Abstract

In this paper, we first establish a kind of weighted space-time $L^2$ estimate, which belongs to Keel-Smith-Sogge type estimates, for perturbed linear elastic wave equations. This estimate refines the corresponding one established by the second author [J. Differential Equations 263 (2017), 1947–1965] and is proved by combining the methods in the former paper, the first author, Wang and Yokoyama’s paper [Adv. Differential Equations 17 (2012), 267–306], and some new ingredients. Then together with some weighted Sobolev inequalities, this estimate is used to show a refined version of almost global existence of classical solutions for nonlinear elastic waves with small initial data. Compared with former almost global existence results for nonlinear elastic waves due to John [Comm. Pure Appl. Math. 41 (1988) 615–666], Klainerman-Sideris [Comm. Pure Appl. Math. 49 (1996) 307–321], the main innovation of our one is that it considerably improves the amount of regularity of initial data, i.e., the Sobolev regularity of initial data is assumed to be the smallest among all the admissible Sobolev spaces of integer order in the standard local existence theory. Finally, in the radially symmetric case, we establish the almost global existence of a low regularity solution for every small initial data in $H^3 \times H^2$.

keywords: Elastic waves; Keel-Smith-Sogge type estimates; regularity; almost global existence

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1 Introduction and main results

As a significant kind of space-time $L^2$ estimates for wave equations, the Keel-Smith-Sogge (KSS for short) type estimate, which was first introduced in Keel, Smith and Sogge [16], is largely recognized as a key to giving a simple proof of long time existence of small solutions to nonlinear wave equations (see, e.g., Section 2.3 of [36]). In particular, since Rodnianski [38] discovered the multiplier method for the proof of this kind of estimates, for studying quasilinear wave equations, the KSS type estimate for linear wave operators with variable coefficients has drawn many attentions (see [27], [28], [7] etc.). For perturbed linear elastic wave equations, the second author [40] proved a KSS type estimate where the weight is a negative power of $\langle x \rangle$. A full history of KSS type estimates can be found in [40].
In this paper, we first revisit [40] to present a refinement in that the KSS estimate thereby obtained in [40] remains to hold even for the weight of the form of a negative power of $|x|^{2\delta} |1|^{-2\delta}$ for small $\delta > 0$. To achieve this goal, we will use the framework in [40], the multiplier method of Rodnianski and the multiplier (suggested by Metcalfe) used in Hidano, Wang and Yokoyama [7], where such weight has been used (see also [9]) in the context of wave equations with low-regularity radial data, and some new ingredients associated with the distinguishing features of elastic waves. We can also find a significant difference from the earlier version of [40] in that the present version of the KSS estimate leads to not only a simple proof, but also considerable improvement of the earlier results due to John [15], Klainerman-Sideris [19], and the second author [40] concerning almost global existence of small solutions to 3-D nonlinear elastic wave equations. Recall that the standard local existence theorem was established in the Sobolev space $H^{s+1} \times H^s$ with $s > 5/2$ (see Hughes, Kato and Marsden [11]), which has motivated us to obtain almost global existence result in $H^4 \times H^3$, the lowest integer order Sobolev space. Our theorem thereby proved requires the smallness of data with respect to some weighted $H^4 \times H^3$ norm, while, say, the results in [19] and [40] required it with respect to some weighted $H^7 \times H^6$ norm and weighted $H^{10} \times H^9$ norm, respectively. Compared with [19], there exists another advantage in the proof of almost global existence on the basis of the KSS estimate: it never uses the scaling operator $S = t \partial_t + x \cdot \nabla$. We benefit from this advantage, especially when studying radially symmetric solutions (see page 6 below concerning the definition of radial symmetry of $\mathbb{R}^3$-valued functions, see also Remark 4.1 below). Owing to $\tilde{\Omega}_{ij} u \equiv 0$ for radial functions (see (1.13) below for the definition of the generators of simultaneous rotations $\tilde{\Omega}_{ij}$), we have almost global existence for small $H^4 \times H^3$ radial data. Note that no additional decay is required. Actually, by our analysis we can go further below the Sobolev space $H^{s+1} \times H^s (s > 5/2)$ where, as mentioned above, Hughes, Kato, and Marsden established the local existence theorem. That is, we prove almost global existence for small $H^3 \times H^2$ radial data.

Before the further formulation of our results, let us first review the history on the topics considered here. The long time existence of classical solutions for nonlinear elastic waves started from Fritz John’s pioneering work on elastodynamics (see Klainerman [17]). John [14] showed that local classical solutions in general develop singularities for radial and small initial data, and they almost globally exist [15]. See also simplified proofs in [19], [40] and a lower bound estimate in [20]. In order to ensure the global existence of classical solutions with small initial data, some structural condition on the nonlinearity which is called the null condition is necessary. We refer the reader to Agemi [1] and Sideris [33] (see also a previous result in Sideris [32]). The exterior domain analogues of John’s almost global existence result and Agemi and Sideris’s global existence result were obtained in [26] and [29], respectively. We note that all the above works are in the framework of classical solutions and very high regularity on the initial data is required.

Now we give a brief review for the low regularity well-posedness for the Cauchy problem of nonlinear wave equations. Studies on local low regularity well-posedness for the Cauchy problem of nonlinear wave equations started from Klainerman-Machedon’s pioneering work [18] in the semilinear case. For general quasilinear equations, we refer the reader to the sharpest result [35], [39] and the references therein. On the other hand, Lindblad constructed some counterexamples to show sharp results of ill-posedness [21], [22]. It should be mentioned
that improvement of regularity in the presence of radial symmetry was first observed in \cite{18} for semilinear equations such as $\Box u = c_1(\partial_t u)^2 + c_2|\nabla u|^2$. (See also page 176 of \cite{31}, Section 5 of \cite{25}, and \cite{9}, \cite{30}, and \cite{6} for related results.) We also mention that it was also observed for the wave equation with a power-type nonlinear term $\Box u = F(u)$, first by Lindbald and Sogge \cite{23}, and then by some authors (see \cite{4}, \cite{12}, \cite{3}), and quasilinear wave equations of the form $\partial_t^2 u - a^2(\cdot) \Delta u = c_1(\partial_t u)^2 + c_2|\nabla u|^2$, by \cite{7} and \cite{41}. In particular, the almost global existence of low regularity radially symmetric solutions with small initial data was showed in \cite{9} and \cite{7} for the semilinear and the quasilinear case in 3-D, respectively. A global regularity result can be found in \cite{41}, which is the first result for global existence of low regularity solutions to 3-D quasilinear wave equations. We also note that the global existence for some 4-D quasilinear wave equations with low regularity was obtained in \cite{24} recently. By the analysis of the present paper, improvement of regularity for almost global solutions can be successfully observed for nonlinear elastic wave equations under the assumption of radial symmetry.

The outline of this paper is as follows. The remainder of this section will be devoted to the exact description of three main results in this paper, i.e., the space-time $L^2$ estimates for perturbed linear elastic wave equations (Theorem 1.1), refined version of almost global existence for nonlinear elastic wave equations with small weighted $H^4 \times H^3$ norm (Theorem 1.2) and low regularity almost global existence in the radially symmetric case for nonlinear elastic wave equations with small $H^3 \times H^2$ norm (Theorem 1.3). In the next section, we will prove Theorem 1.1. The proof of Theorem 1.2 will be given in Section 3 and the proof of Theorem 1.3 will be given in Section 4. They are all based on Theorem 1.1 and some weighted Sobolev inequalities.

### 1.1 Space-time $L^2$ estimates for perturbed linear elastic wave equations

Consider the following operator corresponding to perturbed linear elastic waves

$$L_h u = Lu + Hu,$$

where the linear elastic wave operator

$$Lu = \partial_t^2 u - c_2^2 \Delta u - (c_1^2 - c_2^2) \nabla \cdot u,$$

and the perturbed term

$$(Hu)^i = \partial_t (h_{lm}^{ij}(t,x) \partial_l w^j), \quad i = 1, 2, 3.$$

Here $u(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$ denotes the displacement vector from the reference configuration, and the material constants $c_1$ (pressure wave speed) and $c_2$ (shear wave speed) satisfy $0 < c_2 < c_1$.

The first main result in this paper is the following

**Theorem 1.1.** Assume that $h = (h_{lm}^{ij})$ satisfies the following symmetric condition

$$h_{lm}^{ij} = h_{ml}^{ji},$$

\footnote{We use the summation convention over repeated indices.}
and the smallness condition

\[ |h| = \sum_{i,j,l,m=1}^{3} |h_{lm}^{ij}| \ll 1. \tag{1.5} \]

Denote the strip by \( S_T = [0, T] \times \mathbb{R}^3 \). Suppose that \( u \in C^\infty([0, T]; C^\infty(\mathbb{R}^3; \mathbb{R}^3)) \). Then we have

\[
\sup_{0 \leq t \leq T} \|\partial_t u\|_{L^2(\mathbb{R}^3)} + (\log (2 + T))^{-1/2} \langle r \rangle^{-1/2} \| \partial_t u \|_{L^2_{t,x}(S_T)} + (\log (2 + T))^{-1/2} \langle r \rangle^{-3/2} \| u \|_{L^2_{t,x}(S_T)} \\
\leq C \| \partial_t u(0, \cdot) \|_{L^2(\mathbb{R}^3)} + C \| L_0 u \|_{L^1_t L^2_x(S_T)} + C \| \partial_t h \| \| \nabla u \|_{L^1_t L^2_x(S_T)} + C \| \langle r \rangle^{-1} |h| \| \nabla u \|_{L^3_t L^3_x(S_T)} \tag{1.6}
\]

and for any \( 0 < \delta < 1/2 \),

\[
\sup_{0 \leq t \leq T} \|\partial_t u\|_{L^2(\mathbb{R}^3)} + (\log (2 + T))^{-1/2} \langle r \rangle^{-1/2} \| r \|^{-1/2+\delta} \partial_t u \|_{L^2_{t,x}(S_T)} + (\log (2 + T))^{-1/2} \langle r \rangle^{-3/2+\delta} \| u \|_{L^2_{t,x}(S_T)} \\
\leq C \| \partial_t u(0, \cdot) \|_{L^2(\mathbb{R}^3)} + C \| L_0 u \|_{L^1_t L^2_x(S_T)} + C \| \partial_t h \| \| \nabla u \|_{L^1_t L^2_x(S_T)} + C \| \langle r \rangle^{-1} |h| \| \nabla u \|_{L^3_t L^3_x(S_T)}, \tag{1.7}
\]

where \( r = |x| \) and \( \langle \cdot \rangle = (1 + |\cdot|^2)^{1/2} \), here and in what follows \( C \) denotes a positive constant.

### 1.2 Refined almost global existence for nonlinear elastic wave equations

Consider the following Cauchy problem for homogeneous, isotropic and hyperelastic waves:

\[
\begin{aligned}
\partial_t^2 u - c_0^2 \Delta u - (c_1^2 - c_0^2) \nabla \cdot u &= N(u, u), \\
t = 0 : u = u_0, \quad u_t = u_1.
\end{aligned} \tag{1.8}
\]

Here the nonlinear term \( N \) is taken as

\[ N(u, v)^i = \partial_i (g_{mn}^{ijk} \partial_m u^j \partial_n v^k), \tag{1.9} \]

where the coefficients are constants and are symmetric with respect to pairs of indices

\[ g_{mn}^{ijk} = g_{mn}^{ijk} = g_{nmi}^{kji}, \tag{1.10} \]

and \( g = (g_{mn}^{ijk}) \) is a six order isotropic tensor.

The angular momentum operators (generator of the spatial rotation) are the vector fields

\[ \Omega = (\Omega_{ij} : 1 \leq i < j \leq 3), \tag{1.11} \]

where

\[ \Omega_{ij} = x_i \partial_j - x_j \partial_i. \tag{1.12} \]
Denote the generators of simultaneous rotations by \( \tilde{\Omega} = (\tilde{\Omega}_{ij} : 1 \leq i < j \leq 3) \), where
\[
\tilde{\Omega}_{ij} = \Omega_{ij} I + U_{ij},
\]
and \( U_{ij} = e_i \otimes e_j - e_j \otimes e_i \), \( \{e_i\}_{i=1}^3 \) is the standard basis on \( \mathbb{R}^3 \). Denote the collection of vector fields by \( Z = (\nabla, \tilde{\Omega}) \), which is time-independent. It is easy to verify the following commutation relationship
\[
[\tilde{\Omega}, \nabla] = \nabla.
\]  
(1.14)

By \( Z^a \), \( a = (a_1, \ldots, a_k) \), we denote an ordered product of \( k = |a| \) vector fields \( Z_{a_1} \cdots Z_{a_k} \).

Define the time-independent spaces
\[
H^k_Z = \{ f \in L^2(\mathbb{R}^2; \mathbb{R}^3) : Z^a f \in L^2(\mathbb{R}^2; \mathbb{R}^3), |a| \leq k \}
\]
with the norm
\[
\| f \|_{Z,k} = \left( \sum_{|a| \leq k} \| Z^a f \|^2_{L^2(\mathbb{R}^3)} \right)^{1/2}.
\]  
(1.16)

The solution will be constructed in the space \( X^k_Z(T) \) obtained by closing the set \( C^\infty([0,T]; C^\infty_0 (\mathbb{R}^3; \mathbb{R}^3)) \) in the norm \( \| u \|_{X^k_Z(T)} \), where
\[
\| u \|_{X^k_Z(T)} = \sum_{|a| \leq k-1} \| \partial Z^a u \|_{L^\infty_t L^2_x(S_T)} + (\log(2 + T))^{-1/2} \sum_{|a| \leq k-1} \| (r)^{-\delta} \partial^{1/2+\delta} \partial Z^a u \|_{L^2_t(S_T)}
\]  
(1.17)
\[
+ (\log(2 + T))^{-1/2} \sum_{|a| \leq k-1} \| (r)^{-\delta} \partial^{-3/2+\delta} Z^a u \|_{L^2_t(S_T)},
\]
and \( 0 < \delta \leq 1/4 \) is fixed.

The second main result in this paper is the following

**Theorem 1.2.** There exist constants \( \kappa, \varepsilon_0 > 0 \) such that for any given \( \varepsilon \) with \( 0 < \varepsilon \leq \varepsilon_0 \), if
\[
\| \nabla u_0 \|_{Z,3} + \| u_1 \|_{Z,3} \leq \varepsilon,
\]
then Cauchy problem (1.8) admits a unique classical solution \( u \in X^4_Z(T_\varepsilon) \) with
\[
T_\varepsilon = \exp(\kappa/\varepsilon).
\]  
(1.19)

**Remark 1.1.** Using (3.2) in Klainerman and Sideris [19], following the argument in Sideris and Tu [34] and adapting the argument (3.11)–(3.13) and (4.18)–(4.26) in Hidano [5], we are able to prove almost global existence under the slightly more restrictive condition
\[
\| \nabla u_0 \|_{Z,3} + \| u_1 \|_{Z,3} + \| r \partial_r \nabla u_0 \|_{Z,2} + \| r \partial_r u_1 \|_{Z,2} \leq \varepsilon.
\]  
(1.20)

The last two norms on the left-hand side above come from the use of the scaling operator \( S = \partial_t + r \partial_r \). It is well known that the method in Klainerman and Sideris [19] and Sideris [33] works for the proof of global existence under the null condition.

\(^2\)Note that the notation used for \( Z^a \) differs from the standard multi-index notation.
Remark 1.2. For radially symmetric initial data\(^3\) \(u_0, u_1\), noting that \(\hat{\Omega}u_0 = \hat{\Omega}u_1 = 0\), we easily see the condition (1.18) is satisfied whenever the norm

\[
\|\nabla u_0\|_{H^3} + \|u_1\|_{H^3}
\]

is small enough. Note that we assume no additional decay. The result of almost global existence for nonlinear elastic wave equations is new when smallness is required of only such Sobolev norm of radial data. In fact, we will even lower the regularity requirement on initial data by one derivative in the next subsection.

1.3 Low regularity almost global existence in the radially symmetric case

Noting the nonlinear elastic wave equation is invariant under simultaneous rotations (see page 860 of [33]), we can seek radially symmetric solution for the Cauchy problem (1.8) with radially symmetric initial data.

Consider the space \(X^k(T)\), which is obtained by closing the set \(C^\infty([0,T];C^\infty_0(\mathbb{R}^3;\mathbb{R}^3))\) in the norm \(\|u\|_{X^k(T)}\), where

\[
\|u\|_{X^k(T)} = \sum_{|a| \leq k-1} \|\partial^a \nabla u\|_{L^\infty_t L^2_x(S_T)} + (\log(2 + T))^{-1/2} \sum_{|a| \leq k-1} \|\langle r \rangle^{-\delta} r^{-1/2} \partial^a \nabla u\|_{L^2_t x(S_T)} + (\log(2 + T))^{-1/2} \sum_{|a| \leq k-1} \|\langle r \rangle^{-\delta} r^{-3/2} \partial^a \nabla u\|_{L^2_t x(S_T)},
\]

(1.22)

and \(0 < \delta \leq 1/4\) is fixed. The solution will be constructed in the space \(X^k_{rad}(T) = \{u \text{ is radially symmetric}, \|u\|_{X^k(T)} < +\infty\}\).

(1.23)

The third main result in this paper is the following

**Theorem 1.3.** There exist constants \(\kappa, \varepsilon_0 > 0\) such that for any given \(\varepsilon\) with \(0 < \varepsilon \leq \varepsilon_0\), if the initial data is radially symmetric and satisfies

\[
\|\nabla u_0\|_{H^2} + \|u_1\|_{H^2} \leq \varepsilon,
\]

(1.24)

then Cauchy problem (1.8) admits a unique solution \(u \in X^3_{rad}(T_\varepsilon)\) with

\[
T_\varepsilon = \exp(\kappa/\varepsilon).
\]

(1.25)

2 Proof of Theorem 1.1

The strategy of our proof of Theorem 1.1 is as follows. As in [40], we first use the Hodge decomposition to decompose the solution into its curl-free and divergence-free components, then employ the multiplier method of Rodnianski (see Appendix of [38]) which works for the wave equation with variable coefficients. Here we use the multiplier in [7] which was suggested by Metcalfe. Inevitably, the presence of the inverse of the Laplacian in front of the perturbation terms then becomes a major obstacle to success. Outside of the integration-by-part argument, the boundedness of the Riesz operators on the Lebesgue space \(L^2\) with the

\(\text{A vector function } v \text{ is called radial if it has the form } v(x) = x \phi(r) (r = |x|), \text{ where } \phi \text{ is a scalar radial function. We refer the reader to Definition 4.4 and Lemma 4.5 of [13].}\)
Muckenhoupt $A_2$ weight, together with the divergence form of the perturbation terms, plays a crucial role in handling such terms. Note that on the right-hand side of (1.6) and (1.7), we have to take $L^2$ norm in spatial variables. Particularly in (1.7), for the consideration of nonlinear applications in the following two sections, we have to avoid some terms which are too singular near $r = 0$. See (2.45). This is why a space-time $L^2$ norm is taken in the last term on the right-hand side of (1.7), which is also the main distinction between our KSS estimate (1.7) for elastic waves and the corresponding one for wave equations such as (2.5) in [7].

First we introduce the Hodge decomposition in $\mathbb{R}^3$. For function $u \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$, denote the curl of $u$ by $\nabla \times u$. The following lemma is the Hodge decomposition, which projects any vector field onto its curl-free and divergence-free components. It is widely known and the proof can be found in [2].

**Lemma 2.1.** For any function $u \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ with sufficient decay at infinity, we have

$$u = u_{cf} + u_{df},$$

where

$$u_{cf} = \nabla \nabla \cdot \Delta^{-1} u, \quad u_{df} = -\nabla \times \nabla \times \Delta^{-1} u.$$  

And it holds that

$$\nabla \times u_{cf} = 0, \quad \nabla \cdot u_{df} = 0,$$

$$\|u\|_{L^2(\mathbb{R}^3)}^2 = \|u_{cf}\|_{L^2(\mathbb{R}^3)}^2 + \|u_{df}\|_{L^2(\mathbb{R}^3)}^2,$$

$$\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 = \|\nabla u_{cf}\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_{df}\|_{L^2(\mathbb{R}^3)}^2.$$  

In order to treat the effect of the inverse of the Laplacian, we will also use the following

**Lemma 2.2.** Let $0 < \delta < 1/2$. For the Riesz transformation $R_i = \frac{\partial}{\sqrt{-\Delta}}$, we have

$$\|\langle x \rangle^{-\delta} |x|^{-1/2+\delta} R_i f \|_{L^2(\mathbb{R}^3)} \leq C \|\langle x \rangle^{-\delta} |x|^{-1/2+\delta} f \|_{L^2(\mathbb{R}^3)}.$$  

**Proof.** When $0 < \delta < 1/2$, the weight $\langle x \rangle^{-2\delta} |x|^{-1+2\delta}$ belongs to the Muckenhoupt class $A_1$ (see Lemma 2.5 of [6]). Since $A_1 \subset A_2$, it also belongs to the Muckenhoupt class $A_2$. Thanks to this fact, we enjoy the $L^2(\mathbb{R}^3, \langle x \rangle^{-2\delta} |x|^{-1+2\delta} dx)$ boundedness of the Riesz transformation, i.e., (2.6) holds. See page 205 of Stein [37].

Now we prove Theorem 1.1. We note that (1.6) has been proved in [40]. Actually, from the proof of (1.6), it is easy to find that it also holds that

$$(\log(2 + T))^{-1/2} \|\langle r \rangle^{-1/2} \partial u_{cf} \|_{L^2(\mathbb{R}^3)} + (\log(2 + T))^{-1/2} \|\langle r \rangle^{-3/2} u_{cf} \|_{L^2(\mathbb{R}^3)}$$

$$+ (\log(2 + T))^{-1/2} \|\langle r \rangle^{-1/2} u_{df} \|_{L^2(\mathbb{R}^3)} + (\log(2 + T))^{-1/2} \|\langle r \rangle^{-3/2} u_{df} \|_{L^2(\mathbb{R}^3)}$$

$$\leq C \|\partial h(0, \cdot) \|_{L^2(\mathbb{R}^3)} + C \| L_h u \|_{L^2(\mathbb{R}^3)} + C \| \partial h \|_{L^2(\mathbb{R}^3)} + C \| h \|_{L^2(\mathbb{R}^3)} + C \| \langle r \rangle^{-1} h \|_{L^2(\mathbb{R}^3)}.$$  

(2.7)
which will be useful in the proof of (1.7).

Now we will give the proof of (1.7). For the following classical energy estimate

$$\sup_{0 \leq t \leq T} \|\partial u\|_{L^2(\mathbb{R}^3)}^2 \leq C \|\partial u(0, \cdot)\|_{L^2(\mathbb{R}^3)}^2 + C\|\partial_t h\|\|\nabla u\|_{L^2_t L^2_x(\mathbb{R}^3)}^2 + C\|L_h u\|_{L^2_t L^2_x(\mathbb{R}^3)}^2, \quad (2.8)$$

we omit its proof because it has been given in the literature (see, e.g., page 1951 of [40]).

Now we give the proof of the weighted space-time $L^2$ estimates. By Lemma 2.1, we have

$$u = u_{cf} + u_{df}, \quad Hu = (Hu)_{cf} + (Hu)_{df}. \quad (2.9)$$

Denote the wave operator by $\Box_c = \partial_t^2 - c^2 \Delta$. It can be verified that

$$(L_h u)_{cf} = \Box_c u_{cf} + (Hu)_{cf}, \quad (L_h u)_{df} = \Box_c u_{df} + (Hu)_{df}. \quad (2.10)$$

Denote the multiplier vector field

$$M = f(r)\partial_r + \frac{1}{r} f(r), \quad (2.11)$$

where $f(r)$ will be determined later. It follows from (2.9) and (2.10) that

$$\langle Mu_{cf}, \Box_c u_{cf} \rangle + \langle Mu_{df}, \Box_c u_{df} \rangle = -\langle Mu_{cf}, (Hu)_{cf} \rangle - \langle Mu_{df}, (Hu)_{cf} \rangle + \langle Mu_{cf}, (L_h u)_{cf} \rangle + \langle Mu_{df}, (L_h u)_{df} \rangle$$

$$= -\langle Mu, Hu \rangle + \langle Mu_{df}, (Hu)_{cf} \rangle + \langle Mu_{cf}, (Hu)_{df} \rangle$$

$$+ \langle Mu_{cf}, (L_h u)_{cf} \rangle + \langle Mu_{df}, (L_h u)_{df} \rangle. \quad (2.12)$$

By the Leibniz rule, we can establish the following differential identity

$$\langle Mu_{cf}, \Box_c u_{cf} \rangle = \partial_t e_1 + \nabla \cdot p_1 + q_1, \quad (2.13)$$

where

$$e_1 = f(r)(\partial_t u_{cf}^i)(\partial_t u_{cf}^i + \frac{1}{r} u_{cf}^i), \quad (2.14)$$

$$p_1 = \frac{1}{2} f(r)\omega (c_t^2 |\nabla u_{cf}|^2 - |\partial_t u_{cf}|^2) - c_t^2 f(r)(\nabla u_{cf}^i)(\partial_t u_{cf}^i + \frac{1}{r} u_{cf}^i)$$

$$+ \frac{1}{2} c_t^2 r^{-1} f'(r) - f(r)\omega |u_{cf}|^2, \quad (2.15)$$

$$q_1 = \frac{f(r)}{2} |\partial_t u_{cf}|^2 + c_t^2 f'(r) |\partial_t u_{cf}|^2$$

$$+ c_t^2 \left( \frac{f(r)}{r} - \frac{f'(r)}{2} \right) |\nabla \omega u_{cf}|^2 - \frac{1}{2} c_t^2 (\Delta \frac{f(r)}{r}) |u_{cf}|^2, \quad (2.16)$$

and

$$|\nabla \omega u_{cf}|^2 = |\nabla u_{cf}|^2 - |\partial_r u_{cf}|^2 \quad (2.17)$$

is the angular component of the gradient. Similarly, we also have

$$\langle Mu_{cf}, \Box_c u_{df} \rangle = \partial_t e_2 + \nabla \cdot p_2 + q_2, \quad (2.18)$$
where
\[ e_2 = f(r)(\partial_r u_{df}^i)(\partial_r u_{df}^i + \frac{1}{r} u_{df}^i), \] (2.19)
\[ p_2 = \frac{1}{2} f(r)\omega (c_2^2 |\nabla u_{df}^i|^2 - |\partial_t u_{df}^i|^2) - c_2^2 f(r)(\nabla u_{df}^i)(\partial_r u_{df}^i + \frac{1}{r} u_{df}^i) \]
\[ + \frac{1}{2} r f'(r)\frac{\omega}{r} |u_{df}^i|^2, \] (2.20)
\[ q_2 = \frac{f'(r)}{2} |\partial_t u_{df}^i|^2 + \frac{c_2^2 f'(r)}{2} |\partial_r u_{df}^i|^2 \]
\[ + \frac{c_2^2}{r} (\frac{f(r)}{r} - \frac{f'(r)}{2}) |\nabla u_{df}^i|^2 - \frac{1}{2} \frac{c_2^2}{r^2} (\Delta \frac{f(r)}{r}) |u_{df}^i|^2. \] (2.21)

Noting the symmetry condition (1.4), we can get
\[ \langle Mu, Hu \rangle = \nabla \cdot p_3 + q_3, \] (2.22)
where
\[ (p_3)_l = f(r)\omega_k h_{lm}^{ij} \partial_k u^i \partial_m u^j - \frac{1}{2} f(r)\omega_k h_{km}^{ij} \partial_k u^i \partial_m u^j \]
\[ + \frac{1}{r} f(r) h_{lm}^{ij} u^i \partial_m u^j, \quad l = 1, 2, \ldots, n, \] (2.23)
\[ q_3 = -\frac{rf'(r)}{r} - \frac{f(r)}{r} \omega_k h_{lm}^{ij} \partial_k u^i \partial_m u^j + \frac{1}{2} f'(r) h_{lm}^{ij} \partial_k u^i \partial_m u^j \]
\[ + \frac{1}{2} f(r) \omega_k (\partial_k h_{lm}^{ij}) \partial_k u^i \partial_m u^j - \frac{1}{r} f(r) h_{lm}^{ij} \partial_k u^i \partial_m u^j \]
\[ - \frac{rf'(r)}{r^2} \omega h_{lm}^{ij} u^i \partial_m u^j. \] (2.24)

We have the following differential identity
\[ \langle Mu_{df}, (Hu)_{cf} \rangle = \nabla \cdot p_4 + q_4, \] (2.25)
where
\[ p_4 = (Mu_{df}) \nabla \cdot \Delta^{-1}(Hu), \] (2.26)
\[ q_4 = -\nabla \cdot (Mu_{df}) \nabla \cdot \Delta^{-1}(Hu). \] (2.27)

Noting that \( \nabla \cdot u_{df} = 0 \), we can verify that
\[ \nabla \cdot (Mu_{df}) = (f'(r) - \frac{f(r)}{r}) \omega \cdot (\partial_r u_{df} + \frac{1}{r} u_{df}). \] (2.28)

We also have the following differential identity
\[ \langle Mu_{cf}, (Hu)_{df} \rangle = \nabla \cdot p_5 + q_5, \] (2.29)
where
\[ p_5 = (Mu_{cf}) \times \nabla \times \Delta^{-1}(Hu), \] (2.30)
\[ q_5 = -\nabla \times (Mu_{cf}) \cdot (\nabla \times \Delta^{-1}(Hu)). \] (2.31)

Noting that \( \nabla \times u_{cf} = 0 \), we can verify that
\[ \nabla \times (Mu_{cf}) = (f'(r) - \frac{f(r)}{r}) \omega \times (\partial_r u_{cf} + \frac{1}{r} u_{cf}). \] (2.32)
By the above discussion, we have
\[
q_1 + q_2 = -\partial_t(e_1 + e_2) + \nabla \cdot (-p_1 - p_2 - p_3 + p_4 + p_5)
- q_3 + q_4 + q_5 + \langle Mu_{cf}, (L_h u)_{cf} \rangle + \langle Mu_{g}, (L_h u)_{g} \rangle. \tag{2.33}
\]
Following [7], we take
\[
f(r) = \left( \frac{r}{1 + r} \right)^{2\delta}. \tag{2.34}
\]
Some simple computations can give
\[
f'(r) = 2\delta r^{2\delta-1}(1 + r)^{-2\delta-1}, \quad f(r) - \frac{f'(r)}{2} = r^{2\delta-1}(1 + r)^{-2\delta}(1 - \frac{\delta}{1 + r}), \tag{2.35}
\]
\[
\Delta \frac{f(r)}{r} \leq -2\delta(1 - 2\delta)r^{2\delta-3}(1 + r)^{-2\delta-2}, \tag{2.36}
\]
\[
f'(r) - \frac{f(r)}{r} = r^{2\delta-1}(1 + r)^{-2\delta} \left( \frac{2\delta}{1 + r} - 1 \right). \tag{2.37}
\]
By (2.16), (2.21), (2.35) and (2.36), we have
\[
q_1 + q_2 \geq C \left( r^{2\delta-1}(1 + r)^{-2\delta-1}|\partial_t u_{cf}|^2 + r^{2\delta-1}(1 + r)^{-2\delta-1}|\partial_t u_{cf}|^2 
+ r^{2\delta-1}(1 + r)^{-2\delta}|\nabla \omega u_{cf}|^2 + r^{2\delta-3}(1 + r)^{-2\delta}|u_{cf}|^2 \right) 
+ C \left( r^{2\delta-1}(1 + r)^{-2\delta-1}|\partial_t u_{df}|^2 + r^{2\delta-1}(1 + r)^{-2\delta-1}|\partial_t u_{df}|^2 
+ r^{2\delta-1}(1 + r)^{-2\delta}|\nabla \omega u_{df}|^2 + r^{2\delta-3}(1 + r)^{-2\delta}|u_{df}|^2 \right). \tag{2.38}
\]
On both sides of (2.33), we integrate over the strip $S_T = [0, T] \times \mathbb{R}^3$. By the divergence theorem, in view of (2.38), we get
\[
\int_0^T \int_{\mathbb{R}^3} \frac{r^{2\delta-1}(1 + r)^{-2\delta-1}|\partial_t u_{cf}|^2 + r^{2\delta-1}(1 + r)^{-2\delta-1}|\partial_t u_{cf}|^2}{\nabla \omega u_{cf}}^2 + r^{2\delta-3}(1 + r)^{-2\delta}|u_{cf}|^2 \, dx \, dt 
+ \int_0^T \int_{\mathbb{R}^3} \frac{r^{2\delta-1}(1 + r)^{-2\delta-1}|\partial_t u_{df}|^2 + r^{2\delta-1}(1 + r)^{-2\delta-1}|\partial_t u_{df}|^2}{\nabla \omega u_{df}}^2 + r^{2\delta-3}(1 + r)^{-2\delta}|u_{df}|^2 \, dx \, dt 
\leq C_0 \left( \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |e_1| \, dx + \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |e_2| \, dx + \int_0^T \int_{\mathbb{R}^3} |q_3| \, dx \, dt 
+ \int_0^T \int_{\mathbb{R}^3} |q_4| \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} |q_5| \, dx \, dt 
+ \int_0^T \int_{\mathbb{R}^3} \langle Mu_{cf}, (L_h u)_{cf} \rangle \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} \langle Mu_{g}, (L_h u)_{g} \rangle \, dx \, dt \right). \tag{2.39}
\]
Now we will estimate all terms on the right-hand side of (2.39). First, by (2.14), (2.34), the Hölder inequality, the Hardy inequality, (2.4) and (2.5), we have
\[
\int_{\mathbb{R}^3} |e_1| \, dx 
\leq C \int_{\mathbb{R}^3} |\partial_t u_{df}| |\partial_t u_{cf}| \, dx + C \int_{\mathbb{R}^3} |\partial_t u_{df}| |\frac{u_{df}}{r}| \, dx 
\leq C \|\partial_t u_{df}\|_{L^2(\mathbb{R}^3)} \|\nabla u_{df}\|_{L^2(\mathbb{R}^3)} + C \|\partial_t u_{df}\|_{L^2(\mathbb{R}^3)} \|\frac{u_{df}}{r}\|_{L^2(\mathbb{R}^3)} 
\leq C \|\partial_t u_{df}\|_{L^2(\mathbb{R}^3)} \|\nabla u_{df}\|_{L^2(\mathbb{R}^3)} \leq C \|\partial_t u\|_{L^2(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)} \leq C \|\partial u\|_{L^2(\mathbb{R}^3)}^2, \tag{2.40}
\]
Similarly, we also have
\[ \int_{\mathbb{R}^3} |e_2| dx \leq C \|\partial u\|_{L^2(\mathbb{R}^3)}^2. \]  

(2.41)

So it holds that
\[ \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |e_1| dx + \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |e_2| dx \leq C \sup_{0 \leq t \leq T} \|\partial u\|_{L^2(\mathbb{R}^3)}^2. \]

(2.42)

By (2.24), (2.35) and (2.37), we have the following pointwise estimate
\[ |q_3| \leq C |\nabla h| |\nabla u|^2 + C r^{2\delta}(1 + r)^{-2\delta} \frac{|h|}{r} \left| \frac{|u|}{r} + |\nabla u| \right| |\nabla u|. \]

Hence by (2.43), the Hölder inequality, the Hardy inequality and the Cauchy-Schwarz inequality, we have
\[ \int_0^T \int_{\mathbb{R}^3} |q_3| dx \]
\[ \leq C \int_0^T \int_{\mathbb{R}^3} |\nabla h| |\nabla u|^2 + r^{2\delta}(1 + r)^{-2\delta} \frac{|h|}{r} \left| \frac{|u|}{r} + |\nabla u| \right| |\nabla u|dxdt \]
\[ \leq C \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2(\mathbb{R}^3)} \|\nabla h| |\nabla u|\|_{L^1_t L^2_x(S_T)} \]
\[ + C \|\langle r \rangle^{-2\delta} r^{-1/2+\delta} |h| |\nabla u|\|_{L^2_x(S_T)} \|\langle r \rangle^{-\delta} r^{-1/2+\delta} |\nabla u|\|_{L^2_x(S_T)} \]
\[ + C \|\langle r \rangle^{-\delta} r^{-1/2+\delta} |h| |\nabla u|\|_{L^2_x(S_T)} \|\langle r \rangle^{-\delta} r^{-3/2+\delta} |u|\|_{L^2_x(S_T)} \]
\[ \leq C \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 + C \|\nabla h| |\nabla u|\|_{L^1_t L^2_x(S_T)}^2 \]
\[ + C \|\langle r \rangle^{-2\delta} r^{-1/2+\delta} |h| |\nabla u|\|_{L^2_x(S_T)} \|\langle r \rangle^{-\delta} r^{-1/2+\delta} |\nabla u|\|_{L^2_x(S_T)} \]
\[ + C \|\langle r \rangle^{-\delta} r^{-1/2+\delta} |h| |\nabla u|\|_{L^2_x(S_T)} \|\langle r \rangle^{-\delta} r^{-3/2+\delta} |u|\|_{L^2_x(S_T)}. \]

(2.44)

By (1.3), (2.27), (2.28), (2.37), the Hölder inequality, the Hardy inequality and Lemma 2.2, we have
\[ C_0 \int_{\mathbb{R}^3} |q_4| dx \]
\[ \leq C \int_{\mathbb{R}^3} \left( r^{2\delta-1}(1 + r)^{-2\delta}|\partial_r u_{df}| + r^{2\delta-2}(1 + r)^{-2\delta}|u_{df}| \right) \left| \Delta^{-1} \partial_r \partial_t (h_{lm}^{ij} \partial_m u^j) \right| dx \]
\[ \leq C \int_{\mathbb{R}^3} \left( \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \partial_r u_{df}\|_{L^2(\mathbb{R}^3)} + \|\langle r \rangle^{-\delta} r^{-3/2+\delta} u_{df}\|_{L^2(\mathbb{R}^3)} \right) \left( \|\langle r \rangle^{-\delta} r^{-1/2+\delta} R_r R_t (h_{lm}^{ij} \partial_m u^j)\|_{L^2(\mathbb{R}^3)} \right) \]
\[ \leq C \left( \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \partial_r u_{df}\|_{L^2(\mathbb{R}^3)} + \|\langle r \rangle^{-\delta} r^{-3/2+\delta} u_{df}\|_{L^2(\mathbb{R}^3)} \right) \left( \|\langle r \rangle^{-\delta} r^{-1/2+\delta} |h| \|\nabla u\|_{L^2(\mathbb{R}^3)} \right) \]
\[ \leq \frac{1}{100} (\log(2 + T))^{-1} \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \partial_r u_{df}\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{100} (\log(2 + T))^{-1} \|\langle r \rangle^{-\delta} r^{-3/2+\delta} u_{df}\|_{L^2(\mathbb{R}^3)}^2 \]
\[ + C \left( \log(2 + T) \right) \|\langle r \rangle^{-\delta} r^{-1/2+\delta} |h| \|\nabla u\|_{L^2(\mathbb{R}^3)}^2. \]

(2.45)

Then we have
\[ C_0 \int_{\mathbb{R}^3} \int_0^T |q_4| dx \]
\[ \leq \frac{1}{100} (\log(2 + T))^{-1} \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \partial_r u_{df}\|_{L^2_x(S_T)}^2 + \frac{1}{100} (\log(2 + T))^{-1} \|\langle r \rangle^{-\delta} r^{-3/2+\delta} u_{df}\|_{L^2_x(S_T)}^2 \]
\[ + C \left( \log(2 + T) \right) \|\langle r \rangle^{-\delta} r^{-1/2+\delta} |h| \|\nabla u\|_{L^2_x(S_T)}^2. \]

(2.46)
Similarly, we can show that

\[
C_0 \int_0^T \int_{\mathbb{R}^3} |q_0| |dxdt
\leq \frac{1}{100} (\log(2 + T))^{-1} \left| \langle r \rangle^{-\delta} r^{1/2+\delta} \partial u_{cf} \right| L^2_{L^2}(S_T) + \frac{1}{100} (\log(2 + T))^{-1} \left| \langle r \rangle^{-\delta} r^{-3/2+\delta} u_{cf} \right| L^2_{L^2}(S_T)
\]

+ C(\log(2 + T)) \left| \langle r \rangle^{-\delta} r^{-1/2+\delta} \| h \| \nabla u \right| L^2_{L^2}(S_T).
\]

It follows from (2.11), (2.34), the Hölder inequality, the Hardy inequality and (2.5) that

\[
\int_{\mathbb{R}^3} \langle Mu_{cf}, (L_h u)_{cf} \rangle dx
\leq \int_{\mathbb{R}^3} \left( |\partial_r u_{cf}| + \frac{|u_{cf}|}{r} \right) |(L_h u)_{cf}| dxdt
\leq (\| \partial_r u_{cf} \| L^2(\mathbb{R}^3) + \| \frac{u_{cf}}{r} \| L^2(\mathbb{R}^3)) \| (L_h u)_{cf} \| L^2(\mathbb{R}^3)
\leq C \| \nabla u_{cf} \| L^2(\mathbb{R}^3) \| (L_h u)_{cf} \| L^2(\mathbb{R}^3) \leq C \| \nabla u \| L^2(\mathbb{R}^3) \| L_h u \| L^2(\mathbb{R}^3).
\]

Then by the Hölder inequality and the Cauchy-Schwarz inequality, we have

\[
\int_0^T \int_{\mathbb{R}^3} \langle Mu_{cf}, (L_h u)_{cf} \rangle dx dt
\leq C \sup_{0 \leq t \leq T} \| \nabla u \| L^2(\mathbb{R}^3) \| L_h u \| L^2_{L^2}(S_T) \leq C \sup_{0 \leq t \leq T} \| \nabla u \| L^2(\mathbb{R}^3) \| L_h u \| L^2_{L^2}(S_T).
\]

Similarly, we also have

\[
\int_0^T \int_{\mathbb{R}^3} \langle Mu_{df}, (L_h u)_{df} \rangle dx dt \leq C \sup_{0 \leq t \leq T} \| \nabla u \| L^2(\mathbb{R}^3) \| L_h u \| L^2_{L^2}(S_T).
\]

By (2.39), (2.42), (2.44), (2.46)–(2.50), we can get

\[
\| r^{-1/2+\delta} \partial u_{cf} \| L^2_{L^2}(0, T; L^2(\mathbb{R}^3)) + \| r^{-3/2+\delta} u_{cf} \| L^2_{L^2}(0, T; L^2(\mathbb{R}^3))
\]

+ \| r^{-1/2+\delta} \partial u_{df} \| L^2_{L^2}(0, T; L^2(\mathbb{R}^3)) + \| r^{-3/2+\delta} u_{df} \| L^2_{L^2}(0, T; L^2(\mathbb{R}^3))
\]

\leq C \| \partial u(0, \cdot) \| L^2(\mathbb{R}^3) + C \| L_h u \| L^2_{L^2}(S_T) + C \| h \| \| \nabla u \| L^2_{L^2}(S_T)
\]

+ C \| \langle r \rangle^{-\delta} r^{-1/2+\delta} \partial u \| L^2_{L^2}(S_T) \| \langle r \rangle^{-\delta} r^{-3/2+\delta} u \| L^2_{L^2}(S_T)
\]

+ \frac{1}{100} (\log(2 + T))^{-1} \| \langle r \rangle^{-\delta} r^{-1/2+\delta} u_{cf} \| L^2_{L^2}(S_T)
\]

+ \frac{1}{100} (\log(2 + T))^{-1} \| \langle r \rangle^{-\delta} r^{-3/2+\delta} u_{cf} \| L^2_{L^2}(S_T)
\]

+ C(\log(2 + T)) \| \langle r \rangle^{-\delta} r^{-1/2+\delta} \| h \| \nabla u \| L^2_{L^2}(S_T)
\]

+ \frac{1}{100} (\log(2 + T))^{-1} \| \langle r \rangle^{-\delta} r^{-1/2+\delta} u_{df} \| L^2_{L^2}(S_T)
\]

+ \frac{1}{100} (\log(2 + T))^{-1} \| \langle r \rangle^{-\delta} r^{-3/2+\delta} u_{df} \| L^2_{L^2}(S_T).
\]
The combination of (2.7) and (2.51) gives

\[
(\log(2 + T))^{-1} \langle r \rangle^{-\delta} r^{-1/2 + \delta} \partial u_{cf} \|_{L^2_{1, t,x}(S_T)}^2 + (\log(2 + T))^{-1} \langle r \rangle^{-\delta} r^{-3/2 + \delta} u_{cf} \|_{L^2_{1, t,x}(S_T)}^2
\]

\[
+ (\log(2 + T))^{-1} \langle r \rangle^{-\delta} r^{-1/2 + \delta} \partial u_{df} \|_{L^2_{1, t,x}(S_T)}^2 + (\log(2 + T))^{-1} \langle r \rangle^{-\delta} r^{-3/2 + \delta} u_{df} \|_{L^2_{1, t,x}(S_T)}^2
\]

\[
\leq C \| \partial u(0, \cdot) \|_{L^2(\mathbb{R}^3)}^2 + C \| L_h u \|_{L^1_t L^2_x(S_T)}^2 + C \| |\partial h| \| \nabla u \|_{L^1_t L^2_x(S_T)}^2 + C \| \langle r \rangle^{-1} h \| \nabla u \|_{L^1_t L^2_x(S_T)}^2
\]

\[
+ C \| \langle r \rangle^{-\delta} r^{-1/2 + \delta} h \| \nabla u \|_{L^1_t L^2_x(S_T)} \langle \| \partial h \| \nabla u \|_{L^2_{1, t,x}(S_T)} + \| \langle r \rangle^{-\delta} r^{-3/2 + \delta} u \|_{L^2_{1, t,x}(S_T)} + C \| \langle r \rangle^{-\delta} r^{-1/2 + \delta} \partial u \|_{L^2_{1, t,x}(S_T)}^2
\]

\[
+ C \| \langle r \rangle^{-\delta} r^{-3/2 + \delta} u \|_{L^2_{1, t,x}(S_T)}^2
\]

(2.52)

The last four terms on the right-hand side of (2.52) can be absorbed into the the left-hand side of (2.52). By (2.1), it is obvious that

\[
|u| \leq |u_{cf}| + |u_{df}|
\]

which implies that

\[
|u|^2 \leq |u_{cf}|^2 + |u_{df}|^2 + 2 |u_{cf}| |u_{df}| \leq 2 (|u_{cf}|^2 + |u_{df}|^2).
\]

(2.53)

(2.54)

Hence it follows from (2.54) and the Cauchy-Schwarz inequality that

\[
(\log(2 + T))^{-1} \langle r \rangle^{-\delta} r^{-1/2 + \delta} \partial u \|_{L^2_{1, t,x}(S_T)}^2 + (\log(2 + T))^{-1} \langle r \rangle^{-\delta} r^{-3/2 + \delta} u \|_{L^2_{1, t,x}(S_T)}^2
\]

\[
\leq C \| \partial u(0, \cdot) \|_{L^2(\mathbb{R}^3)}^2 + C \| L_h u \|_{L^1_t L^2_x(S_T)}^2 + C \| |\partial h| \| \nabla u \|_{L^1_t L^2_x(S_T)}^2 + C \| \langle r \rangle^{-1} h \| \nabla u \|_{L^1_t L^2_x(S_T)}^2
\]

\[
+ C \| \langle r \rangle^{-\delta} r^{-1/2 + \delta} h \| \nabla u \|_{L^1_t L^2_x(S_T)} \langle \| \partial h \| \nabla u \|_{L^2_{1, t,x}(S_T)} + \| \langle r \rangle^{-\delta} r^{-3/2 + \delta} u \|_{L^2_{1, t,x}(S_T)} + C \| \langle r \rangle^{-\delta} r^{-1/2 + \delta} \partial u \|_{L^2_{1, t,x}(S_T)}^2
\]

\[
+ C \| \langle r \rangle^{-\delta} r^{-3/2 + \delta} u \|_{L^2_{1, t,x}(S_T)}^2
\]

(2.55)

Note that the last two terms on the right-hand side of (2.55) can be absorbed into the the left-hand side of (2.55). Using also the energy inequality (2.8), we can get

\[
\sup_{0 \leq t \leq T} \| \partial u \|_{L^2(\mathbb{R}^3)} + (\log(2 + T))^{-1/2} \langle r \rangle^{-\delta} r^{-1/2 + \delta} \partial u \|_{L^2_{1, t,x}(S_T)}
\]

\[
+ (\log(2 + T))^{-1/2} \langle r \rangle^{-\delta} r^{-3/2 + \delta} u \|_{L^2_{1, t,x}(S_T)}
\]

\[
\leq C \| \partial u(0, \cdot) \|_{L^2(\mathbb{R}^3)} + C \| L_h u \|_{L^1_t L^2_x(S_T)} + C \| |\partial h| \| \nabla u \|_{L^1_t L^2_x(S_T)} + C \| \langle r \rangle^{-1} h \| \nabla u \|_{L^1_t L^2_x(S_T)} + C \| \langle r \rangle^{-\delta} r^{-1/2 + \delta} h \| \nabla u \|_{L^2_{1, t,x}(S_T)}.
\]

(2.56)

Now we complete the proof of Theorem 1.1.
3 Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2 concerning the almost global existence for nonlinear elastic waves, which requires the smallness of data with respect to some weighted $H^4 \times H^3$ norm. Our strategy is to combine the KSS type estimate (1.7) and some weight Sobolev inequalities (Lemma 3.1), in which the angular integrability often plays a key role.

We will use the following mixed-norm

$$\|u\|_{L^p_t L^q_x} := \left( \int_0^{+\infty} \|u(rw)\|_{L^p_t(S^2)}^p r^q dr \right)^{1/p},$$

with trivial modification for the case $p = +\infty$.

**Lemma 3.1.** For $u \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$, we have

$$\langle r \rangle^{1/2} |u(x)| \leq C \sum_{|a| \leq 2} \|\langle y \rangle^{-1/2} \hat{\Omega}^a u\|_{L^2(\mathbb{R}^3)},$$

$$\|u\|_{L^2_t L^4_x} \leq C \sum_{|a| \leq 1} \|\hat{\Omega}^a u\|_{L^2(\mathbb{R}^3)}, 2 \leq q < +\infty,$$

$$\|r^{1/2} u\|_{L^\infty_t L^6_x(r \geq 1)} \leq C \sum_{|a| \leq 1} \|\langle r \rangle^{-1/2} \hat{\Omega}^a u\|_{L^2(\mathbb{R}^3)}, 2 \leq p \leq 4,$$

$$\|r^{(3/2) - s} u\|_{L^\infty_t L^6_x} \leq C \|u\|_{H^s(\mathbb{R}^3)}, \frac{1}{2} < s \leq 1, \frac{2}{p} = -s + \frac{3}{2}. $$

**Proof.** It follows from the Sobolev embedding $H^2(B_1) \hookrightarrow L^\infty(B_1)$ and the following weighted Sobolev inequality (see (3.14b) of [33])

$$r |u(x)| \leq C \sum_{|a| \leq 1} \|\partial_r \hat{\Omega}^a u\|_{L^2(\{|y| \geq r\})}^{1/2} \sum_{|a| \leq 2} \|\hat{\Omega}^a u\|_{L^2(\{|y| \geq r\})}^{1/2},$$

that

$$\langle r \rangle |u(x)| \leq C \sum_{|a| \leq 2} \|\hat{\Omega}^a u\|_{L^2(\mathbb{R}^3)}.$$ (3.7)

(3.2) can be proved by replacing $u$ by $\langle r \rangle^{-1/2} u$ in (3.7). (3.3) is a consequence of the Sobolev embedding on $S^2 : H^1(S^2) \hookrightarrow L^q(S^2), 2 \leq q < +\infty$. As for (3.4), we only need to prove the case $p = 4$. By (3.19) of Sideris [33], we have

$$r \|u(rw)\|_{L^4_x} \leq C \|\partial_r u\|_{L^2(|y| \geq r)}^{1/2} \sum_{|a| \leq 1} \|\hat{\Omega}^a u\|_{L^2(|y| \geq r)}^{1/2}. $$

Replacing $u$ by $r^{-1/2} u$ in (3.8), we can get

$$\|r^{1/2} u\|_{L^\infty_t L^4_x(r \geq 1)} \leq C \sum_{|a| \leq 1} \|\langle r \rangle^{-1/2} \hat{\Omega}^a u\|_{L^2(\mathbb{R}^3)}. $$

(3.9)

For (3.5), it follows from the trace inequality due to Hoshiro [10] and the Sobolev embedding on $S^2$ that

$$r^{(3/2) - s} \|u(rw)\|_{L^6_x} \leq C \|u\|_{H^s(\mathbb{R}^3)}, \frac{1}{2} < s < \frac{3}{2}, \frac{2}{p} = -s + \frac{3}{2}. $$

(3.10)

See, e.g., Proposition 2.4 of [8]. (3.5) is a consequence of (3.10) and the interpolational inequality:

$$\|u\|_{H^s(\mathbb{R}^3)} \leq C \|u\|_{L^2(\mathbb{R}^3)}^{1-s} \|u\|_{H^1(\mathbb{R}^3)}^s \leq C \|u\|_{H^1(\mathbb{R}^3)}, 0 < s \leq 1.$$

\(\square\)
As Proposition 3.1 in Sideris [33], we have the following

**Lemma 3.2.** For any solution $u$ of (1.8) in $X^k(T)$, we have

$$LZ^a u = \sum_{b+c=a} N(Z^b u, Z^c u),$$  \hspace{1cm} (3.12)

in which the sum extends over ordered partitions of the sequences $a$, with $|a| \leq k - 1$.

Now we will prove Theorem 1.2. Assume that $u = u(t, x)$ is a local solution of the Cauchy problem (1.8) on $[0, T]$. We will show that there exist positive constants $\kappa, \varepsilon_0$ and $A$ such that for any $T \leq \exp(\kappa/\varepsilon)$, we have $\|u\|_{X^3(T)} \leq A\varepsilon$ under the assumption that (1.18) and $\|u\|_{X^3(T)} \leq 2A\varepsilon$, where $0 < \varepsilon \leq \varepsilon_0$.

By Lemma 3.2 and (1.7) in Theorem 1.1, we can get

$$\|u\|_{X^3(T)} \leq C_1\varepsilon + C \sum_{|a| \leq 3} \|\partial \nabla u \nabla Z^a u\|_{L^1_t L^2_x(S_T)} + C \sum_{|a| \leq 3} \|\nabla u \nabla Z^a u\|_{L^1_t L^2_x(S_T)}$$

$$+ C(\log(2 + T))^{1/2} \sum_{|a| \leq 3} \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \nabla u \nabla Z^a u\|_{L^2_t r(S_T)}$$

$$+ C \sum_{|a| \leq 3} \sum_{b+c=a, b \neq a} \|\nabla Z^b u \nabla^2 Z^c u\|_{L^1_t L^2_x(S_T)}.$$  \hspace{1cm} (3.13)

For $|a| \leq 3$, by the Hölder inequality and (3.2), we can get

$$\|\partial \nabla u \nabla Z^a u\|_{L^1_t L^2_x(S_T)}$$

$$\leq \|\langle r \rangle^{-1/2} \nabla Z^a u\|_{L^1_t r(S_T)} \langle r \rangle^{1/2} \partial \nabla u\|_{L^2_t r(S_T)}$$

$$\leq C \|\langle r \rangle^{-1/2} \nabla Z^a u\|_{L^1_t r(S_T)} \sum_{|a| \leq 3} \|\langle r \rangle^{-1/2} \partial Z^a u\|_{L^2_t r(S_T)}$$

$$\leq C \sum_{|a| \leq 3} \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \partial Z^a u\|_{L^2_t r(S_T)}^2$$

$$\leq C(\log(2 + T)) \|u\|_{X^3(T)}^2.$$  \hspace{1cm} (3.14)

Similarly, it also holds that for $|a| \leq 3$,

$$\|\nabla u \nabla Z^a u\|_{L^1_t L^2_x(S_T)} \leq C(\log(2 + T)) \|u\|_{X^3(T)}^2.$$  \hspace{1cm} (3.15)

For $|a| \leq 3$, it follows from the Hölder inequality and the Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ that

$$(\log(2 + T))^{1/2} \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \nabla u \nabla Z^a u\|_{L^2_t r(S_T)}$$

$$\leq (\log(2 + T))^{1/2} \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \nabla Z^a u\|_{L^2_t r(S_T)} \|\nabla u\|_{L^\infty_r(S_T)}$$

$$\leq C(\log(2 + T))^{1/2} \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \nabla Z^a u\|_{L^2_t r(S_T)} \sum_{|a| \leq 2} \|\nabla Z^a u\|_{L^\infty_r L^2_x(S_T)}$$

$$\leq C(\log(2 + T)) \|u\|_{X^3(T)}^2.$$  \hspace{1cm} (3.16)

Now we will estimate the last term on the right-hand side of (3.13) For $|a| \leq 3, b + c = a, b, c \neq a$, if $|b| \leq 1, |c| \leq 2$, similarly to (3.14), by the Hölder inequality and (3.2), we can
It follows from (3.21) that
\[
\|\nabla Z^b u \nabla^2 Z^c u\|_{L^1_t L^2_x(S_T)} \leq \|\langle r \rangle^{-1/2} \nabla Z^c u\|_{L^1_t L^2_x(S_T)} \left( \|\langle r \rangle^{1/2} \nabla Z^b u\|_{L^\infty_t L^\infty_x(S_T)} \right)
\]
where \(2 \leq p, q < +\infty, 1/p + 1/q = 1/2\). By (3.3), we have
\[
\|\langle r \rangle^{-\delta} r^{-1/2 + \delta} \nabla^2 Z^c u\|_{L^2_t L^\infty_x(S_T)} \leq C \sum_{|\alpha| \leq 3} \|\langle r \rangle^{-\delta} r^{-1/2 + \delta} \nabla Z^\alpha u\|_{L^2_t L^\infty_x(S_T)}.
\]

It is obvious that
\[
\|\langle r \rangle^{\delta} r^{1/2 - \delta} \nabla Z^b u\|_{L^\infty_t L^\infty_x(S_T)} \leq C \|\langle r \rangle^{\delta} r^{1/2 - \delta} \nabla Z^b u\|_{L^\infty_t L^\infty_x(r \geq 1)} + C \|\langle r \rangle^{\delta} r^{1/2 - \delta} \nabla Z^b u\|_{L^\infty_t L^\infty_x(r \leq 1)}
\]
\[
\leq C \|r^{1/2} \nabla Z^b u\|_{L^\infty_t L^2_x(r \geq 1)} + C \|r^{1/2 - \delta} \nabla Z^b u\|_{L^\infty_t L^2_x(r \leq 1)}.
\]

Now we take \(p = 2/(1 - 2\delta)\). Noting that \(0 < \delta < 1/4\) implies that \(2 < p < 4\). By (3.4), we have
\[
\|r^{1/2} \nabla Z^b u\|_{L^\infty_t L^2_x(r \geq 1)} \leq C \sum_{|\alpha| \leq 3} \|\langle r \rangle^{-\delta} r^{-1/2 - \delta} \nabla Z^\alpha u\|_{L^2_t L^\infty_x(S_T)} \leq C \sum_{|\alpha| \leq 3} \|\langle r \rangle^{-\delta} r^{-1/2 + \delta} \nabla Z^\alpha u\|_{L^2_t L^\infty_x(S_T)}.
\]

It follows from (3.5) that
\[
\|r^{1/2 - \delta} \nabla Z^b u\|_{L^\infty_t L^\infty_x(r \leq 1)} \leq C \|r^{1/2 - \delta} \nabla Z^b u\|_{L^\infty_t L^\infty_x(r \leq 1)} \leq C \|r^{-2\delta} \langle r \rangle^{-\delta} r^{-1/2 + \delta} \nabla Z^b u\|_{L^\infty_t L^\infty_x(S_T)} \leq C \|\langle r \rangle^{-\delta} r^{-1/2 + \delta} \nabla Z^b u\|_{H^1(R^3)}
\]
\[
\leq C \sum_{|\alpha| \leq 3} \|\langle r \rangle^{-\delta} r^{-1/2 + \delta} \nabla Z^\alpha u\|_{L^2_t L^\infty_x(S_T)} + C \sum_{|\alpha| \leq 3} \|\langle r \rangle^{-\delta} r^{-3/2 + \delta} \nabla Z^\alpha u\|_{L^2_t L^\infty_x(S_T)}.
\]

By (3.20), (3.21) and (3.22), we have
\[
\|\langle r \rangle^{\delta} r^{1/2 - \delta} \nabla Z^b u\|_{L^\infty_t L^\infty_x(S_T)} \leq C \sum_{|\alpha| \leq 3} \|\langle r \rangle^{-\delta} r^{-1/2 + \delta} \nabla Z^\alpha u\|_{L^2_t L^\infty_x(S_T)} + C \sum_{|\alpha| \leq 3} \|\langle r \rangle^{-\delta} r^{-3/2 + \delta} \nabla Z^\alpha u\|_{L^2_t L^\infty_x(S_T)}.
\]

It follows from (3.18), (3.19) and (3.23) that when \(|b| \leq 2, |c| \leq 1\), it holds that
\[
\|\nabla Z^b u \nabla^2 Z^c u\|_{L^1_t L^2_x(S_T)} \leq C (\log(2 + T)) \|u\|_{X^2_T(S_T)}.
\]
Proof. (3.25) then it holds that
\[ |u|^2(T) \leq C_1 \varepsilon + C_2(\log(2 + T)) \|u\|_{X^2_\delta(T)}^2 \leq C_1 \varepsilon + 4C_2(\log(2 + T))A^2 \varepsilon^2. \] (3.25)
Take $A = 4C_1$ and $\varepsilon_0 > 0$ sufficiently small. Then for any $0 < \varepsilon \leq \varepsilon_0$, if
\[ 16C_2(\log(2 + T))A\varepsilon \leq 1, \] (3.26)
then it holds that
\[ \|u\|_{X^2_\delta(T)} \leq A\varepsilon. \] (3.27)
Consequently, it follows from (3.26) that we can get the lifespan estimate of smooth solutions to the Cauchy problem (1.8):
\[ T_\varepsilon = \exp(\kappa/\varepsilon), \] (3.28)
where $\kappa$ a positive constant independent of $\varepsilon$. So we complete the proof of Theorem 1.2.

4 Proof of Theorem 1.3

In this section, we will prove Theorem 1.3 concerning the low regularity almost global existence in the radial symmetric case. Our strategy is to combine the KSS type estimate (1.7) and some weight Sobolev inequalities (see Lemma 4.1), and exploit the fact $\Omega u = 0$ sufficiently.

Remark 4.1. We note that in the radial symmetric case, the nonlinear elastic wave equation can be reduced to a system of wave equations, but as far as the proof of almost global existence is concerned, this reduction is not necessary.

Lemma 4.1. For $u \in C^\infty_0(\mathbb{R}^3; \mathbb{R}^3)$, we have
\[
\|r^{1/2}u\|_{L^\infty_{(r \geq 1)}} \leq C \sum_{|a| \leq 1} \|r^{-1/2} \nabla \tilde{\Omega}^a u\|_{L^2(\mathbb{R}^3)} + C \sum_{|a| \leq 2} \|r^{-1/2} \tilde{\Omega}^a u\|_{L^2(\mathbb{R}^3)},
\] (4.1)
\[
\|r^{1/2-\delta}u\|_{L^\infty_{(r \leq 1)}} \leq C \sum_{|a| \leq 1} \|r^{-\delta} r^{-1/2+\delta} \nabla \tilde{\Omega}^a u\|_{L^2(\mathbb{R}^3)} + C \sum_{|a| \leq 1} \|r^{-\delta} r^{-1/2+\delta} \tilde{\Omega}^a u\|_{L^2(\mathbb{R}^3)}
\] + C \sum_{|a| \leq 1} \|r^{-\delta} r^{-3/2+\delta} \tilde{\Omega}^a u\|_{L^2(\mathbb{R}^3)}, \quad 0 < \delta \leq 1/4.
\] (4.2)

Proof. (4.1) follows from replacing $u$ by $r^{-1/2}u$ in (3.6). For (4.2), first we have (see (3.14a) of [33])
\[
\|r^{1/2}u\|_{L^\infty(\mathbb{R}^3)} \leq C \sum_{|a| \leq 1} \|\nabla \tilde{\Omega}^a u\|_{L^2(\mathbb{R}^3)}.
\] (4.3)
Noting that $0 < \delta \leq 1/4$, we have
\[
\|r^{1/2-\delta}u\|_{L^\infty_{(r \leq 1)}} \leq C \|r^{1/2} \langle r \rangle^{-\delta} r^{-1/2+\delta} u\|_{L^\infty(\mathbb{R}^3)},
\] (4.4)
We can prove (4.2) by replacing $u$ by $\langle r \rangle^{-\delta} r^{-1/2+\delta} u$ in (4.3). 

Similarly to Lemma 3.2, we have the following
Lemma 4.2. For any solution $u$ of (1.8) in $X^k(T)$, we have

$$L \nabla^a u = \sum_{b+c=a} N (\nabla^b u, \nabla^c u),$$

(4.5)

in which the sum extends over ordered partitions of the sequences $a$, with $|a| \leq k - 1$.

In order to prove Theorem 1.3, the key point is the following a priori estimate.

Proposition 4.1. There exist positive constants $\kappa$, $\varepsilon_0$ and $A$ such that for any given $\varepsilon$ with $0 < \varepsilon \leq \varepsilon_0$, if the initial data is radially symmetric and satisfies

$$\|\nabla u_0\|_{H^2} + \|u_1\|_{H^2} \leq \varepsilon,$$

(4.6)

and $u$ is a smooth and radially symmetric solution to Cauchy problem (1.8), then for any $T \leq \exp(\kappa/\varepsilon)$,

$$\|u\|_{X^3(T)} \leq A \varepsilon.$$

(4.7)

Based on the method of proving Proposition 4.1, by some density argument and contraction-mapping argument, we can show Theorem 1.3. Because this procedure is routine and in order to keep the paper to a moderate length, we will omit it and refer the reader to [7] and [9].

Now we will prove Proposition 4.1. Assume that $u = u(t, x)$ is a smooth and radially symmetric solution of the Cauchy problem (1.8) on $[0, T]$. We will show that there exist positive constants $\kappa$, $\varepsilon_0$ and $A$ such that for any $T \leq \exp(\kappa/\varepsilon)$, we have $\|u\|_{X^3(T)} \leq A \varepsilon$ under the assumption that (4.6) and $\|u\|_{X^3(T)} \leq 2A \varepsilon$, where $0 < \varepsilon \leq \varepsilon_0$.

By Lemma 4.2 and (1.7) in Theorem 1.1, we can get

$$\|u\|_{X^3(T)} \leq C_1 \varepsilon + C \sum_{|a| \leq 2} \|\partial \nabla u \nabla^a u\|_{L^1_t L^2_x(S_T)} + C \sum_{|a| \leq 2} \|\nabla u \nabla^a u\|_{L^1_t L^2_x(S_T)}$$

$$+ C (\log(2 + T))^{1/2} \sum_{|a| \leq 2} \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \nabla^a u\|_{L^2_t L^2_x(S_T)}$$

$$+ C \sum_{|a| \leq 2} \sum_{b+c=a, \ b,c \neq a} \|\nabla^b u \nabla^c u\|_{L^1_t L^2_x(S_T)}.$$

(4.8)

For $|a| \leq 2$, we have

$$\|\partial \nabla u \nabla^a u\|_{L^1_t L^2_x(S_T)} \leq \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \nabla^a u\|_{L^2_t L^2_x(S_T)} \leq \|\langle r \rangle^{-\delta} r^{1/2-\delta} \partial \nabla u\|_{L^2_t L^2_x(S_T)}.$$

(4.9)

Noting that $\tilde{\Omega} u = 0$ and the commutation relationship (1.14), it follows from (4.1) that

$$\|\langle r \rangle^{\delta} r^{1/2-\delta} \partial \nabla u\|_{L^\infty(r \geq 1)} \leq C \|r^{1/2} \partial \nabla u\|_{L^\infty(r \geq 1)} \leq C \sum_{|a| \leq 2} \|\langle r \rangle^{-1/2} \partial \nabla^a u\|_{L^2(\mathbb{R}^3)}$$

$$\leq C \sum_{|a| \leq 2} \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \partial \nabla^a u\|_{L^2(\mathbb{R}^3)}.$$

(4.10)

And by (4.2) we can get

$$\|\langle r \rangle^{\delta} r^{1/2-\delta} \partial \nabla u\|_{L^\infty(r \leq 1)} \leq C \|r^{1/2-\delta} \partial \nabla u\|_{L^\infty(r \leq 1)}$$

$$\leq C \sum_{|a| \leq 2} \|\langle r \rangle^{-\delta} r^{-1/2+\delta} \partial \nabla^a u\|_{L^2(\mathbb{R}^3)} + C \|\langle r \rangle^{-\delta} r^{-3/2+\delta} \partial \nabla u\|_{L^2(\mathbb{R}^3)}.$$

(4.11)
It follows from (4.9), (4.10) and (4.11) that
\[ \| \partial \nabla u \nabla^a u \|_{L^1_t L^2_x(S_T)} \leq C (\log (2 + T)) \| u \|^2_{H^3(T)}. \] (4.12)

By similar argument, we can also show that for \(|a| \leq 2\), it holds that
\[ \| \nabla u \nabla^a u \|_{L^1_t L^2_x(S_T)} \leq C (\log (2 + T)) \| u \|^2_{H^3(T)}, \] (4.13)
and for \(|a| \leq 2, b + c = a, b, c \neq a\) (i.e., \(|b| \leq 1, |c| \leq 1\)), we also have
\[ \| \nabla \nabla^b u \nabla^c u \|_{L^1_t L^2_x(S_T)} \leq C (\log (2 + T)) \| u \|^2_{H^3(T)}. \] (4.14)

For \(|a| \leq 2\), it follows from the H"{o}lder inequality and the Sobolev embedding \(H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)\) that
\[
(\log(2 + T))^{1/2} \| \langle r \rangle^{-\delta} r^{-1/2 + \delta} \nabla \nabla^a u \|_{L^1_{t,x}(S_T)} \\
\leq (\log(2 + T))^{1/2} \| \langle r \rangle^{-\delta} r^{-1/2 + \delta} \nabla \nabla^a u \|_{L^2_{t,x}(S_T)} \| \nabla u \|_{L^\infty_{t,x}(S_T)} \sum_{|\alpha| \leq 2} \| \nabla \nabla^\alpha u \|_{L^\infty_{t,x} L^2_x(S_T)} \\
\leq C (\log(2 + T)) \| u \|^2_{H^3(T)}. \] (4.15)

Hence the above argument gives
\[ \| u \|_{H^3(T)} \leq C_1 \varepsilon + C_2 (\log(2 + T)) \| u \|^2_{H^3(T)} \leq C_1 \varepsilon + 4C_2 (\log(2 + T)) A^2 \varepsilon^2. \] (4.16)
Take \(A = 4C_1\) and \(\varepsilon_0 > 0\) sufficiently small. Then for any \(0 < \varepsilon \leq \varepsilon_0\), if
\[ 16C_2 (\log(2 + T)) A \varepsilon \leq 1, \] (4.17)
then it holds that
\[ \| u \|_{H^3(T)} \leq A \varepsilon. \] (4.18)

Consequently, it follows from (4.17) that we can get the lifespan estimate:
\[ T_\varepsilon = \exp(\kappa/\varepsilon), \] (4.19)
where \(\kappa\) a positive constant independent of \(\varepsilon\). So we complete the proof of Proposition 4.1.

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