Plane algebraic curves in fancy balls

N. G. Kruzhilin and S. Yu. Orevkov

Abstract. Boileau and Rudolph [1] called an oriented link \( L \) in the 3-sphere a \( \mathbb{C} \)-boundary if it can be realized as the intersection of an algebraic curve \( A \) in \( \mathbb{C}^2 \) and the boundary of a smooth embedded closed 4-ball \( B \). They showed that some links are not \( \mathbb{C} \)-boundaries. We say that \( L \) is a strong \( \mathbb{C} \)-boundary if \( A \setminus B \) is connected. In particular, all quasipositive links are strong \( \mathbb{C} \)-boundaries.

In this paper we give examples of non-quasipositive strong \( \mathbb{C} \)-boundaries and non-strong \( \mathbb{C} \)-boundaries. We give a complete classification of (strong) \( \mathbb{C} \)-boundaries with at most five crossings.

Keywords: quasipositive link, \( \mathbb{C} \)-boundary, Thom conjecture.

§ 1. Introduction

Let \( B \subset \mathbb{C}^2 \) be diffeomorphic to a closed 4-ball and let \( A \) be a complex analytic curve in a neighbourhood of \( B \) which is transverse to \( \partial B \) (since we are considering only topological properties, we may assume that \( A \) is a piece of an algebraic curve). Let \( L = A \cap \partial B \) be a link in the 3-sphere \( \partial B \) endowed with the boundary orientation from \( A \cap B \). Which links can be obtained in this way? (All links in this paper are assumed to be oriented.)

If we impose no additional restrictions, then, as Lee Rudolph showed in [2], the answer is ‘any link’. Moreover, any embedded oriented surface without closed components can be realized as \( A \cap B \).

When \( B \) is strictly pseudoconvex (for example, an ordinary round ball), it was shown in [3] that a link is realizable in this way if and only if it is quasipositive (the ‘if’ part was earlier proved in [4]), that is, equal to the braid closure of a quasipositive braid (an \( n \)-braid is said to be quasipositive if it is a product of conjugates of the standard generators \( \sigma_1, \ldots, \sigma_{n-1} \) of the braid group \( B_n \)). This is a rather strong restriction on the class of possible links (see [1], [5]).

In [1], a link is called a \( \mathbb{C} \)-boundary if it is realizable as \( A \cap \partial B \), where \( B \) is diffeomorphic to a closed 4-ball (without any pseudoconvexity assumptions) and \( A \) is a whole algebraic curve in \( \mathbb{C}^2 \) (not just a piece of one as in [2]). It is also natural to distinguish the case when \( L \) is realizable in the form \( A \cap \partial B \) as above and, moreover, \( A \setminus B \) is connected. We call such links strong \( \mathbb{C} \)-boundaries. It was observed in [1]...
that a result of Kronheimer and Mrowka [6] imposes some restrictions on this class of links and, in particular, there exist knots and links that are not concordant to any $\mathbb{C}$-boundary. In fact, some of the results stated in [1] for arbitrary $\mathbb{C}$-boundaries are true only for strong $\mathbb{C}$-boundaries; for more details see §3.

Michel Boileau (private communication) asked whether non-quasipositive $\mathbb{C}$-boundaries exist. Here we give an affirmative answer to this question. Moreover, we show that all the following inclusions are strict:

$$Q := \{\text{quasipositive links}\} \subset SB := \{\text{strong $\mathbb{C}$-boundaries}\} \subset B := \{\mathbb{C}\text{-boundaries}\} \subset \{\text{all links}\}.$$  

To prove that some $\mathbb{C}$-boundaries are not quasipositive, we use the following facts.

**Theorem 1.1** ([5], Corollary 1.5). *If a link and its mirror image are both quasipositive, then the link is trivial (see Remark 1.3).*

**Theorem 1.2** ([7], Theorems 1.1 and 1.2). *If the split sum or the connected sum of two links $L_1$ and $L_2$ is quasipositive, then $L_1$ and $L_2$ are quasipositive.*

Another necessary condition for the quasipositivity of links follows from the Franks–Williams–Morton inequality (see Theorem 6.1 below).

**Remark 1.3.** Let $-L$ be the link $L$ with opposite orientation. Let $\mathcal{C}$ be one of the classes $Q$, $SB$, $B$. Then $L \in \mathcal{C}$ if and only if $-L \in \mathcal{C}$. Indeed, write $L = \partial (A \cap B)$ as above and let $\overline{A}$, $\overline{B}$ and $\overline{L}$ be the images of $A$, $B$ and $L$ under the complex conjugation $c: (z, w) \mapsto (\bar{z}, \bar{w})$. Then $\overline{A}$ is an algebraic curve and we endow $\overline{L}$ with the boundary orientation induced by the complex orientation of $\overline{A} \cap \overline{B}$. Since $c|_A$ is antiholomorphic, we deduce that $(\partial B, \overline{L})$ has the oriented link type of $(\partial B, -L)$. Notice that the equivalence $L \in Q \Leftrightarrow -L \in Q$ can also be obtained algebraically: if $L$ is represented by a braid $\sigma_{i_1}^\pm \sigma_{i_2}^\pm \ldots \sigma_{i_n}^\pm$, then $-L$ is represented by the braid $\sigma_{i_n}^\pm \sigma_{i_2}^\pm \ldots \sigma_{i_1}^\pm$.

The definition of a strong $\mathbb{C}$-boundary can be restated by replacing the condition that $A \setminus B$ is connected by the condition that $A \setminus B$ has no bounded components. If $B$ is strictly pseudoconvex, bounded components may appear; see Wermer’s example in [8], p. 34 (but no component of $A \setminus B$ can be a disc by Nemirovski’s result in [9]). Nonetheless, when $B$ is strictly pseudoconvex, the link $A \cap B$ is a strong $\mathbb{C}$-boundary because it is quasipositive by [3] and hence realizable in the standard sphere [4], where the absence of bounded components follows from the maximum principle. Thus, $Q \subset SB$. Notice that Wermer’s example also gives rise to an example of a non-quasipositive $\mathbb{C}$-boundary; see Remark 4.1.

**Plan of the paper.** In §2 we give the simplest examples of non-quasipositive $\mathbb{C}$-boundaries. They are obtained using a fancy 4-ball which is a small thickening of a 3-ball embedded in the standard 3-sphere.

In §3 we present a number of tools for proving that some links are not (strong) $\mathbb{C}$-boundaries. All of them are based on the Kronheimer–Mrowka theorem.

In §§4, 5 we discuss some links cut out by a complex line on an embedded 3-sphere. If $L$ is such a link, then $L$ and its mirror image $L^*$ are $\mathbb{C}$-boundaries and, therefore, one of them is a non-quasipositive $\mathbb{C}$-boundary by Theorem 1.1.
In § 5 we show that these links are iterated torus links and we describe their Eisenbud–Neumann splice diagrams.

In § 6 we give a complete classification of $\mathbb{C}$-boundaries and strong $\mathbb{C}$-boundaries with at most five crossings. In particular, this shows that all the inclusions $Q \subset SB \subset B \subset \{\text{all links}\}$ are strict. This classification follows easily from the general facts established in the previous sections except for the $\mathbb{C}$-boundary representation of the link $5\sharp_1$, which is rather tricky.

**§ 2. The simplest examples of non-quasipositive $\mathbb{C}$-boundaries**

For a link $L$, let $L^*$ denote its mirror image and let $-L$ denote $L$ with opposite orientation.

**Theorem 2.1.** Let $B$ and $B_0$ be closed 4-balls smoothly embedded in $\mathbb{C}^2$ in such a way that $B_0$ is contained in the interior of $B$. Let $A$ be an algebraic curve in $\mathbb{C}^2$ which is transverse to $\partial B$ and $\partial B_0$. Let $L$ and $L_0$ be the links cut out by $A$ on $\partial B$ and $\partial B_0$ respectively. Then the split sum $L \sqcup (-L_0^*)$ and a connected sum $L \# (-L_0^*)$ (see Remark 2.2) are $\mathbb{C}$-boundaries.

Moreover, if $B_0$ is strictly pseudoconvex and $L_0$ is non-trivial, then $L \sqcup (-L_0^*)$ and $L \# (-L_0^*)$ are non-quasipositive $\mathbb{C}$-boundaries.

**Remark 2.2.** A connected sum $L = L_1 \# L_2$ of two oriented links usually depends on the choice of the components which are joined to a single component of $L$. In Theorem 2.1, the components of $A \cap \partial B$ and $A \cap \partial B_0$ that we choose should be adjacent to the same connected component of $A \cap (B \setminus B_0)$.

**Proof.** We claim that the links discussed are $\mathbb{C}$-boundaries. Indeed, let $I$ be an embedded line segment in $B \setminus B_0$ which connects $A \cap \partial B$ with $A \cap \partial B_0$. Let $U$ be a small tubular neighbourhood of $I$. Then $B \setminus (B_0 \cup U)$ is a 4-ball and the link cut out on it by $A$ is $L \sqcup (-L_0^*)$ (if $I$ is disjoint from $A$) or $L \# (-L_0^*)$ (if $I \subset A$).

If $B_0$ is strictly pseudoconvex, then $L_0$ is a quasipositive link. If it is non-trivial, Theorem 1.1 (see also Remark 1.3) implies that $-L_0^*$ is not quasipositive and the result follows from Theorem 1.2. $\square$

**Corollary 2.3.** Let $L$ be a non-trivial quasipositive link. Then $L \sqcup (-L^*)$ and $L \# (-L^*)$ are non-quasipositive $\mathbb{C}$-boundaries. Moreover, if $L$ is a knot, then $L \# (-L^*)$ is a non-quasipositive strong $\mathbb{C}$-boundary.

This construction admits the following generalization.

**Theorem 2.4.** Let $L$ be a $\mathbb{C}$-boundary in $S^3$ transverse to an equatorial 2-sphere $S^2 \subset S^3$. Let $H$ be one of the halves of $S^3 \setminus S^2$, and $\xi: S^3 \to S^3$ the symmetry with respect to $S^2$. Then the link $(L \cap H) \cup \xi(-L \cap H)$ is a non-quasipositive $\mathbb{C}$-boundary unless it is trivial.

**Proof.** Let $(A, B)$ be the realization of $L$ as a $\mathbb{C}$-boundary, $f: S^3 \to \partial B$ a diffeomorphism that maps $L$ onto $A \cap \partial B$, and $B'$ a small thickening of $f(H)$. Then $(A, B')$ is a $\mathbb{C}$-boundary realization of the required link. Being amphicheiral, it is either trivial or quasipositive by Theorem 1.1. $\square$
This theorem enables us to construct a lot of non-quasipositive $\mathbb{C}$-boundaries. In Fig. 1 we give an example of such a link. Many others can be constructed starting with any quasipositive braid.

**§ 3. Restrictions on (strong) $\mathbb{C}$-boundaries**

All the known restrictions on (strong) $\mathbb{C}$-boundaries are more or less immediate consequences of the Kronheimer–Mrowka theorem [6] (also known as the Thom conjecture or the adjunction inequality) and its version for immersed surfaces in $\mathbb{CP}^2$ with negative double points (see [11], [12], §2), which was actually proved in [6] but was not explicitly stated there.

**Theorem 3.1** (the immersed Thom conjecture). *Let $\Sigma$ be a connected oriented closed surface of genus $g$ and let $j: \Sigma \to \mathbb{CP}^2$ be an immersion with only negative ordinary double points as self-intersections. Suppose that $j_*([\Sigma]) = d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2)$, where $d > 0$. Then $g$ is bounded below by the genus of a smooth algebraic curve of degree $d$, that is, $g \geq (d-1)(d-2)/2$.*

Given a link $L$ in $S^3 = \partial B^4$, we define the *slice Euler characteristic* of $L$ by $\chi_s(L) = \max_{\Sigma} \chi(\Sigma)$, where the maximum is taken over all embedded smooth oriented surfaces $\Sigma$ without closed components and such that $\partial \Sigma = L$.

Similarly, we define the *slice negatively immersed Euler characteristic* of $L$ by $\chi_s^{-}(L) = \max_{(\Sigma, j)} \chi(\Sigma)$, where the maximum is taken over all immersions $j: (\Sigma, \partial \Sigma) \to (B^4, S^3)$ of oriented surfaces $\Sigma$ without closed components such that $j(\Sigma)$ has only negative double points and $j(\partial \Sigma) = L$.

Theorem 3.1 immediately yields the following assertion.

**Proposition 3.2** (compare with [1], Theorem 1.3). *Let $A$ be a smooth algebraic curve in $\mathbb{C}^2$ which is transverse to the boundary of a 4-ball $B$ smoothly embedded in $\mathbb{C}^2$ and such that $A \setminus B$ is connected. Put $L = A \cap \partial B$. Then $\chi_s^{-}(L) = \chi_s(L) = \chi(A \cap B)$.*

The connectedness condition in Proposition 3.2 can be replaced by the condition that $A \setminus B$ has no bounded components. Indeed, in this case $A \setminus B$ becomes connected after a perturbation of the union of $A$ with a generic line lying far from $B$. 

---

Figure 1. L10m36(1) in [10]
Proof. We replace $A \setminus B$ by a negatively immersed surface $j(\Sigma)$ of maximum $\chi(\Sigma)$ and apply Theorem 3.1. □

Remark 3.3. The connectedness condition is missing in [1], Theorem 1.3. Without this condition, Proposition 3.2 is false. Indeed, take $A = \{w = 0\}$ and let $B$ be the unit ball ‘drilled’ along the line segment $[(0,0), (0,1)]$. Then $\chi(A \cap B) = 0$ whereas $\chi_s(L) = 2$. The proof fails because when we replace $A \cap B$ by $\Sigma$, the Euler characteristic increases on account of splitting out a 2-sphere while the Euler characteristic of the unbounded component does not change. Note that Proposition 1.4 in [1] is false even for strong $\mathbb{C}$-boundaries if both links are multi-component. A correct version is given in Proposition 3.6 below.

Remark 3.4. A similar inaccuracy appears in [12]: the connectedness of the auxiliary surface (an analogue of $(A \setminus B) \cup \Sigma$ in the proof of Theorem 3.1) has not been checked. Hence the conclusion of Theorem 1 in [12] is false, for example, in the case when both curves are real conics with non-empty disjoint sets of real points. However, this inaccuracy can easily be corrected and does not affect the most interesting case when the curves have common real points.

Definition 3.5. A component of a $\mathbb{C}$-boundary $L$ is said to be outer if it is adjacent to an unbounded component of $A \setminus B$, where $A$ and $B$ are the same as in the definition of a $\mathbb{C}$-boundary. In particular, all components of a strong $\mathbb{C}$-boundary are outer.

Proposition 3.6 (compare with [2], Proposition 1.4). Let $K_1$ and $K_2$ be outer components of $\mathbb{C}$-boundaries $L_1$ and $L_2$ respectively. Then $L_1 \cap L_2$ and $L_1 \# L_2 = L_1 \# (K_1, K_2) L_2$ are $\mathbb{C}$-boundaries. Moreover, if $L_1$ and $L_2$ are strong $\mathbb{C}$-boundaries, then so are $L_1 \cup L_2$ and $L_1 \# L_2$, and $\chi_s(L_1 \# L_2) + 1 = \chi_s(L_1 \cup L_2) = \chi_s(L_1) + \chi_s(L_2)$.

Proof. For $j = 1, 2$ let $(A_j, B_j)$ be the realization of $L_j$ as a (strong) $\mathbb{C}$-boundary. By translating $A_1$ and $B_1$ sufficiently far away, we can achieve that $A_1 \cap B_2 = A_2 \cap B_1 = B_1 \cap B_2 = \emptyset$. Perturbing $A_1 \cup A_2 \cup L$ for a suitable line $L$, we can achieve that $L_j = A \cap \partial B_j$, $j = 1, 2$, for a smooth projective algebraic curve $A$. Set $B = B_1 \cup B_2 \cup T$, where $T$ is a small tubular neighbourhood of an embedded arc connecting a point in $K_1$ with a point in $K_2$. Then $(A, B)$ realizes $L_1 \cup L_2$ (resp. $L_1 \# L_2$) provided that this arc does not lie (resp. lies) on $A$. The required formula for $\chi_s(\cdot)$ follows easily from Proposition 3.2. □

Proposition 3.7. If $L$ is a strong $\mathbb{C}$-boundary and $-L^*$ is a $\mathbb{C}$-boundary (not necessarily strong), then $\chi_s(L) = \chi^*_s(L) \geq 1$.

Proof. Put $\hat{L} = \#(K, -K^*) (-L^*)$, where $-K^*$ is an outer component of $-L^*$ and $K$ is the corresponding component of $L$. Let $A \cap B$ and $A^* \cap B^*$ be the realizations of $L$ and $-L^*$ as (strong) $\mathbb{C}$-boundaries. The construction in the proof of Proposition 3.6 provides a curve $\hat{A}$ and a smooth ball $\hat{B}$ such that $\hat{A} \cap \hat{B}$ realizes $\hat{L}$, and all the components of $\hat{L}$ inherited from $L$ (including $K \# (-K^*)$) lie on the boundary of a single unbounded component of $\hat{A} \setminus \hat{B}$. It is well known (and easily verifiable) that $\hat{L}$ bounds a surface $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$ in $\hat{B}$, where $\Sigma_i$ is a disc bounded by $K \# (-K^*)$ and, for every $i \geq 2$, $\Sigma_i$ is an annulus bounded by $K_i \cup (-K_i^*)$, where
$K, K_2, \ldots, K_r$ are the components of $L$. Thus $\chi(\Sigma) = 1$. Put $A' = (\hat{A} \setminus \hat{B}) \cup \Sigma$. By construction, $A'$ is connected. Hence $\chi(A') \leq \chi(\hat{A})$ by Theorem 3.1. Again by construction, we have $\chi(\hat{A} \cap \hat{B}) = \chi(A \cap B) + \chi(A^* \cap B^*) - 1 \leq 2\chi_s(L) - 1$. Thus,

$$0 \leq \chi(\hat{A}) - \chi(A') = \chi(\hat{A} \cap \hat{B}) - \chi(\Sigma) \leq (2\chi_s(L) - 1) - 1.$$ 

Finally, $\chi_s(L) = \chi_s(L)$ by Proposition 3.2. □

**Lemma 3.8.** Let $L$ be a $\mathbb{C}$-boundary which is not a strong $\mathbb{C}$-boundary. Then there is a proper sublink of $L$ which has zero linking number with its complement in $L$.

**Proof.** Write $L = A \cap B$ as in the definition of a $\mathbb{C}$-boundary. There is no loss of generality in assuming that $A$ is smooth. Then $A \setminus B$ has a bounded connected component since otherwise a perturbation of $A \cup C$ for an appropriate line $C$ would realize $L$ as a strong $\mathbb{C}$-boundary. Put $A_1 = (A \setminus B) \setminus A_0$ and let $B'$ be the complement of $B$ in the one-point compactification of $\mathbb{C}^2$. Then $B'$ is a ball and $A_0$ is disjoint from the closure of $A_1$ in $B'$. Hence the linking number of $\partial A_0$ and $\partial A_1$ is equal to zero. □

§ 4. A non-quasipositive $\mathbb{C}$-boundary coming from Wermer’s example

Suppose that $S^3 = \{|z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$ and $0 < \varepsilon \ll 1$. Let $G_f$ be the graph of the function

$$f(z) = \begin{cases} 
2\varepsilon, & |z| \geq \varepsilon, \\
\frac{2z}{\varepsilon}, & |z| \leq \varepsilon.
\end{cases}$$

We endow it with the orientation induced by projection on the $z$-axis. It is easy to check that $L_f := G_f \cap S^3$ is the link shown in Fig. 2, where the horizontal circle represents the component of $L_f$ close to the $z$-axis.

![Figure 2. Example of a non-quasipositive $\mathbb{C}$-boundary](image-url)

This link is evidently non-trivial. Hence, either $L_f$ or its mirror image $L_\mathcal{T}$ is non-quasipositive by Theorem 1.1. The mapping $T_f: (z, w) \mapsto (z, w - f(z))$ transforms $G_f$ into the line $\{w = 0\}$. Similarly, $T_\mathcal{T}(G_\mathcal{T}) = \{w = 0\}$. Thus, either the pair $(T_f(B^4), \{w = 0\})$ or its image under $(z, w) \mapsto (z, \overline{w})$ is the desired example of a fancy ball in $\mathbb{C}^2$ and an algebraic curve which cuts out a non-quasipositive link on its boundary sphere. In other words, either $L_f$, or $L_\mathcal{T}$ is a non-quasipositive $\mathbb{C}$-boundary.
So far this is a non-constructive proof of existence since we do not know which link is not quasipositive (we will establish this later). However, if we replace $f(z)$ by $f(z + 1/2) + \bar{f}(z - 1/2)$, the resulting link will be amphicheiral because it is invariant under the involution $(z, w) \mapsto (-z, -\bar{w})$. Thus it is not quasipositive (again by Theorem 1.1) and it is a $\mathbb{C}$-boundary for the same reasons as above.

We claim that the link $L_T$ is quasipositive (and hence, by Theorem 1.1, $L_f$ is not). Indeed, $L_T$ is isotopic to the braid closure of the 3-braid $(\sigma_1\sigma_2\sigma_1^{-1})(\sigma_1^{-1}\sigma_2\sigma_1)$. We do not know whether $L_f$ is a strong $\mathbb{C}$-boundary.

The quasipositivity of $L_T$ can also be seen geometrically as follows. The components of $L_f$ can be parametrized (preserving their orientation) by the map

$$t \mapsto (e^{it}, 2\varepsilon e^{it}), \quad t \mapsto (2\varepsilon e^{-it}, e^{-it}), \quad t \mapsto \left(\frac{1}{2}\varepsilon e^{it}, e^{it}\right)$$

(here we approximate the coefficients with accuracy $O(\varepsilon^2)$). Therefore,

$$L_f = S^3 \cap \{(w = 2\varepsilon z) \cup \{z = 2\varepsilon w\}^{\text{op}} \cup \{2z = \varepsilon w\}\},$$

where $\{\cdots\}^{\text{op}}$ means reversion of the orientation on a complex line. Any two triples of distinct complex lines through the origin are isotopic to each other. Hence,

$$L_f \sim S^3 \cap \{(w = 0) \cup \{w = \varepsilon z\} \cup \{z = \varepsilon w\}^{\text{op}}\}. \quad (1)$$

Thus $L_T$ is isotopic to the image of the right-hand side of (1) under the map $(z, w) \mapsto (z, \bar{w})$. This image can be parametrized as $t \mapsto (e^{it}, 0), \quad t \mapsto (e^{it}, \varepsilon e^{-it}), \quad t \mapsto (\varepsilon e^{-it}, e^{it})$. But this is equal (again up to $O(\varepsilon^2)$) to a parametrization of $S^3 \cap \{w(zw - \varepsilon) = 0\}$. Thus, $L_T \sim S^3 \cap \{w(zw - \varepsilon) = 0\}$ is quasipositive.

Remark 4.1. A third way to see that $L_T$ is quasipositive is to observe that it can be obtained from Wermer’s example (see [8], p.34), which explicitly gives a function $F(z) = (1 + i)z - iz\bar{z}^2 - z^2\bar{z}^3$ with the following properties: $F'_z \neq 0$ in the unit disc $\Delta$ and $F|_{\partial\Delta} = 0$. Then the graph of $F$ is totally real and, therefore, has a small neighbourhood which is a smooth strictly pseudoconvex ball $B$. One can check that the link cut out by the $z$-axis on $\partial B$ is isotopic to $L_T$. Thus $L_T$ is quasipositive by [3].

§ 5. Further examples of $\mathbb{C}$-boundaries cut out by a complex line and their properties

It is clear that in § 4 we could have taken any function $f: \mathbb{C} \to \mathbb{C}$ whose graph $G_f$ is transverse to $S^3$ and cuts out a non-trivial link $L_f$ on it. In this case, Hayden’s theorem (Theorem 1.1) guarantees that either $L_f$, or its mirror image $L_T$ is non-quasipositive. This is, however, a very small family of links, which we are going to describe in this section. Since they are all iterated torus knots, an appropriate language for their description is that of EN-diagrams (also known as splice-diagrams). These are certain graphs introduced by Eisenbud and Neumann in [13]. More precisely, regarding diagrams obtained from one another by certain simple operations as equivalent (see [13], Theorem 8.1), each iterated torus link corresponds to a unique equivalence class of diagrams.
Computation of the iterated torus link structure of \( L_f \) in terms of initial data is very similar to that in [14] (in both cases, the initial data are given by an arrangement of disjoint circles on the plane equipped with some extra information).

So let \( f, G_f \) and \( L_f \) be as above and let \( \text{pr}_1: \mathbb{C}^2 \to \mathbb{C} \) be the projection \( (z, w) \mapsto z \). There is no loss of generality in assuming that \( L_f \) is disjoint from the \( z \)-axis. Then \( \text{pr}_1(L_f) \) is a disjoint union of smooth embedded circles \( C_1 \cup \cdots \cup C_n \). Let \( D_1, \ldots, D_n \) be the bounded connected components of \( \mathbb{C} \setminus \text{pr}_1(L_f) \) enumerated in such a way that \( C_j \) is the exterior component of the boundary of \( D_j \). We say that \( D_j \) is positive or negative according to the sign of \(|f(z)|^2 + |z|^2 - 1\) for \( z \in D_j \) (thus \( \text{pr}_1(G_f \cap B^4) \) is the union of all the negative domains \( D_j \)). We endow each \( C_j \) with the boundary orientation induced by the adjacent negative component of \( \mathbb{C} \setminus \text{pr}_1(L_f) \) (it is also the orientation induced by the projection of \( L_f \)). Let \( a_j \) be the increment of \((\text{Arg } f)/(2\pi)\) along \( C_j \). Then the link \( L_f \) is uniquely determined by the following combinatorial data: the partial order \( \prec \) on \( \{C_1, \ldots, C_n\} \) defined by requiring that \( C_i \prec C_j \) if \( C_i \) lies inside \( C_j \), and the numbers \( a_j \) corresponding to all non-maximal circles \( C_j \) with respect to this order. These data are realizable if and only if \( \sum_{k \in I(j)} a_k = 0 \) for every positive domain \( D_j \); here \( I(j) = \{k \mid C_k \subset \partial D_j\} \).

**Definition 5.1.** Let \( K \) be a component of an oriented link \( L \). Let \( p, q \) and \( d \) be integers such that \( \text{GCD}(p, q) = 1 \) and \( d \geq 1 \). We say that \( L \cup L' \) (resp. \((L \setminus K) \cup L'\)) is a \((pd, qd)\)-cable of \( L \) along \( K \) with the core retained (resp. with the core removed) if, for some tubular neighbourhood \( T \) of \( K \) disjoint from \( L \setminus K \), we have

- \( L' \subset \partial T \) and \( L' \) is a disjoint union of knots: \( L' = K_1 \cup \cdots \cup K_d \);
- \([K_j] = p[K] \) in \( H_1(T) \) and \( \text{lk}(K, K_j) = q \) for all \( j = 1, \ldots, d \).

An **iterated torus link** is a link obtained from the unknot by repeated cabling of either kind.

**Remark 5.2.** Reversal of the orientation of some components of an iterated torus link yields an iterated torus link. Indeed, reversal of the orientation of a component \( K \) is the \((-1, 0)\)-cable along \( K \) with the core removed.

**Proposition 5.3.** \( L_f \) is an iterated torus link.

**Proof.** This follows from Lemma 5.4 below. \( \Box \)

**Lemma 5.4.** Let \( L = K_1 \cup \cdots \cup K_n \) be a link in \( S^3 = \partial B^4 \) such that \( \text{pr}_1|_L \) is injective. Then \( L \) is an iterated torus link.

**Proof.** We can assume that \( L \) is disjoint from the \( z \)-axis. Let \( K_1, \ldots, K_n \) be the components of \( L \). We put \( C_j = \text{pr}_1(K_j) \) and call the sets \( C_j \) ovals. By Remark 5.2, we can choose any orientation on the components. So we fix the orientation on \( K_j \) induced by the anti-clockwise orientation of \( C_j \). Let \( a_j \) be the linking number of \( K_j \) and the \( z \)-axis (the increment of \( \text{Arg } F_j/(2\pi) \) where \( K_j \) is regarded as the graph of a function \( F_j : C_j \to \mathbb{C} \)). The link \( L \) is uniquely determined by the projection \( \text{pr}_1(L) \) and the numbers \( a_1, \ldots, a_n \) (if \( C_j \) is an outermost oval, then \( L \) is independent of \( a_j \) up to isotopy).

We shall prove the lemma for a larger class of links. Namely, we allow some components of \( L \) to be fibres of \( \text{pr}_1 \) positively linked with the \( z \)-axis (note that, up to an isotopy, the link does not change if we replace such a component by a small oval with \( a_j = 1 \)).
There is no loss of generality in assuming that \( pr_1(L) \) has a single outermost oval. Otherwise \( L \) is a split sum of sublinks each of which corresponds to an outermost oval and all the ovals surrounded by it. If \( pr_1(L) \) consists of a single oval and a point inside it, then \( L \) is the Hopf link which is the \((1,1)\)-cable over the unknot. Therefore it suffices to check that the following operations (i)–(iii) are cablings (see the first row of Fig. 3). Let \( K \) be a component of \( L \) of the form \( pr^{-1}_1(P) \) for some point \( p \) and let \( D \) be a disc such that \( D \cap pr_1(L) = \{p\} \). The operations are:

(i) adding a new component whose projection is \( \partial D \) and whose linking number with the \( z \)-axis is any given integer \( a \);
(ii) the operation (i) followed by the removal of \( K \);
(iii) replacing \( K \) by \( pr^{-1}_1(P) \), where \( P = \{p_1, \ldots, p_k\} \subset D \).

Then (i) (resp. (ii)) is the \((a,1)\)-cable along \( K \) with the core retained (resp. removed) and (iii) is the \((k,0)\)-cable along \( K \) with the core removed. □

The second row of Fig. 3 represents the evolution of the EN-diagrams under the iterated cablings considered in the proof of Lemma 5.4.

In Fig. 4 we show two EN-diagrams of a link \( L_f \) for the arrangement of ovals and the linking numbers as on the left-hand side of this figure. The grey area is \( pr_1(G_f \cap B^4) \) (we recall that the sum of the linking numbers along the boundary of every bounded white, that is, positive domain should be equal to zero). The EN-diagram on the left corresponds to the proof of Lemma 5.4, and that on the right is obtained from it by admissible operations described in [13], Theorem 8.1 (3) and (6). In general, such operations can be applied to each piece of an EN-diagram corresponding to an annular component of \( pr_1(G_f \setminus B^4) \) (the white annular component in the colouring as in Fig. 4).

**Remark 5.5.** We do not know whether any of the non-quasipositive links considered in this section is a strong \( \mathbb{C} \)-boundary.
§ 6. Links with at most five crossings

In this section, given any link admitting a projection with at most five crossings, we determine whether it belongs to the classes $Q$, $SB$ or $B$. Table 1 contains the answers for all such links without unknot split components (that is, not of the form $L \sqcup O$, where $O$ is the unknot) but the answer for links of the form $L \sqcup O \sqcup \cdots \sqcup O$ with $\leq 5$ crossings follows easily. The implication $L \in C \Rightarrow L \sqcup O \in C$ (where $C$ is $Q$, $SB$ or $B$) is evident. The reverse implication is true for $Q$ (see [7]), but we do not know whether it holds in general for $SB$ and $B$. However, the nature of our proofs is such that whenever we prove that $L \not\in C$ (where $C$ is $SB$ or $B$), the same argument can easily be adapted to prove that $L \sqcup O \sqcup \cdots \sqcup O \not\in C$.

The list of prime links (that is, indecomposable into a connected or split sum) with at most five crossings is taken from [15], [10] (but we abbreviate $2_2^1$ to $2_1^1$). We write $4_2^1 −$ for the link $4_2^1$ (oriented as in [10]) with reversed orientation of one component. Note that any choice of orientations of the components of $5_1^2$ provides the same oriented link type. In the second column we give the braid notation. It serves to identify the link as well as to confirm its quasipositivity (when applicable). The braid words also help us to bound $\chi_s(L)$ below using the observation that if a braid $\beta'$ is obtained from $\beta$ by replacing some $\sigma_i^{-1}$ by $\sigma_i$, then $\chi_s(\beta) \geq \chi_s(\beta')$. (In fact we only need lower bounds for $\chi_s$ and upper bounds for $\chi_s$ in our proofs, and the reader can assume that Table 1 contains just these bounds for the Euler characteristics.) For example, $\chi_s(4_2^1) = \chi_s(\sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1}) \geq \chi_s(\sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2) = \chi_s(\sigma_1) = 2$ (here “$\sigma_1$” in “$\chi_s(\sigma_1)$” is regarded as a 3-braid).

We also use Murasugi’s inequality in estimates for the links $4_2^1$, $4_1^2$, and $5_1^2$.

The comments at the end of this section are referred to in Table 1 as “(a)”, “(b)”, and so on. Almost all proofs are based on general results in §§2, 3. Here we present some of the specific results used in the table.

The following theorem is an immediate corollary of the Franks–Williams–Morton inequality [16], [17] combined with Proposition 3.2 (in [15] this result is used in most cases to prove the non-quasipositivity of knots).

**Theorem 6.1** ([1], Theorem 3.2). If $L$ is a quasipositive link, then $\text{ord}_v P_L \geq 1 - \chi_s(L)$, where $P_L(v, z)$ is the HOMFLY polynomial normalized by the conditions $P_O = 1$ and $P_{L^+} = vzP_{L_0} + v^2 P_{L_-}$.
Table 1

| L    | braid     | L ∈ Q | L ∈ SB | L ∈ B | χ_s(L) | χ_s^{-}(L) |
|------|-----------|-------|--------|-------|--------|------------|
| 2_1  | σ_{1}^2   | yes   | yes    | yes   | 0      | 2         |
| 2_1^*| no        | no (a)| no (b) | 0     | 0      | 2         |
| 3_1  | σ_{1}^3   | yes   | yes    | yes   | -1     | 1         |
| 3_1^*| no        | no (a)| no (b) | -1    | 1      |           |
| 4_1  | (σ_{1}^{-1}σ_2)^2 | no | no (a) | no (b) | -2    | 2         |
| 4_1^*| σ_{1}^4   | yes   | yes    | yes   | 0      | 2         |
| 4_1^*| no        | no (a)| no (b) | 0     | 2      |           |
| 4_1^*| σ_{1}^{-1}σ_2σ_1^2σ_2 | yes | yes    | yes   | -1    | 3         |
| 2_1 # 2_1| yes   | yes    | yes   | -1    | 3      |           |
| 2_1 # 2_1^*| no (c)| yes (d)| yes   | 0     | 2      |           |
| 2_1^* # 2_1^*| no | no (f,e)| -1    | 3      |           |
| 2_1 ∪ 2_1| yes   | yes    | yes   | 0     | 2      |           |
| 2_1 ∪ 2_1^*| no (a)| yes (d)| yes   | 0     | 4      |           |
| 2_1^* ∪ 2_1^*| no | no (f,e)| 0     | 2      |           |
| 5_1  | σ_{1}^5   | yes   | yes    | yes   | -1     | 1         |
| 5_1^*| no        | no (a)| no (b) | -3    | 1      |           |
| 5_2  | σ_{1}^2σ_2σ_1σ_2^{-1} | yes | yes    | yes   | 0      | 2         |
| 5_2^*| no        | no (a)| no (b) | -1    | 1      |           |
| 5_2^*| (σ_1σ_2^{-1})^2σ_1 | no (i)| yes (j)| yes   | 0      | 2         |
| 3_1 # 2_1| yes   | yes    | yes   | 0      | 2      |           |
| 3_1 # 2_1^*| no (c)| yes (d)| yes   | 0     | 2      |           |
| 3_1^* # 2_1| no    | no (f,g)| 0     | 2      |           |
| 3_1^* # 2_1^*| no | no (f,e)| -2    | 2      |           |
| 3_1 ∪ 2_1| yes   | yes    | yes   | -2    | 2      |           |
| 3_1 ∪ 2_1^*| no (c)| no (a,g)| yes (d)| -1    | 1      |           |
| 3_1^* ∪ 2_1| no    | no (f,g)| no (h)| -1    | 1      |           |
| 3_1^* ∪ 2_1^*| no | no (f,e)| -1    | 3      |           |

**Proposition 6.2.** The link 3_1^* ∪ 2_1 is not a C-boundary.

**Proof.** Put L = L_1 ∪ L_2, where L_1 realizes 3_1^* and L_2 realizes 2_1. Suppose that L is a C-boundary ∂(A ∩ B), where A is a smooth algebraic curve in CP^2 and B is a 4-ball smoothly embedded in C^2 which is regarded as an affine chart in CP^2. There is no loss of generality in assuming that A \ B has only one unbounded (in C^2) component. Let Σ be a disjoint union Σ_1 ∪ Σ_2 of two surfaces and let j: (Σ, ∂Σ) → (B, ∂B) be an immersion with negative crossings such that j(∂Σ_i) = L_i, i = 1, 2. We can assume that Σ_1 is a disc and Σ_2 is an annulus. Let A' be the
result of gluing $A \setminus B$ and $\Sigma$ along the boundary. We extend $j$ to $A'$ in such a way that $j(A') = (A \setminus B) \cup j(\Sigma)$.

We have $\chi_s(L) = -1$ and $\chi_s^-(L) = 1$ (see Table 1). Hence $\chi^-(L) > \chi_s(L)$ and, therefore, $\chi(A') > \chi(A)$. Thus $A'$ is disconnected by Theorem 3.1. We recall that $A \setminus B$ has only one unbounded component. Hence there is a component $A'_0$ of $A'$ such that $j(A'_0)$ is bounded in $\mathbb{C}^2$. Since $j(A' \setminus A'_0)$ also intersects $B$, we have $j(A'_0) \cap B = j(\Sigma_k)$ for some $k \in \{1, 2\}$ and, therefore, $j(A'_0) \cap \partial B = L_k$. Since $[j(A' \setminus A'_0)] = [j(\Sigma)]$ in $H_2(\mathbb{CP}^2)$, it follows from Theorem 3.1 that $\chi(A) \geq \chi(A' \setminus A'_0)$ and, therefore,

$$\chi(A) + \chi(A'_0) \geq \chi(A') = \chi(A \setminus B) + \chi(\Sigma) \geq \chi(A) - \chi_s(L) + \chi(\Sigma),$$

whence $\chi(A'_0) \geq \chi(\Sigma) - \chi_s(L) = 2$. We claim that this is impossible. Indeed, put $\Sigma_0 = A'_0 \setminus \Sigma_k$. Then $j(\Sigma_0)$ can be regarded as a smooth surface with boundary $-L_k$ which is embedded in the one-point compactification of $\mathbb{C}^2$. It follows that $\chi(\Sigma_0) \leq \chi_s(L_k) \leq 0$ (because $\chi_s(3_1) = -1$ and $\chi_s(2_1) = 0$) while we have assumed that $\chi(\Sigma_1) = 1$ and $\chi(\Sigma_2) = 0$. $\square$

**Proposition 6.3.** The link $5_2^2$ (see Fig. 5) is a strong $\mathbb{C}$-boundary.

![Figure 5](image1.png) \hspace{1cm} \![Figure 6](image2.png)

**Proof.** Put $A = \{(z, w) \mid w^2 = z^2 + z^3\}$. Suppose that $1 \ll r \ll R$ and let $\Delta_r, \Delta_R \subset \mathbb{C}$ be discs of the respective radii. Put $U_t = ([t, r] \times \Delta_R) \cup \partial(\Delta_r \times \Delta_R)$. Take $z_1 = r \exp(\pi i/3)$ and $z_2 = z_1$ and let $w_j, j = 1, 2$, be the solutions of $w^2 = z_j^2 + z_j^3$ with $\text{Im} w_j > 0$. Then $w_1 \approx w_2 \approx r^{3/2} i$. Put $p_j = (z_j, w_j)$ and $p'_j = (z_j, Ri)$. Let $\gamma$ be an embedded arc in $\Delta_r$ which connects $z_1$ and $z_2$ avoiding the interval $[0, r]$. We put $\Gamma = [p_1, p'_1] \cup (\gamma \times \{Ri\}) \cup [p_2, p_2]$. Given a set $X \subset \mathbb{C}^2$ and a number $\varepsilon > 0$, we denote the $\varepsilon$-neighbourhood of $X$ in $\mathbb{C}^2$ by $X^\varepsilon$. Finally, for $0 \ll \delta \ll \varepsilon$, let $B_\varepsilon$ be a small smoothing of $(U_t \setminus \Gamma^\varepsilon)^\delta$.

Then $A \cap \partial B_0$ is a strong $\mathbb{C}$-boundary isotopic to $5_2^2$ in the embedded 3-sphere $\partial B_0$. Indeed, we have $U_r = \partial(\Delta_r \times \Delta_R)$ and $A \cap U_r$ is a trefoil knot sitting in the ‘vertical’ solid torus $T = (\partial\Delta_r) \times \Delta_R$; see Fig. 6, where we represent the piecewise smooth 3-sphere $U_r$ in terms of its central projection onto the unit sphere followed by a suitable stereographic projection onto 3-space.
Hence the link $A \cap B_r$ is as in the left picture in Fig. 7 (compare with Theorem 2.4 and its proof). Consider the family of links $(B_t, A \cap B_t)$, where $t$ varies continuously from $r$ to 0. Under this deformation, the link changes only in a small area on the ‘inner part’ of the sphere. Namely, as $t$ varies, nothing changes except for the portion of the link in the sector $-\eta < \text{Arg} z < \eta$ ($0 < \eta \ll 1$) of the solid torus $(\partial \Delta_{r-\delta}) \times \Delta_R$. When the parameter $t$ crosses the value $t = \delta$, the sphere $\partial B_t$ passes through the double point of $A$ (at the origin) and the link bifurcates as shown in the middle picture in Fig. 7. The resulting link is exactly $5_2^2$ (see the right picture in Fig. 7).

**Comments on Table 1.**
(a) Because $\chi_s(L) \neq \chi_s^-(L)$ (see Proposition 3.2).
(b) By Lemma 3.8 combined with the relation $L \notin SB$.
(c) By Theorem 1.2.
(d) By Theorem 2.1 applied to the nodal or cuspidal cubic $y^2 = ax^2 + x^3$ ($a = 0$ or 1), where $B_0$ is a small ball centred at the origin and $B$ is a small (for $2_1 \# 2_1^*$ and $2_1 \sqcup 2_1^*$) or large (in the other cases) ball containing $B_0$. One easily checks that the resulting link is a strong $\mathbb{C}$-boundary in the corresponding cases.
(e) Use the equality $\chi_s(L) = \chi_s(L^*)$ and apply Proposition 3.6 to compute $\chi_s(L^*)$.
(f) By Proposition 3.7.
(g) An embedded surface with boundary $L$ in the 4-ball cannot contain a disc as a component. Hence $\chi_s(L) \leq 0$. (In the case of $3_1^* \neq 2_1$ one can also compute $\chi_s(L) = \chi_s(L^*)$ using Proposition 3.2.)
(h) See Proposition 6.2.
(i) By Theorem 6.1.
(j) See Proposition 6.3.

**Bibliography**

[1] M. Boileau and L. Rudolph, “Nœuds non concordants à un $\mathbb{C}$-bord”, *Vietnam J. Math.* 23 (1995), 13–28.

[2] L. Rudolph, “Plane curves in fancy balls”, *Enseign. Math.* (2) 31:1-2 (1985), 81–84.

[3] M. Boileau and S. Orevkov, “Quasi-positivité d’une courbe analytique dans une boule pseudo-convexe”, *C. R. Acad. Sci. Paris Sér. I Math.* 332:9 (2001), 825–830.

[4] L. Rudolph, “Algebraic functions and closed braids”, *Topology* 22:2 (1983), 191–201.

[5] K. Hayden, “Minimal braid representatives of quasipositive links”, *Pacific J. Math.* 295:2 (2018), 421–427.
[6] P. B. Kronheimer and T. S. Mrowka, “The genus of embedded surfaces in the projective plane”, Math. Res. Lett. 1:6 (1994), 797–808.
[7] S. Yu. Orevkov, “Quasipositive links and connected sums”, Funkts. Anal. Prilozhen. 54:1 (2020), 81–86; English transl., Funct. Anal. Appl. 54:1 (2020), 64–67.
[8] R. Nirenberg and R. O. Wells, “Approximation theorems on differentiable submanifolds of a complex manifold”, Trans. Amer. Math. Soc. 142 (1969), 15–35.
[9] S. Yu. Nemirovski, “Complex analysis and differential topology on complex surfaces”, Uspekhi Mat. Nauk 54:4(328) (1999), 47–74; English transl., Russian Math. Surveys 54:4 (1999), 729–752.
[10] C. Livingston and A. H. Moore, LinkInfo: table of link invariants, June 17, 2020, http://linkinfo.sitehost.iu.edu.
[11] R. Fintushel and R. J. Stern, “Immersed spheres in 4-manifolds and the immersed Thom conjecture”, Turkish J. Math. 19:2 (1995), 145–157.
[12] G. Mikhalkin, “Adjunction inequality for real algebraic curves”, Math. Res. Lett. 4:1 (1997), 45–52.
[13] D. Eisenbud and W. Neumann, Three-dimensional link theory and invariants of plane curve singularities, Ann. of Math. Stud., vol. 110, Princeton Univ. Press, Princeton, NJ 1985.
[14] P. M. Gilmer and S. Yu. Orevkov, “Signatures of real algebraic curves via plumbing diagrams”, J. Knot Theory Ramifications 27:3 (2018), 1840003.
[15] C. Livingston and A. H. Moore, KnotInfo: table of knots, June 17, 2020, http://www.indiana.edu/~knotinfo.
[16] J. Franks and R. F. Williams, “Braids and the Jones polynomial”, Trans. Amer. Math. Soc. 303:1 (1987), 97–108.
[17] H. R. Morton, “Seifert circles and knot polynomials”, Math. Proc. Cambridge Philos. Soc. 99:1 (1986), 107–109.

Nikolai G. Kruzhilin
Steklov Mathematical Institute,
Russian Academy of Sciences, Moscow
E-mail: kruzhil@mi-ras.ru

Stepan Yu. Orevkov
Steklov Mathematical Institute,
Russian Academy of Sciences, Moscow
E-mail: orevkov@mi-ras.ru

Received 29/JUN/20
Translated by THE AUTHORS