On signed diagonal flip sequences

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Abstract

Eliahou [1] and Kryuchkov conjectured a proposition that Gravier and Payan [4] proved to be equivalent to the Four Color Theorem. It states that any triangulation of a polygon can be transformed into another triangulation of the same polygon by a sequence of signed diagonal flips. It is well known that any pair of polygonal triangulations are connected by a sequence of (non-signed) diagonal flips. In this paper we give a sufficient and necessary condition for a diagonal flip sequence to be a signed diagonal flip sequence. We also give further results relating diagonal flips with simplicial decompositions of disks.

In the study of the Four Color Problem it was observed early on that the problem could be restricted to graphs that are the 1-skeleton of a sphere triangulation, without lost of generality. Due to a result of Whitney’s we can improve the restriction to graphs of this type that are also Hamiltonian. Thus the 4CT is equivalent to the following statement:

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Theorem 1 Given two triangulations of a polygon there exists a coloring of the vertices that is possible for each triangulation.

Eliahou and Kryuchkov observed that some moves on triangulations preserve the coloring (in fact, Kryuchkov worked on trees instead of polygon triangulations). A proper 4-coloring on the vertices of a polygon induces a 3-coloring on the edges of the triangulation such that the boundary of each triangle is colored by the three colors. This can be done by considering the four colors as the four elements of the field $\mathbb{F}_4$ of order 4, and then the 3-coloring on the edges of the triangulation is obtained by coloring each edge with the sum (or difference) of the colors of its end points. By fixing an order on the three colors we get a signing on the triangles, $+$ if the colors on the boundary are ordered in the counterclockwise sense and $-$ if the colors on the boundary are ordered in the clockwise sense. In summary, a proper coloring on the vertices gives a signing for the triangles of a triangulation of the polygon. If two adjacent triangles have the same sign then the vertices opposite to the common edge have different colors and then we can flip the diagonal on the quadrilateral formed by the two triangles. The sign of the new triangles is the opposite of the sign of the previous triangles.

\[ \begin{array}{c}
\begin{array}{c}
+ \\
\downarrow \\
+ \\
\end{array}
\end{array} \quad \longleftrightarrow \quad \begin{array}{c}
\begin{array}{c}
- \\
\downarrow \\
- \\
\end{array}
\end{array} \]

Since this move does not change the coloring on vertices we have that the existence, for any pair of triangulations of the same polygon, of signs such that it is possible to transform one to the other by a sequence of signed diagonal flips implies the Four Color Theorem, as was observed Eliahou and Kryuchkov. This led them to conjecture the following:

Conjecture 2 Given two triangulations of the same polygon there exist signs for them such that it is possible to transform one to the other by a finite sequence of signed diagonal flips.

In Gravier and Payan proved that this conjecture is, in fact, equivalent to the Four Color Theorem.

It is well known that for any pair of triangulations of a polygon it is possible to go from one to the other by a finite sequence of (non-signed) diagonal flips (in the language of binary trees these are reassociation moves).
However, not all sequences of non-signed diagonal flips can be transformed into a sequence of signed diagonal flips.

We now go on to study when it is possible to transform a sequence of (non-signed) diagonal flips into a sequence of signed diagonal flips.

Given a sequence of diagonal flips \( \varphi(1), \varphi(2), \ldots, \varphi(k) \) from one triangulation of a \( n \)-polygon to another, we will construct a graph \( G(\varphi) \) in the following way. Given an enumeration on the vertices of the polygon we represent a diagonal flip \( \varphi(i) \) by a triple \( (i, X(i), Y(i)) \) where \( i \) is the order in the sequence that \( \varphi(i) \) appears, \( X(i) = \{abc, acd\} \) is the set of the triangles in the triangulation that have been removed by the flip and \( Y(i) = \{abd, bcd\} \) is the set of the triangles in the triangulation that have been inserted. The diagonal flips will be the vertices of the graph \( G(\varphi) \) and, for \( i < j \), the flip \( \varphi(i) = (i, X(i), Y(i)) \) is adjacent to the flip \( \varphi(j) = (j, X(j), Y(j)) \) if and only if \( Y(i) \cap X(j) \neq \emptyset \) and

\[
Y(i) \cap X(j) \not\subseteq \bigcup_{i<k<j} X(k).
\]

Thus, an edge in the graph between the \( i \)th and \( j \)th vertex occurs when one or both triangles in the output of the \( i \)th flip is in the input of the \( j \)th (there is a "flip interaction"), and the same triangle(s) are not involved in an intermediate flip interaction (with flip \( k \), for \( i < k < j \)).

This graph gives us a criterion for a sequence of (non-signed) diagonal flips to be able to be transformed into a sequence of signed diagonal flips as is shown by the following result

**Theorem 3** A sequence \( \varphi \) of (non-signed) diagonal flips can be lifted to a sequence of signed diagonal flips if and only if the graph \( G(\varphi) \) is 2-colorable (i.e. bipartite).

**Proof.**

Let \( T(k) \) be the set of triangles of the polygonal triangulation that precede the flip \( \varphi(k) \) and/or succeed the flip \( \varphi(k-1) \). This means that, for each flip \( \varphi(k) \), \( X(k) \subseteq T(k) \) and \( Y(k) \subseteq T(k+1) \). Given a 2-coloring on the graph \( G(\varphi) \), for each \( k \), we sign each triangle \( t \in T(k) \) by one of the following rules:
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R1: search for the last flip $\varphi(i)$ with $i < k$ such that $t \in Y(i)$ and, if it exists, give $t$ the sign $+$ if the color of $\varphi(i)$ is 1 and $-$ if the color is 2;

R2: search for the first flip $\varphi(j)$ with $j \geq k$ such that $t \in X(j)$ and, if it exists, give $t$ the sign $-$ if the color of $\varphi(j)$ is 1 and $+$ if the color is 2;

R3: if $t$ doesn’t belong to $X(j)$ for any $j \geq k$ or $Y(i)$ for any $i < k$ we choose an arbitrary sign for $t$.

We need to show that these rules are consistent (i.e. if we apply rules R1 and R2 we obtain the same sign for the triangle $t$) and that the signing produced makes the flips become signed flips.

First we observe that if rules R1 and R2 can be both applied then

$$t \in Y(i) \cap X(j), \quad t \notin \bigcup_{i<l<k} Y(l) \quad \text{and} \quad t \notin \bigcup_{k<l<j} X(l).$$

Also by observing that if a triangle $t$ is not in $T(l)$ nor in $Y(l)$ then it is not in $T(l+1)$, we can deduce that

$$t \in T(k) \quad \text{and} \quad t \notin \bigcup_{i<l<k} Y(l) \quad \Rightarrow \quad t \notin \bigcup_{i<l<k} X(l)$$

because if $t \in X(l)$ for some $i < l < k$ then $t \notin T(l+1)$ with $l+1 \leq k$ and, if $l+1 < k$, we can use the fact that $t \notin Y(l+1)$ for any $i < l + 1 < k$ to obtain the contradiction $t \notin T(k)$.

Thus by putting together (1) and (2) we have that

$$t \in Y(i) \cap X(j) \quad \text{and} \quad t \notin \bigcup_{i<l<j} X(l)$$

which means that $Y(i) \cap X(j) \notin \bigcup_{i<l<j} X(l)$.

So $\varphi(i)$ and $\varphi(j)$ have different colors (are adjacent) and rules R1 and R2 induce the same sign on $t$.

Now, let us consider an arbitrary flip $\varphi(i)$ with $X(i) = \{t_1, t_2\}$ and $Y(i) = \{t_3, t_4\}$. By rule R2 $t_1$ and $t_2$ receive the same sign ($-$ if the color of $\varphi(i)$ is 1
and + if the color is 2) and by rule R1 $t_3$ and $t_4$ receive the opposite sign (+ if the color of $\varphi(i)$ is 1 and − if the color is 2). Thus $\varphi(i)$ becomes a signed flip.

Conversely, if a sequence of diagonal flips $\varphi$ is a sequence of signed diagonal flips then this induces a 2-coloring on the graph $G(\varphi)$ by coloring each vertex $\varphi(i)$ with the sign of the triangles on $Y(i)$. □

**Example 1** Consider the following sequence of flips:

\[
\begin{align*}
\varphi(1) &= (1, \{236, 356\}, \{235, 256\}) \\
\varphi(2) &= (2, \{235, 345\}, \{234, 245\}) \\
\varphi(3) &= (3, \{256, 267\}, \{257, 567\}) \\
\varphi(4) &= (4, \{127, 257\}, \{125, 157\}) \\
\varphi(5) &= (5, \{125, 245\}, \{124, 145\})
\end{align*}
\]

This gives a non-2-colorable graph:

Thus the sequence of flips is not signable.

**Example 2** Now we consider another sequence of flips:
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This gives a 2-colorable graph:

which gives a sequence of signed flips:
The motivation for the graph used in the last theorem was the following. We can see a diagonal flip as the gluing of a tetrahedron onto the polygonal triangulation. Thus a sequence of diagonal flips $\varphi$ produces a 3-dimensional simplicial complex $K(\varphi)$ and the graph $G(\varphi)$ is just the graph where the vertices are the tetrahedra and two tetrahedra are adjacent if they share a common face. For a sequence of signed diagonal flips we are considering two types of tetrahedra, one on which the faces are signed positively, and another on which the faces are signed negatively, and two tetrahedra are allowed to be glued together only if the faces have opposite signs. Thus for a sequence of signed diagonal flips we have a 2-colorable graph.

The same idea gives us an alternative for theorem 1.3 of [2]:

**Theorem 4** Suppose we are given a triangulation $T$ of the sphere with signed faces. Then the signing comes from a strict 4-coloring of $T$ (i.e. a 4-coloring that uses the four colors) if and only if $T$ comes from the tetrahedron with the same sign on all its faces, by means of a sequence of signed diagonal flips (move I) and/or divisions of a triangle into three triangles (by adding a vertex $v$ inside the triangle and edges joining $v$ to the vertices of the triangle) with opposite signs (move II).

Since these two moves are equivalent to gluing signed tetrahedra, a sequence of moves I and II is equivalent to a 2-colorable, 3-dimensional simplicial complex whose boundary is the sphere triangulation $T$ and which has no vertices apart from those on the boundary (i.e. inside the complex). We will show that the previous theorem is a corollary of the following theorem:
Theorem 5 If $\psi$ is a strict 4-coloring of a triangulation $T$ of the sphere $S^2$ then there exists a triangulation $T'$ of the disk $D^3$ such that $\partial T' = T$, the vertices of $T'$ are in $T$ and $\psi$ is a 4-coloring of $T'$.

To prove this theorem we will make use of the following lemma:

Lemma 6 If $\psi$ is a strict 3-coloring of a triangulation $T$ of the circle $S^1$ then there exists a triangulation $T'$ of the disk $D^2$ such that $\partial T' = T$, the vertices of $T'$ are in $T$ and $\psi$ is a 3-coloring of $T'$.

Proof.
In this case $T$ is just a cycle graph, thus if $\psi$ is a strict 3-coloring then there are three consecutive vertices $v_1, v_2$ and $v_3$ with distinct colors $a$, $b$ and $c$. Suppose that the middle vertex $v_2$ is colored by $b$. If $v_2$ is the only vertex in $T$ colored by $b$ then we can add edges linking $v_2$ with all vertices of $T$ and therefore we get a triangulation $T'$ of the disk $D^2$ with the desired properties.

If $v_2$ is not the only vertex in $T$ colored by $b$ then we can add an edge linking $v_1$ with $v_3$, and then complete, by induction, the triangulation on the disk whose boundary is the cycle $v_1, v_3, \ldots, v_n$.

Proof of theorem 5.
First we prove that in the triangulation $T$ there exists a vertex $v$ that is adjacent to a cycle\footnote{Indeed, we are considering the link of $v$, regarding $T$ as a simplicial complex and $v$ as a simplex. Although the link of $v$ need not be a cycle in general, it can be considered as such for the purpose of our argument.} colored by three colors. We take a triangle colored by (say) $a$, $b$ and $c$ and consider the region formed by the triangles colored by the vertices of $T$ and therefore we get a triangulation $T'$ of the disk $D^2$ with the desired properties.
the same colors. Since \( \psi \) is a strict 4-coloring, this region has a non-empty boundary and any vertex of the boundary satisfies the required condition.

Now, if \( v \) is the only vertex in \( T \) colored by its color \( \psi(v) \) then we can add edges linking \( v \) with all other vertices of \( T \) and therefore we get a triangulation \( T' \) of the disk \( \mathbb{D}^3 \) with the desired properties.

If, on the contrary, \( v \) is not the only vertex in \( T \) colored by \( \psi(v) \), we remove \( v \), use the lemma to triangulate the region bounded by the link of \( v \), use induction to get a triangulation of \( \mathbb{D}^3 \) and join again the vertex \( v \) to obtain the desired triangulation \( T' \) of \( \mathbb{D}^3 \).

\[ \blacksquare \]

Note that, mutatis mutandis, this proof can be used recursively to prove a more general theorem:

**Theorem 7** If \( \psi \) is a strict \( n+2 \)-coloring of a triangulation \( T \) of the \( n \)-sphere \( S^n \) then there exists a triangulation \( T' \) of the disk \( \mathbb{D}^{n+1} \) such that \( \partial T' = T \), the vertices of \( T' \) are in \( T \) and \( \psi \) is a \( n+2 \)-coloring of \( T' \).

Now let us give our alternative proof of theorem 4:

**Proof.**

For the sufficient condition the proof is equal to the proof in [2]. Move I does not change the coloring and move II extends the coloring in a unique way.

For the other implication, we take the triangulation of \( \mathbb{D}^3 \) obtained in theorem 5, we choose one tetrahedron and by adding the adjacent tetrahedra one by one we get a sequence of moves I and II (the signs of the faces are determined by the coloring as was observed in the paragraph following theorem 1).

\[ \blacksquare \]

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