Sharp phase transitions for exact support recovery under local differential privacy

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Abstract

We address the problem of variable selection in the Gaussian mean model in $\mathbb{R}^d$ under the additional constraint that only privatised data are available for inference. For this purpose, we adopt a recent generalisation of classical minimax theory to the framework of local $\alpha-$differential privacy. We provide lower and upper bounds on the rate of convergence for the expected Hamming loss over classes of at most $s$-sparse vectors whose non-zero coordinates are separated from 0 by a constant $a > 0$. As corollaries, we derive necessary and sufficient conditions (up to log factors) for exact recovery and for almost full recovery. When we restrict our attention to non-interactive mechanisms that act independently on each coordinate our lower bound shows that, contrary to the non-private setting, both exact and almost full recovery are impossible whatever the value of $a$ in the high-dimensional regime such that $n \alpha^2/d^2 \lesssim 1$. However, in the regime $n \alpha^2/d^2 \gg \log(n \alpha^2/d^2) \log(d)$ we can exhibit a sharp critical value $a^*$ (up to a logarithmic factor) such that exact and almost full recovery are possible for all $a \gg a^*$ and impossible for $a \leq a^*$. We show that these results can be improved when allowing for all non-interactive (that act globally on all coordinates) locally $\alpha-$differentially private mechanisms in the sense that phase transitions occur at lower levels.

Keywords : Hamming loss, Local differential privacy, Minimax rates, Phase transition, Support recovery, Variable selection

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1 Introduction

1.1 Problem statement

Nowadays, a large amount of data, such as internet browsing history, social media activity, location information from smartphones, or medical records, are collected and stored. On the one hand, the analysis of these data can benefit to individuals, companies, or communities such as the scientific one. For instance, companies can use data to improve their products and services, or health data can be used for medical research. On the other hand, people are more and more concerned with the protection of their privacy and may be reluctant to share their sensitive data. In this context, it seems essential to be able to understand the tradeoffs between the statistical utility of the collected data and the privacy of individuals from whom these data are obtained. This requires a formal definition of privacy and differential privacy has been adopted by researchers in the computer science, machine learning, and statistics communities as a natural one.

Two kinds of differential privacy are discussed in the literature: central differential privacy which has been introduced by Dwork et al. in $\mathbb{R}$, and local differential privacy. We will focus in this paper on the second setting but we briefly discuss the difference between central and local privacy. In both settings, $n$ individuals want their privacy to be preserved while their data, which

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will be denoted \( X_1, \ldots, X_n \), are used for statistical analyses. In the central setting, the \( n \) data-holders share confidence in a common curator who has access to the original data \( X_1, \ldots, X_n \) and use them to generate a private release \( Z \). In a nutshell, central differential privacy ensures that the probability of observing an output does not change much when a single data point of the original database is modified. We refer to \cite{10} for the formal definition of differential privacy in the central setting. In the local setting, data is privatized before it is shared with a data collector: for all \( i \in [1, n] \), \( X_i \) is transformed into a private data \( Z_i \) directly on the \( i \)th individual's machine and the data collector or the statistician only have access to the private sample \( Z_1, \ldots, Z_n \). However, some interaction between the different data-holders is allowed. Formally, the privatized data \( Z_1, \ldots, Z_n \) are obtained by successively applying suitable Markov kernels: given \( X_i = x_i \) and \( Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1} \), the \( i \)-th dataholder draws

\[
Z_i \sim Q_i(\cdot \mid X_i = x_i, Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1})
\]

for some Markov kernel \( Q_i : \mathcal{X} \times \mathcal{Z}^{i-1} \to [0, 1] \) where the measure spaces of the non-private and private data are denoted with \((\mathcal{X}, \mathcal{Z})\) and \((\mathcal{Z}, \mathcal{Z})\), respectively. Such randomizations are known as sequentially interactive. We say that the sequence of Markov kernels \( (Q_i)_{i=1, \ldots, n} \) provides \( \alpha \)-local differential privacy or that \( Z_1, \ldots, Z_n \) are \( \alpha \)-local differentially private views of \( X_1, \ldots, X_n \) if

\[
\sup_{A \in \mathcal{X}} \frac{Q_i(A \mid X_i = x_i, Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1})}{Q_i(A \mid X_i = x', Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1})} \leq \exp(\alpha) \quad \forall i \in [1, n], \forall x, x' \in \mathcal{X}.
\]

(1)

In this paper, we will focus on the special case of non-interactive local differential privacy where \( Z_i \) depends only on \( X_i \) but not on \( Z_k \) for \( k < i \). In this scenario, we have

\[
Z_i \sim Q_i(\cdot \mid X_i = x_i),
\]

and condition (1) becomes

\[
\sup_{A \in \mathcal{X}} \frac{Q_i(A \mid X_i = x)}{Q_i(A \mid X_i = x')} \leq \exp(\alpha) \quad \forall i \in [1, n], \forall x, x' \in \mathcal{X}.
\]

We consider the problem of support recovery with separation assumptions under local differential privacy as a special instance of the general problem described above. Precisely, for \( i = 1, \ldots, n \), the \( i \)th data holder observes a random vector \( X^i = (X^i_j)_{j=1, \ldots, d} \in \mathbb{R}^d \) distributed according to the normal distribution \( \mathcal{N}(\theta, \sigma^2 I_d) \), where the mean vector \( \theta \) is assumed to be \((s, a)\)-sparse in the sense that \( \theta \) belongs to one of the following sets:

\[
\Theta^+_d(s, a) = \{ \theta \in \mathbb{R}^d : \text{there exists a set } S \subseteq \{1, \ldots, d\} \text{ with at most } s \text{ elements such that } \theta_j \geq a \text{ for all } j \in S, \text{ and } \theta_j = 0 \text{ for all } j \notin S \},
\]

or

\[
\Theta_d(s, a) = \{ \theta \in \mathbb{R}^d : \text{there exists a set } S \subseteq \{1, \ldots, d\} \text{ with at most } s \text{ elements such that } |\theta_j| \geq a \text{ for all } j \in S, \text{ and } \theta_j = 0 \text{ for all } j \notin S \}.
\]

The aim is that every data holder releases a private view \( Z^i \) of \( X^i \) such that the notion of local differential privacy is satisfied and that the support of \( \theta \) can be estimated from the data \( Z^1, \ldots, Z^n \) in an optimal way.

**Notation.** For two sequences \( \{a_d\}_d \) and \( \{b_d\}_d \) of non-negative real numbers, we write \( a_d \lesssim b_d \) if there exists some constant \( C > 0 \) such that \( a_d \leq C b_d \). If \( b_d > 0 \) we write \( a_d \sim b_d \) if \( a_d/b_d \to 1 \) as \( d \to \infty \) and we write \( a_d \gg b_d \) if \( a_d/b_d \to \infty \) as \( d \to \infty \). We recall that a centred Laplace distribution with parameter \( \lambda > 0 \) has the probability density function defined by \( f_\lambda(x) = \frac{1}{\lambda} \exp \left( -\frac{|x|}{\lambda} \right) \) on \( \mathbb{R} \).

1.2 Motivation

The problem of high-dimensional sparse vectors estimation has recently been studied in the framework of local differential privacy in \cite{2}. For the 1-sparse means problem, the authors proved that
the private minimax risk for non-interactive α-locally differentially private mechanisms over the set of distributions \( P \) supported on \( \mathbb{B}_\infty(r) \subset \mathbb{R}^d \) with \( \| E_P[X] \|_0 \leq 1 \) is bounded from below by

\[
\min \left\{ r^2, \frac{r^2 d \log(2d)}{n(e^\alpha - 1)^2} \right\},
\]

proving that high-dimensional 1-sparse mean estimation is impossible in this setting when both \( r^2 \gtrsim 1 \) and \( r^2 d \log(2d) \gtrsim n(e^\alpha - 1)^2 \).

Obvious applications of variable selection are the estimation of the set that supports the non-null coefficients in the mean vector, or the estimation of its size. We may use our procedure to build a private mean estimator of \( s \)-sparse vectors in two steps: use one part of the sample to recover the support and the other part to estimate the mean values of the selected variables, that is a vector of reduced size. Moreover these results are a benchmark for working on more realistic models such as high-dimensional linear regression and clustering of high-dimensional vectors, see \[13\] and \[12\].

1.3 Minimax framework

Let \( X^i, i = 1, \ldots, n \) be i.i.d \( \mathcal{N}(\theta, \sigma^2 I_d) \) random vectors of \( \mathbb{R}^d \). We assume that the vectors \( X^i = (X^i_j)_{j=1, \ldots, d} \) for \( i = 1, \ldots, n \) are observed by \( n \) distinct data holders who refuse to share their respective observations. The statistician does not have access to these data but only to \( \alpha \)-locally differentially private views \( Z^1, \ldots, Z^n \). We assume that \( \theta \) belongs to one of the sets \( \Theta^*_F(s, a) \) or \( \Theta_d(s, a) \) introduced in Section \[14\] and we study the problem of selecting the relevant components of \( \theta \), that is, of estimating the vector \( \eta = \eta(P_\theta) = (I(\theta_j \neq 0))_{j=1, \ldots, d} \), where \( I(\cdot) \) is the indicator function. Our goal is to estimate the vector \( \eta \) by a selector \( \hat{\eta} \), that is a measurable function \( \hat{\eta} = \hat{\eta}(Z^1, \ldots, Z^n) \) taking values in \( \{0, 1\}^d \), where \( Z^1, \ldots, Z^n \) are \( \alpha \)-locally differentially private views of \( X^1, \ldots, X^n \). We judge the quality of a selector \( \hat{\eta} \) as an estimator of \( \eta \) by the Hamming loss between \( \hat{\eta} \) and \( \eta \) which counts the number of positions at which \( \hat{\eta} \) and \( \eta \) differ:

\[
|\hat{\eta} - \eta| := \sum_{j=1}^d |\hat{\eta}_j - \eta_j| = \sum_{j=1}^d I(\hat{\eta}_j \neq \eta_j).
\]

For the support recovery problem, we consider only \( \alpha \)-locally differentially private mechanisms which transform each \( X^i \in \mathbb{R}^d \) into a private release \( Z^i \) taking also values in \( \mathbb{R}^d \), that are known as non-interactive privacy mechanisms. However, we distinguish between privacy mechanisms that act on each coordinate of \( X^i \) either separately, locally or globally. More specifically, we will consider the two following scenarios:

- **Coordinate Local (CL) Privacy Mechanisms**: there is a sequence \( Q = (Q^i)_{i=1, \ldots, n} \) of Markov kernels providing \( \alpha \)-local differential privacy such that \( Z^i \sim Q^i(\cdot \mid X^i = x^i) \) for all \( i \in [1, n] \), and \( Q^i \) is obtained as product of coordinate-wise kernels as follows:

  for all \( i \in [1, n] \) and all \( j \in [1, d] \), \( Z^i_j \sim Q^i_j(\cdot \mid X^i_j = x) \)

  for some \((\alpha/d)\)-differentially private mechanism \( Q^i_j \). We denote by \( Q^\alpha_{CL} \) the set of all privacy mechanisms \( Q = (Q^1, \ldots, Q^n) \) satisfying these assumptions.

- **Coordinate Global (CG) Privacy Mechanisms**: there is a sequence \( Q = (Q^i)_{i=1, \ldots, n} \) of Markov kernels providing \( \alpha \)-local differential privacy such that \( Z^i \sim Q^i(\cdot \mid X^i = x^i) \) for all \( i \in [1, n] \). We denote by \( Q^\alpha \) the set of all privacy mechanisms \( Q = (Q^1, \ldots, Q^n) \) satisfying this assumption.

In other words, in the Coordinate Local case, we consider only non-interactive \( \alpha \)-locally differentially private mechanisms that act coordinates by coordinates. This scenario is easier to study than the second one for which any non-interactive \( \alpha \)-locally differentially private mechanism is allowed to be used.
We will say that value \( a \) conditions for almost full recovery and exact recovery to be possible. They highlight a critical the non-private version of minimax risk (2). As corollaries, they derive necessary and sufficient results are affected by the privacy constraints.

Almost full recovery is impossible

We say that a selector \( \hat{\eta} = (\hat{\eta}_1, \ldots, \hat{\eta}_d) \) is separable if for all \( j = 1, \ldots, d \) its \( j \)th component \( \hat{\eta}_j \) depends only on \( \{Z_j\}_i=1,\ldots,n \). We denote by \( \mathcal{T} \) the set of all separable selectors. We are interested in the study of the following private minimax risks

\[
R_n^{CL}(\alpha, \Theta) = \inf_{Q \in \mathcal{Q}_n^{CL}} \inf_{\hat{\eta} = \hat{\eta}(Z^1, \ldots, Z^n) \in \mathcal{T}} \sup_{\alpha} \frac{1}{Q(P^\otimes n)} \mathbb{E}_Q(\frac{1}{n} |\hat{\eta}(Z^1, \ldots, Z^n) - \eta|),
\]

in the coordinate local case, and

\[
R_n(\alpha, \Theta) = \inf_{Q \in \mathcal{Q}_n^\alpha} \inf_{\hat{\eta} = \hat{\eta}(Z^1, \ldots, Z^n) \in \mathcal{T}} \sup_{\alpha} \frac{1}{Q(P^\otimes n)} \mathbb{E}_Q(\frac{1}{n} |\hat{\eta}(Z^1, \ldots, Z^n) - \eta|),
\]

in the coordinate global case, for \( \Theta = \Theta_d^+(s,a) \) and \( \Theta = \Theta_d(s,a) \).

We are interested in the study of two asymptotic properties: almost full recovery and exact recovery, that we define here. Let \( (\Theta_d^+(s, d, a_d))_{d \geq 1} \) be a sequence of classes of sparse vectors. We will say that almost full recovery is possible for \( (\Theta_d^+(s, d, a_d))_{d \geq 1} \) in the Coordinate Local case if there exists \( Q \in \mathcal{Q}_n^{CL} \) and a selector \( \hat{\eta} \) such that

\[
\lim_{d \to \infty} \sup_{\theta \in \Theta_d^+(s, d, a_d)} \frac{1}{Q(P^\otimes n)} |\hat{\eta} - \eta| = 0.
\]

We will say that almost full recovery is impossible for \( (\Theta_d^+(s, d, a_d))_{d \geq 1} \) in the Coordinate Local case if

\[
\lim_{d \to \infty} \inf_{Q \in \mathcal{Q}_n^{CL}} \inf_{\hat{\eta} = \hat{\eta}(Z^1, \ldots, Z^n) \in \mathcal{T}} \sup_{\alpha} \frac{1}{Q(P^\otimes n)} |\hat{\eta} - \eta| > 0.
\]

We will say that exact recovery is possible for \( (\Theta_d^+(s, d, a_d))_{d \geq 1} \) in the Coordinate Local case if there exists \( Q \in \mathcal{Q}_n^{CL} \) and a selector \( \hat{\eta} \) such that

\[
\lim_{d \to \infty} \sup_{\theta \in \Theta_d^+(s, d, a_d)} |\hat{\eta} - \eta| = 0.
\]

We will say that exact recovery is impossible for \( (\Theta_d^+(s, d, a_d))_{d \geq 1} \) in the Coordinate Local case if

\[
\lim_{d \to \infty} \inf_{Q \in \mathcal{Q}_n^{CL}} \inf_{\hat{\eta} = \hat{\eta}(Z^1, \ldots, Z^n) \in \mathcal{T}} \sup_{\theta \in \Theta_d^+(s, a)} |\hat{\eta} - \eta| > 0.
\]

We use similar definitions in the Coordinate Global case with \( \mathcal{Q}_n^{CL} \) replaced by \( \mathcal{Q}_n^\alpha \).

### 1.4 Related work

Variable selection with Hamming loss in the Gaussian mean model in \( \mathbb{R}^d \) has been studied in the non-private setting in [2]. The authors provide non-asymptotic lower and upper bounds on the non-private version of minimax risk [2]. As corollaries, they derive necessary and sufficient conditions for almost full recovery and exact recovery to be possible. They highlight a critical value \( a^* = (\sigma/\sqrt{n})\sqrt{2 \log(d/s - 1)}(1 + o(1)) \) such that almost full recovery is possible for \( a \geq a^* \) and impossible for \( a < a^* \). Similar results have been obtained for exact recovery with the greater critical value \( a^* = (\sigma/\sqrt{n})(\sqrt{2 \log(d - s)} + \sqrt{2 \log s}) \). In the present paper, we will see how these results are affected by the privacy constraints.

A few papers tackle selection problems under privacy constraints. In [13], the authors study top- \( k \) selection under a relaxation of central differential privacy called \((\varepsilon, \delta)\)-approximate differential privacy (see for instance [8]). However, they use a weighted Hamming loss as described below. Precisely, if \( X_1, \ldots, X_n \) are drawn i.i.d. from some distribution \( P \) on \( \{0, 1\}^d \), they want to find the \( k \) greatest coordinates of the mean vector \( \theta = \mathbb{E}_P[X_1] \) while respecting \((\varepsilon, \delta)\)-differential privacy.
\[ a \leq 2\sigma \] possible, as soon as
\[ N \gg \log(N)\log(d) \]
if moreover
\[ N \gg \log(N)\log(d) \]

Table 1: Exact recovery in the Coordinate Local case. Similar results hold for almost full recovery with \( \log(d) \) replaced by \( \log(d/s) \).

constraints. They prove that the existence of a \((1, 1/(nd))\)-differentially private mechanism that outputs \( Z \in \{0, 1\}^d \) with \( k \) non-zero coordinates such that

\[
E \left[ \sum_{j=1}^{d} \theta_j 1(Z_j = 1) \right] \geq \max_{\eta \in \{0, 1\}^d} \sum_{j=1}^{d} \theta_j 1(\eta_j = 1) - \beta
\]

requires \( n \gtrsim \sqrt{k}\log d \) samples in the low accuracy regime where \( \beta = k/10 \). Moreover, repeated use of the classical exponential mechanism solves this problem with \( n = O(\sqrt{k}\log d) \) samples. In [1], the authors study an empirical version of the problem studied in [15]: they want to find the top-\( k \) coordinates of the vector \( q \in \mathbb{R}^d \) defined by \( q_j = (1/n) \sum_{i=1}^n X_{i,j}, j = 1, \ldots, d \) while respecting \((\varepsilon, \delta)\)-differential privacy constraints. Let \( \tau \) be the \( k \)-th largest value among the coordinates \( \{q_1, \ldots, q_k\} \). They prove that the existence of a \((\varepsilon, \delta)\)-differentially private mechanism that outputs a set \( S \subset [1, d] \) of \( k \) elements such that \( q_j \geq \tau - \beta \) for all \( j \in S \) requires \( n \gtrsim k\log(d) \) samples in the high-accuracy regime where \( \beta \approx \sqrt{\log d/n} \). In [17], Ullman studies the same problem as [15] for \( k = 1 \) under non-interactive \( \alpha \)-local differential privacy constraints. If we consider the low-accuracy regime considered by [15], the result by Ullman shows that solving the problem under non-interactive \( \alpha \)-local differential privacy requires \( n \gtrsim d\log d/\alpha^2 \) samples, highlighting an exponentially worse dependence on the dimension compared to the result obtained in the central model of \((\varepsilon, \delta)\)-approximate differential privacy.

1.5 Organisation of the paper

In Section 2, we study the minimax risk \( (2) \). We first provide a lower bound which enables us to obtain necessary conditions for almost full recovery and exact recovery to be possible in the case where only coordinate local privacy mechanisms are used. In particular, we prove that almost full recovery is impossible in this case as soon as the quantity \( n\alpha^2/d^2 \) is bounded from above. We then provide non-asymptotic upper bounds on the minimax risks and state more explicit asymptotic sufficient conditions for almost full recovery and exact recovery to be possible in our corollaries. These conditions and associated results are summarised in Table 1. In Section 3, we study the minimax risk \( (3) \) and prove that the results of Section 2 can be improved when any non-interactive (coordinate global) \( \alpha \)-locally differentially private mechanism is allowed. See Table 2 for a summary of these results. Detailed proofs can be found in the Appendix.

2 Minimax risk using coordinate local non-interactive privacy mechanisms

In this section, we provide a lower bound on the private minimax risk \( (2) \). This enables us to obtain necessary conditions for almost full recovery and exact recovery to be possible in the Coordinate Local scenario. In particular, we prove that almost full recovery is impossible in the private setting of the Coordinate Local case if the quantity \( N := n\alpha^2/d^2 \) is bounded from above. We then provide
\( a \lesssim \sigma \sqrt{\frac{\log d}{N^2}} \) & \( \sigma \sqrt{\frac{\log d}{N^2}} \ll a \leq 2\sigma \) & impossible if \\
\( \frac{Nd}{\log d} \lesssim 1 \) & impossible & impossible \\
\( \frac{Nd}{\log d} \gg 1 \) & possible, as soon as \\
\( a \gtrsim \sigma \sqrt{\frac{\log d}{N^2}} \sqrt{\log(Nd)} \), if moreover \\
\( Nd \gg \log(Nd) \log(d) \) & possible

Table 2: Exact recovery in the Coordinate Global case. We have set \( N = na^2/d^2 \) for a better comparison with the Coordinate Local case.

upper bounds on the minimax risk that entail sufficient conditions for almost full recovery and exact recovery to be possible.

### 2.1 Lower bound

We first state our lower bound.

**Theorem 2.1.** For any \( a > 0 \), \( \alpha > 0 \), \( 1 \leq s \leq d \), \( n \geq 1 \), we have

\[
R^C_{n}(\alpha, \Theta^+_d(s, a)) \geq \left( 1 - \frac{8}{d} \right) \exp \left( -4n(e^{\alpha/d} - 1)^2 \min \left\{ \frac{a^2}{4\sigma^2}, 1 \right\} \right).
\] (4)

The proof of Theorem 2.1 can be found in Appendix A.2. Some auxiliary results used for the proof of Theorem 2.1 can be found in Appendix A.1. Note that since \( \Theta^+_d(s, a) \subset \Theta_d(s, a) \) we have \( R^C_{n}(\alpha, \Theta^+_d(s, a)) \leq R^C_{n}(\alpha, \Theta_d(s, a)) \), thus the right hand side of (4) is also a lower bound for \( R^C_{n}(\alpha, \Theta_d(s, a)) \).

For better confidentiality in practice, the parameter \( \alpha \) must not be too large. In particular, we assume that \( \alpha/d \to 0 \) when \( d \to +\infty \). We thus have \( n(e^{\alpha/d} - 1)^2 \sim na^2/d^2 \) and Theorem 2.1 immediately shows the following.

**Corollary 2.2.** Let \( \alpha > 0 \), \( 1 \leq s \leq d \), \( n \geq 1 \) be such that \( s/d \leq C_0 \) for some constant \( C_0 \in (0, 1) \), and \( \alpha/d \to 0 \) when \( d \to +\infty \). Then, if \( na^2/d^2 \leq C_1 \) for some constant \( C_1 > 0 \) or if \( na^2/d^2 \to \infty \) as \( d \to \infty \) and \( a^2 \leq C_2\sigma^2d^2/(na^2) \), it holds

\[
R^C_{n}(\alpha, \Theta) \geq C
\]

for some constant \( C > 0 \), where \( \Theta = \Theta^+_d(s, a) \) or \( \Theta = \Theta_d(s, a) \).

Corollary 2.2 shows that almost full recovery is impossible under local differential privacy constraints if the quantity \( na^2/d^2 \) is bounded from above. In particular, almost full recovery is impossible under local differential privacy constraints in the high-dimensional setting, that is when \( n \leq d \), whatever the value of \( a \). Corollary 2.2 also proves that if \( na^2/d^2 \to +\infty \) then almost full recovery is impossible if \( a \lesssim (\sigma d)/(\sqrt{n}) \).

This underlines a strong difference between the private setting and the classical setting, since [5] proved that in the non-private setting almost full recovery is possible for \( a \) large enough even if \( n = 1 \). Transposed to this setting, variable selection is possible for \( a \gg \sigma/\sqrt{n} \) and any \( n \geq 1 \) without privacy constraints, whereas it is impossible for any signal value \( a \) when the effective size \( N = na^2/d^2 \lesssim 1 \) under privacy constraints.
2.2 Privacy mechanism

In this section, we introduce a non-interactive privacy mechanism creating private views \(Z^1, \ldots, Z^n\) of the original data \(X^1, \ldots, X^n\) that satisfy the local differential privacy constraint of level \(\alpha\). These privatized data will then be used to define a private selector whose risk will be studied in Section 2.3.

To obtain the privatized data, we first censor the unbounded random variables \(X^i_j\), for \(i = 1, \ldots, n, j = 1, \ldots, d\), and then make use of an appropriately scaled version of the classical Laplace mechanism. For all \(i \in [1, n]\) and \(j \in [1, d]\) define

\[
Z^i_j = [X^i_j]_T + \frac{2Td}{\alpha}W^i_j,
\]

where \([\cdot]_T = \max\{-T, \min\{\cdot, T\}\}\) denotes the censoring operator at level \(T\), the \(W^i_j\)'s are i.i.d Laplace(1) random variables, and \(W^i_j\) is independent from \(X^i_j\). The censoring level \(T\) needs to be properly chosen and will be specified later.

Note that the privacy mechanism defining \((Z^i_j)_{i=1, \ldots, n}\) is non-interactive since \(Z^i\) does only depend on \(X^i\) and not on \(Z^k\) for \(k \neq i\). This is also a coordinate local mechanism since \(Z^i_j\) depends on \(X^i_j\) but not on the \(X^i_l\) for \(l \neq j\).

The following Proposition shows that it satisfies the condition of \(\alpha\)-local differential privacy.

**Proposition 2.3.** For all \(i \in [1, n]\) and \(j \in [1, d]\), \(Z^i_j\) is an \(\alpha/d\)-differentially private view of \(X^i_j\). Consequently, for all \(i \in [1, n]\) \(Z^i = (Z^i_j)_{j=1, \ldots, d}\) is an \(\alpha\)-differentially private view of \(X^i\).

**Proof.** Set \(r = 2Td/\alpha\). By definition of the privacy mechanism (5), the conditional density of \(Z^i_j\) given \(X^i = x\) can be written as

\[
q_{Z^i_j | X^i = x}(z) = \frac{1}{2r} \exp \left( -\frac{||z - [x]_T||}{r} \right).
\]

Thus, by the reverse and the ordinary triangle inequality it holds for all \(i \in [1, n]\), \(j \in [1, d]\) and all \(x, x', z \in \mathbb{R}\),

\[
\frac{q_{Z^i_j | X^i = x}(z)}{q_{Z^i_j | X^i = x'}(z)} = \exp \left( \frac{||z - [x']_T||}{r} - \frac{||z - [x]_T||}{r} \right)
\leq \exp \left( \frac{||[x']_T - [x]_T||}{r} \right)
\leq \exp \left( \frac{2T}{r} \right) \leq \exp \left( \frac{\alpha}{d} \right).
\]

This proves that \(Z^i_j\) is an \(\alpha/d\)-differentially private view of \(X^i_j\). Let us check that \(Z^i\) is an \(\alpha\)-differentially private view of \(X^i\). Denote by \(q_{Z^i | X^i = x}\) the conditional density of \(Z^i\) given \(X^i = x\) and note that for all \(x, x', z \in \mathbb{R}^d\) it holds

\[
\frac{q_{Z^i | X^i = x}(z)}{q_{Z^i | X^i = x'}(z)} = \prod_{j=1}^d \frac{q_{Z^i_j | X^i_j = x_j}(z_j)}{q_{Z^i_j | X^i_j = x'_j}(z_j)} \leq e^\alpha,
\]

using the independence of the coordinates \(X^i_1, \ldots, X^i_d\) and the conditional independence of \(Z^i_1, \ldots, Z^i_d\) given \(X^i\). \qed

2.3 Upper bounds

Using these privatized data, we define two selectors that will provide upper bounds on the minimax risk (2). For the class \(\Theta^+_d(s, a)\), we will use the selector \(\hat{\eta}^+_j\) with the components

\[
\hat{\eta}^+_j = I \left( \frac{1}{n} \sum_{i=1}^n Z^i_j \geq \tau \right), \quad j = 1, \ldots, d,
\]
where the threshold $\tau$ has to be properly chosen, later on. For the class $\Theta_d(s,a)$, we will use the selector $\hat{\eta}$ with the components

$$
\hat{\eta}_j = I \left( \frac{1}{n} \sum_{i=1}^{n} Z_{ij}^2 \geq \tau \right), \quad j = 1, \ldots, d,
$$

where $\tau$ to be defined later on. Note that $\hat{\eta}^+$ and $\hat{\eta}$ are separable selectors since $\hat{\eta}_j^+$ and $\hat{\eta}_j$ depend only on $(Z_j^i)_{i=1,\ldots,n}$ and not on the $Z_k^i$ for $k \neq j$. We now study the performances of these selectors. Recall that $\Phi$ denotes the standard Gaussian cumulative distribution function.

**Proposition 2.4.** Assume that $a > b \sigma$ for some constant $b > 0$. Set $C_1 := 1 - 2\Phi(-b)$. If $\tau, T$ are chosen such that

$$
T \leq a - b\sigma, \quad C_1 T - \tau \geq 0, \quad \tau \alpha/(8Td) \leq 1 \text{ and } \alpha (C_1 T - \tau)/(8Td) \leq 1,
$$

then it holds for all $\theta \in \Theta_d^+(s,a)$,

$$
\mathbb{E}\left[ \frac{1}{s} |\hat{\eta}^+ - \eta| \right] \leq \frac{d - |S|}{s} \left[ \exp \left( - \frac{n\tau^2}{2^4 \min\{T^2, \sigma^2\}} \right) + \exp \left( - \frac{\tau^2 n\alpha^2}{2^7 T^2 d^2} \right) \right]
$$

$$
+ \frac{|S|}{s} \left[ \exp \left( - \frac{n(C_1 T - \tau)^2}{2^4 \min\{T^2, \sigma^2\}} \right) + \exp \left( - \frac{(C_1 T - \tau)^2 n\alpha^2}{2^7 T^2 d^2} \right) \right],
$$

and for all $\theta \in \Theta_d(s,a)$ it holds

$$
\mathbb{E}\left[ \frac{1}{s} |\hat{\eta} - \eta| \right] \leq 2 \left\{ \frac{d - |S|}{s} \left[ \exp \left( - \frac{n\tau^2}{2^4 \min\{T^2, \sigma^2\}} \right) + \exp \left( - \frac{\tau^2 n\alpha^2}{2^7 T^2 d^2} \right) \right]
$$

$$
+ \frac{|S|}{s} \left[ \exp \left( - \frac{n(C_1 T - \tau)^2}{2^4 \min\{T^2, \sigma^2\}} \right) + \exp \left( - \frac{(C_1 T - \tau)^2 n\alpha^2}{2^7 T^2 d^2} \right) \right] \right\},
$$

where $S$ denotes the support of $\theta$.

The proof of Proposition 2.4 is given in section A.4 in the Appendix. Some auxiliary results used in the proof of Proposition 2.4 can be found in Appendix A.3. The following Corollary gives sufficient conditions so that almost full recovery and exact recovery are possible under local differential privacy in the Coordinate Local case when $a \geq 2\sigma$.

**Corollary 2.5.** Set $C_1 = 1 - 2\Phi(-1)$. Assume that

$$
\alpha/d \to 0, n\alpha^2/d^2 \to +\infty \text{ and } \limsup \frac{\log(d/s)}{n\alpha^2/d^2} < \frac{C_1^2}{2^9}.
$$

Then the selector $\hat{\eta}^+$ defined by (6) with $T = \sigma$ and $\tau = C_1 T/2$ satisfies

$$
\sup_{\theta \in \Theta} \frac{1}{s} \mathbb{E}_{Q_{(p_\theta^{\#})}} |\hat{\eta}^+(Z^1, \ldots, Z^d) - \eta| \to 0,
$$

for all $a \geq 2\sigma$, where $\Theta = \Theta_d^+(s,a)$ or $\Theta = \Theta_d(s,a)$. If, in addition, $\limsup \frac{\log(d/s)}{n\alpha^2/d^2} < \frac{C_1^2}{2^9}$, then

$$
\sup_{\theta \in \Theta} \frac{1}{s} \mathbb{E}_{Q_{(p_\theta^{\#})}} |\hat{\eta}^+(Z^1, \ldots, Z^d) - \eta| \to 0,
$$

for all $a \geq 2\sigma$.

The proof of Corollary 2.5 is given in section A.5 in the Appendix. Since we have seen that almost full recovery is impossible when $n\alpha^2/d^2$ is bounded from above or when $n\alpha^2/d^2 \to +\infty$ and $a \lesssim (\sigma d)/(\sqrt{n}\alpha)$, it remains to study the case where $n\alpha^2/d^2 \to +\infty$ and $\sigma d/(\sqrt{n}\alpha) \ll a \leq 2\sigma$. This is done below.
Proposition 2.6. Let $a > 0$. If $T$ and $\tau$ are chosen such that $T \geq a + \sqrt{2\log \left( \max \left\{ e, \frac{d}{a \sqrt{2\pi}} \right\} \right)}$, $\tau \leq a/2$, $\tau a/(8Td) < 1$ and $a(a/2 - \tau)/(8Td) \leq 1$ then it holds for all $\theta \in \Theta^+_d(s,a)$,

$$
\mathbb{E}\left[ \frac{1}{s} |\hat{\eta}^+ - \eta| \right] \leq \frac{d - |S|}{s} \left[ \exp \left( - \frac{n\tau^2}{2^3 \min\{T^2, \sigma^2\}} \right) + \exp \left( - \frac{\tau^2 na^2}{2^3 T^2 d^2} \right) \right] + \frac{|S|}{s} \left[ \exp \left( - \frac{n(a/2 - \tau)^2}{2^3 \min\{T^2, \sigma^2\}} \right) + \exp \left( - \frac{(a/2 - \tau)^2 na^2}{2^3 T^2 d^2} \right) \right],
$$

where $S$ denotes the support of $\theta$.

The proof of Proposition 2.6 can be found in section A.6 in the Appendix. Note that as for the case $a \geq b\sigma$, if $\theta \in \Theta_d(s,a)$ we use $\hat{\eta}$ instead of $\hat{\eta}^+$ and we can prove the same result with an extra multiplicative factor 2. The next corollary gives new sufficient conditions so that almost full recovery and exact recovery are possible.

Corollary 2.7. Assume that $a/d \to 0$, $na^2/d^2 \to +\infty$ and $\sigma d/(\sqrt{n}a) \ll a \leq 2\sigma$. The selector $\hat{\eta}^+$ defined by (4) with $T = a + \sigma \sqrt{2\log \left( \sqrt{n}a \right)}$ and $\tau = a/4$ satisfies for $d$ large enough

$$
\sup_{\theta \in \Theta^+_d(s,a)} \frac{1}{s} \mathbb{E}_{Q(P^\otimes n)} |\hat{\eta}^+(Z^1, \ldots, Z^d) - \eta| \leq 2 \exp \left( \log \left( \frac{d}{s} \right) - \frac{a^2 n a^2}{2^{13} \sigma^2 \log \left( \frac{na^2}{d^2} \right) d^2} \right).
$$

In particular, if $a \gg \frac{\sigma d}{a \sqrt{n}} \log^{1/2} \left( \frac{na^2}{d^2} \right) \log^{1/2} \left( \frac{d}{s} \right)$ it holds

$$
\sup_{\theta \in \Theta^+_d(s,a)} \frac{1}{s} \mathbb{E}_{Q(P^\otimes n)} |\hat{\eta}^+(Z^1, \ldots, Z^d) - \eta| \to 0. \tag{12}
$$

Moreover, if $a \gg \frac{\sigma d}{a \sqrt{n}} \log^{1/2} \left( \frac{na^2}{d^2} \right) \log^{1/2} (d)$

$$
\sup_{\theta \in \Theta^+_d(s,a)} \mathbb{E}_{Q(P^\otimes n)} |\hat{\eta}^+(Z^1, \ldots, Z^d)| - \eta| \to 0. \tag{13}
$$

If $na^2/d^2 \to \infty$ with $(na^2/d^2) \gg \log(na^2/d^2) \log(d/s)$, then Corollary 2.7 combined with Corollary 2.6 and with the lower bound [3] prove a phase transition result (up to log factors) at the value $a^* = a^*(n, a, d, \sigma) = \sigma d/(a \sqrt{n})$. Indeed, we get that almost full recovery is impossible in the Coordinate Local case for all $a \leq Ca^*$ and is possible for all $a \gg a^* \log^{1/2}(na^2/d^2) \log^{1/2}(d/s)$.

3 Minimax risk using coordinate global non-interactive privacy mechanisms

In this section, we study the minimax risk [3]. We prove that in the Coordinate Global case, almost full recovery and exact recovery are possible under weaker assumptions than the one we obtained for the Coordinate Local case.

3.1 Privacy mechanism

We describe in this section the privacy mechanism we use to obtain private data that will be used to design a private selector and to obtain upper bounds on the minimax risk [3] in the Coordinate Global case.

For all $i \in [1, n]$, the private view $Z^i$ of $X^i$ is obtained using the following steps:

- Compute $f(T(X^i)) = ([X^i]_T)_{j=1, \ldots, d}$ where $[.]_T = \max\{T, \min\{\cdot, T\}\}$ denotes the censoring operator of level $T$ and $T$ will be specified later.
• Define a random vector \( \tilde{X}^i \in \mathbb{R}^d \) with coordinates
\[
\tilde{X}^i_j \sim \begin{cases} 
T & \text{with probability } \frac{1}{2} + \frac{|X^i_j|}{2T} \\
-T & \text{with probability } \frac{1}{2} - \frac{|X^i_j|}{2T}
\end{cases}
\]

• Sample \( Y^i \sim B(\pi_\alpha) \) where \( \pi_\alpha = e^\alpha/(e^\alpha + 1) \) and generate
\[
\tilde{Z}^i \sim \begin{cases}
\mathcal{U}\left( \tilde{z} \in \{-B, B\}^d \mid \langle \tilde{z}, \tilde{X}^i \rangle > 0 \text{ or } (\langle \tilde{z}, \tilde{X}^i \rangle = 0 \text{ and } \tilde{z}_1 = (B/T)\tilde{X}^i_1) \right) & \text{if } Y^i = 1 \\
\mathcal{U}\left( \tilde{z} \in \{-B, B\}^d \mid \langle \tilde{z}, \tilde{X}^i \rangle < 0 \text{ or } (\langle \tilde{z}, \tilde{X}^i \rangle = 0 \text{ and } \tilde{z}_1 = -(B/T)\tilde{X}^i_1) \right) & \text{if } Y^i = 0.
\end{cases}
\]

with
\[
B = T \frac{e^\alpha + 1}{e^\alpha - 1} K_d, \quad \text{where } \frac{1}{K_d} = \begin{cases} 
\frac{1}{2^d-1} & \text{if } d \text{ is odd} \\
\frac{1}{2^{d-1}(d-2)} & \text{if } d \text{ is even.}
\end{cases}
\]

• Define the vector \( Z^i \) by \( Z^i = \tilde{Z}^i \) if \( d \) is odd, and by its components
\[
Z^i_j = \begin{cases} 
\frac{d-2}{2^{d-1}} \tilde{Z}^i_j & \text{if } j = 1 \\
\tilde{Z}^i_j & \forall j \in [2, d],
\end{cases}
\]

if \( d \) is even.

This mechanism is strongly inspired by the one proposed by Duchi et al. \[7\] for mean estimation on the set of distributions \( P \) supported on \( \mathbb{B}_\infty(r) \subset \mathbb{R}^d \) with \( \|E[X]\|_0 \leq s \). In particular, if \( d \) is odd, the event \( \{\langle \tilde{z}, \tilde{X}^i \rangle = 0\} \) has probability zero for all \( \tilde{z} \in \{-B, B\}^d \) and our mechanism coincides with the one proposed by Duchi et al. \[7\] at the exception that we added a censoring step. Since in our framework the \( X^i \) are not compactly supported, this censoring step is needed so that the probabilities appearing in the definition of \( \tilde{X}^i \) are non-negative.

**Proposition 3.1.** For all \( i \in [1, n] \), \( Z^i \) is an \( \alpha \)-differentially private view of \( X^i \).

The following proposition will be useful in the analysis of the selector proposed in Section 3.2.

**Proposition 3.2.** For all \( i \in [1, n] \), it holds
\[
\mathbb{E}[Z^i \mid X^i] = f_T(X^i).
\]

The proofs of Proposition 3.1 and Proposition 3.2 can be found respectively in Section 3.1 and 3.2 of the Appendix. Note that it also holds \( \mathbb{E}[Z^i \mid X^i] = f_T(X^i) \) when \( Z^i \) is produced via the Laplace mechanism described in Subsection 2.2. However the variance \( \text{Var}[Z^i \mid X^i] \) is slower by a multiplicative factor \( d \) when \( Z^i \) is produced with the Laplace mechanism than when it is obtained with the above coordinate global mechanism. Indeed, if \( Z^i \) is produced with the above mechanism, then we have \( \text{Var}(Z^i_j \mid X^i) \leq B^2 \). Stirling’s approximation yields \( K_d^2 \lesssim d \) for \( d \) large enough, see Lemma B.1 in Appendix B.3 for details. Thus, if \( \alpha \) is bounded, we obtain \( \text{Var}(Z^i_j \mid X^i) \leq T^2 d/\alpha^2 \). Now, if \( Z^i \) is produced with the Laplace mechanism then it holds \( \text{Var}(Z^i_j \mid X^i) = 8T^2 d^2/\alpha^2 \). This faster variance explains that we will obtain better results when allowing for coordinate global mechanisms.

### 3.2 Upper bounds

Using the privatized data of the previous subsection, we define two selectors that will enable us to obtain upper bounds on the minimax risk \[3\]. As in the Coordinate Local case, for the class \( \Theta^+_d(s, \alpha) \), we will use the selector \( \tilde{\eta}^+ \) with the components
\[
\tilde{\eta}^+_j = I\left(\frac{1}{n} \sum_{i=1}^n Z^i_j \geq \tau\right), \quad j = 1, \ldots, d,
\]

(15)
where the threshold \( \tau \) has to be chosen. For the class \( \Theta_d(s, a) \), we will use the selector \( \hat{\eta} \) with the components

\[
\hat{\eta}_j = I \left( \frac{1}{n} \sum_{i=1}^{n} Z_j \geq \tau \right), \quad j = 1, \ldots, d.
\]

(16)

We now study the performances of these selectors.

The following result gives an upper bound on the risk of selector (15) when \( a \geq C \sigma \) and will enable us to obtain sufficient conditions so that almost full recovery is possible when \( a \geq 2 \sigma \) in the Coordinate Global case.

**Proposition 3.3.** Assume that \( a > b \sigma \) for some constant \( b > 0 \). Set \( C_1 := 1 - 2 \Phi(-b) \). If \( \tau, T \) are chosen such that

\[
T \leq a - \sigma b, \text{ and } C_1 T - \tau > 0,
\]

then it holds for all \( \theta \in \Theta_d^+(s, a) \),

\[
\mathbb{E} \left[ \frac{1}{s} |\hat{\eta}^+ - \eta| \right] \leq \frac{d - |S|}{s} \exp \left( -\frac{n \tau^2}{2B^2} \right) + \frac{|S|}{s} \exp \left( -\frac{n(C_1 T - \tau)^2}{2B^2} \right),
\]

where \( S \) denotes the support of \( \theta \). In particular, choosing \( T = \sigma \) and \( \tau = C_1 T/2 \) with \( C_1 = 1 - 2 \Phi(-1) \) yields

\[
\sup_{\theta \in \Theta_d^+(s, a)} \mathbb{E} \left[ \frac{1}{s} |\hat{\eta}^+ - \eta| \right] \leq \frac{d}{s} \exp \left( -\frac{C_1^2 n(\sigma - 1)^2}{8(e^\alpha + 1)^2 K_d^2} \right)
\]

for all \( a \geq 2 \sigma \).

The proof of Proposition 3.3 can be found in section 3.4 of the Appendix. Note that we can provide similar results on the class \( \Theta_d(s, a) \) considering the selector \( \hat{\eta} \). The upper bounds are the same than for the class \( \Theta^+_d(s, a) \) up to a multiplicative factor 2 that comes from the use in the proof of the two-sided Hoeffding’s inequality instead of the one-sided inequality. Since \( K_d \leq C \sqrt{d} \) for \( d \) large enough, we obtain that a sufficient condition for almost full recovery to be possible when \( a \geq 2 \sigma \) is that \( \frac{n(\sigma - 1)^2}{(e^\alpha + 1)^2 d} \gtrsim \log(d/s) \). Moreover, using that \( (\sigma - 1)^2/(e^\alpha + 1)^2 \geq 0.2a^2 \) if \( \alpha \leq 1 \), we obtain that a sufficient condition for almost full recovery to be possible when \( a \geq 2 \sigma \) and \( \alpha \leq 1 \) is that \( n \alpha^2/d \gtrsim \log(d/s) \). This improves the result we obtained when we considered only privacy mechanisms acting coordinates by coordinates for which we needed \( n \alpha^2/d \gtrsim \log(d/s) \). We now deal with the case \( a << \sigma \).

**Proposition 3.4.** Let \( a > 0 \). If \( T \) and \( \tau \) are chosen such that

\[
T \geq a + \sigma \sqrt{2 \log \left( \max \left\{ e, \frac{4 \sigma}{a \sqrt{2 \pi}} \right\} \right)}, \quad \text{and } \tau < a/2,
\]

then it holds for all \( \theta \in \Theta_d^+(s, a) \),

\[
\mathbb{E} \left[ \frac{1}{s} |\hat{\eta}^+ - \eta| \right] \leq \frac{d - |S|}{s} \exp \left( -\frac{n \tau^2}{2B^2} \right) + \frac{|S|}{s} \exp \left( -\frac{n(a/2 - \tau)^2}{2B^2} \right),
\]

where \( S \) denotes the support of \( \theta \).

The proof of Proposition 3.4 can be found in Appendix B.5

**Corollary 3.5.** Assume that \( \alpha/d \to 0 \), \( n \alpha^2/d \to +\infty \) and \( \sigma \sqrt{d}/(a \sqrt{n}) \ll a \leq 2 \sigma \). The selector \( \hat{\eta}^+ \) defined by (15) with \( T = a + \sigma \sqrt{2 \log \left( \sqrt{\frac{n \alpha^2}{d}} \right)} \) and \( \tau = a/4 \) satisfies for \( n, d \) large enough

\[
\sup_{\theta \in \Theta_d^+(s, a)} \mathbb{E} \left[ \frac{1}{s} |\hat{\eta}^+(Z^1, \ldots, Z^d) - \eta| \right] \leq \frac{d}{s} \exp \left( -\frac{n(\alpha - 1)^2 a^2}{2^7 \sigma^2 (e^\alpha + 1)^2 K_d^2 \log(n \alpha^2/d)} \right).
\]
In particular, if \( \alpha \in (0, 1] \), if \( n\alpha^2/d \to +\infty \) with \((n\alpha^2/d)/\log(n\alpha^2/d) \gg \log(d)\) then it holds
\[
\sup_{\theta_0 \in \Theta_{s,a}^1(s,a)} \mathbb{E}_{Q(P_{s,a}^\circ)}[\hat{\eta}^+ \left( Z_1, \ldots, Z_d \right) - \eta] \to 0,
\]
for all \( \sigma \) satisfying \( \sigma \sqrt{\frac{n\alpha^2}{d}} \sqrt{\log \left( \frac{n\alpha^2}{d} \right)} / \log(d) \ll a \leq 2\sigma \).

The first statement in Corollary 3.5 is a direct consequence of Proposition 3.4. The second statement is a direct consequence of the first one where we have used \((e^\alpha - 1)^2 / (e^\alpha + 1)^2 \geq 0.2\alpha^2\) for \( \alpha \in (0, 1] \) and \( K_d \leq C\sqrt{d} \) for \( d \) large enough. In the next subsection, we complement these results with a lower bound. This will enable us to exhibit a value \( a^* \) such that exact recovery is impossible for all \( a \leq a^* \) and possible for \( a > a^* \) under the assumptions \( \alpha \in (0, 1] \) and \( n\alpha^2/d \to \infty \) with \((n\alpha^2/d)/\log(n\alpha^2/d) \gg \log(d)\).

### 3.3 Lower bound

**Proposition 3.6.** For any \( \alpha > 0 \), \( \alpha > 0 \), \( d \geq 4 \), \( 1 \leq s \leq d \), \( n \geq 1 \), we have
\[
\inf_{Q \in \mathcal{Q}_s} \inf_{\eta \in \Theta} \sup_{\theta_0 \in \Theta_{s,a}^1(s,a)} \mathbb{E}_{Q(P_{s,a}^\circ)}[\hat{\eta}^+ - \eta] \geq \frac{1}{4} \left( 1 - 2n(e^\alpha - 1)^2 / d \log(d) \right) \left[ \exp \left( \frac{a^2}{\sigma^2} \right) - 1 \right].
\]

The proof of Proposition 3.6 is based on a private version of Fano’s method, see Proposition 2 in [7]. It can be found in Section B.6 of the Appendix. Using that \((e^\alpha - 1)^2 \leq 4\alpha^2\) for \( \alpha \in (0, 1) \) and \( \exp(x^2) - 1 \leq 14x^2 \) for \( 0 \leq x \leq 2 \), Proposition 3.6 immediately shows the following.

**Corollary 3.7.** Let \( \alpha \in (0, 1) \). If \( n\alpha^2/(d\log d) \leq C/448 \) for some constant \( C \in (0, 1) \) then it holds
\[
\inf_{Q \in \mathcal{Q}_s} \inf_{\eta \in \Theta} \sup_{\theta_0 \in \Theta_{s,a}^1(s,a)} \mathbb{E}_{Q(P_{s,a}^\circ)}[\hat{\eta}^+ - \eta] \geq \frac{1}{4} \left( 1 - C \right) > 0,
\]
for all \( a \leq 2\sigma \).

This shows that exact recovery is impossible for all \( a \leq 2\sigma \) if \( n\alpha^2/(d\log d) \leq C/448 \) for some constant \( C \in (0, 1) \). Proposition 3.6 also implies that exact recovery is impossible if \( a \leq \sigma \sqrt{\log(1 + Cd\log d)/(8na^2)} \) for some constant \( C \in (0, 1) \). However, unlike the coordinate local case, the lower bound provided by Proposition 3.6 does not allow us to say that exact recovery is also impossible for \( a \geq \max \{ 2\sigma, \sigma \sqrt{\log(1 + Cd\log d)/(8na^2)} \} \) when \( n\alpha^2/(d\log d) \) is bounded from above. The following corollary is also a direct consequence of Proposition 3.6. It shows that when \( n\alpha^2/(d\log d) \to \infty \), exact recovery is still impossible if \( a \) is too small.

**Corollary 3.8.** If \( \alpha \in (0, 1) \), \( n\alpha^2/d \to +\infty \) with \( n\alpha^2/d \gg \log d \) and \( a \leq (\sigma/224) \sqrt{d\log d/(n\alpha^2)} \) it holds
\[
\liminf_{d \to +\infty} \inf_{Q \in \mathcal{Q}_s} \inf_{\eta \in \Theta} \sup_{\theta_0 \in \Theta_{s,a}^1(s,a)} \mathbb{E}_{Q(P_{s,a}^\circ)}[\hat{\eta}^+ - \eta] \geq \frac{1}{8}.
\]

The lower bound of Proposition 3.6 combined with the upper bounds of Subsection 3.2 exhibit a phase transition at the value \( a^* \) (up to a logarithmic factor) such that exact recovery is impossible for all \( a \leq a^* \) and possible for \( a > a^* \) under the assumptions \( \alpha \in (0, 1] \) and \( n\alpha^2/d \to +\infty \) with \((n\alpha^2/d)/\log(n\alpha^2/d) \gg \log(d) \). Precisely, set
\[
a^* = a^*(n, \alpha, d, \sigma) = \frac{\sigma}{224} \sqrt{\frac{d\log d}{n\alpha^2}}.
\]

Proposition 3.6 combined with Corollary 3.7 and Proposition 3.8 give the following result.

**Corollary 3.9.** Assume that \( \alpha \in (0, 1] \) and \( n\alpha^2/d \to +\infty \) with \((n\alpha^2/d)/\log(n\alpha^2/d) \gg \log(d) \). Then, exact recovery is impossible for all \( a \leq a^* \) and possible for all \( a \gg a^* \sqrt{\log(n\alpha^2/d)} \).
Note that Proposition 3.6 does not allow us to obtain impossibility results for almost full recovery in the regime \( na^2/(d \log d) \gg 1 \). Its proof relies on a private Fano’s method (Proposition 2 in [7]) applied with the family of distributions \( \{ N(a\omega_i, \sigma^2 I_d), i = 1, \ldots, d \} \) where \( \omega_i \in \{0, 1\}^d \) is defined by \( \omega_{ij} = \delta_{ij} \) and \( \delta \) is the Kronecker delta. The same proof with \( \omega_i \) defined by \( \omega_{ij} = 1 \) if \( j \in [i-1]s+1, [i]s \) and \( 0 \) otherwise for \( i = 1, \ldots, [d/s] \), provides the following lower bound.

**Proposition 3.10.** For any \( a > 0 \), \( \alpha > 0 \), \( n \geq 1 \). If \( d/s \leq 4 \) then we have

\[
\inf_{Q \in \Omega_n} \inf_{\eta \in T} \sup_{\theta \in \Theta_n^{(s,a)}} \frac{1}{s} \mathbb{E}_{Q_{\{\theta\}^s}}[|\hat{\eta} - \eta|] \geq \frac{1}{4} \left( 1 - \frac{2n(e^a - 1)^2}{[d/s] \log([d/s])} \right) \left[ \exp \left( \frac{sa^2}{\sigma^2} \right) - 1 \right].
\]

However, this bound turns out to be suboptimal in the sense that when it holds \((na^2/d)/(\log(na^2/d) \gg \log(d/s))\) the combination of this bound with upper bounds in Proposition 3.3 and Corollary 3.5 allows us to exhibit the critical value \( a^* \) for almost full recovery only up to a logarithmic factor times the sparsity \( s \). Indeed, on the one hand Proposition 3.3 and Corollary 3.5 prove that almost full recovery is possible for all \( a \gg \sigma \sqrt{d/(na^2)} \sqrt{\log(d/s)} \) in the regime \((na^2/d)/(\log(na^2/d) \gg \log(d/s))\). On the other hand Proposition 3.10 proves that, in the same regime, almost full recovery is impossible for \( a \lesssim (\sigma/s) \sqrt{d/(na^2)} \sqrt{\log(d/s)} \) but does not allow us to say what happen for \((\sigma/s) \sqrt{d/(na^2)} \sqrt{\log(d/s)} \ll a \lesssim (\sigma/s) \sqrt{d/(na^2)} \sqrt{\log(na^2/d) \log(d/s)}\).

4 Discussion

We addressed the problem of variable selection in the Gaussian mean model in \( \mathbb{R}^d \) under local differential privacy constraints. We have provided lower and upper bounds on the rate of convergence for the expected Hamming loss over classes of at most \( s \)-sparse vectors whose non-zero coordinates are separated from \( 0 \) by a constant \( a > 0 \). When we restrict our attention to non-interactive mechanisms that act independently on each coordinate (coordinate local privacy mechanisms) we have proved that, contrary to the non-private setting, almost full recovery and exact recovery are impossible whatever the value of \( a \) in the high-dimensional regime. This is due to the fact that the loss of information due to privacy may reduce the effective sample size \( N := na^2/d^2 \) under the value \( 1 \), and this does not allow support recovery neither exact nor almost full.

However, in the regime \( na^2/d^2 \gg \log(na^2/d^2) \log(d) \) we have exhibited a critical value \( a^* \) (up to a logarithmic factor) such that exact recovery is possible for all \( a \gg a^* \) and impossible for all \( a \lesssim a^* \). We have also proved that these results can be improved when allowing for all non-interactive locally differentially private mechanisms, that we also call coordinate global. The effective sample size is \( Nd \) in this case and it is larger than \( N \).

For many estimation problems allowing for sequentially interactive privacy mechanisms do not improve substantially over non-interactive minimax rates. This includes for instance density estimation [1], one-dimensional mean estimation [7], and estimation of a linear functional of the true distribution [14]. However, for some estimation problems (see for instance the estimation of the integrated square of a density, [6]) and some testing problems (see [2] and [6]) allowing for sequentially interactive interaction between data-holders can substantially improve over non-interactive minimax rates of estimation or non-interactive minimax rates of testing. The change of regime from coordinate local to coordinate global privacy mechanisms suggests that support recovery will be improved for interactive privacy mechanisms, but it is left for future work to study whether that is indeed the case.

A Proofs of Section 2

A.1 Some auxiliary results for the proof of the lower bound

The proof of Theorem 2.1 strongly relies on the following result known as the Bayesian version of the Neyman-Pearson lemma.

**Theorem A.1** ([11], Problem 3.10). Let \( P_0 \) and \( P_1 \) be probability distributions possessing densities \( p_0 \) and \( p_1 \) with respect to a measure \( \mu \). Consider the problem of testing \( H_0 : P = P_0 \) against
$H_1 : P = P_1$, and suppose that known probabilities $\pi$ and $1 - \pi$ can be assigned to $H_0$ and $H_1$ prior to the experiment. Then the test $T^*$ given by

$$T^*(X) = I((1 - \pi)p_1(X) > \pi p_0(X))$$

is a minimizer of the overall probability of error resulting from the use of a test $T$,

$$\pi \mathbb{E}_0[T(X)] + (1 - \pi)\mathbb{E}_1[1 - T(X)].$$

The following lemmas are also useful to prove the lower bound.

**Lemma A.2.** Let $b, c > 0$. Let $P$ and $Q$ be two probability measures having densities $p$ and $q$ with respect to some measure $\mu$. It holds

$$\int \min\{bp(x), cq(x)\} d\mu(x) \geq \frac{bc}{b + c} \left(\int \sqrt{p(x)q(x)} d\mu(x)\right)^2.$$

The case $b = c = 1$ can be found in [16] (lemma 2.3). We generalize the proof for any $b, c > 0$.

**Proof.** Cauchy-Schwarz inequality yields

$$bc \left(\int \sqrt{p(x)q(x)} d\mu(x)\right)^2 = \left(\int \sqrt{bp(x) \cdot cq(x)} d\mu(x)\right)^2 = \left(\int \sqrt{\min\{bp(x), cq(x)\}} \sqrt{\max\{bp(x), cq(x)\}} d\mu(x)\right)^2 \leq \int \min\{bp(x), cq(x)\} d\mu(x) \int \max\{bp(x), cq(x)\} d\mu(x).$$

Set $A = \{x : bp(x) \geq cq(x)\}$ and note that

$$\int \min\{bp, cq\} d\mu + \int \max\{bp, cq\} d\mu = \int_A cq d\mu + \int_{A^c} bpd\mu + \int_A bpd\mu + \int_{A^c} cq d\mu = b + c.$$

Thus,

$$bc \left(\int \sqrt{p(x)q(x)} d\mu(x)\right)^2 \leq \int \min\{bp(x), cq(x)\} d\mu(x) \left[b + c - \int \min\{bp(x), q(x)\} d\mu(x)\right] \leq (b + c) \int \min\{bp(x), q(x)\} d\mu(x).$$

In the proof of the lower bound, Lemma A.2 will be combined with the following result whose proof can be found in [16].

**Lemma A.3.** Let $P$ and $Q$ be two probability measures having densities $p$ and $q$ with respect to some measure $\mu$. It holds

$$\left(\int \sqrt{p(x)q(x)} d\mu(x)\right)^2 \geq \exp(-\text{KL}(P, Q)).$$

**A.2 Proof of Theorem 2.1**

Let $Q \in Q^C_\alpha$ and let $\hat{\eta}$ be a separable selector. Since $\hat{\eta}_j$ depends only on $(Z_j)_{j=1,\ldots,n}$, it holds

$$\mathbb{E}_{Q^0 \otimes_{j=1}^d \mathbb{P}_{\hat{\eta}_j}^0} |\hat{\eta}(Z) - \eta| = \sum_{j=1}^d \mathbb{E}_{Q^0 \otimes_{j=1}^d \mathbb{P}_{\hat{\eta}_j}^0} |\hat{\eta}_j(Z_1, \ldots, Z_n) - \eta_j|.$$
Following the proof of Theorem 2.2 in [10], we denote by $\Theta'$ the set of all $\theta$ in $\Theta_\theta$ such that exactly $s$ components of $\theta$ are equal to $a$ and the remaining $d-s$ components are equal to 0. Since $\Theta'$ is a subset of $\Theta_\theta$, it holds

$$
\sup_{\theta \in \Theta_\theta} \frac{1}{n} \mathbb{E}_{Q^\theta} \left[ \hat{\theta}(Z) - \eta \right] \geq \frac{1}{n} \left| \sum_{\theta \in \Theta'} \sum_{j=1}^{d} E_{\otimes_{i=1}^{n} Q^\theta_j P_\theta} [\hat{\eta}_j(Z^1, \ldots, Z_n^1) - \eta_j] \right|
$$

$$
= \frac{1}{n} \sum_{j=1}^{d} \left( \sum_{\theta \in \Theta'} E_{\otimes_{i=1}^{n} Q^\theta_j P_\theta} (\hat{\eta}_j) + \sum_{\theta \in \Theta'} E_{\otimes_{i=1}^{n} Q^\theta_j P_\theta} (1 - \hat{\eta}_j) \right)
$$

$$
= \frac{1}{s} \sum_{j=1}^{d} \left( \left(1 - \frac{s}{d}\right) E_{\otimes_{i=1}^{n} Q^\theta_j P_\theta} (\hat{\eta}_j) + \frac{s}{d} E_{\otimes_{i=1}^{n} Q^\theta_j P_\theta} (1 - \hat{\eta}_j) \right)
$$

$$
\geq \frac{1}{s} \sum_{j=1}^{d} \inf_{T \in [0,1]} \left( \left(1 - \frac{s}{d}\right) E_{\otimes_{i=1}^{n} Q^\theta_j P_\theta} (T) + \frac{s}{d} E_{\otimes_{i=1}^{n} Q^\theta_j P_\theta} (1 - T) \right).
$$

Set

$$
L^*_s = \inf_{T \in [0,1]} \left( \left(1 - \frac{s}{d}\right) E_{\otimes_{i=1}^{n} Q^\theta_j P_\theta} (T) + \frac{s}{d} E_{\otimes_{i=1}^{n} Q^\theta_j P_\theta} (1 - T) \right).
$$

Since $Q^\theta_j$ provides $\alpha_j$-differential privacy, the channel probabilities $Q^\theta_j(\cdot | x)$ have densities $z \mapsto q^\theta_j(z | x)$ with respect to some measure $\mu^\theta_j$. Therefore, $dQ^\theta_j P_\theta(z) = m^\theta_{j,a}(z) d\mu^\theta_j(z)$, and $dQ^\theta_j P_\theta(z) = m^\theta_j(z) d\mu^\theta_j(z)$, where $m^\theta_{j,a}(z) = \int_{\mathbb{R}} q^\theta_j(z | x) dP(x)$, $b \in \{0, a\}$. Thus, for $b \in \{0, a\}$, it holds

$$
d(\otimes_{i=1}^{n} Q^\theta_j P_\theta)(y_1, \ldots, y_n) = \prod_{i=1}^{n} m^\theta_j(b)(y_i) \mu_j(y_1, \ldots, y_n),
$$

where $\mu_j = \mu^\theta_{j,0} \cdots \mu^\theta_{j,a}$. According to Theorem 3.1, the infimum $L^*_s$ is thus attained for $T = T^*_j$ given by

$$
T^*_j(Y_1, \ldots, Y_n) = I \left( \frac{s}{d} \prod_{i=1}^{n} m^\theta_{j,a}(Y_i) > \left(1 - \frac{s}{d}\right) \prod_{i=1}^{n} m^\theta_{j,0}(Y_i) \right).
$$

Set $A_j = \{(y_1, \ldots, y_n) \in \mathbb{R}^n : \frac{s}{d} \prod_{i=1}^{n} m^\theta_{j,a}(y_i) > \left(1 - \frac{s}{d}\right) \prod_{i=1}^{n} m^\theta_{j,0}(y_i)\}$.

$$
\sup_{\theta \in \Theta_\theta} \frac{1}{n} \mathbb{E}_{Q^\theta} \left[ \hat{\theta}(Z) - \eta \right] \geq \frac{1}{s} \sum_{j=1}^{d} \left( \int_{A_j} \prod_{i=1}^{n} m^\theta_{j,0}(y_i) d\mu_j(y_1, \ldots, y_n) + \frac{s}{d} \int_{\mathbb{R}^n} \prod_{i=1}^{n} m^\theta_{j,a}(y_i) d\mu_j(y_1, \ldots, y_n) \right)
$$

$$
= \frac{1}{s} \sum_{j=1}^{d} \int_{\mathbb{R}^n} \min \left\{ \left(1 - \frac{s}{d}\right) \prod_{i=1}^{n} m^\theta_{j,0}(y_i), \frac{s}{d} \prod_{i=1}^{n} m^\theta_{j,a}(y_i) \right\} d\mu_j(y_1, \ldots, y_n)
$$

$$
\geq \left(1 - \frac{s}{d}\right) \frac{1}{d} \sum_{j=1}^{d} \left( \int_{\mathbb{R}^n} \prod_{i=1}^{n} m^\theta_{j,0}(y_i) \left( \prod_{i=1}^{n} m^\theta_{j,a}(y_i) \right)^{2} d\mu_j(y_1, \ldots, y_n) \right)
$$

$$
\geq \left(1 - \frac{s}{d}\right) \frac{1}{d} \sum_{j=1}^{d} \exp \left( -\text{KL}(\otimes_{i=1}^{n} Q^\theta_j P_\theta, \otimes_{i=1}^{n} Q^\theta_j P_\theta) \right),
$$

where the two last inequalities follow from lemma 3.2 and lemma 3.3. Using successively a property of the Kullback-Leibler divergence and Theorem 1 of [10] on the contractive effects of privacy on
pairs of distribution, we obtain
\[
\sup_{\theta \in \Theta_\mathcal{F}(s,a)} \frac{1}{S} \mathbb{E}_{Q(\mathcal{P}^{\infty}_{\theta})} |\hat{\theta}(Z) - \eta| \geq \left(1 - \frac{s}{d}\right) \frac{1}{\sqrt{d}} \sum_{j=1}^{d} \exp \left(- \sum_{i=1}^{n} \text{KL}(Q_j^T P_0, Q_j^T P_0)\right)
\]
\[
= \left(1 - \frac{s}{d}\right) \frac{1}{\sqrt{d}} \sum_{j=1}^{d} \exp \left(-4n(e^{\alpha/d} - 1)^2 \text{TV}(P_0, P_0)^2\right).
\]
Since this result holds for all \(Q \in \mathcal{Q}^{CL}_{\alpha}\) and all separable selector \(\hat{\eta}\), we obtain
\[
\inf_{Q \in \mathcal{Q}^{CL}_{\alpha}} \inf_{\theta \in \Theta_\mathcal{F}(s,a)} \sup_{\theta \in \Theta_\mathcal{F}(s,a)} \frac{1}{S} \mathbb{E}_{Q(\mathcal{P}^{\infty}_{\theta})} |\hat{\theta} - \eta| \geq \left(1 - \frac{s}{d}\right) \frac{1}{\sqrt{d}} \exp \left(-4n(e^{\alpha/d} - 1)^2 \text{TV}(P_0, P_0)^2\right).
\]
The inequality \(\text{TV}(P_0, P_0) \leq 1\) and Pinsker’s inequality
\[
\text{TV}(P_0, P_0) \leq \sqrt{\frac{\text{KL}(P_0, P_0)}{2}} = \frac{\alpha}{2\sigma},
\]
then imply the statement of Theorem 2.1.

A.3 Some auxiliary results for the upper bounds

Lemma A.4. For all \(a \geq 0\), \(\frac{1}{n} \sum_{i=1}^{n} \left([a + \sigma \xi_i]^T - \mathbb{E} \left[[a + \sigma \xi_i]^T\right]\right)\) is a sub-gaussian variable with parameter at most \(\frac{T}{\sqrt{n}}\).

Proof. First, observe that \([a + \sigma \xi_i]^T \in [-T, T]\) almost surely. Then, according to Exercise 2.4 in [18], the random variable \([a + \sigma \xi_i]^T - \mathbb{E} \left[[a + \sigma \xi_i]^T\right]\) is sub-gaussian with parameter at most \(T\) for all \(i \in \{1, n\}\). Since \((\xi_i)_{i=1}^{n}\) are independent, we obtain that \(\frac{1}{n} \sum_{i=1}^{n} \left([a + \sigma \xi_i]^T - \mathbb{E} \left[[a + \sigma \xi_i]^T\right]\right)\) is sub-gaussian with parameter at most \(T/\sqrt{n}\). We now prove that it is also subgaussian with parameter at most \(\sigma/\sqrt{n}\). To aim, define \(g : x \in \mathbb{R}^n \mapsto \frac{1}{n} \sum_{i=1}^{n} [a + \sigma x_i]^T \in \mathbb{R}\). It holds
\[
|g(x) - g(y)| \leq \frac{\sigma}{\sqrt{n}} \|x - y\|_2 \quad \forall x, y \in \mathbb{R}^n.
\]
Set \(Y = (\xi_1^1, \ldots, \xi_n^1)\). According to Theorem 2.26 in [18], \(g(Y) - \mathbb{E}[g(Y)]\) is sub-gaussian with parameter at most \(\sigma/\sqrt{n}\). \(\square\)

We now recall Bernstein’s inequality (cf. [3] Corollary 2.11).

Theorem A.5. Let \(Y_1, \ldots, Y_n\) be independent real valued random variables. Assume that there exist some positive numbers \(v\) and \(c\) such that
\[
\sum_{i=1}^{n} \mathbb{E}[Y_i^2] \leq v,
\]
and for all integers \(m \geq 3\)
\[
\sum_{i=1}^{n} \mathbb{E}[|Y_i|^m] \leq \frac{m!}{2} v c^{m-2}.
\]
Let \(S = \sum_{i=1}^{n} (Y_i - \mathbb{E}[Y_i])\), then for every positive \(t\)
\[
\mathbb{P}(S \geq t) \leq \exp \left(- \frac{t^2}{2(v + ct)}\right).
\]
Note that if \(v \leq ct\) then [19] yields \(\mathbb{P}(S \geq t) \leq \exp(-t^2/4c)\). If \(ct \leq v\) then [19] yields \(\mathbb{P}(S \geq t) \leq \exp(-t^2/4v)\).
A.4 Proof of Proposition 2.4

It holds

\[
|\hat{\eta}^+ - \eta| = \sum_{j: \eta_j = 0} \hat{\eta}_j^+ + \sum_{j: \eta_j = 1} (1 - \hat{\eta}_j^+)
\]

\[
= \sum_{j: \eta_j = 0} I \left( \frac{1}{n} \sum_{i=1}^n [\sigma \xi_i^T_T + \frac{2T d}{n\alpha} \sum_{i=1}^n W_j^i \geq \tau} \right)
\]

\[
+ \sum_{j: \eta_j = 1} I \left( \frac{1}{n} \sum_{i=1}^n [\theta_j + \sigma \xi_i^T_T + \frac{2T d}{n\alpha} \sum_{i=1}^n W_j^i < \tau} \right).
\]

Thus,

\[
E \left[ \frac{1}{s} |\hat{\eta}^+ - \eta| \right] = \frac{1}{s} \sum_{j: \eta_j = 0} P \left( \frac{1}{n} \sum_{i=1}^n [\sigma \xi_i^T_T + \frac{2T d}{n\alpha} \sum_{i=1}^n W_j^i \geq \tau} \right) = T_{1,j}
\]

\[
+ \frac{1}{s} \sum_{j: \eta_j = 1} P \left( \frac{1}{n} \sum_{i=1}^n [\theta_j + \sigma \xi_i^T_T + \frac{2T d}{n\alpha} \sum_{i=1}^n W_j^i < \tau} \right) = T_{2,j}.
\]

We first study \(T_{1,j}\). It holds

\[
T_{1,j} \leq P \left( \frac{1}{n} \sum_{i=1}^n [\sigma \xi_i^T_T \geq \frac{\tau}{2}} \right) + P \left( \sum_{i=1}^n W_j^i \geq \frac{\tau n\alpha}{4TI} \right).
\]

Note that \(E [\sigma \xi_i^T_T] = 0\). Using Lemma A.3 to bound from above the first term and Bernstein’s inequality (19) with \(v = 2n\) and \(c = 1\) to bound from above the second term, we obtain if \(\tau\alpha/(8T d) < 1\)

\[
T_{1,j} \leq \exp \left( -\frac{n\tau^2}{2^3 \min\{T^2, \sigma^2\}} \right) + \exp \left( -\frac{n\tau^2 \alpha^2}{2^7 T^2 d^2} \right).
\]

Since \(x \mapsto [x]_T\) is a non-decreasing function and since \(\theta_j \geq \alpha\) for all \(j\) such that \(\eta_j = 1\), it holds

\[
T_{2,j} \leq P \left( \frac{1}{n} \sum_{i=1}^n [a + \sigma \xi_i^T_T + \frac{2T d}{n\alpha} \sum_{i=1}^n W_j^i < \tau} \right)
\]

\[
= P \left( \frac{1}{n} \sum_{i=1}^n ([a + \sigma \xi_i^T_T] - E [a + \sigma \xi_i^T_T]) + E [a + \sigma \xi_i^T_T] + \frac{2T d}{n\alpha} \sum_{i=1}^n W_j^i < \tau} \right)
\]

\[
= P \left( -\frac{1}{n} \sum_{i=1}^n ([a + \sigma \xi_i^T_T] - E [a + \sigma \xi_i^T_T]) - \frac{2T d}{n\alpha} \sum_{i=1}^n W_j^i > E [a + \sigma \xi_i^T_T] - \tau} \right).
\]
Now, if $\xi \sim \mathcal{N}(0, 1)$ then

\[
E[|a + \sigma \xi|_T] = TP \left( \frac{T - a}{\sigma} > \xi \right) - TP \left( \frac{T - a}{\sigma} < -\frac{T - a}{\sigma} \right) + E[(a + \sigma \xi)I(|a + \sigma \xi| \leq T)]
\]

\[
= TP \left( \frac{T - a}{\sigma} > \xi \right) - TP \left( \frac{T - a}{\sigma} < -\frac{T - a}{\sigma} \right) + \int_{-\frac{T - a}{\sigma}}^{\frac{T - a}{\sigma}} (a + \sigma x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx
\]

\[
\geq TP \left( \frac{T - a}{\sigma} > \xi \right) - TP \left( \frac{T - a}{\sigma} < -\frac{T - a}{\sigma} \right) + aP \left( \frac{T - a}{\sigma} \leq \xi \leq \frac{T - a}{\sigma} \right)
\]

\[
= T \left[ P \left( \frac{T - a}{\sigma} > \xi \right) - P \left( \xi \leq \frac{T - a}{\sigma} \right) \right]
\]

\[
= T \left[ 1 - 2\Phi \left( \frac{T - a}{\sigma} \right) \right],
\]

where $\Phi$ denotes the standard Gaussian cumulative distribution function. Thus, if $a \geq T + \sigma b$ for some $b > 0$, it holds $E[|a + \sigma \xi|_T] \geq C_1 T$ with $C_1 = 1 - 2\Phi(-b)$, and

\[
T_{2,j} \leq P \left( -\frac{1}{n} \sum_{i=1}^{n} (a + \sigma \xi_j)_T - E \left[ (a + \sigma \xi_j)_T \right] - \frac{2Td}{n\alpha} \sum_{i=1}^{n} W_j > C_1 T - \tau \right)
\]

\[
\leq P \left( -\frac{1}{n} \sum_{i=1}^{n} (a + \sigma \xi_j)_T - E \left[ (a + \sigma \xi_j)_T \right] > \frac{C_1 T - \tau}{2} \right)
\]

\[
+ P \left( \sum_{i=1}^{n} (-W_j) > \frac{n\alpha(C_1 T - \tau)}{4Td} \right).
\]

We can now bound from above the first term using lemma [A.3] and the second term using Bernstein’s inequality. This gives, if $C_1 T \geq \tau$ and $\alpha(C_1 T - \tau)/(8Td) \leq 1$

\[
T_{2,j} \leq \exp \left( -\frac{n(C_1 T - \tau)^2}{2\min\{T^2, \sigma^2\}} \right) + \exp \left( -\frac{(C_1 T - \tau)^2n\alpha^2}{2T^2d^2} \right).
\]

This ends the proof of [6]. We now prove [6]. If $\theta \in \Theta_d(s,a)$, we use the estimator $\hat{\eta}$ instead of $\hat{\eta}^+$ and it holds

\[
E \left[ \frac{1}{s} |\hat{\eta} - \eta| \right] = \frac{1}{s} \sum_{j, \eta_j = 0} P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \sigma \xi_j \right| T - \frac{2Td}{n\alpha} \sum_{i=1}^{n} W_j \geq \tau \right)
\]

\[
\leq \frac{1}{s} \sum_{j, \eta_j = 1} P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \theta_j + \sigma \xi_j \right| T + \frac{2Td}{n\alpha} \sum_{i=1}^{n} W_j \leq \tau \right).
\]

We first study $\tilde{T}_{1,j}$. It holds

\[
\tilde{T}_{1,j} \leq P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \sigma \xi_j \right| T + \frac{2Td}{n\alpha} \sum_{i=1}^{n} W_j \geq \tau \right)
\]

\[
\leq \frac{1}{s} \sum_{j, \eta_j = 0} P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \sigma \xi_j \right| T \geq \frac{\tau}{2} \right) + P \left( \sum_{i=1}^{n} W_j \geq \frac{\tau n\alpha}{4Td} \right)
\]

\[
\leq 2 \left[ \exp \left( -\frac{n\tau^2}{2\min\{T^2, \sigma^2\}} \right) \right] + \exp \left( -\frac{-\tau^2 n\alpha^2}{2T^2d^2} \right).
\]
if \( \tau a/(8Td) < 1 \), where we have used the two-sided versions of the concentration inequalities we used to prove (3). We now study \( \tilde{T}_{2,j} \). For all \( j \) such that \( \eta_j = 1 \), it holds

\[
\tilde{T}_{2,j} = P\left( \frac{1}{n} \sum_{i=1}^{n} (\theta_i + \sigma_\xi^j)_{T} - E \left[ (\theta_i + \sigma_\xi^j)_{T} \right] \geq - \log(1/H) \right) \leq \frac{4Td}{\alpha a} \sum_{i=1}^{n} W_j^i < \tau \]

Now, observe that

\[
|E \left[ (\theta_i + \sigma_\xi^j)_{T} \right] | \geq E \left[ (\theta_i + \sigma_\xi^j)_{T} \right] \geq E \left[ a + \sigma_\xi^j \right] ,
\]

if \( \theta_j \geq a \) since \( x \rightarrow (x)_{T} \) is non-decreasing, and if \( \theta_j \leq -a \) we have

\[
|E \left[ (\theta_i + \sigma_\xi^j)_{T} \right] | \geq -E \left[ (\theta_i + \sigma_\xi^j)_{T} \right] \geq -E \left[ a + \sigma_\xi^j \right] ,
\]

where we have used that \( x \rightarrow (x)_{T} \) is a non-decreasing and odd function and that \( -\xi^j \) and \( \xi^j \) have the same distribution. Moreover, we have seen in the proof of (9) that \( \xi \sim N(0,1) \) then it holds

\[
E \left[ a + \sigma_\xi \right] \geq T \left[ 1 - 2\Phi \left( \frac{T-a}{\sigma} \right) \right] ,
\]

where \( \Phi \) denotes the standard Gaussian cumulative distribution function. Thus, if \( a \geq T + \sigma b \) for some \( b > 0 \), it holds \( E \left[ a + \sigma_\xi \right] \geq C_1 T \) for all \( j \) such that \( \eta_j = 1 \) with \( C_1 = 1 - 2\Phi(-b) \), and

\[
\tilde{T}_{2,j} \leq \frac{1}{n} \sum_{i=1}^{n} \left( (\theta_i + \sigma_\xi^j)_{T} - E \left[ (\theta_i + \sigma_\xi^j)_{T} \right] \right) + \frac{2Td}{\alpha a} \sum_{i=1}^{n} W_j^i < C_1 T - \tau \]

Using the two-sided version of the concentration inequalities that we used to bound \( T_{2,j} \) in the proof of (8), we obtain if \( C_1 T \geq \tau \) and \( \alpha(C_1 T - \tau)/(8Td) \leq 1 \)

\[
\tilde{T}_{2,j} \leq 2 \left[ \exp \left( - \frac{n(C_1 T - \tau)^2}{2^3 \min\{T^2, \sigma^2\}} \right) + \exp \left( - \frac{(C_1 T - \tau)^2 n a^2}{2^5 T^2 d^2} \right) \right] .
\]

This ends the proof of (10).

### A.5 Proof of Corollary 2.5

Let prove (10). Note that if the assumptions of corollary 2.5 are satisfied, and it \( T = \sigma \) and \( \tau = C_1 T/2 \) then the assumptions of proposition 2.4 are also satisfied and for all \( a \geq 2\sigma \) we have

\[
\sup_{\theta \in \Theta} E \left[ \frac{1}{s} |\eta^+ - \eta| \right] \leq 2 \cdot \frac{d}{s} \left[ \exp \left( - \frac{C_1^2 n}{2^5} \right) + \exp \left( - \frac{C_1^2 n a^2}{2^9 d^2} \right) \right] = 2 \left[ \exp \left( - \frac{\log(d/s)}{s} - \frac{C_1^2 n}{2^5} \right) + \exp \left( - \frac{\log(d/s)}{s} - \frac{C_1^2 n a^2}{2^9 d^2} \right) \right] .
\]
The two terms appearing in the last inequality both tend to 0 as $d \to +\infty$ under the assumptions of Corollary 2.5, which gives (10). The proof of (11) is similar.

### A.6 Proof of Proposition 2.6

The beginning of the proof is similar to the proof of Proposition 2.4. It holds

$$
E \left[ \frac{1}{s} \frac{\partial}{\partial s} \tilde{\eta}^{+} \right] = \frac{1}{s} \sum_{j: \eta_j = 0} \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} [\xi_i^j] + \frac{2T}{n\alpha} \sum_{i=1}^{n} W_i^j < \tau \right)
$$

and we have

$$
T_{1,j} \leq \exp \left( -\frac{n\tau^2}{2\beta \min \{T^2, \sigma^2\}} \right) + \exp \left( -\frac{\tau^2n\alpha^2}{2T^2d^2} \right)
$$

if $\tau \alpha / (2Td) < 1$, and

$$
T_{2,j} \leq \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} (a + \sigma \xi_i^j) - E \left[ (a + \sigma \xi_i^j) \right] \right) = \mathbb{P} \left( \frac{2T}{n\alpha} \sum_{i=1}^{n} W_i^j > E \left[ (a + \sigma \xi_i^j) \right] - \tau \right).
$$

Now, we bound from below $E \left[ (a + \sigma \xi_i^j) \right] $ in a different way than in the proof of Proposition 2.4

$$
E \left[ (a + \sigma \xi_i^j) \right] = TP \left( \xi > \frac{T - a}{\sigma} \right) - TP \left( \xi < \frac{-T - a}{\sigma} \right) + E \left[ (a + \sigma \xi) I (|a + \sigma \xi| \leq T) \right]
$$

$$
= TP \left( \xi > \frac{T - a}{\sigma} \right) - TP \left( \xi < \frac{-T - a}{\sigma} \right) + \int_{-\frac{T - a}{\sigma}}^{\frac{T - a}{\sigma}} (a + \sigma x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx
$$

$$
= TP \left( \xi > \frac{T - a}{\sigma} \right) - TP \left( \xi < \frac{-T - a}{\sigma} \right) + a \mathbb{P} \left( \frac{T - a}{\sigma} \right) \leq \xi \leq \frac{T - a}{\sigma} \right)
$$

$$
+ \frac{\sigma}{\sqrt{2\pi}} \left[ \left. \exp \left( \frac{(T + a)^2}{2\sigma^2} \right) \right| \right] - \exp \left( \frac{(T - a)^2}{2\sigma^2} \right) \right]
$$

$$
\geq a \mathbb{P} \left( \frac{T - a}{\sigma} \right) \leq \xi \leq \frac{T - a}{\sigma} \right) = \frac{\sigma}{\sqrt{2\pi}} \exp \left( \frac{(T - a)^2}{2\sigma^2} \right)
$$

$$
\geq a \mathbb{P} \left( \frac{T - a}{\sigma} \right) \leq \xi \leq \frac{T - a}{\sigma} \right) = \frac{\sigma}{\sqrt{2\pi}} \exp \left( \frac{(T - a)^2}{2\sigma^2} \right)
$$

$$
= a \left[ 1 - 2 \mathbb{P} \left( \xi > \frac{T - a}{\sigma} \right) \right] = \frac{\sigma}{\sqrt{2\pi}} \exp \left( \frac{(T - a)^2}{2\sigma^2} \right)
$$

If $T \geq a + \sqrt{2}\sigma$, then

$$
2 \mathbb{P} \left( \xi \geq \frac{T - a}{\sigma} \right) \leq 2 \sqrt{\frac{\pi}{2\sigma}} \left. \exp \left( \frac{(T - a)^2}{2\sigma^2} \right) \right| \right| \leq 1.
$$

If $T \geq a + \sigma \sqrt{2 \log(\delta)}$ with $\delta \geq \frac{4\sigma}{a \sqrt{2\pi}}$, then

$$
\frac{\sigma}{\sqrt{2\pi}} \exp \left( \frac{(T - a)^2}{2\sigma^2} \right) \leq \frac{\sigma}{\sqrt{2\pi \delta}} \leq \frac{a}{4}.
$$
Thus, if $T \geq a + \sigma \sqrt{2 \log \left( \max \left\{ e, \frac{4a}{a\sqrt{2\pi}} \right\} \right)}$, then $\mathbb{E} \left[ [a + \sigma \xi]_r \right] \geq a/2$, and

$$T_{2,j} \leq \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} \left[ (a + \sigma \xi_j^i)_r - \mathbb{E} \left[ (a + \sigma \xi_j^i)_r \right] \right] - \frac{2Td}{n\alpha} \sum_{i=1}^{n} W_j^i > \frac{a}{2} - \tau \right)$$

$$\leq \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} \left[ (a + \sigma \xi_j^i)_r - \mathbb{E} \left[ (a + \sigma \xi_j^i)_r \right] \right] > \frac{a/2 - \tau}{2} \right)$$

$$+ \mathbb{P} \left( \sum_{i=1}^{n} (-W_j^i) > \frac{n\alpha(a/2 - \tau)}{4Td} \right).$$

We can now bound from above the first term using lemma A.4 and the second term using Bernstein’s inequality. This gives, if $T \geq a + \sqrt{2 \log \left( \max \left\{ e, \frac{4a}{a\sqrt{2\pi}} \right\} \right)}$, $\tau \leq a/2$ and $o(a/2 - \tau)/(8Td) \leq 1$

$$T_{2,j} \leq \exp \left( -\frac{n(a/2 - \tau)^2}{2t \min \{T^2, \sigma^2\}} \right) + \exp \left( \frac{(a/2 - \tau)^2n\alpha^2}{2tT^2d^2} \right).$$

### A.7 Proof of Corollary 2.7

Let prove (12). The chosen values of $T$ and $\tau$ satisfy the assumptions of Proposition 2.6 for $d$ large enough and yield

$$\sup_{\theta \in \Theta} \mathbb{E} \left[ \frac{1}{s} |\tilde{y}^+ - \eta| \right] \leq \frac{d}{s} \left[ \exp \left( -\frac{n\alpha^2}{2t \min \{T^2, \sigma^2\}} \right) + \exp \left( -\frac{a^2\alpha^2}{2t^2T^2d^2} \right) \right]$$

$$= \exp \left( \log \left( \frac{d}{s} \right) - \frac{n\alpha^2}{2t \sigma^2} \right) + \exp \left( \log \left( \frac{d}{s} \right) - \frac{a^2\alpha^2}{2t^2T^2d^2} \right)$$

$$\leq 2 \exp \left( \log \left( \frac{d}{s} \right) - \frac{a^2\alpha^2}{2t^2T^2d^2} \right).$$

Conclude using $T = a + \sigma \sqrt{2 \log \left( \frac{\sqrt{n\alpha}}{d} \right)} \leq 2\sigma + \sqrt{2 \log \left( \frac{n\alpha^2}{d^2} \right)} \leq 2\sigma \sqrt{\log \left( \frac{n\alpha^2}{d^2} \right)}$. The proof of (13) is similar.

### B Proofs of Section 3

#### B.1 Proof of Proposition 3.1

Note that it is sufficient to prove that $\tilde{Z}^i$ is an $\alpha$-LDP view of $X^i$. Indeed, if $\tilde{Z}^i$ is an $\alpha$-LDP view of $X^i$ then it holds for all $z \in Z$ and $x, x' \in \mathbb{R}^d$ (we omit the superscript $i$)

$$\mathbb{P} \left( Z = z \mid X = x \right) \leq \mathbb{P} \left( \tilde{Z} = \tilde{z} \mid \tilde{X} = \tilde{x} \right) \mathbb{P} \left( \tilde{Z} = \tilde{z} \mid X = x \right) \mathbb{P} \left( \tilde{X} = \tilde{x} \right) \mathbb{P} \left( Z = z \mid X = x \right) \mathbb{P} \left( \tilde{Z} = \tilde{z} \mid \tilde{X} = \tilde{x} \right) \mathbb{P} \left( \tilde{Z} = \tilde{z} \mid \tilde{X} = \tilde{x} \right) \mathbb{P} \left( \tilde{X} = \tilde{x} \right)$$

$$\leq e^\alpha,$$

where we have used that $Z$ is independent from $X$ conditionally to $\tilde{Z}$ and the fact that $\mathbb{P} \left( \tilde{Z} = \tilde{z} \mid X = x \right) \leq e^\alpha \mathbb{P} \left( \tilde{Z} = \tilde{z} \mid X = x \right)$ for all $\tilde{z} \in \left\{ \tilde{B}, \tilde{B} \right\}^d$ if $\tilde{Z}$ is an $\alpha$-LDP view of $X$. So, let’s prove that $\tilde{Z}^i$ is an $\alpha$-LDP view of $X^i$. In what follows, we omit once again the superscript $i$. We have to prove that for all $\tilde{z} \in \left\{ \tilde{B}, \tilde{B} \right\}^d$ and all $x, x' \in \mathbb{R}^d$ it holds

$$\mathbb{P} \left( \tilde{Z} = \tilde{z} \mid X = x \right) \mathbb{P} \left( \tilde{Z} = \tilde{z} \mid X = x' \right) \leq e^\alpha.$$
Let $\tilde{z} \in \{-B, B\}^d$ and $x \in \mathbb{R}^d$. It holds

$$
P (\tilde{Z} = \tilde{z} \mid X = x) = \sum_{\tilde{x} \in \{-T,T\}^d} P (\tilde{Z} = \tilde{z} \mid X = x, \tilde{X} = \tilde{x}) \cdot P (\tilde{X} = \tilde{x} \mid X = x)
$$

and since $Y$ and $\tilde{X}$ are independent we have

$$P (\tilde{Z} = \tilde{z} \mid \tilde{X} = \tilde{x}) = P (\tilde{Z} = \tilde{z} \mid \tilde{X} = \tilde{x}, Y = 1) \cdot P (Y = 1) + P (\tilde{Z} = \tilde{z} \mid \tilde{X} = \tilde{x}, Y = 0) \cdot P (Y = 0) = \pi \alpha P (\tilde{Z} = \tilde{z} \mid \tilde{X} = \tilde{x}, Y = 1) + (1 - \pi) P (\tilde{Z} = \tilde{z} \mid \tilde{X} = \tilde{x}, Y = 0) .$$

Moreover, since for $\tilde{x} \in \{-T,T\}^d$

$$\text{Card} \left( \left\{ \tilde{z} \in \{-B, B\}^d \mid \langle \tilde{z}, \tilde{x} \rangle > 0 \text{ or } \langle \tilde{z}, \tilde{x} \rangle = 0 \text{ and } \tilde{z} = (B/T)\tilde{x}_1 \right\} \right) = 2^{d-1} ,$$

it holds

$$P (\tilde{Z} = \tilde{z} \mid \tilde{X} = \tilde{x}, Y = 1) = \begin{cases} 0 & \text{if } \langle \tilde{z}, \tilde{x} \rangle < 0 \text{ or } \langle \tilde{z}, \tilde{x} \rangle = 0 \text{ and } \tilde{z} = (B/T)\tilde{x}_1 \\ \frac{1 - \pi}{2^{d-1}} & \text{if } \langle \tilde{z}, \tilde{x} \rangle > 0 \text{ or } \langle \tilde{z}, \tilde{x} \rangle = 0 \text{ and } \tilde{z} = (B/T)\tilde{x}_1 , \end{cases}$$

and

$$P (\tilde{Z} = \tilde{z} \mid \tilde{X} = \tilde{x}, Y = 0) = \begin{cases} 0 & \text{if } \langle \tilde{z}, \tilde{x} \rangle < 0 \text{ or } \langle \tilde{z}, \tilde{x} \rangle = 0 \text{ and } \tilde{z} = (B/T)\tilde{x}_1 \\ \frac{1 - \pi}{2^{d-1}} & \text{if } \langle \tilde{z}, \tilde{x} \rangle > 0 \text{ or } \langle \tilde{z}, \tilde{x} \rangle = 0 \text{ and } \tilde{z} = (B/T)\tilde{x}_1 . \end{cases}$$

We thus have

$$P (\tilde{Z} = \tilde{z} \mid \tilde{X} = \tilde{x}) = \begin{cases} 0 & \text{if } \langle \tilde{z}, \tilde{x} \rangle < 0 \text{ or } \langle \tilde{z}, \tilde{x} \rangle = 0 \text{ and } \tilde{z} = (B/T)\tilde{x}_1 \\ \frac{\pi}{2^{d-1}} & \text{if } \langle \tilde{z}, \tilde{x} \rangle > 0 \text{ or } \langle \tilde{z}, \tilde{x} \rangle = 0 \text{ and } \tilde{z} = (B/T)\tilde{x}_1 , \end{cases}$$

and, if we set

$$A_\tilde{z} = \{ \tilde{x} \in \{-T,T\}^d : \langle \tilde{z}, \tilde{x} \rangle > 0 \text{ or } \langle \tilde{z}, \tilde{x} \rangle = 0 \text{ and } \tilde{z} = (B/T)\tilde{x}_1 \} ,$$

and

$$C_\tilde{z} = \{ \tilde{x} \in \{-T,T\}^d : \langle \tilde{z}, \tilde{x} \rangle < 0 \text{ or } \langle \tilde{z}, \tilde{x} \rangle = 0 \text{ and } \tilde{z} = (B/T)\tilde{x}_1 \} ,$$

we obtain

$$P (\tilde{Z} = \tilde{z} \mid X = x) = \frac{\pi}{2^{d-1}} \sum_{\tilde{x} \in A_\tilde{z}} P (\tilde{X} = \tilde{x} \mid X = x) + \frac{1 - \pi}{2^{d-1}} \sum_{\tilde{x} \in C_\tilde{z}} P (\tilde{X} = \tilde{x} \mid X = x) .$$

Consequently, it holds for all $\tilde{z} \in \{-B, B\}^d$ and all $x \in \mathbb{R}^d$,

$$\min_{\tilde{x} \in A_\tilde{z}} \frac{\pi}{2^{d-1}} \leq P (\tilde{Z} = \tilde{z} \mid X = x) \leq \max_{\tilde{x} \in C_\tilde{z}} \frac{1 - \pi}{2^{d-1}},$$

where we have used that $A_\tilde{z} \cup C_\tilde{z} = \{-T,T\}^d$ and $\sum_{\tilde{x} \in \{-T,T\}^d} P (\tilde{X} = \tilde{x} \mid X = x) = 1$. We finally obtain for all $\tilde{z} \in \{-B, B\}^d$ and all $x, x' \in \mathbb{R}^d$,

$$\frac{P (\tilde{Z} = \tilde{z} \mid X = x)}{P (\tilde{Z} = \tilde{z} \mid X = x')} \leq \frac{\max_{\tilde{x} \in \{-B, B\}^d} \frac{\pi}{2^{d-1}}}{\min_{\tilde{x} \in \{-B, B\}^d} \frac{1 - \pi}{2^{d-1}}} = \frac{\pi}{1 - \pi} = e^\alpha.$$
B.2 Proof of Proposition 3.2
Let $x \in \mathbb{R}^d$. We first compute $\mathbb{E} [\tilde{Z} | X = x]$. It holds
\[
\mathbb{E} [\tilde{Z} | X = x] = \sum_{\tilde{x} \in \{-T, T\}^d} \mathbb{P} (\tilde{X} = \tilde{x} | X = x) \cdot \mathbb{E} [\tilde{Z} | X = x, \tilde{X} = \tilde{x}]
\]
and since $Y$ and $\tilde{X}$ are independent we have
\[
\mathbb{E} [\tilde{Z} | X = x] = \mathbb{P} (Y = 1) \cdot \mathbb{E} [\tilde{Z} | X = \tilde{x}, Y = 1] + \mathbb{P} (Y = 0) \cdot \mathbb{E} [\tilde{Z} | X = \tilde{x}, Y = 0]
\]

Define
\[
A_\varepsilon := \{ \tilde{z} \in \{-B, B\}^d | \langle \tilde{z}, \tilde{x} \rangle > 0 \text{ or } (\tilde{z}, \tilde{x}) = 0 \text{ and } \tilde{z}_1 = (B/T)\tilde{x}_1 \},
\]
\[
C_\varepsilon := \{ \tilde{z} \in \{-B, B\}^d | \langle \tilde{z}, \tilde{x} \rangle < 0 \text{ or } (\tilde{z}, \tilde{x}) = 0 \text{ and } \tilde{z}_1 = -(B/T)\tilde{x}_1 \}.
\]
Conditionally on \{ $\tilde{X} = \tilde{x}, Y = 1$ \}, it holds $Z \sim \mathcal{U}(A_\varepsilon)$. Thus,
\[
\mathbb{E} [\tilde{Z} | \tilde{X} = \tilde{x}, Y = 1] = \sum_{\tilde{z} \in A_\varepsilon} \mathbb{P} (\tilde{Z} = \tilde{z} | \tilde{X} = \tilde{x}, Y = 1) \tilde{z} = \frac{1}{\text{Card}(A_\varepsilon)} \sum_{\tilde{z} \in A_\varepsilon} \tilde{z}.
\]
Similarly,
\[
\mathbb{E} [\tilde{Z} | \tilde{X} = \tilde{x}, Y = 0] = \frac{1}{\text{Card}(C_\varepsilon)} \sum_{\tilde{z} \in C_\varepsilon} \tilde{z} = -\mathbb{E} [\tilde{Z} | \tilde{X} = \tilde{x}, Y = 1],
\]
where we have used $\text{Card}(C_\varepsilon) = \text{Card}(A_\varepsilon)$. We thus obtain
\[
\mathbb{E} [\tilde{Z} | \tilde{X} = \tilde{x}] = \frac{2\pi_{\alpha} - 1}{\text{Card}(A_\varepsilon)} \sum_{\tilde{z} \in A_\varepsilon} \tilde{z},
\]
and, using that $\text{Card}(A_\varepsilon) = 2^{d-1}$ for all $\tilde{x} \in \{-T, T\}^d$ we obtain
\[
\mathbb{E} [\tilde{Z} | X = x] = \frac{2\pi_{\alpha} - 1}{2^{d-1}} \sum_{\tilde{z} \in \{-T, T\}^d} \left[ \mathbb{P} (\tilde{X} = \tilde{x} | X = x) \cdot \sum_{\tilde{z} \in A_\varepsilon} \tilde{z} \right].
\]
We now compute $\sum_{\tilde{z} \in A_\varepsilon} \tilde{z}$ for all $\tilde{x} \in \{-T, T\}^d$. Note that for $\tilde{z} \in \{-B, B\}^d$ and $\tilde{x} \in \{-T, T\}^d$, $\langle \tilde{z}, \tilde{x} \rangle$ is a sum of $d$ terms, each equal to $-BT$ or $BT$. If $a$ denotes the number of elements of this sum equal to $-BT$ and $b$ denotes the number of elements of this sum equal to $-BT$, then it holds $a + b = d$ and $\langle \tilde{z}, \tilde{x} \rangle = aBT - bBT = BT(d - 2b)$. Thus we can only have $\langle \tilde{z}, \tilde{x} \rangle = kBT$, with $k \in [-d, d]$ and $|k|$ has the same parity as $d$. We thus have
\[
\sum_{\tilde{z} \in A_\varepsilon} \tilde{z} = \sum_{p=0}^{(d-1)/2} \sum_{\tilde{z} \in \{-B, B\}^d, \langle \tilde{z}, \tilde{x} \rangle = (2p+1)BT} \tilde{z},
\]
if $d$ is odd, and
\[
\sum_{\tilde{z} \in A_\varepsilon} \tilde{z} = \sum_{p=1}^{d/2} \sum_{\tilde{z} \in \{-B, B\}^d, \langle \tilde{z}, \tilde{x} \rangle = 2pBT} \tilde{z} + \sum_{\tilde{z} \in \{-B, B\}^d, \langle \tilde{z}, \tilde{x} \rangle = 0} \tilde{z}
\]
if $d$ is even.
if $d$ is even. Now, observe that for all $\tilde{x} \in \{-T, T\}^d$, for all $j \in [1, d]$ and for all $k \in \{0, \ldots, d\}$ with the same parity as $d$, it holds

$$
\sum_{z \in \{-B, B\}^d : \langle \tilde{z}, \tilde{x} \rangle = k} \tilde{z} = \frac{B}{T} \left\lfloor \left( \frac{d-1}{d+k} - 1 \right) \tilde{x}_j \right\rfloor.
$$

(22)

Indeed, for all $\tilde{z} \in \{-B, B\}^d$, for all $\tilde{x} \in \{-T, T\}^d$, and for all $k \in \{0, \ldots, d\}$ with the same parity as $d$, it holds

$$
\langle \tilde{z}, \tilde{x} \rangle = k \cdot BT \iff \begin{cases} 
\tilde{z}_j = \frac{B}{T} \tilde{x}_j & \text{for } \frac{d+k}{2} \text{ elements } j \in [1, d] \\
\tilde{z}_j = -\frac{B}{T} \tilde{x}_j & \text{for } \frac{d-k}{2} \text{ elements } j \in [1, d].
\end{cases}
$$

Setting $D_{k, \tilde{z}} = \{\tilde{z} \in \{-B, B\}^d : \langle \tilde{z}, \tilde{x} \rangle = k \cdot BT\}$, it thus holds

$$
\sum_{\tilde{z} \in D_{k, \tilde{z}}} \tilde{z}_j = \sum_{\tilde{z} \in D_{k, \tilde{z}}} \frac{B}{T} \tilde{x}_j \chi \left( \tilde{z}_j = \frac{B}{T} \tilde{x}_j \right) - \sum_{\tilde{z} \in D_{k, \tilde{z}}} \frac{B}{T} \tilde{x}_j \chi \left( \tilde{z}_j = -\frac{B}{T} \tilde{x}_j \right)
$$

$$
= \frac{B}{T} \left\lfloor \left( \frac{d-1}{d+k} - 1 \right) \tilde{x}_j \right\rfloor.
$$

We now end the proof of Proposition 3.2 when $d$ is odd. Combining (22) with (20), we obtain for $d$ odd

$$
\sum_{\tilde{z} \in A_e} \tilde{z} = \frac{B}{T} \left( \frac{d-1}{d} \right) \tilde{x},
$$

and the choice of $B$ yields

$$
\mathbb{E} \left[ \tilde{Z} \mid X = x \right] = \frac{2\pi e - 1}{2^{d-1}} \frac{B}{T} \left( \frac{d-1}{d} \right) \sum_{\tilde{z} \in \{-T, T\}^d} P(\tilde{X} = \tilde{x} \mid X = x) \cdot \tilde{x}
$$

$$
= \mathbb{E} \left[ \tilde{X} \mid X = x \right]
$$

Since for all $j \in [1, d]$ it holds

$$
\mathbb{E} \left[ \tilde{X}_j \mid X = x \right] = T \left( \frac{1}{2} + \frac{|x_j|T}{2T} \right) - T \left( \frac{1}{2} - \frac{|x_j|T}{2T} \right) = [x_j]_T,
$$

we obtain for $d$ odd

$$
\mathbb{E} [Z \mid X = x] = \mathbb{E} [\tilde{Z} \mid X = x] = \mathbb{E} [\tilde{X} \mid X = x] = f_T(x),
$$

which proves Proposition 3.2 when $d$ is odd. From now on, we assume that $d$ is even. Combining (22) with (21), we obtain

$$
\sum_{\tilde{z} \in A_e} \tilde{z} = \frac{B}{T} \left( \frac{d-1}{d} \right) \tilde{x} + \sum_{\tilde{z} \in \{-B, B\}^d : \langle \tilde{z}, \tilde{x} \rangle = 0} \tilde{z}.
$$

Now, observe that for $\tilde{z} \in \{-B, B\}^d$ and $\tilde{x} \in \{-T, T\}^d$ it holds $\langle \tilde{z}, \tilde{x} \rangle = 0$ if and only if $\tilde{z}_j = (B/T)\tilde{x}_j$ for exactly $d/2$ subscripts $j \in [1, d]$ and $\tilde{z}_j = -(B/T)\tilde{x}_j$ for exactly $d/2$ subscripts $j \in [1, d]$. We thus have

$$
\sum_{\tilde{z} \in \{-B, B\}^d : \langle \tilde{z}, \tilde{x} \rangle = 0} \tilde{z}_1 = \frac{B}{T} \tilde{x}_1 \cdot \text{Card} \left( \left\{ \tilde{z} \in \{-B, B\}^d : \langle \tilde{z}, \tilde{x} \rangle = 0 \text{ and } \tilde{z}_1 = \frac{B}{T} \tilde{x}_1 \right\} \right)
$$

$$
= \frac{B}{T} \left( \frac{d-1}{d} \right) \tilde{x}_1,
$$

24
Lemma B.1. Asymptotic analysis of the value $K_d$ defined in (14)

The value $K_d$ defined in (14) behaves asymptotically in $d$ as

$$K_d \rightarrow \infty \sqrt{\frac{\pi}{2}\sqrt{d}}.$$ 

In particular, it holds $K_d \lesssim \sqrt{d}$ for $d$ large enough.

The proof relies on Stirling’s approximation. We first deal with the case where $d$ is odd. In this case, Stirling’s approximation yields

$$K_d = 2^{d-1} \frac{[(d-1)!]}{(d-1)!} \pi^{(d-1)} \left( \frac{d-1}{2e} \right)^{d-1} \cdot \left[ \sqrt{\frac{2\pi(d-1)}{(d-1)}} \left( \frac{d-1}{e} \right)^{d-1} \right]^{-1}.$$ 

The right-hand side of this asymptotic equivalence is equal to $\sqrt{\pi/2}\sqrt{d-1}$. We thus obtain

$$K_d \rightarrow \infty \sqrt{\frac{\pi}{2}\sqrt{d}}.$$ 

We now assume that $d$ is even. In this case, Stirling’s approximation yields

$$K_d = \frac{2^{d-1}(d-1)!}{(d-2)!} \pi^{(d-2)} \left( \frac{d-2}{2e} \right)^{d-2} \cdot \left[ \sqrt{\frac{2\pi(d-2)}{(d-2)}} \left( \frac{d-2}{e} \right)^{d-2} \right]^{-1}.$$ 

The right-hand side of this asymptotic equivalence is equal to

$$\sqrt{\frac{\pi}{e\sqrt{2}}} d(d-2)^{-\frac{d}{2}} = \sqrt{\frac{\pi}{e\sqrt{2}}} \sqrt{d} \exp \left( -\frac{d}{2} \log \left( 1 - \frac{2}{d} \right) \right) \rightarrow \sqrt{\frac{\pi}{2}\sqrt{d}},$$

which ends the proof.
B.4 Proof of Proposition 3.3

The proof is similar to the one we made in the Coordinate Local case (Proposition 2.3). However, in the Coordinate Global case, for all \( j \in [1, d] \) the \( (Z_j) \) are bounded random variables, which will enable us to use Hoeffding’s inequality instead of Lemma A.4 and Bernstein’s inequality.

Writing

\[
|\hat{\eta} - \eta| = \sum_{j: \eta_j = 0} \hat{\eta}_j + \sum_{j: \eta_j = 1} (1 - \hat{\eta}_j),
\]

we have

\[
E \left[ \frac{1}{s} |\hat{\eta} - \eta| \right] = \frac{1}{s} \sum_{j: \eta_j = 0} \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} Z_i^j \geq \tau \right) + \frac{1}{s} \sum_{j: \eta_j = 1} \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} Z_i^j < \tau \right).
\]

We first study \( T_{1,j} \). For \( j \) satisfying \( \eta_j = 0 \), it holds

\[
E [Z_j^1] = E [E [Z_j^1 \mid \theta_j^1]] = E [X_j^1] = E [\sigma \xi_j^1] = 0,
\]

where we have used Proposition 3.2. Thus, Hoeffding’s inequality yields

\[
T_{1,j} = \mathbb{P} \left( \sum_{i=1}^{n} (Z_i^j - E[Z_i^j]) \geq n\tau \right) \leq \exp \left( -\frac{n\tau^2}{2B^2} \right).
\]

We now study \( T_{2,j} \). Let \( j \in [1, d] \) such that \( \eta_j = 1 \). It holds

\[
T_{2,j} = \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} (Z_i^j - E[Z_i^j]) + \frac{1}{n} \sum_{i=1}^{n} E[Z_i^j] < \tau \right)
\]

\[
= \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} (-Z_i^j - E[-Z_i^j]) > E[Z_j^1] - \tau \right).
\]

Proposition 3.2 gives

\[
E [Z_j^1] = E [\mid X_j^1 \mid \theta_j] = E [\theta_j + \sigma \xi_j^1] \geq E [a + \sigma \xi_j^1],
\]

and we have proved in Appendix A.4 that if \( \xi \sim \mathcal{N}(0, 1) \) then it holds

\[
E [a + \sigma \xi] \geq T \left[ 1 - 2\Phi \left( \frac{T-a}{\sigma} \right) \right],
\]

where \( \Phi \) denotes the standard Gaussian cumulative distribution function. Thus, if \( a \geq T + \sigma b \) for some \( b > 0 \), it holds \( E [(a + \sigma \xi)^1] \geq C_1 T \) with \( C_1 = 1 - 2\Phi(-b) \), and

\[
T_{2,j} \leq \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} (-Z_i^j - E[-Z_i^j]) > C_1 T - \tau \right)
\]

\[
\leq \exp \left( -\frac{n(C_1 T - \tau)^2}{2B^2} \right)
\]

according to Hoeffding’s inequality if \( C_1 T - \tau > 0 \). This yields

\[
E \left[ \frac{1}{s} |\hat{\eta} - \eta| \right] \leq \frac{d - |S|}{s} \exp \left( -\frac{n\tau^2}{2B^2} \right) + \frac{|S|}{s} \exp \left( -\frac{n(C_1 T - \tau)^2}{2B^2} \right).
\]

The proof of the second statement of Proposition 3.3 is straightforward.
B.5 Proof of Proposition 3.4

The beginning of the proof is similar to the proof of Proposition 3.3. It holds

$$
\mathbb{E} \left[ \frac{1}{s} |\bar{\eta}^+ - \eta| \right] = \frac{1}{s} \sum_{j : \eta_j = 0} \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} Z_j^i \geq \tau \right) + \frac{1}{s} \sum_{j : \eta_j = 1} \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} Z_j^i < \tau \right),
$$

with

$$T_{1,j} \leq \exp \left( - \frac{n\tau^2}{2B^2} \right),$$

and

$$T_{2,j} \leq \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} (-Z_j^i - \mathbb{E} [-Z_j^i]) > \mathbb{E} [(a + \sigma \xi_j^i)|T] - \tau \right).$$

Moreover, we have proved in Appendix A.6 that if $\xi \sim \mathcal{N}(0, 1)$ and $T \geq a + \sigma \sqrt{2 \log \left( \max \{ e, \frac{\tau}{\sigma^2} \} \right)}$, then $\mathbb{E} [(a + \sigma \xi)|T] \geq a/2$. Thus, if $T$ and $\tau$ are chosen such that $T \geq a + \sigma \sqrt{2 \log \left( \max \{ e, \frac{\tau}{\sigma^2} \} \right)}$, and $\tau < a/2$, Hoeffding’s inequality yields

$$T_{2,j} \leq \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} (-Z_j^i - \mathbb{E} [-Z_j^i]) > \frac{a}{2} - \tau \right) \leq \exp \left( \frac{n(a/2 - \tau)^2}{2B^2} \right).$$

B.6 Proof of Proposition 3.6

For $i = 1, \ldots, d$, define the vector $\omega_i \in \{0, 1\}^d$ by $\omega_{i,j} = 1$ if $j = i$, $\omega_{i,j} = 0$ if $j \neq i$ and define $P_{\omega_i}$ as the multivariate normal distribution $\mathcal{N}(a\omega_i, \sigma^2 I_d)$. For $i \neq j$ if holds

$$|\eta(P_{\omega_i}) - \eta(P_{\omega_j})| = |\omega_i - \omega_j| = 2.$$

The private Fano method (Proposition 2 in [7]) thus yields

$$\inf_{Q \in \mathbb{Q}_1} \inf_{\theta \in \Theta_{\delta}(s,a)} \sup_{\theta \in \Theta_{\delta}(s,a)} \mathbb{E}_{Q(P_{\omega}^{\infty})} |\eta - \bar{\eta}| \geq \frac{1}{2} \left\{ 1 - \frac{n(e^\alpha - 1)^2}{d \log(d)} \right\} \left[ \sup_{\gamma \in \mathbb{B}_{\infty}(\mathbb{R}^d)} \sum_{i=1}^{d} (\varphi_{\omega_i}(\gamma))^2 \right] - \frac{\log(2)}{\log(d)},$$

with

$$\mathbb{B}_{\infty}(\mathbb{R}^d) = \{ \gamma \in L_{\infty}(\mathbb{R}^d) \mid \| \gamma \|_{\infty} \leq 1 \},$$

$$\varphi_{\omega_i}(\gamma) = \int_{\mathbb{R}^d} \gamma(x)(dP_{\omega_i} - d\tilde{P})(x) = \int_{\mathbb{R}^d} \gamma(x)(f_{\omega_i}(x) - \bar{f}(x))dx,$$

where $f_{\omega_i}$ is the density of $P_{\omega_i}$ and $\tilde{f} = (1/d) \sum_{i=1}^{d} f_{\omega_i}$. We have

$$\sum_{i=1}^{d} (\varphi_{\omega_i}(\gamma))^2 = \sum_{i=1}^{d} \left( \int_{\mathbb{R}^d} \gamma(x)(f_{\omega_i}(x) - \bar{f}(x))dx \right) \left( \int_{\mathbb{R}^d} \gamma(y)(f_{\omega_i}(y) - \bar{f}(y))dy \right)$$

$$= \int_{\mathbb{R}^d} \gamma(x) \left[ \int_{\mathbb{R}^d} \left( \sum_{i=1}^{d} (f_{\omega_i}(x) - \bar{f}(x))(f_{\omega_i}(y) - \bar{f}(y)) \right) \gamma(y)dy \right] dx.$$

Let $\Phi$ denote the density of the $\mathcal{N}(0, \sigma^2 I_d)$ distribution. If $\gamma \in \mathbb{B}_{\infty}(\mathbb{R}^d)$ then $\gamma \in L_2(\mathbb{R}^d, d\Phi)$, $\| \gamma \|_{L_2(\mathbb{R}^d, d\Phi)} \leq 1$, and we can write

$$\sum_{i=1}^{d} (\varphi_{\omega_i}(\gamma))^2 = \int_{\mathbb{R}^d} \gamma(x) \left[ \int_{\mathbb{R}^d} \left( \sum_{i=1}^{d} \frac{f_{\omega_i}(x) - \bar{f}(x)}{\Phi(x)} \cdot \frac{f_{\omega_i}(y) - \bar{f}(y)}{\Phi(y)} \right) \gamma(y)\Phi(y)dy \right] \Phi(x)dx$$

$$= \langle \gamma, K\gamma \rangle_{L_2(\mathbb{R}^d, d\Phi)},$$

27
where

\[ K : L_2(\mathbb{R}^d, d\Phi) \to L_2(\mathbb{R}^d, d\Phi) \]

\[ \gamma \mapsto \int_{\mathbb{R}^d} \left( \sum_{i=1}^{d} \frac{f_{\omega_i} - \bar{f}}{\Phi} (\cdot) \cdot \frac{f_{\omega_j} - \bar{f}}{\Phi} (\cdot) \right) \gamma(y) \Phi(y) dy \]

Note that we can rewrite

\[ K\gamma = \sum_{i=1}^{d} \left[ \left\langle \frac{f_{\omega_i} - \bar{f}}{\Phi} \right\rangle_{L_2(\mathbb{R}^d, d\Phi)}, \frac{f_{\omega_j} - \bar{f}}{\Phi} \right\rangle_{L_2(\mathbb{R}^d, d\Phi)} \right]. \]

This expression implies that \( K \) is an operator of finite rank (it is thus a compact operator), \( K \) is self-adjoint, and \( \langle K\gamma, \gamma \rangle \geq 0 \) for all \( \gamma \in L_2(\mathbb{R}^d, d\Phi) \). In particular, the last point implies that the eigenvalues of \( K \) are non-negative. We have

\[
\sup_{\gamma \in \mathbb{R}^d} \sum_{i=1}^{d} (\varphi_{\omega_i}(\gamma))^2 \leq \sup_{\gamma \in L_2(\mathbb{R}^d, d\Phi)} \frac{\|\gamma\|^2}{L_2(\mathbb{R}^d, d\Phi)} \langle \gamma, K\gamma \rangle_{L_2(\mathbb{R}^d, d\Phi)}
=
\sup_{\gamma \in L_2(\mathbb{R}^d, d\Phi)} \frac{\|\gamma\|^2}{L_2(\mathbb{R}^d, d\Phi)} \langle \gamma, K\gamma \rangle_{L_2(\mathbb{R}^d, d\Phi)}
=
\sup_{\gamma \in L_2(\mathbb{R}^d, d\Phi)} \frac{\|\gamma\|^2}{L_2(\mathbb{R}^d, d\Phi)} \left| \langle \gamma, K\gamma \rangle_{L_2(\mathbb{R}^d, d\Phi)} \right|
= \|K\|,
\]

where the last equality follows from the fact that \( (L_2(\mathbb{R}^d, d\Phi), \langle \cdot, \cdot \rangle_{L_2(\mathbb{R}^d, d\Phi)}) \) is an Hilbert space and \( K \) is self-adjoint. Since \( K \) is also compact and since the eigenvalues of \( K \) are non-negative it follows

\[
\sup_{\gamma \in \mathbb{R}^d} \sum_{i=1}^{d} (\varphi_{\omega_i}(\gamma))^2 \leq \|K\| = \max\{|\lambda| : \lambda \in VP(T)\} = \max\{|\lambda| : \lambda \in VP(T)\},
\]

where \( VP(T) \) is the set of all the eigenvalues of \( K \). It remains to compute this maximum. By definition, \( \lambda \) is an eigenvalue of \( K \) if \( \lambda I - K \) is not injective. For \( \lambda = 0 \), the Fredholm alternative for compact self-adjoint operators (see for instance \( [10] \), p.209) implies that \( \lambda I - K \) is not injective if and only if \( \lambda I - K \) is not surjective. Thus, the non-zero eigenvalues of \( K \) are the values of \( \lambda \in \mathbb{R}^* \) such that the operator \( \lambda I - K \) is not surjective. For \( \lambda \in \mathbb{R} \), let \( A_\lambda \) be the matrix with coefficients

\[(A_\lambda)_{ij} = \left\langle \frac{f_{\omega_i} - \bar{f}}{\Phi}, \frac{f_{\omega_j} - \bar{f}}{\Phi} \right\rangle_{L_2(\mathbb{R}^d, d\Phi)} - \lambda \delta_{ij}, \quad i, j \in [1, d], \]

where \( \delta \) is the Kronecker delta. The following result proves that if \( \lambda \) is a non-zero eigenvalue of \( K \) then it holds \( \det(A_\lambda) = 0 \)

**Lemma B.2.** Let \( \lambda \in \mathbb{R}, \lambda \neq 0 \). If \( \det(A_\lambda) \neq 0 \) then \( \lambda I - K \) is surjective.

**Proof.** To lighten the notations, set \( \langle \cdot, \cdot \rangle_{2,\Phi} = \langle \cdot, \cdot \rangle_{L_2(\mathbb{R}^d, d\Phi)} \). Let \( \lambda \in \mathbb{R}, \lambda \neq 0 \) and assume that \( \det(A_\lambda) \neq 0 \). We prove that for all \( g \in L_2(\mathbb{R}^d, d\Phi) \) there exists \( \gamma \in L_2(\mathbb{R}^d, d\Phi) \) such that \( g = (\lambda I - K)\gamma \). Consider \( g \in L_2(\mathbb{R}^d, d\Phi) \). Since \( \det(A_\lambda) \neq 0 \), the matrix \( A_\lambda \) is invertible and for all \( v \in \mathbb{R}^d \) there exists \( \xi \in \mathbb{R}^d \) such that \( v = A_\lambda \xi \). In particular, for

\[ v = \left( \left\langle \frac{f_{\omega_1} - \bar{f}}{\Phi}, g \right\rangle_{2,\Phi}, \ldots, \left\langle \frac{f_{\omega_d} - \bar{f}}{\Phi}, g \right\rangle_{2,\Phi} \right)^T, \]

there exists \( \xi \in \mathbb{R}^d \) such that \( v = A_\lambda \xi \), that is

\[
\left\langle \frac{f_{\omega_i} - \bar{f}}{\Phi}, g \right\rangle_{2,\Phi} = (A_\lambda)_{ii} \xi_i = \sum_{j=1}^{d} \left\langle \frac{f_{\omega_i} - \bar{f}}{\Phi}, \frac{f_{\omega_j} - \bar{f}}{\Phi} \right\rangle_{2,\Phi} \xi_j - \lambda \xi_i
\]
for all $i \in [1, d]$. Define
\[
\gamma = \frac{1}{\lambda} g - \frac{1}{\lambda} \sum_{j=1}^{d} \xi_j \frac{f_{\omega_j} - \bar{f}}{\Phi}.
\]
We have
\[
(\lambda I - K)\gamma = \lambda \gamma - K\gamma
= g - \sum_{i=1}^{d} \xi_i \frac{f_{\omega_i} - \bar{f}}{\Phi}
- \sum_{i=1}^{d} \left[ \frac{\langle f_{\omega_i} - \bar{f}, \gamma \rangle_{L_2(\mathbb{R}^d, d\Phi)}}{\Phi} \right]
= g
\]
which concludes the proof of the Lemma.

We now find the values of $\lambda$ for which we have $\text{Det}(A_\lambda) = 0$. To do so, we first make explicit the coefficients of $A_\lambda$. It holds
\[
\left\langle \frac{f_{\omega_i} - \bar{f}}{\Phi}, \frac{f_{\omega_j} - \bar{f}}{\Phi} \right\rangle_{2, \Phi}
= \left\langle \frac{f_{\omega_i}}{\Phi}, \frac{f_{\omega_j}}{\Phi} \right\rangle_{2, \Phi}
- \left\langle \frac{f_{\omega_j}}{\Phi}, \frac{f_{\omega_i}}{\Phi} \right\rangle_{2, \Phi}
- \left\langle \bar{f}, \frac{f_{\omega_j}}{\Phi} \right\rangle_{2, \Phi}
+ \left\langle \bar{f}, \frac{f_{\omega_i}}{\Phi} \right\rangle_{2, \Phi}
\]
and, if we denote by $\langle \cdot, \cdot \rangle_2$ the usual scalar product on $\mathbb{R}^d$, we have
\[
\left\langle \frac{f_{\omega_i}}{\Phi}, \frac{f_{\omega_j}}{\Phi} \right\rangle_{2, \Phi}
= \int_{\mathbb{R}^d} \frac{1}{(2\pi\sigma^2)^d} \exp \left( -\frac{|x-a_\omega|^2}{2\sigma^2} \right) \exp \left( -\frac{|x-a_{\omega_j}|^2}{2\sigma^2} \right) \, dx
= \frac{1}{(2\pi\sigma^2)^{d/2}} \int_{\mathbb{R}^d} \exp \left( -\frac{|x|^2}{2\sigma^2} + \frac{|a_\omega|^2}{2} - 2 \langle x, a_\omega \rangle + |a_{\omega_j}|^2 - 2 \langle x, a_{\omega_j} \rangle \right) \, dx
= \exp \left( \frac{a_{\omega_j}^2}{2\sigma^2} \right)
\]
and
\[
\left\langle \frac{f_{\omega_i}}{\Phi}, \frac{f_{\omega_j}}{\Phi} \right\rangle_{2, \Phi}
= \begin{cases} 1 & \text{if } j = i \\ \frac{1}{\pi} \left[ \exp \left( \frac{a_i^2}{\sigma^2} \right) - 1 \right] & \text{if } j \neq i. \end{cases}
\]

We thus obtain
\[
\left\langle \frac{f_{\omega_i} - \bar{f}}{\Phi}, \frac{f_{\omega_j} - \bar{f}}{\Phi} \right\rangle_{2, \Phi}
= \begin{cases} 1 - \frac{1}{\pi} \left[ \exp \left( \frac{a_i^2}{\sigma^2} \right) - 1 \right] & \text{if } j = i \\ \frac{1}{\pi} \left[ 1 - \exp \left( \frac{a_i^2}{\sigma^2} \right) \right] & \text{if } j \neq i. \end{cases}
\]
Write
\[
C_1 = \left( 1 - \frac{1}{\pi} \right) \left[ \exp \left( \frac{a_i^2}{\sigma^2} \right) - 1 \right],
\]
and 
\[ C_2 = \frac{1}{d} \left[ 1 - \exp \left( \frac{a^2}{\sigma^2} \right) \right]. \]

The matrix \( A_\lambda \) has its diagonal elements equal to \( C_1 - \lambda \) and the other coefficients equal to \( C_2 \). Operations on the rows and columns of \( A_\lambda \) yield
\[
\det(A_\lambda) = (C_1 + (d-1)C_2 - \lambda) (C_1 - C_2 - \lambda)^{d-1}
\]
\[ = -\lambda \left( \exp \left( \frac{a^2}{\sigma^2} \right) - 1 - \lambda \right)^{d-1} \]

Thus, the operator \( K \) has only one non-zero eigenvalue and it is equal to \( \exp \left( \frac{a^2}{\sigma^2} \right) - 1 \). We finally obtain
\[
\inf_{Q \in \mathcal{Q}_n} \inf_{\hat{\eta} \in \mathcal{T}} \sup_{\theta \in \Theta^* \{s,a\}} E_Q(P^{\otimes n}) |\hat{\eta} - \eta| \geq \frac{1}{2} \left( 1 - \frac{n(\epsilon^a - 1)^2}{d \log(d)} \left[ \exp \left( \frac{a^2}{\sigma^2} \right) - 1 \right] - \frac{\log(2)}{\log(d)} \right)
\]
\[
\geq \frac{1}{4} \left( 1 - \frac{2n(\epsilon^a - 1)^2}{d \log(d)} \left[ \exp \left( \frac{a^2}{\sigma^2} \right) - 1 \right] \right),
\]
if \( d \geq 4 \).

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