Euler observers for the perfect fluid without vorticity

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Abstract. There are many astrophysical problems that require the use of relativistic hydrodynamics, in particular ideal perfect fluids. Fundamentally, problems associated with intense gravitational fields. In this context, there are evolution problems that need to be addressed with techniques that are both as simple as possible and computer time inexpensive. A new technique using a new kind of tetrads was developed for the case where there is vorticity in order to locally and covariantly diagonalize the perfect fluid stress-energy tensor. The perfect fluid was already studied for the case where there is vorticity and several sources of simplification were already found. In this manuscript, we will analyze the case where there is no vorticity. We will show how to implement for this case the diagonalization algorithm that will differ from the previously developed for the case with vorticity. A novel technique to build tetrads using Killing vector fields will be introduced. We implement this new technique using only covariant and local manipulations of an algebraic nature, which will not add more substantial computational time and nonetheless bring about simplification in further applications like the construction of Euler observers for example. Precisely, as an application in spacetime dynamical evolution, a new algorithm will be formulated with the aim of finding Euler observers for this case without vorticity. It will be shown that the Einstein equations with the perfect fluid stress–energy tensor get substantially simplified through the use of these new tetrads. We will also show that the Frobenius theorem is covertly encoded in these new tetrads when Killing vector fields do exist.

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1. Introduction

There are many astrophysical problems that require the use of general relativity. Fundamentally, many problems are associated with intense gravitational fields, e.g., gravitational collapse, fast rotation pulsars, neutron star–neutron star and neutron star–black hole mergers, gravitational radiation, stability of different objects, ultrarelativistic flows, etc. Many of these studies [1–9] need the use of relativistic hydrodynamics, in particular, ideal perfect fluids. A relevant object to carry out these analyses is the stress–energy tensor,

\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \]

(1)

where \( \rho \) is the energy density of the fluid, \( p \) the isotropic pressure and \( u^\mu \) its four-velocity field, \( g_{\mu\nu} \) is the metric tensor. In many schemes or algorithms that confront these complicated relativistic hydrodynamic equations, it is necessary to modify the stress–energy tensor (1) in order to include artificial viscosity \( Q \) as [4],

\[ T_{\mu\nu} = (\rho + p + Q) u_\mu u_\nu + (p + Q) g_{\mu\nu}. \]

(2)

In order to understand the relevance of Eq. (2), we quote from section 2.1.2 in references [4,5] “However, reference [10] proposes that one add the Q terms in a consistent way, in order to consider the artificial viscosity as a real viscosity...In general, Q is a nonlinear function of the velocity”. We are using the artificial viscosity in this sense. Our goal is to find local and covariant geometrical structures that
enable a geometrical understanding of the different problems with simplification in both, the numerical and physical fronts. To this end, we carry out the program of finding new tetrads that locally and covariantly diagonalize the stress–energy tensor (1) or the variant (2) in order to develop a covariant algorithm for the construction of Euler observers and Cauchy surfaces [11,12], always in situations with no vorticity. A new technique was already developed in four-dimensional Lorentzian spacetimes where electromagnetic fields are present [13–16]. This technique was already analyzed for the case with vorticity [17] as well. In order to apply our method for diagonalizing the stress–energy tensor, we need a second rank antisymmetric tensor. If there is vorticity, then we can apply our diagonalizing method for the stress–energy tensor just using the very antisymmetric tensor of the vorticity itself, like we did in reference [17]. In our present manuscript, we do not have vorticity, hence we need to find another antisymmetric tensor in order to apply our procedure. We will do this by introducing antisymmetric tensors built only out of covariant derivatives of Killing vector fields. We will introduce the basic elements needed for its development in our case without vorticity in Sect. 2. This new technique for the case without vorticity will be novel and a relevant contribution of this paper, since we will make use of purely geometrical objects only, when building the extremal fields. The technique to find Eulerian observers for the Einstein–Maxwell case was introduced in manuscript [18]. We will extend this technique to the perfect fluid case with no vorticity in Sect. 3 as an application of our new method to build tetrads that locally and covariantly diagonalize the stress–energy tensor (1) or its modification (2). A second application of our new tetrad vectors will be presented in Sect. 4 with the goal of building timelike Killing vector fields when only spacelike Killing vector fields are available.

2. New tetrads for the case without vorticity

The new technique implemented with the goal of finding locally and covariantly new tetrads that diagonalize the perfect fluid stress–energy tensor in the case with vorticity was founded on the antisymmetric nature of the fluid extremal field or the velocity curl extremal field second rank tensor itself [17]. In our present case, there is no velocity curl, however we can proceed to introduce other second rank antisymmetric tensors that support the construction of new tetrads that on one hand diagonalize locally and covariantly the perfect fluid stress–energy tensor and on the other hand allow for the construction of Euler vector fields. Let us consider the following objects,

\[ C_{\mu\nu} = \xi_{\mu;\nu} \] (3)

Since we consider \( \xi_{\mu} \) to be a Killing vector field, then the object \( \xi_{\mu;\nu} \) is an antisymmetric tensor field in spacetime satisfying \( \xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \). As usual we will call \( R_{\mu\nu\rho\lambda} \) the Riemann tensor, \( R_{\mu\nu} \) the Ricci tensor, \( R \) the Ricci scalar, etc. The symbol \( ; \) stands for covariant derivative with respect to the metric tensor \( g_{\mu\nu} \). We introduce an extremal field \( \xi_{\mu\nu} \) and a local complexion scalar \( \alpha \) exactly as we did for instance in references [13–16] and specially in manuscript [17] and follow the necessary analogous steps to obtain the tetrad that locally and covariantly diagonalizes the stress–energy tensor (1). We raise the attention to possible nomenclature confusions. Following the notation in previous works [13–16], we call \( \xi_{\mu\nu} \) the extremal field. However, \( \xi_{\mu} \) is a Killing vector field according to notation in reference [19] and \( \xi_{\mu;\nu} \) are the covariant derivatives of the Killing vector field. As in manuscript [17] vorticity was present, the extremal field and the complexion found through a local duality transformation made use of this fact. They were found through a local duality transformation of the velocity curl. In the present paper, there is no vorticity and the duality transformation is performed on the fields (3), instead. As an example, the extremal field could be defined as \( \xi_{\mu\nu} = \cos \alpha \ C_{\mu\nu} - \sin \alpha \ * C_{\mu\nu} \). Extremal fields satisfy the following equation,

\[ \xi_{\mu\nu} * \xi_{\mu\nu} = 0 \] (4)

The object \( * C_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\sigma\tau} g^{\rho\sigma} g^{\tau\lambda} C_{\rho\lambda} \) is the dual tensor of \( C_{\mu\nu} \). Equation (4) is imposed on \( \xi_{\mu\nu} = \cos \alpha \ C_{\mu\nu} - \sin \alpha \ * C_{\mu\nu} \) and its dual \( * \xi_{\mu\nu} = \cos \alpha \ * C_{\mu\nu} + \sin \alpha \ C_{\mu\nu} \) and then the explicit expression for
the complexion is found. In this example, after imposing the condition (4) the complexion is \( \tan(2\alpha) = -(C_{\mu\nu} g^{\rho\mu} g^{\sigma\nu} * C_{\sigma\tau}) / (C_{\lambda\rho} g^{\lambda\alpha} g^{\rho\beta} C_{\alpha\beta}) \). It can be proved that condition (4) through the use of the general identity,

\[ A_{\mu\alpha} B^{\nu\alpha} - *B_{\mu\alpha} * A^{\nu\alpha} = \frac{1}{2} \delta_{\mu}^{\nu} A_{\alpha\beta} B^{\alpha\beta} , \]

which is valid for every pair of antisymmetric tensors in a four-dimensional Lorentzian spacetime [15], when applied to the case \( A_{\mu\alpha} = \xi_{\mu\alpha} \) and \( B^{\nu\alpha} = *\xi^{\nu\alpha} \) yields the equivalent condition to condition (4),

\[ \xi_{\mu\rho} * \xi^{\mu\lambda} = 0 . \]

As antisymmetric fields, the extremal fields also verify the identity,

\[ \xi_{\mu\alpha} \xi^{\nu\alpha} - *\xi_{\mu\alpha} * \xi^{\nu\alpha} = \frac{1}{2} \delta_{\mu}^{\nu} Q , \]

which is a particular case of (5). \( Q = \xi_{\mu\nu} \xi^{\mu\nu} \) is assumed not to be zero. One last point of consistency. We might wonder how it is possible to obtain the tetrad that diagonalizes the stress–energy tensor, starting from any of the antisymmetric fields (3) for possible different Killing vector fields. The answer lies in the expression for the complexion found by imposing Eq. (4). For the example analyzed, the complexion was

\[ \xi_{\mu\alpha} \xi^{\nu\alpha} - *\xi_{\mu\alpha} * \xi^{\nu\alpha} = \frac{1}{2} \delta_{\mu}^{\nu} Q , \]

where \( V_{\alpha} = \xi_{\mu\alpha} u_{\mu} / \sqrt{g_{\mu\nu} \xi^{\mu\nu} u^\nu} \), \( Z_{\alpha} = *\xi^{\alpha\lambda} u_{\lambda} / \sqrt{u_{\mu} * \xi^{\mu\nu} * \xi^{\nu\sigma} u^\sigma} \), and \( W_{\alpha} = (V_{(4)}^{\alpha} (V_{(1)}^{\rho} u_{\rho}) - V_{(1)}^{\alpha} (V_{(4)}^{\rho} u_{\rho})) / \sqrt{V_{(5)}^{\beta} V_{(5)}^{\beta}} \),

where \( V_{(4)}^{\beta} \) and \( V_{(5)}^{\beta} \) are the extremal electromagnetic fields. The only variant so far is the initial antisymmetric field used to find the extremal field and the complexion through a local duality transformation like \( \xi_{\mu\nu} = \cos \alpha \, C_{\mu\nu} - \sin \alpha \, *C_{\mu\nu} \). In manuscript [17], the antisymmetric field was the velocity curl, and in our present manuscript, it is the antisymmetric tensors (3) for Killing vector fields. These vectors (8–11) satisfy,

\[ T_{\alpha \beta} = \begin{pmatrix} -p & U^\beta \\ V^\alpha & p \end{pmatrix} \]

where \( p = \frac{1}{2} \delta_{\mu}^{\nu} Q, \) and

\[ T_{\alpha \beta} = \begin{pmatrix} -p & U^\beta \\ V^\alpha & p \end{pmatrix} . \]
We introduce the equations satisfied by the hypersurface orthogonal [18–22] unit vector fields

$$n_\alpha n_{\beta;\gamma} + n_\beta n_{\gamma;\alpha} + n_\gamma n_{\alpha;\beta} - n_\alpha n_{\gamma;\beta} - n_\gamma n_{\beta;\alpha} - n_\beta n_{\alpha;\gamma} = 0.$$  \hfill (16)

We will name $\hat{U}^\mu$ the Euler unit timelike vector field that satisfies Eq. (16). We will name the other three vectors in the new orthonormal tetrad as $\hat{V}^\mu$, $\hat{Z}^\mu$ and $\hat{W}^\mu$. Then, the hypersurface orthogonal vector $\hat{U}^\mu$ must satisfy the equation,

$$\hat{U}_\alpha \hat{U}_{\beta;\gamma} + \hat{U}_\beta \hat{U}_{\gamma;\alpha} + \hat{U}_\gamma \hat{U}_{\alpha;\beta} - \hat{U}_\alpha \hat{U}_{\gamma;\beta} - \hat{U}_\gamma \hat{U}_{\beta;\alpha} - \hat{U}_\beta \hat{U}_{\alpha;\gamma} = 0.$$  \hfill (17)

Next, when we project Eq. (17) using the four tetrad vectors ($\hat{U}_\alpha, \hat{V}_\alpha, \hat{Z}_\alpha, \hat{W}_\alpha$) we get only three meaningful equations,

$$\hat{U}_{[\alpha;\beta]} \hat{V}^\alpha \hat{Z}^\beta = 0$$  \hfill (18)
$$\hat{U}_{[\alpha;\beta]} \hat{V}^\alpha \hat{W}^\beta = 0$$  \hfill (19)
$$\hat{U}_{[\alpha;\beta]} \hat{Z}^\alpha \hat{W}^\beta = 0.$$  \hfill (20)

Equations (18–20) are three conditions on the vector field $\hat{U}_\alpha$. Our intention is to use the tetrad (8–11) that locally and covariantly diagonalizes the perfect fluid stress–energy tensor, and introduce three local scalars that will solve the three Eqs. (18–20). To this end, first we perform a rotation on the local plane determined by ($V_\alpha, W_\alpha$) using the local scalar $\phi$,

$$V_\alpha^{(\phi)} = \cos(\phi) V_\alpha - \sin(\phi) W_\alpha$$  \hfill (21)
$$W_\alpha^{(\phi)} = \sin(\phi) V_\alpha + \cos(\phi) W_\alpha.$$  \hfill (22)

Second, we perform another local rotation in the plane ($Z_\alpha, W_\alpha^{(\phi)}$) by the local angle $\varphi$,

$$Z_\alpha^{(\varphi)} = \cos(\varphi) Z_\alpha - \sin(\varphi) W_\alpha^{(\phi)}$$  \hfill (23)
$$W_\alpha^{(\varphi)} = \sin(\varphi) Z_\alpha + \cos(\varphi) W_\alpha^{(\phi)}.$$  \hfill (24)

Finally a boost by the local angle $\psi$ in the plane ($U_\alpha, W_\alpha^{(\varphi)}$),

$$U_\alpha = \cosh(\psi) U_\alpha^{(\varphi)} + \sinh(\psi) W_\alpha^{(\varphi)}$$  \hfill (25)
$$W_\alpha = \sinh(\psi) U_\alpha^{(\varphi)} + \cosh(\psi) W_\alpha^{(\varphi)}.$$  \hfill (26)

Three local scalars ($\phi, \varphi, \psi$) become through these succession of local Lorentz transformations in three local variables that will be the solution to the system (18–20). The final orthonormal tetrad that has as a timelike vector field $\hat{U}_\alpha$ the hypersurface orthogonal vector field that will function as an input for our evolution algorithms is given by,

$$U_\alpha = \cosh(\psi) U_\alpha^{(\varphi)} + \sinh(\psi) W_\alpha^{(\varphi)}$$  \hfill (27)
$$V_\alpha = V_\alpha^{(\phi)}$$  \hfill (28)
$$Z_\alpha = Z_\alpha^{(\varphi)}$$  \hfill (29)
$$W_\alpha = \sinh(\psi) U_\alpha^{(\varphi)} + \cosh(\psi) W_\alpha^{(\varphi)}.$$  \hfill (30)
The algorithm would not work if the vector that involves the three local Lorentz transformations, and therefore the three local scalars \((\phi, \varphi, \psi)\), were not \(\hat{U}^\alpha\). If we would have considered Lorentz transformations only involving the original vectors \((V^\alpha, Z^\alpha, W^\alpha)\), then we would only have produced combinations of the original Eqs. (18–20), and since these can be algebraically decoupled, we would not have introduced any new information. It is through the inclusion of the three local scalars \((\phi, \varphi, \psi)\) inside the derivatives of the vector \(\hat{U}^\alpha\) that we get Eqs. (18–20) to be meaningful. Next, we contract the tetrad vectors \((\hat{U}^\alpha, \hat{V}^\alpha, \hat{Z}^\alpha, \hat{W}^\alpha)\) with the stress–energy tensor (1),

\[
\hat{U}^\alpha T_{\alpha \beta} = -\rho \cosh(\psi) U^\beta + p \sinh(\psi) W_{(\varphi)}^\beta. 
\]

(31)

\[
\hat{V}^\alpha T_{\alpha \beta} = p \hat{V}^\beta 
\]

(32)

\[
\hat{Z}^\alpha T_{\alpha \beta} = p \hat{Z}^\beta 
\]

(33)

\[
\hat{W}^\alpha T_{\alpha \beta} = -\rho \sinh(\psi) U^\beta + p \cosh(\psi) W_{(\varphi)}^\beta.
\]

(34)

Therefore, the only nonzero components of the stress–energy tensor in terms of the new tetrad are:

\[
\hat{U}^\alpha T_{\alpha \beta} \hat{U}_\beta = \rho \cosh^2(\psi) + p \sinh^2(\psi) 
\]

(35)

\[
\hat{V}^\alpha T_{\alpha \beta} \hat{V}_\beta = p 
\]

(36)

\[
\hat{Z}^\alpha T_{\alpha \beta} \hat{Z}_\beta = p 
\]

(37)

\[
\hat{W}^\alpha T_{\alpha \beta} \hat{W}_\beta = \rho \sinh^2(\psi) + p \cosh^2(\psi) 
\]

(38)

\[
\hat{U}^\alpha T_{\alpha \beta} \hat{W}_\beta = (\rho + p) \frac{1}{2} \sinh(2\psi). 
\]

(39)

By performing the three local Lorentz transformations in our algorithm, we have the following result. First, we ended up with a new local tetrad that adds only one off-diagonal component to the stress–energy tensor, the minimum possible. Second, we found the Euler hypersurface orthogonal congruence. We have found an algorithm that provides both a hypersurface orthogonal congruence and a maximum simplification of the stress–energy tensor given that the tetrad that diagonalized the tensor underwent three Lorentz transformations. When we take the limit \(\psi \rightarrow 0\), it can be readily seen from expressions (35–39) that we recover the results for the old tetrad that diagonalizes the stress–energy tensor.

4. Application: timelike Killing vector fields

In this section, we have the goal of building timelike Killing vector fields when only spacelike Killing vector fields are available. We will follow in this section the notation in reference [21] Appendix B.3 and Appendix C.3 as well. From these sections, we learn that a Killing vector field \(\xi^\mu\) in a four-dimensional Lorentzian curved spacetime satisfies the equation (C.3.1)

\[
\xi^\mu_{;\nu} + \xi^\nu_{;\mu} = 0 
\]

(40)

and it also satisfies equation (C.3.6),

\[
\xi^\mu_{;\nu;\lambda} = -R^\rho_{\nu \mu \lambda} \xi^\rho. 
\]

(41)

where ; stands for the covariant derivative \(\nabla = ;\) with respect to the metric tensor \(g_{\mu\nu}\). We also learn that a Killing field is completely determined by the values of \(\xi^\mu\) and \(\xi^\mu_{;\nu}\) at any point \(p\), since we can always connect \(p\) with any other point \(q\) through a curve with tangent \(v^\alpha\). Let us say as an introduction that we can always use the three Lorentz transformations of Sect. 3 in order to find timelike Killing vector fields. Because these local transformations involve three local scalars \((\phi, \varphi, \psi)\) and we can always write a timelike Killing vector field as \(\xi^\alpha = \Omega \hat{U}^\alpha\) where \(\Omega\) is a local additional scalar. Therefore, let us also make clear that the three local scalars involved in the three local Lorentz transformations (21–22), (23–24) and (25–26) will be in this new problem new unknowns which will be found through the Killing Eq. (40) in
addition to the local scalar $\Omega$. Now, we have turned through the use of our new tetrads the search for Killing vector fields, timelike in this example, into a search for four local scalars. Let us review the whole framework for the search of timelike Killing vector fields. We started with a Killing vector field $\xi_\mu$ in Eq. (3) which does not have to be necessarily timelike and built the tetrad (8–11). Then we proceeded to carry out the Lorentz transformations (21–22), (23–24) and (25–26) with local scalars ($\phi, \varphi, \psi$). Finally, adding a new local scalar $\Omega$ in $\xi^\alpha = \Omega \hat{U}^\alpha$ we solve the Eq. (40) for the four local scalars in order to find a timelike Killing vector field if it exists, naturally. We can also investigate the converse problem where the covariant derivatives $\xi_{\mu;\lambda}$ are given and see how far we can go through our new method to build tetrads developed in references [13,18]. This final problem we set out to solve will not be only for perfect fluid geometries, it is general for any geometries. Let us consider the special case where the Killing vector field $\chi^\mu$ is hypersurface orthogonal, and then by Frobenius theorem B.3.2 in reference [21], there exists a vector field $\nu^{\sigma}$ such that [21] (C.3.10),

$$\chi_{\mu;\lambda} = \chi_{[\mu;\lambda]} = \chi_{[\lambda \nu]} \nu_{\mu}.$$ \hfill (42)

Assuming that $\chi^\lambda$ is not null, we may choose $\nu^\alpha$ to be orthogonal to $\chi^\lambda$. We would like to find the result (42) alternatively and independently through the use of our new method to build a tetrad and see if the results match each other. Therefore, at this point, we proceed to apply the method that we introduced in references [13,18] and summarized in Sect. 6 for antisymmetric fields in four-dimensional curved Lorentzian spacetimes to the second rank antisymmetric tensor $\chi_{\mu;\lambda}$. This method has many similarities to the method developed in Sect. 2 even though they are not identical. We repeat one more time, this is now a general situation, not necessarily the perfect fluid. The antisymmetric tensor $\chi_{\mu;\lambda}$ and its dual $\star \chi_{\mu;\lambda}$ can be expressed through a local duality transformation by local angle scalar $\alpha_k$ through an extremal field $\xi^k_{\mu\nu}$ and its dual $\star \xi^k_{\mu\nu}$, that is,

$$\chi_{\mu;\lambda} = \cos \alpha_k \xi^k_{\mu\lambda} + \sin \alpha_k \star \xi^k_{\mu\lambda}.$$ \hfill (43)

Conversely,

$$\xi^k_{\mu\lambda} = \cos \alpha_k \chi_{\mu;\lambda} - \sin \alpha_k \star \chi_{\mu;\lambda}$$ \hfill (44)

$$\star \xi^k_{\mu\lambda} = \cos \alpha_k \star \chi_{\mu;\lambda} + \sin \alpha_k \chi_{\mu;\lambda}.$$ \hfill (45)

Next we impose the local condition $\xi^k_{\mu\lambda} \star \xi^{k\mu\lambda} = 0$ which because of identities (5) and (7) becomes $\xi^k_{\mu\lambda} \star \xi^{k\mu\lambda} = 0$. Let us remind ourselves that the $k$ in $\xi^k_{\mu\lambda}$ or $\cos \alpha_k$ stands for Killing in order to establish a difference from the notation in Sect. 2. Then, by imposing this condition, we obtain the expression for the local Killing complexion scalar $\tan(2\alpha_k) = - \left( \chi_{\mu;\nu} g^\sigma_\mu g^\nu_\sigma * \chi_{\rho;\tau} \right) / \left( \chi_{\lambda;\rho} g^\gamma_\lambda g^\rho_\beta \chi_{\alpha;\beta} \right)$. If we introduce the local scalar $Q^k = \xi^k_{\mu\lambda} \xi^{k\mu\lambda}$ and assuming that it is non-trivial, then, using the method of references [13, 18] we obtain an orthonormal tetrad $(V^{(1)}_{\kappa\alpha}, \ldots, V^{(4)}_{\kappa\alpha})$ that can be normalized into $(U^{\kappa\alpha}, V^{\kappa\alpha}, Z^{\kappa\alpha}, W^{\kappa\alpha})$. We must not confuse the method in Sect. 2 with the method in references [13,18], they have similarities but are not exactly the same. For example, the vectors $(U^{\kappa\alpha}, V^{\kappa\alpha}, Z^{\kappa\alpha}, W^{\kappa\alpha})$ will be given by,

$$U^{\kappa\alpha} = \xi^{\kappa\alpha \lambda} \xi_{\kappa\rho} X^\rho / \left( \sqrt{-Q^k/2} \sqrt{X_\mu \xi^{k\mu\sigma} \xi_{k\nu} X^\nu} \right) \hfill (46)$$

$$V^{\kappa\alpha} = \xi^{\kappa\alpha \lambda} X^\lambda / \sqrt{X_\mu \xi^{k\mu\sigma} \xi_{k\nu} X^\nu} \hfill (47)$$

$$Z^{\kappa\alpha} = \xi^{k\alpha \lambda} Y^\lambda / \sqrt{Y_\mu \star \xi^{k\mu\sigma} \xi_{k\nu} Y^\nu} \hfill (48)$$

$$W^{\kappa\alpha} = \xi^{k\alpha \lambda} \star \xi_{\kappa\rho} Y^\rho / \left( \sqrt{-Q^k/2} \sqrt{Y_\mu \star \xi^{k\mu\sigma} \xi_{k\nu} Y^\nu} \right). \hfill (49)$$


The two vector fields $X^\rho$ and $Y^\rho$ can be chosen arbitrarily as long as the four-vector fields (46–49) are not trivial. Just to give one example, it could be $X^\rho = Y^\rho = g^{\rho\lambda} Q_k^\lambda = g^{\rho\lambda} (\xi^k_{\mu\lambda} \xi^{k\mu})\lambda$. There are many possible non-trivial choices. There is a further discussion on the tetrad vectors (46–49) in Sect. 6.

We proceed then to re-express in papers [13,18] the extremal object as:

$$\chi_{\mu;\lambda} = -2 \sqrt{-Q_k^k/2} \cos \alpha_k \ U^k_{[\alpha} V^k_{\beta]} + 2 \sqrt{-Q_k^k/2} \sin \alpha_k \ Z^k_{[\alpha} W^k_{\beta]}.$$  \hfill (50)

The method to find Eq. (50) is developed in detail in reference [13], and in Sect. 6 we summarize the main elements of this procedure. We are assuming for simplicity that $U^k_\alpha$ is the timelike tetrad vector $U^k_\alpha U^{k\alpha} = -1$. When comparing Eq. (50) with equation C.3.10 in reference [21], we notice that if the case is that the hypersurface orthogonal Killing vector field $\chi^\mu$ is timelike, then we must consider the case where the local scalar complexion $\alpha_k = 0$ making $\xi^k_{\mu\lambda} = \chi_{\mu;\lambda}$ according to Eq. (43), and from comparison with Eq. (42) we can define,

$$\chi^\mu = \cosh(\omega) \ U^{k\mu} + \sinh(\omega) \ V^{k\mu}$$  \hfill (51)

$$v^{k\mu} = -\sqrt{-Q_k^k/2} (\sinh(\omega) \ U^{k\mu} + \cosh(\omega) \ V^{k\mu}).$$  \hfill (52)

It is clear that our method took us one step forward from Eq. (42) since we have been able to use the $\chi_{\mu;\lambda}$ antisymmetric tensor to find the expression for $\chi^\mu$ in Eq. (51) such that the local scalar $\omega$ solves the equation $\chi^\mu;_\mu = 0$. We could alternatively define

$$\chi^\mu = -\sqrt{-Q_k^k/2} (\cosh(\omega) \ U^{k\mu} + \sinh(\omega) \ V^{k\mu})$$  \hfill (53)

$$v^{k\mu} = \sinh(\omega) \ U^{k\mu} + \cosh(\omega) \ V^{k\mu},$$  \hfill (54)

such that once again $\chi^\mu;_\mu = 0$. In fact, through Eq. (50), we have found the most general way of expressing $\xi^k_{\mu\lambda}$ (whether $\chi^\mu$ is timelike or not, or hypersurface orthogonal or not) without having to invoke Frobenius theorem at all. In four-dimensional Lorentzian curved spacetimes, our tetrad method is more precise than Frobenius theorem itself, that in this particular case is just an existence theorem.

5. Conclusions

Many relativistic hydrodynamical problems [1–9] consist of a system of coupled differential equations that possess as a source term a stress–energy tensor of the perfect fluid nature as in Eq. (1). The resolution of a problem involving artificial viscosity in a relativistic consistent way as in Eq. (2) would involve similar techniques and we just focus on Eq. (1). Eulerian observers have proved to be useful in numerous dynamical problems, and we set out to find a local and covariant technique to produce them in a geometrical fashion, using the tensors that play a fundamental role in these schemes. There is at each point in spacetime only one tetrad diagonalizing the stress–energy tensor with or without artificial viscosity up only to possible local Lorentz transformations as can be noticed in several cases (21–22) and (23–24) put forward in Sect. 3. We find it using only covariant and local manipulations of an algebraic nature, which will not add more substantial computational time and nonetheless bring about simplification in further applications like the construction of Euler observers therefore reducing computational time by other means. The same notion applies to the finding of timelike Killing vector fields as in Sect. 4. This search is synthesized in the construction of local extremal fields through local duality transformations of second rank antisymmetric tensors as with (3) which are covariant derivatives of Killing vector fields. In previous works, these second rank antisymmetric tensors have been the electromagnetic field, the velocity curl, etc. Therefore, this new way of building extremal fields by using purely geometrical objects like in Eq. (3) is a decisive contribution of this manuscript. These new tetrads enjoy several useful properties due to its very construction. They diagonalize the stress–energy tensor locally and covariantly. They introduce important simplifications in the Einstein equations; see Sect. 7. They allow through three local Lorentz
transformations to find the Euler hypersurface orthogonal vector fields, minimizing the number of off-diagonal nonzero stress–energy tensor additional components just to one more. They allow in a natural way to find Cauchy surfaces and study the dynamical evolution of a myriad of astrophysical problems. We quote from [4] “With the exception of the vacuum two-body problem (i.e., the coalescence of two black holes), all realistic astrophysical systems and sources of gravitational radiation involve matter. Not surprisingly, the joint integration of the equations of motion for matter and geometry was in the minds of theorists from the very beginning of numerical relativity. Nowadays, there is a large body of numerical investigations in the literature dealing with hydrodynamical integrations in static background spacetimes. Most of those are based on Wilson’s Eulerian formulation of the hydrodynamic equations and use schemes based on finite differences with some amount of artificial viscosity”. Our new Euler observers built with the tetrads that locally and covariantly diagonalize the stress-energy tensors point into this direction of research.

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6. Appendix I

Given the perfect fluid stress–energy tensor (1), it is not difficult to prove the eigenvector Eqs. (12–15). The tetrad vector (8) is evidently an eigenvector of the stress–energy tensor

\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu} \]

with eigenvalue \(-\rho\). The orthogonalities with vectors (9–10)

\[ u_\alpha V^\alpha = u_\alpha Z^\alpha = 0 \]

are the result of the antisymmetry of both the extremal tensor \(\xi_{\alpha\lambda}\) and its dual \(\ast \xi_{\alpha\lambda}\). The orthogonality \(u_\alpha W^\alpha = 0\) arises because of the construction of the vector (11). In order to prove the other eigenvalue equations, we use systematically and iteratively Eqs. (6–7).

With regards to the different tetrad presented in Eqs. (46–49), we can say that the building principles are different from those in tetrad (8–11). For example, in vector (46)

\[ U^k\alpha = \xi^{k\alpha\lambda} \xi_{k\beta\lambda} X_\beta / (\sqrt{\frac{-Q^k}{2}} \sqrt{X_{\mu} \xi^{k\mu\sigma} \xi_{k\nu\sigma} X^\nu}) \]

there are three elements of construction. The skeleton \(\xi^{k\alpha\lambda} \xi_{k\beta\lambda}\), the gauge vector \(X_\beta\) and the normalizing factor \(1 / (\sqrt{\frac{-Q^k}{2}} \sqrt{X_{\mu} \xi^{k\mu\sigma} \xi_{k\nu\sigma} X^\nu})\) assuming \(X_\mu \xi^{k\mu\sigma} \xi_{k\nu\sigma} X^\nu > 0\). In vector (48)

\[ Z^{k\alpha} = \ast \xi^{k\alpha\lambda} Y_\lambda / (\sqrt{\frac{-Q^k}{2}} \sqrt{Y_{\mu} \ast \xi^{k\mu\sigma} \ast \xi_{k\nu\sigma} Y^\nu}) \]

the skeleton is \(\ast \xi^{k\alpha\lambda}\), the gauge vector \(Y_\lambda\) and the normalizing factor \(1 / (\sqrt{\frac{-Q^k}{2}} \sqrt{Y_{\mu} \ast \xi^{k\mu\sigma} \ast \xi_{k\nu\sigma} Y^\nu})\) assuming \(Y_\mu \ast \xi^{k\mu\sigma} \ast \xi_{k\nu\sigma} Y^\nu > 0\). The gauge vectors are a choice as long as the vectors are not trivial. For example, if \(\alpha_\kappa\) is not trivial, we can choose \(X_\mu = Y_\mu = \partial_\mu \alpha_k\). It is again very simple to prove that (46–49) are orthonormal by using iteratively Eqs. (6–7) plus the antisymmetry of extremal fields and its duals. It is also relevant to find the expression for the extremal field \(\xi^{k\alpha\lambda}\) and its dual \(\ast \xi^{k\alpha\lambda}\) in terms of the tetrad vectors (46–49). The detailed deduction is given in reference [13]. Here we provide the main elements. For example, using Eqs. (6–7) iteratively, we find,

\[ U^{k\alpha} \xi^k_{\alpha\beta} = \sqrt{-Q^k/2} V^k_{\beta} \]  \hspace{1cm} (55)
\[ V^{k\alpha} \xi^k_{\alpha\beta} = \sqrt{-Q^k/2} U^k_{\beta} \]  \hspace{1cm} (56)
\[ Z^{k\alpha} \ast \xi^k_{\alpha\beta} = \sqrt{-Q^k/2} W^k_{\beta} \]  \hspace{1cm} (57)
\[ W^{k\alpha} \ast \xi^k_{\alpha\beta} = -\sqrt{-Q^k/2} Z^k_{\beta} \]  \hspace{1cm} (58)
We also find,
\begin{align}
U^{k\alpha} \xi_{\alpha\beta}^{k} V^{k\beta} &= \sqrt{-Q^{k}/2} \\
Z^{k\alpha} \xi_{\alpha\beta}^{k} W^{k\beta} &= \sqrt{-Q^{k}/2}
\end{align}
(59)
(60)

\[ \xi_{\alpha\beta}^{k} = -2 \sqrt{-Q^{k}/2} U_{[\alpha}^{k} V_{\beta]}^{k} \]
(61)

\[ \xi_{\alpha\beta}^{k} = 2 \sqrt{-Q^{k}/2} Z_{[\alpha}^{k} W_{\beta]}^{k} \].
(62)

From Eqs. (55–58) and (59–62), we can obtain the Eq. (50) starting from Eq. (43).

7. Appendix II

In this appendix, we will study the components of the Ricci tensor when a Killing vector field exists. This task will be carried out with independence of our algorithm to find the tetrad that diagonalizes the stress–energy tensor. This tetrad will be used along with the Killing vector field in order to find the components of the left hand side of the Einstein equations and see whether there is a possible simplification also on the right-hand side of the Einstein equations. Let us consider the following equation that can be found in the literature [2,19],
\[ v_{\mu;\nu;\rho} - v_{\mu;\rho;\nu} = -R^{\sigma}_{\mu\nu} v_{\sigma} \].
(63)

This Eq. (63) is valid in general for any vector field \( v_{\sigma} \). Let us consider this generic vector field for the case where it is the Killing vector field \( \xi_{\sigma} \). Let us also contract this equation on both sides with \( g^{\mu\rho} \). We observe that in this case we find,
\[ \xi^{\mu;\rho}_{\nu;\mu} - \xi^{\mu;\nu}_{\mu;\nu} = -R^{\sigma}_{\nu} \xi_{\sigma} \].
(64)

Because of Eq. (40) the second term on the left hand side is zero, that is, \( \xi^{\mu;\rho}_{\mu;\nu} = 0 \). This is a first general important simplification. Then we contract both sides with the four vectors (8–11) in the tetrad (\( U^{\alpha}, V^{\alpha}, Z^{\alpha}, W^{\alpha} \)). In order to show an example, let us contract with vector (8) \( U^{\nu} = u^{\nu} \). We would obtain after elementary algebra,
\[ (u^{\nu} \xi^{\mu;\nu}_{\nu;\mu})_{;\mu} - u_{\nu;\mu} \xi^{\mu;\nu} = -u^{\nu} R^{\sigma}_{\nu} \xi_{\sigma} \].
(65)

If the main assumption of this paper holds, that is, there is no vorticity, then \( u_{\nu;\mu} = u_{\mu;\nu} \) and \( \xi^{\mu;\nu} = -\xi^{\nu;\mu} \); then the second term in the left-hand side of Eq. (64) will be zero, another simplification. Therefore, in this case, where we contracted with the Killing vector field and the first vector in the tetrad that diagonalizes the stress–energy tensor, in this case the four velocity and see Eq. (12), the resulting Einstein equation will be,
\[ (u^{\nu} \xi^{\mu;\nu}_{;\mu})_{;\mu} = -u^{\nu} R^{\sigma}_{\nu} \xi_{\sigma} = -u^{\nu} \left( T^{\sigma}_{\nu} - \frac{1}{2} \delta^{\sigma}_{\nu} T^{\mu}_{\mu} \right) \xi_{\sigma} = - \left( -\rho - \frac{1}{2} T^{\mu}_{\mu} \right) u^{\sigma} \xi_{\sigma} \].
(66)

We can see just through this example that both the Killing vector field and the tetrad that diagonalizes locally and covariantly the stress–energy tensor produce simplification in the Einstein equations. Let us see one more example and use the previous algorithm starting with Eq. (63) applied to the vector (9) after contracting this equation on both sides with \( g^{\rho\mu} \),
\[ V^{\mu}_{;\mu} - V^{\mu}_{;\mu} = -R^{\sigma}_{\nu} V_{\sigma} \].
(67)

Next, let us contract with vector (10),
\[ Z^{\nu} V^{\mu}_{;\nu;\mu} - Z^{\nu} V^{\mu}_{;\mu;\nu} = -Z^{\nu} R^{\sigma}_{\nu} V_{\sigma} = -Z^{\nu} \left( T^{\sigma}_{\nu} - \frac{1}{2} \delta^{\sigma}_{\nu} T^{\mu}_{\mu} \right) V_{\sigma} = 0 \],
(68)
since \( Z^\alpha T_{\alpha \beta} = p Z^\beta \) according to Eq. (14) and also using the orthogonality \( Z^\alpha V_\alpha = 0 \). Equation (68) is another source of simplification.

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