RECONSTRUCTION OF PIECEWISE CONSTANT LAYERED CONDUCTIVITIES IN ELECTRICAL IMPEDANCE TOMOGRAPHY

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Abstract. This work presents a new constructive uniqueness proof for Calderón’s inverse problem of electrical impedance tomography, subject to local Cauchy data, for a large class of piecewise constant conductivities that we call piecewise constant layered conductivities (PCLC). The resulting reconstruction method only relies on the physically intuitive monotonicity principles of the local Neumann-to-Dirichlet map, and therefore the method lends itself well to efficient numerical implementation and generalization to electrode models [14, 13]. Several direct reconstruction methods exist for the related problem of inclusion detection, however they share the property that “holes in inclusions” or “inclusions-within-inclusions” cannot be determined. One such method is the monotonicity method of Harrach, Seo, and Ullrich [21, 22], and in fact the method presented here is a modified variant of the monotonicity method which overcomes this problem. More precisely, the presented method abuses that a PCLC type conductivity can be decomposed into nested layers of positive and/or negative perturbations that, layer-by-layer, can be determined via the monotonicity method. The conductivity values on each layer are found via basic one-dimensional optimization problems constrained by monotonicity relations.

Keywords: electrical impedance tomography, partial data reconstruction, piecewise constant coefficient, monotonicity principle.

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1. Introduction and setting

Let \( \Omega \subset \mathbb{R}^d \), \( d \geq 2 \), be a bounded domain with piecewise \( C^\infty \)-smooth boundary \( \partial \Omega \) (without cusps), for which \( \mathbb{R}^d \setminus \overline{\Omega} \) is connected. We denote by \( \nu \) an outer unit normal on \( \partial \Omega \), and \( \Gamma \subseteq \partial \Omega \) is a non-empty relatively open subset whose role is to employ local Cauchy data. For an electrical conductivity coefficient \( \sigma \in L^\infty_+(\Omega) := \{ \varsigma \in L^\infty(\Omega; \mathbb{R}) | \text{ess inf}_{x \in \Omega} \varsigma(x) > 0 \} \) and boundary current density \( f \in L^2_\Gamma(\Gamma) := \{ g \in L^2(\Gamma) | \int_{\Gamma} g \, dS = 0 \} \) we consider the partial data conductivity problem

\[
\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad \nu \cdot \sigma \nabla u|_{\partial \Omega} = \begin{cases} f & \text{on } \Gamma, \\ 0 & \text{on } \partial \Omega \setminus \Gamma. \end{cases}
\]

From standard elliptic theory there is a unique solution \( u = u^f \) to (1.1), representing the interior electric potential, belonging to the “\( \Gamma \)-mean free” Sobolev space

\[
H^1_\Gamma(\Omega) := \{ w \in H^1(\Omega) | \int_{\Gamma} w \, dS = 0 \}.
\]

This gives rise to a well-defined local Neumann-to-Dirichlet (ND) operator \( \Lambda(\sigma) : f \mapsto u|_\Gamma \), which in this work is interpreted as a compact self-adjoint operator in \( \mathcal{L}(L^2_\Gamma(\Gamma)) \), the space of bounded linear operators on \( L^2_\Gamma(\Gamma) \).
The inverse problem of electrical impedance tomography (EIT), in the sense of Calderón’s formulation [8], is:

\[ \text{Reconstruct } \sigma \text{ from knowledge of } \Lambda(\sigma). \]

In the practical setting, this corresponds to finding the conductivity coefficient in the interior of an object from indirect measurements of current–voltage pairs (injected current and measured voltage) recorded at electrodes placed on the object’s surface. Hence, \( \Lambda(\sigma) \) represents the ideal datum for such a problem. This paper will provide a new simple reconstruction method for recovering a large class of piecewise constant conductivities from their corresponding local ND map. However, first we review some known results on uniqueness and reconstruction in EIT.

For full boundary data, \( \Gamma = \partial \Omega \), unique recovery of \( \sigma \) from \( \Lambda(\sigma) \), i.e. injectivity of \( \sigma \mapsto \Lambda(\sigma) \), has been solved in high generality. See e.g. [1] for general \( L^\infty(\Omega) \)-conductivities in dimension two, and [9] for Lipschitz conductivities in dimension three and beyond. For full boundary data there are also reconstruction methods, based on the works of e.g. [38, 39, 5], such as the \( \bar{\partial} \)-method which has received much attention regarding theoretical development and practical implementation [42, 35, 36, 11, 17, 43, 24]. The motivation behind this paper stems from the expectation that, with enough restrictions on the considered class of conductivities, more straightforward and intuitive reconstruction methods will emerge. This expectation is supported by recent promising computational results in [2], based on shape optimization for piecewise constant conductivities on polygonal partitions.

For the different types of partial data problems in EIT (partial Dirichlet and/or Neumann data on various parts of the boundary) we refer to the review paper [33] and the references therein. Here we will focus on local Cauchy data, in the sense of the local ND map defined above. The uniqueness problem is treated in [28, 29] in two dimensions and for certain three-dimensional geometric shapes in [30, 32]. Although for piecewise analytic conductivities the uniqueness result holds in all reasonable geometric shapes via [37, 20]. Even when uniqueness holds for the partial data problem, exact reconstruction methods are scarce. In fact to the author’s knowledge, the only other proven reconstruction method (besides the one given in this paper) is found in [40] which does not apply to local Cauchy data, but requires Dirichlet and Neumann data to be applied on a (slightly overlapping) partition of \( \partial \Omega \).

We refer to the review papers [3, 10, 45] and references therein for more information on the theoretical and practical aspects of EIT, and refer to the list of references in the next section on the related problem of inclusion detection.

In this paper, we will consider a class of piecewise constant conductivity coefficients that can be decomposed into a sum of piecewise constant functions on nested sets (layers) with connected complement. We call such a conductivity coefficient of type \textit{piecewise constant layered conductivity} (PCLC), formally defined in Definition 1.3 below. As illustrated by the example in Figure 1.1, this type of decomposition is in fact possible for many piecewise constant functions. The purpose of this paper is to provide a reconstruction method, based on a short and comparatively non-technical proof, that determines any PCLC type conductivity \( \gamma \) from its local ND map \( \Lambda(\gamma) \) via the monotonicity relations of \( \sigma \mapsto \Lambda(\sigma) \).

Before giving a precise definition of PCLC type conductivities, we will start by defining the (closed) \( \tau \)-thinning and the outer \( \tau \)-layer of a set \( E \subseteq \mathbb{R}^d \) as

\[
H_\tau(E) := \{ x \in E \mid \text{dist}(x, \partial E) \geq \tau \},
\]

\[
F_\tau(E) := \{ x \in E \mid \text{dist}(x, \partial E) < \tau \}.
\]

We now state a list of assumptions on a family of sets that will be used to represent layers of a conductivity coefficient.

\textbf{Assumption 1.1.} Let \( \tau > 0 \), \( N \in \mathbb{N} \), and \( \{D_j\}_{j=1}^N \) be sets satisfying:

(i) \( D_j \) is the closure of a non-empty open set with piecewise \( C^\infty \)-smooth boundary.

(ii) \( D_j \) has connected complement \( \mathbb{R}^d \setminus D_j \).

(iii) \( D_{j+1} \subseteq H_\tau(D_j) \) for \( j = 1, \ldots, N-1 \) and \( D_1 \subseteq \Omega \).

(iv) Each set \( D_j \) consists of finitely many connected components \( \{D_{j,n}\}_{n=1}^{N_j} \).
Before continuing, we give a few remarks on these assumptions.

**Remark 1.2 (Related to Assumption 1.1).**

1. The case \( \Gamma = \partial \Omega \) allows \( D_1 \subseteq \Omega \) with only minor modifications to the proof of Theorem 2.3.
2. Using \( D_j \) as the closure of an open set, compared to a more general closed set, has the following immediate advantage: \( B \cap D_j \) contains a non-empty open set for every open neighborhood \( B \) of \( x \in D_j \). This avoids some obvious pathological cases in the proof of Theorem 2.3.
3. Each connected component \( D_{j,n} \) obviously also satisfies (i) and (ii) of Assumption 1.1 and \( \text{dist}(\partial D_{j,n}, D_{j+1}) \geq \tau \).
4. We will refer to \( \tau > 0 \) as the *minimal thickness* related to \( \{D_j\}_{j=1}^N \).

For a set \( E \subseteq \mathbb{R}^d \) let \( \chi_E \) denote the characteristic function on \( E \). We now define the PCLC type conductivities.

**Definition 1.3.** Suppose \( \{D_j\}_{j=1}^N \) satisfy Assumption 1.1 with minimal thickness \( \tau > 0 \), then we call \( \gamma \) a piecewise constant layered conductivity (PCLC), provided that

\[
\gamma = c_0 + \sum_{j=1}^N \sum_{n=1}^{N_j} c_{j,n} \chi_{D_{j,n}},
\]

where \( c_0 > 0 \) and \( c_{j,n} \in \mathbb{R} \setminus \{0\} \) satisfy \( 0 < \beta_L \leq \gamma \leq \beta_U \) in \( \Omega \) for scalars \( \beta_L \) and \( \beta_U \). Here \( D_j \) is called the \( j \)'th layer of \( \gamma \), with \( D_0 := \Omega \) denoting the 0'th layer.

![Figure 1.1. Decomposition of a PCLC type conductivity (top left) into each of its layers. The numbers represent function values in each of the colored regions.](image)

For \( k \in \{0, 1, \ldots, N\} \) we define

\[
\gamma_k := c_0 + \sum_{j=1}^k \sum_{n=1}^{N_j} c_{j,n} \chi_{D_{j,n}},
\]

where in particular \( \gamma = \gamma_N \). Note that Assumption 1.1 implies \( \gamma_k \) is piecewise analytic (see e.g. [22, Definition 2.1] and [37, Section 3]). In the following we will devise an iterative reconstruction method that at its \( k \)'th iteration exactly reconstructs \( \gamma_k \), and naturally terminates at \( k = N \). Purely from a notational point of view, in the following section we will use \( D_{N+1} := \emptyset \), which naturally is the conclusion from the \((N+1)\)'th iteration. For this method the following is assumed known/unknown a priori:

- The following is assumed to be known a priori: \( \Omega, \Gamma, \Lambda(\gamma), c_0, \) and \( \gamma \) is of type PCLC with known lower and upper bounds \( \beta_L \) and \( \beta_U \) and minimal thickness \( \tau \).
- The following is unknown a priori: \( c_{j,n}, D_{j,n}, N_j, \) and \( N \).
Remark 1.4. Here we assume \( c_0 \) is known a priori. Such an assumption is also often imposed on other reconstruction methods such as the \( \hat{\partial} \)-method, which can be circumvented by first applying another method to reconstruct \( \gamma \) on \( \Gamma \), see e.g. [41].

In section 2 we state and prove the two results Theorem 2.3 and Theorem 2.4 that combined gives the reconstruction method for recovering \( \gamma \); the actual reconstruction method is summarized in section 2.1 at the end of the paper. First, however, we give a few general notational remarks.

1.1. Notational remarks. For brevity we denote the essential infimum/supremum \( \text{ess inf} \) and \( \text{ess sup} \) of a function \( \varsigma \in L^\infty(\Omega; \mathbb{R}) \) by \( \text{inf}(\varsigma) \) and \( \text{sup}(\varsigma) \), respectively. \( \langle \cdot, \cdot \rangle \) will always denote the usual \( L^2(\Gamma) \)-inner product.

Let \( \mathcal{L}(X,Y) \) be the space of bounded linear operators between Banach spaces \( X \) and \( Y \), with the shorthand notation \( \mathcal{L}(X) := \mathcal{L}(X,X) \). For a self-adjoint operator \( T \in \mathcal{L}(L^2_0(\Gamma)) \) then \( T \geq 0 \) denotes that \( T \) is a positive semi-definite operator, i.e. \( \langle Tf,f \rangle \geq 0 \) for all \( f \in L^2_0(\Gamma) \).

We will often use the symbols “+”/“−” to associate sets and operators to positive/negative perturbations. To avoid excessive repetition, “±” will indicate that a statement holds for both the “+” and “−” version of the set/operator. For example, \( T_{k,n_0}^+ \geq 0 \) means that both \( T_{k,n_0}^+ \geq 0 \) and \( T_{k,n_0}^- \geq 0 \) hold true.

As additional notation we define the index sets \( I_j := \{1, \ldots, N_j\} \) for \( j \in \{1, \ldots, N\} \) and \( I_0 := \{1\} \) as \( D_{0,1} = D_0 := \overline{\Omega} \). Moreover,

\[
I_j^+ := \{ n \in I_j \mid c_{j,n} > 0 \}, \quad D_j^+ := \bigcup_{n \in I_j^+} D_{j,n},
I_j^- := \{ n \in I_j \mid c_{j,n} < 0 \}, \quad D_j^- := \bigcup_{n \in I_j^-} D_{j,n},
\]

such that \( D_j = D_j^+ \cup D_j^- \) decomposes the set into parts with only positive and only negative perturbations, respectively.

Since each connected component \( D_{j,n_0} \) of \( D_j \) can contain several connected components of \( D_{j+1} \), it can swiftly become notationally demanding to have a hierarchical structure of such sets. For this reason we define a function \( n_j : I_{j+1} \to I_j, \; m \mapsto n \), where \( n \in I_j \) is the unique integer such that \( D_{j+1,m} \subset D_{j,n} \) for given \( j \in \{0, \ldots, N-1\} \) and \( m \in I_{j+1} \).

2. Monotonicity-based reconstruction of PCLC conductivities

The reconstruction method will be derived based on the following two results, the monotonicity principle and localized potentials, both of which are well-known results for monotonicity-based reconstruction of the support of perturbations (inclusion detection) and for non-constructive uniqueness and stability proofs in EIT, cf. e.g. [31, 25, 44, 15, 21, 22, 23, 20, 13, 14, 12]. The main idea is to determine the sets \( D_j \) iteratively using the monotonicity principle, which in some circumstances can be reduced to a local condition by the use of localized potentials. After a layer \( D_j \) is determined, the monotonicity principle is used once more to find each of the constants \( c_{j,n} \) through a basic one-dimensional optimization problem.

It is also expected that other inclusion detection methods, such as the factorization method [6, 7, 34, 18, 19, 16] or the enclosure method [26, 27, 4], can lead to similar reconstruction methods under stronger assumptions on the constants \( c_{j,n} \) and sets \( D_j \).

Lemma 2.1 (Monotonicity principle). For \( f \in L^2_0(\Gamma) \) and \( \sigma_1, \sigma_2 \in L^\infty(\Omega) \) it holds

\[
\int_{\Omega} \frac{\sigma_2}{\sigma_1} (\sigma_1 - \sigma_2) |\nabla u_n^2|^2 \, dx \leq \langle \Lambda(\sigma_2) - \Lambda(\sigma_1) \rangle f, f \rangle \leq \int_{\Omega} (\sigma_1 - \sigma_2) |\nabla u_n^2|^2 \, dx.
\]

Proof. This type of result goes back to [31, 25]. See [22, Lemma 3.1] or [21, Lemma 2.1] for a proof of this version of the result, that is readily modified to the local ND map using the variational form of (1.1). See also [22, Section 4.3] for remarks on such extensions. \( \square \)

Lemma 2.2 (Localized potentials). Let \( U \subset \overline{\Omega} \) be a relatively open connected set, which intersects \( \Gamma \), and has connected complement. Let \( B \subset U \) be an open non-empty set and \( \sigma \in L^\infty(\Omega) \) piecewise
analytic, then there are sequences \((f_i) \subset L^2_\sigma(\Gamma)\) and \((u_i) \subset H^1_\sigma(\Omega)\) with \(u_i = u^V_i\) satisfying
\[
\lim_{i \to \infty} \int_B |\nabla u_i|^2 \, dx = \infty \quad \text{and} \quad \lim_{i \to \infty} \int_{\Omega \setminus U} |\nabla u_i|^2 \, dx = 0.
\] (2.1)

**Proof.** This result and its generalizations, ultimately based on unique continuation, is the main topic of [15]. Furthermore, this result is a special case of [22, Theorem 3.6 and Section 4.3], which is also stated for locally supported Neumann conditions in [20, Lemma 2.7].

The map \(\sigma \mapsto \Lambda(\sigma)\) is nonlinear, however it is Fréchet differentiable with derivative \(D\Lambda(\sigma; \cdot) \in \mathcal{L}(L^\infty(\Omega; \mathbb{R}), \mathcal{L}(L^2_\lambda(\Gamma)))\). For each \(\sigma \in L^\infty_\sigma(\Omega)\), \(\eta \in L^\infty(\Omega)\), and \(f \in L^2_\sigma(\Gamma)\) then \(D\Lambda(\sigma; \eta)\) is compact, self-adjoint, and satisfies the well-known quadratic formula (cf. e.g. [20, Lemma 2.5])
\[
\langle D\Lambda(\sigma; \eta) f, f \rangle = -\int_\Omega \eta |\nabla u_\eta|^2 \, dx.
\] (2.2)

While we could completely avoid \(D\Lambda\) in this work by changing the conductivities used for the monotonicity principles, \(D\Lambda\) does lead to a fast numerical method that may be of much higher practical value, without lengthening any of the proofs.

From this point onwards it is assumed \(\gamma_k\) is known for some \(k \in \{0, \ldots, N-1\}\) and we will obtain results that determine \(\gamma_{k+1}\). Denoting the constants
\[
\alpha_{k,n} := \gamma_k |_{D_{k,n}} \quad n \in I_k, \quad \hat{\alpha}_{k,m} := \alpha_{k,n_k(m)} \quad m \in I_{k+1},
\]
these constants will be used to define *conservative* upper bounds on the possible perturbations inside the connected components of \(D_k\). Thereby we avoid having to consider the actual conductivity value on all connected components simultaneously when applying the monotonicity relations. Due to Definition 1.3 and Assumption 1.1 it clearly holds \(\beta_L \leq \alpha_{k,n} \leq \beta_U\) for all \(n \in I_k\). Moreover, from (1.4), Definition 1.3, and Assumption 1.1(iii) we obtain the following bounds for any \(n_0 \in I_k\):
\[
\gamma - \gamma_k \leq \sum_{m \in I_{k+1}} (\beta_U - \hat{\alpha}_{k,m}) \chi_{D_{k+1,m}} \leq \sum_{n \in I_k \setminus \{n_0\}} (\beta_U - \alpha_{k,n}) \chi_{D_{k,n}} (\beta_U - \alpha_{k,n_0}) \chi_{D_{k+1} \cap D_{k,n_0}},
\] (2.3)
\[
\gamma - \gamma_k \geq \sum_{m \in I_{k+1}} (\beta_L - \hat{\alpha}_{k,m}) \chi_{D_{k+1,m}} \geq \sum_{n \in I_k \setminus \{n_0\}} (\beta_L - \alpha_{k,n}) \chi_{D_{k,n}} + (\beta_L - \alpha_{k,n_0}) \chi_{D_{k+1} \cap D_{k,n_0}}.
\] (2.4)

In particular, \(\beta_L - \alpha_{k,n}\) represents the largest possible (signed) negative perturbation that can occur within \(D_{k,n}\) when determining \(\gamma_{k+1}\) from \(\gamma_k\), and likewise \(\beta_U - \alpha_{k,n}\) is the largest possible positive perturbation.

We now define for \(n_0 \in I_k\) and measurable \(C \subseteq \Omega\) some operators based on \(\gamma_k\) and \(\Lambda(\gamma)\):
\[
T_{k,n_0}^+(C) := \Lambda(\gamma) - \Lambda(\gamma_k) - \sum_{n \in I_k \setminus \{n_0\}} (\beta_U - \alpha_{k,n}) \Lambda(\gamma_k; \chi_{D_{k,n}}) - (\beta_U - \alpha_{k,n_0}) \Lambda(\gamma_k; \chi_C),
\]
\[
T_{k,n_0}^-(C) := \Lambda(\gamma_k) - \Lambda(\gamma) + \sum_{n \in I_k \setminus \{n_0\}} \frac{\alpha_{k,n}}{\beta_L} (\beta_L - \alpha_{k,n}) \Lambda(\gamma_k; \chi_{D_{k,n}}) + \frac{\alpha_{k,n_0}}{\beta_L} (\beta_L - \alpha_{k,n_0}) \Lambda(\gamma_k; \chi_C).
\]

In fact, we will consider sets \(C\) that belong to families of *admissible test inclusions* relative to some subset \(E \subseteq \Omega\):
\[
\mathcal{A}(E) := \{C \subseteq \overline{E} \mid C\text{ is closed and } \mathbb{R}^d \setminus C \text{ is connected}\}.
\]

In what follows these test inclusions will be used to determine \(D_{k+1}\) from \(\gamma_k\). Note that Theorem 2.3 below essentially corresponds to a modified version of the usual monotonicity method for indefinite inclusions, applied separately on each connected component of \(D_k\); cf. [14, Theorem 2.3] and [22, Section 4.2].

**Theorem 2.3.** Let \(n_0 \in I_k\), then for all \(C \in \mathcal{A}(D_{k,n_0})\) it holds
\[
D_{k+1} \cap D_{k,n_0} \subseteq C \quad \text{if and only if} \quad T_{k,n_0}^+(C) \geq 0.
\] (2.5)

In particular, \(D_{k+1} \cap D_{k,n_0} = \cap \{C \in \mathcal{A}(D_{k,n_0}) \mid T_{k,n_0}^+(C) \geq 0\}\).
Lemma 2.2 that there are sequences \((U, T, D)\) for all \(D\) we have by Lemma 2.1, (2.2), and (2.1),

\[-(T_{k,n_0}^+(C)f, f) = \int_{\Omega} \gamma - \kappa - \sum_{n \in I_k \setminus \{n_0\}} (\beta - \alpha, \kappa) D_{k,n} - (\beta - \alpha, \kappa) C |\nabla u_k^2| \, dx\]

\[\leq (\alpha, \kappa, \beta, \kappa) \int_{C \setminus (D_{k+1} \cup D_{k,n_0})} |\nabla u_k^2| \, dx \leq 0\]

for all \(f \in L^2(\Gamma), \) i.e. \(T_{k,n_0}^+(C) \geq 0\).

Likewise, since \(\frac{2k}{\gamma} \leq \frac{\alpha, \kappa, \beta}{\beta} \) in \(D_{k,n_1} \) and \(\beta \leq \alpha, \kappa \) thenLemma 2.1, (2.2), and (2.4) imply

\[\langle T_{k,n_0}^-(C)f, f \rangle \geq \int_{\Omega} \gamma (\gamma - \kappa) - \sum_{n \in I_k \setminus \{n_0\}} \frac{\alpha, \kappa, \beta}{\beta} (\beta - \alpha, \kappa) D_{k,n} - \frac{\alpha, \kappa, \beta}{\beta} (\beta - \alpha, \kappa) C |\nabla u_k^2| \, dx\]

\[\geq \frac{\alpha, \kappa, \beta}{\beta} (\alpha, \kappa, \beta, \kappa) \int_{C \setminus (D_{k+1} \cup D_{k,n_0})} |\nabla u_k^2| \, dx \geq 0\]

for all \(f \in L^2(\Gamma), \) i.e. \(T_{k,n_0}^-(C) \geq 0\). This concludes the first part of the proof.

The proof of the other direction “⇐” of the if and only if statement is shown as a contrapositive, i.e. assume \(D_{k+1} \cap D_{k,n_0} \not\subseteq C\) then we will in the following contradict one of the inequalities \(T_{k,n_0}^\pm \geq 0\).

We now pick a relatively open connected set \(U \subseteq \Omega, \) which intersects \(\Gamma, \) has connected complement, and satisfies: \(D_{k+1,m_0} \cap U\) contains an open ball \(B\) for some \(m_0 \in I_{k+1}\) with \(n_0(m_0) = n_0\) and

\[U \cap [(D_{k, m_0} \cup C) \cup (D_{k+1} \setminus D_{k+1,m_0}) \cup D_{k+2}] = \emptyset.\]

The reasoning behind the properties of \(U\) is: Assumption 1.1 and \(C \in \mathcal{A}(D_{k,n_0})\) imply the set \(\Omega \setminus [(D_{k, m_0} \cup C) \cup D_{k+1} \cap D_{k,n_0}]\) is connected and contains \(\Gamma.\) Moreover, \((D_{k+1} \cap D_{k,n_0}) \cap C\) contains a non-empty open set due to (i) and (iii) of Assumption 1.1 (cf. Remark 1.2). Since \(D_{k+1}\) comprise finitely many closed connected components (Assumption 1.1) implies a strictly positive distance between these connected components. Thus \(U\) can be chosen to only intersect one connected component of \(D_{k+1} \cap D_{k,n_0},\) and furthermore avoid \(D_{k+2}\) due to Assumption 1.1(iii).

This splits the rest of the proof into two possible cases, related to which one of the inequalities \(T_{k,n_0}^\pm \geq 0\) that will be contradicted:

- Case (a): \(m_0 \in I_{k+1}^+\)
- Case (b): \(m_0 \in I_{k+1}^-.\)

Denoting

\[\gamma_k := \frac{\gamma}{\gamma} (\gamma - \kappa) - \sum_{n \in I_k \setminus \{n_0\}} (\beta - \alpha, \kappa) D_{k,n} - (\beta - \alpha, \kappa) C,\]

we have by Lemma 2.1, (2.2), and (2.1)

\[-(T_{k,n_0}^+(C)f, f) \geq \int_B \frac{\gamma_k}{\gamma} (\gamma - \kappa)|\nabla u_i|^2 \, dx + \int_{U \setminus B} \frac{\gamma_k}{\gamma} (\gamma - \kappa)|\nabla u_i|^2 \, dx + \int_{\Omega \setminus U} \gamma_k |\nabla u_i|^2 \, dx\]

\[\geq \frac{\alpha, \kappa, \beta, \kappa}{\alpha, \kappa, \beta, \kappa} \int_B |\nabla u_i|^2 + \inf(\gamma_k) \int_{\Omega \setminus U} |\nabla u_i|^2 \, dx \to \infty \text{ for } i \to \infty,\]

from which we conclude \(T_{k,n_0}^+(C) \not\leq 0.\)
Case (b). In this case we have \( \gamma = \alpha_{k,n_0} + c_{k+1,m_0} \) in \( B \) with \( c_{k+1,m_0} < 0 \) and \( \gamma \leq \gamma_k \) in \( U \).

Denote

\[
\tilde{\gamma} := \gamma - \gamma_k - \sum_{n \in I_{k-1}\setminus\{k\}} \frac{\alpha_{k,n}}{\beta_L} (\beta_L - \alpha_{k,n}) \chi_{D_k,n} - \frac{\alpha_{k,n_0}}{\beta_L} (\beta_L - \alpha_{k,n_0}) \chi_{C}.
\]

Applying the above construction of localized potentials satisfying (2.1), we contradict the inequality \( T_{k,m_0}^- \geq 0 \) using Lemma 2.1 and (2.2):

\[
\langle T_{k,m_0}^- (C)f_1, f_1 \rangle \leq \int_B (\gamma - \gamma_k) |\nabla u_i|^2 \, dx + \int_{U \setminus B} (\gamma - \gamma_k) |\nabla u_i|^2 \, dx + \int_{\Omega \setminus U} \tilde{\gamma} |\nabla u_i|^2 \, dx \leq c_{k+1,m_0} \int_B |\nabla u_i|^2 \, dx + \sup(\tilde{\gamma}) \int_{\Omega \setminus U} |\nabla u_i|^2 \, dx \to -\infty \text{ for } i \to \infty,
\]

hence concluding \( T_{k,m_0}^- (C) \not\geq 0 \).

The equality \( D_{k+1} \cap D_{k,m_0} = \cap \mathcal{M} \) with \( \mathcal{M} := \{ C \in A(D_{k,m_0}) \mid T_{k,m_0}^-(C) \geq 0 \} \) is satisfied via (2.5) since \( D_{k+1} \cap D_{k,m_0} \subseteq C \) for each \( C \in \mathcal{M} \) and that \( D_{k+1} \cap D_{k,m_0} \) itself is a member of \( \mathcal{M} \). \( \square \)

Now that Theorem 2.3 gives a way of determining \( D_{k+1} \) from \( \gamma_k \), the next step is to determine the constant \( c_{k+1,m_0} \) for each \( m_0 \in I_{k+1} \) in order to obtain \( \gamma_{k+1} \). For this purpose we define for \( m_0 \in I_{k+1} \), \( s \in [0, \beta_U - \dot{\alpha}_{k,m_0}] \), and \( t \in [\beta_L - \dot{\alpha}_{k,m_0}, 0] \) the operators

\[
S_{k,m_0}^+(s) := \Lambda(\gamma) - \Lambda(\gamma_{k,m_0}, \beta_U + s \chi_{F_{\gamma}(D_{k+1,m_0})}),
\]

\[
S_{k,m_0}^-(t) := \Lambda(\gamma_{k,m_0}, \beta_L + t \chi_{F_{\gamma}(D_{k+1,m_0})}) - \Lambda(\gamma),
\]

for which \( \gamma_{k,m_0,\beta} \) with \( \beta \in (\beta_L, \beta_U) \) is defined as

\[
\gamma_{k,m_0,\beta} := \gamma_k + \sum_{m \in I_{k+1}\setminus\{m_0\}} (\beta - \dot{\alpha}_{k,m}) \chi_{D_{k+1,m}} + (\beta - \dot{\alpha}_{k,m_0}) \chi_{H_{\gamma}(D_{k+1,m_0})}.
\]

Recall the definition of \( H_{\gamma} \) and \( F_{\gamma} \) in (1.2) and (1.3). As we shall see in Theorem 2.4, there are two equivalent ways of determining if \( m_0 \in I_{k+1} \) belongs to \( I_{k+1}^+ \) or \( I_{k+1}^- \). Afterwards, we may find the constant \( c_{k+1,m_0} \in [\beta_L - \dot{\alpha}_{k,m_0}, 0] \cup (0, \beta_U - \dot{\alpha}_{k,m_0}] \) via an optimization problem, by varying \( s \) and \( t \) on the outer \( \tau \)-layer of \( D_{k+1,m_0} \), constrained by positive semi-definiteness of \( S_{k,m_0}^\pm \).

**Theorem 2.4.** Let \( m_0 \in I_{k+1} \) then it holds

\[
[0, \beta_U - \dot{\alpha}_{k,m_0}] \ni s \geq c_{k+1,m_0} \quad \text{if and only if} \quad S_{k,m_0}^+(s) \geq 0, \quad \text{(2.6)}
\]

\[
[\beta_L - \dot{\alpha}_{k,m_0}, 0] \ni t \leq c_{k+1,m_0} \quad \text{if and only if} \quad S_{k,m_0}^-(t) \geq 0. \quad \text{(2.7)}
\]

As a direct consequence it holds

\[
m_0 \in I_{k+1}^+ \quad \text{if and only if} \quad S_{k,m_0}^+(0) \geq 0 \quad \text{if and only if} \quad S_{k,m_0}^+(0) \not\leq 0,
\]

\[
m_0 \in I_{k+1}^- \quad \text{if and only if} \quad S_{k,m_0}^-(0) \geq 0 \quad \text{if and only if} \quad S_{k,m_0}^-(0) \not\leq 0,
\]

and \( c_{k+1,m_0} \) is determined via:

\[
m_0 \in I_{k+1}^+ \quad \text{implies} \quad c_{k+1,m_0} = \inf \{ s \in (0, \beta_U - \dot{\alpha}_{k,m_0}] \mid S_{k,m_0}^+(s) \geq 0 \},
\]

\[
m_0 \in I_{k+1}^- \quad \text{implies} \quad c_{k+1,m_0} = \sup \{ t \in [\beta_L - \dot{\alpha}_{k,m_0}, 0] \mid S_{k,m_0}^-(t) \geq 0 \}.
\]

**Proof.** Note that \( \gamma = \dot{\alpha}_{k,m_0} + c_{k+1,m_0} \) in the set \( F_{\gamma}(D_{k+1,m_0}) \) due to Assumption 1.1(iii). Moreover, \( \gamma_k = \dot{\alpha}_{k,m_0} \) in \( F_{\gamma}(D_{k+1,m_0}) \), so writing

\[
\gamma - \gamma_{k,m_0,\beta_U} = (\gamma - \gamma_{k,m_0,\beta_U}) \chi_{\Omega \setminus F_{\gamma}(D_{k+1,m_0})} + (\gamma - \gamma_{k,m_0,\beta_U}) \chi_{F_{\gamma}(D_{k+1,m_0})}
\]

we may apply (2.3) to bound the first term from above by 0. Likewise for \( \gamma - \gamma_{k,m_0,\beta_L} \) we obtain a lower bound using (2.4), which results in

\[
\gamma - \gamma_{k,m_0,\beta_U} \leq C_{k+1,m_0} \chi_{F_{\gamma}(D_{k+1,m_0})} \leq \gamma - \gamma_{k,m_0,\beta_L}. \quad \text{(2.8)}
\]

We begin by proving (2.6), hence denote the piecewise analytic \( L_+^\infty(\Omega) \)-function

\[
\tilde{\gamma} := \gamma_{k,m_0,\beta_U} + s \chi_{F_{\gamma}(D_{k+1,m_0})},
\]
and assume \( s \geq c_{k+1, m_0} \). By virtue of Lemma 2.1 and (2.8)

\[
-S^+_{k,m_0}(s)f, f) \leq \int_{\Omega} \left[ \gamma - \gamma_{k,m_0, \beta_U} - s \chi_{F^c_r(D_{k+1, m_0})} \right] |\nabla u^\gamma_j|^2 \, dx
\]

\[
\leq (c_{k+1, m_0} - s) \int_{F_r(D_{k+1, m_0})} |\nabla u^\gamma|^2 \, dx \leq 0
\]

for all \( f \in L^2_0(\Gamma) \), i.e. \( S^+_{k,m_0}(s) \geq 0 \) for \( s \geq c_{k+1, m_0} \).

For the opposite implication we assume \( s < c_{k+1, m_0} \). In a similar way to the proof of Theorem 2.3, we pick a relatively open connected set \( U \subset \Omega \), which intersects \( \Gamma \), has connected complement, satisfies \((D_{k+1} \setminus D_{k+1, m_0}) \cap U = H_r(D_{k+1, m_0}) \cap U = \emptyset \), and \( F_r(D_{k+1, m_0}) \cap U \) contains an open ball \( B \). Once again this is possible due to Assumption 1.1. Hence \( \gamma - \tilde{\gamma} = c_{k+1, m_0} - s > 0 \) in \( B \) and \( \gamma \geq \tilde{\gamma} \) in \( U \).

Now let \((f_i) \subset L^2_0(\Gamma) \) and \((u_i) \subset H^1_0(\Omega) \) be chosen via Lemma 2.2 with respect to the sets \( U \) and \( B \) for the conductivity \( \tilde{\gamma} \). Lemma 2.1 gives

\[
-S^+_{k,m_0}(s)f_i, f_i) \geq \int_{\Omega} \frac{\tilde{\gamma}}{\gamma} (\gamma - \tilde{\gamma}) |\nabla u_{i}^{\tilde{\gamma}}|^2 \, dx
\]

\[
\geq \frac{\alpha_{k,m_0} + s}{\alpha_{k,m_0} + c_{k+1, m_0} - s} \int_B |\nabla u_i|^2 \, dx + \max(\frac{\gamma}{\gamma - \tilde{\gamma}})(\gamma - \tilde{\gamma}) \int_{\Omega \setminus U} |\nabla u_i|^2 \, dx.
\]

Since \( 0 \leq s < c_{k+1, m_0} \) then (2.1) implies \( \lim_{i \to \infty} (S^+_{k,m_0}(s)f_i, f_i) = -\infty \). We conclude \( S^+_{k,m_0}(s) \not\geq 0 \) for \( s < c_{k+1, m_0} \).

Next we prove (2.7) in an analogous way. Denote the piecewise analytic \( L^\infty_\beta(\Omega) \)-function

\[
\tilde{\gamma} := \gamma_{k,m_0, \beta_L} + t \chi_{F^c_r(D_{k+1, m_0})}
\]

First we assume \( t \leq c_{k+1, m_0} \), and since \( t \in [\beta_L - \alpha_{k,m_0}, 0] \) it holds \( \tilde{\gamma} \geq \frac{\beta_L}{\beta_U} \) in \( \Omega \). Thus from Lemma 2.1 and (2.8) it holds

\[
-S^+_{k,m_0}(t)f, f) \geq \int_{\Omega} \frac{\tilde{\gamma}}{\gamma} (\gamma - \tilde{\gamma}) |\nabla u_{i}^{\tilde{\gamma}}|^2 \, dx \geq \frac{\beta_L}{\beta_U} (c_{k+1, m_0} - t) \int_{F_r(D_{k+1, m_0})} |\nabla u_i|^2 \, dx \geq 0
\]

for all \( f \in L^2_0(\Gamma) \), i.e. \( S^+_{k,m_0}(t) \geq 0 \) for \( t \leq c_{k+1, m_0} \).

For the opposite implication we assume \( t > c_{k+1, m_0} \) and pick the sets \( U \) and \( B \) in exactly the same way as in the proof of (2.6). In particular, \( \gamma - \tilde{\gamma} = c_{k+1, m_0} - t < 0 \) in \( B \) and \( \gamma \geq \tilde{\gamma} \) in \( U \).

Now let \((f_i) \subset L^2_0(\Gamma) \) and \((u_i) \subset H^1_0(\Omega) \) be chosen according to Lemma 2.2 for the sets \( U \) and \( B \) and with conductivity \( \tilde{\gamma} \).

Applying Lemma 2.1 and (2.1) yields

\[
-S^-_{k,m_0}(t)f_i, f_i) \leq \int_{\Omega} (\gamma - \tilde{\gamma}) |\nabla u_{i}^{\tilde{\gamma}}|^2 \, dx
\]

\[
\leq (c_{k+1, m_0} - t) \int_B |\nabla u_i|^2 \, dx + \max(\gamma - \tilde{\gamma})(\gamma - \tilde{\gamma}) \int_{\Omega \setminus U} |\nabla u_i|^2 \, dx \to -\infty \text{ as } i \to \infty,
\]

whence \( S^-_{k,m_0}(t) \not\leq 0 \) for \( t > c_{k+1, m_0} \). \( \square \)

Remark 2.5. Based on the proofs of Theorem 2.3 and Theorem 2.4, it is straightforward to show that the conclusion of whether \( m_0 \in I_{k+1} \) belongs to \( I_{k+1}^+ \) or \( I_{k+1}^- \) in Theorem 2.4 is preserved when replacing \( S^+_{k,m_0}(0) \) with \( \tilde{S}^+_{k,m_0} \) defined below, where \( \tilde{D} := H_r(D_{k+1, m_0}) \):

\[
\tilde{S}^+_{k,m_0} := \Lambda(\gamma) - \Lambda(\gamma_k) - \sum_{m \in I_{k+1} \setminus \{m_0\}} (\beta_U - \alpha_{k,m}) \Delta(\gamma_k; \chi_{D_{k+1, m}}) - (\beta_U - \alpha_{k,m_0}) \Delta(\gamma_k; \chi_{\tilde{D}}),
\]

\[
\tilde{S}^-_{k,m_0} := \Lambda(\gamma_k) - \Lambda(\gamma) + \sum_{m \in I_{k+1} \setminus \{m_0\}} (\beta_L - \alpha_{k,m}) \Delta(\gamma_k; \chi_{D_{k+1, m}}) + (\beta_L - \alpha_{k,m_0}) \Delta(\gamma_k; \chi_{\tilde{D}}).
\]

It is tempting to also use \( DA \) to apply the variation of \( s \) and \( t \) on \( F_r(D_{k+1, m_0}) \) in Theorem 2.4. However, the set \( U \) for the localized potentials will intersect part of the set on which \( DA \) is applied
(unlike in the proof of Theorem 2.3, where this is specifically avoided), and the resulting integrals do not lead to a proof of the desired assertion.

2.1. The reconstruction method. We can now summarize the reconstruction method based on Theorem 2.3 and Theorem 2.4 in the following way:

1. Let $\gamma_k$ for some $k \in \{0, 1, \ldots, N\}$ be given (initially $\gamma_0 = c_0$ with $D_0 := \overline{\Omega}$).
2. Determine $D_{k+1}$ via: for each $m_0 \in I_{k}$ we find $D_{k+1} \cap D_{k,m_0} = \cap \{ C \in \mathcal{A}(D_{k,m_0}) | T^+_k(C) \geq 0 \}$.
3. For each $m_0 \in I_{k+1}$ we employ Theorem 2.4/Remark 2.5 to determine if $m_0 \in I_{k+1}^+$ or $m_0 \in I_{k+1}^-$ by the positive semi-definiteness (or lack thereof) of either $S^+_k(m_0)$, $S^-_k(m_0)$, or $\hat{S}^+_k(m_0)$.
4. For $m_0 \in I_{k+1}^+$ we find $c_{k+1,m_0}$ via Theorem 2.4 as $c_{k+1,m_0} = \inf \{ s \in (0, \beta_U - \hat{\alpha}_{k,m_0} ) | S^+_k(m_0)(s) \geq 0 \}$.
   and for $m_0 \in I_{k+1}^-$ we find $c_{k+1,m_0}$ as $c_{k+1,m_0} = \sup \{ t \in [\beta_L - \hat{\alpha}_{k,m_0}, 0] | S^-_k(m_0)(t) \geq 0 \}$.
5. The above steps determine $\gamma_{k+1}$. Repeat the above steps iteratively, until we reach $\gamma_{N+1} = \gamma_N$ by finding $D_{N+1} = \emptyset$ in step (2), which concludes the reconstruction method.

Remark 2.6. Note that numerical implementation of step (2) above can be handled, both in terms of regularization theory and practical implementation, via a layer peeling approach (14, Theorem 3.1 and Algorithm 1). For other considerations in this direction see also (13, 23, 12). Step (4) can be handled straightforwardly via bisection due to (2.6) and (2.7) in Theorem 2.4.

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