A description of a Drinfeld module with class number \( h = 1 \) and rank 1

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Abstract

We work with detail the Drinfeld module over the ring

\[ A = \mathbb{F}_2[x, y]/(y^2 + y = x^3 + x + 1). \]

The example in question is one of the four examples that come from quadratic imaginary fields with class number \( h = 1 \) and rank one.

We develop specific formulas for the coefficients \( d_k \) and \( \ell_k \) of the exponential and logarithmic functions and relate them with the product \( D_k \) of all monic elements of \( A \) of degree \( k \). On the Carlitz module, \( D_k \) and \( d_k \) coincide, but this is not true in general Drinfeld modules. On this example, we obtain a formula relating both invariants. We prove also using elementary methods a theorem due to Thakur that relate two different combinatorial symbols important in the analysis of solitons.

1 Introduction.

Let \( \mathbb{F}_q \) be a finite field of characteristic \( p \) and \( K \) a function field over \( \mathbb{F}_q \). After we choose \( \infty \), a fixed infinite place of \( K \), let \( A \) be the ring
of regular functions outside of $\infty$ and let $K_{\infty}$ be its completion. Now take $C_{\infty}$ to be the completion of an algebraic closure of $K_{\infty}$.

Let $C_{\infty}\{\tau\}$ be the ring of twisted polynomials, i.e., the noncommutative ring of polynomials $\sum a_i \tau^i$ with coefficients in $C_{\infty}$ such that $\tau z = z^q \tau$. A twisted polynomial $f = a_0 + a_1 \tau + \cdots + a_d \tau^d \in C_{\infty}$ is identified with the $F_q$-linear endomorphism of $C_{\infty}$,

$$z \mapsto f(z) = a_0 z + a_1 z^q + \cdots + a_d z^{q^d}.$$  

A Drinfeld $A$-module of rank one is a $F_q$-algebra homomorphism $\rho: A \to C_{\infty}\{\tau_p\}$ injective, for which $\rho(a) = a \tau^0 + \text{higher order terms in } \tau$. The action $a \cdot z = \rho(a)(z)$ of $A$ in $C_{\infty}$ makes $C_{\infty}$ into an $A$-module, and hence the name “Drinfeld module”.

For each Drinfeld module $\rho$ we associate an exponential entire function $e$ defined for a power series in all $C_{\infty}$ by

$$e(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{d_i}.$$  

The linear term in this exponential function satisfy the following fundamental functional equation

$$e(az) = \rho_a(e(z)), \quad (1)$$  

for $z \in C_{\infty}$ and $a \in A$, where $\rho_a$ stands for $\rho(a)$.

The Carlitz module, defined by Carlitz [1] in 1935, is given by the $F_q$-algebra homomorphism $C : F_q[t] \to C_{\infty}\{\tau\}$ determined by $C_t = t + \tau^q$. Equation (1) produces $e(tz) = te(z) + e(z)^q$. It follows that

$$\sum_{i=0}^{\infty} \frac{(t^{q^i} - t)z^{q^i}}{d_i} = \sum_{i=0}^{\infty} \frac{z^{q^{i+1}}}{d_i^i}.$$  

By equating coefficients we get a unique solution $d_n = [n]d_{n-1}^q$ where $[n] = (t^{q^n} - t)$ and $d_0 = 1$. Therefore, $d_n = [n][n-1]^q \cdots [1]^q$ and it is easily seen that $d_n$ is the product of all monic polynomials of degree $n$.

Since $e(z)$ is periodic, it can not have a global inverse, but we may formally derive an inverse $\log(z)$ for $e(z)$ as a power series around the origin. By definition $e(\log(z)) = z$. Since $e(z)$ satisfies the functional equation $e(tz) = te(z) + e(z)^q$, it follows that $tz = \log(te(z)) +$
log(e(z)^q). Replacing log(z) for z we obtain t log(z) = log(tz) + log(z^q).
Let log(z) = \sum z^q/\ell_i. Then
\[
\sum_{i=0}^{\infty} \frac{(t - t^q)}{\ell_i} \cdot z^{q^i} = \sum_{i=0}^{\infty} \frac{z^{q^i+1}}{\ell_i}
\]
It follows that \( \ell_{i+1} = -[i + 1]\ell_i \). Therefore \( \ell_i = (-1)^i[i][i - 1] \cdots [1] \).

We follow the ideas developed in the Carlitz module case, but applied to the Drinfeld module over \( A = F_2[x, y]/(y^2 + y = x^3 + x + 1) \). We explore specific ways to understand the mentioned example, which is one of four examples provided from imaginary quadratic fields with class number \( h = 1 \) [4] and rank 1. The formulas obtained are compared with the Theorem 4.15.4 of [5] and are related to solitons, as exposed in Chapter 8 of the same reference, and Theorem 3 of the article [6].

2 Action of the Drinfeld module on the variables \( x \) and \( y \).

In our example, we have \( d_\infty = 1 \), \( v_\infty(x) = -2 \), \( v_\infty(y) = -3 \), and using that \( \deg(a) = -v_\infty(a)d_\infty \forall a \in A \), it follows that \( \deg(x) = 2 \) and \( \deg(y) = 3 \).

Based on it, the Drinfeld Module \( \rho \) is determined by its values in \( x \) and \( y \) (actually, it is enough to know its value in one element \( a \in A \), see 2.5 in [5]). According to the aforementioned degrees and that the unique sign in our example is +1, we obtained that
\[
\rho_x = x + x_1 \tau + \tau^2,
\rho_y = y + y_1 \tau + y_2 \tau^2 + \tau^3
\]
with \( x_1, y_1, y_2 \in A \). Now, using the commutative property of the Drinfeld module \( \rho_x \rho_y = \rho_y \rho_x \) and equaling on degree 1, we get
\[
x_1(y^2 + y) = y_1(x^2 + x).
\]
Next, using the equation on the curve \( y^2 + y = x^3 + x + 1 \) and dividing, we obtain
\[
y_1 = x_1 \left( x + 1 + \frac{1}{x^2 + x} \right).
\]
This implies that $x^2 + x \mid x_1$ and $y^2 + y \mid y_1$. Assuming that $x_1 = x^2 + x$, it is also obtained that $y_1 = y^2 + y$. Now, equaling on degree 2, one has the equation

$$ (x^4 + x)y_2 = -y_1x_1^2 + y_1^2x_1 + (y^4 + y). \quad (2) $$

But, we can use the identities

$$ y^4 + y = (y^2 + y)^2 + y^2 + y $$
$$ = (y^2 + y)(y^2 + y + 1) $$
$$ = (y^2 + y)(x^3 + x) $$
$$ = (y^2 + y)(x^2 + x)(x + 1) $$

and

$$ x^4 + x = (x^2 + x)(x^2 + x + 1). $$

So dividing the equation (2) by $x_1 = x^2 + x$, and substituting the values $x_1$ and $y_1$, we get

$$ y_2(x^2 + x + 1) = (y^2 + y)(x^2 + x + y^2 + y) + (y^2 + y)(x + 1) $$
$$ = (y^2 + y)(y^2 + y + x^2 + 1) $$
$$ = (y^2 + y)(x^3 + x^2 + x). $$

Thus, clearing $y_2$, we have $y_2 = x(y^2 + y)$, as it is known in the literature \[3\].

### 3 Exponential and Logarithm coefficients.

We find recursive formulas for the coefficients of both the exponential $e(z)$ and the logarithmic $\log(z)$ functions associated to Drinfeld module in $A$.

Write

$$ e(z) = \sum_{i=0}^{\infty} \frac{z^{2i}}{d_i} = \sum_{i=0}^{\infty} a_i z^{2^i} $$

and

$$ \log(z) = \sum_{i=0}^{\infty} \frac{z^{2^i}}{\ell_i} = \sum_{i=0}^{\infty} b_i z^{2^i} $$

where $a_i = d_i^{-1} y$ $b_i = \ell_i^{-1}$. Using that

$$ e(xz) = \rho_x(e(z)) $$
$$ = xe(z) + [1]_x e^2(z) + e^4(z) $$
where \([1]_x = x^2 + x\). Then, working both sides of the equality:

\[
e(xz) + xe(z) = [1]_x e^2(z) + e^4(z),
\]

we have on the left side:

\[
e(xz) + xe(z) = \sum_{j=0}^{\infty} (x^{2j} + x) a_j z^{2j}
= \sum_{j=0}^{\infty} [j]_x a_j z^{2j}
= [1]_x a_1 z^2 + \sum_{j=2}^{\infty} [j]_x a_j z^{2j},
\]

where \([j]_x := x^{2j} + x\). Now, developing the right side, we get:

\[
[1]_x e^2(z) + e^4(z) = [1]_x \sum_{i=0}^{\infty} a_i^2 z^{2i+1} + \sum_{i=0}^{\infty} a_i^4 z^{2i+2}.
\]

From where, by setting \(j = i + 1\) in the first sum, and \(j = i + 2\) in the second sum, we obtain:

\[
[1]_x e^2(z) + e^4(z) = [1]_x \sum_{j=1}^{\infty} a_{j-1}^2 z^{2j} + \sum_{j=2}^{\infty} a_{j-2}^4 z^{2j}
= [1]_x a_0^2 z^2 + \sum_{j=2}^{\infty} ([1]_x a_{j-1}^2 + a_{j-2}^4) z^{2j}.
\]

Comparing equations (3) and (4), recursive formulas are obtained

\[
a_1 = a_0^2
a_j = \frac{[1]_x a_{j-1}^2 + a_{j-2}^4}{[j]_x} \text{ for } j \geq 2.
\]

Subsequently, we assume that \(a_0 = 1\), i.e., the exponential is normalized. Notice that if we do not normalize the coefficients, the exponential function varies by a factor given by the initial term. If we denote \(e(z, a_0)\) to this exponential function, it is easy to see that

\[
e(z, a_0) = a_0 e(z),
\]

where \(e(z)\) is the normalized exponential.
Now, in terms of the $d_j$’s (assuming also, the normalization of the exponential), the recursive formula is as follows:

\[
d_1 = d_0^2 = 1
\]

\[
d_j = \frac{[j]_x d_{j-1} d_{j-2}}{[1]_x d_{j-2}^2 + d_{j-1}^2} \quad \text{for} \quad j \geq 2.
\] (7)

Similarly, for the logarithm function, we have that

\[
x \log(z) = \log (\rho_x(z)) = \log (xz + [1]_x z^2 + z^4) = \log(xz) + \log([1]_x z^2) + \log(z^4),
\]

from which it follows that

\[
x \log(z) + \log(xz) = \log([1]_x z^2) + \log(z^4).
\]

So, we developed the left side to

\[
x \log(z) + \log(xz) = \sum_{j=0}^{\infty} (x^{2j} + x) b_j z^{2j} = \sum_{j=1}^{\infty} [j]_x b_j z^{2j}.
\] (8)

Note that $[0]_x = 0$. The right side must be

\[
\log([1]_x z^2) + \log(z^4) = \sum_{i=0}^{\infty} [1]_x^{2i} b_i z^{2i+1} + \sum_{i=0}^{\infty} b_i z^{2i+2}.
\]

Again, by setting $j = i + 1$ in the first sum, and $j = i + 2$ in the second sum, we obtain

\[
\log([1]_x z^2) + \log(z^4) = [1]_x b_1 z^2 + \sum_{j=2}^{\infty} ( [1]_x^{2j-1} b_{j-1} + b_{j-2} ) z^{2j}.
\] (9)

Comparing the terms in the equations (8) and (9), we obtain the recursive formulas:

\[
b_1 = b_0
\]

\[
b_j = \frac{[1]_x^{2j-1} b_{j-1} + b_{j-2}}{[j]_x} \quad \text{for} \quad j \geq 2.
\] (10)
Now again, if \( \log(z, b_0) \) is the logarithmic function with initial term \( b_0 \), and \( \log(z) = \log(z, 1) \) is the normalized logarithm, by the recursion formula, we deduce the relation:

\[
\log(z, b_0) = b_0 \log(z).
\]

(11)

In terms of values \( \ell_i \)'s, the recursions are as follows:

\[
\ell_1 = \ell_0
\]

\[
\ell_j = \frac{\lfloor j \rfloor \ell_{j-1} \ell_{j-2}}{[1]^2_j \ell_{j-2} + \ell_{j-1}} \text{ for } j \geq 2.
\]

4 Formulae for computing \( \rho_a \).

The first formula is recursive and is in the spirit of the proposition 3.3.10 in [2].

Assume that \( \rho_a = \sum_{k=0}^d \rho_{a,k} \tau^k \) with \( d = \deg(a) \). We will use again commutativity \( \rho_x \rho_a = \rho_a \rho_x \) and the explicit expression: \( \rho_x = x + [1]_x \tau + \tau^2 \). Then, multiplying

\[
\rho_x \rho_a = (x + [1]_x \tau + \tau^2) \left( \sum_{k=0}^d \rho_{a,k} \tau^k \right)
\]

\[
= \sum_{k=0}^d \left( x \rho_{a,k} \tau^k + [1]_x \rho_{a,k}^2 \tau^{k+1} + \rho_{a,k}^4 \tau^{k+2} \right)
\]

and multiplying

\[
\rho_a \rho_x = \left( \sum_{k=0}^d \rho_{a,k} \tau^k \right) (x + [1]_x \tau + \tau^2)
\]

\[
= \sum_{k=0}^d \left( x^{2^k} \rho_{a,k} \tau^k + [1]_x^{2^k} \rho_{a,k} \tau^{k+1} + \rho_{a,k}^4 \tau^{k+2} \right).
\]

By comparing terms a recursive formula is obtained

\[
\rho_{a,0} = a \quad \text{ (first term in recursion)}
\]

\[
\rho_{a,1} = a^2 + a \quad \text{ (comparing degree } k = 1 \text{)}
\]

\[
\rho_{a,k} = \frac{[1]_x^{2^{k-1}} \rho_{a,k-1} + \rho_{a,k-2}}{[k]_x} + \frac{[1]_x \rho_{a,k-1}^2 + \rho_{a,k-2}}{[k]_x}, \text{ for } k \geq 2.
\]
Note the similarity to the recursive formulas for \( a_j \)'s and \( b_j \)'s in the previous section, equations (5) and (10). The same phenomenon occurs in the Carlitz module, but in such a case, there is only a single summand.

Another way to calculate \( \rho_a \), is based on the use of the exponential and the logarithm functions and their formal development as power series. We know that

\[
\exp(a \log(z)) = \rho_a(\log(z)) = \rho_a(z).
\]

Using power series as in the previous section, we get to

\[
\rho_a(z) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} a_j b_{k-j}^2 a_j^2 \right) z^{2k} = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} \frac{a_j^2}{d_j \ell^{2j}_{k-j}} \right) z^{2k}.
\]

The combinatorial terms in the sum, are the ones that D. Thakur used to develop his alternative perspective on solitons [6].

We introduce the following notation for the following pages:

\[
p_k(w) := \left\{ \frac{w^j}{q^k} \right\} := \sum_{j=0}^{k} \frac{w^j}{d_j \ell^{2j}_{k-j}}.
\]

Hence, using that \( \rho_a = \sum \rho_{a,k} \tau^k \) is a monic polynomial in \( \tau \) of degree \( \deg(a) \), we have that

\[
p_k(a) = \rho_{a,k} = \begin{cases} 0, & \text{if } \deg(a) < k \\ 1, & \text{if } \deg(a) = k. \end{cases}
\]

## 5 Comparing the polynomials \( p_k(w) \) and \( e_k(w) \).

We define the following sets

\[
A_{<k} := \{ a \in A : \deg(a) < k \}
\]
\[
A_k := \{ a \in A : \deg(a) = k \}
\]
and the polynomial

\[ e_k(w) = \prod_{a \in A_{<k}} (w + a). \]

Clearly, as every element of \( A_{<k} \) is a root of \( p_k(w) \), we have that

\[ R_k(w) := \frac{p_k(w)}{e_k(w)} \]

is a polynomial. In addition, as \( p'_k(w) = a_k = \ell_k^{-1} \neq 0 \), \( p_k(w) \) and \( R_k(w) \) have no double roots.

In order to calculate the polynomial \( R_k(w) \), suppose

\[ p_k(w) = \sum_{i=0}^{k} A_{k,i} w^{2^i} \]

and

\[ e_k(w) = \sum_{i=0}^{k-1} B_{k,i} w^{2^i}. \]  \( \tag{12} \)

Then, we have the following result:

**Theorem 5.1.** \( R_k(w) = \frac{1}{d_k} e_k(w) + C \), where \( C = \frac{1}{d_{k-1}} + \frac{B_{2,k-2}^2}{d_k} \).

**Proof.** Only for the purpose of this proof, suppose \( k \) is fixed and write \( A_i = A_{k,i} \) and \( B_i = B_{k,i} \). Now, directly dividing \( p_k \) between \( e_k \), using that \( e_k \) is monic, the first term of the quotient ratio is \( A_k w^{2k-1} \). Then, in the first line of the waste division, we have:

\[
A_k B_{k-2} w^{2k-1+2^{k-2}} + A_k B_{k-3} w^{2k-1+2^{k-3}} + \cdots + A_k B_0 w^{2k-1+1} + A_{k-1} w^{2k-1} + \text{lower terms}.
\]

This implies that the next term of the quotient is \( A_k B_{k-2} w^{2k-2} \), and therefore, multiplying by the summands of \( e_k \), after cancelation of the term \( A_k B_{k-2} w^{2k-1+2^{k-2}} \), new summands will be incorporated into the residue in the positions corresponding to the powers:

\[ w^{2k-1}, w^{2k-2+2^{k-3}}, \ldots, w^{2k-2+1}. \]

Hence, all the new terms fall into the “lower terms” of the waste division with exception of the coefficient on \( w^{2k-1} \). This coefficient is \( A_{k-1} + A_k B_{2,k-2}^2 \).
When continuing the division and cancelling the terms of the form $A_k B_j w^{2^{k-1} + 2^j}$ for $j < k - 2$, the terms equal or higher to $w^{2^{k-1}}$ are not affected. This ensures that the obtained quotient is:

$$A_k w^{2^{k-1}} + A_k B_{k-2} w^{2^{k-2}} + A_k B_{k-3} w^{2^{k-3}} + \cdots + A_k B_0 w + A_{k-1} + A_k B_{k-2}^2.$$ 

The result follows, using that $A_k = d_k^{-1}$ and $A_{k-1} = d_{k-1}^{-1}$. 

## 6 Coefficient Formulas for $e_k(w)$.

For $k \geq 2$, set

$$t_k = \begin{cases} x^{\frac{k}{2}}, & \text{if } k \text{ is even} \\ y x^{\frac{k-3}{2}}, & \text{if } k \text{ is odd}. \end{cases}$$

Now, it is clear that $\deg(t_k) = k$ and that the set $\{1, t_2, \ldots, t_{k-1}\}$ is a basis of the vector space $A_{<k}$. Define $D_k := e_k(t_k) = \prod_{a \in A_k} a$. Thus for $k \geq 3$,

$$e_k(w) = \prod_{a \in A_{<k}} (w + a) = \prod_{a \in A_{<k-1}} (w + a) \prod_{a \in A_{k-1}} (w + a)$$

$$= \prod_{a \in A_{<k-1}} (w + a) \prod_{a \in A_{<k-1}} (w + t_{k-1} + a)$$

$$= e_{k-1}(w) e_{k-1}(w + t_{k-1}) = e_{k-1}^2(w) + D_{k-1} \cdot e_{k-1}(w).$$

Developing the right side of the equation (13), we find recursive formulas for the coefficients $B_{k,i}$ in (12):

$$e_{k-1}^2(w) + D_{k-1} \cdot e_{k-1}(w) = \left( \sum_{i=0}^{k-2} B_{k-1,i} w^{2^i} \right)^2 + D_{k-1} \left( \sum_{i=0}^{k-2} B_{k-1,i} w^{2^i} \right)$$

$$= \sum_{i=1}^{k-1} B_{k-1,i-1}^2 w^{2^i} + \sum_{i=0}^{k-2} D_{k-1} B_{k-1,i} w^{2^i}.$$ 

Indeed, we have

$$B_{k,0} = D_{k-1} B_{k-1,0} = D_{k-1} D_{k-2} \cdots D_2$$

$$B_{k,i} = D_{k-1} B_{k-1,i} + B_{k-1,i-1}^2$$

$$B_{k,k-1} = B_{k-1,k-2} = \cdots = B_{2,1} = 1.$$
Before developing explicit formulas for the coefficients $B_{k,i}$, we introduce the following symbols:

$$[1]_w = w^2 + w$$
$$[k]_w = w^{2k} + w.$$ 

It is not difficult to prove that these symbols satisfy the following:

**Lemma 6.1.** Properties of the symbol $[k]_w$.

1) $[k]_{2i}^2 = [k]_{w2i}$
2) $[1][k]_w = [k][1]_w$
3) $[k]_{w_1+w_2} = [k]_{w_1} + [k]_{w_2}$
4) $[k + 1]_w = [k]_{w}^2 + [1]_w$
5) $[k]_w = \sum_{i=0}^{k-1} [1]_{2i}^2$.

Notice that $e_k(w)$ is a polynomial on $[1]_w$ of degree $2^{k-2}$. Set

$$e_k(w) = \sum_{i=0}^{k-2} T_{k,i}[1]_{w}^2.$$ 

Next, we will find specific formulas for the coefficients $T_{k,i}$’s. First, define the following functions:

$$S_{n,r}(x_1, x_2, \cdots, x_n) = \sum_{n \geq i_1 > i_2 > \cdots > i_r \geq 1} \prod_{j=1}^{r} x_{i_j}^{n-j+1-i_j}.$$ 

We have the following lemma:

**Lemma 6.2.** Properties of the sums $S_{n,r}(x_1, x_2, \cdots, x_n)$.

1) $S_{n,0}(x_1, \ldots, x_n) = 1$
2) $S_{n,1}(x_1, \ldots, x_n) = x_n + x_{n-1}^2 + \cdots + x_1^{2^{n-1}}$
3) $S_{n+1,r}(x_1, \ldots, x_{n+1}) = S_{n,r}^2(x_1, \ldots, x_n) + x_{n+1}S_{n,r-1}(x_1, \ldots, x_n)$

Proof. The first two assertions are immediate.

For the third, note that:

$$S_{n,r}^2(x_1, \ldots, x_n) = \left( \sum_{n \geq i_1 > i_2 > \cdots > i_r \geq 1} \prod_{j=1}^{r} x_{i_j}^{n-j+1-i_j} \right)^2$$

$$= \sum_{n \geq i_1 > i_2 > \cdots > i_r \geq 1} \prod_{j=1}^{r} x_{i_j}^{2n+1-j+1-i_j}.$$ 

(14)
On the other hand,

\[ x_{n+1} S_{n,r-1}(x_1, \ldots, x_n) = x_{n+1} \sum_{n \geq i_1 > i_2 > \cdots > i_{r-1} \geq 1} \prod_{j=1}^{r-1} x_{i_j}^{n-j+1-i_j} \]

\[ \sum_{n \geq i_1 > i_2 > \cdots > i_{r-1} \geq 1} x_{n+1} \prod_{j=1}^{r-1} x_{i_j}^{n-j+1-i_j}. \]

(15)

Now, making \( i_1 = n + 1 \) and \( i_{j+1} = i_j \) (moving the variable \( j \) to \( j+1 \)), we obtain that (15) becomes

\[ \sum_{n+1=i_1 > i_2 > \cdots > i_{r-1} \geq 1} \prod_{j=1}^{r} x_{i_j}^{n+1-j+1-i_j}. \]

(16)

Notice that the variable \( x_{i_j} \) with exponent \( n-j+1-i_j \) in (15) coincide with the variable \( x_{i_{j+1}} \) with exponent \( n+1-j+1-i_{j+1} \) in (16).

Now, clearly the sum of (14) and (16) proves the lemma.

Proposition 6.3. For

\[ e_k(w) = \sum_{i=0}^{k-2} T_{k,i}[1]^2 \]

is satisfied that

\[ T_{k,i} = S_{k-2,k-2-1}(D_2, D_3, \ldots, D_{k-1}), \]

where \( D_i = e_i(t_i) \).

Proof. Using the identity

\[ e_{k+1}(w) = e_k^2(w) + D_k e_k(w), \]

for \( k \geq 2 \),

we obtain the following recursive equations

\[ T_{k+1,0} = D_k T_{k,0} \]

\[ T_{k+1,i} = T_{k,i-1}^2 + D_k T_{k,i} \]

\[ T_{k+1,k-1} = 1 \]

Then, from induction suppose that the proposition is valid for \( T_{k,i} \),

using the recursive form we get
\[ T_{k+1,i} = T_{k,i-1}^2 + D_k T_{k,i} \]
\[ = S_{k-2,k-2-(i-1)}^2 (D_2, D_3, \ldots, D_{k-1}) + D_k S_{k-2,k-2-i-1} (D_2, D_3, \ldots, D_{k-1}) \]
\[ = S_{k-2,k-1-i}^2 (D_2, D_3, \ldots, D_{k-1}) + D_k S_{k-2,k-1-i-1} (D_2, D_3, \ldots, D_{k-1}) \]
\[ = S_{k-1,k-1-i} (D_2, D_3, \ldots, D_k). \]

The last equality follows from lemma (6.2). Now, the result follows from verifying that the coefficients \( T_{k,i} \) coincide with \( S_{k-2,k-2-i} (D_2, D_3, \ldots, D_{k-1}) \) for some first small values of \( k \).

For simplicity, set \( S_{k-2,k-2} := S_{k-2,k-2-i} (D_2, D_3, \ldots, D_{k-1}). \)

**Corollary 6.4.** The coefficients of the polynomial
\[ e_k(w) = \sum_{i=0}^{k-1} B_{k,i} w^{2^i} \]
are given by the formulas
\[ B_{k,k-1} = T_{k,k-2} = S_{k-2,0} = 1 \]
\[ B_{k,i} = T_{k,i} + T_{k,i-1} = S_{k-2,k-2-i} + S_{k-2,k-1-i}, \text{ for } 1 \leq i \leq k-2 \]
\[ B_{k,0} = T_{k,0} = S_{k-2,k-2} = D_{k-1} D_{k-2} \cdots D_2. \]

**Proof.** Note that
\[ e_k(w) = \sum_{i=0}^{k-2} t_{k,i} [1^2] w \]
\[ = \sum_{i=0}^{k-2} T_{k,i} \left( w + w^2 \right)^{2^i} \]
\[ = T_{k,k-2} w^{2^{k-1}} + \sum_{i=1}^{k-2} (T_{k,i} + T_{k,i-1}) w^{2^i} + T_{k,0} w. \]

\[ \square \]

7. Relationship among the values \( d_k, \ell_k \) y \( D_k \).

Basically, these relationships are corollary of theorem (5.1) and the explicit expression of the coefficients \( B_{k,i} \) developed in the previous section.
If we evaluate the polynomial equality
\[ p_k(w) = \frac{e_2^2(w)}{d_k} + C e_k(w) \]  
(17)
in \( w = t_k \), we get that
\[ 1 = \frac{D_k^2}{d_k} + C D_k. \]
Solving for \( C \), we obtain
\[ C = \frac{1}{D_k} + \frac{D_k}{d_k} = \frac{d_k + D_k^2}{D_k d_k}. \]  
(18)
Now, using the definition of \( C \) in (5.1), we also have that
\[ C = \frac{1}{d_{k-1}} + \frac{1 + D_{k-1}^2 + D_{k-2}^4 + \cdots + D_{2}^{2k-2}}{d_k}, \]
since
\[ B_{k,2} = (1 + D_{k-1}^2 + D_{k-2}^4 + \cdots + D_{2}^{2k-3})^2, \]
from corollary 6.4 and part 2) of lemma 6.2.
Multiplying by \( D_k d_k \), we obtain
\[ C D_k d_k = \frac{D_k d_k}{d_{k-1}} + D_k \left( 1 + D_{k-1}^2 + D_{k-2}^4 + \cdots + D_{2}^{2k-2} \right) \]
and using (18), we have
\[ d_k + D_k^2 = \frac{D_k d_k}{d_{k-1}} + D_k \left( 1 + D_{k-1}^2 + D_{k-2}^4 + \cdots + D_{2}^{2k-2} \right). \]
From where,
\[ d_k \left( 1 + \frac{D_k}{d_{k-1}} \right) = D_k(1 + D_k + D_{k-1}^2 + \cdots + D_{2}^{2k-2}), \]
so eventually we get
\[ d_k = \frac{D_k d_{k-1}}{d_{k-1} + D_k} \left( 1 + D_k + D_{k-1}^2 + \cdots + D_{2}^{2k-2} \right) \]
(19)
\[ = \frac{D_k d_{k-1}}{d_{k-1} + D_k} \cdot B_{k+1,k-1}. \]
Now, using the recursive formula (7) is easy to see that
\[ d_2 = [1]_x \]
and also

\[ D_2 = e_2(t_2) = [1]_{t_2} = [1]_x, \]

equation \((19)\) gives a recursive procedure to calculate \(d_k\), in terms of values \(D_i\)’s with \(2 \leq i \leq k\).

Now, equating the coefficients of the linear terms of the polynomials in \((17)\), we obtain that

\[ \frac{1}{\ell_k} = CD_{k-1}D_{k-2} \cdots D_2 \]

and using \((18)\), we conclude that

\[ \ell_k = \frac{d_k d_k}{(d_k + D_2^2)(D_{k-1}D_{k-2} \cdots D_2)}. \]

We summarize the above discussion in the main result of the article.

**Theorem 7.1.** Recursive formulas to compute \(\ell_k\) and \(d_k\) values in terms of \(D_k\)'s.

1) \(d_2 = D_2\).

2) \(d_k = \frac{d_k d_{k-1}}{d_{k-1} + D_k} \left(1 + D_k + D_{k-1}^2 + \cdots + D_k^2 \cdots D_2^2\right)\).

3) \(\ell_k = \frac{d_k d_k}{(d_k + D_2^2)(D_{k-1}D_{k-2} \cdots D_2)}\).

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**References**

[1] L. Carlitz. On certain functions connected with polynomials in a Galois field. *Duke Math. J.*, 1(2):137–168, 1935.

[2] D. Goss, *Basic Structures of Function Field Arithmetic*. 2nd Edition, Springer-Verlag, (1998).

[3] D. R. Hayes, *Explicit Class Field Theory in Global Function Fields*, in “Studies in Algebra and Number Theory”, (G.C. Rota, Ed.) pp. 173-217, Academic Press, San Diego 1979.
[4] R. E. MacRae, *On unique factorization in certain rings of algebraic functions* J. Algebra 17 (1971), 77-91.

[5] D. S. Thakur, *Function Field Arithmetic*, World Scientific Publishing Co. Pte. Ltd. (2004).

[6] D. S. Thakur, *An Alternate Approach to Solitons for $F_q[t]$* Journal of Number Theory 76, 301-319 (1999).