Universality of Fedosov’s Construction for
Star Products of Wick Type on
Pseudo-Kähler Manifolds

Nikolai Neumaier *

Fakultät für Mathematik und Physik
Universität Freiburg
Hermann-Herder-Straße 3
D-79104 Freiburg i. Br.
Germany

April 2002
FR-THEP-2002/07

Abstract

In this paper we construct star products on a pseudo-Kähler manifold \((M, \omega, I)\) using a
modification of the Fedosov method based on a different fibrewise product similar to the Wick
product on \(\mathbb{C}^n\). Having fixed the used connection to be the pseudo-Kähler connection these
star products shall depend on certain data given by a formal series of closed two-forms on \(M\)
and a certain formal series of symmetric contravariant tensor fields on \(M\). In a first step we
show that this construction is rich enough to obtain star products of every equivalence class by
computing Deligne’s characteristic class of these products. Among these products we uniquely
characterize the ones which have the additional property to be of Wick type which means
that the bidifferential operators describing the star products only differentiate with respect to
holomorphic directions in the first argument and with respect to anti-holomorphic directions in
the second argument. These star products are in fact strongly related to star products with
separation of variables introduced and studied by Karabegov. This characterization gives rise
to special conditions on the data that enter the Fedosov procedure. Moreover, we compare
our results that are based on an obviously coordinate independent construction to those of
Karabegov that were obtained by local considerations and give an independent proof of the fact
that star products of Wick type are in bijection to formal series of closed two-forms of type \((1, 1)\)
on \(M\). Using this result we finally succeed in showing that the given Fedosov construction is
universal in the sense that it yields all star products of Wick type on a pseudo-Kähler manifold.
Due to this result we can make some interesting observations concerning these star products; we
can show that all these star products are of Vey type and in addition we can uniquely characterize
the ones that have the complex conjugation incorporated as an anti-automorphism.

*Nikolai.Neumaier@physik.uni-freiburg.de
1 Introduction

The concept of deformation quantization as introduced in the pioneering articles [1] by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer has proved to be an extremely useful framework for the problem of quantization: the question of existence of star products $\star$ (i.e. formal, associative deformations of the classical Poisson algebra of complex-valued functions $C^\infty(M)$ on a symplectic or more generally, on a Poisson manifold $M$, such that in the first order of the formal parameter $\nu$ the commutator of the star product yields the Poisson bracket) has been answered positively by DeWilde and Lecomte [8], Fedosov [9], Omori, Maeda and Yoshioka [25] in the case of a symplectic phase space as well as by Kontsevich [20] in the more general case of a Poisson manifold. Moreover, star products have been classified up to equivalence in terms of geometrical data of the phase space by Nest and Tsygan [24], Bertelson, Cahen and Gutt [2], Weinstein and Xu [27] on symplectic manifolds and the classification on Poisson manifolds is due to Kontsevich [20]. Comparisons between the different results on classification and reviews can be found in articles of Deligne [7], Gutt and Rawnsley [12, 13], Neumaier [21] and the thesis of Halfout [14].

In the case of deformation quantizations with separation of variables on Kähler manifolds Karabegov proved existence and gave a classification using a formal deformation of the Kähler form in [15, 16]. Moreover, he has shown in [17] that the Fedosov approach to star products of Wick type that only differ from those with separation of variables by interchanging the roles of holomorphic and anti-holomorphic directions and a minus sign in front of the symplectic resp. Kähler form considered by Bordemann and Waldmann in [5] corresponds to the trivial deformation in the sense of his classification. This result naturally raises the question, whether this construction can be generalized as to obtain all star products of Wick type on a Kähler manifold.

The aim of this paper is to give a positive answer to this question stressing the power and beauty of Fedosov’s method and allowing for a global, geometrical description of all star products of Wick type that avoids local considerations in coordinates. In addition the further investigation of additional properties of the star products of Wick type is extremely simplified due to the simple, lucid description of these star products encoded in the Fedosov derivation. Moreover, our result represents the justification to restrict to the Fedosov framework when addressing further questions like the construction of representations for star products of Wick type.
The paper is organized as follows: In Section 2 we establish our notation and remember some basic definitions that are needed in the context of star products of Wick type on pseudo-Kähler manifolds. Section 3 is devoted to the presentation of the Fedosov constructions using different fibrewise products. Moreover, we give explicit constructions for equivalence transformations relating the resulting star products that enable us using our result in [21, Thm. 4.4] to show that the given construction with \( \omega_\text{Wick} \) is rich enough to obtain star products in every equivalence class. After this first important result we give a unique characterization of the star products of Wick type in Section 4 obtained from our construction in terms of the data that enter the Fedosov construction. In addition we can prove that the star products of Wick type only depend on a formal series of closed two-forms of type \((1,1)\). In Section 5 we remember some results of Karabegov obtained in [15] and give elementary proofs of the important statements. Using these results we are in the position to prove the main result of the paper in Section 6 stating that every star product of Wick type (not only such a star product in every possible equivalence class) can be obtained by some adapted Fedosov construction. Using this fact we can draw some additional conclusions concerning the structure of such products, in particular we show that each such product is of Vey type. In Appendix A we have collected some notations and results concerning the splittings into holomorphic and anti-holomorphic part of the mappings that are essential for the Fedosov construction that make use of the presence of the complex structure \( I \) and that are important for the proofs and statements given in Sections 4 and 6. A further Appendix B is added for completeness providing some details on the application of Banach’s fixed point theorem in the framework of the Fedosov construction which constitutes an important and simplifying tool for several proofs in Sections 3 and 4.

**Conventions:** By \( C^\infty(M) \) we denote the complex-valued smooth functions and similarly \( \Gamma^\infty(T^*M) \) stands for the complex-valued smooth one-forms et cetera. Moreover, we use Einstein’s summation convention in local expressions.

**Acknowledgements:** I would like to express my thanks to Stefan Waldmann for a careful reading of the manuscript and many comments and useful discussions. Moreover, I should like to thank Alexander V. Karabegov for the inspiring discussions that initiated the investigations of the topic of the paper. Finally, I want to thank the DFG-Graduiertenkolleg “Nichtlineare Differentialgleichungen – Modellierung, Theorie, Numerik, Visualisierung” for financial support.

### 2 Some Notations and Basic Definitions

Let \((M, \omega, I)\) denote a pseudo-Kähler manifold with \(\dim_\mathbb{R}(M) = 2n\), i.e. \((M, g)\) is a pseudo-Riemannian (no positivity is required) manifold such that the almost complex structure \( I \) is an isometry with respect to \( g \) (\( g \) is Hermitian) and the almost complex structure is flat with respect to the Levi-Civita connection \( \nabla \) corresponding to \( g \). Under these conditions it is known that \( M \) is a complex manifold such that the almost complex structure \( I \) coincides with the canonical complex structure of \( M \) and that \( \omega \) defined by \( \omega(X,Y) := g(IX,Y) \) for \( X,Y \in \Gamma^\infty(TM) \) is a closed non-degenerate two-form and hence a symplectic form on \( M \), the so-called pseudo-Kähler form. Moreover, one obviously has that \( \nabla \) defines a torsion free symplectic connection since \( \omega \) is also flat with respect to the connection \( \nabla \) which is called the pseudo-Kähler connection. As \( g \) is Hermitian both \( g \) and \( \omega \) are of type \((1,1)\) and in a local holomorphic chart of \( M \) one can write \( \omega = \frac{1}{2}g_{\overline{k}l}dz^k \wedge d\overline{z}^l \) with a Hermitian non-degenerate matrix \( g_{\overline{k}l} = 2g(Z_{\overline{k}},Z_l) \). Here \( Z_k = \partial_{z^k} \) and \( Z_l = \partial_{\overline{z}^l} \) denote local base vector fields of type \((1,0)\) and of type \((0,1)\) (cf. Appendix A) that locally span the \(+i\) and \(−i\) eigenspaces \( TM^{1,0} \) and \( TM^{0,1} \) of the complex structure \( I \).

The most simple example of a (pseudo-)Kähler manifold is given by \( \mathbb{C}^n \) endowed with the canonical (pseudo-)Kähler form \( \omega_0 = \frac{1}{2}\delta_{\overline{k}l}dz^k \wedge d\overline{z}^l \). In this case it is well-known that one can define an associative product on the functions on \( \mathbb{C}^n \) that are polynomials in the coordinates \((z, \overline{z})\)
using the Wick resp. normal ordering of creation and annihilation operators. Replacing \( h \) by \( \frac{\nu}{1} \) in the resulting formula for the product of two polynomial functions \( F, G \) on \( \mathbb{C}^n \) one obtains

\[
F \ast_{\text{Wick}} G = \sum_{l=0}^{\infty} \frac{1}{l!} \left( \frac{2\nu}{i} \right)^l \delta^{i_1 \bar{j}_1} \ldots \delta^{i_l \bar{j}_l} \frac{\partial^l F}{\partial z^{i_1} \ldots \partial z^{i_l}} \frac{\partial^l G}{\partial \bar{z}^{j_1} \ldots \partial \bar{z}^{j_l}}. \tag{1}
\]

From this explicit expression it is obvious that one obtains a star product, the so-called Wick product, also denoted by \( \ast_{\text{Wick}} \) on \( (\mathbb{C}^n, \omega_0) \) using this formula for \( f, g \in C^\infty(\mathbb{C}^n)[[\nu]] \) instead of \( F, G \). Considering the explicit shape of \( \ast_{\text{Wick}} \) one immediately notices that \( f \ast_{\text{Wick}} h \) coincides with the pointwise product of \( f \) and \( h \) for all holomorphic functions \( h \in \mathcal{O}(\mathbb{C}^n) \) and all \( f \in C^\infty(\mathbb{C}^n) \). Moreover, we have \( h' \ast_{\text{Wick}} g = h' g \) for all anti-holomorphic functions \( h' \in \mathcal{O}(\mathbb{C}^n) \) and all \( g \in C^\infty(\mathbb{C}^n) \).

In order to generalize this property of the star product \( \ast_{\text{Wick}} \) to star products on an arbitrary pseudo-Kähler manifold the following definition offers itself: A star product on \( (M, \omega, I) \) is called star product of Wick type if in a local holomorphic chart the describing bidifferential operators \( C_r \) for \( r \geq 1 \) have the shape

\[
C_r(f, g) = \sum_{\kappa, \ell} C_{r}^{K, \ell} \partial^{[K]} f \partial^{[\ell]} g \tag{2}
\]

with certain coefficient functions \( C_{r}^{K, \ell} \). Obviously this characterization of star products of Wick type is a global notion on pseudo-Kähler manifolds which indeed is equivalent to the following (cf. [5 Thm. 4.7]):

**Definition 2.1** A star product \( \ast \) on a pseudo-Kähler manifold \( (M, \omega, I) \) is called star product of Wick type if for all open subsets \( U \subseteq M \) and all \( f, g, h, h' \in C^\infty(M) \) with \( h|_U \in \mathcal{O}(U) \), \( h'|_U \in \mathcal{O}(U) \) the equations

\[
f \ast h|_U = f h|_U \quad \text{and} \quad h' \ast g|_U = h' g|_U \tag{3}
\]

are valid.

Compared to the definition of a star product with separation of variables as introduced by Karabegov the rôles of holomorphic and anti-holomorphic directions are interchanged here and according to our convention for the sign of the Poisson bracket the star products with separation of variables that are considered by Karabegov are star products on \( (M, -\omega, I) \). Briefly, this means that a star product \( \ast \) on \( (M, \omega, I) \) is a star product of Wick type if and only if the opposite star product \( \ast_{\text{opp}} \) which is defined by \( f \ast_{\text{opp}} g := g \ast f \) for \( f, g \in C^\infty(M)[[\nu]] \) is a star product with separation of variables on \( (M, -\omega, I) \). In physics literature the star products with separation of variables on \( (M, \omega, I) \) are usually called star products of anti-Wick type since in the example \( (\mathbb{C}^n, \omega_0) \) they arise from the anti-Wick resp. anti-normal ordering of creation and annihilation operators. In this case an explicit formula can be obtained very easily and is given by

\[
f \ast_{\text{Wick}} g = \sum_{l=0}^{\infty} \frac{1}{l!} \left( \frac{-2\nu}{i} \right)^l \delta^{i_1 \bar{j}_1} \ldots \delta^{i_l \bar{j}_l} \frac{\partial^l f}{\partial z^{i_1} \ldots \partial z^{i_l}} \frac{\partial^l g}{\partial \bar{z}^{j_1} \ldots \partial \bar{z}^{j_l}}. \tag{4}
\]

This example motivates the definition:

**Definition 2.2** A star product \( \ast \) on a pseudo-Kähler manifold \( (M, \omega, I) \) is called star product of anti-Wick type if for all open subsets \( U \subseteq M \) and all \( f, g, h, h' \in C^\infty(M) \) with \( h|_U \in \mathcal{O}(U) \), \( h'|_U \in \mathcal{O}(U) \) the equations

\[
h \ast f|_U = h f|_U \quad \text{and} \quad g \ast h'|_U = gh'|_U \tag{5}
\]

are valid.
There is a very simple relation between star products of Wick type and star products of anti-Wick type that can be established using the parity operator \( \mathcal{P} \) defined by \( \mathcal{P} := (-1)^{\text{deg}_v} \) where \( \text{deg}_v := \nu \partial_v \), that obviously satisfies \( \mathcal{P}^2 = \text{id} \). For any given star product \( \star \) on \( (M, \omega, I) \) we consider \( \ast_{\mathcal{P}, \text{opp}} \) defined by \( f \ast_{\mathcal{P}, \text{opp}} g := \mathcal{P}(\mathcal{P} f \star \mathcal{P} g) = \mathcal{P}((\mathcal{P} g) \ast (\mathcal{P} f)) \) that again is a star product on \( (M, \omega, I) \). Obviously the mapping \( \star \mapsto \ast_{\mathcal{P}, \text{opp}} \) establishes a bijection on the set of star products on \( (M, \omega, I) \) and from \( \mathcal{P}^2 = \text{id} \) it is evident that \( \ast_{\mathcal{P}, \text{opp}} \mathcal{P} \ast_{\mathcal{P}, \text{opp}} = \star \). It follows from the very definitions that \( \star \) is of Wick type resp. anti-Wick type if and only if \( \ast_{\mathcal{P}, \text{opp}} \) is of anti-Wick type resp. Wick type. Due to this relation we shall mostly restrict our attention to the investigation of star products of Wick type and deduce the respective analogous statements for the star products of anti-Wick type using the aforementioned mapping.

### 3 Fedosov Star Products, Equivalences and Characteristic Classes

In this section we shall briefly recall Fedosov’s construction of star products using some different fibrewise products that are adapted to the special geometric situation on a pseudo-Kähler manifold and are modelled on the examples \( \star_{\text{Wick}} \) resp. \( \star_{\text{Wick}} \) on \( \mathbb{C}^n \). Moreover, we shall relate the obtained star products to star products that are obtained using the usual fibrewise Weyl product as in Fedosov’s original construction by explicitly constructing equivalence transformations between the different products. Using our result in \cite{21} Thm. 4.4 and these equivalences it is trivial to compute the characteristic classes of the constructed star products. In addition we shall establish some further relations between the obtained star products.

In the following \( (M, \omega, I) \) is a pseudo-Kähler manifold and \( \nabla \) denotes the corresponding pseudo-Kähler connection. The basis for Fedosov’s construction (cf. \cite{9, 10}) is the following \( \mathbb{C}[\nu] \)-module

\[
W \otimes \Lambda(M) := (X_{a=0}^{\infty} \Gamma_{\text{symm}}(\nu^s T^* M \otimes \wedge T^* M))[[\nu]].
\]

If there is no possibility for confusion we simply write \( W \otimes \Lambda \). By \( W \otimes \Lambda^k \) we denote the elements of anti-symmetric degree \( k \) and set \( W := W \otimes \Lambda^0 \). For \( a, b \in W \otimes \Lambda \) one defines their pointwise (undeformed) product denoted by \( \mu(a \otimes b) = ab \) by the symmetric \( \vee \)-product in the first factor and the anti-symmetric \( \wedge \)-product in the second factor which turns \( W \otimes \Lambda \) into a super-commutative associative algebra. Then the degree-maps \( \text{deg}_s, \text{deg}_a, \text{deg}_\nu := \nu \partial_v \) and \( \text{Deg} := \text{deg}_s + 2 \text{deg}_a \) are defined in the usual way and yield derivations with respect to \( \mu \). For a vector field \( X \in \Gamma^\infty(TM) \) the symmetric resp. anti-symmetric insertion maps are denoted by \( i_a(X) \) resp. \( i_a(X) \) which are super-derivations of symmetric degree \( -1 \) resp. 0 and anti-symmetric degree 0 resp. \( -1 \). In local coordinates \( x^1, \ldots, x^{2n} \) for \( M \) as a real \( 2n \)-dimensional manifold we define \( \delta := (1 \otimes dx^i) i_s(\partial_i) \) and \( \delta^* := (dx^i \otimes 1) i_s(\partial_i) \) satisfying the relations \( \delta^2 = (\delta^*)^2 = 0 \) and \( \delta \delta^* + \delta^* \delta = \text{deg}_s + \text{deg}_a \). Denoting the projection onto the part of symmetric and anti-symmetric degree 0 by \( \sigma : W \otimes \Lambda \to \mathcal{C}^\infty(M)[[\nu]] \) one has

\[
\delta \delta^{-1} + \delta^{-1} \delta + \sigma = \text{id},
\]

where \( \delta^{-1}a := \frac{1}{\delta + \delta^*} \delta^*a \) for \( \text{deg}_s a = ka \), \( \text{deg}_a a = la \) with \( k + l \neq 0 \) and \( \delta^{-1}a := 0 \) else.

Now we consider different fibrewise associative deformations of the pointwise product \( \mu \) for \( a, b \in W \otimes \Lambda \) the usual fibrewise Weyl product \( \circ \) which was used in Fedosov’s original construction is defined by

\[
a \circ \delta b := \mu \circ \exp \left( \frac{\nu}{2} \Lambda_{ij}(\partial_i) \otimes i_s(\partial_j) \right) (a \otimes b),
\]

where \( \Lambda_{ij} \) denotes the components (in real coordinates) of the Poisson tensor corresponding to the symplectic form \( \omega \) that are related to the ones of \( \omega \) by the equation \( \omega_{kj} \Lambda_{ij} = \delta_{ik} \). Moreover, we can
define the fibrewise Wick product \( \circ_{\text{Wick}} \) by
\[
a \circ_{\text{Wick}} b := \mu \circ \exp \left( \frac{2\nu}{i} g^{kl} i_s(Z_k) \otimes i_s(Z_l) \right) (a \otimes b)
\]
and the fibrewise anti-Wick product by
\[
a \circ_{\text{Wick}} b := \mu \circ \exp \left( - \frac{2\nu}{i} g^{kl} i_s(Z_l) \otimes i_s(Z_k) \right) (a \otimes b),
\]
where we have used local holomorphic coordinates of \( M \) as a \( n \)-dimensional complex manifold and the Poisson tensor is written as \( \Lambda = \frac{2}{i} g^{kd} Z_k \wedge \overline{Z}_l \) with \( g^{kd} g_{k\overline{n}} = \delta_{n}^{\overline{n}} \). Using these products we define deg\_\(\circ\)-graded super-commutators with respect to these product s such that
\[
\delta \text{ super-derivation of anti-symmetric degree 1.}
\]
Moreover, (I1) imply that
\[
S^{-1}((Sa) \circ_{\text{Wick}} (Sb)) = a \circ_{\text{F, Wick}} b = S (S^{-1}a) \circ_{\text{Wick}} (S^{-1}b) \quad \forall a, b \in W \otimes \Lambda,
\]
where the fibrewise equivalence transformation is given by
\[
S := \exp \left( \frac{\nu}{i} \Delta_{ab} \right) \quad \text{with} \quad \Delta_{ab} := g^{kl} i_s(Z_k) i_s(Z_l).
\]
Because of the explicit shape of the fibrewise products \( \circ_{\text{F, Wick}} \), it is obvious that \( \delta \) is a super-derivation of anti-symmetric degree 1. In addition one immediately verifies that the total degree-map \( \text{Deg} \) is a derivation with respect to these products such that \( (W \otimes \Lambda, \circ_{\text{Wick}}) \) is a formally \text{Deg}-graded algebra, where again \( \circ \) stands for \( F, \text{Wick} \) or \( \overline{\text{Wick}} \). Using the pseudo-Kähler connection \( \nabla \) on \( TM \) that extends in the usual way to a connection on \( T^*M \) and symmetric resp. anti-symmetric products thereof we define (using the same symbol as for the connection) the map \( \nabla \) on \( W \otimes \Lambda \) by \( \nabla := (1 \otimes dx^i)\nabla_{x^i} \). Since the pseudo-Kähler connection is symplectic \( \nabla \) turns out to be a super-derivation of anti-symmetric degree 1 and symmetric degree 0 with respect to the fibrewise product \( \circ_{\text{F, Wick}} \). From the additional property of the pseudo-Kähler connection that \( \nabla I = 0 \) one directly finds that \( [\nabla, \Delta_{ab}] = 0 \) and hence \( \nabla = S\nabla S^{-1} = S^{-1}\nabla S \). These relations together with equation (I1) imply that \( \nabla \) is also a super-derivation with respect to the fibrewise products \( \circ_{\text{Wick}} \) and \( \overline{\circ}_{\text{Wick}} \). Moreover, \( [\delta, \nabla] = 0 \) since the pseudo-Kähler connection is torsion free and \( \nabla^2 = -\frac{1}{\nu} \text{ad}_F(R) \), where \( R := \frac{i}{\nu} \omega_R R^{jk} dz^j \otimes dz^k \wedge dz^l = \frac{i}{2} g^{kl} R^{jk} dz^j \otimes dz^k \wedge dz^l \wedge dz^j \wedge dz^k \in W \otimes \Lambda^2 \) involves the curvature of the connection. As consequences of the Bianchi identities \( R \) satisfies \( \delta R = \nabla R = 0 \). Since \( \nabla \) commutes with \( S \) we also have \( \nabla^2 = S\nabla^2 S^{-1} = -\frac{1}{\nu} S \text{ad}_F(R) S^{-1} = -\frac{1}{\nu} \text{ad}_{\text{Wick}}(R) = -\frac{1}{\nu} \text{ad}_{\text{Wick}}(R) \). Here we again have used equation (I1) and \( SR = R + \frac{1}{\nu} \Delta_{ab} R \) which implies the above equation since \( \text{Deg}_s \Delta_{ab} R = 0 \) and hence \( \text{ad}_{\text{Wick}}(\Delta_{ab} R) = 0 \). Analogously one finds \( \nabla^2 = -\frac{1}{\nu} \text{ad}_{\text{Wick}}(R) \).

Now remember the following two theorems which are just slight generalizations of Fedosov’s original theorems in [3 Thm. 3.2, 3.3], where \( \circ \) stands for \( F, \text{Wick} \) or \( \overline{\text{Wick}} \).

**Theorem 3.1** Let \( \Omega_o = \sum_{i=1}^\infty \nu^i \Omega_o,i \) denote a formal series of closed two-forms on \( M \) and let \( s_o = \sum_{k=3}^\infty s_o^{(k)} \in W \) be given with \( \sigma(s_o) = 0 \) and \( \text{Deg}_{s_o}^{(k)} = ks_o^{(k)} \). Then there exists a unique element \( r_o \in W \otimes \Lambda^1 \) of total degree \( \geq 2 \) such that
\[
\delta r_o = \nabla r_o - \frac{1}{\nu} r_o \circ_{\text{Wick}} r_o + R + 1 \otimes \Omega_o \quad \text{and} \quad \delta^{-1} r_o = s_o.
\]
Moreover, \( r_o = \sum_{k=2}^\infty r_o^{(k)} \) with \( \text{Deg}_{r_o}^{(k)} = kr_o^{(k)} \) satisfies the formula
\[
r_o = \delta s_o + \delta^{-1} \left( \nabla r_o - \frac{1}{\nu} r_o \circ_{\text{Wick}} r_o + R + 1 \otimes \Omega_o \right)
\]
from which \( r_\circ \) can be determined recursively. In this case the Fedosov derivation

\[
\mathfrak{D}_\circ := -\delta + \nabla - \frac{1}{\nu} \text{ad}_r(r_\circ)
\]  

(15)

is a super-derivation of anti-symmetric degree 1 with respect to \( \circ_\circ \) and has square zero: \( \mathfrak{D}_\circ^2 = 0 \).

**Theorem 3.2** Let \( \mathfrak{D}_\circ = -\delta + \nabla - \frac{1}{\nu} \text{ad}_r(r_\circ) \) be given as in (15) with \( r_\circ \) as in (13).

i.) Then for any \( f \in C^\infty(M)[[\nu]] \) there exists a unique element \( \tau_\circ(f) \in \ker(\mathfrak{D}_\circ) \cap \mathcal{W} \) such that \( \sigma(\tau_\circ(f)) = f \) and \( \tau_\circ : C^\infty(M)[[\nu]] \to \ker(\mathfrak{D}_\circ) \cap \mathcal{W} \) is \( \mathbb{C}[[\nu]] \)-linear and referred to as the Fedosov-Taylor series corresponding to \( \mathfrak{D}_\circ \).

ii.) For \( f \in C^\infty(M) \) we have \( \tau_\circ(f) = \sum_{k=0}^{\infty} \tau_\circ(f) \) where \( \text{Deg} \) \( \tau_\circ(k) = k \tau_\circ(k) \) which can be obtained by the following recursion formula

\[
\tau_\circ(f) = f \delta^{-1} \left( \nabla \tau_\circ(f) - \frac{1}{\nu} \sum_{l=0}^{k-1} \text{ad}_r(r_\circ(l+2)) \tau_\circ(f)(k-l) \right).
\]

(16)

iii.) Since \( \mathfrak{D}_\circ \) is a \( \circ_\circ \)-super-derivation of anti-symmetric degree 1 as constructed in Theorem 3.1 \( \ker(\mathfrak{D}_\circ) \cap \mathcal{W} \) is a \( \circ_\circ \)-subalgebra and a new associative product \( \circ_\circ \) for \( C^\infty(M)[[\nu]] \) is defined by pull-back of \( \circ_\circ \) via \( \tau_\circ \), which turns out to be a star product on \( (M, \omega, I) \).

One first result concerning all the star products constructed above is the following:

**Proposition 3.3** The star products \( \ast_F, \ast_{\text{Wick}} \) and \( \ast_{\text{Vey}} \) on \( (M, \omega, I) \) are of Vey type, i.e. the bidifferential operators \( C_{\circ,r} \) describing the star product in order \( r \) of the formal parameter via \( f \ast \circ g = \sum_{r=0}^{\infty} \nu^r C_{\circ,r}(f, g) \) for \( f, g \in C^\infty(M) \) are of order \( r, r \).

**Proof:** We write the term of total degree \( k \neq 0 \) in the Fedosov-Taylor series as \( \tau_\circ(f)(k) = \sum_{l=0}^{k-1} \nu^l \tau_\circ(f)(k-2l) \) where \( \tau_\circ(f)(k-2l) \in \Gamma^\infty(\nu^{k-2l} T^* M) \) for \( f \in C^\infty(M) \). A straightforward computation yields that

\[
f \ast \circ g = fg + \sum_{r=1}^{\infty} \nu^r \sum_{l=0}^{r-1} \sum_{k=0}^{l} \mu_{r-l}(\tau_\circ(f)(r-l+2k), \tau_\circ(g)(r-l-2k)),
\]

where we have written \( \mu_{r-k}(\cdot, \cdot) \) for the fibrewise bidifferential operator occurring at order \( k \geq 1 \) of the formal parameter in the fibrewise product \( \circ_\circ \). Now a lengthy but straightforward induction on the total degree yields:

**Lemma 3.4** For all \( k \geq 1 \) and \( 0 \leq l \leq \left[ \frac{k-1}{2} \right] \) the mapping

\[
C^\infty(M) \ni f \mapsto \tau_\circ(f)(k) \in \Gamma^\infty(\nu^{k-2l} T^* M)
\]

(17)

is a differential operator of order \( k - l \).

According to this lemma \( \tau_\circ(r-l+2k) \) is a differential operator of order \( r-l+k \) and \( \tau_\circ(r-l-2k) \) is a differential operator of order \( r-k \). Observing the ranges of summation in the expression for \( C_{\circ,r} \) given above it is easy to see that the highest occurring order of differentiation is \( r \) in both arguments, proving the assertion. \( \Box \)

In fact an analogous statement to the one of the above proposition which shall be published elsewhere [22] holds in even more general situations, where one can construct Fedosov-like products.

Another observation that is indeed rather technical but shall turn out to be useful and important in the next section is that one can restrict to elements \( s_{\circ} \) that contain no part of symmetric degree 1 without omitting any of the possible star products in case one allows for all possible formal series of closed two-forms \( \Omega_\circ \) .
Lemma 3.5 Let \( \mathcal{D}_o \) and \( \mathcal{D}_o' \) be two Fedosov derivations for \( o_0 \) constructed from different data \( (\Omega_o, s_o) \) and \( (\Omega_o', s_o') \), then we have

\[
\mathcal{D}_o = \mathcal{D}_o' \iff r'_o - r_o = 1 \otimes B_0 \iff s'_o - s_o = B_0 \otimes 1 \quad \text{and} \quad \Omega_o - \Omega'_o = dB_0, \tag{18}
\]

where \( B_0 = \sum_{i=1}^{\infty} \nu^i B_{o,i} \in \Gamma^\infty(T^*M)[[\nu]] \) is a formal series of one-forms on \( M \). In case one of the conditions in \( \mathbf{[13]} \) is fulfilled obviously the star products \( \ast_o \) and \( \ast'_o \) coincide.

PROOF: Obviously \( \mathcal{D}_o = \mathcal{D}_o' \iff \frac{1}{\nu} \text{ad}_o(r'_o - r_o) = 0 \) but this is equivalent to \( r'_o - r_o \) being central and hence \( r'_o - r_o = 1 \otimes B_0 \) with a formal series \( B_0 \) of one-forms that vanishes at order zero of the formal parameter since the total degree of \( r'_o \) and \( r_o \) is at least 2. For the proof of the second equivalence first let \( r'_o - r_o = 1 \otimes B_0 \) from which we obtain applying \( \delta^{-1} \) to this equation and using \( \mathbf{[13]} \) that \( s'_o - s_o = B_0 \otimes 1 \).

To obtain the second equation we apply \( \delta \) to \( r'_o - r_o = 1 \otimes B_0 \) and a straightforward computation using \( \mathbf{[13]} \) and \( \nabla(1 \otimes B_0) = 1 \otimes dB_0 \) yields \( 0 = dB_0 + \Omega'_o - \Omega_o \) proving the first direction of the second equivalence. Now assume that the relations between the data \( (\Omega_o, s_o) \) and \( (\Omega'_o, s'_o) \) are satisfied and define \( A_o := r_o - r'_o + 1 \otimes B_0 \) then an easy computation using \( \mathbf{[13]} \) yields that \( A_o \) fulfills the equations

\[
-\delta A_o + \nabla A_o - \frac{1}{\nu} \text{ad}_o(r_o)A_o + \frac{1}{\nu} A_o \circ o_0 = 0 \quad \text{and} \quad \delta^{-1} A_o = 0.
\]

From these equations one obtains with \( \sigma(A_o) = 0 \) using the decomposition \( \delta \delta^{-1} + \delta^{-1} \delta + = \text{id} = A_o \) is a fixed point of the mapping \( L \) defined by \( La := \delta^{-1}(\nabla a - \frac{1}{\nu} \text{ad}_o(r_o)a + \frac{1}{\nu} a \circ o_0 a) \) for \( a \in W \otimes \Lambda^1 \) which is at least of total degree 2. Now \( L \) raises the total degree at least by 1 and using Banach’s fixed point theorem (cf. Corollary \( \mathbf{[33]} \) ii.) we obtain that \( L \) has a unique fixed point. But as well \( A_o \) as (trivially) 0 are fixed points of \( L \) yielding \( A_o = 0 \) by uniqueness and hence \( r'_o - r_o = 1 \otimes B_0 \). \( \square \)

The statement of the preceding lemma now means the following: Starting with a formal series \( \Omega_o \) of closed two-forms and an element \( s_o \) with \( \sigma(s_o) = 0 \) that in addition contains no part of symmetric degree 1 and passing over to the normalization condition \( \tilde{s}_o = s_o + B_0 \otimes 1 \) is equivalent to passing over to \( \Omega_o + dB_0 \) sticking to the normalization condition \( s_o \). Hence we may without loss of generality restrict to normalization conditions shaped like \( s_o \) provided that we allow for arbitrarily varying closed two-forms \( \Omega_o \).

Now we shall compute Deligne’s characteristic class of the star products \( \ast_F, \ast_{\text{Wick}} \) and \( \ast_{\text{Wick}} \) constructed above showing that one can obtain star products of every given equivalence class with each of the above fibrewise products \( o_F, o_{\text{Wick}} \) and \( o_{\text{Wick}} \) by suitable choice of the closed two-forms \( \Omega_F, \Omega_{\text{Wick}} \) and \( \Omega_{\text{Wick}} \). For details on the definition of Deligne’s characteristic class and methods to compute it the reader is referred to \( \mathbf{[7, 12, 13, 21]} \).

Proposition 3.6 Consider the star products constructed according to Theorem \( \mathbf{[7, 12, 13, 21]} \) iii.)

i.) Deligne’s characteristic classes \( c(\ast_F), c(\ast_{\text{Wick}}), c(\ast_{\text{Wick}}) \in \frac{[\omega]}{\nu} + H^2 \text{fib}(M, \mathbb{C})[[\nu]] \) of the star products \( \ast_F, \ast_{\text{Wick}}, \ast_{\text{Wick}} \) on \((M, \omega, I)\) are given by

\[
c(\ast_F) = \frac{[\omega]}{\nu} + \frac{[\rho]}{\nu}, \quad c(\ast_{\text{Wick}}) = \frac{[\omega]}{\nu} + \frac{[\Omega_{\text{Wick}}]}{\nu}, \quad c(\ast_{\text{Wick}}) = \frac{[\omega]}{\nu} + \frac{[\rho]}{\nu}, \quad c(\ast_{\text{Wick}}) = \frac{[\rho]}{\nu}, \tag{19}
\]

where \( \rho \) denotes the Ricci form which is the closed two-form given by \( \rho = -\frac{1}{2} R_{\bar{i}j} dz^i \wedge d\bar{z}^j \) resp. \( \Delta_{\text{fib}} R = 1 \otimes \rho \) and \( R_{\bar{i}j} \) denotes the components of the Ricci tensor in a local holomorphic chart.

ii.) Because of the properties of the characteristic class i.) implies that two star products \( \ast_o \) and \( \ast'_o \) constructed from different data \( (\Omega_o, s_o) \) and \( (\Omega'_o, s'_o) \) using the same fibrewise product \( o_o \)
are equivalent if and only if $\{\Omega_0\} = [\Omega_0']$. In this case $\Omega_0 - \Omega_0' = dC_0$ with a formal series $C_0 = \sum_{i=1}^{\infty} \nu^i C_{i,0} \in \Gamma^\infty(T^* M)[[\nu]]$ of one-forms on $M$ and an equivalence transformation can be constructed in the following way: There is a uniquely determined element $h_0 = \sum_{k=3}^{\infty} h_0^{(k)} \in \mathcal{W}$ with $\text{Deg} h_0^{(k)} = kh_0^{(k)}$ and $\sigma(h_0) = 0$ such that

$$r'_o - r_o - \frac{\exp \left( \frac{1}{\nu} \text{ad}_\nu(h_0) \right) - \text{id}}{\frac{1}{\nu} \text{ad}_\nu(h_0)} (\mathcal{D}_o h_0) = 1 \otimes C_0. \quad (20)$$

For this $h_0$ one has $\mathcal{D}'_o = \exp \left( \frac{1}{\nu} \text{ad}_\nu(h_0) \right) \mathcal{D}_o \exp \left( - \frac{1}{\nu} \text{ad}_\nu(h_0) \right)$ and $A_{h_0}$ defined by

$$A_{h_0} f := \sigma \left( \exp \left( \frac{1}{\nu} \text{ad}_\nu(h_0) \right) \tau_0(f) \right) \quad (21)$$

for $f \in C^\infty(M)[[\nu]]$ the inverse of which is given by $A_{h_0}^{-1} f = \sigma \left( \exp \left( - \frac{1}{\nu} \text{ad}_\nu(h_0) \right) \right) \tau'_0(f)$ is an equivalence from $*_o$ to $*_o'$, i.e. $A_{h_0}(f \ast_o g) = (A_{h_0} f) \ast'_o (A_{h_0} g) \forall f, g \in C^\infty(M)[[\nu]]$.

**Proof:** The characteristic class of $*_p$ has been determined in [21, Thm. 4.4] and is given by the above expression. We consider a Fedosov derivation $\mathcal{D}_{\text{Wick}}$ for $\circ_{\text{Wick}}$ then obviously $\mathcal{D}'_p := S^{-1} \mathcal{D}_{\text{Wick}} S = -\delta + \nabla - \frac{1}{\nu} \text{ad}_\nu(S^{-1} r_{\text{Wick}})$ defines a Fedosov derivation for $\circ_p$. We define $s'_p := \delta^{-1} S^{-1} r_{\text{Wick}}$ and observe that $\sigma(s'_p) = 0$ and that $s'_p$ contains no terms of total degree lower than 3. Moreover, the equation that is satisfied by $r_{\text{Wick}}$ yields that $r'_p := S^{-1} r_{\text{Wick}}$ obeys the equation $\delta r'_p = \nabla r'_p - \frac{1}{\nu} r'_p \circ_p r'_p + R - \frac{1}{\nu} 1 \otimes \rho + 1 \otimes \Omega_{\text{Wick}}$ and hence from the first statement of the proposition we get $c(s'_p) = \frac{\left[\Omega_{\text{Wick}}\right]}{\nu^3} + \frac{\left[\Omega \right]}{\nu^4}$. But this implies the result on the characteristic class of $*_\text{Wick}$ as it is straightforward using the relation between $\mathcal{D}_{\text{Wick}}$ and $\mathcal{D}'_p$ to see that $S$ defined by $S f := \sigma \left( S_r(\delta f) \right)$ for $f \in C^\infty(M)[[\nu]]$ yields an equivalence transformation from $*_p$ to $*_p'$ (cf. [4, Prop. 2.3]) implying that their characteristic classes coincide. For the proof of the formula for $c(\omega_{\text{Wick}})$ one proceeds completely analogously replacing $S$ by $S^{-1}$ in the above argument causing the opposite sign in front of the Ricci form $\rho$. For the proof of ii.) first note that two star products are equivalent if and only if their characteristic classes coincide yielding that $*_o$ and $*_o'$ are equivalent if and only if $[\Omega_0] = [\Omega_0']$ by i.). For the explicit construction of the equivalence transformation similar to the one in [4, Thm. 5.2] one finds by easy computation that $\mathcal{D}'_o = \exp \left( \frac{1}{\nu} \text{ad}_\nu(h_0) \right) \mathcal{D}_o \exp \left( - \frac{1}{\nu} \text{ad}_\nu(h_0) \right)$ if and only if (20) is satisfied. Applying $\mathcal{D}_o$ to this equation and using (13) together with $\mathcal{D}'_o = 0$ one finds that the necessary condition for (20) to be solvable is $\Omega_0 - \Omega_0' = dC_0$ which is fulfilled by assumption. Solving (20) for $\mathcal{D}_o h_0$ and applying $\delta^{-1}$ to the obtained equation one gets using $\delta \delta^{-1} = \delta^{-1} \delta + \sigma = \text{id}$ and $\sigma(h_0) = 0$ that $h_0$ has to satisfy the equation

$$h_0 = C_0 \otimes 1 + \delta^{-1} \left( \nabla h_0 - \frac{1}{\nu} \text{ad}_\nu(r_o) h_0 - \frac{\frac{1}{\nu} \text{ad}_\nu(h_0)}{\exp \left( \frac{1}{\nu} \text{ad}_\nu(h_0) \right) - \text{id}} (r'_o - r_o) \right). \quad (22)$$

Taking into account the total degrees of the involved mappings it is easy to see that the right-hand side of this equation is of the form $L h_o$, where $L$ is a contracting mapping and again from Banach’s fixed point theorem (cf. Corollary [5.3 i.)]) we get the existence and uniqueness of the element $h_o$ satisfying $h_o = L h_o$. Now it remains to prove that this fixed point $h_o$ actually solves (20). To this end one defines $A_o := \frac{\exp \left( \frac{1}{\nu} \text{ad}_\nu(h_o) \right) - \text{id}}{\exp \left( \frac{1}{\nu} \text{ad}_\nu(h_o) \right) - \text{id}} (r'_o - r_o) - \mathcal{D}_o h_0 - 1 \otimes C_0$ and applying $\mathcal{D}_o$ to this equation one derives an equation for $A_o$ of the shape $\mathcal{D}_o A_o = R_{h_o, r'_o, r_o}(A_o)$, where $R_{h_o, r'_o, r_o}$ is a linear mapping that does not decrease the total degree. The explicit shape of $R_{h_o, r'_o, r_o}$ is of minor importance. Using $\delta^{-1} A_o = \sigma(A_o) = 0$ this yields that $A_o$ satisfies

$$A_o = \delta^{-1} \left( \nabla A_o - \frac{1}{\nu} \text{ad}_\nu(r_o) A_o - R_{h_o, r'_o, r_o}(A_o) \right).$$

Again this is a fixed point equation with a unique solution, but as $R_{h_o, r'_o, r_o}$ is linear 0 also solves this fixed point equation implying $A_o = 0$ and hence $h_o$ with $L h_o = h_o$ actually solves (20). Now as in i.) it is straightforward to conclude from $\mathcal{D}'_o = \exp \left( \frac{1}{\nu} \text{ad}_\nu(h_o) \right) \mathcal{D}_o \exp \left( - \frac{1}{\nu} \text{ad}_\nu(h_o) \right)$ that $A_{h_o}$ defined in (21) is an equivalence transformation from $*_o$ to $*_o'$.

$\square$
Using the preceding proposition we are in the position to decide under which conditions on the respective data \((s_p, \Omega_p), (s_{\text{Wick}}, \Omega_{\text{Wick}})\) and \((s_{\text{Wick}}, \Omega_{\text{Wick}})\) the star products \(*_p, *_{\text{Wick}}\) and \(*_{\text{Wick}}\) are equivalent and in this case we can moreover explicitly give an equivalence transformation using suitable combinations of the constructed transformations given in the proof of the proposition.

To conclude this section we shall now give a condition on the data \((s_{\text{Wick}}, \Omega_{\text{Wick}})\) and \((s_{\text{Wick}}, \Omega_{\text{Wick}})\) under which we have that \(*_{\text{Wick}} = (\ast_{\text{Wick}})_{\text{opp}}\) which shall enable us to restrict all our further considerations to star products of Wick type, since from this relation we can immediately deduce the corresponding statement about the corresponding star product of anti-Wick type.

**Lemma 3.7** For \(s_{\text{Wick}} = P s_{\text{Wick}}\) and \(\Omega_{\text{Wick}} = P \Omega_{\text{Wick}}\) we have \(*_{\text{Wick}} = (\ast_{\text{Wick}})_{\text{opp}}, i.e. for all \(f, g \in \mathcal{C}^{\infty}(M)[[\nu]]\) we have

\[ f \ast_{\text{Wick}} g = P((P g) \ast_{\text{Wick}} (P f)). \]

**Proof:** First we observe that \(P(a \ast_{\text{Wick}} b) = (-1)^{|a||b|}(Pa) \ast_{\text{Wick}} (Pb)\) because of the explicit shape of the fibrewise products. Using equation (13) for \(r_{\text{Wick}}\) one obtains that \(\delta^{-1}P r_{\text{Wick}} = Ps_{\text{Wick}}\) and \(\nu P r_{\text{Wick}} = \nabla P r_{\text{Wick}} - \frac{1}{\nu}(P r_{\text{Wick}}) \ast_{\text{Wick}} (P r_{\text{Wick}}) + R + 1 \otimes P \Omega_{\text{Wick}}\) and hence for \(\Omega_{\text{Wick}} = P \Omega_{\text{Wick}}\) and \(s_{\text{Wick}} = Ps_{\text{Wick}}\) we have \(\nabla r_{\text{Wick}} = P r_{\text{Wick}}\) implying that \(D_{\text{Wick}} P = P D_{\text{Wick}}\). From this relation it is easy to deduce that \(P r_{\text{Wick}}(f) = \nabla_{\text{Wick}}(P f)\) for all \(f \in \mathcal{C}^{\infty}(M)[[\nu]]\) and using this equation the proof of (23) is straightforward.

\(\square\)

### 4 Unique Characterization of the Star Products of Wick Type

In this section we shall uniquely characterize the star products of Wick type among the star products \(*_{\text{Wick}}\) constructed in the preceding section, which actually not all are of Wick type although the fibrewise product \(\ast_{\text{Wick}}\) used to obtain them is of this shape. Throughout the whole section we shall assume that \(D_{\text{Wick}}\) is a Fedosov derivation constructed as in Theorem 3.1 using an element \(s_{\text{Wick}}\) as normalization condition for \(r_{\text{Wick}}\) that contains no part of symmetric degree 1, which according to Lemma 3.3 represents no restriction for the obtained star products. Throughout this section we shall extensively make use of the notations and results given in Appendix A that are crucial for our investigations.

In [5] Bordemann and Waldmann have considered the star product \(*_{\text{Wick}}\) in case \(s_{\text{Wick}} = \Omega_{\text{Wick}} = 0\) and have shown that the resulting star product is of Wick type. In their proof some special property of the element \(r_{\text{Wick}}\) determining the Fedosov derivation was crucial. In this special case one has according to [5] Lemma 4.5) for all \(q, p \geq 0\) that

\[ \pi^p a_r = \pi^q a_r = 0. \]

Particularly this means that in this case \(r_{\text{Wick}}\) contains no part of symmetric degree 1. But in case \(\Omega_{\text{Wick}} \neq 0\) one immediately finds that \(r_{\text{Wick}}^{(2k+1)}\) contains for some \(k \geq 1\) a summand of the shape \(\nu^k \delta^{-1}(1 \otimes \Omega_{\text{Wick},k})\). On the other hand one cannot do without allowing for \(\Omega_{\text{Wick}} \neq 0\) as to obtain star products of arbitrary characteristic class because of Proposition 3.6. The only way out can consist in showing that a weaker condition on \(r_{\text{Wick}}\) than (24) which can also be satisfied for \(\Omega_{\text{Wick}} \neq 0\) suffices to guarantee that the resulting star product is of Wick type. The results of Karabegov on the classification of star products of Wick type now suggest that the case, where \(\Omega_{\text{Wick}}\) is of type \((1,1)\) should be of special interest.

10
Lemma 4.1 Let \( r_{Wick} \in W \otimes \Lambda^1 \) be constructed as in Theorem 3.1 from \( s_{Wick} \) containing no part of symmetric degree 1, then we have:

\[
\begin{align*}
\pi_z r_{Wick} &= 0 \iff \pi_z (1 \otimes \Omega_{Wick}) = 0 \quad \text{and} \quad \pi_z s_{Wick} = 0, \quad (25) \\
\pi_{\overline{z}} r_{Wick} &= 0 \iff \pi_{\overline{z}} (1 \otimes \Omega_{Wick}) = 0 \quad \text{and} \quad \pi_{\overline{z}} s_{Wick} = 0. \quad (26)
\end{align*}
\]

Hence \( \pi_z r_{Wick} = \pi_{\overline{z}} r_{Wick} = 0 \) if and only if \( \Omega_{Wick} \) is of type \((1, 1)\) and \( \pi_z s_{Wick} = \pi_{\overline{z}} s_{Wick} = 0 \).

Proof: We first show the equivalence \((25)\) and first let \( \pi_z (1 \otimes \Omega_{Wick}) = \pi_z s_{Wick} = 0 \). Applying \( \pi_z \) to the equations that determine \( r_{Wick} \) we obtain using the formulas given in Appendix A that \( \pi_z r_{Wick} \) satisfies the equations

\[
\delta_z \pi_z r_{Wick} = \nabla_z \pi_z r_{Wick} - \frac{1}{\nu} \pi_z ((\pi_z r_{Wick}) \circ_W r_{Wick}) \quad \text{and} \quad \delta_z^{-1} \pi_z r_{Wick} = 0.
\]

Using \( \delta_z^{-1} \delta_z + \delta_z \delta_z^{-1} + \pi_{\overline{z}} = \text{id} \) these equations yield after some easy manipulations that \( \pi_z r_{Wick} \) is a fixed point of the mapping \( L : \pi_z (W \otimes \Lambda^1) \ni a \mapsto \delta_z^{-1} (\nabla_z a - \frac{1}{\nu} \pi_z (\text{ad}_{Wick} (r_{Wick}) a)) \in \pi_z (W \otimes \Lambda^1) \) that raises the total degree at least by 1 and hence has a unique fixed point according to Corollary [3.3 iii.). Obviously \( L \) has 0 as trivial fixed point implying that \( \pi_z r_{Wick} = 0 \) by uniqueness. For the other direction of the equivalence let \( \pi_z r_{Wick} = 0 \) and again apply \( \pi_z \) to the equations that determine \( r_{Wick} \) then it is very easy to see that \( \pi_z r_{Wick} = 0 \) implies that \( \pi_z (1 \otimes \Omega_{Wick}) = 0 \) and \( \pi_z s_{Wick} = 0 \). For the proof of \((26)\) one proceeds completely analogously. \( \square \)

An important consequence of the presence of the complex structure on \( M \) is that the explicit expression for \( f \ast_{Wick} g \) does not depend on the complete Fedosov-Taylor series of \( f \) and \( g \) but only on the holomorphic part of \( \tau_{Wick}(f) \) and the anti-holomorphic part of \( \tau_{Wick}(g) \). To see this fact we just observe that the properties of \( \pi_z \) and \( \pi_{\overline{z}} \) given in Appendix A imply that

\[
f \ast_{Wick} g = \sigma(\tau_{Wick}(f) \circ_W \tau_{Wick}(g)) = \sigma((\pi_z \tau_{Wick}(f)) \circ_W (\pi_{\overline{z}} \tau_{Wick}(g))). \quad (27)
\]

This observation raises the question whether it is possible to find more simple recursion formulas for \( \pi_z \tau_{Wick}(f) \) and \( \pi_{\overline{z}} \tau_{Wick}(g) \) than the one for the whole Fedosov-Taylor series, since these would completely determine the star product \( \ast_{Wick} \). Under the preconditions \( \pi_z r_{Wick} = 0 \) and \( \pi_{\overline{z}} r_{Wick} = 0 \) this question can be answered positively as we state it in the following proposition.

Proposition 4.2 Let \( r_{Wick} \in W \otimes \Lambda^1 \) be constructed as in Theorem 3.1 from \( s_{Wick} \) containing no part of symmetric degree 1 and let \( \tau_{Wick} \) denote the corresponding Fedosov-Taylor series according to Theorem 3.2.

i.) If \( \pi_z r_{Wick} = 0 \) then \( \pi_z \tau_{Wick}(f) \) satisfies the equation

\[
\delta_z \pi_z \tau_{Wick}(f) = \nabla_z \pi_z \tau_{Wick}(f) + \frac{1}{\nu} \pi_z ((\pi_z \tau_{Wick}(f)) \circ_W r_{Wick}) \quad (28)
\]

for \( f \in \mathcal{C}^\infty(M[[\nu]]) \). In this case \( \pi_z \tau_{Wick}(f) \) is uniquely determined from this equation since \( \sigma(\pi_z \tau_{Wick}(f)) = f \) and can be computed recursively for \( f \in \mathcal{C}^\infty(M) \) from

\[
\pi_z \tau_{Wick}(f) = f + \delta_z^{-1} \left( \nabla_z \pi_z \tau_{Wick}(f) + \frac{1}{\nu} \pi_z ((\pi_z \tau_{Wick}(f)) \circ_W r_{Wick}) \right). \quad (29)
\]

ii.) If \( \pi_{\overline{z}} r_{Wick} = 0 \) then \( \pi_{\overline{z}} \tau_{Wick}(f) \) satisfies the equation

\[
\delta_{\overline{z}} \pi_{\overline{z}} \tau_{Wick}(f) = \nabla_{\overline{z}} \pi_{\overline{z}} \tau_{Wick}(f) - \frac{1}{\nu} \pi_{\overline{z}} (r_{Wick} \circ_W (\pi_{\overline{z}} \tau_{Wick}(f))) \quad (30)
\]
for \( f \in \mathcal{C}^\infty(M)[[\nu]] \). In this case \( \pi_\text{Wick}(f) \) is uniquely determined from this equation since \( \sigma(\pi_\text{Wick}(f)) = f \) and can be computed recursively for \( f \in \mathcal{C}^\infty(M) \) from

\[
\pi_\text{Wick}(f) = f + \delta_\nu^{-1}\left(\nabla_\pi \pi_\text{Wick}(f) - \frac{1}{\nu} \pi_\nu (\pi_\text{Wick} \circ \pi_\text{Wick}(f))\right).
\]  

(31)

**Proof:** Applying \( \pi_z \) to the equation \( \mathcal{D}_\text{Wick}(\pi_\text{Wick}(f)) = 0 \) that is valid for all \( f \in \mathcal{C}^\infty(M)[[\nu]] \) we obtain using \( \pi_z \pi_\text{Wick} = 0 \) that

\[
\delta_\nu \pi_z \pi_\text{Wick}(f) = \nabla_\pi \pi_z \pi_\text{Wick}(f) + \frac{1}{\nu} \pi_z \left((\pi_z \pi_\text{Wick}(f)) \circ \pi_\text{Wick} \pi_\text{Wick}(f)\right)
\]

since \( \pi_z \pi_\text{Wick} \circ \pi_\text{Wick}(f) = \pi_z((\pi_z \pi_\text{Wick}) \circ \pi_\text{Wick} \pi_\text{Wick}(f)) = 0 \). To this equation we apply \( \delta_\nu^{-1} \) and with equation (33) this yields the stated equation (29). Now the mapping \( T_f : \pi_z(W) \to \pi_z(W) \) defined by \( T_f a := f + \delta_\nu^{-1}(\nabla_\pi a + \frac{1}{\nu} \pi_z(a \circ \pi_\text{Wick} \pi_\text{Wick})) \) turns out to be contracting and hence according to Corollary B.3 there is a unique fixed point \( b_f \in \pi_z(W) \) with \( T_f b_f = b_f \) that obviously satisfies \( \sigma(b_f) = f \). It remains to show that this fixed point actually solves the equation \( \delta_\nu b_f = \nabla_\pi b_f + \frac{1}{\nu} \pi_z(b_f \circ \pi_\text{Wick} \pi_\text{Wick}) \) since then \( b_f = \pi_z \pi_\text{Wick}(f) \). To this end one defines \( A := -\delta_\nu b_f + \nabla_\pi b_f + \frac{1}{\nu} \pi_z(b_f \circ \pi_\text{Wick} \pi_\text{Wick}) \) and an easy computation using Lemma A.3 yields that \( A \) satisfies \( \delta_\nu A = \nabla_\pi A - \frac{1}{\nu} \pi_z(A \circ \pi_\text{Wick} \pi_\text{Wick}) \) from which we obtain again using (33) that \( A \) is a fixed point of the mapping \( L : \pi_z(W \otimes \Lambda^1) \ni a \to \delta_\nu^{-1}(\nabla_\pi a - \frac{1}{\nu} \pi_z(a \circ \pi_\text{Wick} \pi_\text{Wick})) \in \pi_z(W \otimes \Lambda^1) \). But again this fixed point is unique and so we get \( A = 0 \) implying that \( b_f \) solves the equation that is solved by \( \pi_z \pi_\text{Wick}(f) \) proving that \( \pi_z \pi_\text{Wick}(f) \) is uniquely determined by (28) and can be obtained recursively from (28). For the proof of ii.) one proceeds completely analogously.

An easy consequence of the preceding proposition is:

**Lemma 4.3** Let \( U \subseteq M \) be an open subset of \( M \).

1. If \( \pi_z \pi_\text{Wick} = 0 \) then we have for all \( h' \in \mathcal{C}^\infty(M) \) with \( h'|_U \in \mathcal{O}(U) \) that

\[
\pi_z \pi_\text{Wick}(h')|_U = h'|_U.
\]

(32)

2. If \( \pi_\text{Wick} = 0 \) then we have for all \( h \in \mathcal{C}^\infty(M) \) with \( h|_U \in \mathcal{O}(U) \) that

\[
\pi_\text{Wick}(h)|_U = h|_U.
\]

(33)

**Proof:** To prove (32) it suffices to show that \( h'|_U \) satisfies the equation that determines \( \pi_z \pi_\text{Wick}(h')|_U \). But this is an easy computation using that \( \nabla_\pi h'|_U = 0 \) since \( h' \in \mathcal{O}(U) \) and again that \( \pi_z \pi_\text{Wick} = 0 \). The proof of ii.) is completely analogous.

With the results collected up to now we are already in the position to give a sufficient condition on which the star product *_Wick is of Wick type.

**Proposition 4.4** Let \( \mathcal{D}_\text{Wick} \) be the Fedosov derivation constructed from the data \( (\Omega_\text{Wick}, s_\text{Wick}) \) and let *_Wick be the resulting star product on a pseudo-Kähler manifold \( (M, \omega, I) \). Moreover, let \( U \subseteq M \) be an open subset of \( M \).

1. If \( \pi_z(1 \otimes \Omega_\text{Wick}) = \pi_z(s_\text{Wick}) = 0 \) then we have for all \( g, h' \in \mathcal{C}^\infty(M) \) with \( h'|_U \in \mathcal{O}(U) \) that

\[
h' *_\text{Wick} g|_U = h'|_U.
\]

(34)

2. If \( \pi_z(1 \otimes \Omega_\text{Wick}) = \pi_z(s_\text{Wick}) = 0 \) then we have for all \( f, h \in \mathcal{C}^\infty(M) \) with \( h|_U \in \mathcal{O}(U) \) that

\[
f *_\text{Wick} h|_U = fh|_U.
\]

(35)
Consequently the star product $\ast_{\text{Wick}}$ is of Wick type in case we have $\pi_z s_{\text{Wick}} = \pi s_{\text{Wick}} = 0$ and $\Omega_{\text{Wick}}$ is of type $(1,1)$.

**Proof:** i.) According to Lemma 4.4 the conditions on $s_{\text{Wick}}$ and $\Omega_{\text{Wick}}$ imply that $\pi z r_{\text{Wick}} = 0$. But with Lemma 4.5 i.) this directly implies the equation (34). The proof of ii.) is analogous.

**Remark 4.5** It is remarkable that the statement of the above proposition shows that the two characterizing properties of a star product of Wick type can in fact be realized separately. In case equation (34) is satisfied it is easy to see that the bidifferential operators describing $\ast_{\text{Wick}}$ in a local holomorphic chart are of the shape $C_r(f,g) = \sum_{K,M} C_r^{K,M} \frac{\partial |f|^{(K+1)|M|}}{\partial z^k \partial \bar{z}^l}$ for $r \geq 1$. Analogously (35) implies that we have $C_r(f,g) = \sum_{K,M} C_r^{K,M} \frac{\partial |f|^{(K+1)|M|}}{\partial z^k \partial \bar{z}^l}$.

Our next aim is to see that the conditions given in the above proposition are not only sufficient but even necessary for the equations (34) and (35) to hold. To prove this we need the following two lemmas.

**Lemma 4.6** Let $\tau_{\text{Wick}}$ denote the Fedosov-Taylor series corresponding to $D_{\text{Wick}}$ and let $U \subseteq M$ be an open subset of $M$.

i.) If we have for all $h' \in C^\infty(M)$ with $h'|_U \subseteq \mathcal{O}(U)$ that the Fedosov-Taylor series satisfies $\pi_z \tau_{\text{Wick}}(h')|_U = h'|_U$ then we have $\pi_z r_{\text{Wick}} = 0$.

ii.) If we have for all $h \in C^\infty(M)$ with $h|_U \subseteq \mathcal{O}(U)$ that the Fedosov-Taylor series satisfies $\pi \tau_{\text{Wick}}(h)|_U = h|_U$ then we have $\pi z r_{\text{Wick}} = 0$.

**Proof:** We apply $\pi_z$ to the equation $D_{\text{Wick}} \tau_{\text{Wick}}(h') = 0$ and restrict the obtained equation to $U$ and get using $\pi_z \tau_{\text{Wick}}(h')|_U = h'|_U$ and considering the term of total degree $l \geq 1$ that

$$0 = \frac{1}{l} \sum_{r=1}^{l} \pi_z((\pi_z r_{\text{Wick}}^{(l+2-r)}) \circ_{\text{Wick}} \tau_{\text{Wick}}(h')^{(r)})|_U$$

for all $l \geq 1$. Now we shall show by induction on the total degree that this equation actually implies $\pi_z r_{\text{Wick}} = 0$. Let $(z, V)$ be a local holomorphic chart of $M$ and let $\chi \in C^\infty(M)$ be a function with $\text{supp}(\chi) \subseteq V$ and $\chi|_U = 1$, where $U \subseteq V$ then for all $1 \leq k \leq \dim_{\mathbb{C}}(M)$ the function $\chi^{\bar{z}_k}$ belongs to $C^\infty(M)$ satisfies $\chi^{\bar{z}_k}|_U \subseteq \mathcal{O}(U)$. We now use the above equation for $h' = \chi^{\bar{z}_k}$ and for $l = 1$ yielding $0 = \frac{1}{2} g \chi^{\bar{z}_k} (Z_n) \pi_z r_{\text{Wick}}^{(1)}|_U$ since $\tau_{\text{Wick}}(\chi^{\bar{z}_k})^{(1)}|_U = d\bar{z}_k|_U$. Consequently the only non-vanishing terms in $\pi_z r_{\text{Wick}}^{(2)}|_U$ can only be of symmetric degree 0 as $g$ is non-degenerate. But from the recursion formula for $r_{\text{Wick}}$ and the fact that $s_{\text{Wick}}$ contains no term of symmetric degree 1 it is obvious that $r_{\text{Wick}}$ contains no part of symmetric degree 0 and hence we have $\pi_z r_{\text{Wick}}^{(2)}|_U = 0$. Repeating this argument with all holomorphic charts of $M$ this finally implies $\pi_z r_{\text{Wick}}^{(2)} = 0$. We now assume that $\pi_z r_{\text{Wick}}^{(n)} = 0$ for $2 \leq s \leq m$ then the equation derived above yields for $l = m$ that $0 = \frac{1}{l} \pi_z((\pi_z r_{\text{Wick}}^{(m+1)}) \circ_{\text{Wick}} \tau_{\text{Wick}}(h')^{(1)})|_U$. Repeating the above argument this now implies $\pi_z r_{\text{Wick}}^{(m+1)} = 0$ and hence $\pi_z r_{\text{Wick}} = 0$ by induction. For the proof of ii.) one proceeds analogously using the functions $\chi^{\bar{z}_k}$ instead of $\chi^{\bar{z}_k}$.

**Lemma 4.7** Let $U \subseteq M$ be an open subset of $M$.

i.) If we have $h' \ast_{\text{Wick}} g|_U = h'|_U g|_U$ for all $g, h' \in C^\infty(M)$ with $h'|_U \subseteq \mathcal{O}(U)$ then $\pi_z \tau_{\text{Wick}}(h')|_U = h'|_U$ for all $h \in C^\infty(M)$ with $h|_U \subseteq \mathcal{O}(U)$.

ii.) If we have $f \ast_{\text{Wick}} h|_U = f h|_U$ for all $f, h \in C^\infty(M)$ with $h|_U \subseteq \mathcal{O}(U)$ then $\pi \tau_{\text{Wick}}(h)|_U = h|_U$ for all $h \in C^\infty(M)$ with $h|_U \subseteq \mathcal{O}(U)$.
Proof: It is easy to compute that $h'|_{\text{Wick}} g = h'g + \sum_{s=1}^{\infty} \sum_{r=1}^{2s-1} \sigma((\pi_z \tau_{\text{Wick}}(h')^{(r)}) \circ_{\text{Wick}} (\pi_{\tau} \tau_{\text{Wick}}(g)^{(2s-r)}))$. Since the summand corresponding to the summation index $s$ is of $\nu$-degree $s$ the condition $h'|_{\text{Wick}} g|_U = h'g|_U$ implies that we have $\sum_{s=1}^{2s-1} \sigma((\pi_z \tau_{\text{Wick}}(h')^{(r)}) \circ_{\text{Wick}} (\pi_{\tau} \tau_{\text{Wick}}(g)^{(2s-r)}))|_U = 0$ for all $s \geq 1$. We first consider the case $s = 1$. Now let $(z, V)$ be a holomorphic chart of $M$ and let $\chi \in C^\infty(M)$ with $\text{supp}(\chi) \subseteq V$ and $\chi|_U = 1$ for some open subset $U' \subseteq V$ with $U \cap U' \neq \emptyset$. Then we consider the functions $\chi^{\pi_{\tau}} \in C^\infty(M)$ and from $\tau_{\text{Wick}}(\chi^{\pi_{\tau}})^{(1)}|_{U \cap U'} = \nu \cdot \chi^{\pi_{\tau}}|_{U \cap U'}$ we get from the above equation using $g = \chi^{\pi_{\tau}}$ that $0 = 2\nu \cdot \nu \cdot \chi^{\pi_{\tau}}(Z_n) \pi_z \tau_{\text{Wick}}(h')^{(1)}|_{U \cap U'}$. As the symmetric degree of $\pi_z \tau_{\text{Wick}}(h')^{(1)}$ is 1 and $\nu$ is non-degenerate we conclude that $\pi_z \tau_{\text{Wick}}(h')^{(1)}|_{U \cap U'} = 0$. Repeating this argument with all holomorphic charts such that $U \cap U' \neq \emptyset$ we finally obtain $\pi_z \tau_{\text{Wick}}(h')^{(1)}|_U = 0$. Now we want to use induction on the total degree to show that the assumption actually implies $\pi_z \tau_{\text{Wick}}(h')^{(r)}|_U = 0$ for all $r \geq 1$. To this end we need the following:

**Sublemma 4.8** Let $(z, V)$ be a holomorphic chart of $M$ around $p \in M$ with $z(p) = 0$. Further let $\chi \in C^\infty(M)$ with $\text{supp}(\chi) \subseteq V$ and $\chi|_U = 1$, where $p \in U' \subseteq V$. Moreover, let $\star_{\text{Wick}}$ satisfy $h'|_{\text{Wick}} g|_U = h'g|_U$ for all $U \subseteq M$ and all $g, h' \in C^\infty(M)$ with $h' \in \overline{\mathcal{O}}(U)$. Then we have for all $r \geq 1$ that $\tau_{\text{Wick}}(\chi^{\pi_{\tau}}(\ldots \pi_{\tau})^{(s)}(\pi_{\pi_{\tau}})^{(l_1)}(\ldots (\pi_{\pi_{\tau}})^{(l_s)}(\chi^{\pi_{\tau}})|_{U'}) = 0$ for $0 \leq s < r$ and $\tau_{\text{Wick}}(\chi^{\pi_{\tau}}(\ldots \pi_{\tau})^{(s)}(\pi_{\pi_{\tau}})^{(l_1)}(\ldots (\pi_{\pi_{\tau}})^{(l_s)}(\chi^{\pi_{\tau}})|_{U'}) = 0$ for all $r \geq 1$. To this end we need the following:

Proof: As the functions $\chi^{\pi_{\tau}}$ are locally anti-holomorphic on $U'$ and as $\pi_{\pi_{\tau}}^r|_U = 1$ for $r \in \mathbb{N}$ we have $\tau_{\text{Wick}}(\chi^{\pi_{\tau}}(\ldots \pi_{\tau})^{(s)}(\pi_{\pi_{\tau}})^{(l_1)}(\ldots (\pi_{\pi_{\tau}})^{(l_s)}(\chi^{\pi_{\tau}})|_{U'}) = 0$. To this we apply $\tau_{\text{Wick}}$ and consider the term of total degree $s$ and get

$$\tau_{\text{Wick}}(\chi^{\pi_{\tau}}(\ldots \pi_{\tau})^{(s)}(\pi_{\pi_{\tau}})^{(l_1)}(\ldots (\pi_{\pi_{\tau}})^{(l_s)}(\chi^{\pi_{\tau}})|_{U'}) = \sum_{l_1 + \ldots + l_s = s} \tau_{\text{Wick}}(\chi^{\pi_{\tau}}(\pi_{\pi_{\tau}})^{(l_1)}(\ldots (\pi_{\pi_{\tau}})^{(l_s)}(\chi^{\pi_{\tau}})|_{U'})$$

For $s < r$ at least one $l_i$ is equal to 0 and we get the first assertion since $z(p) = 0$ and $\tau_{\text{Wick}}(\chi^{\pi_{\tau}}(\ldots \pi_{\tau})^{(s)}(\pi_{\pi_{\tau}})^{(l_1)}(\ldots (\pi_{\pi_{\tau}})^{(l_s)}(\chi^{\pi_{\tau}})|_{U'}) = 0$. For $s = r$ the only summand that does not vanish at $p$ is the one for $l_1 = \ldots = l_s = 1$ and observing that $\tau_{\text{Wick}}(\chi^{\pi_{\tau}}(\pi_{\pi_{\tau}})^{(l_1)}(\ldots (\pi_{\pi_{\tau}})^{(l_s)}(\chi^{\pi_{\tau}})|_{U'}) = \nu \cdot \chi^{\pi_{\tau}}|_{U'}$ the explicit shape of $\tau_{\text{Wick}}$ implies the second sentence of the sublemma.

Now we assume that $\pi_z \tau_{\text{Wick}}(h')^{(r)}|_U = 0$ for $1 \leq r \leq m - 1$, where $m \geq 2$. Using the equation $\sum_{r=1}^{2s-1} \sigma((\pi_z \tau_{\text{Wick}}(h')^{(r)}) \circ_{\text{Wick}} (\pi_{\tau} \tau_{\text{Wick}}(g)^{(2s-r)}))|_U = 0$ for all $s$ with $2s \geq m + 1$ this induction hypothesis implies that $0 = \sum_{r=1}^{2s-1} \sigma((\pi_z \tau_{\text{Wick}}(h')^{(r)}) \circ_{\text{Wick}} (\pi_{\tau} \tau_{\text{Wick}}(g)^{(2s-r)}))|_U$. Now let $(z, V)$ be a chart as in the sublemma then we use this equation for $g = \chi^{\pi_{\tau}}(\ldots \pi_{\pi_{\tau}})^{(s)}(\pi_{\pi_{\tau}})^{(l_1)}(\ldots (\pi_{\pi_{\tau}})^{(l_s)}(\chi^{\pi_{\tau}})|_{U'})$ and obtain

$$0 = \left(\frac{2\nu}{1}\right)^{2s-m} \sum_{r=1}^{2s-m} \sigma((\pi_z \tau_{\text{Wick}}(h')^{(r)}) \circ_{\text{Wick}} (\pi_{\tau} \tau_{\text{Wick}}(g)^{(2s-r)}))|_U$$

for all $s$ with $2s \geq m + 1$. Considering the equation for $m$ even and $m$ odd separately it is easy to see that this implies $\pi_z \tau_{\text{Wick}}(h')^{(m)}|_U = 0$. But if $p \in U$ was arbitrary we may conclude that $\pi_z \tau_{\text{Wick}}(h')^{(r)}|_U = 0$ implying i.) For the proof of ii.) one proceeds quite analogously.

Altogether we have now shown:

Theorem 4.9 Let $\mathcal{D}_{\text{Wick}}$ be the Fedosov derivation constructed for $\circ_{\text{Wick}}$ from the data $(\Omega_{\text{Wick}}, s_{\text{Wick}})$, where $s_{\text{Wick}}$ contains no terms of symmetric degree 1 and let $\star_{\text{Wick}}$ be the corresponding star product on a pseudo-Kähler manifold $(M, \omega, I)$ then we have the following equivalences:

i.)

$$\pi_z s_{\text{Wick}} = 0 \quad \pi_z (1 \otimes \Omega_{\text{Wick}}) = 0 \quad \iff \quad \pi_z \tau_{\text{Wick}}(h')|_U = h'|_U \quad \forall h' \in C^\infty(M) \quad (36)$$

$$\iff \quad h'|_{\text{Wick}} g|_U = h'g|_U \quad \forall g, h' \in C^\infty(M) \quad (37)$$

ii.)

$$\pi_z s_{\text{Wick}} = 0 \quad \pi_z (1 \otimes \Omega_{\text{Wick}}) = 0 \quad \iff \quad \pi_z \tau_{\text{Wick}}(h)|_U = h|_U \quad \forall h \in C^\infty(M) \quad (38)$$

$$\iff \quad f \star_{\text{Wick}} h|_U = fh|_U \quad \forall f, h \in C^\infty(M) \quad (39)$$
iii.) Hence the star product $\ast_{\text{Wick}}$ on $(M, \omega, I)$ is of Wick type if and only if $\Omega_{\text{Wick}}$ is of type $(1, 1)$ and $\pi_z s_{\text{Wick}} = \pi_z \pi s_{\text{Wick}} = 0$.

After having uniquely characterized the star products $\ast_{\text{Wick}}$ that are of Wick type it is almost trivial to characterize the star products $\ast_{\text{Wick}}$ that are of anti-Wick type.

**Corollary 4.10** Let $\Omega_{\text{Wick}}$ be the Fedosov derivation obtained for $\sqrt{\Omega_{\text{Wick}}}$ from the data $(\Omega_{\text{Wick}}, s_{\text{Wick}})$, where $s_{\text{Wick}}$ contains no terms of symmetric degree 1 and let $\ast_{\text{Wick}}$ be the corresponding star product on a pseudo-Kähler manifold $(M, \omega, I)$ then we have the following equivalences:

i.)
\[
\frac{\pi_z s_{\text{Wick}}}{\pi_z(1 \otimes \Omega_{\text{Wick}})} = 0 \iff \frac{\pi_z \tau_{\text{Wick}}}{\pi_z(1 \otimes \Omega_{\text{Wick}})} = 0 \iff \frac{\pi_z \tau_{\text{Wick}}(h')|_U = h'|_U}{\pi_z(1 \otimes \Omega_{\text{Wick}})} \forall h' \in C^\infty(M) \quad (40)
\]
\[
\iff g \ast_{\text{Wick}} h'|_U = gh'|_U \forall g, h' \in C^\infty(M) \quad (41)
\]

ii.)
\[
\frac{\pi_z s_{\text{Wick}}}{\pi_z(1 \otimes \Omega_{\text{Wick}})} = 0 \iff \frac{\pi_z \tau_{\text{Wick}}}{\pi_z(1 \otimes \Omega_{\text{Wick}})} = 0 \iff \frac{\pi_z \tau_{\text{Wick}}(h)|_U = h|_U}{\pi_z(1 \otimes \Omega_{\text{Wick}})} \forall h \in C^\infty(M) \quad (42)
\]
\[
\iff h \ast_{\text{Wick}} f|_U = hf|_U \forall f, h \in C^\infty(M) \quad (43)
\]

iii.) Hence the star product $\ast_{\text{Wick}}$ on $(M, \omega, I)$ is of anti-Wick type if and only if $\Omega_{\text{Wick}}$ is of type $(1, 1)$ and $\pi_z s_{\text{Wick}} = \pi_z \pi s_{\text{Wick}} = 0$.

**Proof:** We consider the star product $\ast_{\text{Wick}}$ constructed from the data $(\Omega_{\text{Wick}}, s_{\text{Wick}})$ then we have from Lemma 3.37 that the star product $\ast_{\text{Wick}}$ constructed from $(\Omega_{\text{Wick}} = P\Omega_{\text{Wick}}, s_{\text{Wick}} = Ps_{\text{Wick}})$ satisfies $f \ast_{\text{Wick}} g = P(fg)$ for all $f, g \in C^\infty(M)[[\nu]]$. Moreover, in this case $r_{\text{Wick}} = P \tau_{\text{Wick}}$ and $\tau_{\text{Wick}}(f) = P \tau_{\text{Wick}}(Pf)$ for all $f \in C^\infty(M)[[\nu]]$. Using these relations the proof of the corollary is trivial using the corresponding statements of Theorem 4.9.

After having uniquely characterized the star products $\ast_{\text{Wick}}$ resp. $\ast'_{\text{Wick}}$ that are of Wick resp. anti-Wick type by some conditions on the data $(\Omega_{\text{Wick}}, s_{\text{Wick}})$ resp. $(\Omega_{\text{Wick}}, s_{\text{Wick}})$ we can show another interesting property of these star products namely that they actually do not depend on $s_{\text{Wick}}$ resp. $s'_{\text{Wick}}$. To this end we consider two Fedosov star products $\ast_{\text{Wick}}$ and $\ast'_{\text{Wick}}$ of Wick type that are obtained from the same formal series of closed two-forms $\Omega_{\text{Wick}} = \Omega'_{\text{Wick}}$ of type $(1, 1)$ but different elements $s_{\text{Wick}}$ and $s'_{\text{Wick}}$. From Proposition 3.6 we already know that these star products are equivalent and we even have an explicit construction for an equivalence transformation. Letting $h_{\text{Wick}} \in W$ be the element determined by the equation (22), where $\circ = \text{w}ick$ and $C_{\text{Wick}} = 0$ we have that $A_{\text{Wick}}$ defined by $A_{\text{Wick}} f := \sigma(\exp(\frac{1}{\nu} \text{ad}_{\text{Wick}}(h_{\text{Wick}})) \tau_{\text{Wick}}(f))$ is an equivalence from $\ast_{\text{Wick}}$ to $\ast'_{\text{Wick}}$ and our aim is to show that the fact that both star products are of Wick type implies that this equivalence transformation is equal to the identity.

**Lemma 4.11** Let $\ast_{\text{Wick}}, \ast'_{\text{Wick}}$ and $h_{\text{Wick}}$ be given as above then we have
\[
\pi_z h_{\text{Wick}} = \pi_z \pi h_{\text{Wick}} = 0. \quad (44)
\]
PROOF: We want to prove the statement by induction on the total degree and hence we explicitly write the term of total degree $k \geq 3$ of equation \[22\] as

$$h_{\text{Wick}}^{(k)} = \delta^{-1} \left( \nabla h_{\text{Wick}}^{(k-1)} - \frac{1}{\nu} \sum_{k_1+l=k+1, l \geq 2} \text{ad}_{\text{Wick}}(r_{\text{Wick}}(l)) h_{\text{Wick}}^{(k_1)} \right)$$

$$- \sum_{r=0}^{\infty} \frac{1}{r!} B_r \left( \frac{1}{\nu} \right)^r \sum_{k_1 + \ldots + k_r = k+1+2r, 3 \leq k_1 \leq k-1} \text{ad}_{\text{Wick}}(h_{\text{Wick}}^{(k_1)}) \ldots \text{ad}_{\text{Wick}}(h_{\text{Wick}}^{(k_r)})(r_{\text{Wick}} - r_{\text{Wick}}(l))^0 \right),$$

where $B_r$ denotes the $r$th Bernoulli number that arises from the Taylor expansion of \[\frac{\text{ad}_{\text{Wick}}(h_{\text{Wick}})}{\exp(\frac{1}{\nu} \text{ad}_{\text{Wick}}(h_{\text{Wick}})) - \text{id}}\] which is given by \[\sum_{r=0}^{\infty} \frac{1}{r!} B_r \left( \frac{1}{\nu} \right)^r \text{ad}_{\text{Wick}}(h_{\text{Wick}})^r \]. For $k = 3$ this yields using $B_0 = 1$ that $h_{\text{Wick}}^{(3)} = \delta^{-1}((r_{\text{Wick}} - r_{\text{Wick}}^{(2))})$ and hence we get using Lemma \[A.2\] and $\pi_z r_{\text{Wick}} = \pi_z r_{\text{Wick}}^{(0)} = \pi_{\text{Wick}} = \pi_{\text{Wick}}^{(0)} = 0$ which holds since both star products are of Wick type that $\pi_z h_{\text{Wick}}^{(3)} = \pi_{\text{Wick}}^{(3)} = 0$. Let us now assume that $\pi_z h_{\text{Wick}}^{(m)} = 0$ for $3 \leq m \leq k - 1$, where $k \geq 4$ and consider

$$\pi_z h_{\text{Wick}}^{(k)} = \delta^{-1} \left( \nabla z \pi_z h_{\text{Wick}}^{(k-1)} - \frac{1}{\nu} \sum_{k_1+l=k+1, l \geq 2} \pi_z(\text{ad}_{\text{Wick}}(r_{\text{Wick}}(l)) h_{\text{Wick}}^{(k_1)}) \right)$$

$$- \sum_{r=0}^{\infty} \frac{1}{r!} B_r \left( \frac{1}{\nu} \right)^r \sum_{k_1 + \ldots + k_r = k+1+2r, 3 \leq k_1 \leq k-1} \pi_z(\text{ad}_{\text{Wick}}(h_{\text{Wick}}^{(k_1)}) \ldots \text{ad}_{\text{Wick}}(h_{\text{Wick}}^{(k_r)})(r_{\text{Wick}} - r_{\text{Wick}}(l))^0 \right).$$

From the induction hypothesis we have $\delta^{-1}(\nabla z \pi_z h_{\text{Wick}}^{(k-1)}) = 0$. Moreover, we have using equation \[81\] that $\pi_z(\text{ad}_{\text{Wick}}(r_{\text{Wick}}(l)) h_{\text{Wick}}^{(k_1)}) = \pi_z(\pi_z(r_{\text{Wick}}^{(0)}) \circ_{\text{Wick}} h_{\text{Wick}}^{(k_1)} - (\pi_z h_{\text{Wick}}^{(k_1)}) \circ_{\text{Wick}} r_{\text{Wick}}(l))$. Here the first summand vanishes since $*_{\text{Wick}}$ is of Wick type whereas the second one vanishes by induction observing that $3 \leq k_1 \leq k - 1$. Now the contribution $\pi_z(\text{ad}_{\text{Wick}}(h_{\text{Wick}}^{(k_1)}) \ldots \text{ad}_{\text{Wick}}(h_{\text{Wick}}^{(k_r)})(r_{\text{Wick}} - r_{\text{Wick}}(l))^0)$ consists of three types of terms. In case $r = 0$ this contributes $\delta^{-1}\pi_z((r_{\text{Wick}} - r_{\text{Wick}}^{(k-1)}))$ which is 0 since both star products are of Wick type. For $r \neq 0$ the whole expression consists of terms of the shape $\pi_z(h_{\text{Wick}}^{(k_1)} \circ_{\text{Wick}} A)$ and $\pi_z((r_{\text{Wick}} - r_{\text{Wick}}(l)) \circ_{\text{Wick}} B) = \pi_z((\pi_z(r_{\text{Wick}} - r_{\text{Wick}}(l))) \circ_{\text{Wick}} B)$, with certain elements $A \in \mathcal{W} \otimes \Lambda^1$ and $B \in \mathcal{W}$. Again observing the range of summation the terms of the first kind vanish by the induction hypothesis and the others vanish since $\pi_z r_{\text{Wick}} = \pi_z r_{\text{Wick}} = 0$ proving $\pi_z h_{\text{Wick}}^{(k)} = 0$ and hence $\pi_z h_{\text{Wick}} = 0$ by induction. For the proof of $\pi_{\text{Wick}}^{(k)}$ one proceeds analogously. 

Using this result we can show that the star products $*_{\text{Wick}}$ and $*_{\text{Wick}}^{(k)}$ are not only equivalent but even equal.

**Proposition 4.12** Let $*_{\text{Wick}}$ and $*_{\text{Wick}}^{(k)}$ be two star products of Wick type on $(M, \omega, I)$ constructed from the data $(\Omega_{\text{Wick}}, *_{\text{Wick}})$ and $(\Omega_{\text{Wick}}, *_{\text{Wick}}^{(k)})$. Then the equivalence transformation $A_{h_{\text{Wick}}}$ from $*_{\text{Wick}}$ to $*_{\text{Wick}}^{(k)}$ is equal to the identity and hence the star products $*_{\text{Wick}}$ and $*_{\text{Wick}}^{(k)}$ coincide.

**Proof:** We have $A_{h_{\text{Wick}}} f = f + \sigma(\sum_{r=0}^{\infty} \frac{1}{r!} B_r \left( \frac{1}{\nu} \right)^r \text{ad}_{\text{Wick}}(h_{\text{Wick}}(r_{\text{Wick}}(l))))$ for all $f \in C^\infty(M)[[\nu]]$. Now the terms occurring in the sum are of the shape

$$\sigma(h_{\text{Wick}} \circ_{\text{Wick}} A) = \sigma(\pi_z h_{\text{Wick}} \circ_{\text{Wick}} A) \quad \text{and} \quad \sigma(B \circ_{\text{Wick}} h_{\text{Wick}}) = \sigma(B \circ_{\text{Wick}} (\pi_z h_{\text{Wick}}))$$

with certain elements $A, B \in \mathcal{W}$. From the preceding lemma we have that both types of terms vanish and hence $A_{h_{\text{Wick}}} = \text{id}$. \[\square\]
Corollary 4.13 The two star products of anti-Wick type \( \ast_{\text{Wick}} \) and \( \ast_{\text{Wick}}' \) on \((M, \omega, I)\) constructed from the data \((\Omega_{\text{Wick}}, s_{\text{Wick}})\) and \((\Omega_{\text{Wick}}', s_{\text{Wick}}')\) coincide.

Proof: Considering the star products \( \ast_{\text{Wick}} \) and \( \ast_{\text{Wick}}' \) of Wick type obtained from \((\Omega_{\text{Wick}} = P\Omega_{\text{Wick}}, s_{\text{Wick}} = P s_{\text{Wick}})\) and \((\Omega_{\text{Wick}} = P\Omega_{\text{Wick}}, s_{\text{Wick}}' = P s_{\text{Wick}}')\) we have from Proposition 3.22 that \( \ast_{\text{Wick}} = \ast_{\text{Wick}}' \). But then the relation \( (23) \) implies that \( \ast_{\text{Wick}} = \ast_{\text{Wick}}' \).

One should observe that the star products \( \ast_{\text{Wick}} \) resp. \( \ast_{\text{Wick}}' \) that are not of Wick resp. anti-Wick type actually do depend on \( s_{\text{Wick}} \) resp. \( s_{\text{Wick}}' \). But in view of the above proposition and the corollary we can always choose the simplest normalization condition \( s_{\text{Wick}} = 0 \) resp. \( s_{\text{Wick}}' = 0 \) when considering star products \( \ast_{\text{Wick}} \) resp. \( \ast_{\text{Wick}}' \) of Wick resp. anti-Wick type. Hence the Fedosov construction induces mappings

\[
\nu Z_{\ast_{\text{Wick}}}^2(M, C)^1,1([\nu]) \ni \Omega_{\text{Wick}} \mapsto \ast_{\text{Wick}} \in \{ \text{star products of Wick type on } (M, \omega, I) \} \tag{45}
\]

\[
\nu Z_{\ast_{\text{Wick}}'}^2(M, C)^1,1([\nu]) \ni \Omega_{\text{Wick}} \mapsto \ast_{\text{Wick}}' \in \{ \text{star products of anti-Wick type on } (M, \omega, I) \} \tag{46}
\]

where \( \nu Z_{\ast_{\text{Wick}}}^2(M, C)^1,1([\nu]) = \{ \Omega \in \nu \Gamma^\infty((\bigwedge^2 T^*M)[[\nu]]) | d\Omega = 0, \pi^1,1\Omega = \Omega \} \) shall turn out to be bijections. In order to prove this we shall translate some of the results of Karabegov obtained in \[15\] to the situation of star products of Wick type in the next section.

5 Identification of Star Products of Wick Type using Karabegov’s Method

In this section we shall briefly report some general facts on star products of Wick type that shall enable us to identify the Fedosov star products of Wick type. Moreover, we give a direct elementary proof (independent of a construction of such star products) of the fact that star products of Wick type can be uniquely characterized by some local equations.

Let \( \ast_{\text{Wick}} \) be a star product of Wick type on \((M, \omega, I)\) and let \((z, V)\) denote a local holomorphic chart of \(M\) such that \( V \subseteq M \) is a contractible open subset of \(M\). It has been shown in \[15\] Prop. 1 that there are locally defined formal functions \( u_k \in C^\infty(V)[[\nu]] \) for \(1 \leq k \leq \dim C(M)\) such that

\[
u Z_{\ast_{\text{Wick}}}^2(M, C)^1,1([\nu]) \ni \Omega_{\text{Wick}} \mapsto \ast_{\text{Wick}} \in \{ \text{star products of Wick type on } (M, \omega, I) \}
\]

\[
u Z_{\ast_{\text{Wick}}'}^2(M, C)^1,1([\nu]) \ni \Omega_{\text{Wick}} \mapsto \ast_{\text{Wick}}' \in \{ \text{star products of anti-Wick type on } (M, \omega, I) \}
\]

\[
u Z_{\ast_{\text{Wick}}}^2(M, C)^1,1([\nu]) = \{ \Omega \in \nu \Gamma^\infty((\bigwedge^2 T^*M)[[\nu]]) | d\Omega = 0, \pi^1,1\Omega = \Omega \} \]

\[
\nu Z_{\ast_{\text{Wick}}}^2(M, C)^1,1([\nu]) = \{ \Omega \in \nu \Gamma^\infty((\bigwedge^2 T^*M)[[\nu]]) | d\Omega = 0, \pi^1,1\Omega = \Omega \}
\]

Observe that we are using a different sign convention for the Poisson bracket causing the opposite sign in the above formula compared to the one in \[15\]. Analogously one has that there are locally defined formal functions \( \overline{u}_l \in C^\infty(V)[[\nu]] \) for \(1 \leq l \leq \dim C(M)\) such that

\[
\nu Z_{\ast_{\text{Wick}}}^2(M, C)^1,1([\nu]) = \{ \Omega \in \nu \Gamma^\infty((\bigwedge^2 T^*M)[[\nu]]) | d\Omega = 0, \pi^1,1\Omega = \Omega \}
\]

\[
\nu Z_{\ast_{\text{Wick}}}^2(M, C)^1,1([\nu]) = \{ \Omega \in \nu \Gamma^\infty((\bigwedge^2 T^*M)[[\nu]]) | d\Omega = 0, \pi^1,1\Omega = \Omega \}
\]

\[
\nu Z_{\ast_{\text{Wick}}}^2(M, C)^1,1([\nu]) = \{ \Omega \in \nu \Gamma^\infty((\bigwedge^2 T^*M)[[\nu]]) | d\Omega = 0, \pi^1,1\Omega = \Omega \}
\]

Using the fact that \( \ast_{\text{Wick}} \) is a star product of Wick type one can show (cf. \[15\] Lemma 2) that the local functions \( u_k, \overline{u}_l \in C^\infty(V)[[\nu]] \) satisfy the equations

\[
f \ast_{\text{Wick}} u_k = f u_k + \nu Z_k(f) \quad \text{and} \quad \overline{u}_l \ast_{\text{Wick}} f = \overline{u}_l f + \nu Z_l(f) \tag{49}
\]

for all \( f \in C^\infty(V)[[\nu]] \). Moreover, Karabegov considers locally defined formal series of one-forms \( \alpha, \beta \in \Gamma^\infty(T^*V)[[\nu]] \) given by \( \alpha := -u_k dz^k \) which is of type \((1,0)\) and \( \beta := \overline{u}_l d\overline{z}^l \) which is of type \((0,1)\). As the \( \ast_{\text{Wick}} \)-right-multiplication with \( u_k \) obviously commutes with the \( \ast_{\text{Wick}} \)-left-multiplication with \( \overline{u}_l \) one in addition obtains from \[19\] that \( \overline{\partial} \alpha = \partial \beta \). Moreover, one can show that this procedure yields a formal series of closed two-forms of type \((1,1)\) on \(M\) that does not depend on any of the choices made. In the following the so-defined formal two-form that can
be associated to any star product $\ast_{\text{Wick}}$ of Wick type shall be denoted by $K(\ast_{\text{Wick}})$ and is referred to as Karabegov's characterizing form. It is easy to see from the very definition that $K(\ast_{\text{Wick}}) \in \omega + \nu Z_{\text{dr}}^2(M, \mathbb{C})^{1,1}[[\nu]]$ and hence from the $\partial \overline{\partial}$-Poincaré lemmas one has that there exist formal local functions $\varphi \in C^\infty(V)[[\nu]]$ on every contractible domain $V$ of holomorphic coordinates such that $K(\ast_{\text{Wick}})[V] = \partial \overline{\partial} \varphi$ and $\varphi$ is called a formal local Kähler potential of $K(\ast_{\text{Wick}})$. Writing $\varphi = \varphi_0 + \varphi_+$, where $\varphi_+ \in \nu C^\infty(V)[[\nu]]$ one evidently has that $\varphi_0$ is a local Kähler potential for the pseudo-Kähler form $\omega$. With such a formal local Kähler potential the equations

$$f \ast_{\text{Wick}} Z_k(\varphi) = f Z_k(\varphi) + \nu Z_k(f) \quad\text{and}\quad \overline{Z}_l(\varphi) \ast_{\text{Wick}} f = \overline{Z}_l(\varphi) f + \nu \overline{Z}_l(f)$$

hold for all $f \in C^\infty(V)[[\nu]]$. Altogether one has the following theorem:

**Theorem 5.1** ([15 Thm. 1]) Let $\ast_{\text{Wick}}$ be a star product of Wick type on a pseudo-Kähler manifold $(M, \omega, I)$. Then $K(\ast_{\text{Wick}}) \in \omega + \nu Z_{\text{dr}}^2(M, \mathbb{C})^{1,1}[[\nu]]$ associates a formal series of closed two-forms of type $(1,1)$ on $M$ which is a deformation of the pseudo-Kähler form $\omega$ to this star product. In case $\varphi \in C^\infty(V)[[\nu]]$ is a formal local Kähler potential of $K(\ast_{\text{Wick}})$ we have the equations (50) for all $f \in C^\infty(V)[[\nu]]$.

Conversely in [15 Sect. 4] Karabegov has shown that to each such form $K$ as in the preceding theorem one can assign a star product of Wick type such that the characterizing form of this star product actually coincides with this given $K$. To this end Karabegov has given an explicit construction of such a star product extensively using local considerations. We now want to show that a star product of Wick type in fact is completely determined by its characterizing form which shall turn out to be the key result for the proof that one can construct all star products of Wick type on a pseudo-Kähler manifold using Fedosov's method with a suitably chosen $\Omega_{\text{Wick}}$.

**Theorem 5.2** Let $\ast_{\text{Wick}}$ and $\ast'_{\text{Wick}}$ be two star products of Wick type on a pseudo-Kähler manifold $(M, \omega, I)$ and let $\{U_\alpha\}_{\gamma \in J}$ be a good open cover of $M$, where the $U_\alpha$ are the domains of holomorphic coordinates of $M$. Moreover, let $K \in \omega + \nu Z_{\text{dr}}^2(M, \mathbb{C})^{1,1}[[\nu]]$ and let $\varphi_\gamma \in C^\infty(U_\alpha)[[\nu]]$ be formal local Kähler potentials of $K$, i.e. $K|U_\alpha = \partial \overline{\partial} \varphi_\gamma$.

i.) If we have for all $\gamma \in J$ and respectively all $f \in C^\infty(U_\alpha)[[\nu]]$ that

$$f \ast_{\text{Wick}} Z_k(\varphi_\gamma) = f \ast'_{\text{Wick}} Z_k(\varphi_\gamma)$$

then $\ast_{\text{Wick}}$ and $\ast'_{\text{Wick}}$ coincide.

ii.) If we have for all $\gamma \in J$ and respectively all $f \in C^\infty(U_\alpha)[[\nu]]$ that

$$\overline{Z}_l(\varphi_\gamma) \ast_{\text{Wick}} f = \overline{Z}_l(\varphi_\gamma) \ast'_{\text{Wick}} f$$

then $\ast_{\text{Wick}}$ and $\ast'_{\text{Wick}}$ coincide.

**Proof:** We assume that the bidifferential operators $C_i$ and $C'_i$ describing the star products $\ast_{\text{Wick}}$ and $\ast'_{\text{Wick}}$ coincide for $0 \leq i \leq k-1$, where $k \geq 1$ and want to show that this implies that $C_k = C'_k$ proving by induction that the star products coincide since all star products coincide in zeroth order of $\nu$. From the associativity of both star products at the order $k$ of $\nu$ one gets using the induction hypothesis that $C_k - C'_k$ is a Hochschild cocycle. According to the Hochschild-Kostant-Rosenberg-Theorem we thus have that there is a differential operator $B$ on $C^\infty(M)$ and a two-form $A$ on $M$ such that $(C_k - C'_k)(f, g) = (\delta_{\overline{\nu}} B)(f, g) + A(X_f, X_g)$ for all $f, g \in C^\infty(M)$. Here $X_f$ denotes the Hamiltonian vector field corresponding to the function $f$, $\delta_{\overline{\nu}}$ denotes...
the Hochschild differential and as $C_k$ and $C'_k$ vanish on constants $\delta_kB$ also vanishes on constants. Taking the anti-symmetric part of the obtained equation and using that $\delta_kB$ is symmetric we get

$$A(X_1, X_2) = \frac{1}{2} (C_k(f, g) - C_k(g, f) - C'_k(f, g) + C'_k(g, f)).$$

Now let $z$ denote a holomorphic chart on $\mathcal{U}_z$. Since the bidifferential operator $(f, g) \mapsto A(X_1, X_2)$ is obviously one-differential it is enough to evaluate it on the coordinate functions $z^m$ and $\overline{z}^n$ to determine $A$. Using the fact that $\ast_{\text{Wick}}$ and $\ast_{\text{Wick}}$ are of Wick type we evidently have $C_k(z^l, z^m) = C'_k(z^l, z^m) = 0$ and using $X_z = \frac{\partial g}{\partial z} \frac{\partial g}{\partial z} Z_n$ and $X_{\overline{z}} = \frac{\partial g}{\partial \overline{z}} \frac{\partial g}{\partial \overline{z}} Z_n$ we get

$$0 = A(X_z, X_{\overline{z}}) = 4g^{m\overline{n}} \frac{\partial}{\partial z^m} A(Z_n, Z_{\overline{n}}) \quad \text{and} \quad 0 = A(X_{\overline{z}}, X_z) = 4g^{m\overline{n}} \frac{\partial}{\partial \overline{z}^n} A(Z_n, Z_{\overline{z}}).$$

Repeating the same argument for all $\gamma \in J$ and a corresponding local holomorphic chart this yields with the fact that $g$ is non-degenerate that $A$ is of type $(1,1)$ and in a local chart we write $A = A g^{i\overline{j}} \frac{\partial g}{\partial z^i} \frac{\partial g}{\partial \overline{z}^j}$ such that $A(X_1, X_2) = 4A g^{\gamma\overline{\gamma}} g^{|\gamma|}(Z_l(f)Z_n(g) - Z_l(g)Z_n(f))$. To proceed we also need some special property of the symmetric part $\delta_kB$ of $C_k - C'_k$.

**Sublemma 5.3** Let $\delta_kB$ be a bidifferential operator vanishing on constants that in local holomorphic coordinates $(z, \mathcal{U}_z)$ has the shape

$$\sum_{k, \mathcal{T}} C^k, \mathcal{T} \left( \frac{\partial^{|K|} f}{\partial z^K} \frac{\partial^{|\mathcal{T}|} g}{\partial \mathcal{T}^L} + \frac{\partial^{|K|} g}{\partial z^K} \frac{\partial^{|\mathcal{T}|} f}{\partial \mathcal{T}^L} \right)$$

with local functions $C^k, \mathcal{T} \in C^\infty(\mathcal{U}_z)$. Then in local holomorphic coordinates $\delta_kB$ has the shape

$$\sum_{k, \mathcal{T}} C^k, \mathcal{T} \left( Z_k(f)Z_l(g) + Z_k(g)Z_l(f) \right),$$

where $C^k, \mathcal{T}$ denotes the components of a tensor field $C \in \Gamma^\infty(TM \otimes TM)$ of type $(1,1)$.

**Proof:** From $\delta^0_k = 0$ we get $0 = f(\delta_kB)(g, h) - (\delta_kB)(f, g, h) + (\delta_kB)(f, gh) - (\delta_kB)(f, g)h$ for all $f, g, h \in C^\infty(M)$. Evaluating this equation for $g|_U, h|_U \in \mathcal{O}(U)$, where $U \subseteq M$ is an open subset with $U \cap \mathcal{U}_z \neq \emptyset$ one obtains with the above shape of $\delta_kB$ that

$$0 = \sum_{k, \mathcal{T}} C^k, \mathcal{T} \left( \frac{\partial^{|K|}(gh)}{\partial z^K} \frac{\partial^{|\mathcal{T}|} f}{\partial \mathcal{T}^L} - g \frac{\partial^{|K|} h}{\partial z^K} \frac{\partial^{|\mathcal{T}|} f}{\partial \mathcal{T}^L} - h \frac{\partial^{|K|} g}{\partial z^K} \frac{\partial^{|\mathcal{T}|} f}{\partial \mathcal{T}^L} \right)$$

on $U \cap \mathcal{U}_z$. Here one has to observe that $|K|$ and $|\mathcal{T}|$ are at least one since $\delta_kB$ vanishes on constants.

From this we may conclude that $\sum_{k, \mathcal{T}} C^k, \mathcal{T} \left( \frac{\partial^{|K|} f}{\partial z^K} \frac{\partial^{|\mathcal{T}|} g}{\partial \mathcal{T}^L} \right) = \sum_{k, \mathcal{T}} C^k, \mathcal{T} Z_k(f)Z_l(g)$ repeating the above argument with $f, g, h \in C^\infty(M)$ such that $f|_U, g|_U \in \mathcal{O}(U)$. Altogether the proven local equations prove the statement since it is easy to see that the local functions $C^k, \mathcal{T}$ define a tensor field.

From the fact that both star products are of Wick type it is obvious that $(\delta_kB)(f, g) = \frac{1}{2} ((C_k - C'_k)(f, g) + (C_k - C'_k)(g, f))$ is of the shape as in the sublemma yielding

$$(C_k - C'_k)(f, g) = C^k, \mathcal{T}(Z_k(f)Z_l(g) + Z_k(g)Z_l(f)) + 4Ag^{\gamma\overline{\gamma}} g^{|\gamma|}(Z_l(f)Z_n(g) - Z_l(g)Z_n(f)).$$

Again using that the star products are of Wick type this implies $C^k, \mathcal{T} = -4Ag^{\gamma\overline{\gamma}} g^{|\gamma|}$ and finally

$$(C_k - C'_k)(f, g) = -8Ag^{\gamma\overline{\gamma}} g^{|\gamma|} Z_n(f)Z_l(g).$$

But now we need the conditions \[31\] resp. \[32\] to proceed with the proof. Considering \[31\] in order $k$ of the formal parameter we get from the induction hypothesis that $C_k(f, Z_n(\varphi_0)) - C'_k(f, Z_n(\varphi_0)) = 0$ for
$f \in C^\infty(U)$, where $\varphi_{\gamma 0}$ denotes a Kähler potential for $\omega$ on $U$. Using the expression for $C_k - C'_k$ derived above this yields $0 = -4\partial A_{\gamma j}^{mn} Z_m(f)$ and hence $A_{\gamma j} = 0$. As $\gamma \in J$ was arbitrary we get $A = 0$ proving i.). The final step to prove ii.) is analogous. 

The statement of this theorem means that a star product of Wick type is uniquely determined by one of the equations (50) in case it is valid in every holomorphic chart $(z, V)$. For completeness we give the important relations of this theorem means that a star product of Wick type is uniquely determined by one of these local equations valid in every holomorphic chart $(z, V)$ and hence it is determined by $K$.

**Remark 5.4** All the statements about star products of Wick type made in this section can be easily transferred to star products of anti-Wick type. For completeness we give the important relations but omit the proofs since they are easily done using the relation between star products of Wick type and those of anti-Wick type explained in Section 5. For a star product $\ast_{\text{Wick}}$ one defines the characterizing form $K(\ast_{\text{Wick}}) \in \omega + \nu Z^2_{\text{dist}}(M, \mathbb{C})^{1,1}[[\nu]]$ by $K(\ast_{\text{Wick}}) := P K((\ast_{\text{Wick}})_{P,\text{opp}})$. Denoting by $\varphi \in C^\infty(V)[[\nu]]$ a formal Kähler potential of $K(\ast_{\text{Wick}})$ on $V$ that is obtained from a formal Kähler potential $\varphi$ of $K((\ast_{\text{Wick}})_{P,\text{opp}})$ by $\varphi = P \varphi$ one has

$$Z_k(\varphi) \ast_{\text{Wick}} f = Z_k(\varphi)f - \nu Z_k(f) \quad \text{and} \quad f \ast_{\text{Wick}} Z_l(\varphi) = f Z_l(\varphi) - \nu Z_l(f),$$

(55)

where $(z, V)$ is a holomorphic chart and $f \in C^\infty(V)[[\nu]]$. As in Theorem 7, the form $K(\ast_{\text{Wick}})$ uniquely determines the star product of anti-Wick type by one of these local equations valid in every holomorphic chart $(z, V)$ of $M$.

6 **Universality of Fedosov’s Construction for Star Products of Wick Type**

In this section we want to compute Karabegov’s characterizing form $K(\ast_{\text{Wick}})$ of the Fedosov star products of Wick type constructed in this paper. Furthermore, our result shall enable us to prove that the mapping defined in equation (55) is a bijection. To determine $K(\ast_{\text{Wick}})$ we need according to Section 5 local formal functions $u_k \in C^\infty(V)[[\nu]]$ with

$$u_k \ast_{\text{Wick}} z^j - z^j \ast_{\text{Wick}} u_k = -\nu \delta^j_k,$$

where $(z, V)$ is a local holomorphic chart of $M$. Denoting by $\varphi_0$ a local Kähler potential of $\omega$ on $V$ we obviously have

$$-\delta^j_k = \{Z_k(\varphi_0), z^j\} = -\sigma(\mathcal{L}_{Z_k} \tau_{\text{Wick}}(z^j)),$$

since the Lie derivative commutes with $\sigma$. Defining $u_{k 0} := Z_k(\varphi_0)$ we thus have to show the existence of local formal functions $u_{k+} \in \nu C^\infty(V)[[\nu]]$ such that

$$-\sigma(\mathcal{L}_{Z_k} \tau_{\text{Wick}}(z^j)) = \frac{1}{\nu} \sigma(\text{ad}_{\text{Wick}}(\tau_{\text{Wick}}(u_{k 0} + u_{k+})) \tau_{\text{Wick}}(z^j)).$$

For the further computations we need the following formula:

**Lemma 6.1** For all vector fields $X \in \Gamma^\infty(TM)$ the Lie derivative $\mathcal{L}_X : \mathcal{W} \otimes \Lambda \to \mathcal{W} \otimes \Lambda$ can be expressed in the following way:

$$\mathcal{L}_X = \mathcal{D}_{\text{Wick}} i_a(X) + i_a(X) \mathcal{D}_{\text{Wick}} + \frac{1}{\nu} \text{ad}_{\text{Wick}}(i_a(X) \tau_{\text{Wick}}) + i_a(X) + (dz^m \otimes 1)i_s(\nabla_{Z_m} X) + (dz^m \otimes 1)i_s(\nabla_{Z_{m'}} X).$$

(56)
Proof: The proof of this formula is the same as for the analogous formula using $\omega_F$ and $\mathcal{D}_F$ instead of $\omega_{\text{Wick}}$ and $\mathcal{D}_{\text{Wick}}$ and can be found in \cite{Koszul1950} Appx. A.

Using the preceding lemma for $X = Z_k$ we get

$$-\sigma(L_{Z_k} \tau_{\text{Wick}}(z^l)) = -\sigma(\frac{1}{\nu} \text{ad}_{\text{Wick}}(i_a(Z_k)r_{\text{Wick}})\tau_{\text{Wick}}(z^l) + i_a(Z_k)\tau_{\text{Wick}}(z^l)),$$

where we have used that $\nabla^m Z_k = 0$ and the obvious equations $\mathcal{D}_{\text{Wick}} \tau_{\text{Wick}}(z^l) = 0 = i_a(Z_k)\tau_{\text{Wick}}(z^l)$ and $\sigma((dz^n \otimes 1)i_a(\nabla Z_n Z_k)\tau_{\text{Wick}}(z^l)) = 0$. Defining $a_k := i Z_k \omega \otimes 1 \in \mathcal{W}(V)$ one easily finds $i_a(Z_k) = -\frac{1}{\nu} \text{ad}_{\text{Wick}}(a_k)$ and we get

$$-\sigma(L_{Z_k} \tau_{\text{Wick}}(z^l)) = -\frac{1}{\nu} \sigma((\pi_z(i_a(Z_k)r_{\text{Wick}} - a_k)) \text{ad}_{\text{Wick}} (\pi_z \tau_{\text{Wick}}(z^l))) + \frac{1}{\nu} \sigma((\pi_z \tau_{\text{Wick}}(z^l)) \text{ad}_{\text{Wick}} (\pi_z(i_a(Z_k)r_{\text{Wick}} - a_k))),$$

where we have used $\sigma = \pi_z \pi_{\text{Wick}}$ and $\Sigma$. Obviously we have $\pi_z a_k = 0$ and $\pi_z a_k = a_k$ since $\omega$ is of type $(1,1)$ and $\pi_z i_a(Z_k) r_{\text{Wick}} = i_a(Z_k) \pi_z r_{\text{Wick}} = 0$ from Theorem 4.9 such that we finally obtain

$$-\nu \delta_k = \sigma((\pi_z \tau_{\text{Wick}}(z^l)) \text{ad}_{\text{Wick}} (\pi_z i_a(Z_k)r_{\text{Wick}} - a_k)).$$

From this equation it is obvious that $\pi_z i_a(Z_k)r_{\text{Wick}} - a_k$ has to be strongly related to the Fedosov-Taylor series of the function $u_k$ that is to be found and we have:

Lemma 6.2 Let $\varphi_0$ be a local Kähler potential of $\omega$ on the open contractible domain $V$ of a holomorphic chart $z$ of $M$ and let $u_{k0} \in \mathcal{C}^\infty(V)$ be defined by $u_{k0} := Z_k(\varphi_0)$. In case there exist local formal functions $u_{k+} \in \nu \mathcal{C}^\infty(V)[[\nu]]$ such that we have

$$\pi_z \tau_{\text{Wick}}(u_k) = u_k + i Z_k \omega \otimes 1 - \pi_z i_a(Z_k)r_{\text{Wick}}$$

for $u_k = u_{k0} + u_{k+}$ then $u_k *_{\text{Wick}} z^l - z^l *_{\text{Wick}} u_k = -\nu \delta_k$.

Proof: For the proof we compute using equation (58)

$$-\nu \delta_k = \sigma((\pi_z \tau_{\text{Wick}}(z^l)) \text{ad}_{\text{Wick}} (u_k - \pi_z \tau_{\text{Wick}}(u_k))) = u_k z^l - z^l *_{\text{Wick}} u_k = u_k *_{\text{Wick}} z^l - z^l *_{\text{Wick}} u_k,$$

where the last equality follows from the fact that $*_{\text{Wick}}$ is of Wick type.

So it remains to show that there exist such local functions as in (58) and to get more information about their concrete shape.

Proposition 6.3 With the notation of Lemma 6.2 one has the following statements:

i.) For all $u_{k+} \in \nu \mathcal{C}^\infty(V)[[\nu]]$ one has $\sigma(u_k + i Z_k \omega \otimes 1 - \pi_z i_a(Z_k)r_{\text{Wick}}) = u_k$ and

$$-\delta A_k + \nabla_z A_k - \frac{1}{\nu} \pi_z \tau_{\text{Wick}}(u_k) = 1 \otimes (\overline{\mathcal{D}} u_{k+} - i Z_k \Omega_{\text{Wick}}),$$

where $A_k := u_k + i Z_k \omega \otimes 1 - \pi_z i_a(Z_k)r_{\text{Wick}} \in \mathcal{W}(V)$.

ii.) One has $\pi_z \tau_{\text{Wick}}(u_k) = u_k + i Z_k \omega \otimes 1 - \pi_z i_a(Z_k)r_{\text{Wick}}$ if and only if $\overline{\mathcal{D}} u_{k+} - i Z_k \Omega_{\text{Wick}} = 0$. In case $u_{k+} = Z_k(\varphi_+)$, where $\varphi_+ \in \nu \mathcal{C}^\infty(V)[[\nu]]$ is a local formal Kähler potential for $\Omega_{\text{Wick}}$, i.e. $\partial \overline{\partial} \varphi_+ = \Omega_{\text{Wick}}|_V$ one has $\pi_z \tau_{\text{Wick}}(u_k) = u_k + i Z_k \omega \otimes 1 - \pi_z i_a(Z_k)r_{\text{Wick}}$. 

21
iii.) Karabegov’s characterizing form $K(*_{\text{Wick}})$ of the star product $*_{\text{Wick}}$ of Wick type is given by

$$K(*_{\text{Wick}}) = \omega + \Omega_{\text{Wick}}.$$  \hfill (60)

iv.) For all $f \in \mathcal{C}^\infty(V)[[\nu]]$ and $u_k$ as in ii.) one has

$$f *_{\text{Wick}} u_k = f u_k + \nu Z_k(f).$$  \hfill (61)

**Proof:** Obviously we have $\sigma(u_k + iz_\omega \otimes 1 - \pi_{\text{Wick}}(Z_k) r_{\text{Wick}}) = u_k$ for all $u_{k+} \in \mathcal{C}^\infty(V)[[\nu]]$ since $\sigma = \pi_{\text{Wick}}$ and $\pi_z(i_a(Z_k)r_{\text{Wick}}) = 0$. Further we compute for the proof of i.)

$$\delta_z(u_k + iz_\omega \otimes 1 - \pi_{\text{Wick}}(Z_k) r_{\text{Wick}}) = 1 \otimes iz_\omega - \pi_z \delta_z(Z_k) r_{\text{Wick}} = 1 \otimes iz_\omega - \pi_z(i_a(Z_k) r_{\text{Wick}} - i_a(Z_k) \delta r_{\text{Wick}}),$$

where we have used $\delta_z(Z_k) + i_a(Z_k) \delta = i_a(Z_k)$ and $i_a(Z_k)$ and $\pi_{\text{Wick}} r_{\text{Wick}} = 0$ and

$$\pi_z(i_a(Z_k))(r_{\text{Wick}} \omega \circ \text{Wick} r_{\text{Wick}})$$

the intermediate result

$$\pi_z((i_a(Z_k) r_{\text{Wick}})) - \pi_z(r_{\text{Wick}} \circ \text{Wick} (i_a(Z_k) r_{\text{Wick}})) = -\pi_z(r_{\text{Wick}} \circ \text{Wick} (i_a(Z_k) r_{\text{Wick}}))$$

since $\Omega_{\text{Wick}}$ is of type $(1,1)$. Further we have from $\nabla_{\text{Wick}} Z_k = 0$ and $\nabla_{\text{Wick}} i_\omega = 0$ that $\nabla_{\text{Wick}} i_z \omega$ vanishes also, such that we obtain using $\nabla_i a(Z_k) + i_a(Z_k) \nabla = \mathcal{L}_{Z_k}^\text{Wick} - (dz^n \otimes 1)i_z(\nabla_{\text{Wick}} Z_k)$ and $\nabla_{\text{Wick}} Z_k = 0$

$$\nabla_{\text{Wick}} (u_k + iz_\omega \otimes 1 - \pi_{\text{Wick}}(Z_k) r_{\text{Wick}})$$

$$= 1 \otimes \nabla_{\text{Wick}} (u_k + iz_\omega \otimes 1 - \pi_{\text{Wick}}(Z_k) r_{\text{Wick}}) = 1 \otimes \nabla_{\text{Wick}} (u_k + i_z \omega - \pi_{\text{Wick}}(Z_k) r_{\text{Wick}})$$

where besides $\delta_z u_k$ and $\pi_{\text{Wick}} a(Z_k) r_{\text{Wick}}$ we have used the trivial but helpful identity $0 = \pi_z((i_z \omega \otimes 1) \circ \text{Wick} r_{\text{Wick}})$ and $i_a(Z_k) = -\frac{1}{\nu} \text{ad}_{\text{Wick}}(i_z \omega \otimes 1)$ Combining our three intermediate results one finds $u_{k+} = Z_k(\varphi_0)$ and hence $\delta_z u_k = i_z \omega$ proving i.) According to Proposition 3.2 ii.) we thus have that $A_k = \pi_{\text{Wick}}(u_k)$ and if only if $\delta_z u_k = i_z \omega$ proving i.) According to Lemma 4.2 we then have $u_k *_{\text{Wick}} z^l - z^l *_{\text{Wick}} u_k = -\nu \delta_t^l$ and we get using the definition of the characterizing form $K(*_{\text{Wick}})$ with $u_k = Z_k(\varphi) = Z_k(\varphi_0 + \varphi_0)$ that

$$K(*_{\text{Wick}}) = -\delta(u_k dz^k) = -\delta(u_k) \wedge dz^k = dz^k \wedge i_z \omega - \Omega_{\text{Wick}} = \omega + \Omega_{\text{Wick}}$$

proving iii.) For the proof of iv.) we compute for $f \in \mathcal{C}^\infty(V)[[\nu]]$ using $\pi_{\text{Wick}} r_{\text{Wick}}(u_k) = u_k + iz_\omega \otimes 1 - \pi_{\text{Wick}}(Z_k) r_{\text{Wick}}$, $\mathcal{O}_{\text{Wick}} r_{\text{Wick}}(f) = 0$ and (51)

$$f *_{\text{Wick}} u_k = \sigma(r_{\text{Wick}}(f)) \circ \text{Wick} (u_k + iz_\omega \otimes 1 - i_a(Z_k) r_{\text{Wick}})$$

$$= f u_k + \frac{2\nu}{\nu} \pi_{\text{Wick}} (i_a(Z_k) r_{\text{Wick}})(u_k) + \sigma(i_a(Z_k) \text{ad}_{\text{Wick}}(r_{\text{Wick}}) r_{\text{Wick}}(f))$$

$$- \sigma((i_a(Z_k) r_{\text{Wick}})) \circ \text{Wick} r_{\text{Wick}}(f)$$

$$= f u_k + \nu \sigma(i_a(Z_k) r_{\text{Wick}}(f)) + \nu \sigma(i_a(Z_k)(\nabla - \delta)) r_{\text{Wick}}(f) - \sigma((i_a(Z_k) r_{\text{Wick}})) \circ \text{Wick} r_{\text{Wick}}(f).$$
Further we obtain by reason of $\pi_z i_a (Z_k) r_{\text{Wick}} = i_a (Z_k) \pi_z r_{\text{Wick}} = 0$ and $i_a (Z_k) (\nabla - \delta) r_{\text{Wick}} (f) = (\nabla z_k - i_s (Z_k)) r_{\text{Wick}} (f)$ that
\[
f *_{\text{Wick}} u_k = fu_k + \nu \sigma (\nabla z_k r_{\text{Wick}} (f)) = fu_k + \nu Z_k (f),
\]
and the proposition is proven. \(\Box\)

Slightly modifying the proofs of Lemma 6.2 and Proposition 6.3 one can also show the existence of locally defined formal functions $\tau_l \in C^\infty (V) [[\nu]]$ such that $\tau_l *_{\text{Wick}} \tau_l - \tau_l *_{\text{Wick}} \tau_l = \nu \delta_{\tau_l}$. For completeness we just give the results and omit the proof since by Theorem 5.2 it is enough to know the properties of one of the functions $u_k$ resp. $\tau_l$ to be able to identify the star product $*_{\text{Wick}}$. Moreover, the proofs given above can easily be transferred to the following statements.

**Lemma 6.4** Let $\varphi_0$ be a local Kähler potential of $\omega$ on the open contractible domain $V$ of a holomorphic chart $z$ of $M$ and let $\tau_0 \in C^\infty (V)$ be defined by $\tau_0 := Z_l (\varphi_0)$. In case there exist local formal functions $\tau_l+ \in \nu C^\infty (V) [[\nu]]$ such that we have
\[
\pi_z r_{\text{Wick}} (\tau_l) = \tau_l - i Z_l \omega \otimes 1 + \pi_z i_a (Z_l) r_{\text{Wick}} (\tau_l)\] (62)
for $\tau_l = \tau_0 + \nu \tau_l$ then $\tau_l *_{\text{Wick}} \tau_l - \tau_l *_{\text{Wick}} \tau_l = \nu \delta_{\tau_l}$.

**Proposition 6.5** With the notation of Lemma 6.4 one has the following statements:

i.) For all $\tau_l+ \in \nu C^\infty (V) [[\nu]]$ one has $\sigma (\tau_l - i \nu Z (\tau_l) r_{\text{Wick}} (\tau_l)) = \tau_l$ and
\[
- \delta \overline{\tau_l} \nu + \overline{\nu} \nu Z_l + 1 + \pi_z i_a (Z_l) r_{\text{Wick}} (\tau_l) = 1 \otimes (\nu \tau_l + i \nu Z_l) r_{\text{Wick}} (\tau_l),\] (63)
where $\overline{\tau_l} := \tau_l - i \nu Z_l \omega \otimes 1 + \pi_z i_a (Z_l) r_{\text{Wick}} (\tau_l) \in \mathcal{W} (V)$.

ii.) One has $\pi_z r_{\text{Wick}} (\tau_l) = \tau_l - i \nu Z_l \omega \otimes 1 + \pi_z i_a (Z_l) r_{\text{Wick}} (\tau_l)$ if and only if $\delta \overline{\tau_l} + i \nu Z_l = 0$. In case $\tau_l+ := Z_l (\varphi_0)$, where $\varphi_0 \in C^\infty (V) [[\nu]]$ is a local formal Kähler potential for $\Omega_{\text{Wick}}$, i.e. $\partial \overline{\nu} \varphi_0 = \Omega_{\text{Wick}} | V$ one has $\pi_z r_{\text{Wick}} (\tau_l) = \tau_l - i \nu Z_l \omega \otimes 1 + \pi_z i_a (Z_l) r_{\text{Wick}} (\tau_l)$.

iii.) Here also Karabegov’s characterizing form $K (\star_{\text{Wick}})$ of the star product $\star_{\text{Wick}}$ of Wick type is given by
\[
K (\star_{\text{Wick}}) = \omega + \Omega_{\text{Wick}}.\] (64)

iv.) For all $f \in C^\infty (V) [[\nu]]$ and $\tau_l$ as in ii.) one has
\[
\tau_l *_{\text{Wick}} f = \tau_l f + \nu Z_l (f).\] (65)

The assertions iii.) and iv.) of Propositions 6.3 and 6.5 represent the generalizations of the theorem proven by Karabegov in [17, Sect. 4] for the special case $\Omega_{\text{Wick}} = 0$. In the following corollary we just collect the more or less trivial analogous statements for the star products $\star_{\text{Wick}}$ of anti-Wick type.

**Corollary 6.6** Let $\star_{\text{Wick}}$ be the Fedosov star product of anti-Wick type constructed from $\Omega_{\text{Wick}} \in \nu Z_{\text{dR}} (M, \mathbb{C})^{1,1} [[\nu]]$ then its characterizing form is given by
\[
K (\star_{\text{Wick}}) = \omega + \Omega_{\text{Wick}}.\] (66)
Denoting by $\varphi$ a local formal Kähler potential of $\overline{K (\star_{\text{Wick}})}$, i.e. $\partial \overline{\nu} \varphi | V = \overline{K (\star_{\text{Wick}})}$ and defining $\varpi_l := Z_l (\varphi), v_k := Z_k (\varphi) \in C^\infty (V) [[\nu]]$ we have
\[
v_k *_{\text{Wick}} f = v_k f - \nu Z_k (f)\] and \[
f *_{\text{Wick}} \varpi_l = f \varpi_l - \nu Z_l (f),\] (67)
where $(z, V)$ is a holomorphic chart and $f \in C^\infty (V) [[\nu]]$. 

23
Theorem 6.7 (Universality of Fedosov’s construction for star products of Wick type)
For all star products \(*_{Wick}\) of Wick type on a pseudo-Kähler manifold \((M, \omega, I)\) there is a Fedosov construction with \(\circ_{Wick}\) such that the obtained star product \(*_{Wick}\) coincides with \(*_{Wick}\). Moreover, the mapping
\[
\nu Z^{2}(M, \mathbb{C})^{1,1}[[\nu]] \ni \Omega_{Wick} \mapsto *_{Wick} \in \{ \text{star products of Wick type on } (M, \omega, I) \}
\] (68)
induced by the Fedosov construction is a bijection and its inverse is given by \(*_{Wick} \mapsto K(*_{Wick}) - \omega\), where \(K(*_{Wick})\) denotes Karabegov’s characterizing form of \(*_{Wick}\).

Proof: The universality of the Fedosov construction just means that the mapping in equation (68) is surjective and hence we have to show that for any given star product \(*_{Wick}\) of Wick type there is a \(\Omega_{Wick}\) such that \(*_{Wick} = *_{Wick}\). Given \(*_{Wick}\) we consider \(K(*_{Wick})\) according to Theorem 5.1 and have that the equations (54) are fulfilled for a local formal Kähler potential \(\varphi\) of \(K(*_{Wick})\) on \(V \subseteq M\) and \(f \in \mathcal{C}^{\infty}(V)[[\nu]]\).

Using the Fedosov construction for \(\Omega_{Wick} = K(*_{Wick}) - \omega\) we have from Propositions 6.3 and 6.5 that \(f *_{Wick} Z_{k}(\varphi) = f Z_{k}(\varphi) + \nu Z_{k}(f)\) and \(Z_{l}(\varphi) *_{Wick} f = Z_{l}(\varphi) f + \nu Z_{l}(f)\). With Theorem 5.2 each of these equations implies \(*_{Wick} = *_{Wick}\) proving surjectivity. From \(K(*_{Wick}) = \omega + \Omega_{Wick}\) the fact that the mapping in (68) is injective and the shape of its inverse are obvious.

The preceding theorem can immediately be transferred to the star products of anti-Wick type that can all be obtained from a Fedosov construction using \(\circ_{Wick}\) and the mapping according to equation (68) also turns out to be a bijection.

A quite remarkable consequence of this theorem is:

Deduction 6.8 All star products \(*_{Wick}\) (\(*_{Wick}\)) of (anti-)Wick type on a pseudo-Kähler manifold \((M, \omega, I)\) are of Vey type.

Proof: By the above theorem all star products of (anti-)Wick type can be obtained by some Fedosov construction using \(\circ_{Wick}\) (\(\circ_{Wick}\)). But from Proposition 6.8 we have that all star products \(*_{Wick}\) (\(*_{Wick}\)) even those that are not of (anti-)Wick type are of Vey type.

Deduction 6.9 ([16, Thm. 3]) The characteristic classes \(c(*_{Wick})\) resp. \(c(*_{Wick})\) of star products of Wick resp. anti-Wick type on a pseudo-Kähler manifold \((M, \omega, I)\) are related to the characterizing forms \(K(*_{Wick})\) resp. \(K(*_{Wick})\) by \(c(*_{Wick}) = \frac{K(*_{Wick})}{\rho} - \frac{\rho}{4}\) resp. \(c(*_{Wick}) = \frac{K(*_{Wick})}{\rho} + \frac{\rho}{4}\), where \(\rho\) denotes the Ricci form.

Proof: With the results of Propositions 6.3 and Corollaries 6.1 and Proposition 3.1 i.) the statements are obviously true for the star products \(*_{Wick}\) and \(*_{Wick}\) of Wick resp. anti-Wick type. But as the Fedosov constructions are universal for these types of star products they in fact hold in general.

Another application of our results is the unique characterization of the Hermitian star products of Wick and of anti-Wick type that is those star products that have the complex conjugation denoted by \(C\) incorporated as an anti-automorphism (cf. [21 Sect. 5]).

Proposition 6.10 A star product \(*_{Wick}\) (\(*_{Wick}\)) of (anti-)Wick type on a pseudo-Kähler manifold \((M, \omega, I)\) is Hermitian if and only if the characterizing form \(K(*_{Wick})\) \(\frac{K(*_{Wick})}{\rho}\) is real, i.e. \(CK(*_{Wick}) = K(*_{Wick})\) \(\frac{CK(*_{Wick})}{\rho}\) = \(\frac{K(*_{Wick})}{\rho}\).
Proof: Let $\ast_{\text{Wick}}$ be a Hermitian star product of Wick type and let $u_k \in \mathcal{C}^\infty(V)[[\nu]]$ such that $f \ast_{\text{Wick}} u_k = f u_k + \nu Z_k(f)$ for all $f \in \mathcal{C}^\infty(V)[[\nu]]$. Using $C\nu = -\nu$ since the formal parameter is assumed to be purely imaginary this implies that $C(f)C(u_k) - \nu Z_k(Cf) = C(f \ast_{\text{Wick}} u_k) = (Cu_k) \ast_{\text{Wick}} (Cf)$ and hence we have $\overline{\nu}_k \ast_{\text{Wick}} g = \overline{\nu}_k g + \nu Z_k(g)$ for all $g \in \mathcal{C}^\infty(V)[[\nu]]$, where $\overline{\nu}_k := -Cu_k$. Using the definition of the characterizing form and $C^2 = i\theta$ this yields

$$K(\ast_{\text{Wick}}) = \partial(\overline{\nu}_k d\overline{\nu}) = C\partial C(-u_k dz^k) = C\overline{C}(-u_k dz^k) = CK(\ast_{\text{Wick}}).$$

Now let $\ast_{\text{Wick}}$ be a star product of Wick type such that $CK(\ast_{\text{Wick}}) = K(\ast_{\text{Wick}})$ and consider a star product $\ast_{\text{Wick}} = \ast_{\text{Wick}}$ which exists according to Theorem 6.7. Using $K(\ast_{\text{Wick}}) = \omega + \Omega_{\text{Wick}}$ we obviously have $C\Omega_{\text{Wick}} = \Omega_{\text{Wick}}$. We now want to show that this implies that $\ast_{\text{Wick}}$ is Hermitian. To this end we first observe that the fibrewise product $\circ_{\text{Wick}}$ satisfies $C(a \circ_{\text{Wick}} b) = (-1)^{l} (Cb \circ_{\text{Wick}} (Ca))$ for $a \in \mathcal{W} \otimes \Lambda^k, b \in \mathcal{W} \otimes \Lambda^l$. Using this it is straightforward to see that $\tau_{\text{Wick}}$ satisfies the same equations as $\tau_{\text{Wick}}$ does yielding $\tau_{\text{Wick}} = \tau_{\text{Wick}}$. But then we have $\mathcal{D}_{\text{Wick}} C = C\mathcal{D}_{\text{Wick}}$ and hence $\tau_{\text{Wick}}(f) = \tau_{\text{Wick}}(Cf)$ for all $f \in \mathcal{C}^\infty(M)[[\nu]]$ implying by an easy computation that $\ast_{\text{Wick}}$ and therefore $\ast_{\text{Wick}}$ is Hermitian. The proof of the analogous statements for the star products of anti-Wick type is completely analogous.

To conclude this section we want to establish a relation to the star products with separation of variables considered by Karabegov. To this end we have to drop our convention to use the purely imaginary formal parameter $\nu$ and have to replace it by $i\lambda$ and consider the star product defined by

$$f \ast_{\text{K}} g := gf + \sum_{l=1}^{\infty} (i\lambda)^l C_l(g, f)$$

(69)

for $f, g \in \mathcal{C}^\infty(M)$, where the bidifferential operators $C_l$ describe the star product $\ast_{\text{Wick}}$ of Wick type by $f \ast_{\text{Wick}} g = fg + \sum_{l=1}^{\infty} \nu^l C_l(f, g)$. Obviously $\ast_{\text{K}}$ is a star product with separation of variables on $(M, -\omega, I)$ and using Karabegov’s original definition of the characterizing form (cf. [15 Sect. 3]) it is an easy task to compute $K(\ast_{\text{K}}) = \omega + \Omega_{\text{Wick}}(i\lambda)$ using Proposition 6.3. Moreover, it is easy to conclude from Proposition 4.6 and the properties of the characteristic class (cf. [21 Lemma 5.2 i.)] and [12 Thm. 6.4]) that $c(\ast_{\text{K}}) = \frac{1}{i\lambda}[K(\ast_{\text{K}})] - i[\rho]$. It stands to reason that one can also give a Fedosov construction that directly yields the star product $\ast_{\text{K}}$. We just give a sketch of the necessary modifications: Consider $(\mathcal{W} \otimes \Lambda_{\lambda}, \circ_{\text{K}})$, where $\mathcal{W} \otimes \Lambda_{\lambda}$ is the same object as $\mathcal{W} \otimes \Lambda$, where $\nu$ has been replaced by $\lambda$ and $\circ_{\text{K}}$ is defined by

$$a \circ_{\text{K}} b := \mu \circ \exp \left(2\lambda g^{i\lambda} i_{\lambda}(Z_l) \otimes i_{\lambda}(Z_k)\right) (a \otimes b).$$

(70)

For this fibrewise product one obtains a Fedosov derivation $\mathcal{D}_{\text{K}} := -\delta + \nabla - \frac{1}{i\lambda} \text{ad}_{r_K}$ with $\mathcal{D}^2_{\text{K}} = 0$, where $r_K$ is the solution of the equations

$$\delta r_K = \nabla r_K - \frac{1}{i\lambda} r_K \circ_{\text{K}} r_K - R - 1 \otimes \Omega_{\text{Wick}}(i\lambda) \quad \text{and} \quad \delta^{-1} r_K = 0.$$

(71)

It is easy to see that $r_K = -r_{\text{Wick}}(i\lambda)$ and that $\tau_{\text{K}}(f) = \tau_{\text{Wick}}(f)(i\lambda)$ for all $f \in \mathcal{C}^\infty(M)$, but this actually implies that the obtained star product coincides with the one defined in equation (69).

Outlook and Further Questions

Let us conclude with a few remarks on this approach to star products of Wick type. First we would like to point out that the Fedosov construction has the advantage that one is only dealing with global geometric objects and is not forced to use expressions of bidifferential operators in local charts. Moreover, the Fedosov setting allows for very detailed investigations of algebraic properties
of the obtained star products just using their construction and there is no need to have explicit closed formulas for the star products. So it might be useful to consider the following questions and topics in this framework:

i.) It should be possible to study representations of the star products of Wick type and of anti-Wick type within Fedosov’s framework. Moreover, the question of Morita equivalence of these star products can be discussed in this setting by explicitly constructing equivalence bimodules which is subject of the work [23].

ii.) With a more detailed understanding of the representation theory of the considered star products it should be possible to establish a relation to geometric quantization and to Berezin-Toeplitz quantization starting from the formal framework (cf. [18]).

iii.) For a star product *Wick of Wick type one can define the formal Berezin transform B as the unique formal series of differential operators on C∞(M) such that for any open subset U ⊆ M and f, g ∈ C∞(M) with f|U ∈ O(U), g|U ∈ \( {\overline{O}}(U) \) the relation B(fg) = f *Wick g holds. Then the star product *Wick defined by f *Wick g := B⁻¹((Bf) *Wick (Bg)) turns out to be a star product of anti-Wick type and it would be interesting to find a relation between K(*Wick) and \( \prod \)(*Wick) in this case and to obtain a more explicit description of B.

iv.) Finally it should be possible to consider different ways of obtaining star products (possibly of Wick resp. anti-Wick type) by phase space reduction (for instance BRST quantization as in [3], or the analogue of the procedure in [11]) in case there is a group acting on \((M, \omega, I)\) not only preserving the symplectic form but also the complex structure. Again this will be discussed in a forthcoming project.

A Consequences of the Complex Structure

Using the complex structure on \((M, \omega, I)\) one can define several splittings of the mappings which are involved into the Fedosov construction which were of great advantage in Sections 4 and 5. First it is obvious from \( I^2 = -\text{id} \) that the bundle isomorphism I of the complexified tangent bundle TM has the fibrewise eigenvalues \( \pm i \) and that \( TM = TM^{1,0} \oplus TM^{0,1} \), where \( TM^{1,0} \) denotes the bundle of the +i-eigenspaces and \( TM^{0,1} \) denotes the bundle of \( -i \)-eigenspaces. Moreover, all sections \( X \in \Gamma^\infty(TM) \) can be uniquely decomposed into a section of type \((1, 0)\) and a section of type \((0, 1)\). Considering the dual bundles \( T^*M^{1,0} \) and \( T^*M^{0,1} \) to \( TM^{1,0} \) and \( TM^{0,1} \) it is obvious that the cotangent bundle can also be written as \( T^*M = T^*M^{1,0} \oplus T^*M^{0,1} \). This decomposition naturally gives rise to analogous decompositions of sections \( T \in \Gamma^\infty(V^s T^*M) \) and \( \beta \in \Gamma^\infty(\wedge^{a'} T^*M) \) such that we have

\[
\Gamma^\infty(\wedge T^*M) = \bigoplus_{a' = 0}^{\dim_{\mathbb{R}}(M)} \bigoplus_{p' + q' = a'} \Gamma^\infty(\wedge^{p', q'} T^*M). \tag{72}
\]

Here \( \wedge^{p', q'} T^*M \) denotes the bundle with the characteristic fibre \( \wedge^{p', q'} T^*xM, \) \( x \in M \) and \( \wedge^{p', q'} T^*xM \) denotes the subspace of \( \wedge^{p', q'} T^*xM \) with \( a' = p' + q' \) that is generated by elements of the shape \( v' \wedge w' \) with \( v' \in \wedge^{p'} T^*xM^{1,0} \) and \( w' \in \wedge^{q'} T^*xM^{0,1} \). In addition this decomposition induces natural projections

\[
\pi^{p', q'} : \Gamma^\infty(\wedge^{a'} T^*M) \to \Gamma^\infty(\wedge^{p', q'} T^*M), \text{ where } a' = p' + q' \tag{73}
\]

onto the part of type \((p', q')\). Analogously we have

\[
X^s = \bigoplus_{p + q = s} \Gamma^\infty(V^{p-q} T^*M). \tag{74}
\]
Here $\mathcal{V}^{p,q}T^*M$ denotes the bundle with the characteristic fibre $\mathcal{V}^{p,q}T_+T^*M$, $x \in M$ and $\mathcal{V}^{p,q}T^*_x M$ denotes the subspace of $\mathcal{V}^{p,q}T^*M$ with $s = p + q$ that is generated by elements of the shape $v \vee w$ with $v \in \mathcal{V}^{p,q}T_+T^*M_1$ and $w \in \mathcal{V}^{p,q}T^*_0M$. Again there are the induced projections

$$
\pi^{p,q} : \Gamma^\infty(\mathcal{V}^{p,q}T^*M) \to \Gamma^\infty(\mathcal{V}^{p,q}T_+T^*M), \text{ where } s = p + q
$$

onto the sections of type $(p,q)$. The defined projections extend in a natural way to $\Gamma^\infty(\wedge T^*M)$ and $\mathcal{X}^{\infty}_s = \Gamma^\infty(\mathcal{V}^{p,q}T^*M)$ setting $\pi^{p,q}(\beta) := 0$ for $\beta \in \Gamma^\infty(\wedge T^*M)$ with $a' \neq p' + q'$ and $\pi^{p,q}(T) := 0$ for $T \in \Gamma^\infty(\mathcal{V}^{p,q}T^*M)$ with $s \neq p + q$. Using these projections we define the mappings $\pi_{s,z}^{p,q}, \pi_{a,z}^{p,q} : \mathcal{W} \otimes \Lambda$ on factorized sections $T \otimes \beta \in \mathcal{W} \otimes \Lambda$ by

$$
\pi_{s,z}^{p,q}(T \otimes \beta) := (\pi^{p,q}T) \otimes \beta \quad \text{and} \quad \pi_{a,z}^{p,q}(T \otimes \beta) := T \otimes (\pi^{p,q}\beta).
$$

From these mappings we obtain further projections onto the purely holomorphic resp. purely anti-holomorphic part of the symmetric and the anti-symmetric part of $\mathcal{W} \otimes \Lambda$ by

$$
\pi_{s,z} := \sum_{p=0}^{\infty} \pi_{s,z}^{p,0} \quad \text{resp.} \quad \pi_{s,\bar{z}} := \sum_{q=0}^{\infty} \pi_{s,\bar{z}}^{0,q}
$$

and

$$
\pi_{a,z} := \sum_{p'=0}^{\dim_{\mathbb{R}}(M)} \pi_{a,z}^{p',0} \quad \text{resp.} \quad \pi_{a,\bar{z}} := \sum_{q'=0}^{\dim_{\mathbb{R}}(M)} \pi_{a,\bar{z}}^{0,q'}.
$$

With these projections we moreover define the projections onto the totally holomorphic resp. totally anti-holomorphic part of $\mathcal{W} \otimes \Lambda$ by

$$
\pi_z := \pi_{s,z} \pi_{a,z} = \pi_{a,z} \pi_{s,z} \quad \text{resp.} \quad \pi_{\bar{z}} := \pi_{s,\bar{z}} \pi_{a,\bar{z}} = \pi_{a,\bar{z}} \pi_{s,\bar{z}}.
$$

**Lemma A.1** The mappings defined in equations (77), (78) and (79) are projections, i.e. we have $\pi_{s,z} \pi_{s,z} = \pi_{s,z}, \pi_{a,z} \pi_{a,z} = \pi_{a,z}, \pi_{s,\bar{z}} \pi_{s,\bar{z}} = \pi_{s,\bar{z}}, \pi_{a,\bar{z}} \pi_{a,\bar{z}} = \pi_{a,\bar{z}}$ and hence we also have $\pi_z \pi_z = \pi_z$ and $\pi_{\bar{z}} \pi_{\bar{z}} = \pi_{\bar{z}}$. Furthermore, these projections are homomorphisms of the undeformed product on $\mathcal{W} \otimes \Lambda$. In addition $\pi_{a,z}$ and $\pi_{a,\bar{z}}$ are homomorphisms of the fibre-wise Wick product $\circ_{\text{Wick}}$. For all $a, b \in \mathcal{W} \otimes \Lambda$ we moreover have the equations

$$
\pi_{s,z}(a \circ_{\text{Wick}} b) = \pi_{s,z}((\pi_{s,z} a) \circ_{\text{Wick}} b) \quad \text{and} \quad \pi_{s,\bar{z}}(a \circ_{\text{Wick}} b) = \pi_{s,\bar{z}}((\pi_{s,\bar{z}} a) \circ_{\text{Wick}} b),
$$

and

$$
\pi_{s}(a \circ_{\text{Wick}} b) = \pi_{s}((\pi_{s} a) \circ_{\text{Wick}} b) \quad \text{and} \quad \pi_{\bar{z}}(a \circ_{\text{Wick}} b) = \pi_{\bar{z}}((\pi_{\bar{z}} a) \circ_{\text{Wick}} b).
$$

In addition we have $\sigma = \pi_z \pi_{\bar{z}} = \pi_{\bar{z}} \pi_z$.

**Proof:** The proof of this lemma is straightforward using the very definitions and the explicit shape of $\circ_{\text{Wick}}$. \]

Now we consider the mappings $\delta, \delta^*$ and $\nabla$ more closely. Using a local holomorphic chart of $M$ the mappings $\delta_z, \delta_{\bar{z}}, \delta_z^*, \delta_{\bar{z}}^*, \nabla_z, \nabla_{\bar{z}}, \nabla^* : \mathcal{W} \otimes \Lambda \to \mathcal{W} \otimes \Lambda$ are obviously well-defined by

$$
\delta_z := (1 \otimes dz^k) i_a(Z_k), \quad \delta_z^* := (dz^k \otimes 1) i_a(Z_k), \quad \nabla_z := (1 \otimes dz^k) \nabla Z_k, \quad \delta_{\bar{z}} := (1 \otimes d\bar{z}^k) i_a(Z_k), \quad \delta_{\bar{z}}^* := (d\bar{z}^k \otimes 1) i_a(Z_k), \quad \nabla_{\bar{z}} := (1 \otimes d\bar{z}^k) \nabla \bar{Z}_k
$$

and we have $\delta = \delta_z + \delta_{\bar{z}}, \delta^* = \delta_z^* + \delta_{\bar{z}}^*$ and $\nabla = \nabla_z + \nabla_{\bar{z}}$. Completely analogously to the definition of $\delta^{-1}$ we now define $\delta_z^{-1}$ for $a \in \mathcal{W} \otimes \Lambda$ with $(dz^k \otimes 1) i_s(Z_k) a = ka$ and $(d\bar{z}^l \otimes 1) i_s(Z_l) a = la$ by

$$
\delta_z^{-1} a := \begin{cases} 
\frac{1}{k+l} \delta_z^* a & \text{in case } k + l \neq 0 \\
0 & \text{in case } k + l = 0.
\end{cases}
$$
For $a \in \mathcal{W} \otimes \Lambda$ with $(d_{\mathbb{C}}^2 \otimes 1)i_s(\bar{Z}_i)a = ka$ and $(1 \otimes d_{\mathbb{C}})i_a(Z_i)a = la$ we analogously define

$$
\delta^{-1}_Z a := \begin{cases} 
\frac{1}{k+l} \delta_a^* a & \text{in case } k + l \neq 0 \\
0 & \text{in case } k + l = 0
\end{cases}.
$$

(84)

In the following two lemmas we just collect some properties of the defined splittings of the mappings $\delta$, $\delta^{-1}$ and $\nabla$.

Lemma A.2 For all $a \in \mathcal{W} \otimes \Lambda$ we have the decompositions

$$
\delta^{-1}_Z \delta_z a + \delta_z \delta^{-1}_Z a + \pi_z a = a \quad \text{and} \quad \delta^{-1}_\pi \delta_\pi a + \delta_\pi \delta^{-1}_\pi a + \pi_\pi a = a.
$$

(85)

Furthermore, we have the following relations:

$$
\delta_z \pi = \delta_z \pi_z, \quad \pi_z \nabla = \nabla_z \pi_z, \quad \pi_z \delta_z = \delta_z^{-1}_z \pi_z, \quad \pi_\pi \delta_\pi = \delta_\pi^{-1}_\pi \pi_\pi.
$$

(86)

**Proof:** Again the proof is straightforward using the definitions and the property of the pseudo-Kähler connection to be compatible with the complex structure.

Lemma A.3 The holomorphic and anti-holomorphic parts of the mappings $\delta$ and $\nabla$ satisfy the following identities:

$$
\delta^2_Z = \delta^2_\pi = [\delta_z, \delta_\pi] = 0, \quad [\delta_z, \nabla_z] = [\delta_\pi, \nabla_\pi] = 0, \quad \frac{1}{2} \text{ad}_{\text{Wick}}(R), \quad [\delta_z, \nabla_\pi] = [\delta_\pi, \nabla_z] = 0.
$$

(87)

For all $b \in \pi_z(\mathcal{W} \otimes \Lambda^k), c \in \pi_\pi(\mathcal{W} \otimes \Lambda^l), a \in \mathcal{W} \otimes \Lambda^m$ we have:

$$
\begin{align*}
\delta_z \pi_z(b, \text{Wick} a) &= \pi_z(\delta_z b, \text{Wick} a) + (-1)^k \pi_z(b, \text{Wick} \delta a), \\
\nabla_z \pi_z(b, \text{Wick} a) &= \pi_z(\nabla_z b, \text{Wick} a) + (-1)^k \pi_z(b, \text{Wick} \nabla a), \\
\delta_\pi \pi_\pi(a, \text{Wick} c) &= (-1)^m \pi_\pi(a, \text{Wick} \delta c) + \pi_\pi(\delta a, \text{Wick} c), \\
\nabla_\pi \pi_\pi(a, \text{Wick} c) &= (-1)^m \pi_\pi(a, \text{Wick} \nabla c) + \pi_\pi(\nabla a, \text{Wick} c).
\end{align*}
$$

(88)

**Proof:** The statements in equation (87) directly follow from the properties of $\delta$ and $\nabla$ considering the equations $\delta^2 = [\delta, \nabla] = 0$ and $\nabla^2 = -\frac{1}{2} \text{ad}_{\text{Wick}}(R)$ sorted with respect to the holomorphic and anti-holomorphic degrees. Using Lemma A.2, it is easy to verify (88) and (89) since $\delta$ and $\nabla$ are super-derivations with respect to $\text{Wick}$.

**B** An Application of Banach’s Fixed Point Theorem

In this appendix we collect some consequences of Banach’s fixed point theorem that were very useful for divers proofs in Sections 4 and 5. Our presentation which is slightly more general than it is actually needed mainly follows [26, Appx. 1.2].

First let us recall Banach’s fixed point theorem in its usual formulation. We consider a metric space $(\mathcal{M}, d)$ and a mapping $L : \mathcal{M} \to \mathcal{M}$ from this space to itself, then $L$ is called contracting in case there is a $q \in \mathbb{R}$ with $0 \leq q < 1$ such that we have $d(L(x), L(y)) \leq q d(x, y)$ for all $x, y \in \mathcal{M}$. In other words this means that $L$ is Lipschitz continuous with Lipschitz constant $q$. In case $(\mathcal{M}, d)$ is in addition complete we have that such a contracting mapping $L$ has a unique fixed point according to Banach’s fixed point theorem:
Lemma B.1 Let \((\mathcal{M}, d)\) be a complete metric space and let \(L : \mathcal{M} \to \mathcal{M}\) be a contracting mapping, then \(L\) has a unique fixed point \(x_0 \in \mathcal{M}\) that can be obtained by iteration \(x_0 = \lim_{n \to \infty} L^n(x)\), where \(x \in \mathcal{M}\) is arbitrary.

The idea for the following application of this statement on the one hand is to find an appropriate metric such that the considered space becomes a complete metric space and on the other hand to guarantee that the relevant mappings are contracting with respect to this metric. Now let \(R\) denote a ring. Then we consider the Cartesian product \(V := X_{k=0}^\infty V_k\) of \(R\)-modules \(V_k\) for \(k \in \mathbb{N}\) which again is a \(R\)-module. We define the order of an element \(v = (v_k)_{k \in \mathbb{N}} \in V\) by \(o(v) := \min\{k \in \mathbb{N} \mid v_k \neq 0\}\) for \(v \neq 0\) and \(o(0) := +\infty\). Using this definition we define the valuation \(\varphi : V \to \mathbb{Q}\) by \(\varphi(v) := 2^{-o(v)}\) for \(v \neq 0\) and \(\varphi(0) := 0\) and get the following:

Lemma B.2 With the notations from above \((V = X_{k=0}^\infty V_k, d_\varphi)\), where \(d_\varphi\) is defined by \(d_\varphi(v, w) := \varphi(v - w)\) for \(v, w \in V\), is a complete ultrametric space. Moreover, a mapping \(L : V \to V\) is contracting with respect to \(d_\varphi\) if and only if there is a \(0 < k \in \mathbb{N} \cup \{+\infty\}\) such that

\[
   o(L(v) - L(w)) \geq k + o(v - w) \quad \forall v, w \in V.
\]

In this case \(2^{-k}\) is a Lipschitz constant for \(L\), where we have set \(2^{-\infty} := 0\).

Proof: The proof is just a slight modification of the proof in case all the \(R\)-modules \(V_k\) coincide with \(V_0\). The proof for this situation, where \(V = V_0[[\nu]]\) is the space of formal power series with values in \(V_0\) and the topology induced by \(d_\varphi\) is the \(\nu\)-adic topology, can be found in several textbooks e.g. [19] p. 388 or in the article [8] Prop. 2.

Since the Fedosov algebra \(W \otimes \Lambda\) is equal to the Cartesian product \(W \otimes \Lambda = X_{k=0}^\infty (W \otimes \Lambda)^{(k)}\), where \((W \otimes \Lambda)^{(k)}\) denotes the subspace of the elements that are homogeneous of degree \(k\) with respect to the total degree, i.e. \(\text{Dega} = ka\) for all \(a \in (W \otimes \Lambda)^{(k)}\), we can apply the above lemma and obtain:

Corollary B.3 i.) A mapping \(L : W \otimes \Lambda \to W \otimes \Lambda\) has a unique fixed point in \(W \otimes \Lambda\), in case there is a \(0 < k \in \mathbb{N}\) such that for all \(a, b \in W \otimes \Lambda\) the total degree of the term with lowest total degree in \(L(a) - L(b)\) is at least higher by \(k\) than the total degree of the term of lowest total degree in \(a - b\).

ii.) If \(L : W \otimes \Lambda \to W \otimes \Lambda\) raises the total degree at least by 1 then \(L\) is contracting.

iii.) Let \(\pi : W \otimes \Lambda \to W \otimes \Lambda\) denote a projection that is continuous with respect to the topology induced by the above ultrametric. Then the statements analogous to i.) and ii.) hold for a mapping \(L : \pi(W \otimes \Lambda) \to \pi(W \otimes \Lambda)\) and the unique fixed point lies in \(\pi(W \otimes \Lambda)\).

Proof: The statements i.) and ii.) are obvious from the very definitions. For the proof of iii.) one just has to observe that the continuity of \(\pi\) and the fact that \(\pi\) is a projection imply that \(\pi(W \otimes \Lambda) = \ker(\text{id} - \pi) \subseteq W \otimes \Lambda\) is closed and hence a complete ultrametric space with respect to the restriction of the metric on \(W \otimes \Lambda\) to \(\pi(W \otimes \Lambda)\).

References

[1] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation Theory and Quantization. Ann. Phys. 111, Part I: 61–110, Part II: 111–151 (1978).
[2] Bertelson, M., Cahen, M., Gutt, S.: *Equivalence of star products*. Class. Quant. Grav. **14**, A93–A107 (1997).

[3] Bordemann, M., Herbig, H.-C., Waldmann, S.: *BRST Cohomology and Phase Space Reduction in Deformation Quantization*. Commun. Math. Phys. **210**, 107–144 (2000).

[4] Bordemann, M., Neumaier, N., Waldmann, S.: *Homogeneous Fedosov Star Products on Cotangent Bundles I: Weyl and Standard Ordering with Differential Operator Representation*. Commun. Math. Phys. **198**, 363–396 (1998).

[5] Bordemann, M., Waldmann, S.: *A Fedosov Star Product of Wick Type for Kähler Manifolds*. Lett. Math. Phys. **41**, 243–253 (1997).

[6] Bordemann, M., Waldmann, S.: *Formal GNS Construction and States in Deformation Quantization*. Commun. Math. Phys. **195**, 549–583 (1998).

[7] Deligne, P.: *Déformations de l’Algèbre des Fonctions d’une Variété Symplectique: Comparaison entre Fedosov et DeWilde, Lecomte*. Sel. Math., New Series **1** (4), 667–697 (1995).

[8] DeWilde, M., Lecomte, P. B. A.: *Existence of Star-Products and of Formal Deformations of the Poisson Lie Algebra of Arbitrary Symplectic Manifolds*. Lett. Math. Phys. **7**, 487–496 (1983).

[9] Fedosov, B. V.: *A Simple Geometrical Construction of Deformation Quantization*. J. Diff. Geom. **40**, 213–238 (1994).

[10] Fedosov, B. V.: *Deformation Quantization and Index Theory*. Akademie Verlag, Berlin (1996).

[11] Fedosov, B. V.: *Non-Abelian Reduction in Deformation Quantization*. Lett. Math. Phys. **43**, 137–154 (1998).

[12] Gutt, S., Rawnsley, J.: *Equivalence of star products on a symplectic manifold; an introduction to Deligne’s Čech cohomology classes*. J. Geom. Phys. **29**, 347–392 (1999).

[13] Gutt, S.: *Variations on deformation quantization*. in: Dito, G., Sternheimer, D. (eds.): Conférence Moshé Flato 1999, Vol. I. Kluwer Academic Publ., Dordrecht, 217–254 (2000).

[14] Halbout, G.: *Sur la classification des déformations des variété de Poisson*. Thèse, Institut de recherche mathématique avancée, Université Louis Pasteur et C.N.R.S (1999).

[15] Karabegov, A. V.: *Deformation Quantization with Separation of Variables on a Kähler Manifold*. Commun. Math. Phys. **180**, 745–755 (1996).

[16] Karabegov, A. V.: *Cohomological Classification of Deformation Quantization with Separation of Variables*. Lett. Math. Phys. **43**, 347–357 (1998).

[17] Karabegov, A. V.: *On Fedosov’s approach to Deformation Quantization with Separation of Variables*. in: Dito, G., Sternheimer, D. (eds.): Conférence Moshé Flato 1999, Vol. II. Kluwer Academic Publ., Dordrecht, 167–176 (2000).

[18] Karabegov, A. V., Schlichenmaier, M.: *Identification of Berezin-Toeplitz Deformation Quantization*. J. reine angew. Math. **540**, 49–76 (2001).
[19] Kassel, C.: *Quantum Groups*. Graduate Texts in Mathematics 155, Springer Verlag, New York, Berlin, Heidelberg (1995).

[20] Kontsevich, M.: *Deformation Quantization of Poisson Manifolds, I*. Preprint, September 1997, q-alg/9709040.

[21] Neumaier, N.: *Local ν-Euler Derivations and Deligne’s Characteristic Class of Fedosov Star Products and Star Products of Special Type*. Commun. Math. Phys. 230, 271–288 (2002).

[22] Neumaier, N.: *Some General Aspects of Fedosov-like Products*. Preprint in preparation.

[23] Neumaier, N., Waldmann S.: *Morita Equivalence Bimodules for Wick Type Star Products*. Preprint, July 2002, Freiburg FR-THEP-2002/09, math.QA/0207162. To appear in J. Geom. Phys.

[24] Nest, R., Tsygan, B.: *Algebraic Index Theorem*. Commun. Math. Phys. 172, 223–262 (1995).

[25] Omori, H., Maeda, Y., Yoshioka, A.: *Weyl Manifolds and Deformation Quantization*. Adv. Math. 85, 224–255 (1991).

[26] Waldmann, S.: *Zur Deformationsquantisierung in der klassischen Mechanik: Observablen, Zustände und Darstellungen*. Ph.D. thesis, Fakultät für Physik der Albert-Ludwigs-Universität, Freiburg (1999). (available at: http://idefix.physik.uni-freiburg.de/~stefan/)

[27] Weinstein, A., Xu, P.: *Hochschild cohomology and characteristic classes for star-products*. in: Khovanskij, A. ET AL. (eds.): Geometry of differential equations. Dedicated to V. I. Arnol’d on the occasion of his 60th birthday. Providence, Amer. Math. Soc. Transl., Ser. 2, 186 (39), 177–194 (1998).