{1}{u} q_z \left( \frac{1}{u} \right) \). Moreover, by Theorem VI.1 \( \left| \frac{1}{u} q_z \left( \frac{1}{u} \right) \right|^2 \sqrt{u} \leq \frac{u^{1/2}}{(1+u)^{3/2}} \)

then \( \tilde{h} \in L^2(\mathbb{R}^+, \sqrt{u} du) \) and hence, by Plancherel’s identity \( h \in L^2(\mathbb{R}^+, \sqrt{y} dy) \).
Finally, let us denote \( r_0 = \int_{1}^{\infty} U_{t-\tau}(x)q_z(\tau)d\tau \). As before, we have \( r_0(x) = g(\frac{x^2}{4}) \), with \( g(y) = \int_0^\infty e^{-iy/(t-\tau)}(t-\tau)^{-3/2}q_z(\tau)d\tau \). Moreover, we can set \( y = \frac{x^2}{4} \) exploit the change of variables \( u = -\frac{1}{t-\tau} \) in order to get

\[
g(y) = \int_0^\infty e^{-iyu}u^{-3/2}((1+t)^u+1)^{3/2}du.
\]

Again, Theorem VI.1 implies that \( \tilde{g}(u) = \frac{i}{u}q_z(t + \frac{1}{u}) \in L^2(\mathbb{R}^+,\sqrt{u}du) \), for any \( t \geq 0 \). In particular,

\[
\|g\|_L^2(\mathbb{R}^+,\sqrt{u}du) \leq \tilde{c} \int_0^\infty \frac{u^{3/2}}{(1+t)u+1}du \leq c(1+t)^{-5/2},
\]

for any \( t \geq 0 \), with \( \tilde{c}, c > 0 \) independent of time. Summing up, Plancherel’s identity allows us to conclude \( \|r_0\|_L^2 = O(t^{-5/4}) \) as \( t \to +\infty \).

Hence the theorem follows with \( \psi_\infty = z_0 + \psi_0 + \psi_1 \), and \( r_\infty = r_0 + r_1 \).

Appendix A: The generalized kernel of the operator \( L \)

The aim of this appendix is to provide the proof or Theorem VI.1.

Proof. It is easy to see that \( c\Phi_\omega \), with \( c \in \mathbb{C} \), is the unique family of distributional solutions to the equation

\[-\Delta u + \omega u = 0.\]

Furthermore, \( \Phi_\omega \) belongs to \( D(H_\alpha) \) but not to \( D(H_{\alpha_1}) \) since the boundary condition is not satisfied. Hence

\[
\ker(L) = \text{span} \left\{ \begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix} \right\}.
\]

Let us now consider the operator

\[
L^2 = \begin{bmatrix} -L_2L_1 & 0 \\ 0 & -L_1L_2 \end{bmatrix}.
\]

Since the operator \( L_1 \) is invertible, the following holds

\[
u \in \ker(L_1L_2) \Leftrightarrow u \in \ker(L_2), \quad \text{then} \quad \ker(L_1L_2) = \text{span}\{\Phi_\omega\},
\]

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\[ u \in \ker(L_2L_1) \iff \exists u \in D(H_{\alpha_1}) \text{ such that } L_1u = \Phi_\omega. \]

Solving the former equation one gets that \( \ker(L_1L_2) = \text{span} \{\varphi_\omega\} \). From this follows

\[
\ker(L^2) = \text{span} \left\{ \begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}, \begin{pmatrix} \varphi_\omega \\ 0 \end{pmatrix} \right\}.
\]

The operator \( L^3 \) has the following form

\[
L^3 = \begin{bmatrix} 0 & -L_2L_1 \\ L_1L_2L_1 & 0 \end{bmatrix}.
\]

As before

\[
u \in \ker(L_1L_2L_1) \iff L_1u \in \ker(L_1L_2) = \text{span} \{\Phi_\omega\} \iff \ker(L_1L_2L_1) = \text{span} \{\varphi_\omega\},
\]

\[
u \in \ker(L_2L_1L_2) \iff u \in \ker(L_2) = \text{span} \{\Phi_\omega\} \text{ or } L_2u \in \ker(L_2L_1) = \text{span} \{\varphi_\omega\}.
\]

Let us notice that the equation

\[-\triangle u + \omega u = \varphi_\omega\]

has a unique family of distributional solutions given by

\[
u(x) = \left( \frac{\sqrt{\omega}}{4\pi \nu} \right)^{\frac{1}{2}} \left[ \begin{pmatrix} c_2 \\ 2\sqrt{\omega} \end{pmatrix} + \frac{1}{16\sigma^2 \omega^2} \frac{e^{-\sqrt{\omega}|x|}}{4\pi |x|} + \frac{c_1}{2\sqrt{\omega}} \frac{e^{\sqrt{\omega}|x|}}{4\pi |x|} \right.
\]

\[
-\frac{1}{8\omega} \frac{e^{-\sqrt{\omega}|x|}}{4\pi} + \left( \frac{1}{8\sigma \omega^{\frac{3}{2}}} - \frac{1}{8\omega^{\frac{3}{2}}} \right) \frac{e^{-\sqrt{\omega}|x|}}{4\pi} \right].
\]

Notice that one must impose that \( u \) belongs to \( D(H_{\alpha_2}) \) which means that \( u \in L^2(\mathbb{R}^3) \) and satisfies the boundary condition. This is equivalent to ask the following algebraic conditions to be verified

\[
\begin{cases}
    c_1 = 0 \\
    c_2 = \frac{\sigma - 1}{8\sigma \omega^{\frac{3}{2}}}.
\end{cases}
\]

Therefore, if \( \sigma \neq 1 \), then \( \ker(L_2L_1L_2) = \text{span} \{\Phi_\omega\} \). Hence

\[
\ker(L^3) = \ker(L^2),
\]

which concludes the first part of the theorem.
In the case $\sigma = 1$ we get $\ker(L_2L_1L_2) = \{\Phi_\omega, g_\omega\}$, then
\[
\ker(L^3) = \text{span}\left\{\begin{pmatrix} 0 \\ \Phi_\omega \\ 0 \end{pmatrix}, \begin{pmatrix} \varphi_\omega \\ 0 \\ g_\omega \end{pmatrix}\right\}.
\]
With analogous computations one can prove that
\[
\ker(L^4) = \ker(L^5) = \text{span}\left\{\begin{pmatrix} 0 \\ \Phi_\omega \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \varphi_\omega \\ 0 \\ g_\omega \\ 0 \end{pmatrix}, \begin{pmatrix} h_\omega \\ 0 \end{pmatrix}\right\},
\]
which concludes the proof.

Appendix B: Proof of the resolvent formula

In this appendix we prove that the operator $(L - \lambda I)^{-1}$ is given by
\[
R(\lambda) = \begin{bmatrix} -\lambda(\lambda^2 + L_2L_1)^{-1} -L_2(\lambda^2 + L_1L_2)^{-1} \\ L_1(\lambda^2 + L_2L_1)^{-1} -\lambda(\lambda^2 + L_1L_2)^{-1} \end{bmatrix}
\]
for the resolvent of the linear operator $L$. More precisely, we prove the following proposition.

**Proposition B.1.** If $\lambda \in \mathbb{C} \setminus \sigma(L)$, then $R(\lambda)(L - \lambda I)u = u$, $\forall u \in D(L)$, and $(L - \lambda I)R(\lambda)f = f$ for any $f \in (L^2(\mathbb{R}^3))^2$.

Before proving the former proposition, let us prove the following lemma.

**Lemma B.2.** For any $\lambda \in \mathbb{C} \setminus \sigma(L)$ the following identities hold

1. $(\lambda^2 + L_2L_1)^{-1}L_1^{-1} = L_1^{-1}(\lambda^2 + L_1L_2)^{-1},$
2. $(\lambda^2 + L_1L_2)^{-1} = (\lambda^2L_1^{-1} + L_2)^{-1}L_1^{-1},$
3. $(\lambda^2 + L_1\widetilde{L}_2)^{-1}\widetilde{L}_2^{-1} = \widetilde{L}_2^{-1}(\lambda^2 + \widetilde{L}_2L_1)^{-1},$

where $\widetilde{L}_2$ is the restriction of the operator $L_2$ to the projection of its domain onto the subspace of $L^2(\mathbb{R}^3)$ associated to the continuous spectrum of $L_2$.

**Proof.** First of all, let us notice that all the inverse operators are well defined since $\lambda$ is not allowed to be a spectral point of $L$, $L_1$ is invertible and $L_2$ is restricted to a subspace on which it is invertible too.
In order to prove \( I \) we prove the following claim

\[(\lambda^2 + L_2L_1)^{-1}L_1^{-1} = (\lambda^2L_1 + L_1L_2L_1)^{-1} = L_1^{-1}(\lambda^2 + L_1L_2)^{-1}.\]

To this purpose, let us take any \( \xi \in L^2(\mathbb{R}^3) \), then one has

\[(\lambda^2 + L_2L_1)^{-1}L_1^{-1} \xi \in D(L_2L_1) \quad \text{and} \quad L_1^{-1} \xi \in D(L_1).
\]

Hence, the following chain of identities holds

\[(\lambda^2L_1 + L_1L_2L_1)(\lambda^2 + L_2L_1)^{-1}L_1^{-1} \xi = L_1(\lambda^2 + L_2L_1)(\lambda^2 + L_2L_1)^{-1}L_1^{-1} \xi = L_1L_1^{-1} \xi = \xi.
\]

On the other hand, let us take \( \eta \in D(L_1L_2L_1) \), and observe that, in particular, \( \eta \in D(L_2L_1) \). This justifies the following identities

\[(\lambda^2 + L_2L_1)^{-1}L_1^{-1}(\lambda^2L_1 + L_1L_2L_1)\eta =
\]

\[= (\lambda^2 + L_2L_1)^{-1}L_1^{-1}L_1(\lambda^2 + L_2L_1)\eta = (\lambda^2 + L_2L_1)^{-1}(\lambda^2 + L_2L_1)\eta = \eta,
\]

which concludes the proof of the first identity of the claim. The second one is proved in the same way.

The proof of \( II \) can be done in the same way exchanging \( L_1 \) with \( \tilde{L}_2 \) and \( L_2 \) with \( L_1 \).

It is left to prove \( II \). To do that, let \( \xi \) be in \( L^2(\mathbb{R}^3) \), then \((\lambda^2L_1^{-1} + L_2)^{-1}L_1^{-1} \xi \in D((\lambda^2L_1^{-1} + L_2))\) and \( L_1^{-1} \xi \in D(L_1) \). Hence, we have

\[(\lambda^2 + L_1L_2)(\lambda^2L_1^{-1} + L_2)^{-1}L_1^{-1} \xi = L_1(\lambda^2L_1^{-1} + L_2)(\lambda^2L_1^{-1} + L_2)^{-1}L_1^{-1} \xi = \xi.
\]

On the other hand, for any \( \eta \in D(L_1L_2) \) one has \( \eta \in D(L_2) \subset L^2(\mathbb{R}^3) = D(L_1^{-1}) \), which justifies

\[(\lambda^2L_1^{-1} + L_2)^{-1}L_1^{-1}(\lambda^2 + L_1L_2)\eta = (\lambda^2L_1^{-1} + L_2)^{-1}L_1^{-1}L_1(\lambda^2L_1^{-1} + L_2)\eta = \eta.
\]

\[\square\]

We can now now prove the proposition.

**Proof.** I step: proof of the first identity.

Let us recall that for \( u \in D(L) \) holds

\[R(\lambda)(L - \lambda I)u =
\]

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\[
\begin{bmatrix}
-\lambda (\lambda^2 + L_2 L_1)^{-1} & -L_2 (\lambda^2 + L_1 L_2)^{-1} \\
L_1 (\lambda^2 + L_2 L_1)^{-1} & -\lambda (\lambda^2 + L_1 L_2)^{-1}
\end{bmatrix}
\begin{bmatrix}
-\lambda & L_2 \\
-L_1 & -\lambda
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
= \begin{bmatrix} w_1 \\
w_2 \end{bmatrix},
\]
where
\[
w_1 = \lambda^2 (\lambda^2 + L_2 L_1)^{-1} u_1 + L_2 (\lambda^2 + L_1 L_2)^{-1} L_1 u_1 - \lambda (\lambda^2 + L_2 L_1)^{-1} L_2 u_2 + \lambda L_2 (\lambda^2 + L_1 L_2)^{-1} u_2,
\]
and
\[
w_2 = \lambda^2 (\lambda^2 + L_1 L_2)^{-1} u_2 + L_1 (\lambda^2 + L_2 L_1)^{-1} L_2 u_2 + \lambda (\lambda^2 + L_1 L_2)^{-1} L_1 u_1 - \lambda L_1 (\lambda^2 + L_2 L_1)^{-1} u_1.
\]
We will concentrate on the first component \(w_1\), because the second one can be treated in the same way.

The spectrum of the self-adjoint operator \(L_2\) is \([3]\)
\[
\sigma(L_2) = \{0\} \cup \omega, +\infty),
\]
where 0 is a simple eigenvalue and \(\ker(L_2) = \text{span}\{\Phi_\omega\}\). Hence, any \(u_2 \in D(L_2)\) can be decomposed as
\[
u_2 = a \Phi_\omega + g_2,
\]
where \(a \in \mathbb{C}\) and \(g_2\) belongs to the projection of \(D(L_2)\) onto the continuous spectrum of \(L_2\).

Moreover, since \(L_2 \Phi_\omega = 0\), one gets \(\Phi_\omega \in D(L_1 L_2)\) and
\[
\Phi_\omega = \frac{1}{\lambda^2} (\lambda^2 + L_1 L_2) \Phi_\omega = (\lambda^2 + L_1 L_2) \left( \frac{1}{\lambda^2} \Phi_\omega \right),
\]
which is equivalent to \((\lambda^2 + L_1 L_2)^{-1} \Phi_\omega \in \ker(L_2)\).

As a consequence, since \(L_1\) and \(\tilde{L}_2\) are invertible on their domains, one has
\[
w_1 = \lambda^2 (\lambda^2 + L_2 L_1)^{-1} L_1^{-1} L_1 u_1 + L_2 (\lambda^2 + L_1 L_2)^{-1} L_1 u_1 +
\]
\[
-\lambda (\lambda^2 + \tilde{L}_2 L_1)^{-1} \tilde{L}_2 g_2 + \lambda \tilde{L}_2 (\lambda^2 + \tilde{L}_1 L_2)^{-1} \tilde{L}_2^{-1} \tilde{L}_2 g_2,
\]
and, hence, by lemma [B.2] it follows
\[
w_1 = (\lambda^2 L_1^{-1} + L_2) (\lambda^2 + L_1 L_2)^{-1} L_1 u_1 - \lambda (\lambda^2 + \tilde{L}_2 L_1)^{-1} \tilde{L}_2 g_2 + \lambda \tilde{L}_2 (\lambda^2 + \tilde{L}_1 L_2)^{-1} \tilde{L}_2^{-1} \tilde{L}_2 g_2 =
\]
\[
= (\lambda^2 L_1^{-1} + L_2) (\lambda^2 L_1^{-1} + L_2)^{-1} L_1^{-1} L_1 u_1 = u_1.
\]

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Summing up, we proved

\[ R(\lambda)(L - \lambda I)u = u \quad \forall u \in D(L). \]

II step: proof of the second identity.

First of all let us recall that for \( f \in (L^2(\mathbb{R}^3))^2 \) one has

\[(\lambda^2 + L_2L_1)^{-1}f_1 \in D(L_2L_1) \quad \text{and} \quad (\lambda^2 + L_1L_2)^{-1}f_2 \in D(L_1L_2).\]

Hence, the following identities hold

\[(L - \lambda I)R(\lambda)f = \begin{pmatrix} -\lambda & L_2 \\ -L_1 & -\lambda \end{pmatrix} \begin{pmatrix} -\lambda(\lambda^2 + L_2L_1)^{-1} & -L_2(\lambda^2 + L_1L_2)^{-1} \\ L_1(\lambda^2 + L_2L_1)^{-1} & -\lambda(\lambda^2 + L_1L_2)^{-1} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \]

\[= \begin{pmatrix} (\lambda^2 + L_2L_1)(\lambda^2 + L_2L_1)^{-1}f_1 \\ (\lambda^2 + L_1L_2)(\lambda^2 + L_1L_2)^{-1}f_2 \end{pmatrix} = f,\]

which concludes the proof. \( \square \)

Appendix C: The dynamics generated by \( L \) along the generalized kernel

In this appendix we estimate the behaviour of the propagator of \( L \) along the eigenvalue 0. This is achieved in the following theorem in which it is proved that the dynamics has a linear growth in time along the generalized kernel.

**Theorem C.1.** For any \( r \in (0, \omega) \) the following identity holds

\[
\frac{1}{2\pi i} \int_{|\lambda| = r} R(\lambda; x, y)e^{-\lambda t}d\lambda = \\
\begin{pmatrix}
\frac{\sqrt{\sigma}}{1-\sigma} & -\frac{\sqrt{\omega}(|x| + |y|)}{2\pi |x||y|} (2\sigma \sqrt{\omega}|x| - 1) & 0 \\
\frac{\sqrt{\sigma}}{1-\sigma} & -\frac{\sqrt{\omega}(|x| + |y|)}{2\pi |x||y|} t & \frac{\sqrt{\omega}}{1-\sigma} e^{-\sqrt{\omega}(|x| + |y|)t} (2\sigma \sqrt{\omega}|y| - 1)
\end{pmatrix},
\]

for any \( x, y \in \mathbb{R}^3 \).

**Proof.** Since the convolution term of the resolvent \( R(\lambda) \) is continuous in zero it suffices to compute the integral of the multiplication term. First of all, let us note that the function

\[ f(\lambda) = \frac{4\pi i}{W(\lambda^2)} \Lambda_1(\lambda)e^{-\lambda t} = i e^{-\lambda t}. \]
Switching is the sum of a continuous function and a function with a pole of second order in zero, namely

\[
\frac{(4\pi\alpha_2 - i\sqrt{-\omega + i\lambda})e^{i\sqrt{-\omega - i\lambda}|x|} + (4\pi\alpha_2 - i\sqrt{-\omega - i\lambda})e^{i\sqrt{-\omega + i\lambda}|x|}}{8\pi|x||y|} (e^{i\sqrt{-\omega + i\lambda}|y|} - e^{i\sqrt{-\omega - i\lambda}|y|}) + \\
+ \frac{(4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})e^{i\sqrt{-\omega - i\lambda}|x|} + (4\pi\alpha_1 - i\sqrt{-\omega - i\lambda})e^{i\sqrt{-\omega + i\lambda}|x|}}{8\pi|x||y|} (e^{i\sqrt{-\omega + i\lambda}|y|} + e^{i\sqrt{-\omega - i\lambda}|y|})
\]

\[
= \frac{i}{8\pi|x||y|} \left[ 2(4\pi\alpha_2 + \sqrt{\omega})e^{-\sqrt{\omega}|x|} \left( \frac{|y|}{\sqrt{\omega}} e^{\sqrt{\omega}|y|} + o(\lambda^2) \right) + 8\pi\alpha_1 e^{-\sqrt{\omega}|x|} \left( \frac{|x|}{\sqrt{\omega}} e^{\sqrt{\omega}|x|} + o(\lambda^2) \right) + \\
+ 2i e^{-\sqrt{\omega}|y|} \left( \left( \frac{1}{\sqrt{\omega}} + |x| \right) e^{\sqrt{\omega}|x|} + o(\lambda^2) \right) \right] (\frac{1 - \sigma}{2\omega})^{-1} \sim \\
\sim - \frac{\sqrt{\omega}}{1 - \sigma} \frac{e^{-\sqrt{\omega}(|x| + |y|)}}{2\pi|x||y|} \left[ (4\pi\alpha_2 + \sqrt{\omega})|y| + (4\pi\alpha_1 + \sqrt{\omega})|x| + 1 \right] \frac{1}{\lambda}.
\]

as \( \lambda \to 0 \). Hence the function \( f(\lambda) \) has a pole of order one in zero. Then, by the Cauchy theorem one gets

\[
\frac{1}{2\pi i} \int_{|\lambda|=r} \frac{4\pi i \Lambda_1(\lambda)}{W(\lambda^2)} e^{-\lambda t} d\lambda = -\frac{\sqrt{\omega}}{1 - \sigma} \frac{e^{-\sqrt{\omega}(|x| + |y|)}}{2\pi|x||y|} \left[ (4\pi\alpha_2 + \sqrt{\omega})|y| + (4\pi\alpha_1 + \sqrt{\omega})|x| + 1 \right] = \\
= -\frac{\sqrt{\omega}}{1 - \sigma} \frac{e^{-\sqrt{\omega}(|x| + |y|)}}{2\pi|x||y|} [-2\sigma \sqrt{\omega}|x| + 1].
\]

Switching \( \alpha_1 \) to \( \alpha_2 \) and vice versa, it follows

\[
\frac{1}{2\pi i} \int_{|\lambda|=r} \frac{4\pi i \Lambda_2(\lambda)}{W(\lambda^2)} e^{-\lambda t} d\lambda = -\frac{\sqrt{\omega}}{1 - \sigma} \frac{e^{-\sqrt{\omega}(|x| + |y|)}}{2\pi|x||y|} \left[ (4\pi\alpha_1 + \sqrt{\omega})|y| + (4\pi\alpha_2 + \sqrt{\omega})|x| + 1 \right] = \\
= -\frac{\sqrt{\omega}}{1 - \sigma} \frac{e^{-\sqrt{\omega}(|x| + |y|)}}{2\pi|x||y|} [-2\sigma \sqrt{\omega}|y| + 1].
\]

On the other hand, the function

\[
\frac{4\pi i}{W(\lambda^2)} \Sigma_1(\lambda) e^{-\lambda t}
\]

is the sum of a continuous function and a function with a pole of second order in zero, namely

\[
g(\lambda) e^{-\lambda t} = \\
\frac{i(4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})e^{i\sqrt{-\omega - i\lambda}|x|} + (8\pi\alpha_1 - i\sqrt{-\omega - i\lambda})e^{i\sqrt{-\omega + i\lambda}|x|}}{W(\lambda^2)4\pi|x||y|} (e^{i\sqrt{-\omega + i\lambda}|y|} + e^{i\sqrt{-\omega - i\lambda}|y|}) e^{-\lambda t}.
\]
Note that $g(\lambda) = \sum_{k=2}^{+\infty} a_k \lambda^k$ with
\[
a_{-2} = i \frac{\omega}{1 - \sigma} \frac{4\pi\alpha_1 + \sqrt{\omega}}{\pi |x||y|} e^{-\sqrt{\omega}(|x|+|y|)}, \quad a_{-1} = 0,
\]
then, by residue theorem,
\[
\frac{1}{2\pi i} \int_{|\lambda|=r} \frac{4\pi i}{W(\lambda^2)} \Sigma_1(\lambda) e^{-\lambda t} d\lambda = -i \frac{\omega}{1 - \sigma} \frac{4\pi\alpha_1 + \sqrt{\omega}}{\pi |x||y|} e^{-\sqrt{\omega}(|x|+|y|)} t = i \frac{2\sigma \omega^3}{(1 - \sigma)\pi |x||y|} e^{-\sqrt{\omega}(|x|+|y|)} t.
\]
In the same way
\[
\frac{1}{2\pi i} \int_{|\lambda|=r} \frac{4\pi i}{W(\lambda^2)} \Sigma_2(\lambda) e^{-\lambda t} d\lambda = 0,
\]
which concludes the proof.

\[\square\]

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