Almost primes in various settings

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Abstract
Let \( k \geq 3 \) and let \( L_i(n) = A_i n + B_i \) be some linear forms such that \( A_i \) and \( B_i \) are integers. Define \( P(n) = \prod_{i=1}^{k} L_i(n) \). For each \( k \) it is known that \( \Omega(P(n)) \leq \rho_k \) infinitely often for some integer \( \rho_k \). We improve the possible values of \( \rho_k \) for \( 4 \leq k \leq 10 \) assuming the generalized Elliott–Halberstam conjecture. We also show that we can take \( \rho_5 = 14 \) unconditionally. As a by-product of our approach we reprove the \( \rho_3 = 7 \) result which was previously obtained by Maynard who used techniques specifically designed for this case.

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1 INTRODUCTION

1.1 State of the art

One of the most famous problems concerning prime numbers is the so called twin prime conjecture, which can be stated in the following form.

Conjecture 1.1. There are infinitely many primes \( p \) such that \( p + 2 \) is also a prime.

This statement seems to be unachievable by current techniques and can be considered as a major target of studies in analytic number theory. In [3], Chen proved the following result which can be seen as a weak variation of Conjecture 1.1.

Theorem 1.2. There are infinitely many primes \( p \) such that \( p + 2 \) has at most two prime factors, each greater than \( p^{1/10} \).
Unfortunately, neither techniques developed in order to prove Theorem 1.2 nor any other like the circle method are able to find infinitely many twin primes.

Let \( p_n \) denote the \( n \)th consecutive prime number. Goldston et al. in [6] proved the following revolutionary result.

**Theorem 1.3.** We have

\[
\liminf_{n \to \infty} (p_{n+1} - p_n) \leq 16
\]

under the Elliott–Halberstam conjecture (cf. Conjecture 1.5) and

\[
\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log p_n} = 0
\]

unconditionally.

In [6], it was also proven that \( EH[\frac{1}{2} + \eta] \) for any \( \eta > 0 \) implies

\[
\liminf_{n \to \infty} (p_{n+1} - p_n) \leq C,
\]

for some explicit constant \( C \) depending on \( \eta \). On the other hand, proving \( EH[\frac{1}{2} + \eta] \) seems to be unreachable by currently known techniques.

The next giant step towards twin primes came with the work of Zhang [18]. He showed that the inequality (1.1) is unconditionally true for \( C = 70 000 000 \). The main idea was to get around the troubles with the Elliott–Halberstam conjecture by restricting attention only to sufficiently smooth numbers (i.e., numbers \( n \) of the same order of magnitude as \( x \) having only prime divisors smaller than \( x^\delta \) for some small, fixed \( \delta \) and some large \( x \)) and then, by going beyond the 1/2 limit induced by the Bombieri–Vinogradov theorem in this special case. Such a strategy was also proposed in [14].

The result of Zhang was later dramatically improved by Maynard in [12], where he was able to take \( C = 600 \). For \( i = 1, \ldots, k \) put \( L_i(n) = A_in + B_i \), where \( A_i \in \mathbb{N}, B_i \in \mathbb{Z} \) and let \( H = \{L_1, \ldots, L_k\} \). Define

\[
P(n) = \prod_{i=1}^{k} L_i(n),
\]

\[
\nu_p(H) = \#\{1 \leq n \leq p : P(n) \equiv 0 \text{ mod } p\}.
\]

We call a tuple \( H \) admissible if \( \nu_p(H) < p \) for all primes and each two of the \( L_i \) are distinct. We can also assume that each of the coefficients \( A_i \) is composed of the same primes, none of which divides any of the \( B_i \) (which can be done without loss of generality due to the nature of the problems studied here; cf. Conjecture 1.10, Main Theorems 1 and 2). Maynard’s idea was to use a multidimensional variant of the Selberg sieve, that is, he considered the expression

\[
\sum_{N < n \leq 2N} \left( \sum_{i=1}^{k} 1_{p(n + h_i)} - 1 \right) \left( \sum_{d_1, \ldots, d_k} \lambda_{d_1, \ldots, d_k} \prod_{i} d_i \mid L_i(n) \right)^2
\]
for \( 1_p \) being the indicator function of primes, an admissible tuple \( \{n + h_1, \ldots, n + h_k\} \) and the weights \( \lambda_{d_1, \ldots, d_k} \) suitably chosen to make the whole sum greater than 0 for sufficiently large \( N \). In the same paper, Maynard also proved that

\[
\liminf_{n \to \infty} (p_{n+m} - p_n) < \infty
\]

for every positive integer \( m \). Surprisingly, his proof of these two facts does not rely on any distributional claims stronger than the Bombieri–Vinogradov theorem.

Methods developed by Maynard were further investigated and up to this moment the best achievements concerning small gaps between primes were obtained in the Polymath Project [15].

**Theorem 1.4.** We have

1. \( \liminf_{n \to \infty} (p_{n+1} - p_n) \leq 246 \) unconditionally,
2. \( \liminf_{n \to \infty} (p_{n+1} - p_n) \leq 12 \) on \( EH \),
3. \( \liminf_{n \to \infty} (p_{n+1} - p_n) \leq 6 \) on generalized Elliot–Halberstam (GEH) conjecture (cf. Conjecture 1.7).

The second statement is actually due to Maynard [12]. There is a hope that some advances on the numerics will allow us to change 12 into 8 in the nearest future. The third result can be considered to be the most interesting because it is already shown that it reaches a limit of what is potentially provable by the sieve techniques due to the parity problem (see [15] for details).

### 1.2 Distributional claims on arithmetic functions

The question how well the prime numbers are distributed among arithmetic progressions is one of the major problems in analytic number theory. We state a famous conjecture which seems to be true due to heuristic reasoning.

**Conjecture 1.5** (Elliott–Halberstam conjecture). Let \( \theta \in (0, 1) \), \( A \geq 1 \) be fixed. For every \( Q \ll x^\theta \), we have

\[
\sum_{q \leq Q} \max_{(a, q) = 1} \left| \pi(x; a, q) - \frac{\pi(x)}{\varphi(q)} \right| \ll x \log^{-A} x
\]

(cf. Subsection 1.4 for the notation).

We will refer to this conjecture for some specific exponent \( \theta \) by \( EH[\theta] \). The best known result of this kind is due to Bombieri and Vinogradov.

**Theorem 1.6.** \( EH[\theta] \) holds for every \( \theta \in (0, 1/2) \).

One can view the Bombieri–Vinogradov theorem as a “generalized Riemann hypothesis on average”, which works as a very powerful substitute in the sieve-theoretical context.

There exists also a conjecture much more general than \( EH \). It asserts that not only the prime counting function \( \pi \) or von Mangoldt function \( \Lambda \) but all functions equipped with sufficiently
strong bilinear structure and having a good correlation with arithmetic sequences do satisfy the Elliott–Halberstam conjecture.

**Conjecture 1.7** (Generalized Elliott–Halberstam conjecture). Let \( \theta \in (0, 1), \varepsilon > 0, A \geq 1 \) be fixed. Let \( N, M \) be quantities such that \( x^{\varepsilon} \ll N, M \ll x^{1-\varepsilon} \) with \( x \ll NM \ll x \), and let \( \alpha, \beta : \mathbb{N} \to \mathbb{R} \) be sequences supported on \([N, 2N]\) and \([M, 2M]\), respectively, such that one has the pointwise bound

\[
|\alpha(n)| \ll \tau(n)o(1) \log o(1) x; \quad |\beta(m)| \ll \tau(m)o(1) \log o(1) x
\]

for all natural numbers \( n, m \). Suppose also that \( \beta \) obeys the Siegel–Walfisz type bound

\[
\left| \sum_{(n, r)=1} \beta(n) - \frac{1}{\varphi(q)} \sum_{(n, qr)=1} \beta(n) \right| \ll \tau(qr)o(1)M \log^{-A} x
\]

for any \( q, r \geq 1 \), any fixed \( A \), fixed \( C \geq 0 \), and any primitive residue class \( a \) (\( q \)). Then for any \( Q \ll x^{\theta} \), we have

\[
\sum_{q \leq Q} \tau(q)^C \max_{y \leq x} \max_{(a,q)=1} \left| \sum_{n \leq y} (\alpha * \beta)(n) - \frac{1}{\varphi(q)} \sum_{n \leq y} \alpha(n) \beta(n) \right| \ll x \log^{-A} x.
\]

The broad generalization of this kind first appeared in [2]. The best known result in this direction is currently proven by Motohashi [13].

**Theorem 1.8.** \( GEH[\theta] \) holds for every \( \theta \in (0, 1/2) \).

It is possible to get \( EH[\theta] \) easily from \( GEH[\theta] \) by Vaughan’s identity as shown in [15]. The importance of \( GEH[\theta] \) stands on the fact that it allows us to obtain important results of \( EH \) type concerning almost primes as in the following result almost completely analogous to what is proven in [15, Subsection ‘The generalized Elliott–Halberstam case’].

**Theorem 1.9.** Assume \( GEH[\theta] \). Let \( r \geq 1, \varepsilon > 0 \) and \( A \geq 1 \) be fixed, let

\[
\Delta_{r, \varepsilon} = \{(t_1, \ldots, t_r) \in [\varepsilon, 1]^r : t_1 \leq \cdots \leq t_r; t_1 + \cdots + t_r = 1\},
\]

and let \( F : \Delta_{r, \varepsilon} \to \mathbb{R} \) be a fixed smooth function. Let \( \overline{F} : \mathbb{N} \to \mathbb{R} \) be the function defined by setting

\[
\overline{F}(n) = F\left(\frac{\log p_1}{\log n}, \ldots, \frac{\log p_r}{\log n}\right)
\]

whenever \( n = p_1 \cdots p_r \) is the product of \( r \) distinct primes \( p_1 < \cdots < p_r \) with \( p_1 \geq x^{\varepsilon} \) for some fixed \( \varepsilon > 0 \), fixed \( C \geq 0 \), and \( \overline{F}(n) = 0 \) otherwise. Then for every \( Q \ll x^{\theta} \), we have

\[
\sum_{q \leq Q} \tau(q)^C \max_{y \leq x} \max_{(a,q)=1} \left| \sum_{n \leq x} \overline{F}(n) - \frac{1}{\varphi(q)} \sum_{n \leq x} \overline{F}(n) \right| \ll x \log^{-A} x.
\]
This work is focused on almost primes, so that is the reason, why in Main Theorem 2 we have results achieved under GEH, but not under EH.

1.3 The main result and key ingredients of its proof

The twin prime conjecture is equivalent to

$$\liminf_{n \to \infty} \Omega(n(n+2)) = 2.$$  

The above statement with 3 instead of 2 is a slightly weaker version of Chen’s theorem. We can also formulate the Dickson–Hardy–Littlewood $k$-tuple conjecture this way.

Conjecture 1.10. For every admissible $k$-tuple $H$, we have

$$\liminf_{n \to \infty} \Omega(P(n)) = k,$$

where $P(n)$ is defined as in (1.2).

Maynard [10, 11] proved that

$$\liminf_{n \to \infty} \Omega(P(n)) \leq \rho_k,$$

where $\rho_k$ are given as in Table A.

It was an improvement upon the results of Diamond and Halberstam [4] for $3 \leq k \leq 6$, and Ho and Tsang [8] for $7 \leq k \leq 10$. The proof employed the following equality valid for any $n$ square-free and any $y \geq n^{1/2}$:

$$\Omega(n) = \sum_{p \mid n, p \leq y} \left( 1 - \frac{\log p}{\log y} \right) + \frac{\log n}{\log y} + \sum_{r=1}^{\infty} \chi_r(n),$$

where

$$\chi_r(n) = \begin{cases} -\left( \frac{\log n}{\log y} - 1 - \sum_{i=1}^{r-1} \frac{\log p_i}{\log y} \right), & \text{if } n = p_1 \ldots p_r \text{ with } p_1 < \cdots < p_r \leq y < p_r, \\ 0, & \text{otherwise}. \end{cases}$$

The idea was to consider the sum

$$\sum_{n \leq 2N, \Omega(n) \leq c} \text{weight}(n),$$

Table A

| $k$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|----|
| $\rho_k$ | 7 | 11 | 15 | 18 | 22 | 26 | 30 | 34 |
with the classical Selberg sieve in the place of weights and use the identity (1.4) to prove that (1.5) is positive for sufficiently large $N$ and $c$ suitably chosen.

The objective of the present work is to improve upon the result of Maynard in the case $k = 5$.

**Main Theorem 1.** *Given an admissible 5-tuple $\mathcal{H}$ the inequality (1.3) holds with $\rho_5 = 14$.*

As a by-product of our methods we are also able to give an alternative proof of Maynard’s result for $k = 3$ (it is worth mentioning that previously this case required a specifically devised approach using the Diamond–Halberstam sieve).

We can also draw stronger conclusions for $4 \leq k \leq 10$ assuming $GEH$.

**Main Theorem 2.** *Assuming GEH[2/3], for any admissible $k$-tuple $\mathcal{H}$, the same inequality holds with $\rho_k$ given in Table B.*

Motivated by Maynard’s successful proof concerning small gaps between consecutive primes, we apply the multidimensional sieve to the problem of $k$-tuples of almost primes. The main difficulty in this approach is that the variation of the sieve proposed by Maynard in [12] combined with techniques developed in [11] is not strong enough for Main Theorems 1 and 2 to be proven.

The main parameters for our set-up are denoted $\vartheta$ and $\vartheta_0$. The former is related to the level of distribution in $GEH[\vartheta]$ by $\vartheta = 2\vartheta$. The latter can be viewed as related to $y$ from the identity (1.6) by $N^{\vartheta_0} \approx y$. To produce non-trivial results relying on $GEH$, we need to work with the parameter $\vartheta$ greater than $1/4$. Then, the constraint $\vartheta_0 + 2\vartheta < 1$ in Proposition 1.12 forces us to take $\vartheta_0 < 1/2$. As a result, we need a variation of identity (1.4) which works also for $0 \leq y < n^{1/2}$. We can use the following simple equation

$$1 = \sum_{r=1}^{\infty} \sum_{s=0}^{r} 1_{n=p_1 \ldots p_r} \text{ for some } p_1 < \cdots < p_{r-s} \leq y < p_{r-s+1} < \cdots < p_r,$$

valid for square-free $n$ and get

$$\Omega(n) = \sum_{p|n \atop p \leq y} \left( 1 - \frac{\log p}{\log y} \right) + \frac{\log n}{\log y} + \sum_{r=1}^{\infty} \sum_{s=1}^{r} \chi_{r,s}(n), \quad (1.6)$$

where

$$\chi_{r,s}(n) = \begin{cases} -\frac{\log n}{\log y} - s - \sum_{i=1}^{r-s} \frac{\log p_i}{\log y}, & \text{if } n = p_1 \ldots p_r \text{ with } \quad p_1 < \cdots < p_{r-s} \leq y < p_{r-s+1} < \cdots < p_r, \\ 0, & \text{otherwise.} \end{cases}$$

### Table B

| $k$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|----|
| $\rho_k$ | 10 | 13 | 17 | 20 | 24 | 28 | 32 |
With this modification, we can use Propositions 1.12–1.15 with $\varepsilon_0$ arbitrarily small and the final results will not be damaged in any way. In [11] the author was forced to take $\varepsilon_0$ not smaller than $1/2$ (in his work it was $r_1$ instead of $\varepsilon_0$ and $r_2$ instead of $\varepsilon$) and also $\varepsilon$ not greater than $1/4$ which strictly blocked any possibility of using GEH to amplify the existing results.

The unconditional result, Main Theorem 1, is even harder. The reason for that is well described by Conjecture 2.2, which is supported by some numerical experiments, and the calculations mentioned in Table F. We can enhance our sieve by expanding its support as described in (1.7). This idea made its debut in [15]. This procedure does not allow us to take $\varepsilon_0$ as large as $1/2$ any longer, but thanks to the identity (1.6) this shall not be a major obstacle.

The general strategy of our proof is fairly simple. We consider the sum from (1.5) and just like Maynard we wish to prove that it is greater than 0 for sufficiently large $N$. We need some function $\text{weight}(n)$ which gives preference to the “good candidates” for almost primes. The multidimensional Selberg sieve used in this role allows us to transform the problem about $k$-tuples of almost primes into estimating four sums which can be viewed as multidimensional analogues of sums (5.8)–(5.11) from [11]. Then, we have to recover the results on sums $T_\delta$ and $T_\delta^*$, playing the crucial role in [11], in the multidimensional context. Appropriate lemmas are described in Section 2. In the end, we are left with an optimization problem depending on various complicated integrals, which is studied in the last section.

### 1.4 Notation

The letter $p$ always denotes a prime number and log denotes the natural logarithm. We consider $N$ as a number close to infinity. To avoid any problems with the domains of logarithmic functions, we assume that $N > 16$. We also use the notation $\mathbb{N} = \{1, 2, 3, \ldots\}$. By $\mathcal{H}$ we always denote an admissible $k$-tuple. The letter $k$ always denotes a positive integer greater than or equal to 3.

We use the following functions which are common in analytic number theory:

- $\varphi(n) := |(\mathbb{Z}/n\mathbb{Z})^\times|$ denotes Euler totient function;
- $\tau(n) := \sum_{d|n} 1$ denotes the divisor function;
- $\Omega(n)$ denotes the number of prime factors of $n$;
- $\pi(x) := \{n \in \mathbb{N} : n \leq x, n \text{ is prime}\}$;
- $\pi(x; q, a) := \{n \in \mathbb{N} : n \leq x, n \equiv a \text{ mod } q, n \text{ is prime}\}$;
- $(n_1, \ldots, n_r)$ and $[n_1, \ldots, n_r]$ denote the greatest common divisor and the lowest common multiple, respectively;
- For a logical formula $\phi$ we define the indicator function $1_{\phi(x)}$ which equals 1 when $\phi(x)$ is true and 0 otherwise;
- For two arithmetic functions $\alpha, \beta : \mathbb{N} \rightarrow \mathbb{C}$ we define their Dirichlet convolution by the formula $(\alpha * \beta)(n) = \sum_{d|n} \alpha(d)\beta(n/d)$.

We use the “big $O$” and the “small $o$” notation. The formula $f = O(g)$ or $f \ll g$ means that there exists a constant $C > 0$ such that $|f(x)| \leq C g(x)$ on the domain of $f$. In the second case, $f = o(g)$ means simply $\lim_{x \to \infty} f(x)/g(x) = 0$. We see that the big $O$ makes sense also in the case when the considered functions are multivariate. In the “big $O$” or “$\ll$” notation the dependence on the variables $k, A_i, B_i, \varepsilon, \varepsilon_0$, the functions $W_0, W_r$, and any other parameters declared as fixed, will not be mentioned explicitly. The notation $O_\varepsilon(f(x))$ or $\ll_\varepsilon$ means that the considered constant depends on the variable in the lower index.
1.5  Constructing the sieve

To deal with some small problematic primes, we use a device called the $W$-trick. For some $D_0$ we put

$$W = \prod_{p < D_0} p$$

and we further demand $n$ to lie in some residue class $\nu_0 \mod W$ such that $(\mathcal{P}(\nu_0), W) = 1$. We take

$$D_0 = \log \log \log N$$

and without loss of generality we can assume that $N$ is so large that $D_0 > A_i B_j - A_j B_i$ and $D_0 > A_i$ for all $i, j = 1, \ldots, k$.

We construct the expanded multidimensional Selberg sieve in the following way:

$$\lambda_{d_1, \ldots, d_k} = 0 \quad \text{if} \quad (d, W) > 1 \quad \text{or} \quad \mu(d)^2 \neq 1 \quad \text{or} \quad \exists j \frac{d}{d_j} \geq R,$$  \hspace{1cm} (1.7)

where $d := \prod_{i=1}^k d_i$ and $R$ to be chosen later. In the remaining cases we may choose the weights arbitrarily.

The proper choice of the sieve weights should maximize the value of the sum (1.5). In the one-dimensional case the standard form of a weight is something similar to

$$\lambda_d \approx \mu(d)G\left(\frac{\log d}{\log R}\right)$$

for some smooth function $G$. In the multidimensional case we use weights of the form:

$$\lambda_{d_1, \ldots, d_k} \approx \left(\prod_{i=1}^k \mu(d_i)\right)G\left(\frac{\log d_1}{\log R}, \ldots, \frac{\log d_k}{\log R}\right).$$

The correlation with the Möbius function makes the sums containing sieve weights difficult to evaluate, since there are many positive terms and many negative ones, so we have a lot of cancelations to deal with. To overcome this obstacle we apply the reciprocity law which transfers the $\lambda_{d_1, \ldots, d_k}$ into a set of new variables which are positive. Specifically, in the multidimensional case, we use the following lemma from [7] which follows easily from the Möbius inversion.

**Lemma 1.11.** Suppose that $L(d)$ and $Y(r)$ are two sequences of complex numbers that are indexed by $d, r \in \mathbb{N}^k$ and supported on the $d_i$ and $r_i$ which are square-free, coprime to $W$ and satisfy $\prod_{i=1}^k d_i, \prod_{i=1}^k r_i < R$. Then

$$L(d_1, \ldots, d_k) = \prod_{i=1}^k \mu(d_i) \sum_{r_1, \ldots, r_k} Y(r_1, \ldots, r_k)$$

\forall d_i \mid r_i
if and only if

\[ Y(r_1, \ldots, r_k) = \prod_{i=1}^{k} \mu(r_i) \sum_{d_1, \ldots, d_k \atop \forall i \ r_i | d_i} L(d_1, \ldots, d_k). \]

This lemma lets us define the Selberg weights indirectly in a convenient way.

**Definition 1** (Selberg weights). We put

\[ \lambda_{d_1, \ldots, d_k} = \left( \prod_{i=1}^{k} \mu(d_i) d_i \right) \frac{y_{r_1, \ldots, r_k}}{\prod_{i=1}^{k} \varphi(r_i)} \]  

for some \( y_{r_1, \ldots, r_k} \) being non-negative, square-free, coprime to \( W \), and supported only on the \( r_i \) satisfying \( r/r_j < R \) for each \( j = 1, \ldots, k \), where \( r = \prod_{i=1}^{k} r_i \). By Lemma 1.11, the equality (1.8) is equivalent to

\[ y_{r_1, \ldots, r_k} = \left( \prod_{i=1}^{k} \mu(r_i) \varphi(r_i) \right) \sum_{d_1, \ldots, d_k \atop \forall i \ r_i | d_i} \lambda_{d_1, \ldots, d_k} \prod_{i=1}^{k} d_i. \]  

(1.9)

For Lemma 3.6 and further, we shall take

\[ y_{r_1, \ldots, r_k} = \begin{cases} F \left( \log \frac{r_1}{\log R}, \ldots, \log \frac{r_k}{\log R} \right) & \text{if } \mu(r)^2 = 1, \ (r, W) = 1, \ \text{and } \forall j \ r/r_j < R, \\ 0 & \text{otherwise}, \end{cases} \]  

(1.10)

where \( F \) is a nonzero function \( F : \mathbb{R}^k \to \mathbb{R}_{\geq 0} \), supported on

\[ \mathcal{R}_k' = \{(t_1, \ldots, t_k) \in [0, 1]^k : \sum_{i=1}^{k} t_i \leq 1 \text{ for each } j = 1, \ldots, k \} \]

and differentiable in the interior of this region.

**Remark 2.** We may also note that if the sieve weights \( \lambda_{d_1, \ldots, d_k} \) are supported on the \( d_i \) such that \( \prod_{i=1}^{k} d_i < R \), then by the reciprocity law it is true that \( \text{supp}(F) \subset \mathcal{R}_k \), where

\[ \mathcal{R}_k = \{(t_1, \ldots, t_k) \in [0, 1]^k : \sum_{i=1}^{k} t_i \leq 1 \}. \]

In such a case we assume only the differentiability of \( F \) inside the interior of \( \mathcal{R}_k \). We use this specific support while proving Main Theorem 2. It is less powerful than \( \mathcal{R}_k' \) (it is obvious, because it is strictly smaller) but leads to much simpler calculations.
It is also worth mentioning that this is also the support which was originally used by Maynard in [12], where the multidimensional Selberg sieve appeared for the first time. We also define

$$\lambda_{\text{max}} = \sup_{d_1, \ldots, d_k} |\lambda_{d_1, \ldots, d_k}|, \quad y_{\text{max}} = \sup_{r_1, \ldots, r_k} |y_{r_1, \ldots, r_k}|,$$

(1.11)

$$F_{\text{max}} = \sup_{(t_1, \ldots, t_k) \in [0,1]^k} \left( |F(t_1, \ldots, t_k)| + \sum_{i=1}^k \left| \frac{\partial F}{\partial t_i}(t_1, \ldots, t_k) \right| \right).$$

1.6 | Sketch of the proof

We start from

$$S = S(\sigma; N, R_0, R, \mathcal{H}, \nu_0) = \sum_{N < n \leq 2N} \left( \sigma - \sum_{p \in \mathcal{P}(n)} \left( 1 - \frac{\log p}{\log R_0} \right) \Lambda_{\text{Sel}}^2(n) \right),$$

(1.12)

where

$$\Lambda_{\text{Sel}}^2(n) = \left( \sum_{d_1, \ldots, d_k \forall i \mid d_i \mid L_i(n)} \lambda_{d_1, \ldots, d_k} \right)^2$$

and we choose some \(\nu_0\) coprime to \(W\). We also choose some positive constants \(\delta\) and \(\delta_0\) to be fixed later and put

$$R_0 = N^{\delta_0}, \quad R = N^{\delta}.$$ 

We note that for \(P\) square-free we have

$$\sum_{p \mid P(n)} \left( 1 - \frac{\log p}{\log R_0} \right) = \Omega(P(n)) - \frac{\log P(n)}{\log R_0}.$$  

(1.13)

Now, we follow the reasoning of Maynard described in equations [11, (5.4)–(5.11)]. Note that the precise shape of the sieve, the \(W\)-trick, and the usage of identity (1.6) do not affect these equations so we may just rewrite the results with our choice of sieve weights:

$$S \geq \sigma S_0 - S' - T_0 + \sum_{j=1}^{k} \sum_{r=1}^{h} \sum_{s=1}^{r} \tau^{(j)}_{r,s},$$

(1.14)

for any \(h \in \mathbb{N}\), where

$$S_0 = \sum_{N < n \leq 2N} \Lambda_{\text{Sel}}^2(n),$$

with

$$\Lambda^2_{\text{Sel}}(n) = \left( \sum_{d_1, \ldots, d_k \forall i \mid d_i \mid L_i(n)} \lambda_{d_1, \ldots, d_k} \right)^2.$$
\[ S' = \sum_{\substack{N<n\leq 2N \\mu(P(n))^2 \neq 1}} \left( \sigma - \sum_{p \mid P(n)} \left( 1 - \frac{\log p}{\log R_0} \right) \right) \Lambda^2_{\text{Sel}}(n), \]

\[ T_0 = \sum_{\substack{N<n\leq 2N \\mu(P(n)) \neq 1}} \sum_{p \mid P(n), \ p \leq R_0} \left( 1 - \frac{\log p}{\log R_0} \right) \Lambda^2_{\text{Sel}}(n), \]

\[ T_{r,s}^{(j)} = \sum_{\substack{N<n\leq 2N \\mu(P(n)) \neq 1}} \chi_{r,s}(L_j(n)) \Lambda^2_{\text{Sel}}(n), \quad (1.15) \]

and

\[ \chi_{r,s}(n) = \begin{cases} 
\frac{s - \log N}{\log R_0} + \sum_{i=1}^{r-s} \frac{\log p_i}{\log R_0}, & \text{if } n = p_1 \ldots p_r \text{ with } \varepsilon p_1 < \ldots < p_{r-s} \leq n_0 < p_{r-s+1} < \ldots < p_r, \\
0, & \text{otherwise.} 
\end{cases} \quad (1.16) \]

To calculate \( S_0, S', T_0 \) and \( T_{r,s}^{(j)} \) we need multidimensional analogues of Propositions 5.1–5.4 from [11].

**Proposition 1.12** Analogue of Proposition 5.1 from [11]. Let \( W_0 : [0, \frac{\delta_0}{\delta}] \to \mathbb{R}_{\geq 0} \) be a piecewise smooth non-negative function. Assume one of the following hypotheses:

1. \( \delta_0 + 2 \delta < 1 \) and \( \text{supp} \, (F) \subset \mathcal{R}_k, \)
2. \( \delta_0 + \frac{2k}{k-1} \delta < 1 \) and \( \text{supp} \, (F) \subset \mathcal{R}'_k, \)

Then, for every \( 0 < \varepsilon < \frac{\delta_0}{\delta} \), we have

\[ \Sigma_0 = \sum_{\substack{N<n\leq 2N \\mu(P(n)) \neq 1}} \left( \sum_{p \mid P(n), \ p \leq R_0} \left( \frac{\log p}{\log R_0} \right) \right) \Lambda^2_{\text{Sel}}(n) \]

\[ = \frac{\varphi(W)kN(\log R)^k}{W^{k+1}} \sum_{j=1}^k \frac{I_0^{(j)}}{W^{k+1}} + O \left( \frac{R_0^2 \max \varphi(W)kN(\log R)^k(\varepsilon + |\log \varepsilon|)}{\delta_0^2} \right), \]

where

\[ I_0^{(j)} = \int_{\varepsilon}^{\delta_0/\delta} \frac{W_0(y)}{y} I_0^{(j)}(y) \, dy \]

and

\[ I_0^{(j)}(y) = \int_{0}^{1} \ldots \int_{0}^{1} \left( F(t_1, \ldots, t_k) - F(t_1, \ldots, t_{i-1}, t_j + y, t_{i+1}, \ldots, t_k) \right)^2 \, dt_1 \ldots \, dt_k. \]
Proposition 1.13  Analogue of Proposition 5.2 from [11]. Given $\epsilon > 0$ and $r \in \mathbb{N}$, let

$$A_r := \left\{ x \in [0,1]^{r-1} : \epsilon < x_1 < \cdots < x_{r-1}, \sum_{i=1}^{r-1} x_i < \min(1 - \vartheta, 1 - x_{r-1}) \right\}.$$  

Let $W_r : [0,1]^{r-1} \to \mathbb{R}_{\geq 0}$ be a piecewise smooth function supported on $A_r$, such that

$$\frac{\partial}{\partial x_j} W_r(x) \ll W_r(x) \quad \text{uniformly for } x \in A_r.$$

Let

$$\beta_r(n) = \begin{cases} W_r\left(\frac{\log p_1}{\log n}, \ldots, \frac{\log p_{r-1}}{\log n}\right) & \text{for } n = p_1 \cdots p_r, \text{ with } p_1 \leq \cdots \leq p_r, \\
0 & \text{otherwise.} \end{cases}$$

Assume GEH[$2\vartheta$]. We have

$$\sum_{N < n \leq 2N \atop n \equiv \nu_0 \mod W} \beta_r(L_j(n)) \Lambda_{\text{Sel}}^2(n) = \frac{\varphi(W)^k N (\log R)^{k+1}}{W^{k+1} \log N} J_r^{(j)} + O_{\epsilon} \left( \frac{F_{\max} \varphi(W)^k N (\log R)^k}{W^{k+1} D_0} \right),$$

where

$$J_r^{(j)} = \int_{(x_1, \ldots, x_{r-1}) \in A_r} \frac{W_r(x_1, \ldots, x_{r-1}) I_r^{(j)}(x_1 \vartheta^{-1}, \ldots, x_{r-1} \vartheta^{-1})}{\prod_{i=1}^{r-1} x_i \left( 1 - \sum_{i=1}^{r-1} x_i \right)} \, dx_1 \cdots dx_{r-1},$$

$$I_r^{(j)}(x_1, \ldots, x_{r-1}) = \int_0^1 \cdots \int_0^1 I_r^{(j)}(x_1, \ldots, x_{r-1})^2 \, dt_1 \cdots dt_{j-1} \, dt_j \, dt_{j+1} \cdots dt_k,$$

$$\overline{I}_r^{(j)}(x_1, \ldots, x_{r-1}) = \int_0^1 \sum_{J \subseteq \{1, \ldots, r-1\}} (-1)^{|J|} F \left( t_1, \ldots, t_{j-1}, t_j + \sum_{i \in J} x_i, t_{j+1}, \ldots, t_k \right) \, dt_j.$$

Proposition 1.14  Analogue of Proposition 5.3 from [11]. Assume one of the following hypotheses:

1. $\vartheta < \frac{1}{2} - \eta$ and $\text{supp} (F) \subset R_k$
2. $\vartheta < \frac{k-1}{2k} - \eta$ and $\text{supp} (F) \subset R'_k$

for some positive $\eta$. Then, we have

$$\sum_{N < n \leq 2N \atop n \equiv \nu_0 \mod W} \Lambda_{\text{Sel}}^2(n) \ll_{\eta} \frac{F_{\max} \varphi(W)^k N (\log R)^k}{W^{k+1} D_0}.$$

The first case of the next result is a part of [12, Proposition 4.1]. The proof of the second case is analogous.
Proposition 1.15 (Analogue of Proposition 5.4 from [11]). Assume one of the following hypotheses:

1. \( \vartheta < \frac{1}{2} - \eta \) and \( \text{supp}(F) \subset \mathcal{R}_k \)
2. \( \vartheta < \frac{k-1}{2k} - \eta \) and \( \text{supp}(F) \subset \mathcal{R}'_k \)

for some positive \( \eta \). Then, we have

\[
\sum_{\substack{N < n \leq 2N \\ n \equiv \nu_0 \pmod{\Lambda_2}}} \Lambda^2_{\text{Sel}}(n) = \frac{\varphi(W)^k N \log R)^k}{W^{k+1}} J + O\left( \frac{F_2^2 \varphi(W)^k N \log R)^k}{W^{k+1} D_0} \right),
\]

where

\[
J = \int_0^1 \cdots \int_0^1 F(t_1, \ldots, t_k)^2 \, dt_1 \cdots dt_k.
\]

The shape of conditions from Propositions 1.12–1.15 lead us to two different cases: \( \text{supp}(F) \subset \mathcal{R}_k \) or \( \text{supp}(F) \subset \mathcal{R}'_k \). These considerations motivate the following hypotheses. The first one is related to the unextended multidimensional sieve and is useful while proving Main Theorem 2.

**Hypothesis 1.** Let \( \vartheta_0, \vartheta, \) and \( F \) be such that

1. \( \text{supp}(F) \subset \mathcal{R}_k \),
2. \( \vartheta \leq \vartheta_0 \),
3. \( \vartheta_0 + 2 \vartheta < 1 \),
4. \( \vartheta < \frac{1}{2} \),
5. \( GEH[2\vartheta] \).

The second hypothesis is used in the proof of Main Theorem 1 and is related to the situation when our sieve has extended support.

**Hypothesis 2.** Let \( \vartheta_0, \vartheta, \) and \( F \) be such that

1. \( \text{supp}(F) \subset \mathcal{R}'_k \),
2. \( \vartheta \leq \vartheta_0 \),
3. \( \vartheta_0 + \frac{2k-1}{k-1} \vartheta < 1 \),
4. \( \vartheta < \frac{k-1}{2k} \),
5. \( GEH[2\vartheta] \).

Note that if \( \vartheta < 1/4 \), then the fifth assumption in Hypothesis 2 holds by Theorem 1.8.

### 2 | PROOF OF MAIN THEOREMS 1 AND 2

#### 2.1 | Setup

We apply Propositions 1.12, 1.13, 1.14, and 1.15 to estimate \( S \) from below (accordingly to (1.14)). We assume Hypothesis 1 or Hypothesis 2. Fix \( \varepsilon > 0 \). Let us put

\[
W_0(x) = 1 - \frac{\vartheta}{\vartheta_0} x,
\]
and
\[ W_{r,s}(x_1, \ldots, x_{r-1}) = \begin{cases} \frac{1}{\xi_0} - s - \frac{1}{\xi_0} \sum_{i=1}^{r-s} x_i, & \text{if } \varepsilon < x_1 < \cdots < x_{r-s} \leq \xi_0 < x_{r-s+1} < \cdots < x_{r-1} \text{ and} \sum_{i=1}^{r-1} x_i < 1 - x_{r-1}, \\ 0 & \text{otherwise} \end{cases} \] (2.2)
for any \( r, s \in \mathbb{N} \).

Recall that \( P(n) \) is a product of \( k \) linear forms, each of the size \( O(n) \). Thus, \( \log P(n) / \log R_0 \ll 1 \), and in consequence Proposition 1.14 provides us

\[ S' \leq \sum_{N < n \leq 2N} \left( \sigma + \frac{\log P(n)}{\log R_0} \right) \Lambda^2_{\text{Sel}}(n) \ll \sum_{N < n \leq 2N} \Lambda^2_{\text{Sel}}(n) \ll \frac{F^2_{\max} \varphi(W)^k N (\log R)^k}{W^{k+1} D_0}. \] (2.3)

In Section 1, we defined \( R = N^\delta \) and \( R_0 = N^\delta_0 \). From Proposition 1.12 we obtain

\[ T_0 = \sum_{N < n \leq 2N} \sum_{\mu \equiv \nu_0 \mod W \atop n \equiv \nu_0 \mod W} \left( 1 - \frac{\log p}{\log R_0} \right) \Lambda^2_{\text{Sel}}(n) \]
\[ = \frac{\varphi(W)^k N (\log R)^k}{W^{k+1}} \sum_{j=1}^{k} F^{(j)}_{0} + O \left( \frac{F^2_{\max} \varphi(W)^k N (\log R)^k (\varepsilon + \frac{|\log \varepsilon|}{D_0})}{W^{k+1}} \right). \] (2.4)

We apply Proposition 1.13 with \( \beta_r(n) = \chi_{r,s}(n) \), where \( \chi_{r,s}(n) \) is defined as in (1.16) and where \( J_r \) is relabeled as \( J_{r,s} \). We get

\[ T_{r,s}^{(j)} = \sum_{N < n \leq 2N} \chi_{r,s}(L_j(n)) \Lambda^2_{\text{Sel}}(n) \]
\[ = \frac{\varphi(W)^k N (\log R)^{k+1}}{W^{k+1} \log N} F^{(j)}_{r,s} + O \left( \frac{F^2_{\max} \varphi(W)^k N (\log R)^k}{W^{k+1} D_0} \right). \] (2.5)

Proposition 1.15 gives

\[ S_0 = \sum_{N < n \leq 2N \atop n \equiv \nu_0 \mod W} \Lambda^2_{\text{Sel}}(n) = \frac{\varphi(W)^k N (\log R)^k}{W^{k+1}} F^{(j)}_{r,s} + O \left( \frac{F^2_{\max} \varphi(W)^k N (\log R)^k}{W^{k+1} D_0} \right). \] (2.6)
Combining (1.14), (2.3), (2.4), (2.5), and (2.6), we obtain
\[
S \geq \frac{\varphi(W)^k N(\log R)^k}{W^{k+1}} \left( \sigma J - \sum_{j=1}^{k} J_{0}^{(j)} + \partial \sum_{j=1}^{k} \sum_{r=1}^{h} \sum_{s=1}^{r} J_{r,s}^{(j)} \right) - O\left( \varepsilon \frac{F^2 \max \varphi(W)^k N(\log R)^k}{W^{k+1}} \right) - O_{\varepsilon} \left( \frac{F^2 \max \varphi(W)^k N(\log R)^k}{W^{k+1} D_0} \right).
\]

From this we can see that if
\[
\sigma = \frac{\sum_{j=1}^{k} J_{0}^{(j)} - \partial \sum_{j=1}^{k} \sum_{r=1}^{h} \sum_{s=1}^{r} J_{r,s}^{(j)}}{J} + \varepsilon C_F,
\]
where \(C_F\) is a sufficiently large constant depending only on \(F\), then \(S > 0\). We put
\[
\bar{\Upsilon}_k(F; \partial, \partial_0, \varepsilon) = \frac{\sum_{j=1}^{k} J_{0}^{(j)} - \partial \sum_{j=1}^{k} \sum_{r=1}^{h} \sum_{s=1}^{r} J_{r,s}^{(j)}}{J} + \frac{k}{\partial_0}.
\]

Therefore, from (1.12) and (1.13), we have
\[
\Omega(P(n)) \leq \bar{\Upsilon}_k(F; \partial, \partial_0, \varepsilon) + O_F(\varepsilon)
\]
infinitely often. We also note that \(\bar{\Upsilon}_k\) is actually well defined for every \(\partial \in [0, \frac{1}{2}]\), \(\partial_0 \in [0, 1]\), \(\varepsilon \in [0, 1]\). Let us put
\[
\bar{\Omega}_k(\partial, \partial_0, \varepsilon) = \inf\{\bar{\Upsilon}_k(F; \partial, \partial_0, \varepsilon) : F \text{ is smooth and supported on } R_k\},
\]
\[
\bar{\Omega}^{\text{ext}}_k(\partial, \partial_0, \varepsilon) = \inf\{\bar{\Upsilon}_k(F; \partial, \partial_0, \varepsilon) : F \text{ is smooth and supported on } R'_k\}.
\]

We wish to find good upper bounds for \(\bar{\Omega}_k\) and \(\bar{\Omega}^{\text{ext}}_k\) by suitable choices of functions \(F\). We restrict our attention to multivariate polynomials. Moreover, if our polynomial is symmetric, then the main term from (2.7) simplifies to
\[
\frac{k(J_0 - \partial \sum_{r=1}^{h} \sum_{s=1}^{r} J_{r,s})}{J} + \frac{k}{\partial_0},
\]
where \(J_0 = J_0^{(1)}\) and \(J_{r,s} = J_{r,s}^{(1)}\) for every \(r, s \in \mathbb{N}\). Observe that \(\bar{\Upsilon}_k\) is continuous and differentiable with respect to each variable. This implies
\[
\bar{\Upsilon}_k(F; \partial, \partial_0, \varepsilon) = \bar{\Upsilon}_k(F; \partial, \partial_0, 0) + O_F(\varepsilon).
\]

We usually take \(\varepsilon\) very close to 0, so (2.10) motivates the following definitions.
\[
Y_k(F; \partial, \partial_0) = \bar{\Upsilon}_k(F; \partial, \partial_0, 0),
\]
\[
\Omega_k^{\text{ext}}(\partial, \partial_0) = \bar{\Omega}_k^{\text{ext}}(\partial, \partial_0, 0),
\]
\[
\Omega_k(\partial, \partial_0) = \bar{\Omega}_k(\partial, \partial_0, 0).
\]

Therefore, we can summarize the above discussion in the following result.
Theorem 2.1. Let $k$ be an integer greater than or equal to 3. Let also $\varepsilon > 0$, and $\mathcal{H}$ be an admissible $k$-tuple. Then, we have

1. $\Omega(\mathcal{P}(n)) \leq \Omega_k + O(\varepsilon)$ infinitely often, if $\vartheta$ and $\vartheta_0$ satisfy Hypothesis 1;
2. $\Omega(\mathcal{P}(n)) \leq \Omega^\text{ext}_k + O(\varepsilon)$ infinitely often, if $\vartheta$ and $\vartheta_0$ satisfy Hypothesis 2.

2.2 The GEH case

Let us begin this subsection with an observation concerning the integrals $J_0$ and $J_{r,s}$. The precise shape and the very existence of these integrals can be perceived as a consequence of the relations between (1.4) and (2.7).

\[
\sum_{p|n, \, p \leq R_0} \left(1 - \frac{\log p}{\log R_0}\right) \text{ transforms into } J_0 / J,
\]

\[
\sum_{r=1}^{\infty} \sum_{s=1}^{r} \chi_{r,s}(n) \text{ transforms into } \frac{\vartheta \sum_{r=1}^{\infty} \sum_{s=1}^{r} J_{r,s}}{J},
\]

\[
\frac{\log n}{\log R_0} \text{ transforms into } \frac{k}{\vartheta_0}.
\]

The $J$ integral is basically a normalizing term. Note that it is relatively easy to calculate $\Omega_k$ or $\Omega^\text{ext}_k$ upon $\vartheta_0 = 1$, because in such a case we would have $J_{r,s} = 0$ for every permissible pair $r, s$. Thus, if $\vartheta_0 = 1$ and $F$ is symmetric, then (2.9) reduces to

\[
\frac{k J_0}{J} + \frac{k}{\vartheta_0}.
\]

We can view $\sum_{r=1}^{\infty} \sum_{s=1}^{r} \chi_{r,s}(n)$ as a correction term which appears to give back the contribution taken away by the $p \leq R_0$ restriction in the upper sum from (2.12). Therefore, we can propose the following.

Conjecture 2.2. The function $\Omega_k(\vartheta, \vartheta_0)$ is constant with respect to $\vartheta_0$. The same applies to $\Omega^\text{ext}_k$.

In Main Theorem 2 we apply the first part of Theorem 2.1 with $\vartheta = \vartheta_0 = 1/3$. Conjecture 2.2 is probably not exactly correct the way it is stated, but it has been very useful for choosing good (conjecturally near-optimal) polynomials $F$ for various $k$. In general, it is just much easier to minimize the expression (2.13) rather than (2.9). Our choices are listed in Table C.

Recall that $P_j = \sum_{i=1}^{k} t_j^i$. The $F_1$ polynomials are chosen to be almost best from all polynomials of the form

\[ a_0 + a_1(1 - P_1) + a_2(1 - P_1)^2 + a_3(1 - P_1)^3 \]

with $a_0, a_1, a_2, a_3 > 0$. One may note that polynomials listed in the $F_1$ column resemble $(1 - P_1)^3$ quite much, up to constant multipliers. On the other hand, we can also take into account all other possible terms of order greater than or equal to 3 constructed from $(1 - P_1), P_2$ and find the almost
**Table C**

| \(k\) | \(F_1\) | \(F_2\) |
|---|---|---|
| 3 | \(529 - 877P_1 + 567P_1^2 - 189P_1^3\) | \(1846 - 3225P_1 + 2203P_1^2 - 727P_1^3 + 228P_2 - 223P_1P_2\) |
| 4 | \(17950 - 36681P_1 + 28786P_1^2 - 9510P_1^3\) | \(20875 - 43615P_1 + 33273P_1^2 - 10000P_1^3 + 4867P_2 - 4649P_2P_1\) |
| 5 | \(15566 - 35617P_1 + 30136P_1^2 - 9807P_1^3\) | \(17195 - 40385P_1 + 33413P_1^2 - 10000P_1^3 + 5366P_2 - 5148P_2P_1\) |
| 6 | \(12739 - 31508P_1 + 28087P_1^2 - 9178P_1^3\) | \(11908 - 30242P_1 + 26486P_1^2 - 8071P_1^3 + 4310P_2 - 4162P_2P_1\) |
| 7 | \(11754 - 30703P_1 + 28386P_1^2 - 9354P_1^3\) | \(11091 - 29705P_1 + 27075P_1^2 - 8420P_1^3 + 4322P_2 - 4197P_2P_1\) |
| 8 | \(11131 - 30235P_1 + 28687P_1^2 - 9531P_1^3\) | \(10523 - 2941P_1 + 27419P_1^2 - 8679P_1^3 + 4232P_2 - 4128P_2P_1\) |
| 9 | \(6710 - 18690P_1 + 18003P_1^2 - 6001P_1^3\) | \(9528 - 27175P_1 + 26009P_1^2 - 8351P_1^3 + 3857P_2 - 3775P_2P_1\) |
| 10 | \(6573 - 18606P_1 + 18072P_1^2 - 6204P_1^3\) | \(9726 - 1513P_1 + 712P_2 - P_3 + 4548P_2 - 3828P_2P_1\) |

**Table D**

| \(k\) | Upper bound on \(\Omega_k\) (contribution from \(J_4,1\) included) | Upper bound on \(\Omega_k\) (contribution from \(J_4,1\) included) | True value of \(\Upsilon_k(F_1)\) foreseen by Conjecture 2.2 | True value of \(\Upsilon_k(F_2)\) foreseen by Conjecture 2.2 |
|---|---|---|---|---|
| 3 | 7.530... | 7.415... | 7.38120... | 7.38096... |
| 4 | 10.750... | 10.523... | 10.44612... | 10.44486... |
| 5 | 14.192... | 13.828... | 13.68862... | 13.68492... |
| 6 | 17.822... | 17.301... | 17.07933... | 17.07180... |
| 7 | 21.615... | 20.921... | 20.58687... | 20.58499... |
| 8 | 25.550... | 24.672... | 24.22809... | 24.20929... |
| 9 | 29.614... | 28.541... | 27.95889... | 27.93302... |
| 10 | 33.795... | 32.519... | 31.78061... | 31.74691... |

Optimal polynomials \(F_2\) of the form

\[
b_0 + b_1(1 - P_1) + b_2(1 - P_1)^2 + b_3(1 - P_1)^3 + b_4P_2 + b_5(1 - P_1)P_2
\]

with \(b_0, b_1, b_2, b_3, b_4, b_5 > 0\). We are able to calculate the upper bound on \(Y(F_1; \frac{1}{3}, \frac{1}{3})\) thanks to the specific shape of \(F_1\) – as pointed out in Remark 3 we are able to convert the integrals \(J_0\) and the \(J_{r,s}\) into simpler ones.

Considerations mentioned above lead us to the results listed in Table D. In the first column, we present the upper bound on \(\Omega_k\) given by calculating the contributions from \(J_0, J_{1,1}, J_{2,1}, J_{3,1}, J_{2,2},\) and \(J_{3,2}\) into \(Y(F_1)\). In the second column, we extract also the contribution incoming from \(J_{4,1}\), which can be considered as a black box due to its complicated shape. The third column contains the possible true values of \(Y(F_1)\) predicted by Conjecture 2.2. In the fourth column we have the same thing for \(F_2\). As we can see in Table D, the differences between \(Y(F_1)\) and \(Y(F_2)\) are probably minuscule. As numerical experiments suggest, we are also unable to win much by considering higher powers of \((1 - P_1), P_2\) or even the negative \(a_i, b_i\).

By (2.10) we have

\[
\tilde{\Upsilon}_k(F_1; \frac{1}{3} - \varepsilon, \frac{1}{3}, \varepsilon) = Y_k(F_1; \frac{1}{3}, \frac{1}{3}) + O(\varepsilon),
\]

The choice \(\tilde{\varepsilon} = 1/3 - \varepsilon\) and \(\tilde{\varepsilon}_0 = 1/3\) satisfies all assertions of Hypothesis 1 under \(GEH[2/3]\), so fixing \(\varepsilon\) sufficiently close to 0 enables us to use the results from Table D to prove Main Theorem 2.
As we can see, there should be a possibility to improve the result for \( k = 9 \) and \( k = 10 \) by considering the contributions from integrals like \( J_{5,1} \) or \( J_{4,2} \) (some experiments with Monte Carlo method suggest that \( J_{5,1} \) should contribute much more than \( J_{4,2} \); unfortunately, calculating this expression is extremely onerous). We also have no improvement for \( k = 3 \) in view of what is already proven unconditionally (see [10]).

There is a problem with using the full strength of \( GEH \). Namely, we are limited by the inequality \( \vartheta \leq \vartheta_0 \). If \( \vartheta > \vartheta_0 \), then we cannot apply Proposition 1.13 to our choices of \( W_{r,s} \) (which are forced by the specific shape of functions \( \chi_{r,s}(n) \)). On the other hand, we can perform a little trick to overcome this issue. Take a look at the following inequality:

\[
\Omega(n) \geq \sum_{p \mid n \atop p \leq y} \left( 1 - \frac{\log p}{\log y} \right) + \frac{\log n}{\log y} + \sum_{r=1}^{\infty} \zeta_r(n),
\]

where

\[
\zeta_r(n) = \begin{cases} 
\left( \frac{\log n}{\log y} - 1 - \sum_{i=1}^{r-1} \frac{\log p_i}{\log y} \right), & \text{if } n = p_1 \ldots p_r \text{ with } 
p_1 < \ldots < p_{r-1} < y < p_r < n^{\vartheta}, \\
0, & \text{otherwise}.
\end{cases}
\]

Put

\[
W^\flat_{r,s}(x_1, \ldots, x_{r-1}) = \begin{cases} 
\frac{1}{\vartheta_0} - s - \frac{1}{\vartheta_0} \sum_{i=1}^{r-s} x_i, & \text{if } \varepsilon < x_1 < \ldots < x_{r-s} \leq \vartheta_0 < x_{r-s+1} < \ldots < x_{r-1} \text{ and } 
\sum_{i=1}^{r-1} x_i < \min(1 - x_{r-1}, 1 - \vartheta), \\
0 & \text{otherwise}
\end{cases}
\]

for all \( r, s \in \mathbb{N} \) in the place of \( W_{r,s} \). Now, we repeat the reasoning from (1.12)–(1.16) with \( \zeta_r \) instead of \( \chi_r \) and an analogous choice of \( y \). Note that we are allowed to apply Propositions 1.12–1.15 assuming only points 1,3,4,5 from Hypothesis 1. Proceeding like in (2.1)–(2.7) we conclude that if \( F \) is a symmetric polynomial of \( k \) variables, then we have

\[
\Omega(P(n)) \leq \frac{k(J_0 - \vartheta \sum_{r=1}^{h} \sum_{s=1}^{r} J^\flat_{r,s})}{J} + \frac{k}{\vartheta_0} + O(\varepsilon),
\]

where the \( J^\flat_{r,s} \) are the same as \( J_{r,s} \) but with \( W^\flat_{r,s} \) instead of \( W_{r,s} \). Now, the condition \( \vartheta \leq \vartheta_0 \) can be discarded, so we can try to use \( GEH[\vartheta] \) with an exponent greater than 2/3, without harming the condition \( \vartheta_0 + 2\vartheta < 1 \). Unfortunately, this possibility has its price. Note that (2.14) is not an equality like (1.11). That difference implies that some part of contribution given previously by \( J_{r,s} \) disappears (it is not surprising because \( \text{supp}(W^\flat_{r,s}) \subset \text{supp}(W_{r,s}) \)). If \( \vartheta > \vartheta_0 \), then the bigger the difference \( \vartheta - \vartheta_0 \) is, the stronger that phenomenon is going to be. Up to some point we are able to make some little progress over the results listed in Table D. Put \( \vartheta_0 = 1 - 2\vartheta \). Define \( Y^\flat_k \), \( \Omega^\flat_k \) and \( \Omega^\flat_k \) the same way as their “non-\( \flat \)” analogues, but with \( J^\flat_{r,s} \) instead of \( J_{r,s} \). Taking polynomials \( F_1 \)
from Table C and considering the contribution from $\mathcal{J}_{1,1}^{\flat}, \mathcal{J}_{2,1}^{\flat}, \mathcal{J}_{3,1}^{\flat}, \mathcal{J}_{4,1}^{\flat}, \mathcal{J}_{5,2}^{\flat}, \mathcal{J}_{5,3}^{\flat}$ integrals we get Table E.

The improvements over bounds listed in Table D are rather subtle. Numerical experiments suggest that taking greater $\vartheta$ worsens the results because the rising gap between $J_{r,s}$ and $\mathcal{J}_{r,s}^{\flat}$ takes from us more contribution to $\Omega_{k}^{\flat}(F_1)$ than the increasing $\vartheta$ can possibly offer. To use the full power of GEH one needs to find a way around this issue.

Conjecture 2.2 can also be used to predict the limits of Maynard’s original method from [11]. He used the identity (1.4) instead of (1.6), so $\vartheta_0 = \frac{1}{2}$ and $\vartheta = \frac{1}{2}$ is the optimal choice in this situation. We find close to best polynomials $F_1$ and $F_2$ (which are not written explicitly here) in the same way as these listed in Table C.

Our heuristic predicts that Maynard’s results from [11] cannot be improved purely by his method. Moreover, even using multidimensional variation of his techniques does not make the situation much better. It is a good question whether it is possible to cross the 15 barrier in the case $k = 5$ relying only on better choice of polynomial than $F_2$, but without using the extended sieve support like in the next subsection. It turns out that even if we choose the optimal polynomial $F$ of the form:

$$
c_0 + c_1(1 - P_1) + c_{11}(1 - P_1)^2 + c_2P_2 + c_{111}(1 - P_1)^3 + c_{21}(1 - P_1)P_2 + c_3P_3$$
$$+ c_{1111}(1 - P_1)^4 + c_{211}(1 - P_1)^2P_2 + c_{22}P_2^2 + c_{31}(1 - P_1)P_3 + c_4P_4
$$

with all the coefficients being real (so we can also consider negative values) then by Conjecture 2.2 we can expect only $Y_5(F) = 15.01185\ldots$. This discussion shows that the extended sieve support was necessary to prove Main Theorem 1.

Conjecture 2.2 also allows us to predict the optimal upper bounds on $\Omega_{k}^{\text{ext}}(\frac{1}{4}, \frac{3}{8})$ incoming from polynomials $F$ of the form $a(1 - P_1)^2 + b(1 - P_1) + c$. The results are listed in Table G.

We can foresee that the extending of sieve support allows us to prove (1.3) with $\rho_5 = 14$ and reprove the main result from [10], that is, that (1.3) is true with $\rho_3 = 7$.

The discussion in this subsection leads us to the conclusion that both Main Theorems 1 and 2 cannot be improved only by optimizing parameters except possibly the cases $k = 9, 10$ under GEH. One needs to rely on some sort of new ideas in order to set new records. Perhaps, the technology developed in [15] can be used to expand the sieve support even further and obtain new results.
2.3  The upper bound for $\Omega_5^\text{ext}(\frac{1}{4}, \frac{3}{8})$

Recall that due to (2.11) we consider integrals from Propositions 1.12 and 1.13 with $\epsilon = 0$. For $k = 5$ we put

$$F(t_1, t_2, t_3, t_4, t_5) = \begin{cases} 
11 + 85(1 - P_1) + 170(1 - P_1)^2, & \text{if } (t_1, t_2, t_3, t_4, t_5) \in \mathcal{R}'_5, \\
0, & \text{otherwise,}
\end{cases}$$

and take

$$\vartheta = \frac{1}{4}, \quad \vartheta_0 = \frac{3}{8}.$$ 

Note that we have $F(t_1, t_2, t_3, t_4, t_5) = f(t_1 + t_2 + t_3 + t_4 + t_5)$ for $(t_1, t_2, t_3, t_4, t_5) \in \mathcal{R}'_5$, where

$$f(x) = 11 + 85(1 - x) + 170(1 - x)^2.$$ 

In the unextended variation of multidimensional Selberg sieve (i.e., when supp $(F) \subset \mathcal{R}_k$) this symmetry transforms our sieve into its one-dimensional analogue (see Remark in [12, Section 6]).
However, our function $F$ does not satisfy it for all points $(t_1, t_2, t_3, t_4, t_5) \in \mathbb{R}^5_{\geq 0}$. For this reason it should not be surprising that we are able to make a progress compared to what is achievable by a one-dimensional sieve.

With our choices of $k, F, \vartheta,$ and $\vartheta_0$ we shall calculate (2.9) to get the desired result. The precise values of integrals appearing in this subsection are found by Mathematica 11. All integrals that appear in the following part of this section are defined in Propositions 1.12, 1.13, and 1.15.

2.3.1 Calculating $J_{1,1}$

For the sake of generality, we perform the calculations for an arbitrary $k$. We also assume that the symmetry $F(t_1, \ldots, t_k) = f(t_1 + \cdots + t_k)$ is satisfied for $(t_1, \ldots, t_k) \in \mathbb{R}^k_{\geq 0}$, where $f : \mathbb{R} \to \mathbb{R}$ is a piecewise smooth function. We have

$$J_{1,1} = \frac{1 - \vartheta_0}{\vartheta_0} \int_{\mathbb{R}^k_{\geq 0}} \cdots \int_{\mathbb{R}^k_{\geq 0}} \left( \int_0 f(t_1 + \cdots + t_k) dt_1 \right)^2 dt_2 \ldots dt_k \quad (2.15)$$

where $\rho(t_2, \ldots, t_k) = \sup \{ t_1 \in \mathbb{R} : (t_1, \ldots, t_k) \in \mathbb{R}^k_{\geq 0} \}$. We observe that any permutation of the variables $t_2, \ldots, t_k$ does not change the integrand. We also note that $0 \leq t_2 \leq \cdots \leq t_k$ implies

$$\rho(t_2, \ldots, t_k) = 1 - t_3 - \cdots - t_k.$$

Therefore, the outer integral from (2.15) equals

$$(k - 1)! \int_{\mathbb{R}^k_{\geq 0}} \cdots \int_{\mathbb{R}^k_{\geq 0}} \left( \int_0 \frac{1 + t_2}{t} f(x) dx \right)^2 dt_2 \ldots dt_k. \quad (2.16)$$

We make a substitution $t = t_2 + \cdots + t_k$ and transform the integral over $\mathbb{R}^k_{\geq 0}$ from expression (2.16) into

$$\int_0^1 \int_{t \leq t_2 \leq \cdots \leq t_{k-1} \leq t - \sum_{i=2}^{k-1} t_i} \left( \int_0 \frac{1 + t_2}{t} f(x) dx \right)^2 dt_2 \ldots dt_{k-1} dt. \quad (2.17)$$

For the sake of clarity, we put $s$ in the place of $t_2$. We can rewrite (2.17) as

$$\int_0^1 \int_0^{t_3} \left( \int_0^{t_2} \left( \int_0^{t_3} \left( \cdots \int_0^{t_{k-2}} \left( \int_0^{t_{k-1}} \left( \int_0^{t_k-2} \left( \int_0^{t_{k-1}} dt_k \right)^2 \cdots dt_3 \right)^2 \right)^2 \right)^2 \right)^2 ds dt \quad (2.18)$$
By induction we calculate that the expression in the right-hand side parentheses from (2.18) equals

$$
\frac{(t - (k - 1)s)^{k-3}}{(k - 2)!(k - 3)!}.
$$

(2.19)

Combining (2.15)–(2.19) we conclude that

$$
J_{1,1} = \frac{1 - \vartheta_0}{\vartheta_0(k - 3)!} \int_0^1 \int_0^t \left( \int f(x) \, dx \right)^2 (t - s)^{k-3} \, ds \, dt.
$$

(2.20)

In the $k = 5$ case, the expression from (2.20) equals

$$
\frac{5}{6} \int_0^1 \int_0^t \left( \int f(x) \, dx \right)^2 (t - s)^2 \, ds \, dt > 9.4661240888.
$$

(2.21)

**Remark 3.** Note that in the $\Omega_k$ case, when the function $F$ is supported on $R_k$, we have $1 - t_2 - \cdots - t_k$ instead of $\rho(t_2, \ldots, t_k)$ in (2.15). By performing the same procedures as in (2.15)–(2.20) we get the same integral as in (2.20) but with $1 + s$ replaced by 1 in the limit of integration. That would lead to

$$
J_{1,1} = \frac{1 - \vartheta_0}{\vartheta_0} \int_0^1 \left( \int f(x) \, dx \right)^2 t^{k-2} \frac{1}{(k - 2)!} \, dt
$$

which is known from [11, (5.30)]. The same observation applies to $J_0$ and the other $J_{r,s}$ integrals.

### 2.3.2 Calculating $J$

We proceed as in the previous subsection. We have

$$
J = \int_{R_k'} \cdots \int f(t_1 + \cdots + t_k)^2 \, dt_1 \cdots dt_k = \int_{R_{k-1}} \cdots \int f(t_1 + \cdots + t_k)^2 \, dt_1 \cdots dt_k.
$$

(2.22)

Taking the lower expression from (2.22) and proceeding like in (2.15)–(2.19) we can easily get that in the $k = 5$ case we have

$$
J = \frac{1}{2} \int_0^1 \int_0^t \int_{1+\frac{s}{t}} f(x)^2 (t - s)^2 \, dx \, ds \, dt > 14.3115286045.
$$

(2.23)
2.3.3 Calculating $J_0$

The function $F$ is assumed to be symmetric, so we have

$$J_0 = \int_{0}^{\partial_0/\partial} \frac{\partial_0 - \partial y}{\partial_0 y} \int_{\mathcal{R}_k'} \cdots \int (F(t_1, \ldots, t_k) - F(t_1 + y, t_2, \ldots, t_k))^2 \, dt_1 \cdots dt_k \, dy. \quad (2.24)$$

As mentioned at the beginning of this subsection, we integrate over the interval $[0, \partial_0/\partial]$ here (instead of $[\epsilon, \partial_0/\partial]$) at the cost of introducing the $O(\epsilon)$ term in Theorem 2.1. The inner integral equals

$$\int_{\mathcal{R}_{k-1}} \cdots \int_{0}^{1} (F(t_1, \ldots, t_k) - F(t_1 + y, t_2, \ldots, t_k))^2 \, dt_1 \, dt_2 \cdots dt_k$$

$$= \int_{\mathcal{R}_{k-1}} \cdots \int_{0}^{\varpi(y, t_2, \ldots, t_k)} (f(t_1 + \cdots + t_k) - f(y + t_1 + \cdots + t_k))^2 \, dt_1 \, dt_2 \cdots dt_k$$

$$+ \int_{\mathcal{R}_{k-1}} \cdots \int_{\varpi(y, t_2, \ldots, t_k)} \rho(t_2, \ldots, t_k) \varpi(y, t_2, \ldots, t_k) \int_{0}^{0} (f(t_1 + \cdots + t_k))^2 \, dt_1 \, dt_2 \cdots dt_k =: \int_1 + \int_2, \quad (2.25)$$

where $\varpi(y, t_2, \ldots, t_k) = \max(0, \sup\{t_1 \in \mathbb{R} : (t_1 + y, t_2, \ldots, t_k) \in \mathcal{R}_k'\})$. We have $\int_1 = 0$ for $y > 1$. Note that if $0 \leq t_2 \leq \cdots \leq t_k$, then

$$\varpi(y, t_2, \ldots, t_k) = \max(0, 1 - y - t_3 - \cdots - t_k). \quad (2.26)$$

We again use the fact, that any permutation of $t_2, \ldots, t_k$ does not change the integrand, so for $0 \leq y \leq 1$ it is true that $\int_1$ times $1/(k-1)!$ equals

$$\int_{\mathcal{R}_{k-1}} \cdots \int_{0 \leq t_2 \leq \cdots \leq t_k} (f(t_1 + \cdots + t_k) - f(y + t_1 + \cdots + t_k))^2 \, dt_1 \, dt_2 \cdots dt_k$$

$$= \int_{0 \leq t_2 \leq \cdots \leq t_{k-1} \leq t_1 - \sum_{i=2}^{k-1} t_i} \int_{l} \cdots \int_{l} (f(x) - f(x + y))^2 \, dx \, dt_2 \, dt_3 \cdots dt_{k-1} \, dt$$

$$= \int_{0}^{1} \left( \int_{l_1} \cdots \int_{l_{k-2}} (f(x) - f(x + y))^2 \, dx \right) \int_{l_1} \cdots \int_{l_{k-3}} \cdots \int_{l_{k-2}} dt_{k-1} \cdots dt_3 \, ds \, dt$$
\[
\frac{1}{(k-1)!(k-3)!} \int_0^1 \int_0^t \int_t \max\left(t,1-y+\frac{s}{k-1}\right) (f(x) - f(x + y))^2 (t - s)^{k-3} \, dx \, ds \, dt. \tag{2.27}
\]

Let us move on to the \(f_2\) case. By the same argument as above, this integral times \(1/(k-1)!\) equals
\[
\frac{1}{(k-1)!(k-3)!} \int_0^1 \int_0^t \int_t \max\left(t,1-y+\frac{s}{k-1}\right) f(\frac{t_1}{1+\frac{s}{k-1}}) \, dt_1 \, dt_2 \ldots \, dt_k
\]
\[
= \frac{1}{(k-1)!(k-3)!} \int_0^1 \int_0^t \int_t \max\left(t,1-y+\frac{s}{k-1}\right) f(\frac{t_1}{1+\frac{s}{k-1}}) \, dt_1 \, dt_2 \ldots \, dt_k \, dy. \tag{2.28}
\]

Note that the expression above equals \(J\) for \(y > 1\).

From (2.24) and (2.25), we have
\[
J_0 = \int_0^\delta \frac{\partial^2}{\partial y^2} \left( \int_1^2 \right) dy. \tag{2.29}
\]

In the \(k = 5\) case, we can write that the integral in the expression above equals
\[
\frac{1}{2} (J_{0;1} + J_{0;2}),
\]
where
\[
J_{0;1} = \left(\int_0^{1/2} \int_0^{1-y} \int_0^{1-y+\frac{s}{4}} \int_0^{1/2} \int_0^t (f(x) - f(x + y))^2 (t - s)^2 \, dx \, ds \, dt \right)
\]
\[
\times \frac{3-2y}{3y} (f(x) - f(x + y))^2 (t - s)^2 \, dx \, ds \, dt \, dy < 11.3104037062,
\]
and
\[
J_{0;2} = \left(\int_0^{1/2} \int_0^{1-y} \int_0^{1-y+\frac{s}{4}} \int_0^{1/2} \int_0^t (f(x) - f(x + y))^2 (t - s)^2 \, dx \, ds \, dt \right)
\]
\[
\times \frac{3-2y}{3y} (f(x) - f(x + y))^2 (t - s)^2 \, dx \, ds \, dt \, dy < 11.3104037062,
\]
\[
\begin{align*}
&+ \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \\
&\times \frac{3}{3y} f(x)^2 (t - s)^2 dx \, ds \, dy < 20.2508453206.
\end{align*}
\]

A direct calculation shows that
\[
J_0 < 15.7806245134. \tag{2.30}
\]

### 2.3.4 Calculating \( J_{2,1} \)

We have
\[
J_{2,1} = \frac{1}{(k - 3)!} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} (t - \frac{1}{3} - \frac{1}{3}y) \left( \frac{1}{3} \int_0^{\frac{1}{3}} f(x) dx \right)^2 dt \, dt_1 \, ... \, dt_{k-3} \, dt dy. \tag{2.32}
\]

The inner integral equals
\[
\int_0^y \max(y, \rho(t_2, ..., t_k)) F(t_1, ..., t_k) \, dt_1 = \int_0^0 f(t_1 + ... + t_k) \, dt_1.
\]

Proceeding very much like in the \( J_{1,1} \) case, we deduce that the integral over \( \mathcal{R}_{k-1} \) from (2.31) equals

\[
(k - 1)! \int_0^{\frac{1}{k-1}} \int_0^{\frac{1}{k-1}} ... \int_0^{\frac{1}{k-1}} \left( \max(t_1, t_2, ..., t_{k-1}) \right)^2 \int_0^t \max(t + y, 1 + t_2) \left( \int_0^t f(x) \, dx \right) \, dt_2 \, ... \, dt_{k-1} dt.
\]

We put
\[
\text{Int}_a^b(u) = \int_{\min(a,1+u)}^{\max(b,1+u)} f(x) \, dx.
\]

Continuing the reasoning from the \( J_{1,1} \) case we get
\[
J_{2,1} = \frac{1}{(k - 3)!} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \left( \text{Int}_a^b \left( \frac{s}{k-1} \right) \right)^2 (t - s)^{k-3} ds \, dt dy. \tag{2.32}
\]
We decompose the outer integral from (2.32) as follows:

\[
\int_0^{3/2} \int_0^1 \int_0^t = \int_{R_{2;1}} + \int_{R_{2;2}},
\]

where

\[
R_{2;1} = \{(y, t, s) \in \mathbb{R}^3 : 0 < y < \frac{3}{2}, 0 < t < 1, 0 < s < t, t + y > 1 + \frac{s}{k-1}\},
\]

\[
R_{2;2} = \{(y, t, s) \in \mathbb{R}^3 : 0 < y < \frac{3}{2}, 0 < t < 1, 0 < s < t, t + y < 1 + \frac{s}{k-1}\}.
\]

We see that

\[
\text{Int}_{\frac{t+y}{t}} \left( \frac{s}{k-1} \right) = \begin{cases} 
\int_{\frac{t+y}{t}} f(x) \, dx & \text{if } (y, t, s) \in R_{2;1}, \\
\int_{\frac{t+y}{t}} f(x) \, dx & \text{if } (y, t, s) \in R_{2;2}.
\end{cases}
\]

For \( k = 5 \) we have

\[
\int_{R_{2;1}} = \left( \int_0^{1/4} \int_0^{1/2} \int_0^t + \int_0^{1/4} \int_0^{4/5} \int_0^t + \int_0^{1/2} \int_0^{4/5} \int_0^t + \int_0^{1/4} \int_0^{4/5} \int_0^t \right) \frac{1 - \hat{v}_0 - \hat{v}_y}{\hat{v}_0^2y(1 - \hat{v}_y)} \\
\times \left( \int_{\frac{t+y}{t}} f(x) \, dx \right)^2 (t - s)^{k-3} \, ds \, dt \, dy > 15.2749404974
\]  

(2.33)

and

\[
\int_{R_{2;2}} = \left( \int_0^{1/4} \int_0^{1/2} \int_0^t + \int_0^{1/4} \int_0^{4/5} \int_0^t + \int_0^{1/2} \int_0^{4/5} \int_0^t \right) \frac{1 - \hat{v}_0 - \hat{v}_y}{\hat{v}_0^2y(1 - \hat{v}_y)} \\
\times \left( \int_{\frac{t+y}{t}} f(x) \, dx \right)^2 (t - s)^{k-3} \, ds \, dt \, dy > 16.5050961382.
\]  

(2.34)

Combining (2.33) and (2.34), we get

\[
J_{2,1} > 15.8900183178.
\]  

(2.35)
2.3.5 Calculating $J_{3,1}$

Let us define

$$A'_r := \left\{ x \in [0, 4]^{r-1} : 0 < x_1 < \cdots < x_{r-1} < 4 \theta_0, \sum_{i=1}^{r-1} x_i < \min (4(1 - \theta_0), 4 - x_{r-1}) \right\}. \tag{2.3.5}$$

Performing calculations analogous to the $J_{2,1}$ case we get

$$J_{3,1} = \frac{1}{(k-3)!} \iint \int_{A'_3} 1 \, d\theta_0 \, \int_{0}^{1} \int_{0}^{t} \frac{1 - \theta_0 - \theta(y+z)}{\theta_0yz(1-\theta(y+z))} \left( \text{Int}^{t+z}_{t} - \text{Int}^{t+y+z}_{t+y} \right)^2 (t-s)^{k-3} \, ds \, dz \, dy,$$

where we write $\text{Int}^h_u(u)$ instead of $\text{Int}^h_u\left(\frac{s-1}{k}\right)$ for the sake of clarity. We decompose the integral as follows:

$$\iint \int_{A'_3} 1 \, d\theta_0 \, \int_{0}^{1} \int_{0}^{t} = \int_{R_{3;1}} + \int_{R_{3;2}} + \int_{R_{3;3}} + \int_{R_{3;4}},$$

where

$$R_{3;1} = R_3 \cap \{(y, z, t, s) \in \mathbb{R}^4 : t < 1 + \frac{s}{k-1} < t + z\},$$

$$R_{3;2} = R_3 \cap \{(y, z, t, s) \in \mathbb{R}^4 : t + z < 1 + \frac{s}{k-1} < t + y\},$$

$$R_{3;3} = R_3 \cap \{(y, z, t, s) \in \mathbb{R}^4 : t + y < 1 + \frac{s}{k-1} < t + y + z\},$$

$$R_{3;4} = R_3 \cap \{(y, z, t, s) \in \mathbb{R}^4 : t + y + z < 1 + \frac{s}{k-1}\}$$

with

$$R_3 = \{(y, z, t, s) \in \mathbb{R}^4 : 0 < z < y < \frac{3}{2}, \quad y + z < \frac{5}{2}, \quad 0 < t < 1, \quad 0 < s < t\}.$$

We have

$$\text{Int}^{t+z}_{t} - \text{Int}^{t+y+z}_{t+y} = \begin{cases} \int_{t}^{t+1} \frac{s}{k-1} f(x) \, dx, & \text{if } (y, z, t, s) \in R_{3;1}, \\ \int_{t}^{t+z} f(x) \, dx, & \text{if } (y, z, t, s) \in R_{3;2}, \\ \int_{t}^{t+y} f(x) \, dx, & \text{if } (y, z, t, s) \in R_{3;3}, \\ \int_{t}^{t+y+z} f(x) \, dx, & \text{if } (y, z, t, s) \in R_{3;4}. \end{cases} \tag{2.3.6}$$
By (2.36) we are able to decompose the integrals \( \int_{R_{3j}} \) into integrals with explicit limits. For \( k = 5 \) and different cases \( i = 1, 2, 3, 4 \) we get 12, 24, 29, 10 such integrals, respectively, which gives 75 of them in total. The results are

\[
\int_{R_{31}} > 4.968, \quad \int_{R_{32}} > 12.158, \\
\int_{R_{33}} > 4.227, \quad \int_{R_{34}} > 1.831,
\]

which implies

\[
J_{3,1} > 11.592. \quad (2.37)
\]

2.3.6 \hspace{1em} Calculating \( J_{4,1} \)

We follow the steps from the last subsection. We have

\[
J_{4,1} = \frac{1}{(k-3)!} \iiint_{A'_{4}} \int_{0}^{1} \int_{0}^{t} \frac{1 - \vartheta_{0} - \vartheta(y + z + w)}{\vartheta_{0}yzw(1 - \vartheta(y + z + w))} \times \left( \text{Int}_{t+y+w}^{t} - \text{Int}_{t+z+w}^{t} - \text{Int}_{t+y}^{t+y+w} + \text{Int}_{t+y+z+w}^{t+y+z+w} \right)^{2} (t-s)^{k-3} ds \, dt \, dw \, dz \, dy.
\]

We decompose

\[
\iiint_{A'_{4}} \int_{0}^{1} \int_{0}^{t} = \sum_{i=1}^{11} \int_{R_{4j}},
\]

where

\[
R_{4:1} = R_{4} \cap \left\{ (y, z, w, t, s) \in \mathbb{R}^{5} : t < 1 + \frac{s}{k-1} < t + w \right\}, \\
R_{4:2} = R_{4} \cap \left\{ (y, z, w, t, s) \in \mathbb{R}^{5} : t + w < 1 + \frac{s}{k-1} < t + z \right\}, \\
R_{4:3} = R_{4} \cap \left\{ (y, z, w, t, s) \in \mathbb{R}^{5} : t + z < 1 + \frac{s}{k-1} < t + z + w, \, y > z + w \right\}, \\
R_{4:4} = R_{4} \cap \left\{ (y, z, w, t, s) \in \mathbb{R}^{5} : t + z + w < 1 + \frac{s}{k-1} < t + y, \, y > z + w \right\}, \\
R_{4:5} = R_{4} \cap \left\{ (y, z, w, t, s) \in \mathbb{R}^{5} : t + y < 1 + \frac{s}{k-1} < t + y + w, \, y > z + w \right\}, \\
R_{4:6} = R_{4} \cap \left\{ (y, z, w, t, s) \in \mathbb{R}^{5} : t + z + w < 1 + \frac{s}{k-1} < t + y, \, y < z + w \right\}, \\
R_{4:7} = R_{4} \cap \left\{ (y, z, w, t, s) \in \mathbb{R}^{5} : t + y < 1 + \frac{s}{k-1} < t + z + w, \, y < z + w \right\}.
\]
\[ R_{4;8} = R_4 \cap \{(y, z, w, t, s) \in \mathbb{R}^5 : t + z + w < 1 + \frac{s}{k-1} < t + y + w, \; y < z + w\}, \]

\[ R_{4;9} = R_4 \cap \{(y, z, w, t, s) \in \mathbb{R}^5 : t + y + w < 1 + \frac{s}{k-1} < t + y + z\}, \]

\[ R_{4;10} = R_4 \cap \{(y, z, w, t, s) \in \mathbb{R}^5 : t + y + z < 1 + \frac{s}{k-1} < t + y + z + w\}, \]

\[ R_{4;11} = R_4 \cap \{(y, z, w, t, s) \in \mathbb{R}^5 : t + y + z + w < 1 + \frac{s}{k-1}\} \]

with

\[ R_4 = \{(y, z, w, t, s) \in \mathbb{R}^5 : 0 < w < z < y < \frac{3}{2}, \; y + z + w < \frac{5}{2}, \; 0 < t < 1, \; 0 < s < t\}. \]

The expression \((\text{Int}_{t+w}^{t+z+w} - \text{Int}_{t}^{t+z} - \text{Int}_{t+y}^{t+z} + \text{Int}_{t+y+z}^{t+y+z+w})\) equals

\[
\begin{align*}
&\int_{t}^{t+w} f(x) \, dx, \\
&\int_{t}^{t+z} f(x) \, dx, \\
&\left(\int_{t}^{t+w} - \int_{t+z}^{t+w} - \int_{t+y}^{t+z} + \int_{t+y+z}^{t+y+z+w}\right) f(x) \, dx,
\end{align*}
\]

for \((y, z, w, t, s)\) belonging to \(R_{4;1}, \ldots, R_{4;11}\) respectively. For \(k = 5\) this allows us to decompose the integrals \(\int_{R_{4,j}}\) into integrals with explicit limits and there are 1337 of them in total. The
results are

\[
\begin{align*}
\int_{R_{4,1}} & > 0.392, & \int_{R_{4,2}} & > 1.538, & \int_{R_{4,3}} & > 0.851, \\
\int_{R_{4,4}} & > 0.608, & \int_{R_{4,5}} & > 0.939, & \int_{R_{4,6}} & > 0.410, \\
\int_{R_{4,7}} & > 0.307, & \int_{R_{4,8}} & > 0.073, & \int_{R_{4,9}} & > 0.066, \\
\int_{R_{4,10}} & > 0.017, & \int_{R_{4,11}} & > 0.010
\end{align*}
\]

so the overall contribution satisfies

\[
J_{4,1} > 4.365. \quad (2.38)
\]

### 2.3.7 Calculating \( J_{2,2} \) and \( J_{3,2} \)

Proceeding as in the \( J_{1,1} \) case we calculate

\[
J_{2,2} = \frac{\int_{\frac{1}{2}}^{2} \int_{0}^{1} \int_{0}^{t} \int_{0}^{t} \int_{0}^{t} (F(t_1, \ldots, t_k)) \, dt_1 \, dt_2 \ldots \, dt_k \, dy \, dz}{(k-3)!} \geq 1.9342154969.
\]

Let us move our attention to the \( J_{3,2} \) case. Define

\[
\mathcal{A}_r'' = \left\{ x \in [0,4]^{r-1} : 0 < x_1 < \ldots < x_{r-s} < 4 \vartheta_0 < x_{r-s+1} < \ldots < x_{r-1}, \sum_{i=1}^{r-1} x_i < \min(4(1 - \vartheta_0), 4 - x_{r-1}) \right\}.
\]

We have

\[
J_{3,2} = \frac{1}{(k-3)!} \int_{\mathcal{A}_r''} \int_{0}^{1} \int_{0}^{t} \int_{0}^{t} \int_{0}^{t} (F(t_1, \ldots, t_k)) \, dt_1 \, dt_2 \ldots \, dt_k \, dy \, dz \geq \ldots
\]
Note that \( \text{Int}_{t+y+z}^{t+y} \) vanishes because \( y > 3/2 \). We again decompose the integral in order to simplify the integrand.

\[
\iint_{A_2'} 1 = \iiint_{R'_{3,1}} + \iiint_{R'_{3,2}},
\]

where

\[
R'_{3,1} = R_3' \cap \{ (y, z, t, s) \in \mathbb{R}^4 : t < 1 + \frac{s}{k-1} < t + z \},
\]

\[
R'_{3,2} = R_3' \cap \{ (y, z, t, s) \in \mathbb{R}^4 : t + z < 1 + \frac{s}{k-1} \},
\]

with

\[
R_3' = \{ (y, z, t, s) \in \mathbb{R}^4 : 0 < z < \frac{3}{2} < y, y + z < \frac{5}{2}, 2y + z < 4, 0 < t < 1, 0 < s < t \}.
\]

We get

\[
\text{Int}_{t}^{t+z} - \text{Int}_{t+y}^{t+y+z} = \begin{cases} 
1 + \frac{s}{k-1} & \text{if } (y, z, t, s) \in R'_{3,1}, \\
\int_{t}^{t+z} f(x) \, dx & \text{if } (y, z, t, s) \in R'_{3,2}.
\end{cases}
\]

Therefore, for \( k = 5 \) we have

\[
\int_{R'_{3,1}} = \left( \int_{\frac{3}{2}}^{\frac{15}{8}} \int_{\frac{1}{4}}^{\frac{4-2y}{3}} \int_{\frac{1}{4}}^{\frac{1-2y}{3}} \int_{0}^{t} \frac{15}{8} 4-2y \, dx \right) + \left( \int_{\frac{3}{2}}^{\frac{15}{8}} \int_{\frac{1}{4}}^{\frac{4-2y}{3}} \int_{0}^{t} \frac{15}{8} 4-2y \, dx \right) + \left( \int_{\frac{3}{2}}^{\frac{15}{8}} \int_{\frac{1}{4}}^{\frac{4-2y}{3}} \int_{0}^{t} \frac{15}{8} 4-2y \, dx \right)
\]

\[
\int_{R'_{3,2}} = \left( \int_{\frac{3}{2}}^{\frac{15}{8}} \int_{\frac{1}{4}}^{\frac{4-2y}{3}} \int_{\frac{1}{4}}^{\frac{1-2y}{3}} \int_{0}^{t} \frac{15}{8} 4-2y \, dx \right) + \left( \int_{\frac{3}{2}}^{\frac{15}{8}} \int_{\frac{1}{4}}^{\frac{4-2y}{3}} \int_{0}^{t} \frac{15}{8} 4-2y \, dx \right) + \left( \int_{\frac{3}{2}}^{\frac{15}{8}} \int_{\frac{1}{4}}^{\frac{4-2y}{3}} \int_{0}^{t} \frac{15}{8} 4-2y \, dx \right)
\]

\[
\frac{1 - 2\vartheta_0 - \vartheta z}{\vartheta_0 y z (1 - \vartheta (y + z))} \left( \int_{t}^{t+z} f(x) \, dx \right)^2 (t - s)^{k-3} \, ds \, dt \, dz \, dy > 0.347,
\]

\[
\frac{1 - 2\vartheta_0 - \vartheta z}{\vartheta_0 y z (1 - \vartheta (y + z))} \left( \int_{t}^{t+z} f(x) \, dx \right)^2 (t - s)^{k-3} \, ds \, dt \, dz \, dy > 1.564,
\]
which gives
\[ J_{3,2} > 0.9555. \]  
(2.40)

### 2.4 Proof of main theorem 1

We combine (2.9), (2.21), (2.23), (2.30), (2.35), (2.37), (2.38), (2.39), (2.40) and prove that
\[ \Psi_{\text{ext}}^{5} \left( F; \frac{1}{4}, \frac{3}{8} \right) \leq \frac{5(J_{0-\frac{1}{2}}(J_{1}+J_{2}+J_{3}+J_{4}+J_{2,2}+J_{3,2}))}{f} + \frac{40}{3} < 14.98582. \]  
(2.41)

Hence, equation (2.10) implies
\[ \bar{\Psi}_{\text{ext}}^{5} \left( F; \frac{1}{4} - \epsilon, \frac{3}{8}, \epsilon \right) < 14.98582 + O(\epsilon). \]  
(2.42)

The inequality above combined with Theorem 2.1 implies Main Theorem 1.

Remark 4. Taking \( k = 3 \) and
\[ F(t_1, t_2, t_3) = \begin{cases} 256 + 819(1 - P_1) + 833(1 - P_1)^2, & \text{if } (t_1, t_2, t_3) \in R_3', \\ 0, & \text{otherwise}, \end{cases} \]
we can perform analogous calculations in order to show that
\[ \bar{\Psi}_{\text{ext}}^{3} \left( F; \frac{1}{4} - \epsilon, \frac{3}{8}, \epsilon \right) < 7.928 + O(\epsilon). \]

This reproves the main theorem from [10].

### 3 LEMMAS

#### 3.1 The quantities \( T_{\delta_1, \ldots, \delta_k} \) and \( T^{(\varphi)}_\delta \)

In the next sections, we frequently apply a useful lemma proven in [12, (5.9)] enabling us to estimate the Selberg weights (in [12] the author assumes supp \( F \subset R_k \), but in the supp \( F \subset R'_k \) case the proof is analogous).

**Lemma 3.1.** We have
\[ \lambda_{\text{max}} \ll y_{\text{max}} (\log R)^k. \]

In [11], the estimation of the two sums \( T_\delta \) and \( T^{\varphi}_\delta \) makes up the crucial part of the proof. We propose their analogues for the multidimensional sieve.

\[ T_{\delta_1, \ldots, \delta_k} = \sum' \frac{\lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k}}{\prod_{i=1}^{k} \delta_i}, \]  
(3.1)
Here and from now on $\Sigma'$ denotes that $[d_1, e_1], \ldots, [d_k, e_k]$ are pairwise coprime.

In the next lemma, we find a representation of $T_{\delta_1, \ldots, \delta_k}$ in almost diagonal form. To this end, we combine the methods from the proofs of [12, Lemma 5.1] and [5, Lemma 6].

**Lemma 3.2.** For $\delta_1, \ldots, \delta_k \in \mathbb{N}$ pairwise coprime satisfying $\sum_{i=1}^{k} \omega(\delta_i) \ll 1$ we have

$$T_{\delta_1, \ldots, \delta_k} = \sum_{u_1, \ldots, u_k} \frac{1}{\prod_{i=1}^{k} \varphi(u_i)} \left( \sum_{s_1, \ldots, s_k} \mu(s_1) \cdots \mu(s_k) \prod_{i=1}^{k} g_{u_i s_1, \ldots, u_k s_k} \right)^2 + O\left( \frac{y^2 \varphi(W)^k (\log R)^k}{W^k D_0} \right).$$

**Proof.** From the support of $y$ we observe that those $u_i$ and $s_i$ that are not coprime to $W$ provide no contribution to the sum. Hence, we may replace $\delta_i$ by $(\delta_i, W)$ freely, and consequently, we can add an extra assertion that $(\prod_{i=1}^{k} \delta_i, W) = 1$ without losing generality. For each $i = 1, \ldots, k$ put

$$g_i(d) = \frac{d}{(d, \delta_i)}.$$

With this notation, we get

$$T_{\delta_1, \ldots, \delta_k} = \sum_{d_1, \ldots, d_k} \frac{\lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k}}{\prod_{i=1}^{k} g_i((d_i, e_i))} \prod_{i=1}^{k} g_i(d_i).$$

Consider the Dirichlet convolutions

$$\overline{g}_i = g_i * \mu.$$

Note that

$$\overline{g}_i(p) = \begin{cases} \varphi(p) & \text{for } p \nmid \delta_i, \\ 0 & \text{if } p | \delta_i. \end{cases}$$

By Möbius inversion, we have

$$T_{\delta_1, \ldots, \delta_k} = \sum_{d_1, \ldots, d_k} \frac{\lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k}}{\prod_{i=1}^{k} g_i((d_i, e_i))} \prod_{i=1}^{k} \overline{g}_i(u_i)$$

$$= \sum_{u_1, \ldots, u_k} \left( \prod_{i=1}^{k} \varphi(u_i) \right) \sum_{d_1, \ldots, d_k} \frac{\lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k}}{\prod_{i=1}^{k} g_i((d_i, e_i))} \prod_{i=1}^{k} \overline{g}_i(u_i).$$
Note that all the $u_i$ are square-free, since $u_i | d_i$, $e_i$ and both $d_i$ and $e_i$ are also square-free for each $i$. We wish to drop the dependencies between the $d_i$ and the $e_j$ variables in the inner sum. The requirement that $[d_1, e_1], ..., [d_k, e_k]$ are pairwise coprime is equivalent to $(d_i, e_j) = 1$ for all $i \neq j$. This is because $\lambda_{d_1, ..., d_k}$ is supported only on integers $d_1, ..., d_k$ satisfying $(d_i, d_j) = 1$ for all $i \neq j$. We can also remove the restriction $(d_i, e_j) = 1$ for all $i \neq j$ by noticing that

$$1_{(d_i, e_j) = 1} = \sum_{s_i, j | d_i, e_j} \mu(s_i, j).$$

Hence, $T_{\delta_1, ..., \delta_k}$ equals

$$\sum_{u_1, ..., u_k} \left( \prod_{i=1}^k \varphi(u_i) \right) \sum_{s_1, j, s_2, k-1} \left( \prod_{1 \leq i, j \leq k} \mu(s_i, j) \right) \sum_{d_1, ..., d_k, e_1, ..., e_k} \left( \prod_{i=1}^k g_i(d_i) \right) \left( \prod_{i=1}^k g_i(e_i) \right).$$

(3.4)

Note that $\lambda_{d_1, ..., d_k} = 0$, unless $(d_i, d_j) = 1$ for all $i \neq j$, so the $s_i, j$ that are not coprime to $u_i$ or $u_j$ give no contribution to the sum. We can also impose further constraint that $s_i, j$ has to be coprime to $s_i, j$, if these two share any coordinate. We denote this condition by writing $*$ next to the sum. We define

$$w_{r_1, ..., r_k} = \left( \prod_{i=1}^k \mu(r_i) \varphi(r_i) \right) \sum_{d_1, ..., d_k} \lambda_{d_1, ..., d_k} \frac{\lambda_{e_1, ..., e_k}}{\prod_{i=1}^k g_i(d_i)}.$$  

(3.5)

By Lemma 1.11 and the definition of $w_{r_1, ..., r_k}$ we can change the variables as follows:

$$\lambda_{d_1, ..., d_k} = \left( \prod_{i=1}^k \mu(d_i) g_i(d_i) \right) \sum_{r_1, ..., r_k} \frac{w_{r_1, ..., r_k}}{\prod_{i=1}^k \varphi(r_i)}.$$  

(3.6)

From (3.6), by the Möbius inversion we obtain

$$\sum_{d_1, ..., d_k} \frac{\lambda_{d_1, ..., d_k}}{\prod_{i=1}^k g_i(d_i)} = \sum_{r_1, ..., r_k} w_{r_1, ..., r_k} \sum_{d_1, ..., d_k} \frac{\lambda_{d_1, ..., d_k}}{\prod_{i=1}^k \varphi(r_i)} = \sum_{a_1, ..., a_k} \frac{\mu(a_1)}{\prod_{i=1}^k \varphi(a_i)}.$$  

(3.7)

where $a_j = u_j \prod_{i \neq j} s_{j, i}$. An analogous calculation leads us to

$$\sum_{e_1, ..., e_k} \frac{\lambda_{e_1, ..., e_k}}{\prod_{i=1}^k g_i(e_i)} = w_{b_1, ..., b_k} \sum_{b_1, ..., b_k} \frac{\mu(b_1)}{\prod_{i=1}^k \varphi(b_i)}.$$  

(3.8)
where \( b_j = u_j \prod_{i \neq j} s_{i,j} \). We observe that when \( a_j \), respectively, \( b_j \), are not all square-free, the right-hand side of (3.7), respectively, (3.9), vanishes, so we can rewrite \( \mu(a_j) \) as \( \mu(u_j) \prod_{i \neq j} \mu(s_{j,i}) \) and do the same thing with \( \mu(b_j), \varphi(a_j) \) and \( \varphi(b_j) \). Combining this fact with (3.4), (3.7), and (3.9), we find

\[
T_{\delta_1,...,\delta_k} = \sum_{u_1,...,u_k} \left( \prod_{i=1}^{k} \frac{\mu(u_i)^2}{\varphi(u_i)} \right) \sum_{s_{1,2,...,k,k-1}} \left( \prod_{1 \leq i, j \leq k, i \neq j} \frac{\mu(s_{i,j})}{\varphi(s_{i,j})^2} \right) w_{a_1,...,a_k} w_{b_1,...,b_k}. \tag{3.9}
\]

Again, note that the summands of the inner sum vanish when the \( a_j \) and \( b_j \) are not all square-free, by the definition of \( w_{r_1,...,r_k} \).

The relation \( s_{i,j} | d_i \) for all \( i \neq j \) implies that there is no contribution from \( s_{i,j} \) satisfying \((s_{i,j}, W) \neq 1\). Therefore, we can restrict our considerations to the \( s_{i,j} \) such that \( s_{i,j} = 1 \) for all \( i \neq j \) or \( s_{l,m} > D_0 \) for some pair \( l \neq m \).

Consider the case when \( s_{i,j} = 1 \) for all \( i \neq j \). We have to calculate the expression

\[
\sum_{u_1,...,u_k} \frac{w^2_{u_1,...,u_k}}{\prod_{i=1}^{k} \varphi(u_i)}. \tag{3.10}
\]

We have

\[
w_{r_1,...,r_k} = \left( \prod_{i=1}^{k} \frac{\mu(r_i)\varphi(r_i)}{g_i(r_i)} \right) \sum_{d_1,...,d_k} \frac{\lambda_{d_1,...,d_k,r_k}}{\prod_{i=1}^{k} g_i(d_i)}
= \left( \prod_{i=1}^{k} \frac{\mu(r_i)\varphi(r_i)}{g_i(r_i)} \right) \sum_{d_1,...,d_k} \left( \prod_{i=1}^{k} \frac{\mu(d_i r_i) d_i r_i}{g_i(d_i)} \right) \sum_{t_1,...,t_k} \frac{y_{t_1 d_1 r_1,...,t_k d_k r_k}}{\prod_{i=1}^{k} \varphi(t_i d_i r_i)}. \tag{3.11}
\]

For \( i = 1, \ldots, k \) and square-free \( m \) we consider the sum

\[
\sum_{d | m} \frac{\mu(d)d}{g_i(d)} = \sum_{d | m} \mu(d)(d, \delta_i).
\]

We split \( d = d' d'' \), where \((d', \delta_i) = 1\) and \(d'' | \delta_i\). Then the sum above equals

\[
\sum_{d' | m} \mu(d') \sum_{d'' | m} \mu(d'')(d'', \delta_i). \tag{3.12}
\]

The first factor of (3.12) equals

\[
\sum_{d' | m} \mu(d') = \prod_{p | \delta_i} \left( 1 - \frac{1}{p} \right) = 1_{m | \delta_i}.
\]
The second factor of (3.12) equals 
\[ \sum_{d''|m} \mu(d'')(d'', \delta_i) = \prod_{p|m, \delta_i} (1 - p) = \mu((m, \delta_i)) \varphi((m, \delta_i)). \]

Therefore, we find that 
\[ w_{r_1, \ldots, r_k} = \left( \prod_{i=1}^{k} (r_i, \delta_i) \right) \sum_{m_1, \ldots, m_k} \forall_i m_i | \delta_i y_{m_1, \ldots, m_k r_k} \prod_{i=1}^{k} \mu(m_i). \]  
(3.13)

Note that after substituting (3.13) into (3.10) the product of \((r_i, d_i)\) disappears.

Recall that the number of prime factors of \(\delta_1 \cdots \delta_k\) is bounded. In the case when \(s_{l,m} > D_0\) for some pair \(l \neq m\) the contribution of (3.9) is 
\[ \ll (y_{\text{max}})^2 \left( \prod_{u<R} \sum_{(u, \delta_i W)=1} \frac{\mu(u)^2}{\varphi(u)} \right) \sum_{s|m, \delta_i W=1} \frac{\mu(s_{l,m})^2(s_{l,m}, \delta_i)(s_{l,m}, \delta_m)}{\varphi(s_{l,m})^2} \]
\[ \times \left( \prod_{i,j=1}^{k} \sum_{s=1}^{\infty} \frac{\mu(s)^2(s, \delta_i)(s, \delta_j)}{\varphi(s)^2} \right) \]  
(3.14)

where we used identity (3.13) and \((a_i, \delta_i) = \prod_{i \neq j} (s_{j,i}, \delta_i)\) and an analogous one for the \(b_i\). By factoring \(s = s's''\), where \((s', \delta_i) = 1\) and \(s'' | \delta_i\), we find that 
\[ \sum_{s=1}^{\infty} \frac{\mu(s)^2(s, \delta_i)(s, \delta_j)}{\varphi(s)^2} = \sum_{s'' | \delta_i} s'' \mu(s'')^2(s'', \delta_j) \sum_{s'=1}^{\infty} \frac{\mu(s')^2(s', \delta_j)}{\varphi(s')^2}. \]  
(3.15)

A trivial estimation gives 
\[ \sum_{s'' | \delta_i} s'' \mu(s'')^2(s'', \delta_j) \leq \prod_{\rho | \delta_i} \left( 1 + \frac{p^2}{(p - 1)^2} \right) \ll 1. \]  
(3.16)

We repeat this procedure for \((s', \delta_j)\) in order to get 
\[ \sum_{s=1}^{\infty} \frac{\mu(s)^2(s, \delta_i)(s, \delta_j)}{\varphi(s)^2} \ll 1. \]  
(3.17)
By the same method, we also have

\[
\sum_{s > D_0} \frac{\mu(s)^2(s, \delta)(s, \delta_m)}{\varphi(s)^2} = \sum_{s'', \delta_l} \frac{\mu(s'')^2(s'', \delta_m)}{\varphi(s'')^2} \sum_{s' > D_0, (s', \delta_l W) = 1} \frac{\mu(s')^2(s', \delta_m)}{\varphi(s')^2} \ll \sum_{s > D_0} \frac{\mu(s)^2(s, \delta_m)}{\varphi(s)^2} + \frac{1}{D_0} \sum_{s = 1}^{\infty} \frac{\mu(s)^2(s, \delta_m)}{\varphi(s)^2},
\]

(3.18)

where we used the fact that

\[
\sum_{s \mid \delta_l, s \neq 1} \frac{\mu(s)^2 s}{\varphi(s)^2} = \left( \prod_{p \mid \delta_l} \left( 1 + \frac{p}{(p - 1)^2} \right) \right) - 1 \ll \left( 1 + O \left( \frac{1}{D_0} \right) \right)^{\omega(\delta_l)} - 1 \ll \frac{1}{D_0}.
\]

(3.19)

We repeat the splitting procedure for \( s' \) and combining the result with (3.18) we obtain

\[
\sum_{s_{1, m} > D_0} \frac{\mu(s_{1, m})^2(s_{1, m}, \delta)(s_{1, m}, \delta_m)}{\varphi(s_{1, m})^2} \ll \frac{1}{D_0}.
\]

(3.20)

Combining (3.15)–(3.20) we get that the total error term coming from (3.14) is

\[
\ll O \left( \frac{y_{\max}^2 \varphi(W)^k (\log R)^k}{W^k D_0} \right).
\]

Note that in the inequality above we take advantage of the fact that

\[
\sum_{u < R} \frac{1}{\varphi(u)} \ll \prod_{D_0 < p < R} \left( 1 + \frac{1}{p - 1} \right) = \prod_{D_0 < p < R} \left( 1 - \frac{1}{p} \right)^{-1} \ll \frac{\varphi(W)}{W} \log R
\]

by the Mertens’ third theorem. We will use this estimation a few times throughout the paper without mentioning it explicitly.

We define a function which is going to be useful in the next lemma.

\[
\omega_{r_1, \ldots, r_k}^{(X)} = \left( \prod_{i=1}^{k} \mu(r_i) \right) \left( \prod_{i=1}^{k} \psi(r_i) \right) \sum_{d_{r_1, \ldots, d_k} \mid d \mid \delta} \lambda_{d_{r_1, \ldots, d_k}} \frac{\varphi(d_i)}{\prod_{i \neq \ell \mid d} \varphi(d_i)},
\]

(3.21)

where \( \psi \) is the totally multiplicative function defined by \( \psi(p) = p - 2 \) for all primes. Let us put \( w_{\max}^{(X)} = \sup_{r_1, \ldots, r_k} \omega_{r_1, \ldots, r_k}^{(X)} \). The function \( w_{r_1, \ldots, r_k}^{(X)} \) can be perceived as a generalization of \( y_{r_1, \ldots, r_k}^{(X)} \) from [12].
We now state a lemma, similar to the previous one, referring to $T_\delta^{(\varepsilon)}$ (cf. (3.2)).

**Lemma 3.3.** For $\delta \in \mathbb{N}$ satisfying $\omega(\delta) \ll 1$ we have

$$T_\delta^{(\varepsilon)} = \sum_{u_1, \ldots, u_k} \left( \prod_{i=1}^k \psi(u_i) \right)^2 + O \left( \frac{(w_{\text{max}}^{(\varepsilon)})^2 \varphi(W)^{k-1} \log^{k-1} R}{W^{k-1} D_0} \right).$$

**Proof.** We proceed very much like in the proof of the previous lemma. Again, we wish to drop the dependencies between the $d_i$ and the $e_j$ variables. This time we proceed by using the following identity valid for square-free numbers:

$$\frac{1}{\varphi([d_1, e_1])} = \frac{1}{\varphi(d_1 \varphi(e_1)} \sum_{u_1 \mid [d_1, e_1]} \psi(u_1).$$

Therefore,

$$T_\delta^{(\varepsilon)} = \sum_{u_1, \ldots, u_k} \left( \prod_{i=1}^k \psi(u_i) \right) \sum_{d_1, \ldots, d_k} \lambda_{d_1, \ldots, d_k} \frac{\lambda_{e_1, \ldots, e_k}}{\varphi(d_1) \varphi(e_1)} \left( \prod_{1 \leq i, j \leq k \neq \ell} \mu(a_i) \right).$$

Recall that $\Sigma'$ denotes that $[d_1, e_1], \ldots, [d_k, e_k]$ are pairwise coprime. To slightly simplify our notation we will introduce an extra variable $u_\varepsilon$ under the exterior sum and we make it equal 1. We rewrite the conditions $(d_i, e_j) = 1$ by multiplying the expression by $\sum s_{i,j} \mid d_i, e_j$ $\mu(s_{i,j})$ for each $i \neq j$. We also note that if $i$ or $j$ equals $\varepsilon$, then $s_{i,j} \mid \delta$. We again impose the same conditions on $s_{i,j}$ as in Lemma 3.2. Hence,

$$T_\delta^{(\varepsilon)} = \sum_{u_1, \ldots, u_k} \left( \prod_{i=1}^k \psi(u_i) \right) \sum_{s_{1,2}, \ldots, s_{k-1}} \lambda_{s_{1,2}, \ldots, s_{k-1}} \left( \prod_{1 \leq i, j \leq k \neq \ell} \mu(a_i) \right) \sum_{d_1, \ldots, d_k} \lambda_{d_1, \ldots, d_k} \frac{\lambda_{e_1, \ldots, e_k}}{\varphi(d_1) \varphi(e_1)}.$$  

(3.22)

Again, $\Sigma'$ denotes that $s_{i_1, j_1}$ is coprime to $s_{i_2, j_2}$, if these two share any coordinate. Transforming (3.21) a bit, we obtain

$$\sum_{d_1, \ldots, d_k} \lambda_{d_1, \ldots, d_k} \frac{\lambda_{e_1, \ldots, e_k}}{\varphi(d_1) \varphi(e_1)} = \frac{w_{\text{max}}^{(\varepsilon)}}{\prod_{i \neq \ell} \psi(a_i)} \prod_{i=1}^k \mu(a_i).$$  

(3.23)
and a similar equation for the $b_i$ with $a_i$ and $b_i$ defined as in Lemma 3.2. Substituting this into (3.22), we obtain

$$T^{(\ell)}_\delta = \sum_{u_1, \ldots, u_k \atop u_r = 1} \left( \prod_{i=1}^k \frac{\mu(u_i)^2}{\psi(u_i)} \right) \sum_{s_1, \ldots, s_{k-1}}^{*} \left( \prod_{1 \leq i, j \leq k} \frac{\mu(s_{i,j})}{\psi(s_{i,j})^2} \right) \left( \prod_{1 \leq i, j \leq k} \frac{\mu(s_{i,j})}{\psi(s_{i,j})} \right) w^{(\ell)}_{a_1, \ldots, a_k} w^{(\ell)}_{b_1, \ldots, b_k}. \quad (3.24)$$

We can see that the contribution coming from $s_{i,j} \neq 1$ for $i, j \neq \ell$ is

$$\ll (w^{(\ell)}_{\max})^2 \left( \sum_{u < R \atop (u, W) = 1} \frac{\mu(u)^2}{\psi(u)^k} \right) \left( \sum_{s_{i,j} > D_0} \frac{\mu(s_{i,j})^2}{\psi(s_{i,j})^2} \right) \left( \sum_{s \mid \delta} \frac{\mu(s)^2}{\psi(s)} \right)^{2k-2} \phi(W)^{k-1} \log^{k-1} R / W^{k-1}D_0.$$ 

On the other hand, if $i$ or $j$ equals $\ell$ then the contribution from $s_{i,j}$ is

$$\ll (w^{(\ell)}_{\max})^2 \phi(W)^{k-1} \log^{k-1} R / W^{k-1}D_0.$$ 

Thus, the main term equals

$$\sum_{u_1, \ldots, u_k \atop u_r = 1} \left( \prod_{i=1}^k \frac{\mu(u_i)^2}{\psi(u_i)} \right) \frac{w^{(\ell)}_{a_1, \ldots, a_k} w^{(\ell)}_{b_1, \ldots, b_k}}{\prod_{i \neq \ell} \psi(u_i)}. \quad (3.25)$$

By (3.25) we need to calculate $w^{(\ell)}_{r_1, \ldots, r_k}$ in terms of $y_{r_1, \ldots, r_k}$ only when the $\ell$th coordinate of $w^{(\ell)}_{r_1, \ldots, r_k}$ equals 1. However, we can just as easily deal with a slightly more general situation.

**Lemma 3.4.** For $r_1, \ldots, r_k \in \mathbb{N}$ and $\delta$ satisfying $\omega(\delta) \ll 1$, we have

$$w^{(\ell)}_{r_1, \ldots, r_k} = \mu(r_\ell) \left( \sum_{r_r \mid \delta} \mu(r_r) \sum_{a \mid \delta} \frac{y_{r_1, \ldots, r_{\ell-1}, a, r_{\ell+1}, \ldots, r_k}}{\varphi(a)} \right) \left( 1 + O \left( \frac{1}{D_0} \right) \right) + O \left( y_{\max} \phi(W) \log R / WD_0 \right).$$
Proof. We wish to calculate $w_{r_1, \ldots, r_k}(\ell)$ in the terms of $y_{r_1, \ldots, r_k}$. Note that by the definition of $w_{r_1, \ldots, r_k}$, it equals 0 if $r_\ell \nmid \delta$, so we will further assume that $r_\ell | \delta$. We obtain

$$w_{r_1, \ldots, r_k}(\ell) = \left( \prod_{i=1}^{k} \mu(r_i) \right) \left( \prod_{i=1}^{k} \psi(r_i) \right) \sum_{d_1, \ldots, d_k \atop \forall i \neq \ell \atop d_\ell | \delta} \lambda_{d_1, \ldots, d_k} \prod_{i \neq \ell} \varphi(d_i)$$

$$= \left( \prod_{i=1}^{k} \mu(r_i) \right) \left( \prod_{i=1}^{k} \psi(r_i) \right) \sum_{a_1, \ldots, a_k} y_{a_1, \ldots, a_k} \sum_{d_1, \ldots, d_k \atop \forall i \neq \ell} \prod_{i=1}^{k} \frac{\varphi(a_i)}{\varphi(d_i)} \left( \prod_{i=1}^{k} \frac{\mu(d_i)d_i}{\varphi(d_i)} \right) \mu(d_\ell)d_\ell,$$

where the $a_i$ are new variables and are not related to these from equation (3.24). We have

$$\sum_{d_1, \ldots, d_\ell-1, d_\ell+1, \ldots, d_k \atop \forall i \neq \ell \atop d_\ell | \delta} \prod_{i=1}^{k} \frac{\mu(d_i)d_i}{\varphi(d_i)} = 1_{[\ell \neq \ell]} \prod_{i=1}^{k} \frac{\mu(a_i)r_i}{\varphi(a_i)},$$

and

$$\sum_{d_\ell | (a_\ell, \delta)} \mu(d_\ell)d_\ell = 1_{r_\ell | (a_\ell, \delta)} \mu(r_\ell) r_\ell \prod_{p | (a_\ell, \delta)/r_\ell} (1-p) = 1_{r_\ell | a_\ell} \frac{r_\ell}{\varphi(r_\ell)} \mu((a_\ell, \delta)) \varphi((a_\ell, \delta)).$$

We see that $y_{a_1, \ldots, a_k}$ is supported only on the $a_i$ satisfying $(a_1, W) = 1$, so either $a_j = r_j$ for all $j \neq \ell$ or $a_j > D_0 r_j$ for at least one of them. The total contribution of the latter case is

$$\ll y_{\max} \left( \prod_{i=1}^{k} \frac{\psi(r_i)}{\varphi(r_i)} \right) \frac{r_\ell}{\varphi(r_\ell)} \left( \sum_{a_1, \ldots, a_k \atop (a_1, W) = 1} \frac{\mu(a_1)^2}{\varphi(a_1)^2} \right) \left( \sum_{a_\ell < R \atop (a_\ell, W) = 1} \frac{\mu(a_\ell)^2}{\varphi(a_\ell)^2} \right) \prod_{i=1}^{k} \left( \sum_{a_i | a_\ell \atop i \neq j, \ell} \frac{\mu(a_i)^2}{\varphi(a_i)^2} \right) \prod_{i=1}^{k} \frac{\mu(a_i)^2}{\varphi(a_i)^2} \frac{\varphi(W) \log R}{WD_0} \ll y_{\max} \frac{\varphi(W) \log R}{WD_0},$$

$$\ll \tau(\delta) r_\ell \left( \prod_{i=1}^{k} \frac{\psi(r_i)}{\varphi(r_i)^2} \right) \frac{y_{\max} \varphi(W) \log R}{WD_0} \ll y_{\max} \frac{\varphi(W) \log R}{WD_0},$$
where in the summation over $a_\ell$, we factorized $a = a'$'s, where $s|\delta$ and $(a', \delta) = 1$, to obtain

$$\sum_{a_\ell < R} \frac{\mu(a_\ell)^2 \varphi((a_\ell, \delta))}{\varphi(a_\ell)} = \sum_{s|\delta} \frac{\mu(s)^2 \varphi(s)}{\varphi(s)} \sum_{a' < R/s} \frac{\mu(a')^2}{\varphi(a')} \ll \frac{\tau(\delta) \varphi(W) \log R}{W}.$$ 

Therefore, we have

$$w_{r_1, \ldots, r_k}^{(\ell)} = \left( \prod_{i=1 \atop i \neq \ell}^k \frac{\psi(r_i)r_i}{\varphi(r_i)^2} \right) \left( \prod_{p > D_0} \left( 1 - \frac{1}{p^2 - 2p + 1} \right) \right) \exp \left( -O \left( \frac{1}{D_0} \right) \right) = 1 - O \left( \frac{1}{D_0} \right)$$

and

$$1 \leq \frac{r_\ell}{\varphi(r_\ell)} \leq \prod_{p|r_\ell} \left( 1 + \frac{1}{D_0 - 1} \right) \leq 1 + O \left( \frac{1}{D_0} \right).$$

Decomposing $a = a'$'s, where $s|\delta$ and $(a', \delta) = 1$, we obtain

$$\sum_{a_\ell < R} \frac{y_{r_1, \ldots, r_k} \psi((a_\ell, \delta)) \varphi((a_\ell, \delta))}{\varphi(a_\ell)} = \sum_{s|\delta} \frac{\mu(s)^2 \varphi(s)}{\varphi(s)} \sum_{a' < R/s} \frac{\mu((a_\ell, \delta)) \varphi((a_\ell, \delta))}{\varphi(a')}.$$

To convert certain sums into integrals, we use the following lemma.

**Lemma 3.5.** Let $\kappa, A_1, A_2, L > 0$. Let $\gamma$ be a multiplicative function satisfying

$$0 \leq \frac{\gamma(p)}{p} \leq 1 - A_1$$

and

$$-L \leq \sum_{w < p \leq z} \frac{\gamma(p) \log p}{p} - \kappa \log(z/w) \leq A_2$$

for any $2 \leq w \leq z$. Let $g$ be the completely multiplicative function defined on primes by

$$g(p) = \frac{\gamma(p)}{p - \gamma(p)}.$$
Finally, let $G : [0, 1] \to \mathbb{R}$ be a piecewise differentiable function and let

$$G_{\text{max}} = \sup_{t \in [0,1]} (|G(t)| + |G'(t)|).$$

Then

$$\sum_{d<z} \mu(d)^2 g(d)G(\frac{\log d}{\log z}) = \mathfrak{E} (\frac{\log z}{\Gamma(\kappa)}) \int_0^1 G(x)x^{\kappa-1} \, dx + O_{A_1,A_2,\kappa}(\mathfrak{E}L_{G_{\text{max}}} \log^{\kappa-1} z),$$

where

$$\mathfrak{E} = \prod_p \left( 1 - \frac{\gamma(p)}{p} \right)^{-1} \left( 1 - \frac{1}{p} \right)^{\kappa}.$$  

Here the constant implied by the “$O$” term is independent of $G$ and $L$.

**Proof.** This is [5, Lemma 4] with the notation from [12]. \qed

For the sake of convenience, we denote the main terms from Lemmas 3.2, 3.3, and 3.4 as follows:

$$T_{\delta_1,...,\delta_k} = \sum_{u_1,...,u_k} \frac{1}{\prod_{i=1}^k \varphi(u_i)} \left( \sum_{s_1,...,s_k} \mu(s_1) ... \mu(s_k) y_{u_1s_1,...,u_ks_k} \right)^2,$$  

$$T_{\delta}^{(\ell)} = \sum_{r_1,...,r_k} \frac{(w_{r_1,...,r_k}^{(\ell)})^2}{\prod_{i \neq \ell} \psi(r_i)},$$  

$$w_{r_1,...,r_k}^{(\ell)} = \sum_{r_\ell | s | \delta} \mu(s) \sum_{(u, \delta) = 1} y_{r_1,...,r_{\ell-1} s u_{r_\ell+1},...,r_k} \psi(u).$$

**Lemma 3.6.** Fix a positive integer $r$. Let $\delta = p_1 \cdots p_{r-1}$ for some distinct primes $p_1, \ldots, p_{r-1}$ greater than $D_0$. Then we have

$$T_{1,1,\delta,1,1,1} = \frac{\varphi(W)^k (\log R)^k}{W^k} I^{(j)}_0 \left( \log p_1 / \log R, \ldots, \log p_{r-1} / \log R \right) + O \left( \frac{F_{\text{max}} \varphi(W)^k (\log R)^k}{W^k D_0} \right),$$

where $\delta$ appears on the $m$th coordinate and

$$I^{(j)}_0(x_1, \ldots, x_{r-1}) = \int_0^1 \cdots \int_0^1 \left( \sum_{J \subset \{1, \ldots, r-1\}} (-1)^{|J|} F \left( t_1, \ldots, t_{m-1}, t_m + \sum_{j \in J} x_j, t_{m+1}, \ldots, t_k \right) \right)^2 \, dt_1 \ldots dt_k.$$
Proof. The expression (3.29) simplifies to

$$\tilde{T}_{1,\ldots,1,\delta,1,\ldots,1} = \sum_{u_1,\ldots,u_k \mid u_m,\delta=1} \frac{1}{\varphi(u)} \left( \sum_{s \mid \delta} \mu(s) \gamma_{u_1,\ldots,u_{m-1},su_m,u_{m+1},\ldots,u_k} \right)^2.$$ 

To replace the weights $y$ by a function $F$ we need the following conditions to be satisfied: $\mu(sU)^2 = 1$ and $(sU, W) = 1$, where $U = \prod_{i=1}^k u_i$. To achieve such a situation, we wish to impose the condition $(U, \delta) = 1$. We can do this at the cost of the error of size

$$\ll F^2 \max_{p \mid \delta} \sum_{u_1,\ldots,u_k < R \mid u} \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} \ll F^2 \max_{p \mid \delta} \sum_{u < R \mid u} \left( \sum_{u_1,\ldots,u_k \mid u} \mu(u)^2 \varphi(u) \right)^k \ll \frac{F^2 \varphi(W)^k (\log R)^k}{W^k D_0}.$$ 

On the other hand, the support of $y_{u_1,\ldots,u_r}$ forces $(u_i, u_j) = 1$ for each $i \neq j$. Note that if $(u_i, u_j) \neq 1$, then the common factor of $u_i$ and $u_j$ has to be greater than $D_0$. Thus, the discussed requirement can be dropped at the cost of

$$\ll F^2 \max_{p > D_0} \sum_{u_1,\ldots,u_k < R \mid u} \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} \ll F^2 \max_{p > D_0} \sum_{u < R \mid u} \left( \sum_{u_1,\ldots,u_k \mid u} \mu(u)^2 \varphi(u) \right)^k \ll \frac{F^2 \varphi(W)^k (\log R)^k}{W^k D_0}.$$ 

Hence, we can rewrite $\tilde{T}_{1,\ldots,1,\delta,1,\ldots,1}$ as

$$\sum_{u_1,\ldots,u_k \mid u_1,\delta W = 1} \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} \left( \sum_{s \mid \delta} \mu(s) F \left( \frac{\log u_1}{\log R}, \ldots, \frac{\log u_{m-1}}{\log R}, \frac{\log s}{\log R}, \frac{\log u_m}{\log R}, \ldots, \frac{\log u_k}{\log R} \right) \right)^2 + O \left( \frac{F^2 \varphi(W)^k (\log R)^k}{W^k D_0} \right). \tag{3.32}$$

For the sake of convenience, we put

$$F^{(m)}_\delta(t_1, \ldots, t_k) = \left( \sum_{s \mid \delta} \mu(s) F \left( t_1, \ldots, t_{m-1}, t_m + \frac{\log s}{\log R}, t_{m+1}, \ldots, t_k \right) \right)^2. \tag{3.33}$$

Note that we may slightly rearrange the sum at the right-hand side of (3.33) as

$$\sum_{J \subset \{1, \ldots, r-1\}} (-1)^{|J|} F \left( t_1, \ldots, t_{m-1}, t_m + \sum_{j \in J} \frac{\log p_j}{\log R}, t_{m+1}, \ldots, t_k \right). \tag{3.34}$$
We apply Lemma 3.5 to the sum in (3.32) with \( \kappa = 1 \) and

\[
\gamma(p) = \begin{cases} 
1, & p \nmid \delta W, \\
0, & \text{otherwise},
\end{cases}
\]

\[
L \ll 1 + \sum_{p \mid \delta W} \frac{\log p}{p} \ll \log D_0,
\]

and with \( A_1, A_2 \) suitable and fixed. Applying (3.34) and dealing with the sum over each \( u_i \) in turn gives

\[
\sum_{u_1, \ldots, u_k} \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} F^{(m)} \left( \log \frac{u_1}{\log R}, \ldots, \log \frac{u_k}{\log R} \right) = \frac{\varphi(W)^k (\log R)^k}{W^k} I^{(j)}_0 \left( \log \frac{p_1}{\log R}, \ldots, \log \frac{p_{r-1}}{\log R} \right) + O \left( \frac{F_{\max} \varphi(W)^k (\log R)^k}{W^k D_0} \right),
\]

where we took into account that

\[
1 \geq \frac{\varphi(\delta)}{\delta} = \prod_{p \mid \delta} \left( 1 - \frac{1}{p} \right) \geq \left( 1 - \frac{1}{D_0} \right)^{\omega(\delta)} \geq 1 - O \left( \frac{1}{D_0} \right).
\]

□

**Lemma 3.7.** For a square-free \( \delta \) coprime to \( W \) and pairwise coprime square-free \( r_1, \ldots, r_k \) such that \((\prod_{i=1}^k r_i, W) = 1, r_\ell \mid \delta \) and \((\prod_{i \neq \ell} r_i, \delta) = 1\) we have

\[
\tilde{w}_{(r_1, \ldots, r_k)}^{(\epsilon)} = \varphi(W) W \prod_{i=1}^k \frac{\varphi(r_i)}{r_i} (\log R) I^{(\epsilon)}_{\delta; r_1, \ldots, r_k} + O \left( \frac{F_{\max} \varphi(W) \log R}{WD_0} \right),
\]

where

\[
I^{(\epsilon)}_{\delta; r_1, \ldots, r_k} = \int_0^1 \sum_{r_\ell \mid \delta} \mu(s) F \left( \log \frac{r_1}{\log R}, \ldots, \log \frac{r_{\ell-1}}{\log R}, t_\ell + \log \frac{s}{\log R}, \log \frac{r_{\ell+1}}{\log R}, \ldots, \log \frac{r_k}{\log R} \right) dt_\ell.
\]

**Proof.** Again, we wish to substitute weights \( y_{r_1, \ldots, r_k} \) by a function \( F \). We are allowed to do so under the conditions \( \mu(us \prod_{i \neq \ell} r_i)^2 = 1 \) and \((u, W) = 1\), otherwise the weight vanishes. We have

\[
\tilde{w}_{(r_1, \ldots, r_k)}^{(\epsilon)} = \sum_{(u, \delta W \prod_{i \neq \ell} r_i) = 1} \frac{\mu(u)^2}{\varphi(u)} \sum_{r_\ell \mid \delta} \mu(s) F \left( \log \frac{r_1}{\log R}, \ldots, \log \frac{r_{m-1}}{\log R}, \log \frac{su}{\log R}, \log \frac{r_{m+1}}{\log R}, \ldots, \log \frac{r_k}{\log R} \right).
\]

We put

\[
F^{(\epsilon)}_{\delta, r_\ell}(t_1, \ldots, t_k) = \sum_{r_\ell \mid \delta} \mu(s) F \left( t_1, \ldots, t_{m-1}, t_m + \frac{\log s}{\log R}, t_{m+1}, \ldots, t_k \right).
\]
By Lemma 3.5 applied with $\kappa = 1$,

$$\gamma(p) = \begin{cases} 1, & p \nmid W \prod_{i \neq \ell} r_i, \\ 0, & \text{otherwise}, \end{cases}$$

$$L \ll 1 + \sum_{p \mid W} \frac{\log p}{p} \ll \sum_{p < \log R} \frac{\log p}{p} + \sum_{p \mid W} \frac{\log \log R}{\log R} \ll \log \log N,$$

and $A_1, A_2$ suitable and fixed, we get

$$\tilde{w}_{\ell_1, \ldots, \ell_k}(\ell) = \sum_{u \mid W} \psi(u) F_{\ell, r_1}^{(\ell)}(\log r_1 \log R, \ldots, \log r_k \log R)$$

$$= \phi(\delta) \phi(W) \prod_{i = 1}^{k} \left( \frac{\varphi(r_i)}{r_i} \log R \right) \int_0^1 \int_0^1 \cdots \int_0^1 \tilde{I}(\ell, x_1, \ldots, x_{r-1})^2 dt_1 \cdots dt_{\ell-1} dt_{\ell+1} \cdots dt_k,$$

$$+ O \left( \frac{F_{\max} \phi(W) \log \log N}{W} \right). \tag{3.37}$$

Again, we have to apply the inequality $\phi(\delta) / \delta \geq 1 + O(D_0^{-1})$ to complete the proof. \qed

Note that (3.37) implies

$$w_{\max}^{(\ell)} \ll \frac{F_{\max} \phi(W) \log R}{W}. \tag{3.38}$$

**Lemma 3.8.** Fix a positive integer $r$. Let $\delta = p_1 \cdots p_{r-1}$ for some distinct primes $p_1, \ldots, p_{r-1}$ greater than $D_0$. Then we have

$$\tilde{T}_\delta^{(\ell)} = \frac{\phi(W)^{k+1} (\log R)^{k+1}}{W^{k+1}} \mathcal{I}^{(\ell)}(\log p_1 \log R, \ldots, \log p_{r-1} \log R) + O \left( \frac{F_{\max}^2 \phi(W)^{k+1} (\log R)^{k+1}}{W^{k+1} D_0} \right),$$

where

$$\mathcal{I}^{(\ell)}(x_1, \ldots, x_{r-1}) = \int_0^1 \cdots \int_0^1 \mathcal{I}^{(\ell)}(x_1, \ldots, x_{r-1})^2 dt_1 \cdots dt_{\ell-1} dt_{\ell+1} \cdots dt_k,$$

$$\mathcal{T}(x_1, \ldots, x_{r-1}) = \sum_{J \subseteq \{1, \ldots, r-1\}} (-1)^{|J|} \left( t_1, \ldots, t_{\ell-1}, t_\ell + \sum_{j \in J} x_j, t_{\ell+1}, \ldots, t_k \right) dt_\ell.$$

**Proof.** Recall

$$\tilde{T}_\delta^{(\ell)} = \sum_{(r_1, \ldots, r_k) \mid W} \left( \frac{w_{\max}^{(\ell)} r_1 \cdots r_k}{\prod_{i \neq \ell} \psi(r_i)} \right). \tag{3.39}$$
We need to impose the condition \((\prod_{i=1}^{k} r_i, \delta) = 1\) under the sum. It creates the error term of the following size

\[
\ll \frac{F_{\text{max}}^2 \varphi(W)^2 \log^2 R}{W^2} \left( \sum_{p \mid \delta} \sum_{t \in R \cap (t, W) = 1 \atop p \mid t} \frac{1}{\psi(t)} \right) \left( \sum_{r \in R \cap (r, W) = 1} \frac{1}{\psi(r)} \right)^{k-2} \ll \frac{F_{\text{max}}^2 \varphi(W)^{k+1} (\log R)^{k+1}}{W^{k+1} D_0}. \quad (3.40)
\]

By Lemma 3.7 and (3.39)–(3.40) we have

\[
\tilde{T}_\delta^{(\epsilon)} = \frac{\varphi(W)^2 \log^2 R}{W^2} \sum_{r_1, \ldots, r_k} \prod_{i=1}^{k} \left( \frac{\varphi(r_i)^2}{\psi(r_i) r_i^2} \right) \left( f^{(\epsilon)}_{\delta; r_1, \ldots, r_k} \right)^2 + O \left( \frac{F_{\text{max}}^2 \varphi(W)^{k+1} (\log R)^{k+1}}{W^{k+1} D_0} \right).
\]

We can restrict our attention to the sum

\[
\sum_{r_1, \ldots, r_k} \prod_{i=1}^{k} \left( \frac{\varphi(r_i)^2}{\psi(r_i) r_i^2} \right) \left( f^{(\epsilon)}_{\delta; r_1, \ldots, r_k} \right)^2.
\]

We wish to drop the condition \((r_i, r_j) = 1\) for all \(i \neq j\) in the summation in order to use Lemma 3.7. We can do this at the cost of introducing an error which is of size

\[
\ll F_{\text{max}}^2 \left( \sum_{p > D_0} \frac{\varphi(p)^4}{\psi(p)^2 p^4} \right) \left( \sum_{r \in R \cap (r, W) = 1} \frac{\varphi(r)^2}{\psi(r) r^2} \right)^{k-1} \ll \frac{F_{\text{max}}^2 \varphi(W)^{k-1} \log^{k-1} R}{W^{k-1} D_0}.
\]

Afterwards, we apply Lemma 3.5 individually to each summand of

\[
\sum_{r_1, \ldots, r_k} \prod_{i=1}^{k} \left( \frac{\varphi(r_i)^2}{\psi(r_i) r_i^2} \right) \left( f^{(\epsilon)}_{\delta; r_1, \ldots, r_k} \right)^2.
\]

We always take \(x = 1\) and

\[
\gamma(p) = \begin{cases} 
1 + \frac{p^2 - p + 1}{p^3 - 3 p^2 + 2 p - 1}, & p \nmid \delta W, \\
0, & \text{otherwise},
\end{cases}
\]

\[
L \ll 1 + \sum_{p \mid \delta W} \frac{\log p}{p} \ll \log D_0,
\]
and $A_1, A_2$ being suitable fixed constants. Notice that we may rearrange $I^{(\varepsilon)}_{\delta, r_1, \ldots, r_k}$ as

$$
\int_0^1 \sum_{s \leq \delta} \mu(s) F \left( \log r_1 \log R, \ldots, \log r_{\ell - 1} \log R, t_\varepsilon + \sum_{j \in J} \log p_j \log R, \log r_{\ell} \log R, \ldots, \log r_k \log R \right) dt_\varepsilon
$$

$$
= \int_0^1 \sum_{J \subseteq \{1, \ldots, r - 1\}} (-1)^{|J|} F \left( \log r_1 \log R, \ldots, \log r_{\ell - 1} \log R, t_\varepsilon + \sum_{j \in J} \log p_j \log R, \log r_{\ell} \log R, \ldots, \log r_k \log R \right) dt_\varepsilon.
$$

Now, in a similar fashion to (3.35) we calculate

$$
\overline{T}_\delta^{(\varepsilon)} = \frac{\varphi(W)^{k+1}(\log R)^{k+1}}{W^{k+1}} I^{(\varepsilon)} \left( \log p_1 \log R, \ldots, \log p_{r-1} \log R \right) + O \left( \frac{F^2_{\max} \varphi(W)^{k+1}(\log R)^{k+1}}{W^{k+1}D_0} \right).
$$

Lemma 3.9. Fix $\varepsilon > 0$. For $p > D_0$ we have

$$
T_{1, \ldots, 1, p, 1, \ldots, 1} \ll \frac{F^2_{\max} \varphi(W)^k(\log p)(\log R)^{k-1}}{W^k},
$$

where $p$ appears on the $m$th coordinate of $T$.

Proof. First, note that if $p \nmid d_m, e_m$, then $[d_m, e_m, p]/p = [d_m, e_m]$. Also, if $p | d_m$ and $p \nmid e_m$, then

$$
\frac{\lambda_{d_1, \ldots, d_k, e_1, \ldots, e_k}}{[d_m, e_m, p]/p \prod_{i \neq m} [d_i, e_i]} = \frac{\lambda_{d_1, \ldots, d_{m-1}, p^e_d, d_{m+1}, \ldots, d_k, e_1, \ldots, e_k}}{[d'_m, e_m] \prod_{i \neq m} [d_i, e_i]},
$$

where $d_m = pd'_m$. Applying the same observation to the remaining cases (where $p$ divides both $d_m, e_m$ and where $p$ divides only $e_m$), we arrive at

$$
T_{1, \ldots, 1, p, 1, \ldots, 1} = \sum' \frac{\lambda_{d_1, \ldots, d_k, e_1, \ldots, e_k}}{[d_m, e_m, p]/p \prod_{i \neq m} [d_i, e_i]} \prod_{i = 1}^k [d_i, e_i]
$$

$$
= \sum' \frac{\lambda_{d_1, \ldots, d_k} + \lambda_{d_1, \ldots, d_{m-1}, pd_m, d_{m+1}, \ldots, d_k}}{p | d_m \neq m} \prod_{i = 1}^k [d_i, e_i] \left( \lambda_{e_1, \ldots, e_k} + \lambda_{e_1, \ldots, e_{m-1}, p^e_m, e_{m+1}, \ldots, e_k} \right).
$$

We put

$$
\lambda_{d_1, \ldots, d_k} = \left( \lambda_{d_1, \ldots, d_k} + \lambda_{d_1, \ldots, d_{m-1}, pd_m, d_{m+1}, \ldots, d_k} \right) 1_{p \nmid d_m}
$$
and get

\[ T_{1,\ldots,1,p,1,\ldots,1} = \sum' \frac{\lambda_{d_1,\ldots,d_k} \epsilon_{e_1,\ldots,e_k}}{\prod_{i=1}^k |d_i, e_i|}. \]

Note that \( \lambda_{d_1,\ldots,d_k} \) satisfies the conditions imposed on the Selberg weights by Definition 1. After sequence of operations analogous to what was presented in Lemma 3.2 we get

\[ T_{1,\ldots,1,p,1,\ldots,1} = \sum^* s_{1,2,\ldots,k,k-1} \left( \prod_{1 \leq i,j \leq k} \frac{1}{\varphi(s_{i,j})^2} \right) \sum_{u_1,\ldots,u_k} \left( \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} \right) y_{a_1,\ldots,a_k} y_{b_1,\ldots,b_k} \]

\[ \ll F_{\text{max}} \sum^* s_{1,2,\ldots,k,k-1} \left( \prod_{1 \leq i,j \leq k} \frac{1}{\varphi(s_{i,j})^2} \right) \sum_{u_1,\ldots,u_k} \left( \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} \right) \left| y_{a_1,\ldots,a_k} \right|, \quad (3.42) \]

where the \( a_i, b_i \) and \( \Sigma^* \) are the same as in Lemma 3.2 and

\[ y_{r_1,\ldots,r_k} = \left( \prod_{i=1}^k \frac{\mu(r_i)\varphi(r_i)}{\varphi(r_i)} \right) \sum_{d_1,\ldots,d_k \in \mathbb{N}} \lambda_{d_1,\ldots,d_k} \prod_{i=1}^k \frac{d_i}{d_i} \]

\[ \lambda_{d_1,\ldots,d_k} = \left( \prod_{i=1}^k \frac{\mu(r_i)\varphi(r_i)}{\varphi(r_i)} \right) \sum_{d_1,\ldots,d_k} \frac{\lambda_{d_1,\ldots,d_k}}{\prod_{i=1}^k d_i} + p \left( \prod_{i=1}^k \frac{\mu(r_i)\varphi(r_i)}{\varphi(r_i)} \right) \sum_{d_1,\ldots,d_k \in \mathbb{N}} \lambda_{d_1,\ldots,d_k} \frac{d_i}{d_i} \]

\[ \lambda_{d_1,\ldots,d_k} = \left( \prod_{i=1}^k \frac{\mu(r_i)\varphi(r_i)}{\varphi(r_i)} \right) \sum_{d_1,\ldots,d_k} \frac{\lambda_{d_1,\ldots,d_k}}{\prod_{i=1}^k d_i} + (p-1) \left( \prod_{i=1}^k \frac{\mu(r_i)\varphi(r_i)}{\varphi(r_i)} \right) \sum_{d_1,\ldots,d_k \in \mathbb{N}} \lambda_{d_1,\ldots,d_k} \frac{d_i}{d_i} \]

\[ = y_{r_1,\ldots,r_k} - y_{r_1,\ldots,r_{m-1},pr_m,r_{m+1},\ldots,r_k}. \]

From (1.10) we have (if we assume supp \((F) \subset R_k\), then we perform analogously)

\[ y_{r_1,\ldots,r_k} - y_{r_1,\ldots,r_{m-1},pr_m,r_{m+1},\ldots,r_k} \ll \begin{cases} \frac{F_{\text{max}} \log p}{\log R}, & \text{if } \forall i \ r_1 \cdots r_k < Rr_i/p, \\ F_{\text{max}}, & \text{otherwise.} \end{cases} \]

We split the inner sum from (3.42) in the following manner:

\[ \sum_{u_1,\ldots,u_k} = \sum_{u_1,\ldots,u_k \in \mathbb{N}} + \sum_{u_1,\ldots,u_k \in \mathbb{N}} \sum_{a_1 \cdots a_k < Ra_j/p} \sum_{\exists j \ r_j/p \leq a_1 \cdots a_k < Ra_j}. \]
The contribution of the first sum to the bottom expression in (3.42) can be easily estimated to be

$$\ll \frac{F_{\max} \varphi(W)^k (\log p)(\log R)^{k-1}}{W^k}.$$ 

In the second case, we fix some index $j$ and obtain that the analogous contribution is

$$\ll F_{\max} \sum_{s_1, s_2, \ldots, s_k, k-1}^\ast \left( \prod_{1 \leq i, j \leq k} \frac{1}{\varphi(s_{i, j})^2} \right) \sum_{u_1, \ldots, u_k} \left( \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} \right) \left| \hat{y}_{a_1, \ldots, a_k} \right|$$

$$\ll \frac{F_{\max} \varphi(W)^k (\log p)(\log R)^{k-1}}{W^k}. \quad \square$$

4 | PROOFS OF PROPOSITIONS 1.12, 1.13, AND 1.14

4.1 | Proof of Proposition 1.12

Proof. Recall that

$$\Sigma_0 = \sum_{N < n \leq 2N} \sum_{n \equiv \nu_0 \mod W} \left( \sum_{p | \lambda(n)} W_0 \left( \frac{\log p}{\log R} \right) \right) \Lambda_{\text{Sel}}^2(n).$$

If $p$ divides $L_i(n)$ and $L_j(n)$ for some indices $i \neq j$, then $p | A_i B_j - A_j B_i$, which implies $p < D_0$. This contradicts $n \equiv \nu_0 \mod W$, which means that each $p$ in the sum above divides exactly one linear form $L_i(n)$. Hence, it is reasonable to decompose $\Sigma_0$ into $k$ sums according to which factor $L_j(n)$ of $P(n)$ is divisible by $p$. Therefore, for each $j = 1, \ldots, k$ we have to calculate

$$\Sigma_0^{(j)} := \sum_{N < n \leq 2N} \sum_{n \equiv \nu_0 \mod W} \left( \sum_{p | \lambda_j(n)} W_0 \left( \frac{\log p}{\log R} \right) \right) \sum_{d_1, \ldots, d_k} \lambda_{d_1, \ldots, d_k} \left| \hat{y}_{e_1, \ldots, e_k} \right|^2$$

$$= \sum_{D_0 \leq p \leq R_0} W_0 \left( \frac{\log p}{\log R} \right) \sum_{d_1, \ldots, d_k} \lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k} \sum_{N < n \leq 2N} \sum_{n \equiv \nu_0 \mod W} \sum_{d_j \leq \lambda_j(n)} W_0 \left( \frac{\log p}{\log R} \right) \left( \prod_{j \neq i} \frac{\lambda_{d_i, \ldots, d_k, e_j, \ldots, e_k}}{|d_j, e_j, p| \prod_{i \neq j} |d_i, e_i|} \right)$$

$$= \frac{N}{W} \sum_{D_0 \leq p \leq R_0} W_0 \left( \frac{\log p}{\log R} \right) \sum_{d_1, \ldots, d_k} \lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k} \sum_{d_j \leq \lambda_j(n)} \sum_{n \equiv \nu_0 \mod W} \sum_{p | \lambda_j(n)} \frac{\lambda_{d_1, \ldots, d_k, e_j, \ldots, e_k}}{|d_j, e_j, p| \prod_{i \neq j} |d_i, e_i|}$$

$$+ O \left( \sum_{D_0 \leq p \leq R_0} \sum_{d_1, \ldots, d_k} \sum_{e_1, \ldots, e_k} |\lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k}| \right).$$
We may add the 'condition without losing anything, because the sum equals 0 otherwise. Indeed, if \( p |[d_i, e_i], [d_j, e_j] \) for some \( p, i, j \) satisfying \( i \neq j \), then also \( p |L_i(n), L_j(n) \), which is a contradiction.

Let us estimate the error term of (4.1) first. We note that the value of the product \( d_1 \ldots d_k e_1 \ldots e_k \) is never greater than \( R^2 \). On the other hand, every value \( r \leq R^2 \) can be obtained in no more than \( \tau_{2k}(r) \) different ways. Thus, from Lemma 3.1 and definitions described in (1.11), we get

\[
\sum_{D_0 \leq p \leq R_0} \sum_{\lambda \in \mathbb{Z}} |\lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k}| \ll \lambda_{\max} \sum_{r \leq R^2} \tau_{2k}(r) \ll F_{\max}^2 \left( \log^{4k} N \right) R_0 R^2.
\]

We can assume that \( R_0 R^2 \ll N \log^{-10k} N \) to obtain a satisfactory bound.

The main term from (4.1) equals

\[
\frac{N}{W} \sum_{D_0 \leq p \leq R_0} W_0 \left( \frac{\log p}{\log R} \right) \frac{T_1, \ldots, p, 1, \ldots, 1}{p}.
\]

(4.2)

We decompose the sum above as follows:

\[
\sum_{D_0 \leq p \leq R_0} = \sum_{D_0 \leq p \leq N^c} + \sum_{N^c < p \leq R_0}.
\]

(4.3)

By Lemma 3.9 the contribution of the first sum to (4.2) is

\[
\ll \frac{F_{\max}^2 \varphi(W)^k N (\log R)^{k-1}}{W^{k+1}} \sum_{D_0 < p \leq N^c} \frac{\log p}{p} \ll \frac{\varepsilon F_{\max}^2 \varphi(W)^k N (\log R)^{k}}{W^{k+1}}.
\]

(4.4)

By Lemma 3.6 the contribution of the second sum to (4.2) equals

\[
\frac{\varphi(W)^k N (\log R)^{k}}{W^{k+1}} \sum_{N^c < p \leq R_0} \frac{1}{p} W_0 \left( \frac{\log p}{\log R} \right) f_{(j)} \left( \frac{\log p}{\log R} \right) + O \left( \frac{F_{\max}^2 \varphi(W)^k N (\log R)^{k}}{W^{k+1} D_0} \right),
\]

(4.5)

where we estimated the error by the inequality

\[
\sum_{N^c < p \leq R_0} \frac{1}{p} \ll |\log \varepsilon|.
\]

(4.6)

We use summation by parts (for example, [11, Lemma 6.8]) and get

\[
\sum_{N^c < p \leq R_0} \frac{1}{p} W_0 \left( \frac{\log p}{\log R} \right) f_{(j)} \left( \frac{\log p}{\log R} \right) = \int_{\varepsilon/\hat{\varepsilon}}^{\varepsilon_0/\hat{\varepsilon}} W_0(u) f_{(j)}(u) \frac{du}{u} + O \left( \frac{\log \log N}{\log N} \right).
\]

(4.7)

Combining (4.1)–(4.7) we get the desired result. The last minor detail is to rescale \( \varepsilon/\hat{\varepsilon} \mapsto \varepsilon$.
4.2 Proof of Proposition 1.13

We apply a Bombieri–Vinogradov style lemma for numbers with exactly $r$ prime factors weighted by $W_r$, which is a minor variation of [11, Lemma 8.1].

Lemma 4.1. Assume $GEH[\theta]$. Let $A \geq 1$, $\epsilon > 0$, and $h > 0$ be fixed. Let

$$\beta_r(n) = \begin{cases} W_r \left( \frac{\log p_1}{\log n}, \ldots, \frac{\log p_{r-1}}{\log n} \right) & \text{for } n = p_1 \ldots p_r, \text{ with } \epsilon < p_1 \leq \cdots \leq p_r, \\ 0 & \text{otherwise} \end{cases}$$

for some piecewise smooth function $W_r : [0,1]^{r-1} \to \mathbb{R}_{\geq 0}$. Put

$$\Delta_{\beta_r}(x; q) = \max_{y \leq x} \max_{\nu : (\nu, q) = 1} \left| \sum_{y < n \leq 2y} \frac{\beta_r(n) - \frac{1}{\phi(q)} \sum_{y < n \leq 2y} \beta_r(n)}{(n, q) = 1} \right|.$$

Then, for $Q \ll x^{\theta}$ we have

$$\sum_{q \leq Q} \mu(q)^2 h_{\psi(q)} \Delta_{\beta_r}(x; q) \ll \left( \max_{[0,1]^{r-1}} |W_r| \right) x \log^{-A} x.$$

Proof. Follows from the reasoning described in [11, Lemma 8.1] combined with Theorem 1.9. □

To prove the next lemma we proceed similarly to Thorne [16] with a few minor differences. The main disparity is that we are able to focus on the prime divisors of $L_i(n)$ directly, so we do not have to “search” for them among $[d,e]$ (see (4.3) in [16]).

Lemma 4.2. Assume $GEH[\vartheta]$. Fix $\epsilon > 0$. We require $W_r$ to be fixed and supported on

$$A_r = \left\{ x \in [0,1]^{r-1} : \epsilon < x_1 < \cdots < x_{r-1}, \sum_{i=1}^{r-1} x_i < \min(1-\vartheta, 1-x_{r-1}) \right\}.$$

For every $U \geq 1$ we have

$$S_j = \sum_{\substack{N < n \leq 2N \\ n \equiv b \mod W}} \beta_r(L_j(n)) \left( \sum_{d_1,\ldots,d_k} \lambda_{d_1,\ldots,d_k} \right)^2 \left( \sum_{\nu \mid d_1,\ldots,d_k} \frac{T_{\nu}(j)}{\varphi(W) \log N} \frac{\log p_1}{\log R}, \ldots, \frac{\log p_{r-1}}{\log R} \right) \left( 1 + O \left( \frac{1}{\log N} \right) \right) \left( \frac{N}{\log^U N} \right).$$
where $q$ is the product of primes appearing under the summation and

$$
\alpha(x_1, \ldots, x_{r-1}) = \frac{W_r(\vartheta x_1, \ldots, \vartheta x_{r-1})}{1 - \vartheta \sum_{i=1}^{r-1} x_i}.
$$

**Proof.** Switching the order of summation, we get

$$
S_j = \sum_{\substack{d_1, \ldots, d_k \atop e_1, \ldots, e_k}} \lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k} \sum_{\substack{N < n \leq 2N \atop n \equiv \nu_0 \mod W \atop \forall i \ [d_i, e_i] | L_i(n)}} \beta_r(L_j(n)).
$$

Similarly to (4.1), we may add the $'$-condition here freely. In the first step, we apply the trick devised by Thorne [16], that is, we decompose the inner sum according to how many prime factors of $L_j(n)$ divide $[d_j, e_j]$:

$$
S_j = S_0 + \cdots + S_r,
$$

where (here we write $q$ instead of $p_1 \cdots p_{r-h}$ for the sake of brevity)

$$
S_h = \sum_{\substack{d_1, \ldots, d_k \atop e_1, \ldots, e_k}} \lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k} \sum_{\substack{p_1 < \cdots < p_{r-h} \atop q = [d_j, e_j]}} \sum_{\substack{N < n \leq 2N \atop n \equiv \nu_0 \mod W \atop \forall i \neq j \ [d_i, e_i] | L_i(n) \atop q | L_j(n)}} \beta_r(L_j(n)).
$$

For $r = h$ we simply consider $q = 1$. We substitute $L_j(n) = m$. Observe that for any $i \neq j$ the condition $[d_i, e_i] | L_i(n)$ is equivalent to $m \equiv (A_iB_j - A_jB_i)A^{-1} \mod [d_i, e_i]$, where $A^{-1}$ denotes the inverse element in the multiplicative residue class group (which in this case actually exists because $(d_i e_i, W) = 1$). Moreover, $(A_iB_j - A_jB_i)$ is also coprime to $d_i e_i$, so we can combine all these congruences into one: $m \equiv m_0 \mod \prod_{i \neq j} [d_i, e_i]$, where $m_0$ is coprime to the modulus. We also have to transform the congruence $n \equiv \nu_0 \mod W$. To do so, we split it into two:

$$
n \equiv \nu_0 \mod [A_j, W]/A_j \text{ and } n \equiv \nu_0 \mod \text{rad } A_j,
$$

where $\text{rad } A_j$ denotes the radical (the square-free part) of $A_j$. The latter congruence is equivalent to $m \equiv A_j \nu_0 + B_j \mod A_j \text{rad } A_j$ and we can see that it is compatible with $m \equiv B_j \mod A_j$, which is forced by our substitution. This gives

$$
S_h = \sum_{\substack{d_1, \ldots, d_k \atop e_1, \ldots, e_k}} \lambda_{d_1, \ldots, d_k} \lambda_{e_1, \ldots, e_k} \sum_{\substack{p_1 < \cdots < p_{r-h} \atop q = [d_j, e_j]}} \sum_{\substack{m \equiv m_0 \mod \prod_{i \neq j} [d_i, e_i] \atop m \equiv m_0 \mod [A_j, W]/A_j \atop m \equiv A_j \nu_0 + B_j \mod A_j \text{rad } A_j \atop \nu_0 \mod W}} \beta_r(m),
$$

$$
\beta_r(m).
$$
where \((\mu_0, [A_j, W]/A_j) = 1\). We can simplify the situation further by substituting \(m = qt\). The inner sum from the equation above satisfies

\[
\sum_{\substack{A_jN+B_j < m \leq 2A_jN+B_j \\ m \equiv m_0 \mod \prod_{i \neq j} [d_i, e_i] \\ q|m \\ m \equiv \mu_0 \mod [A_j, W]/A_j \\ m \equiv A_j \gamma_0 + B_j \mod A_j \text{rad} A_j}} \beta_r(m) = \sum_{\substack{(A_jN+B_j)/q < t \leq (2A_jN+B_j)/q \\ t \equiv 0 \mod u}} \beta_r(qt) \quad (4.9)
\]

for some residue \(t_0\) coprime to its modulus \(u := [A_j, W] \text{rad} A_j \prod_{i \neq j} [d_i, e_i]\). We also put \(\beta_r(qt) = \bar{\beta}_h(t)\) for some function \(\bar{\beta}_h\) that satisfies

\[
\bar{\beta}_h(t) = \begin{cases} \widetilde{W}_h \left( \frac{\log p_1}{\log t}, \ldots, \frac{\log p_{h-1}}{\log t} \right) & \text{for } t = p_1 \cdots p_h, \text{ with } \epsilon < p_1 \leq \cdots \leq p_h, \\ 0 & \text{otherwise} \end{cases}
\]

for some piecewise smooth function \(\widetilde{W}_h : [0, 1]^{h-1} \to \mathbb{R}_{\geq 0}\). It can be done because the largest prime divisor of \(qt\) does not divide \(q\) as long as it is greater than \(R\). This is implied by the fact that \(W_r\) is supported on \((x_1, \ldots, x_{r-1})\) satisfying \(\sum_{i=1}^{r-1} x_i < 1 - \delta\) together with \(n > N\). Note that \(\bar{\beta}_h\) is still depending on \(q\), so we have to be careful. The sum on the left-hand of the Equation (4.9) has the form required by Lemma 4.1. We have

\[
\sum_{\substack{(A_jN+B_j)/q < t \leq (2A_jN+B_j)/q \\ t \equiv 0 \mod u}} \beta_r(qt) = \frac{1}{\varphi([A_j, W] \text{rad} A_j) \prod_{i \neq j} \varphi([d_i, e_i])} \sum_{A_jN/q < t \leq 2A_jN/q} \bar{\beta}_h(t) + O\left( \Delta_{\bar{\beta}, h}(A_jN; u) + 1 \right).
\]

We write

\[
S_h = M_h + E_h,
\]

where

\[
M_h = \frac{1}{\varphi([A_j, W] \text{rad} A_j)} \sum_{d_1, \ldots, d_k, s_1, \ldots, s_k}^\prime \lambda_{d_1, \ldots, d_k} \lambda_{s_1, \ldots, s_k} \prod_{i \neq j} \varphi([d_i, e_i]) \sum_{p_1 < \cdots < p_{r-h}} \sum_{q = [d_j, e_j]} \bar{\beta}_h(t).
\]

By Lemma 4.1 we may obtain

\[
E_h \ll \sum_{d_1, \ldots, d_k, s_1, \ldots, s_k}^\prime |\lambda_{d_1, \ldots, d_k} \lambda_{s_1, \ldots, s_k}| \sum_{p_1 < \cdots < p_{r-h}} \sum_{q = [d_j, e_j]} \left( \Delta_{\bar{\beta}, h}(A_jN; u) + O(1) \right)
\]
\[ F^2_{\max} \left( \log^2 R \right) \sum_{d_j, e_j} \sum_{p_1 \cdots p_{r-h}} \sum_{q=[d_j, e_j]} \left( \Delta \tilde{\beta}_h (A_j N; u) + O(1) \right) \]

\[ \ll U F^2_{\max} N \left( \log^{-U+O(1)} N \right) \sum_{d, e} \frac{1}{[d, e]} \]

\[ \ll F^2_{\max} N \log^{-U+O(1)} N. \]

for any \( U \geq 1 \). After rescaling \( U \) here a bit we are done.

Now, we concentrate on the main term of \( S_h \). It is convenient to put back \( \tilde{\beta}_h(t) = \beta_r(qt) \). We find

\[ \sum_{p_1 \cdots p_{r-h}} \beta_r(qt) = \sum_{p_1 \cdots p_{r-h}} \beta_r(qt) \]

\[ = \sum_{p_1 \cdots p_{r-h}} \beta_r(qt') A_j N/q < q' \leq 2A_j N/q', \]

where \( q' = \prod_i p_i \). We wish to extract the greatest prime divisor of \( qq' \) from the summations. We remember that it has to be larger than \( R \), so the only possibility is \( p_h' \). We set this index apart and name it \( p \). Therefore, we put \( q'' = q'/p \) and the expression from (4.11) equals

\[ \sum_{p_1 \cdots p_{r-h}} \sum_{q''} \beta_r(qq'' p) A_j N/qq'' < p \leq 2A_j N/qq'' \]

\[ = \sum_{p_1 \cdots p_{r-h}} \sum_p \sum_{q''} W_r \left( \frac{\log p_1}{\log pq''}, \ldots, \frac{\log p_{r-1}}{\log pq'''}, \frac{\log pq}{\log pq'''} \right) A_j N/qq'' < p \leq 2A_j N/qq'', \]

where \( q''' = \prod_{i=1}^{r-1} p_i \). The condition \( p > \max(p_h'^{-1}, R) \) could be erased due to the definition of \( W_r \). Note that the “old” collection of \( p_1, \ldots, p_{r-h} \) is not necessarily equivalent to the “new” collection – there is a possibility that all of the \( p_i \) and \( p_i' \) were mixed with each other. From now on, we relabel \( q''' \) as \( q \) for the sake of convenience. We conclude that \( M_0 + \cdots + M_r \) equals

\[ \frac{1}{\varphi([A_j, W] \text{rad } A_j)} \sum_{d_1, \ldots, d_k} \sum_{e_1, \ldots, e_k} \frac{\lambda_{d_1, \ldots, d_k}}{\varphi([d_1, e_1])} \sum_{p_1 \cdots p_{r-1}} \sum_{q''} W_r \left( \frac{\log p_1}{\log pq''}, \ldots, \frac{\log p_{r-1}}{\log pq'''}, \frac{\log pq}{\log pq'''} \right) A_j N/qq'' < p \leq 2A_j N/qq''. \]

(4.12)
Combining (4.10), (4.12) with (8.13) and (8.14) from [11], we obtain for any $U \geq 1$

\[
\sum_{\substack{N < n \leq 2N \\ n \equiv \nu_0 \mod W}} \beta_r(L_j(n)) \left( \sum_{d_1, \ldots, d_k \mid L_j(n)} \lambda_{d_1, \ldots, d_k} \right)^2 = \frac{N}{\varphi(W) \log N} \sum_{p_1 < \cdots < p_{r-1}} \frac{T_q^{(j)}}{q} \alpha \left( \frac{\log p_1}{\log R}, \ldots, \frac{\log p_{r-1}}{\log R} \right) \left( 1 + O \left( \frac{1}{\log N} \right) \right) + O_U(N \log^{-U} N),
\]

where we noted that

\[
\frac{A_j}{\varphi([A_j, W] \text{rad } A_j)} = \frac{1}{\varphi(W)}.
\]

Now, to finish the proof of Proposition 1.13 we have to perform the reasoning completely analogous to what appears in the proof of [11, Lemma 8.3]. Let us provide a sketch of such a proof. By Lemma 4.2, we need to prove that

\[
\sum_{p_1 < \cdots < p_{r-1}} \frac{T_q^{(j)}}{q} \alpha \left( \frac{\log p_1}{\log R}, \ldots, \frac{\log p_{r-1}}{\log R} \right) = \frac{\varphi(W)^{k+1}(\log R)^{k+1}}{W^{k+1}} J_r + O_{\varepsilon} \left( \frac{F_0^2 \varphi(W)^{k+1}(\log R)^{k+1}}{W^{k+1} D_0} \right).
\]

By Lemma 3.3, Lemma 3.8, and (3.38), the main term arising from expanding $T_q^{(j)}$ equals

\[
\frac{\varphi(W)^{k+1}(\log R)^{k+1}}{W^{k+1}} I^{(j)} \left( \frac{\log p_1}{\log R}, \ldots, \frac{\log p_{r-1}}{\log R} \right)
\]

and the error term is manageable. Thus, we are left with proving that

\[
\sum_{p_1 < \cdots < p_{r-1}} \frac{1}{q} \alpha \left( \frac{\log p_1}{\log R}, \ldots, \frac{\log p_{r-1}}{\log R} \right) J^{(j)} \left( \frac{\log p_1}{\log R}, \ldots, \frac{\log p_{r-1}}{\log R} \right) = J_r^{(j)} + O_{\varepsilon} \left( \frac{F_0^2}{D_0} \right).
\]

For $u_1, \ldots, u_m \in [0, 1/\delta]$ we put

\[
V_m(u_1, \ldots, u_m) := \int \ldots \int_{\sum_{i=1}^{r-1} u_i \leq \min \left( \frac{1}{\delta} - 1, \frac{1}{\delta} - u_{r-1} \right)} \alpha(u_1, \ldots, u_{r-1}) f^{(j)}(u_1, \ldots, u_{r-1}) du_{m+1} \ldots du_{r-1}.
\]

We prove that

\[
I^{(j)}(u_1, \ldots, u_{r-1}) \ll F_0^2 u_m^2
\]
very much the same way as in [11, (8.25)]. Now, from $\alpha(u_1, \ldots, u_{r-1}) \ll 1$ we get

$$V_m(u_1, \ldots, u_m) \ll F_{\max}^2 u_m^2 \int \cdots \int \frac{1}{\prod_{i=m+1}^{r-1} u_i} \prod_{i=m+1}^{r-1} u_i \cdots d u_{r-1}$$

$$\ll F_{\max}^2 u_m^2 \left(1 + \log \left|\frac{1}{u_m}\right|^r\right)$$

$$\ll F_{\max}^2 u_m.$$

Following [11] further, for $u_1, \ldots, u_m \in [0, 1/\vartheta]$ we have

$$\frac{\partial}{\partial u_m} I^{(j)}(u_1, \ldots, u_{r-1}) \ll F_{\max}^2 u_j,$$

$$\frac{\partial}{\partial u_m} \alpha(u_1, \ldots, u_{r-1}) \ll 1.$$

Consequently, we also obtain

$$\frac{\partial}{\partial u_m} V_m(u_1, \ldots, u_m) \ll F_{\max}^2 u_m \int \cdots \int \frac{1}{\prod_{i=m+1}^{r-1} u_i} \prod_{i=m+1}^{r-1} u_i \cdots d u_{r-1} \ll F_{\max}^2,$$

so the key condition of [11, Lemma 6.8] is satisfied for all the functions $V_m$. Applying it in turn to each of the $V_1, \ldots, V_{r-1}$ provides the result with an acceptable error term.

### 4.3 Proof of Proposition 1.14

**Proof.** The proof is the same regardless of whether we assume Hypothesis 1 or Hypothesis 2, so we describe only the first case. Define $A := 2 \max(A_1, \ldots, A_k)$. Recall that $n \equiv \nu_0 \mod W$ implies $(P(n), W) = 1$. We wish to estimate from above the following expression:

$$\sum_{D_0 \leq p < AN^{1/2}} \sum_{N \leq n \leq 2N} \left[ \sum_{n \equiv \nu_0 \mod W} \sum_{p \mid \lambda_{d_1, \ldots, d_k}} \frac{\lambda_{d_1, \ldots, d_k}}{p^2 |L_j(n)|} \right]^2.$$

We split the outer sum as follows:

$$\sum_{D_0 \leq p < AN^{1/2}} = \sum_{D_0 \leq p < AN^\eta} + \sum_{N^\eta \leq p < AN^{1/2}}.$$

(4.14)
To calculate the second sum from (4.14) we apply Lemma 3.1 and the divisor bound \( \tau(n) \ll n^{o(1)} \). Recalling that \( \lambda_{\text{max}} \ll y_{\text{max}} \log^{2k} N \ll F_{\text{max}} N^{o(1)} \) we obtain

\[
\sum_{N \leq p < AN^{1/2}} \sum_{n \equiv \nu_0 \mod W \atop p^2 \mid L_j(n)} \left( \sum_{d_1, \ldots, d_k} \lambda_{d_1, \ldots, d_k} \right)^2 \ll F_{\text{max}}^2 N^{o(1)} \sum_{N \leq p < AN^{1/2}} \sum_{n \equiv \nu_0 \mod W \atop p^2 \mid L_j(n)} 1 \ll F_{\text{max}}^2 N^{1+o(1)} \sum_{p \geq N^\eta} \frac{1}{p^2} \ll N^{1-\eta+o(1)}.
\]

In the case of the first sum from (4.14) we can rearrange the order of summation and get

\[
\sum_{N \leq p < AN^{1/2}} \sum_{d_1, \ldots, d_k} \sum_{\epsilon_1, \ldots, \epsilon_k} \lambda_{d_1, \ldots, d_k} \lambda_{\epsilon_1, \ldots, \epsilon_k} \sum_{N \leq n \leq 2N} \frac{1}{n \equiv \nu_0 \mod W} \lambda_{d_1, \ldots, d_k} \frac{1}{\lambda_{\epsilon_1, \ldots, \epsilon_k}} \ll \sum_{N \leq p < AN^{1/2}} \sum_{n \equiv \nu_0 \mod W \atop p^2 \mid L_j(n)} \frac{1}{p^2} \ll N^{1-\eta+o(1)}.
\]

As before in similar cases, we may add the \( \epsilon \)-condition freely, because the sum equals 0 otherwise. By Chinese remainder theorem the last summand equals

\[
\frac{N}{W[d_j, \epsilon_j, p^2] \prod_{i \neq j} [d_i, \epsilon_i]} + O(1).
\]

Therefore, the expression from (4.15) is

\[
\frac{N}{W} \sum_{D_0 \leq p < N^\eta} \sum_{d_1, \ldots, d_k} \sum_{\epsilon_1, \ldots, \epsilon_k} \lambda_{d_1, \ldots, d_k} \lambda_{\epsilon_1, \ldots, \epsilon_k} \frac{1}{[d_j, \epsilon_j, p^2]} \prod_{i \neq j} [d_i, \epsilon_i] + O \left( \sum_{D_0 \leq p < N^\eta} \sum_{d_1, \ldots, d_k} \sum_{\epsilon_1, \ldots, \epsilon_k} \frac{1}{\lambda_{d_1, \ldots, d_k}} \frac{\lambda_{\epsilon_1, \ldots, \epsilon_k}}{\lambda_{d_1, \ldots, d_k}} \right).
\]

To calculate the error term we again apply Lemma 3.1 to estimate the Selberg weights. We also see that the product \( d_1 \ldots d_k \epsilon_1 \ldots \epsilon_k \) can take only values lower than \( R^2 \), so we obtain

\[
\sum_{D_0 \leq p < N^\eta} \sum_{d_1, \ldots, d_k} \sum_{\epsilon_1, \ldots, \epsilon_k} \frac{1}{\lambda_{d_1, \ldots, d_k}} \frac{\lambda_{\epsilon_1, \ldots, \epsilon_k}}{\lambda_{d_1, \ldots, d_k}} \ll y_{\text{max}}^2 \sum_{r < R^2 N^\eta} \tau_{2k+1}(r) \ll F_{\text{max}}^2 N^{\eta+o(1)} R^2,
\]

which is negligible for \( 2\hat{\eta} < 1 - \eta \).

The main term equals

\[
\frac{N}{W} \sum_{D_0 \leq p < N^\eta} T_{1, \ldots, p, 1, \ldots, 1, \frac{1}{p^2}},
\]

where \( p \) appears on the \( j \)th coordinate. From Lemma 3.6 we get

\[
\tilde{T}_{1, \ldots, 1, p, 1, \ldots, 1} = \frac{\varphi(W)^k \log^k R}{W^k} \int_0^1 \cdots \int_0^1 G(t_1, \ldots, t_k) \, dt_1 \cdots dt_k + O \left( \frac{F_{\text{max}}^2 \varphi(W)^k (\log R)^k}{W^k D_0} \right).
\]
where
\[ G(t_1, \ldots, t_k) = \left( F(t_1, \ldots, t_k) - F(t_1, \ldots, t_{j-1}, t_j + \frac{\log p}{\log R}, t_{j+1}, \ldots, t_k) \right)^2. \]

Note that
\[ G(t_1, \ldots, t_k) \ll \frac{F_{\max}^2 \log p}{\log R} \]
for \( t_j \leq 1 - \frac{\log p}{\log R} \), since
\[ \frac{\partial F}{\partial t_j} \ll F_{\max}. \]

For \( t_j \geq 1 - \frac{\log p}{\log R} \) we obtain simply
\[ G(t_1, \ldots, t_k) \ll F_{\max}^2. \]

Combining (4.16)--(4.18), we get
\[ \bar{T}_{1,\ldots,1,p,1,\ldots,1} \ll \frac{\varphi(W)^k (\log R)^k}{W^k} \left( \frac{F_{\max}^2 \log p}{\log R} + \frac{F_{\max}^2}{\log R} \right) \ll \frac{\varphi(W)^k \log^{k-1} R \log p}{W^k}. \]

Thus,
\[ \frac{N}{W} \sum_{D_0 \leq p < N^\eta} \frac{T_{1,\ldots,1,p,1,\ldots,1}}{p^2} = \frac{N}{W} \sum_{D_0 \leq p < N^\eta} \frac{\bar{T}_{1,\ldots,1,p,1,\ldots,1}}{p^2} + O \left( \frac{F_{\max}^2 \varphi(W)^k N (\log R)^k}{W^{k+1} D_0} \sum_{D_0 \leq p < N^\eta} \frac{1}{p^2} \right). \]

To sum up,
\[ \sum_{D_0 \leq p < N^\eta} \sum_{N < n \leq 2N} \left( \sum_{d_1, \ldots, d_k \mid \lambda_{d_1, \ldots, d_k}} \left( \sum_{n \equiv 0 \mod W} \frac{\lambda_{d_1, \ldots, d_k}}{p^k \mid \lambda_i(n)} \right)^2 \right) \ll \frac{F_{\max}^2 \varphi(W)^k N \log^{k-1} R}{W^{k+1}} \left( \sum_{D_0 \leq p < N^\eta} \frac{\log p}{p^2} + \frac{\log R}{D_0} \right) \ll \frac{F_{\max}^2 \varphi(W)^k N (\log R)^k}{W^{k+1} D_0}. \]

\( \square \)

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