Geometric intersection in representations of mapping class groups of surfaces

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Abstract

We show that the detection of geometric intersection in an arbitrary representation of the mapping class group of surface implies the injectivity of that representation up to center, and vice versa. As an application, we discuss the geometric intersection in the Johnson filtration. Also, we further consider the problem of detecting the geometric intersection between separating simple closed curves in a representation.

1 Introduction

Detecting geometric intersection can be a powerful tool for the study of representations of the mapping class group of surface. For instance, a certain kind of such detection in the Lawrence–Krammer representation by Bigelow [2] led to an affirmative solution to the linearity problem for Artin’s braid group which is nothing but the mapping class group of a punctured disk. Conversely, the impossibility of detecting a similar kind of geometric intersection had led to the unfaithfulness results for the Burau representation of the braid group as shown by Moody [12], Long–Paton [11], and Bigelow [1]. As for the mapping class group of a surface of higher genus, this type of result was given by Suzuki [16] for the Magnus representation of the Torelli group. In each of all these works, it was fundamental to establish a criterion that the representation in question can detect the geometric intersection if and/or only if its kernel is small.

In this paper, instead of considering any particular representation, we derive a similar criterion applicable to an arbitrary group homomorphism of the mapping class group of a surface of genus at least one, by focusing our attention on the following fact:

The geometric intersection number between two simple closed curves is zero if and only if the commutator of the two Dehn twists along them represents the identity in the mapping class group.

We now describe our main result. Let $\Sigma_{g,n}$ be an oriented compact connected surface of genus $g \geq 1$ with $n \geq 0$ boundary components. The mapping class group $\mathcal{M}_{g,n}$ of $\Sigma_{g,n}$ is defined as the group of all the isotopy classes of orientation preserving homeomorphisms of $\Sigma_{g,n}$ where all homeomorphisms and isotopies are assumed to preserve the boundary of $\Sigma_{g,n}$ pointwise. Let $\mathcal{S}$ be the set of all the isotopy classes of essential simple closed curves on $\Sigma_{g,n}$. Here, essential is meant to be not homotopic to a point nor parallel to any of the boundary components. For $c \in \mathcal{S}$, we denote by $t_c$ the right-handed Dehn twist along $c$. We denote by $\mathcal{S}_{\text{nonsep}}$ the subset of
\(\mathcal{I}\) consisting of all the isotopy classes of nonseparating simple closed curves. The commutator of two elements \(x\) and \(y\) in a group is defined by \([x,y] = xyx^{-1}y^{-1}\).

Our criterion states that the triviality of geometric intersection number for all pairs of essential simple closed curves can be detected by a homomorphisms of \(\mathcal{M}_{g,n}\) if and only if its kernel is small:

**Theorem 1.1.** Let \(G\) be a group and \(\rho : \mathcal{M}_{g,n} \to G\) an arbitrary homomorphism. If \([t_{c_1}, t_{c_2}] = 1\) for those \(c_1, c_2 \in \mathcal{I}_{\text{nonsep}}\) which satisfy \(\rho ([t_{c_1}, t_{c_2}]) = 1\), then the kernel of \(\rho\) is contained in the center \(Z(\mathcal{M}_{g,n})\) of \(\mathcal{M}_{g,n}\). Conversely, if \(\ker \rho \subset Z(\mathcal{M}_{g,n})\), then \([t_{c_1}, t_{c_2}] = 1\) for any \(c_1\) and \(c_2 \in \mathcal{I}\) which satisfy \(\rho ([t_{c_1}, t_{c_2}]) = 1\).

Note that the curves \(c_1\) and \(c_2\) need not be nonseparating in the latter half of Theorem 1.1.

**Remark 1.2.** The structure of the center \(Z(\mathcal{M}_{g,n})\) is well-known due to Paris–Rolfsen [14]. If \(n = 0\), \(Z(\mathcal{M}_{g,n})\) is trivial except for the case \(g \leq 2\), where the center is generated by the class of hyperelliptic involution. For the case of \(g = 1\) and \(n = 1\), the center is an infinite cyclic group generated by the “half-twist” along the unique boundary component. For all the other cases, \(Z(\mathcal{M}_{g,n})\) is a free abelian group of rank \(n\) and is generated by the Dehn twists along the boundary components of \(\Sigma_{g,n}\).

The organization of this paper is as follows. The proof of Theorem 1.1 is given in Section 2 after necessary preparation. In Section 3 as an application, we discuss the geometric intersection in the Johnson filtration and pose a certain problem. Also, in Section 4 we further consider the geometric intersection between separating simple closed curves and provide a criterion similar to Theorem 1.1.

## 2 Proof of Theorem 1.1

We first prepare some necessary results. We refer to [4] as basic reference for mapping class groups of surfaces. We also need some results in our previous work [9] with certain modification.

For \(a, b \in \mathcal{I}\), the geometric intersection number, denoted by \(i_{\text{geom}}(a,b)\), is the minimum of the number of the intersection points of the simple closed curves \(\alpha\) and \(\beta\) where \(\alpha\) and \(\beta\) vary the isotopy classes of \(a\) and \(b\), respectively. It defines a function
\[
i_{\text{geom}} : \mathcal{I} \times \mathcal{I} \to \mathbb{Z}_{\geq 0}.
\]

The following is the precise statement of the fact mentioned in Introduction (c.f. Fact 3.9 in [4]).

**Lemma 2.1.** For \(c_1, c_2 \in \mathcal{I}\), \(i_{\text{geom}}(c_1,c_2) = 0\) if and only if \([t_{c_1}, t_{c_2}] = 1\) in \(\mathcal{M}_{g,n}\).

The next is also well-known, and will be crucial in our argument.

**Lemma 2.2.** Suppose \(c_1, c_2 \in \mathcal{I}\). If \(c_1 \neq c_2\), then there exists \(d \in \mathcal{I}\) such that \(i_{\text{geom}}(c_1,d) = 0\) and \(i_{\text{geom}}(c_2,d) \neq 0\).

The proof can be found in [4, page 73].

We define the mapping \(t : \mathcal{I} \to \mathcal{M}_{g,n}\) by \(t(c) := t_c\) for \(c \in \mathcal{I}\). It is known that \(t\) is injective (c.f. [4, Fact 3.6]). As explained in [9, Lemma 3.2], the proof of this fact, which depends on Lemma 2.2, actually implies the following.
Lemma 2.3. For any $c_1$ and $c_2 \in \mathcal{S}$, the element $t(c_1)t(c_2)^{-1}$ lies in the center $Z(\mathcal{M}_{g,n})$ if and only if $c_1 = c_2$.

The following is a slight generalization of our previous result [9], where we dealt with $\mathcal{S}$ instead of $\mathcal{S}_{\text{nonsep}}$. We denote by $t_{\text{nonsep}}$ the restriction of $t$ to $\mathcal{S}_{\text{nonsep}}$.

Lemma 2.4. Let $G$ be a group and $\rho : \mathcal{M}_{g,n} \to G$ an arbitrary homomorphism.

1. If the mapping $\rho \circ t_{\text{nonsep}}$ is injective, then $\text{Ker} \rho \subset Z(\mathcal{M}_{g,n})$.
2. If $\text{Ker} \rho \subset Z(\mathcal{M}_{g,n})$, then the mapping $\rho \circ t$ is injective.

The proof of Lemma 2.4 follows from [9, Lemma 3.2]. The subtle point for the first part is to observe that one can choose a generating set for $\mathcal{M}_{g,n}$ due to Gervais [5] from the image of $t_{\text{nonsep}}$ and the Dehn twists along boundary components of $\Sigma_{g,n}$. This follows actually from the construction there.

Proof of Lemma 2.4. It suffices to prove the first part. We first recall the effect of the natural action of $\mathcal{M}_{g,n}$ on $\mathcal{S}$ over the image of $t$. For any $f \in \mathcal{M}_{g,n}$ and $c \in \mathcal{S}$, it holds

$$t(f(c)) = f \cdot t(c) \cdot f^{-1} \quad (2.1)$$

(c.f. [4, Fact 3.7]).

Suppose next that the mapping $\rho \circ t_{\text{nonsep}}$ is injective. Let $f \in \text{Ker} \rho$. Then for any $c \in \mathcal{S}_{\text{nonsep}}$, we have

$$\rho \circ t_{\text{nonsep}}(f(c)) = \rho(f \cdot t_{\text{nonsep}}(c) \cdot f^{-1}) = \rho \circ t_{\text{nonsep}}(c).$$

Hence we have $f(c) = c$ for each $c \in \mathcal{S}_{\text{nonsep}}$. In view of (2.1), this shows that $f$ commutes with each element of $t_{\text{nonsep}}(\mathcal{S}_{\text{nonsep}})$. As mentioned above, the mapping class group $\mathcal{M}_{g,n}$ is generated by $t_{\text{nonsep}}(\mathcal{S}_{\text{nonsep}})$ together with the Dehn twists along boundary components of $\Sigma_{g,n}$. The latter type of mapping classes obviously lie in the center $Z(\mathcal{M}_{g,n})$. Therefore, we have $f \in Z(\mathcal{M}_{g,n})$, and hence $\text{Ker} \rho \subset Z(\mathcal{M}_{g,n})$. This completes the proof of Lemma 2.4.

We are now ready to prove Theorem 1.1. Let $\rho : \mathcal{M}_{g,n} \to G$ be an arbitrary group homomorphism. Suppose $[t_{c_1}, t_{c_2}] = 1$ for those $c_1, c_2 \in \mathcal{S}_{\text{nonsep}}$ which satisfy $\rho([t_{c_1}, t_{c_2}]) = 1$. For the first part of the theorem, by virtue of Lemma 2.4(1), it suffices to show that $\rho \circ t_{\text{nonsep}}$ is injective. For any $c_1, c_2 \in \mathcal{S}_{\text{nonsep}}$ with $c_1 \neq c_2$, by Lemma 2.2, we may choose $d \in \mathcal{S}$ such that $i_{\text{geom}}(c_1, d) = 0$ and $i_{\text{geom}}(c_2, d) \neq 0$. Then by Lemma 2.1 and the assumption of the theorem, respectively, we have

$$\rho([t_{c_1}, t_d]) = 1, \quad \text{and} \quad \rho([t_{c_2}, t_d]) \neq 1$$

This implies $\rho \circ t_{\text{nonsep}}(c_1) \neq \rho \circ t_{\text{nonsep}}(c_2)$, which shows that $\rho \circ t_{\text{nonsep}}$ is injective.

Next, suppose conversely that $\text{Ker} \rho \subset Z(\mathcal{M}_{g,n})$. Then $\rho \circ t$ is injective by Lemma 2.4(2). For $c_1, c_2 \in \mathcal{S}$, assume $\rho([t_{c_1}, t_{c_2}]) = 1$. Since $[t_{c_1}, t_{c_2}] = t(t_{c_1}(c_2)) \cdot t(c_2)^{-1}$ in view of (2.1), we have $t(t_{c_1}(c_2)) \cdot t(c_2)^{-1} \in Z(\mathcal{M}_{g,n})$. We can then conclude $t_{c_1}(c_2) = c_2$ by Lemma 2.3. This implies $[t_{c_1}, t_{c_2}] = 1$. This completes the proof of Theorem 1.1.

3
3 The Johnson filtration

Let $g \geq 2$. We now consider the relation of geometric intersection with the Johnson filtration. We consider only the case of $n = 1$ in order to avoid the problem that there are no canonical choices of the filtration for $n > 1$.

Let $\Gamma$ be the fundamental group of the surface $\Sigma_{g,1}$ with a fixed base point on the boundary, which is a free group of rank $2g$. The lower central series of $\Gamma$, denoted by $\{\Gamma_k\}_{k \geq 0}$, is defined recursively by $\Gamma_0 = \Gamma$, and $\Gamma_k = [\Gamma, \Gamma_{k-1}]$ for $k \geq 1$. For each $k \geq 1$, $\Gamma_k$ is a characteristic subgroup of $\Gamma$, so that the natural action of $\mathcal{M}_{g,1}$ on $\Gamma$ gives rise to the one on the quotient nilpotent group $N_k := \Gamma/\Gamma_k$. The latter action induces a homomorphism, which we denote by $\rho_k : \mathcal{M}_{g,1} \to \operatorname{Aut}(N_k)$.

We denote the kernel of $\rho_k$ by $\mathcal{N}(k)$. These $\mathcal{N}(k)$’s form a descending central filtration of $\mathcal{M}_{g,1}$ which is called the Johnson filtration. It follows, by definition, that $\mathcal{N}(k)$ is the domain of the $k$th Johnson homomorphism for $k \geq 1$, and is also the kernel of the $k - 1$st Johnson homomorphism for $k \geq 2$.

The following shows, in view of Lemma 2.1, that the Johnson filtration in any finite depth does not detect the geometric intersection.

Corollary 3.1. For any integer $k \geq 1$, there exists a pair of simple closed curves $c_1$ and $c_2 \in \mathcal{S}$ such that the commutator of the Dehn twists along them lies in $\mathcal{N}(k)$ but is not the identity. Furthermore, one can always choose such $c_1$ and $c_2$ from $\mathcal{S}_{\text{nonsep}}$.

Proof. Suppose to the contrary that there exists some $k_0 \geq 1$ such that the condition $[t_{c_1}, t_{c_2}] \in \mathcal{M}(k_0)$ for any $c_1, c_2 \in \mathcal{S}$ implies $[t_{c_1}, t_{c_2}] = 1$ in $\mathcal{M}_{g,1}$. Then we can apply Theorem 1.1 for $\rho = \rho_k$ to see that $\mathcal{N}(k_0)$ must be contained in the center $Z(\mathcal{M}_{g,1})$. Therefore, it is sufficient to confirm:

Claim. For each $k \geq 1$, $\mathcal{N}(k)$ is not contained in the center $Z(\mathcal{M}_{g,1})$.

This is well-known, but we provide a short proof for completeness. Let $a, b$ be two separating essential simple closed curves with $i_{\text{geom}}(a, b) \geq 2$. Then the two Dehn twists along $a$ and $b$ generate a free group $F$ of rank 2, due to the work by Ishida [6]. Take the lower central series of $F$, each term of which is obviously non-trivial and is not contained in $Z(\mathcal{M}_{g,1})$. On the other hand, by the work of Johnson [7], the Dehn twist along any separating essential simple closed curve lies in $\mathcal{M}(2)$, and therefore $F$ is contained in $\mathcal{M}(1)$. By Morita [13], we have $[\mathcal{M}(k), \mathcal{M}(l)] \subset \mathcal{M}(k + l)$. Hence for each $k \geq 1$, the $k - 1$st term of the lower central series of $F$ is contained in $\mathcal{M}(k)$. This proves the claim, and hence Corollary 3.1. 

Towards refinement

In spite of Corollary 3.1, the totality of the Johnson filtration can detect the geometric intersection, due to the following:

Theorem 3.2 (Johnson [8]).

$$\bigcap_{k \geq 1} \mathcal{M}(k) = \{1\}.$$
Let $\mathcal{S}$ denote the complement of $\mathcal{S}_{\text{nonsep}}$ in $\mathcal{S}$, which consists of the isotopy classes of essential separating simple closed curves on $\Sigma_{g,n}$. By using the results by Brendle–Margalit [3] and Kida [10], we can prove that Theorem 1.1 still holds true if $i_{\mathcal{F}}(c_1, c_2) = 0$ if and only if $i_{\text{geom}}(c_1, c_2) = 0$, which is due to Theorem [3,2] and Lemma [2,1]. $i_{\mathcal{F}}(c_1, c_2) \geq 2$ if and only if the algebraic intersection number of $c_1$ and $c_2$ with respect to arbitrarily fixed orientations on them is zero. It would be an interesting problem to study a further relation.

4 The case of separating curves

Let $\mathcal{S}_{\text{sep}}$ denote the complement of $\mathcal{S}_{\text{nonsep}}$ in $\mathcal{S}$, which consists of the isotopy classes of essential separating simple closed curves on $\Sigma_{g,n}$. By using the results by Brendle–Margalit [3] and Kida [10], we can prove that Theorem 1.1 still holds true if $\mathcal{S}_{\text{nonsep}}$ is replaced by $\mathcal{S}_{\text{sep}}$. For simplicity, we assume $g \geq 3$ and $n \geq 0$ in this section.

Let $t_{\text{sep}} : \mathcal{S}_{\text{sep}} \to \mathcal{M}_{g,n}$ denote the restriction of $t$ to $\mathcal{S}_{\text{sep}}$.

Theorem 4.1. Assume $g \geq 3$ and $n \geq 0$. Let $G$ be a group, and $H$ an arbitrary subgroup of $\mathcal{M}_{g,n}$ which contains the image of $t_{\text{sep}}$. Suppose $\rho : H \to G$ is an arbitrary homomorphism. Then the following holds.

1. If $[t_{c_1}, t_{c_2}] = 1$ for those $c_1, c_2 \in \mathcal{S}_{\text{sep}}$ which satisfy $\rho([t_{c_1}, t_{c_2}]) = 1$, then $\text{Ker}\ \rho \subset Z(\mathcal{M}_{g,n})$.

2. If Ker $\rho \subset Z(\mathcal{M}_{g,n})$, then $[t_{c_1}, t_{c_2}] = 1$ for those $c_1, c_2 \in \mathcal{S}_{\text{sep}}$ which satisfy $\rho([t_{c_1}, t_{c_2}]) = 1$.

Remark 4.2. Theorem 4.1 seems to shed some light on the significance of the work by Suzuki. He gave in [15] explicit elements, as commutators of two Dehn twists along separating essential simple closed curves, of the kernel of the representation mentioned in Introduction, the linear representation of the Torelli group $\mathcal{M}(1)$ for $n = 1$ which is defined as the Magnus representation associated with the abelianization of $\Gamma$.

The proof of Theorem 4.1 is essentially the same as that of Theorem 1.1 and here we give just a sketch of it. First, a little care in proving Lemma 2.2 obtains:

Lemma 4.3. Suppose $c_1, c_2 \in \mathcal{S}_{\text{sep}}$. If $c_1 \neq c_2$, then there exists $d \in \mathcal{S}_{\text{sep}}$ such that $i_{\text{geom}}(c_1, d) = 0$ and $i_{\text{geom}}(c_2, d) \neq 0$.

On the other hand, by the works of Brendle–Margalit and Kida mentioned above, specifically Theorem 1.2 (i) in [10], we see that any mapping class of $\mathcal{M}_{g,n}$ which acts trivially on $\mathcal{S}_{\text{sep}}$ lies in the center $Z(\mathcal{M}_{g,n})$. This fact and Lemma 4.3 implies the following.

Lemma 4.4. Let $G$ be a group, and $H$ an arbitrary subgroup of $\mathcal{M}_{g,n}$ which contains the image of $t_{\text{sep}}$. For any homomorphism $\rho : H \to G$, the mapping $\rho \circ t_{\text{sep}}$ is injective if and only if Ker $\rho \subset Z(\mathcal{M}_{g,n})$.

Now the same argument for Theorem 1.1 completes the proof of Theorem 4.1.  

\footnote{Note that the quotient of our mapping class group $\mathcal{M}_{g,n}$ by its center is contained in Kida’s extended mapping class group $\text{Mod}^*(S)$ as the pure mapping class group provided $p = n$.}
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References

[1] S. Bigelow, *The Burau representation is not faithful for n = 5*, Geom. Topol. 3 (1999), 397–404.

[2] ———, *Braid groups are linear*, J. Amer. Math. Soc. 14 (2001), no. 2, 471–486.

[3] T. E. Brendle and D. Margalit, *Commensurations of the Johnson kernel*, Geom. Topol. 8 (2004), 1361–1384.

[4] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012.

[5] S. Gervais, *A finite presentation of the mapping class group of a punctured surface*, Topology 40 (2001), no. 4, 703–725.

[6] A. Ishida, *The structure of subgroup of mapping class groups generated by two Dehn twists*, Proc. Japan Acad. Ser. A Math. Sci. 72 (1996), no. 10, 240–241.

[7] D. Johnson, *An abelian quotient of the mapping class group $J_g$*, Math. Ann. 249 (1980), no. 3, 225–242.

[8] ———, *A survey of the Torelli group*, Low-dimensional topology (San Francisco, Calif., 1981), Contemp. Math., vol. 20, Amer. Math. Soc., Providence, RI, 1983, pp. 165–179.

[9] Y. Kasahara, *On visualization of the linearity problem for mapping class groups of surfaces*, to appear in Geom. Dedicata.

[10] Y. Kida, *Automorphisms of the Torelli complex and the complex of separating curves*, J. Math. Soc. Japan 63 (2011), no. 2, 363–417.

[11] D. D. Long and M. Paton, *The Burau representation is not faithful for n ≥ 6*, Topology 32 (1993), no. 2, 439–447.

[12] J. Moody, *The faithfulness question for the Burau representation*, Proc. Amer. Math. Soc. 119 (1993), no. 2, 671–679.

[13] S. Morita, *Abelian quotients of subgroups of the mapping class group of surfaces*, Duke Math. J. 70 (1993), no. 3, 699–726.

[14] L. Paris and D. Rolfsen, *Geometric subgroups of mapping class groups*, J. Reine Angew. Math. 521 (2000), 47–83.

[15] M. Suzuki, *The Magnus representation of the Torelli group $J_{g,1}$ is not faithful for g ≥ 2*, Proc. Amer. Math. Soc. 130 (2002), no. 3, 909–914.
[16] ______. *On the kernel of the Magnus representation of the Torelli group*, Proc. Amer. Math. Soc. 133 (2005), no. 6, 1865–1872.

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