FLUCTUATIONS OF SUBGRAPH COUNTS IN RANDOM GRAPHONS

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ABSTRACT. Given a graphon \( W \) and a finite simple graph \( H \), denote by \( X_n(H, W) \) the number of copies of \( H \) in a \( W \)-random graph on \( n \) vertices. The asymptotic distribution of \( X_n(H, W) \) was recently obtained by Hladký, Pelekis, and Šileikis [14] in the case where \( H \) is a clique. In this paper, we extend this result to any fixed graph \( H \). Towards this we introduce a notion of \( H \)-regularity of graphons and show that if the graphon \( W \) is not \( H \)-regular, then \( X_n(H, W) \) has Gaussian fluctuations. On the other hand, if \( W \) is \( H \)-regular, the limiting distribution of \( X_n(H, W) \) can have a non-Gaussian component which is a (possibly) infinite weighted sum of centered chi-squared random variables, where the weights are determined by the spectral properties of a graphon derived from \( W \). Our proofs use the asymptotic theory of generalized \( U \)-statistics developed by Janson and Nowicki [15].

1. Introduction

A graphon is a measurable function \( W : [0, 1]^2 \rightarrow [0, 1] \) which is symmetric, that is, \( W(x, y) = W(y, x) \), for all \( x, y \in [0, 1] \). Graphons arise as the limit objects of sequences of large graphs and have received phenomenal attention over the last few years. It provides a bridge between combinatorics and analysis, and has found applications in several disciplines including statistical physics, probability, and statistics [2, 7–10]. For a detailed exposition of the theory of graph limits refer to Lovász [17]. Graphons provide a natural sampling procedure for generating inhomogeneous variants of the classical Erdős-Rényi random graph, a concept that has been proposed independently by various authors (see [5, 6, 12, 18] among others). Formally, given a graphon \( W : [0, 1]^2 \rightarrow [0, 1] \), a \( W \)-random graph on the set of vertices \( [n] := \{1, 2, \ldots, n\} \), hereafter denoted by \( G(n, W) \), is obtained by connecting the vertices \( i \) and \( j \) with probability \( W(U_i, U_j) \) independently for all \( 1 \leq i < j \leq n \), where \( \{U_i : 1 \leq i \leq n\} \) is an i.i.d. sequence of \( U[0,1] \) random variables. An alternative way to achieve this sampling is to generate i.i.d. sequences \( \{U_i : 1 \leq i \leq n\} \) and \( \{Y_{ij} : 1 \leq i < j \leq n\} \) of \( U[0,1] \) random variables and then assigning the edge \((i,j)\) whenever \( Y_{ij} \leq W(U_i, U_j) \), for \( 1 \leq i < j \leq n \). Observe that setting \( W = W_p = p \in [0,1] \) gives the classical (homogeneous) Erdős-Rényi random graph model, where every edge is present independently with constant probability \( p \).

Counts of subgraphs encode important structural information about the geometry of a network. In fact, the convergence of a sequence finite graphs to a graphon is precisely determined by the convergence of its subgraph densities. As a consequence, understanding the asymptotic properties of subgraph counts in \( W \)-random graphs is a problem of central importance in graph limit theory. To this end, given a finite simple graph \( H = (V(H), E(H)) \) denote by \( X_n(H, W) \) the number of copies of \( H \) in the \( W \)-random graph \( G(n, W) \). More formally,

\[
X_n(H, W) = \sum_{1 \leq i_1 < \cdots < i_{|V(H)|} \leq n} \sum_{H' \in \mathcal{H}(\{i_1, \ldots, i_{|V(H)|}\})} \prod_{(i_a, i_b) \in E(H')} 1\{Y_{i_a i_b} \leq W(U_{i_a} U_{i_b})\}, \quad (1.1)
\]

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where, for any set $S \subseteq [n]$, $\mathcal{G}_H(S)$ denotes the collection of all subgraphs of the complete graph on the vertex set $S$ which are isomorphic to $H$. The asymptotic distribution of $X_n(H,W)$ in the Erdős-Rényi model, where $W = W_p \equiv p$, has been classically studied, using various tools such as U-statistics [20], method of moments [21], and Stein’s method [1], and the precise conditions under which $X_n(H,W)$ is asymptotically normal are well-understood [1]. In fact, when $p \in (0,1)$ is fixed, $X_n(H,W_p)$ is asymptotically normal for any finite graph $H$.

In this paper we study the asymptotic distribution of $X_n(H,W)$ for general graphons $W$. This problem has received significant attention recently, beginning with the work of Féray, Méliot, and Nikeghbali [13], where the asymptotic normality for homomorphism densities in general $W$-random graphs was derived using the framework of mod-Gaussian convergence (see also [11] and the references therein). Using this machinery the authors also obtained moderate deviation principles and local limit theorems for the homomorphism densities in this regime. Very recently, using Stein’s method, rates of convergence to normality in Kolmogorov distance [16] and Berry-Esseen type bounds [22] have been derived as well. However, interestingly, the limiting normality of the subgraph counts obtained in [13] can be degenerate depending on the structure of the graphon $W$. This phenomenon was first explored in the remarkable recent paper of Hladký, Peleki, and Šileikis [14] for the case where $H = K_r$, the $r$-clique, for some $r \geq 2$. They showed that the usual Gaussian limit is degenerate when a certain regularity function, which encodes the homomorphism density of $K_r$ incident on a given ‘vertex’ of $W$, is constant almost everywhere. In this case, the graphon $W$ is said to be $K_r$-regular and the asymptotic distribution of $X_n(K_r,W)$ has both Gaussian and non-Gaussian components. In this paper we extend this result to any fixed graph $H$. To this end, we introduce the analogous notion of $H$-regularity and show that the fluctuations of $X_n(H,W)$ depends on whether or not $W$ is $H$-regular. In particular, if $W$ is not $H$-regular, then $X_n(H,W)$ is asymptotically Gaussian. However, if $W$ is $H$-regular, then the limiting distribution of $X_n(H,W)$ has a Gaussian component and another independent (non-Gaussian) component which is a (possibly) infinite weighted sum of centered chi-squared random variables. Here, the weights are determined by spectrum of the graphon obtained from the 2-point conditional densities of $H$ in $W$, that is, the density of $H$ in $W$ when two vertices of $H$ are mapped to two ‘vertices’ of $W$, averaged over all pairs of vertices of $H$. Unlike in [14] which use the method of moments, our proofs employ the orthogonal decomposition for generalized U-statistics developed by Janson and Nowicki [15]. This avoids cumbersome moment calculations and provides a more streamlined framework for dealing with the asymmetries of general subgraphs. The formal statements of the results are given in the section below.

2. Asymptotic Distribution of Subgraph Counts in $W$-Random Graphs

In this section we will state our main result on the asymptotic distribution $X_n(H,W)$. The section is organized as follows: In Section 2.1 we recall some basic definitions about graphons. The notion of conditional homomorphism density and $H$-regularity are introduced in Section 2.2. The spectral properties of graphons and the operator which arise in the limiting distribution of $X_n(H,W)$ are described in Section 2.3. The result is formally stated in Section 2.4.

2.1. Preliminaries. A quantity that will play a central role in our analysis the homomorphism density of a fixed multigraph $F = (V(F), E(F))$ (without loops) in a graphon $W$, which is defined as:

$$t(F, W) = \int_{[0,1]|V(F)|} \prod_{(s,t) \in E(F)} W(x_a, x_b) \prod_{a=1}^{V(F)} dx_a. \quad (2.1.1)$$
Note that this is the natural continuum analogue of the homomorphism density of a fixed graph \( F = (V(F), E(F)) \) into finite (unweighted) graph \( G = (V(G), E(G)) \) which is defined as:

\[
t(F, G) := \frac{|\text{hom}(F, G)|}{|V(G)||V(F)|},
\]

where \( |\text{hom}(F, G)| \) denotes the number of homomorphisms of \( F \) into \( G \). In fact, it is easy to verify that \( t(F, G) = t(F, W^G) \), where \( W^G \) is the empirical graphon associated with the graph \( G \) which defined as:

\[
W^G(x, y) = \mathbf{1}\{([|V(G)|x], [V(G)|y]) \in E(G)\}.
\]

(In other words, to obtain the empirical graphon \( W^G \) from the graph \( G \), partition \([0, 1]^2\) into \(|V(G)|^2\) squares of side length \( 1/|V(G)| \), and let \( W^G(x, y) = 1 \) in the \((i, j)\)-th square if \((i, j) \in E(G)\), and 0 otherwise.)

For a non-empty simple graph \( H = (V(H), E(H)) \), with vertices labeled \( V(H) = \{1, 2, \ldots, |V(H)|\} \), the homomorphism density defined \((2.1.1)\) can also interpreted as the probability that \( W\)-random graph on \(|V(H)|\) vertices is isomorphic to \( H \), that is,

\[
t(H, W) = \mathbb{P}(G(|[V(H)]|), W) = H).
\]

To see this, recall the construction of a \( W\)-random graph and note from \((2.1.1)\) that,

\[
t(H, W) = \mathbb{E}\left[ \prod_{(a,b) \in E(H)} W(U_a, U_b) \right] = \mathbb{E}\left[ \prod_{(a,b) \in E(H)} \mathbf{1}\{Y_{ab} \leq W(U_a, U_b)\} \right] \quad \text{=} \quad \mathbb{E}\left[ \mathbf{1}\{|G(|V(H)|), W) = H\}\right].
\]

Next, recalling \((1.1)\) note that

\[
\mathbb{E}X_n(H, W) = \sum_{1 \leq i_1 < \cdots < i_{|V(H)|}} \sum_{H' \in \mathcal{G}_H([i_1, \ldots, i_{|V(H)|}])} t(H, W)
\]

\[
= \binom{n}{|V(H)|} |\mathcal{G}_H([1, \ldots, |V(H)|])| \cdot t(H, W) \quad \text{(2.1.2)}
\]

where the last equality follows since the number of subgraphs of \( K_{|V(H)|} \) on \( \{i_1, \ldots, i_{|V(H)|}\} \) isomorphic to \( H \) is same for any collection of distinct indices \( 1 \leq i_1 < \cdots < i_{|V(H)|} \leq n \). Clearly,

\[
|\mathcal{G}_H([1, \ldots, |V(H)|])| = \frac{|V(H)|!}{|\text{Aut}(H)|}, \quad \text{(2.1.3)}
\]

where \( \text{Aut}(H) \) is the collection of all automorphisms of \( H \), that is, the collection of permutations \( \sigma \) of the vertex set \( V(H) \) such that \((x, y) \in E(H)\) if and only if \((\sigma(x), \sigma(y)) \in E(H)\). This implies, from \((2.1.2)\),

\[
\mathbb{E}X_n(H, W) = \frac{(n)^{|V(H)|}}{|\text{Aut}(H)|} t(H, W),
\]

where \( (n)^{|V(H)|} := n(n - 1) \cdots (n - |V(H)| + 1) \).

### 2.2. Conditional Homomorphism Densities and \( H\)-Regularity

In this section we will formalize the notion of \( H\)-regularity of a graphon \( W \). To this end, we need to introduce the notion of conditional homomorphism densities. Throughout, we will assume \( H = (V(H), E(H)) \) is a non-empty simple graph with vertices labeled \( V(H) = \{1, 2, \ldots, |V(H)|\} \).
Definition 2.1. Fix \( 1 \leq K \leq |V(H)| \) and an ordered set \( \mathbf{a} = (a_1, a_2, \ldots, a_K) \) of distinct vertices \( a_1, a_2, \ldots, a_K \in V(H) \). Then the \( K \)-point conditional homomorphism density function of \( H \) in \( W \) given \( \mathbf{a} \) is defined as:

\[
t_a(x, H, W) := \mathbb{E} \left[ \prod_{(a, b) \in E(H)} W(U_a, U_b) \bigg| U_{a_j} = x_j, \text{ for } 1 \leq j \leq K \right]
\]

\[
= \mathbb{P} \left( G(V(H), W) = H \big| U_{a_j} = x_j, \text{ for } 1 \leq j \leq K \right),
\]

where \( x = (x_1, x_2, \ldots, x_K) \). In other words, \( t_a(x, H, W) \) is the homomorphism density of \( H \) in the graphon \( W \) when the vertex \( a_j \in V(H) \) is marked with the value \( x_j \in [0, 1] \), for \( 1 \leq j \leq K \).

The conditional homomorphism densities will play a crucial role in the description of the limiting distribution of \( X_n(H, W) \). In particular, the \( H \)-regularity of a graphon \( W \) is determined by the 1-point conditional homomorphism densities, which we formalize below:

Definition 2.2 \((H\text{-regularity of a graphon})\). A graphon \( W \) is said to be \( H \)-regular if

\[
\tilde{t}(x, H, W) := \frac{1}{|V(H)|} \sum_{a=1}^{|V(H)|} t_a(x, H, W) = t(H, W),
\]

(2.2.1)

for almost every \( x \in [0, 1] \).

Note that in (2.2.1) it is enough to assume that \( \tilde{t}(x, H, W) \) is a constant for almost every \( x \in [0, 1] \). This is because

\[
\int_0^1 t_a(x, H, W) \, dx = t(H, W),
\]

for all \( a \in V(H) \). Hence, if \( \tilde{t}(x, H, W) \) is a constant almost everywhere, then the constant must be \( t(H, W) \). Therefore, in other words, a graphon \( W \) is \( H \)-regular if the homomorphism density of \( H \) in \( W \) when one of the vertices of \( H \) is marked, is a constant independent of the value of the marking.

Remark 2.1. Note that when \( H = K_r \) is the \( r \)-clique, for some \( r \geq 2 \), then \( t_a(x, H, W) = t_b(x, H, W) \), for all \( 1 \leq a \neq b \leq r \). Hence, (2.2.1) simplifies to

\[
t_1(x, K_r, W) = \mathbb{E} \left[ \prod_{1 \leq a < b \leq r} W(U_a, U_b) \bigg| U_1 = x \right] = t(H, W), \text{ for almost every } x \in [0, 1],
\]

(2.2.2)

which is precisely the notion of \( K_r \)-regularity defined in [14]. Also, note that the notion of \( K_2 \)-regularity coincides with the standard notion of degree regularity, where the degree function \( d_W(x) := \int_{[0,1]} W(x, y) \, dy \) is constant almost everywhere. This is because, from (2.2.2),

\[
t_1(x, K_2, W) = \mathbb{E} \left[ W(U_1, U_2) | U_1 = x \right] = \int_{[0,1]} W(x, y) \, dy := d_W(x).
\]

2.3. Spectrum of Graphons and 2-Point Conditional Densities. Hereafter, we denote by \( \mathcal{W}_0 \) the space of all graphons, which is the collection all symmetric, measurable functions \( W : [0, 1]^2 \to [0, 1] \). Every graphon \( W \in \mathcal{W}_0 \) defines an operator \( T_W : L^2[0,1] \to L^2[0,1] \) as follows:

\[
(T_W f)(x) = \int_0^1 W(x, y) f(y) \, dy,
\]

(2.3.1)
for each $f \in L^2[0,1]$. $T_W$ is a Hilbert-Schmidt operator, which is compact and has a discrete spectrum, that is, it has a countable multiset of non-zero real eigenvalues, which we denote by $\text{Spec}(W)$. In particular, we can write

$$
(T_W f)(x) = \sum_{\lambda \in \text{Spec}(W)} \lambda \langle f, \phi_\lambda \rangle \phi_\lambda(x)
$$

and $W(x, y) = \sum_{\lambda \in \text{Spec}(W)} \lambda \phi_\lambda(x) \phi_\lambda(y)$, where $\{\phi_\lambda\}_{\lambda \in \text{Spec}(W)}$ denotes the orthonormal system of eigenfunctions associated with $\text{Spec}(W)$. For a more detailed discussion on the spectral properties of graphons and its connection to graph limit theory, see [17, Chapters 7, 11].

To describe the limiting distribution of $X_n(H, W)$ when $W$ is $H$-regular, we will need to understand the spectral properties of the following graphon obtained from the 2-point conditional graphon induced by $W$.

**Definition 2.3.** Given a graphon $W \in \mathcal{W}_0$ and simple graph $H = (V(H), E(H))$, the 2-point conditional graphon induced by $H$ is defined as:

$$
W_H(x, y) = \frac{1}{2|\text{Aut}(H)|} \sum_{a \in \mathbb{Z}, b \in [V(H)]} t_{(a,b)}((x, y), H, W),
$$

(2.3.2)

where $t_{(a,b)}((x, y), H, W)$ is the 2-point conditional homomorphism density function of $H$ in $W$ given the vertices $(a, b)$, as in Definition 2.1.

Intuitively, $W_H(x, y)$ can be interpreted as the homomorphism density of $H$ in $W$ containing the ‘vertices’ $x, y \in [0, 1]$. Note that a graphon $W$ is $H$-regular (in the sense of Definition 2.2) if and only if the 2-point conditional graphon $W_H$ is degree regular. This is because, for all $x \in [0, 1]$,

$$
\int_0^1 W_H(x, y) dy = \frac{|V(H)| - 1}{2|\text{Aut}(H)|} \sum_{a=1}^{\lambda} t_a(x, H, W),
$$

and the RHS of (2.3.3) is a constant if and only if $W$ is $H$-regular. In fact, if $W$ is $H$-regular, then

$$
\frac{1}{|V(H)|} \sum_{a=1}^{\lambda} t_a(x, H, W) = t(H, W) \text{ almost everywhere, hence, the degree of } W_H \text{ becomes}
$$

$$
\int_0^1 W_H(x, y) dy = \frac{|V(H)|(|V(H)| - 1)}{2|\text{Aut}(H)|} \cdot t(H, W) := d_{W_H},
$$

(2.3.4)

for almost every $x \in [0, 1]$. This implies, when $W$ is $H$-regular, then $d_{W_H}$ is an eigenvalue of the operator $T_{W_H}$ (recall (2.3.1)) and $\phi = 1$ is a corresponding eigenvector. In this case, we will use $\text{Spec}^-(W_H)$ to denote the collection $\text{Spec}(W_H)$ with multiplicity of the eigenvalue $d_{W_H}$ decreased by 1.

**Remark 2.2.** Note that if $W$ is complete, that is, $W \equiv 1$, then almost surely $X_n(H, W) = \frac{(n)^{|V(H)|}}{|\text{Aut}(H)|}$ and $\bar{t}(x, H, W) = t(H, W) = 1$. Similarly, if $W$ is $H$-free, that is, $t(H, W) = 0$, then almost surely $X_n(H, W) = 0$ and $\bar{t}(x, H, W) = t(H, W) = 0$. Therefore, we will hereafter assume that $W$ is not complete and not $H$-free. This implies, when $W$ is $H$-regular, then $\bar{t}(x, H, W) = t(H, W) \in (0, 1)$, almost everywhere.

24. Statement of the Main Result. To state our results on the asymptotic distribution of $X_n(H, W)$, we need to define a few basic graph operations.

**Definition 2.4.** Fix $r \geq 1$ and consider two graphs $H_1$ and $H_2$ on the vertex set $\{1, 2, \ldots, r\}$ and edge sets $E(H_1)$ and $E(H_2)$, respectively.
- **Vertex Join**: For \( a, b \in \{1, 2, \ldots, r\} \), the \((a, b)\)-vertex join of \( H_1 \) and \( H_2 \) is the graph obtained by identifying the \( a \)-th vertex of \( H_1 \) with the \( b \)-th vertex of \( H_2 \) (see Figure 1 for an illustration). The resulting graph will be denoted by 
\[
H_1 \oplus_{a,b} H_2.
\]

- **Weak Edge Join**: For edges \( (a, b) \in E(H_1) \) and \( (c, d) \in E(H_2) \), with \( 1 \leq a < b \leq r \) and \( 1 \leq c < d \leq r \), the \((a, b), (c, d)\)-weak edge join of \( H_1 \) and \( H_2 \) is the graph obtained identifying the vertices \( a \) and \( c \) and the vertices \( b \) and \( d \) and keeping a single edge between the two identified vertices (see Figure 2 for an illustration). The resulting graph will be denoted by 
\[
H_1 \ominus_{(a,b),(c,d)} H_2.
\]

- **Strong Edge Join**: For edges \( (a, b) \in E(H_1) \) and \( (c, d) \in E(H_2) \), with \( 1 \leq a < b \leq r \) and \( 1 \leq c < d \leq r \), the \((a, b), (c, d)\)-strong edge join of \( H_1 \) and \( H_2 \) is the multi-graph obtained identifying the vertices \( a \) and \( c \) and the vertices \( b \) and \( d \) and keeping both the edges between the two identified vertices (see Figure 2 for an illustration). The resulting graph will be denoted by 
\[
H_1 \oplus_{(a,b),(c,d)} H_2.
\]

Having introduced the framework and the relevant definitions, we are now ready to state our main result regarding the asymptotic distribution of \( X_n(H, W) \), the number of copies of \( H \) in the \( W \)-random graph \( G(n, W) \). Recall from Remark 2.2 that it suffices to assume \( W \) is not \( H \)-regular.

**Theorem 2.1.** Fix a graphon \( W \in \mathcal{W}_0 \) and a non-empty finite simple graph \( H = (V(H), E(H)) \) with vertices labeled \( V(H) = \{1, 2, \ldots, |V(H)|\} \). Then for \( X_n(H, W) \) as defined in (1.1) the following hold:

1. If \( W \) is not \( H \)-regular, then
\[
\frac{X_n(H, W) - \frac{(n)^{|V(H)|}}{|\text{Aut}(H)|} t(H, W)}{\sqrt{n^{|V(H)| - \frac{1}{2}}}} \xrightarrow{D} N(0, \tau_{H,W}^2),
\]
where
\[
\tau_{H,W}^2 := \frac{1}{|\text{Aut}(H)|^2} \sum_{1 \leq a, b \leq |V(H)|} t(H \oplus_{a,b} H, W) - |V(H)|^2 t(H, W)^2.
\]

\[\]

**Figure 1.** The \((a, b)\)-vertex join of the graphs \( H_1 \) and \( H_2 \).
Figure 2. The weak and strong edge joins of the graphs $H_1$ and $H_2$.

(2) If $W$ is $H$-regular, then

$$\frac{X_n(H, W) - \frac{(n)_{|\text{Aut}(H)|}}{|\text{Aut}(H)|} t(H, W)}{\sqrt{|V(H)|-1}} \xrightarrow{D} \sigma_{H,W} \cdot Z + \sum_{\lambda \in \text{Spec}^{-}(W_H)} \lambda (Z^2_{\lambda} - 1),$$

where $Z, \{Z_{\lambda}\}_{\lambda \in \text{Spec}^{-}(W_H)}$ are all independent standard Gaussians,

$$\sigma^2_{H,W} := \frac{2}{|\text{Aut}(H)|^2} \sum_{(a,b),(c,d) \in E(H)} \left[ t \left( H \bigoplus_{(a,b),(c,d)} H, W \right) - t \left( H \bigoplus_{(a,b),(c,d)} H, W \right) \right],$$

and $\text{Spec}^{-}(W_H)$ is the collection $\text{Spec}(W_H)$ with multiplicity of the eigenvalue $d_{W_H}$ (recall (2.3.4)) decreased by 1.

The proof of this result uses the projection method for generalized $U$-statistics developed in Janson and Nowicki [15], which allows us to decompose $X_n(H, W)$ over sums of increasing complexity. The terms in the expansion are indexed by the vertices and edges subgraphs of the complete graph of increasing sizes, and the asymptotic behavior of $X_n(H, W)$ is determined by the non-zero terms indexed by the smallest size graphs. Details of the proof are given in Section 4. Various examples are discussed in Section 3.

Remark 2.3. As mentioned earlier, the result in Theorem 2.1 (1) has been proved recently by Féray, Méliot, and Nikeghbali (see [13, Theorem 21]) using the machinery of mod-Gaussian convergence. However, the limiting distribution in [13, Theorem 21] is degenerate, that is, $\tau_{H,W} = 0$, whenever the graphon $W$ is $H$-regular. The regular regime falls within the purview of Theorem 2.1 (2), which is the main emphasis of this paper, where the more interesting non-Gaussian fluctuation emerges. Interestingly, in an article posted few days ago, Méliot [19] introduced the notion of singular graphons, which are graphons $W$ for which $\tau_{H,W} = 0$, for all
graphs $H$. For such graphons the author derived the order of fluctuations for the homomorphism densities, but did not prove any limit theorems.

Finally, it is worth mentioning that limiting distributions very similar to that in Theorem 2.1 (2) also appears in the context of counting monochromatic subgraphs in uniform random colorings of sequences of dense graphs [3, 4]. Although this is a fundamentally different problem, the appearance of similar limiting objects in both situations is nevertheless interesting.

3. Examples

In this section we compute the limiting distribution of $X_n(H, W)$ for various specific choices of $H$ and $W$ using Theorem 2.1.

Example 3.1. (Cliques) Suppose $H = K_r$, the complete graph on $r$ vertices, for some $r \geq 2$. This is the case that was studied in [14]. To see that Theorem 2.1 indeed recovers the main result in [14] first recall Remark 2.1, which shows that our notion of $H$-regularity matches with the notion of $K_r$-regularity defined in [14]. Next, note by the symmetry of the vertices of a clique,

$$t \left( H \bigoplus_{a,b} H, W \right) = t \left( H \bigoplus_{1,1} H, W \right),$$

for $1 \leq a, b \leq |V(H)|$, and $|Aut(K_r)| = r!$. Therefore, Theorem 2.1 (1) implies, when $W$ is not $K_r$-regular,

$$\frac{X_n(K_r, W) - \binom{n}{r} t(K_r, W)}{n^{r-\frac{1}{2}}} \xrightarrow{D} N \left( 0, \frac{1}{(r-1)!^2} \left[ t \left( K_r \bigoplus_{1,1} K_r, W \right) - t(K_r, W)^2 \right] \right),$$

which is precisely the result in [14, Theorem 1.2(b)]. For the regular case, note by the symmetry of the edges of a clique, the 2-point conditional graphon induced by $K_r$ (recall Definition 2.3) simplifies to,

$$W_{K_r}(x, y) = \frac{1}{2(r-2)!} t_{(1,2)}((x, y), K_r, W).$$

Moreover, for all $(a, b), (c, d) \in E(K_r),

$$t \left( K_r \ominus_{(a,b),(c,d)} K_r, W \right) = t \left( K_r \ominus_{(1,2),(1,2)} K_r, W \right),$$

and similarly for the strong edge-join operation. Hence, Theorem 2.1 (2) implies

$$\frac{X_n(K_r, W) - \binom{n}{r} t(K_r, W)}{n^{r-1}} \xrightarrow{D} \sigma_{K_r, W} \cdot Z + \sum_{\lambda \in \text{Spec} -(W_{K_r})} \lambda(Z^2 - 1)$$

with

$$\sigma_{K_r, W}^2 = \frac{1}{2(r-2)!^2} \left\{ t \left( H \bigominus_{(1,2),(1,2)} H, W \right) - t \left( H \bigoplus_{(1,2),(1,2)} H, W \right) \right\},$$

as shown in [14, Theorem 1.2(c)].

Example 3.2. (2-Star) Suppose $H = K_{1,2}$ with the vertices labelled $1, 2, 3$ as shown in Figure 3. In this case, for any graphon $W \in \mathcal{W}_0$,

$$t_1(x, K_{1,2}, W) = \int_0^1 W(x, y)W(x, z)dydz = d_W(x)^2,$$  (3.1)
and

\[
t_2(x, K_{1,2}, W) = t_3(x, K_{1,2}, W) = \int_0^1 W(x,y)W(y,z)dydz = \int_0^1 W(x,y)dW(y)dy,
\]

where \( d_W(x) := \int_{[0,1]} W(x,y)dy \) is the degree function of \( W \). It follows from (3.1) and (3.2) that a graphon \( W \) is \( K_{1,2} \)-regular if only if \( d_W(\cdot) \) is constant almost everywhere, that is, \( W \) is degree regular. Therefore, from Theorem 2.1 we have the following:

- If \( d_W(\cdot) \) is not constant almost everywhere, then
  \[
  \frac{X_n(K_{1,2}, W) - 3(\frac{n}{3})t(H,W)}{n^{\frac{1}{2}}} \overset{D}{\to} N(0, \tau_{K_{1,2},W}^2)
  \]
  with
  \[
  \tau_{K_{1,2},W}^2 := \frac{1}{4} \left\{ t(K_{1,4}, W) + 4t(P_4, W) + 4t(B_4, W) - 9t(K_{1,2}, W)^2 \right\},
  \]
  where the graphs \( K_{1,4}, P_4, \) and \( B_4 \) are as shown in Figure 3. Note that \( K_{1,4} \) is the 4-star (obtained by joining the two central vertices of the 2-stars), \( P_4 \) is the path with 4 edges (obtained by joining a leaf vertex of one 2-star with a leaf vertex of another), and \( B_4 \) is the graph obtained by joining the central vertex of one 2-star with a leaf vertex of another. For a concrete example of a graphon which is not \( K_{1,2} \)-regular consider \( W_0(x,y) := xy \). In this case, \( d_{W_0}(x) = x \), for all \( x \in [0,1] \), is non-constant, hence, \( W_0 \) is not \( K_{1,2} \)-regular.

- If \( d_W(\cdot) \) is a constant almost everywhere, then
  \[
  \frac{X_n(K_{1,2}, W) - 3(\frac{n}{3})t(H,W)}{n^{\frac{1}{2}}} \overset{D}{\to} \sigma_{K_{1,2},W} \cdot Z + \sum_{\lambda \in \text{Spec}^-(W_{K_{1,2}})} \lambda(Z_\lambda - 1),
  \]
  with
  \[
  \sigma_{K_{1,2},W}^2 := 2(t(K_{1,3}, W) - t(K_{1,3}^+, W)),
  \]
  where \( K_{1,3} \) is the 3-star as shown in Figure 4(a) (obtained by the strong edge-join of two copies of \( K_{1,2} \)) and the \( K_{1,3}^+ \) is the multigraph shown in Figure 4(b) (obtained by the weak edge-join of two copies of \( K_{1,2} \)). Moreover, in this case the 2-point conditional graphon \( W_{K_{1,2}} \) simplifies to:

\[
W_{K_{1,2}}(x,y) = \frac{1}{2} \left\{ W(x,y)(d_W(x) + d_W(y)) + \int W(x,z)W(y,z)dz \right\},
\]
Figure 4. (a) The weak edge join of two copies of $K_{1,2}$ and (b) the strong edge join of two copies of $K_{1,2}$.

Fix a graphon $W$ such that $t_{(1,2)}(x, y, K_{1,2}, W) = t_{(1,3)}(x, y, K_{1,2}, W) = W(x, y)dW(x)$ and $t_{(2,3)}(x, y, K_{1,2}, W) = \int_{[0, 1]} W(x, z)W(y, z)dz$, and similarly for the others. For a concrete example of graphon which is $K_{1,2}$-regular consider

$$\tilde{W}(x, y) := \begin{cases} p & \text{if } (x, y) \in \left[0, \frac{1}{2}\right]^2 \cup \left[\frac{1}{2}, 1\right]^2, \\ 0 & \text{otherwise}. \end{cases}$$

Note that this is a 2-block graphon (with equal block sizes) taking value $p$ in the diagonal blocks and zero in the off-diagonal blocks. (One can think of this as the ‘disjoint union of two Erdős-Rényi graphons’.) It is easy to check that this graphon is degree regular, hence $K_{1,2}$-regular. In fact, in this case

$$\tilde{W}_{K_{1,2}}(x, y) = \begin{cases} \frac{3p^2}{4} & \text{if } (x, y) \in \left[0, \frac{1}{2}\right]^2 \cup \left[\frac{1}{2}, 1\right]^2, \\ 0 & \text{otherwise}. \end{cases}$$

and $\sigma_{K_{1,2}, W}^2 = \frac{1}{4}p^3(1 - p)$. Moreover,

$$\text{Spec}(\tilde{W}_{K_{1,2}}) = \{3p^2/8, 3p^2/8\},$$

with the eigenfunctions $1$ and $1[0, 1/2] - 1[1/2, 1]$, respectively. Hence, $\text{Spec}^-(\tilde{W}_{K_{1,2}}) = \{3p^2/8\}$.

4. Proof of Theorem 2.1

Fix a graphon $W \in \mathcal{W}_0$ and a non-empty simple graph $H = (V(H), E(H))$ with vertices labeled $V(H) = \{1, 2, \ldots, |V(H)|\}$, and recall the definition of $X_n(H, W)$ from (1.1). To express $X_n(H, W)$ as a generalized $U$-statistic note that

$$X_n(H, W) = \sum_{1 \leq i_1 < \cdots < i_{|V(H)|} \leq n} f(U_{i_1}, \ldots, U_{i_{|V(H)|}}, Y_{i_1i_2}, \ldots, Y_{i_{|V(H)|-1}i_{|V(H)|}})$$

where $\mathcal{G}_H := \mathcal{G}_H(\{1, 2, \ldots, |V(H)|\})$ and

$$f(U_1, \ldots, U_{|V(H)|}, Y_{12}, \ldots Y_{|V(H)|-1}) = \sum_{H' \in \mathcal{G}_H} \prod_{(a, b) \in E(H')} 1\{Y_{ab} \leq W(U_a, U_b)\}. \quad (4.1)$$
This is exactly in the framework of generalized \(U\)-statistics considered in [15]. Therefore, we can now orthogonally expand the function \(f\) as a sum over subgraphs of the complete graph as explained in the section below.

4.1. Orthogonal Decomposition of Generalized \(U\)-Statistics. Suppose \(\{U_i : 1 \leq i \leq n\}\) and \(\{Y_{ij} : 1 \leq i < j \leq n\}\) are i.i.d. sequences of \(U[0,1]\) random variables. Denote by \(K_n\) the complete graph on the set of vertices \(\{1, 2, \ldots, n\}\) and let \(G = (V(G), E(G))\) be a subgraph of \(K_n\). Let \(\mathcal{F}_G\) be the \(\sigma\)-algebra generated by the collections \(\{U_i\}_{i \in V(G)}\) and \(\{Y_{ij}\}_{ij \in E(G)}\). Denote by \(L^2(G)\) the space of all square integrable random variables that are functions of \(\{U_i : 1 \leq i \leq n\}\) and \(\{Y_{ij} : 1 \leq i < j \leq n\}\). Now, consider the following subspace of \(L^2(G)\),

\[
M_G := \{Z \in L^2(G) : \mathbb{E}[ZV] = 0 \text{ for every } V \in L^2(H) \text{ such that } H \subset G\}.
\]

(For the empty graph, \(M_G\) is the space of all constants.) One important implication of the above definition is that, \(Z \in M_G\) if and only if \(Z \in L^2(G)\) and

\[
\mathbb{E}[Z|X_i, Y_{ij} : i \in V(H), (i, j) \in E(H)] = 0, \quad \text{for all } H \subset G.
\]

Janson and Nowicki [15, Lemma 1] showed that

\[
L^2(G) = \bigoplus_{H \subseteq G} M_H,
\]

that is, \(L^2(G)\) is the orthogonal direct sum of \(M_H\) for all \(H \subseteq G\). The above decomposition allows us to decompose the \(L^2\) norm of any function \(L^2(G)\) as the sum of \(L^2\) norms of its projections on to \(M_H\) for \(H \subseteq G\). In particular, for \(f\) as in (4.1), we have the following decomposition

\[
f = \sum_{H \subseteq G} f_H,
\]

where \(f_H\) is the projection of \(f\) on to \(M_H\). Further, for \(1 \leq s \leq |V(H)|\), define,

\[
f_{(s)} = \sum_{H \subseteq G: |V(H)| = s} f_H.
\]

The smallest positive \(d\) such that \(f_{(d)} \neq 0\) is called the principle degree of \(f\). The asymptotic distribution of \(X_n(H, W)\) depends on the principle degree of \(f\) and the geometry of the subgraphs which appear in its decomposition.

4.2. Proof of Theorem 2.1 (1). Recall the definition of the function \(f\) from (4.1) and consider its decomposition as in (4.1.3). Then (4.1.4) for \(s = 1\) gives,

\[
f_{(1)} = \sum_{a=1}^{|V(H)|} f_{K(a)},
\]

where \(K(a)\) is the graph with the single vertex \(a\) and \(f_{K(a)}\) is the projection of \(f\) on to the space \(M_{K(a)}\), for \(1 \leq a \leq |V(H)|\).

**Lemma 4.1.** For \(1 \leq a \leq |V(H)|\), the projection of \(f\) on to the space \(M_{K(a)}\) is given by,

\[
f_{K(a)} = \mathbb{E}[f|U_a] - \mathbb{E}[f].
\]

**Proof.** Define \(g_a = \mathbb{E}[f|U_a] - \mathbb{E}[f]\). Note that \(g_a \in M_{K(a)}\), since \(V(K(a)) = 1\) and \(\mathbb{E}[g_a] = 0\). Now, consider any \(h \in M_{K(a)}\). Then

\[
\mathbb{E}[(f - g_a)h] = \mathbb{E}[fh] - \mathbb{E}[E[f|U_a]h] - \mathbb{E}[f]\mathbb{E}[h] = 0,
\]

where the last equality follows since \(\mathbb{E}[h] = 0\) and \(h\) is only a function of \(U_a\). Hence, \(f_{K(a)} = g_a\). \(\square\)
Using Lemma 4.1 and recalling the definition of \( f \) from (4.1),
\[
 f_{K(a)} = \sum_{H' \in \mathcal{G}} \mathbb{E} \left[ \prod_{(b,c) \in E(H')} 1 \{ Y_{bc} \leq W(U_b, U_c) \} \middle| U_a \right] - \mathbb{E}[f]
\]
\[
 = \sum_{H' \in \mathcal{G}} t_a(U_a, H, W) - \mathbb{E}[f],
\]
where the last step follows from the definition of the 1-point conditional homomorphism density function (recall Definition 2.1). Then from (4.2.1),
\[
f^{(1)} = \sum_{a=1}^{\lfloor V(H) \rfloor} \left( \sum_{H' \in \mathcal{G}} t_a(U_a, H, W) - \mathbb{E}[f] \right). \tag{4.2.2}
\]

We now proceed to show that \( f^{(1)} \neq 0 \) when \( W \) is not \( H \)-regular. For this, we need the following combinatorial identity. To this end, for any permutation \( \phi : [V(H)] \to [V(H)] \) the permuted graph \( \phi(H) = (\phi(V(H)), \phi(E(H))) \), where \( \phi(V(H)) = \{ \phi(a) : 1 \leq a \leq |V(H)| \} \) and \( \phi(E(H)) = \{ (\phi(a), \phi(b)) : (a,b) \in E(H) \} \).

**Lemma 4.2.** For the vertex join operation \( \oplus_{a,b} \) as in Definition 2.4 the following holds:
\[
|\mathcal{G}_H|^2 \sum_{1 \leq a, b \leq |V(H)|} t \left( H \oplus_{a,b} H, W \right) = |V(H)|^2 \sum_{H_1, H_2 \in \mathcal{G}_H} t \left( H_1 \oplus_{1,1} H_2, W \right). \tag{4.2.3}
\]

**Proof.** First, fix \( (a,b) \in [|V(H)|]^2 \) and consider two permutations, \( \phi_a : [|V(H)|] \to [|V(H)|] \) and \( \phi_b : [|V(H)|] \to [|V(H)|] \) such that \( \phi_a(a) = \phi_b(b) = 1 \). Then
\[
\sum_{1 \leq a, b \leq |V(H)|} \sum_{H_1, H_2 \in \mathcal{G}_H} t \left( H_1 \oplus_{a,b} H_2, W \right) = \sum_{1 \leq a, b \leq |V(H)|} \sum_{H_1, H_2 \in \mathcal{G}_H} t \left( \phi_a(H_1) \oplus_{1,1} \phi_b(H_2), W \right)
\]
\[
= \sum_{1 \leq a, b \leq |V(H)|} \sum_{H_1, H_2 \in \mathcal{G}_H} t \left( H_1 \oplus_{1,1} H_2, W \right)
\]
\[
= |V(H)|^2 \sum_{H_1, H_2 \in \mathcal{G}_H} t \left( H_1 \oplus_{1,1} H_2, W \right), \tag{4.2.4}
\]
where the second equality follows, since the map \( (H_1, H_2) \to (\phi_a(H_1), \phi_b(H_2)) \) is a bijection from \( \mathcal{G}_H^2 \) to \( \mathcal{G}_H^2 \), for all \( 1 \leq a, b \leq |V(H)| \).

Next, fix \( H_1, H_2 \in \mathcal{G}_H \). Then consider isomorphisms \( \phi_1, \phi_2 : [|V(H)|] \to [|V(H)|] \) such that \( \phi_1(H_1) = H \) and \( \phi_2(H_2) = H \). Thus,
\[
\sum_{H_1, H_2 \in \mathcal{G}_H} \sum_{1 \leq a, b \leq |V(H)|} t \left( H_1 \oplus_{a,b} H_2, W \right) = \sum_{H_1, H_2 \in \mathcal{G}_H} \sum_{1 \leq a, b \leq |V(H)|} t \left( H \oplus_{\phi_1(a), \phi_2(b)} H, W \right)
\]
\[
= \sum_{H_1, H_2 \in \mathcal{G}_H} \sum_{1 \leq a, b \leq |V(H)|} t \left( H \oplus_{a,b} H, W \right)
\]
\[
= |\mathcal{G}_H|^2 \sum_{1 \leq a, b \leq |V(H)|} t \left( H \oplus_{a,b} H, W \right). \tag{4.2.5}
\]
Here, once again the second equality follows since \( (a,b) \to (\phi_1(a), \phi_2(b)) \) is a bijection from \( [|V(H)|]^2 \) to \( [|V(H)|]^2 \).
Combining (4.2.4) and (4.2.5) the identity in (4.2.3) follows. □

Now we are ready to show that \( f_{(1)} \neq 0 \) (recall (4.2.2)) when \( W \) is not \( H \)-regular.

**Lemma 4.3.** If \( W \) is not \( H \)-regular, then \( f_{(1)} \neq 0 \).

**Proof.** To show that \( f_{(1)} \neq 0 \), it suffices to show that \( \text{Var}[f_{(1)}] \neq 0 \). Towards this, recalling (4.2.2) gives,

\[
\text{Var}[f_{(1)}] = \sum_{a=1}^{\lvert V(H) \rvert} \text{Var} \left[ \sum_{H' \in \mathcal{G}_H} t_a(U_a, H', W) \right]. \tag{4.2.6}
\]

Consider the term corresponding to \( a = 1 \) in the sum above. Then for any \( H_1, H_2 \in \mathcal{G}_H \),

\[
\mathbb{E} \left[ t_1(U_1, H_1, W) t_1(U_1, H_2, W) \right] = t \left( H_1 \bigoplus_{1,1} H_2, W \right).
\]

Hence,

\[
\text{Var} \left[ \sum_{H' \in \mathcal{G}_H} t_1(U_1, H', W) \right] = \sum_{H_1, H_2 \in \mathcal{G}_H} \text{Cov} \left[ t_1(U_1, H_1, W), t_1(U_1, H_2, W) \right] = \sum_{H_1, H_2 \in \mathcal{G}_H} \left( t \left( H_1 \bigoplus_{1,1} H_2, W \right) - t(H, W)^2 \right). \tag{4.2.7}
\]

Now, an argument similar to Lemma 4.2 shows that

\[
\sum_{H' \in \mathcal{G}_H} t_a(x, H', W) = \sum_{H' \in \mathcal{G}_H} t_b(x, H', W),
\]

for all \( x \in [0, 1] \) and \( 1 \leq a, b \leq \lvert V(H) \rvert \). Hence, (4.2.6) and (4.2.7) implies,

\[
\text{Var}[f_{(1)}] = \lvert V(H) \rvert \sum_{H_1, H_2 \in \mathcal{G}_H} \left( t \left( H_1 \bigoplus_{1,1} H_2, W \right) - t(H, W)^2 \right) = \lvert V(H) \rvert \sum_{1 \leq a, b \leq \lvert V(H) \rvert} \left( H \bigoplus_{a,b} W - t(H, W)^2 \right) \tag{4.2.8}
\]

where the last step uses Lemma 4.2. This shows \( \text{Var}[f_{(1)}] \) is zero if and only if

\[
\frac{1}{\lvert V(H) \rvert^2} \sum_{1 \leq a, b \leq \lvert V(H) \rvert} t \left( H \bigoplus_{a,b} W \right) = t(H, W)^2. \tag{4.2.9}
\]

Now observe,

\[
\frac{1}{\lvert V(H) \rvert^2} \sum_{1 \leq a, b \leq \lvert V(H) \rvert} t \left( H \bigoplus_{a,b} W \right) = \sum_{1 \leq a, b \leq \lvert V(H) \rvert} \int t_a(x, H, W) t_b(x, H, W) dx = \int \left( \sum_{1 \leq a \leq \lvert V(H) \rvert} t_a(x, H, W) \right)^2 dx.
\]
Thus (4.2.9) becomes,
\[
\int \left( \sum_{1 \leq a \leq |V(H)|} t_a(x, H, W) \right)^2 \, dx - \left( \int \sum_{1 \leq a \leq |V(H)|} t_a(x, H, W) \, dx \right)^2 = 0 \iff \text{Var} [\Delta(U)] = 0,
\]
where \( \Delta(x) = \sum_{1 \leq a \leq |V(H)|} t_a(x, H, W) \) and \( U \sim \text{Uniform}[0, 1] \). Then \( \text{Var}[f_{(1)}] = 0 \) if and only if \( \Delta(\cdot) \) is constant a.s. Therefore, since \( \mathbb{E}\Delta(U) = |V(H)|t(H, W) \), then \( \text{Var}[f_{(1)}] = 0 \) if and only if
\[
\frac{1}{r} \sum_{1 \leq a \leq r} t_a(x, H, W) = t(H, W) \text{ for almost every } x \in [0, 1].
\]
This shows \( f_{(1)} \neq 0 \) if \( W \) is not \( H \)-regular.

The above lemma shows that the principle degree of \( f \) in 1. Thus, by [15, Theorem 1] we have,
\[
X_n(H, W) - \frac{(n)^{|V(H)|} t(H, W)}{n^{|V(H)| - \frac{1}{2}}} \xrightarrow{D} \mathcal{N}(0, \tau^2),
\]
where
\[
\tau^2 = \frac{1}{|V(H)|!(|V(H)| - 1)!} \text{Var}[f_{(1)}]
\]
\[
= \frac{1}{|\text{Aut}(H)|^2} \left( \sum_{1 \leq a, b \leq |V(H)|} t(H \oplus a, b, H, W) - |V(H)|^2 t(H, W)^2 \right) := \tau_{H, W}^2.
\]

Here, in the last step we use (4.2.8) and (2.1.3). This completes the proof of Theorem 2.1 (1).

4.3. Proof of Theorem 2.1 (2). In this case, \( W \) is \( H \)-regular, hence \( f_{(1)} \equiv 0 \) by Lemma 4.3. Therefore, we consider \( f_{(2)} \) (recall (4.1.4)) which can be written as
\[
f_{(2)} = \sum_{1 \leq a < b \leq |V(H)|} \left( f_{K(a,b)} + f_{Q(a,b)} \right),
\]
where \( K_{a,b} = (\{a, b\}, \emptyset) \) is the graph with two vertices \( a \) and \( b \) and no edges, and \( Q_{a,b} = (\{a, b\}, \{(a,b)\}) \) is the complete graph with vertices \( a \) and \( b \).

**Lemma 4.4.** For \( 1 \leq a < b \leq |V(H)| \) the projection of \( f \) on to the space \( M_{K_{a,b}} \) is given by,
\[
f_{K_{a,b}} = \mathbb{E}[f|U_a, U_b] - \mathbb{E}[f|U_a] - \mathbb{E}[f|U_b] + \mathbb{E}[f].
\]

**Proof.** Observe that,
\[
\mathbb{E}[f_{K_{a,b}}|U_a] = \mathbb{E} [\mathbb{E}[f|U_a, U_b]|U_a] - \mathbb{E}[f|U_a] - \mathbb{E}[f] + \mathbb{E}[f] = 0.
\]
Similarly, \( \mathbb{E}[f_{K_{a,b}}|U_b] = 0 \) and, of course, \( \mathbb{E}[f_{K_{a,b}}] = 0 \). Then by (4.1.1) we have \( f_{K_{a,b}} \in M_{K_{a,b}} \). Now, consider \( g \in M_{K_{a,b}} \). Then,
\[
\mathbb{E} \left[ (f - f_{K_{a,b}})g \right] = \mathbb{E}[fg] - \mathbb{E}[\mathbb{E}[f|U_a, U_b]|U_a] g + \mathbb{E} [\mathbb{E}[f|U_a]|g] + \mathbb{E} [\mathbb{E}[f|U_b]|g] - \mathbb{E}[f] \mathbb{E}[g].
\]
(4.3.1)

Now, observe that \( \mathbb{E}[g] = 0 \) and
\[
\mathbb{E} [\mathbb{E}[f|U_a]|g] = \mathbb{E} [\mathbb{E}[f|U_a]|U_a] = \mathbb{E} [\mathbb{E}[f|U_a]|g|U_a]] = 0,
\]
and
where the last equality follows since \(E[g|U_a] = 0\). Similarly, \(E[E[f|U_b]g] = 0\). Moreover, since \(g\) is a function of only \(U_a\) and \(U_b\), \(E[f|g] = E[E[f|U_a, U_b]|g]\). Hence, from (4.3.1) we conclude that \(E[(f - f_K_{(a,b)})g] = 0\). Thus, \(f_K_{(a,b)}\) is the projection of \(f\) on to \(M_{K_{(a,b)}}\).

Now we focus on finding \(f_{Q_{(a,b)}}\), the projection of \(f\) on \(M_{Q_{(a,b)}}\).

**Lemma 4.5.** For \(1 \leq a < b \leq |V(H)|\) the projection of \(f\) on to \(M_{Q_{(a,b)}}\) is given by,

\[
f_{Q_{(a,b)}} = E[f|U_a, U_b, Y_{ab}] - \sum_{H \in \mathcal{H}} t_{(a,b)}(U_a, U_b, H', W).
\]  

(4.3.2)

**Proof.** It suffices to prove the result for \(a = 1, b = 2\). Towards this, note that

\[
E[t_{(1,2)}(U_1, U_2, H, W)] = t(H', W).
\]

This implies, \(E[f_{Q_{(1,2)}}] = 0\). Next, observe that

\[
E[t_{(1,2)}(U_1, U_2, H, W)|U_1] = t_1(U_1, H', W).
\]

This implies, \(E[f_{Q_{(1,2)}|U_1}] = 0\), since by definition,

\[
E[f|U_1] = \sum_{H' \in \mathcal{H}} t_1(U_1, H', W).
\]

Similarly, we have \(E[f_{Q_{(1,2)}|U_2}] = 0\). Finally, using \(E[f|U_1, U_2] = t_{(1,2)}(U_1, U_2, H', W)\), shows that \(E[f_{Q_{(1,2)}|U_1, U_2}] = 0\). Thus, \(f_{Q_{(1,2)}} \in M_{Q_{(1,2)}}\).

Now, take \(g \in M_{Q_{(1,2)}}\), then

\[
E[(f - f_{Q_{(1,2)}})g] = E[g] - E[E[f|U_1, U_2, Y_{12}]g] + E \left[ g \sum_{H \in \mathcal{H}} t_{(1,2)}(U_1, U_2, H', W) \right]
\]

\[
= E[E[g|U_1, U_2, Y_{12}]] - E[E[g|U_1, U_2, Y_{12}] + E[E[g|U_1, U_2]h(U_1, U_2)] = 0,
\]

where \(h(U_1, U_2) := \sum_{H \in \mathcal{H}} t_{(1,2)}(U_1, U_2, H', W)\) and the last equality follows since \(g \in M_{Q_{(1,2)}}\) implies that \(E[g|U_1, U_2] = 0\). This shows that \(f_{Q_{(1,2)}}\) is the projection of \(f\) on to \(M_{Q_{(1,2)}}\). \(\square\)

Now, we will to compute the variance of \(f_{Q_{(1,2)}}\). For this we need a few definitions. Let \(V^2_H = \{(a, b) \in V(H)^2 : a < b\}\). For \((a, b), (c, d) \in V^2_H\) define,

\[
t \left( H_1 \bigotimes_{(a,b),(c,d)} H_2, W \right) = t \left( H_1 \bigotimes_{(a,b),(c,d)} H_2, W \right) \mathbf{1}\{(a, b) \in E(H_1) \text{ and } (c, d) \in E(H_2)\}.
\]  

(4.3.3)

an similarly,

\[
t \left( H_1 \bigoplus_{(a,b),(c,d)} H_2, W \right) = t \left( H_1 \bigoplus_{(a,b),(c,d)} H_2, W \right) \mathbf{1}\{(a, b) \in E(H_1) \text{ and } (c, d) \in E(H_2)\}.
\]  

(4.3.4)

Then we have the following identities:

**Lemma 4.6.** Let \(V^2_H\) be as defined above, then

\[
\sum_{(a, b), (c, d) \in V^2_H} t \left( H_1 \bigotimes_{(a,b),(c,d)} H_2, W \right) = |\mathcal{H}|^2 \sum_{(a, b), (c, d) \in V^2_H} t \left( H \bigotimes_{(a,b),(c,d)} H, W \right).
\]  

(4.3.5)
Similarly,

\[
\sum_{(a,b),(c,d) \in V_2} \sum_{H_1, H_2 \in \mathcal{G}_{H}} t \left( H_1 \bigoplus_{(a,b),(c,d)} H_2, W \right) = \left| \mathcal{G}_{H} \right|^2 \sum_{(a,b),(c,d) \in V_2} t \left( H \bigoplus_{(a,b),(c,d)} H, W \right), \tag{4.3.6}
\]

Proof. We will first show that

\[
\sum_{(a,b),(c,d) \in V_2} \sum_{H_1, H_2 \in \mathcal{G}_{H}} t \left( H_1 \bigoplus_{(a,b),(c,d)} H_2, W \right) = K_H \sum_{H_1, H_2 \in \mathcal{G}_{H}} t \left( H_1 \bigoplus_{(1,2),(1,2)} H_2, W \right), \tag{4.3.7}
\]

where \( K_H := \frac{|V(H)|^2 (|V(H)| - 1)^2}{4} \). For this consider permutations \( \phi_{(a,b)}, \phi_{(c,d)} : [|V(H)|] \rightarrow [|V(H)|] \) such that \( \phi_{(a,b)}(a) = 1 \) and \( \phi_{(a,b)}(b) = 2 \), and \( \phi_{(c,d)}(c) = 1 \) and \( \phi_{(c,d)}(d) = 2 \). Hence

\[
\sum_{(a,b),(c,d) \in V_2} \sum_{H_1, H_2 \in \mathcal{G}_{H}} t \left( H_1 \bigoplus_{(a,b),(c,d)} H_2, W \right) = \sum_{(a,b),(c,d) \in V_2} \sum_{\mathcal{G}_{H}} t \left( \phi_{(a,b)}(H_1) \bigoplus_{(1,2),(1,2)} \phi_{(c,d)}(H_2), W \right) = K_H \sum_{\mathcal{G}_{H}} t \left( H_1 \bigoplus_{(1,2),(1,2)} H_2 \right),
\]

where the last equality follows from the observation that \( (H_1, H_2) \rightarrow (\phi_{(a,b)}(H_1), \phi_{(c,d)}(H_2)) \) is an isomorphism from \( \mathcal{G}_{H} \) to \( \mathcal{G}_{H} \), for all \( (a,b), (c,d) \in V_2 \).

Now by considering isomorphisms \( \phi_1 \) and \( \phi_2 \) such that \( \phi_1(H_1) = H \) and \( \phi_2(H_2) = H \) and a similar argument as above shows that

\[
K_H \sum_{H_1, H_2 \in \mathcal{G}_{H}} t \left( H_1 \bigoplus_{(1,2),(1,2)} H_2, W \right) = \left| \mathcal{G}_{H} \right|^2 \sum_{(a,b),(c,d) \in V_2} t \left( H \bigoplus_{(a,b),(c,d)} H, W \right). \tag{4.3.8}
\]

Combining (4.3.7) and (4.3.8) the identity in (4.3.5) follows. The identity in (4.3.6) can be proved similarly.

With the above definitions and identities we now proceed to compute the variance of \( f_{Q(1,2)} \).

**Lemma 4.7.** For \( f_{Q(1,2)} \) as defined in (4.3.2),

\[
\text{Var}[f_{Q(1,2)}] = \frac{4(|V(H)| - 1)^2}{|Aut(H)|^2} \sum_{(a,b),(c,d) \in E(H)} t \left( H \bigoplus_{(a,b),(c,d)} H, W \right) - t \left( H \bigoplus_{(a,b),(c,d)} H, W \right).
\]

**Proof.** Note that \( E[f_{Q(1,2)}] = 0 \). Hence, \( \text{Var}[f_{Q(1,2)}] = E[f_{Q(1,2)}^2] \). Now, define

\[
Z_{H'}(Y_{12}, U_1, U_2) = \begin{cases} 1 & \text{if } (1,2) \notin E(H') \\ 1 \{Y_{12} \leq W(U_1, U_2)\} & \text{otherwise} \end{cases}
\]

and consider \( \mathcal{G}_{H,(1,2)} = \{ H' \in \mathcal{G}_{H} : (1,2) \in E(H') \} \). Then observe that,

\[
E[f|U_1, U_2, Y_{12}] = \sum_{H' \in \mathcal{G}_{H}} t_{(1,2)}(U_1, U_2, H'\{\{1,2\}\}, W) Z_{H'}(Y_{12}, U_1, U_2),
\]

where \( H'\{\{1,2\}\} \) is the graph obtained from \( H' \) with the edge \( (1,2) \) removed, if the edge exists. Next, note that

\[
t_{(1,2)}((x,y), H', W) = t_{(1,2)}((x,y), H'\{\{1,2\}\}, W) W_{H',\{1,2\}}(x,y).
\]
where
\[
W_{H',(1,2)}(x, y) := \begin{cases} 
W(x, y) & \text{if } (1, 2) \in E(H') \\
1 & \text{otherwise.}
\end{cases}
\]

Thus,
\[
\mathbb{E}[f_{Q,(1,2)}^2] = \sum_{H_1, H_2 \in \mathcal{G}_H} \mathbb{E} \left[ h(U_1, U_2, H_1, H_2) Y_{H_1}(U_1, U_2) Y_{H_2}(U_1, U_2) \right],
\]
where \( h(U_1, U_2, H_1, H_2) := t_{(1,2)}(U_1, U_2, H_1 \setminus \{(1, 2)\}, W) t_{(1,2)}(U_1, U_2, H_2 \setminus \{(1, 2)\}, W) \), and
\[
Y_{H'}(U_1, U_2) := Z_{H'}(U_1, U_2) - W_{H',(1,2)}(U_1, U_2),
\]
for \( H' \in \mathcal{G}_H \). Observe that if \((1, 2) \notin H_1 \cup H_2\), then
\[
Y_{H_1}(U_1, U_2) Y_{H_2}(U_1, U_2) = 0.
\]

Hence, recalling the notions of weak and strong edge joins from Definition 2.4 gives,
\[
\mathbb{E}[f_{Q,(1,2)}^2] = \sum_{H_1, H_2 \in \mathcal{G}_H} \mathbb{E} \left[ h(U_1, U_2, H_1, H_2) \left( 1 \{Y_{12} \leq W(U_1, U_2)\} - W(U_1, U_2) \right)^2 \right]
\]
\[
= \sum_{H_1, H_2 \in \mathcal{G}_H} \mathbb{E} \left[ h(U_1, U_2, H_1, H_2, W) W(U_1, U_2)(1 - W(U_1, U_2)) \right]
\]
\[
= \sum_{H_1, H_2 \in \mathcal{G}_H} t \left( H_1 \ominus_{(1,2),(1,2)} H_2, W \right) - t \left( H_1 \oplus_{(1,2),(1,2)} H_2, W \right). \tag{4.3.9}
\]

Now, using the notations introduced in (4.3.3) and (4.3.4), the identity (4.3.9) can be written as,
\[
\mathbb{E}[f_{Q,(1,2)}^2] = \sum_{H_1, H_2 \in \mathcal{G}_H} t \left( H_1 \ominus_{(1,2),(1,2)} H_2, W \right) - t \left( H_1 \oplus_{(1,2),(1,2)} H_2, W \right)
\]
\[
= \frac{4(|V(H)| - 2)!^2}{|\text{Aut}(H)|^2} \sum_{(a,b),(c,d) \in V_H^2} t \left( H \ominus_{(a,b),(c,d)} H, W \right) - t \left( H \oplus_{(a,b),(c,d)} H, W \right)
\]
\[
= \frac{4(|V(H)| - 2)!^2}{|\text{Aut}(H)|^2} \sum_{(a,b),(c,d) \in E(H)} t \left( H \ominus_{(a,b),(c,d)} H, W \right) - t \left( H \oplus_{(a,b),(c,d)} H, W \right),
\]
where the second equality uses the identities from Lemma 4.6 and the third equality follows from the definitions in (4.3.3) and (4.3.4).

Next, we compute the Hilbert-Schmidt operator \( T \) as defined in [15, Theorem 2]. Note that in our case this operator is defined on the space \( M_{K_{(1)}} \). Then considering \( g, h \in M_{K_{(1)}} \) and following the definitions given in [15, Theorem 2] we get
\[
\langle Tg, h \rangle = \frac{1}{2(|V(H)| - 2)!} \mathbb{E} \left[ f g(U_1) h(U_2) \right] = \frac{1}{2(|V(H)| - 2)!} \mathbb{E} \left[ f_{K,(1,2)} g(U_1) h(U_2) \right], \tag{4.3.10}
\]
where the second step uses the orthogonal projection of \( f \) on to the space \( M_{K_{(1,2)}} \). Now, from Lemma 4.4,
\[
f_{K,(1,2)} = \mathbb{E} \left[ f |U_1, U_2 \right] - \mathbb{E} \left[ f |U_1 \right] - \mathbb{E} \left[ f |U_2 \right] + \mathbb{E} [f]
\]
Combining (4.3.12) and (4.3.13), we have

$$= \sum_{H' \in \mathcal{G}_H} \{t_{(1,2)}(U_1, U_2, H', W) - t_1(U_1, H', W) - t_2(U_2, H', W) + \mathbb{E}[f]\}.$$  

This implies, from (4.3.10) and using $\mathbb{E}[g] = \mathbb{E}[h] = 0$, that

$$\langle Tg, h \rangle = \frac{1}{2(|V(H)| - 2)!} \mathbb{E} \left[ \sum_{H' \in \mathcal{G}_H} t_{(1,2)}(U_1, U_2, H', W)g(U_1)h(U_2) \right]$$

$$= \left\langle \frac{1}{2(|V(H)| - 2)!} \int \sum_{H' \in \mathcal{G}_H} t_{(1,2)}(x, \cdot, H', W)g(x)dx, h(\cdot) \right\rangle.$$  

This implies that the operator $T : M_{K(1)} \to L^2[0, 1]$ is given by,

$$(Tg)(y) = \frac{1}{2(|V(H)| - 2)!} \int \sum_{H' \in \mathcal{G}_H} t_{(1,2)}(x, y, H', W)g(x)dx, \quad (4.3.11)$$

for $g \in M_{K(1)}$. The following lemma shows that the this operator is in fact same as the operator $T_{W_H}$ (as defined in (2.3.1)) associated with the 2-point conditional graphon $W_H$ induced by $H$ (recall (2.3.2)) restricted to the space $M_{K(1)}$. For this, also recall that when $W$ is $H$-regular $\text{Spec}^{-}(W_H)$ denotes collection $\text{Spec}(W_H)$ with multiplicity of the eigenvalue $d_{W_H}$ (as defined in (2.3.4)) decreased by 1.

**Lemma 4.8.** The operator $T$ as defined in (4.3.11) is same as the operator $T_{W_H}$ restricted to the space $M_{K(1)}$. Moreover, if $W$ is $H$-regular, then the multiset of eigenvalues of $T$ is equal to $\text{Spec}^{-}(W_H)$.

**Proof.** Denote by $S_{|V(H)|}$ the set of all $|V(H)|!$ permutations of $|V(H)|$. Then it is easy to observe that

$$\sum_{\phi \in S_{|V(H)|}} t_{(1,2)}(x, y, \phi(H), W) = |\text{Aut}(H)| \sum_{H' \in \mathcal{G}_H} t_{(1,2)}(x, y, H', W) \quad (4.3.12)$$

Also,

$$\sum_{\phi \in S_{|V(H)|}} t_{(1,2)}(x, y, \phi(H), W) = \sum_{1 \leq a \neq b \leq r} \sum_{\phi \in S_{|V(H)|}} t_{(1,2)}(x, y, \phi(H), W)$$

$$= \sum_{1 \leq a \neq b \leq r} \sum_{\phi \in S_{|V(H)|}} t_{\phi^{-1}(1)\phi^{-1}(2)}(x, y, H, W)$$

$$= \sum_{1 \leq a \neq b \leq r} \sum_{\phi \in S_{|V(H)|}} t_{(a,b)}(x, y, H, W)$$

$$= (|V(H)| - 2)! \sum_{1 \leq a \neq b \leq r} t_{ab}(x, y, H, W) \quad (4.3.13)$$

Combining (4.3.12) and (4.3.13), we have

$$\frac{1}{2(|V(H)| - 2)!} \sum_{H' \in \mathcal{G}_H} t_{(1,2)}(x, y, H', W) = \frac{1}{2|\text{Aut}(H)|!} \sum_{1 \leq a \neq b \leq r} t_{(a,b)}(x, y, H, W) = W_H(x, y)$$

Thus (4.3.11) becomes,

$$(Tg)(x) = \int W_H(y, x)g(y)dy = \int W_H(x, y)g(y)dy,$$
where the last equality follows since $W_H$ is symmetric. Hence, recalling the definition of $T_{W_H}$ from (2.3.1), it follows that the operator $T$ is same as the kernel operator $T_{W_H}$ restricted to $M_{K(1)}$.

To find the spectrum of $T$ recall the following decomposition of $L^2[0, 1]$ from (4.1.2):

$$L^2[0, 1] = M_\emptyset \oplus M_{K(1)},$$

where $M_\emptyset$ is the space of all constants. Now, the calculation leading up to (2.3.4) shows that if $W$ is $H$-regular, then the constant function 1 is an eigenvector of the operator $T_{W_H}$ (corresponding to the eigenvalue $d_{W_H}$). Clearly, $M_\emptyset = \text{span}(\{1\})$. Thus, using the orthogonality of eigenfunctions, all other eigenfunctions of $T_{W_H}$ belongs to $M_{K(1)}$. Since the operator $T$ is the restriction of $T_{W_H}$ to $M_{K(1)}$, it follows that the multiset of eigenvalues of $T$ is equal to $\text{Spec}^-(W_H)$.

Now, invoking [15, Theorem 2] and using Lemma 4.8 it follows that

$$X_n(H, W) - \frac{(n)|V(H)|}{|\text{Aut}(H)|} t(H, W) \xrightarrow{\mathcal{D}} \sigma \cdot Z + \sum_{\lambda \in \text{Spec}^-(W_H)} \lambda(Z^2_{\lambda} - 1),$$

where $Z, \{Z_\lambda\}_{\lambda \in \text{Spec}^-(W_H)}$ are all independent standard Gaussians and

$$\sigma^2 = \frac{1}{2(|V(H)| - 2)!^2} \text{Var}[f_{Q(1,2)}] = \frac{2}{|\text{Aut}(H)|^2} \sum_{(a,b), (c,d) \in E(H)} \left[ t \left( H \begin{array}{c} \oplus \ H, W \end{array} \begin{array}{c} \ominus \ \ominus \ \ominus \ \ominus \end{array} \begin{array}{c} (a,b), (c,d) \end{array} \right) - t \left( H \begin{array}{c} \ominus \ \ominus \ \ominus \ \ominus \end{array} \begin{array}{c} (a,b), (c,d) \end{array} \end{array} \right) \right] := \sigma^2_{H, W},$$

where the last equality uses Lemma 4.7. This completes the proof of Theorem 2.1 (2).

References

[1] A. D. Barbour, M. Karoński, and A. Ruciński. A central limit theorem for decomposable random variables with applications to random graphs. Journal of Combinatorial Theory, Series B, 47(2):125–145, 1989.

[2] A. Basak and S. Mukherjee. Universality of the mean-field for the potts model. Probability Theory and Related Fields, 168(3):557–600, 2017.

[3] B. B. Bhattacharya and S. Mukherjee. Monochromatic subgraphs in randomly colored graphons. European Journal of Combinatorics, 81:328–353, 2019.

[4] B. B. Bhattacharya, P. Diaconis, and S. Mukherjee. Universal limit theorems in graph coloring problems with connections to extremal combinatorics. Annals of Applied Probability, 27(1):337–394, 2017.

[5] M. Boguná and R. Pastor-Satorras. Class of correlated random networks with hidden variables. Physical Review E, 68(3):036112, 2003.

[6] B. Bollobás, S. Janson, and O. Riordan. The phase transition in inhomogeneous random graphs. Random Structures & Algorithms, 31(1):3–122, 2007.

[7] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi. Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing. Advances in Mathematics, 219(6):1801–1851, 2008.

[8] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi. Convergent sequences of dense graphs II. multiway cuts and statistical physics. Annals of Mathematics, pages 151–219, 2012.

[9] S. Chatterjee and P. Diaconis. Estimating and understanding exponential random graph models. The Annals of Statistics, 41(5):2428–2461, 2013.

[10] S. Chatterjee and S. S. Varadhan. The large deviation principle for the Erdős-Rényi random graph. European Journal of Combinatorics, 32(7):1000–1017, 2011.
[11] J.-F. Delmas, J.-S. Dherin, and M. Sciauveau. Asymptotic for the cumulative distribution function of the degrees and homomorphism densities for random graphs sampled from a graphon. *Random Structures & Algorithms*, 58(1):94–149, 2021.

[12] P. Diaconis and D. Freedman. On the statistics of vision: the Julesz conjecture. *Journal of Mathematical Psychology*, 24(2):112–138, 1981.

[13] V. Féray, P.-L. Méliot, and A. Nikeghbali. Graphons, permutations and the Thoma simplex: three mod-Gaussian moduli spaces. *Proceedings of the London Mathematical Society*, 121(4):876–926, 2020.

[14] J. Hladký, C. Pelekis, and M. Šileikis. A limit theorem for small cliques in inhomogeneous random graphs. *Journal of Graph Theory (to appear)*, 2021.

[15] S. Janson and K. Nowicki. The asymptotic distributions of generalized $U$-statistics with applications to random graphs. *Probability Theory and Related Fields*, 90(3):341–375, 1991.

[16] G. Kaur and A. Röllin. Higher-order fluctuations in dense random graph models. *arXiv preprint arXiv:2006.15805*, 2020.

[17] L. Lovász. *Large networks and graph limits*, volume 60. American Mathematical Soc., 2012.

[18] L. Lovász and B. Szegedy. Limits of dense graph sequences. *Journal of Combinatorial Theory, Series B*, 96(6):933–957, 2006.

[19] P.-L. Méliot. A central limit theorem for singular graphons. *arXiv preprint arXiv:2103.15741*, 2021.

[20] K. Nowicki and J. C. Wierman. Subgraph counts in random graphs using incomplete $U$-statistics methods. *Discrete Mathematics*, 72(1-3):299–310, 1988.

[21] A. Ruciński. When are small subgraphs of a random graph normally distributed? *Probability Theory and Related Fields*, 78(1):1–10, 1988.

[22] Z.-S. Zhang. Berry–Esseen bounds for generalized $U$-statistics. *arXiv preprint arXiv:2104.03479*, 2021.