Completeness of the cubic and quartic Hénon-Heiles Hamiltonians*

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Abstract

The quartic Hénon-Heiles Hamiltonian
\[ H = \frac{(P_1^2 + P_2^2)}{2} + \frac{(\Omega_1 Q_1^2 + \Omega_2 Q_2^2)}{2} + \frac{(CQ_1^4 + BQ_1^2 Q_2^2 + AQ_2^4)}{2} + \frac{1}{2}(\alpha/Q_1^4 + \beta/Q_2^4) - \gamma Q_1 \]
passes the Painlevé test for only four sets of values of the constants. Only one of these, identical to the traveling wave reduction of the Manakov system, has been explicitly integrated (Wojciechowski, 1985), while the three others are not yet integrated in the generic case \((\alpha, \beta, \gamma) \neq (0, 0, 0)\). We integrate them by building a birational transformation to two fourth order first degree equations in the classification (Cosgrove, 2000) of such polynomial equations which possess the Painlevé property. This transformation involves the stationary reduction of various partial differential equations (PDEs). The result is the same as for the three cubic Hénon-Heiles Hamiltonians, namely, in all four quartic cases, a general solution which is meromorphic and hyperelliptic with genus two.

As a consequence, no additional autonomous term can be added to either the cubic or the quartic Hamiltonians without destroying the Painlevé integrability (completeness property).

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1 Introduction

The considered Hamiltonian originates from celestial mechanics, as a system describing the motion of a star in the axisymmetric potential of the galaxy. Denoting \( q_1 \) the radius and \( q_2 \) the altitude, this “Hénon-Heiles Hamiltonian” (HH) \[ H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1 q_2^2 - \frac{1}{3} q_1^3, \] it is nonintegrable and displays a strange attractor. However, if one changes the numerical coefficients in the potential, the system may become integrable, and this question (to find all the integrable cases and to integrate them) has attracted a lot of activity in the last three decennia.

A prerequisite is to define the word integrability, and in section 2 we briefly recall its three main acceptations in the context of Hamiltonian systems.

In section 3 we recall all the cases (three “cubic” plus four “quartic”) for which the most general two-degree of freedom classical time-independent Hamiltonian may have a single valued general solution.

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Then, discarding the integrated cases (see [10] for a review of the current state of this problem), we focus on the three cases (all “quartic”) for which the general solution is still missing, with the aim of finding this general solution.

In section 4 we build an equivalent fourth order ordinary differential equation (ODE) for \( q_1(t) \), in the hope of finding it listed in one of the classical tables of explicitly integrated ODEs. This hope is deceived because these tables are not yet finished.

This is why, in the last two sections, we adopt a different strategy. In front of the difficulty to perform the separation of variables in the sense of Arnol’d and Liouville, we establish a birational transformation between the two second order Hamilton equations and a fourth order ODE listed in a classical table established by Cosgrove [11], whose general solution is single valued.

2 Integrability for Hamiltonian systems

Given a Hamiltonian system with a finite number \( N \) of degrees of freedom, three main definitions of integrability are known,

1. the one in the sense of Liouville, that is the existence of \( N \) independent invariants \( K_j \) whose pairwise Poisson brackets vanish, \( \{ K_j, K_l \} = 0 \),

2. the one in the sense of Arnol’d-Liouville [2, chap. 9], which is to find explicitly some canonical variables \( s_j, r_j, j = 1, N \) which “separate” the Hamilton-Jacobi equation for the action \( S \), which for two degrees of freedom writes as,

\[
H(q_1, q_2, p_1, p_2) - E = 0, \ p_1 = \frac{\partial S}{\partial q_1}, \ p_2 = \frac{\partial S}{\partial q_2}, \tag{2}
\]

3. the one in the sense of Painlevé [8] i.e. the representation of the general solution \( q_j(t) \) by an explicit, closed form, single valued expression of the time \( t \).

3 The seven Hénon-Heiles Hamiltonians

Given the most general two-degree of freedom classical time-independent Hamiltonian

\[
H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2) = E, \tag{3}
\]

the requirement that the system made of the two Hamilton equations passes the Painlevé test [8] (for at least some integer powers \( q_1^{n_1}, q_2^{n_2} \)) selects seven and only seven potentials \( V \) depending on a finite number of constants, namely

1. three “cubic” potentials (HH3 case) [7, 15, 9],

\[
H = \frac{1}{2}(p_1^2 + p_2^2 + \omega_1 q_1^2 + \omega_2 q_2^2) + \alpha q_1 q_2 - \frac{1}{3} \beta q_1^3 + \frac{1}{2} \gamma q_2^{-2}, \ \alpha \neq 0 \tag{4}
\]

in which the constants \( \alpha, \beta, \omega_1, \omega_2, \gamma \) can only take three sets of values,

\[
\text{(SK)} : \ \beta/\alpha = -1, \omega_1 = \omega_2, \tag{5}
\]

\[
\text{(KdV5)} : \ \beta/\alpha = -6, \tag{6}
\]

\[
\text{(KK)} : \ \beta/\alpha = -16, \omega_1 = 16 \omega_2. \tag{7}
\]
2. four “quartic” potentials (HH4 case) [24, 17],

\[
H = \frac{1}{2}(P_1^2 + P_2^2 + \Omega_1 Q_1^2 + \Omega_2 Q_2^2) + CQ_1^4 + BQ_1^2 Q_2^2 + AQ_2^4 + \frac{1}{2}\left(\frac{\alpha}{Q_1^2} + \frac{\beta}{Q_2^2}\right) - \gamma Q_1, \quad B \neq 0, \quad (8)
\]

in which the constants \(A, B, C, \alpha, \beta, \gamma, \Omega_1, \Omega_2\) can only take the four values (the notation \(A : B : C = p : q : r\) stands for \(A/p = B/q = C/r = \text{arbitrary}\)),

\[
\begin{cases}
A : B : C = 1 : 2 : 1, \quad \gamma = 0, \\
A : B : C = 1 : 6 : 1, \quad \gamma = 0, \quad \Omega_1 = \Omega_2, \\
A : B : C = 1 : 6 : 8, \quad \alpha = 0, \quad \Omega_1 = 4\Omega_2, \\
A : B : C = 1 : 12 : 16, \quad \gamma = 0, \quad \Omega_1 = 4\Omega_2.
\end{cases} \quad (9)
\]

All seven cases are integrable in the sense of Liouville, with a second constant of the motion \(K\) [12, 4, 19] [20, 3, 4] either quadratic or quartic in the momenta \(p_1, p_2\).
In the sense of Arnol’d-Liouville, the separation of variables has been performed [12, 31, 25, 28, 6, 26], except in three cases,

1. HH4 1:6:1 \(\alpha \neq \beta\),
2. HH4 1:6:8 \(\beta\gamma \neq 0\),
3. HH4 1:12:16 \(\alpha\beta \neq 0\).

What is remarkable is the fact that, in all cases when the separation of variables is achieved, the equations of Hamilton have the Painlevé property, the general solution being a hyperelliptic function of genus two. The purpose of this work is to prove equally the Painlevé property in the three remaining cases where the separation of variables is not yet performed.

4 Equivalent fourth order ODEs

In the cubic case, the two Hamilton equations

\[
q_1'' + \omega_1 q_1 - \beta q_1^2 + \alpha q_2^2 = 0, \quad (10)
\]
\[
q_2'' + \omega_2 q_2 + 2\alpha q_1 q_2 - \gamma q_2^{-3} = 0, \quad (11)
\]

together with the Hamiltonian [44], are equivalent [15] to a single fourth order ODE for \(q_1(t)\),

\[
q_1'''' + (8\alpha - 2\beta)q_1'' - 2(\alpha + \beta)q_1'^2 - \frac{20}{3}\alpha\beta q_1^3 + (\omega_1 + 4\omega_2)q_1'' + (6\alpha\omega_1 - 4\beta\omega_2)q_1' + 4\omega_1\omega_2 q_1 + 4\alpha E = 0, \quad (12)
\]

independent of the coefficient \(\gamma\) of the nonpolynomial term \(q_2^{-2}\) and depending on the constant value \(E\) of the Hamiltonian \(H\). In the three HH3 cases [45–47], this ODE belongs to a list [11] (“classification”) of equations enjoying the Painlevé property, whose general solution is hyperelliptic with genus two.

In the quartic case, the similar fourth order equation is built by eliminating \(Q_2\) and \(Q_1'''\) between the two Hamilton equations,

\[
Q_1'' + \Omega_1 Q_1 + 4CQ_1^3 + 2BQ_1 Q_2^2 - \alpha Q_1^{-3} + \gamma = 0, \quad (13)
\]
\[
Q_2'' + \Omega_2 Q_2 + 4AQ_2^3 + 2BQ_2 Q_1^2 - \beta Q_2^{-3} = 0, \quad (14)
\]
and the Hamiltonian \( H \), which results in

\[
-Q_1''' + 2 \frac{Q_1 Q_1'''}{Q_1} + \left(1 + 6 \frac{A}{B}\right) \frac{Q_1'^2}{Q_1} - 2 \frac{Q_1'^2 Q_1'''}{Q_1^2} + 8 \left( \frac{6AC}{B} - B - C \right) Q_1^2 Q_1' + 4(B - 2C)Q_1 Q_1'^2 + 24C \left( \frac{4AC}{B} - B \right) Q_1^5 \\
+ 12 \frac{A}{B} \omega_1 - 4\omega_2 + \left(1 + 12 \frac{A}{B}\right) \frac{\gamma}{Q_1} - 4 \left(1 + 3 \frac{A}{B}\right) \frac{\alpha}{Q_1} \right) Q_1'' \\
+ 6A \frac{\alpha^2}{B Q_1^3} + 20 \frac{\alpha}{Q_1} Q_1'^2 - 12A \frac{\gamma \alpha}{B Q_1^2} + 4 \left(3 \frac{A}{B} \alpha - \omega_2\right) \left(\gamma - \frac{\alpha}{Q_1}\right) - 2\gamma Q_1'^2 Q_1' \\
+ 6 \frac{A}{B} \gamma^2 + 2B \alpha - 5 \frac{AC}{B} \alpha \right) \frac{1}{Q_1} + \left(6 \frac{A}{B} \alpha - 4\omega_2 - 8BE\right) Q_1 \\
+ 48 \frac{AC}{B} \gamma Q_1^3 + 4 \left(12 \frac{AC}{B} - B - 4C\right) \alpha Q_1''
\]

(15)

This ODE depends on \( E \) but not on \( \beta \) and, as opposed to the cubic case, it does not belong to a classified set of equations, because \( Q_1''' \) is not polynomial in \( Q_1 \).

In the three remaining cases, since one is yet unable either to perform the separation of variables or to establish a direct link to a classified ODE, let us build an indirect link to such a classified ODE. This link, which involves soliton equations, is the following.

For each of the seven cases, the two Hamilton equations are equivalent \([15, 16, 3]\) to the traveling wave reduction of a soliton system made either of a single PDE (HH3) or of two coupled PDEs (HH4), most of them appearing in lists established from group theory \([13]\). Among the various soliton equations which are equivalent to them via a Bäcklund transformation, some of them admit a traveling wave reduction to a classified ODE. This property defines a path \([22, 30]\) which starts from one of the three remaining HH4 cases, goes up to a soliton system of two coupled 1+1-dimensional PDEs admitting a reduction to the considered case, then goes to another 1+1-dim PDE system equivalent under a Bäcklund transformation, finally goes down by reduction to an already integrated ODE or system of ODEs.

5 General solution of the quartic 1:6:1 and 1:6:8 cases

Let us denote the two constants of the motion of the 1:6:1 and 1:6:8 cases as,

\[
1:6:1 \quad \begin{cases} 
H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\Omega}{2}(Q_1^2 + Q_2^2) - \frac{1}{32}(Q_1^4 + 6Q_1^2 Q_2^2 + Q_2^4) \\
- \frac{1}{2}(\kappa_1^2 Q_1^2 + \kappa_2^2 Q_2^2) = E, \\
K = \left( P_1 P_2 + Q_1 Q_2 \left( -\frac{Q_1^2 + Q_2^2}{8} + \Omega \right) \right)^2 \\
- \frac{P_2^2 \kappa_2^2}{Q_1^2} - \frac{P_1^2 \kappa_1^2}{Q_2^2} + \frac{1}{4}(\kappa_1^2 Q_2^2 + \kappa_2^2 Q_1^2) + \frac{\kappa_1^2 \kappa_2^2}{Q_1^2 Q_2^2}, 
\end{cases}
\]

(16)

and

\[
1:6:8 \quad \begin{cases} 
H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\omega}{2}(4q_1^2 + q_2^2) - \frac{1}{16}(8q_1^4 + 6q_1^2 q_2^2 + q_2^4) \\
- \gamma q_1 + \frac{\beta}{2q_2^2} = E, \\
K = \left( \frac{p_2^2 - \frac{q_2^2}{16}(2q_2^2 + 4q_2^2 + \omega) + \frac{\beta}{q_2^2}}{16} - \frac{1}{4}q_2^2(2q_2 p_1 - 2q_1 p_2)^2 \right) \\
+ \gamma \left( -2\gamma q_2^2 - 4q_2 p_1 p_2 + \frac{1}{2}q_1 q_2^2 + q_1^3 q_2^2 + 4q_1 p_2^2 - 4\omega q_1 q_2^2 + 4q_1 \frac{\beta}{q_2^2} \right). 
\end{cases}
\]

(17)
There is a canonical transformation \[3\] between the 1:6:1 and 1:6:8 cases, mapping the constants as follows,

\[\begin{align*}
E_{1:6:8} &= E_{1:6:1}, \\
K_{1:6:8} &= K_{1:6:1}, \\
\omega &= \Omega, \\
\gamma &= \frac{\kappa_1 + \kappa_2}{2}, \\
\beta &= -(\kappa_1 - \kappa_2)^2,
\end{align*}\] (18)

therefore one only needs to integrate either case.

The path to an integrated ODE comprises the following three segments.

The coordinate \(q_1(t)\) of the 1:6:8 case can be identified \[4, 3\] to the component \(F\) of the traveling wave reduction \(f(x, \tau) = F(x - c\tau)\), \(g(x, \tau) = G(x - c\tau)\) of a soliton system of two coupled KdV-like equations (c-KdV system) denoted c-KdV \[4, 3\]

\[
\begin{align*}
&f_{\tau} + \left( f_{xx} + \frac{3}{2} ff_x - \frac{1}{2} f^3 + 3fg \right) = 0, \\
&-2g_{\tau} + g_{xxx} + 6gg_x + 3fg_{xx} + 9f_{xx}g_x + 9f_xg_x - 3f^2g_x \\
&+ \frac{3}{2} f_{xxxx} + \frac{3}{2} f_{xxx} + 9f_xf_{xx} - 3f^2f_x - 3ff_x = 0,
\end{align*}\] (19)

with the identification

\[
\begin{align*}
&\begin{cases}
q_1 = F, \\
c = -\omega, \\
K_1 = \gamma, \\
K_2 = E,
\end{cases}
\end{align*}\] (20)

in which \(K_1\) and \(K_2\) are two constants of integration.

There exists a Bäcklund transformation between this soliton system and another one of the c-KdV type, denoted bi-SH system \[13\],

\[
\begin{align*}
&-2u_{\tau} + \left( u_{xx} + u^2 + 6v \right) = 0, \\
v_{\tau} + v_{xxx} + uv_x = 0.
\end{align*}\] (21)

This BT is defined by the Miura transformation \[22\]

\[
\begin{align*}
&\begin{cases}
U = \frac{3}{2} \left( 2g - f_x - f^3 \right), \\
V = \frac{3}{4} \left( 2f_{xxx} + 4f_{xx} + 8fg_x + 4f g_x + 3f_x^2 - 2f^2 f_x - f^4 + 4gf^2 \right).
\end{cases}
\end{align*}\] (22)

Finally, the traveling wave reduction \(u(x, \tau) = U(x - c\tau), v(x, \tau) = V(x - c\tau)\) can be identified \[30\] to the autonomous F-VI equation (a-F-VI) in the classification of Cosgrove \[11\],

\[
a-F-VI : y''' = 18yy'' + 9y'^2 - 24y^3 + \alpha_{V1}y^2 + \frac{\alpha_{V1}^2}{9} y + \kappa_{V1}t + \beta_{V1}, \quad \kappa_{V1} = 0,
\] (23)

an ODE whose general solution is meromorphic, expressed with genus two hyperelliptic functions \[11\] Eql. (7.26)]. The identification is

\[
\begin{align*}
&\begin{cases}
U = -6 \left( y + \frac{c}{18} \right), \\
V = y''' - 6y^2 + \frac{4}{3} cy + \frac{16}{27} c^2 - \frac{K_A}{2}, \\
\alpha_{V1} = -4c, \\
\beta_{V1} = K_B - 2cK_A + \frac{512}{243} c^3,
\end{cases}
\end{align*}\] (24)

in which \(K_A, K_B\) are two constants of integration.

In order to perform the integration of both the 1:6:1 and the 1:6:8 cases, it is sufficient to express \((F, G)\)
rationally in terms of \((U, V, U', V')\). The result is

\[
\begin{aligned}
F &= \frac{W'}{2W} + \frac{K_1}{24W} \left[ -3U'^2 - 2(U - 3c) \left( 12V + (U + 3c)^2 \right) + 36K_B - 54K_1^2 \right], \\
G &= \frac{U}{3} + \frac{1}{8W} \left[ (2V + 3K_2) (2V'' + K_1U' - 3K_1^2) - 2(U - 3c) (2K_1V' + K_1^2(U + 3c)) \right], \\
W &= \left( V + \frac{3}{2} K_2 \right)^2 + \frac{3}{2} K_1^2 (U - 3c), \\
K_A &= K_2.
\end{aligned}
\]  

(25)

Making the product of the successive transformations \([20], [25], [24]\), one obtains a meromorphic general solution for \(Q_1, Q_2, q_1, q_2\).

\[
\begin{aligned}
q_1 &= \frac{W'}{2W} + \frac{\gamma}{W} \left[ 9j - 3 \left( y + \frac{4}{9} \omega \right) (h + E) - \frac{9}{4} \gamma^2 \right], \\
q_2 &= -16 \left( y - \frac{5}{9} \omega \right), \\
W &= (h + E)^2 - 9\gamma^2 \left( y - \frac{5}{9} \omega \right), \\
\alpha_{V1} &= 4\omega, \quad \beta_{V1} = \frac{3}{4} \gamma^2 + 2\omega E - \frac{3}{16} \beta - \frac{512}{243} \omega^3, \\
K_{1,V1} &= \frac{3}{32} K - \frac{1}{2} E^2, \quad K_{2,V1} = \frac{3}{32} E K - \frac{9}{64} \omega E^3 + \frac{9}{64} \beta \gamma^2, \\
K_1 &= \gamma, \quad K_2 = E, \quad K_A = E, \quad K_B = -\frac{3}{16} \beta + \frac{3}{4} \gamma^2.
\end{aligned}
\]  

(26)

in which \(h\) and \(j\) are convenient auxiliary variables [11] Eqs. (7.4)–(7.5)],

\[
\begin{aligned}
y &= \frac{Q(s_1, s_2) + \sqrt{Q(s_1)Q(s_2)}}{2 \left( s_1^2 - C_{V1} + \sqrt{s_2^2 - C_{V1}} \right)} + \frac{5}{36} \alpha_{V1}, \\
h &= -\frac{3}{4} E_{V1} s_1 s_2 + C_{V1} + \sqrt{(s_1^2 - C_{V1})(s_2^2 - C_{V1})} - \frac{F_{V1}}{2}, \\
j &= \frac{1}{6} \left( 2h + F_{V1} \right) \left\{ y + \frac{\alpha_{V1}}{9} - \frac{E_{V1}}{s_1 + s_2} \right\}.
\end{aligned}
\]  

(27)

In the above, the variables \(s_1, s_2\) are defined by the hyperelliptic system [11]

\[
\begin{aligned}
(s_1 - s_2)s_2' &= \sqrt{P(s_1)}, \quad (s_2 - s_1)s_1' = \sqrt{P(s_2)}, \\
P(s) &= (s^2 - C_{V1})Q(s), \\
Q(s, t) &= (s^2 - C_{V1})(t^2 - C_{V1}) - \frac{\alpha_{V1}}{2} (s^2 + t^2 - 2C_{V1}) + \frac{E_{V1}}{2} (s + t) + F_{V1}, \\
Q(s) &= Q(s, s).
\end{aligned}
\]  

(28)

The expressions \([27]\) cannot be written as rational functions of \(s_1, s_2, s_1', s_2'\) and are nevertheless meromorphic [14] [21].

The coefficients \((\alpha, C, E, F)_{V1}\) of the hyperelliptic curve depend algebraically on the parameters of the
Hamiltonians $\beta, \gamma, \omega, E, K$ \[ Eqs. (7.9)-(7.12) \] \[
\begin{align*}
A_{VI} &= 4 \omega, \\
E_{VI}^2 &= -\frac{16}{3} \omega (F_{VI} - 2E) - \beta + 4\gamma^2, \\
C_{VI} E_{VI}^2 &= \frac{3}{4} (F_{VI} - 4E^2) + K, \\
(F_{VI} - 2E)^2 (F_{VI} + 4E) + \frac{9K}{4} (F_{VI} - 2E) - \frac{27}{4} \beta \gamma^2 &= 0,
\end{align*}
\] (29)

and this algebraic dependence could explain the difficulty to separate the variables in the Hamilton-Jacobi equation. Note that, in the particular case $\beta \gamma = 0$, i.e. $\kappa_1^2 = \kappa_2^2$, these coefficients become rational, see \[29\].

Remark. The F-VI ODE can be written in Hamiltonian form,

\[
\begin{align*}
H &= P_2^2 + Q_2 P_1 - \frac{Q_1^4}{6} + \frac{3}{2} Q_1 Q_2^2 - \frac{13}{347} \alpha_{VI}^3 Q_1 + \frac{1}{16} \alpha_{VI}^2 Q_1^2 - \frac{1}{8} \alpha_{VI} Q_1^2 \\
&\quad - 6 \kappa_{VI} Q_1 - 6 \kappa_{VI} t Q_1 + \frac{347}{2^{13} 3^5} \alpha_{VI}^3 + \frac{7}{2} \alpha_{VI} \beta_{VI}, \\
Q_1 &= -6 \left( y - \frac{\alpha_{VI}}{72} \right), \quad Q_2 = -6 y', \quad P_1 = 6 y'' - 108 y y', \quad P_2 = -6 y''.
\end{align*}
\] (30)

In the autonomous case $\kappa_{VI} = 0$, the Hamiltonian $H$ is a first integral (equal to $36 K_{1,VI}$), and the other constant of the motion is cubic in the momenta. However, because of the nonlinear link between $K_{1,VI}$ and the two first integrals of the 1:6:8 case, see \[20\], there exists no canonical transformation between the variables $(q_j, p_j)$ of 1:6:8 and the above canonical variables of a-F-VI.

6 General solution of the quartic 1:12:16 case

Let us denote the two constants in involution as, \(1:12:16\)

\[
\begin{align*}
H &= \frac{1}{2} (P_1^2 + P_2^2) + \frac{\Omega}{8} (4Q_1^2 + Q_2^2) - \frac{1}{32} (16Q_1^4 + 12Q_1^2Q_2^2 + Q_2^4) \\
&\quad - \frac{1}{2} \left( \frac{\kappa_1^2}{Q_1^2} + \frac{4 \kappa_2^2}{Q_2^2} \right) = E, \\
K &= \frac{1}{16} \left( 8 (Q_2 P_1 - Q_1 P_2) P_2 - Q_1 Q_2^4 - 2Q_1^3Q_2^2 + 2Q_1 Q_2^2 + 32 Q_1 \frac{\kappa_2^2}{Q_2^2} \right)^2.
\end{align*}
\] (31)

Similarly to the 1:6:1-1:6:8 couple, there exists a canonical transformation between the 1:12:16 Hamiltonian and another Hamiltonian \[33\ \[34\], which is however not the sum of a kinetic energy and a potential energy, which we denote similarly as \(5:9:4\), \(5:9:4\)

\[
\begin{align*}
H &= \frac{1}{2} \left( p_1^2 + \left( p_2 - \frac{3}{2} q_1 q_2 \right)^2 \right) - \frac{1}{8} (4q_1^4 + 9q_1^2 q_2^2 + 5q_2^4) + \frac{\omega}{2} (q_1^2 + q_2^2) - \kappa q_1 + \frac{\zeta}{2 q_2^2} = E, \\
K &= \frac{1}{q_2^2} \left( 2q_2^2 p_1 + 2q_1^2 q_2^2 - 2q_1 q_2 p_2 - q_1^4 - 4 \kappa q_1 \right)^2 \\
&\quad \times \left( 2q_2^2 p_1 + 2q_1^2 q_2^2 + p_2^2 - 4q_1 q_2 p_2 - 2q_1^4 + \omega q_2^2 + 4 \kappa_2 q_1 - 4 \kappa_2 p_2 q_2 \right) \\
&\quad + 4 (\zeta + 4 \kappa) \left( -2q_1 \frac{P_2}{q_2^2} + (q_1^2 + q_2^2) + q_1 \frac{\kappa}{q_2^2} \right) p_1 - \frac{1}{q_2^2} (q_1^2 q_2^2 + q_2^4 + 2 \kappa q_1)^2 \\
&\quad + 2 \frac{q_2^2}{q_2^2} \left( p_2 - \frac{3}{2} q_1 q_2 \right)^2 + \frac{(q_1^2 + q_2^2)^2}{2} + q_1^2 (\zeta) \right),
\end{align*}
\] (32)

\[
E_{5:9:4} = E_{1:12:16}, \quad K_{5:9:4} = K_{1:12:16}, \quad \omega = \Omega, \quad \kappa = \kappa_1 + \kappa_2, \quad \zeta = -(\kappa_1 - \kappa_2)^2.
\]
The path to an integrated ODE is also quite similar and is made of the following three segments \([4, 27, 22]\).

Firstly, the coordinate \(q_1(t)\) of \([32]\) is identified \([31, 3]\) to the component \(F\) of the traveling wave reduction \(f(x, \tau) = F(x - c\tau), g(x, \tau) = G(x - c\tau)\) of a soliton system of two coupled KdV-like equations denoted c-KdVa(f, g) \([4, 3]\).

\[
\begin{align*}
q_1 &= F, \quad q_2^2 = \frac{2}{5} (F' - 2F^2 - G + \omega), \\
c &= -\omega.
\end{align*}
\]  

(33)

Secondly, there exists a Bäcklund transformation between this soliton system and another one of the c-KdV type, denoted bi-SK system \([23]\), transformation defined by the Miura map

\[
\begin{align*}
u &= \frac{3}{10} (3f_x - f^2 + 2g), \\
v &= \frac{3}{10} \left(f_{xxx} + g_{xx} + f_xg - fg_x - f_{fxx} + g^2\right).
\end{align*}
\]  

(34)

Finally, the traveling wave reduction \(u(x, \tau) = U(x - c\tau), v(x, \tau) = V(x - c\tau)\) is identified \([30]\), to the F-IV equation (or to the F-III as well) in the classification of Cosgrove \([11]\),

\[
\begin{align*}
U &= -3 \left(y - \frac{\omega}{30}\right), \\
V &= -6y'' + 18y^2 - \frac{9}{5} \omega y + \frac{1}{10} \omega^2 - \frac{3}{5} E,
\end{align*}
\]  

(35)

to the F-IV equation (or to the F-III as well) in the classification of Cosgrove \([11]\),

\[
\begin{align*}
y''' &= 30yy'' - 60y^3 + \alpha_{1IV} y + \beta_{1IV}, \\
y &= \frac{1}{2} \left(s_1 + s_2 + s_1^2 + s_1s_2 + s_2^2 + A\right), \\
(s_1 - s_2)s_1' &= \sqrt{P(s_1)}, \quad (s_2 - s_1)s_2' = \sqrt{P(s_2)}, \\
P(s) &= (s^2 + A)^3 - \frac{\alpha_{1IV}}{3}(s^2 + A) + Bs + \frac{\beta_{1IV}}{3}, \\
K_{1,IV} &= \left(\frac{3B}{4}\right)^2, \quad K_{2,IV} = -\frac{9AB^2}{64},
\end{align*}
\]  

(36)

in which \((K_{1,IV}, K_{2,IV})\) denote two polynomial first integrals of F-IV. The general solution of this ODE is meromorphic, expressed with genus two hyperelliptic functions \([11]\).

In order to perform the integration of both Hamiltonians \([34, 32]\) and \([32]\), it is sufficient to express \((F, G)\) rationally in terms of \((U, V, U', V')\). The result is

\[
\begin{align*}
F &= -\frac{W'}{2W} + K_{1,a} X_2, \\
G &= -F^2 - X_1 X_2 + K_{1,a} \frac{54U'}{X_1} - 54K_{1,a} \left(U + \frac{3\omega}{20}\right) \frac{W'}{WX_1} + 2 \left(U + \frac{9\omega}{10}\right), \\
W &= X_1^2 + 108K_{1,a}^2 \left(U + \frac{3\omega}{20}\right), \\
X_1 &= V + 2U^2 - 3\omega U + \frac{9}{50} \omega^2 - \frac{27}{5} E, \\
X_2 &= 9 \left(-4U'' + \frac{8}{3} UV - \frac{8}{25} \omega U^2 + \frac{2}{5} \omega V + \frac{48}{5} EU - \frac{42}{25} \omega^2 U - \frac{9}{2} \left(\kappa_1^2 + \kappa_2^2\right) - \frac{9}{2} K_{1,a}^2 + \frac{36}{25} \omega E - \frac{27}{125} \omega^3\right), \\
K_{1,a} &= \kappa_1 - \kappa_2.
\end{align*}
\]  

(37)

From the point of view of the separation of variables, one should first exhibit a Hamiltonian representation of F-IV. One such structure is that of the cubic SK case. However, since the constant value of the Hamiltonian of the cubic SK case, when expressed only in terms of the parameters \((E, K, \omega, \kappa_1, \kappa_2)\) of the 1:12:16, is not an affine function of \(E\), there exists no canonical transformation between the cubic SK case and the 1:12:16 case.
7 Conclusion, remaining work

The explicit integration of all the seven cases is now achieved in the Painlevé sense (finding a closed form single valued expression for the general solution), and the common features are the following.

1. In all cases, the general solution is hyperelliptic with genus two, and therefore meromorphic.

2. Each case is birationally equivalent to a fourth order ODE which is complete in the Painlevé sense, i.e. which accepts no additional term, under penalty of losing its Painlevé property. Consequently, for each of the seven Hamiltonians, it is impossible to add any term to the Hamiltonian without destroying the Painlevé property, and the seven Hénon-Heiles Hamiltonians are complete.

About the integration in the Arnold-Liouville sense (finding the separating variables of the Hamilton-Jacobi equation), two problems remain open.

1. In the $1:6:1$-$1:6:8$ case, the hyperelliptic curve $y^2 = P(s)$ of F-VI (see (28)) reduces in the separated cases $\beta \gamma = 0$ to the hyperelliptic curve of the separating variables. Therefore, F-VI is the good ODE to consider, and the only missing item is to find a Hamiltonian structure of F-VI, necessarily distinct from (30), admitting a canonical transformation to $1:6:1$-$1:6:8$.

2. In the $1:12:16$-$5:9:4$ case, the hyperelliptic curve $y^2 = P(s)$ of F-IV (see (36)) does not reduce in the separated cases $\kappa_1 \kappa_2 = 0$ to the hyperelliptic curve of the separating variables, which is

$$
\kappa_1 \kappa_2 = 0 : \quad P(s) = s^6 - \omega s^3 + 2Es^2 + \frac{K}{20}s + \kappa_1^2 + \kappa_2^2 = 0.
$$

Therefore, F-IV (as well as its birationally equivalent ODE F-III) is not the good ODE to consider, and it should be quite instructive to integrate the fourth order equivalent ODE (15) in that case.

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