Initial value problem for the $(2 + 1)$-dimensional time-fractional generalized convection–reaction–diffusion wave equation: invariant subspaces and exact solutions

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Abstract
This work investigates how we can extend the invariant subspace method to $(2 + 1)$-dimensional time-fractional non-linear PDEs. More precisely, the systematic study has been provided for constructing the various dimensions of the invariant subspaces for the $(2 + 1)$-dimensional time-fractional generalized convection–reaction–diffusion wave equation along with the initial conditions for the first time. Additionally, the special types of the above-mentioned equation are discussed through this method separately such as reaction–diffusion wave equation, convection–diffusion wave equation and diffusion wave equation. Moreover, we explain how to derive variety of exact solutions for the underlying equation along with initial conditions using the obtained invariant subspaces. Finally, we extend this method to $(2 + 1)$-dimensional time-fractional non-linear PDEs with time delay. Also, the effectiveness and applicability of the method have been illustrated through the $(2 + 1)$-dimensional time-fractional cubic non-linear convection–reaction–diffusion wave equation with time delay. In addition, we observe that the obtained exact solutions can be viewed as the combinations of the Mittag-Leffler function and polynomial, exponential and trigonometric type functions.

Keywords Fractional convection–reaction–diffusion wave equations · Initial value problems · Fractional non-linear PDEs · Invariant subspaces · Exact solutions

Mathematics Subject Classification 35R11 · 35Bxx · 35-XX · 35Cxx

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1 Introduction

In recent years, the study of fractional differential equations (FDEs) has gained considerable popularity and importance due to the exact description of widespread applications in various areas of science and engineering (Podlubny 1999; Diethelm 2010; Kilbas et al. 2006; Tarasov 2011; Sun et al. 2018; Machado et al. 2010) such as physics (Tarasov 2011), mechanics (Mainardi 1997), electrodynamics (Tarasov and Trujillo 2013), biology (Ionescu et al. 2017), visco-elasticity (Bagley and Torvik 1984) and so on Tarasov (2013). Unlike the integer-order differential operator, fractional-order (non-integer order) differential operators are non-local and depend not only on the immediate past but also on all the past values of the function. Thus, the fractional derivative operators are typically used to add memory in a complex system. Hence the fractional-order models help to understand the behaviour of the system and phenomena that are classified by power-law non-locality, power-law long term memory and fractal properties (Tarasov 2013). Also, we know that many complex systems depend not only on an instantaneous time but also on all the past time history of the system which can be successfully modelled using the theory of non-integer order derivatives and non-integer order integrals (Podlubny 1999; Diethelm 2010; Kilbas et al. 2006; Tarasov 2011; Sun et al. 2018; Mainardi 1997; Tarasov and Trujillo 2013; Ionescu et al. 2017; Bagley and Torvik 1984; Tarasov 2013; Machado et al. 2010; Ma and Xiu 2020; Ma 2020, 2021a, b; Ma et al. 2021).

In addition, it is important to note that one of the advantages of fractional PDEs is helpful to study both the hyperbolic and parabolic types of PDEs simultaneously. For example, let us consider the fractional-order PDE

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = c \frac{\partial^2 u}{\partial x^2}, \quad c \in \mathbb{R}, \quad t > 0, \quad \alpha \in (0, 2].
\]

Note that the above Eq. (1.1) is referred to as a time-fractional diffusion wave equation (Bakkyaraj and Sahadevan 2015; Lukashchuk 2015). Additionally, we observe that

(i) If \( \alpha = 1 \), then Eq. (1.1) is popularly known as the standard diffusion equation.

(ii) When \( \alpha = 2 \), Eq. (1.1) is familiarly called a standard wave equation.

(iii) If \( \alpha < 1 \), Eq. (1.1) describes the sub-diffusion phenomenon (Bakkyaraj and Sahadevan 2015; Lukashchuk 2015; Luo et al. 2016) which represents the slow movement of particles. Also, if \( \alpha > 1 \), Eq. (1.1) can be viewed as a super-diffusion phenomenon (Bakkyaraj and Sahadevan 2015) that represents the fast movement of particles.

(iv) It is important to note that when \( \alpha \in (1, 2) \), equation (1.1) helps to study the intermediate process between diffusion and wave phenomena (Bakkyaraj and Sahadevan 2015). For this case, the above Eq. (1.1) is known as fractional diffusion wave equation (Bakkyaraj and Sahadevan 2015; Lukashchuk 2015).

It is well known that the generalized non-linear convection–reaction–diffusion (CRD) equations (Gilding and Kersner 2004; Molati and Murakawa 2019; Cherniha et al. 2021; Lapinska-Chrzaszcz and Matus 2014; Harko and Mak 2015; Qiao et al. 2019; Prakash 2019) have a wide range of applications in the fields of biology, physics, chemistry and engineering such as heat transfer processes, adsorption in porous medium, chemical reactions, population dynamics, lubrication and viscosity of fluids, turbulence of heterogeneous physical system, amount of contamination transported through ground water and so on. The CRD equation can be classified into convection, reaction and diffusion processes. Convection refers to the movement of a substance from one region to another, reaction process is due to adsorption, decay and reaction of substances with other components. Finally, the key process is diffusion which is due to the movement of substances from an area of high concentration to
an area of low concentration. The CRD equation is a mathematical model that describes how the concentration of the substance distributed in the medium changes under the influence of the above-mentioned three processes. Also, we note that the generalized CRD equation can be categorized into three important special types that are (i) diffusion, (ii) reaction–diffusion and (iii) convection-diffusion or absorption–diffusion phenomena. For instance, the Newell–Whitehead equation or amplitude equation (Harko and Mak 2015), the Fisher–KPP equation or the logistic equation (Harko and Mak 2015), the FitzHugh–Nagumo equation or the bistable equation (Harko and Mak 2015) and the Zeldovich equation or Huxley equation (Harko and Mak 2015) are reaction–diffusion equations. The Richards equation (Harko and Mak 2015), the foam drainage equation (Harko and Mak 2015) and the Burgers equation (Harko and Mak 2015) are non-linear convection–diffusion equations.

The well-known generalized \((n+1)\)-dimensional convection–reaction–diffusion equation (Cherniha et al. 2021)

\[
\frac{\partial u}{\partial t} = \nabla \cdot (A(u) \nabla u) + B(u) \cdot \nabla u + C(u), \tag{1.2}
\]

where \(u = u(x_1, x_2, \ldots, x_n, t), \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)\), ‘.’ means scalar product and the functions \(A(u), B(u), C(u)\) are related to three most common types of transport mechanisms occurring in the real-world process. The diffusivity \(A(u) > 0\) is the main characteristic of the diffusion (heat conductivity) process. The vector \(B(u)\) typically means velocity, which can be positive and/or negative and describes the convective transport (in contrast to diffusion one is not random) and the reaction term \(C(u)\) describes the kinetics process (for example, this function represents interaction of the population \(u\) with the environment and its birth–death rate).

We know that fractional-order derivative operators violate some standard fundamental properties such as the semigroup property, chain rule and Leibniz rule. The violation of the above-mentioned properties is the fundamental properties of the fractional-order derivatives that allow us to describe the memory (Tarasov 2013). Using the fractional-order derivative, the present article deals with the systematic study for finding the exact solutions of the initial value problem for the \((2+1)\)-dimensional time-fractional generalized non-linear convection–reaction–diffusion wave equation in the form

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(A_i(u) \frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{2} B_i(u) \frac{\partial u}{\partial x_i} + C(u), \quad \alpha \in (0, 2], \tag{1.3}
\]

along with the initial conditions

- \(u(x_1, x_2, 0) = \varphi_1(x_1, x_2)\) if \(\alpha \in (0, 1],\)
- \(u(x_1, x_2, 0) = \varphi_1(x_1, x_2)\) and \(\frac{\partial u}{\partial t} |_{t=0} = \varphi_2(x_1, x_2)\) if \(\alpha \in (1, 2],\)

where \(u = u(x_1, x_2, t)\) and \(\frac{\partial^\alpha u}{\partial t^\alpha}\) denotes the Caputo fractional derivative of order \(\alpha > 0\) is defined by Diethelm (2010), Podlubny (1999)

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} 
\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\partial^m u(x_1, x_2, \tau)}{\partial \tau^m} (t-\tau)^{m-\alpha-1} d\tau, & m-1 < \alpha < m, \ m \in \mathbb{N}; \\
\frac{\partial^m u}{\partial t^m}, & \alpha = m \in \mathbb{N}
\end{cases} \tag{1.4}
\]
and the functions \( A_i(u) \) denote the process of diffusion, \( B_i(u) \) denote convection, and \( C(u) \) denotes reaction of a system under consideration with space variables \( x_1, x_2 \) and time-variable \( t \). Note that the Lie symmetries of the above Eq. (1.3) with \( \alpha = 1 \) is discussed in Cherniha et al. (2021). Also, the numerical solution of (1.3) with linear case was discussed in Lapinska-Chrzczonowicz and Matus (2014). Additionally, we would like to point out that when \( A_i(u) = 2u, B_i(u) = 0, i = 1, 2 \) and \( C(u) = ku^2 + \delta u \), Eq. (1.3) contains three kinds of \((2 + 1)\)-dimensional models such as time-fractional biological model (Rui 2020), time-fractional reaction–diffusion model (Rui 2020) and time-fractional fluid model in porous media (Rui 2020). Rui (2020) derived exact solutions of the time-fractional biological population model using the dynamical system method. In Wu and Rui (2018), using method of separation of variables combined with the homogeneous balance principle, Wu and Rui studied exact solutions of time-fractional biological population model. Also, Abdel Kader et al. (2021) investigated exact solutions of time-fractional biological population model with variable coefficients through the invariant subspace method combined with wave transformation technique.

Many analytic techniques have been developed to deal with nonlinear integer-order PDEs during recent decades. Very recently, \( N \)-soliton solutions to nonlinear integrable equations are systematically studied by the Hirota bilinear method for both \((1 + 1)\)-dimensional integrable equations (Ma and Xiu 2020) and and \((2 + 1)\)-dimensional integrable equations (Ma 2020) and also some novel interesting \((2 + 1)\)-dimensional nonlinear PDEs (Ma 2020a, b; Ma et al. 2021). In the literature, there is no well-defined method for solving the FDEs due to the violations of the fundamental properties of fractional-order derivatives. Hence finding the exact solutions of fractional non-linear differential equations are a very challenging and toughest job. However, the derivation of exact solutions of FDEs is an important task because it will help to understand the behaviour of the complex systems. Due to these reasons, in recent years, many mathematicians and physicists have spent much attention on developing the exact and numerical methods for solving the FDEs such as invariant subspace method (Sahadevan and Bakkyaraj 2015; Sahadevan and Prakash 2016, 2017a, b; Choudhary et al. 2019; Gazizov and Kasatkin 2013; Artale Harris and Garra 2013; Choudhary and Daftardar-Gejji 2017; Prakash 2020; Prakash et al. 2020; Rui 2018; Prakash 2021; Hashemi 2018), Lie group method (Prakash 2021; Sahadevan and Prakash 2017b; Gazizov et al. 2009; Prakash and Sahadevan 2017; Sahadevan and Bakkyaraj 2012; Bakkyaraj 2020; Sahadevan and Prakash 2019; Nass 2019a; Sethukumarasamy et al. 2021; Nass 2019b; Hashemi and Baleanu 2016), element free Galerkin (EFG) method (Abbaspadeh and Dehghan 2019), quadratic spline collocation method (Luo et al. 2016), Adomian decomposition method (Daftardar-Gejji and Jafari 2005; Momani and Odibat 2006), differential transform method (Odibat and Momani 2008), iterative method (Alkahtani and Atangana 2016), new hybrid method (Ma et al. 2020), difference methods (Owolabi and Atangana 2018), method separation of variables combined with the homogeneous balanced principles (Rui 2020; Ren et al. 2020; Rui 2018, 2017; Rui and Zhang 2020), dynamical system method (Wu and Rui 2018) and so on. Recent studies have shown that the invariant subspace method is a very effective and powerful analytical approach for deriving the exact solutions of fractional non-linear PDEs.

The invariant subspace method is also called the generalized separation of variables method. This method was initially studied by Galaktionov and Svirshchevskii (2007). After that, it was developed for scalar and coupled non-linear PDEs by Ma (2012) and many others (Ma et al. 2012; Ma and Liu 2012; Ye et al. 2014; Liu 2018). Recently, the invariant subspace method was extended for scalar and coupled non-linear fractional PDEs (Prakash 2021; Sahadevan and Bakkyaraj 2015; Sahadevan and Prakash 2016, 2017a, b; Choudhary et al. 2019; Gazizov and Kasatkin 2013; Artale Harris and Garra 2013; Choudhary and Daftardar-
Zhu and Qu (2016) have extended this method to (2 + 1)-dimensional non-linear PDEs. However, to the best of our knowledge, no one has been extended the invariant subspace method to (2 + 1)-dimensional fractional non-linear PDEs. Hence, the main purpose of the article is to show how to extend the invariant subspace method for (2 + 1)-dimensional time-fractional non-linear PDEs. More specifically, this article investigates for finding the invariant subspaces for the special types of the given equation that are the reaction–diffusion wave equation, reaction–diffusion wave equation and diffusion wave equation.

The structure of the paper is organized as follows. In Sect. 2, we give a detailed study for finding the invariant subspaces of the (2 + 1)-dimensional time-fractional non-linear PDEs. Also, we explain how to construct exact solutions for the given PDEs using the obtained invariant subspaces. Thus, we consider the generalized (2 + 1)-dimensional time-fractional non-linear PDE in the form

\[
\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \hat{K}[u] \equiv \hat{K}\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}, \ldots, \frac{\partial^k u}{\partial x_1^{k_1} \partial x_2^{k_2}}\right),
\]

(2.1)

where \( u = u(x_1, x_2, t) \), \( \alpha > 0 \), \( t > 0 \), \( x_1, x_2 \in \mathbb{R} \) and \( \frac{\partial^{\alpha}}{\partial t^{\alpha}}(\cdot) \) denotes Caputo fractional derivative (1.4) of order \( \alpha \), and \( \hat{K}[u] \) is the sufficiently given smooth differential operator of order \( k \), and \( k_1 + k_2 = k, k_1, k_2 \in \mathbb{N} \).
Now, we assume the linearly independent set
\[ S = \{ \xi_m(x_1, x_2) : m = 1, 2, \ldots, n \} \]  
(2.2)

Then, we can define the \( n \)-dimensional linear space
\[ \mathcal{V}_n = \text{Span}(S) = \text{Span}\{\xi_m(x_1, x_2) : m = 1, 2, \ldots, n\} = \left\{ \sum_{m=1}^{n} \kappa_m \xi_m(x_1, x_2) : \kappa_m \in \mathbb{R}, m = 1, 2, \ldots, n \right\} \]  
(2.3)

where \( n \) denotes the dimension of the linear space and it is denoted by \( \dim(\mathcal{V}_n) = n \) (\(< \infty \)).

The linear space \( \mathcal{V}_n \) given in (2.3) is said to be invariant under the differential operator \( \hat{\mathcal{L}}[u] \) given in (2.1) if for every \( u \in \mathcal{V}_n \) implies \( \hat{\mathcal{L}}[u] \in \mathcal{V}_n \), which can be written as
\[ \hat{\mathcal{L}} \left[ \sum_{m=1}^{n} \kappa_m \xi_m(x_1, x_2) \right] = \sum_{m=1}^{n} \Psi_m(\kappa_1, \kappa_2, \ldots, \kappa_n) \xi_m(x_1, x_2), \]  
(2.4)

where \( \kappa_m \in \mathbb{R} \) and \( \Psi_m \) denotes coefficient of expansion \( \hat{\mathcal{L}}[u] \) with respect to the basis set given in (2.2), \( m = 1, 2, \ldots, n \).

**Theorem 2.1** Suppose that the differential operator \( \hat{\mathcal{L}}[u] \) given in (2.1) admits an \( n \)-dimensional linear space \( \mathcal{V}_n \) given in (2.3), then the underlying \( (2 + 1) \)-dimensional time-fractional PDE has admits an exact solution of the form
\[ u(x_1, x_2, t) = \sum_{m=1}^{n} \Phi_m(t) \xi_m(x_1, x_2), \]  
(2.5)

where the functions \( \Phi_m(t), m = 1, 2, \ldots, n \) are the solutions of the following system of ODEs of fractional-order
\[ \frac{d^\alpha \Phi_m(t)}{dt^\alpha} = \Psi_m(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t)), \quad m = 1, 2, \ldots, n. \]  
(2.6)

**Proof** Let \( \mathcal{V}_n \) be an \( n \)-dimensional invariant subspace admitted by the given differential operator \( \hat{\mathcal{L}}[u(x_1, x_2, t)] \). Also, we assume that
\[ u(x_1, x_2, t) = \sum_{m=1}^{n} \Phi_m(t) \xi_m(x_1, x_2). \]  
(2.7)

Computing the Caputo fractional derivative of \( u(x_1, x_2, t) \) of order \( \alpha > 0 \) with respect to \( t \) gives
\[ \frac{\partial^\alpha u}{\partial t^\alpha} = \sum_{m=1}^{n} \frac{d^\alpha \Phi_m(t)}{dt^\alpha} \xi_m(x_1, x_2). \]  
(2.8)

Since the linear space \( \mathcal{V}_n \) is invariant under \( \hat{\mathcal{L}}[u] \). Thus, we have
\[ \hat{\mathcal{L}}[u(x_1, x_2, t)] = \sum_{m=1}^{n} \Psi_m(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t)) \xi_m(x_1, x_2). \]  
(2.9)

Substituting (2.8) and (2.9) in (2.1), we get
\[
\sum_{m=1}^{n} \left[ \frac{d^m \Phi_m(t)}{dt^m} - \Psi_m(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t)) \right] \xi_m(x_1, x_2) = 0. \tag{2.10}
\]

Since the functions \( \xi_m(x_1, x_2), m = 1, 2, \ldots, n \) are linearly independent. Thus, the above equation (2.10) reduces to the system of fractional ODEs (2.6).

\[ \square \]

### 2.1 Estimation of invariant subspaces for the given Eq. (2.1)

In this subsection, we would like to explain how to find the invariant subspaces for the given differential operator \( \hat{K}[u] \). Thus, let us first define the following two linear spaces:

\[ \mathcal{V}_{n_1} = \text{Span} \left\{ v_1(x_1), v_2(x_1), \ldots, v_{n_1}(x_1) \right\} = \left\{ y_1 \left| \mathcal{D}_{x_1}^{n_1}[y_1] = \frac{d^{n_1} y_1}{dx_1^{n_1}} + a_{n_1-1} \frac{d^{n_1-1} y_1}{dx_1^{n_1-1}} + \cdots + a_0 y_1 = 0 \right\} \right., \]

\[ \mathcal{V}_{n_2} = \text{Span} \left\{ \xi_1(x_2), \xi_2(x_2), \ldots, \xi_{n_2}(x_2) \right\} = \left\{ y_2 \left| \mathcal{D}_{x_2}^{n_2}[y_2] = \frac{d^{n_2} y_2}{dx_2^{n_2}} + b_{n_2-1} \frac{d^{n_2-1} y_2}{dx_2^{n_2-1}} + \cdots + b_0 y_2 = 0 \right\} \right., \tag{2.11} \]

where \( \{v_i(x_1) : i = 1, 2, \ldots, n_1\} \) and \( \{\xi_j(x_2) : j = 1, 2, \ldots, n_2\} \) are the set of \( n_k \)-linearly independent solutions of the linear homogeneous ODEs \( \mathcal{D}_{x_k}^{n_k}[y_k] = 0 \), for \( k = 1, 2 \), respectively and \( a_r, b_r \in \mathbb{R}, r = 0, 1, 2, \ldots, n_k - 1, k = 1, 2 \). Here \( \mathcal{D}_{x_k}^{n_k}[y_k] = 0 \) denotes the \( n_k \)th order homogeneous linear ODEs. Now, we can formally define the type I and type II invariant linear spaces for the non-linear differential operator \( \hat{K}[u] \) given in (2.1) that are discussed below in detail.

1. Type I linear space: First, we define the type I invariant subspace for the differential operator \( \hat{K}[u] \) in the form

\[ \mathcal{V}_{n_1n_2} = \text{Span} \left\{ v_1(x_1)\xi_1(x_2), \ldots, v_1(x_1)\xi_{n_2}(x_2), \ldots, v_{n_1}(x_1)\xi_1(x_2), \ldots, v_{n_1}(x_1)\xi_{n_2}(x_2) \right\} = \left\{ \xi(x_1, x_2) = \sum_{j=1}^{n_2} \left( \sum_{i=1}^{n_1} \kappa_{ij} v_i(x_1) \right) \xi_j(x_2) \left| \mathcal{D}_{x_1}^{n_1}[\xi] = 0, \mathcal{D}_{x_2}^{n_2}[\xi] = 0 \right\} \right., \tag{2.12} \]

where \( \kappa_{ij} \in \mathbb{R}, \ i = 1, \ldots, n_1, \ j = 1, 2, \ldots, n_2 \).

If the differential operator \( \hat{K}[u] \) given in (2.1) admits the linear space \( \mathcal{V}_{n_1n_2} \), then we have

\[ \hat{K} \left[ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \kappa_{ij} v_i(x_1)\xi_j(x_2) \right] = \sum_{j=1}^{n_2} \sum_{i=1}^{n_1} \Psi_{ij}(\kappa_{11}, \ldots, \kappa_{n_1n_2}) v_i(x_1)\xi_j(x_2). \]

This suggests an exact solution of the given Eq. (2.1) in the form

\[ u(x_1, x_2, t) = \sum_{j=1}^{n_2} \left( \sum_{i=1}^{n_1} \Phi_{ij}(t) v_i(x_1) \right) \xi_j(x_2) = \sum_{i=1}^{n_1} \left( \sum_{j=1}^{n_2} \Phi_{ij}(t) \xi_j(x_2) \right) v_i(x_1). \]
For this case, we obtain the invariance conditions for type I linear space of the given differential operator $\hat{K}[u]$ in the form
\[
D^n_{x_1}[\hat{K}] = \frac{d^{n_1}u}{dx_1^{n_1}} + a_{n_1-1} \frac{d^{n_1-1}u}{dx_1^{n_1-1}} + \cdots + a_1 \frac{d\hat{K}}{dx_1} + a_0 \hat{K} = 0,
\]
\[
D^n_{x_2}[\hat{K}] = \frac{d^{n_2}u}{dx_2^{n_2}} + b_{n_2-1} \frac{d^{n_2-1}u}{dx_2^{n_2-1}} + \cdots + b_1 \frac{d\hat{K}}{dx_2} + b_0 \hat{K} = 0
\]
along with
\[
D^n_{x_1}[u] = \frac{d^{n_1}u}{dx_1^{n_1}} + a_{n_1-1} \frac{d^{n_1-1}u}{dx_1^{n_1-1}} + \cdots + a_1 \frac{du}{dx_1} + a_0 u = 0,
\]
\[
D^n_{x_2}[u] = \frac{d^{n_2}u}{dx_2^{n_2}} + b_{n_2-1} \frac{d^{n_2-1}u}{dx_2^{n_2-1}} + \cdots + b_1 \frac{du}{dx_2} + b_0 u = 0
\]
and their differential consequences with respect to $x_1$ and $x_2$ if the differential operator $\hat{K}[u]$ admits the type I linear space $\mathcal{V}_{n_1n_2}$. Some important consequences of type I invariant subspaces for the given differential operator $\hat{K}[u]$ are as follows:

(i) If $1 \in \mathcal{V}_{n_1}$, then $a_0 = 0$ in $D^n_{x_1}[\xi] = 0$. Suppose $v_1(x_1) = 1 \in \mathcal{V}_{n_1}$. For this case, we obtain $\mathcal{V}_{n_1n_2} = \text{Span}\{\xi_1(x_2), \ldots, \xi_{n_2}(x_2), v_2(x_1)\xi_1(x_2), \ldots, v_{n_1}(x_1)\xi_{n_2}(x_2)\}$. Thus, we have $\dim(\mathcal{V}_{n_1n_2}) = n_1n_2$.

(ii) Similarly, $b_0 = 0$ in $D^n_{x_2}[\xi] = 0$ when $1 \in \mathcal{V}_{n_2}$. Suppose $\xi_1(x_2) = 1 \in \mathcal{V}_{n_2}$. Thus, we have $\mathcal{V}_{n_1n_2} = \text{Span}\{\xi_1(x_1), v_1(x_1), v_1(x_1)\xi_2(x_2), \ldots, v_{n_1}(x_1)\xi_{n_2}(x_2)\}$ and $\dim(\mathcal{V}_{n_1n_2}) = n_1n_2$.

(iii) If $1 \in \mathcal{V}_{n_1} \cap \mathcal{V}_{n_2}$, then $a_0 = 0$ in $D^n_{x_1}[\xi] = 0$ and $b_0 = 0$ in $D^n_{x_2}[\xi] = 0$. In this case, we obtain $\mathcal{V}_{n_1n_2} = \text{Span}\{1, v_2(x_1), \ldots, v_{n_1}(x_1), \xi_2(x_2), \ldots, \xi_{n_2}(x_2), v_2(x_1)\xi_2(x_2), \ldots, v_{n_1}(x_1)\xi_{n_2}(x_2)\}$ and $\dim(\mathcal{V}_{n_1n_2}) = n_1n_2$.

2. Type II linear space: The type II invariant subspace $\mathcal{V}_{n_1+n_2-1}$ for the given differential operator $\hat{K}[u]$ is defined as follows:

\[
\mathcal{V}_{n_1+n_2-1} = \text{Span}\{1, v_1(x_1), \ldots, v_{n_1-1}(x_1), \xi_1(x_2), \ldots, \xi_{n_2-1}(x_2)\} = \left\{\xi(x_1, x_2) = \kappa + \sum_{i=1}^{n_1-1} \kappa_i v_i(x_1) + \sum_{j=1}^{n_2-1} \delta_j \xi_j(x_2) \mid D^n_{x_1}[\xi] = 0, \quad (2.13)\right\}
\]

\[
D^n_{x_2}[\xi] = 0, \quad \frac{\partial^2 \xi}{\partial x_1 \partial x_2} = 0, \quad a_0 = b_0 = 0
\]

If the given differential operator $\hat{K}[u]$ admits the linear space $\mathcal{V}_{n_1+n_2-1}$, then we can write

\[
\hat{K} \left[ \kappa + \sum_{i=1}^{n_1-1} \kappa_i v_i(x_1) + \sum_{j=1}^{n_2-1} \delta_j \xi_j(x_2) \right] = \kappa + \sum_{i=1}^{n_1-1} \Psi_i(\kappa_1, \ldots, \kappa_{n_1-1}, \delta_1, \ldots, \delta_{n_2-1}) v_i(x_1)
\]

\[
+ \sum_{j=1}^{n_2-1} \Psi_{n_1+j-1}(\kappa_1, \ldots, \kappa_{n_1-1}, \delta_1, \ldots, \delta_{n_2-1}) \xi_j(x_2).
\]
This gives the exact solution of the given Eq. (2.1) in the form

$$u(x_1, x_2, t) = \Phi(t) + \sum_{i=1}^{n_1-1} \Phi_i(t) v_i(x_1) + \sum_{j=1}^{n_2-1} \Phi_{n_1+j-1}(t) \xi_j(x_2).$$

Thus, the invariance conditions of type II linear spaces for the given differential operator $\mathcal{K}[u]$ take the form

$$D^{n_1}_{x_1}[\mathcal{K}] = \frac{d^{n_1} \mathcal{K}}{dx_1^{n_1}} + a_{n_1-1} \frac{d^{n_1-1} \mathcal{K}}{dx_1^{n_1-1}} \cdots + a_1 \frac{d \mathcal{K}}{dx_1} = 0,$$

$$D^{n_2}_{x_2}[\mathcal{K}] = \frac{d^{n_2} \mathcal{K}}{dx_2^{n_2}} + b_{n_2-1} \frac{d^{n_2-1} \mathcal{K}}{dx_2^{n_2-1}} \cdots + b_1 \frac{d \mathcal{K}}{dx_2} = 0 \& \frac{\partial^{2} \mathcal{K}}{\partial x_1 \partial x_2} = 0$$

along with

$$D^{n_1}_{x_1}[u] = \frac{d^{n_1} u}{dx_1^{n_1}} + a_{n_1-1} \frac{d^{n_1-1} u}{dx_1^{n_1-1}} \cdots + a_1 \frac{du}{dx_1} = 0,$$

$$D^{n_2}_{x_2}[u] = \frac{d^{n_2} u}{dx_2^{n_2}} + b_{n_2-1} \frac{d^{n_2-1} u}{dx_2^{n_2-1}} \cdots + b_1 \frac{du}{dx_2} = 0 \& \frac{\partial^{2} u}{\partial x_1 \partial x_2} = 0$$

and their differential consequences with respect to $x_1$ and $x_2$ if the differential operator $\mathcal{K}[u]$ admits the type II linear space $\mathcal{V}_{n_1+n_2-1}$.

We would like to point out that the type II linear space $\mathcal{V}_{n_1+n_2-1}$ is the subspace of type I linear space $\mathcal{V}_{n_1n_2}$, that is, $\mathcal{V}_{n_1+n_2-1} \subseteq \mathcal{V}_{n_1n_2}$. Additionally, we observe that type II linear space $\mathcal{V}_{n_1+n_2-1}$ is smallest non-trivial (both variables involving $x_1$ and $x_2$) invariant subspace for the given differential operator $\mathcal{K}[u]$.

For example, let us consider the type I and type II linear spaces $\mathcal{V}_4 = \text{Span}\{1, x_1, x_2, x_1x_2\}$ and $\mathcal{V}_3 = \text{Span}\{1, x_1, x_2\}$ and also, the differential operator is

$$\mathcal{K}[u] = \frac{\partial}{\partial x_1} \left[ (c_2 u^2 + c_1 u + c_0) \left( \frac{\partial u}{\partial x_1} \right) \right] + \frac{\partial}{\partial x_2} \left[ (\beta_2 u^2 + \beta_1 u + \beta_0) \left( \frac{\partial u}{\partial x_2} \right) \right]$$

$$+(d_1 u + d_0) \left( \frac{\partial u}{\partial x_1} \right) + (\lambda_1 u + \lambda_0) \left( \frac{\partial u}{\partial x_2} \right) + k_1 u + k_0,$$

where $c_2, \beta_2, c_1, d_i, a_1, b_1, \beta_i, \lambda_i, k_i \in \mathbb{R}, \quad i = 0, 1$. Note that $\mathcal{V}_3 \nsubseteq \mathcal{V}_4$. Clearly, we check that the type I linear space is not invariant under the given operator $\mathcal{K}[u]$ because $\mathcal{K}[\delta_1 + \delta_2 x_1 + \delta_3 x_2 + \delta_4 x_1x_2] \notin \mathcal{V}_4$. But, the type II linear space $\mathcal{V}_3 = \text{Span}\{1, x_1, x_2\}$ is invariant under the given differential operator $\mathcal{K}[u]$, since $\mathcal{K}[\delta_1 + \delta_2 x_1 + \delta_3 x_2] \in \mathcal{V}_3$.

From this, we obtain smallest non-trivial invariant subspace $\mathcal{V}_3$ which is proper subspace of $\mathcal{V}_4$. Next, we would like to explain how to find the invariant subspaces for the given (2+1)-dimensional time-fractional generalized convection–reaction–diffusion wave equation (1.3).

### 3 Invariant subspaces for the given Eq. (1.3) and its special types

In this section, we explain how to construct the invariant subspaces for the generalized (2+1)-dimensional time-fractional convection–reaction–diffusion wave equation (1.3) and its special kind of equations such as convection–diffusion wave equation, reaction–diffusion wave equation, and diffusion wave equation.
3.1 Invariant subspaces for the given Eq. (1.3)

In this subsection, we give a detailed systematic study for finding the invariant subspaces of the given Eq. (1.3). Thus, the differential operator $\hat{K}[u]$ can be considered for the given Eq. (1.3) as follows:

$$\hat{K}[u] = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( A_i(u) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{2} B_i(u) \frac{\partial u}{\partial x_i} + C(u).$$

(3.1)

Here, we can discuss various dimensions of invariant subspaces for the above-mentioned differential operator $\hat{K}[u]$. For example, in this work, we consider the following possibilities of $(n_1, n_2)$ as

$$\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}.$$

From this, let us first consider $n_1 = n_2 = 2$. Thus, we present a detailed study for finding the invariant subspaces of the given differential operator (3.1).

**Estimation of invariant subspaces for the given Eq. (1.3):** Here we would like to point out that two-types of linear spaces are available for the given differential operator (3.1). Now, we define the linear spaces as

$$\mathcal{V}_2 = \left\{ y_1 | \mathcal{D}_x^2[y_1] = \frac{d^2y_1}{dx_1^2} + a_1 \frac{dy_1}{dx_1} + a_0 y_1 = 0 \right\},$$

$$\mathcal{V}_2 = \left\{ y_2 | \mathcal{D}_x^2[y_2] = \frac{d^2y_2}{dx_2^2} + b_1 \frac{dy_2}{dx_2} + b_0 y_2 = 0 \right\},$$

where the functions $y_i, i = 1, 2$, are the linear combinations of the two-linearly independent solutions of $\mathcal{D}_x^2[y_i] = 0$. Then, let us assume that the functions $v_1(x_1)$ and $v_2(x_1)$ are two-linearly independent solutions of $\mathcal{D}_x^2[y_1] = 0$. Similarly, we assume that another set of functions $\zeta_1(x_2)$ and $\zeta_2(x_2)$ are two-linearly independent solutions of $\mathcal{D}_x^2[y_2] = 0$. Then we can define two-types of linear spaces for the given differential operator (3.1) that are discussed below. For this case, we obtain the dimensions of type I and type II invariant subspaces as follows:

1. $\dim(\mathcal{V}_{n_1n_2}) = 4$ if either $1 \notin \mathcal{V}_{n_1}$ and $1 \notin \mathcal{V}_{n_2}$ or $1 \in \mathcal{V}_{n_1}$ (or $1 \in \mathcal{V}_{n_2}$),
2. $\dim(\mathcal{V}_{n_1+n_2-1}) = 3$ if $1 \in \mathcal{V}_{n_1}$ & $1 \in \mathcal{V}_{n_2}$,

which are discussed below.

**Type I and Type II linear spaces:** First, we consider the type I linear space in the following form

$$\mathcal{V}_{2,2} = \mathcal{V}_4 = \text{Span} \{ v_1(x_1) \zeta_1(x_2), v_1(x_1) \zeta_2(x_2), v_2(x_1) \zeta_1(x_2), v_2(x_1) \zeta_2(x_2) \} = \left\{ \xi(x_1, x_2) = \sum_{j=1}^{2} \left( \sum_{i=1}^{2} c_{ij} v_i(x_1) \right) \zeta_j(x_2) | \mathcal{D}_{x_1}^2[\xi] = 0, \mathcal{D}_{x_2}^2[\xi] = 0 \right\}$$

(3.2)

and type II linear space is considered as

$$\mathcal{V}_{2+2-1} = \mathcal{V}_3 = \text{Span} \{ 1, v_1(x_1), \zeta_1(x_2) \}$$

$$\left\{ \xi(x_1, x_2) = \kappa + \kappa_1 v_1(x_1) + \delta_1 \zeta_1(x_2) | \mathcal{D}_{x_1}^2[\xi] = 0, \frac{\partial^2 \xi}{\partial x_1 \partial x_2} = 0, \quad a_0 = b_0 = 0, k = 1, 2 \right\}$$

(3.3)

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Now we wish to find the type I and type II invariance conditions for the differential operator \( \hat{K}[u] \) given in (3.1).

**Type I invariant conditions:** If the given differential operator \( \hat{K}[u] \) admits type I linear space \( V_3 \), then the type I invariant conditions of \( \hat{K}[u] \) read in the form

\[
D_{x_1}^2 [\hat{K}[u]] = \frac{d^2 \hat{K}[u]}{dx_1^2} + a_1 \frac{d \hat{K}[u]}{dx_1} + a_0 \hat{K}[u] = 0,
\]

along with

\[
D_{x_2}^2 [\hat{K}[u]] = \frac{d^2 \hat{K}[u]}{dx_2^2} + b_1 \frac{d \hat{K}[u]}{dx_2} + b_0 \hat{K}[u] = 0,
\]

where \( a_i \), \( i = 0, 1, 2 \) and \( b_j \), \( j = 0, 1, 2 \) are constants to be determined and the differential operator \( \hat{K}[u] \) given in (3.1). In a similar manner, we can find the type II invariant conditions for the given differential operator \( \hat{K}[u] \).

**Type II invariant conditions:** If the given differential operator \( \hat{K}[u] \) admits type II linear space \( V_3 \), then the type II invariant conditions of \( \hat{K}[u] \) take the form

\[
D_{x_1}^2 [\hat{K}[u]] = \frac{d^2 \hat{K}[u]}{dx_1^2} + a_1 \frac{d \hat{K}[u]}{dx_1} = 0, \tag{3.4}
\]

along with

\[
D_{x_2}^2 [\hat{K}[u]] = \frac{d^2 \hat{K}[u]}{dx_2^2} + b_1 \frac{d \hat{K}[u]}{dx_2} = 0 \quad \text{and} \quad \frac{\partial^2 \hat{K}[u]}{\partial x_1 \partial x_2} = 0
\]

(3.5)

where \( a_i \), \( i = 1, 2 \) and \( b_j \), \( j = 1, 2 \) are constants to be determined and the differential operator \( \hat{K}[u] \) given in (3.1).

Now, we give a detailed explanation for finding the type II invariant subspaces \( V_3 \) of the given differential operator \( \hat{K}[u] \) using the above-discussed case \( n_1 = n_2 = 2 \).

Thus, we substitute the given differential operator \( \hat{K}[u] \) into (3.4), which gives

\[
\begin{align*}
2(A_2)_u u_{x_1}^2 + 4(A_2)_u u_{x_2} u_{x_1} u_{x_1 x_2} + & [-2(A_2)_u b_1 + 2(B_2)_u] u_{x_1 x_2} + (A_1)_{uu} u_{x_1}^4 \\
+ \frac{[-5(A_1)_{uu} a_1 + (B_1)_{uu}] A_{x_1}^3 + u_{x_2}^2 (A_2)_{uu} u_{x_1}^2 + [-2(A_1)_{uu} b_1 + (B_2)_{uu}] u_{x_2}^2 u_{x_1}^2}{2(B_1)_u a_1 + C_{uu}} u_{x_1}^2 = 0, \tag{3.6}
\end{align*}
\]

\[
\begin{align*}
[-A_1 a_1 + B_1] u_{x_1 x_2} + 2(A_1)_{uu} u_{x_1} u_{x_2} x_2 + & 2(A_1)_{uu} u_{x_1 x_2} + [(4(A_1)_{uu} u_{x_2} + 2(A_1)_u b_1) u_{x_1} \\
+ (-2(A_1)_u a_1 + 2(B_2)_u)] u_{x_2} - A_1 b_1 a_1 + B_1 b_1] u_{x_1 x_2} + (A_1)_{uu} u_{x_2}^2 u_{x_1}^2 + (A_2)_{uu} u_{x_1}^4 \tag{3.7}
\end{align*}
\]

\[
+ [4(A_2)_u b_1^2 - 2(B_2)_u b_1 + C_{uu}] u_{x_2}^2 = 0
\]
and
\[ A_2 b_1 + B_2 + 2(A_2) u x_2 ] u_{x_2} x_2 + 3(A_1) u a u^2_{x_1} x_{1,x_2} + [-6(A_1) a a_1 + 2(B_1) u ] u_{x_1} x_{1,x_2} + 3(A_2) u u u x_{1,x_2} + [2(B_2) u - 4(A_2) u b_1 ] u_{x_2} x_{1,x_2} + [A_1 a^2 - B_1 a_1 + C_a u_{x_2} ] (3.8) \\
(3.8)
\]

Substituting Eq. (3.5) into Eqs. (3.6)–(3.8), which are reduced to the following form:
\[ (A_1) u u u u^4_{x_1} + [-5(A_1) u a a_1 + (B_1) u u u ] u^2_{x_1} + u^2_{x_2} (A_2) u u a u^2_{x_1} + [- (A_2) u a b_1 + (B_2) u u u ] u_{x_2} x_{1,x_2} + 4(A_1) a a_1^2 - 2(B_1) a_1 + C_a u ] u_{x_1} x_{1,x_2} = 0, (3.9) \]

\[ (A_1) u u u u^2_{x_2} x_{1,x_2} + [- (A_1) u u a_1 + (B_1) u u u ] u^2_{x_2} x_{1,x_2} + (-A_1) u u a_1 + (B_1) u u ] u^2_{x_2} x_{1,x_2} = 0, (3.10) \]

and
\[ (A_1) u u u u^3_{x_1} x_{1,x_2} + [(B_1) u u - 3(A_1) u u a_1 ] u^2_{x_1} x_{1,x_2} + [(B_2) u u - 3(A_2) u u a_1 ] u^2_{x_2} x_{1,x_2} + (A_2) u u u u_3_{x_1,x_2} + [(A_1) a a_1^2 + (A_2) b_1 b_1 - (B_1) a_1 - (B_2) b_1 + C_a u ] u_{x_1} x_{1,x_2} = 0, (3.11) \]

where \( A_i \) = \( \frac{dA_i}{du} \), \( A_i u u = \frac{d^2 A_i}{du^2} \), \( B_i u u = \frac{d^2 B_i}{du^2} \), \( u_{x_1} \) = \( \frac{\partial u}{\partial x_1} \), \( i = 1, 2 \)

etc. The above Eqs. (3.9)–(3.11) are reduced to the system of over-determined equations. In general, the obtained system of over-determined equations may not be soluble. Various cases are discussed below (Refer Table 1). Thus, let us first consider the functions \( A_1(u) = c_2 u^2 + c_1 u + c_0, A_2(u) = \beta_2 u^2 + \beta_1 u + \beta_0, B_1(u) = d_2 u^2 + d_1 u + d_0, B_2(u) = \lambda_2 u^2 + \lambda_1 u + \lambda_0 \) and \( C(u) = k_3 u^3 + k_2 u^2 + k_1 u + k_0 \).

For this case, we obtain the cubic non-linear differential operator \( \hat{K}[u] \) of the form
\[ \hat{K}[u] = \frac{\partial}{\partial x_1} [ (c_2 u^2 + c_1 u + c_0) \left( \frac{\partial u}{\partial x_1} \right) ] + \frac{\partial}{\partial x_2} [ (\beta_2 u^2 + \beta_1 u + \beta_0) \left( \frac{\partial u}{\partial x_2} \right) ] + (d_2 u^2 + d_1 u + d_0) \left( \frac{\partial u}{\partial x_1} \right) + (\lambda_2 u^2 + \lambda_1 u + \lambda_0) \left( \frac{\partial u}{\partial x_2} \right) + k_3 u^3 + k_2 u^2 + k_1 u + k_0, \]

where \( c_i, \beta_i, d_i, \lambda_i, k_i, k_3 \in \mathbb{R}, i = 0, 1, 2 \). Substituting the functions \( A_1(u), B_1(u) (i = 1), 2 \) and \( C(u) \) into the above Eqs. (3.9)–(3.11), which yield the following system of over-determined equations
\[ 6a_1 c_2 - 2d_2 = 0, 6b_1 \beta_2 - 2\lambda_2 = 0, 2d_2 - 2a_1 c_2 = 0, \]
\[ 2\lambda_2 - 10b_1 \beta_2 = 0, 2d_2 - 10a_1 c_2 = 0, \]
\[ 8a_1^2 c_2 - 4a_1 d_2 + 6k_3 = 0, 4a_1^2 c_1 - 2a_1 d_1 + 2k_2 = 0, 8b_1^2 \beta_2 - 4b_1 \lambda_2 + 6k_3 = 0, \]
\[ 2\lambda_2 - 2b_1 \beta_2 = 0, 4b_1^2 \beta_1 - 2b_1 \lambda_1 + 2k_2 = 0, a_1^2 c_1 + b_1^2 \beta_1 - a_1 d_1 - b_1 \lambda_1 + 2k_2 = 0, \]
\[ a_1^2 c_2 + b_1^2 \beta_2 - a_1 d_2 - b_1 \lambda_2 + 3k_3 = 0. \]

Solving the above over-determined system of equations, we get three-dimensional type II invariant subspaces \( V_3 \) with their corresponding differential operators that are given in Tables 2 and 3. Proceeding the above similar procedure, we can find various dimensions of
the invariant subspaces for the given differential operator (3.1) using the other possible values of \((n_1, n_2)\) with different non-linearities that are listed from Tables 4, 5, 6, 7, 8 and 9.

### 3.2 Special kinds of the given Eq. (1.3) with their invariant subspaces

In this subsection, we show how to find the invariant subspaces for the special types of the given \((2+1)\)-dimensional convection–reaction–diffusion wave equation (1.3). It is interesting to note that the given Eq. (1.3) is reduced into three different special kinds of equations that are discussed below.

- When \(C(u) = 0\), the given Eq. (1.3) becomes the generalised \((2+1)\)-dimensional time-fractional convection–diffusion wave equation

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( A_i(u) \frac{\partial u}{\partial x_i} \right) + \frac{\partial}{\partial x_i} B_i(u) \frac{\partial u}{\partial x_i}, \quad \alpha \in (0, 2].
\]  

(3.13)

- If \(B_1(u) = B_2(u) = 0\), then the given Eq. (1.3) can be viewed as the generalized \((2+1)\)-dimensional time-fractional reaction–diffusion wave equation

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( A_i(u) \frac{\partial u}{\partial x_i} \right) + C(u), \quad \alpha \in (0, 2].
\]  

(3.14)

- If \(B_1(u) = B_2(u) = C(u) = 0\), then Eq. (1.3) is reduced into

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( A_i(u) \frac{\partial u}{\partial x_i} \right), \quad \alpha \in (0, 2]
\]  

(3.15)

which is familiarly known as the generalized \((2+1)\)-dimensional time-fractional diffusion wave equation.

Proceeding the above similar procedure, we can find the various dimensions of the type I and type II invariant subspaces for the above-mentioned equations with different power-law non-linearities (Table 10). The obtained invariant subspaces with their corresponding differential operators are listed from Tables 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 and 21.
Table 2  Differential operator (3.12) with quadratic non-linearities and their corresponding invariant subspaces

| Cases | $A_i(u)$, $B_i(u)$ & $C(u)$, $i = 1, 2$ | Invariant subspaces |
|-------|-----------------------------------|---------------------|
| 1.    | \[
\begin{align*}
    A_1(u) &= \frac{-b_2^2 \beta_1}{a_1} u + c_0 \\
    A_2(u) &= \beta_1 u + \beta_0 \\
    B_1(u) &= \frac{-4b_1^2 \beta_1 + b_1 \lambda_1}{a_1} u + d_0 \\
    B_2(u) &= \lambda_1 u + \lambda_0 \\
    C(u) &= (-2b_2^2 \beta_1 + b_1 \lambda_1)u^2 + k_1 u + k_0
\end{align*}
\] | $\mathcal{V}_3 = \text{Span}\{1, e^{-a_1 x_1}, e^{-b_1 x_2}\}$ |
Table 3  Differential operator (3.12) with cubic non-linearities and their corresponding invariant subspaces

| Cases | $A_i(u), B_i(u) $ & $C(u), i = 1, 2$ | Invariant subspaces |
|-------|--------------------------------|-------------------|
| 1.    | \( A_1(u) = c_2u^2 + c_1u + c_0 \) \( B_1(u) = d_1u + d_0 \) \( C(u) = k_1u + k_0 \) | \( \mathcal{V}_3 = \text{Span}\{1, x_1, x_2\} \) |
| 2.    | \( A_1(u) = c_0 \) \( B_2(u) = \lambda_1u + \lambda_0 \) \( C(u) = k_1u + k_0 \) | \( \mathcal{V}_3 = \text{Span}\{1, e^{-\alpha_1x_1}, x_2\} \) |
| 3.    | \( A_1(u) = c_2u^2 + c_1u + c_0 \) \( B_1(u) = d_1u + d_0 \) \( B_2(u) = \lambda_0 \) \( C(u) = k_1u + k_0 \) | \( \mathcal{V}_3 = \text{Span}\{1, e^{-\beta_1x_2}, x_1\} \) |

4 Exact solutions of (1.3) along with initial conditions

In this section, we would like to explain how to derive exact solutions for the initial value problem of the given Eq. (1.3) with different types of non-linearities using the obtained invariant subspaces that are discussed in the previous section. Hence, let us first explain how to construct exact solutions for the initial value problem of the given Eq. (1.3) with cubic non-linearities using the obtained exponential subspaces.

4.1 Exact solutions of (1.3) with cubic non-linearities using the exponential subspaces

Now, we first discuss how to derive the exact solutions for the initial value problem of the Eq. (1.3) with cubic non-linearities using the two-dimensional exponential subspace which was discussed in Table 6 of case 3. Thus, we consider the time-fractional convection–reaction–diffusion wave equation with cubic non-linearities in the form

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial x_1} \left( \left( \frac{d_2}{3a_0} u^2 + c_1u + c_0 \right) \frac{\partial u}{\partial x_1} \right) + \beta_0 \frac{\partial^2 u}{\partial x_2^2} + (d_2u^2 + d_1u + d_0) \frac{\partial u}{\partial x_1}
\]

\[+ \lambda_0 \frac{\partial u}{\partial x_2} + (-2a_0^2c_1 + a_0d_1)u^2 + k_1u, \quad \alpha \in (0, 2], \ t \geq 0, \ x_1, x_2 \in \mathbb{R} \tag{4.1} \]

along with the initial conditions

- If \( \alpha \in (0, 1) \), then \( u(x_1, x_2, 0) = v_1e^{-a_0x_1} + v_2e^{-(a_0x_1 + b_1x_2)} \).
- If \( \alpha \in (1, 2) \), then \( u(x_1, x_2, 0) = v_1e^{-a_0x_1} + v_2e^{-(a_0x_1 + b_1x_2)} \) & \( \frac{\partial u}{\partial t} |_{t=0} = \mu_1e^{-a_0x_1} + \mu_2e^{-(a_0x_1 + b_1x_2)} \).

For this case, we obtain the cubic non-linear differential operator

\[
\tilde{A}[u] = \left( \frac{d_2}{3a_0} u^2 + c_1u + c_0 \right) \frac{\partial^2 u}{\partial x_1^2} + \left( \frac{2d_2}{3a_0} u + c_1 \right) \left( \frac{\partial u}{\partial x_1} \right)^2 + \beta_0 \frac{\partial^2 u}{\partial x_2^2} + (d_2u^2 + d_1u + d_0) \frac{\partial u}{\partial x_1} + \lambda_0 \frac{\partial u}{\partial x_2} + (-2a_0^2c_1 + a_0d_1)u^2 + k_1u.
\]
Table 4  Differential operator (3.1) with quadratic non-linearities and their corresponding invariant subspaces

| Cases | \( \tilde{K}_i[u] = \sum_{i=1}^{2} (A_i(u)u_x)_x + \sum_{i=1}^{2} B_i(u)u_x + C(u), \ A_i(u), \ B_i(u) \& C(u), \ i = 1, 2 \) | Invariant subspaces |
|-------|-----------------------------------------------------------------|------------------------|
| 1. \( \begin{align*} A_1(u) &= c_1 u + c_0, \\ A_2(u) &= \frac{-c_1 a_1}{b_1} u + \beta_0, \\ B_1(u) &= d_1 u + d_0, \\ B_2(u) &= \lambda_1 u + \lambda_0, \\ C(u) &= a_1 d_1 u^2 + k_1 u + k_0 \end{align*} \) | \( \mathcal{V}_3 = \text{Span}\{1, e^{-a_1 x_1}, e^{-b_1 x_2}\} \) |
| 2. \( \begin{align*} A_1(u) &= c_0, \\ A_2(u) &= \beta_0, \\ B_1(u) &= d_1 u + d_0, \\ B_2(u) &= \lambda_1 u + \lambda_0, \\ C(u) &= a_1 d_1 u^2 + k_1 u + k_0 \end{align*} \) | \( \mathcal{V}_3 = \text{Span}\{1, e^{-a_1 x_1}, e^{-\frac{a_1}{x_1}} x_2\} \) |
| 3. \( \begin{align*} A_1(u) &= c_1 u + c_0, \\ A_2(u) &= \beta_1 u + \beta_0, \\ B_1(u) &= d_0, \\ B_2(u) &= \lambda_0, \\ C(u) &= k_0 \end{align*} \) | \( \mathcal{V}_3 = \text{Span}\{1, x_1, x_2\} \) |
| 4. \( \begin{align*} A_1(u) &= c_1 u + c_0, \\ A_2(u) &= \beta_1 u + \beta_0, \\ B_1(u) &= d_0, \\ B_2(u) &= \lambda_1 u + \lambda_0, \\ C(u) &= k_1 u + k_0 \end{align*} \) | \( \mathcal{V}_4 = \text{Span}\{1, x_1, \frac{x_1^2}{2}, x_2\} \) |
| 5. \( \begin{align*} A_1(u) &= c_1 u + c_0, \\ A_2(u) &= \beta_1 u + \beta_0, \\ B_1(u) &= d_0, \\ B_2(u) &= \lambda_0, \\ C(u) &= k_1 u + k_0 \end{align*} \) | \( \mathcal{V}_5 = \text{Span}\{1, x_1, \frac{x_1^2}{2}, x_2, \frac{x_2^2}{2}\} \) |
| 6. \( \begin{align*} A_1(u) &= c_0, \\ A_2(u) &= \beta_1 u + \beta_0, \\ B_1(u) &= d_0, \\ B_2(u) &= \lambda_1 u + \lambda_0, \\ C(u) &= k_1 u + k_0 \end{align*} \) | \( \mathcal{V}_3 = \text{Span}\{1, e^{-a_1 x_1}, x_2\} \) \( \mathcal{V}_4 = \text{Span}\{1, e^{-a_2 x_1}, x_1, x_2\} \) |
Table 5 Differential operator (3.1) with quadratic non-linearities and their corresponding invariant subspaces

| Cases | \( \hat{K}[u] = \frac{2}{i=1} (A_i(u)u_x)_x + \sum_{i=1}^2 B_i(u)u_x + C(u), A_i(u), B_i(u) & C(u), i = 1, 2 \) | Invariant subspaces |
|-------|---------------------------------------------------------------------------------|---------------------|
| 7.    | \[ \begin{align*} A_1(u) &= c_0 \\ A_2(u) &= \beta_1 u + \beta_0 \end{align*} \] | \[ V_5 = \text{Span}\{1, e^{\frac{1}{2}(a_2 + a_1^2)x_1}, e^{\frac{1}{2}(a_2 - a_1^2)x_1}, x_2, x_2^2\} \] |
| 8.    | \[ \begin{align*} A_1(u) &= c_1 u + c_0 \\ A_2(u) &= \beta_1 u + \beta_0 \end{align*} \] | \[ V_5 = \text{Span}\{1, e^{\frac{1}{2}(a_2 - b_1^2)x_2}, e^{\frac{1}{2}(b_2 + b_1^2)x_2}, x_2, x_2^2\} \] |
| 9.    | \[ \begin{align*} A_1(u) &= \frac{d_1}{4a_1} u + c_0 \\ A_2(u) &= \beta_1 u + \beta_0 \end{align*} \] | \[ V_4 = \text{Span}\{1, e^{-\beta_1 x_1}, \sin\left(\frac{1}{2} \sqrt{\frac{a_1 d_1}{\beta_1^2}} x_2\right), \cos\left(\frac{1}{2} \sqrt{\frac{a_1 d_1}{\beta_1^2}} x_2\right)\} \] |
| 10.   | \[ \begin{align*} A_1(u) &= c_1 u + c_0 \\ A_2(u) &= \frac{\lambda_1}{4b_1} u + \beta_0 \end{align*} \] | \[ V_4 = \text{Span}\{1, e^{-b_1 x_2}, \sin\left(\frac{1}{2} \sqrt{\frac{b_1 c_1}{\lambda_1}} x_1\right), \cos\left(\frac{1}{2} \sqrt{\frac{b_1 c_1}{\lambda_1}} x_1\right)\} \] |
| 11.   | \[ \begin{align*} A_1(u) &= \frac{a_1 c_1}{b_1^2} u + c_0 \\ A_2(u) &= \beta_1 u + \beta_0 \end{align*} \] | \[ V_4 = \text{Span}\{1, e^{-b_1 x_2}, \sin\left(\sqrt{a_1} x_1\right), \cos\left(\sqrt{a_1} x_1\right)\} \] |
| 12.   | \[ \begin{align*} A_1(u) &= c_1 u + c_0 \\ A_2(u) &= \beta_0 \end{align*} \] | \[ V_5 = \text{Span}\{1, x_1, \frac{x_1^2}{2}, \sin(\sqrt{\beta_1} x_2), \cos(\sqrt{\beta_1} x_2)\} \] |
| 13.   | \[ \begin{align*} A_1(u) &= c_0 \\ A_2(u) &= \beta_1 u + \beta_0 \end{align*} \] | \[ V_5 = \text{Span}\{1, x_2, \frac{x_2^2}{2}, \sin(\sqrt{\beta_1} x_1), \cos(\sqrt{\beta_1} x_1)\} \] |
| Cases | \( \tilde{K}[u] = \sum_{i=1}^{2} (A_i(u)u_x)_i + \sum_{i=1}^{2} B_i(u)u_x + C(u), A_i(u), B_i(u) & C(u), i = 1, 2 \) | Invariant subspaces |
|-------|-------------------------------------------------------------------|-------------------|
| 1.    | \[
\begin{align*}
A_1(u) &= \left( \frac{-3b_0^2\beta_2 + a_0d_2 + b_0\lambda_2}{3a_0^2} \right) u^2 \\
A_2(u) &= \beta_2 u^2 + \beta_1 u + \beta_0 \\
B_1(u) &= d_2 u^2 + d_1 u + d_0 \\
B_2(u) &= \lambda_2 u^2 + \lambda_1 u + \lambda_0 \\
C(u) &= k_2 u^2 + k_1 u
\end{align*}
\] | \( \mathcal{V}_1 = \text{Span}\{e^{-a_0x_1-b_0x_2}\} \) |
| 2.    | \[
\begin{align*}
A_1(u) &= \frac{d_2}{3a_0} u^2 + c_0 \\
A_2(u) &= \beta_0 \\
B_1(u) &= d_2 u^2 + d_1 u + d_0 \\
B_2(u) &= \lambda_0 \\
C(u) &= a_0d_1 u^2 + k_1 u
\end{align*}
\] | \( \mathcal{V}_2 = \text{Span}\{e^{-a_0x_1}, e^{-a_0x_1-b_1x_2}\} \) |
| 3.    | \[
\begin{align*}
A_1(u) &= \frac{d_2}{3a_0} u^2 + c_1 u + c_0 \\
A_2(u) &= \beta_0 \\
B_1(u) &= d_2 u^2 + d_1 u + d_0 \\
B_2(u) &= \lambda_0 \\
C(u) &= (-2a_0^2c_1 + a_0d_1) u^2 + k_1 u
\end{align*}
\] | \( \mathcal{V}_2 = \text{Span}\{e^{-a_0x_1}, e^{-a_0x_1-b_1x_2}\} \) |
| 4.    | \[
\begin{align*}
A_1(u) &= \frac{d_2}{3d_0} u^2 + c_0 \\
A_2(u) &= \beta_0 \\
B_1(u) &= d_2 u^2 + d_1 u + d_0 \\
B_2(u) &= \lambda_0 \\
C(u) &= a_0d_1 u^2 + k_1 u
\end{align*}
\] | \( \mathcal{V}_2 = \text{Span}\{e^{-a_0x_1}-\frac{1}{2}(b_1-\sqrt{b_1^2-4b_0})x_2, e^{-a_0x_1}-\frac{1}{2}(b_1+\sqrt{b_1^2-4b_0})x_2\} \) |
| 5.    | \[
\begin{align*}
A_1(u) &= c_2 u^2 + c_0 \\
A_2(u) &= \beta_0 \\
B_1(u) &= 3a_0c_2 u^2 + d_1 u + d_0 \\
B_2(u) &= \lambda_0 \\
C(u) &= a_0d_1 u^2 + k_1 u
\end{align*}
\] | \( \mathcal{V}_2 = \text{Span}\{e^{-a_0x_1}sin(\sqrt{b_0}x_2), e^{-a_0x_1}cos(\sqrt{b_0}x_2)\} \) |
| 6.    | \[
\begin{align*}
A_1(u) &= c_0 \\
A_2(u) &= \beta_2 u^2 + \beta_1 u + \beta_0 \\
B_1(u) &= d_0 \\
B_2(u) &= 3b_0 \beta_0 u^2 + \lambda_1 u + \lambda_0 \\
C(u) &= (-2b_0^2 \beta_1 + b_0 \lambda_1) u^2 + k_1 u
\end{align*}
\] | \( \mathcal{V}_2 = \text{Span}\{e^{-b_0x_2}sin(\sqrt{a_0}x_1), e^{-b_0x_2}cos(\sqrt{a_0}x_1)\} \) |
Table 7  Differential operator (3.1) with cubic non-linearities and their corresponding invariant subspaces

| Cases | $A_1(u)$, $B_i(u)$ & $C(u)$, $i = 1, 2$ | Invariant subspaces |
|-------|---------------------------------|---------------------|
| 7.    | $A_1(u) = c_2u^2 + c_1u + c_0$   | $B_1(u) = 3a_0c_2u^2 + d_1u + d_0$ |
|       | $A_2(u) = \beta_0$               | $B_2(u) = \lambda_0$   |
|       | $C(u) = (-2a_0^2c_1 + a_0d_1)u^2 + k_1u$ | $\mathcal{V}_3 = \text{Span}\{e^{-a_0x_1}, e^{-a_0x_1} \sin(\sqrt{b_1}x_2), e^{-a_0x_1} \cos(\sqrt{b_1}x_2)\}$ |
| 8.    | $A_1(u) = c_2u^2 + c_1u + c_0$   | $B_1(u) = d_1u + d_0$   |
|       | $A_2(u) = \beta_1u + \beta_0$   | $B_2(u) = \lambda_1u + \lambda_0$ |
|       | $C(u) = k_1u + k_0$               | $\mathcal{V}_3 = \text{Span}\{1, x_1, x_2\}$ |
| 9.    | $A_1(u) = c_1u + c_0$            | $B_1(u) = d_1u + d_0$   |
|       | $A_2(u) = \beta_2u^2 + \beta_1u + \beta_0$ | $B_2(u) = \lambda_1u + \lambda_0$ |
|       | $C(u) = k_1u + k_0$               | $\mathcal{V}_4 = \text{Span}\{1, x_1, x_2, \sqrt{2}\}$ |
| 10.   | $A_1(u) = c_2u^2 + c_1u + c_0$   | $B_1(u) = d_1u + d_0$   |
|       | $A_2(u) = \beta_1u + \beta_0$   | $B_2(u) = \lambda_0$   |
|       | $C(u) = k_1u + k_0$               | $\mathcal{V}_4 = \text{Span}\{1, x_1, x_2, \sqrt{2}\}$ |
| 11.   | $A_1(u) = c_0$                    | $B_1(u) = d_0$         |
|       | $A_2(u) = \beta_2u^2 + \beta_1u + \beta_0$ | $B_2(u) = \lambda_1u + \lambda_0$ |
|       | $C(u) = k_1u + k_0$               | $\mathcal{V}_4 = \text{Span}\{1, e^{-\frac{1}{2}(a_2 - \sqrt{a_2^2 - 4a_1})x_1}, e^{-\frac{1}{2}(a_2 + \sqrt{a_2^2 - 4a_1})x_1}, x_2\}$ |
| 12.   | $A_1(u) = c_2u^2 + c_1u + c_0$   | $B_1(u) = d_1u + d_0$   |
|       | $A_2(u) = \beta_0$               | $B_2(u) = \lambda_0$   |
|       | $C(u) = k_1u + k_0$               | $\mathcal{V}_4 = \text{Span}\{1, e^{-\frac{1}{2}(b_2 - \sqrt{b_2^2 - 4b_1})x_2}, e^{-\frac{1}{2}(b_2 + \sqrt{b_2^2 - 4b_1})x_2}, x_1\}$ |
| 13.   | $A_1(u) = c_2u^2 + c_1u + c_0$   | $B_1(u) = d_1u + d_0$   |
|       | $A_2(u) = \beta_0$               | $B_2(u) = \lambda_0$   |
|       | $C(u) = k_1u + k_0$               | $\mathcal{V}_4 = \text{Span}\{1, x_1, \sin(\sqrt{b_1}x_2), \cos(\sqrt{b_1}x_2)\}$ |
| 14.   | $A_1(u) = c_0$                    | $B_1(u) = d_0$         |
|       | $A_2(u) = \beta_2u^2 + \beta_1u + \beta_0$ | $B_2(u) = \lambda_1u + \lambda_0$ |
|       | $C(u) = k_1u + k_0$               | $\mathcal{V}_4 = \text{Span}\{1, x_2, \sin(\sqrt{a_1}x_1), \cos(\sqrt{a_1}x_1)\}$ |
Table 8: Differential operator (3.1) with other non-linearities and their corresponding invariant subspaces

| Cases | $\hat{K}[u] = \sum_{i=1}^{2} (A_i(u) u x_i)_x + \sum_{i=1}^{2} B_i(u) u x_i + C(u)$, $A_i(u), B_i(u)$ & $C(u), i = 1, 2$ | Invariant subspaces |
|-------|---------------------------------------------------------------------------------------------------------------------------------|-------------------|
| 1.    | $A_1(u) = \frac{d^3}{dx_0} u^3 + c_2 u^2 + c_0$  
$A_2(u) = \beta_0$  
$B_1(u) = d_3 u^3 + d_2 u^2 + d_1 u + d_0$  
$B_2(u) = \lambda_0$  
$C(u) = (-3a_0^2 c_2 + a_0 d_2) u^3 + (a_0 d_1) u^2 + k_1 u$ | $V_2 = \text{Span}\{e^{-a_0 x_1} - \frac{1}{2}(b_1 - \sqrt{b_1^2 - 4b_0}) x_2, e^{-a_0 x_1} - \frac{1}{2}(b_1 + \sqrt{b_1^2 - 4b_0}) x_2\}$ |
| 2.    | $A_1(u) = \frac{d^3}{dx_0} u^3 + c_2 u^2 + \beta_1 u + \beta_0$  
$A_2(u) = \frac{1}{3} \lambda_0 u^3 + \beta_1 u + \beta_0$  
$B_1(u) = d_0$  
$B_2(u) = \lambda_3 u^3 + \lambda_2 u^2 + \lambda_1 u + \lambda_0$  
$C(u) = b_0 \lambda_2 u^3 + (-2b_0^2 \beta_1 + b_0 \lambda_1) u^2 + k_1 u$ | $V_2 = \text{Span}\{e^{-b_0 x_2} - \frac{1}{2}(a_1 - \sqrt{a_1^2 - 4a_0}) x_1, e^{-b_0 x_2} - \frac{1}{2}(a_1 + \sqrt{a_1^2 - 4a_0}) x_1\}$ |
| 3.    | $A_1(u) = \frac{d^3}{dx_0} u^3 + c_2 u^2 + c_1 u + c_0$  
$A_2(u) = \beta_0$  
$B_1(u) = d_0$  
$B_2(u) = \lambda_0$  
$C(u) = (-3a_0^2 c_2 + a_0 d_2) u^3 + (a_0 d_1) u^2 + k_1 u$ | $V_3 = \text{Span}\{e^{-a_0 x_1}, e^{-a_0 x_1} - \frac{1}{2}(b_1 - \sqrt{b_1^2 - 4b_0}) x_2, e^{-a_0 x_1} - \frac{1}{2}(b_1 + \sqrt{b_1^2 - 4b_0}) x_2\}$ |
| 4.    | $A_1(u) = \frac{d^3}{dx_0} u^3 + c_2 u^2 + c_0$  
$A_2(u) = \beta_0$  
$B_1(u) = 4c_0 x_3 u^3 + d_2 u^2 + d_1 u + d_0$  
$B_2(u) = \lambda_0$  
$C(u) = (-3a_0^2 c_2 + a_0 d_2) u^3 + (a_0 d_1) u^2 + k_1 u$ | $V_2 = \text{Span}\{e^{-a_0 x_1} \sin(\sqrt{b_0} x_2), e^{-a_0 x_1} \cos(\sqrt{b_0} x_2)\}$ |
Table 9  Differential operator (3.1) with other non-linearities and their corresponding invariant subspaces

| Cases | $\hat{K}[u] = \sum_{i=1}^{2} (A_i(u)u_{x_i})_{x_j} + \sum_{i=1}^{2} B_i(u)u_{x_i} + C(u), A_i(u), B_i(u) & C(u), i = 1, 2$ | Invariant subspaces |
|-------|-----------------------------------------------------------------|------------------|
| 5.    | $A_1(u) = c_0$  
$A_2(u) = \beta_3 u^3 + \beta_1 u + \beta_0$  
$B_1(u) = d_0$  
$B_2(u) = 4b_0 \beta_3 u^3 + \lambda_2 u^2 + \lambda_1 u + \lambda_0$  
$C(u) = \lambda_3 b_0 u^3 + (-2b_0 \beta_1 + b_0 \lambda_1) u^2 + k_1 u$ | $\mathcal{V}_2 = \text{Span}\{e^{-b_0 x_2} \sin(\sqrt{a_0} x_1), e^{-b_0 x_2} \cos(\sqrt{a_0} x_1)\}$ |
| 6.    | $A_1(u) = c_0$  
$A_2(u) = \beta_3 u^3 + \beta_2 u^2 + \beta_1 u + \beta_0$  
$B_1(u) = d_0$  
$B_2(u) = 4b_0 \beta_3 u^3 + \lambda_2 u^2 + \lambda_1 u + \lambda_0$  
$C(u) = (-3b_0^2 \beta_2 + b_0 \lambda_2) u^3$  
$+ (-2b_0 \beta_1 + b_0 \lambda_1) u^2 + k_1 u$ | $\mathcal{V}_3 = \text{Span}\{e^{-b_0 x_2}, e^{-b_0 x_2} \sin(\sqrt{a_1} x_1), e^{-b_0 x_2} \cos(\sqrt{a_1} x_1)\}$ |
| 7.    | $A_1(u) = c_3 u^3 + c_2 u^2 + c_1 u + c_0$  
$A_2(u) = \beta_0$  
$B_1(u) = 4a_0 c_3 u^3 + d_2 u^2 + d_1 u + d_0$  
$B_2(u) = \lambda_0$  
$C(u) = (-3a_0^2 c_2 + a_0 d_2) u^3$  
$+ (-2a_0^2 c_1 + a_0 d_1) u^2 + k_1 u$ | $\mathcal{V}_3 = \text{Span}\{e^{-a_0 x_1}, e^{-a_0 x_1} \sin(\sqrt{b_1} x_2), e^{-a_0 x_1} \cos(\sqrt{b_1} x_2)\}$ |
| Cases | Differential operators for equations (3.13)-(3.15) | Type of invariant subspaces $n_k = 1, 2, 3, k = 1, 2$ | Reference tables |
|-------|--------------------------------------------------|--------------------------------------------------|------------------|
| 1.    | $\hat{K}[u] = \sum_{i=1}^{2} (A_i(u)u x_i) x_i + \sum_{i=1}^{2} B_i (u) u x_i$ | Type I : $V_{n_1 n_2}$  
Type II : $V_{n_1 + n_2 - 1}$ | Tables 11, 12, 13, 14, 15 |
| 2.    | $\hat{K}[u] = \sum_{i=1}^{2} (A_i(u)u x_i) x_i + C(u)$ | Type I : $V_{n_1 n_2}$  
Type II : $V_{n_1 + n_2 - 1}$ | Tables 16, 17, 18 |
| 3.    | $\hat{K}[u] = \sum_{i=1}^{2} (A_i(u)u x_i) x_i$ | Type I : $V_{n_1 n_2}$  
Type II : $V_{n_1 + n_2 - 1}$ | Tables 19, 20, 21 |
Table 11 Differential operator of (3.13) with quadratic non-linearities and their corresponding invariant subspaces

| Cases | $\hat{K}[u] = \sum_{i=1}^{2} (A_i(u)u_{x_i})x_i + \sum_{i=1}^{2} B_i(u)u_{x_i}$, $A_i(u)$ & $B_i(u)$, $i = 1, 2$ | Invariant subspaces |
|-------|-----------------------------------------------------------------|---------------------|
| 1.    | $\begin{cases} A_1(u) = \frac{-b_1 \lambda_1}{2a_1} u + c_0 \\ A_2(u) = \frac{\lambda_1}{2b_1} u + \beta_0 \end{cases}$ | $\mathcal{V}_3 = \text{Span}\{1, e^{-a_1 x_1}, e^{-b_1 x_2}\}$ |
| 2.    | $\begin{cases} A_1(u) = c_0 \\ A_2(u) = \beta_1 u + \beta_0 \end{cases}$ | $\mathcal{V}_3 = \text{Span}\{1, x_2, e^{-a_1 x_1}\}$ |
| 3.    | $\begin{cases} A_1(u) = c_0 \\ A_2(u) = \beta_1 u + \beta_0 \end{cases}$ | $\mathcal{V}_4 = \text{Span}\{1, e^{-a_1 x_1}, x_2, \frac{x_2^2}{2}\}$ |
|       | $\begin{cases} B_1(u) = \frac{-b_1 \lambda_1}{a_1} u + d_0 \\ B_2(u) = \lambda_1 u + \lambda_0 \end{cases}$ | $\mathcal{V}_5 = \text{Span}\{1, e^{-\frac{1}{2} \sqrt{a_2 - 4a_1} x_1}, e^{-\frac{1}{2} \sqrt{a_2 + 4a_1} x_1}, x_2, \frac{x_2^2}{2}\}$ |
Table 12  Differential operator of (3.13) with quadratic non-linearities and their corresponding invariant subspaces

| Cases | \( \hat{K}[u] = \sum_{i=1}^{2} (A_i(u)u_{x_i})_{x_i} + \sum_{i=1}^{2} B_i(u)u_{x_i} \), \( A_i(u) \) & \( B_i(u) \), \( i = 1, 2 \) | Invariant subspaces |
|-------|-----------------------------------------------------------------|-----------------|
| 4.    | \[
\begin{align*}
A_1(u) &= c_1 u + c_0 \\
A_2(u) &= \beta_0 \\
B_1(u) &= d_0 \\
B_2(u) &= \lambda_0
\end{align*}
\] | \( \mathcal{V}_4 = \text{Span}\{1, e^{-b_1 x_2}, x_1, \frac{x_1^2}{2}\} \) |
|       | \( \mathcal{V}_5 = \text{Span}\{1, x_1, \frac{x_1^2}{2}, \sin(\sqrt{b_1} x_2), \cos(\sqrt{b_1} x_2)\} \) | \( \mathcal{V}_5 = \text{Span}\{1, e^{-\frac{1}{2}(b_2-\sqrt{b_2^2-4b_1})x_2}, e^{-\frac{1}{2}(b_2+\sqrt{b_2^2-4b_1})x_2}, x_1, \frac{x_1^2}{2}\} \) |
| 5.    | \[
\begin{align*}
A_1(u) &= c_1 u + c_0 \\
A_2(u) &= \beta_1 u + \beta_0 \\
B_1(u) &= d_0 \\
B_2(u) &= \lambda_1 u + \lambda_0
\end{align*}
\] | \( \mathcal{V}_4 = \text{Span}\{1, x_1, \frac{x_1^2}{2}, x_2\} \) |
| 6.    | \[
\begin{align*}
A_1(u) &= c_1 u + c_0 \\
A_2(u) &= \beta_1 u + \beta_0 \\
B_1(u) &= d_0 \\
B_2(u) &= \lambda_0
\end{align*}
\] | \( \mathcal{V}_5 = \text{Span}\{1, x_1, \frac{x_1^2}{2}, x_2, \frac{x_2^2}{2}\} \) |
| 7.    | \[
\begin{align*}
A_1(u) &= c_0 \\
A_2(u) &= \beta_1 u + \beta_0 \\
B_1(u) &= d_0 \\
B_2(u) &= \lambda_0
\end{align*}
\] | \( \mathcal{V}_5 = \text{Span}\{1, x_2, \frac{x_2^2}{2}, \sin(\sqrt{a_1} x_1), \cos(\sqrt{a_1} x_1)\} \) |
### Table 13: Differential operator of (3.13) with cubic non-linearities and their corresponding invariant subspaces

| Cases | \( \mathcal{K}[u] = \sum_{i=1}^{2} (A_i(u)u_{x_i})_{x_i} + \sum_{i=1}^{2} B_i(u)u_{x_i}, A_i(u) & B_i(u), i = 1, 2 \) | Invariant subspaces |
|-------|-------------------------------------------------------------------------------------------------------------------------------------|------------------|
| 1.    | \[
A_1(u) = \frac{1}{3} \frac{-3b_0^2 \beta_2 + a_0 d_2 + b_0 \lambda_2}{a_0} u^2 \\
+ \frac{1}{2} \frac{-2b_0^2 \beta_1 + a_0 d_1 + b_0 \lambda_1}{a_0^2} u + c_0 \\
A_2(u) = \beta_2 u^2 + \beta_1 u + \beta_0 \\
B_1(u) = d_2 u^2 + d_1 u + d_0 \\
B_2(u) = \lambda_2 u^2 + \lambda_1 u + \lambda_0
\] | \( \mathcal{V}_1 = \text{Span}\{e^{-a_0 x_1 - b_0 x_2}\} \) |
| 2.    | \[
A_1(u) = \frac{1}{3} \frac{d_2}{a_0} u^2 + \frac{1}{2} \frac{d_1}{a_0} u + c_0 \\
A_2(u) = \beta_0 \\
B_1(u) = d_2 u^2 + d_1 u + d_0 \\
B_2(u) = \lambda_0
\] | \( \mathcal{V}_2 = \text{Span}\{e^{-a_0 x_1}, e^{-a_0 x_1 - b_0 x_2}\} \) |
| 3.    | \[
A_1(u) = \frac{c_0}{3b_0} u^2 + \frac{\lambda_1}{2b_0} u + \beta_0 \\
A_2(u) = \frac{\lambda_2}{3b_0} u^2 + \frac{\lambda_1}{2b_0} u + \beta_0 \\
B_1(u) = d_0 \\
B_2(u) = \lambda_2 u^2 + \lambda_1 u + \lambda_0
\] | \( \mathcal{V}_2 = \text{Span}\{e^{-\frac{1}{2}(a_1 - \sqrt{a_1^2 - 4a_0}) x_1 - b_0 x_2}, e^{-\frac{1}{2}(a_1 + \sqrt{a_1^2 - 4a_0}) x_1 - b_0 x_2}\} \) |
### Table 14  Differential operator of (3.13) with cubic non-linearities and their corresponding invariant subspaces

| Cases | $\hat{\lambda}[u] = \sum_{i=1}^{2} (A_i(u)u_{x_i})x_i + \sum_{i=1}^{2} B_i(u)u_{x_i}$, $A_i(u)$ & $B_i(u)$, $i = 1, 2$ | Invariant subspaces |
|-------|-----------------------------------------------------------------|---------------------|
| 4.    | $A_1(u) = c_2 u^2 + c_0$  \[ A_2(u) = \beta_0 \]  \[ B_1(u) = 3a_0 c_2 u^2 + d_0 \]  \[ B_2(u) = \lambda_0 \] | $V_2 = \text{Span}\{e^{-a_0 x_1} \sin(\sqrt{b_0} x_2), e^{-a_0 x_1} \cos(\sqrt{b_0} x_2)\}$ |
| 5.    | $A_1(u) = c_0$  \[ A_2(u) = \beta_2 u^2 + \beta_1 u + \beta_0 \]  \[ B_1(u) = d_0 \]  \[ B_2(u) = 3b_0 \beta_2 \]  \[ 2b_0 \beta_1 u + \lambda_0 \] | $V_2 = \text{Span}\{e^{-b_0 x_2} \sin(\sqrt{a_0} x_1), e^{-b_0 x_2} \cos(\sqrt{a_0} x_1)\}$ |
| 6.    | $A_1(u) = c_2 u^2 + c_1 u + c_0$  \[ A_2(u) = \beta_0 \]  \[ B_1(u) = d_1 u + d_0 \]  \[ B_2(u) = \lambda_0 \] | $V_3 = \text{Span}\{1, e^{-b_1 x_2}, x_1\}$ |
| 7.    | $A_1(u) = c_0$  \[ A_2(u) = \beta_2 u^2 + \beta_1 u + \beta_0 \]  \[ B_1(u) = d_0 \]  \[ B_2(u) = \lambda_1 u + \lambda_0 \] | $V_3 = \text{Span}\{1, e^{-a_1 x_1}, x_2\}$ |
| 8.    | $A_1(u) = c_2 u^2 + c_1 u + c_0$  \[ A_2(u) = \beta_1 u + \beta_0 \]  \[ B_1(u) = d_1 u + d_0 \]  \[ B_2(u) = \lambda_1 u + \lambda_0 \] | $V_4 = \text{Span}\{1, e^{-\frac{1}{2} (a_2 - \sqrt{a_1^2 - 4a_1})x_1}, e^{-\frac{1}{2} (a_2 + \sqrt{a_1^2 - 4a_1})x_1}, x_2\}$ |
| 9.    | $A_1(u) = c_2 u^2 + c_1 u + c_0$  \[ A_2(u) = \beta_0 \]  \[ B_1(u) = d_1 u + d_0 \]  \[ B_2(u) = \lambda_0 \] | $V_4 = \text{Span}\{1, e^{-\frac{1}{2} (b_2 - \sqrt{b_1^2 - 4b_1})x_2}, e^{-\frac{1}{2} (b_2 + \sqrt{b_1^2 - 4b_1})x_2}, x_1\}$ |
| 10.   | $A_1(u) = c_2 u^2 + c_1 u + c_0$  \[ A_2(u) = \beta_1 u + \beta_0 \]  \[ B_1(u) = d_1 u + d_0 \]  \[ B_2(u) = \lambda_0 \] | $V_4 = \text{Span}\{1, x_1, x_2, \frac{x_2^2}{2}\}$ |
| 11.   | $A_1(u) = c_1 u + c_0$  \[ A_2(u) = \beta_2 u^2 + \beta_1 u + \beta_0 \]  \[ B_1(u) = d_0 \]  \[ B_2(u) = \lambda_1 u + \lambda_0 \] | $V_4 = \text{Span}\{1, x_1, \frac{x_1^2}{2}, x_2\}$ |
| 12.   | $A_1(u) = c_0$  \[ A_2(u) = \beta_2 u^2 + \beta_1 u + \beta_0 \]  \[ B_1(u) = d_0 \]  \[ B_2(u) = \lambda_1 u + \lambda_0 \] | $V_4 = \text{Span}\{x_1, \sin(\sqrt{a_1} x_1), \cos(\sqrt{a_1} x_1)\}$ |
| 13.   | $A_1(u) = c_2 u^2 + c_1 u + c_0$  \[ A_2(u) = \beta_0 \]  \[ B_1(u) = d_1 u + d_0 \]  \[ B_2(u) = \lambda_0 \] | $V_4 = \text{Span}\{1, x_1, \sin(\sqrt{b_1} x_2), \cos(\sqrt{b_1} x_2)\}$ |
Table 15  Differential operator of (3.13) with other non-linearities and their corresponding invariant subspaces

| Cases | \( \hat{K}[u] = \sum_{i=1}^{2} (A_{i}(u)u_{x_{i}})_{x_{i}} + \sum_{i=1}^{2} B_{i}(u)u_{x_{i}} \) & Invariant subspaces |
|-------|-------------------------------------------------|------------------------------------------------------|
| 1.    | \[
A_1(u) = \frac{d_1}{4a_0} u^3 + \frac{d_2}{3a_0} u^2 + c_0 \\
A_2(u) = \beta_0
\]
|       | \[
B_1(u) = d_2 u^3 + d_2 u^2 + d_0 \\
B_2(u) = \lambda_0
\] & \( V_2 = \text{Span}\{e^{-a_0 x_1 - \frac{1}{2}(b_1 - \sqrt{b_1^2 - 4b_2})x_2}, e^{-a_0 x_1 - \frac{1}{2}(b_1 + \sqrt{b_1^2 - 4b_2})x_2}\} \)
| 2.    | \[
A_1(u) = \lambda_3 \frac{u^3}{4b_0} + \frac{\lambda_1}{2b_0} u + \beta_0 \\
A_2(u) = \lambda_0
\]
|       | \[
B_1(u) = d_0 \\
B_2(u) = \lambda_3 u^3 + \lambda_1 u + \lambda_0
\] & \( V_2 = \text{Span}\{e^{-b_0 x_2 - \frac{1}{2}(a_1 - \sqrt{a_1^2 - 4a_0})x_1}, e^{-b_0 x_2 - \frac{1}{2}(a_1 + \sqrt{a_1^2 - 4a_0})x_1}\} \)
| 3.    | \[
A_1(u) = \frac{d_1}{4a_0} u^3 + \frac{d_2}{3a_0} u^2 + \frac{d_4}{2a_0} u + c_0 \\
A_2(u) = \beta_0
\]
|       | \[
B_1(u) = d_3 u^3 + d_2 u^2 + d_1 u + d_0 \\
B_2(u) = \lambda_0
\] & \( V_3 = \text{Span}\{e^{-a_0 x_1 - \frac{1}{2}(b_1 - \sqrt{b_1^2 - 4b_2})x_2}, e^{-a_0 x_1 - \frac{1}{2}(b_1 + \sqrt{b_1^2 - 4b_2})x_2}, e^{-a_0 x_1}\} \)
| 4.    | \[
A_1(u) = c_1 u^3 + c_2 u^2 + c_0 \\
A_2(u) = \beta_0
\]
|       | \[
B_1(u) = 4 a_0 c_3 u^3 + 3 a_0 c_2 u^2 + d_0 \\
B_2(u) = \lambda_0
\] & \( V_2 = \text{Span}\{e^{-a_0 x_1} \sin(\sqrt{b_0} x_2), e^{-a_0 x_1} \cos(\sqrt{b_0} x_2)\} \)
| 5.    | \[
A_1(u) = c_3 u^3 + c_2 u^2 + c_1 u + c_0 \\
A_2(u) = \beta_0
\]
|       | \[
B_1(u) = 4 a_0 c_3 u^3 + 3 a_0 c_2 u^2 + 2 a_0 c_1 u + d_0 \\
B_2(u) = \lambda_0
\] & \( V_3 = \text{Span}\{e^{-a_0 x_1} \sin(\sqrt{b_1} x_2), e^{-a_0 x_1} \cos(\sqrt{b_1} x_2)\} \)
| 6.    | \[
A_1(u) = c_0 \\
A_2(u) = \beta_3 u^3 + \beta_1 u + \beta_0
\]
|       | \[
B_1(u) = d_0 \\
B_2(u) = 4 b_0 \beta_3 u^3 + 2 b_0 \beta_1 u + \lambda_0
\] & \( V_2 = \text{Span}\{e^{-b_0 x_2} \sin(\sqrt{a_1} x_1), e^{-b_0 x_2} \cos(\sqrt{a_1} x_1)\} \)
| 7.    | \[
A_1(u) = c_0 \\
A_2(u) = \beta_3 u^3 + \beta_2 u^2 + \beta_1 u + \beta_0
\]
|       | \[
B_1(u) = d_0 \\
B_2(u) = 4 b_0 \beta_3 u^3 + 3 b_0 \beta_2 u^2 + 2 b_0 \beta_1 u + \lambda_0
\] & \( V_3 = \text{Span}\{e^{-b_0 x_2} \sin(\sqrt{a_1} x_1), e^{-b_0 x_2} \cos(\sqrt{a_1} x_1)\} \)
## Table 16
Differential operator of (3.14) with quadratic non-linearities and their corresponding invariant subspaces

| Cases | 
|-------|
| 
| Invariant subspaces |
| 
| \( \hat{\mathcal{K}}[u] = \sum_{i=1}^{2} (A_i(u)u_x) + C(u) \) & \( A_1(u) \) & \( A_2(u) \) & \( C(u) \), \( i = 1, 2 \) |
| 
| \( V_1 = \text{Span} \{ e^{-(a_0 x_1 - b_0 x_2)} \} \) |
| 
| \( V_1 = \text{Span} \{ e^{-(a_0 x_1 - b_0 x_2)} \} \) |

\[ A_1(u) = -\frac{1}{2} \beta_1 u + b_0 \]
\[ A_2(u) = \beta_1 u + b_0 \]
\[ C(u) = k_2 u_x^2 + k_1 u \]
### Table 17

Differential operator of (3.14) with quadratic non-linearities and their corresponding invariant subspaces

| Cases | $\hat{K}[u] = \sum_{i=1}^{2} (A_i(u)u_{x_i})x_j + C(u)$, $A_i(u)$ & $C(u)$, $i = 1, 2$ | Invariant subspaces |
|-------|--------------------------------------------------------------------------------|-------------------|
| 2.    | $\begin{align*} A_1(u) &= c_1u + c_0 \\
                        A_2(u) &= \beta_0 \\
                        C(u) &= (-2a_1^2c_1)u^2 + k_1u \end{align*}$ | $V_2 = \text{Span}\{e^{-a_0x_1}, e^{-a_0x_1-b_1x_2}\}$ |
| 3.    | $\begin{align*} A_1(u) &= c_0 \\
                        A_2(u) &= \beta_1u + \beta_0 \\
                        C(u) &= (-2b_0^2\beta_1)u^2 + k_1u \end{align*}$ | $V_2 = \text{Span}\{e^{-b_0x_2}, e^{-a_1x_1-b_0x_2}\}$ |
| 4.    | $\begin{align*} A_1(u) &= c_1u + c_0 \\
                        A_2(u) &= \beta_1u + \beta_0 \\
                        C(u) &= k_1u + k_0 \end{align*}$ | $\begin{align*} V_4 &= \text{Span}\{1, e^{-a_1x_1}, x_2, x_2^2\} \\
                        V_4 &= \text{Span}\{1, e^{-a_2x_2}, x_1, x_1^2\} \\
                        V_5 &= \text{Span}\{1, e^{-\frac{1}{2}(a_2-\sqrt{a_2^2-4a_1})x_1}, e^{-\frac{1}{2}(a_2+\sqrt{a_2^2-4a_1})x_1}, x_1^2, \frac{x_2^2}{2}\} \end{align*}$ |
| 5.    | $\begin{align*} A_1(u) &= c_1u + c_0 \\
                        A_2(u) &= \beta_0 \\
                        C(u) &= k_1u + k_0 \end{align*}$ | $\begin{align*} V_4 &= \text{Span}\{1, e^{-b_1x_1}, x_1^2, x_1\} \\
                        V_5 &= \text{Span}\{1, e^{-\frac{1}{2}(b_2-\sqrt{b_2^2-4b_1})x_1}, x_2, x_2^2\} \end{align*}$ |
| 6.    | $\begin{align*} A_1(u) &= c_1u + c_0 \\
                        A_2(u) &= \beta_1u + \beta_0 \\
                        C(u) &= k_1u + k_0 \end{align*}$ | $\begin{align*} V_4 &= \text{Span}\{1, x_1, \frac{x_1^2}{2}, x_2\} \\
                        V_5 &= \text{Span}\{1, x_1, x_2, \frac{x_2^2}{2}, \frac{x_2^2}{2}\} \end{align*}$ |
| 7.    | $\begin{align*} A_1(u) &= c_1u + c_0 \\
                        A_2(u) &= \beta_0 \\
                        C(u) &= k_1u + k_0 \end{align*}$ | $\begin{align*} V_5 &= \text{Span}\{1, x_1, x_2, \frac{x_2^2}{2}, \frac{x_2^2}{2}\} \\
                        V_5 &= \text{Span}\{1, x_1, x_2, \frac{x_2^2}{2}, \sin(\sqrt{a_1}x_2), \cos(\sqrt{a_1}x_2)\} \end{align*}$ |
| 8.    | $\begin{align*} A_1(u) &= c_0 \\
                        A_2(u) &= \beta_1u + \beta_0 \\
                        C(u) &= k_1u + k_0 \end{align*}$ | $\begin{align*} V_4 &= \text{Span}\{1, e^{-a_1x_1}, x_2, \frac{x_2^2}{2}\} \\
                        V_5 &= \text{Span}\{1, x_2, \frac{x_2^2}{2}, \sin(\sqrt{a_1}x_1)\}, \cos(\sqrt{a_1}x_1)\} \\
                        V_5 &= \text{Span}\{1, x_2, \frac{x_2^2}{2}, e^{-\frac{1}{2}(a_2-\sqrt{a_2^2-4a_1})x_1}, e^{-\frac{1}{2}(a_2+\sqrt{a_2^2-4a_1})x_1}\} \end{align*}$ |
### Table 18: Differential operator of (3.14) with cubic non-linearities and their corresponding invariant subspaces

| Cases | \( \hat{\mathcal{K}}[u] = \sum_{i=1}^{2} (A_i(u)u_{x_i})_{x_i} + C(u), A_1(u) & C(u), i = 1, 2 \) | Invariant subspaces |
|-------|----------------------------------------------------------------------------------------------------------------------------------|---------------------|
| 1.    | \( \begin{align*} A_1(u) &= c_2u^2 + c_1u + c_0 \\ A_2(u) &= \beta_0 \\ C(u) &= k_1u + k_0 \end{align*} \) | \( \mathcal{V}_3 = \text{Span}\{1, x_1, e^{-\beta_1x_2}\} \) |
| 2.    | \( \begin{align*} A_1(u) &= c_0 \\ A_2(u) &= \beta_2u^2 + \beta_1u + \beta_0 \\ C(u) &= k_1u + k_0 \end{align*} \) | \( \mathcal{V}_3 = \text{Span}\{1, x_2, e^{-\beta_1x_1}\} \) |
| 3.    | \( \begin{align*} A_1(u) &= c_2u^2 + c_1u + c_0 \\ A_2(u) &= \beta_0 \\ C(u) &= k_1u + k_0 \end{align*} \) | \( \mathcal{V}_4 = \text{Span}\{1, x_1, e^{-\frac{1}{2}(b_2 - \sqrt{b_2^2 - 4b_1})x_1}, e^{-\frac{1}{2}(b_2 + \sqrt{b_2^2 - 4b_1})x_1}\} \) |
| 4.    | \( \begin{align*} A_1(u) &= c_0 \\ A_2(u) &= \beta_2u^2 + \beta_1u + \beta_0 \\ C(u) &= k_1u + k_0 \end{align*} \) | \( \mathcal{V}_4 = \text{Span}\{1, x_2, e^{-\frac{1}{2}(a_2 - \sqrt{a_2^2 - 4a_1})x_1}, e^{-\frac{1}{2}(a_2 + \sqrt{a_2^2 - 4a_1})x_1}\} \) |
| 5.    | \( \begin{align*} A_1(u) &= c_2u^2 + c_1u + c_0 \\ A_2(u) &= \beta_0 \\ C(u) &= k_1u + k_0 \end{align*} \) | \( \mathcal{V}_3 = \text{Span}\{1, x_1, x_2\} \) |
| 6.    | \( \begin{align*} A_1(u) &= c_0 \\ A_2(u) &= \beta_1u + \beta_0 \\ C(u) &= k_1u + k_0 \end{align*} \) | \( \mathcal{V}_4 = \text{Span}\{1, x_1, x_2, \frac{x^2}{2}\} \) |
| 7.    | \( \begin{align*} A_1(u) &= c_1u + c_0 \\ A_2(u) &= \beta_2u^2 + \beta_1u + \beta_0 \\ C(u) &= k_1u + k_0 \end{align*} \) | \( \mathcal{V}_4 = \text{Span}\{1, x_1, x_2, \frac{x^2}{2}\} \) |
| 8.    | \( \begin{align*} A_1(u) &= c_2u^2 + c_1u + c_0 \\ A_2(u) &= \beta_0 \\ C(u) &= k_1u + k_0 \end{align*} \) | \( \mathcal{V}_4 = \text{Span}\{1, x_1, \sin(\sqrt{b_1}x_2), \cos(\sqrt{b_1}x_2)\} \) |
| 9.    | \( \begin{align*} A_1(u) &= c_0 \\ A_2(u) &= \beta_2u^2 + \beta_1u + \beta_0 \\ C(u) &= k_1u + k_0 \end{align*} \) | \( \mathcal{V}_4 = \text{Span}\{1, x_2, \sin(\sqrt{a_1}x_1), \cos(\sqrt{a_1}x_1)\} \) |
Table 19  Differential operator of (3.15) with quadratic non-linearities and their corresponding invariant subspaces

| Cases | $\mathcal{K}[u] = \sum_{i=1}^{2} (A_i(u)u_{x_i})x_i$, $A_i(u)$, $i = 1, 2$ | Invariant subspaces |
|-------|-------------------------------------------------------------------------------------------------|---------------------|
| 1.    | $\begin{cases} A_1(u) = c_0 \\ A_2(u) = \beta_1 u + \beta_0 \end{cases}$ | $\begin{cases} V_4 = \text{Span}\{1, e^{-a_1 x_1}, x_2, \frac{x_2^2}{2}\} \\ V_4 = \text{Span}\{1, e^{-a_2 x_1}, x_1, x_2\} \\ V_5 = \text{Span}\{1, e^{-\frac{1}{2}(a_2-\sqrt{a_2^2-4a_1})x_1}, e^{-\frac{1}{2}(a_2+\sqrt{a_2^2-4a_1})x_1}, x_2, \frac{x_2^2}{2}\} \end{cases}$ |
| 2.    | $\begin{cases} A_1(u) = c_1 u + c_0 \\ A_2(u) = \beta_0 \end{cases}$ | $\begin{cases} V_4 = \text{Span}\{1, e^{-b_1 x_2}, x_1, \frac{x_1^2}{2}\} \\ V_5 = \text{Span}\{1, e^{-\frac{1}{2}(b_2-\sqrt{b_2^2-4b_1})x_2}, e^{-\frac{1}{2}(b_2+\sqrt{b_2^2-4b_1})x_2}, x_1, \frac{x_1^2}{2}\} \end{cases}$ |
| 3.    | $\begin{cases} A_1(u) = c_1 u + c_0 \\ A_2(u) = \beta_1 u + \beta_0 \end{cases}$ | $\begin{cases} V_4 = \text{Span}\{1, x_1, x_2, \frac{x_2^2}{2}\} \\ V_5 = \text{Span}\{1, x_1, \frac{x_1^2}{2}, x_2, \frac{x_2^2}{2}\} \end{cases}$ |
| 4.    | $\begin{cases} A_1(u) = c_0 \\ A_2(u) = \beta_1 u + \beta_0 \end{cases}$ | $V_5 = \text{Span}\{1, x_2, \frac{x_2^2}{2}, \sin(\sqrt{a_1} x_1), \cos(\sqrt{a_1} x_1)\}$ |
| 5.    | $\begin{cases} A_1(u) = c_1 u + c_0 \\ A_2(u) = \beta_0 \end{cases}$ | $V_5 = \text{Span}\{1, x_1, \frac{x_1^2}{2}, \sin(\sqrt{b_1} x_2), \cos(\sqrt{b_1} x_2)\}$ |
Table 20  Differential operator of (3.15) with cubic non-linearities and their corresponding invariant subspaces

| Cases | Differential operator | Invariant subspaces |
|-------|-----------------------|---------------------|
| 1.    | $A_1(u) = \frac{-b_0^2b_2}{a_0^2} u^2 + c_0$  \[ A_2(u) = \beta_2 u^2 + \beta_0 \] | $V_1 = \text{Span}\{e^{-a_0x_1-b_0x_2}\}$ |
| 2.    | $A_1(u) = c_2 u^2 + c_1 u + c_0$  \[ A_2(u) = \beta_0 \] | $V_1 = \text{Span}\{1, e^{-b_1 x_2}, x_1\}$  \[ V_2 = \text{Span}\{1, e^{-\frac{1}{2}(b_2-\sqrt{b_2^2-4b_1})x_2}, e^{-\frac{1}{2}(b_2+\sqrt{b_2^2-4b_1})x_2}, x_1\}$ |
| 3.    | $A_1(u) = c_0$  \[ A_2(u) = \beta_2 u^2 + \beta_1 u + \beta_0 \] | $V_3 = \text{Span}\{1, e^{-a_1 x_1}, x_2\}$  \[ V_4 = \text{Span}\{1, e^{-\frac{1}{2}(a_2-\sqrt{a_2^2-4a_1})x_1}, e^{-\frac{1}{2}(a_2+\sqrt{a_2^2-4a_1})x_1}, x_2\}$ |
which admits a two-dimensional exponential subspace \( \mathcal{V}_2 = \text{Span}\{e^{-a_0 x_1}, e^{-(a_0 x_1 + b_1 x_2)}\} \), since for some constants \( \delta_1, \delta_2 \in \mathbb{R} \)

\[
\hat{K}[\delta_1 e^{-a_0 x_1} + \delta_2 e^{-(a_0 x_1 + b_1 x_2)}] = (a_0^2 c_0 - d_0 a_0 + k_1) \delta_1 e^{-a_0 x_1} + (k_1 + \beta_0 b_1^2 - \lambda_0 b_1 + c_0 a_0^2 - a_0 d_0) \delta_2 e^{-(a_0 x_1 + b_1 x_2)} \in \mathcal{V}_2.
\]

Hence, Eq. (4.1) possesses an exact solution in the form

\[
u(x_1, x_2, t) = \Phi_1(t)e^{-a_0 x_1} + \Phi_2(t)e^{-(a_0 x_1 + b_1 x_2)},
\]

where the unknown functions \( \Phi_1(t) \) and \( \Phi_2(t) \) are to be determined.

Substituting (4.2) into (4.1), we get the system of fractional ODEs

\[
\frac{d^{\alpha} \Phi_1(t)}{dt^{\alpha}} = (a_0^2 c_0 - d_0 a_0 + k_1) \Phi_1(t),
\]

\[
\frac{d^{\alpha} \Phi_2(t)}{dt^{\alpha}} = (k_1 + \beta_0 b_1^2 - \lambda_0 b_1 + c_0 a_0^2 - a_0 d_0) \Phi_2(t), \quad \alpha \in (0, 2].
\]

The above equations can be written in the following form:

\[
\frac{d^{\alpha} \Phi_i(t)}{dt^{\alpha}} = \gamma_i \Phi_i(t), \quad i = 1, 2, \quad \alpha \in (0, 2],
\]

where \( \gamma_1 = (a_0^2 c_0 - d_0 a_0 + k_1) \) and \( \gamma_2 = (k_1 + \beta_0 b_1^2 - \lambda_0 b_1 + c_0 a_0^2 - a_0 d_0) \). First, let us consider \( \alpha \in (0, 1] \). It should be noted that the Laplace transformation of the Caputo fractional derivative of order \( \alpha > 0 \) is given by Diethelm (2010), and Podlubny (1999)

\[
L \left[ \frac{d^{\alpha} w(\xi)}{d \xi^{\alpha}} \right] = s^{\alpha} L[w(\xi)] - \sum_{m=0}^{n-1} s^{\alpha-(m+1)} w^{(m)}(0), \quad n - 1 < \alpha \leq n, \quad n \in \mathbb{N}, \quad Re(s) > 0,
\]

where \( w^{(m)}(0) = \frac{d^m w(\xi)}{d \xi^m} |_{\xi=0}, \xi \in [0, \infty) \) and \( L[w(\xi)] = \tilde{w}(s) = \int_0^\infty e^{-st} w(t) dt \). Thus applying the Laplace transformation to Eq. (4.3), we obtain

\[
s^{\alpha} \tilde{\Phi}_i(s) - s^{\alpha-1} \Phi_i(0) = \gamma_i \tilde{\Phi}_i(s).
\]

Applying the inverse Laplace transformation for the above equation, we get

\[
\Phi_i(t) = v_i E_{\alpha,1}(\gamma_i t^{\alpha}), \quad \alpha \in (0, 1], \quad i = 1, 2,
\]

where \( v_i = \Phi_i(0) \) and \( E_{\alpha,\beta}(t^\alpha) \) is the two-parameter Mittag-Leffler function (Mathai and Haubold 2008), defined as

\[
E_{\alpha,\beta}(t^\alpha) = \sum_{m=0}^{\infty} \frac{(t^\alpha)^m}{\Gamma(m \alpha + \beta)}.
\]

Hence, for the case \( \alpha \in (0, 1] \), the obtained exact solution for the cubic non-linear Eq. (4.1) associated with two-dimensional exponential subspace \( \mathcal{V}_2 = \text{Span}\{e^{-a_0 x_1}, e^{-(a_0 x_1 + b_1 x_2)}\} \) is as follows

\[
u(x_1, x_2, t) = v_1 e^{-a_0 x_1} E_{\alpha,1}(\gamma_1 t^{\alpha}) + v_2 e^{-(a_0 x_1 + b_1 x_2)} E_{\alpha,1}(\gamma_2 t^{\alpha}), \quad \alpha \in (0, 1],
\]

where \( \gamma_1 = (a_0^2 c_0 - d_0 a_0 + k_1) \), \( \gamma_2 = (k_1 + \beta_0 b_1^2 - \lambda_0 b_1 + c_0 a_0^2 - a_0 d_0) \) and \( a_0, b_1, v_i \in \mathbb{R}, i = 1, 2 \). Also, it should be noted that the obtained exact solution (4.4) satisfies the
Table 21  Differential operator of (3.15) with cubic non-linearities and their corresponding invariant subspaces

| Cases | $\hat{K}[u] = \sum_{i=1}^{2} (A_i(u)u_{x_i})_{x_i}, A_i(u), i = 1, 2$ | Invariant subspaces |
|-------|----------------------------------------------------------------|---------------------|
| 4.    | \[
\begin{align*}
    A_1(u) &= c_2u^2 + c_1u + c_0 \\
    A_2(u) &= \beta_1u + \beta_0,
\end{align*}
\] | $\mathcal{V}_3 = \text{Span}\{1, x_1, x_2\}$ |
| 5.    | \[
\begin{align*}
    A_1(u) &= c_2u^2 + c_1u + c_0 \\
    A_2(u) &= \beta_1u + \beta_0,
\end{align*}
\] | $\mathcal{V}_4 = \text{Span}\{1, x_1, x_2, x_2^2\}$ |
| 6.    | \[
\begin{align*}
    A_1(u) &= c_1u + c_0 \\
    A_2(u) &= \beta_2u^2 + \beta_1u + \beta_0,
\end{align*}
\] | $\mathcal{V}_4 = \text{Span}\{1, x_1, \frac{x_2}{2}, x_2\}$ |
| 7.    | \[
\begin{align*}
    A_1(u) &= c_2u^2 + c_1u + c_0 \\
    A_2(u) &= \beta_0,
\end{align*}
\] | $\mathcal{V}_4 = \text{Span}\{1, x_1, \text{sin}(\sqrt{a_1}x_2), \text{cos}(\sqrt{a_1}x_2)\}$ |
| 8.    | \[
\begin{align*}
    A_1(u) &= c_0 \\
    A_2(u) &= \beta_2u^2 + \beta_1u + \beta_0,
\end{align*}
\] | $\mathcal{V}_4 = \text{Span}\{1, x_2, \text{sin}(\sqrt{a_1}x_1), \text{cos}(\sqrt{a_1}x_1)\}$ |
different values of \( \alpha \) which gives (4.3), we obtain \( u \).

\[ \begin{align*}
    (a) & \quad \nu_1 = \nu_2 = \gamma_1 = \gamma_2 = b_1 = a_0 = 1, \quad x_1 = -2, x_2 = 3 \\
    (b) & \quad \nu_1 = \nu_2 = \gamma_1 = \gamma_2 = a_0 = b_1 = 1, x_2 = 3
\end{align*} \]

Fig. 1 2D and 3D graphical representations of solution (1.3) for different values of \( \alpha \)

initial condition \( u(x_1, x_2, 0) = v_1 e^{-a_0 x_1} + v_2 e^{-(a_0 x_1 + b_1 x_2)} \), since \( E_{\alpha, 1}(\gamma t^{\alpha}) \mid_{t=0} = 1 \). Two-dimensional (2D) and three-dimensional (3D) graphical representations of solution (4.4) for different values of \( \alpha \) with \( \nu_1 = \nu_2 = \gamma_1 = \gamma_2 = a_0 = b_1 = 1 \) are shown in Fig. 1.

Now, we consider the case \( \alpha \in (1, 2] \). Then, applying the Laplace transformation to Eq. (4.3), we obtain

\[ \mathcal{L} \left[ \frac{d^\alpha \Phi_i(t)}{dt^\alpha} \right] = \mathcal{L} \left[ \gamma_i \Phi_i(t) \right], \quad \alpha \in (1, 2], \]

which gives

\[ s^\alpha \tilde{\Phi}_i(s) - s^{\alpha-1} \Phi_i(0) - s^{\alpha-2} \Phi'_i(0) = \gamma_i \tilde{\Phi}_i(s). \]

The above equation can be written as

\[ \tilde{\Phi}_i(s) = \Phi_i(0) \left( \frac{s^{\alpha-1}}{s^{\alpha-\gamma_i}} \right) + \Phi'_i(0) \left( \frac{s^{\alpha-2}}{s^{\alpha-\gamma_i}} \right). \]

Applying the inverse Laplace transformation of the above equation, we get

\[ \Phi_i(t) = v_i E_{\alpha, 1}(\gamma t^{\alpha}) + t \mu_i E_{\alpha, 2}(\gamma t^{\alpha}), \quad \alpha \in (1, 2], \]

where \( v_i = \Phi_i(0) \) and \( \mu_i = \Phi'_i(0) = \frac{d\Phi_i(t)}{dt} \mid_{t=0}, \ i = 1, 2. \)

For this case, we obtain an exact solution for Eq. (4.1) as

\[ \begin{align*}
    u(x_1, x_2, t) = (v_1 E_{\alpha, 1}(\gamma_1 t^{\alpha}) + t \mu_1 E_{\alpha, 2}(\gamma_1 t^{\alpha})) e^{-a_0 x_1} \\
    + (v_2 E_{\alpha, 1}(\gamma_2 t^{\alpha}) + t \mu_2 E_{\alpha, 2}(\gamma_2 t^{\alpha})) e^{-(a_0 x_1 + b_1 x_2)}, \quad \alpha \in (1, 2],
\end{align*} \]

where \( \gamma_1 = (a_0^2 c_0 - d_0 a_0 + k_1), \gamma_2 = (k_1 + \beta_0 b_1^2 - \lambda_0 b_1 + c_0 a_0^2 - a_0 d_0) \) and \( a_0, b_1, \mu_i, v_i \in \mathbb{R}, \ i = 1, 2. \) 2D and 3D graphical representations of solution (4.5) for different values of \( \alpha \) with \( \mu_1 = \mu_2 = v_1 = v_2 = \gamma_1 = \gamma_2 = b_1 = 1 \) and \( a_0 = -1 \) are shown in Fig. 2. Additionally, we observe that the obtained exact solution (4.5) satisfies the given initial conditions for \( \alpha \in (1, 2] \) as

\[ u(x_1, x_2, 0) = v_1 e^{-a_0 x_1} + v_2 e^{-(a_0 x_1 + b_1 x_2)} \]
Note that Eq. (4.6) admits a one-dimensional invariant subspace \( V_1 = \text{Span}\{e^{-a_0 x_1 + b_0 x_2}\} \) which was discussed in Table 6 of case 1. Following the above similar procedure, the obtained exact solutions for Eq. (4.6) are as follows:

\[
\begin{align*}
\frac{\partial^\alpha u}{\partial t^\alpha} &= \left( \frac{\partial}{\partial x_1} \left[ \left( -3b_0^2\beta_2 + a_0d_2 + b_0\lambda_2 \right) u^2 + \frac{-2b_0^2\beta_1 + a_0d_1 + b_0\lambda_1 - k_2}{2a_0^2} u + c_0 \right] \frac{\partial u}{\partial x_1} \right) \\
&\quad + \frac{\partial}{\partial x_2} \left[ (\beta_2 u^2 + \beta_1 u + \beta_0) \frac{\partial u}{\partial x_2} \right] + (d_2 u^2 + d_1 u + d_0) \frac{\partial u}{\partial x_1} + (\lambda_2 u^2 + \lambda_1 u + \lambda_0) \frac{\partial u}{\partial x_2} + k_2 u^2 + k_1 u,
\end{align*}
\]

subject to the initial conditions

\[
\begin{align*}
\alpha &\in (0, 1). & (4.7) \\
\alpha &\in (1, 2). & (4.8)
\end{align*}
\]

Note that Eq. (4.6) admits a one-dimensional invariant subspace \( V_1 = \text{Span}\{e^{-a_0 x_1 + b_0 x_2}\} \) which was discussed in Table 6 of case 1. Following the above similar procedure, the obtained exact solutions for Eq. (4.6) are as follows:

\[
\begin{align*}
\alpha &\in (0, 1). \\
\alpha &\in (1, 2).
\end{align*}
\]

where \( \gamma = a_0^2c_0 + b_0^2\beta_0 - \lambda_0a_0 - d_0b_0 + k_1 \) and \( a_0, b_0, \mu, \in \mathbb{R} \). In addition, note that the obtained exact solutions satisfy the given initial conditions (4.7) and (4.8).
Finally, let us consider the cubic non-linear equation in the form
\[
\frac{\partial^\alpha u}{\partial t^\alpha} = c_0 \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial}{\partial x_2} \left( \frac{\lambda_2}{3b_0} u^2 + \beta_1 u + \beta_0 \right) + d_0 \frac{\partial u}{\partial x_1} + (\lambda_2 u^2 + \lambda_1 u + \lambda_0) \frac{\partial u}{\partial x_2} + b_0 \lambda_1 u^2 + k_1 u, \quad \alpha \in (0, 2], \quad t \geq 0,
\]  
(4.9)
subject to the initial conditions
\[
\text{If } \alpha \in (0, 1], \text{ then } u(x_1, x_2, 0) = v_1 e^{-b_0 x_2} + v_2 e^{-(a_1 x_1 + b_0 x_2)}. \tag{4.10}
\]
\[
\text{If } \alpha \in (1, 2], \text{ then } \left\{ \begin{array}{l}
\frac{\partial u}{\partial t} |_{t=0} = \mu_1 e^{-b_0 x_2} + \mu_2 e^{-(a_1 x_1 + b_0 x_2)}.
\end{array} \right. \tag{4.11}
\]
Above Eq. (4.9) admits an invariant subspace \( V_2 = \text{Span} \{e^{-b_0 x_2}, e^{-(a_1 x_1 + b_0 x_2)}\} \) that was discussed in case 2 of Table 6. In a similar manner, we can find the exact solutions for Eq. (4.9) in the form
\[
u(x_1, x_2, t) = v_1 e^{-b_0 x_2} E_{\alpha, 1}(\gamma_1 t^\alpha) + v_2 e^{-(a_1 x_1 + b_0 x_2)} E_{\alpha, 1}(\gamma_2 t^\alpha) \text{ if } \alpha \in (0, 1],
\]
\[
u(x_1, x_2, t) = [v_1 E_{\alpha, 1}(\gamma_1 t^\alpha) + t \mu_1 E_{\alpha, 2}(\gamma_1 t^\alpha)] e^{-b_0 x_2}
+ [v_2 E_{\alpha, 1}(\gamma_2 t^\alpha) + t \mu_2 E_{\alpha, 2}(\gamma_2 t^\alpha)] e^{-(a_1 x_1 + b_0 x_2)} \text{ if } \alpha \in (1, 2],
\]
where \( \gamma_1 = (b_0^2 \beta_0 - \lambda_0 b_0 + k_1), \gamma_2 = (k_1 + \beta_0 b_0 - \lambda_0 b_0 + c_0 a_2^2 - a_1 d_0) \) and \( a_1, b_0, v_1, \mu_i \in \mathbb{R}, i = 1, 2. \) Note that the above obtained exact solutions satisfy the given initial conditions (4.10) and (4.11).

### 4.2 Exact solutions of (1.3) with cubic non-linearities using the combination of exponential and trigonometric subspaces

In this subsection, we consider the \((2 + 1)\)-dimensional time-fractional convection–reaction–diffusion wave equation (1.3) with cubic non-linearities in the form
\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial x_1} \left( c_2 u^2 + c_0 \right) \frac{\partial u}{\partial x_1} + \beta_0 \frac{\partial^2 u}{\partial x_2^2} + (3a_0 c_2 u^2 + d_1 u + d_0) \frac{\partial u}{\partial x_1} + a_0 d_1 u^2 + k_1 u, \quad \alpha \in (0, 2], \quad t \geq 0,
\]  
(4.12)
subject to the initial conditions
\[
\text{If } \alpha \in (0, 1], \text{ then } u(x_1, x_2, 0) = e^{-a_0 x_1} \left( v_1 \sin(\sqrt{b_0} x_2) + v_2 \cos(\sqrt{b_0} x_2) \right). \tag{4.13}
\]
\[
\text{If } \alpha \in (1, 2], \text{ then } \left\{ \begin{array}{l}
\frac{\partial u}{\partial t} |_{t=0} = e^{-a_0 x_1} \left( \mu_1 \sin(\sqrt{b_0} x_2) + \mu_2 \cos(\sqrt{b_0} x_2) \right).
\end{array} \right. \tag{4.14}
\]
Equation (4.12) admits an invariant subspace \( V_2 = \text{Span} \{e^{-a_0 x_1} \sin(\sqrt{b_0} x_2), e^{-a_0 x_1} \cos(\sqrt{b_0} x_2)\} \) that was discussed in case 5 of Table 6. For this case, we can find the exact solutions for Eq. (4.12) in the form
\[
u(x_1, x_2, t) = e^{-a_0 x_1} E_{\alpha, 1}(\gamma t^\alpha) \left[ v_1 \sin(\sqrt{b_0} x_2) + v_2 \cos(\sqrt{b_0} x_2) \right] \text{ if } \alpha \in (0, 1], \tag{4.15}
\]
\[
u(x_1, x_2, t) = e^{-a_0 x_1} E_{\alpha, 1}(\gamma t^\alpha) \left[ v_1 \sin(\sqrt{b_0} x_2) + v_1 \cos(\sqrt{b_0} x_2) \right]
\]
\[
\text{if } \alpha \in (1, 2),
\]
Fig. 3 2D and 3D graphical representations of solution (4.15) for different values of $\alpha$ with $\nu_1 = \nu_2 = \gamma = a_0 = 1$ and $b_1 = 25$.

Fig. 4 2D and 3D graphical representations of solution (4.16) for different values of $\alpha$ with $\nu_1 = 0.5, \mu_1 = 2, \nu_2 = \mu_2 = \gamma = a_0 = 1$ and $b_1 = 25$.
Now, we consider the another initial value problem for \((2+1)\)-dimensional time-fractional convection–reaction–diffusion wave equation (1.3) with cubic non-linearities in the form

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = c_0 \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial}{\partial x_2} \left[ (\beta_2 u^2 + \beta_1 u + \beta_0) \frac{\partial u}{\partial x_1} \right] + (3b_0 \beta_2 u^2 + \lambda_1 u + \lambda_0) \frac{\partial u}{\partial x_2} \tag{4.17}
\]

with the initial conditions

If \(\alpha \in (0, 1)\), then

\[
u(x_1, x_2, 0) = e^{-b_0 x_2^2} \left( v_1 \sin(\sqrt{a_0} x_1) + v_2 \cos(\sqrt{a_0} x_1) \right) \tag{4.18}
\]

If \(\alpha \in (1, 2)\), then

\[
u(x_1, x_2, 0) = e^{-b_0 x_2^2} \left( v_1 \sin(\sqrt{a_0} x_1) + v_2 \cos(\sqrt{a_0} x_1) \right) \tag{4.19}
\]

We note that the above Eq. (4.17) admits a two-dimensional invariant subspace \(V_2 = \text{Span}\{e^{-b_0 x_2^2} \sin(\sqrt{a_0} x_1), e^{-b_0 x_2^2} \cos(\sqrt{a_0} x_1)\}\) which was discussed in case 6 of Table 6 (with \(d_0 = 0\)). Then using the above-similar procedure, the obtained exact solutions for Eq. (4.17) are in the form

\[
u(x_1, x_2, t) = e^{-b_0 x_2^2} E_{\alpha, 1}(\gamma t^{\alpha}) \left( v_1 \sin(\sqrt{a_0} x_1) + v_2 \cos(\sqrt{a_0} x_1) \right) \text{ if } \alpha \in (0, 1],
\]

\[
u(x_1, x_2, t) = e^{-b_0 x_2^2} E_{\alpha, 1}(\gamma t^{\alpha}) \left( v_1 \sin(\sqrt{a_0} x_1) + v_2 \cos(\sqrt{a_0} x_1) \right)
+ e^{-b_0 x_2^2} t E_{\alpha, 2}(\gamma t^{\alpha}) \left( \mu_1 \sin(\sqrt{a_0} x_1) + \mu_2 \cos(\sqrt{a_0} x_1) \right) \text{ if } \alpha \in (1, 2],
\]

where \(\gamma = (-a_0 c_0 + b_0^2 \beta_0 - b_0 \lambda_0 + k_1)\) and \(a_0, b_0, c_0, \beta_0, \lambda_0, v_1, \mu_i \in \mathbb{R}, i = 1, 2\). Observe that the above solutions (4.20) satisfy the given initial conditions (4.18) and (4.19).

### 4.3 Exact solutions for the given Eq. (1.3) with quadratic non-linearities using the polynomial subspace

Now, let us consider the initial value problem for \((2+1)\)-dimensional time-fractional convection–reaction–diffusion wave Eq. (1.3) with quadratic non-linearities

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial x_1} \left[ (c_1 u + c_0) \frac{\partial u}{\partial x_1} \right] + \frac{\partial}{\partial x_2} \left[ (\beta_1 u + \beta_0) \frac{\partial u}{\partial x_2} \right]
+ d_0 \frac{\partial u}{\partial x_1} + \lambda_0 \frac{\partial u}{\partial x_2} + k_0, \quad \alpha \in (0, 2], \quad t \geq 0,
\]

along with the initial conditions

\[
u(x_1, x_2, 0) = v_1 + v_2 x_1 + v_3 x_2 \text{ if } \alpha \in (0, 1],
\]

\[
u(x_1, x_2, 0) = v_1 + v_2 x_1 + v_3 x_2 \text{ and } \frac{\partial u}{\partial t} \big|_{t=0} = \mu_1 + \mu_2 x_1 + \mu_3 x_2 \text{ if } \alpha \in (1, 2].\tag{4.23}
\]

We know that Eq. (4.21) admits a three-dimensional invariant subspace \(V_3 = \text{Span}\{1, x_1, x_2\}\) which is discussed in case 3 of Table 4. Thus, the obtained exact solutions for the quadratic non-linear Eq. (4.21) are in the form

\[
u(x_1, x_2, t) = \gamma_0 \frac{t^\alpha}{\Gamma(\alpha + 1)} + v_1 + v_2 x_1 + v_3 x_2 \text{ if } \alpha \in (0, 1],
\]

\[
u(x_1, x_2, t) = \gamma_0 \frac{t^\alpha}{\Gamma(\alpha + 1)} + \gamma_1 \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + 2\gamma_2 \frac{t^{\alpha+2}}{\Gamma(\alpha + 3)} + t \mu_1 + v_1.
\]
\( \gamma_0 = 4, \nu_1 = 100, \nu_2 = 20, \nu_3 = 3, x_1 = 120, x_2 = 30 \)

\( \gamma_0 = 40, \nu_1 = 10, \nu_3 = 3, \mu_1 = 100, \mu_2 = \nu_2 = \gamma_2 = 20, x_1 = 12, x_2 = 30 \)

**Remark 1** We would like to point out that when \( \alpha \in (0, 1] \), \( A_i(u) = 2u \), \( B_i(u) = 0 \), \( i = 1, 2 \) and \( C(u) = pu^2 + qu \), Eq. (1.3) becomes a \((2+1)\)-dimensional Caputo time-fractional biological population model

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2}{\partial x_1^2}(u^2) + \frac{\partial^2}{\partial x_2^2}(u^2) + pu^2 + qu, \quad \alpha \in (0, 1]
\]  

\((4.26)\)
where the functions \( \Phi_1 \) and \( \Phi_2 \) satisfy the following system of fractional-order ODEs

\[
\frac{d^\alpha \Phi_1}{dt^\alpha} = p\Phi_1^2(t) + q\Phi_1(t) + 2p\Phi_2(t)\Phi_3(t),
\]

\[
\frac{d^\alpha \Phi_2}{dt^\alpha} = \frac{3p}{2}\Phi_1(t)\Phi_2(t) + q\Phi_2(t),
\]

\[
\frac{d^\alpha \Phi_3}{dt^\alpha} = \frac{3p}{2}\Phi_1(t)\Phi_3(t) + q\Phi_3(t).
\]

For \( p \neq 0 \), the above system (4.31) with the Caputo sense may not be solvable exactly in general. Now, let \( p = 0 \). For this case, we obtain an exact solution of (4.26) in the form

\[
u(x_1, x_2, t) = \Phi_1(t) + \Phi_2(t)e^{a_0x_1 + \frac{1}{2}\sqrt{-p-4a_0^2x_2}} + \Phi_3(t)e^{a_0x_1 - \frac{1}{2}\sqrt{-p-4a_0^2x_2}},
\]

(4.30)
In addition, it should be noted that the analytical solutions of (4.26) were discussed through various methods in (Rui 2020; Wu and Rui 2018; Srivastava et al. 2014; Veeresha and Prakasha 2021). Rui (2020) derived the exact solution of (4.26) using the dynamical system method. Also, Wu and Rui (2018) investigated exact solutions of (4.26) through the method separation of variables combined with homogeneous balanced principle. Using the fractional reduced differential transform method, Srivastava et al. (2014) derived the exact solution of the given equation (4.26). In Veeresha and Prakasha (2021), Veeresha and Prakasha obtained an exact solution of (4.26) through the fractional natural decomposition method. Using these methods, the exact solutions of (4.26) in the Caputo sense were derived in Rui (2020), Wu and Rui (2018), Veeresha and Prakasha (2021), Srivastava et al. (2014) that are similar to the above-obtained solutions (4.28) and (4.29). Finally, we would like to mention that for dynamical system method and method separation of variables combined with homogeneous balanced principle, Rui (2020), and Wu and Rui (2018) considered the exact solution of (4.26) in the Caputo sense as

$$u(x_1, x_2, t) = [a_0 + a_1 v]E_{\alpha, 1}(\lambda t^{\alpha}),$$

where $v = v(x_1, x_2)$ is an undetermined function defined by space variables $x_1, x_2$ and $a_0, a_1, \lambda$ are undetermined constants along with the transformation

$$v(x_1, x_2) = v(\eta), \quad \eta = x_1 + \omega x_2.$$

The idea of these methods are similar to the invariant subspace method but way of finding the exact solutions are different. Also, it is important to note that in the sense of Abdel Kader et al. (2021) derived a variety of exact solutions of Eq. (4.26) with variable coefficients using the invariant subspace method along with the transformation $u(x_1, x_2, t) = F(c_1 x + c_2 y, t), c_1, c_2 \in \mathbb{R}$.

5 Invariant subspace method to (2 + 1)-dimensional time-fractional non-linear PDE with time delay

In this section, we explain how to extend the invariant subspace method to (2 + 1)-dimensional time-fractional non-linear delay PDEs. In addition, we also explain how to derive the exact solution for the initial value problem of the (2 + 1)-dimensional time-fractional convection–reaction–diffusion wave equation with linear term involving time delay.

5.1 Invariant subspace method to (2 + 1)-dimensional time-fractional non-linear PDE with linear time delay

In this subsection, we give a detailed study for constructing the invariant subspaces of the (2 + 1)-dimensional time-fractional non-linear PDEs with linear time delay. Thus, we consider the generalized (2 + 1)-dimensional time-fractional non-linear PDE with linear terms involving several time delay

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \tilde{K}[u, \hat{u}_i] \equiv \tilde{K}[u] + \sum_{i=1}^{q} \mu_i \hat{u}_i, \quad \alpha > 0, \quad t > 0,$$

$$u(x_1, x_2, t) = \vartheta(x_1, x_2, t) \quad \text{if} \quad t \in [-\hat{t}, 0],$$
where \( u = u(x_1, x_2, t), \) \( \hat{u}_i = u(x_1, x_2, t - \tau_i), \) \( \tau_i > 0, \hat{\tau} = \max\{\tau_i : i = 1, 2, \ldots, q\}, \) \( x_1, x_2 \in \mathbb{R}, \mu_i > 0, i = 1, 2, \ldots, q, q \in \mathbb{N} \), \( \frac{\partial^\alpha}{\partial t^\alpha} (\cdot) \) denotes the Caputo fractional derivative (1.4) of order \( \alpha \), and \( \hat{K}[u] \) is the sufficiently given smooth differential operator of order \( k \), that is

\[
\hat{K}[u] = \hat{K} \left( x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}, \ldots, \frac{\partial^k u}{\partial x_1^k}, \frac{\partial^k u}{\partial x_2^k}, \frac{\partial^k u}{\partial x_1^k \partial x_2^k} \right) \quad (5.2)
\]

and \( k_1 + k_2 = k, k_1, k_2 \in \mathbb{N} \).

Then the finite-dimensional linear space \( V_n \) given in (2.3) is said to be invariant under the differential operator \( \hat{K}[u, \hat{u}_i] \) if for every \( u \in V_n \) implies \( \hat{K}[u, \hat{u}_i] \in V_n \), which can be written as

\[
\hat{K} \left[ \sum_{m=1}^n \kappa_m \xi_m(x_1, x_2), \sum_{m=1}^n \tilde{\kappa}_m \tilde{\xi}_m(x_1, x_2) \right] = \sum_{m=1}^n \Psi_m(\kappa_1, \kappa_2, \ldots, \kappa_n) \xi_m(x_1, x_2) \quad (5.3)
\]

+ \[
\sum_{i=1}^q \sum_{m=1}^n \mu_i \tilde{\kappa}_m \tilde{\xi}_m(x_1, x_2),
\]

where \( \tilde{\kappa}_m, \kappa_m \in \mathbb{R} \) and \( \Psi_m \) denotes co-efficient of expansion \( \hat{K}[u, \hat{u}_i] \) with respect to the basis in (2.2), \( m = 1, 2, \ldots, n \).

**Theorem 5.1** Suppose that the finite-dimensional linear space \( V_n \) defined in (2.3), is invariant under the non-linear differential operator \( \hat{K}[u, \hat{u}_i] \) given in (5.1), then the (2+1)-dimensional non-linear time-fractional PDE with several linear time delays (5.1) admits an exact solution of the form

\[
u(x_1, x_2, t) = \sum_{m=1}^n \Phi_m(t) \xi_m(x_1, x_2), \quad (5.4)
\]

where the functions \( \Phi_m(t) \) satisfy the system of ODEs of fractional-order

\[
\frac{d^\alpha \Phi_m(t)}{dt^\alpha} = \Psi_m(\Phi_1(t), \ldots, \Phi_n(t)) + \sum_{i=1}^q \mu_i \Phi_m(t - \tau_i), \quad m = 1, 2, \ldots, n. \quad (5.5)
\]

**Proof** Suppose that \( V_n \) be an \( n \)-dimensional invariant subspace admitted by the given differential operator \( \hat{K}[u, \hat{u}_i] \). Now, let us assume that \( u(x_1, x_2, t) = \sum_{m=1}^n \Phi_m(t) \xi_m(x_1, x_2) \). Computing the Caputo fractional derivative of \( u(x_1, x_2, t) \) of order \( \alpha > 0 \) with respect to \( t \) gives

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \sum_{m=1}^n \frac{d^\alpha \Phi_m(t)}{dt^\alpha} \xi_m(x_1, x_2). \quad (5.6)
\]
Since $\mathcal{V}_n$ is invariant under $\tilde{\mathcal{K}}[u, \hat{u}_i]$. Thus, we have

$$
\tilde{\mathcal{K}}[u, \hat{u}_i] = \tilde{\mathcal{K}}[u(x_1, x_2, t), u(x_1, x_2, t - \tau_i)]
$$

$$
= \tilde{\mathcal{K}} \left[ \sum_{m=1}^{n} \Phi_m(t) \xi_m(x_1, x_2), \sum_{m=1}^{n} \Phi_m(t - \tau_i) \xi_m(x_1, x_2) \right]
$$

$$
= \sum_{m=1}^{n} \Psi_m(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t)) \xi_m(x_1, x_2)
$$

$$
+ \sum_{m=1}^{n} \sum_{i=1}^{q} \mu_i \Phi_m(t - \tau_i) \xi_m(x_1, x_2).
$$

Substituting (5.6) and (5.7) in (5.1), we get

$$
\sum_{m=1}^{n} \left[ \frac{d^\alpha \Phi_m(t)}{dt^\alpha} - \Psi_m(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t)) - \sum_{i=1}^{q} \mu_i \Phi_m(t - \tau_i) \right] \xi_m(x_1, x_2) = 0.
$$

Since the functions $\xi_m(x_1, x_2)$, $m = 1, \ldots, n$ are linearly independent. Thus, the above Eq. (5.8) reduces to system of fractional ODEs (5.5).

**Estimation of type I and type II invariant subspaces for (5.1):** Here we explain how to find the type I and type II linear spaces for the differential operator $\tilde{\mathcal{K}}[u, \hat{u}_i]$ that is given in (5.1). First, we consider the linear spaces which are defined as the solution space of homogeneous linear ODEs

$$
\mathcal{V}_{n_1} = \left\{ y_1 \mid D_{x_1}^{n_1}[y_1] = \frac{d^{n_1} y_1}{dx_1^{n_1}} + a_{n_1-1} \frac{d^{n_1-1} y_1}{dx_1^{n_1-1}} + \cdots + a_0 y_1 = 0 \right\},
$$

$$
\mathcal{V}_{n_2} = \left\{ y_2 \mid D_{x_2}^{n_2}[y_2] = \frac{d^{n_2} y_2}{dx_2^{n_2}} + b_{n_2-1} \frac{d^{n_2-1} y_2}{dx_2^{n_2-1}} + \cdots + b_0 y_2 = 0 \right\},
$$

where the functions $y_i$ are the linear combinations of the $n_i$-linearly independent solutions of $D_{x_i}^{n_i}[y_i] = 0$, $i = 1, 2$. Then, let us assume that the functions $v_1(x_1), v_2(x_1), \ldots, v_{n_1}(x_1)$ are $n_1$-linearly independent solutions of $D_{x_1}^{n_1}[y_1] = 0$. Similarly, we assume that another set of functions $\zeta_1(x_2), \zeta_2(x_2), \ldots, \zeta_{n_2}(x_2)$ are $n_2$-linearly independent solutions of $D_{x_2}^{n_2}[y_2] = 0$. Then we can find two-types of linear spaces for the given differential operator $\tilde{\mathcal{K}}[u, \hat{u}_i]$. Suppose the given differential operator $\tilde{\mathcal{K}}[u, \hat{u}_i]$ admits type I and type II linear spaces $\mathcal{V}_{n_1n_2}$ and $\mathcal{V}_{n_1+n_2-1}$ that are given in (2.12) and (2.13). Then the invariant conditions for the type I and type II invariant subspaces are obtained as follows.
1. Type I invariance conditions for $\tilde{K}[u, \hat{u}_i]$ are obtained in the following form:

$$\mathcal{D}^n_{x_1}[\tilde{K}] = \frac{d^n\tilde{K}}{dx_1^n} + a_{n-1} \frac{d^{n-1}\tilde{K}}{dx_1^{n-1}} + \cdots + a_1 \frac{d\tilde{K}}{dx_1} + a_0 \tilde{K} = 0$$

along with

$$\mathcal{D}^n_{x_2}[\tilde{K}] = \frac{d^n\tilde{K}}{dx_2^n} + b_{n-1} \frac{d^{n-1}\tilde{K}}{dx_2^{n-1}} + \cdots + b_1 \frac{d\tilde{K}}{dx_2} + b_0 \tilde{K} = 0$$

where $a_i, (i = 0, 1, \ldots, n_1 - 1)$ and $b_j, (j = 0, 1, \ldots, n_2 - 1)$ are constants to be determined.

2. Type II invariance conditions for $\tilde{K}[u, \hat{u}_i]$ take the form

$$\mathcal{D}^n_{x_1}[\tilde{K}] = \frac{d^n\tilde{K}}{dx_1^n} + a_{n-1} \frac{d^{n-1}\tilde{K}}{dx_1^{n-1}} + \cdots + a_1 \frac{d\tilde{K}}{dx_1} = 0$$

along with

$$\mathcal{D}^n_{x_2}[\tilde{K}] = \frac{d^n\tilde{K}}{dx_2^n} + b_{n-1} \frac{d^{n-1}\tilde{K}}{dx_2^{n-1}} + \cdots + b_1 \frac{d\tilde{K}}{dx_2} = 0 \quad \text{and} \quad \frac{\partial^2\tilde{K}}{\partial x_1 \partial x_2} = 0$$

where $a_i, (i = 1, \ldots, n_1 - 1)$ and $b_j, (j = 1, \ldots, n_2 - 1)$ are constants to be determined. Next, the applicability and effectiveness of the method have been illustrated through the $(2 + 1)$-dimensional time-fractional cubic non-linear convection–reaction–diffusion wave equation with linear time delay.

$$\square$$

5.2 Exact solution for the $(2 + 1)$-dimensional time-fractional cubic non-linear convection–reaction–diffusion wave equation with linear time delay

In this subsection, let us consider the $(2 + 1)$-dimensional time-fractional cubic non-linear convection–reaction–diffusion wave equation with linear time delay

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial x_1} \left[ \left( \frac{d_2}{3a_0} u^2 + c_1 u + c_0 \right) \frac{\partial u}{\partial x_1} + \rho_0 \frac{\partial^2 u}{\partial x_2^2} + (d_2 u^2 + d_1 u + d_0) \frac{\partial u}{\partial x_1} \right. \right.$$  

$$+ \lambda_0 \frac{\partial u}{\partial x_2} + (-2a_0 c_1 + a_0 d_1) u^2 + k_1 u + \mu \hat{u}, \quad \alpha \in (0, 2), \quad t \geq 0,$$

$$\left. u(x_1, x_2, t) = \vartheta(x_1, x_2, t) = \phi_1(t) e^{-a_0 x_1} + \phi_2(t) e^{-(a_0 x_1 + b_1 x_2)}, \quad t \in [-\tau, 0], \right)$$
along with the initial conditions

\[
\begin{align*}
    u(x_1, x_2, 0) &= \kappa_1 e^{-\alpha x_1} + \kappa_2 e^{-(\alpha x_1 + b_1 x_2)} \quad \text{if} \quad \alpha \in (0, 1], \quad t \in [-\tau, 0], \\
    u(x_1, x_2, 0) &= \kappa_1 e^{-\alpha x_1} + \kappa_2 e^{-(\alpha x_1 + b_1 x_2)} \quad \text{if} \quad \alpha \in (1, 2], \\
    \frac{\partial u}{\partial t}
\end{align*}
\]

(5.10)

(5.11)

where \( \hat{u} = u(x_1, x_2, t - \tau), \tau > 0, \kappa_i, \hat{k}_i, \mu \in \mathbb{R}, \quad i = 1, 2. \)

Here, we consider the differential operator \( \hat{\mathcal{K}}[u, \hat{u}] \) for the given Eq. (5.9) as

\[
\hat{\mathcal{K}}[u, \hat{u}] = \frac{\partial}{\partial x_1} \left[ \left( \frac{d_2}{2a_0} u^2 + c_1 u + c_0 \right) \frac{\partial u}{\partial x_1} \right] + \beta_0 \frac{\partial^2 u}{\partial x_2^2} + (d_2 u^2 + d_1 u + d_0) \frac{\partial u}{\partial x_1}
\]

(5.12)

It should be noted that the cubic non-linear differential operator \( \hat{\mathcal{K}}[u, \hat{u}] \) admits a two-dimensional exponential linear space \( \mathcal{V}_2 = \text{Span}\{e^{-q_{0} x_1}, e^{-(q_{0} x_1 + q_{1} x_2)}\} \) since for some constants \( \delta_k, \hat{k}_k \in \mathbb{R}, \quad k = 1, 2 \)

\[
\hat{\mathcal{K}}[\delta_1 e^{-q_{0} x_1} + \delta_2 e^{-(q_{0} x_1 + q_{1} x_2)}, \hat{\delta_1} e^{-q_{0} x_1} + \hat{\delta_2} e^{-(q_{0} x_1 + q_{1} x_2)}]
\]

\[
= (a_0^2 c_0 - d_0 a_0 + k_1) \delta_1 e^{-q_{0} x_1} + (k_1 + \beta_0 b_1^2 - \lambda_0 b_1 + c_0 d_0^2 - a_0 d_0) \delta_2 e^{-(q_{0} x_1 + q_{1} x_2)} + \mu [\delta_1 e^{-q_{0} x_1} + \hat{\delta_2} e^{-(q_{0} x_1 + q_{1} x_2)}] \in \mathcal{V}_2,
\]

which suggests an exact solution of (5.9) in the form

\[
u(x_1, x_2, t) = \Phi_1(t) e^{-q_{0} x_1} + \Phi_2(t) e^{-(q_{0} x_1 + q_{1} x_2)},
\]

(5.13)

where the functions \( \Phi_1(t) \) and \( \Phi_2(t) \) satisfy the following system:

\[
\frac{d^{\alpha} \Phi_i(t)}{dt^{\alpha}} = \gamma_i \Phi_i(t) + \mu \Phi_i(t - \tau), \quad i = 1, 2, \quad \alpha \in (0, 2],
\]

(5.14)

where \( \gamma_1 = (a_0^2 c_0 - d_0 a_0 + k_1) \) and \( \gamma_2 = (k_1 + \beta_0 b_1^2 - \lambda_0 b_1 + c_0 d_0^2 - a_0 d_0) \). Now, let us first consider the case \( \alpha \in (0, 1] \). Applying the Laplace transformation to Eq. (5.14), we obtain

\[
\mathcal{L} \left[ \frac{d^{\alpha} \Phi_i(t)}{dt^{\alpha}} \right] = \mathcal{L} [\gamma_i \Phi_i(t)] + \mu \mathcal{L} [\Phi_i(t - \tau)], \quad i = 1, 2.
\]

From the given initial data (5.9), we get \( \Phi_i(t) = \phi_i(t), \forall t \in [-\tau, 0], i = 1, 2. \) Thus, we obtain

\[
\phi^{\alpha}_i(s) - s^{\alpha-1} \Phi_i(0) = \gamma_i \Phi_i(s) + \mu e^{\tau s} \Phi_i(s) + \mu e^{\tau s} \int_{-\tau}^{0} e^{-\tau \xi} \phi_i(\xi) d\xi.
\]

The above equation can be written as

\[
\Phi_i(s) = \Phi_i(0) \left( \frac{s^{\alpha-1}}{s^{\alpha} - \gamma_i - \mu e^{\tau s}} \right) + \left( \frac{\mu e^{\tau s}}{s^{\alpha} - \gamma_i - \mu e^{\tau s}} \right) \int_{-\tau}^{0} e^{-\tau \xi} \phi_i(\xi) d\xi.
\]

(5.15)
Taking the inverse Laplace transformation of (5.15), we get

\[ \Phi_i(t) = \Phi_i(0) L^{-1} \left[ \frac{s^{\alpha-1}}{s^\alpha - \gamma_i - \mu e^{-\tau s}} \right] + \mu L^{-1} \left[ \left( \frac{e^{-\tau s}}{s^\alpha - \gamma_i - \mu e^{-\tau s}} \right) \int_{-\tau}^{0} e^{-s\xi} \Phi_i(\xi) d\xi \right]. \tag{5.16} \]

Let \( 0 < |\frac{\mu e^{-\tau s}}{s^\alpha - \gamma_i}| < 1 \). Now, we can simplify the first term of (5.16) as

\[ L^{-1} \left[ \frac{s^{\alpha-1}}{s^\alpha - \gamma_i - \mu e^{-\tau s}} \right] = L^{-1} \left[ \sum_{m=0}^{\infty} \frac{\mu^m e^{-\tau ms}}{(s^\alpha - \gamma_i)^{m+1}} \right] = \sum_{m=0}^{\infty} \mu^m U_{m\tau}(t)(t - m\tau)^{\alpha m} E^{m+1}_{\alpha,\alpha m+1}(\gamma_i(t - m\tau)^\alpha), \]

where \( U_a(t) \) denotes the unit step function that can be defined as \( U_a(t) = \begin{cases} \ 1 & t \geq a \\ \ 0 & t < a \end{cases} \) and \( E^r_{p,q}(\cdot) \) is the generalized three-parameter Mittag-Leffler function (Mathai and Haubold 2008), which is defined as \( E^r_{p,q}(w) = \sum_{m=0}^{\infty} \frac{(r)_m w^m}{\Gamma(pm + q)m!}, w \in \mathbb{R}, (r)_m = \frac{\Gamma(r + m)}{\Gamma(r)} \), \( p, q, r > 0 \).

Next, using the Laplace convolution theorem which states that any two piecewise continuous functions \( f_1(t) \) and \( f_2(t) \) defined on \([0, \infty)\) and of exponential order \( v > 0 \), then Laplace transformation of convolution of \( f_1(t) \) and \( f_2(t) \) is given by

\[ \mathcal{L}[(f_1 * f_2)(t)] = \mathcal{L}(f_1(t))\mathcal{L}(f_2(t)), \]

where \( (f_1 * f_2)(t) = \int_{0}^{t} f_1(\xi)f_2(t - \xi)d\xi = \int_{0}^{t} f_1(\xi)f_2(t - \xi)d\xi \). Also, consider the extension of \( \phi_i(t) \) on \([-\tau, \infty)\) of the form \( \phi_i(t) = \begin{cases} \Phi_i(t) & t \in [-\tau, 0] \\ \Phi_i(0) & t \geq 0 \end{cases} \), \( i = 1, 2 \) and define the function \( f(t) = \begin{cases} 0 & t \geq 0 \\ 1 & t < 0 \end{cases} \), then we can simplify the second term

\[ L^{-1} \left[ \left( \frac{e^{-\tau s}}{s^\alpha - \gamma_i - \mu e^{-\tau s}} \right) \int_{-\tau}^{0} e^{-s\xi} \Phi_i(\xi) d\xi \right] \] of (5.16) for \( 0 < |\frac{\mu e^{-\tau s}}{s^\alpha - \gamma_i}| < 1, i = 1, 2 \), as follows:

\[ L^{-1} \left[ \left( \frac{e^{-\tau s}}{s^\alpha - \gamma_i - \mu e^{-\tau s}} \right) \int_{-\tau}^{0} e^{-s\xi} \Phi_i(\xi) d\xi \right] = \sum_{m=0}^{\infty} L^{-1} \left[ \frac{\mu^m e^{-\tau(m+1)s}}{(s^\alpha - \gamma_i)^{m+1}} \right] * L^{-1} \left[ e^{-\tau s} \int_{-\tau}^{0} e^{-s\xi} \Phi_i(\xi) d\xi \right] = \sum_{m=0}^{\infty} \mu^m L^{-1} \left[ \frac{e^{-\tau ms}}{(s^\alpha - \gamma_i)^{m+1}} \right] * \int_{-\tau}^{0} e^{-s\xi} \Phi_i(\xi) d\xi = \sum_{m=0}^{\infty} \mu^m L^{-1} \left[ \frac{e^{-\tau ms}}{(s^\alpha - \gamma_i)^{m+1}} \right] * \int_{0}^{\infty} e^{-s\xi} \phi_i(\xi - \tau) f(\xi - \tau) d\xi = \sum_{m=0}^{\infty} \mu^m U_{m\tau}(t)(t - m\tau)^{\alpha(m+1)-1} E^{m+1}_{\alpha,\alpha m+1}(\gamma_i(t - m\tau)^\alpha) * \int_{0}^{\infty} e^{-s\xi} \phi_i(\xi - \tau) f(\xi - \tau) d\xi. \]
Thus, we obtain the function $\Phi_i(t)$, $i = 1, 2$, in the form

$$\Phi_i(t) = \Phi_i(0) \sum_{m=0}^{\infty} \mu^m U_{m\tau}(t)(t - m\tau)^{\alpha m} E_{\alpha,\alpha m+1}^{m+1}(\gamma_i(t - m\tau)^{\alpha})$$

$$+ \sum_{m=0}^{\infty} \mu^{m+1} U_{m\tau}(t)(t - m\tau)^{\alpha(m+1)-1} E_{\alpha,\alpha(m+1)}^{m+1}(\gamma_i(t - m\tau)^{\alpha})$$

$$* [\phi_i(t - \tau)f(t - \tau)].$$

Hence, the obtained exact solution of the Eq. (5.9) for $0 < |\frac{\mu e^{-\tau}}{s^\alpha - \gamma_i}| < 1$, $i = 1, 2$, is as follows:

$$u(x_1, x_2, t) = e^{-a_0 x_1} \left\{ \kappa_1 \sum_{n=0}^{\infty} \mu^n (t - m\tau)^{\alpha n} E_{\alpha,\alpha n+1}^{n+1}(\gamma_1(t - m\tau)^{\alpha})$$

$$+ \sum_{m=0}^{\infty} \mu^{m+1} (t - m\tau)^{\alpha(m+1)-1} E_{\alpha,\alpha(m+1)}^{m+1}(\gamma_1(t - m\tau)^{\alpha})$$

$$* [\phi_1(t - \tau)f(t - \tau)] \right\}$$

$$+ e^{-(a_0 x_1 + b_1 x_2)} \left\{ \kappa_2 \sum_{m=0}^{\infty} \mu^n (t - m\tau)^{\alpha n} E_{\alpha,\alpha n+1}^{n+1}(\gamma_2(t - m\tau)^{\alpha})$$

$$+ \sum_{m=0}^{\infty} \mu^{m+1} (t - m\tau)^{\alpha(m+1)-1} E_{\alpha,\alpha(m+1)}^{m+1}(\gamma_2(t - m\tau)^{\alpha})$$

$$* [\phi_2(t - \tau)f(t - \tau)] \right\},$$

(5.17)

where $\alpha \in (0, 1], n - 1 < \frac{t}{\tau} \leq n$, $t > 0$, $\tau > 0$, $\kappa_i = \Phi_i(0), i = 1, 2$, $n \in \mathbb{N}$, $f(t) = \begin{cases} 0 & t \geq 0 \\ 1 & t < 0 \end{cases}$, $\phi_i(t) = \Phi_i(t)$, $\forall t \in [-\tau, 0], i = 1, 2$, $\gamma_1 = (a_0^2 c_0 - d_0 a_0 + k_1)$ and $\gamma_2 = (k_1 + \beta_0 b_1 + \lambda_0 b_1 + c_0 a_0^2 - a_0 d_0)$.

It is important to note that the above solution (5.17) satisfies the given initial condition (5.10). The obtained solution (5.17) can be viewed as

$$u(x_1, x_2, t) = e^{-a_0 x_1} \left\{ \kappa_1 \left( E_{\alpha,\alpha n+1}^{n+1}(\gamma_1^\alpha) + \sum_{m=1}^{n} \mu^m (t - m\tau)^{\alpha m} E_{\alpha,\alpha m+1}^{m+1}(\gamma_1(t - m\tau)^{\alpha}) \right)$$

$$+ \int_0^t \sum_{m=0}^{\infty} \mu^{m+1} (\xi - m\tau)^{\alpha(m+1)-1} E_{\alpha,\alpha(m+1)}^{m+1}(\gamma_1(\xi - m\tau)^{\alpha}) \phi_1(t - \tau - \xi)f(t - \tau - \xi)d\xi \right\}$$

$$+ e^{-(a_0 x_1 + b_1 x_2)} \left\{ \kappa_2 \left( E_{\alpha,\alpha n+1}^{n+1}(\gamma_2^\alpha) + \sum_{m=1}^{n} \mu^m (t - m\tau)^{\alpha m} E_{\alpha,\alpha m+1}^{m+1}(\gamma_2(t - m\tau)^{\alpha}) \right)$$

$$+ \int_0^t \sum_{m=0}^{\infty} \mu^{m+1} (\xi - m\tau)^{\alpha(m+1)-1} E_{\alpha,\alpha(m+1)}^{m+1}(\gamma_2(\xi - m\tau)^{\alpha}) \phi_2(t - \tau - \xi)f(t - \tau - \xi)d\xi \right\}.$$
Substitute $t = 0$ in (5.18), we have

$$u(x_1, x_2, 0) = \kappa_1 e^{-a_0 x_1} + \kappa_2 e^{-(a_0 x_1 + b_1 x_2)}$$

since $n = 0$ and $E_{\alpha,1}^1(\gamma t^\alpha) = 1$ at $t = 0, i = 1, 2$.

Now we consider the second case $\alpha \in (1, 2)$, proceeding in the similar way as explained above for $\alpha \in (0, 1)$. For this case, applying the Laplace transformation to Eq. (5.14) which yields

$$\mathcal{L} \left[ \frac{d^\alpha \Phi_i(t)}{dt^\alpha} \right] = \mathcal{L}\left[ \gamma_i \Phi_i(t) \right] + \mu \mathcal{L}[\Phi_i(t - \tau)], i = 1, 2.$$

The above equation can be simplified as

$$s^\alpha \tilde{\Phi}_i(s) - s^{\alpha-1} \Phi_i(0) - s^{\alpha-2} \Phi_i'(0) = \gamma_i \tilde{\Phi}_i(s) + \mu e^{-\tau s} \tilde{\Phi}_i(s) + \mu e^{-\tau s} \int_{-\tau}^0 e^{-s \xi} \Phi_i(\xi) d\xi, i = 1, 2,$$

which can be written in the form

$$\tilde{\Phi}_i(s) = \Phi_i(0) \left( \frac{s^{\alpha-1}}{s^\alpha - \gamma_i - \mu e^{-\tau s}} \right) + \Phi_i'(0) \left( \frac{s^{\alpha-2}}{s^\alpha - \gamma_i - \mu e^{-\tau s}} \right) + \left( \frac{\mu e^{-\tau s}}{s^\alpha - \gamma_i - \mu e^{-\tau s}} \right) e^{-\tau s} \int_{-\tau}^0 e^{-s \xi} \Phi_i(\xi) d\xi.$$

Thus, the obtained function $\Phi_i(t)$, for $0 < |\frac{\mu e^{-\tau s}}{s^\alpha - \gamma_i}| < 1, i = 1, 2$, is of the form

$$\Phi_i(t) = \Phi_i(0) \sum_{m=0}^{\infty} \mu^m U_{m \tau}(t)(t - m \tau)^{\alpha m} E_{\alpha, \alpha m+1}^{m+1}(\gamma_i(t - m \tau)^\alpha)$$

$$+ \Phi_i'(0) \sum_{m=0}^{\infty} \mu^m U_{m \tau}(t)(t - m \tau)^{\alpha m+1} E_{\alpha, \alpha m+2}^{m+1}(\gamma_i(t - m \tau)^\alpha)$$

$$+ \left[ \sum_{m=0}^{\infty} \mu^{m+1} U_{m \tau}(t)(t - m \tau)^{\alpha m+1} E_{\alpha, \alpha m+1}^{m+1}(\gamma_i(t - m \tau)^\alpha) \right] * [\phi_i(t - \tau) f(t - \tau)].$$

Hence, the exact solution of Eq. (5.9), for $0 < |\frac{\mu e^{-\tau s}}{s^\alpha - \gamma_i}| < 1, i = 1, 2$, is as follows

$$u(x_1, x_2, t) = e^{-a_0 x_1} \left\{ \kappa_1 \sum_{m=0}^{n} \mu^m (t - m \tau)^{\alpha m} E_{\alpha, \alpha m+1}^{m+1}(\gamma_i(t - m \tau)^\alpha)$$

$$+ \hat{\kappa}_1 \sum_{m=0}^{n} \mu^{m+1} (t - m \tau)^{\alpha m+1} E_{\alpha, \alpha m+2}^{m+1}(\gamma_i(t - m \tau)^\alpha)$$

$$+ \left[ \sum_{m=0}^{n} \mu^{m+1} (t - m \tau)^{\alpha m+1} E_{\alpha, \alpha m+1}^{m+1}(\gamma_i(t - m \tau)^\alpha) \right] * [\phi_i(t - \tau) f(t - \tau)] \right\}$$

$$+ e^{-(a_0 x_1 + b_1 x_2)} \left\{ \kappa_2 \sum_{m=0}^{n} \mu^m (t - m \tau)^{\alpha m} E_{\alpha, \alpha m+1}^{m+1}(\gamma_2(t - m \tau)^\alpha)$$

$$+ \hat{\kappa}_2 \sum_{m=0}^{n} \mu^{m+1} (t - m \tau)^{\alpha m+1} E_{\alpha, \alpha m+2}^{m+1}(\gamma_2(t - m \tau)^\alpha)$$

$$+ \left[ \sum_{m=0}^{n} \mu^{m+1} (t - m \tau)^{\alpha m+1} E_{\alpha, \alpha m+1}^{m+1}(\gamma_2(t - m \tau)^\alpha) \right] * [\phi_2(t - \tau) f(t - \tau)] \right\}.$$
\[ + \kappa_2 \sum_{m=0}^{n} \mu^m(t-m\tau)^{am+1} E_{\alpha,am+2}^{m+1}(\gamma_2(t-m\tau)\^\alpha) + \left[ \sum_{m=0}^{n} \mu^{m+1}(t-m\tau)^{a(m+1)-1} E_{\alpha,a(m+1)}^{m+1}(\gamma_2(t-m\tau)\^\alpha) \right] * \left[ \phi_2(t-\tau)f(t-\tau) \right] \right\}, \]

(5.19)

where \( \phi_i(t) = \Phi_i(t), \forall t \in [-\tau,0], \kappa_i = \Phi_i(0), \hat{\kappa}_i = \Phi'_i(0), i = 1, 2, \) and \( n \in \mathbb{N}, \gamma_1 = (\alpha_0 c_0 - d_0 a_0 + k_1) \) and \( \gamma_2 = (k_1 + \beta_0 b_1^2 - \lambda_0 b_1 + c_0 \alpha_0^2 - a_0 d_0). \) Additionally, we observe that the obtained exact solution (5.19) satisfies the given initial conditions (5.11). The above solution (5.19) can be written as

\[
\begin{align*}
\quad u(x_1,x_2,t) &= e^{-a_0 x_1} \left\{ \kappa_1 \left( E_{\alpha,1}^1(\gamma_1 t^{\alpha}) + \sum_{m=1}^{n} \mu^m(t-m\tau)^{am} E_{\alpha,am+1}^{m+1}(\gamma_1(t-m\tau)^{\alpha}) \right) \\
\quad &+ \hat{\kappa}_1 \left( t E_{\alpha,2}^1(\gamma_1 t^{\alpha}) + \sum_{m=1}^{n} \mu^m(t-m\tau)^{am+1} E_{\alpha,am+2}^{m+1}(\gamma_1(t-m\tau)^{\alpha}) \right) \\
\quad &+ \int_0^t \sum_{m=0}^{n} \mu^{m+1}(\xi-m\tau)^{a(m+1)-1} E_{\alpha,a(m+1)}^{m+1}(\gamma_1(\xi-m\tau)^{\alpha}) \phi(t-\tau-\xi) f(t-\tau-\xi) d\xi \right\} \\
\quad &+ e^{-(a_0 x_1+b_1 x_2)} \left\{ \kappa_2 \left( \gamma_2 \right) + \sum_{m=1}^{n} \mu^m(t-m\tau)^{am} E_{\alpha,am+1}^{m+1}(\gamma_2(t-m\tau)^{\alpha}) \right) \\
\quad &+ \hat{\kappa}_2 \left( t E_{\alpha,2}^1(\gamma_2 t^{\alpha}) + \sum_{m=1}^{n} \mu^m(t-m\tau)^{am+1} E_{\alpha,am+2}^{m+1}(\gamma_2(t-m\tau)^{\alpha}) \right) \\
\quad &+ \int_0^t \sum_{m=0}^{n} \mu^{m+1}(\xi-m\tau)^{a(m+1)-1} E_{\alpha,a(m+1)}^{m+1}(\gamma_2(\xi-m\tau)^{\alpha}) \phi(t-\tau-\xi) f(t-\tau-\xi) d\xi \right\}, \end{align*}
\]

(5.20)

which satisfies the initial condition at \( t = 0 \)

\[
\quad u(x_1,x_2,0) = \kappa_1 e^{-a_0 x_1} + \kappa_2 e^{-(a_0 x_1+b_1 x_2)},
\]

since \( n = 0 \) and \( E_{\alpha,1}^1(\gamma_1 t^{\alpha}) = 1 \) at \( t = 0, i = 1, 2. \) And also, the solution (5.20) satisfies second initial condition

\[
\quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \hat{\kappa}_1 e^{-a_0 x_1} + \hat{\kappa}_2 e^{-(a_0 x_1+b_1 x_2)},
\]

because

\[
\quad \left. \frac{d}{dt} \left[ E_{\alpha,1}^1(\gamma_1 t^{\alpha}) \right] \right|_{t=0} = 0, \quad \left. \frac{d}{dt} \left[ t E_{\alpha,2}^1(\gamma_1 t^{\alpha}) \right] \right|_{t=0} = 1, \ i = 1, 2 \text{ and}
\]

\[
\quad \left. \frac{d}{dt} \left[ \int_0^t \sum_{m=0}^{n} \mu^{m+1}(\xi-m\tau)^{a(m+1)-1} E_{\alpha,a(m+1)}^{m+1}(\gamma_1(\xi-m\tau)^{\alpha}) \phi(t-\tau-\xi) f(t-\tau-\xi) d\xi \right] \right|_{t=0} = 0,
\]

Now, we explain how to extend the invariant subspace method to generalized \((2 + 1)\)-dimensional time-fractional non-linear PDE with time delay.
5.3 Extension of the invariant subspaces to generalized (2 + 1)-dimensional
time-fractional non-linear PDE with time delay

Consider the generalized (2 + 1)-dimensional time-fractional time delay non-linear PDE in the form

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \tilde{H}[u, \dot{u}], \quad \alpha > 0, \quad t > 0
\]

(5.21)

\[u(x_1, x_2, t) = \omega(x_1, x_2, t) \text{ if } t \in [-\tau, 0], \quad \tau > 0,
\]

where \(u = u(x_1, x_2, t), \dot{u} = u(x_1, x_2, t - \tau), x_1, x_2 \in \mathbb{R}, \frac{\partial^\alpha}{\partial t^\alpha}()\) denotes Caputo fractional derivative (1.4) of order \(\alpha\), and \(\tilde{H}[u, \dot{u}]\) is the sufficiently given smooth differential operator of order \(k\), that is,

\[
\tilde{H}[u, \dot{u}] = \tilde{H}\left( x_1, x_2, u, \dot{u}, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \frac{\partial^3 u}{\partial x_1^3}, \frac{\partial^3 u}{\partial x_2^3}, \frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3}, \ldots, \frac{\partial^k u}{\partial x_1^k}, \frac{\partial^k u}{\partial x_2^k}, \frac{\partial^k u}{\partial x_1 \partial x_2^k}, \frac{\partial^k u}{\partial x_1^k \partial x_2^k}, \frac{\partial^k u}{\partial x_1^k \partial x_2^k} \right)
\]

(5.22)

and \(k_1 + k_2 = k, k_1, k_2 \in \mathbb{N}\). Suppose the given differential operator (5.22) admits the invariant subspace \(\mathcal{V}_n\) given in (2.3). Then there exists \(n\) functions \(\Psi_m (m = 1, 2, \ldots, n)\) such that

\[
\tilde{H}[u, \dot{u}] = \tilde{H}\left[ \sum_{m=1}^{n} \kappa_m \xi_m(x_1, x_2), \sum_{m=1}^{n} \hat{\kappa}_m \hat{\xi}_m(x_1, x_2) \right]
\]

(5.23)

\[= \sum_{m=1}^{n} \Psi_m (\kappa_1, \ldots, \kappa_n, \hat{\kappa}_1, \ldots, \hat{\kappa}_m) \xi_m(x_1, x_2),
\]

where \(\kappa_m, \hat{\kappa}_m \in \mathbb{R}, \ m = 1, 2, \ldots, n\).

**Theorem 5.2** Suppose that the linear space \(\mathcal{V}_n\) given in (2.3) is invariant under the non-linear differential operator \(\tilde{H}[u, \dot{u}]\) given in (5.22), then the generalized (2 + 1)-dimensional non-linear time-fractional PDE with time delay (5.21) has admits an exact solution in the form

\[
u(x_1, x_2, t) = \sum_{m=1}^{n} \Phi_m(t) \xi_m(x_1, x_2),
\]

(5.24)

where the functions \(\Phi_m(t)\) satisfy the system of ODEs of fractional-order

\[
\frac{d^\alpha \Phi_m(t)}{d\tau^\alpha} = \Psi_m (\Phi_1(t), \ldots, \Phi_n(t), \Phi_1(t - \tau), \ldots, \Phi_n(t - \tau)), \ m = 1, 2, \ldots, n.
\]

(5.25)

**Proof** Proof of the theorem is similar to above Theorem 5.1. \(\square\)

6 Concluding remarks and discussion

The presented work was investigated how we can extend the invariant subspace method to (2 + 1)-dimensional time-fractional non-linear PDEs. More precisely, the systematic study...
was given for constructing the various dimensions of the invariant subspaces for the $(2 + 1)$-dimensional time-fractional generalized convection–reaction–diffusion wave equation along with the initial conditions (1.3) for the first time. Also, we have shown explicitly that the time-fractional convection–reaction–diffusion wave equation attains more than one invariant subspaces in the same dimension of linear spaces under consideration. Additionally, the special types of the above-mentioned equation were discussed through this method separately such as convection–diffusion wave equation (3.13), reaction–diffusion wave equation (3.14) and diffusion wave equation (3.15). Moreover, we explained how to derive the exact solutions for the underlying equation along with initial conditions using the obtained invariant subspaces. Several kinds of the obtained exact solutions for the given equation were graphically shown for various values of $\alpha$. Finally, we extended this method to $(2 + 1)$-dimensional time-fractional non-linear PDEs with time delay (5.1). Also, the effectiveness and applicability of the method were illustrated through the $(2 + 1)$-dimensional time-fractional convection–reaction–diffusion wave equation with time delay (5.9). In addition, we observe that the obtained exact solutions can be viewed as the combinations of the Mittag-Leffler function and polynomial, exponential and trigonometric type functions. We would like to point out that the obtained results are new and interesting, also observe that the given equation has not been discussed anywhere in the literature. This study shows that the discussed method and the obtained results are a very useful and efficient mathematical method to derive a variety of exact solutions for various types of integer-order and non-integer order scalar and coupled non-linear PDEs in the fields of science and engineering for future research. In the literature, the dynamical system method (Rui 2020) and method separation of variables combined with the homogeneous balanced principles (Wu and Rui 2018; Ren et al. 2020) are available for deriving the exact solutions of higher dimensional time-fractional nonlinear PDEs. These two analytical methods are based on the idea of the invariant subspace method but the ways of finding exact solutions are different, which will be studied in the future.

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