Long range frequency chirping of Alfvén eigenmodes

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Abstract
A theoretical framework has been developed for an NBI scenario to model the hard non-linear evolution of global Alfvén eigenmodes (GAEs) where the adiabatic motion of phase-space structures (holes and clumps), associated with frequency chirping, occurs in generalised phase-space of slowing down energetic particles. The radial profile of the GAE is expanded using finite elements which allows update of the mode structure as the mode frequency chirps. Constants of motion are introduced to track the dynamics of energetic particles during frequency chirping by implementing proper action-angle variables and canonical transformations which reduce the dynamics essentially to 1D. Consequently, we specify whether the particles are drifting inward/outward as the frequency deviates from the initial MHD eigenfrequency. Using the principle of least action, we have derived the non-linear equation describing the evolution of the radial profile by varying the total Lagrangian of the system with respect to the weights of the finite elements. For the choice of parameters in this work, it is shown that the peak of the radial profile is shifted and also broadens due to frequency chirping. The time rate of frequency change is also calculated using the energy balance and we show that the adiabatic condition remains valid once it is satisfied. This model clearly illustrates the theoretical treatment to study the long range adiabatic frequency sweeping events observed for Alfvén gap modes in real experiments.

Keywords: nonlinear wave-particle interaction, energetic particle instabilities in fusion plasmas, long range frequency chirping/sweeping, adiabatic frequency chirping in tokamaks, frequency chirping of Alfvén eigenmodes

(Some figures may appear in colour only in the online journal)

1. Introduction

Alfvén waves can be unstable as a result of their interaction with energetic particles (EPs) which satisfy the resonance condition during the slowing down process [1]. In magnetic fusion devices e.g. tokamaks, Alfvén eigenmodes (AEs) [2, 3], located outside the shear Alfvén continuum, are subject to weak continuum damping and therefore can be destabilised by supra-thermal particles and fusion products. These modes are potentially dangerous for particle transport. The feedback between unstable waves and enhanced particle diffusion would degrade the EP confinement (see the review article [4] and the references therein and also [5–7]), influence the fuel burnup [8–11] and subject the material of the containment vessel to increased erosion. On the other hand, destabilization of Alfvén waves may have some beneficial effects e.g. diagnostic purposes of the plasma core [12, 13], achieving higher confinement regime due to redistribution of injected ions in DIII-D [14] and energy channeling of fusion born alpha particles [15]. Therefore, an ability to model and control these kinetically driven instabilities is crucial to the design and operation of a fusion power plant.

The wave-particle interaction, which is essentially one dimensional, may result in frequency sweeping behaviors
References [22, 23], which successfully explain the chirping observed in experiments [24, 25], describe the possible formation of phase–space structures, namely holes and clumps, whose motions are associated with frequency sweeping events. In these models, the radial structure of the MHD mode is fixed, a logical assumption as long as the frequency remains close to the initial eigenfrequency. Subsequently, a nonperturbative model [26] was presented to investigate the long range sweeping events [27–29] in the hard nonlinear regime where the structure of the mode is considerably affected by the nonthermal fast particles population. Inclusion of collision operators into this model was accomplished in references [30, 31]. More recently, a 1D theoretical framework was developed in reference [32], which investigates the impact of different EPs orbit topologies (magnetically trapped/passing) on long range frequency chirping of BGK modes. It should be noted that these models consider the adiabatic evolution of phase–space structures and therefore the Vlasov equation can be bounce averaged to find the adiabatic evolution of phase–space structures and there-

In this paper, we develop a model to describe the hard nonlinear evolution of a global Alfvén eigenmode (GAE), which is destabilised outside the Alfvén continuum, using a Lagrangian formalism. We make two main assumptions in this work:

- For a GAE, where poloidal components are weakly coupled, the toroidal effects are neglected in the description of the bulk plasma. However, we retain toroidal effects on EP dynamics, which determine the non-linear behavior of an energetic particle mode (EPM) which is established and evolved inside the shear Alfvén continuum.

- In a tokamak, all the components of the mode structure namely toroidal, poloidal and radial, need to be updated during frequency chirping due to the non-linearity of the EPs current. The formalism of the problem presented in section 2 explains the roadmap to update all the components of the mode structure during chirping. As mentioned in [26], when the frequency deviates from the initial eigenfrequency significantly, the mode preserves its periodic behavior but does not remain sinusoidal. Hence, the poloidal and toroidal components of the mode structure, which represent periodic behavior, can be treated using the Fourier expansion method implemented in [32]. The main challenge left is to treat the radial profile, which does not have a periodic behavior. Accordingly, we focus on the evolution of the radial profile and the non-linear contribution of EPs current updates only the radial profile of the mode.

Therefore, the eigenfunction is presented by a single poloidal ($m$) and toroidal ($n$) mode number except for section 2. The initial eigenfrequency lies just below the shear Alfvén continuum and we study the dynamics associated with a downward branch of frequency chirping. There is no continuum crossing in this case, which is a requirement for the model to remain valid as the frequency chirps. We consider the total Lagrangian of the system and use finite element method to expand the radial structure of the eigenmode. Varying the total Lagrangian with respect to the finite element weights gives non-linear equations describing the evolution of the mode radial profile, which is analyzed by invoking the adiabatic condition.

Section 2 describes the general picture of the problem in tokamaks. In section 3, the equation of the mode driven by EPs is presented. The linear growth rate is calculated by finding an explicit expression for the perturbed EPs phase–space density. Afterwards, we introduce an adiabatic Hamiltonian describing the dynamics of EPs during frequency chirping, which together with bounce averaging the Vlasov equation, allows us to solve the non-linear equation for the evolving radial profile of the mode and the rate of frequency chirping. Section 4 describes a numerical procedure implemented to solve the non-linear equations. Section 5 presents the results for a specific poloidal and toroidal mode number. This includes equilibrium profiles, dynamics of EPs during frequency chirping, the evolution of the radial profile, the rate of frequency chirping and validation of the adiabatic condition. Section 6 is a summary.

2. General formalism for non-linear GAEs

We consider a saturated MHD eigenmode with an already established structure in the case of a near-threshold instability. In the presence of weak damping, the coherent group of EPs locked in the mode results in signals with adiabatic frequency chirping in the EPs phase-space associated with the slow evolution of the saturated structure. These signals represent the non-linear BGK modes with a chirping frequency. In tokamak geometry the general form of a non-linear chirping mode whose radial profile is evolving slowly/adiabatically can be presented by

$$\Phi \left( r; t; t_0 \right) = \sum_{h} \phi_h (r; t_0) e^{-ih\alpha(t)} + c.c \quad (1)$$

with

$$\phi_h = \sum_{m} \phi_{m,h} (r; t_0) e^{ih(m\theta + n\phi)} = \sum_{m,l} \lambda (t_0) Y_l (r) e^{ih(m\theta + n\phi)}, \quad (2)$$

where $m$ and $n$ are the poloidal and toroidal mode numbers, respectively, $Y_l (r)$ are base functions and the corresponding weights $\lambda$ used to describe the finite-element expansion of the radial mode structure, $t_0$ represents fast time scale on the order of the inverse eigenfrequency, $t_0$ represents the slow time scale on which the BGK mode evolves i.e. much longer than the bounce period of the particles trapped in the mode and $\alpha (t_0) = \int_{t_0}^{t} \omega (t') dt'$. For simplicity, the indices are dropped from $t$ in the following.
The total Lagrangian describing the system reads

\[ L = L_{\text{wave}} + \sum_{\text{fast particles}} L_{\text{particles}} + \sum_{\text{fast particles}} L_{\text{int}}, \]  

(3)

where \( L_{\text{wave}} \) is the MHD wave Lagrangian, \( L_{\text{particles}} \) is the EP Lagrangian describing the equilibrium motion and the interaction Lagrangian is denoted by \( L_{\text{int}} \). The total Lagrangian contains all the dynamical variables of the system i.e. particle variables and field variables. In principle, one should vary the Lagrangian with respect to each dynamical variable and follow each variable individually. But instead, we use a kinesthetic description. Nevertheless, each particle trajectory is still a characteristic of the kinetic equation which needs to be solved. However, we do not need to follow each particle trajectory because of the rapid phase mixing in the field of the wave. Therefore, we can characterise the EPs by adiabatic invariants assuming that the mode frequency is evolving adiabatically on much slower time scales compared to the bounce period time of the EPs trapped in the mode. Within the adiabatic chirping approximation, we describe such an adiabatic response of the particles analytically by bounce-averaging the kinetic equation to find the perturbed phase-space density [26, 30, 32]. Hence, we do not vary the total Lagrangian with respect to the fast particle variables but only with respect to the field variables. Starting from the Littlejohns Lagrangian [34] for the guiding center motion of the fast particles, we have

\[ L_{\text{Littlejohn}} = e (A + \rho \| B \| \cdot X) + m_i e \mu \Omega - H, \]  

(4)

where \( B = \nabla \times A \) is the total magnetic field with \( A \) the vector potential, \( \rho_i \) is the gyroradius, \( m_i \) is the ion mass, \( e \) is the electron charge, \( \mu \) is the magnetic moment, \( \Omega \) is the time rate of change of gyrophase and \( H = \frac{1}{2} m v_i^2 + \mu B \) is the particle Hamiltonian with \( v_i \) being the particle velocity parallel to the equilibrium magnetic field. It should be noted that we have considered a gauge where the perturbed electrostatic potential is zero. Using equation (4), the particle and interaction Lagrangian can be found as

\[ L_{\text{particles}} = P_\theta \dot{\theta} + P_\varphi \dot{\varphi} + P_\Omega \dot{\Omega} - H_0 (P_\theta, P_\varphi, P_\Omega, \theta), \]  

(5)

and

\[ L_{\text{int}} = e \vec{A} \cdot \dot{X} = -e \left[ \frac{\partial \Phi}{\partial t} + v_i b \cdot \nabla \Phi \right], \]  

(6)

respectively, where \( H_0 \) is the equilibrium Hamiltonian written in terms of the canonical variables: canonical momenta \( P_\theta, P_\varphi \) and \( P_\Omega \) conjugated to \( \theta, \varphi \) and \( \Omega \), respectively, \( b \) is the unit vector in the direction of the equilibrium magnetic field \( B_0 \) and \( \vec{A}_\perp = \vec{A} - \frac{\vec{B}_0}{B_0^2} (\vec{A} \cdot B_0) \) is the perturbed vector potential that can be represented by two independent scalar functions \( \Phi \) and \( \Psi \) as

\[ \vec{A}_\perp = \nabla \Phi - B_0 (B_0 \cdot \nabla \Phi) / B_0^2 + B_0 \times \nabla \Psi / B_0. \]  

(7)

As mentioned in [35], the compressional perturbation \( \Psi \) is almost decoupled from the shear Alfvén perturbation \( \Phi \). We, therefore only consider the \( \Phi \) term for a GAE. The periodicity of the unperturbed motion allows us to implement action-angle (AA) variables \( (\tilde{\theta}, \tilde{\varphi}, \tilde{\Omega}, P_\tilde{\theta}, P_\tilde{\varphi}, P_\tilde{\Omega}) \) for EPs dynamics. The transformation to the AA variables is governed by the following type-2 generating function

\[ G_2 = \tilde{\varphi} P_\tilde{\varphi} + \Omega P_\tilde{\Omega} + \int_0^\theta P_\tilde{\theta} d\theta. \]  

(8)

The interaction Lagrangian can be written in a more explicit form by substituting equations (1) and (2) in equation (6). The perturbed/interaction Hamiltonian reads

\[ H_{\text{int}} = -L_{\text{int}}. \]  

(9)

By neglecting the toroidal coupling between poloidal components of the mode and expanding in the AA variables of the unperturbed motion, we find

\[ H_{\text{int}} = -\sum_{l,h,p} \lambda_{l,h} (t) V_{P_{\theta},P_{\varphi}} (P_\theta, P_\varphi, P_\Omega) \]  

\[ \times e^{i \theta \hat{\theta} + i h [\tilde{\varphi} - \alpha(i)]} + c.c., \]  

(10)

where \( \hat{\theta} \) and \( \tilde{\varphi} \) are poloidal and toroidal angles, respectively, corresponding to the AA variables of the unperturbed motion, \( V \) is the coefficient of the Fourier expansion in \( \theta \), and \( p \) is the indice of resonances in the linear stage, whereas in the nonlinear problem \( \frac{\delta}{\partial \lambda} \) labels different resonances.

The equations governing the evolution of the mode structure and the frequency can be derived by varying the total lagrangian of the system with respect to the two dynamical field variables, namely \( \lambda \) and \( \alpha \). We write the system of equations for the evolution of the mode structure by varying the total Lagrangian with respect to \( \lambda \), however, in the limit of adiabatic frequency chirping and the slow evolution of the radial profile \( \frac{d m \lambda}{d t} \ll \dot{\alpha} \), the energy balance principle is used to track the evolution of the frequency \( \dot{\alpha} \) during chirping. Therefore, we find the equation corresponding to each harmonic \( (h) \) of the evolving mode structure by varying equation (3) with respect to \( \lambda \), which gives

\[ (h^2 \dot{\alpha}^2 M_h - N_h) \cdot \chi_h = -\frac{1}{4} \int d^3 p d^3 q \delta f (q,p,t) \]  

\[ \times \sum_p \left[ \begin{array}{c} V_{P_{\theta},P_{\varphi}} (P_{\theta},P_{\varphi}) \\ \vdots \\ V_{P_{\theta},P_{\varphi}} (P_{\theta},P_{\varphi}) \end{array} \right] e^{i \theta \hat{\theta} + i h [\tilde{\varphi} - \alpha(i)]} + c.c., \]  

(11)

where matrices \( M_h \) and \( N_h \), corresponding to each harmonic \( (h) \), are constructed by discretising the field and integrating over the plasma volume and the integration on the RHS is over the phase-space of the EPs. It is noteworthy that the derivation of equation (11) and the matrices are detailed in section 3. During the adiabatic evolution, the BGK mode acts like a bucket and the trapped EPs inside this mode will be moved slowly in phase-space, associated with the adiabatic frequency chirping. Hence, the perturbed phase-space density \( \delta f \) is mainly due to the EPs trapped in the mode and can be found by solving the kinetic equation in the adiabatic regime.
In what follows, the theoretical picture is developed for a single harmonic \((h = 1)\). In other words, this model focuses on the evolution of the radial component of the mode during the adiabatic frequency chirping.

3. The model

We consider a GAE in the following calculations and therefore retain only one poloidal harmonic in the linear response of the cold particles. We, however, take into account the poloidal variation of the confining magnetic field in our calculations of the EP trajectories and use a high aspect ratio approximation for that.

3.1. MHD wave Lagrangian

The kinetic \((K)\) and potential energy \((W)\) of MHD waves are given by

\[
K = \frac{1}{2} \int \rho \dot{\boldsymbol{\xi}}^2 dV \quad (12a)
\]

\[
W = -\frac{1}{2} \int \boldsymbol{\xi} \cdot \boldsymbol{F} (\boldsymbol{\xi}) dV, \quad (12b)
\]

where \(\rho\) is the mass density, \(\boldsymbol{\xi}\) is the displacement vector, \(\boldsymbol{F}\) is the force operator, \(dV\) denotes the differential volume element and the integration is performed over the whole plasma volume. For a low-\(\beta\) system and the linearised force operator and in the aforementioned gauge where the perturbed scalar electrostatic potential is zero, the wave Lagrangian reads

\[
L_w = \frac{1}{2} \int \rho \left( \frac{\dot{\boldsymbol{A}}}{B_0} \right)^2 - \frac{[\nabla \times \boldsymbol{A}]^2}{\mu_0} + \frac{J_\parallel}{B_0} \nabla \times \dot{\boldsymbol{A}} dV \quad (13)
\]

where \(\dot{\boldsymbol{A}}\) is the perturbed vector potential, \(B_0\) is the equilibrium magnetic field, \(\mu_0\) is the magnetic permeability, \(J_\parallel\) is the unperturbed plasma current parallel to the equilibrium field and we have neglected the non-linear bulk plasma response. Considering only the \(\Phi\) term in equation (7) for a GAE, equation (13) reduces to

\[
L_w = \frac{1}{2 \mu_0} \int \frac{\rho \dot{\Phi}}{B_0^2} [\nabla \times \dot{\Phi}]^2 - \left( B_0 \nabla \cdot \frac{\nabla \Phi}{B_0} \right)^2 - \left( \frac{[\nabla \times B_0](\mathbf{B}_0 \cdot \nabla \Phi)}{B_0} \right)^2 - \left( \frac{B_0 \cdot \nabla \Phi (\nabla \Phi \cdot \Delta \mathbf{B}_0)}{B_0} \right) dV, \quad (14)
\]

which can be varied with respect to \(\Phi\) to obtain the linear dispersion relation. For a single GAE, \(\Phi\) can be written as

\[
\Phi (\mathbf{r},t) = \sum_{i=1}^{s} \lambda_i (t) Y_l (r) e^{i \theta + i \varphi - i \alpha (t)} + c.c., \quad (15)
\]

where \((r, \theta, \varphi)\) are cylindrical coordinates, \(\alpha (t)\) represents rapid oscillations, \(s\) is the total number of finite elements and \(\lambda_i\) and \(\alpha\) are real quantities with \(\lambda_i\) being assumed to change slowly compared to the mode frequency, \(\frac{d \ln \lambda_i}{d\tau} \ll \dot{\alpha}\). This implies a proper set of the base functions that can represent a smooth radial profile of the global eigenmode. Substituting equation (15) into equation (14) gives

\[
L_w = 2 \alpha^2 \left[ \lambda^T \cdot \mathbf{M} \cdot \lambda \right] - 2 \left[ \mathbf{X}^T \cdot \mathbf{N} \cdot \lambda \right], \quad (16)
\]

where the fast time varying part is integrated out and the superscript \(\tau\) denotes the transpose operation, \(\{\lambda\} \in \mathbb{R}^{s \times 1}\), \(\{\mathbf{M}, \mathbf{N}\} \in \mathbb{R}^{s \times s}\) whose elements are given by

\[
M_{j,k} = \frac{1}{2 \mu_0} \int \frac{r R_0}{V_A^2} \left[ \frac{d Y_j}{dr} \frac{d Y_k}{dr} + \frac{m^2}{r^2} Y_j Y_k \right] dr d\theta d\varphi \quad (17a)
\]

\[
N_{j,k} = \frac{1}{2 \mu_0} \int \left[ B_0^2 \left( \frac{d k_i}{dr} \frac{Y_j}{B_0} \right) \left( \frac{d k_i}{dr} \frac{Y_k}{B_0} \right) + \left( \frac{m^2}{r^2} k_i^2 \right) \right.
+ \left. \frac{\mu_0^2 k_i^2}{B_0^2} + \mu_0 m k_i \frac{J||}{B_0} \right] Y_j Y_k r R_0 dr d\theta d\varphi, \quad (17b)
\]

where \(R_0\) is the major radius, \(V_A\) is the Alfvén velocity and \(k_i\) is the wavenumber parallel to the equilibrium magnetic field.

3.2. Energetic particle and interaction Lagrangian

We write the unperturbed particle Lagrangian part of equation (4) using the high aspect ratio tokamak limit where the flux surfaces are approximated by the contours of constant \(r\) [36]. In these coordinates, one can write

\[
A_0 = \psi \nabla \theta - \chi \nabla \varphi + \nabla \eta, \quad (18)
\]

and

\[
B_0 = \nabla \psi \times \nabla \theta - \nabla \chi \times \nabla \varphi = B_0 \nabla \theta + B_\varphi \nabla \varphi, \quad (19)
\]

where \(A_0\) is the equilibrium part of the vector potential and \(\chi\) is the poloidal flux. We have \(\nabla \varphi = R^{-1} \mathbf{e}_\varphi\) and according to Amperes law \(B_0 \propto R^{-1}\). Hence, \(B_0 \approx B_0(r)\) is a constant with \(B_0\) being the equilibrium magnetic field at the center of the plasma. Using equations (18) and (19), we have

\[
\frac{\partial \psi}{\partial r} \approx r B_0 (1 - \epsilon \cos \varphi), \quad (20)
\]

where \(\epsilon\) is the inverse aspect ratio.

Using a proper gauge to cancel \(\nabla \eta\) in \(A_0\), \(L_{\text{particles}}\) can be written in the canonical form given by equation (5) and the canonical variables are given by

\[
P_\psi = e \psi (r, \theta) + m_\parallel v_\parallel b_\psi (r, \theta), \quad (21)
\]

\[
P_\varphi = - e \chi (r) + m_\parallel v_\parallel b_\varphi (r, \theta), \quad (22)
\]

\[
P_\Omega = \frac{m_i}{c} \mu, \quad (23)
\]
where \( b_\theta \) and \( b_\varphi \) are covariant components of \( \mathbf{b} \), with \( b_\theta \approx r^2/qR \) and \( b_\varphi \approx R \), and \( \chi (r) \) can be found using equation (20) and the safety factor \( q (r, \theta) = \frac{\partial \psi}{\partial \nu} = \frac{\partial \psi}{\partial \rho} \approx q (r) \). The conversion from \( r \) and \( v_\parallel \) to the canonical variables are now implicitly given by equations (21) and (22).

The large aspect ratio assumption allows us to drop the \( \theta \) dependency in equation (20). Taking into account that the toroidal equilibrium field is dominant, \( b_\theta \) can also be neglected. Hence, we have

\[
P_\theta = \frac{1}{2} e X_\parallel^2 B_0, \tag{24}
\]

and

\[
P_\varphi = - e \chi (X_r (P_\theta)) + m_i v_\parallel R, \tag{25}
\]

where \( X_r \) is the radial position of the EPs. By implementing the canonical equations of motion, we find

\[
\frac{\partial H_0}{\partial P_\theta} = m_i v_\parallel = \mu B_0 \frac{1}{R_0} \cos \theta \frac{d X_\parallel}{dP_\theta} = \dot{\theta}, \tag{27}
\]

\[
\frac{\partial H_0}{\partial P_\varphi} = m_i v_\parallel \mu B_0 X_r \frac{\sin \theta}{R_0} = \dot{\varphi}, \tag{28}
\]

\[
\frac{\partial H_0}{\partial \theta} = m_i v_\parallel \mu B_0 X_r \frac{\sin \theta}{R_0} = - \dot{P}_\theta. \tag{29}
\]

Using equation (25), we get

\[
\dot{P}_\theta = - \left[ \frac{m_i v_\parallel^2}{R} + \frac{\mu B_0}{R_0} \right] X_r \sin \theta, \tag{30}
\]

\[
\dot{\varphi} = \frac{v_\parallel}{R}, \tag{31}
\]

\[
\dot{\theta} = \frac{v_\parallel}{R q(X_r)} \left[ \frac{m_i v_\parallel^2}{R} + \frac{\mu B_0}{R_0} \right] \cos \theta \frac{1}{e B_0 X_r}. \tag{32}
\]

It should be mentioned that in the high aspect ratio tokamak limit, the safety factor can be considered only as a function of the radius.

The conjugate momenta corresponding to \( \varphi \) and \( \Omega \) are constants of motion. Therefore, in terms of the AA variables introduced by (8), one can set \( P_\varphi = P_\varphi \), \( P_\Omega = P_\Omega \) and considering the definition of the angle variables from 0 to \( 2\pi \), it is found that \( \dot{\varphi} = \dot{\varphi} + \Delta \varphi \) for motion in the direction of the field line. In this work, we consider the case of a neutral beam injection (NBI) where the majority of the EPs are deeply passing (\( \mu_0 \approx 0 \)) inside the equilibrium field. Consequently, \( v_\parallel \) becomes a constant of motion. We also assume the maximum orbit width (\( \Delta r \)) to be much smaller than the width of the radial mode structure and let \( r_0 (P_\theta, P_\varphi, P_\Omega) \) be the average position of a drift orbit. These conditions together with the large aspect ratio assumption make \( \varphi \) and \( \theta \) approximately linear in time. Therefore, we find

\[
\dot{\varphi} = \frac{V_\parallel (P_\theta, P_\varphi, P_\Omega)}{q(r_0) R_0} \tag{33a}
\]

\[
\dot{\theta} = \frac{V_\parallel (P_\theta, P_\varphi, P_\Omega)}{R_0}, \tag{33b}
\]

where \( \omega_\varphi \) and \( \omega_\theta \) represent the poloidal and toroidal guiding center frequency of the deeply passing EPs.

Substituting equation (15) into equation (6) results in

\[
L_{\text{int}} = ie \left[ \hat{\chi} - v_\parallel \left( \frac{m_i}{q(X_r)} + n \right) \frac{R_0}{R_0} \right] \times \sum \lambda_i (t) Y_1 (X_r) e^{in \varphi - in \phi(t)} + c.c. \tag{34}
\]

Now, we express the above Lagrangian in terms of the AA variables of the unperturbed motion to find

\[
L_{\text{int}} = \sum_{p} \sum \lambda_i (t) V_{p,n} e^{ip \theta + in \varphi - in \phi(t)} + c.c., \tag{35}
\]

where the coupling strength, \( V_{p,n} \), is determined by

\[
V_{p,n} = \frac{1}{2\pi} \int ie \left[ \hat{\chi} (t) - v_\parallel \left( \frac{m_i}{q(X_r)} + n \right) \frac{R_0}{R_0} \right] \times Y_1 (X_r) e^{in \varphi - in \phi(t)} d\theta, \tag{36}
\]

whose detailed calculation for the deeply co-passing orbit types of the EPs is presented in appendix A.

3.3. Mode equation

The total Lagrangian for one eigenmode can be expressed by

\[
L = 2i \lambda^2 [\lambda \lambda] - 2 [\lambda^\parallel \lambda^\parallel] + \sum_{\text{fast particles}} P_\theta \dot{\theta} + P_\varphi \dot{\varphi} + P_\Omega \dot{\Omega} - H_0 (P_\theta, P_\varphi, P_\Omega) + \sum_{\text{fast particles}} \lambda^\parallel (t) D + c.c., \tag{37}
\]

where the elements of \([D] \in \mathbb{R}^{5 \times 1}\) are \( D_{1,1} = \sum_{p} V_{p,n} e^{ip \theta + in \varphi - in \phi(t)} \). Varying the above Lagrangian with respect to \( \lambda \) gives the following expression for the non-linear mode structure

\[
(4\lambda^2 M - 4N) \lambda + \sum_{\text{fast particles}} D + c.c. = 0 \tag{38}
\]
The sum over the fast particles can be replaced by integration over the initial phase–space. The canonicity of the transformation from the initial phase–space coordinates to the instant coordinates \([\{q_0, p_0\} \rightarrow \{q, p\}]\) allows us to write the phase-space integration in terms of the instant coordinates. In addition, as mentioned in section 2, we take \(h = 1\) and therefore we find the non-linear mode equation in the form given by equation (11) with \(h = 1\), where \(\delta f = \sum_p \delta f^p\) is the perturbed part of the total distribution function \((f = \delta f + F_0)\) of the EPs with \(F_0\) being the equilibrium part. In this model, krook type collisions inside the bulk plasma provide the damping mechanism to the wave amplitude at a rate \(\gamma_d\), which is implicitly included in equation (11) (see [37] and [35]).

3.4. Mode evolution

The total Hamiltonian of the EPs during the hard-non-linear evolution reads,

\[
H = H_0(P_{\delta}, P_{\tilde{\varphi}}, P_{\tilde{\Omega}}) + H_{\text{int}} \tag{39}
\]

where \(H_{\text{int}} = -L_{\text{int}} = \sum_{\lambda} H_{\text{int}, \lambda}\) written in terms of the AA variables of the unperturbed motion. In order to simplify the dynamics, we consider the canonical transformation using the type-2 generating function

\[
F_2(q, p_{\text{new}}, t) = P_1 \left[ \delta \tilde{\varphi} + n \delta \tilde{\varphi} - \alpha(t) \right] + P_2 \tilde{\varphi} + P_3 \tilde{\Omega}, \tag{40}
\]

for the \(p\)-th resonance. The new variables are defined as follows

\[
\begin{align*}
P_1 &= \frac{1}{p} P_{\delta} \\
P_2 &= P_{\tilde{\varphi}} - \frac{n}{p} P_{\delta} \\
P_3 &= P_{\tilde{\Omega}}
\end{align*}
\]

which shows that the wave-particle interaction is effectively one-dimensional in an isolated resonance, i.e. \(P_2\) and \(P_3\) corresponding to ignorable coordinates, are constants of the motion. The above canonical transformation is defined for a specific value of \(p\) corresponding to the \(p\)-th resonance. This can be emphasized by considering a subscript \(p\) on the new variables. However, such subscripts are neglected for simplicity.

In what follows, we first calculate an analytic expression for the perturbed phase–space density of EPs in the linear limit. Afterwards, the dynamics of the resonant particles during the adiabatic chirping of the GAE are identified, followed by the perturbed distribution function during the evolution of the holes/clumps.

3.4.1. Linear regime. The linearised Vlasov equation for the \(p\)-th resonance

\[
\frac{\partial \delta f^p}{\partial t} + \frac{\partial \delta f^p}{\partial \zeta} \frac{\partial H_0}{\partial P_1} + \frac{\partial F_0}{\partial P_1} \frac{\partial H_{\text{int}, \lambda}}{\partial P_{\lambda, \mu}} \bigg|_{P_{\lambda}, P_{\mu}}, \tag{42}
\]

where \(\delta f^p = \tilde{f}^p(P_1) \exp(\tilde{\varphi} + c.c.\) and \(H_{\text{int}, \lambda} = -\sum_{\lambda} \lambda V_{p,\pi, \lambda} \exp(\tilde{\varphi} + c.c., can be used to derive an analytic expression for \(\delta f^p\). During the linear evolution, we set \(\alpha(t) = \omega t\), with \(\omega\) being the complex frequency having the real part \(\omega_r\) and imaginary part \(\gamma_d\). By substituting the relevant expressions in equation (42), we find

\[
\tilde{f} = \frac{\sum_{\lambda} \lambda V_{p,\pi, \lambda} \left( \frac{\partial F_0}{\partial P_{\lambda}} n + \frac{\partial F_0}{\partial P_{\mu}} p \right)}{\omega - \rho \omega_d - n \omega_d}, \tag{43}
\]

which gives the resonance condition

\[
\omega_r = \rho \omega_d - n \omega_d. \tag{44}
\]

In the limit of deeply passing particles, equations (11) (with \(h = 1\)) and (43) are used to find the linear dispersion relation of the mode given by

\[
(\omega^2 M - N) \lambda = 4 \pi^3 \sum_p \int dp_1 dp_2 \left| \frac{\partial F_0}{\partial P_1} \right| P_1 P_2 - T \Lambda, \tag{45}
\]

where \(G(P_1) = \frac{\partial H_{\text{int}, \lambda}}{\partial P_{\lambda, \mu}} \mid_{P_{\lambda, \mu}}\), \(\{\Lambda\} \in C^{S \times S}\) whose elements are given by \(T_{j,k} = V_{p,\pi, l,j}^l V_{p,\pi, l,k}^l\) and \(\Lambda\) represents the initial MHD eigenvector. Neglecting the infinitesimal contribution from the principal value allows us to set \(\omega = \omega_{\text{GAE}},\) where \(\omega_{\text{GAE}}\) is the initial MHD frequency of the eigenmode. Therefore, the linear growth rate of the mode is found to be

\[
\gamma_l = \left[ \sum_p \int dp_2 \frac{\partial F_0}{\partial P_2} \left( \frac{\partial G}{\partial P_1} \right)^{-1} \right. \times T \Lambda \mid_{P_1 = P_{1,\text{res}}} \frac{2 \pi^2 \chi_i^2}{\omega_{\text{GAE}} \lambda^\Lambda M^\chi}, \tag{46}
\]

where \(P_{1,\text{res}}\) is the value of \(P_1\) at resonance denoted by \(\Pi\) throughout this manuscript and we have assumed \(\gamma_l \ll \omega_{\text{GAE}}\).

3.4.2. Nonlinear chirping GAE. The existence of the damping mechanism introduced in subsection 3.3 leads to an unstable plateau in the phase–space density of EPs which supports sideband oscillations that evolve into chirping modes [22, 38]. For the purpose of investigating the mode during frequency sweeping, we consider a marginal instability case where mode overlap is neglected and we take the limit where phase–space structures (holes and clumps) move adiabatically. Hence, we have

\[
\frac{d \omega_b}{d t}, \quad \frac{d \alpha}{d t} \ll \omega_b^2 \sim \gamma_d^2 \sim \gamma_d^2, \tag{47}
\]

where \(\omega_b\) is the bounce frequency of EPs trapped inside the separatrix. Therefore, the finite element amplitudes, \(\lambda_i(t)\), evolve on a slow time scale; however, \(\alpha(t)\) includes a fast time scale on the order of \(\omega_b^{-1}\) which corresponds to the periodic behavior of the field. The canonical transformation
presented by equation (41) can be implemented to cancel this fast time scale dependency from the Hamiltonian given by equation (39). Therefore, for the $p$-th resonance, the total Hamiltonian converts to

$$K = H_0(P_1, P_2, P_3) - \dot{\alpha} P_1$$

$$- \sum_{p' \neq p, q} \lambda_i V_{p', n, i}(P_1, P_2, P_3) e^{i \zeta} + c.c.$$

where highly oscillating terms corresponding to other resonances ($p' \neq p$) have been neglected. Assuming the separatrix width to be small compared with the characteristic width of the distribution function, we can Taylor expand the quantities around the middle of the separatrix ($\Pi(t)$), so we have

$$K \approx H_0(P_1, P_2, P_3) + \frac{\partial H_0}{\partial P_1} (P_1, P_2, P_3) |P_1 - \Pi| - \dot{\alpha} P_1$$

$$+ \frac{1}{2} \sum_{i} \lambda_i V_{p, n, i}(P_1, P_2, P_3) e^{i \zeta} + c.c.$$

The higher order terms in the expansion of the equilibrium Hamiltonian have been neglected due to the smallness of the separatrix width. $\Pi$ satisfies: $\frac{\partial H_0}{\partial P_1} (P_1, P_2, P_3) = \dot{\alpha} (t)$, consequently $\Pi = \Pi (P_2, P_3, t)$. Therefore, the new Hamiltonian is

$$K \approx \frac{1}{2} \frac{\partial^2 H_0}{\partial P_1^2} |P_1 - \Pi|^2$$

$$- \sum_{i} \lambda_i V_{p, n, i}(P_1, P_2, P_3) e^{i \zeta} + c.c.$$

(50)

which evolves adiabatically during frequency sweeping. It is noteworthy to mention that for $\frac{\partial^2 H_0}{\partial P_1^2} (P_1, P_2, P_3) > 0$, substituting $K$ in equation (50) with the maximum value of $\sum_{i} \lambda_i V_{p, n, i}(P_1, P_2, P_3) e^{i \zeta}$ gives the dynamics on the separatrix. The preserved adiabatic invariant corresponding to the above slowly evolving Hamiltonian is

$$I = \frac{1}{2\pi} \int P_1 d\zeta$$

(51)

and we denote the corresponding angle by $\eta$. The above equation can be solved for each $P_2$, corresponding to a separatrix, by substituting for $P_1$ and integrating from 0 to $2\pi$ over the angle variable $\zeta$. The perturbed distribution of the passing particles traveling around the separatrix remains approximately close to the equilibrium distribution [26, 30]. Hence, the perturbed density is assumed to be dominantly from the trapped particles inside the separatrix. Considering the small separatrix width assumption mentioned above and bounce-averaging the Vlasov equation (see section 3 in [30] and appendix B in [32]), we find

$$\delta f = \begin{cases} f_0 - F_0(t) = F_0(t = 0) - F_0(t), & \text{trapped passing} \\ 0, & \end{cases}$$

(52)

where $\delta f$ is the perturbed distribution function of the particles inside the holes/clumps, $f_0$ is the lowest order term in the expansion of $f$ around the small parameter $\eta = \frac{\lambda}{\gamma}$ with $\gamma$ and $\tau_1$ being the bounce period and the slow time scale of mode evolution, respectively. It should be noted that $t = 0$ denotes the initial stage of chirping in this paper. However, for the case of an expanding separatrix and for newly trapped particles during chirping, $t = 0$ in the above expression implies the time when EPs are trapped inside the separatrix.

3.4.3. Chirping rate. According to [39], the dissipated power ($Q$) via weak collisions due to the work of friction force is $2\gamma_0 E_{\text{wave}}$, with $E_{\text{wave}}$ being the MHD energy of the mode, which consists the perturbed energy of the cold plasma and the perturbed electromagnetic field. This absorbed power ($Q$) should be equal to the power ($P$) released by the phase–space structures energy. Therefore, we have

$$2\gamma_0 E_{\text{wave}} = -\sum_{p_i} \frac{dE}{dt}$$

(53)

where $N_{p_i}$ is the perturbed number of EPs inside each coherent phase–space structure (hole/clump) in the interval $\Delta P_2$, given by

$$N_{p_i} = \int \delta f dP_1 d\zeta \Delta P_2$$

(54)

and $E$ is the energy of each fast particle inside the hole/clump. This energy consists the kinetic energy and the potential energy of the EPs. Compared to the change in their kinetic energy, we neglect the contribution from the small change in their potential energy which is proportional to the change in the width of the separatrix (See appendix C in [32]). Hence, $E$ can be replaced by $H_0$. So we find

$$\frac{dE}{dt} = \frac{dH_0}{dt} = \frac{\partial H_0}{\partial P_1} (P_2, P_3) G'(\Pi)^{-1} \alpha,$$

(55)

where $G = \frac{\partial H_0}{\partial P_1} (P_2, P_3) = p\omega_\parallel + n\omega_\perp = \alpha$ previously defined in subsection 3.4.1 and the last factor on RHS is the rate of chirping of the mode which is the same for all the separatrices corresponding to different $P_2$ in this model. We also have

$$\frac{\partial^2 H_0}{\partial P_1^2} (P_2, P_3) = \frac{1}{2mR_0} \left( \frac{p + n}{g} \right)^2 - \frac{p v_\parallel \delta E_0 p}{R_0 \delta p^2 \sqrt{2\pi eB_0 \delta p}}.$$  

(56)

The MHD energy of the mode ($E_{\text{wave}}$) is the sum of equations (12a) and (12b), which gives

$$E_{\text{wave}} = W + K = \frac{1}{2m} \int \frac{\dot{\alpha}^2}{\eta} + \left| \nabla \times \vec{A} \right|^2$$

$$- \left( \vec{A} \cdot \nabla \times \vec{A} \right) \frac{\Sigma eB \cdot B dV}{\delta B},$$

(57)

which gives

$$E_{\text{wave}} = 2\alpha^2 \left[ \lambda^T \cdot M \cdot \lambda \right] + 2 \left[ \lambda^T \cdot N \cdot \lambda \right].$$

(58)
Substituting the relevant terms into equation (53) yields
\[
\frac{\partial (\dot{\alpha} - \dot{\alpha}_{t=0})^2}{\partial t} = -8\gamma_{2d} [\dot{\alpha}^2 \lambda^T \cdot M \cdot \lambda + \lambda^T \cdot N \cdot \lambda] (\dot{\alpha} - \dot{\alpha}_{t=0}),
\]
where \(\dot{\alpha}_{t=0} = \omega_{GAE}\).

4. Numerical approach

In this section, the numerical approach implemented to solve for the rate of chirping along with the non-linear mode structure is presented. We have used cubic Hermite elements as the base functions. It is noteworthy that sufficient number of elements should be implemented in order to ensure that the weight/coefficient of each element \((\lambda_i)\) varies slowly \((\frac{\partial \ln \lambda_i}{\partial t} \ll \dot{\alpha})\) during frequency chirping and the radial structure is smooth. The MHD eigenfrequency \((\omega_{GAE})\) and eigenvector \((\lambda_{GAE})\) of the mode are derived separately by solving the MHD eigenvalue problem by setting the fast particle contribution in equation (11) to zero. The equilibrium profiles used to solve the MHD problem and the resonance condition are given in section 5 (see figure 3).

The general roadmap is as follows

- A 5th order Rungge-Kutta method is used to solve the differential equation for the chirping rate.
- The resonance condition is solved for each new frequency (see section 5).
- At each time step of the Rungge-Kutta method, the non-linear mode structure is calculated by solving equation (11) with \(h = 1\) for \(\lambda\) iteratively. This stage is visualised in figure 1 where we choose a fixed number of iterations. In our numerical experiment, we found that after 14 iterations, the maximum relative error in the convergence of the elements of \(\lambda\) vector is on the order of \(10^{-4}\).

The explanation of this diagram including the integration over phase-space in RHS of equation (11) or equation (59), how to treat a/an shrinking/expanding separatrix and the special treatment for the initial stage are detailed below.

4.1. Integration over phase-space

Investigation of the hard non-linear evolution of the mode structure requires one to consider the contribution of different groups of particles that are simultaneously in resonance with the mode and provide equation (11) with the corresponding perturbed densities during frequency sweeping. It should be noted that in this model \((P_3 = 0)\), each \(P_2\) corresponds to a slice of resonance line (a specific group of particles in resonance with the mode) associated with a separatrix. For a specific value of \(P_2\), there exists a corresponding separatrix in \((P_2, \zeta)\) space whose dynamics affects the mode behavior during chirping.

4.1.1. Integration over \(P_2\). The integration over \(P_2\) is performed by the Trapezoidal rule. As the frequency of the mode begins to deviate from the initial value, there may be some groups of the EPs that lose resonance with the mode. On the other hand, there are other groups of the EPs whose dynamics satisfy the new resonance condition associated with the updated frequency and will contribute to the interaction. Consequently, after each time step where the frequency is updated, new values of \(P_2\) are added to the domain over which the trapezoidal rule is performed.

4.1.2. Integration over \((P_1, \zeta)\). Provided that all the separatrices shrink during the evolution, one can integrate over \(P_1\) analytically. Nevertheless, it is shown in [32] that even for a constant trend in frequency sweeping i.e. upward or downward, the value of the adiabatic invariant can have different behaviors depending on the initial equilibrium orbits. Therefore, even for deeply passing energies and for a constant trend in frequency sweeping, the adiabatic invariant corresponding to different groups of the resonant particles may exhibit different behaviors in terms of the expansion/shrinkage. This needs to be considered in developing a numerical treatment for the evolution of each separatrix [31, 40, 41]. Accordingly, the procedure designed to calculate the integral over each separatrix includes the following main steps:

- Calculating the energy related to the EPs dynamics on the separatrix for the given \(\lambda\) and the corresponding value of the adiabatic invariant \((I_{max})\).
• Calculating the ambient phase-space density using the slowing-down distribution given by equation (62).
• Calculating δf using equation (62) and
  * the special treatment presented in 4.2 at the early stage.
  * the stored phase-space data together with equation (52) during the later evolution.
• Storing/updating the phase-space data.
  * At the initial stage: discretising the separatrix using different adiabatic invariants and assigning an ambient phase-space density value to each region and storing the corresponding data.
  * At the later evolution: updating the stored information: Identifying the shrinkage/expansion by comparing each new \( I_{\text{max}} \) to the saved data and updating the data accordingly.

The first two steps can be accomplished by using equation (51) and the notes thereafter for a constant value of \( P_2 \). As mentioned above, the numerical scheme should be able to resolve a shrinking separatrix as well as an expanding one. In the first instance, this may imply that a fully numerical method should be implemented to perform the integration over the phase–space \((P_1, \zeta)\) since the perturbed phase–space density term can not be taken out of the integral to allow further analytic calculations/simplifications. However, we implement the following justification to further simplify the integral over each region inside the separatrix with a constant value of the distribution function and speedup the calculations: For the case of an expanding separatrix, it is necessary to chirp continuously between the initial and final frequency in order to derive an exact non-linear structure at a specific frequency after chirping. This means that the corresponding frequency/time step of the numerical approach is chosen to be sufficiently small so that each group of the newly trapped particles will carry the value of their distribution function prior to becoming trapped inside the separatrix. Subsequently, sufficiently small time steps result in a sufficiently small phase–space area added around the previous separatrix. Hence, this enables us to consider a flat-top phase-space density over each newly added region/ring around the previous separatrix and simplify the integral for each region having a constant density. Figure 2 shows the phase–space area of an expanding separatrix after two time steps during chirping. Regions b and c represent the small areas added to the initial phase-space area (a) and the value of the distribution function is taken to be the same over each region.

At the initial stage, we discretise the phase-space area surrounded by the separatrix using different values of the adiabatic invariant inside the range \( I = [0, I_{\text{max}}] \). This is achieved by substituting different energy values for \( K \) in equation (50). Each adiabatic invariant is assigned a corresponding value of the phase-space density, which represents the value of the distribution function inside the separatrix between two neighbouring discretised adiabatic invariants. Therefore, we define one adiabatic invariant and one phase-space density vector for each separatrix to track its evolution in the numerical scheme. At each time step, depending on whether the value of the new adiabatic invariant at the separatrix \( (I_{\text{max}}) \) is greater or smaller than its value at the previous time step, both vectors are being updated.

During the evolution of each phase-space structure, the value of the ambient phase-space density is the same as the equilibrium distribution function. The difference between this value and the phase-space density inside the separatrix, stored from the previous steps, gives the perturbed distribution function across the separatrix, which can be associated with the height of the coherent structure (hole/clump) in a 3D picture.

### 4.2. Early stage of chirping

In order to investigate the sweeping rate and the mode structure at the early stage of chirping, we rewrite the differential equation (59) and equation (11) (with \( h = 1 \)) at \( t = 0 \). If the separatrix does not trap new particles (a shrinking separatrix) during chirping, \( f_0 \) remains the same as \( F_{0,t=0} \) [30, 32]. For an expanding separatrix the phase–space density of newly trapped particles should be set to the value of the ambient distribution function at the point where the particles are trapped. However, for the very initial stage, one can still set \( \delta f = F_0(P_{1,\text{res}t=0}, P_2) - F_0(P_{1,\text{res}}(t), P_2) \). Using the expansion of \( F_0(P_{1,\text{res}}(t), P_2) \) around \( t = 0 \) and \( \frac{\Pi - \Pi(0)}{\alpha - \alpha(0)} \approx (\frac{\partial P_{1,\text{res}}}{\partial t})^{-1} \), we find

\[
\frac{\partial (\hat{\alpha} - \alpha(0))}{\partial t} = 8\gamma_d \left[ \hat{\alpha} \hat{\alpha}^T - \mathbf{M} \cdot \hat{\lambda} + \hat{\lambda}^T \cdot \mathbf{N} \cdot \hat{\lambda} \right]
\]

\[
\sum_{P_2} \left| \int \frac{dP_1 d\zeta dP_{2}}{dP_1} \bigg|_{P_1 = \Pi(t=0)} \left( \frac{\partial P_{1,\text{res}}}{\partial t} \bigg|_{P_1 = \Pi(t=0)} \right) \hat{\alpha} \Delta P_2 \right|
\]
fast particles density
number density of bulk plasma ions

α
eigenvalue problem code with the radial component of the position. We have benchmarked the eigenvectors of the MHD shows the equilibrium parameters as a function of the radial

\text{the contribution of fast particles) in the low-

\text{limit. Figure 3. MHD equilibrium profiles: (a) the poloidal component of the magnetic field (b) the black and red lines represent the q and the density profiles, respectively (c) the parallel current density and (d) Alfvén continuum for } m = 3 \text{ and } n = 9.

\text{at the initial stage of frequency chirping. For analysing the saturated mode structure at the early stage of chirping, we write } \dot{\alpha} = \alpha \omega_{00} + \Delta \dot{\alpha} \text{ and substitute in equation (11) (with } h = 1 \text{) for } \dot{\alpha}, \text{ to have}

\begin{align*}
\frac{\partial^2 H_0}{\partial P_1^2} &\approx \frac{\lambda_{\text{GAE}}^T}{8 \alpha \lambda_{\text{GAE}}^\lambda \mathbf{M} \cdot \lambda_{\text{GAE}}^\lambda} \int d^3p \frac{\partial F_0}{\partial P_1} \\
&\times \left( \frac{\partial^2 H_0}{\partial P_1^2} \right)^{-1} |p_1=\Pi(t=0) \sum_p \left[ V_{p,n,t=1} : V_{p,n,t=8} \right] e^{i\omega_0} + c.c. \quad (61)
\end{align*}

\text{where we have considered the saturated mode structure to be a linear factor of the MHD eigenvector } (\lambda_{\text{GAE}}^\lambda, \lambda = \alpha \lambda_{\text{GAE}}^\lambda). \text{For an eigenmode growing outside the shear Alfvén continuum, the structure of the radial profile remains almost the same as the initial eigenvector[35].}

5. Results

\text{We set the values of physical parameters as follows: the axial magnetic field at the center } B_z (r = 0) = 2 \text{ T, } R_0 = 35 \text{ m, the minor radius } r_m = 1 \text{ m, the ion mass } m_i \approx 3 \times 10^{-27} \text{ kg, the number density of bulk plasma ions } n_{\text{Bulk}} = 5 \times 10^{20} \text{ m}^{-3}. \text{The fast particles density } n_f \text{ is taken to be } (1 - 10)\% \text{ of } n_{\text{Bulk}}.

5.1. Equilibrium profiles and resonance condition

\text{For the purpose of this work, we consider the density and current profiles mentioned in [42] to solve the equilibrium condition and the MHD eigenmode problem (equation (11) without the contribution of fast particles) in the low-β limit. Figure 3 shows the equilibrium parameters as a function of the radial position. We have benchmarked the eigenvectors of the MHD eigenvalue problem code with the radial component of the displacement vectors reported in [42]. In this model, the GAE exists just below the shear Alfvén continuum since the singularity in the eigenfunction no longer occurs at } r = r_{\text{wmin}} = \text{ due to the inclusion of the current dependent terms. The choice of the mode numbers is based on two factors: an MHD eigenmode should exist for the corresponding mode numbers and also there should be sufficient drive for the corresponding mode with respect to a realistic description of the EPs distribution. Consequently, we have considered the mode numbers } m = 3, n = 9 \text{ for the case of a slowing down distribution of the energetic ions presented by}

\begin{equation}
F_0 = \frac{n_0 A}{v_{i}^2} e^{\frac{\omega_{\text{GAE}}}{2\pi}} \delta (P_3 - 0^+) \quad (62)
\end{equation}

\text{where } n_0 \text{ is the density of the fast particles at the center, } A = \frac{3\nu_{\text{cGAE}}}{4\pi^2} \text{ is the normalisation constant with } v_{i} \text{ being the critical velocity and } \delta P_3 \text{ the width of } F_0 \text{ on } P_3. \text{The aforementioned mode numbers correspond to an eigenfrequency of } \omega_{\text{GAE}} = 4.73 \times 10^6 \text{ rad s}^{-1} \text{ where the radial wavenumber is } 1. \text{The fast ion parameters are chosen to satisfy } \gamma_1 \ll \omega_{\text{GAE}}. \text{ By setting } v_i = 2.6 \times 10^6, \Delta P_3 = 0.47 \times 10^{-20} \text{ and } n_f = 10\% n_{\text{Bulk}}, \text{ we find the linear growth rate } \gamma_1 = 1.13 \times 10^5 \text{ s}^{-1} \text{ for } p = 2. \text{It is noteworthy that for the highly co-passing energetic ions studied in this model, the coupling strength is nonzero for } p = m \pm 1 \text{ (see appendix A). Prior to investigating the evolution of the non-linear structure, the resonance condition equation (44) should be solved to find the values of the action variables, equation (41), at the resonance and also to track the dynamics of the resonant particles. For a downward trend in}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{MHD equilibrium profiles: (a) the poloidal component of the magnetic field (b) the black and red lines represent the q and the density profiles, respectively (c) the parallel current density and (d) Alfvén continuum for } m = 3 \text{ and } n = 9.
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Resonance line of } v_i (a), P_1 (b) \text{ and } P_2 (c) \text{ for different wave frequencies. Points } A \text{ and } A' \text{ on (c) represent the initial position of a separatrix that moves to points } B, C, D \text{ and } B', C', D' \text{, respectively, during frequency chirping while the value of } P_2 \text{ is preserved.}
\end{figure}
frequency chirping, figure 4(a), (b) and (c) illustrate the resonance line for \( v_\parallel \), \( P_1 \) and \( P_2 \) respectively, versus the radial position for different frequencies. The conservation of conjugate momenta \( P_3 \) and \( P_5 \), with the latter being \( \approx 0 \) for deeply passing EPs, allows us to track the dynamics of EPs and identify whether the motion of the non-linear structures (holes/clumps) in the phase–space results in an inward or outward flux of the fast ions in resonance with the chirping GAE. Since \( P_2 \) corresponds to an ignorable coordinate, its value must be preserved during the motion of the holes/clumps. Therefore, the motion of the corresponding separatrix in the radial direction occurs in a way that the value of \( P_2 \) remains the same during chirping. As an example, the group of particles that satisfy the resonance condition at point A on figure 4(c) should move to point D in order to conserve the value of \( P_2 \) while satisfying the resonance condition. Therefore, this results in an inward flux of the fast ions towards the plasma core during frequency sweeping of the eigenmode.

Figure 5 shows the equilibrium phase–space density of the EPs as well as the resonance line for different frequencies. Initially, the value of the total phase-space density inside each separatrix is the same as the equilibrium distribution function. This is illustrated by using circles on the initial resonance line (1) in figure 5. As the frequency chirps, the separatrices (phase–space structures) preserve the initial value of the phase-space density during their motion and carry the initially-in-resonance EPs with the mode to new regions in the phase-space. Therefore, depending on whether the value of \( F_0 \) at these new regions is lower or higher than the value of \( F_0 \) at the initial resonance i.e. \( \delta f > 0 \) or \( \delta f < 0 \) (see equation (52)), a clump or hole will be developed inside the separatrix.

If the separatrices trap new EPs on their way due to the expansion (see figure 2) and carry them, the preservation of the phase–space density of the newly trapped EPs should also be taken into account. The separatrix, which is initially located at point A on figure 5, will move to points B, C and D at each corresponding frequency. It can be observed that the separatrices move to regions where the value of the ambient equilibrium phase-space density is lower. Therefore, a clump will be developed inside the phase-space structure to preserve the distribution function value inside the separatrix. Figure 6 illustrates the total distribution function inside the separatrix at point A which moves down in \( P_1 \) to point B as a result of frequency chirping in this model.

Further explanations can be given to identify the phase-space structures as clumps: In this case, the value of \( \delta P_\theta \), \( P_1 \approx P_{1,0} \), namely the drive, remains positive for all the separatrices during frequency chirping of the mode. For deeply passing particles, we have

\[
X_r = r_0 + \Delta r \cos \theta,
\]

and using equations (24) and (41), we find

\[
P_\theta = \frac{1}{2} e B_0 r_0^2.
\]
damping where the linear structure is not mainly identified by the fast particles as opposed to energetic particle modes (EPMs). Therefore, crossing the continuum edge should be avoided during chirping.

5.2. chirping rate, structure evolution and adiabaticity validation

For the case of a near threshold instability $|\gamma - \gamma_d| \ll \gamma_d \leq \gamma_l$, we choose $\gamma_d = 1.1 \times 10^4 \text{ s}^{-1}$ and solve the differential equation (59) coupled with the integral equation (11) with the approach mentioned in section 4 to determine the non-linear behavior during long range frequency chirping.

The radial current created by the population of the energetic ions modifies the structure of the MHD eigenmode in the hard non-linear regime. Figure 7 demonstrates the evolution of the radial profile of $\hat{\phi}(r,t)$ while the frequency of the mode deviates from the initial eigenfrequency. The peak of the initial eigenmode structure, located at the point where the extremum of the Alfvén continuum occurs, will be shifted inward towards the center of the plasma and the mode becomes more localized close to the plasma center. This inward displacement is in compliance with the inward drift of the EPs explained above. In addition, it can be observed that the radial profile is broadened as the frequency moves away from the shear Alfvén continuum. As the frequency decreases, the amplitude of the radial profile initially grows and then starts to decrease. It is noteworthy that the amplitude value at $\tilde{\omega} = 1$ represents the saturated amplitude corresponding to the aforementioned linear growth rate. In this model, the axial current resolves a pole in the MHD equations and allows weakly damped GAEs with smooth radial profiles as opposed to highly damped continuum modes with spiky radial structures. However, the eigenfrequency of this GAE lies just below the shear Alfvén continuum and the initial frequency is very close to the value corresponding to the pole in the MHD equations. Therefore, for a fixed frequency change, the mode structure changes more when the frequency change occurs closer to the initial eigenfrequency. This can be investigated using figure 7.

It has been shown that the radial profile changes more when the frequency changes from $\tilde{\omega} = 1$ to $\tilde{\omega} = 0.98$ as opposed to the case where it changes from $\tilde{\omega} = 0.98$ to $\tilde{\omega} = 0.95$.

The change in the radial component of various plasma quantities during chirping can also be analyzed using $\Phi (r,t)$. The displacement vector reads,

$$\xi = -\frac{1}{B_0} \left( \hat{A}_\perp \times B_0 \right).$$

Using equation (7), we have

$$\xi B_0 = \frac{\partial \Phi (r,t)}{\partial r} - \hat{e}_r \hat{r} \frac{m}{r} \left( \Phi (r,t) e^{i(\theta t + n \phi - \alpha t)} - e\right) \hat{e}_r,$$

where the poloidal component of the equilibrium magnetic field has been neglected compared to the toroidal component. Equation (66) clearly demonstrates the relation between radial component of the displacement vector and the radial mode structure plotted in figure 7(a). Figure 8 illustrates the rate at which the frequency chirps. It is shown that the square root dependency holds for the very early stages of frequency chirping.

The adiabatic condition represented in subsection 3.4.2, which is implemented for the formalism, needs to be validated if it remains satisfied [23, 40, 43]. Equation (50) can be written as

$$K = \frac{1}{2} \frac{\partial^2 H_0}{\partial P_1} (\Pi, P_2, P_3) [P_1 - \Pi]^2 - \sum_i 2 |\lambda_i V_{p,n|i}| \cos (\zeta + \sigma),$$
where \( \sigma = \tan^{-1} \left( \frac{3(\lambda_l V_{p,n,l})}{\Re(\lambda_l V_{p,n,l})} \right) \) and for the case of the EPs with highly passing orbit types, we have \( \sigma = \pm \frac{\pi}{2} \). It is worth noting that in this case we have \( \Re(\lambda_l V_{p,n,l}) = 0 \). Using canonical equations of motion, one finds

\[
P_1 = - \sum_l 2|\lambda_l V_{p,n,l}| \cos(\zeta + \sigma), \tag{68a}
\]

\[
\dot{\zeta} = - \frac{\partial^2 H_0}{\partial P_1^2} |P_1 - \Pi|^2. \tag{68b}
\]

The motion of the deeply trapped EPs inside the separatrix satisfies the pendulum equation

\[
\frac{d^2}{dt^2} (\zeta + \sigma) = - \frac{\partial^2 H_0}{\partial P_1^2} \sum_l 2|\lambda_l V_{p,n,l}| \times \sin(\zeta + \sigma), \tag{69}
\]

where we have used \( \sin(\zeta + \sigma) \approx (\zeta + \sigma) \) at the center of the separatrix, the so-called O-point. As shown in figure 2, we have \( \sigma = -\frac{\pi}{2} \) for the results reported in this paper and the O-point is located at \( \zeta = \frac{\pi}{2} \). Subsequently, the bounce frequency of the deeply trapped EPs inside the separatrix is

\[
\omega_b = \sqrt{\frac{2\frac{d^2 H_0}{dP_1^2}}{\sum_l |\lambda_l V_{p,n,l}|}}. \tag{70}
\]

The RHS of the adiabatic condition \( 1 \gg \frac{|\omega_b|}{\omega_G} \) is plotted in figure 9 for separatrix with different initial radial positions. Consistent with the previously reported results [32, 43], we also observe that the adiabatic condition is never formally satisfied at the very early stage of chirping. However, it is shown that once the adiabatic condition is satisfied, it remains valid for later evolution of the mode. In addition, it was discussed in [32] that the assumption of \( \gamma \ll \omega_{GAE} \) implies that the period during which the adiabatic limit is not satisfied is very short.

### 6. Concluding remarks

A theoretical description has been developed to study the hard non-linear evolution of a global Alfvén eigenmode (GAE) in resonance with co-passing energetic particles (EPs) in an NBI scenario during the adiabatic frequency chirping behavior of the mode. Constructing appropriate constants of motion allows us to track the dynamics of the EPs as the frequency of the mode changes. In addition, a finite element method using cubic Hermite base functions, has been implemented to represent the radial profile of the GAE. This enables the derivation of an analytic expression for the non-linear radial structure of the mode by varying the total Lagrangian of the system with respect to the weight of the finite elements. Hence, the radial structure can be updated as the frequency deviates from the initial MHD eigenfrequency. During chirping, the possibility of both the shrinkage and the expansion of a phase-space structure (a hole/clump) has been taken into account. The phase-space structures, identified to be clumps, move in order to extract energy from the EPs distribution function and deposit it into the bulk plasma. This energy balance is used to derive an expression for the time rate of the change in the frequency. The adiabatic condition is also evaluated which remains valid once it is satisfied.

Energetic-ion parameters, such as orbit width or pressure, can cause a shift and a broadening in the radial profile of the mode [44]. In addition, for the case of a near threshold instability, we have shown how the deviation of the frequency from the initial eigenfrequency can also result in shifting the peak location of the radial profile and also radial broadening during the hard non-linear stage. For the case presented in this manuscript, the slowing down EPs move radially inward as clumps when the frequency chirps downward. The orbit width of the EPs follows

\[
\Delta \tau = \frac{q m_i v_n}{e B}, \tag{71}
\]

where \( q \) is the safety factor, \( m_i \) is the ion mass, \( v_n \) is the velocity of the EPs parallel to the equilibrium magnetic field, \( e \) is the electron charge and \( B \) is the magnetic field. The range of orbit widths from the plasma center to the boundary is 0.01-0.08 m. This corresponds to an energy range of 23–32 kev for the EPs initially in resonance with the mode.

With respect to applications to actual geometries, the presented formalism can be applied to shear Alfvén eigenmodes without or with very weak effect of mode coupling e.g. GAEs, in an NBI scenario. The calculation of the EPs dynamics is done for the deeply passing particles where the coupling strength is nonzero for only two values of \( m \), i.e. \( m \pm 1 \). A comprehensive description of the problem is aimed in our research plan. This includes
• calculation of action-angle variables for more general EP orbits and
• Adding the effect of toroidal coupling for the mode which does not require any fundamental difficulties as all the main ingredients are already introduced in this work. It should be noted that the effect of toricity on the EPs dynamics is already included in the formalism, and
• allowing the EPs non-linearity to update all the components of the mode structure simultaneously, namely poloidal, toroidal and radial using the general formalism presented in section 2.

In this manuscript, we have taken a short-cut, associated with some assumptions, to the above roadmap to build the presented model along the way and produce the core of the full problem which requires more effort but is feasible. To our knowledge, this is the first attempt to present a technique for the evolution of the radial structure in the non-linear regime and this technique can be generalised which requires heavier computations. A further study is to allow the frequency to cross the continuum edge and behave as realistically as possible inside the continuum.

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Appendix A. Calculation of the coupling strength

Using equation (31) and the resonance condition \( \dot{\theta} = \dot{\psi} (t) + n \dot{\theta} \), equation (36) can be written as

\[
V_{p,n,l} = \frac{\pi}{2} \int_0^{2\pi} \left[ \left( \frac{m}{q(X_r)} \right) \dot{Y}_l(X_r) e^{im\theta - in\Delta \varphi - \psi_0} \right] d\theta.
\]  

(A1)

The integral over the first term of the integrant can be performed using integration by parts. Therefore,

\[
V_{p,n,l} = \frac{\pi}{2} \int_0^{2\pi} \left[ \left( \frac{m}{q(X_r)} \right) \dot{Y}_l(X_r) e^{im\theta - in\Delta \varphi - \psi_0} \right] d\theta.
\]  

(A2)

Equations (24) and (30) can be used to find

\[
\dot{X}_r = \frac{\partial X_r}{\partial \theta} = - \left[ \frac{m v_\parallel^2}{R} + \frac{\mu B_0}{R_0} \right] \sin \theta \frac{1}{e B_0}.
\]  

(A3)

Simple implementation of equations (31) and (32) gives

\[
\dot{\theta} - \frac{\dot{\psi}}{q(X_r)} = - \left[ \frac{m v_\parallel^2}{R} + \frac{\mu B_0}{R_0} \right] \cos \theta \frac{1}{e B_0}.
\]  

(A4)

Under the small orbit width assumption, \( V_{p,n,l} \) reads

\[
V_{p,n,l} = - \frac{1}{2} \int_0^{2\pi} \left[ \frac{m v_\parallel^2}{RB_0} + \frac{\mu}{R_0} \right] \frac{d Y_l(r_0)}{dr} \sin \theta
\]

\[+ im Y_l(r_0) \cos \theta \]  

\[e^{im\theta - in\Delta \varphi - \psi_0} d\theta. \]  

(A5)

For deeply passing EPs inside the equilibrium field, one can neglect the infinitesimal perpendicular velocity of the particles to the magnetic field and set \( \mu \approx 0 \). In this limit, \( v_\parallel \) becomes a constant of motion and we can set \( \theta \approx \dot{\theta} \) and \( \Delta \varphi = \text{cte} \). Using Euler’s formula and the orthogonality of trigonometric functions, one finds

\[
V_{p,n,l} = - \frac{im v_\parallel^2}{2B_0 R_0} \left[ \pm Y_l' + \frac{m}{r_0} Y_l \right], \quad p = (m \mp 1)
\]  

(A6)

where we have set \( \Delta \varphi = 0 \). Non-zero values of \( \Delta \varphi \) result in a shift of the separatrix in phase–space compared to the existing model.

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