UNIVERSAL INEQUALITIES FOR DIRICHLET EIGENVALUES ON DISCRETE GROUPS

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Abstract. We prove universal inequalities for Laplacian eigenvalues with Dirichlet boundary condition on subsets of certain discrete groups. The study of universal inequalities on Riemannian manifolds was initiated by Weyl, Polya, Yau, and others. Here we focus on a version by Cheng and Yang.

Specifically, we prove Yang-type universal inequalities for Cayley graphs of finitely generated amenable groups, as well as for the d-regular tree (simple random walk on the free group).

1 Introduction

The spectral theory of Laplace-Beltrami operators on Riemannian manifolds was extensively studied in the literature, see e.g. [CH53, Cha84, SY94, Li12]. For a bounded domain Ω in a Riemannian manifold, we denote by

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \uparrow \infty$$

the spectrum of the Laplace-Beltrami operator with Dirichlet boundary condition on Ω, counting the multiplicity of eigenvalues.

For the Euclidean space, Weyl [Wey12] proved the asymptotic behavior of eigenvalues that

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n \text{vol}(\Omega))^\frac{2}{n}} k^\frac{2}{n}, \quad k \to \infty,$$

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where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \) and \( \text{vol}(\Omega) \) is the volume of \( \Omega \). It was conjectured by Pólya [P61] that
\[
\lambda_k \geq \frac{4\pi^2}{(\omega_n \text{vol}(\Omega))^{\frac{2}{n}}} k^\frac{2}{n}, \quad k = 1, 2, 3, \ldots.
\]

Li and Yau [LY83] proved that
\[
\lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \text{vol}(\Omega))^{\frac{2}{n}}} k^\frac{2}{n}, \quad k = 1, 2, 3, \ldots.
\]

Payne, Pólya and Weinberger [PPW56] proved the gap estimate of consecutive eigenvalues for a bounded domain in \( \mathbb{R}^2 \), generalized to \( \mathbb{R}^n \) by Thompson [Tho69], that for any \( k \geq 1 \),
\[
\lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{i=1}^{k} \lambda_i.
\]

This was improved by Hile and Protter [HP80]. A sharp inequality was proved by Yang [Yan91, CY07] that
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} \lambda_i (\lambda_{k+1} - \lambda_i).
\]

As is well-known, see e.g. [Ash99], Yang’s inequality implies the Payne-Pólya-Weinberger inequality etc. These are called universal inequalities for eigenvalues since they are independent of the domain \( \Omega \). See [AB91, AB92, AB94, AB96, HS97, Ash99, Ash02, CY05, AB07] for more results regarding Euclidean spaces.

Universal inequalities have been generalized to eigenvalues of Laplace-Beltrami operators on Riemannian manifolds. In particular, Yang’s inequality has been proved for space forms. For the unit \( n \)-sphere, Cheng and Yang [CY05] proved that
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) (\lambda_i + \frac{n^2}{4}).
\]

For \( \mathbb{H}^n \), the \( n \)-dimensional hyperbolic space of sectional curvature \(-1\), Cheng and Yang [CY09] proved that
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) (\lambda_i - \frac{(n-1)^2}{4}).
\]

Note that \( \frac{(n-1)^2}{4} \) is the bottom of the spectrum of \( \mathbb{H}^n \). For a general Riemannian manifold, Chen and Cheng [CC08] proved a variant of Yang’s inequality using related geometric quantities via isometric embedding into the Euclidean space. For universal inequalities on manifolds, we refer the readers to [Li80, YY80, Leu91, Har93, HM94, CY06, Har07, SCY08, CY09, ESH109, CZL12, CP13, CY16].
In this paper, we study universal inequalities for eigenvalues on graphs, in particular Cayley graphs of discrete groups. We recall the setting of general networks. A network is a pair \((V, c)\) where \(V\) is a countable set and \(c : V \times V \to [0, \infty)\) is called the conductance. The conductance must satisfy \(0 \leq c(x, y) = c(y, x) < \infty\) (symmetric) and and \(\pi(x) := \sum_y c(x, y) < \infty\) for every \(x\). We write \(x \sim y\) to indicate \(c(x, y) > 0\) (in which case we say that \(x \sim y\) is an edge in the network). A network naturally provides a reversible Markov chain, whose transition matrix is given by \(P(x, y) = \frac{c(x, y)}{\pi(x)}\).

The (normalized) Laplacian is the operator \(\Delta = I - P\), where \(I\) denotes the identity operator, i.e.

\[
\Delta f(x) = \sum_y P(x, y)(f(x) - f(y)).
\]

We denote by \(L^2(V, \pi)\) the Hilbert space of \(L^2\) summable functions on \(V\), equipped with the inner product

\[
\langle f, g \rangle = \langle f, g \rangle_{\pi} := \sum_x \pi(x) f(x)g(x).
\]

It is well-known, the Laplacian \(\Delta\) is a bounded self-adjoint operator on \(L^2(V, \pi)\), whose spectrum is contained in \([0, 2]\). We write \(\lambda_{\min}\) for the bottom of the spectrum of \(\Delta\).

The Laplacian with Dirichlet boundary condition on finite subsets of networks has been investigated in the literature, see e.g. \([\text{Dod84}, \text{Fri93}, \text{CG98}, \text{CY00}, \text{BHJ14}]\). For finite \(\Omega \subset V\), the Laplacian with Dirichlet boundary conditions on \(\Omega\), denote by \(\Delta_{\Omega}\), is defined as the Laplacian \(\Delta\) restricted to the subspace

\[
L^2(\Omega) := \{f \in L^2(V, \pi) : f|_{G\setminus \Omega} \equiv 0\}.
\]

The eigenvalues of \(\Delta_{\Omega}\), called Dirichlet eigenvalues on \(\Omega\), are ordered by

\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|\Omega|},
\]

where \(|\cdot|\) denotes the cardinality of the subset. We are interested in proving universal inequalities on graphs, in particular Yang-type inequalities (1) and (2). Due to the discrete nature of graphs, some modification is required.

**Definition 1** We say that the network \((V, c)\) satisfies Yang’s inequality (resp. the Yang-type inequality) with constant \(C_Y\) (resp. \(C_{YT}\)) if the following holds for any finite subset \(\Omega \subset G\):

Let \(0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|\Omega|}\) be the Dirichlet eigenvalues of \(\Omega\). Then, for any \(k < |\Omega|\),

\[
\sum_{i=1}^{k} |\lambda_{k+1} - \lambda_i|^2 \leq C_Y \cdot \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_{\min}).
\]

(3) (resp. \(\sum_{i=1}^{k} |\lambda_{k+1} - \lambda_i|^2 (1 - \lambda_i) \leq C_{YT} \cdot \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_{\min})\).
Since $\lambda_i \leq 2$, for any $i \geq 1$, one easily sees that in case of $\lambda_{\min} = 0$, the Yang-type inequality implies Yang’s inequality with $C_Y = C_{YT} + 2$. Following the arguments in [Yan91, Ash99, CY07], the first author et al. [HLS17] proved that the integer lattice $\mathbb{Z}^n$, a discrete analog of $\mathbb{R}^n$, satisfies Yang-type inequality, with constant $C_{YT} = \frac{4}{n^2}$. Recently, Kobayashi [Kob20] proved the Yang-type inequality for the eigenvalues of the Laplacian (not Dirichlet eigenvalues) of a finite edge-transitive graph.

Note that $\mathbb{Z}^n$ can be regarded as a Cayley graph of a free Abelian group. In this paper, we prove Yang-type inequalities for more general Cayley graphs of finitely generated infinite groups.

### 1.1 Amenable groups

Our first result regards amenable groups. Let $G$ be a finitely generated amenable group. Consider some probability measure $\mu$ on $G$ (which we think of as a non-negative function $\mu : G \to [0, 1]$ such that $\sum_x \mu(x) = 1$). Assume that $\mu$ is symmetric, i.e. $\mu(x) = \mu(x^{-1})$ for all $x \in G$. Then $\mu$ induces a corresponding Cayley graph (or network) by setting the conductances $c(x, y) = \mu(x^{-1}y)$. This network corresponds to the $\mu$-random walk on $G$. This network is denoted by $(G, \mu)$.

**Theorem 2** Let $G$ be a finitely generated infinite amenable group. Let $\mu$ be a symmetric probability measure on $G$, and consider the Cayley network $(G, \mu)$ of $G$ with respect to $\mu$. Set $\mu_* := \inf_{\mu \neq 0} \mu(y) \in \text{supp}(\mu) \mu(y)$.

Then, the network $(G, \mu)$ satisfies Yang’s inequality, with constant $C_Y = \frac{6}{\mu_*}$.

For finitely generated groups with Abelian quotients, i.e. those groups which admit homomorphisms onto $\mathbb{Z}^n$ for some $n$, we prove the Yang-type inequality with $C_{YT} = \frac{4}{n^2}$ for specific $\mu$-random walks, see Theorem 6. This extends the result for $\mathbb{Z}^n$ from [HLS17].

### 1.2 Free groups

Next, we consider Yang-type inequalities on regular trees, which can be regarded as Cayley graphs of free groups. Let $\mathbb{T}_d$, $d \geq 3$, be a $d$-regular tree with the conductances of the edges $c(x, y) = 1_{\{x \sim y\}} \frac{1}{d}$, which is a discrete analog of hyperbolic space $\mathbb{H}^d$. The Laplacian corresponds to the generator of the simple random walk on $\mathbb{T}_d$. As is well-known, the bottom of the spectrum of $\mathbb{T}_d$ is $1 - \frac{2\sqrt{d-1}}{d}$. Following the arguments in [CY09], we prove the following result.
The network given by the simple random walk on the $d$-regular tree $T_d$ (where $d > 2$) satisfies the Yang-type inequality with constant $C_{YT} = \frac{8\sqrt{d-1}}{d}$.

We sketch the proof strategies of Theorem 2 and Theorem 3: By the variational principle, for an upper bound estimate of eigenvalues, it suffices to construct appropriate test functions. Following the arguments in [Yan91, CY06], for any network and any test function $\alpha : V \to \mathbb{R}$, we prove the Dirichlet eigenvalues satisfy some crucial estimate involving $\alpha$, see Lemma 4, a discrete analog of [CY06, Proposition 1]. This enables us to derive the Yang-type inequality with choice of $\alpha$ with nice properties for $\Delta \alpha$ and the gradient of $\alpha$. For $\mathbb{R}^n$ or $\mathbb{Z}^n$, as in [Yan91, CY07, HLS17], linear functions are good candidates for test functions.

In order to generalize the result to Cayley graphs of amenable groups, i.e. Theorem 2, we use harmonic cocycles as test functions. The existence of harmonic cocycles for amenable groups was proved by [Mok95, KS97].

For $\mathbb{H}^n$, Cheng and Yang [CY09] used Busemann functions of geodesic rays to prove Yang-type inequality (2). To extend the result to $T_d$, i.e. Theorem 3, we use the discrete analogs of Busemann functions as test functions.

The paper is organized as follows: In next section, we introduce some basic facts on networks. In Section 3, we prove the useful estimate of eigenvalues for general networks, Lemma 4. Section 4 is devoted to the proofs of main results, Theorem 2 and Theorem 3. In the last section, we derive some applications of the Yang-type inequality, such as the Paley-Polya-Weinberger inequality and the Hile-Protter inequality, etc.

2 Notation and basic operators

2.1 $\Gamma$ calculus

Let $(V, c)$ be a network on the set of vertices $V$ with the conductance $c$. We allow $c(x, x) > 0$, which corresponds to a self-edge at $x \in V$.

Recall the inner product on functions defined in the introduction

$$\langle f, g \rangle = \sum_x \pi(x) f(x) g(x).$$

Accordingly we write $||f||^2 = ||f||_\pi^2 := \langle f, f \rangle$, and the space of $L^2$ summable functions is given by $L^2(V, \pi) := \{ f : V \to \mathbb{C} : ||f|| < \infty \}$.

The Dirichlet energy is defined to be

$$\mathcal{E}(f, g) := \sum_{x, y} c(x, y) (f(x) - f(y))(g(x) - g(y)).$$
and $\mathcal{E}(f) := \mathcal{E}(f, f)$. If $f, g \in L^2(V, \pi)$, then it is not difficult to prove the “integration by parts” formula,

$$\mathcal{E}(f, g) = 2 \langle \Delta f, g \rangle = 2 \langle f, \Delta g \rangle.$$  

Define the so called *carré du champ* operator (at $x \in V$) as follows:

$$2\Gamma(f, g)(x) := (f \Delta \bar{g} + \bar{g} \Delta f - \Delta(f \bar{g}))(x) \sum_y P(x, y)(f(x) - f(y))(g(x) - g(y)),$$

and $\Gamma(f) := \Gamma(f, f)$. Note that $\Gamma$ is symmetric and bi-linear.

Finally we define the scalar-valued (non-linear) functional:

$$\Lambda(f, g) = \frac{1}{4} \sum_{x, y} c(x, y)|f(x) - f(y)|^2 \cdot |g(x) - g(y)|^2.$$

### 2.2 Identities

In this section we summarize a few identities which we will require in the analysis below. All are straightforward and easy to prove, and hold for all $f, g \in L^2(V, \pi)$.

\begin{equation}
\mathcal{E}(f, g) = 2 \sum_x \pi(x) \Gamma(f, g)(x) = 2 \langle \Gamma(f, g), 1 \rangle. \tag{4}
\end{equation}

Also, note that

$$\langle \Gamma(f, g) , g \rangle = \frac{1}{2} \sum_{x, y} P(x, y)(f(x) - f(y))(\overline{g(x) - g(y)})g(x)\pi(x).$$

Since $\pi(x)P(x, y) = c(x, y) = \pi(y)P(y, x)$,

$$\mathcal{E}(f, g^2) = \sum_{x, y} c(x, y)(f(x) - f(y))(\overline{g(x)^2 - g(y)^2})$$

$$= \sum_{x, y} P(x, y)(f(x) - f(y))(\overline{g(x) - g(y)})g(x)\pi(x)$$

$$+ \sum_{x, y} P(x, y)(f(x) - f(y))(\overline{g(x) - g(y)})g(y)\pi(x)$$

$$= 4 \langle \Gamma(f, g), g \rangle. \tag{5}$$

So in conclusion

$$\langle 2\Gamma(f, g), g \rangle = \langle \Delta f, g^2 \rangle. \tag{5}$$
We also may compute,
\[\langle 2\Gamma(f, g), f \cdot g \rangle = \sum_{x,y} c(x, y)(f(x) - f(y))(g(x) - g(y))f(x)g(x)\]
\[= \sum_{x,y} c(x, y)(f(x) - f(y))(g(x) - g(y)) \cdot \frac{f(x)g(x) + f(y)g(y)}{2}\]
\[= \sum_{x,y} c(x, y)(f(x) - f(y))(g(x) - g(y)) \cdot \frac{(f(x) + f(y))(g(x) + g(y)) + (f(x) - f(y))(g(x) - g(y))}{4}\]
\[= \frac{1}{4} \sum_{x,y} c(x, y)(f(x)^2 - f(y)^2)(g(x)^2 - g(y)^2)\]
\[+ \frac{1}{4} \sum_{x,y} c(x, y)|f(x) - f(y)|^2 \cdot |g(x) - g(y)|^2,\]
which culminates in
\[\langle 2\Gamma(f, g), f \cdot g \rangle = \frac{1}{4} \mathcal{E}(f^2, g^2) + \Lambda(f, g).\]

3 Universal inequality

The following is an analogue of [CY06, Proposition 1]. It is the main estimate which will imply our results.

Let \((V, c)\) be a network. Let \(\Omega \subset V\) be a finite subset of size \(n = |\Omega|\). Let \(u_1, \ldots, u_n\) be an orthonormal basis of eigenvectors for \(\Delta_\Omega\) defined on the subspace \(L^2(\Omega)\) of \(L^2(V, \pi)\); that is,
\[\begin{align*}
\text{• } & \lambda_{\text{min}} \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n, \\
\text{• } & \Delta u_i = \lambda_i u_i, \\
\text{• } & u_i|_{G\setminus \Omega} \equiv 0, \\
\text{• } & \langle u_i, u_j \rangle = 1_{\{i=j\}}.
\end{align*}\]

Since the Laplacian is self-adjoint, such an orthonormal basis exists, \(\lambda_i \in \mathbb{R}\) and \(u_i\) are real valued.

We call such a collection \((\lambda_i, u_i)_{i=1}^n\) the Dirichlet system for \(\Omega\).

**Lemma 4** Let \((V, c)\) be a network. Let \(\Omega \subset V\) be a finite subset of size \(n = |\Omega|\). Let \((\lambda_i, u_i)_{i=1}^n\) be the Dirichlet system for \(\Omega\).
Then, for any \( k < n \) and any \( \alpha : V \to \mathbb{R} \) we have
\[
\sum_{i=1}^{k} |\lambda_{k+1} - \lambda_{i}|^2 \left( \langle \Gamma(\alpha), u_i^2 \rangle - \Lambda(\alpha, u_i) \right) \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_{i}) ||u_i \cdot \Delta \alpha - 2\Gamma(\alpha, u_i)||^2.
\]

**Proof.** Let \( \alpha : G \to \mathbb{R} \). Fix some \( 1 \leq k < n \). Set
\[
a_{ij} = \langle u_i \cdot \alpha, u_j \rangle,
\]
\[
\varphi_i = u_i \cdot \alpha - \sum_{j=1}^{k} a_{ij} \cdot u_j,
\]
\[
\alpha_i = u_i \cdot \Delta \alpha - 2\Gamma(u_i, \alpha),
\]
\[
b_{ij} = \langle \alpha_i, u_j \rangle,
\]
\[
w_i = \langle \alpha_i, \varphi_i \rangle,
\]
\[
z_i = \langle \alpha_i, u_i \cdot \alpha \rangle,
\]
\[
y_i = \Lambda(\alpha, u_i).
\]

We collect a few observations regarding these quantities:

For all \( 1 \leq i,j \leq k \),
\[
\langle \varphi_i, u_j \rangle = \langle u_i \cdot \alpha, u_j \rangle - \sum_{\ell=1}^{k} \langle u_\ell, u_j \rangle a_{i\ell} = a_{ij} - a_{ji} = 0.
\]

Also, \( a_{ij} = a_{ji} \) and since the Laplacian is self-adjoint,
\[
\lambda_j \cdot a_{ij} = \langle u_i \cdot \alpha, \Delta u_j \rangle = \langle \Delta(u_i \cdot \alpha), u_j \rangle \\
= \langle \Delta u_i \cdot \alpha + u_i \cdot \Delta \alpha - 2\Gamma(u_i, \alpha), u_j \rangle \\
= \lambda_i \cdot a_{ij} + \langle \alpha_i, u_j \rangle = \lambda_i \cdot a_{ij} + b_{ij},
\]
which proves that for all \( 1 \leq i,j \leq k \),
\[
b_{ij} = -b_{ji} = (\lambda_j - \lambda_i) \cdot a_{ij}
\]
\[
\Delta \varphi_i = \Delta(u_i \cdot \alpha) - \sum_{j=1}^{k} \Delta u_j \cdot a_{ij} = \lambda_i u_i \cdot \alpha + \alpha_i - \sum_{j=1}^{k} \lambda_j u_j \cdot a_{ij}.
\]
Since \( \langle u_i, u_j \rangle = \mathbf{1}_{\{i = j\}} \),

\[
||\alpha_i - \sum_{j=1}^{k} b_{ij} \cdot u_j||^2 = ||\alpha_i||^2 + \sum_{j=1}^{k} ||b_{ij} \cdot u_j||^2 - 2 \sum_{j=1}^{k} b_{ij} \cdot \langle \alpha_i, u_j \rangle
\]

(10)

\[
= ||\alpha_i||^2 - \sum_{j=1}^{k} |b_{ij}|^2.
\]

By (8) we know that \(- \langle \alpha_i, u_j \rangle = -b_{ij} = (\lambda_i - \lambda_j)a_{ij}\), so

(11)

\[
w_i = z_i - \sum_{j=1}^{k} \langle \alpha_i, a_{ij} \cdot u_j \rangle = z_i + \sum_{j=1}^{k} (\lambda_i - \lambda_j)|a_{ij}|^2.
\]

By (6) we have that

\[
\langle 2\Gamma(u_i, \alpha), u_i \cdot \alpha \rangle = \frac{1}{2} \langle \Delta(\alpha^2), u_i^2 \rangle + \Lambda(\alpha, u_i).
\]

Thus,

(12)

\[
z_i + y_i = \langle u_i \cdot \Delta \alpha - 2\Gamma(u_i, \alpha), u_i \cdot \alpha \rangle + \Lambda(\alpha, u_i)
\]

\[
= \langle u_i \cdot \Delta \alpha, u_i \cdot \alpha \rangle - \frac{1}{2} \langle \Delta(\alpha^2), u_i^2 \rangle
\]

\[
= \langle \Delta \alpha \cdot \alpha - \frac{1}{2} \Delta(\alpha^2), u_i^2 \rangle = \langle \Gamma(\alpha), u_i^2 \rangle.
\]

By (7) we get that \( \langle \varphi_i, u_i \cdot \alpha \rangle = ||\varphi_i||^2 \). Also, since \( \varphi_i \) is orthogonal to \( \{u_1, \ldots, u_k\} \), using (9),

\[
\lambda_{k+1} ||\varphi_i||^2 \leq \langle \Delta \varphi_i, \varphi_i \rangle
\]

\[
= \langle \lambda_i u_i \cdot \alpha + a_i - \sum_{j=1}^{k} \lambda_j u_j \cdot a_{ij}, \varphi_i \rangle
\]

\[
= w_i + \lambda_i \langle u_i \cdot \alpha, \varphi_i \rangle = w_i + \lambda_i ||\varphi_i||^2.
\]

Using the Cauchy-Schwarz inequality and (10),

\[
(\lambda_{k+1} - \lambda_i)|w_i|^2 = (\lambda_{k+1} - \lambda_i)\left|\langle \alpha_i - \sum_{j=1}^{k} b_{ij} \cdot u_j, \varphi_i \rangle \right|^2
\]

\[
\leq (\lambda_{k+1} - \lambda_i)||\varphi_i||^2 \cdot \left(||\alpha_i||^2 - \sum_{j=1}^{k} |b_{ij}|^2 \right)
\]

\[
\leq w_i \cdot \left(||\alpha_i||^2 - \sum_{j=1}^{k} |b_{ij}|^2 \right).
\]
Thus,

(13) \[(\lambda_{k+1} - \lambda_i)w_i \leq ||\alpha_i||^2 - \sum_{j=1}^{k} |\lambda_i - \lambda_j|^2 \cdot |a_{ij}|^2.\]

By (11),

\[
\sum_{i=1}^{k} |\lambda_{k+1} - \lambda_i|^2 w_i = \sum_{i=1}^{k} |\lambda_{k+1} - \lambda_i|^2 z_i + \sum_{i,j=1}^{k} |\lambda_{k+1} - \lambda_i|^2 (\lambda_i - \lambda_j)|a_{ij}|^2
\]

\[
= \sum_{i=1}^{k} |\lambda_{k+1} - \lambda_i|^2 z_i + \frac{1}{2} \sum_{i,j=1}^{k} (|\lambda_{k+1} - \lambda_i|^2 - |\lambda_{k+1} - \lambda_j|^2)(\lambda_i - \lambda_j)|a_{ij}|^2
\]

\[
= \sum_{i=1}^{k} |\lambda_{k+1} - \lambda_i|^2 z_i - \sum_{i,j=1}^{k} (\lambda_{k+1} - \frac{\lambda_i + \lambda_j}{2})|\lambda_i - \lambda_j|^2 |a_{ij}|^2
\]

\[
= \sum_{i=1}^{k} |\lambda_{k+1} - \lambda_i|^2 z_i - \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i)|\lambda_i - \lambda_j|^2 |a_{ij}|^2.
\]

Multiplying (13) by \(\lambda_{k+1} - \lambda_i\) and summing over \(i\), we obtain

(14) \[
\sum_{i=1}^{k} |\lambda_{k+1} - \lambda_i|^2 z_i \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)||\alpha_i||^2.
\]

The proof is now complete using \(z_i = \langle \Gamma(\alpha), u_i^2 \rangle - \Lambda(\alpha, u_i)\) by (12). \(\square\)

Let \(\mathcal{H}\) be a Hilbert space and \(\alpha : V \to \mathcal{H}\). We extend the definitions of the inner product and of \(\Gamma, \Lambda\) by defining

\[
2\Gamma(\alpha, u) = \sum_y P(x, y)(u(x) - u(y)) \cdot (\alpha(x) - \alpha(y)),
\]

\[
2\Gamma(\alpha)(x) = \sum_y P(x, y)||\alpha(x) - \alpha(y)||^2_{\mathcal{H}},
\]

\[
\langle \alpha, u \rangle = \sum_x \pi(x)u(x) \cdot \alpha(x),
\]

\[
||\alpha||^2 = \langle \alpha, \alpha \rangle = \sum_x \pi(x)||\alpha(x)||^2_{\mathcal{H}},
\]

\[
\Lambda(\alpha, u) = \frac{1}{4} \sum_{x, y} c(x, y)|u(x) - u(y)|^2 \cdot ||\alpha(x) - \alpha(y)||^2_{\mathcal{H}}.
\]

Here \(u : V \to \mathbb{R}\) is any (finitely supported) real valued function. With this notation, we have the following theorem generalizing Lemma 4.
Theorem 5 Let \((V, c)\) be a network. Let \(\Omega \subset V\) be a finite subset of size \(n = |\Omega|\). Let \((\lambda_i, u_i)_{i=1}^n\) be the Dirichlet system for \(\Omega\). Let \(H\) be a Hilbert space and let \(\alpha : V \to H\). Then for any \(k < n\),
\[
\sum_{i=1}^k |\lambda_{k+1} - \lambda_i|^2 \cdot \left( \langle \Gamma(\alpha), u_i^2 \rangle - \Lambda(\alpha, u_i) \right) \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \cdot ||u_i \cdot \Delta \alpha - 2\Gamma(\alpha, u_i)||^2.
\]

Note that when \(H = \mathbb{R}\) this is exactly Lemma 4.

Proof. Let \(h \in H\) be any non-zero vector. Define the function \(\alpha' : V \to \mathbb{R}\) by \(\alpha'(x) = \langle \alpha(x), h \rangle_H\). Plugging this into Lemma 4 we see that we only need to compute \(\Gamma(\alpha'), \Lambda(\alpha', u_i), \Gamma(\alpha', u_i), \Delta \alpha'\). It is simple to verify that
\[
\Delta \alpha' = \langle \Delta \alpha, h \rangle_H,
\]
\[
\Lambda(\alpha', u_i) = \frac{1}{4} \sum_{x,y} c(x,y)|u_i(x) - u_i(y)|^2 \cdot |\langle \alpha(x) - \alpha(y), h \rangle_H|^2,
\]
\[
2\Gamma(\alpha')(x) = \sum_y P(x,y)|\langle \alpha(x) - \alpha(y), h \rangle_H|^2,
\]
\[
2\Gamma(\alpha', u_i)(x) = \sum_y P(x,y)(u_i(x) - u_i(y)) \cdot \langle \alpha(x) - \alpha(y), h \rangle_H.
\]
Summing this over \(h\) in an orthonormal basis for \(H\), we have the theorem. \(\square\)

4 The proof of main results

4.1 Amenable groups

One application of Theorem 5 is for the case of amenable groups. Given a finitely generated group, there is a natural network one may define. Actually, the initial data is a finitely generated group \(G\) and a probability measure \(\mu\) on \(G\), which is assumed to be symmetric, i.e. \(\mu(x) = \mu(x^{-1})\). This measure is used to construct the random walk on \(G\), which is just the Markov chain with transition matrix \(P(x,y) = \mu(x^{-1}y)\). This Markov chain is precisely the reversible Markov chain associated to the network on \(G\) given by conductances \(c(x,y) = \mu(x^{-1}y)\). We denote this network by \((G, \mu)\), and call it the Cayley network of \(G\) with respect to \(\mu\). (Since \(\mu\) is a probability measure, in this case \(\pi(x) = 1\) for all \(x\).)

For a probability measure \(\mu\) on \(G\), define
\[
\mu_* := \inf_{1 \neq y \in \text{supp}(\mu)} \mu(y).
\]
Note that \(\mu\) has finite support if and only if \(\mu_* > 0\).
Recall that Kesten’s amenability criterion [Kes59] states that the bottom of the spectrum of $\Delta$ is 0 if and only if $G$ is an amenable group.

We are now ready to prove Theorem 2.

Proof of Theorem 2. Since $G$ is amenable and infinite, it does not have Kazhdan property (T). (This is very well known, and an easy exercise following the definitions of property (T) and amenability. See e.g. [Pet17, Chapter 7].) It follows from [Mok95, KS97] that there exists a Hilbert space $\mathcal{H}$ on which the group $G$ acts by unitary operators, with a harmonic cocycle $\alpha : G \to \mathcal{H}$. That is, $\alpha(xy) = \alpha(x) + x.\alpha(y)$ for all $x, y \in G$ and $\Delta \alpha \equiv 0$. (For a short proof see e.g. [Oza18].)

Since the $G$-action is unitary, we may compute that

$$||\alpha(x) - \alpha(xy)||_{\mathcal{H}}^2 = ||\alpha(y)||_{\mathcal{H}}^2,$$

so

$$2\Gamma(\alpha)(x) = \sum_y \mu(y)||\alpha(y)||_{\mathcal{H}}^2,$$

is a constant function.

Now, if $u$ is an eigenfunction of unit length, with $\Delta u = \lambda u$, then

$$\langle \Gamma(\alpha), u^2 \rangle = \Gamma(\alpha) \cdot \sum_x \pi(x)u(x)^2 = \Gamma(\alpha).$$

Also,

$$4\Lambda(\alpha, u) = \sum_{x,y} c(x,y)|u(x) - u(y)|^2 \cdot ||\alpha(x) - \alpha(y)||_{\mathcal{H}}^2$$

$$= \sum_{x,y} \mu(y)|u(x) - u(xy)|^2 \cdot ||\alpha(y)||_{\mathcal{H}}^2.$$

since for any $1 \neq y \in \text{supp}(\mu)$,

$$||\alpha(y)||_{\mathcal{H}}^2 \leq \frac{1}{\mu_*} \sum_y \mu(y)||\alpha(y)||_{\mathcal{H}}^2 \leq \frac{1}{\mu_*} \cdot 2\Gamma(\alpha),$$

we get that

$$4\Lambda(\alpha, u) \leq \frac{1}{\mu_*} \cdot 2\Gamma(\alpha) \cdot \sum_{x,y} \mu(y)|u(x) - u(xy)|^2 = \frac{4}{\mu_*} \Gamma(\alpha) \cdot \lambda.$$

Finally,

$$2\Gamma(\alpha, u)(x) = \sum_y \mu(y)(u(x) - u(xy)) \cdot (\alpha(x) - \alpha(xy)) = -\sum_y \mu(y)(u(x) - u(xy)) \cdot x.\alpha(y).$$
Since $G$ acts unitarily on $\mathcal{H}$, we have by Jensen’s inequality,
\[
||2\Gamma(\alpha, u)||^2 = \sum_x ||\sum_y \mu(y)(u(x) - u(xy)) \cdot \alpha(y)||^2_{\mathcal{H}} \\
\leq \sum_{x,y} \mu(y)||u(x) - u(xy)||^2 \cdot ||\alpha(y)||^2_{\mathcal{H}} = 4\Lambda(\alpha, u).
\]

Plugging all the above into Theorem 5 we arrive at
\[
k \sum_{i=1}^k |\lambda_{k+1} - \lambda_i|^2 \Gamma(\alpha) \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \cdot \Lambda(\alpha, u_i) \cdot (4 + \lambda_{k+1} - \lambda_i) \\
\leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i \cdot \frac{6}{\mu_s} \cdot \Gamma(\alpha),
\]
where we have used that $\lambda_{k+1} - \lambda_i \leq 2$. This completes the proof. \hfill \qed

4.2 Groups with Abelian quotients

For general groups with Abelian quotients, we can prove the Yang-type inequality, analogous to the result in [HLS17].

**Theorem 6** Let $G$ be a finitely generated group. Let $\alpha : G \to \mathbb{Z}^n$ be a surjective homomorphism. Let $S = \{s_1, \ldots, s_n, k_1, \ldots, k_m\}$ be a generating set for $G$ so that $(\alpha(s_j))_{j=1}^n$ is the standard basis of $\mathbb{Z}^n$, and such that $\alpha(k_j) = 0$ for all $j = 1, \ldots, m$. Let $\mu$ be a symmetric measure supported on $S \cup S^{-1}$. Let $0 < \varepsilon = 1 - \sum_{j=1}^n (\mu(s_j) + \mu(s_j^{-1}))$. (e.g. one may take $\mu(k_j) = \mu(k_j^{-1}) = \frac{\varepsilon}{2m}$ and $\mu(s_j) = \mu(s_j^{-1}) = \frac{1-\varepsilon}{2m}$.)

Then, the network $(G, \mu)$ satisfies the following: For any finite $\Omega \subset G$ and $k < |\Omega|$,
\[
\sum_{i=1}^k |\lambda_{k+1} - \lambda_i|^2 \cdot (1 - \varepsilon - \lambda_i) \leq 8 \max_j \mu(s_j) \cdot \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \cdot \lambda_i.
\]

**Remark 7** When we choose $\mu(k_j) = \mu(k_j^{-1}) = \frac{\varepsilon}{2m}$ and $\mu(s_j) = \mu(s_j^{-1}) = \frac{1-\varepsilon}{2m}$, we get the Yang-type inequality up to an $\varepsilon$-defect, with constant at most $\frac{4}{n}$.

**Remark 8** The case $G \cong \mathbb{Z}^n$ was already treated in [HLS17], where the same result was shown, using similar methods. This is the case $\varepsilon = 0$ and $\mu(s_j) = \mu(s_j^{-1}) = \frac{1}{2m}$ in the above theorem.

**Proof.** The main advantage of $\alpha$ being a homomorphism is that
\[
\mu(y)\alpha(y) = \begin{cases} 
\pm \mu(s_j) e_j & y = (s_j)^{\pm 1}, \\
0, & \text{otherwise}. 
\end{cases}
\]
where \( \{e_j\}_{j=1}^n \) is the standard basis of \( \mathbb{Z}^n \). Thus, for the Euclidean Hilbert space \( \mathcal{H} = \mathbb{R}^n \),

\[
2\Gamma(\alpha)(x) = \sum_{y} \mu(y)||\alpha(x) - \alpha(xy)||_H^2 = \sum_{j=1}^n \left( \mu(s_j) + \mu(s_j^{-1}) \right) = 1 - \varepsilon,
\]

for any \( x \in G \). Also, \( \Delta \alpha = 0 \). Now, if \( u \) is an eigenfunction of unit length, with \( \Delta u = \lambda u \), then

\[
\langle \Gamma(\alpha), u^2 \rangle = \Gamma(\alpha) = \frac{1}{2}(1 - \varepsilon).
\]

We may bound

\[
4\Lambda(\alpha, u) = \sum_{x,y} \mu(y)|u(x) - u(xy)|^2 \cdot ||\alpha(y)||_H^2
\]

\[
= \sum_{x} \sum_{j=1}^n \mu(s_j) \left( |u(x) - u(xs_j)|^2 + |u(x) - u(xs_j^{-1})|^2 \right)
\]

\[
\leq \sum_{x,y} \mu(y)|u(x) - u(xy)|^2 = 2\lambda.
\]

As in the proof of Theorem 2,

\[
2\Gamma(\alpha, u)(x) = \sum_{j=1}^n \mu(s_j)(u(x) - u(xs_j) - u(x) + u(xs_j^{-1})) \cdot \alpha(s_j),
\]

\[
||2\Gamma(\alpha, u)||^2 = \sum_{x} \sum_{j=1}^n \mu(s_j)^2 |u(xs_j^{-1}) - u(xs_j)|^2
\]

\[
\leq 2 \sum_{x} \sum_{j=1}^n \mu(s_j)^2 (|u(x) - u(xs_j)|^2 + |u(x) - u(xs_j^{-1})|^2)
\]

\[
\leq 2 \max_j \mu(s_j) \sum_{x,y} \mu(y)|u(x) - u(xy)|^2 = \max_j \mu(s_j) \cdot 4\lambda.
\]

Plugging all of this into Theorem 5, we arrive at

\[
\sum_{i=1}^k |\lambda_{k+1} - \lambda_i|^2 \cdot (1 - \varepsilon - \lambda_i) \leq 8 \max_j \mu(s_j) \cdot \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \cdot \lambda_i.
\]

\[ \Box \]

4.3 Trees

In this section, we prove the Yang-type inequality for \( d \)-regular tree \( \mathbb{T}_d \), \( d \geq 3 \), with the conductances of the edges \( c(x, y) = 1_{\{x \sim y\}} \frac{1}{d} \).

Proof of Theorem 3. Fix a ray to infinity, and an origin \( o \). Let \( b \) be the Buseman function corresponding to the ray with \( b(o) = 0 \). That is: let \( o = x_0 \sim x_1 \sim \cdots \sim x_n \sim \)
\(x_{n+1} \sim \cdots\) be an infinite simple path, so \(x_i \neq x_j\) for all \(i \neq j\). Because \(\mathbb{T}_d\) is a tree, this path is necessarily a geodesic: the distance between \(x_j, x_i\) in the graph is always \(|j - i|\). This path is the ray mentioned above. Now, for any \(j \geq 0\) set \(b(x_j) := -j\). Furthermore, for any vertex \(z\), let \(z_\ast\) be the closest vertex to \(z\) from the above path. Set \(b(z) = b(z_\ast) + \text{dist}(z, z_\ast)\).

The important properties of \(b\) are thus: \(b : \mathbb{T}_d \to \mathbb{Z}\) is a function such that \(b(o) = 0\) and such that every vertex \(x\) has \(d - 1\) neighbors \(y \sim x\) with \(b(y) = b(x) + 1\), and exactly one neighbor \(\overline{x} \sim x\) with \(b(\overline{x}) = b(x) - 1\). One easily sees that

\[
2\Gamma(b)(x) = 1 \quad \forall x \in \mathbb{T}_d.
\]

It is also simple to check that the function \(f(x) = (\frac{\xi}{\sqrt{d-1}})^{b(x)}\) satisfies

\[
\Delta f(x) = f(x) \cdot (1 - \frac{d-1}{d} \cdot (\xi + \xi^{-1})�).
\]

Hence, if \(\lambda = 1 - \frac{2\sqrt{d-1}}{d}\) (which corresponds to choosing \(\xi = 1\), maximizing the above expression) then \(\Delta f = \lambda f\). Coincidentally, this is the bottom of the \(L^2\) spectrum of \(\Delta\), i.e. \(\lambda_{\text{min}} = 1 - \frac{2\sqrt{d-1}}{d}\).

For any \(x\) let \(\overline{x}\) be the unique vertex with \(b(\overline{x}) = b(x) - 1\). For a function \(f\) let \(\tilde{f}(x) := f(\overline{x})\). Note that as \(x\) ranges over the whole graph, the pair \((x, \overline{x})\) ranges over all edges in the graph, each edge counted exactly once in the direction of decreasing the Buseman function \(b\). Thus,

\[
||f - \tilde{f}||^2 = \sum_x |f(x) - f(\overline{x})|^2 = \frac{1}{2} \sum_{x \sim y} |f(x) - f(y)|^2 = \frac{1}{2} \sum_{x, y} c(x, y) |f(x) - f(y)|^2 = d \langle \Delta f, f \rangle.
\]

Also, the map \(x \mapsto \overline{x}\) is a \((d - 1)\)-to-1 map. So,

\[
||\tilde{f}||^2 = \sum_x |f(\overline{x})|^2 = \sum_y \sum_{x : \overline{x} = y} |f(y)|^2 = (d - 1) ||f||^2.
\]

Thus,

\[
d \langle \Delta f, f \rangle = ||f - \tilde{f}||^2 = d \cdot ||f||^2 - 2 \langle f, \tilde{f} \rangle.
\]

Note that the Buseman function satisfies:

\[
\Delta b(x) = \sum_y P(x, y)(b(x) - b(y)) = -\frac{d-2}{d} =: -\gamma,
\]

and also \(|b(x) - b(y)| = 1\) for any \(x \sim y\).

Let \(u\) be an eigenfunction \(\Delta u = \lambda u\). Note that

\[
\langle 2\Gamma(b, u), u \rangle = \frac{1}{2} \mathcal{E}(b, u^2) = \langle \Delta b, u^2 \rangle = -\gamma ||u||^2.
\]
Thus,
\[ ||2\Gamma(b, u) - u\Delta b||^2 = 4||\Gamma(b, u)||^2 + \gamma^2 \cdot ||u||^2 + 2\gamma \langle 2\Gamma(b, u), u \rangle \]
(17)
\[ = 4||\Gamma(b, u)||^2 - \gamma^2 \cdot ||u||^2 \]
Also,
\[ 2\Gamma(b, u)(x) = \sum_y c(x, y)(b(x) - b(y))(u(x) - u(y)) \]
\[ = -\sum_{y \neq \bar{x}} c(x, y)(u(x) - u(y)) + c(x, \bar{x})(u(x) - u(\bar{x})) \]
\[ = -\Delta u(x) + \frac{2}{d}(u(x) - u(\bar{x})) = (\frac{2}{d} - \lambda)u(x) - \frac{2}{d}\bar{u}(x), \]
so using (15) and (16), assuming that ||u|| = 1,
\[ ||2\Gamma(b, u)||^2 = (1 - \lambda - \gamma)^2 ||u||^2 + \frac{4}{d^2} ||\bar{u}||^2 - \frac{4}{d}(1 - \lambda - \gamma) \langle u, \bar{u} \rangle \]
\[ = (1 - \lambda)^2 + \gamma^2 - 2\gamma(1 - \lambda) + \frac{4}{d^2}(d - 1) - 2(1 - \lambda - \gamma)(1 - \lambda) \]
(18)
\[ = \gamma^2 + (1 - \lambda_{\min})^2 - (1 - \lambda)^2. \]
Finally,
\[ 4\Lambda(b, u) = \sum_{x,y} c(x, y)|b(x) - b(y)|^2 \cdot |u(x) - u(y)|^2 = 2\lambda. \]
Combining this with (17), (18), and plugging into Lemma 4, we have that:
\[ \sum_{i=1}^k |\lambda_{k+1} - \lambda_i|^2 \cdot (1 - \lambda_i) \leq 2 \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \cdot (\lambda_i - \lambda_{\min}) \cdot (1 - \lambda_i + 1 - \lambda_{\min}) \]
\[ \leq \frac{8\sqrt{d-1}}{d} \cdot \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \cdot (\lambda_i - \lambda_{\min}), \]
where we used \( \lambda_i \geq \lambda_{\min} = 1 - \frac{2\sqrt{d-1}}{d}. \)

5 Applications of Yang-type inequalities

In this section, we derive some applications of the Yang-type inequality on graphs.

Let \((V, c)\) be the network with the bottom of the spectrum \(\lambda_{\min}\). For any finite subset \(\Omega\), let \(\{\lambda_i\}_{i=1}^{||\Omega||}\) be the Dirichlet eigenvalues of the Laplace on \(\Omega\). Set
\[ \mu_i := \lambda_i - \lambda_{\min} \geq 0, \quad 1 \leq i \leq ||\Omega||. \]
(20)
By the trace of the Laplacian,
\[ \sum_{i=1}^{||\Omega||} \lambda_i \leq ||\Omega||. \]
Hence for any $1 \leq k \leq |\Omega|$, 
\[ \sum_{i=1}^{k} (1 - \lambda_i) \geq 0. \]

**Corollary 9** Suppose that the network $(V, c)$ satisfies the Yang-type inequality (3). Then for any finite subset $\Omega$, 
\[ \lambda_2 - \lambda_{\min} \leq \left( \frac{C_{YT}}{1 - \lambda_1} + 1 \right) (\lambda_1 - \lambda_{\min}). \]

**Proof.** This follows from the Yang-type inequality (3) for $k = 1$. $\square$

The Yang-type inequality implies the following result, which is a discrete analog of Yang’s second inequality.

**Corollary 10** Suppose that the network $(V, c)$ satisfies the Yang-type inequality (3). Then for any finite subset $\Omega$, if $\lambda_k \leq 1 + C_{YT}$ for some $1 \leq k < |\Omega|$, then 
\[ \lambda_{k+1} - \lambda_{\min} \leq \frac{\sum_{i=1}^{k} (\lambda_i - \lambda_{\min})(1 + C_{YT} - \lambda_i)}{\sum_{i=1}^{k} (1 - \lambda_i)}. \]

**Proof.** Let $C = C_{YT}$. Without loss of generality, we may assume that $\lambda_{k+1} > \lambda_1$, otherwise the result is trivial. By the Yang-type inequality (3), 
\[ \frac{1}{k} \sum_{i} (\mu_{k+1} - \mu_i)(\mu_{k+1} - \mu_i)(1 - \mu_i - \lambda_{\min}) - C_{\mu_i} \leq 0, \]
where $\{\mu_i\}_i$ is defined in (20) and $C = C_{YT}$. Set $a_i := \mu_{k+1} - \mu_i$ and 
\[ b_i := (\mu_{k+1} - \mu_i)(1 - \mu_i - \lambda_{\min}) - C_{\mu_i}. \]

Note that the function 
\[ f(x) := (\mu_{k+1} - x)(1 - x - \lambda_{\min}) - Cx \]
is non-increasing in $(-\infty, \frac{1}{2}(1 + C + \mu_{k+1} - \lambda_{\min})]$. Moreover, the assumption $\lambda_k \leq 1 + C$ yields that 
\[ \mu_i \leq \frac{1}{2}(1 + C + \mu_{k+1} - \lambda_{\min}), \]
which implies that $b_i$ is non-increasing. Using Chebyshev’s inequality, i.e. 
\[ \sum_{i} a_i b_i \geq k \sum_{i} a_i \sum_{i} b_i, \]
we have
\[ \left( \mu_{k+1} - \frac{1}{k} \sum_{i=1}^{k} \mu_i \right) \left[ \mu_{k+1} \cdot \frac{1}{k} \sum_{i=1}^{k} (1 - \lambda_i) - \frac{1}{k} \sum_{i=1}^{k} \mu_i (1 + C - \lambda_i) \right] \leq 0. \]

Note that by \( \lambda_{k+1} > \lambda_1 \),
\[ \lambda_{k+1} > \frac{1}{k} \sum_{i=1}^{k} \lambda_i. \]
Thus,
\[ \mu_{k+1} \leq \frac{\sum_{i=1}^{k} \mu_i (1 + C - \lambda_i)}{\sum_{i=1}^{k} (1 - \lambda_i)}, \]
which proves the theorem. \( \square \)

By the above result, we derive the following inequality, a discrete analog of the Hille-Protter inequality.

**Corollary 11** Suppose that the network \((V, c)\) satisfies the Yang-type inequality (3). Then for any finite subset \(\Omega\), if \(\lambda_k \leq 1 + C_{YT}\) for some \(1 \leq k < |\Omega|\), then
\[ \sum_{i=1}^{k} \frac{\lambda_i - \lambda_{\min}}{\lambda_{k+1} - \lambda_i} \geq \frac{1}{C_{YT}} \sum_{i=1}^{k} (1 - \lambda_i). \]

**Proof.** Without loss of generality, we may assume that \(\lambda_k < \lambda_{k+1}\). Let \(C = C_{YT}\). Set \(g(x) := \frac{x}{k_{k+1} - x}\), which is convex in \(x \in (-\infty, \mu_{k+1})\). Hence
\[
\frac{1}{k} \sum_{i=1}^{k} \frac{\lambda_i - \lambda_{\min}}{\lambda_{k+1} - \lambda_i} = \frac{1}{k} \sum_{i=1}^{k} \frac{\mu_i}{\mu_{k+1} - \lambda_i} = \frac{1}{k} \sum_{i=1}^{k} g(\mu_i) \geq g \left( \frac{1}{k} \sum_{i=1}^{k} \mu_i \right) = \frac{1}{k} \sum_{i=1}^{k} \mu_i, \]
where we used Jensen’s inequality for \(g(x)\). By Corollary 10,
\[ \mu_{k+1} \leq \frac{\sum_{i=1}^{k} \mu_i (1 + C - \lambda_i)}{\sum_{i=1}^{k} (1 - \lambda_i)} = \frac{C \sum_{i=1}^{k} \mu_i}{\sum_{i=1}^{k} (1 - \lambda_i)} + \frac{\sum_{i=1}^{k} \mu_i (1 - \lambda_i)}{\sum_{i=1}^{k} (1 - \lambda_i)} \leq \frac{C \sum_{i=1}^{k} \mu_i}{\sum_{i=1}^{k} (1 - \lambda_i)} + \frac{1}{k} \sum_{i=1}^{k} \mu_i, \]
where we used Chebyshev’s inequality in the last line.
By plugging it into (21), we prove the result.

This result yields a discrete analog of the Paley-Polya-Weinberger inequality.

\textbf{Corollary 12} Suppose that the network \((V, c)\) satisfies the Yang-type inequality (3). Then for any finite subset \(\Omega\), if \(\lambda_k \leq 1 + C_{YT}\) for some \(1 \leq k < |\Omega|\), then

\[ \lambda_{k+1} - \lambda_k \leq C_{YT} \frac{\sum_{i=1}^{k} (\lambda_i - \lambda_{\min})}{\sum_{i=1}^{k} (1 - \lambda_i)}. \]

\textit{Proof.} Without loss of generality, we assume that \(\lambda_k < \lambda_{k+1}\). By Corollary 11,

\[ \frac{\sum_{i=1}^{k} (\lambda_i - \lambda_{\min})}{\lambda_{k+1} - \lambda_k} \leq \sum_{i=1}^{k} \frac{\lambda_i - \lambda_{\min}}{\lambda_{k+1} - \lambda_i} \geq \frac{1}{C_{YT}} \sum_{i=1}^{k} (1 - \lambda_i), \]

which yields the result. \qed

We remark that for amenable groups, groups with Abelian quotients, and \(d\)-trees, the discrete analogs of the Paley-Polya-Weinberger inequality and the Hile-Protter inequality, as in Corollary 12 and Corollary 11 without the assumption that \(\lambda_k \leq 1 + C_{YT}\) for some \(1 \leq k < |\Omega|\), can be derived using same arguments in [HLS17, Theorem 1.1 and Theorem 1.3].

We recall a recursion formula proved by Cheng and Yang [CY07], see also [HLS17, Theorem 4.2].

\textbf{Proposition 13} Let \(a_1 \leq a_2 \leq \cdots \leq a_{k+1}\) be any positive numbers and \(\theta > 0\) such that

\[ (22) \quad \sum_{i=1}^{k} (a_{k+1} - a_i)^2 \leq \theta \sum_{i=1}^{k} a_i (a_{k+1} - a_i). \]

Define

\[ F_k = \left(1 + \frac{\theta}{2}\right) \left(\frac{1}{k} \sum_{i=1}^{k} a_i\right)^2 - \frac{1}{k} \sum_{i=1}^{k} a_i^2. \]

Then we have

\[ F_{k+1} \leq \left(\frac{k+1}{k}\right)^\theta F_k. \]

Now we prove an upper bound estimate for \(\lambda_k\).
Corollary 14. Suppose that the network \((V, c)\) satisfies the Yang-type inequality (3). Then for any finite subset \(\Omega\), if \(\lambda_k \leq 1 - \delta\) for some \(\delta > 0\), then
\[
\lambda_{k+1} - \lambda_{\text{min}} \leq (1 + \theta) k^{\theta} (\lambda_1 - \lambda_{\text{min}}),
\]
where \(\theta = \frac{1}{\delta} C_{YT}\).

Proof. Let \(\mu_i := \lambda_i - \lambda_{\text{min}}\). By the Yang-type inequality (3), we have
\[
\sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^2 (1 - \lambda_i) \leq C \sum_{i=1}^{k} (\mu_{k+1} - \mu_i) \mu_i,
\]
where \(C = C_{YT}\). Since \(\lambda_k \leq 1 - \delta\), \(1 - \lambda_i \geq \delta\) for any \(1 \leq i \leq k\). This yields that
\[
\sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^2 \leq \theta \sum_{i=1}^{k} (\mu_{k+1} - \mu_i) \mu_i,
\]
where \(\theta = \frac{C}{\delta}\). By the recursion formula in Proposition 13, setting \(a_i = \mu_i\),
\[
F_{k+1} \leq \left(\frac{k+1}{k}\right)^{\theta} F_k.
\]
Since the above result holds for all small \(k\), we have
\[
\frac{F_{k+1}}{(k+1)^{\theta}} \leq \frac{F_k}{k^{\theta}} \leq \cdots \leq F_1 = \frac{\theta}{2} a_1^2.
\]
By (24),
\[
\left( a_{k+1} - (1 + \frac{\theta}{2}) A_k \right)^2 \leq \left(1 + \frac{\theta}{2}\right)^2 A_k^2 - (1 + \theta) B_k = (1 + \theta) F_k - \frac{\theta}{2} (1 + \theta) A_k^2.
\]
This yields that
\[
\frac{\theta}{2(1 + \theta)} a_{k+1}^2 + (a_{k+1} - (1 + \theta) A_k)^2 \leq (1 + \theta) F_k.
\]
Hence
\[
a_{k+1}^2 \leq \frac{2(1 + \theta)^2}{\theta} F_k \leq (1 + \theta)^2 k^{\theta} a_1^2.
\]
This proves the result. \(\square\)

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