Some spectral bounds for Schrödinger operators with Hardy-type potentials

Douglas Lundholm*

Department of Mathematics, Royal Institute of Technology
SE-100 44 Stockholm, Sweden

Abstract

This note points out some bounds for the number of negative eigenvalues of Schrödinger operators with Hardy-type potentials, which follow from a simple coordinate transformation, and could prove useful in a spectral analysis of certain supersymmetric quantum mechanical models.

1 Introduction

In a recent approach [1] to study the spectrum of a class of quantum mechanical models, called supersymmetric matrix models and described by matrix-valued Schrödinger operators (see e.g. [2, 3]), it is relevant to consider the negative spectrum of Schrödinger operators with critical Hardy terms, i.e. operators of the form

\[ H = -\Delta_{\mathbb{R}^d} - \frac{(d-2)^2}{4|x|^2} + V(x), \]

where \( V \) is a real- or operator-valued potential. This approach has so far only been applied to a simplified model, where a bound for the number of negative eigenvalues of a one-dimensional Schrödinger operator with Hardy term, following from a simple coordinate transformation, turned out to be very important. The aim of this note is to extend this transformation to higher dimensions and derive corresponding bounds which could be useful in an extension of the technique to the higher-dimensional matrix models.

*e-mail: dogge@math.kth.se

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It also allows for generalizations of some statements in [4, 5, 6, 7] regarding the one-dimensional, and higher-dimensional, operators. After searching the literature, we found that the transformation we use and some of its consequences have been considered before (see e.g. [8, 9] for the one-dimensional case, and [10] for higher dimensions), however, we are not aware of any reference stating these explicit bounds. In Section 2 we recall some Hardy-type inequalities, while the essential coordinate transformation is considered in Section 3, and the bounds for the negative eigenvalues are stated and proved in Section 4.

2 Some Hardy-type inequalities

In the following we will denote by $\bar{B}_r(x)$ the closed ball of radius $r \geq 0$ at $x \in \mathbb{R}^d$. We also use the conventions $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$.

For $x \in \mathbb{R}^d \setminus \{0\}$, $d = 1, 2, 3, \ldots$, let

$$
\Psi_d(x) := |x|^{-(d-2)}, \quad \Phi(x) := \ln |x|.
$$

These are the fundamental solutions of the Laplace operator on $\mathbb{R}^d \neq 2$ and $\mathbb{R}^2$, respectively, since in the sense of distributions

$$
-\Delta_{\mathbb{R}^d} \Psi_d(x) = c_d \delta(x), \quad \text{and} \quad -\Delta_{\mathbb{R}^2} \Phi(x) = c_2 \delta(x),
$$

for some constants $c_d$ and $c_2$. By considering the square root of these functions, we can prove the following Hardy-type inequalities.

**Proposition 1.** We have

$$
- \frac{(d-2)^2}{4|x|^2} \geq 0, \quad (2)
$$

considered as a quadratic form on $C_0^\infty(\mathbb{R}^d \setminus \{0\})$, and

$$
- \frac{(d-2)^2}{4|x|^2} - \frac{1}{4|x|^2} \ln |x| \geq 0, \quad (3)
$$

considered as a quadratic form on $C_0^\infty(\mathbb{R}^d \setminus \bar{B}_1(0))$. In other words,

$$
\frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^d} |\nabla u|^2 \, dx, \quad (4)
$$

(the standard Hardy inequality in $L^2(\mathbb{R}^d)$) for all $u \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$, and

$$
\frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \, dx + \frac{1}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2 \ln |x|} \, dx \leq \int_{\mathbb{R}^d} |\nabla u|^2 \, dx, \quad (5)
$$

for all $u \in C_0^\infty(\mathbb{R}^d \setminus \bar{B}_1(0))$. 2
Proof. Let us consider the first inequality (2). It is straightforward to check that
\[ \nabla \ln \Psi_d(x) \frac{1}{2} = \frac{1}{2} \Psi_d(x)^{-1}(\nabla \Psi_d(x)) = -\frac{d-2}{2} \frac{x}{|x|^2}, \]
and
\[ \Delta \ln \Psi_d(x) \frac{1}{2} = -(d-2)^2 \frac{1}{2|x|^2}. \]
Now, define the vector-valued operator
\[ Q := \nabla + \nabla \ln \Psi_d(x) \frac{1}{2} = \nabla + \frac{1}{2} \Psi_d(x)^{-1} \nabla \Psi_d(x). \]
Then, considered in the sense of quadratic forms on \( C_0^\infty(\mathbb{R}^d \setminus \{0\}) \), we have
\[ 0 \leq Q \cdot Q^\dagger = (\nabla + \nabla \ln \Psi_d(x) \frac{1}{2}) \cdot (-\nabla + \nabla \ln \Psi_d(x) \frac{1}{2}) \]
\[ = -\nabla \cdot \nabla + \nabla \cdot \nabla \ln \Psi_d(x) \frac{1}{2} - \nabla \ln \Psi_d(x) \frac{1}{2} \cdot \nabla + |\nabla \ln \Psi_d(x) \frac{1}{2}|^2 \]
\[ = -\Delta + \Delta \ln \Psi_d(x) \frac{1}{2} + |\nabla \ln \Psi_d(x) \frac{1}{2}|^2 \]
\[ = -\Delta - (d-2)^2 \frac{1}{2|x|^2} + \frac{(d-2)^2}{4|x|^2}, \]
which gives (3).

For the second inequality (3), we observe that
\[ \nabla \ln \Phi(x) \frac{1}{2} = \frac{1}{2} \Phi(x)^{-1}(\nabla \Phi(x)) = \frac{1}{2(\ln |x|)} \frac{x}{|x|^2}, \]
and
\[ \Delta \ln \Phi(x) \frac{1}{2} = \nabla \cdot \frac{1}{2(\ln |x|)} \frac{x}{|x|^2} = -\frac{1}{2|\ln |x||^2} + \frac{d-2}{2|\ln |x||^2} + \frac{d-2}{2|x|^2 \ln |x|}. \]
Hence, defining
\[ \tilde{Q} := \nabla + \nabla \ln(\Psi_d(x) \Phi(x)) \frac{1}{2} = \nabla + \frac{1}{2} \Psi_d(x)^{-1} \nabla \Psi_d(x) + \frac{1}{2} \Phi(x)^{-1} \nabla \Phi(x), \]
we obtain, in the sense of quadratic forms on \( C_0^\infty(\mathbb{R}^d \setminus \bar{B}_1(0)) \),
\[ 0 \leq \tilde{Q} \cdot \tilde{Q}^\dagger \]
\[ = (\nabla + \nabla \ln(\Psi_d(x)^{1/2} + \nabla \ln \Phi(x)^{1/2}) \cdot (-\nabla + \nabla \ln(\Psi_d(x)^{1/2} + \nabla \ln \Phi(x)^{1/2}) \]
\[ = -\Delta + \Delta \ln(\Psi_d(x)^{1/2} + \nabla \ln \Phi(x)^{1/2} + 2 \nabla \ln \Psi_d(x)^{1/2} \cdot \nabla \ln \Phi(x)^{1/2} \]
\[ + |\nabla \ln \Psi_d(x)^{1/2}|^2 + |\nabla \ln \Phi(x)^{1/2}|^2 \]
\[ = -\Delta - \frac{(d-2)^2}{2|x|^2} - \frac{1}{2|\ln |x||^2} + \frac{d-2}{2|x|^2 \ln |x|} - \frac{2}{2|x|^2} \frac{1}{2|x| \ln |x|} \]
\[ + \frac{(d-2)^2}{4|x|^2} + \frac{1}{4(\ln |x|)^2|x|^2}, \]
which proves (3). \qed
Remark. Note that if $\tilde{Q}$ is considered as taking values in the grade-one part of $G(\mathbb{R}^d)$, the Clifford algebra over $\mathbb{R}^d$, (hence a Dirac-type operator) then also the Clifford product $\tilde{Q}\tilde{Q}^\dagger = Q\cdot Q^\dagger + \tilde{Q}\wedge \tilde{Q}^\dagger$ (decomposed in terms of inner and outer products) is a non-negative operator on e.g. $C_0^\infty(\mathbb{R}^d \setminus B_1(0)) \otimes \mathcal{S}$, where $\mathcal{S}$ denotes a representation of $G(\mathbb{R}^d)$.

3 Transformation of quadratic forms

Combining the so-called ground state representation of the operator $\Pi$ (which is implicitly used in Proposition $\Pi$), with a coordinate transformation, we can relate a Schrödinger operator with a Hardy term defined on a domain in $\mathbb{R}^d$ to a corresponding operator without the term on a transformed domain. More precisely, denoting $B^c_R := \mathbb{R}^d \setminus \bar{B}_R(0)$, for $R \geq 0$, and for $R < 0$ generalizing this to denote the cone parametrized by $(r, \omega) \in (R, \infty) \times S^{d-1}$ (for $d = 1$ we write $B^c_R := (R, \infty)$), we have the following simple result.

Lemma 2. For any $u \in C_0^\infty(B^c_R)$, $R \geq 0$ for $d$ odd, $R \geq 1$ for $d$ even, we have

$$\langle u, \left( -\Delta_{\mathbb{R}^d} - \frac{(d - 2)^2}{4|x|^2} + V(x) \right) u \rangle_{L^2(B^c_R)} = \langle \psi, \left( -\Delta_{\mathbb{R}^d} - \frac{1}{|\mathbf{x}|^2} \right) \Delta_{S^{d-1}} - \frac{(d - 1)(d - 3)}{4|\mathbf{x}|^2} + e^{2V(e^r \omega)} \rangle_{L^2(B^c_{\ln R})},$$

where $\psi(r, \omega) := r^{-\frac{d+2}{2}} e^{\frac{d-2}{2} r} u(e^r \omega)$, and $(r, \omega) \in (\ln R, \infty) \times S^{d-1}$.

Proof. In spherical coordinates, the l.h.s., denote it $I$, is

$$I = \int_{\mathbb{R}} \int_{\omega \in S^{d-1}} \frac{u(r, \omega)}{u(r, \omega)} \left( -\frac{1}{r^{d-1}} \partial_r r^{d-1} \partial_r - \frac{1}{r^2} \Delta_\omega - \frac{(d - 2)^2}{4r^2} + V(r, \omega) \right) u(r, \omega) r^{d-1} dr d\omega.$$

First, put $u(x) := \Psi_d(x) \frac{1}{r} v(x)$, i.e $v(r, \omega) := r^{\frac{d-2}{2}} u(r, \omega)$, and arrive via partial integration at the ground state representation,

$$I = \int_{\mathbb{R}} \int_{\omega \in S^{d-1}} \left( \left| \partial_r u(r, \omega) \right|^2 - \frac{(d - 2)^2}{4r^2} |u|^2 + \overline{\nu} \left( -\frac{1}{r^2} \Delta_\omega + V(r, \omega) \right) u \right) r^{d-1} dr d\omega.$$

Because of the form of the integral measure here, this expression actually possesses two-dimensional features, which explains why the function $\Phi$ enters in the proof of Proposition $\Pi$. Next, change variables, $r = e^s$, $dr = r ds$, $w(s, \omega) := v(e^s, \omega)$, which in a sense lowers the dimension by one:

$$I = \int_{\ln R}^{\infty} \int_{S^{d-1}} \left( |\partial_s w(s, \omega)|^2 + \overline{\nu} \left( -\Delta_\omega + e^{2s} V(e^s \omega) \right) w \right) ds d\omega$$

$$= \langle w, \left( -\partial_s^2 - \Delta_{S^{d-1}} + e^{2s} V(e^s \omega) \right) w \rangle_{L^2((\ln R, \infty)) \otimes L^2(S^{d-1})}. \quad (6)$$
Finally, transform back from this corresponding ground state representation, by taking \( \psi(s, \omega) := s^{-\frac{d-1}{2}} w(s, \omega) \), resulting in

\[
I = \int_{\ln R}^{\infty} \int_{S^{d-1}} \overline{\psi} \left( -\frac{1}{s^{d-1}} \partial_s s^{d-1} \partial_s - \frac{(d-1)(d-3)}{4s^2} - \Delta_q + e^{2s}V(q) \right) \psi \ s^{d-1} ds d\omega
= \left\langle \psi, \left( -\Delta_R^q + \frac{1}{s^{d-1}} \Delta_q - \frac{(d-1)(d-3)}{4s^2} + e^{2s}V(q) \right) \psi \right\rangle_{L^2(B_{\ln R}^\infty)} ,
\]

which is the r.h.s. of the claimed identity. \( \square \)

In particular, we have the following special cases and consequences.

**Proposition 3.** Consider \( d = 1 \). For all \( u \in C_0^\infty(\mathbb{R}_+) \), we have

\[
\left\langle u, \left( -\frac{d^2}{dx^2} - \frac{1}{4x^2} + V(x) \right) u \right\rangle_{L^2(\mathbb{R}_+)} = \left\langle \psi, \left( -\frac{d^2}{dx^2} + e^{2x} V(e^x) \right) \psi \right\rangle_{L^2(\mathbb{R})} .
\]

Furthermore, if \( V(x) = -\frac{1}{4x^2} \psi(x) + W(x) \) then we have for all \( u \in C_0^\infty((1, \infty)) \)

\[
\left\langle u, \left( -\frac{d^2}{dx^2} - \frac{1}{4x^2} - \frac{1}{4x^2 (\ln x)^2} + W(x) \right) u \right\rangle_{L^2((1, \infty))} = \left\langle \phi, \left( -\frac{d^2}{dx^2} + e^{2x} e^{2x} W(e^{2x}) \right) \phi \right\rangle_{L^2(\mathbb{R})} ,
\]

with \( \phi(x) = e^{-x/2} \psi(x) = e^{-x/2} e^{-e^x/2} u(e^{e^x}) \). This procedure can be iterated further to the interval \( e, \infty \), and so on.

**Proposition 4.** Consider \( d = 2 \), with polar coordinates \((r, \varphi)\). For all \( u \in C_0^\infty(B_1^r) \), we have

\[
\left\langle u, \left( -\Delta_{\mathbb{R}^2} + V(x) \right) u \right\rangle_{L^2(B_1^r)} = \left\langle \psi, \left( -\Delta_{\mathbb{R}^2} - (1 - r^{-2}) \frac{d^2}{dr^2} + \frac{1}{4r^2} + e^{2r} V(e^r, \varphi) \right) \psi \right\rangle_{L^2(\mathbb{R}^2)} ,
\]

so that, with \( V(x) = -\frac{1}{4|x|^2 (\ln |x|)^2} + W(x) \) we have for all \( u \in C_0^\infty(B_1^r) \)

\[
\left\langle u, \left( -\Delta_{\mathbb{R}^2} - \frac{1}{4r^2 (\ln r)^2} + W(x) \right) u \right\rangle_{L^2(B_1^r)} = \left\langle \psi, \left( -\Delta_{\mathbb{R}^2} - (1 - r^{-2}) \frac{d^2}{dr^2} + e^{2r} W(e^r, \varphi) \right) \psi \right\rangle_{L^2(\mathbb{R}^2)} .
\]

**Proposition 5.** For general \( d = 1, 2, 3, \ldots \), we have

\[
\left\langle u, \left( -\Delta_{\mathbb{R}^d} - \frac{(d-2)^2}{4|x|^2} - \frac{1}{4|x|^2 (\ln |x|)^2} + V(x) \right) u \right\rangle_{L^2(B_1^r)} = \left\langle \psi, \left( -\Delta_{\mathbb{R}^d} - \left( 1 - \frac{1}{|x|^2} \right) \Delta_{S^{d-1}} - \frac{(d-2)^2}{4|x|^2} + e^{2r} V(e^r, \omega) \right) \psi \right\rangle_{L^2(\mathbb{R}^d)}.
\]
for all \( u \in C_0^\infty(B_1^c) \), where \( \psi(r\omega) = r^{-\frac{d-1}{2}} e^{\frac{d-3}{2} r^2} u(e^r \omega) \).

**Proof.** This follows immediately from Lemma 2 because \((d-1)(d-3) + 1 = (d-2)^2\).

**Remark.** The above transformations all extend to the case when \( V \) is operator-valued (cp. [11]).

In the following, denote

\[
\ln^{(n)} x := \underbrace{\ln \circ \ln \circ \ldots \circ \ln}_n(x) \quad \text{and} \quad \exp^{(n)} x := \underbrace{\exp \circ \exp \circ \ldots \circ \exp}_n(x).
\]

Then we also obtain by iteration of Lemma 2 the following generalization of Proposition 1:

**Proposition 6.** For general \( d = 1, 2, 3, \ldots \), we have

\[
-\Delta_{\mathbb{R}^d} - \frac{(d-2)^2}{4|x|^2} - \frac{1}{4|x|^2 (\ln |x|)^2} - \ldots - \frac{1}{4|x|^2 (\ln |x|)^2 \ldots (\ln^{(n)} |x|)^2} \geq 0
\]

in the sense of quadratic forms on \( C_0^\infty \left( B_0^{c_{\exp(n)}(0)} \right) \).

### 4 Bounds for the number of negative eigenvalues

Denote by \( N(A) \) the rank of the spectral projection on \((-\infty, 0)\) of a self-adjoint operator \( A \), and by \( V_\pm \) the positive/negative parts of a function \( V \).

In the one-dimensional case we have the following (cp. e.g. Proposition 3.2 in [4] and Theorem 9 in Chapter 8 of [9]):

**Theorem 7.** Let \( n \in \mathbb{N} \), and \( V \) be a real-valued potential such that \( x^2 (\ln x)^2 \ldots (\ln^{(n)} x)^2 V(x) \) is bounded from below. Then the self-adjoint operator

\[
H_{1,n} := -\frac{d^2}{dx^2} - \frac{1}{4x^2} - \ldots - \frac{1}{4x^2 (\ln x)^2 \ldots (\ln^{(n)} x)^2} + V(x),
\]

defined by Friedrichs extension on \( C_0^\infty ((\exp(n) 0, \infty)) \), has at least one negative eigenvalue for all negative (non-zero) potentials \( V \). Furthermore, the number of negative eigenvalues is bounded by

\[
N(H_{1,n}) \leq 1 + \int_{\exp(n) 0}^{\infty} |V(x)| |\ln x| \ldots |\ln^{(n+1)} x| \, dx.
\]

On the other hand, \( H_{1,n} \) defined by Friedrichs extension on \( C_0^\infty ((\exp(n) 1, \infty)) \) satisfies the bound

\[
N(H_{1,n}) \leq \int_{\exp(n) 1}^{\infty} |V(x)| |\ln x| \ldots |\ln^{(n+1)} x| \, dx.
\]
Remark. For a different version of the latter bound (for \( n = 0 \)) in the case of operator-valued potentials, and an application, see [1].

proof. We will use that Bargmann’s bound in three dimensions, together with Dirichlet boundary conditions, implies (see e.g. [12])

\[
N \left( -\frac{d^2}{dx^2} |_\mathbb{R} + V(x) \right) \leq 1 + \int_{-\infty}^{\infty} |V(x) - |x|| \, dx.
\]

By Proposition 3 we have for any \( u \in C^\infty_0((\exp(n), 0, \infty)) \)

\[
\langle u, H_{1,n} u \rangle_{L^2((\exp(n), 0, \infty))} = \left\langle \phi, \left( -\frac{d^2}{dx^2} |_\mathbb{R} + e^{2x} \ldots e^{2\exp(n)} x V(\exp(n+1)x) \right) \phi \right\rangle_{L^2(\mathbb{R})},
\]

for some \( \phi \in C^\infty_0(\mathbb{R}) \). From this expression one immediately obtains the first statement of the theorem by relating to the case for a one-dimensional Schrödinger operator. Furthermore, linearly independent sets of such functions \( u \) correspond to linearly independent sets of \( \phi \). Hence, since \( N(H_{1,n}) \) is equal to the maximal dimension of a subspace of functions \( u \in C^\infty_0((\exp(n), 0, \infty)) \) s.t. \( \langle u, H_{1,n} u \rangle < 0 \), and correspondingly for the operator on the r.h.s. of (7), we have

\[
N(H_{1,n}) = N \left( -\frac{d^2}{dx^2} |_\mathbb{R} + e^{2x} \ldots e^{2\exp(n)} x V(\exp(n+1)x) \right)
\]

\[
\leq 1 + \int_{-\infty}^{\infty} e^{2x} \ldots e^{2\exp(n)} x |V(\exp(n+1)x) - |x|| \, dx
\]

\[
= 1 + \int_{\exp(n) 0}^{\exp(n) \infty} |V(y) - |y|| |\ln y| \ldots |\ln(n+1)y| \, dy.
\]

The second bound is proved analogously, using that

\[
N \left( -\frac{d^2}{dx^2} |_\mathbb{R} + V(x) \right) \leq \int_{0}^{\infty} |V(x) - |x|| \, dx.
\]

For higher dimensions we have instead the following version of the above bounds:

Theorem 8. Let \( n \in \mathbb{N} \), \( V \) be real-valued and s.t. \( |x|^2(\ln |x|)^2 \ldots (\ln(n) |x|)^2 V(x) \)

is bounded from below, and let

\[
H_{d,n} := -\Delta_{\mathbb{R}^d} - \frac{(d - 2)^2}{4|x|^2} - \ldots - \frac{1}{4|x|^2(\ln |x|)^2 \ldots (\ln(n) |x|)^2} + V(x)
\]

be defined as a self-adjoint operator by Friedrichs extension on \( C^\infty_0(B_{\exp(n+2)}^c 0) \).

For \( d \geq 3 \), and some universal positive constant \( C_d \), we have the following
bound for the number of negative eigenvalues:

\[ N(H_{d,n}) \leq C_d \int_{|x| > \exp(n+2)} \left( \frac{(d-1)(d-3)}{4|x|^2(\ln |x|)^2 \cdots (\ln(n+1)|x|)^2} - V(x) \right)^{\frac{4}{d}} + \]

\[-(\ln |x|)^{d-1} \cdots (\ln(n+1)|x|)^{d-1} dx.\]

On the other hand, \( H_{d,n} \) defined by Friedrichs extension on \( C_0^\infty(B_{\exp(n+2)}^c) \) satisfies the bound

\[ N(H_{d,n}) \leq C_d \int_{|x| > \exp(n+2)} \left( \frac{(d-1)(d-3) - (\ln(n+2)|x|)^2}{4|x|^2(\ln |x|)^2 \cdots (\ln(n+2)|x|)^2} - V(x) \right)^{\frac{4}{d}} + \]

\[-(\ln |x|)^{d-1} \cdots (\ln(n+2)|x|)^{d-1} dx.\]

Remark. These bounds also extend to operator-valued potentials according to [11], where \( \left( \cdot \right)^{\frac{4}{d}} \) is replaced by \( \text{tr} \left( \cdot \right)^{\frac{4}{d}} \), and \( C_d \) is slightly larger. Also, by a monotonicity argument (see e.g. Remark 2.2 in [11]), they imply corresponding Lieb-Thirring inequalities for non-zero moments of the eigenvalues (cp. [6]).

Remark. Note that there is always an extra contribution to the above bound for the number of negative eigenvalues of \( H_{d,n} \) for all \( d \geq 4 \), but not so in the case of \( d = 1 \) (Theorem 7) and \( d = 3 \). This is quite interesting when related with the fact that supersymmetric matrix models, split into coordinates of \( \mathbb{R}^d \times \mathbb{R}^2 \) (cp. [3]), are conjectured to have zero energy states for \( d = 7 \), but not for \( d = 0, 1, 3 \).

Proof. Here we apply the Cwikel-Lieb-Rozenblum bound for \( d \geq 3 \) (see e.g. [12, 9]):

\[ N(-\Delta_{\mathbb{R}^d} + V(x)) \leq C_d \int_{\mathbb{R}^d} |V(x)| - \frac{d}{2} dx.\]

By iterating the bound obtained from Lemma 2

\[ \langle u, \left( -\Delta_{\mathbb{R}^d} - \frac{(d-2)^2}{4|x|^2} + V(x) \right) u \rangle_{L^2(B_{\epsilon})} \geq \langle \psi, \left( -\Delta_{\mathbb{R}^d} - \frac{(d-1)(d-3)}{4|x|^2} + e^{2|x|} V(e^{2|x|} \omega) \right) \psi \rangle_{L^2(B_{\epsilon})}, \]

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with \( \psi(r\omega) = r^{-\frac{d+1}{2}} e^{\frac{d+3}{2}r}(r\omega) \), we have as in the one-dimensional case

\[
N(H_{d,n}) \leq N \left( -\Delta_B^n - \frac{(d-1)(d-3)}{4|x|^2} + e^{2|\omega|} \exp \left( (\exp(n+1) |x|)\omega \right) \right)
\]

\[
\leq C_d \int_{B_1^n} \left( \frac{(d-1)(d-3)}{4|x|^2} - e^{2|\omega|} \exp(n+1) |x| \exp(n+1) |x| \right) \frac{dx}{dx}
\]

\[
= C_d \int_{B_1^n} \exp(n+1) \left( \frac{(d-1)(d-3)}{4|x|^2 \exp(n+1) |x|^2} - V(x) \right) \frac{dx}{dx} + \cdot (\ln |x|)_{n-1} \exp(n+1) |x|_{n-1} dx.
\]

For the operator on the domain \( B_{\exp(n+2)}^c \), we can add and subtract a term \( 1/(4|x|^2) \) and iterate one step further to obtain

\[
N(H_{d,n}) \leq C_d \int_{B_1^n} \left( \frac{(d-1)(d-3)}{4|x|^2} - \frac{1}{4} - e^{2|\omega|} \exp(n+1) |x| \exp(n+1) |x| \right) \frac{dx}{dx}.
\]

The stated bound then follows as above. \( \square \)

We expect that it is possible to find analogous bounds on the larger domains \( B_{\exp(n)}^c \) and \( B_{\exp(n+1)}^c \). Indeed, for central potentials we have the following:

**Theorem 9.** If \( V(x) = \tilde{V}(|x|) \) is a central potential s.t. \( r^2 \ln r^2 \exp(n+1) \tilde{V}(r) \) is bounded from below, then for \( H_{d,n} \) defined on the domain \( B_{\exp(n)}^c \)

\[
N(H_{d,n}) \leq \sum_{l=0}^{l_{\max}} D_{d,l} \left( 1 + \int_{\exp(n)0}^{\infty} \left( -\frac{l(l+d-2)}{r^2} - \tilde{V}(r) \right) + |r| \ln r \cdots |\ln(n+1) r| dr \right),
\]

while on \( B_{\exp(n+1)}^c \)

\[
N(H_{d,n}) \leq \sum_{l=0}^{l_{\max}} D_{d,l} \int_{\exp(n)1}^{\infty} \left( -\frac{l(l+d-2)}{r^2} - \tilde{V}(r) \right) + |r| \ln r \cdots |\ln(n+1) r| dr,
\]

where

\[
D_{d,l} := \frac{(2l + d - 2)\Gamma(d + l - 2)}{\Gamma(d-1)\Gamma(l+1)},
\]

and \( l_{\max} \) is the maximal integer \( l \geq 0 \) s.t. the negative part of \( \frac{l(l+d-2)}{r^2} + \tilde{V}(r) \) is non-zero on the respective domain.
Proof. For central potentials, we can split the Hilbert space \( \mathcal{H} = \bigoplus_{l=0}^{\infty} \mathcal{H}_l \) into eigenspaces of the angular laplacian, where 
\[-\Delta_{S^{d-1}}|_{\mathcal{H}_l} = l(l+d-2)\]
with degeneracy \( D_{d,l} \) (see e.g. [13]; cp. [8]). Using (6) and iterating, we have for 
\[u = \tilde{u} \otimes \psi \in C^\infty_0((\exp(n)0, \infty)) \otimes L^2(S^{d-1}) \cap \mathcal{H}_l\]
\[
\langle u, H_{d,n} u \rangle_{L^2(\mathbb{R}^d)}
\]
\[= \left\langle w, \left( -\partial_s^2 + e^{2s} \ldots e^{2\exp(n)s} \left( \frac{l(l+d-2)}{\exp((n+1)s)^2} + \tilde{V}(\exp((n+1)s)) \right) \right) w \right\rangle_{L^2(\mathbb{R})}
\]
\[\cdot \|\psi\|_{L^2(S^{d-1})}^2,
\]
with \( w \in C^\infty_0(\mathbb{R}) \). Hence, by reasoning as in the proof of Theorem 7,
\[N(H_{d,n}|_{\mathcal{H}_l}) \leq D_{d,l} \left( 1 + \int_{-\infty}^{\infty} e^{2s} \ldots e^{2\exp(n)s} \left( -\frac{l(l+d-2)}{\exp((n+1)s)^2} - \tilde{V}(\exp((n+1)s)) + |s| \right) ds \right)
\]
if \( l \leq l_{\text{max}} \), and \( N(H_{d,n}|_{\mathcal{H}_l}) = 0 \) otherwise. The first statement of the theorem then follows by a change of variables, and similar reasoning gives the second statement.

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