On certain supercuspidal representations of symplectic groups associated with tamely ramified extensions: the formal degree conjecture and the root number conjecture

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1 Introduction

1.1 Let $F/\mathbb{Q}_p$ be a finite extension with $p \neq 2$ whose integer ring $O_F$ has unique maximal ideal $\mathfrak{p}_F$ which is generated by $\varpi_F$. The residue class field $F = O_F/\mathfrak{p}_F$ is a finite field of $q$-elements. The Weil group of $F$ is denoted by $W_F$ which is a subgroup of the absolute Galois group $\text{Gal}(\overline{F}/F)$ where $\overline{F}$ is a fixed algebraic closure of $F$ in which we will take the algebraic extensions of $F$.

Let $G$ be a connected semi-simple linear algebraic group defined over $F$. For the sake of simplicity, we will assume that $G$ splits over $F$. Then the $L$-group $L^\times G$ of $G$ is equal to the dual group $\hat{G}$ of $G$. An admissible representation $\varphi : W_F \times SL_2(\mathbb{C}) \rightarrow L^\times G$

of the Weil-Deligne group of $F$ is called a discrete parameter of $G$ over $F$ if the centralizer $A_\varphi = Z_{L^\times G}(\text{Im} \varphi)$ of the image of $\varphi$ in $L^\times G$ is a finite group. Let us denote by $D_F(G)$ the $\hat{G}$-conjugacy classes of the discrete parameters of $G$ over $F$. The conjectural parametrization of $\text{Irr}_2(G)$ (resp. $\text{Irr}_s(G)$), the set of the equivalence classes of the irreducible admissible square-integrable (resp. supercuspidal) representations of $G$, by $D_F(G)$ is (see [7, p.483, Conj.7.1] for the details)

**Conjecture 1.1.1** For every $\varphi \in D_F(G)$, there exists a finite subset $\Pi_\varphi$ of $\text{Irr}_2(G)$ such that

1) $\text{Irr}_2(G) = \bigsqcup_{\varphi \in D_F(G)} \Pi_\varphi$,

2) there exists a bijection of $\Pi_\varphi$ onto the equivalence classes $\mathcal{A}_\varphi$ of the irreducible complex linear representations of $A_\varphi$,

3) $\Pi_\varphi \subset \text{Irr}_s(G)$ if $\varphi|_{SL_2(\mathbb{C})} = 1$.

The finite set $\Pi_\varphi$ is called a $L$-packet of $\varphi$.

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According to this conjecture, any $\pi \in \text{Irr}(G)$ is determined by $\varphi \in \mathcal{D}_F(G)$ and $\sigma \in \mathcal{A}_\varphi$. So the formal degree of $\pi$ should be determined by these data. The formal degree conjecture due to Hiraga-Ichino-Ikeda [8] is (with the formulation of [7])

**Conjecture 1.1.2** The formal degree $d_\pi$ of $\pi$ with respect to the absolute value of the Euler-Poincaré measure (see [12, §3] for the details) on $G(F)$ is equal to

$$d_\pi = \frac{\dim \sigma}{|\mathcal{A}_\varphi|} \cdot \frac{\gamma(\varphi, \text{Ad}, \psi, d(x), 0)}{\gamma(\varphi_0, \text{Ad}, \psi, d(x), 0)}. $$

Here

$$\gamma(\varphi, \text{Ad}, \psi, d(x), s) = \varepsilon(\varphi, \text{Ad}, d(x), s) \cdot \frac{L(\varphi^\vee, \text{Ad}, 1 - s)}{L(\varphi, \text{Ad}, s)}$$

is the gamma-factor associated with the $\varphi$ combined with the adjoint representation $\text{Ad}$ of $G^\circ$ on its Lie algebra $\mathfrak{g}$, and a continuous additive character $\psi$ of $F$ such that $\{ x \in F \mid \psi(xO_F) = 1 \} = O_F$ and the Haar measure $d(x)$ on the additive group $F$ such that $\int_{O_F} d(x) = 1$. See [7, pp.440-441] for the details.

$\varphi_0 : W_F \times \text{SL}_2(\mathbb{C}) \xrightarrow{\text{proj.}} \text{SL}_2(\mathbb{C}) \to G^\circ$

is the principal parameter (see [7, p.447] for the definition).

The formal degree conjecture concerns with the absolute value of the epsilon-factor

$$\varepsilon(\varphi, \text{Ad}, d(x), s) = w(\varphi, \text{Ad}) \cdot q^{a(\varphi, \text{Ad})(\frac{x}{2} - s)}$$

where $a(\varphi, \text{Ad})$ is the Artin-conductor and $w(\varphi, \text{Ad})$ is the root number.

In order to state the root number conjecture, we need some notations. Let $T \subset G$ be a maximal torus split over $F$ with respect to which the root datum $(X(T), \Phi(T), X^\vee(T), \Phi^\vee(T))$

is defined. Then the dual group $G^\circ$ is, by the definition, the connected reductive complex algebraic group with a maximal torus $T^\circ$ with which its root datum is isomorphic to

$$(X^\vee(T), \Phi^\vee(T), X(T), \Phi(T)).$$

Put $2 \cdot \rho = \sum_{0 < \alpha \in \Phi(T)} \alpha$, then $\epsilon = 2 \cdot \rho(-1) \in T$ is a central element of $G$. Now the root number conjecture says that

**Conjecture 1.1.3** [7, p.493, Conj.8.3]

$$\frac{w(\varphi, \text{Ad})}{w(\varphi_0, \text{Ad})} = \pi(\epsilon)$$

where $\epsilon$ is the central element of $G$ defined above (see [7, p.492, (65)] for the details).

Since $G$ is assumed to be splits over $F$, we have $w(\varphi_0, \text{Ad}) = 1$ (see [7, p.448]).
In this paper, we will construct quite explicitly supercuspidal representations of $G(F) = Sp_{2n}(F)$ associated with a tamely ramified extension $K/F$ of degree $2n$ (Theorem 2.3.1). Here $K$ is a quadratic extension of over field $K_{+}$ of $F$. When $K/F$ is normal, we will also give candidates of Langlands parameters of the supercuspidal representations (the section 3), and will verify the validity of the formal degree conjecture (Theorem 4.3.1) and the root number conjecture (Theorem 5.3.1) with them. Surprisingly the root number conjecture is valid only if $K/F$ is not totally ramified or $K/F$ is totally ramified and

$$q - 1 \over 2 \cdot (n - 1) \equiv 0 \pmod{4}.$$

Our supercuspidal representations, denoted by $\pi_{\beta,\theta}$, are given by the compact induction $\text{ind}_{G(O_F)}^{G(F)} \delta_{\beta,\theta}$ from irreducible unitary representations $\delta_{\beta,\theta}$ of the hyperspecial compact subgroup $G(O_F) = Sp_{2n}(O_F)$. Here $\pi_{\beta,\theta}$ and $\delta_{\beta,\theta}$ are characterized each other by the conditions

1) $\delta_{\beta,\theta}$ factors through the canonical morphism $G(O_F) \to G(O_F/p_{F}^{r})$ with $r \geq 2$, and the multiplicity of $\delta_{\beta,\theta}|_{G(O_F)}$ is one,

2) any irreducible unitary representation $\delta$ of $G(O_F)$ which factors through the canonical morphism $G(O_F) \to G(O_F/p_{F}^{r})$, and a constituent of $\pi_{\beta,\theta}|_{G(O_F)}$, then $\delta = \delta_{\beta,\theta}$.

The parameters $\beta$ and $\theta$ are associated with the tamely ramified extension $K/F$, that is, $O_K = O_F[\beta]$ and $\theta$ is a certain continuous unitary character of $U_{K/K_{+}} = \{ x \in K^{\times} \mid N_{K/K_{+}}(x) = 1 \}$ (see the subsection 2.2 for the precise definitions). We have the irreducible representation $\delta_{\beta,\theta}$ by the general theory given by [16].

The candidate of Langlands parameter is given by the method of Kaletha [10]. Regard the compact group $U_{K/K_{+}}$ as the group of $F$-rational points of an elliptic torus of $Sp_{2n}$. Then, by the local Langlands correspondence of tori (see [20]) and the Langlands-Schelstad procedure ([11]) gives a group homomorphism $\varphi$ of the Weil group $W_F$ of $F$ to the dual group $G = SO_{2n+1}(\mathbb{C})$ of $Sp_{2n}$ over $F$.

Although a general theory of construction of the supercuspidal representation is given by [10, 10], that of ours is based upon a method of [13] which has an advantage of being more direct and explicit.

Note that the formal degree conjecture is proved by [15] between the supercuspidal representations of [10] and the Langlands parameters of Kaletha. In this paper, supercuspidal representations are constructed by a method different from that of [10], so it is of some interest.

The section 2 is devoted to the construction of the supercuspidal representation $\pi_{\beta,\theta}$ of $Sp_{2n}(F)$. After recalling, in the subsection 2.1 the general theory of the regular irreducible representations of the finite group $G(O_F/p_{F}^{r})$ ($r \geq 2$) given by [16], we will define the irreducible unitary representation $\delta_{\beta,\theta}$ of $Sp_{2n}(O_F)$ in the subsection 2.2. The construction of the supercuspidal representation $\pi_{\beta,\theta}$ is given in the subsection 2.3.
The candidate of Langlands parameter is given in the section 3. The local Langlands correspondence of elliptic torus (Proposition 3.1.1) and the Langlands-Schelstad procedure (the subsection 3.2) are given quite explicitly. They give a candidate of Langlands parameter

\[ \varphi : W_F \xrightarrow{\text{canonical}} W_{K/F} \xrightarrow{\varphi_1 \oplus \det \varphi_1} SO_{2n+1}(\mathbb{C}) \]

where \( \varphi_1 = \text{Ind}_{W_{K/F}}^{W_{K^s}} \tilde{\theta} \) is the induced representation from a character \( \tilde{\theta} \) of \( K^s \) to the relative Weil group \( W_{K/F} = W_F/\overline{W_K}W_K \). The character \( \tilde{\theta} \) is defined by \( \tilde{\vartheta}(x) = \vartheta(x^{1-\tau}) \) where \( \text{Gal}(K/K^+) = \langle \tau \rangle \) and \( \vartheta = c \cdot \theta \) with the character \( c \) of \( U_{K/K^+} \) which is generated by the Langlands-Schelstad procedure.

Using the explicit description of the parameter \( \varphi \), we will verify the formal degree conjecture in the section 4, and the root number conjecture in the section 5.

In section 6, we will discuss the case of \( n = 2 \) where we can define another “natural” candidate for the Langlands parameter of \( \pi_{\beta, \theta} \). The representation space of \( \text{Ind}_{W_{K/F}}^{W_{K^s}} \tilde{\theta} \), with \( \tilde{\vartheta}(x) = \theta(x^{1-\tau}) \), has \( W_{K/F} \)-quasi invariant symplectic form. Then the candidate is given by

\[ W_F \xrightarrow{\text{can.}} W_{K/F} \xrightarrow{\text{Ind}_{W_{K/F}}^{W_{K^s}} \tilde{\theta}} GSp_4(\mathbb{C}) \xrightarrow{(*)} SO_5(\mathbb{C}) \]  

(1.1)

where \( (*) \) is the accidental surjection. With respect to this parameter

1) the formal degree conjecture is valid only if \( K/F \) is unramified or totally ramified, and in this case

2) the root number conjecture is valid only if

\[ \theta(-1) = \begin{cases} 1 & : K/F \text{ is unramified,} \\ \overline{(-1)^{n+1}} & : K/F \text{ is totally ramified.} \end{cases} \]

This means that the parameter (1.1) is not the Langlands parameter of \( \pi_{\beta, \theta} \), in general.

Several basic facts on the local factor associated with representations of the Weil group are given in the appendix A.

The quasi-invariant symmetric or symplectic form in the induced representation on \( W_{K/F} \) from the characters of \( K^s \) is discussed in the appendix B.

2 Supercuspidal representations of \( Sp_{2n}(F) \)

2.1 Regular irreducible characters of hyperspecial compact subgroup

Let us recall the main results of \[16\].

Fix a continuous unitary additive character \( \psi : F \to \mathbb{C}^1 \) such that

\[ \{ x \in F \mid \psi(xO_F) = 1 \} = O_F. \]
Let $G = S_{D2n}$ be the $O_F$-group scheme such that, for any $O_F$-algebra $R$, the group of the $A$-valued point $G(A)$ is a subgroup of $GL_{2n}(R)$ defined by

$$G(R) = \{g \in GL_{2n}(R) \mid gJ_n \cdot g = J_n\}$$

where

$$J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \quad \text{where} \quad I_n = \begin{bmatrix} & \ddots & 1 \\ \vdots & \ddots & \ddots \\ 1 & \ddots & \ddots \end{bmatrix}.$$

For a matrix $A \in M_{m,n}(R)$, put $A^t = I_n \cdot A^t I_m \in M_{n,m}(R)$. Let $\mathfrak{g}$ the Lie algebra scheme of $G$ which is a closed affine $O_F$-subscheme of $\mathfrak{gl}_n$ the Lie algebra scheme of $GL_n$ defined by

$$\mathfrak{g}(R) = \{X \in \mathfrak{gl}_{2n}(R) \mid XJ_n + J_n \cdot X = 0\}$$

for all $O_F$-algebra $R$. Let

$$B : \mathfrak{gl}_{2n} \times_{O_F} \mathfrak{gl}_{2n} \to A^1_{O_F}$$

be the trace form on $\mathfrak{gl}_{2n}$, that is $B(X,Y) = \text{tr}(XY)$ for all $X,Y \in \mathfrak{gl}_{2n}(R)$ with any $O_F$-algebra $R$. Since $G$ is smooth $O_F$-group scheme, we have a canonical isomorphism

$$\mathfrak{g}(O_F)/\varpi^r \mathfrak{g}(O_F) \to \mathfrak{g}(O_F)/\mathfrak{p}^r = \mathfrak{g}(O_F) \otimes_{O_F} O_F/\mathfrak{p}^r$$

([2 Chap.II, §4, Prop.4.8]) and the canonical group homomorphism $G(O_F) \to G(O_F/\mathfrak{p}^r)$ is surjective, due to the formal smoothness [2 p.111, Cor. 4.6], whose kernel is denoted by $K_l(O_F)$. For any $0 < l < r$, let us denote by $K_l(O_F/\mathfrak{p}^r)$ the kernel of the canonical group homomorphism $G(O_F/\mathfrak{p}^r) \to G(O_F/\mathfrak{p}^r)$ which is surjective.

The following basic assumptions on $G$ are satisfied:

I) $B : \mathfrak{g}(F) \times \mathfrak{g}(F) \to F$ is non-degenerate,

II) for any integers $r = l + l'$ with $0 < l' \leq l$, we have a group isomorphism

$$\mathfrak{g}(O_F/\mathfrak{p}^{l'}) \to K_l(O_F/\mathfrak{p}^{r})$$

defined by $X (\mod \mathfrak{p}^{l'}) \mapsto 1 + \varpi^{l'}X (\mod \mathfrak{p}^{r})$,

III) if $r = 2l - 1 \geq 3$ is odd, then we have a mapping

$$\mathfrak{g}(O_F) \to K_{l-1}(O_F/\mathfrak{p}^{r})$$

defined by $X \mapsto (1 + \varpi^{-l-1}X + 2^{-l-2}\varpi^{2l-2}X^2) (\mod \mathfrak{p}^{r})$.

The condition I) implies that $B : \mathfrak{g}(O_F/\mathfrak{p}^{l}) \times \mathfrak{g}(O_F/\mathfrak{p}^{l}) \to O_F/\mathfrak{p}^{l}$ is non-degenerate for all $l > 0$, and so $B : \mathfrak{g}(O_F) \times \mathfrak{g}(O_F) \to O_F$ is also non-degenerate. By the condition II), $K_l(O_F/\mathfrak{p}^{r})$ is a commutative normal subgroup of $G(O_F/\mathfrak{p}^{r})$, and its character is

$$\chi_\beta(1 + \varpi^lX \mod \mathfrak{p}^{l'}) = \psi\left(\varpi^{-l'}B(X, \beta)\right) \quad (X \mod \mathfrak{p}^{l'} \in \mathfrak{g}(O_F/\mathfrak{p}^{l'}))$$

In this paper, an $O_F$-algebra means an unital commutative $O_F$-algebra.
with $\beta \pmod{p^r} \in \mathfrak{g}(O_F/p^r)$.

Since any finite dimensional complex continuous representation of the compact group $G(O_F)$ factors through the canonical group homomorphism $G(O_F) \to G(O_F/p^r)$ for some $0 < r \in \mathbb{Z}$, we want to know the irreducible complex representations of the finite group $G(O_F/p^r)$. Let us assume that $r > 1$ and put $r = l + l'$ with the minimal integer $l$ such that $0 < l' \leq l$, that is

$$l' = \begin{cases} l & : r = 2l, \\ l-1 & : r = 2l-1. \end{cases}$$

Let $\delta$ be an irreducible complex representation of $G(O_F/p^r)$. The Clifford’s theorem says that the restriction $\delta|_{K_i(O_F/p^r)}$ is a sum of the $G(O_F/p^r)$-conjugates of characters of $K_i(O_F/p^r)$:

$$\delta|_{K_i(O_F/p^r)} = \left( \bigoplus_{\beta \in \Omega} \chi_{\beta} \right)^m$$

with an adjoint $G(O_F/p^r)$-orbit $\Omega \subset \mathfrak{g}(O_F/p^r)$. In this way the irreducible complex representations of $G(O_F/p^r)$ correspond to adjoint $G(O_F/p^r)$-orbits in $\mathfrak{g}(O_F/p^r)$.

Fix an adjoint $G(O_F/p^r)$-orbit $\Omega \subset \mathfrak{g}(O_F/p^r)$ and let us denote by $\Omega^-$ the set of the equivalence classes of the irreducible complex representations of $G(O_F/p^r)$ correspond to $\Omega$. Then \[11\] gives a parametrization of $\Omega^-$ as follows:

**Theorem 2.1.1** Take a representative $\beta \pmod{p^r} \in \Omega$ $(\beta \in \mathfrak{g}(O_F))$ and assume that

1) the centralizer $G_{\beta} = Z_G(\beta)$ of $\beta \in \mathfrak{g}(O_F)$ in $G$ is smooth over $O_F$,

2) the characteristic polynomial $\chi_{\overline{\beta}}(t) = \det(t \cdot 1_{2n} - \overline{\beta})$ of $\overline{\beta} = \beta \pmod{p}$ $\in \mathfrak{g}(F) \subset \mathfrak{gl}_{2n}(F)$ is the minimal polynomial of $\overline{\beta} \in M_{2n}(F)$.

Then there exists a bijection $\theta \mapsto \delta_{\beta, \theta}$ of the set

$$\{ \theta \in G_{\beta}(O_F/p^r)^- \text{ s.t. } \theta = \chi_{\beta} \text{ on } G_{\beta}(O_F/p^r) \cap K_i(O_F/p^r) \}$$

onto $\Omega^-$. The correspondence $\theta \mapsto \delta_{\beta, \theta}$ is given by the following procedure. The second condition in the theorem implies

$$G_{\beta}(O_F/p^r) = G(O_F/p^r) \cap (O_F/p^r) [\beta \pmod{p^r}],$$

in particular $G_{\beta}(O_F/p^r)$ is commutative. So $G_{\beta}(O_F/p^r)^-$ means the character group of $G_{\beta}(O_F/p^r)$.

$\Omega^-$ consists of the irreducible complex representations whose restriction to $K_i(O_F/p^r)$ contains the character $\chi_{\beta}$. Then the Clifford’s theory says the followings: put

$$G(O_F/p^r; \beta) = \{ g \in G(O_F/p^r) \mid \chi_{\beta}(g^{-1}hg) = \chi_{\beta}(h) \forall h \in K_i(O_F/p^r) \}$$

$$= \{ g \in G(O_F/p^r) \mid \text{Ad}(g) \beta \equiv \beta \pmod{p^r} \}$$
and let us denote by $\text{Irr}(G(O_F/p^r; \beta), \chi_\beta)$ the set of the equivalence classes of the irreducible complex representations $\sigma$ of $G(O_F/p^r; \beta)$ such that the restriction $\sigma|_{K_i(O_F/p^r)}$ contains the character $\chi_\beta$. Then $\sigma \mapsto \text{Ind}_{G(O_F/p^r; \beta)}^{G(O_F/p^r)} \sigma$ gives a bijection of $\text{Irr}(G(O_F/p^r; \beta), \chi_\beta)$ onto $\Omega^r$.

Since $G_\beta$ is smooth over $O_F$, the canonical homomorphism $G_\beta(O_F/p^r) \to G_\beta(O_F/p^r)$ is surjective. Hence we have

$$G(O_F/p^r; \beta) = G_\beta(O_F/p^r) \cdot K_i(O_F/p^r).$$

If $r = 2l$ is even, then $l' = l$ and, for any character $\theta \in G_\beta(O_F/p^r)$ such that $\theta = \chi_\beta$ on $G_\beta(O_F/p^r) \cap K_i(O_F/p^r)$, the character

$$\sigma_{\theta, \beta}(gh) = \theta(g) \cdot \chi_\beta(h) \quad (g \in G_\beta(O_F/p^r), h \in K_i(O_F/p^r))$$

is well-defined, and $\theta \mapsto \sigma_{\theta, \beta}$ is a surjection onto $\text{Irr}(G(O_F/p^r; \beta), \chi_\beta)$. Hence

$$\theta \mapsto \delta_{\theta, \beta} = \text{Ind}_{G(O_F/p^r; \beta)}^{G(O_F/p^r)} \sigma_{\theta, \beta}$$

is the bijection of Theorem [2.1.1].

If $r = 2l - 1$ is odd, then $l' = l - 1$. Let us denote by $g_\beta = \text{Lie}(G_\beta)$ the Lie algebra $O_F$-scheme of the smooth $O_F$-group scheme $G_\beta$. Then

$$\mathcal{V}_\beta = g(F)/g_\beta(F)$$

is a symplectic $F$-space with a symplectic $F$-form

$$D_\beta(X, Y) = B([X, Y], \overline{\mathbf{F}}) \in F \quad (X, Y \in g(F)).$$

Let $H_\beta = \mathcal{V}_\beta \times \mathbb{C}^1$ be the Heisenberg group associated with $(\mathcal{V}_\beta, D_\beta)$ and $(\sigma^\beta, L^2(\mathcal{W}))$ the Schrödinger representation of $H_\beta$ associated with a polarization $\mathcal{V}_\beta = \mathcal{W} \oplus \mathcal{W}$. More explicitly the group operation of $H_\beta$ is defined by

$$(u, s) \cdot (v, t) = (u + v, s \cdot \hat{\psi}(2^{-1} D_\beta u, v))$$

where $\hat{\psi}(\overline{x}) = \psi(x^{-1})$ for $\overline{x} = x \mod \mathfrak{p} \in F$, and the action of $h = (u, s) \in H_\beta$ on $f \in L^2(\mathcal{W}')$ (a complex-valued function on $\mathcal{W}'$) is defined by

$$(\sigma^\beta(h)f)(w) = s \cdot \hat{\chi}(2^{-1} D_\beta(u_-, u_+) + D_\beta(w, u_+)) \cdot f(w + u_-)$$

where $u = u_- + u_+ \in \mathcal{V}_\beta = \mathcal{W}' \oplus \mathcal{W}$.

Take a character $\theta : G_\beta(O_F/p^r) \to \mathbb{C}^\times$ such that

$$\theta = \chi_\beta \text{ on } G_\beta(O_F/p^r) \cap K_i(O_F/p^r).$$

Then an additive character $\rho_\theta : g_\beta(F) \to \mathbb{C}^\times$ is defined by

$$\rho_\theta(X \mod \mathfrak{p}) = \chi (-\varpi^{-1} B(X, \beta) \cdot \theta (1 + \varpi^{l-1} X + 2^{-1} \varpi^{2l-2} X^2 \mod \mathfrak{p}^r))$$

with $X \in g_\beta(O_F)$. Fix a $F$-vector subspace $V \subset g(F)$ such that $g(F) = V \oplus g_\beta(F)$. Then an irreducible representation $(\sigma^{\beta, \theta}, L^2(\mathcal{W}))$ of $K_{l-1}(O_F/p^r)$ is defined by the following proposition:
Proposition 2.1.2 Take a $g = 1 + \varpi^{l_1}T \mod p^r \in K_{l_1}(O_F/p^r)$ with $T \in g_{l_1}(O_F)$. Then we have $T \mod p^{r-1} \in g(O_F/p^{r-1})$ and

$$\sigma^\beta,\varrho(g) = \tau(\varpi^{-l}B(T,\beta) - 2^{-1}\varpi^{-1}B(T^2,\beta)) \cdot \rho_0(Y) \cdot \sigma^\beta(v,1)$$

where $T = [v] + Y \in g(F)$ with $v \in V_\beta$ and $Y \in g_\beta(F)$.

Then main result shown in [16], under the assumptions of Theorem 2.1.1, is that there exists a group homomorphism (not unique)

$$U : G_\beta(O_F/p^r) \to GL_C(L^2(\mathcal{W}))$$

such that

1) $\sigma^\beta,\varrho(h^{-1}gh) = U(h)^{-1} \circ \sigma^\beta,\varrho(g) \circ U(h)$ for all $h \in G_\beta(O_F/p^r)$ and $g \in K_{l_1}(O_F/p^r)$, and

2) $U(h) = 1$ for all $h \in G_\beta(O_F/p^r) \cap K_{l_1}(O_F/p^r)$.

Now an irreducible representation $(\sigma_\beta,\theta, L^2(\mathcal{W}))$ is defined by

$$\sigma_\beta,\theta(hg) = \theta(h) \cdot U(h) \circ \sigma^\beta,\varrho(g)$$

for $hg \in G(O_F/p^r;\beta) = G_\beta(O_F/p^r) \cdot K_{l_1}(O_F/p^r)$ with $h \in G_\beta(O_F/p^r)$ and $g \in K_{l_1}(O_F/p^r)$, and $\theta \mapsto \sigma_\beta,\theta$ is a surjection onto $\text{Irr}(G(O_F/p^r;\beta), \chi_\beta)$. Then

$$\theta \mapsto \delta_{\beta,\theta} = \text{Ind}_{G(O_F/p^r)}^{G_\beta(O_F/p^r)}(\sigma_\beta,\theta)$$

is the bijection of Theorem 2.1.1.

Because the connected $O_F$-group scheme $G = Sp_{2n}$ is reductive, that is, the fibers $G \otimes_{O_F} K$ ($K = F, \mathbb{F}$) are reductive $K$-algebraic groups, the dimension of a maximal torus in $G \otimes_{O_F} K$ is independent of $K$ which is denoted by $\text{rank}(G)$.

For any $\beta \in g(O_F)$ we have

$$\dim_K g_\beta(K) = \dim g_\beta \otimes_{O_F} K \geq \dim G_\beta \otimes_{O_F} K \geq \text{rank}(G). \ (2.2)$$

We say $\beta$ is smoothly regular over $K$ if $\dim_K g_\beta(K) = \text{rank}(G)$ (see [14] (5.7)).

In this case $G_\beta \otimes_{O_F} K$ is smooth over $K$.

Let $G_{\beta}^\alpha$ be the neutral component of $O_F$-group scheme $G_\beta$ which is a group functor of the category of $O_F$-scheme (see §3 of Exposé VIb in [3]). The following statements are equivalent;

1) $G_{\beta}^\alpha$ is representable as an smooth open $O_F$-group subscheme of $G_\beta$,

2) $G_\beta$ is smooth at the points of unit section,

3) each fibers $G_\beta \otimes_{O_F} K$ ($K = F, \mathbb{F}$) are smooth over $K$ and their dimensions are constant

(see Th. 3.10 and Cor. 4.4 of [3]). So if $\beta$ is smoothly regular over $F$ and $\mathbb{F}$, then $G_{\beta}^\alpha$ is smooth open $O_F$-group subscheme of $G_\beta$. So we have

Proposition 2.1.3 The centralizer $G_\beta = Z_G(\beta)$ of $\beta$ in $G$ is smooth over $O_F$ if the following two conditions are fulfilled:
1) \( \beta \in \mathfrak{g}(O_F) \) is smoothly regular over \( F \) and \( \mathbb{F} \), and

2) \( G_{\beta \otimes O_F} \) and \( G_{\beta \otimes O_F} \) are connected.

Let us assume the two conditions of the preceding proposition. Since we have canonical isomorphisms

\[
\mathfrak{g}(\mathbb{F}) \rightarrow K_{m-1}(O_F/p^m), \quad \mathfrak{g}_{\beta}(\mathbb{F}) \rightarrow G_{\beta}(O_F/p^m) \cap K_{m-1}(O_F/p^m)
\]

and the canonical morphism \( G_{\beta}(O_F) \rightarrow G_{\beta}(O_F/p^m) \) is surjective for any \( m > 1 \), we have

\[
|G(O_F/p^m)| = |G(\mathbb{F})| \cdot q^{(m-1)\dim G}, \quad |G_{\beta}(O_F/p^m)| = |G_{\beta}(\mathbb{F})| \cdot q^{(m-1)\rank G}
\]

for all \( m > 0 \). Then we have

\[
\sharp \Omega = \sharp \{ \theta \in G_{\beta}(O_F/p^r) \text{ s.t. } \theta = \psi_{\beta} \text{ on } G_{\beta}(O_F/p^r) \cap K_{l}(O_F/p^r) \}
\]

\[
= (G_{\beta}(O_F/p^r) : G_{\beta}(O_F/p^r) \cap K_{l}(O_F/p^r)) = |G_{\beta}(O_F/p^r)|
\]

\[
= |G_{\beta}(\mathbb{F})| \cdot q^{(l-1)\rank G} = \frac{|G(\mathbb{F})|}{\sharp \Omega} \cdot q^{(l-1)\rank G}
\]

where \( \Omega \subset \mathfrak{g}(\mathbb{F}) \) is the image of \( \Omega \subset \mathfrak{g}(O_F/p^r) \) under the canonical morphism \( \mathfrak{g}(O_F/p^r) \rightarrow \mathfrak{g}(\mathbb{F}) \). On the other hand we have

\[
\dim \sigma_{\beta, \theta} = \begin{cases} 1 & \text{if } r \text{ is even}, \\ q^{\frac{1}{2} \dim_{\mathbb{F}}(\mathfrak{g}(\mathbb{F})/\mathfrak{g}_{\beta}(\mathbb{F}))} & \text{if } r \text{ is odd}, \end{cases}
\]

so we have

\[
\dim \delta_{\beta, \theta} = (G(O_F/p^r) : G(O_F/p^r; \beta)) \cdot \dim \sigma_{\beta, \theta} = \frac{\sharp \Omega}{\sharp \Omega} \cdot q^{(r-2)(\dim G - \rank G)/2}.
\] (2.3)

In our case of \( G = Sp_{2n} \), the following two statements are equivalent for a \( \beta \in \mathfrak{g}(O_F) \):

1) \( \overline{\beta} \in \mathfrak{g}(K) \) is smoothly regular over \( K \),

2) the characteristic polynomial of \( \overline{\beta} \in \mathfrak{g}(K) \subset \mathfrak{g}l_{2n}(K) \) is equal to its minimal polynomial

where \( \overline{\beta} \in \mathfrak{g}(K) \) is the image of \( \beta \in \mathfrak{g}(O_F) \) by the canonical morphism \( \mathfrak{g}(O_F) \rightarrow \mathfrak{g}(K) \) with \( K = F \) or \( \mathbb{F} \). If further \( \beta \in \mathfrak{g}(K) \subset \mathfrak{g}l_{2n}(K) \) is nonsingular, then \( G_{\beta \otimes O_F} K \) is connected.

Now let \( \Omega \subset \mathfrak{g}(O_F/p^r) \) be a \( G(O_F/p^r) - \text{adjoint orbit of } \beta \pmod{p^r} \in \mathfrak{g}(O_F/p^r) \) with \( \beta \in \mathfrak{g}(O_F) \) such that \( \beta \pmod{p} \in \mathfrak{g}(F) \subset \mathfrak{g}l_{2n}(F) \) is nonsingular and smoothly regular over \( F \). Then Theorem 2.1.1 gives a parametrization of \( \Omega \) by a subset of the character group \( G_{\beta}(O_F/p^r) \).

**Remark 2.1.4** The assumption in Theorem 2.1.1 that the centralizer \( G_{\beta} \) to be smooth \( O_F \)-group scheme can be replaced by the surjectivity of the canonical morphisms

\[
G_{\beta}(O_F) \rightarrow G_{\beta}(O_F/p^l), \quad \mathfrak{g}_{\beta}(O_F) \rightarrow \mathfrak{g}_{\beta}(O_F/p^l),
\]

for all \( l > 0 \).
2.2 Symplectic spaces associated with tamely ramified extensions

Let $K/F$ be a tamely ramified field extension of degree $n > 1$ and $K/K_+$ a quadratic field extension with $\Gal(K/K_+) = \langle \tau \rangle$. Let

$$e = e(K/F), \quad f = f(K/F)$$

be the ramification index and the inertial degree of $K/F$ respectively. Similarly put

$$e_+ = e(K_+/F), \quad f_+ = f(K_+/F).$$

Then we have $ef = 2n$ and $e_+f_+ = n$. There exists a $\omega \in \mathcal{O}_K$ such that $\omega^2 = -\omega$ and $\mathcal{O}_K = \mathcal{O}_{K_+} \oplus \omega \cdot \mathcal{O}_{K_+}$. Then we have

$$\ord_{\mathcal{O}_K}(\omega) = e(K/K_+) - 1.$$

Let $K_0/F$ be the maximal unramified subextension of $K/F$. Then $K_0/F$ is a cyclic Galois extension whose Galois group is generated by the geometric Frobenius automorphism $F\tau$ which induces the inverse of the Frobenius automorphism $[x \mapsto x^q]$ of the residue field $K_0$ over $F$. Since $K/K_0$ is totally ramified, there exists a prime element $\omega_0 \in K_0$ such that $\omega_0^e \in K_0$. Then $\{1, \omega_0, \omega_0^2, \cdots, \omega_0^{e-1}\}$ is an $\mathcal{O}_{K_0}$-basis of $\mathcal{O}_K$. The following two propositions are proved by Shintani [13, Lemma 4-7, Cor.1, Cor.2, pp.545-546]:

**Proposition 2.2.1** Put $\beta = \sum_{i=0}^{c-1} a_i \omega_0^i \in \mathcal{O}_K$ ($a_i \in \mathcal{O}_{K_0}$). Then $\mathcal{O}_K = \mathcal{O}_F[\beta]$ if and only if the following two conditions are satisfied:

1) $a_0^F \not\equiv a_0 (\mod \mathfrak{p}_{K_0})$ if $f > 1$,

2) $a_1 \in \mathcal{O}_{K_0}^{\times}$ if $e > 1$.

**Proposition 2.2.2** Let $\chi_\beta(t) \in \mathcal{O}_F[t]$ be the characteristic polynomial of $\beta \in \mathcal{O}_K \subset M_n(\mathcal{O}_F)$ via the regular representation with respect to an $\mathcal{O}_F$-basis of $\mathcal{O}_K$. If $\mathcal{O}_K = \mathcal{O}_F[\beta]$, then

1) $\chi_\beta(t) (\mod \mathfrak{p}_F) \in \mathbb{F}[t]$ is the minimal polynomial of $\overline{\beta} \in M_n(\mathbb{F})$,

2) $\chi_\beta(t) (\mod \mathfrak{p}_F) = p(t)^e$ with an irreducible polynomial $p(t) \in \mathbb{F}[t]$,

3) if $e > 1$, then $\chi_\beta(t) (\mod \mathfrak{p}_F^2)$ is irreducible over $\mathcal{O}_F/\mathfrak{p}_F^2$.

We can prove the following

**Proposition 2.2.3** Take a $\beta \in M_n(\mathcal{O}_F)$ whose the characteristic polynomial be

$$\chi_\beta(t) = t^n - a_n t^{n-1} - \cdots - a_2 t - a_1.$$

If $\chi_\beta(t) (\mod \mathfrak{p}_F) \in \mathbb{F}[t]$ is the minimal polynomial of $\beta (\mod \mathfrak{p}_F) \in M_n(\mathbb{F})$, then

1) $\{X \in M_n(\mathcal{O}_F) \mid [X, \beta] = 0\} = \mathcal{O}_F[\beta]$,
2) for any \( m > 0 \), put \( \overline{\beta} = (\text{mod } \mathfrak{p}_F^m) \in M_n(O_F/\mathfrak{p}_F^m) \), then
\[
\{ X \in M_n(O_F/\mathfrak{p}_F^m) \mid [X, \overline{\beta}] = 0 \} = O_F/\mathfrak{p}_F^m[\overline{\beta}],
\]
3) there exists a in\( GL_n(O_F) \) such that
\[
g \beta g^{-1} = \begin{bmatrix}
0 & 0 & \cdots & 0 & a_1 \\
1 & 0 & \cdots & 0 & a_2 \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & 0 & a_{n-1} \\
& & & 1 & a_n
\end{bmatrix}
\]

Then we have

**Proposition 2.2.4** There exists a \( \beta \in O_K \) such that \( O_K = O_F[\beta] \) and \( \beta + \beta^T = 0 \) if and only if \( K/K_+ \) is unramified or \( K/F \) is totally ramified.

**Proof** Assume that there exists a \( \beta \in O_K \) such that \( O_K = O_F[\beta] \) and \( \beta + \beta^T = 0 \). Then \( K = K_+(\beta^2) \). If \( K/F \) is not totally ramified, we have \( \beta \in O_K^\times \) by Proposition 2.2.1 and hence \( K/K_+ \) is an unramified extension.

Assume that \( K/F \) is totally ramified. Then \( K_0 = F \) and \( \omega_K^e \in O_F \). Since the quadratic extension \( K/K_+ \) is ramified, there exists a prime element \( \beta \) of \( K \) such that \( \beta^2 \in K_+ \). Then
\[
\beta = e \cdot \omega_K \quad \text{with} \quad e \in O_K^\times.
\]
Put \( e = \sum_{i=0}^{e-1} a_i \omega_K^i \) with \( a_i \in O_F \).

Then
\[
\beta = a_{e-1} \omega_K^e + \sum_{i=1}^{e-1} a_{i-1} \omega_K^i
\]
with \( a_0 \in O_K^\times \). Now we have \( \beta^T = -\beta \) and \( O_K = O_F[\beta] \) by Proposition 2.2.1.

Assume that \( K/K_+ \) is unramified. Let \( K_{+0}/F \) be the maximal unramified subextension of \( K_/F \). Since \( (K_{+0} : F) = f_+ \) divides \( (K_0 : F) = f \), we have \( K_{+0} \subseteq K_0 \). We can choose \( \omega_K \) in \( K_+ \) so that \( \omega_K^e \in K_{+0} \). For the residue fields, we have
\[
(K_0 : K_{+0}) = (K_0 : F) = f_+ = 2.
\]

Put \( K_{+0} = \mathbb{F}(\overline{\alpha}) \) with \( \alpha \in O_{K_{+0}}^\times \) such that \( \overline{\alpha} \notin (K_{+0}^\times)^2 \). Since \( K_0 \) is the splitting filed of \( f(X) = X^2 - \overline{\alpha} \in \mathbb{K}_{+0}[X] \), there exists \( \gamma \in O_{K_0}^\times \) such that
\[
f(\gamma) \equiv 0 \pmod{p_{K_0}} \quad \text{and} \quad f'(\gamma) \not\equiv 0 \pmod{p_{K_0}}.
\]
Hence there exists \( a \in O_{K_0}^\times \) such that \( f(a) = 0 \) and \( a \equiv \gamma \pmod{p_{K_0}} \). Since
\[
K_{+0} = \mathbb{F}(\overline{\alpha}) = \mathbb{F}(\overline{\alpha}),
\]
we have \( a^{f_+} \equiv a \pmod{p_{K_0}} \) and \( a^T = -a \). Put \( \beta = a(1 + \omega_K) \in O_K \), then \( O_K = O_F[\beta] \) by Proposition 2.2.1 and \( \beta^T = -\beta \). □

From now on let us assume that \( K/K_+ \) is unramified or \( K/F \) is totally ramified, and take a \( \beta \in O_K \) such that \( O_K = O_F[\beta] \) and \( \beta^T + \beta = 0 \). Fix a
prime element \( \varpi_{K^+} \) of \( K^+ \). Then a symplectic form on \( F \)-vector space \( K \) is defined by

\[
D(x, y) = \frac{1}{2} T_{K/F} \left( \omega^{-1} \varpi_{K^+}^{1-x^+} x^T y \right) \quad (x, y \in K).
\]

For any \( a \in K \), we have \( D(xa, y) = D(x, ya^T) \) for all \( x, y \in K \). In particular

\[
\beta \in \mathfrak{sp}(K, D) = \{ X \in \text{End}_F(K) \mid D(xX, y) + D(x, yX) = 0 \ \forall x, y \in K \}
\]

if we put \( K \subset \text{End}_F(K) \) by the regular representation.

Let \( \{ u_i \}_{1 \leq i \leq n} \) be an \( O_F \)-basis of \( O_{K^+} \). Then \( x \mapsto (T_{K^+/F}(u_1x), \cdots, T_{K^+/F}(u_nx)) \) gives an isomorphism \( p_{K^+}^{1-x^+} \to O_F^d \) of \( O_F \)-module. Hence there exists an \( O_F \)-basis \( \{ u_i^+ \}_{1 \leq i \leq n} \) of \( p_{K^+}^{1-x^+} \) such that \( T_{K^+/F}(u_iu_j^+) = \delta_{ij} \). Put \( v_i = \omega \cdot \varpi_{K^+}^{1-x^+} u_i^+ \) \((1 \leq i \leq n)\). Then \( \{ u_1, \cdots, u_n, v_n, \cdots, v_1 \} \) is an \( O_F \)-basis of \( O_K \) and a symplectic \( F \)-basis of \( K \), that is

\[
D(u_i, u_j) = D(v_i, v_j) = 0, \quad D(u_i, v_j) = \delta_{ij} \quad (1 \leq i, j \leq n).
\]

This means that our \( O_F \)-group scheme \( G = Sp_{2n} \) is defined by the symplectic \( F \)-space \( (K, D) \) and the symplectic basis \( \{ u_i, v_j \}_{1 \leq i, j \leq n} \).

By Proposition 2.2.2 the characteristic polynomial of \( F \beta \) (mod \( p_F \)) \( \in M_{2n}(F) \) is equal to its minimal polynomial. Then, by Proposition 2.2.2 we have

\[
\{ X \in M_{2n}(O_F) \mid [X, \beta] = 0 \} = O_F[\beta] = O_K
\]

and

\[
\{ X \in M_{2n}(O_F/p_F^l) \mid [X, \beta] = 0 \} = O_F/p_F^l[\beta] = O_K/p_K^l
\]

for any \( m > 0 \). Put

\[
U_{K/K^+} = \{ \varepsilon \in O_K^\times \mid N_{K/K^+}(\varepsilon) = 1 \}.
\]

Then we have

\[
G_\beta(O_F) = G(O_F) \cap O_K = U_{K/K^+}.
\]

We have also

\[
g_\beta(O_F) = g(O_F) \cap O_K = \{ X \in O_K \mid T_{K/K^+}(X) = 0 \}
\]

and

\[
G_\beta(O_F/p_F^l) = \{ \varepsilon \in (O_K/p_K^l)^\times \mid N_{K/K^+}(\varepsilon) \equiv 1 \pmod{p_K^{l+1}} \},
\]

\[
g_\beta(O_F/p_F^l) = \{ \overline{X} \in O_K/p_K^l \mid T_{K/K^+}(X) \equiv 0 \pmod{p_K^{l+1}} \}
\]

for all \( l > 0 \). Then the canonical morphisms

\[
G_\beta(O_F) \to G_\beta(O_F/p_F^l), \quad g_\beta(O_F) \to g_\beta(O_F/p_F^l)
\]

are surjective for all \( l > 0 \). In fact, Take a \( \varepsilon \in O_K^\times \) such that \( N_{K/K^+}(\varepsilon) \equiv 1 \pmod{p_K^{l+1}} \). Because \( K/K^+ \) is tamely ramified, we have \( N_{K/K^+}(1 + p_K^l) = 1 + p_K^{l+1} \). Hence there exists a \( \eta \in 1 + p_K^l \) such that \( N_{K/K^+}(\eta) = \varepsilon \). Then \( \alpha = \varepsilon \eta^{-1} \in O_K^\times \) such that \( N_{K/K^+}(\alpha) = 1 \) and \( \alpha = \varepsilon \pmod{p_K^l} \). Take a \( X \in O_K \).
such that $T_{K/K_+}(X) \equiv 0 \pmod{p_{K_+}^{e_1}}$. If we put $X = s + \omega t$ with $s, t \in O_{K_+}$, then $s \in p_{K_+}^{e_1} \subset O_{K_+}^{e_1}$. Hence we have $\omega t \in g_\beta(O_F)$ and $\omega t \equiv X \pmod{p_{K_+}^{e_1}}$.

Due to Remark 2.1.4 we can apply the general theory of subsection 2.1 to our $\beta \in g(O_F)$. Take an integer $r > 1$ and put $r = l + l'$ with minimal integer $l$ such that $0 < l' \leq l$. Let $\Omega \subset g(O_F/p_{F}^{r})$ be the adjoint $G(O_F/p_{F}^{r})$-orbit of $\beta$ (mod $p_{F}^{r}$) $\in g(O_F/p_{F}^{r})$, and $\Omega^*$ the set of the equivalent classes of the irreducible representations of $G(O_F/p_{F}^{r})$ corresponding to $\Omega$ via Clifford’s theory described in subsection 2.1. Then we have a bijection $\theta \mapsto \delta_{\beta, \theta}$ of the continuous unitary character $\theta$ of $U_{K/K_+}$ such that

1) $\theta$ factors through the canonical morphism $U_{K/K_+} \to (O_K/p_{K}^\omega)^\times$,

2) for an $\alpha \in U_{K/K_+}$ such that $\alpha \equiv 1 + \omega_F x (\pmod{p_{K}^{r}})$ with $x \in O_K$ such that $T_{K/K_+}(x) \equiv 0 (\pmod{p_{K_+}^{e_1} + l})$, we have $\theta(\alpha) = \psi \left( \frac{1}{\omega_F} T_{K/K_+}(x) \right)$.

onto $\Omega^*$. Here $\psi : F \to \mathbb{C}^\times$ is a continuous unitary character of the additive group $F$ such that $\{ x \in F | \psi(x_{O_F}) = 1 \} = O_F$. Then we have

**Proposition 2.2.5**

\[
\dim \delta_{\beta, \theta} = q^{2r} \prod_{k=1}^{n} (1 - q^{-2k}) \times \begin{cases} \frac{1}{2} & : K/F \text{ is totally ramified}, \\ \frac{1}{1 + q^{-f}} & : K/K_+ \text{ is unramified}. \end{cases}
\]

**Proof** For the dimension formula (2.3), we have

\[
\dim G = n(2n + 1), \quad \text{rank } G = n, \quad \frac{\prod_{k=1}^{n} (1 - q^{-2k})}{2} = \frac{|G(F)|}{|G_\beta(F)|}
\]

and

\[
|G(F)| = |Sp_{2n}(F)| = q^{n(n+1)} \prod_{k=1}^{n} (1 - q^{-2k}).
\]

On the other hand $G_\beta(F)$ is the kernel of

\[
(*) : (O_K/p_{K}^\omega)^\times \to \left( O_{K_+}/p_{K_+}^{e_1} \right)^\times, \quad \tau \mapsto N_{K/K_+}(\tau).
\]

Since $K/K_+$ is tamely ramified quadratic extension, we have

\[
1 + p_{K_+}^{e_1} = N_{K/K_+}(1 + p_{K}^{e_1}) \subset N_{K/K_+}(O_K^\times) \subset O_{K_+}^\times,
\]

and $(O_{K_+}^\times : N_{K/K_+}(O_{K_+}^\times)) = e/e_+$, hence

\[
|G_\beta(F)| = \frac{|O_K/p_{K}^\omega|^\times |O_{K_+}/p_{K_+}^{e_1}|}{|O_{K_+}/p_{K_+}^{e_1}|} = \frac{e \cdot q^n \cdot 1 - q^{-f}}{1 - q^{-f}}.
\]

\[
= q^n \times \begin{cases} 2 & : K/F \text{ is totally ramified}, \\ 1 + q^{-f} & : K/K_+ \text{ is unramified}. \end{cases}
\]
2.3 Construction of supercuspidal representations

We will keep the notations of the preceding subsection. The purpose of this subsection is to prove the following theorem:

**Theorem 2.3.1** If \( l' = \left\lceil \frac{r}{2} \right\rceil \geq \max\{2, 2(e - 1)\} \), then the compactly induced representation \( \pi_{\beta, \theta} = \text{ind}_{G(O_{F})}^{G(F)} \delta_{\beta, \theta} \) is an irreducible supercuspidal representation of \( G(F) = \text{Sp}_{2n}(F) \) such that

1) the multiplicity of \( \delta_{\beta, \theta} \) in \( \pi_{\beta, \theta}|_{G(O_{F})} \) is one,

2) \( \delta_{\beta, \theta} \) is the unique irreducible unitary constituent of \( \pi_{\beta, \theta}|_{G(O_{F})} \) which factors through the canonical morphism \( G(O_{F}) \to G(O_{F}/p^{r}) \),

3) with respect to the Haar measure on \( G(F) \) such that the volume of \( G(O_{F}) \) is one, the formal degree of \( \pi_{\beta, \theta} \) is equal to

\[
\dim \delta_{\beta, \theta} = q^{n^2 r} \prod_{k=1}^{n} (1 - q^{-2k}) \times \begin{cases} 
\frac{1}{2} : K/F \text{ is totally ramified,} \\
\frac{1}{1 + q^{-f}} : K/K_{+} \text{ is unramified.}
\end{cases}
\]

The rest of this subsection is devoted to the proof.

We have the Cartan decomposition

\[
G(F) = \bigsqcup_{m \in \mathbb{M}} G(O_{F}) t(m) G(O_{F})
\]

where

\[
\mathbb{M} = \{(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n \mid m_1 \geq m_2 \geq \cdots \geq m_n \geq 0\}
\]

and

\[
t(m) = \begin{bmatrix}
w_F^m & 0 \\
0 & w_F^{-m}
\end{bmatrix}
\]

with \( w_F = \begin{bmatrix} w^{m_1}_F \\
& \ddots \\
& & w^{m_n}_F \end{bmatrix} \)

for \( m = (m_1, \ldots, m_n) \in \mathbb{M} \).

For an integer \( 1 \leq i \leq n \), let \( L_i \) and \( U_i \) be \( O_{F} \)-group subscheme of \( G = \text{Sp}_{2n} \) defined by

\[
L_i = \left\{ \begin{bmatrix} a & g \\ A & g A^{-1} \end{bmatrix} \mid a \in GL_i, g \in \text{Sp}_{2(n-i)} \right\},
\]

\[
U_i = \left\{ \begin{bmatrix} 1_i & A & B \\ 1_{n-i} & C & 0 \\ 0 & 1 & 1_{n-i} \end{bmatrix} \in \text{Sp}_{2n} \right\}
\]

so that \( P_i = L_i \cdot U_i \) is a maximal parabolic subgroup of \( G = \text{Sp}_{2n} \) and \( U_i \) (resp. \( L_i \)) is the unipotent (resp. Levi) part of \( P_i \). Put \( U_i(\mathfrak{p}_F^a) = U_i(O_{F}) \cap K_{a}(O_{F}) \) for a positive integer \( a \).
Proposition 2.3.2 If \( K/F \) is unramified or \( r \geq 4 \), then the compactly induced representation \( \pi_{\beta, \theta} = \text{ind}^{G(F)}_{G(O_F)} \delta_{\beta, \theta} \) is an admissible representation of \( G(F) \).

[Proof] It is enough to show that \( \dim C \text{Hom}_{K_a(O_F)}(1, \pi_{\beta, \theta}) < \infty \) for all \( a > 0 \), where \( 1 \) is the trivial one-dimensional representation of \( K_a(O_F) \). We have

\[
G(F) = \bigsqcup_{s \in S} K_a(O_F)sG(O_F)
\]

where

\[
S = \{ k \cdot t(m) \mid \hat{k} \in K_a(O_F)\backslash G(O_F), m \in \mathbb{M} \}.
\]

Then, by the restriction formula of induced representations and by the Frobenius reciprocity, we have

\[
\text{Hom}_{K_a(O_F)}(1, \pi_{\beta, \theta}) = \bigsqcup_{s \in S} \text{Hom}_{K_a(O_F)}(1, \text{ind}^{G(F)}_{K_a(O_F) \cap \alpha(F,O_F)s^{-1}}\delta_{\beta, \theta})
\]

\[
= \bigsqcup_{s \in S} \text{Hom}_{K_a(O_F) \cap \alpha(F,O_F)s^{-1}}(1, \delta_{\beta, \theta})
\]

\[
= \bigsqcup_{s \in S} \text{Hom}_{s^{-1}K_a(O_F)sG(O_F)}(1, \delta_{\beta, \theta}).
\]

So it is enough to show that the number of \( s \in S \) such that \( \text{Hom}_{s^{-1}K_a(O_F)sG(O_F)}(1, \delta_{\beta, \theta}) \neq 0 \) is finite. Take such a \( s = k \cdot t \in S \) with \( k \in G(O_F) \) and \( t = t(m) \) \((m \in \mathbb{M})\).

Suppose

\[
\max\{m_k - m_{k+1} \mid 1 \leq k < n\} = m_i - m_{i+1} \geq a.
\]

Then we have \( tU_i(p_F^a)t^{-1} \subset K_a(O_F) \) and hence

\[
U_i(p_F^a) \subset U_i(O_F) \subset s^{-1}K_a(O_F)sG(O_F)
\]

and

\[
\text{Hom}_{U_i(p_F^a)}(1, \delta_{\beta, \theta}) \supset \text{Hom}_{s^{-1}K_a(O_F)sG(O_F)}(1, \delta_{\beta, \theta}) \neq 0.
\]

This means, by (2.21), that there exists a \( g \in G(O_F) \) such that \( \chi_{\lambda \delta(t)\delta}(h) = 1 \) for all \( h \in U_i(p_F^a) \), that is \( \psi\left( \varpi_F^{-l'} \text{tr}(g \delta g^{-1}X) \right) = 1 \) for all \( X \in \text{Lie}(U_i(O_F)) \). Hence we have \( g \delta g^{-1} \in \text{Lie}(P_\delta(O_F/p_F^{l'})) \). Then the characteristic polynomial \( \chi_\delta(t) \) \((\text{mod } p_F) \in \mathbb{F}[t] \) is reducible. Hence \( e > 1 \) by Proposition 2.2.2. Then \( l' = \left\lfloor \frac{p}{2} \right\rfloor \geq 2 \) and \( \chi_\delta(t) \) \((\text{mod } p_F^{l'}) \) is reducible over \( O_F/p_F^{l'} \), contradicting to Proposition 2.2.2. Hence we have

\[
\max\{m_i - m_{i+1} \mid 1 \leq i < n\} < a.
\]

Similar arguments using the parabolic subgroup \( P_n \) shows that \( 2m < a \). This shows the required finiteness of \( s \in S \). ■

Lemma 2.3.3

1) If \( \text{Hom}_{U_i(p_F^{e-1})}(1, \delta_{\beta, \theta}) \neq 0 \) for some \( 1 \leq i \leq n \), then \( i \equiv 0 \) \((\text{mod } f) \) and \( e > 1 \). If further \( i < n \), then \( e \geq 3 \).
2) If $\frac{r}{2} \geq 2$, then $\text{Hom}_{U_i(p_F^{-3})}(1, \delta_{\beta, \theta}) = 0$ for all $1 \leq i \leq n$.

[Proof] Assume that $\text{Hom}_{U_i(p_F^{-k})}(1, \delta_{\beta, \theta}) \neq 0$ with some $0 < k \leq l'$. Then $U_i(p_F^{-k}) \subset K_i(O_F)$ and (2.1) implies that there exists a $g \in G(O_F)$ such that $\chi_{\lambda_i(G)}(h) = 1$ for all $h \in U_i(p_F^{-k})$, that is $\psi(t_f^{-k} tr(g \delta g^{-1}X)) = 1$ for all $X \in \text{Lie}(U_i)(O_F)$. Then

$$g \delta g^{-1} \equiv \begin{bmatrix} A & * & * \\ 0 & X & * \\ 0 & 0 & -tA \end{bmatrix} \quad (\text{mod } p_F^k)$$

with $A \in \mathfrak{gl}_i(O_F)$ and $X \in \mathfrak{sp}_{2(n-i)}(O_F)$. So the characteristic polynomial is

$$\chi_{\beta}(t) \equiv \det(t_1I - A) \det(t_2(n-1) - X) \det(1 - A) \quad (\text{mod } p_F^k).$$

If $k = 1$, then the first statement of Proposition 2.3.4 implies that

$$i = \text{deg det}(t_1I - A) \equiv 0 \quad (\text{mod } f) \quad \text{and} \quad e > 1.$$ 

If $l' \geq 2$ and $k = 2$, then $\chi_{\beta}(t) (\text{mod } p_F^2)$ is reducible over $O_F/p_F^2$, contradicting to the third statement of Proposition 2.3.4. $\blacksquare$

Proposition 2.3.4 Assume that $l' = \left\lfloor \frac{r}{2} \right\rfloor \geq \text{Max}\{2, 2(e - 1)\}$. Then

1) $\dim_{\mathbb{C}} \text{Hom}_{G(O_F)}(\delta_{\beta, \theta}, \pi_{\beta, \theta}) = 1$,

2) for any irreducible representation $(\delta, V_\delta)$ of $G(O_F)$ which factors through the canonical morphism $G(O_F) \rightarrow G(O_F/p_F^2)$, if $\text{Hom}_{G(O_F)}(\delta, \pi_{\beta, \theta}) \neq 0$, then $\delta = \delta_{\beta, \theta}$.

[Proof] Let $(\delta, V_\delta)$ be an irreducible unitary representation of $G(O_F)$ which factors through the canonical morphism $G(O_F) \rightarrow G(O_F/p_F^2)$. Then we have

$$\text{Hom}_{G(O_F)}(\delta, \pi_{\beta, \theta}) = \bigoplus_{m \in M} \text{Hom}_{G(O_F)}(\delta, \text{ind}_{G(O_F)}^{G(O_F)}(\delta, \pi_{\beta, \theta})) = \bigoplus_{m \in M} \text{Hom}_{G(O_F) \cap \Gamma(m)G(O_F)t(m)^{-1}}(\delta_{\beta, \theta}) = \bigoplus_{m \in M} \text{Hom}_{G(O_F) \cap \Gamma(m)G(O_F)t(m)^{-1}}(\delta_{\beta, \theta}).$$

Assume that $\text{Hom}_{\Gamma(m)^{-1}G(O_F)t(m)^{-1}}(\delta_{\beta, \theta}) \neq 0$ for a $m = (m_1, \cdots, m_n) \in M$. If

$$\text{Max}\{m_k - m_{k+1} \mid 1 \leq k < n\} = m_i - m_{i+1} \geq 2$$

then we have $t(m)U_i(p_F^{-2})t(m)^{-1} \subset U_i(p_F^{-2})$. Since $K_i(O_F) \subset \text{Ker}(\beta)$, the restriction of $\delta_{\beta, \theta}$ to $U_i(p_F^{-2})$ is trivial. On the other hand, we have

$$U_i(p_F^{-2}) \subset t(m)^{-1}U_i(p_F) \cap U_i(O_F) \subset t(m)^{-1}G(O_F)t(m) \cap G(O_F).$$
Now we have $\text{Hom}_{U_i}(\mathcal{F}^{-1}_i(1, \delta_{\beta, \theta}) \neq 0$ contradicting to the second statement of Lemma 2.3.3. Hence we have

$$\text{Max}\{m_k - m_{k+1} \mid 1 \leq k < n\} \leq 1.$$  

Similarly we have $2m_n \leq 1$, that is $m_n = 0$. If there exists $1 \leq i < n$ such that $m_i - m_{i+1} = 1$. Then, with the similar arguments as above, we have $\text{Hom}_{U_i}(\mathcal{F}^{-1}_i(1, \delta_{\beta, \theta}) \neq 0$. The first statement of Lemma 2.3.3 implies that $i \equiv 0 \pmod{f}$. Since $ef = 2n$, this means $m_1 < \frac{e}{2}$, hence

$$4m_1 \leq 2(e - 1) \leq l'.$$

Since $t(m)K_{i+2m_1}(O_F)t(m)^{-1} \subset K_l(O_F)$ and hence

$$K_{i+2m_1}(O_F) \subset t(m)^{-1}G(O_F)t(m) \cap G(O_F),$$

we have

$$\text{Hom}_{K_{i+2m_1}}(\delta^{t(m)^{-1}}, \delta_{\beta, \theta}) \supset \text{Hom}_{t(m)^{-1}G(O_F)t(m)\cap G(O_F)}(\delta^{t(m)^{-1}}, \delta_{\beta, \theta}) \neq 0.$$  

Assume that $\delta$ corresponds, as explained in subsection 2.1, to an adjoint $G(O_F/\mathcal{F}^{-1}_i)$-orbit $\Omega' \subset g(O_F/\mathcal{F}^{-1}_i)$ of $\beta'$ (mod $\mathcal{F}^{-1}_i$) ($\beta' \in g(O_F)$). Then there exists $k, h \in G(O_F)$ such that

$$\chi_{\text{Ad}(k)}(x) = \chi_{\text{Ad}(h)}(t(m)xt(m)^{-1})$$

for all $x \in K_{i+2m_1}(O_F)$. This means

$$k\beta k^{-1} \equiv t(m)^{-1}h\beta'h^{-1}t(m) \pmod{\mathcal{F}^{-2m_1}}.$$  

Then, because of (2.4), the matrix $t(m)k\beta k^{-1}t(m)^{-1}$ belongs to

$$h\beta'h^{-1} + t(m)M_{2n}(\mathcal{F}^{-2m_1})t(m)^{-1} \subset h\beta'h^{-1} + M_{2n}(\mathcal{F}^{-4m_1}) \subset M_{2n}(O_F).$$

Since the characteristic polynomials of $t(m)k\beta k^{-1}t(m)^{-1}$ and $\beta$ are identical, there exists, by the third statement of Proposition 2.2.3, a $g \in GL_{2n}(O_F)$ such that $t(m)k\beta k^{-1}t(m)^{-1} = g\beta g^{-1}$. Then $g^{-1}t(m)k \in F[\beta] = K$ and

$$N_{K/F}(g^{-1}t(m)k) = \det(g^{-1}t(m)k) \in O_F^\times.$$  

Hence $g^{-1}t(m)k \in O_K \subset M_{2n}(O_F)$ and $t(m) \in M_{2n}(O_F)$, that is $m = (0, \ldots, 0)$. So we have proved

$$\text{Hom}_{G(O_F)}(\delta, \pi_{\beta, \theta}) = \text{Hom}_{G(O_F)}(\delta, \delta_{\beta, \theta})$$

which clearly implies the statements of the proposition. 

The admissible representation $\pi_{\beta, \theta} = \text{ind}_{G(O_F)}^{G(F)}(\delta_{\beta, \theta})$ of $G(F)$ is irreducible. In fact, if there exists a $G(F)$-subspace $0 \leq W \leq \text{ind}_{G(O_F)}^{G(F)}(\delta_{\beta, \theta}$, we have

$$0 \neq \text{Hom}_{G(F)}(W, \text{ind}_{G(O_F)}^{G(F)}(\delta_{\beta, \theta}) \subset \text{Hom}_{G(O_F)}(W, \text{Ind}_{G(O_F)}^{G(F)}(\delta_{\beta, \theta}) = \text{Hom}_{G(O_F)}(W, \delta)$$

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by Frobenius reciprocity. Hence $\delta \mapsto W|_{G(O_F)}$. On the other hand, we have

$$0 \neq \text{Hom}_{G(F)} \left( \text{ind}_{G(O_F)}^{G(F)} \delta_{\beta, \theta}, \left( \text{ind}_{G(O_F)}^{G(F)} \delta_{\beta, \theta} \right) / W \right) = \text{Hom}_{G(O_F)} \left( \delta_{\beta, \theta}, \left( \text{ind}_{G(O_F)}^{G(F)} \delta_{\beta, \theta} \right) / W \right),$$

hence $\delta \mapsto \left( \text{ind}_{G(O_F)}^{G(F)} \delta_{\beta, \theta} \right) / W$. Now $\text{ind}_{G(O_F)}^{G(F)} \delta_{\beta, \theta}$ is semi-simple $G_F$-module, we have

$$\dim_C \text{Hom}_{G(O_F)}(\delta_{\beta, \theta}, \text{ind}_{G(O_F)}^{G(F)} \delta_{\beta, \theta}) \geq 2$$

which contradicts to the first statement of Proposition 2.3.4.

Now $\pi_{\beta, \theta}$ is a supercuspidal representation of $G(F)$ whose formal degree with respect to the Haar measure $d_{G(F)}(x)$ of $G(F)$ such that $\int_{G(O_F)} d_{G(F)}(x) = 1$ is equal to $\dim \delta_{\beta, \theta}$. We have completed the proof of Theorem 2.3.1

### 3 Kaleta’s $L$-parameter

#### 3.1 Local Langlands correspondence of elliptic tori

Let $K_+/F$ be a finite extension, $K/K_+$ a quadratic extension with a non-trivial element $\tau$ of $\text{Gal}(K/K_+)$. Let us denote by $L$ an arbitrary Galois extension over $F$ containing $K$ for which let us denote by

$$\text{Emb}_F(K, L) = \{ \sigma_K \mid \sigma \in \text{Gal}(L/F) \}$$

the set of the embeddings over $F$ of $K$ into $L$.

Put $O_K = O_{K_+} \oplus \omega O_{K_+}$ with $\omega^2 + \omega = 0$. Then $\text{ord}_K(\omega) = e(K/K_+) - 1$. Let us denote by $\mathcal{V}$ the $F$-algebra of the functions $v$ on $\text{Emb}_F(K, F)$ with values in $F$ which is endowed with a symplectic $F$-form

$$D(u, v) = \frac{1}{2} \sum_{\gamma \in \text{Emb}_F(K, F)} \left( \omega^{-1} \frac{-d_{K_+}}{\omega} \right)^{\gamma} u(\tau \gamma) \cdot v(\gamma)$$

($u, v \in \mathcal{V}$) where $D(K_+/F) = \frac{d_{K_+}}{\omega}$ is the difference of $K_+/F$. The action of $\sigma \in \text{Gal}(F/F)$ on $v \in \mathcal{V}$ is defined by $v^\sigma(\gamma) = v(\gamma \sigma^{-1})^\sigma$. Then fixed point subspace $\mathcal{V}^{\text{Gal}(F/F)} = \mathcal{V}(L)$ is the set of the functions on $\text{Emb}_F(K, L)$ with values in $L$, and $\forall^{\text{Gal}(F/F)} = \mathcal{V}(F)$ is identified with $K$ via $v \mapsto v(1_K)$.

The action of $\sigma \in \text{Gal}(F/F)$ on $g \in \text{Sp}(\mathcal{V}, D)$ is defined by $v \cdot g^\sigma = (v^\sigma^{-1})^\sigma$. Then the fixed point subgroup $\text{Sp}(\mathcal{V}, D)^{\text{Gal}(F/F)}$ is identified with $\text{Sp}(K, D)$ via $g \mapsto g|_K$.

Put $S = \text{Res}_{K/F} \mathbb{G}_m$ which is identified with the multiplicative group $\mathcal{V}^\times$. Then $S(F)$ is identified with the multiplicative group $K^\times$. Let $T$ be a subtorus of $S$ which is identified with the multiplicative subgroup of $\mathcal{V}^\times$ consisting of the functions $s$ on $\text{Emb}_F(K, F)$ to $F^\times$ such that $s(\tau \gamma) = s(\gamma)^{-1}$ for all $\gamma \in \text{Emb}_F(K, F)$. In other words $T$ is a maximal torus of $\text{Sp}(\mathcal{V}, D)$ by identifying $s \in T$ with $[v \mapsto v \cdot s] \in \text{Sp}(\mathcal{V}, D)$. The fixed point subgroup $T^{\text{Gal}(F/F)} = T(F)$ is identified with

$$U_{K/K_+} = \{ \varepsilon \in O_K^\times \mid N_{K/K_+}(\varepsilon) = 1 \} \quad \text{by} \quad s \mapsto s(1_K).$$
The group $X(S)$ of the characters over $\overline{F}$ of $S$ is a free $\mathbb{Z}$-module with $\mathbb{Z}$-basis $\left\{ b_{s} \right\}_{s \in \text{Emb}_{F}(K, \overline{F})}$, where $b_{s}(s) = s(\delta)$ for $s \in S$. The dual torus $S^{-} = X(S) \otimes_{\mathbb{Z}} \mathbb{C}^{\times}$ is identified with the group of the functions $s$ on $\text{Emb}_{F}(K, \overline{F})$ with values in $\mathbb{C}^{\times}$. The action of $\sigma \in W_{F} \subset \text{Gal}(\overline{F}/F)$ on $S$ induces the action on $X(S)$ such that $b_{\sigma}^{\delta} = b_{\delta_{\sigma}}$, and hence the action on $s \in S^{-}$ is defined by $s(\gamma) = s(\gamma^{-1})$.

Since we have a bijection $\hat{\rho} \mapsto \rho|_{K}$ of $W_{K} \setminus W_{F}$ onto $\text{Embr}_{F}(K, \overline{F})$, the $\overline{F}$-algebra $V$ (resp. the torus $S$) is identified with the set of the left $W_{K}$-invariant functions on $W_{F}$ with values in $\overline{F}$ (resp. $\overline{F}^{\times}$, $\mathbb{C}^{\times}$). If we denote by $\tilde{\tau} \in W_{K}$, a pull back of $\tau \in \text{Gal}(K/K_{+})$ by the restriction mapping $W_{K} \rightarrow \text{Gal}(K/K_{+})$, the torus $T$ is identified with the set of $s \in S$ such that $s(\tilde{\tau} \rho) = s(\rho)^{-1}$ for all $\rho \in W_{F}$. Note that

$$\tilde{\tau}^{2} \pmod{[W_{K}, W_{K}]} = \delta_{K}(\alpha_{K/K_{+}}(\tau, \tau))$$

where $[\alpha_{K/K_{+}}] \in H^{2}(\text{Gal}(K/K_{+}), K^{\times})$ is the fundamental class which gives the isomorphism

$$\text{Gal}(K/K_{+}) \rightarrow K^{\times}_{+}/N_{K/K_{+}}(K^{\times}) \quad (\sigma \mapsto \alpha_{K/K_{+}}(\sigma, \tau)).$$

The local Langlands correspondence for the torus $S$ is the isomorphism

$$H^{1}(W_{F}, S^{-}) \rightarrow \text{Hom}(W_{K}, \mathbb{C}^{\times}) \quad (3.1)$$

given by $[\alpha] \mapsto [\rho \mapsto \alpha(\rho)(1_{K})]$. The inverse mapping is defined as follows. Let $l : \text{Embr}_{F}(K, \overline{F}) \rightarrow W_{F}$ be a section of the restriction mapping $W_{F} \rightarrow \text{Embr}_{F}(K, \overline{F})$, that is $l(\gamma)|_{K} = \gamma$ for all $\gamma \in \text{Embr}(K, \overline{F})$ and $l(1_{K}) = 1$, and put

$$J(\gamma, \sigma) = l(\gamma)\sigma l(\gamma\sigma)^{-1} \in W_{K} \text{ for } \gamma \in \text{Embr}(K, \overline{F}), \sigma \in W_{F}.$$ 

Take a $\psi \in \text{Hom}(W_{K}, \mathbb{C}^{\times})$ and define $\alpha \in \mathbb{Z}^{1}(W_{F}, S^{-})$ by

$$\alpha(\gamma)(\rho) = \alpha(\sigma\rho^{-1})(1) \cdot \alpha(\rho^{-1})(1) \text{ with } \alpha(\sigma)(1) = \psi(1_{K}, \sigma^{-1})^{-1})$$

for all $\sigma, \rho \in W_{F}$. Then $\psi \mapsto [\alpha]$ is the inverse mapping of the isomorphism $\mathbb{Z}^{1}(W_{F}, S^{-}) \rightarrow \text{Hom}(W_{K}, \mathbb{C}^{\times})$.

If we restrict the isomorphism $(3.1)$ to continuous group homomorphisms, we have an isomorphism

$$H^{1}_{\text{cont}}(W_{F}, S^{-}) \rightarrow \text{Hom}_{\text{cont}}(K^{\times}, \mathbb{C}^{\times}) \quad (3.2)$$

via $(3.1)$ combined with the isomorphism of the local class filed theory

$$\delta_{K} : K^{\times} \rightarrow W_{K}/[W_{K}, W_{K}].$$

The surjection $x \mapsto x^{1-\gamma}$ of $K^{\times}$ onto $U_{K/K_{+}}$ gives a canonical inclusion

$$\text{Hom}_{\text{cont}}(U_{K/K_{+}}, \mathbb{C}^{\times}) \subset \text{Hom}_{\text{cont}}(K^{\times}, \mathbb{C}^{\times}). \quad (3.3)$$

The restriction from $S$ to $T$ gives a surjection $X(S) \rightarrow X(T)$ whose kernel is the subgroup of $X(S)$ generated by $\{b_{\delta} + b_{\gamma, \delta} \mid \delta \in \text{Embr}(K, L)\}$. Then the dual torus $T^{-} = X(T) \otimes_{\mathbb{Z}} \mathbb{C}^{\times}$ is identified with the group of the functions $s$ on $\text{Embr}_{F}(K, \overline{F})$ with values in $\mathbb{C}^{\times}$ such that $s(\gamma) = s(\gamma^{-1})$ for all $\gamma \in \text{Embr}_{F}(K, \overline{F})$. As above $T^{-}$ is identified with the set of the left $W_{K}$-invariant functions $s$ of $W_{F}$ with values in $\mathbb{C}^{\times}$ such that $s(\tilde{\tau} \rho) = s(\rho)^{-1}$ for all $\rho \in W_{F}$.

Then we have
Proposition 3.1.1  
1) The inclusion \( T^\ast \subset S^\ast \) gives a canonical inclusion
\[
H^1_{\text{conti.}}(W_F, T^\ast) \subset H^1_{\text{conti.}}(W_F, S^\ast).  \tag{3.4}
\]

2) The restriction of the isomorphism \((\ast.2)\) to these included subgroups \((\ast.4)\) gives the isomorphism
\[
H^1_{\text{conti}}(W_F, T^\ast) \rightarrow \text{Hom}^*_{\text{conti}}(U_{K/K^+}, \mathbb{C}^\times). \tag{3.5}
\]

[Proof] See \((2.1)\) for the arguments with general tori. A direct proof for our specific setting is as follows.

1) Take a \( \beta \in Z^1(W_F, T^\ast) \subset Z^1(W_F, S^\ast) \) such that \( \beta \in B^1(W_F, S^\ast) \), that is, there exists a \( s S^\ast \) such that \( \beta(\sigma) = s^{\sigma - 1} \) for all \( \sigma \in W_F \). Chose a \( \varepsilon \in \mathbb{C}^\times \) such that \( \varepsilon^2 = s(1_K) \cdot s(\tau) \). The relation \( \beta(\sigma)(\tau) = \beta(\sigma)(1_K)^{-1} \) for all \( \sigma \in W_F \) implies
\[
s(\sigma) \cdot s(\tau^{-1}) = s(1_K) \cdot s(\tau) = \varepsilon^2
\]
for all \( \sigma \in W_F \). Then \( t = [\sigma \mapsto s(\sigma)\varepsilon^{-1}] \) is an element of \( T^\ast \) such that \( t^{\sigma - 1} = \beta(\sigma) \) for all \( \sigma \in W_F \).

2) Put
\[
\text{Emb}_F(K, \mathbb{T}) = \{ \gamma_i, \tau \gamma_i \mid 1 \leq i \leq n \}
\]
and let \( l : \text{Emb}_F(K, \mathbb{T}) \rightarrow W_F \) be a section of the restriction mapping \( W_F \rightarrow \text{Emb}_F(K, \mathbb{T}) \) such that \( l(\tau \gamma_i) = \tau l(\gamma_i) \) \( (1 \leq i \leq n) \). Take a \( \theta \in \text{Hom}^*_{\text{conti}}(K^\times, \mathbb{C}^\times) \) which corresponds to \( \alpha \in Z^1(W_F, S^\ast) \), that is
\[
\alpha(1_K) = \theta(x)
\]
for \( \sigma \in W_F \) with \( x \in K^\times \) such that \( J(1_K, \sigma^{-1})^{-1} \) \( (\mod \frac{W_K}{W_K}) \) \( = \delta_K(x) \).

For any \( \sigma \in W_F \), we have
\[
\alpha(\sigma^{-1}) = \begin{cases} 
\theta(x^\tau) : \sigma^{-1} | K = \gamma_i, \\
\theta(\alpha_{K/K^+}(\tau, \tau) \cdot x^\tau) : \sigma^{-1} | K = \tau \gamma_i.
\end{cases}
\]
Since
\[
\alpha(\sigma)(\rho) = \alpha(\sigma \rho^{-1})(1_K) \cdot \alpha(\rho^{-1})(1_K)^{-1}
\]
for all \( \sigma, \rho \in W_F \) and \( K^+ = N_{K/K^+}(K^\times) \sqcup \alpha_{K/K^+}(\tau, \tau) N_{K/K^+}(K^\times) \), we have \( \alpha \in Z^1(W_F, T^\ast) \) if and only if \( \theta(N_{K/K^+}(K^\times)) = 1 \), that is, there exists \( c \in \text{Hom}^*_{\text{conti}}(U_{K/K^+}, \mathbb{C}^\times) \) such that \( \theta(x) = c(x^{1+\tau}) \) \( (x \in K^\times) \).

Put \( T^* = W_F \rtimes T^\ast \). Then a cohomology class \( [\alpha] \in H^1_{\text{conti}}(W_F, T^\ast) \) defines a continuous group homomorphism
\[
\bar{\alpha} : W_F \rightarrow L^T \quad (\sigma \mapsto (\sigma, \alpha(\sigma))) \tag{3.6}
\]
and \( [\alpha] \mapsto \bar{\alpha} \) induces a well-defined bijection
\[
H^1_{\text{conti}}(W_F, T^\ast) \rightarrow \text{Hom}^*_{\text{conti}}(W_F, L^T)/"T^\ast\text{-conjugate}"
\]
where \( \text{Hom}^*_{\text{conti}}(W_F, L^T) \) denotes the set of the continuous group homomorphisms \( \psi \) of \( W_F \) to \( L^T \) such that \( W_F \xrightarrow{\psi} L^T \xrightarrow{\text{proj}} W_F \) is the identity map.
3.2 $\chi$-datum

In this subsection, let us assume that $K/F$ is a Galois extension and put $\Gamma = \text{Gal}(K/F)$. For a $\gamma \in \Gamma$ of order two, let us denote by $K_\gamma$ the intermediate subfield of $K/F$ such that $\text{Gal}(K/K_\gamma) = \langle \gamma \rangle$.

Let us denote by $SO_{2n+1}(\mathbb{C})$ the complex special orthogonal group with respect to the symmetric matrix

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & -2 \end{bmatrix} \text{ with } S_1 = \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}$$

and put

$$T^- = \left\{ \begin{bmatrix} t & t^{-1} \\ t & 1 \end{bmatrix} \bigg| \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} \in \mathbb{C}^n \right\}$$

a maximal torus of $SO_{2n+1}(\mathbb{C})$. We have an isomorphism $T^- \to \mathbb{T}^-$ given by

$$s \mapsto \text{diag}(s(\gamma_1), \ldots, s(\gamma_n), s(\gamma_{n+1}), \ldots, s(\gamma_{2n}), 1)$$

where $\text{Emb}_F(K, \mathcal{F}) = \{ \gamma_i \}_{1 \leq i \leq 2n}$ where $\gamma_1 = 1_K$ and $\gamma_{n+i} = \tau \gamma_i$ ($1 \leq i \leq n$).

The action of $W_F$ on $T^-$ induces the action on $\mathbb{T}^-$ which factors through $\Gamma$.

The Weyl group $W(\mathbb{T}) = N_{SO_{2n+1}(\mathbb{C})}(\mathbb{T})/\mathbb{T}$ on $\mathbb{T}^-$ is identified with a subgroup of the permutation group $S_{2n}$ generated by

$$\left( \begin{array}{cccccc} 1 & \cdots & n & n+1 & \cdots & 2n \\ \sigma(1) & \cdots & \sigma(n) & n+\sigma(1) & \cdots & n+\sigma(n) \end{array} \right) \text{ with } \sigma \in S_n$$

and

$$\left( \begin{array}{cccccc} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n \\ n+1 & 2 & \cdots & n & 1 & n+2 & \cdots & 2n \end{array} \right).$$

Then any $w \in W(\mathbb{T})$ is represented by

$$\tilde{w} = \begin{bmatrix} [w] & 0 \\ 0 & \text{det}[w] \end{bmatrix} \in N_{SO_{2n+1}(\mathbb{C})}(\mathbb{T}),$$

where $[w] \in GL_{2n}(\mathbb{Z})$ is the permutation matrix corresponding to $w \in W(\mathbb{T}) \subset S_{2n}$.

For any $\gamma \in \text{Emb}_F(K, \mathcal{F}) = \Gamma$, let us denote by $a_\gamma$ an element of $X(T^-)$ such that $a_\gamma(s) = s(\gamma)$ for all $s \in T^-$. Then

$$\Phi(T^-) = \{ a_\gamma \cdot a_{\gamma'}, a_\gamma | \gamma, \gamma' \in \Gamma, \gamma \neq \gamma' \}.$$

is the set of the roots of $SO_{2n+1}(\mathbb{C})$ with respect to $T^- = \mathbb{T}^-$ with the simple roots

$$\Delta = \{ \alpha_i = a_{\gamma_i} \cdot a_{\gamma_{i+1}}, \alpha_n = a_{\gamma_n} | 1 \leq i < n \}.$$

Let $\{X_\alpha, X_{-\alpha}, H_\alpha\}$ be the standard triple associate with a simple root $\alpha \in \Delta$. Then $s_\alpha \in W(\mathbb{T})$ is represented by

$$n(s_\alpha) = \exp(X_\alpha) \cdot \exp(-X_{-\alpha}) \cdot \exp(X_\alpha) \in N_{SO_{2n+1}(\mathbb{C})}(\mathbb{T})$$

and $W(\mathbb{T})$ is generated by $S = \{ s_\alpha \}_{\alpha \in \Delta}$. For any $w \in W(\mathbb{T})$, let $w = s_1 s_2 \cdots s_r$ ($s_i \in S$) be a reduced presentation and put

$$n(w) = n(s_1)n(s_2) \cdots n(s_r) \in N_{SO_{2n+1}(\mathbb{C})}(\mathbb{T}).$$
Then $r(w) = \hat{w}^{-1} w(\nu) \in T^\gamma$.

The action of $a \in W_F$ on $X(T^\gamma)$ induced from the action on $T^\gamma$ is such that $a^\gamma = a_{n\gamma}$ for all $a \in \text{Emb}_F(K, T)$, and it determines an element $w(\sigma) \in W(T^\gamma)$. Then $[11]$ shows that the 2-cocycle $t \in Z^2(W_F, T)$ defined by

$$t(\sigma, \sigma') = n(w(\sigma\sigma'))^{-1} n(w(\sigma)) \cdot n(w(\sigma')) \quad (\sigma, \sigma' \in W_F)$$

is split by $r_p : W_F \to \hat{T}$ defined by $\chi$-data as follows.

For any $\lambda \in \Phi(T^\gamma)$, put

$$\Gamma_\lambda = \{ \sigma \in \Gamma \mid \lambda^\sigma = \lambda \}, \quad \Gamma_{\pm\lambda} = \{ \sigma \in \Gamma \mid \lambda^\sigma = \pm \lambda \}$$

and put $F_\lambda = L^{\Gamma_\lambda}$, $F_{\pm\lambda} = L^{\Gamma_{\pm\lambda}}$. Then $(F_\lambda : F_{\pm\lambda}) = 1$ or 2, and $\lambda$ is called symmetric if $(F_\lambda : F_{\pm\lambda}) = 2$.

The Galois group $\Gamma$ acts on $\Phi(T^\gamma)$ and

$$\Phi(T^\gamma)/\Gamma = \{ a_{1K}, a_{\gamma}, a_{1K} \mid 1 \neq \gamma \in \Gamma \}.$$

If $\lambda = a_{1K} a_{\gamma}$, then $\lambda$ is symmetric if and only if $\gamma \neq \tau$. If further $\gamma^2 \neq 1$, then $F_\lambda = K$ and $F_{\pm\lambda} = K_+$ and choose a continuous character $\chi_\lambda : F_\lambda^\times = K_+^\times \to \mathbb{C}_x^\times$ such that $\chi_\lambda|_{F_\lambda^\times} : K_+^\times \to \{ \pm 1 \}$ is the character of the quadratic extension $K/K_+$. We may assume that $\chi_{a_{1K} a_{\gamma-1}} = \chi_{a_{1K} a_\gamma}^{-1}$.

If $\gamma^2 = 1$, then $F_\lambda = K_+$ and $F_{\pm\lambda} = E = K_+ \cap K_+$ and choose a continuous character $\chi_\lambda : F_\lambda^\times = K_+^\times \to \mathbb{C}_x^\times$ such that $\chi_\lambda|_{F_\lambda^\times} : E_+^\times \to \{ \pm 1 \}$ is the character of the quadratic extension $K/E$.

If $\lambda = a_{1K}$, then $F_\lambda = K$ and $F_{\pm\lambda} = K_+$ and choose a continuous character $\chi_\lambda : F_\lambda^\times = K_+^\times \to \mathbb{C}_x^\times$ such that $\chi_\lambda|_{F_\lambda^\times} : K_+^\times \to \{ \pm 1 \}$ is the character of the quadratic extension $K/K_+$.

These characters are parts of a system of $\chi$-data $\chi_\lambda : F_\lambda \to \mathbb{C}_x^\times (\lambda \in \Phi(T^\gamma))$ such that

1) $\chi_{-\lambda} = \chi_\lambda^{-1}$ and $\chi_{-\lambda} = \chi_\lambda(x^{-1})$ for all $\sigma \in \Gamma$, and

2) $\chi_\lambda = 1$ if $\lambda$ is not symmetric.

With this $\chi$-data and the gauge

$$p : \Phi(T^\gamma) \to \{ \pm 1 \} \text{ s.t. } p(\lambda) = \begin{cases} 1 & : \lambda > 0, \\ -1 & : \lambda < 0, \\ \end{cases}$$

the mechanism of $[11]$ gives a $r_p : W_F \to \hat{T}$ such that

$$t(\sigma, \sigma') = r_p(\sigma) r_p(\sigma')^{-1} r_p(\sigma')$$

for all $\sigma, \sigma' \in W_F$.

and

$$r_p(\sigma) = \prod_{\gamma \in \Gamma, \gamma^2 = 1} \prod_{0 < \lambda \in \{ a_{1K} a_{\gamma} \} \Gamma} \chi_\lambda(x)^{\lambda} \times \prod_{\gamma \in \Gamma, \gamma^2 = 1} \prod_{0 < \lambda \in \{ a_{1K} a_{\gamma} \} \Gamma} \chi_\lambda(N_K/F_\lambda(x))^\lambda \times \prod_{0 < \lambda \in \{ a_{1K} \} \Gamma} \chi_\lambda(x)^{\lambda}$$

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if $\hat{\sigma} = (1, x) \in W_{K/F} = \Gamma \rtimes \alpha_{K/F} K^\times$, where $\{\alpha\}_T$ is the $\Gamma$-orbit of $\alpha \in \Phi(T)$ and $\hat{\lambda}$ is the co-root of $\lambda$. Then we have a group homomorphism

$$L^T = W_F \rtimes T^e \rightarrow SO_{2n+1}(C) \quad ((\sigma, s) \mapsto n(w) (\sigma^e - \rho \cdot \sigma^{-1}) \cdot s).$$

(3.7)

If we put $r(\sigma) = r(w(\sigma))$ for $\sigma \in W_F$, we have

$$t(\sigma, \sigma') = r(\sigma') r(\sigma)^{-1} r(\sigma') \quad (\sigma, \sigma' \in W_F).$$

Now $\chi_p(\sigma) = r(\sigma) \cdot r_p(\sigma)^{-1}$ $(\sigma \in W_F)$ define an element of $Z^1(W_F, T)$ and the group homomorphism $\{3.7\}$ is

$$L^T = W_F \rtimes T^e \rightarrow SO_{2n+1}(C) \quad ((\sigma, s) \mapsto \widetilde{w}(\sigma) \chi_p(\sigma) \cdot s).$$

(3.8)

Let $c \in \text{Hom}_{\text{cont}}(U_{K/K}, C^\times)$ be the character corresponding to the cohomology class $[\chi_p] \in H^1_{\text{cont}}(W_F, T)$ by the local Langlands correspondence of torus $\{3.5\}$. 

### 3.3 Explicit value of $c(-1)$

From now on, we will assume that $K/F$ is a tamely ramified Galois extension and put $\Gamma = \text{Gal}(K/F)$.

The structure of the Galois group $\text{Gal}(K/F)$ is well understood:

$$\text{Gal}(K/F) = \langle \delta, \rho \rangle$$

(3.9)

where $\text{Gal}(K/K_0) = \langle \delta \rangle$ with the maximal unramified subextension $K_0/F$ of $K/F$ and $\rho|_{K_0} \in \text{Gal}(K_0/F)$ is the inverse of the Frobenius automorphism. There exists a prime element $\varpi_K$ of $K$ such that $\varpi_K \in K_0$. Then $\sigma \mapsto \varpi_K^{\sigma}$ (mod $p_K$) is an injective group homomorphism of $\text{Gal}(K/K_0)$ into $\mathbb{F}_p^\times$, and hence $e | q^f - 1$. Put $\rho^f = \delta^m$ with $0 \leq m < e$. We have a relation $\rho^{-1} \delta \rho = \delta^e$ due to Iwasawa $[9]$ and hence

$$\delta^m = \rho^{-1} \delta^m \rho = \delta^{em}$$

that is $m(q - 1) \equiv 0 \pmod{e}$. So we have

$$\rho^{f(q-1)} = 1 \quad (3.10)$$

Since $f$ divides ord($\rho$), we have

$$\text{ord}(\rho) = f \cdot \frac{e}{\text{GCD}(e, m)}.$$

The structure of the elements of order two in $\text{Gal}(K/F)$ plays an important role in our arguments, and we have

**Proposition 3.3.1** $H = \{ \gamma \in \text{Gal}(K/F) \mid \gamma^2 = 1 \} \subset Z(\text{Gal}(K/F))$ and

$$H = \begin{cases} \{1, \delta^e\} & : f = \text{odd or} \\
\{1, \rho^e \delta^{-e} \} & : e = \text{odd, } m = \text{odd} \\
\{1, \rho^e \delta^{-e} \} & : e = \text{odd, } m = \text{odd} \\
\{1, \delta^e, \rho^e \delta^{-e}, \rho^e \delta^{-e} \} & : f = \text{even, } e = \text{even, } m = \text{even.} \end{cases}$$

For $\gamma \in \text{Gal}(K/F)$ of order two, the quadratic extension $K/K_\gamma$ is ramified if and only if $\gamma \in \text{Gal}(K/K_0)$. 

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[Proof] Take a $1 \neq \gamma \in \text{Gal}(K/F)$ such that $\gamma^2 = 1$.

If $\gamma \in \text{Gal}(K/K_0)$, then $e$ is even and $\gamma = \delta^2$ is the unique element of order 2 of the normal subgroup $\text{Gal}(K/K_0)$. So $\gamma \in Z(\text{Gal}(K/F))$. In this case $K_0 \subset K_\gamma$ and $K/K_\gamma$ is ramified extension.

Assume that $\gamma \not\in \text{Gal}(K/K_0)$. Then $\gamma|_{K_0} \in \text{Gal}(K_0/F)$ is of order two (hence $f = 2f'$ is even), and $\gamma = \rho f' \delta^a$ with $0 \leq a < e$. Then $K/K_\gamma$ is unramified extension, because if it was not the case we have $f(K_\gamma/F) = f(K/F)$ and hence $K_0 \subset K_\gamma$ which means

$$\gamma \in \text{Gal}(K/K_\gamma) \subset \text{Gal}(K/K_0),$$

contradicting to the assumption $\gamma \not\in \text{Gal}(K/K_0)$. Then $f(K_\gamma/F) = f'$ and $e(K_\gamma/F) = e$, and hence $e|qf' - 1$. So we have

$$1 = \gamma^2 = \rho f' \rho f' \delta^a \delta^a = \delta^{m+aqf'} + a = \delta^{e+a},$$

hence $2a \equiv -m (\text{mod } e)$. Then $a \equiv -\frac{m}{2}$ or $\frac{e}{2} - \frac{m}{2}$ (mod $e$) if $e$ is even (hence $m$ is even), and

$$a \pmod{e} = \begin{cases} -\frac{m}{2} & : \text{if } m \text{ is even}, \\ \frac{e}{2} - \frac{m}{2} & : \text{if } m \text{ is odd} \end{cases}$$

if $e$ is odd. We have $e|qf' - 1$ hence

$$\delta \gamma = \rho f' \delta^{qf'+a} = \rho f' \delta^{1+a} = \gamma \delta.$$

Now we have

$$\rho f'(q-1) = 1. \quad (3.11)$$

In fact $\text{Gal}(K_{\gamma}/F) = \langle \delta', \rho' \rangle$ with $\delta' = \delta|_{K_{\gamma}}, \rho' = \rho|_{K_{\gamma}}$. Then $(\rho')^{f(q-1)} = 1,$ that is

$$\rho f'(q-1) \in \text{Gal}(K/K_{\gamma}) = \langle \gamma \rangle.$$

If $\rho f'(q-1) \neq 1$, then $\rho f'(q-1) = \gamma = \rho f' \delta^a$, therefore

$$\rho f' = \rho f' \delta^a = \delta^{m+n} \in \text{Gal}(K/K_0)$$

and hence $f$ divides $f'q$, contradicting to the assumption that $q$ is odd. Now we have

$$\gamma \rho = \rho f' + \delta^{qa} = \rho \gamma \cdot \delta^{a(q-1)}.$$

For $a = -\frac{m}{2}$ or $a = \frac{e}{2} - \frac{m}{2}$, we have $a(q-1) \equiv 0 (\text{mod } e)$ if and only if

$$\frac{q-1}{2} \equiv 0 \pmod{\frac{e}{\gcd(e,m)}}$$

which is equivalent to $\rho f' \delta^{qa} = \rho f'(q-1) = 1$. Then (3.11) implies $\gamma \rho = \rho \gamma$. Then we have $\gamma$ is an element of the center of $\text{Gal}(K/F)$. ■

Put $\tilde{c}(x) = c(x^{1-\tau})$ for $x \in K^\times$. Then we have

$$\tilde{c}(x) = \chi_p(1,x)(1_K)$$

$$= \prod_{\gamma \in \Gamma, \gamma^2 \neq 1} \chi_{a_1K^\times}(x) \times \prod_{\gamma \in \Gamma, \gamma^2 \neq 1} \chi_{a_1K^{\times}}(N_{K/K_{\gamma}}(x)) \times \chi_{a_1K}(x)^2.$$
Since $\chi_{a_1 K} a_{\gamma^{-1}} = \chi_{a_1 K}^{-1}$ for $\gamma \in \Gamma$, we have
\[
\overline{c}(x) = \chi_{a_1 K}(x^{1-\tau}) \quad (x \in K^\times)
\]
if $H = \{1, \tau\}$, and
\[
\overline{c}(x) = \chi_{a_1 K a_{\rho'}}(N_{K/K_{\rho'}}(x)) \cdot \chi_{a_1 K a_{\rho'}}(N_{K/K_{\rho'}}(x)) \cdot \chi_{a_1 K}(x^{1-\tau}) \quad (x \in K^\times)
\]
if $H = \{1, \tau, \delta' = \delta^\tau, \tau \delta'\}$. In this case, since $f$ is even, $K/K_+$ is unramified so that $\tau \not\in \text{Gal}(K/K_0) = (\delta)$. We have

**Proposition 3.3.2** If $|H| = 2$, then
\[
c(-1) = \begin{cases} 
(\overline{\mathbb{Z}}/2\mathbb{Z}) : & \text{if } K/K_+ \text{ is ramified}, \\
1 : & \text{if } K/K_+ \text{ is unramified},
\end{cases}
\]
If $|H| = 4$, then
\[
c(-1) = -(-1)^{\frac{q - 1}{2}}.
\]
Note that $K/F$ is totally ramified if $K/K_+$ is ramified.

**Proof** If $|H| = 2$, we have $c(x) = \chi_{a_1 K}(x)$ for $x \in U_{K/K_+}$ so that
\[
c(-1) = (-1, K/K_+) = \begin{cases} 
1 : & \text{if } K/K_+ \text{ is unramified}, \\
(-1)^{\frac{q - 1}{2}} : & \text{if } K/K_+ \text{ is ramified}.
\end{cases}
\]
From now on, we will consider the case of $|H| = 4$. Put $H = \{1, \tau, \delta', \tau \delta'\}$ and let $E = K^H$ be the fixed subfield of $K/F$. Then we have
\[
\text{Gal}(K_{\delta'}/E) = \ker(\tau|_{K_{\delta'}}), \quad \text{Gal}(K_{\tau \delta'}/E) = \ker(\tau|_{K_{\tau \delta'}}).
\]
Put $K_{\delta'} = E(\eta)$ with $\eta^2 \in E$, or equivalently $\eta^2 = -\eta$. Then we have
\[
c(-1) = \overline{c}(\eta) = \chi_{a_1 K a_{\rho'}}(N_{K/K_{\rho'}}(\eta)) \cdot \chi_{a_1 K a_{\rho'}}(N_{K/K_{\rho'}}(\eta)) \cdot \chi_{a_1 K}(\eta^{1-\tau}) = (\eta^2, K_{\delta'}/E) \cdot (-\eta^2, K_{\delta'}/E) = (-1, K/K_+).
\]
Since $K/K_+$ is unramified, we have $(-1, K/K_+) = 1$. Since $N_{K_{\delta'}/E}(\eta) = -\eta^2$, we have $(-\eta^2, K_{\delta'}/E) = 1$. Since $K_{\delta'}/E$ is unramified, we have $(-1, K_{\delta'}/E) = 1$. Hence $(\eta^2, K_{\delta'}/E) = 1$. By the standard formula of the norm residue symbol, we have
\[
(\eta, K/K_{\delta'}) = (-\eta^2, K/E) \in \text{Gal}(K/K_{\delta'}) \subset \text{Gal}(K/E) \quad (3.12)
\]
since $N_{K_{\delta'}/E}(\eta) = -\eta^2$, and
\[
(-\eta^2, K/E) = ((-\eta^2, K_{\delta'}/E), (-\eta^2, K_{\tau \delta'}/E))
\]
in $\text{Gal}(K/E) = \text{Gal}(K_{\delta'}/E) \times \text{Gal}(K_{\tau \delta'}/E)$ by $\sigma = (\sigma|_{K_{\delta'}}, \sigma|_{K_{\tau \delta'}})$. Note that we have $(-\eta^2, K_{\delta'}/E) = 1$. So if $(\eta, K/K_{\delta'}) = 1$, then $(-\eta^2, K_{\tau \delta'}/E) = 1$. If $(\eta, K/K_{\delta'}) \neq 1$, then $(\eta, K/K_{\delta'}) = \delta'$, hence
\[
(-\eta^2, K_{\tau \delta'}/E) = \delta'|_{K_{\tau \delta'}} \neq 1.
\]
So we have \((-\eta^2, K_{r+}\mathcal{O}/E) = (\eta, K/K_{r'})\). The restriction mapping \(\text{Gal}(K/E) \to \text{Gal}(K_/E)\) sends \((-\eta^2, K/E)\) to \((-\eta^2, K_/E)\). Since the restriction mapping gives the isomorphism
\[
\text{Gal}(K/K_{r'}) \to \text{Gal}(K_/E).
\]
Hence (3.12) shows \((\eta, K/K_{r'}) = (-\eta^2, K_/E)\). Since \(K_/E\) is a ramified quadratic extension and \(\eta^2 \in E\) is not square in \(E\), we have
\[
(-\eta^2, K_/E) = (-1, K_/E) \cdot (\eta^2, K_/E) = (-1)^{\frac{j_e-1}{2}} \cdot (-1).
\]

The following proposition will be used in the next two sections.

**Proposition 3.3.3** We can choose the \(\chi\)-data \(\{\chi_{\lambda}\}_{\lambda \in \Phi(T)}\) so that \(c(x) = 1\) for all \(x \in U_{K/K_+} \cap (1 + p^2_K)\).

**Proof** If \(K/K_+\) is ramified, then \(K/F\) is totally ramified and \(c(x) = \chi_{a_1}^{\chi}(x)\) for \(x \in U_{K/K_+}\). Since
\[
(1 + p_{K_+}, K/K_+) = 1 \text{ and } (1 + p^2_K) \cap K_+^\times = 1 + p_{K_+},
\]
we can assume that \(\chi_{a_1}\) is trivial on \(1 + p^2_K\). Then \(c(x) = 1\) for all \(x \in U_{K/K_+} \cap (1 + p^2_K)\).

Assume that \(K/K_+\) is unramified. Since
\[
(1 + p_{K_+}, K/K_+) = 1 \text{ and } (1 + p_K) \cap K_+^\times = 1 + p_{K_+},
\]
we can choose \(\chi_{a_1}^{\chi}\) so that \(\chi_{a_1}^{\chi}(1 + p_K) = 1\). If further \(|H| = 4\), then \(K_{r'}/E\) is unramified, and \(K_{r'}/E\) is ramified. Since
\[
(1 + p_{K_{r'}}) \cap E^\times = (1 + p^2_{K_{r'}}) \cap E = 1 + p_E
\]
and \((x, K_{r'}/E) = (x, K_{r'}/E) = 1\) for all \(x \in 1 + p_E\), we can assume that
\[
\chi_{a_1}^{\chi_{a_1}^{\chi}}(1 + p_{K_{r'}}) = 1, \quad \chi_{a_1}^{\chi_{a_1}^{\chi}}(1 + p_{K_{r'}}) = 1.
\]
Since \(K/K_{r'}\) is ramified and \(K/K_{r'}\) is unramified, we have
\[
N_{K/K_{r'}}(1 + p^2_K) = 1 + p_{K_{r'}}, \quad N_{K/K_{r'}}(1 + p^2_K) = 1 + p^2_{K_{r'}}.
\]
Hence \(\tilde{c}(x) = 1\) for all \(x \in 1 + p^2_K\). Because \(K/K_+\) is unramified, we can prove by induction on \(k\) that \(x \mapsto x^{1/k}\) is surjection of \(1 + p^k_K\) onto \(U_{K/K_+} \cap (1 + p^k_K)\).

Then \(c(x) = 1\) for all \(x \in U_{K/K_+} \cap (1 + p^k_K)\). ☐

### 3.4 \(L\)-parameters associated with characters of tame elliptic tori

By local Langlands correspondence of tori described in Proposition 3.1.1, the continuous character \(\theta\) of \(U_{K/K_+}\) which parametrizes the irreducible representation \(\delta_{\beta, \theta}\) of \(Sp_{2n}(O_F)\) determines the cohomology class \([\alpha] \in H^1_{\text{cont}}(W_F, T)\). Then we have a group homomorphism
\[
\varphi: W_F \xrightarrow{\tilde{\alpha}} L^T \xrightarrow{\oplus} SO_{2n+1}^+(C).
\]

(3.13)
The construction of $\varphi$ shows that $\varphi(\sigma) \in SO_{2n+1}(\mathbb{C})$ is of the form

$$\varphi(\sigma) = \begin{bmatrix} \varphi_1(\sigma) & 0 \\ 0 & \det \varphi_1(\sigma) \end{bmatrix}$$

with $\varphi_1(\sigma) \in O(S_1, \mathbb{C})$ (3.14)

for $\sigma \in W_F$. The definition of (3.8) shows that $\text{tr} \varphi_1(\sigma) = \sum_{\gamma \in \text{Emb}_{W_F}(K,F), \gamma \sigma = \gamma} \chi_p(\sigma)(\gamma) \cdot \alpha(\sigma)(\gamma)$

for $\sigma \in W_F$. Here $\psi_c$ (resp. $\psi_\theta$) is the element of $\text{Hom}_{\text{conti}}(W_K, \mathbb{C})$ corresponding to $c$ (resp. $\theta$) by

$$\text{Hom}_{\text{conti}}(U_{K/K}, \mathbb{C}) \rightarrow \text{Hom}_{\text{conti}}(K^\times, \mathbb{C}) \delta_{K} \rightarrow \text{Hom}_{\text{conti}}(W_K, \mathbb{C})$$

This shows that $\varphi_1$ is the induced representation of $W_F$ from the character $\psi_c \cdot \psi_\theta$ of $W_K$. So $\varphi_1$ factors through the canonical surjection $W_F \rightarrow W_{K/F} = W_F/[W_K, W_K]$ and, if we put $\tilde{\vartheta} = c \cdot \vartheta$ and $\tilde{\vartheta}(x) = \vartheta(x^{1-\tau})$ ($x \in K^\times$), we have

$$\text{tr} \varphi_1(\sigma, x) = \begin{cases} 0 : & \sigma \neq 1, \\ \sum_{\gamma \in \text{Gal}(K/F)} \tilde{\vartheta}(x^{\gamma}) : & \sigma = 1 \end{cases}$$

(3.15)

for $(\sigma, x) \in W_{K/F} = \text{Gal}(K/F) \ltimes_{\alpha_{K/F}} K^\times$ with the fundamental class $[\alpha_{K/F}] \in H^2(\text{Gal}(K/F), K^\times)$.

The representation space $V_{\vartheta}$ of the induced representation $\text{Ind}_{K^\times/F} W_{K/F} \tilde{\vartheta}$ is the complex vector space of the $\mathbb{C}$-valued function $v$ on $\text{Gal}(K/F)$ with the action of $(\sigma, x) \in W_{K/F}$

$$(x \cdot v)(\gamma) = \tilde{\vartheta}(x^{\gamma}) \cdot v(\gamma), \quad (\sigma \cdot v)(\gamma) = \tilde{\vartheta}(\alpha_{K/F}(\sigma, \gamma) \cdot x^{-1}) \cdot v(\sigma^{-1})$$

A $\mathbb{C}$-basis $\{v_\rho\}_{\rho \in \text{Gal}(K/F)}$ of $V_{\vartheta}$ is defined by

$$v_\rho(\gamma) = \begin{cases} 1 : & \gamma = \rho, \\ 0 : & \gamma \neq \rho. \end{cases}$$

Then

$$x \cdot v_\rho = \tilde{\vartheta}(x^{\rho}) \cdot v_\rho, \quad \sigma \cdot v_\rho = \tilde{\vartheta}(\alpha_{K/F}(\sigma, \rho)) \cdot v_{\sigma \rho}$$

for $(\sigma, x) \in W_{K/F}$. The following proposition will be used to analyze $\text{Ind}_{K^\times/F} W_{K/F} \tilde{\vartheta}$ in detail.

**Proposition 3.4.1** Assume $l \geq 2$, then
1) Min \( \left\{ 2 \leq k \in \mathbb{Z} \mid \tilde{\vartheta}(\alpha) = 1 \forall \alpha \in 1 + p_K^k \right\} = \begin{cases} e(r-1) + 1 & : K/K_+ \text{ is unramified}, \\ e(r-1) & : K/K_+ \text{ is ramified}. \end{cases} \)

2) For an integer \( k \geq 2 \)

\[
\left\{ \sigma \in \text{Gal}(K/F) \mid \tilde{\vartheta}(x) = \tilde{\vartheta}(x) \text{ for } \forall x \in 1 + p_K^k \right\} = \begin{cases} \text{Gal}(K/F) & : k > e(r-1), \\ \text{Gal}(K/K_0) & : k = e(r-1), \\ \{1\} & : k < e(r-1). \end{cases}
\]

**Proof** Note that \( \vartheta(x) = \vartheta(x) \) for all \( x \in U_{K/K_+} \cap (1 + p_K^2) \) (by Proposition 3.3.3) and \( \vartheta(x) = 1 \) for all \( x \in U_{K/K_+} \cap (1 + p_K^2) \). Take an integer \( k \) such that \( 0 \leq k \leq e! \), and hence \( 2 \leq e! - k \). Then, for any \( x \in O_K \), we have

\[
(1 + \omega_F \omega_K^k x)^{1-e} \equiv 1 + \omega_F (\omega_K^k x - \omega_K^{-e} x^e) \quad (\text{mod } p_K^e)
\]

since \( 2(k-e) \geq e! \). Hence, for \( \alpha = 1 + \omega_F \omega_K^k x \in 1 + p_K^{e-k} (x \in O_K) \), we have

\[
\tilde{\vartheta}(\alpha) = \psi(T_{K/F}((\omega_K^k x - \omega_K^{-e} x^e)\beta)) = \psi(2T_{K/F}(\omega_K^k x\beta)).
\]  

1) The statement \( \tilde{\vartheta}(\alpha) = 1 \) for all \( \alpha \in 1 + p_K^{e-k} \) is equivalent to the statement \( T_{K/F}(\omega_K^k x \beta) \in O_F \) for all \( x \in O_K \), or to the statement \( \omega_K^k (\beta) \in \mathcal{D}(K/F)^{-1} = p_K^{1-e} \), and hence \( \text{ord}_K(\beta) \geq k - e - 1 \). Since

\[
\text{ord}_K(\beta) = \begin{cases} 0 & : K/K_+ \text{ is unramified}, \\ 1 & : K/K_+ \text{ is ramified} \end{cases}
\]

the proof is completed.

2) Because \( K/F \) is tamely ramified, we have

\[
V_t(K/F) = \{ \sigma \in \text{Gal}(K/F) \mid \text{ord}_K(x^e - x) \geq t + 1 \forall x \in O_K \}
\]

\[
= \begin{cases} \text{Gal}(K/F) & : t < 0, \\ \text{Gal}(K/K_0) & : 0 \leq t < 1, \\ \{1\} & : 1 \leq t. \end{cases}
\]  

(3.17)

Take a \( \sigma \in \text{Gal}(K/F) \). Then, by (3.16), we have

\[
\tilde{\vartheta}(\sigma^e) = \psi(2T_{K/F}(\omega_K^{-e} x^e \beta)) = \psi(2T_{K/F}(\omega_K^k x\beta)).
\]

So the statement \( \tilde{\vartheta}(\sigma^e) = \tilde{\vartheta}(\alpha) \) for all \( \alpha \in 1 + p_K^{e-k} \) is equivalent to the statement \( \omega_K^k (\beta^e - \beta) \in \mathcal{D}(K/F)^{-1} = p_K^{1-e} \), or to the statement

\[
\text{ord}_K(x^e - x) \geq k - e + 1 \quad \text{for all } x \in O_K
\]

since \( O_K = O_F[\beta] \), which is equivalent to \( \sigma \in V_{k-e} \). Then (3.17) completes the proof. ■
Proposition 3.4.2 The induced representation \( \text{Ind}_{W^\times} K/F \) is irreducible.

[Proof] Take a \( 0 \neq T \in \text{End}_{W^\times} V_0 \). Since \( T v_\rho = T(\rho \cdot v_1) = \rho \cdot T v_1 \) for all \( \rho \in \text{Gal}(K/F) \), we have \( T v_1 \neq 0 \). If \( (T v_1)(\gamma) \neq 0 \) for a \( \gamma \in \text{Gal}(K/F) \), then we have

\[
\tilde{\psi}(x^\gamma) \cdot (T v_1)(\gamma) = (x \cdot T v_1)(\gamma) = T(x \cdot v_1)(\gamma) = (T(\tilde{\psi}(x) \cdot v_1))(\gamma) = \tilde{\psi}(x) \cdot (T v_1)(\gamma),
\]

and hence \( \tilde{\psi}(x^\gamma) = \tilde{\psi}(x) \) for all \( x \in K^\times \). Then \( \gamma = 1 \) by Proposition 3.4.1. This means \( T v_1 = c \cdot v_1 \) with a \( c \in C^\times \). Then \( T v_\rho = T(v_1) = c \cdot v_\rho \) for all \( \rho \in \text{Gal}(K/F) \), and hence \( T \) is a homothety. \( \blacksquare \)

Remark 3.4.3 The proof of Proposition 3.4.2 shows that the induced representation \( \text{Ind}_{W^\times} K/F \) is irreducible if \( \tilde{\psi} \) is a character of \( K^\times \) such that \( \tilde{\psi}(x^\sigma) = \tilde{\psi}(x) \) for all \( x \in K^\times \) with \( \sigma \in \text{Gal}(K/F) \) implies \( \sigma = 1 \).

4 Formal degree conjecture

In this section, we will assume that \( K/F \) is a tamely ramified Galois extension of degree \( 2n \) and put \( \Gamma = \text{Gal}(K/F) \). We will keep the notations of the preceding sections.

4.1 \( \gamma \)-factor of adjoint representation

The admissible representation of the Weil-Deligne group \( W_F \times SL_2(C) \) to \( SO_{2n+1}(C) \) corresponding to the triple \( (\varphi, SO_{2n+1}(C), 0) \) as explained in the appendix A.6 is

\[
W_F \times SL_2(C) \xrightarrow{\text{projection}} W_F \xrightarrow{\varphi} SO_{2n+1}(C)
\]

which is also denoted by \( \varphi \). The purpose of this subsection is to determine the \( \gamma \)-factor \( \gamma(\varphi, \text{Ad}, \psi, d(x), s) \) whose definition and the basic properties are presented in the appendix A.6. Our result is

Theorem 4.1.1

\[
\gamma(\varphi, \text{Ad}, \psi, d(x), 0) = w(\text{Ad} \circ \varphi) \cdot q^{\frac{n^2 r}{2}} \times \begin{cases} 
1 & : K/K_+ \text{ is ramified,} \\
\frac{2}{1 + q^{-r}} & : K/K_+ \text{ is unramified}
\end{cases}
\]

where \( \psi \) is a continuous unitary additive character of \( F \) such that

\[
\{ x \in F \mid \psi(xO_F) = 1 \} = O_F
\]

and \( d(x) \) is the Haar measure on \( F \) such that \( \int_{O_F} d(x) = 1 \).
The rest of this subsection is devoted to the proof of the theorem. Let us use the notation of (3.9)
\[ \Gamma = \text{Gal}(K/F) = \langle \delta, \rho \rangle, \]
that is, \( \text{Gal}(K/K_0) = \langle \delta \rangle \) with the maximal unramified subextension \( K_0/F \) of \( K/F \) and \( \rho|_{K_0} \in \text{Gal}(K_0/F) \) is the inverse of the Frobenius automorphism. Put
\[ \rho \delta \rho^{-1} = \delta^{l_i}, \quad \rho f = \delta^m \quad (0 \leq i, m < e, \text{ql} \equiv 1 \pmod{e}). \]
By the canonical surjection
\[ W_F \to W_F/[W_K, W_K] = W_{K/F} = \text{Gal}(K/F) \ltimes_{\alpha_{K/F}} K^\times \subset \text{Gal}(K^{ab}/F), \]
\( I_F = \text{Gal}(F^{alg}/F^{ur}) \subset W_F \) is mapped onto
\[ \text{Gal}(K/K_0) \ltimes_{\alpha_{K/F}} O_K^\times = \text{Gal}(K^{ab}/F^{ur}). \]
The representation space \( V_\vartheta \) of \( \varphi_1 = \text{Ind}^{W_{K/F}}_{K^\times} \tilde{\vartheta} \) has a \( W_{K/F} \)-invariant non-degenerate symmetric form
\[ S_1(u, v) = \sum_{\gamma \in \Gamma} \tilde{\vartheta}(\alpha_{K/F}(\gamma, \tau))^{-1} \cdot u(\gamma \tau) v(\gamma \tau) \quad (u, v \in V_\vartheta) \]
which is unique up to constant multiple, by Proposition B.0.1. Put
\[ u_\sigma = \tilde{\vartheta}(\alpha_{K/F}(\sigma, \tau)) \cdot v_\sigma \in V_\vartheta \]
for \( \sigma \in \Gamma \). Then we have
\[ \alpha \cdot v_\beta = \tilde{\vartheta}(\alpha_{K/F}(\alpha, \beta)) \cdot v_\alpha \beta, \quad \alpha \cdot u_\beta = \tilde{\vartheta}(\alpha_{K/F}(\alpha, \beta))^{-1} \cdot u_\alpha \beta \quad (4.2) \]
and
\[ S_1(v_\alpha, v_\beta) = S_1(u_\alpha, u_\beta) = 0, \quad S_1(v_\alpha, u_\alpha) = \begin{cases} 1 & : \alpha = \beta, \\ 0 & : \alpha \neq \beta \end{cases} \]
for \( \alpha, \beta \in \Gamma \). Fixing a representatives \( S \) of \( \Gamma/\langle \tau \rangle \), we will identify the orthogonal group \( O(V, S_1) \) of the symmetric form \( S_1 \) with the matrix group \( O(S_1, \mathbb{C}) \) of \( 3.14 \) by means of the \( \mathbb{C} \)-basis \( \{ v_\sigma, u_\sigma \}_{\sigma \in \mathcal{S}} \) of \( V_\vartheta \) which we will call the canonical basis associated with \( S \). Then we have
\[ \varphi(x) = \begin{bmatrix} [x] & [x]^{-1} \\ 1 \end{bmatrix} \in SO_{2n+1}(\mathbb{C}) \text{ with } [x] = \text{diag}(\tilde{\vartheta}(x^\sigma))_{\sigma \in \mathcal{S}} \]
for \( x \in K^\times \subset W_{K/F} \) so that the centralizer of \( \varphi(O_K^\times) \) in \( SO_{2n+1}(\mathbb{C}) \) is
\[ Z_{SO_{2n+1}(\mathbb{C})}(\varphi(O_K^\times)) = \left\{ \begin{bmatrix} a & a^{-1} \\ a^{-1} & 1 \end{bmatrix} \bigg| a = \text{diagonal} \in GL_n(\mathbb{C}) \right\} \quad (4.3) \]
and the space \( \hat{\mathfrak{g}}^{O_K^\times} \) of the \( \text{Ad} \circ \varphi(O_K^\times) \)-fixed vectors in
\[ \hat{\mathfrak{g}} = \mathfrak{so}_{2n+1}(\mathbb{C}) = \{ X \in \mathfrak{gl}_{2n+1}(\mathbb{C}) \mid XS + S^T X = 0 \} \]
is
\[ \hat{\mathfrak{g}}^O_K = \left\{ \begin{bmatrix} A & -A \\ 0 & \end{bmatrix} \bigg| A=\text{diagonal} \in \mathfrak{gl}_n(\mathbb{C}) \right\} \] (4.4)
by Proposition 3.4.1.

Let us denote by \( A_\varphi \) the centralizer of \( \text{Im}(\varphi) \) in \( SO_{2n+1}(\mathbb{C}) \). We have

**Proposition 4.1.2**

\[ L(\varphi, \text{Ad}, s) = \begin{cases} 1 & : K/K_+ \text{ is ramified,} \\ \frac{1}{1 + q^{-f+s}} & : K/K_+ \text{ is unramified} \end{cases} \]

and

\[ A_\varphi = \left\{ \begin{bmatrix} \pm 1_{2n} \\ 0 \\ 0 \end{bmatrix} \right\}. \]

**Proof** Assume that \( K/K_+ \) is ramified. Then \( K/F \) is totally ramified and \( \text{Gal}(K/F) = \langle \delta \rangle \) a cyclic group of order \( 2n \) with \( \tau = \delta^n \). Put \( S = \{ \delta^i \}_{0 \leq i < n} \) which is a representatives of \( \Gamma/\langle \tau \rangle \). Since \( K/F \) is a cyclic extension, we have \( \alpha_{K/F}(\alpha, \beta) \in F^\times \) for all \( \alpha, \beta \in \Gamma \), the canonical basis associated with \( S \) is \( v_i = v_{\delta^i-1}, \quad u_i = u_{\delta^i-1} \quad (1 \leq i \leq n) \).

Then (4.2) shows

\[ \varphi(\delta) = \begin{bmatrix} 0 & 1 & 0 \\ 1_{2n-1} & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in SO_{2n+1}(\mathbb{C}). \] (4.5)

Then the centralizer \( A_\varphi \) of \( \text{Im}(\varphi) \) in \( SO_{2n+1}(\mathbb{C}) \) is

\[ A_\varphi = \left\{ \begin{bmatrix} \pm 1_{2n} \\ 0 \\ 0 \end{bmatrix} \right\}, \]

and the space \( \hat{\mathfrak{g}}^{f_F} \) of the \( \text{Ad} \circ \varphi(I_F) \)-fixed vectors in \( \hat{\mathfrak{g}} \) is \( \{0\} \) so that we have

\[ L(\varphi, \text{Ad}, s) = 1. \]

Now assume that \( K/K_+ \) is unramified. Then \( \tau = \delta^a \rho^j \) with \( 0 \leq a < e \) by Proposition 3.3.1. Put \( S = \{ \delta^i \rho^j \}_{0 \leq i, 0 \leq j < f_+} \) which is a representatives of \( \Gamma/\langle \tau \rangle \). The associated basis

\[ v_{ij} = v_{\delta^i \rho^j-1}, \quad u_{ij} = u_{\delta^i \rho^j-1} \quad (1 \leq i \leq e, 1 \leq j \leq f_+) \]

is ordered lexicographically. Then

\[ \varphi(\delta) = \begin{bmatrix} \Delta \\ \Delta^{-1} \\ 1 \end{bmatrix} \in SO_{2n+1}(\mathbb{C}) \text{ with } \Delta = \begin{bmatrix} \Delta_1 & & \\ & \ddots & \\ & & \Delta_{f_+} \end{bmatrix}, \]

where

\[ \Delta_j = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \alpha_{e,j} \\ \alpha_{1,j} & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{2,j} & 0 & \cdots & 0 & 0 \\ & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & 0 & 0 & \alpha_{e-1,j} \\ & & & & \alpha_{e-1,j} & 0 \end{bmatrix} \]

is
with $\alpha_{i,j} = \tilde{\vartheta} (\alpha_{K/F}(\delta, \delta^{-1} \rho^{j-1}))$. The action of $\varphi(\delta)$ on (4.4) shows that the space $\hat{g}_{I_{F}}^{\varphi}$ of the $\text{Ad} \circ \varphi(I_{F})$-fixed vectors in $\hat{g}$ is

$$
\hat{g}_{I_{F}}^{\varphi} = \left\{ \begin{bmatrix} A & -A \\ 0 & 0 \end{bmatrix} \mid A = \begin{bmatrix} a_{11}e & a_{21}e & \ldots & a_{f_{+}1}e \\ 0 & 0 & \ldots & 0 \end{bmatrix} \right\}.
$$

Since

$$
\rho \cdot \delta^{i-1} \rho^{j-1} = \begin{cases} 
\delta^{i(i+1)} \rho^{j} = \delta^{i'-1} \rho - j & : 1 \leq j < f_{+}, \\
\delta^{i(i+1)} - a_{\tau} \rho^{j} = \delta^{i''-1} \rho & : j = f_{+} 
\end{cases} \quad (1 \leq i', i'' \leq e)
$$

for $1 \leq i \leq e$, let $[l]$ and $[l, a]$ be the permutation matrices of the permutations

$$
\begin{pmatrix} 1 & 2 & \ldots & e \\ l' & l'' & \ldots & e' \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & \ldots & e \\ 1'' & 2'' & \ldots & e'' \end{pmatrix}
$$

respectively. Then we have

$$
\varphi(\rho) = \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & (-1)^{e} \end{bmatrix} \in SO_{2n+1}({\mathbb{C}})
$$

with

$$
A = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ P_{1} & 0 & 0 & \ldots & 0 & 0 \\ & P_{1} & 0 & \ldots & 0 & 0 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & 0 & 0 \\ & & & & P_{f_{+}1} & 0 
\end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 
\end{bmatrix},
$$

$$
C = \begin{bmatrix} 0 & 0 & \ldots & P_{f_{+}} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 0 
\end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ Q_{1} & 0 & \ldots & 0 & 0 \\ Q_{2} & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & 0 & Q_{f_{+}1} 
\end{bmatrix}.
$$
where

\[
P_j = \begin{cases}
(12 \ldots e) & \begin{bmatrix}
\alpha_{1j} \\
\alpha_{2j} & \ddots \\
& \alpha_{ej}
\end{bmatrix} : 1 \leq j < f_+,
\end{cases}
\]

\[
Q_j = \begin{cases}
(12 \ldots e) & \begin{bmatrix}
\beta_1 \\
\beta_2 & \ddots \\
& \beta_e
\end{bmatrix} : j = f_+,
\end{cases}
\]

with

\[
\alpha_{ij} = \tilde{\vartheta} (\alpha_{K/F}(\rho, \delta^{-1} \rho^{j-1})),
\]

\[
\beta_i = \tilde{\vartheta} \left( \alpha_{K/F}(\rho, \delta^{-1} \rho^{f_+}) \cdot \alpha_{K/F}(\delta^{j-1}, \tau) \right).
\]

Then the adjoint action of \( \varphi(\rho) \) on \( \tilde{\mathfrak{g}}^{f_+} \) gives

\[
\det \left( 1_{f_+} - t \cdot \text{Ad} \circ \varphi(\rho) |_{\tilde{\mathfrak{g}}^{f_+}} \right) = 1 + t^{-f_+}
\]

so that we have \( L(\varphi, \text{Ad}, s) = (1 + q^{-f_+})^{-1} \). Finally the centralizer of \( \text{Im}(\varphi) \) in \( SO_{2n+1}(\mathbb{C}) \) is

\[
A_\varphi = \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.
\]

Next we will calculate the Artin conductor of \( \text{Ad} \circ \varphi \). Note that the complex vector space \( \tilde{\mathfrak{g}} \) is isomorphic to the space of alternating matrices

\[
\text{Alt}_{2n+1}(\mathbb{C}) = \{ X \in M_{2n+1}(\mathbb{C}) \mid X + \dagger X = 0 \}
\]

and \( \text{Ad} \circ \varphi \) on \( \tilde{\mathfrak{g}} \) is isomorphic to \( \bigwedge^2 \varphi \) on \( \text{Alt}_{2n+1}(\mathbb{C}) \). Since \( \varphi = \varphi_1 \oplus \det \varphi_1 \) and \( \varphi_1 = \text{Ind}_{K/F}^{W_{K/F}} \tilde{\vartheta} \) with \( (\det \varphi_1)|_{K^s} = 1 \), we have

\[
\bigwedge^2 \varphi = (\bigwedge^2 \varphi_1) \oplus (\varphi_1 \oplus \det \varphi_1) = (\bigwedge^2 \varphi_1) \oplus \varphi_1.
\]
Then $\chi_{\text{Ad} \circ \varphi} = \chi_{\lambda^2 \varphi_1} + \chi_{\varphi_1}$ and the character formula

$$\chi_{\varphi_1}(g) = \begin{cases} 0 : & \sigma \neq 1, \\
\sum_{\gamma \in \Gamma} \tilde{\vartheta}(x^\gamma) : & \sigma = 1 
\end{cases}$$

for $g = (\sigma, x) \in W_{K/F} = \Gamma \times K/F$. Then we have

$$\chi_{\lambda^2 \varphi_1}(g) = \frac{1}{2} \left\{ \chi_{\varphi_1}(g)^2 - \chi_{\varphi_1}(g^2) \right\}$$

$$= \begin{cases} 0 : & \sigma^2 \neq 1, \\
\frac{1}{2} \sum_{\gamma \in \Gamma} \tilde{\vartheta} \left( \alpha_{K/F}(\sigma, \sigma)^\gamma \cdot x^{(1+\sigma)\gamma} \right) : & \sigma^2 = 1, \sigma \neq 1, \\
\frac{1}{2} \sum_{\sigma \neq 1} \tilde{\vartheta} \left( x^{\alpha(1+\gamma)} \right) : & \sigma = 1. 
\end{cases} \quad (4.6)$$

Now we have

**Proposition 4.1.3** The Artin conductor of $\text{Ad} \circ \varphi$ is

$$a(\text{Ad} \circ \varphi) = 2n^2 r.$$  

**[Proof]** Let us denote by $K^{(k)} = K_{\varpi_{K,k}}$ ($k = 1, 2, \ldots$) the field of $\varpi_{K,k}$-th division points of Lubin-Tate theory over $K$. Then we have an isomorphism

$$\delta_K : 1 + \mathfrak{p}_K^k \rightarrow \text{Gal}(K^{ab}/K^{(k)}K^{ur}).$$

Because the character $\tilde{\vartheta} : K^\times \rightarrow \mathbb{C}^\times$ comes from a character of

$$G_\beta(O_F/p^r) \subset (O_K/p_{K,F}^r)^\times,$$

$\varphi$ is trivial on $\text{Gal}(K^{ab}/K^{(cr)}K^{ur})$. Note that $K^{(cr)}K^{ur} = K^{(cr)}F^{ur}$ is a finite extension of $F^{ur}$. If we use the upper numbering

$$V^s = V_t(K^{(cr)}F^{ur}/F^{ur})$$

of the higher ramification group, where $t \mapsto s$ is the inverse of Hasse function whose graph is
then $\delta_K$ induces the isomorphism

$$(1 + p^K)/(1 + p^\text{cr}) \cong \text{Gal}(K^{\text{cr}}K_{\text{ur}}/K) = \mathcal{V}_s$$

for $k-1 < s < k$ ($k = 1, 2, \cdots$), and hence, for $V_t = V_t(K^{\text{cr}}F_{\text{ur}}/F_{\text{ur}})$, we have

$$|V_t| = \begin{cases} 
2 \cdot q^{nr}(1 - q^{-f}) : t = 0, \\
q^{nr-fk} : q^{f(k-1)} - 1 < t \leq q^{fk} - 1.
\end{cases}$$

By the definition

$$a(\text{Ad} \circ \varphi) = \sum_{r=0}^{\infty} (\dim_{\mathbb{C}} \hat{\mathcal{g}} - \dim_{\mathbb{C}} \hat{\mathcal{g}}^{V_t}) \cdot (V_0 : V_t)^{-1}.$$  

We have

$$\dim_{\mathbb{C}} \hat{\mathcal{g}}^{V_t} = \dim_{\mathbb{C}} \hat{\mathcal{g}}^{I^r} = \begin{cases} 
0 : K/K_+ \text{ is ramified}, \\
f_+ : K/K_+ \text{ is unramified}
\end{cases}$$

as shown in the proof of Proposition 4.1.2. For $q^{f(k-1)} - 1 < t \leq q^{fk} - 1$ with $k > 0$, we have

$$|V_t| \cdot \dim_{\mathbb{C}} \hat{\mathcal{g}}^{V_t} = \sum_{g \in V_t} \chi_{\text{Ad}_\varphi}(g) = \frac{1}{2} \sum_{x \in V_t} \sum_{\alpha, \gamma \in \Gamma} \tilde{\vartheta}(x^\alpha(1+\gamma)) + \sum_{x \in V_t} \tilde{\vartheta}(x^\gamma)$$

$$= n \cdot \sum_{x \in V_t} \sum_{\tau \neq \gamma \in \Gamma} \tilde{\vartheta}(x^{1-\gamma}) + 2n \cdot \sum_{x \in V_t} \tilde{\vartheta}(x)$$

where $V_t$ is identified with $(1 + p^K)/(1 + p^K_{\text{cr}})$.

If $K/K_+$ is unramified, then $\tau \notin \text{Gal}(K/K_0)$, and Proposition 3.3.1 gives

$$\sum_{x \in V_t} \sum_{\tau \neq \gamma \in \Gamma} \tilde{\vartheta}(x^{1-\gamma}) = |V_t| \times \begin{cases} 
2n - 1 : k > e(r-1), \\
e : k = e(r-1), \\
1 : k < e(r-1)
\end{cases}$$
and
\[ \sum_{\bar{x} \in \mathcal{V}_t} \tilde{\vartheta}(\bar{x}) = \begin{cases} |\mathcal{V}_t| & : k > e(r - 1), \\ 0 & : k \leq e(r - 1). \end{cases} \]

So we have
\[ \text{dim}_{\mathcal{C}} \mathcal{V}_t = \begin{cases} n(2n + 1) & : k > e(r - 1), \\ ne & : k = e(r - 1), \\ n & : k < e(r - 1). \end{cases} \]

Then we have
\[ a(\text{Ad} \circ \varphi) = n(2n + 1) - f_+ + \{n(2n + 1) - n\} \cdot e^{-1} \cdot \{e(r - 1) - 1\} \\
+ \{n(2n + 1) - ne\} \cdot e^{-1} = 2n^2 r. \]

If $K/K_+$ is ramified, then $K/F$ is totally ramified and we have
\[ \sum_{x \in \mathcal{V}_t} \sum_{\tau \neq \gamma \in \Gamma} \tilde{\vartheta}(x^{1-\gamma}) = |\mathcal{V}_t| \times \begin{cases} 2n - 1 & : k \geq e(r - 1), \\ 1 & : k < e(r - 1) \end{cases} \]
and
\[ \sum_{x \in \mathcal{V}_t} \tilde{\vartheta}(x) = \begin{cases} |\mathcal{V}_t| & : k \geq e(r - 1), \\ 0 & : k < e(r - 1) \end{cases} \]
by Proposition 3.4.1. The we have
\[ a(\text{Ad} \circ \varphi) = n(2n + 1) + \{n(2n + 1) - n\} \cdot (2n)^{-1} \cdot \{2n(r - 1) - 1\} \\
= 2n^2 r. \]

Since
\[ \gamma(\varphi, \text{Ad}, \psi, d(x), 0) = \varepsilon(\varphi, \text{Ad}, \psi, d(x)) \cdot \frac{L(\varphi, \text{Ad}, 1)}{L(\varphi, \text{Ad}, 0)} \]
and
\[ \varepsilon(\varphi, \text{Ad}, \psi, d(x)) = w(\text{Ad} \circ \varphi) \cdot q^{a(\text{Ad} \circ \varphi)/2}, \]
Proposition 4.1.2 and Proposition 4.1.3 give the proof of Theorem 4.1.1.

4.2 $\gamma$-factor of principal parameter

Let $\text{Sym}_{2n}$ be the symmetric tensor representation of $SL_2(\mathbb{C})$ on the space $\mathcal{P}_{2n}$ of the complex coefficient homogeneous polynomials of $X, Y$ of degree $2n$. Then
\[ \langle f, g \rangle = f \left( -\frac{\partial}{\partial Y}, \frac{\partial}{\partial X} \right) g(X, Y) \bigg|_{(X, Y) = (0, 0)} \quad (f, g \in \mathcal{P}_{2n}) \]
defines a $SL_2(\mathbb{C})$-invariant non-degenerate symmetric complex bilinear form on the complex vector space $\mathcal{P}_{2n}$. For the $\mathbb{C}$-basis \( \left\{ v_k = \frac{1}{(k - 1)!} X^{2n+1-k} Y^{k-1} \right\}_{k=1, 2, \ldots, 2n+1} \) of $\mathcal{P}_{2n}$, we have
\[ \langle v_k, v_l \rangle = \begin{cases} 0 & : k + l = 2n + 2, \\ (-1)^{k-1} = (-1)^{l-1} & : k + l = 2n + 2. \end{cases} \]
and the identification

\[ SO(\mathcal{P}_2n, (\cdot, \cdot)) = SO_{2n+1}(\mathbb{C}) = \{ g \in SL_{2n+1}(\mathbb{C}) \mid gJ_{2n+1}J = J_{2n+1} \} \]

where

\[
J_{2n+1} = \begin{bmatrix} 1 & & & & 0 \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \\ 0 & & & & 1 \end{bmatrix}.
\]

The Lie algebra of \( SO_{2n+1}(\mathbb{C}) \) is

\[ \mathfrak{so}_{2n+1}(\mathbb{C}) = \{ X \in \mathfrak{gl}_{2n+1}(\mathbb{C}) \mid XJ_{2n+1} + J_{2n+1}X = 0 \} \]

and

\[
d\text{Sym}_{2n} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = N_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 1 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \in \widehat{\mathfrak{g}}
\]

is the nilpotent element in \( \mathfrak{so}_{2n+1}(\mathbb{C}) \) associated with the standard épininglage of the standard root system of \( \mathfrak{so}_{2n+1}(\mathbb{C}) \). Then

\[
\varphi_0 : W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{proj.}} SL_2(\mathbb{C}) \xrightarrow{\text{Sym}_{2n}}} SO_{2n+1}(\mathbb{C}) \quad (4.7)
\]

is a representation of Weil-Deligne group with the associated triplet \( (\rho_0, SO_{2n+1}(\mathbb{C}), N_0) \) such that \( \rho_0|_{\tilde{I}_F} \) is trivial and

\[
\rho_0(\tilde{\text{Fr}}) = \begin{bmatrix} q^{-n} & q^{-(n-1)} & \cdots & q^{-(1)} & q \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 1 & q^{-2} & \cdots & q^{-1} & q^n \end{bmatrix} \in SO_{2n+1}(\mathbb{C}).
\]

Now

\[ \{ N_0^{2k-1} \mid k = 1, 2, \cdots, n \} \quad (4.8) \]

is a \( \mathbb{C} \)-basis of

\[ \widehat{\mathfrak{g}} N_0 = \{ X \in \widehat{\mathfrak{g}} \mid [X, N_0] = 0 \}. \]

The representation matrix of \( \text{Ad} \circ \rho_0(\tilde{\text{Fr}}) \in GL(\widehat{\mathfrak{g}}) \) is

\[
\begin{bmatrix} q^{-1} & q^{-3} & \cdots & q^{-(2n-1)} \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}
\]

so that we have

\[
L(\varphi_0, \text{Ad}, s) = \det \left( (1 - q^{-s} \cdot \text{Ad} \circ \rho_0(\tilde{\text{Fr}}))|_{\widehat{\mathfrak{g}} N_0} \right)^{-1} = \prod_{k=1}^{n} \left( 1 - q^{-(s+2k-1)} \right)^{-1}.
\]
On the other hand [7, p.448] shows
\[ \varepsilon(\varphi_0, \text{Ad}, \psi, d(x)) = q^{n^2}. \]
Since the symmetric tensor representation \( \text{Sym}_{2n} \) is self-dual, we have
\[
\gamma(\varphi_0, \text{Ad}, \psi, d(x), 0) = \varepsilon(\varphi_0, \text{Ad}, \psi, d(x)) \frac{L(\varphi_0, \text{Ad}, 1)}{L(\varphi_0, \text{Ad}, 0)} = q^{n^2} \prod_{k=1}^{n} \frac{1 - q^{-(2k-1)}}{1 - q^{-2k}}.
\]
\[ (4.9) \]

4.3 Verification of formal degree conjecture

Let \( d_{G(F)} \) be the Haar measure on \( G(F) \) such that \( \int_{G(O_F)} d_{G(F)}(x) = 1 \). Then the Euler-Poincaré measure \( \mu_{G(F)} \) on \( G(F) = \text{Sp}_{2n}(F) \) is (see [12, p.150, Th.7])
\[
d\mu_{G(F)}(x) = (-1)^n q^{n^2} \prod_{k=1}^{n} \left( 1 - q^{-(2k-1)} \right) \cdot d_{G(F)}(x).
\]
Then Theorem 2.3.1 implies that the formal degree of the supercuspidal representation \( \pi_{\beta, \theta} = \text{ind}_{G(O_F)}^{G(F)} \delta_{\beta, \theta} \) with respect to the absolute value of the Euler-Poincaré measure on \( G(F) \) is
\[
q^{n^2(r-1)} \prod_{k=1}^{n} \frac{1 - q^{-2k}}{1 - q^{-(2k-1)}} \times \begin{cases} 
\frac{1}{2} & : K/K_+ \text{ is ramified,} \\
\frac{1}{1 + q^{J_+}} & : K/K_+ \text{ is unramified.} 
\end{cases}
\]
\[ (4.10) \]

Since the order of the centralizer \( A_\varphi \) of \( \text{Im}(\varphi) \) in \( SO_{2n+1}(\mathbb{C}) \) is two (Proposition 4.1.2), Theorem 4.1.1 and (4.9) gives the following

Theorem 4.3.1 The formal degree of the supercuspidal representation \( \pi_{\beta, \theta} = \text{ind}_{G(O_F)}^{G(F)} \delta_{\beta, \theta} \) with respect to the absolute value of the Euler-Poincaré measure on \( G(F) \) is
\[
\frac{1}{|A_\varphi|} \begin{vmatrix} \frac{\gamma(\varphi, \text{Ad}, \psi, d(x), 0)}{\gamma(\varphi_0, \text{Ad}, \psi, d(x), 0)} \end{vmatrix}.
\]

Since \( A_\varphi \) is a finite abelian group, all the irreducible representation of \( A_\varphi \) is one-dimensional. So Theorem 4.3.1 says that the formal degree conjecture is valid if we consider \( (\varphi, \text{Ad}, \psi, d(x), 0) \) as the Arthur-Langlands parameter of the supercuspidal representation \( \pi_{\beta, \theta} \) and \( (\varphi_0, \text{Ad}, \psi, d(x), 0) \) as the principal parameter of \( G(F) = \text{Sp}_{2n}(F) \).

5 Root number conjecture

In this section, we will assume that \( K/F \) is a tamely ramified Galois extension of degree \( 2n \) and put \( \Gamma = \text{Gal}(K/F) \). We will keep the notations of the preceding sections.
5.1 Structure of adjoint representation

We will identify the representations of $W_{K/F}$ with the representations of $W_F$ which factor through the canonical surjection

$$W_F \to W_F/[W_K,W_K] = W_{K/F}.$$ 

We will also regard a representation of $\Gamma$ as the representation of $W_{K/F}$ via the projection $W_{K/F} \to \Gamma$.

As we have seen in the subsection 4.1

$$\text{Ad} \circ \varphi = \bigwedge^2 \varphi = \bigwedge^2 \varphi_1 \oplus \varphi_1$$

with $\varphi_1 = \text{Ind}_{W_{K/F}}^{W_K} \vartheta$. Now we have

**Theorem 5.1.1**

$$\bigwedge^2 \varphi_1 = \bigoplus_{\pi(\tau) \neq 1} \pi \dim \pi \oplus \bigoplus_{\gamma \neq \gamma^{-1} \in \Gamma} \text{Ind}_{K_{\gamma}}^{W_{K/F}} \vartheta_{\gamma} \oplus \bigoplus_{1 \neq \gamma \in \Gamma} \oplus \text{Ind}_{W_{K/F}}^{W_{K/K_{\gamma}}} \chi_{\gamma}.$$ 

Here $\bigoplus$ denotes the direct sum over the equivalence classes $\pi$ of the irreducible representations of $\Gamma$ such that $\pi(\tau) \neq 1$. The direct sum $\bigoplus_{\gamma \neq \gamma^{-1} \in \Gamma}$ is over the subsets $\{\gamma, \gamma^{-1}\} \subset \Gamma$ such that $\gamma^2 \neq 1$, and $\vartheta_{\gamma}(x) = \vartheta(x^{1+\gamma})$ ($x \in K^\times$). For a $\gamma \in \Gamma$ of order two, the unitary character $\chi_{\gamma}$ of $W_{K/K_{\gamma}}$ is defined by

$$\chi_{\gamma} : W_{K/K_{\gamma}} = W_{K_{\gamma}}/[W_K,W_K] \longrightarrow W_{K_{\gamma}}/[W_K,W_K] \longrightarrow W_{K_{\gamma}}/W_{K/K_{\gamma}}$$

with the subfield $F \subset K_{\gamma} \subset K$ such that $\text{Gal}(K/K_{\gamma}) = \langle \gamma \rangle$ and

$$(x, K/K_{\gamma}) = \begin{cases} 1 & : x \in N_{K/K_{\gamma}}(K^\times), \\ -1 & : x \notin N_{K/K_{\gamma}}(K^\times). \end{cases}$$

The rest of this subsection is devoted to the proof of the theorem.

5.1.1 Take a $\gamma \in \Gamma$ of order two. Note that the group homomorphism $x \mapsto (x, K/K_{\gamma})$ induces the inverse of the isomorphism

$$\text{Gal}(K/K_{\gamma}) \to K_{\gamma}^\times / N_{K/K_{\gamma}}(K^\times) W_K[[K_{\gamma}/[W_K,W_K]]] \to W_{K_{\gamma}}/[W_K,W_K] \to W_{K_{\gamma}}/W_K$$

if we identify $\text{Gal}(K/K_{\gamma})$ with $\{\pm 1\}$. Then the commutative diagram

$$\begin{array}{cccc}
W_K/[W_K,W_K] & W_{K_{\gamma}}/[W_K,W_K] & W_{K_{\gamma}}/W_K \\
\text{Gal}(K/K_{\gamma}) & \quad & \quad & \quad
\end{array}$$

with $K_{\gamma}^\times / N_{K/K_{\gamma}}(K_{\gamma})$
implies that we have
\[ \chi_{\gamma}(\sigma, x) = \text{sign}(\sigma) \cdot \tilde{\theta}(\alpha_{K/F}(\sigma, \gamma) \cdot x^{1+\gamma}) \]
for \((\gamma, x) \in W_{K/K_{\gamma}} = \text{Gal}(K/K_{\gamma}) \times_{\alpha_{K/F}K} W_{K/F}\) where
\[ \text{sign}(\sigma) = \begin{cases} 1 & : \sigma = 1, \\ -1 & : \sigma = \gamma. \end{cases} \]

By means of the cocycle relation of \(\alpha_{K/F}\), we have
\[ \chi_{\gamma}(\alpha^{-1}(\sigma, x)\alpha) = \begin{cases} \tilde{\theta}(x^{\alpha(1+\gamma)}) & : \sigma = 1, \\ -\tilde{\theta}(\alpha_{K/F}(\gamma, \gamma)\alpha x^{\alpha(1+\gamma)}) & : \sigma = \gamma \end{cases} \]
for any \((\sigma, x) \in W_{K/K_{\gamma}}\) and \(\alpha \in \sigma\). Note that the elements of \(\Gamma\) of order two are central as shown by Proposition 3.3.1. The character of the induced representation \(\pi_{\gamma} = \text{Ind}_{W_{K/K_{\gamma}}}^{W_{K/K}} \chi_{\gamma}\) is
\[ \chi_{\pi_{\gamma}}(\sigma, x) = \begin{cases} 0 & : \sigma \notin \text{Gal}(K/K_{\gamma}), \\ \sum_{\alpha \in \Gamma/(\gamma)} \chi_{\gamma}(\alpha^{-1}(\sigma, x)\alpha) & : \sigma \in \text{Gal}(K/K_{\gamma}) \end{cases} \\
= \begin{cases} 0 & : \sigma \neq 1, \gamma, \\ \sum_{\alpha \in \Gamma/(\gamma)} \tilde{\theta}(x^{\alpha(1+\gamma)}) & : \sigma = 1, \\ -\sum_{\alpha \in \Gamma/(\gamma)} \tilde{\theta}(\alpha_{K/F}(\gamma, \gamma)\alpha x^{\alpha(1+\gamma)}) & : \sigma = \gamma. \end{cases} \tag{5.2} \]

5.1.2 Since \(\tau \in \Gamma\) is a central element, the character of the induced representation \(R_{\tau} = \text{Ind}_{\langle \tau \rangle}^{\Gamma} 1_{\langle \tau \rangle}\) is
\[ \chi_{R_{\tau}}(\sigma) = \begin{cases} 0 & : \sigma \neq 1, \tau, \\ (\Gamma : \langle \tau \rangle) = n & : \sigma = 1, \tau. \end{cases} \]

For an irreducible representation \(\pi\) of \(\Gamma\), we have \(\pi(\tau) = \pm 1\), and we have
\[ \langle \pi, R_{\tau} \rangle = |\Gamma|^{-1}(n \cdot \chi_{\pi}(1) + n \cdot \chi_{\pi}(\tau)) = \begin{cases} \dim \pi & : \pi(\tau) = 1, \\ 0 & : \pi(\tau) = -1. \end{cases} \]

Hence \(R_{\tau} \oplus \bigoplus_{\pi(\tau) \neq \pm 1} \pi_{\dim \pi}\) is the regular representation \(R_{\Gamma} = \text{Ind}_{\langle 1 \rangle}^{\Gamma} 1_{\langle 1 \rangle}\), and we have
\[ (\chi_{R_{\Gamma}} - \chi_{R_{\tau}})(\sigma) = \begin{cases} n & : \sigma = 1, \\ -n & : \sigma = \tau, \\ 0 & : \sigma \neq 1, \tau. \end{cases} \tag{5.3} \]
5.1.3 Recall the character formula (4.6). Since \( \tilde{\vartheta}(x) = \vartheta(x^{1-\gamma}) \) (\( x \in K^\times \)) and the elements of \( \Gamma \) of order two are central (Proposition 3.3.1), we have

\[
\frac{1}{2} \sum_{\alpha \in \Gamma} \tilde{\vartheta}(x^{\alpha(1+\tau)}) = \frac{1}{2} |\Gamma| = n.
\]

Since

\[
\sum_{\alpha \in \Gamma} \tilde{\vartheta}(x^{\alpha(1+\gamma^{-1})}) = \sum_{\alpha \in \Gamma} \tilde{\vartheta}(x^{\alpha(1+\gamma)})
\]

for any \( \gamma \), we have

\[
\chi_{\lambda^2 \varphi_1}(1, x) = \frac{1}{2} \sum_{\alpha, \gamma \in \Gamma, \gamma \neq 1} \tilde{\vartheta}(x^{\alpha(1+\gamma)})
\]

\[
= n + \sum_{\{\gamma \neq \gamma^{-1}\} \subset \Gamma} \sum_{\alpha \in \Gamma} \tilde{\vartheta}(x^{\alpha(1+\gamma)}) + \sum_{1, \tau \neq \gamma \in \Gamma} \sum_{\alpha \in \Gamma / \langle \gamma \rangle} \tilde{\vartheta}(x^{\alpha(1+\gamma)}).
\]

Take a \( \sigma \in \Gamma \) of order two. Then the cocycle relation of \( \alpha_{K/F} \) gives \( \alpha_{K/F}(\sigma, \sigma) = \alpha_{K/F}(\sigma, \sigma) \).

Since \( \sigma \) is a central element of \( \Gamma \), we have

\[
\alpha_{K/F}(\sigma, \sigma)^{\sigma_\alpha} = \alpha_{K/F}(\sigma, \sigma)^{\alpha}
\]

for any \( \alpha \in \Gamma \). If \( \sigma = \tau \), we have

\[
\chi_{\lambda^2 \varphi_1}(\tau, x) = -\frac{1}{2} \sum_{\alpha \in \Gamma} \tilde{\vartheta}(x^{\alpha(1+\tau)}) = -\frac{1}{2} |\Gamma| = -n.
\]

If \( \sigma \neq \tau \), then we have

\[
\chi_{\lambda^2 \varphi_1}(\sigma, x) = -\sum_{\alpha \in \Gamma / \langle \sigma \rangle} \tilde{\vartheta}(\alpha_{K/F}(\sigma, \sigma)^{\alpha} \cdot x^{\alpha(1+\sigma)}).
\]

Since the character of the induced representation \( \rho_\gamma = \text{Ind}_{K/F}^{W_{K/F}} \tilde{\vartheta}_\gamma \) for \( \gamma \in \Gamma \) such that \( \gamma^2 \neq 1 \) is

\[
\chi_{\rho_\gamma}(\sigma, x) = \begin{cases} 0 : \sigma \neq 1, \\ \sum_{\alpha \in \Gamma} \tilde{\vartheta}(x^{\alpha(1+\gamma)}) : \sigma = 1, \end{cases}
\]

the formulae (5.2) and (5.3) gives

\[
\chi_{\lambda^2 \varphi_1} = \chi_{R_\sigma} - \chi_{R_\tau} + \sum_{\{\gamma \neq \gamma^{-1}\} \subset \Gamma} \chi_{\sigma_\gamma} + \sum_{1, \tau \neq \gamma \in \Gamma} \chi_{\tau_\gamma}
\]

which complete the proof of Theorem 5.1.1.

5.2 Root number of adjoint representation

By the decomposition 5.1.1 and Theorem 5.1.1, the adjoint representation \( \text{Ad} \circ \varphi \) of the Weil group \( \text{W}_F \) on \( \tilde{g} \) is written as a direct sum of representations induced from abelian characters. Using this decomposition, we can calculate the \( \varepsilon \)-factor of the adjoint representation. The result is...
Theorem 5.2.1  With respect to a additive character $\psi$ of $F$ such that
\[ \{ x \in F \mid \psi(xO_F) = 1 \} = O_F \]
and the Haar measure $d(x)$ on $F$ such that $\int_{O_F} d(x) = 1$, we have
\[ \varepsilon(\varphi, \Ad, \psi, d(x)) = w(\Ad \circ \varphi) \cdot q^{2r} \]
with the root number
\[ w(\Ad \circ \varphi) = \vartheta(-1) \times \begin{cases} (-1)^{\frac{q-1}{2n}} \cdot \frac{n(q+1)}{2} & : K/K_+ \text{ is ramified}, \\ 1 & : K/K_+ \text{ is unramified and } |H| = 2, \\ -(-1)^{\frac{q^2-1}{2}} & : K/K_+ \text{ is unramified and } |H| = 4. \end{cases} \]

Here $H = \{ \gamma \in \Gamma \mid \gamma^2 = 1 \}$ whose structure is given in Proposition 3.3.1.

Note that if $K/K_+$ is ramified, then $K/F$ is totally ramified and hence $2n = (K:F)$ divides $q - 1$.

The rest of this devoted to the proof of the theorem.

5.2.1  To begin with
\[ \varepsilon(\varphi, \Ad, \psi, d(x)) = \varepsilon(\Ad \circ \varphi, \psi, d(x)) \]
by the definition. Define the additive character $\psi_F$ of $F$ by
\[ \psi_F : F \rightarrow \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \exp(\pi \sqrt{-1}) \rightarrow \mathbb{C}^\times. \]
Then
\[ \{ x \in F \mid \psi_F(xO_F) = 1 \} = \mathcal{D}(F/\mathbb{Q}_p)^{-1} = p_F^{-d(F)} \]
and $\psi_K = \psi_F \circ T_{K/F}$. Let $d_F(x)$ be the Haar measure on $F$ such that
\[ \int_{O_F} d_F(x) = q^{-d(F)}. \]
Then
\[ \varepsilon(\Ad \circ \varphi, \psi, d(x)) = q^{-n(2n+1) - d(F)/2} \varepsilon(\Ad \circ \varphi, \psi_F, d_F(x)). \]
Put
\[ \varepsilon(\ast, \psi_F) = \varepsilon(\ast, \psi_F, d_F(X)), \quad \lambda(K/F, \psi_F) = \lambda(K/F, \psi_F, d_F(x), d_K(x)) \]
for the sake of simplicity. By (5.1) and Theorem 5.1.1 we have
\[ \Ad \circ \varphi = \Pi_1 \oplus \Pi_2 \oplus \Pi_3 \]
with $\Pi_1 = \bigoplus_{\pi(\tau) \neq 1} \pi^{\dim \pi}$ and
\[ \Pi_2 = \Ind_{K_+^\times}^{W_K^+} \widetilde{\psi} \oplus \bigoplus_{(\gamma \neq \gamma^{-1}) \subset \Gamma} \Ind_{K_+^\times}^{W_K^+} \widetilde{\psi}_\gamma, \quad \Pi_3 = \bigoplus_{1 \neq \gamma \in \Gamma, \gamma^2 = 1} \Ind_{W_K/K_+}^{W_K} \chi_\gamma, \]

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were $\Pi_1$ appears only if $|H| = 4$. Note also that we can change in $\Pi_2$, the definition of $\vartheta_\gamma$ to $\vartheta_\gamma(x) = \vartheta(x^{1-\gamma})$ for $\gamma \in \Gamma$ such that $\gamma \neq \gamma^{-1}$ by replacing $\gamma$ with $\tau\gamma$. Now we have

$$\varepsilon(\text{Ad} \circ \varphi, \psi_F) = \varepsilon(\Pi_1, \psi_F) \cdot \varepsilon(\Pi_2, \psi_F) \cdot \varepsilon(\Pi_3, \psi_F).$$

Since $\Pi_1 = \text{Ind}_{W_K}^{W_F} 1_{W_K} - \text{Ind}_{W_{K^+}}^{W_F} 1_{W_{K^+}}$, we have

$$\varepsilon(\Pi_1, \psi_F) = \varepsilon \left( \text{Ind}_{W_K}^{W_F} 1_{W_K}, \psi_F \right) \cdot \varepsilon \left( \text{Ind}_{W_{K^+}}^{W_F} 1_{W_{K^+}}, \psi_F \right)^{-1} = \lambda(K/F, \psi_F) \varepsilon(1_{W_K}, \psi_K) \cdot \lambda(K_+/F, \psi_F)^{-1} \varepsilon(1_{W_{K^+}}, \psi_{K^+})^{-1},$$

and

$$\varepsilon(1_{W_K}, \psi_K) = q_{K}^{d(K) / 2} = q^{f(e-d(F) + e-1)/2},$$

where $q_K = |O_K/p_K| = q^f$. Since $K/F$ is tamely ramified, we have $d(K) = e \cdot d(F) + e - 1$. Similarly we have

$$\varepsilon(1_{W_{K^+}}, \psi_{K^+}) = q^{f_e + d(F) + e_{+1} - 1)/2}.\] On the other hand, we have

$$\varepsilon(\Pi_2, \psi_F) = \varepsilon(\bar{\vartheta}, \psi_K) \prod_{\gamma \neq \gamma^{-1} \in \Gamma} \varepsilon(\bar{\vartheta}, \psi_K) \times \begin{cases} \lambda(K/F, \psi_F)^n : |H| = 2, \\
\lambda(K/F, \psi_F)^{n-1} : |H| = 4. \end{cases}$$

Now we have

$$\varepsilon(\bar{\vartheta}, \psi_K) = G_{\psi_K} \left( \vartheta^{-1}, \varpi_K^{-(d(K)+f(\bar{\vartheta}))} \right) \cdot \vartheta(\varpi_K)^{d(K)+f(\bar{\vartheta})} \cdot \varpi_K^{(d(K)+f(\bar{\vartheta})) / 2}.\] Since $\bar{\vartheta}|_{K^+} = 1$ and $K = K_+(\beta)$, Theorem 3 of [3] says that

$$G_{\psi_K} \left( \vartheta^{-1}, \varpi_K^{-(d(K)+f(\bar{\vartheta}))} \right) \cdot \vartheta(\varpi_K)^{d(K)+f(\bar{\vartheta})} = \vartheta(\beta) = \vartheta(-1).$$

So we have

$$\varepsilon(\bar{\vartheta}, \psi_K) = \vartheta(-1) \cdot q^{f(e-d(F) + e-1 + f(\bar{\vartheta})) / 2}.\] Similarly we have

$$\varepsilon(\bar{\vartheta}, \psi_K) = q^{f(e-d(F) + e-1 + f(\bar{\vartheta})) / 2}$$

for $\gamma \in \Gamma$ such that $\gamma^2 \neq 1$, since $\bar{\vartheta}_{\gamma}(\beta) = \vartheta((-1)^{1-\gamma}) = 1$.

### 5.2.2

Assume that $K/K_+$ is ramified. Then $K/F$ is totally ramified and $H = \{1, \tau = \delta^0\}$. Since $e = 2n$ is even, we have

$$\lambda(K/F, \psi_F) = (-1)^{\frac{n+1}{2}} G_{\psi_F} \left( \frac{\delta}{F} , \varpi_F^{-(d(F)+1)} \right)$$

by Proposition [A.3.5]. Similarly we have

$$\lambda(K_+/F, \psi_F) = \begin{cases} (-1)^{\frac{n+1}{2}} G_{\psi_F} \left( \frac{\delta}{F} , \varpi_F^{-(d(F)+1)} \right) : n \text{ is even,} \\
1 : n \text{ is odd.} \end{cases}$$

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So we have
\[
\varepsilon(I_1, \psi_F) = q^{(n \cdot d(F) + n)/2} \times \begin{cases} 
(-1)^{\frac{n+1}{2}} & : n \text{ is even,} \\
(-1)^{\frac{n-1}{2}} \cdot \frac{n+1}{2} G_{\psi_F}(\left(\frac{*}{F}\right), \varpi_F^{-(d(F)+1)}) & : n \text{ is odd.}
\end{cases}
\]
By Proposition 3.4.1 we have
\[
f(\tilde{\theta}) = \min\{0 < k \in \mathbb{Z} \mid \tilde{\theta}(1 + p_K^k) = 1\} = 2n(r - 1)
\]
and
\[
f(\tilde{\gamma}) = \min\{0 < k \in \mathbb{Z} \mid \tilde{\gamma}(1 + p_K^k) = 1\} = 2n(r - 1)
\]
for \(\gamma \in \Gamma\) such that \(\gamma^2 \neq 1\). Then we have
\[
\varepsilon(I_2, \psi_F) = \tilde{\theta}(-1) \cdot q^{n^2d(F) + n^2r - n/2} \lambda(K/F, \psi_F)^n
\]
and
\[
\lambda(K/F, \psi_F)^n = (-1)^{\frac{n-1}{2}} \cdot \tilde{\gamma}(-1)^{\frac{n}{2}} \cdot (-1)^{\frac{n+1}{2}} \frac{n+1}{2} G_{\psi_F}(\left(\frac{*}{F}\right), \varpi_F^{-(d(F)+1)})^n
\]
Since
\[
G_{\psi_F}(\left(\frac{*}{F}\right), \varpi_F^{-(d(F)+1)}) = \left(\frac{-1}{F}\right) = (-1)^{\frac{n-1}{2}},
\]
we have
\[
\lambda(K/F, \psi_F)^n = (-1)^{\frac{n-1}{2}} \frac{n+1}{2} \cdot (-1)^{\frac{n}{2}} \frac{n+1}{2} = 1
\]
if \(n\) is even, and
\[
\lambda(K/F, \psi_F)^n = (-1)^{\frac{n-1}{2}} \cdot \frac{n+1}{2} \cdot (-1)^{\frac{n}{2}} \frac{n+1}{2} G_{\psi_F}(\left(\frac{*}{F}\right), \varpi_F^{-(d(F)+1)})
\]
\[
= G_{\psi_F}(\left(\frac{*}{F}\right), \varpi_F^{-(d(F)+1)})
\]
if \(n\) is odd. So we finally get
\[
\varepsilon(Ad \circ \varphi, \psi_F) = \varepsilon(I_1, \psi_F) \cdot \varepsilon(I_2, \psi_F)
\]
\[
= \tilde{\theta}(-1) \cdot q^{n(2n+1)d(F)/2 + n^2r} \times \begin{cases} 
(-1)^{\frac{n+1}{2}} & : n \text{ is even,} \\
(-1)^{\frac{n-1}{2}} \cdot \frac{n+1}{2} & : n \text{ is odd.}
\end{cases}
\]

5.2.3 Assume that \(K/K_+\) is unramified and \(|H| = 2\). In this case, Proposition 3.5.1 shows that \(e = e_+\) is odd, and \(H = \{1, \tau\}\). Then \(f = 2f_+\) is even, since \(ef = 2n\). By Proposition 3.4.1 we have
\[
\lambda(K/F, \psi_F) = (-1)^{(f-1)d(F)} = (-1)^{d(F)}, \quad \lambda(K_+/F, \psi_F) = (-1)^{(f_+-1)d(F)}.
\]
So we have
\[
\varepsilon(I_1, \psi_F, d_F(x)) = (-1)^{f_+ \cdot d(F)} q^{(n \cdot d(F) + n - f_+)/2}.
\]
By Proposition 3.4.1 we have
\[
f(\tilde{\theta}) = \min\{0 < k \in \mathbb{Z} \mid \tilde{\theta}(1 + p_K^k) = 1\} = e(r - 1) + 1,
\]
and
\[
f(\tilde{\psi}) = \min \{ 0 < k \in \mathbb{Z} \mid \tilde{\psi}(1 + p_k^e) = 1 \}
\]
\[
= \begin{cases} 
\varepsilon(r-1) : & \gamma \in \{ \delta^{\pm 1}, \delta^{\pm 2}, \cdots, \delta^{\pm \frac{e-1}{2}} \}, \\
\varepsilon(r-1) + 1 : & \text{otherwise}
\end{cases}
\]
for a \( \gamma \in \Gamma \) such that \( \gamma \neq \gamma^{-1} \). So we have
\[
\varepsilon(\Pi_2, \psi_F, d_F(x)) = \vartheta(-1) \cdot (-1)^n d(F) q^{n^2 (d(F) + r - 1)n - f_+/2}.
\]
Then finally we have
\[
\varepsilon(\text{Ad} \circ \varphi, \psi_F) = \varepsilon(\Pi_1, \psi_F) \cdot \varepsilon(\Pi_2, \psi_F)
\]
\[
= \vartheta(-1) \cdot q^{n(2n+1)d(F)/2 + n^2 r}.
\]

5.2.4 Assume that \( K/K^+ \) is unramified and \( |H| = 4 \). In this case, Proposition \[53.7\] shows that \( e = e_+, f = 2f_+ \) and \( m \) are all even, and
\[
H = \{ 1, \tau, \delta' = \delta^{\pm 2}, \tau' = \delta' \tau \}.
\]
Put
\[
E = K_+ \cap K'_\tau = K_\tau' \cap K_\delta' = K_\delta' \cap K_+.
\]
Then \( K_+ / E, K_\tau'/E \) and \( K_\delta'/E \) are unramified quadratic extension, on the other hand \( K_+ / E, K_\tau'/E \) and \( K_\delta'/E \) are ramified quadratic extension. \( K_0 \subset K_\delta' \subset K \) and \( E_0 = E \cap K_0 \) is the maximal unramified subextension of \( E/F \).

By Proposition \[A.3.6\] we have
\[
\lambda(K/F, \psi_F) = -(-1)^{f_+/2} \frac{q_{f_+/2}}{2}.
\]
By Proposition 3.4.1 we have
\[ f(\tilde{\vartheta}) = \min\{0 < k \in \mathbb{Z} \mid \tilde{\vartheta}(1 + p_{K}^{k}) = 1\} = e(r - 1) + 1, \]
and
\[ f(\tilde{\vartheta}) = \min\{0 < k \in \mathbb{Z} \mid \tilde{\vartheta}(1 + p_{K}^{k}) = 1\} \]
\[ = \begin{cases} e(r - 1) & : \gamma \in \{\delta^{\pm 1}, \delta^{\pm 2}, \cdots, \delta^{\pm \frac{r - 1}{2}}\}, \\ e(r - 1) + 1 & : \text{otherwise} \end{cases} \]
for a \( \gamma \in \Gamma \) such that \( \gamma \neq \gamma^{-1} \). So we have
\[ \varepsilon(\Pi_{1}, \psi_{F}) \cdot \varepsilon(\Pi_{2}, \psi_{F}) = \vartheta(-1) \cdot q^{n(2n-1)d(F)/2 + n(n-1)r + f_{+}/2} \cdot \lambda(K_{+}/F, \psi_{F})^{-1}. \]

(5.4)

Proposition 5.2.2
\[ \varepsilon(\Pi_{3}, \psi_{F}) = q^{n-d(F)+nr-f_{+}/2} \cdot \lambda(K_{r}/F, \psi_{F}) \]
\[ \times \begin{cases} -1 & : e/2 \text{ is even}, \\ (-1)^{d(F)+\frac{e_{f}+1}{2}} & : e/2 \text{ is odd} \end{cases} \]

[Proof] We have
\[ \varepsilon(\Pi_{3}, \psi_{F}) = \prod_{\gamma \in \{\vartheta', \tau'\}} \lambda(K_{r}/F, \psi_{F}) \cdot \varepsilon(\tilde{\chi}_{\gamma}, \psi_{K_{\gamma}}) \]
where \( \tilde{\chi}_{\gamma}(x) = (x, K/K_{\gamma}) \cdot \tilde{\vartheta}(x) (x \in K_{\gamma}^{\times}). \)

1) The case \( \gamma = \tau'. \) Since \( K/K_{\tau'} \text{ is unramified}, \) we have
\[ (x, K/K_{\tau'}) = (-1)^{\text{ord}_{K_{\tau'}}(x)} (x \in K_{\tau'}^{\times}) \]
and \( N_{K/K_{\tau'}}(1 + p_{K_{\tau'}}^{k}) = 1 + p_{K_{\tau'}}^{k} (0 < k \in \mathbb{Z}). \) Then we have
\[ f(\chi_{\gamma}) = \min\{0 < k \in \mathbb{Z} \mid \chi_{\gamma}(1 + p_{K_{\gamma}}^{k}) = 1\} = e(r - 1) \]
because \( \chi_{\gamma}(1 + p_{K_{\gamma}}^{k}) = 1 \) if and only if \( \tilde{\vartheta}(x^{1-\delta'}) = 1 \) for all \( x \in 1 + p_{K}^{k} \) which is equivalent to \( k \geq e(r - 1) \) by Proposition 3.4.1. Since \( K_{\gamma}/E \text{ is ramified quadratic extension, we have} \)
\( K_{\gamma} = E(\sqrt{\varpi_{E}}) \text{ where } \varpi_{E} \text{ is a prime element of } E. \) Then \( \tilde{\chi}_{\tau'}|_{E^{\times}} = 1 \) and
\[ \tilde{\chi}_{\gamma}(\sqrt{\varpi_{E}}) = (\sqrt{\varpi_{E}}, K/K_{\gamma}) \cdot \tilde{\vartheta}(\sqrt{\varpi_{E}}) = -\vartheta(-1), \]
hence we have
\[ G_{\psi_{K_{\gamma}}}(\tilde{\chi}_{\gamma}^{-1}, -\omega_{K_{\gamma}}^{d(K_{\gamma})+f(\tilde{\chi}_{\gamma})}) \cdot \tilde{\chi}_{\gamma}(\varpi_{K_{\gamma}})^{d(K_{\gamma})+f(\tilde{\chi}_{\gamma})} = -\vartheta(-1) \]
by Theorem 3 of [8]. Then we have
\[ \varepsilon(\tilde{\chi}_{\gamma}, \psi_{K_{\gamma}}) = -\vartheta(-1) \cdot q^{n-d(F)+nr-f_{+}/2}. \]
2) The case $\gamma = \delta'$. In this case $K/K_\gamma$ is ramified quadratic extension. Then we have
\[
f(\tilde{\gamma}) = \min \{0 < k \in \mathbb{Z} | \tilde{\gamma}(1 + p_{K_\gamma}^k) = 1\} = \frac{e}{2} \cdot (r - 1) + 1
\]
because $\tilde{\gamma}(1 + p_{K_\gamma}^k) = 1$ if and only if $\tilde{\delta}((x^{1-r'})) = 1$ for all $x \in 1 + p_{K_\gamma}^{2k}$ which is equivalent to $k \geq \frac{e}{2} \cdot (r - 1) + 1$. There exists a prime element $\varpi_{K_\gamma}$ of $K_\gamma$ such that $K = \sqrt{\varpi_{K_\gamma}}$, and we have
\[
(\varpi_{K_\gamma}, K/K_\gamma) = (-1, K/K_\gamma) = (-1)^{\frac{e}{2} - 1} = 1
\]
since $f = 2f_\varepsilon$ is even. On the other hand, we have
\[
\text{Proposition 5.2.3}
\]
\[
\[\text{Proof}\]
\]
\[
\lambda(K_{\gamma}/F, \psi_F) = \begin{cases} 
-(-1)^{\frac{e}{2} - 1} \cdot \vartheta(-1) \cdot \vartheta(\varepsilon) & : e/2 \text{ is even,} \\
(-1)^{\frac{e}{2}} & : e/2 \text{ is odd.}
\end{cases}
\]
by Proposition A.3.6.}

**Proposition 5.2.3**

\[
\lambda(K_{\gamma}/F, \psi_F) \cdot \lambda(K_{\gamma}/F, \psi_F)^{-1} = \begin{cases} 
1 & : e/2 \text{ is even,} \\
(-1)^{d(F)+1} & : e/2 \text{ is odd.}
\end{cases}
\]

[**Proof**] Since $K_{\gamma}/E$ is an unramified quadratic extension, put $K_{\gamma} = E(\sqrt{\varepsilon})$ with $\varepsilon \in O_E^{\times}$. Then $\sqrt{\varepsilon} = -\sqrt{\varepsilon}$. Since $K_{\gamma}/E$ is a ramified quadratic extension, we have $K_{+} = E(\varpi_{K_+})$ with a prime element $\varpi_{K_+}$ of $K_+$ such that $\varpi_{K_+}^2 \in E$. Then $\varpi_{K_+}^{-1} = -\varpi_{K_+}$, and hence $\varpi_{K_{\gamma}} = \sqrt{\varepsilon} \cdot \varpi_{K_+}$ is a prime element of $K_{\gamma}$, such that $K_{\gamma} = E(\varpi_{K_{\gamma}})$ and $\varpi_{K_{\gamma}}^2 \in E$. Then
\[
\varpi_+ = N_{K_+/E}(\varpi_{K_+}) \quad \text{and} \quad \varpi_{\gamma} = N_{K_{\gamma}/E_0}(\varpi_{K_{\gamma}})
\]
are prime elements of $E_0$, since $K_+/E_0$ and $K_{\gamma}/E_0$ are totally ramified extensions. On the other hand, we have
\[
N_{K_{\gamma}/E}(\varpi_{K_{\gamma}}) = -\varepsilon \cdot \varpi_{K_+}^2
\]
and
\[
N_{K_{\gamma}/E_0}(\varpi_{K_{\gamma}}) = \varepsilon \cdot n_{K_+/E}(\varpi_{K_+}),
\]
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and hence \( \varpi^r = N_{E_0/E}(\varepsilon) \cdot \varpi_+ \). Now we have

\[
\lambda(K_{+}/F, \psi_F) \cdot \lambda(K_{+}/F, \psi_F)^{-1} = G_{\psi_{E_0}}\left( \frac{*}{E_0}, -\varpi_{+}^{-(d(E_0)+1)} \right) \cdot G_{\psi_{E_0}}\left( \frac{*}{E_0}, -\varpi_{+}^{-(d(E_0)+1)} \right)^{-1} = \left( \frac{N_{E/E_0}(\varepsilon)}{E_0} \right)^{d(F)+1}
\]

by Proposition [A.3.5]. Since \( K_{+}/E_0 \) is a tamely totally ramified extension, and hence a cyclic extension, let \( E_0 \subset M \subset K_{+} \) be the intermediate field such that \( (M : E_0) = 2 \). Then

\[
\left( \frac{N_{E/E_0}(\varepsilon)}{E_0} \right) = (N_{E/E_0}(\varepsilon), M/E_0).
\]

If \( e/2 \) is even, then \( M \subset E \) because \( (E : E_0) = e/2 \), and hence

\[
\left( \frac{N_{E/E_0}(\varepsilon)}{E_0} \right) = (N_{M/E_0}(N_{E/M}(\varepsilon)), M/E_0) = 1.
\]

Assume that \( e/2 \) is odd. Since \( K_{+}/E \) is a ramified quadratic extension and \( \varepsilon \in O_{E}^\times \) is not square, we have

\[
(\varepsilon, K_{+}/E) = \left( \frac{\varepsilon}{E} \right) = -1.
\]

On the other hand, we have

\[
(\varepsilon, K_{+}/E) = (N_{E/E_0}(\varepsilon), K_{+}/E_0) \in \text{Gal}(K_{+}/E) \subset \text{Gal}(K_{+}/E_0)
\]

and \((N_{E/E_0}(\varepsilon), K_{+}/E_0)\) is mapped to \((N_{E/E_0}(\varepsilon), M/E_0)\) by the restriction mapping

\[
\text{Gal}(K_{+}/E_0) \to \text{Gal}(M/E_0).
\]

\( M \not\subset E \) since \( (E : E_0) = e/2 \) is odd, hence \( K_{+} = ME \) and \( M \cap E = E_0 \). Then the restriction mapping gives the isomorphism

\[
\text{Gal}(K_{+}/E) \cong \text{Gal}(M/E_0),
\]

hence we have

\[
\left( \frac{N_{E/E_0}(\varepsilon)}{E_0} \right) = (N_{E/E_0}(\varepsilon), M/E_0) = (\varepsilon, K_{+}/E) = -1.
\]

\( \square \)

Proposition \([5.2.2]\) and Proposition \([5.2.3]\) combined with Proposition \([5.2.1]\) gives

\[
\varepsilon(\text{Ad} \circ \varphi, \psi_F) = \varepsilon(\Pi_1, \psi_F) \cdot \varepsilon(\Pi_2, \psi_F) \cdot \varepsilon(\Pi_3, \psi_F)
\]

\[
= \vartheta(-1) \cdot \varrho^{n(2n+1) \cdot d(F)/2 + n^2 r} \times \left\{ \begin{array}{ll}
-1 & : e/2 \text{ is even}, \\
(-1)^{\frac{f_+}{2}} & : e/2 \text{ is odd}.
\end{array} \right.
\]

Since \( K_{+}/F \) is a tamely ramified extension such that \( e(K_{+}/F) = e \) and \( f(K_{+}/F) = f_+ \), and \( e \) is even, \( e/2 \) divides \((q^{f_+} - 1)/2\). Hence \( (-1)^{\frac{f_+}{2}} = 1 \) if \( e/2 \) is even.

The proof of the formula of Theorem \([5.2.1]\) is completed.
5.3 Verification of root number conjecture

Let $D$ be the maximal torus of $Sp_{2n}$ consisting of the diagonal matrices. The group $X^\vee(D)$ of the one-parameter subgroups of $D$ is identified with $\mathbb{Z}^n$ by $m \to u_m$ where

$$u_m(t) = \begin{bmatrix} t^m & 0 \\ 0 & t^{-m} \end{bmatrix} \text{ with } t^m = \begin{bmatrix} t^{m_1} & \cdots \\ \cdots & \cdots \\ t^{m_n} \end{bmatrix} \in GL_n,$$

or we will denote by $u_m = \sum_{i=1}^n m_i \cdot u_i$. Then the set of the co-roots of $Sp_{2n}$ with respect to $D$ is

$$\Phi^\vee(D) = \{ \pm (u_i \pm u_j), \pm u_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n \}.$$

Now we have

$$2 \cdot \rho = \sum_{1 \leq i < j \leq n} (u_i - u_j) + \sum_{1 \leq i < j \leq n} (u_i + u_j) + \sum_{k=1}^n 2u_k$$

$$= 2 \sum_{i=1}^n \{ 2(n - i) + 1 \} \cdot u_i.$$

So the special central element is

$$\epsilon = 2 \cdot \rho(-1) = -12n \in Sp_{2n}(F).$$

If we recall

$$\pi_{\beta, \theta} = \text{ind}_{G(O_F/\mathcal{P}_F)}^{G(F)} \delta_{\beta, \theta} \text{ with } \delta_{\beta, \theta} = \text{Ind}_{G(O_F/\mathcal{P}_F; \beta)}^{G(O_F/\mathcal{P}_F)} \sigma_{\beta, \theta}$$

and the construction of $\sigma_{\beta, \theta}$, we have

$$\pi_{\beta, \theta}(\epsilon) = \delta_{\beta, \theta}(\epsilon) = \sigma_{\beta, \theta}(\epsilon) = \theta(-1).$$

Since $\vartheta = c \cdot \theta$, Theorem 5.2.1 and Proposition 3.3.2 show that

$$w(\text{Ad} \circ \varphi) = \theta(-1) \times \begin{cases} 1 & : K/K_+ \text{ is unramified,} \\ (-1)^{\frac{n-1}{2}} \cdot \frac{q-1}{q} & : K/K_+ \text{ is ramified.} \end{cases}$$

So we have proved the following theorem.

**Theorem 5.3.1** If $K/F$ is not totally ramified or $K/F$ is totally ramified and

$$\frac{q-1}{2} \cdot (n-1) \equiv 0 \pmod{4},$$

then we have $w(\text{Ad} \circ \varphi) = \pi_{\beta, \theta}(\epsilon)$.

This theorem says that the root number conjecture is valid if we consider $\varphi$ as the Langlands parameter of the supercuspidal representation $\pi_{\beta, \theta}$ under the required conditions.
6 The case of $Sp_4(F)$

In this section, let us assume that $K/F$ is a quintic Galois extension, and consider a candidate of the Langlands parameter of the supercuspidal representation $\pi_{\beta,\theta}$ of $Sp_4(F)$ different from the parameter considered in the subsection 3.4. Note that $\Gamma = Gal(K/F)$ is a cyclic group if and only if $K/F$ is unramified or totally ramified.

The proofs are omitted because they are quite similar to those of the preceding sections.

6.1 Another candidate for the Langlands parameter

The character $\theta$ of $U_{K/K}$ which parametrizes the supercuspidal representation $\pi_{\beta,\theta}$ defines the character $\tilde{\theta}$ of $K^\times$ by $\tilde{\theta}(x) = \theta(x^{1-\tau})$. Then the representation space $V_{\theta}$ of the induced representation $\text{Ind}_{W_{K/F}}^{W_{K/F} \times \tilde{\theta}}$ has $W_{K/F}$-quasi invariant anti-symmetric form

$$D_{\nu}(\varphi,\psi) = \sum_{\gamma \in \text{Gal}(K/F)} \nu(\gamma) \cdot \tilde{\theta}(\alpha_{K/F}(\gamma,\tau))^{-1} \varphi(\gamma) \psi(\gamma \tau)$$

where $\nu$ is a character of $\text{Gal}(K/F)$ such that $\nu(\tau) = -1$ (c.f. appendix B).

Let us identify $GSp_4(\mathbb{C})$ with $GSp_4(C)$ by means of the symplectic basis $\{ u_\rho, v_\rho \}_{\rho \in \Gamma/\langle \tau \rangle}$. Then we have a group homomorphism

$$\varphi : W_F \xrightarrow{\text{can.}} W_{K/F} \xrightarrow{\text{Ind}_{W_{K/F}^\times \tilde{\theta}}} GSp_4(\mathbb{C}) \xrightarrow{(\ast)} SO_5(\mathbb{C})$$

where $(\ast)$ is the accidental surjection. The admissible representation of the Weil-Deligne group $W_F \times SL_2(\mathbb{C})$ to $SO_5(\mathbb{C})$ corresponding to the triple $(\varphi, SO_5(\mathbb{C}), 0)$ as explained in appendix A.6 is

$$W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{proj.}} W_F \xrightarrow{\varphi} SO_5(\mathbb{C})$$

which is also denoted by $\varphi$.

6.2 Formal degree conjecture

By writing down the parameter (6.2) explicitly as in the subsection 4.1, we have

$$Z_{SO_5(\mathbb{C})}(\text{Im}\varphi) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & : K/F \text{ is unramified or totally ramified}, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & : \text{otherwise} \end{cases}$$

and

$$L(\varphi, \text{Ad}, s) = \begin{cases} 1 & : K/K_+ \text{ is ramified}, \\ \frac{1}{1 + q^{-1}s} & : K/K_+ \text{ is unramified} \end{cases}$$

The Artin conductor of $\text{Ad} \circ \varphi$ is

$$c(\text{Ad} \circ \varphi) = 8r.$$
Then
\[ \frac{1}{|Z_{SO_5(\mathbb{C})}(\text{Im}\varphi)|} \left| \gamma(\varphi, \text{Ad}, 0) \right| \right| \left| \gamma(\varphi_0, \text{Ad}, 0) \right| \]
(6.6)
gives the formal degree of the supercuspidal representation \( \pi_{\beta,\theta} \) given by (4.10) if \( K/F \) is unramified or totally ramified. If \( K/F \) is ramified but not totally ramified, this is not the case, that is, the order of the centralizer \( Z_{SO_5(\mathbb{C})}(\text{Im}\varphi) \) is twice as big as required, in other words, the image \( \text{Im}(\varphi) \) of the parameter is too small.

### 6.3 The root number conjecture

Since the parameter (6.2) failed the formal degree conjecture if \( K/F \) is ramified but not totally ramified, we will consider in this subsection, the root number conjecture in the case of \( K/F \) being unramified or totally ramified.

In this case \( K/F \) is a cyclic extension. So we put \( \text{Gal}(K/F) = \langle \rho \rangle \) so that \( \tau = \rho \cdot 2 \). Then the representation (6.1) has a decomposition
\[ \varphi = \varphi_1 \oplus \text{det} \varphi_1 \]
with
\[ \varphi_1 : W_F \xrightarrow{\text{can.}} W_{K/F} \xrightarrow{\text{Ind}_{K/F}^W} O_4(\mathbb{C}) \]
where \( \tilde{\theta}(x) = \tilde{\theta}(x^{-1}) \) (\( x \in K^\times \)). Then we have
\[ \text{Ad} \circ \varphi = \bigoplus_{\chi \neq 1}^\chi \text{Ind}_{K/F}^W \tilde{\theta} \]

The epsilon factor with respect to the additive character and the Haar measure normalized as in Theorem 5.2.1 is
\[ \varepsilon(\varphi, \text{Ad}, \psi, d(x)) = w(\text{Ad} \circ \varphi) \cdot q^{4r} \]
with the root number
\[ w(\text{Ad} \circ \varphi) = \begin{cases} 1 & : K/F \text{ is unramified}, \\ (-1)^{4r} & : K/F \text{ is totally ramified}. \end{cases} \]
This means that the root number conjecture is valid if and only if
\[ \theta(-1) = \begin{cases} 1 & : K/F \text{ is unramified}, \\ (-1)^{4r} & : K/F \text{ is totally ramified}. \end{cases} \]
In other words, the parameter (6.2) is not the Langlands parameter of the supercuspidal representation \( \pi_{\beta,\theta} \) in general.

### A Local factors

Fix an algebraic closure \( F^{\text{alg}} \) of \( F \) in which we will take every algebraic extensions of \( F \). Put
\[ \nu_F(x) = (F(x) : F)^{-1} \text{ord}_F(N_F(x)/F(x)) \text{ for all } x \in F^{\text{alg}} \]
and
\[ O_K = \{ x \in F^{\text{alg}} | \nu_F(x) \geq 0 \}, \quad p_K = \{ x \in F^{\text{alg}} | \nu_F(x) > 0 \}. \]
Then \( K = O_K/p_K \) is an algebraic extension of \( \mathbb{F} = O_F/p_F \). If \( K/F \) is a finite extension, fix a generator \( \varpi_K \in O_K \) of \( p_K \).

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A.1 Weil group

Let $F_{ur}$ be the maximal unramified extension of $F$ and $Fr \in \text{Gal}(F_{ur}/F)$ the inverse of the Frobenius automorphism of $F_{ur}$ over $F$. The Weil group $W_F$ of $F$ is

$$W_F = \{ \sigma \in \text{Gal}(F^{alg}/F) \mid \sigma|_{F_{ur}} \in \langle Fr \rangle \}$$

The group $W_F$ is a locally compact group with respect to the topology such that $IF = \text{Gal}(F^{alg}/F_{ur})$ is an open compact subgroup of $W_F$.

Let $F_{ab}$ be the maximal abelian extension of $F$ in $F^{alg}$. Then $[W_F, W_F] = \text{Gal}(F^{alg}/F_{ab})$ and $W_F/[W_F, W_F] \xrightarrow{\text{res.}} \{ \sigma \in \text{Gal}(F^{ab}/F) \mid \sigma|_{F_{ur}} \in \langle Fr \rangle \}$.

So, by the local class field theory, there exists a topological group isomorphism $\delta_F : F^\times \rightarrow W_F/[W_F, W_F]$ such that $\delta_F(\pi)|_{F_{ur}} = Fr$. Fix a $\tilde{Fr} \in \text{Gal}(F^{alg}/F)$ such that $\tilde{Fr}|_{F_{ab}} = \delta_F(\pi)$. Then $W_F = \langle \tilde{Fr} \rangle \rtimes \text{Gal}(F^{alg}/F_{ur})$.

Let $K/F$ be a finite extension in $F^{alg}$. Then $K_{ur} = K \cap F_{ur}$ and $W_K = \{ \sigma \in \text{Gal}(F^{alg}/K) \mid \sigma|_{F_{ur}} \in \langle Fr' \rangle \} = \{ \sigma \in W_F \mid \sigma|_K = 1 \}$, where $f = (K : F)$, is a closed subgroup of $W_F$. If further $K/F$ is a Galois extension, then $[W_K, W_K] \triangleleft W_F$ and

$$W_{K/F} = W_F/[W_K, W_K] = \{ \sigma \in \text{Gal}(K^{ab}/F) \mid \sigma|_{F_{ur}} \in \langle Fr \rangle \}$$

is called the relative Weil group of $K/F$. Then we have an exact sequence

$$1 \rightarrow K^\times \xrightarrow{\delta_K} W_{K/F} \xrightarrow{\text{res.}} \text{Gal}(K/F) \rightarrow 1$$

which is the group extension associated with the fundamental class $[\alpha_{K/F}] \in H^2(\text{Gal}(K/F), K^\times)$, that is, we can identify $W_{K/F} = \text{Gal}(K/F) \times K^\times$ with the group operation

$$(\sigma, x) \cdot (\tau, y) = (\sigma \tau, \alpha_{K/F}(\sigma, \tau) \cdot xy).$$

Let $K_0 = K \cap F^{ur}$ be the maximal unramified subextension of $K/F$. Then the fundamental class can be chosen so that $\alpha_{K/F}(\sigma, \tau) \in O_K$ for all $\sigma, \tau \in \text{Gal}(K/K_0)$, and the image $I_{K/F}$ of $IF = \text{Gal}(F^{alg}/F^{ur}) \subset W_F$ under the canonical surjection $W_F \rightarrow W_{K/F}$ is identified with $\text{Gal}(K/K_0) \times O_K$. 

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A.2 Artin conductor of representations of Weil group

Let $(\Phi, V)$ be a finite dimensional continuous complex representation of the Weil group $W_F$. Since $I_F \cap \text{Ker}(\Phi)$ is an open subgroup of $I_F = \text{Gal}(F^{\text{alg}}/F^{\text{ur}})$, there exists a finite Galois extension $K/F^{\text{ur}}$ such that $\text{Gal}(F^{\text{alg}}/K) \subset \text{Ker}(\Phi)$. Let

$$V_k = V_k(K/F^{\text{ur}}) = \{ \sigma \in \text{Gal}(K/F^{\text{ur}}) \mid x^\sigma \equiv x \pmod{p_K^{k+1}} \text{ for } \forall x \in O_K \}$$

be the $k$-th ramification group of $K/F^{\text{ur}}$ put

$$\tilde{V}_k = \left[ \text{Gal}(F^{\text{alg}}/F^{\text{ur}}) \xrightarrow{\text{res.}} \text{Gal}(K/F^{\text{ur}}) \right]^{-1} V_k$$

for $k = 0, 1, 2, 3, \ldots$. So $\tilde{V}_0 = I_F$. The Artin conductor $a(\Phi) = a(V)$ is defined by

$$a(\Phi) = a(V) = \sum_{k=0}^{\infty} \dim \mathbb{C}(V/V_{\Phi(\tilde{V}_k)}) \cdot |V_0/V_k|^{-1}$$

where

$$V_{\Phi(\tilde{V}_k)} = \{ v \in V \mid \Phi(\tilde{V}_k)v = v \} \quad (k = 0, 1, 2, 3, \ldots).$$

A.3 $\varepsilon$-factor of representations of Weil group

Fix a continuous unitary character $\psi : F \to \mathbb{C}^\times$ of the additive group $F$ and a Haar measure $d(x)$ of $F$.

Langlands and Deligne [1] show that, for every finite dimensional continuous complex representation $(\Phi, V)$ of $W_F$, there exists a complex constant $\varepsilon(\Phi, \psi, d(x)) = \varepsilon(V, \psi, d(x))$ which satisfies the following relations:

1) an exact sequence

$$1 \to V' \to V \to V'' \to 1$$

implies

$$\varepsilon(V, \psi, d(x)) = \varepsilon(V', \psi, d(x)) \cdot \varepsilon(V'', \psi, d(x)),$$

2) for a positive real number $r$

$$\varepsilon(\Phi, \psi, r \cdot d(x)) = r^{\dim \Phi} \cdot \varepsilon(\Phi, \psi, d(x)),$$

3) for any finite extension $K/F$ and a finite dimensional continuous complex representation $\phi$ of $W_K$, we have

$$\varepsilon \left( \text{Ind}_{W_K}^{W_F} \phi, \psi, d(x) \right) = \varepsilon \left( \phi, \psi \circ T_{K/F}, d^{(K)}(x) \right) \cdot \lambda(K/F, \psi)^{\dim \phi}$$

where $d^{(K)}(x)$ is a Haar measure of $K$

$$\lambda(K/F, \psi) = \lambda(K/F, \psi, d(x), d^{(K)}(x)) = \frac{\varepsilon \left( \text{Ind}_{W_K}^{W_F} 1_K, \psi, d(x) \right)}{\varepsilon \left( 1_K, \psi \circ T_{K/F}, d^{(K)}(x) \right)}.$$
4) if \( \dim \Phi = 1 \), then \( \Phi \) factors through \( W_F/[W_F,W_F] \) and put
\[
\chi : F^\times \xrightarrow{\delta_F} W_F/[W_F,W_F] \xrightarrow{\Phi} \mathbb{C}^\times.
\]
Then we have
\[
\varepsilon(\Phi, \psi, d(x)) = \varepsilon(\chi, \psi, d(x))
\]
where the right hand side is the \( \varepsilon \)-factor of Tate [18].

By the definition of \( \lambda(K/F, \psi) \), we have the following chain rule for the finite extensions:

**Proposition A.3.1** For finite extensions \( F \subset K \subset L \), we have
\[
\lambda(L/F, \psi) = \lambda(L/K, \psi \circ T_{K/F}) \cdot \lambda(K/F, \psi)^{(L/K)}.
\]

If the Haar measure \( d(x) \) of \( F \) is normalized so that the Fourier transform
\[
\hat{\varphi}(y) = \int_F \varphi(x) \cdot \psi(-xy) d(x)
\]
has inverse transform
\[
\varphi(x) = \int_F \hat{\varphi}(y) \cdot \psi(xy) d(y),
\]
in other words
\[
\int_{O_F} d(x) = q^{-n(\psi)/2} \text{ with } \{ x \in F \mid \psi(xO_F) = 1 \} = p_F^{-n(\psi)},
\]
then the explicit value of the \( \varepsilon \)-factor \( \varepsilon(\chi, \psi, d(x)) \) is

1) if \( |\chi|_{O_F^\times} = 1 \), then
\[
\varepsilon(\chi, \psi, d(x)) = \chi(\varpi)^{n(\psi)} \cdot q^{n(\psi)/2},
\]
(A.1)

2) if \( |\chi|_{O_F^\times} \neq 1 \), then
\[
\varepsilon(\chi, \psi, d(x)) = G_\psi(\chi^{-1}, -\varpi^{-(n(\psi)+f(\chi))}) \cdot \chi(\varpi)^{n(\psi)+f(\chi)} \cdot q^{-(n(\psi)+f(\chi))/2}
\]
(A.2)

where \( f(\chi) = \text{Min}\{0 < n \in \mathbb{Z} \mid \chi(1 + p_F^n) = 1\} \)

\[
G_\psi(\chi^{-1}, -\varpi^{-(n(\psi)+f(\chi))}) = q^{-n/2} \sum_{t \in (O_F/p_F^{f(\chi)})^\times} \chi(t)^{-1} \psi\left(-\varpi^{-(n(\psi)+f(\chi))} t \right)
\]
is the Gauss sum.

**Remark A.3.2** The definition of the Gauss sum is normalized so that
\[
\left| G_\psi(\chi^{-1}, -\varpi^{-(n(\psi)+f(\chi))}) \right| = 1.
\]

We have
Proposition A.3.3  
1) Put $\psi_a(x) = \psi(ax)$ for $a \in F^\times$. Then 
$$\varepsilon(\Phi, \psi_a, d(x)) = \det \Phi(a) \cdot |a|_{F^\times}^{-\dim \Phi} \cdot \varepsilon(\Phi, \psi, d(x))$$
where 
$$\det \Phi : F^\times \xrightarrow{\delta_F} W_F/[W_F, W_F] \xrightarrow{\det \varepsilon_F} C^\times.$$ 

2) For any $s \in C$ 
$$\varepsilon(\Phi, \psi, d(x), s) = \varepsilon(\Phi \otimes | \cdot |^s_F, \psi, d(x)) = \varepsilon(\Phi, \psi, d(x)) \cdot q^{-s(\varepsilon(\psi) \cdot \dim \Phi + \varepsilon(\Phi))}.$$ 

Proposition A.3.4 If $n(\psi) = 0$ and the Haar measure $d(x)$ is normalized so that 
$$\int_{O_F} d(x) = 1,$$ 
then 
$$\varepsilon(\Phi, \psi, d(x)) = w(\Phi) \cdot q^{\varepsilon(\Phi)/2} = w(V) \cdot q^{\varepsilon(V)/2}$$
with $w(\Phi) \in C$ of absolute value one.

When $K/F$ is a finite tamely ramified Galois extension, the maximal unramified subextension $K_0 = K \cap F^{ur}$ is a cyclic extension of $F$ and $K/K_0$ is also cyclic extension. So, by means of Proposition [A.3.1] we can give the explicit value of $\lambda(K/F, \psi)$.

Let $\psi_F : F \to C^\times$ be a continuous unitary character such that 
$$\{x \in F \mid \psi_F(xOF) = 1\} = D(F/Q_p)^{-1} = p_F^{-d(F)}$$
and the Haar measure $d_F(x)$ on $F$ is normalized so that 
$$\int_{O_F} d_F(x) = q^{-d(F)}.$$ 

Let $K/F$ be a tamely ramified finite Galois extension, and put $\psi_K = \psi_F \circ T_{K/F}$. Put 
$$e = e(K/F) = (K : K_0), \quad f = f(K/F) = (K_0 : F)$$
where $K_0 = K \cap F^{ur}$ is the maximal unramified subextension of $K/F$. Let 
$$\left( \frac{\varepsilon}{K_0} \right) = \begin{cases} 1 & : \varepsilon \equiv \text{square (mod } p_{K_0}), \\ 1 & : \text{otherwise} \end{cases} \quad (\varepsilon \in O_{K_0}^\times)$$
be the Legendre symbol of $K_0$. Then we have

Proposition A.3.5
$$\lambda(K/F, \psi_F) = \lambda(K/F, \psi_F, d_F(x), d_K(x)) = \begin{cases} (-1)^{\frac{e-1}{2} \frac{e+2}{2}} \cdot G_{\psi_{K_0}} \left( \frac{-1}{K_0} \right) \cdot \varepsilon_0^{-(d(K_0)+1)} & : e = \text{even}, \\ (-1)^{(f-1)d(F)} & : e = \text{odd} \end{cases}$$
where $\varepsilon_0$ is a prime element of $K_0$ such that $\varepsilon_0 \in N_{K/K_0}(K^\times)$.

Proposition A.3.6 If there exists an intermediate field $F \subset E \subset K$ such that $K/E$ is unramified quadratic extension, then $f = 2f_+$ is even and 
$$\lambda(K/F, \psi_F) = \begin{cases} (-1)^{\frac{f_+}{2}} & : e = \text{even}, \\ (-1)^{d(F)} & : e = \text{odd}. \end{cases}$$
A.4 \(\gamma\)-factors of admissible representations of Weil group

Definition A.4.1 The pair \((\Phi, V)\) is called an admissible representation of \(W_F\) if

1) \(V\) is a finite dimensional complex vector space and \(\Phi\) is a group homomorphism of \(W_F\) to \(GL_C(V)\),

2) \(\text{Ker}(\Phi)\) is an open subgroup of \(W_F\),

3) \(\Phi(\tilde{Fr}) \in GL_C(V)\) is semisimple.

Let \((\Phi, V)\) be an admissible representation of \(W_F\). Since \(I_F = \text{Gal}(F_{\text{alg}}/F_{\text{ur}})\) is a normal subgroup of \(W_F\), \(\Phi(\tilde{Fr}) \in GL_C(V)\) keeps \(V_{I_F} = \{v \in V \mid \Phi(\sigma)v = v \forall \sigma \in I_F\}\) stable. Then the \(L\)-factor of \((\Phi, V)\) is defined by

\[
L(\Phi, s) = L(V, s) = \det \left( 1 - q^{-s} \cdot \Phi(\tilde{Fr})|_{V_{I_F}} \right)^{-1}.
\]

Since \(\Phi : W_F \to GL_C(V)\) is a continuous group homomorphism, we have the \(\varepsilon\)-factor \(\varepsilon(\Phi, \psi, d(x), s)\) of \(\Phi\). Then the \(\gamma\)-factor of \((\Phi, V)\) is defined by

\[
\gamma(\Phi, \psi, d(x), s) = \gamma(V, \psi, d(x), s) = \varepsilon(\Phi, \psi, d(x), s) \cdot \frac{L(\Phi^\wedge, 1 - s)}{L(\Phi, s)}
\]

where \(\Phi^\wedge\) is the dual representation of \(\Phi\).

A.5 Symmetric tensor representation of \(SL_2(\mathbb{C})\)

The complex special linear group \(SL_2(\mathbb{C})\) acts on the polynomial ring \(\mathbb{C}[X, Y]\) of two variables \(X, Y\) by

\[
g \cdot \varphi(X, Y) = \varphi((X, Y)g) \quad (g \in SL_2(\mathbb{C}), \varphi(X, Y) \in \mathbb{C}[X, Y]).
\]

Let

\[
\mathcal{P}_n = \langle X^n, X^{n-1}Y, \ldots, XY^{n-1}, Y^n \rangle_{\mathbb{C}}
\]

be the subspace of \(\mathbb{C}[X, Y]\) consisting of the homogeneous polynomials of degree \(n\). The action of \(SL_2(\mathbb{C})\) on \(\mathcal{P}_n\) defines the symmetric tensor representation \(\text{Sym}_n\) of degree \(n + 1\). The complex vector space \(\mathcal{P}_n\) has a non-degenerate bilinear form defined by

\[
\langle \varphi, \psi \rangle = \varphi \left( -\frac{\partial}{\partial Y}, \frac{\partial}{\partial X} \right) \psi(X, Y) \bigg|_{(X, Y) = (0, 0)} \in \mathbb{C}
\]

for \(\varphi, \psi \in \mathcal{P}_n\). This bilinear form is \(SL_2(\mathbb{C})\)-invariant

\[
(\text{Sym}_n(g) \varphi, \text{Sym}_n(g) \psi) = \langle \varphi, \psi \rangle \quad (g \in SL_2(\mathbb{C}), \varphi, \psi \in \mathcal{P}_n)
\]

and

\[
(\psi, \varphi) = (-1)^n \langle \varphi, \psi \rangle \quad (\varphi, \psi \in \mathcal{P}_n).
\]

So we have group homomorphisms

\[
\text{Sym}_n : SL_2(\mathbb{C}) \to SO(\mathcal{P}_n) \quad \text{if } n \text{ is even}
\]

and

\[
\text{Sym}_n : SL_2(\mathbb{C}) \to Sp(\mathcal{P}_n) \quad \text{if } n \text{ is odd}.
\]
A.6 Admissible representations of Weil-Deligne group

Fix a complex Lie group $\mathcal{G}$ such that the connected component $\mathcal{G}^o$ is a reductive complex algebraic linear group. Then the $\mathcal{G}^o$-conjugacy class of the group homomorphisms

$$\varphi : W_F \times SL_2(\mathbb{C}) \to \mathcal{G}$$

such that

1) $I_F \cap \ker(\varphi)$ is an open subgroup of $I_F$,
2) $\varphi(\tilde{F}r) \in \mathcal{G}$ is semi-simple,
3) $\varphi|_{SL_2(\mathbb{C})} : SL_2(\mathbb{C}) \to \mathcal{G}^o$ is a morphism of complex linear algebraic group

corresponds bijectively the equivalence classes of the triples $(\rho, \mathcal{G}, N)$ where $N \in \text{Lie}(\mathcal{G})$ is a nilpotent element and

$$\rho : W_F \to \mathcal{G}$$

is a group homomorphism such that

1) $\rho|_{I_F} : I_F \to \mathcal{G}$ is continuous,
2) $\rho(\tilde{F}r) \in \mathcal{G}$ is semi-simple,
3) $\rho(g)N = |g|_F \cdot N$ for $\forall g \in W_F$ where

$$| \cdot |_F : W_F \xrightarrow{\can} W_F/[W_F, W_F] \xrightarrow{\text{l.c.f.t.}} \mathbb{F}^{\times} \xrightarrow{q^{-\ord_F(\cdot)}} \mathbb{Q}^{\times}$$

by the relations

$$\rho|_{I_F} = \varphi|_{I_F}, \quad \rho(\tilde{F}r) = \varphi(\tilde{F}r) : \varphi \left( \begin{array}{cc} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{array} \right), \quad N = d\varphi \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)$$

(see [7, Prop.2.2]). Here two triples $(\rho, \mathcal{G}, N)$ and $(\rho', \mathcal{G}, N')$ is equivalent if there exists a $g \in \mathcal{G}$ such that $\rho' = gpg^{-1}$ and $N' = \text{Ad}(g)N$.

The couple $(\varphi, \mathcal{G})$ or the triple $(\rho, \mathcal{G}, N)$ is called an admissible representation of the Weil-Deligne group.

Let $(r, V)$ be a continuous finite dimensional complex representation of $\mathcal{G}$ which is algebraic on $\mathcal{G}^o$. Then the $L$-factor associated with $(\varphi, \mathcal{G})$ and $(r, V)$ is defined by

$$L(\varphi, r, s) = \det \left( 1 - q^{-s}r \circ \rho(\tilde{F}r)|_{V_N^{I_F}} \right)^{-1},$$

where $V_N = \{ v \in V \mid dr(N)v = 0 \}$ and

$$V_N^{I_F} = \{ v \in V_N \mid r \circ \rho(\sigma)v = v \ \forall \sigma \in I_F \}.$$

The $\varepsilon$-actor is defined by

$$\varepsilon(\varphi, r, \psi, d(x), s) = \varepsilon(r \circ \rho, \psi, d(x), s) \cdot \det \left( -q^{-s}r \circ \rho(\tilde{F}r)|_{V_N^{I_F}/V_N^{I_F}} \right)$$

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where $\varepsilon(r \circ \rho, \psi, d(x), s)$ is the $\varepsilon$-factor of the representation $(r \circ \rho, V)$ of $W_F$ defined in the subsection A.4. Finally the $\gamma$-factor is defined by

$$
\gamma(\varphi, r, \psi, d(x), s) = \varepsilon(\varphi, r, \psi, d(x), s) \cdot \frac{L(\varphi, r^\vee, 1 - s)}{L(\varphi, r, s)}
$$

where $r^\vee$ is the dual representation of $r$.

Let $\text{Sym}_n$ be the symmetric tensor representation of $SL_2(\mathbb{C})$ of degree $n + 1$. Then the $W_F \times SL_2(\mathbb{C})$-module $V$ has a decomposition

$$
V = \bigoplus_{n=0}^{\infty} V_n \otimes \text{Sym}_n
$$

where $V_n$ is a $W_F$-module. Then we have

$$
V'_n = \bigoplus_{n=0}^{\infty} V'_n \otimes \text{Sym}_{n,N}
$$

where $\text{Sym}_{n,N}$ is the highest part of $\text{Sym}_n$. Since $r \circ \rho(\widetilde{Fr})$ act on $V_n \otimes \text{Sym}_{n,N}$ by $q^{-n/2} r \circ \varphi(\widetilde{Fr})$, we have

$$
L(\varphi, r, s) = \prod_{n=0}^{\infty} \det \left( 1 - q^{-(s+n/2)} r \circ \varphi(\widetilde{Fr}) | V'_n \right)^{-1}.
$$

If the Haar measure $d(x)$ on the additive group $F$ and the additive character $\psi : F \to \mathbb{C}^\times$ are normalized so that $\int_{O_F} d(x) = 1$ and

$$
\{ x \in F | \psi(xO_F) = 1 \} = O_F,
$$

then we have

$$
\varepsilon(\varphi, r, \psi, d(x), s) = w(\varphi, r) \cdot q^{a(\varphi, r)(1/2 - s)}
$$

where

$$
w(\varphi, r) = \prod_{n=0}^{\infty} w(V_n)^n \cdot \prod_{n=1}^{\infty} \det \left( -\varphi(\widetilde{Fr}) | V'_n \right)^n
$$

and

$$
a(\varphi, r) = \sum_{n=0}^{\infty} (n + 1) a(V_n) + \sum_{n=1}^{\infty} n \cdot \dim V'_n.
$$

If $\varphi|_{SL_2(\mathbb{C})} = 1$, then $V_n = 0$ for all $n > 0$ and we have

$$
w(\varphi, r) = w(r \circ \varphi) = w(r \circ \rho), \quad a(\varphi, r) = a(r \circ \varphi) = a(r \circ \rho).
$$

**B Symmetric or anti-symmetric forms on induced representations of Weil group**

Let $K/F$ be a finite Galois extension of even degree. We will assume that the elements of $\Gamma = \text{Gal}(K/F)$ of order two are central. Fix an element $\tau \in \Gamma$ of degree $2$. This is the case if $K/F$ is tamely ramified extension. See Proposition 3.3.1.

If $\varphi|_{SL_2(\mathbb{C})} = 1$, then $V_n = 0$ for all $n > 0$ and we have

$$
w(\varphi, r) = w(r \circ \varphi) = w(r \circ \rho), \quad a(\varphi, r) = a(r \circ \varphi) = a(r \circ \rho).$$

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order two. Let $K_+$ be the intermediate field of $K/F$ such that $\text{Gal}(K/K^+) = \langle \tau \rangle$, and put

$$U_{K/K_+} = \{ \varepsilon \in O_K^\times \mid N_{K/K_+}(\varepsilon) = 1 \}.$$ 

Take a continuous unitary character $\vartheta : U_{K/K_+} \to \mathbb{C}^\times$ and put $\tilde{\vartheta}(x) = \vartheta(x^{1-\tau})$ ($x \in K^\times$). The representation space $V_\vartheta = \text{Ind}^{W_{K/F}}_{K^\times} \tilde{\vartheta}$ is the complex vector space of the $\mathbb{C}$-valued functions $v$ on $\Gamma$ on which $(\sigma, x) \in W_{K/F} = \Gamma \ltimes_{\alpha_{K/F}} K^\times$ acts by

$$(x \cdot v)(\gamma) = \tilde{\vartheta}(x^\gamma) \cdot v(\gamma), \quad (\sigma \cdot v)(\gamma) = \tilde{\vartheta}(\alpha_{K/F}(\sigma, \sigma^{-1}\gamma)) \cdot v(\sigma^{-1}\gamma)$$

with the fundamental class $[\alpha_{K/F}] \in H^2(\Gamma, K^\times)$. The character $\chi_\vartheta$ of $V_\vartheta$ is

$$\chi_\vartheta(\sigma, x) = \begin{cases} 0 & : \sigma \neq 1, \\
\sum_{\gamma \in \Gamma} \tilde{\vartheta}(x^\gamma) & : \sigma = 1 \end{cases}$$

for $(\sigma, x) \in W_{K/F}$, which is self-conjugate, that is $\overline{\chi_\vartheta} = \chi_\vartheta$.

Let $\nu : W_{K/F} \to \mathbb{C}^\times$ be a continuous group homomorphism. We will look for the $\nu$-invariant $\nu$-symmetric bilinear form on $V_\vartheta$, that is, the non-zero complex bilinear form $B$ on $V_\vartheta$ such that

1) $B(g \cdot u, g \cdot v) = \nu(g) \cdot B(u, v)$ for all $g \in W_{K/F}$,

2) $B(v, u) = \nu(\tau) \cdot B(u, v)$ for all $u, v \in V_\vartheta$.

Note that, in this case, we have $\nu(\tau) = \pm 1$.

If $\nu|_{K^\times} = 1$, then

$$B_\nu(u, v) = \sum_{\gamma \in \Gamma} \nu(\gamma) \cdot \tilde{\vartheta}(\alpha_{K/F}(\gamma, \tau))^{-1} \cdot u(\gamma)v(\gamma \tau) \quad (u, v \in V_\vartheta)$$

is a non-degenerate $\nu$-invariant $\nu$-symmetric bilinear form on $V_\vartheta$. For a $\rho \in \Gamma$, define $w_\rho \in V_\vartheta$ by

$$w_\rho(\gamma) = \begin{cases} 1 & : \gamma = \rho, \\
0 & : \gamma \neq \rho \end{cases}$$

and $u_\rho, v_\rho \in V_\vartheta$ by

$$u_\rho = \nu(\rho)^{-1}w_\rho, \quad v_\rho = \tilde{\vartheta}(\alpha_{K/F}(\rho, \tau)) \cdot w_{\rho \tau}.$$ 

If we fix a complete system of representatives $\mathcal{S}$ of $\Gamma/\langle \tau \rangle$, then $\{u_\rho, v_\rho\}_{\rho \in \mathcal{S}}$ is a $\mathbb{C}$-basis of $V_\vartheta$ such that

$$B_\nu(u_\rho, u_{\rho'}) = B_\nu(v_\rho, v_{\rho'}) = 0, \quad B_\nu(u_\rho, v_{\rho'}) = \begin{cases} 1 & : \rho = \rho', \\
0 & : \rho \neq \rho' \end{cases}.$$ 

**Proposition B.0.1** Assume that

1) $\nu$ is of finite order,

2) $\{\sigma \in \Gamma \mid \tilde{\vartheta}(x^\sigma) = \tilde{\vartheta}(x) \forall x \in 1 + \mathfrak{p}_K\} = \{1\}$. 

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Then \( V_\vartheta \) has \( \nu \)-invariant \( \nu \)-symmetric bilinear form if and only if \( \nu|_{K^\times} = 1 \). In this case, the form is a constant multiple of \( B_\nu \).

**Proof** Due to the second assumption and Remark 3.4.3, the induced representation \( V_\vartheta = \text{Ind}^{W_{K/F}}_{K/F} \tilde{\vartheta} \) is irreducible. Since \( \nu \) is of finite order, we can choose positive integers \( s, t \) such that \( M = \langle \varpi_K \rangle \times (1 + \mathfrak{p}_K^t) \) is a \( \Gamma \)-subgroup of \( K^\times \) on which \( \vartheta \) and \( \nu \) are trivial. Then the induced representation \( \text{Ind}^{W_{K/F}}_{K/F} \vartheta \) and the character \( \nu \) factor through the canonical morphism

\[
W_{K/F} \to G = \Gamma \rtimes \alpha_{K/F} K^\times / M.
\]

So we will consider them on the finite group \( G \). The it is well-known that

\[
\dim_{\mathbb{C}} \text{Hom}_G(\varpi, \text{Hom}_{\mathbb{C}}(V_0, V_0^*)) = \begin{cases} 1 & \text{if } \nu \cdot \vartheta = \vartheta, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( V_0^* \) is the dual representation of \( V_0 \). Since \( T \in \text{Hom}_{\mathbb{C}}(V_0, V_0^*) \) gives a complex bilinear form \( B_T(u, v) = \langle u, T v \rangle \) with the canonical pairing \( \langle , \rangle : V_0 \times V_0^* \to \mathbb{C} \), and

\[
\text{Hom}_{\mathbb{C}}(V_0, V_0^*) = \text{Sym}(V_0, V_0^*) \oplus \text{Alt}(V_0, V_0^*),
\]

there exists \( \nu \)-invariant \( \nu \)-symmetric bilinear form on \( V_\vartheta \) if and only if \( \nu \cdot \vartheta = \vartheta \), and in this case

\[
\dim_{\mathbb{C}} \text{Hom}_G(\varpi, \text{Sym}(V_0, V_0^*)) = \frac{1}{2} \left( 1 + \frac{1}{|G|} \sum_{g \in G} \nu(g) \vartheta(g^2) \right),
\]

\[
\dim_{\mathbb{C}} \text{Hom}_G(\varpi, \text{Alt}(V_0, V_0^*)) = \frac{1}{2} \left( 1 - \frac{1}{|G|} \sum_{g \in G} \nu(g) \vartheta(g^2) \right),
\]

that is

\[
\frac{1}{|G|} \sum_{g \in G} \nu(g) \vartheta(g^2) = \nu(\tau).
\]

Let us assume \( \nu \cdot \vartheta = \vartheta \). Then the prime element \( \varpi_K \) of \( K \) can be chosen so that \( \nu(\varpi_K) = 1 \). In fact there exists a prime element \( \varpi_K \) of \( K \) such that \( \varpi_K^\tau = \pm \varpi_K \). Then

\[
\tilde{\vartheta}(\varpi_K^\gamma) = \vartheta(\varpi_K^{\gamma(1-\gamma)}) = \vartheta(\varpi_K^{1-\gamma}) = \vartheta(\pm 1)
\]

for all \( \gamma \in \Gamma \), and hence

\[
\vartheta(1, \varpi_K) = \sum_{\gamma \in \Gamma} \tilde{\vartheta}(\varpi_K^\gamma) = |\Gamma| \cdot \vartheta(\pm 1) \neq 0.
\]

Then \( \nu \cdot \vartheta = \vartheta \) implies \( \nu(\varpi_K) = 1 \).

Note also that \( \nu(x^\gamma) = \nu(x) \) for all \( x \in K^\times \) and \( \gamma \in \Gamma \), since \( (\gamma, 1)^{-1}(1, x)(\gamma, 1) = (1, x^\gamma) \).
Since
\[
\chi^\sigma((\sigma, x)^2) = \begin{cases} 
0 & : \sigma^2 \neq 1, \\
\sum_{\gamma \in \Gamma} \tilde{\vartheta} \left( x^{(1+\sigma)\gamma} \alpha_{K/F}(\sigma, \sigma)^\gamma \right) & : \sigma^2 = 1,
\end{cases}
\]
we have
\[
\sum_{g \in G} \nu(g) \chi^\sigma(g^2) = \sum_{\sigma, \gamma \in \Gamma, \tilde{x} \in K^{x/M}} \sum_{\sigma^2 = 1} \nu(\sigma, x) \cdot \tilde{\vartheta} \left( x^{(1+\sigma)\gamma} \alpha_{K/F}(\sigma, \sigma)^\gamma \right)
\]
\[
= \sum_{\sigma, \gamma \in \Gamma, \tilde{x} \in K^{x/M}} \sum_{\sigma^2 = 1} \nu(\sigma, x^{\gamma^{-1}}) \cdot \tilde{\vartheta} \left( x^{(1+\sigma)} \alpha_{K/F}(\sigma, \sigma)^\gamma \right)
\]
\[
= \sum_{\sigma, \gamma \in \Gamma, \tilde{x} \in K^{x/M}} \nu(\sigma) \cdot \tilde{\vartheta} \left( \alpha_{K/F}(\sigma, \sigma)^\gamma \right) \sum_{\tilde{x} \in K^{x/M}} \nu(x) \cdot \tilde{\vartheta}(x^{1+\sigma}).
\]
Since \(\nu(\varpi_K) = 1\) and \(\varpi_K^{1-\tau} = \pm 1\), we have
\[
\tilde{\vartheta}(\varpi_K^{1+\sigma}) = \vartheta(\varpi_K^{(1+\sigma)(1-\tau)}) = \vartheta((\pm 1)^{1+\sigma}) = 1
\]
for all \(\sigma \in \Gamma\), we have
\[
\sum_{\tilde{x} \in K^{x/M}} \nu(x) \cdot \tilde{\vartheta}(x^{1+\sigma}) = s \sum_{\tilde{x} \in (O_K/p_K)^x} \nu(x) \tilde{\vartheta}(x^{1+\sigma}).
\]
If \(\nu(x) \tilde{\vartheta}(x^{1+\sigma}) = 1\) for all \(x \in O_K^x\), then we have
\[
1 = \nu(x^\tau) \tilde{\vartheta}(x^{(1+\sigma)^\tau}) = \nu(x) \tilde{\vartheta}(x^{1+\sigma})^{-1},
\]
and hence
\[
\tilde{\vartheta}(x^{2\sigma}) = \tilde{\vartheta}(x^{-2}) = \tilde{\vartheta}(x^{2\tau})
\]
for all \(x \in O_K^x\). Since \(x \mapsto x^2\) gives a surjection of \(1 + p_K\) onto \(1 + p_K\), we have
\[
\tilde{\vartheta}(x^\sigma) = \tilde{\vartheta}(x^\tau)
\]
for all \(x \in 1 + p_K\). Since \(\alpha_{K/F}(\tau, \tau)^\tau \alpha_{K/F}(\tau, 1)^{-1} \alpha_{K/F}(1, \tau) \alpha_{K/F}(\tau, \tau)^{-1} = 1\).

and \(\alpha_{K/F}(1, \tau) = \alpha_{K/F}(\tau, 1) = 1\), we have
\[
\tilde{\vartheta} \left( \alpha_{K/F}(\tau, \tau)^\gamma \right) = \vartheta \left( \alpha_{K/F}(\tau, \tau)^{(1-\gamma)\gamma} \right) = 1
\]
for all \(\gamma \in \Gamma\). Then we have
\[
|G|^{-1} \sum_{g \in G} \nu(g) \chi^\sigma(g^2) = |(O_K/p_K^x)^x|^{-1} \nu(\tau) \sum_{\tilde{x} \in (O_K/p_K)^x} \nu(x)
\]
\[
= \begin{cases} 
0 & : \nu|_{O_K^x} \neq 1, \\
\nu(\tau) & : \nu|_{O_K^x} = 1.
\end{cases}
\]
This completes the proof. \(\blacksquare\)
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