ON THE $S$–TRANSFORM OVER A BANACH ALGEBRA

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Abstract. The $S$–transform is shown to satisfy a specific twisted multiplicativity property for free random variables in a $B$–valued Banach noncommutative probability space, for an arbitrary unital complex Banach algebra $B$. Also, a new proof of the additivity of the $R$–transform in this setting is given.

1. Introduction and statement of the main result

Let $B$ be a unital complex Banach algebra. (In this paper, all Banach algebras will be over the complex numbers.) A $B$–valued Banach noncommutative probability space is a pair $(A, E)$ where $A$ is a unital Banach algebra containing an isometrically embedded copy of $B$ as a unital subalgebra and where $E : A \to B$ is a bounded projection satisfying the conditional expectation property

$$E(b_1ab_2) = b_1E(a)b_2 \quad (a \in A, b_1, b_2 \in B).$$

In the free probability theory of Voiculescu, see [7] and [10], elements $x$ and $y$ of $A$ are said to be free if their mixed moments $E(b_1a_1 \cdots b_na_n)$, where $a_j \in \{x, y\}$ and $b_j \in B$, are determined in a specific way from the moments of $x$ and of $y$. Of particular interest, for example to garner spectral data, are the symmetric moments

$$E(bxybx y \cdots bxy)$$

of the product $xy$, for $b \in B$.

In the case $B = \mathbb{C}$, Voiculescu [8] invented the $S$–transform of an element $x \in A$ satisfying $E(x) \neq 0$. The $S$–transform can be used to find the generating function for the symmetric moments [11] of $xy$ in terms of those for $x$ and $y$ individually, when $x$ and $y$ are free and when $E(x) \neq 0$ and $E(y) \neq 0$. In particular, Voiculescu showed that the $S$–transform is multiplicative:

$$S_{xy} = S_xS_y$$

when $x$ and $y$ are free.

In [9], Voiculescu gave a definition of an $S$–transform in the context of an arbitrary noncommutative probability space. However, this definition was quite complicated and involved differential equations.

Recently, Aagaard [11] took the straightforward extension of Voiculescu’s definition [8] of the scalar–valued $S$–transform to the Banach algebra situation and generalized Voiculescu’s result [2] to the case when $B$ is a commutative unital Banach algebra and $E(x)$ and $E(y)$ are invertible elements of $B$.

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In this paper, we treat the case when $B$ is an arbitrary unital Banach algebra. We make an improvement in Aagaard’s definition of the $S$–transform. For us, $S_x$ is a $B$–valued analytic function defined in a neighborhood of 0 in $B$. We write $S_{xy}$ in terms of $S_x$ and $S_y$ (again assuming $E(x)$ and $E(y)$ are invertible). Instead of simple multiplicativity (2), we have in general a twisted multiplicativity, as stated in our main theorem immediately below, which reduces to (2) when $B$ is commutative.

**Theorem 1.1.** Let $B$ be a unital complex Banach algebra and let $(A, E)$ be a $B$–valued Banach noncommutative probability space. Let $x, y \in A$ be free in $(A, E)$ and assume both $E(x)$ and $E(y)$ are invertible elements of $B$. Then

$$S_{xy}(b) = S_y(b)S_x(S_y(b)^{-1}bS_y(b)).$$

(3)

Our definition of the $S$–transform and our proof of Theorem 1.1 rely on the theory of analytic functions between Banach spaces – see for example Chapters III and XXVI of [5] and papers cited there.

In [3], Haagerup gave two new proofs of the multiplicativity of the $S$–transform in the case $B = C$. Our proof of Theorem 1.1 is very much inspired by one of Haagerup’s proofs, namely Theorem 2.3 of [3], which uses creation and annihilation operators in the full Fock space. In particular, we consider a $B$–valued Banach algebra analogue of the full Fock space and we construct random variables having arbitrary moments up to a given finite order, using analogues of the creation and annihilation operators. These are reminiscent of, though slightly different from, Voiculescu’s constructions in [9].

In §2 below, we define the $S$–transform $S_a$ (assuming the expectation of $a$ is invertible). Then, considering Taylor expansions about zero, we show that the $n$th order term in the expansion for $S_a$ depends only on the moments up to $n$th order of $a$. In §3 we construct operators analogous to the creation and annihilation operators on full Fock space, and we use these to prove the main result, Theorem 1.1. In §4 we offer a new proof of additivity of the $R$–transform over a Banach space, using the operators and techniques introduced in the preceding sections.

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**2. The $S$–transform in a Banach noncommutative probability space**

Let $B$ be a unital Banach algebra. For $n \geq 1$ we will let $B_n(B)$ denote the set of all bounded $n$–multilinear maps

$$\alpha_n : B \times \cdots \times B \rightarrow B,$$

where multilinearity means over $C$ and a multilinear map $\alpha_n$ is bounded if

$$\|\alpha_n\| := \sup\{\|\alpha_n(b_1, \ldots, b_n)\| : b_j \in B, \|b_1\|, \ldots, \|b_n\| \leq 1\} < \infty.$$ 

We say $\alpha_n$ is *symmetric* if it is invariant under arbitrary permutations of its $n$ arguments.
From the theory of analytic functions between complex Banach spaces, any $B$-valued analytic function $F$ defined on a neighborhood of zero in $B$ has an expansion

$$F(b) = F(0) + \sum_{n=1}^{\infty} F_n(b, \ldots, b),$$

(4)

for some symmetric multilinear functions $F_n \in B_n(B)$, with $\lim \sup_{n \to \infty} \|F_n\|^{1/n} < \infty$; see, for example, Theorem 3.17.1 of [5] and its proof. Here $F_1$ is just the Fréchet derivative of $F$ at 0 and the multilinear function $F_n$ appearing in (4) is $1/n!$ times the $n$th variation of $F$, i.e. $n!F_n(h_1, \ldots, h_n)$ is the $n$-fold Fréchet derivative taken with respect to increments $h_1, \ldots, h_n$. For convenience we will write $F_0$ for $F(0)$. We will refer to (4) as the power series expansion of $F(b)$ around 0 and to $F_n(b, \ldots, b)$ as the $n$th term in this power series expansion. Note that the full symmetric multilinear function $F_n$ can be recovered from knowing its diagonal $b \mapsto F_n(b, \ldots, b)$; for example, $n!F_n(b_1, \ldots, b_n)$ is the obvious partial derivative of $F_n(t_1b_1 + \cdots + t_n b_n, \ldots, t_1b_1 + \cdots + t_n b_n)$ at $(0, \ldots, 0)$, where $t_1, \ldots, t_n$ are real variables.

Let $(A, E)$ be a Banach noncommutative probability space over $B$, let $a \in A$ and suppose $E(a)$ is an invertible element of $B$. Consider the function

$$\Psi_a(b) = E((1 - ba)^{-1}) - 1 = \sum_{n=1}^{\infty} E((ba)^n),$$

(5)

defined for $\|b\| < \|a\|^{-1}$. Then $\Psi_a$ is Fréchet differentiable on its domain, i.e. is analytic there. We also have

$$\Psi_a(b) = b \Phi_a(b),$$

(6)

where

$$\Phi_a(b) = E(a(1 - ba)^{-1});$$

(7)

clearly $\Phi_a$ is analytic on the domain of $\Psi_a$. The Fréchet differential of $\Psi_a$ at $b = 0$ is easily found to be the bounded linear map

$$h \mapsto hE(a)$$

(8)

from $B$ to itself. By hypothesis, this linear map has bounded inverse $h \mapsto hE(a)^{-1}$. By the usual Banach space inverse function theorem, there are neighborhoods $U$ and $V$ of zero in $B$ such that $U$ lies in the domain of $\Psi_a$ and the restriction of $\Psi_a$ to $U$ is a homeomorphism onto $V$. Moreover, letting $\Psi_a^{(-1)}$ denote the inverse with respect to composition of the restriction of $\Psi_a$ to $U$, the function $\Psi_a^{(-1)}$ is Fréchet differentiable on its domain and is, therefore, analytic there.

**Lemma 2.1.** Assuming $E(a)$ is invertible, there is an open neighborhood of 0 in $B$ and unique analytic $B$-valued function $H_a$ defined there such that $\Psi_a^{(-1)}(b) = bH_a(b)$.
Proof. Uniqueness of $H_a$ is clear by uniqueness of power series expansions about zero. Let us show existence. Using (6), we seek $H_a$ such that $b H_a(b) \Phi_a(b H_a(b)) = b$, and it will suffice to find $H_a$ such that

$$H_a(b) \Phi_a(b H_a(b)) = 1.$$  

(9) 

The existence of $H_a$ follows from an easy application of the implicit function theorem for functions between Banach spaces, which is a result of Hildebrandt and Graves [4] (see also the discussion on p. 655 of [2]). Indeed, $H_a(0) = E(a) - 1$ is a solution of (9) at $b = 0$ and the Fréchet differential of the function $x \mapsto x \Phi_a(b x)$ at $b = 0$ is the map (8), which has bounded inverse. □

Definition 2.2. Let $a \in A$ and assume $E(a)$ is invertible. The $S$–transform of $a$ is the $B$–valued analytic function

$$S_a(b) = (1 + b) H_a(b),$$

which defined in some neighborhood of 0 in $B$, where $H_a$ is the function from Lemma 2.1.

Note that $S_a(0) = E(a)^{-1}$.

We may write

$$S_a(b) = (1 + b) b^{-1} \Psi_a^{-1}(b),$$

(11) which is the same formula given by Voiculescu [8] and used by Aagaard [1]. In the case $B = C$, the definition (11) yields, of course, the same function as Voiculescu’s S–transform. Moreover, the only difference between the definition (11) and the one appearing in [1] is that we have used the implicit function theorem to show that (11) makes sense for all $b$ in a neighborhood of zero.

If $F$, $G$ and $H$ are $B$–valued analytic functions defined on neighborhoods of 0 in $B$, then the product $FG$ is analytic and, if $H(0) = 0$, also the composition $F \circ H$ is analytic in some neighborhood of 0 in $B$. Straightforward asymptotic analysis yields the following formulas for the diagonals of the multilinear functions appearing in the power series expansions of $FG$ and $F \circ H$.

Lemma 2.3. We have for $n \geq 0$

$$(FG)_n(b, \ldots, b) = \sum_{k=0}^{n} F_k(b, \ldots, b) G_{n-k}(b, \ldots, b)$$

(12) and for $n \geq 1$

$$(F \circ H)_n(b, \ldots, b) = \sum_{k=1}^{n} \sum_{p_1, \ldots, p_k \geq 1 \atop p_1 + \cdots + p_k = n} F_k(H_{p_1}(b, \ldots, b), \ldots, H_{p_k}(b, \ldots, b)).$$

(13)

Lemma 2.4. Let $F$ be analytic in a neighborhood of 0. If $F(0)$ is an invertible element of $B$, then $G(b) = F(b)^{-1}$ defines a function that is analytic in a neighborhood of 0, and the $n$th term of its power series expansion is $G_n = F_n^{-1}$ and, for $n \geq 1$,

$$G_n(b, \ldots, b) = -F_0^{-1} \sum_{k=1}^{n} F_k(b, \ldots, b) G_{n-k}(b, \ldots, b).$$

(14)
On the other hand, if $F(0) = 0$ and if $F_1$ has a bounded inverse, then $F$ has an inverse with respect to composition, denoted $F^{(-1)}$, that is analytic in a neighborhood of 0. Taking $H = F^{(-1)}$, we have $H_1 = (F_1)^{(-1)}$ and, for $n \geq 2$,

$$H_n(b, \ldots, b) = - (F_1)^{(-1)} \left( \sum_{k=2}^{n} \sum_{p_1, \ldots, p_k \geq 1} F_k(H_{p_1}(b, \ldots, b), \ldots, H_{p_k}(b, \ldots, b)) \right). \quad (15)$$

**Proof.** Assuming $F(0)$ is invertible, that $G(b) = F(b)^{-1}$ is analytic is clear, and we have $(FG)_0 = 1$ and $(FG)_n = 0$ for $n \geq 1$. Now the expression (14) results from solving (12) for $G_n$.

If $F(0) = 0$ and the Fréchet derivative $F_1$ of $F$ at 0 has bounded inverse, then by the inverse function theorem for Banach spaces, $F$ has an inverse with respect to composition $F^{(-1)}$ that is analytic in a neighborhood of 0. Taking $H = F^{(-1)}$, we have $(F \circ H)_1 = \text{id}_B$ and $(F \circ H)_n = 0$ for all $n \geq 2$. Solving in (13) for $H_n$ yields the expression (15).

Consider an element $a \in A$ as at the beginning of this section. We say the $n$th moment function of $a$ is the multilinear function $\mu_{a,n} \in B_n(B)$ given by

$$\mu_{a,n}(b_1, \ldots, b_n) = E(b_1a b_2 a \cdots b_n a).$$

**Proposition 2.5.** Assume $E(a)$ is an invertible element of $B$. Then the $n$th term $(S_a)_n(b, \ldots, b)$ in the power series expansion of the $S$–transform $S_a$ of $a$ about zero depends only on the first $n$ moment functions $\mu_{a,1}, \mu_{a,2}, \ldots, \mu_{a,n}$ of $a$.

**Proof.** The symmetric $n$–multilinear function $(\Psi_a)_n$ appearing in the power series expansion of $\Psi_a$ is the symmetrization of $\mu_{a,n}$. Using Lemma 2.4, we see that the $n$th term $(\Psi_a^{(-1)})_n(b, \ldots, b)$ in the power series expansion of $\Psi_a^{(-1)}(b)$ around 0 depends only on $\mu_{a,1}, \ldots, \mu_{a,n}$. But

$$(\Psi_a^{(-1)})_n(b, \ldots, b) = b (H_a)_n(b, \ldots, b)$$

and

$$(S_a)_n(b, \ldots, b) = (1 + b) (H_a)_n(b, \ldots, b)$$

and the result is proved.

### 3. Twisted multiplicativity of the $S$–transform

Let $B$ be a unital Banach algebra over $\mathbb{C}$ and let $I$ be a set. Let $D = \ell^1(I, B)$ be the Banach space of all functions $d : I \to B$ such that $\|d\| := \sum_{i \in I} \|d(i)\| < \infty$. For $i \in I$, $\delta_i \in D$ will denote the function taking value 1 at $i$ and 0 at all other elements of $I$. We have the obvious left action of $B$ on $D$ by $(bd)(i) = bd(i)$, and the resulting algebra homomorphism $B \to \mathcal{B}(D)$ is isometric. (Whenever $X$ is a Banach space, we denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators from $X$ to itself.) For $k \geq 1$, let $D^\otimes k = D \otimes \cdots \otimes D$ be the $k$–fold Banach space projective tensor product of $D$ with itself (over the complex field). Consider the Banach space

$$\mathcal{F} = B \Omega \oplus \bigoplus_{k=1}^{\infty} D^\otimes k \otimes B, \quad (16)$$

where
where also \( \hat{\otimes} \) is the Banach space projective tensor product and where we take the direct sum with respect to the \( \ell^1 \)-norm. Here, \( B\Omega \) signifies just a copy of \( B \) and \( \Omega \) denotes the identity element of this copy of \( B \), considered as a vector in \( \mathcal{F} \). Let \( \lambda : B \to \mathcal{B}(\mathcal{F}) \) be the map defined by

\[
\lambda(b)(b_0\Omega) = (bb_0)\Omega \\
\lambda(b)(d_1 \otimes \cdots \otimes d_k \otimes b_0) = (bd_1) \otimes d_2 \otimes \cdots \otimes d_k \otimes b_0
\]

for \( k \in \mathbb{N}, d_1, \ldots, d_k \in D \) and \( b_0 \in B \). Then \( \lambda \) is an isometric algebra homomorphism. We will often omit to write \( \lambda \), and just think of \( B \) as included in \( \mathcal{B}(\mathcal{F}) \) by this left action.

**Remark 3.1.** For specificity, we took the \( \ell^1 \) norms in the definitions of \( D \) and \( \mathcal{F} \), but we actually have considerable flexibility. For \( D \) we need only a Banach space completion of the set of all functions \( d : I \to B \) vanishing at all but finitely many elements in \( I \) with the property \( \|b\delta_i\| = \|b\| \), and similarly for \( \mathcal{F} \). Moreover, we could replace the projective tensor norm \( \hat{\otimes} \) in \( B\Omega \) with any tensor norm so that \( \|d \otimes B\| = \|d\| \|b\| \) for all \( d \in D^{\hat{\otimes}k} \) and \( b \in B \).

Let \( P : \mathcal{F} \to B \) be the projection onto the summand \( B\Omega = B \) that sends all summands \( D^{\hat{\otimes}k} \otimes B \) to zero and let \( \mathcal{E} : \mathcal{B}(\mathcal{F}) \to B \) be \( \mathcal{E}(X) = P(X\Omega) \). Then \( \mathcal{E} \) has norm 1 and satisfies \( \mathcal{E} \circ \lambda = \text{id}_B \). Let \( \rho : B \to \mathcal{B}(\mathcal{F}) \) be the map defined by

\[
\rho(b)(b_0\Omega) = (bb_0)\Omega \\
\rho(b)(d_1 \otimes \cdots \otimes d_k \otimes b_0) = (d_1 \otimes \cdots \otimes d_k \otimes (bb_0)).
\]

Then \( \rho \) is an isometric algebra isomorphism from the opposite algebra \( B^{\text{op}} \) into \( \mathcal{B}(\mathcal{F}) \). Let \( \mathcal{B}(\mathcal{F}) \cap \rho(B)' \) denote the set of all bounded operators on \( \mathcal{F} \) that commute with \( \rho(b) \) for all \( b \in B \). Note that \( \lambda(B) \subseteq \mathcal{B}(\mathcal{F}) \cap \rho(B)' \).

**Proposition 3.2.** The restriction of \( \mathcal{E} \) to \( \mathcal{B}(\mathcal{F}) \cap \rho(B)' \) satisfies the conditional expectation property

\[
\mathcal{E}(b_1Xb_2) = b_1\mathcal{E}(X)b_2 \quad (X \in \mathcal{B}(\mathcal{F}) \cap \rho(B)', b_1, b_2 \in B).
\]

**Proof.** We have

\[
\mathcal{E}(b_1Xb_2) = P(\lambda(b_1)X\lambda(b_2)\Omega) = P(\lambda(b_1)X\rho(b_2)\Omega) \\
= P(\rho(b_2)\lambda(b_1)X\Omega) = P(\lambda(b_1)X\Omega)b_2 = b_1P(X\Omega)b_2 = b_1\mathcal{E}(X)b_2.
\]

\[\Box\]

For \( i \in I \), let \( L_i \in \mathcal{B}(\mathcal{F}) \) be defined by

\[
L_i(b_0\Omega) = \delta_i \otimes b_0 \\
L_i(d_1 \otimes \cdots \otimes d_k \otimes b_0) = \delta_i \otimes d_1 \otimes \cdots \otimes d_k \otimes b_0.
\]

Thus,

\[
b_1\delta_{i_1} \otimes b_2\delta_{i_2} \otimes \cdots \otimes b_k\delta_{i_k} \otimes b_0 = b_1L_{i_1}b_2L_{i_2} \cdots b_kL_{i_k}b_0\Omega.
\]
Recall that $\mathcal{B}_n(B)$ denotes the set of all bounded multilinear functions from the $n$-fold product of $B$ to $B$. We will also let $\mathcal{B}_0(B) = B$. If $i \in I$, $n \in \mathbb{N}$ and $\alpha_n \in \mathcal{B}_n(B)$, define $V_{i,n}(\alpha_n)$ and $W_{i,n}(\alpha_n)$ in $\mathcal{B}(\mathcal{F})$ by

$$V_{i,n}(\alpha_n)(b_0 \Omega) = 0$$

$$V_{i,n}(\alpha_n)(d_1 \otimes \cdots \otimes d_k \otimes b_0) = \begin{cases} 0, & k < n \\ \alpha_n(d_1(i), \ldots, d_n(i))b_0 \Omega, & k = n \\ \alpha_n(d_1(i), \ldots, d_n(i))d_{n+1} \otimes \cdots \otimes d_k \otimes b_0, & k > n \end{cases}$$

and

$$W_{i,n}(\alpha_n)(b_0 \Omega) = 0$$

$$W_{i,n}(\alpha_n)(d_1 \otimes \cdots \otimes d_k \otimes b_0) = \begin{cases} 0, & k < n \\ \alpha_n(d_1(i), \ldots, d_n(i))\delta_i \otimes b_0, & k = n \\ \alpha_n(d_1(i), \ldots, d_n(i))\delta_i \otimes d_{n+1} \otimes \cdots \otimes d_k \otimes b_0, & k > n \end{cases}$$

Finally, taking $n = 0$ and $\alpha_0 \in B$, let

$$V_{i,0}(\alpha_0) = \alpha_0 \quad W_{i,0}(\alpha_0) = \alpha_0 L_i.$$ 

These formulas are guaranteed to define bounded operators on $\mathcal{F}$, because we took the projective tensor product in $D^{\otimes k}$. The expression $V_{i,n}(\alpha_n)$, $n \geq 1$, is a sort of $n$-fold annihilation operator, while $W_{i,n}(\alpha_n)$ is $n$-fold annihilation combined with single creation, and, of course, $W_{i,0}$ is a single creation operator. Note that in all cases we have $V_{i,n}(\alpha_n), W_{i,n}(\alpha_n) \in \mathcal{B}(\mathcal{F}) \cap \rho(B)'$.

The relations gathered in the following lemma are easily verified.

**Lemma 3.3.** Let $n, m \in \mathbb{N}$ and $\alpha_n \in \mathcal{B}_n(B)$, $\beta_m \in \mathcal{B}_m(B)$ and take $b \in B$. Then

(i) $V_{i,n}(\alpha_n)\lambda(b) = V_{i,n}(\tilde{\alpha}_n)$ \quad $W_{i,n}(\alpha_n)\lambda(b) = W_{i,n}(\tilde{\alpha}_n),$ 

where

$$\tilde{\alpha}_n(b_1, \ldots, b_n) = \alpha_n(bb_1, b_2, \ldots, b_n);$$

(ii) if $n = 1$, then

$$V_{i,1}(\alpha_1)L_i = \lambda(\alpha_1(1)), \quad W_{i,1}(\alpha_1)L_i = \lambda(\alpha_1(1))L_i$$

and for $n \geq 2$ we have

$$V_{i,n}(\alpha_n)L_i = V_{i,n-1}(\tilde{\alpha}_{n-1}), \quad W_{i,n}(\alpha_n)L_i = W_{i,n-1}(\tilde{\alpha}_{n-1}),$$

where here

$$\tilde{\alpha}_{n-1}(b_1, \ldots, b_{n-1}) = \alpha_n(1, b_1, \ldots, b_{n-1});$$

(iii) we have

$$V_{i,n}(\alpha_n)V_{i,m}(\beta_m) = V_{i,n+m}(\gamma_{n+m}), \quad W_{i,n}(\alpha_n)V_{i,m}(\beta_m) = W_{i,n+m}(\gamma_{n+m}),$$

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where
\[ \gamma_{n+m}(b_1, \ldots, b_{m+n}) = \alpha_n(\beta_m(b_1, \ldots, b_m)b_{m+1}, b_{m+2}, \ldots, b_{m+n}); \]

(iv)
\[ V_{i,n}(\alpha_n)W_{i,m}(\beta_m) = V_{i,n+m-1}(\gamma_{n+m-1}), \]
\[ W_{i,n}(\alpha_n)W_{i,m}(\beta_m) = W_{i,n+m-1}(\gamma_{n+m-1}); \]

where
\[ \gamma_{n+m-1}(b_1, \ldots, b_{m+n-1}) = \alpha_n(\beta_m(b_1, \ldots, b_m), b_{m+1}, b_{m+2}, \ldots, b_{m+n-1}); \]

(v)
\[ \lambda(b)V_{i,n}(\alpha_n) = V_{i,n}(b\alpha_n), \]

(vi) if \( i' \neq i \) and \( n \geq 1 \), then
\[ V_{i,n}(\alpha_n)L_{i'} = 0 = W_{i,n}(\alpha_n)L_{i'}. \]

**Proposition 3.4.** For \( i \in I \) let \( \mathfrak{A}_i \subseteq \mathcal{B}(\mathcal{F}) \cap \rho(B)' \) be the subalgebra generated by
\[ \lambda(B) \cup \{ L_i \} \cup \{ V_{i,n}(\alpha_n) \mid n \in \mathbb{N}, \alpha_n \in \mathcal{B}_n(B) \} \cup \{ W_{i,n}(\alpha_n) \mid n \in \mathbb{N}, \alpha_n \in \mathcal{B}_n(B) \}. \]
Then the family \( \{ \mathfrak{A}_i \}_{i \in I} \) is free with respect to \( \mathcal{E} \).

**Proof.** Using Lemma 3.3 we see that every element of \( \mathfrak{A}_i \) can be written as a sum of finitely many terms of the following forms:

(i) \( \lambda(b) \)
(ii) \( \lambda(b_0)L_i\lambda(b_1) \cdots L_i\lambda(b_n) \)
(iii) \( V_{i,n}(\alpha_n) \)
(iv) \( \lambda(b_0)L_i\lambda(b_1)L_i \cdots L_i\lambda(b_k)L_iV_{i,n}(\alpha_n) \)
(v) \( \lambda(b)W_{i,n}(\alpha_n) \)
(vi) \( \lambda(b_0)L_i\lambda(b_1)L_i \cdots L_i\lambda(b_{k-1})L_i\lambda(b_k)V_{i,n}(\alpha_n). \)

Now all terms of the forms (ii)–(vi) lie in \( \ker \mathcal{E} \), while \( \mathcal{E}(\lambda(b)) = b \). Therefore, \( \mathfrak{A}_i \cap \ker \mathcal{E} \) is the set of all finite sums of terms of the forms (ii)–(vi).

Let \( p \in \mathbb{N} \) with \( p \geq 2 \) and take \( i_1, \ldots, i_p \in I \) with \( i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{p-1} \neq i_p \). Suppose \( a_j \in \mathfrak{A}_i \cap \mathcal{E} \) (\( 1 \leq j \leq p \)) and let us show \( \mathcal{E}(a_1 \cdots a_p) = 0 \). From Lemma 3.3 part (vi), we see \( a_1a_2 \cdots a_p = 0 \) unless either \( \forall j \ b_j \) is of the form (ii) or \( \forall j \ a_j \) is of the form (iii) or (v). But \( V_{i,n}(\alpha_n)\Omega = 0 = W_{i,n}(\alpha_n)\Omega \) when \( n \geq 1 \), so if \( a_p \) is of the form (iii) or (v), then \( \mathcal{E}(a_1 \cdots a_p) = 0 \). We are left to consider the case when \( a_1 \cdots a_p \) can be written as
\[ (\lambda(b_0)L_i\lambda(b_1^{(1)})L_iL_i\lambda(b_2^{(1)})L_i \cdots L_i\lambda(b_{k_1}^{(1)}))(L_iL_i\lambda(b_1^{(2)})L_i \cdots L_iL_i\lambda(b_{k_2}^{(2)})) \cdots \]
\[ \cdots (L_iL_i\lambda(b_1^{(p)})L_i \cdots L_iL_i\lambda(b_{k_p}^{(p)})), \]
where all \( k(j) \geq 1 \). But in this case, clearly \( \mathcal{E}(a_1 \cdots a_p) = 0. \) \( \Box \)
Lemma 3.5. Let \( N \in \mathbb{N} \) and for every \( n \in \{0,1,\ldots,N\} \) let \( \alpha_n \in \mathcal{B}_n(B) \). Fix \( i \in I \) and let
\[
X = \sum_{n=0}^{N-1} (V_{i,n}(\alpha_n) + W_{i,n}(\alpha_n))
\]
\[
Y = X + V_{i,N}(\alpha_N) + W_{i,N}(\alpha_N).
\]
Then for any \( b_0,\ldots,b_N \in B \), we have
\[
\mathcal{E}(b_0Yb_1Y\cdots b_NY) = b_0\alpha_N(b_1\alpha_0, b_2\alpha_0, \ldots, b_N\alpha_0) + \mathcal{E}(b_0Xb_1X\cdots b_NX).
\]

Proof. To evaluate \( \mathcal{E}(b_0Yb_1Y\cdots b_NY) \), first write
\[
Y = \sum_{n=0}^{N} (V_{i,n}(\alpha_n) + W_{i,n}(\alpha_n))
\]
and distribute. Now using the creation and annihilation properties of the \( W_{i,n}(\alpha_n) \) and \( V_{i,n}(\alpha_n) \) operators, we see that the only term involving \( \alpha_N \) to contribute a possibly nonzero quantity to \( \mathcal{E}(b_0Yb_1Y\cdots b_NY) \) is
\[
\mathcal{E}(b_0V_{i,N}(\alpha_N)b_1W_{i,0}(\alpha_0)\cdots b_NW_{i,0}(\alpha_0)),
\]
whose value is \( b_0\alpha_N(b_1\alpha_0, b_2\alpha_0, \ldots, b_N\alpha_0) \). The other terms involve only \( \alpha_0,\ldots,\alpha_{N-1} \) and their sum is \( \mathcal{E}(b_0Xb_1X\cdots b_NX) \).

Proposition 3.6. Let \((A, E)\) be a \( B \)-valued Banach noncommutative probability space and let \( a \in A \), \( N \in \mathbb{N} \). Suppose \( E(a) \) is an invertible element of \( B \). Let \( \alpha_0 = E(a) \). Then there are \( \alpha_1,\ldots,\alpha_N \), with \( \alpha_n \in \mathcal{B}_n(B) \), such that if
\[
X = \sum_{n=0}^{N} (V_{i,n}(\alpha_n) + W_{i,n}(\alpha_n)) \in \mathcal{B}(\mathcal{F}),
\]
then
\[
\mathcal{E}(b_0Xb_1X\cdots b_kX) = E(b_0ab_1a\cdots b_ka) \tag{17}
\]
for all \( k \in \{1,\ldots,N\} \) and all \( b_0,\ldots,b_N \in B \).

Proof. Using Lemma 3.5. The maps \( \alpha_k \) can be chosen recursively in \( k \) so that (17) holds.

For the remainder of this section, we take \( I = \{1,2\} \).

Lemma 3.7. Let \( \alpha_0 \in B \) be invertible. Let \( N \in \mathbb{N} \) and choose \( \alpha_n \in \mathcal{B}_n(B) \) for \( n \in \{1,\ldots,N\} \), and let
\[
F(b) = \alpha_0 + \sum_{n=1}^{N} \alpha_n(b,\ldots,b).
\]
Note that \( F(b) \) is invertible for \( \|b\| \) sufficiently small. Let
\[
X = \sum_{n=0}^{N} (V_{i,n}(\alpha_n) + W_{1,n}(\alpha_n)) \in \mathcal{B}(\mathcal{F}). \tag{18}
\]
Then the $S$-transform of $X$ is $S_X(b) = F(b)^{-1}$.

**Proof.** For $b \in B$, $\|b\| < 1$, let

$$\omega_b = \Omega + \sum_{k=1}^{\infty} (b\delta_1)^{\otimes k} \otimes 1 \in \mathcal{F}$$

We have $V_{1,0}(\alpha_0)\omega_b = \alpha_0 \omega_b$ and, for $n \geq 1$,

$$V_{1,n}(\alpha_n)\omega_b = \alpha_n(b, \ldots, b)\Omega + \sum_{k=n+1}^{\infty} \alpha_n(b, \ldots, b)(b\delta_1)^{\otimes (k-n)} \otimes 1 = \alpha_n(b, \ldots, b)\omega_b.$$ 

Moreover, $W_{1,0}(\alpha_0)\omega_b = \alpha_0 L_1 \omega_b$ and, for $n \geq 1$,

$$W_{1,n}(\alpha_n)\omega_b = \alpha_n(b, \ldots, b) \delta_1 \otimes 1 + \sum_{k=n+1}^{\infty} \alpha_n(b, \ldots, b) \delta_1 \otimes (b\delta_1)^{\otimes (k-n)} \otimes 1 = \alpha_n(b, \ldots, b)L_1 \omega_b.$$ 

Thus,

$$X\omega_b = F(b)(1 + L_1)\omega_b.$$ 

For $\|b\|$ sufficiently small, we get

$$F(b)^{-1}X\omega_b = \omega_b + L_1 \omega_b$$

$$bF(b)^{-1}X\omega_b = b\omega_b + (\omega_b - \Omega)$$

$$\Omega = (1 + b)\omega_b - bF(b)^{-1}X\omega_b$$

$$\Omega = (1 - bF(b)^{-1}X(1 + b)^{-1})(1 + b)\omega_b$$

$$(1 - bF(b)^{-1}X(1 + b)^{-1})^{-1}\Omega = (1 + b)\omega_b$$

$$\mathcal{E}((1 - bF(b)^{-1}X(1 + b)^{-1})^{-1}) = P((1 + b)\omega_b)$$

$$= 1 + b.$$ 

Conjugating with $(1 + b)$ yields

$$1 + b = \mathcal{E}((1 - (1 + b)^{-1}bF(b)^{-1}X)^{-1}) = 1 + \Psi_X((1 + b)^{-1}bF(b)^{-1}).$$ 

Hence,

$$\Psi_X^{(-1)}(b) = (1 + b)^{-1}bF(b)^{-1}$$

and $S_X(b) = F(b)^{-1}$. \hfill $\square$

**Lemma 3.8.** Let $\alpha_0, \ldots, \alpha_n$, $F$ and $X$ be as in Lemma 3.7. Let $\beta_0 \in B$ be invertible and let $\beta_n \in \mathcal{B}_n(B)$ for $n \in \{1, \ldots, N\}$. Let

$$G(b) = \beta_0 + \sum_{n=1}^{N} \beta_n(b, \ldots, b)$$

and let

$$Y = \sum_{n=0}^{N} (V_{2,n}(\alpha_n) + W_{2,n}(\alpha_n)) \in \mathcal{B}(\mathcal{F}). \quad (19)$$
Then the $S$–transform of $XY$ is

$$S_{XY}(b) = G(b)^{-1}F(G(b)bG(b)^{-1})^{-1} = S_{Y}(b)S_{X}(S_{Y}(b)^{-1}bS_{Y}(b)).$$ \hspace{1cm} (20)

**Proof.** From Lemma 3.7, we have $S_{Y}(b) = G(b)^{-1}$ and $S_{X}(b) = F(b)^{-1}$, so the final equality in (20) is true. For $b \in B$ let

$$Z_{b} = bL_{2} + bG(b)^{-1}L_{1}G(b) + bG(b)^{-1}L_{1}G(b)L_{2} \in \mathcal{B}(\mathcal{F})$$

and insist that $\|b\|$ be so small that $\|Z_{b}\| < 1$. Let

$$\sigma_{b} = (1 - Z_{b})^{-1}\Omega = \Omega + \sum_{k=1}^{\infty} Z_{b}^{k} \Omega.$$ 

Using Lemma 3.3 we find for $n, k \geq 0$,

$$V_{2,n}(\beta_{n})Z_{b}^{k} = \begin{cases} 
V_{2,n-k}(\tilde{\beta}_{n-k}), & k < n, \\
\beta_{n}(b, \ldots, b), & k = n, \\
\beta_{n}(b, \ldots, b)Z_{b}^{k-n}, & k > n
\end{cases}$$

and

$$W_{2,n}(\beta_{n})Z_{b}^{k} = \begin{cases} 
W_{2,n-k}(\tilde{\beta}_{n-k}), & k < n, \\
\beta_{n}(b, \ldots, b)L_{2}, & k = n, \\
\beta_{n}(b, \ldots, b)L_{2}Z_{b}^{k-n}, & k > n
\end{cases}$$

where

$$\tilde{\beta}_{n-k}(b_{1}, \ldots, b_{n-k}) = \beta_{n}(b, \ldots, b, b_{1}, \ldots, b_{n-k}).$$

Therefore,

$$V_{2,n}(\beta_{n})Z_{b}^{k}\Omega = \begin{cases} 
0, & k < n, \\
\beta_{n}(b, \ldots, b)\Omega, & k = n, \\
\beta_{n}(b, \ldots, b)Z_{b}^{k-n}\Omega, & k > n
\end{cases}$$

and

$$W_{2,n}(\beta_{n})Z_{b}^{k}\Omega = \begin{cases} 
0, & k < n, \\
\beta_{n}(b, \ldots, b)L_{2}\Omega, & k = n, \\
\beta_{n}(b, \ldots, b)L_{2}Z_{b}^{k-n}\Omega, & k > n
\end{cases}$$

and we get

$$Y\sigma_{b} = G(b)(1 + L_{2})\sigma_{b}.$$ 

Letting $b' = G(b)bG(b)^{-1}$, we similarly find for $n, k \geq 0$,

$$V_{1,n}(\alpha_{n})G(b)Z_{b}^{k} = \begin{cases} 
V_{1,n-k}(\tilde{\alpha}_{n-k})G(b), & k < n, \\
\alpha_{n}(b', \ldots, b')G(b)(1 + L_{2}), & k = n, \\
\alpha_{n}(b', \ldots, b')G(b)(1 + L_{2})Z_{b}^{k-n}, & k > n
\end{cases}$$

and

$$W_{1,n}(\alpha_{n})G(b)Z_{b}^{k} = \begin{cases} 
W_{1,n-k}(\tilde{\alpha}_{n-k})G(b), & k < n, \\
\alpha_{n}(b', \ldots, b')L_{1}G(b)(1 + L_{2}), & k = n, \\
\alpha_{n}(b', \ldots, b')L_{1}G(b)(1 + L_{2})Z_{b}^{k-n}, & k > n
\end{cases}$$
\[
\tilde{\alpha}_{n-k}(b_1, \ldots, b_{n-k}) = \alpha_n(b'_1, \ldots, b'_{k}, b_1, \ldots, b_{n-k}).
\]

Therefore, we get
\[
XY\sigma_b = F(b')(1 + L_1)G(b)(1 + L_2)\sigma_b.
\]

Thus, for \(\|b\|\) sufficiently small we get
\[
F(b')^{-1}XY = (1 + L_1)G(b)(1 + L_2)\sigma_b \\
F(b')^{-1}XY = G(b)\sigma_b + (G(b)L_2 + L_1G(b) + L_1G(b)L_2)\sigma_b \\
bG(b)^{-1}F(b')^{-1}XY\sigma_b = b\sigma_b + Z_b\sigma_b \\
bG(b)^{-1}F(b')^{-1}XY\sigma_b = b\sigma_b + (\sigma_b - \Omega) \\
\Omega = ((1 + b) - bG(b)^{-1}F(b')^{-1}XY)\sigma_b \\
\Omega = (1 - bG(b)^{-1}F(b')^{-1}XY(1 + b)^{-1})(1 + b)\sigma_b \\
(1 - bG(b)^{-1}F(b')^{-1}XY(1 + b)^{-1})\Omega = (1 + b)\sigma_b \\
E((1 - bG(b)^{-1}F(b')^{-1}XY(1 + b)^{-1}) = P((1 + b)\sigma_b) \\
= 1 + b.
\]

Conjugating with \((1 + b)\) yields
\[
\Psi_{XY}((1 + b)^{-1}bG(b)^{-1}F(b')^{-1}) = E((1 - (1 + b)^{-1}bG(b)^{-1}F(b')^{-1}XY)^{-1}) - 1 = b.
\]

Hence,
\[
\Psi_{XY}^{-1}(b) = (1 + b)^{-1}bG(b)^{-1}F(b')^{-1}
\]

and (20) holds. \(\square\)

**Proof of Theorem 1.1** The formula (3) asserts the equality of the germs of two analytic \(B\)-valued functions. This is equivalent to asserting the equality of the \(n\)th terms in their respective power series expansions around zero, for every \(n \geq 0\). By Lemmas 2.3 and 2.4 the \(n\)th term, call it RHS\(_n\), in the expansion for the right hand side of (3) depends only on the 0th through the \(n\)th terms of the power series expansions for \(S_x(b)\) and \(S_y(b)\). Hence, by Proposition 2.5 RHS\(_n\) depends only on the moment functions \(\mu_{x,1}, \ldots, \mu_{x,n}\) and \(\mu_{y,1}, \ldots, \mu_{y,n}\). On the other hand, again by Proposition 2.5 the \(n\)th term in the power series expansion for the left hand side of (3), call it LHS\(_n\), depends only on \(\mu_{xy,1}, \ldots, \mu_{xy,n}\). But by freeness of \(x\) and \(y\), for each \(k \geq 1\) the moment function \(\mu_{xy,k}\) depends only on \(\mu_{x,1}, \ldots, \mu_{x,k}\) and \(\mu_{y,1}, \ldots, \mu_{y,k}\). Thus, both LHS\(_n\) and RHS\(_n\) depend only on \(\mu_{x,1}, \ldots, \mu_{x,n}\) and \(\mu_{y,1}, \ldots, \mu_{y,n}\).

Hence, in order to prove (3) at the level of the \(n\)th terms in the power series expansion, it will suffice to prove (3) for some free pair \(X\) and \(Y\) of elements in a Banach noncommutative probability space over \(B\), whose first \(n\) moment functions agree with those of \(x\) and \(y\), respectively. However, by Propositions 3.3 and 3.6 such \(X\) and \(Y\) can be chosen of the forms (18) and (19). By Lemma 3.8 the equality (3) holds for these operators. \(\square\)
4. A proof of the additivity of the $R$–transform over a Banach algebra

The $R$–transform over a general unital algebra $B$ has been well understood since Voiculescu’s work [9] (and see also Speicher’s approach in [6]). However, for completeness, in this section we offer a new proof, using the techniques and constructions of the previous two sections, of the additivity of the $R$–transform for free random variables in a Banach noncommutative probability space. This proof is, of course, analogous to Haagerup’s proof of Theorem 2.2 of [3] in the scalar–valued case.

Let $(A, E)$ be a Banach noncommutative probability space over $B$ and let $a \in A$. Consider the function

$$C_a(b) = E((1 - ba)^{-1}b) = \sum_{n=0}^{\infty} E((ba)^n b),$$

defined and analytic for $\|b\| < \|a\|^{-1}$. We have $C_a(b) = b + b\Phi_a(b)b$, where $\Phi_a$ is as in [7]. Since the Fréchet differential of $C_a$ at $b = 0$ is the identity map, $C_a$ is invertible with respect to composition in a neighborhood of zero.

**Proposition 4.1.** There is a unique $B$–valued analytic function $R_a$, defined in a neighborhood of $0$ in $B$, such that

$$C_a^{(-1)}(b) = (1 + bR_a(b))^{-1}b = b(1 + R_a(b)b)^{-1}. \tag{21}$$

**Proof.** Again, uniqueness is clear by the power series expansions.

The right–most equality in (21) holds for any analytic function $R_a$. We seek a function $R_a$ such that

$$C_a((1 + bR_a(b))^{-1}b) = b.$$

But

$$C_a((1 + bR_a(b))^{-1}b) = (1 + bR_a(b))^{-1}b$$

$$+ (1 + bR_a(b))^{-1}b \Phi_a((1 + bR_a(b))^{-1}b) (1 + bR_a(b))^{-1}b,$$

so it will suffice to find $R_a$ so that any of the following hold:

$$(1 + bR_a(b))^{-1} + (1 + bR_a(b))^{-1}b \Phi_a((1 + bR_a(b))^{-1}b) (1 + bR_a(b))^{-1} = 1,$$

$$1 + b \Phi_a((1 + bR_a(b))^{-1}b) (1 + bR_a(b))^{-1} = 1 + bR_a(b),$$

$$b \Phi_a((1 + bR_a(b))^{-1}b) (1 + bR_a(b))^{-1} = bR_a(b),$$

$$\Phi_a((1 + bR_a(b))^{-1}b) (1 + bR_a(b))^{-1} = R_a(b). \tag{22}$$

However, $R_a(0) = E(a)$ is a solution of (22) at $b = 0$, and the Fréchet differential of the function $x \mapsto \Phi_a((1 + bx)^{-1}b)(1 + bx)^{-1} - x$ at $b = 0$ is the negative of the identity map, hence is invertible. The implicit function theorem of Hildebrandt and Graves [4] (see also the discussion on p. 655 of [2]) guarantees the existence of $R_a$. □

The $R$–transform of $a$ is defined to be the analytic function $R_a$ from Proposition 4.1. Analogously to Proposition 2.5, we have the following.

**Proposition 4.2.** The $n$th term $(R_a)_n(b, \ldots, b)$ in the power series expansion for $R_a$ about zero depends only on the first $n + 1$ moment functions $\mu_{a,1}, \ldots, \mu_{a,n+1}$ of $a$. 
Here is the analogue to Lemma 3.5, which can be proved similarly.

**Lemma 4.3.** Let $N \in \mathbb{N}$ and for every $n \in \{0, 1, \ldots, N\}$ let $\alpha_n \in \mathcal{B}_n(B)$. Fix $i \in I$ and let

$$X = L_i + \sum_{n=0}^{N-1} V_{i,n}(\alpha_n)$$

$$Y = X + V_{i,N}(\alpha_N).$$

Then for any $b_0, \ldots, b_N \in B$, we have

$$\mathcal{E}(b_0Yb_1Y \cdots b_NY) = b_0\alpha_N(b_1, b_2, \ldots, b_N) + \mathcal{E}(b_0Xb_1X \cdots b_NX).$$

We immediately get the following analogue of Proposition 3.6.

**Proposition 4.4.** Let $(A, E)$ be a $B$–valued Banach noncommutative probability space and let $a \in A$, $N \in \mathbb{N}$. Then there are $\alpha_0, \alpha_1, \ldots, \alpha_N$, with $\alpha_n \in \mathcal{B}_n(B)$, such that if

$$X = L_1 + \sum_{n=0}^{N} V_{1,n}(\alpha_n) \in \mathcal{B}(\mathcal{F}),$$

then

$$\mathcal{E}(b_0Xb_1X \cdots b_kX) = E(b_0ab_1a \cdots b_ka)$$

for all $k \in \{1, \ldots, N\}$ and all $b_0, \ldots, b_N \in B$.

Now we have the following analogues of Lemmas 3.7 and 3.8.

**Lemma 4.5.** Let $N \in \mathbb{N}$ and choose $\alpha_n \in \mathcal{B}_n(B)$ for $n \in \{0, 1, \ldots, N\}$, and let

$$F(b) = \alpha_0 + \sum_{n=1}^{N} \alpha_n(b, \ldots, b).$$

Let

$$X = L_1 + \sum_{n=0}^{N} V_{1,n}(\alpha_n) \in \mathcal{B}(\mathcal{F}).$$

Then the $R$–transform of $X$ is $R_X(b) = F(b)$.

**Proof.** With $\omega_b$ defined as in the proof of Lemma 3.7, we have

$$X\omega_b = L_1\omega_b + F(b)\omega_b$$

$$bX\omega_b = (\omega_b - \Omega) + bF(b)\omega_b$$

$$(1 + bF(b) - bX)\omega_b = \Omega$$

$$(1 - bX(1 + bF(b)^{-1})^{-1}\Omega = (1 + bF(b))\omega_b$$

$$\mathcal{E}((1 - bX(1 + bF(b)^{-1})^{-1}) = P((1 + bF(b))\omega_b)$$

$$= 1 + bF(b).$$

Conjugating yields

$$\mathcal{E}((1 - (1 + bF(b)^{-1}bX)^{-1}) = 1 + bF(b),$$
so
\[ C_X((1 + bF(b))^{-1}b) = \mathcal{E}((1 - (1 + bF(b))^{-1}bX)^{-1})(1 + bF(b))^{-1}b = b. \]
Thus,
\[ C_X^{(-1)}(b) = (1 + bF(b))^{-1}b \]
and \( R_a(b) = F(b) \). □

**Lemma 4.6.** Let \( \alpha_0, \ldots, \alpha_n, F \) and \( X \) be as in Lemma 4.3. Let \( \beta_n \in \mathcal{B}_n(B) \) for \( n \in \{0, 1, \ldots, N\} \). Let
\[ G(b) = \beta_0 + \sum_{n=1}^N \beta_n(b, \ldots, b) \]
and let
\[ Y = L_2 + \sum_{n=0}^N V_{2,n}(\alpha_n) \in \mathcal{B}(\mathcal{F}). \]
Then the \( R \)-transform of \( X + Y \) is
\[ R_{X+Y}(b) = F(b) + G(b) = R_X(b) + R_Y(b). \]

**Proof.** For \( b \in B \) with \( \|b\| < 1/2 \), let
\[ \sigma_b = (1 - b(L_1 + L_2))^{-1}\Omega = \Omega + \sum_{k=1}^\infty (b\delta_1 + b\delta_2)^\otimes k \otimes 1 \in \mathcal{F}. \]
Then
\[ (X + Y)\sigma_b = (L_1 + L_2)\sigma_b + (F(b) + G(b))\sigma_b \]
\[ b(X + Y)\sigma_b = (\sigma_b - \Omega) + b(F(b) + G(b))\sigma_b. \]
Now arguing as in the proof of Lemma 4.5 above yields \( R_{X+Y}(b) = F(b) + G(b) \). □

Finally, we get a proof, which is analogous to our proof of Theorem 1.1 of the additivity of the \( R \)-transform in a Banach noncommutative probability space.

**Theorem 4.7** ([9]). Let \( B \) be a unital complex Banach algebra and let \( (A, E) \) be a \( B \)-valued Banach noncommutative probability space. Let \( x, y \in A \) be free in \( (A, E) \). Then
\[ R_{x+y}(b) = R_x(b) + R_y(b). \]

**Proof.** By Proposition 4.2, it will suffice to show that given \( n \in \mathbb{N} \) we have \( R_{X+Y} = R_X + R_Y \) for some free pair \( X \) and \( Y \) of elements in a Banach noncommutative probability space over \( B \) whose first \( n \) moment functions agree with those of \( x \) and \( y \), respectively. Precisely this fact follows from Proposition 4.1, Proposition 3.3 and Lemmas 4.5 and 4.6 □
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