Pseudo Polynomial-Time Top-$k$ Algorithms for d-DNNF Circuits

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Abstract

We are interested in computing $k$ most preferred models of a given d-DNNF circuit $C$, where the preference relation is based on an algebraic structure called a monotone, totally ordered, semigroup $(K, \otimes, <)$. In our setting, every literal in $C$ has a value in $K$ and the value of an assignment is an element of $K$ obtained by aggregating using $\otimes$ the values of the corresponding literals. We present an algorithm that computes $k$ models of $C$ among those having the largest values w.r.t. $<$, and show that this algorithm runs in time polynomial in $k$ and in the size of $C$. We also present a pseudo polynomial-time algorithm for deriving the top-$k$ values that can be reached, provided that an additional (but not very demanding) requirement on the semigroup is satisfied. Under the same assumption, we present a pseudo polynomial-time algorithm that transforms $C$ into a d-DNNF circuit $C'$ satisfied exactly by the models of $C$ having a value among the top-$k$ ones. Finally, focusing on the semigroup $(\mathbb{N}, +, <)$, we compare on a large number of instances the performances of our compilation-based algorithm for computing $k$ top solutions with those of an algorithm tackling the same problem, but based on a partial weighted MaxSAT solver.

1 Introduction

In this paper, we are interested in optimization problems under compiled constraints. Roughly, the goal is to derive most preferred solutions among the feasible ones, where the set of feasible solutions is of combinatorial nature and represented implicitly as valid assignments, i.e., those truth assignments satisfying some given constraints. Such optimization questions are key issues in a number of applications about configuration, recommendation, and e-commerce (see e.g., (Chen and Feng 2018; Jannach et al. 2021; Ricci, Rokach, and Shapira 2015; Khabbaz and Lakshmanan 2011)).

Unlike the preference relation at hand that is user-specific, the set of constraints representing the valid assignments is typically independent of the user, so that it does not often change. In such a case, taking advantage of a knowledge compilation approach can be useful, since compiling the constraints during an offline phase may allow polynomial-time algorithms for optimization tasks, whilst getting a single optimal solution already is NP-hard when no assumptions are made on the representations of the constraints. The discrepancy between the two approaches is amplified when computing multiple best solutions. Indeed, in practice, the computation of several best solutions for a set of uncompiled constraints generally requires successive calls to a NP oracle (see e.g., (Jabbour, Sais, and Salhi 2013)), while one can expect polynomial-time algorithms for this task too when the constraints have been compiled first.

When dealing with propositional constraints, the language of deterministic, decomposable Negation Normal Form circuits (d-DNNF) (Darwiche 2001a) appears as a valuable language for compiling constraints because it supports in polynomial time a number of queries and transformations that are NP-hard in general (Darwiche and Marquis 2002). Among them are queries and transformations about optimization. Thus, (Darwiche and Marquis 2004) has shown how to derive in polynomial time a most preferred, yet feasible solution where the set of feasible solutions is the set of models of a given d-DNNF circuit and the value of a solution is the sum of the weights (numbers) associated with the literals it satisfies. The authors have also presented a polynomial-time transformation that returns a d-DNNF circuit whose models are precisely the preferred, feasible solutions of the d-DNNF circuit one started with.

Such an approach has been extended to a much more general, algebraic model counting setting in (Kimmig, Van den Broeck, and De Raedt 2017). The extension that has been achieved is threefold: first, the value of a solution is not necessarily a number, but an element of an abstract set, the carrier $K$ of an algebraic structure called a commutative semiring; then, the aggregation operator used to define the value of a solution is not restricted to summation, but can be any abstract binary operator $\otimes$ over $K$; finally, the authors take advantage of an additional aggregation operator $\oplus$, which is not necessarily equal to max or min. The goal is to compute the algebraic model count of a given d-DNNF circuit $C$, defined as the aggregation using $\oplus$ of the values of all models of $C$, where the value of a solution is the aggregation using $\otimes$ of the values of the literals satisfied by the solution (such values are elements of $K$). Algebraic model counting generalizes a number of problems of interest, including satisfiability (SAT), (possibly weighted) model counting ($\#$SAT-WMC), and probabilistic inference (PROB) (see Theorem 1 in (Kimmig, Van den Broeck, and De Raedt 2017)).

Following (Kimmig, Van den Broeck, and De Raedt
we extend the approach to optimization under d-DNNF constraints considered in (Darwiche and Marquis 2004) but the generalization made here relies on a different perspective, as reflected by the queries and transformation we focus on. Whilst (Darwiche and Marquis 2004) aims to compute a single, most preferred solution (and the corresponding value), we are interested in computing $k$ most preferred models of a given d-DNNF circuit $C$, where $k$ is a preset bound given by the user. The returned assignments must be valid and their values must be among the largest possible ones, i.e., for any top-$k$ assignment $\omega$, there cannot exist $k$ (or more) valid assignments having a value strictly greater than the one of $\omega$.

Considering top-$k$ solutions is important to handle situations when the user is not satisfied by the top-1 solution that is provided (maybe he/she would finally prefer another solution reaching the top value, or even a solution with value slightly smaller than the value of a top-1 solution). We are also interested in computing the $k$ most preferred values, thus extending the issue of computing a top-1 solution and the top-1 value as considered in (Darwiche and Marquis 2004) to the computation of top-$k$ solutions and top-$k$ values. Finally, we investigate the corresponding transformation problem.

As (Kimmig, Van den Broeck, and De Raedt 2017), we consider a more general algebraic setting than the one in (Darwiche and Marquis 2004) where, implicitly, $\odot$ is the summation operator and $K$ is the set of real numbers. We focus here on an algebraic structure called a monotone, totally ordered, semigroup $(K, \odot, <)$. Notwithstanding the $\odot$ operator (which is implicitly $\max$ in our case) the structure used is less demanding than commutative semirings; especially, in the general case, the existence of a neutral element for $\odot = \max$ (i.e., a least element in $K$ w.r.t. $<$) that is an annihilator for $\odot$ is not required. Given a mapping $\nu$ associating with every literal $\ell$ of $C$ an element of $K$, the value $\nu(\omega)$ of an assignment $\omega$ is defined as $\nu(\omega) = \odot_{\ell \in \text{Var}(C)} \nu(\ell)$. On this ground, a top-$k$ value of $C$ given $(K, \odot, <)$ and $\nu$ is one of the $k$-largest values $v$ of $K$ w.r.t. $<$ such that $v = \nu(\omega)$ is the value of a valid assignment $\omega$ of $C$. A top-$k$ solution of $C$ given $(K, \odot, <)$ and $\nu$ is a model $\omega$ of $C$ such that there is strictly less than $k$ valid assignments of $C$ having a value strictly greater w.r.t. $<$ than $\nu(\omega)$.

Our contribution is as follows. We first present an algorithm that computes top-$k$ solutions of $C$ given $(K, \odot, <)$ and $\nu$ and that runs in time polynomial in $k$ and in the size of the d-DNNF circuit $C$. We also present a pseudo-polynomial-time algorithm for deriving the top-$k$ values of $C$ given $(K, \odot, <)$ and $\nu$, provided that an additional (but not very demanding) requirement on the semigroup $(K, \odot, <)$, namely almost strict monotony, is satisfied. Under the same assumption, we present a pseudo-polynomial-time algorithm that transforms $C$ into a d-DNNF circuit $C'$ satisfied exactly by the models of $C$ having a value among the top-$k$ values of $C$ given $(K, \odot, <)$ and $\nu$. Whenever $k$ is small enough so that it can be considered as bounded by a constant (which is a reasonable assumption in practice), each of our top-$k$ algorithms runs in time linear in the size of the d-DNNF circuit $C$. Finally, focusing on the semigroup $(\mathbb{N}, +, <)$, we present the results of an empirical comparison of our compilation-based algorithm for computing top-$k$ solutions with an algorithm tackling the same problem, but based on the partial weighted MaxSAT solver MaxHS (Davies and Bacchus 2011; Davies 2013; Davies and Bacchus 2013a; 2013b). The obtained results show that in practice taking advantage of the compilation-based algorithm makes sense for many instances.

The rest of the paper is organized as follows. We give some preliminaries in Section 2. Then we present our algorithm for computing $k$ top solutions of a d-DNNF circuit in Section 3 and our algorithm for computing its top-$k$ values in Section 4. Our algorithm for achieving the top-$k$ transformation of a d-DNNF circuit is presented in Section 5. The results of the empirical evaluation are given in Section 6. Possible extensions of our results are briefly discussed in Section 7. Finally, Section 8 concludes the paper. The code of our algorithms can be found at https://gitlab.inria.fr/jdusart/knowledge-compilation-xp.

## 2 Preliminaries

Let $X$ be a set of propositional variables. The set of literals over $X$ is the union of $X$ with the set of negated variables over $X$, and it and denoted by $\text{Lit}(X)$. An assignment $\omega$ is a mapping from $X$ to $\{1, 0\}$. A Boolean function $f$ over $X$ is a mapping from the assignments over $X$ to $\{1, 0\}$. An assignment $\omega$ such that $f(\omega) = 1$ is called a valid assignment, or a model of $f$. The set of valid assignments for $f$ is noted $\text{ValidA}(f)$.

**d-DNNF circuits.** Circuits are convenient representations of Boolean functions. A deterministic, Decomposable Negation Normal Form (d-DNNF) circuit (Darwiche 2001a) is a directed acyclic graph (DAG) where internal nodes are labelled by connectives in $\{\land, \lor\}$ and leaves are labelled by literals from $\text{Lit}(X)$ or Boolean constants. The two main properties of d-DNNF circuits are that the sets of variables appearing in the subcircuits of any $\land$ node are pairwise disjoint (decomposability) and the valid assignments of the subcircuits of any $\lor$ node are pairwise disjoint (determinism). Figure 1a gives an example of a d-DNNF circuit.

The language of d-DNNF circuits includes several interesting languages as subsets, namely the set of Decision-DNNF circuits (Oztok and Darwiche 2014) (where every $\lor$-node is a decision node), the set of FBDD of free binary decision diagrams (Gervov and Meinel 1994), and its subset OBDD, the set of ordered binary decision diagrams (Bryant 1986). Interestingly, Decision-DNNF is more succinct than FBDD and OBDD (Razgon 2016). Furthermore, the language of d-DNNF circuits (and its subsets listed above) supports in polynomial time many queries that are intractable (NP-hard) when no restriction is put on the circuit (see (Darwiche and Marquis 2002; 2004) for details).

**Top-$k$ problems.** In order to present in formal terms the three main top-$k$ computation problems over d-DNNF circuits $C$ we are interested in, we first need to make precise
the algebraic structure over which the values of the satisfying assignments of $C$ are evaluated:

**Definition 1** A monotone, totally ordered semigroup is a triple $(K, \otimes, \prec)$ where $K$ is a set that is totally ordered by $\prec$ (a strict, total ordering), $\otimes$ is a binary operator over $K$ that is commutative, associative, and monotone, i.e., for any $p, q, r, s \in K$, if $p \leq q$ and $r \leq s$ then $p \otimes r \leq q \otimes s$ (where $x \leq y$ iff $x < y$ or $x = y$). The semigroup is strictly monotone if it is monotone and for any $p, q, r, s \in K$, if $p \leq q$ and $r < s$ then $p \otimes r < q \otimes s$. $(K, \otimes, \prec)$ is said to have a least absorptive element $a$ whenever $a$ is the least element of $K$ w.r.t. $\prec$ and $a$ is absorptive for $\otimes$, i.e., $\forall x \in K, x \otimes a = a \otimes x = a$. $(K, \otimes, \prec)$ is said to be almost strictly monotone if either $(K, \otimes, \prec)$ is strictly monotone or $K$ has a least absorptive element $a$ and $(K \setminus \{a\}, \otimes, \prec)$ is strictly monotone.

Clearly enough, whenever $(K, \otimes, \prec)$ has a least absorptive element $a$, $a$ is neutral for $\oplus = \max$. Furthermore, when $\otimes$ is monotone, it distributes over $\oplus = \max$ (which is obviously commutative). Thus, in this case, provided that $\otimes$ has a neutral element $n$, $(K, \max, \otimes, a, n)$ is a commutative semiring. However, the existence of such a neutral element $n$ is not mandatory in our setting.

Here are two examples. $([\mathbb{R}, +, <])$ and $([0, 1], \times, <)$ are monotone, totally ordered semigroups. In $([\mathbb{R}, +, <])$, the elements of $\mathbb{R}$ may denote utilities and in $([0, 1], \times, <)$, the elements of $[0, 1]$ may denote probabilities. It is easy to check that $([\mathbb{R}, +, <])$ is strictly monotone (just like its restriction $([\mathbb{N}, +, <])$), which implies that it does not have a least absorptive element, and that $([0, 1], \times, <)$ has a least absorptive element (namely, 0) and it is almost strictly monotone.

Provided a monotone, totally ordered semigroup $(K, \otimes, \prec)$, evaluating an assignment $\omega$ over $X$ requires to indicate how the literals from $\text{Lit}(X)$ are interpreted in $K$. This calls for a notion of value function over $X$ onto $K$:

**Definition 2** Given a $X$ be a set of propositional variables and a monotone, totally ordered semigroup $(K, \otimes, \prec)$, a value function $\nu$ over $X$ onto $K$ is a mapping from the literals over $X$ to $K$, assigning to each literal $\ell$ an element from $K$ noted $\nu(\ell)$ and called the value of $\ell$.

When $\otimes$ is a binary operator over $K$, the value of a literal $\ell$ as given by a value function $\nu$ over $X$ onto $K$ can then be extended to the value of an assignment $\omega$ over $X$, defined as the $\otimes$-aggregation of the values of the literals (as given by $\nu$) satisfied by $\omega$ (the order with which they are taken does not matter as soon as $\otimes$ is commutative and associative). This value is noted $\nu(\omega)$. We denote by $\text{ValidV}(C)$ the subset of values from $K$ that are reached by the valid assignments of $C$. Formally, $\text{ValidV}(C) = \{ \nu(\omega) \mid \omega \in \text{ValidA}(C) \}$.

We can now define the three top-$k$ problems of $C$ given $(K, \otimes, \prec)$ and $\nu$ we focus on:

**Definition 3** Let $(K, \otimes, \prec)$ be a monotone, totally ordered semigroup. Let $X$ be a set of variables. Let $\nu$ be a value function over $X$ onto $K$. Let $C$ be a Boolean circuit over $X$. The problem $\text{TopVal}_k(\nu, C)$ consists in computing the set of the $k$ largest values w.r.t. $\prec$ in $\text{ValidV}(C)$. When $\text{ValidV}(C)$ contains less than $k$ elements, the set is defined as $\text{ValidV}(C)$.

Note the set of top-$k$ values of $C$ given $(K, \otimes, \prec)$ and $\nu$ is unique since $K$ is totally ordered by $\prec$.

To define the problem of generating top-$k$ solutions of $C$ given $(K, \otimes, \prec)$, one first lifts the notion of value of a model $\omega$ of $C$ to the notion of value of a set of models of $C$, defined as follows: the value of a set $S = \{\omega_1, \ldots, \omega_m\}$ of models of $C$ is the list of values $(\nu(\omega_{\pi(1)}), \ldots, \nu(\omega_{\pi(m)}))$ where $\pi$ is a permutation over $\{1, \ldots, m\}$ such that $\nu(\omega_{\pi(1)}) \geq \ldots \geq \nu(\omega_{\pi(m)})$. The values of such sets $S$ can then be compared w.r.t. the lexicographic ordering $\succ$ induced by $\succ$.

**Definition 4** Let $(K, \otimes, \prec)$ be a monotone, totally ordered semigroup. Let $X$ be a set of variables. Let $\nu$ be a value function over $X$ onto $K$. Let $C$ be a Boolean circuit over $X$. A set $S$ of $k$ models of $C$ is a set of top-$k$ models of $C$ if and only if its value is the maximal value w.r.t. $\succ$ reached by sets of $k$ models of $C$. As an exception, when $\text{ValidA}(C)$ has less than $k$ elements, the (unique) set of top-$k$ models of $C$ is defined as $\text{ValidA}(C)$. Finally, the problem $\text{TopSol}_k(\nu, C)$ consists in computing a set of top-$k$ models of $C$.

Unlike the set of top-$k$ values, the set of top-$k$ models of $C$ is not unique in general (for instance, it may exist strictly
more than \(k\) models \(\omega\) of \(C\) having a maximal value \(\nu(\omega)\).

**Definition 5** Let \((K, \otimes, \prec)\) be a monotone, totally ordered semigroup. Let \(X\) be a set of variables. Let \(\nu\) be a value function over \(X\) onto \(K\). Let \(C\) be a Boolean circuit over \(X\) from a circuit language \(L\). The problem \(\text{TopTra}_k(\nu, C)\) consists in computing from \(C\) a circuit \(C'\) in the same language \(L\) as \(C\) and whose models are precisely those of ValidA\((C)\) having a value in \(\text{TopVal}_k(\nu, C)\).

Obviously enough, such a circuit \(C'\) is not unique in general. Let us now illustrate on a simple example the top-\(k\) problems we consider.

**Example 1** Consider the \(d\)-DNNF circuit \(C\) over \(X = \{b, c, h, s\}\) given at Figure 1a. \(C\) encodes an E-Shop security system where at least one payment between bank transfer (b) and credit card (c) is required, exactly one security policy between high (h) and standard (s) must be chosen, with the constraint that \(c\) implies \(h\). Suppose that \(\nu\) gives value 2 to literal \(b\) and to literal \(c\), value 1 to literal \(h\), and value 0 to every other literal and that \((N, +, \prec)\) is the monotone, totally ordered semigroup under consideration.

\(C\) has 4 models. The models of \(C\) are reported in the following table and for each of them, we indicate its value according to \(\nu\).

| \((b, c, h, s)\) \(\in\) ValidA\((C)\) | \(\nu(b, c, h, s)\) |
|---------------------------------|-----------------|
| \((0, 1, 1, 0)\)                | 3               |
| \((1, 0, 0, 1)\)                | 2               |
| \((1, 0, 1, 0)\)                | 3               |
| \((1, 1, 1, 0)\)                | 5               |

For this example, the set of top-2 values is \(\{5, 3\}\). There are two possible sets of top-2 solutions, namely \(\{(1, 1, 1, 0), (0, 1, 1, 0)\}\) and \(\{(1, 1, 1, 0), (1, 0, 1, 0)\}\). Any circuit in the \(d\)-DNNF language over \(X = \{b, c, h, s\}\) having as models \(\{(1, 1, 1, 0), (0, 1, 1, 0), (1, 0, 1, 0)\}\) is an admissible result for the top-2 transformation of \(C\).

### 3 Computing Top-\(k\) Solutions

For keeping the presentation simple enough, we assume in the following that the \(d\)-DNNF circuits that are considered as inputs satisfy a few assumptions. First, we suppose that every internal node in such a circuit (whatever it is a \(\land\) node or a \(\lor\) node) is binary (it has two children). We also assume that the circuit is smooth, which means that the sets of variables associated with the two children of any \(\lor\) node are the same one. We finally assume that those \(d\)-DNNF circuits are reduced, in the sense that no leaf node labelled by a Boolean constant occurs in the circuit, unless the circuit is such a leaf (in which case the optimization tasks trivialize). Those three assumptions are computationally harmless: any \(d\)-DNNF circuit can be binarized in linear time (every internal \(N\) node having \(m > 2\) children \(N_1, \ldots, N_m\) can be replaced by a binary tree with \(m - 1\) internal nodes of the same type as \(N\) and \(N_1, \ldots, N_m\) as children; and every internal node with a single child can be replaced by its child), smoothed in quadratic time (Darwiche 2001b) (and even more efficiently for structured \(d\)-DNNF (Shih et al. 2019)), and reduced in linear time (just applying the elementary rules of Boolean calculus). Another reasonable assumption concerns the representation of the values in \(K\) and the costs of computing \(\otimes\) and of comparing elements of \(K\) using \(\prec\): one supposes that the size of the representations of the values in \(K\) are bounded by a constant and that \(\otimes\) and \(\prec\) are constant-time operations.

We have obtained the following result:

**Theorem 1** Let \((K, \otimes, \prec)\) be a monotone, totally ordered semigroup. Let \(X\) be a set of variables. Let \(\nu\) be a value function over \(X\) onto \(K\). Let \(C\) be a \(d\)-DNNF circuit over \(X\). The problem \(\text{TopSol}_k(\nu, C)\) can be solved in time \(O(|C| \cdot k \cdot \log k)\).

**Algorithm 1:** \(\text{TopSol}_k(\nu, N)\).

**Input:** \(N\) : a node in a \(d\)-DNNF circuit, \(k\) : a positive integer, \(\nu\) : the value function

**Result:** a list of top-\(k\) solutions of the \(d\)-DNNF circuit rooted at \(N\)

if \(N\) is a leaf node labelled by literal \(\ell\)
| return \([\ell]\];
end

\(\text{top}_c0 = \text{TopSol}_k(\nu, N.\text{children}(0));\)

\(\text{top}_c1 = \text{TopSol}_k(\nu, N.\text{children}(1));\)

if \(N\) is a \(\lor\) node
| return sorted fusion(\(\text{top}_c0\), \(\text{top}_c1\), \(k\), \(\nu\));
else if \(N\) is a \(\land\) node
| return sorted product(\(\text{top}_c0\), \(\text{top}_c1\), \(k\), \(\nu\));
end

**An algorithm for the top-\(k\) solutions problem.** As a constructive proof of this theorem, we present Algorithm 1 that solves \(\text{TopSol}_k(\nu, C)\) within the expected amount of time. This algorithm computes in a bottom-up fashion the values of two synthesized attributes (representing top-\(k\) solutions and their values) for the \(d\)-DNNF circuits rooted at the nodes of the input circuit \(C\). When \(N\) is an internal node of \(C\), \(\text{top}_c0\) (resp. \(\text{top}_c1\)) denotes the list of top-\(k\) solutions (and the corresponding values for \(\nu\)) that has been computed for the \(d\)-DNNF circuit rooted at the left (resp. right) child of \(N\). Depending on the label of \(N\) (\(\land\) or \(\lor\)), a different procedure is run to derive the values of the two attributes at \(N\): \(\text{sorted fusion}(\text{top}_c0, \text{top}_c1, \ell, \nu)\) when \(N\) is a \(\lor\) node and \(\text{sorted product}(\text{top}_c0, \text{top}_c1, \ell, \nu)\) when \(N\) is a \(\land\) node. When called at a \(\lor\) node, \(\text{sorted fusion}\) returns the ordered list of top-\(k\) solutions extracted from the sorted union of \(\text{top}_c0\) and \(\text{top}_c1\). When called at a \(\land\) node, \(\text{sorted product}\) returns an ordered list of top-\(k\) solutions generated from the cross product of \(\text{top}_c0\) and \(\text{top}_c1\).

**Correctness and complexity.** In order to prove the correctness of our bottom-up algorithms for computing \(k\) top solutions of a \(d\)-DNNF circuit \(C\), one must show that at any internal node \(N\) of \(C\) computing \(k\) top solutions of the \(d\)-DNNF circuit \(C_N\) rooted at \(N\) can be achieved when \(k\) top
solutions of the \(d\)-DNNF circuits \(C_{N_0}\) and \(C_{N_1}\), rooted respectively at the children \(N_0\) and \(N_1\) of \(N\), have been computed first. Note that each of \(C_{N_0}\) and \(C_{N_1}\) has at least one model since \(C\) is supposed to be simplified.

The expected result is rather obvious when \(N\) is a \(\lor\) node: in this case, by definition, the set of valid assignments of the circuit rooted at \(N\) is the union of the sets of valid assignments of the circuits rooted at \(N_0\) and at \(N_1\). Thus, to derive \(k\) top solutions of the \(d\)-DNNF circuit rooted at \(N\), it is enough to make the union of the sets of \(k\) top solutions associated with its two children, to sort them in decreasing order w.r.t. their values, and to keep the first \(k\) elements of the sorted list of solutions.

Things are a bit more tricky when considering \(\land\) nodes \(N\). In that case, the monotony assumption about \(\otimes\) is useless.

Towards a contradiction, suppose that the set \(\{\omega \cdot \omega' : \omega \in \text{TopSol}_k(\nu, C_{N_0}), \omega \in \text{TopSol}_k(\nu, C_{N_1})\}\) does not include \(k\) top solutions of \(C_N\). Thus, there exist \(\omega = C_{N_0}\) and \(\omega' = C_{N_1}\) such that \(\omega \not\in \text{TopSol}_k(\nu, C_{N_0}) = \{\omega_1, \ldots, \omega_k\}\) and for each \(i \in [k]\), \(\nu(\omega \cdot \omega') > \nu(\omega_i \cdot \omega')\), where \(\omega'_i \in \text{TopSol}_k(\nu, C_{N_1})\). In such a case, since \(\nu(\omega') \leq \nu(\omega'_1)\) and \(\nu(\omega) \leq \nu(\omega_i)\) (\(i \in [k]\)), the monotony of the semigroup ensures that for each \(i \in [k]\), \(\nu(\omega) \otimes \nu(\omega') \leq \nu(\omega_i) \otimes \nu(\omega'_i)\), or equivalently \(\nu(\omega \cdot \omega') \leq \nu(\omega_i \cdot \omega'_1)\), a contradiction.

As to simplicity, let us first consider a simple, yet naive implementation of the two functions \text{sorted fusion} and \text{sorted product}. Function \text{sorted fusion}(top\_c0, top\_c1, \(k, \nu\)) computes the sorted union of two disjoint lists of size \(k\). It is well-known that this can be done in time \(O(k \cdot \log k)\).

Function \text{sorted product}(top\_c0, top\_c1, \(k, \nu\)) can be implemented by computing explicitly the cross product of the two lists of \(k\) solutions, ordering them and picking up top-\(k\) elements. Through this implementation, the algorithm runs in time \(O(|C| \cdot k^2 \cdot \log k \cdot |X|)\).

However, a better implementation can be obtained by avoiding to sort at each \(\land\) node \(N\) all the assignments resulting from the cross product of the two lists, \(\text{top\_c0}\) and \(\text{top\_c1}\). Our algorithm takes advantage of a max-heap implementation of a priority queue \(Q\), i.e., a data structure allowing to add elements to \(Q\) and remove elements from \(Q\) in time logarithmic in the size of \(Q\), and also to retrieve an element of maximal value from \(Q\) in constant time. In our implementation of \text{sorted product}, one stores in \(Q\) pairs of indices \((i, j)\) together with the corresponding value for \(\nu\) denoted by \(\nu(i, j)\). Each pair \((i, j)\) represents the concatenation of the \(i\)th assignment of \text{top\_c0} with the \(j\)th assignment of \text{top\_c1}. Because \((K, \otimes, <)\) is monotone, the first pair \((0, 0)\) of \(Q\) is one of top value. Then the following treatment is iterated for \(k - 1\) steps. At each step, the first pair stored in \(Q\), \((i, j)\) is retrieved, then deleted from \(Q\), and the corresponding assignment is added to the list of top-\(k\) solutions under construction for node \(N\). Then the pairs \((i, j + 1)\) and \((i + 1, 0)\) are added to the queue if they were not added yet. We use additional structures to check these in a decent time (this test can be done in constant time using a hashmap or in time \(\log k\) using B+ trees). By construction, after \(k\) steps, the list at node \(N\) contains \(k\) top assignments of the cross product between \text{top\_c0} and \text{top\_c1}. Since at each step, one element is removed from \(Q\) and at most two elements are added, the size of \(Q\) increases linearly in the number of steps and therefore the size of \(Q\) remains linear in \(k\). Therefore, the time complexity of computing \(k\) top assignments for a \(\land\) node \(N\) is in \(O(k \cdot \log k \cdot |X|)\).

The \(|X|\) factor in the complexity evaluation comes from the computational cost of concatenating the two assignments represented in a naive manner. It is possible to remove this factor through a more efficient representation of assignments. In our implementation, sets of literals are represented as binary trees where leaves are labelled by literals. Then, the concatenation of partial assignments at \(\land\) nodes can be achieved in constant time by taking the roots of the two sets and creating a new root having them as children. The decomposability of \(\land\) nodes ensures the correctness of the approach (i.e., the resulting tree is guaranteed to correspond to a partial assignment). At \(\lor\) nodes, where unions of sets of partial assignments must be done, there is no need for equality tests to avoid duplicates: by construction, the determinism of \(\lor\) nodes ensures that those unions are disjoint ones. As a consequence, the construction of solutions can be done efficiently through the tree representation of assignments and the multiplicative factor \(|X|\) can be removed from the time complexity of our algorithm.

Finally, the time used by \text{sorted fusion}(\text{top\_c0}, \text{top\_c1}, \(k, \nu\)) can be improved given that \text{top\_c0} and \text{top\_c1} are sorted. It is well-known that sorting two sorted lists can be done in time linear in the sum of the sizes of the lists. This has no impact on Theorem 1 but in practice, it leads to significant time savings.

Example 2 Let us consider Example 1 again. The nodes \(N\) of the \(DAG\) reported in Figure 1b correspond in a bijective way with those of the \(d\)-DNNF circuit in Figure 1a (the nodes and the arcs of the two \(DAGs\) are the same ones, only the labels change). The label of each node \(N\) of the \(DAG\) at Figure 1b is a list of top-2 solutions of the \(d\)-DNNF circuit in Figure 1a rooted at the same node (for the sake of readability, the values of those solutions are not reported on the figure).

4 Computing Top-\(k\) Values

From the user perspective, computing top-\(k\) solutions requires first to decide which value of \(k\) should be retained. To make an informed choice, deriving first top-\(k\) values (with possibly another value for \(k\) than the one representing the number of solutions) can be very useful. Indeed, to make things simple, suppose that the top-5 values for a given scenario are 100, 99, 98, 10, 2. Here, there is a huge gap between the first top-3 values and the two remaining ones. The user can then be tempted to ask first for the computation of top-3 solutions, and then look at their values. If the values are respectively 100, 99, 98, he/she may decide to stop the computation because he/she knows that there are only one top-1 solution, and only two top-2 solutions and he/she is fine with them. Contrastingly, if the computed values are respectively 100, 100, 100, he/she may ask for the computation of more top solutions.

To deal with this issue, we have designed a pseudo
polynomial-time algorithm for solving the top-\(k\) values problem, provided that the semigroup \((K, \otimes, <)\) is almost strictly monotone. Though more demanding than the monotony condition, this restriction is met by several semigroups that are useful for modeling utilities or probabilities (as sketched in Section 2). We have obtained the following result:

**Theorem 2** Let \((K, \otimes, <)\) be an almost strictly monotone, totally ordered semigroup. Let \(X\) be a set of variables. Let \(\nu\) be a value function over \(X\) onto \(K\). Let \(C\) be a d-DNNF circuit over \(X\). The problem \(\mbox{TopVal}_k(\nu, C)\) can be solved in time \(O(|C| \cdot k^2 \cdot \log k)\).

An algorithm for the top-\(k\) values problem. Our algorithm to solve \(\mbox{TopVal}_k(\nu, C)\) is a variant of Algorithm 1 for computing top-\(k\) solutions. A main difference is that it is sufficient to store values and thus, the procedures sorted_fusion and sorted_product take as inputs tables of values and not tables of pairs (assignment, value) and they output tables of values. Those procedures must be updated to handle (respectively) \(\lor\) nodes and \(\land\) nodes in a satisfying way since it is possible in the top-\(k\) values context that duplicates appear. When computing top-\(k\) solutions, the decomposability and the determinism conditions on d-DNNF circuits ensure that the assignments generated at each node of \(C\) when applying Algorithm 1 are distinct, but it is not the case for values.

Thus, when calling sorted_fusion(top_c0, top_c1, k, \(\nu\)), it may happen that the same value appears both in top_c0 and top_c1 so that one of the duplicates has to be removed after sorting. Thus the update of sorted_fusion(top_c0, top_c1, k, \(\nu\)) simply consists in sorting the values of the union of top_c0 and top_c1 and then removing the duplicates from the resulting sorted table. This has no impact on the complexity of sorted_fusion. When calling sorted_product(top_c0, top_c1, k, \(\nu\)) at a \(\land\) node, it is also possible to get duplicates as distinct \(\otimes\)-combinations of the values of the solutions of its two children. The update of sorted_product(top_c0, top_c1, k, \(\nu\)) can be done as follows. The values that are obtained are kept in memory, using a binary search tree \(S\) allowing us to add an element and to check whether an element is already stored in \(S\) in time logarithmic in the size of \(S\). Whenever we pop a pair \((i, j)\), we check using \(S\) whether \(\nu(i, j)\) has already been outputted. \(\nu(i, j)\) is then outputted and added to \(S\) if it has not been outputted before. The remaining instructions of the algorithm, i.e., adding \((i+1, 0)\) and \((i, j+1)\) to \(Q\), are the same ones as those in the algorithm for computing top-\(k\) solutions. This treatment is repeated until \(k\) distinct values have been found or we went through all the pairs \((i, j)\).

**Correctness and complexity.** The main point for proving the correctness of our algorithm is the correctness of sorted_product. For this, we can prove that if a value \(v\) belongs to the top-\(k\) values of a subcircuit rooted at a \(\land\) node \(N\) with children \(N_0\) and \(N_1\), then either \(v\) is equal to the least absorptive element of \(K\) if it exists or there exist \(u\) in the top-\(k\) values of \(N_0\) and \(w\) in the top-\(k\) values of \(N_1\) such that \(u \otimes w = v\). This property is a consequence of the fact that \((K, \otimes, <)\) is almost strictly monotone.

The time complexity of our algorithm comes from doing \(|C|\) times a call to the procedure sorted_fusion(top_c0, top_c1, \(\nu\)) or to the procedure sorted_product(top_c0, top_c1, \(\nu\)). For sorted_fusion, the complexity bound is the same one as for the top-\(k\) solutions case. For sorted_product, one may need to consider the full set of \(k^2\) pairs of values coming from the \(\otimes\)-combinations of the top-\(k\) values associated with the children of the \(\land\) node at hand. Therefore, the time complexity of a call to this procedure is in \(O(k^2 \cdot \log k)\).

**Example 3** Let us step back to Example 1 once more. The nodes \(N\) of the DAG reported in Figure 1c correspond in a bijective way with those of the d-DNNF circuit in Figure 1a. The label of each node \(N\) of the DAG reported in Figure 1c is the list of top-2 values of the d-DNNF circuit in Figure 1a rooted at the corresponding node.

## 5 Top-\(k\) Transformation

Interestingly, the top-\(k\) transformation problem can also be solved in polynomial time when \(C\) is a d-DNNF circuit provided that the totally ordered semigroup at hand is almost strictly monotone.

**Theorem 3** Let \((K, \otimes, <)\) be an almost strictly monotone, totally ordered semigroup. Let \(X\) be a set of variables. Let \(\nu\) be a value function over \(X\) onto \(K\). Let \(C\) be a d-DNNF circuit over \(X\). The problem \(\mbox{TopTran}_k(\nu, C)\) can be solved in time \(O(|C| \cdot k^2 \cdot \log k)\) and the resulting d-DNNF circuit has a size in \(O(|C| \cdot k^2)\).

An algorithm for the top-\(k\) transformation problem. Our algorithm for \(\mbox{TopTran}_k(\nu, C)\) starts by running \(\mbox{TopVal}_k(\nu, C)\) so that the top-\(k\) values of the d-DNNF circuits rooted at the nodes \(N\) of \(C\) are stored as additional labels of the corresponding nodes. The generation of a d-DNNF circuit \(C'\) as a result of \(\mbox{TopTran}_k(\nu, C)\) is achieved by parsing the nodes \(N\) of \(C\) (together with the list \(L_N\) of values associated with them) in a bottom-up manner. For the sake of simplicity, let us suppose first that \((K, \otimes, <)\) does not have a least absorptive element. Let \(N_0\) and \(N_1\) be the children of \(N\) when \(N\) is an internal node. The treatment is as follows. Every leaf node of \(C\) is kept unchanged. For each internal node \(N\) together with the associated list \(L_N\) of values, we create one new node for each value \(v\) in the list \(L_N\). The node created from \(N\) with value \(v\) is noted \((N, v)\).

If \(N\) is a \(\lor\) node, then \((N, v)\) is a \(\lor\) node, and for each value \(v\), arcs connecting \((N, v)\) to \((N_0, v)\) and/or \((N_1, v)\) are added if those last nodes exist, i.e., if \(v\) belongs to the list of the top-\(k\) values associated with \(N_0\) and/or \(N_1\). If the root \(N\) of \(C\) is a \(\lor\) node, then the root of \(C'\) is a new \(\lor\) node having as children the nodes \((N, v)\) where \(v\) varies in \(L_N\). Now, for a \(\land\) node \(N\), for each value \(v\) in the list of the top-\(k\) values associated with \(N\), let \(L(v)\) be the list of pairs \((u, w)\) of values from the lists of the top-\(k\) values associated respectively with \(N_0\) and \(N_1\), such that \(u \otimes w = v\). We construct a subcircuit rooted at \((N, v)\) that encodes the disjunction over
the values $v$ of the conjunctions of $(N_0, u)$ and $(N_1, w)$ such that $u \otimes w = v$.

Once the root of $C$ has been processed, the resulting circuit is simplified in linear time by removing every node and every arc that cannot be reached from the root, and by shunting every node that has a single child.

When the semigroup has a least absorptive element $a$, the construction of $C'$ is similar to the previous one, except when $a$ belongs to the top-$k$ values of a subcircuit rooted at a $\&$ node $N$. Indeed, in such a case, $a$ also belongs to the top-$k$ values associated with one of the children $N_0$, $N_1$ of $N$. Suppose that $a$ is a top-$k$ value associated with $N_0$. Because $a$ is absorptive, for any valid assignment $\omega$ of the subcircuit rooted at $N_1$ such that $\nu(\omega) = v$ (whatever $v$ is), we have $a \otimes v = a$. Therefore, $\omega$ can be among the valid assignments of the subcircuit rooted at $N_1$ that produce (once concatenated with a valid assignment of the subcircuit rooted at $N_0$) a top-$k$ solution at node $N$. Accordingly, all the valid assignments $\omega$ at $N_1$ must be stored, even if they are not among those having a top-$k$ value at $N_1$. To deal with this case, we copy $C$ into a new circuit $C'$ so that for each node $N$ of $C$ we have an associated node in $C'$, denoted $N^C$. The subcircuit associated with each node $(N, v)$ where $v$ is different of $a$ is built as explained before. We now explain how to build the subcircuits associated with the nodes $(N, a)$.

The construction depends on the type of $N$:

- if $N$ is a $\lor$ node, then the subcircuit associated with $(N, a)$ is equal to the disjunction of the nodes $(N_0, a)$ and $(N_1, a)$ when they exist. If there exists only one such node, $(N, a)$ is equal to this node.

- if $N$ is a $\land$ node, then there are three cases: if both $(N_0, a)$ and $(N_1, a)$ exist, then the subcircuit associated with $(N, a)$ is the disjunction of the conjunction of $(N_0, a)$ and $N^C_1$ with the conjunction of $N^C_0$ and $(N_1, a)$; if only $(N_0, a)$ exists, then the subcircuit associated with $(N, a)$ is the conjunction of $(N_0, a)$ and $N^C_1$; in the remaining case, i.e., when $(N_1, a)$ exists, then the subcircuit associated with $(N, a)$ is the conjunction of $N^C_0$ and $(N_1, a)$.

Correctness and complexity. By construction, the decomposability of the $\land$ nodes is preserved by the transformation algorithm. Furthermore, the $\lor$ nodes that are created in the resulting circuit are deterministic ones (each child of such a node corresponds to a set of satisfying assignments having a value different of those of its sibling or obtained by concatenating distinct partial assignments, therefore those sets of assignments are pairwise disjoint).

The main part of the correctness of the algorithm is a generalization of the property used for proving the correctness of our algorithm for solving $\text{TopVal}_k(\nu, C)$. Let $N$ be a $\land$ node with children $N_0$ and $N_1$. Let $v$ be a value among the top-$k$ values of the circuit rooted at $N$. Then, if there exist $u$ and $w$ values of assignments from $N_1$ and $N_2$ such that $u \otimes w = v$ then $v$ is the absorptive element or $u$ and $v$ are in the top-$k$ values of $N_1$ and $N_2$.

The multiplicative factor $k^2$ in the size of the resulting $d$-$\text{DNNF}$ circuit comes directly from the fact that in the worst case, every $L_N$ contains $k^2$ elements.

**Example 4** Let us consider again Example 1 once more. Figure 2a illustrates the computation achieved for deriving $\text{TopTr}^2(\nu, C)$ where $C$ is the $d$-$\text{DNNF}$ circuit presented at Figure 1a. An equivalent, yet simplified circuit is reported on Figure 2b.

Interestingly, our transformation algorithm leads to the generation of a $d$-$\text{DNNF}$ circuit where all the assignments satisfying a subcircuit rooted at a $\land$ node have the same values. This property opens the door for a number of additional tractable treatments that could be useful, like counting the number of valid assignments having a given value among the top-$k$ ones, or uniformly sampling such assignments using results of (Sharma et al. 2018a).

## 6 Experimental Results

Clearly, all the top-$k$ algorithms presented in the previous sections prove practical when $k$ is small enough (they run in linear time in the size of the $d$-$\text{DNNF}$ circuit $C$ when $k$ is bounded by a constant). Notably, considering that $k$ is small enough is a reasonable assumption since the generation of top-$k$ solutions is typically triggered by a human user who will not be able to encompass a large set of solutions as a whole due to his/her cognitive limitations (see e.g., (Miller 1956)). However, the efficiency of our approach deeply relies on the assumption that the constraints considered at start have been compiled into a $d$-$\text{DNNF}$ circuit, and it is well-known that such a compilation step can be computationally expensive (the size of the resulting $d$-$\text{DNNF}$ circuit can be exponential in the size of the input constraints).

**Empirical protocol.** In order to figure out the benefits that can be reached by taking advantage of such a compilation-based approach, we focused on the strictly monotone, totally ordered semigroup $(\mathbb{N}, +, <)$ and, in this setting, implemented an approach to the computation of top-$k$ solutions based on an algorithm for the NP-hard problem called **Weighted Partial MAXSAT** problem. As evoked previously, the semigroup $(\mathbb{N}, +, <)$ is suited to the (quite general) class of scenarios where the values under consideration represent utilities.

Let us recall that an instance of **Weighted Partial MAXSAT** consists of a pair $(C_{\text{soft}}, C_{\text{hard}})$ where $C_{\text{soft}}$ and $C_{\text{hard}}$ are (finite) sets of weighted clauses, and a weighted clause is an ordered pair $(c, w)$ where $w$ is a natural number or $\infty$. Intuitively, $w$ gives the cost of falsifying $c$. If $w$ is infinite, the clause is hard, otherwise it is soft. The objective is to determine a truth assignment that maximizes the sum of the weights of the clauses $c$ in $C_{\text{soft}}$ that are satisfied, while satisfying all clauses $c$ such that $(c, \infty) \in C_{\text{hard}}$. Now, starting with a CNF formula $C$ over $X$ and a value function $\nu$ over $X$ onto $\mathbb{N}$, every clause $c$ of $C$ can be turned into a hard clause $(c, \infty) \in C_{\text{hard}}$ and every literal $\ell$ over $X$ can be turned into a soft clause $(\ell, \nu(\ell)) \in C_{\text{soft}}$. The truth assignment $\omega$ over $X$ that is obtained as a solution of $(C_{\text{soft}}, C_{\text{hard}})$ is by construction a top-1 solution of $C$ given $(\mathbb{N}, +, <)$ and $\nu$. In order to leverage the approach so as to
compute top-$k$ solutions, once $\omega$ has been generated, it is enough to add a hard clause equivalent to $(\neg \omega, \infty)$ to $C_{\text{hard}}$ in order to block the further generation of $\omega$ and to solve the resulting instance of Weighted Partial MaxSAT. Once $k$ solutions have been generated (when this is possible), the procedure stops.

In order to compare the performances of our d-DNNF-driven algorithm for generating top-$k$ solutions with those of the Weighted Partial MaxSAT-based procedure sketched above for tackling the same issue, we made some experiments. In our experimental evaluation, we took advantage of d4\footnote{www.cril.univ-artois.fr/KC/d4.html} (Lagniez and Marquis 2017) to compile CNF formulae into d-DNNF circuits. A Java library called Winston has been developed. It includes software for loading the d-DNNF circuit $C$ computed using d4, for smoothing it, and for computing top-$k$ solutions from it. For the top-$k$ solutions approach based on Weighted Partial MaxSAT, our procedure was empowered by one of the best solvers from the 2020 MaxSAT competition, namely MaxHS (Davies and Bacchus 2013b; 2013a; Davies 2013).

We considered the dataset reported in (Sharma et al. 2018b). This dataset contains 1424 CNF formulae coming from various fields, including probabilistic reasoning, bounded model checking, circuit, product configuration, SMTLib benchmarks, planning, quantified information flow and bug synthesis.

We have run the compilation-based top-$k$ algorithm and the one based on MaxHS for different values of $k$ (1, 5, 10, 20) and, for each of these algorithms, we measured the time required to get $k$ solutions. For every literal $\ell$ over the variables $X$ of the input CNF formula, an integer between 0 and 1,000,000 has been picked up at random following a uniform distribution as the value of $\ell$. Following this approach, five value functions $\nu_1, \ldots, \nu_5$ have been generated per instance. An instance has been viewed as solved when the corresponding algorithm for top-$k$ solutions succeeded in deriving $k$ top solutions for each of the five value functions before the timeout was reached. For every instance solved, the mean time used to get $k$ top solutions when the value function varies has been considered. For the compilation-based approach, the run time includes the time needed to compile the input CNF formula into a d-DNNF circuit.

The code of the algorithms and the data used in our experiments are available online. All the experiments have been run on a cluster of computers based on bi-processors Intel Xeon E5-2680 v4 (2.2 GHz) with 768 GB of memory. For the experiments, we used a timeout set to 20 minutes.

**Empirical results.** The results are reported on the scatter plot at Figure 3 and in Table 1. Figure 3 shows a sharp separation about the run times required by the two approaches when $k = 10$: the instances that were computationally easy for the approach based on MaxHS were typically hard for compilation-based approach. As reported in Table 1, the top-$k$ approach based on MaxHS solved 94.0% of the instances of the dataset for $k = 1$ to 87.2% for $k = 20$, while the compilation-based approach solved 81.5% of the instances, whatever the value of $k$ among those considered in the experiments. When $k$ increased, the number of instances solved only by the MaxSAT-based approach diminished from 196 for $k = 1$ to 158 for $k = 20$, while the number of instances solved only by the d-DNNF approach went up from 18 to 78. As to run times, the fastest method was the MaxSAT-based approach for $k = 1$, but around $k = 10$ both approaches were tied, and for $k = 20$, the compilation-based approach was still better but not by much.

![Figure 2: TopTra$_k(\nu, C)$ at work.](image1)

![Figure 2: The resulting d-DNNF circuit (once simplified).](image2)
based approach was clearly faster. This is not surprising since the main computational effort in the compilation-based approach is the time (and space) spent in the compilation phase, but this phase has to be performed once only, and the resources used are independent of the value of $k$. This contrasts with the MaxSAT-based approach to computing $k$ top solutions which requires to solve $k$ instances of an NP-hard problem (the instances being possibly harder and harder).

Accordingly, the compilation-based approach appears as the more interesting option to derive $k$ top solutions when $k$ is large enough. Our experiments show that even for a small value of $k$, it can be a challenging method.

![Figure 3: Comparison of the runtimes of the compilation-based approach and the MaxSAT-based approach for computing 10 top solutions.](image)

### 7 Discussion

Before concluding, we would like to mention that some of the restrictions considered in the previous sections could be questioned. First, we have focused on the $d$-DNNF language mainly because existing compilers targeting the DNNF language, including c2d (Darwiche 2001a; 2004), Dsharp (Muise et al. 2012), and d4 (used in our experiments), actually target a subset of $d$-DNNF, namely the language of Decision-DNNF circuits. Nonetheless, our top-$k$ algorithms could be extended to DNNF, i.e., removing the determinism condition on circuits $C$. This would not have any impact on the complexity of the algorithms, especially those for computing the top-$k$ values and making the top-$k$ transformation, and only a slight impact on the complexity of the algorithm for deriving $k$ top solutions (equality tests should be implemented at $\lor$ nodes, thus one could not get rid of the $|X|$ factor in the complexity assessment).

The assumption according to which the size of the representations of the values in $K$ are bounded by a constant and that $\otimes$ and $<$ are constant-time operations could also be relaxed. In that case, an extra factor in the time complexity of the top-$k$ algorithms has to be added. This factor depends not only on the cost of performing the $\otimes$-operation over values from $K$ represented using $m$ bits, but also of the size of the resulting value. For instance, summation combines two $m$-bits numbers into an $m + 1$-bit number in linear time, while product combines two $m$-bits numbers into a $2m$-bit number in quadratic time – if the naive schoolbook algorithm for multiplication is used. Accordingly, if $d$ is the depth of $C$, the extra complexity factor to be considered is in $O(m + d)$ when $\otimes = +$, and this is not that much. However, it is in $O(2^{2d} \cdot m^2)$ when $\otimes = \times$, which cannot be neglected when the circuit $C$ is deep.

### 8 Conclusion

We have presented three top-$k$ algorithms for $d$-DNNF circuits $C$ given a totally ordered, semigroup $(K, \otimes, <)$. Provided that some assumptions about monotony w.r.t. $<$ are

### Table 1: Performances of MaxHS and of the $d$-DNNF approach for computing $k$ top solutions, depending on $k$

| $k$ | MaxHS | $d$-DNNF |
|-----|-------|---------|
| 1   | 94%   | 81.5%   |
| 5   | 90.3% | 81.5%   |
| 10  | 88.6% | 81.5%   |
| 20  | 87.2% | 81.5%   |

Note: The success rate is the percentage of instances solved.

Success rate for MaxHS

Success rate for the $d$-DNNF approach

# instances solved only by MaxHS

# instances solved only by the $d$-DNNF approach

# instances where MaxHS was faster

# instances where the $d$-DNNF approach was faster
satisfied by \( \otimes \), these algorithms can be used, respectively, to compute in pseudo polynomial-time \( k \) top solutions of \( C \), the top-\( k \) values met by solutions of \( C \), and a d-DNNF circuit \( C' \) satisfied exactly by the models of \( C \) having a value among the top-\( k \) ones. We have also presented the results of an empirical evaluation, showing that the d-DNNF compilation-based approach can prove valuable to address the top-\( k \) solutions problem.

Among the perspectives for further research, we plan to extend our top-\( k \) algorithms to a multicriteria setting, i.e., when several value functions \( v_i \) onto distinct sets \( K_i \) are considered at the same time. This would be useful to handle in a better way many applications about configuration, recommendation, and e-commerce. From the technical side, making such an extension would require a drastic update of the top-\( k \) algorithms since the algebraic structure that underlies the present framework would not be preserved.

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