Half-vortices, confinement transition and chiral Josephson effect in superconducting Weyl/Dirac semimetals

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We start by showing that the most generic spin-singlet pairing in a superconducting Weyl/Dirac semimetal is specified by a U(1) phase $e^{i\phi}$ and two real numbers $(\Delta_s, \Delta_5)$ that form a representation of complex algebra. Such a "complex" superconducting state realizes a $Z_2 \times U(1)$ symmetry breaking in the matter sector where $Z_2$ is associated with the chirality. The resulting effective XY theory of the fluctuations of the U(1) phase $\phi$ will be now augmented by coupling to another dynamical variable, the chiral angle $\chi$ that defines the polar angle of the complex number $(\Delta_s, \Delta_5)$. We compute this coupling by considering a Josephson set up. Our energy functional of two phase variables $\phi$ and $\chi$ allows for the realization of a half-vortex (or double Cooper pair) state and its BKT transition. The half-vortex state is sharply characterized by a flux quantum which is half of the ordinary superconductors. Such a $\pi$-periodic Josephson effect can be easily detected as doubled ac Josephson frequency. We further show that the Josephson current $I$ is always accompanied by a chiral Josephson current $I_5$. Strain pseudo gauge fields that couple to the $\chi$, destabilize the half-vortex state. We argue that our "complex superconductor" realizes an extension of XY model that supports confinement transition from half-vortex to full vortex excitations.

Introduction: Weyl semimetals (WSMs) are a novel class of topological materials that host chiral fermions as their low-energy excitations [1]. The chirality is an additional attribute of the electrons in WSMs that defines whether the momentum and the spin are parallel ($\tau = +1$) or anti-parallel ($\tau = -1$) [2]. We consider the superconductivity in a simplest spin-singlet pairing channel of a WSM. Now, with respect to the exchange of the chirality attribute $\tau$ of the electrons, the pairing potential can be either even or odd. Therefore in WSMs, the most general form of the pairing potential is a combination of even- or odd-chirality parts:

$$e^{i\phi} \{ \Delta_s \left( \psi_{s,\tau} \psi_{s,\tau}^* - \psi_{s,-\tau} \psi_{s,-\tau}^* \right) + i\Delta_5 \left( \psi_{s,\tau} \psi_{s,\tau}^* - \psi_{s,-\tau} \psi_{s,-\tau}^* \right) \} + \alpha,$$

where $\alpha = \pm \frac{1}{2}$ denote the chirality and spin of the electrons, respectively. The parts of the above paring that are even (odd) with respect to chirality reversal, are represented with the amplitudes $\Delta_s$ ($\Delta_5$). In the absence of the chirality attribute, the second line of the generic pairing (the odd part) in Eq. (1) is absent. Choosing $\Delta_s$ in Eq. (1) to be real and positive, it appears that $\Delta_s$ can in general be arbitrary complex numbers. In the following we show that in the above representation, $\Delta_s$ can only be a real number. Assume that $\Delta_5$ has a complex phase difference $\alpha$ with respect to $\Delta_s$. Then for right/left chirality pairing one would obtain the gap parameters $|\Delta_s|^2 = \Delta_5^2 + \Delta_5^2 = 2\Delta_5 \sin \alpha$. For a generic $\alpha$, this would lead to two superconducting transitions at two different temperatures $T_{c}^s$ and $T_{c}^s$. Demanding to characterize the superconducting phase of Weyl superconductors by a unique transition temperature, $T_c$, one obtains $\pm \Delta_s \Delta_5 \sin \alpha = 0$. This means that either (i) $\Delta_5 = 0$ (namely the case of the superconductors that do not posses chirality attribute), or (ii) if $\Delta_5$ is non-zero, then one must have $\sin \alpha = 0$ which gives $\alpha = 0$ or $\alpha = \pi$.

One therefore finds that for a unique superconducting transition in a Weyl material the phase of the chirality odd part pairing $\Delta_5$ can only satisfy $\arg (\Delta_5) = \arg (\Delta_s) + \nu \pi$, where the integer $\nu$ defines a $Z_2$ valued quantity $e^{i\nu \pi}$. The pair of numbers $(\Delta_s, \pm \Delta_5)$ are real numbers, and both represent energetically equivalent superconducting states.

$Z_2 \times U(1)$ symmetry breaking: What is the meaning of the emergent Ising variable $\tau$? In the Weyl basis where $\Psi_e = (\psi_{s,\tau}^e \psi_{s,-\tau}^e, \psi_{s,-\tau}^e \psi_{s,\tau}^e)^T$ and $\Psi_h = (-\psi_{s,\tau}^h, \psi_{s,\tau}^h, \psi_{s,-\tau}^h, -\psi_{s,-\tau}^h)^T$, the matrix representation of $\Delta$ will become

$$e^{i\phi} (\Delta_s + i\Delta_5 \gamma^5),$$

where the $\gamma^5$ matrix in the Weyl basis is diag(1, 1, -1, -1) [2, 3]. In the above representations, the diagonal elements of $\gamma^5$ are the chirality eigenvalues and for real $\Delta_s$ and $\Delta_5$ the prefactor $i$ guarantees the Hermiticity of the resulting Weyl-Bogoliubov-De Gennes (WBDG) Hamiltonian. Therefore the amplitude $\Delta_5$ is nothing but the previously found pseudoscalar superconductivity [4]. Such form of pseudoscalar superconductivity, spontaneously breaks not only the global $U(1)$ symmetry (that is broken by every SC and is equivalent to a non-zero $\Delta_s$), but also the parity symmetry that corresponds to picking one of the eigenvalues of $\gamma^5$ and giving the eigenvalues $e^{i\phi}(\Delta_s \pm i\Delta_5)$. Therefore the Ising variable $\tau = e^{i\nu \pi} = \pm 1$ found above actually corresponds to the eigenvalues of $\gamma^5$, and specifies whether the relative phase of $i\Delta_5$ with respect to $\Delta_s$, is positive or negative. Here the $Z_2$ symmetry means that for any given $\Delta_5$, there is another energetically degenerate state with $-\Delta_5$, and therefore picking among the eigenvalues of $\gamma^5$ in Eq. (2) is tantamount to breaking this $Z_2$ symmetry.

Complex $(\Delta_s, \Delta_5)$ plane: Now that we have established that $\Delta_s$ and $\Delta_5$ are both real numbers, consider the geometric algebra constructed from scalar $\mathbb{1}$ and the pseudoscalar $\gamma^0\gamma^1\gamma^2\gamma^3 = -\mathbb{1}$ of the Clifford algebra [5]: $\Delta_s \mathbb{1} = -\Delta_5 \gamma^0 \gamma^1 \gamma^2 \gamma^3$. This is exactly the superconducting order parameter (2) of Dirac/Weyl superconductors. Hence
Dirac/Weyl superconductor realizes a graded algebra which is isomorphic to complex algebra. Therefore, apart from the $U(1)$ phase $e^{i\phi}$, the most generic form of superconductivity in WSMs can be represented by a number in the complex plane $(\Delta_s, \Delta_5)$. The conventional superconductors are confined to the real axis of this plane, while the pure pseudoscalar superconductors [6, 7] are confined to the imaginary axis of this plane. Eq. (2) can be alternatively represented as

$$e^{i\phi}(\Delta_s + i\Delta_5) = e^{i\phi}e^{i\gamma}e^{i\chi}.$$  

(3)

where $\Delta = \sqrt{\Delta_s^2 + \Delta_5^2}$ and the *chiral angle* (CA) is defined by $\chi = \mp \arctan(\Delta_5/\Delta_s)$. After the $\mathbb{Z}_2$ breaking, either of the signs for $\pm \chi$ in the Argand diagram of Fig. 1 is spontaneously picked.

In order to appreciate the importance of the notion of the complex $(\Delta_s, \Delta_5)$ plane, let us consider a purely pseudoscalar superconductor [6], $(0, \Delta_5)$. The complex plane structure allows to immediately understand the topologically non-trivial structure of such this superconductors. The coefficient $"i"=e^{i\pi/2}$ in front of $\Delta_5$ of Eq. (2), directly enters the amplitude of Andreev reflection at a superconductor-normal interface. This amounts to an additional phase change of $\pi/2$ upon every Andreev reflection (For more details refer to supplementary material). Therefore in a superconductor-normal-superconductor (S/N/S) Josephson junction, a total phase of $\pi$ is accumulated at the two interfaces of the S/N/S junction. Such a $\pi$ phase corresponds to change in the number parity. Therefore in a closed loop geometry of a Josephson junction, the electron has to traverse the loop once again, giving rise to the $4\pi$ (two rounds) periodic Josephson effect. The $4\pi$-periodic Josephson effect is a hallmark of topological superconductivity and its associated Majorana modes [6]. This can be the most natural explanation for the observed $4\pi$—periodic Andreev bound states [7]. So the Josephson physics on the real axis $(\Delta_s)$ and imaginary axis $(\Delta_5)$ are significantly different. Now we are going to explore the rest of this complex plane and show that it contains a remarkably rich physics of half-vortices and confinement transition.

*Josephson coupling*: Alternative way of expressing the complex algebra $(\Delta_s, \Delta_5)$ of superconducting Dirac/Weyl materials is to specify it by a strength $\Delta$ and two angles $(\phi, \chi)$, $\phi$ being the $U(1)$ phase couples to external $U(1)$ gauge fields (EM fields), while $\chi$ being the axial phase couples to pseudo gauge fields (that can be produced by strain fields [8–13].

The natural question will be, given two superconductors $\Delta_s e^{i\phi} e^{ix_\phi}$ where $\phi = l, r$ corresponds to left/right superconductor as in Fig. 1, how will the CAs, $\chi_a$ modify the Josephson effect? The answer to this basic question will provide us with the effective Hamiltonian governing the dynamics of phase fields $(\phi, \chi)$ in Josephson arrays. The chemical potential between the left $(l)$ and right $(r)$ superconductors is set by the voltage $V$ across the barrier. We assume that the barrier layer is sufficiently thin for electrons to tunnel through, and that the tunneling process can be regarded as a small perturbation. The tunneling current is given by [14]

$$I = \frac{e}{\hbar V(V_t)} \sum_{k, q} \int_{-\infty}^{\infty} dt' e^{i(t+V_t)}$$

$$\text{Tr} \left[ \langle \hat{c}_k(t) \hat{c}_k^\dagger(t') \rangle \hat{T}_{kj} \hat{n}_z \langle \hat{d}_q(t) \hat{d}_q^\dagger(t') \rangle^T \hat{T}_{qk} \right]$$

$$-\langle \hat{c}_k^\dagger(t') \hat{c}_k(t) \rangle^T \hat{T}_{kj} \hat{n}_z \langle \hat{d}_q(t') \hat{d}_q^\dagger(t) \rangle \hat{T}_{qk},$$

where $V(V_t)$ is the volume of the $l$ $(r)$ superconductor, $\hat{c}_k$ and $\hat{d}_q$ are the electron field operators that annihilate in $l$ and $r$ superconductors at wave vectors $k$ and $q$, respectively and $\hat{T}_{kj}$ is the tunneling matrix element between them. We further assume that the tunneling is independent of spin and chirality and does not break the time-reversal symmetry, $\hat{T}_{qk} = T_{qk}^T e^{-i\chi}$. Therefore the tunneling matrix in the chirality, spin and Nambu space is specified by the unit matrices, $\tau_0, \sigma_0$ and $\gamma_0$, respectively: $\hat{T}_{kj} = t\delta_0 \gamma_0 \eta_0$. In situations where some form of boundary conditions [15–18] impose a chirality reversal during the tunneling, one has to make a replacement $\tau_0 \rightarrow \tau_-^z$, where $\tau_-^z$ is the third Pauli matrix in the chirality space. We find that the later forms of tunneling processes, do not change our main result. Hence, in what follows we focus on the simplest case given above. The factor $e^{i\phi(t)}$, guarantees that the integrand vanishes for $t' \rightarrow -\infty$ and thereby ensures the convergence of the $t'$ integral.

It is straightforward to calculate the trace terms in current formula (4) by evaluating the expectation values as functions of $V$. It has two parts. One is the single particle part which is not relevant to Josephson current. Here we only focus on the second term that describes the transport of the Cooper pairs.
Here the total Josephson current becomes, such that at zero voltage the $I_{2}$ becomes zero and the second term disappears. The phase configuration in Fig. 2(c) and (d) where in addition to the $U(1)$ phase variables (red arrows), the CA variables (gray arrows) also enter the game. The important feature of Eq. (9) is a locking between the rotor variable $\phi$ of the XY model and another axial rotor variable $\chi$. While in the XY model, every phase (red) flip $\phi \to \phi + \pi$ such as the one in Fig. 2(b) entails an energy cost of $2\lambda_f$, in the extended XY model of Eq. (9) this can be compensated by a corresponding flip in the chiral angle (gray) $\chi$ of the two Weyl superconductors. The ability of CA $\chi$ to adjust itself with the phase flip is due to the fact that, in the absence of strain field $A_{y}^{p}$, nothing is coupled to $\chi$ field and $\chi$ has no preferences. Therefore in addition to Fig. 2(c), many other configurations such as the one in Fig. 2(d) also belong to the ground state. The same phase flip can take place for any other superconductor. One might think that, the proliferation of such phase flip can destroy the phase coherence as it looks like an Ising-disordered arrangement of red arrows. But a closer inspection reveals that this in fact corresponds to half-vortices [22, 23] that are fundamental topological excitations of a degenerate Bose gas with a spinor structure [22] and can be regarded as dual to a boson-pair condensation [24]. To see how can this take place, let us assume that the internal rotor $\chi$ is integrated out, and we wish to construct an effective Hamiltonian in terms of the $\phi$ rotor only. The simplest XY potential that fits together with
the phase flip $\phi \rightarrow \phi + \pi$ is:

$$E_{\text{eff}}[\phi_{ij}] = -\lambda^{\text{eff}}_{ij} \sum_{(i,j)} \cos(2\phi_{ij}). \tag{10}$$

From the continuum limit of the above Hamiltonian one can immediately recognize that, in order to accumulate a phase $\pi$ round a half-vortex, the $V\phi$ must be smaller by a factor of two compared to a full vortex. Therefore the BKT temperature required to proliferate half-vortices is nearly $(1/2)^2 = 1/4$ of the BKT of the full vortices. Furthermore, the substitution $\phi \rightarrow 2\phi$ allows to interpret the superfluid phase of the new XY model as a condensate of the \textit{dual boson pairs} [24].

Now imagine a 2D array of Josephson junctions. In the absence of $\Delta_\gamma$, a configuration of red arrows that round a closed path is identified as $\phi \sim \phi \pm \pi$ defines a vortex/anti-vortex (excited) state. When both $\Delta_\gamma$ and $\Delta_5$ are present, the superconducting phase round a circle can also be identified as $\phi \sim \phi \pm \pi$, which is accompanied by a corresponding half-vortex in $\chi \sim \chi \mp \pi$. Therefore when an external EM gauge field is coupled to our system, half-vortex configurations of the $\phi$-field can be excited. Since a half-vortex in $\phi$ is always accompanied by a half-vortex in $\chi$, and that $\varphi_+ = \phi \pm \chi$, the half-vortex state can be alternatively interpreted as chirality polarized vortex state where the population of vortices in $\varphi_+$ and $\varphi_-$ fields differs.

Irrespective of our interpretation, the EM field only couples to $\phi$ and not to $\chi$. This will then provide a very sharp and definite experimental signature for the half-vortex state as follows: Straightforwardly following the arguments of Weinberg [25], instead of obtaining the flux quantum $2\pi\hbar/(2e) = h/(2e) - \gamma$ as in ordinary superconductors one finds the flux quantum $\pi\hbar/(2e)$. Interpreting this quantization rule as $h/(2e)$ as in conventional superconductors, one is lead to conclude that $2\hbar = 4e$. This "doubling" can be immediately detected in the ac Josephson frequency of a single junction. Although it appears that we are dealing with a condensate of a \textit{pair of Cooper pairs} (remember the boson doubling contained in $\phi \rightarrow 2\phi$), but in fact the root cause of this effect is the presence of the CA $\chi$ that plays a compensating role in Eq. (9) and allows for the formation of half-vortices. The half-vortices clearly show up as doubling of the frequency in ac Josephson effect. In fact such $\pi$ periodic Josephson effect has been seen in the Shapiro steps of Ref. [26]. The authors of Ref. [26] attribute the observed $\pi$ period to an interference between the surface and bulk Josephson currents. But the observation of such interference requires a very large inductance in their circuit model that can not be expected from a non-solenoidal superconductors. Our explanation is as follows: As we have shown in [4], proximitizing a Dirac material induces both $(\Delta_\gamma, \Delta_5)$ [4], and therefore the compensation effect of the chiral angle gives rise to a half-vortex state. As for the additional $4\pi$ period observed in Ref. [26] it can be attributed to the surface states as follows: The surface is $2 + 1$ dimensional and therefore no $\gamma_5$ can be defined for it. We have shown that proximitizing the surface itself induces a Bogoliubov Fermi contour (BFC) [27]. The states outside/inside the BFC are more electron/hole-like, while those at the BFC are equal superposition of electron and hole. Therefore the BFC is actually a Majorana Fermi surface that explains the $4\pi$ periodicity. Within our scenario, the $4\pi$ periodicity observed in Ref. [7] has a different origin. According to [4], if both $(\Delta_\gamma, \Delta_5)$ were non-zero, one would expect the $\pi$-period as explained above. But since in Ref. [7], only the $4\pi$ period has been observed, and that they have ruled out a possible contribution from the surface states, it must be that the induced superconductivity is specified by $(0, \Delta_5)$ from which as pointed out below Eq. (3), the $4\pi$ periodicity immediate follows. Explicit calculations also support the $4\pi$ periodic Josephson effect for a purely pseudoscalar superconductivity [6].

The effective model (10) is the $\Delta = 1$ or $K = 0$ limit of the model considered in Ref. [24]. Let us see, how can we realize deviations from this limit in our Josephson array of superconducting Weyl/Dirac semimetals. In an strain field $A_{\mu}^5$ that couples antisymmetrically to the two chiralities, the dynamics of CA $\chi$ will be restricted and therefore it may not be able to compensate the $\phi$ flips anymore. This amounts to a term that wants more full vortices, rather than half-vortices. Therefore the strain gives rise to a non-zero $(1 - \Delta)$ in the extended XY model of Ref. [24]. Hence a Josephson array of strained superconducting Weyl semimetals in which both $\Delta_\gamma$ and $\Delta_5$ are non-zero, is an experimental realization of the extended XY model of Ref. [24]. Applying a moderate strain places the system in the $\Delta \rightarrow 1 - \Delta$ limit. This model has a remarkably rich phase diagram. The simulations of Ref. [24] on the square lattice shows that in the $\Delta \rightarrow 1 - \Delta$ limit of interest to us, upon increasing the temperature (decreasing $J$ in that reference), the superfluid phase of the effective (dual boson) XY model undergoes a BKT transition to a disordered phase of double bosons caused by proliferation of half-vortices. Upon further increase in temperature, the entropic increase arising from the freedom in placing the broken boson pairs wins and establishes a disordered state of single bosons. Disordered phase of double bosons (caused by proliferation of half-vortices) is separated from the disordered phase of single bosons by a confinement transition that can be interpreted as the merging of half-vortices to form full vortices [24]. Therefore the Josephson array of $(\Delta_\gamma, \Delta_5)$ superconductors in a strained background features a confining transition above the BKT transition of double bosons.

Such a confining transition is absent in a pure $U(1)$ gauge theory in two space dimensions [28–30]. How do we understand the confining transition of model (10) in the presence of strain in terms of the effective description of $(\Delta_\gamma, \Delta_5)$ superconductors? As pointed out, the superconducting Weyl semimetal breaks a $Z_2 \times U(1)$ symmetry. Breaking the $U(1)$ alone by matter gives the XY model as an effective low-energy theory that is dual to a $U(1)$ gauge theory whose charges correspond to the vortices of the original XY model [28]. Breaking an additional $Z_2$ gives rise to an Ising model in terms of variable $\tau$ representing the chirality. The resulting Ising model is dual to an Ising gauge theory [29, 30]. Therefore, the confining transition of underlying Ising gauge theory is inherited. Hence, the confining transition found in Ref. [24] when interpreted within our physical realization, implies a confining transition for coupled $Z_2$ and $U(1)$ gauge theories in two space dimensions.
Chiral Josephson current: So far, the chiral angle $\chi$ has lead to the formation of half-vortices and the associated confinement transition that separates this phase from the full vortex state. In a similar way that spatial variations of $\phi$ in a conventional superconductor leads to the conventional supercurrent, let us show that the spatial variations of $\chi$ leads to a chiral Josephson current. For this purpose we return again to a single Josephson junction setup of Fig. 1. To compute the chiral Josephson current, the only generalization we need to perform in Eq. (4) is to insert an additional tensor product of Pauli matrices $\tau_z \tau_z$. The $\tau_z$ encodes the fact that right- and left-handed chiral fermions have to enter the chiral current with opposite signs. The $\eta_z$ encodes the fact that time reversal operation mapping the electrons and holes in the BdG equation indeed flips the chirality. Therefore,

$$I_5 = \frac{e}{\hbar^2 V_l V_r} \sum_{k,q} \int_{-\infty}^{t_f} dt' \, c_{k}^\dagger (t') c_{k} (t') T_{kq} \eta_z (\tau_z \eta_z) \left( \hat{d}_q (t') \hat{d}_q^T (t') \right)^T \hat{T}_{qk}$$

$$\text{Tr} \left[ \left( \hat{c}_k (t) \hat{c}_k^\dagger (t') \right) T_{kq} \eta_z (\tau_z \eta_z) \left( \hat{d}_q (t') \hat{d}_q^T (t') \right)^T \hat{T}_{qk} \right]$$

This equation yields the chiral Josephson current when the applied voltage $V = 0$ as:

$$I_5 = I_{ss} \sin (\phi_l - \phi_r) \sin (\chi_l - \chi_r). \quad (12)$$

Note that the pair $(I_4, I_5)$ are both determined by the quantity $I_{ss}$ introduced in Eq. (7). The only difference is that, $I_4$ is proportional to $\cos (\chi_l - \chi_r)$, while $I_5$ is proportional to the $\sin (\chi_l - \chi_r)$. In both cases the phase difference $\phi_{lr}$ is needed to drive the current of Cooper pairs. When the CA difference $\chi_{lr}$ is non-zero, in addition to the electric current, a net chirality is also carried by the Cooper pairs. It is useful to view the pair of numbers $(I_4, I_5)$ in a complex plane and note that the modulus of this complex numbers will be $I_{ss} \sin (\phi_{lr})$ of non-chiral superconductors. Therefore, for a generic chiral angle difference $\chi_{lr}$, the Josephson current has both non-chiral $(I_4)$ and chiral $(I_5)$ “components”. In the special case where the difference in the chiral angles of the two superconductors is $\chi_{lr} = \pi/2$, the non-chiral Josephson current will be zero, and the entire Josephson current will become chiral (i.e. along the imaginary axis in the complex plane of $I_4$ and $I_5$).

Taking the continuum limit of the above results immediately reveals that the spatial variations of the CA, $\chi$ generates chiral currents. In 1+1 dimensional spacetime the above chiral current acquires a nice interpretation as follows: The continuum limit of the Josephson lattice of superconductors of Weyl semimetals according to Eq. (12) will give $I_5 \propto \chi (x + \delta x_0) - \chi (x) \propto \partial_\alpha \chi$, where $\alpha$ denotes a spatial direction. Lorentz boosting this result gives, $I_{s\mu} \propto \partial_\mu \chi$. In 1+1 spacetime dimensions, using the fact that $\gamma^\mu \propto \epsilon^{\mu\nu\gamma}$, the above result immediately gives $I^\mu = \epsilon^{\mu\nu} \partial_\nu \chi$ which has a manifest Goldston-Wilczek current form [21, 31, 32] and satisfies the conservation $\partial_\mu I^\mu = 0$. This qualifies the chiral Josephson current in 1+1 dimension as a direct manifestation of Goldstone-Wilczek current.

Summary and outlook. In this paper, we have shown that the most generic form of a superconducting Weyl semimetal is specified by a single $U(1)$ phase $\phi$ and a pair of real numbers $(\Delta_s, \Delta_\ell)$ that faithfully represent a complex algebra. The chiral angle defined to be the polar angle in the Argand diagram of complex plane $(\Delta_s, \Delta_\ell)$ starts to play a significant role in the Josephson coupling. While the spatial variations of $\phi$ generates current of Cooper pairs across a junction, depending on the difference in the CA of the two superconductors, this current can be purely non-chiral, purely chiral or a co-existence of both.

When it comes to a Josephson lattice, the presence of chiral angle $\chi$, remarkably enriches the physics of vortices. The phase flips $\phi_i \to \phi_i + \pi$ can be compensated by a corresponding flip in the CA, $\chi_i \to \chi_i \pm \pi$. This stabilizes half-vortices when the Josephson array is coupled to the $U(1)$ vector potential $A_\mu$. Half-vortices directly lead to doubled-Josephson frequency. This can be a natural explanation of the $\pi$-periodic Josephson effect [26]. Application of strain field $A^\gamma_{\tau}$ restricts the dynamics of $\chi$ and its ability to compensate the $\pi$ phase flips of the $\phi$ field. The strain therefore corresponds to non-zero $1 - \Delta$ (and also non-zero $K$) in Ref. [24]. Having realized the extended XY model of this reference, a confinement transition will separate the disordered phase of double bosons (occurring $\sim T_{BKT}/4$) from the disordered phase of the single bosons. The vortex Nernst-effect that continues to be present at temperatures above the BKT transition has proven to be a reliable signature of vorticity [33]. Therefore the above confining transition is expected to be imprinted in the vortex Nernst measurements. Since the above confinement transition takes place for any non-zero strain and is absent otherwise, it can be used to accurately detect small strain (pseudo magnetic fields), in the same way that SQUIDs are employed to detect small magnetic fields.

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S1. ANDREEV REFLECTION FROM PURELY PSEUDOSCALAR SUPERCONDUCTOR

We calculate the Andreev reflection at the interface of a Dirac/Weyl semimetal with a pseudoscalar superconductor. We show that such a peculiar SN interface gives rise to an additional phase change of $\pi/2$ compared to scalar superconductivity. The general Hamiltonian is of the Dirac-Bogoliubov-de Gennes type given by

$$H_W = \begin{pmatrix} \hat{H}_+ & \hat{\Delta} \\ \hat{\Delta}^\dagger & \hat{H}_- \end{pmatrix}$$

where, $\hat{H}_\pm = \pm i\hbar v_\perp (\hat{\sigma} \cdot \vec{v}) + \mu \tau_0 \sigma_0$ represents the Hamiltonian for massless Weyl fermions. In this equation, $v$ is the Fermi velocity and $\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are a set of Pauli matrices and $\sigma_0$ is the $2 \times 2$ identity matrix that act on the spin degree of freedom, while $\hat{\tau} = (\hat{\tau}_x, \hat{\tau}_y, \hat{\tau}_z)$ and $\hat{\tau}_0$ the unit $2 \times 2$ matrix act on the chirality degree of freedom. $\mu$ is the chemical potential of the whole system. The Hamiltonian acts on the Nambu spinor $\psi = (\psi_{+\uparrow}, \psi_{+\downarrow}, -\psi_{-\downarrow}, \psi_{-\uparrow})^T$ which encompasses the electron and hole components with positive ($+$) or negative ($-$) chirality and spin up ($\uparrow$) or down ($\downarrow$). In this Nambu basis, $\hat{\Delta}$ stands for the superconducting pair potential. On the normal side we set it equal to $\hat{\Delta} = 0$. On the superconducting side it can be $\hat{\Delta} = (\Delta_\text{e}^\text{Sc}) \hat{\sigma}_0$ for a conventional (scalar) superconductor, or $\hat{\Delta} = (i\Delta_\text{pC}) \hat{\sigma}_z$ for pseudoscalar superconductivity, or a combination thereof. To diagonalize the above Hamiltonian, we solve the eigen equation,

\begin{equation}
\begin{pmatrix}
-\mu + i\partial_y & \Delta_{++} & \Delta_{+-} & \psi_{+\uparrow} \\
-i\partial_x - \mu & \Delta_{--} & \Delta_{-+} & \psi_{-\downarrow} \\
\Delta_{++} & \mu + i\partial_y & -i\partial_x - \mu & \psi_{-\uparrow} \\
\Delta_{--} & \Delta_{-+} & \mu - i\partial_y & \psi_{+\downarrow}
\end{pmatrix} = \varepsilon
\begin{pmatrix}
\psi_{+\uparrow} \\
\psi_{-\downarrow} \\
\psi_{+\downarrow} \\
\psi_{-\uparrow}
\end{pmatrix}.
\end{equation}

If we assume that the interface is in the $xy$ plane, the $k_x$ and $k_y$ still remain good quantum numbers. Therefore the eigenstates are of the plane wave $(u_1, u_2, v_1, v_2) \times \exp(ik_xx + ik_yy)$ form whose energy $\varepsilon$ depends on the matrix elements of $\Delta$. In the normal region ($z < 0$) where the pair potential vanishes, the electron and hole parts decouple. Defining $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$, $\varepsilon_e = -\mu \pm k$ and $\varepsilon_h = \mu \pm k$, the decoupled equations are,

\begin{equation}
\begin{pmatrix}
k_z - \mu & k_x - ik_y \\
-k_x + ik_y & -k_z - \mu
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix} = \varepsilon_e
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix},
\end{equation}

\begin{equation}
\begin{pmatrix}
\mu + k_z \\
-(k_x + ik_y) & \mu - k_z
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} = \varepsilon_h
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix},
\end{equation}

The corresponding eigenstates for electrons and holes are,

$$\psi_{e+}^+ = \begin{pmatrix}
\cos(\theta/2) \\
\sin(\theta/2)
\end{pmatrix} e^{ik_z z}, \quad \psi_{e-}^+ = \begin{pmatrix}
\sin(\theta/2) \\
\cos(\theta/2)
\end{pmatrix} e^{-ik_z z},$$

$$\psi_{h+}^+ = \begin{pmatrix}
0 \\
\cos(\theta/2)
\end{pmatrix} e^{ik_z z}, \quad \psi_{h-}^+ = \begin{pmatrix}
0 \\
\sin(\theta/2)
\end{pmatrix} e^{-ik_z z},$$

where $\theta = \arccos[k_z/(\varepsilon + \mu)]$ is the angle that the incidence electron subtends with $k_z$ axis while $\gamma$ is the azimuthal angle.

\footnote{We have used $\phi$ for superconducting phase. So we use the unusual symbol $\gamma$ for azimuthal angle.}
the eigenstates simplify to, Andreev reflection amplitude for electrons and holes which in dispersion relation. It is straightforward to compute the

$$\psi^+_S = \begin{pmatrix} e^{i\phi} \\ 0 \\ 0 \end{pmatrix}, \quad \psi^-_S = \begin{pmatrix} e^{i\phi} \\ 0 \\ e^{-i\beta_s} \end{pmatrix}, \quad (S7)$$

with $\beta_s = \arccos(-\varepsilon/\Delta_s)$. The superscripts + and − in above wave functions refer to ± sign under the square root in dispersion relation. It is straightforward to compute the Andreev reflection amplitude for electrons and holes which

$$\begin{pmatrix} k_x - \mu & k_x - ik_y & \Delta_s e^{i\phi} \\ k_x + ik_y & -k_x - \mu & 0 \\ \Delta_s e^{-i\phi} & 0 & -\mu - k_z \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_2 \end{pmatrix} = \varepsilon \begin{pmatrix} u_1 \\ u_2 \\ v_1 \end{pmatrix}, \quad (S6)$$

are given by,

$$r^a_e = -e^{-i\phi+i\beta_s}, \quad r^a_h = e^{2i\phi}r^a_e. \quad (S8)$$

To develop an intuition, one can simplify the corresponding expressions by taking the $\theta \to 0$ limit and considering retro Andreev reflection. In the following paragraphs we give details of this computation for pseudoscalar superconductivity that will manifest an extra factor "i", providing a hint that the pseudoscalar superconductivity acts like imaginary part of a more generic "complex" object as discussed in the main text.

Now let us solve the Dirac/Weyl-BdG equation for the superconducting side where the superconducting parameter is non-zero. To set the stage, first we consider a scalar superconductor on the top side ($z > 0$) of the NS junction. The corresponding BdG Hamiltonian is,

$$\begin{pmatrix} k_z - \mu & k_x - ik_y & \Delta_s e^{i\phi} \\ k_x + ik_y & -k_x - \mu & 0 \\ \Delta_s e^{-i\phi} & 0 & -\mu - k_z \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_2 \end{pmatrix} = \varepsilon \begin{pmatrix} u_1 \\ u_2 \\ v_1 \end{pmatrix}, \quad (S9)$$

whose eigenvalues are $\varepsilon_{ps} = \pm \sqrt{(\mu + k)^2 + \Delta_s^2}$. To bring out the essential physics of the imaginary "i", we only consider the vertical incidences with $k_x = k_y = 0$. In this case the eigenstates simplify to,

$$\psi^+_S = \begin{pmatrix} 0 \\ e^{i\phi} \\ 0 \end{pmatrix}, \quad \psi^-_S = \begin{pmatrix} e^{i\phi} \\ 0 \\ -i e^{-i\beta_5} \end{pmatrix}. \quad (S10)$$

where $\beta_5 = \arccos(-\varepsilon/\Delta_5)$. Comparing these eigenstates with those in Eq. (S7) for scalar superconductors, clearly shows an additional $i$ prefactor in the hole part of the Nambu spinor which comes directly from $i$ that appears in the $i\Delta_5$ combination.

Now we consider an incident electron from the normal WSM side ($\psi^+_N$) with energy $\varepsilon < \Delta_5$ and wave vector $k = (k_x, k_y, k_z)$. Again because of the translational invariance, $k_x$ and $k_y$ are good quantum numbers, and so they do not change upon Andreev reflection. The $k_z$ does however change in the reflection process. The incident electron will be Andreev reflected as a hole ($\psi^-_N$) and normal reflected as an electron ($\psi^+_N$). The amplitude of each reflection can be obtained by demanding the continuity of the wave functions at the interface ($z = 0$). The wave functions at the two sides can be expressed as,
\[ \psi_N = \left( \begin{array}{c} e^{-i\gamma \cos(\theta/2)} \\
 \sin(\theta/2) \\
 0 \\
 0 \end{array} \right) + r \left( \begin{array}{c} \sin(\theta/2) \\
 -e^{i\gamma \cos(\theta/2)} \\
 0 \\
 0 \end{array} \right) + r_a \left( \begin{array}{c} 0 \\
 0 \\
 e^{-i\gamma' \cos(\theta'/2)} \\
 \sin(\theta'/2) \end{array} \right) \]

\[ \psi_S = a \left( \begin{array}{c} 0 \\
 e^{i\phi} \\
 0 \\
 -ie^{-i\beta_5} \end{array} \right) + b \left( \begin{array}{c} e^{i\phi} \\
 0 \\
 -ie^{-i\beta_5} \\
 0 \end{array} \right) \]  

(S11)

where \( r \) and \( r_a \) are the normal and Andreev reflection amplitudes, respectively. \( a \) and \( b \) are the coefficients of the quasiparticles in the superconductor region. \( \theta' \) is the reflection angle and \( \gamma' \) is the azimuthal angle of the Andreev reflected hole. The continuity equations give,

\[
\begin{align*}
\frac{e^{-i\gamma} \cos(\theta/2) + r \sin(\theta/2)}{\sin(\theta/2)} &= b e^{i\phi} \\
\sin(\theta/2) - re^{i\gamma} \cos(\theta/2) &= a e^{i\phi} \\
r_a e^{-i\gamma'} \cos(\theta'/2) &= -ib e^{-i\beta_5} \\
r_a \sin(\theta'/2) &= -ia e^{-i\beta_5}
\end{align*}
\]

\[ \Rightarrow \begin{cases} 
\frac{e^{-i\gamma} \cos(\theta/2) + r \sin(\theta/2)}{\sin(\theta/2)} = b \\
\frac{e^{-i\gamma'} \cos(\theta'/2)}{\sin(\theta'/2)} = \frac{b}{a} 
\end{cases} \]  

(S12)

In the limit of retro-reflection where \( \theta = -\theta' \) and \( \gamma = \gamma' \) one has,

\[ e^{-i\gamma} \sin(\theta/2) \cos(\theta'/2) + e^{i(\gamma-\gamma')} \cos(\theta/2) \cos(\theta'/2) = \sin(\theta/2) \cos(\theta/2) + e^{-i\gamma} \tan \theta, \]

\[ r_a = -i(2 \cos \theta - 1)e^{-i(\phi-\beta_5)}. \]  

(S13)

For vertical incidence with \( \theta = 0 \) the life becomes simpler and one has \( r = 0 \) and

\[ r_a = -ie^{-i\phi+i\beta_5} = e^{i\pi/2-i\phi+i\beta_5}, \]

\[ r'_a = e^{2i\phi} r_a, \]  

(S14)

where \( r'_a \) is the Andreev reflection amplitude which has been obtained from similar equations for an incident hole, that can be either normal reflected as a hole or Andreev reflected as an electron.

Comparing (S14) and (S8) shows clearly that each electron-hole or hole-electron Andreev reflection at the interface of a normal Weyl/Dirac semimetal with a pseudoscalar superconductor, generates a \( \pi/2 \) phase change in addition to conventional phase changes occurring at the interface of a normal conductor with the standard scalar superconductor. It can be further seen that the additional \( i \) factor in \( r_a \) comes from the prefactor of \( i \) in our Nambu spinor which as we noticed below Eq. (S7), comes from \( \Delta_\nu \) and hence is tied with the pseudoscalar superconductivity and its parity breaking nature.

This can be summarized in the jargon of A. Zee in his book *Quantum Field Theory in a Nutshell*, by stating that, nature Wick rotates the superconductivity (and Andreev reflections) when it breaks the parity symmetry.

### S2. CHIRAL TUNNELING CURRENT

As pointed out in the main text, the same way that separating two conventional (scalar) superconducting materials by a barrier gives rise to Josephson current, separating two pseudoscalar superconductors will give rise to a form of Josephson current that carries a net chirality, and hence the name chiral Josephson current or axial Josephson current can describe it. In this section we provide the detailed derivation of this phenomenon.

The basic formulation of this section is based on the textbook of Kita [14] that has been expanded for the case of Dirac/Weyl Hamiltonians. Consider two superconducting Weyl semimetals described by the most generic form of “complex superconducting order parameter in Eq. (2) of the main text. We further give subscript \( l \) and \( r \) to denote the superconducting WSM in the left and right sides of the S[N]S junction as depicted in Fig. 1 of the main text. The superconducting orders are therefore, \( \Delta_l = \Delta_l e^{i\phi_l} e^{\chi_l} \) and \( \Delta_r = \Delta_r e^{i\phi_r} e^{i\chi_r} \)
where $\phi_{l,r}$ and $\chi_{l,r}$ are the the U(1) and axial phases, respectively and $\Delta_{l,r}$ denotes the strength of the superconductivity for the left/right superconductor. The entire system may be described by the Hamiltonian,

$$H = H_l + H_r + T$$

where $H_l (H_r)$ is the Hamiltonian of the right (left) superconductor as described in (S1). We consider a potential difference between two superconductors, namely $\mu_l = \mu_r + eV$. To illustrate the basic physics, a weak tunneling is enough as it will allow a straightforward perturbative treatment. We therefore regard the tunneling Hamiltonian $T$ as a perturbation which is given by,

$$T = \frac{1}{\sqrt{V_l V_r}} \sum_{kq} \hat{c}_k^\dagger \hat{T}_{kq} \hat{\sigma}_z \hat{d}_q,$$  \hspace{1cm} (S16)

where $V_l (V_r)$ is the volume of the left (right) side. $\hat{c}_k = (\hat{c}_{k+}, \hat{c}_{k-}, \hat{c}^\dagger_{-k+}, \hat{c}^\dagger_{-k-})$ introduces the basis of field operators of the left side while $\hat{d}_q = (\hat{d}_{q+}, \hat{d}_{q-}, \hat{d}^\dagger_{-q+}, \hat{d}^\dagger_{-q-})$ is for the right side. $\hat{T}_{kq} = i\delta_0 \hat{T}_{kq}$ is the tunneling matrix which can be assumed independent of spin, chirality and charge attributes. One can show that the phenomenon of axial/chiral Josephson current is robust against boundary conditions, and peculiar boundary conditions flipping the chirality ($\tau_0 \to \tau_z$) or spin ($\sigma_0 \to \sigma_z$), do not alter the main result.

The current operator can be expressed as the loss of positive charge on e.g. the left side and appearance of the same charge in the right side enabled by the tunneling process as follows $^2$:

$$\hat{I} = \frac{i}{\hbar} \sum_{k,q} \hat{d}_q^\dagger T_{kq} \hat{c}_k$$  \hspace{1cm} (S17)

where $\hat{T}_{kq} = \hat{T}_{kq}^*$ and the Hermitian nature of the above current operator is implicit in the definition of $\hat{c}_k$ that includes both creation and annihilation operators. According to linear response theory, when a perturbation such as tunneling is turned on, a non-equilibrium current is driven in the system. The equilibrium current is zero and hence for small $T$ the current is,

$$I(t) = \frac{e}{\hbar} \int_{-\infty}^{t} dt' \langle \hat{I}(t), \hat{T}(t) \rangle e^{0 + t'},$$  \hspace{1cm} (S18)

where $\hat{I}(t) = e^{iH_0 t/\hbar} \hat{I} e^{-iH_0 t/\hbar}$ and $\hat{T}(t) = e^{iH_0 t/\hbar} \hat{T} e^{-iH_0 t/\hbar}$ with $H_0 = H_l + H_r$ being the unperturbed Hamiltonian. The factor $e^{0 + t'}$ guarantees the convergence of the Fourier integral.

Using the definitions (S16) and (S17), and employing the Wick theorem, the current can be expressed as,

$$I(t) = \frac{e}{\hbar^2} \frac{1}{V_l V_r} \sum_{kq} \int_{-\infty}^{t} dt' e^{0 + t'} \text{Tr} \left[ T_{kq} \hat{\sigma}_z \langle \hat{c}_k(t)\hat{c}_k^\dagger(t') \hat{d}_q(t)\hat{d}_q^\dagger(t') \rangle T_{kq} - \langle \hat{c}_k^\dagger(t')\hat{c}_k^\dagger(t) \hat{d}_q^\dagger(t)\hat{d}_q(t) \rangle T_{kq} \hat{\sigma}_z \langle \hat{d}_q(t')\hat{d}_q^\dagger(t) \rangle T_{kq} \right],$$  \hspace{1cm} (S19)

denoting spin and chirality, respectively. Then the field operators $\hat{c}_k(t)$ thereby gains an extra time-evolution phase (proportional to the driving voltage $V$) with respect to $\hat{d}_q(t)$. Substituting this extra phase, we can express the current as

$$I = \frac{e}{\hbar^2} \frac{|t|^2}{V_l V_r} \sum_{k,q} \int_{-\infty}^{t} dt' \text{Tr} \left[ \hat{\Gamma}(t) \langle \hat{c}_k(t)\hat{c}_k^\dagger(t') \rangle \hat{\Gamma}^+(t') \hat{\sigma}_z \langle \hat{d}_q(t)\hat{d}_q^\dagger(t') \rangle - \hat{\Gamma}^+(t') \langle \hat{c}_k^\dagger(t')\hat{c}_k^\dagger(t) \hat{d}_q^\dagger(t)\hat{d}_q(t) \rangle \hat{\Gamma}(t) \hat{\sigma}_z \right] e^{0 + t'},$$  \hspace{1cm} (S20)

where all the expectations should be calculated at the unperturbed ground state corresponding to $V = 0$ and $\hat{\Gamma}(t)$ incorporates the extra phase described above,

$$\hat{\Gamma}(t) = \begin{pmatrix} e^{iVt/\hbar} \hat{\sigma}_0 \hat{\sigma}_0 & 0 \\ 0 & e^{-iVt/\hbar} \hat{\sigma}_0 \hat{\sigma}_0 \end{pmatrix}. \hspace{1cm} (S21)$$

To calculate expectation values needed in Eq. (S20), we expand $\hat{c}_k$ and $\hat{d}_q$ in terms of quasiparticle fields with relevant unitary matrices. Then we can express our results in terms of the occupation numbers $n(\varepsilon_{k/q}) = \sum_{k/q} \langle \gamma^\dagger_{k/q\lambda} \gamma^\dagger_{k/q\lambda} \rangle$. 

\textsuperscript{2}
For each superconductor the explicit matrix form of Dirac/Weyl-BdG Hamiltonian is,

\[
H = \begin{pmatrix}
    k_z & k_x - ik_y & 0 & 0 & \Delta e^{i(\phi - \chi)} & 0 & 0 & 0 \\
    k_x + ik_y & -k_z & 0 & 0 & \Delta e^{i(\phi + \chi)} & 0 & 0 & 0 \\
    0 & 0 & -k_z & -(k_x - ik_y) & 0 & 0 & 0 & 0 \\
    0 & 0 & -(k_z + ik_y) & k_z & 0 & 0 & 0 & 0 \\
    \Delta e^{-i(\phi - \chi)} & 0 & 0 & 0 & \Delta e^{-i(\phi + \chi)} & 0 & 0 & 0 \\
    0 & \Delta e^{-i(\phi - \chi)} & 0 & 0 & 0 & \Delta e^{-i(\phi + \chi)} & 0 & 0 \\
    0 & 0 & \Delta e^{-i(\phi + \chi)} & 0 & 0 & 0 & \Delta e^{i(\phi - \chi)} & 0 \\
    0 & 0 & 0 & \Delta e^{i(\phi + \chi)} & 0 & 0 & 0 & \Delta e^{i(\phi - \chi)} \\
\end{pmatrix}, \quad (S22)
\]

which is written in the basis,

\[
\Psi = (\psi_{-\uparrow}, \psi_{-\downarrow}, \psi_{+\uparrow}, \psi_{+\downarrow}, -\psi^*_{-\downarrow}, \psi^*_{+\downarrow}, -\psi^*_{-\uparrow}, \psi^*_{+\uparrow})^T = (\Psi_{\psi}, \Psi_s)^T. \quad (S23)
\]

Diagonalizing the Hamiltonian, gives \( \pm \varepsilon_k \) where

\[
\varepsilon_k = \sqrt{k^2 + \Delta^2}, \quad \text{(S24)}
\]

where, regardless of the plane wave prefactor, the eigenvectors corresponding to \(+ \varepsilon_k\) are,

\[
\begin{pmatrix}
    \sqrt{\varepsilon_k + k_z} \\
    \varepsilon_k + k_z \\
    \sqrt{2\varepsilon_k(\varepsilon_k + k_z)} \\
    \sqrt{2\varepsilon_k(\varepsilon_k + k_z)} \\
    \sqrt{2\varepsilon_k(\varepsilon_k + k_z)} \\
    \sqrt{2\varepsilon_k(\varepsilon_k + k_z)} \\
    \sqrt{2\varepsilon_k(\varepsilon_k + k_z)} \\
    \sqrt{2\varepsilon_k(\varepsilon_k + k_z)}
\end{pmatrix}
\]

while those for \(- \varepsilon_k\) are

\[
\begin{pmatrix}
    \sqrt{-\varepsilon_k + k_z} \\
    -\varepsilon_k + k_z \\
    \sqrt{2\varepsilon_k(-\varepsilon_k + k_z)} \\
    \sqrt{2\varepsilon_k(-\varepsilon_k + k_z)} \\
    \sqrt{2\varepsilon_k(-\varepsilon_k + k_z)} \\
    \sqrt{2\varepsilon_k(-\varepsilon_k + k_z)} \\
    \sqrt{2\varepsilon_k(-\varepsilon_k + k_z)} \\
    \sqrt{2\varepsilon_k(-\varepsilon_k + k_z)}
\end{pmatrix}
\]

So we construct the unitary matrix \( \hat{U} \) that diagonalizes the unperturbed Dirac/Weyl-BdG Hamiltonian:

\[
\hat{U} = \begin{pmatrix}
    & & & & & & & \\
    & \sqrt{\varepsilon_k + k_z} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} \\
    & \sqrt{\varepsilon_k + k_z} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} \\
    & & & & & & & \\
    & \sqrt{\varepsilon_k + k_z} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} \\
    & \sqrt{\varepsilon_k + k_z} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} \\
    & & & & & & & \\
    & \sqrt{\varepsilon_k + k_z} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} \\
    & \sqrt{\varepsilon_k + k_z} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} & \sqrt{\varepsilon_k + k_x} & \sqrt{\varepsilon_k + k_y} \\
    & & & & & & & \\
\end{pmatrix}
\]
That satisfies \( \hat{H} = \tilde{U} \hat{\varepsilon} \) with \( \varepsilon = \varepsilon_k \sigma_0 \tilde{\eta}_0 \).

Using \( \tilde{c}_k = \tilde{U} \gamma_k \) and substituting \( \tilde{U} \) in \( \langle \tilde{c}_k(t') \tilde{c}_k^\dagger(t) \rangle \) expression, we can express the correlation functions in (S20) as, \( \langle \tilde{c}_k(t) \tilde{c}_k^\dagger(t') \rangle = \tilde{U}_k (\gamma_k(t) \gamma_k^T(t')) \tilde{U}_k^\dagger \) and \( \langle \tilde{c}_k(t) \tilde{c}_k^\dagger(t') \rangle = [\tilde{U}_k (\gamma_k(t) \gamma_k^T(t')) \tilde{U}_k^\dagger]^T \). Also using \( \langle \gamma_{k,\alpha,\lambda} \gamma_{k,\alpha,\lambda} \rangle = n(\varepsilon_k) \), we obtain

\[
\langle \tilde{c}_k(t) \tilde{c}_k^\dagger(t') \rangle = \left( g_1(\varepsilon_k) f(\varepsilon_k) \right) g_2(\varepsilon_k),
\]

where,

\[
g_1(\varepsilon_k) = \left[ 1 + \frac{\Delta_1^2}{2(\varepsilon_k - k_z^2)} \right] n_+(\varepsilon_k) \sigma_0 \tilde{\eta}_0 \tag{S26}
\]

\[
- \frac{n_-(\varepsilon_k)}{2\varepsilon_k} \left[ -i k_z n_-(\varepsilon_k) (k_x \sigma_y + k_y \sigma_x) + k_z n_-(\varepsilon_k) (k_x \sigma_y - k_y \sigma_x) \right] \Delta_1 e^{i(\phi - \chi z)},
\]

while,

\[
f(\varepsilon_k) = \varepsilon_k n_+(\varepsilon_k) (k_x \sigma_x + k_y \sigma_y) \Delta_1 e^{i(\phi - \chi z)},
\]

with,

\[
n_\pm(\varepsilon_k) = n(\varepsilon_k) e^{\pm i k t / h} \pm n(-\varepsilon_k) e^{-i k t / h}.
\] \( \tag{S29} \)

Finally the correlator \( \langle \hat{d}^T_q(t) \hat{d}^T_q(t') \rangle \) can be obtained by replacements \( n_-(\varepsilon_k) \to -n_-(\varepsilon_q) \), \( k \to q \) and \( i \to -i \).

\[
I = \frac{e^2}{\hbar^2 V |V_r|} \sum_{k,q} \int_0^\infty dt \left\{ -e^{i e V t_1 / h} \tilde{g}_1(t_1) - \tilde{g}_1(-t_1) + e^{-i e V t_1 / h} [\tilde{g}_2(t_1) - \tilde{g}_2(-t_1)] \right. + e^{i e V (2t_1 - t_1) / h} \left[ \tilde{f}(t_1) - \tilde{f}(-t_1) \right] + e^{-i e V (2t_1 - t_1) / h} \left[ \tilde{f}^*(t_1) - \tilde{f}^*(-t_1) \right] \} e^{-0 + t_1},
\]

where we safely set \( e^{0 + t} \to 1 \). As we can see, the contributions of single particle and two particle currents are well separated in above current equation. Now, based on our goal of deriving the Josephson current, we focus only on the two particle part:

\[
I_J = \frac{e^2}{\hbar^2 V |V_r|} \sum_{k,q} \int_0^\infty dt \left\{ e^{i e V (2t_1 - t_1) / h} \left[ \tilde{f}(t_1) - \tilde{f}(-t_1) \right] + e^{-i e V (2t_1 - t_1) / h} \left[ \tilde{f}^*(t_1) - \tilde{f}^*(-t_1) \right] \right\},
\]

with,

\[
[\tilde{f}(t_1) - \tilde{f}(-t_1)] = e^{i (\phi - \phi_r)} \Delta_1 \Delta_r \cos(\chi_l - \chi_r)
\]

\[
\times \left\{ [1 - n(\varepsilon_k) - n(\varepsilon_q)] e^{-i (\varepsilon_k \varepsilon_q) t_1 / h} - e^{i (\varepsilon_k \varepsilon_q) t_1 / h} \right\}.
\] \( \tag{S36} \)

To integrate on \( t_1 \), we use \( \int_0^\infty dt e^{-i(x - i 0^+)} t_1 = -i / x \). The integral is thereby given by,

\[
I_J = \frac{e^2}{\hbar^2 V |V_r|} \Delta_1 \Delta_r \cos(\chi_l - \chi_r) \sum_{k,q} \left\{ e^{i (\phi - \phi_r + 2e V t/h)} \right. \]

\[
\times \left\{ [1 - n(\varepsilon_k) - n(\varepsilon_q)] e^{-i h (\varepsilon_k + \varepsilon_q) / e V} - e^{i h (\varepsilon_k + \varepsilon_q) / e V} \right\} - i h \left\{ n(\varepsilon_k) - n(\varepsilon_q) \right\} e^{-i h (\varepsilon_k - \varepsilon_q) / e V} - e^{i h (\varepsilon_k - \varepsilon_q) / e V} \right\}.
\] \( \tag{S37} \)
Sum over $k$ and $q$ are turned into integrations over the corresponding energy variables $\xi_k = \hbar k$ and $\xi_q = h q$ as $(1/V_{Ir}) \sum_{k,q} \to \int_{-\infty}^{+\infty} d\xi_k N(\xi_k) \int_{-\infty}^{+\infty} d\xi_q N(\xi_q)$, where the density of states at any volume, spin and chirality component is $N(\xi_k/q) = (1/V_{Ir}) \sum_{k,q} \delta(\xi_k/q)$. Then we change the variables $\xi_k$ and $\xi_q$ to $\varepsilon_k$ and $\varepsilon_q$ as, $d\xi_k/q = d\varepsilon_k/q \Theta(\varepsilon_k/q - \Delta_{Ir}) - \frac{e\varepsilon_k/q}{\sqrt{\xi_k - \Delta_{Ir}}}$. Upto here, it seems that in calculation of $I_J$ in Eq. (S37) we deal with four different integrals, but we show that they are the same. At the first step we can replace $1 - n(\varepsilon_k)$ in the first term, with $n(-\varepsilon_k)$ and then change variable $-\varepsilon_k \to \varepsilon$ and $\varepsilon_q \to \varepsilon'$. Since the integration is over $\xi_k$ and $\xi_q$, the variable change $-\varepsilon_k \to \varepsilon$ produces a negative sign. Then we have, $[n(\varepsilon') - n(\varepsilon)][1/(eV_{Ii} - e\varepsilon') - 1/(eV_{Ii} + e\varepsilon')]$ which is the same as the second two terms in (S37). On the other hand the first term $[n(\varepsilon') - n(\varepsilon)][1/(eV_{Ii} - e\varepsilon') - 1/(eV_{Ii} + e\varepsilon')]$ is transformed to the second one by changing variable $\varepsilon \leftrightarrow \varepsilon'$. Hence, we are only left with one two-fold integral that carries a multiplicative factor of 4:

\[
I_J = \frac{4 e |t|^2}{\hbar} \Delta_I \cos(\chi_I - \chi) \int_{-\infty}^{+\infty} d\varepsilon \Theta(\varepsilon - \Delta_I) N(\xi_I) \int_{-\infty}^{+\infty} d\varepsilon \Theta(\varepsilon - \Delta_I) N(\xi_I) \int_{-\infty}^{+\infty} \varepsilon \Theta(\varepsilon - \Delta_I) N(\xi_I) \int_{-\infty}^{+\infty} \varepsilon \Theta(\varepsilon - \Delta_I) N(\xi_I)
\]

\[
\times \text{Re} \left\{ -i e^{i(\phi_I - \phi_I + 2 eV t/\hbar)} n(\varepsilon') - n(\varepsilon) \right\} \frac{e\varepsilon}{eV_\theta + \varepsilon - \varepsilon'}.
\]

(S38)

In order to evaluate the above integral, for simplicity we replace $N(\xi_I)$ and $N(\xi_I)$ with the density of states at the Fermi energy $N(\xi_I)$. Let us define $\int_{-\infty}^{+\infty} d\xi F_I(\varepsilon)$ as a shorthand for $\int_{-\infty}^{+\infty} d\xi \Theta(\varepsilon - \Delta_I) N(\xi_I)$, then replacing and $\frac{1}{eV_\theta + (k^2 + p^2 q^2)}$ with $P \frac{1}{eV_\theta + (k^2 + p^2 q^2)}$, we then multiply a minus sign in front of the terms in the integral.

\[
I_J = \frac{4 e |t|^2}{\hbar} \Delta_I \cos(\chi_I - \chi) \int_{-\infty}^{+\infty} d\varepsilon \Theta(\varepsilon - \Delta_I) \int_{-\infty}^{+\infty} d\varepsilon \Theta(\varepsilon - \Delta_I) \int_{-\infty}^{+\infty} \varepsilon \Theta(\varepsilon - \Delta_I) \int_{-\infty}^{+\infty} \varepsilon \Theta(\varepsilon - \Delta_I)
\]

\[
\times \left\{ P \frac{n(\varepsilon') - n(\varepsilon)}{eV_\theta + \varepsilon - \varepsilon'} \sin(\phi_I - \phi_I + 2 eV t/\hbar) + \pi [n(\varepsilon') - n(\varepsilon)] \frac{e\varepsilon}{eV_\theta + \varepsilon - \varepsilon'} \cos(\phi_I - \phi_I + 2 eV t/\hbar) \right\}.
\]

(S39)

As the supercurrent splits to two distinct parts, one part proportional to $\sin(\phi_I - \phi_I + 2 eV t/\hbar)$ and the other one proportional to $\cos(\phi_I - \phi_I + 2 eV t/\hbar)$, it is appropriate to rewrite the supercurrent equation as:

\[
I_J = I_s \cos(\chi_I - \chi),
\]

(S40)

where $I_s$ is the usual supercurrent ($\Delta \phi = \phi_I - \phi_I$),

\[
I_s = I_{ss} \sin(\Delta \phi + \frac{2 eV t}{\hbar}) + I_{sc} \cos(\Delta \phi + \frac{2 eV t}{\hbar}),
\]

(S41)

\[
I_5 = \frac{e}{\hbar^2} \frac{|t|^2}{V_{Ir}} \sum_{k,q} \int_{-\infty}^{+\infty} dt' e^{0 \pm t'}
\]

\[
\times \text{Tr} \left[ \hat{G}(t) \hat{c}_k(t) \hat{c}_k^\dagger(t') \hat{G}^\dagger(t) \hat{c}_k^\dagger(t) \right].
\]

(S44)

Then we instead of (S30), we should compute,

\[
\text{Tr} \left[ \hat{G}(t) \hat{c}_k(t) \hat{c}_k^\dagger(t') \hat{G}^\dagger(t) \hat{c}_k^\dagger(t') \right].
\]

There are some differences between this equation and

\[
\text{Tr} \left[ \hat{G}(t) \hat{c}_k(t) \hat{c}_k^\dagger(t') \hat{G}^\dagger(t) \hat{c}_k^\dagger(t') \right].
\]
Eq. (S30), that leads to a characteristic and nice distinction between the chiral current and the total current. To reveal these fine differences, since we are eventually interested in two-particle currents, we focus on the \( \tilde{I} \) part of (S30) and explain how it changes upon insertion of a \( \hat{\tau}_z \hat{n}_z \) matrix within the brackets \( \langle \hat{c}_k(t)\hat{c}_k^\dagger(t') \rangle \) and \( \langle \hat{d}_q(t)\hat{d}_q^T(t') \rangle^T \). The expression \( \tilde{\Gamma}(t)\langle \hat{c}_k(t)\hat{c}_k^\dagger(t') \rangle \tilde{\Gamma}^*(t') \) has the following matrix form in Nambu space,

\[
\begin{pmatrix}
  e^{i\varepsilon_k(t-t')/\hbar} g_1(\varepsilon_k) & e^{i\varepsilon_k(t+t')/\hbar} f(\varepsilon_k) \\
  e^{-i\varepsilon_k(t+t')/\hbar} f^*(\varepsilon_k) & e^{-i\varepsilon_k(t-t')/\hbar} g_2(\varepsilon_k)
\end{pmatrix},
\]  

while \( \langle \hat{d}_q(t)\hat{d}_q^T(t') \rangle^T \) is,

\[
\begin{pmatrix}
  g_1'(\varepsilon_q) & f'(\varepsilon_q) \\
  f^*(\varepsilon_q) & g_2'(\varepsilon_q)
\end{pmatrix},
\]  

where \( g_1' \) is the same as \( g_1 \) except that \( n_- \rightarrow -n_- \) and \( i \rightarrow -i \). If we have a \( \hat{n}_z \) between matrices (S45) and (S46) and follow the steps in (S30), it causes a negative sign for \( f^*(\varepsilon_q) \), so instead of what we have in Eq. (S30), in the first step we would obtain 

\[
-
\]

\[
e^{i\varepsilon_k(t+t')/\hbar} f(\varepsilon_k)\hat{n}_z f^*(\varepsilon_q) + e^{-i\varepsilon_k(t+t')/\hbar} f^*(\varepsilon_k)\hat{n}_z f'(\varepsilon_q).
\]

In the second step, one needs to insert \( \hat{\tau}_z \hat{n}_z \). But since the current itself already has a \( \hat{n}_z \) matrix, multiplication with an additional \( \hat{\tau}_z \hat{n}_z \) gives \( \hat{\tau}_z \hat{n}_z \). Hence the negative sign in the first term is cancelled out and a \( \hat{\tau}_z \) is left behind, giving finally 

\[
e^{i\varepsilon_k(t+t')/\hbar} f(\varepsilon_k)\hat{\tau}_z f^*(\varepsilon_q) + e^{-i\varepsilon_k(t+t')/\hbar} f^*(\varepsilon_k)\hat{\tau}_z f'(\varepsilon_q).\]

Since just the first term of \( f \) and \( f^* \) functions produce nonzero terms (because they are even in \( k_x \) and \( k_y \)), we are left with

\[
\text{Tr}
\left[
\frac{-n_- n_-(\varepsilon_q)}{4\varepsilon_k\varepsilon_q} \Delta I_5 \Delta e^{i(\phi_l + \chi_l \hat{\tau}_z)\hat{\tau}_z e^{-i(\phi_r + \chi_r \hat{\tau}_z)}}
\right],
\]

where

\[
\text{Tr}
\left[
\Delta I_5 \Delta e^{i\chi_l \hat{\tau}_z e^{-i\chi_r \hat{\tau}_z}}
\right]
\quad= \text{Tr}
\left[
(\cos \chi_l + i \sin \chi_l \hat{\tau}_z)(\cos \chi_r \hat{\tau}_z - i \sin \chi_r)
\right]
\quad= i \sin(\chi_l - \chi_r).
\]

This should be contrasted with the case of Eq. (S30) where we had \( \cos(\chi_l - \chi_r) \). The rest of calculations is the same as the derivation of \( I_f \) and the chiral supercurrent obtains as,

\[
I_5 = I_s \sin(\chi_l - \chi_r).
\]  

It is nice to notice that the forms of Eq. (S40) and (S48) suggest regard them as real and imaginary parts of another entity of the form

\[
I_{\text{complex}} = I_s e^{i(\chi_l - \chi_r)},
\]

thereby the standard/chiral Josephson current becomes the real/imaginary part of (S49). This means that the standard and chiral Josephson currents are locked to each other by a phase lag of \( \pi/2 \) in the chiral angle difference \( \chi_l - \chi_r \).