Bounds for the boxicity of Mycielski graphs

*Akira Kamibeppu

March 13, 2014

Abstract

A box in Euclidean $k$-space is the Cartesian product $I_1 \times I_2 \times \cdots \times I_k$, where $I_j$ is a closed interval on the real line. The boxicity of a graph $G$, denoted by box($G$), is the minimum integer $k$ such that $G$ can be represented as the intersection graph of a family of boxes in Euclidean $k$-space.

In this paper, we observe behavior of the boxicity of generalized Mycielski graphs. Here we consider whether the boxicity of the generalized Mycielski graph $M_r(G)$ is more than that of $G$. Actually, the inequality $\text{box}(M_r(G)) \geq \text{box}(G)$ holds for a graph $G$. We give a lower bound for the boxicity of generalized Mycielski graphs. We prove that $\text{box}(M_r(G)) \geq \text{box}(G) + \lceil \frac{l}{2} \rceil$ holds for a graph $G$ with $l$ focal vertices. This indicates that the boxicity of the generalized Mycielski graph $M_r(G)$ is more than that of $G$ if $G$ has a focal vertex. Moreover, we also present an upper bound for the boxicity of Mycielski graphs. For a graph $G$ with $l$ focal vertices, we prove that $\text{box}(M_2(G)) \leq \theta(G) + \lceil \frac{l}{2} \rceil + 1$ holds, where $\theta(G)$ is the edge clique cover number of the complement of $G$. These results determine the boxicity of Mycielski graphs of some of complete $k$-partite graphs.

Keywords: boxicity; chromatic number; cointerval graph; edge clique cover number; (generalized) Mycielski graph

1 Introduction

The notion of boxicity of graphs was introduced by Roberts [10]. It has applications in some research fields, like niche overlap in ecology and fleet maintenance in operations research (see [9, 11]). Roberts [10] proved that the maximal boxicity of graphs with $n$ vertices is $\lfloor \frac{n}{2} \rfloor$ (also see [4]), where $\lfloor x \rfloor$ denotes the largest integer at most $x$. Cozzens [3] proved that the task of computing boxicity of graphs is NP-hard. Some researchers have
attempted to calculate or bound boxicity of graph classes with special structure. Roberts [10] showed that the boxicity of a complete \( k \)-partite graph \( K_{n_1,n_2,...,n_k} \) is the number of \( n_i \) which is at least 2. Scheinerman [12] proved that the boxicity of outer planner graphs is at most 2. Thomassen [13] proved that the boxicity of planer graphs is at most 3. Cozzens and Roberts [4] investigated the boxicity of split graphs. As Chandran et al. [1] say, not much is known about the boxicity of most of the well-known graph classes. They proved that the boxicity of a graph \( G \) is at most \( \text{tw}(G) + 2 \), where \( \text{tw}(G) \) is the treewidth of \( G \).

By using this bound, they presented upper bounds for chordal graphs, circular arc graphs, AT-free graphs, co-comparability graphs, and permutation graphs.

Recently, Chandran et al. [2] found the following relation between boxicity and chromatic number.

**Theorem 1.1** ([2], Theorem 6.1). Let \( G \) be a graph with \( n \) vertices. If \( \text{box}(G) = \frac{n}{2} - s \) for \( s \geq 0 \), the inequality \( \chi(G) \geq \frac{n}{2s+2} \) holds, where \( \chi(G) \) is the chromatic number of \( G \).

Theorem 1.1 implies that, if the boxicity of a graph with \( n \) vertices is very close to the maximum boxicity \( \lfloor \frac{n}{2} \rfloor \), the chromatic number of the graph must be very large. The converse does not hold in general; there is a graph whose boxicity is small, even if the chromatic number of the graph is large, like a complete graph. Also there is a graph whose boxicity is large and whose chromatic number is small (see Section 5.1 in [2]). However, there is not much information about structure of graphs with large boxicity.

One of the purpose of this paper is to consider whether behavior of boxicity is similar to that of chromatic number under the generalized Mycielski’s construction \( M_r(\cdot) \) given in [5]. This construction is a generalization of the focalization of graphs. It is well-known that the chromatic number of the Mycielski graph \( M_2(G) \) of a graph \( G \) is more than that of \( G \). We see that the boxicity of the generalized Mycielski graph \( M_r(G) \) is at least that of \( G \) since \( M_r(G) \) contains \( G \) as an induced subgraph (see the definition). Here we observe whether the boxicity of the generalized Mycielski graph \( M_r(G) \) is more than that of \( G \). Note that the focalization of a graph does not increase boxicity.

We give an upper bound and a lower bound for the boxicity of Mycielski graphs. As a consequence, our observation determine the boxicity of Mycielski graphs of some of complete \( k \)-partite graphs.

## 2 Preliminary

In this paper, all graphs are finite, simple and undirected. We use \( V(G) \) for the vertex set of a graph \( G \). We use \( E(G) \) for the edge set of a graph \( G \). An edge of a graph with endpoints \( u \) and \( v \) is denoted by \( uv \). A graph is said to be trivial if \( E(G) \) is empty. For a subset \( V \) of \( V(G) \), let \( G - V \) be the subgraph induced by \( V(G) \setminus V \). For a subset \( E \) of \( E(G) \), let \( G - E \) be the subgraph on \( V(G) \) with \( E(G) \setminus E \) as its edge set. For a graph \( G \),
its complement is denoted by $\overline{G}$. The intersection graph of a non-empty family $F$ of sets is the graph whose vertex set is $F$ and $F_1$ is adjacent to $F_2$ if and only if $F_1 \cap F_2 \neq \emptyset$ for $F_1, F_2 \in F$. Especially, the intersection graph of a non-empty family of sets on the real line is called an interval graph. A graph $G$ can be represented as the intersection graph of a family $F$ if and only if the corresponding sets in $F$ have non-empty intersection. A box in Euclidean $k$-space is the Cartesian product $I_1 \times I_2 \times \ldots \times I_k$, where $I_j$ is a closed interval on the real line. The boxicity of a graph $G$, denoted by $\text{box}(G)$, is the minimum integer $k$ such that $G$ can be represented as the intersection graph of a family of boxes in Euclidean $k$-space. The boxicity of a complete graph is defined to be 0. If $G$ is an interval graph, $\text{box}(G) \leq 1$. If $H$ is an induced subgraph of $G$, $\text{box}(H) \leq \text{box}(G)$ holds by the definition.

A graph is a cointerval graph if its complement is an interval graph. Lekkerkerker and Boland [6] presented the forbidden subgraph characterization of interval or cointerval graphs. Cointerval graphs do not contain the complement of a cycle of length at least 4 as an induced subgraph, for example. It is easy to see that the union of a cointerval graph and isolated vertices is also a cointerval graph. A cointerval edge covering $C$ of a graph $G$ is a family of cointerval spanning subgraphs of $G$ such that each edge of $G$ is in some graph of $C$. For a set $X$, the cardinality of $X$ is denoted by $|X|$. The following theorem is useful to calculate of the boxicity of graphs.

**Theorem 2.1** ([4], Theorem 3). Let $G$ be a graph. Then, $\text{box}(G) \leq k$ if and only if there is a cointerval edge covering $C$ of $G$ with $|C| = k$.

Let $G$ be a graph and $r$ an integer at least 2. Let $V(G)_i$ be a copy of $V(G)$, where $i = 1, 2, \ldots, r$. For each vertex $v \in V(G)$, $v_i$ denotes the vertex in $V(G)_i$ corresponding to $v$. The generalized Mycielski graph of a graph $G$, denoted by $M_r(G)$, is the graph whose vertex set is $\{z\} \cup \bigcup_{i=1}^{r+1} V(G)_i$, the disjoint union of $\{z\}$ and $V(G)_1, \ldots, V(G)_{r+1}$, and whose edge set is $\bigcup_{i=1}^{r+1} E_i$, where

$$E_1 = \{u_1v_1 \mid uv \in E(G)\},$$

$$E_i = \{u_{i-1}v_i, v_{i-1}u_i \mid uv \in E(G)\} \quad \text{for } i = 2, 3, \ldots, r,$$

$$E_{r+1} = \{zu_r \mid u \in V(G)\}.$$ 

The construction $M_2(\cdot)$ was invented by Mycielski [8] (see Fig. 1) in order to construct a triangle-free graph with arbitrary large chromatic number. Here the graph $M_2(G)$ is simply called Mycielski graph of a graph $G$. It is not difficult to show that $\chi(M_2(G)) = \chi(G) + 1$ holds for a graph $G$ (see [2], Problem 9.18). Note that the subgraph of $M_r(G)$ induced by $V(G)_1$ is isomorphic to the graph $G$, and hence $\text{box}(M_r(G)) \geq \text{box}(G)$.
3 A lower bound for the boxicity of generalized Mycielski graphs

For a complete graph $K_n$, it is easy to see that $\text{box}(M_r(K_n)) \geq 1 > 0 = \text{box}(K_n)$ since $M_r(K_n)$ is not complete by the definition. We determine the boxicity of $M_2(K_n)$ next section (see Lemma 4.1). First we consider if the boxicity of the generalized Mycielski graph of a graph $G$ is more than that of $G$ in general.

Observation 1. For a graph $G$, does the inequality $\text{box}(M_r(G)) > \text{box}(G)$ hold?

Let $V_n$ be the graph with $n$ vertices and without edges. Note that $M_r(V_n)$ is the disjoint union of a star $K_{1,n}$ and $(r - 1)n$ isolated vertices. It is easy to see that its boxicity is 1, the same as the boxicity of $V_n$ if $n \geq 2$. Hence, this is an example of a graph $G$ such that $\text{box}(M_r(G)) = \text{box}(G)$ holds.

Observation 2. For a connected graph $G$, does the inequality $\text{box}(M_r(G)) > \text{box}(G)$ hold?

A vertex $v$ of $G$ is called a focal vertex if $v$ is adjacent to all vertices in $V(G) \setminus \{v\}$. The following example shows that there is a connected graph $G$ such that the equality $\text{box}(M_2(G)) = \text{box}(G)$ holds.

Example 3.1. The boxicity of the Mycielski graph of a cycle $C_4$ is equal to 2. To check this, we give a cointerval edge covering of the complement $\overline{M_2(C_4)}$ (see Fig. 1).

Let $H_1$ and $H_2$ be the graphs appeared in Fig. 2. Both graphs are cointerval spanning subgraphs of $\overline{M_2(C_4)}$. Note that the disjoint union of a cointerval graph and isolated vertices is also cointerval since these isolated vertices become pairwise adjacent focal vertices in the complement. Hence, we may prove that $H_1 - \{v_1, y_1\}$ and $H_2 - \{u_1, u_2, x_1\}$ are cointerval, instead of $H_1$ and $H_2$, respectively. A family of intervals on the real line with intersection graph isomorphic to $\overline{H_1 - \{v_1, y_1\}}$ can be found as in the bottom of Fig. 2. The almost same argument works for $H_2 - \{u_1, u_2, x_1\}$. Also see that $H_1$ and $H_2$ cover all
edges of $M_2(C_4)$. The family $\{H_1, H_2\}$ is a desired cointerval edge covering of $M_2(C_4)$, and hence, $\text{box}(M_2(C_4)) \leq 2$ by Theorem 2.1. Also note that $\text{box}(M_2(C_4)) \geq \text{box}(C_4) = 2$.

![Diagram of $H_1$ and $H_2$]

**Observation 3.** For a connected graph $G$ and $r \geq 3$, does the inequality $\text{box}(M_r(G)) > \text{box}(G)$ hold?

The *distance between two vertices* $u$ and $v$ in a graph $G$ is defined by length of the shortest path from $u$ to $v$ in $G$ and is denoted by $d_G(u, v)$. If there exist no paths from $u$ to $v$ in $G$, define $d_G(u, v) = \infty$. Let $H_1$ and $H_2$ be subgraphs of $G$. The *distance between two subgraphs* $H_1$ and $H_2$ in $G$, denoted by $d_G(H_1, H_2)$, is defined to be $\min\{d_G(v_1, v_2) \mid v_1 \in V(H_1), v_2 \in V(H_2)\}$. The following lemma is a generalization of Corollary 3.6 in [4].

**Lemma 3.2.** Let $G$ be a graph and $H_1$, $H_2$ induced subgraphs of the complement $\overline{G}$. If $d_G(H_1, H_2) \geq 2$, the following inequality holds:

$$\text{box}(G) \geq \text{box}(\overline{H_1}) + \text{box}(\overline{H_2}).$$

**Proof.** If either $H_1$ or $H_2$ is trivial, say $H_1$, then $\overline{H_1}$ is complete. Hence, $\text{box}(\overline{H_1}) = 0$. Since $\overline{H_2}$ is an induced subgraph of $G$, we see that

$$\text{box}(G) \geq \text{box}(\overline{H_2}) = \text{box}(\overline{H_1}) + \text{box}(\overline{H_2}).$$
holds. In what follows, we may assume that $H_1$ and $H_2$ are non-trivial.

The assumption $d_{\overline{G}}(H_1, H_2) \geq 2$ means that $d_{\overline{G}}(v_1, v_2) \geq 2$ for a vertex $v_1$ of $H_1$ and a vertex $v_2$ of $H_2$. Hence, an edge of $H_1$ and an edge of $H_2$ form $2K_2$, the disjoint union of two edges, as an induced subgraph of $\overline{G}$. Moreover, we claim the following.

Claim (1): no cointerval spanning subgraphs of $\overline{G}$ contain an edge of $H_1$ and an edge of $H_2$, and

Claim (2): we need at least $\text{box}(\overline{H}_i)$ cointerval spanning subgraphs of $\overline{G}$ to cover all edges of $H_i$, where $i = 1, 2$.

Claim (1) follows from the forbidden subgraph characterization of cointerval graphs. Actually, cointerval graphs do not contain $2K_2$ as an induced subgraph. Claim (2) follows from Theorem 2.1. A cointerval graph with edges of $H_1$ does not contain edges of $H_2$. Thus, the inequality $\text{box}(G) \geq \text{box}(\overline{H}_1) + \text{box}(\overline{H}_2)$ holds.

The focalization $G^f$ of a graph $G$ is the graph obtained from $G$ by adding a new vertex $x$ adjacent to all vertices of $G$. Let $n$ be a natural number. The $n$th focalization of a graph $G$ is $\underbrace{(...(G^{f^1})^{f^1}...)}^n$, the graph obtained from $G$ by iteration of $n$ times focalization and is denoted by $G^{f^n}$. Here, $G^{f^1} = G$. Note that $\text{box}(G^{f^n}) = \text{box}(G)$.

It is possible for the generalized Mycielski’s construction $M_r(\cdot)$ to make boxicity of graphs arbitrary large. To show this, the following lemma is useful. Here, $\lfloor x \rfloor$ denotes the smallest integer at least $x$.

Lemma 3.3 ([4], Lemma 3). Let $G$ be a graph. Let $S_1 = \{a_1, a_2, \ldots, a_n\}$ and $S_2 = \{b_1, b_2, \ldots, b_n\}$ be disjoint subsets of $V(G)$ such that the only edges between $S_1$ and $S_2$ in $\overline{G}$ are the edges $a_ib_i$, where $i = 1, 2, \ldots, n$. Then, $\text{box}(G) \geq \lfloor \frac{n}{2} \rfloor$.

The following partially answers Observation 3. Also see Corollary 3.6.

Theorem 3.4. For a graph $G$ and a non-negative integer $n$, the following inequality holds:

$$\text{box}(M_2(G^{f^n})) \geq \text{box}(G^{f^n}) + \left\lceil \frac{n}{2} \right\rceil.$$  

Proof. Let $G$ be a graph. Let $V(G^{f^n}) = V(G) \cup \{x_1, x_2, \ldots, x_n\}$, where $x_i$ is an additional focal vertex of $G^{f^n}$. We consider the Mycielski graph $M_2(G^{f^n})$ and its complement $\overline{M_2(G^{f^n})}$. Let $X_j = \{(x_1)_j, (x_2)_j, \ldots, (x_n)_j\}$, the set of vertices in $V(G^{f^n})_j$ corresponding to additional focal vertices of $G^{f^n}$. Let $H$ be the subgraph of $\overline{M_2(G^{f^n})}$ induced by $V(G^{f^n})_1 \setminus X_1$, which is isomorphic to $\overline{G}$. Note that $\text{box}(\overline{H}) = \text{box}(G)$. Let $D_n$ be the subgraph of $\overline{M_2(G^{f^n})}$ induced by the union of $X_1$ and $X_2$. Note that $X_1$ and $X_2$ are disjoint by their definition. It is not difficult to check that the only edges between $X_1$ and $X_2$ in $D_n$ are the edges $(x_i)_1(x_i)_2$, where $i = 1, 2, \ldots, n$. Actually, the vertex $(x_i)_1 \in X_1$ is adjacent to all vertices in $V(G^{f^n})_2 \setminus \{(x_i)_2\}$ in $M_2(G^{f^n})$ and the vertex $(x_i)_2 \in X_2$ is
adjacent to all vertices in \(V(G^{f^n})_1 \setminus \{(x_i)_1\}\) in \(M_2(G^{f^n})\) since \(x_i\) is a focal vertex of \(G^{f^n}\). We see that \(\text{box}(\overline{D_n}) \geq \lceil \frac{n}{2} \rceil\) by Lemma 3.3.

We prove that \(d_{M_2(G^{f^n})}(H, D_n) \geq 2\) holds. Let \(v\) be a vertex of \(H\) and \(x\) a vertex of \(D_n\). The vertex \(v\) is in \(V(G^{f^n})_1 \setminus X_1\) and the vertex \(x\) is in \(X_1\) or \(X_2\). We may represent \(x\) as \((x_i)_j\), where \(j = 1, 2\). Since \(x_i\) is an additional focal vertex of \(G^{f^n}\), \((x_i)_j\) is not adjacent to \(v\) in \(M_2(G^{f^n})\). This implies that \(d_{M_2(G^{f^n})}(v, x) \geq 2\) for a vertex \(v\) of \(H\) and a vertex \(x\) of \(D_n\), that is, \(d_{M_2(G^{f^n})}(H, D_n) \geq 2\). Thus, the inequality

\[
\text{box}(M_2(G^{f^n})) \geq \text{box}(\overline{H}) + \text{box}(\overline{D_n}) = \text{box}(G) + \text{box}(\overline{D_n}) \geq \text{box}(G^{f^n}) + \left\lceil \frac{n}{2} \right\rceil
\]

holds by Lemma 3.2.

\[\square\]

**Remark 3.5.** We note the proof of Theorem 3.4 works on the generalized Mycielski graph \(M_r(G^{f^n})\), that is, \(\text{box}(M_r(G^{f^n})) \geq \text{box}(G^{f^n}) + \left\lceil \frac{n}{2} \right\rceil\) holds.

In the proof of Theorem 3.4, we prove that \(\text{box}(\overline{D_n}) \geq \left\lceil \frac{n}{2} \right\rceil\) by using Lemma 3.3. Actually, note that \(\text{box}(\overline{D_n}) = \left\lceil \frac{n}{2} \right\rceil\). Any two vertices in \(X_1\) are not adjacent in \(M_2(G^{f^n})\) since they are adjacent in \(M_2(G^{f^n})\). Hence, \(X_1\) is independent in \(D_n\). Also note that \(X_2\) is a clique in \(M_2(G^{f^n})\) by the definition of \(M_2(G^{f^n})\), that is, in \(D_n\). Also see the argument behind the proof of Theorem 5 in [4].

If we restrict our attention to the focalization \(G^f\) in the proof of Theorem 3.4, then Lemma 3.3 is superfluous. Note that \(\text{box}(\overline{D_1}) = 1\) since \(D_1\) is the graph with two vertices \((x_1)_1\) and \((x_1)_2\) and the edge \((x_1)_1(x_1)_2\).

**Corollary 3.6.** For a graph \(G\) with \(l\) focal vertices, the inequality \(\text{box}(M_r(G)) \geq \text{box}(G) + \left\lceil \frac{l}{2} \right\rceil\) holds. If \(G\) has a focal vertex, the inequality \(\text{box}(M_r(G)) > \text{box}(G)\) holds.

## 4 An upper bound for the boxicity of Mycielski graphs

In this section, we give an upper bound for the boxicity of Mycielski graphs. Moreover we calculate the boxicity of Mycielski graphs of some of complete \(k\)-partite graphs. First we determine the boxicity of Mycielski graphs of complete graphs.

**Lemma 4.1.** For a complete graph \(K_n\), the following equality holds:

\[
\text{box}(M_2(K_n)) = \begin{cases} 
\left\lceil \frac{n}{2} \right\rceil & \text{if } n \text{ is odd}, \\
\left\lceil \frac{n}{2} \right\rceil + 1 & \text{if } n \text{ is even}.
\end{cases}
\]

**Proof.** Let \(H_0\) be the subgraph of \(M_2(K_n)\) induced by \(V(M_2(K_n)) \setminus \{z\}\). We have the inequality \(\text{box}(M_2(K_n)) \geq \text{box}(H_0) \geq \left\lceil \frac{n}{2} \right\rceil\) by Lemma 3.3.

Let \(V(K_n) = \{v_1, v_2, \ldots, v_n\}\). To see \(\text{box}(M_2(K_n)) \leq \left\lceil \frac{n}{2} \right\rceil + 1\), we give cointerval subgraphs of \(M_2(K_n)\). Let \(G_0\) be the subgraph of \(M_2(K_n)\) induced by \(\{z, (v_n)_2\} \cup V(K_n)_1\).
We define $G_i$ to be the subgraph of $\overline{M_2(K_n)}$ induced by $\{(v_{2i-1}), (v_{2i})\} \cup V(K_n)_2$, where $i = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil - 1$. Moreover, let $G_{\left\lfloor \frac{n}{2} \right\rfloor}$ be the subgraph of $\overline{M_2(K_n)}$ induced by $\{(v_{n-1}), (v_n)\} \cup V(K_n)_2$. It is easy to see that the family $\{G_0, G_1, \ldots, G_{\left\lceil \frac{n}{2} \right\rceil}\}$ is a cointerval edge covering of $\overline{M_2(K_n)}$, and hence $\text{box}(M_2(K_n)) \leq \left\lceil \frac{n}{2} \right\rceil + 1$ holds.

If $n$ is odd, the family $\{G_0, G_1, \ldots, G_{\left\lceil \frac{n}{2} \right\rceil-1}\}$ is a cointerval edge covering of $\overline{M_2(K_n)}$. Hence we have the equality $\text{box}(M_2(K_n)) = \left\lceil \frac{n}{2} \right\rceil$.

If $n$ is even, that is, $n = 2k$, we show that $\text{box}(M_2(K_{2k})) > k$. Suppose to the contrary that $\overline{M_2(K_{2k})}$ can be covered with $k$ cointerval (spanning) subgraphs $H_1, H_2, \ldots, H_k$ of $\overline{M_2(K_{2k})}$. Let $e_j = (v_j)_1(v_j)_2$ for $j = 1, 2, \ldots, 2k$. The graph $H_i$ contains at most two edges in $\{e_j\}$ since $H_i$ is cointerval. Actually, the graph $H_i$ must contain two edges in $\{e_j\}$. Otherwise there is a graph $H$ in $\{H_i\}$ which contains only one edge in $\{e_j\}$ or which contains no edges in $\{e_j\}$. Hence the family $\{H_i\} \setminus \{H\}$ of $k - 1$ cointerval subgraphs of $\overline{M_2(K_{2k})}$ must cover at least $2k - 1$ edges in $\{e_j\}$, but this is impossible. On the other hand, there is a cointerval graph $H_*$ in $\{H_i\}$ which contains an edge $z(v)_1$, where the vertex $v$ is in $V(K_{2k})$. We may assume that the graph $H_*$ contains two edges $e_s$ and $e_t$ in $\{e_j\}$. Hence we see $V(H_*) \supset \{(v_s)_1, (v_s)_2, (v_t)_1, (v_t)_2, z\}$. We note that

$$(v_s)_1(v_s)_1, (v_s)_1(v_s)_2, (v_t)_1(v_t)_2, z(v_s)_2, z(v_t)_2 \not\in E(\overline{M_2(K_{2k})})$$

by the definition of Mycielski’s construction. If $v \neq v_s, v_t$, it follows from Lemma 3.3 that $\text{box}(H_*) \geq 2$ since $(v)_1(v)_1, (v)_1(v)_1 \not\in E(\overline{M_2(K_{2k})})$, which is a contradiction. Hence we may assume that $v = v_s$. We reach the following four cases on the graph $H_*$ (see Fig. 3).

![Fig. 3: The subgraph $H_*$ of $\overline{M_2(K_{2k})}$ containing edges $e_s$ and $e_t$.](image)
These cases imply that \( \text{box}(\overline{H}_*) \geq 2 \), which contradicts our assumption that \( H_* \) is cointerval. Thus we have \( \text{box}(M_2(K_{2k})) > k \). Hence we have the equality \( \text{box}(M_2(K_n)) = \left\lceil \frac{n}{2} \right\rceil + 1 \) if \( n \) is even. \( \square \)

The edge clique cover number of a graph \( G \), denoted by \( \theta(G) \), is the minimum cardinality of a family of cliques that covers all edges of \( G \). The following theorem gives us an upper bound for the boxicity of Mycielski graphs.

**Theorem 4.2.** For a graph \( G \) with \( l \) focal vertices, the inequality

\[
\text{box}(M_2(G)) \leq \theta(\overline{G}) + \left\lceil \frac{l}{2} \right\rceil + 1
\]

holds. If \( l \) is odd or zero, we have the inequality

\[
\text{box}(M_2(G)) \leq \theta(\overline{G}) + \left\lceil \frac{l}{2} \right\rceil.
\]

**Proof.** Let \( \{A_1, A_2, \ldots, A_{\theta(\overline{G})}\} \) be a family of cliques in \( \overline{G} \) that covers all edges of \( \overline{G} \). Let \( v_1, v_2, \ldots, v_l \) be all isolated vertices of \( \overline{G} \) and write \( J = \{v_1, v_2, \ldots, v_l\} \). Note that \( V(G) = A_1 \cup A_2 \cup \ldots \cup A_{\theta(\overline{G})} \cup J \). We define \( H_i \) to be the subgraph of \( M_2(G) \) induced by \( (A_i)_1 \cup V(G)_2 \cup \{z\} \) and let \( E_i = \{xy \mid x, y \in V(G)_2 \setminus (A_i)_2\} \) and \( F_i = \{xy \mid x \in (A_i)_1, y \in V(G)_2 \setminus (A_i)_2\} \), where \( i = 1, 2, \ldots, \theta(\overline{G}) \). We can check that \( H_i - (E_i \cup F_i) \) is a cointerval graph (see Fig. 4).

![Diagrams showing subgraphs of \( M_2(G) \) and their relationships](image)

Fig. 4: The subgraph \( H_i - (E_i \cup F_i) \) and an interval representation of \( \overline{H}_i - (E_i \cup F_i) \).
Note that the subgraph of $M_2(G)$ induced by $J_1 \cup J_2 \cup \{z\}$ is isomorphic to $M_2(K_1)$. Hence the edge set of the subgraph of $M_2(G)$ isomorphic to $M_2(K_1)$ can be covered with at most $\lceil \frac{l}{2} \rceil + 1$ cointerval subgraphs as in the proof of Lemma 4.1. Let $G_0$ be the subgraph of $M_2(G)$ induced by $\{z, (v_1)\}$ and $G_i$ the subgraph of $M_2(G)$ induced by $\{(v_{2i-1})_1, (v_{2i})_1\} \cup J_2$ for $i = 1, 2, \ldots, \lceil \frac{l}{2} \rceil - 1$. Moreover, let $G_{\lceil \frac{l}{2} \rceil}$ be the subgraph of $M_2(G)$ induced by $\{(v_{2i-1})_1, (v_{2i})_1\} \cup J_2$. We can check that $\theta(G) + \lceil \frac{l}{2} \rceil + 1$ cointerval subgraphs $H_1 - (E_1 \cup F_1), \ldots, H_{\theta(G)} - (E_{\theta(G)} \cup F_{\theta(G)})$, $G_0, G_1, \ldots, G_{\lceil \frac{l}{2} \rceil}$ cover all edges of $M_2(G)$.

Let $e$ be an edge of $E(M_2(G))$. If $e \cap \{z\} \neq \emptyset$, we see $e \cap V(G_1) \neq \emptyset$. Hence there is an $i \in \{1, 2, \ldots, \theta(G)\}$ such that $e \in E(H_i - (E_i \cup F_i))$ or $e \in E(G_0)$. If $e \cap \{z\} = \emptyset$, we have $e \subset V(G_1) \cup V(G_2)$. Hence, if $e \subset V(G_1)$, especially, $e \cap (A_i)_2 \neq \emptyset$, we see $e \in E(H_i - (E_i \cup F_i))$. If $e \subset V(G_2)$ and $e \cap (A_i)_2 = \emptyset$ for any $i$, we see $e \subset J_2$, and hence $e \in E(G_i)$ for $i \neq 0$. If $e \cap V(G_1) \neq \emptyset$, we reach the following two cases:

(i) $e \subset V(G_1)$ or (ii) $e \cap V(G_2) \neq \emptyset$.

In the case (i), the edge $e$ is in some $(A_i)_1$, since the family $\{A_1, A_2, \ldots, A_{\theta(G)}\}$ of cliques covers all edges of $C_1$, and hence we have $e \in E(H_i - (E_i \cup F_i))$.

Now we focus on the case (ii). Let $u$ be a vertex in $V(G)$ and $C_u$ the union of cliques in $\{A_1, A_2, \ldots, A_{\theta(G)}\}$ containing the vertex $u$. If $u$ is an isolated vertex in $G$, let $C_u$ be the set $\{u\}$. Then we note $u_1 \in V(G_1)$ is never adjacent to vertices in $V(G_2) \setminus (C_u)_2$ on $M_2(G)$ by the definition of Mycielski graphs. Hence the following two cases occur:

(ii-1) the edge $e$ connects a vertex of $(A_i)_1$ and a vertex of $(A_i)_2$ for some $i$ or

(ii-2) the edge $e$ connects a vertex $(v_{1i})$ and a vertex $(v_{2i})$, where $v_{1i} \in J$.

Under the case (ii-1), we notice that $e \in E(H_i - (E_i \cup F_i))$. Under the case (ii-2), we see that $e \in E(G_i)$. These arguments complete the proof of our first statement.

If $l = 0$, the graphs $H_1 - (E_1 \cup F_1), \ldots, H_{\theta(G)} - (E_{\theta(G)} \cup F_{\theta(G)})$ cover all edges of $M_2(G)$. If $l$ is odd, $H_1 - (E_1 \cup F_1), \ldots, H_{\theta(G)} - (E_{\theta(G)} \cup F_{\theta(G)})$, $G_0, G_1, \ldots, G_{\lceil \frac{l}{2} \rceil}$ cover all edges of $M_2(G)$, because the edge $(v_{1i})_1(v_{2i})$ is covered with the graph $G_0$. Our second statement follows from similar arguments as above.

Recall that the boxicity of a complete $k$-partite graph $K_{n_1, n_2, \ldots, n_k}$ is the number of $n_i$ which is at least 2. If $K_{n_1, n_2, \ldots, n_k}$ has $l$ focal vertices, for example, $n_i = 1$ for $i = k - l + 1, \ldots, k$, we have

$$\text{box}(M_2(K_{n_1, n_2, \ldots, n_k})) \geq \text{box}(K_{n_1, n_2, \ldots, n_{k-1}}) + \left\lceil \frac{l}{2} \right\rceil = \left\lceil \frac{2k - l}{2} \right\rceil$$

by Corollary 3.6. Hence we can determine the boxicity of Mycielski graphs of some of complete $k$-partite graphs by using Theorem 4.2.
**Corollary 4.3.** For a complete $k$-partite graph $K_{n_1,n_2,...,n_k}$ with $l$ focal vertices, the inequality

$$\left\lceil \frac{2k-l}{2} \right\rceil \leq \text{box}(M_2(K_{n_1,n_2,...,n_k})) \leq \min \left\{ k, \left\lceil \frac{2k-l}{2} \right\rceil + 1 \right\}$$

holds. Especially, if $l$ is odd or zero, the equality $\text{box}(M_2(K_{n_1,n_2,...,n_k})) = \left\lceil \frac{2k-l}{2} \right\rceil$ holds.

**Proof.** Note that $\theta(K_{n_1,n_2,...,n_k}) = k - l$. Our statement follows from Theorem 4.2.

In [10], among all graphs with $n$ vertices, the boxicity of either $K_{2,2,...,2}$ or $K_{2,2,...,2,1}$ attains the maximal boxicity. These graphs contain a lot of induced cycles $C_3$ and $C_4$. Hence, it seems like containing a lot of these induced subgraphs for a graph makes its boxicity large. Note that complete graphs have small boxicity with a lot of $C_3$. The Mycielski graph $M_2(K_{1,n})$ contains a lot of $C_4$ as induced subgraphs and its boxicity is equal to 2 by Corollary 4.3. Then, take the disjoint union of $K_n$ and $M_2(K_{1,n})$, and add the edge set $\{uv \mid u \in V(K_n), v \in V(M_2(K_{1,n}))\}$ into the graph. The boxicity of the resulting graph is equal to 2 by Lemma 3.2. We note that containing a lot of $C_3$ and $C_4$ as induced subgraphs for a graph does not make its boxicity large in general.

## 5 Concluding Remarks

We proved that the boxicity of the generalized Mycielski graph of a graph $G$ with focal vertices is more than that of $G$. This statement partially answers Observation 3, but it is still open. As examples of complete $k$-partite graphs without focal vertices, one may expect that the equality $\text{box}(M_r(G)) = \text{box}(G)$ holds for a graph $G$ without focal vertices. However, we note that there is a graph $G$ without focal vertices such that $\text{box}(M_2(G)) > \text{box}(G)$. The graph $P_4$, a path with four vertices, is the desired one. The graph $P_4$ is interval, but $M_2(P_4)$ is not interval.

## Acknowledgments

This work was supported by Grant-in-Aid for Young Scientists (B), No.25800091.

## References

[1] L. S. Chandran and N. Sivadasan, Boxicity and treewidth, J. Combin. Theory Ser. B 97 (2007) 733-744.

[2] L. S. Chandran, A. Das, and C. D. Shah, Cubicity, boxicity, and vertex cover, Discrete Math. 309 (2009) 2488-2496.
[3] M. B. Cozzens, Higher and multidimensional analogues of interval graphs, Ph.D. thesis, Rutgers University, New Brunswick, NJ, 1981.

[4] M. B. Cozzens and F. S. Roberts, Computing the boxicity of a graph by covering its complement by cointerval graphs, Discrete Appl. Math. 6 (1983) 217-228.

[5] A. Gyárfás, T. Jensen, and M. Stiebitz, On graphs with strongly independent color-classes, J. Graph Theory 46 (2004) 1-14.

[6] C. G. Lekkerkerker and J. C. Boland, Representation of a finite graph by a set of intervals on the real line, Fund. Math. 51 (1962) 45-64.

[7] L. Lovász, Combinatorial Problems and Exercises (2nd edition), AMS Chelsea Publishing, 2007.

[8] J. Mycielski, On graph coloring (in French), Colloq. Math. 3 (1955) 161-162.

[9] R. J. Opsut, and F. S. Roberts, On the fleet maintenance, mobile radio frequency, task assignment, and traffic phasing problems, in: G. Chartrand et al., eds., The Theory and Applications of Graphs, Wiley, New York (1981) 479-492.

[10] F. S. Roberts, On the boxicity and cubicity of a graph, in: Recent Progress in Combinatorics, Academic Press, New York (1969) 301-310.

[11] F. S. Roberts. Food webs, competition graphs, and the boxicity of ecological phase space, Theory and Applications of Graphs, Lecture Notes in Mathematics 642 (1978), Y. Alavi and D. Lick, eds., Springer-Verlag, 447-490.

[12] E. R. Scheinerman, Intersection classes and multiple intersection parameters, Ph.D. thesis, Princeton University, 1984.

[13] C. Thomassen, Interval representations of planer graphs, J. Combin. Theory Ser. B 40 (1986) 9-20.