Wavelets Defined Over Non Nested Tetrahedral Grids: A Theoretical Approach

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Abstract

The main contribution of this paper is the definition of wavelets over non nested tetrahedral grids, allowing the representation of functions defined on an irregular tetrahedralization. In this way, it is possible to represent different attributes of a 3D object such as its color, brightness, density, etc. This representation consists of a set of coefficients corresponding to a coarse resolution followed by a set of detail coefficients that measures the error between two successive approximations. In this work the analysis matrix that allows going from a fine to a coarser resolution and the synthesis matrix needed for going from a coarse resolution to a finer one, are presented. All this is within the framework of non nested tetrahedral grids.

Keywords: non nested multiresolution analysis, irregular tetrahedralizations, nested multiresolution analysis, wavelets.

1 Introduction

Wavelets are basis functions which allow to decompose functions in different levels of detail. They enable any function to be described in terms of a coarse approximation plus detail coefficients. Wavelets were first used in processing or approximation of signals but afterwards they began to be used in other areas like cartography, computer vision, computer graphics, statistics, etc. The first wavelets to appear were defined in \( \mathbb{R} \) or in \( \mathbb{R}^n \) and they were obtained as contractions or dilations of a fixed function called "mother wavelet". Several different methods were used for constructing them until in 1989, Stéphane Mallat [1] introduced the idea of multiresolution analysis which gives a natural unified framework and a useful tool for their construction. Basically, a multiresolution analysis enables to treat a function in a hierarchical way.

The traditional works on wavelets were mainly done by Daubechies [2], Mallat [1] and Chui [3]. Examples of first generation wavelets are Haar wavelets, Shannon wavelet, Daubechies wavelets and B-wavelets linked to B-splines. The essential characteristic of these wavelets called first generation wavelets is that they can only represent functions whose data are equally spaced. Although signals and statistic data are so distributed, several other data are irregularly distributed and this motivated a first extension of the wavelet theory.

At the beginning of the nineties the second generation wavelets appeared. Their name is due to Wim Sweldens and their main feature is that they are not translations and dilations of a fixed function “mother wavelet”. Examples of this kind of wavelets are wavelets over an interval, wavelets over bounded domains in \( \mathbb{R}^n \), wavelets over curves and surfaces and wavelets adapted to irregular sampling. The most general techniques for building second generation wavelets are the lifting scheme and the surfaces subdivision. The first of them was developed by Wim Sweldens [4] in 1996 but the results he obtained with this technique are similar to those obtained before by
D. Donoho [5, 6, 7] and by M. Lounsbry, T. De Rose and Warren [8]. The surfaces subdivision is a technique developed by Lounsbry [8] who extended the wavelet theory to arbitrary topological surfaces. He also found out the relation between multiresolution analysis and subdivision schemes. Both techniques together, lifting scheme and surfaces subdivision, have been useful and versatile tools for building second generation wavelets. In this way, generalized orthogonal wavelets [9] and spherical wavelets [10] were built. Wavelet theory is based on the knowledge of a sequence of functional spaces where data are successively approximated. But this sequence has to be nested. In all the works aforementioned it was possible to establish such sequence and then the wavelet theory was successful applied. A really flexible construction of second generation wavelets was presented in [11] where using a non-nested framework [12], A. Gerussi presents a method for representing piecewise constant functions over a surface but relaxing the nesting property.

The need to address problems in three dimensional spaces motivated researchers to study subdivision schemes in those spaces because a 3D object is in general represented by a tetrahedral mesh on which its attributes are defined. Several refinement methods in n dimensional spaces are discussed in [13].

The first approach for defining a multiresolution analysis for piecewise constant functions defined over a tetrahedral mesh is to define wavelets over a tetrahedron which is the basic building block of the mesh. We define these wavelets in [14, 15] and we consider the data regularly distributed. In this work the volume’s tetrahedralization consists of a regular tetrahedral mesh subdivided using the Bey’s subdivision scheme [16]. This method allows to define a nested sequence of subspaces such as those required in a multiresolution analysis. Afterwards, in [17] we extend this result to a semiregular tetrahedral mesh.

However, data points are often irregularly distributed. In this case, irregular tetrahedral meshes are the right meshes to deal with them. But such meshes generally do not generate a sequence of nested spaces and in consequence the non nested framework should be applied.

The main contribution of this paper is the multiresolution representation of piecewise constant functions defined over non nested tetrahedral meshes. For doing this, we first apply the half collapsed technique [18, 19] to simplify tetrahedral irregular meshes and then we find the analysis matrix which is needed for going from a fine resolution to a coarser one and the synthesis matrix which makes possible to go from a fine to a coarser resolution.

The paper is organized as follows. In Section 2 we describe the general framework for working with non nested spaces. In Section 3, full edge collapse and half edge collapse are briefly described and two operators, orthogonal and average operators are presented. In Section 4 we demonstrate two theorems which allow to find the analysis matrix in order to go from a fine to a coarse resolution and the synthesis matrix needed to go from a coarse resolution to a finer one. We cope with two cases: the operator P is surjective (subsection 4.1) and the operator P is not surjective (subsection 4.2).

In Section 5 we give an example where the theoretical results developed in the previous sections are applied to a simple mesh. Next, in Section 6 the conclusions are detailed. Finally, in in Section 7 we present the matrices used in the example of Section 5.

## 2 Multiresolution analysis with non nested approximating spaces

In this section, we present the general framework for constructing multiresolution analysis schemes based on a non nested sequence of approximating spaces. This extension of the classic wavelet theory was presented in [12].

Let \( \Omega \) be a measurable domain in \( \mathbb{R}^3 \) and \( V' \) a sequence of finite dimension subspaces of \( L^2(\Omega) \). These spaces, called approximating spaces do not have to be nested but they have to fulfill the condition \( \dim(V') \leq \dim(V' + 1) \). These sets are the domain of the linear operators \( P^k \), called approximating operators.

Now let \( f = f_N \) be a function defined in the finest space \( V^N \). In the classic multiresolution analysis, the spaces are nested and the link between \( V^N \) and \( V^{N-1} \) is made by taking a complementary space \( W^{N-1} \) of \( V^{N-1} \) in \( V^N \), that is:

\[
V^N = V^{N-1} \oplus W^{N-1}
\]

This allows to write:

\[
f_N = f_{N-1} + g_{N-1},
\]

being \( f_{N-1} \) a coarse approximation of \( f_N \) in \( V^{N-1} \) and \( g_{N-1} \) the detail needed to recover the original function from its approximation. By repeating this decomposition, one obtains:

\[
f_N = f_0 + g_0 + g_1 + \ldots + g_{N-1}
\]

which corresponds to the space decomposition:

\[
V^N = V^0 \oplus (\oplus_{i=0}^{N-1} W^i).
\]

Notice that in this case \( f_i = P_{V'}(f_{i+1}) \), where \( P_{V'} \) is a projector on \( V' \) in the \( W^i \) direction, and usually it is the orthogonal projector.

Returning to the general framework we describe, we also have linear approximating operators \( P:V'^{i+1} \rightarrow V^i \). Nevertheless the approximating spaces are not nested so it is necessary to capture the details in a different way.
Let \( W^i \) be the kernel of \( P^i \) and \( \tilde{V}^i \) a complementary space of \( W^i \) in \( V^{i+1} \), so that:

\[
V^{i+1} = \tilde{V}^i \oplus W^i.
\]

In [11], Chapter 2, Lemma II.1, it is proven that the restriction of \( P^i \) to \( V^i \) is a bijective operator having the same range as \( P^i \), so the operator \( P^i \) restricted to \( \tilde{V}^i \) has inverse which we note \( \text{Inv}(P_i|_{\tilde{V}^i}) \). In this way, if \( f_i = P^i(f_{i+1}) \) and \( g_i = Q^i(f_{i+1}) \), where \( Q^i \) is the projector on \( W^i \) parallel to \( V^i \), the reconstruction formula given by:

\[
f_{i+1} = \tilde{f}_i + g_i = \text{Inv}(P^i|_{\tilde{V}^i})(f_i) + g_i, \tag{5}
\]

allows to reconstruct \( f_{i+1} \) using its approximation \( \tilde{f}_i \) and the detail coefficient \( g_i \) in \( W^i \). To reconstruct the function it is necessary to repeat this decomposition on each level.

Let us introduce the following notation.

**Notation.**

For each \( k \in \mathbb{N} \), it is defined:

i) the detail spaces \( W^k \cong \text{Ker}(P^k) \);

ii) the subspaces \( \tilde{V}^k \) for which: \( V^{k+1} = \tilde{V}^k \oplus W^k \);

iii) the operators \( Q^k : V^{k+1} \to W^k \) which are the projectors on \( W^k \) parallel to \( \tilde{V}^k \);

iv) the operators \( R^k : V^{k+1} \to \tilde{V}^k \), which are the projectors on \( \tilde{V}^k \) parallel to \( W^k \). The items ii), iii) and iv) allow to write: \( f = R^k(f) + Q^k(f) \);

v) the synthesis operators \( S^k = \text{Inv}(P^k|_{\tilde{V}^k}) \).

**Definition:** A multiresolution analysis is said to be **semiorthogonal** if for each \( k \), the spaces \( W^k \) are orthogonal to the spaces \( \tilde{V}^k \), that is to say:

\[
W^k \perp \tilde{V}^k, \text{ for all } k \in \mathbb{N}.
\]

### 2.1 Bases, Analysis and Synthesis Matrices

We give now the bases and the notation for the different spaces we defined above.

**Notation.** For all \( k \in \mathbb{N} \), it be noted:

i) \( n_k \) the dimension of \( V^k \) and \( \phi^k = \{ (\phi^k_i), i = 1, \ldots, n_k \} \), a basis for \( V^k \);

ii) \( m_k \) the dimension of \( \tilde{V}^k \) and \( \tilde{\phi}^k = \{ (\tilde{\phi}^k_i), i = 1, \ldots, m_k \} \), a basis for \( \tilde{V}^k \);

iii) \( r_k \) the dimension of \( W^k \) and \( \psi^k = \{ (\psi^k_i), i = 1, \ldots, r_k \} \), a basis for \( W^k \);

iv) \( (a^k) \), \( (\tilde{a}^k) \) and \( (b^k) \) the coefficients of the functions \( f_k \in V^k \), \( \tilde{f}_k \in \tilde{V}^k \) and \( g_k \in W^k \) respectively in the bases indicated in the previous items. The coefficients \( (b^k) \) are also called wavelet coefficients of the function \( f_k \).

As in the classical wavelet theory, in order to describe a step in the analysis and a step in the synthesis it is necessary to obtain the analysis and synthesis matrices respectively.

**Analysis or decomposition**

In this case, the \( n_{k+1} \) coefficients \( (a^{k+1}) \) of a function \( f_{k+1} \in V^{k+1} \) relates to the basis \( \phi^{k+1} \) are known and we look for the \( n_k \) coefficients of \( P^k(f_{k+1}) \) relative to the basis \( \phi^k \) and the \( m_k \) coefficients of \( Q^k(f_{k+1}) \) relative to the basis \( \psi^k \). The matrix that allows to determine those coefficients is called **analysis matrix** and we note it \( A^k \). Then we can write:

\[
\begin{bmatrix}
  a^k \\
  b^k
\end{bmatrix}
= A^k \begin{bmatrix}
  a^{k+1} \\
  b^{k+1}
\end{bmatrix} = \begin{bmatrix}
  p^k_{\phi^{k+1}:\phi^k} \\
  q^k_{\phi^{k+1}:\psi^k}
\end{bmatrix} \begin{bmatrix}
  a^{k+1} \\
  b^{k+1}
\end{bmatrix}, \tag{6}
\]

where:

i) \( A^k \) is \( (n_k + r_k) \times n_{k+1} \);

ii) \( p^k_{\phi^{k+1}:\phi^k} \) is the \( P^k \) operator’s matrix in the bases \( \phi^{k+1} \) and \( \phi^k \);

iii) \( q^k_{\phi^{k+1}:\psi^k} \) is the \( Q^k \) operator’s matrix in the bases \( \phi^{k+1} \) and \( \psi^k \).

**Synthesis or reconstruction**

The inverse operation \( (f_k, g_k) \to S^k(f_k) + g_k \) is made by means of the **synthesis matrix** \( B^k \) given by the equation:

\[
\begin{bmatrix}
  a^{k+1} \\
  b^{k+1}
\end{bmatrix} = B^k \begin{bmatrix}
  a^k \\
  b^k
\end{bmatrix} = \begin{bmatrix}
  \Phi^k & \Psi^k
\end{bmatrix} \begin{bmatrix}
  a^k \\
  b^k
\end{bmatrix}, \tag{7}
\]

where \( \Psi^k \) is the matrix whose columns are the coefficients of the functions of the basis \( \psi^k \) written in the basis \( \phi^{k+1} \) and \( \Phi^k \) is the matrix that represents a surjective transformation from \( V^k \) in \( \tilde{V}^k \).

The link between the analysis and synthesis matrices is given by:

\[
B^k A^k = \text{Id}_{n_{k+1}},
\]

being \( \text{Id}_{n_{k+1}} \) the identity matrix of dimension \( n_{k+1} \). For calculating these matrices it is necessary to know if the operator \( P^k \) is surjective or not.

When the operator \( P^k \) is surjective, \( \text{dim}(\text{Im}P^k) = \text{dim}(V^k) = n_k \); on the other hand, \( \text{dim}(\text{Ker}P^k) + \text{dim}(\text{Im}P^k) = \text{dim}(V^{k+1}) \) so that \( r^k + n^k = n^{k+1} \), that is to say, the analysis and synthesis matrices are square and they are the inverse of each other.
3 Approximation operators and spaces

After giving the general framework for working with non nested spaces, we present two examples of approximation operators and their respective approximation spaces. This will be done in subsections 3.2 and 3.3. But in order to define these approximation spaces, we need to have techniques for simplifying irregular tetrahedral meshes [19, 20]. In the subsection 3.1 we describe two of these techniques: full edge collapse and half edge collapse and present the approximation spaces we will work with. In subsection 3.2 and 3.3 we present the orthogonal operator and the average operator respectively. Both operators are used to simplify irregular meshes and constitute an essential stage to define wavelets over non nested tetrahedral grids.

3.1 Edge collapsing

A half edge collapse is a simplifying technique for tetrahedral meshes that consists of shrinking a tetrahedron’s edge \( e \) whose vertices are \( v \) and \( w \) into one of its extremes, for example \( w \). The reverse modification of a half-edge collapse is a vertex split, which expands a vertex \( w \) into an edge \( e \) by inserting the other extreme vertex \( v \) of \( e \), Figure 1.

A full edge collapse is a simplifying technique for tetrahedral irregular meshes that consists of shrinking a tetrahedron’s edge \( e \), whose vertices are \( v \) and \( w \), into a new vertex, (often the midpoint of \( e \)), Figure 2. We choose the half-edge collapse technique because it produces fewer updates in comparison with those generated by a full-edge collapse.

Now we will describe the approximation spaces we want to work with and afterwards the operators. Let \( \tau^V \) be an arbitrary tetrahedralization of a domain \( \Omega \subset \mathbb{R}^3 \). Let \( \tau^i, i = 0, \ldots, N - 1 \) be the tetrahedralizations obtained by successively applying the half-edge collapse method to the tetrahedralization \( \tau^V \). We will use the following notation.

**Notation**

1. \( T_j^i, \ i = 0, \ldots N, \ j = 1, \ldots J \): the tetrahedron \( T_j \) of the tetrahedralization \( \tau^i \).
2. \( \chi_{T_j^i}, \ j = 1, \ldots J \) the characteristic function of \( T_j^i \), \( i = 0, \ldots N \).
3. \( C^i \) the approximation space of constant functions over each tetrahedron of the \( \tau^i \) tetrahedralization.

The functions \( \chi_{T_j^i}, \ j = 1, \ldots J \) constitute an orthogonal basis for \( C^i \).

3.2 Orthogonal Operator

In this subsection we present the orthogonal operator.

**Definition:** For all \( k = 0, \ldots, N - 1 \), we define the approximation operators:

\[
P^k: C^{k+1} \rightarrow C^k \quad f^{k+1} \mapsto P^k_{C^k}(f^{k+1}),
\]

where \( P^k_{C^k} \) is the orthogonal projector over \( C^k \).

**Observations.**

1) The matrix of the orthogonal operator is given by:

\[
P^k_{\chi^{k+1}; \chi^k} = G^{-1}_{\chi^k} G_{\chi^k; \chi^{k+1}},
\]

where:

1. \( G_{\chi^k; \chi^{k+1}} \) is the matrix whose entries are the scalar products relatives to the bases \( \{ \chi^k \} \) y \( \{ \chi^{k+1} \} \); that is:

\[
G_{\chi^k; \chi^{k+1}} = \langle \chi_i^k, \chi_j^{k+1} \rangle_{\tau^i} = \int_{T_i^k \cap T_j^{k+1}} dx = \text{volume}(T_i^k \cap T_j^{k+1}).
\]

2. \( G_{\chi^k; \chi^k} = G_{\chi^k; \chi^k} \) is the matrix whose entries are the scalar products relative to the orthogonal basis \( \{ \chi^k \} \); that is:

\[
G_{\chi^k; \chi^k} = \left\{ \begin{array}{ll}
0, & \text{if } i \neq j \\
\text{volume}(T_i^k), & \text{if } i = j
\end{array} \right. .
\]

2) The above defined operator \( P^k \) guarantees that \( f_k \) is the best approximation of \( f_{k+1} \) in the mean squares sense.
3) The \((j_i, j_h)\) entry of the matrix (8) is:

\[
P_{j_i, j_h} = \left( \frac{\text{volume}(T^k_{j_i} \cap T^k_{j_h})}{\text{volume}(T^k_{j_i})} \right).
\]

From now on and just to simplify the notation, we will write:

i) \(P\) instead of \(P^k\) and \(Q\) instead of \(Q^k\),

ii) \(\Phi\) instead of \(\Phi^k\) and \(\Psi\) instead of \(\Phi^k\), so that the analysis and synthesis matrices will be:

\[
A = \begin{bmatrix}
P \\
Q
\end{bmatrix}
\]

and

\[
B = \begin{bmatrix}
\Phi & \Psi
\end{bmatrix},
\]

respectively.

3.3 Average Operator

In this section we present the average operator that we note \(P^a\). In first place we detail how this linear operator is defined and then we obtain its associated matrix. Applying half edge collapse to a set of \(n\) tetrahedra covering a domain, another set of \(m\) tetrahedra, \(m < n\), covering the same domain is obtained. If there exist data defined on the finest mesh, we describe next how to compute the data on the new simplified mesh. For each tetrahedron \(T^i_{j_i}\), \(k = 1, \ldots, K\) belonging to the \(\tau^i\) tetrahedralization, we look for all the tetrahedra \(T^{j_i+1}_k\), \(j = 1, \ldots, J\) belonging to the \(\tau^{j_i+1}\) tetrahedralization such that \(T^i_k \cap T^{j_i+1}_k \neq \emptyset\). The value we assign to the function on the \(T^i_k\) tetrahedron is an average of the functions’ values over all the tetrahedra \(T^{j_i+1}_k\) which intersect \(T^i_k\). Now let’s see which is the matrix associated to the average operator. Let \(u_j(k)\) be:

\[
u_j(k) = \begin{cases} 
1, & T^i_{j_i} \cap T^k_k \neq \emptyset \\
0, & T^i_{j_i} \cap T^k_k = \emptyset
\end{cases}
\]

that is to say \(u_j(k)\) is zero if and only if the tetrahedron \(T^{j_i+1}_k\) belonging to the \((i + 1)\)-tetrahedralization and the tetrahedron \(T^i_k\) belonging to the \(i\)-tetrahedralization, have no intersection. Besides, for each \(k = 1, \ldots, K\) we will note \(l_k\):

\(l_k\): number of \(T^{j_i+1}_k\) such that \(T^i_k \cap T^{j_i+1}_k \neq \emptyset\).

Taking into account the above definitions, the average operator \(P^a\):

\[
P^a : C^{i+1} \rightarrow C^i, \quad f^{i+1}_i \mapsto P^a(f^{i+1}_i),
\]

has the following matricial representation:

\[
\begin{pmatrix}
u_{1(1)} / l_1 & \cdots & \nu_{1(l)} / l_1 \\
u_{2(1)} / l_2 & \cdots & \nu_{2(l)} / l_2 \\
\vdots & \ddots & \vdots \\
u_{k(1)} / l_k & \cdots & \nu_{k(l)} / l_k
\end{pmatrix}.
\]

4 Analysis and synthesis matrices for arbitrary frames and operators \(P\)

Depending on the tetrahedralizations, the orthogonal operator and the average operator may be surjective or not. In the this section we present two theorems. Both of them allow to obtain the analysis matrix when the synthesis matrix is known and viceversa. The first one, in subsection 4.1, is for surjective operators \(P\) and arbitrary multiresolution analysis and it generalizes the result obtained by Gerussi in [11], chapter V, Lemma VI, for semi-orthogonal multiresolution analysis. The second one, in subsection 4.2, is for non surjective operators and arbitrary multiresolution analysis.

4.1 Surjective operator \(P\)

Let us suppose the operator \(P\) is surjective and the multiresolution analysis in not necessarily semiorthogonal. The following theorem provides a method for calculating the matrices \(P\) and \(Q\) when \(\Phi\) and \(\Psi\) are known and viceversa. The matrix \(G_{2^k, 2^k+1}\), already defined in (9), will be note by \(H\). In what follows, we will note \(P\) the approximation operator as well as the matrix which represents that operator.

**Theorem:** Let \(P : V^{k+1} \rightarrow V^k\) be a surjective operator and let \(P^\ast\) be the transpose matrix of \(P\). Then:

1) \(PH^{-1}P^\ast = \left( (\Phi^H \Phi)^{-1} \right) \left( (\Psi^H \Psi)^{-1} \right) \left( (\Phi^H \Psi)^{-1} \right) \left( (\Phi^H \Phi)^{-1} \right)
\)

2) \(PH^{-1}Q^\ast = - \left( (\Phi^H \Phi)^{-1} \right) \left( (\Phi^H \Psi)^{-1} \right) \left( (\Phi^H \Phi)^{-1} \right) \left( (\Phi^H \Psi)^{-1} \right)
\)

3) \(QH^{-1}P^\ast = \left( (\Phi^H \Psi)^{-1} \right) \left( (\Phi^H \Phi)^{-1} \right) \left( (\Phi^H \Psi)^{-1} \right)
\)

Moreover:

4) \(P = \left( (\Phi^H \Phi)^{-1} \right) \left( (\Phi^H \Psi)^{-1} \right) \left( (\Phi^H \Phi)^{-1} \right) \left( (\Phi^H \Psi)^{-1} \right) \left( (\Phi^H \Phi)^{-1} \right)|H|
\)

5) \(Q = \left( (\Phi^H \Psi)^{-1} \right) \left( (\Phi^H \Phi)^{-1} \right) \left( (\Phi^H \Psi)^{-1} \right) \left( (\Phi^H \Phi)^{-1} \right) \left( (\Phi^H \Psi)^{-1} \right)|H|
\)

6) \(\Phi = H^{-1}P R + H^{-1}Q S
\)

7) \(\Psi = H^{-1}P T + H^{-1}Q U
\)

being:

\[
R = \left( (PH^{-1}P)^{-1} \right) \left( (QH^{-1}Q)^{-1} \right) \left( (QH^{-1}P)^{-1} \right)
\]

\[
S = - \left( (PH^{-1}Q)^{-1} \right) \left( (QH^{-1}Q)^{-1} \right) \left( (QH^{-1}P)^{-1} \right)
\]

\[
T = - \left( (QH^{-1}P)^{-1} \right) \left( (PH^{-1}P)^{-1} \right) \left( (QH^{-1}Q)^{-1} \right)
\]

\[
U = \left( (QH^{-1}Q)^{-1} \right) \left( (PH^{-1}P)^{-1} \right) \left( (QH^{-1}Q)^{-1} \right)
\]

Furthermore:

8) \(R = \Phi^H \Phi, \quad S = \Phi^H \Psi, \quad T = \Psi^H \Phi, \quad U = \Psi^H \Psi
\)

**Proof:** By hypothesis,

\[
\begin{bmatrix}
P \\
Q
\end{bmatrix} = \begin{bmatrix}
\Phi & \Psi
\end{bmatrix}^{-1}, \quad (9)
\]
then:
\[
\begin{bmatrix}
P \\
Q
\end{bmatrix} H^{-1} \begin{bmatrix}
P \\
Q
\end{bmatrix}
= \begin{bmatrix}
\Phi \\
\Psi
\end{bmatrix}^{-1} H^{-1} \begin{bmatrix}
P \\
Q
\end{bmatrix}
= \begin{bmatrix}
\Phi \\
\Psi
\end{bmatrix}^{-1} H^{-1} \begin{bmatrix}
P \\
Q
\end{bmatrix}
= \begin{bmatrix}
\Phi \\
\Psi
\end{bmatrix}^{-1} H^{-1} (\begin{bmatrix}
\Phi \\
\Psi
\end{bmatrix})^{-1}.
\]

that is to say:
\[
\begin{bmatrix}
P \\
Q
\end{bmatrix} H^{-1} \begin{bmatrix}
P \\
Q
\end{bmatrix} = \begin{bmatrix}
\Phi^* H \Phi & \Phi^* H \Psi \\
\Psi^* H \Phi & \Psi^* H \Psi
\end{bmatrix}^{-1}.
\]

Let:
\[
\begin{bmatrix}
A \\
C
\end{bmatrix}
\begin{bmatrix}
B \\
D
\end{bmatrix}
\begin{bmatrix}
\Phi^* H \Phi & \Phi^* H \Psi \\
\Psi^* H \Phi & \Psi^* H \Psi
\end{bmatrix} = \text{Id}.
\]

Then, the following equations have to verify:
\[
\begin{align*}
A \Phi^* H \Phi + B \Psi^* H \Phi &= \text{Id}, \\
A \Phi^* H \Psi + B \Psi^* H \Psi &= 0, \\
C \Phi^* H \Phi + D \Psi^* H \Phi &= 0, \\
C \Phi^* H \Psi + D \Psi^* H \Psi &= \text{Id},
\end{align*}
\]

from which it can be concluded:
\[
\begin{align*}
A &= \left[ (\Phi^* H \Phi) - (\Phi^* H \Psi)(\Psi^* H \Psi)^{-1}(\Psi^* H \Phi) \right]^{-1}, \\
B &= -\left[ (\Phi^* H \Phi) - (\Phi^* H \Psi)(\Psi^* H \Psi)^{-1}(\Psi^* H \Phi) \right]^{-1}(\Psi^* H \Phi)(\Phi^* H \Phi)^{-1}, \\
C &= -\left[ (\Psi^* H \Psi) - (\Psi^* H \Phi)(\Phi^* H \Phi)^{-1}(\Phi^* H \Psi) \right]^{-1}(\Psi^* H \Phi)(\Phi^* H \Phi)^{-1}, \\
D &= \left[ (\Psi^* H \Psi) - (\Psi^* H \Phi)(\Phi^* H \Phi)^{-1}(\Phi^* H \Psi) \right]^{-1}(\Phi^* H \Psi)^{-1}.
\end{align*}
\]

Besides, from equation (12) results:
\[
\begin{bmatrix}
A \\
C
\end{bmatrix}
\begin{bmatrix}
B \\
D
\end{bmatrix}
\begin{bmatrix}
\Phi^* H \\
\Psi^* H
\end{bmatrix} = \begin{bmatrix}
\Phi^* H \\
\Psi^* H
\end{bmatrix} = \text{Id}
\]

and, as a consequence of equation (9):
\[
P = \Phi^* H + B \Psi^* H \quad \text{and} \quad Q = C \Phi^* H + D \Psi^* H.
\]

Replacing \(A, B, C, D\) the lemma’s items 4) and 5) are proved.

Also from equations (9) and (12) results:
\[
\begin{bmatrix}
PH^{-1} P \\
QH^{-1} Q
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}.
\]

This last identity proves items 1), 2) and 3) of the theorem. In order to prove items 6) and 7) we have to notice that from equation (10), we can write:
\[
\begin{bmatrix}
P \\
Q
\end{bmatrix} H^{-1} \begin{bmatrix}
P \\
Q
\end{bmatrix} = \begin{bmatrix}
\Phi^* H \Phi & \Phi^* H \Psi \\
\Psi^* H \Phi & \Psi^* H \Psi
\end{bmatrix}^{-1}.
\]

Let:
\[
\begin{bmatrix}
R \\
T
\end{bmatrix}
\begin{bmatrix}
S \\
U
\end{bmatrix}
\begin{bmatrix}
PH^{-1} P \\
QH^{-1} Q
\end{bmatrix} = \text{Id}.
\]

Then the following equations have to verify:
\[
R PH^{-1} P + SQH^{-1} P = \text{Id}, \\
R PH^{-1} Q + SQH^{-1} Q = 0,
\]

from which it can be deduced:
\[
R = \left[ (PH^{-1} P) - (PH^{-1} Q)(QH^{-1} Q)(QH^{-1} P) \right]^{-1}, \\
U = \left[ (QH^{-1} Q) - (QH^{-1} P)(PH^{-1} P)^{-1}(PH^{-1} Q) \right]^{-1}, \\
S = \left[ (PH^{-1} P) - (PH^{-1} Q)(QH^{-1} Q)QH^{-1} P \right]^{-1}, \\
T = \left[ (QH^{-1} Q) - (QH^{-1} P)(PH^{-1} P)^{-1}(PH^{-1} Q) \right]^{-1}.
\]

Besides, from equation (17) it results:
\[
\begin{bmatrix}
R \\
T
\end{bmatrix}
\begin{bmatrix}
S \\
U
\end{bmatrix}
\begin{bmatrix}
PH^{-1} \\
QH^{-1}
\end{bmatrix} \begin{bmatrix}
P \\
Q
\end{bmatrix} = \text{Id},
\]

and as a consequence of equation (9):
\[
\Phi^* = R PH^{-1} + SQH^{-1} \quad \text{and} \quad \Psi^* = T PH^{-1} + UQH^{-1}.
\]

Replacing \(R, S, U, V\) it can be deduced theorem’s items 6) and 7). Also from equations (15) and (17) it results:
\[
\begin{bmatrix}
\Phi^* H \Phi & \Phi^* H \Psi \\
\Psi^* H \Phi & \Psi^* H \Psi
\end{bmatrix} = \begin{bmatrix}
R \\
T
\end{bmatrix}
\begin{bmatrix}
S \\
U
\end{bmatrix}.
\]

This last equality proves 8). □

**Corollary.** If the multiresolution analysis is semiorthogonal, then \(P, Q\), \(\Phi, \Psi\) are given by:
\[
i) \quad P = (\Phi^* H \Phi)^{-1} \Phi^* H.
\]
Proof.
If the multiresolution analysis is semiorthogonal, then
\[ \Phi^* H \Psi = \Psi^* H \Phi = 0 \]
and replacing this condition in the above theorem, it results: i) and ii) of the corollary. Besides, if the multiresolution analysis is semiorthogonal,

a) \( S = T = 0 \), (by 8) of Theorem 4.1),
b) \( PH^* P^* = (\Phi^* H \Phi)^{-1} \), (by 1) of Theorem 4.1),
c) \( QH^* Q^* = (\Psi^* H \Psi)^{-1} \), (by 3) of Theorem 4.1).

Then replacing these conditions in 6) and 7) of Theorem 4.1, result:

a) \[ \Phi = H^{-1} P^* R^* = H^{-1} P^* (\Phi^* H \Phi)^{-1} \]
b) \[ \Psi = H^{-1} Q^* U^* = H^{-1} Q^* (\Psi^* H \Psi)^{-1} \]

which are items iii) and iv) of corollary. \( \square \)

The above theorem provides a method for calculating the analysis matrix \( A = \begin{bmatrix} P \\ Q \end{bmatrix} \) when the synthesis matrix \( B = \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} \) is known and vice versa. But for a given surjective operator \( P \), how can we get the analysis matrix \( A = \begin{bmatrix} P \\ Q \end{bmatrix} \)? In what follows we propose two different methods for determining the \( Q \)-operator’s matrix when the surjective operator \( P \) is known and the multiresolution analysis is non semi-orthogonal. The first one is a constructive method and the second one is a matrix method.

4.1.1 Constructive method

This method describes how to obtain the \( Q \)-operator’s matrix when the surjective operator \( P \) is known. It consists of building a space \( \tilde{V}^k \) with the same dimension of \( V^k \), not orthogonal to \( W^k \) and such that \( V^{k+1} = \tilde{V}^k \oplus W^k \). This construction is made as follows.

Let \( T_j^{i+1} \), \( j = 1, ..., J \), be the tetrahedra of the tetrahedralization \( \tau^{i+1} \), \( T_j^k \), \( k = 1, ..., K \), the tetrahedra of the tetrahedralization \( \tau^k \). For each one of these tetrahedralization, we define the following spaces:

\[ V^{i+1} = \{ f : f \text{ is constant over each tetrahedron of } \tau^{i+1} \}, \]

and

\[ V^i = \{ f : f \text{ is constant over each tetrahedron of } \tau^i \} \]

From these definitions it follows that:

\[ V^{i+1} = \text{span} \{ X_{T_j^{i+1}}, j = 1, ..., J \} \] and

\[ V^i = \text{span} \{ X_{T_j^k}, k = 1, ..., K \} \]

where \( \text{span} \{ h_i, i = 1, ..., R \} \) means subspace spanned by the \( h_i \) functions.

For a surjective approximation operator \( P^i : V^{i+1} \rightarrow V^i \), let \( W^i \) be its kernel and \( \{ \psi_l^i \}, l = 1, ... , L \) a basis for \( W^i \) written in the basis of \( V^{i+1} \). It’s clear that \( L + K = J \). We now choose a subspace \( \tilde{V}^i \subset V^{i+1} \) such that:

i) \( \tilde{V}^i \) has the same dimension of \( V^i \),

ii) \( \tilde{V}^i \) is not orthogonal to \( W^i \),

\[ \tilde{V}^{i+1} = \tilde{V}^i \oplus W^i \]

For constructing this space \( \tilde{V}^i \) it suffices to look for a basis \( \tilde{\phi}_l^i \) for \( \tilde{V}^i \), \( k = 1, ..., K \) not orthogonal to the \( W^i \)’s basis. Let \( M \) be the matrix whose \( L \) first rows are the vectors corresponding to the basis \( \{ \psi_l^i \}, l = 1, ..., L \) of \( W^i \) and the \( K \) following rows are the vectors of the basis \( \tilde{\phi}_k \), \( k = 1, ..., K \) of \( \tilde{V}^i \), all of them written in the basis of \( V^{i+1} \). It’s clear that \( L + K = J \). Then, \( M \) is the square matrix:
\[ \psi = H^{-1}Q^* - H^{-1}P^*(PH^{-1}P^*)^{-1}(PH^{-1}Q^*) \]  

This is a linear system whose unknowns are the entries of \( Q \). If the determinant of this system is zero, we need another condition for determining \( Q \). Let \( \tilde{Q} \) be the matrix of the operator \( Q \) in the basis of \( V^{k+1} \), i.e \( \tilde{Q} = Q_{k+1}g_{k+1}^t ; \tilde{Q} \) is square and it must verify \( \tilde{Q} \tilde{Q} = \tilde{Q} \) in order to be a matrix corresponding to a projector. If some degrees of freedom are still left, \( Q \) can be chosen so that it has minimum norm.

### 4.2 Analysis and synthesis matrices for an arbitrary frame

In this section, we state a theorem that allows to find \( P \) and \( Q \) when \( \Phi \) and \( \Psi \) are known and vice versa for non surjective operators and non semiorthogonal multiresolution analysis. We previously introduce the following notation.

**Notation.**

- (i) \( \mathcal{C}(A) \): linear subspace spanned by the columns of the matrix \( A \).
- (ii) \( A_{L}^{-1} \): an arbitrary left inverse of the matrix \( A \).
- (iii) \( B_{R}^{-1} \): an arbitrary right inverse of the matrix \( B \).
- (iv) \( r(X) \): rank of the matrix \( X \).

**Theorem:** Let \( P \) be an approximation operator, \( A = [P \quad Q] \) and \( B = [\Phi \quad \Psi] \) the analysis and synthesis matrices, respectively. Then:

1. \( P = \Phi^r(\Phi \Phi^r + \Psi \Psi^r)^{-1} \)
2. \( Q = \Psi^r(\Phi \Phi^r + \Psi \Psi^r)^{-1} \)

Moreover:

3. \( \Phi = (P^*P + Q^*Q)^{-1}P^* \)
4. \( \Psi = (P^*P + Q^*Q)^{-1}Q^* \)

**Proof**

Given the synthesis matrix \( B \) we want to solve the system:

\[
[a_{k+1}^1] = B \begin{bmatrix} a^k_1 \\ b^k_1 \end{bmatrix} = [\Phi \quad \Psi] \begin{bmatrix} a^k_1 \\ b^k_1 \end{bmatrix}, \quad (21)
\]

being \( \begin{bmatrix} a^k_1 \\ b^k_1 \end{bmatrix} \) the unknown. The synthesis matrix \( B = [\Phi \quad \Psi] \) is of full file rank; then it has right inverse, the analysis matrix \( A \). All its right inverses are given by \( B_{R}^{-1} = VB(VBV^*)^{-1} \), being \( V \) an arbitrary matrix such that \( r(VBV^*) = r(B) \). We notice that whatever right inverse \( B_{R}^{-1} \) of \( B \) gives a solution:

\[
\begin{bmatrix} a^k_1 \\ b^k_1 \end{bmatrix} = B_{R}^{-1}a_{k+1}
\]

of the system (21). Really, \( BB_{R}^{-1}a_{k+1} = a_{k+1}, \forall a_{k+1} \).

For the implementation, we can choose \( A \) as the following \( B^r \)’s right inverse:

\[
\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} \Phi^r \\ \Psi^r \end{bmatrix} \begin{bmatrix} [\Phi] & [\Psi] \end{bmatrix}^{-1} = \begin{bmatrix} \Phi^r \\ \Psi^r \end{bmatrix} \begin{bmatrix} \Phi^r \\ \Psi^r \end{bmatrix}^{-1} = \begin{bmatrix} \Phi^r \\ \Psi^r \end{bmatrix} \begin{bmatrix} \Phi \Phi^r & \Psi \Psi^r \end{bmatrix}^{-1},
\]

From the last equation, results:

1. \( P = \Phi^r(\Phi \Phi^r + \Psi \Psi^r)^{-1} \)
2. \( Q = \Psi^r(\Phi \Phi^r + \Psi \Psi^r)^{-1} \).

Reciprocally, given the analysis matrix we want to solve the system:

\[
\begin{bmatrix} a^k_1 \\ b^k_1 \end{bmatrix} = A[a_{k+1}^1] = \begin{bmatrix} P_{\phi \psi},,^1,^1 \phi \psi \\ Q_{\phi \psi},,^1,^1 \phi \psi \end{bmatrix} [a_{k+1}^1], \quad (22)
\]

where \( a_{k+1} \) is the unknown and \( \begin{bmatrix} a^k_1 \\ b^k_1 \end{bmatrix} \) are the data.

We notice that the matrix \( A \) is \( (n_k + r_k) \times n_{k+1} \) and its rank is \( n_{k+1} \) then it has left inverse; in fact it has infinite left inverses. All of them are given by \( (A^*KA)^{-1}A^*K \), where \( K \) is an arbitrary matrix such that \( r(A^*KA) = r(A) \). In this case, all the left inverses of \( A \) give the same solution because:

\[
(A_L)^{-1}A[a_{k+1}^1] = (A_L)^{-1} \begin{bmatrix} a^k_1 \\ b^k_1 \end{bmatrix} \Rightarrow a_{k+1} = (A_L)^{-1} \begin{bmatrix} a^k_1 \\ b^k_1 \end{bmatrix} \quad (23)
\]

This \( a_{k+1} \) is a solution of the system (22) because:

\[
\begin{bmatrix} a^k_1 \\ b^k_1 \end{bmatrix} = Ac
\]

and then:

\[
A[a_{k+1}^1] = A(A_L)^{-1}Ac = Ac = \begin{bmatrix} a^k_1 \\ b^k_1 \end{bmatrix} \quad (24)
\]
As all the left inverses give the same solution, we choose the following left inverse:

\[ B = (A_L)^{-1} = (A^*A)^{-1}A^* \]

\[ = \left( \begin{bmatrix} P \\ Q \end{bmatrix}^* \begin{bmatrix} P \\ Q \end{bmatrix} \right)^{-1} \begin{bmatrix} P \\ Q \end{bmatrix}^* \]

\[ = \left( \begin{bmatrix} P^* \\ Q^* \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} \right)^{-1} \begin{bmatrix} P^* \\ Q^* \end{bmatrix} \]

\[ = \left( P^*P + Q^*Q \right)^{-1} \begin{bmatrix} P^* \\ Q^* \end{bmatrix} \]

\[ = \left( \begin{bmatrix} (P^*P + Q^*Q)^{-1} \\ (P^*P + Q^*Q)^{-1}Q^* \end{bmatrix} \right), \]

that is to say:

\[ \begin{align*}
\Phi &= (P^*P + Q^*Q)^{-1}P^* \\
\Psi &= (P^*P + Q^*Q)^{-1}Q^*
\end{align*} \]

5 Example

In this section we present an example where the theoretical results above described are used. We consider a tetrahedral initial grid consisting of sixteen tetrahedra (fine resolution). We applied twice the collapsed half edge and we obtain two meshes at two different resolutions, that is, the first one consisting of fourteen tetrahedra (intermediate resolution) and the second one consisting of ten tetrahedra (coarse resolution).

In order to have a good visualization of the three grids, in Figure 3 it can be seen in different colours and from top to bottom, a grid with fewer tetrahedra and its three resolutions, fine, intermediate and coarse. For these tetrahedralizations we followed the following steps. All matrices and operators mentioned below are detailed in the Appendix.

1) We determined the following matrices:

i) \( H_{16} \): matrix whose entries are the volumes of the tetrahedra belonging to the initial tetrahedralization,

ii) \( H_{14} \): matrix whose entries are the volumes of the tetrahedra of the intermediate tetrahedralization,

iii) \( H_{1416} \): matrix whose entries are the volumes of the intersections between the tetrahedra of the initial tetrahedralization and the tetrahedra of the intermediate tetrahedralization,

iv) \( H_{1014} \): matrix whose entries are the volumes of the intersections between the tetrahedra of the intermediate tetrahedralization and the tetrahedra of the coarsest tetrahedralization.

The matrices \( H_{16} \) and \( H_{14} \) are needed for applying the results given in Theorem 4.2 while the matrices \( H_{1416} \) and \( H_{1014} \) are necessary for defining the average operators.

2) We constructed average operators just as described in Subsection 3.3. We called them:

i) \( P_{1416} \): average operator for going from the initial tetrahedralization to the intermediate one; it is obtained using the matrix \( H_{1416} \).
ii) \( P_{1014} \): average operator for going from the intermediate tetrahedralization to the coarse one; it is obtained using the matrix \( H_{1014} \).

3) For each one of those average operators, we found the respective operators \( Q_{1416} \) and \( Q_{1014} \). Then we got the following analysis and synthesis matrices:

i) \( A_{1416} = \begin{bmatrix} P_{1416} \\ Q_{1416} \end{bmatrix} \), analysis matrix which allows to go from the fine resolution to the intermediate one.

ii) \( A_{1014} = \begin{bmatrix} P_{1014} \\ Q_{1014} \end{bmatrix} \), analysis matrix which allows to go from the intermediate resolution to the coarsest one.

4) The average operator \( P_{1416} \) was no surjective; that’s why it was necessary to applied Theorem 4.2 for finding the synthesis matrix \( B_{1416} = \begin{bmatrix} \Phi_{1416} \\ \Psi_{1416} \end{bmatrix} \).

On the other hand, supposing we know the synthesis matrix \( B_{1416} \), we applied the same Theorem 4.2 to recover the analysis matrix \( A_{1416} \).

5) The average operator \( P_{1014} \) was surjective, so we applied Theorem 4.1 for finding synthesis matrix \( B_{1416} = \begin{bmatrix} \Phi_{1014} \\ \Psi_{1014} \end{bmatrix} \). On the other hand, supposing we know the synthesis matrix \( B_{1014} \), we applied Theorem 4.1 to recover the analysis matrix \( A_{1014} \).

6 Conclusions

Hierarchical decomposition of datasets based on wavelet theory has been growing up in several disciplines (Cartography, Computer Graphics, Finite Elements, Approximation Theory and Computational Geometry). When the data points are regularly distributed over a surface, regular triangular mesh subdivision schemes and classic wavelet theory together have been proved to be powerful tools for giving a hierarchical decomposition of those data points. In this way, spherical wavelets were constructed in [10] and wavelets for surfaces of arbitrary topological type were presented in [8]. When the data points are irregularly distributed over a surface, irregular triangular mesh subdivision schemes and the framework for non nested wavelets were the key for giving a multiresolution decomposition of functions defined over the surface, [11].

Once the problem for data on a surface was solved, the new challenge for researchers was how to define wavelets over a volume (surface and interior). Typically, this volume is modeled by a 3D grid of tetrahedra and the issue is to represent, at different level of details, a function (color, density, brightness) defined over it. In [14] we have constructed a wavelet basis over a tetrahedron which enables the multiresolution representation of functions defined over a tetrahedron and, consequently, the multiresolution representation of functions defined over a tetrahedralized volume. Afterwards, in [15] we give a simpler expression for our wavelets that allows an easier numerical implementation of our methodology. In those cases we suppose that data are regularly distributed on the grid and therefore we have used nested tetrahedral grids generated by a recursive method (Bey’s method) that are suitable to deal with that kind of data. In [17] we extend that result to semi-regular tetrahedral meshes.

In this work we present an innovative solution to the problem of the multiresolution analysis for functions defined over a tetrahedralized volume (surface and interior) considering that data are irregularly distributed on the grid. For giving a multiresolution analysis of those data, we select the half edge collapsed for refining the mesh and we work in the non nested framework for wavelets. In this context we find the analysis matrix that allows to go from a fine resolution to a coarser one and the synthesis matrix which allows to go from a coarser resolution to a finer one. Moreover, in each stage of the multiresolution analysis the approximation operator may result surjective or non surjective, that is why we give a method for calculating the analysis and synthesis matrices for each one of these cases.

7 Appendix

The actual values of the matrices and operators we have mentioned in Section 5 are listed below.
Matrix which entries are the tetrahedra’s volumes of the initial tetrahedralization. \( a = 166.667, b = 83.333. \)

\[
H_{i_0} = \begin{bmatrix}
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b \\
\end{array}
\end{bmatrix}
\]

Matrix which entries are the tetrahedra’s volumes of the intermediate tetrahedralization. \( a = 166.667, b = 83.333. \)

\[
H_{i_1} = \begin{bmatrix}
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b \\
\end{array}
\end{bmatrix}
\]

Matrix which entries are the tetrahedra’s volumes of the coarsest tetrahedralization. \( a = 166.667, b = 83.333, c = 333.333. \)

\[
H_{i_0} = \begin{bmatrix}
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b \\
\end{array}
\end{bmatrix}
\]

Matrix whose entries are the volumes of the intersections among the tetrahedra of the finest and intermediate tetrahedralization. \( a = 166.667, b = 83.333. \)

\[
H_{i_{1416}} = \begin{bmatrix}
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b \\
\end{array}
\end{bmatrix}
\]

Matrix whose entries are the volumes of the intersections among the tetrahedra of the coarsest and intermediate tetrahedralization. \( a = 166.667, b = 83.333. \)

\[
H_{i_{014}} = \begin{bmatrix}
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b \\
\end{array}
\end{bmatrix}
\]
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Competing interests

The authors have declared that no competing interests exist.

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