The effect of measurement noise on the typical quantum teleportation protocol

Zhang Zhaolei¹, Ma Zhiqiang², Shi Lei²*

¹Air Traffic Control and Navigation College, Air Force Engineering University, 710076, China
²Information and Navigation College, Air Force Engineering University, 710076, China
*Email: youxiang_wangyi@126.com

Abstract—Quantum teleportation is an important research direction of quantum communication. Quantum measurement is one of necessary physical implementation for quantum teleportation. Thus, this paper explored the influence of measurement noise on quantum teleportation. The realization steps of quantum teleportation with measurement noise are presented. From the density operation and fidelity value, the performance of quantum teleportation is analyzed. The research results about measurement noise might be utilized for quantum communication.

1. INTRODUCTION
Quantum entanglement is an intriguing property of composite quantum systems[1], and can be used to perform some quantum communication tasks, for example, quantum teleportation[2], dense coding[3], remote state preparation[4], and so on. Quantum teleportation is a feasible technique for transmit quantum states from senders to remote receivers via quantum entanglement and classical communication. The first theoretical scheme of quantum teleportation was presented by Bennett et al. [2], and this protocol is experimentally proved by Bouwmeester et al. [5].

Due to its potential applications of quantum information, quantum teleportation has been studied[6-11]. Rieber et al[6] reported deterministic quantum-state teleportation between a pair of trapped calcium ions. Bowen et al[7] report the experimental demonstration of quantum teleportation of the quadrature amplitudes of a light field. Li et al[8] presented two different quantum channels of two three-qubit GHZ states and the six-qubit entangled state can be used for quantum teleportation of an arbitrary two-qubit. Podoshvedov et al[9] considered the implementation of a quantum teleportation protocol of an unknown qubit with an entangled hybrid state. Wei et al[10] presented an efficient protocol for probabilistic teleportation is presented with multi-parameter measurements via a non-maximally entangled state. Nevertheless, there are also some important and open subjects to be investigated in the quantum teleportation.

Quantum measurement is one of necessary physical implementation for quantum teleportation. However, quantum measurement would be influenced by human beings and instruments inevitably for quantum teleportation. Thus, this paper explored the influence of measurement noise on quantum teleportation. The detailed realization procedures of quantum teleportation with measurement noise are elaborated. Additionally, the performance of quantum teleportation is analyzed based on the density

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operation and fidelity value. The results of this paper about measurement noise might be useful for quantum communication.

The rest of this paper are organized as follows: In Section II, the original quantum teleportation scheme would be presented. And then, the implementation procedures of quantum teleportation with measurement noise are presented in Section III. The performance of quantum teleportation will be discussed from the density operation and fidelity value in Section IV. The paper concludes with Section V.

2. TYPICAL QUANTUM TELEPORTATION SCHEME

The brief statement for original quantum teleportation scheme would be presented in this section. Suppose that the sender Alice want to transmit an unknown single-qubit quantum state to the receiver Bob via quantum channel. Without loss of generality, the unknown state transmitted from Alice to Bob can be shown as follow

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]  

(1)

here \( \alpha \) is a real number, and \( \beta \) is complex. For the teleportation of single-qubit state, the two-qubit entanglement state can be used as quantum channel

\[ |\text{Bell}^{\alpha}\rangle_{23} = \frac{\sqrt{2}}{2} (|00\rangle + |11\rangle)_{23} \]  

(2)

Qubits 2 and 3 are belong to Alice and Bob, respectively. Thus, the state of total system can be obtained as follow

\[ |\psi\rangle_{\text{total}} = |\psi\rangle \otimes |\text{Bell}^{\alpha}\rangle_{23} = \frac{\sqrt{2}}{2} (\alpha |0\rangle + \beta |1\rangle) \otimes (|00\rangle + |11\rangle)_{23} \]  

(3)

It is noting that Alice has qubits 1 and 2, and Bob has qubit 3. For transmitting the unknown quantum state, Alice need to perform the Bell-state measurement \( \{ |\text{Bell}^{i}\rangle |i, j = 1, 2\} \)

\[ |\text{Bell}^{i}\rangle = \frac{\sqrt{2}}{2} (|00\rangle + |11\rangle) \]  

\[ |\text{Bell}^{il}\rangle = \frac{\sqrt{2}}{2} (|00\rangle - |11\rangle) \]  

\[ |\text{Bell}^{ii}\rangle = \frac{\sqrt{2}}{2} (|01\rangle + |10\rangle) \]  

\[ |\text{Bell}^{ii}\rangle = \frac{\sqrt{2}}{2} (|01\rangle - |10\rangle) \]  

(4)

According to the Bell-state measurement, the total state can be rewritten as follow
\[ |\psi\rangle_{\text{total}} = \frac{\sqrt{2}}{2} (\alpha |0\rangle + \beta |1\rangle) \otimes (|00\rangle + |11\rangle) \]
\[ = \frac{1}{2} \left[ \text{Bell}^{00}_{12} \otimes (\alpha |0\rangle + \beta |1\rangle) \right] + \frac{1}{2} \left[ \text{Bell}^{01}_{12} \otimes (\alpha |0\rangle - \beta |1\rangle) \right] + \frac{1}{2} \left[ \text{Bell}^{10}_{12} \otimes (\alpha |1\rangle + \beta |0\rangle) \right] + \frac{1}{2} \left[ \text{Bell}^{11}_{12} \otimes (\alpha |1\rangle - \beta |0\rangle) \right] \]  

After the measurement performed by Alice, the state of qubit 3 held by Bob would be collapse into
\[ \left[ \text{Bell}^{00}_{12} \right] |\psi\rangle_{\text{total}} = \frac{1}{2} (\alpha |0\rangle + \beta |1\rangle) \]
\[ \left[ \text{Bell}^{01}_{12} \right] |\psi\rangle_{\text{total}} = \frac{1}{2} (\alpha |0\rangle - \beta |1\rangle) \]
\[ \left[ \text{Bell}^{10}_{12} \right] |\psi\rangle_{\text{total}} = \frac{1}{2} (\alpha |1\rangle + \beta |0\rangle) \]
\[ \left[ \text{Bell}^{11}_{12} \right] |\psi\rangle_{\text{total}} = \frac{1}{2} (\alpha |1\rangle - \beta |0\rangle) \]  

In order to realize the teleportation, Alice would inform Bob of her measurement result via classical channel. Based on these results, Bob can perform corresponding unitary operations to reconstruct the original quantum state shown as Eq. (1). The special unitary transformations relative with measurement outcomes can be obtained from Table 1. These unitary transformations on Table 1 could be presented as follow
\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  

| Alice’s results | Classical information | Bob’s gate |
|-----------------|----------------------|------------|
| Bell^{00} \rangle | 00                   | I          |
| Bell^{01} \rangle | 01                   | \sigma_x   |
| Bell^{10} \rangle | 10                   | \sigma_y   |
| Bell^{11} \rangle | 11                   | i\sigma_z  |

Based on classical channel and quantum entanglement, we can use quantum teleportation to transmit unknown quantum states via some special unitary transformations. It is noting that quantum measurement is one of essential operations for quantum teleportation.

3. TELEPORTATION WITH MEASUREMENT NOISE

From the above discussion of Section II, it could be obtained that the Bell-state measurement is the first step of physical realization for original quantum teleportation. In the real system, some human beings and instruments could inevitably affect the physical implementation of quantum measurement. Therefore, the performance of quantum measurement would influence the teleportation. The concrete processes for quantum teleportation with measurement noise would be elaborated in detail.
For the sake of the realization of the original teleportation scheme, the sender Alice need to perform the Bell-state measurement shown as Eq. (4). Suppose that the measurement with noise could be presented as follow

\[
\begin{align*}
    |\text{Bell}^{0}\rangle &= \cos \frac{\theta_1}{2} |00\rangle + e^{i\phi_1} \sin \frac{\theta_1}{2} |11\rangle \\
    |\text{Bell}^{1}\rangle &= e^{-i\phi_1} \sin \frac{\theta_1}{2} |00\rangle - \cos \frac{\theta_1}{2} |11\rangle \\
    |\text{Bell}^{0}\rangle &= \cos \frac{\theta_2}{2} |01\rangle + e^{i\phi_2} \sin \frac{\theta_2}{2} |10\rangle \\
    |\text{Bell}^{1}\rangle &= e^{-i\phi_2} \sin \frac{\theta_2}{2} |01\rangle - \cos \frac{\theta_2}{2} |10\rangle
\end{align*}
\] (8)

The factors $\theta_1$ and $\theta_2$ can be considered as the amplitude noise parameters, and $\phi_1$ and $\phi_2$ are in the value region $[0,\pi]$. If and only if $\theta_1=\theta_2=\pi/2$ and $\phi_1=\phi_2=0$, the measurement operations $|\text{Bell}^{m}\rangle_{m,n=1,2}$ will equal to the Bell-state measurement $|\text{Bell}^{m}\rangle_{i,j=1,2}$ in Eq. (4). In terms of Eq. (8), the total system state could be presented as follow

\[
\begin{align*}
|\psi\rangle_{\text{total}} &= \frac{\sqrt{2}}{2} (\alpha |0\rangle_i + \beta |1\rangle_i) \otimes (|00\rangle + |11\rangle)_{3} \\
&= \frac{\sqrt{2}}{2} |\text{Bell}^{0}\rangle_{12} \otimes \left( \alpha \cos \frac{\theta_1}{2} |0\rangle + e^{i\phi_1} \sin \frac{\theta_1}{2} |1\rangle \right)_{3} \\
&+ \frac{\sqrt{2}}{2} |\text{Bell}^{1}\rangle_{12} \otimes \left( \alpha e^{-i\phi_1} \sin \frac{\theta_1}{2} |0\rangle - \cos \frac{\theta_1}{2} |1\rangle \right)_{3} \\
&+ \frac{\sqrt{2}}{2} |\text{Bell}^{0}\rangle_{12} \otimes \left( \alpha \cos \frac{\theta_2}{2} |1\rangle + e^{i\phi_2} \sin \frac{\theta_2}{2} |0\rangle \right)_{3} \\
&+ \frac{\sqrt{2}}{2} |\text{Bell}^{1}\rangle_{12} \otimes \left( \alpha e^{-i\phi_2} \sin \frac{\theta_2}{2} |1\rangle - \cos \frac{\theta_2}{2} |0\rangle \right)_{3}
\end{align*}
\] (9)

Based on the above measurement states in Eq. (9), qubit 3 hold by the receiver Bob will become to

\[
\begin{align*}
|\psi\rangle_{\text{total}} &= \frac{\sqrt{2}}{2} (\alpha \cos \frac{\theta_1}{2} |0\rangle_i + \beta e^{i\phi_1} \sin \frac{\theta_1}{2} |1\rangle_i) \\
|\psi\rangle_{\text{total}} &= \frac{\sqrt{2}}{2} (\alpha e^{-i\phi_1} \sin \frac{\theta_1}{2} |0\rangle_i - \cos \frac{\theta_1}{2} |1\rangle_i) \\
|\psi\rangle_{\text{total}} &= \frac{\sqrt{2}}{2} (\alpha \cos \frac{\theta_2}{2} |1\rangle_i + \beta e^{i\phi_2} \sin \frac{\theta_2}{2} |0\rangle_i) \\
|\psi\rangle_{\text{total}} &= \frac{\sqrt{2}}{2} (\alpha e^{-i\phi_2} \sin \frac{\theta_2}{2} |1\rangle_i - \cos \frac{\theta_2}{2} |0\rangle_i)
\end{align*}
\] (10)

After the measurement, Alice would send the measurement results to Bob with classical communication, and then Bob will perform the relative unitary gates on qubit 3. The relationship between the after-measurement state of qubit 3 and the measurement results of qubit 1 and 2 could be found from Table 2. The probability parameters could be presented as follow
\[ p_i = \frac{1}{2} \left( \alpha^2 \cos^2 \theta_i + |\beta|^2 \sin^2 \theta_i + |\beta| \alpha \cos \frac{\theta_i}{2} \right) \]
\[ p_i = \frac{1}{2} \left( \alpha^2 \sin^2 \theta_i + |\beta|^2 \cos^2 \theta_i + |\beta| \alpha \cos \frac{\theta_i}{2} \right) \]

(11)

Table 2 The relationship between the after-measurement state of qubit 3 and the measurement results of qubits 1 and 2

| Case | The measurement results of qubits 1 and 2 | Classical information | The after-measurement state of qubit 3 | Probability | Gate |
|------|-----------------------------------------|----------------------|--------------------------------------|-------------|------|
| 1    | Bell 00                                 | 00                   | \[|\psi\rangle_i = \frac{\sqrt{2}}{2} \left( \alpha \cos \frac{\theta_1}{2} |0\rangle + \beta e^{i\phi_1} \sin \frac{\theta_1}{2} |1\rangle \right)\] | \[p_i\] | \[I\] |
| 2    | Bell 01                                 | 01                   | \[|\psi\rangle_i = \frac{\sqrt{2}}{2} \left( \alpha e^{-i\phi_1} \sin \frac{\theta_1}{2} |0\rangle - \beta \cos \frac{\theta_1}{2} |1\rangle \right)\] | \[p_i\] | \[\sigma_z\] |
| 3    | Bell 10                                 | 10                   | \[|\psi\rangle_i = \frac{\sqrt{2}}{2} \left( \alpha \cos \frac{\theta_1}{2} |1\rangle + \beta e^{i\phi_1} \sin \frac{\theta_1}{2} |0\rangle \right)\] | \[p_i\] | \[\sigma_x\] |
| 4    | Bell 11                                 | 11                   | \[|\psi\rangle_i = \frac{\sqrt{2}}{2} \left( \alpha e^{-i\phi_1} \sin \frac{\theta_1}{2} |1\rangle - \beta \cos \frac{\theta_1}{2} |0\rangle \right)\] | \[p_i\] | \[i\sigma_y\] |

4. THE TELEPORTATION PERFORMANCE WITH MEASUREMENT NOISE

The density operation language provides an efficient method to describe quantum system. According to the above table, qubit 3 is on one of states \{"|\psi\rangle_1, \sigma_z, |\psi\rangle_2, \sigma_x, |\psi\rangle_3, i\sigma_y, |\psi\rangle_4\}\ with the respective probabilities \{"p_1, p_2, p_3, p_4\}\. Thus, the density operation of qubit 3 after measurement could be defined by the equation

\[
\rho = p_1 |\psi\rangle_1 \langle \psi| + p_2 (|\sigma_z\rangle \langle \sigma_z|) (|\psi\rangle_1 \langle \psi|)
+ p_3 (|\sigma_x\rangle \langle \sigma_x|) (|\psi\rangle_1 \langle \psi|)
+ p_4 (i|\sigma_y\rangle \langle \sigma_y|) (|\psi\rangle_1 \langle \psi|) \]

(12)

From Table 2 and Eq. (11), the after-measurement states and probability parameters can be obtained. If and only if \(\theta_1 = \theta_2 = \pi/2\) and \(\phi_1 = \phi_2 = 0\), this density operation in Eq. (11) equal to the corresponding operation for the original state shown as Eq. (1).

The fidelity could be used to measure the distance of different quantum system. The fidelity between the after-measurement states \(\rho\) and the original state \(|\psi\rangle\) could be presented as follow

\[
F(\rho, |\psi\rangle) = tr \sqrt{\rho \sqrt{|\psi\rangle \langle \psi|} \rho} = \sqrt{\rho \rho |\psi\rangle \langle \psi|} = \sqrt{\rho \rho |\psi\rangle \langle \psi|} = \sqrt{\rho \rho |\psi\rangle \langle \psi|} = \sqrt{\rho \rho |\psi\rangle \langle \psi|} \]

(13)

It should be emphasized that this fidelity on Eq. (13) is relative without the phase noise parameters \(\phi_1\) and \(\phi_2\). If and only if \(\theta_1 = \theta_2 = \pi/2\), the fidelity reach up to one, which is the biggest value. This result about fidelity is the same as the original quantum teleportation protocol.
5. CONCLUSION
Quantum communication has lots of research directions, one of which is quantum teleportation. It is noting that quantum measurement is one of necessary physical implementation for quantum teleportation. In this paper, we explored the effect of measurement noise on quantum teleportation. The implementation procedures of quantum teleportation with measurement noise are presented in detail. According to the density operation and fidelity value, the performance of quantum teleportation is analyzed. The results about measurement noise on quantum teleportation might be useful for quantum communication.

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