A Cutting Plane Method based on Redundant Rows for Improving Fractional Distance

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Abstract—In this paper, an idea of the cutting plane method is employed to improve the fractional distance of a given binary parity check matrix. The fractional distance is the minimum weight (with respect to \(\ell_1\)-distance) of vertices of the fundamental polytope. The cutting polytope is defined based on redundant rows of the parity check matrix and it plays a key role to eliminate unnecessary fractional vertices in the fundamental polytope. We propose a greedy algorithm and its efficient implementation for improving the fractional distance based on the cutting plane method.

I. INTRODUCTION

Linear programming (LP) decoding proposed by Feldman [3] is one of the promising decoding algorithms for low density-parity check (LDPC) codes. The invention of LP decoding opened a new research field of decoding algorithms for binary linear codes. Recently, a number of studies on LP decoding have been made such as [13][7][8].

LP decoding has several virtues that belief propagation (BP) decoding does not possess. One of the advantages of LP decoding is that the behavior of the LP decoder can be clearly understood from a viewpoint of optimization. The decoding process of an LP decoder is just a minimization process of a linear function subject to linear inequalities corresponding to the parity check conditions. The feasible set of this linear programming problem is called the fundamental polytope [3]. The fundamental polytope is a relaxed polytope that includes the convex hull of all the codewords of a binary linear code. Decoding performance of LP decoding is thus closely related to geometrical properties of the fundamental polytope.

Another advantage of LP decoding is that its decoding performance can be improved by including additional constraints in the original LP problem. The additional constraints tighten the relaxation and they lead to improved decoding performance. Of course, additional constraints increase the decoding complexity of LP decoding but we can obtain flexibility to choose a trade-off between performance and complexity. The goal of the paper is to design an efficient method to find additional constraints that improve this trade-off for a given parity check matrix.

In this paper, we will propose a new greedy type algorithm to enhance the fractional distance based on the cutting polytope based on the redundant rows that eliminates unnecessary fractional vertices in the fundamental polytope. The additional constraints generated by the proposed method is based on redundant rows of a parity check matrix. In other words, the proposed method can be considered as a method to generate redundant parity check matrices from a given original parity check matrix. Therefore, the present work has close relationship to the works on elimination of stopping sets (SS) by using redundant parity check matrices and on stopping redundancy such as [12][1][6].

The cutting plane method is a well established technique for solving an integer linear programming (ILP) problem based on LP [10]. The basic idea of the cutting plane method is simple. In the first phase, an ILP problem is relaxed to an LP problem and then it is solved by an LP solver. If we get a fractional solution (i.e., a vector with elements of fractional number), a cutting plane (i.e., an additional linear constraint) matched to the fractional solution is added to the LP problem in the second phase. The cutting plane is actually a half space which excludes the fractional solution but it includes all the ILP solutions. In the third phase, the extended LP problem is solved and the above process continues until we obtain an integral solution.

The fractional distance of a binary linear code is the \(\ell_1\)-weight of the minimum weight vertex of the fundamental polytope. The fractional distance is known to be an appropriate geometrical property which indicates the decoding performance of LP decoding for the binary symmetric channel (BSC). It is proved that the LP decoder can successfully correct bit flip errors if the number of errors is less than half of the fractional distance [3]. In contrast to the minimum distance of a binary linear code, we can evaluate the fractional distance efficiently with an LP solver. Efficient evaluation of the fractional distance is especially important in the proposed method.

The idea for improving LP decoding performance using redundant rows of a given parity check matrix was discussed in [13][4]. In their methods, the redundant rows are efficiently found based on a short cycle of a Tanner graph of the parity check matrix. Their results indicate that addition of redundant rows is a promising technique to improve LP decoding performance and they suggest that further studies on this subject are meaningful to pursue more systematic ways to find appropriate redundant rows that achieves better trade-offs between decoding performance and complexity.
II. PRELIMINARIES

In this section, notations and definitions required throughout the paper are introduced.

A. Notations and definitions

Let $H$ be a binary $m \times n$ matrix where $n > m \geq 1$. The binary linear code defined by $H$, $C(H)$, is defined by

$$C(H) \equiv \{ x \in F_2^n : xH^T = 0^n \},$$  

where $F_2$ is the Galois field with two elements $\{0, 1\}$ and $0^n$ is the zero vector of length $m$. In the present paper, the bold face letter, like $x$, denotes a row vector. The elements of a vector is expressed by corresponding normal face letter with subscript; e.g., $x = (x_1, x_2, \ldots, x_n)$. The following definition of the fundamental polytope is due to Feldman [3].

Definition 1: (Fundamental polytope) Assume that $t$ is a binary row vector of length $n$. Let

$$X(t) \equiv \{ S \subseteq \text{Supp}(t) : |S| \text{ is odd} \},$$  

where $\text{Supp}(t)$ denotes the support set of the vector $t$ defined by

$$\text{Supp}(t) \equiv \{ i \in \{1, \ldots, n \} : t_i \neq 0 \}.$$  

The single parity polytope of $t$ is the polytope defined by

$$U(t) \equiv \{ f \in [0, 1]^n : \forall S \in X(t), \sum_{j \in S} f_j + \sum_{j \in (\text{Supp}(t) \setminus S)} (1 - f_j) \leq |\text{Supp}(t)| - 1 \},$$  

where $[a, b] \equiv \{ v \in \mathbb{R} : a \leq v \leq b \}$. The symbol $\mathbb{R}$ denotes the set of real numbers. For a given binary $m \times n$ parity check matrix $H$, the fundamental polytope $\mathcal{P}(H)$ is defined by

$$\mathcal{P}(H) \equiv \bigcap_{i=1}^{m} U(h_i),$$  

where $h_i = (h_{i1}, h_{i2}, \ldots, h_{in})$ is the $i$-th row vector of $H$. The convex hull of $C(H)$ is the intersection of all convex sets including $C(H)$ [3]. It is known that the fundamental polytope $\mathcal{P}(H)$ contains the convex hull of $C(H)$ as a subset and that $\mathcal{P}(H) \cap \{0, 1\}^n = C(H)$ holds.

Let $M$ denote the number of the linear constraints (inequalities) that forms $\mathcal{P}(H)$. Assume that these linear constraints are numbered from 1 to $M$ and that the $k$-th linear constraint has the form:

$$\alpha_{k1}f_1 + \cdots + \alpha_{kn}f_n \leq \beta_k,$$  

for $k \in \{1, \ldots, M\}$. We call the $k$-th constraint $\text{Const}_k$. The hyper plane corresponding to $\text{Const}_k$ is given by

$$\mathcal{F}_k \equiv \{ f \in \mathbb{R}^n : \alpha_{k1}f_1 + \cdots + \alpha_{kn}f_n = \beta_k \},$$  

and the half space satisfying $\text{Const}_k$ is defined by

$$\mathcal{H}_k \equiv \{ f \in \mathbb{R}^n : \alpha_{k1}f_1 + \cdots + \alpha_{kn}f_n \leq \beta_k \}.$$  

The fundamental polytope $\mathcal{P}(H)$ is thus the intersection of the half spaces such that

$$\mathcal{P}(H) = \mathcal{H}_1 \cap \mathcal{H}_2 \cap \cdots \cap \mathcal{H}_M.$$  

Let $\mathcal{F}_{\text{act}}(H)$ be the set of indices of the active constraints defined by

$$\mathcal{F}_{\text{act}}(H) \equiv \{ k \in \{1, \ldots, M\} : 0^n \in \mathcal{F}_k \}. $$  

The active constraints are the linear constraints whose hyper plane contains the origin $0^n$. In a similar manner, we define $\mathcal{F}_{\text{inact}}(H)$, which is the set of indices of the inactive constraints, by

$$\mathcal{F}_{\text{inact}}(H) \equiv \{ k \in \{1, \ldots, M\} : 0^n \notin \mathcal{F}_k \}. $$  

The fundamental cone $\mathcal{K}(H)$ is the cone defined by the active constraints:

$$\mathcal{K}(H) \equiv \{ f \in [0, 1]^n : \forall k \in \mathcal{F}_{\text{act}}(H), f_k \in \mathcal{H}_k \}. $$

III. CUTTING PLANE METHOD

In this section, the main idea of the cutting plane method based on redundant rows will be introduced and then an application to Hamming code is shown as an example.

A. Fractional distance

The fractional distance $d_{\text{frac}}$ is the $\ell_1$-distance between a codeword and the nearest vertex of $\mathcal{P}(H)$ [3].

Definition 2: (Fractional distance) Let $\mathcal{V}(X)$ be the set of all vertices of a polytope $X$. For a given binary $m \times n$ parity check matrix $H$, the fractional distance of $H$ is defined by

$$d_{\text{frac}}(H) \equiv \min_{\substack{x \in C(H) \setminus \{0, 1\}^n \ \text{such that} \ x \neq f \ \text{for some} \ f \in \mathcal{V}(\mathcal{P}(H))}} \frac{1}{n} \sum_{i=1}^{n} |x_i - f_i|.$$  

The importance of the fractional distance is stated in the following lemma due to Feldman [3].

Lemma 1: Assume that the channel is BSC. Let $e$ be the number of the bit flip errors. If

$$\left\lfloor \frac{d_{\text{frac}}(H)}{2} \right\rfloor - 1 \geq e,$$  

holds, then all the bit flip errors can be corrected by the LP decoder. (Proof) The proof is given in [3].

Based on the geometrical uniformity of the fundamental polytope (called C-symmetry in [3]), it has been proved that $d_{\text{frac}}(H)$ is the $\ell_1$-weight of the minimum weight vertex of $\mathcal{P}(H)$ except for the origin, which is expressed by

$$d_{\text{frac}}(H) = \min_{f \in \mathcal{V}(\mathcal{P}(H)) \setminus \{0\} \setminus \{0^n\}} \frac{1}{n} \sum_{i=1}^{n} f_i.$$  

Let $\Gamma(H)$ be a set of the minimum weight vertices of $\mathcal{P}(H)$:

$$\Gamma(H) \equiv \left\{ p \in \mathcal{V}(\mathcal{P}(H)) : \sum_{i=1}^{n} p_i = d_{\text{frac}}(H) \right\}.$$
and let \( d_{\min} \) be the minimum distance of \( C(H) \). Since any codeword of \( C(H) \) is a vertex of \( \mathcal{P}(H) \), it is obvious that the inequality \( d_{\text{trac}}(H) \leq d_{\min} \) holds for any \( H \). The fractional distance \( d_{\text{trac}}(H) \) depends on the representation of a given binary linear code (i.e., the parity check matrix) and there are a number of parity check matrices that define the same binary linear code. This means that the parity check matrices of a binary linear code can be ranked in terms of its fractional distance. It is hoped to find a better parity check which achieves larger fractional distance for a given binary linear code because such a parity check matrix may improve the performance/complexity trade-off of LP decoding for the target code.

**B. Cutting polytope**

In this subsection, we will define the cutting polytope based on the redundant rows. The next definition gives the definition of the redundant row.

**Definition 3:** (Redundant row) Let \( H \) be a binary \( m \times n \) parity check matrix of the target code \( C \). A redundant row \( h \) is a linear combination of the row vectors of \( H \) such that

\[
h = a_1 h_1 + a_2 h_2 + \cdots + a_m h_m,
\]

where \( a_i \in \mathbb{F}_2 (i \in \{1, \ldots, m\}) \).
The single parity polytope of a redundant row \( h \) includes all the codewords of \( C(H) \) because any codeword \( x \in C(H) \) satisfies \( xh^T = 0 \) [12].

The next lemma asserts a cutting property of a single parity polytope satisfying a certain condition.

**Lemma 2:** Let \( H \) be a binary \( m \times n \) parity check matrix. Assume that a point \( p = (p_1, \ldots, p_n) \in \mathbb{R}^n \) and \( t \in \mathbb{F}_2^m \) are given. If

\[
\exists j \in \text{Supp}(t), \quad p_j > \sum_{l \in \text{Supp}(t) \setminus \{j\}} p_l,
\]

holds, then \( p \notin \mathcal{U}(t) \) holds.

(Proof) Let \( j^* \) be the index satisfying

\[
p_{j^*} > \sum_{l \in \text{Supp}(t) \setminus \{j^*\}} p_l.
\]

The above inequality is equivalent to the following inequality:

\[
\sum_{j \in \{j^*\}} p_j + \sum_{l \in \text{Supp}(t) \setminus \{j^*\}} (1 - p_l) > |\text{Supp}(t)| - 1.
\]

From the definition of \( \mathcal{U}(t) \), it is evident that \( p \notin \mathcal{U}(t) \).

The cutting polytope defined below is used to cut a fractional vertex in the proposed method for improving the fractional distance.

**Definition 4:** (Cutting polytope) Assume that \( p = (p_1, p_2, \ldots, p_n) \in \mathcal{P}(H) \). If a redundant row \( h \) with the form:

\[
h = a_1 h_1 + a_2 h_2 + \cdots + a_m h_m
\]

satisfies

\[
p_j > \sum_{l \in (\text{Supp}(h) \setminus \{j\})} p_l,
\]

where \( j = \arg \max_{i \in \text{Supp}(h)} p_i \), then \( \mathcal{U}(h) \) is called a cutting polytope of \( p \).

The next theorem introduce a tighter relaxation of the convex hull of \( C(H) \) which may improve the fractional distance.

**Theorem 1:** Let \( p = (p_1, p_2, \ldots, p_n) \in \mathcal{P}(H) \) and \( \mathcal{U}(h) \) be a cutting polytope of \( p \). The following relations hold:

\[
C(H) \subset \mathcal{P}(H) \cap \mathcal{U}(h)
\]

and

\[
p \notin \mathcal{P}(H) \cap \mathcal{U}(h).
\]

(Proof) The claim of the theorem is directly derived from \( C(H) \subset \mathcal{U}(t) \) and Lemma 2.

The theorem implies that the cutting polytope of \( p \) also contains \( C(H) \) but excludes \( p \). In other words, the intersection \( \mathcal{P}(H) \cap \mathcal{U}(h) \) is a tighter relaxation of the convex hull of \( C(H) \) compared with \( \mathcal{P}(H) \). Let \( H' \) be the parity check matrix obtained by stacking \( H \) and \( h \), namely,

\[
H' = \left( \begin{array}{cc} H & h \end{array} \right).
\]

Note that \( \mathcal{P}(H') = \mathcal{P}(H) \cap \mathcal{U}(h) \) has also geometrical uniformity because \( \mathcal{P}(H') \) is a fundamental polytope. This means that the cutting polytope cuts not only \( p \) but also some non-codeword vertices of \( \mathcal{P}(H) \) which are geometrically equivalent to \( p \). The fractional distance of \( H' \), \( d_{\text{trac}}(H') \), thus can be larger than the fractional distance \( d_{\text{trac}}(H) \) because the point \( p \in \Gamma(H) \) is excluded from the new fundamental polytope \( \mathcal{P}(H') \). Furthermore, the multiplicity (i.e., size of \( \Gamma(H) \)) can be reduced as well by eliminating a fractional vertex with the minimum \( \ell_1 \)-weight. Figure 1 illustrates the idea of the cutting polytope.

**C. Cutting plane method: an example**

In this subsection, we will examine the idea of the cutting plane method described in the previous subsection thorough a concrete example. Let \( H \) be a parity check matrix of \((7, 4, 3)\)
Hamming code:

\[
H = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1
\end{pmatrix}.
\] (26)

In this case, we have the index sets:

\[
\text{Supp}(h_1) = \{1, 3, 4, 5\}, \quad \text{Supp}(h_2) = \{1, 2, 4, 6\}, \quad \text{Supp}(h_3) = \{2, 3, 4, 7\}
\]

and

\[
X(h_1) = \{\{1\}, \{3\}, \{4\}, \{5\}, \{3, 4\}, \{1, 4, 5\}, \{1, 3, 5\}, \{1, 3, 4\}\}, \\
X(h_2) = \{\{1\}, \{2\}, \{4\}, \{6\}, \{2, 4, 6\}, \{1, 4, 6\}, \{1, 2, 6\}, \{1, 2, 4\}\}, \\
X(h_3) = \{\{2\}, \{3\}, \{4\}, \{7\}, \{3, 4, 7\}, \{2, 4, 7\}, \{2, 3, 7\}, \{2, 3, 4\}\}.
\]

The fundamental polytope of \(H\) is the set of points in \([0, 1]^7\) satisfying

\[
\sum_{j \in S} f_j + \sum_{j \in (\text{Supp}(h_i) \setminus S)} (1 - f_j) \leq 3 \tag{27}
\]

for any \(i \in \{1, 2, 3\}\) and any \(S \in X(h_i)\). From some computations (details of computation are described in the next section), we can obtain the set of \(\ell_1\)-minimum weight vertices of \(\mathcal{P}(H)\)(i.e., \(\Gamma(H)\)):

\[
\left(0, \frac{2}{3}, 0, \frac{2}{3}, 0, 0, 0\right), \left(\frac{2}{3}, 0, \frac{2}{3}, 0, 0, 0\right), \left(0, 0, \frac{2}{3}, 0, \frac{2}{3}, 0, 0\right).
\]

Therefore, in this case, \(d_{\text{frac}}(H)\) is equal to 2.

Assume that we choose \(h = (1, 0, 1, 0, 0, 1, 1)\) as a redundant row that is the sum of the second and third rows of \(H\). Let \(p = (0, 2/3, 2/3, 2/3, 0, 0, 0) \in \Gamma(H)\). It is easy to check that

\[
p_3 = \frac{2}{3} > \frac{1}{\ell_n(S)} \quad \text{s.t.} \quad p \in \text{Supp}(h_i) \quad (31)
\]

holds where \(\text{Supp}(h) = \{1, 3, 6, 7\}\). This means that \(U(h)\) is a cutting polytope of \(p\). By stacking \(H\) and \(h\), we get a new parity check matrix \(H'\) whose fundamental polytope does not contain \(p\) as its vertex. In a similar manner, continuing the above process (appending redundant rows to \(H\) for cutting the vertices in \(\Gamma(H)\)), we eventually obtain a parity check matrix \(H^*\):

\[
H^* = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0
\end{pmatrix}.
\] (29)

The fractional distance of \(H^*\) is equal to 3 which is strictly larger than the fractional distance of \(H\) (\(d_{\text{frac}}(H) = 2\)). The details of a way to find an appropriate redundant rows will be discussed in the subsequent sections. It has been observed that the vectors in \(\Gamma(H^*)\) are integral; namely, they are the minimum weight codewords of Hamming code.

D. A greedy algorithm for cutting plane method

The previous example on Hamming code suggests that iterative use of the cutting plane method for a given parity check matrix may yield a parity check matrix with redundant rows which is better than the original one in terms of the fractional distance. The following greedy algorithm, called greedy cutting plane algorithm, is naturally obtained from the above observation.

| Greedy cutting plane algorithm |
|-------------------------------|
| **Step 1** Evaluate \(\Gamma(H)\). |
| **Step 2** Pick up \(p \in \Gamma(H)\). |
| **Step 3** Find a redundant row \(h\) of \(H\) which gives a cutting polytope of \(p\). If such \(h\) does not exist, exit the procedure. |
| **Step 4** Update \(H\) by |
| \(H := \begin{pmatrix} H \\ h \end{pmatrix} \) \tag{30} |
| **Step 5** Return to Step 1. |

Details on the process of finding a redundant row \(h\) is shown in the next section. The most time consuming parts of the above algorithm are evaluation of \(\Gamma(H)\) and search for a redundant row. In the next section, we will discuss efficient implementations for these parts that are indispensable to deal with codes of long length.

IV. EFFICIENT IMPLEMENTATION

A. Efficient computation of \(d_{\text{frac}}(H)\) and \(\Gamma(H)\)

As described in the previous section, evaluation of \(d_{\text{frac}}(H)\) and \(\Gamma(H)\) is required for finding a redundant parity check matrix with better fractional distance. An algorithm for computing \(d_{\text{frac}}(H)\) has been proposed by Feldman [3].

Firstly, we review the Feldman’s method. For any \(k \in \{1, \ldots, M\}\), Let \(d_k\) be

\[
d_k \equiv \text{minimize } \sum_{i=1}^{n} f_i \quad \text{s.t.} \quad f \in (\mathcal{P}(H) \cap \mathcal{F}_k) \tag{31}
\]

where this LP problem is denoted by \(LP_k\). Thus, \(d_k\) can be considered as the \(\ell_1\)-weight of the minimum weight vertex on the facet \(\mathcal{F}_k\) of \(\mathcal{P}(H)\). Since there exists at least one facet of \(\mathcal{P}(H)\) which includes \(p\) for any vector \(p\) in \(\Gamma(H)\), it is evident that

\[
d_{\text{frac}}(H) = \min_{k=1}^{M} \delta_k \tag{32}
\]

holds where \(\delta_k = d_k\) if \(d_k > 0\); otherwise \(\delta_k = \infty\). The LP problems \(LP_k\) can be efficiently solved with an LP solver based on the simplex algorithm or the interior point algorithm. From the solution of these LP problems, we can obtain the fractional distance of \(H\) by [32].
The number of constraints related to a fundamental polytope defined by $\mathbf{H}$ is an exponential function of the row weight of $\mathbf{H}$. Thus the number of executions of the LP solver rapidly increases as the row weight of $\mathbf{H}$ grows. Another formulation of the fundamental polytope proposed by Yang et al.\cite{Yang2013} can be used to reduce the number of constraints. In their formulation, high weight rows of $\mathbf{H}$ are divided to some low weight rows by introducing auxiliary variables. Although their method is effective for evaluating the fractional distance as well, we here propose another efficient method for evaluating $d_{\text{frac}}(\mathbf{H})$ in this section. In our method, a fundamental polytope is relaxed to a fundamental cone. This method can be combined with Yang et al.’s formulation.

We need to prepare a relaxed version of the fractional distance before discussing another expression of the fractional distance. For $k \in \mathbf{F}_{\text{inact}}(\mathbf{H})$, let $d_k^{\text{(relax)}}$ be

$$d_k^{\text{(relax)}} \equiv \min \sum_{i=1}^{n} f_i \quad \text{s.t.} \quad f \in (\mathcal{K}(\mathbf{H}) \cap \mathcal{F}_k).$$

This relaxed LP problem is denoted by $LP_k^{\text{(relax)}}$. The relaxed fractional distance $d_k^{\text{frac}}$ is defined by

$$d_k^{\text{frac}}(\mathbf{H}) \equiv \min_{k \in \mathbf{F}_{\text{inact}}(\mathbf{H})} d_k^{\text{(relax)}}.$$  \hspace{1cm} (34)

The next theorem states a useful equivalence relation.

**Theorem 2:** For a given $\mathbf{H}$, the following equality holds:

$$d_k^{\text{frac}}(\mathbf{H}) = d_{\text{frac}}(\mathbf{H}).$$  \hspace{1cm} (35)

(Proof) See appendix. \hfill \Box

A merit of Theorem 2 is that the evaluation of $d_k^{\text{frac}}(\mathbf{H})$ takes less computational complexity than that of the evaluation of $d_{\text{frac}}(\mathbf{H})$ using the Feldman’s method. The reduction on the computational complexity comes from the following two reasons.

One reason is that the feasible region of $LP_k^{\text{(relax)}}$ is based on the fundamental cone $\mathcal{K}(\mathbf{H})$ (instead of $\mathcal{P}(\mathbf{H})$) which can be expressed with fewer linear constraints than the fundamental polytope. In the case of a regular LDPC code with row weight $w_r$, the number of linear constraints required to define $LP_k^{\text{(relax)}}$ is $m w_r + 1$. On the other hand, to express the fundamental polytope required in $LP_k$, $m 2^{w_r-1}$ linear constraints are needed.

Another reason of the reduction on complexity is that fewer executions of the LP solver are required for evaluating $LP_k^{\text{(relax)}}$ because we can focus only on the inactive linear constraints in the case of $LP_k^{\text{(relax)}}$.

**B. Search for redundant rows**

A straightforward way to obtain a redundant row $\mathbf{h}$ that gives a cutting polytope of a given point $\mathbf{p}$ is the exhaustive search. Namely, each redundant row is checked whether it satisfies the condition $j^\ast$ or not. However, this naive approach is prohibitively slow even for codes of moderate length because there are $2^m$ redundant rows. We thus need a remedy to narrow the search space. The following theorem gives the basis of the reduction on the search space.

**Theorem 3:** Assume that there exists a cutting polytope of $\mathbf{p} \in \mathcal{F}(\mathbf{H})$. The polytope $U(\mathbf{h}^\ast)$ is such a cutting polytope of $\mathbf{p}$ and the redundant row $\mathbf{h}^\ast$ is given by $\mathbf{h}^\ast = \sum_{i=1}^{m} a_i^\ast \mathbf{h}_i$. Then, $U(\mathbf{h})$ is also a cutting polytope of $\mathbf{p}$ where $\mathbf{h} = \sum_{i=1}^{m} a_i \mathbf{h}_i$ and

$$a_i = \left\{ \begin{array}{ll} a_i^\ast & i \in Q \\ 0 & i \notin Q. \end{array} \right. \hspace{1cm} (36)$$

The index set $Q$ is defined by

$$Q \equiv \{ i \in \{1, \ldots, m\} : \exists j \in \text{Supp}(\mathbf{p}), h_{ij} \neq 0 \}. \hspace{1cm} (37)$$

(Proof) Let $j^\ast \in \text{Supp}(\mathbf{h}^\ast) \cap \text{Supp}(\mathbf{p})$ be the index satisfying

$$p_{j^\ast} > \sum_{l \in \text{Supp}(\mathbf{h}^\ast) \setminus \{j^\ast\}} p_l. \hspace{1cm} (38)$$

Since $p_{j^\ast} = 0$ if $j \notin \text{Supp}(\mathbf{p})$, the above condition is equivalent to

$$p_{j^\ast} > \sum_{l \in \text{Supp}(\mathbf{h}^\ast) \setminus \text{Supp}(\mathbf{p}) \setminus \{j^\ast\}} p_l. \hspace{1cm} (39)$$

From the definition of $\mathbf{h}$ and $Q$, we have

$$\text{Supp}(\mathbf{h}^\ast) \cap \text{Supp}(\mathbf{p}) = \text{Supp}(\mathbf{h}) \cap \text{Supp}(\mathbf{p}). \hspace{1cm} (40)$$

This equality leads to the inequality

$$p_{j^\ast} > \sum_{l \in \text{Supp}(\mathbf{h}) \setminus \text{Supp}(\mathbf{p}) \setminus \{j^\ast\}} p_l = \sum_{l \in \text{Supp}(\mathbf{h}) \setminus \{j^\ast\}} p_l. \hspace{1cm} (41)$$

The above inequality implies that $U(\mathbf{h})$ is a cutting polytope of $\mathbf{p}$.

The significance of the above theorem is that we can fix $a_i = 0$ for $i \notin Q$ in a search process without loss of the chance to find a redundant row generating a cutting polytope. Therefore, computational complexity to find a redundant row can be reduced by using this property. Let $V$ be a sub-matrix of $H$ composed from the columns of $H$ corresponding to the support of $\mathbf{p}$. The index set $Q$ consists of the row indices of non-zero rows of $V$. Thus, in the case of LDPC codes, the size of $Q$ is expected to be small when the size of $\text{Supp}(\mathbf{p})$ is small because of sparseness of the parity check matrices. In such a case, the search space of the redundant rows are limited in a reasonable size. For example, in the case of the LDPC code “96,33,964” $[9](n = 96, m = 48$, row weight 6, column weight 3), the size of $\text{Supp}(\mathbf{p})$ is 7(p \in 2 \mathcal{F}(\mathbf{H})) and the size of $Q$ is 8.

Let $H^Q$ be the $|Q| \times n$ sub-matrix of $H$ composed from a row vectors whose indices are included in $Q$. From Theorem 3, we can limit the search space to the linear combinations of rows of $H^Q$. In the following, we will present an efficient search algorithm for a redundant row. Let $\mathbf{r} = (r_1, \ldots, r_n)$ denote an indices vector that satisfies $p_{r_1} \geq p_{r_2} \geq \cdots \geq p_{r_n}$. 


Redundant row search algorithm
Step 1 Construct $H^Q$.
Step 2 Permute columns of $H^Q$ to the following form:

$$H^Q \Pi = \begin{pmatrix} v_{r_2} & v_{r_3} & \ldots & v_{r_n} & v_{r_1} \end{pmatrix},$$

where $\Pi$ denotes a column permutation matrix and $v_{r_j}, j \in \{1, \ldots, n\}$ denotes the $j$-th column vector of $H^Q$.
Step 3 Convert $H^Q \Pi$ into $U$ of row echelon form by applying elementary row operations.
Step 4 Let

$$i^* = \arg \min \{ i \in \{1, \ldots, |Q|\} : \ u_i \text{ satisfies } (22) \},$$

where $u_i = (u_{i1}, \ldots, u_{in})$ denotes the $i$-th row vector of $U\Pi^{-1}$.
Step 5 Output $u_{i^*}$.

The idea of the redundant row search algorithm is based on the fact that $u_i$ (a candidate of desirable redundant rows) tends to satisfy condition (22). This can be explained as follows. Assume that $u_{i, r_1} = 1$. From the definition of row echelon form,

$$u_{i, r_2} = u_{i, r_3} = \ldots = u_{i, r_K} = 0$$

holds where $K$ is an integer larger than or equal to $i - 1$. This means that

$$|\text{Supp}(u_i) \cap \text{Supp}(p)| \leq |\text{Supp}(p)| - i + 1 \quad (42)$$

holds for $i \in \{1, \ldots, |Q|\}$. Let $\eta_i$ be

$$\eta_i = \sum_{l \in (\text{Supp}(u_i) \cap \text{Supp}(p)) \setminus \{r_1\}} p_l \quad (43)$$

From the inequality (42), it is evident that $\{\eta_1, \eta_2, \ldots\}$ is a decreasing sequence. We thus can expect that condition (22), i.e., $p_{r_1} > \eta_i$, eventually holds as $i$ grows. It may be reasonable to choose the smallest index $i$ satisfying (22) because such $u_i$ would be sparser in the case of a low density matrix. A sparse redundant row is advantageous since it is able to cut other fractional vertices with small weight.

V. RESULTS

A. Application to Hamming, Golay and LDPC codes

In this subsection, we applied the cutting plane method to Hamming code ($n = 7, m = 4$), Golay code ($n = 24, m = 12$), and a short regular LDPC code “204.33.484” [9] ($n = 204, m = 102$, row weight 6, column weight 3). We here use a parity check matrix of Golay code described in [12]. Let $d_{\text{frac}}$ be the fractional distance of the original parity check matrices and $d_{\text{frac}}^{\text{after}}$ be the fractional distance of parity check matrices generated by the cutting plane method. Let $N_d$ be the number of the appended redundant rows. The results are shown in Table I. For example, addition of 100 rows to the original parity check matrix of Golay code increases the fractional distance from 2.625 to 3.895. It is expected that these matrices constructed by the cutting plane method shows better LP decoding performance compared with the original matrices because $d_{\text{frac}}^{\text{after}}$ is greater than $d_{\text{frac}}$ for all the cases.

B. Decoding performances

In this subsection, we will present decoding performance of redundant parity check matrices obtained by the cutting plane method. We here assume BSC as a target channel and LP decoding [3] as a decoding algorithm used in a receiver. Figure 2 shows block error probabilities of the 24 × 12 parity check matrix of Golay code given in [12] (labeled “original”), the 24 × 52 parity check matrix with 40 redundant rows (labeled “+40rows”), and the 24 × 112 parity check matrix with 100 redundant rows (labeled “+100rows”).

From Figure 2 we can see that the block error probability of the redundant matrix (+100rows) is approximately two order of magnitude lower than that of the original matrix when a crossover probability is $10^{-2}$. Figure 3 shows block error probabilities of the original parity check matrix of “204.33.484” [9] and that the parity check matrix with 9 redundant rows (labeled “+9rows”). From Figure 3 it can be observed that the slope of the error curve of “+9rows” is steeper than that of “original”.

VI. CONCLUSION

In this paper, the cutting plane method based on redundant rows of a parity check matrix for improving the fractional distance has been presented. In order to reduce the search

![Table I: Fractional distances of redundant parity check matrices obtained by the cutting plane method](image)

![Figure 2: Comparison on block error probabilities of parity check matrices of Golay code (original 24 × 12 matrix, redundant 24 × 52 matrix, redundant 24 × 112 matrix)](image)
complexity to find an appropriate redundant row, we introduced an efficient technique to compute the fractional distance and proved that the limited search space indicated in Theorem 3 is sufficient to find a redundant row generating a cutting polytope. Some numerical results obtained so far are encouraging. The redundant parity check matrices constructed by the cutting plane method have larger fractional distance than that of the original matrices. The simulation results support that improvement on the fractional distance actually leads to better decoding performance under LP decoding.

APPENDIX

Proof 1 (Theorem 1): For proving Theorem 1, we will prove the following two inequalities:

\[ d_{\text{frac}}^{(\text{relax})}(H) \leq d_{\text{frac}}(H), \quad (44) \]
\[ d_{\text{frac}}^{(\text{relax})}(H) \geq d_{\text{frac}}(H). \quad (45) \]

We first assume \( d_{\text{frac}}^{(\text{relax})}(H) > d_{\text{frac}}(H) \) for proving inequality (44) by contradiction. Assume that the index \( i \in \{1, \ldots, M\} \) is given by \( i = \arg \min_{j=1}^{M} \delta_k \). Let \( p \) be the solution of \( LP_i \). This means that \( p \in \Gamma(H) \) and \( p \in (P(H) \cap F_i) \) where \( P(H) \cap F_i \) is the feasible set of \( LP_i \). Note that \( p \) is not the origin \( 0^n \) because \( \delta_k = \infty \) holds when \( p = 0^n \). Thus, \( \text{Const}_i \) is an inactive constraint, namely \( i \in F_{\text{inact}}(H) \). Since the fundamental corn \( K(H) \) contains the fundamental polytope \( P(H) \) as a subset, the relation \( p \in (K(H) \cap F_i) \) holds as well. The intersection \( K(H) \cap F_i \) is the feasible set of \( LP_i^{(\text{relax})} \). Let the solution of \( LP_i^{(\text{relax})} \) be \( p^* \). Since the objective function of \( LP_i \) is identical to that of \( LP_i^{(\text{relax})} \) and the feasible set of \( LP_i^{(\text{relax})} \) includes that of \( LP_i \), the \( \ell_1 \)-weight of \( p^* \) is smaller than or equal to the \( \ell_1 \)-weight of \( p \). This implies \( d_{\text{frac}}^{(\text{relax})}(H) \leq d_{\text{frac}}(H) \) but it contradicts the assumption. The proof of the inequality (44) is completed.

We next assume \( d_{\text{frac}}^{(\text{relax})}(H) < d_{\text{frac}}(H) \) for proving inequality (45) by contradiction. Assume that \( i \in \arg \min_{k \in F_{\text{inact}}(H)} \delta_k^{(\text{relax})} \). Let \( p^{(\text{relax})} \) be the solution of \( LP_i^{(\text{relax})} \). From the definition of \( LP_i^{(\text{relax})} \), it is evident that \( p^{(\text{relax})} \in (K(H) \cap F_i) \) holds. In the following, we will discuss the two cases: (i) \( p^{(\text{relax})} \in P(H) \), (ii) \( p^{(\text{relax})} \notin P(H) \). We start from case (i). If \( p^{(\text{relax})} \in P(H) \) holds, then \( p^{(\text{relax})} \in (P(H) \cap F_i) \) holds because of the relation \( P(H) \subset K(H) \). Due to almost the same argument, we obtain \( d_{\text{frac}}^{(\text{relax})}(H) = d_{\text{frac}}(H) \) which contradicts the assumption. We then move to case (ii). If \( p^{(\text{relax})} \notin P(H) \) holds, there exists \( \text{Const}_i \) satisfying\( p^{(\text{relax})} \notin \mathcal{H}_i, \quad 0^l \in F_{\text{inact}}(H). \quad (46) \)

This is because \( p^{(\text{relax})} \in K(H) \) but \( p^{(\text{relax})} \notin P(H) \). Let \( \text{Seg}(p^{(\text{relax})}) \) be the line segment between \( p^{(\text{relax})} \) and the origin:

\[ \text{Seg}(p^{(\text{relax})}) = \{ p \in R^n : 0 < t < 1, \quad p = tp^{(\text{relax})} \}. \quad (47) \]

The line segment \( \text{Seg}(p^{(\text{relax})}) \) passes through \( F_i \). Thus, there exists the point \( p_i \in (F_i \cap \text{Seg}(p^{(\text{relax})})) \). Note that the line segment \( \text{Seg}(p_i) \) is totally included in \( K(H) \) because \( p_i \in K(H) \). This leads to \( p_i \in (K(H) \cap F_i) \) and it means that \( p_i \) is included in the feasible set of \( LP_i^{(\text{relax})} \). The \( \ell_1 \)-weight of \( p_i \) is smaller than that of \( p^{(\text{relax})} \). This implies that the \( \ell_1 \)-weight of \( p_i \) is smaller than \( d_{\text{frac}}^{(\text{relax})}(H) \). However, it contradicts the definition of \( d_{\text{frac}}^{(\text{relax})}(H) \). The proof of the inequality (45) is completed.

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