On Some Idempotent and Non-Associative Convex Structure.

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Abstract

\(\mathbb{B}\)-convexity was defined in [7] as a suitable Kuratowski-Painlevé upper limit of linear convexities over a finite dimensional Euclidean vector space. Excepted in the special case where convex sets are subsets of \(\mathbb{R}^n_+\), \(\mathbb{B}\)-convexity was not defined with respect to a given explicit algebraic structure. This is done in that paper, which proposes an extension of \(\mathbb{B}\)-convexity to the whole Euclidean vector space. An unital idempotent and non-associative magma is defined over the real set and an extended \(n\)-ary operation is introduced. Along this line, the existence of the Kuratowski-Painlevé limit of the convex hull of two points over \(\mathbb{R}^n\) is shown and an explicit extension of \(\mathbb{B}\)-convexity is proposed.

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1 Introduction

\(\mathbb{B}\)-convexity was defined in [7]. One can say, loosely speaking, that this \(\mathbb{B}\)-convexity is obtained from the usual linear convexity making the formal transformation \(+\to\max\). By definition, a \(\mathbb{B}\)-convex subset of \(\mathbb{R}^n_+\) is a connected upper semilattice. \(\mathbb{B}\)-convex functions were analyzed in [1]. Hanh-Banach like separation properties [9] as well as fixed point results [8] -see also [11]- have been established. The standard form of \(\mathbb{B}\)-convexity is defined on the nonegative Euclidean orthant \(\mathbb{R}^n_+\) and is linked to Max-Plus algebra via a suitable homeomorphism. In finite dimensional space, Max-Plus convexity and \(\mathbb{B}\)-convexity are isomorphic topological Maslov’s semi-modules [10] and, consequently, a proposition that is true in the framework of \(\mathbb{B}\)-convexity holds, with obvious lexical modifications, in Max-Plus convexity. Though \(\mathbb{B}\)-convexity was initially defined over \(\mathbb{R}^n\) as

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a Kuratowski-Painlevé upper limit of linear convexities, it was not described in term of an explicit algebraic structure, excepted in the case where convex sets are subsets of $\mathbb{R}^n$. More recently, $\mathbb{B}^{-1}$-convex sets were introduced in [2] and [3].

This paper introduces a suitable algebraic structure extending $\mathbb{B}$-convexity to the whole Euclidean vector space. However, there do not exist non trivial algebraic structures being both idempotent, associative, and having inverse elements. Therefore, a special class of idempotent magma is considered in which associativity is relaxed to preserve symmetry and idempotence. This binary operation is based upon absolute value function. An $n$-ary extension of this algebraic structure is proposed and related to the pointwise limit of a generalized Hölder sum. Some algebraic properties are established and an extended definition of $\mathbb{B}$-convexity is then proposed, including as a special case that one proposed in [7]. It is shown that such a notion of convexity is equivalently characterized from the Kuratowski-Painlevé limit of the generalized convex hull of two points defined in [7].

The paper unfolds as follows. Section 2 focusses on a special class of symmetrical idempotent magmas. An $n$-ary extension of this operation is proposed and is related to the limit of a generalized Hölder sum. In section 3, it is established that such an algebraic structure yields a very simple extension of $\mathbb{B}$-convexity over $\mathbb{R}^n$.

## 2 Pointwise Limit of a Generalized Sum and Algebraic Structure

For all $p \in \mathbb{N}$, we consider a bijection $\varphi_p : \mathbb{R} \rightarrow \mathbb{R}$ defined by: $\varphi_p : \lambda \rightarrow \lambda^{2^p+1}$ and $\Phi_p(x_1, \ldots, x_n) = (\varphi_p(x_1), \ldots, \varphi_p(x_n))$. We can induce a field structure on $\mathbb{R}$ for which $\varphi_p$ becomes a field isomorphism. Given this change of notation via $\varphi_p$ and $\Phi_p$ we can define a $\mathbb{R}$-vector space structure on $\mathbb{R}^n$ by: $\lambda \Phi_p x = \Phi_p^{-1}(\varphi_p(\lambda) \cdot \Phi_p(x)) = \lambda x$ and $x_1 + x_2 = \Phi_p^{-1}(\Phi_p(x_1) + \Phi_p(x_2))$; we call these two operations the indexed scalar product and the indexed sum (indexed by $\varphi$ of course). For all natural number $n \geq 1$, let us denote $[n] = \{1, \ldots, n\}$. Suppose that $A = \{x_1, \ldots, x_m\}$. The $\varphi_p$-sum - denoted $\sum_{i \in [m]} x_i = \Phi_p^{-1}\left(\sum_{i \in [m]} \Phi_p(x_i)\right)$ is defined by $\sum_{i \in [m]} x_i = \Phi_p^{-1}\left(\sum_{i \in [m]} \Phi_p(x_i)\right)$. In the remainder of the paper we denote for all $x, y \in \mathbb{R}^n$ $x + y = \varphi_p x + y$.

### 2.1 On some Idempotent, Symmetrical and Non-associative Algebraic Structure

If $x, y \in \mathbb{R}_+$ then one has $\lim_{p \rightarrow +\infty} x^p + y = \max\{|x|, |y|\}$. In the case where $x$ and $y$ belong to the whole real set, it is easy to establish the following property.
Fact 2.1.1 For all \( x, y \in \mathbb{R} \) we have:

\[
\lim_{p \to +\infty} x^p + y = \lim_{p \to +\infty} (x^{2p+1} + y^{2p+1})^{1/(2p+1)} = \begin{cases} 
  x & \text{if } |x| > |y| \\
  \frac{1}{2}(x + y) & \text{if } |x| = |y| \\
  y & \text{if } |x| < |y|.
\end{cases}
\]

Proof: Suppose that \( x = -y \). Then for all \( p \in \mathbb{N} \) we have \( (x^{2p+1} + y^{2p+1})^{1/(2p+1)} = (x^{2p+1} - x^{2p+1})^{1/(2p+1)} = 0 \), which proves this case. If \( x = y \), then \( \lim_{p \to +\infty} (x^{2p+1} + y^{2p+1})^{1/(2p+1)} = x \lim_{p \to +\infty} 2^{1/(2p+1)} = x \). To end the proof, suppose, for example, that \( |x| > |y| \). The map \( t \mapsto \ln(1 + t) \) is continuous at point 0. Thus, since \( |\frac{y}{x}| < 1 \), we have \( \lim_{p \to +\infty} \left( (1 + (\frac{y}{x})^{2p+1})^{1/(2p+1)} \right) = 0 \). Consequently,

\[
\lim_{p \to +\infty} (x^{2p+1} + y^{2p+1})^{1/(2p+1)} = x \left[ \lim_{p \to +\infty} \left( 1 + (\frac{y}{x})^{2p+1} \right)^{1/(2p+1)} \right] = x. \quad \square
\]

Let \((M, \boxplus)\) be a magma or groupoid that is a set \( M \) equipped with a single closed binary operation \( M \times M \to M \) defined by \((x, y) \to x \boxplus y\). \( M \) is unital if it has a neutral element 0. This binary operation is idempotent if for all \( x \in M \) \( x \boxplus x = x \). It is associative if for all \((x, y, z) \in M^3\), one has \((x \boxplus y) \boxplus z = x \boxplus (y \boxplus z) = x \boxplus y \boxplus z\). \( M \) has inverse elements if for all \( x \in M \) there is some \(-x\) in \( M \) such that \( x \boxplus (-x) = (-x) \boxplus x = 0 \). We say that \( x \) and \(-x\) are symmetrical if \(-x\) is the inverse of \( x \) and conversely. It is a standard fact that a nontrivial group is not idempotent. In general idempotence is compatible with a semigroup structure that is an algebraic structure consisting of a set together with an associative binary operation. A semigroup generalizes a group to a type where every element did not have to have a symmetrical element. In the following a non-associative and idempotent magma is considered. The real set \( \mathbb{R} \) is endowed with the binary operation \( \boxplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) defined by

\[
x \boxplus y = \lim_{p \to +\infty} x^p + y = \lim_{p \to +\infty} (x^{2p+1} + y^{2p+1})^{1/(2p+1)} \tag{2.1}
\]

The map \((x, y) \to x \boxplus y\) is not continuous over \( \mathbb{R}^2 \). Moreover, this operation is not associative. For example, one has \((1 \boxplus 1) \boxplus (-1) = 1 \boxplus (-1) = 0 \) and \( 1 \boxplus (1 \boxplus (-1)) = 1 \boxplus 0 = 1 \) which contradicts associativity. Associativity is replaced with a weakened assumption which only requires that associativity works for any pair of non symmetrical elements. By definition, for all \( x, y \in \mathbb{R}_+ \) one has \( x \boxplus y = \max\{x, y\} \). Moreover, if \( x, y \in \mathbb{R}_- \) then \( x \boxplus y = \min\{x, y\} \). It follows that the operation \( \boxplus \) defines a total order on \( \mathbb{R}_+ \) and \( \mathbb{R}_- \), but not on the whole real set. In the remainder \((\mathbb{R}, \boxplus)\) denotes the set \( \mathbb{R} \) equipped with the operation \( \boxplus \). The proof of the following lemma is left to the reader.

Proposition 2.1.2 The set \( \mathbb{R} \) equipped with the operation \( \boxplus \) and the scalar multiplication satisfies the following properties:

(a) For all \( x \in \mathbb{R} \), \( x \boxplus x = x \) (idempotence).
(b) For all \( x \in \mathbb{R} \), \( x \oplus 0 = 0 \oplus x = x \) (neutral element).

(c) For all \( x \in \mathbb{R} \), there exists a uniqueness symmetrical element \( -x \in \mathbb{R} \) such that \( x \oplus (-x) = (-x) \oplus x = 0 \) (symmetrical element).

(d) For all \( x, y \in \mathbb{R} \), we have \( x \oplus y = y \oplus x \) (commutativity).

(e) For all \((x, y, z) \in \mathbb{R}^3\), if \( x, y \) and \( z \) are mutually non symmetrical then \( (x \oplus y) \oplus z = x \oplus (y \oplus z) \) (weakened form of associativity).

(f) For all \((x, y, z) \in \mathbb{R}^3\), one has \( z(x \oplus y) = (x \oplus y)z = (zx) \oplus (zy) \) (distributivity).

The properties above show that \((\mathbb{R}, \oplus, \cdot)\) is endowed with some kind of “scalar field like” algebraic structure. It is not a scalar field because \((\mathbb{R}, \oplus)\) is not a group. The next statement is an immediate consequence of Lemma 2.1.1.

**Lemma 2.1.3** For all \( x, y \in \mathbb{R} \), the following inequalities are equivalent: (a) \( x \leq y \); (b) \( 0 \leq (-x) \oplus y \); (c) \( x \oplus (-y) \leq 0 \).

**Proof:** First, note that the distributivity of the scalar multiplication on the operation \( \oplus \) implies that (b) and (c) are equivalent. All we need to prove is that (a) and (b) are equivalent. Let us prove the first implication. If \( x \leq y \), then for all natural number \( p \geq 1 \), we have \(((-x)^{2p+1} + y^{2p+1})^{1/(2p+1)} \geq 0 \). It follows that \( \lim_{p \to +\infty} ((-x)^{2p+1} + y^{2p+1})^{1/(2p+1)} = (-x) \oplus y \geq 0 \), which proves (b). Conversely, suppose that (b) holds. By hypothesis \( 0 \leq (-x) \oplus y \). If \((-x) \oplus y = 0\) then one has \( x = y \) and (a) is immediate. Suppose now that \((-x) \oplus y = y \geq 0\). This implies that one has \(|y| \geq |-x| = |x|\). Since \( y \geq 0 \), we have \( y \geq x \), and we deduce condition (a). Finally, if \((-x) \oplus y = -x \geq 0\), from the distributivity of the scalar multiplication on the operation \( \oplus \), we have \( x \oplus (-y) = x \leq 0 \). Since \(|x| \geq |y|\), this implies that \( x \leq y \), which ends the proof. \(\square\)

### 2.2 Construction of a n-ary Operation

In the following it is established that, though the operation \( \oplus \) does not satisfy associativity, it can be extended by constructing a non-associative algebraic structure which returns to a given \( n \)-tuple a real value. For all \( x \in \mathbb{R}^n \) and all subset \( I \) of \([n]\), let us consider the map \( \xi_I[x] : \mathbb{R} \to \mathbb{Z} \) defined for all \( \alpha \in \mathbb{R} \) by

\[
\xi_I[x](\alpha) = \text{Card}\{i \in I : x_i = \alpha\} - \text{Card}\{i \in I : x_i = -\alpha\}.
\]

This map measures the symmetry of the occurrences of a given value \( \alpha \) in the components of a vector \( x \). This map satisfies the following properties whose the proofs are obvious and left to the reader.

**Lemma 2.2.1** For all \( x \in \mathbb{R}^n \) and for all subset \( I \) of \([n]\) the map \( \xi_I[x] \) defined in (2.2) satisfies the following properties:
(a) $\xi_I[x]$ is an impair map, that is for all $\alpha \in \mathbb{R}$, $\xi_I[x](\alpha) = -\xi_I[x](-\alpha)$.
(b) For all $\alpha \in \mathbb{R}$ the map $x \mapsto \xi_I[x](\alpha)$ is impair.
(c) If $\{\alpha, -\alpha\} \cap \{x_i : i \in I\} = \emptyset$ then $\xi_I(\alpha) = 0$.
(d) For all $i \in I$, $\xi_{I\setminus\{i\}}(x_i) = \xi_I(x_i) - 1$.
(e) If $\xi_I[x](\max_{i \in I} |x_i|) > 0$ then $\max_{i \in I} x_i = \max_{i \in I} |x_i|$.
(f) If $\xi_I[x](\max_{i \in I} |x_i|) < 0$ then $\min_{i \in I} x_i = -\max_{i \in I} |x_i|$.
(g) For all subsets $I$ and $J$ of $[n]$ and all $\alpha \in \mathbb{R}$, $\xi_{I\cup J}[x](\alpha) = \xi_I[x](\alpha) + \xi_J[x](\alpha) - \xi_{I\cap J}[x](\alpha)$.
(h) $\xi_0[x](\alpha) = 0$, for all $\alpha \in \mathbb{R}$.

For all $x \in \mathbb{R}^n$ let $\mathcal{J}_I(x)$ be a subset of $I$ defined by

$$\mathcal{J}_I(x) = \left\{ j \in I : \xi_I[x](x_j) \neq 0 \right\} = I \setminus (\xi_I[x]^{-1}(0)). \quad (2.3)$$

$\mathcal{J}_I(x)$ is called the residual index set of $x$. It is obtained by dropping from $I$ all the $i$’s such that $\text{Card}\{ j \in I : x_j = x_i \} = \text{Card}\{ j \in I : x_j = -x_i \}$.

**Definition 2.2.2** For all positive natural number $n$ and for all subset $I$ of $[n]$, let $F_I : \mathbb{R}^n \rightarrow \mathbb{R}$ be the map defined for all $x \in \mathbb{R}^n$ by

$$F_I(x) = \begin{cases} \max_{i \in \mathcal{J}_I(x)} x_i & \text{if } \xi_I[x](\max_{i \in \mathcal{J}_I(x)} |x_i|) > 0 \\ \min_{i \in \mathcal{J}_I(x)} x_i & \text{if } \xi_I[x](\max_{i \in \mathcal{J}_I(x)} |x_i|) < 0 \\ 0 & \text{if } \xi_I[x](\max_{i \in \mathcal{J}_I(x)} |x_i|) = 0. \end{cases} \quad (2.4)$$

where $\xi_I[x]$ is the map defined in (2.2) and $\mathcal{J}_I(x)$ is the residual index set of $x$. The operation that takes an $n$-tuple $(x_1, ..., x_n)$ of $\mathbb{R}^n$ and returns a single real element $F_I(x_1,...,x_n)$ is called a $n$-ary extension of the binary operation $\oplus$.

Notice that, if $\mathcal{J}_I(x) = \emptyset$ if and only if $\xi_I[x](\max_{i \in \mathcal{J}_I(x)} |x_i|) = 0$. To see the key idea of the definition above let us define the generalized sum of $n$ real numbers $x_1, ..., x_n$ as $S_p(x_1, ..., x_n) = \left( \sum_{i \in [n]} x_i^p \right)^{\frac{1}{p}}$, say a Hölder sum. When one consider the subsequence of pair natural numbers, this generalized sum has the limit: $\lim_{p \rightarrow +\infty} S_{2p}(x_1, ..., x_n) = \lim_{p \rightarrow +\infty} \left( \sum_{i \in [n]} x_i^{2p} \right)^{\frac{1}{2p}} = \max_{i \in [n]} |x_i|$. The case where the generalized sum is defined with respect to the impair natural numbers is analyzed in this section. It is shown below that $F_I(x)$ is the limit of the generalized sum $S_{2p+1}(x_1, ..., x_n)$.

**Proposition 2.2.3** For all natural number $n \geq 1$ and all $x \in \mathbb{R}^n$, if $I$ is a nonempty subset of $[n]$ then:

$$F_I(x) = \lim_{p \rightarrow \infty} \left( \sum_{i \in I} x_i^{2p+1} \right)^{\frac{1}{2p+1}} = \lim_{p \rightarrow \infty} \sum_{i \in I} x_i^{2p+1}.$$
Proof: Let $J_I(x)$ be the residual index set of $x$. We have
\[
\left( \sum_{i \in I} x_i^{2p+1} \right)^{\frac{1}{2p+1}} = \left( \sum_{i \in I \setminus J_I(x)} x_i^{2p+1} + \sum_{i \in J_I(x)} x_i^{2p+1} \right)^{\frac{1}{2p+1}}.
\]
By definition, there exists a partition of $I \setminus J_I(x)$ whose any block contains two symmetric elements. Hence it follows that
\[
\sum_{i \in I \setminus J_I(x)} x_i^{2p+1} = - \sum_{i \in I \setminus J_I(x)} x_i^{2p+1} = 0.
\]
Hence, we deduce that
\[
\left( \sum_{i \in I} x_i^{2p+1} \right)^{\frac{1}{2p+1}} = \left( \sum_{i \in J_I(x)} x_i^{2p+1} \right)^{\frac{1}{2p+1}}.
\]
Suppose that $J_I(x) = \emptyset$. In such a case
\[
\left( \sum_{i \in I} x_i^{2p+1} \right)^{\frac{1}{2p+1}} = \left( \sum_{i \in J_I(x)} x_i^{2p+1} \right)^{\frac{1}{2p+1}} = 0.
\]
Consequently $\lim_{p \to \infty} \left( \sum_{i \in I} x_i^{2p+1} \right)^{\frac{1}{2p+1}} = 0 = \Gamma_I(x)$ which proves this case.

Suppose now that $J_I(x) \neq \emptyset$. Let us denote
\[
M_I(x) = \{ i \in J_I(x) : |x_i| = \max_{i \in J_I(x)} |x_i| \}.
\]
Then, from the definition of map $\xi_I[x]$ in equation (2.2), we have
\[
\sum_{i \in M_I(x)} \left( \frac{x_i}{\max_{i \in J_I(x)} |x_i|} \right)^{2p+1} = \xi_I[x] \left( \max_{i \in J_I(x)} |x_i| \right).
\]
It follows that for all $x \in \mathbb{R}^n$,
\[
\left( \sum_{i \in J_I(x)} x_i^{2p+1} \right)^{\frac{1}{2p+1}} = \max_{i \in J_I(x)} |x_i| \left( \sum_{i \in J_I(x)} \frac{x_i^{2p+1}}{\max_{i \in J_I(x)} |x_i|^{2p+1}} \right)^{\frac{1}{2p+1}} \leq \left( \max_{i \in J_I(x)} |x_i| \right)^{\frac{1}{2p+1}}
\] (2.5)
where
\[
\alpha_p(x) = \left( \xi_I[x] \left( \max_{i \in J_I(x)} |x_i| \right) \right) + \sum_{i \notin M_I(x)} \left( \frac{x_i}{\max_{i \in J_I(x)} |x_i|} \right)^{2p+1} \left( \frac{1}{2p+1} \right)
\]
We need to compute the limit of $\alpha_p(x)$, when $p \to \infty$. Clearly, for all $i \notin M_I(x)$, we have
\[
\left| \frac{x_i}{\max_{i \in J_I(x)} |x_i|} \right| < 1.
\] (2.6)
Since \( \mathcal{J}_I(x) \neq \emptyset \), \( \xi_I[x](\max_{i \in \mathcal{J}_I(x)} |x_i|) \neq 0 \), hence we consider two cases:

(i) \( \xi_I[x](\max_{i \in \mathcal{J}_I(x)} |x_i|) > 0 \).

For the sake of simplicity, define \( a = \xi_I[x](\max_{i \in \mathcal{J}_I(x)} |x_i|) \) and \( b_i = |\max_{i \in \mathcal{J}_I(x)} x_i| \) for each \( i \). We then obtain \( \alpha_p(x) = (a + \sum_{i \notin \mathcal{M}_I(|x|)} b_i^{2^{p+1}})^{\frac{1}{2p+1}} \). Moreover, from (2.6), we have \( |b_i| < 1 \) for all \( i \notin \mathcal{M}_I(|x|) \). By hypothesis \( a > 0 \) and we deduce that \( \lim_{p \to +\infty} \ln(a + \sum_{i \notin \mathcal{M}_I(|x|)} b_i^{2^{p+1}}) = \ln(a) \). Hence, we have

\[
\lim_{p \to +\infty} \ln \alpha_p(x) = \lim_{p \to +\infty} \frac{\ln(a + \sum_{i \notin \mathcal{M}_I(|x|)} b_i^{2^{p+1}})}{2p + 1} = 0.
\]

Thus, \( \lim_{p \to +\infty} \alpha_p(x) = 1 \). Hence, from (2.5), we deduce that:

\[
\lim_{p \to +\infty} \left( \sum_{i \in \mathcal{J}_I(x)} x_i^{2p+1} \right)^{\frac{1}{2p+1}} = \max_{i \in \mathcal{J}_I(x)} |x_i| = \varGamma_I(x).
\]

(ii) \( \xi_I[x](\max_{i \in \mathcal{J}_I(x)} |x_i|) < 0 \).

Applying (i), we then obtain

\[
- \lim_{p \to +\infty} \left( \sum_{i \in \mathcal{J}_I(x)} x_i^{2p+1} \right)^{\frac{1}{2p+1}} = \lim_{p \to +\infty} \left( \sum_{i \in \mathcal{J}_I(x)} (-x_i)^{2p+1} \right)^{\frac{1}{2p+1}} = \max_{i \in \mathcal{J}_I(x)} |x_i|.
\]

Thus

\[
\lim_{p \to +\infty} \left( \sum_{i \in \mathcal{J}_I(x)} x_i^{2p+1} \right)^{\frac{1}{2p+1}} = - \max_{i \in \mathcal{J}_I(x)} |x_i| = \min_{i \in \mathcal{J}_I(x)} x_i = \varGamma_I(x). \quad \square
\]

Let us introduce for all \( n \)-tuple \( x = (x_1, \ldots, x_n) \) the operation defined by:

\[
\bigoplus_{i \in I} x_i = \lim_{p \to +\infty} \left( \sum_{i \in I} x_i^{2p+1} \right)^{\frac{1}{2p+1}} = \lim_{p \to +\infty} \sum_{i \in I} x_i. \quad (2.7)
\]

Clearly, this operation encompasses as a special case the binary operation defined in equation (2.1). From Fact [2.1.1] and Definition [2.2.2] if \( n = 2 \) and \( I = \{1, 2\} \), then, for all \( (x_1, x_2) \in \mathbb{R}^2 \)

\[
\bigoplus_{i \in \{1, 2\}} x_i = x_1 \bigoplus x_2.
\]

**Example 2.2.4** Suppose that \( x = (2, 3, -2, -3, \frac{3}{2}, -3, 3, -\frac{1}{2}) \). First, note that \( [8] = \{1, \cdots, 8\} \) and \( \{x_i : i \in [8]\} = \{1, -2, 2, -3, 3, -\frac{1}{2}\} \). We have \( \{i : x_i = 3\} = \{2, 7\} \) and \( \{i : x_i = -3\} = \{4, 6\} \). Therefore, \( \text{Card}\{i : x_i = 3\} = \text{Card}\{i : x_i = -3\} = 2 \),
\[\xi_{[8]}(x)(i) = 0 \text{ for } i \in \{2, 4, 6, 7\}. \text{ Moreover: } \{i : x_i = 2\} = \{1\} \text{ and } \{i : x_i = -2\} = \{3\}.
Consequently, Card\{i : x_i = 2\} = Card\{i : x_i = -2\} = 1 \text{ and } \xi_{[8]}(x)(i) = 0 \text{ for } i \in \{1, 3\}.
Hence, we have
\[J_{[8]}(x) = \{8\} \setminus \{\{2, 4, 6, 7\} \cup \{1, 3\}\} = \{5, 8\}.
\]

Therefore \[J_{[8]}(x) = \{5, 8\}.\] Hence
\[\bigoplus_{i \in \{5, 8\}} x_i = \frac{3}{2} \bigoplus_{i \in \{5, 8\}} \left(-\frac{1}{2}\right) = x_5 = \frac{3}{2}.
\]

### 2.3 Some Algebraic Properties

A few immediate properties whose the proofs are obvious are established in the next Lemma.

**Proposition 2.3.1** For all \(x \in \mathbb{R}^n\) and all nonempty subset \(I\) of \([n]\), we have:

(a) If \(J_I(x) \neq \emptyset\) then there is some \(i_0 \in I\) such that \(x_{i_0} = \bigoplus_{i \in I} x_i\). Moreover, \(\xi(x)(i_0) > 0\).

(b) If \(\alpha \in \mathbb{R}\) and \(x_i = \alpha\) for all \(i \in I\) then \(\bigoplus_{i \in I} x_i = \alpha\).

(c) Moreover, if all the elements of the family \(\{x_i\}_{i \in I}\) are mutually non symmetrical, then:
\[\bigoplus_{i \in I} x_i = \arg \max_{x_i \in I} \{x_i, i \in I\}.\]

(d) For all \(\alpha \in \mathbb{R}\), one has:
\[\alpha \left(\bigoplus_{i \in I} x_i\right) = \bigoplus_{i \in I} (\alpha x_i).\]

(e) Suppose that \(x \in \mathbb{R}^n_+\) where \(\epsilon\) is +1 or −1. Then \(\bigoplus_{i \in I} x_i = \epsilon \max_{i \in I} \{\epsilon x_i\}\).

(f) We have \(\bigoplus_{i \in I} x_i \leq \bigoplus_{i \in I} |x_i|\).

(g) For all permutation \(\sigma : I \rightarrow I\), we have \(\bigoplus_{i \in I} x_i = \bigoplus_{i \in I} x_{\sigma(i)}\).

(h) If there exists \(j, k \in I\) with \(j \neq k\) and \(x_j + x_k = 0\), then \(\bigoplus_{i \in I \setminus \{j, k\}} x_i = \bigoplus_{i \in I} x_i\). Moreover \(\bigoplus_{i \in I} x_i = \bigoplus_{i \in J_I(x)} x_i\).

**Proof:** (a) By hypothesis, the subset \(J = \{i \in I : \xi_I(x)(i) \neq 0\}\) is nonempty. Therefore, there exists some \(i_0 \in J_I(x)\) such that \(x_{i_0} \geq |x_i|\) for all \(i \in J_I(x)\). There are two possibilities. If \(\xi_I(x)(\max_{i \in J_I(x)} |x_i|) > 0\), then, from Lemma 2.2.1(e), \(\max_{i \in J_I(x)} |x_i| = \max_{i \in J_I(x)} x_i = x_{i_0}\). Thus, \(\xi_I(x)(x_{i_0}) > 0\). If \(\xi_I(x)(\max_{i \in J_I(x)} |x_i|) < 0\), from Lemma 2.2.1(f), we have \(\min_{i \in J_I(x)} x_i = -\max_{i \in J_I(x)} |x_i| = x_{i_0}\). Since \(\xi_I(x)\) is an impair map, this also implies that \(\xi_I(x)(x_{i_0}) > 0\). (b) is immediate setting \(x_i = \alpha\) for all \(i \in I\). (c) If all the \(x_i\)'s are mutually non symmetrical, then \(\xi_I(x)(x_i) \neq 0\) for all \(i \in I\). Hence, \(\xi_I(x)(\max_{i \in J_I(x)} |x_i|) \neq 0\) and there is some \(\epsilon \in \{-1, 1\}\) such that \(F_I(x) = \epsilon \max_{j \in J_I(x)} |x_i|\), we deduce (c). (d) Since the scalar multiplication is distributive on addition, it is an immediate consequence of Proposition 2.2.3 (e). If \(x \in \mathbb{R}^n_+\) then \(\bigoplus_{i \in I} x_i = \max_{i \in I} x_i\). \(x \in -\mathbb{R}^n_+\) implies that \(\xi_I(x)(\max_{i \in I} |x_i|) < 0\) and \(\bigoplus_{i \in I} x_i = \min_{i \in I} x_i = -\max_{i \in I} (-x_i)\). (f) For all \(x \in \mathbb{R}^n\) and all \(I \subset [n]\), we have \(F_I(|x|) = \max_{i \in I} |x_i|\). Moreover, by definition there is some \(\delta \in \{-1, 0, 1\}\) such that \(F_I(x) = \delta \max_{j \in J_I(x)} \delta x_i\). Therefore \(F_I(|x|) \geq |F_I(x)|\). (g)
Given a nonempty subset $I$ of $[n]$, the H"{o}lder sum is independent of any permutation of the index set $I$. Therefore, from Proposition 2.2.3, we deduce $(g)$. (h) In such a case 
\[ \{ i \in I \setminus \{ j, k \} : \xi_I[x](x_i) \neq 0 \} = \{ i \in I : \xi_I[x](x_i) \neq 0 \} \]. Therefore $J_{f_{I\setminus\{j,k\}}}(x) = J_f(x)$, which proves the first part of the statement. Since $J_{\xi_I(x)}(x) = J_f(x)$, the second part is immediate. \[ \square \]

In the following we introduce the operation \[ \langle \cdot, \cdot \rangle_\infty : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \] defined for all $x, y \in \mathbb{R}^n$ by \[ \langle x, y \rangle_\infty = \bigoplus_{i \in [n]} x_i y_i. \] Let $\| \cdot \|_\infty$ be the Tchebychev norm defined by \[ \| x \|_\infty = \max_{i \in [n]} |x_i|. \] The next result is an immediate consequence of Proposition 2.3.1.

**Proposition 2.3.2** For all $x, y \in \mathbb{R}^n$, we have:

(a) $\sqrt{\langle x, x \rangle_\infty} = \| x \|_\infty$.

(b) $|\langle x, y \rangle_\infty| \leq \| x \|_\infty \| y \|_\infty$.

(c) For all $\alpha \in \mathbb{R}$, $\alpha \langle x, y \rangle_\infty = \langle \alpha x, y \rangle_\infty = \langle x, \alpha y \rangle_\infty$.

**Proof:**

(a) By definition \[ \langle x, x \rangle_\infty = \bigoplus_{i \in [n]} x_i^2 = \max_{i \in [n]} x_i^2 = \| x \|_\infty^2, \] which ends the proof. (b) From Proposition 2.3.1, \[ |\langle x, y \rangle_\infty| = |\bigoplus_{i \in [n]} x_i y_i| \leq \bigoplus_{i \in [n]} |x_i y_i| \leq (\max_{i \in [n]} |x_i|) (\max_{i \in [n]} |y_i|), \] which proves (b). (c) is immediate from Proposition 2.3.1d. \[ \square \]

The next statement establishes a key property resulting from Proposition 2.3.3.

**Proposition 2.3.3** Suppose that $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For all nonempty subset $I$ of $[n]$ and all $i \in I$:

\[ x_i \bigoplus_{j \in I \setminus \{ i \}} x_j \in \{ 0, \bigoplus_{j \in I} x_j \} \]

and

\[ \bigoplus_{i \in I} x_i = \bigoplus_{i \in I} \left[ x_i \bigoplus_{j \in I \setminus \{ i \}} x_j \right]. \]

**Proof:** If $J_f(x) = \emptyset$, then this property is immediate. In such a case, since $I$ is nonempty, from Proposition 2.3.1a, there is some $i_0 \in I \setminus \{ i \}$ such that $x_{i_0} = -x_i$. Therefore $x_i \bigoplus_{j \in I \setminus \{ i_0 \}} x_j = 0$. Suppose now that $J_{f_{I \setminus \{ i \}}}(x) \neq \emptyset$ and let us consider four cases:

(i) $|x_i| > \bigoplus_{j \in I} x_j$. This implies that $i \in I \setminus J_f(x)$. Thus $\xi_I[x](x_i) = 0$. Hence, $\xi_{I \setminus \{ i \}}[x](x_i) < 0$ and from Proposition 2.3.1a $\bigoplus_{j \in I \setminus \{ i \}} x_j \neq x_i$. Moreover $\xi_{I \setminus \{ i \}}[x](x_i) > 0$, and by hypothesis, $|x_i| \geq |x_j|$ for all $j \in J_{f_{I \setminus \{ i \}}}(x)$. Therefore $\bigoplus_{j \in I \setminus \{ i \}} x_j = -x_i$. It follows that \[ x_i \bigoplus_{j \in I \setminus \{ i \}} x_j = x_i \bigoplus_{j \in I \setminus \{ i \}} (-x_i) = 0, \] which proves this case.
(ii) \( x_i = \bigoplus_{j \in I} x_j \). By definition, this implies that \( \xi_L[x](x_i) > 0 \). Since \( \xi_{L\setminus\{i\}}[x](x_i) = \xi_L[x](x_i) - 1 \), one has \( \xi_{L\setminus\{i\}}[x](x_i) \geq 0 \) and, consequently, \( \xi_{L\setminus\{i\}}[x](x_i) \leq 0 \). Thus, from Proposition 2.3.1a, \( \bigoplus_{j \in I \setminus \{i\}} x_j \neq -x_i \). Moreover, \( |\bigoplus_{j \in I \setminus \{i\}} x_j| \leq |x_i| \) and we have

\[
x_i \oplus (\bigoplus_{j \in I \setminus \{i\}} x_j) = x_i = \bigoplus_{j \in I} x_j.
\]

(iii) \( x_i = -\bigoplus_{j \in I} x_j \). Equivalently, we have \( -x_i = \bigoplus_{j \in I} x_j \) and, from Proposition 2.3.1a, this implies that there is some \( i_0 \in I \) such that \( x_{i_0} = \bigoplus_{j \in I} x_j = -x_i \) with \( \xi_L[x](x_{i_0}) > 0 \). Since \( J_L(x) \neq \emptyset \), \( x_{i_0} \neq 0 \) and \( x_{i_0} \neq x_i \). It follows that \( i_0 \neq i \). Thus \( \xi_{L \setminus \{i\}}[x](x_{i_0}) > 0 \). Therefore, \( \bigoplus_{j \in I \setminus \{i\}} x_j = x_{i_0} = -x_i \). Consequently:

\[
x_i \oplus (\bigoplus_{j \in I \setminus \{i\}} x_j) = x_{i_0} = \bigoplus_{j \in I} x_j,
\]

which ends the proof of the first part of the statement.

To prove the second part of the statement, we need to establish that there exists some \( i \in I \) such that \( x_i \oplus (\bigoplus_{i \in I \setminus \{i\}} x_i) = x_i \). If \( J_L(x) = \emptyset \) then \( \bigoplus_{j \in I} x_j = 0 \) and from the statement above \( x_i \oplus (\bigoplus_{j \in I \setminus \{i\}} x_j) \in \{0\} \) for all \( i \). In such a case, this property is obviously true. Suppose that \( J_L(x) \neq \emptyset \). Recall that from Proposition 2.3.1a there is some \( i_0 \in I \) such that \( x_i = \bigoplus_{j \in I} x_j \). Then, using (ii), the second statement follows. \( \square \)

For example, for all \( x, y, z \in \mathbb{R} \), we have the identities: \( x \oplus y = x \oplus y \oplus x \oplus y \) and \( x \oplus y \oplus z = [x \oplus (y \oplus z)] \oplus [y \oplus (z \oplus x)] \oplus [z \oplus (x \oplus y)] \).

Let \( \Lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be the map defined by:

\[
\Lambda(x_1, ..., x_n) = \left(x_1 \oplus \left(\bigoplus_{j \in [n] \setminus \{1\}} x_j\right), ..., x_n \oplus \left(\bigoplus_{j \in [n] \setminus \{n\}} x_j\right)\right).
\]

(2.8)

**Lemma 2.3.4** Suppose that \( x = (x_1, ..., x_n) \in \mathbb{R}^n \). Let \( \epsilon \in \{-1, 1\} \) such that \( \bigoplus_{i \in I} x_i \in \epsilon \mathbb{R}_+ \). Then:

(a) \( \bigoplus_{i \in I} x_i = \epsilon \max_{i \in I} \{\epsilon \Lambda_i(x)\} \);

(b) For all \( i, j, k \in [n] \), we have:

\[
\Lambda_i(x) \oplus \Lambda_j(x) \oplus \Lambda_k(x) = \left(\Lambda_i(x) \oplus \Lambda_j(x)\right) \oplus \Lambda_k(x) = \Lambda_i(x) \oplus \left(\Lambda_j(x) \oplus \Lambda_k(x)\right).
\]

**Proof:** (a) From Proposition 2.3.3 we have \( \Lambda_i(x) \in \{0, \bigoplus_{i \in I} x_i\} \) for all \( i \in I \). Hence \( \Lambda_i(x) \in \epsilon \mathbb{R}_+ \) for all \( i \) and from Proposition 2.3.3c, the result follows. (b) is an immediate
Example 2.3.5 Let $x = (4, -3, -4, 2, 3, 2, -2) \in \mathbb{R}^7$. We have $\mathcal{J}_7(x) = \{4, 6, 7\}$ and $[7] \setminus \mathcal{J}_7(x) = \{1, 2, 3, 5\}$. We have $4 \boxplus (-3) \boxplus (-4) \boxplus 2 \boxplus 3 \boxplus 2 \boxplus (-2) = 2$. Moreover, $\Lambda_1(x) = 4 \boxplus ((-3) \boxplus (-4) \boxplus 2 \boxplus 3 \boxplus 2 \boxplus (-2)) = 4 \boxplus (-4) = 0$; Similarly we obtain $\Lambda_2(x) = (-3) \boxplus 3 = 0$; $\Lambda_3(x) = (-4) \boxplus 4 = 0$; $\Lambda_4(x) = 2 \boxplus 0 = 2$; $\Lambda_5(x) = 3 \boxplus (-3) = 0$; $\Lambda_6(x) = 2 \boxplus 0 = 2$; $\Lambda_7(x) = (-2) \boxplus 2 = 0$. It follows that $\Lambda(x) = (0, 0, 0, 0, 2, 0, 0) = (0, 0, 0, \bigoplus_{i \in [7]} x_i, 0, \bigoplus_{i \in [7]} x_i, 0)$.

Lemma 2.3.6 Let $n$ be a positive natural number and $I$ be a nonempty subset of $[n]$. Let $\mathcal{Q}(I) = \{I_j : j \in [m]\}$ be a partition of $I$ with $m$ nonempty subsets $I_j$. If for all $(j, k) \in [m] \times [m]$ \((\bigoplus_{i \in I_j} x_i) + (\bigoplus_{i \in I_k} x_i) \neq 0\) then \(\bigoplus_{j \in [m]} (\bigoplus_{i \in I_j} x_i) = \bigoplus_{i \in I} x_i\).

Proof: For all $j \in [m]$, let us denote $y_j = \bigoplus_{i \in I_j} x_i$. By hypothesis the $y_j$’s are mutually non symmetrical, it follows that there exists some $j_0 \in [m]$ such that $y_{j_0} = \bigoplus_{j \in [m]} y_j = \arg \max_{y_j} \{|y_j| : j \in [m]\}$.

Therefore, for all $i \in I$ such that $|x_i| > |y_{j_0}|$, and all $j \in [m]$ $\xi_{I_j}[x](x_i) = 0$. However, by hypothesis $\mathcal{Q}(I)$ is a partition of $I$. Hence $I = \bigcup_{j \in [m]} I_j$ with $I_j \cap I_k = \emptyset$, for all $j \neq k$. Thus, for all $i \in I$ such that $|x_i| > |y_{j_0}|$ we have from Lemma 2.2.1g $\xi_I[x](x_i) = \sum_{j \in [m]} \xi_{I_j}[x](x_i) = 0$. It follows that $\mathcal{J}_I(x) \subset \{i \in I : |x_i| \leq |y_{j_0}|\}$.

Therefore, $|\bigoplus_{j \in [m]} (\bigoplus_{i \in I_j} x_i)| = |\bigoplus_{j \in [m]} y_j| = |y_{j_0}| \geq |\bigoplus_{i \in I} x_i|$. From Proposition 2.3.1a, $y_{j_0} \neq 0$ implies that there is some $i_0 \in I_{j_0}$ and such that $x_{i_0} = \bigoplus_{i \in I_{j_0}} x_i = y_{j_0}$. Since $|x_i| \leq |x_{i_0}|$ for all $i \in \mathcal{J}_I(x)$ and the $y_j$’s are not symmetrical it follows that $\xi_I[x](x_{i_0}) > 0$ which ends the proof. □

2.4 Euclidean Orthant, Absolute Value and Upper Semi-Lattice Structure

The algebraic structure $(\mathbb{R}, \boxplus, \cdot)$ can be extended to $\mathbb{R}^n$. Suppose that $x, y \in \mathbb{R}^n$, and let us denote

$$x \boxplus y = (x_1 \boxplus y_1, \ldots, x_n \boxplus y_n).$$

(2.9)
Moreover, let us consider $m$ vectors $x_1, ..., x_m \in \mathbb{R}^n$, and define

$$
\bigsqcup_{j \in [m]} x_j = \left( \bigsqcup_{j \in [m]} x_{j,1}, \ldots, \bigsqcup_{j \in [m]} x_{j,n} \right).
$$

(2.10)

Let the triple $(\mathbb{R}^n, \bigsqcup, \cdot)$ denotes the $n$-dimensional Euclidean vector space equipped with the operation binary operation $(x, y) \mapsto x \bigsqcup y$ and the external scalar multiplication of vectors by real numbers $\cdot$.

For all $(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ we have $x \bigsqcup y = x \lor y$. Moreover, for all $(x, y) \in \mathbb{R}_-^n \times \mathbb{R}_-^n$, $x \bigsqcup y = x \land y$ where $\lor$ and $\land$ respectively denote the maximum and the minimum with respect to partial order of $\mathbb{R}^n$ associated to the positive cone, that is, the coordinatewise supremum and infimum. For all $x$ and $y$ in $\mathbb{R}^n$, $x \leq y$ means $y - x \in \mathbb{R}_+^n$. It follows that given a subset $\{x_1, ..., x_m\}$ of $\mathbb{R}_+^n$, we have

$$
\bigsqcup_{j \in [m]} x_j = \bigvee_{j \in [m]} x_j.
$$

(2.11)

If $\{x_1, ..., x_m\}$ is a subset of $\mathbb{R}_+^n$ then

$$
\bigsqcup_{j \in [m]} x_j = \bigwedge_{j \in [m]} x_j.
$$

(2.12)

For all $x, y \in \mathbb{R}^n$, let us denote $x \boxplus y = (x_1 y_1, ..., x_n y_n)$. In the following we say that two vectors $x, y \in \mathbb{R}^n$ are copositive if

$$
x \boxplus y \in \mathbb{R}_+^n.
$$

(2.13)

We say that a subset $K$ of $\mathbb{R}^n$ is copositive if for all $x, y \in K$ one has $x \boxplus y \geq 0$. For all subset $L$ of $\mathbb{R}^n$, $K$ is copositive and maximal in $L$ if there does not exists a copositive subset $K' \subset L$ which contains $K$. A $n$-dimensional orthant in $\mathbb{R}^n$ is copositive and maximal in $\mathbb{R}^n$. Equivalently, a $n$-dimensional orthant in $\mathbb{R}^n$ is a subset defined by a system of inequalities: $\epsilon_i x_i \geq 0$ for any $i \in [n]$, where each $\epsilon_i$ is $+1$ or $-1$. A $n$-dimensional closed orthant $K$ of $\mathbb{R}^n$ can be written $K = \prod_{i=1}^{n} (+\mathbb{R}^n)$. Let $\Psi_K : \mathbb{R}^n_+ \longrightarrow K$ be the map defined by $\Psi_K(x) = (\epsilon_1 x_1, ..., \epsilon_n x_n)$ with $|\epsilon_i| = 1$, for all $i \in [n]$. $\psi_K$ is a linear homeomorphism such that $\psi_K(\mathbb{R}_+^n) = K$ and one has $\psi_K^{-1} = \psi_K$, which implies that $\psi_K(K) = \mathbb{R}_+^n$.

For all $x \in \mathbb{R}^n$, let us denote $|x| = (|x_1|, ..., |x_n|)$. Let $K$ be a $n$-dimensional orthant and let us consider the binary relation defined by $x \leq y \iff |x| \leq |y|$. $\leq$ is a partial order over $K$. For all $x, y,$ and $z$ in $K$, we have $x \leq x$ (reflexivity); if $x \leq y$ and $y \leq x$ then $x = y$ (antisymmetry); if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity). A $n$-dimensional closed orthant $K$ equipped with the partial order $\leq$ is a partially ordered set (or a poset). Then $\bigsqcup$ is a join on $K$, and the triple $(K, \bigsqcup, \geq)$ is an upper-semilattice.

If $\{x_1, ..., x_m\}$ is a subset of $K$ then

$$
\bigsqcup_{j \in [m]} x_j = \Psi_K \left( \bigvee_{j \in [m]} \Psi_K(x_j) \right).
$$

(2.14)

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3 On Some Idempotent Convex Structure

A subset $C$ of a $\mathbb{R}_+^n$ is $\mathbb{B}$-convex if and only if for all $t \in [0, 1]$ and all $x, y \in C$, $x \vee ty \in C$. Equivalently, we say that a subset $C$ of a $n$-dimensional orthant $K$ is $\mathbb{B}$-convex if and only if for all $t \in [0, 1]$ and all $x, y \in C$, $x \oplus ty \in C$. Such a definition is equivalent to that one proposed in [7]. It is also the definition proposed further in the paper to define $\mathbb{B}$-convex sets on the whole Euclidean vector space. Equivalently, a subset $C$ of $K$ is $\mathbb{B}$-convex if and only if $\Psi_K(C)$ is a $\mathbb{B}$-convex subset of $\mathbb{R}_+^n$. For all finite subset $A = \{x_1, \ldots, x_m\}$ of $K$ the smallest $\mathbb{B}$-convex set which contains it is $\mathbb{B}[A] = \left\{ \sum_{i=1}^{m} t_i x_i : t_i \in [0, 1], \max_{i \in [m]} t_i = 1 \right\}$. For the sake of simplicity, let $\mathbb{B}[x, y]$ denote the $\mathbb{B}$-convex hull of $\{x, y\}$ for all $x, y \in K$.

The binary operation $\boxplus$ yields a simple formulation of $\mathbb{B}$-convexity on each orthant. However, the problem to solve is much more complex over $\mathbb{R}^n$. Suppose for example that $x, y \in \mathbb{R}^n$, $|x| = |y|$ and $x \neq y$, then $\{tx \oplus sy : \max\{t, s\} = 1, t, s \geq 0\} = \{x, x \oplus y, y\}$ that is not a path-connected subset of $\mathbb{R}^n$.

3.1 An Extended Definition of $\mathbb{B}$-convexity

In [7] $\mathbb{B}$-convexity is introduced as a limit of linear convexities. More precisely, for all $p \in \mathbb{N}$ the $\Phi_p$-convex hull of a finite set $A \subset \mathbb{R}^n$ is defined by:

$$Co^{\Phi_p}(A) = \left\{ \sum_{i \in [m]} t_i \varphi_p x_i : \sum_{i \in [m]} t_i = \varphi_p^{-1}(1), \varphi_p(t_i) \geq 0, i \in [m] \right\}$$ (3.1)

which can be rewritten:

$$Co^{\Phi_p}(A) = \left\{ \Phi_p^{-1}\left( \sum_{i \in [m]} t_i^{2p+1} \Phi_p(x_i) \right) : \left( \sum_{i \in [m]} t_i^{2p+1} \right)^{\frac{1}{2p+1}} = 1, t_i \geq 0, i \in [m] \right\}.$$ 

This is basically the approach of Ben-Tal [6] and Avriel [4].

Equivalently, one has $Co^{\Phi_p}(A) = \Phi_p^{-1}\left( Co\left(\Phi_p(A)\right)\right)$. Recall that, for all $x, y \in \mathbb{R}^n$, $x + y = x + y$. For simplicity, throughout the remainder of the paper we denote for all subset $L$ of $\mathbb{R}^n$ $Co^{\Phi_p}(L) = Co^{\Phi_p}(L)$.

From Briec and Horvath [7] a subset $L$ of $\mathbb{R}^n$ is $\mathbb{B}$-convex if for all finite subset $A \subset L$ the $\mathbb{B}$-polytope $Co^{\infty}(A) = Ls_{p \to \infty} Co^p(L)$ is contained in $L$. In the following, we show that the Painlevé-Kuratowski limit of the $\Phi_p$-convex hull of two points $x, y$ exists in $\mathbb{R}^n$ and we give an algebraic characterization.

1 The Kuratowski-Painlevé lower limit of the sequence of sets $\{A_n\}_{n \in \mathbb{N}}$, denoted $Li_{n \to \infty} A_n$, is the set of points $p$ for which there exists a sequence $\{p_n\}$ of points such that $p_n \in A_n$ for all $n$ and $p = \lim_{n \to \infty} p_n$; a sequence $\{A_n\}_{n \in \mathbb{N}}$ of subsets of $\mathbb{R}^m$ is said to converge, in the Kuratowski-Painlevé sense, to a set $A$ if $Ls_{n \to \infty} A_n = A = Li_{n \to \infty} A_n$, in which case we write $A = \lim_{n \to \infty} A_n$. 

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In this paper, a weaker definition is proposed in line with the algebraic structure above introduced.

**Definition 3.1.1** A subset $C$ of $\mathbb{R}^n$ is $B^z$-convex if and only if for all $t \in [0,1]$ and all $(x,y) \in C \times C$, $x \oplus ty \in C$.

Notice that for all $n$-dimensional orthant $K$ of $\mathbb{R}^n$, a $B$-convex subset of $K$ is $B^z$-convex. It is shown that the following definitions of $B^z$-convexity are equivalent.

**Proposition 3.1.2** For all subset $C$ of $\mathbb{R}^n$, the following claims are equivalent:

(a) $C$ is a $B^z$-convex subset of $\mathbb{R}^n$.

(b) For all $(x_1,\ldots,x_m) \in C^m$ we have:

$$\left\{ \bigoplus_{i \in [m]} t_i x_i : \max_{i \in [m]} t_i = \max_{i \in [m]} x_i = 1, t \in [0,1]^m \right\} \subseteq C.$$

**Proof:** Let us prove that (a) implies (b). If $C$ is $B^z$-convex, this property is true for $m = 2$. Suppose it is true at rank $m$ and let us prove that it is true at rank $m + 1$. In other words, assume that for all $(x_1,\ldots,x_m) \in C^m$ we have:

$$\left\{ \bigoplus_{i \in [m]} t_i x_i : \max_{i \in [m]} t_i = 1, t \in [0,1]^m \right\} \subseteq C,$$

we need to prove that if $(x_1,\ldots,x_m,x_{m+1}) \in C^{m+1}$ then for all $t \in [0,1]^{m+1}$ such that $\max_{i \in [m+1]} t_i = 1$ we have $\bigoplus_{i \in [m+1]} t_i x_i \in C$. To establish this property, we use Proposition 2.3.3 which implies that if $(x_1,\ldots,x_m,x_{m+1}) \in C^{m+1}$ then, for all $t \in [0,1]^{m+1}$ such that $\max_{i \in [m+1]} t_i = 1$, we have

$$\bigoplus_{i \in [m+1]} t_i x_i = \bigoplus_{i \in [m+1]} \left[ t_i x_i \oplus \left( \bigoplus_{j \in [m+1] \setminus \{i\}} t_j x_j \right) \right]. \tag{*}$$

For all $i$ set $t_i^* = \max\{t_j : j \in [m+1] \setminus \{i\}\}$. It follows that

$$\bigoplus_{i \in [m+1]} t_i x_i = \bigoplus_{i \in [m+1]} \left[ t_i x_i \oplus t_i^* \left( \bigoplus_{j \in [m+1] \setminus \{i\}} \left( \frac{t_j}{t_i^*} \right) x_j \right) \right].$$

(i) By definition, if $t_i^* = \max\{t_j : j \in [m+1] \setminus \{i\}\} < 1$ then, since $\max_{j \in [m+1]} t_j = 1$, we have $t_i = 1$. Moreover, $\max_{j \in [m+1] \setminus \{i\}} \left( \frac{t_j}{t_i^*} \right) = 1$ and since, by hypothesis, the property is assumed to be true at rank $m$, it follows that $\bigoplus_{j \in [m+1] \setminus \{i\}} \left( \frac{t_j}{t_i^*} \right) x_j \in C$. Hence, we deduce that

$$x_i \oplus t_i^* \left( \bigoplus_{j \in [m+1] \setminus \{i\}} \left( \frac{t_j}{t_i^*} \right) x_j \right) \in C.$$  

(ii) If $t_i^* = 1$ then there is some $i_0 \in [m+1] \setminus \{i\}$ such that $t_{i_0} = 1$ and, by hypothesis, it follows that $\bigoplus_{j \in [m+1] \setminus \{i\}} t_j x_j \in C$. Furthermore, since $t_i \in [0,1]$ we deduce from (a) that $t_i x_i \oplus \left( \bigoplus_{j \in [m+1] \setminus \{i\}} t_j x_j \right) \in C$. For all $i$, set $\Lambda_i = t_i x_i \oplus \left( \bigoplus_{j \in [m+1] \setminus \{i\}} t_j x_j \right)$. We have
proven that, for each \( i, \Lambda_i \in C \). Moreover, from Propositions 2.3.3 and Lemma 2.3.4, the \( \Lambda_i \)'s belong to a \( n \)-dimensional orthant \( K \) and, then, can be composed associatively using the operation \( \boxplus \). Thus, we deduce from (\( \star \)) that \( \bigboxplus_{i \in [m+1]} t_i x_i \in C \) which ends the proof of (b). The converse inclusion is immediate. \( \square \)

**Proposition 3.1.3** (a) The emptyset, \( \mathbb{R}^n \), as well as all the singletons are \( \mathbb{B}^\sharp \)-convex. (b) If \( \{ D_\delta : \delta \in \Delta \} \) is an arbitrary family of \( \mathbb{B}^\sharp \)-convex sets then \( \bigcap_\lambda D_\delta \) is \( \mathbb{B}^\sharp \)-convex. (c) If \( \{ D_\lambda : \delta \in \Delta \} \) is a family of \( \mathbb{B}^\sharp \)-convex sets such that \( \forall \delta_1, \delta_2 \in \Delta \exists \delta_3 \in \Delta \) such that \( D_{\delta_1} \cup D_{\delta_2} \subset D_{\delta_3} \) then \( \bigcup_\delta D_\delta \) is \( \mathbb{B}^\sharp \)-convex. (d) If \( C \) a \( \mathbb{B} \)-convex subset of \( \mathbb{R}^n_+ \) then it is \( \mathbb{B}^\sharp \)-convex.

Given a set \( S \subset \mathbb{R}^n \) there is, according to (a) above, a \( \mathbb{B}^\sharp \)-convex set which contains \( S \); by (b) the intersection of all such \( \mathbb{B}^\sharp \)-convex sets is \( \mathbb{B} \)-convex; we call it the \( \mathbb{B}^\sharp \)-convex hull of \( S \) and we write \( \mathbb{B}^\sharp[S] \) for the \( \mathbb{B}^\sharp \)-convex hull of \( S \).

**Proposition 3.1.4** The following properties hold:

(a) \( \mathbb{B}^\sharp[0] = \emptyset, \mathbb{B}^\sharp[\mathbb{R}^n] = \mathbb{R}^n \), for all \( x \in \mathbb{R}^n \), \( \mathbb{B}^\sharp[\{x\}] = \{x\} \).

(b) For all \( S \subset \mathbb{R}^n \), \( S \subset \mathbb{B}^\sharp[S] \) and \( \mathbb{B}^\sharp[\mathbb{B}^\sharp[S]] = \mathbb{B}^\sharp[S] \).

(c) For all \( S_1, S_2 \subset \mathbb{R}^n \), if \( S_1 \subset S_2 \) then \( \mathbb{B}^\sharp[S_1] \subset \mathbb{B}^\sharp[S_2] \).

(d) For all \( S \subset \mathbb{R}^n \), \( \mathbb{B}^\sharp[S] = \bigcup \{ \mathbb{B}^\sharp[A] : A \text{ is a finite subset of } S \} \).

(e) A subset \( L \subset \mathbb{R}^n \) is \( \mathbb{B}^\sharp \)-convex if and only if, for all finite subset \( A \) of \( L \), \( \mathbb{B}^\sharp[A] \subset L \).

### 3.2 Intermediate Points and Copositivity

A set of points we term the **intermediate points** is introduced to characterize the \( \mathbb{B} \)-convex hull of two points on the whole Euclidean vector space. For all \( (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \), let us consider the map \( \gamma : \mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty) \longrightarrow \mathbb{R} \) defined by:

\[
\gamma(x, y, t) = (\max\{1, t\})^{-1}(x \boxplus ty), \quad \text{for all } t \geq 0 \tag{3.2}
\]

and by \( \gamma(x, y, +\infty) = y \). For all \( (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \), let \( I(x, y) = \{ i \in [n] : x_i y_i < 0 \} \) and let \( n(x, y) \) be its cardinal. Remark that \( \gamma(x, y, 0) = x \).

For all \( i \in I(x, y) \) and all \( t_i^* \in \mathbb{R}_{++} \) a point \( \gamma \in \mathbb{R}^n \) is called an \( i \)-**intermediate point** between \( x \) and \( y \) if there is some \( t_i^* \in [0, +\infty) \) such that

\[
\gamma_i(x, y, t_i^*) := (\gamma(x, y, t_i^*))_i = 0. \tag{3.3}
\]

**Lemma 3.2.1** Let \( \gamma : \mathbb{R}^n \times \mathbb{R}^n \times (\mathbb{R}_+ \cup \{+\infty\}) \longrightarrow \mathbb{R}^n \) be the map defined in (3.2). Suppose that \( I(x, y) \neq \emptyset \). We have the following properties:
(a) For all $i \in \mathcal{I}(x, y)$, the map $t \mapsto \gamma_i(x, y, t)$ has a uniqueness zero $t^*_i = -\frac{x_i}{y_i} = |\frac{x_i}{y_i}| > 0$ and there is a uniqueness $i$-intermediate point

$$
\gamma(x, y, t^*_i) = \left(\frac{|y_i|}{\max\{|x_i|, |y_i|\}} x\right) \oplus \left(\frac{|x_i|}{\max\{|x_i|, |y_i|\}} y\right).
$$

(b) For all $i \in \mathcal{I}(x, y)$ and all $t \geq 0$

$$
\gamma_i(x, y, t) = \begin{cases} 
\max\{1, t\}^{-1}x_i & \text{if } t < -\frac{x_i}{y_i} \\
0 & \text{if } t = -\frac{x_i}{y_i} \\
t \max\{1, t\}^{-1}y_i & \text{if } t > -\frac{x_i}{y_i}
\end{cases}
$$

(c) If $t > \max\{1, -\frac{x_i}{y_i}\}$ then $\gamma_i(x, y, t) = y_i$. Moreover, if $t < \min\{1, -\frac{x_i}{y_i}\}$ then $\gamma_i(x, y, t) = x_i$.

(d) $\lim_{t \to 0} \gamma(x, y, t) = x$ and $\lim_{t \to +\infty} \gamma(x, y, t) = y$.

**Proof:** (a) For all $i \in \mathcal{I}(x, y)$, we have $x_i y_i < 0$, which implies that $-\frac{x_i}{y_i} = |\frac{x_i}{y_i}| > 0$. Moreover, $\gamma_i(x, y, t) = 0$ if and only if $\left(\max\{1, t\}^{-1}x_i\right) \oplus \left(\max\{1, t\}^{-1}y_i\right) = 0$. Since this is equivalent to $x_i \oplus t y_i = 0$, we deduce that $t^*_i = -\frac{x_i}{y_i}$ is the uniqueness positive zero of the equation $\gamma_i(x, y, t) = 0$. Moreover $\gamma_i(x, y, 0) = x_i \neq 0$ and $\gamma_i(x, y, +\infty) = y_i \neq 0$, which ends the proof. (b) The case $t = -\frac{x_i}{y_i}$ is an immediate consequence of (a). Assume that $t \neq -\frac{x_i}{y_i}$. In such a case, one has

$$
\gamma_i(x, y, t) = \begin{cases} 
\max\{1, t\}^{-1}x_i & \text{if } \max\{1, t\}^{-1}x_i > t \max\{1, t\}^{-1}y_i \\
 \max\{1, t\}^{-1}t \max\{1, t\}^{-1}y_i & \text{if } \max\{1, t\}^{-1}x_i < t \max\{1, t\}^{-1}y_i
\end{cases}
$$

Since $|\frac{x_i}{y_i}| = -\frac{x_i}{y_i}$, it follows that

$$
\gamma_i(x, y, t) = \begin{cases} 
\max\{1, t\}^{-1}x_i & \text{if } t < -\frac{x_i}{y_i} \\
 \max\{1, t\}^{-1}t \max\{1, t\}^{-1}y_i & \text{if } t > -\frac{x_i}{y_i}
\end{cases}
$$

(c) If $t > \max\{1, -\frac{x_i}{y_i}\}$, then, from (b) one has $\gamma_i(x, y, t) = t \max\{1, t\}^{-1}y_i$. Moreover $\max\{1, t\} = t$. Therefore $\gamma_i(x, y, t) = y_i$. If $t < \min\{1, -\frac{x_i}{y_i}\}$, then $\gamma_i(x, y, t) = \max\{1, t\}^{-1}x_i$. Moreover $\max\{1, t\} = 1$. Therefore $\gamma_i(x, y, t) = x_i$. (d) Suppose that $j \notin \mathcal{I}(x, y)$, then there is $\epsilon \in \{-1, 1\}$ such that $\left(\max\{1, t\}^{-1}x\right)_j \oplus \left(t \max\{1, t\}^{-1}y\right)_j \in \epsilon \mathbb{R}_+$. It follows that the map $t \mapsto \gamma_j(x, y, t)$ is continuous. Hence, we clearly have $\lim_{t \to 0} \gamma_j(x, y, t) = x_j$ and $\lim_{t \to +\infty} \gamma_j(x, y, t) = y_j$. Suppose now that $i \in \mathcal{I}(x, y)$. From (c) $\lim_{t \to 0} \gamma_i(x, y, t) = x_i$ and $\lim_{t \to +\infty} \gamma_i(x, y, t) = y_i$ which ends the proof. □

Notice that it may happen that there are two indexes $i, k \in \mathcal{I}(x, y)$ such that $\gamma(x, y, t^*_i) = \gamma(x, y, t^*_k)$. Let $\Theta(x, y) = \{0, +\infty, -\frac{x_i}{y_i} : i \in \mathcal{I}(x, y)\}$. If $\mathcal{I}(x, y) = \emptyset$, then $\Theta(x, y) = \{x, y\}$. 

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Example 3.2.2 Let $x = (4, 2)$, $x' = (3, 4)$, $x'' = (-\frac{7}{2}, 3)$ and $y = (-2, -3)$ four points of $\mathbb{R}^2$. Clearly, we have $\mathcal{I}(x, y) = \mathcal{I}(x', y) = \{1, 2\}$. Let us denote $\gamma_i$ the intermediate points between $x$ and $y$. We have $\gamma_1 = \max\{4, 2\}^{-1}\left((2, 4) \bigoplus (4(-2, -3))\right) = (0, -3)$. The second intermediate point between $x$ and $y$ is $\gamma_2 = \max\{2, 3\}^{-1}\left((3, 4) \bigoplus 2(-2, -3)\right) = (4, 0)$. Let us denote $\gamma_i'$ the intermediate points between $x'$ and $y$. We have $\gamma_1' = \max\{3, 2\}^{-1}\left((2, 3) \bigoplus 3(-2, -3))\right) = (0, -3).$ Following a similar procedure, we get $\gamma_2' = (\frac{3}{4}, 0)$. Finally, we have $\mathcal{I}(x'', y) = \{2\}$, and using similar notations we obtain $\gamma_2'' = (-\frac{21}{8}, 0)$.

Lemma 3.2.3 For all $x, y \in \mathbb{R}^n$ such that $\mathcal{I}(x, y) \neq \emptyset$, let $\{i_m\}_{m \in [n(x,y)]}$ be an index sequence of $\mathcal{I}(x, y)$ such that

$$i_m \in \arg \max_{i \in \mathcal{I}(x, y)} \{-\frac{x_i}{y_i} : i \geq m\} \quad \text{for all} \quad m \in [n(x, y)].$$

We have for all $m \in [n(x, y) - 1]$:

$$\gamma(x, y, -\frac{x_{i_m}}{y_{i_m}}) \bigoplus \gamma(x, y, -\frac{x_{i_m+1}}{y_{i_m+1}}) \geq 0.$$  

Moreover $x \bigoplus \gamma(x, y, -\frac{x_{i_m}}{y_{i_m}}) \geq 0$, $\gamma(x, y, -\frac{x_{i_m(x,y)}}{y_{i_m(x,y)}}) \bigoplus y \geq 0$.

Proof: Suppose that $j \notin \mathcal{I}(x, y)$. In such a case $x_j y_j \geq 0$. It follows that there is some $\epsilon \in \{-1, 1\}$ such that $x_j, y_j \in \epsilon \mathbb{R}_+$. Therefore $\gamma_j(x, y, t) = [(\max\{1, t\}^{-1} x) \bigoplus (t \max\{1, t\}^{-1} y)]_j \in \epsilon \mathbb{R}_+$, for all $t \geq 0$. Hence if $\gamma(x, y, t^*_m)$ and $\gamma(x, y, t^*_{m+1})$ are two intermediate points, then for all $j \in [n] \setminus \mathcal{I}(x, y)$ one has

$$(\gamma_j(x, y, t^*_m))(\gamma_j(x, y, t^*_{m+1})) \geq 0.$$  

Suppose now that $i \in \mathcal{I}(x, y)$. Set $t^*_i = -\frac{x_i}{y_i}$ for all $i \in \mathcal{I}(x, y)$. By construction $\{t^*_m, m \in [n(x,y)]\}$ is a nondecreasing sequence of $\mathbb{R}_+$. Since $t^*_m$ and $t^*_{m+1}$ are two consecutive terms of this sequence, for all $i \in \mathcal{I}(x, y)$ we have $t^*_m \notin [t^*_m, t^*_{m+1}]$. Thus one has either $t^*_i \leq t^*_m \leq t^*_m + 1$ or $t^*_i \geq t^*_m + 1 \geq t^*_m$. From Lemma 3.2.1b,

$$\gamma_i(x, y, t^*_m) \gamma_i(x, y, t^*_{m+1}) = \begin{cases} \max\{1, t^*_m\}^{-1} \max\{1, t^*_{m+1}\}^{-1}(x_i)^2 \geq 0 & \text{if } t^*_i > t^*_{m+1} \geq t^*_m \\ 0 & \text{if } t^*_i \in \{t^*_m, t^*_{m+1}\} \\ t^*_m t^*_{m+1} \max\{1, t^*_m\}^{-1} \max\{1, t^*_{m+1}\}^{-1}(y_i)^2 \geq 0 & \text{if } t^*_i < t^*_m \leq t^*_{m+1} \end{cases}$$

(3.4)

It follows that for all $m \in [n(x, y) - 1]$, $\gamma(x, y, t^*_m) \bigoplus \gamma(x, y, t^*_{m+1}) \geq 0$. Let us prove that $x \bigoplus \gamma(x, y, t^*_i) \geq 0$ and $\gamma(x, y, t^*_{m(x,y)+1}) \bigoplus y \geq 0$. Since for all $i \in \mathcal{I}(x, y)$ $t^*_i \leq t^*_i \leq t^*_m$, we have

$$x_i \gamma_i(x, y, t^*_i) = \begin{cases} \max\{1, t^*_i\}^{-1}(x_i)^2 \geq 0 & \text{if } t^*_i > t^*_i \\ 0 & \text{if } t^*_i = t^*_i \end{cases}$$
and
\[
\gamma_i(x, y, t_{i_{n(x,y)}}^*) y_i = \begin{cases} 
0 & \text{if } t_i^* = t_{i_{n(x,y)}}^* \\
 t_{i_{n(x,y)}}^* \max\{1, t_{i_{n(x,y)}}^*\}^{-1}(y_i)^2 \geq 0 & \text{if } t_i^* < t_{i_{n(x,y)}}^* 
\end{cases}
\]

Since \( t_i^* = -\frac{x_i}{y_i} > 0 \) for all \( i \in \mathcal{I}(x, y) \), this ends the proof. \( \Box \)

Set \( t_{i_0}^* = 0 \), \( t_{n(x,y)+1}^* = +\infty \) and \( t_m^* = -\frac{x_m}{y_m} \) for all \( m \in [n(x,y)] \). A sequence \( \{t_m^*\}_{m=0}^{\gamma(x,y)+1} \) of \( \Theta(x, y) \) satisfying the conditions of Lemma 3.2.3 is called an intermediate sequence.

One can then establish the following inclusion, whose the proof is immediate.

**Proposition 3.2.4** Let \( C \) be a \( \mathbb{B}^k \)-convex set of \( \mathbb{R}^n \). Suppose that \( x, y \in C \), and let \( \{t_m^*\}_{m=0}^{n(x,y)+1} \) be an intermediate sequence of \( \Theta(x, y) \). Then, for all \( m \in [n(x,y)] \), \( \gamma(x, y, t_m^*) \in C \). Furthermore
\[
\bigcup_{m=0}^{n(x,y)} \mathbb{B} \left[ \gamma(x, y, t_m^*), \gamma(x, y, t_{m+1}^*) \right] \subset C.
\]

**Proof:** For all \( t \geq 0 \), \( \max\{\max\{1, t\}^{-1}, t \max\{1, t\}^{-1}\} = 1 \). Consequently, since \( x, y \in C \), all the intermediate points lie in \( C \). Since for all \( m \in \{0,1,\ldots,n(x,y)\} \) \( \gamma(x, y, t_m^*) \) and \( \gamma(x, y, t_{m+1}^*) \) are copositive, it follows that \( \mathbb{B} \left[ \gamma(x, y, t_m^*), \gamma(x, y, t_{m+1}^*) \right] \subset C \) for all \( m \), which ends the proof. \( \Box \)

The next results will be useful in the remainder of the paper.

**Lemma 3.2.5** For all \( (a, b, c, d) \in \mathbb{R}^4 \), if \( (a \boxplus b) (c \boxplus d) \geq 0 \) then
\[
(a \boxplus b) \boxplus (c \boxplus d) = F_4(a, b, c, d).
\]

**Proof:** We first assume that \( (a \boxplus b)(c \boxplus d) = 0 \). In such a case, one has either \( a \boxplus b = 0 \) or \( c \boxplus d = 0 \). Suppose, for example, that \( a \boxplus b = 0 \). Then \( (a \boxplus b) \boxplus (c \boxplus d) = 0 \boxplus (c \boxplus d) = c \boxplus d \). Moreover, \( F_4(a, b, c, d) = F_4(a, -a, c, d) = c \boxplus d \) and the equality holds true. The proof is similar in the case where \( c \boxplus d = 0 \).

Suppose now that \( (a \boxplus b)(c \boxplus d) > 0 \). Then \( a \boxplus b \neq 0 \) and \( c \boxplus d \neq 0 \) and from Lemma 2.3.6 we deduce the result. \( \Box \)

For all \( u, v, w, z \in \mathbb{R}^n \), let us denote
\[
u \boxplus v \boxplus w \boxplus z = (F_4(u_1, v_1, w_1, z_1), \ldots, F_4(u_n, v_n, w_n, z_n)).
\tag{3.5}
\]
Proposition 3.2.6 For all \( x, y \in \mathbb{R}^n \), let \( \{t^*_m\}_{m=0}^{n(x,y)+1} \) be an intermediate sequence of \( \Theta(x,y) \). Then
\[
\bigcup_{m=0}^{n(x,y)} \mathbb{B} \left[ \gamma(x, y, t^*_m), \gamma(x, y, t^*_{m+1}) \right] \\
\subset \left\{ tx \uplus rx \uplus sy \uplus wy : \max\{t, r, s, w \} = 1, t, r, s, w \geq 0 \right\}.
\]

Proof: We have just to show that if \( \gamma, \gamma' \in \Gamma(x, y) \) and \( \gamma \uplus \gamma' \geq 0 \), then \( \mathbb{B}[\gamma, \gamma] \subset \left\{ tx \uplus rx \uplus sy \uplus wy : \max\{t, r, s, w \} = 1, t, r, s, w \geq 0 \right\} \). Suppose that \( z \in \mathbb{B}[\gamma, \gamma] \). Hence, by hypothesis, there are \( \alpha, \alpha' \in \mathbb{R}_+ \) such that \( \max\{\alpha, \alpha'\} = 1 \) and \( z = \alpha \gamma \uplus \alpha' \gamma' \).

Since \( \gamma \) and \( \gamma' \) are two intermediate points there exists \( s, t, s', t' \geq 0 \) with \( \max\{s, t\} = 1 \) and \( \max\{s', t'\} = 1 \) and such that \( \gamma = sx \uplus ty \) and \( \gamma' = s'x \uplus t'y \). It follows that \( z = [\alpha(sx \uplus ty)] \uplus [\alpha'(s'x \uplus t'y)] = [(\alpha sx) \uplus (\alpha ty)] \uplus [(\alpha' s'x) \uplus (\alpha't'y)] \). Since \( \gamma \) and \( \gamma' \) are copositive, for all \( i \in [n] \), \([(\alpha sx)_i \uplus (\alpha ty)_i][(\alpha' s'x)_i \uplus (\alpha't'y)_i]) \geq 0 \). We deduce from Lemma 3.2.5 that
\[
[(\alpha sx)_i \uplus (\alpha ty)_i] \uplus [(\alpha' s'x)_i \uplus (\alpha't'y)_i] \leq (\alpha sx)_i \uplus (\alpha ty)_i \uplus (\alpha' s'x)_i \uplus (\alpha't'y)_i.
\]
It follows that \( z = (\alpha s)x \uplus (\alpha t)y \uplus (\alpha' s')x \uplus (\alpha' t')y \). Moreover, one has \( \max\{\alpha s, \alpha t, \alpha' s', \alpha' t'\} = 1 \). Hence, \( z \in \left\{ tx \uplus rx \uplus sy \uplus wy : \max\{t, r, s, w \} = 1, t, r, s, w \geq 0 \right\} \), which ends the proof. \( \square \)

3.3 Some Topological Properties

In the following we show that \( \mathbb{B} \)-convex sets have a path-connected structure. This we do using the intermediate function and focusing on the copositive case.

Proposition 3.3.1 Let \( \gamma : \mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty] \longrightarrow \mathbb{R}^n \) be the map defined in (3.2). Suppose that \( x \) and \( y \) are copositive. Then:

(a) The map \( t \mapsto \gamma(x, y, t) \) is continuous on \( \mathbb{R}_+ \).
(b) One has \( \lim_{t \to 0} \gamma(x, y, t) = \gamma(x, y, 0) = x \) and \( \lim_{t \to +\infty} \gamma(x, y, t) = \gamma(x, y, +\infty) = y \).
(c) For all copositive pairs \( (x, y) \), we have \( \gamma(x, y, [0, +\infty]) = \mathbb{B}[x, y] \).

Proof: (a) The maps \( t \mapsto \max\{1, t\}^{-1} \) is continuous over \( \mathbb{R}_+ \). Since \( x \) and \( y \) are copositive, for all \( i \) there is some \( \epsilon_i \in \{-1, 1\} \) such that
\[
\gamma_i(x, y, t) = \epsilon_i \max\{\epsilon_i \max\{1, t\}^{-1} x_i, \epsilon_i t \max\{1, t\}^{-1} y_i\}.
\]
Consequently, each map \( \gamma_i(x, y, t) \) is continuous in \( t \) and the result follows.

(b) Clearly \( \lim_{t \to 0} \max\{1, t\}^{-1} = 1 \) and \( \lim_{t \to 0} t \max\{1, t\}^{-1} = 0 \). Hence, we have \( \lim_{t \to 0} \gamma(x, y, t) = x \). Moreover, \( \lim_{t \to +\infty} \max\{1, t\}^{-1} = 0 \) and \( \lim_{t \to +\infty} t \max\{1, t\}^{-1} = 1 \). Consequently, \( \lim_{t \to +\infty} \gamma(x, y, t) = y \).
(c) Since $x$ and $y$ are copositive, we have $\mathbb{B}[x, y] = \{tx \oplus sy : t, s \in [0, 1], \max\{t, s\} = 1\}$. Since for all $t \geq 0$ \(\max\{t^1, t\}^{-1}, t \max\{1, t\}^{-1}\} = 1\), we deduce that $\gamma(x, y, \mathbb{R}_+) \subset \mathbb{B}[x, y]$. However, from [?] $\mathbb{B}[x, y]$ is a closed subset of $\mathbb{R}^n$. From (a) and (b), we deduce that $\gamma(x, y, [0, +\infty)) \subset \mathbb{B}[x, y]$. Let us show the converse inclusion. By definition, we have

\[
\mathbb{B}[x, y] = \{tx \oplus y : t \in [0, 1]\} \cup \{x \oplus sy : s \in [0, 1]\}.
\]

Suppose that $0 < t \leq 1$, and set $t' = t^{-1}$. We have $tx \oplus y = (\max\{1, t\}^{-1})x \oplus (t' \max\{1, t\}^{-1})y \in \gamma(x, y, [0, +\infty])$. If $t = 0$ then $x = \gamma(x, y, 0)$. Furthermore, if $0 \leq s \leq 1$, then $x \oplus sy = (\max\{1, s\}^{-1}x) \oplus (s \max\{1, s\}^{-1}y) \in \gamma(x, y, [0, +\infty])$, which proves the converse inclusion. □

**Corollary 3.3.2** Let $a$ and $b$ two real numbers with $a < b$. Let $h : [a, b] \rightarrow \mathbb{R}_+$ be an homeomorphism such that $h(a) = 0$ and $h(b) = \infty$. Let $\xi_h : \mathbb{R}^n \times \mathbb{R}^n \times [a, b]$ be the map defined by $\xi_h(x, y, s) = \gamma(x, y, h(s))$. If $x$ and $y$ are copositive, then the map $s \mapsto \xi_h(x, y, s)$ is continuous. Moreover $\xi_h(x, y, a) = x$, $\xi_h(x, y, b) = y$ and $\xi_h(x, y, [a, b]) = \mathbb{B}[x, y]$.

It is shown below that a $\mathbb{B}^2$-convex set is path-connected.

**Proposition 3.3.3** A non empty $\mathbb{B}^2$-convex of $\mathbb{R}^n$ is path-connected.

**Proof:** We first establish that for all $x, y \in \mathbb{R}^n$, there exists a continuous map $\xi(x, y, \cdot) : [0, 1] \rightarrow \mathbb{R}^n$ such that

\[
\xi(x, y, [0, 1]) = \bigcup_{m=0}^{n(x,y)} \mathbb{B}[\gamma(x, y, t^*_m), \gamma(x, y, t^*_{m+1})],
\]

with $\xi(x, y, 0) = x$ and $\xi(x, y, 1) = y$. Let $\{t^*_m\}_{m=0}^{n(x,y)+1}$ be an intermediate sequence of $\Theta(x, y)$ from $x$ to $y$. Set $x = \gamma(x, y, 0)$ and $y = \gamma(x, y, \infty)$. From corollary 3.3.2, for all $m \in \{0, ..., n(x,y)\}$ there is collection of continuous maps

\[
\eta_m : [t^*_m, t^*_m] \rightarrow \mathbb{B}[\gamma(x, y, t^*_m), \gamma(x, y, t^*_{m+1})]
\]

such that $\eta_m([t^*_m, t^*_m]) = \mathbb{B}[\gamma(x, y, t^*_m), \gamma(x, y, t^*_{m+1})]$, $\eta(t_m) = \gamma(x, y, t^*_m)$ and $\eta(t_{m+1}) = \gamma(x, y, t^*_{m+1})$. Let $\xi : \mathbb{R}^n \times \mathbb{R}^m \times [0, 1] \rightarrow \mathbb{R}^n$ be the map defined by $\xi(x, y, t) = \eta_m(t)$ for all $t \in [t_m, t_{m+1}]$. Clearly, the map $t \mapsto \xi(x, y, t)$ is continuous and we have $\xi(x, y, 0) = x$ and $\xi(x, y, 1) = y$. Moreover, from Proposition 4.2.2, one has $\xi(x, y, [0, 1]) = \bigcup_{m=0}^{n(x,y)} \mathbb{B}[\gamma(x, y, t^*_m), \gamma(x, y, t^*_{m+1})]$. Since for all $x, y \in L$, $\xi(L \times L \times [0, 1]) \subset L$, this ends the proof. □
3.4 Separation of Copositive $\mathbb{B}$-Convex Sets

We say that two subsets $C_1$ and $C_2$ of $\mathbb{R}^n$ are copositive if for all $(x_1, x_2) \in C_1 \times C_2$, $x_1 \sqcup x_2 \in \mathbb{R}_+^n$. In this subsection, it is shown that the inner product $(x, y) \mapsto \langle x, y \rangle_\infty = \bigoplus_{i \in [n]} x_i y_i$ can be used to separate two copositive $\mathbb{B}$-convex sets. For all $u, v \in \mathbb{R}$, let us define the binary operation

$$u \ast v = \begin{cases} v & \text{if } |v| > |u| \\ \min\{u, v\} & \text{if } |u| = |v| \\ v & \text{if } |u| < |v|. \end{cases}$$

An elementary calculus shows that $\mathbb{R} \ast \mathbb{R} = \frac{1}{2}(u \ast v - [(-u) \ast (-v)])$. It has been established in [9] that the set $\mathbb{R}$ equipped with the semilattice operation $\ast$ and the usual multiplication $\cdot$ by positive real numbers is a semimodule over the semifield of positive real numbers $\mathbb{R}_+$. Furthermore, both $(\mathbb{R}_+, \ast, \cdot)$ and $(\mathbb{R}_-, \ast, \cdot)$ are sub-semimodules isomorphic to $(\mathbb{R}_+, \max, \cdot)$; the isomorphisms are, respectively, given by the inclusion, $u \mapsto u$, and the negative of the inclusion, $u \mapsto -u$.

Given $m$ elements $u_1, \ldots, u_m$ of $\mathbb{R}$, not all of which are 0, let $I_+$, respectively $I_-$, be the set of indices for which $0 < u_i$, respectively $u_i < 0$. We can then write $u_1 \ast \cdots \ast u_m = (\ast_{i \in I_+} u_i) \ast (\ast_{i \in I_-} u_i) = (\max_{i \in I_+} u_i) \ast (\min_{i \in I_-} u_i)$ from which we have

$$u_1 \ast \cdots \ast u_m = \begin{cases} \max_{i \in I_+} u_i & \text{if } I_- = \emptyset \text{ or } \max_{i \in I_-} |u_i| < \max_{i \in I_+} u_i \\ \min_{i \in I_-} u_i & \text{if } I_- = \emptyset \text{ or } \max_{i \in I_+} u_i < \max_{i \in I_-} |u_i| \\ \min_{i \in I_+} u_i & \text{if } \max_{i \in I_-} |u_i| = \max_{i \in I_+} u_i. \end{cases} \quad (3.6)$$

We define a $\mathbb{B}$-form on $\mathbb{R}_+^n$ as a map $f : \mathbb{R}_+^n \to \mathbb{R}$ such that, for all $u_1, \ldots, u_m$ in $\mathbb{R}_+^n$ and all $t_1, \ldots, t_m$ in $\mathbb{R}_+$ $f(t_1 u_1 \lor \cdots \lor t_m u_m) = t_1 f(u_1) \lor \cdots \lor t_m f(u_m)$. It has been shown in [9] that a map $f : \mathbb{R}_+^n \to \mathbb{R}$ is a $\mathbb{B}$-form if and only if there exists $(a_1, \ldots, a_n) \in \mathbb{R}^n$, necessarily unique, such that, for all $(x_1, \ldots, x_n) \in \mathbb{R}_+^n$,

$$f(x_1, \ldots, x_n) = a_1 x_1 \lor \cdots \lor a_n x_n. \quad (3.7)$$

Moreover for all $\mathbb{B}$-forms $f : \mathbb{R}_+^n \to \mathbb{R}$ and all real numbers $c$:

$$\text{if } 0 \leq c \text{ then } f(x) \leq c \text{ if and only if } \max_{i \in I_+} \{a_i x\} \leq \max_{i \in I_-} \{-a_i x, c\} \quad (3.8)$$

and

$$\text{if } c \leq 0 \text{ then } f(x) \leq c \text{ if and only if } \max_{i \in I_+} \{a_i x, -c\} \leq \max_{i \in I_-} \{-a_i x\}. \quad (3.9)$$

For all $c \in \mathbb{R}$ and all sunset $I$ of $[n]$ the map $y \mapsto \max_{i \in I} \{y_i, c\}$ is continuous over $\mathbb{R}^n$. Therefore for all $c \in \mathbb{R}$, $f^{-1}([-\infty, c]) = \{x \in \mathbb{R}^n : f(x) \leq c\}$ is closed. It follows that a $\mathbb{B}$-form is lower semi-continuous.

The largest (smallest) lower (upper) semi-continuous minorant (majorant) of a map $h$ is said to be the lower (upper) semi-continuous regularization of $h$. 

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Proposition 3.4.1 Let $f$ be a $\mathbb{B}$-form defined by $f(x_1, \ldots, x_n) = a_1 x_1 \sim \cdots \sim a_n x_n$, for some $a \in \mathbb{R}^n$. Then $f$ is the lower semi-continuous regularization of the map $x \mapsto \langle a, x \rangle_\infty = \bigoplus_{i \in [n]} a_i x_i$.

Proof: Suppose that $\varphi_a : \mathbb{R}^n \rightarrow \mathbb{R}$ is the lower semi-continuous regularization of $\langle a, x \rangle_\infty$. First, remark that for all $x \in \mathbb{R}^n$:

$$f(x) \leq \langle a, x \rangle_\infty.$$ 

Therefore, all we need to prove is that $\varphi_a(x) = f(x)$. By definition, since $f$ is lower semi-continuous, we have for all $x \in \mathbb{R}^n$

$$f(x) \leq \varphi_a(x) \leq \langle a, x \rangle_\infty.$$ 

Let $I^a = \{ i \in [n] : a_i x_i < 0 \}$. If $I^a = \emptyset$ then $\langle a, x \rangle_\infty = \max_{i=1 \ldots n} |a_i x_i| = \max_{i=1 \ldots n} \{ a_i x_i \}$. Moreover by definition $f(x) = \max_{i=1 \ldots n} \{ a_i x_i \} = \langle a, x \rangle_\infty$. Consequently, since $f(x) \leq \varphi_a(x) \leq \langle a, x \rangle_\infty$, we deduce that $f(x) = \langle a, x \rangle_\infty = \varphi_a(x)$.

Suppose now that $I^a \neq \emptyset$ and pick some $i_0 \in I^a$. By hypothesis, we have $a_{i_0} \neq 0$. Now, let $\{ x_k \}_{k \in \mathbb{N}}$ be the sequence defined as:

$$x_{k,i} = \begin{cases} x_i & \text{if } i \neq i_0 \\ x_{i_0} + \frac{1}{a_{i_0} k} & \text{if } i = i_0. \end{cases}$$

Hence, since $a_{i_0} x_{i_0} < 0$ and $i_0 \in I^a$ we have $\langle a, x_k \rangle_\infty = a_{i_0} x_{i_0} - \frac{1}{k} = - \max_{i=1 \ldots n} |a_i x_i| - \frac{1}{k}$. Thus:

$$\lim_{k \to \infty} \langle a, x \rangle_\infty = \lim_{k \to \infty} \left( - \max_{i=1 \ldots n} |a_i x_i| - \frac{1}{k} \right) = - \max_{i=1 \ldots n} |a_i x_i|.$$ 

Moreover, since $\varphi_a$ is lower semi-continuous and $\lim_{k \to \infty} x_k = x$:

$$\liminf_{k \to \infty} \varphi_a(x_k) \geq \varphi_a(x).$$

By hypothesis $\varphi_a$ is the lower semi-continuous regularization of $\langle a, \cdot \rangle_\infty$, thus, by definition, $\langle a, x_k \rangle_\infty \geq \varphi_a(x_k)$. Therefore:

$$- \max_{i=1 \ldots n} |a_i x_i| = \lim_{k \to \infty} \langle a, x_k \rangle_\infty \geq \liminf_{k \to \infty} \varphi_a(x_k) \geq \varphi_a(x)$$ 

Hence $\varphi_a(x) \leq - \max_{i=1 \ldots n} |a_i x_i|$. However, since $I^a \neq \emptyset$, $f(x) = - \max_{i=1 \ldots n} |a_i x_i|$, and we deduce that:

$$\varphi_a(x) \leq f(x).$$

But since $\varphi_a$ is the lower semi-continuous regularization of the map $x \mapsto \langle a, x \rangle_\infty$ and $f(x) \leq \langle a, x \rangle_\infty \forall x \in \mathbb{R}^n$, we also have:

$$\varphi_a(x) \geq f(x).$$
Consequently, \( \varphi_a(x) = f(x) \) which ends the proof. \( \square \)

In \([9]\) it was established that if \( C_1 \) and \( C_2 \) are nonproximate \( \B \)-convex subsets of \( \mathbb{R}_+^n \) then there exists a \( \B \)-form \( f : \mathbb{R}_+^n \to \mathbb{R} \) such that \( \sup_{x \in C_1} f(x) < \inf_{x \in C_2} f(x) \). In the following, this result is extending to the inner product \( \langle x, y \rangle \to \langle x, y \rangle_\infty = \bigoplus_{i \in [n]} x_i y_i \).

**Proposition 3.4.2** If \( C_1 \) and \( C_2 \) are nonproximate copositive \( \B \)-convex subsets of \( \mathbb{R}^n \) then there exists some \( a \in \mathbb{R}^n \) such that

\[
\sup_{x \in C_1} \langle a, x \rangle_\infty < \inf_{x \in C_2} \langle a, x \rangle_\infty.
\]

**Proof:** If \( C_1 \) and \( C_2 \) are copositive, then they belong to the same \( n \)-dimensional orthant \( K \) that is homeomorphic to \( \mathbb{R}_+^n \) using a suitable linear homeomorphism. Therefore, for sake of simplicity, we shall assume that \( K = \mathbb{R}_+^n \). From \([9]\), there is some \( a \in \mathbb{R}^n \) such that the map \( x \mapsto a_1 x_1 \cdots a_n x_n \) separates \( C_1 \) and \( C_2 \). This implies that \( \inf_{x \in C_2} f(x) > \sup_{x \in C_1} f(x) \). Since \( f \) is the lower semi-continuous regularization of \( \langle a, \cdot \rangle_\infty \), it follows that \( \langle a, x \rangle_\infty \geq f(x) \) for all \( x \in C_2 \). Therefore

\[
\inf_{x \in C_2} \langle a, x \rangle_\infty \geq \inf_{x \in C_2} f(x). \quad (*)
\]

Let us consider the map \( g : \mathbb{R}^n \to \mathbb{R} \) defined for all \( x \in \mathbb{R}^n \) by \( g(x) = -f(-x) \). Since \( x \mapsto -x \) is continuous, \( g \) is upper semi-continuous. Moreover, for all \( x \in \mathbb{R}^n \), \( f(-x) \leq \langle a, -x \rangle_\infty = -\langle a, x \rangle_\infty \) implies that \( g(x) \geq \langle a, x \rangle_\infty \). From equations (3.8) and (3.11), for all real numbers \( c \):

- if \( c \leq 0 \) then \( g(x) \geq c \) if and only if \( \max_{i \in I_+} \{-a_i x_i\} \leq \max_{i \in I_-} \{a_i x_i\} \)

and

- if \( c \geq 0 \) then \( g(x) \geq c \) if and only if \( \max_{i \in I_+} \{-a_i x_i, c\} \leq \max_{i \in I_-} \{a_i x_i\} \).

Hence \( g(x) < c \) if and only if \( f(x) < c \) and \( \sup_{x \in C_1} g(x) < \inf_{x \in C_2} f(x) \leq \inf_{x \in C_2} \langle a, x \rangle_\infty \). Since \( \langle a, x \rangle_\infty \leq g(x) \) for all \( x \in C_1 \), this ends the proof from (*) \( \square \).

## 4 Relation to a Limit of Linear Convexities

### 4.1 Intermediate Points of Order \( p \)

For all natural number \( p \), let us consider the map \( \gamma^{(p)} : \mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty] \to \mathbb{R} \) defined by:

\[
\gamma^{(p)}(x, y, t) = \left( \frac{1}{1 + t} \right)^p x + \left( \frac{t}{1 + t} \right)^p y, \quad \text{for all } t \geq 0 \quad (4.1)
\]

and by \( \gamma^{(p)}(x, y, +\infty) = y \).
Lemma 4.1.1 For all \( p \in \mathbb{N} \), the map defined in (4.1) satisfies the following properties.

For all \( x, y \in \mathbb{R}^n \):

(a) The map \( t \mapsto \gamma^{(p)}(x, y, t) \) is continuous over \( \mathbb{R}_+ \);
(b) We have \( \lim_{t \to 0} \gamma^{(p)}(x, y, t) = \gamma^{(p)}(x, y, 0) = x \) and \( \lim_{t \to +\infty} \gamma^{(p)}(x, y, t) = \gamma^{(p)}(x, y, +\infty) = y \);
(c) We have \( \gamma^{(p)}(x, y, [0, +\infty]) = Co^{(p)}(x, y) \);
(d) For all \( t \in [0, +\infty] \), we have \( \lim_{p \to +\infty} \gamma^{(p)}(x, y, t) = \gamma(x, y, t) \);
(e) For all \( t \in [0, +\infty] \), \( \gamma^{(p)}(x, y, t) \) and \( \gamma(x, y, t) \) are copositive.

Proof: (a) The \( \varphi_p \) generalized sum is continuous. Moreover, for all \( t \geq 0 \) the map \( t \mapsto (1 + t)^{-1} \) is continuous and positive. (b) follows from the continuity and using the fact that \( \lim_{t \to +\infty} \frac{x}{1 + t} = 1 \). (c) Let us show that for all \( t \geq 0 \) \( \gamma^{(p)}(x, y, t) \in Co^{(p)}(x, y) \). It is easy to see that \( \left( \frac{1}{1 + t} \right) = 1 \) and it follows that \( \gamma^{(p)}(x, y, t) \in Co^{(p)}(x, y) \) for all \( t \geq 0 \). Since \( \gamma^{(p)}(x, y, \infty) = y \), we deduce that \( \gamma^{(p)}(x, y, [0, +\infty]) \subset Co^{(p)}(x, y) \). Conversely, suppose that \( z \in Co^{(p)}(x, y) \). By hypothesis there is some \( \theta \in [0, 1] \) such that \( z = \theta x + (1 - \theta) y \). If \( \theta \in \{0, 1\} \) then either \( z = x \) or \( z = y \). Suppose that \( \theta \in (0, 1] \), then setting \( t = \frac{1}{1 + \theta} \), we obtain \( \theta = \frac{1}{1 + t} \) and \( 1 - \theta = \frac{t}{1 + t} \). Consequently, \( z \in (x, y, \mathbb{R}^+) \). Therefore, the converse inclusion is true and we deduce that \( \gamma^{(p)}(x, y, [0, +\infty]) = Co^{(p)}(x, y) \). (d) If either \( t = 0 \) or \( t = +\infty \) then this property obviously holds true. Suppose that \( t \in \mathbb{R}^+ \). For all \( j \in [n] \), we have

\[
\lim_{p \to +\infty} \gamma_j(x, y, t) = \lim_{p \to +\infty} \left( \frac{x_j}{1 + t} \right)^p + \left( \frac{y_j}{1 + t} \right)^p = \lim_{p \to +\infty} \left( \frac{2^{2p+1} \theta_j + (ty_j)^{2p+1}}{1 + t^{2p+1}} \right)^\frac{1}{2p+1} = \frac{x_j \theta_j + ty_j}{\max\{1, t\}} = \gamma_j(x, y, t).
\]

(e) From Lemma 2.1.3, we have for all \( j \in [n] \), \( \left( x_j^{2p+1} + (ty_j)^{2p+1} \right)^\frac{1}{2p+1} \geq 0 \) if and only if \( x_j \theta_j + ty_j \geq 0 \). Using distributivity of scalar multiplication, we deduce (e). □

For all \( i \in \mathcal{I}(x, y) \), we say that \( \gamma^{(p)}(x, y, t_i^*) \) is a \( i \)-intermediate point of order \( p \) between \( x \) and \( y \) if \( \gamma_i^{(p)}(x, y, t_i^*) = 0 \).

Corollary 4.1.2 Let \( x, y \in \mathbb{R}^n \) and suppose that \( \mathcal{I}(x, y) \neq \emptyset \). Then, for all \( i \in \mathcal{I}(x, y) \) one has the following properties:

(a) For all \( p \in \mathbb{N} \), there is a uniqueness \( i \)-intermediate point of order \( p \)

\[
\gamma^{(p)}(x, y, t_i^*) = \left( \frac{|y_i|^p}{|x_i|^p + |y_i|^p} \right)^x + \left( \frac{|x_i|^p}{|x_i|^p + |y_i|^p} \right)^y,
\]

with \( t_i^* = -\frac{x_i}{y_i} + \frac{x_i}{y_i} \).

(b) For all \( p \in \mathbb{N} \) and all \( i \in \mathcal{I}(x, y) \), \( \gamma^{(p)}(x, y, -\frac{x_i}{y_i}) \) is a \( i \)-intermediate point of order \( p \) if
and only if \( \gamma(x, y, -\frac{x_i}{y_i}) \) is an \( i \)-intermediate point.

(c) Let \( \gamma(x, y, t^*_i) \) be a \( i \)-intermediate point and let \( \{\gamma^{(p)}(x, y, t^*_i)\}_{p \in \mathbb{N}} \) be a sequence of \( i \)-intermediate points of order \( p \). Then \( \lim_{p \to \infty} \gamma^{(p)}(x, y, t^*_i) = \gamma(x, y, t^*_i) \).

(d) If \( \{t^*_i\}_{m=0}^{n(x,y)+1} \) is an intermediate sequence of \( \Theta(x, y) \) satisfying the conditions of Lemma 3.2.3 with \( t^*_i = 0 \), \( t^*_i = +\infty \) and \( t^*_i = -\frac{x_i}{y_i} \) for all \( m \in [n(x,y)] \), then:

\[
Co^p(x, y) = \bigcup_{m=0}^{n(x,y)} Co^p(\gamma^{(p)}(x, y, t^*_i), \gamma^{(p)}(x, y, t^*_i+1)).
\]

Proof: (a) \( \gamma_i(x, y, t_i) = 0 \) if and only if \( \left(\frac{1}{1+t}\right)x_i + \left(\frac{t}{1+t}\right)y_i = 0 \) which is equivalent to \( t_i = -\frac{x_i}{y_i} = |\frac{x_i}{y_i}| \). (b) and (c) are two immediate consequences of (a) and Proposition 3.1.1(d). (d) By definition, for all \( m \) \( \gamma^{(p)}(x, y, t^*_i), \gamma^{(p)}(x, y, t^*_i+1) \in Co^p(x, y) \). Therefore \( Co^p(x, y) \supset \bigcup_{m=0}^{n(x,y)} Co^p(\gamma^{(p)}(x, y, t^*_i), \gamma^{(p)}(x, y, t^*_i+1)) \). Moreover, since \( x = \gamma^{(p)}(x, y, t^*_i) \) and \( y = \gamma^{(p)}(x, y, t^*_i+1) \), the converse inclusion holds. \( \Box \)

4.2 Painlevé-Kuratowski Limit

From Lemma 3.2.3 the \( \Phi^p \)-convex hull of \( x \) and \( y \) is the finite union of the \( \Phi^p \)-convex hull of two consecutive intermediate points of order \( p \). The sequence which these intermediate points of order \( p \) are arranged is identical to the copositive sequence of the intermediate points.

For future reference, we gather in the lemma below some elementary facts, which are a slight extension of a result established in [7].

**Lemma 4.2.1** Let \( K \) be a \( n \)-dimensional orthant of \( \mathbb{R}^n \). Let \( A = \{x_1, \ldots, x_m\} \) be a finite subset of \( K \). For all natural number \( p \) let \( A^{(p)} = \{x^{(p)}_1, \ldots, x^{(p)}_m\} \) be a finite collection of \( m \) vectors in \( K \).

(a) If there exists an an increasing sequence of natural numbers \( \{p_k\}_{k \in \mathbb{N}} \) such that for \( i = 1, \ldots, m \) \( \lim_{k \to \infty} x^{(p_k)}_i = x_i \), then:

\[
\lim_{k \to \infty} \sum_{i \in [m]} x^{(p_k)}_i = \bigoplus_{i \in [m]} x_i.
\]

(b) If for \( i = 1, \ldots, m \) \( \lim_{p \to \infty} x^{(p)}_i = x_i \), then

\[
\text{Lim}_{p \to \infty} Co^p(A^{(p)}) = \left\{ \bigoplus_{i \in [m]} t_i x_i : \max t_i = 1, t_i \geq 0 \right\} = \mathbb{B}[A].
\]

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Proof: (a) Let $\Psi_K : K \to \mathbb{R}_+^n$ be the function characterizing the $n$-dimensional orthant $K$. By definition for all $x \in \mathbb{R}^n$ one has $\Psi_K(x) = (\epsilon_1 x_1, \ldots, \epsilon_n x_n)$ where $\epsilon_j \in \{-1, 1\}$ for all $j \in [n]$. By definition one has for all $j$:

$$\sum_{i \in [m]} x_{i,j}^{(p_k)} = \epsilon_j \sum_{i \in [m]} |x_{i,j}^{(p_k)}|.$$  

From the Lemma 2.0.1.b established in [7], we have $\lim_{k \to +\infty} \sum_{i \in [n]} |x_{i,j}^{(p_k)}| = \max_i |x_{i,j}|$. Consequently, for all $j \in [n]$

$$\lim_{k \to +\infty} \sum_{i \in [m]} x_{i,j}^{(p_k)} = \epsilon_j \max_i |x_{i,j}| = \bigoplus_{i \in [m]} x_{i,j},$$

which ends the proof.

(b) We first establish that $\mathbb{B}[A] = \left\{ \bigoplus_{i=1}^{m} t_i x_i : t_i \in [0, 1], \max_{i \in [m]} t_i = 1 \right\} \subset L_{i \to +\infty} \mathcal{C}o^p(A)$. Let $y = t_1 x_1 \oplus \cdots \oplus t_m x_m$ with $t_1, \ldots, t_m \in [0, 1]$ and $\max_{i \in [m]} t_i = 1$. Define $y_p \in \mathcal{C}o^p(A)$ by

$$y^{(p)} = \frac{1}{p} \left( t_1^p x_1^{(p)} + \cdots + t_m^p x_m^{(p)} \right).$$

Since $x_1^{(p)}, \ldots, x_m^{(p)} \in K$ and $\lim_{p \to \infty} \left( t_1^p + \cdots + t_m^p \right) = \max_{i \in [m]} t_i = 1$ we deduce from (a) that

$$\lim_{p \to \infty} y^{(p)} = \lim_{p \to \infty} \left( t_1 x_1^{(p)} + \cdots + t_m x_m^{(p)} \right) = t_1 x_1 \oplus \cdots \oplus t_m x_m = y.$$  

This completes the first part of the proof. Next, we establish that $L_{s \to +\infty} \mathcal{C}o^p(A) \subset \mathbb{B}[A]$. 

Take $z \in L_{s \to +\infty} \mathcal{C}o^p(A)$; there is an increasing sequence $\{p_k\}_{k \in \mathbb{N}}$ and a sequence of points $\{z_k\}_{k \in \mathbb{N}}$ such that $z_k \in \mathcal{C}o^{p_k}(A^{(p_k)})$ and $\lim_{k \to \infty} z_k = z$. Each $z_k$ being in $\mathcal{C}o^{p_k}(A^{(p_k)})$, we can write

$$z_k = t_{k,1} x_1^{(p_k)^{p_k}} + \cdots + t_{k,m} x_m^{(p_k)^{p_k}}.$$  

Since $t_k = (t_{k,1}, \ldots, t_{k,m}) \in [0, 1]^m$ one can extract a subsequence $(t_{k_i})_{i \in \mathbb{N}}$ converges to a point $t^* = (t_{i}^*, \ldots, t_{m}^*) \in [0, 1]^m$. It follows that for all $i \in [m]$ $\lim_{k \to +\infty} t_{k,i} x_i^{(p_k)} = x_i$. Furthermore, from (a), $\lim_{i \to +\infty} \left( \frac{m}{2^{p_k+1}} \right)^{1/(2^{p_k+1})} = \max_i t_{i}^* = 1$. It follows that for all $i \in [m]$ $\lim_{k \to +\infty} t_{k,i} x_i^{(p_k)} = t_{i}^* x_i$. From (a) we deduce that $x = \bigoplus_{i=1}^{m} t_{i}^* x_i$ with $\max_{i \in [m]} \{t_{i}^*\} = 1$. The first and the second part of the proof show that

$$L_{s \to +\infty} \mathcal{C}o^p(A^{(p)}) \subset \mathbb{B}[A] \subset L_{i \to +\infty} \mathcal{C}o^p(A^{(p)})$$

and this completes the proof since we always have the inclusion $L_{i \to +\infty} \mathcal{C}o^p(A^{(p)}) \subset L_{s \to +\infty} \mathcal{C}o^p(A^{(p)})$.
In the following, it is proven that $\text{Co}^\infty(\{x, y\}) = \operatorname{Lim}_{p \to +\infty} \text{Co}^p(\{x, y\})$. This means that, given two points in the whole Euclidean vector space, the Painlevé-Kuratowski limit of their generalized convex hull exists. Moreover it is established that it has an algebraic description. For the sake of simplicity let $\text{Co}^\infty(x, y)$ and $\text{Co}^p(x, y)$ denote these convex hulls for all $p \in \mathbb{N}$. Let us consider $\ell$ sequences of subsets of $\mathbb{R}^n \{A_m^{(p)}\}_{p \in \mathbb{N}}, m \in [\ell]$. If there exists a subset $A_m$ of $\mathbb{R}^n$ such that $\operatorname{Lim}_{p \to +\infty} A_m^{(p)} = A_m$ for all $m \in [\ell]$, then it is easy to show that:

$$\operatorname{Lim}_{p \to +\infty} \left( \bigcup_{m \in [\ell]} A_m^{(p)} \right) = \bigcup_{m \in [\ell]} A_m. \quad (4.2)$$

**Proposition 4.2.2** For all $x, y \in \mathbb{R}^n$, let $\{t^{*}_{im}\}_{m=0}^{n(x,y)+1}$ be an intermediate sequence of $\Theta(x, y)$. Then

$$\text{Co}^\infty(x, y) = \operatorname{Lim}_{p \to +\infty} \text{Co}^p(x, y) = \bigcup_{m=0}^{n(x,y)} \mathbb{B} \left[ \gamma(x, y, t^{*}_{im}), \gamma(x, y, t^{*}_{im+1}) \right].$$

**Proof:** From Lemma 4.1.1.d, we have for all $i \in \mathcal{I}(x, y)$

$$\lim_{p \to +\infty} \gamma(p)^{(p)}(x, y, -\frac{x_i}{y_i}) = \gamma(x, y, -\frac{x_i}{y_i}).$$

Moreover, from Lemma 4.1.1.e, for all $p \in \mathbb{N}$ and all $i \in \mathcal{I}(x, y)$, $\gamma^{(p)}(x, y, -\frac{x_i}{y_i})$ and $\gamma(x, y, -\frac{x_i}{y_i})$ are copositive. Recall that two vectors are copositive if their components have the same sign. From Proposition 3.2.3, for all $m \in [n(x,y)] \gamma(x, y, t^{*}_{im})$ and $\gamma(x, y, t^{*}_{im+1})$ are copositive. Hence it follows that for all $m$, $\gamma^{(p)}(x, y, t^{*}_{im})$ and $\gamma^{(p)}(x, y, t^{*}_{im+1})$ are copositive.

From Proposition 4.2.1 we have for all $m$

$$\operatorname{Lim}_{p \to +\infty} \text{Co}^p \left( \gamma^{(p)}(x, y, t^{*}_{im}), \gamma^{(p)}(x, y, t^{*}_{im+1}) \right) = \mathbb{B} \left[ \gamma(x, y, t^{*}_{im+1}), \gamma(x, y, t^{*}_{im+1}) \right].$$

Moreover, we have from Corollary 4.1.2.d

$$\text{Co}^p(x, y) = \bigcup_{m=0}^{n(x,y)} \text{Co}^p \left( \gamma^{(p)}(x, y, t^{*}_{im}), \gamma^{(p)}(x, y, t^{*}_{im+1}) \right).$$

Hence, from equation (4.2), and Corollary 4.1.2.c, the result follows. □

This property has an immediate consequence.

**Proposition 4.2.3** A subset $C$ of $\mathbb{R}^n$ is $\mathbb{B}^\sharp$-convex if and only if for all $x, y \in \mathbb{R}^n$ $\text{Co}^\infty(x, y) \subset C$. 

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Proof: This is an immediate consequence of Propositions 3.2.6 and 3.2.6. □

Notice that it is not clear from the definition of \( \mathcal{B}^\sharp \)-convex sets that, for an arbitrary couple \((x, y)\) of \( \mathbb{R}^n \times \mathbb{R}^n \), \( Co^\infty(x, y) \) is \( \mathcal{B}^\cdot \)-convex.

**Corollary 4.2.4** A \( \mathcal{B} \)-convex subset of \( \mathbb{R}^n \) is \( \mathcal{B}^\sharp \)-convex.

In [7] it was established that a \( \mathcal{B} \)-convex subset of \( \mathbb{R}^n \) is connected. A stronger property is established below. It is shown that \( \mathcal{B} \)-convex sets are path-connected.

**Proposition 4.2.5** A non empty \( \mathcal{B} \)-convex of \( \mathbb{R}^n \) is path-connected.

Proof: By definition a subset \( L \) of \( \mathbb{R}^n \) is \( \mathcal{B} \)-convex if for all finite subset \( A \subset L \) we have \( Co^\infty(\{x, y\}) \subset L \). This implies that \( Co^\infty(\{x, y\}) \subset L \) for all \( x, y \in L \), which yields the result from Proposition 3.3.3. □

In the next statement, an algebraic characterization of \( Co^\infty(x, y) \). To prove this, we use the fact that the convex hull is not modified whenever one consider several occurrences of a given point. For example, for all \( p \in \mathbb{N} \), one can equivalently write:

\[
Co^p(x, y) = \left\{ \alpha x + \beta y : \alpha + \beta = 1, \alpha, \beta \geq 0 \right\} \quad (4.3)
\]

\[
= \left\{ \alpha_1 x + \alpha_2 x + \beta_1 y + \beta_2 y : \alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 1, \alpha_i, \beta_i \geq 0 \right\}.
\]

**Lemma 4.2.6** For all \( x, y \in \mathbb{R}^n \),

\[
Co^\infty(x, y) = \left\{ tx \boxplus rx \boxplus sy \boxplus wy : \max\{t, r, s, w\} = 1, t, r, s, w \geq 0 \right\}.
\]

Proof: Suppose that \( z \in \left\{ tx \boxplus rx \boxplus sy \boxplus wy : \max\{t, r, s, w\} = 1, t, r, s, w \geq 0 \right\} \). By definition there exists \( \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0 \) with \( \max\{\alpha_1, \alpha_2, \beta_1, \beta_2\} = 1 \) and such that

\[z = \alpha_1 x \boxplus \alpha_2 x \boxplus \beta_1 y \boxplus \beta_2 y.\]

Define

\[z^{(p)} = \frac{1}{\alpha_1 + \alpha_2 + \beta_1 + \beta_2} \left( \alpha_1 x + \alpha_2 x + \beta_1 y + \beta_2 y \right).\]

By construction \( z^{(p)} \in Co^p[x, y] \). Taking the limit on both sides yields from Proposition 2.2.3

\[
\lim_{p \to \infty} z^{(p)} = \frac{1}{\max\{\alpha_1, \alpha_2, \beta_1, \beta_2\}} \left( \alpha_1 x \boxplus \alpha_2 x \boxplus \beta_1 y \boxplus \beta_2 y \right)
\]

\[= \alpha_1 x \boxplus \alpha_2 x \boxplus \beta_1 y \boxplus \beta_2 y = z.\]

Consequently, \( z \in Li_{\lim p \to \infty}Co^p(x, y) \). □
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