Discontinuous phase transition in an open-ended Naming Game

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Abstract. In this work we study on a 2D square lattice a recent version of the Naming Game, an agent-based model used for describing the emergence of linguistic structures. The system is open-ended and agents can invent new words throughout the evolution of the game, picking them up from a pool characterised by a Gaussian distribution with standard deviation $\sigma$. The model displays a nonequilibrium phase transition at a critical point $\sigma_c \approx 25.6$, which separates an absorbing consensus state from an active fragmented state where agents continuously exchange different words. The finite-size scaling analysis of our simulations strongly suggests that the phase transition is discontinuous.

Keywords: finite-size scaling, phase transitions into absorbing states (theory), interacting agent models
1. Introduction

In recent years, an original model for describing spreading of conventions, generally denominated Naming Game [1], has attracted the attention of the scientific community interested in modelling linguistic related phenomena. Despite—or perhaps because of—its simplicity, agent-based models directly inspired by the Naming Game have shown enough power to unravel the key factors involved in such phenomena as the birth of neologisms [2], the self-organisation of a hierarchical category structure in a population of individuals [3], the emergence of universality in colour naming patterns [4], the rise of a protosyntactic structure [5] and the so-called duality of patterning in human communication [6].

The key ingredients present in this model, which make it well suited to cope with the description of linguistic phenomena, can be roughly summarised in quite realistic but simple rules of interaction conjugated with the ability to describe a flexible and rich individual memory.

In the Naming Game the interaction is built up on a reinforcement rule which fixes a linguistic convention throughout the selective exclusion of unshared variants. This process is a memory-based negotiation strategy, made up of a sequence of trials that shape and reshape the system memories. Memory and feedback are naturally encompassed by these rules which result to be more widespread than simple imitation mechanisms that can be described by Ising-like dynamics.

The memory of the different linguistic variants is guaranteed by an individual inventory which can contain an infinite number of symbols. In fact, it is structured by an array of potentially infinite cells where each cell is set on one of an infinite number of possible numerable states. It follows that the number of possible states an agent can experience is limited only by the dynamics of symbols invention and exchange. This is a really important fact, considering that arbitrariness and variation is a structural characteristic of linguistic
units which forces toward a model scheme which must encompass the open-ended nature of the linguistic system [7]. For this reason, agents must be characterised by a memory capable of accumulating an unlimited number of possible traits, each one characterised by an infinite number of possible choices.

In the original model, the problem of limiting this variety, necessary for reaching consensus, was solved by allowing the introduction of different words just in the first time steps. In this first stage of the simulation the initial random conditions in terms of variant diversity are fixed. In the following of the dynamics, words diffuse throughout agreement and learning mechanisms. Learning allows the memorisation of a new word in the case of a failed communication, causing a rapid increase in the overlap between the words exchanged by the community.

Recently, variants of this implementation of the Naming Game [8, 9] explored the possibility of a constant introduction of new words with the intention of defining a really open-ended system, where convention creation is always a possibility, and not reduced to the definition of the initial conditions. The negotiation dynamics are able to order the system toward consensus only if some type of restriction on the freedom of invention is introduced. In fact, also in nature, linguistic conventions are not totally arbitrary. The continuity with the history of the linguistic community restricts the freedom of invention [7], limiting the open-ended character of symbols production. These considerations suggest to describe the reservoir of words by means of a probability distribution which models the presence of some constraints, differentiating between more and less probable words.

The introduction of these different ingredients [9], which at a first sight can seem a minor modification of the classical implementation, are quite interesting for two reasons. First, because this variant of the model is able to explore the nature and effects of ordering in an open-ended system. Second, because it naturally introduces a nontrivial phase transition between a consensus absorbing state, characterised by the use of the same single word by all the players, and an active stationary state of disordered and fragmented clusters, where agents continuously exchange different words. This transition is driven by the tuning of a control parameter which limits the degree of variance of the invented words. A similar transition is obtained also in the original Naming Game, at the expense of introducing a noise term. In that work [10], a parameter modulates the level of noise present in communications (no noise corresponds to the original Naming Game), generating a limitation in the efficiency of the negotiation dynamics. For a sufficiently low level of noise, consensus is reached.

The principal purpose of this work is to study and characterise the mentioned transition, throughout an accurate numerical analysis. The first work in this direction [9] presented such transition for the model implemented on the fully-connected graph. Here, we look at the model on a 2D square lattice, with the purpose of giving a description of this behaviour with the help of Monte Carlo simulations and finite-size scaling analysis. In other words, we are interested in analysing the impact of short-range interactions on the dynamics of the model and on the phase transition that occurs between the above-mentioned states.

Nonequilibrium phase transitions in ordering dynamics associated with the description of opinion formation have attracted increasing theoretical interest among the community of the statistical physicists [11]. Even in the absence of a general theory for nonequilibrium critical behaviour, their characterisation in terms of different universality classes and
the definition of the prominent and robust ones, is a very important issue [12]. In this classification effort, because of the lack of theoretical basis, it is not obvious which parameters are relevant. Inspired by the equilibrium systems and based on general grounds, loosely speaking one expects to have to pay attention to the dimensionality of the system, the number of absorbing states, the symmetry of the problem and in some cases to the details of the dynamics, with the hope of this last aspect being relevant only in accidental cases, rather than in generic.

Considering systems embedded in a 2D space, common results exist for models which display the equilibrium Ising class [13]. Another widespread class is the Directed Percolation universality class for some models which exhibit a single absorbing state or double absorbing states with some asymmetry in the dynamics which favours one of the two possible absorbing states [14–16]. Finally, associated with critical phenomena for systems with two symmetric absorbing states, recently appeared the Generalised Voter universality class [17,18], which seems to be an important and solid class, characterised by the logarithmic decay of the density of interfaces and others characteristic critical exponents.

Clearly, our system has fundamentally different ingredients from the ones listed above. In particular, a proliferation of possible absorbing states appears with a lack of symmetry in the dynamics. The presence of a large number of possible final absorbing states is a trait commonly observed in the Axelrod’s model for the formation of cultural domains [19]. In such a model, the nature of the consensus-fragmentation phase transition turns from continuous to discontinuous, increasing the number of cultural features above two [20]. This behaviour also occurs in other models with more than two symmetric absorbing states that display a discontinuous phase transition [21]. Also in our system, the transition between the ordered consensus state and the disordered phase with fragmented clusters characterised by heterogeneous memories clearly seems to be a discontinuous transition.

In the following we will describe the details of the model (section 2), the behaviour of the dynamics on a 2D lattice compared with the implementation on the fully-connected graph (section 3), and finally the numerical analysis for the characterisation of the phase transition (section 4).

2. The model

We simulate a naming game played by $P$ agents embedded on a regular 2D square lattice with $P = L \times L$ sites and periodic boundary conditions. This implementation defines a short-range-interaction system, where agents can interact only with their four nearest neighbours.

In the initial state each player starts with an empty inventory. At each time step, a pair of agents is randomly selected and their actions are governed by the following rules, which are the same used in the former implementation on the fully-connected graph [9]:

1. The first agent (speaker) selects one of his/her words or, if his/her inventory is empty, he/she creates a new one. After that, the word is transmitted to the second agent (hearer).
(2) a. If the hearer possesses the transmitted word, the communication is a success. The two agents update their inventories so as to keep only the word involved in the interaction.

b. Otherwise, the communication is a failure. The speaker invents a new word which must be different from all the other ones present in his/her inventory.

New words are invented in the early stages of the simulation, when inventories are empty, or at anytime, if the communication is a failure. This second option is not present in the original Naming Game [1], where, in the case of a failed communication, the hearer learns the speaker’s selected word.

Invention is performed picking up a word from a pool of different words which are sorted out with a probability characterised by a Gaussian distribution centred in zero with standard deviation $\sigma$. This distribution was chosen because of the efficiency of its numerical implementation. Words are determined by taking the integer part of the absolute value of the sorted real number. We can observe that the reservoir must be unlimited for allowing that any invented new word can be always different from the ones present in the speaker’s memory. Anyway, it must be characterised by a distribution which falls to zero very quickly, like the Gaussian one, otherwise consensus can be hardly attained, as can be seen for the implementation realised on the fully-connected graph [9].

3. Dynamical behaviour

In the following we unfold a comparison between the typical results of the fully-connected graph implementation [9], with the ones obtained in the 2-dimensional lattice (see figure 1).

We describe the time evolution of our system on the basis of some usual global quantities [1], namely the total number of words ($N_t(t)$), the number of different words ($N_d(t)$) present in the population and the success rate ($S(t)$), which measures the average rate of success of communications. This is obtained by evaluating the frequency of successful communications in a given time interval. In figure 1 we can appreciate the time evolution of these quantities for the lattice implementation compared with the fully-connected graph one. In both situations $N_t$ shows a fast initial transient, during which many different words are created. In fact, agents start with an empty inventory and, in each interaction, each speaker invents at least a new word and each hearer can possibly learn one. After this early stage, a maximum is reached. In the fully-connected graph implementation this stage is followed by a long plateau slightly decreasing along the time evolution. In contrast, the lattice simulations, after having crossed the maximum, abruptly diminish the number of total words, directly starting a long way towards the ordered phase. These different behaviours can be easily understand. On the fully-connected graph each agent can interact with all other players. This fact increases the possibility of unsuccessful interactions which determine the invention of new words. During this phase, a success in communication is rare and words are hardly eliminated, generating the observed long and declining plateau. Only when the redundancy of words reaches a sufficiently high level does the system undergo the ordering transition. The number of successful games
increases and $N_t$ changes its concavity and begins a decay towards the consensus state, corresponding to one common word for all the players. In contrast, on the lattice, players only communicate locally with the same four neighbours. This means that words spread locally and, in general, the number of success in communications is significant and agents have a limited necessity to invent new words. It follows that the maximum value reached by the memory is smaller and no plateau is observed, since the coarsening of the single-state clusters soon reduces the number of words. However, in this condition, the final convergence to the consensus state is much slower. These considerations are supported by and coherent with the dynamics displayed by $N_a(t)$ and $S(t)$ (see figure 1).

**Figure 1.** Time evolution in the fully-connected graph (black) and in the 2D lattice model (red). The convergence of the simulations on the lattice is reached only after more than $1.4 \times 10^6$ time steps. Top: temporal evolution for the total number of words divided by the total population. In the inset, the maximum number of different words. Bottom: the success rate $S(t)$. Data are averaged over 100 simulations, $P = 1600$, $\sigma = 5$. 

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Figure 2. Top: Maximum number of total words (the dashed line has slope 1) and, in the inset, maximum number of different words for different population sizes ($\sigma = 18$). Bottom: Maximum number of total words (circles) and maximum number of different words (triangles) for different $\sigma$ values ($L = 40$). Data are averaged over 100 simulations.

Figure 2 describes the scaling behaviour of the system memory size. The maximum number of total words linearly scales with the population size $P = L^2$. So, like in the mean-field case, the number of total words of each player is not dependent on the community size [9]. The fact that the same behaviour is maintained is quite obvious, considering that we cannot have a dependence slower than linear. The same scaling law was observed in the standard Naming Game implemented on a 2D lattice, where the total memory scales as $P$ [22].

We also studied the scaling behaviour in dependence of the $\sigma$ value: $N_d$ follows a linear scaling and $N_t \sim \sigma^{1/2}$. In summary, no modification of the scaling behaviour of the mean-field case [9] is observed.

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Finally, we explored the behaviour of the convergence time $T_c$ (the average time needed to reach consensus), which is clearly a quantity of particular interest in the context of the social dynamics applications. We analysed the dependence of $T_c$ on $P$, obtaining $T_c \sim P^{1.5}$ (see the main plot of figure 3). As expected, the time needed to reach consensus grows in relation to the fully-connected graph case, where, to a first approximation, $T_c$ scales linearly with $P$. It is interesting to remember that the original Naming Game simulated on a 2D lattice has a convergence time equal to $P^2$, with an exponent 0.5 larger than the fully-connected graph case [22], as it is found in our implementation.

In the inset of figure 3, we report $T_c$ as a function of the parameter $\sigma$ for a fixed population size ($L = 30$). For low $\sigma$ values, $T_c$ increases exponentially and, for higher values, it points towards a divergence, signalling the fact that for large $\sigma$ convergence is not attained. A similar behaviour was observed in the fully-connected graph case, and is associated with the proximity of the phase transition point. This transition will be analysed in more details in the next section.

4. Phase transition

In this section we study the stationary states of the system with the aim of analysing the transition properties of the model. The phase transition is characterised by the passage from a consensus absorbing state, characterised by the use of the same single word by all the players, to an active stationary state of disordered and fragmented clusters, where agents continuously interchange different words. Consensus is reached only if the variance of the distribution of the invented words is sufficiently small. In this case, a unique cluster characterised by the same word emerges. In contrast, for large $\sigma$, convergence is not
attained and small clusters with one or more different words are present in the system. In this regime the invention of words outperforms the capacity of the population to reach consensus because the variety of the new words is too high and generates too much diversity among the players.

Inspired by previous studies [18, 20], we decided on the relative size of the largest cluster ($C_m/L^2$) for the role of the order parameter of the model. We define $C_m$ as the size of the largest cluster composed by agents sharing the same unique word and we measure it at the end of each simulation. In addition, we estimate the fluctuations (or the susceptibility) $\chi$ of this order parameter and its fourth-order cumulant $U$, which can be obtained from the simulations by evaluating

$$\chi = L^2(\langle C_m^2 \rangle - \langle C_m \rangle^2)$$
$$U = 1 - \frac{\langle C_m^4 \rangle}{3\langle C_m^2 \rangle^2},$$

where $\langle \rangle$ stands for averages over different simulations taken at the steady states. We generate simulations with $L$ ranging from 20 up to 60. For these $L$ values the system presents a remarkable slow relaxation time close to the transition, which obliges us to run very long simulations which last $4 \times 10^{10}$ Monte Carlo steps.

In figure 4 we exhibit the behaviour of the order parameter (top) and its fluctuations (bottom) as functions of $\sigma$ for some lattice sizes $L$. One can observe the typical behaviour of a phase transition, i.e. the order parameter goes to zero at critical values $\sigma_c(L)$ and the susceptibility peaks $\chi_{\text{max}}$ grow for increasing values of $L$. Following the standard procedure for finite-size scaling (FSS) analysis [11, 14, 16, 26], one can consider that the above-mentioned quantities present power-law scalings near the transition point $\sigma_c$ of the form

$$\langle C_m \rangle / L^2 \sim L^{-\beta/\nu},$$
$$\chi \sim L^{\gamma/\nu},$$
$$U \sim b L^a,$$

where $\beta$, $\gamma$, $\nu$, $a$ are critical exponents and $b$ is a constant. In continuous phase transitions these exponents present values that satisfy the relation $2\beta + \gamma = d\nu$, where $d$ is the dimension of the system, $a = 0$ and $b > 0$. In the case of discontinuous (first-order) phase transitions, we have $\beta = 0$, $\nu = 1/d$, $\gamma/\nu = d$, $a = d$ and $b < 0$ [12, 23–27]. In other words, for discontinuous transitions the susceptibility peaks diverge to $+\infty$ as $\chi_{\text{max}} \sim L^d$, whereas the fourth-order cumulant presents a well-defined minimum that diverges to $-\infty$ as $U_{\text{min}} \sim -L^d$.

Taking into account the above FSS relations, in the insets of figure 4 we exhibit the scaling plot of the order parameter and the susceptibility peaks as functions of $L$ in the log–log scale. The former is obtained for $\sigma_c \approx 25.6$ and $1/\nu \approx 1.95$, which give us an estimate of the critical point $\sigma_c = \sigma_c(L \to \infty)$. In addition, we obtain the results $\beta/\nu = 0$ and $1/\nu \approx d$, which suggests a discontinuous transition. The value of $\chi_{\text{max}}$ grows with the system size with $\gamma/\nu \approx 2.02$, which corresponds to the system dimension $d = 2$ within error bars. All these results strongly suggest the occurrence of a discontinuous transition. Furthermore, as can be seen in figure 5, the fourth-order cumulant presents well-defined minima near the transition. Looking for the minima values, the modulus $|U_{\text{min}}|$ grows
with an exponent $a \approx 1.97$, which is also compatible with a discontinuous transition ($a \approx d$) [25, 27].

Finally, we plot the probability distribution function (PDF) of the order parameter near the transition. In the case of a continuous transition, this quantity presents a one-peak structure. On the other hand, for discontinuous transitions the PDF is double peaked. In figure 6 we show the results for $L = 30$ and $\sigma = 26.7$, i.e. near the transition for the mentioned size (see figure 4). In this case, we clearly observe the two-peak structure, which confirms the discontinuous character of this transition.
5. Conclusions

In this work, we studied on a regular 2D lattice the open-ended Naming Game introduced in [9], where it was implemented on the fully-connected graph.

In a first step we explored the differences which the topology can generate in the dynamics of the model. In general, we found that the implementation on a regular lattice limits the necessity of invention of new words, causing a smaller memory effort. However, the scaling behaviour presents no modification from the mean-field case for the maximum number of total and different words. On the other side, the convergence to the consensus state is remarkably slower and the time needed to reach consensus grows with $P$ with an exponent 0.5 larger than the case of the fully-connected graph.

Next, we directed our attention to the principal purpose of our work: the analysis of the phase transition properties of the system. In fact, this model is characterised by a transition between a consensus absorbing state, where the same single world is used by all the players, and an active stationary state of fragmented clusters distinguished by the presence of different words. Consensus is reached if the variance of the invented words is small enough.

The presence of a double-peaked probability distribution function of the order parameter near the transition suggests a phase transition of discontinuous type. We tested this supposition by means of a finite-size scaling analysis. The estimation of the $\beta$ and $\nu$ exponents, the dependence of $\chi_{\text{max}}$ on the system size, and the behaviour of the fourth-order cumulant, were all compatible with a discontinuous transition. From this analysis we also estimated the value of the critical point $\sigma_c \approx 25.6$, which is significantly larger than in the fully-connected graph implementation ($6 < \sigma_{c}^{\text{FC}} < 7$), a fact that can be ascribed to the minor propensity for inventing new words.
Figure 6. Probability distribution function of the normalised largest cluster size. A clear bimodal distribution appears near $\sigma_c(L)$. Incidentally, we can observe that, in general, if consensus is not attained, single-state clusters sharing the same word are small. Data are obtained from the final state of 500 different simulations running with $\sigma = 26.7$ and $L = 30$.

Interestingly, these results can be compared with the transition obtained in the original Naming Game with a noise term [10]. Also in that work, a discontinuous phase transition between consensus and polarised states was found and analytically described in the case of a limited number of opinions.

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