Consistent holographic description of boost-invariant plasma

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Prior attempts to construct the gravity dual of boost-invariant flow of $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory plasma suffered from apparent curvature singularities in the late time expansion. This Letter shows how these problems can be resolved by a different choice of expansion parameter. The calculations presented correctly reproduce the plasma energy-momentum tensor within the framework of second order viscous hydrodynamics.

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Introduction

Gauge/string theory duality [1] has proved to be a valuable tool in describing the properties of strongly coupled gauge theory at finite temperature. Most activity has focused on static situations or linearized dynamics. Only recently some progress was made in the studies of strongly coupled nonlinear gauge theory dynamics using string theory methods [2, 3, 4, 5, 6, 7, 8, 9, 10]. Such studies are currently of great practical importance due to experimental investigations of strongly coupled QCD plasma at RHIC and soon at the LHC [13]. While the AdS/CFT correspondence is well understood only in the supersymmetric setting, it is believed that this case can capture some essential features of real-world QCD above the deconfinement temperature.

Experimental data suggest that a boost invariant description of the expansion of the fireball seen at RHIC should provide a useful model. This is based on Bjorken’s observation [14] that multiplicity spectra when expressed in proper time and rapidity variables are approximately independent of rapidity in the mid-rapidity region. The pioneering work of Janik and Peschanski [2] established the gravity dual of the boost-invariant flow of $\mathcal{N} = 4$ plasma in the regime of large proper time. These authors showed that asymptotic large time behavior of gauge theory plasma, when matter is locally well equilibrated, is given by hydrodynamics. The large proper time expansion is equivalent to a gradient expansion of hydrodynamics. In a series of follow-up papers [3, 4, 5] sub-asymptotic corrections were calculated, corresponding to various dissipative terms in the hydrodynamical description. These results were of great interest to the heavy ion community, because they provided numerical values of strongly coupled gauge theory transport coefficients starting from first principles. Obtaining the energy-momentum tensor of boost-invariant flow up to second order also helped to establish the correct theory of causal conformal hydrodynamics [15].

The approach explored in [2, 3] relied entirely on demanding that coefficients in the expansion of curvature invariants

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in powers of inverse proper time should be regular. This determined the viscosity coefficient in a way consistent with results obtained by other means [16] and for the first time provided information about second order viscous hydrodynamics for strongly coupled plasma [5]. However, this regularity condition turned out to be violated at third order in the large proper time expansion [3]. It was suggested [6, 7] that the singularities encountered cannot be canceled within the supergravity approximation and they indicate either the need for additional string theory degrees of freedom, or a genuine instability. The intention of this Letter is to readdress this issue.

Recently another framework describing gauge theory plasma hydrodynamics was developed [9], where the gravity dual is determined order by order in a gradient expansion starting from a locally boosted black brane geometry. This approach utilizes generalized incoming Eddington-Finkelstein coordinates and yields a manifestly regular metric [3, 9, 10, 11, 12]. In [3] the energy-momentum tensor was found explicitly up to second order in the gradient expansion; this approach utilizes generalized incoming Eddington-Finkelstein coordinates and yields a manifestly regular metric dual is determined order by order in a gradient expansion starting from a locally boosted black brane geometry.

Boost-invariant flow from the black brane solution

Bjorken expansion [14] of $N = 4$ SYM plasma is a one-dimensional flow with boost invariance along the expansion axis and rotational and translational symmetries in the perpendicular plane [14]. Proper time $\tau$ and rapidity $y$ are related to the usual lab-frame coordinates by $x^0 = \tau \cosh y$, $x^1 = \tau \sinh y$. Proper time $\tau$ is invariant under boosts along the collision axis, whereas rapidity $y$ is not. Thus boost invariance implies that physical quantities can depend only on proper time $\tau$, not on rapidity $y$.

Following the ideas of [3], boost-invariant perfect fluid flow can be obtained locally from the 5-dimensional boosted black brane solution

$$ds^2 = -2u_\mu dx^\mu dr - r^2 \left(1 - \frac{1}{b^4} r^4\right) u_\mu u_\nu dx^\mu dx^\nu + r^2 (\eta_{\mu\nu} + u_\mu u_\nu) dx^\mu dx^\nu,$$

where $u^\mu$ is the boost velocity parameter and $b$ is a dilatation parameter related to the black brane temperature $T$ by $b = 1/\pi T$. The key ingredient of this approach is the introduction the incoming Eddington-Finkelstein time coordinate $\tilde{\tau}$. At the boundary this coordinate reduces to the usual Minkowski time. Gauge/gravity duality maps physics at the boundary to the bulk of the AdS spacetime. For an effective description at long wavelengths (relativistic hydrodynamics) the map is provided by decreasing $r$ while keeping the Eddington-Finkelstein time coordinate fixed [10]. For a Bjorken expansion it is natural to use proper time $\tau$ instead of the usual Minkowski time, so an analogous Eddington-Finkelstein type proper time coordinate $\tilde{\tau}$ is introduced. Specifically, $u = \partial_{\tau}$ at the boundary, but is now taken as $u = \partial_\tilde{\tau}$ in the bulk of AdS space. Furthermore, for a boost invariant flow the temperature $T$ is asymptotically proportional to $\tau^{-1/3}$, which translates to $b = 3^{1/4} 2^{-1/2} \tau^{1/3}$ (to ensure agreement with [2]). Thus finally

$$ds^2 = -r^2 (1 - \frac{4}{3^{3/4} \tau^4} \tilde{\tau}) d\tilde{\tau}^2 + 2d\tilde{\tau} dr + r^2 \tilde{\tau}^2 dy^2 + r^2 dz_1^2. \quad (2)$$

It is straightforward to verify that metric (2) is related to the Janik-Peschanski metric [2]

$$ds_{JP}^2 = \frac{1}{z^2} \left( \frac{1 - \frac{b^4}{3^{3/4} \tau^4} \tilde{\tau}^2}{1 + \frac{b^4}{3^{3/4} \tau^4}} \right) d\tilde{\tau}^2 + \tilde{\tau}^2 \left(1 + \frac{\tilde{\tau}^4}{3^{3/4} \tau^{4/3}}\right) dy^2 + \left(1 + \frac{\tilde{\tau}^4}{3^{3/4} \tau^{4/3}}\right) dz_1^2 + dz_2^2, \quad (3)$$

by a coordinate transformation

$$\tilde{\tau} = \tau \left(1 - \frac{1}{\tau^{2/3}} \left[\frac{3^{1/4}}{4\sqrt{2}} + \frac{3^{1/4}}{2\sqrt{2}} \arctan \left(\frac{3^{1/4}}{\sqrt{2}} r \cdot \tau^{1/3}\right) + \frac{3^{1/4}}{4\sqrt{2}} \log \frac{r \cdot \tau^{1/3} - \sqrt{2}}{r \cdot \tau^{1/3} + \sqrt{2}}\right]\right)^{1/2}, \quad \frac{1}{z} \sqrt{1 + \frac{z^4}{3 \cdot \tau^{4/3}}}, \quad (4)$$
Because of the nontrivial dependence of $\tilde{\tau}$ on $r$ the limits $\tau \to \infty$ and $\tilde{\tau} \to \infty$ (corresponding to equilibration of gauge theory plasma) differ in the bulk. Moreover the metric \([2]\) is regular and invertible up to the black brane singularity, which is not the case with \([3]\). These observations are crucial for the present approach to work. Note also that the relation between $\tau$ and $\tilde{\tau}$ is singular when $z = 3^{1/4} r^{1/3}$. This is precisely the locus where the singularities found in \([3]\) were encountered.

The metric \([2]\) is not an exact solution of Einstein equations $R_{MN} + 4G_{MN} = 0$ – there are subleading corrections coming from derivatives of the velocity $u$ and temperature $T$. They correspond to the gradient expansion of the boundary energy momentum tensor \([17]\). For a boost-invariant flow the energy-momentum tensor $< T_{\mu\nu} >$ is determined by the energy density $\epsilon(\tau)$ \([2]\). In the gradient expansion each covariant derivative $\nabla u$ is damped by $\frac{1}{\tilde{\tau}}$ ($L$ is the characteristic length scale of a perturbation and $T$ is the fluid temperature). Because boost-invariant flow is characterized by $T \sim \tau^{-1/3}$ and more, $\nabla u \sim \tau^{-1}$, we expect the following expansion of the energy density $\epsilon(\tau)$ \([17]\):

$$\epsilon(\tau) = \frac{1}{\tau^{4/3}} \left\{ 1 - 2\eta_0 \frac{1}{\tau^{2/3}} + \left( \frac{3}{2} \eta_0^2 - \frac{2}{3}\eta_0 \tau_0^0 + \frac{2}{3} \lambda_0^0 \right) \frac{1}{\tau^{4/3}} + \ldots \right\},$$

where $\eta_0$ and $\tau_0^0$ are the viscosity and relaxation time coefficients, whereas $\lambda_0^0$ is a new transport coefficient introduced in \([15]\). The above expansion is written in terms of $\tau$, since the energy density is defined at the boundary.

The apparent curvature singularities in AdS encountered in \([3]\) appear at third order in the large $\tau$ (gradient) expansion. It is difficult to check what happens for a general flow at this order. However the situation is much simpler for a boost-invariant flow, since all the symmetries can be imposed from the outset. This leads to the following ansatz for the metric \([21]\)

$$ds^2 = G_{MN} dx^M dx^N = -r^2 N(\tilde{\tau}, r) d\tilde{\tau}^2 + 2d\tilde{r} dr + r^2 \tau^2 e^{b(\tilde{\tau}, r)} dy^2 + r^2 e^{c(\tilde{\tau}, r)} dx_\perp^2.$$  

Introducing the scaling variable $v = r \cdot \tilde{\tau}^{1/3}$ in analogy with what is done in \([2, 3, 4, 5, 6, 7, 8]\) one obtains the natural expansion of the metric components in $\tilde{\tau}^{-2/3}$ on the gravity side

$$N(\tilde{\tau}, r) = A(v) \exp \left( \sum_{k>0} a_k(v) \tilde{\tau}^{-2k/3} \right),$$

$$e^{b(\tilde{\tau}, r)} = B(v) \exp \left( \sum_{k>0} b_k(v) \tilde{\tau}^{-2k/3} \right),$$

$$e^{c(\tilde{\tau}, r)} = C(v) \exp \left( \sum_{k>0} c_k(v) \tilde{\tau}^{-2k/3} \right).$$

Note that the scaling variable $v$ introduced here is different from the one used in \([2, 3, 4, 5, 6, 7, 8]\), and so the scaling limit considered here must be regarded as different. To obtain a uniform expansion of the Einstein equations $E_{MN} \equiv R_{MN} + 4G_{MN} = 0$ one needs to rescale them (see \([3]\)) according to $\hat{E} = (\tilde{\tau}^{2/3} E_{\tilde{\tau}\tilde{\tau}}, E_{\tilde{\tau}r}, \frac{1}{\tilde{\tau}^{1/3}} E_{rr}, \frac{1}{\tilde{\tau}^{1/3}} E_{yy}, \tilde{\tau}^{2/3} E_{x_\perp x_\perp})$. This leads to

$$\hat{E}(\tilde{\tau}, r) = \hat{E}_0(r \cdot \tilde{\tau}^{1/3}) + \frac{1}{\tilde{\tau}^{2/3}} \hat{E}_1(r \cdot \tilde{\tau}^{1/3}) + \frac{1}{\tilde{\tau}^{4/3}} \hat{E}_2(r \cdot \tilde{\tau}^{1/3}) + \frac{1}{\tilde{\tau}^{2}} \hat{E}_3(r \cdot \tilde{\tau}^{1/3}) + \ldots$$

The curvature invariants (e.g. $R_{MNOP} R^{MNOP}$) defined recursively in \([6]\) can be likewise expanded. The crucial difference between the present approach and the one introduced in \([2]\) is that the expansion parameter involves $\tilde{\tau}$ instead of $\tau$. Einstein equations can be solved order by order in $\tilde{\tau}^{-2/3}$ expansion starting from

$$A(v) = 1 - \frac{4}{3v^2},$$

$$B(v) = C(v) = 1,$$

which simply reproduces the boosted black brane solution \([2]\). Thus the zeroth order solution entails large but finite $\tilde{\tau}$; it is to be expected that the singularity at $r = 0$ should be shielded by an event horizon, however it is difficult to demonstrate this explicitly \([22]\).
Gravity dual of the gradient expansion

The equations of motion are a system of ordinary second order differential equations for the 3 functions \( a_k(v), b_k(v) \) and \( c_k(v) \). Each solution involves two integration constants. On the other hand, two of the equations of motion are constraints. At each order \( k > 0 \) one of the constraints fixes one of the integration constants appearing at that order, and the other one fixes an integration constant left undetermined at order \( k - 1 \). The 4 remaining integration constants can be fixed by order by imposing metric regularity (up to the usual black brane singularity at \( v = 0 \)). It turns out that the potential singularity is located only at \( v = \sqrt{2}/3^{1/4} \); thus the functions \( b_k(v), c_k(v) \) must remain finite as \( v \to \sqrt{2}/3^{1/4} \). In case of \( a_k(v) \) the requirement should be that the product with \( A(v) \) must be finite. However one can take advantage of residual diffeomorphism invariance preserved by the ansatz, which can effectively be fixed by requiring that \( a_k(v) \) itself be regular. Furthermore asymptotic AdS behavior of the metric requires that these functions vanish as \( v \to \infty \) (in the late proper time regime). These conditions together with the constraints fix 5 of the 6 integration constants at a given order \( k > 0 \) and lead to a regular metric with no poles or logarithmic singularities apart from \( v = 0 \). As an example, the first order solution (dual to viscous hydrodynamics) reads

\[
a_1 = -\frac{2}{3} \cdot \frac{2 \cdot 3^{1/2} - 2^{1/2}3^{1/4}v + v^2}{(2^{1/2}3^{1/4} + v) \cdot (2 \cdot 3^{1/2} + v^2)},
\]

\[
b_1 = \frac{\pi}{\sqrt{23}3^{1/4}} - \frac{\sqrt{2}}{3^{1/4}} \arctan \left( \frac{3^{1/4}}{\sqrt{2}} \right) - \frac{2\sqrt{2}}{3^{1/4}} \log v + \frac{\sqrt{2}}{3^{1/4}} \log \left( \frac{\sqrt{2}}{3^{1/4} + v} \right) + \frac{1}{\sqrt{23}3^{1/4}} \log \left( \frac{2}{3^{1/2} + v^2} \right),
\]

and \( c_1 = -b_1/2 \). Higher order formulae (up to the third order) can be found in a Mathematica notebook available online. Holographic renormalization correctly reproduces the energy density for the boost-invariant flow up to second order in derivatives.

Absence of singularities and relation to Fefferman-Graham coordinates

The assumption of non-singularity of coefficients of curvature invariants in the late proper-time expansion was crucial in establishing transport properties of \( \mathcal{N} = 4 \) SYM plasma in [4,5]. The present approach to boost-invariant flow starts from a manifestly regular metric in the leading order (no logarithmic and power-like singularities at \( v = \sqrt{2}/3^{1/4} \)) and produces regular solutions up to the third order. Since the components of the metric as well as its inverse are regular (as well as their derivatives), all curvature invariants are non-singular. Indeed, from (6) it follows that the non-vanishing components of the inverse are \( G^{rr} = r^2 e^{a(p,r)}, \quad G^{\tau r} = 1, \quad G^{yy} = r^{-2} e^{-b(p,r)}, \quad G^{zz} = r^{-2} e^{-c(p,r)} \). If \( e^{-a(p,r)} \) had been present, singularities would have appeared as a consequence of (6).

It is natural to ask how these results are related to those obtained using the original approach of [2,5]. Clearly, they should be related by a coordinate transformation order by order in the large proper-time expansion:

\[
\begin{align*}
\tilde{r} &= r(t_0 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{2/3}} t_1 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{1/3}} t_2 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{2/3}} t_3 \left( \frac{z}{\tau^{1/3}} \right) + \ldots), \\
r &= \left( \frac{z}{\tau^{1/3}} \right) (r_0 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{2/3}} r_1 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{1/3}} r_2 \left( \frac{z}{\tau^{1/3}} \right) + \frac{1}{\tau^{2/3}} r_3 \left( \frac{z}{\tau^{1/3}} \right) + \ldots).
\end{align*}
\]

It is straightforward (though tedious) to determine this transformation explicitly; the results up to third order are online. This transformation defines a map between the two approaches and explains how the results obtained in [3] were correct despite the singularities which were encountered there.

Summary

This Letter shows how to construct a consistent gravity dual to boost-invariant flow of \( \mathcal{N} = 4 \) SYM plasma in a gradient expansion. This solution reproduces known results up to second order viscous hydrodynamics. A manifestly regular metric up to third order was described and arguments were given why all curvature invariants have non-singular expansions in \( \tau^{-2/3} \). The relation with the approach based on Fefferman-Graham coordinates was made explicit by the coordinate transformation discussed above.

In conclusion, the AdS/CFT dual to boost-invariant flow can be realized within the supergravity approximation and is completely free of singularities apart from the black brane singularity at \( r = 0 \).
Note added: Shortly after the original version of this Letter was posted to arXiv.org a very interesting paper appeared which deals with the same subject and along similar lines as the approach discussed here. Apart from demonstrating the presence of an apparent horizon, the authors also give an all-orders argument for the absence of singularities in the large proper time expansion.

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The coordinates transformation in the third order is checked up to $O(v^2)$, where $v$ is the Fefferman-Graham scaling variable.

http://www.stanford.edu/~headrick/physics/index.html

[20] Very recently it was shown in [18] that an apparent horizon is present.