DISPERSION RELATIONS IN STRING THEORY*

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ABSTRACT

We analyze the analytic continuation of the formally divergent one-loop amplitude for scattering of the graviton multiplet in the Type II Superstring. In particular we obtain explicit double and single dispersion relations, formulas for all the successive branch cuts extending out to $+\infty$, as well as for the decay rate

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of a massive string state of arbitrary mass $2N$ into two string states of lower mass. We compare our results with the box diagram in a superposition of $\phi^3$-like field theories. The stringy effects are traced to a convergence problem in this superposition.
1. Introduction

A fundamental direction in the exploration of superstring theory is the understanding of its perturbative expansion in the number of string loops. Already substantial progress was made in obtaining a consistent formulation of perturbation theory as a sum over Riemann surfaces. In particular, it was shown that amplitudes defined this way are Lorentz invariant and perturbatively unitary [1], and it is generally believed that order by order, superstring loop amplitudes are finite. In particular, it was argued long ago that the Type II and heterotic amplitudes do not exhibit the tachyon and massless dilaton tadpole divergences that are known to occur in the bosonic string [2][3][4].

Our understanding is, however, still rather incomplete compared to the situation in quantum field theory. There, simple scaling arguments and recursive combinatorics guarantee a simple physical picture of renormalizability in perturbation theory. Feynman’s $i\epsilon$ prescription on the propagators, combined with Cutkovsky and Landau cutting rules, provides a simple picture of unitarity and causality. In superstring theory, the rules of perturbation theory do not exhibit the properties of renormalizability (or finiteness) and unitarity (or causality) manifestly. Instead unitarity of the amplitudes was established only indirectly, by showing equivalence between the covariant and the light-cone formulations. Renormalizability or finiteness have not even reached such a level of indirect understanding.

In an earlier work [5], we had shown that the formally real and divergent one-loop amplitude for the scattering of four external states in the graviton multiplet of the Type II superstring can be analytically continued to a finite, complex, amplitude consistent with the optical theorem. We also gave a simple explicit formula for the leading term in the forward scattering amplitude. Here we extend this work further by producing explicit formulas for all the successive branch cuts extending out to $+\infty$ as well as for the decay rate of a massive string state of arbitrary mass $2N$ into two string states of lower mass. We illustrate our results by comparing
them with those in an infinite superposition of $\phi^3$-like field theories. The stringy effects are traced to a convergence problem in this superposition. The key ingredients in our approach are dispersion relations, which occur in the form of both double and single relations. Our methods apply in more general set-ups, including the heterotic string and certain toroidal, orbifold, and Calabi-Yau compactifications. We shall however present these extensions and other applications elsewhere.

2. The Analytic Continuation Problem for Superstring Scattering Amplitudes

We begin by discussing the Type II superstring amplitudes for the scattering of four massless on-shell states in the graviton multiplet, their domain of convergence, and the physical meaning of their analytic continuation. They are of the form

\begin{equation}
  s = g^4 \delta^{(10)} \left( \sum_{i=1}^{4} k_i \right) \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4
\end{equation}

(2.1)

Here $k_i, \epsilon_i, i = 1, \cdots, 4$ are respectively the momenta and polarization tensors of the incoming particles, $g$ is the string coupling, and $K\bar{K}$ is a purely kinematical factor analytic (actually polynomial) in the momenta. The expression $A(s, t, u)$ is the reduced Lorentz invariant amplitude, and $s, t,$ and $u$ are the Mandestam variables defined by

\begin{align*}
  s &= s_{12} = s_{34}, \\
  t &= s_{23} = s_{14}, \\
  u &= s_{31} = s_{24}
\end{align*}

with $s_{ij} \equiv -(k_i + k_j)^2$. Momentum conservation implies that

\begin{equation}
  s + t + u = 0
\end{equation}

(2.2)

At tree level, the reduced amplitude is given by the well-known expression in terms
of Green’s functions on the complex plane

\[ A_0(s, t, u) = \frac{1}{t^2} \int_{\mathcal{C}} d^2 z |z|^{-s-2} |1 - z|^{-t} \]  

(2.3)

The above integral converges only when

\[ \text{Re } s < 0, \text{ Re } t < 2, \text{ Re } (s + t) > 0 \]

so that even this case requires an analytic continuation. It is however readily available since (2.3) can be rewritten in terms of \( \Gamma \) functions as

\[ A_0(s, t, u) = \frac{\Gamma\left(-\frac{s}{2}\right) \Gamma\left(-\frac{t}{2}\right) \Gamma\left(-\frac{u}{2}\right)}{\Gamma\left(1 + \frac{s}{2}\right) \Gamma\left(1 + \frac{t}{2}\right) \Gamma\left(1 + \frac{u}{2}\right)} \]  

(2.4)

Thus the amplitude can be extended as a meromorphic function of \( s \) and \( t \) throughout the complex plane, with as only singularities simple poles in \( s, t, \) and \( u = -(s + t) \) at positive even integers. We observe that the integral representation for the amplitude produces only a holomorphic function in its region of convergence, and that physically important singularities such as poles (corresponding to massive intermediate states going on mass shell) must arise through analytic continuation.

To one-loop order, the reduced amplitude is given by an integral over the moduli space of tori

\[ A_1(s, t, u) = \int_{F} \int_{M_{\tau}} \int_{\tau_2} d^2 \tau \prod_{i<j} \exp(s_{ij} G(z_i, z_j)/2) \]  

(2.5)

Here \( M_{\tau} \) is the torus represented in the plane by the parallelogram with corners \( 0, 1, \tau, \) and \( 1 + \tau \) and opposite sides identified, \( F \) is the fundamental domain for the
modulus $\tau$

$$F = \{ \tau \in \mathbb{C}; \tau_2 > 0, |\tau_1| \leq 1/2, |\tau| \geq 1 \}$$

and $G(z, w)$ is the scalar Green’s function on $M_\tau$. It can be expressed in terms of Jacobi $\vartheta$-functions as

$$G(z, w) = -\ln \left| \frac{\vartheta_1(z - w|\tau)}{\vartheta_1'(0|\tau)} \right|^2 + \frac{2\pi}{\tau_2} (\text{Im}(z - w))^2$$

It is easy to see that the integral in (2.5) converges exactly when

$$\text{Re} \ s = \text{Re} \ t = \text{Re} \ u = 0 \quad (2.6)$$

in which case the exponentials in (2.5) are all phases. In fact (2.5) exhibits two types of singularities. When $z_i \sim z_j$, $G(z_i, z_j) \sim -\ln|z_i - z_j|^2 \to +\infty$, and integrability in $z_i$ requires $\text{Re} \ s_{ij} < 2$. On the other hand, when $z_i$ and $z_j$ are well-separated, and $\tau_2 \to \infty$, $G(z_i, z_j)$ tends to $-\infty$, and integrability in $\tau_2$ requires $\text{Re} \ s_{ij} \geq 0$. Our assertion follows from the fact that these conditions must hold for all $i, j$ and that $s + t + u = 0$. Thus the amplitude in both the deep Euclidian region $s < 0$ and the physical region $s > 0$ must be obtained by analytic continuation.

To appreciate the physical meaning of such analytic continuations, we compare the situation with quantum field theory. There, causality is equivalent to locality and, combined with Lorentz invariance, implies analyticity of the Green’s functions in momentum space. The standard way to establish this is to use locality and the fact that observable fields commute at space-like separations to derive a spectral representation for the Green’s functions. The simplest one is the Källen-Lehmann representation for the two-point function, say for a scalar field,

$$= \int_0^\infty dM^2 \frac{\rho(M^2)}{s - M^2 + i\epsilon}$$

Note that the integration is over $M^2 \geq 0$, as the existence of tachyonic states would automatically violate causality. Provided this spectral representation is convergent,
it immediately implies that the two-point function is an analytic function of $s$, with only singularities along the positive real axis. These singularities may be poles or branch cuts, resulting respectively from one-particle or multiparticle intermediate states. In analytic S-matrix theory, the analyticity is assumed as one of the starting points of the theory [6][7].

On the other hand, string theory does not have at present a formulation in terms of vacuum expectation values of observables. It is not known whether spectral representations analogous to those of quantum field theory can be derived solely from the requirements of causality (i.e. locality of the string interactions) and Lorentz invariance. Thus the analyticity of string amplitudes must be established. Our goal is to do so, starting from the integral representations (2.5) obtained via string perturbation theory. More precisely, we address the following questions:

(a) Does an analytic continuation exist?

(b) Is it unique?

(c) Is the analytic continuation physically acceptable?

We have answered (a) in the affirmative by constructing explicitly a suitable analytic continuation to the cut plane $s, t, u \in \mathbb{C} \setminus \mathbb{R}_+, s + t + u = 0$ [5]. In particular, there are no lacunae in the domain of holomorphy. The answer to (b) is evidently yes, since in each variable (say $s$), two analytic continuations would have to agree on a line in the complex plane (imaginary $s$), and thus must be equal. To answer (c), we note that the singularities for the four point function in Type II superstring amplitudes should consist of branch cuts along the positive real axis, and simple and double poles at even integers. The appearance of any other type of singularity in addition to the above, such as poles or branch cuts or lacunae, would be inconsistent with unitarity, causality and Lorentz invariance. We shall indeed see the physically correct singularity structure emerge from the analytic continuation, essentially in the following manner:
• the region $\tau_2 \to \infty$ produces branch cut singularities in $s, t, u$, corresponding to two physical string intermediate states, for example for the box diagram;
• $z_i \sim z_j$ but all other $z$’s far away, leads to a simple pole in the $s_{ij}$ channel;
• $z_i$’s coming pairwise close together leads to double poles.

The singularities which arise from three $z_i$’s coming close together correspond to self-energy graphs for on-shell massless particles which vanish by space-time supersymmetry. The singularities with all four $z_i$’s close together correspond to one-loop tadpole graphs, producing double poles in each channel. For massless (dilaton) tadpoles, these contributions vanish to one-loop, but there are also scalar string states at higher mass levels which produce non-zero double pole contributions with no branch cuts.

3. One-loop Superstring Amplitudes and $\phi^3$ Box Diagrams

We give now a detailed analysis of the integral representation (2.5). In view of the product formula for Jacobi $\vartheta$-functions, the integrand in (2.5) involves an infinite number of factors. It is crucial to recognize which factors can be expanded in uniformly convergent series and treated perturbatively, and which ones cannot. Mathematically, the resulting series need to be uniformly convergent. Physically, the factors which cannot be expanded are the ones responsible for the ”stringy” aspects of the theory, i.e., the ones not present in a mere infinite superposition of quantum field theories.

We begin by decomposing the $z_i$ domain of integration into 6 regions according to the various orderings of $\text{Im } z_i$’s (c.f. [8][9][10]). The contributions are the same for pairs of regions, and we get

$$A(s, t, u) = 2A(s, t) + 2A(t, u) + 2A(u, s)$$ (3.1)

where $A(s, t)$ is still defined by (2.5), but the range of integration is now restricted
to

\[ \text{Im } z_1 \leq \text{Im } z_2 \leq \text{Im } z_3 \leq \text{Im } z_4 \quad (3.2) \]

and \( u \) has been set to \( u = -s - t \). It is convenient to change variables to

\[ z_i - z_{i-1} = \frac{\alpha_i}{2\pi} + i\tau u_i, \quad i = 1, \cdots, 4 \]
\[ z_4 - z_0 = \tau, \quad q = e^{2\pi i \tau} \quad (3.3) \]

Physically the parameters \( \tau u_i \) correspond to evolution time between two vertex operators in the interaction picture of quantum mechanics, and the \( \alpha_i \) angular integrations to enforcing the constraint \( L_0 = \bar{L}_0 \) on each string propagator. In terms of the new variables \( u_i, \alpha_i \), the amplitude \( A(s, t) \) becomes

\[
A(s, t) = \int_F \frac{d^2 \tau}{\tau_2} \int_0^{2\pi} \frac{d\alpha_i}{2\pi} \int_0^1 du_i \delta(1 - \sum_{i=1}^4 u_i)|q|^{-su_1u_3-tu_2u_4} R(|q|^{u_i}, \alpha_i; s, t) \quad (3.4)
\]

where the function \( R \) is given by an infinite product

\[
R(|q|^{u_i}, \alpha_i; s, t) = \prod_{i \neq j} \prod_{n=0}^{\infty} |1 - w_{ij} q^n|^{-s_{ij}} \quad (3.5)
\]

Here \( w_{ij} \) is defined as follows

\[
w_{ij} = \begin{cases} 
eq 0 & \text{for } i \neq j, \quad \text{This definition has been arranged so that } |w_{ij}| \leq 1 \text{ and } w_{ij}w_{ji} = q \text{ for any } i \neq j \text{. It is also convenient to identify the index } i = 1 \text{ with a fifth index } i = 5. 
\end{cases}
\]

We have then \( w_{(j+1)j} = |q|^{u_{j+1}}e^{i\alpha_{j+1}} \) for \( j = 1, \cdots, 4 \). For future reference, we note
also that \( s_{54} = s_{14} = t \). The integral representation (3.4) for \( A(s, t) \) shows that it is absolutely convergent for an entire strip in \( s \) and \( t \)

\[
\text{Re} \, s, \text{Re} \, t < 0, \text{Re} \, (s + t) > -2 \quad (3.7)
\]

Note, however, that all three expressions on the right hand side of (3.1) are simultaneously convergent only when \( \text{Re} \, s = \text{Re} \, t = \text{Re} \, u = 0 \) when the constraint \( s + t + u = 0 \) is enforced, in agreement with our earlier discussion. Thus we need to analytically continue \( A(s, t) \) to the full cut plane \( s, t \in \mathbb{C} \setminus \mathbb{R}_+ \).

Naively, we should expand the factor \( \mathcal{R} \) into a series in \( |q|^{u_i}, \, i = 1 \cdots 4 \), pointwise convergent in the region \( |w_{ij}| < 1 \)

\[
\mathcal{R}(|q|^{u_i}, \alpha; s, t) = \sum_{n_i=0}^{\infty} \sum_{|\nu_i| \leq n_i} P^*_\{n_i\nu_i\}(s, t) \prod_{i=1}^{4} |q|^{u_in_i} e^{i\nu_i \alpha_i} \quad (3.8)
\]

with the expansion coefficients \( P^*_\{n_i\nu_i\}(s, t) \) polynomials in \( s \) and \( t \), and construct an analytic continuation for expressions of the form (3.4) with \( \mathcal{R} \) replaced by a monomial of the form \( \prod_{i=1}^{4} |q|^{u_in_i} e^{i\nu_i \alpha_i} \). However, this would not produce the intermediate poles on top of cuts peculiar to string theory, as the following comparison with \( \phi^3 \) field theory illustrates.

First consider the contribution of a single monomial term in the expression (3.8) inserted into the integral representation (3.4). We shall see later that we may, without affecting the poles on top of cuts, truncate the fundamental domain \( F \) to the simpler region \( \{ \tau_2 \geq 1, |\tau_1| \leq 1/2 \} \). Thus \( \tau_1 \) becomes an angular variable, and the four angular integrations in (3.4) pick out the terms with \( \nu_i = 0 \) for \( i = 1, \cdots, 4 \) in (3.8). The remaining integral is of the form

\[
\int_1^{\frac{1}{\tau_2}} \frac{d\tau_2}{\tau_2} \int_0^{1} du_1 \delta(1 - \sum_{i=1}^{4} u_i) e^{2\pi \tau_2(su_1u_3 + tu_2u_4 - \sum_{i=1}^{4} n_i u_i)} \quad (3.9)
\]

where the integers \( n_i \) are actually positive and even.
Consider now a $\phi^3$-like box diagram in $d$ space-time dimensional quantum field theory with arbitrary masses $m_i$ for each of the propagators, and massless external on-shell states. Couplings are assumed to be identical for any choice of $m_i$ and are non-derivative $\phi^3$. The box diagram (Euclidean) Feynman integral can be performed as usual after introducing Feynman parameters $u_i$ and exponentiating the denominator. We obtain

\[
\int d^d k \prod_{i=1}^{4} \frac{1}{(k + p_i)^2 + m_i^2}
\]

\[
\int_0^{\infty} \frac{d\tau}{\tau^{d-3}} \int_0^1 du_5 \delta (1 - \sum_{i=1}^4 u_i) e^{2\pi \tau (su_1u_3 + tu_2u_4 - \sum_{i=1}^4 u_i m_i^2)}
\]

The $\phi^3$-like box diagram is almost identical to the partial superstring amplitude provided the following identifications are made. Clearly the space-time dimension should be $d = 10$, as expected, and the masses $m_i^2$ should be positive even integers, consistent with the superstring spectrum. A well-understood difference is that the $d\tau_2$ integral in the superstring amplitude is truncated away from 0. This is a remnant of duality, which results in modular invariance and restricts the moduli integration to a fundamental domain for $SL(2, \mathbb{Z})$, thus making the theory finite in the ultraviolet. The more important point is that there must be subtleties in the resummation of the monomial contributions (3.9), since otherwise we would have no poles. These subtleties are manifestations of the differences between string theory and an infinite superposition of field theories.

4. Single and Double Dispersion Relations

*Double Dispersion Relations*
The analytic continuation of the amplitude $A(s, t)$ is based on a double dispersion relation for each term in the following expansion of the factor $\mathcal{R}(|q|^{u_i}, \alpha_i; s, t)$

$$\mathcal{R}(|q|^{u_i}, \alpha_i; s, t) = \frac{4}{\pi^2} \prod_{i=1}^{4} \left(1 - e^{i\alpha_i |q|^{u_i}}\right)^{-s_i} \sum_{n=0}^{\infty} \sum_{|\nu_i| \leq n_i} P_{\{n_i, \nu_i\}}(s, t) \prod_{i=1}^{4} |q|^{n_i u_i} e^{i\nu_i \alpha_i} \quad (4.1)$$

where we have set $s_i = s$ for $i$ even, and $s_i = t$ for $i$ odd. The expansion (4.1) is obtained from (3.5) by keeping the first four factors $\prod_{i=2}^{5} |1 - w_i(i-1)|^{-s_i(i-1)}$ and expanding all the remaining ones. We shall see that the first four factors are precisely the ones responsible for stringy effects, and in particular poles on top of cuts. As in (3.6), the coefficients $P_{\{n_i, \nu_i\}}(s, t)$ are polynomials in $s$ and $t$. They can be obtained explicitly by simple recursive formulas, but we shall not need these here. The desired analytic continuation of $A(s, t)$ to an arbitrary half-space $\text{Re} s, \text{Re} t < N$ can now be obtained as follows.

**Lemma 1.** For any positive integer $N$, we can write

$$A(s, t) = \sum_{n_1 + \cdots + n_4 \leq 4N} \sum_{|\nu_i| \leq n_i} P_{\{n_i, \nu_i\}}(s, t) A_{\{n_i, \nu_i\}}(s, t) + M_N(s, t) \quad (4.2)$$

where $M_N(s, t)$ is a meromorphic function in the region $\text{Re} s, \text{Re} t < N$, and the amplitudes $A_{\{n_i, \nu_i\}}(s, t)$ are defined by

$$A_{\{n_i, \nu_i\}}(s, t) = \int_1^\infty \frac{d\tau_2}{\tau_2^2} \int_0^{2\pi} \frac{d\alpha_i}{2\pi} \int_0^1 du_i \delta(1 - \sum_{i=1}^{4} u_i) e^{-su_1 u_3 - tu_2 u_4} \prod_{i=1}^{4} \left(1 - e^{i\alpha_i |q|^{u_i}}\right)^{-s_i}$$

$$\times \prod_{i=1}^{4} |q|^{n_i u_i} e^{i\nu_i \alpha_i} \quad (4.3)$$

The lemma is established by isolating from the integral (3.4) contributions which can be given an independent meromorphic continuation in the region $\text{Re} s, \text{Re} t < N$. Particular care is required to treat the eight factors in (3.5) corresponding to
\( n = 0 \) which have been expanded into series. Details can be found in [5]. We turn to the analytic continuation of \( A_{\{n, \nu\}}(s, t) \). The angular integrations are decoupled now, and may be evaluated in terms of Gauss’ hypergeometric functions \( F(a, b; c; x) \)

\[
\int_0^{2\pi} \frac{d\alpha}{2\pi} e^{i\alpha} |1 - xe^{i\alpha}|^{-s} = C_{|\nu|}(s) x^{|\nu|} F\left(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; x^2\right)
\]

Here we have denoted by \( C_n(s) \) the following expression closely related to the tree-level amplitude of an intermediate state of mass \( 2n \) with two external massless state \( s \)

\[
C_n(s) = \frac{\Gamma\left(\frac{s}{2} + n\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma(n + 1)}
\]

Thus the amplitude \( A_{\{n, \nu\}}(s, t) \) becomes

\[
A_{\{n, \nu\}}(s, t) = \int_1^{\infty} d\tau \int_0^{\frac{\tau^2}{2}} du_i \delta(1 - \sum_{i=1}^{4} u_i) |q|^{-su_1u_3 - tu_2u_4 + \sum_{i=1}^{4} u_i(n_i + |\nu_i|)}
\times \prod_{i=1}^{4} C_{|\nu_i|}(s_i) F\left(\frac{s_i}{2}, \frac{s_i}{2} + |\nu_i|; |\nu_i| + 1; |q|^{2u_i}\right) \tag{4.4}
\]

The main ingredients needed for the analytic continuation of (4.4) are the Mellin and the inverse Laplace transforms of the hypergeometric functions. The former is defined by

\[
f_{n\nu}(s; \alpha) = C_{|\nu|}(s) \int_0^1 dx x^{-1-\alpha+n+|\nu|} F\left(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; x^2\right)
\]

and can be shown, using the reflection formula for hypergeometric functions, to admit a meromorphic extension in both \( s \) and \( \alpha \) throughout the full complex plane
, with simple poles in $\alpha$ at evenly spaced integers starting from $-n - |\nu|$ and in $s$ at positive integers. The latter is defined by

$$C_{|\nu|}(s)x^{n+|\nu|}F\left(\frac{s}{2}, \frac{s}{2} + |\nu|; |\nu| + 1; x^2\right) = \int_0^\infty d\beta x^\beta \varphi_{n\nu}(s; \beta)$$

We set $\varphi_{n\nu}(s; \beta) = 0$ for $\beta < 0$. The Mellin and the inverse Laplace transforms are related by

$$\varphi_{n\nu}(s, \beta) = \int_0^\infty d\beta \frac{\varphi_{n\nu}(s; \beta)}{\beta - \alpha}$$

Since $f_{n\nu}(s; \beta)$ is a meromorphic function of $s$, its discontinuity is also meromorphic in $s$, and in fact one can easily see that $\varphi_{n\nu}(s; \beta)$ is an entire function of $s$. Finally we set

$$\Psi_{n\nu_i}(s, t; \beta_i) = \prod_{i=1}^4 \varphi_{n\nu_i}(s_i; \beta_i)$$

**Theorem 1.** The amplitudes $A_{\{n, \nu\}}(s, t)$ can be expressed as

$$A_{\{n, \nu\}}(s, t) = \int_0^\infty \int_0^\infty \frac{\rho_{\{n, \nu\}}(s, t; \sigma, \tau)}{(s - \sigma)(t - \tau)} d\tau d\sigma + M_{\{n, \nu\}}(s, t) \quad (4.5)$$

where the density $\rho_{\{n, \nu\}}$ is given by

$$\rho_{\{n, \nu\}}(s, t; \sigma, \tau) = \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^1 du_1 \int_0^1 du_2 (1 - u_1 - u_2)^2 \int_{x_0}^\infty dx (x - x_0)^2$$

$$\times \Psi_{n\nu_i}(s, t; \beta_1, \beta_2, u_1 \sigma - x, u_2 \tau - x) \quad (4.6)$$

Here $x_0$ denotes $(u_1 \beta_1 + u_2 \beta_2)(1 - u_1 - u_2)^{-1}$ and $M_{\{n, \nu\}}(s, t)$ is a globally meromorphic function of $s$ and $t$. The integral (4.5) defines (after a subtraction of a
meromorphic function) a function of $s, t$ meromorphic in the cut plane $s, t \in \mathbb{C} \setminus \mathbb{R}_+$. More precisely, the domain of holomorphy of $A_{\{n_i \nu_i\}}(s, t)$ is given by

$$s \in \mathbb{C} \setminus [(\sqrt{n_1 + |\nu_1|} + \sqrt{n_3 + |\nu_3|})^2, +\infty]$$

$$t \in \mathbb{C} \setminus [(\sqrt{n_2 + |\nu_2|} + \sqrt{n_4 + |\nu_4|})^2, +\infty]$$

Together with Lemma 1, Theorem 1 provides a complete description of all the branch cut singularities in the analytic continuation of the amplitude $A(s, t, u)$. Representations of the form (4.5) are usually referred to as double dispersion relations. They were proposed by Mandelstam and constituted a fundamental tool of particle physics in the 1950’s [11]. They are analogues of the Källen-Lehmann representation we discussed earlier for the two-point function. The function $\rho_{\{n_i \nu_i\}}(s, t; \sigma, \tau)$ is referred to as a double spectral density and is an entire function of $s$ and $t$ for fixed $\sigma$ and $\tau$. To see this, we note that the range of integration in the expression (4.6) for $\rho_{\{n_i \nu_i\}}$ is actually finite. Indeed $\varphi_{n \nu}(s; \beta)$ vanishes when $\beta < 0$, and thus the regions contributing are

$$u_1 \sigma - x > 0, u_2 \tau - x > 0$$

Thus the $x$ range is bounded, and since $x > x_0 > (u_1 \beta_1 + u_2 \beta_2)$, it follows also that $\beta_1 < \sigma$ and $\beta_2 < \tau$. The domain of analyticity is a familiar one, as can be seen by introducing the mass value of the lowest branch cut : $n_i + |\nu_i| = m_i^2$, so that the region becomes $s \in \mathbb{C} \setminus [(m_1 + m_3)^2, \infty]; t \in \mathbb{C} \setminus [(m_2 + m_4)^2, \infty]$.

**Single Dispersion Relations and Emergence of Poles**

In earlier discussions, we had emphasized that string amplitudes have 1-particle reducible contributions as well. The resulting poles occur both in $M_{\{n_i \nu_i\}}$ and in the double spectral representation part. The latter ones are of particular importance since they lead to decay rates of massive strings, and can be recovered as follows.
Say we want to exhibit the poles in $s$ on top of the cut. We regard the double dispersion (4.5) as a single dispersion relation

$$ A_{\{n, \nu\}}(s, t) = \int_0^\infty d\sigma \frac{R_{\{n, \nu\}}(s, t; \sigma)}{s - \sigma} \tag{4.7} $$

with simple spectral density

$$ R_{\{n, \nu\}}(s, t; \sigma) = \int_0^\infty dt \frac{\rho_{\{n, \nu\}}(s, t; \sigma, \tau)}{\tau - t} $$

To understand its dependence on $s$ and $t$, we rewrite it as

$$ R_{\{n, \nu\}}(s, t; \sigma) = \int_0^\infty d\beta_1 \varphi_{n_1 \nu_1}(t; \beta_1) \int_0^\infty d\beta_2 \varphi_{n_2 \nu_2}(s; \beta_2) \int_0^1 du_1 \int_0^{1-u_1} du_2 \times (1 - u_1 - u_2)^2 \int_{x_0}^\infty (x - x_0)^2 \varphi_{n_3 \nu_3}(t; u_1 \sigma - x) f_{n_4 \nu_4}(s; u_2 t - x) $$

This shows that $R_{\{n, \nu\}}(s, t; \sigma)$ is an entire function of $t$. This $t$-analyticity may be used to expand $f_{n, \nu}$ in terms of $t$, giving

$$ R_{\{n, \nu\}}(s, t; \sigma) = \sum_{k_1,k_3=0}^{\infty} C_{k_1}(t) C_{k_1+|\nu_1|}(t) C_{k_3}(t) C_{k_3+|\nu_3|}(t) R_{\{n, \nu\}, k_1 k_3}(s, t; \sigma) $$

where

$$ R_{\{n, \nu\}, k_1 k_3}(s, t; \sigma) = \sum_{p=0}^{\infty} \frac{2tp}{p!(p+3)!} \int_0^1 du \theta(\sigma - \frac{m_1^2}{1-u} - \frac{m_2^2}{u}) \times (\sigma u(1-u) - m_3^2(1-u) - m_4^2 u)^{p+3} \times f_{n_2 \nu_2}^{(p)}(s; \frac{m_3^2}{u} - u \sigma) f_{n_4 \nu_4}^{(p)}(s; \frac{m_4^2}{u} - u \sigma) \tag{4.8} $$

with $m_2^2 \equiv 2k_i + n_i + |\nu_i|$. This expression shows that $R_{\{n, \nu\}}(s, t; \sigma)$ is a meromorphic function of $s$, with possibly simple and double poles at even positive integers.
We may represent the situation pictorially as follows. Recall that at tree level, the amplitudes may be expanded in poles, say in the $s$ channel (see Fig. 1).

But the series expansion on the right hand side of Fig. 1, which is equivalent to a sum over field theory graphs in the $s$-channel, is not in general absolutely and uniformly convergent. As a result, the summed series must be analytically continued, and in this analytically continued answer, we find poles in the $t$ channel. What is taking place here is the one-loop generalization of this phenomenon. We had established previously a very close connection with a summation over all masses of $\phi^3$-like field theory box diagrams. But again this sum was not convergent, and forced us to treat the entire hypergeometric functions exactly. The analytically continued sum as a result exhibits pole singularities as well, as shown in Fig. 2 below.

5. The $i\epsilon$ Prescription and Decay Rates of Massive Strings

We sketch here only two of the most immediate applications.
The $i\epsilon$ prescription

A physical amplitude for real values of $s$, $t$, and $u$ can now be obtained by introducing the $i\epsilon$ prescription in a straightforward way

$$A_{i\epsilon}(s, t) = \int_0^\infty d\sigma \int_0^\infty d\tau \frac{\rho(s, t; \sigma, \tau)}{(s - \sigma + i\epsilon)(t - \tau + i\epsilon)} + M(s + i\epsilon, t + i\epsilon)$$

so that the full amplitude is defined by

$$A(s, t, u) = 2A_{i\epsilon}(s, t) + 2A_{i\epsilon}(t, u) + 2A_{i\epsilon}(u, s)$$

We note that this expression is different from a naive extrapolation when $s, t, u \to s + i\epsilon, t + i\epsilon, u + i\epsilon$, which leads to an incorrect singularity structure.

Decay Rates of Massive Strings

A practical application is the calculation of decay rates for massive string states into 2 body decays of strings of lesser mass to lowest order. As already pointed out, this decay rate is simply related to the imaginary part of the residue (i.e. coefficient in the Laurent expansion) of the 4-point amplitude at a double pole, say in $s$. This problem has been considered in the Veneziano dual model already long ago [12]. The residual $t$ dependence may be resolved in terms of the spin values of the decaying string state. Both $M$ and the double dispersion relation produce double poles, but it is straightforward to see that the residues at the double poles in $M$ are real, and do not contribute to the decay rates. The residue at $s = 2N$ of the Mellin transform $f_{n\nu}(s; \alpha)$

$$\lim_{s \to 2N} (s - 2N)f_{n\nu}(s; \alpha)$$

can be evaluated to be $F_{\nu}(2N, \alpha - n)$, with

$$F_{\nu}(2N, \alpha) = -\frac{1}{2} \frac{\Gamma(-\frac{\nu}{2} - \frac{n}{2})\Gamma(\frac{\nu}{2} - \frac{n}{2})}{\Gamma(N)^2\Gamma(-\frac{\nu}{2} - \frac{n}{2} + 1 - N)\Gamma(\frac{\nu}{2} - \frac{n}{2} + 1 - N)}$$

(5.1)

It is a polynomial of degree $2N - 2$ in $\alpha$. Substituting in (4.8), we find
Theorem 2. The decay rate

\[ \Gamma(2N, t) g^2 C_{2N}(t) \equiv \lim_{s \to 2N} 2\pi (s - 2N)^2 \int_0^\infty d\tau \frac{\rho(s, t; 2N, \tau)}{t - \tau} \]

is given by the expression

\[ \Gamma(2N, t) g^2 C_{2N}(t) = \sum_{n_2, n_4=0}^{\infty} \sum_{|\nu_i| \leq n_i} P^{(2)}_{\{n_i, \nu_i\}}(s, t) \frac{2^{tp}}{p!(p + 3)!} \]

\[ \times \int_0^1 du \, \theta(2N - \frac{m_3^2}{u} - \frac{m_1^2}{1 - u}) (2N u (1 - u) - (1 - u)m_3^2 - um_1^2)^{p+3} \]

\[ \times F_{\sigma_2}^{(p)}(2N, m_3^2 - 2Nu - n_2) F_{\sigma_4}^{(p)}(2N, m_3^2 - 2Nu - n_4) \] (5.2)

where \( P^{(2)}_{n_i, \nu_i}(s, t) \) are polynomials in \( s \) and \( t \) which can be obtained recursively through the following relations

\[ \int \int \frac{d\alpha_1 d\alpha_3}{4\pi^2} R(|q|^{u_1}, \alpha_i; s, t) = |1 - e^{i\alpha_2} |q|^{u_2}|^{-s} |1 - e^{i\alpha_4} |q|^{u_4}|^{-s} \]

\[ \times \sum_{n_i = 0}^{\infty} \sum_{|\nu_j| \leq n_j} P^{(2)}_{\{n_i, \nu_2\nu_4\}}(s, t) |q|^{n_i u_i} e^{i\alpha_2 \nu_2 + i\alpha_4 \nu_4} \]

Note that all sums in (5.2) are finite, actually bounded by \( 2N - 2 \), signalling the fact that for fixed \( N \), the number of channels is finite.

We conclude with a few remarks on some alternative approaches for calculating decay rates [13]. It would of course be natural to calculate decay rates by squaring the corresponding tree level amplitudes and summing over all final polarizations and masses. Two problems arise then: first, we must normalize string states correctly, which becomes prohibitively difficult for higher spin states. Second, we must
actually perform the sum over final polarizations which again is highly non-trivial. In fact, even in quantum field theory, one normally prefers to calculate such decay rates by immediately considering the loop amplitude, and taking its imaginary part. Another method is to construct decay rates directly as a one-loop amplitude with two massive vertex operators inserted. The problem with this approach is that both vertex operators must have on-shell momenta, which are fixed, and thus the amplitude has no more free parameters. But this amplitude is as divergent as the one-loop four-point function that we have been considering, with the difference that there is no variable left to analytically continue in. It is not possible to make sense directly out of such amplitudes, without using the result that an analytic continuation exists, as derived in this paper.

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