Modified gravity theories have received increased attention lately due to combined motivation coming from high-energy physics, cosmology and astrophysics. Among numerous alternatives to Einstein’s theory of gravity, theories which include higher order curvature invariants, and specifically the particular class of $f(R)$ theories, have a long history. In the last five years there has been a new stimulus for their study, leading to a number of interesting results. We review here $f(R)$ theories of gravity in an attempt to comprehensively present their most important aspects and cover the largest possible portion of the relevant literature. All known formalisms are presented — metric, Palatini and metric-affine — and the following topics are discussed: motivation; actions, field equations and theoretical aspects; equivalence with other theories; cosmological aspects and constraints; viability criteria; astrophysical applications.

I. INTRODUCTION

A. Historical

As we are approaching the closing of a century after the introduction of General Relativity (GR) in 1915, questions related to its limitations are becoming more and more pertinent. However, before coming to the contemporary reasons for challenging a theory as successful as Einstein’s theory, it is worth mentioning that it took only four years from its introduction for people to start questioning its unique status among gravitation theories. Indeed, it was just 1919 when Weyl, and 1923 when Eddington (the very man that three years earlier had provided the first experimental verification of GR by measuring light bending during a solar eclipse) started considering modifications of the theory by including higher order invariants in its action \cite{Eddington, Weyl}.

B. Contemporary Motivation

4. Ghost fields
C. The Cauchy problem

VI. Confrontation with particle physics and astrophysics

A. Metric $f(R)$ gravity as dark matter
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VII. Summary and Conclusions

A. Summary
B. Extensions and new perspectives on $f(R)$ gravity
C. Concluding remarks

Acknowledgments

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I. INTRODUCTION

A. Historical

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B. Contemporary Motivation

4. Ghost fields
C. The Cauchy problem
These early attempts were triggered mainly by scientific curiosity and a will to question and, therefore, understand the then newly proposed theory. It is quite straightforward to realize that complicating the action and, consequently, the field equations with no apparent theoretical or experimental motivation is not very appealing. However, the motivation was soon to come.

Beginning in the 1960’s, there appeared indications that complicating the gravitational action might indeed have its merits. GR is not renormalizable and, therefore, can not be conventionally quantized. In 1962, Utiyama and De Witt showed that renormalization at one-loop demands that the Einstein–Hilbert action be supplemented by higher order curvature terms (Utiyama and DeWitt, 1962). Later on, Stelle showed that higher order actions are indeed renormalizable (but not unitary) (Stelle, 1977). More recent results show that when quantum corrections or string theory are taken into account, the effective low energy gravitational action admits higher order curvature invariants (Birrell and Davies, 1982; Buchbinder et al., 1992; Vilkovisky, 1992).

Such considerations stimulated the interest of the scientific community in higher-order theories of gravity, i.e., modifications of the Einstein–Hilbert action in order to include higher-order curvature invariants with respect to the Ricci scalar [see Schmidt (2007) for a historical review and a list of references to early work]. However, the relevance of such terms in the action was considered to be restricted to very strong gravity regimes and they were expected to be strongly suppressed by small couplings, as one would expect when simple effective field theory considerations are taken into account. Therefore, corrections to GR were considered to be important only at scales close to the Planck scale and, consequently, in the early universe or near black hole singularities — and indeed there are relevant studies, such as the well-known curvature-driven inflation scenario (Starobinsky, 1980) and attempts to avoid cosmological and black hole singularities (Brandenberger, 1992; 1993; 1995; Brandenberger et al., 1993; Mukhanov and Brandenberger, 1992; Shahid-Saless, 1990; Trodden et al., 1993). However, it was not expected that such corrections could affect the gravitational phenomenology at low energies, and consequently large scales such as, for instance, the late universe.

B. Contemporary Motivation

More recently, new evidence coming from astrophysics and cosmology has revealed a quite unexpected picture of the universe. Our latest datasets coming from different sources, such as the Cosmic Microwave Background Radiation (CMBR) and supernovae surveys, seem to indicate that the energy budget of the universe is the following: $4\%$ ordinary baryonic matter, $20\%$ dark matter and $76\%$ dark energy (Astier et al., 2006; Eisenstein et al., 2005; Riess et al., 2004; Spergel et al., 2007). The term dark matter refers to an unknown form of matter, which has the clustering properties of ordinary matter but has not yet been detected in the laboratory. The term dark energy is reserved for an unknown form of energy which not only has not been detected directly, but also does not cluster as ordinary matter does. More rigorously, one could use the various energy conditions to distinguish dark matter and dark energy: Ordinary matter and dark matter satisfy the Strong Energy Condition, whereas Dark Energy does not. Additionally, dark energy seems to resemble in high detail a cosmological constant. Due to its dominance over matter (ordinary and dark) at present times, the expansion of the universe seems to be an accelerated one, contrary to past expectations.\footnote{Recall that, from GR in the absence of the cosmological constant and under the standard cosmological assumptions (spatial homogeneity and isotropy etc.), one obtains the second Friedmann equation

$$\frac{\dot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3P),$$

where $a$ is the scale factor, $G$ is the gravitational constant and $\rho$ and $P$ are the energy density and the pressure of the cosmological fluid, respectively. Therefore, if the Strong Energy Condition $\rho + 3P \geq 0$ is satisfied, there can be no acceleration (gravity is attractive).}

Note that this late time speed-up comes to be added to an early time accelerated epoch as predicted by the inflationary paradigm (Guth, 1981; Kolb and Turner, 1992; Linde, 1990). The inflationary epoch is needed to address the so-called horizon, flatness and monopole problems (Kolb and Turner, 1992; Linde, 1990; Misner, 1968; Weinberg, 1972), as well as to provide the mechanism that generates primordial inhomogeneities acting as seeds for the formation of large scale structures (Mukhanov, 2003). Recall also that, in between these two periods of acceleration, there should be a period of decelerated expansion, so that the more conventional cosmological eras of radiation domination and matter domination can take place. Indeed, there are stringent observational bounds on the abundances of light elements, such as deuterium, helium and lithium, which require that Big Bang Nucleosynthesis (BBN), the production of nuclei other than hydrogen, takes place during radiation domination (Burles et al., 2001; Carroll and Kaplinghat, 2002). On the other hand, a matter-dominated era is required for structure formation to take place.

Puzzling observations do not stop here. Dark matter does not only make its appearance in cosmological data but also in astrophysical observations. The “missing mass” question had already been posed in 1933 for galaxy clusters (Zwicky, 1933) and in 1959 for individual galaxies (Kahn and Woltjer, 1959) and a satisfactory final answer has been pending ever since (Bosma, 1978; Ellis, 2002; Moore, 2001; Persic et al., 1996; Rubin and Ford, 1970; Rubin et al., 1980).

One, therefore, has to admit that our current picture
of the evolution and the matter/energy content of the universe is at least surprising and definitely calls for an explanation. The simplest model which adequately fits the data creating this picture is the so called concordance model or ΛCDM (Λ-Cold Dark Matter), supplemented by some inflationary scenario, usually based on some scalar field called inflaton. Besides not explaining the origin of the inflaton or the nature of dark matter by itself, the ΛCDM model is burdened with the well known cosmological constant problems (Carroll, 2001a; Weinberg, 1989): the magnitude problem, according to which the observed value of the cosmological constant is extravagantly small to be attributed to the vacuum energy of matter fields, and the coincidence problem, which can be summed up in the question: since there is just an extremely short period of time in the evolution of the universe in which the energy density of the cosmological constant is comparable with that of matter, why is this happening today that we are present to observe it?

These problems make the ΛCDM model more of an empirical fit to the data whose theoretical motivation can be regarded as quite poor. Consequently, there have been several attempts to either directly motivate the presence of a cosmological constant or to propose dynamical alternatives to dark energy. Unfortunately, none of these attempts are problem-free. For instance, the so-called anthropic reasoning for the magnitude of Λ (Barrow and Tipler, 1986; Carter, 1974), even when placed into the firmer grounds through the idea of the “anthropic or string landscape” (Susskind, 2003), still makes many physicists feel uncomfortable due to its probabilistic nature. On the other hand, simple scenarios for dynamical dark energy, such as quintessence (Bahcall et al., 1998; Caldwell et al., 1998; Carroll, 1998; Ostriker and Steinhardt, 1995; Peebles and Ratra, 1988; Ratra and Peebles, 1988; Wang et al., 2000; Wetterich, 1988) do not seem to be as well motivated theoretically as one would desire.2

Another perspective towards resolving the issues described above, which might appear as more radical to some, is the following: gravity is by far the dominant interaction at cosmological scales and, therefore, it is the force governing the evolution of the universe. Could it be that our description of the gravitational interaction at the relevant scales is not sufficiently adequate and stands at the root of all or some of these problems? Should we consider modifying our theory of gravitation and if so, would this help in avoiding dark components and answering the cosmological and astrophysical riddles?

It is rather pointless to argue whether such a perspec-

tive would be better or worse than any of the other solutions already proposed. It is definitely a different way to address the same problems and, as long as these problems do not find a plausible, well accepted and simple, solution, it is worth pursuing all alternatives. Additionally, questioning the gravitational theory itself definitely has its merits: it helps us to obtain a deeper understanding of the relevant issues and of the gravitational interaction, it has high chances to lead to new physics and it has worked in the past. Recall that the precession of Mercury’s orbit was at first attributed to some unobserved (“dark”) planet orbiting inside Mercury’s orbit, but it actually took the passage from Newtonian gravity to GR to be explained.

C. f(R) theories as toy theories

Even if one decides that modifying gravity is the way to go, this is not an easy task. To begin with, there are numerous ways to deviate from GR. Setting aside the early attempts to generalize Einstein’s theory, most of which have been shown to be non-viable (Will, 1981), and the most well known alternative to GR, scalar-tensor theory (Bergmann, 1968; Brans and Dicke, 1961; Dicke, 1962; Faraoni, 2004a; Nordtvedt, 1970; Wagoner, 1970), there are still numerous proposals for modified gravity in contemporary literature. Typical examples are DGP (Dvali-Gabadadze-Porrati) gravity (Dvali et al., 2000), brane-world gravity (Maartens, 2004), TeVeS (Tensor-Vector-Scalar) (Bekenstein, 2004) and Einstein-Aether theory (Jacobson and Mattingly, 2001). The subject of this review is a different class of theories, f(R) theories of gravity. These theories come about by a straightforward generalization of the Lagrangian in the Einstein–Hilbert action,

\[ S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \, R, \quad (2) \]

where \( \kappa \equiv 8\pi G \), \( G \) is the gravitational constant, \( g \) is the determinant of the metric and \( R \) is the Ricci scalar \((c = \hbar = 1)\), to become a general function of \( R \), i.e.,

\[ S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \, f(R). \quad (3) \]

Before going further into the discussion of the details and the history of such actions — this will happen in the forthcoming section — some remarks are in order. We have already mentioned the motivation coming from high-energy physics for adding higher order invariants to the gravitational action, as well as a general motivation coming from cosmology and astrophysics for seeking generalizations of GR. There are, however, still two questions that might be troubling the reader. The first one is: Why specifically \( f(R) \) actions and not more general ones, which include other higher order invariants, such as \( R_{\mu\nu}R^{\mu\nu} \)?

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2 We are referring here to the fact that, not only the mass of the scalar turns out to be many orders of magnitude smaller than any of the masses of the scalar fields usually encountered in particle physics, but also to the inability to motivate the absence of any coupling of the scalar field to matter (there is no mechanism or symmetry preventing this) (Carroll, 2001b).
The answer to this question is twofold. First of all, there is simplicity: \( f(R) \) actions are sufficiently general to encapsulate some of the basic characteristics of higher-order gravity, but at the same time they are simple enough to be easy to handle. For instance, viewing \( f \) as a series expansion, i.e.,

\[
f(R) = \cdots + \frac{\alpha_2}{R^2} + \frac{\alpha_1}{R} - 2\Lambda + R + \frac{R^2}{\beta_2} + \frac{R^3}{\beta_3} + \cdots , \tag{4}
\]

where the \( \alpha_i \) and \( \beta_i \) coefficients have the appropriate dimensions, we see that the action includes a number of phenomenologically interesting terms. In brief, \( f(R) \) theories make excellent candidates for toy-theories—tools from which one gains some insight in such gravity modifications. Second, there are serious reasons to believe that \( f(R) \) theories are unique among higher-order gravity theories, in the sense that they seem to be the only ones which can avoid the long known and fatal Ostrogradski instability (Woodard, 2007).

The second question calling for an answer is related to a possible loophole that one may have already spotted in the motivation presented: How can high-energy modifications of the gravitational action have anything to do with late-time cosmological phenomenology? Wouldn’t effective field theory considerations require that the coefficients in eq. (4) be such, as to make any corrections to the standard Einstein–Hilbert term important only near the Planck scale?

Conservatively thinking, the answer would be positive. However, one also has to stress two other serious factors: first, there is a large ambiguity on how gravity really works at small scales or high energies. Indeed there are certain results already in the literature claiming that terms responsible for late time gravitational phenomenology might be predicted by some more fundamental theory, such as string theory [see, for instance, Nojiri and Odintsov (2003)]. On the other hand, one should not forget that the observationally measured value of the cosmological constant corresponds to some energy scale. Effective field theory or any other high-energy theory consideration has thus far failed to predict or explain it. Yet, it stands as an experimental fact and putting the number in the right context can be crucial in explaining its value. Therefore, in any phenomenological approach, its seems inevitable that some parameter will appear to be unnaturally small at first (the mass of a scalar, a coefficient of some expansions, etc. according to the approach). The real question is whether this initial “unnaturalness” still has room to be explained.

In other words, in all sincerity, the motivation for infrared modifications of gravity in general and \( f(R) \) gravity in particular is, to some extent, hand-waving. However, the importance of the issues leading to this motivation and our inability to find other, more straightforward and maybe better motivated, successful ways to address them combined with the significant room for speculation which our quantum gravity candidates leave, have triggered an increase of interest in modified gravity that is probably reasonable.

To conclude, when all of the above is taken into account, \( f(R) \) gravity should neither be over- nor underestimated. It is an interesting and relatively simple alternative to GR, from the study of which some useful conclusions have been derived already. However, it is still a toy-theory, as already mentioned; an easy-to-handle deviation from Einstein’s theory mostly to be used in order to understand the principles and limitations of modified gravity. Similar considerations apply to modifying gravity in general: we are probably far from concluding whether it is the answer to our problems at the moment. However, in some sense, such an approach is bound to be fruitful since, even if it only leads to the conclusion that GR is the only correct theory of gravitation, it will still have helped us to both understand GR better and secure our faith in it.

II. ACTIONS AND FIELD EQUATIONS

As can be found in many textbooks — see, for example, Misner et al. (1973); Wald (1984) — there are actually two variational principles that one can apply to the Einstein–Hilbert action in order to derive Einstein’s equations: the standard metric variation and a less standard variation dubbed Palatini variation [even though it was Einstein and not Palatini who introduced it (Ferraris et al., 1982)]. In the latter the metric and the connection are assumed to be independent variables and one varies the action with respect to both of them (we will see how this variation leads to Einstein’s equations shortly), under the important assumption that the matter action does not depend on the connection. The choice of the variational principle is usually referred to as a formalism, so one can use the terms metric (or second order) formalism and Palatini (or first order) formalism. However, even though both variational principles lead to the same field equation for an action whose Lagrangian is linear in \( R \), this is no longer true for a more general action. Therefore, it is intuitive that there will be two version of \( f(R) \) gravity, according to which variational principle or formalism is used. Indeed this is the case: \( f(R) \) gravity in the metric formalism is called metric \( f(R) \) gravity and \( f(R) \) gravity in the Palatini formalism is called Palatini \( f(R) \) gravity (Buchdahl, 1970).

Finally, there is actually even a third version of \( f(R) \) gravity: metric-affine \( f(R) \) gravity (Sotiriou and Liberati, 2007a,b). This comes about if one uses the Palatini variation but abandons the assumption that the matter action is independent of the connection. Clearly, metric affine \( f(R) \) gravity is the most general of these theories and reduces to metric or Palatini \( f(R) \) gravity if further assumptions are made. In this section we will present the actions and field equations of all three versions of \( f(R) \) gravity and point out their difference. We will also clarify the physical meaning behind the assumptions that discriminate
them.

For an introduction to metric $f(R)$ gravity see also [Nojiri and Odintsov, 2007a], for a shorter review of metric and Palatini $f(R)$ gravity see [Capozziello and Francaviglia, 2008] and for an extensive analysis of all versions of $f(R)$ gravity and other alternative theories of gravity see [Sotiriou, 2007b].

\section{A. Metric formalism}

Beginning from the action $S_{\text{met}}$ and adding a matter term $S_M$, the total action for $f(R)$ gravity takes the form

$$S_{\text{met}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_M(g_{\mu\nu}, \psi),$$

where $\psi$ collectively denotes the matter fields. Variation with respect to the metric gives, after some manipulations and modulo surface terms

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box] f'(R) = \kappa T_{\mu\nu},$$

where, as usual,

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{met}}}{\delta g^{\mu\nu}},$$

a prime denotes differentiation with respect to the argument, $\nabla_\mu$ is the covariant derivative associated with the Levi-Civita connection of the metric, and $\Box \equiv \nabla^\mu \nabla_\mu$. Metric $f(R)$ gravity was first rigorously studied in [Buchdahl, 1970].

It has to be stressed that there is a mathematical jump in deriving eq. (3) from the action (5) having to do with the surface terms that appear in the variation: as in the case of the Einstein–Hilbert action, the surface terms do not vanish just by fixing the metric on the boundary. For the Einstein–Hilbert action, however, these terms gather into a total variation of a quantity. Therefore, it is possible to add a total divergence to the action in order to “heal” it and arrive to a well-defined variational principle (this is the well known Gibbons–Hawking–York surface term [Gibbons and Hawking, 1977; York, 1972]). Unfortunately, the surface terms in the variation of the action (3) do not consist of a total variation of some quantity (the reader is urged to calculate the variation in order to verify this fact) and it is not possible to “heal” the action by just subtracting some surface term before performing the variation.

The way out comes from the fact that the action includes higher order derivatives of the metric and, therefore, it should be possible to fix more degrees of freedom on the boundary than those of the metric itself. There is no unique prescription for such a fixing in the literature so far. Note also that the choice of fixing is not void of physical meaning, since it will be relevant for the Hamiltonian formulation of the theory. However, the field equations would be unaffected by the fixing chosen and from a purely classical perspective, such as the one followed here, the field equations are all that one needs [see [Sotiriou, 2007b] for a more detailed discussion on these issues].

Setting aside the complications of the variation we can now focus on the field equations (6). These are obviously fourth order partial differential equations in the metric, since $R$ already includes second derivatives of the latter. For an action which is linear in $R$, the fourth order terms — the last two on the left hand side — vanish and the theory reduces to GR.

Notice also that the trace of eq. (6)

$$f'(R)R - 2f(R) + 3\Box f' = \kappa T,$$

where $T = g^{\mu\nu}T_{\mu\nu}$, relates $R$ with $T$ differentially and not algebraically as in GR, where $R = -\kappa T$. This is already an indication that the field equations of $f(R)$ theories will admit a larger variety of solutions than Einstein’s theory. As an example, we mention here that the Jebsen-Birkhoff’s theorem, stating that the Schwarzschild solution is the unique spherically symmetric vacuum solution, no longer holds in metric $f(R)$ gravity. Without going into details, let us stress that $T = 0$ no longer implies that $R = 0$, or is even constant.

Eq. (5) will turn out to be very useful in studying various aspects of $f(R)$ gravity, notably its stability and weak-field limit. For the moment, let us use it to make some remarks about maximally symmetric solutions. Recall that maximally symmetric solutions lead to a constant Ricci scalar. For $R = \text{constant}$ and $T_{\mu\nu} = 0$, eq. (6) reduces to

$$f'(R)R - 2f(R) = 0,$$

which, for a given $f$, is an algebraic equation in $R$. If $R = 0$ is a root of this equation and one takes this root, then eq. (3) reduces to $R_{\mu\nu} = 0$ and the maximally symmetric solution is Minkowski spacetime. On the other hand, if the root of eq. (3) is $R = C$, where $C$ is a constant, then eq. (6) reduces to $R_{\mu\nu} = g_{\mu\nu}C/4$ and the maximally symmetric solution is de Sitter or anti-de Sitter space depending on the sign of $C$, just as in GR with a cosmological constant.

Another issue that should be stressed is that of energy conservation. In metric $f(R)$ gravity the matter is minimally coupled to the metric. One can, therefore, use the usual arguments based on the invariance of the action under diffeomorphisms of the spacetime manifold (coordinate transformations $x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu$ followed by a pullback, with the field $\xi^\mu$ vanishing on the boundary of the spacetime region considered, leave the physics unchanged, see [Wald, 1984]) to show that $T_{\mu\nu}$ is divergence-free. The same can be done at the level of the field equations: a “brute force” calculation reveals that

\footnote{Specific attention to higher-dimensional $f(R)$ gravity was paid in [Gunther et al., 2002, 2003, 2005; Saidov and Zhuk, 2004, 2007].}
Finally, let us note that it is possible to write the field equations in the form of Einstein equations with an effective stress-energy tensor composed of curvature terms. Specifically, eq. (6) can be written as

\[ G_{\mu\nu} = \frac{\kappa}{f'(R)} \left( T_{\mu\nu} + T^{(eff)}_{\mu\nu} \right) = \frac{\kappa}{f'(R)} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} g_{\mu\nu} f'(R) \right) \]

where the quantity \( G_{\mu\nu} = G/f'(R) \) can be regarded as the effective gravitational coupling strength in analogy to what is done in scalar-tensor gravity — positive of \( G_{\mu\nu} \) (equivalent to the requirement that the graviton is not a ghost) imposes that \( f'(R) > 0 \). Moreover,

\[ T^{(c,eff)}_{\mu\nu} = \frac{1}{\kappa} \left[ \frac{f'(R)}{2} g_{\mu\nu} + \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \Box f'(R) \right] \]

is an effective stress-energy tensor which does not have the canonical form quadratic in the first derivatives of the field \( f'(R) \), but contains terms linear in the second derivatives. The effective energy density derived from this is not positive-definite and none of the energy conditions holds. Again, this situation is analogous to that occurring in scalar-tensor gravity. The effective stress-energy tensor (12) can be put in the form of a perfect fluid energy-momentum tensor, which will turn out to be useful in Sec. IV.

B. Palatini formalism

We have already mentioned that the Einstein equations can be derived using, instead of the standard metric variation of the Einstein–Hilbert action, the Palatini formalism, i.e., an independent variation with respect to the metric and an independent connection (Palatini variation). The action is formally the same but now the Riemann tensor and the Ricci tensor are constructed with the independent connection. Note that the metric is not needed to obtain the latter from the former. For clarity of notation, we denote the Ricci tensor constructed with this independent connection as \( R_{\mu\nu} \) and the corresponding Ricci scalar \( f(R) \) is \( R = g^\mu_\nu R_{\mu\nu} \). The action now takes the form

\[ S_{pal} = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} f(R) + S_M(g_{\mu\nu}, \psi). \]

GR will come about, as we will see shortly, when \( f(R) = R \). Note that the matter action \( S_M \) is assumed to depend only on the metric and the matter fields and not on the independent connection. This assumption is crucial for the derivation of Einstein’s equations from the linear version of the action (13) and is the main feature of the Palatini formalism.

It has already been mentioned that this assumption has consequences for the physical meaning of the independent connection (Sotiriou, 2006a; Sotiriou and Liberati, 2007a). Let us elaborate on this: recall that an affine connection usually defines parallel transport and the covariant derivative. On the other hand, the matter action \( S_M \) is supposed to be a generally covariant scalar which includes derivatives of the matter fields. Therefore, these derivatives ought to be covariant derivatives for a general matter field. Exceptions exist, such as a scalar field, for which a covariant and a partial derivative coincide, and the electromagnetic field, for which one can write a covariant action without the use of the covariant derivative [it is the exterior derivative that is actually needed, see next section and (Sotiriou and Liberati, 2007b)]. However, \( S_M \) should include all possible fields. Therefore, assuming that \( S_M \) is independent of the connection can imply one of two things (Sotiriou, 2006a): either we are restricting ourselves to specific fields, or we are implicitly assuming that it is the Levi-Civita connection of the metric that actually defines parallel transport. Since the first option is implausibly limiting for a gravitational theory, we are left with the conclusion that the independent connection \( \Gamma^\lambda_\mu_\nu \) does not define parallel transport or the covariant derivative and the geometry is actually pseudo-Riemannian. The covariant derivative is actually defined by the Levi-Civita connection of the metric \( \{^\lambda_\mu_\nu \} \).

This also implies that Palatini \( f(R) \) gravity is a metric theory in the sense that it satisfies the metric postulates (Will, 1981). Let us clarify this: matter is minimally coupled to the metric and not coupled to any other fields. Once again, as in GR or metric \( f(R) \) gravity, one could use diffeomorphism invariance to show that

The term “\( f(R) \) gravity” is used generically for a theory in which the action is some function of some Ricci scalar, not necessarily \( R \).

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4 Energy-momentum complexes in the spherically symmetric case have been computed in (Multamaki et al., 2008).

5 The term “\( f(R) \) gravity” is used generically for a theory in which the action is some function of some Ricci scalar, not necessarily \( R \).
the stress energy tensor is conserved by the covariant derivative defined with the Levi-Civita connection of the metric, i.e., $\nabla_{\mu} T^{\mu\nu} = 0$ (but $\nabla_{\mu} T^{\mu\nu} \neq 0$). This can also be shown by using the Levi-Civita connection of the metric and show that it vanishes (Barraco et al., 1999, Koivisto, 2006a). Clearly, then, Palatini $f(R)$ gravity is a metric theory according to the definition of Will (1981) (not to be confused with the term “metric” in “metric $f(R)$ gravity”, which simply refers to the fact that one only varies the action with respect to the metric). Conventionally thinking, as a consequence of the covariant conservation of the matter energy-momentum tensor, test particles should follow geodesics of the metric in Palatini $f(R)$ gravity. This can be seen by considering a dust fluid with $T_{\mu\nu} = \rho u_{\mu} u_{\nu}$ and projecting the conservation equation $\nabla^\beta f_{\mu\nu} = 0$ onto the fluid four-velocity $u^\nu$. Similarly, theories that satisfy the metric postulates are supposed to satisfy the Einstein Equivalence Principle as well (Will, 1981). Unfortunately, things are more complicated here and, therefore, we set this issue aside for the moment. We will return to it and attempt to fully clarify it in Secs. VI.B and VI.C.2. For now, let us proceed with our discussion of the field equations.

Varying the action (13) independently with respect to the metric and the connection, respectively, and using the formula

$$\delta R_{\mu\nu} = \bar{\nabla}_\lambda \delta \Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta \Gamma^\lambda_{\mu\nu}. \tag{14}$$

yields

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{15}$$

$$-\bar{\nabla}_\lambda (\sqrt{-g}f'(R)g^{\nu\lambda}) + \nabla_\nu (\sqrt{-g}f'(R)g^{\nu\lambda}) \delta^\nu_\lambda = 0, \tag{16}$$

where $T_{\mu\nu}$ is defined in the usual way as in eq. (7), $\bar{\nabla}_\mu$ denotes the covariant derivative defined with the independent connection $\Gamma^\lambda_{\mu\nu}$, and $\delta^\nu_\lambda$ denote symmetrization or anti-symmetrization over the indices $\mu$ and $\nu$, respectively. Taking the trace of eq. (16), it can be easily shown that

$$\bar{\nabla}_\sigma (\sqrt{-g}f'(R)g^{\nu\mu}) = 0, \tag{17}$$

which implies that we can bring the field equations into the more economical form

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} = \kappa GT_{\mu\nu}, \tag{18}$$

$$\bar{\nabla}_\lambda (\sqrt{-g}f'(R)g^{\nu\lambda}) = 0. \tag{19}$$

It is now easy to see how the Palatini formalism leads to GR when $f(R) = R$; in this case $f'(R) = 1$ and eq. (19) becomes the definition of the Levi-Civita connection for the initially independent connection $\Gamma^\lambda_{\mu\nu}$. Then, $R_{\mu\nu} = R_{\mu\nu}$, $R = R$ and eq. (18) yields Einstein’s equations. This reproduces the result that can be found in textbooks [Misner et al., 1973, Wald, 1984]. Note that in the Palatini formalism for GR, the fact that the connection turns out to be the Levi-Civita one is a dynamical feature instead of an a priori assumption.

It is now evident that generalizing the action to be a general function of $R$ in the Palatini formalism is just as natural as it is to generalize the Einstein–Hilbert action in the metric formalism. Remarkably, even though the two formalisms give the same results for linear actions, they lead to different results for more general actions (Buchdahl, 1970, Burton and Mann, 1998, Ekiratif and Sheikh-Jabbari, 2008, Querella, 1998, Shahid-Salessi, 1987).

Finally, let us present some useful manipulations of the field equations. Taking the trace of eq. (18) yields

$$f'(R)R - 2f(R) = \kappa T. \tag{20}$$

As in the metric case, this equation will prove very useful later on. For a given $f$, it is an algebraic equation in $R$. For all cases in which $T = 0$, including vacuum and electrovacuum, $R$ will therefore be a constant and a root of the equation

$$f'(R)R - 2f(R) = 0. \tag{21}$$

We will not consider cases for which this equation has no roots since it can be shown that the field equations are then inconsistent (Ferraris et al., 1992). Therefore, choices of $f$ that lead to this behaviour should simply be avoided. Eq. (21) can also be identically satisfied if $f(R) \propto R^2$. This very particular choice for $f$ leads to a conformally invariant theory (Ferraris et al., 1992). As is apparent from eq. (20), if $f(R) \propto R^2$ then only conformally invariant matter, for which $T = 0$ identically, can be coupled to gravity. Matter is not generically conformally invariant though, and so this particular choice of $f$ is not suitable for a low energy theory of gravity. We will, therefore, not consider it further [see Sotiriou, 2006b for a discussion].

Next, we consider eq. (19). Let us define a metric conformal to $g_{\mu\nu}$ as

$$h_{\mu\nu} = f'(R)g_{\mu\nu}. \tag{22}$$

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6 Energy super-tensors and pseudotensors in Palatini $f(R)$ gravity were studied in [Barraco et al., 1999, Borowiec et al., 1994, 1998, Ferraris et al., 1992], and alternative energy definitions were given in [Deser and Tekin, 2002, 2003a, 2007].

7 See, however, (Sotiriou, 2007b) for further analysis of the $f(R)$ action and how it can be derived from first principles in the two formalisms.
It can easily be shown that it can easily be shown that
\[ \sqrt{-h} \, h^{\mu \nu} = \sqrt{-g} \, f'(\mathcal{R}) g^{\mu \nu}. \] (23)

Then, eq. (19) becomes the definition of the Levi-Civita connection of \( h_{\mu \nu} \) and can be solved algebraically to give
\[ \Gamma^\lambda_{\mu \nu} = \frac{1}{2} h^\lambda_{\sigma} (\partial_\mu h_{\nu \sigma} + \partial_\nu h_{\mu \sigma} - \partial_\sigma h_{\mu \nu}), \] (24)
or, equivalently, in terms of \( g_{\mu \nu} \),
\[ \Gamma^\lambda_{\mu \nu} = \frac{1}{2} \frac{1}{f'(\mathcal{R})} g^\lambda_{\sigma} \left[ \partial_\mu (f'(\mathcal{R}) g_{\nu \sigma}) + \partial_\nu (f'(\mathcal{R}) g_{\mu \sigma}) - \partial_\sigma (f'(\mathcal{R}) g_{\mu \nu}) \right], \] (25)

Given that eq. (20) relates \( \mathcal{R} \) algebraically with \( T \), and since we have an explicit expression for \( \Gamma^\lambda_{\mu \nu} \) in terms of \( \mathcal{R} \) and \( g^{\mu \nu} \), we can in principle eliminate the independent connection from the field equations and express them only in terms of the metric and the matter fields. Actually, the fact that we can algebraically express \( \Gamma^\lambda_{\mu \nu} \) in terms of the latter two already indicates that this connection act as some sort of auxiliary field. We will explore this further in Sec. III. For the moment, let us take into account how the Ricci tensor transforms under conformal transformations and write
\[ \mathcal{R}_{\mu \nu} = R_{\mu \nu} + \frac{3}{2} \frac{1}{f'(\mathcal{R})^2} \left( \nabla_\mu f'(\mathcal{R}) \right) \left( \nabla_\nu f'(\mathcal{R}) \right) - \frac{1}{f'(\mathcal{R})} \left( \nabla_\mu \nabla_\nu - \frac{1}{2} g_{\mu \nu} \Box \right) f'(\mathcal{R}). \] (26)

Contraction with \( g^{\mu \nu} \) yields
\[ \mathcal{R} = R + \frac{3}{2} \frac{1}{f'(\mathcal{R})^2} \left( \nabla_\mu f'(\mathcal{R}) \right) \left( \nabla^\mu f'(\mathcal{R}) \right) + \frac{3}{2} \frac{1}{f'(\mathcal{R})^2} \Box f'(\mathcal{R}). \] (27)

Note the difference between \( \mathcal{R} \) and the Ricci scalar of \( h_{\mu \nu} \) due to the fact that \( g_{\mu \nu} \) is used here for the contraction of \( R_{\mu \nu} \).

Replacing eqs. (26) and (27) in eq. (18), and after some easy manipulations, one obtains
\[ G_{\mu \nu} = \frac{\kappa}{f'} T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \left( \mathcal{R} - \frac{f''}{f'} \right) \] (28)
\[ + \frac{1}{f'} \left( \nabla_\mu \nabla_\nu - g_{\mu \nu} \Box \right) f' - \frac{3}{2} \frac{1}{f'^2} \left[ \nabla_\mu f' \left( \nabla_\nu f' \right) - \frac{1}{2} g_{\mu \nu} \left( \nabla f' \right)^2 \right]. \]

Notice that, assuming that we know the root of eq. (20), \( \mathcal{R} = \mathcal{R}(T) \), we have completely eliminated the independent connection from this equation. Therefore, we have successfully reduced the number of field equations to one and at the same time both sides of eq. (28) depend only on the metric and the matter fields. In a sense, the theory has been brought to the form of GR with a modified source.

We can now straightforwardly deduce the following:

- When \( f(\mathcal{R}) = \mathcal{R} \), the theory reduces to GR, as discussed previously.

- For matter fields with \( T = 0 \), due to eq. (21), \( \mathcal{R} \) and consequently \( f(\mathcal{R}) \) and \( f'(\mathcal{R}) \) are constants and the theory reduces to GR with a cosmological constant and a modified coupling constant \( G/f' \). If we denote the value of \( \mathcal{R} \) when \( T = 0 \) as \( \mathcal{R}_0 \), then the value of the cosmological constant is
\[ \frac{1}{2} \left( \mathcal{R}_0 - \frac{f(\mathcal{R}_0)}{f'(\mathcal{R}_0)} \right) = \frac{\mathcal{R}_0}{4}, \] (29)
where we have used eq. (21). Besides vacuum, \( T = 0 \) also for electromagnetic fields, radiation, and any other conformally invariant type of matter.

- In the general case \( T \neq 0 \), the modified source on the right hand side of eq. (28) includes derivatives of the stress-energy tensor, unlike in GR. These are implicit in the last two terms of eq. (28), since \( f' \) is in practice a function of \( T \), given that \( f' = f'(\mathcal{R}) \) and \( \mathcal{R} = \mathcal{R}(T) \).

The serious implications of this last observation will become clear in Sec. VI.C.1.

C. Metric-affine formalism

As we already pointed out, the Palatini formalism of \( f(\mathcal{R}) \) gravity relies on the crucial assumption that the matter action does not depend on the independent connection. We also argued that this assumption relieves this connection to the role of some sort of auxiliary field and the connection carrying the usual geometrical meaning — parallel transport and definition of the covariant derivative — remains the Levi-Civita connection of the metric. All of these statements will be supported further in the forthcoming sections, but for the moment let us consider what would be the outcome if we decided to be faithful to the geometrical interpretation of the independent connection \( \Gamma^\lambda_{\mu \nu} \): this

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8 This calculation holds in four dimensions. When the number of dimensions \( D \) is different from 4 then, instead of using eq. (22), the conformal metric \( h_{\mu \nu} \) should be introduced as \( h_{\mu \nu} = \left[ f'(\mathcal{R}) \right]^{2/(D-2)} g_{\mu \nu} \) in order for eq. (23) to still hold.

9 Note that, apart from special cases such as a perfect fluid, \( T_{\mu \nu} \) and consequently \( T \) already include first derivatives of the matter fields, given that the matter action has such a dependence. This implies that the right hand side of eq. (28) will include at least second derivatives of the matter fields, and possibly up to third derivatives.
would imply that we would define the covariant derivat-
ives of the matter fields with this connection and, there-
fore, we would have \( S_M = S_M(g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}, \psi) \). The ac-
tion of this theory, dubbed metric-affine \( f(R) \) gravity
(Sotiriou and Liberati, 2007b), would then be [note the

definition of this theory, dubbed metric-affine
connection is symmetric. It is useful to define
related to the metric, we will also drop the assumption
besides dropping the assumption that the connection is
arbitrary (Cartan, 1922, 1923, 1924; Hehl et al., 1976; Kibble,
1961; Sciama, 1964). In this theory, as well as in other
theories with an independent connection, some part of
the connection is still related to the metric (e.g., the non-
metricity is set to zero). In our case, the connec-
tion is left completely unconstrained and is to be deter-
dined by the field equations. Metric-affine gravity with
the linear version of the action \( (30) \) was initially pro-
cised in (Hehl and Kerling, 1978) and the generalization
to \( f(R) \) actions was considered in (Sotiriou and Liberati,
2007a,b). Un-
fortunately, leaving the connection completely uncon-
strained comes with a complication. Let us consider
the projective transformation
\[
\Gamma^{\lambda}_{\mu\nu} \rightarrow \Gamma^{\lambda}_{\mu\nu} + \delta^{\lambda}_{\mu} \xi_{\nu},
\]
where \( \xi_{\nu} \) is an arbitrary covariant vector field. One can
easily show that the Ricci tensor will correspondingly
transform like
\[
R_{\mu\nu} \rightarrow R_{\mu\nu} - 2\partial_{[\mu} \xi_{\nu]}.
\]
However, given that the metric is symmetric, this implies
that the curvature scalar does not change
\[
R \rightarrow R,
\]
i.e., \( R \) is invariant under projective transformations.
Hence the Einstein–Hilbert action or any other action
built from a function of \( R \), such as the one used here,
is projective invariant in metric-affine gravity. However,
the matter action is not generically projective invariant
and this would be the cause of an inconsistency in the
field equations.
One could try to avoid this problem by generalizing
the gravitational action in order to break projective
invariance. This can be done in several ways, such as
allowing for the metric to be non-symmetric as well,
adding higher order curvature invariants or terms in-
cluding the Cartan torsion tensor [see (Sotiriou, 2007b;
Sotiriou and Liberati, 2007b) for a more detailed dis-
cussion]. However, if one wants to stay within the framework
of \( f(R) \) gravity, which is our subject here, then there is
only one way to cure this problem: to somehow constrain
the connection. In fact, it is evident from eq. (35) that,
if the connection were symmetric, projective invariance
would be broken. However, one does not have to take
such a drastic measure.
To understand this issue further, we should re-examine
the meaning of projective invariance. This is very similar
to gauge invariance in electromagnetism (EM). It tells us
that the corresponding field, in this case the connections
\( \Gamma^\lambda_{\mu\nu} \), can be determined from the field equations up to a
projective transformation [eq. (33)]. Breaking this invar-
iance can therefore come by fixing some degrees of free-
dom of the field, similarly to gauge fixing. The number
of degrees of freedom which we need to fix is obviously
the number of the components of the four-vector used
for the transformation, i.e., simply four. In practice, this means that we should start by assuming that the connection is not the most general which one can construct, but satisfies some constraints.

Since the degrees of freedom that we need to fix are four and seem to be related to the non-symmetric part of the connection, the most obvious prescription is to demand that \( \sigma_{\mu} = \sigma_{\nu} \) be equal to zero, which was first suggested in (Sandberg, 1975) for a linear action and shown to work also for an \( f(R) \) action in (Sotiriou and Liberati, 2007b). Note that this does not mean that \( \Gamma_{\mu\nu} \) should vanish, but merely that \( \Gamma_{\mu\nu} = \Gamma_{\sigma\mu} \). Imposing this constraint can easily be done by adding a Lagrange multiplier \( B^\mu \). The additional term in the action will be

\[
S_{LM} = \int d^4x \sqrt{-g} B^\mu \sigma_{\mu}.
\] (38)

The action (30) with the addition of the term in eq. (38) is, therefore, the action of the most general metric-affine \( f(R) \) theory of gravity.

2. Field Equations

We are now ready to vary the action and obtain field equations. Due to space limitations, we will not present the various steps of the variation here. Instead we merely give the formula

\[
\delta R_{\mu\nu} = \nabla_\lambda \delta \Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta \Gamma^\lambda_{\mu\nu} + 2 \Gamma^\sigma_{(\nu\lambda)} \delta \Gamma^\lambda_{\mu\sigma},
\] (39)

which is useful to those wanting to repeat the variation as an exercise, and we also stress our definitions for the covariant derivative

\[
\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda - \Gamma^\nu_{\lambda\mu} A^\lambda.
\] (40)

and for the Ricci tensor of an independent connection

\[
R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\gamma_{\nu\lambda} \Gamma^\lambda_{\mu\gamma} - \Gamma^\gamma_{\mu\lambda} \Gamma^\lambda_{\nu\gamma}.
\] (41)

The outcome of varying independently with respect to the metric, the connection and the Lagrange multiplier is, respectively,

\[
f'(R) R_{(\mu\nu)} - \frac{1}{2} f(R) g_{\mu\nu} = \kappa T_{\mu\nu},
\] (42)

\[
\frac{1}{\sqrt{-g}} \left[ \nabla_\lambda \left( \sqrt{-g} f'(R) g^{\mu\nu} \right) + \nabla_\sigma \left( \sqrt{-g} f'(R) g^{\sigma\nu} \right) \delta^\nu_\lambda \right]
+ 2 f'(R) \left( g^{\mu\sigma} \sigma_{\mu\sigma} - g^{\mu\sigma} S_{\sigma\lambda}^\gamma \delta^\nu_\lambda + g^{\mu\sigma} S_{\sigma\lambda}^\nu \right)
= \kappa (\Delta_{\lambda}^{\mu\nu} - B_{[\mu} g^\nu_{\lambda]}),
\] (43)

\[
S_{\mu\sigma} = 0.
\] (44)

Taking the trace of eq. (13) over the indices \( \mu \) and \( \lambda \) and using eq. (44) yields

\[
B^\mu = \frac{2}{3} \Delta_{\sigma}^{\mu\sigma}.
\] (45)

Therefore, the final form of the field equations is

\[
f'(R) R_{(\mu\nu)} - \frac{1}{2} f(R) g_{\mu\nu} = \kappa T_{\mu\nu},
\] (46)

\[
\frac{1}{\sqrt{-g}} \left[ \nabla_\lambda \left( \sqrt{-g} f'(R) g^{\mu\nu} \right) + \nabla_\sigma \left( \sqrt{-g} f'(R) g^{\sigma\nu} \right) \delta^\nu_\lambda \right]
+ 2 f'(R) g^{\mu\sigma} S_{\sigma\lambda}^\nu = \kappa (\Delta_{\lambda}^{\mu\nu} - \frac{2}{3} \Delta_{\sigma}^{[\nu} g^\nu_{\lambda]}),
\] (47)

\[
S_{\mu\sigma} = 0.
\] (48)

Next, we examine the role of \( \Delta_{\lambda}^{\mu\nu} \). By splitting eq. (47) into a symmetric and an antisymmetric part and performing contractions and manipulations it can be shown that (Sotiriou and Liberati, 2007b)

\[
\Delta_{\lambda}^{[\mu\nu]} = 0 \Rightarrow S_{\mu\nu} = 0.
\] (49)

This straightforwardly implies two things: a) Any torsion is introduced by matter fields for which \( \Delta_{\lambda}^{\mu\nu} \) is non-vanishing; b) torsion is not propagating, since it is given algebraically in terms of the matter fields through \( \Delta_{\lambda}^{[\mu\nu]} \). It can, therefore, only be detected in the presence of such matter fields. In the absence of the latter, spacetime will have no torsion.

In a similar fashion, one can use the symmetrized version of eq. (47) to show that the symmetric part of the hypermomentum \( \Delta_{\lambda}^{\mu\nu} \) is algebraically related to the non-metricity \( Q_{\mu\nu}\lambda \). Therefore, matter fields with non-vanishing \( \Delta_{\lambda}^{[\mu\nu]} \) will introduce non-metricity. However, in this case things are slightly more complicated because part of the non-metricity is also due to the functional form of the Lagrangian itself [see (Sotiriou and Liberati, 2007b)].

We will not perform a detailed study of different matter fields and their role in metric-affine gravity. We refer the reader to the more exhaustive analysis of (Sotiriou and Liberati, 2007b) for details and we restrict ourselves to the following remarks: Obviously, there are certain types of matter fields for which \( \Delta_{\lambda}^{\mu\nu} = 0 \). Characteristic examples are

- A scalar field, since in this case the covariant derivative can be replaced with a partial derivative. Therefore, the connection does not enter the matter action.
- The electromagnetic field (and gauge fields in general), since the electromagnetic field tensor \( F_{\mu\nu} \) is defined in a covariant manner using the exterior derivative. This definition remains unaffected when torsion is included [this can be related to gauge invariance, see (Sotiriou and Liberati, 2007b) for a discussion].
On the contrary, particles with spin, such as Dirac fields, generically have a non-vanishing hypermomentum and will, therefore, introduce torsion. A more complicated case is that of a perfect fluid with vanishing vorticity. If we set torsion aside, or if we consider a fluid describing particles that would initially not introduce any torsion then, as for a usual perfect fluid in GR, the matter action can be written in terms of three scalars: the energy density, the pressure, and the velocity potential \cite{Schakel1996}. Consequently, in vacuo, where also \( T^\mu{}_{\nu} \) is not divergence-free with respect to the covariant derivative defined with the Levi-Civita connection (nor with \( \nabla_\mu \) actually). However, the physical meaning of this last statement is questionable and deserves further analysis, since in metric-affine gravity \( T^\mu{}_{\nu} \) does not really carry the usual meaning of a stress-energy tensor (for instance, it does not reduce to the special relativistic tensor at an appropriate limit and at the same time there is also another quantity, the hypermomentum, which describes matter characteristics).

III. EQUVALENCE WITH BRANS–DICKE THEORY AND CLASSIFICATION OF THEORIES

In the same way that one can make variable redefinitions in classical mechanics in order to bring an equation describing a system to a more attractive, or easy to handle, form (and in a very similar way to changing coordinate systems), one can also perform field redefinitions in a field theory, in order to rewrite the action or the field equations.

There is no unique prescription for redefining the fields of a theory. One can introduce auxiliary fields, perform renormalizations or conformal transformations, or even simply redefine fields to one’s convenience.

It is important to mention that, at least within a classical perspective such as the one followed here, two theories are considered to be dynamically equivalent if, under a suitable redefinition of the gravitational and matter fields, one can make their field equations coincide. The same statement can be made at the level of the action. Dynamically equivalent theories give exactly the same results when describing a dynamical system which falls within the purview of these theories. There are clear advantages in exploring the dynamical equivalence between theories: we can use results already derived for one theory in the study of another, equivalent, theory.

The term “dynamical equivalence” can be considered misleading in classical gravity. Within a classical perspective, a theory is fully described by a set of field equations. When we are referring to gravitation theories, these equations describe the dynamics of gravitating systems. Therefore, two dynamically equivalent theories are actually just different representations of the same theory (which also makes it clear that all allowed representations can be used on an equal footing).

The issue of distinguishing between truly different theories and different representations of the same theory (or dynamically equivalent theories) is an intricate one. It has serious implications and has been the cause of many misconceptions in the past, especially when conformal transformations are used in order to redefine the fields (e.g., the Jordan and Einstein frames in scalar-tensor theory). It goes beyond the scope of this review to present a detailed analysis of this issue. We refer the reader to the literature, and specifically to \cite{Sotiriou2007} and references therein for a detailed discussion. Here, we simply mention that, given that they are handled carefully, field redefinitions and different representations of the same theory are perfectly legitimate and constitute very useful tools for understanding gravitational theories.

In what follows, we review the equivalence between
metric and Palatini $f(R)$ gravity with specific theories within the Brans–Dicke class with a potential. It is shown that these versions of $f(R)$ gravity are nothing but different representations of Brans–Dicke theory with Brans–Dicke parameter $\omega_0 = 0$ and $\omega_0 = -3/2$, respectively. We comment on this equivalence and on whether preference to a specific representation should be an issue. Finally, we use this equivalence to perform a classification of $f(R)$ gravity.

A. Metric formalism

It has been noticed quite early that metric quadratic gravity can be cast into the form of a Brans–Dicke theory and it did not take long for these results to be extended to more general actions which are functions of the Ricci scalar of the metric (Barrow, 1988; Barrow and Cotsakis, 1988; Teyssandier and Tourrenc, 1983; Wands, 1994) [see also (Flanagan, 2004a) and (Cecotti, 1987; Wands, 1994) for the extension to theories of the type $f(R, \Box^k R)$ with $k \geq 1$ of interest in supergravity]. This equivalence has been re-examined recently due to the increased interest for the extension to theories of the type $f(R)$ gravity (Chiba, 2003; Flanagan, 2004a; Davidson, 2003; Desel, 1970; Fujii, 1982; O’Hanlon, 1972a, 1972b; Olmo, 2007). By looking at eq. (52), it is seen that the equality $\omega = f''(\chi)$ is quite different from a matter field; for example, like all nonminimally coupled scalars, it can violate all of the energy conditions (Faraoni, 2004a).

The field equations corresponding to the action (54) are

$$G_{\mu\nu} = \frac{\kappa}{\phi} T_{\mu\nu} - \frac{1}{2\phi} g_{\mu\nu} V(\phi) + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi), \quad (55)$$

$$R = V'(\phi). \quad (56)$$

These field equations could have been derived directly from eq. (6) using the same field redefinitions that were mentioned above for the action. By taking the trace of eq. (55) in order to replace $R$ in eq. (55), one gets

$$3\Box \phi + 2V(\phi) - \phi \frac{dV}{d\phi} = \kappa T. \quad (57)$$

This last equation determines the dynamics of $\phi$ for given matter sources.

The condition $f'' \neq 0$ for the scalar-tensor theory to be equivalent to the original $f(R)$ gravity theory can be seen as the condition that the change of variable $\phi = f'(R)$ needed to express the theory as a Brans–Dicke one (54) be invertible, i.e., $d\phi/dR = f'' \neq 0$. This is a sufficient but not necessary condition for invertibility: it is only necessary that $f''(R)$ be continuous and one-to-one (Olmo, 2007). By looking at eq. (52), it is seen that $f'' \neq 0$ implies $\phi = f'(R)$ and the equivalence of the actions (54) and (51). When $f''$ is not defined, or it vanishes, the equality $\phi = f'(R)$ and the equivalence between the two theories can not be guaranteed (although this it is not a priori excluded by $f'' = 0$).

Finally, let us mention that, as usual in Brans–Dicke theory and more general scalar-tensor theories, one can perform a conformal transformation and rewrite the action (54) in what is called the Einstein frame (as opposed to the Jordan frame). Specifically, by performing the conformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = f'(R) g_{\mu\nu} \equiv \phi g_{\mu\nu} \quad (58)$$

and the scalar field redefinition $\phi = f'(R)$ with

$$d\tilde{\phi} = \sqrt{2\omega_0 + 3} \frac{d\phi}{\phi}. \quad (59)$$

This is the Jordan frame representation of the action of a Brans–Dicke theory with Brans–Dicke parameter $\omega_0 = 0$. An $\omega_0 = 0$ Brans–Dicke theory [sometimes called “massive dilaton gravity” (Wands, 1994)] was originally proposed by (O’Hanlon, 1972a, 1972b) in order to generate a Yukawa term in the Newtonian limit and has been occasionally considered in the literature (Anderson, 1971; Barbeau, 2003; Dabrowski et al., 2007; Davidson, 2003; Desel, 1970; Fujii, 1982; O’Hanlon, 1972a).
a scalar-tensor theory is mapped into the Einstein frame in which the “new” scalar field \( \phi \) couples minimally to the Ricci curvature and has canonical kinetic energy, as described by the gravitational action

\[
S^{(g)} = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2\kappa} - \frac{1}{2} \partial^\alpha \tilde{\phi} \partial_\alpha \tilde{\phi} - U(\tilde{\phi}) \right].
\]  

(60)

For the \( \omega_0 = 0 \) equivalent of metric \( f(R) \) gravity we have

\[
\phi \equiv f'(R) = e^{\sqrt{2\kappa} \tilde{\phi}},
\]

(61)

\[
U(\tilde{\phi}) = \frac{Rf'(R) - f(R)}{2\kappa (f'(R))^2},
\]

(62)

where \( R = R(\tilde{\phi}) \), and the complete action is

\[
S'_{\text{met}} = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2\kappa} - \frac{1}{2} \partial^\alpha \tilde{\phi} \partial_\alpha \tilde{\phi} - U(\tilde{\phi}) \right] + S_M(e^{-\sqrt{2\kappa/3} \tilde{\phi}} g_{\mu\nu}, \psi).
\]

(63)

A direct transformation to the Einstein frame, without the intermediate passage from the Jordan frame, has been discovered in [Barrow and Cotsakis, 1988; Whitt, 1984].

We stress once more that the actions \((5), (51), \) and \((63)\) are nothing but different representations of the same theory. Additionally, there is nothing exceptional about the Jordan or the Einstein frame of the Brans–Dicke representation, and one can actually find infinitely many conformal frames [Flanagan, 2004a; Sotiriou et al., 2007].

### B. Palatini formalism

Palatini \( f(R) \) gravity can also be cast in the form of a Brans–Dicke theory with a potential [Flanagan, 2004b; Olmo, 2005a; Sotiriou, 2006]. As a matter of fact, beginning from the Palatini \( f(R) \) action, which we repeat here for the reader’s convenience

\[
S_{\text{pal}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_M(g_{\mu\nu}, \psi),
\]

(64)

and following exactly the same steps as before, i.e., introducing a scalar field \( \chi \) which we later redefine in terms of \( \phi \), yields

\[
S_{\text{pal}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [\phi R - V(\phi)] + S_M(g_{\mu\nu}, \psi).
\]

(65)

Even though the gravitational part of this action is formally the same as that of the action \((51)\), this action is not a Brans–Dicke one with \( \omega_0 = 0 \); \( R \) is not the Ricci scalar of the metric \( g_{\mu\nu} \). However, we have already seen that the field equation \((18)\) can be solved algebraically for the independent connection yielding eq. \((25)\). This implies that we can replace the connection in the action without affecting the dynamics of the theory (the independent connection is practically an auxiliary field). Alternatively, we can directly use eq. \((27)\), which relates \( R \) and \( \mathcal{R} \). Therefore, the action \((65)\) can be rewritten, modulo surface terms, as

\[
S_{\text{pal}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left( \phi R + \frac{3}{2\phi} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right) + S_M(g_{\mu\nu}, \psi).
\]

(66)

This is the action of a Brans–Dicke theory with Brans–Dicke parameter \( \omega_0 = -3/2 \). The corresponding field equations obtained from the action \((66)\) through variation with respect to the metric and the scalar are

\[
G_{\mu\nu} = \frac{\kappa}{\phi} T_{\mu\nu} - \frac{3}{2\phi^2} \left( \nabla_{\mu} \phi \nabla_{\nu} \phi - \frac{1}{2} g_{\mu\nu} \nabla^\lambda \phi \nabla^\lambda \phi \right) + \frac{1}{\phi} \left( \nabla_{\mu} \phi \nabla_{\nu} \phi - g_{\mu\nu} \Box \phi \right) - \frac{V}{2\phi^2} g_{\mu\nu},
\]

(67)

\[
\Box \phi = \frac{\phi}{3} (R - \mathcal{R}) + \frac{1}{2\phi} \nabla^\mu \phi \nabla_{\mu} \phi.
\]

(68)

Once again, we can use the trace of eq. \((67)\) in order to eliminate \( \mathcal{R} \) in eq. \((68)\) and relate \( \phi \) directly to the matter sources. The outcome is

\[
2V - \phi V'^2 = \kappa T.
\]

(69)

Finally, one can also perform the conformal transformation \((58)\) in order to rewrite the action \((66)\) in the Einstein frame. The result is

\[
S'_{\text{pal}} = \int d^4x \sqrt{-g} \left[ \frac{\tilde{R}}{2\kappa} - U(\tilde{\phi}) \right] + S_M(\phi^{-1} g_{\mu\nu}, \psi),
\]

(70)

where \( U(\tilde{\phi}) = V(\phi)/(2\kappa \phi^2) \). Note that here we have not used any redefinition for the scalar.

To conclude, we have established that Palatini \( f(R) \) gravity can be cast into the form of an \( \omega_0 = -3/2 \) Brans–Dicke theory with a potential.

### C. Classification

The scope of this section is to present a classification of the different versions of \( f(R) \) gravity. However, before doing so, some remarks are in order.

Let us, first of all, use the Brans–Dicke representation of both metric and Palatini \( f(R) \) gravity to comment on the dynamics of these theories. This representation makes it transparent that metric \( f(R) \) gravity has just

\[
\text{This has been an issue of debate and confusion, see for example the references in Faraoni and Nadeau, 2007.} \]


one extra scalar degree of freedom with respect to GR. The absence of a kinetic term for the scalar in the action \( \phi \) or in eq. (58) should not mislead us to think that this degree of freedom does not carry dynamics. As can be seen by eq. (57), \( \phi \) is dynamically related to the matter fields and, therefore, it is a dynamical degree of freedom. Of course, one should also not fail to mention that eq. (54) does constrain the dynamics of \( \phi \).

In this sense metric \( f(R) \) gravity and \( \omega_0 = 0 \) Brans–Dicke theory differs from the general Brans–Dicke theories and constitutes a special case. On the other hand, in the \( \omega_0 = -3/2 \) case which corresponds to Palatini \( f(R) \) gravity, the scalar \( \phi \) appears to have dynamics in the action (66) or in eq. (68). However, once again this is misleading since, as is clear from eq. (69), \( \phi \) is in this case algebraically related to the matter and, therefore, carries no dynamics of its own [indeed the field eqs. (67) and (69) could be combined to give eq. (28), eliminating \( \phi \) completely]. As a remark, let us state that the equivalence between Palatini \( f(R) \) gravity and \( \omega_0 = -3/2 \) Brans–Dicke theory and the clarifications just made highlight two issues already mentioned: the fact that Palatini \( f(R) \) gravity is a metric theory according to the definition of Will (1981), and the fact that the independent connection is actually some sort of auxiliary field.

The fact that the dynamics of \( \phi \) are not transparent at the level of the action in both cases should not come as a big surprise: \( \phi \) is coupled to the derivatives of the metric (through the coupling with \( R \)) and, therefore, partial integrations to “free” \( \delta \phi \) or \( \delta g_{\mu\nu} \) during the variation are bound to generate dynamical terms even if they are not initially present in the action. The \( \omega_0 = -3/2 \) case is even more intricate because the dynamical terms generated through this procedure exactly cancel the existing one in the action.

We already saw an example of how different representations of the theory can highlight some of its characteristics and be very useful for our understanding of it. The equivalence between \( f(R) \) gravity and Brans–Dicke theory will turn out to be very useful in the forthcoming sections.

Until now we have not discussed any possible equivalence between Brans–Dicke theory and metric-affine \( f(R) \) gravity. However, it is quite straightforward to see that there cannot be any. Metric-affine \( f(R) \) gravity is not a metric theory and, consequently, it can not be cast into the form of one, such as Brans–Dicke theory. For the sake of clarity, let us state that one could still start from the action (30) and follow the steps of the previous section to bring its gravitational part into the form of the action (66). However, the matter action would have an explicit dependence from the connection. Additionally, one would not be able to use eq. (27) to eliminate \( R \) in favour of \( R \) since this only holds in Palatini \( f(R) \) gravity.

In conclusion, metric-affine \( f(R) \) gravity is the most general case of \( f(R) \) gravity. Imposing further assumptions can lead to both metric or Palatini \( f(R) \) gravity, which can be cast into the form of \( \omega_0 = 0 \) and \( \omega_0 = -3/2 \) Brans–Dicke theories with a potential. In both cases, restricting the functional form of the action leads to GR. These results are summarized in the schematic diagram of Fig. 1.

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**FIG. 1** Classification of \( f(R) \) theories of gravity and equivalent Brans–Dicke theories. The flowchart shows the list of assumptions that are needed to arrive to the various versions of \( f(R) \) gravity and GR beginning from the the general \( f(R) \) action. It also includes the equivalent Brans–Dicke classes. Taken from Sotiriou (2006).

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### D. Why \( f(R) \) gravity then?

Since \( f(R) \) gravity in both the metric and the Palatini formalisms can acquire a Brans–Dicke theory represen-

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work with the Brans–Dicke one, and second, why, since we know a lot about Brans–Dicke theory, should we regard \( f(R) \) gravity as unexplored or interesting?

The answer to the first question is quite straightforward. There is actually no reason to prefer either of the two representations — at least as far as classical gravity is concerned. There can be applications where the \( f(R) \) representation can be more convenient and applications where the Brans–Dicke representation is more convenient. One should probably mention that habit affects our taste and, therefore, an \( f(R) \) representation seems more appealing to relativists due to its more apparent geometrical nature, whereas the Brans–Dicke representation seems more appealing to particle physicists. This issue can have theoretical implications. To give an example: if \( f(R) \) gravity is considered as a step towards a more complicated theory, which generalisation would be more straightforward will depend on the chosen representation [see also \( \text{Sotiriou et al.} \, 2007 \) for a discussion].

Whether \( f(R) \) theories of gravity are unexplored and interesting or just an already-studied subcase of Brans–Dicke theory, is a more practical question that certainly deserves a direct answer. It is indeed true that scalar-tensor theories and, more precisely, Brans–Dicke theory are well-studied theories which have been extensively used in many applications, including cosmology. However, the specific choices \( \omega_0 = 0, -3/2 \) for the Brans–Dicke parameter are quite exceptional, as already mentioned in the previous section. It is also worthwhile pointing out the following: a) As far as the \( \omega_0 = 0 \) case is concerned, one can probably speculate that it is the apparent absence of the kinetic term for the scalar in the action which did not seem appealing and prevented the study of this theory. b) The \( \omega_0 = -3/2 \) case leads to a conformally invariant theory in the absence of the potential [see \( \text{Sotiriou} \, 2006b \) and references therein], which constituted the initial form of Brans–Dicke theory, and hence it was considered non-viable (a coupling with non-conformally invariant matter is not feasible). However, in the presence of a potential, the theory no longer has this feature. Additionally, most calculations which are done for a general value of \( \omega_0 \) in the literature actually exclude \( \omega_0 = -3/2 \), mainly because, merely for simplicity purposes, they are done in such a way that the combination \( 2\omega_0 + 3 \) appears in a denominator (see also Sec. \( \text{V-A} \)).

In any case, the conclusion is that the theories in the Brans–Dicke class that correspond to metric and Palatini \( f(R) \) gravity had not yet been explored before the recent re-introduction of \( f(R) \) gravity and, as will also become clear later, several of their special characteristics when compared with more standard Brans–Dicke theories were revealed through studies of \( f(R) \) gravity.

IV. COSMOLOGICAL EVOLUTION AND CONSTRAINTS

We now turn our attention to cosmology, which motivated the recent surge of interest in \( f(R) \) gravity in order to explain the current cosmic acceleration without the need for dark energy. Before reviewing how \( f(R) \) gravity might provide a solution to the more recent cosmological riddles, let us stress that the following criteria must be satisfied in order for an \( f(R) \) model to be theoretically consistent and compatible with cosmological observations and experiments. The model must:

- have the correct cosmological dynamics;
- exhibit the correct behaviour of gravitational perturbations;
- generate cosmological perturbations compatible with the cosmological constraints from the cosmic microwave background, large scale structure, Big Bang Nucleosynthesis, and gravity waves.

These are independent requirements to be studied separately, and they must all be satisfied.

A. Background evolution

In cosmology, the identification of our universe with a Friedmann–Lemaitre–Robertson–Walker (FLRW) spacetime is largely based on the high degree of isotropy measured in the cosmic microwave background; this identification relies on a formal result known as the Ehlers–Germin-Sachs (EGS) theorem \( \text{[Ehlers et al.} \, 1968 \) which is a kinematical characterization of FLRW spaces stating that, if a congruence of timelike freely falling observers see an isotropic radiation field, then (assuming that isotropy holds about every spatial point) the spacetime is spatially homogeneous and isotropic and, therefore, a FLRW one. This applies to a universe filled with any perfect fluid that is geodesic and barotropic \( \text{[Clarkson and Barrett} \, 1999 \); \( \text{Ellis et al.} \, 1983b \)\). Moreover, an “almost-EGS theorem” holds: spacetimes that are close to satisfying the EGS conditions are close to FLRW universes in an appropriate sense \( \text{[Stoeger et al.} \, 1995 \). One would expect that the EGS theorem be extended to \( f(R) \) gravity; indeed, its validity for the (metric) theory

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} \right] + S_M \tag{71}
\]

was proved in \( \text{[Maartens and Taylor} \, 1994 \); \( \text{Taylor and Maartens} \, 1995 \) and the generalization to arbitrary metric \( f(R) \) gravity was given by \( \text{[Rippl et al.} \, 1996 \). The validity of the EGS theorem can also be seen through the equivalence between \( f(R) \) and Brans–Dicke theory: the theorem was extended to scalar-tensor theories in \( \text{[Clarkson et al.} \, 2001 \); \( \text{2003 \). Since metric and Palatini \( f(R) \) gravities are equivalent to \( \omega = 0 \) and \( \omega_0 = -3/2 \) Brans–Dicke theories respectively, it seems that the results of \( \text{[Clarkson et al.} \, 2001 \); \( \text{2003 \) can be considered as straightforward generalizations of the EGS theorem in both versions of \( f(R) \) gravity.

as well. However, in the case of Palatini $f(R)$ gravity there is still some doubt regarding this issue due to complications in averaging (Flanagan 2004b).

1. Metric $f(R)$ gravity

Considering the discussion above, it is valid to use the FLRW line element

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]$$

(72)

as a local description of spacetime at cosmological scales, where $(t, r, \theta, \phi)$ are comoving coordinates. We remind the reader that $k = -1, 0, 1$ according to whether the universe is hyperspherical, spatially flat, or hyperbolic and that $a(t)$ is the scale factor. Part of the standard approach, which we follow here as well, is to use a perfect fluid description for matter with stress-energy tensor

$$T^{\mu\nu} = (\rho + P) u^{\mu} u^{\nu} + P g^{\mu\nu},$$

(73)

where $u^{\mu}$ denotes the four-velocity of an observer comoving with the fluid and $\rho$ and $P$ are the energy density and pressure of the fluid, respectively.

Note that the value of $k$ is an external parameter. As in many other works in the literature, in what follows we choose $k = 0$, i.e., we focus on a spatially flat universe. This choice in made in order to simplify the equations and should be viewed sceptically. It is sometimes claimed in the literature that such a choice is favoured by the data. However, this is not entirely correct. Even though the data [e.g. Spergel et al. 2007] indicate that the current value of $\Omega_k$ is very close to zero, it should be stressed that this does not really reveal the value of $k$ itself. Since

$$\Omega_k = -\frac{k}{a^2 H^2},$$

(74)

the current value of $\Omega_k$ is sensitive to the current value of $a(t)$, i.e. to the amount of expansion the universe has undergone after the Big Bang. A significant amount of expansion can easily drive $\Omega_k$ very close to zero. The success of the inflationary paradigm is exactly that it explains the flatness problem — how did the universe become so flat — in a dynamical way, allowing us to avoid fine-tuning the parameter $k$ (the value $k = 0$ is statistically exceptional).

The above having been said, choosing $k = 0$ for simplicity is not a dramatic departure from generality when it comes to late time cosmology. If it is viewed as an approximation and not as a choice of an initial condition, then one can say that, since $\Omega_k$ as inferred from observations is very close to zero at current times, the terms related to $k$ will be subdominant in the Friedmann or generalised Friedmann equations and, therefore, one could choose to discard them by setting $k = 0$, without great loss of accuracy. In any case, results derived under the assumption that $k = 0$ should be considered preliminary until the influence of the spatial curvature is precisely determined, since there are indications that even a very small value of $\Omega_k$ may have an effect on them [see, for instance Clarkson et al. 2007].

Returning to our discussion, inserting the flat FLRW metric in the field equations (6) and assuming that the stress-energy tensor is that of eq. (73) yields

$$H^2 = \frac{\kappa}{3f'} \left[ \rho + \frac{R'f - f}{2} - 3H \dot{R}f'' \right],$$

(75)

$$2\dot{H} + 3H^2 = \frac{\kappa}{f'} \left[ P + \frac{(\dot{R})^2 f'' + 2H \ddot{R}f'' + \dddot{R}f'' + \frac{3}{2} (f - R f')}{f'} \right].$$

(76)

With some hindsight, we assume that $f' > 0$ in order to have a positive effective gravitational coupling and $f'' > 0$ to avoid the Dolgov-Kawasaki instability (Dolgov and Kawasaki 2003a; Faraoni 2006a) discussed in Sec. V.B.

A significant part of the motivation for $f(R)$ gravity is that it can lead to accelerated expansion without the need for dark energy (or an inflaton field). An easy way to see this is to define an effective energy density and pressure of the geometry as

$$\rho_{\text{eff}} = \frac{R f'}{2f'} - \frac{3H \dot{R}f''}{f'},$$

(77)

$$P_{\text{eff}} = \frac{\dot{R}^2 f'' + 2H \ddot{R}f'' + \dddot{R}f'' + \frac{3}{2} (f - R f')}{f'},$$

(78)

where $\rho_{\text{eff}}$ has to be non-negative in a spatially flat FLRW spacetime, as follows from the inspection of eq. (75) in the limit $\rho \to 0$. Then, in vacuo eqs. (75) and (76) can take the form of the standard Friedmann equation

$$H^2 = \frac{\kappa}{3} \rho_{\text{eff}},$$

(79)

$$\frac{\dot{a}}{a} = \frac{\kappa}{6} \left[ \rho_{\text{eff}} + 3P_{\text{eff}} \right].$$

(80)

Hence, in vacuo the curvature correction can be viewed as an effective fluid.\textsuperscript{14}

The effective equation of state parameter $w_{\text{eff}}$ of modified gravity can be expressed as

$$w_{\text{eff}} = \frac{P_{\text{eff}}}{\rho_{\text{eff}}} = \frac{\dot{R}^2 f'' + 2H \ddot{R}f'' + \dddot{R}f'' + \frac{3}{2} (f - R f')}{\frac{R f'}{2f'} - \frac{3H \dot{R}f''}{f'}}.$$  

(81)

\textsuperscript{14} Note the following subtlety though: should we have included matter it would enter the Friedmann equations with a modified coupling $\kappa / f'$. In general this effective fluid representation is used only for demonstrative purposes and should not be overestimated or misinterpreted.
Since the denominator on the right hand side of eq. (81) is strictly positive, the sign of \( w_{eff} \) is determined by its numerator. In general, for a metric \( f(R) \) model to mimic the de Sitter equation of state \( w_{eff} = -1 \), it must be

\[
f''(R) = \frac{RH - \dot{R}}{(R)^2}. \tag{82}
\]

Let us also give two simple examples that can be found in the literature for demonstrative purposes and without considering their viability: First, one can consider the function \( f \) to be of the form \( f(R) \propto R^n \). It is quite straightforward to calculate \( w_{eff} \) as a function of \( n \) if the scale factor is assumed to be a generic power law \( a(t) = a_0(t/t_0)^\alpha \) (a general \( a(t) \) would lead to a time dependent \( w_{eff} \)) (Capozziello et al., 2003). The result is

\[
w_{eff} = -\frac{6n^2 - 7n - 1}{6n^2 - 9n + 3} \tag{83}
\]

for \( n \neq 1 \), and \( \alpha \) is given in terms of \( n \) as

\[
\alpha = \frac{-2n^2 + 3n - 1}{n - 2}. \tag{84}
\]

A suitable choice of \( n \) can lead to a desired value for \( w_{eff} \). For instance, \( n = 2 \) yields \( w_{eff} = -1 \) and \( \alpha = \infty \), as expected, considering that quadratic corrections to the Einstein-Hilbert Lagrangian were used in the well known Starobinsky inflation (Starobinsky, 1980).

The second example which we will refer to is a model of the form \( f(R) = \mu (n+1) f^n \), where \( \mu \) is a suitably chosen parameter (Carroll et al., 2004). In this case, and once again if the scale factor is assumed to be a generic power law, \( w_{eff} \) can again be written as a function of \( n \) (Carroll et al., 2004):

\[
w_{eff} = -1 + \frac{2(n+2)}{3(2n+1)(n+1)}. \tag{85}
\]

The most typical model within this class is that with \( n = 1 \) (Carroll et al., 2004), in which case \( w_{eff} = -2/3 \). Note that in this class of models, a positive \( n \) implies the presence of a term inversely proportional to \( R \) in the action, contrary to the situation for the \( R^n \) models.

In terms of the quantity \( \phi(R) \equiv f'(R) \) one can rewrite eq. (81) as

\[
w_{eff} = -1 + \frac{1}{R \phi - f - 6H \phi} \overset{\text{eq. (81)}}{=} -1 + \frac{\dot{\phi} - H \phi}{3\phi H^2} \tag{86}
\]

and

\[
\rho_{eff} + P_{eff} = \frac{\ddot{\phi} - H \dot{\phi}}{\dot{\phi}} = \dot{\phi} \frac{d}{dt} \left[ \ln \left( \frac{\phi}{a} \right) \right]. \tag{87}
\]

An exact de Sitter solution corresponds to \( \ddot{\phi} = f''(R) \dot{R} = 0 \), or to \( \dot{\phi} = - \frac{C \alpha}{a} e^{H_0 t} \) for some constant \( C \neq 0 \) on an integration constant. However, the second solution for \( \dot{\phi}(t) \) is not acceptable because it leads to \( f''(R) \dot{R} = C a_0 e^{H_0 t} \), which is absurd because the left hand side is time-independent (for a de Sitter solution), while the right hand side depends on time.

One could impose energy conditions for the effective stress-energy tensor (12) of \( f(R) \) gravity. However, this is not very meaningful from the physical point of view since it is well known that effective stress-energy terms originating from the geometry by rewriting the field equations of alternative gravities as effective Einstein equations do, in general, violate all the energy conditions (e.g., Carroll, 2004a). Also, the concept of gravitational energy density is, anyway, ill-defined in GR and in all metric theories of gravity as a consequence of the Equivalence Principle. Moreover, the violation of the energy conditions makes it possible to have \( H > 0 \) and bouncing universes (Carloni et al., 2006; Novello and Bergliaffa, 2008).

The field equations are clearly of fourth order in \( a(t) \). When matter is absent (a situation of interest in early time inflation or in a very late universe completely dominated by \( f(R) \) corrections), \( a(t) \) only appears in the combination \( H = a/a \). Since the Hubble parameter \( H \) is a cosmological observable, it is convenient to adopt it as the (only) dynamical variable; then the field equations (75) and (76) are of third order in \( H \). This elimination of \( a \) is not possible when \( k \neq 0 \), or when a fluid with density \( \rho = \rho(a) \) is included in the picture.

Regarding the dynamical field content of the theory, the fact that quadratic corrections to the Einstein–Hilbert action introduce a massive scalar field was noted in (Bychbinder et al., 1992; Stelle, 1978; 1977; Strominger, 1984; Utvama and DeWitt, 1962; Vilkovisky, 1992); this applies to any \( f(R) \) gravity theory in the metric formalism (see, e.g., (Ferraris et al., 1988; Hindawi et al., 1996; Olmedo, 2007)). The metric tensor contains, in principle, various degrees of freedom: spin 2 modes, and vector and scalar modes, which can all be massless or massive. In GR we find only the massless graviton but, when the action is allowed to include terms that depend on \( R, R_{\mu\nu} R^\mu\nu, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \), other modes show up. In \( f(R) \) gravity, a massive scalar mode can appear, which is evident in the equivalence with scalar-tensor theory (see Sec. III). As discussed in Sec. III.C, the scalar field \( \phi = R \) is dynamical in the metric formalism and non-dynamical in the Palatini formalism.

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15 In Santos et al., 2007, the Null Energy Condition and the Strong Energy Condition for metric \( f(R) \) gravity have been derived by using the Raychaudhuri equation and imposing that gravity be attractive, whereas for the Weak Energy Condition and the Dominant Energy Condition an effective stress-energy tensor which includes the matter was used. In Bergliaffa, 2006, a different approach was followed, in which the standard energy conditions on matter were used in an attempt to constrain \( f(R) \) gravity.
2. Palatini $f(R)$ gravity

As already mentioned, some concerns have been expressed on whether the homogeneity approximation can justify the use of the FLRW metric as a cosmological solution in Palatini $f(R)$ gravity (Flanagan 2004a) [see also Li et al., 2008]. Therefore, even though it is standard practice in the literature to assume a FLRW background and a perfect fluid description for matter when studying cosmology in Palatini $f(R)$ gravity (e.g. Allemandi et al., 2004; 2005a; Amarzguioui et al., 2003; 2004; 2005; Sotiriou, 2006a; Vollick, 2003), and we are going to review this approach here, the reader should approach it with some reasonable skepticism until this issue is clarified further.

Under the assumptions that the spacetime is indeed described at cosmological scales by the FLRW metric, eq. (72), that the stress-energy tensor of matter is that of eq. (43), and that $k = 0$, easy manipulations reveal that the field eqs. (18) and (19) yield the following modified Friedmann equation [see for instance Meng and Wang, 2003; Sotiriou, 2006a]:

$$
(H + \frac{f}{2} f')^2 = \frac{1}{6} \kappa (\rho + 3P) + \frac{1}{6} f,$$

where the overdot denotes differentiation with respect to coordinate time. Note that when $f$ is linear, $f' = 1$ and, therefore, $f' = 0$. Taking into account eq. (20), one can easily show that in this case eq. (88) reduces to the standard Friedmann equation.

We will avoid representing the extra terms in eq. (88) with respect to the standard Friedmann equation as a an effective stress energy density and pressure since, as it is not that difficult to see, the former equation does not carry more dynamics than the latter. Indeed, assume as usual that the cosmological fluid is composed by pressureless dust ($P_m = 0$) and radiation ($P_r = \rho_r/3$) and $\rho = \rho_m + \rho_p$ and $P = P_m + P_r$ where $\rho_m$, $\rho_p$ and $P_m$, $P_r$ denote the energy density and the pressure of dust and radiation, respectively. Due to eq. (20) and the fact that for radiation $T = 0$, it is quite straightforward to derive an algebraic relation between $R$ and the energy density of the dust. Combining this with energy conservation, one obtains (Sotiriou 2006a)

$$\dot{R} = -\frac{3H(R f' - 2f)}{R f'' - f'}.$$

This equation can be used to replace $\dot{R}$ in eq. (88), yielding

$$H^2 = \frac{1}{6 f'} \left[ 2 \kappa \rho + R f' - f \right] \left( \frac{1}{1 - 3 f (R f' - 2f)} \right)^2.$$

Considering now that, due to eq. (20), $R$ is just an algebraic function of $P_m$, it is easy to realize that eq. (90) is actually just the usual Friedmann equation with a modified source. The functional form of $f$ will determine how the dynamics will be affected by this modification.

It seems, therefore, quite intuitive that by tampering with the function $f$ one can affect the cosmological dynamics in a prescribed way. Indeed it has been shown that for $f(R) = R - \alpha^2/(3R)$ one approaches a de Sitter expansion as the density goes to zero (Vollick, 2003). In order to match observations of the expansion history, one needs to choose $\alpha \sim 10^{-67} (eV)^2 \sim 10^{-53} m^{-2}$. Additionally, in regimes for which $\kappa \rho \gg \alpha$, eq. (90) reduces to high precision to the standard Friedmann equation. The above can very easily be verified by replacing this particular choice of $f$ in eq. (90). We refer the reader to the literature for more details.

One could, of course, consider more general functions of $R$. Of particular interest would be having positive powers of $R$ higher than the first power added in the action (since one could think of the Lagrangian as a series expansion). Indeed this has been considered (Meng and Wang, 2003; 2004a; 2005; Sotiriou, 2006a). However, it can be shown that such terms do not really lead to interesting phenomenology as in metric $f(R)$ gravity: for instance they cannot drive inflation as, unlike in the scenario proposed by Starobinsky (1980) in the metric formalism, here there are no extra dynamics and inflation cannot end gracefully (Meng and Wang, 2004a; Sotiriou, 2006a). As a matter of fact, it is more likely that positive powers of $R$ will lead to no interesting cosmological phenomenology unless their coefficients are large enough to make the models non-viable (Sotiriou, 2006a).

B. Cosmological eras

As stated in the Introduction, the recent flurry of theoretical activity on $f(R)$ models derives from the need to explain the present acceleration of the universe discovered with supernovae of type Ia (Astier et al., 2006; Barris et al., 2004; Filippenko and Riess, 1998; Knop et al., 2003; Perlmutter et al., 1998; Riess et al., 1998; 1999; 2004; Schmidt et al., 1998; Tonry et al., 2003). We have seen in the previous section how $f(R)$ gravity can achieve cosmic acceleration and an effective equation of state parameter $w_{eff} \sim -1$; on the other hand, it was already known from $R^2$-inflationary scenarios of the early universe that this is possible, so we are actually witnessing a resurrection of this theoretical possibility in models of the late universe — this parallels the use of scalar fields to drive early inflation or late-time acceleration in quintessence models. There are also attempts to unify early inflation and late time acceleration in modified gravity (Bamba and Odintsov, 2008; Nojiri and Odintsov, 2007a, 2008a, b, c, d). However, any model attempting to explain the cosmic speed-up at late times should not spoil the successes of the standard cosmological model which requires a definite sequence of eras.
to follow each other, including:

1. early inflation
2. a radiation era during which Big Bang Nucleosynthesis occurs;
3. a matter era;
4. the present accelerated epoch, and
5. a future era.

Big Bang Nucleosynthesis is well constrained — see (Brookfield et al., 2006) Clifton and Barrow, 2005a; Evans et al., 2007; Kneller and Steinman, 2004; Lambiase and Scarpetta, 2006; Nakamura et al., 2006) for such constraints on \( f(R) \) models. The matter era must last long enough to allow the primordial density perturbations generated during inflation to grow and become the structures observed in the universe today. The future era is usually found to be a de Sitter attractor solution, or to be truncated at a finite time by a Big Rip singularity.

Furthermore, there must be smooth transitions between consecutive eras, which may not happen in all \( f(R) \) models. In particular, the exit from the radiation era has been studied and claimed to originate problems for many forms of \( f(R) \) in the metric formalism, including \( f = R - \mu^{2(n+1)}/R^n, \ n > 0 \) (Amendola et al., 2007a,b; Brookfield et al., 2006; Capozziello et al., 2006c; Nojiri and Odintsov, 2006) [but not in the Palatini formalism (Carvalho et al., 2008; Fay et al., 2007b)]. However, the usual model \( f(R) = R - \mu^4/R \) with “bad” behaviour was studied using singular perturbation methods (Evans et al., 2007), definitely finding a matter era which is also sufficiently long.

Moreover, one can always find choices of the function \( f(R) \) with the correct cosmological dynamics in the following way: one can prescribe the desired form of the scale factor \( a(t) \) and integrate a differential equation for \( f(R) \) that produces the desired scale factor (Capozziello et al., 2005b, 2006c; de la Cruz-Dombriz and Dobado, 2006; Faulkner et al., 2007; Fay et al., 2007a,b; Hu and Sawicki, 2007a,b; Multamaki and Viola, 2006a; Nojiri and Odintsov, 2006, 2007b, 2007d; Song et al., 2007). In general, this “designer \( f(R) \) gravity” produces forms of the function \( f(R) \) that are rather contrived. Moreover, the prescribed evolution of the scale factor \( a(t) \) does not determine uniquely the form of \( f(R) \) but, at best, only a class of \( f(R) \) models (Multamaki and Viola, 2006a; Sokolowski, 2007; Starobinsky, 2002). Therefore, the observational data providing information on the history of \( a(t) \) are not sufficient to reconstruct \( f(R) \); one needs additional information, which may come from cosmological density perturbations. There remains a caveat on being careful to terminate the radiation era and allowing a matter era that is sufficiently long for scalar perturbations to grow.

While sometimes it is possible to find exact solutions to the cosmological equations, the general behaviour of the solutions can only be assessed with a phase space analysis, which constitutes a powerful tool in cosmology (Coles, 2003; Wainwright and Ellis, 1997). In a spatially flat FLRW universe the dynamical variable is the Hubble parameter \( H \), and a convenient choice of phase space variables in this case is \( (H, R) \). Then, for any form of the function \( f(R) \), the phase space is a two-dimensional curved manifold embedded in the three-dimensional space \( (H, R, \dot{R}) \) with de Sitter spaces as fixed points (de Souza and Faraoni, 2007); the structure of the phase space is simplified with respect to that of general scalar-tensor cosmology (Faraoni, 2005a).

Studies of the phase space of \( f(R) \) cosmology (not limited to the spatially flat FLRW case) were common in the pre-1998 literature on \( R^2 \)-inflation (Amendola et al., 1992; Capozziello et al., 1993; Muller et al., 1994; Starobinsky, 1980). The presence or absence of chaos in metric \( f(R) \) gravity was studied in (Barrow and Cotsakis, 1989, 1991). Such studies with dynamical system methods have become widespread with the recent surge of interest in \( f(R) \) gravity to explain the present cosmic acceleration. Of course, detailed phase space analyses are only possible for specific choices of the function \( f(R) \) (Abdelwahab et al., 2008; Amendola et al., 2007a,b,c; Amendola and Tsujikawa, 2008; Carloni et al., 2008a; Carloni and Dunsby, 2007; Carloni et al., 2005, 2007; Carroll et al., 2005; Clifton, 2006a; Clifton and Barrow, 2005b; Easson, 2004; Fay et al., 2007a,b; Goheer et al., 2007; Leach et al., 2006, 2007; Li and Barrow, 2007; Nojiri and Odintsov, 2004b, 2005a).

C. Dynamics of cosmological perturbations

Obtaining the correct dynamics of the background cosmological model is not sufficient for the theory to be viable: in fact, the FLRW metric can be obtained as a solution of the field equations of most gravitation theories, and it is practically impossible to discriminate between \( f(R) \) gravity and dark energy theories (or between different \( f(R) \) models) by using only the unperturbed FLRW cosmological model, i.e., by using only probes that are sensitive to the expansion history of the universe. By contrast, the growth of cosmological perturbations is sensitive to the theory of gravity adopted and constitutes a possible avenue to discriminate between dark energy and modified gravity. Changing the theory of gravity affects the dynamics of cosmological perturbations and, among other things, the imprints that these leave in the cosmic microwave background (which currently provide the most sensitive cosmological probe) and in galaxy surveys (Knox et al., 2006; Koivisto, 2006; Kovács and Maartens, 2006; Li and Barrow, 2007; Li and Cui, 2008; Sealfon et al., 2005; Shirata et al., 2005, 2007; Skordis et al.
2006; Song et al., 2007; Stabenau and Jain, 2006; Tsujikawa, 2007; White and Kochanek, 2001; Zhang et al., 2007). This originated various efforts to constrain $f(R)$ gravity with cosmic microwave background data (Amendola and Tsujikawa, 2008; Appleby and Battye, 2007; Carloni et al., 2008; Hu and Sawicki, 2007; Li and Barrow, 2007; Li et al., 2007; Lue et al., 2007; Pogosian and Silvestri, 2008; Starobinsky, 2007; Tsujikawa, 2008; Tsujikawa et al., 2008; Wei and Zhang, 2008).

Most of these works are restricted to specific choices of the function $f(R)$, but a few general results have also been obtained. The growth and evolution of local scalar perturbations, which depends on the theory of gravity employed, was studied in metric $f(R)$ gravity theories which reproduce GR at high curvatures in various papers (Carroll et al., 2006; de la Cruz-Dombriz et al., 2008; Song et al., 2007) by assuming a scale factor evolution typical of a ΛCDM model. Vector and tensor modes are unaffected by $f(R)$ corrections. It is found that $f''(R) > 0$ is required for the stability of scalar perturbations (Song et al., 2007), which matches the analysis of Sec. (V.B.2) in a locally de Sitter background. The corrections to the Einstein–Hilbert action produce qualitative differences with respect to Einstein gravity: they lower the large angle anisotropy of the cosmic microwave background and may help explain the observed low quadrupole; and they produce different correlations between the cosmic microwave background and galaxy surveys (Song et al., 2007). Further studies challenge the viability of $f(R)$ gravity in comparison with the ΛCDM model: in (Bean et al., 2007) it is found that large scale density fluctuations are suppressed in comparison to small scales by an amount incompatible with the observational data. This makes it impossible to fit simultaneously large scale data from the cosmic microwave background and small scale data from galaxy surveys. Also, a quasi-static approximation used in a previous analysis (Zhang, 2007) is found to be invalid.

In (de la Cruz-Dombriz et al., 2008), the growth of matter density perturbations is studied in the longitudinal gauge using a fourth order equation for the density contrast $\delta \rho/\rho$, which reduces to a second order one for sub-horizon modes. The quasi-static approximation, which does not hold for general forms of the function $f(R)$, is however found to be valid for those forms of this function that describe successfully the present cosmic acceleration and pass the Solar System tests in the weak-field limit. It is interesting that the relation between the gravitational potentials in the metric which are responsible for gravitational lensing, and the matter overdensities depends on the theory of gravity; a study of this relation in $f(R)$ gravity (as well as in other gravitational theories) is contained in (Zhang et al., 2007).

Cosmological density perturbations in the Palatini formalism have been studied in (Amarzguioui et al., 2006; Carroll et al., 2006; Koivisto, 2006a, 2007; Koivisto and Kurki-Suonio, 2006; Lee, 2007, 2008; Li et al., 2007; Uddin et al., 2007). Two different formalisms developed in (Hwang and Noh, 2002; Koivisto and Kurki-Suonio, 2006) and (Lue et al., 2004) were compared for the model $f(R) = R - \mu^{2(n+1)}/R^n$ and it was found that the two models agree for scenarios that are “close” (in parameter space) to the standard concordance model, but give different results for models that differ significantly from the ΛCDM model. Although this is not something to worry about in practice (all models aiming at explaining the observational data are “close” to the standard concordance model), it signals the need to test the validity of perturbation analyses for theories that do differ significantly from GR in some aspects.

V. OTHER STANDARD VIABILITY CRITERIA

In addition to having the correct cosmological dynamics and the correct evolution of cosmological perturbations, the following criteria must be satisfied in order for an $f(R)$ model to be theoretically consistent and compatible with experiment. The model must:

- have the correct weak-field limit at both the Newtonian and post-Newtonian levels, i.e., one that is compatible with the available Solar System experiments;
- be stable at the classical and semiclassical level (the checks performed include the study of a matter instability, of gravitational instabilities for de Sitter space, and of a semiclassical instability with respect to black hole nucleation);
- not contain ghost fields;
- admit a well-posed Cauchy problem;

These independent requirements are discussed separately in the following.

A. Weak-field limit

It is obvious that a viable theory of gravity must have the correct Newtonian and post-Newtonian limits. Indeed, since the modified gravity theories of current interest are explicitly designed to fit the cosmological observations, Solar System tests are more stringent than the cosmological ones and constitute a real testbed for these theories.

1. The scalar degree of freedom

It is clear from the equivalence between $f(R)$ and Brans–Dicke gravities discussed in Sec. III that the former contains a massive scalar field $\phi$ [see eqs. (54) and (60)]. While in the metric formalism this scalar is dynamical and represents a genuine degree of freedom, it
is non-dynamical in the Palatini case. Let us, therefore, consider the role of the scalar field in the metric formalism as it will turn out to be crucial for the weak-field limit. Using the notations of Sec. [III.A] the action is given by eq. ([55]) and the corresponding field equations by eq. ([55]).

Equation ([52]) for $\chi$ has no dynamical content because it only enforces the equality $\chi = R$. However, $\chi = R$ is indeed a dynamical field that satisfies the wave equation,

$$3f''(\chi)\Box \chi + 3f''(\chi)\nabla_\alpha \chi \nabla^\alpha \chi + \chi f'(\chi) - 2f(\chi) = \kappa T.$$  

(91)

When $f'' \neq 0$ a new effective potential $W(\chi) \neq V(\chi)$ can be introduced, such that

$$\frac{dW}{d\chi} = \frac{\kappa T - \chi f'(\chi) + 2f(\chi)}{3f''(\chi)}.$$  

(92)

The action can be seen as a Brans–Dicke action with derivative of the potential evaluated at the minimum (or $f$ and $f'$, 0 if the field $\phi \equiv f'(\chi) = f'(R)$ is used instead of $\chi$ as the independent Brans–Dicke field:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [\phi R - V(\phi)] + S^{(m)},$$  

(93)

where $V(\phi)$ is given by eq. ([53]).

Now one may think of studying the dynamics and stability of the model by looking at the shape and extrema of the effective potential $V(\chi)$ but this would be misleading because the dynamics of $\chi$ are not regulated by $V(\chi)$ (indeed, the wave equation (91) does not contain $V$), but are subject the strong constraint $\chi = R$, and $R$ (or $f'(R)$) is ruled by the trace equation (8). The following example shows how the use of the potential $V(\chi)$ can be misleading. As is well known, the effective mass of a scalar field (corresponding to the second derivative of the potential evaluated at the minimum) controls the range of the force mediated by this field. Thus, when studying the weak-field limit of the theory it is important to know the range of the dynamical scalar field $\chi = R$ present in the metric formalism in addition to the metric field $g_{\mu\nu}$, as this field can potentially violate the post-Newtonian constraints obtained from Solar System experiments if the scalar field gives observable effects at the relevant scales. One way to avoid Solar System constraints, however, is to have $\chi$ have a sufficiently short range (see Sec. [V.A.2] for more details). Consider the example $f(R) = R + aR^2$, with $a$ a positive constant. By naively taking the potential, one obtains

$$V(\chi) = a \chi^2 \equiv \frac{m^2}{2} \chi^2.$$  

(94)

with effective mass squared $m^2 = 2a$. Then, the small values of $a$ generated by quantum corrections to GR imply a small mass $m_1$ and a long range field $\chi$ might be detectable at Solar System scales (Chiba et al., 2007; Jin et al., 2004; Olmo, 2007). However, this conclusion is incorrect because $m_1$ is not the physical mass of $\chi$. The true effective mass is obtained from the trace equation ruling the evolution of $R$ which, for $f(R) = R + aR^2$, reduces to

$$\Box R - \frac{R}{6a} = \frac{\kappa T}{6a},$$  

(95)

and the identification of the mass squared of $\chi = R$ as

$$m^2 = \frac{1}{6a}.$$  

(96)

is straightforward. A small enough value of $a$ now leads to a large value of $m$ and a short range for $\chi$. The situation is, however, more complicated; the chameleon effect due to the dependence of the effective mass on the curvature may change the range of the scalar (Faulkner et al., 2007; Starobinsky, 2007).

For a general $f(R)$ model, the effective mass squared of $\chi = R$ is obtained in the weak-field limit by considering a small, spherically symmetric, perturbation of de Sitter space with constant curvature $R_0$. One finds

$$m^2 = \frac{1}{3} \left( f''(R_0) - R_0 \right).$$  

(97)

This equation coincides with eq. (6) of Muller et al., 1990, with eq. (26) of Olmo, 2007, and with eq. (17) of Navarro and Van Acoleyen, 2007. It also appears in a calculation of the propagator for $f(R)$ gravity in a locally flat background (eq. (8) of Nunez and Solganik, 2004). The same expression is recovered in a gauge-invariant stability analysis of de Sitter space (Faraoni and Nadeau, 2005) reported in Sec. [V.B.2] below.

Another possibility is to consider the field $\phi \equiv f'(R)$ instead of $\chi = R$, and to define the effective mass of $\phi$ by using the Einstein frame scalar-tensor analog of $f(R)$ gravity instead of its Jordan frame cousin already discussed (Chiba, 2002). By performing the conformal transformation

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = f'(R) g_{\mu\nu} \equiv \phi g_{\mu\nu}$$  

(98)

and the scalar field redefinition $\phi = f'(R) \rightarrow \tilde{\phi}$ with

$$d\tilde{\phi} = \sqrt{\frac{2\omega_0 + 3}{2\kappa}} \frac{d\phi}{\phi},$$  

(99)

a scalar-tensor theory is mapped to the Einstein frame in which the “new” scalar field $\tilde{\phi}$ couples minimally to

\[\text{References}\]

\[\text{References}\]
the Ricci curvature and has canonical kinetic energy, as described by the action

\[ S = \int d^4x \sqrt{-g} \left[ \tilde{R} - \frac{1}{2} \partial^a \tilde{\phi} \partial_a \tilde{\phi} - U(\tilde{\phi}) \right] + S_M(e^{-2\kappa/3} \tilde{g}_{\mu\nu}, \psi), \tag{100} \]

(note once more the non-minimal coupling of the matter in the Einstein frame). For the \( \omega_0 = 0 \) equivalent of metric \( f(R) \) gravity we have

\[ \phi = f'(R) = e^{\sqrt{\frac{2}{\kappa}} \phi}, \tag{101} \]

\[ U(\tilde{\phi}) = \frac{R f'(R) - f(R)}{2\kappa (f'(R))^2}, \tag{102} \]

where \( R = R(\tilde{\phi}) \). By using \( d\tilde{\phi}/d\phi = \sqrt{\frac{3}{2\kappa}} f'(R) \), the effective mass of \( \tilde{\phi} \) is defined by

\[ \tilde{m}^2_{\text{eff}} \equiv \frac{d^2U}{d\tilde{\phi}^2} = \frac{1}{3} \left[ \frac{1}{f''(R)} + \frac{\phi}{f'(R)} - \frac{2f'(R)}{(f'(R))^2} \right], \tag{103} \]

[this equation appears in the footnote on p. 2 of \( \text{Chiba, 2003} \)]. By assuming a de Sitter background with constant curvature \( R_0 = 12H_0^2 = f_0/(6f_0') \), this turns into

\[ \tilde{m}^2_{\text{eff}} = \frac{1}{3f_0'} \left( \frac{f_0'}{f_0} - R_0 \right) = \frac{m^2_{\text{eff}}}{m_u^2}, \tag{104} \]

In the Einstein frame, it is not the mass \( m_\phi \) of a particle or a field that is measurable, but rather the ratio \( \tilde{m}/m_u \) between \( \tilde{m} \) and the Einstein frame unit of mass \( m_u \), which is varying, scaling as \( \tilde{m}_u = [f'(R)]^{-1/2} m_u = \phi^{-1/2} m_u \), where \( m_u \) is the constant unit of mass in the Jordan frame \( \text{Dick, 1962; Faraoni et al., 1994; Faraoni and Nadeau, 2007} \). Therefore,

\[ \frac{\tilde{m}^2_{\text{eff}}}{m_u^2} = \frac{m^2_{\text{eff}}}{m_u^2}. \tag{105} \]

In practice, \( \phi = f'(R) \) is dimensionless and its value must be of order unity in order to obtain the gravitational coupling strength measured in the Solar System; as a result, the Einstein frame metric \( g_{\mu\nu} \) and the Jordan frame metric \( g_{\mu\nu} \) are almost equal, and the same applies to \( \tilde{m}_u, m_u \) and to \( \tilde{m}_{\text{eff}}, m_{\text{eff}} \), respectively. Then, the only relevant difference between Einstein and Jordan frames is the scalar field redefinition \( \phi \to \tilde{\phi} \).

2. Weak-field limit in the metric formalism

Having discussed the field content of the theory, we are now ready to discuss the weak-field limit. Having the correct weak-field limit at the Newtonian and post-Newtonian levels is essential for theoretical viability.

From the beginning, works on the weak-field (Newtonian and post-Newtonian) limit of \( f(R) \) gravity led to opposite results appearing in the literature \( \text{[Accioy et al., 1995; Baghram et al., 2007; Barrow and Clifton, 2006a; Capozziello et al., 2006; 2007a; 2007b; Capozziello and Troisi, 2005; Capozziello and Tsujikawa, 2007; Celabranos, 2006; Clifton and Barrow, 2005a; 2006; Dick, 2004; Easson, 2004; Hu and Sawicki, 2007b; Iorio, 2007; Multamaki and Vilja, 2006b; 2007a; Navarro and Van Acoleyen, 2005, 2006; Olma, 2005; Rajaraman, 2003; Ruggero and Iorio, 2007; Shao et al., 2008; Soussa and Woodard, 2004; Zhang, 2007]. Moreover, a certain lack of rigour in checking the convergence of series used in the expansion around a de Sitter background often left doubts even on results that, a posteriori, turned out to be correct \( \text{[Sotiriou, 2006d]}. \)

By using the equivalence between \( f(R) \) and scalar-tensor gravity, Chiba originally suggested that all \( f(R) \) theories are ruled out \( \text{[Chiba, 2003]}. \) This claim was based on the fact that metric \( f(R) \) gravity is equivalent to an \( \omega_0 = 0 \) Brans–Dicke theory, while the observational constraint is \( |\omega_0| > 40000 \) \( \text{[Bertotti et al., 2003]}. \) This is not quite the case and the weak-field limit is more subtle than it appears, as the discussion of the previous section might have already revealed: The value of the parametrized post-Newtonian (PPN) parameter \( \gamma \), on which the observational bounds are directly applicable, is practically independent of the mass of the scalar only when the latter is small \( \text{[Wagoner, 1970]}. \) In this case, the constraints on \( \gamma \) can indeed be turned into constraints on \( \omega_0 \). However, if the mass of this scalar is large, it dominates over \( \omega_0 \) in the expression of \( \gamma \) and drives its value to unity. The physical explanation of this fact, as mentioned previously, is that the scalar becomes short-ranged and, therefore, has no effect at Solar System scales. Additionally, there is even the possibility that the effective mass of the scalar field itself is actually scale-dependent. In this case, the scalar may acquire a large effective mass at terrestrial and Solar System scales, shielding it from experiments performed there while being effectively light at cosmological scales. This is the \textit{chameleon mechanism}, well-known in quintessence models \( \text{[Khoury and Weltman, 2004a,b]}. \)

Given the above, it is worth examining these issues in more detail. Even though early doubts about the validity of the dynamical equivalence with scalar-tensor theory in the Newtonian limit \( \text{[Faraoni, 2006a; Kainulainen et al., 2007a; 2007b; Olma, 2005; Rajaraman, 2003; Rajaraman et al., 2003; Ruggero and Iorio, 2007; Shao et al., 2008; Soussa and Woodard, 2004; Zhang, 2007]. Moreover, a certain lack of rigour in checking the convergence of series used in the expansion around a de Sitter background often left doubts even on results that, a posteriori, turned out to be correct \( \text{[Sotiriou, 2006d]}. \)

...
Isotropic coordinates are usually employed in the study of the universe, in particular, the spherical symmetry of metric $f(R)$ gravity [see also (Olmo, 2007)]. Chiba’s result based on the Palatini version of $f(R)$ gravity.

In what follows we adhere to, but streamline, the discussion of (Chiba et al., 2007) with minor modifications, in order to compute the PPN parameter $\gamma$ for metric $f(R)$ gravity [see also (Olmo, 2007)]. We consider a spherically symmetric, static, non-compact body embedded in a background de Sitter universe; the latter can exist in an adiabatic approximation in which the evolution of the universe is very slow in comparison with local dynamics.

The condition for the existence of a de Sitter space with $R_{\mu\nu} = R_0 g_{\mu\nu}/4$ and constant curvature $R_0 = 12H_0^2$ is

$$f'_0 R_0 - 2 f_0 = 0, \quad H_0 = \sqrt{\frac{f'_0}{6f_0}},$$

(106)

The line element is

$$ds^2 = -\left[1 + 2\Psi(r) - H_0^2 r^2\right] dt^2 + \left[1 + 2\Phi(r) + H_0^2 r^2\right] dr^2 + r^2 d\Omega^2$$

(107)

in Schwarzschild coordinates, where the post-Newtonian potentials $\Psi(r)$ and $\Phi(r)$ are treated as small perturbations. The goal is to compute the PPN parameter $\gamma = -\Psi/\Phi$ by solving the equations satisfied by these potentials. A linearized analysis is performed assuming

$$|\Psi(r)|, |\Phi(r)| \ll 1, \quad r \ll H_0^{-1},$$

(108)

and

$$R(r) = R_0 + R_1(r),$$

(109)

where the deviation $R_1(r)$ of the Ricci curvature from the constant $R_0$ is also a small perturbation.

Three assumptions are made:

Assumption 1: $f(R)$ is analytical at $R_0$.

Assumption 2: $mr << 1$, where $m$ is the effective mass of the scalar degree of freedom of the theory. In other words, this scalar field (the Ricci curvature, which is an extra dynamical quantity in the metric formalism) must have a range longer than the size of the Solar System—if it is much shorter than, say, 0.2 mm (Hoyle et al., 2001), the presence of this scalar is effectively hidden from Solar System and terrestrial experiments. In this case, this field could not have cosmological effects at late times, but could only be important in the very early universe at high curvatures, e.g., in Starobinsky-like inflation (Starobinsky, 1980).

Assumption 3: the pressure $P \equiv 0$ for the energy-momentum of the local star-like object. The trace of the corresponding energy-momentum tensor reduces to $T_1 \simeq -\rho$.

By expanding $f(R)$ and $f'(R)$ around $R_0$, the trace equation (110) reduces to

$$3f''_0 \Box R_1 + (f''_0 R_0 - f'_0) R_1 = \kappa T_1,$$

(110)

where $T = T_1$, since $T$ is zero in the background. For a static, spherically symmetric body, $R_1 = R_1(r)$ and $\Box R_1 = \nabla^2 R_1 = -\frac{\kappa}{3f''_0} \left(\frac{r^2}{d^2r^2} \right)$, which yields

$$\nabla^2 R_1 = -\frac{\kappa}{3f''_0} \rho,$$

(111)

where

$$m^2 = \frac{f''_0 - f'''_0 R_0}{3f''_0}.$$  

(112)

By using $R_0 = 12H_0^2 = 2f_0/f'_0$, this reduces to

$$m^2 = \left(\frac{f'_0}{f''_0}\right)^2 - \frac{2f_0 f''_0}{3f''_0}.$$  

(113)

This equation is found in various other treatments of perturbations of de Sitter space (Farooqi and Nadeau, 2005; Navarro and Van Acoleyen, 2004; Nunez and Solganik, 2004; Olmo, 2006).

Assumption 2 that the scalar $R_1$ is light, which enables the $f(R)$ theory to produce significant cosmological effects at late times, also allows one to neglect the term $m^2 R_1$ in eq. (111). The Green function of the equation $
abla^2 R_1 = -\frac{\kappa}{3f''_0} \rho$ is then

$$R_1 \simeq \frac{\kappa}{12\pi f''_0} \int d^3x \frac{\rho(r')}{3f''_0} G(r-r'),$$

(114)

where $G(r-r')$ is the Green function of the harmonic oscillator, which is defined as

$$G(r-r') = \frac{1}{\sqrt{4\pi f''_0 r}} e^{-\frac{\rho(r')}{3f''_0}}.$$

(115)

Although (Chiba et al., 2007) provide Green functions in both cases $m^2 > 0$ and $m^2 < 0$, the latter corresponds to a spacetime instability and is unphysical. This is irrelevant in the end because only the case $m^2 \rightarrow 0$ is necessary and used in the calculation (Farooqi and Lanahan-Tremblay, 2007).
Now, the condition \( m^2 r^2 \ll 1 \) yields
\[
\left| \frac{1}{3} \frac{f_0'}{f_0} - R_0 \right| r^2 \ll 1 \tag{115}
\]
and, using \( H_0 r \ll 1 \),
\[
\left| \frac{f_0'}{f_0} \right| r^2 \ll 1. \tag{116}
\]

Let us use now the full field equations (6); by expanding \( f(R) \) and \( f'(R) \) and using \( f_0 = 6H_0 f'_0 \) we get
\[
\delta^\alpha_\beta f_0' \Box R_1 + f_0' (R_0^3 - 3H_0^2 \delta^\alpha_\beta) - \frac{f_0'}{2} R_1 \delta^\alpha_\beta - f''_0 \nabla^\alpha \nabla_\beta R_1 + f''_0 R_1 R_0^\beta = \kappa T_0^\alpha. \tag{117}
\]

By using again the assumption \( H_0 r \ll 1 \), the d’Alembertian \( \Box \) becomes \( \nabla^2 \) and, for \( (\mu, \nu) = (0, 0) \),
\[
f_0' (R_0^3 - 3H_0^2) - \frac{f_0'}{2} R_1 + f''_0 R_1 R_0^\beta + f''_0 \nabla^2 R_1 = - \kappa \rho. \tag{118}
\]

Recalling that \( \nabla^2 R_1 \approx - \frac{\kappa \rho}{r^2} \) for \( mr \ll 1 \), one obtains
\[
f_0' \nabla^2 \Psi(r) + \frac{f_0'}{2} R_1 - f''_0 \nabla^2 R_1 = \kappa \rho. \tag{119}
\]

Eq. (120) can be integrated from \( r = 0 \) to \( r > r_0 \) (where \( r_0 \) is the radius of the star-like object) to obtain, using Gauss’ law,
\[
\frac{d \Psi}{dr} = \frac{\kappa M}{6 \pi f'_0} \frac{\kappa M}{48 \pi f''_0 r^2} - C_1, \tag{121}
\]

where \( M(r) = 4\pi \int_0^r dr' (r')^2 \rho(r') \). The integration constant \( C_1 \) must be set to zero to guarantee regularity of the Newtonian potential at \( r = 0 \). The potential \( \Psi(r) \) then becomes
\[
\Psi(r) = - \frac{\kappa M}{6 \pi f'_0 r} - \frac{\kappa M}{48 \pi f''_0} r. \tag{122}
\]

The second term on the right hand side is negligible; in fact,
\[
\left| \frac{\kappa M}{48 \pi f''_0} \right| = \left| \frac{f_0'}{f_0} \right| r^2 \ll 1, \tag{123}
\]
and
\[
\Psi(r) \approx - \frac{\kappa M}{6 \pi f'_0} r. \tag{124}
\]

Let us now find the second potential \( \Phi(r) \) appearing in the line element (107). By using the field equations (6) with \( (a, b) = (1, 1) \),
\[
f_0' (R_1^3 - 3H_0^2) - \frac{f_0'}{2} R_1 - f''_0 \nabla^1 \nabla_1 R_1 + f''_0 R_1 R_0^1 = \kappa T_1^1 \tag{125}
\]
with \( T_1^1 \approx 0 \) outside the star, and
\[
R_1^3 \approx 3H_0^2 \frac{2 \Phi}{dr} + \frac{2 d \Phi}{r}, \tag{126}
\]
\[
g^{11} \nabla_1 \nabla_1 R_1 \approx \frac{d^2 R_1}{dr^2}. \tag{127}
\]

and neglecting higher order terms, one obtains (eq. (22) of (Chiba et al., 2007))
\[
f_0' \left( \frac{d^2 \Psi}{dr^2} + \frac{2 d \Phi}{r} \right) - \frac{f_0'}{2} R_1 + 2 f''_0 \frac{d R_1}{dr} \approx 0. \tag{128}
\]

Now, using eq. (114) for \( R_1 \), one concludes that the third term in eq. (128) is negligible in comparison with the fourth term. In fact,
\[
\left| \frac{f_0' R_1}{f_0^2} \right| r^2 \ll 1. \tag{129}
\]

Then, using again the expression (114) for \( dR_1/dr \) and eq. (121) for \( \Psi(r) \), one obtains
\[
\frac{d \Phi}{dr} = - \frac{\kappa M}{12 \pi f'_0} r, \tag{130}
\]
which is immediately integrated to
\[
\Phi(r) = \frac{\kappa M}{12 \pi f'_0} r. \tag{131}
\]

The post-Newtonian metric (107) therefore gives the PPN parameter \( \gamma \) as
\[
\gamma = \frac{\Phi(r)}{\Psi(r)} = \frac{1}{2}. \tag{132}
\]

This is a gross violation of the experimental bound \( |\gamma - 1| < 2.3 \times 10^{-5} \) (Bertotti et al., 2003) and agrees with the calculation of the PPN parameter \( \gamma = \frac{\omega_0}{2} \approx 0 \) Brans–Dicke theory (Chiba, 2003).

The results of (Chiba et al., 2007) have been reproduced by (Olmeda, 2007), who works in isotropic coordinates with a slightly different approach. (Kaimulainen et al., 2007) have obtained spherically symmetric interior solutions matched to the exterior solutions of metric \( f(R) \) gravity and have confirmed the result \( \gamma = 1/2 \).

**Limits of validity of the previous analysis:** One can contemplate various circumstances in which the assumptions above are not satisfied and the previous analysis
breaks down. It is important to ascertain whether these are physically relevant situations. There are three main cases to consider.

The case of non-analytic $f(R)$: While Chiba et al. (2007) consider functions $f(R)$ that are analytic at the background value $R_0$ of the Ricci curvature, the situation in which this function is not analytical has been contemplated briefly in Jin et al. (2006). Assuming that $f(R)$ has an isolated singularity at $R = R_s$, it can be expressed as the sum of a Laurent series,

$$f(R) = \sum_{n=0}^{\infty} a_n (R - R_s)^n.$$  \hspace{1cm} (133)

Jin et al. (2006) note that it must be $R \neq R_s$ in the dynamics of the universe because a constant curvature space with $R = R_s$ can not be a solution of the field equations. Therefore, one can approximate the solution adiabatically with a de Sitter space with constant curvature $R_0 \neq R_s$. The function $f(R)$ is analytical here and the previous discussion applies. This is not possible if $f(R)$ has an essential singularity, for example, if $f(R) = R - \mu^2 \sin \left(\frac{R}{\mu^2}\right)$ (Jin et al. 2006). There is, of course, no reason other than Occam’s razor to exclude this possibility.

Short range scalar field: If the assumption $mr << 1$ is not satisfied, the scalar is massive. If its range is sufficiently short, it is effectively hidden from experiments probing deviations from Newton’s law and from other Newtonian and post-Newtonian experiments in the solar neighbourhood. This is the case of quadratic quantum corrections to Einstein’s gravity, e.g., $f(R) = R + \alpha R^2$. If the effective mass is $m \geq 10^{-3}$ eV (corresponding to a fifth force range less than ~0.2 mm, the shortest scale currently accessible to weak-field experiments), this correction is undetectable and yet it can still have large effects in the early inflationary universe Starobinsky (1980). However, it can not work as a model for late time acceleration.

Chameleon behaviour: The chameleon effect (Khoury and Weltman, 2004a,b), originally discovered in scalar field models of dark energy, consists of the effective mass $m$ of the scalar degree of freedom being a function of the curvature (or, better, of the energy density of the local environment), so that $m$ can be large at Solar System and terrestrial curvatures and densities, and small at cosmological curvatures and densities — effectively, it is short-ranged in the Solar System and it becomes long-ranged at cosmological densities thus causing the acceleration of the universe. The chameleon effect can be applied to metric $f(R)$ gravity (Cembranos 2006; Faulkner et al. 2007; Navarro and Van Acoleyen 2007; Starobinsky 2007), with the result that theories of the kind Amendola et al. (2007a,b,c, 2008) and Carroll et al. (2004)

$$f(R) = R - (1 - n)\mu^2 \left(\frac{R}{\mu^2}\right)^n.$$  \hspace{1cm} (134)

are compatible with the observations in the region of the parameter space $0 < n \leq 0.25$ with $\mu$ sufficiently small Faulkner et al. (2007). Precisely, using the Cassini bound on the PPN parameter $\gamma$ Bertotti et al. (2003), the constraint

$$\frac{\mu}{H_0} \leq \sqrt{3} \left[ \frac{2}{n(1 - n)} \right]^{\frac{1}{2(1 - n)}} \frac{10^{6 - 5n}}{10^{1 - 12n}}$$

is obtained Faulkner et al. (2007). Fifth force experiments give the bounds

$$\frac{\mu}{H_0} \leq \sqrt{1 - n} \left[ \frac{2}{n(1 - n)} \right]^{\frac{1}{2(1 - n)}} \frac{10^{1 - 12n}}{10^{2 - 12n}}.$$  \hspace{1cm} (136)

Preferred values seem to be $m \simeq 10^{-50}$ eV ~ $10^{-17} H_0$ Faulkner et al. (2007). Note that $n > 0$, which guarantees $f'' > 0$, is required for Ricci scalar stability ($n = 0$ reduces the model to GR with a cosmological constant, but avoiding the latter was exactly the reason why dark energy and modified gravity were introduced in the first place).

These models work to explain the current cosmic acceleration because, for small curvatures $R$, the correction in $R^n$ with $n < 1$ is larger than the Einstein–Hilbert term $R$ and comes to dominate the dynamics. On the negative side, these theories are observationally indistinguishable from a cosmological constant and they have been dubbed “vanilla $f(R)$ gravity” Amendola et al. (2007a,b, 2008) and Amendola and Tsujikawa (2008). However, they still have the advantage of avoiding a fine-tuning problem in $\Lambda$ at the price of a much smaller fine-tuning of the parameter $\mu$. As for all modified gravity and dark energy models, they do not address the cosmological constant problem.

The weak-field limit of metric $f(R)$ theories which admit a global Minkowski solution around which to linearize, was studied by Clifton (2008). These theories (including, e.g., analytic functions $f(R) = \sum_{n=1}^{\infty} a_n R^n$ are not motivated by late time cosmology and the Minkowski global solution, although present, may not be stable Clifton and Barrow (2005a), which in practice detracts from the usefulness of this analysis. Several new post-Newtonian potentials are found to appear in addition to the two usual ones Clifton (2008).

3. Weak-field limit in the Palatini formalism

Early works on the weak-field limit of Palatini $f(R)$ gravity often led to contradictory results and to several technical problems as well Allemandi et al. (2005a,b). Allemandi and Ruggiero (2007), Barraco and Hamity (2006), Bustelo and Barraco (2007), Dominguez and Baracco (2004), Kainulainen et al. (2007a,b), Meng and Wang (2004a,b), Olmedo (2005a,b), Ruggiero (2007), Ruggiero and Iorio (2007), Sotiriou (2006a) which seem to have been clarified by now.
First of all, there seems to have been some confusion in the literature about the fact that Palatini $f(R)$ gravity reduces to GR with a cosmological constant in vacuum and the consequences that this can have on the weak-field limit and Solar System tests. It is, of course, true (see Sec. 4.I.B) that in vacuo Palatini $f(R)$ gravity will have the same solutions of GR plus a cosmological constant and, therefore, the Schwarzschild-(anti-)de Sitter solution will be the unique vacuum spherically symmetric solution (see also Sec. VII.C.1 for a discussion of the Jebsen-Birkhoff theorem). This was interpreted in (Allemandi and Ruggiero, 2007; Ruggiero and Iorio, 2007) as an indication that the only parameter that can be constrained is the effective cosmological constant and, therefore, models that are cosmologically interesting (for which this parameter is very small) trivially satisfy Solar System tests. However, even if one sets aside the fact that a weak gravity regime is possible inside matter as well, such claims cannot be correct: they would completely defeat the purpose of performing a parametrized post-Newtonian expansion for any theory for which one can establish uniqueness of a spherically symmetric solution, as in this case we would be able to judge Solar System viability just by considering this vacuum solution (which would be much simpler).

Indeed, the existence of a spherically symmetric vacuum solution, irrespective of its uniqueness, does not suffice to guarantee a good Newtonian limit. For instance, the Schwarzschild-de Sitter solution has two free parameters; one of them can be associated with the effective cosmological constant in a straightforward manner (using the asymptotics). However, it is not clear how the second parameter, which in GR is identified with the mass of the object in the Newtonian regime, is related to the internal structure of the object in Palatini $f(R)$ gravity. The assumption that it represents the mass defined in the usual way is not, of course, sufficient. One would have to actually match the exterior solution to a solution describing the interior of the Sun within the realm of the theory in order to express the undetermined parameter in the exterior solution in terms of known physical quantities, such as Newton’s constant and the Newtonian mass. The essence of the derivation of the Newtonian limit of the theory consists also in deriving such an explicit relation for this quantity and showing that it agrees with the Newtonian expression. The parametrized post-Newtonian expansion is nothing but an alternative way to do that without having to solve the full field equations. Therefore, it is clear that more information than the form of the vacuum solution is needed in order to check whether the theory can satisfy the Solar System constraints.

However, some early attempts towards a Newtonian and post-Newtonian expansion were also flawed. In (Meng and Wang, 2004a) and (Barraco and Hamity, 2004), for instance, a series expansion around a de Sitter background was performed in order to derive the Newtonian limit. Writing

$$R = R_0 + R_1,$$  \hspace{1cm} (137)

where $R_0$ is the Ricci curvature of the background and $R_1$ is a correction, one is tempted to expand in powers of $R_1/R_0$ regarding the latter as a small quantity. Since one needs the quantities $f(R_0 + R_1)$ and $f'(R_0 + R_1)$, the usual approach is to Taylor-expand around $R = R_0$ and keep only the leading order terms in $R_1$. However, it has been shown in (Sotiriou, 2006c) that this can not be done for most cosmologically interesting models because $R_1/R_0$ is not small.

Take as an example the model (Vollick, 2003)

$$f(R) = R - \frac{\epsilon_2}{R},$$  \hspace{1cm} (138)

and $\epsilon_2 \sim 10^{-67} (eV)^2 \sim 10^{-53} m^{-2}$. Expanding as

$$f(R) = f(R_0) + f'(R_0)R_1 + \frac{1}{2} f''(R_0)R_1^2 + \ldots$$  \hspace{1cm} (139)

and using eq. (138) yields

$$f(R) = f(R_0) + \left(1 + \frac{\epsilon_2^2}{R_0^2}\right) R_1 - \frac{1}{2} \frac{\epsilon_2^2}{R_0^3} R_1^2 + \ldots,$$  \hspace{1cm} (140)

where now $R_0 = \epsilon_2$. It is then easy to see that the second term on the right-hand side is of the order of $R_1$, whereas the third term is of the order of $R_1^2/\epsilon_2$. Therefore, in order to truncate before the third term, one needs $R_1 \gg R_1^2/\epsilon_2$ or

$$\epsilon_2 \gg R_1.$$  \hspace{1cm} (141)

This is not a stringent constraint: $R_0 \sim \epsilon_2$ and so this is the usual condition for linearization.

Let us return now to the trace equation (20). For the model under consideration,

$$\mathcal{R} = \frac{1}{2} \left( -\kappa T \pm \sqrt{\kappa^2 T^2 + 12\epsilon_2^2} \right).$$  \hspace{1cm} (142)

According to eq. (142), the value of $\mathcal{R}$, and consequently, $R_1$, is algebraically related to $T$ and, whether or not the condition (141) is satisfied or not critically depends on the value of the energy density. To demonstrate this, pick the mean density of the Solar System, $\rho \sim 10^{-11} gr/cm^3$, which satisfies the weak-field limit criteria. For this value, $|\epsilon_2/\kappa T| \sim 10^{-21}$, where $T \sim -\rho$. The “physical” branch of the solution (142) is the one with positive sign because, given that $T < 0$, it ensures that matter leads to a standard positive curvature in strong gravity. Then,

$$\mathcal{R} \sim -\kappa T - \frac{3\epsilon_2^2}{\kappa T}$$  \hspace{1cm} (143)

and $R_1 \sim -\kappa T \sim \kappa \rho$. Thus, $\epsilon_2/R_1 \sim 10^{-21}$ and it is evident that the required condition does not hold for some typical densities related to the Newtonian limit.
The situation does not improve even with the “unphysical” branch of eq. (142) with a negative sign. In fact, in this case, \( R_3 \sim \epsilon_2 \frac{3\epsilon_2}{(\kappa T) + \sqrt{3}} \) and the correction to the background curvature is of the order \( \epsilon_2 \) and not much smaller than that, as it would be required in order to truncate the expansion (140). In (Barraco and Hamity, 2001), this fact was overlooked and only linear terms in \( R_1 \) were kept in the expansion of \( f(R) \) and \( f'(R) \) around \( R_0 \). In (Meng and Wang, 2004a), even though this fact is noticed in the final stages of the analysis and is actually used, the authors do not take it into account properly from the outset, keeping again only linear terms [see, e.g., eq. (11) of (Meng and Wang, 2004a)].

However, the algebraic dependence of \( R \) on the density does not only signal a problem for the approaches just mentioned. It actually implies that the outcome of the post-Newtonian expansion itself depends on the density, as shown in (Olmo, 2005a,b; Sotiriou, 2006c). Consider post-Newtonian expansion itself depends on the density, due to eq. (20), the deviation of \( \Omega(T) \) from \( \Omega(t, \vec{x}) \) will be large deviations from \( \Omega(T) \) to high precision. However, for the same function \( f' \), there will be large deviations from \( f' = 1 \) at a different density range. This dependence of the weak-field limit on the energy density is a novel characteristic of Palatini \( f(R) \) gravity.

This dependence can be made explicit if the problem is approached via the equivalent Brans–Dicke theory (Olmo, 2005a,b). Note that the usual bounds coming from Solar System experiments do not apply in the \( \omega_0 = -3/2 \) case, which is equivalent to Palatini \( f(R) \) gravity. This is because the standard treatment of the post-Newtonian expansion of Brans–Dicke theory, which one uses to arrive to such bounds, is critically based on the assumption that \( \omega_0 \neq -3/2 \) and the term \( 2\omega_0 + 3 \) frequently appears as a denominator. Making this assumption is not necessary, of course, in order to derive a post-Newtonian expansion, but is a convenient choice, which allows for this otherwise general treatment. Therefore, a different approach, such as the one followed in (Olmo, 2005b), was indeed required for the \( \omega_0 = -3/2 \) case. Following the standard assumptions of a post-Newtonian expansion around a background specified by a cosmological solution (Will, 1981), the following relations were derived for the post-Newtonian limit:

\[
- \frac{1}{2} \nabla^2 [h_{00} - \Omega(T)] = \frac{\kappa \rho - V(\phi)}{2 \phi}, \tag{146}
\]

\[
- \frac{1}{2} \nabla^2 [h_{ij} + \delta_{ij} \Omega(T)] = \left[ \frac{\kappa \rho + V(\phi)}{2 \phi} \right], \tag{147}
\]

where \( V \) is the potential of the scalar field \( \phi \) and \( \Omega(T) \equiv \log[\phi_0/\phi_0] \). The subscript 0 in \( \phi_0 \), and in any other quantity in the rest of this subsection, denotes that it is evaluated at \( T = 0 \).

The solutions of eqs. (146) and (147) are

\[
h_{00}^{(1)}(t, \vec{x}) = 2G_{\text{eff}}M_\odot \frac{V_0}{\phi_0} r^2 + \Omega(T), \tag{148}
\]

\[
h_{ij}^{(1)}(t, \vec{x}) = \left[ 2G_{\text{eff}}M_\odot \frac{V_0}{\phi_0} r^2 - \Omega(T) \right] \delta_{ij}, \tag{149}
\]

where \( M_\odot \equiv \phi_0 \int d^3 \vec{x}' \rho(t, \vec{x}') / \phi \). The effective Newton constant \( G_{\text{eff}} \) and the post-Newtonian parameter \( \gamma \) are defined as

\[
G_{\text{eff}} \equiv \frac{G}{\phi_0} \left( 1 + \frac{M_V}{M_\odot} \right), \tag{150}
\]

\[
\gamma \equiv \frac{M_\odot - M_V}{M_\odot + M_V}, \tag{151}
\]

where \( M_V \equiv \kappa^{-1} \phi_0 \int d^3 \vec{x}' [V_0/\phi_0 - V(\phi)/\phi] \).

As stated in different words in (Olmo, 2005b), if the Newtonian mass is defined as \( M_N \equiv \int d^3 \vec{x}' \rho(t, \vec{x}') \), the requirement that a theory has a good Newtonian limit is that \( G_{\text{eff}}M_\odot \) equals \( GM_N \), where \( N \) denotes Newtonian, and \( \gamma \approx 1 \) to very high precision. Additionally, the second term on the right hand side of both eqs. (148) and (149) should be negligible, since it plays the role of a cosmological constant term. \( \Omega(T) \) should also be small and have a negligible dependence on \( T \).

Even though it is not impossible, as mentioned before, to prescribe \( f \) such that all of the above are satisfied for some range of densities within matter (Sotiriou, 2006c), this does not seem possible over the wide range of densities relevant for the Solar System tests. As a matter of fact, \( \Omega \) is nothing but an algebraic function of \( T \) and, therefore, of the density (since \( \phi \) is an algebraic function of \( R \)). The presence of the \( \Omega(T) \) term in eqs. (148) and (149) signals an algebraic dependence of the post-Newtonian metric on the density. This direct dependence of the metric on the matter field is not only surprising but also seriously problematic. Besides the fact that it is evident that the theory cannot have the proper Newtonian limit for all densities (the range of densities for which it will fail depends on the functional form of \( f \)), consider the following: What happens to the post-Newtonian metric if a very weak point source (approximated by a delta function) is taken into account as a perturbation? And will the post-Newtonian metric be continuous when going from the interior of a source to the exterior, as it should?
We will refrain from further analysis of these issues here, since evidence coming from considerations different than the post-Newtonian limit, which we will review shortly, will be of significant help. We will, therefore, return to this discussion in Sec. [VLC.2]

### B. Stability issues

In principle, several kinds of instabilities need to be considered to make sure that \( f(R) \) gravity is a viable alternative to GR (Calcagni et al., 2006a; De Felice et al., 2006; Søndergaard et al., 2007). The Dolgov-Kawasaki instability was rediscovered in (Baghram et al., 2007) for a specific form of the function \( f(R) \), and evaluating \( \Box f' \),

\[
\Box R + \frac{\varphi''}{\varphi'} \nabla^a R \nabla_a R + \left( \frac{\epsilon \varphi' - 1}{3 \epsilon \varphi''} \right) R = \frac{\kappa T}{3 \epsilon \varphi''} + \frac{2 \varphi}{3 \epsilon \varphi''},
\]

We assume that \( \varphi'' \neq 0 \): if \( \varphi'' = 0 \) on an interval then the theory reduces to GR. Isolated zeros of \( \varphi'' \), at which the theory is “instantaneously GR”, are in principle possible but will not be considered here.

Consider a small region of spacetime in the weak-field regime and approximate locally the metric and the curvature by

\[
g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}, \quad R = -\kappa T + R_1,
\]

where \( \eta_{\mu \nu} \) is the Minkowski metric and \( |R_1/\kappa T| \ll 1 \). This inequality excludes the case of conformally invariant matter with \( T = 0 \), a situation considered later. Equation (155) yields, to first order in \( R_1 \),

\[
\tilde{R}_1 - \nabla^2 \tilde{R}_1 - 2 \kappa \varphi'' \varphi'' \tilde{T} \tilde{R}_1 + 2 \kappa \varphi'' \varphi'' \varphi'' \tilde{\nabla} \cdot \tilde{\nabla} \tilde{R}_1 + 
\]

\[
\frac{1}{3 \varphi''} \left( \frac{1}{\epsilon} - \varphi' \right) R_1 = \kappa \tilde{T} - \kappa \nabla^2 \tilde{T} - \frac{(\kappa T \varphi' + 2 \varphi)}{3 \varphi''},
\]

where \( \tilde{\nabla} \) and \( \nabla^2 \) are the gradient and Laplacian in Euclidean three-dimensional space, respectively, and an overdot denotes differentiation with respect to time. The function \( \varphi \) and its derivatives are now evaluated at \( \tilde{R}_1 = -\kappa T \). The coefficient of \( R_1 \) in the fifth term on the left hand side is the square of an effective mass and is dominated by the term \( (3 \epsilon \varphi'')^{-1} \) due to the extremely small value of \( \epsilon \) needed for these theories to reproduce the correct cosmological dynamics. Then, the scalar mode \( R_1 \) of the \( f(R) \) theory is stable if \( \varphi'' = f'' > 0 \), and unstable if this effective mass is negative, i.e., if \( \varphi'' = f'' < 0 \). The time scale for this instability to manifest is estimated to be of the order of the inverse effective mass \( \sim 10^{-26} \) s in the example \( \epsilon \varphi(R) = -\mu^4/R \) (Dolgov and Kawasaki, 2003a). The small value of \( \varphi'' \) gives a large effective mass and is responsible for the small time scale over which the instability develops.

Let us consider now matter with vanishing trace \( T \) of the stress-energy tensor. In this case eq. (156) becomes

\[
\tilde{R}_1 + \frac{\varphi''}{\varphi'} \tilde{T}_1^2 - \tilde{\nabla}^2 \tilde{R}_1 - \frac{\varphi''}{\varphi'} \left( \tilde{\nabla} \tilde{R}_1 \right)^2 + 
\]

\[
\frac{1}{3 \varphi''} \left( \frac{1}{\epsilon} - \varphi' \right) \frac{\tilde{T}_1}{2} = \frac{2 \varphi}{3 \epsilon \varphi''}. \tag{157}
\]
Again, the effective mass term is \( \sim (3\epsilon^f''')^{-1} \), which has the sign of \( f'' \) and the previous stability criterion is recovered. The stability condition \( f''(R) \geq 0 \) is useful to veto \( f(R) \) gravity models.\(^{22}\)

When \( f'' < 0 \), the instability of these theories can be interpreted, following eq. (159), as an instability in the gravity sector. Equivalent, since it appears inside matter when \( R \) starts deviating from \( T \) [see eq. (153)], it can be seen as a matter instability [this is the interpretation taken in (Dolgov and Kawasaki, 2003a)]. Whether the instability arises in the gravity or matter sector seems to be a matter of interpretation.

The instability of stars made of any type of matter in theories with \( f'' < 0 \) and sufficiently small is confirmed, with a different approach (a generalized variational principle) in (Seifert, 2007), in which the time scale for instability found by Dolgov and Kawasaki in the 1/R model is also recovered. The stability condition \( f'' \geq 0 \) is recovered in studies of cosmological perturbations (Sawicki and Hu, 2007).

The stability condition \( f''(R) \geq 0 \), expressing the fact that the scalar degree of freedom is not a ghost, can be given a simple physical interpretation (Faraoni, 2007). Assume that the effective gravitational coupling \( G_{eff}(R) = G/f'(R) \) is positive; then, if \( G_{eff} \) increases with the curvature, i.e.,

\[
\frac{dG_{eff}}{dR} = \frac{-f''(R)G}{(f'(R))^2} > 0,
\]

(158)

at large curvature the effect of gravity becomes stronger, and since \( R \) itself generates larger and larger curvature via eq. (153), the effect of which becomes stronger and stronger because of an increased \( G_{eff}(R) \), a positive feedback mechanism acts to destabilize the theory. There is no stable ground state if a small curvature grows and grows without limit and the system runs away. If instead the effective gravitational coupling decreases when \( R \) increases, which is achieved when \( f''(R) > 0 \), a negative feedback mechanism operates which compensates for the increase in \( R \) and there is no running away of the solutions. These considerations have to be inverted if \( f' < 0 \), which can only happen if the effective energy density \( \rho_{eff} \) also becomes negative. This is not a physically meaningful situation because the effective gravitational coupling becomes negative and the tensor field and the scalar field of metric \( f(R) \) gravity become ghosts (Nunez and Solganik, 2004).

GR, with \( f''(R) = 0 \) and \( G_{eff} = \) constant, is the borderline case between the two behaviours corresponding to stability \( (f'' > 0) \) and instability \( (f'' < 0) \), respectively.

Remarkably, besides the Dolgov-Kawasaki instability which manifest itself in the linearized version of equation (153), there are also recent claims that \( R \) can be driven to infinity due to strong non-linear effects related to the same equation (Appleby and Battye, 2008; Frolov, 2008; Tsujikawa, 2008). More specifically, in (Tsujikawa, 2008) an oscillating mode is found as a solution to the perturbed version of eq. (153). This mode appears to dominate over the matter-induced mode as one goes back into the past and, therefore, it can violate the stability conditions. In (Frolov, 2008), eq. (153) was studied, with the use of a convenient variable redefinition but without resorting to any perturbative approach. It was found that there exists a singularity at a finite field value and energy level. The strongly non-linear character of the equation allows \( R \) to easily reach the singularity in the presence of matter. As noticed in (Appleby and Battye, 2008), since when it comes to cosmology the singularity lies in the past, it can in principle be avoided by choosing appropriate initial conditions and evolving forward in time. This, of course, might result in a hidden fine-tuning issue.

All three studies mentioned consider models in which \( f(R) \) includes, besides the linear term, only terms which become important at low curvatures. It is the form of the effective potential governing the motion of \( R \), which depends on the functional form of \( f(R) \), that determines how easy it is to drive \( R \) to infinity (Frolov, 2008). Therefore, it seems interesting to study how the presence of terms which become important at large curvatures, such as positive powers of \( R \), could affect these results. Finally, it would be interesting to see in detail how these findings manifest themselves in the case of compact objects, and whether there is any relation between this issue and the Dolgov-Kawasaki instability.

2. Gauge-invariant stability of de Sitter space in the metric formalism

One can consider the generalized gravity action

\[
S = \int d^4x \sqrt{-g} \left[ \frac{f(\phi, R)}{2} - \frac{\omega^2}{2} \nabla^\alpha \phi \nabla_\alpha \phi - V(\phi) \right].
\]

(159)

incorporating both scalar-tensor gravity (if \( f(\phi, R) = \psi(\phi)R \)) and modified gravity (if the scalar field \( \phi \) is absent and \( f_{RR} \neq 0 \)). In a spatially flat FLRW universe the vacuum field equations assume the form

\[
H^2 = \frac{1}{3f'} \left( \frac{\omega^2}{2} + \frac{R_f}{2} - \frac{f}{2} + V - 3H \dot{f} \right),
\]

(160)

\[
\dot{H} = -\frac{1}{2f'} \left( \omega \dot{\phi}^2 + \ddot{F} - H \dot{\phi} \right),
\]

(161)

\[
\dot{\phi} + 3H \dot{\phi} + \frac{1}{2\omega} \left( \frac{d\omega}{d\phi} \dot{\phi}^2 - \frac{\partial f}{\partial \phi} + 2 \frac{dV}{d\phi} \right) = 0,
\]

(162)

where \( f' \equiv \partial f/\partial \phi, F \equiv \partial f/\partial R \), and an overdot denotes differentiation with respect to \( t \). We choose \( (H, \dot{\phi}) \) as dynamical variables; then, the stationary points of the dynamical system (160)-(162) are de Sitter spaces with

\(^{22}\) Refs. (Multamaki and Vilia, 2006a; Nojiri, 2004) hinted towards the stability criterion, but did not fully derive it because a decomposition in orders of \( \epsilon^{-1} \) was not performed.
constant scalar field \((H_0, \phi_0)\). The conditions for these de Sitter solutions to exist are
\[
6H_0^2 f'_0 - f_0 + 2V_0 = 0, \tag{163}
\]
\[
\frac{df}{d\phi}|_{\phi_0} - 2 \frac{dV}{d\phi}|_{\phi_0} = 0, \tag{164}
\]
where \(f'_0 \equiv f'(\phi_0, R_0), f_0 \equiv f(\phi_0, R_0), V_0 \equiv V(\phi_0),\) and \(R_0 = 12H_0^2\). The phase space is a curved two-dimensional surface embedded in a three-dimensional space (de Souza and Faraoni, 2007).

Inhomogeneous perturbations of de Sitter space have been studied using the covariant and gauge-invariant formalism of Bardeen, 1980; Ellis and Bruni, 1989; Ellis et al., 1990a, 1990b, 1997, 1998; Hwang and Noh, 1996) for general-relativistic gravity, and (Seifert, 2007) for a variational approach applied to D-dimensional space-times.

The generalization of the condition (173) to D spacetime dimensions, derived in (Rador, 2007) for homogeneous perturbations, is
\[
\frac{(f'_0)^2 - 2f_0f''_0}{f'_0^2} \geq 0. \tag{173}
\]

The only term containing the comoving wave vector \(k\) in eq. (174) becomes negligible at late times and/or for zero-momentum modes and thus the spatial dependence effectively disappears. In fact, eq. (173) coincides with the stability condition that can be obtained by a straightforward homogeneous perturbation analysis of eqs. (163) and (161). As a result, in the stability analysis of de Sitter space in modified gravity, inhomogeneous perturbations can be ignored and the study can be restricted to the simpler homogeneous perturbations, which are free of the notorious gauge-dependence problems. This result, which could not be reached a priori but relies on the inhomogeneous perturbation analysis, holds only for de Sitter spaces and not for different attractor (e.g., power-law) solutions that may be present in the phase space. The stability condition (173) is equivalent to the condition that the scalar field potential in the Einstein frame of the equivalent Brans–Dicke theory has a minimum at the configuration identified by the de Sitter space of curvature \(R_0\) (Sokolowski, 2007).

As an example, let us consider the prototype model
\[
f(R) = R - \frac{\mu^4}{R}. \tag{174}
\]

The background de Sitter space has \(R_0 = 12H_0^2 = \sqrt{3} \mu^2\) and the stability condition (173) is never satisfied: this de Sitter solution is always unstable. An improvement is obtained by adding a quadratic correction to this model:
\[
f(R) = R - \frac{\mu^4}{R} + aR^2. \tag{175}
\]

Then, the condition for the existence of a de Sitter solution is again \(R_0 = \sqrt{3} \mu^2\), while the stability condition (173) is satisfied if \(a > \frac{1}{\sqrt{3} \mu^2}\), in agreement with (Nojiri and Odintsov, 2003a, 2004b) who use independent methods.

Different definitions of stability lead to different, albeit close, stability criteria for de Sitter space [see (Cognola et al., 2003, 2008) for the semiclassical stability of modified gravity, (Bertolami, 1987) for scalar-tensor gravity, and (Seifert, 2007) for a variational approach applicable to various alternative gravities].

\[23\] The generalization of the condition (173) to D spacetime dimensions, derived in (Rador, 2007) for homogeneous perturbations, is
\[
\frac{(D - 2)(f'_0)^2 - Df_0f''_0}{f'_0^3} \geq 0. \tag{172}
\]
3. Ricci stability in the Palatini formalism

For Palatini $f(R)$ gravity the field equations (18) and (19) are of second order and the trace equation (20) is

$$f'(R) R - 2f(R) = \kappa T,$$

(176)

where $R$ is the Ricci scalar of the non-metric connection $\Gamma_{\mu\nu}^\rho$ (and not that of the metric connection $\{g_{\mu\nu}\}$ of $g_{\mu\nu}$). Contrary to the metric case, eq. (176) is not an evolution equation for $R$; it is not even a differential equation, but rather an algebraic equation in $R$ once the function $f(R)$ is specified. This is also the case in GR, in which the Einstein field equations are of second order and taking their trace yields $R = -\kappa T$. Accordingly, the scalar field $\phi$ of the equivalent $\omega_0 = -3/2$ Brans–Dicke theory is not dynamical. Therefore, the Dolgov-Kawasaki instability and (19) are of second order and the trace equation (20) is specified. This is also the case in GR, in which the

4. Ghost fields

Ghost fields (massive states of negative norm that cause apparent lack of unitarity) appear easily in higher order gravities. A viable theory should be ghost-free: the presence of ghosts in $f(R)$ gravity (Sotiriou, 2007a).

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C. The Cauchy problem

A physical theory must have predictive power and, to this extent, a well-posed initial value problem is a required feature. GR satisfies this requirement for most reasonable forms of matter (Wald, 1984). The well-posedness of the Cauchy problem for $f(R)$ gravity is an open issue. Using harmonic coordinates, Noakes showed that theories with action

$$S = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} \left( R + \alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2 \right) + S_M$$

(177)

in the metric formalism have a well posed initial value problem in vacuo (Noakes, 1983). By using the dynamical equivalence with the scalar-tensor theory (54) when $f''(R) \neq 0$, the well-posedness of the Cauchy problem can be reduced to the analogous problem for Brans–Dicke gravity with $\omega_0 = 0$ (metric formalism) or $\omega_0 = -3/2$ (Palatini formalism). The fact that the initial value problem is well-posed was demonstrated for particular scalar-tensor theories in (Cocke and Cohen, 1968; Noakes, 1983) and a general analysis has recently been presented in (Salgado, 2006; Salgado et al., 2008). This work, however, does not cover the $\omega_0 = 0, -3/2$ cases.

A system of $3 + 1$ equations of motion is said to be well-formulated if it can be rewritten as a system of equations which are of only first order in both time and space derivatives. When this set can be put in the full first order form

$$\partial_i \bar{u} + M^i \nabla_i \bar{u} = \tilde{S} (\bar{u}),$$

(178)

where $\bar{u}$ collectively denotes the fundamental variables $h_{ij}, K_{ij}, \text{etc.}$ introduced below, $M^i$ is called the characteristic matrix of the system, and $\tilde{S} (\bar{u})$ describes source terms and contains only the fundamental variables but not their derivatives. The initial value formulation is well-posed if the system of partial differential equations is symmetric hyperbolic (i.e., the matrices $M^i$ are symmetric) and strongly hyperbolic if $s_i M^i$ has a real set of eigenvalues and a complete set of eigenvectors for any 1-form $s_i$, and obeys some boundedness conditions [see Solin, 2006].

The Cauchy problem for metric $f(R)$ gravity is well-formulated and is well-posed in vacuo and with matter, as shown below. For Palatini $f(R)$ gravity, instead, the Cauchy problem is unlikely to be well-formulated or well-posed unless the trace of the matter energy-momentum tensor is constant, due to the presence of higher derivatives of the matter fields in the field equations and to the impossibility of eliminating them (see below).

A systematic covariant approach to scalar-tensor theories of the form

$$S = \int d^4 x \sqrt{-g} \left[ \frac{\psi(\phi) R}{2\kappa} - \frac{1}{2} \partial^\alpha \phi \partial_\alpha \phi - W(\phi) \right] + S_M$$

(179)

is due to (Saldagá, 2000), who showed that the Cauchy problem of these theories is well-posed in the absence of

Furthermore, $R_{\mu\nu} R^{\mu\nu}$ can be expressed in terms of $R^2$ in a FLRW background (Wands, 1994).
matter and well-formulated otherwise. With the exception of \( \omega_0 = -3/2 \), as we will see later, most of Salgado’s results can be extended to the more general action

\[
S = \int d^4 x \sqrt{-g} \left[ \frac{\psi'(\phi) R - \omega(\phi)}{2} \partial^\alpha \phi \partial_\alpha \phi - W(\phi) \right] + S_M,
\]

which contains the additional coupling function \( \omega(\phi) \) (which is different from the Brans–Dicke parameter \( \omega_0 \)) (Lanahan-Tremblay and Faraoni, 2007).

The field equations, after setting \( \kappa = 1 \) for this section, are

\[
G_{\mu\nu} = \frac{1}{\psi} \left[ \psi'' (\nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi) + \psi' (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi) \right] + \frac{1}{\psi^2} \left[ \omega (\nabla_\mu \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi) - W(\phi) g_{\mu\nu} + T^{(m)}_{\mu\nu} \right],
\]

where a prime denotes differentiation with respect to \( \phi \). Eq. (183) can be cast in the form of the effective Einstein equation \( G_{\mu\nu} = T^{(eff)}_{\mu\nu} \), with the effective stress-energy tensor (Salgadó, 2006)

\[
T^{(eff)}_{\mu\nu} = \frac{1}{\psi(\phi)} \left( T^{(\psi)}_{\mu\nu} + T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu} \right),
\]

and

\[
T^{(\psi)}_{\mu\nu} = \psi'' (\nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \nabla^\beta \phi \nabla_\beta \phi) + \psi' (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi),
\]

\[
T^{(\phi)}_{\mu\nu} = \omega (\nabla_\mu \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi) - W(\phi) g_{\mu\nu}.
\]

The trace of the effective Einstein equations yields

\[
\Box \phi = \left\{ \psi \left[ \omega + \frac{3 (\psi')^2}{2 \psi} \right] \right\}^{-1} \left\{ \frac{\psi' T^{(m)}}{2} - 2 \psi W(\phi) \right\} \psi' W(\phi) + \psi' W(\phi) \psi'' (\omega + 3 \psi') \nabla^\nu \phi \nabla_\nu \phi.
\]

The 3 + 1 Arnowitt–Deser–Misner (ADM) formulation of the theory proceeds by introducing lapse, shift, extrinsic curvature, and gradients of \( \phi \) (Reula, 1998; Salgadó, 2006; Wald, 1984). Assume that a time function \( t \) exists such that the spacetime \( (M, g_{\mu\nu}) \) admits a foliation with hypersurfaces \( \Sigma_t \) of constant \( t \) with unit timelike normal \( n^\alpha \). The 3-metric and projection operator on \( \Sigma_t \) are \( h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \) and \( h^{\alpha\beta} \), respectively. Moreover,

\[
n^\alpha n_\mu = -1, \quad h_\alpha^\beta n_\gamma = h_\alpha^\gamma, \quad n^\alpha = 0, \quad h_\alpha^\beta h_\beta^\gamma = h_\alpha^\gamma.
\]

The metric is then

\[
ds^2 = -(N^2 - N^i N_i) dt^2 - 2 N_i dt dx^i + h_{ij} dx^i dx^j
\]

(188)

\[(i, j = 1, 2, 3), \text{ where } N > 0, \quad n_\alpha = -N \nabla_\alpha t, \quad N^\alpha = -h_\alpha^\beta t^\beta \text{ is the shift vector, while } t^\alpha = -N^\alpha + N n^\alpha \text{ so that } N = -n_\alpha t^\alpha \text{ and } N^\alpha n_\alpha = 0.
\]

The extrinsic curvature of \( \Sigma_t \) is

\[
K_{\alpha\beta} = -h^\gamma_\alpha h^\delta_\beta \nabla_\gamma n_\delta
\]

(189)

and the 3D covariant derivative of \( h_{\alpha\beta} \) on \( \Sigma_t \) is defined by

\[
D_{\alpha\beta} = \delta_{\alpha\beta} - h_{\alpha1} h_{\beta2} - h_{\alpha2} h_{\beta1} - h_{\alpha3} h_{\beta3} - h_{\alpha3} h_{\beta3} \equiv \eta_{\alpha\beta}
\]

(190)

for any 3-tensor \( (3) T^{\alpha\beta\gamma} ... \). With \( D_{\alpha\beta} h_{\gamma\delta} = 0 \). The spatial gradient of the scalar \( \phi \) is \( \phi_\mu \equiv D_\mu \phi \) (where \( D_\mu \) denotes the covariant derivative of \( h_{\mu\nu} \), while its momentum is \( \Pi = N_\alpha \phi = n^\nu \nabla_\nu \phi \)

\[
K_{ij} = -\nabla_i n_j = \frac{1}{2N} \left( \frac{\partial h_{ij}}{\partial t} + D_i N_j + D_j N_i \right),
\]

(191)

\[
\Pi = \frac{1}{N} \left( \partial_\alpha \phi + N^\gamma Q_\gamma \right),
\]

(192)

\[
\partial_i Q_i + N^\gamma \partial_\gamma Q_i + Q_\mu \partial_\mu N^\psi = D_\gamma (N^\mu).
\]

(193)

The effective stress-energy tensor \( T^{(eff)}_{\alpha\beta} \) is decomposed as

\[
T^{(eff)}_{\alpha\beta} = \frac{1}{\psi} \left( S_{\alpha\beta} + J_{\alpha} n_{\beta} + J_{\beta} n_{\alpha} + E_{\alpha} n_{\beta} \right),
\]

(194)

where

\[
S_{\alpha\beta} \equiv h^\gamma_\alpha h^\delta_\beta T^{(eff)}_{\gamma\delta} = \frac{1}{\psi} \left( S_{\alpha\beta}^{(\psi)} + S_{\alpha\beta}^{(\phi)} + S_{\alpha\beta}^{(m)} \right),
\]

(195)

\[
J_{\alpha} \equiv -h^\gamma_\alpha T^{(eff)}_{\gamma\delta} n_\delta = \frac{1}{\psi} \left( J_{\alpha}^{(\psi)} + J_{\alpha}^{(\phi)} + J_{\alpha}^{(m)} \right),
\]

(196)

\[
E \equiv n^\alpha n^\beta T^{(eff)}_{\alpha\beta} = \frac{1}{\psi} \left( E^{(\psi)} + E^{(\phi)} + E^{(m)} \right).
\]

(197)

Its trace is \( T^{(eff)} = S - E \), where \( S \equiv S^{\mu\nu} \). The Gauss–Codazzi equations then yield the Hamiltonian constraint (Salgadó, 2006; Wald, 1984)

\[
(3) R + K^2 - K_{ij} K^{ij} = 2E,
\]

(198)

the vector constraint

\[
D_1 K_{1i} - D_i K = J_i,
\]

(199)

and the dynamical equations

\[
\partial_i K^{ij} + N^i \partial_j K^{ij} + K^i \partial_j N^i - K^{ij} \partial_i N^j + D^i D_j N^i - (3) R^i_j N - NK K^i_j = \frac{N}{2} \left[ (S - E) \delta^i_j - 2S^i_j \right],
\]

(200)

where \( K \equiv K^{ij} \). The trace of eq. (200) yields

\[
\partial_i K + N^i \partial_k K + (3) \Delta N - NK K^{ij} = \frac{N}{2} (S - E),
\]

(201)
The Hamiltonian and the vector constraints become

\[
E^{(v)} = \psi' (D_v Q_\mu + K \Pi) + \psi'' Q^2, \\
J^{(v)}_\mu = -\omega \Pi Q_\mu, \\
S^{(v)}_{\alpha\beta} = \psi' (D_\alpha Q_\beta + \Pi K_{\alpha\beta} - h_{\alpha\beta} \Box \phi) - \psi'' [h_{\alpha\beta} (Q^2 - \Pi^2) - Q_\alpha Q_\beta],
\]

where \( Q^2 \equiv Q^\mu Q_\mu \), while

\[
S^{(\phi)} = \psi' (D_v Q^\nu + K \Pi - 3 \Box \phi) + \psi'' (3 \Pi^2 - 2 Q^2),
\]

and

\[
E^{(\phi)} = \frac{\omega}{2} (\Pi^2 + Q^2) + W(\phi), \\
J^{(\phi)}_\mu = -\omega \Pi Q_\mu, \\
S^{(\phi)}_{\alpha\beta} = \omega Q_\alpha Q_\beta - h_{\alpha\beta} \left[ \frac{\omega}{2} (Q^2 - \Pi^2) + W(\phi) \right], \\
S^{(\phi)} = \frac{\omega}{2} (3 \Pi^2 - 2 Q^2) - 3 W(\phi).
\]

The Hamiltonian and the vector constraints become

\[
(3) \left[ R + K^2 - K_{ij} K^{ij} - \frac{2}{\psi} \left[ \psi' (D_\nu Q^\mu + K \Pi) + \omega \Pi^2 \right] + \frac{Q^2}{2} (\omega + 2 \psi'') \right] = \frac{2}{\psi} \left( E^{(m)} + W(\phi) \right),
\]

\[
D_i K^i - D_i K + \frac{1}{\psi} \left[ \psi' (K_i Q_e + D_i \Pi) + (\omega + \psi'') \Pi Q_i \right] = \frac{j^{(m)}}{\psi},
\]

respectively, and the dynamical equation (200) is

\[
\partial_t K + N^i \partial_i K + N^i \partial_i N^j - K^j_i \partial_i N^i + D^i D_j N - (3) R^j_i j - N K K^j_i + \frac{N}{2 \psi} \left[ \psi'' (Q^2 - \Pi^2) + 2 W(\phi) + \psi' \Box \phi \right] \delta_j^i + \frac{N \psi'}{\psi} \left( D^i_j + \Pi K^i_j \right) \frac{N}{\psi} (\omega + \psi'') Q^i Q_j = \frac{N}{2 \psi} \left[ \left( S^{(m)} - E^{(m)} \right) \delta_j^i - 2 S^{(m)} \right] \delta_j^i,
\]

with trace

\[
\partial_t K + N^i \partial_i K + (3) \Delta N - N K_{ij} K^{ij} - \frac{N \psi'}{\psi} (D^\mu Q_\mu + \Pi K) + \frac{N}{2 \psi} \left[ \psi'' Q^2 - (2 \omega + 3 \psi'') \Pi^2 \right] = \frac{N}{2 \psi} \left( -2 W(\phi) - 3 \psi' \Box \phi + S^{(m)} + E^{(m)} \right)
\]

where \( \Box \phi \) is present, which is present in eqs. (193-197), which contains only first-order derivatives in both space and time once the d’Alembertian \( \Box \phi \) is written in terms of \( \phi, \nabla^\mu \phi \nabla_\mu \phi, \psi, \) and its derivatives by means of eqs. (189) or (214). As mentioned earlier, this can be done whenever \( \omega \neq -3 (\psi^2) / (2 \psi) \).
with
\[
\left(\omega_0 + \frac{3}{2}\right) \square \phi = \frac{T^{(m)}}{2} - 2V(\phi) + \phi V'(\phi) + \frac{\omega_0}{\phi} (\Pi^2 - Q^2).
\]  
(219)

The condition \( \omega \neq -3(\psi')^2/(2\psi) \), which needs to be satisfied in order for one to be able to use eq. \(189\) in order to eliminate \( \square \phi \) can be written in the Brans–Dicke theory notation as \( \omega_0 \neq -3/2 \). One could of course had guessed that by looking at eq. \(219\). Therefore, metric \( f(R) \) gravity, which is equivalent to \( \omega_0 = 0 \) Brans–Dicke gravity, has a well-formulated Cauchy problem in general and is well-posed in vacuo. Further work by [Olmo and Singh, 2009], considering that Palatini gravity have a well-formulated and well-posed Cauchy problem for scalar-tensor gravity with \( \omega = 1 \) in the presence of matter; this can be translated into the well-posedness of metric \( f(R) \) gravity with matter along the lines established above.

How about Palatini \( f(R) \) gravity, which, corresponding to \( \omega_0 = -3/2 \), is exactly the case that the constraint \( \omega_0 \neq -3/2 \) excludes? Actually, for this value of the Brans–Dicke parameter, eq. \(69\), and consequently at eq. \(219\), include no derivatives of \( \phi \). Therefore, one can actually solve algebraically for \( \phi \). [The same could be done using eq. \(189\) in the more general case where \( \omega \) is a function of \( \phi \), when \( \omega = -3(\psi')^2/(2\psi) \).] We will not consider cases for which eq. \(69\) has no roots or when it is identically satisfied in vacuo. These cases lead to theories for which, in the Palatini \( f(R) \) formulation, eq. \(219\) has no roots or when it is identically satisfied in vacuo respectively. As already mentioned in Sec. II.B the first case leads to inconsistent field equations, and the second to a conformally invariant theory (Ferraris et al. 1992), see also (Sotiriou 2006a) for a discussion.

Now, in vacuo one can easily show that the solutions of eq. \(69\) or \(219\) will be of the form \( \phi = \text{constant} \). Therefore, all derivatives of \( \phi \) vanish and one conclude that \( \omega_0 = -3/2 \) Brans–Dicke theory or Palatini \( f(R) \) gravity have a well-formulated and well-posed Cauchy problem.\(^{25}\) This could have been expected, as noticed in [Olmo and Singh, 2009], considering that Palatini \( f(R) \) gravity reduces to GR with a cosmological constant in vacuo.

In the presence of matter, things are more complicated. The solutions of eq. \(69\) or \(219\) will give \( \phi \) as a function of \( T \), the trace of the stress-energy tensor. This can still be used to replace \( \phi \) in all equations but it will lead to terms such as \( \square \phi \). Therefore, for the Cauchy problem to be well-formulated in the presence of matter, one does not only have to assume that the matter is “reasonable”, in the sense that the matter fields satisfy a quasilinear, diagonal, second order hyperbolic system of equations [see (Wald, 1984)], but also to require that the matter field equations are such that allow to express all derivatives of \( T \) present in eq. \(219\) to \(218\) for \( \omega_0 = -3/2 \) in terms of only first derivatives of the matter fields. It seems highly implausible that this requirement can be fulfilled for generic matter fields. This seems to imply that \( \omega_0 = -3/2 \) Brans–Dicke theory and Palatini \( f(R) \) gravity are unlikely to have a well-formulated Cauchy problem in the presence of matter fields. However, more precise conclusions can only be drawn if specific matter fields are considered on a case by case basis. The complication arising from the appearance of derivatives of \( T \), and consequently higher derivatives of the matter fields in the equations, and seem to be critical on whether the Cauchy problem can be well-formulated in the presence of matter will be better understood in Sec. VI.C.2.

VI. CONFRONTATION WITH PARTICLE PHYSICS AND ASTROPHYSICS

A. Metric \( f(R) \) gravity as dark matter

Although most recent motivation for \( f(R) \) gravity originates from the need to find alternatives to the mysterious dark energy at cosmological scales, several authors adopt the same perspective at galactic and cluster scales, using metric \( f(R) \) gravity as a substitute for dark matter (Capozziello et al. 2004, 2005a, 2005b, 2007a, 2006b, 2007b; Iorio and Ruggeri, 2007a,b; Jhingan et al. 2008; Martin and Sahni, 2007; Nojiri and Odintsov, 2008a; Saffari and Sobouti, 2007; Zhao and Li, 2008). Given the equivalence between \( f(R) \) and scalar-tensor gravity, these efforts resemble previous attempts to model dark matter using scalar fields (Alcubierre et al. 2002a,b; Bernal and Guzmán, 2006a,b, 2007a,b; Bernal and Mateo, 2005; Cervantes-Cota et al. 2007a,b, 2008a; Matos and Guzmán, 2006; Matos et al., 2004; Matos and Urena-Lopez, 2001, 2007; Rodriguez-Meza and Cervantes-Cota, 2004; Rodriguez-Meza et al., 2005, 2007).

Most works concentrate on models of the form \( f(R) = R^\alpha \). A theory of this form with \( n = 1 - \alpha/2 \) was studied in [Mendoza and Rosas-Guevara, 2007; Saffari and Sobouti, 2007] by using spherically symmetric solutions to approximate galaxies. The fit to galaxy samples yields

\[
\alpha = (3.07 \pm 0.18) \times 10^{-17} \left( \frac{M}{10^{10} M_\odot} \right)^{0.494},
\]  
(220)

where \( M \) is the mass appearing in the spherically symmetric metric (the mass of the galaxy). Notice that having \( \alpha \) depending on the mass of each individual galaxy straightforwardly implies that one cannot fit the data for all galactic masses with the same choice of \( f(R) \). This make the whole approach highly implausible.

\(^{25}\) This has been missed in [Lanahan-Tremblay and Faraoni, 2007], where it is claimed that the Cauchy problem is not well-posed because the constraint \( \omega_0 \neq -3/2 \) does not allow for the use of eq. \(219\) in order to eliminate \( \square \phi \). Note also that in the absence of a potential (there is no corresponding Palatini \( f(R) \) gravity) \( \omega_0 = -3/2 \) Brans–Dicke theory does not have a well-posed Cauchy problem, as noticed in [Noak 1984], because this theory is conformally invariant and \( \phi \) is indeterminate.
(Capozziello et al. 2004, 2005a, 2006a, 2007a,b) computed weak-field limit corrections to the Newtonian galactic potential and the resulting rotation curves; when matched to galaxy samples, a best fit yields \( n \approx 1.7 \). Martin and Salucci (2007) performed a \( \chi^2 \) fit using two broader samples, finding \( n \approx 2.2 \) [see also (Boehmer et al. 2007a) for a variation of this approach focusing on the constant velocity tails of the rotation curves]. All these values of the parameter \( n \) are in violent contrast with the bounds obtained by Barrow and Clifton (2006), Clifton and Barrow (2005a), Zakharov et al. (2006) and have been shown to violate also the current constraints on the precession of perihelia of several Solar System planets (Iorio and Ruggiero 2007a,b). In addition, the consideration of vacuum metrics used in these works in order to model the gravitational field of galaxies is highly questionable.

The potential obtained in the weak-field limit of \( f(R) \) gravity can affect other aspects of galactic dynamics as well: the scattering probability of an intruder star and the relaxation time of a stellar system were studied by Hadjimichef and Kokubun (1997), originally motivated by quadratic corrections to the Einstein–Hilbert action.

### B. Palatini \( f(R) \) gravity and the conflict with the Standard Model

One very important and unexpected shortcoming of Palatini \( f(R) \) gravity is that it appears to be in conflict with the Standard Model of particle physics, in the sense that it introduces non-perturbative corrections to the matter action (or the field equations) and strong couplings between gravity and matter in the local frame and at low energies. The reason why we call this shortcoming unexpected is that, judging by the form of the action (13), Palatini \( f(R) \) gravity is, as we mentioned, a metric theory of gravity in the sense that matter is only coupled minimally to the metric. Therefore, the stress energy tensor is divergence-free with respect to the metric covariant derivative, the metric postulates (Will 1981) are fulfilled, the theory apparently satisfies the Einstein Equivalence principle, and the matter action should trivially reduce to that of Special Relativity locally.

Let us see how this conflict comes about. This issue was first pointed out in (Flanagan 2004b) using Dirac particles for the matter action as an example, and later on studied again in (Iglesias et al. 2007) by assuming that the matter action is that of the Higgs field [see also (Olmo 2008)]. Both calculations use the equivalent Brans–Dicke theory and are performed in the Einstein frame. Although the use of the Einstein frame has been criticized (Vollick 2004), this frame is equivalent to the Jordan frame and both are perfectly suitable for performing calculations (Flanagan 2004a) [see also the relevant discussion in Sec. III and (Faraoni and Nadeau 2007, Sotiriou et al. 2007)].

Nevertheless, since test particles are supposed to follow geodesics of the Jordan frame metric, it is this metric which becomes approximately flat in the laboratory reference frame. Therefore, when the calculations are performed in the Einstein frame they are less transparent since the actual effects could be confused with frame effects and vice-versa. Consequently, for simplicity and clarity, we present the calculation in the Jordan frame, as it appears in (Barausse et al. 2008b). We begin from the action (66), which is the Jordan frame equivalent of Palatini \( f(R) \) gravity, and we take matter to be represented by a scalar field \( H \) (e.g., the Higgs boson), the action of which reads

\[
S_M = \frac{1}{2\hbar} \int d^4x \sqrt{g} \left( g^{\mu\nu} \partial_\mu H \partial_\nu H - \frac{m_H^2}{\hbar^2} H^2 \right) \tag{221}
\]

(in units in which \( G = c = 1 \)). As an example, we choose \( f(R) = R - \mu^4/R \) (Carroll et al. 2004, Vollick 2003). For this choice of \( f \), the potential is \( V(\phi) = 2\mu^2(\phi - 1)^{1/2} \). To go to the local frame, we want to expand the action to second order around vacuum. The vacuum of the action (66) with (221) as a matter action is \( H = 0, \phi = 4/3 \) [using eq. (69)] and \( \eta_{\mu\nu} \simeq \eta_{\mu\nu} \) (\( \mu^2 \sim \Lambda \) acts as an effective cosmological constant, so its contribution in the local frame can be safely neglected).

However, when one tries to use a perturbative expansion for \( \phi \), things stop being straightforward: \( \phi \) is algebraically related to the matter fields as is obvious from eq. (69). Therefore, one gets \( \delta\phi \sim T/\mu^2 \sim \beta_1^2/4 \beta_3^2/12 \) at energies lower than the Higgs mass (\( m_H \sim 100 - 1000 \) GeV). Replacing this expression in the action (66) perturbed to second order, one immediately obtains that the effective action for the Higgs scalar is

\[
S_M^{\text{effective}} \simeq \int d^4x \sqrt{-g} \left[ \frac{1}{2\hbar} \left( g^{\mu\nu} \partial_\mu \delta H \partial_\nu \delta H - \frac{m_H^2}{\hbar^2} \delta H^2 \right) \right. \times \left[ 1 + \frac{m_H^2}{\beta_1^2} \frac{\delta H^2}{\mu^2 \hbar^3} + \frac{m_H^2 (\delta H)^2}{\beta_3^2 \mu^4 \hbar^3} \right] \tag{222}
\]

at energies \( E \ll m_H \). Taking into account the fact that \( \mu^2 \sim \Lambda \sim H_0^2 \), where \( H_0 = 4000\) Mpc is the Hubble radius and \( \delta H \sim m_H \) because \( E \ll m_H \), it is not difficult to estimate the order of magnitude of the corrections: at an energy \( E = 10^{-3} \) eV (corresponding to the length

\[ \sim 10^{27} \text{m} \](see also the relevant discussion in Sec. III and (Faraoni and Nadeau 2007, Sotiriou et al. 2007)).

\[ \sim 10^{27} \text{m} \]

This is not at all in agreement with the predictions of the Standard Model and of cosmology at low energy. Consequently, the Jordan frame potential must be represented as an effective potential, e.g.,

\[ \phi(\delta H) \sim 4/3 + \beta_1^2/4 \beta_3^2/12 \frac{\delta H^2}{\mu^2 \hbar^3} + \frac{m_H^2}{\beta_3^2 \mu^4 \hbar^3} (\delta H)^2 \]

Note that in the case of (Flanagan 2004b) in which fermions are used as the matter fields, one could decide to couple the indepen-
scale \( L = \hbar/E = 2 \times 10^{-4} \) m, the first correction is of the order \( m_\text{H}^2 \delta H^2/(\mu^2 \hbar^3) \sim (H_0^2/\lambda_\text{H})^2(m_\text{H}/M_\text{P})^2 \gg 1 \), where \( \lambda_\text{H} = \hbar/m_\text{H} \sim 2 \times 10^{-19} - 2 \times 10^{-18} \) m is the Compton wavelength of the Higgs and \( M_\text{P} = (h \delta^2/G)^{1/2} = 1.2 \times 10^{19} \) GeV is the Planck mass. The second correction is of the order \( m_\text{H}^2 (\partial^2 \delta H^2)/(\mu^4 \hbar^3) \sim (H_0^{-1}/\lambda_\text{H})^2(m_\text{H}/M_\text{P})^2(H_0^{-1}/L)^2 \gg 1 \). Clearly, having such non-perturbative corrections to the local frame matter action is unacceptable.

An alternative way to see the same problem would be to replace \( \delta \phi \sim m_\text{H}^2 \delta H^2/(\hbar^2 \mu^2) \) directly in (6). Then the coupling of matter to gravity is described by the interaction Lagrangian

\[
\mathcal{L}_{\text{int}} \sim \frac{m_\text{H}^2 \delta H^2}{\hbar^3} \left( \delta g + \partial_\alpha \delta g \partial_\beta \right) \\
\sim \frac{m_\text{H}^2 \delta H^2}{\hbar^3} \delta g \left[ 1 + \left( \frac{H_0^{-1}}{L} \right)^2 \right].
\]

(223)

This clearly exhibits the fact that gravity becomes non-perturbative at microscopic scales.

It is obvious that the algebraic dependence of \( \phi \) on the matter fields stands at the root of this problem. We have still not given any explanation for the “paradox” of seeing such a behavior in a theory which apparently satisfies the metric postulates both in the \( f(R) \) and the Brans–Dicke representation. However, this will become clear in Sec. VI.C.2.

C. Exact solutions and relevant constraints

1. Vacuum and non-vacuum exact solutions

Let us now turn our attention to exact solutions starting from metric \( f(R) \) gravity. We have already mentioned in Sec. IIA that, as can be seen easily from the form of the field equations (4), the maximally symmetric vacuum solution will be either Minkowski spacetime, if \( R = 0 \) is a root of eq. (9), or de Sitter and anti-de Sitter spacetime, depending on the sign of the root of the same equation. Things are slightly more complicated for vacuum solutions with less symmetry: by using eq. (9) it is easy to verify that any vacuum solution \( (\mathcal{R}_\mu = \Lambda g_\mu, \mathcal{T}_\mu = 0) \) of Einstein’s theory with a (possibly vanishing) cosmological constant, including black hole solutions, is a solution of metric \( f(R) \) gravity (except for pathological cases for which eq. (9) has no roots). However, the converse is not true.

For example, when spherical symmetry is imposed, the Schwarzschild metric is a solution of metric \( f(R) \) gravity if \( R = 0 \) in vacuum. If \( R \) is constant in vacuum, then Schwarzschild-(anti-)de Sitter spacetime is a solution. As we have already mentioned though, the Jeeves-Birkhoff theorem (Wald, 1984; Weinberg, 1972) does not hold in metric \( f(R) \) gravity unless, of course one wishes to impose further conditions, such as that \( R \) is constant (Capozziello et al., 2007b). Therefore, other solutions can exist as well. An interesting finding is that the cosmic no-hair theorem valid in GR and in pure \( f(R) \) gravity is not valid, in general, in theories of the form

\[
S = \frac{1}{2K} \int d^4x \sqrt{-g} \left( R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda \right),
\]

(224)

for which exact anisotropic solutions that continue to inflate anisotropically have been found (Barrow and Hervik, 2006a,b) [see also (Kluske and Schmidt, 1996; Maeda, 1988)]. However, isotropization during inflation occurs in mixed \( f(\phi, R) \) models (Maeda et al., 1989).

In addition to the exact cosmological solutions explored for the purpose of explaining the current cosmic acceleration [see, e.g., (Abdalla et al., 2005; Barrow and Clifton, 2006; Clifton, 2006a,b; Clifton and Barrow, 2005a,b, 2006) and (Capozziello and De Felice, 2008; Capozziello et al., 2007c, 2007d; Modak et al., 2005; Vakili, 2008) for an approach based on Noether symmetries; see (Carloni et al., 2006) for bouncing solutions and the conditions that they satisfy], exact spherically symmetric solutions of metric \( f(R) \) gravity have been explored in the literature, with most recent studies being motivated by the need to understand the weak-field limit of cosmologically-motivated theories.

Regarding non-vacuum solutions, the most common matter source is a perfect fluid. Fluid dynamics in metric \( f(R) \) gravity was studied in (Maartens and Taylor, 1994; Mohseni Sadjadi, 2007; Rippl et al., 1996; Taylor and Maartens, 1995). Spherically symmetric solutions were found in (Bromnikov and Chernikova, 2005a,b; Bustelo and Barraco, 2007; Capozziello et al., 2007c, 2007d; Clifton, 2006a,b; Mignemi and Wiltshire, 1992; Miltamaki and Vilja, 2006b, 2007a; Whitt, 1985). We regret not being able to present these solutions here due to space limitations and refer the reader to the literature for more details.

Stability issues for spherically symmetric solutions were discussed in (Seferis, 2004). In the theory

\[
S = \int d^4x \sqrt{-g} \left[ R - \alpha R^2 - \beta R_{\mu\nu}R^{\mu\nu} \right] + \epsilon \chi,
\]

(225)

where \( \alpha, \beta, \) and \( \epsilon \) are constants and \( \chi \) is the Gauss–Bonnet invariant, the Schwarzschild metric is a solution and the stability of Schwarzschild black holes was studied in (Whitt, 1985). Surprisingly, it was found that the massive ghost graviton (“poltergeist”) present in this theory stabilizes small mass black holes against quantum instabilities [see also (Mizers and Simon, 1988, 1989)]. In the case \( \beta = \epsilon = 0 \), which reduces the theory (225) to a quadratic \( f(R) \) gravity, the stability criterion found in (Whitt, 1985) reduces to \( \alpha < 0 \), which corresponds again to \( f''(R) > 0 \). For \( \alpha = 0 \) we recover GR, in which black holes are stable classically (but not quantum-mechanically, due to Hawking radiation and their negative specific heat, a feature that persists in \( f(R) \) gravity),
so the classical stability condition for Schwarzschild black holes can be enunciated as $f''(R) \geq 0$.

Let us now turn our attention to Palatini $f(R)$ gravity. In this case things are simpler in vacuo: as we saw in Sec. 11.13 the theory reduces in this case (or more precisely even for matter fields with $T = \text{const.}$, where $T$ is the trace of the stress energy tensor) to GR with a cosmological constant, which might as well be zero for some models [Barraco et al., 1998, Borowiec et al., 1998, Ferraris et al., 1992, 1994]. Therefore, it is quite straightforward that Palatini $f(R)$ gravity will have the same vacuum solutions as GR with a cosmological constant. Also, the Jebsen-Birkhoff theorem (Wald, 1984; Weinberg, 1972) is valid in the Palatini formalism [Barraso et al., 2008a,b, Kainulainen et al., 2007b].

Cosmological solutions in quadratic gravity were obtained in [Shahid-Saless, 1990, 1991]. Spherically symmetric interior solutions in the Palatini formalism can be found by using the generalization of the Tolman-Oppenheimer-Volkoff equation valid for these theories, which was found in [Barraco and Hamity, 2000, Bustelo and Barraco, 2007, Kainulainen et al., 2007b]. Indeed, such solutions have been found and matched with the unique exterior (anti-)de Sitter solution [Barraco and Hamity, 1998, 2004, Bustelo and Barraco, 2007, Kainulainen et al., 2007a,b]. Nevertheless, a matching between exterior and interior which can lead to a sensible solution throughout spacetime is not always feasible and this seems to have serious consequences for the viability of $f(R)$ gravity [Barraso et al., 2008a,b, 2009]. This is discussed extensively in the next section.

Let us close this section with some remarks on black hole solutions. As is clear from the above discussion, all black hole solutions of GR (with a cosmological constant) will also be solutions of $f(R)$ in both the metric and the Palatini formalism [see also Barausse and Sotiriou, 2008, Psaltis et al., 2008]. However, in the Palatini formalism they will constitute the complete set of black hole solutions of the theory, whereas in the metric formalism other black hole solutions can exist in principle, as the Jebsen-Birkhoff theorem does not hold. For a discussion on black hole entropy in $f(R)$ gravity see Jacobson et al., 1994, 1995, Vollick, 2007.27

2. Surface singularities and the incompleteness of Palatini $f(R)$ gravity

In Secs. 11.11, 11.12 and 11.13 we already spotted three serious shortcomings of Palatini $f(R)$ gravity, namely the algebraic dependence of the post-Newtonian metric on the density, the complications with the initial value problem in the presence of matter, and a conflict with particle physics. In this section we will study static spherically symmetric interior solutions and their matching to the unique exterior with the same symmetries, the Schwarzschild-de Sitter solution, along the lines of [Barausse et al., 2008a,b,c]. As we will see, the three problems mentioned earlier are actually very much related and stem from a very specific characteristic of Palatini $f(R)$ gravity, which the discussion of this section will help us pin down.

A common way of arriving to a full description of a spacetime which includes matter is to solve separately the field equations inside and outside the sources, and then match the interior and exterior solutions using appropriate junction conditions [called Israel junction condition in GR [Israel, 1966]]. This is what we are going to attempt here. We already know the exterior solution so, for the moment, let us focus on the interior. Since we assume that the metric is static and spherically symmetric, we can write it in the form

$$ds^2 = -e^{A(r)}dt^2 + e^{B(r)}dr^2 + r^2d\Omega^2.$$  \hspace{1cm} (226)

We can then replace this metric in the field equations of Palatini $f(R)$ gravity, preferably in eq. (225) which is the simplest of all the possible reformulations. Assuming also a perfect fluid description for matter with $\Gamma_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$, where $\rho$ is the energy density, $P$ is the pressure, and $u^\mu$ is the fluid 4-velocity, and representing $d/dr$ with a prime,28 one arrives at the equations

$$A' = \frac{-1}{1 + \gamma} \left( 1 - e^B \frac{r}{F} - e^B 8\pi G r P \frac{\alpha}{F} \right),$$  \hspace{1cm} (227)

$$B' = \frac{1}{1 + \gamma} \left( 1 - e^B \frac{r}{F} + e^B 8\pi G r \rho \frac{\alpha + \beta}{F} \right),$$  \hspace{1cm} (228)

$$\alpha = r^2 \left[ \frac{3}{4} \left( \frac{F'}{F} \right)^2 + 2F' \frac{F}{F} \right],$$  \hspace{1cm} (229)

$$\beta = r^2 \left[ \frac{F''}{F} - 3 \left( \frac{F'}{F} \right)^2 \right],$$  \hspace{1cm} (230)

where $F = \partial f/\partial R$. To determine an interior solution we need a generalization of the Tolman-Oppenheimer-Volkoff (TOV) hydrostatic equilibrium equation. This has been derived for Palatini $f(R)$ gravity in [Barraco and Hamity, 2004, Bustelo and Barraco, 2007, Kainulainen et al., 2007b]: Defining $m_{\text{tot}}(r) \equiv r(1 - e^{-B})/2$ and using Euler’s equation

$$P' = -\frac{A'}{2}(P + \rho),$$  \hspace{1cm} (231)

27 See also [Eling et al., 2000] for a derivation of the field equations of metric $f(R)$ gravity based on thermodynamical arguments applied to local Rindler horizons.

28 In this section we modify our standard notation for economy and a prime denotes differentiation with respect to the radial coordinate instead of differentiation with respect to the argument of the function.
one can use eqs. (227) and (228) to arrive to the generalised TOV equations:

\[ P' = -\frac{1}{1 + \gamma} \frac{(\rho + P)}{r(r - 2m_{\text{tot}})} \times \]

\[ \times \left( m_{\text{tot}} + \frac{4\pi r^3 P}{F} - \frac{\alpha}{2}(r - 2m_{\text{tot}}) \right), \]

\[ m'_{\text{tot}} = \frac{1}{1 + \gamma} \left( \frac{4\pi r^2 \rho}{F} + \frac{\alpha + \beta}{2} - \frac{m_{\text{tot}}}{r}(\alpha + \beta - \gamma) \right). \] (233)

We now have three differential equations, namely (231), (232) and (233), and four unknown functions, namely \( A \), \( m_{\text{tot}} \) (or \( B \)), \( P \), and \( \rho \). The missing piece is the information about the microphysics of the matter configuration under investigation. In the case of a perfect fluid, this is effectively given by an equation of state (EOS). A one-parameter EOS relates the pressure directly to the energy density, i.e., \( P = P(\rho) \). A simple form of such an EOS is a polytropic equation of state \( P = k\rho^\gamma \), where \( \rho_{0} \) is the rest-mass density and \( k \) and \( \Gamma \) are constants. This is the case that we will consider here. Note that the rest-mass density can be expressed in terms of the energy density \( \rho \) and the internal energy \( U \) as \( \rho_{0} = \rho - U \). Assuming an adiabatic transformation and using the first law of thermodynamics, one can express the internal energy in terms of the pressure, i.e., \( U = P/(\Gamma - 1) \). Therefore, the polytropic EOS can be rewritten as

\[ \rho = \left( \frac{P}{k} \right)^{1/\Gamma} + \frac{P}{\Gamma - 1}, \] (234)

giving a direct link between \( P \) and \( \rho \).

Without specifying the interior solution, we can already examine the appropriate matching conditions needed. One needs continuity of the metric and of \( A' \) on the surface of the matter configuration (\( A \) is given by a second order differential equation). Since we know that the exterior solution is unique and it is the Schwarzschild-de Sitter solution with a cosmological constant equal to \( \mathcal{R}_{0}/4 \), where \( \mathcal{R}_{0} \) is the vacuum value of \( \mathcal{R} \) (see Sec. II.B), we can directly write for the exterior

\[ \exp(-B(r)) = \ell \exp(A(r)) = 1 - 2m/r - \mathcal{R}_{0}r^2/12, \] (235)

where \( \ell \) and \( m \) are integration constants to be fixed by requiring continuity of the metric coefficients across the surface, which is implicitly defined by \( r = r_{\text{out}} \) where \( P = \rho = 0 \). Using the definition of \( m_{\text{tot}}(r) \) this gives, in the exterior,

\[ m_{\text{tot}}(r) = m + \frac{r^3}{24}\mathcal{R}_{0}. \] (236)

On the other hand, based on the exterior solution, one gets on the surface

\[ A'(r_{\text{out}}) = \frac{2(r_{\text{out}}^3\mathcal{R}_{0} - 12m)}{r_{\text{out}}(\mathcal{R}_{0}r_{\text{out}}^2 - 12r_{\text{out}} + 24m)}. \] (237)

Assuming that, approaching the surface from the interior, \( A \) and \( m_{\text{tot}} \) indeed take the correct values required for the matching, it can be shown that continuity of \( A' \) across the surface requires \( F'(r_{\text{out}}) = 0 \) for \( r \to r_{\text{out}} \) (Barausse et al., 2008a). Additionally, if this is the case then \( m_{\text{tot}}'(r_{\text{out}}) \) is

\[ m_{\text{tot}}'(r_{\text{out}}) = \frac{2F_{0}\mathcal{R}_{0}r_{\text{out}}^2 + (r_{\text{out}}^3\mathcal{R}_{0} - 8m_{\text{tot}})C'}{16F_{0}}, \] (238)

where

\[ C = \frac{dF}{dP}(P + \rho) = \frac{dF}{d\rho} \frac{d\rho}{dP}(P + \rho). \] (239)

Let us now examine the behavior of \( m_{\text{tot}}' \) at the surface for different values of the polytropic index \( \Gamma \). For \( 1 < \Gamma < 3/2 \), \( C' = dC/dP P' \propto d\mathcal{C}/dP (P + \rho) \to 0 \) at the surface so that the expression (238) is finite and it gives continuity of \( m_{\text{tot}}' \) across the surface [cf. eq. (236)]. However, for \( 3/2 < \Gamma < 2, \) \( C' \to \infty \) as the surface is approached, provided that \( dF/d\mathcal{R}(\mathcal{R}_{0}) \neq 0 \) and \( d\mathcal{R}/dT(T_{0}) \neq 0 \) (note that these conditions are satisfied by generic forms of \( f(\mathcal{R}) \), i.e., whenever an \( \mathcal{R}^2 \) term or a term inversely proportional to \( \mathcal{R} \) is present). Therefore, even though \( m_{\text{tot}} \) remains finite (as can be shown using the fact that \( P' = 0 \) at the surface), the divergence of \( m_{\text{tot}}' \) drives to infinity the Riemann tensor of the metric, \( \mathcal{R}_{\mu\nu\sigma\lambda} \), and curvature invariants, such as \( \mathcal{R} \) or \( \mathcal{R}^{\mu\nu\sigma\lambda}R_{\mu\nu\sigma\lambda} \), as can easily be checked.\(^{29}\) Clearly, such a singular behaviour is bound to give rise to unphysical phenomena, such as infinite tidal forces at the surface (cf. the geodesic deviation equation) which would destroy anything present there. We are, therefore, forced to conclude that no physically relevant solution exists for any polytropic EOS with \( 3/2 < \Gamma < 2 \).

The following points about the result just presented should be stressed:

- The sufficient condition for the singularity to occur is that a polytropic EOS with \( 3/2 < \Gamma < 2 \) should adequately describe just the outer layer of the matter configuration (and not necessarily the whole configuration).

- In practice, there is no dependence of the result on the functional form of \( f(\mathcal{R}) \) [a few unrealistic exceptions can be found in (Barausse et al., 2008a)], so what is revealed is a generic aspect of Palatini \( f(\mathcal{R}) \) gravity as a class of theories.

- The singularities discussed are not coordinate, but true singularities, as can be easily verified by checking that curvature invariants diverge.

\(^{29}\) This fact seems to have been missed in (Barraco and Hamity, 2004).
• The only assumptions made regard the EOS and the symmetries. Thus, the result applies to all regimes ranging from Newtonian to strong gravity.

Let us now interpret these results. Obviously, one could object to the use of the polytropic EOS. Even though it is extensively used for simple stellar models, it is not a very realistic description for stellar configurations. However, one does not necessarily have to refer to stars in order to check whether the issue discussed here leaves an observable signature. Consider two very well known matter configurations which are exactly described by a polytropic EOS: a monotonic isentropic gas and a degenerate non-relativistic electron gas. For both of those cases $\Gamma = 5/3$, which is well within the range for which the singularities appear. Additionally, both of these configuration can be very well described even with Newtonian gravity. Yet, Palatini $f(R)$ gravity fails to provide a reasonable description. Therefore, one could think of such matter configurations as gedanken experiments which reveal that Palatini $f(R)$ gravity is at best incomplete (Barausse et al., 2008a,b,c).

On the other hand, the use of the polytropic EOS requires a perfect fluid approximation for the description of matter. One may, therefore, wish to question whether the length scale on which the tidal forces become important is larger than the length scale for which the fluid approximation breaks down (Kainulainen et al., 2007a). However, quantitative results for tidal forces have been given in (Barausse et al., 2008a), and it has been shown that the length scales at which the tidal forces become relevant are indeed larger than it would be required for the fluid approximation to break down. The observable consequences on stellar configurations have also been discussed there. To this, one could also add that a theory which requires a full description of the microscopic structure of the system in order to provide a macroscopic description of the dynamics is not very appealing anyway.

In any case, it should be stated that the problem discussed is not specific to the polytropic EOS. The use of the latter only simplifies the calculation and allows an analytic approach. The root of the problem actually lies with the differential structure of Palatini $f(R)$ gravity.

Consider the field equations in the form (28): it is not difficult to notice that these are second order partial differential equations in the metric. However, since $f$ is a function of $\mathcal{R}$, and $\mathcal{R}$ is an algebraic function of $T$ due to eq. (20), the right hand side of eq. (28) includes second derivatives of $T$. Now, $T$, being the trace of the stress energy tensor, will include up to first order derivatives of the matter fields (assuming that the matter action has to lead to second order field equations when varied with respect to the matter fields). Consequently, eq. (28) can be up to third order in the matter fields!

In GR and most of its alternatives, the field equations are only of first order in the matter fields. This guarantees that gravity is a cumulative effect: the metric is generated by an integral over the matter sources and, therefore, any discontinuities (or even singularities) in the latter and their derivatives, which are allowed, will not become discontinuities or singularities of the metric, which are not allowed [see (Barausse et al., 2008b) for a detailed discussion]. This characteristic is not present in Palatini $f(R)$ gravity and creates an algebraic dependence of the metric on the matter fields.

The polytropic description not only does not cause this problem but, as a matter of fact, it makes it less acute than it is, simply because in the fluid approximation the stress-energy tensor does not include derivatives of the matter fields and effectively “smoothes out” the matter distribution. Actually, the fact that the metric is extremely sensitive to rapid changes of the matter field has been exhibited also in the interior of stars described by realistic tabulated EOSs in (Barausse et al., 2008a).

One should not be puzzled by the fact that this awkward differential structure of Palatini $f(R)$ gravity is not manifest in the $f(\mathcal{R})$ formulation of the theory (and the field eqs. (18) and (19)). We have already mentioned that the independent connection is actually an auxiliary field and the presence of auxiliary fields can always be misleading when it comes to the dynamics. In fact, it just takes a closer look to realize that the Palatini $f(R)$ action does not contain any derivatives of the metric and is of only first order in the derivatives of the connection. Now, given that the connection turns out to be an auxiliary field and can be algebraically related to derivatives of the matter and of the metric, it no longer comes as a surprise that the outcome is a theory with higher differential order in the matter than the metric.

By now, the fact that the post-Newtonian metric turns out to be algebraically dependent on the density, as discussed in Sec. V.A.3 should no longer sound surprising: it is merely a manifestation of the problem discussed here in the weak field regime. The fact that it is unlikely that the Cauchy problem will be well-formulated in the presence of matter also originates from the same feature of Palatini $f(R)$ gravity, as already mentioned. Similarly, the fact that a theory which manifestly satisfies the metric postulates and, therefore, is expected to satisfy the Equivalence Principle, actually exhibits unexpected phenomenology in local non-gravitational experiments and conflicts with the Standard Model, as shown in Sec. V.B.3 also ceases to be a puzzle: the algebraic dependence of the connection on the derivatives of matter fields (as the former is an auxiliary field) makes the matter enter the gravitational action through the “back door”. This introduces strong couplings between matter and gravity and self-interactions of the matter fields which manifest themselves in the local frame. Alternatively, if one completely eliminates the connection (or the scalar field in the equivalent Brans–Dicke representation) at the level of the action, or attempts to write down an action which leads to the field eqs. (28) directly through metric variation, then this action would have to include higher order derivatives of the matter fields and self-interactions in the matter sector. In this sense, the $f(\mathcal{R})$ representation is simply misleading [see also (Sotiriou et al., 2007) for
D. Gravitational waves in $f(R)$ gravity

By now it is clear that the metric tensor of $f(R)$ gravity contains, in addition to the usual massless spin 2 graviton, a massive scalar that shows up in gravitational waves in the metric version of these theories (in the Palatini version, this scalar is not dynamical and does not propagate). A scalar gravitational wave mode is familiar from scalar-tensor gravity (Will, 1981), to which $f(R)$ gravity is equivalent. Because this scalar is massive, it propagates at a speed lower than the speed of light and of massless tensor modes and is, in principle, detectable in the arrival times of signals from an exploding supernova when gravitational wave detectors are sufficiently sensitive [this possibility has been pointed out as a discriminator between Tensor-Vector-Scalar theories and GR (Kahya and Woodard, 2007)]. This massive scalar discriminator between Tensor-Vector-Scalar theories and dark gravity is equivalent. Because this scalar is massive, propagation). A scalar gravitational wave mode is familiar

\[ h_{ij}(t) = h(t) e^{\frac{c_k}{t_0} c_{ij}} \]

in a background FLRW universe with scale factor $a(t) = a_0 \left( \frac{t}{t_0} \right)^n$:

\[ \ddot{h} + \frac{(3n - 2\delta)}{t} \dot{h} + k^2 a_0 \left( \frac{t}{t_0} \right)^2 n h = 0. \]

This can be solved in terms of Bessel functions: plots of these wave amplitudes are reported in (Capozziello et al., 2008) for various values of the parameter $\delta$, but the limit $0 \leq \delta < 7.2 \cdot 10^{-19}$ obtained by (Barrow and Clifton, 2008; Clifton and Barrow, 2005a, 2005b) leaves little hope of detecting $f(R)$ effects in the gravitational wave background.

(Ananda et al., 2008) give a covariant and gauge-invariant description of gravitational waves in a perturbed FLRW universe filled with a barotropic perfect fluid in the toy model $f(R) = R^n$. The perturbation equations are solved (again, in terms of Bessel functions of the first and second kind) in the approximation of scales much larger or much smaller than the Hubble radius $H^{-1}$, finding a high sensitivity of the tensor modes evolution to the value of the parameter $n$. In particular, a tensor mode is found that grows during the radiation-dominated era, with potential implications for detectability in advanced space interferometers. This study, and others of this kind expected to appear in future literature, are in the spirit of discriminating between dark energy and dark gravity, or even between different $f(R)$ theories (if this class is taken seriously), when gravitational wave observations will be available: as already remarked, this is not possible by considering only unperturbed FLRW solutions.

VII. SUMMARY AND CONCLUSIONS

A. Summary

While we have presented $f(R)$ gravity as a class of toy theories, various authors elevate modified gravity, in one or the other of its incarnations corresponding to specific choices of the function $f(R)$, to the role of a fully realistic model to be compared in detail with cosmological observations, and to be distinguished from other models. A large fraction of the works in the literature is actually devoted to specific models corresponding to definite choices of the function $f(R)$, and to specific parametrizations.

Besides the power law and power series of $R$ models which we have already mentioned extensively, some other typical examples are functions which contain terms such as $\ln(\lambda R)$ (Nojiri and Odintsov, 2004); $R^n$ (Perez Bergliaffa, 2006) or $e^{\lambda R}$ (Abdelwahab et al., 2008); $a R$ (Bean et al., 2007); (Carloni et al., 2006; Song et al., 2007) or are more involved functions of $R$, such as $f(R) = R - a (R - \Lambda_1)^m + b (R - \Lambda_2)^n$ with $n, m, a, b > 0$ (Nojiri and Odintsov, 2003a). Some models have actually been tailored to pass all or most of the known constraints, such as the one proposed in (Starobinsky, 2007) where $f(R) = R + \ldots$
\( \lambda R_0 [1 + R^2 / R_0^2]^{-n} - 1 \) with \( n, \lambda > 0 \) and \( R_0 \) being of the order of \( H_0^2 \). Here we have tried to avoid considering specific models and we have attempted to collect general, model-independent results, with the viewpoint that these theories are to be seen more as toy theories than definitive and realistic models.

We are now ready to summarize the main results on \( f(R) \) gravity. On the theoretical side, we have explored all three versions of \( f(R) \) gravity: metric, Palatini and metric-affine. Several issues concerning dynamics, degrees of freedom, matter couplings, etc. have been extensively discussed. The dynamical equivalence between both metric/Palatini \( f(R) \) gravity and Brans–Dicke theory has been, and continues to be, a very useful tool to study these theories given some knowledge of the aspects of interest in scalar-tensor gravity. At the same time, the study of \( f(R) \) gravity itself has provided new insight in the two previously unexplored cases of Brans–Dicke theory with \( \omega_0 = 0 \) and \( \omega_0 = -3/2 \). We have also considered most of the applications of \( f(R) \) gravity to both cosmology and astrophysics. Finally, we have explored a large number of possible ways to constrain \( f(R) \) theories and check their viability. In fact, many avatars of \( f(R) \) have been shown to be subject to potentially fatal troubles, such as a grossly incorrect post-Newtonian limit, short time scale instabilities, the absence of a matter era, conflict with particle physics or astrophysics, etc.

To avoid repetition, we will not attempt to summarize here all of the theoretical issues, the applications or the constraints discussed. This, besides being redundant, would not be very helpful to the reader, as, in most cases, the insight gained cannot be summarized in a sentence or two. Specifically, some of the constraints that have been derived in the literature are not model or parametrization independent (and the usefulness of some parametrizations is questionable). This does not allow for them to be expressed in a straightforward manner through simple mathematical equations applicable directly to a general function \( f(R) \). Particular examples of such constraints are those coming from cosmology (background evolution, perturbations, etc.).

However, we have encountered cases in which clear-cut viability criteria are indeed easy to derive. We would therefore, like to make a specific mention of those. A brief list of quick-and-easy-to-use results is:

- In metric \( f(R) \) gravity, the Dolgov-Kawasaki instability is avoided if and only if \( f''(R) \geq 0 \). The stability condition of de Sitter space is expressed by eq. 1.73.
- Metric \( f(R) \) gravity might pass the weak-field limit test and at the same time constitute an alternative to dark energy only if the chameleon mechanism is effective—this restricts the possible forms of the function \( f(R) \) in a way that cannot be specified by a simple formula.
- Palatini \( f(R) \) gravity suffered multiple deaths, due to the differential structure of its field equations. These conclusions are essentially model-independent. (However, this theory could potentially be fixed by adding extra terms quadratic in the Ricci and/or Riemann tensors, which would raise the order of the equations.)
- Metric-affine gravity as an extension of the Palatini formalism is not sufficiently developed yet. At the moment of writing, it is not clear whether it suffers or not of the same problems that afflict the Palatini formalism.

Of course, as already mentioned, the situation is often more involved and cannot be summarized with a quick recipe. We invite the reader to consult the previous sections and especially the references that they contain.

B. Extensions and new perspectives on \( f(R) \) gravity

We have treated \( f(R) \) gravity here as a toy theory and, as stated in the Introduction, one of its merits is its relative simplicity. However, we have seen a number of viability issues related to such theories. One obvious way to address this issue is to generalize the action even further in order to avoid these problems, at the cost of increased complexity. Several extensions of \( f(R) \) gravity exist. Analyzing them in detail goes beyond the scope of this review, but let us make a brief mention of the most straightforward of them.

We have already discussed the possibility of having higher order curvature invariants, such as \( R_{\mu\nu}R^{\mu\nu} \), in the action. In fact, from a dimensional analysis perspective, the terms \( R^2 \) and \( R_{\mu\nu}R^{\mu\nu} \) should appear at the same order. However, theories of this sort seem to be burdened with what is called the Ostrogradski instability (Woodard, 2007). Ostrogradski’s theorem states that there is a linear instability in the Hamiltonians associated with Lagrangians which depend upon higher than first order derivatives in such a way that the dependence cannot be eliminated by partial integration (Ostrogradski, 1850). \( f(R) \) gravity seems to be the only case that manages to avoid this theorem (Woodard, 2007) and it obviously does not seem very appealing to extend it in a way that will spoil this.\(^{30}\)

The alert reader has probably noticed that the above holds true only for metric \( f(R) \) gravity. In Palatini \( f(R) \) gravity (and metric-affine \( f(R) \) gravity), as it was mentioned earlier, one could add more dynamics to the action without having to worry about making it second order in the fields. Recall that, in practice, the independent connection is an auxiliary field. For instance, the term \( R_{\mu\nu}R^{\mu\nu} \) still contains only first derivatives of the connection. In fact, since we have traced the root of some of

\(^{30}\) However, one could consider adding a function of the Gauss–Bonnet invariant \( G = R^2 - 4R_{\mu\nu}R^{\mu\nu} - R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \) (Cognola et al., 2004; Nojiri and Odintsov, 2003).
the most crucial viability issues of Palatini \( f(R) \) gravity to the lack of dynamics in the gravity sector, such generalizations could actually help by promoting the connection from the role of an auxiliary field to that of a truly dynamical field (Barausse et al., 2008b). Such generalizations have been considered in (Li et al., 2007a).

Another extension of metric \( f(R) \) gravity that appeared recently is that in which the action includes also an explicit coupling between \( R \) and the matter fields. In (Bertolami et al., 2007; Bertolami and Paramos, 2007; Boehmer et al., 2008) the following action was considered:

\[
S = \int d^4 x \sqrt{-g} \left\{ \frac{f_1(R)}{2} + [1 + \lambda f_2(R)] L_m \right\}, \quad (241)
\]

where \( L_m \) is the matter Lagrangian and \( f_{1,2} \) are \( (a \textit{ priori} \) arbitrary) functions of the Ricci curvature \( R \). Since the matter is not minimally coupled to \( R \), such theories will not lead to energy conservation and will generically exhibit a violation of the Equivalence Principle (which could potentially be controlled by the parameter \( \lambda \)).

The motivation for considering such an action spelled out in (Bertolami et al., 2007) was that the nonconservation of energy could lead to extra forces, which in turn might give rise to phenomenology similar to Modified Newtonian Dynamics (MOND) (Milgrom, 1983) on galactic scales. Other variants of this action have also been considered elsewhere: in (Nojiri and Odintsov, 2004a), as an alternative to dark energy by setting \( f_1(R) = R \) and keeping only the nonminimal coupling of matter to the Ricci curvature; in (Dolgov and Kawasaki, 2003b; Mukohyama and Randall, 2004), where the idea of making the kinetic term of a (minimally coupled) scalar field dependent on the curvature, while keeping \( f_1(R) = R \) was exploited in attempts to cure the cosmological constant problem; in (Bertolami and Paramos, 2007) the consequences of such a theory for stellar equilibrium were studied; finally, generalized constraints in order to avoid the instability discussed in Sec. 3.B.1 were derived in (Faraoni, 2007a). The viability of theories with such couplings between \( R \) and matter is still under investigation. However, the case in which both \( f_1 \) and \( f_2 \) are linear has been shown to be non-viable (Sotiriou, 2008) and, for the more general case of the action (241), serious doubts have been expressed (Sotiriou and Faraoni, 2008) on whether extra forces are indeed present in galactic environments and, therefore, whether this theory can really account for the MOND-like phenomenology that initially motivated its use in (Bertolami et al., 2007) as a substitute for dark matter.

One could also consider extensions of \( f(R) \) gravity in which extra fields appearing in the action couple to different curvature invariants. A simple example with a scalar field is the action (159), which is sometimes dubbed \textit{extended quintessence} (Perrotta et al., 2000), similarly to the extended inflation realized with Brans–Dicke theory. However, such generalizations lie beyond the scope of this review.

Finally, it is worth mentioning a different perspective on \( f(R) \) gravity. It is common in the literature that we reviewed here to treat \( f(R) \) gravity as an \textit{exact} theory: the generalized action is used to derive field equations, the solutions of which describe the exact dynamics of the gravitational field (in spite of the fact that the action might be only an approximation and the theory merely a toy theory). A different approach (Bel and Zia, 1985; Simon, 1990) which was recently revived in (DeDeo and Psaltis, 2007) is that of treating metric \( f(R) \) as an effective field theory. That is, to assume that the extra terms are an artifact of some expansion of which we are considering only the leading order terms. Now, when we consider a correction to the usual Einstein–Hilbert term, this correction has to be suppressed by some coefficient. This approach assumes that this coefficient controls the order of the expansion and, therefore, the field equations and their solutions are only to be trusted to the order with which that coefficient appears in the action (higher orders are to be discarded). Such an approach is based on two assumptions: first, some power (or function) of the coefficient of the correction considered should be present in all terms of the expansion; second, the extra degrees of freedom (which manifest themselves as higher order derivatives in metric \( f(R) \) gravity) are actually an artifact of the expansion (and there would be a cancellation if all orders where considered). This way, one can do away with these extra degrees of freedom just by proper power counting. Since many of the viability issues troubling higher order actions are related to the presence of such degrees of freedom (e.g., classical instabilities), removing these degrees of freedom could certainly alleviate many problems (DeDeo and Psaltis, 2007). However, the assumptions on which this approach is based should not be underestimated either. For instance, early results that showed renormalization of higher order actions are related to the presence of an \textit{exact} treatment, \textit{i.e.}, it is fourth order gravity that is renormalizable (Stelle, 1977). Even though, from one hand, the effective field theory approach seems very reasonable (these actions are regarded as low energy limits of a more fundamental theory anyway), there is no guarantee that extra degrees of freedom should indeed not be present in a non-perturbative regime.

C. Concluding remarks

Our goal was to present a comprehensive but still thorough review of \( f(R) \) gravity in order to provide a starting point for the reader less experienced in this field and a reference guide for the expert. However, even though we have attempted to cover all angles, no review can replace an actual study of the literature itself. It seems inevitable that certain aspects of \( f(R) \) might have been omitted, or analyzed less than rigorously and, therefore, the reader is urged to resort to the original sources.

Although many shortcomings of \( f(R) \) gravity have been presented which may reduce the initial enthusiasm

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with which one might have approached this field, the fact that such theories are mostly considered as toy theories should not be missed. The fast progress in this field, especially in the last five years, is probably obvious by now. And very useful lessons, which have helped significantly in the understanding of (classical) gravity, have been learned in the study of \( f(R) \) gravity. In this sense, the statement made in the Introduction that \( f(R) \) gravity is a very useful toy theory seems to be fully justified. Remarkably, there are still unexplored aspects of \( f(R) \) theories or their extensions, such as those mentioned in the previous section, which can turn out to be fruitful.

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References

Abdalla, M. C. B., S. Nojiri, and S. D. Odintsov, 2005, Class. Quant. Grav. 22, L35.
Abdelwahab, M., S. Carloni, and P. K. S. Dunsby, 2008, Class. Quant. Grav. 25, 135002.
Accioly, A., S. Ragusa, E. C. de Rey Neto, and H. Mukai, 1999, N. Cimento B114, 595.
Acquapendente, M., et al., 2002a, Class. Quant. Grav. 19, 5017.
Acquapendente, M., et al., 2002b, eprint astro-ph/0204307.
Allemendi, G., A. Borowiec, and M. Francaviglia, 2004, Phys. Rev. D70, 043524.
Allemendi, G., A. Borowiec, M. Francaviglia, and S. D. Odintsov, 2005a, Phys. Rev. D72, 063505.
Allemendi, G., M. Francaviglia, M. L. Ruggiero, and A. Tartaglia, 2005b, Gen. Rel. Grav. 37, 1891.
Allemendi, G., and M. L. Ruggiero, 2007, Gen. Rel. Grav. 39, 1381.
Amarzguioui, M., O. Elgaroy, D. F. Mota, and T. Multamaki, 2006, Astron. Astrophys. 454, 507.
Amendola, L., S. Capozziello, M. Litterio, and F. Occhioni, 1992, Phys. Rev. D45, 417.
Amendola, L., R. Gannouji, D. Polarski, and S. Tsujikawa, 2007a, Phys. Rev. D75, 083504.
Amendola, L., D. Polarski, and S. Tsujikawa, 2007b, Phys. Rev. Lett. 98, 131302.
Amendola, L., D. Polarski, and S. Tsujikawa, 2007c, Int. J. Mod. Phys. D16, 1555.
Amendola, L., and S. Tsujikawa, 2008, Phys. Lett. B660, 125.
Ananda, K. N., S. Carloni, and P. K. S. Dunsby, 2008, Phys. Rev. D77, 024033.
Anderson, J. L., 1971, Phys. Rev. D3, 1689.
Appleby, S. A., and R. A. Battye, 2007, Phys. Lett. B654, 7.
Appleby, S. A., and R. A. Battye, 2008, eprint 0803.1081.
Astier, P., et al. (The SNLS), 2006, Astron. Astrophys. 447, 31.
Baghram, S., M. Farhang, and S. Rahvar, 2007, Phys. Rev. D75, 044024.
Bahcall, N. A., J. P. Ostriker, S. Perlmutter, and P. J. Steinhardt, 1999, Science 284, 1481.
Bamba, K., and S. D. Odintsov, 2008, JCAP 0804, 024.
Barausse, E., and T. P. Sotiriou, 2008, eprint 0803.3433.
Barausse, E., T. P. Sotiriou, and J. C. Miller, 2008a, Class. Quant. Grav. 25, 062001.
Barausse, E., T. P. Sotiriou, and J. C. Miller, 2008b, Class. Quant. Grav. 25, 105008.
Barausse, E., T. P. Sotiriou, and J. C. Miller, 2008c, eprint 0801.4852.
Barber, G. A., 2003, eprint gr-qc/0302088.
Bardeen, J. M., 1980, Phys. Rev. D22, 1882.
Barraco, D. E., E. Dominguez, and R. Guibert, 1999, Phys. Rev. D60, 044012.
Barraco, D. E., and V. H. Hamity, 1998, Phys. Rev. D57, 954.
Barraco, D. E., and V. H. Hamity, 2000, Phys. Rev. D62, 044027.
Barris, B. J., et al., 2004, Astrophys. J. 602, 571.
Barrow, J. D., 1988, Nucl. Phys. B296, 697.
Barrow, J. D., and T. Clifton, 2006, Class. Quant. Grav. 23, L1.
Barrow, J. D., and S. Cotsakis, 1988, Phys. Lett. B214, 515.
Barrow, J. D., and S. Cotsakis, 1989, Phys. Lett. B232, 172.
Barrow, J. D., and S. Cotsakis, 1991, Phys. Lett. B258, 290.
Barrow, J. D., and S. Hervik, 2006a, Phys. Rev. D73, 023007.
Barrow, J. D., and S. Hervik, 2006b, Phys. Rev. D74, 124017.
Barrow, J. D., and A. C. Ottewill, 1983, J. Phys. A16, 2757.
Barrow, J. D., and F. J. Tipler, 1986, The Anthropic Cosmological Principle (Clarendon Press, Oxford).
Beane, R., D. Bernat, L. Pogosian, A. Silvestri, and M. Trodden, 2007, Phys. Rev. D75, 064020.
Bekenstein, J. D., 2004, Phys. Rev. D70, 083509.
Bel, L., and H. S. Zia, 1985, Phys. Rev. D32, 3128.
Bergmann, P. G., 1968, Int. J. Theor. Phys. 1, 25.
Bernal, A., and F. S. Guzman, 2006a, AIP Conf. Proc. 841, 441.
Bernal, A., and F. S. Guzman, 2006b, Phys. Rev. D74, 103002.
Bernal, A., and F. S. Guzman, 2006c, Phys. Rev. D74, 063504.
Bernal, A., and T. Matos, 2005, AIP Conf. Proc. 758, 161.
Bertolami, O., 1987, Phys. Lett. B186, 161.
Bertolami, O., C. G. Boehmer, T. Harko, and F. S. N. Lobo, 2007, Phys. Rev. D75, 104016.
Bertolami, O., and J. Paramos, 2007, eprint arXiv:0709.3988 [astro-ph].
Berti, T., L. Iess, and P. Tortora, 2003, Nature 425, 374.
Birrell, N. D., and P. C. W. Davies, 1982, *Quantum Fields in Curved Spacetime* (Cambridge University Press, Cambridge).
Boehmer, C. G., T. Harko, and F. S. N. Lobo, 2007a, eprint 0709.0046.
Boehmer, C. G., T. Harko, and F. S. N. Lobo, 2008, JCAP 0803, 024.
