COMPLEX SYMMETRY AND DYNAMICS OF COMPOSITION OPERATORS ON $H^2(\mathbb{C}_+)$

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Abstract. In this article, we completely characterize the complex symmetry, cyclicity and hypercyclicity of composition operators $C_\phi f = f \circ \phi$ induced by linear fractional self-maps $\phi$ of the right half-plane $\mathbb{C}_+$ on the Hardy-Hilbert space $H^2(\mathbb{C}_+)$. We also provide new proofs for the normal, self-adjoint and unitary cases and for an adjoint formula discovered by Gallardo-Gutiérrez and Montes-Rodríguez.

Introduction

A bounded operator $T$ on a separable Hilbert space $\mathcal{H}$ is complex symmetric if there exists an orthonormal basis for $\mathcal{H}$ with respect to which $T$ has a self-transpose matrix representation. An equivalent definition also exists. A conjugation is a conjugate-linear operator $C : \mathcal{H} \to \mathcal{H}$ that satisfies the conditions

(a) $C$ is isometric: $\langle Cf, Cg \rangle = \langle g, f \rangle \ \forall \ f, g \in \mathcal{H}$,

(b) $C$ is involutive: $C^2 = I$.

We say that $T$ is $C$-symmetric if $CT = T^*C$, and complex symmetric if there exists a conjugation $C$ with respect to which $T$ is $C$-symmetric.

Complex symmetric operators on Hilbert spaces are natural generalizations of complex symmetric matrices, and their general study was initiated by Garcia, Putinar, and Wogen ([7], [8], [9], [10]). The class of complex symmetric operators includes a large number of concrete examples including all normal operators.

An operator $T$ on $\mathcal{H}$ is said to be cyclic if there exists a vector $f \in \mathcal{H}$ for which the linear span of its orbit $(T^n f)_{n \in \mathbb{N}}$ is dense in $\mathcal{H}$. If the orbit $(T^n f)_{n \in \mathbb{N}}$ itself is dense in $\mathcal{H}$, then $T$ is said to be hypercyclic. In these cases $f$ is called a cyclic or hypercyclic vector for $T$ respectively. If we assume that $T$ is both complex symmetric and cyclic (hypercyclic), then the relation $CT = T^*C$ implies that $T^*$ must also be cyclic (hypercyclic). The conjugation $C$ acts as a bijection between cyclic (hypercyclic) vectors of $T$ and $T^*$. Two monographs [1] and [11] on the dynamics of linear operators have appeared recently.

If $X$ is a Banach space of holomorphic functions on an open set $U \subset \mathbb{C}$ and if $\phi$ is a holomorphic self-map of $U$, the composition operator with symbol $\phi$ is defined by $C_\phi f = f \circ \phi$ for any $f \in X$. The emphasis here is on the comparison of properties of $C_\phi$ with those of symbol $\phi$. If $X$ is the Hardy space $H^2(\mathbb{C}_+)$ of the open right half-plane $\mathbb{C}_+$, then a holomorphic self-map $\phi$ of $\mathbb{C}_+$ induces a bounded

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$C_\phi$ on $H^2(\mathbb{C}_+)$ if and only if $\phi$ has a finite angular derivative at the fixed point $\infty$. That is, if $\phi(\infty) = \infty$ and if the non-tangential limit

$$\phi'(\infty) := \lim_{w \to \infty} \frac{w}{\phi(w)}$$

exists and is finite. This was proved by Matache in [13]. Then Elliot and Jury [3] prove that the norm of $C_\phi$ on $H^2(\mathbb{C}_+)$ is given by $\|C_\phi\| = \sqrt{\phi'(\infty)}$. Matache [12] also showed that the only linear fractional selfmaps of $\mathbb{C}_+$ that induce bounded composition operators on $H^2(\mathbb{C}_+)$ are of the form

$$\phi(w) = aw + b$$

where $a > 0$ and $\text{Re}(b) \geq 0$. In this case, $C_\phi$ is normal on $H^2(\mathbb{C}_+)$ if and only if $\phi(w) = aw + b$ with $a = 1$ or $\text{Re}(b) = 0$. This was first proved by Gallardo-Gutiérrez and Montes-Rodríguez [4] and then again with a different proof by Matache [13].

The study of complex symmetry of composition operators on the Hardy-Hilbert space of the unit disk $H^2(\mathbb{D})$ was initiated by Garcia and Hammond [6]. They showed that involutive disk automorphisms induce non-normal complex symmetric composition operators. Then Narayan, Sieveright and Thompson [15] discovered non-automorphic symbols with the same property. The general problem in the disk case is far from being solved. On the other hand the cyclity and hypercyclicity phenomena for composition operators in the linear fractional disk case have been characterized (see [2] and [5]). The objective here is to characterize the complex symmetry, cyclicity and hypercyclicity of $C_\phi$ in the linear fractional half-plane case. The interplay between complex symmetry and linear dynamics will play a key role in our analysis.

The plan of the paper is as follows. In Section 1, after some preliminaries, we provide a different proof of the adjoint formula for linear fractional composition operators first discovered by Gallardo-Gutiérrez and Montes-Rodríguez [4]. This is used to give new and shorter proofs for the normal, self-adjoint and unitary cases. In Section 2 we characterize complex symmetry of $C_\phi$ on $H^2(\mathbb{C}_+)$. In particular we show that these are precisely the normal ones. In Section 3 we consider the cyclicity of $C_\phi$ proving that this occurs only when $\phi$ is a non-automorphism with no fixed points in $\mathbb{C}_+$. Finally in Section 4 we prove that $H^2(\mathbb{C}_+)$ supports no hypercyclic linear fractional $C_\phi$. Our main results are summarized in the following table.

| Symbol $\phi(w) = aw + b$ | Comp. Symmetric $C_\phi$ | Cyclic $C_\phi$ | Hypercyclic $C_\phi$ |
|--------------------------|--------------------------|----------------|---------------------|
| Re($b$) = 0              | ✓                        | X              | X                   |
| $a = 1$ & Re($b$) > 0    | ✓                        | ✓              | X                   |
| $a < 1$ & Re($b$) > 0    | X                        | X              | X                   |
| $a > 1$ & Re($b$) > 0    | X                        | ✓              | X                   |

The authors of [4] work with the upper half-plane $\Pi$ whereas [13] is concerned with the right half-plane $\mathbb{C}_+$. Hence the necessary translation of results must be made.
1. Preliminaries

1.1. The Hardy space $H^2(\mathbb{C}_+)$.

Let $\mathbb{C}_+$ be the open right half-plane. The Hardy space $H^2(\mathbb{C}_+)$ is the Hilbert space of analytic functions on $\mathbb{C}_+$ for which the norm
\[
||f||_2^2 = \sup_{0<x<\infty} \int_{-\infty}^{\infty} |f(x+iy)|^2 dy
\]
is finite. For each $\alpha \in \mathbb{C}_+$, let $k_{\alpha}$ denote the reproducing kernel for $H^2(\mathbb{C}_+)$ at $\alpha$; that is,
\[
k_{\alpha}(w) = \frac{1}{w + \alpha}.
\]
These kernels satisfy the fundamental relation $\langle f, k_{\alpha} \rangle = f(\alpha)$ for all $f \in H^2(\mathbb{C}_+)$. If $\phi$ is a holomorphic self-map of $\mathbb{C}_+$, then a simple computation gives
\[
C^*_\phi k_{\alpha} = k_{\phi(\alpha)}
\]
for each $\alpha \in \mathbb{C}_+$.

1.2. Linear fractional composition operators.

The linear fractional self-maps $\phi$ of $\mathbb{C}_+$ that induce bounded composition operators on $H^2(\mathbb{C}_+)$ are of the form
\[
\phi(w) = aw + b
\]
where $a > 0$ and $\text{Re}(b) \geq 0$. Such a map $\phi$ is said to be of parabolic type if $a = 1$ and is a parabolic automorphism if additionally $\text{Re}(b) = 0$. Similarly $\phi$ is of hyperbolic type if $a \neq 1$ and is a hyperbolic automorphism if additionally $\text{Re}(b) = 0$. Gallardo-Gutiérrez and Montes-Rodríguez \cite{4}, Theorem 7.1] proved a formula for the adjoint of linear fractional composition operators. We provide a short proof of this result.

**Proposition 1.** If $\phi$ is as in (1.2), then $C^*_\phi = a^{-1}C_\psi$, where $\psi(w) = a^{-1}w + a^{-1}b$.

**Proof.** We observe that for each $w \in \mathbb{C}_+$, we have
\[
(C_\phi k_{\alpha})(w) = \frac{1}{aw + b + \alpha} = \frac{1}{a(w + a^{-1}\alpha + a^{-1}b)} = \frac{1}{aw + \psi(\alpha)}
\]
\[
= a^{-1}k_{\psi(\alpha)}(w) = (a^{-1}C^*_\psi k_{\alpha})(w).
\]
The completeness of the reproducing kernels $(k_{\alpha})_{\alpha \in \mathbb{C}_+}$ in $H^2(\mathbb{C}_+)$ implies that $C_\phi = a^{-1}C^*_\psi$, or equivalently $C^*_\phi = a^{-1}C_\phi$.

This allows us to obtain new and shorter proofs for the normal, self-adjoint and unitary composition operators (see also Theorems 2.4, 3.1 and 3.4 of \cite{14}).

**Theorem 2.** Let $\phi(w) = aw + b$ with $a > 0$ and $\text{Re}(b) \geq 0$. Then
\begin{enumerate}
\item $C_\phi$ is normal if and only if $a = 1$ or $\text{Re}(b) = 0$,
\item $C_\phi$ is self-adjoint if and only if $a = 1$ and $b \geq 0$,
\item $C_\phi$ is unitary if and only if $a = 1$ and $\text{Re}(b) = 0$.
\end{enumerate}

**Proof.** By Proposition 1 the operator $C_\phi$ is normal if and only if $C_\phi C_\psi = C_\psi C_\phi$. This is equivalent to the equality $\phi \circ \psi = \psi \circ \phi$. For $w \in \mathbb{C}_+$, we have
\[
(\phi \circ \psi)(w) = a(a^{-1}w + a^{-1}b) + b = w + 2\text{Re}(b)
\]
and similarly
\[
(\psi \circ \phi)(w) = a^{-1}(aw + b) + a^{-1}b = w + 2a^{-1}\text{Re}(b).
\]
In this case we shall say \( \phi \) not prove that type I composition operators are non-automorphisms. Composition operators that are complex symmetric we must consider the hyperbolic since normal operators are complex symmetric, it follows that to characterize all corresponding composition operators \( C_\psi \) is normal if and only if \( \psi \) is complex symmetric when \( \psi \) is cyclic. Hence if one of them is both complex symmetric and cyclic, then so must be the other. So to show that \( C_\psi \) is a scalar multiple of a \( C_\phi \) and using (1.3) gives

\[
\frac{1}{\phi(w) + 1} = \frac{a^{-1}}{\psi(w) + 1} \iff \psi(w) + 1 = a^{-1}\phi(w) + a^{-1}
\]

\[
\iff a^{-1}w + a^{-1}b + 1 = w + a^{-1}b + a^{-1}.
\]

The last equality holds precisely when \( a = 1 \) and \( b \geq 0 \). Finally if \( C_\phi \) is unitary then \( a^{-1}C_\phi C_\psi = I = a^{-1}C_\psi C_\phi \) and in particular \( C_{\psi \circ \psi} = aI \). Applying the latter identity to \( k_1 \) and using (1.3) gives

\[
\frac{1}{w + 2\text{Re}(b) + 1} = \frac{a}{w + 1} \iff w + 1 = aw + 2a\text{Re}(b) + a
\]

which clearly holds precisely when \( a = 1 \) and \( \text{Re}(b) = 0 \). \( \square \)

In the next section we address the first main theme of this work which is the complex symmetry of composition operators on \( H^2(\mathbb{C}_+) \).

2. Complex Symmetry of \( C_\phi \)

According to Theorem 2, a linear fractional composition operator \( C_\phi \) on \( H^2(\mathbb{C}_+) \) is normal if and only if \( \phi \) is an automorphism or a parabolic non-automorphism. Since normal operators are complex symmetric, it follows that to characterize all composition operators that are complex symmetric we must consider the hyperbolic non-automorphisms. These are precisely the symbols

\( \phi(w) = aw + b \) with \( a \in (0, 1) \cup (1, \infty) \) and \( \text{Re}(b) > 0 \).

In this case we shall say \( \phi \) is of type I if \( a \in (0, 1) \) and of type II if \( a \in (1, \infty) \). The corresponding composition operators \( C_\phi \) shall also be called type I and II respectively. Note that according to Proposition 1 the adjoint of each \( C_\phi \) of type I is a scalar multiple of a \( C_\psi \) of type II and vice versa. Hence if one of them is both complex symmetric and cyclic, then so must be the other. So to show that \( C_\phi \) is not complex symmetric when \( \phi \) is a hyperbolic non-automorphism, it is enough to prove that type I composition operators are not cyclic whereas those of type II are cyclic.

2.1. Symbols of type I. Bourdon and Shapiro [2 Proposition 2.7] proved that if the adjoint of a bounded linear operator \( T \) on a Hilbert space has a multiple eigenvalue, then \( T \) is not cyclic. Let \( \psi(w) = aw + b \) with \( a \in (1, \infty) \) and \( \text{Re}(b) > 0 \), in which case \( \psi \) is of type II. For each complex \( \lambda \) define the function

\[
f_\lambda(w) = \left( w + \frac{b}{a - 1} \right)^\lambda
\]

which is holomorphic in \( \mathbb{C}_+ \) since \( \text{Re}\left(\frac{b}{a - 1}\right) > 0 \), and \( f_\lambda \in H^2(\mathbb{C}_+) \) if and only if \( \text{Re}(\lambda) < -1/2 \). Hence for \( \text{Re}(\lambda) < -1/2 \), we see that

\[
C_\psi f_\lambda(w) = \left( aw + b + \frac{b}{a - 1} \right)^\lambda = \left( aw + \frac{ab}{a - 1} \right)^\lambda = a^\lambda f_\lambda(w).
\]

\[\text{We cannot use constants and monomials as they do not belong to } H^2(\mathbb{C}_+).\]
This implies that each such $a^\lambda$ is an eigenvalue of infinite multiplicity since

$$C_\psi f_{\lambda+r\frac{2\pi n}{\log a}} = a^\lambda e^{2\pi in} f_{\lambda+r\frac{2\pi n}{\log a}} = a^\lambda f_{\lambda+r\frac{2\pi n}{\log a}}$$

for each integer $n$. It follows that $C_\psi^*$ is not cyclic. But type I operators are scalar multiples of the adjoints of those of II. Therefore type I operators are not cyclic.

**Proposition 3.** If $\phi$ is a hyperbolic non-automorphism of type I, then $C_\phi$ is not cyclic.

2.2. **Symbols of type II.** Let $\psi(w) = aw + b$ with $a \in (0, 1)$ and $\Re(b) > 0$. Hence $\psi$ is of type I. It is easy to see that $\psi$ has a fixed point $w = \frac{b}{1-a}$ which belongs to $\mathbb{C}_+$. Now since type II operators are scalar multiples of adjoints of type I operators, the following general result suffices to show that all type II operators are cyclic.

**Lemma 4.** Let $\phi$ be an analytic self-map of $\mathbb{C}_+$ with $\phi(\alpha) = \alpha$ for some $\alpha \in \mathbb{C}_+$ such that $C_\phi$ is bounded on $H^2(\mathbb{C}_+)$. Then $C_\phi^*$ is cyclic.

**Proof.** We first note that $\phi$ cannot be an automorphism of $\mathbb{C}_+$. Indeed, if $\phi$ were an automorphism then we must have $\phi(w) = aw + ir$ with $a > 0$ and $r \in \mathbb{R}$ and it is easy to see that $\phi$ has no fixed point when $a = 1$ and pure imaginary fixed point $\frac{ir}{-a}$ when $a \neq 1$. Now let $\gamma(z) = \frac{1+iz}{-z}$ be the Cayley transform of the open unit disk $\mathbb{D}$ onto $\mathbb{C}_+$. Hence $\Phi = \gamma^{-1} \circ \phi \circ \gamma$ is a non-automorphic self-map of $\mathbb{D}$ with an interior fixed point $\beta = \gamma^{-1}(\alpha) \in \mathbb{D}$. The well-known Denjoy-Wolff Theorem says that the composition iterates $\Phi^{[n]} \to \beta$ locally uniformly in $\mathbb{D}$ as $n \to \infty$. It also follows that the iterates $\Phi^{[n]}(z)$ are distinct for each $z \neq \beta$ (see [10, Lemma 1]). Hence $\phi^{[n]} \to \alpha$ locally uniformly in $\mathbb{C}_+$ as $n \to \infty$ and $\phi^{[n]}(w)$ is a sequence of distinct points for each $w \neq \alpha$. Our goal is to prove that each reproducing kernel $k_w$ for $w \neq \alpha$ is a cyclic vector for $C_\phi^*$. Suppose $f \in H^2(\mathbb{C}_+)$ is orthogonal to $(C_\phi^*)^nk_w$ for all $n \in \mathbb{N}$. Then

$$0 = \langle f, (C_\phi^*)^nk_w \rangle = \langle C_\phi^n f, k_w \rangle = \langle f \circ \phi^{[n]}, k_w \rangle = f(\phi^{[n]}(w))$$

implies that $f$ vanishes on a sequence of distinct points with limit $\alpha$ in $\mathbb{C}_+$. Hence $f \equiv 0$ and $k_w$ is a cyclic vector for $C_\phi^*$ for each $w \neq \alpha$. \[\square\]

This concludes the proof of the cyclicity of type II composition operators.

**Proposition 5.** If $\phi$ is a hyperbolic non-automorphism of type II, then $C_\phi$ is cyclic.

Therefore Propositions 3 and 5 imply that $C_\phi$ is not complex symmetric when $\phi$ is a hyperbolic non-automorphism. We therefore obtain a characterization for complex symmetry.

**Theorem 6.** Let $\phi$ be a linear fractional self-map of $\mathbb{C}_+$. Then $C_\phi$ is complex symmetric on $H^2(\mathbb{C}_+)$ if and only if $\phi$ is an automorphism or a parabolic non-automorphism. That is, precisely when $C_\phi$ is normal.

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3In the proof of [4, Theorem 7.4] it is claimed that $(z-b)^\lambda$ is an eigenvector for $C_\psi$ which is clearly incorrect.
3. CYCLICITY OF $C_\phi$

The goal of this section is to completely characterize the cyclic $C_\phi$ on $H^2(\mathbb{C}_+)$. We already showed in the previous section that hyperbolic non-automorphisms $\phi$ of type I and type II induce non-cyclic and cyclic $C_\phi$ respectively. In contrast the next result shows that all parabolic non-automorphisms $\phi$ induce cyclic $C_\phi$ on $H^2(\mathbb{C}_+)$. 

**Proposition 7.** Let $\phi(w) = w + b$ with $\text{Re}(b) > 0$. Then $C_\phi$ is cyclic on $H^2(\mathbb{C}_+)$. 

**Proof.** First note that the compositional iterates of $\phi$ are given by $\phi^{[n]}(w) = w + nb$. This implies that we have 

$$(C_{\phi}^nk_1)(w) = k_1(\phi^{[n]}(w)) = k_1(w + nb) = \frac{1}{1 + nb + w} = k_{bn}(w)$$

where $b_n = 1 + nb$. Now if we assume some $f \in H^2(\mathbb{C}_+)$ is orthogonal to the span of the orbit $(C_{\phi}^nk_1)_{n \in \mathbb{N}}$, then

$$0 = \langle f, C_{\phi}^nk_1 \rangle = \langle f, k_{bn} \rangle = f(b_n).$$

To conclude the proof, it is enough to show that the sequence $(b_n)_{n \in \mathbb{N}}$ does not satisfy the so-called Blaschke condition for zeros of $H^2(\mathbb{C}_+)$ functions. That is, we must prove that

$$\sum_{n=1}^{\infty} \frac{\text{Re}(b_n)}{1 + |b_n|^2} = \infty.$$ 

First note that

$$1 + |b_n|^2 = 1 + (1 + n\text{Re}(b))^2 + (n\text{Im}(b))^2 \leq 2(1 + n|b|^2)^2 \leq 2(1 + |b|^2)^2n^2$$

which implies that

$$\frac{\text{Re}(b_n)}{1 + |b_n|^2} \geq \frac{1 + n\text{Re}(b)}{2(1 + |b|^2)^2n^2} \geq \frac{\text{Re}(b)}{2(1 + |b|^2)2n^2}. $$

Therefore (3.1) clearly holds and hence $(b_n)_{n \in \mathbb{N}}$ cannot be a zero sequence for $f$ unless $f \equiv 0$. 

Hence the only case remaining is the cyclicity of $C_\phi$ where $\phi(w) = aw + b$ with $a > 0$ and $\text{Re}(b) = 0$, that is precisely when $\phi$ is an automorphism of $\mathbb{C}_+$. This will be achieved with the help of the following result about the non-cyclicity of certain multiplication operators on $L^2$ spaces of the real line. The idea of the proof is inspired by that of [5, Theorem 3.13]. 

**Lemma 8.** Suppose $s \in \mathbb{R}$ and let $M := M_{e^{ist}}$ be the operator of multiplication by $e^{ist}$ on $L^2(\mathbb{R}_+, dt)$ or $L^2(\mathbb{R}, dt)$. Then $M$ is not cyclic on either space. 

**Proof.** If $s = 0$ then $M = I$ is clearly non-cyclic. So assume $s \neq 0$. Since $L^2(\mathbb{R}_+, dt)$ is clearly a reducing subspace for $M$ acting on $L^2(\mathbb{R}, dt)$, it is enough to prove the result for $L^2(\mathbb{R}_+, dt)$. Consider any function $f \in L^2(\mathbb{R}_+, dt)$. Then we have

$$\text{span}\{M^nf : n \in \mathbb{N}\} = \{pf : \text{where } p \text{ is a polynomial in } e^{ist}\}.$$ 

First suppose that $f$ vanishes on a set $A \subset \mathbb{R}_+$ of positive measure. Then each $pf$ vanishes on $A$ and hence sequences of these $pf$ can approximate only functions that vanish almost everywhere on $A$. Therefore $M$ is not cyclic in this case. For the other case, suppose $f \neq 0$ on any set of positive measure. That $M$ is non-cyclic will follow from the fact that any polynomial in $e^{ist}$ is $2\pi/s$ periodic.
Let $\chi_{[0,1]}$ be the characteristic function of $[0,1]$ and suppose $p_n f \to \chi_{[0,1]}$ in $L^2(\mathbb{R}^+, dt)$ for some sequence $p_n$ of polynomials in $e^{i \theta t}$. Then some subsequence $p_{n_k} f \to \chi_{[0,1]}$ pointwise almost everywhere. So on the one hand $p_{n_k} \to 1/f$ almost everywhere on $[0,1]$ and $p_{n_k} \to 0$ almost everywhere on $(1, \infty)$ since $f \neq 0$ almost everywhere. But on the other hand the periodicity of $p_{n_k}$ implies that we also have $p_{n_k} \to 0$ almost everywhere on $[0,1]$ and hence that $1/f = 0$ almost everywhere on $[0,1]$. This contradiction proves that $M$ is not cyclic in this case also. □

We are now ready to complete the characterization of linear fractional cyclicity.

**Theorem 9.** Let $\phi(w) = aw + b$ be a linear fractional self-map of $\mathbb{C}_+$. Then $C_\phi$ is cyclic on $H^2(\mathbb{C}_+)$ if and only if $\phi$ is a parabolic non-automorphism or a hyperbolic non-automorphism of type II. That is precisely when $a \geq 1$ and $\text{Re}(b) > 0$.

**Proof.** Since the only case that remains is when $\phi$ is an automorphism, we may assume $\text{Re}(b) = 0$. We first note that if $\Pi$ denotes the upper half-plane, then $(Uf)(w) = f(\overline{w})$ defines a unitary map of the Hardy space $H^2(\Pi)$ of the upper half-plane onto $H^2(\mathbb{C}_+)$ and $C_\phi$ on $H^2(\mathbb{C}_+)$ is unitarily equivalent to $C_\psi$ on $H^2(\Pi)$ where $\psi(w) = aw + ib$ is a self map of $\Pi$. It follows from Gallardo-Gutiérrez and Montes-Rodríguez [11 Theorem 7.1] that if $a = 1$ then $C_\phi$ is similar to $M_{e^{-it\log a}}$ on $L^2(\mathbb{R}^+, dt)$, and hence so is $C_\phi$. Therefore $C_\phi$ is not cyclic by Lemma 8 when $\phi$ is a parabolic automorphism. Similarly, when $\phi$ is a hyperbolic automorphism ($a \neq 1$), then $C_\phi$ is similar to $M_{e^{-it \log a}}$ on $L^2(\mathbb{R}^+, dt)$ which is again not cyclic by Lemma 8. This completes the proof of the theorem. □

4. HYPERCYCLICITY OF $C_\phi$

In this final section we show that $H^2(\mathbb{C}_+)$ does not support any hypercyclic composition operator with linear fractional symbols. This is in sharp contrast to various weighted Hardy spaces of the open unit disk (see [5 page 8]).

**Theorem 10.** $C_\phi$ is not hypercyclic on $H^2(\mathbb{C}_+)$ for any linear fractional $\phi$.

**Proof.** If $\phi$ is an automorphism or a parabolic non-automorphism, then $C_\phi$ is normal and hence is not hypercyclic (see [11 Theorem 5.30]). Similarly if $\phi$ is a hyperbolic non-automorphism of type I, then $C_\phi$ is not cyclic by Proposition 3 and hence is not hypercyclic either. The case that remains is when $\phi$ is hyperbolic non-automorphic of type II, that is $\phi(w) = aw + b$ where $a > 1$ and $\text{Re}(b) > 0$. By induction it is easy to show that the $n$-th iterate of $\phi$ is given by

$$\phi^n(w) = a^n w + \frac{(1 - a^n)b}{1 - a}.$$ 

Then $||C_\phi^n||_{H^2(\mathbb{C}_+)} = \sqrt{\phi^n(\infty)} = \sqrt{1/a^n}$ (see [11] and [12]). Since $a \in (1, \infty)$, the sequence $||C_\phi^n||_{H^2(\mathbb{C}_+)} \to 0$ as $n \to \infty$. This implies that $C_\phi$ cannot be hypercyclic. □

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