POISSON SPECTRA IN POLYNOMIAL ALGEBRAS

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Abstract. A significant class of Poisson brackets on the polynomial algebra \( \mathbb{C}[x_1, x_2, \ldots, x_n] \) is studied and, for this class of Poisson brackets, the Poisson prime ideals and Poisson primitive ideals are determined. Moreover it is established that these Poisson algebras satisfy the Poisson Dixmier-Moeglin equivalence.

In [9], the authors analyzed Poisson brackets on the polynomial algebra \( \mathbb{C}[x, y, z] \) in three indeterminates \( x, y, z \), including a class of Poisson brackets determined by Jacobians. In particular, for an arbitrary rational function \( s/t \in \mathbb{C}(x, y, z) \), they analyzed the prime and primitive Poisson ideals for the Poisson bracket such that, for \( f, g \in \mathbb{C}[x, y, z] \),

\[
\{f, g\} = t^2 \text{Jac}(f, g, s/t),
\]

where Jac denotes the Jacobian determinant.

The main purpose of this paper is to generalize the results in [9] to the general polynomial algebra \( A := \mathbb{C}[x_1, x_2, \ldots, x_n] \), \( n \geq 3 \), equipped with a Poisson bracket which is determined by \( n - 2 \) rational functions and which generalizes (0.1). As in [9], the results will be illustrated using particular examples.

Fix \( s_1, t_1, \ldots, s_{n-2}, t_{n-2} \in A \) such that \( s_i \) and \( t_i \neq 0 \) are coprime for each \( i = 1, 2, \ldots, n - 2 \). In Section 1 it is shown that there is a Poisson bracket on the quotient field \( B \) of \( A \) such that, for all \( f, g \in B \),

\[
\{f, g\} = (t_1 \ldots t_{n-2})^2 \text{Jac}(f, g, s_1/t_1, s_2/t_2, \ldots, s_{n-2}/t_{n-2}).
\]

The purpose of the factor \((t_1 \ldots t_{n-2})^2\) is to ensure that this restricts to a Poisson bracket on \( A \).

The Poisson prime ideals of \( A \) for the above bracket are determined in Section 2 where Definition 2.2 uses the terminology residually null, respectively proper, for Poisson prime ideals \( P \) where the induced Poisson bracket on \( A/P \) is zero, respectively non-zero. The residually null Poisson prime ideals of \( A \) form a Zariski closed set of the prime spectrum of \( A \) and can often be found explicitly using elementary commutative algebra. We shall determine the proper Poisson prime ideals of \( A \) in terms of a finite set of localizations \( A_\gamma \) of \( A \), each of which has a subalgebra \( C_\gamma \) that is a polynomial ring in \( n - 2 \) variables and is contained in the Poisson centre of \( A_\gamma \). As the Poisson bracket on \( C_\gamma \) is trivial, any prime ideal \( Q \) of \( C_\gamma \) is Poisson. Although \( QA_\gamma \) need not be prime, it is a Poisson ideal and the finitely many minimal prime ideals of \( A_\gamma \) over \( QA_\gamma \) are Poisson prime ideals of \( A_\gamma \). Taking the intersection of each of these with \( A \), we obtain finitely many Poisson...
prime ideals of $A$. The main result is that every proper Poisson prime ideal $P$ of $A_\gamma$ occurs in this way with $Q = P A_\gamma \cap C_\gamma$. The passage between Poisson prime ideals of $A_\gamma$ and those of $A$ can then be handled by standard localization techniques. This will be illustrated using examples with $n = 4$ in which case the algebras $C_\gamma$ are polynomial algebras in two indeterminates. The main example is the Poisson bracket associated with $2 \times 2$ quantum matrices with which the reader may be familiar. We also consider actions on $A$, as Poisson automorphisms, of subgroups of the multiplicative group $(\mathbb{C}^*)^n$.

In Section 3, we determine the Poisson primitive ideals of $A$ and show that $A$ satisfies the Poisson Dixmier-Moeglin equivalence discussed in [11, 2.4] and [6]. Here, as indeed is the case with the Poisson prime ideals, the varieties determined by $n - 2$ polynomials of the form $\lambda_i s_i - \mu_i t_i$, $i = 1, 2, \ldots, n - 2$, where $(\lambda_i, \mu_i) \in \mathbb{C}^2 \setminus \{(0,0)\}$ for all $i$, play an important role.

1. Poisson Brackets

**Definition 1.1.** A Poisson algebra is a $\mathbb{C}$-algebra $A$ with a Poisson bracket, that is a bilinear product $\{-,-\} : A \times A \rightarrow A$ such that $A$ is a Lie algebra under $\{-,-\}$ and, for all $a \in A$, the Hamiltonian $\text{ham}(a) := \{a,-\}$ is a $\mathbb{C}$-derivation of $A$.

**Notation 1.2.** Let $A$ denote the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ in $n$ indeterminates and let $B$ denote the quotient field $\mathbb{C}(x_1, \ldots, x_n)$ of $A$. For $1 \leq i \leq n$, let $\partial_i$ be the derivation $\frac{\partial}{\partial x_i}$ of $B$. For $b_1, b_2, \ldots, b_n \in B$, let $\text{Jac}_M(b_1, b_2, \ldots, b_n)$ denote the Jacobian matrix $(\partial_j(b_i))$ and let $\text{Jac}(b_1, b_2, \ldots, b_n)$ denote the Jacobian determinant $|\text{Jac}_M(b_1, b_2, \ldots, b_n)|$. Thus the $i$th row of $\text{Jac}_M(b_1, b_2, \ldots, b_n)$ is $\nabla(b_i)$ where $\nabla = (\partial_1, \partial_2, \ldots, \partial_n)$ is the gradient.

Let $a, f_1, f_2, f_3, \ldots, f_{n-2} \in B$ and, for $f, g \in B$, let

$$\{f, g\} = a \text{Jac}(f, g, f_1, f_2, \ldots, f_{n-2}). \tag{1.1}$$

Poisson brackets of this form, with $a = 1$, appear in the literature of mathematical physics, for example see [8] [15]. Our aim in this section is to give an algebraic proof that (1.1) defines a Poisson bracket on the rational function field $B$.

For an $(n-2) \times n$ matrix $M$ over $B$ and $1 \leq i < j \leq n$, let $M_{ij}$ be the $(n-2) \times (n-2)$ minor obtained by deleting columns $i$ and $j$ of $M$ and taking the determinant. Let $D$ be the $(n-2) \times n$ matrix with ith row $\nabla(f_i)$. Then

$$\{x_i, x_j\} = (-1)^{i+j-1}aD_{ij}.$$  

Also, if $u_1, u_2, \ldots, u_{n-2} \in B$ are such that $a = u_1 u_2 \ldots u_{n-2}$ then

$$\{x_i, x_j\} = (-1)^{i+j-1}E_{ij},$$

where $E$ is the $(n-2) \times n$ matrix with ith row $u_i \nabla(f_i)$.

**Lemma 1.3.** Let $1 \leq i \leq n$ and let $a, \phi_1, \phi_2, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_n \in B$. The map $\delta : B \rightarrow B$ given by

$$\delta(b) = a \text{Jac}(\phi_1, \phi_2, \ldots, \phi_{i-1}, b, \phi_{i+1}, \ldots, \phi_n)$$

is a derivation of $B$.

**Proof.** Denoting the $ij$-minor of the Jacobian matrix $\text{Jac}_M(\phi_1, \phi_2, \ldots, \phi_n)$ by $m_{ij}$,

$$\delta = a((-1)^{i+1}m_{i1}\partial_1 + (-1)^{i+2}m_{i2}\partial_2 + \cdots + (-1)^{i+n}m_{in}\partial_n).$$

Hence $\delta$ is a derivation of $B$. \qed
Theorem 1.4. Let $a, f_1, f_2, f_3, \ldots, f_{n-2} \in B$ and let $\{-, -\}$ be as in (1.1). Then $\{-, -\}$ is a Poisson bracket on $B$.

Proof. Applying Lemma 1.3 with $(\phi_1, \phi_2, \ldots, \phi_n) = (f, g, f_1, f_2, \ldots, f_{n-2})$ and $i = 1, 2$, we see that, if $\{-, -\}$ is defined as in (1.1), then $\{f, -\}$ and $\{-, g\}$ are derivations. Also $\{-, -\}$ is clearly antisymmetric so it remains to show that it satisfies the Jacobi identity.

We begin with the case $a = 1$ where we can exploit the $n$-Jacobi identity for the Jacobian (1.4). Given an ordered set $F = \{f_0, f_1, f_2, f_3, \ldots, f_{n-2}\}$ of $n - 1$ elements of $B$, define $\partial_F : B \to B$

$$\partial_F(h) = \text{Jac}(h, f_0, f_1, f_2, \ldots, f_{n-2}).$$

There is a minor difference here to (1.4) where $h$ appears in the rightmost argument. Note that $\partial_F(f_i) = 0$ for $0 \leq i \leq n - 2$. When $a = 1$, $\{g, f\} = \partial_F(g)$, where $F = \{f, f_1, f_2, f_3, \ldots, f_{n-2}\}$. The $n$-Jacobi identity for the Jacobian says that, for $h_1, h_2, \ldots, h_n \in B$,

$$\partial_F(\text{Jac}(h_1, h_2, \ldots, h_{n-1}, h_n)) = \text{Jac}(\partial_F(h_1), h_2, \ldots, h_{n-1}, h_n)$$

$$+ \text{Jac}(\partial_F(h_2), h_1, h_3, \ldots, h_{n-1}, h_n) + \cdots + \text{Jac}(h_1, h_2, \ldots, h_{n-1}, \partial_F(h_n)).$$

The proof of this in (1.4) is presented for the algebra of $C^\infty$-functions on a real manifold but it is valid for the rational function field $B$. It is first checked when each $h_i = x_i$ and then, using derivation properties as in the proof of (9, Proposition 1.14], extended first to the polynomial algebra and then to the rational function field.

Let $f, g, h \in B$. Then

$$\{\{g, h\}, f\} = \partial_F(\{g, h\}) = \partial_F(\text{Jac}(g, h, f_1, f_2, \ldots, f_{n-2}))$$

$$= \text{Jac}(\partial_F(g), h, f_1, f_2, \ldots, f_{n-2}) + \text{Jac}(g, \partial_F(h), f_1, f_2, \ldots, f_{n-2})$$

(by the $n$-Jacobi identity, the other summands being 0)

$$= \{\{g, f\}, h\} + \{g, \{h, f\}\} = -\{\{f, g\}, h\} - \{\{h, f\}, g\}.$$  

Thus $\{-, -\}$ satisfies the Jacobi identity and is a Poisson bracket on $B$.

Now let $a \in B$. We need to show that the bracket $a\{-, -\}$ satisfies the Jacobi identity. As

$$a\{f, a\{g, h\}\} = a^2\{f, \{g, h\}\} + a\{g, h\}\{f, a\}$$

and $\{-, -\}$ satisfies the Jacobi identity, it suffices to show that

$$\{g, h\}\{f, a\} + \{f, g\}\{h, a\} + \{h, f\}\{g, a\} = 0$$

for all $a, f, g, h \in B$. As $\{g, h\}\{f, -\} + \{f, g\}\{h, -\} + \{h, f\}\{g, -\}$ and the similar maps, where three of $a, f, g, h$ are fixed, are derivations, it suffices to show that

$$\{x_i, x_j\}\{x_k, x_\ell\} + \{x_k, x_i\}\{x_j, x_\ell\} + \{x_j, x_k\}\{x_i, x_\ell\} = 0$$

for $1 \leq i \leq j \leq k \leq \ell \leq n$. Clearly (1.2) holds when any two of $i, j, k, \ell$ are equal so we may assume that $i < j < k < \ell$. In this case (1.2) is, using Notation (1.2)

$$D_{ij}D_{k\ell} - D_{ik}D_{j\ell} + D_{jk}D_{i\ell} = 0.$$  

This is a Plücker relation for the $(n - 2) \times n$ matrix $D$, see (11, Theorem 1.3), or (11, Chapter VII §6], where Plücker relations are called $p$-relations. Indeed it is one
of the three-term Plücker relations stated explicitly in [7] foot of p311]. In the
notation of [7], where subscripts indicate included rather than excluded rows, it is
\[ p_{i_1 \ldots i_{n-4} j} + p_{i_1 \ldots i_{n-4} k} p_{i_1 \ldots i_{n-4} k} + p_{i_1 \ldots i_{n-4} k} i \in_{i_{n-4} k} = 0 , \]
where \( \{ i_1 , \ldots , i_{n-4} \} = \{ 1 , 2 , 3 , \ldots , n \} \backslash \{ i , j , k , \ell \} . \)
\[ \square \]

**Theorem 1.5.** If \( f_1 , f_2 , \ldots , f_{n-2} \) are algebraically dependent over \( \mathbb{C} \) then \( \{ - , - \} = 0 . \)

**Proof.** Let \( 0 \neq G = G( y_1 , \ldots , y_{n-2} ) \in \mathbb{C}[ y_1 , \ldots , y_{n-2} ] \) be of minimal total degree such that \( G( f_1 , f_2 , \ldots , f_{n-2} ) = 0 . \) Without loss of generality, we may assume that the
degree in \( y_1 \) of \( G \) is at least one. Let
\[
G = \sum_{r= ( r_1 , \ldots , r_{n-2} )} \alpha_r y_1^{r_1} y_2^{r_2} \cdots y_{n-2}^{r_{n-2}} .
\]
Let \( f , g \in B \) and let \( \delta \) be the derivation in Lemma [33] in the case where \( i = 3 , \)
\( \phi_1 = f , \phi_2 = g \) and, for \( 4 \leq j \leq n , \phi_j = f_{j-2} . \) Then \( \delta ( f_k ) = 0 \) for \( 2 \leq k \leq n-2 , \)
wheras \( \delta ( f_{1} ) = \{ f , g \} . \) Then
\[
0 = \delta ( G( f_1 , f_2 , \ldots , f_{n-2} ) ) = \delta \left( \sum_{r= ( r_1 , \ldots , r_{n-2} )} \alpha_r f_1^{r_1} f_2^{r_2} \cdots f_{n-2}^{r_{n-2}} \right) = \left( \sum_{r} r_1 \alpha_r f_1^{r_1-1} f_2^{r_2} \cdots f_{n-2}^{r_{n-2}} \right) \{ f , g \} .
\]
By the minimality of \( G , \sum_{r} r_1 \alpha_r f_1^{r_1-1} f_2^{r_2} \cdots f_{n-2}^{r_{n-2}} \neq 0 \) so \( \{ f , g \} = 0 . \)

\[ \square \]

2. **Poisson spectra**

The following definitions and the claims made for them are well-known. Appropriate
references include [3] [6].

**Definitions 2.1.** Let \( A \) be a Poisson algebra with bracket \( \{ - , - \} . \) The Poisson centre of \( A , \) denoted \( PZ(A) , \) of \( A \) is \( \{ a \in A : \{ a , r \} = 0 \) for all \( r \in A \} . \)

An ideal \( I \) of \( A \) is a Poisson ideal if \( \{ i , r \} \subset I \) for all \( i \in I \) and \( r \in A . \) If \( I \) is a
Poisson ideal of \( A \) then \( A/I \) is a Poisson algebra with \( \{ a + I , b + I \} = \{ a , b \} + I \) for
all \( a , b \in A . \) A Poisson ideal \( P \) of \( A \) is Poisson prime if, for all Poisson ideals \( I \) and
\( J \) of \( A , \) \( I J \subset P \) implies \( I \subset P \) or \( J \subset P . \) If \( A \) is Noetherian then this is equivalent
to saying that \( P \) is both a prime ideal and a Poisson ideal. The Poisson spectrum
of \( A , \) written \( \operatorname{Spec} A , \) is the set of all Poisson prime ideals of \( A . \) A maximal ideal
\( M \) of \( A \) is said to be a Poisson maximal ideal if it is also a Poisson ideal. This is
not equivalent to saying that \( M \) is maximal as a Poisson ideal.

The Poisson core of an ideal \( I \) of \( A , \) denoted \( P(I) , \) is the largest Poisson ideal of
\( A \) contained in \( I . \) If \( P \) is a prime ideal of \( A \) then \( P(I) \) is Poisson prime. A Poisson
ideal \( P \) of \( A \) is Poisson primitive if \( P = P(M) \) for some maximal ideal \( M \) of \( A . \)
Every Poisson primitive ideal is Poisson prime.

If \( S \) is a multiplicatively closed subset of a Poisson algebra \( A \) then the localization
\( A_S \) is also a Poisson algebra, with \( \{ as^{-1} , br^{-1} \} \) computed using the quotient rule
for derivations. If \( P \) is a Poisson prime ideal of \( A \) then the quotient field \( Q(A/P) \) is
a Poisson algebra and \( P \) is said to be rational if \( PZ(Q(A/P)) = \mathbb{C} . \) For a Poisson
prime ideal $P$ of an affine Poisson algebra $A$, there is, by [11] 1.7(i) and 1.10, a sequence of implications:

$$P$$ is locally closed $\Rightarrow$ $P$ is Poisson primitive $\Rightarrow$ $P$ is rational.

To establish the Poisson Dixmier-Moeglin equivalence, it is enough to show that if $P$ is a rational Poisson prime ideal of $A$ then $P$ is locally closed. For further discussion of this, see [11, 9].

A $\mathbb{C}$-algebra automorphism $\theta$ of a Poisson algebra $A$ is a Poisson automorphism of $A$ if $\theta(\{a,b\}) = \{\theta(a),\theta(b)\}$ for all $a,b \in A$ and is a Poisson anti-automorphism of $A$ if $\theta(\{a,b\}) = \{\theta(b),\theta(a)\}$ for all $a,b \in A$. Under composition, the set of all Poisson automorphisms and Poisson anti-automorphisms of $A$ is a group in which the Poisson automorphisms form a normal subgroup of index 2.

The height of a prime ideal $P$ of $A$ will be denoted $\text{ht} P$.

**Definition 2.2.** Let $A$ be a Poisson algebra and $I$ be a Poisson ideal of $A$. Following [9, Definition 1.8], we say that $I$ is residually null if the induced Poisson bracket on $A/I$ is zero. This is equivalent to saying that $I$ contains all elements of the form $\{a,b\}$ where $a,b \in A$, or that $I$ contains all such elements where $a,b \in G$ for some generating set $G$ for $A$. We shall also say that a Poisson ideal is a proper Poisson ideal if it is not residually null.

**Lemma 2.3.** Let $A$ be a Poisson algebra.

1. Every residually null Poisson primitive ideal $P$ is a Poisson maximal ideal.
2. A Poisson algebra $A$ is Poisson simple if and only if there does not exist a nonzero Poisson primitive ideal of $A$.
3. Let $A = \mathbb{C}[x_1,\ldots,x_n]$ be a polynomial algebra with a Poisson bracket. Let $P$ be a proper Poisson prime ideal in $A$ of height $\geq n - 2$. Then $P$ is locally closed and Poisson primitive.

**Proof.** (1) Let $M$ be a maximal ideal of $A$ such that $P = \mathcal{P}(M)$. Suppose that $P$ is residually null. Then $M$ is Poisson and $P = \mathcal{P}(M) = M$ is maximal.

(2) The “only if” part is clear. For the converse, suppose that $A$ is not simple, let $I$ be a proper ideal of $A$ that is Poisson and let $M$ be a maximal ideal of $A$ containing $I$. Then $\mathcal{P}(M)$ is Poisson primitive and $0 \not\subset \mathcal{P}(M)$.

(3) Since $P$ is proper, $\{x_k,x_j\} \notin P$ for some pair $k,j$. Let $Q$ be a Poisson prime ideal containing $P$ properly. Then $\text{ht} Q > n - 2$ and hence $Q$ is residually null by [9, Proposition 3.2]. It follows that $\{x_k,x_j\} \in Q$. Thus $P$ is locally closed and hence, by [11, 1.7(i)], $P$ is Poisson primitive.

**Notation 2.4.** For the remainder of the paper, let $A = \mathbb{C}[x_1,\ldots,x_n]$ and $B = \mathbb{C}(x_1,\ldots,x_n)$, where $n \geq 3$. Let $s_1,t_1,\ldots,s_{n-2},t_{n-2} \in A$ be such that $s_i$ and $t_j \neq 0$ are coprime for $1 \leq i \leq n-2$. Let $f_i = s_it_i^{-1} \in B$, $1 \leq i \leq n-2$, and let $a = t_1^2 t_2^2 \ldots t_{n-2}^2$. By Theorem 1.4 there is a Poisson bracket on $B$ such that $\{f,g\} = \text{Jac}(f,g,f_1,f_2,\ldots,f_{n-2})$ for all $f,g \in B$. Thus $\{f,g\} = \text{det} J$, where $J$ is the $n \times n$ matrix with first row $\nabla f$, second row $\nabla g$ and, for $3 \leq i \leq n$, ith row $t_{i-2}^2 \nabla(s_{i-2}t_{i-2})$. In the notation of 1.2 with each $u_i = t_i^2$,

$$\{x_i,x_j\} = (-1)^{i+j-1} a D_{ij} = (-1)^{i+j-1} E_{ij}.$$

Note that $t_i^2 \partial_j(s_i t_i^{-1}) \in A$ for $1 \leq i \leq n-2$ and $1 \leq j \leq n$. It follows that $\{f,g\} \in A$ for all $f,g \in A$ and hence that $A$ is a Poisson subalgebra of $B$. 
If $f_1, \ldots, f_{n-2}$ are algebraically dependent over $\mathbb{C}$ then the Poisson bracket \(\{-,-\} = 0\), by Theorem \ref{thm:dependence}, so, henceforth, we shall assume that $f_1, \ldots, f_{n-2}$ are algebraically independent over $\mathbb{C}$.

**Example 2.5.** Let $n = 4$ and let
\[
s_1 = x_1x_4 - x_2x_3, \quad t_1 = 1, \quad s_2 = x_2, \quad t_2 = x_3.
\]
Then, in the notation of 1.2,
\[
E = \begin{pmatrix}
\left(t_1^2 \nabla (f_1)\right) & 0 & 0 & 0 \\
0 & \left(t_2^2 \nabla (f_2)\right) & x_3 & -x_2
\end{pmatrix}
\]
and the resulting Poisson bracket on $\mathbb{C}[x_1, x_2, x_3, x_4]$ is such that:
\[
\{x_1, x_2\} = x_1x_2, \quad \{x_1, x_3\} = x_1x_3, \quad \{x_1, x_4\} = 2x_2x_3,
\]
\[
\{x_2, x_3\} = 0, \quad \{x_2, x_4\} = x_2x_4, \quad \{x_3, x_4\} = x_3x_4.
\]
This is the well-known Poisson bracket associated with $2 \times 2$ quantum matrices, see [11. 2.9]. This example will be used to illustrate our methods and results.

**Example 2.6.** Let $n = 4$ and let $s_1 = x_1 + x_2 + x_3 + x_4$, $s_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$ and $t_1 = t_2 = 1$. In the notation of 1.2
\[
E = \begin{pmatrix}
1 & 1 & 1 & 1 \\
x_2 + x_3 + x_4 & x_1 + x_3 + x_4 & x_1 + x_2 + x_4 & x_1 + x_2 + x_3
\end{pmatrix}
\]
and, for the resulting Poisson bracket on $\mathbb{C}[x_1, x_2, x_3, x_4],
\[
\{x_1, x_2\} = x_3 - x_4, \quad \{x_1, x_3\} = x_4 - x_2, \quad \{x_1, x_4\} = x_2 - x_3,
\]
\[
\{x_2, x_3\} = x_1 - x_4, \quad \{x_2, x_4\} = x_3 - x_1, \quad \{x_3, x_4\} = x_1 - x_2.
\]
Here the elementary symmetric polynomials $s_1$ and $s_2$ are Poisson central. The prime ideal generated by $x_1 - x_2$, $x_1 - x_3$ and $x_1 - x_4$ is residually null Poisson as are all the maximal ideals of the form $(x_1 - \lambda, x_2 - \lambda, x_3 - \lambda, x_4 - \lambda)$.

As $\{x_1, x_2\}$ is homogeneous of degree one, the Poisson bracket here is the Kirillov-Kostant-Souriau bracket, [11. III.5.5], for a 4-dimensional Lie algebra $\mathfrak{g}$ in which $z := x_1 + x_2 + x_3 + x_4$ is central. If $\mathfrak{s} = \mathfrak{g}/\mathbb{C}z$ then it is a routine calculation to check that $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ whence, by [3. 3.2.4], $\mathfrak{s} \simeq \mathfrak{sl}_2$.

**Examples 2.7.** The examples in 2.5 and 2.6 exhibit very different symmetry properties. In Example 2.6 there is alternating symmetry in the following sense. For each $\alpha \in S_4$, there are $C$-automorphisms $\phi_\alpha$ and $\theta_\alpha$ of $A$ such that, for $1 \leq i \leq 4$,
\[
\phi_\alpha(x_i) = x_{\alpha(i)} \quad \text{and} \quad \theta_\alpha(x_i) = (-1)^{\text{sgn} \alpha} x_{\alpha(i)}.
\]
Then $\theta_\alpha$ is a Poisson automorphism. It is enough to check this for the generators $(1 2)$ and $(1 2 3 4)$ of $S_4$, for which
\[
\theta_{(1 2)} : x_1 \mapsto -x_2, \quad x_2 \mapsto -x_1, \quad x_3 \mapsto -x_3, \quad x_4 \mapsto -x_4
\]
and
\[
\theta_{(1 2 3 4)} : x_1 \mapsto -x_2, \quad x_2 \mapsto -x_3, \quad x_3 \mapsto -x_4, \quad x_4 \mapsto -x_1.
\]
Note that $\theta_{\alpha}(s_2) = s_2$ for all $\alpha \in S_4$, whereas $\theta_{\alpha}(s_1) = \text{sgn} \alpha s_1$. If $\alpha$ is even then $\phi_\alpha = \theta_\alpha$ and if $\alpha$ is odd then $\phi_\alpha$ is a Poisson anti-automorphism.

In Example 2.5 there is a well-known action of the group
\[
H := \{(h_1, h_2, h_3, h_4) \in (\mathbb{C}^*)^4 : h_1h_4 = h_2h_3\}
\]
on $A = \mathbb{C}[x_1, x_2, x_3, x_4]$, acting as Poisson automorphisms with $x_i \mapsto h_1x_i$ for $1 \leq i \leq n$. If, in [11. 1.2], we take $A^P_{2,1} = \mathbb{C}[y_1, x_1, y_2, x_2]$ with $q_1 = q_2 = \ldots = 2$. 
0, \rho_1 = p_2 = -2 and \gamma_{12} = -1 then \(A_{2,1}^{p,q}\) is Poisson isomorphic to \(A\) via the map 
\(y_1 \mapsto (x_2, x_1, y_2) \mapsto (x_1, x_2, x_4)\). The above action of \(H\) on \(A_{2,1}^{p,q}\) corresponds to the action of \(H\) on \(A_{2,1}^{p,q}\) specified in [13] and, by [13, 2.6], the \(H\)-prime Poisson ideals of \(A\) are as follows:

\[
0, \\
x_2A, \quad x_3A, \quad DA, \\
x_1A + x_2A, \quad x_2A + x_4A, \quad x_2A + x_3A, \quad x_1A + x_3A, \quad x_3A + x_4A, \\
x_1A + x_2A + x_3A, \quad x_1A + x_2A + x_4A, \quad x_1A + x_3A + x_4A, \quad x_2A + x_3A + x_4A, \\
x_1A + x_2A + x_3A + x_4A,
\]

where \(D = x_1x_4 - x_2x_3\), the determinant.

This is a special case of a general situation. The multiplicative group \((\mathbb{C}^*)^n\) acts, as algebra automorphisms, on \(A\) by the rule

\[
(h_1, \ldots, h_n), f = f(h_1x_1, \ldots, h_nx_n).
\]

With \(E_{ij}\) as in Notation 1, let \(H\) be the subgroup

\[
\{(h_1, \ldots, h_n) : h_i E_{ij} = h_i h_j E_{ij} \text{ for } 1 \leq i, j \leq n\}.
\]

If \(h = (h_1, \ldots, h_n) \in H\) then, for \(1 \leq i, j \leq n\),

\[
\{h.x, h.x\} = h_h h_j \{x_i, x_j\} = h_h h_j (h^{-1}i^{-1+j-j}E_{ij} = (-1)^{i+j-1}h_i h_j E_{ij} = h_i \{x_i, x_j\}.
\]

Thus \(H\) acts on \(A\) by Poisson automorphisms. In Example 2.5, \(H \simeq (\mathbb{C}^*)^3\), the 3-torus. Note that \(H\) might be trivial.

**Notation 2.8.** Consider the group

\[
H' = \{h \in (\mathbb{C}^*)^n : h.s_i \in \mathbb{C}s_i \text{ and } h.t_i \in \mathbb{C}t_i \text{ for all } i\}.
\]

This group is readily calculated from the data and its elements sometimes, but not always, act on \(A\) as Poisson automorphisms. In Example 2.5, all elements of \(H'\) act as Poisson automorphisms. However, in Example 2.6 where \(H' = \{(h_1, h_1, h_1, h_1)\}\), only \((1, 1, 1, 1)\) acts as a Poisson automorphism. For \(1 \leq i \leq n - 2\), let \(\sigma_i : H' \to \mathbb{C}\) and \(\tau_i : H' \to \mathbb{C}\) be such that \(h.s_i = \sigma_i(h)s_i\) and \(h.t_i = \tau_i(h)t_i\). Let \(\rho : H' \to \mathbb{C}\) be such that, for \(h \in H', \rho(h) = \sigma_1(h)\tau_1(h)\sigma_{n-2}(h)\tau_{n-2}(h)\). The next result gives a criterion for an element of \(H'\) to act as a Poisson automorphism of \(A\).

**Proposition 2.9.** Let \(h = (h_1, h_2, \ldots, h_n) \in H'\). Then \(h\) acts as a Poisson automorphism if and only if \(\rho(h) = h_1 \ldots h_n\).

**Proof.** For all \(g \in A\) and \(1 \leq i \leq n - 2\), \(h_\partial_j(g) = h_j^{-1}\partial_j(h.g)\) so, for \(1 \leq j \leq n\), \(h.\partial_j^2(s_i/t_i) = h_j^{-1}\partial_j(h.t_i) = h_j^{-1}\sigma_i(h)\tau_i(h)\partial_j(s_i/t_i)\). In other words, when \(h\) acts on the \((n-2) \times n\) matrix \(E\) whose rows are \(t_i \nabla(s_i/t_i)\), the \(j\)th column gets multiplied by \(h_j^{-1}\) and the \(i\)th row by \(\sigma_i(h)\tau_i(h)\). It follows that, for \(1 \leq k, \ell \leq n\),

\[
h_i \{x_k, x_\ell\} = h_i^{-1}h_k h_\ell \rho(h) \{x_k, x_\ell\},
\]

As \(\{h.x, h.x\} = h_k h_\ell \{x_k, x_\ell\}\), the result follows. \(\square\)

**Examples 2.10.** Proposition 2.9 is nicely illustrated in Examples 2.5 and 2.6. In Example 2.5 for \(h = (h_1, h_2, h_3, h_1^{-1}h_2 h_3) \in H', \sigma_1(h) = h_1 h_4 = h_2 h_3, \sigma_2(h) = h_2, \tau_1(h) = 1\) and \(\tau_2(h) = h_3\) so \(\rho(h) = h_1 h_2 h_2 h_3 h_4\) for all \(h \in H'.\) Here \(H' = H'.\) In Example 2.6 for \(h = (h_1, h_1, h_1, h_1) \in H', \sigma_1(h) = h_1, \sigma_2(h) = h_1^2, \tau_1(h) = 1\) and \(\tau_2(h) = 1\) so, unless \(h_1 = 1\), \(\rho(h) = h_1^3 \neq h_1 h_2 h_3 h_4 = h_1^4\).
Lemma 2.11. Let $R$ be a commutative noetherian $\mathbb{C}$-algebra that is a domain and let $\delta$ be a $\mathbb{C}$-derivation of $R$. Let $K$ denote the subring of constants, that is $K = \{ r \in R : \delta(r) = 0 \}$. Then $K$ is algebraically closed in $R$.

Proof. The proof is essentially the same as that of [9, Lemma 3.1] but with the word ‘algebraic’ replacing the word ‘integral’ and inserting a leading coefficient $k_n$ that need not be 1. □

Lemma 2.12. Let $P$ be a proper Poisson prime ideal of $A$. Then $s_i \notin P$ or $t_i \notin P$ for each $i = 1, \ldots, n - 2$.

Proof. If $s_i \in P$ and $t_i \in P$ for some $i$ then $P$ is residually null since $t_i^2 \nabla s_i = t_i \nabla s_i - s_i \nabla t_i \in P$. □

The proof of our main result, Theorem 2.19, involves the relationship between transcendence degree and heights of prime ideals.

Notation 2.13. Let $K$ be a field, $A$ be an integral domain which is also an affine $K$-algebra, $Q(A)$ be the field of quotients of $A$ and $L$ be a field extension of $K$. We shall denote the Krull dimension of an affine $K$-algebra $A$ by $\dim(A)$ and the transcendence degree of $L$ by $\deg_K(L)$. Following [16, Chapter 6] we extend the latter notation to $A$ by taking $\deg_K(A)$ to be the number of elements in any maximal algebraically independent set of elements in $A$. By [17, Corollary 14.29] and [16, Theorem 6.35], $\deg_K(Q(A)) = \dim(A) = \deg_K(A)$. Note also that any algebraically independent set of elements in $A$ can be extended to a maximally algebraically independent set in $A$, see [16, Example 6.4 and Remark 6.6]. We shall simply write $\deg(A)$ for $\deg_K(A)$ if no confusion arises. By [17, Corollary 14.32],

$$\text{ht}(P) + \dim(A/P) = \dim(A).$$

Notation 2.14. Let $\Gamma$ be the set of all sequences $\gamma = (\gamma_1, \delta_1), \ldots, (\gamma_{n-2}, \delta_{n-2})$, of length $n - 2$, in $\{0, 1\} \times \{0, 1\}$. Call an element $\gamma$ of $\Gamma$ dense if, for each $i = 1, \ldots, n - 2$, $(\gamma_i, \delta_i) \neq (0, 0)$. To each $\gamma \in \Gamma$, we associate a finite subset $S_\gamma$ of $\{s_1, \ldots, s_{n-2}, t_1, \ldots, t_{n-2}\}$, a finite subset $V_\gamma = \{v_1, \ldots, v_{n-2}\}$ of $B$, a multiplicatively closed subset $M_\gamma$ of $A$ and a localization $A_\gamma$ of $A$ as follows:

$$s_i \in S_\gamma \iff \gamma_i = 1 \quad \text{and} \quad t_i \in S_\gamma \iff \delta_i = 1,$$

$$v_i = \begin{cases} s_i/t_i & \text{if } \delta_i = 1 \\ t_i/s_i & \text{otherwise,} \end{cases}$$

$M_\gamma$ is the multiplicative closed subset of $A$ generated by the elements of $S_\gamma$, and $A_\gamma$ is the localization $M_\gamma^{-1}A$. Note that if $\gamma$ is dense then $v_i \in A_\gamma$ for each $i = 1, \ldots, n - 2$. In this case, denote by $C_\gamma$ the subalgebra of $A_\gamma$ generated by $v_1, \ldots, v_{n-2}$. Since $s_1/t_1, \ldots, s_{n-2}/t_{n-2}$ are algebraically independent over $\mathbb{C}$, the transcendence degree of $C_\gamma$ is $n - 2$.

For example, in Example 2.5 if $\gamma = (0, 1)$, then $S_\gamma = \{t_1, s_2\} = \{1, x_2\}$ and $v_\gamma = \{s_1/t_1, t_2/s_2\} = \{x_1x_4 - x_2x_3, x_3/x_2\}$.

Notation 2.15. For $P \in \text{Spec}(A)$, let $\gamma(P) = (\gamma_1, \delta_1), \ldots, (\gamma_{n-2}, \delta_{n-2})$ be the sequence such that, for each $i$, $\gamma_i = 0 \iff s_i \in P$ and $\delta_i = 0 \iff t_i \in P$. For example, in Example 2.5 if $P = x_1A + x_3A$ then $\gamma(P) = (0, 1)$, $S_{\gamma(P)} = \{s_2, t_1\} = \{x_2, 1\}$ and $V_{\gamma(P)} = \{s_1/t_1, t_2/s_2\} = \{x_1x_4 - x_2x_3, x_3/x_2\}$. 
The next lemma amounts to observing some restrictions on $\gamma(P)$.

**Lemma 2.16.** Let $P$ be a Poisson prime ideal of $A$.

(1) If $P$ is proper then $\gamma(P)$ is dense.

(2) If $t_i = 1$ for some $i$, then, in $\gamma(P)$, $\delta_i = 1$ and, in $V_{\gamma(P)}$, $v_i = s_i$.

**Proof.** (1) holds because, by Lemma 2.12, we cannot have $s_i \in P$ and $t_i \in P$ for any $i$ and (2) holds because $t_i \notin P$. □

**Remark 2.17.** The converse to (1) is false as can be seen from Example 2.5 where, for the residually null Poisson prime ideal $P = x_1 A + x_2 A + x_4 A$, $\gamma(P) = \{(0, 1), (0, 1)\}$ is dense.

**Notation 2.18.** The Poisson spectrum $P.\text{Spec} A$ can be partitioned using $\Gamma$. For $\gamma \in \Gamma$, let

$$ P.\text{Spec}_\gamma A = \{ P \in P.\text{Spec} A | S_\gamma = S_{\gamma(P)} \}. $$

The set $P.\text{Spec}_\gamma A$ may be empty. Indeed, by Lemma 2.16(2), if $t_i = 1$ for some $i$ then $P.\text{Spec}_\gamma A = \emptyset$ whenever $\delta_i = 0$.

Our strategy in attempting to understand $P.\text{Spec} A$ is based on the following:

(1) $P.\text{Spec} A$ is the disjoint union of the subsets $P.\text{Spec}_\gamma A$ taken over $\gamma \in \Gamma$.

(2) By standard localization theory, if $P \in P.\text{Spec}_\gamma A$ then $PA_\gamma$ is a Poisson prime ideal of $A_\gamma$ and $P = A \cap PA_\gamma$.

(3) When $\gamma$ is dense, Theorem 2.19 below determines the Poisson prime ideals of $A_\gamma$ in terms of prime ideals of the polynomial algebra $C_\gamma$.

(4) When $\gamma$ is not dense, every Poisson prime ideal in $P.\text{Spec}_\gamma A$ is residually null.

The next result determines the proper Poisson prime ideals in $A_\gamma$ when $\gamma$ is dense.

**Theorem 2.19.** Let $\gamma \in \Gamma$ be dense.

(1) Let $I$ be an ideal of $C_\gamma$ and let $Q$ be a prime ideal of $A_\gamma$ that is minimal over $IA_\gamma$. Then $Q$ is a Poisson prime ideal of $A_\gamma$.

(2) If $Q$ is a nonzero proper Poisson prime ideal of $A_\gamma$ then $\text{ht}(Q) = \text{ht}(Q \cap C_\gamma)$.

(3) If $Q$ is a nonzero proper Poisson prime ideal of $A_\gamma$ then $Q$ is a minimal prime ideal over $(Q \cap C_\gamma)A_\gamma$.

**Proof.** (1) Since $v_i = s_i/t_i$ or $v_i = t_i/s_i$ and $\nabla_{t_i} = -s_i^{-2}(t_i^2 \nabla_{s_i})$, the subalgebra $C_\gamma$ is contained in the Poisson centre of $A_\gamma$. Hence $IA_\gamma$ is a Poisson ideal of $A_\gamma$ and, by [12, 1.4], every prime ideal of $A_\gamma$ minimal over $IA_\gamma$ is Poisson.

(2) By Noether’s Normalization Theorem, as stated in [17, 14.14], there exist non-negative integers $m, d$, with $d \leq m$, and $y_1, \ldots, y_m \in C_\gamma$ such that $y_1, \ldots, y_m$ are algebraically independent over $\mathbb{C}$, $C_\gamma$ is integral over $\mathbb{C}[y_1, \ldots, y_m]$ and $(Q \cap C_\gamma) \cap \mathbb{C}[y_1, \ldots, y_m] = \sum_{i=d+1}^m \mathbb{C}[y_1, \ldots, y_m]y_i$. Note that $m = \text{tr. deg}_C(C_\gamma) = n - 2$ and $d = \text{tr. deg}_{C_\gamma}(C_\gamma/Q \cap C_\gamma)$. The algebraically independent subset $\{y_1, \ldots, y_{n-2}\}$ of $A_\gamma$ can be extended to a maximal algebraically independent subset $\{y_1, \ldots, y_{n-2}, z_1, z_2\}$ of $A_\gamma$. Thus $A_\gamma$ is algebraic over $\mathbb{C}[y_1, \ldots, y_{n-2}, z_1, z_2]$. As $y_{d+1}, \ldots, y_{n-2} \in Q$, $A_\gamma/Q$ is algebraic over $\mathbb{C}[y_1 + Q, \ldots, y_d + Q, z_1 + Q, z_2 + Q]$. It follows that

$$ \text{tr. deg}(A_\gamma/Q) \leq d + 2 = \text{tr. deg}(C_\gamma/Q \cap C_\gamma) + 2, $$

and, from Notation 2.13 that

$$ \text{ht}(Q) \geq \text{ht}(Q \cap C_\gamma). $$

(2.1)
Now suppose that \( \text{ht}(Q) > \text{ht}(Q \cap C_\gamma) \). Then \( \text{tr. deg}(A_\gamma/Q) \leq \text{tr. deg}(C_\gamma/Q \cap C_\gamma) + 1 \). Let \( T \) be the set of all nonzero elements of the integral domain \( C_\gamma/Q \cap C_\gamma \) and let \( \gamma = T^{-1}(C_\gamma/Q \cap C_\gamma) \) be the quotient field of \( C_\gamma/Q \cap C_\gamma \). Thus \( T^{-1}(A_\gamma/Q) \) is an affine \( K \)-algebra. Let \( L \) be the quotient field of \( A_\gamma/Q \) which is also the quotient field of \( T^{-1}(A_\gamma/Q) \). Then \( \text{tr. deg}_L(L) \leq \text{tr. deg}_L(K) + 1 \), by [17, 12.56], \( \text{tr. deg}_K L \leq 1 \). Hence there exists \( w \in T^{-1}(A_\gamma/Q) \) such that \( T^{-1}(A_\gamma/Q) \) is algebraic over \( K[w] \). Moreover \( C_\gamma \) is contained in the Poisson centre of \( A_\gamma \), whence the Poisson bracket in \( K[w] \) is trivial. The constant subring of the hamiltonian \( \text{ham} w \) contains \( K[w] \) and, by Lemma 2.11, it contains \( T^{-1}(A_\gamma/Q) \). Hence, for any \( b \in T^{-1}(A_\gamma/Q) \), the constant subring of \( \text{ham} b \) contains \( K[w] \), and, again by Lemma 2.11, \( \text{ham} b = 0 \). Thus the Poisson bracket in \( T^{-1}(A_\gamma/Q) \) is trivial, which is impossible since \( Q \) is proper. Therefore we must have equality in (2.1), that is, \( \text{ht}(Q) = \text{ht}(Q \cap C_\gamma) \).

(3) Let \( Q' \) be a minimal prime ideal of \((Q \cap C_\gamma)A_\gamma \) such that \( Q' \subseteq Q \). Then \( Q' \) is Poisson prime, by (1), and is proper. Suppose that \( Q' \neq Q \). Then \( \text{ht}(Q') < \text{ht}(Q) \) and

\[
\text{ht}(Q') = \text{ht}(Q' \cap C_\gamma) = \text{ht}(Q \cap C_\gamma) = \text{ht}(Q),
\]
a clear contradiction. \( \square \)

**Remark 2.20.** Let \( P \) be a proper Poisson prime ideal of \( A \) and let \( \gamma = \gamma(P) \). By Theorem 2.19, \( PA_\gamma \) is a minimal prime ideal over \( PA_\gamma \cap C_\gamma \) and \( \text{ht}(P) = \text{ht}(PA_\gamma) = \text{ht}(PA_\gamma \cap C_\gamma) \). Denote by \( \text{Pht}(P) \) the maximal length \( \ell \) of a chain of distinct Poisson prime ideals

\[
0 = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_\ell = P
\]
of \( A \). Clearly \( \text{Pht}(P) \leq \text{ht}(P) \). But, if \( j = \text{ht} P = \text{ht} PA_\gamma = \text{ht}(PA_\gamma \cap C_\gamma) \) then a chain of prime ideals of \( C_\gamma \) of length \( j \) inside \( PA_\gamma \cap C_\gamma \) gives rise to a chain of Poisson prime ideals of \( A \) of length \( j \) inside \( P \) so \( \text{Pht}(P) \geq \text{ht}(PA_\gamma \cap C_\gamma) = \text{ht}(P) \), whence \( \text{Pht}(P) = \text{ht}(P) \). There are many examples of residually null Poisson prime ideals \( P \) for which \( \text{Pht}(P) < \text{ht}(P) \), for example the symmetric algebra of \( sl_2 \) with the Kirillov-Kostant-Souriau bracket, where the unique Poisson maximal ideal \( M \) has \( \text{Pht}(M) = 2 \) and \( \text{ht}(M) = 3 \), see [9] Example 4.1. For an example where \( \text{Pht}(P) < \text{ht}(P) \) and \( P \) is not residually prime, see [10] Remark 5.13.

**Example 2.21.** To illustrate Theorem 2.19 and the strategy outlined in Notation 2.18, we return to Example 2.5 and describe all Poisson prime ideals. It is easy to see that the residually null Poisson prime ideals in this case are the height two prime ideal \( x_2A+x_3A \), the height three prime ideals \( x_1A+x_2A+x_4A \) and \( x_1A+x_3A+x_4A \) and all prime ideals containing one, or more, of these.

Let \( D = x_1x_4 - x_2x_3 = s_1 \), the determinant. The dense subsets of \( \Gamma \) for which \( \text{P. Spec} \ A \) can be non-empty and the corresponding sets \( S_\gamma \) and \( V_\gamma \) are:

\[
\begin{align*}
\gamma_1 &= \{(0,1),(0,1)\}, & S_{\gamma_1} &= \{1,x_3\}, & V_{\gamma_1} &= \{D,x_2/x_3\}, \\
\gamma_2 &= \{(0,1),(1,1)\}, & S_{\gamma_2} &= \{x_2,1,x_3\}, & V_{\gamma_2} &= \{D,x_2/x_3\}, \\
\gamma_3 &= \{(1,1),(0,1)\}, & S_{\gamma_3} &= \{D,1,x_3\}, & V_{\gamma_3} &= \{D,x_2/x_3\}, \\
\gamma_4 &= \{(1,1),(1,1)\}, & S_{\gamma_4} &= \{D,x_2,1,x_3\}, & V_{\gamma_4} &= \{D,x_2/x_3\}, \\
\gamma_5 &= \{(1,1),(1,0)\}, & S_{\gamma_5} &= \{D,x_2,1\}, & V_{\gamma_5} &= \{D,x_3/x_2\}, \\
\gamma_6 &= \{(0,1),(1,0)\}, & S_{\gamma_6} &= \{x_2,1\}, & V_{\gamma_6} &= \{D,x_3/x_2\}.
\end{align*}
\]

Consequently \( C_\gamma = C_1 := \mathbb{C}[D,x_2/x_3] \) or \( C_\gamma = C_2 := \mathbb{C}[D,x_3/x_2] \).
Let $P$ be a proper Poisson prime ideal of $A$ and let $\gamma = \gamma(P)$. Then $PA_{\gamma}$ is minimal over $PA_{\gamma} \cap C_{\gamma}$. Suppose that $C_{\gamma} = C_1$ so that $\gamma = \gamma_i$ for some $i$ with $1 \leq i \leq 4$. The prime ideals of $C_1$ are 0, the principal ideals $fC_1$, where $f$ is irreducible in $C_1$ and the maximal ideals $(D - \lambda)C_1 + (x - \mu)C_1$, where $x = x_2/x_3$.

If $i = 1$ then $D, x_2 \in P$ and thus $PA_{\gamma} \cap C_1 = xC_1 + DC_1$ so $P$ is minimal over $x_2A + DA = x_2A + x_1x_4A$. It follows that $P = x_2A + x_1A$ or $P = x_2A + x_4A$. In both cases $A/P \simeq \mathbb{C}[y, z]$ with $\{y, z\} = yz$.

If $i = 2$ then $D \in P$, and $PA_{\gamma} \cap C_1 = DC_1$ or $PA_{\gamma} \cap C_1 = DC_1 + (x - \lambda)C_1$ for some non-zero $\lambda \in \mathbb{C}$. In this case either $P = DA$ or $P = DA + (x_2 - \lambda x_3)A$. In the latter case, $A/P$ is isomorphic to $\mathbb{C}[x_1, x_3, x_4]/(x_1x_4 - \lambda x_3^2)$ with the bracket induced by the Poisson bracket on $\mathbb{C}[x_1, x_3, x_4]$ such that $\{f, g\} = x_3 \text{Jac}(f, g, x_1x_4 - \lambda x_3^2)$ for all $f, g \in \mathbb{C}[x_1, x_3, x_4]$. It is easy to see, using [3] Theorem 3.8, that the non-zero Poisson prime ideals of $A/P$ are residually null.

If $i = 3$ then $PA_{\gamma} \cap C_1 = xC_1$ or $PA_{\gamma} \cap C_1 = xC_1 + (D - \lambda)C_1$ for some non-zero $\lambda \in \mathbb{C}$ so either $P = x_2A$ or $P = x_2A + (D - \lambda)A$ and, in the latter case, $A/P$ is isomorphic to $\mathbb{C}[x_1, x_3]/\{x_1, x_3\}$ with $\{x_1, x_3\} = x_1x_3$.

If $i = 4$ then $PA_{\gamma} \cap C_1 = 0$ or $PA_{\gamma} \cap C_1 = fC_1$, for some irreducible $f \in C_1$ that is not an associate of $D$ or $x$, or $PA_{\gamma} \cap C_1 = (x - \mu)C_1 + (D - \lambda)C_1$ for some non-zero $\mu, \lambda \in \mathbb{C}$. In the third of these cases, $P = (x_2 - \mu x_3)A + (D - \lambda)A$ and $A/P \simeq \mathbb{C}[x_1, x_3, x_4]/(x_1x_4 - \mu x_3^2 - \lambda)$. In the second case, $f$ remains irreducible in the polynomial extensions $\mathbb{C}[x, D, x_4] = \mathbb{C}[x, x_2, x_3, x_4]$ and $\mathbb{C}[x, x_2x_4, x_3, x_1]$ and in the localization $T$ of the latter at the multiplicatively closed subset generated by $x_1$ and $s_5$. It follows that $f$ is irreducible in $A_{\gamma}$ as $A_{\gamma}$ is a subalgebra of $T$. Hence if $j$ is the minimal non-negative integer such that $fx_j^2 \in A$ and $g = fx_j^2$ then $g$ is irreducible in $A$ and $P = fA_{\gamma} \cap A = gA$. Examples of Poisson prime ideals arising in this way include the principal ideals generated by $g_0 = D - \lambda$, $g_1 = (x_2 - \lambda x_3) = x_3(x - \lambda)$, where $\lambda \notin \mathbb{C}\{0\}$, $g_2 = x_1x_3x_4 - x_2x_3^2 + x_2 = Dx_3 + x_2 = (D + 1)x_3$, $g_3 = x_1x_2x_4 - x_2^2 x_3 + x_2 = Dx_2 + x_3 = (D + 1)x_3$, $g_4 = (D^2 - x_3^2)x_3^2 = D^2 x_3^2 - x_3$ and $g_5 = D^2 x_2 - x_3 = x_3(D^2 - x_3 - 1)$. Here the pairs $g_2, g_3$ and $g_4, g_5$ show how the choice of $v_2$, which is not symmetric between $x_2$ and $x_3$, takes account of the inherent symmetry between $x_2$ and $x_3$. In general, if $f(D, x^{-1})$ is irreducible in $C_2$, where $x^{-1} = x_3x_2^{-1}$, then there is an irreducible polynomial $g(D, x)$ such that $g(D, x) = x^k f(D, x^{-1})$ for some $k \geq 0$.

The symmetry between $x_2$ and $x_3$ is more explicit in the analysis for $\gamma_6$ and $\gamma_4$, which are analogous to $\gamma_5$ and $\gamma_1$ respectively. Here the Poisson prime ideals are $P = x_3A$ or $P = x_3A + (D - \lambda)A$, $\lambda \notin \mathbb{C}\{0\}$, for $\gamma_5$, and $P = x_3A + x_1A$ or $P = x_3A + x_4A$ for $\gamma_6$.

In the case where $t_1 = t_2 = \cdots = t_{n-2} = 1$, if $\gamma$ is such that $P, \text{Spec } A_{\gamma}$ is non-empty, then, by Lemma 2.16 each $\delta_i = 1$, each $v_i = s_i$ and $C_{\gamma} = \mathbb{C}[s_1, s_2, \ldots, s_{n-2}]$ which, under our working assumption, is a polynomial subalgebra of $A$.

**Corollary 2.22.** Suppose that $t_1 = t_2 = \cdots = t_{n-2} = 1$, let $C = \mathbb{C}[s_1, s_2, \ldots, s_{n-2}]$ and let $P$ be a proper Poisson prime ideal of $A$. Then there exists a prime ideal $Q$ of $C$ such that $P$ is a minimal prime ideal over $QA$.

**Proof.** Let $\gamma = \gamma(P)$. In this case, $C \subset A \subset A_{\gamma}$. By Theorem 2.19 $PA_{\gamma}$ is a minimal prime of $A_{\gamma}$ over $PA_{\gamma} \cap C$ and it follows easily that $P$ is a minimal prime of $A$ over $(P \cap C)A$. \qed
Example 2.23. In Example 2.21 each irreducible factor \( f \) of \( C_1 \) leads to a single principal Poisson prime ideal. This is not always the case. For example, consider the case where \( n = 4 \), \( s_1 = x_1 x_4 - x_2 x_3 \), \( s_2 = x_2 x_3 \) and \( t_1 = t_2 = 1 \). Thus Corollary 2.22 applies. Note that \( s_2 \) and \( s_1 + s_2 \) are irreducible in \( C \) but not in \( A \). This gives rise to four height one Poisson prime ideals of \( A, x_i A \) for \( 1 \leq i \leq 4 \).

Example 2.24. This example illustrates the situation where \( A \) has an element of the form \( \lambda_i s_i - \mu_i t_i \) for two different values of \( i \). This gives rise to residually null Poisson prime ideals. Let \( n = 4 \) and let \( s_1 = x_1 + x_2 + x_3 + x_4 \), \( t_1 = 1 \), \( s_2 = x_1 + x_4 \) and \( t_2 = x_2 + x_3 \). Then \( s_1 = \lambda_1 s_1 - \mu_1 t_1 = \lambda_2 s_2 - \mu_2 t_2 \), where \( \lambda_1 = \lambda_2 = 1 \), \( \mu_1 = 0 \) and \( \mu_2 = -1 \). The Poisson bracket on \( A \) in this example is given by

\[
\begin{align*}
\{x_1, x_2\} &= s_1, & \{x_1, x_3\} &= -s_1, & \{x_1, x_4\} &= 0, \\
\{x_2, x_3\} &= 0, & \{x_2, x_4\} &= s_1, & \{x_3, x_4\} &= -s_1.
\end{align*}
\]

Here the height one prime ideal \( P = s_1 A \) is residually null Poisson and \( g(P) = ((0, 1), (1, 1)) \) is dense. Notice that \( PA_\gamma \cap C_\gamma \) contains both \( v_1 = s_1 \) and \( v_2 + 1 = (x_2 + x_3)^{-1} s_1 \) and that \( \text{ht}(PA_\gamma \cap C_\gamma) = 2 \) whereas \( \text{ht}(PA_\gamma) = 1 \).

3. Poisson primitive spectra

Notation 3.1. For each \( i = 1, \ldots, n - 2 \) and for each \( (\lambda_i, \mu_i) \in \mathbb{C}^2 \setminus \{(0, 0)\} \), set

\[ f_{\lambda_i, \mu_i}^i = \lambda_i s_i - \mu_i t_i. \]

Observe that

\[ t_i^2 \nabla \frac{s_i}{t_i} = \begin{cases} 
\lambda_i^{-1} (ti \nabla f_{\lambda_i, \mu_i}^i - f_{\lambda_i, \mu_i}^i \nabla t_i) & \text{if } \lambda_i \neq 0, \\
\mu_i^{-1} (s_i \nabla f_{\lambda_i, \mu_i}^i - f_{\lambda_i, \mu_i}^i \nabla s_i) & \text{if } \mu_i \neq 0.
\end{cases} \tag{3.1}
\]

Lemma 3.2. For \( p = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n \), let \( M_p = (x_1 - \alpha_1) A + (x_2 - \alpha_2) A + \cdots + (x_n - \alpha_n) A \) and let \( g_i = f_{\lambda, \mu}^i |_{p} \), \( 1 \leq i \leq n - 2 \). Then \( M_p \) is a Poisson ideal if and only if one of the following conditions holds:

1. \( p \) is a common zero of \( s_i \) and \( t_i \) for some \( i \);
2. \( g_1, \ldots, g_{n-2} \) are algebraically dependent over \( \mathbb{C} \);
3. \( p \) is a singular point of the affine variety determined by \( g_1, \ldots, g_{n-2} \).

Proof. It suffices to show that \( M_p \) is a Poisson ideal if (1) holds or if (1) fails and (2) holds and that if (1) and (2) fail then \( M_p \) is a Poisson ideal if and only if (3) holds.

Suppose that (1) holds. Thus \( s_i(p) = t_i(p) = 0 \) for some \( i = 1, \ldots, n - 2 \) and

\[ (t_i^2 \nabla \frac{s_i}{t_i})(p) = (t_i \nabla s_i - s_i \nabla t_i)(p) = 0. \]

Let \( 1 \leq k, \ell \leq n \). Then

\[
\{x_k, x_\ell\}(p) = \begin{vmatrix}
\frac{e_k}{t_1} \\
\frac{e_\ell}{t_1} \\
\vdots \\
\frac{e_{n-2}}{t_{n-2}} \\
\frac{\frac{s_i}{t_i} - \frac{s_{i-2}}{t_{i-2}}}{t_i - t_{i-2}}
\end{vmatrix} = 0
\]

so \( \{x_k, x_\ell\} \in M_p \) which is therefore a Poisson ideal.
We can now assume that (1) fails. Let \( i = 1, \ldots, n - 2 \) and set \( \lambda_i = t_i(p) \) and \( \mu_i = s_i(p) \). Thus \( (\lambda_i, \mu_i) = (t_i(p), s_i(p)) \neq (0, 0) \) and \( g_i(p) = 0 \). By (3.1),
\[
(n^2 \nabla s_i \nabla t_i)(p) = (\nabla g_i)(p).
\]

Now suppose that (1) fails but (2) holds. Thus \( (\lambda_i, \mu_i) \neq (0, 0) \) for \( 1 \leq i \leq n - 2 \) and \( g_1, \ldots, g_{n-2} \) are algebraically dependent over \( \mathbb{C} \). For \( 1 \leq k, \ell \leq n - 2 \),
\[
\{x_k, x_{\ell}\}(p) = \begin{vmatrix}
    e_k & e_{\ell} \\
    (t_1^2 \nabla s_i t_i)(p) & (\nabla g_1)(p) \\
    \vdots & \vdots \\
    (t_{n-2}^2 \nabla s_{n-2} t_{n-2})(p) & (\nabla g_{n-2})(p)
\end{vmatrix} = 0
\]
by Theorem [1.3] applied with the algebraic dependent elements \( g_1, \ldots, g_{n-2} \) in place of \( f_1, \ldots, f_{n-2} \). Thus \( M_p \) is a Poisson ideal.

Finally, suppose that (1) and (2) fail. As (1) fails, \( (\lambda_i, \mu_i) \neq (0, 0) \) for \( 1 \leq i \leq n - 2 \). As (2) fails, \( g_1, \ldots, g_{n-2} \) are algebraically independent over \( \mathbb{C} \) and so the dimension of the affine variety \( V(g_1, \ldots, g_{n-2}) \) that they determine is two. Then \( M_p \) is a Poisson ideal of \( A \) if and only if, for all \( k, \ell \),
\[
0 = \{x_k, x_{\ell}\}(p) = \begin{vmatrix}
    e_k & e_{\ell} \\
    (t_1^2 \nabla s_i t_i)(p) & (\nabla g_1)(p) \\
    \vdots & \vdots \\
    (t_{n-2}^2 \nabla s_{n-2} t_{n-2})(p) & (\nabla g_{n-2})(p)
\end{vmatrix}
\]
if and only if the determinants of all \( (n - 2) \times (n - 2) \)-submatrices of
\[
\begin{pmatrix}
    \nabla g_1 \\
    \vdots \\
    \nabla g_{n-2}
\end{pmatrix}
\]
vanish at \( p \) if and only if \( p \) is a singular point of \( V(g_1, \ldots, g_{n-2}) \). Note that here we are using the Jacobian Criterion in a more general form, for example [2, Corollary 16.20], than a form which applies to generators of a prime or reduced ideal.

This completes the proof. \( \square \)

**Lemma 3.3.** Suppose that the parameters \( \lambda_i \) and \( \mu_i \) are such that the ideal \( I := f_1^{\lambda_1, \mu_1} A + \cdots + f_{n-2}^{\lambda_{n-2}, \mu_{n-2}} A \) is a proper ideal of \( A \) and let \( P \) be a minimal prime ideal of \( I \).

1. \( P \) is a Poisson prime ideal with height less than or equal to \( n - 2 \).
2. If \( P \) is residually null then it is not Poisson primitive.
3. If \( P \) is proper then \( htP = n - 2 \) and \( P \) is locally closed and Poisson primitive.

**Proof.** (1) It follows from (3.1) that \( I \) is a Poisson ideal so \( P \) is a Poisson prime ideal. The height of \( P \) is less than or equal to \( n - 2 \) by [17, 15.4].

(2) By Lemma 2.3(1), any residually null Poisson primitive ideal is maximal and hence has height \( n \). By (1), \( P \) is not Poisson primitive.

(3) Let \( \gamma = \gamma(P) \) which is dense by Lemma 2.16(1). Let \( 1 \leq i \leq n - 2 \). Suppose that \( t_i \notin P \). Then \( t_i \in S_\gamma \), \( v_i = s_i/t_i \), and \( \lambda_i v_i - \mu_i = t_i^{-1} f_i^{\lambda_i, \mu_i} \in PA_\gamma \). Note
that \( \lambda_i \neq 0 \), otherwise \( 0 \neq \mu_i t_i = -f_{\lambda_i, \mu_i}^1 \in P \). Similarly if \( t_i \in P \) then \( s_i \in S_{\gamma_i} \), 
\( v_i = t_i/s_i \), \( \lambda_i - \mu_i v_i = s_i^{-1} f_{\lambda_i, \mu_i}^1 \in PA_i \), and \( \mu_i \neq 0 \). Therefore \( PA_i \cap C_{\gamma_i} \) must be the maximal ideal of \( C_{\gamma_i} \) generated by the elements \( m_i \), where \( m_i = v_i - \frac{n_i}{\mu_i} \) if \( t_i \notin P \) and \( m_i = v_i - \frac{n_i}{\mu_i} \) if \( t_i \in P \). By Theorem 2.19(2), \( n - 2 = \text{ht} PA_i = \text{ht} P \), and hence \( P \) is locally closed and Poisson primitive by Lemma 2.3(3).

We next determine the Poisson primitive ideals of \( A \) and establish the Poisson Dixmier-Moeglin equivalence.

**Theorem 3.4.** The Poisson primitive ideals of \( A \) are the Poisson maximal ideals, as specified in Lemma 3.2, and the proper Poisson ideals that are minimal prime ideals of a proper ideal \( f_{1, \mu_1} A + \cdots + f_{n-2, \mu_{n-2}} A \), as specified in Lemma 3.3. Moreover \( A \) satisfies the Poisson Dixmier-Moeglin equivalence.

**Proof.** Poisson maximal ideals are always Poisson primitive, so it follows from Lemma 3.3(3) that the listed ideals are Poisson primitive. Let \( P \) be any Poisson primitive ideal and let \( 1 \leq i \leq n - 2 \). Since \( s_i/t_i \) is a Poisson central element of the quotient field \( Q(A) \) of \( A \), it follows from [11, 1.10], that there exists \( (\lambda_i, \mu_i) \in \mathbb{C}^2 \setminus \{(0,0)\} \) such that \( P \) contains \( f_{\lambda_i, \mu_i}^1 = \lambda_i s_i - \mu_i t_i \). Thus \( P \) contains the ideal \( f_{1, \mu_1} A + \cdots + f_{n-2, \mu_{n-2}} A \). If \( \text{ht} P \geq n - 1 \) then \( P \) is residually null by [9, Proposition 3.2]. If \( P \) is residually null then, by Lemma 2.3(1), \( P \) is a Poisson maximal ideal. Hence we may assume that \( \text{ht} P \leq n - 2 \) and that \( P \) is proper. Let \( Q \) be a minimal prime ideal of \( f_{1, \mu_1} A + \cdots + f_{n-2, \mu_{n-2}} A \) such that \( Q \subseteq P \). By [12, 1.4], \( Q \) is Poisson. If \( Q \) is residually null then so is \( P \), a contradiction. Hence \( Q \) is proper and \( \text{ht} Q = n - 2 \) by Lemma 3.3(2). It follows that \( P = Q \) is a minimal prime ideal of the ideal \( f_{1, \mu_1} A + \cdots + f_{n-2, \mu_{n-2}} A \), as specified in Lemma 3.3.

To establish the Poisson Dixmier-Moeglin equivalence, let \( P \) be a rational Poisson prime ideal. Let \( 1 \leq i \leq n - 2 \). As \( s_i/t_i \in \text{PZ}(Q(A)) \), \( P \) contains \( f_{\lambda_i, \mu_i}^1 \) for some \( (\lambda_i, \mu_i) \in \mathbb{C}^2 \setminus \{(0,0)\} \), and therefore \( P \) contains a proper ideal of the form \( f_{1, \mu_1} A + \cdots + f_{n-2, \mu_{n-2}} A \). If \( P \) is residually null then \( C = \text{PZ}(Q(A/P)) = Q(A/P) \) so \( P \) is a Poisson maximal ideal and hence is locally closed. If \( P \) is proper then \( \text{ht} P \leq n - 2 \) by [9, Proposition 3.2] and, by Lemma 3.3(3), \( P \) is locally closed.

**Corollary 3.5.** Suppose that \( t_i = 1 \) for each \( i \) and that \( s_1, \ldots, s_{n-2} \) are algebraically independent. Let \( (\mu_1, \ldots, \mu_{n-2}) \in \mathbb{C}^{n-2} \) be such that \( P := (s_1 - \mu_1)A + \cdots + (s_{n-2} - \mu_{n-2})A \) is a prime ideal of \( A \). Let \( X \subset \mathbb{C}^n \) be the variety determined by \( P \). Then \( P \) is Poisson prime. Moreover \( X \) is nonsingular if and only if \( A/P \) is Poisson simple.

**Proof.** In Notation 3.1 let \( \lambda_i = 1 \) so that \( f_{1, \mu_i}^1 = s_i - \mu_i \). By Lemma 3.3(1), \( P \) is Poisson. By Lemma 2.11(2), \( C_{\gamma_i}(P) = \mathbb{C}[s_1, \ldots, s_{n-2}] \) so it follows, by Theorem 2.19(2), that \( \text{ht} P = n - 2 \). Hence \( \dim X = 2 \). Let \( Q \) be a Poisson primitive ideal of \( A \) such that \( P \subseteq Q \). By Theorem 3.4 and Lemma 3.3 either \( \text{ht} Q = n - 2 \), in which case \( Q = P \), or \( Q \) is the maximal ideal corresponding to a singularity of \( X \). Hence \( A/P \) has no nonzero Poisson primitive ideal if and only if \( X \) is nonsingular. The result now follows from Lemma 2.3(2).

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