NECESSARY AND SUFFICIENT CONDITIONS FOR \( n \)-TIMES FRÉCHET DIFFERENTIABILITY ON \( S^p \), \( 1 < p < \infty \).

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Abstract. Let \( 1 < p < \infty \) and let \( n \geq 1 \). It is proved that a function \( f : \mathbb{R} \to \mathbb{C} \) is \( n \)-times Fréchet differentiable on \( S^p \) at every self-adjoint operator if and only if \( f \) is \( n \)-times differentiable, \( f', f'' \), \ldots, \( f^{(n)} \) are bounded and \( f^{(n)} \) is uniformly continuous.

1. Introduction

Let \( H \) be an infinite dimensional separable Hilbert space. For \( 1 < p < \infty \), denote by \( S^p := S^p(H) \) the \( p \)-Schatten class on \( H \), with norm \( \| \cdot \|_p \). For \( n \geq 1 \), denote by \( C^n_b(\mathbb{R}) \) the space of \( n \)-times continuously differentiable functions \( f : \mathbb{R} \to \mathbb{C} \) such that \( f', f'', \ldots, f^{(n)} \) are all bounded\(^1\) and \( C^n_0(\mathbb{R}) := \bigcap_{n \geq 1} C^n_b(\mathbb{R}) \). We denote the self-adjoint (real) subspace of \( S^p \) by \( S^p_{sa} \).

A famous result of Potapov and Sukcochev [5] states that if \( f : \mathbb{R} \to \mathbb{C} \) is Lipschitz continuous, \( A \) is a (potentially unbounded) self-adjoint operator on \( H \) and \( X \in S^p_{sa} \), then
\[
\varphi(A, X) := f(A + X) - f(A) \in S^p.
\]

It therefore makes sense to define a function
\[
\varphi(a, p, X) : S^p_{sa} \to S^p, \quad \varphi(a, p, X) = f(A + X) - f(A).
\]

The following definitions are in line with [4, Definition 3.2].

Definition 1.1. Let \( 1 < p < \infty \), let \( n \geq 1 \) and let \( f : \mathbb{R} \to \mathbb{C} \) be Lipschitz continuous.

1. Let \( A \) be a potentially unbounded self-adjoint operator on \( H \).
   
   We say that \( f \) is 1-time Fréchet differentiable on \( S^p \) at \( A \) if there exists a bounded linear map
   \[
   D^1_p f(A) \in \mathcal{B}(S^p, S^p), \quad X \mapsto D^1_p f(A)[X]
   \]
   such that
   \[
   \|f(A + X) - f(A) - D^1_p f(A)[X]\|_p = o(\|X\|_p)
   \]
as \( \|X\|_p \to 0 \), \( X \in S^p_{sa} \).

   For \( n \geq 2 \), we say that \( f \) is \( n \)-times Fréchet differentiable on \( S^p \) at \( A \) if \( f \) is \( (n - 1) \)-times Fréchet differentiable on \( S^p \) at \( A + X \) for every \( X \) in an \( S^p_{sa} \)-neighborhood of 0 and there exists a bounded \( n \)-multilinear operator
   \[
   D^n_p f(A) \in \mathcal{B}_n(S^p \times \cdots \times S^p, S^p), \quad (X_1, \ldots, X_n) \mapsto D^n_p f(A)[X_1, \ldots, X_n]
   \]
such that
   \[
   \|D^{n-1}_p f(A + X)[X_1, \ldots, X_{n-1}] - D^{n-1}_p f(A)[X_1, \ldots, X_{n-1}] - D^n_p f(A)[X_1, \ldots, X_{n-1}, X]\|_p = o(\|X\|_p)\|X_1\|_p \cdots \|X_{n-1}\|_p
   \]
as \( \|X\|_p \to 0 \), \( X \in S^p_{sa} \), uniformly for all \( X_1, \ldots, X_{n-1} \in S^p \).

2. We say that \( f \) is \( n \)-times continuously Fréchet differentiable on \( S^p \) if it is \( n \)-times Fréchet differentiable on \( S^p \) at every self-adjoint operator \( A \) and for any such \( A \), the mapping
   \[
   S^p_{sa} \to \mathcal{B}_n(S^p \times \cdots \times S^p, S^p), \quad X \mapsto D^n_p f(A + X),
   \]
is continuous.

^1Note that \( f \) itself is not assumed to be bounded.

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We note that the function \( \varphi_{A,p} \) from (1.1) is an \( n \)-times Fréchet differentiable map between the Banach spaces \( S^p_{sa} \) and \( S^p \) if and only if \( f \) is \( n \)-times Fréchet differentiable on \( S^p \) at \( A + X \) for any \( X \in S^p_{sa} \). In this case, for any \( 1 \leq m \leq n \), the \( m \)th order Fréchet derivative of \( \varphi_{A,p} \) is the mapping
\[
D^m \varphi_{A,p} = D^m_p f(A + \cdot): S^p_{sa} \to B_m(S^p \times \cdots \times S^p, S^p).
\]
Consequently \( f \) is \( n \)-times continuously Fréchet differentiable on \( S^p \) if and only if \( \varphi_{A,p} \) is \( n \)-times continuously Fréchet differentiable for any self-adjoint operator \( A \) on \( H \).

The aim of this note is to prove the following necessary and sufficient characterisation of such functions.

**Theorem 1.2.** Let \( f \) be a Lipschitz continuous function on \( \mathbb{R} \), let \( 1 < p < \infty \) and let \( n \geq 1 \). Then the following assertions are equivalent:

(i) \( f \) is \( n \)-times continuously Fréchet differentiable on \( S^p \),

(ii) For all self-adjoint \( A \), \( f \) is \( n \)-times Fréchet differentiable on \( S^p \) at \( A \),

(iii) \( f \in C^n_b(\mathbb{R}) \) and \( f^{(n)} \) is uniformly continuous on \( \mathbb{R} \).

The existence of \( f \in C^n_b(\mathbb{R}) \) which is not \( 1 \)-time Fréchet differentiable on \( S^p \) at all self-adjoint \( A \) was established in [3, Example 7.20]. It is shown in [4, Theorem 3.6] that a Lipschitz continuous function \( f : \mathbb{R} \to \mathbb{C} \) is \( n \)-times continuously Fréchet differentiable at every bounded self-adjoint operator if and only if \( f \in C^n(\mathbb{R}) \) (the case \( n = 1 \) goes back to [3, Theorem 7.17]). Theorem 1.2 is a similar characterization in the unbounded case. We refer the reader to the above mentioned papers and to [1] for more results on this theme.

2. Proof of the main result

It is clear that (i) implies (ii). We prove Theorem 1.2 by first showing that (ii) implies (iii), and secondly that (iii) implies (i).

**Proof that (ii) implies (iii).** Assume that \( f \) satisfies (ii). Then it follows from [4, Proposition 3.9] that \( f \in C^n_b(\mathbb{R}) \). Hence, we only need to prove that \( f^{(n)} \) is uniformly continuous. Let \( \{\lambda_k\}_{k=0}^\infty \) be a dense sequence in \( \mathbb{R} \). Let \( \{e_k\}_{k=0}^\infty \) be an orthonormal basis for \( H \) and define an unbounded self-adjoint operator on \( H \) by
\[
Ax = \sum_{k=0}^\infty \lambda_k \langle e_k, x \rangle e_k, \quad \text{dom}(A) = \left\{ x \in H : \sum_{k=0}^\infty |\langle e_k, x \rangle \lambda_k|^2 < \infty \right\}.
\]
For \( k \geq 0 \), let \( Q_k \) be the rank one projection on \( H \) defined by
\[
Q_k x = \langle e_k, x \rangle, \quad x \in H.
\]
We denote \( \varphi_A := \varphi_{A,p} \) for brevity. By assumption, \( \varphi_A \) is \( n \)-times Fréchet differentiable on \( S^p_{sa} \). Repeating identically the argument of [4, Proposition 3.9], we obtain that
\[
\varphi_A(tQ_k) = (f(\lambda_k + t) - f(\lambda_k))Q_k \quad k \geq 0, \quad t \in \mathbb{R},
\]
and for any \( 1 \leq m \leq n \),
\[
D^m \varphi_A(tQ_k)[Q_k, Q_k, \ldots, Q_k] = f^{(m)}(\lambda_k + t)Q_k, \quad k \geq 0, \quad t \in \mathbb{R},
\]
where \( D^m \varphi_A \) is the \( m \)th order Fréchet derivative of \( \varphi_A \), see (1.2). We write \( D^0 \varphi_A = \varphi_A \) by convention. Applying (2.2) with \( m = n \), and either (2.1) if \( n = 1 \) or (2.2) with \( m = n - 1 \) if \( n \geq 2 \), we obtain
\[
D^n \varphi_A(tQ_k)[Q_k, \ldots, Q_k] - D^n \varphi_A(0)[Q_k, \ldots, Q_k] = f^{(n)}(\lambda_k + t) - f^{(n)}(\lambda_k))Q_k.
\]
Since \( \varphi_A \) is \( n \)-times Fréchet differentiable, we deduce that
\[
\|f^{(n)}(\lambda_k + t) - f^{(n)}(\lambda_k))Q_k\|_p = o(\|t\|\|Q_k\|_p)\|Q_k\|_p^{n-1}, \quad |t| \to 0,
\]
uniformly in \( k \).
Using \( \|Q_k\|_p = 1 \), it follows that for every \( \varepsilon > 0 \) there exists \( \eta > 0 \) such that if \( 0 < |t| < \eta \) then
\[
|f^{(n-1)}(\lambda_k + t) - f^{(n-1)}(\lambda_k) - t f^{(n)}(\lambda_k)| \leq \varepsilon |t|, \quad k \geq 0.
\]
Hence, for \( 0 < |t| < \eta \) and all \( k \geq 0 \) we have
\[
\left| \frac{f^{(n-1)}(\lambda_k + t)}{t} - f^{(n-1)}(\lambda_k) \right| \leq \varepsilon.
\]
Since \( k \) is arbitrary, \( \{\lambda_k\}_{k=0}^\infty \) is dense and both \( f^{(n)} \) and \( f^{(n-1)} \) are continuous, it follows that
\[
\left| \frac{f^{(n-1)}(s + t)}{t} - f^{(n-1)}(s) \right| \leq \varepsilon, \quad s \in \mathbb{R}, \ 0 < |t| < \eta.
\]
Equivalently,
\[
\sup_{0 < |s - r| < \eta} \left| \frac{f^{(n-1)}(s) - f^{(n-1)}(r)}{s - r} \right| - f^{(n)}(s) \leq \varepsilon.
\]
By the triangle inequality, whenever \( 0 < |s - r| < \eta \) we have
\[
|f^{(n)}(s) - f^{(n)}(r)| \leq \left| f^{(n)}(s) - \frac{f^{(n-1)}(s) - f^{(n-1)}(r)}{s - r} \right| + \left| \frac{f^{(n-1)}(s) - f^{(n-1)}(r)}{s - r} - f^{(n)}(r) \right| \leq 2\varepsilon.
\]
This shows that for every \( \varepsilon > 0 \) there exists \( \eta > 0 \) such that
\[
\sup_{0 < |s - r| < \eta} |f^{(n)}(s) - f^{(n)}(r)| \leq 2\varepsilon.
\]
That is, \( f^{(n)} \) is uniformly continuous. \( \square \)

The following simple lemma is well-known, but we supply a proof for convenience.

**Lemma 2.1.** Let \( \phi \) be a smooth compactly supported function supported in the interval \((-1, 1)\) such that \( \int_{-\infty}^{\infty} \phi(s) \, ds = 1 \). Denote
\[
\phi_\varepsilon(t) := \varepsilon^{-1} \phi(\varepsilon^{-1} t), \quad \varepsilon > 0.
\]
If \( f \in C_b^0(\mathbb{R}) \), then for all \( \varepsilon > 0 \) we have \( \phi_\varepsilon \ast f \in C_b^0(\mathbb{R}) \). If in addition \( f^{(n)} \) is uniformly continuous, then
\[
\lim_{\varepsilon \to 0} \|f^{(n)} - (\phi_\varepsilon \ast f)^{(n)}\|_\infty = 0.
\]
**Proof.** The assertion that \( \phi_\varepsilon \ast f \in C_b^\infty(\mathbb{R}) \) is clear from the assumption on \( \phi \). Note further that for all \( f \in C_b^0(\mathbb{R}) \),
\[
(\phi_\varepsilon \ast f)^{(n)} = \phi_\varepsilon \ast f^{(n)}.
\]
Hence, it suffices to take \( n = 0 \) and prove only that if \( f \) is uniformly continuous then
(2.3)
\[
\lim_{\varepsilon \to 0} \|f - \phi_\varepsilon \ast f\|_\infty = 0.
\]
Since \( \int_{-\infty}^{\infty} \phi(s) \, ds = 1 \), it follows that \( \int_{-\infty}^{\infty} \phi_\varepsilon(s) \, ds = 1 \) and hence
\[
f(t) - (\phi_\varepsilon \ast f)(t) = \int_{-\infty}^{\infty} \phi_\varepsilon(t - s)(f(t) - f(s)) \, ds, \quad t \in \mathbb{R}.
\]
Since \( \phi \) is supported in the set \((-1, 1)\), \( \phi_\varepsilon \) is supported in \((-\varepsilon, \varepsilon)\) and therefore
\[
f(t) - (\phi_\varepsilon \ast f)(t) = \varepsilon^{-1} \int_{t-\varepsilon}^{t+\varepsilon} \phi(\varepsilon^{-1}(t - s))(f(t) - f(s)) \, ds, \quad t \in \mathbb{R}.
\]
By the triangle inequality,
\[
|f(t) - (\phi_\varepsilon \ast f)(t)| \leq 2 \sup_{s \in \mathbb{R}, |t-s| < \varepsilon} |f(s) - f(t)|, \quad t \in \mathbb{R}.
\]
Therefore,
\[
\|f - \phi_\varepsilon \ast f\|_\infty \leq 2 \sup_{t,s \in \mathbb{R}, |t-s| < \varepsilon} |f(s) - f(t)|.
\]
Due to the uniform continuity of $f$, the right hand-side goes to 0 when $\varepsilon \to 0$. Therefore (2.3) holds true.

**Proof that (iii) implies (i).** This result is an improvement of [4, Theorem 3.4]. Our approach is based on the proofs of the latter theorem and of [4, Lemma 3.12]. We use the multiple operator integrals $T^A_{f_i}[n]$ from the latter paper. We let $S_k$ denote the symmetric group of degree $k$, for all $k \geq 1$.

Let $f \in C^m_b(\mathbb{R})$ be such that $f^{(n)}$ is uniformly continuous, let $A$ be a self-adjoint operator on $H$ and let $\phi$ be as in Lemma 2.1. For brevity, denote $f_\varepsilon := \phi_\varepsilon * f$ for all $\varepsilon > 0$. Then the statement of the lemma implies that for every $\varepsilon > 0$ the function $f_\varepsilon$ belongs to $C^\infty_b(\mathbb{R})$. Hence, by [4, Theorem 3.3] $f_\varepsilon$ is $n$-times continuously Fréchet differentiable, and

$$D^n_{p,n}f_\varepsilon(A)[X_1, X_2, \ldots, X_n] = \sum_{\sigma \in S_n} T^n_{f_\varepsilon}[n](X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}), \quad X_1, \ldots, X_n \in S^p.$$ 

Since $f_\varepsilon$ is $n$-times Fréchet differentiable, for every $\delta > 0$ there exists a $\delta'_\varepsilon > 0$ such that if $\|X\|_p < \delta'_\varepsilon$, $X \in S^p_{sa}$, then

$$\|(D^n_{p}f_\varepsilon(A + X) - D^n_{p}f_\varepsilon(A))[X_1, X_2, \ldots, X_{n-1}] - D^n_{p}f_\varepsilon(A)[X_1, \ldots, X_{n-1}, X]\|_p$$

$$\leq \delta\|X\|_p\|X_1\|_p \cdots \|X_{n-1}\|_p,$$

for all $X_1, \ldots, X_{n-1} \in S^p$.

Define

$$(2.4) \quad \Gamma(f)[X_1, \ldots, X_n] := \sum_{\sigma \in S_n} T^n_{f_\varepsilon}[n](X_{\sigma(1)}, \ldots, X_{\sigma(n)}), \quad X_1, \ldots, X_n \in S^p.$$ 

It follows from above that $\Gamma(f_\varepsilon) = D^n_{p,n}f_\varepsilon(A)$ for all $\varepsilon > 0$. Thus we have

$$\|(D^n_{p}f_\varepsilon(A) - \Gamma(f))[X_1, \ldots, X_{n-1}, X]\|_p = \|(\Gamma(f_\varepsilon) - \Gamma(f))[X_1, \ldots, X_{n-1}, X]\|_p$$

$$\leq C_{p,n}\|(f_\varepsilon - f)^{(n)}\|_\infty\|X\|_p\|X_1\|_p \cdots \|X_{n-1}\|_p,$$

by [4, Theorem 2.2], where $C_{p,n} > 0$ is a constant only depending on $p$ and $n$.

Likewise, using [4, (3.24) & (3.25)], we have a similar estimate

$$\|(D^n_{p}f_\varepsilon(A + X) - D^n_{p}f_\varepsilon(A))[X_1, \ldots, X_{n-1}]\|_p$$

$$\leq C_{p,n}\|(f_\varepsilon - f)^{(n)}\|_\infty\|X\|_p\|X_1\|_p \cdots \|X_{n-1}\|_p.$$ 

Therefore, for every $\delta > 0$ and every $\varepsilon > 0$, there exists a $\delta'_\varepsilon > 0$ such that for every $X \in S^p_{sa}$ with $\|X\|_p < \delta'_\varepsilon$, and for all $X_1, \ldots, X_{n-1} \in S^p$, we have

$$\|(D^n_{p}f_\varepsilon(A + X) - D^n_{p}f_\varepsilon(A))[X_1, \ldots, X_{n-1}] - \Gamma(f)[X_1, \ldots, X_{n-1}, X]\|_p$$

$$\leq \|(D^n_{p}f_\varepsilon(A + X) - D^n_{p}f_\varepsilon(A))[X_1, \ldots, X_{n-1}] - D^n_{p}f_\varepsilon(A)[X_1, \ldots, X_{n-1}, X]\|_p$$

$$+ \|(D^n_{p}f_\varepsilon(A + X) - D^n_{p}f_\varepsilon(A))[X_1, \ldots, X_{n-1}] - \Gamma(f)[X_1, \ldots, X_{n-1}, X]\|_p$$

$$+ \|(D^n_{p}f_\varepsilon(A) - \Gamma(f))[X_1, \ldots, X_{n-1}, X]\|_p$$

$$\leq \delta\|X\|_p\|X_1\|_p \cdots \|X_{n-1}\|_p + 2C_{p,n}\|(f_\varepsilon - f)^{(n)}\|_\infty\|X\|_p\|X_1\|_p \cdots \|X_{n-1}\|_p.$$ 

Let $\delta > 0$. From Lemma 2.1, we may select $\varepsilon > 0$ sufficiently small so that $\|(f_\varepsilon - f)^{(n)}\|_\infty < \delta C_{p,n}^{-1}$. Then we find $\delta' := \delta'_\varepsilon$ such that

$$\|(D^n_{p}f_\varepsilon(A + X) - D^n_{p}f_\varepsilon(A))[X_1, \ldots, X_{n-1}] - \Gamma(f)[X_1, \ldots, X_{n-1}, X]\|_p$$

$$\leq 3\|X\|_p\|X_1\|_p \cdots \|X_{n-1}\|_p$$

for $X \in S^p_{sa}$ with $\|X\|_p < \delta'$, and for all $X_1, \ldots, X_{n-1} \in S^p$. Since $\delta > 0$ is arbitrary, we arrive at

$$\|(D^n_{p}f_\varepsilon(A + X) - D^n_{p}f_\varepsilon(A))[X_1, \ldots, X_{n-1}] - \Gamma(f)[X_1, \ldots, X_{n-1}, X]\|_p$$

$$= o(\|X\|_p\|X_1\|_p \cdots \|X_{n-1}\|_p)$$

for $X \in S^p_{sa}$ with $\|X\|_p < \delta'$, and for all $X_1, \ldots, X_{n-1} \in S^p$. Since $\delta > 0$ is arbitrary, we arrive at

$$\|(D^n_{p}f_\varepsilon(A + X) - D^n_{p}f_\varepsilon(A))[X_1, \ldots, X_{n-1}] - \Gamma(f)[X_1, \ldots, X_{n-1}, X]\|_p$$

$$= o(\|X\|_p\|X_1\|_p \cdots \|X_{n-1}\|_p)$$
as \( \|X\|_p \to 0 \), uniformly for \( X_1, \ldots, X_{n-1} \in S^p \). Hence \( f \) is \( n \)-times Fréchet differentiable at \( A \) in \( S^p \), with

\[
(2.5) \quad \Gamma(f) = D^n_p f(A).
\]

Let us now check that \( X \mapsto D^n_p f(A + X) \) is continuous on \( S^p_x \). It suffices to prove continuity at 0. It follows from (2.5), (2.4) and [4, Theorem 2.2] that

\[
\|D^n_p f(A + X) - D^n_p f(A)\| \leq C_{p,n}\|(f_\varepsilon - f)^{(n)}\|_\infty,
\]

where the norm in the left hand-side is computed in the Banach space \( B_n(S^p \times \cdots \times S^p, S^p) \). We have

\[
\|D^n_p f(A + X) - D^n_p f(A)\| \leq \|D^n_p(f_\varepsilon - f)(A + X) - D^n_p(f_\varepsilon - f)(A)\| + \|D^n_p f_\varepsilon(A + X) - D^n_p f_\varepsilon(A)\|,
\]

and hence

\[
\|D^n_p f(A + X) - D^n_p f(A)\| \leq C_{p,n}\|(f_\varepsilon - f)^{(n)}\|_\infty + \|D^n_p f_\varepsilon(A + X) - D^n_p f_\varepsilon(A)\|.
\]

By [4, Theorem 3.3], the mapping \( X \mapsto D^n_p f_\varepsilon(A + X) \) is continuous for any \( \varepsilon > 0 \). Recall that \( \|(f_\varepsilon - f)^{(n)}\|_\infty \to 0 \) when \( \varepsilon \to 0 \). We deduce that

\[
\lim_{X \to 0} \|D^n_p f(A + X) - D^n_p f(A)\| = 0,
\]

which completes the proof. \( \square \)

3. Final Comments

We provide two additional comments.

Comment 1. It follows from the proof of Theorem 1.2 that if \( f \) satisfies the conditions of this theorem, then

\[
(3.1) \quad D^n_p f(A)[X_1, \ldots, X_n] = \sum_{\sigma \in S_n} T^A_1A, \ldots, A(X_{\sigma(1)}, \ldots, X_{\sigma(n)}), \quad X_1, \ldots, X_n \in S^p,
\]

for all self-adjoint operators \( A \) on \( H \). Further the argument in [4, (3.53)] shows the following Taylor formula:

\[
(3.2) \quad f(A + X) = f(A) + D^1_p f(A)[X] + \frac{1}{2!} D^2_p f(A)[X, X] + \cdots
\]

\[
(3.3) \quad + \frac{1}{(n - 1)!} D^{n-1}_p f(A)[X, \ldots, X] + T^A_1A, \ldots, A(X, \ldots, X),
\]

Comment 2. The Potapov-Sukochev result [5, Theorem 1] holds true on all non-commutative \( L^p \)-spaces (not only on Schatten classes). However the results discussed in this note cannot be extended beyond Schatten classes. The lack of Fréchet differentiability in a general context was already mentioned in [2, Introduction]. For the sake of completeness, we provide a simple example in the commutative case. Fix \( 1 < p < \infty \) and let \( f \in C^1_b(\mathbb{R}) \) such that \( f(t) = t^2 \) for \( |t| \leq 2 \). Then for any \( X \in L^\infty[0, 1] \subset L^p[0, 1] \) with \( \|X\|_\infty \leq 1 \), we have \( f(1 + X) - f(1) = 2X + X^2 \). If \( f \) was 1-time Fréchet differentiable in \( L^p[0, 1] \), \( D^1_p f(1) \) would be the mapping \( X \mapsto 2X \) and we would have

\[
(3.4) \quad \|X^2\|_p = o(\|X\|_p).
\]

For all \( \varepsilon \in (0, 1) \), the indicator function \( X = \chi_{[0, \varepsilon]} \) satisfies \( \|\chi_{[0, \varepsilon]}^2\|_p = \|\chi_{[0, \varepsilon]}\|_p = \varepsilon^\frac{p}{2} \). This contradicts (3.4).

Comment 3. Repeating the arguments in the proof that (ii) implies (iii) of Theorem 1.2, we can also see that uniform continuity of \( f^{(n)} \) is a necessary condition for \( f \) to be \( n \)-times Fréchet differentiable in \( S^1 \) at every self-adjoint operator.

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