ON THE CHENG-YAU GRADIENT ESTIMATE FOR CARNOT GROUPS AND SUB-RIEMANNIAN MANIFOLDS

FABRICE BAUDOIN⋆, MARIA GORDINA†‡, AND PHANUEL MARIANO†

Abstract. In this note we show how results in [4, 6, 11] yield the Cheng-Yau estimate on two classes of sub-Riemannian manifolds: Carnot groups and sub-Riemannian manifolds satisfying a generalized curvature-dimension inequality.

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1. Introduction

Let $M$ be a $d$-dimensional Riemannian complete non-compact manifold with the Ricci curvature bounded below by $-(d-1)K$. Let $u$ be a positive harmonic function in a Riemannian ball $B(x_0, 2r)$, then we say that $u$ satisfies the Cheng-Yau estimate if

\begin{equation}
\sup_{B(x_0,r)} |\nabla \log u| \leq C_d \left( \frac{1}{r} + \sqrt{K} \right),
\end{equation}

where $C_d$ is a global constant depending only on the dimension $d$. In particular, when $K = 0$ this estimate shows that positive harmonic functions are constant.

This estimate was formulated in a more general form in [10, 28], and stated as in (1.1) in [24] Theorem 3.1. Sharp versions of the Cheng-Yau inequality were given...
in [21, 22]. The stability of (1.1) under certain perturbations of the metric was considered in [29].

The standard curvature arguments are not easily available in the case when \( M \) is replaced by a sub-Riemannian manifold. Nevertheless, there has been significant progress in geometric analysis on sub-Riemannian manifolds in [1, 4, 5, 7]. Even in the absence of a Riemannian structure, [7] developed new techniques to prove a number of results which in the Riemannian setting go back to the work of Yau and Li-Yau. The main tool in [7] relied on a generalized curvature-dimension inequality on a class of sub-Riemannian manifolds with transverse symmetries.

Carnot groups also form a large and interesting class of sub-Riemannian manifolds. These are Lie groups whose Lie algebra admits a stratified structure. This stratified structure allows for Hörmander’s condition [18, Theorem 1.1] to be satisfied. Hörmander’s theorem guarantees that the sub-Laplacian associated with the structure of a Carnot group is hypoelliptic. In particular, this gives us the existence of a smooth heat kernel for this Laplacian. Most Carnot groups do not satisfy a generalized curvature-dimension inequality, so one needs to employ different techniques than in [7].

The Cheng-Yau estimate was proved in [2, Corollary 4.6] for the simplest non-commutative Carnot group, the Heisenberg group, using probabilistic (coupling) techniques. The purpose of this note is to show that this estimate can be proven on two classes of sub-Riemannian manifolds, namely, sub-Riemannian manifolds satisfying a generalized curvature-dimension inequality and Carnot groups, by relying on results from [4, 6, 11]. In particular, this recovers the known fact that global non-negative harmonic functions in these two settings have to be constant (see [8, Theorem 5.8.1] and [5, Theorem 5.1] in more generality).

2. Carnot Groups

2.1. Preliminaries. We recall that a Carnot group of step \( N \) is a simply connected Lie group \( G \) whose Lie algebra can be written as

\[ \mathfrak{g} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_N, \]

where

\[ [\mathcal{V}_i, \mathcal{V}_j] = \mathcal{V}_{i+j} \]

and \( \mathcal{V}_k = 0 \) for \( k > N \). In particular, Carnot groups are nilpotent.

Let \( \mathcal{V}_1, \ldots, \mathcal{V}_d \) be a linear basis for the vector space \( \mathcal{V}_1 \). The \( \mathcal{V}_i \)s can be viewed as left-invariant vector fields on \( G \). The left-invariant sub-Laplacian on \( G \) is the operator

\[ L = \sum_{i=1}^{d} \mathcal{V}_i^2. \]

Let \( \mu \) be the bi-invariant Haar measure on \( G \). Since Carnot groups are complete the operator \( L \) in (2.2) is essentially self-adjoint on \( L^2(G, \mu) \), with domain being the space of smooth and compactly supported functions \( f : G \to \mathbb{R} \) denoted by \( C_c^\infty(G) \).

We abuse notation and denote by \( L \) the Friedrichs extension of this operator to a unique non-positive self-adjoint operator on \( L^2(G, \mu) \). Then the heat semigroup \( (P_t)_{t \geq 0} \) on \( G \) can be defined through the spectral theorem. As \( L \) is hypoelliptic,
Denote by $\nabla = (V_1, \ldots, V_d)$ the gradient determined by this basis, and denote by $\| \cdot \|$ the usual Euclidean norm. The carré du champ operator of $L$ is defined by
\[
\Gamma(f, f) := \frac{1}{2} (Lf^2 - 2fLF) = \|\nabla f\|^2 = \sum_{i=1}^d (V_i f)^2,
\]
and it is often thought of as the square of the length of the gradient $\nabla$. We let $d$ be the Carnot-Carathéodory distance on $G$ making $(G, d)$ a metric space. We refer the reader to [8] for more details and results on Carnot groups.

2.2. The Cheng-Yau estimate. We say a function $u : G \to \mathbb{R}$ is harmonic in a domain $D \subset G$ if $Lu = 0$ on $D \subset G$.

**Theorem 2.1.** If $u$ is any positive harmonic function for $L$ in a ball $B(x, 2r) \subset G$, then there exists a constant $C > 0$ not dependent on $u, r$ and $x$ such that
\[
\sup_{B(x, r)} \|\nabla \log u\| \leq \frac{C}{r}.
\]
Moreover, if $u$ is a positive harmonic function on $G$, then $u$ must be equal to a constant.

**Proof.** In the proof, $C$ will denote a generic positive constant that does not depend on $u, r$ and $x$ whose value might change from line to line. First recall the reverse Poincaré inequality for the heat semigroup obtained for Carnot groups in [4, Proposition 2.5]
\[
\|\nabla P_t f\|^2 \leq \frac{C}{t} \left( P_t f^2 - (P_t f)^2 \right)
\]
for functions $f \in C^\infty_c(G)$. Note that for functions $f \in L^\infty(G)$ we have
\[
P_t f^2 \leq \|f\|_{L^\infty}^2,
\]
therefore combining (2.4) and (2.5) we obtain
\[
\|\nabla P_t f\|^2 \leq \frac{C}{t} \|f\|_{L^\infty}^2.
\]
Taking a square root in (2.6) implies that
\[
\|\nabla P_t\|_{\infty \to \infty} \leq \frac{C}{\sqrt{t}}.
\]
Applying [11] Theorem 1.2], in particular that (iii) implies (i), shows that (2.7) implies that there exists a $C > 0$ such that, for every ball $B(x, r)$ and every function $u$ that is harmonic in $B(x, 2r)$ we have
\[
\|\nabla u\|_{L^\infty(B(x, r))} \leq \frac{C}{r \mu(B(x, 2r))} \int_{B(x, 2r)} |u| \, d\mu.
\]
Then we can apply [11] Lemma 2.3 to show that (2.8) implies the Cheng-Yau estimate (2.3).

We rely on results in [11] that require several assumptions that are satisfied for Carnot groups as follows. Their first assumption is that the underlying space is
a non-compact doubling Dirichlet metric measure space. The space \( G \) is doubling since by \([16, \text{Proposition 11.15}] \) (or more classically by \([14, 20]\)) there exists a \( C > 0 \) independent of \( x \in G, r > 0 \) such that \( |B(x, r)| = Cr^Q \) where \( Q = \sum_{j=1}^{N} j \dim(V_j) \) is the homogeneous dimension of \( G \). Here \( |E| \) denotes the Lebesgue measure of the set \( E \) and recall that the Haar measure \( \mu \) is Lebesgue measure up to a constant. We also have that \( G \) is Dirichlet space as described in \([25, \text{Section 3}, \text{pp.233-234}] \). This is actually true in general for Hörmander’s type operators with bounded measurable coefficients on Lie groups having polynomial volume growth in the sense of \([23]\). Another ingredient in \([11]\) is upper Gaussian bounds on the heat kernel which follow from \([27, \text{Theorem VIII2.9}] \). The space also supports a local scale-invariant \( L^2\)-Poincaré inequality by \([16, \text{Proposition 11.17}] \).

3. Sub-Riemannian manifolds

3.1. Preliminaries. In this section we study the setting similar to \([5]\). We state relevant details here for completeness. Let \((M, \mu)\) be a measure space, where \( M \) is an \( n \)-dimensional \( C^\infty \) connected manifold endowed with a smooth measure \( \mu \). Recall (e.g. \([26, \text{p.85}] \)) that the measure \( \mu \) on a smooth manifold \( M \) is called a smooth measure if \( \mu \) is a Radon measure which has a smooth Radon-Nikodym derivative with respect to the Lebesgue measure when viewed in coordinates, that is, for any smooth coordinate chart \( \varphi : U \to V, U \subset M, V \subset \mathbb{R}^n \), the pushforward measure \( \varphi_* (\mu) \) has a smooth Radon-Nikodym derivative with respect to the Lebesgue measure on \( V \). Let \( L \) be a second order diffusion operator on \( M \) which is locally subelliptic (in the sense of \([13,19]\)). We refer to \([3, \text{Section 1}] \) for a detailed account on properties of locally subelliptic operators and associated distances that we are going to use in the sequel. In addition, we assume that

\[
L1 = 0,
\]

\[
\int_M f L g d\mu = \int_M g L f d\mu,
\]

\[
\int_M f L f \leq 0
\]

for every \( f, g \in C^\infty_c(M) \), where as before \( C^\infty_c(M) \) denotes the space of smooth compactly supported functions on \( M \).

The space \( M \) is endowed with a carré du champ operator defined by

\[
\Gamma(f, g) := \frac{1}{2} (L(fg) - fLg - gLf), \quad f, g \in C^\infty(M).
\]

We denote \( \Gamma(f) = \Gamma(f, f) \). It is not too hard to see that \( \Gamma(f) \geq 0 \) for all \( f \in C^\infty(M) \). We will also assume the existence of a symmetric, first-order differential bilinear form \( \Gamma^Z : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \) that satisfies

\[
\Gamma^Z(f, g, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h),
\]

\[
\Gamma^Z(f) = \Gamma^Z(f, f) \geq 0,
\]
for all \( f, g, h \in C^\infty(M) \). Given the first order bi-linear forms \( \Gamma \) and \( \Gamma^Z \) on \( M \), we can introduce the following second-order differential forms

\[
\Gamma_2(f, g) = \frac{1}{2} (L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf))
\]

and

\[
\Gamma^Z_2(f, g) = \frac{1}{2} (L\Gamma^Z(f, g) - \Gamma^Z(f, Lg) - \Gamma^Z(g, Lf)) .
\]

Similar to \( \Gamma \), we will use the notation \( \Gamma_2(f) := \Gamma_2(f, f), \Gamma^Z_2(f) := \Gamma^Z_2(f, f) \).

We suppose the following assumptions to hold throughout this section.

(I) There exists an increasing sequence \( h_k \in C^\infty_c(M) \) such that \( h_k \uparrow 1 \) on \( M \), and

\[
\|\Gamma(h_k)\|_\infty + \|\Gamma^Z(h_k)\|_\infty \to 0, \text{ as } k \to \infty.
\]

(II) For any \( f \in C^\infty(M) \) one has

\[
\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)) .
\]

(III) The generalized curvature-dimension inequality \( \text{CD}(\rho_1, \rho_2, \kappa, d) \) is satisfied with \( \rho_1 \geq 0 \). That is, there exist constants \( \rho_1 \geq 0, \rho_2 > 0, \kappa \geq 0 \), and \( d \geq 2 \) such that the following inequality holds

\[
\Gamma_2(f) + \nu \Gamma^Z_2(f) \geq \frac{1}{d} (Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right) \Gamma(f) + \rho_2 \Gamma^Z(f),
\]

for all \( f \in C^\infty(M) \) and every \( \nu > 0 \).

(IV) The heat semigroup generated by \( L \), which will be denoted \( P_t \), is stochastically complete, that is, for \( t \geq 0, P_t1 = 1 \) and for every \( f \in C^\infty_c(M) \) and \( T \geq 0 \), one has

\[
\sup_{t \in [0, T]} \|\Gamma(P_t f)\|_\infty + \|\Gamma^Z(P_t f)\|_\infty < +\infty.
\]

(V) Given any two points \( x, y \in M \), there exists a subunit curve (in the sense of \([12]\)), joining them.

(VI) The metric space \( (M, d) \) is complete with respect to the intrinsic distance defined by

\[
d(x, y) := \sup \{|f(x) - f(y)| : f \in C^\infty(M), \|\Gamma(f)\|_\infty \leq 1\},
\]

for all \( x, y \in M \) and where we define \( \|g\|_\infty = \text{ess sup}_{M} |g| \).

Note that by \([9\text{, Lemma 5.29}]\) and \([3\text{, Equation (2.4)}]\) we know that Assumption \( (V) \) implies that \( d \) is indeed a metric on \( M \).

As a consequence of Assumption \( (VI) \), the operator \( L \) is essentially self-adjoint on \( C^\infty_c(M) \) (e.g., \([3\text{, Proposition 1.20, Proposition 1.21}]\)). Thus, the pre-Dirichlet form \( E(f, g) \) defined on \( C^\infty_c(M) \) by

\[
E(f, g) = \int_M \Gamma(f, g)d\mu,
\]

has a unique closure as a Dirichlet form, and the generator of this Dirichlet form is the Friedrichs extension of \( L \). We define the Sobolev space \( W^{1,2}(M) \) to be the domain \( D \) of \( E \) with the norm on \( W^{1,2}(M) \) given by

\[
\|f\|_{W^{1,2}(M)} = \sqrt{\|f\|_2^2 + E(f, f)}.
\]

It is a consequence of Assumption \( (III) \) that the metric measure space \( (M, d, \mu) \) satisfies the volume doubling property and supports a scale invariant \( L^2\)-Poincaré
inequality on metric balls (see [5]). In particular, by [17, Chapter 8] locally Lipschitz continuous functions form a dense subclass in $W^{1,2}(\mathcal{M})$.

For an open set $U \subset \mathcal{M}$, one can define the local Sobolev space $W^{1,2}_{\text{loc}}(U)$ to be the space of all functions $f$ such that for any compact set $K \subset U$ there exists $F \in \mathcal{D}$ satisfying $f = F$ a.e. on $K$. For each $p \geq 2$, we define $W^{1,p}(U)$ to be the space of functions $f \in W^{1,2}_{\text{loc}}(U)$ satisfying $f, \sqrt{\Gamma(f)} \in L^p(U)$.

### 3.2. Examples

We remark that so far the approach has been purely analytical as we have not mentioned any geometric structure of these sub-Riemannian manifolds. In fact $\mathcal{M}$ and $L$ are rather general, even though we have sub-Riemannian manifolds in mind for $\mathcal{M}$.

#### 3.2.1. Sum of squares operators

We start by recalling a natural setting where assumption [V] is satisfied. Let us consider $L$ that are sums of squares operators in the form of

$$L = \sum_{i=1}^{m} X_i^2 + X_0, \quad (3.10)$$

where the $X_i$ are $C^\infty$ vector fields. We refer the reader to [15] for more details on operators of the form given in (3.10) in the context of sub-Riemannian manifolds. Consider the following assumption.

**Assumption 3.1.** (Hörmander’s condition) We will say that $L$ satisfies Hörmander’s (bracket generating) condition if the vector fields $\{X_1, \ldots, X_m\}$ with their Lie brackets span the tangent space $T_x \mathcal{M}$ at every point $x \in \mathcal{M}$.

Hörmander’s condition guarantees analytic and topological properties such as hypoellipticity of $L$ and topological properties of $\mathcal{M}$. The Chow-Rashevski theorem says that Hörmander’s condition is sufficient to ensure that any two points in $\mathcal{M}$ can be connected by a finite length sub-unit curve. Thus, operators $L$ of the form (3.10) that satisfy Hörmander’s condition automatically satisfy assumption [V].

#### 3.2.2. Other examples

We note that the assumptions (I)-(VI) are satisfied for a large class of sub-Riemannian manifolds. Such a class includes all Sasaki manifolds whose horizontal Webster-Tanaka-Ricci curvature is bounded below, a wide subclasses of principal bundles over Riemannian manifolds whose Ricci curvature is bounded below, and Carnot groups of step 2. We remark that in general we do not know if $CD(\rho_1, \rho_2, \kappa, d)$ is satisfied for Carnot groups of an arbitrary step. This shows the need to treat Carnot groups separately in Theorem 2.1. We refer the reader to [7] for a comprehensive treatment on sub-Riemannian manifolds satisfying assumptions (I)-(VI).

### 3.3. The Cheng-Yau estimate

Recall that we say a function $u : \mathcal{M} \to \mathbb{R}$ is harmonic in a domain $D \subset \mathcal{M}$ if $Lu = 0$ on $D \subset \mathcal{M}$. We can now state the main result of this section.

**Theorem 3.2.** Suppose assumptions (I)-(VI) hold for $\mathcal{M}$ and $L$. If $u$ is any positive harmonic function for $L$ in a ball $B(x, 2r) \subset \mathcal{M}$, then there exists a constant $C > 0$ not dependent on $u, r$ and $x$ such that

$$\sup_{B(x,r)} \sqrt{\Gamma(\log u)} \leq \frac{C}{r}. \quad (3.11)$$
Moreover, if $u$ is any positive harmonic function on $\mathbb{M}$, then $u$ must be equal to a constant.

**Proof.** In the proof, $C, C_2$ will denote generic positive constants that do not depend on $u, r$ and $x_0$, whose values might change from line to line. First we check that the assumptions of the results in [11, Theorem 1.2, Lemma 2.3] are satisfied. The results in [11] require the assumption that the underlying space is a non-compact doubling Dirichlet metric measure space. Doubling is shown in (1.11) of [5, Theorem 1.5]. Further, we need $\mathbb{M}$ to satisfy upper Gaussian bounds on the heat kernel, which is given in [5, Theorem 4.1]. Finally we need $\mathbb{M}$ to support a local $L^2$-Poincaré inequality of the form

\begin{equation}
\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f_B| \, d\mu \leq C_2 r \left( \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \Gamma(f) \, d\mu \right)^{1/2},
\end{equation}

for some $C_2 > 0$, for every ball $B(x,r)$ and each $f \in W^{1,2}(B(x,r))$. Here we used the notation $f_B = \mu(B)^{-1} \int_B f \, d\mu$. To see this, by (1.12) in [5, Theorem 1.5] we have that there exists constant $C_2 > 0$, depending only on $\rho_1, \rho_2, \kappa, d$, for which one has for every $x \in \mathbb{M}$ and every $r > 0$,

\begin{equation}
\int_{B(x,r)} |f - f_B|^2 \leq C_2 r^2 \int_{B(x,r)} \Gamma(f) \, d\mu,
\end{equation}

for every $f \in C^1(\overline{B(x,r)})$. Using Cauchy-Schwarz inequality, followed by (3.13) we have that

\begin{align*}
\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f_B| \, d\mu &\leq \frac{1}{\mu(B(x,r))} \left( \int_{B(x,r)} |f - f_B|^2 \, d\mu \right)^{1/2} \sqrt{\mu(B(x,r))} \\
&\leq \frac{1}{\sqrt{\mu(B(x,r))}} \left( C_2 r^2 \int_{B(x,r)} \Gamma(f) \, d\mu \right)^{1/2} \\
&= \sqrt{C_2} r \left( \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \Gamma(f) \, d\mu \right)^{1/2}.
\end{align*}

This shows (3.12) holds for all $f \in C^1(\overline{B(x,r)})$. By a density argument we can show that (3.12) holds for all $f \in W^{1,2}(B(x,r))$, as needed.

To finish off the proof, we note that Corollary 3.5 in [6] shows that

\begin{equation}
\left\| \sqrt{\Gamma(P_t f)} \right\|_{\infty \to \infty} \leq \frac{C}{\sqrt{t}},
\end{equation}

where $C = n \sqrt{\frac{(2\kappa + \rho_2)}{2\rho_2}}$. Using the estimate (3.14), the rest of the proof becomes similar to the proof of Theorem 2.1.

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Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA
E-mail address: fabrice.baudoin@uconn.edu

Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA
E-mail address: maria.gordina@uconn.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA
E-mail address: pmariano@purdue.edu