Weakly Nonlocal Hydrodynamics and The Origin of Viscosity in the Adhesion Model

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(Dated: October 26, 2018)

We use the weakly nonlocal hydrodynamics approach to obtain a dynamical equation for the peculiar velocity field in which the viscosity term is physically motivated. Based on some properties of the Ginzburg-Landau equation and the wave mechanics analog of hydrodynamics we find the nonlocal adhesion approximation taking into account the internal structures of the Zeldovich pancakes. If the internal structures correspond to significant mesoscopic fluctuations, viscosity is probably driven by a stochastic force and dynamics is given by the noisy Burgers equation.

PACS numbers: 98.80.-k, 95.30.Lz, 45.20.Jj

I. INTRODUCTION

The complexity of the nonlinear structure formation process is not possible to be studied just using analytical methods. But the comparison of analytical approximations and N-body experiments may help us to understand the physics of highly nonlinear collective phenomena. The Zeldovich approximation was the first analytical try to predict large-scale structures like pancakes, filaments and voids. The problem with this kinematical model is that particles pass through the pancakes instead of clustering into smaller objects like groups and galaxies. An important improvement of the Zeldovich model is the adhesion approximation, in which a viscous term is introduced to prevent particle crossing. However, this modification is not intended to describe the internal structure of pancakes and, then, its form has no constraint. Usually, a simple ad-hoc term, $\nu \nabla^2 \mathbf{u}$, is assumed and hence viscosity is considered as an artificial effect in the large-scale fluid, despite the good agreement of the model with numerical simulations (e.g. [3]). Actually, there are some alternative approaches (without introducing viscosity) to the nonlinear regime [4, 5]. But a systematic comparison of several statistics for the distributions resulting from these approximations shows that the adhesion model works better over the nonlinear regime. This situation suggests the need of a more physical motivation to invoke the viscous term. In this work, we use weakly nonlocal hydrodynamics to obtain a dynamical equation for the large-scale fluid in which the viscosity term is physically motivated.

II. HYDRODYNAMICS AND ADHESION APPROXIMATION

Let us define the peculiar velocity field as the vector field $\mathbf{u} = \dot{\mathbf{r}} - H\mathbf{r}$ (where $H = \dot{a}/a$ and $a$ is the scale factor). Consider the fluid motion near a point $0$. Then

$$u_i(r) = u_i(0) + r_j \frac{\partial u_i}{\partial r_j} \bigg|_0 + \frac{1}{2} r_j r_k \frac{\partial^2 u_i}{\partial r_j \partial r_k} \bigg|_0 + ...$$

(1)

Thus,

$$u_i(r) - u_i(0) \approx r_j \frac{\partial u_i}{\partial r_j} \bigg|_0$$

(2)

The quantity $\frac{\partial u_i}{\partial r_j} \equiv (\nabla \mathbf{u})_{ij}$ is called the velocity gradient tensor. The symmetric part of the velocity gradient is the rate of the strain tensor

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} \right)$$

(3)

and the antisymmetric part is the vorticity tensor

$$\Omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial r_j} - \frac{\partial u_j}{\partial r_i} \right) = -\frac{1}{2} \epsilon_{ijk} \omega_k$$

(4)

where the vector $\omega$ is the vorticity $\omega = \nabla \times \mathbf{u}$. Thus

$$u_i(r) - u_i(0) \approx r_j e_{ij} + \frac{1}{2} (\omega \times r)_i$$

(5)

We assume an irrotational large-scale astrophysical fluid, which is modelled as an incompressible Newtonian fluid with constitutive equation given by

$$\sigma_{ij} = -p \delta_{ij} + \sigma'_{ij}$$

(6)
where $\sigma_{ij}$ is the stress tensor, $p$ is the thermal pressure and $\sigma'_{ij}$ is the shear stress tensor. Assuming that the shear tensor is linearly related to the velocity gradient tensor, we have, for an isotropic fluid

$$\sigma'_{ij} = A_{ijkl} \frac{\partial u_k}{\partial x_l}, \quad (7)$$

with $A_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \mu' \delta_{il} \delta_{jk}$ and $\mu = \mu'$ for a symmetric tensor. Thus, the constitutive equation becomes

$$\sigma_{ij} = -p \delta_{ij} + 2\mu \varepsilon_{ij}. \quad (8)$$

Introducing this into the momentum equation

$$\rho \frac{Du_i}{Dt} = -\rho \nabla \phi + \frac{\partial \sigma_{ij}}{\partial x_j} \quad (9)$$

and making the spatial derivatives with respect to $x = x/a(t)$, we find the Euler equation for the peculiar velocity field

$$\frac{\partial u}{\partial t} + \frac{1}{a} (u \cdot \nabla) u = -\frac{1}{a} \nabla \phi - \frac{\dot{a}}{a} u + \frac{\mu}{\rho a^2} \nabla^2 u \quad (10)$$

where $\mu/\rho = \nu$ is the kinematic viscosity, $\phi$ is the gravitational potential and, for the scales of interest, we have neglected the thermal pressure.

Now, changing the time variable from $t$ to $b$ (the growing mode of the linear theory) and rescaling the peculiar velocity and the potential gravitational as

$$v = \frac{u}{ab} = \frac{dx}{db} \quad \text{and} \quad \varphi = \left(\frac{3}{2} \Omega_0 a^2 b\right)^{-1} \phi \quad (11)$$

we find

$$\frac{\partial v}{\partial b} + (v \cdot \nabla)v = \frac{3}{2} \Omega_0 \frac{\nabla^2}{f^2} \nabla (\theta - \varphi) + \nu \nabla^2 v. \quad (12)$$

where $f(t) = \frac{d\ln b}{d\ln a}$, $\Omega_0 = 8\pi G \rho_0 / 3H_0^2$ and we have taken the velocity field as the gradient of a velocity potential ($v = \nabla \theta$) as long as the motion is linear and streams of particles do not cross.

Assuming that over the mildly nonlinear regime the gravitational potential is approximately equal to the velocity potential, we finally get

$$\frac{\partial v}{\partial b} + (v \cdot \nabla)v = \nu \nabla^2 v \quad (13)$$

which is known as the adhesion approximation. For small $\nu$ it reduces to the Zeldovich model which corresponds to a simple inertial motion. The viscosity term in the right side of (13) is introduced just to prevent the inertial broadening of the first nonlinear structures – the Zeldovich pancakes. Hence, it is not a real viscosity, in the sense of a significant velocity gradient in the fluid, but a term which forces particles to stick together.

Despite the use of a mock viscosity, the adhesion model shows a remarkable agreement with the large-scale structure produced in N-body simulations. Thus, any simple improvement to this model does not need to change its mathematical structure very much, but just find a less artificial way to introduce the term in the right side of (13). Particularly, in this work, we want to find a physically motivated ‘viscosity’ term presenting a dependence on the internal structure of the pancakes.

### III. WEAKLY NONLOCAL HYDRODYNAMICS

The correct framework to study continuum physics dealing with internal structures is the weakly nonlocal (coarse grained) hydrodynamics. In this context, we can use the Ginzburg-Landau equation as a first weakly nonlocal extension of a homogeneous relaxation equation of an internal variable. The Ginzburg-Landau equation is a modulational equation for a complex function, $\psi$, which describes weakly nonlinear phenomena in continuous media with linear dispersion of general type. It is given by

$$\frac{\partial \psi}{\partial t} = \psi + (1 + ic)\nabla^2 \psi - (1 + id)|\psi|^2 \psi \quad (14)$$

where the parameters $c$ and $d$ are the linear and nonlinear dispersions. Some important properties of this equation deserve special attention.

i) It can be reduced to a dissipative extension of the nonlinear Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = \nabla^2 \psi \pm V(\psi)\psi. \quad (15)$$

ii) It admits amplitude-phase representation in the form

$$\psi = R \exp i\zeta. \quad (16)$$

iii) It can be derived from a variational principle. In particular, weakly nonlocal fluids respecting the second law of thermodynamics depends on the nonlocal potential given by

$$U = \delta \int \rho s(\rho, \nabla \rho) \, dV \quad (17)$$

where $s$ is the entropy density.
of stationary solutions, is the Schrödinger-Madelung potential energy \[ U = -\frac{\hbar^2}{2m} \nabla^2 R \] (18)

IV. NONLOCAL ADHESION MODEL

The wave mechanics analog of classical hydrodynamics is a well known subject in the literature (see e.g. \[ 11 \], \[ 12 \]), but the use of this equivalence as an approach to study structure formation in the universe is relatively new (e.g. \[ 13 \], \[ 14 \]). The basic idea is very simple: to describe collisionless matter as a classical field \( \psi(x, t) \) obeying the Schrödinger and Poisson equations

\[ i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V \right) \psi, \quad \nabla^2 \phi = 4\pi G \psi^* \psi; \quad (19) \]

where \( V = m\phi \) and \( \hbar \) is a parameter controlling the spatial resolution \( \lambda \), according to the de Broglie relation \( \lambda = \hbar/mv \). This approach has the advantage of working only with three spatial coordinates (plus time) and following multiple streams in the phase space. Promising results of this technique were found for the collapse of a self-gravitating object and to simulate a two-dimensional Cold Dark Matter universe \[ 15 \].

Here, we make the wave analog of a nonlocal fluid under a smooth gravitational field plus a coarse grained field by introducing the amplitude-phase ansatz \( \psi = R \exp \left( i S \right) \) into the Schrödinger equation (19). Equating real and imaginary parts separately, we find two coupled equations for the classical action \( S \) and the real amplitude \( R \):

\[ \frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V + U = 0 \quad (20) \]

\[ \frac{\partial R}{\partial t} = -\frac{1}{m} \nabla R \cdot \nabla S - \frac{1}{2m} R \nabla^2 S, \quad (21) \]

where \( U \) is the potential previously defined in (18). Note that equation (21) can be rewritten as

\[ \frac{\partial R^2}{\partial t} = -\frac{1}{m} \nabla (R^2 \nabla S). \quad (22) \]

But \( p = \nabla S \) in the Hamilton-Jacobi canonical transformation. It then follows that \( u = \nabla S/m \) and (22) takes the form of the hydrodynamical continuity equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0. \quad (23) \]

We can therefore attribute to the square of the amplitude of the wave function the interpretation of a fluid density whose flux is conserved over time: \( \rho = \psi^* \psi = |\psi|^2 \).

Now, note that Equation (20) can be rewritten as

\[ m \frac{\partial u}{\partial t} + m (u \cdot \nabla) u + \nabla (V + U) = 0 \quad (25) \]

or

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla (\phi + \phi_{vis}) = 0 \quad (26) \]

where the quantity

\[ \phi_{vis} = \frac{U}{m} = -\frac{\nu^2}{2} \left[ \frac{\nabla^2 \rho}{\rho} - \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 \right] \quad (27) \]

is the “viscosity” (or coarse grained) potential. Note that \( \hbar/m \) has the dimensions of a kinematical viscosity, such that we have defined \( \nu = \hbar/m \).

Using the same variable transformation we made in Section II for an expanding background, we finally come to

\[ \frac{\partial \nu}{\partial b} + (\nu \cdot \nabla) \nu = \frac{3}{2b} \nabla (\theta - \varphi - \phi_{vis}) \quad (28) \]

Again, for the weakly nonlinear regime, we have \( \varphi \approx \theta \), and then

\[ \frac{\partial \nu}{\partial b} + (\nu \cdot \nabla) \nu = -\frac{3}{2b} \nabla \phi_{vis} \quad (29) \]

which is the weakly nonlocal adhesion approximation. It is important to note that the origin of viscosity should be related to the divergence of some “shear” tensor. Indeed, if we define

\[ \sigma_{ij}^{kin} \equiv \frac{\nu^2}{4} \left[ \frac{\nabla_i \nabla_j \rho}{\rho} - \frac{(\nabla_i \rho)(\nabla_j \rho)}{\rho} \right] = \frac{\nu^2}{4} \nabla_i \nabla_j \ln \rho \quad (30) \]

we have that the divergence of this tensor is equal to the density force associated with \( \phi_{vis} \)

\[ \nabla \cdot \sigma = -\nabla \phi_{vis}. \quad (31) \]

Now, defining a “kinematical velocity” as

\[ u_{kin} \equiv \frac{\nu}{2} \nabla \ln \rho, \quad (32) \]
the symmetric part of the velocity gradient \((\nabla \mathbf{u}_{\text{kin}})_{ij}\) is
\[
\epsilon^{\text{kin}}_{ij} = \frac{\nu}{\tau} \left( \frac{\partial (u_{\text{kin}})_i}{\partial r_j} + \frac{\partial (u_{\text{kin}})_j}{\partial r_i} \right) \quad (33)
\]
and
\[
\sigma^{\text{kin}}_{ij} = \rho \nu \left( \frac{\partial (u_{\text{kin}})_i}{\partial r_j} + \frac{\partial (u_{\text{kin}})_j}{\partial r_i} \right) \quad (34)
\]
thus showing the connection between coarse grained viscosity and nonlocal phenomena through the kinematical shear tensor. The kinematical velocity was first introduced by [11] and it is not related to the organized mechanical motion, but to the perturbations due to the coarse grained nature of the fluid. From (34), we have
\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \nu \nabla^2 \mathbf{v}, \quad (35)
\]
The kinematical velocity introduces a deviation with respect to the field \(\mathbf{v}\). If we define \(\Delta \mathbf{v}_{\text{kin}} \equiv \mathbf{v} - \mathbf{v}_{\text{kin}}\), we find the general expression
\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \nu \nabla^2 \mathbf{v} + \nabla \eta \quad (36)
\]
where \(\nabla \eta \equiv -\nu \nabla^2 \langle \Delta \mathbf{v}_{\text{kin}} \rangle\) is a noise term. Hence, the weakly nonlocal adhesion model corresponds to the noisy Burgers equation. Note that we recover the Zeldovich approximation if \(\mathbf{v}_{\text{kin}} \rightarrow 0\), which points out the importance of internal structures to prevent inertial broadening.

V. DISCUSSION

The wave-mechanical analog of hydrodynamics seems to be a promising framework to find alternative solutions to both analytical and numerical problems over the nonlinear regime of structure formation. In this work, we have investigated the origin of the viscosity term in the adhesion model. Viscosity is the property which makes particles stick together once they enter into the caustics predicted by the Zeldovich approximation, just mimicking gravitational effects on smaller scales. This approximation describes the general nonlinear structures but does not trace the internal dynamics of the pancakes. In this work, we assume the nonlocal hydrodynamics description in the amplitude-phase representation to introduce a physically motivated “viscosity” term into the adhesion model. Using the Schrödinger-Madelung potential, we come to a new equation for the adhesion model as a function of a viscosity potential, which can be written as the divergence of a new shear tensor.

Our model identifies the origin of viscosity as a nonlocal effect. Notice that the fluid equations are only valid when the average of the microscopic linear momentum equals the product of the averages of density and velocity. However, due to the mesoscopic nature of the averaging procedure, this is not guaranteed. Generally, local fluctuations in the fluid are small enough to fade away over averaging. But nonlocal (configurational) noise can add on the hydrodynamics noise, leading to non-selfaveraging properties in the matter flows. Mesoscopic arrangements in the system are sources of fluctuations, since the averaging procedure in the free volumes around them can lead to different results. This effect is a source of nonlocal noise [11]. In the case of pancakes, the internal arrangements could be the spatial configuration of protogalactic gas clouds developed over the path followed by matter (inside pancakes) to form filaments and knots. These intermediate structures can be considered as sources of the noisy mesoscopic fluctuations.

Applications of (36) to the nonlinear regime and comparisons to N-body simulations are beyond the scope of this paper and will be presented in a future contribution. For the moment, we conclude that the origin of viscosity in the adhesion model is probably associated with weakly nonlocal effects in the internal structure of the pancakes. These effects are given by some viscosity potential. In the case of significant mesoscopic fluctuations, we assume that the potential has a stochastic nature and the dynamics is given by the noisy Burgers equation.

Acknowledgments

This work has the financial support of CNPq (grants 470185/2003-1) and UESC (grants 220.1300.357 and 220.1300.324).

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