Analytic functions of the annihilation operator

Aleksandar Petrović
M. Gorkog 27, 11000 Belgrade, Serbia

A method for construction of a non-entire function $f$ of the annihilation operator $a$ is given for the first time. $f(z)$ is analytic on some compact domain that does not separate the complex plane. A new form of the identity is given, which is well suited for the function’s domain. Using Runge’s polynomial approximation theorem, such a function $f$ of the annihilation operator is defined on the whole domain. The constructed operators are given in terms of dyads formed of Fock states.

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I. INTRODUCTION

The annihilation operator in Hilbert space is operator $a$ which satisfies relation

$$[a, a^\dagger] = 1.$$  (1)

A self-adjoint Number operator $N$ and the corresponding Fock states are constructed using $a$:

$$N = a^\dagger a, \quad N|n\rangle = n|n\rangle, \quad n = 0, 1, 2, 3, ...$$  (2)

Eigenstates of $a$ are called coherent states:

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad |\alpha\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \alpha \in \mathbb{C}.$$  (3)

The coherent states are denoted with Greek letters ($|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$), and the Fock states with Latin letters ($|l\rangle$, $|k\rangle$, $|n\rangle$, $|m\rangle$).

The identity can be formed using eigenvectors $|\alpha\rangle$:  

$$I = \frac{1}{\pi} \int d^2 \alpha |\alpha\rangle \langle \alpha|. \quad (4)$$

$a$ is not a normal operator, so The Spectral Theorem cannot be applied to construct $f(a)$. Nevertheless, every entire function of the operator $a$ can be represented in a form analogous to the resolution of a normal operator with respect to its projective measure:

$$f(a) = \frac{1}{\pi} \int d^2 \alpha f(\alpha)|\alpha\rangle \langle \alpha|. \quad (5)$$

Contrary to the normal operator case, non-entire functions of $a$ cannot be represented in such a way [1], [2]. Previous unsuccessful attempts to use $\ln a$ were pointed out in [1]. Related questions concerning this issue were also studied by Perelomov [3]. The underlying problem is the fact that until now non-entire functions of the annihilation operator have not been studied in detail, and some straightforward assumptions arising from resolution [3] have lead to erroneous conclusions. In this paper, we construct a function $f(a)$ of the annihilation operator, where $f$ is analytic on a compact domain that does not separate the complex plane. The operators constructed here are given in terms of dyads formed of Fock states.

*Electronic address: a.petrovic.phys@gmail.com
II. RESULTS

A. New identity resolution

Standard resolution of identity (11) is not suitable for non-entire functions, so a new resolution of identity is given. To do that, non-normalized coherent states (NCS) are defined:

\[ |\tilde{\alpha}\rangle = e^{\alpha a^+} |0\rangle = e^{\frac{|\alpha|^2}{2}} |\alpha\rangle. \] (6)

Translation operator for NCS is:

\[ T(\beta) = e^{\beta a^+}, \quad T(\beta)|\tilde{\alpha}\rangle = |\tilde{\alpha} + \beta\rangle, \quad \alpha, \beta \in \mathbb{C}. \] (7)

Now, an identity whose eigenstates are on the circle of radius \( R \) centered at the origin can be formed from NCS:

\[ I = -i \oint_{|\gamma| = R} d\gamma |\tilde{\gamma}\rangle \langle \tilde{\gamma}| J, \] (8)

where

\[ J = \frac{1}{2\pi} \sum_{n=0}^{\infty} n! |n\rangle \langle n|. \] (9)

B. Runge’s approximation theorem and functions of the annihilation operator

In this subsection, \( f(a) \) where \( f \) is an analytic function defined on a compact domain \( \Omega \) that does not separate the complex plane, is constructed. Two classes of such functions are considered. The first class consist of functions \( f \) defined on domains containing 0: \( 0 \in \Omega \). The second class consists of functions which are not defined at 0: \( 0 \not\in \Omega \). A convenient theoretical tool for this is Runge’s polynomial approximation theorem.

**Theorem:** If \( f \) is an analytic function on a compact domain \( \Omega \) that does not separate the complex plane, then there exists a sequence \( P_l(z) \) of polynomials such that converges uniformly to \( f(z) \) on \( \Omega \) [4],[5],[6]:

\[ f(z) = \sum_{l=0}^{\infty} P_l(z - z_0) = \sum_{l=0}^{\infty} \sum_{k=0}^{d_l} c_k^{(l)} (z - z_0)^k, c_k^{(l)} \in C, z, z_0 \in \Omega. \] (10)

Let us consider the first class of analytic function \((0 \in \Omega)\) and let

\[ f(z) = \sum_{l=0}^{\infty} \sum_{k=0}^{d_l} c_k^{(l)} z^k, z \in \Omega \] (11)

be an expansion of \( f \) in a series of polynomials. Then [11] and [8] can be used to construct the function \( f(a) \) of annihilation operator:

\[ f(a) = \sum_{l=0}^{\infty} \sum_{k=0}^{d_l} c_k^{(l)} a^k \cdot I = -i \sum_{l=0}^{\infty} \sum_{k=0}^{d_l} c_k^{(l)} \oint_{|\gamma| = R} d\gamma \gamma^{k-1} |\tilde{\gamma}\rangle \langle \tilde{\gamma}| J. \] (12)

Using definition (6) of \( |\tilde{\gamma}\rangle \), and after performing integration, \( f(a) \) becomes:
\[ f(a) = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=n}^{d_l+n} c_{m-n}^{(l)} \sqrt{\frac{m!}{n!}} |n\rangle \langle m|, \]  

which for \( \alpha \in \Omega \) gives:

\[ f(a) |\alpha\rangle = f(a) |\alpha\rangle. \]  

Let us now examine the second class of \( f \). Again we can use (10), but requiring \( z_0 \neq 0, z_0 \in \Omega \). However, a repetition of the above procedure using (8) leads to a divergence problem. To resolve this issue, a different resolution of identity, one involving translation (7) is chosen:

\[ I = e^{z_0 a^\dagger} I e^{-z_0 a^\dagger}. \]  

If \( I \) in the right hand side of equation (15) is substituted with (8), and using (7) we get:

\[ I = -i \oint_{|\gamma|=R} d\gamma \gamma^{k-1} |\gamma + z_0\rangle \langle \gamma| J e^{-z_0 a^\dagger}. \]  

Now we apply the operator, obtained by polynomial approximation, to the identity (16):

\[ f(a) = \sum_{l=0}^{\infty} \sum_{k=0}^{d_l} c_k^{(l)} (a - z_0)^k \cdot I = -i \sum_{l=0}^{\infty} \sum_{k=0}^{d_l} c_k^{(l)} \oint_{|\gamma|=R} d\gamma \gamma^{k-1} |\gamma + z_0\rangle \langle \gamma| J e^{-z_0 a^\dagger}. \]  

Again, using definition (10) of \( |\gamma\rangle \), and after performing integration, \( f(a) \) becomes:

\[ f(a) = \sum_{l=0}^{\infty} \sum_{k=0}^{d_l} c_k^{(l)} \sum_{n=0}^{n+k} \sum_{m=n}^{m+k} \left( \begin{array}{c} n \\ m-k \end{array} \right) \sqrt{\frac{m!}{n!}} z_0^{n-m+k} |n\rangle \langle m| e^{-z_0 a^\dagger}, \]  

which for \( \alpha \in \Omega \) gives:

\[ f(a) |\alpha\rangle = f(a) |\alpha\rangle. \]  

We can simplify (18), collecting coefficients at dyads, as

\[ f(a) = \chi e^{-z_0 a^\dagger}, \]

\[ \chi = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \chi_{nm}^{(l)} |n\rangle \langle m|, \]

\[ \chi_{nm}^{(l)} = \sqrt{\frac{m!}{n!}} \sum_{k=p}^{s_l} c_k^{(l)} \left( \begin{array}{c} n \\ m-k \end{array} \right) z_0^{n-m+k}, \]

\( p = \max\{0, m-n\}, \)

\( s_l = \min\{m, d_l\}. \)
III. DISCUSSION AND CONCLUSION

In this paper, for the first time an expression for a non-entire function $f(a)$ of the annihilation operator is obtained. $f(z)$ is an analytic function on a compact domain $\Omega \subset \mathbb{C}$ that does not separate the complex plane. A very significant application of our result is its use in a construction of $\ln a$. Since $\ln z$ and $1/z$ can be defined on a domain which satisfies previous conditions, our method allows us to obtain $\ln a$ and $1/a$. Since, formally, $[\ln a, a^\dagger] = 1/a$, it follows

$$[a^\dagger a, -i \ln a] = i. \quad (21)$$

Operator $-i \ln a$ is conjugate to the Number operator. It is not self-adjoint, but due to commutation relation (21) it can serve as a good starting point for construction of the Phase Operator $[1,2]$. Considering the significance of this result, it will be topic of a separate paper.

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[1] Lj. Davidović, D. Arsenović, M. Davidović and D.M. Davidović, J. Phys. A: Math. Theor. 42, 235302 (2009).
[2] M. Davidović, D. Arsenović and D.M. Davidović, J. Phys.: Conf. Ser. 36, 46 (2006).
[3] A.M. Perelomov, Theor. Math. Phys. 6 (Engl. Transl.), 213 (1971).
[4] E.B. Saff, “Proceedings of Symposia in Applied Mathematics Vol.36”, (1986).
[5] A.I. Markushevich, ”Theory of function of complex variable”, (1977).
[6] L.V. Ahlfors, ”Complex Analysis” (1979).