Flux-tube Structure, Sum Rules and Beta-functions in SU(2)

A.M. Green, P. Pennanen\textsuperscript{a} and C. Michael\textsuperscript{b} \textsuperscript{†}

\textsuperscript{a}Department of Physics and The Helsinki Institute of Physics, University of Helsinki, Finland
\textsuperscript{b}Theoretical Physics Division, Dept. of Math. Sciences, University of Liverpool, Liverpool, UK

Action and energy flux-tube profiles are computed, in SU(2) with $\beta = 2.4, 2.5$, for two quarks up to 1 fm apart and for which the colour fields are in their ground state ($A_{1g}$) and the first ($E_u$) and higher ($A'_{1g}$) excited gluonic states. When these profiles are integrated over all space, a scaling comparison is made between the $\beta = 2.4$ and 2.5 data. Using sum rules, these integrated forms also permit an estimate to be made of generalised $\beta$-functions giving $b(2.4) = -0.312(15)$, $b(2.5) = -0.323(9)$, $f(2.4) = 0.65(1)$ and $f(2.5) = 0.68(1)$. When the profiles are integrated only over planes transverse to the interquark line and assuming underlying string features, scaling comparisons are again made near the centres of the interquark line for the largest interquark distances. For the $A_{1g}$ case, some of the profiles exhibit a ‘dip-like’ structure characteristic of the Isgur-Paton model.

Energy and action profiles for two quarks a distance $R$ apart are calculated by measuring the correlation $< WP^{\mu\nu} >$ between the Wilson loop $W(R)$ and different orientations of the plaquette $P^{\mu\nu}$. One of the new features compared with earlier such studies is that here the colour fields are not only in the ground state ($A_{1g}$) but also in the excited states ($E_u$) and ($A'_{1g}$) – more details being given in refs. \textsuperscript{[3,4]}. The colour fields are measured using a plaquette, whose physical size changes with $\beta$. As only observables with the same physical size at different values of the coupling have a continuum limit, there will be no naive scaling but we are able to make use of the sum-rules presented below (after subtracting divergent self-energy contributions) to control the normalisation of the three-dimensional sums of the fields. More microscopic observables, such as planar sums or transverse profiles, do not have a well defined continuum limit, so our comparisons at these at the two values of coupling should be taken as exploratory.

Figs. 1, 2 and 3 show for the three gluonic states the action ($S$), Longitudinal and Transverse energies ($E_{L,T}$) profiles. Here the $\beta = 2.4$ data has been scaled by the factor $2.4/(2.5\rho^4)$, where $\rho = a(2.4)/a(2.5) = 1.410(13)$. It is seen that, except for $S(A_{1g})$ and $E_L(A_{1g})$, ‘scaling’ is not very evident. The effect of discretization means that any underlying profile structure tends to get ‘smoothed out’ as $\beta$ decreases, since the lattice spacing is larger. This is seen in the figures as sharper peaks for the $\beta = 2.5$ data.

Figs. 4, 5 and 6 show the profiles, integrated over transverse planes – again for the three gluonic states. Several features are apparent:

i) The peaks at $R_L \approx 6$ are the self-energies at the quark positions. They diverge in a manner suggested by leading order perturbation theory.

ii) For $0 \leq R_L \leq 4$ clear flux-tubes emerge each with a constant radius.

iii) ‘Scaling’ between the $\beta = 2.4$ and 2.5 data is now clearer than in Figs. 1 and 2.

iv) The $E_T$ sum increases when going from the $A_{1g}$ to $E_u$ to the $A'_{1g}$ gluonic states – a feature expected from a vibrating string model.

When the transverse sums are themselves integrated over $R_L$ to give the total energy or action, a comparison can be made with the sum rules:

\begin{align}
-\frac{1}{b} \left( V + R \frac{\partial V}{\partial R} \right) + S_0 &= \sum S \quad (1) \\
\frac{1}{4 \beta f} \left( V + R \frac{\partial V}{\partial R} \right) + E_0 &= \sum E_L \quad (2) \\
\frac{1}{4 \beta f} \left( V - R \frac{\partial V}{\partial R} \right) + E_0 &= \sum E_T. \quad (3)
\end{align}

*Presented by A.M. Green, green@phcu.helsinki.fi

petrus@hip.fi, cmi@liv.ac.uk

arXiv:hep-lat/9708012v1 19 Aug 1997
Here $V(R)$ is the interquark potential and $b, f$ are generalised $\beta$-functions, which show how the bare couplings of the theory vary with the generalised lattice spacings $a_\mu$ in four directions. The main interest is to extract $b, f$ by first calculating $V$ and the $\sum$’s on a lattice. Unfortunately, this strategy is complicated by two features – the unknown self-energies $(S_0, E_0)$ and the value of $\partial V/\partial R$. Here three methods are attempted to estimate $b, f$.

**Method 1.** Since $V$ is known numerically, $V \pm R \partial/\partial R$ can be calculated and plotted against, say $\sum S$. This is a linear plot and the slope gives $b$. The extraction of $f$ is less clean and necessitates a simultaneous fit using both the $\sum E_L$ and $\sum E_T$ sum-rules. This arises because the $E_{L,T}$ are differences between the electric and magnetic fields – unlike the action – and this leads to a numerical inaccuracy problem. The outcome yields our best estimates of $b, f$.

**Method 2.** In the large $R$ limit $V(R) = -e/R + b_s R + V_0$, so that $\partial V/\partial R$ is readily estimated. Furthermore, the self-energies can be removed by using the sum-rules at two different values of $R = R_1, R_2$ to give:

\[
b = \frac{-2b_s(R_1 - R_2)}{\sum S_{R_1} - \sum S_{R_2}} \quad \text{(4)}
\]

\[
f(I) = \frac{b_s(R_1 - R_2)}{2\beta [\sum (E_L)_{R_1} - \sum (E_L)_{R_2}]} \quad \text{(5)}
\]

\[
f(II) = \frac{-e(1/R_1 - 1/R_2)}{2\beta [\sum (E_T)_{R_1} - \sum (E_T)_{R_2}]} \quad \text{(6)}
\]

Using directly the 3-D sums $(S, E)_R$ gives results consistent with Method 1 but having larger error bars. The further approximation of a constant longitudinal flux tube profile gives consistent results at the largest $R$’s, supporting a string-like picture.

**Method 3.** It is possible to combine the three sum-rules using two values of $R$ in such a way as to completely eliminate both $\partial V/\partial R$ and the self-energies. This would seem ideal, since it involves quantities that can be measured directly. However, in practice, there is a problem, since now even $b$ depends on the energy differences, leading to large uncertainties.

Our best values for $b$ shown in the abstract can be compared with other recent estimates in Table 1, most importantly the finite $T$ approach of Ref. [3]. Based on the agreement of non-perturbative estimates we conclude that order $a^2$ effects in the extraction of the $\beta$-function are small at the $\beta$-values studied using the methods described. Thus a unique $\beta$-function describes the deviations from asymptotic scaling at these values of the coupling.

| $\beta$ | Ref. [3] | Ref. [4] | 3-loop PT |
|---------|---------|---------|-----------|
| 2.4     | -0.3018 | -0.3056 | -0.3893   |
| 2.5     | -0.3115 | -0.3122 | -0.3889   |

Table 1

Comparison between values of $b = \partial \beta/\partial \ln a$

A Viennese group has measured field distributions around a static quark pair in dually transformed U(1) on a lattice [3]. Their results for the three-dimensional sums vs. quark separation show slopes and ordering of the transverse and longitudinal components of electric and magnetic fields similar to ours.

At present theories [3,4] only calculate the total energy profile for the $A_{1g}$ state and for this they show some success. However, no one has a profile model for $S$ or $E_{L,T}$ or for the excited gluon fields. Such comparisons will be very demanding for any model proposed.

**REFERENCES**

1. A.M. Green, C. Michael and P. Spencer, Phys. Rev. D55 (1997) 1216.
2. P. Pennanen, A.M. Green and C. Michael, accepted for Phys. Rev. D, [hep-lat/9705033]
3. J. Engels, F. Karsch and K. Redlich, Nucl. Phys. B 435, (1995) 295.
4. P. Pennanen, Phys. Rev. D 55, (1996) 3958.
5. M. Zach, M. Faber and P. Skala, [hep-lat/9705015]
6. N. Isgur and J. Paton, Phys. Rev. D31 (1985) 2910.
7. M. Baker, J.S. Ball and F. Zachariasen, Phys. Rev. D51 (1995) 1968 and Int. J. Mod. Phys. A11 (1996) 343.
Figure 1. The colour flux contributions for $S$, $E_{L,T}$. These are shown in units of $a(2.5)$ versus transverse distance $R_T$ at the mid-point ($R_L = R/2$) for $\beta = 2.4, 2.5$ and separation $R = 8, 12$ i.e. 0.95, 1.01 fm respectively. The data are for the symmetric ground state ($A_{1g}$ representation), multiplied with a factor of $10^3$.

Figure 2. As Fig. 1 but for the first gluonic excitation ($E_u$ representation).

Figure 3. As Fig. 1 but for $R = 4, 6$ and the higher gluonic excitation (first excited state of $A_{1g}$ representation).

Figure 4. The dependence on longitudinal position ($R_L$) of the sum over the transverse plane of the colour flux contributions for $S, E_{L,T}$. Here $R_L$ is measured from the mid-point for the same case as in Fig. 1.

Figure 5. As Fig. 4 but for the first gluonic excitation. For each data set one error bar is shown.

Figure 6. As Fig. 5 but for the higher gluonic excitation.