Cavity atom optics and the ‘free atom laser’

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The trap environment in which Bose-Einstein condensates are generated and/or stored strongly influences the way they interact with light. The situation is analogous to cavity QED in quantum optics, except that in the present case, one tailors the matter-wave mode density rather than the density of modes of the optical field. Just as in QED, for short times, the atoms do not sense the trap and propagate as in free space. After times long enough that recoiling atoms can probe the trap environment, however, the way condensates and light fields are mutually influenced differs significantly from the free-space situation. We use as an example the condensate collective atomic recoil laser, which is the atomic matter-wave analog of the free-electron laser.

I. INTRODUCTION

The manipulation of atomic trajectories by optical fields forms the basis of atom optics [1], a topic of considerable current interest. Many atom optics experiments proceed by simply reversing the roles of light and matter from the situation in conventional optics. That is, the “optical elements” used to refract, diffract, reflect, focus, trap, etc. matter waves are oftentimes — but not always — made of light, which transfers momentum to the atoms to achieve the desired goal. In many situations, it is sufficient to treat the optical fields as constant and imposed from the outside. In addition, for low enough densities, collisions can be ignored.

In general, it is necessary to describe both the Maxwell field describing light and the Schrödinger field describing the atoms as dynamically coupled. Under certain circumstances, one can then formally eliminate the dynamics of the atomic field, resulting in effective interactions between the light fields. Under a different set of conditions, one can alternatively eliminate all or part of the electromagnetic field, resulting in effective atom-atom interactions (collisions). These are the regimes of nonlinear optics [2] and nonlinear atom optics [3,4], respectively. But they represent limiting cases, where either the atomic or the optical field is not dynamically independent, following instead the other field in some adiabatic manner that allows for its effective elimination. Outside of these two regimes the atomic and optical fields are dynamically independent. Neither field is readily eliminated.

In the past, there have been numerous examples where the dynamical coupling between the optical and matter-wave fields was important. Much of quantum optics and of laser physics deals with such situations. However, in the spirit of atom optics, we wish to concentrate only on those cases where the center-of-mass motion of the material system is central to the problem. An early example of such a situation is the free-electron laser (FEL) [5–7], where a periodic magnetic field (the wiggler) and a running wave optical probe field conspire to spatially modulate the density of a relativistic electron beam. This density modulation results in an oscillating current which amplifies the optical field. As a result, the electron density modulation is itself increased, leading to a runaway amplification process. A similar mechanism governs the collective atomic recoil laser (CARL) [8], except that in that case the relativistic electrons are replaced by atoms, and the periodic magnetic field by an optical pump field. Here, runaway amplification results from the stimulated scattering of the pump field off the density grating imposed on the atomic sample by the combined action of the pump and probe fields. Similarly to the free-electron laser nomenclature, one would therefore be justified in calling the CARL a “free-atom laser.”

Both the FEL and the CARL operate under conditions such that the electrons or the atoms can be described as classical particles. Indeed, in analogy with the situation in optics, one can expect that the wave nature of the particles involved is irrelevant if the associated de Broglie wavelength is small compared to the characteristic length of the “optical elements” involved in their manipulation. For the case of the CARL, for example, this implies that a classical description of the atomic trajectories is appropriate only for temperatures large compared to the recoil temperature.

The situation becomes in many ways much more interesting when the wave properties of the atoms become essential. Early examples of such situations were discussed by Meystre et al. [9] who studied the diffraction of an atom by a quantized field mode, by Haroche et al. [10] who discussed the possible trapping of an atom in the vacuum field of a high-Q microwave cavity, and by Englert et al. [11] who considered the reflection of an ultracold atom at the entrance of such a cavity. Scully and coworkers subsequently extended this work to the theory of the micromaser, an ultracold-atom version of the micromaser [12].

In parallel to these efforts, extensions of the CARL theory into the regime of ultracold atoms have also been
carried out. They bring to light both the impact of atomic diffraction on the threshold behavior of the system and the role of the vacuum fluctuations of the Schrödinger field on the build-up from noise of the signal. Recently, this work was further extended to analyze the superradiant Rayleigh scattering from a condensate observed by Ketterle’s group. [15]

These brief remarks touch upon just one of the many contributions of Marlan Scully, yet they clearly illustrate his impact on modern quantum optics. It is particularly noteworthy that his “ancient” work on the FEL and cavity QED has now gained a second youth, and is central to much of the exciting work taking place in Bose-Einstein condensation, atom lasers, and the atom optics of ultracold atom samples. The present paper discusses some new results in matter-wave optics that build on the kind of ideas which have been central to Marlan Scully’s research, namely laser physics [16], amplifier theory, and cavity QED [17]. The new twist here is that instead of tailoring the density of modes of the electromagnetic field, we consider a situation where it is the matter-wave density of states that is manipulated.

 Atomic Bose-Einstein condensates are always formed in traps, a feature that has of course important implications for many of their properties, such as e.g. the dependence of the critical temperature on particle number, the quasi-excitation spectrum, etc. In optical experiments involving the buildup of condensate momentum side modes, however, one usually considers a plane-wave expansion rather than the actual matter-wave modes in the trap. This assumes implicitly that the side modes behave as if in free space. The question, then, is to which extent this is appropriate and when a proper description of the matter-wave modes of the trap is necessary.

The problem at hand is similar to the situation in cavity QED: for instance, it is known that spontaneous emission always first occurs at the free space rate, until the wave packet emitted by the atom has a chance to probe the cavity boundaries and be reflected back to the atom. [13] This leads to interference effects that can result either in enhanced or inhibited spontaneous emission, or, for sufficiently high-Q resonators, to reversible spontaneous emission. Likewise, it is expected that matter waves will behave as in free space for times short compared to their propagation time to the trap boundaries. For longer times, however, cavity effects are expected to play an important role. In situations where matter-wave side modes result from an interaction with light, the characteristic atomic velocity is the recoil velocity $v_{rec} = \hbar k/m$, which is typically of the order of centimeters per second. For trap sizes of 100 microns or so, this corresponds to characteristic times of the order of tens of milliseconds, after which the atoms will have probed their environment and determined that they are in a trap.

This paper illustrates the role of cavity effects in matter-wave optics on the example of the ultracold CARL. We first briefly present our model, and analyze the system’s dynamics in the small signal regime in a way that exposes the differences between the free-space and trap description of the system. We then compare the instabilities associated with the two descriptions, showing the appearance of a new kind of instabilities in the “cavity matter-wave optics” analysis.

II. MODEL

Our model consists of a Schrödinger field of non-interacting bosonic two-level atoms coupled via the electric-dipole interaction to two single-mode running wave light fields. We consider only the case where the detuning between the optical fields and the atomic resonance is many orders of magnitude larger than the natural linewidth of the atomic transition. While single photon processes are therefore non-resonant, the atoms may still undergo two-photon virtual transitions in which their internal state remains unchanged, but due to recoil may result in a change in their center-of-mass motion. In the far-off resonant regime, the excited state population, and therefore spontaneous emission, may be neglected, and the ground state atomic field then evolves coherently under the effective Hamiltonian

$$\hat{H} = \int d^3r \hat{\psi}(r) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) + \hbar \frac{g_1^2 g_2}{\Delta} a_2 \hat{a}_1 e^{-iK \cdot r} + \hbar \frac{g_1^2 g_2}{\Delta} \hat{a}_1 \hat{a}_2 e^{iK \cdot r} \right]$$

where $m$ is the atomic mass, $V(r)$ is the trap potential, $g_1$ and $g_2$ are the probe and pump coupling coefficients, respectively, and $K = k_1 - k_2$ is the difference between the probe and pump wavevectors. The operator $\hat{a}_1$ is the photon annihilation operator for the probe mode, taken in the frame rotating at the pump frequency $\omega_2$, hence the energy of the probe mode is the difference between the probe frequency $\omega_1$ and that of the pump. The pump is treated classically, and assumed to remain undepleted. Thus $a_2$ is simply a constant, related to the pump intensity $I_2$ by $|g_2|^2 |a_2|^2 = d^2 I_2/2\hbar \omega_2 \hbar c$, where $d$ is the magnitude of the atomic dipole moment. We remark that we have neglected terms corresponding to the spatially independent light shift potential. These terms do not influence the dynamics, as all atoms inside the light field undergo the same phase shift. Note that this shift modifies the dispersion relation for the light fields, hence by ignoring it we are taking the index of refraction of the atomic sample to be the same as that of the vacuum.

We assume that the atomic field is initially a Bose-Einstein condensate with mean number of condensed atoms $N$, and that this condensate is well described by a number state so that the initial state of the atomic field may be taken as

$$|\psi\rangle(t=0) = \frac{1}{\sqrt{N!}} \left( \hat{c}_0^\dagger \right)^N |0\rangle,$$
where $|0\rangle$ is the vacuum state, and

$$\hat{c}_0^\dagger = \int d^3r \varphi_0(r) \hat{\psi}^\dagger(r)$$

is the creation operator for atoms in the condensate state $\varphi_0(r)$, which in the absence of atom-atom interactions is the ground state of the potential $V(r)$.

The Heisenberg equation of motion for the condensate field operator is readily derived from (3), giving

$$\frac{d}{dt} \hat{c}_0 = -i \omega_0 \hat{c}_0 - i \frac{\chi}{\sqrt{N}} \int d^3r \hat{r} \varphi_0^*(r) \left[ \hat{a}^\dagger e^{-i K \cdot r} + \hat{a} e^{i K \cdot r} \right] \hat{\psi}(r),$$

where $\hbar \omega_0$ is the ground state energy, $\chi = |g_1||g_2|/\sqrt{|N|/|\Delta|}$ is the effective coupling constant between the condensate and the probe field, and $\hat{a} = (g_1g_2^* \Delta/|g_1||g_2|)|\Delta|) \hat{a}_1$ is simply the probe annihilation operator times a phase factor related to the phase of the pump laser and the sign of the detuning.

A. Free-propagation regime

From Eq. (3) we see that the condensate mode is optically coupled to two new states whose field operators are given by

$$\hat{c}_\pm = \int d^3r \varphi_0^*(r)e^{i K \cdot r} \hat{\psi}(r).$$

These states retain the spatial probability distribution of the condensate state, but propagate at the recoil velocity $v_r = \hbar |K|/m$, hence we refer to them as ‘momentum side modes’ of the original condensate. With these definitions, Eq. (3) becomes

$$\frac{d}{dt} \hat{c}_0 = -i \omega_0 \hat{c}_0 - i \frac{\chi}{\sqrt{N}} \left[ \hat{a}^\dagger \hat{c}_+ + \hat{a} \hat{c}_- \right].$$

Due to the fact that the recoil velocity is extremely slow ($\sim 1$ cm/s), the propagation of the side mode wavepackets can be neglected for reasonably long times, in which case the side mode field operators obey

$$\frac{d}{dt} \hat{c}_- = -i(\omega_0 + \omega_r) \hat{c}_- - i \frac{\chi}{\sqrt{N}} \hat{a}^\dagger \hat{c}_0,$$

and

$$\frac{d}{dt} \hat{c}_+ = -i(\omega_0 + \omega_r) \hat{c}_+ - i \frac{\chi}{\sqrt{N}} \hat{a} \hat{c}_0,$$

where $\omega_r = \hbar |K|^2/2m$ is the recoil frequency. We note that we have neglected coupling to higher order side modes, an approximation valid in the small-signal regime. Lastly, we note that in this free propagation model the probe field operator satisfies the equation of motion

$$\frac{d}{dt} \hat{a} = -i \delta \hat{a} - i \frac{\chi}{\sqrt{N}} \left[ \hat{c}_-^\dagger \hat{c}_0 + \hat{c}_0^\dagger \hat{c}_+ \right],$$

where $\delta = \omega_1 - \omega_2$ is the pump-probe detuning, and we again neglect higher-order momentum side modes.

B. Cavity regime

The free propagation model is valid for times short enough that atoms at the recoil velocity propagate only over distances which are short compared to the dimensions of the initial condensate. As we have shown, in this regime the atomic field can be expanded onto momentum side modes, which are simply plane waves with a slowly varying spatial envelope. A second well-defined regime occurs at much longer time scales, when the recoil velocity atoms propagate over distances which are larger than the trap dimensions. In this ‘cavity atom optics’ regime, the atomic field operator is best expressed in terms of the trap eigenmodes $\{\varphi_n(r)\}$ according to

$$\hat{\psi}(r) = \sum_{n=0}^\infty \varphi_n(r) \hat{c}_n,$$

where $\hat{c}_n$ is the annihilation operator for atoms in mode $n$. With this expansion, the equation of motion for the condensate field operator becomes

$$\frac{d}{dt} \hat{c}_0 = -i \omega_0 \hat{c}_0 - i \frac{\chi}{\sqrt{N}} \sum_{n=0}^\infty A_{nn'} \left[ \hat{a}^\dagger + \hat{a} \right] \hat{c}_n,$$

where

$$A_{nn'} = \int d^3r \varphi_{n'}^*(r) e^{i K \cdot r} \varphi_n(r)$$

is the matrix element for the optical transition, and we have assumed without loss of generality that $A_{nn'}$ is a real number.

We assume for simplicity that the matrix elements $A_{nn'}$ are sharply peaked, so that $|n - n'| = m$ where $m$ is some number. This could be the case, e.g. in a Fabry-Pérot-type matter-wave resonator, where the absolute value of the momentum is relatively well defined for each trap energy level. This allows us to neglect all excited states except $n = m$, which gives

$$\frac{d}{dt} \hat{c}_0 = -i \omega_0 \hat{c}_0 - i \frac{\chi}{\sqrt{N}} A_{mm} \left[ \hat{a}^\dagger + \hat{a} \right] \hat{c}_m.$$
Equations (13-15) differ from (12) in that the two running wavepacket momentum side modes have now been combined into a single standing wave atomic field mode. In the free-propagation regime, which state the atom jumps into depends on whether a probe photon was absorbed or emitted, a result of momentum conservation. In contrast, in the cavity regime both processes transfer atoms to the same quantum state, due to the uncertainty in the momentum direction of the trap levels. We will see shortly that this distinction can lead to remarkably different physical effects. A similar situation is familiar in conventional atom optics, where the diffraction of an atom by a standing wave is in general different from the diffraction by two counterpropagating running waves.

**C. Linear response**

The condensate mode is assumed to be initially occupied by a very large number \(N\) of atoms, in which case it can be treated classically. Introducing a spontaneous symmetry breaking ansatz and replacing the operator \(\hat{c}_0\) with the \(c\)-number order parameter \(c(t)\), and considering furthermore times short enough that the side mode populations are small compared to \(N\), so that we can neglect condensate depletion, we have

\[
c_0(t) = \sqrt{N} e^{-i\omega_0 t}.
\]

It is natural to choose the zero of energy to coincide with the ground state energy, so that \(\omega_0 = 0\) without loss of generality. Substituting expression (16) into Eqs. (13-15) and linearizing in the side mode and probe field operators yields a closed 3 \(\times\) 3 system of equations,

\[
\frac{d}{dt} \begin{pmatrix} \hat{c}_\uparrow \cr \hat{c}_\downarrow \cr \hat{a} \end{pmatrix} = i \begin{pmatrix} \omega_r & 0 & \chi \\ 0 & -\omega_r & -\chi \\ -\chi & -\chi & -\delta \end{pmatrix} \begin{pmatrix} \hat{c}_\uparrow \\ \hat{c}_\downarrow \\ \hat{a} \end{pmatrix},
\]

which describes the linear response of the system in the free propagation regime. We note that while there are six independent operators \(\hat{c}_\uparrow, \hat{c}_\downarrow, \hat{a}\), and their Hermitian conjugates, the linearized equations have decoupled into two sets of three coupled equations due to the momentum selection rules.

In the cavity regime, there are only four coupled operators, \(\hat{c}_m, \hat{c}_m^\dagger, \hat{a}\) and \(\hat{a}^\dagger\). The equations of motion, however, do not decouple and the dynamics is determined by the 4 \(\times\) 4 system of equations

\[
\frac{d}{dt} \begin{pmatrix} \hat{c}_m \\ \hat{c}_m^\dagger \\ \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = i \begin{pmatrix} -\omega_m & 0 & -\chi_m & -\chi_m \\ 0 & \omega_m & \chi_m & \chi_m \\ -\chi_m & -\chi_m & -\delta & 0 \\ \chi_m & \chi_m & 0 & \delta \end{pmatrix} \begin{pmatrix} \hat{c}_m \\ \hat{c}_m^\dagger \\ \hat{a} \\ \hat{a}^\dagger \end{pmatrix},
\]

where \(\chi_m = \chi A_{0m}\).

Equation (13) for the free propagation regime, and Eq. (18) for the cavity regime can be derived from the effective Hamiltonians

\[
\hat{H}_{\text{free}} = \hbar \omega_r \left( \hat{c}_\uparrow^\dagger \hat{c}_\downarrow + \hat{c}_\downarrow^\dagger \hat{c}_\uparrow + \hbar \delta \hat{a}^\dagger \hat{a} \right) + \hbar \chi \left( \hat{a}^\dagger \hat{c}_\uparrow \hat{c}_\downarrow + \hat{c}_\downarrow^\dagger \hat{c}_\uparrow \hat{a} + \hat{c}_\uparrow^\dagger \hat{c}_\downarrow \hat{a} \right),
\]

and

\[
\hat{H}_{\text{cav}} = \hbar \omega_m \hat{c}_m^\dagger \hat{c}_m + \hbar \delta \hat{a}^\dagger \hat{a} + \hbar \chi_m \left( \hat{a}^\dagger \hat{c}_m + \hat{a}^\dagger \hat{c}_m + \hat{c}_m^\dagger \hat{a} + \hat{c}_m^\dagger \hat{a} \right),
\]

respectively. Terms like \(\hat{a}^\dagger \hat{c}_\uparrow\) correspond to the generation of correlated atom-photon pairs. This is analogous to the optical parametric amplifier (OPA), except that in that case it is correlated photon pairs that are generated. The OPA is currently the primary device for the generation of entangled and/or nonclassical states, and plays a role in many fundamental experiments in quantum physics, such as tests of Bell’s inequality, quantum cryptography, and quantum teleportation. This analogy, and the possibility of generating entanglement between atomic and optical fields is discussed in detail in [2,14].

Both linear systems can be solved analytically, all that is required are the eigenvalues and eigenvectors of the 3 \(\times\) 3 and 4 \(\times\) 4 matrices which appear on the right-hand-side of equations (13) and (14) respectively. The characteristic frequencies in the free space regime satisfy the cubic equation

\[
\omega^3 + \delta \omega^2 - \omega_r^2 \omega - \omega_r^2 \delta + 2 \omega_r \chi^2 = 0.
\]

This equation either has three real solutions, in which case the system is stable and experiences only small oscillations about the initial state, or it has one real and a pair of complex conjugate solutions. In the latter case the eigenvalue with the negative imaginary part corresponds to an exponentially growing solution, and hence the system is unstable. In this case any small input, even quantum fluctuations, results in a large output, i.e. the system acts as an amplifier for the optical probe and atomic side mode fields.

In the cavity regime, the characteristic equation is a quartic equation, given by

\[
\omega^4 - \left( \delta^2 + \omega_r^2 \right) \omega^2 + \delta^2 \omega_r^2 - 4 \delta \omega_m \chi_m^2 = 0,
\]

which is simply a quadratic equation in \(\omega^2\). The solutions to Eq. (22) fall into three categories. The first category is of course when all solutions are purely real, in which case the system is stable. A second possibility is that there are two purely real and two purely imaginary solutions of the form \(\{ \omega_1 = \Omega, \omega_2 = -\Omega, \omega_3 = i\Gamma, \omega_4 = -i\Gamma \}\), where \(\Omega\) and \(\Gamma\) are both real quantities. In this case there is only one exponentially growing solution, at the imaginary frequency \(\omega_4\). A final possibility is that the solutions are complex numbers of the form \(\{ \omega_1 = \Omega + i\Gamma, \omega_2 = -\Omega + i\Gamma, \omega_3 = \Omega - i\Gamma, \omega_4 = -\Omega - i\Gamma \}\). This last case is
interesting in that there are two exponentially growing solutions, $\omega_1$ and $\omega_2$, which grow at the same rate $\Gamma$, but rotate at equal and opposite frequencies $\pm \Omega$. This leads to the temporal modulation of the exponential growth rate.

In Fig. 1 we plot the domains of parameter space corresponding to the various types of solutions. The shaded areas correspond to the unstable regions where exponential growth occurs. Figure 1a shows the free propagation model, where the stable and unstable regions are depicted in the $\delta - \chi^2$ plane. Both $\delta$ and $\chi$ are taken in units of $\omega_r$. In Fig 1b, we plot the corresponding results for the cavity model in the $\delta - \chi^2_m$ plane. Here $\delta$ and $\chi_m$ are taken in units of $\omega_m$. In the figures, roman numeral I corresponds to a single exponentially growing solution and roman numeral II corresponds to two exponentially growing solutions.

III. MEAN INTENCITIES

In this section we use the linearized description of section II to derive expressions for the mean intensities of the probe and side mode fields, and compare the results for the two models. We focus in particular on the 'exponential growth regime', which occurs for times long enough that all but the exponentially growing solutions of section II can be safely neglected, yet short enough that condensate depletion can be ignored. We assume for concreteness that the atomic side modes begin in the vacuum state, while the probe field is initially in a coherent state $\alpha$, corresponding to the injection of a weak laser field into the ring cavity.

A. Free propagation regime

In the free propagation regime, the solutions to Eq. (17) is given by

$$
\begin{pmatrix}
\hat{c}_m(t) \\
\hat{c}_m(t) \\
\hat{a}(t)
\end{pmatrix}
= A(t)
\begin{pmatrix}
\hat{c}_m(0) \\
\hat{c}_m(0) \\
\hat{a}(0)
\end{pmatrix}.
$$

The $3 \times 3$ matrix $A(t)$ is given by

$$
A(t) = U \exp(iWt)U^{-1},
$$

where $U_{ik}$ is the $i$th component of the $k$th eigenvector of the matrix on the right-hand-side of Eq. (17), and $\mathbf{W}$ is the corresponding diagonal matrix of eigenvalues. For times long compared to the exponential doubling rate, we can neglect all but the exponentially growing terms, in which case we have

$$
A_{ij}(t) \approx a_{ij}e^{\delta t + \Gamma t},
$$

where $a_{ij}$ is a constant which depend only on the control parameters $\delta$ and $\chi$.

From expression (24), together with the specified initial condition, we find that the mean intensities $I_1 = \langle \hat{c}_m^\dagger \hat{c}_m \rangle$, $I_2 = \langle \hat{c}_m^\dagger \hat{c}_m \rangle$, and $I_3 = \langle \hat{a}^\dagger \hat{a} \rangle$ take the form $I_j(t) = I_j \exp(2\Gamma t)$, where

$$
I_j = |a_{j1}|^2 + |a|^2|a_{j3}|^2.
$$

This expression contains a spontaneous contribution, present even in the case $\alpha = 0$, as well as a stimulated term proportional to the injected signal strength $|a|^2$. The former is due to the amplification of vacuum fluctuations in the atomic density and the probe electromagnetic field, whereas the latter is due to amplification of the injected probe field.

B. Cavity atom optics regime

In the cavity regime, the solution to Eq. (18) is given by

$$
\begin{pmatrix}
\hat{c}_m(t) \\
\hat{c}_m(t) \\
\hat{a}(t)
\end{pmatrix}
= B(t)
\begin{pmatrix}
\hat{c}_m(0) \\
\hat{c}_m(0) \\
\hat{a}(0)
\end{pmatrix},
$$

where the $4 \times 4$ matrix $\mathbf{B}$ is defined in the same manner as $\mathbf{A}$, but the eigenvalues and eigenvectors are those of the $4 \times 4$ matrix on the right-hand-side of Eq. (18).

Again for moderately long times we keep only exponentially growing terms, which in region I of Fig. 1b allows us to take

$$
B_{ij}(t) \approx b_{ij}e^{\Gamma t},
$$

where $b_{ij}$ is a constant which depends only on the parameters $\delta$ and $\chi_m$. With this approximation, we find that the mean intensities $I_1 = \langle \hat{c}_m^\dagger \hat{c}_m \rangle$, and $I_3 = \langle \hat{a}^\dagger \hat{a} \rangle$, are all of the form $I_j(t) = I_j \exp(2\Gamma t)$, where

$$
I_j = |b_{j2}|^2 + |b_{j4}|^2 + |a|^2(|b_{j3}|^2 + |b_{j4}|^2)
+ 2Re \left( b_{j3}b_{j4}^\ast \alpha^2 \right).
$$

As in the free propagation model, there is a spontaneous component, present even for $\alpha = 0$, and a stimulated component, which is present only for $\alpha \neq 0$. One important difference, however, is that in contrast to the free propagation regime of Eq. (28), where the intensities depend only on $|\alpha|$, in the cavity situation the intensities are also sensitive to the phase of the initial probe field. We remark that the intensities in region I of the cavity model and those in the free propagation model are very similar in that the time-dependence is a pure exponential. This is no longer the case in unstable region II of the cavity model.

In region II of Fig. 1b, there are two independent exponentially growing solutions. Keeping only these terms gives
The mean intensities in region II take the form
\[ \bar{I}_j(t) = X_j + 2\text{Re} \left[ x_j e^{i2\Omega t} + |\alpha|^2 \right] + 2\text{Re} \left[ (Z_j + z_j e^{i2\Omega t} + \bar{z}_j e^{-i2\Omega t}) \alpha^2 \right], \quad (31) \]

where the various parameters are
\[
\begin{align*}
X_j &= |c_{j2}|^2 + |d_{j2}|^2 + |c_{j4}|^2 + |d_{j4}|^2, \\
x_j &= c_{j2}d_{j2}^* + c_{j4}d_{j4}^*, \\
Y_j &= |c_{j3}|^2 + |d_{j3}|^2 + |c_{j4}|^2 + |d_{j4}|^2, \\
y_j &= c_{j3}d_{j3}^* + c_{j4}d_{j4}^*, \\
Z_j &= c_{j3}c_{j4}^* + d_{j3}d_{j4}^*, \\
\bar{z}_j &= d_{j3}c_{j4}^*, \\
z_j &= c_{j3}d_{j4}^*. 
\end{align*}
\]

From Eq. (31) we see that in addition to purely exponential terms, there are now contributions with superimposed oscillations. These beats occur even when quantum noise alone is amplified, as can be seen in the first and second terms in Eq. (31). These terms are spontaneous contributions, present even in the absence of an injected probe field \( \alpha = 0 \). In contrast, the third term is a stimulated contribution which is independent of the phase of the injected field and the fourth term, which gives additional oscillatory and non-oscillatory terms, is sensitive to the phase of the injected field \( \alpha \).

We remark that when the transcendental equation
\[ Z_j + z_j e^{i2\Omega t} + \bar{z}_j e^{-i2\Omega t} = 0 \quad (39) \]
appearing in that last contribution is satisfied, say at a time \( t = t_0 \), then the phase sensitivity of the solution disappears. This property recurs for the times \( t = t_0 + n\pi/\Omega \), where \( n \) is any integer. At these times, the intensities corresponding to initial states having the same \( |\alpha| \) but different phases intersect.

The growth of the matter-wave intensity is illustrated in Fig. 2 for two examples. Figure 2a shows \( \ln |I_3| \) versus \( t \) for the case \( \alpha = 1 \) for \( \chi_m = \omega_m^{-1} \) and \( \delta = -\omega_m^{-1} \), a set of system parameters in region I. Here, there is only one exponentially growing solution and no oscillations. Figure 2b shows the result for \( \chi_m = \omega_m^{-1} \) and \( \delta = \omega_m^{-1} \). This example demonstrates the beating which occurs in region II as a result of the interplay between the two counterrotating exponential terms. In both figures \( t \) is taken to be in units of \( \omega_m \).

IV. SUMMARY AND CONCLUSION

The fact that Bose-Einstein condensates of low density atomic vapors are generated in traps has important consequences on their intrinsic properties. For instance, in a harmonic trap the dependence of the number of condensate atoms on the critical temperature scales as \( 1-(T/T_c)^3 \), rather than the free-space result \( 1-(T/T_c)^{3/2} \). The condensate ground state and the spectrum of quasi-excitation also depends drastically on the presence and form of a trap potential, so much so that it is even possible to create stable small trapped condensates of atoms subject to an attractive interaction.

In this paper, we have discussed how the trap environment also strongly influences the way it interacts with light. We have shown that after times long enough that the condensate atoms can probe their environment, the way condensate side modes and light fields are amplified differs significantly from the free-space situation. This paper has concentrated on the growth rate of the intensity of the side modes of both the condensate and the optical field. Further work will investigate in which way such trap environments also modify the quantum statistics of these fields and their entanglement.

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FIG. 1. Exponential instability regions for the free propagation and cavity regimes are shown in Figs. 1a and 1b respectively. Region I corresponds to one exponentially growing solution, whereas in region II there are two counterrotating exponential solutions. In Fig. 1a, $\delta$ and $\chi$ are taken in units of $\omega_r$, while in Fig. 1b $\delta$ and $\chi_m$ are in units of $\omega_m$.

FIG. 2. Figure 2a shows $\ln |I_3|$ versus $t$ for the case $\alpha = 1$ for $\chi_m = \omega_m^{-1}$ and $\delta = -\omega_m^{-1}$, a set of system parameters in region I. Figure 2b shows the result for $\chi_m = \omega_m^{-1}$ and $\delta = \omega_m^{-1}$. This example demonstrates the beating which occurs in region II as a result of the interplay between the two counterrotating exponential terms. In both figures $t$ is taken to be in units of $\omega_m$. 

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