Log-Harnack Inequality for Mild Solutions of SPDEs with Strongly Multiplicative Noise

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Abstract

Due to technical reasons, existing results concerning Harnack type inequalities for SPDEs with multiplicative noise apply only to the case where the coefficient in the noise term is an Hilbert-Schmidt perturbation of a fixed bounded operator. In this paper we investigate a class of semi-linear SPDEs with strongly multiplicative noise whose coefficient is even allowed to be unbounded which is thus no way to be Hilbert-Schmidt. Gradient estimates, log-Harnack inequality and applications are derived. Applications to stochastic reaction-diffusion equations driven by space-time white noise are presented.

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1 Introduction

Let \((\mathbb{H}, \langle \cdot, \cdot \rangle, | \cdot |)\) be a separable Hilbert space. Let \(\mathcal{L}(\mathbb{H})\) be the set of all densely defined linear operators on \(\mathbb{H}\). We will use \(\| \cdot \|\) and \(\| \cdot \|_{HS}\) to denote the operator norm and the Hilbert-Schmidt norm for linear operators on \(\mathbb{H}\) respectively. Let

\[ b : \mathbb{H} \to \mathbb{H}, \quad \sigma : \mathbb{H} \to \mathcal{L}(\mathbb{H}) \]

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be two given measurable maps.

Consider the following SPDE on $\mathbb{H}$:

\[(1.1) \quad dX_t = \{AX_t + b(X_t)\}dt + \sigma(X_t)dW_t,\]

where $W_t, t \geq 0$ is a cylindrical Brownian motion on $H$ admitting the representation:

\[(1.2) \quad W_t = \sum_{i=1}^{\infty} \beta_i(t)e_i.\]

Here $\beta_i(t), i \geq 1$ is a sequence of independent real-valued Brownian motions on a complete filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Recall that an $\mathbb{H}$-valued adapted process $(X_t)_{t \geq 0}$ is called a mild solution to (1.1) if

\[(1.3) \quad \int_0^t E(|T_{t-s}b(X_s)| + \|T_{t-s}\sigma(X_s)\|_{HS}^2)ds < \infty, \quad t \geq 0\]

and almost surely

\[X_t = T_tX_0 + \int_0^t T_{t-s}b(X_s)ds + \int_0^t T_{t-s}\sigma(X_s)dW_s, \quad t \geq 0.\]

To ensure the existence and uniqueness of the mild solution, and to derive regularity estimates of the associated semigroup, we shall make use of the following conditions.

(A1) There exists a positive function $K_b \in C((0, \infty))$ such that

\[\phi_b(t) := \int_0^t K_b(s)ds < \infty, \quad |T_t(b(x) - b(y))|^2 \leq K_b(t)|x - y|^2, \quad t > 0, x, y \in \mathbb{H}.\]

(A2) $|\sigma v|^2 \geq \lambda(\sigma)|v|^2$ holds for some constant $\lambda(\sigma) > 0$ and all $v \in \mathbb{H}$.

(A3) There exists $x \in \mathbb{H}$ such that for any $s > 0$, $T_s\sigma(x)$ extends to an unique Hilbert-Schmidt operator which is again denote by $T_s\sigma(x)$ such that $\int_0^t \|T_s\sigma(x)\|^2_{HS}ds < \infty, t > 0$; and there exists a positive measurable function $K_\sigma \in C((0, \infty))$ such that

\[\phi_\sigma(t) := \int_0^t K_\sigma(s)ds < \infty, \quad \|T_t(\sigma(x) - \sigma(y))\|^2_{HS} \leq K_\sigma(t)|x - y|^2, \quad t > 0, x, y \in \mathbb{H}.\]

(A4) The operator $A$ admits a complete orthonormal system of eigenvectors, that is, there exists an orthonormal basis $\{e_n, n \geq 1\}$ of $\mathbb{H}$ such that $-Ae_n = \lambda_n e_n, n \geq 1$, where $\lambda_n \geq 0, n \geq 0$ are the corresponding eigenvalues.

Replacing (A.3) and (B.1) in the proof of Theorem A.1] by the current (A1) and (A3), we see that for any $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ the equation has a unique mild solution and $E|X_t|^2$ is locally bounded in $t$. Let

\[P_t f(x) = \mathbb{E} f(X_t^x), \quad x \in \mathbb{H}, f \in \mathcal{B}_b(\mathbb{H}), t \geq 0,\]
where $X^x_t$ denotes the unique mild solution to (1.1) with $X_0 = x$. Under (A1)-(A3) the proof of [11, Theorem 1.2] implies that $P_t f$ is Lipschitz continuous for any $t > 0$ and $f \in B_b(H)$; consequently, $P_t$ is strong Feller.

In this paper we aim to investigate Harnack type inequalities for $P_t$, which implies not only the strong Feller property but also some concrete estimates on the heat kernel. As the process is infinite-dimensional, the Harnack inequality we shall establish will be dimension-free. The following type dimension-free Harnack inequality with a power $\alpha > 1$

$$|P_t f|^\alpha(x) \leq P_t |f|^\alpha(y) e^{C(t)\rho(x,y)^2}$$

was first found in [13] for diffusion semigroups on Riemannian manifolds, where $\rho$ is the Riemannian distance. Because of the new coupling argument introduced in [1], this inequality has been established for a large class of SDEs and SPDEs (see [6, 8, 9, 10, 13, 17, 21, 22] and references therein). When this type of inequality is invalid, the following weaker version, known as log-Harnack inequality, was investigated as a substitution (see [12, 14, 16, 20]):

$$P_t \log f(x) \leq \log P_t f(y) + C(t)|x - y|^2, \quad t > 0, x, y \in H, \quad f > 0, f \in B_b(H).$$

However, when SPDEs with multiplicative noise is considered, existing results on Harnack type inequalities work only for the case that the coefficient in the noise term is an Hilbert-Schmidt perturbation of a constant operator; i.e. the conditions imply that $\|\sigma(x) - \sigma(y)\|_{HS} < \infty$ for $x, y \in H$. Although this assumption comes out naturally by applying the Itô formula to the distance of the two martingale processes of the coupling, it however excludes many important models; for instance $A$ being the Dirichlet Laplacian on $[0, 1]$ and $\sigma(x) = (\phi \circ x)I\!d$ for a Lipschitz function $\phi$ on $\mathbb{R}$ as studied in [7, 22] where a reflection is also considered (see Section 4 for details).

To get ride of the condition on $\|\sigma(x) - \sigma(y)\|_{HS}$, we will not make use of the coupling method, but follow the line of [12] by establishing the gradient estimate of type $|\nabla P_t f|^2 \leq C(t)P_t|\nabla f|^2$, which, along with a finite-dimensional approximation argument, will enable us to derive the log-Harnack inequality. Here, for any function $f$ on $H$ and $x \in H$, we let

$$|\nabla f|(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|x - y|}.$$

Moreover, it is easy to see that (A1) and (A3) imply

$$t_0 := \sup \left\{ t > 0 : \phi_b(t) + \phi_a(t) \leq \frac{1}{6} \right\} > 0.$$

The following is the main result of the paper, where when $t_0 = \infty$, we set

$$t_0(6^{-\frac{1}{\rho}} - 1) = t_0(1 - 6^{-\frac{1}{\rho}}) = \lim_{r \to \infty} r(6^{-\frac{1}{\rho}} - 1) = \lim_{r \to \infty} r(1 - 6^{-\frac{1}{\rho}}) = t \log 6.$$

**Theorem 1.1.** Assume (A1), (A3) and (A4).
For any \( f \in C_b^1(\mathbb{H}) \),
\[
|\nabla P_t f|^2 \leq 6^{1 + \frac{1}{t_0}} P_t |\nabla f|^2, \quad t \geq 0.
\]

(2) If (A2) holds, then for any strictly positive \( f \in B_b(\mathbb{H}) \),
\[
P_t \log f(y) \leq \log P_t f(x) + \frac{3 \log 6}{\lambda(\sigma)t_0(1 - 6^{-\frac{1}{t_0}})} |x - y|^2, \quad x, y \in \mathbb{H}, t > 0.
\]

(3) If (A2) holds, then for any \( f \in B_b(\mathbb{H}) \),
\[
|\nabla P_t f|^2 \leq \frac{3 \log 6}{t_0 \lambda(\sigma)(1 - 6^{-\frac{1}{t_0}})} \{P_t f^2 - (P_t f)^2\}, \quad t > 0.
\]

(4) If \(|\sigma v|^2 \leq \bar{\lambda}(\sigma)|v|^2 \) holds for some constant \( \bar{\lambda}(\sigma) > 0 \) and all \( v \in \mathbb{H} \), then
\[
P_t f^2 - (P_t f)^2 \leq \frac{12 \bar{\lambda}(\sigma)t_0(6^{\frac{1}{t_0}} - 1)}{\log 6} P_t |\nabla f|^2, \quad f \in C_b^1(\mathbb{H}), t \geq 0.
\]

As application of Theorem 1.1, (3) implies that \( P_t \) sends bounded measurable functions to Lipschitz continuous functions and is thus strong Feller; (4) provides a Poincaré inequality for \( P_t \); and the log-Harnack inequality in (3) implies the following assertions on the quasi-invariant measure and heat kernel estimates (see Corollary 1.2 in [17, 12, 16]). Recall that a \( \sigma \)-finite measure \( \mu \) is called quasi-invariant for \( \mu \) if \( \mu P_t \) is absolutely continuous with respect to \( \mu \).

**Corollary 1.2.** Assume (A1)-(A4). Let \( \mu \) be a quasi-invariant measure of \( (P_t)_{t>0} \). Then

1. \( P_t \) has a density \( p_t \) with respect to \( \mu \) and
\[
\int_{\mathbb{H}} p_t(x, z) \log \frac{p_t(x, z)}{p_t(y, z)} \mu(\mathrm{d}z) \leq \frac{3 \log 6}{\lambda(\sigma)t_0(1 - 6^{-\frac{1}{t_0}})} |x - y|^2, \quad x, y \in \mathbb{H}, t > 0.
\]

2. For any \( x, y \in \mathbb{H} \) and \( t > 0 \),
\[
\int_{\mathbb{H}} p_t(x, z)p_t(y, z) \mu(\mathrm{d}z) \geq \exp \left[ -\frac{3 \log 6}{\lambda(\sigma)t_0(1 - 6^{-\frac{1}{t_0}})} |x - y|^2 \right].
\]

3. If \( \mu \) is an invariant probability measure of \( P_t \), then \( \mu \) has full support and it is the unique invariant probability measure. Moreover, letting \( P_t^* \) be the adjoint operator of \( P_t \) in \( L^2(\mu) \) and let \( W_2 \) be the quadratic Wasserstein distance with respect to \( |\cdot| \), the following entropy-cost inequality holds:
\[
\mu((P_t^* f) \log P_t^* f) \leq \frac{3 \log 6}{\lambda(\sigma)t_0(1 - 6^{-\frac{1}{t_0}})} W_2(f \mu, \mu)^2, \quad t > 0, f \geq 0, \mu(f) = 1.
\]
To apply Corollary 1.2 in particular the third assertion, we need to verify the existence of the invariant probability measure of $P_t$. This can be done by using e.g. [5, Theorem 6.1.2] (see the proof of Theorem 4.1(3) below).

The rest of the paper is organized as follows. In Section 2, we establish the finite dimensional approximations to the mild solutions of the SPDEs. Section 3 is devoted to the proofs of Theorem 1.1 and Corollary 1.2. In Section 4, we apply our results to stochastic reaction-diffusion equations driven by space-time white noise.

## 2 Finite dimensional approximations

In this section, we will prove a finite dimensional approximation result for the mild solution of equation (1.1) which will be used later. Let $\{e_n, n \geq 1\}$ be the eigenbasis of the operator $A$. Set $H_n = \text{span}\{e_1, e_2, ..., e_n\}$. Denote by $P_n$ the projection operator from $H$ into $H_n$. Note that $P_n$ commutes with the semigroup $T_t, t \geq 0$. Define for $x = \sum_{i=1}^n \langle x, e_i \rangle e_i \in H_n$,

$$A_n x = -\sum_{i=1}^n \lambda_i \langle x, e_i \rangle e_i.$$ 

Then $A_n$ is a bounded linear operator on $H_n$. Introduce

$$(2.1) \quad b_n(x) = P_n b(x), \quad \sigma_n(x)y = P_n(\sigma(x)y), \quad x, y \in H.$$ 

Consider the following system of stochastic differential equations in $H_n$:

$$(2.2) \quad \begin{cases} 
\mathrm{d}X_t^n = A_n X_t^n \mathrm{d}t + b_n(X_t^n) \mathrm{d}t + \sigma_n(X_t^n) \mathrm{d}W_t^n \\
X_0^n = P_n X_0,
\end{cases}$$

where $W_t^n = \sum_{i=1}^n \beta_i(t) e_i$. It is well known that under (A1) and (A3) the above equation admits a unique strong solution.

**Theorem 2.1.** Let $X^n_t, X_t$ be the (mild) solutions to equation (2.2) and (1.1). Assume (A1), (A3) and (A4). If $\mathbb{E}|X_0|^2 < \infty$ then

$$(2.3) \quad \lim_{n \to \infty} \mathbb{E}|X_t^n - X_t|^2 = 0, \quad t \geq 0.$$ 

**Proof.** Fix an arbitrary positive constant $T > 0$. We will prove (2.3) for $t \leq T$. Let $T_t^n$ denote the semigroup generated by $A_n$. The following representation holds:

$$T_t^n x = \sum_{i=1}^n \mathrm{e}^{-\lambda_i t} \langle x, e_i \rangle e_i, \quad x \in H_n.$$ 

In a mild form, we have

$$(2.4) \quad X_t^n = T_t^n X_0^n + \int_0^t T_{t-s}^n b_n(X_s^n) \mathrm{d}s + \int_0^t T_{t-s}^n \sigma_n(X_s^n) \mathrm{d}W_s^n.$$
Subtracting $X$ from $X^n$ and taking expectation we get
\[ \mathbb{E}|X^n_t - X^n_t|^2 \leq 3\mathbb{E}|T^n_t X^n_0 - T^n_0 X_0|^2 \]
(2.5)
\[ + 3\mathbb{E} \int_0^t |T^n_{t-s}b_n(X^n_s) - T^n_{t-s}b(X_s)|^2 ds \]
\[ + 3\mathbb{E} \int_0^t \|T^n_{t-s}\sigma_n(X^n_s) - T^n_{t-s}\sigma(X_s)\|^2_{HS} ds. \]

Now, since $T^n_n X^n_0 = \mathcal{P} T_n X_0$ and $T^n_{t-s} b_n = \mathcal{P} T_{t-s} b$, we have
\[ \mathbb{E}|T^n_t X^n_0 - T^n_0 X_0|^2 = \mathbb{E} \sum_{k=n+1}^{\infty} \langle X_0, e_k \rangle^2 e^{-2\lambda_k t} \leq \mathbb{E} \sum_{k=n+1}^{\infty} \langle X_0, e_k \rangle^2, \]
(2.6) and
\[ \mathbb{E} \int_0^t |T^n_{t-s}b_n(X^n_s) - T^n_{t-s}b(X_s)|^2 ds \]
\[ \leq 2\mathbb{E} \int_0^t |T^n_{t-s}b_n(X^n_s) - T^n_{t-s}b_n(X_s)|^2 ds + 2\mathbb{E} \int_0^t |T^n_{t-s}b_n(X_s) - T^n_{t-s}b(X_s)|^2 ds \]
\[ \leq 2\mathbb{E} \int_0^t |T^n_{t-s}b(X^n_s) - T^n_{t-s}b(X_s)|^2 ds + 2\mathbb{E} \int_0^t \sum_{k=n+1}^{\infty} e^{-2\lambda_k(t-s)} \langle b(X_s), e_k \rangle^2 ds \]
\[ \leq 2\mathbb{E} \int_0^t K_b(t-s)|X^n_s - X_s|^2 ds + 2\mathbb{E} \int_0^t \sum_{k=n+1}^{\infty} e^{-2\lambda_k(t-s)} \langle b(X_s), e_k \rangle^2 ds. \]

To get an upper bound for the last term in (2.5), we observe that
\[ \mathbb{E} \int_0^t \|T^n_{t-s}\sigma_n(X^n_s) - T^n_{t-s}\sigma(X_s)\|^2_{HS} ds = \mathbb{E} \int_0^t \|T^n_{t-s}(\sigma(X^n_s) - \sigma(X_s))\|^2_{HS} ds \]
(2.8)
\[ \leq \mathbb{E} \int_0^t \|T^n_{t-s}\sigma(X^n_s) - T^n_{t-s}\sigma(X_s)\|^2_{HS} ds \leq \mathbb{E} \int_0^t K_{\sigma}(t-s)|X^n_s - X_s|^2 ds, \]
where condition (A3) was used. Moreover,
\[ \mathbb{E} \int_0^t \|T^n_{t-s}\sigma_n(X_s) - T^n_{t-s}\sigma(X_s)\|^2_{HS} ds = \mathbb{E} \int_0^t \| (\mathcal{P} - \text{Id}) T^n_{t-s}\sigma(X_s)\|^2_{HS} ds \]
(2.9)
\[ = \mathbb{E} \int_0^t \sum_{m=1}^{\infty} \sum_{k=n+1}^{\infty} e^{-2\lambda_k(t-s)} \langle \sigma(X_s) e_m, e_k \rangle^2 ds. \]

It follows from (2.8) and (2.9) that
\[ \mathbb{E} \int_0^t \|T^n_{t-s}\sigma_n(X^n_s) - T^n_{t-s}\sigma(X_s)\|^2_{HS} ds \]
(2.10)
\[ \leq 2\mathbb{E} \int_0^t K_{\sigma}(t-s)|X^n_s - X_s|^2 ds + 2\mathbb{E} \int_0^t \sum_{m=1}^{\infty} \sum_{k=n+1}^{\infty} e^{-2\lambda_k(t-s)} \langle \sigma(X_s) e_m, e_k \rangle^2 ds. \]
Putting (2.5), (2.6), (2.7), (2.10) together we arrive at

\[ \mathbb{E}[|X_t^n - X_t|^2] \]

(2.11)

\[ \leq C \{ a_n + c_n(t), d_n(t) \} + C \int_0^t (K_\sigma(t-s) + K_b(t-s)) \mathbb{E}[X_s^n - X_s]^2 ds \]

for some constant \( C > 0 \)

and

\[ a_n := \mathbb{E} \sum_{k=n+1}^{\infty} \langle X_0, e_k \rangle^2, \]

\[ c_n(t) := \mathbb{E} \int_0^t \sum_{k=n+1}^{\infty} e^{-2\lambda_k(t-s)} \langle b(X_s), e_k \rangle^2 ds, \]

\[ d_n(t) := \mathbb{E} \int_0^t \sum_{m=1}^{\infty} \sum_{k=n+1}^{\infty} e^{-2\lambda_k(t-s)} \langle \sigma(X_s)e_m, e_k \rangle^2 ds. \]

By (1.3) and the dominated convergence theorem we see that \( a_n + c_n(t), d_n(t) \to 0 \) as \( n \to \infty \). On the other hand, for any \( T > 0 \) our assumptions imply (see [4]) that

(2.12)

\[ \sup_{n \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}[X_t^n]^2 + \sup_{0 \leq t \leq T} \mathbb{E}[X_t]^2 < \infty. \]

So, the function

\[ g(t) := \limsup_{n \to \infty} \mathbb{E}[X_t^n - X_t]^2, \quad t \geq 0 \]

is locally bounded. We will complete the proof of the theorem by showing \( g(t) = 0 \). Taking \( \limsup \) in (2.11) we obtain

(2.13)

\[ g(t) \leq C \int_0^t (K_\sigma(t-s) + K_b(t-s)) g(s) ds. \]

Given any \( \beta > 0 \). Multiplying (2.13) by \( e^{-\beta t} \) and integrating from 0 to \( T \) we get

(2.14)

\[ \int_0^T g(t)e^{-\beta t} dt \leq C \int_0^T dt e^{-\beta t} \int_0^t (K_\sigma(t-s) + K_b(t-s)) g(s) ds \]

\[ = C \int_0^T e^{-\beta s} g(s) ds \int_s^T (K_\sigma(t-s) + K_b(t-s)) e^{-\beta(t-s)} dt \]

\[ = C \int_0^T e^{-\beta s} g(s) ds \int_0^{T-s} (K_\sigma(u) + K_b(u)) e^{-\beta u} du \]

\[ \leq \left( C \int_0^T (K_\sigma(u) + K_b(u)) e^{-\beta u} du \right) \int_0^T e^{-\beta s} g(s) ds. \]

Choosing \( \beta > 0 \) sufficiently big so that \( C \int_0^T (K_\sigma(u) + K_b(u)) e^{-\beta u} du < 1 \), we deduce from (2.14) that \( \int_0^T g(t)e^{-\beta t} dt = 0 \) and hence \( g(t) = 0 \), a.e. By virtue of (2.13), we further conclude \( g(t) = 0 \) for every \( t \in [0, T] \), and thus finish the proof. \( \square \)
3 Proof of Theorem 1.1

According to Theorem 2.1 and using the monotone class theorem, it would be sufficient to prove Theorem 1.1 for the finite-dimensional setting, i.e. to prove the following result.

**Theorem 3.1.** Let $\mathbb{H} = \mathbb{R}^n$ and assume that (A1) and (A3) hold.

1. For any $f \in C^1_b(\mathbb{R}^n)$,
   $$|\nabla P_t f|^2 \leq 6^{1+\frac{r}{v_0}} P_t |\nabla f|^2, \quad t \geq 0.$$

2. If (A2) holds, then for any strictly positive $f \in \mathcal{B}_b(\mathbb{R}^n)$,
   $$P_t \log f(y) \leq \log P_t f(x) + \frac{3 \log 6}{\lambda(\sigma)t_0(1-6^{-\frac{r}{v_0}})} |x-y|^2, \quad x, y \in \mathbb{R}^n, t > 0.$$

3. If (A2) holds, then for any $f \in \mathcal{B}_b(\mathbb{R}^n)$,
   $$|\nabla P_t f|^2 \leq \frac{3 \log 6}{t_0 \lambda(\sigma)(1-6^{-\frac{r}{v_0}})} \{P_t f^2 - (P_t f)^2\}, \quad t > 0.$$

4. If $|\sigma v|^2 \leq \bar{\lambda}(\sigma)|v|^2$ holds for some constant $\bar{\lambda}(\sigma) > 0$ and all $v \in \mathbb{R}^n$, then
   $$P_t f^2 - (P_t f)^2 \leq \frac{12 \bar{\lambda}(\sigma)t_0(6^{-\frac{r}{v_0}} - 1)}{\log 6} P_t |\nabla f|^2, \quad f \in C^1_b(\mathbb{R}^n), t \geq 0.$$

**Proof.** In the present finite-dimensional setting, (A1) and (A3) imply that $b$ and $\sigma$ are Lipschitz continuous. By a standard approximation argument we may and do assume that they are smooth with bounded gradients, such that

$$\sup_{s \in [0,t]} E|\nabla_v X_s|^2 < \infty, \quad t \geq 0, \quad v \in \mathbb{R}^n,$$

where $\nabla_v$ denotes the directional derivative along $v$. In this case the derivative process

$$\nabla_v X_t := \lim_{\varepsilon \downarrow 0} \frac{X_{t+\varepsilon v} - X_t}{\varepsilon}$$

solves the equation

$$d\nabla_v X_t = \left\{ A \nabla_v X_t + (\nabla_{\nabla_v X_t} b)(X_t) \right\} dt + (\nabla_{\nabla_v X_t} \sigma)(X_t) dW_t, \quad \nabla_v X_0 = v.$$

Since $\nabla b$ and $\nabla \sigma$ are bounded, this implies that

$$\sup_{s \in [0,t]} E|\nabla_v X_s|^2 < \infty, \quad t \geq 0.$$
We aim to find an upper bound of $\mathbb{E}|\nabla v X_t|^2$ independent of the dimension $n$ so that it can be passed to the infinite-dimensional setting. To this end, let us observe that for any $s_0 \geq 0$ we have
\[
\nabla v X_t = T_{t-s_0} \nabla v X_{s_0} + \int_{s_0}^t T_{t-s} (\nabla \nabla v X_s b)(X_s) ds + \int_{s_0}^t T_{t-s} (\nabla \nabla v X_s \sigma)(X_s) dW_s, \quad t \geq s_0.
\]
Combining this with (3.1) we obtain
\[
\mathbb{E}|\nabla v X_t|^2 \leq 3 \mathbb{E}|\nabla v X_{s_0}|^2 + 3 \int_{s_0}^t K_b(s-s_0) \mathbb{E}|\nabla v X_s|^2 ds + 3 \int_{s_0}^t K_\sigma(s-s_0) \mathbb{E}|\nabla v X_s|^2 ds \\
\leq 3 \mathbb{E}|\nabla v X_{s_0}|^2 + \{3\phi_b(t-s_0) + 3\phi_\sigma(t-s_0)\} \sup_{s \in [s_0,t]} \mathbb{E}|\nabla v X_s|^2
\]
for $t \geq s_0$. Since the resulting upper bound is increasing in $t \geq s_0$, it follows that
\[
\sup_{s \in [s_0,t]} \mathbb{E}|\nabla v X_s|^2 \leq 3 \mathbb{E}|\nabla v X_{s_0}|^2 + \{3\phi_b(t-s_0) + 3\phi_\sigma(t-s_0)\} \sup_{s \in [s_0,t]} \mathbb{E}|\nabla v X_s|^2
\]
holds for $t \geq s_0$. Taking $t = s_0 + t_0$ in this inequality leads to
\[
\sup_{s \in [s_0, s_0 + t_0]} \mathbb{E}|\nabla v X_s|^2 \leq 6 \mathbb{E}|\nabla v X_{s_0}|^2, \quad s_0 \geq 0.
\]
Therefore,
\[
(3.2) \quad \mathbb{E}|\nabla v X_t|^2 \leq 6 ^t_0 |v|^2, \quad t \geq 0, v \in \mathbb{R}^n.
\]
With this estimate in hand, we are able to complete the proof easily.
(1) follows from (3.2) and the Schwarz inequality, more precisely
\[
|\nabla v P_t f|^2 = |\nabla v \mathbb{E} f(X_t)|^2 = |\mathbb{E} \langle \nabla f(X_t), \nabla v X_t \rangle|^2 \leq 6 ^t_0 |v|^2 |\nabla f|^2.
\]
(2) follows from (1) according to the argument in [12], see Proposition 3.2 below for details.
(3) follows from (1) by noting that
\[
\frac{d}{ds} P_s (P_{t-s} f)^2 = 2P_s |\sigma \nabla P_{t-s} f|^2 \geq 2\lambda(\sigma) P_s |\nabla P_{t-s} f|^2 \geq 2\lambda(\sigma) 6 ^{t-s}_0 |\nabla f|^2, \quad s \in [0, t].
\]
(4) follows from (1) since
\[
\frac{d}{ds} P_s (\bar{P}_{t-s} f)^2 = 2\bar{P}_s |\sigma \nabla \bar{P}_{t-s} f|^2 \leq 2\bar{\lambda}(\sigma) P_s |\nabla \bar{P}_{t-s} f|^2 \leq 2\bar{\lambda}(\sigma) 6 ^{t-s}_0 P_1 |\nabla f|^2, \quad s \in [0, t].
\]
\[\square\]
Proposition 3.2. If there exists a positive function $\Phi \in C([0, \infty))$ such that
\[
|\nabla P_t f|^2 \leq \Phi(t) P_t |\nabla f|^2, \quad t \geq 0, f \in C^1_b(\mathbb{R}^n),
\]
then for any strictly positive $f \in \mathcal{B}_b(\mathbb{R}^n)$,
\[
P_t \log f(x) \leq \log P_t f(y) + \frac{|x-y|^2}{2\lambda(\sigma) \int_0^t \Phi(s)^{-1} ds}, \quad t > 0, x, y \in \mathbb{R}^n.
\]

Proof. For fixed $x, y \in \mathbb{R}^n, t > 0$ and $h \in C^1([0, t]; \mathbb{R})$ with $h_0 = 0$ and $h_t = 1$, let $x_s = (x-y)h_s + y$, $s \in [0, t]$. Combining (3.3), (A2) and (see (2.3) in [12])
\[
\frac{d}{ds} P_s \log P_{t-s} f = -\frac{1}{2} P_s |\sigma \nabla \log P_{t-s} f|^2,
\]
we obtain
\[
P_t \log f(x) - \log P_t f(y) = \int_0^t \frac{d}{ds} [(P_s \log P_{t-s} f)(x_s)] ds
\]
\[
= \int_0^t \left\{ h'_s (x-y, \nabla P_s \log P_{t-s} f) - \frac{1}{2} P_s |\sigma \nabla \log P_{t-s} f|^2 \right\} (x_s) ds
\]
\[
\leq \int_0^t \left\{ |h'_s| \cdot |x-y| \cdot |\nabla P_s \log P_{t-s} f| - \frac{\lambda(\sigma)}{2\Phi(s)} |\nabla P_s \log P_{t-s} f|^2 \right\} (x_s) ds
\]
\[
\leq \frac{|x-y|^2}{2\lambda(\sigma)} \int_0^t \Phi(s)(h'_s)^2 ds.
\]
Taking
\[
h_s = \frac{\int_0^s \Phi(u)^{-1} du}{\int_0^t \Phi(u)^{-1} du}, \quad s \in [0, t],
\]
we complete the proof. \hfill \Box

Proof of Theorem 1.7. Let $P^n_t$ be the semigroup for $X^n_t$ solving the equation (2.2). By Theorem 2.1 we have
\[
P^n_t f(x) = \lim_{n \to \infty} P^n_t f(P^n_n x), \quad t \geq 0, f \in C_b(\mathbb{H}).
\]
Let $f \in C^1_b(\mathbb{H})$. It is easy to see that if (Ai) $(1 \leq i \leq 3)$ holds for $A, \sigma, b$ on $\mathbb{H}$, it also holds for $A_n, \sigma_n, b_n$ on $\mathbb{H}_n$. By Theorem 3.1 we have
\[
\frac{|P^n_t f(P^n_n x) - P^n_t f(P^n_n y)|^2}{|x-y|^2} \leq 6^{1+\frac{2}{\sigma_0}} \int_0^1 (P^n_t |\nabla f|^2)(sP^n_n x + (1-s)P^n_n y) ds, \quad x \neq y.
\]
Letting $n \to \infty$ and using (3.4), we arrive at
\[
\frac{|P_t f(x) - P_t f(y)|^2}{|x-y|^2} \leq 6^{1+\frac{2}{\sigma_0}} \int_0^1 (P_t |\nabla f|^2)(sx + (1-s)y) ds, \quad x \neq y.
\]
This is equivalent to the gradient inequality in (1). (4) can be proved similarly. By the same reason, it is easy to see that the inequalities in (2) and (3) hold for all $f \in C_b(\mathbb{H})$. Noting that the inequality in (3) is equivalent to

$$
\frac{|P_t f(x) - P_t f(y)|^2}{|x - y|^2} \leq \frac{3 \log 6}{t_0 \lambda(\sigma)(1 - 6^{-\frac{1}{d}})} \int_0^1 \{P_t f^2 - (P_t f)^2\} (sx + (1 - s)y) ds, \quad x \neq y,
$$

by the monotone class theorem, if inequalities in (2) and (3) hold for all $f \in C_b(\mathbb{H})$, they also hold for all $f \in \mathcal{B}_b(\mathbb{H})$.

\[ \square \]

## 4 Application to white noise driven SPDEs

In this section, we will apply our results to stochastic reaction-diffusion equations driven by space-time white noise which are extensively studied in the literature, see [5] and references therein.

Consider the stochastic reaction-diffusion equation on a bounded closed domain $D \subset \mathbb{R}^d (d \geq 1)$:

$$
\begin{align*}
\frac{\partial u_t(\xi)}{\partial t} & = -(-\Delta)^\alpha u_t(\xi) + \psi(u_t(\xi)) + \phi(u_t(\xi)) \frac{\partial^{1+d}}{\partial \xi_1 \cdots \partial \xi_d} W(t, \xi), \\
u_0 & = g, \quad u_t|_{\partial D} = 0, \quad \xi = (\xi_1, \cdots, \xi_d) \in D,
\end{align*}
$$

(4.1)

where $\alpha > 0$ is a constant, $W(t, \xi)$ is a Brownian sheet on $\mathbb{R}^{d+1}$, $\Delta$ is the Dirichlet Laplacian on $D$, and $\phi, \psi$ are Lipschitz functions on $\mathbb{R}$, i.e. there exists a constant $C > 0$ such that

$$
|\psi(r) - \psi(s)| \leq c|r - s|, \quad |\phi(r) - \phi(s)| \leq c|r - s|, \quad r, s \in \mathbb{R}.
$$

The equivalent integral equation is (see [19])

$$
\begin{align*}
\begin{aligned}
u_t(\xi) & = T_t g(\xi) + \int_{[0,t] \times D} T_{t-s}(\xi, \eta) \psi(u_s(\eta)) ds d\eta \\
& + \int_{[0,t] \times D} T_{t-s}(\xi, \eta) \phi(u_s(\eta)) W(ds, d\eta), \quad t \geq 0.
\end{aligned}
\end{align*}
$$

(4.3)

where $T_t, T_t(x, \eta)$ are the semigroup and the heat kernel associated with the Dirichlet Laplacian on $D$.

To apply our main results to the present model, we reformulate the equation by using the cylindrical Brownian motion on $\mathbb{H} := L^2(D)$. Let $A = -(-\Delta)^\alpha$. Then $-A$ has discrete spectrum with eigenvalues $\{\lambda_n\}_{n \geq 1}$ satisfying

$$
\frac{n^{2\alpha/d}}{C^2} \leq \lambda_n \leq C n^{2\alpha/d}, \quad n \geq 1
$$

(4.4)

for some constant $C > 1$. Let $\{e_n\}_{n \geq 1}$ be the corresponding unit eigenfunctions. Since $e_m$ is independent of $\alpha$, letting $\alpha = 1$ and using the classical Dirichlet heat kernel bound, we obtain

$$
\|e_m\|_\infty = e\|T_{h_m}^{-1} e_m\|_\infty \leq e\|T_{h_m}^{-1}\|_{L^2 \to L^\infty} \leq c_1 \lambda_m^{d/4} \leq c_2 \sqrt{m}, \quad m \geq 1
$$

(4.5)
for some constants $c_1, c_2 > 0$.

Now, define a sequence of independent Brownian motions by

$$
\beta_n(t) = \int_{[0,t] \times \mathbb{D}} e_n(\eta) W(ds, d\eta), \quad n \geq 1.
$$

Then

$$
W_t := \sum_{n=1}^{\infty} \beta_n(t)e_n
$$

is a cylindrical Brownian motion on $\mathbb{H}$. Let

$$
b(u)(\xi) = \psi(u(\xi)), \quad \{\sigma(u)x\}(\xi) = \phi(u(\xi)) \cdot x(\xi), \quad u, x \in \mathbb{H}, \xi \in D.
$$

It is easy to see that the reaction-diffusion diffusion equation (4.1) can be reformulated as

$$
du_t = Au_t dt + b(u_t) dt + \sigma(u_t) dW_t.
$$

Obviously, $\sigma$ takes values in the space of bounded linear operators on $\mathbb{H}$ if and only if $\phi$ is bounded. So, in general $\sigma$ is not a Hilbert-Schmidt perturbation of any bounded linear operator as indicated in the Introduction.

**Theorem 4.1.** Let $A, b$ and $\sigma$ be given above such that $\phi^2 \geq \lambda$ for some constant $\lambda > 0$.

1. If $\alpha > d$ then conditions (A1)-(A4) hold for $\lambda(\sigma) = \lambda, K_b \equiv C$ and

$$
K_\sigma(t) = C \sum_{m=1}^{\infty} m e^{-\delta t m^{2\alpha/d}}
$$

for some constants $C, \delta > 0$, so that Theorem 1.1 and Corollary 1.2 apply to the semigroup associated to solutions of equation (4.1).

2. Let $D = \prod_{i=1}^{d}[a_i, b_i]$ for some $b_i > a_i, 1 \leq i \leq d$. If $\alpha > d / 2$ then (A1)-(A4) hold for $\lambda(\sigma) = \lambda, K_b \equiv C$ and

$$
K_\sigma(t) = C \sum_{m=1}^{\infty} e^{-\delta t m^{2\alpha/d}}
$$

for some constants $C, \delta > 0$, so that Theorem 1.1 and Corollary 1.2 apply to the semigroup associated to solutions of equation (4.1).

3. In the situations of (1) and (2), there exists a constant $\varepsilon_0 > 0$ such that $P_t$ has a unique invariant probability measure provided

$$
|\phi(s)|^2 + |\psi(s)|^2 \leq \varepsilon_0 |s|^2 + C_0, \quad s \in \mathbb{R}
$$

holds for some constant $C_0 > 0$. 

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Proof. Since (A4) is obvious due to (4.4) and (A2) with $\lambda(\sigma) = \lambda$ follows from $\phi^2 \geq \lambda$, it suffices to verify (A1) and (A3) for the desired $K_b$ and $K_\sigma$. By the contraction of $T_t$ and (4.2), we have

$$|T_t(b(x) - b(y))| \leq c|x - y|, \quad x, y \in \mathbb{H},$$

where $|\cdot|$ is now the $L^2$-norm on $D$. Then (A1) holds for $K_b \equiv c^2$.

Below, we verify (A3) and the existence of the invariant probability measure respectively.

(1) By the definition of $\sigma$, (4.2) and (4.5), we have

$$\|T_t(\sigma(x) - \sigma(y))\|^2_{HS} = \sum_{n=1}^{\infty} \|T_t(\sigma(x) - \sigma(y))e_n\|^2$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle T_t(\sigma(x) - \sigma(y))e_n, e_m \rangle^2$$

$$= \sum_{m=1}^{\infty} e^{-2t\lambda_m} \sum_{n=1}^{\infty} \langle (\sigma(x) - \sigma(y))e_n, e_m \rangle^2$$

$$= \sum_{m=1}^{\infty} e^{-2t\lambda_m} \int_D \left| (\sigma(x) - \sigma(y))^* e_m \right|^2 d\xi$$

$$= \sum_{m=1}^{\infty} e^{-2t\lambda_m} \int_D \left| (\phi(x(\xi)) - \phi(y(\xi))) e_m(\xi) \right|^2 d\xi$$

$$\leq c^2 |x - y|^2 \sum_{m=1}^{\infty} \|e_m\|^2_\infty e^{-\delta tm^2\alpha/d}.$$  

for some constant $\delta > 0$. Combining this with (4.4) we obtain

$$\|T_t(\sigma(x) - \sigma(y))\|^2_{HS} \leq C \sum_{m=1}^{\infty} m e^{-\delta tm^2\alpha/d}$$

for some constant $C > 0$. Moreover,

$$\int_0^t \|T_s\sigma(0)\|^2_{HS} = \phi(0)^2 \int_0^t \|T_s\|^2_{HS} ds \leq C' \sum_{m=1}^{\infty} e^{-\delta tm^2\alpha/d}$$

holds for some constant $C' > 0$. Therefore, if $\alpha > d$ then (A3) holds for $K_\sigma$ given in (4.6) since in this case

$$\int_0^\infty \sum_{m=1}^{\infty} m e^{-\delta tm^2\alpha/d} dt = \frac{1}{\delta} \sum_{m=1}^{\infty} \frac{1}{m^{(2\alpha-d)/d}} < \infty.$$

(2) When $D = \prod_{i=1}^d [a_i, b_i]$ for some $b_i > a_i, 1 \leq i \leq d$, the eigenfunctions $\{e_m\}_{m \geq 1}$ are uniformly bounded, i.e. $\|e_m\|_\infty \leq C$ holds for some constant $C > 0$ and all $m \geq 1$. Combining this with (4.4), we obtain

$$\|T_t(\sigma(x) - \sigma(y))\|^2_{HS} \leq C |x - y|^2 \sum_{m=1}^{\infty} e^{-\delta tm^2\alpha/d}$$
for some constants $C, \delta > 0$. Combining this with (4.10), we conclude that (A3) holds for $K_{\sigma}$ given in (4.7) provided $\alpha > \frac{d}{2}$.

(3) The uniqueness of the invariant probability measure follows from Corollary 1.2(3), it suffices to prove the existence by verifying conditions (i)-(iv) in [5, Theorem 6.1.2]. By (4.2), (4.4) and (A3), conditions (i) and (iii) hold. It remains to verify condition (ii), i.e.

$$\int_0^1 s^{-\varepsilon}K_{\sigma}(s)ds < \infty$$

for some $\varepsilon \in (0, 1)$;

and condition (iv), which is implied by

$$\sup_{t \geq 0} \mathbb{E}|u_t|^2 < \infty.$$ 

Let $K_{\sigma}$ be in (4.6) with $\alpha > d$. Then for $\varepsilon \in (0, \frac{a-d}{a})$,

$$\int_0^1 s^{-\varepsilon}K_{\sigma}(s)ds = C \sum_{m=1}^{\infty} m \int_0^1 s^{-\varepsilon}e^{-\delta sm^{2\alpha/d}}ds$$

$$\leq C \sum_{m=1}^{\infty} m \int_0^{m^{-2\alpha/d}} s^{-\varepsilon}ds + m^{2\alpha\varepsilon/d} \int_{m^{-2\alpha/d}}^1 e^{-\delta sm^{2\alpha/d}}ds$$

$$\leq C(\varepsilon) \sum_{m=1}^{\infty} m^{1-2(1-\varepsilon)\alpha/d} < \infty,$$

where $C(\varepsilon) > 0$ is a constant depending on $\varepsilon$. Similarly, (4.11) holds for $K_{\sigma}$ in (4.7) with $\alpha > \frac{d}{2}$ and $\varepsilon \in (0, \frac{2\alpha-d}{2\alpha})$.

Next, by (4.8) we have

$$\mathbb{E}|u_t|^2 \leq C_1|g|^2 + C_1 \int_0^t \sum_{m=1}^{\infty} \|e_m\|^2_{\infty}e^{-2\lambda_m(t-s)}(C_0 + \varepsilon_0 \mathbb{E}|u_s|^2)ds$$

$$\leq C_1|g|^2 + C_2 \left(C_0 + \varepsilon_0 \sup_{s \in [0,t]} \mathbb{E}|u_s|^2\right) \int_0^t K_{\sigma}(s)ds, \ t \geq 0$$

for some constants $C_1, C_2 > 0$, where $K_{\sigma}$ is in (4.6) with $\alpha > d$ or in (4.7) with $\alpha > \frac{d}{2}$ such that $\int_0^{\infty} K_{\sigma}(s)ds < \infty$ as observed above. So, there exist constants $C_3, C_4 > 0$ such that

$$\mathbb{E}|u_t|^2 \leq C_3 + C_4\varepsilon_0 \sup_{s \in [0,t]} \mathbb{E}|u_s|^2, \ t \geq 0.$$ 

Taking $\varepsilon_0 = \frac{1}{2C_4}$, we obtain

$$\sup_{s \in [0,t]} \mathbb{E}|u_s|^2 \leq C_3 + C_4\varepsilon_0 \sup_{s \in [0,t]} \mathbb{E}|u_t|^2 \leq C_3 + \frac{1}{2} \sup_{s \in [0,t]} \mathbb{E}|u_s|^2, \ t \geq 0.$$ 

Since by (2.12) $\sup_{s \in [0,t]} \mathbb{E}|u_s|^2 < \infty$, this implies (4.12).
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