Birational models for varieties of
Poncelet curves

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We consider Poncelet pairs \((S, C)\), where \(S\) is a smooth conic and \(C\) is a degree\(-\)c plane curve having the Poncelet property with respect to \(S\). We prove that for \(c > 4\) the projection \((S, C) \mapsto C\) is generically one-to-one and use this to describe a birational model of the variety of Poncelet curves for \(c\) odd.

Introduction

A pair \((S, C)\) consisting of a smooth conic \(S\) and a curve \(C\) of degree \(c\) in the complex projective plane will be called a Poncelet pair if there exist \(c + 1\) tangents to \(S\) whose \(\binom{c+1}{2}\) intersection points lie on \(C\). \(C\) will then be called Poncelet related to \(S\) or simply a Poncelet curve. A theorem of Darboux \((\text{I})\) states that in this case there is an infinity of such sets of \(c + 1\) tangents.

The Poncelet curves reappeared in a natural way in the study of stable vector bundles on projective spaces, e.g. in \([\text{I}], [\text{II}], [\text{III}], [\text{IV}])\). Motivated by these developments, Trautmann gave a modern description of their geometry in \([\text{VI}]\).

In this paper we consider the variety \(Pon_c\) of Poncelet curves of degree \(c\). It is the image of the space of Poncelet pairs \((S, C)\) through the second projection. We show that for \(c \geq 5\), this projection is birational. This fails for \(c \leq 3\), simply because of dimension reasons, and has been proved for \(c = 4\) by Le Potier via the study of divisors on the corresponding moduli space of
rank – 2 semi-stable sheaves on \( \mathbb{P}^2 \), ([8]). As a corollary, we get the rationality of \( Pon_c \) for every \( c \).

The generic injectivity of the above mentioned projection brings some evidence for (and maybe could help proving) its generalizations to:

- Barth’s morphism which associates to a stable \( rank – 2 \) vector bundle of type \((0, c)\) on \( \mathbb{P}^2 \) its curve of jumping lines ([4]), and
- the restriction of the previous to Hulsbergen bundles. (The image will be the variety of Darboux curves, cf. [1], [4]).

In the last paragraph we describe a projective birational model of \( Pon_c \) when \( c \) is odd.

§ 1 Preliminaries

For the proofs of the facts stated in this paragraph as well as for further descriptions we refer the reader to [12].

Let \( W \) be a 3-dimensional complex vector space, \( S \) a smooth conic in \( \mathbb{P}^2 = \mathbb{P}(W) \), \( c \) a positive integer and \( \Lambda \) a pencil of degree-\((c + 1)\) divisors on \( S \). We consider the intersection points of the tangents to \( S \) at the points of some member \( D \) of \( \Lambda \). When \( D \) moves in \( \Lambda \), these intersection points describe a curve \( C = C(\Lambda) \) in \( \mathbb{P}^2 \). It is clear that the tangents at the base points of \( \Lambda \) will be components of \( C \). We require that their multiplicity in \( C \) be equal to that of the base points in \( D \). Then \( C \) has degree \( c \), is called Poncelet related to \( S \) or simply a Poncelet curve and \((S, C)\) is called a Poncelet pair.

That this definition is equivalent to that of the introduction follows from the

**Theorem.** (Darboux) Let \( S \) be a smooth conic and \( C \) a curve of degree \( c \) in \( \mathbb{P}^2 \). If there exist \( c + 1 \) tangents to \( S \) such that their intersection points lie on \( C \), then \( C \) is Poncelet related to \( S \).

The base points of \( \Lambda \) correspond to tangents to \( S \) which are components of \( C(\Lambda) \) and the residual curve (obtained by eliminating these components) is again Poncelet related to \( S \) and associated to the pencil obtained from \( \Lambda \) by subtracting its base points.
For a generic $C(\Lambda)$ is smooth.

Let’s denote by $G_S(1, c+1)$ the Grassmannian of pencils of degree-$(c+1)$ effective divisors on $S$. The map

$$G_S(1, c+1) \to \mathbb{P}(S^cW^*) = \mathbb{P}^{(c+2)-1},$$

$$\Lambda \mapsto C(\Lambda)$$

coincides with the Plücker embedding (for a suitable choice of coordinates, [12]). Thus, there is a 1:1 correspondence between $G_S(1, c+1)$ and the variety $Pon_{c,S}$ of curves $C$ which are Poncelet related to $S$.

Let $U$ be the open set of $\mathbb{P}(S^2W^*)$ which parameterizes smooth conics in $\mathbb{P}^2$, $I_U \subset U \times \mathbb{P}^2$ the tautological conic bundle over $U$, $I_U^{(c+1)}$ the Hilbert scheme of degree-$(c+1)$ effective divisors in the fibers of $I_U \to U$ (which is a $\mathbb{P}^{c+1}$-fibration over $U$) and $G_U(1, c+1)$ the relative Grassmannian of lines in the fibers of $I_U^{(c+1)} \to U$. We shall identify $G_U(1, c+1)$ with the subvariety $Pon_{c,U}$ of Poncelet pairs in $U \times \mathbb{P}(S^cW^*)$. (The equations of $Pon_{c,U}$ are given in [12]).

We shall examine the second projection $pr_2 : Pon_{c,U} \to \mathbb{P}(S^cW^*)$. Its image is the variety of degree-$c$ Poncelet curves $Pon_c$.

We end this paragraph with some elementary remarks on Poncelet curves to be used later:

**Remarks.** Let $S$ be a smooth conic.

1.1 If $C$ is Poncelet related to $S$, then the only components of $C$ which may be multiple are tangent lines to $S$. (Apply Bertini’s theorem to the associated pencil, $\Lambda$.)

1.2 If $C_1 + C_2$ and $C_1 + C_2'$ are two elements of $Pon_{c,S}$ such that $C_2$ and $C_2'$ have no common component, then $C_1$ is Poncelet related to $S$. (Start from a point on $C_1$, draw tangents to $S$ and continue this procedure from the intersection points with $C_1$. The process stops after a finite number of steps. If there exist intersection points of the drawn tangents not on $C_1$, then these points lie on a common component of $C_2$ and $C_2'$).

1.3 Let $C_1, C_1 + C_2$ be Poncelet related to $S$ with associated pencils $\Lambda_1$ and $\Lambda$, respectively. If $\Lambda$ has no base points ,
then the induced morphism \( \varphi_\Lambda : S \to \Lambda^* \cong \mathbb{P}^1 \) factorizes through \( \varphi_{\Lambda'} : S \to \Lambda'^* \). In particular, \( \deg C_1 + 1 \) divides \( \deg(C_1 + C_2) + 1 \).

1.4 If \( C \) is a singular conic Poncelet related to \( S \), then one of its components is tangent to \( S \). (Follows from 1.3).

§ 2 Generic injectivity of \( pr_2 \)

**Theorem.** When \( c \geq 5 \), the projection \( Pon_{c,U} \to Pon_c \) is birational.

**Remarks.**

2.1 The corresponding statement for \( c = 4 \) (and more) is proven in \[8\].

2.2 \( Pon_{c,U} \) is rational (\[8\], Prop.2.2), hence so is \( Pon_c \) too.

The theorem follows by a dimension count out of the following

**Proposition.** Let \( S, S' \) be distinct smooth conics. Then

(A) for \( c \geq 3 \), \( \dim(Pon_{c,S} \cap Pon_{c,S'}) \leq c \).

(B) \( Pon_{5,S} \cap Pon_{5,S'} \) is at most 4-dimensional at points represented by smooth quintics.

Indeed, let us estimate the dimension of \( \tilde{G}_S := pr_2^{-1}(Pon_{c,S}) \setminus G_S(1, c + 1) \). By part (A) of the Proposition, \( \dim \tilde{G}_S \leq c + 5 \). Thus, for \( c \geq 6 \) we have \( \dim pr_2(\tilde{G}_S) < 2c = \dim Pon_{c,S} \) proving the theorem in this case.

When \( c = 5 \), we use part (B) for a slightly modified \( \tilde{G}_S \), \( \tilde{G}_S := pr_2^{-1}(Reg_{c,S}) \setminus G_S(1, c + 1) \), where \( Reg_{c,S} \) is the set of smooth Poncelet curves related to \( S \).

In order to prove the Proposition we look at the linear projection of \( Pon_{c,S} \cap Pon_{c,S'} \) with center \( L_d := \{ \text{degree-c curves in } \mathbb{P}(W) \text{ allowing } d \text{ as a component} \} \subset \mathbb{P}(S^*W^*) \), for some common tangent line \( d \) to \( S \) and \( S' \).
Let \( d \) be the tangent at \( P \) to the smooth conic \( S \). Then \( \text{Pon}_{c,S} \cap L_d \cong \text{Pon}_{c-1,S} \) corresponds to degree-\( c \) pencils on \( S \) having \( P \) as base point (or equivalently, to the Schubert variety on \( G_S(1, c+1) \) of lines contained in a hyperplane).

Let further \( D \) be an element of \(|\mathcal{O}_S(c+1)| \) containing \( P \) and \( L_D \subset G_S(1, c+1) \) the Schubert variety of lines through \( D \). Then \( L_D \cong \mathbb{P}^c \) is linear in \( \mathbb{P}(S^cW^*) \) and cuts on \( L_d \) a \((c-1)\)-dimensional linear subspace, (the \( L_{D-P} \) of \( G_S(1,c) \)). Thus \( L_D \cup L_d \) spans a \( c \)-dimensional linear subspace of \( \mathbb{P}(S^cW^*) \).

Conversely, for every \( c \)-codimensional linear subspace \( F \) which contains \( L_d \), there exists some \( D \) as above such that \( F \cap \text{Pon}_{c,S} = L_D \cup (\text{Pon}_{c,S} \cap L_d) \).

For two distinct divisors \( D, D' \) as above, \( L_D \) and \( L_{D'} \) cut themselves in exactly one point lying on \( L_d \). In particular, the projection of center \( L_d \) displays \( \text{Pon}_{c,S} \setminus L_d \) as a vector bundle over \(|\mathcal{O}_S(c)| \).

Let now \( V \) be an irreducible component of \( \text{Pon}_{c,S} \cap \text{Pon}_{c,S'} \) not contained in \( L_d \).

Let \( \pi : V \setminus L_d \to |\mathcal{O}_S(c)| \) be the restriction to \( V \) of the above projection, \( t := \dim \pi(V \setminus L_d) \), and \( f \) the dimension of a generic fiber of \( \pi \). Note that the fibers of \( \pi \) are linear.

**Lemma 2.1** When \( c \geq 2 \), up to finitely many exceptions, the fibers of \( \pi \) are at most \((c-1)\)-dimensional.

**Proof.** We choose a general hyperplane in \(|\mathcal{O}_S(c)| \) and show that the fibers of \( \pi \) over its points cannot be \( c \)-dimensional. Indeed, take \( P_1 \) on \( S \) such that it doesn’t belong to a common tangent to \( S \) and \( S' \). The divisors containing \( P_1 \) form a hyperplane in \(|\mathcal{O}_S(c)| \). Take any such divisor and add \( P \) to it in order to obtain a divisor \( D = P + P_1 + P_2 + \ldots + P_c \) in \(|\mathcal{O}_S(c+1)| \). Take further a pencil with base points \( P_1, P_2, \ldots, P_{c-1} \) and containing \( D \). Its associated Poncelet curve is the union of the tangents at \( P_1, \ldots, P_{c-1} \) to \( S \) and a line through the intersection point of \( d \) with the tangent to \( S \) at \( P_c \). By Remarks 1.2 and 1.3 not all such curves are Poncelet related to \( S' \). \( \Box \)

**Proof of part (A) of the Proposition**

We argue by induction on \( c \).
Let \( c = 3 \) and suppose that \( t + f > 3 \). Then \( t \geq 2 \) by Lemma 2.1. We pick then a divisor \( D = P + P_1 + P_2 + P_3 \) on \( S \) and look at the fiber of \( \pi \) contained in \( L_D \).

We get a divisor \( D' = P' + P'_1 + P'_2 + P'_3 \) on \( S' \) such that this fiber is exactly \( L_D \cap L_{D'} \setminus L_d \). (\( L_{D'} \) is to be considered with respect to \( S' \)). Note that \( P_1, P_2 \) or \( P_1, P'_2 \) may be freely chosen so that this fiber be non empty.

Denote by \( d_i, d'_i \) the tangents at \( P_i \) to \( S \) and at \( P'_i \) to \( S' \), respectively, \( i \in \{1, 2, 3\} \). We index our \( P'_i \)'s so that \( d \cap d_i = d \cap d'_i \).

Choose now \( P_1, P'_2 \) such that the point \( d_1 \cup d'_2 \) does not lie on a common tangent to \( S \) and \( S' \). In particular, it will not lie on both \( d_3 \) and \( d'_3 \), say not on \( d'_3 \).

The Poncelet curves in \( L_D \cap L_{D'} \) touch \( d'_2 \) at its intersection points with \( d, d'_1, \ldots, d'_t \). (If one of the \( P'_i \) had multiplicity \( (k+1) \) in \( D' \) then the condition on the corresponding Poncelet curves would be to touch \( S \) at \( P'_i \) with multiplicity \( k \), by [12], Prop. 2.6). If we impose to such a curve that \( d_1 \) be a component (which we may do since \( f \geq 1 \)), we get that \( d'_2 \) must be a component too. By Remark 1.4 the third component of our cubic is tangent to both \( S \) and \( S' \), so \( f = 1 \) and \( t = 3 \). This allows a free choice on \( P'_3 \) too and we get that the tangents through \( P'_3 \) and \( P'_t \) to \( S' \) form a conic which is Poncelet related to \( S \). But this contradicts Remark 1.4.

Let now \( c > 3 \).

**Case (A1) \( t > \frac{c}{2}, f \geq 1 \).**

We choose as above \( D = P + P_1 + \ldots + P_c \) and get in the same way a \( D' = P' + P'_1 + \ldots + P'_c \) such that \( L_D \cap L_{D'} \setminus L_s \neq \emptyset \). In doing this \( P_1, \ldots, P_t \) are free.

We claim that the choice may be done such that \( d_1 \) intersects \( d'_2, \ldots, d'_t \) away from the vertices of the polygon \( d, d'_1, \ldots, d'_c \) (with the obvious extension of notation). If this were not true then there would exist a point, say \( P'_{t+1} \), in a “bad position”, e.g. such that \( d_1 \cap d'_2 \cap d'_{t+1} \neq \emptyset \). We look then at \( P_2 \) instead of \( P_1 \) and may find a second point \( P'_{t+2} \), in a bad position, and continue in this way. The assumption \( t > \frac{c}{2} \) shows now that our claim holds.

Next, from \( f \geq 1 \) it follows that we may degenerate a curve
in our fiber, so that $d_1$ and hence also $d'_2, \ldots, d'_t$ become its components. By Remarks 1.2-4 the other components of $C$ are not all tangent lines to $S$, otherwise we move the lines $d'_i$, $2 \leq i \leq t$. In particular $t < c$.

If $f \geq 2$, then we may still move our curve $C$ (in its class), fact which combined with Remark 1.3 shows the existence of a pair of degrees $(a, b)$ such that $t - 1 \leq a < b < c$ and $a + 1$ divides $b + 1$. But this would imply $t \leq \frac{c}{2}$, which proves that $f = 1$ and $f + t \leq c$.

**Case (A2) $f > \frac{c}{2}$**

The proof is similar to that of Lemma 1.

We may first assume that the general $D = P + P_1 + \ldots + P_c$ in the image of $\pi$ doesn’t contain any tangency point of a common tangent to $S$ and $S'$ excepting $P$. Otherwise we restrict our attention to the subvariety of Poncelet curves containing this tangent and apply the induction hypothesis.

In the fiber of $\pi$ over $D$ we require that the tangents $d_1, \ldots, d_{f-1}$ at $P_1, \ldots, P_{f-1}$ to $S$ be components of the Poncelet curves $C$. Then by Remarks 1.3 and 1.4 there exists a pair of degrees $(a, b)$ with $f - 1 \leq a < b < c$ and $a + 1$ divides $b + 1$. This forces $f = \frac{c + 1}{2}$ proving part (A) of the Proposition.

**Proof of part (B) of the Proposition**

Keeping our previous notations we assume that $t + f = c = 5$.

**Case (B1), $t = 5$.**

Since in this case $\pi$ is dominant we find for any choice of points $P_1, \ldots, P_c$ on $S$, an element in $L_D \cap V$, where $D = P + P_1 + \ldots + P_c$.

Take $P_1$ on $S$ and $P'_1$ the corresponding point on $S'$ (draw the tangent $d'_1$ from $d_1$ to $S'$). Choose further a point $Q$ on $d'_1$ and draw tangents from $Q$ to $S$. Let these tangents be $d_2, d_3$. A curve in $L_D \cap V$ has to contain $Q$ and the intersection points of $d'_1$ with $d, d'_2, \ldots, d'_t$. Thus this curve has $d'_1$ as a component. In the same way, choosing $d_4$ through $d'_2 \cap d_1$ fixes $d'_2$ as a component of $C$. We may still move our point $P_5$, and since $d'_1 \cup d'_2$ is not in $Pon_2, s$, we get by Remarks 1.2 and 1.3 degrees $a, b$ with $3 \leq a < b \leq 6$ and $a + 1 \mid b + 1$. This is absurd so case (B1) cannot occur.

**Case (B2), $t = 4$.**
As in case (A1) we may fix the components $d_1, d'_2, \ldots, d'_t$ as we wish. The last component of our Poncelet curve in not tangent to $S$ as already remarked in (A1). Since we may consider $d'_2, d'_4$ fixed and move $d'_4$ we get again a contradiction of Remarks 1.2 and 1.4. Case (B2) is thus excluded.

Case (B3), $t = 3$, reduces itself to (A1).

Case (B4) $t = 2$.

We pick as usual two arbitrary tangent lines to $S$ and look at the curves in $V$ containing them. They form an at least 1-dimensional subvariety. Since these two tangents do not form a Poncelet conic with respect to $S'$ we get degrees $3 \leq a < b \leq c$ such that $a + 1$ divides $b + 1$. This implies $c \geq 7$ and excludes this case too.

Case (B5), $t = 1$.

If we find some $D = P + P_1 + \ldots + P_c$ in the “image” of $\pi$ containing two points which don’t lie on common tangents to $S$ and $S'$ we shall argue as in (B4).

Let’s assume then that $P_2, \ldots, P_c$ are fixed, all lying on common tangents. We have free choice on $P_1$. Imposing that $d_1$ become a component, we are left with an $(f - 1)$-dimensional family of curves $C$ in $L_{D-P_1}$, Poncelet related to $S$, and such that $C + d_1$ belongs to $P_{onc,S'}$. Applying to this family the same procedure with a different $P_1$, we get a 1-dimensional family of curves $C$ with the above properties with respect to both choices of $P_1$. The curves $C$ do not consist only of tangents to $S'$, since these would have to touch $S'$ only at $P'_2, \ldots, P'_c$. We thus get a contradiction of Remarks 1.2-4 when applied to $P_{onc,S'}$.

Case (B6), $t = 0$.

In this case there exist a divisor $D = P + P_1 + \ldots + P_c$ on $S$ such that $V = L_D$. The argument from (B4) shows that at most one point, $P_1$ say, is not on a common tangent to $S$ and $S'$. By requiring that $d_3, \ldots, d_6$ be components we reduce ourselves to the case $c = 2 = f$, $D = P + P_1 + P_2$. But this is excluded by showing that $d_2$ has to be a component of $C$.

If all points of $D$ are tangency points of common tangents to $S$ and $S'$, then either two of them are multiple in $D$ or one of them has multiplicity bigger than 3 in $D$. In both cases the associated Poncelet curves are all singular by [12], Proposition 5.1. □
§ 3 A projective birational model of $Pon_c$ for $c$ odd

For odd $c$, $G_U(1, c+1)$ is the relative Grassmannian of 2-dimensional subspaces of a rank-$(c + 2)$ vector bundle over $U$. This vector bundle and thus also $G_U(1, c+1)$ may be extended to the whole of $\mathbb{P}(S^2W^*)$. When $c \geq 5$ one obtains a birational map from this extended relative Grassmannian to $Pon_c$.

In this paragraph we describe the linear system which induces this birational map and its base locus as a set. (Note that the knowledge of the full scheme - structure of this base-locus would allow one to compute the degree of $Pon_c$). Our method is to consider the stable rank-two vector bundles on $\mathbb{P}(W)$ associated to Poncelet curves and their curves of jumping lines.

Throughout this paragraph $c$ is assumed to be odd and bigger than 4.

The extension of $G_U(1, c+1)$ to $\mathbb{P}(S^2W^*)$ is obvious. Take $I \subset \mathbb{P}(S^2W^*) \times \mathbb{P}(W)$ the incidence variety “points of conics”, $p_1 : \mathbb{P}(S^2W^*) \times \mathbb{P}(W) \to \mathbb{P}(S^2W^*)$ the first projection, $\mathcal{V} := p_{1,*}\mathcal{O}_I\left(0, \frac{c+1}{2}\right)$ and $G$ the relative Grassmannian of 2-dimensional subspaces in the fibers of $\mathcal{V}$. $G$ is then a compactification of $G_U(1, c+1)$.

Consider now a Poncelet pair $(S, C)$ with $S$ smooth and $C = C(\Lambda)$ with base-point-free pencil $\Lambda \subset \mathcal{O}_S\left(\frac{c+1}{2}\right)$. (In this paragraph we use the notation $\mathcal{O}_S\left(\frac{c+1}{2}\right) := \mathcal{O}_{\mathbb{P}(W)}\left(\frac{c+1}{2}\right)|_S$). $\Lambda$ induces a surjective morphism

$$\mathcal{O}_{\mathbb{P}(W)}^2 \to \mathcal{O}_S\left(\frac{c+1}{2}\right).$$

Let $F$ be its kernel. Then $F$ is a stable rank-2 vector bundle on $\mathbb{P}(W)$ with $c_1(F) = -2$, $c_2(F) = c + 1$ and its jumping lines are exactly the lines joining the points of some divisor of $\Lambda$ (see [12], 4.1). Thus, the curve $C' \subset \mathbb{P}(W^*)$ of jumping lines of $F$ is Poncelet related to the conic $S'$ dual to $S$.

We shall make $F$ fit into a flat family of coherent rank-2 sheaves.
over $G$, and examine the birational map associating the curves of jumping lines to the fibers of this family.

We start with some Lemmata.

**Lemma 3.1.** Let $0 \to E' \to O_{\mathbb{P}^2}^N \to E'' \to 0$ be an exact sequence with $E'$, $E''$ coherent torsion-free, indecomposable sheaves on $\mathbb{P}^2$ and $c_1(E') = -1$. Then $N \leq 3$ and $E'$, $E''$ are slope-stable.

*Proof.* $E''$ torsion-free implies $E'$ locally-free. If $E'$ were not slope-stable, there would exist a locally free subsheaf $F$ of $E'$ with $c_1(F) = 0$. Standard arguments on slope-stable vector bundles (cf. [7], V 8.3) show that $F$ would be a direct summand in $E'$, contradicting the hypothesis.

Similarly, $E''$ must be slope-stable.

Let $r' := \text{rank } E'$, $r'' := \text{rank } E$.

Suppose $r' = 1$. Then $E' = O_{\mathbb{P}^2}(-1)$ and the morphism $O_{\mathbb{P}^2}(-1) \to O_{\mathbb{P}^2}^N$ is induced by $N$ sections in $O_{\mathbb{P}^2}(1)$. Since $\Gamma(O_{\mathbb{P}^2}(1))$ is 3-dimensional we may choose when $N > 3$ a basis for $O_{\mathbb{P}^2}(1)^N$ such that $N - 3$ components of $O_{\mathbb{P}^2} \to O_{\mathbb{P}^2}(1)^N$ vanish. But then $E''$ would split.

The case $r'' = 1$ is treated in the same way.

We are left with the situation $r' > 1$, $r'' > 1$. But now the Bogomolov inequality gives

$$c_2(E') \geq \frac{r' - 1}{2r'} \cdot c_1(E')^2 > 0$$

$$c_2(E'') \geq \frac{r'' - 1}{2r''} \cdot c_1(E'')^2 > 0$$

hence $c_2(O_{\mathbb{P}^2}^N) = -1 + c_2(E') + c_2(E'') > 0$, a contradiction. □

**Lemma 3.2.** Let $\mathcal{K}$ be the kernel of the natural surjective morphism

$$p_1^*\mathcal{V} \to O_I \left(0, \frac{c + 1}{2}\right)$$

on $\mathbb{P}(S^2W^*) \times \mathbb{P}(W)$. Then $\mathcal{K}$ is locally free and its restrictions to the fibers of $p_1$ are slope-stable.
Proof. The restriction to a fiber of $p_1$ over some point $s$ representing a conic $S$ gives an exact sequence

$$0 \to K\vert_{p^{-1}(s)} \to \Gamma \left( \mathcal{O}_S \left( \frac{c+1}{2} \right) \right) \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_S \left( \frac{c+1}{2} \right) \to 0.$$  

It is enough to check that $K := K\vert_{p^{-1}(s)}$ is locally free and slope stable.

Checking the locally freeness is easy (compare $K$ to its double dual or just use the fact that $K$ appears from an elementary transformation!).

Suppose that $K$ were not slope-stable.

Since $c_1(K) = -2$, there would exist a destabilizing subsheaf $K'$ of $K$ with $c_1(K') = -1$. By Lemma 3.1, $K'$ would have at most rank 2, hence the rank of $K$ could not exceed 4.

But rank $K = c+2$ and the Lemma follows from our assumption on $c$. \hfill \square

Let $S$ and $Q$ be the tautological sub- and quotient bundle on the relative Grassmannian $G$ of 2-dimensional subspaces in the fibers of $\mathcal{V}$. We denote by $S_{\mathbb{P}(W)}$, $Q_{\mathbb{P}(W)}$, $K_G$, $\mathcal{V}_{G \times \mathbb{P}(W)}$ the pullbacks to $G \times \mathbb{P}(W)$ of $S$, $Q$, $K$ and $\mathcal{V}$, respectively. (Subscript will be used in the sequel to indicate pullbacks through obvious maps). Denote further by $\alpha$ the composite morphism

$$Q_{\mathbb{P}(W)}^\vee \to \mathcal{V}_{G \times \mathbb{P}(W)}^\vee \to K_G^\vee$$

on $G \times \mathbb{P}(W)$ and by $E$ its cokernel.

Remark. One sees immediately that the fiber of $E$ over a point of $G$ represented by a Poncelet pair $(S, C(\Lambda))$ with $S$ smooth and $\Lambda$ base-point-free is just the dual of the kernel $F$ of the natural morphism

$$\mathcal{O}_{\mathbb{P}^2(W)}^2 \to \mathcal{O}_S \left( \frac{c+1}{2} \right).$$

In particular, it is stable and locally free.

We claim that $E$ is flat over $G$. This is a consequence of a local flatness criterion ([J]; 2.2.5) and the following

Lemma 3.3 The restrictions of $\alpha$ to the fibers of $G \times \mathbb{P}(W) \to G$ are injective and their cokernels are stable as soon as they are torsion-free.

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Proof. Fix a conic $S$ and a $c$-dimensional subspace $A$ of $\Gamma \left( \mathcal{O}_S \left( \frac{c+1}{2} \right) \right)^* \cong \Gamma(\mathbb{P}(W), K^\vee)$. The slope-stability of $K$ and Lemma 3.1 imply that the natural morphism $s : A \otimes \mathcal{O}_\mathbb{P}(W) \to K^\vee$ is injective.

If $\text{Coker } s$ is torsion-free but not stable, it will admit a rank-1 subsheaf $E'$ with torsion-free quotient and $c_1(E') = 1$, $c_2(E') \leq \frac{c-1}{2}$. In particular, $\dim \text{Ext}^1(E', \mathcal{O}_\mathbb{P}(W)) \leq \frac{c-1}{2}$. We get a subsheaf $K'$ of $K^\vee$ with $c_1(K') = 1$, $\text{rank}(K') = c + 1$, and which sits in an exact sequence

$$0 \to A \otimes \mathcal{O}_\mathbb{P}(W) \to K' \to E' \to 0.$$ 

This leads to a contradiction of the stability of $K$ in view of the following simple fact of homological algebra:

**Lemma 3.4** Let $A$ be a finite-dimensional $\mathbb{C}$-vector space, $\mathcal{C}$ a coherent sheaf on a compact complex manifold $X$ and

$$0 \to A^* \otimes \mathcal{O}_X \to B \to \mathcal{C} \to 0$$

an extension given by an element

$$\eta \in \text{Ext}^1(\mathcal{C}, A^* \otimes \mathcal{O}_X) \cong \text{Hom} \left( A, \text{Ext}^1(\mathcal{C}, \mathcal{O}_X) \right)$$

such that $A = A' \oplus A''$ and $\eta |_{A''} = 0$. Then

$$\eta |_{A'} \in \text{Hom} \left( A', \text{Ext}^1(\mathcal{C}, \mathcal{O}_X) \right) \cong \text{Ext}^1(\mathcal{C}, A'^* \otimes \mathcal{O}_X)$$

gives rise to a commutative diagram

$$\begin{array}{ccc}
0 & \longrightarrow & A'^* \otimes \mathcal{O}_X \\
\downarrow & & \downarrow \\
0 & \longrightarrow & B'
\end{array} \quad \begin{array}{ccc}
& & \longrightarrow \\
\downarrow & & \downarrow \\
& & \longrightarrow \\
\begin{array}{ccc}
0 & \longrightarrow & A^* \otimes \mathcal{O}_X \\
\downarrow & & \downarrow \\
0 & \longrightarrow & B
\end{array} & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

with injective vertical maps. \hfill \Box

Associating to a fiber of $E$ over $G$ its divisor of jumping lines defines a rational map $G \to \mathbb{P}(S^cW)$ in the usual way. Indeed, we consider the standard diagram

$$\begin{array}{ccc}
\mathbb{F} & \longrightarrow & \mathbb{P}(W^*) \\
\downarrow & & \downarrow \\
\mathbb{P}(W)
\end{array}$$
where $\mathbb{F}$ is the incidence variety “points on lines”, then take its product with $G$

$$
G \times \mathbb{F} \xrightarrow{q} G \times \mathbb{P}(W^*) \xrightarrow{p} G \times \mathbb{P}(W)
$$

and look at the relative theta characteristic associated to $\mathcal{E}$,

$$
\Theta(\mathcal{E}) := R^1 q_* p^*(\mathcal{E}(-2)).
$$

One sees immediately that, for the torsion free fibers, $\mathcal{E}_g$ of $\mathcal{E}$, the corresponding fibers of $\Theta(\mathcal{E})$ are the usual theta-characteristics associated to a stable sheaf on $\mathbb{P}^2$ with even first Chern class (cf. [1]). In particular, their supports are the curves of jumping lines of the $\mathcal{E}_g - s$. Let $B$ be the subset of $G$ over which $\mathcal{E}$ has nontrivial torsion in its fibers. Then we get a morphism

$$
G \setminus B \to \mathbb{P}(S^c W),
$$

(see [10]).

From the defining sequence of $\mathcal{E}$ one obtains the following resolution of $\Theta(\mathcal{E})$ on $G \times \mathbb{P}(W^*)$

$$
0 \to Q_{\mathbb{P}(W^*)}^\vee(-1) \xrightarrow{\varphi} (R^1 p_1_* \mathcal{K}^\vee)_{G \times \mathbb{P}(W^*)} \to \Theta(\mathcal{E}) \to 0.
$$

In order to obtain the “support” $Z$ of $\Theta(\mathcal{E})$ one takes the determinant of $\varphi$ and twists it correspondingly to get:

$$
0 \to \mathcal{L}^{-1} \boxtimes \mathcal{O}_{\mathbb{P}(W^*)}(-c) \to \mathcal{O}_{G \times \mathbb{P}(W^*)} \to \mathcal{O}_Z \to 0
$$

where

$$
\mathcal{L} := \det Q \otimes \mathcal{O}_{\mathbb{P}(S^2 W^*)} \left( -\frac{(c-1)(c-3)}{8} \right)_G.
$$

Notice that $Z$ is not flat over $G$. In fact, its fibers over $B$ are 2-dimensional. Thus the rational map $G \to \mathbb{P}(S^c W)$ above is given by a linear subsystem of $|\mathcal{L}|$ and has base locus $B$. The structural sheaf of this base locus may be recovered by taking
the push-down on $G$ of the previous exact sequence twisted by
(-3). One obtains for it the presentation
\[ \mathcal{L}^{-1} \otimes H^0(\mathcal{O}_{\mathbb{P}(W)}(c))^* \to \mathcal{O}_G \to R^2\pi_* (\mathcal{O}_Z(-3)) \to 0, \]
where $\pi : G \times \mathbb{P}(W^*) \to G$ is the projection.

Finally we want to identify $B$.

Let $\psi : \mathbb{P}(W^*) \times \mathbb{P}(W^*) \to \mathbb{P}(S^2W^*)$ be the map induced by the product, $p' : \mathbb{P}(W^*) \times \mathbb{P}(W^*) \times \mathbb{P}(W) \to \mathbb{P}(W^*) \times \mathbb{P}(W^*)$ the projection,
\[ I_i = \{(l_1, l_2, x) \in \mathbb{P}(W^*) \times \mathbb{P}(W^*) \times \mathbb{P}(W) \mid x \in l_i\}, \ i \in \{1, 2\}, \]
the incidence varieties and $\mathcal{W} := p'_* (\mathcal{O}_{I_1(0, -1, \frac{c-1}{2})})$.

**Proposition.** $W$ is a subbundle of $\psi^* \mathcal{V}$ and the natural map
from the Grassmannian of 2-dimensional subspaces in the fibers of $\mathcal{W}$ to $G$ is an embedding whose image is $B$. In particular, $B$ is smooth of dimension $c + 5$.

**Proof.** We examine when a fiber of $\mathcal{E}$ over $G$ admits nontrivial torsion. Fix a conic $S$ and a 2-dimensional subspace $L$ of
\[ \Gamma (\mathcal{O}_S (\frac{c+1}{2})). \]
If $K$ and $E$ are the fibers of $\mathcal{K}$ and $\mathcal{E}$ over the corresponding point of $G$ we have exact sequences:
\[ 0 \to \left( \Gamma \left( \mathcal{O}_S \left( \frac{c+1}{2} \right) \right) \right)^* \otimes \mathcal{O}_{\mathbb{P}(W)} \to K' \to E \to 0, \]
\[ o \to \mathcal{E}x^1(E, \mathcal{O}_{\mathbb{P}(W)}) \to 0. \]

where $\epsilon$ is the evaluation morphism. Since $K$ is slope-stable, $E$ may only have purely 1-dimensional torsion, and this happens exactly when $\mathcal{E}x^1(E, \mathcal{O}_{\mathbb{P}(W)})$ has 1-dimensional support or equivalently, when $\epsilon$ fails to be “generically surjective on some component of $S'$”. More precisely, if this is the case, then $S = S_1 + S_2$ and the composite morphism
\[ L \otimes \mathcal{O}_{\mathbb{P}(W)} \left( \frac{c+1}{2} \right) \to \mathcal{O}_S \left( \frac{c+1}{2} \right) \to \mathcal{O}_{S_2} \left( \frac{c+1}{2} \right) \]
vanishes. Then $\epsilon$ factors through $\mathcal{O}_{S_1} \left( \frac{c+1}{2} \right)$ and the pullback
of $L$ through $\psi$ will be a 2-dimensional subspace in a fiber of $\mathcal{W}$
over $\mathbb{P}(W^*) \times \mathbb{P}(W^*)$. Indeed, pushing the exact sequence
\[ 0 \to \mathcal{O}_{I_1}(0, -1, -1) \to \psi^* \mathcal{O}_I \to \mathcal{O}_{I_2} \to 0 \]
twisted by \( \left( \frac{c+1}{2} \right) \) from \( \mathbb{P}(W^*) \times \mathbb{P}(W^*) \times \mathbb{P}(W) \) down to \( \mathbb{P}(W^*) \times \mathbb{P}(W^*) \) gives

\[
0 \to \mathcal{W} \to \psi^* \mathcal{V} \to p'_*(\mathcal{O}_{l_2} \left( 0, 0, \left( \frac{c+1}{2} \right) \right)) \to 0.
\]

The claims of the Proposition are now easy to check; that the considered morphism is an embedding follows e.g. from a computation of its Jacobian matrix.

\[\square\]

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