FROM N=2 SUPERGRAVITY TO CONSTRAINED MODULI SPACES

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Abstract

In this talk we review some results concerning a mechanism for reducing the moduli space of a topological field theory to a proper submanifold of the ordinary moduli space. Such mechanism is explicitly realized in the example of constrained topological gravity, obtained by topologically twisting the N=2 Liouville theory.
Topological field theories \([1]\) represent an amazing joint-venture between mathematics and physics. They can be divided, according to Witten, in two broad classes: the cohomological, or semiclassical theories, whose prototypes are either the topological Yang-Mills theory \([2]\) or the topological \(\sigma\)-model \([3]\) and the quantum theories, whose prototype is the abelian Chern-Simons theory \([4]\).

In this talk we are concerned with cohomological theories. The basic idea is that a generic correlation function of \(n\) physical observables \(\{O_1, \ldots, O_n\}\) has an interpretation as the intersection number

\[
< O_1 O_2 \cdots O_n > = \# (H_1 \cap H_2 \cap \cdots \cap H_n) \tag{1}
\]

of \(n\) homology cycles \(H_i \subset M\) in the moduli space \(M\) of suitable instanton configurations \(\Im[\phi(x)]\) of the basic fields \(\phi\) of the theory.

Topological field theories can been defined in completely geometrical terms. However, in every topological model, the right hand side of equation (1) should admit an independent definition as a functional integral in a suitable Lagrangian quantum field theory, in order to be of physical interest.

We present the idea of reducing the moduli space to a constrained submanifold of the ordinary moduli space and analyze the field theoretical mechanism that implements such a reduction. The results are based on ref. [5]. The specific model that suggested this idea to us comes from the twist of \(N=2\) Liouville theory and it is called by us constrained topological gravity. The physical correlators are intersection numbers in a proper submanifold \(V_{g,s} \subset M_{g,s}\) of the moduli space \(M_{g,s}\) of genus \(g\) Riemann surfaces \(\Sigma_{g,s}\) with \(s\) marked points. \(V_{g,s}\) is defined as follows. Consider the \(g\)-dimensional vector bundle \(E_{\text{hol}} \rightarrow M_{g,s}\), whose sections \(s(m)\) are the holomorphic differentials \(\omega\) on the Riemann surfaces \(\Sigma_{g,s}\), \(m\) denoting the point of the base-manifold \(M_{g,s}\) (i.e. the polarized Riemann surface). Let \(c(E_{\text{hol}}) = \det(1 + \mathcal{R})\) be the total Chern class of \(E_{\text{hol}}\), \(\mathcal{R}\) being the curvature two-form of a holomorphic connection on \(E_{\text{hol}}\). Then \(V_{g,s}\) is the Poincaré dual of the top Chern class \(c_g(E_{\text{hol}}) = \det \mathcal{R}\). \(V_{g,s}\) is a submanifold of codimension \(g\) which can be described as the locus of those Riemann surfaces \(\Sigma_{g,s}(m)\) where some section \(s(m)\) of \(E_{\text{hol}}\) vanishes \([3]\).

Explicitly, the topological correlators of constrained topological gravity are the intersection numbers of the standard Mumford-Morita cohomology classes \(c_1(\mathcal{L}_i)\) on the constrained moduli space, namely

\[
< O_1(x_1) O_2(x_2) \cdots O_n(x_n) > = \int_{V_{g,s}} [c_1(\mathcal{L}_1)]^{d_1} \wedge \cdots \wedge [c_1(\mathcal{L}_n)]^{d_n} = \int_{M_{g,s}} c_g(E_{\text{hol}}) \wedge [c_1(\mathcal{L}_1)]^{d_1} \wedge \cdots \wedge [c_1(\mathcal{L}_n)]^{d_n}. \tag{2}
\]

Precisely, \(c_1(\mathcal{L}_i)\) are the first Chern-classes of the bundles \(\mathcal{L}_i \rightarrow M_{g,s}\) whose sections are elements of the form \(h(m)dz_i\) of the cotangent bundle \(T^*_x \Sigma_g(m)\) at the marked point \(x_i = (z_i, \bar{z}_i)\).
The origin of a constraint on moduli space is due to the presence of the graviphoton in the N=2 graviton multiplet. The graviphoton is initially a physical gauge-field and after the twist maintains zero ghost-number. Nevertheless, in the twisted theory, it is no longer a physical field, rather it is a Lagrange multiplier (in the BRST sense). Indeed, it appears in the right-hand side of the BRST-variation of suitable antighosts, coming from some components of the gravitini. Since this Lagrange multiplier possesses global degrees of freedom (the g moduli of the graviphoton), it imposes g constraints on the space $\mathcal{M}_g$, which is the space of the global degrees of freedom of the metric tensor. The metric tensor, on the other hand, is the only field that remains physical also after twist. We are lead to conjecture that the inclusion of Lagrange multiplier gauge-fields is a general mechanism producing constraints on the moduli spaces.

The gauge-free BRST algebra $B_{\text{gauge-free}}$ is the same as in the Verlinde and Verlinde model [7], based on the gauge group $SL(2, \mathbb{R})$. The flat $SL(2, \mathbb{R})$ connection $\{e^\pm, e^0\}$ contains the zweibein $e^\pm$ and the spin connection of a constant curvature metric on the imaginary upper half-plane $H$. The BRST quantization of the most general continuous deformation of the $SL(2, R)$ connection is derived in the standard way. The (off-shell) gauge-fixing BRST algebra $B_{\text{gauge-fixing}}$, on the other hand, is of the following type

$$s \bar{\psi} = A - d\gamma, \quad sA = -dc, \quad s\gamma = c, \quad sc = 0,$$

where $A$ is the graviphoton, $\bar{\psi}$ is a one-form of ghost number $-1$, coming from the gravitini, $\gamma$ is a zero-form of ghost number 0 and $c$ is the ordinary gauge ghost (with ghost number 1).

The true (complex) dimension of the subspace $\mathcal{V}_{g,s}$ is $\dim_{\mathbb{C}} \mathcal{V}_{g,s} = 2g - 3 + s$, so that the selection rule for (1) to be nonvanishing is

$$\sum_i d_i = 2g - 3 + s,$$

or

$$\sum_i (d_i - 1) = 2g - 3 = \dim_{\mathbb{C}} \mathcal{V}_g.$$

However, the formal (real) dimension of the moduli space turns out to be $4g - 4$, instead of $4g - 6$, so that one has to satisfy

$$\sum_i g_i = 4g - 4,$$

g_i being the ghost number of $O_i$. The fact that the true dimension is smaller than the formal dimension is only apparently puzzling and can be understood as follows. Antighost zero-modes correspond to local vector fields normal to the constrained surface and ghost zero-modes correspond to possible obstructions to the globalization of such local vector fields. As a consequence, the difference, in the constraint sector of the BRST algebra, of antighost zero-modes minus ghost zero-modes, expresses the minimum number of constraints that are imposed. If the potential obstructions do not occur, then all the
antighosts correspond to actual normal directions to the constrained surface and the true
dimension of the constraint surface is smaller than its formal dimension. As one sees,
rules and roles are reversed with respect to the ordinary case.

At the level of conformal field theories there is also a crucial difference between con-
strained and ordinary topological gravity, which keeps trace of the constraint on moduli
space. Indeed, after gauge-fixing and in the limit where the cosmological constant tends
to zero, our model also reduces to the sum of two topological conformal field theories
$Liouville \oplus Ghost$; the central charges, however, are $c_{Liouville} = 6$ and $c_{Ghost} = -6$, rather
than 3 and $-9$.

1 N=2 D=2 supergravity and its twist

We assume that the Lagrangian of N=2 supergravity is the supersymmetrization of the
following Lagrangian of pure gravity,

$$\mathcal{L}_{Liouville} = \Phi(R[g] + a^2) \sqrt{\det g}, \quad (7)$$

$\Phi$ being an independent field (the dilaton). The result is

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \quad (8)$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the kinetic and de Sitter terms, respectively,

$$\begin{align*}
\mathcal{L}_1 &= (X + \bar{X})R - \frac{i}{2}(X - \bar{X})F - 2\lambda_- \rho^- + 2\lambda_+ \rho^+ + 2\bar{\lambda}^- \bar{\rho}^- - 2\bar{\lambda}^+ \bar{\rho}^+
- 4i\bar{M}He^+ e^- + 4iM\bar{H}e^+ e^-,
\mathcal{L}_2 &= (MX + \bar{M}\bar{X})e^+ e^- + \lambda_- \bar{\zeta}_+ + \lambda_+ \bar{\zeta}_- + \bar{\lambda}^- \zeta_+ e^- - \bar{\lambda}^+ \zeta_- e^-
+ \frac{i}{2}X\zeta_+ \bar{\zeta}_+ + \frac{i}{2}\bar{X}\zeta^- \bar{\zeta}_- + 2i(\bar{H} - \bar{H})e^+ e^-.
\end{align*} \quad (9)$$

$\{e^\pm, \zeta^\pm, \bar{\zeta}^\pm, A, M, \bar{M}\}$ is the graviton multiplet, $M$ and $\bar{M}$ being auxiliary fields, and
$\{X, \bar{X}, \lambda^\pm, \bar{\lambda}^\pm, H, \bar{H}\}$ is the dilaton multilet, $H$ and $\bar{H}$ being auxiliary fields. It is easy to
see that using the equation of motions of $H$, $\bar{H}$ and $X + \bar{X}$, we get precisely a de Sitter
supergravity with cosmological constant $\Lambda = \frac{1}{2}$. The field strength $F$, on the other hand,
is set to zero by the $X - \bar{X}$ field equation.

The superalgebra of the graviton multiplet possesses some interesting features that
we do not think have been previously noticed in the literature and that are related to
the peculiar properties of the graviphoton after twist. In particular, one can show that
the off-shell supersymmetry transformations can be found if and only if the gravitini are
$U(1)$ charged. That is why there is a nontrivial $U(1)$ current. After twist, this current
is viewed as a section of $\mathcal{E}_{hol}$ and is set to zero by the global degrees of freedom of the
Lagrange multiplier $A$, thus realizing the projection onto the Poincaré dual of $c_g(\mathcal{E}_{hol})$.  

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Technically, the A and B twists are performed as follows. To begin with, we have to notice that the Lagrangian $L_1$ of Poincaré gravity, possesses a global $R$-symmetry [which will be denoted by $U(1)'$], under which the fields transform with the following charges: $\zeta^\pm$, $\bar{\zeta}^\pm$, $\lambda_T$ and $\bar{\lambda}^T$ have charge $\pm 1/2$; $M$ and $H$ have charge 1, while $\bar{M}$ and $\bar{H}$ have charge $-1$. $U(1)'$ is not a local symmetry and it is not even a global symmetry for the de Sitter Lagrangian $L_2$. Depending on the choice of the twist (A or B), the new Lorentz group is defined as a combination of the old one with the $U(1)'$ symmetry; viceversa for the ghost number.

We focus here on the A twist. The new assignments and the topological shift are

$$\begin{align*}
\text{spin}' &= \text{spin} + U(1)', \\
\text{ghost}' &= \text{ghost} + 2U_A(1), \\
\Gamma^+ &\rightarrow \Gamma^+ + \alpha, \\
\tilde{\Gamma}_- &\rightarrow \tilde{\Gamma}_- + \beta,
\end{align*}$$

(10)

$\Gamma^+$ and $\tilde{\Gamma}_-$ are the supersymmetry ghosts associated to the gravitini $\zeta^+$ and $\bar{\zeta}_-$, respectively. $\alpha$ and $\beta$ are the so-called brokers [8]. They are to be treated formally as constant ($d\alpha = d\beta = 0$) and their (purely formal) role is to bring the correct contributions of spin and ghost number to the fields.

The shift produces a new BRST operator $s'$ which equals $s + \delta_T$, $\delta_T$ being the topological variation (known as $Q_s$ in conformal field theory) and $s$ is the initial BRST operator. On the graviton multiplet, $\delta_T$ acts as

$$\begin{align*}
\delta_T e^+ &= \frac{i}{2} \alpha \zeta^- \\
\delta_T \zeta^+ &= -\frac{i}{2} \omega \alpha - \frac{i}{4} A\alpha - M\beta e^+ \equiv B_1\alpha \\
\delta_T \bar{\zeta}_+ &= 0 \\
\delta_T M &= -\frac{i}{2} \tau_+ \alpha \\
\delta_T \omega &= \frac{i}{2} M\zeta^- \beta + \frac{i}{2} \bar{M}\alpha \bar{\zeta}_+ + \frac{i}{2} e^- \tau^- \alpha + \frac{i}{2} e^+ \tau_+ \beta \\
\delta_T A &= M\zeta^- \beta - \bar{M}\alpha \bar{\zeta}_+ + \tau^- \alpha e^- - \tau_+ \beta e^+.
\end{align*}$$

(11)

Taking into account that the BRST algebra closes off-shell, we see that $B_1$ and $B_2$ play the role of Lagrange multipliers, since they are the BRST variations of the antighosts $\zeta^+$ and $\bar{\zeta}_-$. $B_1$ and $B_2$ can be considered as redefinitions of $A$, $M$ and $\bar{M}$. Indeed, since $M$ and $\bar{M}$ have spin 1 and $-1$ after the twist, $Me^+ + \bar{M}e^-$ can be considered as a one form. In particular, we have shown that the graviphoton $A$ belongs to $B_{\text{gauge-fixing}}$. On the other hand, it is clear that the gauge-free topological algebra is that of $SL(2, \mathbb{R})$, since the above formulæ show that the topological symmetry is the most continuous deformation of the zweibein.

On the dilaton multiplet $\delta_T$ is

$$\begin{align*}
\delta_T X &= \bar{\lambda}^- \beta \\
\delta_T \lambda_- &= -\frac{i}{2} \nabla_+ X \alpha + H\beta \equiv H_1\alpha \\
\delta_T \lambda_+ &= 0 \\
\delta_T H &= \frac{i}{2} \nabla_+ \bar{\lambda}^- \alpha & \delta_T \bar{X} &= -\lambda_+ \alpha \\
\delta_T \bar{\lambda}^- &= 0 & \delta_T \bar{\lambda}^+ &= \frac{i}{2} \nabla_- \bar{X} \beta - \bar{H} \equiv H_2\beta \\
\delta_T \bar{H} &= -\frac{i}{2} \nabla_- \lambda_+ \beta
\end{align*}$$

(12)
$H_1$ and $H_2$ are also Lagrange multipliers, redefinitions of $H$ and $\bar{H}$.

Finally, the topological variation of the brokers vanishes, but nilpotence of $s'$ and $s$ requires

$$s'\alpha = -\frac{1}{2}C^0\alpha - iC\alpha = s\alpha \quad s'\beta = \frac{1}{2}C^0\beta - iC\beta = s\beta.$$  \hspace{1cm} (13)

In other words, even if formally, $\alpha$ and $\beta$ have to be considered as sections with definite spin and $U(1)$ charge.

Using the above formulae, one can write the full Lagrangian $L$ as the topological variation of a suitable gauge fermion $\Psi$ plus a total derivative term.

The observables of the topological theory are easily derived, as in the case of the Verlinde and Verlinde model, from the descent equations $\hat{d}\hat{R} = 0$, $\hat{R} = R + \psi_0 + \gamma_0$ being the BRST extension of the curvature $R$. In particular, the local observables are

$$\sigma_n^{(0)}(x) = \gamma_0^n(x).$$ \hspace{1cm} (14)

On the other hand, the field strength $F$ does not provide any new observables, due to the fact that $A \in B_{gauge-fixing}$.

2 The conformal field theory associated with N=2 Liouville gravity

Diffeomorphism are fixed by the usual conformal gauge condition:

$$e^+ = e^{\varphi(z,\bar{z})}dz, \quad e^- = e^{\varphi(z,\bar{z})}d\bar{z},$$ \hspace{1cm} (15)

where $\varphi(z,\bar{z})$ is the conformal factor, which is to be identified with the Liouville quantum field.

Supersymmetries are fixed by extending the conformal gauge by means of the conditions

$$\zeta^+ = \eta^{z+}_+ e^\varphi dz, \quad \zeta^- = \eta^{z-}_- e^\varphi dz, \quad \zeta^+ = \eta^{\bar{z}+}_+ e^\varphi d\bar{z}, \quad \zeta^- = \eta^{\bar{z}-}_- e^\varphi d\bar{z},$$ \hspace{1cm} (16)

where $\eta^{\pm}_\pm(z,\bar{z})$ and $\eta^{\pm}_\mp(z,\bar{z})$ are anticommuting fields of spin $1/2$ and $-1/2$ (the superpartners of the Liouville field $\varphi$).

The $U(1)$ gauge transformations have to be treated carefully, in order to reach a complete chiral factorization into two superconformal field theories (left and right moving). This is because the theory that we are now dealing with possesses a single local $U(1)$ symmetry, the $U(1)$' R-symmetry being only global. Let us introduce an additional trivial BRST system (a “one dimensional topological $\sigma$-model”) $\{\xi, C'\}$, $\xi$ being a ghost number zero scalar and $C'$ being a ghost number one scalar. Their BRST algebra is chosen to be trivial, namely

$$s\xi = C', \quad sC' = 0.$$ \hspace{1cm} (17)
The meaning of this BRST system is the gauging of the R-symmetry \( U(1)' \). Indeed, \( U(1)' \), which is only a global symmetry of the starting theory, becomes a local symmetry in the gauge-fixed version of the same theory. We fix both the \( U(1) \) gauge symmetry and the trivial symmetry \((1)\) by choosing the following two gauge-fixings

\[
A_z - \partial_z \xi = 0, \quad A_{\bar{z}} - \partial_{\bar{z}} \bar{\xi} = 0.
\]  

(18)
corresponding to \( A = *d\xi \), where \( A = A_z dz + A_{\bar{z}} d\bar{z} \).

After setting \( \pi = 1/2 (X + \bar{X}) \) and \( \chi = i/2 (X - \bar{X}) \), and performing suitable redefinitions, the total gauge-fixed Poincaré Lagrangian takes the form

\[
\mathcal{L}_{\text{Poincaré}} = -\pi \partial_z \partial_{\bar{z}} \phi + \chi \partial_z \partial_{\bar{z}} \chi + \lambda_+ \partial_z \eta^-_z - \lambda_+ \partial_{\bar{z}} \eta^+_\bar{z} + \lambda_- \partial_{\bar{z}} \eta^-_z - \lambda_- \partial_z \eta^+_\bar{z} + \partial_{\bar{z}} c^z + \partial_z c^\bar{z}.
\]

(19)

It is natural to conjecture that Poincaré N=2 supergravity corresponds to an N=2 superconformal field theory. We derive the energy-momentum tensor \( T_{zz} \), the supercurrents \( G_+ \) and \( G_- \), and the \( U(1) \) current \( J_z \), by first computing the BRST charge \( Q^{BRST} = \oint J_{\gamma}^{BRST} dz \), \( J_z^{BRST} \) denoting the BRST current. Acting with \( Q^{BRST} \) on the various antighost fields it is then simple to get the “gauge-fixings”, which are, in our case, the N=2 currents. We expect to have, on shell and up to total derivative terms, 

\[
J_{\gamma}^{BRST} = -c^z T_{zz} + \frac{1}{2} c J_z + \frac{1}{2} \gamma^+ G_+ - \frac{1}{2} \gamma^- G_-,
\]

(20)
where

\[
T_{zz} = T_{zz}^{grav} + \frac{1}{2} T_{zz}^{gh}, \quad J_z = J_z^{grav} + \frac{1}{2} J_z^{gh}, \quad G_+ = G_+^{grav} + \frac{1}{2} G_+^{gh}, \quad G_- = G_-^{grav} + \frac{1}{2} G_-^{gh}.
\]

(21)
The N=2 currents for the Liouville sector are

\[
T_{zz}^{grav} = -\partial_z \pi \partial_z \phi + \frac{1}{2} \partial^2 z \pi + \partial_z \chi \partial_z \xi + \frac{1}{2} (\partial_z \lambda_+ \eta^-_z - \lambda_+ \partial_z \eta^-_z) + \frac{1}{2} (\lambda_+ \partial_z \eta^-_z - \partial_z \lambda_+ \eta^+_z),
\]

\[
G_+^{grav} = \partial_z \lambda_+ - \lambda_+ \partial_z (\phi + \xi) + \eta^-_z \partial_z (\chi + \pi),
\]

\[
G_-^{grav} = \partial_z \lambda_- - \lambda_- \partial_z (\phi - \xi) + \eta^+_z \partial_z (\chi - \pi),
\]

\[
J_{\gamma}^{grav} = \partial_z \chi - \lambda_- \eta^- - \lambda_+ \eta^+.
\]

(22)

It is easy to check that the N=2 operator product expansions are indeed satisfied by \( \gamma \), with central charge \( c_{grav} = 6 \).

Finally the ghost currents are:

\[
T_{zz}^{gh} = 2b_{zz} \partial_z c^z + \partial_z b_{zz} c^z + \frac{3}{2} \beta_{zz} \partial_z \gamma^+ + \frac{1}{2} \partial_z \beta_{zz} \gamma^+ + \frac{3}{2} \beta_{\bar{z}z} \partial_{\bar{z}} \gamma^- + \frac{1}{2} \partial_{\bar{z}} \beta_{\bar{z}z} \gamma^- + b_z \partial_z c,
\]

\[
G_+^{gh} = 3 \beta_{zz} \partial_z c^z + 2 \partial_z \beta_{zz} c^z - \gamma^- b_{zz} - \gamma^- \partial_z b_{zz} - 2 \partial_z \gamma^- b_{zz} - \beta_{zz} c,
\]

\[
G_-^{gh} = 3 \partial_z c^z \beta_{zz} - b_{zz} \gamma^+ + \partial_z b_{zz} \gamma^+ + 2 b_z \partial_z \gamma^+ + c \beta_{zz},
\]

\[
J_{\gamma}^{gh} = \beta_{zz} \gamma^- - \beta_{zz} \gamma^+ - 2 \partial_z (b_z c^z).
\]

(23)
The ghost contribution to the central charge is $c_{gh} = -6$, so that $c_{tot} = c_{grav} + c_{gh} = 0$, as claimed.

Notice that $\beta$ and $\gamma$ commute among themselves, but anticommute with $b$ and $c$. This is because they carry an odd ghost number together with an odd fermion number, while $b$ and $c$ carry zero fermion number and odd ghost number.

The ghost number charge is

$$Q_{gh} = \oint b_{zz} c^z + \beta_+ \gamma^+ + \beta_- \gamma^- + b_z c,$$

so that $Q_{BRST} = \oint J_{BRST}^z$ has ghost number one:

$$[Q_{gh}, Q_{BRST}] = Q_{BRST}.$$  

We now perform the topological twist on the N=2 gauge-fixed theory.

In order to produce a twisted energy-momentum tensor equal to $T_{zz} + \frac{1}{2} \partial_z J_z$ we can make a redefinition of the ghost $c$ of the form

$$c' = c - \partial_z c^z.$$  

Such a replacement, which changes the spin of the fields, is to be viewed as a redefinition of the $U(1)'$ ghost $C'$ rather than the $U(1)$ ghost $C$, since the new spin is defined by adding the $U(1)'$ charge (not the $U(1)$ charge) to the old spin, as shown in section 1. $c'$ has a nonvanishing operator product expansion with $b_{zz}$ so that it is also necessary to redefine $b_{zz}$, namely

$$b'_{zz} = b_{zz} - \partial_z b_z.$$  

Then, the operator product expansions of the redefined fields are the same as those for the initial fields.

The spin changes justify the following change in notation

$$\eta_+ \rightarrow \eta_z, \quad \lambda_+ \rightarrow \lambda, \quad \beta_+ \rightarrow \beta_z, \quad \gamma^+ \rightarrow \gamma,$$

$$\eta_- \rightarrow \eta, \quad \lambda_- \rightarrow \lambda_z, \quad \beta_- \rightarrow \beta_{zz}, \quad \gamma^- \rightarrow \gamma^z.$$  

Similarly, the supercurrents are changed as $G_{+z} \rightarrow G_z, \quad G_{-z} \rightarrow G_{zz}$.

Redefinitions (24) and (27) produce a new BRST current $J_{BRST}^z$ (equal to the old one apart from a total derivative term) given by

$$J_{BRST}^z = -c^z T_{zz}' + \frac{1}{2} c' J_z + \frac{1}{2} \gamma G_z - \frac{1}{2} \gamma^z G_{zz},$$

where $T_{zz}' = T_{zz} + \frac{1}{2} \partial_z J_z$. As anticipated, $J_{BRST}^z$ generates a new energy-momentum tensor (obtained by acting with the new BRST charge on $b'_{zz}$), which is

$$T_{zz}' = T_{zz} + \frac{1}{2} \partial_z J_z = -\partial_z \pi \partial_z \varphi + \frac{1}{2} \partial_z^2 \pi + \partial_z \chi \partial_z \xi + \frac{1}{2} \partial_z^2 \chi - \lambda_z \partial_z \eta - \partial_z \lambda \eta$$

$$+ 2 b'_{zz} \partial_z c^z + \partial_z b'_{zz} c^z + 2 \beta_{zz} \partial_z \gamma^z + \partial_z \beta_{zz} \gamma^z + \beta_z \partial_z \gamma + b_z \partial_z c'.$$
From this expression, it is immediate to check the new spin assignments. It is interesting to note that the total derivative term in the $U(1)$ current $J_2^{gh}$ combines with redefinitions (26) and (27) to give the correct energy-momentum tensor for the ghosts $T_{zz}^{gh}$.

The other ingredient of the topological twist is the topological shift (8)

$$\gamma \to \gamma + \alpha. \quad (31)$$

Since the spin has been already changed by (26), (31) does not change the spin a second time. Indeed, the new spin of $\gamma$ is zero and so that of $\alpha$. Moreover, after twist $\gamma$ possesses a zero mode (the constant). In this case, $\alpha$ represents a shift of the zero mode of $\gamma$.

The topological shift (31) produces a total BRST current equal to

$$J_{z}^{BRST tot} = J_{z}^{BRST} + \frac{1}{2} \alpha G_z, \quad (32)$$

(again, a total derivative term has been omitted). If we denote, as usual, $Q_{BRST} = \oint J_{z}^{BRST tot}$, $Q_v = \oint J_{z}^{BRST} dz$ and $Q_s = \oint G_z dz$, we see that the BRST charge is precisely shifted by the supersymmetry charge $Q_s$.

Let us now discuss some properties of the twisted theory. It is convenient to write down the $Q_z$ transformation of the fields, that we denote it by $\delta_s$:

$$\begin{align*}
\delta_s(\xi - \varphi) &= 2\eta, & \delta_s\eta &= 0, & \delta_s\lambda_z &= \bar{\partial}_z(\pi + \chi), & \delta_s(\pi + \chi) &= 0, \\
\delta_s(\pi - \chi) &= 2\lambda, & \delta_s\lambda &= 0, & \delta_s\eta_z &= \bar{\partial}_z(\xi + \varphi), & \delta_s(\xi + \varphi) &= 0, \\
\delta_s b'_{zz} &= 0, & \delta_s \beta_{zz} &= -b'_{zz}, & \delta_s c^z &= \gamma^z, & \delta_s \gamma^z &= 0, \\
\delta_s b_z &= \beta_z, & \delta_s \beta_z &= 0, & \delta_s c' &= 0, & \delta_s \gamma &= c'.
\end{align*} \quad (33)$$

These transformations are the analogue, in the gauge-fixed case, of the $\delta_T$ transformations (11) and (12). Notice that, in the last two lines of (33) there are two different $b$-c-$\beta$-$\gamma$ systems. In particular, the last line represents the sector of $B_{\text{gauge-fixing}}$ that is reminiscent of the constraint on the moduli space. The last but one line, on the other hand, represents the usual $b$-c-$\beta$-$\gamma$ ghost for ghost system of topological gravity [7]. It is evident that the roles of $b$ and $\beta$ and the roles of $c$ and $\gamma$ are inverted in the two cases.

The theory is topological, since the energy-momentum tensor $T'_{zz}$ is a physically trivial left moving operator. Indeed, recalling that $G_{zz} = -2 \{Q_v, \beta_{zz}\}$, we have

$$\alpha T'_{zz} = \{ Q, G_{zz} \}, \quad \{ Q_v, G_{zz} \} = 0. \quad (34)$$

Finally we notice that the ghost number current of the twisted theory can be written as the sum of the ghost number charge of the initial N=2 theory plus the $U(1)$ charge. This corresponds to eq. (10):

$$Q_{gh}' = Q_{gh} + \oint J_z = \oint b'_{zz} c^z + 2 \beta_{zz} \gamma^z + b_z c' - \lambda_z \eta - \lambda \eta_z. \quad (35)$$
3 Geometrical Interpretation

We now discuss the moduli space of the twisted theory and the gauge-fixing sector that implements the constraint defining the submanifold $V_g \subset M_g$.

The number of moduli of the twisted theory is $4g - 3$, the same as that of the $N=2$ theory, $3(g - 1)$ moduli $m_i$ corresponding to the metric and $g$ moduli $\nu_j$ corresponding to the $U(1)$ connection $A$. The number of supermoduli, on the other hand, changes by one: it was $4(g - 1)$ for the $N=2$ theory, it is $4g - 3$ for the topological theory, $3(g - 1)$ supermoduli $\hat{m}_i$ corresponding to the zero modes of the spin 2 antighost $\beta_{zz}$ and $g$ supermoduli $\hat{\nu}_j$ corresponding to the zero modes of $\beta_z$. The mismatch of one supermodulus is filled by the presence of one super Killing vector field, corresponding to the (constant) zero mode of $\gamma$.

In particular, after the twist, the number of bosonic moduli equals the number of fermionic moduli, as expected for a topological theory. However, the two kinds of supermoduli $\hat{m}_i$ and $\hat{\nu}_j$ do not carry the same ghost number after the twist. Indeed, $\hat{m}_i$ carry ghost number 1, while $\hat{\nu}_j$ carry ghost number $-1$. Thus, we can interpret $\hat{m}_i$ as the topological variation of $m_i$, but we cannot interpret $\hat{\nu}_j$ as the topological variation of $\nu_j$, rather $\nu_j$ is the topological variation of $\hat{\nu}_j$:

\[
sm_i = \hat{m}_i, \quad s\hat{m}_i = 0, \quad s\hat{\nu}_j = \nu_j, \quad s\nu_j = 0.
\]

This is in agreement with the interpretation of $A$ as a Lagrange multiplier, so that it is only introduced via the gauge-fixing algebra: $m$ and $\hat{m}$ belong to $B_{gauge-free}$, while $\nu$ and $\hat{\nu}$ belong to $B_{gauge-fixing}$.

The amplitudes can be written as

\[
<\prod_k \sigma_{n_k} > = \int d\Phi \int_{M_g} \prod_{i=1}^{3g-3} dm_i \int \Omega^g \prod_{j=1}^g d\nu_j \int d\hat{m} d\hat{\nu} i e^{\bar{q}_i \pi(\bar{z})} e^{-S(m, \hat{m}, \nu, \hat{\nu})} \prod_{k} \sigma_{n_k},
\]

where $\sigma_{n_k}$ are the observables. In this expression, the insertions that remove the zero modes of $b_{zz}$, $\beta_{zz}$, $\beta_z$, $b_z$, $\eta$, $\lambda$, $\bar{\lambda}$ and $\eta_z$ are understood, but attention has to be paid to the fact that a super-Killing-vector-field, corresponding to the zero mode of $\gamma$, forbids one fermionic integration. $e^{q_i \bar{\pi}(\bar{z})}$ are the $\delta$-type insertions that simulate the curvature $R$ such that $\sum_i q_i = 2(1 - g)$, where $\bar{\pi}$ is the BRST invariant extension of $\pi$ [5]. The ghost number of the supermoduli measure adds up to $-2g + 3$. Nevertheless, due to the presence of one super-vector-field, the selection rule is that the total ghost number of $\prod_k \sigma_{n_k}$ must be equal to $2(g - 1)$ and not to $2g - 3$. This is the mismatch between true dimension and formal dimension addressed in the introduction.

To explain why the graviphoton is responsible for the constraints, let us rewrite the action making the dependence on the $U(1)$-moduli $\nu_j$ and the corresponding supermoduli $\hat{\nu}_j$ explicit.

\[
S(m, \hat{m}, \nu, \hat{\nu}) = S(m, \hat{m}, 0, 0) + \nu_j \int_{\Sigma_g} \omega_j^z J_z d^2z + \hat{\nu}_j \int_{\Sigma_g} \omega_j^z G_z d^2z
\]
\[ +\tilde{\nu}_j \int_{\Sigma_g} \omega^j_z J_z d^2z + \hat{\nu}_j \int_{\Sigma_g} \omega^j_z G_z d^2z + \nu \hat{\nu} \text{-terms.} \quad (38) \]

The terms that are quadratic in \( \nu \hat{\nu} \) are due to the fact that the gravitini are initially \( U(1) \)-charged. They have not been reported explicitly, since they can be neglected, as we show in a moment. The coefficient of \( \tilde{\nu}_j \) is the \( U(1) \) current \( J_z \) folded with the \( j \)-th (anti)holomorphic differential \( \omega^j_z \). Similarly, the coefficient of \( \hat{\nu}_j \) is the supercurrent \( G_z \) folded with the same differential.

We want to perform the \( \nu \hat{\nu} \) integrals explicitly. This is allowed, since the observables should not depend on \( \nu \) and \( \hat{\nu} \). Indeed, \( \nu \) and \( \hat{\nu} \) belong to \( B_{\text{gauge-fixing}} \) and not to \( B_{\text{gauge-free}} \), while the observables are constructed entirely from \( B_{\text{gauge-free}} \). Anyway, since \( \nu \) and \( \hat{\nu} \) form a closed BRST subsystem, we can consistently project down to the subset \( \nu = \hat{\nu} = 0 \), while retaining the BRST nilpotence. The \( U(1) \) moduli \( \nu \) are not integrated all over \( C^9 \), which would be nice since the integration would be very easy, rather on the unit cell \( L = C^9/\langle Z^9 + \Omega Z^9 \rangle \) defined by the period matrix \( \Omega \). To overcome this problem, we take the semiclassical limit, which is exact in a topological field theory. We multiply the action \( S \) by a constant \( \kappa \) that has to be stretched to infinity. \( \kappa \) can be viewed as a gauge-fixing parameter, rescaling the gauge-fermion: no physical amplitude depends on it. Let us define \( \nu_j' = \kappa \nu_j \quad \hat{\nu}_j' = \kappa \hat{\nu}_j \). We have

\[ \int_L \prod_{j=1}^9 d\nu_j d\hat{\nu}_j = \int_{\kappa L} \prod_{j=1}^9 d\nu_j' d\hat{\nu}_j, \quad (39) \]

where and \( \kappa L \) is unit cell rescaled. We see that the \( \nu \hat{\nu} \)-terms of (38) are suppressed in the \( \kappa \to \infty \) limit, as claimed. We can replace \( \kappa L \) with \( C^9 \) in this limit. Finally, the integration over the \( U(1) \) moduli and supermoduli produces the insertions

\[ \prod_{j=1}^9 \int_{\Sigma_g} \omega^j_z G_z d^2z \cdot \delta \left( \int_{\Sigma_g} \omega^j_z J_z d^2z \right). \quad (40) \]

The delta-function is the origin of the desired constraint on moduli space. Indeed, the current \( J_z \) can be thought as a (field dependent) section of \( E_{\text{hol}} \). The requirement of its vanishing is equivalent to projecting onto the Poincaré dual of the top Chern class \( c_g(E_{\text{hol}}) \) of \( E_{\text{hol}} \). Changing section only changes the representative in the cohomology class of \( c_g(E_{\text{hol}}) \). Indeed, the Poincaré dual of the top Chern class of a holomorphic vector bundle \( E \to M \) is shown to be the submanifold of the base manifold \( M \) where one holomorphic section \( a \in \Gamma(E, M) \) vanishes identically. In other words, the dual of \( c_g(E_{\text{hol}}) \) is the divisor of some section. For a line bundle \( L \to M \), this is easily seen. Let \( h \) be a fiber metric so that \( ||a||^2 = a(z)\overline{a}(z)h(z, \overline{z}) \) is the norm of the section \( a \). The top Chern class \( c_1(L) \) can be written as the cohomology class of the curvature \( R = \partial \Gamma \) of the canonical holomorphic connection \( \Gamma = h^{-1} \partial h \), so that \( c_1(L) = \partial \Gamma \ln ||a(z)||^2 \). Patchwise, the metric \( h \) can be reduced to the identity, but then \( c_1(L) \) becomes a de Rham current, namely a singular \((1,1)\) form with delta-function support on the divisor \( \text{Div}[a] \), i.e. the locus of zeroes and poles of \( a(z) \). The divisor \( \text{Div}[a] \) is the Poincaré dual of \( c_1(L) \). For a holomorphic vector
bundle $E \to M$ of rank $n$, the same theorem can be understood using the so-called splitting principle, regarding $E$ as the Whitney sum of $n$ line-bundles $L_i$ corresponding naively, to the eigendirections of the curvature matrix two form $R^{jk}$. From the above argument, we can say that $c_n(E)$ has delta-function support on the divisor of $a$. That is why in our derivation of the topological correlators from the functional integral, we do not pay particular attention to the explicit form of $J_z$ and to its dependence on the other fields. What matters is that it is a conserved holomorphic one form, namely a section of $\mathcal{E}_{hol}$. The functional integral imposes its vanishing, so that the Riemann surfaces that effectively contribute lie in the homology class of the Poincaré dual of $c_g(\mathcal{E}_{hol})$.

Summarizing, we argue that the topological observables $\sigma_{nk}$ correspond to the Mumford-Morita classes, as in the case of topological gravity [7], but that in constrained topological gravity the correlation functions are intersection forms on the Poincaré dual $V_g$ of $c_g(\mathcal{E}_{hol})$ and not on the whole moduli space $\mathcal{M}_g$.

It can be convenient to represent $c_g(\mathcal{E}_{hol})$ by introducing the natural fiber metric $h_{jk} = \text{Im} \Omega^{jk} = \int_{\Sigma_g} \omega_j \omega_k^* d^2 z$ on $\mathcal{E}_{hol}$. The canonical connection associated with this metric is then $\Gamma = h^{-1} \partial h = \frac{1}{\Omega - \bar{\Omega}} \partial \Omega$. Then $\mathcal{R} = \bar{\partial} \Gamma$ and

$$c_g(\mathcal{E}_{hol}) = \det \mathcal{R} = \det \left( \frac{1}{\Omega - \bar{\Omega}} \partial \bar{\Omega}, \frac{1}{\bar{\Omega} - \Omega} \partial \Omega \right).$$

(41)

Let $\{\omega^1, \ldots, \omega^g\}$ denote a basis of holomorphic differentials. Locally, we can expand $J_z$ in this basis $J_z = a_j \omega^j$. The field dependent coefficients $a_j$ are the components of the section $J_z \in \Gamma(\mathcal{E}_{hol}, \mathcal{M}_g)$. The constraint then reads $\text{Im} \Omega^{jk} a_k = 0, \forall j$, which, due to the positive definiteness of $\text{Im} \Omega$, is equivalent to

$$a_j = 0, \quad \forall j.$$ 

(42)

These are the equations that (locally) identify the submanifold $V_g \subset \mathcal{M}_g$. It is also useful to introduce the vectors $v_j = \frac{\partial}{\partial a_j}$ that provide a local basis for the normal bundle $\mathcal{N}(V_g)$ to $V_g$. Of course, the vectors $v_j$ commute among themselves: $[v_j, v_k] = 0$. In these explicit local coordinates, the top Chern class $c_g(\mathcal{E}_{hol})$ admits the following representation as a de Rham current:

$$c_g(\mathcal{E}_{hol}) = \delta(V_g) \bar{\Omega}_g,$$

(43)

where

$$\bar{\Omega}_g = \prod_{j=1}^g da_j, \quad \delta(V_g) = \prod_{j=1}^g \delta(a_j).$$

(44)

This explicit notation is useful to trace back the correspondence between the geometrical and field theoretical definition of the correlators.

To begin with, a convenient representation of the BRST operator (36) on the space $\{m, \hat{m}, \nu, \hat{\nu}\}$ is given by

$$Q_{global} = \hat{m}_i \frac{\partial}{\partial m_i} + \nu_j \frac{\partial}{\partial \nu_j}.$$ 

(45)
$Q_{\text{global}}$ is not the total BRST charge, rather it only represents the BRST charge on the sector of the global degrees of freedom. The total BRST charge is the sum of the above operator plus the usual BRST charge $Q = Q_s + Q_v$, that acts only on the local degrees of freedom. Since the total BRST charge acts trivially inside the physical correlation functions, we see that the action of $Q$ inside correlation functions is the opposite of the action of $Q_{\text{global}}$. This means that $Q$ can be identified, apart from an overall immaterial sign, with the operator (45). We know that the geometrical meaning of the supermoduli $\hat{m}_i$ are the differentials $dm_i$ on the moduli space $\mathcal{M}_g$. The ghost number corresponds to the form degree. In view of this, we argue that the geometrical meaning of the $U(1)$ supermoduli $\hat{\nu}_j$ are contraction operators $i_{v_j}$ with respect to the associated vectors $v_j$. Since the $U(1)$ moduli $\nu_j$ are the BRST variations of $\hat{\nu}_j$ and the BRST operation should be identified with the exterior derivative, it is natural to conjecture that $\nu_j$ correspond to the Lie derivatives along the vectors $v_j$.

The correspondence between field theory and geometry is summarized in table 3. We now give arguments in support of this interpretation.

For instance, since $Q \sim d$, $\text{Im } \Omega_{jk} a_k \sim \int \omega^k z d^2 z$ and $[Q, J_z] = -G_z$, then the insertions $\int \omega^k G_z d^2 z$ correspond to $d(\text{Im } \Omega_{jk} a_k)$, so that

$$\prod_{j=1}^{g} \int_{\Sigma_g} \omega^k z d^2 z \cdot \delta \left( \int_{\Sigma_g} \omega^k z d^2 z \right) \sim \tilde{\Omega}_g \delta(\mathcal{V}_g) = c_g(\mathcal{E}_{\text{hol}}).$$  \hspace{1cm} (46)

If $\alpha_k$ denote the Mumford-Morita classes corresponding to the observables $O_k$, the amplitudes are

$$< O_1 \cdots O_n > = \int_{\mathcal{M}_g} \delta(\mathcal{V}_g) \tilde{\Omega}_g \wedge \alpha_1 \wedge \cdots \wedge \alpha_n = \int_{\mathcal{V}_g} \alpha_1 \wedge \cdots \wedge \alpha_n.$$  \hspace{1cm} (47)

From the geometrical point of view, it is immediate to show that the action of (45) on a correlation function is precisely the exterior derivative, as already advocated. Indeed, we can write the $d$-form $\omega_d$ corresponding to a physical amplitude (not necessarily a top form, if we freeze, for the moment, the integration over the global degrees of freedom) as $\omega_d = i_{v_1} \cdots i_{v_g} \Omega_{d+g} = \left( \prod_{j=1}^{g} \hat{\nu}_j \right) \hat{m}_{i_1} \cdots \hat{m}_{i_d} \Omega_{d+g}^{i_1 \cdots i_d}$, where $\Omega_{d+g}$ is a suitable $d + g$-form on $\mathcal{M}_g$ (equal to $\tilde{\Omega}_g \wedge \omega_d$). Now, using the representation (45) of the operator $Q$ and the correspondence given in table II we find precisely $\{Q, \omega_d \} = d\omega_d$. The second piece of (45) replaces a contraction with the vector $v_j$ with the Lie derivative with respect to the same vector.
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| Field Theory | Geometry |
|--------------|----------|
| $\hat{m}_i$ | $d m_i$ |
| $\hat{\nu}_j$ | $i_{v_j}$ |
| $\nu_j$ | $\mathcal{L}_{v_j}$ |
| $Q$ | $d$ |
| $[Q, m_i] = \hat{m}_i$ | $[d, m_i] = d m_i$ |
| $\{Q, m_i\} = 0$ | $\{d, m_i\} = 0$ |
| $\{Q, \hat{\nu}_j\} = \nu_j$ | $\{d, \mathcal{L}_{v_j}\} = 0$ |
| $\{Q, \nu_j\} = 0$ | $\{d, \mathcal{L}_{v_j}\} = 0$ |
| $\{\hat{\nu}_j, \nu_k\} = 0$ | $\{i_{v_j}, \nu_k\} = 0$ |
| $[\hat{\nu}_j, \nu_k] = 0$ | $[i_{v_j}, \mathcal{L}_{v_k}] = 0$ |
| $[\nu_j, \nu_k] = 0$ | $[\mathcal{L}_{v_j}, \mathcal{L}_{v_k}] = 0$ |

\[ \prod_{g=1}^{g} \delta \left( \int \omega_z J_z d^2 z \right) \]
\[ \prod_{g=1}^{g} \int \omega_z^2 G_z d^2 z \]
\[ \prod_{j=1}^{g} \int \omega_z^2 G_z d^2 z \cdot \delta \left( \int \omega_z J_z d^2 z \right) \]

\[ \sigma_{n_j} \]
\[ < \sigma_{n_1} \cdots \sigma_{n_k} > \]
\[ \int_{\gamma_{n}} [c_1(\mathcal{L}_1)]^{n_1} \wedge \cdots \wedge [c_1(\mathcal{L}_k)]^{n_k} \]