The Cauchy interlace theorem for symmetrizable matrices

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Abstract

Symmetrizable matrices are those which are symmetric when multiplied by a diagonal matrix with positive entries. The Cauchy interlace theorem states that the eigenvalues of a real symmetric matrix interlace with those of any principal submatrix (obtained by deleting a row-column pair of the original matrix). In this paper we extend the Cauchy interlace theorem for symmetric matrices to this large class, called symmetrizable matrices. This extension is interesting by the fact that in the symmetric case, the Cauchy interlace theorem together with the Courant-Fischer minimax theorem and Sylvester’s law of inertia, each one can be proven from the others and thus they are essentially equivalent. The first two theorems have important applications in the singular value and eigenvalue decompositions, the third is useful in the development and analysis of algorithms for the symmetric eigenvalue problem. Consequently various and several applications whom are contingent on the symmetric condition may occur for this large class of not necessary symmetric matrices and open the door for many applications in future studies. We note that our techniques are based on the celebrated Dodgson’s identity [4].

Keywords: Symmetrizable matrices, The Cauchy interlace theorem, Eigenvalues.

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1 Introduction.

The Cauchy interlace theorem states that the eigenvalues of a real symmetric matrix of order m interlace with those of any principal submatrix of order m - 1. The idea behind to extend this theorem to symmetrizable matrices came when we calculated, explicitly, the eigenvalues of a class of m order Tridiagonal, Pentadiagonal and Heptadiagonal matrices (see [10], [12] and [11]) whom are not necessary symmetric but symmetrizable. We observed that their eigenvalues
interlace with those of their corresponding m-1 order principal submatrices. We begin by presenting a short and simple proof of the theorem based on the well known Dodgson’s Algorithm, then we prove its extension. At our knowledge, this theorem is applicable only to symmetric matrices and our aim is not to say that our result is new but we hope to provide the reader with another technique to prove the Cauchy interlace theorem.

Proofs of this well known theorem have been based on Sylvester’s law of inertia \[13\], the Courant-Fischer minimax theorem (see \[7\], \[8\], \[9\] and \[10\]) and others more simple are based on some properties of polynomials \[5\].

2 The Cauchy interlace theorem for real symmetric matrices.

The following theorem presents the more simplified form of the Cauchy interlace theorem \[3\].

**Theorem 1** (*The Cauchy interlace theorem*) *If a row-column pair is deleted from a real symmetric matrix, then the eigenvalues of the resulting matrix interlace those of the original one.*

**Proof.** The celebrated Dodgson’s identity \[4\] states, for a square matrix \[A = (a_{i,j})_{1 \leq i,j \leq m}\], the following

\[
\det \left[ (a_{i,j})_{1 \leq i,j \leq m} \right] \det \left[ (a_{i,j})_{i \neq k, j \neq k} \right] = \\
\det \left[ (a_{i,j})_{i \neq l} \right] \det \left[ (a_{i,j})_{i \neq k} \right] - \det \left[ (a_{i,j})_{i \neq l} \right] \det \left[ (a_{i,j})_{i \neq k} \right], \tag{1}
\]

for all \( m > 2 \). If \( \lambda_1 < \lambda_2 < \ldots < \lambda_m \) lists the eigenvalues of \( A \), then by application of the above identity to the matrix \( A - \lambda_p I \) for some fixed \( 1 \leq p \leq m \), we deduce

\[
\det \left[ (a_{i,j})_{i \neq l} \right] - \lambda_p I_{m-1} \right] \det \left[ (a_{i,j})_{i \neq k} \right] - \lambda_p I_{m-1} \right] = \\
\det \left[ (a_{i,j})_{i \neq l} \right] - \lambda_p I_{m-1} \right] \det \left[ (a_{i,j})_{i \neq k} \right] - \lambda_p I_{m-1} \right]. \tag{2}
\]

Since the matrix \( A - \lambda_p I \) is symmetric, then

\[
\det \left[ (a_{i,j})_{i \neq l} \right] - \lambda_p I_{m-1} \right] = \det \left[ (a_{i,j})_{i \neq k} \right] - \lambda_p I_{m-1} \right], \tag{3}
\]

and this shows that all the principal minors of order \( m - 1 \) of the matrix \( A - \lambda_p I \) have the same sign for a fixed \( p = 1, \ldots, m \). By using a well known properties of
the characteristic polynomial of the matrix $A - \lambda I$ which has $p - 1$ negative eigenvalues, one null and $m - p$ positive, we deduce that the product $\prod_{p-1}$ of the non zero eigenvalues is equal to the sum of all its principal minors of order $m - 1$ whom have the same sign as $(-1)^{p-1}$. Then, for a fixed $1 \leq k \leq m$, the characteristic polynomial of the matrix $\left( (a_{i,j})_{i \neq k \land j \neq k} \right)$

$$P_k (\lambda) = \det \left( (a_{i,j})_{i \neq k \land j \neq k} - \lambda I_{m-1} \right),$$

which is of degree $m - 1$, satisfies the following inequalities

$$(-1)^{p-1} P_k (\lambda_p) > 0, \quad p = 1, \ldots, m. \quad (5)$$

This ends the proof of the Cauchy interlace theorem for a real symmetric matrix.

3 Results and Proofs.

To introduce the class of matrices (not necessary symmetric) for whom the Cauchy interlace theorem holds, we begin with the following

**Definition 2** A square matrix $A$ of order $m > 2$ is called symmetrizable if it is sign symmetric, i.e.

$$a_{ij} = a_{ji} = 0 \text{ or } a_{ij}a_{ji} > 0, \quad i \neq j = 1, 2, \ldots, m. \quad (6)$$

and if for all permutation $\sigma$ of the set of integers $\{1, 2, \ldots, m\}$, we have

$$\prod_{i=1}^{m} a_{i,\sigma_i} = \prod_{i=1}^{m} a_{\sigma_i, i}. \quad (7)$$

When $m = 2$, we simply suppose $a_{12}a_{21} > 0$ and when $m = 3$, by application of the definition, $A$ is symmetrizable if $(6)$ is satisfied and $a_{12}a_{23}a_{31} = a_{21}a_{32}a_{13}$; since the other equalities $(7)$ are automatically satisfied. For example, all the matrices

$$A = \begin{pmatrix} a & -3 & 1 \\ -6 & b & 4 \\ 5 & 10 & c \end{pmatrix},$$

are symmetrizable for all real numbers $a$, $b$ and $c$.

In [6], Lemma 3.2, the authors characterized symmetrizable matrices as follows

**Proposition 3** A square matrix of order $m$ is symmetrizable if and only if it is symmetric by sign and for all $k = 3, \ldots, m$, we have

$$a_{i_1,i_2}a_{i_2,i_3} \ldots a_{i_k,i_k} = a_{i_k,i_1}a_{i_1,i_2}a_{i_2,i_3} \ldots a_{i_k,i_k},$$

for all finite sequence $i_1, i_2, \ldots, i_k$.

However, we present a more simplified version of the above proposition and give the following characterization of symmetrizable matrices.
Proposition 4 A square matrix of order m is symmetrizable if and only if all its principal submatrices of order m-1 are symmetrizable.

Proof. First the matrix is symmetric by sign if and only if all its principal submatrices are too. If A is symmetrizable, then by choosing in (7) any permutation \( \sigma \) satisfying \( \sigma_i = i \), for a fixed \( i = 1, 2, ..., m \), we can conclude that any principal submatrix of a symmetrizable matrix A is also. Suppose that all principal submatrices of order m-1 are symmetrizable. By writing \( i_{k+1} = \sigma(i_k), k = 1, ..., m - 1 \) with \( i_1 = 1 \), then \( i_{m+1} = \sigma_1^m = 1 \) and for all \( j = 2, ..., m - 2 \), we have

\[
\prod_{i=1}^{m} a_{i, \sigma_i} = \prod_{k=1}^{j} a_{i_k, i_{k+1}} = \left( \prod_{k=1}^{j} a_{i_k, i_k} \right) \left( \prod_{k=j+1}^{m} a_{i_k, i_{k+1}} \right).
\]

Since the \((j+1)\) order principal submatrix with row-column pairs \( i_1, i_2, ..., i_{j+1} \) is symmetrizable, then we have

\[
\left( \prod_{k=1}^{j} a_{i_k, i_k} \right) a_{i_{j+1}, i_1} = \left( \prod_{k=1}^{j} a_{i_k, i_k} \right) a_{i_1, i_{j+1}}.
\]

Analogously, the \((m-j)\) order principal submatrix matrix with row-column pairs \( i_{j+1}, i_{j+2}, ..., i_m, i_1 \) is symmetrizable, then we have

\[
\left( \prod_{k=j+1}^{m} a_{i_k, i_k} \right) a_{i_1, i_{j+1}} = \left( \prod_{k=j+1}^{m} a_{i_k, i_k} \right) a_{i_{j+1}, i_1},
\]

then, taking in account that \( a_{i_{j+1}, i_1} a_{i_1, i_{j+1}} \neq 0 \), deduce that

\[
\left( \prod_{k=1}^{j} a_{i_k, i_k} \right) \left( \prod_{k=j+1}^{m} a_{i_k, i_k} \right) = \left( \prod_{k=1}^{j} a_{i_{k+1}, i_k} \right) \left( \prod_{k=j+1}^{m} a_{i_k, i_{k+1}} \right)
\]

and this gives (7).

Remark 5 It is shown that a matrix A is symmetrizable if and only if there exists a diagonal matrix D with positive entries (called Symmetrizer), such that the matrix \( D.A \) is symmetric (see [5]).

Our main result is the following

Theorem 6 The eigenvalues of a real symmetrizable matrix A of order m are all real and interlace with those of any principal submatrix of order m - 1.

The proof of the theorem is based on an equality analogous to (3) which is summarized by the following

Lemma 7 For any symmetrizable matrix A, we have

\[
a_{ik} \cdot \text{det} \left[ (a_{i,j})_{i \neq k, j \neq k} - \lambda I_{m-1} \right] = a_{kl} \cdot \text{det} \left[ (a_{i,j})_{i \neq k, j \neq l} - \lambda I_{m-1} \right],
\]

for all real \( \lambda \) and all integers \( k, l = 1, 2, ..., m \).
Proof. Simultaneously permuting rows and columns, if necessary, we may assume that \( k = 1, l = 2 \) and \( a_{12}, a_{21} > 0 \) (i.e. \( \neq 0 \)). We should prove the following
\[
P(\lambda) = Q(\lambda), \text{ for all real } \lambda, \tag{9}
\]
where
\[
P(\lambda) = a_{12} \det \left( (a_{i,j})_{i \neq j \neq 1} - \lambda I_{m-1} \right),
\]
and
\[
Q(\lambda) = a_{21} \det \left( (a_{i,j})_{i \neq j \neq 1} - \lambda I_{m-1} \right).
\]
The left and right parts of formula (9) can be written, respectively, as follows
\[
P(\lambda) = a_{12} \cdot \begin{vmatrix}
a_{21} & a_{23} & \cdots & a_{2,(m-1)} & a_{2,m} \\
a_{31} & a_{33} - \lambda & \cdots & a_{3,(m-1)} & a_{3,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{(m-1),1} & a_{(m-1),3} & \cdots & a_{(m-1),(m-1)} - \lambda & a_{(m-1),m} \\
a_{m,1} & a_{m,3} & \cdots & a_{m,(m-1)} & a_{m,m} - \lambda
\end{vmatrix}, \tag{10}
\]
and
\[
Q(\lambda) = a_{21} \cdot \begin{vmatrix}
a_{12} & a_{13} & \cdots & a_{1,(m-1)} & a_{1,m} \\
a_{32} & a_{33} - \lambda & \cdots & a_{3,(m-1)} & a_{3,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{(m-1),2} & a_{(m-1),3} & \cdots & a_{(m-1),(m-1)} - \lambda & a_{(m-1),m} \\
a_{m,2} & a_{m,3} & \cdots & a_{m,(m-1)} & a_{m,m} - \lambda
\end{vmatrix}.
\]
Since a square matrix and its transpose have the same determinant, then
\[
Q(\lambda) = a_{21} \cdot \begin{vmatrix}
a_{12} & a_{32} & \cdots & a_{(m-1),2} & a_{m,2} \\
a_{13} & a_{33} - \lambda & \cdots & a_{(m-1),3} & a_{m,3} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{1,(m-1)} & a_{3,(m-1)} & \cdots & a_{(m-1),(m-1)} - \lambda & a_{m,(m-1)} \\
a_{1,m} & a_{3,m} & \cdots & a_{(m-1),m} & a_{m,m} - \lambda
\end{vmatrix}. \tag{11}
\]
As \( A \) is symmetrizable, then the matrix \( A - \lambda I_{m} \) and all its principal submatrices are symmetrizable for all real \( \lambda \). By developing the determinants in formulas (10) and (11) each one with respect to its first row for example, we can remark that each coefficient in the \((m-2)\) degree polynomial \( P(\lambda) \) is equal to its analogous in the expression of the same degree polynomial \( Q(\lambda) \):
Indeed, the coefficient of \( \lambda^{m-2} \) is \((-1)^{m-2} a_{12} a_{21} \) for both \( P(\lambda) \) and \( Q(\lambda) \), that of \( \lambda^{m-3} \) is \((-1)^{m-3} \sum_{k=3}^{m} a_{21} a_{2,k} \) for \( P(\lambda) \) and \((-1)^{m-3} \sum_{k=3}^{m} a_{21} a_{2,k} \) for \( Q(\lambda) \), whom are equal since the submatrices
\[
A_k = \begin{pmatrix}
a_{11} & a_{12} & a_{1,k} \\
a_{21} & a_{22} & a_{2,k} \\
a_{k,1} & a_{k,2} & a_{k,k}
\end{pmatrix},
\]
are symmetrizable for all \( k = 3, \ldots, m \). For all integer \( l \geq 4 \), the coefficient of \( \lambda^{m-l} \) is

\[
(-1)^{m-l} a_{12} \sum_{3 \leq k_1 < k_2 < \ldots < k_{l-2} \leq m} a_{21} a_{k_1,1} a_{2,k_1} \cdots a_{2,k_{l-2},k_{l-2}}
\]

for \( P(\lambda) \) and

\[
(-1)^{m-l} a_{21} \sum_{3 \leq k_1 < k_2 < \ldots < k_{l-2} \leq m} a_{12} a_{k_1,1} a_{1,k_1} \cdots a_{1,k_{l-2},k_{l-2}}
\]

for \( Q(\lambda) \), whom are equal since the subamtrices

\[
A_{k_1,k_2,\ldots,k_{l-2}} = \begin{pmatrix}
a_{11} & a_{12} & a_{1,k_1} & a_{1,k_2} & \cdots & a_{1,k_{l-2}} \\
a_{21} & a_{22} & a_{2,k_1} & a_{2,k_2} & \cdots & a_{2,k_{l-2}} \\
a_{k_1,1} & a_{k_1,2} & a_{k_1,k_1} & a_{k_1,k_2} & \cdots & a_{k_1,k_{l-2}} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]

are symmetrizable for all \( 3 \leq k_1 < k_2 < \ldots < k_{l-2} \leq m \). This gives (9) and ends the proof of the Lemma. \( \blacksquare \)

**Proof.** (of Theorem 6) Let \( \lambda_1 < \lambda_2 < \ldots < \lambda_m \) the eigenvalues of the symmetrizable matrix \( A \), then by application of the Lemma and taking in account that \( a_{lk}, a_{lk} > 0 \), for all \( k, l = 1, 2, \ldots, m \), we can conclude that the two determinants in (8) have the same sign for a fixed \( p = 1, 2, \ldots, m \). The Dodgson’s algorithm (1) applied to the matrix \( A - \lambda_l I_m \) gives the formula (2) from which we can deduce, in the same way as when the matrix \( A \) is symmetric, that all principal minors of order \( m-1 \) of the matrix \( A - \lambda_l I \) have the same sign. The rest of the proof can be obtained by following the same reasoning as in the proof of the Cauchy interlacing theorem for symmetric matrices given is the beginning of the above section.

To prove, via the Dodgson’s algorithm, that the eigenvalues of the symmetrizable matrix \( A \) are all real, we shall do this by induction on \( m \):

For \( m = 3 \), the Dodgson’s algorithm (1) gives

\[
\begin{vmatrix}
a_{11} - \lambda & a_{12} & a_{13} \\
a_{21} & a_{22} - \lambda & a_{23} \\
a_{31} & a_{32} & a_{33} - \lambda \\
\end{vmatrix}
= (a_{22} - \lambda) = \begin{vmatrix}
a_{11} - \lambda & a_{12} & a_{13} \\
a_{21} & a_{22} - \lambda & a_{23} \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
a_{11} - \lambda & a_{12} \\
a_{21} & a_{22} - \lambda \\
\end{vmatrix}
= \begin{vmatrix}
a_{21} & a_{22} - \lambda \\
a_{31} & a_{32} \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
a_{21} & a_{22} - \lambda \\
\end{vmatrix}
= \begin{vmatrix}
a_{31} & a_{32} \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
a_{22} - \lambda \\
a_{23} \\
\end{vmatrix}
= \begin{vmatrix}
a_{32} & a_{33} - \lambda \\
\end{vmatrix}
\]

\[
(12)
\]
Let $\mu_1$ and $\mu_2$ the eigenvalues of the submatrix

$$
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}
$$

Since $a_{12}a_{21} > 0$, then $\mu_1$ and $\mu_2$ are real and we have

$$
\mu_1 < a_{22} < \mu_2.
$$

By application of the Lemma, we have

$$
\begin{vmatrix}
  a_{21} & a_{22} - \mu_i \\
  a_{31} & a_{32}
\end{vmatrix}
\begin{vmatrix}
  a_{12} & a_{13} \\
  a_{22} - \mu_i & a_{23}
\end{vmatrix} > 0, \ i = 1, 2,
$$

which gives

$$(a_{22} - \mu_i) P_3(\mu_i) < 0, \ i = 1, 2,$$

where $P_3(\lambda)$ denotes the characteristic polynomial of $A$ for $m = 3$. Using (13), we deduce that $P_3(\mu_1) < 0$ and $P_3(\mu_2) > 0$. Since $\lim_{\lambda \to -\infty} P_3(\lambda) = +\infty$ and $\lim_{\lambda \to +\infty} P_3(\lambda) = -\infty$, we deduce that $P_3(\lambda)$ has three real roots whom are the eigenvalues of the matrix $A$ when $m = 3$.

Suppose that the eigenvalues of any symmetrizable matrix until the order $m - 1$ are real and prove that the property is also true for the order $m$. By application of The Dodgson’s algorithm (1) to the matrix $A - \lambda I_m$ and the formula (8), we have

$$
P_m(\lambda) \det \left[ (a_{i,j} - \lambda I_{m-2})_{i \neq k,l, j \neq k,l} \right] \leq \det \left[ (a_{i,j} - \lambda I_{m-1})_{i \neq l} \right] \det \left[ (a_{i,j} - \lambda I_{m-1})_{j \neq k} \right],
$$

for all real $\lambda$, where $P_m(\lambda)$ denotes the characteristic polynomial of $A$. Let $\mu_1 < \mu_2 < \ldots < \mu_{m-1}$ the eigenvalues of $\left[ (a_{i,j} - \lambda I_{m-1})_{j \neq k} \right]$ which is a symmetrizable submatrix of $A - \lambda I_m$, then

$$
P_m(\mu_p) \det \left[ (a_{i,j} - \mu P_{m-2})_{i \neq k,l, j \neq k,l} \right] < 0, \ p = 1, \ldots, m - 1.
$$

The above inequality is strict because the eigenvalues of $(a_{i,j} - \lambda I_{m-1})_{j \neq l}$ interlace at the same time with those of the two matrices $A$ and $\left[ (a_{i,j} - \lambda I_{m-2})_{j \neq k,l} \right]$. Since this last matrix is a submatrix of the $(m-1)$ order matrix $\left[ (a_{i,j} - \lambda I_{m-1})_{j \neq l} \right]$ and then their corresponding eigenvalues interlace, we conclude that the sequence $\{ P_m(\mu_p) \}_{1 \leq p \leq m-1}$ changes sign $(m-1)$ times. But to deduce that the characteristic polynomial of $A$ has $m$ roots, it must change sign $(m + 1)$ times.
To find the two remaining times we treat two cases:
When \( m \) is odd, then the first and last terms of the sequence \( \{ P_m (\mu_p) \}_{1 \leq p \leq m-1} \) are respectively
\[
P_m (\mu_1) < 0 \text{ and } P_m (\mu_{m-1}) > 0.
\]
Then using the limits
\[
\lim_{\lambda \to -\infty} P_m (\lambda) = +\infty \text{ and } \lim_{\lambda \to +\infty} P_m (\lambda) = -\infty,
\]
we deduce that the characteristic polynomial of \( A \) has \( m \) roots.
When \( m \) is even, they are respectively
\[
P_m (\mu_1) < 0 \text{ and } P_m (\mu_{m-1}) < 0,
\]
then using the limits
\[
\lim_{\lambda \to \pm\infty} P_m (\lambda) = +\infty,
\]
we get the two remaining roots of \( P_m (\lambda) \). This ends the proof of Theorem 6.

\[\square\]

Remark 8 When the symmetrizable matrix \( A \) has a multiple eigenvalue \( \lambda \) of algebraic multiplicity equal to \( r \), then the characteristic polynomial (4) of the matrix
\[
\begin{pmatrix}
(a_{i,j} - \lambda I_{m-k})_{i \neq k, j \neq k}
\end{pmatrix}
\]
satisfies (4) for \( p=1, \ldots, (m-r+1) \) and Theorem 6 remains valid.

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