FREDHOLM ALTERNATIVE, SEMILINEAR ELLIPTIC PROBLEMS, AND WENTZELL BOUNDARY CONDITIONS

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Abstract. We give necessary and sufficient conditions for the solvability of some semilinear elliptic boundary value problems involving the Laplace operator with linear and nonlinear highest order boundary conditions involving the Laplace-Beltrami operator.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with smooth boundary $\Gamma := \partial \Omega$. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a continuous monotone nondecreasing function with $\alpha (0) = 0$ and consider the following boundary value problem:

$$
\begin{cases}
-\Delta u + \alpha (u) = f & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \Gamma,
\end{cases}
$$

where $f \in L^2 (\Omega)$ is a given real function, $\frac{\partial u}{\partial n}$ denotes the outward normal derivative of $u$ on $\Gamma$ and $\Delta$ is the Laplace operator in $\Omega$. Let us denote by $|\Omega|$ the Lebesgue measure of $\Omega$. It is known that a necessary and sufficient condition for the existence of a solution of (1.1) is

$$
|\Omega|^{-1} \int_\Omega f (x) \, dx \in R (\alpha). \tag{1.2}
$$

Here all the functions are real valued. This result is due to J. Mawhin [15]. Earlier, Landesman and Lazer [12] obtained a similar result. This result lead to an enormous body of literature. Landesman and Lazer showed that (1.2) is a necessary condition, while a sufficient condition is

$$
|\Omega|^{-1} \int_\Omega f (x) \, dx \in \text{int}(R (\alpha)), \tag{1.3}
$$

where $\text{int}(I)$ denotes the interior of the set $I$. They also allowed for nonmonotone $\alpha$, which was very important for later developments. Thus for them, $\alpha : \mathbb{R} \to \mathbb{R}$ is continuous, $\alpha (0) = 0$, and

$$
\alpha (-\infty) = \lim_{x \to -\infty} \alpha (x) \leq \alpha (y) \leq \lim_{x \to +\infty} \alpha (x) = \alpha (+\infty) \tag{1.4}
$$

for all $y \in \mathbb{R}$. They proved (1.2) is a necessary condition in this more general context of (1.4), while (1.3) is a sufficient condition. Prior to Mawhin’s work, Brezis and Haraux [2] put

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the [12] result in an abstract context and found a new, elegant proof for it. These works led to very much research, including major contributions by Brezis and Nirenberg [3] and many others. Brezis and Haraux worked in the context of subdifferentials of convex functionals on Hilbert spaces. We will explain the context and the abstract results, used in proving the assertion connecting (1.2) and (1.3), in Sections 2 and 4. But here we emphasize again that these results were inspired by the similar result of Landesman and Lazer [12] who, in giving necessary and sufficient conditions on $f$ for the solvability of certain elliptic problems of the form $Lu + Nu = f$ (with $L$ linear and $N$ nonlinear), established a sort of "nonlinear Fredholm alternative" for the first time. When $\alpha \equiv 0$, the above result reduces to

$$-\Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0, \text{ on } \Gamma$$

has a weak solution if and only if

$$\langle f, 1 \rangle_{L^2(\Omega)} = 0, \text{ i.e., } \int_\Omega f(x) \, dx = 0,$$

which is exactly the Fredholm alternative since the null space of the Neumann Laplacian is the constants. Thus, Mawhin’s result (based on the work in [12]) is an exact nonlinear Fredholm alternative for the nonlinear problem (1.1).

The goal of this paper is to establish similar results (comparable with (1.2), (1.3)) for the following boundary value problem with second order boundary conditions:

$$\begin{cases}
-\Delta u + \alpha_1(u) = f(x) & \text{in } \Omega, \\
b(x) \frac{\partial u}{\partial n} + c(x) u - qb(x) \Delta \Gamma u + \alpha_2(u) = g(x) & \text{on } \Gamma,
\end{cases} \quad (1.5)$$

where the functions appearing in (1.5) are real and satisfy $b \in C(\Gamma), b > 0, c \in C(\Gamma), c \geq 0, q \text{ is a nonnegative constant}; \alpha_1, \alpha_2 : \mathbb{R} \to \mathbb{R}$ are continuous and monotone nondecreasing functions, such that $\alpha_i(0) = 0$. Above, $\Delta \Gamma$ is the Laplace-Beltrami operator on $\Gamma$, $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$ are given real functions. Thus, our emphasis is on the generality of the boundary conditions.

We organize the paper as follows. In Sections 2 and 3, we discuss the auxiliary linear problems corresponding to (1.5), and in Section 4 we show the existence of weak solutions to (1.5) in case certain global constraints (similar to (1.2)) hold. In the same section, we will consider concrete examples as application of our results.

Before we state our main result, we define the notion of weak solutions to (1.5).

**Definition 1.1.** A function $u \in H^1(\Omega)$ is said to be a weak solution to (1.5) if $\alpha_1(u) \in L^1(\Omega), \alpha_2(tr(u)) \in L^1(\Gamma), tr(u) := u|_\Gamma \in H^1(\Gamma), \text{ if } q > 0$, and

$$\int_\Omega f v dx + \int_\Gamma gv \frac{dS}{\beta} = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega \alpha_1(u) v dx + \int_\Gamma \alpha_2(u) v + cv \frac{dS}{\beta} + q \int_\Gamma \nabla \Gamma u \cdot \nabla \Gamma v dS, \quad (1.6)$$

for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$, if $q = 0$ and all $v \in H^1(\Omega) \cap C(\overline{\Omega})$ with $tr(v) \in H^1(\Gamma)$, if $q > 0$. 

Our main result is as follows. Let
\[ \lambda_1 = \int_\Omega dx, \quad \lambda_2 = \int_\Gamma \frac{dS}{b}, \] (1.7)
and let \( \tilde{I} \) be the interval
\[ \tilde{I} = \lambda_1 \mathcal{R} \left( \alpha_1 \right) + \lambda_2 \mathcal{R} \left( \alpha_2 \right). \]
Moreover, for each \( i = 1, 2 \), set
\[ L_i(t) := \int_0^t \alpha_i(s)ds \quad \text{and} \quad \Lambda_i(t) := \max \{ L_i(t), L_i(-t) \}, \quad \text{for all} \ t \in \mathbb{R}. \] (1.8)

**Theorem 1.2.** Let \( c \equiv 0 \) and let \( \alpha_i : \mathbb{R} \to \mathbb{R} \ (i = 1, 2) \) be continuous, monotone nondecreasing functions such that \( \alpha_i(0) = 0 \). If \( (1.5) \) has a weak solution, then
\[ \int_\Omega f(x)dx + \int_\Gamma g(x) dS/(b(x)) \in \tilde{I}. \] (1.9)
Conversely, if there exist positive constants \( t_i, C_i > 0 \), such that the functions \( \Lambda_i : \mathbb{R} \to [0, +\infty], \ i = 1, 2, \) satisfy \( \Lambda_i(2t) \leq C_i \Lambda_i(t) \), for all \( t \geq t_i \), and
\[ \int_\Omega f(x)dx + \int_\Gamma g(x) dS/(b(x)) \in \text{int}(\tilde{I}), \] (1.10)
then \( (1.5) \) has a weak solution.

2. The linear problem

We need to introduce some notation and terminology. We first define the space \( \mathbb{X}_2 \) to be the real Hilbert space \( L^2(\Omega, dx) \oplus L^2(\Gamma, dS/b) \), with norm
\[ \| u \|_{\mathbb{X}_2} = \left( \int_\Omega |u(x)|^2 dx + \int_\Gamma |u(x)|^2 \frac{dS_x}{b(x)} \right)^{\frac{1}{2}} \] (2.1)
for \( u \in C \left( \overline{\Omega} \right) \), where \( dS \) denotes the usual Lebesgue surface measure on \( \Gamma \). Here, if \( u \in C \left( \overline{\Omega} \right) \), we identify \( u \) with the vector \( U = (u|_\Omega, u|_\Gamma)^T \in C \left( \Omega \times C (\Gamma) \right) \). We then note that \( \mathbb{X}_2 = L^2(\Omega, dx) \oplus L^2(\Gamma, dS/b) \) is the completion of \( C \left( \overline{\Omega} \right) \) with respect to the norm (2.1). In general, any vector \( U \in \mathbb{X}_2 \) will be of the form \( (u_1, u_2)^T \) with \( u_1 \in L^2(\Omega, dx) \) and \( u_2 \in L^2(\Gamma, dS/b) \), and there need be no connection between \( u_1 \) and \( u_2 \). Here and below the superscript \( T \) denotes transpose. Let \( \langle \cdot, \cdot \rangle_{\mathbb{X}_2} \) denote the corresponding inner product on \( \mathbb{X}_2 \). For a complete discussion of this space, we refer the reader to [5].

We define the formal operator \( A_0 \) by
\[ A_0 U = ((-\Delta u)|_\Omega, (-\Delta u)|_\Gamma)^T, \] (2.2)
for functions \( U = (u|_\Omega, u|_\Gamma)^T \) with \( u \in C^2 \left( \overline{\Omega} \right) \) that satisfy the Wentzell boundary condition
\[ \Delta u + b(x) \frac{\partial u}{\partial n} + c(x) u - qb(x) \Delta_\Gamma u = 0, \] (2.3)
on $\Gamma$. Here $(\Delta u)_{|\Gamma}$ stands for the trace of the function $\Delta u$ on the boundary $\Gamma$ and it should not be confused with the Laplace-Beltrami operator $\Delta_{\Gamma} u$. From now on, $tr\,(u)$ denotes the trace of $u$ on the boundary. We let

$$D(A_0) = \{U = (u_1, u_2)^T \in X_2 : U \text{ corresponds to } u_1 \in C^2(\overline{\Omega}) , \quad u_2 = u_1|_\Gamma = tr\,(u_1) \text{ and } \text{(2.3) holds} \}. \quad (2.4)$$

For functions $u \in C^2(\overline{\Omega}) \subset X_2$, $A_0 U$ is defined by (2.2). For any functions $u, v$ belonging to $C^2(\overline{\Omega})$, and each satisfying the boundary condition $\Delta_\omega + b(x) \frac{\partial u}{\partial n} + c(x) \omega - qb(x) \Delta_\Gamma \omega = 0$ on $\Gamma$, we identify $u$ and $v$ with $U = (u|_\Omega, u|_\Gamma)^T$ and $V = (v|_\Omega, v|_\Gamma)^T$ and calculate $\langle A_0 U, V \rangle_{X_2}$ as follows:

$$\langle A_0 U, V \rangle_{X_2} = \int_{\Omega} (-\Delta u) v dx + \int_{\Gamma} (-\Delta u) \frac{v}{b(x)} dS, \quad (2.5)$$

$$= \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Gamma} \left( -\Delta u - b(x) \frac{\partial u}{\partial n} \right) \frac{v}{b(x)} dS$$

$$= \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Gamma} (c(x) u - qb(x) \Delta_\Gamma u) \frac{v}{b(x)} dS, \quad (2.6)$$

since $-\Delta u - b(x) \frac{\partial u}{\partial n} = c(x) u - qb(x) \Delta_\Gamma u$ on $\Gamma$. Furthermore, Stokes’ theorem applied in the last term of (2.5) yields

$$\langle A_0 U, V \rangle_{X_2} = \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Gamma} c(x) uv \frac{dS}{b(x)} + q \int_{\Gamma} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} v dS, \quad (2.6)$$

where $\nabla_{\Gamma}$ stands for the tangential gradient on the surface $\Gamma$. Finally, if we denote the right hand side of (2.6) by $\rho(U, V)$, it is now clear that $\rho(U, V) = \rho(V, U) = \langle U, A_0 V \rangle_{X_2}$, therefore $A_0$ is symmetric on $X_2$. Let us now consider a function $f \in C(\overline{\Omega}) \cup H^1(\Omega)$ such that $F = (f_1, f_2)^T$ with $f_1 := f|_\Omega$ and $f_2 := f|_\Gamma$. By the equality $A_0 U = F$, we mean the following boundary value problem:

$$-\Delta u = f_1 \quad \text{in} \quad \Omega, \quad (2.7)$$

$$-\Delta u = f_2 \quad \text{on} \quad \Gamma. \quad (2.8)$$

Using the Wentzell boundary condition (2.3) and replacing $f_2$ by $f_1$, the boundary condition (2.8) becomes

$$b(x) \frac{\partial u}{\partial n} + c(x) u - qb(x) \Delta_\Gamma u = f_2 \quad \text{on} \quad \Gamma. \quad (2.9)$$

Any $u \in H^s(\Omega)$ has a trace $tr\,(u) = u|_\Gamma$ in $H^{s-1/2}(\Gamma)$ for $s > 1/2$. More precisely, we recall that the linear map $tr : H^s(\Omega) \to H^{s-1/2}(\Gamma)$ is bounded and onto for $s > 1/2$. We now define the ”Wentzell version of $A_0$”, $A_0$, by $\tilde{A}_0 U = F = (f_1, f_2)^T$ on

$$D(\tilde{A}_0) = \{U \in X_2 : U \text{ corresponds to } u \in H^2(\Omega) , \quad tr\,(u) \in H^2(\Gamma) \text{ if } q > 0, \text{ and } (2.7), (2.9) \text{ holds} \}. \quad (2.10)$$
In this case, $f_2$ need not be the trace of $f_1$ on $\Gamma$. Then, using the techniques as in [6], we can easily check that $\tilde{A}_0$ is contained in the closure of $A_0$. Let $A = \overline{A_0} = \overline{A_0}$. Then, $A$ is selfadjoint and nonnegative if $c \geq 0$ on $\Gamma$; $A$ is the operator associated with the nonnegative symmetric closed bilinear form $\rho(U, V)$. We have $\langle AU, V \rangle_{X_2} = \rho(U, V)$, for all $U \in D(A)$ and all $V = (v|_\Omega, v|_\Gamma)^T \in D(\rho) := H^1(\Omega) \times H^1(\Gamma)$ (if $q > 0$) and $V \in D(\rho) := H^1(\Omega) \times H^{1/2}(\Gamma)$ if $q = 0$. We emphasize that for $A = \overline{A_0}$, the equations (2.7) and (2.9) hold even if the vector $F = (f_1, f_2)^T$ does not correspond to a function $f$ belonging to $C(\bar{\Omega}) \cup H^1(\Omega)$, that is, $f_2 \neq f_1|_\Gamma$. For $U \in D(A)$, an operator matrix representation of $A$ is given by

$$A = \begin{pmatrix} -\Delta & b \frac{\partial}{\partial n} \\ b \frac{\partial}{\partial n} & cI - qb \Delta \Gamma \end{pmatrix}. \quad (2.11)$$

We will now give a concrete example when $q = 0$ (that is, $\Delta \Gamma$ does not appear in the boundary condition (2.9)). This is a simple example where $f_2 \neq f_1|_\Gamma$.

**Example 2.1.** Let $\Omega = (0, 1) \subset \mathbb{R}$ and let $F = (0, k)$ where $\Gamma = \{0, 1\}$ and $k(0) = a_0$, $k(1) = b_0$ with $(a_0, b_0) \neq (0, 0)$. Take $b(j) = c(j) = 1$ for $j = 0, 1$. Then $AU = F$ means

$$\begin{cases} u'' = 0 \text{ in } [0, 1], \\
-u'(0) + u(0) = a_0, \\
u'(1) + u(1) = b_0, \end{cases} \quad (2.12)$$

since $\partial/\partial n = (-1)^{j+1} d/dx$ at $x = j \in \{0, 1\}$. Solving (2.12) gives

$$u(x) = \frac{1}{3} [(b_0 - a_0)x + (2a_0 + b_0)], \quad x \in [0, 1].$$

**3. The domain of the Wentzell Laplacian**

We recall some facts from the theory of linear elliptic boundary value problems. The standard theory works for uniformly elliptic problems of even order $2m$; we shall restrict ourselves to the second order case, $m = 1$. We shall treat the symmetric case, although this restriction is not needed for the results we present in this section. Our problem takes the form $\tilde{A}u = f \in \Omega$, $\tilde{B}u = g$ on $\Gamma$, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ with boundary $\Gamma$,

$$\begin{cases} Au = -\nabla \cdot \mathcal{A}(x) \nabla u, \\
Bu = b\partial_n^4 u + cu - qb \Delta \Gamma u, \end{cases} \quad (3.1)$$

and $\tilde{A} = A + \lambda I$, $\tilde{B} = B + \lambda I$, for some $\lambda \in \mathbb{R}$. As the theory is based upon pseudo differential operator techniques, we make the standard assumption that $\Omega$, $\mathcal{A}$, $b$ and $c$ are all of class $C^\infty$ in addition to the assumptions that the $N \times N$ matrix function $\mathcal{A}$ is real, symmetric and uniformly positive definite, $b > 0$, $c \geq 0$ and $q \in [0, +\infty)$.

Let $s \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ and $p \in (1, +\infty)$. We refer to Triebel [20] for the general case, where we use his notation; Lions-Magenes [13] treats the Hilbert space case ($p = 2$).

**Theorem 3.1.** Let the above assumptions hold, with $q = 0$. Then for all $\lambda > 0$, with $\tilde{A} = A + \lambda I$, $\tilde{B} = B + \lambda I$, the map

$$\Xi_\lambda : u \mapsto (\tilde{A}u, \tilde{B}u),$$

...
viewed as a map from $W^{s+2}_p(\Omega)$ to $W^2_p(\Omega) \times B^{1+s-1/p}_{p,p}(\Gamma)$, is an isomorphism.

This means that $\Xi_\lambda$ is a linear bijection, and there is a positive constant $C$, independent of $u$, such that

$$
C^{-1} \|u\|_{W^{s+2}_p(\Omega)} \leq \left\| \hat{A}u \right\|_{W^2_p(\Omega)} + \left\| \hat{B}u \right\|_{B^{1+s-1/p}_{p,p}(\Gamma)} \leq C \|u\|_{W^{s+2}_p(\Omega)} $$

(3.2)

for all $u \in W^{s+2}_p(\Omega)$. Thus, the isomorphism is a linear homeomorphism, but it need not be isometric. Here $W^r_p(\Omega)$ is Triebel’s notation for the Sobolev space and $B^r_{p,p}(\Gamma)$ for the Besov space. For $s = 0$ and $p = 2$, this reduces $\Xi_\lambda$ to being an isomorphism from $H^2(\Omega)$ to $L^2(\Gamma, dS) \oplus L^2(\Gamma, dS/b)$, which is equivalent to saying that $\Xi_\lambda$ is an isomorphism from $H^2(\Omega)$ to $X_2$, since $L^2(\Gamma, dS)$ and $L^2(\Gamma, dS/b)$ are the same sets with equivalent inner products.

It follows that, when $q = 0$, the domain of the Wentzell Laplacian $A$ is exactly $H^2(\Omega)$.

**Theorem 3.2.** Let

$$
H^2_*(\Omega) = \{ u \in H^2(\Omega) : u|_\Gamma \in H^2(\Gamma) \}.
$$

The domain of the Wentzell Laplacian $A$, the selfadjoint closure of $A_0$, defined by (2.7), (2.9), is exactly

$$
D(A) = \begin{cases} 
H^2(\Omega) & \text{if } q = 0, \\
H^2_*(\Omega) & \text{if } q > 0.
\end{cases}
$$

(3.3)

The same conclusion holds for the closure of the operator $A$ defined by (3.1).

Before outlining the proof of this theorem we make some remarks. Theorem 3.2 gives the first “simple” explicit characterization of $D(A)$, including the case of $q > 0$. Normally, knowing that $D(A_0)$ is a core for $A$ is enough for most purposes involving linear problems. But we need to know $D(A)$ exactly in order to apply the Brezis-Haraux result (see Proposition 4.14 below). Theorem 3.2 assumes that $\Gamma$, $b$ and $c$ are $C^\infty$. Surely this much regularity is not needed. But the proof is based on pseudo differential operator techniques and this theory is always presented in the $C^\infty$ context, because to do otherwise would entail many complicated calculations requiring a lot of courage. So Theorem 3.2 should be valid if everything is $C^2$, but this is merely an educated guess (however, see Remark 3.1).

We wish to recall the earlier work on this problem by Escher [4] (see also Fila and Quittner [7]). Escher proved Theorem 3.2 in the special case of $b \equiv 1$ and $q = 0$. He worked in the $X_p$ context for $1 < p < +\infty$, but, by focusing on the analytic semigroup aspect of the problem, he did not notice the selfadjointness of $A$. Moreover, his restriction to the case of $b \equiv 1$ avoids many interesting cases, since the coefficient $b$ has physical significance (cf. [8]).

We now recall the strategy of the proof of Theorem 3.1. We outline the proof in several steps:

**Step 1.** Treat the case of constant coefficients and take $\Omega$ to be a half-space.

**Step 2.** Then localizing and using a partition of unity, this breaks the problem down into a large number of problems $\{P_j\}$, where the portion of $\Gamma$ is the subdomain corresponding to $P_j$ is almost flat and the coefficients are almost constants.
Flatten out the boundary and solve each $P_j$, using Step 1, and the theory of pseudo differential operators (see, e.g., Taylor [19]). Finally, put everything together and complete the proof. The proof is quite long, technical and complicated, but it is now well understood and standard. For the moment we focus on Step 1 and, for simplicity, assume that $A$ is the identity matrix, so that $A = -\Delta$. Then our problem \((3.1)\) becomes the constant coefficient problem:

$$
\begin{aligned}
\tilde{A}u = -\Delta u + \lambda u &= f \\
\tilde{B}u = b\partial_z u + cu + \lambda u - qb\Delta_x u &= g
\end{aligned}
$$

(3.4)

Here $\mathbb{R}_+^N = \{x = (y, z) : y \in \mathbb{R}^{N-1}, z \geq 0\}$, $\partial\mathbb{R}_+^N = \{x = (y, 0) : y \in \mathbb{R}^{N-1}\}$ and the boundary condition of \((3.4)\) is equivalent to

$$
\begin{aligned}
b\partial_z u + cu + \lambda u - qb\Delta_y u &= g
\end{aligned}
$$

(3.5)

on $\partial\mathbb{R}_+^N$. For a function $h(y, z)$, let $\hat{h}(\zeta, z)$ be the Fourier transform in the $\mathbb{R}^{N-1}$-variable with $z$ fixed:

$$
\hat{h}(\zeta, z) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^{N-1}} e^{-i\zeta \cdot y} h(y, z) \, dy,
$$

(3.6)

Then, in Fourier space, the first equation of \((3.4)\) and equation \((3.5)\) become

$$
\begin{aligned}
\frac{\partial^2 \hat{u}}{\partial z^2} - (|\zeta|^2 + \lambda) \hat{u} &= \hat{f} \quad \text{in } \mathbb{R}_+^N, \\
\hat{b}\partial_z \hat{u} + (c + \lambda + qb|\zeta|^2) \hat{u} &= \hat{g} \quad \text{on } \partial\mathbb{R}_+^N.
\end{aligned}
$$

(3.7)

We need $u$ to be an $L^2$ function. To solve \((3.6)\), one finds the general solution of the homogeneous equation and adds to it a particular solution of \((3.6)\), obtained by the variation of constants formula. The general solution of the homogenous version of \((3.6)\) is

$$
\hat{u}(\zeta, z) = C_1 e^{\gamma_1 z} + C_2 e^{\gamma_2 z},
$$

(3.8)

where

$$
\gamma_j = (-1)^{j+1} \left(|\zeta|^2 + \lambda\right)^{1/2}, \quad j = 1, 2.
$$

Then for each $\zeta \in \mathbb{R}^{N-1}$, $\gamma_2 < 0 < \gamma_1$. Thus, the general $L^2$ solution of the homogeneous problem is given by \((3.8)\), with $C_2$ an arbitrary constant and $C_1 = 0$.

Next, \((3.7)\) is of the form

$$
\frac{\partial \hat{u}}{\partial z} - p(\zeta) = m(\zeta),
$$

for $z = 0$, where $p \geq \varepsilon_0 > 0$ for all $\zeta$ (For more general problems, the corresponding inequality follows from uniform ellipticity). It follows that \((3.4)\) (as well as \((3.6), (3.7)\)) has a unique $L^2$ solution. Note that this works for $q > 0$ as well as for $q = 0$. For $q > 0$, we require that $|\zeta|^2 \hat{u}$ as well as $\hat{u}$ is in $L^2$. If one studies the proof in [20] in detail, minor modifications of the tedious calculations lead to the proof of Theorem 3.2.

More precisely, for $q > 0$, we conclude that there is a positive constant $C = C(q, b, c, \lambda, A)$, for every $\lambda > 0$, such that

$$
C^{-1} \|u\|_{H^2_\varepsilon(\Omega)} \leq \left\| \begin{pmatrix} \hat{A}u \\ \hat{B}u \end{pmatrix} \right\|_{X_2} \leq C \|u\|_{H^2_\varepsilon(\Omega)}
$$

(3.9)
for all \( u \in H^2_q(\Omega) \). Moreover, the map \( u \mapsto (\hat{A}u, \hat{B}u)^T \) is a surjective linear isomorphism of \( H^2_q(\Omega) \) onto \( \mathbb{R}_2 \), for \( q > 0 \). Above in (3.9), the norm in \( H^2_q(\Omega) \) is defined as

\[
\|u\|_{H^2_q(\Omega)} = \left(\|u\|_{H^2_q(\Omega)}^2 + \|\text{tr}(u)\|_{H^2_q(\Gamma)}^2\right)^{1/2}.
\]

From this, the proof of Theorem 3.2 follows. \( \square \)

Remark 3.1. We note that the first inequality of (3.9) was already obtained in [16, Lemma A.1] for the weak solutions of (3.1), using standard Sobolev inequalities and assuming \( b, c \in C(\Gamma) \), \( b, \lambda > 0 \), \( A = I_{N \times N} \) and \( \Gamma \) is of class \( C^2 \). Observe that (3.1) is also an elliptic boundary value problem in the sense specified in [11, 17], where similar estimates to (3.9) were also obtained. The second inequality of (3.9) is obvious and is based on the definition of \( \hat{A} \) and \( \hat{B} \).

4. Convex analysis

We begin with the following assumptions:

(H1) The functions \( \alpha_i : \mathbb{R} \to \mathbb{R}, i = 1, 2 \), are continuous, monotone nondecreasing with \( \alpha_i(0) = 0 \).

(H2) Let \( \Lambda_i \) be as in (1.8) and suppose that they satisfy the \( \Delta_2 \)-condition near infinity, in the sense that, there are positive constants \( t_i, C_i > 0, i = 1, 2 \), such that

\[
\Lambda_i(2t) \leq C_i \Lambda_i(t), \quad \text{for all } t \geq t_i.
\]

Let \( \hat{\alpha}_i : \mathbb{R} \to \mathbb{R}, (i = 1, 2) \) be the inverse of \( \alpha_i \). Then \( \hat{\alpha}_i \) is a nondecreasing function from \( \mathbb{R} \) to \( \mathbb{R} \), which is multivalued at its jumps and it is in \( L^1_{\text{loc}}(\mathbb{R}) \). Its graph is a connected subset of \( \mathbb{R}^2 \). Let \( \tilde{L}_i : \mathbb{R} \to [0, +\infty), i = 1, 2 \), be defined by

\[
\tilde{L}_i(t) := \int_0^t \hat{\alpha}_i(s)ds \quad \text{and} \quad \tilde{\Lambda}_i := \max \left\{ \tilde{L}_i(t), \tilde{L}_i(-t) \right\}, \quad \text{for all } t \in \mathbb{R}.
\]

All the functions given in (1.8) and (1.2) are convex and continuous on \( \mathbb{R} \), nondecreasing on \( \mathbb{R}_+ \), and all vanish at the origin; \( \Lambda_i \) and \( \tilde{\Lambda}_i \) are even functions and are complementary Young functions in the sense of [18]. Chap. I, Section 1.3, Theorem 3], but they need not be \( N \)-functions. Note that \( L_i'(t) = \alpha_i(t) \) on \( \mathbb{R} \) and \( \tilde{L}_i'(t) = \hat{\alpha}_i(t) \) a.e.; \( |\Lambda_i'(t)| \geq |\alpha_i(t)| \) and \( |	ilde{\Lambda}_i'(t)| \geq |\hat{\alpha}_i(t)| \) almost everywhere. It follows, from [18] Chap. I, Section 1.3, Theorem 3], that for all \( s, t \in \mathbb{R} \),

\[
|st| \leq L_i(t) + \tilde{\Lambda}_i(s) \leq \Lambda_i(t) + \tilde{\Lambda}_i(s).
\]

Suppose that \( \Lambda_i(s) = L_i(\tau) \) and \( \tilde{\Lambda}_i(s) = \tilde{L}_i(\sigma) \), where \( \tau \) is \( s \) or \(-s\) and \( \sigma \) is \( t \) or \(-t\). If \( \tau = \hat{\alpha}_i(\sigma) \) or \( \sigma = \hat{\alpha}_i(\tau) \), then we also have equality, that is,

\[
\tilde{L}_i(\alpha_i(\tau)) = \tilde{\Lambda}_i(\alpha_i(\sigma)) = \tau \alpha_i(\tau) - \Lambda_i(\tau) = \tau \alpha_i(\tau) - L_i(\tau), \quad i = 1, 2.
\]
Let now $\alpha_i : \mathbb{R} \to \mathbb{R}, i = 1, 2$, satisfy (H1). Define the functional $J : \mathbb{X}_2 \to [0, +\infty]$ by

$$J(U) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} L_1(u) \, dx + \int_{\Gamma} j_2(x, u) \frac{dS}{b(x)},$$  \hspace{1cm} (4.5)$$

for $U = (u, tr(u))^T, u \in H^1(\Omega)$ such that all three integrals exist, and $tr(u) \in H^1(\Gamma)$ if $q > 0$. We take

$$j_2(x, u) = c(x) \frac{u^2}{2} + qb(x) \frac{|\nabla u|^2}{2} + L_2(u).$$  \hspace{1cm} (4.6)$$

The effective domain $\mathbb{D}_q := D(J)$ of the functional $J$ is precisely

$$\mathbb{D}_0 = \{U = (u, tr(u))^T : u \in H^1(\Omega), \int_{\Omega} \Lambda_1(u) \, dx + \int_{\Gamma} \Lambda_2(u) \frac{dS}{b(x)} < \infty\}$$  \hspace{1cm} (4.7)$$

if $q = 0$, and

$$\mathbb{D}_q = \{U = (u, tr(u))^T \in \mathbb{D}_0 : tr(u) \in H^1(\Gamma)\}$$  \hspace{1cm} (4.8)$$

if $q > 0$, respectively. Define $J(U) = +\infty$, for all $U \in \mathbb{X}_2 \setminus \mathbb{D}_q, q \geq 0$. As before, for $u \in H^1(\Omega)$, we identify $u$ with $U = (u, tr(u))^T \in \mathbb{X}_2$. Then $J$ is proper, convex and lower semicontinuous on $\mathbb{X}_2$, as can be shown adapting the ideas of Brezis [1] (see also [6]).

Suppose now that $\alpha_i, i = 1, 2$, satisfies assumptions (H1)-(H2). Then, by (4.1), the monotonicity (on $\mathbb{R}_+$ and the convexity of $\Lambda_i, i = 1, 2$, we have that $\mathbb{D}_q, q \geq 0$, is a vector space (see, e.g., [18, Chap. III, Section 3.1, Theorem 2]).

In what follows, we shall compute the subdifferential of $J$. To this end, let $F := (f, g)^T \in \mathbb{X}_2$ and $U = (u, tr(u))^T \in \mathbb{D}_q$. We claim that $F \in \partial J(U)$ if and only if

$$-\Delta u + \alpha_1(u) = f \text{ in } \mathcal{D}'(\Omega),$$  \hspace{1cm} (4.9)$$

$$b(x) \frac{\partial u}{\partial n} + c(x)u - qb(x) \Delta u + \alpha_2(u) = g \text{ on } \Gamma.$$  \hspace{1cm} (\text{4.9})$$

First, assume that $F \in \partial J(U)$. Then, by definition, for every $V = (v, tr(v))^T \in \mathbb{D}_q$, we have

$$\int_{\Omega} f(v-u) \, dx + \int_{\Gamma} g(v-u) \frac{dS}{b} \leq \frac{1}{2} \int_{\Omega} (|\nabla v|^2 - |\nabla u|^2) \, dx$$

$$+ \int_{\Omega} (L_1(v) - L_1(u)) \, dx + \int_{\Gamma} (j_2(x,v) - j_2(x,u)) \frac{dS}{b},$$  \hspace{1cm} (4.10)$$

where, from (4.6), we find that

$$\int_{\Gamma} (j_2(x,v) - j_2(x,u)) \frac{dS}{b} = \frac{1}{2} \int_{\Gamma} c (|v|^2 - |u|^2) \frac{dS}{b} + \frac{q}{2} \int_{\Gamma} (|\nabla v|^2 - |\nabla u|^2) \, dS$$

$$+ \int_{\Gamma} (L_2(v) - L_2(u)) \frac{dS}{b}.$$
Let \( W = (w, tr(w))^T \in \mathbb{D}_q \) be fixed and let \( t \in [0, 1] \). Choosing \( V := tW + (1 - t)U \in \mathbb{D}_q \) in (4.10), dividing by \( t \) and taking the limit as \( t \to 0^+ \), from (4.10), we obtain

\[
\int_{\Omega} f(w - u) \, dx \quad \text{and} \quad \int_{\Gamma} g(w - u) \, dS
\]

\[
\leq \int_{\Omega} \nabla u \cdot \nabla (w - u) \, dx + \int_{\Omega} \alpha_1(u)(w - u) \, dx + \int_{\Gamma} cu(w - u) \, dS
\]

\[
+ q \int_{\Gamma} \nabla u \cdot \nabla (w - u) \, dS + \int_{\Gamma} \alpha_2(u)(w - u) \, dS.
\]

Here we used the definition of the functions \( L_i \) \( (i = 1, 2) \) from (1.8) and the Lebesgue Dominated convergence theorem, which implies

\[
\lim_{t \to 0^+} \int_{\Omega} \frac{L_1(u + t(w - u)) - L_1(u)}{t} \, dx = \int_{\Omega} \alpha_1(u)(w - u) \, dx
\]

and

\[
\lim_{t \to 0^+} \int_{\Gamma} \frac{L_2(u + t(w - u)) - L_2(u)}{t} \, dS = \int_{\Gamma} \alpha_2(u)(w - u) \, dS.
\]

Letting \( W = U \pm \Psi \) in (4.11), where \( \Psi = (\psi, tr(\psi))^T \) is an arbitrary element of \( \mathbb{D}_q \), we easily deduce

\[
\int_{\Omega} f\psi \, dx + \int_{\Gamma} g\psi \, dS = \int_{\Omega} \nabla u \cdot \nabla \psi \, dx + \int_{\Omega} \alpha_1(u)\psi \, dx
\]

\[
+ \int_{\Gamma} cu\psi \, dS + q \int_{\Gamma} \nabla u \cdot \nabla \psi \, dS + \int_{\Gamma} \alpha_2(u)\psi \, dS.
\]

Taking \( \psi \in C_0^\infty(\Omega) \) in (4.12), one obtains the first equation of (4.9). A simple partial integration argument shows that one also has the second equation in (4.12).

We shall now prove the converse. Let \( U = (u, tr(u))^T \in \mathbb{D}_q \) be fixed and let \( V = (v, tr(v))^T \in \mathbb{D}_q \) be arbitrary. On account of (4.3) and (4.4), we have

\[
\alpha_1(u)(v - u) = \alpha_1(u)v - \alpha_1(u)u \quad \text{and} \quad \alpha_2(u)(v - u) = \alpha_2(u)v - \alpha_2(u)u
\]

\[
\leq L_1(v) + \tilde{L}_1(\alpha_1(u)) - \alpha_1(u)u,
\]

\[
\leq L_1(v) - L_1(u)
\]

and

\[
\alpha_2(u)(v - u) = \alpha_2(u)v - \alpha_2(u)u \quad \text{and} \quad \alpha_2(u)(v - u) = \alpha_2(u)v - \alpha_2(u)u
\]

\[
\leq L_2(v) + \tilde{L}_2(\alpha_2(u)) - \alpha_2(u)u,
\]

\[
\leq L_2(v) - L_2(u).
\]
Therefore, by (4.13) and using (4.13)- (4.14), we have
\[
J(V) - J(U) = \frac{1}{2} \int_{\Omega} (|\nabla v|^2 - |\nabla u|^2) \, dx + \int_{\Omega} (L_1(v) - L_1(u)) \, dx \\
+ \frac{1}{2} \int_{\Gamma} c (|v|^2 - |u|^2) \frac{dS}{b} + \frac{q}{2} \int_{\Gamma} (|\nabla \Gamma v|^2 - |\nabla \Gamma u|^2) \, dS \\
+ \int_{\Gamma} (L_2(v) - L_2(u)) \frac{dS}{b} \\
\geq \int_{\Omega} \nabla u \cdot \nabla (v - u) \, dx + \int_{\Omega} \alpha_1(u)(v - u) \, dx \\
+ \int_{\Gamma} cu(v - u) \frac{dS}{b} + q \int_{\Gamma} \nabla \Gamma u \cdot \nabla \Gamma (v - u) \, dS + \int_{\Gamma} \alpha_2(u)(v - u) \frac{dS}{b}.
\]
This inequality is also true for \( V = U + W \in \mathbb{D}_q \), for some arbitrary \( W \in \mathbb{D}_q \). Indeed, let \( W = (w, tr(w))^T \in \mathbb{D}_q \) be fixed, \( w_m := [w \wedge m] \vee (-m) \) and set \( W_m := (w_m, tr(w_m))^T \). Let \( W_{m,n} = (w_{m,n}, tr(w_{m,n}))^T \) be a sequence in \( \mathbb{D}_q \) such that \(-m \leq w_{m,n} \leq m, w_{m,n} \to w_m \) in \( H^1(\Omega) \) and \( tr(w_{m,n}) \to tr(w_m) \) in \( H^1(\Gamma) \), if \( q > 0 \), as \( n \to \infty \). Then,
\[
J(W_{m,n} + U) - J(U) = \lim_{n \to \infty} J(W_{m,n} + U) - J(U) \\
\geq \lim_{n \to \infty} \left( \int_{\Omega} f w_{m,n} \, dx + \int_{\Gamma} g w_{m,n} \frac{dS}{b} \right) \\
\geq \int_{\Omega} f w_m \, dx + \int_{\Gamma} g w_m \frac{dS}{b}.
\]
Passing to the limit as \( m \to \infty \) in (4.16) in a standard way and using the fact \( W \in \mathbb{D}_q \) is arbitrary, we immediately get
\[
J(W + U) - J(U) \geq \int_{\Omega} f w \, dx + \int_{\Gamma} g w \frac{dS}{b}.
\]
Since \( \mathbb{D}_q \) is a vector space, we also obtain the corresponding inequality (4.17) when replacing \( W + U \) by \( V \). Hence, \( F \in \partial J(U) \) and this completes the proof of the claim.

We have shown that the (single-valued) subdifferential of the functional \( J \) at \( U \) is given by
\[
D(\partial J) = \left\{ (u, tr(u))^T \in \mathbb{D}_q : -\Delta u + \alpha_1(u) \in L^2(\Omega), b(x) \frac{\partial u}{\partial n} - qb(x) \Delta \Gamma u + \alpha_2(u) \in L^2(\Gamma) \right\}
\]
and
\[
\partial J(U) = \left( -\Delta u + \alpha_1(u), b(x) \frac{\partial u}{\partial n} + c(x) u - qb(x) \Delta \Gamma u + \alpha_2(u) \right)^T.
\]
Since the functional \( J \) is proper, convex and lower-semicontinuous, it follows from Minty’s theorem [14] that the operator \( B := \partial J \) is maximal monotone (or \( -B \) is m-dissipative),
for our choice of the function $j_2(x,u)$ in (4.6). Thus, the first result of this section is the following.

**Theorem 4.1.** The operator $B$ is the subdifferential of a proper, convex, lower semicontinuous function on $\mathbb{X}_2$.

Theorem 4.1 applies to both $A$, the negative Wentzell Laplacian (by taking both $\alpha_1$ and $\alpha_2$ to be zero) and to the operator governing (1.5) on $\mathbb{X}_2$. We remark that the above construction leads easily to a proof that the Wentzell Laplacian has a compact resolvent. Of course, this follows easily from the results quoted in Section 3, but the compactness does not require $C^\infty$-regularity.

Next, let $A_2U = (\alpha_1(u), \alpha_2(v))^T$, for every $U \in D(A_2)$, where

$$D(A_2) = \{(u,v)^T \in \mathbb{X}_2 : (\alpha_1(u), \alpha_2(v))^T \in \mathbb{X}_2\}. \quad (4.20)$$

Define the functional $J_2 : \mathbb{X}_2 \to [0, +\infty]$ by

$$J_2(U) = \left\{ \begin{array}{ll}
\int_\Omega L_1(u)dx + \int_\Gamma L_2(v)^dS, & \text{if } (u,v)^T \in D(J_2) \\
+\infty & \text{if } (u,v)^T \in \mathbb{X}_2 \setminus D(J_2),
\end{array} \right.$$

with effective domain

$$D(J_2) := \{(u,v)^T \in \mathbb{X}_2 : \int_\Omega \Lambda_1(u)dx + \int_\Gamma \Lambda_2(v)^dS/b(x) < \infty\}.$$ 

It is easy to see that, under the assumption (H1) on $\alpha_i$, the functional $J_2$ is proper, convex and lower-semicontinuous on $\mathbb{X}_2$. We have the following.

**Lemma 4.2.** Let $\alpha_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, satisfy (H1)-(H2). Then the subdifferential $\partial J_2$ and the operator $A_2$ coincide, that is, $D(\partial J_2) = D(A_2)$ and, for all $U := (u,v)^T \in D(A_2)$, we have

$$\partial J_2(U) = A_2U = (\alpha_1(u), \alpha_2(v))^T.$$ 

**Proof.** Note that (H1) implies that $\partial J_2$ is a single valued operator. Let $U = (u,v)^T \in D(J_2)$ and $(f,g)^T = \partial J_2(U)$. Then, by definition, $(f,g)^T \in \mathbb{X}_2$ and for every $V := (u_1,v_1)^T \in D(J_2)$, we get

$$\int_\Omega f(u_1-u)dx + \int_\Gamma g(v_1-v)^dS/b(x) \leq J_2(V) - J_2(U). \quad (4.21)$$

Next, let $W = (u,v)^T + t(u_2,v_2)^T$, with $(u_2,v_2)^T \in D(J_2)$ and $0 < t \leq 1$. Since (H2) implies that $D(J_2)$ is a vector space, then $W \in D(J_2)$. Now, replacing $V$ in (4.21) with $W$, dividing by $t$ and taking the limit as $t \to 0^+$ (where we make use of the Lebesgue Dominated Convergence theorem once again), we obtain

$$\int_\Omega f u_2 dx + \int_\Gamma g v_2 ^dS/b(x) \leq \int_\Omega \alpha_1(u)u_2 dx + \int_\partial \Omega \alpha_2(v)v_2 ^dS/b(x). \quad (4.22)$$

Changing $(u_2,v_2)^T$ to $-(u_2,v_2)^T$ in (4.22) gives

$$\int_\Omega f u_2 dx + \int_\Gamma g v_2 ^dS/b(x) = \int_\Omega \alpha_1(u)u_2 dx + \int_\partial \Omega \alpha_2(v)v_2 ^dS/b(x).$$
In particular, taking \( v_2 = 0 \), for every \( u_2 \in C_0^\infty(\Omega) \), we have
\[
\int_\Omega fu_2 \, dx = \int_\Omega \alpha_1(u)u_2 \, dx,
\]
and this shows that \( \alpha_1(u) = f \). Similarly, one obtains \( \alpha_2(v) = g \). We have shown that
\[
U := (u, v)^T \in D(A_2) \quad \text{and} \quad \partial J_2(U) = (\alpha_1(u), \alpha_2(v))^T.
\]
Conversely, let \( U = (u, v)^T \in D(A_2) \) and set \((f, g)^T := A_2U = (\alpha_1(u), \alpha_2(v))^T\). Observe preliminarily that, owing to (H2), there exist constants \( t_i > 0 \) and \( k_i \in (0, 1] \) such that
\[
k_i t \alpha_i(t) \leq \Lambda_i(t) \leq t \alpha_i(t), \text{ for all } |t| \geq t_i, \ i = 1, 2.
\]
Since \((\alpha_1(u), \alpha_2(v))^T \in X_2\), from (4.23), it follows that
\[
\int_\Omega \Lambda_1(u) \, dx = \int_{\{x \in \Omega: |u(x)| < t_1\}} \Lambda_1(u) \, dx + \int_{\{x \in \Omega: |u(x)| \geq t_1\}} \Lambda_1(u) \, dx
\leq |\Omega| (\Lambda_1(t_1) + \Lambda_1(-t_1)) + \int_\Omega u \alpha_1(u) \, dx < \infty,
\]
where a similar inequality holds for \( \Lambda_2 \). Hence
\[
\int_\Omega \Lambda_1(u) \, dx + \int_{\partial \Omega} \Lambda_2(v) \, \frac{dS}{b(x)} < \infty
\]
and this shows that \((u, v)^T \in D(J_2)\). Let \( V = (u_1, v_1)^T \in D(J_2)\). Note that by (4.13) and (4.14), we have once more that
\[
\alpha_1(u)(u_1 - u) \leq L_1(u_1) - L_1(u)
\]
and
\[
\alpha_2(v)(v_1 - v) \leq L_2(v_1) - L_2(v).
\]
Therefore, on account of (4.24)–(4.26), it follows that
\[
\int_\Omega f(u_1 - u) \, dx + \int_\Gamma g(v_1 - v) \, \frac{dS}{b(x)} = \int_\Omega \alpha_1(u)(u_1 - u) \, dx + \int_{\partial \Omega} \alpha_2(v)(v_1 - v) \, \frac{dS}{b(x)}
\leq J_2(V) - J_2(U).
\]
By definition, we have shown that \((\alpha_1(u), \alpha_2(v))^T = \partial J_2(U)\). Hence, \( U \in D(\partial J_2) \) and \( A_2U = \partial J_2(U) \). This completes the proof. \( \square \)

We will need the following results from semigroup theory and convex analysis.

**Definition 4.3** ([2]). Let \( \mathcal{H} \) be a real Hilbert space. Two subsets \( K_1 \) and \( K_2 \) are almost equal, written as \( K_1 \simeq K_2 \), if \( K_1 \) and \( K_2 \) have the same closure and the same interior, that is, \( \overline{K_1} = \overline{K_2} \) and \( \text{int}(K_1) = \text{int}(K_2) \).

The following result is contained in [2] pp.173–174.

**Theorem 4.4.** Let \( A \) and \( B \) be subdifferentials of proper convex lower semicontinuous functionals \( \varphi_1 \) and \( \varphi_2 \), respectively, on a real Hilbert space \( \mathcal{H} \) with \( D(\varphi_1) \cap D(\varphi_2) \neq \emptyset \). Let \( C \) be the subdifferential of the proper, convex lower semicontinuous functional \( \varphi_1 + \varphi_2 \), that is, \( C = \partial(\varphi_1 + \varphi_2) \). Then
\[
\mathcal{R}(A) + \mathcal{R}(B) \subset \overline{\mathcal{R}(C)} \quad \text{and} \quad \text{Int}(\mathcal{R}(A) + \mathcal{R}(B)) \subset \text{Int}(\mathcal{R}(C)). \tag{4.26}
\]
In particular, if the operator \( A + B \) is maximal monotone, then
\[
\mathcal{R}(A + B) \simeq \mathcal{R}(A) + \mathcal{R}(B)
\]
\text{(4.27)}
and this is the case, if \( \partial(\varphi_1 + \varphi_2) = \partial \varphi_1 + \partial \varphi_2 \).

Here by \( \mathcal{R}(A) + \mathcal{R}(B) \) we mean
\[
\bigcup \{ Af + Bg : f \in D(A), g \in D(B) \}
= \bigcup \{ h + k : (f, h) \in A, (g, k) \in B \text{ for some } f, g \in \mathcal{H} \}.
\]
We use the union symbol since \( A \) and \( B \) may be multi-valued. However, in our applications, \( A \) and \( B \) will be single valued.

Let us recall that we want to solve the following problem:
\[
\begin{cases}
-\Delta u + \alpha_1(u) = f_1(x) \quad \text{in } \Omega, \\
\frac{b(x)}{\partial n} \frac{\partial u}{\partial n} + c(x)u - qb(x)\Delta_{\Gamma}u + \alpha_2(u) = f_2(x) \quad \text{on } \Gamma.
\end{cases}
\]  
\text{(4.28)}

In order to solve \text{(4.28)}, recall that \( A \) is the linear operator, defined in Section 2 (see \text{(2.11)}). More precisely, \( A \) has the following operator representation:
\[
A = \begin{pmatrix}
-\Delta & 0 \\
\frac{b(x)}{\partial n} & cI - qb\Delta_{\Gamma}
\end{pmatrix}.
\]  
\text{(4.29)}

Denote the null space of \( A \) by \( \mathcal{N}(A) \). Then \( U = (u, tr(u))^T \in \mathcal{N}(A) \) if and only if (by definition) \( u \) is a weak solution of
\[
\begin{cases}
-\Delta u = 0 \quad \text{in } \Omega, \\
\frac{b(x)}{\partial n} \frac{\partial u}{\partial n} + c(x)u - qb(x)\Delta_{\Gamma}u = 0 \quad \text{on } \Gamma,
\end{cases}
\]  
\text{(4.30)}
that is, \( u \in H^1(\Omega) \) with \( tr(u) \in H^1(\Gamma) \) if \( q > 0 \) and
\[
\int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Gamma} cuv dS/b + q \int_{\Gamma} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} v dS = 0,
\]  
\text{(4.31)}
for all \( v \in H^1(\Omega) \) with \( tr(v) \in H^1(\Gamma) \) if \( q > 0 \). In this case it is easy to see that \( u \) is a weak solution of \text{(4.30)} if and only if \( u \in H^1(\Omega) \) with \( tr(u) \in H^1(\Gamma) \), if \( q > 0 \), and \text{(4.31)} holds for all \( v \in H^1(\Omega) \) with \( tr(v) \in H^1(\Gamma) \), if \( q > 0 \). Hence, it is clear that the null space of \( A \) is \( \mathcal{N}(A) = \mathbb{R}1 = \{ C1 : C \in \mathbb{R} \} \) if \( c \equiv 0 \) in \text{(4.30)}, that is, \( \mathcal{N}(A) \) consists of all the real constant functions on \( \bar{\Omega} \). We shall discuss this case first.

From now on, let \( A_1 \) be the linear operator \( A \) corresponding to the case of \( c \equiv 0 \). Moreover, let \( A_3 \) be the subdifferential \( \partial J \) of the functional \( J \), defined in \text{(1.5)-(1.6)}, that is, \( A_3 := \partial J = \partial (J_1 + J_2) \) (see \text{(1.18)-(1.20)}). It follows, from the assumptions on the functions \( \alpha_1, \alpha_2 \) and the results of Section 2, that \( A_i = \partial J_i \) for each \( i = 1, 2, 3 \), where each \( J_i \) is a proper, convex and lower semicontinuous functional on \( \mathbb{X}_2 \).

Let us recall the Fredholm alternative, which says that for any selfadjoint operator \( B \) with compact resolvent and \( 0 \notin \rho(B) \), we have that the range \( \mathcal{R}(B) = \overline{\mathcal{R}(B)} = \mathcal{N}(B)^\perp \). This is the case with our operator \( A_1 \), that is, we have
\[
\mathcal{R}(A_1) = \mathcal{N}(A_1)^\perp = 1^\perp = \left\{ F \in \mathbb{X}_2 : \int_{\bar{\Omega}} F d\mu = 0 \right\},
\]  
\text{(4.32)}
where the measure $\mu$ is defined by $d\mu = dx|_{\Omega} \otimes \frac{dS}{b}|_{\Gamma}$ on $\overline{\Omega}$. Let us now define $\lambda_1, \lambda_2 \in \mathbb{R}_+$ by
\[
\lambda_1 = \int_{\Omega} dx, \quad \lambda_2 = \int_{\Gamma} \frac{dS}{b}
\] (4.33)
so that $\mu(\overline{\Omega}) = \lambda_1 + \lambda_2$. We also define the average of $F$ with respect to the measure $\mu$, as follows:
\[
\text{ave}_\mu (F) := \frac{1}{\mu(\overline{\Omega})} \int_{\overline{\Omega}} F d\mu = \frac{1}{\mu(\overline{\Omega})} \left( \int_{\Omega} f_1 dx + \int_{\Gamma} f_2 \frac{dS}{b} \right),
\] (4.34)
for every $F = (f_1, f_2)^T \in \mathbb{X}_2$.

We now restate Theorem 4.2.

**Theorem 4.5.** Let $\alpha_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, satisfy (H1). Let $c \equiv 0$ in (4.28) and let $F = (f_1, f_2)^T \in \mathbb{X}_2$. A necessary condition for the existence of a weak solution of (4.28) is
\[
\text{ave}_\mu (F) \in \frac{\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)}{\lambda_1 + \lambda_2},
\] (4.35)
while a sufficient condition is that $\alpha_i$ satisfies (H2) and
\[
\text{ave}_\mu (F) \in \text{int} \left( \frac{\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)}{\lambda_1 + \lambda_2} \right).
\] (4.36)
Assuming (H2), the condition (4.35) is both necessary and sufficient when $\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)$ is open, which holds if at least one of $\mathcal{R}(\alpha_1), \mathcal{R}(\alpha_2)$ is open.

**Proof.** Let $\alpha_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, satisfy (H1). Let $F = (f_1, f_2)^T \in \mathbb{X}_2$ be given and let $u$ be a weak solution of (4.28) with $c \equiv 0$. Then (see Definition 1.2), $u \in H^1(\Omega)$, $\alpha_1(u) \in L^1(\Omega)$, $\alpha_2(tr(u)) \in L^1(\Gamma)$, $tr(u) \in H^1(\Gamma)$ if $q > 0$ and
\[
\int_{\Omega} f_1 v dx + \int_{\Gamma} f_2 v \frac{dS}{b} = \int_{\Omega} \nabla u \cdot \nabla v dx 
\] (4.37)
\[+ \int_{\Omega} \alpha_1(u) v dx + \int_{\Gamma} \alpha_2(u) v \frac{dS}{b} + q \int_{\Gamma} \nabla_\Gamma u \cdot \nabla_\Gamma v dS,
\]
for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$, if $q > 0$, and all $v \in H^1(\Omega) \cap C(\overline{\Omega})$ with $tr(v) \in H^1(\Gamma)$, if $q > 0$. Taking $v = 1$ in (4.37), we obtain
\[
\int_{\Omega} F d\mu = \int_{\Omega} f_1 dx + \int_{\Gamma} f_2 \frac{dS}{b} = \int_{\Omega} \alpha_1(u) dx + \int_{\Gamma} \alpha_2(u) \frac{dS}{b}
\] \[\in (\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)) ,
\]
and so (4.35) holds. This proves the necessity.

For the sufficiency, let (4.36) hold and assume that $\alpha_i$ satisfies (H2). To show that (4.28), with $c \equiv 0$, has a weak solution $u$, it is enough to prove that $F := (f_1, f_2) \in \mathcal{R}(A_3)$. To this end, we will make use of (4.26) from Theorem 4.4 to show that $F \in \text{int} (\mathcal{R}(A_1) + \mathcal{R}(A_2)) \subset \mathcal{R}(A_3)$. We know that $-A_1, -A_2$ and $-A_3$ are m-dissipative on $\mathbb{X}_2$ and $A_i = \partial J_i$, for every
i = 1, 2, 3, where each \( J_i \), \( i = 2, 3 \), is a proper, convex and lower semicontinuous functional on \( X_2 \). Here, \( J_3 = J_1 + J_2 \) has the effective domain \( D(J_3) = D(J_1) \cap D(J_2) \neq \emptyset \).

Let \( c_1, c_2 \in \mathbb{R}, C = (c_1, c_2)^T \in X_2 \) and let \( C \) be the family of such vectors \( C \) in \( X_2 \). Let

\[
Q := \{ C \in C : c_i \in \mathcal{R}(\alpha_i), \ i = 1, 2 \}.
\]

Clearly \( Q \subset \mathcal{R}(A_2) \), since \( c_i = \alpha_i(d_i) \) for some constant function \( d_i \) on \( \Omega \) (if \( i = 1 \)) or on \( \Gamma \) (if \( i = 2 \)). Now let (4.36) hold for \( F \in X_2 \). We must show \( F \in \mathcal{R}(A_3) \). By (4.36) we may choose \( C = (c_1, c_2)^T \in Q \) such that

\[
\text{ave}_\mu(F) = \frac{\lambda_1 c_1 + \lambda_2 c_2}{\lambda_1 + \lambda_2} \in \text{int} \left( \frac{\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)}{\lambda_1 + \lambda_2} \right),
\]

where \( \lambda_1, \lambda_2 \) are given by (4.33). Then, for \( F \in X_2 \), we have

\[
F = [F - C] + C.
\]

First, \( F - C \in \mathcal{R}(A_1) = \mathcal{N}(A_1)^\perp = 1^\perp \), since

\[
\int_{\Omega} (F - C) \, d\mu = \int_{\Omega} F \, d\mu - (\lambda_1 c_1 + \lambda_2 c_2) = \int_{\Omega} [F - \text{ave}_\mu(F)] \, d\mu = 0.
\]

Next, clearly \( C \in \mathcal{R}(A_2) \). Thus, it is readily seen that \( F \in (\mathcal{R}(A_1) + \mathcal{R}(A_2)) \). Let now \( \varepsilon > 0 \) be given. We want \( \varepsilon > 0 \) to be small enough, in particular, suppose

\[
0 < \varepsilon < \frac{1}{2} \text{dist} \left( \frac{\lambda_1 c_1 + \lambda_2 c_2}{\lambda_1 + \lambda_2}, K \right),
\]

where \( K \) consists of the endpoints of the interval \( I = (\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)) / (\lambda_1 + \lambda_2) \). Let \( \tilde{F} = (\tilde{f}_1, \tilde{f}_2)^T \in X_2 \) satisfy \( \|F - \tilde{F}\|_{X_2} < \varepsilon \). We want to pick \( \tilde{C} = (\tilde{c}_1, \tilde{c}_2)^T \in Q \) such that

\[
\|C - \tilde{C}\|_{X_2} < \varepsilon \quad \text{and} \quad \text{ave}_\mu(\tilde{F}) = \frac{\lambda_1 \tilde{c}_1 + \lambda_2 \tilde{c}_2}{\lambda_1 + \lambda_2}. \tag{4.38}
\]

To see how to do this, let \( \mathcal{J}_i = \mathcal{R}(\alpha_i) \) for \( i = 1, 2 \). Then \( c_i \in \mathcal{J}_i \) and

\[
\text{ave}_\mu(F) = \frac{\lambda_1 c_1 + \lambda_2 c_2}{\lambda_1 + \lambda_2} \in \text{int} \left( \frac{\lambda_1 \mathcal{J}_1 + \lambda_2 \mathcal{J}_2}{\lambda_1 + \lambda_2} \right). \tag{4.39}
\]

We may choose at least one of \( \tilde{c}_1, \tilde{c}_2 \), for \( i \in \mathcal{J}_1 \), cannot be the right hand end point of \( \mathcal{J}_k \) for both \( k = 1, 2 \), because of (4.36). In a similar way, we may choose one of \( \tilde{c}_1, \tilde{c}_2 \), call it \( \tilde{c}_i \), to be larger than \( c_i \). Next,

\[
\left| \text{ave}_\mu(F) - \text{ave}_\mu(\tilde{F}) \right| \leq \|F - \tilde{F}\|_{X_2} < \varepsilon,
\]

by the Schwarz inequality. By this observation and (4.38)-(4.39), we can find \( \tilde{C} = (\tilde{c}_1, \tilde{c}_2) \in Q \) such that (4.38) holds. Thus, \( \mathcal{R}(A_1) + \mathcal{R}(A_2) \) contains an \( \varepsilon \)-ball in \( X_2 \), centered at \( F \), for sufficiently small \( \varepsilon > 0 \). Thus,

\[
F \in \text{int} \left( \mathcal{R}(A_1) + \mathcal{R}(A_2) \right) \subset \text{int} \left( \mathcal{R}(A_3) \right) \subset \mathcal{R}(A_3),
\]
by (4.20). Consequently, problem (4.28) is (weakly) solvable in the sense of Definition 1.1 for any \( f_1 \in L^2(\Omega) \), \( f_2 \in L^2(\Gamma) \), if (4.36) holds. This completes the proof. \(\square\)

We will now give some examples as applications of Theorem 4.5.

**Example 4.6.** Let \( \alpha_1(s) \) or \( \alpha_2(s) \) be equal to \( \alpha(s) = r|s|^{p-1}s \), where \( r, p > 0 \). Then, it is clear that \( \alpha \) satisfies (H1) and that \( L(s) = \Lambda(s) = \frac{r}{p+1}|s|^{p+1} \) also satisfies (H2). Note that \( \mathcal{R}(\alpha) = \mathbb{R} \). Then, it follows that problem (4.28) with \( c \equiv 0 \) is solvable for any \( f_1 \in L^2(\Omega) \), \( f_2 \in L^2(\Gamma) \).

**Example 4.7.** Consider the case when \( c = q = \alpha_2 \equiv 0 \) in (4.28), that is, consider the following boundary value problem:

\[
\begin{cases}
-\Delta u + \alpha_1(u) = f_1(x) \text{ in } \Omega, \\
b(x) \frac{\partial u}{\partial n} = f_2(x) \text{ on } \Gamma.
\end{cases}
\]

Then, by Theorem 4.3, this problem has a weak solution if

\[
\int_{\Omega} f_1 \, dx + \int_{\Gamma} f_2 \frac{dS}{b} \in \lambda_1 \text{int} \left( \mathcal{R}(\alpha_1) \right),
\]

which yields the classical Landesman-Lazer result (see (1.3)) for \( f_2 \equiv 0 \).

**Example 4.8.** Let us now consider the case when \( \alpha_1 \equiv 0 \) and \( \alpha_2 \equiv \alpha \), where \( \alpha \) is a continuous, monotone nondecreasing function on \( \mathbb{R} \) such that \( \alpha(0) = 0 \). The problem

\[
\begin{cases}
-\Delta u = f_1(x) \text{ in } \Omega, \\
q b(x) \frac{\partial u}{\partial n} - q b(x) \Delta \tau u + \alpha(u) = f_2(x) \text{ on } \Gamma,
\end{cases}
\]

has a weak solution if

\[
\int_{\Omega} f_1 \, dx + \int_{\Gamma} f_2 \frac{dS}{b} \in \lambda_2 \text{int} \left( \mathcal{R}(\alpha) \right).
\]

For example, if we choose \( \alpha(s) = \arctan(s) \) in (4.40), (4.41) becomes the necessary and sufficient condition

\[
\left| \frac{1}{\lambda_2} \left( \int_{\Omega} f_1 \, dx + \int_{\Gamma} f_2 \frac{dS}{b} \right) \right| < \frac{\pi}{2}.
\]

Note that \( \alpha(s) = \arctan(s) \) satisfies (H1) and that \( L_2(s) = \Lambda_2(s) = s \arctan(s) - \ln \sqrt{1 + s^2} \) satisfies (H2).

Let us now turn to the case when \( c > 0 \) on a set of positive \( dS \)-measure (that is, \( c(x) \) is not identically zero on the boundary \( \Gamma \)) and consider \( A_1^1 \) to be the linear operator \( A \) of (2.11) corresponding to this case. Since \( A_1^1 = (A_1^1)^* \geq 0 \) and \( A_1^1 \) has compact resolvent, it has a ground state \( Z = (z_{\Omega}, z_{\Gamma})^T \). That is, \( \lambda = \min \sigma(A_1^1) \) is a simple eigenvalue, \( \lambda > 0 \), and \( \mathcal{N}(A_1^1 - \lambda) = \{ CZ : C \in \mathbb{R} \} \) for some positive function \( Z \) on \( \Omega \).

Before proceeding further, we find the ground state of \( A_1^1 \) in a simple one-dimensional example. Let \( \Omega = (0, 1) \), \( \Gamma = \{0, 1\} \), \( b_0 = b_1 = 1 \), \( q = 0 \) and \( c_0, c_1 \) will be specified in the sequel. Here \( b_j = b(j) \) and \( c_j = c(j) \). We will choose \( c_j, j = 0, 1 \) so that the smallest eigenvalue of \( A_1^1 \) is \( \lambda = 1 \). The required positive solution of \( z'' + z = 0 \) has the form
Let \( q \) be a nonnegative function which is positive on \( \Gamma_1 \subset \Gamma \), where \( \int_{\Gamma_1} dS > 0 \).

Let \( q = 0 \) and let \( \alpha \) be a continuous, monotone nondecreasing function on \( \mathbb{R} \) such that \( \alpha(0) = 0 \), \( \alpha(\pm \infty) = \lim_{s \to \pm \infty} \alpha(s) \). Let \( F = (f_1, f_2)^T \in \mathbb{R}_2 \). Also, suppose that \( \lambda > 0 \) is the smallest eigenvalue of \( A_1^1 \) and let \( Z \) be a positive member of the one-dimensional eigenspace of \( A_1 := A_1^1 - \lambda I \). Here we view \( Z \in \mathbb{R}_2 \) as \( Z = (z_1, z_2)^T : \overline{\Omega} \to \mathbb{R} \), and \( Z \) corresponds to a \( z_1 \in C(\overline{\Omega}) \), with \( z_2 = z_1|_{\Gamma} \) and \( z_1 \) is a positive function on \( \overline{\Omega} \). A necessary condition for the existence of a weak solution for

\[
\begin{align*}
-\Delta u - \lambda u + \alpha(u) &= f_1 \text{ in } \Omega, \\
\Delta u + b(x) \frac{\partial u}{\partial n} + (c(x) + \lambda) u &= f_2 \text{ on } \Gamma
\end{align*}
\]

(4.45)
is

\[
\alpha(-\infty) \langle Z, 1 \rangle_{\mathbb{R}_2} \leq \langle F, Z \rangle_{\mathbb{R}_2} \leq \alpha(+\infty) \langle Z, 1 \rangle_{\mathbb{R}_2},
\]

(4.46)

while a sufficient condition is that \( \alpha \) satisfies (H2) and

\[
\frac{\alpha(-\infty)}{\min Z} < \langle F, Z \rangle_{\mathbb{R}_2} < \frac{\alpha(+\infty)}{\max Z}.
\]

(4.47)

Proof. For the necessity part, multiply the first equation of (4.45), the second equation of (4.45) by \( z \) and integrate by parts; here \( Z = (z|_\Omega, z|_\Gamma)^T \). Using the divergence theorem and the fact that \( \mathcal{N}(A_1^1 - \lambda) = \text{span \{Z\}} \), we obtain

\[
\int_{\Omega} \alpha(u) z dx + \int_{\Gamma} \alpha(v) z|_\Gamma \frac{dS}{b} = \int_{\Omega} f_1 z dx + \int_{\Gamma} f_2 z|_\Gamma \frac{dS}{b}.
\]

where \( U = (u, v)^T \) with \( v = \text{tr} (u) \) is the solution of (4.45) with \( F = (f_1, f_2)^T \). Since \( Z > 0 \), this equation becomes

\[
\frac{\langle F, Z \rangle_{\mathbb{R}_2}}{\langle Z, 1 \rangle_{\mathbb{R}_2}} = \frac{\langle \alpha, Z \rangle_{\mathbb{R}_2}}{\langle Z, 1 \rangle_{\mathbb{R}_2}} \in [\alpha(-\infty), \alpha(+\infty)],
\]
and the necessary condition \((4.46)\) follows. If \(\alpha (-\infty) < \alpha (r)\) for all \(r \in \mathbb{R}\), then the endpoint \(\alpha (-\infty)\) can be excluded. A similar remark applies to \(\alpha (+\infty)\).

The sufficiency proof is like that of Theorem \(4.5\), but \(Z\) is not a constant. By the Fredholm alternative, we have

\[
\mathcal{R} (A_1) = \mathcal{N} (A_1)^\perp = \{ F \in \mathbb{X}_2 : \langle F, Z \rangle_{\mathbb{X}_2} = 0 \}. \tag{4.48}
\]

Let us also define the nonlinear operator \(A_5 U = (\alpha (u), 0)^T\), for \((u, v)^T \in D (A_5)\) such that

\[
D (A_5) = \left\{ (u, v)^T \in \mathbb{X}_2 : u \text{ has a trace } tr(u) = v \text{ and } (\alpha (u), 0)^T \in \mathbb{X}_2 \right\}. \tag{4.49}
\]

Let us recall that, due to Theorem \(4.1\), we know that \(-A_4, -A_5\), are m-dissipative on \(\mathbb{X}_2\) and \(A_i = \partial J_i\), for every \(i = 4, 5\) and each \(J_i\) is a proper, convex and lower semicontinuous functional on \(\mathbb{X}_2\). Let \(J_6 := J_4 + J_5\) with domain \(D (J_6) := D (J_4) \cap D (J_5) \neq \emptyset\). Then \(J_6\) is a proper, convex and lower semicontinuous functional on \(\mathbb{X}_2\). Let \(A_6 := \partial (J_4 + J_5)\). Then

\(-A_6\) is m-dissipative on \(\mathbb{X}_2\). It follows, from \((4.26)\), that

\[
\mathcal{R} (A_4) + \mathcal{R} (A_5) \subset \overline{\mathcal{R} (A_6)} \text{ and } int (\mathcal{R} (A_4) + \mathcal{R} (A_5)) \subset int (\mathcal{R} (A_6)). \tag{4.50}
\]

Suppose now that \(Z\) is a positive unit vector in \(\mathcal{N} (A_1)\) (recall that \(A_4 = A_1^\dagger - \lambda I\), that is, \(\lambda = \min \sigma (A_4)\), \(A_1^\dagger Z = \lambda Z\), \(\|Z\|_{\mathbb{X}_2} = 1\) and \(Z > 0\). For \(F \in \mathbb{X}_2\), we have

\[
F = [F - \langle F, Z \rangle_{\mathbb{X}_2} Z] + \langle F, Z \rangle_{\mathbb{X}_2} Z \in \mathcal{R} (A_1) + \mathcal{R} (A_5),
\]

provided that

\[
\alpha (-\infty) < \langle F, Z \rangle_{\mathbb{X}_2} Z < \alpha (+\infty)
\]

holds pointwise on \(\Omega\). But for, \(\bar{F} = (\bar{f}_1, \bar{f}_2)^T \in \mathbb{X}_2\) and \(\|F - \bar{F}\|_{\mathbb{X}_2} < \varepsilon\), we have again

\[
\left\| \langle \bar{F}, Z \rangle_{\mathbb{X}_2} Z - \langle F, Z \rangle_{\mathbb{X}_2} Z \right\|_{\mathbb{X}_2} = \left\| \langle \bar{F} - F, Z \rangle_{\mathbb{X}_2} Z \right\|_{\mathbb{X}_2} \leq \|F - \bar{F}\|_{\mathbb{X}_2} < \varepsilon,
\]

(4.51)

so then \(\alpha (-\infty) < \langle \bar{F}, Z \rangle_{\mathbb{X}_2} Z < \alpha (+\infty)\) on \(\Omega\), for \(\varepsilon > 0\) small enough. It follows that

\[
F \in int (\mathcal{R} (A_1) + \mathcal{R} (A_5)) \subset int (\mathcal{R} (A_6)) \subset \mathcal{R} (A_6),
\]

by \((4.50)\). This completes the proof of our theorem. \(\square\)

**Remark 4.10.** When \(\lambda = 0\) and \(Z \equiv 1\), we have, using a different normalization, \(\|Z\|_{\mathbb{X}_2}^2 = \mu (\Omega) = \lambda_1 + \lambda_2\), \(\min Z = \max Z = 1\); in this case, it turns out that \((4.47)\) reduces to \((4.56)\).

**Remark 4.11.** Of course the result in Theorem \(4.9\) is interesting only when

\[
\frac{\alpha (-\infty)}{\min Z} < \frac{\alpha (+\infty)}{\max Z}.
\]

But this always holds unless \(\alpha \equiv 0\).
Example 4.12. In the context of Theorem 4.9, let us now consider the one dimensional problem:
\[
\begin{align*}
- u'' + u + \alpha(u) &= f_1 \quad \text{in } \Omega = (0,1), \\
- u (j) + (-1)^{j+1} u' (j) + c_j u (j) &= f'_j, \quad j = 0, 1,
\end{align*}
\]

where \(c_j\) are given by (4.44) with \(\delta = 1/2\). It follows from (4.47) that, for (4.52) to have at least one solution, it suffices to have
\[
\alpha (-\infty) \cos (1/2) < \int_0^1 f_1 (x) \cos (x - 1/2) dx + (f'_2 + f'_3) \cos (1/2) < \alpha (+\infty).
\]

Moreover, choosing \(\alpha (u) = r \vert u \vert^{p-1} u, \quad r, p > 0\) in the first equation of (4.52), then (4.53) yields at least one solution to (4.52) for any \(f_1 \in L^2 (0,1)\) and \(f'_2 \in \mathbb{R}, j = 0, 1\).

Finally, let us consider as an application of our main theorems, an example for which \(q > 0\), that is, \(\Delta_{\Gamma}\) is present in the boundary conditions for our nonlinear elliptic problems (4.45). For this purpose, let \(\Omega\) be the two dimensional box \((0,1)^2 \subset \mathbb{R}^2, b (x, y) \equiv 1, \forall (x, y) \in \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, q > 0\) and \(c_i (x, y)\) will be determined in the sequel. The lines \(\Gamma_i\) and \(c_i\) will be defined below. We will choose \(c_i (x, y)\), so that the smallest eigenvalue of \(A^1_i\) is \(\lambda = 2\). The positive solution of \(\Delta z + 2z = 0\) has the form \(z (x, y) = \cos (x - 1/2) \cos (y - 1/2)\) (times a constant, which we take to be 1). Note that \(z (x, y) > 0\) on \(\overline{\Omega} = [0, 1]^2\). Thus, we need to choose positive \(c_i (x, y)\) for each \(i = 1, 2, 3, 4\) such that \(z (x, y)\) satisfies the correct boundary conditions. The boundary conditions are
\[
\left\{
\begin{array}{l}
-2z - z_y + c_1 (x, y) z - q_{yy} = 0 \quad \text{for } (x, y) \in \Gamma_1 = \{(x, 0) : x \in [0,1]\}, \\
-2z + z_x + c_2 (x, y) z - q_{xx} = 0 \quad \text{for } (x, y) \in \Gamma_2 = \{(1, y) : y \in [0,1]\}, \\
-2z + z_y + c_3 (x, y) z - q_{yy} = 0 \quad \text{for } (x, y) \in \Gamma_3 = \{(x, 1) : x \in [0,1]\}, \\
-2z - z_x + c_4 (x, y) z - q_{xx} = 0 \quad \text{for } (x, y) \in \Gamma_4 = \{(0, y) : y \in [0,1]\},
\end{array}
\right.
\]

since \(\partial / \partial n\) equals \(\partial / \partial x\) and \(\partial / \partial y\) along the lines \(\Gamma_2\) and \(\Gamma_3\), respectively and \(\partial / \partial n\) equals \(-\partial / \partial x\) and \(-\partial / \partial y\) along the lines \(\Gamma_4\) and \(\Gamma_1\), respectively. Moreover, we note that \(\Delta_{\Gamma}\) equals \(\partial / \partial y^2\) along \(\Gamma_1 \cup \Gamma_3\) and \(\partial / \partial x^2\) along \(\Gamma_2 \cup \Gamma_4\), respectively. Calculating in (4.54), we obtain, for any \(q \in (0, q_\pm)\), \(q_\pm = 2 \cos (1/2) \pm \tan (1/2)\), the functions
\[
\left\{
\begin{array}{l}
c_1 (x, y) = q_+ - q + d_1 (y), \\
c_2 (x, y) = q_+ - q + d_2 (x), \\
c_3 (x, y) = q_+ - q + d_3 (y), \\
c_4 (x, y) = q_+ - q + d_4 (x),
\end{array}
\right.
\]

where \(d_i\) are nonnegative, continuous functions on \([0,1]\) such that \(d_1 (0) = d_4 (0) = 0\) and \(d_2 (1) = d_3 (1) = 0\). Note that \(c_i > 0\) on \(\Gamma_i\) for each \(i\).

Example 4.13. Let us now consider the boundary value problem in the open rectangle \(\Omega = (0,1)^2:\)
\[
-\Delta u + 2u + \alpha (u) = f_1 \quad \text{in } \Omega,
\]

endowed with the boundary conditions of (4.54), except that now the zero values on the right sides of these equalities are replaced by the functions \(f'_2, f'_3, f'_2\) and \(f'_2\), respectively. Let \(c_i\)
be the functions defined in (4.55). It follows from (4.47) that for (4.56) to have at least one solution, it suffices to have

$$\frac{\alpha(-\infty)}{\cos^2(1/2)} < \mathcal{J} < \alpha(+\infty),$$

(4.57)

where

$$\mathcal{J} = \int_0^1 \int_0^1 f_1(x,y) \cos \left( x - \frac{1}{2} \right) \cos \left( y - \frac{1}{2} \right) dx dy + \sum_{i=1}^4 \int_{\Gamma_i} f_2^i dz dS_i$$

and each $\int dS_i$ denotes the path integral corresponding to each line $\Gamma_i$. Moreover, choosing

$$\alpha(u) = r |u|^{p-1} u, r, p > 0 \text{ in the (4.56)}, \text{ then (4.57) yields at least one solution to (4.56), for any } f_1 \in L^2(\Omega) \text{ and } f_2^i \in L^2(\Gamma_i), i = 1, 2, 3, 4.$$

We conclude the paper by stating sufficient conditions for (4.27) (see Theorem 4.4) to hold. It is worth mentioning, however, that such conditions are not necessary to prove Theorem 4.5, but that the results below have an interest on their own. We consider the following growth conditions for a function $\alpha: \mathbb{R} \to \mathbb{R}$:

\begin{itemize}
  \item [(GC1)] $N = 1$. No growth condition on $\alpha$.
  \item [(GC2)] $N = 2$. The function $\alpha$ is bounded by a power:
    $$|\alpha(s)| \leq C (1 + |s|^r), \text{ for all } s \in \mathbb{R},$$
    (4.58)
    where $C, r$ are positive constants.
  \item [(GC3)] $N = 3$. (4.58) holds with $r = N/(N-2)$.
  \item [(GC4)] This is (GC1), modified by replacing $r = N/(N-2)$ by $r = (N-1)/(N-2)$ in the case $N \geq 3$ and $q > 0$, and replacing $r = N/(N-2)$ by
    $$r = \begin{cases} \text{any number, if } N = 3 \\ \frac{N-1}{N-3}, \text{ if } N \geq 4. \end{cases}$$
\end{itemize}

We start with the following.

**Proposition 4.14.** Let $\alpha_1, \alpha_2 : \mathbb{R} \to \mathbb{R}$ satisfy (H1). Assume that

$$((\alpha_1(u), \alpha_2(u))^T \in \mathbb{X}_2, \text{ for all } u \in H^1(\Omega), \text{ if } q = 0,$$

(4.59)

$$((\alpha_1(u), \alpha_2(u)|_\Gamma))^T \in \mathbb{X}_2, \text{ for all } (u, tr(u))^T \in H^1(\Omega) \times H^1(\Gamma), \text{ if } q > 0.$$

Let $A_1, A_2$ and $A_3$ be as in the proof of Theorem 4.5. Then

$$A_1 + A_2 = A_3 \text{ and } \mathcal{R}(A_1) + \mathcal{R}(A_2) \simeq \mathcal{R}(A_3).$$

(4.61)

**Proof.** Let us first recall that, from Theorem 3.2, $D(A_1)$ equals either $H^2(\Omega)$ or $H^2_\Gamma(\Omega)$, according to whether $q = 0$ or $q > 0$. Moreover,

$$A_1 U = (-\Delta u, b(x)\partial_n u - qb(x)\Delta_\Gamma u)^T.$$

The operators $A_2, A_3$ are given in (4.20) and (4.18)-(4.19), respectively. Since $A_1 = \partial J_1$, $A_2 = \partial J_2$ and $A_3 = \partial J_3 := \partial (J_1 + J_2)$ with $D(J_1) \cap D(J_2) \neq \emptyset$, it follows that $A_1 + A_2 \subset A_3$. Hence, $D(A_1) \cap D(A_2) \subset D(A_3)$. We claim that $A_3 = A_1 + A_2$. To show this we must prove

$$D(A_3) \subset D(A_1) \cap D(A_2).$$
Assume \((4.59)\) and let \(U = (u, u|_\Gamma)^T \in D(A_3)\). Then \(U \in \mathbb{D}_0\), and from \((4.18)\),
\[-\Delta u + \alpha_1(u) \in L^2(\Omega), \quad \frac{\partial u}{\partial n} + \alpha_2(u) \in L^2(\Gamma), \text{ if } q = 0.\]
Therefore, \(u \in H^1(\Omega), \Delta u \in L^2(\Omega)\) and \(\frac{\partial u}{\partial n} \in L^2(\Gamma)\). Since \(\Omega\) is smooth, elliptic regularity implies that \(u \in H^2(\Omega)\). Hence, \(U \in D(A_1) \cap D(A_2)\), if \(q = 0\). If \(q > 0\), one also has that \(\frac{\partial u}{\partial n} - qb(x)\Delta u + \alpha_2(u) \in L^2(\Gamma)\) and \(tr\ (u) \in H^1(\Gamma)\). Since \(u \in H^2(\Omega)\), and \(\alpha_2(u) \in L^2(\Gamma)\), by \((4.60)\), we also have that \(\Delta u \in L^2(\Gamma)\). Elliptic regularity also implies that \(tr\ (u) \in H^2(\Gamma)\). Hence, \(U \in D(A_1) \cap D(A_2)\), if \(q > 0\). It is easy to verify that, for every \(U \in D(A_3) = D(A_1) \cap D(A_2)\), \(A_3 U = A_1 U + A_2 U\). The statement \((4.61)\) is a straightforward consequence of \((4.27)\). The proof is finished. \(\square\)

The following corollary is a consequence of Proposition \(4.14\).

**Corollary 4.15.** Let \(\alpha_1, \alpha_2 : \mathbb{R} \to \mathbb{R}\) be continuous, monotone nondecreasing functions satisfying the growth conditions \((GC1)-(GC2)\). Then \((4.59)-(4.60)\) are fulfilled and therefore, \((4.61)\) holds.

**Proof.** To prove this result, we need the following properties of Sobolev spaces. Since the domain \(\Omega\) has smooth boundary \(\Gamma\), one has the following:

1. If \(N = 1\), \(H^1(\Omega) \hookrightarrow C(\Omega)\).
2. If \(N = 2\), \(H^1(\Omega) \hookrightarrow L^p(\Omega)\), for every \(p \in [1, \infty)\) and \(H^1(\Gamma) \hookrightarrow C(\Gamma)\).
3. If \(N \geq 3\), \(H^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)\).
4. If \(N = 3\), \(H^1(\Gamma) \hookrightarrow L^q(\Gamma)\), for every \(q \in [1, \infty)\).
5. If \(N \geq 4\), \(H^1(\Gamma) \hookrightarrow L^{\frac{2(N-2)}{N-4}}(\Gamma)\).

Now, let \(\tilde{\Omega}\) denote either \(\Omega\) or \(\Gamma\) and suppose that \(q \geq 0\). Then the regularity properties of \(u \in H^1(\Omega)\), if \(q = 0\), \(u|_\Gamma \in H^1(\Gamma)\), if \(q > 0\) given in the five points above, and \(|\alpha (s)| \leq C (1 + |s|^\gamma)\) imply that \(\alpha (u) \in L^2(\tilde{\Omega})\), provided that \((GC1)-(GC2)\) are satisfied.

In particular, it is easy to check that \(\alpha_i (u) \in L^2(\tilde{\Omega})\), for \(i = 1, 2\). This completes the proof. \(\square\)

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**References**

[1] H. Brezis, Propriétés régularisantes de certains semi-groupes non linéaires, *Israel J. Math* 9 (1971), 513–534.
[2] H. Brezis and A. Haraux, Image d’une somme d’opérateurs monotones et applications, *Israel J. Math* 23 (1976), 165–186.
[3] H. Brézis and L. Nirenberg, Image d’une somme d’opérateurs non linéaires et applications, *C. R. Acad. Sci. Paris Sér. A-B* 284 (1977), A1365–A1368.
[4] J. Escher, Quasilinear parabolic systems with dynamical boundary conditions, *Comm. Partial Differential Equations* 18 (1993), 1309–1364.
[5] A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli, The heat equation with generalized Wentzell boundary condition, *J. Evol. Equ.* 2 (2002), 1–19.
[6] A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli, The heat equation with nonlinear general Wentzell boundary condition, *Adv. Differential Equations* 11 (2006), 481–510.

[7] M. Fila and P. Quittner, Large time behavior of solutions of a semilinear parabolic equation with a nonlinear dynamical boundary condition, Topics in Nonlinear Analysis, 251–272, *Progr. Nonlinear Differential Equations Appl.* 35, Birkhäuser, Basel, 1999.

[8] G. Ruiz Goldstein, Derivation and physical interpretation of general boundary conditions, *Adv. Differential Equations* 11 (2006), 457–480.

[9] J. A. Goldstein, Evolution equations with nonlinear boundary conditions, *Nonlinear semigroups, partial differential equations and attractors (Washington, D.C., 1985)*, 78–84, Lecture Notes in Math., 1248, Springer, Berlin, 1987.

[10] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Edition, Springer, Berlin, (1983).

[11] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1976.

[12] E. M. Landesman and A. C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech* 19 (1969/1970), 609–623.

[13] J. L. Lions and E. Magenes, *Problèmes aux limites non Homogènes et Applications. Vol. 2*, Travaux et Recherches Mathématiques, No. 18, Dunod, Paris, 1968.

[14] G. J. Minty, On the solvability of nonlinear functional equations of "monotonic" type, *Pacific J. Math* 14 (1964), 249–253.

[15] J. Mawhin, Semicoercive monotone variational problems, *Bull. Classes Sci. de l’Acad. Roy. Belg.* 73 (1987), 118-130.

[16] A. Miranville and S. Zelik, Exponential attractors for the Cahn-Hilliard equation with dynamic boundary conditions, *Math. Models Appl. Sci.* 28 (2005), 709–735.

[17] J. Peetre, Another approach to elliptic boundary value problems, *Comm. Pure Appl. Math* 14 (1961), 711–731.

[18] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 146. Marcel Dekker, Inc., New York, 1991.

[19] M. Taylor, *Partial Differential Equations*, I, II, III, Springer, New York, 1997.

[20] H. Triebel, *Theory of Function Spaces*, Monographs in Mathematics, 78. Birkhäuser Verlag, Basel, 1983.

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