Relativistic fine structure of the \((^4\text{He}, \mu)\) ion 2p-levels

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Motivated by the recent experimental measures of the 2p-states fine splitting of the muonic He\(^4\) ion \([1]\), we think it may have some interest to present an independent calculation in terms of the two-body relativistic equation for a scalar and a fermion developed in \([2, 3]\) for mesic atoms. This work can thus be considered an addendum to these last papers in a different range for the masses of the two components. It gives theoretical results in good agreement with the recent experimental measures.

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I. INTRODUCTION.

The data of very accurate experimental measures of the fine splitting of the muonic He$^4$ ion have been recently published in [1]. The reported energies for the $2p_{3/2} \rightarrow 2s_{1/2}$ and $2p_{1/2} \rightarrow 2s_{1/2}$ transitions are respectively $368653 \pm 18$ GHz = $1524.6258 \pm 0.0744$ meV and $333339 \pm 15$ GHz = $1378.5789 \pm 0.6534$ meV. The experimental fine splitting $2p_{3/2} \rightarrow 2p_{1/2}$ is finally found to be $146.047 \pm 0.096$ meV. In the same paper the theoretical results, mainly referring to [4–8], have also been presented. The slightly different theoretical $2p_{1/2} \rightarrow 2s_{1/2}$ splittings obtained by different groups are reported in [4] where the chosen value is $1668.4892 \pm 0.0135$. This figure is independent of the nuclear structure and contains the non-relativistic one- and two-loop electron vacuum polarization and the corresponding relativistic corrections. The nuclear contributions are estimated to be $9.340 \pm 0.250$ meV + $0.0112$ meV. The mismatch between theoretical and experimental data of the $2p_{1/2} \rightarrow 2s_{1/2}$ transition is overcome by taking into account the charge radius $r_{\text{nucl}}$ of the He$^4$ nucleus. As shown in [4–8], it is expressed by the relation $\Delta E = A r_{\text{nucl}}^2 + B$. Although large differences may appear in the parameters $A$ and $B$ presented by different research groups, the final values of the He$^4$ nucleus charge radius are however very close. In [4], for instance, the choice is $A = -106.3536$ meV/fm$^2$ and $B = 0.0784$ meV, while in [1] the authors assume $A = -106.220$ meV/fm$^2$, $B = 0.00112$ meV. Since the presented $2p_{1/2} \rightarrow 2s_{1/2}$ splittings are $1677.6792$ meV and $1677.670$ meV respectively, the corresponding results for $r_{\text{nucl}}$ are $1.677$ fm and $1.678$ fm.

The usual and well established method to obtain theoretical results is to start from a non-relativistic atomic framework and calculate the corrections due to relativity, to the two-body recoil, to QED electron vacuum polarization and the contribution of nuclear properties, mainly involving the finite charge radius of the nucleus. When the light particle is a fermion, the Dirac equation has also been used for a better initial description of the system [8]. In last years we have proposed a scheme allowing the formulation of relativistic two-body wave equations, according to their fermionic or bosonic nature. Details and references to previous papers can be found in [2, 3]. Results in good agreement with experimental data have been produced through a wide range of energies, from mesons to atoms. In particular, in [2] we have dealt with mesic atoms constituted by a proton and a scalar meson, $\pi$ or $K$. This last framework applies also to the ($^4$He, $\mu$) ion, with the difference that the
scalar is now the heavy particle. We thus believe that there may be some interest in producing our results, obtained in an independent covariant, two-body scheme and to compare them with the present experimental data.

II. STATEMENT OF THE RESULTS.

We solve the spectral problem for the lowest levels of the relativistic scalar-fermion \((^4\text{He}, \mu)\) ion using the two-body equation given in \cite{2}. The physical parameters we use are the following. The Helion mass is \(m_{^4\text{He}} = 3727.3794066\) MeV, the muon mass is \(m_\mu = 105.6583755\) MeV, the fine structure constant is \(\alpha = 0.0072973525693\) \cite{9}. The results for the lowest levels are reported in the following Table I.

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|}
\hline
State & \(1s_{1/2}\) & \(2s_{1/2}\) & \(2p_{1/2}\) & \(2p_{3/2}\) \\
\hline
Levels & -10943.19011297 & -2735.83861160 & -2735.84873222 & -2735.71095812 \\
\hline
\end{tabular}
\caption{The lowest pure Coulomb levels of the \((^4\text{He}, \mu)\) ion in eV.}
\end{table}

Observe that the the \(2s_{1/2}\) and the \(2p_{1/2}\) are not degenerate as for a Dirac electron, since the Johnson-Lippman symmetry \cite{2} is broken in the two-body equation. We then determine the electron vacuum polarization (eVP) corrections of the levels by calculating the corresponding matrix elements of the Uehling, Källen-Sabry and iterated reducible Uehling potential with the relativistic eigenfunctions. The use of the latter removes the requirement of further relativistic corrections on the result. The expressions of the potentials have been taken from \cite{7, 10}. Their values are given in Table II.

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|}
\hline
State & \(1s_{1/2}\) & \(2s_{1/2}\) & \(2p_{1/2}\) & \(2p_{3/2}\) \\
\hline
Uehling & -18792.31849 & -2077.94106 & -411.74833 & -411.48797 \\
Källen-Sabry & -108.96659 & -11.77264 & -3.88856 & -3.88671 \\
Two loop reducible & -30.37895 & -3.64576 & 0.15541 & 0.15561 \\
\hline
\end{tabular}
\caption{The eVP corrections to the lowest levels of the \((^4\text{He}, \mu)\) ion in meV.}
\end{table}

Thus the total eVP correction to the \(2p_{1/2} \rightarrow 2s_{1/2}\) splitting is 1677.8780 meV and to the \(2p_{3/2} \rightarrow 2p_{1/2}\)
splitting is 0.26241 meV. Including the above QED corrections we therefore have

$$\Delta_{2p_{1/2} \rightarrow 2s_{1/2}} = 1667.7574 \text{ meV}$$

This has to be compared with the value 1668.489(14) meV, reported in [1]. These last two figures must be corrected with nuclear properties. As these are not pure QED effects, we will assume the same formula (11) of [1] in order to evaluate the contributions and consequently we find a Helion charge radius equal to 1.676 fm.

Let us turn to the fine structure of the $p$-states. The value of the $2p_{3/2} \rightarrow 2p_{1/2}$ splitting given in [1] and taken from [4], is determined by the the Dirac splitting of the $p$-levels to which four different groups of corrections are added. The first group collects the contributions due to relativity and recoil. Adding these, the proposed splitting is 145.5833 meV. The second group considers the eVP with the corresponding relativistic corrections and the third one takes into account the $\mu$ anomalous magnetic moment. The fourth and last group deals with the finite charge radius. The final figure proposed for the $2p_{3/2} \rightarrow 2p_{1/2}$ splitting is 148.1828 meV.

We give a short discussion of our results. The previous four groups of contributions can be distinguished in our framework also. The first three of them, however, come from different inputs as we later explain. The correction for the finite nuclear charge is not included in our scheme and we assume for it the same value proposed in [4], namely $-0.0113 \text{ meV}$. The corrections for the muon anomalous magnetic moment and the one for the ‘external’ part of the fine structure are neither included: indeed, while the spin-orbit of the muon with itself is taken into account in the two body relativistic equation [2, 3], the interaction between the spin of the muon and the Helion nucleus orbit is not included. Its contribution is therefore determined by means of the Pauli-Breit perturbative term, as generally done [6, 11]. The anomalous magnetic moment is taken into account in a similar way. The QED corrections for the eVP polarization, calculated using the two-body relativistic wave-functions, are finally added.

The fine splitting of the $2p$-states including the difference of the $p$-levels $\Delta_{2p_{3/2} \rightarrow 2p_{1/2}} = 137.7741 \text{ meV}$, purely quantum mechanical and thus non-perturbative, together with the muon external spin-orbit and its anomalous magnetic moment $a_\mu = 0.001165920$, is

$$\Delta_{QM} = \Delta_{2p_{1/2} \rightarrow 2s_{1/2}} \left( 1 + 2 \frac{m_\mu}{m_{He}} + 2 a_\mu \left( 1 + \frac{m_\mu}{m_{He}} \right) \right) = 145.9153 \text{ meV}$$
The level splitting $\Delta^0$ is given in literature by the approximate perturbative expression $(Z\alpha)^4 \frac{m_R^3}{(32 m_\mu^2)} = 137.75798$ meV \[6\]. The eVP corrections are obtained from Table II and amount to

$$\Delta_{\text{QED}} = 0.2624053133 \text{ meV}$$

Assuming finally the value $\Delta^N = -0.01176$ meV for the finite nuclear charge radius correction \[4\], our final result for the $2p$ fine structure of the $(^4\text{He}, \mu)$ ion is

$$\Delta_{2p_{3/2}\rightarrow 2p_{1/2}} = 146.1660 \text{ meV}$$

in good agreement with experimental measures.

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