Solutions in Self-dual Gravity Constructed Via Chiral Equations

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Abstract

The chiral model for self-dual gravity given by Husain in the context of the chiral equations approach is discussed. A Lie algebra corresponding to a finite dimensional subgroup of the group of symplectic diffeomorphisms is found, and then use for expanding the Lie algebra valued connections associated with the chiral model. The self-dual metric can be explicitly given in terms of harmonic maps and in terms of a basis of this subalgebra.

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1 Introduction

Since the introduction of Ashtekar’s variables in General Relativity \[1\] they were quickly applied to self-dual gravity. Later, Ashtekar, Jacobson and Smolin (AJS) considered a new formulation of half-flat solutions to Einstein’s equations. To be more precise, making a decomposition of a real 4-manifold $\mathcal{M}^4$ into $\mathbb{R} \times \Sigma$, with $\Sigma$ an arbitrary 3-manifold ($\mathcal{M}^4$ has local coordinates $\{x_0, x_1, x_2, x_3\}$), the problem of finding all self-dual metrics was reduced to solving one constraint and one “evolution” equation on a field of triads $V^a_i$ on $\Sigma$. That is,

$$\text{Div} V^a_i = 0,$$

$$\frac{\partial V^a_i}{\partial t} = \frac{1}{2} \epsilon_{ijk} [V^i_j, V^k]_a,$$  \hspace{1cm} (1)

where $i, j, k = 1, 2, 3, \ [2]$. Thus, all self-dual metrics can be described in terms of the triad just as

$$g^{ab} = (\det V)^{-1} [V^a_i V^b_j \delta^{ij} + V^a_0 V^b_0].$$ \hspace{1cm} (2)

where $V^a_0$ is the vector field used in the $3 + 1$ decomposition.

Several authors \[3\][4], beginning with the AJS formulation made contact with the Plebański approach to self-dual gravity \[5\]. In \[3\] Grant has shown that Eqs.(1) are related in a very close way with the first heavenly equation of ref. \[5\]. It was quickly recognized that the relation was only a Legendre transformation on a convenient coordinate chart \[4\]. Here the heavenly equation was brought into a Cauchy-Kovalevski evolution form.

On the other hand in \[4\] Husain gives a chiral formulation for the self-dual gravity. He has shown how self-dual gravity can be derived from a 2-dimensional chiral Model which gauge group corresponds to the group of symplectic diffeomorphisms (area preserving diffeomorphisms of a 2-surface $\mathcal{N}^2$, $\text{SDiff}(\mathcal{N}^2)$).\[4\] Similarly to Grant, starting from Eqs. (1) but using an-

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$^1$As we make only local considerations we assume the space $\mathcal{N}^2$ to be a two-dimensional simply connected manifold with local coordinates $\{p, q\}$. This space has a natural local symplectic structure given by the local area form $\omega = dp \wedge dq$. The group $\text{SDiff}(\mathcal{N}^2)$ is precisely the group of diffeomorphisms on $\mathcal{N}^2$ preserving the symplectic structure $\omega$, i.e. for all $g \in \text{SDiff}(\mathcal{N}^2)$, $g^* (\omega) = \omega$. 

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other choice of the set of vector fields $V_i^a$, Husain derived also the first heavenly equation. However, although the choice of vector fields is different, both formulations are equivalent to that one from Plebański. Thus, we have a class of equations (and therefore the corresponding class of solutions) which will be equivalent. This class of equations we call Grant-Husain-Plebański GHP and they can be seen as equivalent to AJS equations. This because they are only different formulations of the same full theory.

Here, we briefly review the Husain chiral model for self-dual gravity. It is well known that Equations (1) lead to the set

$$[\mathcal{T}, \mathcal{X}] = [\mathcal{U}, \mathcal{V}] = 0,$$

$$[\mathcal{T}, \mathcal{U}] + [\mathcal{X}, \mathcal{V}] = 0,$$

where $\mathcal{T} := V_0 + iV_1$, $\mathcal{U} := V_0 - iV_1$, $\mathcal{X} := V_2 - iV_3$, $\mathcal{V} := V_2 + iV_3$. The vector fields $\mathcal{X}$ and $\mathcal{T}$ can be fixed to be

$$\mathcal{T} = \frac{\partial}{\partial \bar{z}}, \quad \mathcal{X} = \frac{\partial}{\partial z}.$$ (4)

where the $\bar{z} = x_0 + ix_1$, $z = x_2 - ix_3$, $u = x_0 - ix_1$ and $v = x_2 + ix_3$. The bar does not stands for complex conjugation. The choice of vector fields enables four possibilities:

i) Husain [4] (See also [7]):

Taking

$$\mathcal{U} = -\Omega ,_{zp} \frac{\partial}{\partial p} + \Omega ,_{zp} \frac{\partial}{\partial q},$$

$$\mathcal{V} = \Omega ,_{zq} \frac{\partial}{\partial p} - \Omega ,_{zp} \frac{\partial}{\partial q},$$ (5)

where $\Omega$ is a holomorphic function of its arguments and $p,q$ are local coordinates on the two-manifold $\mathcal{N}^2$. Eqs.(3) lead directly to first heavenly equation as usual [3]

$$\Omega ,_{zp} \Omega ,_{zq} - \Omega ,_{zq} \Omega ,_{zp} = 1,$$ (6)

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where $\Omega_{zp} = \frac{\partial^2 \Omega}{\partial z \partial p}$, etc.

**ii) Grant** [3]:

The difference with respect to the Husain’s formalism is just the way in which the vector fields $U$ and $V$ are chosen. Grant takes

$$U = \partial_{\bar{z}} - h_{zp} \frac{\partial}{\partial p} + h_{zp} \frac{\partial}{\partial q},$$

$$V = h_{\bar{z}q} \frac{\partial}{\partial p} - h_{\bar{z}p} \frac{\partial}{\partial q}. \quad (7)$$

Equations (3) lead to the Grant evolution equation, which is of the Cauchy-Kovalevski form

$$h_{\bar{z}z} + h_{zq}h_{\bar{z}q} - h_{zq}h_{\bar{z}p} = 0, \quad (8)$$

where the corresponding metric is

$$g = d\bar{z} \otimes (h_{\bar{z}q} dq + h_{zp} dp) + dz \otimes (h_{zq} dq + h_{zp} dp) + \frac{1}{h_{\bar{z}z}} (h_{\bar{z}q} dq + h_{zp} dp)^2. \quad (9)$$

After a Legendre transformation on the variable $\bar{z}$ we recover the first heavenly equation as usual [3].

**iii) Grant (a variant)** [3]:

This choice leads to a formulation similar to that of Grants. Choosing the vector fields as

$$U = -h_{zq} \frac{\partial}{\partial p} + h_{zp} \frac{\partial}{\partial q},$$

$$V = \frac{\partial}{\partial z} + h_{\bar{z}q} \frac{\partial}{\partial p} - h_{\bar{z}p} \frac{\partial}{\partial q}, \quad (10)$$

and using once again equations (3), one arrives at the Grant evolution equation, which is of the Cauchy-Kovalevski form.
\[ h_{zz} + h_{\bar{z}q}h_{zp} - h_{\bar{z}p}h_{zq} = 0. \]  
(11)

The corresponding metric is of course

\[ g = d\bar{z} \otimes (h_{\bar{z}q}dq + h_{\bar{z}p}dp) + dz \otimes (h_{zq}dq + h_{zp}dp) + \frac{1}{h_{zz}}(h_{zq}dq + h_{zp}dp)^2. \] 
(12)

And, as before, the first heavenly equation is recovered after a Legendre transformation.

\textit{iv) Husain [4]:}

For the self-dual equations (1) there exists another possibility for an appropriate selection of the vector fields. This choice leads to the chiral equations, which appear to be non-equivalent to that of the GHP class of equations. However they might be related to them.

Introducing now two functions \( B_1(\bar{z}, z, p, q) \) and \( B_2(\bar{z}, z, p, q) \), the vector fields \( U \) and \( V \) can be written in a completely general form in terms of these functions as

\[ U = \frac{\partial}{\partial \bar{z}} + \alpha^{\bar{b}} \partial_{\bar{b}} B_1, \]
\[ V = \frac{\partial}{\partial z} + \alpha^b \partial_b B_2, \] 
(13)

where \( \alpha^{\bar{a}} = (\frac{\partial}{\partial p})[a \otimes (\frac{\partial}{\partial q})^b]. \)

Using equations (3), the above choice of vector fields leads directly to the set of equations

\[ B_{2,\bar{z}} - B_{1,z} + \{ B_1, B_2 \} = \mathcal{F}_{\bar{z},z} + \mathcal{G}_{z,\bar{z}}, \]

\[ B_{1,\bar{z}} + B_{2,z} = \mathcal{F}_{z,\bar{z}} - \mathcal{G}_{\bar{z},z}, \] 
(14)

for the arbitrary functions \( \mathcal{F} \) and \( \mathcal{G} \). In the above equation \( \{ , \} \) means the Poisson bracket in the coordinates \( p \) and \( q \).

Redefining
\[ A_1(\bar{z}, z, p, q) = B_1 + \mathcal{G} \]

and

\[ A_2(\bar{z}, z, p, q) = B_2 - \mathcal{F} \]  

(15)

transforms into a two-dimensional chiral model on a two-manifold \( \mathcal{M}^2 \) with local coordinates \( \{\bar{z}, z\} \), having as gauge group the group of area preserving diffeomorphisms of the two-dimensional manifold \( \mathcal{N}^2 \). This two-dimensional chiral model is

\[ F = A_{2,\bar{z}} - A_{1,z} + \{A_1, A_2\} = 0. \]  

(16)

Vanishing curvature \( F = 0 \) implies that the gauge potentials \( A_1 \) and \( A_2 \) are \textit{pure gauge}. Thus, we can write the potentials as

\[ A_1 = (\partial_{\bar{z}} g) g^{-1}, \quad A_2 = (\partial_z g) g^{-1}, \]  

(17)

where \( g : \mathcal{M}^2 \times \mathcal{N}^2 \to \text{SDiff}(\mathcal{N}^2) \) given by \( g(\bar{z}, z, p, q) \in \text{SDiff}(\mathcal{N}^2) \). These potentials satisfy

\[ A_{1,\bar{z}} + A_{2,z} = 0. \]  

(18)

In this paper we work with the chiral formulation for self-dual gravity as given by Husain. In Sec. 2, using the formalism of chiral equations approach to Einstein equations we discuss the chiral equations of the Husain model as harmonic maps in similar philosophy of [8][9]. In Sec. 3 we find a finite dimensional subalgebra of the Lie algebra of \( \text{SDiff}(\mathcal{N}^2) \), and then we use this reduction to find solutions. We also find that the system induced by the Husain formalism is completely integrable at least for this subalgebra. Finally in Sec. 4 we give our final remarks.

2 Chiral Equations as Harmonic Maps

In this section we shall outline the method of harmonic maps for solving the chiral equations. This method consists in applying the harmonic maps ansatz to the chiral equations. Let us explain it.
First we enunciate the following theorem.

**Theorem.** Let \( g \in G \) fulfill the chiral equations. The submanifold of solutions of the chiral equations \( S \subset G \), is a symmetric manifold (the Riemann tensor of \( S \) is covariantly null, i.e. \( \nabla R_S = 0 \)) with metric

\[
1_S = \text{tr}(dg \ g^{-1} \otimes dg \ g^{-1}),
\]

(19)

where \( \otimes \) denotes, the symmetric tensor product. For the proof see refs. [9][10].

The ansatz consists in supposing that \( g \) can be written in terms of harmonic maps. Let \( V_Q \) be a \( Q \)-dimensional Riemannian space with an isometry group \( H \subset G = \text{SDiff}(N^2) \). Suppose that \( \{ \lambda^i \} \) are local coordinates of \( V_Q \). Let \( \{ \phi_s \}, \ s = 1, ..., d = \text{dim}H \leq \text{dim}G = \infty, \) and \( \phi_s = \phi^i_s \partial_{\lambda^i} \) be a basis of the Killing vector space of \( V_Q \) and \( \{ \xi^s \} \) the dual basis of \( \{ \phi_s \} \). We suppose that

\[
g = g(\lambda^i, p, q), \quad i = 1, ..., Q\]

(20)

where \( \lambda^i(z, \bar{z}) \) are affine parameters of the minimal surfaces of \( G \), i.e.

\[
\lambda^i_{,z} + \Gamma^i_{jk} \lambda^j_{,z} \lambda^k_{,\bar{z}} = 0 \quad i, j, k = 1, ..., Q.
\]

(21)

The \( \text{sdiff}(N^2) \)-valued connection 1-form on the two-manifold \( M^2 \) in the basis \( \{ d\lambda^i \} \) can be written as (see ref. [11])

\[
A = a_i(z, \bar{z}, p, q) d\lambda^i = A_1(z, \bar{z}, p, q) dz + A_2(z, \bar{z}, p, q) d\bar{z},
\]

(22)

where \( A_1(z, \bar{z}, p, q) = A_1(\lambda^i, p, q) = a_i(\lambda^i, p, q) \lambda^i_{,z} \) and \( A_2(z, \bar{z}, p, q) = A_2(\lambda^i, p, q) = a_i(\lambda^i, p, q) \lambda^i_{,\bar{z}} \). The functions \( a_i(\lambda^i, p, q) \) can be expanded in terms of a basis of a finite dimensional Lie subalgebra \( H \) of \( \text{sdiff}(N^2) \), \( \{ \sigma_j \}, j = 1, 2, ..., d; \)

that is

\[
a_i(\lambda^i, p, q) = \xi^i(\lambda^i) \sigma_j(p, q),
\]

(23)

(for details of this method see refs. [8][9]).

**Theorem.** The potentials \( A_1(\lambda^i, p, q) = a_i(\lambda^i, p, q) \lambda^i_{,z} \) and \( A_2(\lambda^i, p, q) = a_i(\lambda^i, p, q) \lambda^i_{,\bar{z}} \) are solutions of the chiral equations (16) and (18).
Proof: Using (21), equation (18) implies that the quantities $\xi^s_i(\lambda^i)$ are the components of the Killing vectors of $V_Q$

$$A_{1,\bar{z}} + A_{2,z} = (\xi^s_{i,j} + \xi^s_{j,i})\sigma_s \lambda^i \lambda^j = 0.$$  

where ; means covariant derivative in $V_Q$. Equation (16) implies that $\{\sigma_s\}$ are the corresponding hamiltonian functions of the simplectic form $\omega = dp \wedge dq$ on $N^2$, i.e.

$$\{\sigma_s, \sigma_t\} = C^r_{st}(p,q)\sigma_r,$$  

where $C^r_{st}$ are functions of $p$ and $q$ only.

We shall now use the above approach to Einstein’s equations [8][9], in order to apply them to self-dual gravity. We show that it is possible to translate all relevant tools of the AJS formalism in terms of harmonic maps.

For instance the vector fields $U$ and $V$ are

$$U = \frac{\partial}{\partial \bar{z}} + \xi^s_i \lambda^i_{\bar{z}} (\frac{\partial \sigma_s}{\partial p} \frac{\partial}{\partial q} - \frac{\partial \sigma_s}{\partial q} \frac{\partial}{\partial p}),$$

$$V = \frac{\partial}{\partial z} + \xi^s_i \lambda^i_z (\frac{\partial \sigma_s}{\partial p} \frac{\partial}{\partial q} - \frac{\partial \sigma_s}{\partial q} \frac{\partial}{\partial p}).$$

(25)

The vectors on $R \times \Sigma^3$ are therefore

$$V_0 = \frac{\partial}{\partial \bar{z}} + \frac{1}{2} \xi^s_i \lambda^i_{\bar{z}} (\frac{\partial \sigma_s}{\partial p} \frac{\partial}{\partial q} - \frac{\partial \sigma_s}{\partial q} \frac{\partial}{\partial p}),$$

$$V_1 = \frac{i}{2} \xi^s_i \lambda^i_{\bar{z}} (\frac{\partial \sigma_s}{\partial p} \frac{\partial}{\partial q} - \frac{\partial \sigma_s}{\partial q} \frac{\partial}{\partial p}),$$

$$V_2 = \frac{\partial}{\partial z} + \frac{1}{2} \xi^s_i \lambda^i_z (\frac{\partial \sigma_s}{\partial p} \frac{\partial}{\partial q} - \frac{\partial \sigma_s}{\partial q} \frac{\partial}{\partial p}),$$

$$V_3 = -\frac{i}{2} \xi^s_i \lambda^i_z (\frac{\partial \sigma_s}{\partial p} \frac{\partial}{\partial q} - \frac{\partial \sigma_s}{\partial q} \frac{\partial}{\partial p}).$$

(26)

The self-dual metric (2) can be expressed in terms of harmonic maps
Here it can be observed that similarly to the metric (3.4) of ref. [4], it also appears a singularity for null Poisson brackets (abelian algebra).

For completeness we can write also the inverse of (27)

\[ g^{-1} = \frac{4}{\xi_i^m \xi_j^n \lambda_i^l \lambda_j^m \{\sigma_m, \sigma_n\}} \left[ d\bar{z} \otimes d\bar{\xi} + dz \otimes dz + \xi^k_s (\frac{\partial \sigma_s}{\partial p} dp - \frac{\partial \sigma_s}{\partial q} dq) \otimes d\lambda^k \right] \quad (27) \]

From metric (27) and (28) it is now clear that \( C_{mn}^s \) can not vanish, thus it is not possible to take an abelian algebra in (24).

### 3 Two Dimensional Subspaces

From metric (28) we conclude that it is not possible to take one dimensional subspaces \( V_1 \) since all one dimensional Riemannian spaces contains only abelian groups of motion. We consider a two-dimensional Riemannian space \( V_2 \). In [9] it was shown that the chiral equations imply that \( V_2 \) must be a symmetric space. All two-dimensional Riemannian space is conformally flat. So, the metric of \( V_2 \) can be written as \[ i^*(1_S) = ds^2 = \frac{d\lambda d\tau}{(1 + k\lambda \tau)^2}, \quad (29) \]

where \( i : V_2 \to S \). The symmetry of \( V_2 \) implies that \( k = k(p,q) \) only, i.e. \( k, \lambda = k, \tau = 0 \), (since all two-dimensional symmetric space possesses constant curvature). Thus, \( V_2 \) contains a three-dimensional isometry group \( H \). Three independent Killing vectors of \( V_2 \) are

\[ \xi^1 = \frac{1}{2V_2} [(k\tau^2 + 1)d\lambda + (k\lambda^2 + 1)d\tau], \]
\[ \xi^2 = \frac{1}{V^2}[-\tau d\lambda + \lambda d\tau], \quad V = 1 + k\lambda\tau, \]

\[ \xi^3 = \frac{1}{2V^2}[(k\tau^2 - 1)d\lambda + (1 - k\lambda^2)d\tau]. \] \hspace{1cm} (30)

The three Hamiltonian functions \( \sigma_s \) fulfill the algebra

\[ \{\sigma_1, \sigma_2\} = -4k\sigma_3, \]
\[ \{\sigma_2, \sigma_3\} = 4k\sigma_1, \]
\[ \{\sigma_3, \sigma_1\} = -4\sigma_2, \] \hspace{1cm} (31)

in order to have compatibility with the Killing vectors (30). These Poisson brackets can be seen as three differential equations for the three functions \( \sigma_s \) and the function \( k \), so we can take one of them arbitrarily and determine the other three ones by integration. Knowing the functions \( \sigma_s \) we can determine the potentials \( A_1 \) and \( A_2 \) by means of the formulae

\[ A_1 = \xi^s \sigma_s \lambda_i^{\bar{z}}, \quad A_2 = \xi^s \sigma_s \lambda_i^{\bar{z}}, \]

in terms of the harmonic maps \( \lambda^i \). The harmonic map equation (21) transforms in this case into

\[ \lambda_{,\bar{z}\bar{z}} - \frac{2k\tau}{1 + k\lambda\tau} \lambda_{,\bar{z}} \lambda_{,\bar{z}} = 0, \]
\[ \tau_{,\bar{z}\bar{z}} - \frac{2k\lambda}{1 + k\lambda\tau} \tau_{,\bar{z}} \tau_{,\bar{z}} = 0. \] \hspace{1cm} (32)

In what follows we will solve equations (31). Let us write equation (31) in terms of two new variables \( s = s(p, q) \) and \( t = t(p, q) \) and without loss of generality we can suppose that \( \sigma_2 = s \). The commutation relations (31) transform into

\[ i) \quad \frac{\partial \sigma_1}{\partial t}(s, t) = 4k\sigma_3, \]
\[ ii) \quad \frac{\partial \sigma_3}{\partial t}(s, t) = 4k\sigma_1, \]
\[ iii) \quad \left( \frac{\partial \sigma_3}{\partial s} \frac{\partial \sigma_1}{\partial t} - \frac{\partial \sigma_1}{\partial s} \frac{\partial \sigma_3}{\partial t} \right)(s, t) = -4\sigma_2. \] \hspace{1cm} (33)
If we substitute \((i)\) and \((ii)\) into \((iii)\) we arrive at
\[
k \frac{\partial (\sigma_3^2 - \sigma_1^2)}{\partial s} = -2s, \tag{34}
\]
and by combining \((i)\) and \((ii)\) we conclude that
\[
\frac{\partial (\sigma_3^2 - \sigma_1^2)}{\partial t} = 0, \tag{35}
\]
which imply that \(k\) does not depend on \(t\), that means \(k = k(s)\). Deriving equations \((i)\) and \((ii)\) with respect to \(t\) we find differential equations only for \(\sigma_1\) and \(\sigma_3\)
\[
\frac{\partial^2 \sigma_s l^2}{\partial t^2} + \frac{1}{2} \frac{\partial \sigma_s}{\partial t} \frac{\partial l^2}{\partial t} - 16k^2 \sigma_s = 0, \tag{36}
\]
where we have define \(l = \{s, t\}\). The solution to equation \((36)\) is
\[
\sigma_s = \left[ a_s \ e^{L(t)} + b_s \ e^{-L(t)} \right], \tag{37}
\]
where
\[
L(t) = 4k \int \frac{dt}{l}.
\]
From \((35)\) we find that \(a_1 = a_2 = c_1\) and \(b_1 = -b_2 = c_2\), such that
\[
\sigma_3^2 - \sigma_1^2 = 4 \ c_1 c_2.
\]
The no dependence on \(t\) of \(\sigma_3^2 - \sigma_1^2\) implies that \(c_1 = c_1(s), \ c_2 = c_2(s)\) where \(2k \frac{\partial c_1 c_2}{\partial s} = -s\). So we obtain
\[
\sigma_1 = \left[ c_1(s) \ e^{L(t)} + c_2(s) \ e^{-L(t)} \right],
\]
\[
\sigma_2 = s,
\]
\[
\sigma_3 = \left[ c_1(s) \ e^{L(t)} - c_2(s) \ e^{-L(t)} \right],
\]
\[
L(t) = 4k \int \frac{dt}{l}, \quad 2k \frac{\partial c_1 c_2}{\partial s} = -s, \quad \{s, t\} = l. \tag{38}
\]
Observe that \(k(s), \ c_1(s)\) and \(c_2(s)\) are subjected to only one restriction, therefore two of them are arbitrary. So we have three arbitrary functions of \(p, q\) in general.
4 Final Remarks

In this paper we found an explicit exact class of solutions to self-dual gravity [4]. We used the chiral equations approach in order to obtain explicit solutions. Solving the chiral equations with the harmonic maps method we find that the harmonic maps ansatz can be applied to the chiral equations derived from self-dual gravity. The difference with previous applications of this method is that here we have Poisson brackets in place of matrix brackets in a similar spirit as in [1]. Nevertheless we can solve the corresponding Poisson algebra by making a coordinate transformation and finding the corresponding hamiltonian functions by solving the Poisson algebra as differential equations. We find that there exists a class of such solutions in terms of two arbitrary functions \((s, t)\) of two variables \((p, q)\). The coordinate transformation can be taken also arbitrary, but in the case when the new coordinates are canonical the solution becomes very simple.

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References

[1] A. Ashtekar, Phys. Rev. Lett. 57, 2244, (1986); Phys. Rev. D 36, 1587, (1987).

[2] A. Ashtekar, T. Jacobson and L. Smolin, Commun. Math. Phys. 115, 631, (1988).

[3] J.D.E. Grant, Phys. Rev. D 48, 2606, (1993).
[4] V. Husain, *Phys. Rev. Lett.* **72**, 800, (1994); *Class. Quantum Grav.* **11**, 927, (1994).

[5] J.F. Plebański, *J. Math. Phys.* **16**, 2395, (1975).

[6] J.D. Finley III, J.F. Plebański, M. Przanowski and H. García-Compeán, *Phys. Lett.* **A181**, 435, (1993).

[7] S. Chakravarty, L Mason and E. T. Newman, *J. Math. Phys.* **32**, 1458, (1991).

[8] T. Matos and R. Becerril, *Rev. Mex. Fis.* **38**, 69, (1992).

[9] T. Matos, ‘Exact Solutions of G-invariant Chiral Equations’ to be published in *Math. Notes* (1994).

[10] G. Neugebauer and D. Kramer. Stationary Axisymmetric Einstein-Maxwell fields generated by Bäcklund transformations. *Preprint, Jena Universität* (1990), Germany.

[11] E.G. Floratos, J. Iliopoulos and G. Tiktopoulos, *Phys. Lett.* **B217**, 285, (1989).

[12] T. Matos and J. F. Plebański, *Gen. Rel. Grav.* **26**, 477, (1994).