A Grothendieck-type inequality for local maxima

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March 15, 2016

Abstract

A large number of problems in optimization, machine learning, signal processing can be effectively addressed by suitable semidefinite programming (SDP) relaxations. Unfortunately, generic SDP solvers hardly scale beyond instances with a few hundreds variables (in the underlying combinatorial problem). On the other hand, it has been observed empirically that an effective strategy amounts to introducing a (non-convex) rank constraint, and solving the resulting smooth optimization problem by ascent methods. This non-convex problem has – generically – a large number of local maxima, and the reason for this success is therefore unclear.

This paper provides rigorous support for this approach. For the problem of maximizing a linear functional over the elliptope, we prove that all local maxima are within a small gap from the SDP optimum. In several problems of interest, arbitrarily small relative error can be achieved by taking the rank constraint \( k \) to be of order one, independently of the problem size.

1 Motivation and result

Let \( \text{PSD}(n) \equiv \{ X \in \mathbb{R}^{n \times n} : X \succeq 0 \} \) be the cone of \( n \times n \) symmetric positive semidefinite matrices. The convex set of positive-semidefinite matrices with diagonal entries equal to one will be denoted by

\[
\text{PSD}_1(n) \equiv \{ X \in \mathbb{R}^{n \times n} : X \succeq 0, \ X_{ii} = 1 \forall i \in [n] \}.
\] (1)

The set \( \text{PSD}_1(n) \) is also known as the elliptope. Given a symmetric matrix \( A \), we define

\[
\text{SDP}(A) \equiv \max \{ \langle A, X \rangle : X \in \text{PSD}_1(n) \}.
\] (2)

This semidefinite program (SDP) arises in a large number of applications in particular as a relaxation of max-cut, and min-bisection of graphs. Early references include [GW95, Nes98]. Unfortunately, despite the many remarkable properties of this SDP, generic SDP solvers hardly scale beyond \( n \) of the order of a few hundreds.

Recently, several authors [BM03, JMRT16] found that an effective strategy is to constrain the rank of \( X \) to satisfy \( \text{rank}(X) \leq k \), explicitly solve the PSD constraint and use gradient ascent or coordinate ascent to maximize the resulting non-convex objective.

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1 Here and below \( \langle A, B \rangle = \text{Tr}(A^T B) \) is the usual scalar product between matrices.
Explicitly, the rank-constrained problem and be written as a smooth optimization problem over
the manifold $S(n, k) \subset \mathbb{R}^{n \times k}$, defined by $S(n, k) \equiv \{ \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) : \sigma_i \in \mathbb{R}^k, \|\sigma_i\|_2 = 1 \}$. This can be identified with the product of $n$ $(k-1)$-dimensional spheres
\[ S(n, k) \simeq S^{k-1} \times S^{k-1} \times \cdots \times S^{k-1}. \] (3)
The objective function $F_A : S(n, k) \to \mathbb{R}$ is defined by:
\[ F_A(\sigma) \equiv \sum_{i,j=1}^{n} A_{ij} \langle \sigma_i, \sigma_j \rangle. \] (4)
We then define
\[ \text{OPT}_k(A) \equiv \max \{ F_A(\sigma) : \sigma \in S(n, k) \}. \] (5)
Of course we have $\text{OPT}_k(A) \leq \text{SDP}(A)$, with $\text{OPT}_n(A) = \text{SDP}(A)$. The optimizer of the $k = n$ problem, $\sigma^\star$, corresponds to an optimizer of the SDP through $X^\star = \sigma^\star \sigma^\star^\top$. However, already much smaller values of $k$ give excellent (or exact) approximations of the underlying SDP. For instance, [JMRT16] used this approach to cluster graphs generated according to the sparse stochastic block model. Empirically, the resulting solutions are indistinguishable from the SDP optimum already for $k = 20$. The resulting algorithm can be used to cluster sparse graphs with up to $n = 10^5$ vertices in a matter of minutes.²

Unfortunately, the optimization problem [5] is non-convex and has –generically– a large number of local maxima (but see Section 1.1 for a discussion of related work). It is therefore unclear why coordinate or gradient ascent should find a good approximation of $\text{OPT}_k(A)$, let alone a good approximation of $\text{SDP}(A)$. Our main result shows that in fact all the maxima have value close to the global optimum, and in fact close to $\text{OPT}_k(A)$. (Here and below $\|M\|_2$ denotes the $\ell_2$ operator norm of matrix $M$.)

**Theorem 1.** Let $\sigma$ be a local maximum of $F_A$ over $S(n, k)$, $k \geq 2$. Then
\[ \text{SDP}(A) \geq F_A(\sigma) \geq \text{SDP}(A) - \frac{8}{\sqrt{k}} n\|A\|_2. \] (6)

The proof of this theorem is provided in Section 2.

**Remark 1.1.** In typical applications $\|A\|_2 = \Theta(1)$, while $\text{SDP}(A) = \Theta(n)$. In these cases Theorem 1 guarantees that an arbitrarily small relative error can be achieved by taking $k$ to be a large constant.

A simple example is the so-called $Z_2$ synchronization problem [JMRT16]. In that case $A = (\lambda/n) x_0 x_0^\top + W$ with $x_0 \in \{+1, -1\}^n$, and $W$ a GOE random matrix, i.e. a symmetric matrix with independent entries $(W_{ii})_{1 \leq i \leq n}, W_{ii} \sim N(0, 2/n)$, and $(W_{ij})_{1 \leq i < j \leq n}, W_{ij} \sim N(0, 1/n)$. Basic random matrix theory implies $\|A\|_2 \lesssim 2 + \lambda$ with high probability, while using $X = x_0 x_0^\top$ yields $\text{SDP}(A) \geq n\lambda + o(n)$ (and indeed [MS16] proves $\text{SDP}(A) \geq n\max(2\lambda, o(n))$).

**Remark 1.2.** A naive application of Theorem 1 to the sparse stochastic block model [JMRT16] shows that an arbitrarily small error is achieved for $k = C\sqrt{\log n / \log \log n}$, with $C$ a large constant. In fact this can be improved further to $k = O(1)$, by considering a suitably modified graph.

²Code for the algorithm of [JMRT16] is available at http://web.stanford.edu/~montanar/SDPgraph/.
1.1 Further related work

Burer and Monteiro [BM03] introduced the idea of constraining the rank and solving the PSD constraint thus obtaining a smooth non-convex problem. They also proved that, taking $k \geq \sqrt{2n}$, and under suitable conditions on $A$, the resulting non-convex problem has no local maxima, except for the global one. Their result actually extend to more general SDPs than Eq. (2). While interesting, this result does not clarify the empirical finding that $k = 20$ is sufficient for some problems with $n$ as large as $10^5$ [JMRT16].

Journée, Bach, Absil and Sepulchre [JBAS10] proved sufficient conditions under which a local optimum of the rank-constrained problem is in fact a global optimum of the SDP. In particular they proved that this happens if the local optimum $\sigma \in S(n,k)$ is rank-deficient (i.e. has rank at most $k-1$). It is however unclear a priori when this happens, and even computing the exact rank of $\sigma$ is very difficult (since $\sigma$ is typically produced by an ascent method). In the numerical experiments of [JMRT16], the local optimum was typically full rank.

Bandeira, Boumal and Voroninski recently considered the extreme case $k = 2$, in the specific example of the $\mathbb{Z}_2$ synchronization problem [BBV16]. They proved that, if the signal is strong enough, then this approach can effectively recover the underlying signal. Namely, all local minima are correlated with the signal.

The SDP (2) was recently studied in the context of the so-called community detection problem, namely for recovering vertex labels under the sparse stochastic block model. Among a large number of interesting contributions, the closest to the present work are the ones concerning the so-called detection threshold [GV15, MS16]. In particular, [MS16] proves a Grothendieck-type inequality for $\text{OPT}_k(A)$. In slightly simplified form, this implies

$$\text{OPT}_k(A) \geq \text{SDP}(A) - \frac{C}{k} n\|A\|_2,$$

with $C$ an absolute constant. This can be immediately compared with Theorem 1 which of course implies $\text{OPT}_k(A) \geq \text{SDP}(A) - (9/\sqrt{k}) n\|A\|_2$. This is similar to the result of [MS16], but has a weaker dependence on $k$, thus raising the interesting question of whether the dependence on $k$ in Theorem 1 can be improved.

Grothendieck inequalities have found a broad range of applications in computer science, see [KN12] and references therein. However, they are typically used to approximate a combinatorial problem by solving a semidefinite program. Theorem 1 instead points at a different direction, namely solving the SDP with arbitrarily high accuracy by solving a non-convex rank-constrained problem. Of course this in turn provides an approximation of the original combinatorial problem.

Finally, there has been growing interest in non-convex methods for solving high-dimensional statistical estimation problems. Examples include matrix completion [KMO10], phase retrieval [CC15], regression with missing entries [LW11], and many others. These papers provide rigorous guarantees under the assumption that the noise in the data is ‘small enough.’ Under such conditions, a very good initialization can be constructed, e.g. by a spectral method, and it is sufficient to prove that the optimization problem is well behaved in a neighborhood of the optimum.

The mechanism studied here is dramatically different. Not only do we not require any ‘strong signal’ condition, but there is actually no signal at all (in other words, no ‘signal plus noise’ structure is assumed). We do not establish a property of a neighborhood of the correct solution, but instead a global property of the cost function.
1.2 Notations

Throughout the paper, vectors and matrices are denoted by boldface, e.g. \( \mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots \). We reserve the upper case, e.g. \( \mathbf{A}, \mathbf{B}, \ldots \), for \( n \times n \) matrices. Their coordinates are denoted by non-bold font, e.g. \( \mathbf{x} = (x_1, \ldots, x_n) \). We identify an element \( \boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in S(n, k) \) with a matrix in \( \mathbb{R}^{n \times k} \). The standard scalar product of \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \) is denoted by \( \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i \). Analogously, \( \langle \mathbf{A}, \mathbf{B} \rangle = \text{Tr}(\mathbf{A} \mathbf{B}^T) \) is the scalar product between matrices.

The eigenvalues of a symmetric matrix \( \mathbf{M} \) are denoted by \( \xi_1(\mathbf{M}) \geq \xi_2(\mathbf{M}) \geq \cdots \geq \xi_n(\mathbf{M}) = \xi_{\min}(\mathbf{M}) \). We let \( \| \mathbf{x} \|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \) be the \( \ell_2 \) norm of vector \( \mathbf{x} \), \( \| \mathbf{M} \|_2 \) be the \( \ell_2 \) operator norm of matrix \( \mathbf{M} \), and \( \| \mathbf{M} \|_F \) its Frobenius norm.

2 Proofs

2.1 Preliminary lemmas

Our first lemma collects the classical second order conditions for problem (5). While this is standard, we provide a self-contained proof for the reader’s convenience.

Lemma 2.1. Let \( \boldsymbol{\sigma} \in S(n, k) \) be a local maximum of \( F_{\mathbf{A}} \) on \( S(n, k) \). Then there exists a diagonal matrix \( \Lambda \) of Lagrange multipliers such that the following conditions hold:

1. First order stationarity:

   \[
   (\Lambda - \mathbf{A}) \boldsymbol{\sigma} = 0. \tag{8}
   \]

2. Second-order stationarity: for all \( \mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_n), \mathbf{u}_i \in \mathbb{R}^k \), such that \( \langle \sigma_i, \mathbf{u}_i \rangle = 0 \), we have

   \[
   \sum_{i,j=1}^{n} (\Lambda - \mathbf{A})_{ij} \langle \mathbf{u}_i, \mathbf{u}_j \rangle \geq 0. \tag{9}
   \]

Proof. Fix \( \mathbf{u} \) as in point 2 above an let \( \hat{\mathbf{u}}_i = \mathbf{u}_i/\|\mathbf{u}_i\|_2 \) (with the convention \( \hat{\mathbf{u}}_i = 0 \) if \( \mathbf{u}_i = 0 \)). Define \( \sigma(t) \) for \( t \in \mathbb{R} \), by

   \[
   \sigma_i(t) \equiv \sigma_i \cos \left( \|\mathbf{u}_i\|_2 t \right) + \hat{\mathbf{u}}_i \sin \left( \|\mathbf{u}_i\|_2 t \right). \tag{10}
   \]

Note that \( \sigma(t) \in S(n, k) \) for all \( t \in \mathbb{R} \), and \( \sigma(0) = \sigma \). Also \( t \mapsto \sigma(t) \) is smooth. Since \( \sigma \) is a local maximum of \( F_{\mathbf{A}} \), it follows that \( t = 0 \) must be a local maximum of \( t \mapsto F_{\mathbf{A}}(\sigma(t)) \).

By Taylor expansion

   \[
   \sigma_i(t) = \sigma_i + \mathbf{u}_i \cdot t - \frac{1}{2} \|\mathbf{u}_i\|_2^2 \sigma_i \cdot t^2 + O(t^3), \tag{11}
   \]

and

   \[
   F_{\mathbf{A}}(\sigma(t)) = F_{\mathbf{A}}(\sigma(0)) + 2 \sum_{i=1}^{n} \langle \sigma_i, \mathbf{g}_i \rangle \cdot t - \sum_{i=1}^{n} \|\mathbf{u}_i\|_2^2 \sigma_i \cdot \mathbf{g}_i \cdot t^2 + \sum_{i,j=1}^{n} A_{ij} \langle \mathbf{u}_i, \mathbf{u}_j \rangle \cdot t^2 + O(t^3) \tag{12}
   \]

\[
\mathbf{g}_i \equiv \sum_{j=1}^{n} A_{ij} \sigma_j. \tag{13}
\]
By first order stationarity of $t \mapsto F_A(\sigma(t))$, we must have $\langle u_i, g_i \rangle = 0$ for all $u_i$ orthogonal to $\sigma_i$. Therefore $g_i = \lambda_i \sigma_i$ for some $\lambda_i \in \mathbb{R}$. This yields Eq. (8) with $\Lambda$ the diagonal matrix with $\Lambda_{i,i} = \lambda_i$.

Substituting in Eq. (12), we get
\[
F_A(\sigma(t)) = F_A(\sigma(0)) + \sum_{i,j=1}^{n} (A - \Lambda)_{ij} \langle u_i, u_j \rangle t^2 + O(t^3),
\]
which immediately implies the claim (9), since $t \mapsto F_A(\sigma(t))$ is a local maximum.

Throughout, for $\Lambda$ the matrix of Lagrange multipliers, we will denote by $\lambda_i = \Lambda_{i,i}$ its diagonal entries. The next lemma collects a few simple facts about local maxima.

**Lemma 2.2.** Let $\sigma \in S(n, k)$ be a local maximum of $F_A$ on $S(n, k)$. Then we have the following

1. The associated matrix of multipliers $\Lambda$ is uniquely determined by
\[
\lambda_i = \left\| \sum_{j=1}^{n} A_{ij} \sigma_j \right\|_2.
\]
2. $F_A(\sigma) = \text{Tr}(\Lambda)$.
3. For every $S \subseteq [n], |S| \leq k - 1$, we have $(\Lambda - A)_{S,S} \succeq 0$. In particular, $\lambda_i \geq 0$.
4. $\|\Lambda\|_F^2 = \sum_{i=1}^{n} \lambda_i^2 \leq n\|A\|_2^2$.
5. We have
\[
\xi_{\min}(\sigma^T \sigma) \leq \frac{n}{k}.
\]

**Proof.** For point 1, note that, by Eq. (8), $\lambda_i \sigma_i = \sum_{j=1}^{n} A_{ij} \sigma_j$. The claim follows by taking the norm of both sides since $\|\sigma_i\| = 1$ (and noting that $\lambda_i \geq 0$ as proved in point 3).

Point 2 follows again from $\lambda_i \sigma_i = \sum_{j=1}^{n} A_{ij} \sigma_j$, multiplying both sides by $\sigma_i$, and summing over $i \in \{1, \ldots, n\}$.

For point 3 assume, without loss of generality, that $S = \{1, 2, \ldots, k - 1\}$. Let $u \in \mathbb{R}^k, \|u\|_2 = 1$ be orthogonal to $\sigma_1, \ldots, \sigma_{k-1}$. For any $x \in \mathbb{R}^{k-1}$, define $u_i = x_i u$ for $i \leq k - 1$, and $u_i = 0$ for $i \geq k$. Then, the second order stationarity condition (9) implies
\[
0 \leq \sum_{i,j=1}^{n} (\Lambda - A)_{ij} \langle u_i, u_j \rangle = \langle x, (\Lambda - A)_{S,S} x \rangle,
\]
which proves the claim.

For point 4, note that, since $\Lambda \sigma = A \sigma$ we also have
\[
\sum_{i=1}^{n} \lambda_i^2 = \langle \sigma, \Lambda^2 \sigma \rangle = \langle \Lambda \sigma, \Lambda \sigma \rangle = \langle A \sigma, A \sigma \rangle = \text{Tr}(A^2 \sigma \sigma^T) \leq \|A^2\|_2 \|\sigma \sigma^T\|_*.
\]
Lemma 2.2 (point 5), we can fix $x \mathcal{P}$ where $\sigma$ (The first equality holds because the singular values of $|T|$, $\sigma$ are obtained by squaring the singular values of $\sigma$.)

For the last point let

$$\Sigma = \frac{1}{n} \sigma^T \sigma = \frac{1}{n} \sum_{i=1}^n \sigma_i \sigma_i^T.$$  \hfill (22)

Then $\Sigma \succeq 0$ and $\text{Tr}(\Sigma) = \sum_{i=1}^n \|\sigma_i\|_2^2 / n = 1$. Hence $\sum_{i=1}^k \xi_i(\Sigma) = 1$, and since $\xi_i(\Sigma) \geq 0$, we necessarily have $\xi_{\min}(\Sigma) \leq 1 / k$.  \hfill $\Box$

**2.2 Proof of Theorem 1**

Let $\sigma \in \mathcal{S}(n, k)$ be a local maximum of $F_{\Lambda}$ on $\mathcal{S}(n, k)$, and $\Lambda$ be the associated multipliers. By Lemma 2.2 (point 5), we can fix $x \in \mathbb{R}^k \|x\|_2 = 1$ such that

$$\Delta \equiv \sigma x \in \mathbb{R}^n, \quad \|\Delta\|_2 \leq \sqrt{\frac{n}{k}}.$$  \hfill (23)

Let $S \subseteq [n]$, be the set of indices $S \equiv \{ i \in [n] : |\Delta_i| \geq 1 / \sqrt{2} \}$. By Markov inequality $|S| \leq 2n / k$.

For a vector $z \in \mathbb{R}^n$, let $\tilde{z}$ be the vector obtained by zero-ing the entries in $S$. Define $u = (u_1, \ldots, u_n)$, $u_i \in \mathbb{R}^k$, by

$$u_i = \tilde{z}_i x + \tilde{u}_i, \quad \tilde{u}_i \equiv -\frac{\tilde{z}_i \Delta_i}{1 - \Delta_i^2} P_x^\perp \sigma_i,$$  \hfill (24)

where $P_x^\perp = I - xx^T$ is the projector orthogonal to $x$. Here, it is understood that $\tilde{u}_i = 0$ when $|\Delta_i| = 1$.

Note that $\Delta_i = \langle \sigma_i, x \rangle \in [-1, 1]$. Hence $\|P_x^\perp \sigma_i\|_2^2 = \langle \sigma_i, P_x^\perp \sigma_i \rangle = 1 - \Delta_i^2$. We therefore have

$$\langle \sigma_i, u_i \rangle = \tilde{z}_i \Delta_i - \frac{\tilde{z}_i \Delta_i}{1 - \Delta_i^2} \langle \sigma_i, P_x^\perp \sigma_i \rangle = 0.$$  \hfill (25)

We can therefore apply the second-order stationarity condition (3), to get (with the notation $v_R = (v_i : i \in R)$, for $R \subseteq [n]$):

$$0 \leq \sum_{i,j=1}^n (\Lambda - A)_{i,j} \langle u_i, u_j \rangle$$  \hfill (26)

$$= \sum_{i,j=1}^n (\Lambda - A)_{i,j} \tilde{z}_i \tilde{z}_j + \sum_{i,j=1}^n (\Lambda - A)_{i,j} \langle \tilde{u}_i, \tilde{u}_j \rangle$$  \hfill (27)

$$\leq \langle \tilde{z}, (\Lambda - A) \tilde{z} \rangle + \sum_{i=1}^n \lambda_i \|\tilde{u}_i\|_2^2 + \|A\|_2 \|\tilde{u}\|_F^2$$  \hfill (28)

$$\leq \langle \tilde{z}, (\Lambda - A) \tilde{z} \rangle + \sum_{i=1}^n \lambda_i \|\tilde{u}_i\|_2^2 + \|A\|_2 \|\tilde{u}\|_F^2,$$  \hfill (29)
where in the last inequality $\|M\|_*$ denotes the nuclear norm of matrix $M$ (sum of absolute values of singular values), and we used the fact that $\|MM^T\|_* = \|M\|_F^2$. Now

$$
\|\tilde u\|_F^2 = \sum_{i=1}^n \|\tilde u_i\|_2^2
$$

(30)

$$
= \sum_{i=1}^n \tilde z_i^2 \Delta_i^2
$$

(31)

$$
\leq 2 \sum_{i=1}^n \tilde z_i^2 \Delta_i^2.
$$

(32)

Therefore, continuing from Eq. (29), and using a similar bound for the first sum, we get the inequality

$$
0 \leq \langle z_{S^c}, (\Lambda - A)_{S^c,S^c} z_{S^c} \rangle + 2 \sum_{i \in S^c} \lambda_i z_i^2 \Delta_i^2 + 2 \|A\|_2 \sum_{i \in S^c} \tilde z_i^2 \Delta_i^2,
$$

(33)

where we made explicit the dependence on $S$.

Now consider $v \in S(n, n)$, i.e. $v = (v_1, v_2, \ldots, v_n) = S^{n-1}$. Applying the last inequality to the $a$-th column of $v$, denoted by $v^a$, we get

$$
0 \leq \langle v^a_{S^c}, (\Lambda - A)_{S^c,S^c} v^a_{S^c} \rangle + 2 \sum_{i \in S^c} \lambda_i (v_i^a)^2 \Delta_i^2 + 2 \|A\|_2 \sum_{i \in S^c} (v_i^a)^2 \Delta_i^2.
$$

(34)

Summing over $a \in \{1, \ldots, n\}$ and using $\sum_{a=1}^n (v_i^a)^2 = 1$, we get

$$
0 \leq \langle v_{S^c}, (\Lambda - A)_{S^c,S^c} v_{S^c} \rangle + 2 \sum_{i \in S^c} \lambda_i \Delta_i^2 + 2 \|A\|_2 \sum_{i \in S^c} \Delta_i^2.
$$

(35)

Using Lemma 2.2 points 2 and 3 (in particular, $\lambda_i \geq 0$), we get

$$
\langle v_{S^c}, A_{S^c,S^c} v_{S^c} \rangle \leq F_A(\sigma) + 2 \sum_{i \in S^c} \lambda_i \Delta_i^2 + 2 \|A\|_2 \sum_{i \in S^c} \Delta_i^2
$$

(36)

$$
\leq F_A(\sigma) + \sqrt{2} \sum_{i \in S^c} \lambda_i |\Delta_i| + 2 \|A\|_2 \|\Delta\|_2^2
$$

(37)

$$
\leq F_A(\sigma) + \sqrt{2} \|A\|_F \|\Delta\|_2 + 2 \|A\|_2 \|\Delta\|_2^2,
$$

(38)

where the second inequality follows since $|\Delta_i| \leq 1/\sqrt{2}$ for $i \in S$, and the last one by Cauchy-Schwartz. Finally using Lemma 2.2 point 4, together with Eq. (23), we get

$$
\langle v_{S^c}, A_{S^c,S^c} v_{S^c} \rangle \leq F_A(\sigma) + \sqrt{\frac{8}{k}} n \|A\|_2.
$$

(39)

Next notice that

$$
\langle v_S, A v_{S^c} \rangle \leq \|A_{S^c,S^c}\|_2 \|v_{S^c} v_S^T\|_*
$$

$$
\leq \|A\|_2 \|v_S\|_F \|v_{S^c}\|_F = \|A\|_2 \sqrt{|S| (n - |S|)}
$$

(40)

$$
\leq \sqrt{\frac{2}{k}} n \|A\|_2.
$$

(41)
Proceeding analogously, we get
\[
\langle v_S, Av_S \rangle \leq \frac{2}{k} n \| A \|_2 .
\] (43)

Finally, combining the bounds (39), (42), (43) we get
\[
\langle v, Av \rangle \leq F_A(\sigma) + \frac{5\sqrt{2}}{\sqrt{k}} n \| A \|_2 .
\] (44)

Since this holds for any \( v \in S(n, n) \), it implies the second inequality in Eq. (6), because \( 5\sqrt{2} < 8 \). The first inequality in Eq. (6) is trivial.

Acknowledgments

This work was partially supported by NSF grants CCF-1319979 and DMS-1106627 and the AFOSR grant FA9550-13-1-0036.

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