Moyal Deformation, Seiberg-Witten-Map, and Integrable Models

A. Dimakis
Department of Mathematics, University of the Aegean
GR-83200 Karlovasi, Samos, Greece
dimakis@aegean.gr

F. Müller-Hoissen
Max-Planck-Institut für Strömungsforschung
Bunsenstrasse 10, D-37073 Göttingen, Germany
fmuelle@gwdg.de

Abstract

A covariant formalism for Moyal deformations of gauge theory and differential equations which determine Seiberg-Witten maps is presented. Replacing the ordinary product of functions by the noncommutative Moyal product, noncommutative versions of integrable models can be constructed. We explore how a Seiberg-Witten map acts in such a framework. As a specific example, we consider a noncommutative extension of the principal chiral model.

1 Introduction

Field theory on noncommutative spaces has more and more attracted the attention of researchers during the last years. A major impulse came from the discovery that a noncommutative gauge field theory arises in a certain limit of string theory (see [1] and the references cited there). In [1] a (perturbative) equivalence between ordinary and noncommutative gauge fields was established via a change of variables to which the name Seiberg-Witten map was assigned in subsequent publications [2, 3]. More generally, models on noncommutative space-times obtained by replacing the ordinary product of functions by the noncommutative Moyal product [4] were explored in several recent publications. In particular, Moyal deformations of integrable models were constructed via deformation of an associated bicomplex [5, 6, 7].

Section 2 collects some notes on deformations of products and, in particular, recalls the definition of the Moyal $*$-product. Section 3 deals with a corresponding deformation of gauge theory, develops a covariant differentiation formalism with respect to deformation
parameters, generalizes the Seiberg-Witten map from infinitesimal to finite gauge transformations, and shows that this map describes a parallel transport along a curve in the deformation parameter space. Noncommutativity of covariant derivatives with respect to different deformation parameters is associated with a notion of curvature in section 4. In section 5 we show how the Seiberg-Witten map can be used to generate from solutions of a classical integrable model solutions of the corresponding deformed noncommutative model. As a specific example, we consider a noncommutative version of the principal chiral model. Section 6 contains some concluding remarks.

2 Deformations of products

We take the opportunity to make a few remarks about deformations of products and present some useful formulas and instructive examples. Much of it is not really needed in the following sections, however.

Let \( A \) be an associative unital algebra over a commutative ring \( R \) and let \( m : A \otimes A \to A \) denote the multiplication in \( A \). Given a map \( R : A \otimes A \to A \otimes A \), we define a deformed multiplication in \( A \) by

\[
\hat{m}(f \otimes g) = m(f \otimes g) , \quad \hat{m} = R .
\] (2.1)

In general, this is not an associative deformation. If we assume that \( R \) satisfies:

\[
R m_{12} = m_{12} R_{23} R_{13} , \quad R m_{23} = m_{23} R_{12} R_{13} ,
\] (2.2)

then \( \ast \) is associative if \( R \) also satisfies the Yang-Baxter equation

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} .
\] (2.3)

Indeed, it is easy to check associativity:

\[
\hat{m} \hat{m}_{12} = m R m_{12} R_{12} = m m_{12} R_{23} R_{13} R_{12} = m m_{12} R_{12} R_{13} R_{23} = m R m_{23} R_{12} R_{13} R_{23} = \hat{m} \hat{m}_{23}
\] (2.4)

using the associativity condition \( m m_{12} = m m_{23} \) for \( m \).

Two \( \ast \)-products \( \ast, \ast' \) will be considered to be equivalent, if there is an invertible linear map \( S : A \to A \) such that

\[
S \hat{m} = \hat{m}' S \otimes S
\] (2.5)

which is equivalent to

\[
S(f \ast g) = (Sf) \ast' (Sg) .
\] (2.6)

---

1 Here and in what follows we simply write \( \otimes \) instead of \( \otimes_R \).
2 These are duals of the relations which define quasi-triangular Hopf algebras.
3 Actually, (2.4) shows that the weaker conditions \( m R m_{12} = m_{12} R_{23} R_{13} \), \( m R m_{23} = m_{23} R_{12} R_{13} \)
and \( m m_{12} R_{12} R_{13} R_{23} = m m_{12} R_{23} R_{13} R_{12} \) are sufficient to ensure associativity.
If $S$ is also an $(A, m)$-automorphism, i.e., $S m = m (S \otimes S)$, the above condition reads $m (S \otimes S) R = S m R = m R (S \otimes S)$ which is satisfied if $(S \otimes S) R = R' (S \otimes S)$.

As an example, let us define

$$P = \theta^{ij} \partial_i \otimes \partial_j, \quad R = e^{P/2} = I + \frac{1}{2} P + \cdots \tag{2.7}$$

where $\partial_i : A \rightarrow A$ are commuting derivations of $A$, $\theta^{ij} \in \mathcal{R}$, and $I : A \rightarrow A$ is the identity operator. Using the derivation property

$$\partial_i m = m (\partial_i \otimes I + I \otimes \partial_i) \tag{2.8}$$

we get

$$P m_{12} = \theta^{ij} \partial_i m_{12} \otimes \partial_j = \theta^{ij} m_{12} (\partial_i \otimes I + I \otimes \partial_i) \otimes \partial_j = m_{12} (P_{13} + P_{23}) \tag{2.9}$$

and in the same way

$$P m_{23} = m_{23} (P_{12} + P_{13}) \tag{2.10}.$$

It follows that (2.2) and (2.3) are satisfied. Note that $\partial_i$ are also derivations of $A$ with respect to the product $\hat{m}$. $A$ need not be commutative.

If $A$ is the algebra of smooth functions on $\mathbb{R}^{2n}$ and $\theta^{ij} = i h \epsilon^{ij}$ with real $\epsilon^{ij}$ is antisymmetric and nondegenerate, then $\hat{m}$ is the well-known Moyal product \[3\].

If $A$ is the Heisenberg algebra with $[x^i, x^j] = i h \epsilon^{ij} I$, then $\text{ad}(x^i)y = [x^i, y]$ defines commuting derivations. Let

$$P = \frac{1}{i h} \epsilon_{ij} \text{ad}(x^i) \otimes \text{ad}(x^j) \tag{2.11}$$

where $\epsilon_{ij}$ is antisymmetric and satisfies $\epsilon_{ik} \epsilon^{jk} = \delta_{ij}$. Then one easily verifies that $x^i \ast x^j = x^j \ast x^i$ and $\hat{m}$ is commutative.

As a further example, consider the algebra $A = M(3, \mathbb{C})$ of $3 \times 3$-matrices with complex coefficients. Let $E_{ij}$ denote the matrix with entry 1 in the $i$th row and $j$th column and otherwise 0. Setting $H_1 = E_{11} - E_{22}$ and $H_2 = E_{22} - E_{33}$, the derivations $\text{ad}(H_i), i = 1, 2,$ of $A$ commute with each other. With $P = \vartheta \left[\text{ad}(H_1) \otimes \text{ad}(H_2) - \text{ad}(H_2) \otimes \text{ad}(H_1)\right]$ we get the following associative deformation of the ordinary matrix multiplication,

$$E_{12} * E_{23} = q E_{13}, \quad E_{23} * E_{31} = q E_{21}, \quad E_{31} * E_{12} = q E_{32} \tag{2.12}$$

$$E_{13} * E_{32} = q^{-1} E_{12}, \quad E_{21} * E_{13} = q^{-1} E_{23}, \quad E_{32} * E_{21} = q^{-1} E_{31}$$

where $q = e^{\theta^{ij}/2}$, and $E_{ij} * E_{kl} = \delta_{jk} E_{il}$ for all other combinations. This product is equivalent to the usual product of matrices in the sense of (2.6). A corresponding transformation map $S$ is determined by $S(E_{ii}) = E_{ii}$ and $S(E_{ij}) = q^{-1} E_{ij}, S(E_{ji}) = q E_{ji}$ for $i < j$. Obviously, the above construction can be applied to the universal enveloping algebra of every simple Lie algebra with a symplectic structure on its root space.
3 Moyal deformation of gauge theory and Seiberg–Witten map

In this section $\mathcal{A}$ denotes the algebra of smooth functions on $\mathbb{R}^{2n}$. Let $x^i$, $i = 1, \ldots, 2n$, be coordinate functions and $\partial_i$ the corresponding partial derivatives. The Moyal product is defined as in the previous section with

$$P = \theta^{ij}(\vartheta) \partial_i \otimes \partial_j$$

where $\theta^{ij}$ depends on a deformation parameter $\vartheta$, but not on the coordinates $x^i$. Hence $[x^i, x^j]_\ast = x^i \ast x^j - x^j \ast x^i = \theta^{ij}$. Let $(\Omega(\mathcal{A}), d)$ be the differential calculus over $(\mathcal{A}, \ast)$ such that

$$[d x^i, x^j]_\ast = d x^i \ast x^j - x^j \ast d x^i = 0 .$$

Using the Leibniz rule and $d^2 = 0$, this implies

$$d x^i \ast d x^j + d x^j \ast d x^i = 0 .$$

In the limit $\vartheta \rightarrow 0$ we recover the ordinary differential calculus on $\mathcal{A}$ with the usual product.

Let $\psi$ transform according to

$$\psi \mapsto \psi' = g \ast \psi$$

where $g$ is a map from $\mathbb{R}^{2n}$ into a representation of a Lie group $G$. The exterior covariant derivative of $\psi$ is

$$D \psi = d \psi + A \ast \psi$$

where $A = A_i \ast d x^i$ is a matrix of 1-forms. It transforms in the same way as $\psi$, i.e., $D' \psi' = g \ast D \psi$, if the gauge potential transforms as follows,

$$d g = g \ast A - A' \ast g .$$

One finds that

$$D^2 \psi = F \ast \psi$$

with the field strength (or curvature)

$$F = d A + A \ast A = \frac{1}{2} F_{ij} \ast d x^i \ast d x^j , \quad F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]_\ast$$

which transforms as $F \mapsto F' = g \ast F \ast g_{\ast}^{-1}$ where $g_{\ast}^{-1}$ is the $\ast$-inverse of $g$.

If a field $\varphi$ transforms as $\varphi \mapsto \varphi' = \varphi \ast g_{\ast}^{-1}$, its covariant derivative is $D \varphi = d \varphi - \varphi \ast A$ and we have $D^2 \varphi = -\varphi \ast F$. Furthermore, if $B$ transforms as $B \mapsto B' = g \ast B \ast g_{\ast}^{-1}$, then
\[ DB = dB + A \ast B - B \ast A \text{ and } D^2B = F \ast B - B \ast F. \] Using (3.8) and the Leibniz rule for \( d \), we obtain the Bianchi identity \( DF = 0. \)

In the following we explore the \( \vartheta \)-dependence of the above gauge theoretical formulas. In particular, we are looking for a way to construct a noncommutative gauge transformation from an ordinary (commutative) one. First we note that

\[
\frac{\partial}{\partial \vartheta} (f \ast h) = \frac{\partial f}{\partial \vartheta} \ast h + f \ast \frac{\partial h}{\partial \vartheta} + \tilde{d} f \wedge_\ast \tilde{d} h. \tag{3.9}
\]

In particular, this shows that \( \partial / \partial \vartheta \) is not a derivation of the \( \ast \)-product. Here we introduced, as an auxiliary structure which greatly helps to simplify the following calculations, a differential calculus \( (\tilde{\Omega}(\Omega(A)), \tilde{d}) \) over the algebra \( \Omega(A) \) where \( \tilde{d} \) is defined by

\[
\tilde{d} f = \partial_i f \ast \tilde{d} x^i, \quad [\tilde{d} x^i, x^j]_\ast = 0, \quad \tilde{d}(dx^i) = 0. \tag{3.10}
\]

Furthermore, we define an antisymmetric bilinear form on \( \tilde{\Omega}^1 \) by

\[
\tilde{d} x^i \wedge_\ast \tilde{d} x^j = \frac{1}{2} \epsilon^{ij}, \quad \epsilon^{ij} = \frac{\partial \theta}{\partial \vartheta} 
\]

which means

\[
\tilde{d} f \wedge_\ast \tilde{d} f' = \frac{1}{2} \epsilon^{ij} \partial_i f \ast \partial_j f' \tag{3.12}
\]

for \( f, f' \in A \). The map \( \tilde{d} \) extends to \( \tilde{\Omega}(\Omega(A)) \) as a linear map if we require \( \tilde{d} \tilde{d} = \tilde{d} \tilde{d} \).

Differentiation of (3.4) with respect to \( \vartheta \) leads to

\[
\frac{\partial \psi'}{\partial \vartheta} = \frac{\partial g}{\partial \vartheta} \ast \psi + g \ast \frac{\partial \psi}{\partial \vartheta} + \tilde{d} g \wedge_\ast \tilde{d} \psi. \tag{3.13}
\]

This can be rewritten as follows,

\[
\frac{\partial \psi'}{\partial \vartheta} + \tilde{A}' \wedge_\ast \tilde{d} \psi' = g \ast (\frac{\partial \psi}{\partial \vartheta} + \tilde{A} \wedge_\ast \tilde{d} \psi) + \frac{\partial g}{\partial \vartheta} + \tilde{A}' \wedge_\ast \tilde{d} g \ast \psi \tag{3.14}
\]

where we used

\[
\tilde{d} g = g \ast \tilde{A} - \tilde{A}' \ast g, \quad \tilde{A} = A_i \ast \tilde{d} x^i \tag{3.15}
\]

(which follows from (3.6)), the Leibniz rule for \( \tilde{d} \), and (3.4). Introducing a matrix field \( \Gamma \) with the transformation law

\[
\frac{\partial g}{\partial \vartheta} + \tilde{A}' \wedge_\ast \tilde{d} g = g \ast \Gamma - \Gamma' \ast g, \tag{3.16}
\]

\footnote{The ordinary differentials \( df \) and, moreover, all elements of \( \Omega(A) \) are 0-forms in \( \tilde{\Omega}(\Omega(A)) \).}

\footnote{This should not be confused with the wedge product in \( \tilde{\Omega}(\Omega(A)) \), which is not needed in this work.}
\[ 3.14 \] becomes

\[ \nabla'_\varphi' = g * \nabla_\varphi \psi \]  
(3.17)

with the covariant derivative

\[ \nabla_\varphi \psi = \frac{\partial \psi}{\partial \vartheta} + \tilde{A} \wedge_\ast \tilde{d} \psi + \Gamma * \psi . \]  
(3.18)

For a field \( \varphi \) with \( \varphi' = \varphi * g_{\ast}^{-1} \), an analogous calculation leads to the covariant derivative

\[ \nabla_\varphi \varphi = \frac{\partial \varphi}{\partial \vartheta} - \tilde{d} \varphi \wedge_\ast \tilde{A} - \varphi * \Lambda \]  
(3.19)

with a matrix field \( \Lambda \) which transforms as follows,

\[ \frac{\partial g}{\partial \vartheta} - \tilde{d} g \wedge_\ast \tilde{A}' = g * \Lambda - \Lambda' * g . \]  
(3.20)

Together with \( 3.13 \) and \( 3.16 \), the last equation leads to

\[ g * (\Gamma - \Lambda - \tilde{A} \wedge_\ast \tilde{A}) = (\Gamma' - \Lambda' - \tilde{A}' \wedge_\ast \tilde{A}') * g . \]  
(3.21)

Since \( Q = \Gamma - \Lambda - \tilde{A} \wedge_\ast \tilde{A} \) can be absorbed via a redefinition of \( \Lambda \) in \( 3.21 \), we are allowed to set \( Q = 0 \) and get the following relation between \( \Gamma \) and \( \Lambda \),

\[ \Lambda = \Gamma - \tilde{A} \wedge_\ast \tilde{A} . \]  
(3.22)

Inserting this expression for \( \Lambda \) in \( 3.19 \), we find

\[ \nabla_\varphi \varphi = \frac{\partial \varphi}{\partial \vartheta} - \tilde{D} \varphi \wedge_\ast \tilde{A} - \varphi * \Gamma . \]  
(3.23)

A more symmetric form for the covariant derivatives of \( \psi \) and \( \varphi \) is achieved by setting

\[ \Gamma = \frac{1}{2} \tilde{A} \wedge_\ast \tilde{A} + \gamma , \quad \Lambda = -\frac{1}{2} \tilde{A} \wedge_\ast \tilde{A} + \gamma \]  
(3.24)

with a matrix field \( \gamma \). Then

\[ \nabla_\varphi \psi = \frac{\partial \psi}{\partial \vartheta} + \frac{1}{2} \tilde{A} \wedge_\ast (\tilde{d} \psi + \tilde{D} \psi) + \gamma * \psi \]  
(3.25)

\[ \nabla_\varphi \varphi = \frac{\partial \varphi}{\partial \vartheta} - \frac{1}{2} (\tilde{d} \varphi + \tilde{D} \varphi) \wedge_\ast \tilde{A} - \varphi * \gamma . \]  
(3.26)

For a field \( B \) with \( B' = g * B * g_{\ast}^{-1} \), similar calculations lead to the covariant derivative

\[ \nabla_\varphi B = \frac{\partial B}{\partial \vartheta} + \tilde{A} \wedge_\ast \tilde{d} B - \tilde{d} B \wedge_\ast \tilde{A} - \tilde{A} * B \wedge_\ast \tilde{A} + \Gamma * B - B * \Lambda \]  
(3.27)
Furthermore, we have
\[ \nabla_\theta(B \ast \psi) = (\nabla_\theta B) \ast \psi + B \ast \nabla_\theta \psi + \tilde{D} B \wedge_\ast \tilde{D} \psi , \]  
for example, which is a covariant version of (3.25). The last term destroys the familiar “derivation” property of covariant derivatives.

Differentiation of (3.6) with respect to \( \theta \) and use of (3.15) and (3.16) leads to
\[ \nabla'_\theta A' = g \ast (\nabla_\theta A) \ast g^{-1} \]  
where
\[ \nabla_\theta A = \frac{\partial A}{\partial \theta} + \tilde{A} \wedge_\ast \tilde{d} A - \tilde{F} \wedge_\ast \tilde{A} - D \Gamma \]
\[ = \frac{\partial A}{\partial \theta} + \frac{1}{2} \tilde{A} \wedge_\ast (\tilde{d} A + \tilde{F}) - \frac{1}{2} (\tilde{d} A + \tilde{F}) \wedge_\ast \tilde{A} - D \gamma \]
with \( D \Gamma = d \Gamma + A \ast \Gamma - \Gamma \ast A \), a corresponding definition for \( D \gamma \), and
\[ \tilde{F} = \tilde{d} A - \tilde{A} \ast A - A \ast \tilde{A} . \]  

Moreover, we have
\[ \nabla_\theta(DB) - D\nabla_\theta B = \tilde{F} \wedge_\ast \tilde{D} B - \tilde{D} B \wedge_\ast \tilde{F} + (\nabla_\theta A) \ast B - B \ast \nabla_\theta A \]
from which the corresponding formulas for \( \nabla_\theta(D\psi) \) and \( \nabla_\theta(D\varphi) \), for example, are evident. In particular,
\[ \nabla_\theta(D^2\psi) = D(\nabla_\theta D\psi) + \tilde{F} \wedge_\ast \tilde{D} D\psi + (\nabla_\theta A) \ast D\psi . \]

Using \( \nabla_\theta D\psi = D\nabla_\theta \psi + \tilde{F} \wedge_\ast \tilde{D} \psi + (\nabla_\theta A) \ast \psi \) and the Leibniz rule for \( D \), we obtain
\[ \nabla_\theta F = D(\nabla_\theta A) + \tilde{F} \wedge_\ast \tilde{F} . \]

If we require \( \nabla_\theta A = 0 \), which means
\[ \frac{\partial A}{\partial \theta} = -\frac{1}{2}[\tilde{A} \wedge_\ast (\tilde{d} A + \tilde{F}) - (\tilde{d} A + \tilde{F}) \wedge_\ast \tilde{A}] + D \gamma , \]
then we have also \( \nabla'_\theta A' = 0 \) and thus
\[ \frac{\partial A'}{\partial \theta} = -\frac{1}{2}[\tilde{A}' \wedge_\ast (\tilde{d} A' + \tilde{F}') - (\tilde{d} A' + \tilde{F}') \wedge_\ast \tilde{A}'] + D' \gamma' . \]

Together with (3.16) which reads
\[ \frac{\partial g}{\partial \theta} = \frac{1}{2}(\tilde{d} g \wedge_\ast \tilde{A} - \tilde{A}' \wedge_\ast \tilde{d} g) + g \ast \gamma - \gamma' \ast g , \]
this forms a system of first order differential equations which determines $g(\vartheta), A(\vartheta)$ and $A'(\vartheta)$ (here we suppress the dependence on the coordinates $x^i$, for simplicity) from $g(0), A(0)$ and $A'(0)$ and a choice of $\gamma$ and $\gamma'$. This means that, given a classical gauge transformation, the above equations determine a corresponding noncommutative gauge transformation. This is a Seiberg-Witten map \[1\]. In particular, expanding $g, A$ and $A'$ in powers of $\vartheta$, the coefficients of the $(n + 1)$th power are determined via (3.35), (3.36) and (3.37) by the $n$th order coefficients and thus recursively by the 0th order.

Using $\nabla_\vartheta A = 0$ and (3.27) in (3.34) yields

$$\frac{\partial F}{\partial \vartheta} = \tilde{F} \wedge \tilde{F} - \frac{1}{2} [\tilde{A} \wedge (\tilde{d}F + \tilde{D}F) - (\tilde{d}F + \tilde{D}F) \wedge \tilde{A}] - \gamma \ast F + F \ast \gamma.$$  

This first order differential equation determines the curvature of the noncommutative connection from that of the commutative connection $A$ at $\vartheta = 0$. In particular, $F(0) = 0$ implies $F(\vartheta) = 0$ for all $\vartheta$.

Expressed in components, the equations (3.37), (3.35) and (3.38) with $\gamma = 0 = \gamma'$ take the form

$$\frac{\partial g}{\partial \vartheta} = \frac{1}{4} \epsilon_{ij} (\partial_i g \ast A_j + A'_j \ast \partial_i g)$$  

$$\frac{\partial A_i}{\partial \vartheta} = -\frac{1}{4} \epsilon_{kl} [A_k, \partial_l A_i + F_{li}]_{*,+}$$  

$$\frac{\partial F_{ij}}{\partial \vartheta} = \frac{1}{4} \epsilon_{kl} (2[A_k, D_l F_{ij}]_{*,+} - [A_k, D_l F_{ij}]_{*,+})$$  

where $[f,h]_{*,+} = f \ast h + h \ast f$. From these equations one recovers equations (3.8) in \[1\] for an infinitesimal gauge transformation.

We can extend the Seiberg-Witten map to matter fields like $\psi, \varphi, B$ by setting their covariant $\vartheta$-derivatives to zero. For example, $\nabla_\vartheta \psi = 0$ with $\gamma = 0$ leads to

$$\frac{\partial \psi}{\partial \vartheta} = -\frac{1}{4} \epsilon_{ij} A_i \ast (\partial_j \psi + \nabla_j \psi).$$

4 Parallel transport in deformation space and curvature

In general, $\theta^{ij}$ may depend on several deformation parameters $\vartheta_i$. The deformation elaborated in section 3 can then be performed along any curve in the deformation space $\Theta$ on which $\theta^{ij}$ are functions. We learned that a Seiberg-Witten map has the geometric interpretation of a parallel transport along a curve in $\Theta$. In general, such a parallel transport is path-dependent due to the presence of a curvature associated with the covariant derivatives (see also \[2\]).

\[6\]We may discard the equation for $A'$ and eliminate $A'$ in (3.37) using (3.3).
Instead of $\tilde{d}$, $\tilde{A}$, $\Gamma$ (and other quantities) which refer to a deformation parameter $\vartheta$, we write $d_1, A_1, \Gamma_1$ and $d_2, A_2, \Gamma_2$, referring to deformation parameters $\vartheta_1$ and $\vartheta_2$, respectively. Correspondingly, there are two different antisymmetric bilinear forms replacing (3.11) with $\epsilon_{ij}^1 = \partial \theta_{ij}^1 / \partial \vartheta_1$ and $\epsilon_{ij}^2 = \partial \theta_{ij}^2 / \partial \vartheta_2$. In particular,

$$\nabla_{\vartheta_1} \psi = \frac{\partial \psi}{\partial \vartheta_1} + A_1 \wedge_d d_1 \psi + \Gamma_1 \ast \psi, \quad (4.1)$$

$$\nabla_{\vartheta_2} \psi = \frac{\partial \psi}{\partial \vartheta_2} + A_2 \wedge_d d_2 \psi + \Gamma_2 \ast \psi \quad (4.2)$$

replace (3.18). Now

$$[\nabla_{\vartheta_1}, \nabla_{\vartheta_2}] \psi = F_{12} \wedge_d d_1 d_2 \psi + (F_{12} \wedge_d A_2 - \nabla_{\vartheta_2} A_1) \wedge_d d_1 \psi$$

$$+ (F_{12} \wedge_d A_1 + \nabla_{\vartheta_1} A_2) \wedge_d d_2 \psi$$

$$+ (K_{12} + A_1 \wedge_d d_1 \Gamma_2 - A_2 \wedge_d d_2 \Gamma_1) \ast \psi \quad (4.3)$$

where

$$F_{12} = d_1 A_2 - d_2 A_1 + A_1 \ast A_2 - A_2 \ast A_1, \quad (4.4)$$

$$K_{12} = \frac{\partial}{\partial \vartheta_1} \Gamma_2 - \frac{\partial}{\partial \vartheta_2} \Gamma_1 + \Gamma_1 \ast \Gamma_2 - \Gamma_2 \ast \Gamma_1. \quad (4.5)$$

After several manipulations, one arrives at the following generalized Ricci identity,

$$[\nabla_{\vartheta_1}, \nabla_{\vartheta_2}] \psi = F_{12} \ast \psi + \frac{1}{2} F_{12} \wedge_d (D_1 D_2 + D_2 D_1) \psi$$

$$+ (\nabla_{\vartheta_1} A_2) \wedge_d D_2 \psi - (\nabla_{\vartheta_2} A_1) \wedge_d D_1 \psi \quad (4.6)$$

which is evidently covariant since the generalized curvature

$$F_{12} = K_{12} + A_1 \wedge_d d_1 \Gamma_2 - A_2 \wedge_d d_2 \Gamma_1 + (\nabla_{\vartheta_2} A_1) \wedge_d A_1 - (\nabla_{\vartheta_1} A_2) \wedge_d A_2$$

$$- \frac{1}{2} F_{12} \wedge_d (d_1 A_2 + d_2 A_1 + A_1 \ast A_2 + A_2 \ast A_1)$$

$$= K_{12} + \frac{1}{2} \frac{\partial \theta_{ij}^1}{\partial \vartheta_1} A_i \ast \partial_j \Gamma_2 - \frac{1}{2} \frac{\partial \theta_{ij}^1}{\partial \vartheta_2} A_i \ast \partial_j \Gamma_1$$

$$+ \frac{1}{2} \frac{\partial \theta_{ij}^1}{\partial \vartheta_1} (\nabla_{\vartheta_2} A_i) \ast A_j - \frac{1}{2} \frac{\partial \theta_{ij}^1}{\partial \vartheta_2} (\nabla_{\vartheta_1} A_i) \ast A_j$$

$$- \frac{1}{8} \frac{\partial \theta_{ij}^1 \theta_{kl}}{\partial \vartheta_1} F_{ik} \ast (\partial_j A_l + \partial_l A_j + A_j \ast A_l + A_l \ast A_j) \quad (4.7)$$

transforms as follows,

$$F'_{12} = g \ast F_{12} \ast g^{-1}. \quad (4.8)$$

This result is obtained by a lengthy calculation starting with

$$\left( \frac{\partial}{\partial \vartheta_1} \frac{\partial}{\partial \vartheta_2} - \frac{\partial}{\partial \vartheta_2} \frac{\partial}{\partial \vartheta_1} \right) g = 0, \quad (4.9)$$
using (3.16) in the form

\[ \frac{\partial g}{\partial \vartheta} + A'_1 * g \wedge A_1 = g * \Gamma_1 - \Lambda'_1 * g \]  

(4.10)

and correspondingly with the index 1 replaced by 2, and noting that the generalized curvature also has the following expression,

\[ F_{12} = \frac{\partial}{\partial \vartheta_1} A_2 - \frac{\partial}{\partial \vartheta_2} A_1 + A_1 * A_2 - A_2 * A_1 - d_1 A_2 \wedge_2 A_1 + d_2 A_1 \wedge_2 A_2 \]

\[ - A_1 \wedge_2 \nabla_{\vartheta_2} A_1 + A_2 \wedge_2 \nabla_{\vartheta_1} A_2 \]

\[ + \frac{1}{2} (d_1 A_2 + d_2 A_1 - A_1 * A_2 - A_2 * A_1) \wedge_* F_{12} . \]  

(4.11)

Besides the generalized curvature, there are additional terms on the rhs of the Ricci identity (4.6). Their origin lies in the deviation of the covariant derivative \( \nabla_{\vartheta} \) from a “derivation” (cf (3.28)).

The formula which replaces (4.6) for the field \( \varphi \) is

\[ [\nabla_{\vartheta_1} , \nabla_{\vartheta_2}] \varphi = - \varphi * F_{12} - \frac{1}{2} [(D_1 D_2 + D_2 D_1) \varphi] \wedge_* F_{12} \]

\[ + D_1 \varphi \wedge_* \nabla_{\vartheta_2} A_1 - D_2 \varphi \wedge_* \nabla_{\vartheta_1} A_2 . \]  

(4.12)

The path-dependence of Seiberg-Witten maps leads to the following idea. Consider a closed path through \( \vartheta = 0 \) in the deformation parameter space. We could imagine that, as a consequence of the nonvanishing Ricci identity, parallel transport from \( \vartheta = 0 \) along the path back to \( \vartheta = 0 \) maps a solution of some commutative model to another solution. This may lead to a solution generating method.

### 5 Moyal deformations of integrable models and Seiberg-Witten map

Let \((\Omega, d, \delta)\) be a bi-differential calculus \([6]\) over \((A, *)\) such that the bicomplex conditions \(d^2 = \delta^2 = d\delta + \delta d = 0\) are identically satisfied. Replacing \(d\) with \(D\) defined by

\[ D \psi = g_*^{-1} * d (g * \psi) = d \psi + A * \psi , \quad A = g_*^{-1} * d g \]  

(5.1)

for some \(*\)-invertible matrix-valued function \(g\), the new bicomplex conditions are equivalent to

\[ \delta A = 0 . \]  

(5.2)

Note that, as a consequence of the definition of \(A\), the curvature of \(A\) vanishes, i.e.,

\[ F = dA + A * A = 0 . \]  

(5.3)
For $\vartheta = 0$, such a bicomplex can be associated with many integrable models, including the self-dual Yang-Mills equations, in such a way that \( (5.2) \) is equivalent to the integrable model equation \( [5] \).

Is it possible to obtain solutions of the noncommutative version from those of a classical integrable model via the Seiberg-Witten map? Indeed, we already know that the Seiberg-Witten map preserves the zero curvature condition. It remains to investigate with the help of the formalism of section 3 whether this map also preserves \( (5.2) \). $\tilde{d}$ extends to $\Omega$ with the additional rules $\delta \tilde{d} = \tilde{d} \delta$ and $\tilde{d} \delta x_i = 0$. Applying $\delta$ to \( (3.35) \), using $F = 0$ and $\delta \tilde{d} = \tilde{d} \delta$, we get

$$
\frac{\partial}{\partial \vartheta} \delta A = \delta \frac{\partial A}{\partial \vartheta} = -\frac{1}{2} [\delta \tilde{A} \wedge_{*} \tilde{d} A + \tilde{A} \wedge_{*} \tilde{d} \delta A - \tilde{d} \delta A \wedge_{*} \tilde{A} + \tilde{d} A \wedge_{*} \delta \tilde{A}] .
$$

(5.4)

If the additional condition

$$
\delta \tilde{A} \wedge_{*} \tilde{d} A + \tilde{d} A \wedge_{*} \delta \tilde{A} = 0
$$

(5.5)

holds, then the last equation indeed implies $\delta A(\vartheta) = 0$ if $\delta A(0) = 0$. As a consequence, each solution of the classical integrable model generates a solution of the noncommutative version, if the latter solution satisfies \( (5.5) \).

As an example, let us start with the trivial bi-differential calculus determined by

$$
d\psi = \psi_t * dt + \psi_x * dx, \quad \delta \psi = \psi_x * dt + \psi_t * dx
$$

(5.6)

with coordinates $t$ and $x$, and “dress” the first operator according to \( (5.1) \) so that

$$
D\psi = (\psi_t + U * \psi) dt + (\psi_x + V * \psi) dx
$$

(5.7)

where

$$
U = g_s^{-1} * g_t, \quad V = g_s^{-1} * g_x.
$$

(5.8)

The bicomplex conditions are satisfied if and only if

$$
(g_s^{-1} * g_t)_t - (g_s^{-1} * g_x)_x = 0
$$

(5.9)

which is the noncommutative version of the principal chiral field equation (see also \( [8] \)). The condition \( (5.5) \) becomes

$$
\delta \tilde{A} \wedge_{*} \tilde{d} A + \tilde{d} A \wedge_{*} \delta \tilde{A} = \frac{1}{2} [U_x - V_t, V_x - U_t]_{s,+} * dt * dx
$$

(5.10)

which vanishes as a consequence of the field equation \( (5.9) \). Hence, every solution of the classical principal chiral model generates a solution of the noncommutative model. In practice,

---

7 Most integrable models admit a zero curvature formulation with a parameter ($\lambda$) dependent connection. Since the Seiberg-Witten map is quadratic in the connection, it does not, in general, respect the concrete $\lambda$-dependence of a flat connection. This results in constraints and thus obstructions to construct solutions of the deformed model from the commutative one.
this allows at least the recursive calculation of the coefficients of a power series expansion of the field $g(\vartheta)$ in the deformation parameter $\vartheta$ from a given classical solution $g(0)$. Convergence of the resulting formal power series has still to be investigated.

Using (3.6) and $t^* x - x^* t = \vartheta$, (5.37) leads to

$$\frac{\partial g}{\partial \vartheta} = \frac{1}{4}(g_x \partial_x g_x^{-1} - g_t \partial_t g_t^{-1}) * g = -\frac{1}{2}(\tilde{d}g \wedge \tilde{d}g^{-1}) * g$$

(5.11)

with $\tilde{d}f = f_t \tilde{d}t + f_x \tilde{d}x$. A special class of classical chiral models is defined by $g_0 = I - 2\Pi_0$ where $\Pi_0$ is a projection, i.e., $\Pi_0^2 = \Pi_0$. Let us try to find a deformation of this particular class. If we assume that $g = I - 2\Pi$ with $\Pi \ast \Pi = \Pi$, then we have $g_x^{-1} = I - 2\Pi$. Substituting this in (5.11), we find

$$\frac{\partial \Pi}{\partial \vartheta} = (\tilde{d}\Pi \wedge \ast \tilde{d}\Pi) * (I - 2\Pi).$$

(5.12)

With the help of the equation derived from $\Pi \ast \Pi = \Pi$ by differentiation with respect to $\vartheta$, and using the Leibniz rule for $\tilde{d}$, we obtain

$$\tilde{d}\Pi \wedge \ast \tilde{d}\Pi = 0$$

(5.13)

which is an additional condition for $\Pi$ if $\vartheta \neq 0$. This shows that the Seiberg-Witten map is not necessarily consistent with reductions of the principal chiral model.

6 Final remarks

Let ncEQS stand for a (noncommutative) deformation of a system EQS of field equations. If we can find a system of first order differential equations in the deformation parameter $\vartheta$, as a consequence of which $\partial(ncEQS)/\partial \vartheta = 0$, then (under certain technical conditions) solutions of ncEQS are obtained from solutions of EQS. The Seiberg-Witten map for gauge fields provides us with an example. It allows us, in particular, to construct solutions of the noncommutative zero curvature condition from solutions of the classical zero curvature condition. Moreover, we have shown that this map also works in case of the two-dimensional principal chiral model and its noncommutative version. Another example which fits into the above scheme is the noncommutative KdV equation treated in [7].

References

[1] Seiberg N and Witten E 1999 String theory and noncommutative geometry, [hep-th/9908142], JHEP 09 032.

[2] Asakawa T and Kishimoto I 1999 Comments on gauge equivalence in noncommutative geometry, [hep-th/9909139], JHEP 11 (1999) 024.
[3] Cornalba L 1999 D-brane physics and noncommutative Yang-Mills theory, hep-th/9909081

Ishibashi N 1999 A relation between commutative and noncommutative descriptions of D-branes, hep-th/9909176

Ishibashi N, Iso S, Kawai H and Kitazawa Y 2000 Wilson loops in non-commutative Yang-Mills Nuclear Physics B 573 573–593

Okuyama K 2000 A path integral representation of the map between commutative and noncommutative gauge fields, hep-th/9910138, JHEP 03 016

Chu C-S, Ho P-M and Li M 2000 Matrix theory in a constant C field background, hep-th/9911153, Nucl. Phys. B 574 275-287

Andreev O and Dorn H 1999 On open string sigma-model and noncommutative gauge fields, hep-th/9912070

Jurco B and Schupp P 2000 Noncommutative Yang-Mills from equivalence of star products, hep-th/0001032

Terashima S 2000 On the equivalence between noncommutative and ordinary gauge theories, hep-th/0001111, to appear in JHEP

Madore J, Schraml S, Schupp P and Wess J 2000 Gauge theory on noncommutative spaces, hep-th/0001203

Hashimoto K and Hirayama T 2000 Branes and BPS configurations of non-commutative/commutative gauge theories, hep-th/0002138

Asakawa T and Kishimoto I 2000 Noncommutative gauge theories from deformation quantization, hep-th/0002090

Alekseev A Y and Bytsko A G 2000 Wilson lines on noncommutative tori Phys. Lett. B 482 271–275

Moriyama S 2000 Noncommutative monopole from nonlinear monopole, hep-th/0003234

Benaoum H B 2000 On noncommutative and commutative equivalence for BFYM theory: Seiberg-Witten map, hep-th/0004002

Jurco B, Schupp P and Wess J 2000 Noncommutative gauge theory for Poisson manifolds, hep-th/0005003

Goto S and Hata H 2000 Noncommutative monopole at the second order in $\theta$, hep-th/0005101

Terashima S 2000 The non-Abelian Born-Infeld action and noncommutative gauge theory, hep-th/0006058

Jurco B, Schraml S, Schupp P and Wess J 2000 Enveloping algebra valued gauge transformations for nonabelian gauge groups on noncommutative spaces, hep-th/0006246

Rey S-J and von Unge R 2000 S-duality, noncritical open string and noncommutative gauge theory, hep-th/0007089

[4] Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 Deformation theory and quantization I, II Ann. Phys. 111 61–151.

[5] Dimakis A and Müller-Hoissen F 2000 Bi-differential calculi and integrable models J. Phys. A 33 957-974; 2000 Bicomplexes and integrable models, nlin.SI/0006029.
[6] Dimakis A and Müller-Hoissen F 2000 Bicomplexes, integrable models, and noncommutative geometry, hep-th/0006003; 2000 A noncommutative version of the nonlinear Schrödinger equation, hep-th/0007015.

[7] Dimakis A and Müller-Hoissen F 2000 Noncommutative Korteweg-de-Vries equation, hep-th/0007074.

[8] Takasaki K 2000 Anti-self dual Yang-Mills equations on noncommutative spacetime, hep-th/0005194.