Gravitational perturbation induced by a rotating ring around a Kerr black hole

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The linear perturbation of a Kerr black hole induced by a rotating massive circular ring is discussed by using the formalism by Teukolsky, Chrzanowski, Cohen and Kegeles. In these formalism, the perturbed Weyl scalars, ψ₀ and ψ₄, are first obtained from the Teukolsky equation. The perturbed metric is obtained in a radiation gauge via the Hertz potential. The computation can be done in the same way as in our previous paper [6], in which we considered the perturbation of a Schwarzschild black hole induced by a rotating ring. By adding lower multipole modes such as mass and angular momentum perturbation which are not computed by the Teukolsky equation, and by appropriately setting the parameters which are related to the gauge freedom, we obtain the perturbed gravitational field which is smooth except on the equatorial plane outside the ring.

I. INTRODUCTION

The black hole perturbation method is important to investigate the physical property of black holes, such as the quasinormal modes, and the various astrophysical phenomenon like the orbital evolution of stars around much larger black holes and the gravitational waves induced by them. Especially the later phenomenon include the extreme mass ratio inspirals which are one of the most important sources for the future space laser interferometers, eLISA [1], DECIGO [2, 3], and BBO [4].

When the black holes are non-rotating, the metric perturbation can be analyzed by using the Regge-Wheeler and the Zerilli equations [6, 7], which are the single, decoupled equations for the odd and even parity modes, respectively. On the other hand, if the black holes are rotating, there is no such formalism. The perturbation of Kerr black holes are usually analyzed by using the Teukolsky equation which is the equation for the perturbation of the Weyl scalars, ψ₀ and ψ₄. There is a formalism, by Chrzanowski [8] and Cohen and Kegeles [9, 11], (see also [12, 13]), to compute the metric perturbation from the perturbation of ψ₀ and ψ₄ obtained with the Teukolsky equation (hereafter, this is called the CCK formalism). In this method, a radiation gauge is used to calculate the metric perturbation. Recently, this method becomes more important, mainly because of the necessity to compute the gravitational self-force on the point particle orbiting around a Kerr black hole, and several works have been done [14–17, 19–21, 23].

In spite of these developments, there are still only a few examples of the explicit computation of the metric perturbation by using the CCK formalism. In our previous paper [6] (hereafter, Paper I), we computed the metric perturbation induced by a rotating circular mass ring on the equatorial plane around a Schwarzschild black hole. We found that if we consider only modes which can be derived from ψ₀ or ψ₄ (which contain harmonic modes of l ≥ 2), there appear unphysical discontinuities in the Hertz potential and the metric perturbation at the ring radius. This discontinuities, however, can be removed by adding the lower modes of l = 0 and 1, and by setting free parameters, corresponding to the gauge freedom, appropriately. We obtained metric perturbation which is regular everywhere except for the equatorial plane outside the ring radius.

In this paper, we extend this analysis to the case of the Kerr black hole. As in the Schwarzschild case, this problem is stationary and axisymmetric. Nevertheless, this problem contains both the mass and the angular momentum perturbation. We find that, although some equations must be rederived by using the Kerr metric, we can obtain the metric perturbation in the same way as in the Schwarzschild case in Paper I.

This paper is organized as follows. In Sec. II we derive the perturbed Weyl scalars ψ₀ and ψ₄ by solving the Teukolsky equation. In Sec. III we describe how to obtain the perturbed gravitational field. In Sec. III A we obtain the particular solution of the Hertz potential from ψ₀ and ψ₄. The determination of the homogeneous part of the Hertz potential which contains the lower modes l = 0, 1 is also discussed. In Sec. III B the perturbed Weyl scalars and metric perturbation are shown. Sec. IV is devoted to summary and discussion.

II. SOLUTIONS OF THE TEUKOLSKY EQUATION

The Kerr metric is given in Boyer-Lindquist coordinates as

\[
ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 \\
+ \Sigma d\theta^2 + \sin^2\theta \left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)d\phi^2,
\]

(2.1)
where $\Sigma = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 - 2Mr + a^2$. The Weyl scalars are defined as

\begin{align*}
\Psi_0 &= +C_{abcd}a^m b^l v^m d^l, \\
\Psi_1 &= +C_{abcd}a^m b^l v^m d^l, \\
\Psi_2 &= +C_{abcd}a^m b^l c^m n^d, \\
\Psi_3 &= +C_{abcd}a^m b^l m^m n^d, \\
\Psi_4 &= +C_{abcd}a^m m^m m^m n^d.
\end{align*}

(2.2a)

(2.2b)

(2.2c)

(2.2d)

(2.2e)

where $l^a, n^b, m^d$ are the Kinnersley tetrad [22], which are defined in Appendix A. In the case of the Kerr metric, nonzero Weyl scalar is $\Psi_2 = M\rho^3$, where $\rho = -(r - ia \cos \theta)^{-1}$. The perturbed Weyl scalars are denoted by $\psi_0, \psi_1, \ldots, \psi_4$.

We consider the perturbation of the Kerr metric induced by a rotating ring which is composed of a set of point particles in circular, geodesic orbit on the equatorial plane. The energy-momentum tensor of the rotating circular ring, $T^{ab}$, is given as

\begin{equation}
T^{ab} = \frac{mu^a u^b}{u^0 r_0^2} \delta(r - r_0) \delta(\cos \theta),
\end{equation}

(2.3)

where $r_0$ is the radius of the ring, and $u^a = u^t ((\partial_t)^a + \Omega (\partial_\phi)^a)$ is the four-velocity of the ring. The angular velocity $\Omega$ and $u^t$ are given as

\begin{align*}
\Omega &= \pm M^{1/2} \frac{\pm M^{1/2} \pm a M^{1/2}}{r_0^{3/2} \pm a M^{1/2}}, \\
u^t &= \pm M^{1/2} \frac{r_0^{3/2} \pm \pm M^{1/2} a}{\sqrt{r_0^3 - 3Mr_0^2 + 2a M^{1/2} r_0^{3/2}}},
\end{align*}

(2.4)

where the upper sign is for prograde rotation and the lower sign is for retrograde rotation. The rest mass of the ring becomes $2\pi m$. We assume $m \ll M$.

Since our perturbed space-time is independent of $t$ and $\phi$, we expand $\psi(s = 2) = \psi_0$ and $\psi(s = -2) = \rho^{-1} \psi_4$ as

\begin{equation}
\psi(s)(r, \theta) = \sum_{l=2}^{\infty} R_l^{(s)}(r) \psi_l(\theta).
\end{equation}

(2.5)

Here we have defined $\psi_l(\theta)$ as $\psi_l(\theta) \equiv \psi_{l0}(\theta, 0)$. The Teukolsky equation for $R_l^{(s)}$ becomes

\begin{equation}
\left[ \frac{d}{dr} \left( \Delta^{s+1} \frac{d}{dr} \right) - \Delta^s(l - 2)(l + 3) \right] R_l^{(s)} = -8\pi T_l^{(s)} \Delta^s.
\end{equation}

(2.6)

The source term $T_l^{(s)}$ is given as

\begin{align*}
\frac{T_l^{(s)}}{2\pi} &= -\tilde{T}_{11} \delta(r - r_0) \sqrt{(l + 2)(l - 1)(l + 1)} Y_l(\pi/2) \\
&\quad - 2\tilde{T}_{11} \frac{ia}{\sqrt{2r_0}} \delta(r - r_0) \frac{\sqrt{(l + 2)(l - 1)}}{\sqrt{2}} Y_l(\pi/2) \\
&\quad + 2\tilde{T}_{11} \frac{r_0}{r^2} \frac{d}{dr} \delta(r - r_0) \frac{\sqrt{(l + 2)(l - 1)}}{\sqrt{2}} Y_l(\pi/2) \\
&\quad - \tilde{T}_{33} \frac{r_0}{r^3} \frac{d}{dr} \left( r^4 \frac{d}{dr} \delta(r - r_0) \right) Y_l(\pi/2),
\end{align*}

(2.7)

where $\tilde{T}_{\mu\nu}$ are constants defined as $\tilde{T}_{\mu\nu} = m u_\mu u_\nu / (u^0 r_0^2)$.

The Teukolsky equations are solved by using the Green’s function which is given as

\begin{equation}
G_l^{(s)}(r, r') = \frac{(\Delta')^{-s/2} P_{\Delta l}^2(x'_e)Q_{\Delta l}^2(x'_e)}{\sqrt{M^2 - a^2}(l + 2)(l - 1)},
\end{equation}

(2.8)

where $\Delta' = r'^2 - 2Mr' + a^2$ and

\begin{align*}
x'_e &= \min(r, r') - M \sqrt{M^2 - a^2}, \\
x'_e &= \max(r, r') - M \sqrt{M^2 - a^2}.
\end{align*}

(2.9)

A simple relation $(\Delta')^2 G_l^{(2)}(r, r') = G_{l}^{(-2)}(r, r')$ holds because of symmetries. As a result, we obtain

\begin{align*}
R_l^{(2)} &= -8\pi^2 T_{11} \Delta_0 \epsilon_0 G_l^{(2)}(r, r_0) \sqrt{(l + 2)(l - 1)(l + 1)} Y_l(\pi/2) \\
&\quad - 16\sqrt{\pi}^2 T_{11} \frac{ia\Delta_0^2}{\sqrt{2r_0}} G_l^{(2)}(r, r_0) \sqrt{(l + 2)(l - 1)} Y_l(\pi/2) \\
&\quad - 16\sqrt{\pi}^2 T_{33} \frac{r_0}{r^2} \frac{d}{dr} \left( r^4 \frac{d}{dr} \delta(r - r_0) \right) Y_l(\pi/2) \\
&\quad - 16\pi^2 T_{33} \frac{r_0}{r^3} \frac{d}{dr} \left( r^4 \frac{d}{dr} \delta(r - r_0) \right) Y_l(\pi/2),
\end{align*}

(2.10)

(2.11)

(2.12)

where $\Delta_0 = r_0^2 - 2Mr_0 + a^2$.

As in the Schwarzschild case, we have the symmetry about the equatorial plane,

\begin{align*}
\text{Re}(\psi_0/4(r, \pi - \theta)) &= \text{Re}(\psi_0/4(r, \theta)), \\
\text{Im}(\psi_0/4(r, \pi - \theta)) &= -\text{Im}(\psi_0/4(r, \theta)).
\end{align*}

(2.13)

The plots of $\psi_0$ and $\psi_4$ are shown in Fig. 1 for the case $r_0 = 10M$, $a = 0.99M$, and the ring’s rotation is prograde.

### III. CONSTRUCTION OF THE PERTURBED GRAVITATIONAL FIELDS

#### A. The Hertz potential

Next, we use these solution $\psi_0$ and $\psi_4$ in the CCK formalism to find the Hertz potential. The ingoing radiation gauge (IRG) is defined as $h_{\alpha\beta} = h_{\alpha}^\beta = 0$. The
perturbed metric $h_{ab}$ in IRG is related to the Hertz potential as [8]

$$h_{ab} = -[l_{a} b_{b}(\delta + \alpha + 3\beta - \tau)(\delta + 4\beta + 3\tau)\Psi]$$

$$- l_{a} m_{b}(D + \rho - \pi)(\delta + 4\beta + 3\tau)\Psi$$

$$- l_{a} \bar{m}_{b}(\delta + \alpha + 3\beta - \omega - \tau)(D + 3\rho)\Psi$$

$$+ m_{a}\bar{m}_{b}(D - \pi)(D + 3\rho)\Psi$$

$$+ [c.c.],$$

(3.1)

where [c.c.] represents the complex conjugate of the first term. The bold greek characters are derivative operators defined as

$$D = l^{a}\partial_{a}, \quad \Delta = n^{a}\partial_{a},$$

$$\delta = m^{a}\partial_{a}, \quad \bar{\delta} = \bar{m}^{a}\partial_{a},$$

and the overline denotes the complex conjugate. $\rho, \mu, \alpha, \beta, \gamma, \omega,$ and $\tau$ here are the spin coefficients, which are defined in Appendix A. The Hertz potential $\Psi$ in IRG satisfies the source-free Teukolsky equation with $s = -2$.

$$(\Delta - 2\mu + 2\gamma)D\Psi + 3\rho \partial_{n}\Psi = (\delta - 2\beta - \tau)(\delta + 4\beta)\Psi. \quad (3.3)$$

In IRG, since the perturbed space-time is stationary and axisymmetric, we have

$$\psi_{0} = \frac{1}{2} \left( \frac{\partial}{\partial r} \right)^{4} \Psi,$$

$$4\rho^{-4}\psi_{4} = \frac{1}{2} \sin^{2}\theta \left( \frac{\partial}{\partial \cos\theta} \right)^{4} \sin^{2}\theta \Psi. \quad (3.5)$$

Our task is to find Hertz potential which satisfies (3.4), (3.5) and (3.3). By substituting the solution of the Teukolsky equation into the left hand side of (3.3), $\Psi$ can be integrated as

$$\Psi(r, \theta) = \Psi_{P} + \Psi_{H}, \quad (3.6)$$

where

$$\Psi_{P} = \sum_{l=2}^{\infty} \frac{R_{l}(r r_{2} Y_{l}(\theta))}{l(l + 2)(l - 1)(l + 1)}.$$

$$\Psi_{H} = \frac{2A}{\sin^{2}\theta} \left( \frac{a(r)}{6} \cos^{3}\theta + \frac{b(r)}{2} \cos^{2}\theta + c(r) \cos\theta + d(r) \right). \quad (3.8)$$

$A$ is a constant defined as $A \equiv m/(r_{0} \sqrt{\Delta_{0}})$. It can be shown that $\Psi_{H}$ is a homogeneous solution of (3.3) and (3.5), and satisfies (3.3) when

$$a(r) = a_{1} r^{2} (r - 3M) + 3a_{2} r + a_{2},$$

$$b(r) = b_{1} (r^{2} - a^{2}) + b_{2} (r - M),$$

$$c(r) = \frac{a_{1}}{2} (r^{2} + 4M^{2}) (r - M) - \frac{a_{2}}{2} - \left( c_{1} + \frac{a_{1}}{2} M \right) a^{2} + c_{1} r^{2} + c_{2} (r - M),$$

$$d(r) = \frac{b_{1}}{2} r^{2} + \frac{b_{2}}{2} r + d_{1} (r^{2} - 3M r^{2} + 3a^{2} r) + d_{2}. \quad (3.9)$$

Here $a_{1}, a_{2}, \ldots$ are arbitrary complex constants, and $a$ is the Kerr parameter.

We see that $\Psi_{P}$ satisfies (3.4), (3.5), and (3.3) everywhere except for $r = r_{0}$. At $r = r_{0}$, $\Psi_{P}$ has the singularity. However, this singularity can be removed by appropriately choosing the parameters in $\Psi_{H}$. Here, we explain how to obtain those parameters.

First we demand that the metric perturbation and the Weyl scalars should not diverge at $\theta = 0$ and $\theta = \pi$. These conditions are satisfied if the Hertz potential $\Psi$ does not diverge at $\theta = 0$ and $\pi$. We obtain

$$3d_{1} = \pm a_{1}, \quad \pm c_{1} = \pm Ma_{1} - b_{1},$$

$$\pm c_{2} = \pm \left( 2M^{2} - 3a^{2} \right) a_{1} - b_{2}, \quad (3.10)$$

$$6d_{2} = \pm 2a_{2} - 3a^{2} b_{1} - 3Mb_{2},$$

where the upper sign is for $\theta = 0$, and the lower sign is for $\theta = \pi$. We see that these equations are satisfied if and only if $a_{1} = a_{2} = b_{1} = b_{2} = c_{1} = c_{2} = d_{1} = d_{2} = 0$, i.e., $\Psi_{H} = 0$. This implies that we can not obtain the regular solution globally. Following [8], we divide the space-time into three regions: $I: (M + \sqrt{M^{2} - a^{2}} < r < r_{0}),$ $N: (r > r_{0}, 0 \leq \theta < \pi/2),$ and $S: (r > r_{0}, \pi/2 < \theta \leq \pi)$. We set $\Psi_{H} = 0$ in the inner region $I$, and look for the set of parameters which satisfy (3.10) in $N$ and $S$, respectively. Since there are four equations among eight unknown parameters, the number of parameters we have to determine is four. We adopt $a_{1}, a_{2}, b_{1}, b_{2}$ in region $N$ as independent variables. The parameters in $S$ can be determined by using the symmetry about the equatorial plane $22$. We determine the value of these parameters numerically by using four continuity conditions for $\psi_{1}, \psi_{2}, h_{33},$ and $\Psi$ at $r = r_{0}$. The necessary equations for $\psi_{1}, \psi_{2} h_{33}$ are given in Appendix B. The parameters obtained are shown in Tables I and II for the case, $M = 1$, $m = M/100$, $r_{0} = 10M$, and for various value of the Kerr parameter.

B. Perturbed Weyl scalars and the metric perturbation

The plots of the Weyl scalars $\psi_{1}, \psi_{2},$ and $\psi_{3}$ derived by using $\Psi = \Psi_{P} + \Psi_{H}$ are shown in Fig. 2. The plots of the metric perturbations $h_{22}, h_{23}, h_{33},$ and the Hertz
TABLE I. The parameters \(a_1\) and \(a_2\) in \(N\). This is for the case, \(M = 1, m = M/100, r_0 = 10M\).

| \(a/M\) | \(\text{Re}(a_1)\) | \(\text{Im}(a_1)\) | \(\text{Re}(a_2)\) | \(\text{Im}(a_2)\) |
|---------|----------------|----------------|----------------|----------------|
| 0.3     | -0.299239e-3  | -3.76054      | -106.766       | -1875.23       |
| 0.6     | -0.351477e-3  | -3.29746      | -81.5775       | -1630.99       |
| 0.9     | -0.317026e-3  | -2.85623      | -59.3722       | -1393.31       |
| 0.99    | -0.148324e-3  | -2.72761      | -53.4299       | -1323.53       |

TABLE II. The parameters \(b_1\) and \(b_2\) in \(N\). This is for the case, \(M = 1, m = M/100, r_0 = 10M\).

| \(a/M\) | \(\text{Re}(b_1)\) | \(\text{Im}(b_1)\) | \(\text{Re}(b_2)\) | \(\text{Im}(b_2)\) |
|---------|----------------|----------------|----------------|----------------|
| 0.3     | 64.5010        | 30.0819        | -71.0120       | 0.706601       |
| 0.6     | 62.1322        | 26.3831        | -658.481       | 2.34294        |
| 0.9     | 64.5010        | 30.0819        | -710.120       | 0.706601       |
| 0.99    | 59.4624        | 21.8145        | -651.262       | 5.40521        |

potential \(\Psi\) are shown in Figs. 3 and 4. All plots are for the case of the Kerr parameter \(a = 0.99M\). We find that all of them are continuous at the ring radius, \(r = r_0\).

IV. SUMMARY AND DISCUSSION

We derived the perturbed Weyl scalars and the metric perturbation induced by a rotating circular ring around a Kerr black hole by using the CCK formalism. The computation can be done in the same way as in the Schwarzschild case. However, some equations must be derived in the Kerr case.

In the CCK formalism, the Weyl scalars and the metric perturbation are expressed by the Hertz potential \(\Psi\) in a radiation gauge. We used the ingoing radiation gauge, and derived the \(\Psi_P\) which has discontinuity on the surface of the sphere at the radius of the ring. The homogeneous part of the Hertz potential, \(\Psi_H\), contains the lower multipole modes \(l = 0, 1\) and some gauge freedom. We derived the general form of \(\Psi_H\) in the Kerr case, \([6, 4, 12]\). As in the Schwarzschild case, the lower modes and gauge freedom appear as eight complex parameters. We determined these parameters in \(\Psi_H\) numerically by demanding the continuity of the Weyl scalars, metric perturbation, and the Hertz potential at the ring radius. The expressions for \(\psi_1\) and \(\psi_2\) in terms of the Hertz potential \([12, 10, 6, 8]\) are derived in the Kerr case, and are used to impose the continuity condition. We obtained the Hertz potential which is smooth except for the equatorial plane outside the ring radius. The perturbed Weyl scalars, \(\psi_1, \psi_2, \psi_3\), and the metric perturbation also contain discontinuity on the equatorial plane outside the ring. This is completely the same as in the Schwarzschild case in \([3]\). Note that, as in the Schwarzschild case, in the determination of the parameters in \(\Psi_H\), we did not use the relation between the mass and the angular momentum perturbation, and the parameters in \(\Psi_H\). We only needed to use the continuity condition.

One of the most important extension of this work is the case of a particle orbiting around a Kerr black hole \([12, 10, 19, 21, 23]\). Since the problem becomes nonstationary in such a case, the Teukolsky equation and the spin-weighted spheroidal harmonics must be solved numerically. Thus, the problem becomes much more complicated. But once we obtain the gravitational field in a radiation gauge, it will be possible to compute the gravitational self-force acting on the point particle by using the prescription by Pound et al. \([22]\). We want to work on this problem in the future.

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Appendix A: Definitions of the Kinnersley tetrad and the spin coefficients

The Kinnersley tetrad is defined as

\[
\begin{align*}
I^a &= \frac{r^2 + a^2}{\Delta} (\partial_t)^a + \frac{a}{\Delta} (\partial_\phi)^a, \\
N^a &= \frac{\Delta}{2\Sigma} \left[\frac{r^2 + a^2}{\Delta} (\partial_t)^a - (\partial_r)^a + \frac{a}{\Delta} (\partial_\phi)^a\right], \\
M^a &= \frac{1}{\sqrt{2(r + ia \cos \theta)}} \left[ia \sin \theta (\partial_t)^a + (\partial_\phi)^a + i \csc \theta (\partial_\phi)^a\right], \\
\overline{m^a} &= \frac{1}{\sqrt{2(r - ia \cos \theta)}} \left[-ia \sin \theta (\partial_t)^a + (\partial_\phi)^a - i \csc \theta (\partial_\phi)^a\right].
\end{align*}
\]

These vectors are null, and satisfy normalization and orthogonality conditions.

\[
\begin{align*}
I_a I^a &= n_a n^a = m_a m^a = \overline{m^a} \overline{m^a} = 0, \\
-\overline{m^a} n^a &= m_a m^a = 1, \\
I_a m^a &= I_a \overline{m^a} = n_a m^a = n_a \overline{m^a} = 0.
\end{align*}
\]

The Ricci rotation coefficients are defined as

\[
\gamma_{\mu \nu} \equiv (\nabla_b (e_\mu)_a (e_\nu)^b) (e_\rho)^a = -\gamma_{\mu \rho \nu},
\]

where \(\nabla_a\) is the covariant derivative, and \((e_1)^a = I^a, (e_2)^a = N^a, (e_3)^a = M^a, (e_4)^a = \overline{m^a}\). With the Kerr
metric (2.1) and the Kinnersley tetrad (A.1), non-zero spin coefficients $\rho$, $\mu$, $\beta$, $\gamma$, $\alpha$, $\varpi$, and $\tau$ are given as

$$
\gamma_{24}^3 = \rho = -\frac{1}{r - i a \cos \theta},
$$
(A.4a)

$$
\gamma_{43}^1 = -\mu = -\frac{\Delta}{2\Sigma} \rho,
$$
(A.4b)

$$
\frac{1}{2}(\gamma_{23}^3 + \gamma_{43}^4) = \beta = -\frac{\rho \cot \theta}{2\sqrt{2}},
$$
(A.4c)

$$
\frac{1}{2}(\gamma_{23}^3 + \gamma_{43}^4) = \gamma = \mu + \frac{r - M}{2\Sigma},
$$
(A.4d)

$$
\frac{1}{2}(\gamma_{23}^3 + \gamma_{43}^4) = \alpha = \pi - \beta,
$$
(A.4e)

$$
\gamma_{21}^3 = -\varpi = -\rho \frac{ia \sin \theta}{\sqrt{2}},
$$
(A.4f)

$$
\gamma_{12}^3 = \tau = \frac{\rho}{\rho} \varpi.
$$
(A.4g)

### Appendix B: Weyl scalars $\psi_1, \psi_2$ and the metric perturbation $h_{33}$ in terms of the Hertz potential

In the IRG, expressions for the Weyl scalars in terms of the Hertz potential can be obtained by using (3.1) and (3.4) and

$$
-2C_{abcd} = \nabla_a \nabla_b h_{ac} - \nabla_c \nabla_a h_{bd} - \nabla_d \nabla_a h_{bc} - \nabla_c \nabla_b h_{ad} + C_{abcd}^{(0)} h^e_{ac} - C_{becd}^{(0)} h^e_{ac},
$$
(B.1)

$$
h_{\mu
u;\rho\sigma} = (\nabla_d \nabla_c h_{ab})(e_d)^{a}(e_v)^{b}(e_\sigma)^{c}(e_\eta)^{d}
= (h_{\mu
u,\rho} + 2h_{\kappa\rho}(\gamma^\kappa_{\nu\mu}))
+ (h_{\lambda\mu,\nu} + 2h_{\kappa\nu}(\gamma^\kappa_{\mu\lambda}))
+ (h_{\lambda\nu,\mu} + 2h_{\kappa\mu}(\gamma^\kappa_{\nu\lambda}))
+ (h_{\lambda\mu,\lambda} + 2h_{\kappa\lambda}(\gamma^\kappa_{\mu\lambda}))
$$

where $C_{abcd}$ and $C_{abcd}^{(0)}$ are the first order perturbation and the unperturbed part of the Weyl tensor, respectively. Here the directional derivatives are denoted by ,1, ,2, and so on.

$$
D = .1, \quad \Delta = .2, \quad \delta = .3, \quad \delta = .4.
$$
(B.3)

First, $\psi_1$ is

$$
-2\psi_1 = 2\rho \tau h_{33} = 2\varpi(D + \rho)h_{33} + DDh_{23}
+ \varpi\tau h_{33} - D(7h_{23}) - (D - \rho)(\tau h_{33}).
$$
(B.4)

This equation can be reduced to

$$
2\psi_1 = DDD(\bar{\delta} + 4\bar{\gamma}) \nabla - 3\varpi D(D + 2\rho)D \nabla.
$$
(B.5)

When $\partial_\Psi = \partial_\Psi = 0$, we obtain

$$
2\psi_1 = -\frac{1}{\sqrt{2}} \frac{\partial^3}{\partial r^3} \frac{\rho}{\sin^2 \theta} \frac{\partial \varphi}{\partial \theta} \frac{\partial}{\partial \varrho} \frac{\partial}{\partial \varphi} \nabla.
$$
(B.6)

Therefore, by substituting (3.7) into this, we obtain

$$
\psi_1^p = \frac{1}{2} \sum_{l=2}^{\infty} \left[ -\frac{1}{\sqrt{2}} \frac{\partial^3}{\partial r^3} \frac{\rho}{\sin^2 \theta} \frac{\partial \varphi}{\partial \theta} \frac{\partial}{\partial \varrho} \frac{\partial}{\partial \varphi} \nabla \right] \left[ -3\varpi \frac{\partial}{\partial \varrho} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varrho} \frac{\partial}{\partial \varphi} \nabla \right].
$$
(B.7)

In a similar manner, $\psi_2$ becomes

$$
2\psi_2 = DDD(\bar{\delta} + 2\bar{\beta}) \frac{1}{\rho} (\bar{\delta} + 4\bar{\gamma}) \nabla
- 4\varpi(D + \rho)D(\bar{\delta} + 4\bar{\gamma}) \nabla + 6\varpi D D \nabla.
$$
(B.8)

When $\partial_\Psi = \partial_\Psi = 0$, this equation reduces to

$$
2\psi_2 = \frac{1}{2} \frac{\partial^2}{\partial r^2} \frac{\rho}{\sin^2 \theta} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varrho} \frac{\partial}{\partial \varphi} \sin^2 \theta \nabla + 6\varpi \frac{\partial}{\partial \varrho} \frac{\partial}{\partial \varphi} \varphi \nabla.
$$
(B.9)

By substituting (3.7) into this, we obtain

$$
\psi_2^p = \frac{1}{2} \sum_{l=2}^{\infty} \left[ -\frac{1}{\sqrt{2}} \frac{\partial^2}{\partial r^2} \rho \nu R^2 \sqrt{(l+2)(l-1)l} 0 Y_l \right]
+ 4\varpi \frac{\partial}{\partial \varrho} \frac{\partial}{\partial \varphi} \rho \nu R^2 \sqrt{(l+2)(l-1)l} 1 Y_l
+ 6\varpi \frac{\partial}{\partial \varrho} \frac{\partial}{\partial \varphi} \nu R^2 2 Y_l.
$$
(B.10)

On the other hand, the metric perturbation $h_{13}^p$ can be derived by substituting (3.7) into (3.1). When $\partial_\Psi = \partial_\Psi = 0$, we have

$$
h_{13}^p = -\sum_{l=2}^{\infty} \left[ \frac{1}{\varrho} \frac{\partial}{\partial \varrho} \rho \nu R^2 \right] 2 Y_l.
$$
(B.11)

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FIG. 1. Radial dependence of the Weyl scalars $\psi_0$ and $\psi_4$, at $\theta = \pi/4$ which are obtained by solving the Teukolsky equations. The radius of the ring is $r_0 = 10M$. Solid lines are for the Kerr case with $a = 0.99M$. The ring’s rotation is prograde. Dashed lines are for the Schwarzschild case, $a = 0$. 
FIG. 2. Radial dependence of the Weyl scalars $\psi_1$, $\psi_2$, and $\psi_3$, at $\theta = \pi/4$ which are derived by using $\Psi = \Psi_P + \Psi_H$. The radius of the ring is $r_0 = 10M$. Solid lines are for the Kerr case with $a = 0.99M$. The ring’s rotation is prograde. Dashed lines are for the Schwarzschild case, $a = 0$. We can see the continuity of the fields at $r = r_0$.

FIG. 3. Radial dependence of $h_{22}$ at $\theta = \pi/4$ which are derived by using $\Psi = \Psi_P + \Psi_H$. The radius of the ring is $r_0 = 10M$ and the Kerr parameter is $a = 0.99M$. The radius of the ring is $r_0 = 10M$. Solid lines are for the Kerr case with $a = 0.99M$. The ring’s rotation is prograde. Dashed lines are for the Schwarzschild case, $a = 0$. We can see the continuity of the fields at $r = r_0$. 
FIG. 4. Radial dependence of the metric perturbations $h_{23}$ and $h_{33}$ derived by using $\Psi = \Psi_P + \Psi_H$, and the Hertz potential $\Psi$ at $\theta = \pi/4$. The radius of the ring is $r_0 = 10M$ and the Kerr parameter is $a = 0.99M$. The ring's rotation is prograde. Solid lines are for the Kerr case with $a = 0.99M$. Dashed lines are for the Schwarzschild case, $a = 0$. 