Yang-Mills redux

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Abstract

It is noted that a given pairing of the phase factor and gauge transformation to retain gauge symmetry is not unique. In their seminal paper, when Yang and Mills (YM) discuss the phase factor - gauge transformation relationship, they cite Pauli’s review paper. It is interesting that although Pauli in that paper presents the electromagnetic field strength in terms of a commutator, for whatever reason YM did not extrapolate the commutator’s use to obtain the Yang-Mills field strength – they obtained it by trial and error. Presented is a derivation of this field strength using the commutator approach detailing how certain terms cancel each other. Finally, the Yang-Mills field transformation is derived in a slightly different way than is traditionally done.

1 Introduction

This is an addendum to the article on differential geometry and Feynman diagrams that appeared in the Notices of the American Mathematical Society (Marateck 2006). It expands on some of the topics covered in the original article.

2 Gauge theory

Weyl introduced as a phase factor (Weyl 1929) an exponential in which the phase \( \alpha \) is preceded by the imaginary unit \( i \), e.g., \( e^{+i\alpha(x)} \), in the wave function for the wave equations (for instance, the Dirac equation is \( (i\gamma^\mu \partial_\mu - m)\psi = 0 \)). It is here that Weyl correctly formulated gauge theory as a symmetry principle from which electromagnetism could be derived. It had been shown that for a quantum theory of charged particles interacting with the electromagnetic field, invariance under a gauge transformation of the potentials required multiplication of the wave function by the now well-know phase factor. Yang cites (Yang 1986) Weyl’s gauge theory results as reported (Pauli 1941) by Pauli as a source for Yang-Mills gauge theory; although Yang didn’t find out until much later that these were Weyl’s results. Moreover, Pauli’s article did not mention Weyl’s geometric interpretation. It was only much after Yang and Mills published their article that Yang realized the connection between their work and geometry. In fact, in his selected papers (Yang, 2005), Yang says

What Mills and I were doing in 1954 was generalizing Maxwell’s theory. We knew of no geometrical meaning of Maxwell’s theory, and we were not looking in that direction.

For the wave equations to be gauge invariant, i.e., have the same form after the gauge transformation as before, the local phase transformation \( \psi(x) \rightarrow \psi(x)e^{+ia(x)} \) has to be accompanied by the local gauge transformation

\[
A_\mu \rightarrow A_\mu - q^{-1} \partial_\mu \alpha(x) \tag{1}
\]
This dictates that the $\partial_\mu$ in the wave equations be replaced by the covariant derivative $\partial_\mu + iqA_\mu$ in order for the $\partial_\mu \alpha(\mathbf{x})$ terms to cancel each other. This pair of phase factor-gauge transformation is not unique. Another pair that retains gauge symmetry and results in the same covariant derivative has the $q$ included in the phase factor, i.e., $\psi(\mathbf{x}) \rightarrow \psi(\mathbf{x})e^{+iq\alpha(\mathbf{x})}$ paired with

$$A_\mu \rightarrow A_\mu - \partial_\mu \alpha(\mathbf{x})$$ \(2\)

The fact that this pairing is not unique is not surprising since the phase factor and gauge transformation have no physical significance.

3 Yang-Mills field strength

Pauli, in equation (22a) of Part I of his 1941 review article (Pauli 1941) gives the electromagnetic field strength in terms of a commutator. In present-day usage it is

$$[D_\mu, D_\nu] = i\epsilon F_{\mu\nu}$$ \(3\)

where $D_\mu$ is the covariant derivative $\partial_\mu + i\epsilon A_\mu$. Mathematically, equation \(3\) corresponds to the curvature (the field strength) reflecting the effect of parallel transport of a vector around a closed path, i.e., its holonomic behavior. If the field strength is zero, the vector will return to its point of origin pointing in its original direction. In their seminal paper (Yang 1954) Yang and Mills do not mention this relation, although they do cite Pauli’s 1941 article. They use

$$\psi = S\psi'$$ \(4\)

where $S$ is a local isotopic spin rotation represented by an SU(2) matrix, to obtain the gauge transformation in equation \(3\) of their paper

$$B'_\mu = S^{-1}B_\mu S + iS^{-1}(\partial_\mu S)/\epsilon$$ \(5\)

They then define the field strength as

$$F_{\mu\nu} = (\partial_\nu B_\mu - \partial_\mu B_\nu) + i\epsilon(B_\mu B_\nu - B_\nu B_\mu)$$ \(6\)

This corresponds to Cartan’s second structural equation which in differential geometry notation is $\Omega = d\mathbf{A} + [\mathbf{A}, \mathbf{A}]$, where $\mathbf{A}$ is a connection on a principal fiber bundle.

They introduce equation \(6\) (their equation \(4\)) by saying

In analogy to the procedure of obtaining gauge invariant field strengths in the electromagnetic case, we define \(4\) $F_{\mu\nu} = (\partial_\nu B_\mu - \partial_\mu B_\nu) + i\epsilon(B_\mu B_\nu - B_\nu B_\mu)$ One easily shows from $[B'_\mu = S^{-1}B_\mu S + iS^{-1}(\partial_\mu S)/\epsilon]$ that \(5\) $F'_{\mu\nu} = S^{-1}F_{\mu\nu} S$ under an isotopic gauge transformation. Other simple functions of $B$ than \(4\) do not lead to such a simple transformation property.

*Yang had earlier started studying this problem as a graduate student at the University of Chicago and derived equation \(5\). When he returned to this problem as a visitor at Brookhaven, he in collaboration with Mills obtained (as we will explain) the field strength.
Yang and Mills arrived at the field strength, equation (6), by trial and error. They added terms to the electromagnetic part until they found the commutator part, all the while plugging the resulting field strength into their equation [5] for verification.

Using the Yang-Mills covariant derivative \((\partial_\mu - i\epsilon B_\mu)\) let’s see how the Yang-Mills field strength is obtained from the commutator

\[
[D_\mu, D_\nu] = (\partial_\mu - i\epsilon B_\mu)(\partial_\nu - i\epsilon B_\nu) - (\partial_\nu - i\epsilon B_\nu)(\partial_\mu - i\epsilon B_\mu)
\]  

(7)
operating on the wave function \(\psi\). Note that 

\[-\partial_\mu (B_\nu \psi) = - (\partial_\mu B_\nu) \psi - B_\nu \partial_\mu \psi \quad \text{and} \quad \partial_\nu (B_\mu \psi) = (\partial_\nu B_\mu) \psi + B_\mu \partial_\nu \psi.\]

So we get a needed \(-B_\nu \partial_\mu\) and a \(B_\mu \partial_\nu\) term to cancel \(B_\nu \partial_\mu\) and \(-B_\mu \partial_\nu\) respectively. Thus, expanding (7) we get

\[
\partial_\mu \partial_\nu - i\epsilon \partial_\mu B_\nu - i\epsilon B_\mu \partial_\nu - i\epsilon B_\mu \partial_\mu - \epsilon^2 B_\mu B_\nu - \partial_\nu \partial_\mu + i\epsilon \partial_\nu B_\mu + i\epsilon B_\mu \partial_\nu + \epsilon^2 B_\mu B_\nu
\]  

(8)
which reduces to 

\[-i\epsilon B_\nu \partial_\mu - \epsilon^2 [B_\mu, B_\nu]\]  

or \([D_\mu, D_\nu] = -i\epsilon F_{\mu\nu}\)

4 The field transformation

We present a pedagogical derivation of the gauge transformation by using the transformation

\[\psi' = S\psi\]  

(9)
instead of the traditional \(\psi = S\psi', \) i.e., the one Yang and Mills used. In order to obtain the gauge transformation in equation [3] of the Yang and Mills paper

\[B'_\mu = S^{-1} B_\mu S + iS^{-1}(\partial_\mu S)/\epsilon\]  

(10)
requires you to use \(\partial_\mu S^{-1} = -S^{-1}(\partial_\mu S)S^{-1}.\) Thus, the approach indicated by equation (9) is marginally more straight-forward since it doesn’t require differentiating the inverse of a matrix.

The covariant derivative, \(D_\mu = \partial_\mu - i\epsilon B_\mu,\) transforms the same way as \(\psi\) does

\[D'_\mu \psi' = SD\psi\]  

(11)

The left-hand side of equation (11) becomes

\[(\partial_\mu - i\epsilon B'_\mu)S\psi = (\partial_\mu S)\psi + S\partial_\mu \psi - i\epsilon B'_\mu S\psi\]

(12)
But (12) equals \(S\partial_\mu \psi - i\epsilon SB_\mu \psi.\) Cancelling \(S\partial_\mu \psi\) on both sides we get,

\[(\partial_\mu S)\psi - i\epsilon B'_\mu S\psi = -i\epsilon SB_\mu \psi\]

or

\[\text{†The following can be obtained by differentiating } S^{-1}S = I\]
\[ B'_\mu S = SB_\mu + (\partial_\mu S)/(i\epsilon) \]  \hspace{1cm} (14)

thus

\[ B'_\mu = SB_\mu S^{-1} - i(\partial_\mu S)S^{-1}/\epsilon \]  \hspace{1cm} (15)

We will use \( S = e^{i\alpha(x)\cdot\sigma} \). So for \( \alpha \) infinitesimal, \( S = 1 + i\alpha \cdot \sigma \) which produces

\[ B'_\mu = (1 + i\alpha \cdot \sigma)B_\mu(1 - i\alpha \cdot \sigma) \]

\[ - i(1/\epsilon)\partial_\mu(1 + i\alpha \cdot \sigma)(1 - i\alpha \cdot \sigma) \]  \hspace{1cm} (16)

Remembering that \((a \cdot \sigma)(b \cdot \sigma) = a \cdot b + i\sigma \cdot (a \times b)\), setting \( B_\mu = \sigma \cdot b_\mu \), and since \( \alpha \) is infinitesimal, dropping terms of order \( \alpha^2 \), we get

\[ b'_\mu \cdot \sigma = b_\mu \cdot \sigma \]

\[ + i[(\alpha \cdot \sigma)(b_\mu \cdot \sigma), (b_\mu \cdot \sigma)(\alpha \cdot \sigma)] + (1/\epsilon)\partial_\mu(\alpha \cdot \sigma) \]  \hspace{1cm} (17)

and finally

\[ b'_\mu = b_\mu + 2(b_\mu \times \alpha) + (1/\epsilon)\partial_\mu\alpha \]  \hspace{1cm} (18)

which (because our \( S \) is the inverse of Yang-Mills’ \( S \)) is equation [10] in the Yang-Mills paper.

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**References**

Marateck, Samuel L., 2006. Notic. Amer. Math. Soc. **53** 744.
Pauli, W., 1941. Rev. Mod. Physics. **13** 203.
Weyl, Hermann, 1929. Zeit. f. Physic. **330** 56.
Yang, C. N. and Mills, R. L., 1954. Phys. Rev. **96** 191.
Yang, C.N., 1986 in Hermann Weyl’s contribution to Physics, in Hermann Weyl:1885-1985, ed. Chandrasekharan, K. (Springer-Verlag).
Yang, C.N., 2005 in Selected Papers (1945-1980) With Commentary, World Scientific. p74.