Oscillations of spherical and cylindrical shells

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We have found the complete spectrum and eigenstates for harmonic oscillations of ideal spherical and cylindrical shells, both being infinitely thin. The spectrum of the cylindrical shell has an infinite number of Goldstone modes corresponding to folding deformations. This infrared catastrophe is overcome by accounting for curvature-dependent part of energy.
INTRODUCTION

The oscillating surfaces have numerous applications in many areas of physics, spanning from biophysics and chemical physics to high energy physics.\textsuperscript{1,2,3} In this article, we study the oscillation spectra of spherical and cylindrical surfaces, considering them as infinitely thin elastic membranes. We also calculate their heat capacities. Corresponding calculated values of heat capacity are valid at sufficiently low temperature $T << \Theta_D$, where $\Theta_D$ is Debye temperature, otherwise the effects of discreteness become essential. To our knowledge and big surprise, this problem has not yet been solved, despite of a vast literature on the subject.\textsuperscript{4}

THE SPECTRUM AND EIGENSTATES OF SPHERICAL AND CYLINDRICAL SHELLS

A. Spherical Membrane

The Lagrangian of a spherical membrane reads:

$$
\mathcal{L} = \frac{\rho}{2} \int R^2 d\Omega \left[ \left( \frac{\partial u_r}{\partial t} \right)^2 + \left( \frac{\partial u_\theta}{\partial t} \right)^2 + \left( \frac{\partial u_\phi}{\partial t} \right)^2 \right] - \frac{\lambda}{2} \int R^2 d\Omega (U^2_{\theta\theta} + U^2_{\phi\phi}) - \mu \int R^2 d\Omega (U^2_{\theta\theta} + 2U^2_{\theta\phi} + U^2_{\phi\phi})
$$

(1)

where $R$ is radius of spherical membrane, $d\Omega = \sin \theta d\theta d\phi$ and, $\mu$ and $\lambda$ are the Lame coefficients. $u_r$, $u_\theta$, $u_\phi$ are components of the displacement vector and
$U_{\theta\theta}, U_{\phi\phi}, U_{\theta\phi}$ are components of the strain tensor. The latter are determined by relative change of distance between two points at a deformation: $ds'^2 = ds^2(1 + U_{\alpha\beta}dx_\alpha dx_\beta)$ and can be expressed in terms of the displacement vector and its derivatives as follows: 5

$$U_{\theta\theta} = \frac{1}{R} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{R}$$

(2)

$$U_{\phi\phi} = \frac{1}{R \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{R \cot \theta} + \frac{u_r}{R}$$

(3)

$$2U_{\theta\phi} = \frac{1}{R} \left( \frac{\partial u_\phi}{\partial \theta} - u_\phi \cot \theta \right) + \frac{1}{R \sin \theta} \frac{\partial u_\theta}{\partial \phi}$$

(4)

Equations of motions for oscillations with the frequency $\omega$ read:

$$u_r = \mathcal{K} \left( \frac{\partial u_\theta}{\partial \theta} + u_\theta \cot \theta + \frac{1}{R \sin \theta} \frac{\partial u_\phi}{\partial \phi} \right)$$

(5)

$$\left( \lambda + 2\mu + 2(\lambda + \mu)\mathcal{K} \right) \left( \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial (u_\theta \cot \theta)}{\partial \theta} \right) + \left( \lambda + \mu + 2(\lambda + \mu)\mathcal{K} \right) \frac{1}{\sin \theta} \frac{\partial^2 u_\phi}{\partial \phi \partial \theta}$$

$$- \left( \lambda + 3\mu + 2(\lambda + \mu)\mathcal{K} \right) \frac{\cot \theta \frac{\partial u_\phi}{\partial \theta}}{\sin \theta} + \frac{\mu \frac{\partial^2 u_\phi}{\partial \phi^2}}{\sin^2 \theta} + (\rho \omega^2 R^2 + 2\mu) u_\theta = 0$$

(6)

$$\left( \lambda + \mu + 2(\lambda + \mu)\mathcal{K} \right) \left( \frac{1}{\sin \theta} \frac{\partial^2 u_\phi}{\partial \phi \partial \theta} \right) + \left( \lambda + 3\mu + 2(\lambda + \mu)\mathcal{K} \right) \left( \frac{\cot \theta \frac{\partial u_\phi}{\partial \theta}}{\sin \theta} \right)$$

$$+ \left( \lambda + 2\mu + 2(\lambda + \mu)\mathcal{K} \right) \left( \frac{1}{\sin^2 \theta} \frac{\partial^2 u_\phi}{\partial \phi^2} \right) + \mu \left( \frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{\partial (u_\phi \cot \theta)}{\partial \theta} \right)$$

$$+ (\rho \omega^2 R^2 + 2\mu) u_\phi = 0$$

(7)
where $K = \frac{2(\lambda + \mu)}{\rho \omega^2 R^2 - 4(\lambda + \mu)}$. There exists a symmetric solution (the "breathing" mode), $u_\theta = u_\phi = 0$, $u_r = \text{const} \neq 0$. The frequency of this solution is $\omega_s = \sqrt{\frac{4(\lambda + \mu)}{\mu R^2}}$. From other solutions, we first consider those independent on $\phi$. There exist three branches of such oscillations. The first two ("longitudinal" modes) correspond to $u_\phi = 0$, $u_\theta \neq 0$, $u_r \neq 0$. The frequency of these oscillations are labeled by an integer $l$:

$$\omega_{\pm, l}^2 = \frac{[4\lambda + 2\mu + (\lambda + 2\mu)l(l + 1)]}{2\rho R^2} \pm \sqrt{\frac{2\rho R^2}{(4\lambda + 2\mu + (\lambda + 2\mu)l(l + 1))^2 - 16\mu(\lambda + \mu)(l(l + 1) - 2)}} \quad (8)$$

In the third branch (a "shear" mode) $u_\theta = 0$, $u_r = 0$ and $u_\phi \neq 0$. Its frequencies are

$$\omega_{l,l}^2 = \frac{\mu}{\rho R^2} (l(l + 1) - 2) \quad (9)$$

where $l = 2, 3, ...$ Explicit expressions for the displacements for the longitudinal branches are:

$$u_\phi = 0 \quad u_\theta = C \frac{dP_l(\cos \theta)}{d\theta} \quad (10)$$

$$u_r = -\frac{C(\lambda + \mu)}{\rho \omega^2 R^2 - 4(\lambda + \mu)} l(l + 1) P_l(\cos \theta) \quad (11)$$

where $C$ is constant amplitudes and $P_l(\cos \theta)$ are the Legendre polynomials.

The displacements in the shear branch solution read:
\[ u_r = u_\theta = 0 \quad u_\phi = C \frac{dP_1(\cos \theta)}{d\theta} \]  

(12)

In order to get eigenstates for \( m \neq 0 \), we employ an infinitesimal rotation around \( x \) axis which turns an arbitrary vector \( \vec{r} \) into:

\[ \vec{r}' = O(\omega)\vec{r} = 1 + \omega \vec{g}_x \vec{r} \]  

(13)

Here \(-ig_x\) is the generator of the rotation around \( x \) axis. It generates a transformation of the displacement field \( \vec{u}(\vec{r}) \).

\[ \vec{u}'(\vec{r}) = O(\omega)\vec{u}(O^{-1}(\omega)\vec{r}) = \vec{u}(\vec{r}) - \omega g_x \vec{u}(\vec{r}) + \omega (g_x \vec{r}).\nabla \vec{u}(\vec{r})... \]  

(14)

If \( u_0(\vec{r}) \) was a solution of the system (5-7), then \( \vec{u}'(\vec{r}) \) is also a solution. The operation we performed to find new solutions is readily recognizable as the Lee translation introduced in theory of fiber bundles.

After a simple transformation, we find an operator \( \mathcal{G} \), which, acting on a solution fixed by two components \( \begin{pmatrix} u_\theta \\ u_\phi \end{pmatrix} \), generates another solution with the same frequency.

\[ \mathcal{G} = \begin{pmatrix} \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} & -\frac{\cos \phi}{\sin \theta} \\ \frac{\cos \phi}{\sin \theta} & \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \end{pmatrix} \]  

(15)
The operator $G$ is the analog of the operator of total angular momentum $J$ in quantum mechanics, but it is reduced to a subspace of tangential components of the displacement vector $\vec{u}$. The Lagrangian (1) and equations of motion (6-7) are invariant with respect to the transformation (15). Acting by the operator $G$ onto solutions with $m = 0$ (10-12), we generate solutions with all possible values of $m$. They belong to the same eigen-frequencies as the initial solutions with $m = 0$. The displacements for solutions belonging to the longitudinal branches with $m \neq 0$ read:

$$u_{\theta}^{(m)}(x) = e^{im\phi}i^m \left[ mx(1 - x^2)^{\frac{m-1}{2}} P_l^{(m)}(x) - (1 - x^2)^{\frac{m+1}{2}} P_l^{(m+1)}(x) \right]$$

(16)

$$u_{\phi}^{(m)}(x) = -e^{im\phi}i^{m+1} m(1 - x^2)^{\frac{m-1}{2}} P_l^{(m)}(x)$$

(17)

The displacements for oscillations of the shear branch with $m \neq 0$ are:

$$u_{\theta}^{(m)}(x) = -e^{im\phi}i^{m+1} m(1 - x^2)^{\frac{m-1}{2}} P_l^{(m)}(x)$$

(18)

$$u_{\phi}^{(m)}(x) = e^{im\phi}i^{m} \left[ mx(1 - x^2)^{\frac{m-1}{2}} P_l^{(m)}(x) - (1 - x^2)^{\frac{m+1}{2}} P_l^{(m+1)}(x) \right]$$

(19)

Given the known frequency spectrum $\omega_j$, the thermal energy can be calculated as $U = \sum_j \frac{\hbar \omega_j}{e^{\hbar \omega_j/kT} - 1}$ and the heat capacity can be calculated as $C = (\frac{\partial U}{\partial T})_V$. 

6
As always, low-lying rotational frequencies $\omega_k^{(rot)} = \frac{\hbar k(k+1)}{2MR^2}$ are smaller than low-lying oscillatory frequencies by a factor $\sim \sqrt{\frac{m}{M}} \frac{a}{R}$, where $a$ is characteristic distances between atoms (discreteness effect) and $m$ is the mass of electron. That means that at temperature $T >> T_R = \frac{h^2}{MR^2}$, the rotational degrees of freedom become substantially classical until $k \leq \sqrt{\frac{T}{T_R}}$. Their contribution to the heat capacity is $C_R = \frac{T}{T_R}\zeta(3)$. Elastic oscillations remain frozen till a larger temperature $T_0 \sim \frac{h}{\pi} \frac{\sqrt{\rho}}{\mu} >> T_R$.

B. Cylindrical shells

In order to find the spectrum and corresponding eigenstates of a cylindrical membrane, we follow the same steps as we did for spherical shell. For cylindrical case, the components of strain tensor in terms of cylindrical coordinates $r, \phi, z$ read:

$$ U_{\phi\phi} = \frac{1}{R} \frac{\partial u_{\phi}}{\partial \phi} + \frac{u_r}{R}, \quad U_{zz} = \frac{\partial u_z}{\partial z}, \quad 2U_{\phi z} = \frac{1}{R} \frac{\partial u_z}{\partial \phi} + \frac{\partial u_{\phi}}{\partial z} \quad (20) $$

The Lagrangian reads:

$$ \mathcal{L} = \frac{\rho}{2} \int R d\phi dz \left[ \left( \frac{\partial u_r}{\partial t} \right)^2 + \left( \frac{\partial u_\phi}{\partial t} \right)^2 + \left( \frac{\partial u_z}{\partial t} \right)^2 \right] - \frac{\lambda}{2} \int R d\phi dz (U_{zz} + U_{\phi\phi})^2 - \mu \int R d\phi dz (U_{zz}^2 + 2U_{\phi z}^2 + U_{\phi\phi}^2) \quad (21) $$

where $R$ is the radius of the cylinder. Equations of motions are:
\[ u_r = (\lambda + 2\mu) \tilde{K} \frac{\partial u_\phi}{\partial \phi} + \tilde{K}R \lambda \frac{\partial u_z}{\partial z} \] (22)

\[ u_\phi = - (\lambda + 2\mu) \tilde{K} \frac{\partial^2 u_\phi}{\partial \phi^2} - (\lambda + \mu) \tilde{K} \frac{\partial^2 u_z}{\partial z \partial \phi} + \frac{\mu}{\rho \omega^2 R^2} \left( (\lambda + 2\mu) \tilde{K} R \frac{\partial^2 u_z}{\partial z \partial \phi} - R^2 \frac{\partial^2 u_\phi}{\partial z^2} \right)\] (23)

\[ u_z = - (\lambda + \mu) \tilde{K} R \frac{\partial^2 u_\phi}{\partial z \partial \phi} - (\lambda + 2\mu) \tilde{K} R^2 \frac{\partial^2 u_z}{\partial z^2} + \frac{\mu}{\rho \omega^2 R^2} \left( (\lambda + 2\mu) \tilde{K} R \frac{\partial^2 u_z}{\partial z \partial \phi} + 4(\lambda + \mu) \tilde{K} R^2 \frac{\partial^2 u_z}{\partial z^2} - 4 \frac{\partial^2 u_z}{\partial \phi^2} \right)\] (24)

where \( \tilde{K} = \frac{1}{\rho R^2 \omega^2 (\lambda + 2\mu)} \). Specifying the \( z \) dependence of all displacement components as a plane wave \( e^{ikz} \), and their angular dependence as \( e^{im\phi} \), we find a system of linear equations for the amplitudes \( u_r, u_\theta, u_\phi \). Its secular equation reads :

\[
\left( \omega^2 - \omega_s^2 \right) \left( \omega^6 - \left( \omega_i^2 (R^2 k^2 + m^2) + \omega_s^2 (R^2 k^2 + m^2 + 1) \right) \omega^4 + \left( \omega_i^2 \omega_s^2 \left( (R^2 k^2 + m^2)^2 + 5 R^2 k^2 + m^2 \right) - 4 \omega_i^4 R^2 k^2 \right) \omega^2 + 4 \omega_i^4 (\omega_i^2 - \omega_s^2) R^4 k^4 \right) = 0 \] (25)

where \( \omega_s^2 = \frac{\lambda + 2\mu}{\rho R^2} \) and \( \omega_i^2 = \frac{\mu}{\rho R^2} \). One of its solutions (symmetric) is \( \omega^2 = \omega_s^2 \).

For \( k = 0 \), the remaining cubic equation can be solved explicitly for arbitrary \( m \). The three solutions are :

\[
\omega^2 = 0, \quad \omega^2 = (m^2 + 1) \omega_s^2, \quad \omega^2 = m^2 \omega_i^2 \] (26)
The first of them implies that there exists infinite number of Goldstone modes corresponding to the folding deformations. For \( m = 0 \), the cubic equation gives the following solutions:

\[ \omega^2 = k^2 R^2 \omega_t^2 \]  

\[ \omega^2 = \frac{\omega_s^2 (R^2 k^2 + 1)}{2} \pm \frac{\sqrt{\left(\omega_s^2 (R^2 k^2 + 1)\right)^2 - 16 \omega_t^2 (\omega_s^2 - \omega_t^2) R^2 k^2}}{2} \] 

The physical reason of the peculiarity in the spectrum of cylindrical shell is an opportunity to fold a cylinder conserving its straight-lines and its cross-section contour length. Such a deformation is strainless. The infinite number of folding modes leads to divergence of the heat capacity at any temperature. This difficulty can be avoided by introducing terms with higher derivatives into the Lagrangian. Geometrically, it means that the Gaussian curvature and the mean curvature must be included. In more physical or chemical terms it means that not only the variation of distances between atoms, but also variation of valence angles must be accounted for. Below we write down explicit formulae for the Gaussian curvature \( K \) and the mean curvature \( M \) for cylindrical membranes in linear approximation in displacements.

\[ K = -\frac{1}{R} \frac{\partial^2 u_r}{\partial z^2} \]
\[ M = \frac{1}{R} - \frac{u_r}{R^2} - \frac{1}{R^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{\partial^2 u_r}{\partial z^2} \]  

(30)

The term in Lagrangian depending on curvature reads:

\[ \Delta L = \int \left[ \frac{\nu}{2} (K - K_0)^2 + \frac{\theta}{2} (M - M_0)^2 \right] d^2 S \]  

(31)

where \( K_0 = 0 \) and \( M_0 = \frac{1}{R} \) are the values of \( K \) and \( M \) for undeformed cylinder. With this term in Lagrangian we find the gaps \( \Delta_m \) in the spectrum corresponding to the former folding modes:

\[ \Delta_m = \sqrt{\frac{\theta}{\rho R^4}} (m^2 - 1) \]  

(32)

Note that the gaps in spectrum due to curvature effect are proportional to \( R^{-2} \) and, therefore, are small in comparison to gaps due to strain effects (\( \propto R^{-1} \)).

By perturbation method it is possible to find small \( k \) - dependent corrections to these frequencies:

\[ \delta \omega^2 = \frac{\theta}{\rho R^4} [2k^2 R^2 (m^2 - 1)] - \frac{(\omega_t^2 - 2\omega_m^2) \omega_t^2 (kR)^2}{\Delta_m - \omega_t^2 m^2} \]  

or \( \omega_m^2 (k) = \Delta_m^2 + V_m^2 k^2 \), where \( V_m^2 = \frac{\theta}{\rho R^4} [2R^2 (m^2 - 1)] - \frac{(\omega_t^2 - 2\omega_m^2) \omega_t^2 (R)^2}{\Delta_m - \omega_t^2 m^2} \). A clear tendency is that the band becomes more flat with the growth of \( m \).

For cylindrical shell, we can write the thermal energy in the following form:

\[ U = \int_0^\infty \frac{dk}{2\pi} \sum_{m=2}^{\infty} \frac{\hbar \omega_m (k)}{e^{\frac{\hbar \omega_m (k)}{T}} - 1} \]  

(33)

In the temperature interval \( \hbar \Delta_2 << T << \hbar \omega_t \), we can neglect all "elastic branches" and substitute the summation over the "valence" branches by
integration. It leads to the energy:

\[ U = \frac{0.601}{\hbar^2} \sqrt{\frac{1}{\rho R^2 (\omega_s^2 - 2\omega_t^2)}} T^3 \]  \hspace{1cm} (34)

and the heat capacity:

\[ C_v = \frac{1.803}{\hbar^2} \sqrt{\frac{1}{\rho R^2 (\omega_s^2 - 2\omega_t^2)}} T^2 \]  \hspace{1cm} (35)

For \( T \ll \hbar \Delta_2 \), we can neglect the sum over \( m \), since only \( \Delta_2 \) and \( V_2 \) contribute.

to the energy. The internal energy becomes exponentially small:

\[ U = \sqrt{2\pi \hbar T} \frac{\Delta_2^\frac{3}{2}}{V_2} e^{-\frac{\hbar \Delta_2^2}{4T}} \]  \hspace{1cm} (36)

and the heat capacity:

\[ C_v = \frac{\sqrt{2\pi \hbar T}}{T^{\frac{3}{2}} V_2} \frac{\Delta_2^\frac{3}{2}}{e^{-\frac{\hbar \Delta_2^2}{4T}}} \]  \hspace{1cm} (37)

CONCLUSIONS

We have found complete spectrum of oscillations for spherical and cylindrical membranes and corresponding eigenstates. We have showed that cylindrical shell has an infinite number of Goldstone modes due to folding deformations which do not change the elastic energy of shell. We have also showed that this catastrophe can be avoided by introducing curvature-dependent terms to the Hamiltonian. In addition, heat capacities of spherical and cylindrical
shells are calculated. They can be measured experimentally as an additional
correction to the heat capacity of a solvent in a diluted solution of long
molecules.

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