Erratum: The remainder in the renewal theorem*  

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Abstract

We point out an error in “The remainder in the renewal theorem”, and show that the result is essentially correct in two important special cases.

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The main result in [1] claims that in a renewal process $S = (S_n, n \geq 0)$ whose step distribution $F$ has finite mean $m$ and whose tail $\overline{F}$ is regularly varying with index $-\alpha$ with $\alpha \in (1, 2)$, the renewal function $U$ has the following asymptotic behaviour:

$$W(x) := U(x) - m^{-1} \int_0^x (1 + \Phi(y))dy \sim \frac{|c_\alpha|x\overline{F}(x)^2}{m|2\beta - 1|} \text{ as } x \to \infty. \tag{0.1}$$

Here 

$$\beta = \alpha - 1, c_\alpha = \frac{(1 - 2\beta)\Gamma(1 - \beta)^2}{\Gamma(2 - 2\beta)},$$

and 

$$\overline{F}(x) = \int_x^\infty \phi(y)dy, \text{ with } \phi(y) = \frac{F(y)}{m}, y \geq 0. \tag{0.2}$$

Since $\overline{F} \in RV(-\beta)$ the RHS of (0.1) $\in RV(1 - 2\beta)$, so this is a substantial improvement on the previously known result that $W(x) = o(\int_0^x \Phi(y)dy)$, particularly for the case $\beta > 1/2$.

If $F$ is non-lattice, it is natural for $\phi$ to be involved, since it is the stationary density for the overshoot process of $S$, which fact is used in [1] to derive the following relation. First write $\phi_2$ for the convolution $\phi * \phi$ and define real-valued functions $g$ and $\overline{G}$ on $[0, \infty)$ by

$$g(y) = 2\phi(y) - \phi_2(y),$$

$$\overline{G}(y) = \int_y^\infty g(z)dz, \text{ so that } \overline{G}(0) = 1.$$

Then the relation

$$W(x) = \int_{[0,x]} \overline{G}(x - y)U(dy), \tag{0.3}$$

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which is (2.4) in [1], is key to the results therein. (Note that our $W$ is denoted by $m^{-1}V$ in [1].) The second crucial fact is that although $\overline{G}$ is the difference of two functions which are both in $RV(-\beta)$, it is in $RV(-2\beta)$, and actually

$$
\lim_{x \to \infty} \frac{\overline{G}(x)}{\overline{f}(x)^2} = c_\alpha = \frac{(1 - 2\beta)\Gamma(1 - \beta)^2}{\Gamma(2 - 2\beta)}.
$$

Unfortunately there are mistakes in the proof of (0.1) for the case $\alpha \in (3/2, 2)$. Specifically on P5, L9 of [1], it is claimed that having fixed $x_0 > 0$ such that $g'(x) = -g(x) > 0$ for $x > x_0$, then given $\varepsilon > 0$ we can find $x_1 > x_0$ with

$$
\int_{x_1}^{x} \overline{G}(x - y)dU(y) \leq \frac{1 + \varepsilon}{m} \int_{x_1}^{x} \overline{G}(x - y)dy,
$$

where $\overline{G}(z) := -\overline{F}(z)$. But on $[0, x_0]$ we have no control over the sign of $\overline{G}$, so this statement cannot be justified. It is also unclear how Lemma 2.1 can be applied, since the condition $\int_{0}^{\infty}Q(y)dy = \infty$ fails for $\alpha > 3/2$. A final error is that it is implicitly assumed in [1] that in the lattice case $\phi$ is a stationary density, but of course this is wrong: actually $\{\phi(n), n \in \mathbb{Z}\}$ is a stationary mass function.

Nevertheless the claimed result (0.1) is essentially correct in the two most important situations. In the lattice case the last mentioned error necessitates a slight change in the definition of $W$, (see (1.1) below and compare the LHS of (0.1)), but then we are able to give a simple argument to show that (0.1) holds with this new definition. In the absolutely continuous case, under a minor technical assumption we show that (0.1) follows, after some manipulation and use of (0.3), from a stronger result for the density of $U$ which is established in [2].

1 Lattice case

In this section we assume that $F$ is carried by $\mathbb{Z}$ and has period 1, and we specify that the renewal function $U$ and its modification $W$ are given for $x \geq 0$ by

$$
U(x) = \sum_{r=0}^{[x]} u(r), \quad W(x) = U(x) - m^{-1}\left(\sum_{x=0}^{[x]} (1 + \overline{\Phi}(s))\right),
$$

(1.1)

where $u(r) = \sum_{0}^{\infty} P(S_n = r)$. We start from the observation that the distribution with mass function

$$
\phi(n) = \frac{P(X > n)}{m} = \frac{F(n)}{m}, \quad n = 0, 1, 2, \cdots
$$

is stationary for the overshoot process. $\overline{\Phi}$ is the tail function of this discrete distribution, so

$$
\overline{\Phi}(x) = \overline{\Phi}(n) = \sum_{m=n+1}^{\infty} \phi(m) \text{ for } n \leq x < n + 1, \quad n = 0, 1, 2, \cdots.
$$

With this definition it is clear that $W$ is piece-wise constant, and it follows that (0.1) will hold in general if it holds as $x \to \infty$ through the integers, and we will now establish this.

Again the functions $g$ and $\overline{G}$ are defined by $g(n) = 2\phi(n) - \phi_2(n), n = 0, 1, 2, \cdots$ where $\phi_2(n)$ is the discrete convolution $\sum_{0}^{n} \phi(k) \phi(n-k)$ and for $x \geq 0$

$$
\overline{G}(x) = \sum_{m=[x]+1}^{\infty} g(m) = \overline{G}([x])
$$

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The stationarity of \( \phi \) gives 

\[
\sum_{r=0}^{n} U(r) \phi_2(n - r) = \sum_{r=0}^{n} U(r) \sum_{s=0}^{n-r} \phi(s) \phi(n - r - s) = \sum_{s=0}^{n} \phi(s) \sum_{r=0}^{n-s} \phi(n - r - s)U(r) = m^{-1} \sum_{s=0}^{n} \phi(s)(n + 1 - s) = m^{-1}(n + 1 - \sum_{s=0}^{n} \Phi(s)).
\]

So 

\[
\sum_{r=0}^{n} U(r) g(n - r) = m^{-1}(n + 1 + \sum_{s=0}^{n} \Phi(s)),
\]

then 

\[
W(n) = U(n) - \sum_{r=0}^{n} U(r) g(n - r),
\]

and summation by parts gives

\[
W(n) = \sum_{r=0}^{n} u(r) \mathcal{G}(n - r).
\] (1.3)

This is the discrete analogue of (0.3). Next we see that the proof in [1] of (0.4) is also valid in this lattice case, with minor changes. Finally when \( \beta > 1/2 \) the condition 

\[
f_0 \mathcal{G}(y)dy = 0
\]

also holds, but because of (1.2) it is equivalent to

\[
\sum_{m=0}^{\infty} \mathcal{G}(m) = 0.
\] (1.4)

It is straightforward to see that the results in Theorem 1.1 of [1] when \( \beta \leq 1/2 \) hold in this lattice case with \( x \) restricted to the integers, and we now show that the same is true when \( \beta > 1/2 \). Recalling that \( c_\alpha < 0 \) in this case, so that \( \mathcal{G}^\alpha(n) = -G(n) \) is positive for all large \( n \), we assume we know that

\[
| \sum_{r=0}^{n} (u_n - u_{n-r}) \mathcal{G}(r) | = o(n\mathcal{G}^\alpha(n)) \text{ as } n \to \infty.
\] (1.5)

Then (1.3) and (1.4) give

\[
W(n) = u_n \sum_{r=0}^{n} \mathcal{G}(r) + o(n\mathcal{G}^\alpha(n)) = \frac{1}{m} \sum_{m=0}^{\infty} \mathcal{G}^\alpha(r) + o(n\mathcal{G}^\alpha(n)) = o\left( \frac{|c_\alpha|n\Phi(n)^2}{m(2\beta - 1)} \right),
\]

which is the required result. Next suppose that with \( \Delta_n := u_n - u_{n-1} \) we have \( n\Delta_n \to 0 \) as \( n \to \infty \). For any fixed \( \delta \in (0, 1) \) we can bound the LHS of (1.5) by \( S_1 + S_2 + S_3 \), where

\[
S_1 = \max_{n(1-\delta) \leq m \leq n} |u_n - u_{n-m}| \sum_{n(1-\delta)}^{n} |\mathcal{G}^\alpha(m)| \leq c \sum_{n(1-\delta)}^{n} \mathcal{G}(m),
\]

\[
S_2 = \max_{n\delta \leq m \leq n(1-\delta)} |u_n - u_{n-m}| \sum_{n\delta}^{n(1-\delta)} |\mathcal{G}(m)| = o(1) \sum_{n\delta}^{n(1-\delta)} \mathcal{G}(m),
\]

\[
S_3 = \left| \sum_{0}^{n\delta} \mathcal{G}(m) \sum_{n-m+1}^{n} \Delta_r \right| = o(1) \sum_{0}^{n\delta} \mathcal{G}(m) \frac{m}{n} = o(n\mathcal{G}^\alpha(n)).
\]
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Then (1.5) follows by letting \( n \to \infty \) and then \( \delta \to 0 \). The fact that \( n\Delta_n \to 0 \) can be seen by an application of the Riemann-Lebesgue Lemma: we have the inversion formula

\[
\Delta_n = \sum_{n=0}^{\infty} P(S_m = n) - P(S_m = n - 1) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{p}(t)^m e^{-itn} (1 - e^{it}) \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - e^{it}) e^{-itn} \, dt}{1 - \hat{p}(t)}. \tag{1.6}
\]

Integrating by parts and noting that since everything is periodic with period \( 2\pi \) the contribution from the end points cancel, gives

\[
2\pi \Delta_n = \frac{1}{n} \int_{-\pi}^{\pi} e^{-itn} f_1(t) \, dt, \quad \text{with} \quad f_1(t) = \frac{i(e^{it} - 1)\hat{p}'(t) - e^{it}(1 - \hat{p}(t))}{(1 - \hat{p}(t))^2}. \tag{1.7}
\]

Known results (see e.g. [3]) give the asymptotic behaviour of \( \hat{p}(t) \) and \( \hat{p}'(t) \) as \( |t| \to 0 \) and from them we see that \( |f_1| \) is regularly varying as \( |t| \to 0 \) with index \( a - 2 > -1 \). We deduce that \( f_1 \) is integrable over \([-\pi, \pi]\) and the result follows.

**Remark 1.1.** Alternatively, we could appeal to a stronger result on the asymptotic behaviour of \( \Delta_n \) in [4], but the proof there uses Banach Algebra techniques.

### 1.1 The absolutely continuous case

Assuming that \( F \) has a continuous density and the characteristic function \( \hat{p}(t) = E(e^{itX}) \) is such that \( |\hat{p}(t)|^b \) is integrable for some \( b \geq 1 \), Isozaki [2] has used an inversion theorem to find an asymptotic estimate of the density \( u \) of the renewal measure. This estimate, which is actually valid in the random walk case whenever \( E|X|^\gamma < \infty \) for some \( \gamma \in (3/2, 2) \), when specialised to the renewal case becomes

\[
u(x) = \sum_{j=1}^{N} f_j(x) + \frac{1}{m}(1 + \Phi(x) + \overline{\psi}(x)) + \varepsilon(x), \quad \text{where} \quad \varepsilon(x) = o(x^{-\gamma}) \quad \text{as} \quad x \to \infty. \tag{1.9}
\]

Here \( N \) is the smallest integer \( \geq b + 1 \), \( f_j \) is the density of \( S_j \), and it is necessary to check, by integration by parts, that the function denoted by \( r_j \) in [2] agrees with our \( m^{-1}\overline{\psi} \). Integrating (1.9) and noting that \( \int_0^x f_j(y) \, dy = 1 + o(x^{1-\gamma}) \) for \( 1 \leq j \leq N \) gives

\[
U(x) = \frac{1}{m} x + \int_0^x \Phi(y) dy + \int_0^x \overline{\psi}(y) dy + C + o(x^{1-\gamma}), \quad \text{where} \quad C = N + \int_0^\infty \varepsilon(y) dy. \tag{1.10}
\]

By the same argument as used in [1] the existence of the \( \gamma \)-th moment implies that \( \int_0^\infty \overline{\psi}(y) dy = 0 \), so we can replace \( \int_0^x \overline{\psi}(y) dy \) by \( \int_x^\infty \overline{\psi}(y) dy \), and we also know, from our relation (0.3) and the key renewal theorem, that

\[
W(x) \to \frac{1}{m} \int_0^\infty \overline{\psi}(y) dy = 0,
\]

which means that \( C = 0 \). Then the estimate (1.10) reduces to

\[
W(x) = \frac{1}{m} \int_x^\infty \overline{\psi}(y) dy + o(x^{1-\gamma}), \tag{1.11}
\]

and this is valid whenever the \( \gamma \)-th moment exists, for some \( \gamma \in (3/2, 2) \). In particular under our assumption of asymptotic stability with index \( \alpha \in (3/2, 2) \), we can choose any \( \gamma = \alpha - \delta \) with \( \delta \) sufficiently small that \( 1 - \gamma = \delta - \beta < 1 - 2\beta \) and \( \gamma > 3/2 \). Then (0.1) follows, since we know the first term dominates the RHS of (1.11) and we can read off its asymptotic behaviour from (0.4).
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