AN EXPANDED MIXED FINITE ELEMENT METHOD FOR GENERALIZED FORCHHEIMER FLOWS IN POROUS MEDIA

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Abstract. We study the expanded mixed finite element method applied to degenerate parabolic equations with the Dirichlet boundary condition. The equation is considered a prototype of the nonlinear Forchheimer equation for slightly compressible fluid flow in porous media. The bounds for the solutions are established. In both continuous and discrete time procedures, utilizing the monotonicity properties of Forchheimer equation and the boundedness of solutions, we establish the error estimates in $L^2$-norm for solutions, divergence of the vector variable in several Lebesgue norms. A numerical example using the lowest order Raviart-Thomas ($RT_0$) mixed element confirms our theoretical results regarding convergence rates.

Key words. Expanded mixed finite element, nonlinear degenerate parabolic equations, generalized Forchheimer equations, error estimates

AMS subject classifications. 65M12, 65M15, 65M60, 35Q35, 76S05.

1. Introduction. The paper is dedicated to the analysis of mixed finite element approximations of the solutions of the system of equations modeling the flows of compressible fluid in porous media subject to the generalized Forchheimer law. Forchheimer type flow belongs to the so-called post-Darcy class of flows and is designed to model high velocity filtration in porous media when inertial and friction terms cannot be ignored. In recent years, this phenomenon generated a lot of interest in the research community such as engineering, environmental and groundwater hydrology and in medicine.

An accurate description of fluid flow behavior in porous media is essential to successful forecasting and project design in reservoir engineering. Reservoir engineers often divide flows in the media into three main categories with respect to Darcy law [30]: fast flows near the well and fracture (post-Darcy), linear non-fast/non-slow flows described by Darcy equation in the main domain between the near well zone and the “far away” region, and on the periphery of the media, where the impact of the well is small. In this research, we concentrate on the first type of flow, the fast flows at the wells, when deviation from the linear Darcy is associated with high velocity field.

Engineers commonly use Forchheimer equation to take into account inertial phenomena. In early 1900s, Forchheimer proposed three models for nonlinear flows, the so-called two-term, three-term and power laws [10] to match experimental observations. There is a significant number of papers studying these equations and their variations—the Brinkman-Forchheimer equations for incompressible fluids, see e.g. [5, 6, 11, 12, 31, 29]. Recently, the authors in [2, 14, 15, 17, 16, 18] proposed and studied the generalized Forchheimer equations for slightly compressible fluids in porous media. These papers focus on the theory of existence, stability and qualitative properties of solutions within the framework of non-linear parabolic equations with coefficients degenerating as the gradient of pressure converges to infinity. In order to apply the developed models and methods to practical problems, it is important to investigate the properties and convergences of the approximate numerical solutions of the corresponding degenerate parabolic equations.

The popular numerical methods for modeling flow in porous media are the mixed finite element approximations (e.g. [8, 28, 13, 27, 21]) and the block-centered finite difference method [33]. These methods are widely used because of their inherent conservation properties and because they produce...
accurate flux even for highly homogeneous media with large jumps in the conductivity (permeability) tensor [9]. However according to Arbogast, Wheeler and Yotov in [1], the standard mixed finite element method is not suitable for problems with degenerating tensor coefficients as the tensor needs to be inverted. Our proposed approach reduces the original Forchheimer type equation to the generalized Darcy equation with conductivity tensor $K$, degenerating as gradient of the pressure converges to infinity. Meanwhile, the standard mixed variational formulation requires inverting $K$. We suggest the expanded mixed finite element methods for a nonlinear degenerate equations modeling Forchheimer flow. Compared with the standard mixed finite element method, the expanded mixed finite element method introduces three variables: unknown scalar function, its gradient, and a flux. The expanded mixed finite element method enables us to compute gradient of pressure directly. Woodward and Dawson in [35] studied expanded mixed finite element methods for a nonlinear parabolic equation modeling flow into variably saturated porous media. In their analysis, the Kirchhoff transformations are used to move the nonlinearity from the $K$ term to the gradient and thus analysis of the equations is simplified. This transformation is not applicable to our system (3.1).

In this article, we consider the initial boundary value problem for the pressure in a bounded domain with time-dependent Dirichlet boundary data. Since the system of equations for pressure is of degenerate type, finding the error estimates, in general, is a difficult task and requires much work. The objective of this paper is to propose an expanded mixed finite element approximation to the coupled system (3.1). We show that the error estimates of pressure and gradient for particular flows in the current study can be obtained without overcomplicated calculations thanks to specific structures of the corresponding nonlinear degenerate equations. Additionally, we extend the techniques for parabolic equations in [22, 14] and the expanded mixed finite element method in [1] to deal with the equations’ degeneracy. Such extension enables us (1) to utilize both the special structure of the equation and the advantages of the expanded mixed finite element method to provide, for instance, certain implementable advantages over the standard mixed method especially for the lowest order Raviart-Thomas (RT) space on the rectangular grids, (2) to obtain the error estimates for the solution in several norms of interest.

This paper is organized as follows: In section §2, we recall the main definitions and relevant results from [2], notations and preliminary results. In section §3, we consider the expanded mixed formulation and standard results for the mixed finite element approximations. In section §4, we derive many bounds for solutions to (3.3) and (3.7) in Lebesgue norms in terms of the boundary data and the initial data. In section §5, we consider the semidiscrete mixed finite element method to approximate the solution of the system (3.2). The rate of convergence is established in $L^2$-norms and $L^\infty$-norm for the pressure. In addition to this, the error estimates for the gradient of pressure and the flux variable are also derived under suitable assumptions on the regularity. Also an implicit backward-difference time discretization of the semidiscrete scheme is proposed to solve the system (3.2). We extend the analysis from semidiscrete to fully discrete to obtain the error estimates in a suitable norm for the three relevant variables. In section §6, we give a numerical example using the lowest Raviart-Thomas mixed finite element to support our theoretical analysis.

2. Preliminaries and auxiliaries. In this paper, we consider a fluid in a porous medium in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$ with the boundary $\Gamma$. Let $x \in \mathbb{R}^d$, $0 < T < \infty$, $t \in (0, T]$ be the spatial and time variable respectively. The fluid flow has a velocity $u(x, t) \in \mathbb{R}^d$, and a pressure $p(x, t) \in \mathbb{R}$.

A generalized Forchheimer equation, which is studied in [2], has the form

$$g(|u|)u = -\nabla p,$$  

(2.1)

where the function $g$ is a polynomial with nonnegative coefficients. More precisely, the function $g : \mathbb{R}^r \to \mathbb{R}^r$ is of the form

$$g(s) = a_0 + a_1 s^{\alpha_1} + \cdots + a_N s^{\alpha_N}, \quad s \geq 0, N \geq 1.$$  

(2.2)
0 < α_1 < ... < α_N are fixed numbers, the coefficients α_0, ..., α_N are nonnegative numbers with α_0 > 0, α_N > 0. The number α_N is the degree of g and is denoted by deg(g).

In particular, when \( g(s) = a + \beta s, a + \beta s + \gamma s^2 + \gamma_m s^{m-1} \), where \( a, \beta, \gamma, m, \gamma_m \) are empirical constants, we have Darcy’s law, Forheimer’s two term, three term and power law, respectively.

The monotonicity of the nonlinear term and the non-degeneracy of Darcy’s parts in (2.1) enable us to write \( \mathbf{u} \) implicit in terms of \( \nabla p \):

\[
\mathbf{u} = -K(\|\nabla p\|)\nabla p. \tag{2.3}
\]

Here the function \( K: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is defined by

\[
K(\xi) = \frac{1}{g(s(\xi))} \quad \text{where} \quad s = s(\xi) \geq 0 \quad \text{satisfies} \quad s g(s) = \xi, \quad \text{for} \quad \xi \geq 0. \tag{2.4}
\]

For slightly compressible fluids, the state equation is

\[
\frac{d\rho}{dp} = \kappa^{-1} \rho \quad \text{or} \quad \rho(p) = \rho_0 \exp\left(\frac{p - p_0}{\kappa}\right), \quad \kappa > 0. \tag{2.5}
\]

Other equation governing the fluid’s motion is the equation of continuity

\[
\frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{u}) = 0,
\]

which yields

\[
\frac{d\rho}{dt} + \rho \frac{d\rho}{dp} = 0.
\tag{2.6}
\]

Combining (2.5) and (2.6), we obtain

\[
\frac{d\rho}{dt} + \kappa \nabla \cdot \mathbf{u} = 0. \tag{2.7}
\]

Since the constant \( \kappa \) is very large for most slightly compressible fluids in porous media, see [25]. In most of the practical applications, the third term on the left-hand side of (2.7) is neglected. This results in the following reduced equation

\[
\frac{d\rho}{dt} + \kappa \nabla \cdot \mathbf{u} = 0. \tag{2.8}
\]

By rescaling the time variable, hereafter we assume that \( \kappa = 1 \).

From (2.3) and (2.8), we obtain the system

\[
\begin{cases}
\mathbf{u} + K(\|\nabla p\|)\nabla p = 0, \\
\rho_t + \nabla \cdot \mathbf{u} = 0.
\end{cases} \tag{2.9}
\]

The function \( K(\cdot) \) has the important properties (e.g. [2, 14]):

(i) \( K: [0, \infty) \rightarrow (0, a_0^{-1}] \) and it decreases in \( \xi \).

(ii) Type of degeneracy

\[
\frac{c_1}{(1 + \xi)^a} \leq K(\xi) \leq \frac{c_2}{(1 + \xi)^a}. \tag{2.10}
\]

(iii) For all \( n \geq 1 \),

\[
c_3(\xi^{n-\alpha} - 1) \leq K(\xi)\xi^n \leq c_2\xi^{n-\alpha}. \tag{2.11}
\]
(iv) Relation with its derivative

\[ -a K(\xi) \leq K'(\xi) \xi \leq 0, \quad (2.12) \]

where \( c_1, c_2, c_3 \) are positive constants depending on \( \Omega \) and \( g \). The constant \( a \in (0, 1) \) is defined by

\[ a = \frac{\alpha_N}{\alpha_N + 1} = \frac{\deg(g)}{\deg(g) + 1}. \quad (2.13) \]

We define

\[ H(\xi) = \int_0^{\sqrt{s}} K(\sqrt{s}) ds, \text{ for } \xi \geq 0. \quad (2.14) \]

The function \( H(\xi) \) can be compared with \( \xi \) and \( K(\xi) \) by

\[ K(\xi) \xi^2 \leq H(\xi) \leq 2K(\xi) \xi^2. \quad (2.15) \]

**Notations.** Let \( L^2(\Omega) \) be the set of square integrable functions on \( \Omega \) and \((L^2(\Omega))^d\) the space of \( d \)-dimensional vectors, which have all the components in \( L^2(\Omega) \). We denote \((\cdot, \cdot)\) the inner product in either \( L^2(\Omega) \) or \((L^2(\Omega))^d\) that is

\[ (\xi, \eta) = \int_\Omega \xi \eta dx \quad \text{or} \quad (\xi, \eta) = \int_\Omega \xi \cdot \eta dx. \]

The notation \( \| \cdot \| \) will mean scalar norm \( \| \cdot \|_{L^2(\Omega)} \) or vector norm \( \| \cdot \|_{(L^2(\Omega))^d} \).

For \( 1 \leq q \leq +\infty \) and \( m \) is any nonnegative integer, let

\[ W^{m,q}(\Omega) = \{ f \in L^q(\Omega), D^a f \in L^q(\Omega), |\alpha| \leq m \} \]

denote the Sobolev space, endowed with the norm

\[ \| f \|_{m,q} = \left( \sum_{|\alpha| \leq m} \| D^a f \|_{L^q(\Omega)}^q \right)^{1/q} . \]

We define \( H^m(\Omega) = W^{m,2}(\Omega) \) with the norm \( \| \cdot \|_m = \| \cdot \|_{m,2} \).

For functions \( p, u \) and vector-functions \( v, s, u \), we use short hand notations

\[ \| p(t) \| = \| p(\cdot, t) \|_{0,2}, \quad \| u(t) \| = \| u(\cdot, t) \|_{0,2}, \quad \| s(t) \|_{0,\beta} = \| s(\cdot, t) \|_{0,\beta} \]

and \( u^0(\cdot) = u(\cdot, 0), \quad v^0(\cdot) = v(\cdot, 0). \)

For the monotonicity and continuity of the differential operator in (2.9), we have the following results.

**Proposition 2.1 ([14]).** The following statements hold

(i) For all \( y, y' \in \mathbb{R}^d \),

\[ (K(|y'|) y' - K(|y|) y) \cdot (y' - y) \geq (1 - a) K(\max(|y|, |y'|)) |y' - y|^2. \quad (2.16) \]

(ii) For the vector functions \( s_1, s_2 \), there is a positive constant \( C \) such that

\[ (K(|s_1|) s_1 - K(|s_2|) s_2, s_1 - s_2) \geq C \omega \| s_1 - s_2 \|^2_{L^{2-a}(\Omega)}, \quad (2.17) \]

where

\[ \omega = \left( 1 + \max(\| s_1 \|^2_{L^{2-a}(\Omega)}, \| s_2 \|^2_{L^{2-a}(\Omega)}) \right)^{-a}. \]
Proposition 2.2. For all \( y, y' \in \mathbb{R}^d \). There exists a positive constant \( C \) depending on the polynomial \( g \), the spatial dimension \( d \) and the domain \( \Omega \) such that

\[
|K(|y'|) y' - K(|y|) y| \leq C |y' - y|.
\]  

(2.18)

Proof. Case 1: The origin does not belong to the segment connecting \( y' \) and \( y \). Let

\[
\ell(t) = ty' + (1 - t)y, \quad t \in [0, 1].
\]

For unit \( \mathbf{k} \in \mathbb{R}^d \), we define \( h(t) = K(|\ell(t)|) \ell(t) \cdot \mathbf{k} \).

By the Mean Value Theorem, there is \( t_0 \in [0, 1] \) with \( \ell(t_0) \neq 0 \) such that

\[
|K(|y'|) y' - K(|y|) y \cdot \mathbf{k}| = |h(1) - h(0)| = |h'(t_0)|
\]

\[
= \left| K'(|\ell(t_0)|) \frac{\ell(t_0) \cdot \ell'(t_0)}{|\ell(t_0)|^2} \ell(t_0) \right| K(|\ell(t_0)|). \ell'(t_0) \cdot \mathbf{k}.
\]

Using (2.12), we find that

\[
|K(|y'|) y' - K(|y|) y \cdot \mathbf{k}| \leq K(|\ell(t_0)|) \left( a \left| \frac{\ell(t_0) \cdot \ell'(t_0)}{|\ell(t_0)|^2} \ell(t_0) \right| + |\ell'(t_0)| \right) |\mathbf{k}|,
\]

which implies

\[
|K(|y'|) y' - K(|y|) y| \leq K(|\ell(t_0)|) \left( a \left| \frac{\ell(t_0) \cdot \ell'(t_0)}{|\ell(t_0)|^2} \ell(t_0) \right| + |\ell'(t_0)| \right).
\]

The inequality (2.18) follows by the upper boundedness of \( K() \leq a_0^{-1} \).

Case 2: The origin belongs to the segment connect \( y' \), \( y \). We replace \( y' \) by \( y' + \varepsilon \) then let \( \varepsilon \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \). Then we apply the above inequality for \( y \) and \( y + \varepsilon \), then let \( \varepsilon \rightarrow 0 \). \( \square \)

The following Poincaré-Sobolev inequality with weight is useful in our estimates later.

Lemma 2.3 (cf. [15]). Let \( \xi(x) \geq 0 \) be defined on \( \Omega \). Then for any function \( u(x) \) vanishing on the boundary \( \Gamma \), there is a positive constant \( C \) such that

\[
\| u \|_{L^2(\Omega; a)}^2 \leq C \| K^{1/2} (\xi) \nabla u \|^2 \left( 1 + \| K^{1/2} (\xi) \xi \|^2 \right)^{2d/2d - a},
\]

(2.19)

where \( (2 - a)^* = \frac{d(2 - a)}{d - 2 - a} \) with \( d \) being the dimension of the space.

Throughout this paper, the constants \( \beta = 2 - a, \lambda = \frac{a}{d - a} \). The letters \( C, C_0, C_1,... \) will represent positive generic constants and their values depend on the exponents, coefficients of the polynomial \( g \), the spatial dimension \( d \) and the domain \( \Omega \), independent of the initial and boundary data, size of mesh and time step. These constants may be different from place to place.

3. The expanded mixed finite element method. Next, we introduce the new variable \( s = \nabla p \) to (2.9) and study the initial value boundary problem (IVBP)

\[
\begin{align*}
    p_t + \nabla \cdot u &= f, & (x, t) &\in \Omega \times (0, T), \\
    u + K(|\mathbf{s}|)\mathbf{s} &= 0, & (x, t) &\in \Omega \times (0, T), \\
    \mathbf{s} - \nabla p &= 0, & (x, t) &\in \Omega \times (0, T),
\end{align*}
\]

(3.1)

where \( f : \Omega \times [0, T] \rightarrow \mathbb{R}, f \in C^1([0, T]; L^2(\Omega)) \).

The initial and boundary conditions respectively are \( p(x, 0) = p_0(x) \) in \( \Omega \), and \( p(x, t) = \psi(x, t) \) on \( \Gamma \times [0, T] \). Also, we require \( p_0(x) = \psi(x, 0) \) on the boundary \( \Gamma \).
To deal with the non-homogeneous boundary condition, we extend the Dirichlet boundary data from $\Gamma$ to the whole domain $\Omega$, see in [14, 19, 24]. Let $\Psi(x, t)$ be a such extension. Let $\tilde{p} = p - \Psi$. Then $\tilde{p}(x, t) = 0$ on $\Gamma \times (0, T)$. The system (3.1) will take the form

$$
\begin{aligned}
\begin{cases}
\begin{aligned}
\tilde{p}_t + \nabla \cdot \mathbf{u} &= -\Psi_t + f,
\mathbf{u} + K(|\mathbf{s}|) \mathbf{s} &= 0,
\mathbf{s} - \nabla \tilde{p} &= \nabla \Psi,
\end{aligned}
\end{cases}
\end{aligned}
$$

(3.2)

where $\tilde{p}(x, 0) = p_0(x) - \Psi(x, 0) = \tilde{p}_0(x)$.

Define $W = L^2(\Omega)$, $\tilde{W} = (L^2(\Omega))^d$ and the space

$$
V = H(\text{div}, \Omega) = \left\{ \mathbf{v} \in (L^2(\Omega))^d, \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}
$$

with the norm defined by $\|\mathbf{v}\|_V^2 = \|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2$.

The variational formulation of (3.2) is defined as follows:

Find $(\mathbf{p}, \mathbf{u}, \mathbf{s}) : [0, T] \to W \times V \times \tilde{W}$ such that

$$
\begin{aligned}
(\mathbf{p}_t, \mathbf{w}) + (\nabla \cdot \mathbf{u}, \mathbf{w}) &= (f - \Psi_t, \mathbf{w}), &\forall \mathbf{w} \in W, \\
(\mathbf{u}, \mathbf{z}) + (K(|\mathbf{s}|) \mathbf{s}, \mathbf{z}) &= 0, &\forall \mathbf{z} \in \tilde{W}, \\
(\mathbf{s}, \mathbf{v}) + (\tilde{p}, \nabla \cdot \mathbf{v}) &= (\nabla \Psi, \mathbf{v}), &\forall \mathbf{v} \in V
\end{aligned}
$$

(3.3)

with $\tilde{p}(x, 0) = \tilde{p}_0(x)$.

**Semidiscrete method.** Let $(\mathcal{T}_h)_h$ be a family of quasiuniform triangulations of $\Omega$ with $h$ being the maximum diameter of the mesh element. Let $V_h$ be the Raviart-Thomas-Nédélec spaces [26, 32] of order $k \geq 0$ or the Brezzi-Douglas-Marini spaces [3] of index $k$ over each triangulation $\mathcal{T}_h$. Let $W_h$ the space of discontinuous piecewise polynomials of degree $k$ over $\mathcal{T}_h$ and $\tilde{W}_h$ be the $d$-dimensional vector space of discontinuous piecewise polynomials of degree $k$ over $\mathcal{T}_h$. Let $W_h \times V_h \times \tilde{W}_h$ be the mixed element spaces approximating to $W \times V \times \tilde{W}$.

We use the standard $L^2$-projection operator, see in [7]), $\pi : W \to W_h$, $\pi : \tilde{W} \to \tilde{W}_h$ satisfying

$$
\begin{aligned}
(\pi \mathbf{w}, \nabla \cdot \mathbf{v}_h) = (\mathbf{w}, \nabla \cdot \mathbf{v}_h), &\forall \mathbf{w} \in W, \mathbf{v}_h \in V_h, \\
(\pi \mathbf{z}, \mathbf{z}_h) = (\mathbf{z}, \mathbf{z}_h), &\forall \mathbf{z} \in \tilde{W}, \mathbf{z}_h \in \tilde{W}_h,
\end{aligned}
$$

(3.3a) (3.3b)

Also, we use $H$-div projection $\Pi : V \to V_h$ defined by

$$
(V \cdot \Pi \mathbf{v}, \mathbf{w}_h) = (V \cdot \mathbf{v}, \mathbf{w}_h), &\forall \mathbf{w}_h \in W_h.
$$

(3.3c)

These projections have well-known approximation properties, see in [4, 20],

(i) $\|\pi \mathbf{w}\| \leq \|\mathbf{w}\|$ holds for all $\mathbf{w} \in L^2(\Omega)$.

(ii) There exist positive constants $C_1, C_2$ such that

$$
\|\pi \mathbf{w} - \mathbf{w}\|_{0,a} \leq C_1 h^m \|\mathbf{w}\|_{m,a} &\text{ and } \|\pi \mathbf{z} - \mathbf{z}\|_{0,a} \leq C_2 h^m \|\mathbf{z}\|_{m,a},
$$

(3.4)

for all $\mathbf{w} \in W^{m,a}(\Omega)$, $\mathbf{z} \in (W^{m,a}(\Omega))^d$, $0 \leq m \leq k + 1, 1 \leq \alpha \leq \infty$. In short hand, when $\alpha = 2$ we write (3.4) as

$$
\|\pi \mathbf{w} - \mathbf{w}\| \leq C_1 h^m \|\mathbf{w}\|_m &\text{ and } \|\pi \mathbf{z} - \mathbf{z}\| \leq C_2 h^m \|\mathbf{z}\|_m.
$$

(iii) There exists a positive $C_3$ such that

$$
\|\Pi \mathbf{v} - \mathbf{v}\|_{0,a} \leq C_3 h^m \|\mathbf{v}\|_{m,a}
$$

(3.5)
for any $v \in \{W^{m,a}(\Omega)\}^d$, $1/\alpha \leq m \leq k + 1$, $1 \leq \alpha \leq \infty$.

Because of the commuting relation between $\pi$, $\Omega$ and the divergence, we also have the bound

$$\| \nabla \cdot (\Omega v - v) \|_{0,a} \leq C_1 h^m \| \nabla \cdot v \|_{m,a},$$  \hspace{1cm} (3.6)

provided $\nabla \cdot v \in W^{m,a}(\Omega)$ for $1 \leq m \leq k + 1$.

The semidiscrete expanded mixed formulation of (3.3) can read as follows:

Find $(p_h, u_h, s_h) : [0, T] \rightarrow W_h \times V_h \times \tilde{W}_h$ such that

\begin{align*}
(p_{h,t}, w_h) + (\nabla \cdot u_h, w_h) &= (f - \Psi_t, w_h), &\forall w_h \in W_h, & \text{(3.7a)} \\
(u_h, \zeta_h) + (K(\nabla s_h) \zeta_h, \zeta_h) &= 0, &\forall \zeta_h \in \tilde{V}_h, & \text{(3.7b)} \\
(s_h, v_h) + (\bar{p}_h, \nabla \cdot v_h) &= (\nabla \pi, v_h), &\forall v_h \in \tilde{V}_h, & \text{(3.7c)}
\end{align*}

where $\bar{p}_h(x, 0) = \pi p_0(x)$, $\bar{p}_h = p_h - \pi \Psi$.

**Fully discrete method.** Let $\{t_n\}_{n=1}^N$ be the uniform partition of $[0, T]$ with $t_n = n \Delta t$, for time step $\Delta t > 0$, and we defined $\varphi^n = \varphi(\cdot, t_n)$. The discrete time expanded mixed finite element approximation to (3.3) is defined as follows: For given $\bar{p}_h^0(x) = \pi p_0(x)$, $s_h^0(x) = \pi \nabla \rho^0(x)$, $u_h^0(x) = K(\nabla s_h^0(x)) s_h^0(x)$, $\{f^n\}_{n=1}^N \in L^2(\Omega)$, $\{\pi^n\}_{n=1}^N \in L^\infty(\Omega)$.

Find $(p^n_h, u^n_h, s^n_h) \in W_h \times V_h \times \tilde{W}_h$, $n = 1, 2, \ldots, N$, such that

\begin{align*}
(p^n_h - p^{n-1}_h, w_h) + (\nabla \cdot u^n_h, w_h) &= (f^n - \Psi_t, w_h), &\forall w_h \in W_h, & \text{(3.8a)} \\
(u^n_h, \zeta_h) + (K(\nabla s^n_h) \zeta_h, \zeta_h) &= 0, &\forall \zeta_h \in \tilde{V}_h, & \text{(3.8b)} \\
(s^n_h, v_h) + (\bar{p}_h^n, \nabla \cdot v_h) &= (\nabla \pi^n, v_h), &\forall v_h \in \tilde{V}_h, & \text{(3.8c)}
\end{align*}

4. **Estimates of solutions.** Using the theory of monotone operators e.g. [23, 34, 36], the authors proved in [17] the global existence and uniqueness of a weak solution $p(x, t)$ of equation (3.2).

In this section, we obtain some estimates for the solution of the system (3.3), which are important to our error analysis later. We assume that the solution has sufficient regularities both in $x$ and $t$ variables so that our calculations are valid.

**Theorem 4.1.** Let $(p, u, s)$ be the solution of the problem (3.3). There exists a positive constant $C$ such that for all $t \in (0, T)$

(i)

$$\| \bar{p}(t) \|^2 + \frac{1}{2} \int_0^t \| K^{1/2} (s(\tau)) s(\tau) \|^2 d\tau \leq \| p^0 \|^2 + C \int_0^t \mathcal{A}(\tau) d\tau,$$  \hspace{1cm} (4.1)

where

$$\mathcal{A}(\tau) = \| \nabla \Psi(t) \|^2 + \| (f - \Psi_t)(t) \|_{0,r} + \| (f - \Psi_t)(t) \|_{0,r}^\beta \| \frac{\beta}{\beta(d + 1) - d} \| \phi(t) \|_{0,r}$$

Consequently,

$$\| p(t) \|^2 + \int_0^t \| K^{1/2} (s(\tau)) s(\tau) \|^2 d\tau \leq \| p^0 \|^2 + \| \Psi(t) \|^2 + C \int_0^t \mathcal{A}(\tau) d\tau.$$  \hspace{1cm} (4.2)

(ii)

$$\| u(t) \|^2 + \| s(t) \|_{0,\beta} \leq C \left( \| \bar{p}^0 \|^2 + 1 + \int_0^t e^{-\frac{\tau}{t}} (\Lambda + \mathcal{B})(\tau) d\tau \right),$$
where

\[
\Lambda(t) = \int_0^t A(\tau) \, d\tau, \quad \mathcal{B}(t) = A(t) + \|\nabla \Psi(t)\|^2 + \|(\Psi - f)(t)\|^2. \tag{4.3}
\]

**Proof.** (i) Selecting \( u = \tilde{p}, \ z = s \) and \( v = -u \) in (3.3) and adding three resultant equations give

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{p}\|^2 + \|K^\frac{1}{2}(|s|)s\|^2 = (f - \Psi_t, \tilde{p}) - (\nabla \Psi, u) \leq (f - \Psi_t, \tilde{p}) + \frac{1}{2} \|\nabla \Psi\|^2 + \|u\|^2. \tag{4.4}
\]

Using (3.3b) with \( z = u \in \tilde{W} \), we find that

\[
\|u\| \leq \|K^\frac{1}{2}(|s|)s\|. \tag{4.5}
\]

Thus, (4.4) and (4.5) give

\[
\frac{d}{dt} \|\tilde{p}\|^2 + \|K^\frac{1}{2}(|s|)s\|^2 \leq 2(f - \Psi_t, \tilde{p}) + \|\nabla \Psi\|^2. \tag{4.6}
\]

By Hölder’s inequality and (2.19),

\[
2(f - \Psi_t, \tilde{p}) \leq 2\|f - \Psi_t\|_{0, r} \|\tilde{p}\|_{0, r} \leq C\|f - \Psi_t\|_{0, r} \|K^\frac{1}{2}(|s|)\nabla \tilde{p}\| \left(1 + \|K^\frac{1}{2}(|s|)s\|^2\right)^{\frac{1}{2}} \leq C\|f - \Psi_t\|_{0, r} \|K^\frac{1}{2}(|s|)\nabla \tilde{p}\| \left(1 + \|K^\frac{1}{2}(|s|)s\|^2\right). \tag{4.7}
\]

To estimate the second term on the right hand side of (4.7), we integrate by parts (3.3c) and then select \( v = K(|s|)\nabla \tilde{p} \in V \). It follows that

\[
\|K^\frac{1}{2}(|s|)\nabla \tilde{p}\|^2 = (s - \nabla \Psi, K(|s|)\nabla \tilde{p}) \leq \|K^\frac{1}{2}(|s|)(s - \nabla \Psi)\| \|K^\frac{1}{2}(|s|)\nabla \tilde{p}\|.
\]

The upper boundedness of \( K(\cdot) \) gives

\[
\|K^\frac{1}{2}(|s|)\nabla \tilde{p}\| \leq \|K^\frac{1}{2}(|s|)s\| + C\|\nabla \Psi\|. \tag{4.8}
\]

Combining (4.7), (4.8) and Young’s inequality yield

\[
2(f - \Psi_t, \tilde{p}) \leq C\|f - \Psi_t\|_{0, r} \left(\|K^\frac{1}{2}(|s|)s\| + \|\nabla \Psi\|\right) \left(1 + \|K^\frac{1}{2}(|s|)s\|^2\right)^{\frac{1}{2}} \leq C\|f - \Psi_t\|_{0, r} \left(\|K^\frac{1}{2}(|s|)s\|^2 + \|\nabla \Psi\|^2\right) \left(1 + \|K^\frac{1}{2}(|s|)s\|^4 + \|\nabla \Psi\|^4\right) \leq C\|f - \Psi_t\|_{0, r} \left(1 + \|K^\frac{1}{2}(|s|)s\|^4 + \|\nabla \Psi\|^4\right) \leq \frac{1}{2} \|K^\frac{1}{2}(|s|)s\|^2 + C\mathcal{A}(t). \tag{4.9}
\]

It follows from (4.6) and (4.9) that

\[
\frac{d}{dt} \|\tilde{p}\|^2 + \frac{1}{2} \|K^\frac{1}{2}(|s|)s\|^2 \leq C\mathcal{A}(t). \tag{4.10}
\]
Integrating (4.10) from 0 to $t$, we obtain (4.1).

(ii) Choosing $\mathbf{w} = \tilde{\mathbf{p}}_t$, $\mathbf{z} = \mathbf{s}_t$ in (3.3a), (3.3b), differentiating (3.3c) with respect $t$ and selecting $\mathbf{v} = \mathbf{u}$, we find that

$$\|\tilde{\mathbf{p}}_t\|^2 + (K(\|\mathbf{s}\|)\mathbf{s}, \mathbf{s}_t) = (f - \Psi_t, \tilde{\mathbf{p}}_t) - (\nabla \Psi_t, \mathbf{u}).$$  \hfill (4.11)

Using (3.3b) with $\mathbf{z} = \nabla \Psi_t \in \tilde{W}$ and the boundedness of the $K(\cdot)$ from above we have

$$(\mathbf{u}, \nabla \Psi_t) = -(K(\|\mathbf{s}\|)\mathbf{s}, \nabla \Psi_t) \leq C K^{\frac{1}{2}}(\|\mathbf{s}\|)\|\nabla \Psi_t\|.$$  \hfill (4.12)

The definition of function $H(\cdot)$ in (2.14) gives

$$K(\|\mathbf{s}\|)\mathbf{s} \cdot \mathbf{s}_t = \frac{1}{2} \frac{d}{dt} H(\|\mathbf{s}(x, t)\|).$$

Thus, (4.11) and (4.12) yield

$$\|\tilde{\mathbf{p}}_t\|^2 + \frac{1}{2} \frac{d}{dt} \int_\Omega H(x, t) dx \leq C K^{\frac{1}{2}}(\|\mathbf{s}\|)\|\nabla \Psi_t\| + \|f - \Psi_t\|\|\tilde{\mathbf{p}}_t\|$$

$$\leq \varepsilon \|K^{\frac{1}{2}}(\|\mathbf{s}\|)\|^2 + C \|\nabla \Psi_t\|^2 + \|f - \Psi_t\|\|\tilde{\mathbf{p}}_t\|$$  \hfill (4.13)

for all $\varepsilon > 0$, where $H(x, t) = H(\|\mathbf{s}(x, t)\|)$.

Adding (4.10) and (4.13) and selecting $\varepsilon = \frac{1}{4}$ imply

$$\|\tilde{\mathbf{p}}_t\|^2 + \frac{1}{2} \frac{d}{dt} \int_\Omega H(x, t) dx \leq \frac{1}{4} \int_\Omega H(x, t) dx + C \|\tilde{\mathbf{p}}_t\|^2 + C \mathcal{B}(t).$$

Then by Young’s inequality,

$$\frac{1}{2} \|\tilde{\mathbf{p}}_t\|^2 + \frac{1}{2} \frac{d}{dt} \int_\Omega H(x, t) dx \leq \frac{1}{4} \int_\Omega H(x, t) dx + C \|\tilde{\mathbf{p}}_t\|^2 + C \mathcal{B}(t).$$

This and (2.15) show that

$$\|\tilde{\mathbf{p}}_t\|^2 + \frac{d}{dt} \int_\Omega H(x, t) dx \leq -\frac{1}{4} H(x, t) dx + \mathcal{B}(t).$$  \hfill (4.14)

Dropping the first term and using Gronwall’s inequality, we obtain

$$\int_\Omega H(x, t) dx \leq e^{-\frac{t}{4}} \int_\Omega H(x, 0) dx + C \int_0^t e^{-\frac{t}{4}} (\|\tilde{\mathbf{p}}_t\|^2 + \mathcal{B}(t)) dt.$$  \hfill (4.15)

It follows from (2.11), (2.15) and (4.1) that

$$\|\mathbf{s}\|^2_{0, \beta} \leq -e^{-\frac{t}{4}} \|\mathbf{s}\|^2_{0, \beta} + C + C \int_0^t e^{-\frac{t}{4}} (\|\tilde{\mathbf{p}}_t\|^2 + \mathcal{B}(\tau) + \mathcal{B}(\tau) + \mathcal{B}(\tau)) dt.$$  \hfill (4.16)

Due to (4.5) and $K(\xi) \xi^2 \leq C\xi^{2-\alpha}$$, \quad \|\mathbf{u}\|^2 \leq \|\mathbf{s}\|^2_{0, \beta}.$  \hfill (4.17)

Combining (4.17) and (4.16), we obtain (4.2). The proof is complete. □

Although the solution is considered to be continuous at $t = 0$ in the appropriate Lebesgue or Sobolev space, its time derivative is not. In the following, we prove that the time derivative of the pressure is bounded.
Theorem 4.2. Let $0 < t_0 < T$. Suppose $(\rho, u, s)$ be the solution of the problem (3.3). Then there is a positive constant $C$ such that for all $t_0 < t \leq T$,
\[
\|\check{\rho}(t)\|^2 \leq C \left\{ \|\rho^0\|^2 + t_0^{-1} \left( \|s_0\|_0^2 + \int_0^{t_0} (\Lambda + \mathcal{B}(t)) \, dt \right) + \int_0^t \left( (f_1 - \Psi(t)) \|\nabla s\|^2 + \|\nabla \Psi(t)\|^2 \right) \, dt \right\},
\]
where $\Lambda(t), \mathcal{B}(t)$ are defined as in (4.3).

Proof. We differentiate (3.3) with respect to $t$ to obtain
\[
\begin{align*}
(\check{\rho}_{tt}, w) + (\nabla \cdot u, w) &= (f_1 - \Psi_{tt}, w), \quad \forall w \in W, \\
(u_t, z) + (K(|s|)s, z) + \left\{ \frac{s \cdot s_t}{|s|} s_t - s \right\} &= 0, \quad \forall z \in \tilde{W}, \\
(s_t, v) + (\check{\rho}_{tt}, \nabla \cdot v) &= (\nabla \Psi(t), v), \quad \forall v \in V.
\end{align*}
\]
Selecting $w = \check{\rho}_t$, $z = s_t$, and $v = -u_t$ as a test functions, and summing resultant equations, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\check{\rho}_t\|^2 + \|K^{1/2}(|s|)s_t\|^2 = -\left\{ \frac{s \cdot s_t}{|s|} s_t \left( \frac{\check{\rho}_t}{s_t} s_t - s \right) \right\} + (f_1 - \Psi_{tt}, \check{\rho}_t) - (\nabla \Psi(t), u_t)
\]
\[
= I_1 + I_2 + I_3.
\]
Using (2.12), we can bound $I_1$ by
\[
|I_1| \leq a \|K^{1/2}(|s|)s_t\|^2.
\]
We use Young’s inequality to bound
\[
|I_2| \leq \frac{1}{2} \|f_1 - \Psi_{tt}\|^2 + \frac{1}{2} \|\check{\rho}_t\|^2,
\]
and
\[
|I_3| \leq \frac{1 + a}{2(1 + a)} \|\nabla \Psi(t)\|^2 + \frac{1 - a}{2(1 + a)} \|u_t\|^2.
\]
Then we estimate the last term in (4.23) by taking $z = u_t$ in (4.19b) to get
\[
\|u_t\| \leq \|K^{1/2}(|s|)s_t\| + \|K^{1/2}(|s|)s_t\| \leq (1 + a) \|K^{1/2}(|s|)s_t\|.
\]
Putting (4.24) into (4.23), we find that
\[
|I_3| \leq C \|\nabla \Psi(t)\|^2 + \frac{1 - a}{2} \|K^{1/2}(|s|)s_t\|^2.
\]
Now, substituting (4.21), (4.22) and (4.25) into (4.20) to show that
\[
\frac{d}{dt} \|\rho_t\|^2 + (1 - a) \|K^{1/2}(|s|)s_t\|^2 \leq \|f_1 - \Psi_{tt}\|^2 + \|\check{\rho}_t\|^2 + C \|\nabla \Psi(t)\|^2.
\]
This implies
\[
\frac{d}{dt} \|\check{\rho}_t\|^2 \leq \|\check{\rho}_t\|^2 + C \left( \|f_1 - \Psi_{tt}\|^2 + \|\nabla \Psi(t)\|^2 \right).
\]
For $t \geq t'$, integrating (4.26) from $t'$ to $t$, we find that
\[
\|\check{\rho}_t(t)\|^2 \leq \|\check{\rho}_t(t')\|^2 + C \int_{t'}^t (\|f_1 - \Psi_{tt}\|^2 + \|\nabla \Psi(t)\|^2) \, dt.
\]
Integrating (4.27) in $t'$ from 0 to $t_*$, using (4.15), we obtain
\[ t_* \| \bar{p}_{t'} \|^2 \leq C \left( \int_0^{t_*} \| \bar{p}_{t'} (t') \|^2 + \int_0^{t_*} \int_{t'}^t (\| \Psi_{tt} - f_t \|^2 + \| \nabla \Psi_t \|^2) \, dt \right). \]

To estimate the first term of the RHS of the previous inequality, we drop the negative term of the right hand side in (4.14), then integrate the resultant from 0 to $t_*$ to find the estimate
\[ \int_0^{t_*} \| \bar{p}_{t'} \|^2 \, dt + \int_\Omega H(x, t_*) \, dx \leq \int_\Omega H(x, 0) \, dx + C \int_0^{t_*} (\| \bar{p} \|^2 + \mathcal{B}(t)) \, dt \]
\[ \leq C \| s^0 \|^2 + C \int_0^{t_*} (\| \bar{p} \|^2 + \mathcal{B}(t)) \, dt. \]

Hence,
\[ t_* \| \bar{p}_{t'} \|^2 \leq C \left( \| s^0 \|_{0, \beta} + \int_0^{t_*} (\| \bar{p} \|^2 + \mathcal{B}(t)) \, dt + t_* \int_0^{t_*} (\| f_t - \Psi_{tt} \|^2 + \| \nabla \Psi_t \|^2) \, dt \right). \]

Now using (4.1), we reduce previous inequality to
\[ t_* \| \bar{p}_{t'} \|^2 \leq C \left( t_* \| \bar{p}_0 \|^2 + t_* \int_0^{t_*} (\Lambda(t) + \mathcal{B}(t)) \, dt + t_* \int_0^{t_*} (\| f_t - \Psi_{tt} \|^2 + \| \nabla \Psi_t \|^2) \, dt \right) \]
which proves (4.18). The proof is complete. \[ \Box \]

**Remark 4.3.** Let $(\bar{p}_h, \bar{u}_h, \bar{s}_h)$ solve the semidiscrete problem (3.7). Following the proofs of the Theorem 4.1 and Theorem 4.18, we prove that the discrete solutions $(\bar{p}_h, \bar{u}_h, \bar{s}_h)$ of problem (3.7) and the time derivative $\bar{p}_h$, also hold estimates (4.1), (4.2) and (4.18), respectively.

**Remark 4.4.** The equations (3.7a)–(3.7c) generate the system of ordinary differential equations in the coefficients of $(\bar{p}_h, \bar{u}_h, \bar{s}_h)$ with respect to the basis of $(\Omega h, V_h, \bar{W}_h)$. The stability estimates in (4.1) and (4.2) suffice to establish the existence of $(\bar{p}_h, \bar{u}_h, \bar{s}_h)$ for all $t \in (0, T)$. Furthermore, the uniqueness of the approximation solution is obtained from the monotonicity of the vector field $K(\| s \| s)$, see [2].

5. **Error analysis.** In this section, we use the estimates in the previous section, the techniques in [14], and the expanded mixed finite element method to establish the error estimates between the analytical solution and approximation solution in several norms.

5.1. **Error estimate for semidiscrete method.** We bound the error in the semidiscrete method in various norms by comparing the computed solution to the projections of the exact solution. To do this, we restrict the test functions in (3.3) to finite-dimensional spaces. Let
\[
\bar{p}_h - \bar{p} = (\bar{p}_h - \pi \bar{p}) + (\pi \bar{p} - \bar{p}) \equiv \bar{\theta} + \bar{\vartheta}, \quad \bar{s}_h - s = (\bar{s}_h - \pi s) + (\pi s - s) \equiv \bar{\eta} + \bar{\zeta}, \quad \bar{u}_h - u = (\bar{u}_h - \Pi u) + (\Pi u - u) \equiv \bar{\rho} + \bar{\rho}.
\]

For $t_* > 0$, we put
\[
Y = 1 + \| \bar{p}_0 \|^2 + \sup_{t \in [0, T]} \int_0^t e^{-\frac{\tau^2}{T}} (\Lambda + \mathcal{B}(\tau)) \, d\tau,
\]
\[
\Phi = \Phi(t_*) = \| \bar{p}_0 \|^2 + t_* \left( \| s^0 \|_{0, \beta} + \int_0^{t_*} (\Lambda(t) + \mathcal{B}(t)) \, dt \right) + \int_0^{t_*} (\| f_t - \Psi_{tt}(t) \|^2 + \| \nabla \Psi_t \|^2) \, dt.
\]

**Theorem 5.1.** Let $(p, u, s)$ solve problem (3.3) and $(\bar{p}_h, \bar{u}_h, \bar{s}_h)$ solve the semidiscrete problem (3.7). Suppose $p \in L^\infty(0, T; H^{k+1}(\Omega))$, $s \in L^2(0, T; (W^{k+1, 2}(\Omega))^d)$ then there is a positive constant $C$ such that for all $t \in (0, T), \nabla
\[
\| (p_h - p)(t) \|^2 \leq C (\| \bar{\theta}(t) \|^2 + \| \pi \Psi - \Psi(t) \|^2) + C Y \int_0^t \| \bar{\zeta}(\tau) \|_{0, \beta} \, d\tau.
\]

(5.1)
Consequently,
\[ \| (p_h - p)(t) \| \leq C h^{k+1} \left( \| p(t) \|_{k+1} + \| \Psi(t) \|_{k+1} \right) + C Y^{\frac{1}{2}} h^{\frac{k+1}{2}} \left( \int_0^t \| s(\tau) \|_{k+1} \, d\tau \right)^{\frac{1}{2}}. \]  

(5.2)

Proof. Subtracting (3.7) from (3.3), selecting \( \psi_h = \theta, z_h = \eta, v_h = -\rho \), using the \( L^2 \)-projection and \( H(\text{div}) \)-projection, we obtain

\( (\theta_t, \theta) + (V \cdot \rho, \theta) = 0, \)  

(5.3a)

\( (\rho, \eta) + (K(|s_h|)s_h - K(|s|)s, \eta) = 0, \)  

(5.3b)

\( - (\eta, \rho) - (\theta, V \cdot \rho) = 0. \)  

(5.3c)

Adding three equations (5.3a)–(5.3c) leads to
\[ \frac{1}{2} \frac{d}{dt} \| \theta \|^2 + C_0 \omega \| s_h - s \|_{0, \beta}^2 \leq \left( K(|s_h|)s_h - K(|s|)s, \zeta \right), \]  

(5.4)

As a result of applying (2.17) to the second term of (5.4), we see that
\[ \frac{1}{2} \frac{d}{dt} \| \theta \|^2 + C_0 \omega \| s_h - s \|_{0, \beta}^2 \leq \left( K(|s_h|)s_h - K(|s|)s, \zeta \right), \]  

(5.5)

where
\[ \omega = \omega(t) = \left( 1 + \max(\| s_h(t) \|_{0, \beta}, \| s(t) \|_{0, \beta}) \right)^{-\alpha}. \]  

(5.6)

We use \( K(|\zeta|) \zeta \leq C \zeta^\beta -1 \), Hölder's inequality and (4.2) to estimate the RHS in (5.5),
\[ |(K(|s_h|)s_h - K(|s|)s, \zeta)| \leq C(\| s_h \|^\beta -1 + \| s \|^\beta -1, |\zeta|) \]  

\[ \leq C \left( 1 + \| s_h \|_{0, \beta}^\beta + \| s \|_{0, \beta}^\beta \right) \| \zeta \|_{0, \beta} \]  

\[ \leq CY \| \zeta \|_{0, \beta}. \]

Thus,
\[ \frac{d}{dt} \| \theta \|^2 + \omega \| s_h - s \|_{0, \beta}^2 \leq CY \| \zeta \|_{0, \beta}. \]  

(5.7)

Integrating (5.7) in time from 0 to \( t \), and using the fact that \( \theta(0) = 0 \) show that
\[ \| \theta \|^2 + \int_0^t \omega \| s_h - s \|_{0, \beta}^2 \, dt \leq CY \int_0^t \| \zeta \|_{0, \beta} \, dt. \]  

(5.8)

Since
\[ p_h - p = \theta + \theta + (\pi \Psi - \Psi), \]  

(5.9)

then (5.1) follows by (5.9), Minkowski's inequality and (5.8). The proof is complete. □
Proof. We have from (5.9) and the triangle inequality that
\[ \| p - p_h \|_{0, \infty} \leq \| \theta \|_{0, \infty} + \| \bar{\pi} \|_{0, \infty} + \| \pi \Psi - \Psi \|_{0, \infty}. \] (5.11)

Applying the inverse estimate and using (5.8), we find that
\[ \| \theta \|_{0, \infty} \leq C h^{-1} \| \theta \| \leq C Y \frac{1}{2} h^{-1} \left( \int_0^T \| \zeta \|_{0, \beta} \right)^{\frac{1}{2}}. \] (5.12)

Hence,
\[ \| (p - p_h)(t) \|_{0, \infty} \leq \| \theta(t) \|_{0, \infty} + \| (\pi \Psi - \Psi)(t) \|_{0, \infty} + C Y \frac{1}{2} h^{-1} \left( \int_0^T \| \zeta(r) \|_{0, \beta} \right)^{\frac{1}{2}}, \] (5.13)

which implies (5.10). \[ \square \]

Now, we return to find an error estimate for vector gradient of pressure in \( L^p \)-norm.

**Theorem 5.3.** Let \( 0 < t_* < T \). Under the assumptions of Theorem 5.1, there exists a positive constant \( C \) independent of \( h \) such that for all \( t \in (t_*, T) \),
\[ \| (s_h - s)(t) \|_{0, \beta} \leq C Y \frac{1}{2} h^{-1} \left( \int_0^T \| s(r) \|_{k+1, \beta} \right)^{\frac{1}{2}} + C Y \frac{1}{2} h^{-1} \left( \| s(t) \|_{k+1, \beta} \right)^{\frac{1}{2}}, \] (5.14)

and
\[ \| (u_h - u)(t) \|_{0, \beta} \leq C Y \frac{1}{2} h^{-1} \left( \int_0^T \| s(r) \|_{k+1, \beta} \right)^{\frac{1}{2}} + C Y \frac{1}{2} h^{-1} \left( \| s(t) \|_{k+1, \beta} \right)^{\frac{1}{2}} + Ch^{k+1} \| u(t) \|_{k+1, \beta}. \] (5.15)

**Proof.** Due to (5.5), (5.4) and the \( L^2 \)-projection,
\[ \omega \| s_h - s \|_{0, \beta}^2 \leq \left( K(\| s_h \|) s_h - K(\| s \|) s, s_h - s \right) \]
\[ = -\left( \hat{\rho}_{h, t} - \hat{\rho}_{t}, \hat{\theta} \right) + \left( K(\| s_h \|) s_h - K(\| s \|) s, \zeta \right). \]

This, (5.1), and (5.8) with noticing that \( \| \hat{\rho}_{h, t} \|, \| \hat{\rho}_{t} \| \leq \Phi \) imply
\[ \omega \| s_h - s \|_{0, \beta}^2 \leq C \left( \| \hat{\rho}_{h, t} \| + \| \hat{\rho}_{t} \| \right) \| \theta \| + CY \| \zeta \|_{0, \beta} \]
\[ \leq C \Phi \left( Y \int_0^T \| \zeta \|_{0, \beta} \right)^{\frac{1}{2}} + CY \| \zeta \|_{0, \beta}. \] (5.16)

From (5.6) and the fact that \( \| s_h \|_{0, \beta}, \| s \|_{0, \beta} \leq Y \), we have
\[ \omega^{-1} \leq C \left( 1 + \| s_h \|_{0, \beta}^\beta + \| s \|_{0, \beta}^\beta \right)^4 \leq CY^4. \] (5.17)

Substituting (5.17) into (5.16) yields
\[ \| s_h - s \|_{0, \beta}^2 \leq CY^{4 + \frac{1}{2}} \Phi \left( \int_0^T \| \zeta \|_{0, \beta} \right)^{\frac{1}{2}} + Y^{4 + 1} \| \zeta \|_{0, \beta}. \] (5.18)

This proves (5.14).

To the end of the proof, we use the identity
\[ \| \rho \|_{0, \beta}^\beta = \left( \rho, \rho^{\beta-1} \right) = (u_h - u, \rho^{\beta-1}) = -\left( K(\| s_h \|) s_h - K(\| s \|) s, \rho^{\beta-1} \right) \] (5.19)
Combining (5.20) and Höder’s inequality lead to
\[ \|\rho\|_{L^\beta}^\beta \leq C(\|s_h - s\|, \|\rho\|_{L^\beta}^{\beta - 1}) \leq C\|s - s_h\|_0,\beta \|\rho\|_0,\beta^{\beta - 1} \]
and hence
\[ \|\rho\|_0,\beta \leq C\|s - s_h\|_0,\beta. \] (5.19)

The triangle inequality and (5.19) yield
\[ \|u_h - u\|_{0,\beta} \leq C\left(\|\rho\|_{0,\beta} + \|\varrho\|_{0,\beta}\right) \leq C\left(\|s - s_h\|_{0,\beta} + \|\varrho\|_{0,\beta}\right). \] (5.20)

Combining (5.20) and (5.14) yields (5.15). The proof is complete. \( \square \)

In the previous discussion, we developed error bounds based on the minimal regularity assumptions. In the following discussion, we bound errors in \( L^2 \)-norm. For this, we make assumptions on the regularity of solution
\[ p, \Psi \in L^\infty(0, T; H^{k+1}(\Omega)) \text{ and } s, u \in L^\infty(0, T; \left[L^\infty(\Omega) \cap H^{k+1}(\Omega)\right]^d). \] (5.21)

**Theorem 5.4.** Let \((p, u, s)\) solve problem (3.3) and \((p_h, u_h, s_h)\) solve the semidiscrete mixed finite element approximation (3.7). Then, there is a positive constant \( C \) such that for all \( t \in (0, T) \),
\[ \|p - p(t)\| + \left(\int_0^t \|s(t) - s(t)\|_2^2 \, d\tau\right)^{\frac{1}{2}} \leq C h^{k+1} \left\{ \|\Psi(t)\|_{k+1} + \|p(t)\|_{k+1} + \left(\int_0^t \|s(t)\|_{k+1}^2 + \|u(t)\|_{k+1}^2 + \|\varrho(t)\|_{k+1}^2 \right)^{\frac{1}{2}} \right\}, \] (5.22)
and
\[ \left(\int_0^t \|u_h - u(t)\|_2^2 \, d\tau\right)^{\frac{1}{2}} \leq C h^{k+1} \left\{ \|\Psi(t)\|_{k+1} + \|p(t)\|_{k+1} + \left(\int_0^t \|s(t)\|_{k+1}^2 + \|u(t)\|_{k+1}^2 + \|\varrho(t)\|_{k+1}^2 \right)^{\frac{1}{2}} \right\}. \] (5.23)

**Proof.** Similar to the proof of Theorem 5.1, we bound the second term and the third term of equation (5.4). Using monotonicity with the fact that \( K(\cdot) \) is bounded from below, there is \( C_0 > 0 \) such that
\[ \{K(s_h)|s_h - K(s)|s, s_h - s\} \geq C_0 \|s_h - s\|^2. \] (5.24)

With the help of Höder’s inequality, (2.18) and Young’s inequality, we can show that
\[ \{K(s_h)|s_h - K(s)|s, \zeta\} \leq C\|s_h - s\|_2 \|\zeta\| \leq \frac{C_0}{2} \|s_h - s\|^2 + C\|\zeta\|^2. \] (5.25)

Putting (5.24), (5.25) and (5.4) together, we see that
\[ \frac{d}{dt} \|\theta\|^2 + \|s_h - s\|^2 \leq C\|\zeta\|^2, \]
which gives
\[ \|\theta\|^2 + \int_0^t \|s_h - s\|^2 \, d\tau \leq C \int_0^t \|\zeta\|^2 \, d\tau. \] (5.26)
Therefore,
\[ \|\theta_h - \theta\|^2 + \int_0^t \|s_h - s\|^2 \, d\tau \leq \|\theta\|^2 + C \int_0^t \|\zeta\|^2 \, d\tau. \] (5.27)
Then the result (5.22) follows by combining (5.9) and (5.27).

From (5.3b), we have

\[\|p\|_2^2 = -(K(|s_h|)s_h - K(|s|)s, p) \leq C\|s_h - s\| \|p\|.\]  

(5.28)

We obtain (5.23) by using (5.22) together with estimate

\[\|u_h - u\|_2^2 \leq 2(\|\rho\|_2^2 + \|p\|_2^2) \leq C(\|\rho\|_2^2 + \|s_h - s\|_2^2).\]  

(5.29)

The proof is complete. 

Following directly from (5.22) and the inverse inequality, we have the following.

THEOREM 5.5. Let \((p, u, s)\) be a solution of the problem (3.3) and \((p_h, u_h, s_h)\) solve the semidiscrete mixed finite element approximation (3.7). There exists a positive constant \(C\) independent of \(h\) such that for all \(t \in (0, T)\),

\[\|p - p_h(t)\|_{0,\infty} \leq C h^{k+1} \left(\|p(t)\|_{k+1,\infty} + \|\Psi(t)\|_{k+1,\infty}\right) + C h^k \left(\int_0^t \|s(\tau)\|_{k+1}^2 d\tau\right)^{1/2}.\]

Now, we give the error estimate for gradient of pressure in \(L^2\)-norm.

THEOREM 5.6. Let \(0 < \tau_s < T\). Under the assumption of Theorem 5.5, there are positive constants \(C\) independent of \(h\) such that for all \(\tau_s \leq t \leq T\),

\[\|s_h - s(t)\|_2 \leq C \Phi^2 h^{k+1} \left(\int_0^t \|s(\tau)\|_{k+1}^2 d\tau\right)^{1/2} + C h^{k+1} \|s(t)\|_{k+1},\]

and

\[\|u_h - u(t)\|_2 \leq C \Phi^2 h^{k+1} \left(\int_0^t \|s(\tau)\|_{k+1}^2 d\tau\right)^{1/2} + C h^{k+1} \|s(t)\|_{k+1} + \|u(t)\|_{k+1}.\]

(5.30)

Proof. Thanks to (5.24), (5.4) and \(L^2\)-projection, we see that

\[\|s_h - s\|_2^2 \leq C (K(|s_h|)s_h - K(|s|)s, s_h - s)\]

\[= -C(\tilde{\theta}, \tilde{\theta}) + C(K(|s_h|)s_h - K(|s|)s, \zeta)\]

\[= -C(\tilde{p}_{h,t} - \tilde{p}_t, \tilde{\theta}) + C(K(|s_h|)s_h - K(|s|)s, \zeta)\]

\[\leq C(\|\tilde{p}_{h,t}\| + \|\tilde{p}_t\|) \|\tilde{\theta}\| + C \|\zeta\|^2.\]

This and (5.26) with noting that \(\|\tilde{p}_{h,t}\|, \|\tilde{p}_t\| \leq C \Phi\) give

\[\|s_h - s\|_2^2 \leq C \Phi \left(\int_0^t \|\zeta\|^2 d\tau\right)^{1/2} + C \|\zeta\|^2.\]

(5.32)

Hence (5.30) holds. Then using (5.29) and (5.30), we obtain (5.31). 

5.2. Error analysis for fully discrete scheme. We rewrite (3.3) with \(t = t_n\) as form

\[\left(\frac{\tilde{p}_n - \tilde{p}_{n-1}}{\Delta t}, w_h\right) + (\nabla u^n, w_h) = (f^n - \nabla \Psi^n, w_h) + \left(\frac{\tilde{p}_n - \tilde{p}_{n-1}}{\Delta t} - \tilde{p}_t, w_h\right),\quad \forall w_h \in W_h,\]

\[(u^n, z_h) + (K(|s^n|)s^n, z_h) = 0,\quad \forall z_h \in W_h,\]

\[(s^n, v_h) + (\tilde{p}_n, \nabla \cdot v_h) = (\nabla \Psi^n, v_h),\quad \forall v_h \in W_h.\]

An expanded mixed finite element method for non-Darcy equations

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Using the definitions of projections and the assumption that $\nabla \cdot V_h \subset W_h$, standard manipulations show that the exact solution satisfies the discrete equation

$$
\left( \frac{\pi \tilde{p}^n - \tilde{p}^{n-1}}{\Delta t}, w_h \right) + \left( \nabla \cdot \Pi u^n, w_h \right) = (f^n - \Psi_t^n, w_h) + (e^n, w_h), \quad \forall w_h \in W_h, \quad (5.33a)
$$

\[ (\Pi u^n, z_h) + (K(\|s^n\|)s^n, z_h) = 0, \quad \forall z_h \in \tilde{W}_h, \quad (5.33b) \]

\[ (\pi s^n, v_h) + (\pi \tilde{p}^n, \nabla \cdot v_h) = (\nabla \Psi^n, v_h), \quad \forall v_h \in W_h, \quad (5.33c) \]

where

$$
e^n = \frac{\tilde{p}^n - \tilde{p}^{n-1}}{\Delta t} - \tilde{p}_t^n.
$$

**Theorem 5.7.** Let $(p, u, s)$ solve problem (3.3) and $(p^n_h, u^n_h, s^n_h)$ solve the fully discrete mixed finite element approximation (3.8) for each time step $n$, $n = 1, \ldots, N$. Suppose that $p \in L^1(0, T; H^{k+1}(\Omega))$, $p_{tt} \in L^\infty(0, T; L^2(\Omega))$, then there exists a positive constant $C$ independent of $h$ and $\Delta t$ such that

$$
\| \tilde{p}^n_h - \tilde{p}^m_h \| \leq C \left( \sum_{n=1}^m \Delta t \| \zeta^n \|_{0, \beta}^2 \right)^{1/2} + C \| \theta^m \| + C \Delta t
$$

for all $m = 1, \ldots, N$.

Consequently,

$$
\| p^n_h - p^m_h \| \leq C \left( h^{k+1} + \Delta t \right).
$$

**Proof.** Subtracting (5.33) from (3.8), in the resultant equations using $w_h = \theta^n, z_h = \eta^n, v_h = -\rho^n$ as the test functions, we obtain the error equations:

$$
\left( \frac{\theta^n - \theta^{n-1}}{\Delta t}, \theta^n \right) + (\nabla \cdot \rho^n, \theta^n) = (e^n, \theta^n), \quad (5.36a)
$$

$$
(\rho^n, \eta^n) + (K(\|s^n\|)s^n - K(\|s^n\|)s^n, \eta^n) = 0, \quad (5.36b)
$$

$$
- (\eta^n, \rho^n) - (\rho^n, \nabla \cdot \eta^n) = 0. \quad (5.36c)
$$

It is easy to see that from adding three equations (5.36a)--(5.36c) that

$$
\| \bar{\theta}^n \|^2 + \Delta t \left( K(\|s^n\|)s^n - K(\|s^n\|)s^n, s^n - s^n \right)
$$

\[ = (\bar{\theta}^n, \bar{\theta}^{n-1}) + \Delta t \left( K(\|s^n\|)s^n - K(\|s^n\|)s^n, \zeta^n \right) + (e^n, \bar{\theta}^n). \quad (5.37) \]

Using (2.17), Cauchy’s inequality and (5.1), we find that

$$
\| \bar{\theta}^n \|^2 + C_0 \Delta t \omega^n \| s_h^n - s^n \|^2_{0, \beta} \leq \frac{1}{2} (\| \bar{\theta}^n \|^2 + \| \bar{\theta}^{n-1} \|^2) + \Delta t \left( C \| \zeta^n \|^2_{0, \beta} + \frac{1}{2} (\| \bar{\theta}^n \|^2 + \| e^n \|^2) \right),
$$

where $\omega^n = \omega(t^n)$ is defined in (5.6).

Therefore,

$$
\| \bar{\theta}^n \|^2 - \| \bar{\theta}^{n-1} \|^2 + 2C_0 \Delta t \omega^n \| s_h^n - s^n \|^2_{0, \beta} \leq \Delta t \| \bar{\theta}^n \|^2 + C_0 \Delta t \left( C \| \zeta^n \|^2_{0, \beta} + \| e^n \|^2 \right).
$$
Ignoring the second term of the left hand side of the above inequality and summing over \( n \) give
\[
(1 - \Delta t)\|\theta^n\|^2 \leq \sum_{n=1}^{m-1} \Delta t \|\theta^n\|^2 + C \sum_{n=1}^{m} \Delta t \left( Y\|\xi^n\|_{0,\beta}^2 + \|\epsilon^n\|^2 \right).
\]
Then applying discrete Gronwall’s lemma to the above estimate yields,
\[
\|\theta^m\|^2 \leq C \sum_{n=1}^{m} \Delta t \left( Y\|\xi^n\|_{0,\beta}^2 + \|\epsilon^n\|^2 \right).
\] (5.39)

We bound the \( \epsilon \)-term by
\[
\|\epsilon^n\|^2 = \frac{1}{(\Delta t)^2} \| \int_{t_{n-1}}^{t_n} \tilde{p}_t(\tau)(\tau - t_{n-1}) d\tau d\Omega \|_2^2
\leq \frac{C}{(\Delta t)^2} \left( \int_{t_{n-1}}^{t_n} \tilde{p}_t(\tau) \|d\tau\| \left( \int_{t_{n-1}}^{t_n} (\tau - t_{n-1})^2 d\tau \right) \right)
\leq C \sup_{t\in[0,T]} \| \tilde{p}_t(t) \|_2 (\Delta t)^2. \] (5.40)

Thanks to (5.40) and the triangle inequality, it follows from (5.39) that
\[
\|\tilde{p}_m^h - \bar{p}_m^h\|^2 \leq CY \sum_{n=1}^{m} \Delta t \|\xi^n\|_{0,\beta}^2 + \|\theta^m\|^2 + C(\Delta t)^2.
\]

This proves inequality (5.34).

The result (5.35) follows immediately by using (5.34), the triangle inequality and the project properties.

**Theorem 5.8.** Let \( 0 < t_* < T \). Under the assumption of Theorem 5.7 and additionally assume that \( s \in L^1(0,T; (W^{k+1,2}(\Omega))^d \cap (W^{k+1,\beta}(\Omega))^d) \). Then there exists a positive constant \( C \) and a positive integer \( n_0 \) independent of \( h \) and time step such that for \( m \) between \( n_0 \) and \( N \),
\[
\|s^m_h - s^m\|_{0,\beta} + \|u^m_h - u^m\|_{0,\beta} \leq C \left( h^{-1} + \sqrt{\Delta t} \right).
\] (5.41)

**Proof.** Recall that the exact solution satisfies the discrete equations
\[
(\bar{p}_t^m, w_h) + (\nabla \cdot \Pi u^m, w_h) = (f^m - \Psi^m, w_h), \quad \forall w_h \in W_h, \quad (5.42a)
(\Pi u^m, z_h) + (K(|s^m_h|)s^m_h, z_h) = 0, \quad \forall z_h \in \bar{W}_h, \quad (5.42b)
(\nabla \cdot \nu, v_h) + (\bar{\nu}^m, v_h) = 0, \quad \forall v_h \in V_h, \quad (5.42c)
\]

Subtracting (3.8) from (5.42), choosing \( w_h = \theta^m, z_h = \eta^m, v_h = -\rho^m \), we obtain
\[
\left( \frac{\bar{p}_h^m - \bar{p}_h^{m-1}}{\Delta t} - \bar{p}_t^m, \theta^m \right) + (\nabla \cdot \rho^m, \theta^m) = 0, \quad (5.43a)
(\rho^m, \eta^m) + (K(|s^m_h|)s^m_h - K(|s^m|)s^m, \eta^m) = 0, \quad (5.43b)
- (\eta^m, \rho^m) - (\theta^m, \nabla \cdot \rho^m) = 0. \quad (5.43c)
\]

The above equations yield
\[
\left( \frac{\bar{p}_h^m - \bar{p}_h^{m-1}}{\Delta t} - \bar{p}_t^m, \theta^m \right) + (K(|s^m_h|)s^m_h - K(|s^m|)s^m, \eta^m) = 0. \quad (5.44)
\]
element approximation

We use (5.5), and (5.44) to find that
\[
\omega^n \| s_h^m - s^m \|_{0, \beta}^2 \leq \left( K(\| s_h^m \|) s_h^m - K(\| s^m \|) s_h^m - s^m \right)
\]
\[= -\left( \tilde{p}_h^m - \tilde{p}_h^{m-1} \right) \frac{\Delta t}{\Delta t} - \tilde{p}_h^m \right) + \left( K(\| s_h^m \|) s_h^m - K(\| s^m \|) s_h^m, \xi^m \right). \tag{5.45}
\]

Due to (5.1), Cauchy-Schwartz and triangle inequality, one has
\[
\omega^n \| s_h^m - s^m \|_{0, \beta}^2 \leq C((\Delta t)^{-1} \| \tilde{p}_h^m - \tilde{p}_h^{m-1} \| + \| \tilde{p}_T^2 \| \| \theta^m \| + Y \| \xi^m \|_{0, \beta}. \tag{5.46}
\]

Let \( n_0 = \left[ \frac{t_N}{\tau} \right] + 1 \) then for \( m \geq n_0 \), \( \| \tilde{p}_T^m \| \leq \sup_{(t, T)} \| \tilde{p}_T^m \| \leq \Phi(t) \) and hence
\[
(\Delta t)^{-1} \| \tilde{p}_h^m - \tilde{p}_h^{m-1} \| = (\Delta t)^{-1} \int_{t_{m-1}}^{t_m} \tilde{p}_h^m dt \leq (\Delta t)^{-1} \int_{t_{m-1}}^{t_m} \| \tilde{p}_h^m \| dt \\
\leq \sup_{[t, T]} \| \tilde{p}_h^m \| \leq \Phi(t) .
\]

It follows from (5.39) and (5.46) that
\[
\omega^n \| s_h^m - s^m \|_{0, \beta}^2 \leq C\Phi(t) \| \theta^m \| + Y \| \xi^m \|_{0, \beta}
\]
\[\leq C\Phi(t) \left( \sum_{n=1}^{N} \Delta t(\| \xi^m \|_{0, \beta}^2 + \| \xi^m \|_{0, \beta}^2) \right) + Y \| \xi^m \|_{0, \beta} .
\]

This and (5.17) show that
\[
\| s_h^m - s^m \|_{0, \beta} \leq CY \frac{1}{k+1} \Phi(t) \frac{t}{h} \frac{k+1}{k+1} \left( \sum_{n=1}^{N} \Delta t \| s^m \|_{0, \beta}^2 \right)^{1/2}
\]
\[+ CY \frac{k+1}{k+1} s^m \|_{k+1, \beta} + C\Phi(t) Y \frac{t}{\sqrt{\Delta t}}
\]
\[\leq C(h^{k+1} + \sqrt{\Delta t}) .
\]

Due to (5.20) and (5.47),
\[
\| u_h^m - u^m \|_{0, \beta} \leq C \left( \| s_h^m - s^m \|_{0, \beta} + \| \xi^m \|_{0, \beta} \right) \leq C(h^{k+1} + \sqrt{\Delta t}) .
\]

The result follows by combining (5.47) and (5.48). \( \square \)

To obtain the \( L^2 \)-error estimate for variable \( s \) and \( u \), we make the assumption (5.21) on the solution. Following the ideas of the proofs of Theorems 5.4 and 5.6, we obtain the following results

**Theorem 5.9.** Let \( (p, u, s) \) solve problem (3.3) and \( (p_h^0, u_h^0, s_h^0) \) solve the fully discrete mixed finite element approximation (3.8) for each time step \( n, n = 1 \ldots, N \). Suppose \( p_{1t} \in L^\infty(0, T; L^2(\Omega)) \).

(i) There exists a positive constant \( C \) independent of \( h \) and \( \Delta t \) such that for \( m \) between 1 and \( N \),
\[
\| p_h^m - p^m \| + \left( \sum_{n=1}^{N} \Delta t \| s_h^m - s^m \|^{1/2} \right) + \left( \sum_{n=1}^{N} \Delta t \| u_h^m - u^m \|^{1/2} \right) \leq C \left( h^{k+1} + \Delta t \right) .
\]

(ii) For \( 0 < t_0 < T \), there are a positive constant \( C \) and a positive integer \( n_0 \) independent of \( h \) and \( \Delta t \) such that \( m \) between \( n_0 \) and \( N \),
\[
\| s_h^m - s^m \| + \| u_h^m - u^m \| \leq C \left( h^{k+1} + \sqrt{\Delta t} \right) .
\]
6. Numerical results. In this section, we give a simple numerical result illustrating the convergence theory. For simplicity, the region of examples are unit square $\Omega = [0,1]^2$. The triangularization in region $\Omega$ is uniform subdivision in each dimension. Since we only analyze a first order time discretization, we consider the lowest order Raviart-Thomas mixed finite element on the unit square in two dimensions, which the variable $(p_h, s_h, u_h)$ is approximated in $(P_0, RT_0, RT_0^2)$.

We test the convergence of our method with Forchheimer’s two-term law, $\mathcal{g}(s) = 1 + s, s \geq 0$. Solving equation (2.4), we obtain $s = \frac{-1 + \sqrt{1 + 4\epsilon}}{2}$. Then we find that $K(\xi) = \frac{1}{\mathcal{g}(|\xi|)} = \frac{2}{1 + \sqrt{1 + 4\epsilon}}$. We choose the analytical solution to be

$$p(x, t) = e^{-2t}x_1(1 - x_1)x_2(1 - x_2),$$

$$s(x, t) = e^{-2t}((1 - 2x_1)x_2(1 - x_2), x_1(1 - x_1)(1 - 2x_2)),$$

$$\mathbf{u}(x, t) = \frac{2s(x, t)}{1 + \sqrt{1 + 4|s(x, t)|}}$$

for all $x \in \Omega$, $t \in [0, 1]$. The forcing term $f$ is determined from the equation $p_t - \nabla \cdot \mathbf{u} = f$. The initial data and boundary data respectively are $p_0(x) = x_1(1 - x_1)x_2(1 - x_2)$, and $\psi(x, t) = 0$ for all $(x, t) \in \Gamma \times [0, 1]$.

We divided the unit square into an $\mathcal{N} \times \mathcal{N}$ mesh of squares, each of them subdivided into two right triangles. For each mesh, we solved the generalized Forchheimer equation numerically. The error control in each nonlinear solve is tol $= 10^{-6}$. Our problem is solved at each time level starting at $t = 0$ until the final time $t = 1$. At time $t = 1$, we measured the $L^2$-errors of the pressure, the gradient pressure and the flux due to the regularity of analytical solutions. We obtain the convergence rates $r = \frac{\|p - p_h\|}{\|s - s_h\|}$ of finite approximation at 8 levels with the discretization parameters $h = 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256, 1/512$ respectively and time step $\Delta t = h/2$. The numerical results are listed as the following table:

| $\mathcal{N}$ | $\|p - p_h\|$ Rates | $\|s - s_h\|$ Rates | $\|u - u_h\|$ Rates |
|---------------|----------------------|----------------------|----------------------|
| 4 | 2.530e-03 | – | 1.148e-02 | 9.444e-02 |
| 8 | 2.015e-03 | 0.33 | 9.658e-03 | 0.25 | 7.580e-02 | 0.32 |
| 16 | 1.083e-03 | 0.90 | 7.375e-03 | 0.39 | 5.375e-02 | 0.50 |
| 32 | 5.680e-04 | 0.93 | 5.098e-03 | 0.53 | 3.660e-02 | 0.55 |
| 64 | 2.886e-04 | 0.98 | 3.503e-03 | 0.54 | 2.501e-02 | 0.55 |
| 128 | 1.409e-04 | 1.03 | 2.290e-03 | 0.61 | 1.710e-02 | 0.55 |
| 256 | 6.780e-05 | 1.06 | 1.466e-03 | 0.64 | 1.166e-02 | 0.55 |
| 512 | 3.150e-05 | 1.11 | 9.120e-04 | 0.68 | 7.920e-03 | 0.56 |

Table 1. Convergence study for generalized Forchheimer equation in 2D.

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