MULTIDIMENSIONAL DILATION OPERATORS, BOYD AND SHIMOGAKI INDICES OF BILATERAL WEIGHT GRAND LEBESQUE SPACES

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Abstract

In this paper we compute the norm of dilation operators, multidimensional Boyds and Shimogaki’s indices in the Bilateral Grand Lebesgue Spaces and consider some applications.

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1. INTRODUCTION. STATEMENT OF PROBLEM

Let $X$ be a $d$ dimensional positive Euclidean subspace with a multiplicative decomposition of a view:

$$X = R^{d(1)}_+ \times R^{d(2)}_+ \times \ldots \times R^{d(k)}_+, k = 1, 2, \ldots,$$

so that $d = d(1) + d(2) + \ldots + d(k)$. We will write for $\vec{x} = x \in X$

$$\vec{x} = x = (x(d(1)), x(d(2)), \ldots, x(d(k))), \quad x(d(j)) = \vec{x}(d(j)) \in R^{d(j)}_+, \quad j = 1, 2, \ldots, k.$$

The space $X$ is equipped by the usually Borelian sigma - algebra $\Sigma$ and by the non - negative weight measure

$$\mu(V) = \mu_W(V) = \int_V W(\vec{x}) \, d\vec{x}, \quad V \in \Sigma,$$

$$W(\vec{x}) = \prod_{r=1}^{k} W_r(\vec{x}(d(r))),$$

where all the non - trivial functions $W_r(\cdot)$ are continuous in the domain $X \setminus \{0\}$, non - negative and homogeneous of an order $\theta(r) : \forall \lambda > 0 \Rightarrow$

$$W_r(\lambda x(d(r))) = \lambda^{\theta(r)} W_r(x(d(r))).$$

Notation:

$$\vec{\theta} = \theta = \{\theta(1), \theta(2), \ldots, \theta(k)\}.$$
For $a$ and $b$ constants, $1 \leq a < b \leq \infty$, let $\psi = \psi(p) = \psi(p; a, b), p \in (a, b)$, be a continuous on the open interval $(a, b)$ positive: $\psi(p) \geq 1$ function such that $\psi(a + 0)$ and $\psi(b - 0)$ exist, both may be infinite, and postulate that $\psi(b - 0) = \infty$.

The class such a functions we will denote $E \Psi = E \Psi(a, b); E \Psi = \{\psi\}$.

The Bilateral Grand Lebesgue space (in notation BGL, BGLS) $G_X(\mu; \psi; a, b) = G_X(\psi; a, b) = G(\psi; a, b) = G(\psi)$ is the space of all the measurable functions $h : X \to \mathbb{R}$ endowed with the norm

$$||h||_{G(\psi)} \overset{def}{=} \sup_{p \in (a, b)} |h|_p/\psi(p), \quad |h|_p = |h|_{p, \mu} = \left[\int_X |h(x)|^p \, d\mu(x)\right]^{1/p}.$$ 

By definition, $h(\cdot) \in G_X(\mu; \psi, a, b) = G(\psi)$ if and only if $||h||_{G(\psi)} < \infty$.

**Proposition 1.** The BGL spaces are Banach functional spaces; moreover, they are rearrangement invariant (r.i.) spaces.

**Proof.** We must prove only that the $G(\psi)$ space satisfies the Fatou property, since all the other properties are evident.

Recall at first that the Fatou property of some r.i. space $G$ over source triplet $(X, \Sigma, \mu)$ denotes that for arbitrary non-increasing sequence of non-negative functions $\{f_n\} = \{f_n(x), \ x \in X\}$ belonging to the space $G$ and such that as $n \uparrow \infty$

$$f_n(x) \uparrow f(x), \quad \sup_n ||f_n||_G < \infty$$

it follows

$$||f_n||_G \uparrow ||f||_G.$$ 

Let $G = G(\psi)$ and suppose the sequence of the measurable functions $\{f_n\} = \{f_n : X \to \mathbb{R}\}$ satisfies our condition. As long as the space $L_p(X, \mu)$ satisfies the Fatou property, we have:

$$\sup_n ||f_n||_{G(\psi)} = \sup_n \sup_{p \in (a, b)} |f_n|_p/\psi(p) = \sup_{p \in (a, b)} \sup_n |f|_p/\psi(p) = ||f||_{G(\psi)},$$

Q.E.D.

Notice that in this proof we do not use the concrete view of the measure $\mu$; we assume only that the measure $\mu$ is sigma - finite.

The $G(\psi)$ spaces with $\mu(X) = 1$ appeared in [9]; it was proved that in this case each $G(\psi)$ space coincides with certain exponential Orlicz space, up to norm equivalence. Partial cases of these spaces were intensively studied, in particular, their associate spaces, fundamental functions $\phi(G(\psi; a, b); \delta)$, Fourier and singular operators, conditions for convergence and compactness, reflexivity and separability, martingales in these spaces, etc.; see, e.g., [1], [2]-[7], [8] [13] [14]. These spaces are also Banach and moreover rearrangement invariant (r.i.).

Some classical properties of these spaces (Sobolev embedding theorems, convolution operators etc.) was partially investigated in [11].
The BGLS norm estimates, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in PDE [3]-[6], probability in Banach spaces [10], in the modern non-parametrical statistics, for example, in the so-called regression problems, e.g., for random variables are used in PDE [3]-[6], probability in Banach spaces.

We are going to deal not with all functions \( \psi \) described above but with an essential subset of these functions satisfying certain natural conditions.

Let again \( a \geq 1, \ b \in (a, \infty) \), and let \( \psi = \psi(p) \) be a positive continuous function on the open interval \((a, b)\) such that there exists a measurable function \( f : X \rightarrow \mathbb{R} \) for which

\[
(1) \quad f(\cdot) \in \cap_{p \in (a,b)} L_p(X, \mu), \ \psi(p) = |f|_p, \ p \in (a, b).
\]

and such that \( \max \{ \psi(a + 0), \psi(b - 0) \} = \infty \) and in the case \( b = \infty \) we define \( \psi(b - 0) = \lim_{p \to \infty} \psi(p) \) and postulate again that \( \psi(b - 0) = \infty \).

We say that the equality (1) and the function \( f(\cdot) \) from (1) is the representation of the function \( \psi \). The existence of representation implies, by the way, the log-convexity of \( \psi \).

We denote the subset of all the functions \( \psi \) having representation by \( \Psi = \Psi(a, b) \). For complete description of these functions see, for example, ([13] p.p. 21-27), [14]).

Another definitions. We will say as usually ([1], p. 14 - 16) that the function \( \psi \) is the fundamental function \( \psi \). We will use widely further the notion of fundamental function \( \phi(G, \delta) \), \( \delta \in (0, \infty) \) of the arbitrary r.i. space \( G \) over the triplet \((X, \Sigma, \mu)\). Recall that by definition

\[
\phi(G, \delta) = ||I(A)||G, \ \mu(A) = \delta, \ \delta \in [0, \infty)
\]

and \( I(A) = I(A, x) = 1, x \in A \). \( I(A) = I(A, x) = 0, x \notin A \).

This notion play a very important role in the theory of interpolation of operators, theory of Fourier series, theory of approximation etc. See, for example, [11], [1], [12] etc.

**Lemma 1.**

\( \psi(\cdot) \in E \Psi \Rightarrow \phi(G(\psi), 0+) = 0 \).

**Proof.** Assume at first that \( b < \infty \); then
\[ \phi(G(\psi), \delta) \leq \delta^{1/b} \rightarrow 0, \ \delta \rightarrow 0. \]

Therefore, it remains only to consider the case \( b = \infty \).
Recall that in this case we suppose \( \psi(\infty) = \infty \).
Let \( \epsilon = \text{const} \in (0, 1) \) be arbitrary small number. There exists a number \( Q > 1 \) for which
\[ \forall p > Q \Rightarrow \psi(p) > 1/\epsilon. \]
We have for all sufficient small values \( \delta : \delta \in (0, \delta_0), \delta_0 = \delta_0(\epsilon) \in (0, 1) : \]
\[ \phi(G(\psi), \delta) \leq \delta^{1/Q} + \epsilon < 2\epsilon. \]

As a consequence:

**Lemma 2.** Assume that the metric space \( \Sigma \) under the distance
\[ \rho(A, B) = \arctan(\mu((A \setminus B) \cup (B \setminus A))), \ A, B \in \Sigma \]
is separable. Then the r.i. space \( G^\psi(\psi)(= GA(\psi) = GB(\psi)) \) is also separable.

**Lemma 3.** Let \( \psi(\cdot) \in \Psi(a, b) \). Then
\[ \lim_{s \to 0^+} \frac{\log \phi(G(\psi), s)}{\log s} = \frac{1}{b}; \]
\[ \lim_{s \to 0^+} \frac{\log \phi(G(\psi), s)}{\log s} = \frac{1}{a}. \]

**Proof.** It is enough to consider only the second case \( s \to \infty \) and in the first case \( s \to 0 \) under condition \( b < \infty \).

**A. Upper bound.** We have for the values \( s \in (1, \infty) : \)
\[ \phi(s) = \phi(G(\psi), s) = \sup_{p \in (a, b)} \frac{s^{1/p}}{\psi(p)} \leq s^{1/a}, \]
therefore
\[ \lim_{s \to 0^+} \frac{\log \phi(G(\psi), s)}{\log s} \leq \frac{1}{a}. \]

**B. Low bound.** Let \( \epsilon \in (0, (b - a)) \) be arbitrary number. Tacking into account a simple estimation
\[ \phi(s) \geq \frac{s^{1/(a+\epsilon)}}{\psi(a + \epsilon)}, \]
we conclude
\[ \lim_{s \to 0^+} \frac{\log \phi(G(\psi), s)}{\log s} \geq \frac{1}{a + \epsilon}. \]

This completes the proof of Lemma 3.
Let now \( \vec{s} = s = (s(1), s(2), \ldots, s(k)) \) be an arbitrary positive \( \forall j \Rightarrow s(j) > 0 \) vector. We define for arbitrary measurable function \( f : X \to R \) the following family of linear operators (dilation operators):
\[ \sigma_s f(x) = f\left(\frac{x(d(1))}{s(1)}, \frac{x(d(2))}{s(2)}, \ldots, \frac{x(d(k))}{s(k)}\right). \]

Let also \( V_1 \) and \( V_2 \) be a two rearrangement invariant (r.i.) spaces over \((X, \Sigma, \mu)\) for which the operators \( \sigma_s \) are bounded as operator from the space \( V_1 \) into the (other, in general case) space \( V_2 \):

\[ h(\vec{s}, V_1, V_2) = ||\sigma_s|| (V_1 \rightarrow V_2) < \infty. \]

**Definition 1.** We define the upper particular Boyds index

\[ B^+_j (V_1, V_2) = B^+_j (V_1, V_2; s(1), s(2), \ldots, s(j - 1), s(j + 1), \ldots, s(k)) \]

as a limit (if there exists)

\[ B^+_j (V_1, V_2) \overset{\text{def}}{=} \lim_{s(j) \to \infty} \frac{\log h(\vec{s}, V_1, V_2)}{\log s(j)}. \]

Analogously may be defined the low particular Boyds index:

\[ B^-_j (V_1, V_2) \overset{\text{def}}{=} \lim_{s(j) \to 0} \frac{\log h(\vec{s}, V_1, V_2)}{\log s(j)}. \]

We will write for brevity for the values \( j = 1, 2, \ldots, k \)

\[ B^+(V_1, V_2) = \{B^+_j (V_1, V_2)\}, \quad B^-(V_1, V_2) = \{B^-_j (V_1, V_2)\}. \]

These definitions may be generalized on the case when \( X = R^d \) with at the same weight \( W(\cdot) \) and on the case when \( X = (-\pi, \pi)^d \) relatively the standard weight \( W = 1/(2\pi)^d = \text{const.} \)

Notice that in the case \( d = 1 \), \( W(x) = 1 \) and \( V_1 = V_2 \) we obtain the classical definition of Boyds indices ([1], chapter 3, section 5, p. 149.)

**The main goal of this paper is to compute the upper and low Boyd’s (and Shimogaki’s, see after) indices for pairs of Bilateral Grand Lebesgue spaces.**

The paper is organized as follows. In the next section we calculate the Boyd’s indices for \( G(\psi; a, b) \) spaces.

Further, in the section follows we compute the so - called Shimogaki’s indices in the one - dimensional case \( d = 1 \).

In the fourth section we consider some generalizations of dilation operators (multidimensional matrix dilation operators) in BGL spaces.

In the last section we study and describe some consequences of obtained results.

We use the symbols \( C, C_j, C(X, Y), C(p, g; \psi) \) etc., to denote positive finite constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like \( C_1(X, Y) \) and \( C_2(X, Y) \). The relation \( g(\cdot) \asymp h(\cdot) \), \( p \in (A, B) \), where \( g = g(p) \), \( h = h(p) \), \( g, h : (A, B) \rightarrow \mathbb{R}_+ \), denotes as usually

\[ 0 < \inf_{p \in (A, B)} \frac{h(p)}{g(p)} \leq \sup_{p \in (A, B)} \frac{h(p)}{g(p)} < \infty. \]

The symbol \( \sim \) will denote usual equivalence in the limit sense.
We define as usually for the two \( k \) - dimensional vectors \( s = \vec{s} \) and \( d = \vec{d} \)

\[
s^d = s^d = \prod_{r=1}^{k} s(r)^{d(r)}.\]

2. Boyd’s Indices

In this section we give an expression for the norm of a dilations operators for some pairs of BGL spaces and for the Boyds (and other) indices for these pair of spaces.

2.1. Dilation Operators.

Theorem 1.1.

A. Let \( \psi(\cdot), \zeta(\cdot), \nu(\cdot) \in E \Psi(a,b) \) and let \( \zeta = \psi \cdot \nu \).

We assert:

\[
|||\sigma|||(G(\psi) \rightarrow G(\zeta)) \leq \phi \left( G(\nu), \vec{s}^{d+\theta} \right). \tag{1.A}
\]

B. Let \( \psi(\cdot) \in \Psi, \zeta(\cdot), \nu(\cdot) \in E \Psi(a,b) \) and let \( \zeta = \psi \cdot \nu \).

We assert:

\[
|||\sigma|||(G(\psi) \rightarrow G(\zeta)) = \phi \left( G(\nu), \vec{s}^{d+\theta} \right). \tag{1.B}
\]

Proof. A.

Let \( \psi(\cdot), \zeta(\cdot), \nu(\cdot) \in E \Psi(a,b) \) and let \( \zeta = \psi \cdot \nu \); \( f \in G(\psi) \).

We can assume without loss of generality that \( ||f||G(\psi) = 1 \).

It follows from the definition of the BGL spaces that

\[
|f|^p \leq \left[ ||f||G(\psi) \right]^p \psi^p(p) = \psi^p(p).
\]

We obtain using the formula of changing variables and homogeneity of the weight \( W \):

\[
|\sigma f|^p_p = \int_X \left| f \left( \frac{x(d(1))}{s(1)}, \frac{x(d(2))}{s(2)}, \ldots, \frac{x(d(k))}{s(k)} \right) \right|^p d \mu(x) =
\]

\[
\int_X \left| f \left( \frac{x(d(1))}{s(1)}, \frac{x(d(2))}{s(2)}, \ldots, \frac{x(d(k))}{s(k)} \right) \right|^p W(x) \, dx =
\]

\[
\vec{s}^{d+\theta} \cdot \int_X |f(y(d(1)), y(d(2)), \ldots, y(d(k)))|^p \, W(y) \, dy =
\]

\[
\vec{s}^{d+\theta} \cdot |f|^p_p \leq \vec{s}^{d+\theta} \psi^p(p).
\]

Therefore

\[
|\sigma f|_p \leq \vec{s}^{(d+\theta)/p} \psi(p); \quad \frac{|\sigma f|^p_p}{\zeta(p)} \leq \frac{\vec{s}^{(d+\theta)/p}}{\nu(p)}.
\]

We obtain taking the maximum over \( p \in (a,b) \):

\[
|||\sigma|||G(\zeta) = \sup_{p \in (a,b)} |\sigma f|^p_p \leq \sup_{p \in (a,b)} \frac{\vec{s}^{(d+\theta)/p}}{\nu(p)} = \phi \left( G(\nu), \vec{s}^{d+\theta} \right),
\]

Q.E.D.
Proof. B. It is easy to see that if $\psi(\cdot) \in \Psi = \Psi(a,b)$, i.e. if $|f|_p = \psi(p)$, $p \in (a,b)$, then during all the proof of the first assertion $A$ there is always the equality.

Indeed, let $\psi(\cdot) \in \Psi$, $\zeta(\cdot), \nu(\cdot) \in E\Psi(a,b)$ and let $\zeta = \psi \cdot \nu$; $f \in G(\psi)$ and let $f$ be a representation of $\psi(\cdot)$: $|f|_p = \psi(p)$, $p \in (a,b)$. Hence $||f||G(\psi) = 1$.

It follows from the definition of the BGL spaces and from the equality $|f|_p = \psi(p)$ that

$$|f|_{p,\nu} \leq ||f||G(\psi) \cdot \psi(p) = \psi^p(p).$$

We obtain using the formula of changing variables and homogeneity of the weight $W$:

$$\frac{|\sigma_{\varphi} f|_p}{|f|_p} = \int_X f \left( \frac{x(d(1))}{s(1)}, \frac{x(d(2))}{s(2)}, \ldots, \frac{x(d(k))}{s(k)} \right) \, d \mu(x) =$$

$$\int_X f \left( \frac{x(d(1))}{s(1)}, \frac{x(d(2))}{s(2)}, \ldots, \frac{x(d(k))}{s(k)} \right) \, W(x) \, dx =$$

$$s^{\vec{d}+\vec{\theta}} \cdot \int_X |f(y(d(1)), y(d(2)), \ldots, y(d(k)))|^p \, W(y) \, dy =$$

$$s^{\vec{d}+\vec{\theta}} \cdot |f|_p^p = s^{\vec{d}+\vec{\theta}} \psi^p(p).$$

Therefore

$$|\sigma_{\varphi} f|_p = s^{(\vec{d}+\vec{\theta})/p} \psi(p); \quad |\sigma_{\varphi} f|_\zeta = s^{(\vec{d}+\vec{\theta})/\nu}.$$

We obtain taking the upper bound over $p \in (a,b)$:

$$||\sigma_{\varphi} f||G(\zeta) = \sup_{p \in (a,b)} |\sigma_{\varphi} f|_p \quad = \sup_{p \in (a,b)} s^{(\vec{d}+\vec{\theta})/\nu} = \phi \left( G(\nu), s^{\vec{d}+\vec{\theta}} \right).$$

This completes the proof of Theorem 1.1.

2.2. Boyd’s Indices. Main Result.

Theorem 1.2. Let again $\psi(\cdot) \in \Psi(a,b)$, $\zeta(\cdot), \nu(\cdot) \in E\Psi(a,b)$ and let $\zeta = \psi \cdot \nu$. We assert:

$$B^+(G(\psi), G(\zeta)) = \frac{d + \theta}{a}, \quad (1.C)$$

$$B^-(G(\psi), G(\zeta)) = \frac{d + \theta}{b}. \quad (1.D)$$

Proof it follows from the assertion of theorem 1.1 for the explicit view of Boyds indices and from the Lemma 3. Namely,

$$B^+_{j}(G(\psi), G(\zeta)) = \lim_{s(j) \to \infty} \frac{\log h(s, G(\psi), G(\zeta))}{\log s(j)} =$$

$$\lim_{s(j) \to \infty} \frac{\log \phi(G(\nu), s^{d+\theta})}{\log s(j)} = \frac{d(j) + \theta(j)}{a}. \quad (2.C)$$

Corollary. At the same assertions as in the theorems 1.1 and 1.2 are true for the spaces $G^0(\zeta)$, $G^0(\psi)$, $G^0(\nu)$ and following for the spaces $GA(\zeta)$, $GA(\psi)$, $GA(\nu)$; $GB(\zeta)$, $GB(\psi)$, $GB(\nu)$. 
Namely,

**AA.** Let \( \psi(\cdot), \zeta(\cdot), \nu(\cdot) \in E\Psi(a, b) \) and let \( \zeta = \psi \cdot \nu \). We assert:

\[
||\sigma_{\tilde{s}}|| (G^0(\psi) \rightarrow G^0(\zeta)) \leq \phi \left( G^0(\nu), s^{\tilde{d} + \tilde{\theta}} \right); \tag{1.AA}
\]

**BB.** Let \( \psi(\cdot) \in \Psi(a, b), \zeta(\cdot), \nu(\cdot) \in E\Psi(a, b) \) and let \( \zeta = \psi \cdot \nu \). We assert:

\[
||\sigma_{\tilde{s}}|| (G^0(\psi) \rightarrow G^0(\zeta)) = \phi \left( G^0(\nu), s^{\tilde{d} + \tilde{\theta}} \right).
\]

**Proof.** It is enough to prove that if \( d = 1 \) and \( \psi \in \Psi, \zeta \in E\Psi, \nu \in E\Psi \), then

\[
||\sigma_{\tilde{s}}|| (G^0(\psi) \rightarrow G^0(\zeta)) = \phi \left( G(\nu), s^{\tilde{d} + \tilde{\theta}} \right).
\]

Let \( f : X \rightarrow R \) be the measurable non-negative function for which \( |f|_p = \psi(p), p \in (a, b) \). The existence of this function it follows from the condition \( \psi \in \Psi \). We have from the definition of the norm of a function belonging to the space \( G(\psi) \) or \( G^0(\psi) \) that \( ||f|| G(\psi) = 1 \).

Since the measure \( \mu \) is sigma-finite, there exists a decreasing: \( A(1) \subset A(2) \subset A(3) \ldots \) sequence of measurable sets \( A(n), n = 1, 2, \ldots, A(n) \in \Sigma \) with finite measures \( \mu(A(n)) < \infty \) covering the space \( X : \)

\[
\bigcup_{n=1}^{\infty} A(n) = X.
\]

Let us consider the sequence of a non-negative functions

\[
f_n = f_n(x) = f(x) \cdot I(A(n), x) \cdot I(x : |f(x)| \leq n)).
\]

We observe: \( f_n \uparrow f, f_n \in G^0(\psi) \) as long as \( G^0(\psi) = GB(\psi), ||f_n|| G^0(\psi) = ||f_n|| G(\psi) \uparrow ||f|| G(\psi), \forall p \in (a, b) \Rightarrow |f_n|_p \uparrow |f|_p \).

The inequality \( (1.AA) \) is obvious; let us prove the inverse inequality. Since for all sufficient great values \( n \)

\[
||\sigma_{\tilde{s}}|| (G^0(\psi) \rightarrow G^0(\zeta)) \geq \frac{||\sigma_{\tilde{s}} f_n|| G^0(\zeta)}{||f_n|| G^0(\psi)} \geq \frac{||\sigma_{\tilde{s}} f_n|| G(\zeta)}{||f|| G(\psi)}.
\]

We get using Fatou property of \( G(\psi) \) spaces:

\[
||\sigma_{\tilde{s}}|| (G^0(\psi) \rightarrow G^0(\zeta)) \geq \sup_n \frac{||\sigma_{\tilde{s}} f_n|| G(\zeta)}{||f|| G(\psi)} = \sup_n \frac{\sup_{p \in (a, b)} |\sigma_{\tilde{s}} f_n|_p/|\zeta(p)|}{||f|| G(\psi)} = \sup_{p \in (a, b)} \frac{\sup_n |\sigma_{\tilde{s}} f_n|_p/|\zeta(p)|}{||f|| G(\psi)} = \sup_{p \in (a, b)} \frac{|\sigma_{\tilde{s}} f|_p/|\zeta(p)|}{||f|| G(\psi)} = ||\sigma_{\tilde{s}}|| (G(\psi) \rightarrow G(\zeta)) = \phi \left( G(\nu), s^{\tilde{d} + \tilde{\theta}} \right).
\]
For example, if $\zeta(p) = \psi(p)$, $p \in (a, b)$; $d = 1$, $W(x) = 1$; then (cf. \cite{15})

$$B^+(G(\psi), G(\psi)) = B^+(G^\alpha(\psi), G^\alpha(\psi)) = 1/a,$$

$$B^-(G(\psi), G(\psi)) = B^+(G^\alpha(\psi), G^\alpha(\psi)) = 1/b.$$ 

### 3. Shimogaki’s indices

We consider in this section only the one-dimensional case $d = 1$.

Let $G$ be again r.i. space over $(X, \Sigma, \mu)$ with correspondent fundamental function $\phi(G, \delta) = \phi(\delta)$. Let us denote

$$M_G(t) = \sup_{s > 0} \frac{\phi(st)}{\phi(s)},$$

$$\beta^-(G) = \sup_{t \in (0, 1)} \frac{\log M_G(t)}{\log t} = \lim_{t \to 0+} \frac{\log M_G(t)}{\log t},$$

$$\beta^+(G) = \inf_{t \in (1, \infty)} \frac{\log M_G(t)}{\log t} = \lim_{t \to \infty} \frac{\log M_G(t)}{\log t}.$$ 

The numbers $\beta^-(G)$ and $\beta^+(G)$ are called Shimogaki indices. It is known (see \cite{1}, p. 171-178) that

$$0 \leq B^-(G, G) \leq \beta^-(G) \leq \beta^+(G) \leq B^+(G, G).$$

**Theorem 2.** Let $G = G(\psi; a, b)$, $\psi \in \Psi$; then

$$1/b = \beta^-(G(\psi; a, b)) < \beta^+(G(\psi; a, b)) = 1/a.$$

**Proof.**

**A. Upper bound.** Assume for certainty $t \to \infty$, $t > 1$.

$$\phi(st) = \sup_{p \in (a, b)} \frac{s^{1/p} t^{1/p}}{\psi(p)} \leq t^{1/a} \sup_{p \in (a, b)} s^{1/p} \psi(p) = t^{1/a} \phi(s),$$

hence

$$\lim_{t \to \infty} \frac{\log M_G(t)}{\log t} \leq 1/a.$$

**B. Low bound.** Let $\epsilon = const \in (0, (b - a))$.

$$M_G(t) \geq \frac{\phi((a + \epsilon)t)}{\phi(a + \epsilon)} \geq C(a, \epsilon) \phi((a + \epsilon)t).$$

We find using Lemma 3:

$$\lim_{t \to \infty} \frac{\log M_G(t)}{\log t} \geq 1/(a + \epsilon).$$

Since the number $\epsilon$ is arbitrary, we obtained the proof what is desired.
**Corollary.** The condition of homogeneity of the weight $W = W(\bar{x})$ may be weakened as follows.

Let us denote

$$
K_+^\infty = \sup_{\{\min s(j) \geq 1\}} \frac{W(sy)}{s^\theta W(y)},
$$

$$
K_-^\infty = \inf_{\{\min s(j) \geq 1\}} \frac{W(sy)}{s^\theta W(y)},
$$

$$
K_0^+ = \sup_{\{\max s(j) \leq 1\}} \frac{W(sy)}{s^\theta W(y)},
$$

$$
K_0^- = \inf_{\{\max s(j) \leq 1\}} \frac{W(sy)}{s^\theta W(y)}.
$$

We assert:

I. If $K_+^\infty < \infty$, $\psi, \zeta, \nu \in E\Psi(a,b)$, $\min(s(j)) > 1$, then

$$
||\sigma||((G(\psi) \to G(\zeta)) \leq \max \left[ (K_+^\infty)^{1/a}, (K_0^+)^{1/b} \right] \cdot \phi \left( G(\nu), s^{\bar{d} + \bar{\theta}} \right);
$$

II. If $K_-^\infty < \infty$, $\psi \in \Psi(a,b), \zeta, \nu \in E\Psi(a,b)$, $\min(s(j)) > 1$, then

$$
||\sigma||((G(\psi) \to G(\zeta)) \geq \min \left[ (K_-^\infty)^{1/a}, (K_0^-)^{1/b} \right] \cdot \phi \left( G(\nu), s^{\bar{d} + \bar{\theta}} \right);
$$

III. If $K_0^+ < \infty$, $\psi, \zeta, \nu \in E\Psi(a,b)$, $\max(s(j)) > 1$, then

$$
||\sigma||((G(\psi) \to G(\zeta)) \leq \max \left[ (K_0^-)^{1/a}, (K_0^-)^{1/b} \right] \cdot \phi \left( G(\nu), s^{\bar{d} + \bar{\theta}} \right);
$$

IV. If $K_0^- > 0$, $\psi \in \Psi(a,b), \zeta, \nu \in E\Psi(a,b)$, $\max(s(j)) > 1$, then

$$
||\sigma||((G(\psi) \to G(\zeta)) \geq \min \left[ (K_0^-)^{1/a}, (K_0^-)^{1/b} \right] \cdot \phi \left( G(\nu), s^{\bar{d} + \bar{\theta}} \right);
$$

As a consequence: if all the conditions I - IV are satisfied, then

$$
B^+(G(\psi), G(\zeta)) = \frac{\bar{d} + \bar{\theta}}{a},
$$

$$
B^-(G(\psi), G(\zeta)) = \frac{\bar{d} + \bar{\theta}}{b}.
$$

The conclusion of theorem 2 also holds.
4. Matrix Dilations

In this section we describe briefly some multidimensional matrix dilation operators in BGL spaces and compute its norms.

Let \( X =\mathbb{R}^d \) be usually Euclidean space with Lebesgue measure \( m : m(dx) = dx \). Let \( A \) be a square: \( A = d \times d \) non-degenerate constant matrix. We define alike in \([21]\) the matrix dilation operator \( D_A \) as follows: for arbitrary measurable function \( f : X \rightarrow \mathbb{R} \)

\[
(D_A f)(x) = f(A^{-1}x).
\]

As

\[
|D_A f|^p = \int_X |f(A^{-1}x)|^p \, dx = |\det(A)| \int_X |f(y)|^p \, dy = |\det(A)| |f|^p,
\]

we conclude in the case \( \psi(\cdot), \zeta(\cdot), \nu(\cdot) \in E\Psi(a,b), \zeta(p) = \psi(p) \cdot \nu(p) \):

\[
\frac{|D_A f|^p}{\zeta(p)} \leq \|f\|G(\psi) \frac{|\det(A)|^{1/p}}{\nu(p)},
\]

\[
\|D_A f\|G(\zeta) \leq \|f\|G(\psi) \cdot \sup_{p \in (a,b)} \frac{|\det(A)|^{1/p}}{\nu(p)} = \|f\|G(\psi) \cdot \phi(G(\nu), |\det(A)|).
\]

Therefore,

\[
\|D_A\|G(\psi) \rightarrow G(\zeta) \leq \phi(G(\nu), |\det(A)|)
\]

with equality in the case, e.g. if \( \psi(\cdot) \in \Psi(a,b), \zeta(\cdot), \nu(\cdot) \in E\Psi(a,b), \zeta(p) = \psi(p) \cdot \nu(p) \).

We have as a consequence in the last case:

\[
\lim_{\det(A) \rightarrow \infty} \frac{\log \|D_A\|G(\psi) \rightarrow G(\zeta)}{\log |\det(A)|} = \frac{1}{a},
\]

\[
\lim_{\det(A) \rightarrow 0} \frac{\log \|D_A\|G(\psi) \rightarrow G(\zeta)}{\log |\det(A)|} = \frac{1}{b}.
\]

We consider now some slight generalization. Let \( |||x||| = |||\vec{x}||| \) be some non-degenerate norm in the space \( \mathbb{R}^d \) and let \( |||A||| \) be the correspondence norm of the matrix \( A \):

\[
|||A||| = \sup\{|||A x|||, \ x : |||x||| = 1\}.
\]

Let \( \mu_\sigma \) be a following weight measure:

\[
\mu_\sigma(V) = \int_V |||x|||^\sigma \, dx, \ \sigma = const.
\]

We assert in general case \( \psi(\cdot), \zeta(\cdot), \nu(\cdot) \in E\Psi(a,b), \zeta(p) = \psi(p) \cdot \nu(p) \):

\[
\|D_A\|G(\psi, \mu_\sigma) \rightarrow G(\zeta, \mu_\sigma)) \leq \phi(G(\nu, \mu_\sigma), |\det(A)| \cdot |||A|||^\sigma).
\]
We get as an particular case, i.e. in the case if in addition the norm $||| \cdot |||$ is usually Euclidean norm and $A = s \times U$, where $s = \text{const} > 0$ and $U$ be the unitarian operator:

$$||D_A||(G(\psi, \mu) \rightarrow G(\zeta, \mu\sigma)) \leq \phi(G(\nu, \mu_\sigma), s^{d+\sigma}).$$

Therefore, we conclude in the considered case under additional assumptions $\psi(\cdot) \in \Psi(a, b), \zeta(\cdot), \nu(\cdot) \in E\Psi(a, b)$, $\zeta(p) = \psi(p) \cdot \nu(p)$:

$$\lim_{s \rightarrow \infty} \frac{\log ||D_A||(G(\psi) \rightarrow G(\zeta))}{\log s} = \frac{d + \sigma}{a},$$

$$\lim_{s \rightarrow 0} \frac{\log ||D_A||(G(\psi) \rightarrow G(\zeta))}{\log s} = \frac{d + \sigma}{b}.$$

The case when the operator $A$ is the Croneker product of some non-degenerate linear operators may be considered analogously. See for the definitions and preliminary results, e.g., \[21\].

5. Concluding Remarks

We will show here some applications of obtained results.

In this section we consider only the one-dimensional case $d = 1$, i.e. the cases $X = R$, $X = R^+$ or $X = (0, 2\pi)$; with Lebesgue measure: $W(\vec{x}) = 1$, i.e. $\vec{\theta} = 0$.

1. Conjugate spaces.

The associate spaces $(G(\psi))$ to the BGLS $(G(\psi))$ are described in \[15\]. Using the Corollary 4.2 from \[1\], chapter 1, section 4, we compute the conjugate spaces $(GA(\psi))^*$ to the $GA(\psi)$ spaces:

$$(GA(\psi))^* = (GB(\psi))^* = (G^o(\psi))^* = (G(\psi))^\prime.$$

2. Boyd indices for associate spaces.

It follows from \[1\], chapter 3, section 5, proposition 5.13 that if $\psi \in \Psi(a, b)$, $1 \leq a < b \leq \infty$, then

$$B^+((G(\psi))', (G(\psi))') = 1 - 1/b,$$

$$B^-((G(\psi))', (G(\psi))') = 1 - 1/a.$$

3. Boundedness of some singular operators.

I. Let $X = R^1_+$, $\psi \in \Psi(a, b)$, $1 \leq a < b \leq \infty$, and consider two linear operators of a Hardy-Littlewood type:

$$(P_\alpha)f(t) = t - \alpha \int_0^t s^{\alpha - 1} f(s) \, ds,$$

$$(Q_\beta)f(t) = t - \beta \int_t^\infty s^{\beta - 1} f(s) \, ds,$$

$s, t \in (0, \infty)$, $\alpha, \beta = \text{const} \in (0, 1)$. 
We conclude using the theorem 5.15 from [1], chapter 3, section 5 that the operator \( P_\alpha \) is bounded in the space \( G(\psi; a, b) \) iff \( \alpha > 1/a \); the operator \( Q_\beta \) is bounded in the space \( G(\psi; a, b) \) iff \( \beta < 1/b \).

II. Let \( X = R^d \), \( W(x) = 1, \psi \in \Psi(a, b), 1 \leq a < b \leq \infty \), and consider the (quasi-linear) Hardy-Littlewood maximal operator \( M \) :

\[
(M f)(x) = \sup_Q \left[ \int_Q |f(y)| \frac{dy}{|Q|} \right],
\]

where the supremum extends over all non-degenerate cubes \( Q \) containing \( x \) (cubes will be assumed to have their sides parallel to the coordinate axes), \(|Q|\) denotes the \( d \)-dimensional volume of \( Q \).

We conclude using the theorem 5.17, belonging to G.G.Lorentz and T.Shimogaki, from [1], chapter 3, section 5 that the operator \( M \) is bounded in the space \( G(\psi; a, b) \) iff \( a > 1 \).

III. Let \( X = R^1 \), \( W(x) = 1, \psi \in \Psi(a, b), 1 \leq a < b \leq \infty \), and consider the Hilbert transform \( H \).

We conclude using the theorem 5.18, belonging to D.W.Boyd, from [1], chapter 3, section 5 that the operator \( H \) :

\[
(H f)(x) = \pi^{-1} \lim_{\epsilon \to 0^+} \int_{\{y; |x - y| > \epsilon\}} f(y) \frac{dy}{x - y}
\]

is bounded in the space \( G(\psi; a, b) \) iff \( a > 1, b < \infty \).

4. Norm convergence of Fourier series.

Let here \( X = (0, 2\pi), W(x) = 1, \psi \in \Psi(a, b), 1 \leq a < b \leq \infty \). We consider the usual Fourier series for arbitrary function \( f : X \to R \) and such that \( f \in G^0(\psi)(= GA(\psi) = GB(\psi)) \).

We obtain using the corollary 6.11 from [1], chapter 3, section 6 that the Fourier series for arbitrary function \( f \in G^0(\psi) \) converge in the norm \( G(\psi; a, b) \) if and only if

\[
a > 1, b < \infty.
\]

Note in addition to this section that the other cases, e.g. if \( a = 1 \) or/and \( b = \infty \) are complete investigated in [15]. In this case the considered singular operators: Hilbert, Hardy-Littlewood, Fourier etc. are bounded as an operators from one BGL space into another space.

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