Learning MAX-SAT from Contextual Examples for Combinatorial Optimisation

Mohit Kumar\textsuperscript{a}, Samuel Kolb\textsuperscript{a}, Stefano Teso\textsuperscript{b}, Luc De Raedt\textsuperscript{a}

\textsuperscript{a}KU Leuven, Belgium, firstname.lastname@cs.kuleuven.be
\textsuperscript{b}University of Trento, Italy, firstname.lastname@unitn.it

Abstract

Combinatorial optimization problems are ubiquitous in artificial intelligence. Designing the underlying models, however, requires substantial expertise, which is a limiting factor in practice. The models typically consist of hard and soft constraints, or combine hard constraints with an objective function. We introduce a novel setting for learning combinatorial optimisation problems from \textit{contextual} examples. These positive and negative examples show – in a particular context – whether the solutions are good enough or not. We develop our framework using the MAX-SAT formalism as it is simple yet powerful setting having these features. We study the learnability of MAX-SAT models. Our theoretical results show that high-quality MAX-SAT models can be learned from contextual examples in the realizable and agnostic settings, as long as the data satisfies an intuitive “representativeness” condition. We also contribute two implementations based on our theoretical results: one leverages ideas from syntax-guided synthesis while the other makes use of stochastic local search techniques. The two implementations are evaluated by recovering synthetic and benchmark models from contextual examples. The experimental results support our theoretical analysis, showing that MAX-SAT models can be learned from contextual examples. Among the two implementations, the stochastic local search learner scales much better than the syntax-guided implementation while providing comparable or better models.

\textit{Keywords:} Machine Learning, Constraint Learning, Combinatorial Optimization, Maximum Satisfiability, Soft Constraints, Contextual Examples
1. Introduction

Combinatorial optimisation is an effective and popular class of techniques for solving real-life problems like scheduling [1], routing [2], and planning [3]. However, encoding the underlying models often proves to be time-consuming and complicated, as it requires substantial domain and modeling expertise. Therefore, the question arises as to whether such models can be learned from data. This question is studied in constraint learning [4, 5], where several algorithms have been developed that automatically acquire theories or mathematical models from examples of past working (positive) and non-working (negative) solutions or analogous forms of supervision.

Combinatorial optimisation models have two components: a set of hard constraints $\phi$ defining the feasible region, and an objective function $f$ that measures the quality of candidate solutions, sometimes defined as a set of soft constraints. The task of the solver is then to complete a (potentially empty) partial assignment $x$ into a complete assignment $xy$ that is both feasible and optimal, i.e., $xy \models \phi$ and $y \in \arg\max_{xy \models \phi} f(xy)$. We use the term context to refer to both partial assignments $x$ and to more general temporary constraints that restrict the outcome of optimisation.

Current learning approaches suffer from two limitations. First, to the best of our knowledge, they do not learn from contextual examples. By doing so, they ignore the fact that the optima can be affected drastically by the context. For instance, $\arg\max_{x,xy \models \phi} f(xy)$, where the context fixes $y$, can be very different from $\arg\max_{y,xy \models \phi} f(xy)$, where the context fixes $x$. Furthermore, this is also a less realistic setting in practice, as examples of good and bad solutions will always be relative to a context. The reader may notice a resemblance with structured output prediction (e.g. [6]), where one learns a function $f$ that computes a structured output $y = \arg\max_y f(x, y)$ for a given input $x$. The difference is that in structured prediction the choice of input and output variables is fixed, while in optimisation it is not.

Second, existing approaches do not jointly learn the hard constraints and the objective function: they either learn one or the other, or else learn them sequentially or independently. But this may break down in applications like personnel rostering. Here, past schedules are often stored in a data set, but the reasons why a schedule was found to be unacceptable is usually not tracked. In cases like this, a negative example may be either infeasible (because of the hard constraints) or sub-optimal (because of the objective function). This induces a credit-assignment problem that can only be solved...
by learning the constraints and objective function jointly. See the related work section for a more in-depth discussion.

The key contribution of this paper is that we develop a more realistic setting for learning combinatorial optimisation models from contextual examples that does not suffer from these limitations. Furthermore, we provide foundational results within this setting for one of the simplest but most fundamental models for combinatorial optimisation, maximum satisfiability (MAX-SAT for short). Our theoretical results show that MAX-SAT models can be probably approximately correctly (PAC) andagnostically learned from contextual data using empirical risk minimization (ERM) as long as the contextual examples are “representative enough”, and that if enough data is available the acquired model is guaranteed to output high-quality feasible solutions.

Motivated by this, we introduce two implementations of ERM for MAX-SAT learning, HASSLE-MILP and HASSLE-SLS. HASSLE-MILP relies on ideas from syntax-guided synthesis [7], in that the learning task is encoded as an optimization problem – namely, a mixed-integer linear programming (MILP) problem – and solved using an efficient solver. HASSLE-MILP acquires a MAX-SAT model that is guaranteed to fit the examples (almost) exactly, if one exists. This accuracy, however, comes at the expense of run-time. The second implementation, HASSLE-SLS, uses stochastic local search (SLS) to look for a high-quality MAX-SAT model, and in addition integrates a heuristic to prune the neighborhood of the current candidate model and focus on the most promising neighbors. HASSLE-SLS is not guaranteed to return an (almost) optimal model, but it offers enhanced efficiency. Our experiments show that, on the one hand, HASSLE-MILP successfully recovers both synthetic and benchmark MAX-SAT models from contextual examples, and on the other, that HASSLE-SLS matches the model quality of HASSLE-MILP in a fraction of the time and scales to learning problems beyond the reach of the more exact implementation.

A preliminary version of this work appeared as a conference paper [8]. The present work contributes the following major improvements:

- The theoretical results in the conference paper hold for the realizable, noiseless case only, and only guarantee that, given enough examples, the learned model will perform well at solving time in the presence of the empty context. Here, we show that, under mild assumptions, these results apply to the agnostic and noisy cases. In addition, we
show that the learned model will perform well in *arbitrary* contexts at solving-time and introduce a tighter regret bound, see Theorem 3.

- The only implementation available in the conference paper is HASSLE-MILP. Here we introduce a new implementation based on stochastic local search [9], named HASSLE-SLS. In particular we studied four different versions of HASSLE-SLS, based on some of the most widely used SLS techniques [10]: WalkSAT, Novelty, Novelty+ and Adaptive Novelty+. Compared to the original implementation, HASSLE-SLS scales to larger learning tasks while acquiring models of comparable or better quality in practice. The implementation is non-trivial as evaluating the score of each neighbour requires solving a MAX-SAT model. To keep the run-time low, we designed techniques to restrict the neighbourhood to those models which show some promise of being better than the current candidate.

This paper is structured as follows: Section 2 provides notations and definitions for various terms used throughout the text and Section 3 provides a formal definition of MAX-SAT learning. In Section 4 we first prove PAC learnability of MAX-SAT learning and that this provides guarantees on the quality of the assignment output by the learned model (relative to those output by the ground-truth model). In Section 5 we present the two implementations and evaluate them in Section 6. The related work in discussed in Section 7 and the paper concludes in Section 8. For ease of exposition, all the proofs are deferred to the Appendices. A summary of all the notations is provided in Table 1.

2. Preliminaries

2.1. Maximum Satisfiability

Let \( \mathbf{X} = \{X_1, \ldots, X_n\} \) be a set of Boolean variables and \( \Phi = \{\phi_1, \ldots, \phi_m\} \) a class of Boolean formulas of interest on \( \mathbf{X} \), e.g., the set of conjunctions or disjunctions of up to \( k \) literals (i.e., variables or their negations). An assignment \( \mathbf{x} = (x_1, \ldots, x_n) \) fixes the value of each \( X_i \) to \( x_i \). In practical implementations, we will use 1 and 0 to encode true and false, respectively.

A *partial maximum satisfiability* (abbreviated MAX-SAT) model \( \mathcal{M} \) is a collection of hard and soft constraints taken from \( \Phi \) [11]: the hard constraints define a feasible region while the soft ones define a preference.
relation over feasible assignments. The hard constraints are encoded by a vector $c \in \{0, 1\}^m$ such that $c_j = 1$ if $\phi_j$ appears as a hard constraint in $\mathcal{M}$ and zero otherwise. The conjunction of all hard constraints is indicated as:

$$\phi(c) := ((c_1 = 1) \Rightarrow \phi_1) \land \ldots \land ((c_m = 1) \Rightarrow \phi_m)$$

The soft constraints are encoded as a weight vector $w \in [-1, 1]^m$, where $w_j \neq 0$ if $\phi_j$ appears as a soft constraint with weight $w_j$ and $w_j = 0$ otherwise. Hence, each constraint $\phi_j \in \Phi$ can be hard ($c_j = 1$ and $w_j$ arbitrary), soft ($c_j = 0$ and $w_j \neq 0$), or irrelevant ($c_j = w_j = 0$). All MAX-SAT models can be written in this form by normalizing their weights.

The value $f_w(x)$ of an assignment $x$ is the total weight of the constraints that it satisfies, that is:

$$f_w(x) := \sum_{j=1}^m w_j \mathbb{1}\{x \models \phi_j\}$$

where the indicator function $\mathbb{1}\{cond\}$ evaluates to 1 if $cond$ holds and to 0 otherwise. Solving a MAX-SAT model amounts to finding a feasible assignment that has optimal value, that is:

$$\arg\max_x f_w(x)$$

s.t. $x \models \phi(c)$

or $\arg\max_{x : \models \phi(c)} f_w(x)$ for short. Here $x \models \phi$ indicates that $x$ satisfies $\phi$.

**Example 1.** Consider a nurse rostering problem where $N$ is the set of nurses and $T$ is the set of shifts in a day. Let $X_{n,t}$ be a Boolean variable that takes value 1 if nurse $n \in N$ is scheduled for shift $t \in T$ and 0 otherwise. Consider a set of soft constraints which ensures that the total number of working shifts

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1. The weights of hard constraints can be safely ignored, as they offset the value of all feasible assignments by the same amount.

2. Notice that the range of the weights $w$ is immaterial: it is always possible to rescale the weights to any range $[a, b]$ without altering the ranking of feasible assignments (namely, by offsetting and multiplying the value of all assignments by appropriate amounts). As a matter of fact, our algorithms learn weights in the range $[0, 1]$. We keep negative weights here to stress that they can be used to encode negative preferences.
is minimal, and a set of hard constraints ensuring that each shift has a nurse. These can be encoded as follows:

**Soft Constraints:** \( \forall n \in N, t \in T : \neg X_{n,t} \) with weight \( w_{n,t} = 1 \)

**Hard Constraints:** \( \forall t \in T : \bigvee_{n \in N} X_{n,t} \)

Then the possible assignments to \( X = \{X_{n,t} : n \in N, t \in T\} \) represent the set of schedules for all nurses and shifts in a single day. Clearly, an optimal assignment \( x \) satisfies the hard constraints and maximizes the total weight of the satisfied soft constraints \( f(x) = \sum_{n,t} 1 \cdot 1\{x \models \neg x_{n,t}\} \).

Although solving MAX-SAT is NP-hard in general [12], practical solvers have recently shown impressive performance on highly non-trivial instances.

### 2.2. Contexts

In real-life applications, decisions are influenced by temporary conditions like resource availability and other kinds of restrictions. For instance, in personnel scheduling some employees may be unavailable because they are sick or on leave, while in routing tasks part of the network may be down due to a temporary failure.

Such restrictions are captured by the notion of *context*. As in the previous examples, a context \( \psi \) fixes the value of one or more decision variables and can be viewed as a conjunction of literals. Letting \( \mathcal{M} \) be a MAX-SAT model, solving in a context \( \psi \) amounts to:

\[
\arg\max_{x \models \phi(c) \land \psi} f_w(x)
\]

Note that the context, as well as the set of variables it fixes (if any), is likely to change over time.

Contexts can do more than specifying partial assignment. In rostering applications, for instance, an employee may be unable to work more than four hours a day due to various contingencies (e.g., pregnancy), while in routing some paths may be shut down due to accidents. In the most general sense, a context \( \psi \) specifies an arbitrary Boolean formula that does restrict – but does not necessarily fix – some decision variables. For example, the fact that one out of several radiologists is absent is best represented as a disjunction.

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[3] See for instance: [https://maxsat-evaluations.github.io](https://maxsat-evaluations.github.io)
Example 2. Consider our nurse rostering problem. In this case, a context could be that a nurse $n_1$ was on leave on a particular day, i.e., $\psi = \bigwedge_{t \in T} \neg X_{n_1,t}$ or that the hospital was closed for shift $t_2$, i.e., $\psi = \bigwedge_{n \in N} \neg X_{n,t_2}$.

In the most general terms, contexts specify temporary constraints that a particular (set of) example(s) was obtained in. Notice that, in contrast to the constraints that define the MAX-SAT model, they are supplied together with the data rather than learned from the examples.

Contexts are central in optimization problems as they can significantly alter the properties and quality of optima: optimal schedules may be substantially better if a particularly skilled worker is available, and much worse otherwise. From a learning perspective, contexts change the way a model is evaluated: a model that performs well in a context might perform poorly in other contexts. This means that failing to take contexts into account may lead to evaluate candidate MAX-SAT models poorly and therefore to prefer under-performing models over better ones.

3. Problem Statement

In contrast to existing constraint learning methods, we consider a more realistic setting where example solutions and non-solutions are context-specific. Here and below, $\Psi^*$ indicates the set of possible contexts. More specifically, we assume each example to be generated as follows:

1. A context $\psi \in \Psi^*$ is observed;
2. A assignment $x$ that satisfies context $\psi$ (i.e., $x \models \psi$) is chosen according to some policy, e.g., by asking a domain expert to provide either a high-quality solution or a non-solution;
3. $x$ is labelled as positive, $y = 1$, if it works well enough in practice under the observed temporary restrictions, i.e., context $\psi$, and is labelled negative $y = 0$ otherwise.

This induces a ground-truth distribution $D(y,x,\psi) = D(\psi)D(x \mid \psi)D(y \mid x,\psi)$. We are now ready to define our learning task:

**Definition 1** (MAX-SAT learning). Given a set of Boolean variables $X$, candidate constraints $\Phi$, and context-specific examples $S := \{(\psi_k, x_k, y_k) : k = 1, \ldots, s\}$ sampled independently and identically distributed (i.i.d.) from an unknown distribution $D$, find a MAX-SAT model $M$ with parameters $c, w$ that can be used to obtain high-quality assignments in any context $\psi$. 7
The exact nature of assignment quality will be formalized in the next section using the notion of regret \[13\].

A few remarks are in order. First, we do not know which negative examples are infeasible and which ones are sub-optimal. This introduces a credit assignment problem: if $x$ is a false positive, should $c$ be changed to make it infeasible or should $w$ be adapted to make it sub-optimal? For this reason, the soft and hard constraints have to be acquired simultaneously.

Second, all assignments are expected to include a description of the context in which they were obtained. This is often the case in real-life applications: in rostering, past schedules are annotated with the unavailable employees, while in routing, the conditions under which the congestion occurred (e.g., maintenance work) are kept track of for monitoring purposes. Applications in which context annotations are not available could be addressed by treating the unobserved contexts as latent variables. This extension is non-trivial and therefore left to future work. Below we provide an example of our learning task.

**Example 3.** Consider our nurse rostering problem. Let us assume that $N = \{n_1, n_2\}$ and $T = \{t_1, t_2\}$, then $X = \{X_{n_1, t_1}, X_{n_1, t_2}, X_{n_2, t_1}, X_{n_2, t_2}\}$. Now consider these context specific examples, where each example consists of three elements, first is the observed context, second is an assignment to the variables in $X$ and finally, third is the label specifying whether the assignment is a positive or negative sample in the observed context:

$$\begin{align*}
S &= \{(\neg X_{n_1, t_1} \land \neg X_{n_2, t_2}, \{1, 0, 0, 0\}, 1),
(\neg X_{n_1, t_2} \land \neg X_{n_2, t_2}, \{1, 0, 1, 0\}, 0)\}
\end{align*}$$

Both examples correspond to the context that the hospital is closed for shift $t_2$. In this context, the first example is a positive while the second one is a negative. The aim of our work is to learn the soft and hard constraints, provided such context specific examples.

4. **Learnability of MAX-SAT Models**

In this section, we study the learnability of MAX-SAT models from contextual examples from the perspective of statistical learning theory\[4\]. Basic
knowledge of statistical learning theory is assumed; the necessary material, including the definitions of Vapnik-Chervonenkis dimension and the theorems that link it to generalization outside of the training set, can be found in Shalev-Shwartz and Ben-David [14].

Our analysis relies on a reduction of MAX-SAT learning to learning binary “MAX-SAT classifiers”, to which results from statistical learning theory apply. The analysis is split into three steps:

1. In Section 4.2 we define MAX-SAT classifiers and show that they can be learned from context-specific examples using Empirical Risk Minimization [15].

2. In Section 4.3 we prove that MAX-SAT classifiers with low context-specific risk correspond to MAX-SAT models with high-quality context-specific solutions.

3. Finally, in Section 4.4 we show that, under a suitable “representativeness” condition, MAX-SAT classifiers with low context-specific risk on the contexts annotated in the data enjoy low context-specific risk in any target context.

Taken together, these results entail that MAX-SAT classifiers learned from enough context-specific examples correspond (again, under suitable assump-
tions) to MAX-SAT models that produce high-quality solutions in any target context. This shows that Empirical Risk Minimization (ERM) is a valid solution strategy for MAX-SAT learning and motivates our ERM-based implementations, introduced in the next Section.

4.1. Prerequisites

Before proceeding, we provide a brief summary of some key results from statistical learning theory [14]. Let $\mathcal{H}$ be a class of binary classifiers (aka hypotheses) $h : \mathcal{X} \rightarrow \{0, 1\}$, where $\mathcal{X}$ is the set of assignments to decision variables in $X$, and let $h^*$ be the ground-truth classifier used to label a training set $S$. If $h^* \in \mathcal{H}$ we say that the setting is realizable, otherwise we call the setting agnostic. Given a hypothesis $h \in \mathcal{H}$, the risk $L_D(h)$ of $h$ is the probability that it misclassifies an assignment, that is:

$$L_D(h) := D(h(x) \neq y)$$

where $D$ is the unobserved, ground-truth distribution, while the empirical risk $L_S(h)$ is the probability that it misclassifies an example in $S$, that is:

$$L_S(h) := \frac{1}{s} \sum_{k=1}^{s} 1\{h(x_k) \neq y_k\}$$

Where $s = |S|$. The following definitions are taken from [14]:

**Definition 2** (PAC learnability). A hypothesis class $\mathcal{H}$ is PAC learnable in the realizable setting if there exists a function $s_{\mathcal{H}} : (0, 1)^2 \rightarrow \mathbb{N}$ and a learning algorithm such that, for every $\epsilon, \delta \in (0, 1)$ and distribution $D$ over $\mathcal{X} \times \{0, 1\}$, running the algorithm on $s \geq s_{\mathcal{H}}(\epsilon, \delta)$ i.i.d. examples sampled from $D$ returns a hypothesis $h$ such that:

$$\Pr(L_D(h) > \epsilon) \leq \delta$$

A hypothesis class $\mathcal{H}$ is PAC learnable in the agnostic setting if, under the same conditions as above, it holds that:

$$\Pr(L_D(h) > \min_{h' \in \mathcal{H}} L_D(h') + \epsilon) \leq \delta$$

PAC learnability simply means that, given enough examples, one can identify with high probability a hypothesis whose risk is very close to the risk of the best possible hypothesis.
Definition 3 ($\epsilon$-representative). A training set $S$ is $\epsilon$-representative iff:
\[
\forall h \in \mathcal{H}, \ |L_S(h) - L_D(h)| \leq \epsilon
\] (6)

Definition 4 (Uniform convergence). A hypothesis class $\mathcal{H}$ is said to have the uniform convergence property if there exists a function $s_\mathcal{H} : (0,1)^2 \rightarrow \mathbb{N}$ such that for every $\epsilon, \delta \in (0,1)$ and distribution $D$ over $\mathcal{X}$, if $S$ is a sample of $s \geq s_\mathcal{H}(\epsilon, \delta)$ examples drawn i.i.d. from $D$, then, with probability of at least $1 - \delta$, $S$ is $\epsilon$-representative.

The uniform convergence property simply says that, given enough examples, the empirical risk is very close to the true risk.

It is well known that if a hypothesis class $\mathcal{H}$ is “simple”, i.e., it has finite Vapnik-Chervonenkis (VC) dimension [14], then it has the uniform convergence property:

Theorem 1 (Theorems 6.7 and 6.8 from [14]). Let $\mathcal{H}$ be a hypothesis class with finite VC dimension $d := \text{VC}(\mathcal{H}) < \infty$. Then $\mathcal{H}$ has the uniform convergence property with sample complexity:
\[
\frac{C_1}{\epsilon^2} \left( d + \log \left( \frac{1}{\delta} \right) \right) \leq s_\mathcal{H}(\epsilon, \delta) \leq \frac{C_2}{\epsilon^2} \left( d \log \left( \frac{1}{\epsilon^2} \right) + \log \left( \frac{1}{\delta} \right) \right)
\]

Here $C_1$ and $C_2$ are universal constants.

The theorem entails that searching for a hypothesis with minimal empirical risk – a strategy called Empirical Risk Minimization (ERM) – is guaranteed to find a hypothesis with low risk in the realizable and agnostic settings with high probability, as long as enough examples are available. ERM achieves similar results even if the example labels are flipped (due to noise) independently at random with probability $0 \leq \rho < \frac{1}{2}$, in which case the sample complexity is asymptotically identical to the one for the agnostic setting except for an extra factor $(1 - 2\rho)^{-2}$, cf. [16].

4.2. Learnability in a Fixed Context

We start by introducing the central notion of MAX-SAT classifier.

Definition 5 (MAX-SAT classifier). Given a MAX-SAT model $\mathcal{M}$ with parameters $\mathbf{c}$ and $\mathbf{w}$, the corresponding MAX-SAT classifier $h_{\mathbf{c}, \mathbf{w}} : \mathcal{X} \times \Psi^* \rightarrow \{0,1\}$ is defined as:
\[
h_{\mathbf{c}, \mathbf{w}}(\mathbf{x}, \psi) = 1 \left\{ \mathbf{x} \in \arg\max_{\mathbf{x}' \models \phi(\mathbf{c}) \land \psi} f_{\mathbf{w}}(\mathbf{x}') \right\}
\]
In other words, a MAX-SAT classifier \( h_{c,w}(x, \psi) \) labels assignments \( x \) as positive if and only if they are solutions to the corresponding MAX-SAT model, that is, \( \text{feasible} \) with respect to \( \phi(c) \land \psi \) and \( \text{optimal} \) with respect to \( f_w \).

The set of MAX-SAT classifiers will be henceforth referred to as:

\[
\mathcal{H} := \{h_{c,w} : c \in \{0, 1\}^m, w \in [-1, 1]^m\}
\]

In the following, we momentarily focus on MAX-SAT classifiers restricted to a single context \( \psi \in \Psi \):

\[
\mathcal{H}_\psi := \{h(\cdot, \psi) : h \in \mathcal{H}\}
\]

Our first result is that, regardless of the choice of context \( \psi \), \( \mathcal{H}_\psi \) has finite VC dimension.

**Theorem 2.** For every \( \psi \), it holds that \( \text{VC}(\mathcal{H}_\psi) \leq u(2m + 1) \), where \( u < 5 \).

All proofs can be found in the Appendix. Now, let \( L_{S,\psi} \) be the empirical risk of \( h \) in the context \( \psi \), that is:

\[
L_{S,\psi}(h) := \frac{1}{s} \sum_{k=1}^{s} \mathbb{1}\{h(x_k, \psi) \neq y_k\}
\]

Similarly, let \( L_{D,\psi}(h) \) be the true risk of \( h \) w.r.t. context \( \psi \), that is:

\[
L_{D,\psi}(h) := \mathbb{E}_{y,x|\psi}[\mathbb{1}\{h(x, \psi) \neq y\}] = \sum_{x \mid x = \psi,y} \mathbb{1}\{h(x, \psi) \neq y\} D(x, y | \psi)
\]

Taken together, Theorems 1 and 2 show that \( \mathcal{H}_\psi \) has the uniform convergence property:

**Corollary 1.** For any \( \psi \in \Psi^* \), and \( \epsilon_\psi, \delta_\psi \in [0, 1] \), there exists an integer \( s_{\mathcal{H}_\psi}(\epsilon_\psi, \delta_\psi) \in \mathbb{N} \) such that, given a dataset \( S \) with at least \( s_{\mathcal{H}_\psi}(\epsilon_\psi, \delta_\psi) \) examples specific to \( \psi \), any hypothesis \( h \in \mathcal{H}_\psi \) satisfies the following:

\[
\Pr(L_{D,\psi}(h) < L_{S,\psi}(h) + \epsilon_\psi) \geq 1 - \delta_\psi
\]

The sample complexity \( s_{\mathcal{H}_\psi}(\epsilon_\psi, \delta_\psi) \) is given by Theorem 1.

In turn, this means that \( \mathcal{H}_\psi \) is PAC learnable: given enough examples, with high probability the true risk is bounded by the empirical risk and ERM outputs a low-risk hypothesis.
4.3. From Classification to Optimization

We have just shown that ERM can be used to learn low-risk MAX-SAT classifiers for any given context. Next, we show that these classifiers correspond to good MAX-SAT models, that is, models useful for combinatorial optimization.

The regret refers to the cost incurred by using the solutions output by learned model $M$ in place of those of the ground-truth $M^*$ \[13\]. Let $\text{opt}(\psi) := \{x \in \{0, 1\}^n : h(x, \psi) = 1\}$ be the set of context-specific optima of $h$ in context $\psi$ and $\text{opt}^*(\psi)$ be the same for ground-truth classifier $h^*$. Then, the context-specific and average regret are defined as follows:

**Definition 6 (Regret).** Let $h^*$ be the true hypothesis and $x^* \in \text{opt}^*(\psi)$. The regret $\text{reg}(x, \psi)$ of using assignment $x$ in context $\psi$ is:

$$\text{reg}(x, \psi) := \begin{cases} f_{w^*} - f_w & \text{if } x \models \psi \land \phi(c) \\ r_{\text{max}} & \text{otherwise} \end{cases}$$

where $r_{\text{max}}$ is the regret suffered by using an infeasible assignment. The average regret $\text{reg}(h, \psi)$ of a hypothesis $h$ in context $\psi$ is then the average regret of its optima, i.e.,

$$\text{reg}(h, \psi) := \frac{1}{|\text{opt}(\psi)|} \sum_{x \in \text{opt}(\psi)} \text{reg}(x, \psi)$$

Hence, the average regret boils down to the average difference in quality between the assignments classified as positive by $h$ and by $h^*$.

Notice that our definition of regret covers cases in which the learned model outputs infeasible assignments, in which case the regret is $r_{\text{max}}$. The next theorem depends on the assumption that $r_{\text{max}}$ is finite, which is however reasonable in many applications. For instance, if the learned model produces a schedule in which one nurse too many is asked to go to work, the regret is roughly equivalent to the hourly pay of that nurse.

As long as $r_{\text{max}} < \infty$, the next result links the risk of a MAX-SAT classifier and the regret of the corresponding MAX-SAT model:

**Theorem 3.** Let $\eta = \min_{x : D(x | \psi) > 0} D(x | \psi)$ and $r_{\text{max}}$ be the regret associated to infeasible assignments. Then, for any $h \in \mathcal{H}$ and $\psi \in \Psi^*$, it holds that:

$$\text{reg}(h, \psi) \leq \frac{\|w^*\|_1 + r_{\text{max}}}{\eta |\text{opt}(\psi)|} L_{D, \psi}(h)$$
This inequality holds for both realizable and agnostic settings.\(^5\)

As a consequence, minimizing the (empirical) risk of a MAX-SAT classifier entails minimizing the regret of the corresponding MAX-SAT model.

Clearly, in applications in which infeasible assignments can violate societal or safety requirements, \(r_{\text{max}}\) may be infinite and the bound in Theorem 3 becomes void. This is expected, as in these conditions no worst-case guarantees can be given about the quality of the solutions output by any learned model. This essentially shows that, understandably, extra care must be taken when applying constraint learning to high-stakes domains.

### 4.4. Generalizing across Contexts

So far we have only considered learning and solving within a given context \(\psi\). However, learned models are likely going to be used in previously unobserved contexts. Next, we show that MAX-SAT classifiers learned from context-specific data do generalize, under mild assumptions, to any given target context \(\psi_t\) – potentially distinct from the ones observed in the data set – and therefore have low regret in \(\psi_t\) too.

Intuitively, in order for generalization to occur, the observed contexts have to be collectively representative of the target one. As an example, in nurse rostering, if all the example schedules were collected during a festive period in which most nurses are on holiday, it is unlikely that the learned model generalizes to non-festive periods in which most nurses are available. More formally, we say that a set of contexts \(\Psi\) is representative if whenever \(h(\cdot, \psi_t)\) makes a mistake, there is at least a context in \(\Psi\) that “catches” that mistake.\(^6\)

**Definition 7 (Representativeness).** Fix a ground-truth classifier \(h^* \in \mathcal{H}\) and a hypothesis \(h \in \mathcal{H}\). The formula \(\chi(\psi, x, \psi')\) holds iff whenever \(h\) misclassifies \(x\) in context \(\psi\) it also misclassifies \(x\) in context \(\psi'\), that is:

\[
\chi(\psi, x, \psi') = (x \models \psi \land \psi') \\
\land (h(x, \psi) \neq h^*(x, \psi) \Rightarrow h(x, \psi') \neq h^*(x, \psi'))
\]

\(^5\)This bound improves on the one provided by Kumar et al. \(8\) by a constant factor.

\(^6\)This definition generalizes the one in Kumar et al. \(8\) to arbitrary target contexts \(\psi_t\), and as a consequence Theorem 4 is not limited to the top context \(\psi_t = \top\) anymore, here \(\top\) simply means global context, i.e., the absence of a context.
\begin{table}[h]
\centering
\begin{tabular}{ccc}
$X_1$ & $X_2$ & $X_3$ \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{tabular}
\caption{Example of representative contexts: $\psi_1 = X_1$ (blue), $\psi_2 = \neg X_1$ (violet), $\psi_3 = X_2 \wedge X_3$ (red) are representative for $\Phi = \{X_1, X_2, X_3, \neg X_1, \neg X_2, \neg X_3\}$ and $\mathbf{w}^* = \{1, 1, 1, 0, 0, 0\}$; details in the text. (Best viewed in color.)}
\end{table}

The number of contexts to which misclassification errors propagate to is:

$$\#(\psi, \mathbf{x}) = |\{\psi' \in \Psi : \chi(\psi, \mathbf{x}, \psi')\}|$$

We say that a family of contexts $\Psi$ is representative for a target context $\psi_t$ if and only if $\#(\psi_t, \mathbf{x}) > 0$ for all assignments $\mathbf{x}$.

To see why representativeness is necessary, consider a ground truth model $\mathbf{c}^* = (0, 0, 0, 0, 0, 0)$ and $\mathbf{w}^* = (1, 1, 1, 0, 0, 0)$ with:

$$\Phi = \{X_1, X_2, X_3, \neg X_1, \neg X_2, \neg X_3\}$$

$$\psi_1 = X_1, \quad \psi_2 = \neg X_1, \quad \psi_3 = X_2 \wedge X_3, \quad \psi_t = \top$$

See Figure 1 for an illustration. Notice that under $\psi_1$ the optimum is $(1, 1, 1)$ with value 3, while under $\psi_2$ the optimum is $(0, 1, 1)$ with value 2. Now, take a hypothesis $h$ that misclassifies $\mathbf{x} = (0, 1, 1)$ as a global optimum, i.e., optimum in the target context $\psi_t$. The error does neither show up in $\psi_1$ (because $\mathbf{x}$ does not lie in it) nor in $\psi_2$ (because $\mathbf{x}$ is indeed optimal in it). In other words, $\chi(\psi_t, \mathbf{x}, \psi_1)$ and $\chi(\psi_t, \mathbf{x}, \psi_2)$ do not hold and $\#(\psi_t, \mathbf{x}) = 0$. Therefore, a classifier with arbitrarily low risk on both $\psi_1$ and $\psi_2$ may still misclassify $\mathbf{x}$. Requiring that $h$ performs well also on $\psi_3$ fixes this issue, because $\mathbf{x}$ does lie in $\psi_3$ but is not optimal in it – and $\chi(\psi_t, \mathbf{x}, \psi_3)$ holds.

Crucially, if $\Psi$ is representative and $h$ has low risk on all observed contexts in $\Psi$, then it performs well in the target context too:

**Lemma 1.** Let $\Psi$ be representative for $\psi_t$, and $D(\mathbf{x} \mid \psi) > 0$ for every $\psi \in \Psi$ and $\mathbf{x} \models \psi$. Then there exist finite constants $\beta_\psi \geq 0$ (that only depend on $\Psi$ and $D$) such that for every $h \in \mathcal{H}$:

$$L_{D,\psi_t}(h) \leq \sum_{\psi \in \Psi} \beta_\psi L_{D,\psi}(h)$$
We are now ready to state our main result:

**Theorem 4.** For any $\Psi$ and $D$ that satisfy the conditions of Lemma 1 and for every $\epsilon, \delta \in (0, 1)$, there exist integers $t_{\psi}(\epsilon, \delta)$ for $\psi \in \Psi$ such that, if $S$ contains at least $t_{\psi}(\epsilon, \delta)$ context-specific examples for every $\psi \in \Psi$, then any hypothesis $h$ with minimal empirical risk on $S$ satisfies:

$$
\Pr(L_{D,\psi}(h) > \min_{h' \in H} \sum_{\psi \in \Psi} L_{S,\psi}(h') + \epsilon) < \delta
$$

Notice that, in the realizable setting this reduces to:

$$
\Pr(L_{D,\psi}(h) > \epsilon) < \delta
$$

Summarizing, as long as the observed contexts are representative and there are enough examples, ERM learns a low-risk MAX-SAT model that has low regret in both the observed and in the target context with high probability.

5. Two Implementations

We are finally ready to present our implementations of MAX-SAT learning. Both are based on Empirical Risk Minimization, which, given $X$, $\Phi$, and a context-specific data set $S$ encompassing contexts $\Psi$, amounts to searching for a MAX-SAT model $\mathcal{M}$ that minimizes $\sum_{\psi \in \Psi} L_{S,\psi}(h)$. This equates to solving the following optimization problem:

\[
\text{find } \mathbf{c} \in \{0, 1\}^m, \mathbf{w} \in [-1, 1]^m
\]
\[
\text{s.t. } y_k \Leftrightarrow \left( \mathbf{x}_k \in \arg\max_{\mathbf{x}'} f_w(\mathbf{x}') \right) \quad k = 1, \ldots, s
\]

Intuitively, Eq. 7 searches for two vectors $\mathbf{c}$ and $\mathbf{w}$ that encode a MAX-SAT model, which is constrained by Eq. 8 to classify all examples correctly. More in detail, said model should predict all positive training examples to be both optimal (w.r.t. the learned objective $f_w$) and feasible (w.r.t. the learned hard constraints and the context $\phi(\mathbf{c}) \land \psi_k$) and all the negatives to be either sub-optimal or infeasible.

Different solution strategies can be applied. In the following, we introduce two different implementations, **HASSLE-MILP** and **HASSLE-SLS**, based
respectively on syntax-guided synthesis and on stochastic local search. For both the methods, we also assume the user to provide an upper bound on the number of constraints to be learnt. Some strategies can be used to avoid this, for example, start by learning a few constraints and then increase the number if the learnt model has a low accuracy, we can also learn different number of constraints in parallel and then pick one with the best accuracy. However we leave these extensions for the future work. Now we will discuss both HASSLE-MILP and HASSLE-SLS in turn.

5.1. MAX-SAT Learning with MILP

Following the literature on syntax-guided synthesis [7], we solve the ERM problem directly by encoding it in a suitable optimization formalism and then applying an appropriate solver. This solution offers three main advantages. The main one is that the hard problem of finding a suitable candidate is offloaded to the solver, in a declarative fashion. In addition, using an exact solver guarantees that the empirical error of the learnt model is exactly zero, and will trigger a warning in cases where this is impossible. A third advantage is that, if the solver supports anytime execution, then a sub-optimal solution can be obtained by specifying a time budget.\footnote{Some supposedly anytime solvers are not anytime in practice, as they have a bootstrap stage in which they look for an initial solution and only afterwards attempt to improve upon it. As a consequence, if the cutoff is shorter than the time taken to find this initial solution, then no candidate can be returned by the solver. This is what happens in our experiments.}

However, a naive encoding of Eq. 8 is non-trivial, because checking whether an example is positive (i.e., feasible and optimal) requires to solve a nested partial MAX-SAT model. Furthermore, this has to be done \( s \) times, once for each training example. Instead of dealing with nesting, we find it more convenient to encode the problem into a mixed integer linear programming (MILP) problem, as shown in Figure 2. The encoding can be split into two parts. First, the objective function (Eq 9) maximizes a set of fresh per-context variables \( \gamma_\psi \in \mathbb{R} \) and together with Eq. 10 computes a feasible optimum \( x'_{\psi \ell} \) for every context \( \psi \ell \in \Psi \) appearing in the data. Second, Eq. 11 ensures that \( c \) and \( w \) are chosen so that every positive example \( x_k \) is feasible and optimal in its own context \( \psi_k \) (using the context-specific optimum \( x'_{\psi_k} \) as a reference) and that no negative assignment is. Once solved, the learned MAX-SAT model can be read off of \( c \) and \( w \).
argmax_{c, w, γ} \sum_{ψ_ℓ \in Ψ} γ_ψ_ℓ \quad (9)
\text{s.t. } (x'_ψ \models φ(ψ) \land (γ_ψ \leq f_w(x'_ψ))) \quad (10)
ψ_ℓ \in Ψ
y_k \Leftrightarrow (x_k \models φ(ψ_k) \land (γ_ψ \leq f_w(x_k))) \quad k = 1, \ldots, s \quad (11)
γ \in \mathbb{R}^{|Ψ|}, c \in \{0, 1\}^m, w \in [0, 1]^m

Figure 2: \textsc{hassle-milp} encodes the learning constraint (Eq. 8) as a MILP problem. A simplified encoding is shown here (the full encoding is given in Appendix D). Eq. 11 encodes the main constraint, i.e., an example is positive if-and-only-if it is both feasible and optimal. Equations 9 and 10 solve the inner MAX-SAT optimization problem (γ_ψ_ℓ encodes the value of the optimal solution for context ψ_ℓ).

A couple of important remarks are in order. The first one is that γ_ψ is always bounded thanks to Eq. 10. In addition, for contexts that have at least one positive example, Eq. 10 is superfluous and can be omitted for efficiency.

The second one is that the MILP encoding does not exactly correspond to the encoding in Eq. 8 – it is, instead, a tight approximation. The intuition is as follows. To solve the nested MAX-SAT model, the MILP encoding uses variables γ_ψ_ℓ to represent the value of an optimal solution in a context ψ_ℓ according to the MAX-SAT model encoded by c and w. However, instead of individually maximizing the γ_ψ_ℓ, it has to resort to maximizing the sum \sum_{ψ_ℓ} γ_ψ_ℓ. This summation introduces a dependency that allows the encoding to choose γ sub-optimally for some contexts – allowing it to falsely label examples as positive that are not positive w.r.t. c and w – in order to increase the γ’s for different contexts. A detailed example is given in Appendix D. A simple post-processing step can detect such sub-optimal γ’s and an iterative procedure\footnote{In cases where falsely labeled positives are detected, an iterative procedure could be used to solve the problem to optimality: For every context ψ_ℓ for which there exists an assignment x’ with a value larger than a positive assignment x^+ \models ψ_ℓ: f_w(x') > f_w(x^+), a constraint: x' \neq φ(ψ) \lor f_w(x') \leq f_w(x^+) is added to the encoding. The extended encoding is solved and this procedure is repeated until all assignments are labeled correctly.} could be used to solve the problem, but our experimental results show that this mismatch occurs rarely in practice. Indeed, in our experiments, the MILP encoding makes a mistake on the training set less than 1% of the time.
5.2. MAX-SAT Learning with Stochastic Local Search

HASSLE-MILP presented in the previous section is meant as a near-exact proof of concept implementation, and while it provides guarantees on the empirical risk of the learned model, it is not optimized for efficiency. Using an exact solver in HASSLE-MILP also has a limitation. In an agnostic setting it will be unable to learn a model as the empirical error can not be reduced to zero. This also means that if the training data is noisy it might not be able to find a model. To overcome these limitations, we introduce an alternative implementation, HASSLE-SLS, that makes use of stochastic local search (SLS) [9] techniques.

Given a data set $S$ and a set of candidate constraints $\Phi$, the aim of HASSLE-SLS is to find $c \in \{0,1\}^m$ and $w \in [-1,1]^m$, such that the corresponding classifier $h_{c,w}$ correctly classifies as many examples in $S$ as possible, and it does so using SLS. The backbone of all SLS strategies is a heuristic search procedure that iteratively picks a promising candidate in the neighborhood of the current candidate, while injecting randomness in the search to escape local optima. Many SLS algorithms can be designed by changing the definition of the neighbourhood and the way in which a promising candidate is chosen. For HASSLE-SLS we designed these choices based on ideas borrowed from WalkSAT algorithm and its variants. This choice is based on the simplicity as well as a performance comparison made with other alternatives by Hoos and Tsang [17]. We will discuss the variants of WalksAT later in the section.

The pseudo-code for the WalkSAT version of HASSLE-SLS is shown in Algorithm 1. It starts with a hypothesis $h_{c,w}$ sampled at random from $\mathcal{H}$ (line 2), and then iterates through three steps in a loop. In the first step, a biased coin is tossed and with probability $p$ a random hypothesis is chosen as the next candidate (line 6), which is equivalent to a random restart. If the check fails, a neighbourhood function $\text{nbr}(h_{c,w}) : \mathcal{H} \rightarrow 2^\mathcal{H}$, where $2^\mathcal{H}$ is the power-set of $\mathcal{H}$, is used to generate neighbours of the current candidate hypothesis, a scoring function $\text{score}(h_{c,w}) : \mathcal{H} \rightarrow \mathbb{R}$ assigns a score to each hypothesis in the neighborhood, and finally a neighbour with the highest score is picked as the next candidate, while keeping track of the best candidate model found so far. The loop repeats until either a good enough hypothesis is found or an iteration or time budget is exhausted. The score of a classifier $h_{c,w}$ is defined to be the accuracy on the training set, i.e., the
Algorithm 1 The hasslesls WalkSAT algorithm.

1: procedure hasslesls($H$: candidates, $S$: dataset, $p$: restart probability, $s$: cutoff score, $t$: cutoff time)
2: $h ←$ random candidate from $H$
3: $h' ← h$
4: while score($h$) < $s$ $∧$ runtime < $t$ do
5:   with probability $p$ do
6:     $h ←$ random candidate from $H$ $▷$ random restart
7:   otherwise
8:     $h ←$ argmax$_{h ∈ nbr(h')}$ score($h$) $▷$ depends on SLS strategy
9:    if score($h$) > score($h'$) then
10:       $h' ← h$ $▷$ track best-so-far
11: return $h'$

number of correctly classified examples in the training set $S$:

$$score(h_{c,w}) = \sum_{k=1}^{s} \mathbb{1}\{h_{c,w}(x_k, \psi_k) = y_k\}$$

The choice of neighbourhood function is critical to the success of the learning procedure. In the following, we discuss the choice made for hasslesls.

Neighbourhood Function. Given a candidate hypothesis $h_{c,w}$ as input, the neighbourhood function $nbr(h_{c,w})$ determines the candidate models that can be reached in one search step. Notice that $nbr(h_{c,w})$ takes a hypothesis as input and gives a set of hypotheses as output. The hypotheses $h'_{c',w'} ∈ nbr(h_{c,w})$ can be obtained by either:

- Turning a soft constraint in $h_{c,w}$ into a hard constraint, that is, for some $j$ if $c_j = 0$ and $w_j \neq 0$ then $c'_j$ is set to 1. The corresponding weight $w'_j$ is set to 0, for simplicity.

- Turning a hard constraint in $h_{c,w}$ into a soft constraint, that is, for some $j$ if $c_j = 1$ then $c'_j$ is set to 0. The corresponding weight $w'_j$ is set to 1, for simplicity.

- Decreasing the weight of a soft constraint, that is, for some $j$ if $c_j = 0$ and $w_j \neq 0$ then $w'_j < w_j$, for simplicity we set $w'_j = w_j/2$. 

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• Increasing the weight of a soft constraint, that is, for some \( j \) if \( c_j = 0 \) and \( w_j \neq 0 \) then \( w'_j > w_j \), for simplicity we set \( w'_j = (1 + w_j)/2 \).

• Making a minimal change in an existing constraint. As each constraint is a clause in our case, this means either 1) adding a literal to a clause, 2) removing a literal from a clause or 3) changing the sign of a literal in a clause.

Evaluating each hypothesis in \( \text{nbr}(h_{c,w}) \) requires solving an optimization problem, so a large neighbourhood can significantly impact the efficiency of the algorithm. Moreover, notice that the size of the neighbourhood increases exponentially with the number of variables. Therefore, our algorithm won’t scale if we evaluate each neighbour to select the next candidate. Hence we designed a greedy approach to prune the neighbourhood, and evaluate only those models which show some promise of being better than the current candidate. To do so we take the following steps: 1) Randomly select one of the incorrectly classified examples by the current candidate, 2) Evaluate only those hypotheses in \( \text{nbr}(h_{c,w}) \) which can make the selected example consistent. Doing the first step involves finding the optimal value for the current candidate, i.e., solving just one optimization problem. However, doing the second step is not so trivial, one obvious way would be to evaluate each candidate in \( \text{nbr}(h_{c,w}) \) on the selected example, but it defeats the purpose of not evaluating each candidate. To avoid this, we prune the neighbourhood based on the true and predicted label of the incorrectly classified example selected in step 1. For instance, if the true label is positive while the learnt model predicts it to be infeasible, we evaluate only those neighbours which make changes in the hard constraints which are not satisfied by the example. This makes sense, because only making such change can make the example feasible. See Appendix E for the complete detail of how we prune the neighbourhood based on the true and predicted label.

**WalkSAT variants.** In the WalkSAT implementation we used random restarts, which allows the search to get out of a local optimum, however doing a random restart does not utilise the progress made so far. This shortcoming is overcome by using *tabu search*. It uses memory to prevent the search process from stagnating at a local optimum. The idea is to keep a short-term memory of most recently visited candidates and avoid revisiting them. WalkSAT has three extensions based on this idea: Novelty, Novelty\(^+\) and Adaptive Novelty\(^+\). Novelty keeps track of previous candidates and gives a preference
to the novel ones. Novelty + also does the same but with a probability chooses a random neighbour. Adaptive Novelty + takes it even further by changing the probability to choose a random neighbour: if the improvement achieved in an iteration is high, it keeps the probability low and as the improvement decreases, it increases the probability. In application none of these is a clear winner, their relative performance varies with the problem. Hence we implemented each of the three variants along with the basic version given in Algorithm 1. Through the experiments in the next section we analyze the accuracy of each method and answer various research questions.

6. Experiments

In this section, we empirically answer the following research questions:

Q1 Does ERM succeed in acquiring good quality MAX-SAT models from contextual examples?

Q2 Among the four implementations of hassle-sls, which SLS strategy performs the best and how does it compare to hassle-milp?

Q3 How does hassle-sls scale as the complexity of the ground-truth model increases?

Q4 How good is our strategy of neighbourhood pruning compared to a naive implementation where each neighbour is evaluated?

Q5 Is having both infeasible and sub-optimal negative examples essential to learn a good model?

Q6 Can hassle-sls handle noisy data?

Q7 Is having contextual information essential to learn a good model?

To this end, we implemented both hassle-milp and four different versions of hassle-sls based on different SLS strategies, namely WalkSAT, Novelty, Novelty + and Adaptive Novelty +. Then we used these five implementations to recover both synthetic and benchmark ground truth MAX-SAT models of increasing complexity from contextual examples. In the theory, to represent a MAX-SAT model, we use a vector of all constraints and a weight vector over all possible constraints, however, this is purely a theoretical concept. For
the implementations we restrict ourselves to a fixed number of constraints which are then implicitly (or lazily) generated during the search (finding literals using MILP or manipulating constraints using SLS). The complete experimental setup can be found at: https://github.com/mohitKULeuven/HassleWithLocalSearch

6.1. Datasets

The ground-truth synthetic models were generated by enumerating all possible disjunctions of up to \( n/2 \) literals of \( n \) variables, and then sampling \( c^* \) and \( w^* \) at random so that they have exactly \( m_{\text{hard}} \) hard and \( m_{\text{soft}} \) soft constraints. The weights of the soft constraints were sampled uniformly from \((0, 1]\). To answer Q3, synthetic models were generated by varying the number of variables: \( n = 8, 10, 12, 15 \), hard constraints \( m_{\text{hard}} = 2, 5, 10, 15 \), and soft constraints \( m_{\text{soft}} = 2, 5, 10, 15 \). Five random models were obtained for each assignment by changing the seed for randomization.

To validate our implementations on more realistic combinatorial models, we also chose five benchmark models from a repository of phase-transition SAT instances [18, 19]. Since benchmark models are meant to be used by solvers to test their performance, they are typically huge, with thousands of variables and constraints. We restricted the analysis to models with a reasonable number of variables, namely 20. In addition, the set of constraints given in the benchmark models were reduced from 91 to 20 through random selection. Note that these benchmarks are SAT instances, hence we created MAX-SAT instances by converting half of the randomly selected constraints into soft constraints by adding random weights.

For each ground-truth model \( \mathcal{M}^* \), a dataset \( S \) was collected by first sampling \( |\Psi| = 25, 50, 100 \) random contexts and then taking 2 positive and 2 negative examples from each context. Contexts were chosen to be a conjunction of \( n/2 \) literals, for simplicity. We also make sure that each context impacts the optimal solution of \( \mathcal{M}^* \). By negative example in a context \( \psi \), we refer to an example \( x \) that satisfies the context \( \psi \), but is infeasible or sub-optimal with respect to the model \( \mathcal{M}^* \). Negative examples are generated through rejection sampling, making sure that half of them are sub-optimal while half are infeasible. Generating truly random positive examples, however, is not trivial. Hence, we first enumerated 10 times the required number

\footnote{Taken from: www.cs.ubc.ca/~hoos/SATLIB/benchm.html}
of positive (optimal) examples by solving \( \mathcal{M}^* \) under the given context using a MAX-SAT solver\(^{10}\) and then resorted to reservoir sampling \([21]\) to obtain the required number of samples. For each model \( \mathcal{M}^* \), we generated 5 different set of data by changing the seed of randomization.

6.2. Performance Measures

The performance of the learned models were measured in terms of score (that is, training set accuracy) as well as accuracy in the global context, regret, and infeasibility. The score is the optimization criterion used (either implicitly or explicitly) by \textsc{hassle-milp} and \textsc{hassle-sls}, and it was used here to capture how well the various maximize their objective. The accuracy measures the performance of the acquired MAX-SAT model \( \mathcal{M} \) (when viewed as a classifier) in the global context, while regret measures the quality of its optima in the global context. Finally, infeasibility is defined as the percentage of optimal solutions in \( \mathcal{M} \) which are infeasible in \( \mathcal{M}^* \).

6.2.1. Computing accuracy and infeasibility using model counting

Calculating these performance measures is not trivial. To exactly measure accuracy and infeasibility we rely on a technique called Model Counting (MC or \#SAT). MC is a technique to count the feasible solutions of a propositional formula. If \( \text{MC}(\theta) \) denotes the model count of a formula \( \theta \) over Boolean variables \( X \), then we can compute the accuracy of a learned formula \( \theta^L \) compared to an optimal formula \( \theta^* \) by computing:

\[
\text{acc}_{\theta^*}(\theta^L) = \frac{\text{MC}(\theta^L \land \theta^*)}{2^{\mid X \mid}} + \frac{\text{MC}(-\theta^L \land -\theta^*)}{2^{\mid X \mid}}
\]

In our case, however, we learn MAX-SAT programs with both hard and soft constraints, and we are interested in both the accuracy of the optimal solutions and the infeasibility ratio. To use model counting we need to compute a logic formula that describes the optimal solutions of a MAX-SAT program \( \mathcal{M} \). Let us denote the hard constraints of \( \mathcal{M} \) as \( \mathcal{M}_H \), the weights of \( \mathcal{M} \) as \( w \), an optimal solution as \( x^* \) and its value according to \( \mathcal{M} \) as \( v^* = f_w(x^*) \). To describe the optimal solutions using a formula \( \theta_{\mathcal{M}} \), we find all subsets \( S_i \) of soft constraints of \( \mathcal{M} \) for which it holds that the sum of the weights of the soft constraints in \( S_i \) sums up to \( v^* \). We then obtain \( \theta_{\mathcal{M}} = \mathcal{M}_H \land (\bigvee_{S_i} \bigwedge_{\theta_s \in S_i} \theta_s) \).

\(^{10}\)Using the PySAT solver interface \([20]\).
i.e., a solution to $\theta_M$ must satisfy the hard constraints and satisfy one of the sets of soft constraints that achieves the optimal score. Using $\theta_M$ and $\theta_{M^*}$ we can now compute the accuracy as above and the infeasibility as:

$$\frac{\text{MC}(\theta_M \land \neg M_{H}^*)}{\text{MC}(\theta_M)}$$

6.2.2. Computing regret through sampling

Computing regret is even more tricky, notice that in Definition 6 regret for infeasible examples is a constant. In our experiments, instead of fixing this value we compute regret only for feasible examples. This makes sense as we already compute infeasibility, which gives us the percentage of examples which are infeasible. Hence in all the experiments regret and infeasibility should be seen together to get the complete picture. To calculate regret, we generated 1000 positive examples using $M$ such that these examples are also feasible in $M^*$ and then computed regret using Definition 6. To simplify the comparison, the regret was normalized to $[0, 1]$ by dividing it by $f_{w^*}(x^*)$, where $x^*$ is an optimum of $M^*$. We split the discussion of results in three parts, first we see the performance on synthetic data, then on noisy data and then finally on benchmark instances.

6.3. Results on Synthetic Data

For each experiment we randomly restart the local search if we get stuck at a candidate for more than a quarter of the cutoff time, which is the maximum time an algorithm is allowed to run before producing a learnt model. The probability of randomly selecting a neighbour for Novelty$^+$ and Adaptive Novelty$^+$ is 0.1. Note that here 0.1 is the probability of picking a random neighbour from the neighbourhood, while the probability $p$ in Algorithm 1 is the probability of picking any random candidate, not necessarily in the neighbourhood. The other parameters for Adaptive Novelty$^+$ has been assigned the same values as in the experiments of the paper introducing this method [22].

**Q1** Does ERM succeed in acquiring good quality MAX-SAT models from contextual examples?

We report the performance of all implementations on synthetic data in Figure 3. Different columns in the figure corresponds to different number of contexts used to sample training data. From each context 2 positive and 2
negative examples were sampled, so increasing the number of contexts leads to larger training set. On the $x$-axis of each graph we have cutoff time in minutes. Each row corresponds to different performance measure plotted on the $y$-axis. We can make the following observations from these plots: 1) Increasing the size of the training set leads to a better overall performance (3rd column) proportionally to the cutoff; 2) When using 100 contexts with WalkSAT version of HASSLE-SLS, we achieve excellent performance: 85% accuracy, 15% infeasibility, and less than 6% regret. These numbers clearly show that ERM can learn high-quality MAX-SAT models using only contextual examples, thus answering Q1 in the affirmative.

Figure 3: Performance evaluation on synthetic data. Performance of HASSLE-SLS improves as we increase the size of the training set by increasing $|\Psi|$, with WalkSAT showing the best performance among all the implementations.
Among the four implementations of HASSLE-SLS, which SLS strategy performs the best and how does it compare to HASSLE-MILP?

Looking at Figure 3, it is clear that the overall performance of WalkSAT is better than the other implementations across all performance metrics. For this reason, we focused on the WalkSAT version of HASSLE-SLS in the following experiments and will refer to it simply as HASSLE-SLS, unless stated otherwise. Next, we compared HASSLE-MILP and HASSLE-SLS to measure their scalability as well as quality. The results are shown in Figure 4. In the left plot we show the percentage of cases across all experiments where HASSLE-MILP and HASSLE-SLS were able to learn a model in the provided cutoff time of 1 hour. As expected, HASSLE-MILP fails to learn models due to the cutoff when we increase the size of the training set, while HASSLE-SLS learns a model in every case. Even in the cases where HASSLE-MILP learns a model (Table in Figure 4), the performance of HASSLE-SLS is much better. Hence, HASSLE-SLS outperforms HASSLE-MILP both in terms of quality and scalability.

How does HASSLE-SLS scale as the complexity of the ground-truth model increases?

To answer Q3, we see the change in performance by increasing the complexity of the target model, which is achieved by increasing the number of variables or the number of hard and soft constraints in $\mathcal{M}^\ast$. We report the results for 100 input contexts. The results are shown in Figure 5. When increasing the number of variables, we see a significant dip in the accuracy, this is due to the fact that the size of the feasible region increases exponentially,
making it harder to learn it accurately with a small training set. However, the good thing is that the infeasibility and regret do not suffer, thus the learned model can still be used to generate good quality solutions. Increasing the number of constraints, on the other hand, decreases the size of the feasible region, thus the accuracy improves when we increase either the number of soft constraints (2nd plot) or hard constraints (3rd plot). This also leads to better infeasibility and regret in the 2nd plot, however when increasing the number of hard constraints we observe slight increase in infeasibility, while regret does not show any significant change. However, since increasing the number of contexts and the cutoff time improves the performance, as stressed by our theory and shown in Figure 3, we expect that allowing more contexts, examples, and time to the learner would be enough to improve performance in these more complex models.

Q4 How good is our strategy of neighbourhood pruning compared to a naive implementation where each neighbour is evaluated?

To answer Q4, we compared HASSLE-SLS with a naive version where we do not prune the neighbourhood. When the number of contexts is 50 and cutoff time is 1 hour, HASSLE-SLS explores 56% less neighbours on average and achieves more than 9% improvement in the score, from 89% to 97%. This further translates to 73% drop in infeasibility from 57% to 15% and 36% drop in regret from 8.9% to 5.7%. Most importantly HASSLE-SLS achieves these improvements in just 15 minutes on average, which is much less compared to
Table 2: Having sub-optimal examples is crucial to learn a good model. While, infeasible examples do not seem to be very informative, thus having only sub-optimal examples would also work to get a good model.

35 minutes taken by the naive version. This allows us to confidently answer Q4 in the affirmative.

Q5 Is having both infeasible and sub-optimal negative examples essential to learn a good model?

To investigate the relative impact of sub-optimal and infeasible negative examples, we varied the percentage of the two in the data. We created three scenarios: in the first one all negative examples are infeasible, in the second they are sub-optimal, and in the last one exactly half are sub-optimal and half are infeasible. The performance of HASSLE-SLS in these scenarios is shown in Table 2. The first row clearly shows that the absence of sub-optimal examples in the training set leads to a very inaccurate model, with the accuracy as low as 53% and infeasibility as high as 49%. The second row shows that sub-optimal examples are essential to learn a good model, as the performance improves drastically compared to the first row. In the third row, the performance remains similar to the case when we use only sub-optimal examples, this suggests that infeasible negative examples are not that informative.

6.4. Results on Noisy Data

In this section, we examine the performance of HASSLE-SLS on noisy data. We added noise in the synthetic data using the classification noise process [16], i.e., by randomly flipping the labels of each training example with some probability $p$. One advantage of using HASSLE-SLS is that – in contrast to HASSLE-MILP – it outputs a model even when the training set is inconsistent, i.e., no candidate hypothesis has empirical error zero. Hence, it is suitable for noisy data too.

To generate the training data with noise, we picked the synthetic models generated for the earlier experiments with 8 variables, 10 hard and 10 soft
Table 3: Performance of hassle-sls on noisy data set. The first column specifies the probability of noise in the data, the other columns report performance metrics of the learned model.

| Noise % | Score  | Accuracy | Infeasibility | Regret  |
|---------|--------|----------|---------------|---------|
| 0       | 97.5   | 85.0 ± 2.0 | 15.7 ± 1.6    | 5.6 ± 0.8 |
| 5       | 91.5   | 81.7 ± 2.3 | 18.9 ± 2.5    | 7.6 ± 0.8 |
| 10      | 87.1   | 79.1 ± 2.4 | 28.1 ± 3.1    | 7.7 ± 0.8 |
| 20      | 78.6   | 77.1 ± 2.6 | 31.9 ± 3.3    | 9.4 ± 0.8 |

Q6 Can hassle-sls handle noisy data?

The results for hassle-sls are shown in Table 3. The first row corresponds to performance when there is no noise in the data. In the subsequent rows, as \( p \) increases, the performance of the learnt model decreases across all the performance metrics. However, even with \( p \) as high as 0.2 which entails that around 20% examples in the training set are mislabeled, we achieve \( \sim 77\% \) accuracy, \( \sim 32\% \) infeasibility and \( \sim 9\% \) regret. Hence Q6 can be answered in the affirmative.

Q7 Is having contextual information essential to learn a good model?

To investigate the importance of contexts while learning a model, we designed experiments where we created a data set of contextual examples, but removed the information about the contexts. This replicates the learning setting for most of the constraint learning approaches, as they do not utilise the information about the context while learning. We learn model using this data and compare it with the case where we utilise the contextual information. The results are shown in Figure 6 as expected, using contextual information while learning improves the performance across all metrics. On an average, regret decreases by more than \( \sim 50\% \) when using contextual information, while infeasibility decreases by \( \sim 25\% \). These numbers clearly show that contextual information is quite essential to learn a good model.

6.5. Results on Benchmark Models

Table 4 reports the performance of hassle-sls on benchmark models. The first column represents the number of contexts used to generate the constraints. We generated 100 random contexts to sample 2 positive and 2 negative examples from each. The label for each example was then randomly flipped with probability \( p = 0.05, 0.1, 0.2 \).

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The results for hassle-sls are shown in Table 3. The first row corresponds to performance when there is no noise in the data. In the subsequent rows, as \( p \) increases, the performance of the learnt model decreases across all the performance metrics. However, even with \( p \) as high as 0.2 which entails that around 20% examples in the training set are mislabeled, we achieve \( \sim 77\% \) accuracy, \( \sim 32\% \) infeasibility and \( \sim 9\% \) regret. Hence Q6 can be answered in the affirmative.

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training set. The other columns show the performance of the corresponding learnt model. For each experiment we used a cutoff time of 24 hours, however, in the last column we also report the actual time taken to learn the models for which the performances have been reported. To keep the numbers in perspective, we also report the performance of a random classifier. Increasing the number of contexts from 250 to 500 and therefore the number of examples, leads to a better model, however increasing it further to 1000 does not show any significant change across any metric. With 500 contexts and a cutoff time of 24 hours, we achieved \(\sim 27\%\) infeasibility, which is a 62\% decrease compared to a random classifier. We also observed around 43\% decrease in regret from 12.8\% to 7.3\%. On average it took around 6.3 hours to achieve
Table 4: Performance of hasslesls on benchmark models, compared across increasing number of contexts. From each context, 2 positive and 2 negative examples were sampled for training. A cutoff of 24 hours was used for each experiment.

| Contexts | Score | Accuracy | Infeasibility Hassle | Infeasibility Random | Regret Hassle | Regret Random | Time (hrs) |
|----------|-------|----------|----------------------|----------------------|---------------|---------------|-----------|
| 250      | 97.5  | 72.6 ± 2.6 | 40.0±3.5 | 74 ± 0.5 | 8.4 ± 0.6 | 12.8±0.4 | 4.5 |
| 500      | 96.2  | 80.8 ± 2.1 | 26.8±3.2 | 74 ± 0.5 | 7.3 ± 0.5 | 12.8±0.4 | 6.3 |
| 1000     | 95.8  | 80.0 ± 2.4 | 28.6±3.3 | 74 ± 0.5 | 7.6 ± 0.8 | 12.8±0.4 | 9.6 |

this performance, but considering the time taken by experts to model a problem and also the fact that once learnt, these models can be used again and again, we think this is a very reasonable time to learn a model.

7. Related Work

Our approach is closely related to constraint learning and acquisition [4, 5]. There the goal is to acquire a constraint satisfaction problem (aka constraint network), usually from examples of feasible and infeasible assignments. However, the issue of learning from contextual examples, which are pervasive in real-world decision making, is usually ignored. One exception is QuAcq [23], which acquires hard constraints from membership queries about partial assignments. QuAcq, however, is allowed to choose informative partial assignments, while in our (harder) case the contexts are fixed by the environment. In addition, QuAcq does not support acquiring optimization problems and only handles a restricted class of contexts, namely partial assignments. Context-dependent examples have also been considered in logic-based learning of Answer Set Programs, but the links to combinatorial optimisation and statistical learning are missing [24].

Most approaches in constraint learning acquire hard constraints only by searching the version space with one-directional [25] or bi-directional search [4]. HASSLE-MILP leverages ideas from syntax-guided synthesis [7], where learning is cast as a proper satisfaction or optimization problem and tackled with a solver. This strategy has been used for learning rules [26], hybrid logical-numerical formulas [27], Bayesian networks [28], and causal models [29]. Stochastic local search heuristics have also received some attention as tools for structure learning of machine learning models, see for instance [30, 31].
Approaches that learn soft constraints build on machine learning techniques such as regression [32], learning-to-rank [33], or structured prediction [34], depending on the kind of available supervision. However, these methods require the hard constraints to be given.

Other related areas include structured prediction [35, 36] and contextual combinatorial bandits [37]. In these settings the supervision consists of input-output pairs for which the partitioning of variables into inputs and outputs is fixed. This schema is much more restrictive than general context-specific examples. In addition, in structured prediction there is no support for learning hard constraints.

Maximum a posteriori inference in Bayesian networks and Markov Logic Networks (MLNs) can be cast as weighted MAX-SAT [38, 39]. However, MLNs are not learned from contextual examples where both context $x$ and completion $xy$ are provided. Rather the $xy$ are sampled from the underlying probability distribution and it is not assumed that the examples are optimal (i.e., maximally likely).

One rather recent direction of research are differentiable layers able to implicitly acquire and solve optimization problems within deep networks. Closest to our work, SATNet [40] can be used in neural nets as a module to learn MAX-SAT models in an end-to-end manner. A major difference between our approach and SATNet is that the latter does not really output a proper MAX-SAT model: the model is implicitly captured by the parameters of the layer and cannot be read off directly. Another difference is that SATNet layers are also responsible for solving the acquired model, and it does so using an approximate semi-definite programming relaxation. This heuristic cannot compete in precision or scalability with MAX-SAT solvers. Another downside of SATNet is that it is restricted to inferring a single, fixed solution, even for problems with multiple ones. In contrast, HASSLE produces a full-fledged MAX-SAT model that can be easily used to generate multiple solutions using state-of-the-art solvers, moreover experts can even add new contexts as constraints to generate solutions under new temporary restrictions. SATNet also learns from complete assignments only and has no support for contextual examples.

Finally, in terms of theory, this is the first work in the field of constraint learning that provides PAC learnability guarantees.
8. Conclusion and Future Work

We introduced the novel learning task of acquiring combinatorial optimization models from contextual data, focusing specifically on MAX-SAT models. Our analysis shows that ERM provably learns low-regret MAX-SAT models from context-specific examples in both realizable and agnostic setting. It works even in the noisy setting as long as the noise is random and the probability of occurrence is less than 50%. These results justify our ERM-based implementations, hassle-milp and hassle-sls. The first implementation relies on an approximate but tight MILP encoding to perform learning, while the second one uses stochastic local search. hassle-milp shows good performance when using small number of examples for learning, but fails to scale for large training set. However, increasing the size of the training set shows significant improvement in the quality of the learnt model. hassle-sls utilises large training set much more efficiently to learn better models and hence scales much better compared to hassle-milp. Also, if time is of the essence, hassle-sls provides the flexibility to learn a model given any cutoff time, however this might decrease the quality of the learnt model.

There are three clear directions in which this work can be extended. First, since nothing in our theory relies on the variables being Boolean nor on the constraints being propositional formulas, our results transfer to general combinatorial optimization problems with linear objective functions, e.g., weighted constraint satisfaction problems [32]. With some effort, it should be possible to generalize them even further to hybrid discrete-continuous optimization frameworks like maximum satisfiability modulo linear arithmetic [41], in which the decision variables can be also real valued and the constraints may include linear inequalities. In practice, this would require, first and foremost, to revise the strategy used by hassle-sls for evaluating and exploring the neighborhood of a candidate model. Second, while our theory does apply to the agnostic and the uniform noisy settings, it would be useful to extend it to deal structured noise, like sub-optimal examples. One option is to make use of counterfactual risk minimization [42], an alternative to ERM in which the sampling bias due to irrational annotators is taken into account. Finally, it is worth lifting the restriction that context annotations are always available, complete, and noiseless. This could be handled by treating the context as a latent variable and optimizing it to infer their unobserved values. While very useful, these extensions are highly non-trivial.
and left to future work.

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Appendix A. Learnability in a Fixed Context (Section 4.2)

Here we show that MAX-SAT classifiers applied to a specific context $\mathcal{H}_\psi$ have finite VC dimension and are therefore learnable from context-specific examples. We start by building an easier-to-handle superset $\mathcal{G}_\psi \supseteq \mathcal{H}_\psi$, as follows:

$$
\mathcal{G}_\psi := \{ \mathbb{1} \{ (x \models \phi(c) \land \psi) \land f_w(x) \geq \max_{x' \models \phi(c') \land \psi} f_w(x') \} : w \in [-1,1]^m, \ c, c' \in \{0,1\}^m \}
$$

By construction, $\text{VC}(\mathcal{G}_\psi) \geq \text{VC}(\mathcal{H}_\psi)$. Indeed, fixing $c = c'$ recovers the definition of $\mathcal{H}_\psi$. The VC dimension of $\mathcal{G}_\psi$ can be bounded in terms of two simpler sets defined as:

$$
\mathcal{G}_\psi,\text{hard} := \{ \mathbb{1} \{ x \models \phi(c) \land \psi \} : c \in \{0,1\}^m \},
$$

and:

$$
\mathcal{G}_\psi,\text{soft} := \{ g_{c',w}(\cdot,\psi) : c' \in \{0,1\}^m, w \in [-1,1]^m \},
$$

$$
g_{c',w}(x,\psi) := \mathbb{1} \{ f_w(x) \geq \max_{x' \models \phi(c') \land \psi} f_w(x') \}
$$

Our first result relies on a well-known bound on the VC-dimension of intersections of hypothesis classes, reported here for ease of reference:

**Theorem 5** (Theorem 1.1 of [43]). Let $\mathcal{H}_1, \ldots, \mathcal{H}_l$ be hypothesis classes and $\mathcal{H} = \bigcap_{j=1}^l \mathcal{H}_j$ be their intersection. Then:

$$
\text{VC}(\mathcal{H}) \leq c_1 \log(c_2 l) \sum_{j=1}^l \text{VC}(\mathcal{H}_j)
$$

where $c_1 \approx 2.3$ and $c_2 \approx 3.9$.

**Lemma 2.** For any $\psi$ and $u < 5$, it holds that:

$$
\text{VC}(\mathcal{G}_\psi) \leq u(\text{VC}(\mathcal{G}_{\psi,\text{hard}}) + \text{VC}(\mathcal{G}_{\psi,\text{soft}}))
$$

**Proof.** Each and every hypothesis in $\mathcal{G}_\psi$ is the conjunction of a concept from $\mathcal{G}_{\psi,\text{hard}}$ with a concept from $\mathcal{G}_{\psi,\text{soft}}$. The claim follows by applying Theorem 5. □
The remaining step is to prove that \( \mathcal{G}_{\psi, \text{hard}} \) and \( \mathcal{G}_{\psi, \text{soft}} \) both have bounded VC dimension. It is clear that \( |\mathcal{G}_{\psi, \text{hard}}| \leq 2^m \) and therefore \( \text{VC}(\mathcal{G}_{\psi, \text{hard}}) \leq m \). The answer for \( \mathcal{G}_{\psi, \text{soft}} \) is given by the following lemma:

**Lemma 3.** For any \( \psi \), it holds that \( \text{VC}(\mathcal{G}_{\psi, \text{soft}}) \leq m + 1 \).

**Proof.** \( \mathcal{G}_{\psi, \text{soft}} \) can be rewritten as:

\[
\mathcal{G}_{\psi, \text{soft}} = \{g_{c', w}(\cdot, \psi) : c' \in \{0, 1\}^m, w \in [-1, 1]^m\}
\]

\[
= \{1 \{f_w(x) \geq \max_{x' \in \phi(c') \land \psi} f_w(x')\} : c', w\}
\]

\[
\subset \{1 \{f_w(x) \geq b\} : w \in [-1, 1]^m, b \in \mathbb{R}\}
\]

Here \( c' \) and \( w \) implicitly range over \( \{0, 1\}^m \) and \([-1, 1]^m\), respectively. The final set is a subset of \( \mathcal{L} \), the class of linear classifiers over the \( m \) features \( \{x \mid \phi_j(x)\} \). This shows that \( \text{VC}(\mathcal{G}_{\text{soft}}) \leq \text{VC}(\mathcal{L}) \leq m + 1 \) (Theorem 9.3 of \[14\]). \( \square \)

Lemma 2 clearly implies Theorem 2. Now combining Theorem 1 and Theorem 2 shows that:

**Corollary 1.** For any \( \psi \in \Psi^* \), and \( \epsilon_\psi, \delta_\psi \in [0, 1] \), there exists an integer \( s_{\mathcal{H}_\psi}(\epsilon_\psi, \delta_\psi) \in \mathbb{N} \) such that, given a dataset \( S \) with at least \( s_{\mathcal{H}_\psi}(\epsilon_\psi, \delta_\psi) \) examples specific to \( \psi \), any hypothesis \( h \in \mathcal{H}_\psi \) satisfies the following:

\[
\text{Pr}(L_{D, \psi}(h) < L_{S, \psi}(h) + \epsilon_\psi) \geq 1 - \delta_\psi
\]

The sample complexity \( s_{\mathcal{H}_\psi}(\epsilon_\psi, \delta_\psi) \) is given by Theorem 1.

**Appendix B. From Classification to Optimization (Section 4.3)**

**Theorem 3.** Let \( \eta = \min_{x : D(x | \psi) > 0} D(x | \psi) \) and \( r_{\max} \) be the regret associated to infeasible assignments. Then, for any \( h \in \mathcal{H} \) and \( \psi \in \Psi^* \), it holds that:

\[
\text{reg}(h, \psi) \leq \frac{\|w^*\|_1 + r_{\max}}{\eta |\text{opt}(\psi)|} L_{D, \psi}(h)
\]

This inequality holds for both realizable and agnostic settings\[11\].

\[11\]This bound improves on the one provided by Kumar et al. \[8\] by a constant factor.
Proof. First, notice that:
\[
\sum_{x : h(x, \psi) = 1} \mathbb{1}\{h(x, \psi) \neq h^*(x, \psi)\} \leq \frac{1}{\eta} L_{D, \psi}(h) \tag{B.1}
\]
Indeed, the risk can be lower bounded as:
\[
L_{D, \psi}(h) = \sum_{x \in \{0, 1\}} \mathbb{1}\{h(x, \psi) \neq h^*(x, \psi)\} D(x \mid \psi)
\geq \sum_{x : h(x, \psi) = 1} \mathbb{1}\{h^*(x, \psi) = 0\} D(x \mid \psi)
\geq \eta \sum_{x : h(x, \psi) = 1} \mathbb{1}\{h^*(x, \psi) = 0\}
\]
Second, notice that \(x\) contributes to the average regret iff \(h(x, \psi) = 1 \land h^*(x, \psi) = 0\): if \(x\) is feasible w.r.t. \(\phi(c^*)\), it contributes a factor \((f_{w^*}(x^*) - f_{w^*}(x))\), else it contributes \(r_{\max}\). Thus, ignoring constants, the regret can be written as the sum of two terms. The first one is:
\[
\sum_{x : h(x, \psi) = 1} \mathbb{1}\{h^*(x, \psi) = 0 \land x \models \phi(c^*)\} (f_{w^*}(x^*) - f_{w^*}(x))
\leq \left[ \max_x f_{w^*}(x^*) - f_{w^*}(x) \right] \sum_{x : h(x, \psi) = 1} \mathbb{1}\{h^*(x, \psi) = 0 \land x \models \phi(c^*)\}
= \left[ \max_x \sum_{j=1}^m w_j^* (\mathbb{1}\{x^* \models \phi_j^*\} - \mathbb{1}\{x \models \phi_j^*\}) \right] \sum_{x : h(x, \psi) = 1} \mathbb{1}\{h^*(x, \psi) = 0 \land x \models \phi(c^*)\}
\leq \|w^*\|_1 \sum_{x : h(x, \psi) = 1} \mathbb{1}\{h^*(x, \psi) = 0 \land x \models \phi(c^*)\} \tag{B.2}
\]
where the last step follows from Hölder’s inequality. The second one is:
\[
r_{\max} \sum_{x : h(x, \psi) = 1} \mathbb{1}\{h^*(x, \psi) = 0 \land x \not\models \phi(c^*)\} \tag{B.3}
\]
The summations in Eq. B.2 and Eq. B.3 are upper bounded by the left hand side of Eq. B.1. Summing the two quantities and dividing by \(\|\text{opt}(\psi)\|\) gives the average regret and proves the claim. \(\square\)

Appendix C. Generalizing across Contexts (Section 4.4)

Lemma 1. Let \(\Psi\) be representative for \(\psi_t\), and \(D(x \mid \psi) > 0\) for every \(\psi \in \Psi\) and \(x \models \psi\). Then there exist finite constants \(\beta_\psi \geq 0\) (that only depend on \(\Psi\) and \(D\)) such that for every \(h \in \mathcal{H}\):
\[
L_{D, \psi_t}(h) \leq \sum_{\psi \in \Psi} \beta_\psi L_{D, \psi}(h)
\]
Proof. By definition of $\chi$ and $\#$, it holds that:

$$1 \{h(x, \psi_t) \neq h^*(x, \psi_t)\} = \frac{1}{\#(\psi_t, x)} \sum_{\psi \in \Psi : \chi(\psi_t, x, \psi)} 1 \{h(x, \psi) \neq h^*(x, \psi)\}$$

where $\#(\psi_t, x) > 0$ because $\Psi$ is representative. Plugging this into the definition of risk gives:

$$L_{D,\psi_t}(h) = \sum_{x \in \{0,1\}^n} 1 \{h(x, \psi_t) \neq h^*(x, \psi_t)\} D(x | \psi_t)$$

$$= \sum_x \left( \sum_{\psi \in \Psi : \chi(\psi_t, x, \psi)} 1 \{h(x, \psi) \neq h^*(x, \psi)\} \frac{\#(\psi_t, x)}{\#(\psi_t, x)} \right) D(x | \psi_t)$$

$$= \sum_{\psi \in \Psi} \sum_{x : \chi(\psi_t, x, \psi)} 1 \{h(x, \psi) \neq h^*(x, \psi)\} D(x | \psi_t) \frac{\#(\psi_t, x)}{\#(\psi_t, x)}$$

Letting:

$$\beta_\psi = \max_{x \models \psi} \frac{1}{\#(\psi_t, x)} D(x | \psi_t)$$

we can write:

$$L_{D,\psi_t}(h) \leq \sum_{\psi \in \Psi} \beta_\psi \sum_{x : \chi(\psi_t, x, \psi)} 1 \{h(x, \psi) \neq h^*(x, \psi)\} D(x | \psi_t) = \sum_{\psi \in \Psi} \beta_\psi L_{D,\psi_t}(h)$$

For each $\beta_\psi$, the max runs over those $x$'s that satisfy $\psi$, hence $D(x | \psi_t)$ is always non-zero, implying that each $\beta_\psi$ is finite. \(\square\)

**Theorem 4.** For any $\Psi$ and $D$ that satisfy the conditions of Lemma 1 and for every $\epsilon, \delta \in (0, 1)$, there exist integers $t_\psi(\epsilon, \delta)$ for $\psi \in \Psi$ such that, if $S$ contains at least $t_\psi(\epsilon, \delta)$ context-specific examples for every $\psi \in \Psi$, then any hypothesis $h$ with minimal empirical risk on $S$ satisfies:

$$\Pr(L_{D,\psi_t}(h) > \min_{h' \in \mathcal{H}} \sum_{\psi \in \Psi} L_{S,\psi}(h') + \epsilon) < \delta$$

Notice that, in the realizable setting this reduces to:

$$\Pr(L_{D,\psi_t}(h) > \epsilon) < \delta$$

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Proof. Corollary 1 guarantees that for every $\epsilon$, $\delta \in (0, 1)$, there exists an integer $s(\epsilon, \delta)$ such that if $S$ includes at least $s(\epsilon, \delta)$ examples for context $\psi$, then:

$$
\Pr(L_D(\psi(h)) > L_S(\psi(h)) + \epsilon) \leq \delta
$$

Let $l$ represent the number of mistakes made by the empirical risk minimizer on $S$, that is, $l = \min_h \sum_{\psi \in \Psi} L_S(\psi(h))$. The second part of the proof involves upper bounding the probability that $h$ has large risk in the target context $\psi_t$. Let $\beta = \max_{\psi \in \Psi} \beta$, then:

$$
\Pr(L_D(\psi(h)) > l + \epsilon) \leq \Pr(\sum_{\psi \in \Psi} L_D(\psi(h)) > l + \epsilon)
\leq \Pr(\beta \sum_{\psi \in \Psi} L_D(\psi(h)) > l + \epsilon)
= \Pr(\max_{\psi \in \Psi} L_D(\psi(h)) > \frac{l + \epsilon}{\beta |\Psi|})
\leq \Pr(\bigvee_{\psi \in \Psi}(L_D(\psi(h)) > \frac{l + \epsilon}{\beta |\Psi|})))
\leq \sum_{\psi \in \Psi} \Pr(L_D(\psi(h)) > \frac{l + \epsilon}{\beta |\Psi|})
$$

The first step follows from Lemma 1. By Eq. C.1, the last expression can be made smaller than any $\delta$ by adding enough context-specific examples to $S$. In particular, having at least:

$$
t(\epsilon, \delta) = s(\epsilon, \delta)
$$

examples for every context guarantees that Eq. C.2 is less than any $\delta = \sum_{\psi \in \Psi} \delta$. This concludes the proof.

Appendix D. MILP Encoding

The variables used in the MILP encoding are explained in Table D.5 and the MILP encoding itself if given in Figures D.7 and D.8. Note, that the MILP encoding makes the following assumptions: 1) contexts are partial assignments, i.e., conjunctions of literals; 2) the MAX-SAT problem is in CNF format, i.e., both hard and soft constraints are clauses (disjunctions of literals); and 3) weights are in the interval $[0, 1]$. 44
Table D.5: Names, types and explanations of the variables used in the MILP encoding.

| Symbol | Type   | Description                                                                 |
|--------|--------|------------------------------------------------------------------------------|
| $M$    | int    | Large constant to disable constraints $M = 2 \cdot (m + 1)$                  |
| $\epsilon$ | real  | Small constant to model strict inequality ($\epsilon = 10^{-2}$)             |
| $m$    | int    | Number of constraints                                                       |
| $s$    | int    | Number of examples                                                          |
| $n$    | int    | Number of Boolean variables                                                 |
| $L$    | set    | All literals ($\{X_i | i = 1..n\} \cup \{-X_i | i = 1..n\}$)               |
| $L^+$  | set    | All positive literals ($\{X_i | i = 1..n\}$)                               |
| $L^-$  | set    | All negative literals ($\{-X_i | i = 1..n\}$)                              |
| $\Psi$ | set    | All unique contexts occurring in the examples                               |
| $|\psi|$ | int    | Number of literals in context $\psi$                                       |
| $\psi^+$ | set   | All positive literals in the context $\psi$                                |
| $\psi^-$ | set  | All negative literals in the context $\psi$                                |
| $ccov^\psi$ | bool | True iff context $\psi$ is feasible                                         |
| $x^\psi$ | assignment | The optimal assignment in context $\psi$                                    |
| $x^\ell_\psi$ | bool | True iff literal $\ell$ is true in assignment $x^\psi$                     |
| $cov^j_\psi$ | bool | True iff clause $j$ contains literal $\ell$ and $x^\psi_\ell$ is true      |
| $a^j_\ell$ | bool | True iff clause $j$ contains literal $\ell$                                |
| $cov^j$ | bool | True iff clause $j$ covers $x^\psi_\ell$, i.e., $x^\psi_\ell$ satisfies clause $j$ |
| $c^j$ | bool | True if clause $j$ is a hard constraint, otherwise it is a soft constraint  |
| $cov^j_\ell$ | bool | True iff clause $j$ is a soft constraint or clause $j$ covers $x^\psi_\ell (-c^j \lor cov^j_\ell)$ |
| $cov^\psi$ | bool | True iff $x^\psi$ satisfies all hard constraints and the context is feasible ($cov^\psi \land \bigwedge_{j=1}^m cov^j_\psi$) |
| $w^j$ | real $\in [0, 1]$ | The weight assigned to clause $j$                                      |
| $w_{zj}$ | bool | True iff the weight assigned to clause $j$ is 0 ($\{w^j = 0\}$)              |
| $w^\psi_j$ | real $\in [0, 1]$ | $w^j$ if the clause $j$ is soft and covers $x^\psi_\ell$, 0 otherwise ($w^\psi_j = w^j \cdot 1\{-c^j\} \cdot 1\{cov^\psi_j\}$) |
| $\gamma^\psi_j$ | real | The optimal value of any point ($x^\psi_\ell$) in context $\psi$          |
| $w^\psi_{jk}$ | real $\in [0, 1]$ | $w^j$ if clause $j$ is soft and covers $x_k$, 0 otherwise ($w^\psi_{jk} = w^j \cdot 1\{-c^j\} \cdot 1\{cov^\psi_{jk}\}$) |
| $cov^j_{jk}$ | bool | True iff clause $j$ covers example $k$                                      |
| $opt_k$ | bool | True iff the $k$th example ($x_k$) is optimal                                |
| $cov^j_{jk}$ | bool | True iff clause $j$ is a soft constraint or clause $j$ covers example $k$ ($-c^j \lor cov^j_{jk}$) |
| $x_{k\ell}$ | bool | True iff literal $\ell$ is true in example $k$                             |
| $cov_k$ | bool | True iff $x_k$ satisfies all hard constraints ($\bigwedge_{j=1}^m cov^j_{jk}$) |
\[
\max \sum_{\psi \in \Psi} \gamma_{\psi} \quad \text{(D.1)}
\]

s.t. \[
\text{ccov}_\psi \leq x_\psi^{\ell}\]
\[
\text{ccov}_\psi \leq (1 - x_\psi^{\ell}) \quad \forall \psi \in \Psi, \ell \in \psi^+ \quad \text{(D.2)}
\]
\[
\text{ccov}_\psi \geq \sum_{\ell \in \psi^+} x_\psi^{\ell} + \sum_{\ell \in \psi^-} (1 - x_\psi^{\ell}) - |\psi| + 1 \quad \forall \psi \in \Psi \quad \text{(D.4)}
\]
\[
\text{cov}_\psi^{\ell j} \leq a_{\psi^{\ell j}} \quad \forall \psi \in \Psi, j = 1..m, \ell \in L \quad \text{(D.5)}
\]
\[
\text{cov}_\psi^{\ell j} \leq x_\psi^{\ell} \quad \forall \psi \in \Psi, j = 1..m, \ell \in L^+ \quad \text{(D.6)}
\]
\[
\text{cov}_\psi^{\ell j} \geq a_{\psi^{\ell j}} + x_\psi^{\ell} - 1 \quad \forall \psi \in \Psi, j = 1..m, \ell \in L^+ \quad \text{(D.7)}
\]
\[
\text{cov}_\psi^{\ell j} \leq 1 - x_\psi^{\ell} \quad \forall \psi \in \Psi, j = 1..m, \ell \in L^- \quad \text{(D.8)}
\]
\[
\text{cov}_\psi^{\ell j} \geq a_{\psi^{\ell j}} + (1 - x_\psi^{\ell}) + 1 \quad \forall \psi \in \Psi, j = 1..m, \ell \in L^- \quad \text{(D.9)}
\]
\[
\text{cov}_\psi^{\ell j} \geq \text{cov}_{\psi^{\ell j}} \quad \forall \psi \in \Psi, j = 1..m, \ell \in L \quad \text{(D.10)}
\]
\[
\text{cov}_\psi^{\ell j} \leq \sum_{\ell \in L} \text{cov}_{\psi^{\ell j}} \quad \forall \psi \in \Psi, j = 1..m \quad \text{(D.11)}
\]
\[
\text{cov}_\psi^{\ell j} \geq \text{cov}_\psi^{\ell j} \quad \forall \psi \in \Psi, j = 1..m \quad \text{(D.12)}
\]
\[
\text{cov}_\psi^{\ell j} \geq 1 - c_{\psi^{\ell j}} \quad \forall \psi \in \Psi, j = 1..m \quad \text{(D.13)}
\]
\[
\text{cov}_\psi^{\ell j} \leq \text{cov}_\psi^{\ell j} + (1 - c_{\psi^{\ell j}}) \quad \forall \psi \in \Psi, j = 1..m \quad \text{(D.14)}
\]
\[
\text{cov}_\psi^{\ell j} \leq \text{ccov}_\psi \quad \forall \psi \in \Psi \quad \text{(D.15)}
\]
\[
\text{cov}_\psi \geq \sum_{j=1}^{m} \text{cov}_{\psi^{\ell j}} + \text{ccov}_\psi - m \quad \forall \psi \in \Psi \quad \text{(D.17)}
\]
\[
\text{cov}_\psi^{\ell j} \leq \text{cov}_\psi^{\ell j} \quad \forall \psi \in \Psi, j = 1..m \quad \text{(D.18)}
\]
\[
\text{cov}_\psi^{\ell j} \leq (1 - c_{\psi^{\ell j}}) \quad \forall \psi \in \Psi, j = 1..m \quad \text{(D.19)}
\]
\[
\text{cov}_\psi^{\ell j} \leq w_j + (1 - \text{cov}_\psi^{\ell j}) + c_{\psi^{\ell j}} \quad \forall \psi \in \Psi, j = 1..m \quad \text{(D.20)}
\]
\[
\text{cov}_\psi^{\ell j} \geq w_j - (1 - \text{cov}_\psi^{\ell j}) - c_{\psi^{\ell j}} \quad \forall \psi \in \Psi, j = 1..m \quad \text{(D.21)}
\]
\[
\gamma_{\psi} \leq \sum_{j=1}^{m} w_j^{\psi} \quad \forall \psi \in \Psi \quad \text{(D.22)}
\]
\[
\gamma_{\psi} \leq \text{cov}_\psi \cdot M \quad \forall \psi \in \Psi \quad \text{(D.23)}
\]

Figure D.7: Objective function and first part of the MILP encoding (Eq. 10 in the paper)
\begin{align*}
\text{s.t. } w_j & \leq (1 - wz_j) & \forall j = 1..m \quad (D.25) \\
w_j & \geq 3 \cdot \epsilon - wz_j & \forall j = 1..m \quad (D.26) \\
w_{jk} & \leq \text{cov}_{jk} & \forall j = 1..m, k = 1..s \quad (D.27) \\
w_{jk} & \leq (1 - c_j) & \forall j = 1..m, k = 1..s \quad (D.28) \\
w_{jk} & \leq w_j + (1 - \text{cov}_{jk}) + c_j & \forall j = 1..m, k = 1..s \quad (D.29) \\
w_{jk} & \geq w_j - (1 - \text{cov}_{jk}) - c_j & \forall j = 1..m, k = 1..s \quad (D.30) \\
\gamma_{\psi_k} & \leq \left( \sum_{j=1}^{m} w_{jk} \right) + M \cdot (1 - \text{opt}_k) & \forall k = 1..s \quad (D.31) \\
\gamma_{\psi_k} & \geq \left( \sum_{j=1}^{m} w_{jk} \right) + \epsilon - M \cdot \text{opt}_k & \forall k = 1..s \quad (D.32) \\
\text{cov}'_{jk} & \geq \text{cov}_{jk} & \forall j = 1..m, k = 1..s \quad (D.33) \\
\text{cov}'_{jk} & \geq 1 - c_j & \forall j = 1..m, k = 1..s \quad (D.34) \\
\text{cov}'_{jk} & \leq \text{cov}_{jk} + (1 - c_j) & \forall j = 1..m, k = 1..s \quad (D.35) \\
\text{cov}_{jk} & \geq a_{j\ell} \cdot x_{k\ell} & \forall j = 1..m, k = 1..s, \ell \in L \quad (D.36) \\
\text{cov}_{jk} & \leq \sum_{\ell \in L} a_{j\ell} \cdot x_{k\ell} & \forall j = 1..m, k = 1..s \quad (D.37) \\
a_{j\ell} + a_{j-\ell} & \leq 1 & \forall j = 1..m, \ell \in L^+ \quad (D.38) \\
\text{cov}_k & \leq \text{cov}'_{jk} & \forall j = 1..m, k = 1..s \quad (D.39) \\
\text{cov}_k & \geq \left( \sum_{j=1}^{m} \text{cov}'_{jk} \right) - (m - 1) & \forall k = 1..s \quad (D.40) \\
\text{cov}_k & \leq \text{cov}_{\psi_k} & \forall k = 1..s \quad (D.41) \\
\text{cov}_k + \text{opt}_k + \text{cov}_{\psi_k} & = 3 & \forall k \in \{ k | k = 1..s \land y_k \} \quad (D.42) \\
\text{cov}_k + \text{opt}_k + \text{cov}_{\psi_k} & \leq 2 & \forall k \in \{ k | k = 1..s \land \neg y_k \} \quad (D.43)
\end{align*}

Figure D.8: Second part of the MILP encoding (Eq. 11 in the paper)
Appendix D.1. Mismatch between original and MILP encodings

Example 4. Consider the following target model over three variables \(X_1, X_2\) and \(X_3\):

\[
X_1 \lor \neg X_3 \quad \text{(D.44)}
\]

\[
1.0 \neg X_1 \quad \text{(D.45)}
\]

and the following examples to learn from (literal assignments fixed by the context are in bold):

| \(X_1\) | \(X_2\) | \(X_3\) | \(y\) |
|-------|-------|-------|-----|
| 1     | 1     | 1     | 0   |
| 0     | 1     | 0     | 1   |
| 1     | 1     | 1     | 1   |
| 1     | 0     | 1     | 1   |
| 0     | 1     | 0     | 1   |
| 1     | 1     | 0     | 0   |
| 0     | 0     | 0     | 0   |

In this case, the MILP encoding will learn the following model:

\[
1.0 \neg X_1 \quad \text{(D.46)}
\]

\[
1.0 \neg X_1 \lor X_3 \quad \text{(D.47)}
\]

which mislabels the positive examples \([1, 1, 1]\) and \([1, 0, 1]\). We can examine the gamma values for the true model \(\gamma^T\), the actual values for the learned model \(\gamma^L\) as well as those reported by the MILP solution \(\hat{\gamma}^L\):

| \(\gamma\) \(\psi\) | \(\psi_t\) | \(X_3\) | \(X_2 \land \neg X_3\) | \(\neg X_2 \land \neg X_3\) | \(\sum_{\psi} \gamma^T\psi\) |
|------|-------|-------|----------------|----------------|---------|
| \(\gamma^L\) | 1     | 0     | 1             | 1             | 3       |
| \(\gamma^L\) | 2     | 2     | 2             | 2             | 8       |
| \(\gamma^L\) | 2     | 1     | 2             | 2             | 7       |

Setting gamma correctly (for the learned model) shows that the positive examples \([1, 1, 1]\) and \([1, 0, 1]\) are not labeled correctly, as there exists an example \([0, 1, 1]\) in the same context that is feasible and obtains a higher value \(f_w([0, 1, 1]) = 2\). Therefore:

\[
\gamma^L_{X_3} = \max_{x=\phi \land X_3} f_w(x) = 2 > f_w([1, 1, 1]) = 1
\]
However, by setting $\hat{\gamma}_{X_3}^L$ sub-optimally, those examples are labeled as positives in the MILP encoding:

$$\hat{\gamma}_{X_3}^L \leq f_w([1, 1, 1]) = f_w([1, 0, 1]) = 1$$

and the model can obtain a larger value for $\sum_\psi \gamma_\psi$ than possible when using, for example, the target model.

Appendix E. Generating neighbours for hassle-sls

HASSLE-SLS learns MAX-SAT models, which means the set of constraints $\Phi$ is a set of disjunctions of literals. For any constraint $\phi_i$ and literal $l$, we use $\phi_i - l$ to represent the disjunction after removal of $l$, if present. For example, if $\phi_i = x_1 \lor \neg x_2 \lor x_3$ then $\phi_i - (\neg x_2) = x_1 \lor x_3$. Similarly $\phi_i + l$ represents the disjunction after adding the literal. So, $\phi_i + x_4 = x_1 \lor \neg x_2 \lor x_3 \lor x_4$.

To generate the set of neighbours for any model $M$, first we randomly pick an example $x$, which is incorrectly classified by $M$. The set of neighbours is then generated based on the true and predicted label of $x$. We will use $x^*$ to represent an optimal example for $M$.

- If true label is positive while the predicted label is infeasible:
  1. Make changes in any hard constraint $\phi_i$ such that $x \not\models \phi_i$

- If true label is positive while the predicted label is sub-optimal:
  1. Pick any hard constraint $\phi_i$ such that $x \models \phi_i$. Pick a literal $l$ such that $x^* \models l$ and replace $\phi_i$ by $\phi_i - l$
  2. Pick any soft constraint $\phi_i$ such that $x \models \phi_i$ and $x^* \not\models \phi_i$. Either make it a hard constraint or increase the weight $w_i$ to $(w_i + 1)/2$
  3. Pick any soft constraint $\phi_i$ such that $x \not\models \phi_i$ and $x^* \models \phi_i$. Then do one of these three: 1) Pick a literal $l$ such that $x \models l$ and replace $\phi_i$ by $\phi_i + l$, 2) Pick a literal $l$ such that $x^* \models l$ and replace $\phi_i$ by $\phi_i - l$ or 3) Reduce the weight $w_i$ to $w_i/2$
  4. Pick any soft constraint $\phi_i$ such that $x \not\models \phi_i$ and $x^* \not\models \phi_i$. Pick a literal $l$ such that $x \models l$ but $x^* \not\models l$, then replace $\phi_i$ by $\phi_i + l$
  5. Pick any soft constraint $\phi_i$ such that $x \not\models \phi_i$ and $x^* \models \phi_i$. Pick a literal $l$ such that $x \not\models l$ but $x^* \models l$, then replace $\phi_i$ by $\phi_i + l$

- If true label is negative while the predicted label is positive:
1. Pick any hard constraint $\phi_i$ such that $x \models \phi_i$. Pick a literal $l$ such that $x \models l$ and replace $\phi_i$ by $\phi_i - l$.

2. Pick any soft constraint $\phi_i$ such that $x \not\models \phi_i$ and $x^* \not\models \phi_i$. Either make it a hard constraint or increase the weight $w_i$ to $(w_i + 1)/2$.

3. Pick any soft constraint $\phi_i$ such that $x \models \phi_i$ and $x^* \models \phi_i$. Pick a literal $l$ such that $x \models l$, then replace $\phi_i$ by $\phi_i - l$ or reduce the weight $w_i$ to $w_i/2$. 

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