Gamow vectors and Supersymmetric Quantum Mechanics

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Gamow solutions are used to transform self–adjoint energy operators by means of factorization (supersymmetric) techniques. The transformed non–hermitian operators admit a discrete real spectrum which is occasionally extended by a single complex eigenvalue associated to normalized eigensolutions. These new Hamiltonians are not pseudo–hermitian operators and also differ from those obtained by means of complex–scaling transformations. As an example, Coulomb–like potentials are studied.

Keywords: Factorization method, Gamow vectors, Non–hermitian Hamiltonians

El método de factorización es extendido al caso complejo para construir Hamiltonianos no Hermitianos con espectro real. Algunos de los nuevos Hamiltonianos admiten además un eigenvalor complejo con eigenfunción normalizada. Las funciones de transformación usadas son funciones de Gamow. Los nuevos Hamiltonianos no son pseudo-hermitianos y son diferentes también de aquellos obtenidos con el método de dilatación compleja. Se presenta el caso de potenciales Coulombianos como ejemplo.

Descriptores: Método de factorización, vectores de Gamow, Hamiltonianos no Hermitianos

1. Introduction

Complex energies were studied for the first time in a paper of Gamow concerning the alpha decay (1928) [1]. In a simple picture, a given nucleus is composed in part by alpha particles (3He nuclei) which interact with the rest of the nucleus via an attractive well (obeying the presence of nuclear forces) plus a potential barrier (due, in part, to repulsive electrostatic forces). The former interaction constrains the particles to be bounded while the second holds them inside the nucleus. The alpha particles have a small (non–zero) probability of tunneling to the other side of the barrier instead of remaining confined to the interior of the well. Outside the potential region, they have a finite lifetime. Thus, alpha particles in a nucleus should be represented by quasi–stationary states. For such states, if at time \( t = 0 \) the probability of finding the particle inside the well is unity, in subsequent moments the probability will be a slowly decreasing function of time (see e.g. Secs. 7 and 8 of Ref. 2).

In his paper of 1928, Gamow studied the escape of alpha particles from the nucleus via the tunnel effect. In order to describe eigenfunctions with exponentially decaying time evolution, Gamow introduced energy eigenfunctions \( \psi_G \) belonging to complex eigenvalues \( E_G = \tilde{E}_G \nu_\Gamma \), \( \Gamma_G > 0 \). The real part of the eigenvalue was identified with the energy of the system and the imaginary part was associated with the inverse of the lifetime. Such ‘decaying states’ were the first application of quantum theory to nuclear physics.

Three years later, in 1931, Fock showed that the law of decay of a quasi–stationary state depends only on the energy distribution function \( \omega(E) \) which, in turn, is meromorphic [2]. According to Fock, the analytical expression of \( \omega(E) \) is rather simple and has only two poles \( E = E_0 \pm i\Gamma \), \( \Gamma > 0 \) (see Eq. (8.13) of Ref. 2). A close result was derived by Breit and Wigner in 1936. They studied the cross section of slow neutrons and found that the related energy distribution reaches its maximum at \( E_R \) with a half–maximum width \( \Gamma_R \). A resonance is supposed to take place at \( E_R \) and to have “half–value breath” \( \Gamma_R \) [3]. The resonances can be defined as eigensolutions \( \psi_R \) of the Hamiltonian with complex eigenvalue \( z_R = E_R - i\Gamma_R/2 \). This complex number also corresponds to a first–order pole of the S matrix [4] (for more details see e.g. [5]). However, as the Hamiltonian is a Hermitian operator, then (in the Hilbert space \( \mathcal{H} \)) there can be no eigenstate having a strict complex exponential dependence on time. In other words, decaying states are an approximation within the conventional quantum mechanics framework. This fact is usually taken to motivate the study of the rigged (equipped) Hilbert space \( \mathcal{H} \) [6]. The mathematical structure of \( \mathcal{H} \) lies on the nuclear spectral theorem introduced by Dirac in a heuristic form [7] and studied in formal rigor by Mau- rin [8] and Gelfand and Vilenkin [9].

Some other approaches extend the framework of quantum theory so that quasi–stationary states can be defined in a precise form. For example, the complex–scaling method [10–12] (see also [13]) embraces the transformation \( H \rightarrow \mathcal{S}H\mathcal{S}^{-1} = H_0 \), where \( S \) is the complex–scaling operator \( S = e^{-\theta p} \), \( [r,p] = i \), such that \( Sf(r) = f(re^{i\theta}) \). This transformation converts the description of resonances by non–integrable Gamow states into one by square integrable states (A relevant aspect of the method is that it is possible to construct a resolution to the identity [14]). Thus, the complex–scaled resonance eigenfunctions are \( \theta \)-dependent so they can be normalized. Moreover, as the complex eigenvalues are \( \theta \)-independent, the resonance phenomenon is just associated with the discrete part of the complex–scaled Hamiltonian [15] (but see [13]).

In this paper we show that Gamow (decaying) eigensolutions can be used to transform Hermitian Hamiltonians into non–self adjoint energy operators with purely real spectrum
or admitting a single extra complex eigenvalue with square-integrable wavefunction. The new Hamiltonians could be profitable as testing operators in diverse approaches including complex-scaling and pseudo-hermitian [16] transformations. As we shall see, it is not necessary to work in a equipped Hilbert space framework because the Gamow solutions will be used merely as mathematical tools. Moreover, the exponential growing of the Gamow solutions for large distances will be primordial in order to get well-behaved complex partners while \( \beta(r) \) is the superpotential (see e.g. [22] and references quoted therein).

In general, we want to keep the physical interpretation of \( \Psi \) as connected with the probability density \( \rho(r) = |\Psi(r)|^2 \) in \( \mathcal{H} \) (The dependence of \( \Psi \) on \( \ell \) will be always implicitly considered). Hence, we look for functions

\[
\Psi \propto B \varphi = \frac{W(u, \varphi)}{u}
\]

which are square-integrable in \( \mathcal{H} \) (the symbol \( W(\cdot, \cdot) \) stands for the wronskian of the involved functions). Of course, this last condition is not imperative in Eq. (8). For instance, one could extend the initial boundedness condition \( |\psi(r, \ell)|^2 < \infty \) to better admit another kind of normalization in order to generalize selfadjointness (e.g., in the picture of a equipped Hilbert space \( \mathcal{H} \)). But, in this way, the physical interpretation of either \( \psi(r, \ell) \) or \( \Psi(r) \) as wavefunctions is less

We look for a complex–type factorization [17] of the Hamiltonian (2):

\[
H_{\ell} = AB + \epsilon
\]

with factorization constant \( C \ni \epsilon = \epsilon_1 + i\epsilon_2; \epsilon_1, \epsilon_2 \neq 0 \in \mathbb{R} \) and a couple of not mutually adjoint first order operators

\[
A := -\frac{d}{dr} + \beta, \quad B := \frac{d}{dr} + \beta
\]

where \( \beta \) is a complex–valued function fulfilling the Riccati equation

\[
-\beta''(r) + \beta^2(r) + \epsilon = V_{\ell}(r).
\]

This last equation is easily solved by means of the logarithmic transformation \( \beta(r) = -(d/dr) \ln u(r) \), with \( u(r) \) the eigensolution of \( H_{\ell} \) belonging to the complex eigenvalue \( \epsilon \equiv -k^2, C \ni k = k_1 + ik_2; k_1, k_2 \in \mathbb{R} \).

Remark that \( H_{\ell}^1 = B^\dagger A^\dagger + \check{\epsilon} = H_{\ell} \) (the bar stands for complex conjugation) because the Hamiltonian is assumed to be self-adjoint in the Hilbert space \( \mathcal{H} \). A relevant aspect of the complex factorization (3)-(5) is that the reverse ordering of the factors gives rise to non-hermitian second order differential operators:

\[
BA + \epsilon = H_{\ell} + 2\beta(r) := h_{\ell}.
\]

Conventional factorizations assume \( a \text{ priori} A = B^\dagger \) and real \( \epsilon \) (see e.g. [19]). In counterdistinction, complex factorization is more in the spirit of the ‘refined factorizations’ reported recently [20] (see also [21]). The following intertwining relationships hold

\[
\begin{align*}
A_{\ell}B &= BH_{\ell}, & H_{\ell}A &= Ah_{\ell} \end{align*}
\]

which permit to determine the solutions \( \Psi \propto B \varphi \) of \( h_{\ell} \Psi = \lambda \Psi, \lambda \in \mathbb{C} \), by giving the solutions \( \varphi \) of \( H_{\ell}\varphi = \lambda \varphi \). The operator \( A \) reverses the action of \( B \). In the supersymmetric language, \( H_{\ell} \) and \( h_{\ell} \) are understood as supersymmetric partners while \( \beta(r) \) is the superpotential (see e.g. [22] and references quoted therein).

2. Supersymmetric Gamow transformations

2.1. The complex factorization

Let us consider the time–independent Schrödinger equation for a spherically symmetric potential \( V(r) \). After separation of angular variables, the equation reduces to a differential equation involving only the radial variable:

\[
H_{\ell} \psi(r, \ell) = E \psi(r, \ell),
\]

which can always be integrated numerically. The reduced Hamiltonian reads

\[
H_{\ell} \equiv -\frac{d^2}{dr^2} + V_{\ell}(r) = -\frac{d^2}{dr^2} + \left(\ell(\ell + 1)/r^2\right) + V(r),
\]

where the effective potential \( V_{\ell}(r) \) has the domain \( D_{\ell} = [0, +\infty) \) and the units of energy and coordinates have been properly chosen.

The nature of the energy spectrum of \( H_{\ell} \) may be deduced from the asymptotic behaviour of the solutions \( \psi(r, \ell) \) which are regular at the origin. If \( V(r) \) approaches zero asymptotically faster than \( 1/r \): \( \lim_{r \to \infty} rV(r) = 0 \), then the energy spectrum contains two parts: (a) Negative discrete values \( E_1(\ell), E_2(\ell), \ldots \). To each of them corresponds a radial wavefunction of finite norm. (b) Unbound continuous positive spectrum, with solutions regular at the origin but indefinitely oscillating in the asymptotic region. On the other hand, if \( V(r) \) approaches zero as \( 1/r \) when \( r \to \infty \), the essential result concerning the nature of the spectrum persists [18]. We shall concentrate on the discrete spectrum by assuming that a complete set of normalized wavefunctions \( \psi_{\ell, n}(r, \ell) \in \mathcal{H} \) has been given for each \( V(r) \), otherwise \( H_{\ell} \) would not be an observable.

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clear (one dimension plane waves, for example, are known to be not in $L^2(\mathbb{R})$ but having a probability density which is everywhere finite in the Dirac sense. In other words, the plane waves could be understood as energy Dirac vectors in $\mathcal{H}$. However, if we apply realistic vanishing boundary conditions at $x = 0$ and $x = L$, or $L$-periodic boundary conditions, the plane waves can be normalized in the conventional form. Thus, ‘free particles’ are but an abstraction from the actual quantum world).

As it could be expected, the set of eigenvectors (8) is uncommon in $\mathcal{H}$; though they can be normalized, their elements are not mutually orthogonal [17] (An optional bi–orthogonal basis has been recently discussed in [23]). These vectors are natural in the spaces with an indefinite metric as studied in the Pontrjagin–Krein formalism [24] (see also [25]).

2.2. Gamow transformations

Let us show how the Gamow solutions can be used as transformation functions $u(r)$ in Eq. (8). First, following Gadella–de la Madrid, we define a Gamow function as a solution of the time–independent Schrödinger equation with complex eigenvalue and purely outgoing boundary conditions [26]. Thus, if $u(r)$ is such that $u(r = 0) = 0$, $u(r \to +\infty) \sim e^{-\kappa r}$ ($k_1 < 0$), and solves (1), (2) with $E = \epsilon \in \mathbb{C}$, then $u(r)$ is a Gamow solution (Observe that $\epsilon$ does not necessarily correspond to the poles of the $S$ matrix). In the context of the alpha decay, the condition $u(r = 0) = 0$ describes the ‘creation’ of alpha particles inside the nucleus and obeys the fact that there cannot be any transmission into the region $r < 0$ because the effective potential is infinite there (i.e., this condition avoids the incoming probabilities and is related with the adjointness of the Hamiltonian [26]). On the other hand, the outgoing boundary condition ensures the decay rate obeyed by the particles after tunneling the electrostatic barrier.

Let us take $\Re(\epsilon) \equiv E_R = k_2^2 - k_1^2 > 0$ in $\epsilon = (k_2^2 - k_1^2) - 2ik_1k_2$. Thus $|k_2| > |k_1|$. We can distinguish two general cases:

1) $k_1 < 0, k_2 < 0$. Here $\epsilon^- = E_R - i\Gamma^-/2$, with $\Gamma^- = 4k_1k_2 > 0$, is associated with the decaying part of the solution $U(t)|\phi_-(\cdot) = e^{-itE_R} e^{-it\Gamma^-/2}|\phi_-(\cdot)$.

2) $k_1 < 0$ and $k_2 > 0$. The complex energy $\epsilon^+ = E_R + i\Gamma^+/2$, with $\Gamma^+ = 4|k_1||k_2| > 0$, is associated with the growing part of the solution $U(t)|\phi_+(\cdot) = e^{-itE_R} e^{it\Gamma^+/2}|\phi_+(\cdot)$.

In both cases the roles are interchanged under complex conjugation. Now, if $\epsilon^\pm$ correspond to the poles $\pm \beta$ of the $S$ matrix, then the lifetime $\tau = 1/\Gamma^0$ decreases as the energy increases. Thus, for small widths (large lifetime) the energy resonances are close to the real axis and the Gamow vectors could be considered as bound states for certain physical phenomena. On the other hand, as $\Gamma_R$ increases, the resonances move away from the real axis and the Gamow vectors are far to be considered as representative of bound states.

Now, let us analyze in detail the Eq. (8). Our goal is to characterize the spectrum of $h_\ell$ as well as its eigenfunctions in terms of the analytical behaviour of $\varphi(\ell, t)$ and the boundary conditions of $u(r)$.

A direct calculation shows that $u(r) \propto r^{\ell+1}$ satisfies $u(r = 0) = 0$. Thereby, Eq. (8) reads

$$\Psi(r < l) \sim \varphi(r < l) - \frac{\ell + 1}{r} \varphi(r < 1).$$

It is clear that $\Psi(r)$ will be regular at the origin if $\varphi$ is such that $\varphi(r < 1) \sim r^s$, $s > 1$. In other words, if $\varphi$ is regular at the origin then $\Psi(r = 0) = 0$.

The purely outgoing boundary condition, in turn, is equivalent to the following expression (see [26] p 630):

$$\lim_{r \to \infty} \frac{\partial}{\partial r} \ln u(r) = -\lim_{r \to \infty} \beta(r) = -k.$$

Hence, Eq. (8) reduces to

$$\lim_{r \to \infty} \Psi \sim \lim_{r \to \infty} \varphi + k \lim_{r \to \infty} \varphi.$$  

As the solution $\varphi$ grows indefinitely as one of either $e^{\pm \kappa r}$, $\kappa = \sqrt{-\lambda}$, we can identify the following cases:

I) For a (denumerably infinite) set of negative discrete values $\lambda \in \{E_n(t)\}$, the solution $\varphi$ in (11) behaves as $\varphi \sim e^{-\kappa r}$, $\kappa > 0$. Thus $\Psi \sim (k - \kappa)e^{-\kappa r}$, $\kappa > 0$.

II) If $\lambda > 0$, then $\varphi \sim \sin(\kappa r - \ell \pi/2 + \delta_0)$, with $\delta_0$ the phase shift. Thus, $\varphi$ is an acceptable eigenfunction of $H_{\ell}$ for any $\lambda > 0$ and represents an unbound state [18]. Hence, if $\lambda > 0$ then $\Psi(r)$ indefinitely oscillates when $r \to \infty$.

III) If $\lambda \in \mathbb{C}$ then Eq. (11) gives $\Psi_\pm \sim (\pm \kappa + k)e^{\pm \kappa r}$. Moreover, if $\lambda = \epsilon$ (equivalently $\kappa = k$) then $\Psi_0 = 0$ and $\Psi_+ \sim 2\kappa e^{-\kappa r}$. The former solution is rather trivial as $W(u, u) = 0$ in Eq. (8). Now, as $k_1 < 0$, it seems that $\Psi_+$ could satisfy $\lim_{r \to \infty} |\Psi_+| = 0$. However, in such a case, $\varphi_+$ should also satisfy both conditions $\varphi_+(0) = 0$ and $\varphi_+ \sim e^{\kappa r}$. $k_1 < 0$, which is not possible since $\lambda$ is complex and $H_{\ell}$ is a selfadjoint operator in $H_{\ell} \varphi_+ = \lambda \varphi_+$. A similar situation arises for any complex number $\lambda$ different from $\epsilon$.

In summary, for Gamow transformation functions in (8), if $\varphi \in L^2(\mathbb{R}^+)$ then $\Psi \in L^2(\mathbb{R}^+)$. Furthermore, Eq. (8) does not produce eigenfunctions of the non–hermitian Hamiltonian $h_{\ell}$ belonging to complex eigenvalues. Thereby, the complete discrete spectrum $\sigma_d(H_{\ell})$ of the initial Hamiltonian $H_{\ell}$ is inherited to the Gamow transformed Hamiltonian $h_{\ell}$. In order to exhaust our analysis, let us consider the complex factorization (6). It is easy to verify that the kernel of $A$ provides an eigenfunction $\xi(\rho)$ of $h_{\ell}$ belonging to $\epsilon \in \mathbb{C}$. Thus, $\xi_\ell \sim 1/u$ fulfills $h_{\ell}C_{\ell} = \xi_\ell$. However, as $u$ is a Gamow vector, $\xi_\ell$ diverges at the origin as $r^{-\ell-1}$. In other words, $\xi_\ell$ is out of $H$ and it is deprived of a physical meaning. The same

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2.3. Non-hermitian Hamiltonians with hydrogen–like potential.

The applications of the method by transforming the Coulomb potential by a series of wells and barriers which alternate their positions can be interpreted as 'opaque' in the sense that the particle interacts with the particle behaves as free of interaction.

On the other hand, the intermediate region could be interpreted as an asymptotic region.

A simple calculation shows that the global behaviour of the new potential is as follows:

\[
v_\ell(r) = \begin{cases} 
  V_{\ell+1}(r) & r \sim 0 \\
  0 & r \to \infty.
\end{cases}
\]

Thus, for small distances, a particle with energy \(E_n(\ell)\) interacts with the field as having a quantum number \(\ell + 1\). In the asymptotic region the particle behaves as free of interaction.

The next section elucidates the applications of the method by transforming the Coulomb potential.

3. Generalized Gamow transformations

As it has been shown in the precedent section, though the Gamow vectors \(u(r)\) could have a definite physical meaning as resonant states of \(H_\ell\), we consider them merely as transformation functions to construct the non–hermitian Hamiltonian potential.

\[
\begin{align*}
\ell + 1 \quad &\quad 13.5\text{ eV} \\
\ell + 2 \quad &\quad 0 \quad &\quad \ell \to \infty.
\end{align*}
\]

The solutions (13) have been explicitly derived in [17].

\[
F_\ell(\ell + 1) = \prod_{n=0}^{\infty} \left(1 + n + \ell\right)
\]

with \(r_0\) and \(r_B\) the Bohr radii, for the parameters considered in the figure).

Notice that \(v_\ell(r)\) becomes almost real for small distances and goes to \(+\infty\) on the real branch for \(r = 0\). On the other hand, this potential goes to zero when \(r \to +\infty\). The imaginary part of \(v_\ell(r)\) becomes relevant for intermediate distances (i.e. at distances which are between 2 and 6 Bohr radii, for the parameters considered in the figure).

Finally, Fig. 3 depicts the potential \(V_{\ell+1}(r)\) as well as the real part of \(v_\ell(r)\). Observe the presence of barriers and wells in the intermediate distances. A similar situation occurs for the imaginary part of \(v_\ell(r)\). These ‘partial potentials’ induce local ‘resonance’ effects which are not present in the Hermitian potential \(V_{\ell+1}(r)\). Thus, the spatial distribution of the wave-packets corresponding to \(v_\ell(r)\) differ from that of the wave-packets of \(V_{\ell+1}(r)\) at the same energy.

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In general, all the unphysical (not square–integrable) solutions of the Schrödinger equation are useful to construct new Hamiltonians admitting real spectra and square–integrable eigenfunctions \[28,29\]. In particular, if the factorization constant \(\epsilon\) is a real number, the conventional factorization operators \(A = B^\dagger\) are automatically recovered.

Now, we extend the previous results by opening the chance to incorporate complex eigenvalues with square–integrable solutions in the spectra of the transformed Hamiltonians. First, notice that the general solution of (1) is, for small distances, a linear combination of two particular solutions: \(r^{\ell+1}\) and \(r^{-\ell}\). The second one is usually rejected because it is singular. Moreover, in the context of alpha decay, a vector \(u(r = 0) \neq 0\) does not describe the ‘creation’ of alpha particles. We shall relax the Gamow condition at the origin to include the solution \(r^{-\ell}\) but preserving the purely outgoing condition \(e^{-kr}\). Let us remark that a ‘generalized’ Gamow vector \(\omega(r)\), satisfying these new conditions, still is unphysical in the sense that it is not square–integrable in \(\mathcal{H}\).

The relevance of our generalization lies on the fact that expressions (1)–(8) still hold if \(\omega(r)\) is taken as the transformation function. Equation (9), on the other hand, is slightly modified:

\[
\Psi(r \ll 1) \sim \varphi'(r \ll 1) + \frac{\ell}{r} \varphi(r \ll 1). \tag{14}
\]

Hence, the same conclusion is obtained: if \(\varphi \in L^2(\mathbb{R})\) then \(\Psi \in L^2(\mathbb{R})\). However, for complex eigenvalues of \(h_\ell(r)\), the kernel of \(A\) provides the eigensolution \(\Psi_\epsilon \propto 1/\omega\), which can be normalized in \(\mathcal{H}\) and satisfies \(h_\ell \Psi_\epsilon = e \Psi_\epsilon\). It is easy to check that the corresponding complex conjugate \(\bar{\Psi}_\epsilon\) is neither in the kernel of \(A\) nor that of \(BA\) (see Eq. (6)). In counterpart, if \(\varphi(r)\) is eigensolution of \(h_\ell\) belonging to \(\epsilon\), then \(\bar{\varphi}(r)\) belongs to \(\bar{\epsilon}\) as \(H_\ell\) is selfadjoint.

Therefore, the discrete spectrum of \(h_\ell\) is now given by \(\sigma_d(H_\ell) \cup \{\epsilon\}\). On the other hand, the new potential \(v_\ell(r) = V_\ell(r) - 2\omega'(r)/\omega(r)\) behaves in this case as

\[
v_\ell(r) = \begin{cases} 
V_{\ell-1}(r) & r \sim 0 \\
0 & r \to \infty 
\end{cases}
\tag{15}
\]

with a similar interpretation as for (12) but changing \(\ell + 1\) by \(\ell - 1\).

Figures 4 and 5 show respectively the behaviour of the generalized Gamow vector \(\omega(r)\), the wavefunction \(\Psi_\epsilon(r)\) and the new non–hermitian potential \(v_\ell(r)\) for the Coulomb case \(V(r) = -2/r\). The related transformation function is [17]:

\[
\omega(r) = r^{\ell+1}e^{-kr}[1F_1(\ell + 1 - 1/k, 2\ell + 2, 2kr) + \xi U(\ell + 1 - 1/k, 2\ell + 2, 2kr)] \tag{16}
\]

where \(\xi\) is a complex constant and \(U(a, c, z)\) is the logarithmic hypergeometric function.

4. Concluding remarks

The Gamow (decaying) eigensolutions have been shown to be appropriate transformation functions in the framework of supersymmetric quantum mechanics. Non–hermitian Hamiltonians, which are supersymmetric partners of spherically symmetric self–adjoint energy operators, have been construc-
complex–scaled Hamiltonians rarely considered in the literature. Thus, it seems that this is a remarkable profile of the factorization which is derived in the complex–scaling method. However, the eigensolution is present.

A particular case has been recently reported [30] (see also [31]) by considering conventional factorization operators $e^{\pm} = \mp(d/dr) + \alpha(r)$, $\alpha : \mathbb{R} \rightarrow \mathbb{R}$, real factorization constants $\mathcal{E}$, and the squeezing operator $S = U_r = e^{i(\lambda/2)(r,p)}$, $[r,p] = i$. The so–derived ‘scaled intertwined’ Hamiltonian $\hat{h}_\lambda$ has a real potential $V_\lambda(r) = e^{2\lambda}V(r)e^{-2\lambda} - \alpha'(r)$ and real discrete spectrum $\sigma_{d}(h_\lambda) = \{e^{2\lambda}\mathcal{E}, e^{2\lambda}E_n\}_{n \in \mathbb{N}}$, where $e^{2\lambda}\mathcal{E}$ is the ground state energy and $E_n \in \sigma_{d}(H)$. This procedure allows to deform the excited energy levels of $h_\lambda$ but leaving unaffected the ground state $E_0$. This is a remarkable profile of the factorization which is rarely considered in the literature. Thus, it seems that complex–scaled Hamiltonians $\hat{h}_\lambda$ could be successfully constructed as an application of the technique reported in [30]. Work in this direction is in progress.

On the other hand, the possible connection of our results with other approaches as the $PT$–symmetry [32] or the pseudo–hermitian transformation [16] has been discussed in a previous work [17].

Finally, we have presented the case of first order, not mutually adjoint, intertwining operators $A, B$. However, the method can be iterated at will by considering $\mathcal{H}_\ell$ as the new initial Hamiltonian. The $n$th iterated result can be also obtained by means of intertwining operators of $\mathcal{H}_\ell$ order. In particular, the second order case can be properly used to obtain self–adjoint Hamiltonians with spectrum identical to the initial one [17, 33]. It is also possible to show that second order transformations can produce non–hermitian operators with real spectrum extended by two extra complex eigenvalues $\epsilon$ and $\bar{\epsilon}$. These results will be published elsewhere.

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