Existence Serrin type results for the Dirichlet problem for the prescribed mean curvature equation in Riemannian manifolds

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Abstract

Given a complete $n$-dimensional Riemannian manifold $M$, we study the existence of vertical graphs in $M \times \mathbb{R}$ with prescribed mean curvature $H = H(x, z)$. Precisely, we prove that the Dirichlet problem for the vertical mean curvature equation in a smooth bounded domain $\Omega \subset M$ has solution for arbitrary smooth boundary data if $2020$ AMS Subject Classification: 53C42, 49Q05, 35J25, 35J60.

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The matrix of the operator $\mathcal{M}$ (and $\Omega$) is given by $A = W^2g$, where $g$ is the induced metric on the graph of $u$. This implies that the eigenvalues of $A$ are positive and depend on $x$ and $\nabla u$. Hence, $\mathcal{M}$ is locally uniformly elliptic. Furthermore, if $\Omega$ is bounded and $u \in \mathcal{C}^2(\bar{\Omega})$, then $\mathcal{M}$ is uniformly elliptic in $\Omega$ (see [20] for more details).

We recall that the Dirichlet problem for equation (1) is a classical problem in the intersection between Differential Geometry and Partial Differential Equations. First steps were given by Bernstein [7], Douglas [11] and Radó [18] in domains of $\mathbb{R}^2$ for the minimal case. In 1966 Jenkins-Serrin [14, Th. 1 p. 171] derived a related result in higher dimensions.

Later on, Serrin [19] devoted his attention to study Dirichlet problems for a class of more general elliptic equations within which is the prescribed mean curvature equation. Specifically related to our work, he obtained the following result.

**Theorem 1 (Serrin [19, Th. p. 484]).** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary is of class $\mathcal{C}^2$. Let $H(x) \in \mathcal{C}^1(\bar{\Omega})$ and suppose that

$$\left|\nabla H(x)\right| \leq \frac{n}{n-1}(H(x))^2 \quad \forall \ x \in \Omega. \quad (4)$$

Then the Dirichlet problem in $\Omega$ for surfaces having prescribed mean curvature $H(x)$ is uniquely solvable for arbitrarily given $\mathcal{C}^2$ boundary values if, and only if,

$$(n - 1)\mathcal{H}_{\partial \Omega}(y) \geq n|H(y)| \quad \forall \ y \in \partial \Omega, \quad (5)$$

where $\mathcal{H}_{\partial \Omega}$ denotes the inward mean curvature of $\partial \Omega$.

Since (4) is satisfied by every $H \in \mathbb{R}$, it follows that the Serrin condition (5) is a necessary and sufficient condition for graphs with constant mean curvature to exist over bounded domains of the Euclidean ambient space. Actually, the following classical sharp solvability criteria holds.

**Theorem 2 (Serrin [19, p. 416]).** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary is of class $\mathcal{C}^2$. Then for every constant $H$ the Dirichlet problem for equation $\mathcal{M}u = nH$ in $\Omega$ has a unique solution for arbitrary $\mathcal{C}^2$ boundary data if, and only if, $(n - 1)\mathcal{H}_{\partial \Omega} \geq n|H|$.

The main goal of the present paper is to generalize to more general ambient spaces the existence part in Theorem 1. Specifically, we prove the following.

**Theorem 3 (Main theorem).** Let $\Omega \subset M$ be a bounded domain with $\partial \Omega$ of class $\mathcal{C}^{2,\alpha}$ for some $\alpha \in (0, 1)$. Let $H \in \mathcal{C}^{1,\alpha}(\bar{\Omega} \times \mathbb{R})$ satisfying $\partial_x H \geq 0$ and

$$\text{Ricc}_x \geq n \sup_{z \in \mathbb{R}} \left\|\nabla_x H(x, z)\right\| - \frac{n^2}{n-1} \inf_{z \in \mathbb{R}} (H(x, z))^2 \quad \forall \ x \in \Omega. \quad (6)$$

If

$$(n - 1)\mathcal{H}_{\partial \Omega}(y) \geq n \sup_{z \in \mathbb{R}} |H(y, z)| \quad \forall \ y \in \partial \Omega, \quad (7)$$

then for every $\varphi \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ there exists a unique solution $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ of the Dirichlet problem for equation (1).

In the statement above $\text{Ricc}_x$ is the Ricci curvature\(^1\) of $M$ at $x$. Note that relation (4) is a particular case of (6).

On the other hand, in a previous work we have proved that the strong Serrin condition (7) is sharp in every Hadamard manifold (see [3, Corollary 2 p. 3]). The combination of this non-existence

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\(^1\)The definition of the Ricci curvature we use throughout the text follows [17]. The notation $\text{Ricc}_x \geq f(x)$ means that the Ricci curvature evaluated in any unitary tangent vector at $x$ is bounded below by the function $f(x)$. 

2
result with Theorem 3 fully generalizes Theorem 1 to every Hadamard manifold in the \( C^{2,\alpha} \) class (see also [3, Th. 10 p. 6]). We have also proved that (7) is sharp in every compact and simply connected manifold which is strictly \( 1/4 \)-pinched\(^2\) provided \( \text{diam}(\Omega) < \frac{2}{\sqrt{K_0}} \) (see [3, Corollary 3 p. 4]). A further Serrin type solvability criteria is directly obtained by combining this non-existence result with Theorem 3.

Theorem 3 also generalizes a result proved by Spruck for \( H \in \mathbb{R} \) in the \( M \times \mathbb{R} \) setting (see [20, T 1.4 p. 787]). In this case of constant mean curvature the Serrin condition by itself is not always sufficient for the solvability of the Dirichlet problem for equation (1) as it happens in the Euclidean space. By way of illustrating better this fact notice that when \( M = \mathbb{H}^n \) and \( H \in \mathbb{R} \), relation (6) reduces to \( |H| \geq \frac{n-1}{n} \). In the opposite case \( |H| < \frac{n-1}{n} \), Spruck stated an existence theorem assuming that the strict inequality \((n-1)H_{\partial\Omega} > n|H|\) holds (see [20, Th. 5.4 p. 797]). In this paper we also extend this result of Spruck in the hyperbolic space including the inequality even for not necessarily constant \( H \), as can be seen in the following theorem.

**Theorem 4.** Let \( \Omega \subset \mathbb{H}^n \) be a bounded domain with \( \partial \Omega \) of class \( C^{2,\alpha} \) for some \( \alpha \in (0,1) \) and \( \varphi \in C^{2,\alpha}(\overline{\Omega}) \). Let \( H \in C^{1,\alpha}(\overline{\Omega} \times \mathbb{R}) \) satisfying \( \partial_z H \geq 0 \) and \( \sup_{\Omega \times \mathbb{R}} |H| \leq \frac{n-1}{n} \). If

\[
(n-1)H_{\partial\Omega}(y) \geq n \sup_{z \in \mathbb{R}} |H(y,z)| \quad \forall \ y \in \partial \Omega,
\]

then for every \( \varphi \in C^{2,\alpha}(\overline{\Omega}) \) there exists a unique solution \( u \in C^{2,\alpha}(\overline{\Omega}) \) of the Dirichlet problem for equation (1).

We note that our non-existence result for Hadamard manifolds [3, Cor. 1 p. 3] guaranties that Theorem 4 is sharp (see also [3, Th. 7 p. 5]). Besides, putting together Spruck’s existence theorem [20, Th. 1.4 p. 787] one can deduce that the Serrin sharp solvability criterion for constant \( H \) stated in Theorem 2 also holds in the hyperbolic space (see also [3, Th. 8 p. 5]).

At last, we use the barriers constructed by Galvez-Lozano [12, Th. 6 p. 12] to prove the following result in Hadamard manifolds.

**Theorem 5.** Let \( M \) be a Hadamard manifold with sectional curvature pinched between \(-c^2\) and \(-1\) for some \( c > 1 \). Let \( \Omega \subset M \) be a bounded domain with \( \partial \Omega \) of class \( C^{2,\alpha} \) for some \( \alpha \in (0,1) \) and whose principal curvatures are greater than \( c \). Let \( \varphi \in C^{2,\alpha}(\overline{\Omega}) \) and \( H \in C^{1,\alpha}(\overline{\Omega} \times \mathbb{R}) \) satisfying \( \partial_z H \geq 0 \) and \( \sup_{\Omega \times \mathbb{R}} |H| \leq \frac{n-1}{n} \). Then the Dirichlet problem for equation (1) has a unique solution \( u \in C^{2,\alpha}(\overline{\Omega}) \).

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## 2 The a priori estimates

In order to prove the theorems stated in the introduction we use the classical Leray-Schauder degree theory. In this section we derive the a priori estimates for the solution of the Dirichlet problem

\[
\begin{cases}
\text{div} \left( \frac{\nabla u}{W} \right) = nH(x,u) & \text{in } \Omega, \\
\quad u = \varphi & \text{in } \partial \Omega.
\end{cases}
\]

\(^2\)A Riemannian manifold is said to be strictly \( 1/4 \)-pinched if the sectional curvature \( K \) of \( M \) satisfies \( 0 < \frac{1}{4}K_0 < K \leq K_0 \).
Since (1) and (2) are equivalent, the operator $Ω$ defined in (3) and the equation $Ωu = 0$ are used in this section in order to facilitate the calculations.

Firstly, we establish a lemma that will help us to obtain the a priori height and boundary gradient estimates.

**Lemma 6.** Let $Γ$ be an embedded and compact $C^2$ hypersurface of $M$ oriented by a global unit normal $N$. Let $τ > 0$ be such that

$$\Phi_t : Γ \rightarrow Γ_t \subset M$$

$$x \mapsto \exp^t(x, tN_x)$$

is a diffeomorphism between $Γ$ and $Γ_t$ for each $t \in [0, τ)$. For $y \in Γ$ fixed, let $γ_y(t) = \exp_y(tN_y)$ with $0 ≤ t ≤ τ$. If there exists a function $h \in C^1[0, τ)$ such that

$$H_t(y) ≥ |h(0)|$$

(8)

and

$$\text{Ricc}_{γ_y(t)}(γ_y'(t)) ≥ (n - 1) \left( |h'(t)| - (h(t))^2 \right) \forall t \in [0, τ),$$

(9)

then

$$H_{Γ_t}(γ_y(t)) ≥ |h(t)| \forall t \in [0, τ),$$

(10)

where $H_{Γ_t}(γ_y(t))$ is the mean curvature of $Γ_t$ at $γ_y(t)$ computed with respect to $γ_y'(t)$. Furthermore, $H_{Γ_t}(γ_y(t))$ is increasing as a function of $t$.

**Proof.** First of all note that the hypersurface $Γ_t$ is parallel to $Γ$ for each $t \in [0, τ)$. Let $H(t) := H_{Γ_t}(γ_y(t))$. It is well known that (see [4, Th. 2.7 p. 3] and [2, Cor. B.14 p. 66] for a different proof)

$$H'(t) ≥ \frac{\text{Ricc}_{γ_y(t)}(γ_y'(t))}{n - 1} + (H(t))^2.$$ (11)

From (9) it follows

$$H'(t) ≥ |h'(t)| - (h(t))^2 + (H(t))^2.$$ (12)

Then,

$$(H(t) - h(t))^t ≥ (H(t) + h(t)) (H(t) - h(t))$$ (13)

and

$$(H(t) + h(t))^t ≥ (H(t) - h(t)) (H(t) + h(t)).$$ (14)

Let us define $v(t) = H(t) - h(t)$ and $g(t) = H(t) + h(t)$. Inequality (13) yields

$$\left( \frac{v(t)}{\int_0^1 g(s)ds} \right)' ≥ 0,$$

so $v(t) ≥ v(0)\int_0^1 g(s)ds$ for each $t \in [0, τ)$. As a consequence of (8),

$$H(t) ≥ h(t) \forall t \in [0, τ).$$

Using (14) it follows in a similar way

$$H(t) ≥ -h(t) \forall t \in [0, τ).$$

Therefore,

$$H(t) ≥ |h(t)| \forall t \in [0, τ).$$ (15)

Finally, $H'(t) ≥ 0$ is obtained by substituting (15) in (12).
Remark 7. By choosing the function $h$ appropriately condition (8) becomes the strong Serrin condition (7) and condition (9) turns (6). Roughly speaking, Lemma 6 says that the parallel hypersurfaces to $\Gamma$ inherit the initial condition (8) throughout the orthogonal geodesics provided condition (9) also holds. Moreover, if such a function $h$ exists, necessarily the mean curvature of the parallel hypersurfaces lying inside $\Omega$ increases along the inner normal geodesics. This property is true in mean convex domains of manifolds whose Ricci curvature is non-negative but is not necessarily true in the opposite case (see (11)).

Theorem 8 (A priori height estimate). Let $\Omega \subseteq M$ be a bounded domain with $\partial \Omega$ of class $\mathcal{C}^2$ and $\varphi \in \mathcal{C}^0(\partial \Omega)$. Let $H \in \mathcal{C}^1(\overline{\Omega} \times \mathbb{R})$ satisfying $\partial_z H \geq 0$,

$$\text{Ricc}_x \geq n \sup_{z \in \mathbb{R}} \| \nabla_x H(x, z) \| - \frac{n^2}{n-1} \inf_{z \in \mathbb{R}} (H(x, z))^2 \quad \forall \ x \in \Omega,$$

and

$$(n-1) \mathcal{H}_{\partial \Omega}(y) \geq n |H(y, \varphi(y))| \quad \forall \ y \in \partial \Omega. \quad (17)$$

If $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ is a solution of the Dirichlet problem (P), then

$$\sup_{\Omega} |u| \leq \sup_{\partial \Omega} |\varphi| + \frac{e^{\mu \delta} - 1}{\mu},$$

where $\mu > n \sup \left\{ |H(x, z)|, (x, z) \in \overline{\Omega} \times \left[ -\inf_{\partial \Omega} |\varphi|, \sup_{\partial \Omega} |\varphi| \right] \right\}$ and $\delta = \text{diam}(\Omega)$.

Proof. For $x \in \Omega$ let us define the distance function $d(x) = \text{dist}(x, \partial \Omega)$. Let $\Omega_0$ be the biggest open subset of $\Omega$ having the unique nearest point property; that is, for every $x \in \Omega_0$ there exists a unique $y \in \partial \Omega$ such that $d(x) = \text{dist}(x, y)$. Then $d \in \mathcal{C}^2(\Omega_0)$ (see [15]).

Let

$$\phi(t) = \frac{e^{\mu \delta}}{\mu} \left( 1 - e^{-\mu t} \right)$$

where $\mu$ and $\delta$ are the constant defined in the statement of the theorem. We now define $w = \phi \circ d + \sup_{\partial \Omega} |\varphi|$. The desired estimate follows if $|u| \leq w$ in $\overline{\Omega}$. First, we will prove that $u \leq w$. By contradiction suppose that $u - w$ attains a maximum $m > 0$ at $x_0 \in \Omega$. That is,

$$u \leq w + m \text{ in } \Omega$$

and

$$u(x_0) = w(x_0) + m. \quad (19)$$

Choose $y_0 \in \partial \Omega$ such that $d(x_0) = \text{dist}(x_0, y_0)$ and let $\gamma$ be the minimizing geodesic orthogonal to $\partial \Omega$ joining $x_0$ to $y_0$. Then, $d(x) = \text{dist}(x, y_0)$ for every $x$ lying in $\gamma$ between $x_0$ and $y_0$. By restricting $u$ and $w$ to $\gamma$ we see that $u'(d(x_0)) = w'(d(x_0)) = \phi'(d(x_0)) > 0$. Thus, $\nabla u(x_0) \neq 0$ and the level set $\Gamma_0 = \{ x \in \Omega; u(x) = u(x_0) \}$ is an hypersurface of class $\mathcal{C}^2$ in a neighbourhood of $x_0$. Consequently, there exists a geodesic ball $B_\epsilon(z_0)$ tangent to $\Gamma_0$ in $x_0$ such that

$$u \geq u(x_0) \text{ in } B_\epsilon(z_0). \quad (20)$$

From (18), (19) and (20) we obtain for $x \in B_\epsilon(z_0)$,

$$w(x_0) + m = u(x_0) \leq u(x) \leq w(x) + m.$$

Hence, $w(x_0) \leq w(x)$ which yields

$$d(x_0) \leq d(x) \text{ in } B_\epsilon(z_0) \quad (21)$$
since $\phi$ is increasing.

Therefore, if $\bar{\Gamma}$ lies in the intersection of $\partial B_\epsilon(z_0)$ with a geodesic minimizing the distance between $z_0$ and $y_0$, then
\[
\text{dist}(z_0, y_0) = \text{dist}(z_0, \bar{\Gamma}) + \text{dist}(\bar{\Gamma}, y_0) \\
= \text{dist}(z_0, x_0) + \text{dist}(\bar{\Gamma}, y_0) \\
\geq \text{dist}(z_0, x_0) + d(\bar{\Gamma}) \\
\geq \text{dist}(z_0, x_0) + d(x_0) \text{ from (21)} \\
= \text{dist}(z_0, x_0) + d(x_0, y_0).
\]

Thus, equality holds in the triangle inequality which implies that $x_0 = \bar{\Gamma}$. That is, $z_0 = \gamma(d(x_0) + \epsilon)$.

This ensures that $x_0 \in \Omega_0$ because if there exists $y_1 \neq y_0$ satisfying $d(x_0) = \text{dist}(x_0, y_1)$, then
\[
\text{dist}(z_0, y_1) < \text{dist}(z_0, x_0) + \text{dist}(x_0, y_1) = \text{dist}(z_0, x_0) + \text{dist}(x_0, y_0) = d(z_0),
\]
which contradicts the definition of $d$.

However, $x_0$ can not be in $\Omega_0$ as will be shown in the sequel. Some algebraic computations yields
\[
\mathcal{M}w = \phi'(1 + \phi'^2)\Delta d + \phi'' \quad \text{in } \Omega_0.
\] (22)

For $x \in \Omega_0$, let $y = y(x)$ in $\partial \Omega$ be the nearest point to $x$ and $\gamma_y(t)$ the orthogonal geodesic to $\partial \Omega$ from $y$ to $x$. Let us define
\[
h(t) = \frac{n}{n-1}H(\gamma_y(t), \varphi(y)).
\]

Note that $y$ is now fixed. On account of the Serrin condition (17) one has
\[
|h(0)| = \frac{n}{n-1}|H(y, \varphi(y))| \leq \mathcal{H}_{\partial\Omega}(y) = \mathcal{H}(0).
\]

Besides,
\[
h'(t) = \frac{n}{n-1}\left\langle \nabla_x H(\gamma_y(t), \varphi(y)), \gamma_y'(t) \right\rangle.
\]

The additional hypothesis (16) yields
\[
(n-1)\left( |h'(t)| - (h(t))^2 \right) \leq \text{Ric}_{\gamma_y(t)}(\gamma_y'(t)).
\]

Thus, Lemma 6 can be applied to the function $h(t)$ to obtain
\[
n|H(\gamma_y(t), \varphi(y))| \leq (n-1)\mathcal{H}_{\Gamma_t}(\gamma_y(t)),
\]
where $\Gamma_t$ is parallel to some portion of $\partial \Omega$. By using a well-known formula linking the Laplacian of the distance function with the mean curvature of parallel hypersurfaces, we get
\[
\Delta d(x) \leq -n|H(x, \varphi(y(x)))| \quad \forall \ x \in \Omega_0.
\]

Using this estimate in (22) it follows
\[
\mathcal{M}w \leq -n|H(x, \varphi(y(x)))| \phi'(1 + \phi'^2) + \phi''.
\]

Also,
\[
\phi''(t) = -\mu e^{\mu(t-\epsilon)} = -\mu\phi'(t) < -n|H(x, \varphi(y(x)))| \phi'(t)
\]
and $\phi' \geq 1$, so
\[
\mathcal{M}w < -n|H(x, \varphi(y(x)))| \left(1 + \phi'^2\right)^{3/2}. \quad (23)
\]
On the other hand, the hypothesis \( \partial_z H \geq 0 \) implies that
\[
\mp H(x, \pm w) \leq \mp H(x, \varphi(y(x))) \leq |H(x, \varphi(y(x)))|.
\] (24)

From (23) and (24) we conclude that
\[
\pm \Omega(\pm w) = \mathcal{M}w \mp nH(x, \pm w) \left(1 + \phi'\right)^{3/2} \leq 0.
\]

Therefore,
\[
\Omega(w + m) = \mathcal{M}(w + m) - nH(x, w + m) \left(1 + \phi'\right)^{3/2} \leq \Omega w \leq \Omega u.
\]

Moreover, \( u \leq w + m \) and \( u(x_0) = w(x_0) + m \). The maximum principle implies that \( u = w + m \) in \( \Omega_0 \) which contradicts the fact that \( u < w + m \) in \( \partial \Omega \). This proves that \( u \leq w \) in \( \Omega \).

**Remark 9.** Instead of condition (16), the proof shows that it suffice to assume
\[
\text{Ric}_{x} \geq n \|\nabla_{x} H(x, \varphi(y))\| - \frac{n^2}{n - 1} (H(x, \varphi(y))) \forall x \in \Omega_0,
\]
where \( y \in \partial \Omega \) is the nearest point to \( x \).

**Theorem 10 (Boundary gradient estimate).** Let \( \Omega \in M \) be a bounded domain with \( \partial \Omega \) of class \( \mathcal{C}^2 \) and \( \varphi \in \mathcal{C}^2(\Omega) \). Let \( H \in \mathcal{C}^1(\overline{\Omega} \times \mathbb{R}) \) satisfying \( \partial_z H \geq 0 \),
\[
\text{Ric}_{x} \geq n \sup_{z \in \mathbb{R}} \|\nabla_{x} H(x, z)\| - \frac{n^2}{n - 1} \inf_{z \in \mathbb{R}} (H(x, z))) \forall x \in \Omega,
\] (25)
and
\[
(n - 1) \mathcal{H}_{\partial \Omega}(y) \geq n |H(y, \varphi(y))| \forall y \in \partial \Omega.
\] (26)
If \( u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega}) \) solves problem \( (P) \), then
\[
\sup_{\partial \Omega} \|\nabla u\| \leq \|\varphi\|_1 + e^{C\left(1 + \|H\|_1 + \|\varphi\|_1\right)}(\|u\|_1 + \|\varphi\|_1)^2
\]
(27)
for some \( C = C(n, \Omega) \).

**Proof.** Again we set \( d(x) = \text{dist}(x, \partial \Omega) \) for \( x \in \Omega \). Let \( \tau > 0 \) be such that \( d \) is of class \( \mathcal{C}^2 \) over the set of points in \( \Omega \) for which \( d(x) \leq \tau \). Let \( \psi \in \mathcal{C}^2([0, \tau]) \) be a non-negative function satisfying
\]

P1. \( \psi(t) \geq 1 \), 

P2. \( \psi''(t) \leq 0 \), 

P3. \( t \psi'(t) \leq 1 \).

For \( a < \tau \) to be fixed latter on we consider the set
\[
\Omega_a = \{x \in M; d(x) < a\}.
\]

We now define \( w^\pm = \pm \psi \circ d + \varphi \). Firstly, let us estimate \( \pm \mathcal{M}w^\pm \) in \( \Omega_a \). A straightforward computation yields
\[
\pm \mathcal{M}w^\pm = \psi'W_\pm^2 \Delta d - \psi' \nabla^2 d(\nabla \varphi, \nabla \varphi) + \psi''W_\pm^2 - \psi''(\nabla d, \pm \psi' \nabla d + \nabla \varphi)^2
\]
\[
\pm W_\pm^2 \Delta \varphi \mp \nabla^2 \varphi(\pm \psi' \nabla d + \nabla \varphi, \pm \psi' \nabla d + \nabla \varphi),
\] (28)
where
\[
W_\pm = \sqrt{1 + \|\nabla w^\pm\|^2} = \sqrt{1 + \|\pm \psi' \nabla d + \nabla \varphi\|^2}.
\]

7
Once $\nabla^2 d(x)$ is a continuous bilinear form and $\psi' \geq 1$ we have
\[
\psi' \left| \nabla^2 d(\nabla \varphi, \nabla \varphi) \right| \leq \psi'^2 \|d\|_2 \|\varphi\|^2.
\]
(29)

Since $\psi'' < 0$ and $(\nabla d, \pm \psi' \nabla d + \nabla \varphi)^2 \leq \|\pm \psi' \nabla d + \nabla \varphi\|^2$, then
\[
\psi'' W^2_\pm - \psi'' (\nabla d, \pm \psi' \nabla d + \nabla \varphi)^2 \leq \psi''.
\]
(30)

Also, $\varphi$ is of class $C^2$ in $\Omega_a$ by hypothesis, so
\[
\left| \pm W^2_\pm \Delta \varphi \mp \nabla^2 \varphi(\pm \psi' \nabla d + \nabla \varphi, \pm \psi' \nabla d + \nabla \varphi) \right| \leq 2n \|\varphi\|_2 W^2_\pm.
\]

Note also that
\[
\|\pm \psi' \nabla d + \nabla \varphi\|^2 = \left( \psi'^2 + 2 \psi'(\pm \nabla d, \nabla \varphi) + \|\nabla \varphi\|^2 \right) \leq (1 + \|\varphi\|_1)^2 \psi'^2,
\]
hence
\[
W^2_\pm \leq 1 + (1 + \|\varphi\|_1)\psi'^2 \leq 2 \psi'^2 (1 + \|\varphi\|_1)^2.
\]
(31)

Therefore,
\[
\left| \pm W^2_\pm \Delta \varphi \mp \nabla^2 \varphi(\pm \psi' \nabla d + \nabla \varphi, \pm \psi' \nabla d + \nabla \varphi) \right| \leq 4n \|\varphi\|_2 (1 + \|\varphi\|_1)^2 \psi'^2.
\]
(32)

Substituting (30), (29) and (32) in (28) it follows
\[
\pm \mathcal{M} w^\pm \leq \psi' W^2_\pm \Delta d + \psi'' + c \psi'^2,
\]
where
\[
c = \|d\|_2 \|\varphi\|_1^2 + 4n \|\varphi\|_2 (1 + \|\varphi\|_1)^2.
\]
(34)

On the other hand,
\[
\mp H(x, w^\pm(x)) = \mp H(x, \pm \psi(d(x)) + \varphi(x)) \leq \mp H(x, \varphi(x)) \leq |H(x, \varphi(x))|
\]
since we are assuming that $\partial_z H \geq 0$. Thus,
\[
\pm \mathcal{N} w^\pm = \pm \mathcal{M} w^\pm \mp nH(x, w^\pm)W^3_\pm \leq \pm \mathcal{M} w^\pm + n|H(x, \varphi(x))| W^3_\pm.
\]

Using the estimate (33) we obtain
\[
\pm \mathcal{N} w^\pm \leq \psi' W^2_\pm \Delta d + \psi'' + c \psi'^2 + n|H(x, \varphi(x))| W^3_\pm.
\]
(35)

We now want to estimate $\Delta d$. Let $y \in \partial \Omega$ be fixed and $\gamma_y(t) = \exp_y(tN_y)$ for $0 \leq t \leq a$, where $N$ is the inner normal field to $\partial \Omega$. Again, the hypothesis (25) and (26) guaranty that the function $h(t) = \frac{a}{n+1} H(\gamma_y(t), \varphi(y))$ satisfies the hypothesis of Lemma 6. Hence, if $\Gamma_t$ is parallel to $\partial \Omega$, then
\[
\mathcal{H}_{\Gamma_t}(\gamma_y(t)) \geq \mathcal{H}_{\partial \Omega}(y) \quad [0, \tau].
\]
(36)

Thus,
\[
\Delta d(x) = -(n-1) \mathcal{H}_{\Gamma_t(x)}(x) \leq -(n-1) \mathcal{H}_{\partial \Omega}(y) \quad \forall x \in \Omega_a,
\]
(37)

where $y = y(x) \in \partial \Omega$ is the nearest point to $x$. Using again the Serrin condition (26) it follows
\[
\Delta d(x) \leq -n|H(y, \varphi(y))| \quad \forall x \in \Omega_a.
\]
(38)
Substituting (38) in (35) we obtain
\[ \pm \Omega w^\pm \leq n\psi'W_\pm^2(|H(x, \varphi(x))| - |H(y, \varphi(y))|) + n |H(x, \varphi(x))| W_\pm^2 (W_\pm - \psi') + \psi'' + c\psi'^2. \] (39)

In addition
\[ |H(x, \varphi(x))| - |H(y, \varphi(y))| \leq h_1(1 + \|\varphi\|_1)d(x), \] (40)
where
\[ h_1 = \sup_{\Omega \times \mathbb{R}} \|\nabla M \times R H(x, z)\|. \]

From (31) and (40) one has
\[ n\psi'W_\pm^2(|H(x, \varphi(x))| - |H(y, \varphi(y))|) \leq 2nh_1 (1 + \|\varphi\|_1)^3 d(x)(\psi'(d(x)))^3. \] (41)

On the other hand,
\[ W_\pm - \psi' \leq 1 + \|\pm \psi'\nabla d + \nabla \varphi\| - \psi' \leq 1 + \|\varphi\|_1. \] (42)
From (31) and (42) we obtain
\[ n |H(x, \varphi(x))| (W_\pm - \psi') W_\pm^2 \leq 2nh_0 (1 + \|\varphi\|_1)^3 \psi'^2, \] (43)
where
\[ h_0 = \sup_{\Omega \times \mathbb{R}} |H(x, z)|. \]

Substituting (41) and (43) in (39) we get
\[ \pm \Omega w^\pm \leq \left( c + 2n \|\varphi\|_1^3 \right) \psi'^2 + \psi'' , \]
where \( \|H\|_1 = h_0 + h_1. \)

Remembering the expression for \( c \) given in (34) and making some algebraic computation we infer that
\[ \pm \Omega w^\pm < \nu \psi'^2 + \psi'', \]
where
\[ \nu = 4n (1 + \|d\|_2 + 1/\tau) (1 + \|H\|_1 + \|\varphi\|_2) (1 + \|\varphi\|_1)^3. \] (44)

Defining \( \psi \) explicitly by
\[ \psi(t) = \frac{1}{\nu} \log(1 + kt) \]
we obtain \( \nu \psi'^2 + \psi'' = 0. \) Indeed,
\[ \psi'(t) = \frac{k}{\nu(1 + kt)} \] (45)
and
\[ \psi''(t) = -\frac{k^2}{\nu(1 + kt)^2}. \] (46)
Therefore,
\[ \pm \Omega w^\pm < 0 \text{ in } \Omega_a. \]
Note that property P2 follows from (46). Another consequence of (46) is that \( \psi'(t) > \psi'(a) \) for all \( t \in [0, a] \), thus property P1 is ensured provided
\[
\psi'(a) = \frac{k}{\nu(1 + ka)} = 1. \tag{47}
\]
Also,
\[
t\psi'(t) = \frac{kt}{\nu(1 + kt)} \leq \frac{1}{\nu} < 1
\]
which is property P3. Hence, \( \psi \) thus defined satisfies all the initial requirements.
Furthermore, choosing
\[
\psi(a) = \frac{1}{\nu} \log(1 + ka) = \|u\|_0 + \|\varphi\|_0 \tag{48}
\]
it follows
\[
\pm w^+(x) = \psi(a) \pm \varphi(x) = \|u\|_0 + \|\varphi\|_0 \pm \varphi(x) \geq \pm u(x) \forall x \in \partial \Omega_a \setminus \partial \Omega.
\]
Besides, for \( x \in \partial \Omega \) one has \( w^+(x) = \pm \psi(0) + \varphi(x) = u(x) \). By the maximum principle it can be conclude that \( w^- \leq u \leq w^+ \) in \( \Omega_a \). Thus,
\[
-\psi \circ d \leq u - \varphi \leq \psi \circ d \text{ in } \Omega_a,
\]
being that
\[
-\psi \circ d = u - \varphi = \psi \circ d = 0 \text{ in } \partial \Omega.
\]
Consequently, if \( y \in \partial \Omega \) and \( 0 \leq t \leq a \), then
\[
-\psi(t) + \psi(0) \leq (u - \varphi)(\gamma_y(t)) - (u - \varphi)(\gamma_y(0)) \leq \psi(t) - \psi(0).
\]
Dividing by \( t > 0 \) and passing to the limit as \( t \) goes to zero we infer that
\[
|\langle \nabla u(y), N \rangle| \leq |\langle \nabla \varphi(y), N \rangle| + \psi'(0). \tag{49}
\]
Since \( u = \varphi \) on \( \partial \Omega \), we derive from (49)
\[
\|\nabla u(y)\| \leq \|\nabla \varphi(y)\| + \psi'(0).
\]

The desired estimate (27) follows from this last expression because the combination of (47) and (48) yields
\[
k = \nu e^{\varphi(\|u\|_0 + \|\varphi\|_0)}. \tag{50}
\]
Observ also that from (44), (47) and (50) it follows
\[
a = \frac{e^{\varphi(\|u\|_0 + \|\varphi\|_0)} - 1}{\nu e^{\varphi(\|u\|_0 + \|\varphi\|_0)}} = \frac{1}{\nu} < \tau
\]
as required at the beginning.

Recall now that assumption (25) was also requested in the statement of Theorem 10 because its combination with (26) ensures, in addition, the geometric property (36) (see Remark 7). In order to see that this property does not always happens in mean convex domains of manifolds whose Ricci curvature is non-positive let us consider a mean convex domain \( \Omega \) in the hyperbolic space \( \mathbb{H}^n \). For \( y \in \partial \Omega \) let \( \lambda_i(t) \) be the \( i \)-th principal curvature of \( \Gamma_t \) at \( \gamma_y(t) \), then (see [1, p. 17])
\[
\lambda_i(t) = \frac{-\tanh t + \lambda_i(0)}{1 - \lambda_i(0) \tanh t}, \tag{51}
\]
bounded domain with $\partial\Omega$ domain with a Riemannian manifold whose Ricci curvature satisfies $\text{Ricc} \geq c^2$ for $c > 0$. Let $\Omega$ be a complete Riemannian manifold whose Ricci curvature satisfies $\text{Ricc} \geq -(n-1)c^2$ for $c > 0$. Let $\Omega \subset M$ be a bounded domain with $\partial\Omega$ of class $C^2$ such that $\text{Ricci} \geq c$. Let $\varphi \in C^2(\overline{\Omega})$ and $H \in C^1(\overline{\Omega} \times \mathbb{R})$ satisfying $\partial_z H \geq 0$ and

$$(n-1)\mathcal{H}_{\partial\Omega}(y) \geq n|H(y, \varphi(y))| \quad \forall y \in \partial\Omega.$$
Theorem 13 (Global gradient estimate). Let $\Omega \subset M$ be a bounded domain. If a function $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ is a solution of (1) for $H \in C^1 \left( \Omega \times \left[ -\sup_\Omega |u| , \sup_\Omega |u| \right] \right)$ satisfying $\partial_z H \geq 0$, then

$$\sup_{\Omega} \| \nabla u \| \leq \left( 3 + \sup_{\partial \Omega} \| \nabla u \| \right) \left( 1 + \| H \| + \sup_\Omega |\text{Ricc}| \right) \sup_{\Omega} |u|^{\frac{4n}{n+1}}. $$

Proof. Let $w(x) = \| \nabla u(x) \| e^{Au(x)}$ where $A \geq 1$. Suppose $w$ attains a maximum at $x_0 \in \overline{\Omega}$. If $x_0 \in \partial \Omega$, then

$$w(x) \leq w(x_0) = \| \nabla u(x_0) \| e^{Au(x_0)}. $$

Hence,

$$\sup_{\Omega} \| \nabla u(x) \| \leq \sup_{\partial \Omega} \| \nabla u \| e^{2A \sup_{\Omega} |u|}. $$

(54)

Suppose now that $x_0 \in \Omega$ and that $\nabla u(x_0) \neq 0$. Let us define normal coordinates at $x_0$ in such a way that $\frac{\partial}{\partial x_1} |_{x_0} = \frac{\nabla u(x_0)}{\| \nabla u(x_0) \|}$. Then,

$$\partial_k u(x_0) = \left( \frac{\partial}{\partial x_k} |_{x_0} , \nabla u(x_0) \right) = \| \nabla u(x_0) \| \delta_{k1}. $$

(55)

Besides, if $\sigma$ is the metric in this coordinate system, then

$$\sigma_{ij}(x_0) = \sigma^{ij}(x_0) = \delta_{ij}, $$

(56)

$$\partial_k \sigma_{ij}(x_0) = \partial_k \sigma^{ij}(x_0) = 0, $$

(57)

$$\Gamma^k_{ij}(x_0) = 0. $$

(58)

Observe now that the function $\tilde{w}(x) = \ln w(x) = Au(x) + \ln \| \nabla u(x) \|$ also attains a maximum at $x_0$. Therefore,

$$\partial_k \tilde{w}(x_0) = A \partial_k u(x_0) + \frac{\partial_k \left( \| \nabla u \|^2 \right)(x_0)}{2 \| \nabla u(x_0) \|^2} = 0, $$

(59)

and

$$\partial_{kk} \tilde{w}(x_0) = A \partial_{kk} u(x_0) - \frac{\left( \partial_k \left( \| \nabla u \|^2 \right)(x_0) \right)^2}{2 \| \nabla u(x_0) \|^4} + \frac{\partial_{kk} \left( \| \nabla u \|^2 \right)(x_0)}{2 \| \nabla u(x_0) \|^2} \leq 0. $$

(60)

Let us calculate the derivatives involved in these relations. Recall first that $\nabla u(x) = \sum_i u^i \frac{\partial}{\partial x_i}$, where

$$u^i = \sum_{j=1}^n \sigma^{ij} \partial_j u. $$

(61)

Then

$$\| \nabla u(x) \|^2 = \sum_{i,j=1}^n \sigma^{ij} \partial_i u \partial_j u $$

(62)

and

$$\partial_k \left( \| \nabla u \|^2 \right) = \sum_{i,j=1}^n \left( \partial_k \sigma^{ij} \right) \partial_i u \partial_j u + 2 \sigma^{ij} \partial_k \partial_j u. $$

(63)

Using (55), (56) and (57) one gets

$$\partial_k \left( \| \nabla u \|^2 \right) (x_0) = 2 \| \nabla u(x_0) \| \partial_{kk} u(x_0). $$

(64)
Substituting this last expression and (55) in (59) it follows
\[ \partial_{1k} u(x_0) = -A \| \nabla u(x_0) \|^2 \delta_{k1}. \] (65)

The combination of (64) and (65) finally yields
\[ \partial_k \left( \| \nabla u \|^2 \right) (x_0) = -2A \| \nabla u(x_0) \|^3 \delta_{k1}. \] (66)

Deriving now (63) it follows
\[
\partial_{kk} \left( \| \nabla u \|^2 \right) (x) = \sum_{i,j=1}^n \left( \left( \partial_{kk} \sigma^{ij} \right) \partial_i u \partial_j u + \left( \partial_k \sigma^{ij} \right) \partial_k (\partial_i u \partial_j u) + 2 \left( \partial_k \sigma^{ij} \partial_k u \partial_k \partial_i u + \sigma^{ij} \partial_k u \partial_k \partial_j u \right) \right).
\]

Relations (55), (56) and (57) are used again to obtain
\[
\partial_{kk} \left( \| \nabla u \|^2 \right) (x_0) = \| \nabla u(x_0) \|^2 \left( \partial_{kk} \sigma^{11} \right) (x_0) + 2 \| \nabla u(x_0) \| \partial_{kk1} u(x_0) + 2 \sum_{i=1}^n (\partial_{ki} u(x_0))^2.
\] (67)

Also, from (56), (57) and (58) it can be seen that
\[ \partial_{kk} \sigma^{11}(x_0) = -\partial_{k} \sigma^{11}(x_0) = -2 \left( \nabla \left( \frac{\partial}{\partial x_k} \cdot \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_1} \cdot \frac{\partial}{\partial x_1} \right) \right). \]

Therefore,
\[
\partial_{kk} \left( \| \nabla u \|^2 \right) (x_0) = 2 \left( -\| \nabla u(x_0) \|^2 \left( \nabla \left( \frac{\partial}{\partial x_k} \cdot \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_1} \cdot \frac{\partial}{\partial x_1} \right) + \| \nabla u(x_0) \| \partial_{kk1} u(x_0) + \sum_{i=1}^n (\partial_{ki} u(x_0))^2 \right) \right).
\] (68)

Using (66) and (68) inequality (60) becomes
\[
A \partial_{kk} u(x_0) - 2A^2 \| \nabla u(x_0) \|^2 \delta_{k1} + \frac{\partial_{kk1} u(x_0)}{\| \nabla u(x_0) \|} - \left( \nabla \left( \frac{\partial}{\partial x_k} \cdot \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_1} \cdot \frac{\partial}{\partial x_1} \right) + \sum_{i=1}^n (\partial_{ki} u(x_0))^2 \right) \leq 0.
\]

Since (65) holds it can be inferred that
\[ \partial_{111} u(x_0) \leq 2A^2 \| \nabla u(x_0) \|^3 + \| \nabla u(x_0) \| \left( \nabla \left( \frac{\partial}{\partial x_1} \cdot \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_1} \cdot \frac{\partial}{\partial x_1} \right) \right). \] (69)

and
\[ \partial_{kk1} u(x_0) \leq -A \partial_{kk} u(x_0) \| \nabla u(x_0) \| + \| \nabla u(x_0) \| \left( \nabla \left( \frac{\partial}{\partial x_k} \cdot \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_1} \cdot \frac{\partial}{\partial x_1} \right) \right) \text{ if } k > 1. \] (70)

In the sequel we evaluate at \(x_0\) the mean curvature equation (2). First, recall that
\[
\nabla_{ij}^2 u(x) = \nabla^2 u(x) \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \partial_{ij} u - \sum_{k=1}^n \Gamma^{k}_{ij} \partial_k u,
\]

\[ \Delta u(x) = \sum_{ij} \sigma^{ij} \nabla_{ij}^2 u(x). \] (72)
Using (55), (56) and (58) it can easily be seen that the quantities (61), (71) and (72) at \( x_0 \) have the values

\[
    u^i(x_0) = \partial_i u(x_0) = \| \nabla u(x_0) \| \delta_{i1}, \tag{73}
\]

\[
    \nabla_{ij}^2 u(x_0) = \partial_{ij} u(x_0), \tag{74}
\]

\[
    \Delta u(x_0) = \sum_{i=1}^{n} \partial_i u(x_0). \tag{75}
\]

Using these expressions the mean curvature equation (2) at \( x_0 \) takes the form

\[
    nH_0 W_0^3 = W_0^2 \Delta u(x_0) - \| \nabla u(x_0) \|^2 \partial_{11} u(x_0) = W_0^2 \sum_{i>1} \partial_{ii} u(x_0) + \partial_{11} u(x_0),
\]

where \( H_0 = H(x_0, u(x_0)) \) and \( W_0 = \sqrt{1 + \| \nabla u(x_0) \|^2} \). Using (65) again it follows

\[
    \sum_{i>1} \partial_{ii} u(x_0) = nH_0 W_0 + \frac{A \| \nabla u(x_0) \|^2}{W_0^2}. \tag{76}
\]

Finally let us differentiate (2) with respect to \( x_1 \) and evaluate at \( x_0 \). We have

\[
    \left( \partial_1 \left( W^2 \right) \right) \Delta u + W^2 \left( \partial_1 \Delta u \right) - 2 \sum_{i,j=1}^{n} u^i \left( \partial_1 u^j \right) \nabla_{ij}^2 u - \sum_{i,j=1}^{n} u^i u^j \left( \partial_1 \nabla_{ij}^2 u \right)
\]

\[
    = n \left( \partial_1 H + \partial_2 H \partial_1 u \right) W^3 + nH \left( \partial_1 \left( W^3 \right) \right). \tag{77}
\]

Expression (66) immediately gives

\[
    \partial_1 \left( W^2 \right) (x_0) = \partial_1 \left( \| \nabla u \|^2 \right) (x_0) = -2A \| \nabla u(x_0) \|^3, \tag{78}
\]

\[
    \partial_1 \left( W^3 \right) (x_0) = \frac{3}{2} W_0 \partial_1 \left( W^2 \right) (x_0) = -3AW_0 \| \nabla u(x_0) \|^3. \tag{79}
\]

Deriving (61) and using (56), (57) and (65) one gets

\[
    \partial_1 u^i(x_0) = \partial_{ii} u(x_0) = -A \| \nabla u(x_0) \|^2 \delta_{i1}. \tag{80}
\]

On the other hand, from (71) we deduce

\[
    \partial_1 \nabla_{ij}^2 u(x) = \partial_{1ij} u(x) - \left( \nabla \frac{\partial}{\partial x_i} \nabla u, \nabla \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \right) - \left( \nabla u(x), \nabla \frac{\partial}{\partial x_i} \nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right).
\]

Consequently, since (58) holds,

\[
    \partial_1 \nabla_{ij}^2 u(x_0) = \partial_{1ij} u(x_0) - \left( \nabla \frac{\partial}{\partial x_i} \nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right). \tag{81}
\]

Finally, deriving (72) and using (56), (57) and (81) one can infer

\[
    \partial_1 \Delta u(x_0) = \sum_{i=1}^{n} \left( \partial_{1ii} u(x_0) - \left( \nabla \frac{\partial}{\partial x_i} \nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} \right) \right). \tag{82}
\]
Substituting (55), (73), (74), (78), (79), (80), (81) and (82) in (77) we obtain

\[
n\partial_1 H(x_0)W_0^3 + n\partial_z H(x_0) \| \nabla u(x_0) \| W_0^3 - 3nAH_0W_0 \| \nabla u(x_0) \|^3
\]

\[
= -2A \| \nabla u(x_0) \|^3 \Delta u(x_0) + W_0^2 \sum_{i=1}^n \left( \partial_{1i}u(x_0) - \left< \nabla \frac{\partial}{\partial x_1} \nabla \frac{\partial}{\partial x_i}, \nabla u(x_0) \right> \right)
\]

\[
+ 2A \| \nabla u(x_0) \|^3 \partial_{11} u(x_0) - \| \nabla u(x_0) \|^2 \left( \partial_{111} u(x_0) - \left< \nabla \frac{\partial}{\partial x_1} \nabla \frac{\partial}{\partial x_1}, \nabla u(x_0) \right> \right)
\]

\[
= -2A \| \nabla u(x_0) \|^3 \sum_{i>1} \partial_{ii} u(x_0) + \partial_{111} u(x_0) - \left< \nabla \frac{\partial}{\partial x_1} \nabla \frac{\partial}{\partial x_1}, \nabla u(x_0) \right>
\]

Using (69), (70), (76) and recalling that \( \partial_z H \geq 0 \) we derive

\[
n\partial_1 H(x_0)W_0^3 - 3nAH_0W_0 \| \nabla u(x_0) \|^3
\]

\[
\leq -2A \| \nabla u(x_0) \|^3 \sum_{i>1} \partial_{ii} u(x_0) + 2A^2 \| \nabla u(x_0) \|^3
\]

\[
+ W_0^2 \| \nabla u(x_0) \| \sum_{i>1} \left( -A\partial_{ii} u(x_0) + \left< \nabla \frac{\partial}{\partial x_1} \nabla \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_1} \right> - \left< \nabla \frac{\partial}{\partial x_1} \nabla \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \right> \right)
\]

\[
= -A \| \nabla u(x_0) \| \left( W_0^2 + 2 \| \nabla u(x_0) \|^2 \right) \sum_{i>1} \partial_{ii} u(x_0) + 2A^2 \| \nabla u(x_0) \|^3
\]

\[
+ \| \nabla u(x_0) \| W_0^2 \sum_{i>1} \left< R \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_i} \right), \frac{\partial}{\partial x_1} \right>
\]

\[
= -A \| \nabla u(x_0) \| \left( 1 + 3 \| \nabla u(x_0) \|^2 \right) \left( nH_0W_0 + \frac{A \| \nabla u(x_0) \|^2}{W_0^2} \right)
\]

\[
+ 2A^2 \| \nabla u(x_0) \|^3 - \| \nabla u(x_0) \| W_0^2 \text{Ricc}_x \left( \frac{\partial}{\partial x_1} \right)
\]

\[
= -A \| \nabla u(x_0) \| nH_0W_0 \left( 1 + 3 \| \nabla u(x_0) \|^2 \right) - \frac{A^2 \| \nabla u(x_0) \|^3}{W_0^2} \left( 1 + 3 \| \nabla u(x_0) \|^2 \right)
\]

\[
+ 2A^2 \| \nabla u(x_0) \|^3 - \| \nabla u(x_0) \| W_0^2 \text{Ricc}_x \left( \frac{\partial}{\partial x_1} \right).
\]

Hence,

\[
\frac{A^2 \| \nabla u(x_0) \|^3 \left( \| \nabla u(x_0) \|^2 - 1 \right)}{W_0^2} \leq -AnH_0W_0 \| \nabla u(x_0) \| - n\partial_1 H(x_0)W_0^3 - \| \nabla u(x_0) \| W_0^2 \text{Ricc}_x \left( \frac{\partial}{\partial x_1} \right)
\]

\[
\leq Anh_0W_0 \| \nabla u(x_0) \| + n_h W_0^3 + \| \nabla u(x_0) \| W_0^2 R,
\]

where \( h_0 = \sup_{\Omega \times [-|u|,|u|]} |H|, \ h_1 = \sup_{\Omega \times [-|u|,|u|]} \| \nabla_H \| + \partial_z H \) and \( R = \sup_{\Omega} |\text{Ricc}| \). Therefore,

\[
\frac{A^2 \| \nabla u(x_0) \|^3 \left( \| \nabla u(x_0) \|^2 - 1 \right)}{W_0^2} \leq Anh_0 + n_h + R \leq An (h_0 + h_1 + R)
\]

15
since $W^2 > W > \|u(x_0)\|$ and $A, n > 1$. Choosing $A = 2n (1 + \|H\|_1 + R)$ it follows

$$\frac{\|\nabla u(x_0)\|^3 (\|\nabla u(x_0)\|^2 - 1)}{W^5} \leq \frac{n}{A} (\|H\|_1 + R) < \frac{1}{2},$$

which implies

$$\|\nabla u(x_0)\| < 3.$$

As a consequence,

$$\sup_{\Omega} \|\nabla u\| \leq 3e^{2A \sup_{\Omega} |u|}. \quad (83)$$

The combination of (54) with (83) yields the desired estimate. \qed

**Remark 14.** A related global gradient estimate was obtained independently in [8, Prop. 2.2 p. 5].

### 3 Proof of the existence theorems

**Proof of the main theorem (Theorem 3).** Let $\Omega \subset M$ with $\partial \Omega$ of class $C^2$, for some $\alpha \in (0, 1)$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$. Elliptic theory assures that the solvability of the Dirichlet problem $(P)$ strongly depends on $C^1$ a priori estimates for the family of related problems

$$\begin{cases}
\text{div} \left( \frac{\nabla u}{W} \right) = \tau nH(x, u) \quad \text{in} \quad \Omega, \\
u = \tau \varphi \quad \text{in} \quad \partial\Omega,
\end{cases} \quad (P_\tau)$$

not depending on $\tau$ or $u$.

Let $u$ be a solution of problem $(P_\tau)$ for arbitrary $\tau \in [0, 1]$. Let $w = \phi \circ d + \sup_{\partial\Omega} |\varphi|$ as in the proof of Theorem 8. Then

$$u \leq \sup_{\partial\Omega} |\tau \varphi| \leq \sup_{\partial\Omega} |\varphi| = w \quad \text{on} \quad \partial\Omega.$$

As before, let $\Omega_0$ be the biggest open subset of $\Omega$ having the unique nearest point property. Let $x \in \Omega_0$ and $y = y(x) \in \partial\Omega$ the nearest point to $x$. Once (24) holds and $\tau \in [0, 1]$ we have that

$$\mp n\tau H(x, \pm w) \leq n\tau |H(x, \varphi(y))| \leq n |H(x, \varphi(y))|.$$ 

From (23) we have

$$\pm \Omega_\tau(\pm w) = \mathcal{M}w \mp n\tau H(x, \pm w)(1 + \phi^2)^{3/2} \leq 0.$$

Proceeding as in the proof of Theorem 8, we get that $w$ and $-w$ are supersolution and subsolution in $\Omega_0$, respectively, for the problem $(P_\tau)$. This provides a priori height estimate for any solution of the problems $(P_\tau)$ independently of $\tau$.

In order to prove that Theorem 10 provides a priori gradient estimate for the solutions of the related problems $(P_\tau)$ let us define $w^\pm_\tau = \pm \phi \circ d + \tau \varphi$. Making an examination of the proof of this theorem it can easily be seen that, on account of assumptions (6) and (7), the geometric condition (37) holds for $\Omega$. Using again the strong Serrin condition (7) one has

$$\Delta d(x) \leq -n\tau |H(y, \tau \varphi(y))| \quad \forall \ x \in \Omega_a. \quad (84)$$

Replacing (38) by (84) we can derive in analogous way

$$\pm \Omega_\tau(w^\pm_\tau) \leq \nu \psi' + \psi'' = 0,$$
where $\nu$ is the same defined in (44). Proceeding exactly as before we obtain
\[
\|\nabla u(y)\| \leq \tau \|\nabla \varphi(y)\| + \psi'(0) \leq \|\nabla \varphi(y)\| + \psi'(0),
\]
which yields the same estimate (27), which is independent of $\tau$, for all solutions of the related problems $(P_\tau)$.

On the other hand, elliptic theory guarantees that any solution $u$ of the related problems $(P_\tau)$ belongs to $C^3(\Omega)$. Hence, Theorem 13 can be applied to obtained the desired a priori global gradient estimate independently of $\tau$ and $u$.

The existence of a solution $u \in C^{2,\alpha}(\overline{\Omega})$ for the Dirichlet problem $(P)$ is obtained applying the Leray-Schauder fixed point theorem in an usual way (see [13, Th. 11.4 p. 281]). Uniqueness follows from the maximum principle, in view of the assumption $\partial_z H \geq 0$.

Proof of Theorem 4. We first recall that in $\mathbb{H}^n \times \mathbb{R}$ there exists an entire vertical graph of constant mean curvature $\frac{n-1}{n}$. Explicit formulas were given by Bérard-Sa Earp [6, Th. 2.1 p. 22]. The a priori height estimate for the solutions of the related problems $(P_\tau)$ follows directly from the convex hull lemma [6, Prop. 3.1 p. 41].

The rest of the proof is the same as before, being that the a priori boundary gradient estimate follows from Theorem 11.

Proof of Theorem 5. Under the hypothesis on $M$ and $\Omega$, Galvez-Lozano [12, Th. 6 p. 12] proved the existence of a vertical graph over $\Omega$ with constant mean curvature $\frac{n-1}{n}$ and zero boundary data. As a matter of fact, such a graph constitutes a barrier for the solutions of the related problems $(P_\tau)$.

On the other hand, the strong Serrin condition trivially holds since, for $y \in \partial \Omega$,
\[
(n-1)\mathcal{H}_{\partial \Omega}(y) > (n-1)c > n-1 \geq \sup_{\Omega \times \mathbb{R}} |H(x,z)|.
\]
Besides,
\[
\text{Ricc}_x \geq -(n-1)c^2.
\]
Accordingly, the boundary gradient estimate follows from our Theorem 12.

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