A FIBONACCI’S COMPLEMENT NUMERATION SYSTEM

SÉBASTIEN LABBÉ AND JANA LEPŠOVÁ

Abstract. With the two’s complement notation of signed integers, the fundamental arithmetic operations of addition, subtraction, and multiplication are identical to those for unsigned binary numbers. In this work, we consider a Fibonacci-equivalent of the two’s complement notation allowing to represent nonnegative and negative integers. We show that addition in this numeration system can be done similarly as the sum of Fibonacci representation of nonnegative integers with a deterministic finite-state machine provided by Berstel. Finally, we show that the Fibonacci’s complement numeration system is characterized by the fact of being an increasing bijection between \(\mathbb{Z}\) and some language.

1. Introduction

A nonnegative integer can be written as a sum of powers of 2 which gives rise to its binary expansion over alphabet \(\Sigma = \{0, 1\}\). Binary representations can be added with a standard algorithm - starting from the least significant digit and transferring a carry at each step. In the case that one of the representations is shorter in length, it is padded with the prefix of leading zeroes, as in the following example.

\[
\begin{array}{c|c}
11 & 01011 \\
+17 & 10001 \\
\hline
28 & 11100
\end{array}
\]

(sum of binary representations)

Two’s complement numeration system. Among all the ways to generalize this approach to all integers including negative ones is the two’s complement notation, see [Knu98, §4.1]. In the two’s complement representation of integers, the value of a binary word \(w = w_k \cdots w_1 w_0 \in \Sigma^k\) is

\[
\text{val}_{2c}(w) = \sum_{i=0}^{k-1} w_i 2^i - w_{k-1} 2^{k-1}
\]

which can be simplified to \(\sum_{i=0}^{k-2} w_i 2^i - w_{k-1} 2^{k-1}\). It can be seen that for every \(w \in \Sigma^*\), \(\text{val}_{2c}(00w) = \text{val}_{2c}(0w)\) and \(\text{val}_{2c}(11w) = \text{val}_{2c}(1w)\), thus, 0 and 1 are neutral prefixes which can be used to pad nonnegative and negative representations respectively keeping the two’s complement value invariant. For every \(n \in \mathbb{Z}\) there exists a unique word \(w \in \Sigma^+ \setminus (00\Sigma^* \cup 11\Sigma^*)\) such that \(n = \text{val}_{2c}(w)\). The word \(w\) is called the two’s complement representation of the integer \(n\), and we denote it by \(\text{rep}_{2c}(n)\).

The main interest with the two’s complement notation is that the fundamental arithmetic operations of addition, subtraction, and multiplication are identical to those for unsigned binary representations. For example, we perform below the addition of the representations seen

\[
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\]

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previously, this time interpreting them in the two’s complement notation. The first word has the same value \( \text{val}_{2c}(01011) = 2^3 + 2^1 + 2^0 = 11 \) but this time \( \text{val}_{2c}(10001) = -2^4 + 2^0 = -15 \).

\[
\begin{array}{cccc}
11 & 01011 \\
-15 & 10001 \\
-4 & 11100 \\
\end{array}
\]

(sum of two’s complement representations)

The value of the resulting word is \( \text{val}_{2c}(11100) = -2^4 + 2^3 + 2^2 = -4 \) which confirms that the computation is correct. Notice that the negative integer \(-4\) has a shorter two’s complement representation \( \text{rep}_{2c}(-4) = 100 \) which no longer contains the neutral prefix.

**Fibonacci numeration systems.** Integers can also be expressed in other numeration systems [FS96, FS10]. A typical example uses the Fibonacci numbers instead of the powers of 2.

Let \((F_n)_{n \geq 0}\) be the Fibonacci sequence defined with the recurrence relation \(F_n = F_{n-1} + F_{n-2}\), for all \(n \geq 2\), and the initial conditions \(F_0 = 1\), \(F_1 = 2\), following a convention for the Fibonacci numeration system [Fro88]. In this numeration system, the value of a binary word \(w = w_{k-1}w_{k-2} \cdots w_1w_0 \in \Sigma^k\) is

\[
\text{val}_F(w) = \sum_{i=0}^{k-1} w_i F_i.
\]

A result attributed to Zeckendorf [Lek52, Day60, Car68], published by Zeckendorf much later [Zec72], but which appeared earlier in a more general form in Ostrowski’s work [Ost22] (see also [Knu97, Exercise 1.2.8.34]) says that for every nonnegative integer \(n\), there exists a unique binary word \(w \in \Sigma^* \setminus (\Sigma^*1\Sigma^* \cup 0\Sigma^*)\) such that \(n = \text{val}_F(w)\). In other words, every nonnegative integer can be represented uniquely as a sum of nonconsecutive distinct Fibonacci numbers. For example, \(20 = 13 + 5 + 2 = F_5 + F_3 + F_1 = \text{val}_F(101010)\). The unique word \(w \in \Sigma^* \setminus (\Sigma^*1\Sigma^* \cup 0\Sigma^*)\) such that \(n = \text{val}_F(w)\) is denoted \(\text{rep}_F(n)\), see Table 1. We refer to this unique word \(\text{rep}_F(n)\) as to its Fibonacci representation. Note that the empty word is the Fibonacci representation of the integer 0 and is denoted \(\varepsilon\).

| \(n\) | \(\text{rep}_F(n)\) | \(n\) | \(\text{rep}_F(n)\) | \(n\) | \(\text{rep}_F(n)\) |
|------|----------------|------|----------------|------|----------------|
| 0    | \(\varepsilon\) | 10   | 10010          | 20   | 101010         |
| 1    | 1              | 11   | 10100          | 21   | 1000000        |
| 2    | 10             | 12   | 10101          | 22   | 1000001        |
| 3    | 100            | 13   | 100000         | 23   | 1000010        |
| 4    | 101            | 14   | 100001         | 24   | 1000100        |
| 5    | 1000           | 15   | 100010         | 25   | 1000101        |
| 6    | 1001           | 16   | 100100         | 26   | 1001000        |
| 7    | 1010           | 17   | 100101         | 27   | 1001001        |
| 8    | 10000          | 18   | 101000         | 28   | 1001010        |
| 9    | 10001          | 19   | 101001         | 29   | 1010000        |

**Table 1.** The Fibonacci numeration system \(F\).

One way to adapt the Fibonacci numeration system to represent all integers is to use Fibonacci numbers with negative indices. In [Bun92], it was proved that every integer \(n\), whether positive, negative, or zero, can be written uniquely as a sum of nonconsecutive distinct Fibonacci numbers \(F_i\) where \(i < -2\). Recall that in this article, it is used \(F_0 = 1\) and \(F_1 = 2\), so that \(F_{-1} = 1\), \(F_{-2} = 0\), \(F_{-3} = 1\), \(F_{-4} = -1\), \(F_{-5} = 2\), \(F_{-6} = -3\) and \(F_{-i} = (-1)^{i+1}F_{i-4}\).
This system is called the negaFibonacci number system in [Knu11 §7.1.3] where it is used to navigate efficiently in a pentagrid within the hyperbolic plane.

The addition of integers (including negative integers) based on Fibonacci numbers was considered in [AUFP13] where a bit sign is appended to the Fibonacci representations. The first bit indicates the sign (0 means nonnegative, 1 indicates nonpositive and the integer zero has two representations). Problems arise when adding integers with an opposite sign because it is dependant of which number has the greater magnitude.

### A Fibonacci’s complement numeration system.

Motivated by the study of aperiodic tiling of the plane by Wang tiles, a numeration system representing all integers in \( \mathbb{Z} \) in a unique way based on Fibonacci numbers was introduced recently [LL21]. This numeration system can be defined from the value map \( \text{val}_{\mathcal{F}_c} : \{0, 1\}^* \rightarrow \mathbb{Z} \) defined for every binary word \( w = w_{k-1} \cdots w_0 \in \{0, 1\}^k \) as

\[
\text{val}_{\mathcal{F}_c}(w) = \sum_{i=0}^{k-1} w_i F_i - w_{k-1} F_k
\]

which is an analog of the two’s complement Equation (1), using Fibonacci numbers instead of powers of 2. Note that it simplifies to \( \text{val}_{\mathcal{F}_c}(w) = \sum_{i=0}^{k-2} w_i F_i - w_{k-1} F_{k-2} \). For every \( n \in \mathbb{Z} \), there exists a unique odd-length word \( w \in \Sigma(\Sigma \Sigma)^* \setminus (\Sigma^*11\Sigma^* \cup 000\Sigma^* \cup 101\Sigma^*) \) such that \( n = \text{val}_{\mathcal{F}_c}(w) \), where \( \Sigma = \{0, 1\} \), see Proposition 2.3. We denote this unique word by \( \text{rep}_{\mathcal{F}_c}(n) \). The Fibonacci’s complement \( (\mathcal{F}_c) \) numeration system is defined by the map \( \text{rep}_{\mathcal{F}_c} : \mathbb{Z} \rightarrow \{0, 1\}^* \), see Table 2. Note that the numeration system \( \mathcal{F}_c \) extends naturally to \( \mathbb{Z}^d \) with an appropriate padding of shorter words, see Definition 2.6.

| \( n \) | \( \text{rep}_{\mathcal{F}_c}(n) \) | \( n \) | \( \text{rep}_{\mathcal{F}_c}(n) \) | \( n \) | \( \text{rep}_{\mathcal{F}_c}(n) \) |
|---|---|---|---|---|---|
| -10 | 1000100 | 0 | 0 | 10 | 0010010 |
| -9 | 1000101 | 1 | 001 | 11 | 0010100 |
| -8 | 1001000 | 2 | 010 | 12 | 0010101 |
| -7 | 1001001 | 3 | 00100 | 13 | 0100000 |
| -6 | 1001010 | 4 | 00101 | 14 | 0100001 |
| -5 | 1000000 | 5 | 01000 | 15 | 0100010 |
| -4 | 1000100 | 6 | 01001 | 16 | 0101000 |
| -3 | 1001000 | 7 | 01010 | 17 | 0101001 |
| -2 | 1000000 | 8 | 0010000 | 18 | 0101000 |
| -1 | 1000000 | 9 | 0010001 | 19 | 0101001 |

Table 2. The Fibonacci’s complement numeration system \( \mathcal{F}_c \).

The numeration system \( \mathcal{F}_c \) was used in [LL21] to construct a valid tiling of the plane with a particular aperiodic set of 16 Wang tiles. The tiling of the plane is described by an automaton taking as input the Fibonacci’s complement representation of a position \( (n, m) \in \mathbb{Z}^2 \) and returning as output the tile to place at position \( (n, m) \).

### Addition in the Fibonacci numeration system \( \mathcal{F} \).

Let us recall that, following [Ber86], the addition of Fibonacci representations can be done in terms of a Mealy machine, that is, a deterministic finite-state transducer.

A Mealy machine \( M \) is a labeled directed graph whose vertices are called states and edges are called transitions. The transitions are labeled by pairs \( a/b \) of letters. The first letter \( a \in A \) is the input symbol and the second letter \( b \in B \) is the output symbol. For every state...
s and every letter $a$, there is at most one transition starting from state $s$ with input symbol $a$. One distinguished state is called initial state.

A machine $M$ computes a function $M : A^* \rightarrow B^*$. Let $x = x_0x_1 \ldots x_{k-1} \in A^*$ and $y = y_0y_1 \ldots y_{k-1} \in B^*$ be two words of length $k \in \mathbb{N}$ over the input and output alphabets. The word $y$ is the output of $x$ under the machine $M$ if and only if there is a sequence $\{s_i\}_{0 \leq i \leq k}$ of $k + 1$ states of $M$ such that $s_0$ is the initial state and for every $i$ with $0 \leq i < k$, there is a transition from $s_i$ to $s_{i+1}$ labeled by $x_i/y_i$. The output word $y$ is denoted $M(x)$. The last state $s_k$ is denoted $M_{\text{last}}(x)$ and an extra output word depending on the last state is denoted $M_{\downarrow}(x)$.

A 10-states Mealy machine $B$, called the adder by Berstel [Ber86, p. 22], reading from left to right is shown in Figure 1. The Berstel adder fulfills that for every input $u \in \{0, 1, 2\}^*$, the concatenation of the output word $B(u) \in \{0, 1\}^*$ with a three-letter word $B_{\downarrow}(u) \in \{000, 001, 010, 100, 101\}$ depending on the last state have the same value for the Fibonacci numeration system

$$\text{val}_F(u) = \text{val}_F(B(u) \cdot B_{\downarrow}(u)).$$

This result is stated as Theorem 3.1 herein. Computations done with the Berstel adder are shown in Example 3.2 and Example 3.3.

**Figure 1.** The Berstel adder $B$ is a 10-states Mealy machine with 30 transitions illustrated as solid edges with initial state 000.0. The Mealy machine $T$ is obtained by adding a new state start which replaces the initial state as well as three additional transitions shown with dashed edges.

**Main result.** The goal of this contribution is to prove that the Fibonacci’s complement numeration system $F_c$ behaves like the two’s complement in term of addition of integers. More precisely, addition of integers in the numeration system $F_c$ can be done as the usual addition of nonnegative integers in the Fibonacci numeration system $F$.

The result is based on the Berstel’s adder $B$ to which three transitions

$$\text{start} \xrightarrow{0/\varepsilon} 000.0, \quad \text{start} \xrightarrow{1/\varepsilon} 101.7, \quad \text{start} \xrightarrow{2/\varepsilon} 100.6$$
are added from a new initial state \textbf{start} which replaces the initial state 000.0. The three additional transitions are shown with dashed edges in Figure 1. We denote by \( \mathcal{T} \) this modified Berstel adder consisting of 11 states and 33 transitions.

**Theorem A.** The Mealy machine \( \mathcal{T} \) satisfies that for every nonempty input \( u \in \{0, 1, 2\}^k \), it outputs a binary word \( \mathcal{T}(u) \cdot \mathcal{T}_l(u) \in \{0, 1\}^+ \) of length \( k+2 \) with same value for the Fibonacci’s complement numeration system, i.e.,

\[
\text{val}_{Fc}(u) = \text{val}_{Fc}(\mathcal{T}(u) \cdot \mathcal{T}_l(u)).
\]

Computations done with the Mealy machine \( \mathcal{T} \) illustrating Theorem A are performed in Example 4.3 and Example 4.4.

This result leads to a natural question which is to generalize it to other numeration systems, starting with simple Parry numeration systems which include the Fibonacci numeration system, see [Rén57, Par64, FS10, MPR19].

**Structure of the article.** In Section 2, we recall a few preliminary results on the numeration system \( Fc \) for \( \mathbb{Z} \). In Section 3, we recall the addition of Fibonacci representations on \( \mathbb{N} \). In Section 4, we prove Theorem A. In Section 5, we provide a new proof of the Berstel adder \( B \) for sums of Fibonacci representations of nonnegative integers. In Section 6, we show that the map \( \text{rep}_{Fc} \) defining the Fibonacci’s complement numeration system is characterized by the fact of being an increasing bijection for some total order defined on the language \( \{0, 1\}^* \), see Theorem B.

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### 2. A Fibonacci’s Complement Numeration System for \( \mathbb{Z} \)

In this section, we present the Fibonacci’s complement numeration system which is defined by the value map \( \text{val}_{Fc} : \Sigma^* \to \mathbb{Z} \) given in Equation (2) where \( \Sigma = \{0, 1\} \).

The first observation to make on this value map is given in the next lemma.

**Lemma 2.1.** For every word \( w \in \Sigma^* \), we have

\[
\text{val}_{Fc}(000w) = \text{val}_{Fc}(0w) \quad \text{and} \quad \text{val}_{Fc}(101w) = \text{val}_{Fc}(1w).
\]

**Proof.** The first equation is trivial. Let \( w = w_{k-1} \cdots w_0 \in \Sigma^* \). We have

\[
\text{val}_{Fc}(101w) = \sum_{i=0}^{k-1} w_i F_i + F_k - F_{k+1} = \sum_{i=0}^{k-1} w_i F_i - F_{k-1} = \text{val}_{Fc}(1w). \quad \square
\]

Thus 00 or 10 can be used to pad words without changing their value leading to the following definition.

**Definition 2.2 (Neutral prefix).** Let \( w \in \Sigma^* \). We say that 00 (10 resp.) is the neutral prefix of \( w \) if \( w \in 0\Sigma^* \) (if \( w \in 1\Sigma^* \) resp.). We denote it by \( p_w \).
Representations in the numeration system $\mathcal{F}_c$ of integers $n \in [-13, 21]$ is shown in Figure 2. The neutral prefixes correspond to two loops on the vertices 0 and -1.

Note the following relation between the Fibonacci numeration system and the Fibonacci’s complement numeration system:

$$\text{val}_{\mathcal{F}_c}(w) = \sum_{i=0}^{k-1} w_i F_i - w_{k-1} F_k = \text{val}_F(w) - w_{k-1} F_k.$$  

for every nonempty word $w = w_{k-1} \cdots w_0 \in \Sigma^+$ of length $k$.

The following proposition enables to define representations of integers in the Fibonacci’s complement numeration system. Its proof is in Section 6.

**Proposition 2.3.** The map $\text{val}_{\mathcal{F}_c} : \Sigma(\Sigma^*) \setminus (\Sigma^*11\Sigma^* \cup 000\Sigma^* \cup 101\Sigma^*) \to \mathbb{Z}$ is a bijection.

The inverse of the map $\text{val}_{\mathcal{F}_c}$ defines the Fibonacci’s complement numeration system.

**Definition 2.4** (Fibonacci’s complement numeration system). For every $n \in \mathbb{Z}$, we denote by $\text{rep}_{\mathcal{F}_c}(n)$ the unique word $w \in \Sigma(\Sigma^*) \setminus (\Sigma^*11\Sigma^* \cup 000\Sigma^* \cup 101\Sigma^*)$ such that $\text{val}_{\mathcal{F}_c}(w) = n$.

The Fibonacci’s complement ($\mathcal{F}_c$) numeration system is illustrated in Figure 2. This numeration system was used in [LL21] (and denoted with letter $\mathcal{F}$) to describe some aperiodic Wang tilings of the plane. Herein, we made the choice of denoting the Fibonacci’s complement numeration system by $\mathcal{F}_c$ and the Fibonacci numeration system by $\mathcal{F}$.

The neutral prefix can be used to pad words so that they all have the same length.
**Definition 2.5** (Pad function). Let \( u, v \in \Sigma(\Sigma\Sigma)^* \). We define
\[
\text{pad}\left(\begin{array}{c}u \\ v\end{array}\right) = \left(\begin{array}{c}\text{pad}_k(u) \\ \text{pad}_k(v)\end{array}\right)
\]
where \( k = \max\{|u|, |v|\} \) and \( \text{pad}_k(w) = p_w^{\frac{1}{2}(k-|w|)}w \) for every \( w \in \{u, v\} \) where \( p_w \) is the neutral prefix of the word \( w \).

The padding allows to represent coordinates in \( \mathbb{Z}^d \) in dimension \( d \geq 1 \). Here we consider the case \( d = 2 \). The padding allows also to define the sum of words.

**Definition 2.6** (Numeration system \( \mathcal{F}c \) for \( \mathbb{Z}^2 \)). Let \( \mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2 \). We define
\[
\text{rep}_\mathcal{F}c(\mathbf{n}) = \text{pad}\left(\begin{array}{c}\text{rep}_\mathcal{F}c(n_1) \\ \text{rep}_\mathcal{F}c(n_2)\end{array}\right).
\]

**Definition 2.7** (Sum of two words). Let \( \Sigma = \{0, 1\} \) and \( u, v \in \Sigma^* \). Then we define \( \text{sum} : \Sigma^* \times \Sigma^* \rightarrow \{0, 1, 2\}^* \) as
\[
\text{sum}(u, v) = (u_{k-1} + v_{k-1}) \cdots (u_0 + v_0) \quad \text{where} \quad \left(\begin{array}{c}u_{k-1} \cdots u_0 \\ v_{k-1} \cdots v_0\end{array}\right) = \text{pad}\left(\begin{array}{c}u \\ v\end{array}\right).
\]

As the sum of two words is in the alphabet \( \{0, 1, 2\} \), we extend the maps \( \text{val}_\mathcal{F} \) and \( \text{val}_\mathcal{F}c \) naturally on the alphabet \( \{0, 1, 2\} \). The following lemma is an analogy of Lemma 2.1.

**Lemma 2.8.** For every word \( v \in \{0, 1, 2\}^* \) and every \( a \in \{0, 1, 2\} \), we have
\[
\text{val}_\mathcal{F}c(a0av) = \text{val}_\mathcal{F}c(av).
\]

**Proof.** Let \( v = v_{k-1} \cdots v_0 \in \{0, 1, 2\}^* \) and \( a \in \{0, 1, 2\} \). We have
\[
\text{val}_\mathcal{F}c(a0av) = \sum_{i=0}^{k-1} v_i F_i + a F_k - a F_{k+1} = \sum_{i=0}^{k-1} v_i F_i - a F_{k-1} = \text{val}_\mathcal{F}c(av). \quad \square
\]

3. Addition of Fibonacci representations on \( \mathbb{N} \)

In this section, we recall how the addition of Fibonacci representations of nonnegative integers can be done with the Berstel adder, a finite-state deterministic transducer proposed by Berstel in [Ber86, p. 22].

A Mealy machine with output is a 7-tuple \( M = (S, S_0, A, B, \delta, \eta, \phi) \), where \( S \) is a finite set of states, \( S_0 \in S \) is the initial state, \( A \) is the input alphabet, \( B \) is the output alphabet, \( \delta : S \times A \rightarrow S \) is the transition function mapping pairs of a state and an input symbol to the corresponding next state, \( \eta : S \times A \rightarrow B \) is the output function mapping pairs of a state and an input symbol to the corresponding output symbol and \( \phi : S \rightarrow B^* \) is the function mapping each state to an extra output word in the alphabet \( B \). Reading a word \( u = u_0 \ldots u_{k-1} \in A^* \) for some \( k \in \mathbb{N} \), the Mealy machine \( M \) moves between states \( s_i \in S \), with \( s_0 = S_0 \) and \( s_{i+1} = \delta(s_i, u_i) \), outputting sequentially one letter \( w_i = \eta(s_i, u_i) \in B \) for each input letter \( u_i \in A, i \in \{0, 1, \ldots, k-1\} \). The output word \( w = w_0 \ldots w_{k-1} \) is denoted \( M(u) \). The last state \( s_k \) is denoted \( M_{\text{last}}(u) \) and an extra output word depending on the last state is \( M_{\text{last}}(u) = \phi(M_{\text{last}}(u)) \). In this article, the extra output word needs to be concatenated after the output word \( M(u) \).
The Berstel adder is the Mealy machine $B = (Q, 000.0, \{0, 1, 2\}, \{0, 1\}, \delta_B, \eta_B, \phi_B)$ with the set of 10 states

$$Q = \left\{ 000.0, 001.1, 010.3, 100.5, 101.6, 001.1, 001.2, 010.4, 100.6, 101.7 \right\}$$

with initial state 000.0, input alphabet $\{0, 1, 2\}$, output alphabet $\{0, 1\}$, transition function $\delta_B$ and output function $\eta_B$ as shown in Figure 1. The states of the Berstel adder is a subset of $S \times \{0, 1, \ldots, 7\}$ where $S = \{000, 001, 010, 100, 101\}$. The function $\phi_B$ is the canonical projection $S \times \{0, 1, \ldots, 7\} \rightarrow S$. Our representation of the 10 states (in particular the value in $\{0, 1, \ldots, 7\}$) is not the same but is equivalent to the 10 states provided by Berstel. The value in $\{0, 1, \ldots, 7\}$ is used in Section 5 to prove that there is finitely many states to consider up to an equivalence relation. Reading a word $u \in \{0, 1, 2\}^\ast$, the Berstel adder outputs $B(u) \cdot B_i(u)$.

Berstel introduced the adder $B$ in [Ber86] for the addition of the Fibonacci representations of nonnegative integers in $\mathbb{N}$.

**Theorem 3.1.** [Ber86] The Berstel adder $B$ fulfills that for every input $u \in \{0, 1, 2\}^\ast$, it outputs a word $B(u) \cdot B_i(u) \in \{0, 1\}^\ast$ with same value for the Fibonacci numeration system:

$$\text{val}_F(u) = \text{val}_F(B(u) \cdot B_i(u)).$$

The behavior of the Berstel adder on all ternary words of length $\leq 3$ is listed in Table 3. Below, we illustrate Theorem 3.1 on two examples.

**Example 3.2.** Feeding the Berstel adder $B$ in Figure 1 with the word $u = 2220121$ gives

$$000.0 \xrightarrow{2/0} 010.4 \xrightarrow{2/1} 001.2 \xrightarrow{2/0} 101.7 \xrightarrow{0/1} 010.3 \xrightarrow{1/0} 101.7 \xrightarrow{2/1} 101.7 \xrightarrow{1/1} 100.5,$$

therefore the last state is $B_{\text{last}}(u) = 100.5$ and we obtain $B(u) \cdot B_i(u) = 0101011 \cdot 100$. Thus $\text{val}_F(2220121) = 42 + 26 + 16 + 3 + 4 + 1 = 92 = 55 + 21 + 8 + 5 + 3 = \text{val}_F(010101100)$.

**Example 3.3.** Here is how Berstel adder can be used to compute the sum $33 + 25$. First, we express 33 and 25 by their Fibonacci representation (in general, if representations do not have the same length, the shorter one is padded with leading 0’s). Then, we add them digit by digit to obtain a ternary word in $\{0, 1, 2\}^\ast$:

$$33 \quad 1010101$$

$$+25 \quad 1000101$$

$$58 \quad 2010202$$

Reading from left to right and giving the word $u = 2010202$ as input to the Berstel adder, see Figure 1, we obtain the following path from the initial state 000.0:

$$000.0 \xrightarrow{2/0} 010.4 \xrightarrow{0/0} 101.6 \xrightarrow{1/1} 010.4 \xrightarrow{0/0} 101.6 \xrightarrow{2/1} 100.6 \xrightarrow{0/1} 001.1 \xrightarrow{2/0} 100.6.$$

Therefore, the output word is $B(u) = 0010110$ and the path ends in state $B_{\text{last}}(u) = 100.6$. Removing the last digit (6) of the last state which we can ignore for now (see Definition 5.5 for details), we obtain the three-letter extra output word $B_i(u) = 100$. Their concatenation $0010110 \cdot 100$ has the correct Fibonacci value $33 + 25 = 58$.

$$\text{val}_F(B(u) \cdot B_i(u)) = \text{val}_F(0010110 \cdot 100) = 3 + 8 + 13 + 34 = 58,$$

however, it is not the normal representation as it contains leading 0’s and consecutive 1’s. Indeed, it is known [Ber86, Sak87, Fro91] that no single right-to-left and no single left-to-right transducer can normalize the word $u \in \{0, 1, 2\}^k$. 


The Berstel adder $\mathcal{B}$ was given in [Ber86] without proof. A proof of Theorem 3.1 was provided later in [Fro99, Corollary 4] based on the numeration system in the real base $\tau = \frac{1 + \sqrt{5}}{2}$, see also [Fro88, Fro91]. Another proof follows from [BR10, §2.3.2.3] where it is proved that normalization in the real base $\beta$ can be done with a finite automaton when $\beta$ is a Pisot number. Alternatively, the correctness of the Berstel adder can be proved with a computer program, see [MSS16, Remark 2.1, p. 41].

We provide in Section 5 a new proof of Theorem 3.1 based on linear algebra and identities involving Fibonacci numbers. Theorem 3.1 is used to prove Theorem A.

4. Proof of Theorem A

We derive a Mealy machine $\mathcal{T}$ from the Berstel adder $\mathcal{B}$ as

$$\mathcal{T} = (Q \cup \{\text{start}\}, \text{start}, \{0, 1, 2\}, \{0, 1\}, \delta_\mathcal{T}, \eta_\mathcal{T}, \phi_\mathcal{T})$$

by adding a new initial state start and extending the maps $\delta_\mathcal{B}$ and $\eta_\mathcal{B}$ in the following way:

$$\delta_\mathcal{T}(q, a) = \begin{cases} 
\delta_\mathcal{B}(q, a) & \text{if } q \in Q \text{ and } a \in \{0, 1, 2\}, \\
000.0 & \text{if } q = \text{start} \text{ and } a = 0, \\
101.7 & \text{if } q = \text{start} \text{ and } a = 1, \\
100.6 & \text{if } q = \text{start} \text{ and } a = 2,
\end{cases}$$

$$\eta_\mathcal{T}(q, a) = \begin{cases} 
\eta_\mathcal{B}(q, a) & \text{if } q \in Q \text{ and } a \in \{0, 1, 2\}, \\
\varepsilon & \text{if } q = \text{start} \text{ and } a \in \{0, 1, 2\},
\end{cases}$$

$$\phi_\mathcal{T}(q) = \begin{cases} 
\phi_\mathcal{B}(q) & \text{if } q \in Q, \\
000 & \text{if } q = \text{start}.
\end{cases}$$

The Mealy machine $\mathcal{T}$ is illustrated in Figure 1 with solid and dashed edges.

Before proving the theorem, we need two lemmas. The first lemma shows that the first letter of the output word depends on the first letter of the input word.

**Lemma 4.1.** Let $v \in \{0, 1, 2\}^*$. Then

1. $\mathcal{T}(0v) \cdot \mathcal{T}_0(0v) \in 0\{0, 1\}^*$,
2. $\mathcal{T}(av) \cdot \mathcal{T}_0(av) \in 1\{0, 1\}^*$ for every $a \in \{1, 2\}$.

**Proof.** We use Figure 1 throughout the proof.

1. Let $v = \varepsilon$. Then $\mathcal{T}(0v) \cdot \mathcal{T}_0(0v) = \varepsilon \cdot 000 \in 0\{0, 1\}^*$. Let $v = v_0 \ldots v_{k-1} \in \{0, 1, 2\}^*$ for some $k \geq 1$. We observe $\mathcal{T}(0v_0) \in 0\{0, 1\}^*$. Thus $\mathcal{T}(0v) \cdot \mathcal{T}_0(0v) \in 0\{0, 1\}^*$.

2. Let $v = \varepsilon$. If $a = 1$ then $\mathcal{T}(av) \cdot \mathcal{T}_0(av) = \varepsilon \cdot 101 \in 1\{0, 1\}^*$. If $a = 2$ then $\mathcal{T}(av) \cdot \mathcal{T}_0(av) = \varepsilon \cdot 100 \in 1\{0, 1\}^*$.

Let $v = v_0 \ldots v_{k-1} \in \{0, 1, 2\}^*$ for some $k \geq 1$. If $a = 1$ then $\mathcal{T}(av_0) \in 1\{0, 1\}^*$. If $a = 2$ then $\mathcal{T}(av_0) \in 1\{0, 1\}^*$. Thus for every $a \in \{1, 2\}$ we have $\mathcal{T}(av) \cdot \mathcal{T}_0(av) \in 1\{0, 1\}^*$.

Since the strongly connected component of the Mealy machine $\mathcal{T}$ corresponds to the Berstel adder $\mathcal{B}$, we have the following relations between the two machines.

**Lemma 4.2.** Let $v \in \{0, 1, 2\}^*$. Then

1. $\mathcal{B}(0v) \cdot \mathcal{B}_0(0v) = 0\mathcal{T}(0v) \cdot \mathcal{T}_0(0v)$,
2. $\mathcal{B}(101v) \cdot \mathcal{B}_0(101v) = 000\mathcal{T}(1v) \cdot \mathcal{T}_0(1v)$,
3. $\mathcal{B}(202v) \cdot \mathcal{B}_0(202v) = 001\mathcal{T}(2v) \cdot \mathcal{T}_0(2v)$. 
Proof. In this proof, it is appropriate to introduce the notation $M^s = (S, s, A, B, \delta, \eta, \phi)$ to denote the Mealy machine with a chosen initial state $s \in S$. In particular, it satisfies

\begin{align}
M^s(uv) &= M^s(u) \cdot M^s(v) \\
M_{\text{last}}^s(uv) &= M_{\text{last}}^s(v)
\end{align}

for every $u, v \in A^*$ where $t = M_{\text{last}}^s(u)$.

Moreover, since the strongly connected component of the Mealy machine $\mathcal{T}$ corresponds to the Berstel adder $B$ and both machines have the same function $\phi_{\mathcal{T}} = \phi_B$, the following equations hold for every $q \in Q$ and every $u \in \{0, 1, 2\}^*$:

\begin{align}
\mathcal{B}^q(u) &= \mathcal{T}^q(u), \\
\mathcal{B}^q_{\text{last}}(u) &= \mathcal{T}^q_{\text{last}}(u) \\
\mathcal{B}^q(u) &= \mathcal{T}^q(u)
\end{align}

Using Equations (4), (5) and (6) and the particular form of the machines $\mathcal{B}$ and $\mathcal{T}$, we have

\begin{align*}
\mathcal{B}(0v) \cdot \mathcal{B}_q(0v) &= \mathcal{B}(0) \cdot \mathcal{B}^{000.0}(v) \cdot \mathcal{B}^{000.0}(v) \\
&= 0 \cdot \mathcal{T}^{000.0}(v) \cdot \mathcal{T}^{000.0}(v) \\
&= 0 \cdot \mathcal{T}(0v) \cdot \mathcal{T}(0v).
\end{align*}

Also, we have

\begin{align*}
\mathcal{B}(101v) \cdot \mathcal{B}_q(101v) &= \mathcal{B}(101) \cdot \mathcal{B}^{101.7}(v) \cdot \mathcal{B}^{101.7}(v) \\
&= 000 \cdot \mathcal{T}^{101.7}(v) \cdot \mathcal{T}^{101.7}(v) \\
&= 000 \cdot \mathcal{T}(1v) \cdot \mathcal{T}(1v)
\end{align*}

and similarly,

\begin{align*}
\mathcal{B}(202v) \cdot \mathcal{B}_q(202v) &= \mathcal{B}(202) \cdot \mathcal{B}^{100.6}(v) \cdot \mathcal{B}^{100.6}(v) \\
&= 001 \cdot \mathcal{T}^{100.6}(v) \cdot \mathcal{T}^{100.6}(v) \\
&= 001 \cdot \mathcal{T}(2v) \cdot \mathcal{T}(2v).
\end{align*}

We can now prove the main result.

Proof of Theorem A. Let $u \in \{0, 1, 2\}^+$. We do the proof according to the first letter of $u$.

Let $a \in \{0, 1, 2\}$ and $v \in \{0, 1, 2\}^*$ such that $u = av$. Assume that $|v| = k$.

If $a = 0$, we have

\begin{align*}
\text{val}_{\mathcal{F}_c}(0v) &= \text{val}_{\mathcal{F}}(0v) \quad \text{(Equation (3))} \\
&= \text{val}_{\mathcal{F}}(\mathcal{B}(0v) \cdot \mathcal{B}_q(0v)) \quad \text{(Theorem 3.1)} \\
&= \text{val}_{\mathcal{F}}(\mathcal{T}(0v) \cdot \mathcal{T}_q(0v)) \quad \text{(Lemma 4.2)} \\
&= \text{val}_{\mathcal{F}_c}(\mathcal{T}(0v) \cdot \mathcal{T}_q(0v)) \quad \text{(Equation (3), Lemma 4.1)}.
\end{align*}

If $a = 1$, we have

\begin{align*}
\text{val}_{\mathcal{F}_c}(1v) &= \text{val}_{\mathcal{F}_c}(101v) \quad \text{(Lemma 2.8)} \\
&= \text{val}_{\mathcal{F}}(101v) - F_{k+3} \quad \text{(Equation (3))} \\
&= \text{val}_{\mathcal{F}}(\mathcal{B}(101v) \cdot \mathcal{B}_q(101v)) - F_{k+3} \quad \text{(Theorem 3.1)} \\
&= \text{val}_{\mathcal{F}}(\mathcal{T}(1v) \cdot \mathcal{T}_q(1v)) - F_{k+3} \quad \text{(Lemma 4.2)} \\
&= \text{val}_{\mathcal{F}}(\mathcal{T}(1v) \cdot \mathcal{T}_q(1v)) - F_{k+3}.
\end{align*}
Therefore, the output word is $T_{\text{val}}(\mathcal{T}(1v) \cdot \mathcal{T}_i(1v))$ (Equation 3, Lemma 4.1).

If $a = 2$, we proceed similarly to the previous case and we have

$$\text{val}_{\mathcal{F}_c}(2v) = \text{val}_{\mathcal{F}_c}(202v)$$
$$= \text{val}_{\mathcal{F}}(202v) - 2F_{k+3}$$
$$= \text{val}_{\mathcal{F}}(B(202v) \cdot B_1(202v)) - 2F_{k+3}$$
$$= \text{val}_{\mathcal{F}}(001 \cdot \mathcal{T}(2v) \cdot \mathcal{T}_i(2v)) - 2F_{k+3}$$
$$= \text{val}_{\mathcal{F}}(\mathcal{T}(2v) \cdot \mathcal{T}_i(2v)) - F_{k+3}$$
$$= \text{val}_{\mathcal{F}_c}(\mathcal{T}(2v) \cdot \mathcal{T}_i(2v))$$

(Lemma 2.8, Equation 3, Theorem 3.1, Lemma 4.2, Equation 3, Lemma 4.1).

We illustrate Theorem 3.1 analogically to Example 3.2. See also the behavior of $\mathcal{T}$ on all ternary words of length $\leq 3$ in Table 3.

**Example 4.3.** We feed the modified Berstel adder $\mathcal{T}$ in Figure 7 with the same word $u = 2220121$ as in Example 3.2, this time obtaining

$$\text{start} \xrightarrow{2^1} 106.0 \xrightarrow{2^1} 105.0 \xrightarrow{2^1} 010.4 \xrightarrow{0^0} 101.6 \xrightarrow{1^1} 010.4 \xrightarrow{2^1} 001.2 \xrightarrow{0^1} 100.5,$$

therefore the last state is $\mathcal{T}_{\text{last}}(u) = 100.5$ and we obtain $\mathcal{T}(u) \cdot \mathcal{T}_i(u) = 110110 \cdot 100$. Interpreting the results in the Fibonacci’s complement numeration system, we observe

$$\text{val}_{\mathcal{F}_c}(2220121) = (-42)+42+16+3+4+1 = 24 = (-34)+34+13+8+3 = \text{val}_{\mathcal{F}_c}(110110100).$$

**Example 4.4.** Here is how the Mealy machine $\mathcal{T}$ can be used to compute the sum $-1 + (-9)$. First, we express $(-1)$ and $(-9)$ by their Fibonacci’s complement representation and we pad the shorter word with an appropriate neutral prefix (00’s if it starts with 0 or 10’s if it starts with 1, see Definition 2.2) so that they have the same length. Then, we add them digit by digit to obtain a ternary word in $\{0, 1, 2\}^*$:

$$-1 \quad 1010101$$
$$+(-9) \quad 1000101$$
$$-10 \quad 2010202$$

Note that the resulting word $u = 2010202$ coincides with the one in example showing properties of addition in Fibonacci numeration system. Reading from left to right and giving the word $u = 2010202$ as input to the Mealy machine $\mathcal{T}$, see Figure 7, we obtain the following path from the initial state **start**:

$$\text{start} \xrightarrow{2^1/} 101.7 \xrightarrow{0^1} 001.1 \xrightarrow{1^0} 010.4 \xrightarrow{0^0} 101.6 \xrightarrow{2^1} 100.6 \xrightarrow{0^1} 001.1 \xrightarrow{2^0} 100.6.$$

Therefore, the output word is $\mathcal{T}(u) = 100110$ and the path ends in state $\mathcal{T}_{\text{last}}(u) = 100.6$. Removing the last digit (6) of the last state, we obtain the three-letter extra output word $\mathcal{T}_i(u) = 100$. Their concatenation $100110 \cdot 100$ is a Fibonacci’s complement representation of the sum $-1 + (-9)$ as we confirm that its Fibonacci’s complement value is correct:

$$\text{val}_{\mathcal{F}_c}(\mathcal{T}(u) \cdot \mathcal{T}_i(u)) = \text{val}_{\mathcal{F}_c}(100110 \cdot 100) = 3 + 8 + 13 - 34 = -10.$$
5. An alternate proof of Theorem 3.1

The goal of this section is to provide a new independent proof of the Berstel adder without using numeration system in a real base. The proof provided herein is based solely on linear algebra and identities involving Fibonacci numbers. It proves that there is a canonical way of mapping a ternary word to a binary word having the same Fibonacci value based on the choice made for shorter words, see Lemma 5.3 and Proposition 5.4. Then we introduce an equivalence class on \( \{0,1,2\}^* \) (see Definition 5.7) which depends on a map \( \theta : \{0,1,2\}^* \to \mathbb{Z} \), see Definition 5.5. If two ternary words are equivalent, then their behavior is the same, see Lemma 5.8. Also the range of values under the map \( \theta \) takes only finitely many values which proves that the computation can be done by a finite-state transducer.

First, we prove some Fibonacci identities used afterward. Recall that following the literature on numeration systems, we define the Fibonacci sequence \((F_n)_{n \geq 0}\) with the recurrence relation \(F_n = F_{n-1} + F_{n-2}\), for all \(n \geq 2\), with the initial conditions \(F_0 = 1\) and \(F_1 = 2\).

**Lemma 5.1.** The following identities involving Fibonacci numbers hold for \(k \geq 1\):

1. \[\sum_{i=0}^{2k-1} (-1)^i F_i F_{2k-i} = -F_{2k-2},\]
2. \[\sum_{i=0}^{2k-1} F_i = F_{2k+1} - 2,\]
3. \[\sum_{i=0}^{2k-1} F_i^2 = F_{2k-2} F_{2k+1}.\]

**Proof.**
1. When \(k = 0\), we have \(\sum_{i=0}^{2k-1} (-1)^i F_i F_{2k-i} = 0 = -F_{-2}\). We prove the identity by induction and use the other identity \(F_m F_n + F_{m-1} F_{n-1} = F_{m+n-1}\) (which is listed in [https://en.wikipedia.org/wiki/Fibonacci_number](https://en.wikipedia.org/wiki/Fibonacci_number)). If \(k \geq 1\), we have

\[
\sum_{i=0}^{2k-1} (-1)^i F_i F_{2k-i} = F_0 F_{2k} + \sum_{i=1}^{2k-2} (-1)^i F_i F_{2k-i} - F_{2k-1} F_1
\]

\[
= F_{2k} + \sum_{i=1}^{2k-2} (-1)^i (F_{i-1} F_{2k-i-1} + F_{2k-i}) - 2F_{2k-1}
\]

\[
= F_{2k} + \sum_{j=0}^{2(k-1)-1} (-1)^j F_j F_{2(k-1)-j} + \sum_{i=1}^{2k-2} (-1)^i F_{2k-i} - 2F_{2k-1}
\]

\[
= F_{2k} + (-F_{2(k-1)-2}) + 0 - 2F_{2k-1}
\]

\[
= (F_{2k-1} + F_{2k-2}) - F_{2k-4} - 2F_{2k-1}
\]

\[
= F_{2k-2} - F_{2k-4} - F_{2k-1}
\]

\[
= -F_{2k-3} - F_{2k-4}
\]

\[
= -F_{2k-2}.
\]

2. See [Knu97, Exercise 1.2.8.20].

3. The proof uses the Cassini identity \(F_{n+1} F_{n-1} - F_n^2 = (-1)^n\) and \(\sum_{i=-1}^{n} F_i^2 = F_n F_{n+1}\) whose geometrical proof is well-known. Thus

\[
\sum_{i=0}^{2k-1} F_i^2 = \sum_{i=-1}^{2k-1} F_i^2 - 1
\]

\[
= F_{2k-1} F_{2k} - 1
\]

\[
= (F_{2k-2} + F_{2k-3}) F_{2k} - 1
\]
\[
\begin{align*}
= F_{2k-2}F_k + F_{2k-3}(2F_{2k-2} + F_{2k-3}) - 1 \\
= F_{2k-2}(F_k + 2F_{2k-3}) + F_{2k-3}^2 - 1 \\
= F_{2k-2}(F_k + 2F_{2k-3}) + F_{2k-2}F_{2k-4} - (-1)^{2k-3} - 1 \\
= F_{2k-2}(F_k + 2F_{2k-3} + F_{2k-4}) \\
= F_{2k-2}F_{2k+1}.
\end{align*}
\]

The third identity in Lemma 5.1 was discovered thanks to the existence of the sequence

\[
\left( \sum_{i=0}^{2k-1} F_i^2 \right)_{k \geq 0} = (0, 5, 39, 272, 1869, 12815, 87840, 602069, 4126647, 28284464, \ldots)
\]

in the OEIS available at [https://oeis.org/A003482](https://oeis.org/A003482).

Let \( u, w \in \{0, 1, 2\}^* \) be two finite ternary sequences of integers. If \( u \) and \( w \) have the same value in base 2, then \( u0 \) and \( w0 \) have also the same value in base 2, since their value is simply multiplied by 2. This is no longer true in the Fibonacci numeration system \( \mathcal{F} \). But the next lemma shows that the difference of values of the words \( u0 \) and \( w0 \) is at most 1. Its proof is based on linear algebra and nicely uses three identities involving Fibonacci numbers.

**Lemma 5.2.** Let \( u, w \in \{0, 1, 2\}^* \).

(i) If \( \text{val}_\mathcal{F}(u) = \text{val}_\mathcal{F}(w) \), then \( \text{val}_\mathcal{F}(u0) - \text{val}_\mathcal{F}(w0) \in \{-1, 0, +1\} \).

(ii) If \( \text{val}_\mathcal{F}(u) = \text{val}_\mathcal{F}(w000) \), then \( \text{val}_\mathcal{F}(u0) - \text{val}_\mathcal{F}(w0000) \in \{0, +1\} \).

(iii) If \( \text{val}_\mathcal{F}(u) = \text{val}_\mathcal{F}(w101) \), then \( \text{val}_\mathcal{F}(u0) - \text{val}_\mathcal{F}(w1010) \in \{-1, 0\} \).

**Proof.** (i) The proof is based on linear algebra. Up to padding \( u \) and \( w \) with 0’s, we can assume without loss of generality that \( u \) and \( w \) have the same length and that this length is even. Let \( f = (F_{2k-1}, \ldots, F_1, F_0)^t \in \mathbb{R}^{2k} \) be a vector of Fibonacci numbers where \( 2k \) is the length of \( u \) and \( w \). We extend \( f \) to a base \( \{f, g_1, \ldots, g_{2k-2}, h\} \) of \( \mathbb{R}^{2k} \) with \( g_1 = (1, -1, -1, 0, \ldots, 0)^t, g_2 = (0, 1, -1, -1, 0, \ldots, 0)^t, \ldots, g_{2k-2} = (0, \ldots, 0, 1, -1, 1)^t \in \mathbb{R}^{2k} \) and \( h = (F_0, -F_1, \ldots, (-1)^{2k-2}F_{2k-1})^t \in \mathbb{R}^{2k} \). The base is not orthogonal, but \( f \) and \( h \) are orthogonal to the other vectors of the base. We form the invertible matrix \( K = [f, g_1, \ldots, g_{2k-2}, h] \). There exists a unique vector \( \alpha = (\alpha_0, \ldots, \alpha_{2k-1})^t \in \mathbb{R}^{2k} \) such that \( \vec{u} - \vec{w} = K\alpha \) where we consider \( \vec{u} = (u_{2k-1}, \ldots, u_0)^t \) and \( \vec{w} = (w_{2k-1}, \ldots, w_0)^t \) as column vectors in \( \mathbb{R}^{2k} \). By hypothesis, we have that \( f^t \cdot (\vec{u} - \vec{w}) = 0 \). Since \( f \) is orthogonal to the other vectors of the base, we have

\[
0 = f^t \cdot (\vec{u} - \vec{w}) = f^tK\alpha = f^t\alpha_0.
\]

Therefore \( \alpha_0 = 0 \). Moreover, \( \alpha = K^{-1}(\vec{u} - \vec{w}) \). By definition of the inverse, the last row of \( K^{-1} \) must be orthogonal to \( f \) and to the \( g_i \)'s, thus it must be parallel to \( h^t \). Since \( K^{-1}K \) is the identity matrix, the last row of \( K^{-1} \) must be \( \frac{1}{h^t h}h^t \). This allows to express \( \alpha_{2k-1} \) as

\[
\alpha_{2k-1} = \frac{1}{h^t h}h^t \cdot (\vec{u} - \vec{w}).
\]

Now consider the augmented vector \( f' = (F_{2k}, F_{2k-1}, \ldots, F_1, F_0)^t \in \mathbb{R}^{2k+1} \). Let also \( \alpha' = (\alpha_0, \ldots, \alpha_{2k-1}, 0)^t \in \mathbb{R}^{2k+1} \) and \( K' = \left( \begin{array}{c} K \\ 0 \end{array} \right) \) be the matrix \( K \) augmented by a zero column vector on the right and by a zero row vector at the bottom. We have

\[
f'^t \cdot (u\vec{0} - \vec{w}0) = f'^t \cdot K' \cdot \alpha' = f'^t \cdot \left( \begin{array}{c} f_0 \\ h_0 \end{array} \right) \alpha_0 + 0 + f'^t \cdot \left( \begin{array}{c} h_0 \\ 0 \end{array} \right) \alpha_{2k-1} = f'^t \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \alpha_{2k-1}
\]

\[
= h^t \cdot (\vec{u} - \vec{w}) \sum_{i=0}^{2k-1} (-1)^i F_i F_{2k-i} = h^t \cdot (\vec{u} - \vec{w}) \sum_{i=0}^{2k-1} \frac{(-F_{2k-2})}{F_i^2} = h^t \cdot (\vec{u} - \vec{w})
\]

\[
\frac{F_{2k-1}}{F_{2k+1}}
\]
from which we deduce the upper bound

\[
|f^n \cdot (\bar{u} - \bar{w})| = \left| \frac{h^t \cdot (\bar{u} - \bar{w})}{F_{2k+1}} \right| \leq \| \bar{u} - \bar{w} \|_\infty \frac{\| h \|_1}{F_{2k+1}} = \| \bar{u} - \bar{w} \|_\infty \frac{\sum_{i=0}^{2k-1} F_i}{F_{2k+1}}
\]

\[
= \| \bar{u} - \bar{w} \|_\infty \frac{F_{2k+1} - 2}{F_{2k+1}} < \| \bar{u} - \bar{w} \|_\infty \leq 2
\]

where we used the three following identities involving Fibonacci numbers:

\[
\sum_{i=0}^{2k-1} (-1)^i F_i F_{2k-i} = -F_{2k-2}, \quad \sum_{i=0}^{2k-1} F_i = F_{2k+1} - 2 \quad \text{and} \quad \sum_{i=0}^{2k-1} F_i^2 = F_{2k-2} F_{2k+1}
\]

which are proved in Lemma 5.1. Since \( \text{val}_F(u0) - \text{val}_F(w0) = f^n \cdot (\bar{u} - \bar{w}) \) is an integer, it implies that \( \text{val}_F(u0) - \text{val}_F(w0) \in \{-1, 0, +1\} \).

(ii) Let \( (\bar{u} - \bar{w}) = (d_0, \ldots, d_{k-1}) \) for some integer \( k \geq 2 \). By hypothesis, we have \( d_{2k-3}, d_{2k-2}, d_{2k-1} \geq 0 \). We compute

\[
h^t \cdot (\bar{u} - \bar{w}) = \sum_{i=0}^{2k-4} (-1)^i F_i d_i = \sum_{i=0}^{2k-4} (-1)^i F_i d_i - F_{2k-3} d_{2k-3} + F_{2k-2} d_{2k-2} - F_{2k-1} d_{2k-1}
\]

\[
\leq \| \bar{u} - \bar{w} \|_\infty \left( \sum_{i=0}^{2k-4} F_i + F_{2k-2} \right) \leq \| \bar{u} - \bar{w} \|_\infty (2F_{2k-2} - 2) < 4F_{2k-2}.
\]

Therefore

\[
\text{val}_F(u0) - \text{val}_F(w0000) = -\frac{h^t \cdot (\bar{u} - \bar{w})}{F_{2k+1}} > -4F_{2k-2} \frac{F_{2k+1}}{F_{2k+1}} = -F_{2k+1} + F_{2k-5} \frac{F_{2k+1}}{F_{2k+1}} > -1.
\]

(iii) Let \( (\bar{u} - \bar{w}) = (d_0, \ldots, d_{k-1}) \) for some integer \( k \geq 2 \). By hypothesis, we have \( d_{2k-2} \geq 0 \) and \( d_{2k-1}, d_{2k-3} \in \{-1, 0, 1\} \). We compute

\[
h^t \cdot (\bar{u} - \bar{w}) = \sum_{i=0}^{2k-4} (-1)^i F_i d_i = \sum_{i=0}^{2k-4} (-1)^i F_i d_i - F_{2k-3} d_{2k-3} + F_{2k-2} d_{2k-2} - F_{2k-1} d_{2k-1}
\]

\[
\geq -\| \bar{u} - \bar{w} \|_\infty \sum_{i=0}^{2k-4} F_i - F_{2k-3} - F_{2k-1}
\]

\[
\geq -2(F_{2k-2} - 2) - F_{2k-3} - F_{2k-1} = -F_{2k+1} + 4.
\]

Therefore

\[
\text{val}_F(u0) - \text{val}_F(w0101) = \frac{-h^t \cdot (\bar{u} - \bar{w})}{F_{2k+1}} \leq \frac{F_{2k+1} - 4}{F_{2k+1}} < 1.
\]

This implies there is a unique way of translating a ternary word \( ua \in \{0, 1, 2\}^* \), with \( a \in \{0, 1, 2\} \), to a binary word \( wbt \in \{0, 1\}^* \) with the same Fibonacci value, provided that \( w \) is a prefix of the translation of \( u \) itself. More precisely, let \( \Sigma = \{0, 1\} \) and \( \mathcal{S} \) be the set \( \mathcal{S} = \Sigma^3 \setminus \Sigma^*11\Sigma^* = \{000, 001, 010, 100, 101\} \).

Lemma 5.3. Let \( u \in \{0, 1, 2\}^* \), \( w \in \{0, 1\}^* \) and \( s \in \mathcal{S} \) be such that \( |u| = |w| \) and \( \text{val}_F(u) = \text{val}_F(ws) \). For every \( a \in \{0, 1, 2\} \), there exist unique \( b \in \{0, 1\} \) and unique \( t \in \mathcal{S} \) such that \( \text{val}_F(ua) = \text{val}_F(wbt) \).
Proof. Since \( \text{val}_F(u) = \text{val}_F(ws) \), from Lemma 5.2 we have that \( \text{val}_F(u0) - \text{val}_F(ws0) = \gamma \) for some \( \gamma \in \{-1, 0, +1\} \). We compute

\[
\text{val}_F(ua) - \text{val}_F(u0000) = \text{val}_F(u0) + a - \text{val}_F(w0000) \\
= \text{val}_F(ws0) + \gamma + a - \text{val}_F(w0000) \\
= \text{val}_F(s0) + \gamma + a.
\]

If \( s \in \{001, 010, 100\} \), then \( \text{val}_F(s0) \in \{2, 3, 5\} \) and \( \text{val}_F(s0) + \gamma \in \{1, 2, 3, 4, 5, 6\} \). If \( s = 000 \), then \( \text{val}_F(s0) = 0 \) and \( \gamma \in \{0, +1\} \) from Lemma 5.2(ii) so that \( \text{val}_F(s0) + \gamma \in \{0, 1\} \). Finally if \( s = 101 \), then \( \text{val}_F(s0) = 7 \) and \( \gamma \in \{-1, 0\} \) from Lemma 5.2(iii) so that \( \text{val}_F(s0) + \gamma \in \{6, 7\} \). In summary if \( s \in S \), we have \( \text{val}_F(s0) + \gamma \in \{0, \ldots, 7\} \) and \( \text{val}_F(s0) + \gamma + a \in \{0, \ldots, 9\} \). The map \( \text{val} \) induces a bijection \( 0S \to \{0, 1, 2, 3, 4\} \) and a bijection \( 1S \to \{5, 6, 7, 8, 9\} \). Therefore there exist unique \( b \in \{0, 1\} \) and unique \( t \in S \) such that \( \text{val}_F(s0) + \gamma + a = \text{val}_F(bt) \), which holds if and only if \( \text{val}_F(ua) = \text{val}_F(w0000) + \text{val}_F(bt) = \text{val}_F(wbt) \). □

This implies the existence of a canonical translation of ternary words into binary words having the same value.

**Proposition 5.4.** There exist a unique length-preserving map \( w : \{0, 1, 2\}^* \to \{0, 1\}^* \) and a unique map \( s : \{0, 1, 2\}^* \to S \), that we denote \( w_u := w(u) \) and \( s_u := s(u) \), such that for every \( u \in \{0, 1, 2\}^* \)

- \( w_u \) is a prefix of \( w_{ua} \) for every \( a \in \{0, 1, 2\} \) and,
- \( \text{val}_F(u) = \text{val}_F(w_us_u) \).

Moreover, there exists a unique map \( \lambda : \{0, 1, 2\}^+ \to \{0, 1\} \) such that \( w_{ua} = w_u \lambda_{ua} \) for every \( u \in \{0, 1, 2\}^* \) and \( a \in \{0, 1, 2\} \).

Proof. Let \( w, w' : \{0, 1, 2\}^* \to \{0, 1\}^* \) be two length-preserving maps and \( s, s' : \{0, 1, 2\}^* \to S \) be two maps satisfying the hypothesis. We want to show that \( w = w' \) and \( s = s' \). Let \( u \in \{0, 1, 2\}^* \). If \( |u| = 0 \), then \( |w_u| = 0 = |w'_u| \), thus \( w_u = \varepsilon = w'_u \). The equality

\[
\text{val}_F(s_u) = \text{val}_F(w_us_u) = \text{val}_F(u) = 0 = \text{val}_F(\epsilon) = \text{val}_F(w'_u s'_u) = \text{val}_F(s'_u)
\]

implies that \( s_u = 000 = s'_u \).

Now assume that the hypothesis holds for \( u \in \{0, 1, 2\}^* \) so that \( w_u = w'_u \) and \( s_u = s'_u \) satisfying \( |w_u| = |u| = |w'_u| \) and \( \text{val}_F(w_us_u) = \text{val}_F(u) = \text{val}_F(w'_u s'_u) \). Let \( a \in \{0, 1, 2\} \). Then from the hypothesis \( w_u \) is a prefix of \( w_{ua} \) and of \( w'_{ua} \), thus let \( c, c' \in \{0, 1\} \) such that \( w_{ua} = w_u c \) and \( w'_{ua} = w_u c' \). Also by the definition of \( w, w' \), \( s \) and \( s' \) we have that

\[
\text{val}_F(w_us_u) = \text{val}_F(w_{ua}s_{ua}) = \text{val}_F(u) = \text{val}_F(w'_u s'_u) = \text{val}_F(w'_uc's'_{ua}).
\]

From Lemma 5.3, there exist unique \( b \in \{0, 1\} \) and unique \( t \in S \) such that \( \text{val}_F(ua) = \text{val}_F(w_{ua} bt) \). Therefore, we have \( c = c' \) and \( s_{ua} = s'_{ua} \). We conclude that \( w_{ua} = w_u c = w_u c' = w'_{ua} \) and \( s_{ua} = s'_{ua} \). The existence and unicity of the map \( \lambda \) follows for the existence and unicity of the map \( w \). □

Therefore we have that for every \( u \in \{0, 1, 2\}^* \) and every \( a \in \{0, 1, 2\} \), the following equation holds

\[
(7) \quad \text{val}_F(ua) = \text{val}_F(w_{ua}s_{ua}) = \text{val}_F(w_u \lambda_{ua}s_{ua}).
\]
The values of $w_u$, $\lambda_{ua}$ and $s_{ua}$ and are shown in the table below for small words $u \in \{0, 1, 2\}^*$ and $a \in \{0, 1, 2\}$.

| $u \cdot a$ | $w_u$ | $\lambda_{ua}$ | $s_{ua}$ | $u \cdot a$ | $w_u$ | $\lambda_{ua}$ | $s_{ua}$ |
|-------------|-------|----------------|----------|-------------|-------|----------------|----------|
| $\varepsilon \cdot 0$ | 0 | 0 | 000 | $\varepsilon \cdot 1$ | 0 | 0 | 001 | $\varepsilon \cdot 2$ | 0 | 0 | 010 |
| 0 | 00 | 0 | 000 | 0 | 1 | 000 | 0 | 2 | 00 | 0 | 010 |
| 1 | 00 | 0 | 010 | 1 | 1 | 000 | 1 | 2 | 00 | 1 | 010 |
| 2 | 00 | 1 | 011 | 2 | 1 | 000 | 2 | 2 | 01 | 0 | 000 |

It can be observed that some words $u, v \in \{0, 1, 2\}^*$ for which $s_u = s_v$ behave the same, namely that $\lambda_{ua} = \lambda_{va}$ and $s_{ua} = s_{va}$ for every $a \in \{0, 1, 2\}$. For example, for $u = 1$ and $v = 22$, we have $s_1 = 001 = s_{22}$ and for every $a \in \{0, 1, 2\}$, $s_{1a} = s_{22a}$ and $\lambda_{1a} = \lambda_{22a}$. But this is not always true. For instance $s_{02} = 010 = s_{10}$, but $(s_{020}, s_{021}, s_{022}) = (101, 000, 001) \neq (100, 101, 000) = (s_{100}, s_{101}, s_{102})$.

Thus, to conclude the equivalence between two words $u, v \in \{0, 1, 2\}^*$, the equality $s_u = s_v$ is not enough. We can now introduce the map $\theta$ which is used thereafter to define an equivalence relation on $\{0, 1, 2\}^*$. Its value allows to prove that two ternary words behave the same.

**Definition 5.5.** With the notation above, we define $\theta : \{0, 1, 2\}^* \to \mathbb{Z}$ for every $u \in \{0, 1, 2\}^*$ and $a \in \{0, 1, 2\}$ as

\[
\theta(ua) = \text{val}_F(s_u) + \text{val}_F(s_{ua}) - F_2\lambda_{ua} + a \quad \text{and} \quad \theta(\varepsilon) = 0.
\]

In general, we have $\theta(u) \in \{-3, \ldots, 10\}$. Only after exploring all cases, we will be able to prove that it takes its values only in the interval $0 \leq \theta(u) \leq 7$, in which case we have additional properties.

**Lemma 5.6.** Let $u, v \in \{0, 1, 2\}^*$ be such that both $\theta(u), \theta(v) \in \{0, \ldots, 7\}$. If $\theta(u) = \theta(v)$, then $\lambda_{ua} = \lambda_{va}$ and $s_{ua} = s_{va}$ for every $a \in \{0, 1, 2\}$.

**Proof.** Let $u = u_t \ldots u_2u_1 \in \{0, 1, 2\}^*$ and $wu_{u_1}^{-1} = u_t \ldots u_2$. From Proposition 5.4 and Equation (7), we have $\text{val}_F(wu_{u_1}^{-1}) = \text{val}_F(w_{u_1-1}s_{u_1})$ and $\text{val}_F(u) = \text{val}_F(w_{u_1-1}\lambda_s u_s)$. For convenience with the indices with respect to Fibonacci numbers, we write $w_{u_1}^{-1} = w_t \ldots w_2 \in \{0, 1\}^*$ and $w_1 = \lambda_s \in \{0, 1\}$. Let $a \in \{0, 1, 2\}$. From Proposition 5.4 and Equation (7), we have $\text{val}_F(ua) = \text{val}_F(w_u\lambda_s a s_{ua})$ with $w_u = w_{u_1}^{-1}w_1$. Using the natural property of Fibonacci
numbers and $F_{-1} = 1$, we have
\[
\text{val}_{\mathcal{F}}(ua) = \sum_{i=1}^{\ell} u_i F_i + a = \sum_{i=1}^{\ell} u_i F_{i-1} + \sum_{i=1}^{\ell} u_i F_{i-2} + a
\]
\[
= \text{val}_{\mathcal{F}}(u) + \text{val}_{\mathcal{F}}(uu_1^{-1}) + u_1 F_{-1} + a
\]
\[
= \text{val}_{\mathcal{F}}(w_{uu_1^{-1}}^1 \lambda u_s u) + \text{val}_{\mathcal{F}}(w_{uu_1^{-1}}^1 s_{uu_1^{-1}}) + u_1 + a
\]
\[
= \sum_{i=2}^{\ell} w_i F_{i+2} + \text{val}_{\mathcal{F}}(\lambda u_s u) + \sum_{i=2}^{\ell} w_i F_{i+1} + \text{val}_{\mathcal{F}}(s_{uu_1^{-1}}) + u_1 + a
\]
\[
= \sum_{i=2}^{\ell} w_i F_{i+3} + \lambda u_s u + \text{val}_{\mathcal{F}}(s_u) + \text{val}_{\mathcal{F}}(s_{uu_1^{-1}}) + u_1 + a
\]
\[
= \sum_{i=1}^{\ell} w_i F_{i+3} - \lambda u_s u + \text{val}_{\mathcal{F}}(s_u) + \text{val}_{\mathcal{F}}(s_{uu_1^{-1}}) + u_1 + a
\]
\[
= \text{val}_{\mathcal{F}}(w_{u}0000) + \theta(u) + a.
\] (9)

On the other hand,
\[
\text{val}_{\mathcal{F}}(ua) = \text{val}_{\mathcal{F}}(w_u \lambda u_a s_{ua}) = \text{val}_{\mathcal{F}}(w_u0000) + \text{val}_{\mathcal{F}}(\lambda u_a s_{ua}).
\]
Comparing with (9), we obtain
\[
\theta(u) + a = \text{val}_{\mathcal{F}}(\lambda u_a s_{ua}).
\]

If we repeat the same steps made in (9) with the terms concerning $va$, we arrive at the equation
\[
\theta(v) + a = \text{val}_{\mathcal{F}}(\lambda v a s_{va}).
\]
Since $\theta(u) = \theta(v) \in \{0, \ldots, 7\}$, we have $\text{val}_{\mathcal{F}}(\lambda u_a s_{ua}) = \text{val}_{\mathcal{F}}(\lambda v a s_{va}) \in \{0, \ldots, 9\}$. From the injectivity of $\text{val}_{\mathcal{F}} : \{0,1\}^\mathcal{F} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, it follows that $\lambda_{ua} = \lambda_{va}$ and $s_{ua} = s_{va}$. \qed

**Definition 5.7** (equivalence relation). Let $u, v \in \{0,1,2\}^\star$. We say that $u$ and $v$ are equivalent, denoted $u \equiv v$, if $\theta(u) = \theta(v)$ and $s_u = s_v$.

In the next lemma, still assuming that $\theta(u)$ takes its values in the interval $0 \leq \theta(u) \leq 7$, we prove that two words that are equivalent have children which are equivalent.

**Lemma 5.8.** Let $u, v \in \{0,1,2\}^\star$ such that both $\theta(u), \theta(v) \in \{0, \ldots, 7\}$. If $u \equiv v$ then $ua \equiv va$ for every $a \in \{0,1,2\}$.

**Proof.** The equivalence $u \equiv v$ implies $s_u = s_v$ and $\theta(u) = \theta(v)$. From Lemma 5.6 we have $\lambda_{ua} = \lambda_{va}$ and $s_{ua} = s_{va}$. Then we have
\[
\theta(ua) = \text{val}_{\mathcal{F}}(s_u) + \text{val}_{\mathcal{F}}(s_{ua}) - F_2 \lambda_{ua} + a
\]
\[
= \text{val}_{\mathcal{F}}(s_v) + \text{val}_{\mathcal{F}}(s_{va}) - F_2 \lambda_{va} + a = \theta(va).
\]
Therefore $ua \equiv va$. \qed

As there are finitely many combinations of states $s_u \in \mathcal{S}$ and of values $\theta(u) \in \{-3, \ldots, 10\}$, the quotient $\{0,1,2\}^\star_{\equiv}$ contains at most $\#\mathcal{S} \cdot 14 = 70$ equivalence classes. In each equivalence class, there is a word which is minimal with respect to the radix order. The radix order is
defined as \( u <_{\text{rad}} v \) either if \( u \) is shorter than \( v \), or if \( u \) and \( v \) have the same length and \( u \) is lexicographically smaller than \( v \). This minimal representative is the smallest path from the empty word. It is thus natural to build a graph exploring all equivalence classes which are accessible from the equivalence class of the empty word for which \( s_\varepsilon = 000 \) and \( \theta(\varepsilon) = 0 \). This is done in the following proof.

\[
\text{Figure 3.} \quad \text{The edges represent the directed labeled graph}\ G \text{ which can be folded (after merging equivalent states) into the Mealy machine } Z \text{ equivalent to the Berstel adder } B. \quad \text{A vertex reached from a path } u \text{ has an ellipse shape if and only if } u \text{ is minimal with respect to the radix order within the equivalence class } [u]\equiv \text{ and has a rectangle shape if it } u \equiv v \text{ for some word } v <_{\text{rad}} u.
\]

**Proof of Theorem 3.1.** Let \( G \) be the directed labeled graph \( G = (V, E) \) where

\[
V = \{ [u]_\equiv : u \in \{0, 1, 2\}^*, 0 \leq \theta(u) \leq 7 \},
\]

\[
E = \{ ([u]_\equiv, [ua]_\equiv) \in V \times V : u \in \{0, 1, 2\}^*, a \in \{0, 1, 2\} \},
\]

which is well-defined following Lemma 5.8. The connected component of the equivalence class of the empty word in the graph \( G \) is shown in Figure 3, where vertices \( [u]_\equiv \) are denoted by \( s_u, \theta(u) \) and edges are shown as \( [u]_\equiv \overset{a/\lambda_{ua}}{\longrightarrow} [ua]_\equiv \).

It follows from the exploration of the connected component of the equivalence of the empty word in radix order, see Figure 3, that for every \( u \in \{0, 1, 2\}^* \) and every \( a \in \{0, 1, 2\} \), we have \( 0 \leq \theta(ua) \leq 7 \) and thus \( [ua]_\equiv \in V \).

The cardinality of \( V \) is 10 (the 10 vertices shown with an ellipse shape in Figure 3) and the directed labeled graph \( G \) can be equivalently described by a Mealy machine. Let \( Z = (Q, S_0, A, B, \delta_Z, \eta_Z, \phi_Z) \) be the Mealy machine such that

- \( Q = \{ s_u, \theta(u) : [u]_\equiv \in V \} \), \( S_0 = 000.0 \), \( A = \{0, 1, 2\} \), \( B = \{0, 1\} \),
- for every \( s_u, \theta(u) \in Q \) and \( a \in A \), \( \delta_Z(s_u, \theta(u), a) = s_{ua}, \theta(ua) \) and \( \eta_Z(s_u, \theta(u), a) = \lambda_{ua} \).
- for every \( s_u, \theta(u) \in Q \), \( \phi_Z(s_u, \theta(u)) = s_u \).
The Mealy machine $\mathcal{Z}$ computes $w_u$ and $s_u$ in the sense that $\mathcal{Z}(u) = w_u$ and $\mathcal{Z}_i(u) = s_u$ for every word $u \in \{0, 1, 2\}^*$. Therefore, if $u = \varepsilon$, then $\mathcal{Z}(u) = \varepsilon$, $\mathcal{Z}_i(u) = 000$ and

$$\text{val}_\mathcal{Z}(u) = \text{val}_\mathcal{Z}(\varepsilon) = 0 = \text{val}_\mathcal{Z}(000) = \text{val}_\mathcal{Z}(\mathcal{Z}(u) \cdot \mathcal{Z}_i(u)).$$

If $u \in \{0, 1, 2\}^*$ and $a \in \{0, 1, 2\}$, then $\mathcal{Z}(ua) = w_u \lambda_{ua}$ and $\mathcal{Z}_i(ua) = s_{ua}$, and by Equation 1,

$$\text{val}_\mathcal{Z}(ua) = \text{val}_\mathcal{Z}(w_u \lambda_{ua} s_{ua}) = \text{val}_\mathcal{Z}(\mathcal{Z}(ua) \cdot \mathcal{Z}_i(ua)).$$

This ends the proof as the machine $\mathcal{Z}$ is equivalent to the Berstel adder $\mathcal{B}$ reproduced in Figure 1.

6. Increasing bijections

The goal of this section is to show that the numeration system $\text{rep}_\mathcal{F}: \mathbb{Z} \rightarrow D$ is an increasing map with respect to some total order on its codomain $D = \Sigma(\Sigma \Sigma)^* \setminus (\Sigma^* 11 \Sigma^* \cup 000 \Sigma^* \cup 101 \Sigma^*)$ where $\Sigma = \{0, 1\}$. Furthermore, we show that it is characterized by this total order, see Theorem 2. The proof of the theorem is based on similar results for the maps $\text{val}_\mathcal{F}$ and $\text{rep}_\mathcal{F}$ which we start with.

6.1. Ordering in the Fibonacci numeration system for $\mathbb{N}$. In this short section, we recall known facts about the Fibonacci numeration system also known as the Zeckendorf numeration system. We recall that $\text{rep}_\mathcal{F}$ is increasing with respect to the radix order. For completeness, we also provide the proofs of the results.

**Lemma 6.1.** Let $w \in \Sigma^* \setminus (\Sigma^* 11 \Sigma^* \cup 0 \Sigma^*)$. If $w \neq \varepsilon$ and $k \geq 1$ is an integer, then

$$|w| = k \text{ if and only if } \text{val}_\mathcal{F}(w) < F_k.$$

Moreover, $|w| = 0$ if and only if $\text{val}_\mathcal{F}(w) = 0$.

**Proof.** Let $w \in \Sigma^* \setminus (\Sigma^* 11 \Sigma^* \cup 0 \Sigma^*)$. If $w = \varepsilon$ then $\text{val}_\mathcal{F}(w) = 0$. If $w \neq \varepsilon$ then $w \in \Sigma^* 1 \Sigma^*$ and $\text{val}_\mathcal{F}(w) > 0$. Let $k \in \{1, 2\}$. Then $|w| = k$ if and only if $w = p_k$ where $p_k$ is the prefix of length $k$ of the word 10. Let $n \in \mathbb{N}$. Then $F_{k-1} \leq n < F_k$ if and only if $n = k = \text{val}_\mathcal{F}(p_k)$.

Let $k \geq 3$ and let it hold for all integers $\ell$ up to $k-1$ that $|w| = \ell$ if and only if $F_{\ell-1} \leq \text{val}_\mathcal{F}(w) < F_\ell$. Let $w = w_{k-1} w_{k-2} \cdots w_0 \in \Sigma^* \setminus (\Sigma^* 11 \Sigma^* \cup 0 \Sigma^*)$. Then applying induction hypothesis on the suffix $w_{q-1} \cdots w_1 w_0 \in \Sigma^* \setminus (\Sigma^* 11 \Sigma^* \cup 0 \Sigma^*)$ such that $w = 10^k - q - 1 w_{q-1} \cdots w_0$ it holds that $F_{q-1} \leq \sum_{i=0}^{q-1} w_i F_i < F_q$. As $w_{k-1} = 1$ and $q \leq k - 2$, we have

$$F_{k-1} \leq \text{val}_\mathcal{F}(w) = \sum_{i=0}^{k-1} w_i F_i = F_{k-1} + \sum_{i=0}^{q-1} w_i F_i < F_{k-1} + F_q \leq F_{k-1} + F_{k-2} = F_k. \quad \Box$$

**Lemma 6.2.** The map $\text{val}_\mathcal{F}: \mathbb{N} \setminus (\Sigma^* 11 \Sigma^* \cup 0 \Sigma^*) \rightarrow \mathbb{N}$ is a bijection.

**Proof.** (Injectivity): Let $u, v \in \Sigma^* \setminus (\Sigma^* 11 \Sigma^* \cup 0 \Sigma^*)$ be such that $\text{val}_\mathcal{F}(u) = \text{val}_\mathcal{F}(v)$. Assume by contradiction that $u \neq v$. By Lemma 6.1, $|u| = |v|$. Let $p \in \Sigma^*$ be the longest common prefix of $u$ and $v$. Therefore there exist $\ell \in \mathbb{N}$ and $t, s \in \Sigma^\ell$ such that without loss of generality $u = p0s$ and $v = pt1$. Then by Lemma 6.1,

$$\text{val}_\mathcal{F}(v) - \text{val}_\mathcal{F}(u) = \text{val}_\mathcal{F}(1t) - \text{val}_\mathcal{F}(0s) > F_\ell - F_{\ell-1} = F_{\ell-2} \geq 0$$

which is a contradiction.

(Surjectivity): Denote $S = \Sigma^* \setminus (\Sigma^* 11 \Sigma^* \cup 0 \Sigma^*)$. We proceed by induction on $n \in \mathbb{N}$. If $n = 0$ then $w = \varepsilon \in S$ fulfills $\text{val}_\mathcal{F}(w) = n$. Let $n \geq 1$. Induction hypothesis: for every $m \in \mathbb{N}$ such that $m < n$ there exists $w \in S$ such that $\text{val}_\mathcal{F}(w) = m$. There exists a unique $k \in \mathbb{N}$
such that $F_{k-1} \leq n < F_k$. We have $0 \leq n - F_{k-1} < F_k - F_{k-1} = F_{k-2}$. By Lemma 6.1, $|w| \leq k - 2$. By induction hypothesis, there exists $w \in S$ such that $val_F(w) = n - F_{k-1}$. Let $\ell = k - 1 - |w|$ and $u = 10^\ell w$. Clearly $\ell \geq 1$. Thus $u \in S$ and

$$val_F(u) = val_F(10^\ell w) = F_{k-1} + val_F(w) = F_{k-1} + n - F_{k-1} = n. \quad \square$$

The map $rep_F$ is defined as the inverse map $val_F^{-1}$. It corresponds to the Zeckendorff numeration system [Zec72].

We can now show that $val_F$ is an increasing map. Recall that the radix order is a total order on $\Sigma^*$ such that, for every $u, v \in \Sigma^*$, $u <_{rad} v$ if and only if $|u| < |v|$ or $|u| = |v|$ and $u <_{lex} v$.

**Lemma 6.3.** Let $S = \Sigma^* \setminus (\Sigma^* 11 \Sigma^* \cup 0 \Sigma^*)$. The map $val_F$ is an increasing bijection from $(S, <_{rad})$ to $(\mathbb{N}, <)$.

**Proof.** It follows from Lemma 6.2 that $val_F$ is a bijection. We show that it is increasing.

Let $u, v \in \Sigma^* \setminus (\Sigma^* 11 \Sigma^* \cup 0 \Sigma^*)$ be such that $u <_{rad} v$. If $|u| < |v|$ then by Lemma 6.1, $val_F(u) < val_F(v)$. Assume that $|u| = |v|$. Then $u <_{lex} v$. Then there exists $p \in \Sigma^*$ such that $u = p0s$, $v = pt$ for some $t, s \in \Sigma^* \setminus \Sigma^* 11 \Sigma^*$. Denote $\ell = |t| = |s|$. Thus from Lemma 6.1,

$$val_F(v) - val_F(u) = val_F(pt) - val_F(p0s) = val_F(1t) - val_F(0s) > F_\ell - F_{\ell-1} = 0. \quad \square$$

### 6.2. Proof of Proposition 2.3

The goal of this section is to prove Proposition 2.3, that is that $val_{F_c}$ is a bijection. The following lemma allows to determine whether $val_{F_c}(w)$ is nonnegative or negative based only on the first digit of $w$.

**Lemma 6.4.** For every word $w \in \Sigma^+ \setminus \Sigma^* 11 \Sigma^*$ such that $|w| = k$. We have

1. $w \in 0 \Sigma^*$ if and only if $0 \leq val_{F_c}(w) < F_{k-1}$,
2. $w \in 1 \Sigma^*$ if and only if $-F_{k-2} \leq val_{F_c}(w) < 0$.

**Proof.** Let $w = w_{k-1}w_{k-2}\cdots w_0 \in \Sigma^+ \setminus \Sigma^* 11 \Sigma^*$ for some $k \geq 1$. First we prove the implication from left to right for both cases.

1. Let $\ell$ denote the unique integer $\ell \leq k - 2$ such that $w = v1s$ for some $v \in 0^*$ and $s \in \Sigma^\ell$. Then $0 \leq val_{F_c}(w) = val_F(w) = val_F(1s) = F_{\ell+1} < F_{k-1}$ by Lemma 6.1.
2. Using Equation 3 and part (1), we have $-F_{k-2} \leq -F_{k-2} + \sum_{i=0}^{k-2} w_i F_i = val_{F_c}(w) = val_F(w) - F_k = val_{F_c}(0w) - F_k < F_k - F_k = 0$.

The converse follows from the following observation. Let $X = X_1 \cup X_2$ such that $X_1 \cap X_2 = \emptyset$. Let $F': X \to Y$ be a map, such that $F(X_1) \cap F(X_2) = \emptyset$. Then it holds that $F(x) \in F(X_1)$ for some $x \in X$ implies $x \in X_1$. Indeed, if $x \in X_2$ then $F(x) \in F(X_1) \cap F(X_2)$, a contradiction. It suffices to use this observation with $F = val_F$, $X_1 = 0 \Sigma^*$, $X_2 = 1 \Sigma^*$.

Recall that $D = \Sigma(\Sigma \Sigma)^* \setminus (\Sigma^* 11 \Sigma^* \cup 000 \Sigma^* \cup 101 \Sigma^*)$ is the set of odd-length words $w \in D$ with no consecutive $1$'s and which are not starting with a neutral prefix. The following lemma strengthens Lemma 6.4 for $w \in D$.

**Lemma 6.5.** Let $w \in D$ and $k \in \mathbb{N}$.

1. $w \in 0 \Sigma^{2k}$ if and only if $F_{2k-2} \leq val_{F_c}(w) < F_{2k}$.
2. $w \in 1 \Sigma^{2k}$ if and only if $-F_{2k-2} \leq val_{F_c}(w) < -F_{2k-3}$.
3. Also, $w = 1$ if and only if $val_{F_c}(w) = -1$. 
Proof. Let $w \in D$ and $k \in \mathbb{N}$.

(1): Let $w \in 0\Sigma^*$ and $|w| = 2k + 1$. Then $w = v1s$ for some $v \in \{0, 00\}$ and $s \in \Sigma^*$. Applying Equation (3) we have $\text{val}_F(w) = \text{val}_F(v1s) = \text{val}_F(1s)$. If $v = 0$ then $|s| = 2k - 1$ and from Lemma 6.1, $F_{2k-1} \leq \text{val}_F(1s) < F_{2k}$. If $v = 00$ then $|s| = 2k - 2$ and from Lemma 6.1, $F_{2k-2} \leq \text{val}_F(1s) < F_{2k-1}$. Together, $F_{2k-2} \leq \text{val}_F(w) < F_{2k}$. On the other hand, let $w \in D$ be such that $F_{2k-2} \leq \text{val}_F(w) < F_{2k}$. By Lemma 6.1, $w \in 0\Sigma^{2\ell}$ for some $\ell \leq k$. From the first part of this proof then $k = \ell$ and $w \in 0\Sigma^{2k}$.

(2): Let $w \in 1\Sigma^*$ and $|w| = 2k + 1$. From Lemma 6.1, $F_{2k} \leq \text{val}_F(w)$. From Equation (3),

$$\text{val}_F(w) = \text{val}_F(01v) = \text{val}_F(0v1s) - F_{2k-1} < F_{\ell+1} - F_{2k-1} \leq F_{2k-2} - F_{2k-1} = -F_{2k-3}.$$ 

On the other hand, let $w \in D$ be such that $-F_{2k-1} \leq \text{val}_F(w) < -F_{2k-3}$. By Lemma 6.4, $w \in 1\Sigma^\ell$ for some $\ell \leq 2k$. From the first part of this proof then $\ell = 2k$ and $w \in 1\Sigma^{2k}$.

(3): If $w = 1$ then $\text{val}_F(w) = -1$. The converse follows from previous parts (1) and (2). □

We can now prove that $\text{val}_F$ is a bijection $D \rightarrow \mathbb{Z}$.

Proof of Proposition 2.3 (Injectivity): Let $u, v \in D$ be such that $\text{val}_F(u) = \text{val}_F(v)$. By Lemma 6.5, $u$ and $v$ have the same length $k \in \mathbb{N}$ and the same first digit $d \in \{0, 1\}$. Using Equation (3), $\text{val}_F(u) = \text{val}_F(0u) = \text{val}_F(u) + dF_k = \text{val}_F(v) + dF_k = \text{val}_F(v)$. By Lemma 6.2, $u = v$.

(Surjectivity): It holds that $\mathbb{Z} = \{-1\} \cup \bigcup_{k=0}^{\infty} [F_{2k-2}, F_{2k}) \cup [-F_{2k-1}, -F_{2k-3})$, a disjoint union. If $n = -1$ then $w = 1$ fulfils $\text{val}_F(w) = n$. Let $n \in \mathbb{Z} \setminus \{-1\}$.

Assume $n \geq 0$. Then there exists a unique $k \in \mathbb{N}$ such that $F_{2k-2} \leq n < F_{2k}$. From Lemma 6.2, there exists a unique $w \in \Sigma^* \setminus (\Sigma^*1\Sigma^* \cup 0\Sigma^*)$ such that $\text{val}_F(w) = n$. If $n < F_{2k-1}$, then by Lemma 6.1, $|w| = 2k - 1$ and the word $00w \in D$ fulfils $\text{val}_F(00w) = \text{val}_F(0w) = \text{val}_F(w) = n$. If $F_{2k-1} \leq n$, then by Lemma 6.1, $|w| = 2k$ and the word $0w \in D$ fulfils $\text{val}_F(0w) = \text{val}_F(w) = n$.

Assume $n < -1$. Then there exists a unique $k \in \mathbb{N}$ such that $-F_{2k-1} \leq n < -F_{2k-3}$. From Lemma 6.2, there exists $w \in \Sigma^* \setminus (\Sigma^*1\Sigma^* \cup 0\Sigma^*)$ such that $\text{val}_F(w) = F_{2k-1} + n$. Let $\ell = 2k - |w|$. Let $u = 10^\ell w$. Then $|u| = 2k + 1$ and thus

$$\text{val}_F(u) = -F_{2k+1} + \text{val}_F(10^\ell w) = -F_{2k+1} + F_{2k} + \text{val}_F(w) = -F_{2k+1} + F_{2k} + F_{2k-1} + n = n.$$

6.3. Ordering in the Fibonacci’s complement numeration system. The goal of this section is to prove that $\text{val}_F$ is an increasing bijection with respect to some total order on $D$.

We define reversed-radix order as a total order such that $u <_{\text{rev}} v$ if and only if $|u| > |v|$ or $|u| = |v|$ and $u <_{\text{lex}} v$. We define a total order on $\Sigma^*$ as follows.

**Definition 6.6** (total order $\prec$). For every $u, v \in \Sigma^*$, we define $u \prec v$ if and only if

- $u \in 1\Sigma^*$ and $v \in 0\Sigma^*$, or
- $u, v \in 0\Sigma^*$ and $u <_{\text{rad}} v$, or
- $u, v \in 1\Sigma^*$ and $u <_{\text{rev}} v$.

**Lemma 6.7.** The map $\text{val}_F$ is an increasing bijection from $(D, \prec)$ to $(\mathbb{Z}, <)$.
Proof. Let \( u, v \in D \) be such that \( u \prec v \). Let \( k, \ell \in \mathbb{N} \) be such that \( |u| = 2k+1 \) and \( |v| = 2\ell+1 \).

- If \( u \in 1\Sigma^* \) and \( v \in 0\Sigma^* \), then from Lemma 6.4 we have \( \text{val}_{F_c}(u) < 0 \leq \text{val}_{F_c}(v) \).
- Assume that \( u, v \in 0\Sigma^* \) and \( |u| < |v| \). Thus \( k \leq \ell - 1 \). From Lemma 6.5 we have \( \text{val}_{F_c}(u) < F_{2k} \leq F_{2(\ell-1)} \leq \text{val}_{F_c}(v) \).
- Assume that \( u, v \in 1\Sigma^* \) and \( |u| > |v| \). Thus \( k - 1 \geq \ell \). From Lemma 6.5 we have \( \text{val}_{F_c}(v) < -F_{2k-3} = -F_{2(\ell-1)} \leq \text{val}_{F_c}(v) \).
- Assume that \( u, v \in d\Sigma^* \) for some \( d \in \{0, 1\} \) and \( |u| = |v| \). In this case, we have \( u \prec v \). From Lemma 6.3 we have \( \text{val}_{F_c}(u) < \text{val}_{F_c}(v) \). Since \( u \) and \( v \) start with the same digit we have \( \text{val}_{F_c}(v) - \text{val}_{F_c}(u) = \text{val}_{F_c}(v) - \text{val}_{F_c}(u) > 0 \). Thus \( \text{val}_{F_c}(u) < \text{val}_{F_c}(v) \). \( \square \)

We can now prove that \( \text{rep}_{F_c} \) is characterized by the fact of being an increasing bijection.

Theorem B. Let \( f : \mathbb{Z} \to D \) where \( D = \Sigma(\Sigma \Sigma)^* \setminus (\Sigma^*1\Sigma^* \cup 000\Sigma^* \cup 101\Sigma^*) \). The map \( f \) is an increasing bijection from \((\mathbb{Z}, \prec)\) to \((D, \prec)\) such that \( f(0) = 0 \) if and only if \( f = \text{rep}_{F_c} \).

Proof. Denote \( D = \Sigma(\Sigma \Sigma)^* \setminus (\Sigma^*1\Sigma^* \cup 000\Sigma^* \cup 101\Sigma^*) \). The map \( \text{rep}_{F_c} : \mathbb{Z} \to D \) is the inverse of the map \( \text{val}_{F_c} \) which by Lemma 6.7 is an increasing bijection \( D \to \mathbb{Z} \) with respect to the order \( \prec \). Hence \( \text{rep}_{F_c} : \mathbb{Z} \to D \) is an increasing bijection with respect to the order \( \prec \). Moreover \( \text{rep}_{F_c}(0) = 0 \).

Let \( f : \mathbb{Z} \to D \) be an increasing bijection with respect to the order \( \prec \) such that \( f(0) = 0 \). There is a unique increasing bijection \((\mathbb{Z}, \prec)\) to \((D, \prec)\) such that \( 0 \mapsto 0 \). Thus \( f = \text{rep}_{F_c} \). \( \square \)

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(S. Labbé) Univ. Bordeaux, CNRS, Bordeaux INP, LABRI, UMR 5800, F-33400 Talence, France
Email address: sebastien.labbe@labri.fr
URL: http://www.slabbe.org/

(J. Lepšová) FNSPE, CTU in Prague, Trojanova 13, 120 00 Praha, Czech Republic
Email address: jana.lepsova@labri.fr
Table 3. The table lists for every ternary word $u \in \{0, 1, 2\}^*$ of length $\leq 3$ the output $\mathcal{B}(u) \cdot \mathcal{B}_\downarrow(u) \in \{0, 1\}^*$ under the Berstel adder $\mathcal{B}$ and the output $\mathcal{T}(u) \cdot \mathcal{T}_\downarrow(u) \in \{0, 1\}^*$ under the modified Berstel adder $\mathcal{T}$. The other columns illustrate that $\mathcal{B}$ preserves the value in the Fibonacci numeration system whereas $\mathcal{T}$ preserves the value in the Fibonacci’s complement numeration system.