The Riemannian Penrose inequality and a virtual gravitational collapse

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Abstract

We reinterpret the proof of the Riemannian Penrose inequality by H. Bray. The modified argument turns out to have a nice feature so that the flow of Riemannian metrics appearing Bray’s proof gives a Lorentzian metric of a spacetime. We also discuss a possible extension of our approach to charged black holes.

1 Introduction

The issue of cosmic censorship is still an unsolved problem. Closely related to this, Penrose proposed the following inequality for the black hole

$$\sqrt{A/16\pi} \leq m,$$

where $A$ is the area of the horizon and $m$ is the ADM mass for an asymptotically flat spacetime. This inequality is also yet to be proved and remains an important problem.

In a Riemannian/time-symmetric space, Huisken and Ilmanen proved this inequality where the area $A$ is that of a single black hole by using the inverse mean curvature flow \cite{HuiskenIlmanen}. At almost the same time, Bray proved it for multi black holes using a conformal flow method \cite{Bray}. For the general, non-time-symmetric case, the Penrose inequality is still an open question.

As we review in the next section, Bray’s proof is a bit of a mystery. This is because it is difficult to have a physical reasoning why the proof works. In this paper, we introduce a normalised conformal flow and then we regard it as a model of the time evolution, formulating a Lorentzian metric. As a result, we have a rather natural interpretation of Bray’s proof. We also discuss some implications of our line of reasoning to the charged black hole case \cite{OhashiShiromizuYamada}.

2 Brief sketch of Bray’s proof

We consider a time-symmetric initial data $(\Sigma, q_0)$ where $q_0$ is a Riemannian metric. The time-symmetric initial data is defined by a hypersurface in a spacetime with the zero extrinsic curvature. We suppose that the apparent horizons $H_0$ exist in the spacetime. It is known that the apparent horizon corresponds to the minimal surface in $(\Sigma, q_0)$.

We introduce the following conformal transformation

$$q_t = u_t^4 q_0$$

and define $v_t$ as the “time” derivative of $u_t$, $v_t = u_t$, where dot stands for the derivative with respect to the parameter $t$. We then require that $v_t$ is a harmonic function with respect to $q_0$

$$\Delta_{q_0} v_t = 0$$

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with the boundary condition
\[ v_t(x)|_{H_t} = 0, \quad v_t \rightarrow -e^{-t} \text{ as } r \rightarrow \infty. \] (4)

We require that \( H_t \) is the minimal surface in \((\Sigma, q_t)\). From the definition of \( v_t \), we have
\[ u_t = 1 + \int_0^t v_s(x) ds \rightarrow e^{-t} \text{ (as } r \rightarrow \infty). \] (5)

Now we have a conformal flow defined by the sequence of \((\Sigma_i, q_t, H_t)\).

In this conformal flow, we can show that \( \dot{A}_t = 0 \) and \( \dot{m}_t = 0 \).

Here \( A_t \) is the area of \( H_t \) and \( m_t \) is the ADM mass for \((\Sigma, q_t)\). When we show \( \dot{m}_t = 0 \), an idea of Bunting and Masood-ul-Alam [5] was used in a crucial way. From these we have \( A_\infty = A_0 \) and \( m_\infty \leq m_0 \).

In the limit of \( t = \infty \), we can also show that \((\Sigma, q_t)\) becomes the Schwarzschild slice. Therefore \( \sqrt{A_\infty}/16\pi = m_\infty \) holds. Thus,
\[ \sqrt{A_0}/16\pi = \sqrt{A_\infty}/16\pi = m_\infty \leq m_0 \] (6)
is proven. This is the Riemannian Penrose inequality.

It is difficult to see why this proof works. So we will modify the proof which is just a reformulation of the conformal flow. Although the new argument requires rather minor technical modifications from Bray’s one, we gain a new insight, which in turn offers a physical interpretation to the conformal flow.

3 Normalized conformal flow

Let us introduce the following conformal transformation
\[ \tilde{q}_t = \tilde{u}_t^4 q_0, \] (7)

where \( \tilde{u}_t \) is defined by \( \tilde{u}_t = \left( \frac{m_0}{m_t} \right)^{1/2} u_t \). \( u_t \) is the same with the previous one in Eq. (2). Now we have a new flow \((\Sigma, \tilde{q}_t, H_t)\). Note that the surface \( H_t \) remains minimal after the dilation of the metric. It is easy to show \( \dot{m}_t = 0 \). In addition, it is easy to show \( \dot{A}_t \geq 0 \).

We can show that the space becomes the Schwarzschild slice in the \( t = \infty \) limit as well as the case of the conformal flow. Thus, \( 16\pi \tilde{m}_\infty^2 = \tilde{A}_\infty \) holds. Finally we can show the Riemannian Penrose inequality again as
\[ 16\pi \tilde{m}_0^2 = 16\pi \tilde{m}_\infty^2 = \tilde{A}_\infty \geq A_0. \] (8)

Namely over this normalized conformal flow, the ADM mass is conserved and the area of the apparent horizon is increasing. The former corresponds to the well-known fact that the ADM mass is a conserved quantity in asymptotically flat spacetimes. The latter corresponds to the area theorem of black holes (See Ref. [6] for the area theorem of apparent horizon). These features offers a nice physical interpretation of the normalized conformal flow. In the next section, we will look at this context more closely.

4 Physical Interpretation

From now on, we will regard the normalised conformal flow as a time evolution. We suppose that the time evolution is given by
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2(t, x) dt^2 + \tilde{q}_t \]
\[ = -\alpha^2(t, x) dt^2 + \tilde{q}_{ij} dx^i dx^j, \] (9)

where \( \alpha \) is the lapse function and \( \tilde{q}_{ij} \) is the component of \( \tilde{q}_t \). In this case the extrinsic curvature of \( t = \text{const.} \) hypersurfaces becomes
\[ K_{ij} = \frac{1}{2\alpha} \partial_t \tilde{q}_{ij} = 2 \frac{\dot{\tilde{u}}_t}{\alpha \tilde{u}_t} \tilde{q}_{ij}. \] (10)
Then it turns out that the expansion rate \( \theta \) of the outgoing null geodesic congruence on \( H_t \) is non-negative

\[
\theta |_{H_t} \propto (k + K - K_{ij}r^ir^j)|_{H_t} = -2 \frac{\dot{m}_t}{\alpha m_t} \geq 0.
\]  

(11)

where \( r^i \) is the unit normal vector to \( H_t \) in \( (\Sigma, q) \). This is because of \( \dot{m}_t \leq 0 \). Here \( k \) is the trace of extrinsic curvature of \( H_t \) with respect to \( \tilde{q}_t \) and \( K = K^i_i \). Thus \( H_t \) is located outside an apparent horizon/marginally trapped surface in a virtual spacetime \( (M, g) \).

In the time evolution of \( H_t \), we can see that \( H_t \) approaches to the apparent horizon

\[
\theta |_{H_t} \propto -2 \dot{m}_t/m_t \rightarrow 0,
\]  

(12)

because we know that the final state at \( t = \infty \) is Schwarzschild slice, the convergence implies \( \dot{m}_t \rightarrow 0 \) as \( t \rightarrow \infty \).

Let us suppose that \( (M, g) \) satisfies the four dimensional Einstein equation

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu},
\]  

(13)

where \( R_{\mu\nu} \) and \( R \) are the Ricci curvature and scalar curvature of \( g \). Here we do not yet have the above equation determining the virtual spacetime. The stress tensor \( T_{\mu\nu} \) needs to be chosen so that the above equation is satisfied.

To do so, let us focus on the Hamiltonian and momentum constraints,

\[
\dot{t} \tilde{R} + K^2 - K_{ij}K^{ij} = 16\pi \rho
\]  

(14)

\[
\tilde{D}^iK_{ij} - \tilde{D}_jK = -8\pi J_j,
\]  

(15)

where \( \rho = T_{\mu\nu}t^\mu t^\nu \), \( J_i = T_{\mu i}t^\mu \). \( \dot{t} \tilde{R} \) and \( \tilde{D}_i \) are the Ricci scalar the covariant derivative with respect to \( \tilde{q}_t \), respectively. From the Hamiltonian constraint, we can calculate \( \rho \)

\[
16\pi \rho = 16\pi \left( \frac{m_t}{m_0} \right)^2 u^4 \rho_0 + 24 \frac{1}{\alpha^2} \left( \frac{\dot{u}_t}{u_t} \right)^2 \geq 0.
\]  

(16)

In the above we used \( \dot{t} \tilde{R} = 16\pi \rho_0 \), where \( \rho_0 \) is the energy density of real matters in the physical initial data. Note that \( \rho_0 \) is not one computed from virtual matters \( T_{\mu\nu} \) here. Then we see that \( \rho \) comes out to be non-negative. This is a nice feature in the physical sense.

Next we can calculate \( J_i \) on \( H_t \) and the result is

\[
2\pi J_i|_{H_t} = \partial_t v_i / (\alpha u_t)|_{H_t}.
\]  

(17)

Since \( v_i(x) \) is the harmonic function, the maximum principle tells us \( \partial_t v_i \leq 0 \) outward direction of \( H_t \). More precisely, if one introduces the outward normal vector \( r^i \) of \( H_t \) in \( t = \text{const.} \) slices, \( r^i \partial_r v_i \leq 0 \). Thus we can see the ingoing energy flux of artificial matters, that is, \( r^i J_i \leq 0 \).

As a consequence, we have the following physical picture for the normalised conformal flow. The virtual time evolution corresponds to the gravitational collapse. From the behavior of virtual matters characterized by \( T_{\mu\nu} \), the 3-dimensional hypersurface \( \cap_0 H_t \) looks like a horizon. Moreover, the area of \( H_t \) is increasing with time. We recall in Bray’s construction that the topological type of the surface \( H_t \) may change, as the surface may jump across some singular times. And \( H_t \) approaches to the horizon because the expansion rate of null congruence on \( H_t \) is decaying to zero at \( t = \infty \). Thus, the normalized conformal flow gives us a virtual gravitational collapse. Since the final state is promised to be Schwarzschild slice in this evolution, it is natural to have the Penrose inequality. If we know that the final state is Schwarzschild slice, the area theorem implies the Penrose inequality.

5 Implication to charged black holes

Although our new proof is just a rearrangement of Bray’s proof, there is a possibility to apply it to other issues. For example, one may want to address the Penrose inequality for charged black holes. According
to Ref. [7], Bray’s argument is hoped to be generalized so that

$$m_0 \geq m_\infty = \frac{1}{2} \left( R + \frac{Q^2}{R} \right)$$

holds where $Q$ is the charge of black holes. The Reissner-Nordström slice realizes the equality. Introducing the area radius by $R = \sqrt{A_0}/4\pi = \sqrt{A_\infty}/4\pi$, the above is rewritten by

$$m_0 - \sqrt{m_0^2 - Q_0^2} \leq R \leq m_0 + \sqrt{m_0^2 - Q_0^2}.$$  \hspace{1cm} (19)

However, in Ref. [7], a counterexample to the lower bound was constructed. Because of the evidence, it is unlikely that Bray’s proof works for charged black holes in the way presented above.

On the other hand, we may expect that the upper bound for the area radius holds. Namely we hope to show that the inequality

$$4\pi \left( m_0 + \sqrt{m_0^2 - Q_0^2} \right)^2 = A_\infty \geq A_0 = 4\pi R^2$$

(that is $m_0 + \sqrt{m_0^2 - Q_0^2} \geq R$) holds. The lesson to be learned from the counterexample is that in Bray’s original flow, the area radius was fixed while the mass was decreased via the flow, though physically the area should be increased till it reaches the maximal value set by the fixed mass. This is what we have done with the normalization. So with charge in play, we may hope to prove with $m$ and $Q$ fixed, the area can be increased till it reaches that of Reissner-Nordström’s specified by the parameters $(m_0, Q_0)$.

6 Summary

In this article, we proposed a proof of the Riemannian Penrose inequality which is a modification of Bray’s proof (Ref.[3].) In the original proof by Bray, a conformal flow of the Riemannian metrics was employed, so that the mass is decreasing while the area of the horizon is fixed. However, it is difficult to see the physical reason why the proof works. Hence we proposed a dual viewpoint by normalizing the conformal flow. It is a family of conformal transformations so that now the mass is fixed while the area is increasing. Then we observed that the behaviors of the dual flow enjoy some plausible physical features, that is, the normalised conformal flow corresponds to a virtual time evolution of gravitational collapse, satisfying a non-vacuum Einstein equation. In addition, our new approach may shed some new light to prove the following Penrose type inequality for charged black holes.

$$4\pi \left( m_0 + \sqrt{m_0^2 - Q_0^2} \right)^2 \geq A_0,$$  \hspace{1cm} (21)

which is consistent with a picture (Ref.[8]) resulting from the cosmic censorship as well as the so-called no-hair theorem where an evolving black hole is expected to settle down to a Kerr(-Newman) spacetime with the parameters $(m_0, Q_0)$ specified by the initial slice. This is left for future study.

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