A bicombing that implies a
sub-exponential Isoperimetric Inequality

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1 Definitions

Let $\Gamma_X(G)$ be the Cayley graph of a group $G$ with respect to a finite set of
generators $X$, and let $\Gamma_X(G)$ be equipped with the word metric. Let $F$ be the
free group on $X$. For $v \in F$ let $|v|$ denote the length in the free group.

A bicombing as defined in [1] and [7] is essentially a selection of a path
$\sigma(g, h)$ for every pair of vertices $g, h \in \Gamma_X(G)$, such that the distance between
any two paths which start and end a distance $\leq 1$ apart is uniformly bounded.
We replace the uniform bound for this distance by a bound that is dep endent
on the lengths of the paths. More precisely, we define a bicombing of narrow
shape as follows:

For each $(g, h) \in G \times G$ let $\sigma(g, h) : [0, \infty[ \rightarrow \Gamma_X(G)$ be a path from $g$ to $h$
which is at integer times at vertices (i.e. from $t = n$ to $t = n + 1$ the path either
travels the distance between two adjacent vertices or pauses at a vertex). We
define the length:

$|\sigma(g, h)| = \min\{|t|\sigma(g, h)|t, \infty[ = \text{constant} = h\}$. This is the length of the path
including the pauses which occur before the end of the path is reached. We will
dependently represent such a path by a sequence of elements in $X \cup X^{-1} \cup \{1\}$
which, given the startvertex $g$, completely determines the path. Let $\sigma(h) =$
$\sigma(1, h)$. We call $\sigma$ a bicombing of narrow shape if

1. it is “recursive”, i.e. if there exists an increasing polynomial $f : \mathbb{N} \rightarrow \mathbb{N}$,
such that $|\sigma(g)| \leq f(d(1, g)) \quad \forall g \in G$ (1)

2. there exists an integer $M > 1$ and a real number $k > 2$, such that for any
$g \in G \ |\sigma(g, g)| \leq Mk/2$ and for all $g, h \in G$ and $a, b \in X^{\pm 1} \cup \{1\}$

$|\sigma(\sigma(g, h)(t), \sigma(ga, hb)(t))| \leq \max((|\sigma(g, h)| + |\sigma(ga, hb)|)/k, M/2)$ (2)
holds for all integers $t \in [0, \infty]$.

where $d(1,g)$ denotes the distance in $\Gamma_X(G)$ from 1 to $g$. If possible we will always choose $\sigma(1)$ to be the identical path. A bicombing is called \textit{geodesic} if $f$ is the identity (i.e. the combing lines are geodesics).

Let the group $G$ be finitely generated with generator set $X$. Following Gersten [8], a function $f: \mathbb{N} \to \mathbb{N}$ is called an \textit{isoperimetric function} for $G$ if for any word $w$ in $X$ of length $n$ with $w = 1$ in $G$, the minimum number of 2-cells in a van Kampen diagram for $w$ is at most $f(n)$.

Let $P = < X \mid R >$ be a finite presentation of the group $G$. Following Gersten [8], a function $f: \mathbb{N} \to \mathbb{R}$ is called an \textit{isodiametric function} for $P$, if for any word $w$ in the generators $X$ with $w = 1$ in $G$ there is a van Kampen diagram for $w$, such that any vertex in the diagram has distance at most $f(|w|)$ from the basepoint.

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## 2 An isoperimetric inequality and an isodiametric function

**Theorem 2.1** A group $G$ with finite generator set $X$ and a bicombing of narrow shape is finitely presented and has an isoperimetric function of growth $n^{O(\log n)}$.

**Proof:** Define a presentation $P = < X \mid R >$, where $R$ is the set of all cyclically reduced non-trivial words of length at most $M + 2$ which are trivial in $G$. We proof that $P$ is a presentation for $G$ by constructing a van Kampen diagram for each word which is trivial in $G$, using only 2-cells of $R$.

Let $w \in F$ be a reduced nontrivial word of length $n > M + 2$ which is trivial in $G$. If $w = x_1 \ldots x_n$, $x_i \in X^{\pm 1}$, define $w_i = x_1 \ldots x_i$. Now consider the "fan" of bicombing lines $\sigma(w_i)$ from 1 to $w_i$. $|w| = n$ implies $d(1, w_i) \leq n/2$ and by (1) it follows

$$|\sigma(w_i)| \leq f(n/2) \quad \text{for } 1 \leq i \leq n. \quad (3)$$

If $|\sigma(w_i)| + |\sigma(w_{i+1})| \leq M$, then the closed path $\tau_i = \sigma(w_i)x_{i+1} \sigma(w_{i+1})^{-1}$ in $\Gamma_X(G)$ is of length $\leq M + 2$ and therefore represents up to cyclic reduction an element of $R$.

If $|\sigma(w_i)| + |\sigma(w_{i+1})| > M$ we break up the closed path $\tau_i$ again, using the bicombing paths $\sigma_{i,t} = \sigma(\sigma(w_i)(t), \sigma(w_{i+1})(t))$ that connect $\sigma(w_i)(t)$ to $\sigma(w_{i+1})(t)$ for all positive integers $t \leq \max(|\sigma(w_i)|, |\sigma(w_{i+1})|)$. By (2),

$$|\sigma_{i,t}| \leq \max(2f(n/2)/k, M/2). \quad (4)$$

Let $\sigma(w_i) = a_1 \ldots a_p$, $\sigma(w_{i+1}) = b_1 \ldots b_q$, $a_j, b_l \in X^{\pm 1} \cup \{1\}$. We examine the length of the closed paths $\tau_{i,t}$ that are generated by the connecting paths.
Figure 1: A diagram for $w$

$\tau_{i,t} = \sigma_{i,t} b_{i+1} \sigma_{i,t+1}^{-1} \sigma_{i,t+1} ^{-1}$ (see fig. 1). If $|\sigma_{i,t}| + |\sigma_{i,t+1}| \leq M$, then $|\tau_{i,t}| < M + 2$ and $\tau_{i,t}$ represents up to cyclic reduction an element in $R$. Otherwise, we break $\tau_{i,t}$ up again using bicombing paths $\sigma_{i,t,s} = \sigma(\sigma_{i,t}(s), \sigma_{i,t+1}(s))$ for $s \leq \max(|\sigma_{i,t}|, |\sigma_{i,t+1}|)$.

There is one exception, namely if we are close to the boundary. This is because the path of length one between $w_i$ and $w_{i+1}$ is not (necessarily) a combing line. But the condition $|\sigma(g, g)| \leq Mk/2$ implies

$$|\sigma(\sigma(w_i, w_i)(0), \sigma(w_{i+1}, w_{i+1})(0))| \leq \max \left( \frac{|\sigma(w_i, w_i)| + |\sigma(w_{i+1}, w_{i+1})|}{k}, \frac{M}{2} \right) \leq M,$$

and we have a representation of an element of $R$.

By (4), $|\sigma_{i,t,s}| \leq \max(4f(n/2)/k^2, M/2)$. If $|\sigma_{i,t,s}| + |\sigma_{i,t,s+1}| \leq M$ then the closed path $\tau_{i,t,s}$, using $\sigma_{i,t,s}$, $\sigma_{i,t,s+1}$ and the segments of length $\leq 1$ along $\sigma_{i,t}$ and $\sigma_{i,t+1}$, is of length $\leq M + 2$ and therefore represents an element in $R$. Otherwise, we break up further in the same manner using connecting bicombing paths of length $\leq \max(8f(n/2)/k^3, M/2)$, etc. until $2d f(n/2)/k^d \leq M/2$. In this way we find a van Kampen diagram for $w$. This proves that $G$ is finitely presented. $d$ can be estimated as the smallest integer greater or equal than $\log_{k/2}(2f(n/2)/M)$.

The isoperimetric inequality has the form:

$$\# \text{ (2–cells)} \leq n \cdot (f(n/2) + 1) \cdot 2(f(n/2) + 2) / k \cdots 2d^{-1}(f(n/2) + 2) / k^{d-1} \leq \frac{n(f(n/2) + 2) d^{2d(d-1)/2}}{k^{d(d-1)/2}} = O(\log n)$$

where $d$ is given as above.

\[\n\]

Remark: 1. Condition (1) is not necessary in order to prove that the presentation is finite.
2. The growth of the isoperimetric function is faster than polynomial but slower than exponential; therefore we call it sub-exponential.

**Theorem 2.2** Each group that has a bicombing in the sense of [7] has a bicombing of narrow shape.

**Proof:** By using the notation of the proof above, the bicombing in the sense of Short is a narrow bicombing with $|\sigma_{i,t}| \leq M/2$ and $f(n) = mn$ for a given constant $m \in \mathbb{N}$ and $d = 1$ in this case.  

**Theorem 2.3** Let $P = \langle X \mid R \rangle$ be a finite presentation for the group $G$ with a bicombing of narrow shape $\sigma$ and let $f$ be the polynomial from [7] bounding $|\sigma|$.  
1. There is a polynomial isodiametric function for $P$ of the same degree as $f$.  
2. If $\sigma$ is geodesic, then the isodiametric function is linear.

**Proof:** Let $w \in F$ be a reduced nontrivial word of length $n$, which is trivial in $G$, and let $D$ be the van Kampen diagram for $w$ constructed in the proof of theorem 2.1. One can reach every vertex in the diagram $D$ from the basepoint 1 by traveling part of a bicombing line $\sigma(w_i)$ of the first generation then traveling part of a bicombing line $\sigma_{i,t}$ of the second generation then part of a bicombing line $\sigma_{i,t,s}$ of the third generation etc. The length of a bicombing line of the $l$-th generation is $\leq 2^l f(n/2)/k^l$, and the sum of the lengths of successive generations of bicombing lines therefore is $\leq f(n/2)(1 + 2/k + (2/k)^2 + ... ) = f(n/2)k/(k-2)$. Hence $2^l f(n/2)$ is an isodiametric function for the presentation $P$. If $\sigma$ is geodesic, then $f$ is the identity and the above function is linear.  

The next theorem follows an idea of M. Bridson [8]. It shows that the definition of a bicombing of narrow shape cannot be sharpened.

**Theorem 2.4** Let $X$ be a finite generating set of the group $G$. Choose for every pair $g, h \in G$ a geodesic $\sigma(g, h) \in \Gamma_X(G)$. Then:

$$\forall x, y \in X^{\pm 1}, \forall g, h \in G, \quad |\sigma(\sigma(g, h)(t), \sigma(gx, hy)(t))| \leq (|\sigma(g, h)| + |\sigma(gx, hy)|)/2 + 1$$

holds for all integers $t \in [0, \infty[$.

**Proof:** Let $C = (|\sigma(g, h)| + |\sigma(gx, hy)|)/2$. If $t \leq C/2$, then following $\sigma(g, h)$ backwards from $\sigma(g, h)(t)$ to $g$ then one edge to $gx$ and then going to $\sigma(gx, hy)(t)$
along \( \sigma(gx, hy) \) gives a path of length at most \( C + 1 \). For \( t > C/2 \) follow \( \sigma(g, h) \) from \( \sigma(g, h)(t) \) to the vertex \( h \), then go one edge to \( hy \) and then to \( \sigma(gx, hy)(t) \) backwards along \( \sigma(gx, hy) \). This gives a path of length at most \( C \).

3 A class of Examples

Let \( P_q = \langle x, y, z \mid [x, y^q] = z, [x, z] = [y, z] = 1 \rangle \) be a presentation of the group \( G_q \) where \( q \geq 1 \) and \([a, b]\) denotes the commutator of \( a \) and \( b \). \( G_1 \) is the 3–dimensional integral Heisenberg group. Let \( F \) be the free group on \( \{x, y, z\} \).

Let \( w, v \in F \). If both words are equal in \( F \), we write \( w \equiv v \). If they are the same in \( G_q \), we write \( w = v \).

It is easy to see, that
\[
z^{jl} = x^j y^{ql} x^{-j} y^{-ql}
\]
holds in \( G_q \).

Lemma 3.1 (normal form for \( G_q \)) Let \( w \in F \). Then, for \( q > 1 \), there is a word
\[
\tau(w) \equiv y^s x^{r_1} y^{s_1} x^{r_2} \ldots y^{s_{m-1}} x^{r_m} y^p z^n \in F
\]
with \( r_i, s_i \neq 0 \) and
\[
\text{for } q \text{ even: } s, s_i \in \{-q/2 + 1, \ldots, q/2\},
\]
\[
\text{for } q \text{ odd: } s, s_i \in \{-(q-1)/2, \ldots, (q-1)/2\},
\]
and, for \( q = 1 \), there is a word
\[
\tau(w) \equiv x^r y^p z^n \in F
\]
such that \( \tau(w) = w \) in \( G_q \) and for all \( v \in F \) with \( w = v \) in \( G_q \), \( \tau(w) \equiv \tau(v) \).

Proof: The case \( q = 1 \) is trivial. For \( q > 1 \) it is easy to see that each word \( w \in F \) can be transformed into \( \tau(w) \) using the relations of \( P_q \). In order to prove uniqueness, let \( w \) and \( v \) be two words in \( F \) representing the same element in \( G_q \). Let \( H_q = G_q/\langle \langle z \rangle \rangle \), where \( \langle \langle z \rangle \rangle \) denotes the normal closure of \( z \) in \( G_q \). \( T_q = \langle x, y \mid xy^qx^{-1} = y^q \rangle \) is a presentation for \( H_q \), which is an HNN–extension. Therefore \( w \) and \( v \) have the same normal form (see \( [3] \) \( \tau'(w) = \tau'(v) \) in \( H_q \) which is equal to the normal form in \( G_q \), except that \( n = 0 \). Since \( z \) is central, \( \tau(w) \) and \( \tau(v) \) can only differ by a power of \( z \). But \( z \) has infinite order in \( G_q \) which implies \( \tau(v) \equiv \tau(w) \).
The normal forms (6) and (7) define a path \( \sigma(w) \) from 1 to \( w \) in the Cayley graph \( \Gamma_X(G_q) \) of \( G_q \) for every \( w \in F \). Define paths \( \sigma(g, h) \) by taking equivariant lines; define
\[
\sigma(g, h)(t) := g \cdot \sigma(1, g^{-1}h)(t) = g \cdot \sigma(g^{-1}h)(t) \quad \forall g, h \in G_q
\]

**Theorem 3.2** The paths \( \sigma(g) \) are recursive (i.e. \( |\sigma(g)| \leq f(d(1, g)) \)) with a function \( f(x) = 2x^2 + 3x \) for \( q > 1 \) and \( f(x) = x^2 + x \) for \( q = 1 \).

**Proof:** The relations in \( P_q \) say that \( z \) commutes with \( x \) and \( y \), in particular any power of \( z \) can be shifted to any place in a given word, and that \( x \) commutes with \( y^q \) at the expense of introducing \( z \) or \( z^{-1} \).

For \( q > 1 \), let \( w = \sigma(g) = y^q x^{r_1} y^{s_1} x^{r_2} \ldots y^{s_{m-1}} x^{r_m} y^n z^n \in F \) be the normal form for \( g \). We observe first that
\[
d(1, g) \geq \sum_{i=1}^{m} |r_i| + \sum_{i=1}^{m-1} |s_i| + |s| \quad (9)
\]
This is due to the fact that the exponents of the \( y \)-powers which occur in \( w \) can only be changed by adding multiples of \( q \) (The relations (6) allow to permute powers of \( x \) with powers of \( y^q \)). However, the range for \( s_i \) and \( s \) in the normal form \( w \) is such that \( |s_i| \) and \( |s| \) can not decrease under these changes. The same argument also shows that
\[
d(1, g) \geq \sum_{i=1}^{m} |r_i| + \sum_{i=1}^{m-1} |s_i| + |s| + \max\{ |p| - (\sum |s_i| + |s|), 0 \}, \text{ which implies:}
\]
\[
d(1, g) \geq |p| \quad (10)
\]
Therefore \( \sum_{i=1}^{m} |r_i| + \sum_{i=1}^{m-1} |s_i| + |s| + |p| \leq 2d(1, g) \). In order to prove \( |w| = \sum_{i=1}^{m} |r_i| + \sum_{i=1}^{m-1} |s_i| + |s| + |p| + |n| \leq f(d(1, g)) = 3d(1, g) + 2d^2(1, g) \), we only need to show that \( |n| \leq d(1, g) + 2d^2(1, g) \):

We claim that
\[
d(1, g) \geq \sum_{i=1}^{m} |r_i| + \sum_{i=1}^{m-1} |s_i| + |s| + \min \left\{ \max \left[ |n| - \left( \sum_{i=1}^{m} |r_i| + |r| \right) \left( \sum_{i=1}^{m} |s_i| + |s| + |p| \right) / q + |l|, 0 \right] + 2|r| + 2q|l| \right\}
\]
where the minimum ranges over \( |r| \) and \( |l| \). If \( |n| \leq (\sum_{i=1}^{m} |r_i|)(\sum_{i=1}^{m-1} |s_i| + |s| + |p|) / q \), the minimum term on the right hand will be 0 and the inequality holds by (6). If \( |n| > (\sum_{i=1}^{m} |r_i|)(\sum_{i=1}^{m-1} |s_i| + |s| + |p|) / q \) we observe first that \( |n| \) may decrease by at most \( |k| |l| \) if a power \( y^k \) is pushed across a power \( x^k \) in \( w \).
If we do not introduce new powers of $x$ or $y^q$ by inserting $x^r x^{-r}$ or $y^q y^{-q}$ into the word, the amount by which $|n|$ may be decreased by means of permuting powers of $x$ with powers of $y^q$ is clearly bounded by $\sum_{i=1}^m |r_i| (\sum_{i=1}^{m-1} |s_i| + |s| + |p|)/q$. This coarse estimate stems from the following fact: Among all words in $x$ and $y$ whose sum of absolute values of $x$-exponents and sum of absolute values of $y$-exponents is the same as for $w$, $y \sum |s_i| + |s| + |p| \cdot x \sum |r_i|$ can absorb the largest powers $z^n$ or $z^{-n^q}$ by permuting powers of $x$ with powers of $y^q$.

If we prolong the word by inserting $x^r x^{-r}$ and $y^q y^{-q}$ at suitable places, the amount by which $|n|$ can be decreased by means of $\sum_{i=1}^m |r_i| + |r_i| (\sum_{i=1}^{m-1} |s_i| + |s| + |p|)/q + |l|]$; and, at the same time, the length of the $x$-part of the word increases by $2|r| + 2q|l|$. This explains inequality (11).

Now, let $|r_0|$ and $|l_0|$ be the values for $|r|$ and $|l|$ for which the minimum occurs in (11). Then $d^2(1, g) \geq (\sum_{i=1}^m |r_i| + 2|r_0|) (\sum_{i=1}^{m-1} |s_i| + |s| + 2|l_0|)$, and, by (11), $d^2(1, g) \geq (\sum_{i=1}^m |r_i| + 2|r_0|) |p|$ which implies $2d^2(1, g) \geq (\sum_{i=1}^m |r_i| + |r_0|) (\sum_{i=1}^{m-1} |s_i| + |s| + |p|)/q + |l_0|$. Therefore, by (11) again, $|n| \leq d(1, g) + 2d^2(1, g)$ which proves the Theorem for $q > 1$.

For $q = 1$ the proof is similar, but easier. Let $\sigma(g) \equiv x^r y^s z^n$. It is clear that $d(1, g) \geq |r| + |s|$. If $|n| \leq |r| + |s|$, then $d(1, g) + d(1, g)^2 \geq |\sigma(g)|$; if $|n| > |r| + |s|$, then, by the same ideas as in the proof for $q > 1$, $d(1, g) \geq |r| + |s| + \min \{ \max \{|n| - (|r| + |r'|)(|s| + |s'|), 0\} + 2|r' + 2|s'| \}$ where the minimum ranges over the values of $|r'|$ and $|s'|$. Let $|r_0'|$ and $|s_0'|$ be the values for which the minimum occurs, then $|n| \geq d(1, g) + (|r| + |r_0'|)(|s| + |s_0'|) \geq d(1, g) + d^2(1, g)$.

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**Theorem 3.3** $\sigma(g, h)$ defines a bicombing of narrow shape with constants $M = 24q + 18$ and $k = 11/5$.

**Proof**: Recall that a recursive $\sigma$ is of narrow shape, if there exists an integer $M > 1$ and a real $k > 2$, such that for all $g, h \in G$ and $a, b \in X^{\pm 1} \cup \{1\}$

$$|\sigma(\sigma(g, h)(t), \sigma(ga, bh)(t))| \leq \max(|\sigma(g, h)| + |\sigma(ga, bh)|)/k, M/2)$$

holds for all integers $t \in [0, \infty[$. Since the bicombing is equivariant, it suffices to show this inequality for $g = 1$.

For $q > 1$ let $v \in F$ be in normal form $v \equiv y^s x^{r_1} y^{s_1} x^{r_2} \ldots y^{s_{m-1}} x^{r_m} y^p z^n$, such that $v = h$ in $G_q$ ($\sigma(1, h) \equiv v$). Let $w$ be the group element $a^{-1} v b$ brought into normal form ($\sigma(a, v b) \equiv w$) (see fig. 6).

Now calculate the length of the bicombing lines (the *combing distance*) between these two paths $w, v$ in $\Gamma_X(G_q)$. Call the maximal combing distance between two such paths $\delta(\sigma, w, v)$. 

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If \( a = 1 \) and \( b \in \{1, z^\pm 1\} \), then \( \delta(\sigma, w, v) \leq 1 \). If \( a = 1 \) and \( b \in \{y^{\pm 1}\} \) then 
\( \delta(\sigma, w, v) = 2 \).

If \( a = 1 \) and \( b \in \{x^\epsilon \} (\epsilon = \pm 1) \), then \( \delta(\sigma, w, v) \leq |l| + q + 1 \), where \( l \) is such that 
\(-q/2 + 1 \leq p - lq \leq q/2 \) for \( q \) even and 
\(-q - 1/2 \leq p - lq \leq q - 1/2 \) otherwise.

To see this, observe that \( \nu \) ends with \( y^{p} z^{n} \) but \( w \) ends with \( x^{l} y^{q} z^{n-l} \). Since
\(|w| + |v| \geq 2q|l| \) we get for \( q \geq 2 \) and \( \delta(\sigma, w, v) > M/2 \): 
\(|w| + |v|)/k > \delta(\sigma, w, v).

There are a few more cases which are relatively easy. The most critical case
which requires the sharpest estimates occurs if \( a = y^{s}, b = x^{\alpha} \) with \( \alpha, \epsilon \in \{\pm 1\} \);
in particular if \( y^{s} \) is at the boundary of its range to which it is restricted by the normal form, and the premultiplication by \( a^{-1} = y^{-\epsilon} \) moves it out of this range, as, for example, in the case \( \epsilon = -1, s = q/2 \) and \( q \) even (the other cases can be treated similarly).

In this case \( v \equiv y^{q/2} x^{r_{1}} y^{s_{1}} x r_{2} \ldots y^{s_{m-1}} x r_{m} y^{p} z^{n} \) and
\( w \equiv y^{-q/2 + 1} x^{r_{1}} y^{s_{1}} x r_{2} \ldots y^{s_{m-1}} x r_{m} y^{l} x^{\alpha} y^{(l+1)} q^{z^{n}} - \sum r_{i} - \alpha(l+1), \)
where \( l \) is as above. Using the rule \(|a| + |a - b| \geq |b| \) we obtain the estimate:
\(|w| + |v| \geq 2q|l| + 2 \sum |r_{i}| + |\sum r_{i} + \alpha(l + 1)|. A careful study of the lengths of
the combing distances shows that
\[ \delta(\sigma, w, v) \leq \max \{ \sum_{i=1}^{m} |r_{i}| + |l| + 3q + 2, \} \leq \frac{2}{|l|} \]
\[ \sum_{i=1}^{m} |r_{i}| + |l| + 3q + 2. \]

Since \( q \geq 2 \) and \( k = 11/5 \), \((|w| + |v|)/k \geq 20|l|/11 + 10 \sum |r_{i}|/11 + 5| \sum r_{i} + \alpha(l + 1)/11 \). We will show that the right hand side is \( \geq (\sum_{i=1}^{m} |r_{i}| + |\sum_{i=1}^{m} r_{i}|)/2 + |l| + 3q + 2 \) whenever \( \delta(\sigma, w, v) \geq M/2 \) (which, by the above estimate for \( \delta(\sigma, w, v) \),
proves the Theorem for this case). This is equivalent to:
\[ 9|l| + 10 \sum |r_{i}| + 5| \sum r_{i} + \alpha(l + 1)| \geq 11 \sum |r_{i}|/2 + 11| \sum r_{i}|/2 + 33q + 22. \]

The left hand side can be simplified by the following estimates: 
\( 5|l| + 5| \sum r_{i} + \alpha + \alpha l \geq 5 \sum r_{i}, \alpha \geq 5 \sum r_{i} - 5 \), and 
\( 10 |r_{i}| + 5 \sum r_{i} \)
Therefore the above inequality follows from \(4(|l| + \sum |r_i|) \geq 33q + 27\), which follows from \(\delta(\sigma, w, v) > M/2\) using the value \(M = 24q + 18\) and the estimate \(\delta(\sigma, w, v) \leq (\sum |r_i| + \sum r_i)/2 + |l| + 3q + 2 \leq \sum |r_i| + 3q + 2\).

The proof for \(q = 1\) is much simpler and left to the reader. 

In the following we use Cockcroft 2-complexes to get lower bounds for isoperimetric functions. This idea is due to S. Gersten [4].

**Theorem 3.4** \(G_q\) has no quadratic isoperimetric inequality and therefore no combing in the sense of Short [7].

**Proof:** There is a van Kampen diagram for \(w_n \equiv [x^n, y^m] \cdot [y^{-m}, x^{-n}]\) in \(G_q\), which has \(n^3\) more 2-cells \([x, z]\) of positive then of negative type. W. A. Bogley proved in [2], that the corresponding 2-complex is Cockcroft. So each \(\pi_2\)-element has the same number of positive as of negative 2-cells \([x, z]\), which proves that every van Kampen diagram for \(w_n\) will contain at least \(n^3\) 2-cells \([x, z]\) and so proves the theorem.

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