CHIRAL ALGEBRAS OF CLASS $S$ AND MOORE-TACHIKAWA SYMPLECTIC VARIETIES

TOMOYUKI ARAKAWA

Abstract. We give a functorial construction of the genus zero chiral algebras of class $S$, that is, the vertex algebras corresponding to the theory of class $S$ associated with genus zero pointed Riemann surfaces via the 4d/2d duality discovered by Beem, Lemos, Liendo, Peelaers, Rastelli and van Rees [BLL+]. We show that there is a unique family of vertex algebras satisfying the required conditions and show that they are all simple and conformal. In fact, our construction works for any complex semisimple group $G$ that is not necessarily simply laced. Furthermore, we show that the associated varieties of these vertex algebras are exactly the genus zero Moore-Tachikawa symplectic varieties [MT] that have been recently constructed by Braverman, Finkelberg and Nakajima [BFN2] using the geometry of the affine Grassmannian for the Langlands dual group.

1. Introduction

Let $G$ be a simply connected semisimple linear algebraic group over $\mathbb{C}$, $\mathfrak{g} = \text{Lie}(G)$. In [BFN2], Braverman, Finkelberg and Nakajima have constructed a new family of the (possibly singular) symplectic varieties

$W^b_G = \text{Spec} H^*_G[t]((\text{Gr}_G, i_\Delta^! (\bigotimes_{k=1}^b A_R))), \quad b \in \mathbb{Z}_{\geq 1},$

equipped with a Hamiltonian action of $\prod_{k=1}^b G$. Here $\tilde{G}$ is the Langlands dual group of $G$, $\text{Gr}_G$ is the affine Grassmannian for $\tilde{G}$, $i_\Delta : \text{Gr}_{\tilde{G}} \to \prod_{k=1}^b \text{Gr}_{\tilde{G}}$ is the diagonal embedding and $A_R$ is the perverse sheaf corresponding to the regular representation $O(G)$ of $G$ under the geometric Satake correspondence [MV]. These symplectic varieties satisfy the following properties:

$W^{b=1}_G \cong G \times S, \quad W^{b=2}_G \cong T^*G, \quad (W^b_G \times W^{b'}_G)\big/\big/ \Delta(G) \cong W^{b+b'-2}_G,$

where $S$ is a Kostant-Slodowy slice in $\mathfrak{g}^*$ and the left-hand side in the last isomorphism is the symplectic reduction of $W^b_G \times W^{b'}_G$ with respect to the diagonal action of $G$. The existence of the symplectic varieties satisfying the above conditions was conjectured by Moore and Tachikawa [MT] and $W^b_G$ is called a (genus zero) Moore-Tachikawa symplectic variety. We note that $W^b_G$ is conical for $b \geq 3$.

In this article we perform a chiral quantization of the above construction. More precisely, we construct a family of vertex algebras $V^S_{G,b}, \quad b \in \mathbb{Z}_{\geq 1}$, equipped with a vertex algebra homomorphism (chiral quantum moment map) $\bigotimes_{i=1}^b V_{c_i}(\mathfrak{g}) \to V^S_{G,b}$.

This work is partially supported by JSPS KAKENHI Grant Number No. 20340007 and No. 29650006.
satisfying the following properties:

(1) \( V^{S}_{G,b=1} \cong H^{0}_{DS}(D^{ch}_G), \quad V^{S}_{G,b=2} \cong D^{ch}_{G}, \)

(2) \( H^{\bullet+\bullet}(\widehat{\mathfrak g} \otimes \mathfrak g, V^{S}_{G,b} \otimes V^{S}_{G,b'}) \cong \delta_{b,b'} V^{S}_{G,b=2} \) (associativity),

where \( V^{\kappa,c}(\mathfrak g) \) is the universal affine vertex algebra associated with \( \mathfrak g \) at the critical level \( \kappa, c = \mathbb {Z} \). \( H^{0}_{DS}(\mathfrak g) \) is the quantized Drinfeld-Sokolov reduction functor \( [FF1] \). \( D^{ch}_{G} \) is the algebra of the chiral differential operators \( [MSV] [BD] \) on \( G \) at \( \kappa, c \). \( \mathfrak g - \kappa \) is the affine Kac-Moody algebra associated with \( \mathfrak g \) and the minus of the Killing form \( \kappa \) of \( \mathfrak g \). \( H^{\bullet+\bullet}(\widehat{\mathfrak g} \otimes \mathfrak g, V^{S}_{G,b} \otimes V^{S}_{G,b'}) \) is the relative semi-infinite cohomology with coefficients in \( V^{S}_{G,b} \otimes V^{S}_{G,b'} \), which can be regarded as an affine analogue of the Hamiltonian reduction with respect to the diagonal \( G \)-action, see \( [73] \), \( [74] \), \( [75] \) and Theorem \( \text{11.11} \) for the details. We show that \( V^{S}_{G,b}, b \in \mathbb {Z}_{\geq 1} \) is the unique family of the vertex objects in \( KL^{ob} \) that satisfies the conditions \( \text{1} \) and \( \text{2} \) (Remark \( \text{11.11} \)) and that they are all simple and conformal (Theorem \( \text{11.14} \) and Proposition \( \text{11.7} \)). The central charge of \( V^{S}_{G,b} \) is given by

\[
\hat{b} = \dim \mathfrak g - (b - 2) \dim \mathfrak g - 24(b - 2)(\rho, \rho),
\]

and the character of \( V^{S}_{G,b} \) is given by the following formula (Proposition \( \text{11.3} \)):

(3) \[ \text{tr}_{V^{S}_{G,b}}(q^{L_0}z_1z_2\ldots z_b) = \sum_{\lambda \in P_+} \left( \frac{q^{(\lambda, \rho)}}{\prod_{\alpha \in \Delta_+}(1 - q^{(\lambda + \rho, \alpha)})} \right)^{b-2} \prod_{k=1}^{b} \text{tr}_{\lambda}(q^{-D}z_k), \]

where \( (z_1, \ldots, z_b) \in T^{b} \), \( T \) is the maximal torus of \( G \), \( \Delta_+ \) is the set of positive roots of \( \mathfrak g \), \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha, \rho' = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha', P_+ \) is the set of integrable dominant weights of \( \mathfrak g \), \( \Delta_+ \) is the Weyl module of the affine Kac-Moody algebra \( \widehat{\mathfrak g} \), \( \Delta_+ \) is the Weyl module of the affine Kac-Moody algebra \( \widehat{\mathfrak g} \) at \( \kappa \), with highest weight \( \lambda \), and \( D \) is the standard degree operator of the affine Kac-Moody algebra that acts as zero on the highest weight of \( \mathfrak g \).

Our construction is an affine analogue of that of Ginzburg and Kazhdan \( [GK] \).

In the case that \( G \) is simply laced, the above stated results were conjectured by Beem, Lemos, Liendo, Pelaars, Rastelli and van Rees \( [BLL^+1] \), and Beem, Pelaar, Rastelli and van Rees \( [BPRvR] \) in the theory of the 4d/2d duality in physics discovered in \( [BLL^+] \), as we explain below.

The 4d/2d duality \( [BLL^+] \) associates a conformal vertex algebra \( V_{T} \) to any four-dimensional \( N = 2 \) superconformal field theory (4d \( N = 2 \) SCFT) \( T \). There is a distinguished class of 4d \( N = 2 \) SCFTs called the theory of class \( \mathcal S \) \( [Gal] [GMN] \), which is labeled by a complex semi-simple group \( G \) and a pointed Riemann surface \( \Sigma \). The vertex algebra associated to the theory of class \( \mathcal S \) by the 4d/2d duality is called the chiral algebra of class \( \mathcal S \). The vertex algebra \( V^{S}_{G,b} \) constructed in this article is exactly the chiral algebra of class \( \mathcal S \) corresponding to the group \( G \) and a \( b \)-pointed Riemann surface of genus zero. Chiral algebras of class \( \mathcal S \) associated with higher genus Riemann surfaces are obtained by glueing \( V^{S}_{G,b} \)'s and such a gluing procedure is described in terms of 2d TQFT whose targets are vertex algebras \( [BPRvR] \), which is well-defined by the properties \( \text{1} \) and \( \text{2} \), see \( \text{1} \) \( \text{2} \) \( \text{1} \) \( \text{2} \) for mathematical expositions.

It is known that 4d \( N = 2 \) SCFTs have several important invariants (or observables). One of them is the Schur index, which is a formal series. The 4d/2d duality \( [BLL^+] \) is constructed in such a way that the Schur index of a 4d \( N = 2 \)
SCFT $\mathcal{T}$ is obtained as the character of the corresponding vertex algebra $V_\mathcal{T}$. A recent remarkable conjecture of Beem and Rastelli [BR] states that we can also recover the geometric invariant $Higgs(\mathcal{T})$ of $\mathcal{T}$, called the Higgs branch, which is a possibly singular symplectic variety, from the vertex algebra $V_\mathcal{T}$. More precisely, they expect that we have an isomorphism

$$Higgs(\mathcal{T}) \cong X_{V_\mathcal{T}}$$

of Poisson varieties for any 4d $N = 2$ SCFT $\mathcal{T}$, where $X_V$ is the associated variety [3] of a vertex algebra $V$, see [A6] for a survey.

In this paper we also prove the conjecture (4) for the genus zero class $S$ theories.

For the theory of class $S$, the corresponding Higgs branches are exactly the Moore-Tachikawa symplectic varieties that have been constructed mathematically as explained above ([BFN2]). Therefore showing the isomorphism between the Higgs branches and the associated varieties is equivalent to the following statement (Theorem 11.13):

$$X_{V_{G,b}}^S \cong W_G^b \quad \text{for all } b \geq 1.$$  

In other words, $V_{G,b}^S$ is a chiral quantization (Definition 2.1) of the Moore-Tachikawa symplectic variety $W_G^b$. Note that this in particular proves that the associated variety of $V_{G,b}^S$ is symplectic for all $b \geq 1$.

We remark that there is a close relationship between the Higgs branches of 4d $N = 2$ SCFTs and the Coulomb branches [BFN1] of 3d $N = 4$ SUSY gauge theories. Indeed, it is known [BFN2, Theorem 5.1] in type $A$ that the Moore-Tachikawa variety $W_G^b$ is isomorphic to the Coulomb branch of a star shaped quiver gauge theory.

In view of [BR], we conjecture that $V_{G,b}^S$ is quasi-lisse ([AK]), that is, $W_G^b$ has finitely many symplectic leaves for all $b \geq 1$.

Although it is very difficult to describe the vertex algebra $V_{G,b}^S$ explicitly in general, conjectural descriptions of $V_{G,b}^S$ have been given in several cases in [BLL+], and it is possible to confirm them using Remark 11.12. For instance, $V_{G=SL_2,b=3}^S$ is isomorphic to the $\beta\gamma$-system $SB((\mathbb{C}^2)^{\otimes 3})$ associated with the symplectic vector space $W_{G=SL_2,b=3}^b = (\mathbb{C}^2)^{\otimes 3}$ (Theorem A.1); $V_{G=SL_2,b=4}^S$ is the simple affine vertex algebra $L_{-2}(D_4)$ associated with $D_4$ at level $-2$ (Theorem A.2). The statement (5) for $G = SL_2$, $b = 4$ reproves the fact [AMor1] that $X_{L_{-2}(D_4)}$ is isomorphic to the minimal nilpotent orbit closure $\mathfrak{U}_{\text{min}}$ in $D_4$, and (2) gives a non-trivial isomorphism

$$L_{-2}(D_4) \cong H^{\mathbb{C}}_{+4}((\mathfrak{sl}_2)_{-4},\mathfrak{sl}_2,SB((\mathbb{C}^2)^{\otimes 3})\otimes SB((\mathbb{C}^2)^{\otimes 3}))$$

that was conjectured in [BLL+]. Also, [3] provides a non-trivial identity of the normalized character [KT] of $L_{-2}(D_4)$, whose homogeneous specialization is known to be $E'_4(\tau)/240\eta(\tau)^{10}$ ([AK]). It would be interesting to compare (3) with [KW, Conjecture 3.4].

In general, $V_{G,b}^S$ is a $W$-algebra in the sense that it is not generated by a Lie algebra, and the associativity isomorphism (2) provides non-trivial isomorphisms involving such vertex algebras.

---

1There is a subtlety for higher genus cases due to the non-flatness of the moment map.
Acknowledgments. The author is grateful to Christopher Beem, Boris Feigin, Michael Finkelberg, Davide Gaiotto, Victor Ginzburg, Madalena Lemos, Victor Kac, Anne Moreau, Hiraku Nakajima, Takahiro Nishinaka, Wolfger Peelaers, Leonardo Rastelli, Shu-Heng Shao, Yuji Tachikawa for very useful discussions. The results in this paper has been announced in part in “Conference in Finite Groups and Vertex Algebras dedicated to Robert L. Griess on the occasion of his 71st birthday”, Academia Sinica, Taiwan, August 2016, “60th Annual Meeting of the Australian Mathematical Society”, Canberra, December 2016, “Exact operator algebras in superconformal field theories”, Perimeter Institute for Theoretical Physics, Waterloo, December 2016, “7th Seminar on Conformal Field Theory”, Darmstadt, February 2017, “Representation Theory XV”, Dubrovnik, June 2017, “Affine, Vertex and W-algebras”, Rome, December 2017, “The 3rd KTGU Mathematics Workshop for Young Researchers”, Kyoto, February 2018, “Gauge theory, geometric langlands and vertex operator algebras”, Perimeter Institute for Theoretical Physics, Waterloo, March 2018, “Vertex Operator Algebras and Symmetries”, RIMS, Kyoto, July 2018, “Vertex algebras, factorization algebras and applications”, IPMU, Kashiwa, July 2018, “The International Congress of Mathematics”, Rio de Janeiro, August, 2018, “Workshop on Mathematical Physics”, ICTP-SAIFR, São Paulo, August 2018, and “Geometric and Categorical Aspects of CFTs”, the Casa Matemática Oaxaca, September 2018. He thanks the organizers of these conferences and apologies for the delay of the paper. The author is partially supported in part by JSPS KAKENHI Grant Numbers 17H01086, 17K18724.

2. Vertex algebras and associated varieties

A vertex algebra $\mathcal{B}_{\text{Br}}$ consists of a vector space $V$ with a distinguished vacuum vector $|0\rangle \in V$ and a vertex operation, which is a linear map $V \otimes V \rightarrow V((z))$, written $a \otimes b \mapsto a(z)b = \left(\sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}\right)b$, such that the following are satisfied:

- (Unit axioms) $(|0\rangle(z)) = \text{id}_V$ and $a(z)|0\rangle \in a + zV[[z]]$ for all $a \in V$.
- (Locality) $(z-w)^n[a(z), b(w)] = 0$ for a sufficiently large $n$ for all $a, b \in V$.

The operator $\partial : a \mapsto a_{(-2)}|0\rangle$ is called the translation operator and it satisfies $(\partial a)(z) = [\partial, a(z)] = \partial_z a(z)$. The operators $a_{(n)}$ are called modes.

For elements $a, b$ of a vertex algebra $V$ we have the following Borcherds identity for any $m, n \in \mathbb{Z}$:

\begin{equation}
[a_{(m)}, b_{(n)}] = \sum_{j \geq 0} \binom{m}{j} a_{(j)} b_{(m+n-j)},
\end{equation}

\begin{equation}
(a_{(m)} b)_{(n)} = \sum_{j \geq 0} (-1)^j \binom{m}{j} (a_{(m-j)} b_{(n+j)} - (-1)^m b_{(m+n-j)} a_{(j)}).
\end{equation}

By regarding the Borcherds identity as fundamental relations, representations of a vertex algebra are naturally defined (see [Kac2, FBZ] for the details).

We write (6) as

$$a(z)b(w) \sim \sum_{j \geq 1} \frac{1}{(z-w)^j} (a_{(j)} b)(w)$$

and call it the operator product expansion (OPE) of $a$ and $b$.
For a vertex algebra $V$, let $V^{\text{op}}$ denote the opposite vertex algebra of $V$, that is, the vertex algebra $V$ equipped with the new vertex operation $a(z)^{\text{op}}b = a(-z)b$ and the new translation operator $\partial^{\text{op}} = -\partial$.

A vertex algebra $V$ is called \textit{commutative} if the left side (or the right side) of (6) are zero for all $a,b \in V, m,n \in \mathbb{Z}$. If this is the case, $V$ can be regarded as a \textit{differential algebra} (= a unital commutative algebra with a derivation) by the multiplication $a.b = a(-1)b$ and the derivation $\partial$. Conversely, any differential algebra can be naturally equipped with the structure of a commutative vertex algebra. Hence, commutative vertex algebras are the same\footnote{However, the modules of a commutative vertex algebra are not the same as the modules as a differential algebra.} as differential algebras ([Bor]).

For an affine scheme $X$, let $J_{\infty}X$ be the \textit{arc space} of $X$ that is defined by the functor of points $\text{Hom}(\text{Spec } \mathbb{C}[J_{\infty}X]) = \text{Hom}(\text{Spec } \mathbb{C}[[t]], X)$ for all $\mathbb{C}$-algebra $R$. The ring $\mathbb{C}[J_{\infty}X]$ is naturally a differential algebra, and hence is a commutative vertex algebra. In the case that $X$ is a Poisson scheme, $\mathbb{C}[J_{\infty}X]$ has \cite{A3} the structure of a \textit{Poisson vertex algebra}, see \cite{FBZ, Kac3} for the definition of Poisson vertex algebras.

It is known by Haisheng Li \cite{Li} that any vertex algebra $V$ is canonically filtered, and hence can be regarded as a quantization of the associated graded Poisson vertex algebra $\text{gr} V = \bigoplus_p F_p V/F_{p+1} V$, where $F^\bullet V$ is the canonical filtration of $V$. By definition,

$$F^p V = \text{span}_\mathbb{C} \{(a_1)_{-n_1-1} \cdots (a_r)_{-n_r-1} | a_i \in V, n_i \geq 0, \sum_i n_i \geq p \}.$$  

A vertex algebra is called \textit{good} if the filtration $F^\bullet V$ is separated, that is, $\cap_p F^p V = 0$. For instance, any positively graded vertex algebra is good.

The scheme $\text{Spec}(\text{gr} V)$ is called the \textit{singular support} of $V$ and is denoted by $\text{SS}(V)$.

The subspace 

$$R_V := V/F^1 V = F^0 V/F^1 V \subset \text{gr} V$$

is called \textit{Zhu’s $C_2$-algebra} of $V$. The Poisson vertex algebra structure of $\text{gr} V$ restricts to a Poisson algebra structure of $R_V$, which is given by 

$$\tilde{a}.\tilde{b} = \tilde{a}(-1)\tilde{b}, \quad \{\tilde{a}, \tilde{b}\} = \tilde{a}(0)\tilde{b},$$

where $\tilde{a}$ is the image of $a \in V$ in $R_V$. The \textit{associated variety} of $V$ is by definition the Poisson variety 

$$X_V = \text{Specm}(R_V)$$

and the \textit{associated scheme} of $V$ is the Poisson scheme $\tilde{X}_V = \text{Spec}(R_V)$ (\cite{A3}). We have a surjective homomorphism 

$$(8) \quad \mathcal{O}(J_{\infty} \tilde{X}_V) \twoheadrightarrow \text{gr} V$$

of Poisson vertex algebras which is the identity map on $R_V$ (\cite{Li, A3}).

\textit{Definition 2.1.} Let $X = \text{Spec } R$ be an affine Poisson scheme. A \textit{chiral quantization} of $X$ is a good vertex algebra $V$ such that $X_V \cong X$ as Poisson varieties. A \textit{strict chiral quantization} of $X$ is a chiral quantization of $X$ such that (8) is an isomorphism so that the singular support of $V$ identifies with $J_{\infty}X$. 
Theorem 2.2 ([AMor2]). Let $V$ be a good vertex algebra such that $\tilde{X}_V$ is a reduced, smooth symplectic variety. Then

- $\text{gr} V$ is simple as vertex Poisson algebras;
- $V$ is simple;
- $V$ is a strict chiral quantization of $\tilde{X}_V$.

Let $\phi : V \to W$ be a vertex algebra homomorphism. Then $\phi(F^p V) \subset F^p W$, and $\phi$ induces a homomorphism of vertex Poisson algebras $\text{gr} V \to \text{gr} W$, which we denote by $\text{gr} \phi$. It restricts to a Poisson algebra homomorphism $R_V \to R_W$, and therefore induces a morphism $X_W \to X_V$ of Poisson varieties.

A vertex algebra is called conformal if there exists a vector $\omega$, called the conformal vector, such that the corresponding field $\omega(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfies the following conditions: (1) $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}c \delta_{m+n,0} \text{id}_V$, where $c$ is a constant called the central charge of $V$; (2) $L_0$ acts semisimply on $V$; (3) $L_{-1} = 0$. For a conformal vertex algebra $V$ we set $V^\Delta = \{ v \in V \mid L_0 v = \Delta v \}$, and write $\Delta_a = \Delta$ if $a \in V^\Delta$. A conformal vertex algebra is called conical if $V^\Delta = \{ 0 \}$ unless $\Delta \in \frac{1}{m} \mathbb{Z}_{\geq 0}$ for some $m \in \mathbb{Z}_{\geq 1}$ and $V_0 = \mathbb{C}$ ([AK]).

For a conformal vertex algebra $V = \bigoplus_{\Delta} V_\Delta$, one defines Zhu’s algebra [FYZ] $\text{Zhu}(V)$ of $V$ by

$$\text{Zhu}(V) = V/V \circ V, \quad V \circ V = \text{span}_C \{ a \circ b \mid a, b \in V \},$$

where $a \circ b = \sum_{i \geq 0} \binom{\Delta_a}{i} a(i-2) b$ for $a \in V^\Delta_a$. The algebra $\text{Zhu}(V)$ is a unital associative algebra by the multiplication $a * b = \sum_{i \geq 0} \binom{\Delta_a}{i} a(i-1) b$. The grading of $V$ gives a filtration on $\text{Zhu}(V)$ which makes it almost-commutative, and there is a surjective map

$$R_V \to \text{gr} \text{Zhu}(V)$$

of Poisson algebras. The map [9] is an isomorphism if $V$ admits a PBW basis ([AM2]).

For a vertex subalgebra $W$ of $V$, let

$$\text{Com}(V, W) = \{ v \in V \mid [v_{(m)}, w_{(n)}] = 0 \ \forall w \in W, \ n \in \mathbb{Z} \} = \{ v \in V \mid w_{(n)} v = 0 \ \forall w \in W, \ n \geq 0 \}.$$

Then $\text{Com}(V, W)$ is a vertex subalgebra of $V$, called a coset vertex algebra, of a commutant vertex algebra ([FYZ]). The vertex subalgebras $W_1$ and $W_2$ are said to form a dual pair in $V$ if $W_1 = \text{Com}(W_2, V)$ and $W_2 = \text{Com}(W_1, V)$. The coset vertex algebra $\text{Com}(V, V)$ is called the center of $V$ and is denoted by $Z(V)$.

Let $\mathfrak{a}$ be a finite-dimensional Lie algebra. For an invariant symmetric bilinear form $\kappa$ on $\mathfrak{a}$, let $\hat{\mathfrak{a}}_\kappa$ be the corresponding Kac-Moody affinization of $\mathfrak{a}$:

$$\hat{\mathfrak{a}}_\kappa = \mathfrak{a}((t)) \oplus \mathbb{C} 1,$$

where the commutation relation of $\hat{\mathfrak{a}}_\kappa$ is given by

$$[x f, y g] = [x, y] f g + \text{Res}_{t=0} \langle gdf \rangle \kappa(x, y) 1, \quad [1, \hat{\mathfrak{a}}_\kappa] = 0.$$

Define

$$V^{\kappa}(\mathfrak{a}) = U(\hat{\mathfrak{a}}_\kappa) \otimes U(\mathfrak{a}[[t]]) \oplus \mathbb{C} 1.$$
where $\mathbb{C}$ is the one-dimensional representation of $\mathfrak{a}[t] \oplus \mathbb{C}1$ on which $\mathfrak{a}[t]$ acts trivially and $1$ acts as the identity. There is a unique vertex algebra structure on $V^\kappa(\mathfrak{a})$ such that $|0\rangle = 1 \otimes 1$ is the vacuum vector and

$$x(z) = \sum_{n \in \mathbb{Z}} (xt^n)z^{-n-1},$$

for $z \in \mathfrak{a}$, where on the left-hand-side we have regarded $\mathfrak{g}$ as a subspace of $V^\kappa(\mathfrak{a})$ by the embedding $\mathfrak{g} \hookrightarrow V^\kappa(\mathfrak{a})$, $x \mapsto (xt^{-1})|0\rangle$. The vertex algebra $V^\kappa(\mathfrak{a})$ is a strict chiral quantization of $\mathfrak{a}^*$ and is called the universal affine vertex algebra associated with $\mathfrak{a}$ at level $\kappa$.

Note that we have $V^\kappa(\mathfrak{a})^{op} \rightarrow V^\kappa(\mathfrak{a})$, $x \mapsto -x$, $(x \in \mathfrak{a})$.

A $V^\kappa(\mathfrak{a})$-module is the same as a smooth $\hat{\mathfrak{g}}_\kappa$-module, that is, a $\hat{\mathfrak{g}}_\kappa$-module $M$ such that $xt^m = 0$ for $n \gg 0$ for all $x \in \mathfrak{g}$, $m \in M$.

3. SOME VARIANTS OF CHIRAL HAMILTONIAN REDUCTIONS

For an algebraic group $G$ and an $G$-scheme $X$, let $\text{QCoh}^G(X)$ be the category of quasi-coherent sheaves on $X$ equivariant under the adjoint action of $G$. When $X$ is affine we do not distinguish between a coherent sheaf on an affine scheme and the module of its global sections.

Let $G$ be a simply connected semisimple algebraic group, $\mathfrak{g} = \text{Lie}(G)$. An object of $\text{QCoh}^G(\mathfrak{g}^*)$ is the same as a Poisson $\mathcal{O}(\mathfrak{g}^*)$-module on which the adjoint action of $\mathfrak{g}$ is locally finite.

A Poisson algebra object in $\text{QCoh}^G(\mathfrak{g}^*)$ is a Poisson algebra $R$ equipped with a Poisson algebra homomorphism $\mu_R : \mathcal{O}(\mathfrak{g}^*) \rightarrow R$ such that the adjoint action of $\mathfrak{g}$ is locally finite. If $R$ is a Poisson algebra object in $\text{QCoh}^G(\mathfrak{g}^*)$, $X = \text{Spec}(R)$ is a $G$-scheme and the $G$-equivariant morphism $\mu^*_R : X \rightarrow \mathfrak{g}^*$ is a moment map for the $G$-action. We denote by $R^{op}$ the Poisson algebra object in $\text{QCoh}^G(\mathfrak{g}^*)$ with the opposite Poisson structure, $X^{op} = \text{Spec}R^{op}$. We have $\mu_{R^{op}} = -\mu_R$.

Let $R$ be a Poisson algebra object in $\text{QCoh}^G(\mathfrak{g}^*)$, $X = \text{Spec} R$. Suppose that the moment map $\mu_X : X \rightarrow \mathfrak{g}^*$ is flat, and that there exists a closed subscheme $S$ of $X$ such that the action map $G \times S \rightarrow X$, $(g,s) \mapsto gs$, is an isomorphism of $G$-schemes. Then $\mathcal{O}(S) \cong \mathcal{O}(X)^G$ is a Poisson subalgebra of $\mathcal{O}(X)$, and hence $S$ is a Poisson subvariety of $X$. Let $R'$ be another Poisson algebra object in $\text{QCoh}^G(\mathfrak{g}^*)$, $X' = \text{Spec} R'$, $\mu_{X'} : X' \rightarrow \mathfrak{g}^*$, the moment map. Then $R'^{op} \otimes R$ is a Poisson algebra object in $\text{QCoh}^G(\mathfrak{g}^*)$ with the moment map $\mu : X'^{op} \times X' \rightarrow \mathfrak{g}^*$, $(x,x') \mapsto -\mu_X(x) + \mu_{X'}(x')$ for the diagonal $G$-action. Since $\mu_X$ is flat, so is $\mu$. Moreover, $\mu^{-1}(0) \cong X \times_{\mathfrak{g}^*} X' \cong G \times (S \times_{\mathfrak{g}^*} X')$, where $S \rightarrow \mathfrak{g}^*$ is the restriction of $\mu_X$ to $S$. Hence, the Hamiltonian reduction $(X'^{op} \times X')///\Delta(G)$ is well-defined, and we have

$$\left(X'^{op} \times X'\right)////\Delta(G) = \mu^{-1}(0)/\Delta(G) \cong S \times_{\mathfrak{g}^*} X'.$$

According to [KS, Kuw], this construction is realized by a BRST cohomology as follows. Let $\mathcal{C}(\mathfrak{g})$ be the classical Clifford algebra associated with $\mathfrak{g} \otimes \mathfrak{g}^*$, which is a Poisson superalgebra generated by odd elements $\tilde{\psi}_i, \tilde{\psi}_j^*$, $i = 1, \ldots, \dim \mathfrak{g}$, with the relations $\{\tilde{\psi}_i, \tilde{\psi}_j\} = \delta_{ij}$, $\{\tilde{\psi}_i, \tilde{\psi}_j^*\} = \{\tilde{\psi}_i^*, \tilde{\psi}_j^*\} = 0$. For a Poisson algebra object $A$ in $\text{QCoh}^G(\mathfrak{g}^*)$, set

$$C(\mathfrak{g}^*, A) = A \otimes \mathcal{C}(\mathfrak{g}).$$
Then $C(g^{cl}, A)$ is naturally a Poisson superalgebra. Define the odd element

$$Q = \sum_{i=1}^{\dim g} \mu_A(x_i) \otimes \bar{\psi}_i^* - 1 \otimes \frac{1}{2} c_{ij} \bar{\psi}_i^* \bar{\psi}_j \bar{\psi}_k \in C(g^{cl}, A),$$

where $\{x_i\}$ is a basis of $g$ and $c_{ij}$ is the corresponding structure constant. Then $\{\bar{Q}, \bar{Q}\} = 0$, and hence, $(\ad \bar{Q})^2 = 0$. It follows that $(C(g^{cl}, A), \ad \bar{Q})$ is a differential graded Poisson algebra, where its grading is defined by $\deg a = a \in A$, $\deg \bar{\psi}_i^* = 1$, $\deg \bar{\psi}_i = -1$. The corresponding cohomology is denoted by $H^{\frac{\partial}{\partial \psi}}(g^{cl}, A)$, which is naturally a Poisson superalgebra. We have

(11) $H^{\frac{\partial}{\partial \psi}}(g, R^{op} \otimes R) \cong \mathcal{O}((X^{op} \times X')\//\Delta(G)) \otimes H^*(g, \mathbb{C})$

as Poisson superalgebras ([Kuw]). Note that $H^*(g, \mathbb{C})$ is isomorphic to the De Rham cohomology ring $H^*_d(G)$ of $G$, and to the ring of invariant forms $\Lambda^*(g)^g$. In particular, $H^{\frac{\partial}{\partial \psi}}_i(g, R^{op} \otimes R) = 0$ for $i < 0$ or $i > \dim G$, and $H^{\frac{\partial}{\partial \psi}}(g, R^{op} \otimes R) \cong \mathcal{O}((X^{op} \times X')\//\Delta(G)) \cong \mathcal{O}(S) \otimes \mathcal{O}(\frak{g}^*) R = R|_S$ as Poisson algebras. More generally, for $M \in \text{Coh}^G(\frak{g}^*)$, $C(g^{cl}, R^{op} \otimes M) = R^{op} \otimes M \mathcal{C}(\frak{g})$ is naturally a differential graded Poisson $C(g^{cl}, R^{op} \otimes \mathcal{O}(\frak{g}^*))$-module. The corresponding cohomology $H^{\frac{\partial}{\partial \psi}}(g, R^{op} \otimes M)$ is a Poisson module over $H^{\frac{\partial}{\partial \psi}}(g^{cl}, R^{op} \otimes \mathcal{O}(\frak{g}^*)) \cong \mathcal{O}(S) \otimes H^*(g, \mathbb{C})$, and we have

$$H^*(g^{cl}, R^{op} \otimes M) \cong \left(\mathcal{O}(S) \otimes \mathcal{O}(\frak{g}^*) M\right) \otimes H^*(g, \mathbb{C}).$$

The arc space $J_\infty G$ of $G$ is the proalgebraic group $G[[t]]$, and the Lie algebra $\frak{g}_\infty$ of $J_\infty G$ equals to $\frak{g}[t]$. An object $M$ of $\text{Coh}^G(J_\infty \frak{g}^*)$ is the same as a Poisson vertex $\mathcal{O}(J_\infty \frak{g}^*)$-module on which the action of $J_\infty \frak{g}^*$ is locally finite.

A Poisson vertex algebra object in $\text{Coh}^G(J_\infty \frak{g}^*)$ is a Poisson vertex algebra $V$ equipped with a Poisson vertex algebra homomorphism $\mu_V : \mathcal{O}(J_\infty \frak{g}^*) \rightarrow V$ on which the action of $J_\infty \frak{g}$ is locally finite. If $V$ is a Poisson vertex algebra object in $\text{Coh}^G(J_\infty \frak{g}^*)$, $\text{Spec}(V)$ is a $J_\infty G$-scheme. The $J_\infty G$-equivariant morphism $\mu^*_V : \text{Spec } V \rightarrow J_\infty \frak{g}^*$ is called the chiral moment map. For a Poisson algebra object $R$ in $\text{Coh}^G(\frak{g}^*)$, $J_\infty R := \mathcal{O}(J_\infty \text{Spec } R)$ is a Poisson vertex algebra object in $\text{Coh}^G(J_\infty \frak{g}^*)$ with the chiral moment map $(J_\infty \mu_R)^*$. For a Poisson algebra object $V$ in $\text{Coh}^G(J_\infty \frak{g}^*)$, set

$$C^{cl}(\frak{g}^{cl}, V) = V \otimes \bigwedge^{\frac{\partial}{\partial \psi}}(g),$$

where $\bigwedge^{\frac{\partial}{\partial \psi}}(g)$ is the Poisson vertex superalgebra generated by odd elements $\bar{\psi}_i$, $\bar{\psi}_i^*$, $i = 1, \ldots, \dim g$, with $\lambda$-brackets (see [Kac3]) $\{(\bar{\psi}_i)_\lambda \bar{\psi}_j^*\} = \delta_{ij}$, $\{(\bar{\psi}_i)_\lambda \bar{\psi}_j\} = \{(\bar{\psi}_i^*)_\lambda \bar{\psi}_j^*\} = 0$. Set

$$Q^{cl} = \sum_{i=1}^{\dim g} \mu_V(x_i) \otimes \bar{\psi}_i^* - 1 \otimes \frac{1}{2} c_{ij} \bar{\psi}_i^* \bar{\psi}_j \bar{\psi}_k \in C(\frak{g}^{cl}, V).$$

Then $(Q^{cl}_{(0)})^2 = 0$, and $(C(\frak{g}^{cl}, V), Q^{cl}_{(0)})$ is a differential graded Poisson vertex algebra, where its cohomological grading is defined by $\deg a = a \in V$, $\deg \bar{\psi}_i^* = 1$, $\deg \bar{\psi}_i = -1$, $n \geq 0$. The corresponding cohomology $H^{\frac{\partial}{\partial \psi}}(\frak{g}^{cl}, V)$ is naturally a Poisson vertex superalgebra.

Let $X \cong G \times S$ be as above. Then $J_\infty X \cong J_\infty G \times J_\infty S$. Assume further that the chiral moment map $J_\infty \mu_X : J_\infty X \rightarrow J_\infty \frak{g}^*$ is flat. For a vertex Poisson algebra object $V$ in $\text{Coh}^G(J_\infty \frak{g}^*)$, $\mathcal{O}(J_\infty X^{op}) \otimes V$ is a vertex Poisson algebra
object in $\text{QCoh}^{\leq}(J_* \mathfrak{g}^*)$ with chiral moment map $\hat{\mu}: J_* \mathcal{X}^{\text{op}} \times \text{Spec} V \to J_* \mathfrak{g}^*$, $(x, y) \mapsto -J_* \iota_X(x) + \mu^*(y)$. We have $\hat{\mu}^{-1}(0) \cong J_* \mathcal{X} \times J_* \mathfrak{g}^* \text{Spec} V \cong J_* G \times (J_* S \times J_* \mathfrak{g}^* \text{Spec} V)$. It follows in the same way as (11) that

$$H^{\geq 0}(\mathfrak{g}^\vee, \mathcal{O}(J_* \mathcal{X}^{\text{op}} \otimes \mathcal{O})) \cong (\mathcal{O}(J_* S) \otimes J_* \mathfrak{g}^* \mathcal{O}) \otimes H^*(\mathfrak{g}, \mathbb{C}).$$

In particular, if $R$ is an Poisson algebra object in $\text{Coh}^G(\mathfrak{g}^*)$, then

$$H^{\geq 0}(\mathfrak{g}^\vee, \mathcal{O}(J_* \mathcal{X}^{\text{op}} \times J_* \mathcal{X}')) \cong \mathcal{O}(J_* ((X^{\text{op}} \times X)///\Delta(G))) \otimes H^*(\mathfrak{g}, \mathbb{C}).$$

More generally, the same argument gives that

$$H^{\geq 0}(\mathfrak{g}^\vee, \mathcal{O}(J_* \mathcal{X}^{\text{op}} \otimes \mathcal{O})) \cong (\mathcal{O}(J_* S) \otimes \mathcal{O}(J_* \mathfrak{g}^*)) \otimes H^*(\mathfrak{g}, \mathbb{C})$$

for any $M \in \text{Coh}^{\geq G}(J_* \mathfrak{g}^*)$.

Let $\kappa$ be an invariant symmetric bilinear form on $\mathfrak{g}$. Denote by $\text{KL}_\kappa$ the full subcategory of the category of graded $V^\kappa(\mathfrak{g})$-modules consisting of objects $M$ on which $\mathfrak{g}/[[t]]t$ acts locally nilpotently and $\mathfrak{g}$ acts locally finitely. For $\lambda \in P_+$, let $V^\lambda = U(\mathfrak{g})/U(\mathfrak{g}[t] \oplus \mathfrak{c}L) V^\kappa \in \text{KL}_\kappa$, where $V^\lambda$ is the irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight $\lambda$. Note that $V^0_\lambda \cong V^\kappa(\mathfrak{g})$ as $\mathfrak{g}_\kappa$-modules.

Let $\text{KL}^{\geq}_{\mathfrak{g}^*}$ be the full subcategory of $\text{KL}_\kappa$ consisting of objects that are positively graded and each homogenous subspaces are finite-dimensional. For $M \in \text{KL}^{\geq}_{\mathfrak{g}^*}$, the Li filtration $F^* M$ is separated. Note that any object of $\text{KL}_\kappa$ is an inverse limit of objects of $\text{KL}^{\geq}_{\mathfrak{g}^*}$.

A vertex algebra object in $\text{KL}_\kappa$ is a vertex algebra $V$ equipped with a vertex algebra homomorphism $\mu_V: V^\kappa(\mathfrak{g}) \to V$ such that $V$ is a direct sum of objects in $\text{KL}^{\geq}_{\mathfrak{g}^*}$ as a $V^\kappa(\mathfrak{g})$-module. The map $\mu_V$ is called the chiral quantum moment map.

Let $V$ be a vertex algebra object in $\text{KL}_\kappa$. Then $V$ is good, and $R_V$ and $\text{gr} V$ are a Poisson algebra object in $\text{QCoh}^G(\mathfrak{g}^*)$ and a Poisson vertex algebra object in $\text{QCoh}^{\geq G}(J_* \mathfrak{g}^*)$, respectively. A conformal vertex algebra object in $\text{KL}_\kappa$ is a vertex algebra object $V$ in $\text{KL}_\kappa$ which is conformal with conformal vector $\omega_V$ and $\mu_V(x)$ is primary with respect to $\omega_V$ for all $x \in \mathfrak{g} \subset V^\kappa(\mathfrak{g})$, that is, $(\omega_V)_{(n)} \mu_V(x) = 0$ for $n \geq 2$.

If $V$ is a vertex algebra object in $\text{KL}_\kappa$, $V^{\text{op}}$ is also a vertex algebra object in $\text{KL}_\kappa$ with $\mu_{V^{\text{op}}}(x) = -\mu_V(x)$ for $x \in \mathfrak{g}$.

Let $\kappa_{\mathfrak{g}}$ be the Killing form of $\mathfrak{g}$. For a vertex algebra object in $\text{KL}_{-\kappa_{\mathfrak{g}}}$, let

$$C(\widehat{\mathfrak{g}}_{-\kappa_{\mathfrak{g}}}, V) = V \otimes \bigwedge^{\geq 0} \mathfrak{g},$$

where $\bigwedge^{\geq 0} \mathfrak{g}$ is the vertex superalgebra generated by odd elements $\psi_i, \psi^*_i, i = 1, \ldots, \dim \mathfrak{g}$, with OPEs $\psi_i(z) \psi^*_j(w) \sim \frac{\delta_{ij}}{z-w}, \psi_i(z) \psi_j^*(w) \sim \psi^*_i(z) \psi^*_j(w) \sim 0$. The vertex algebra $\bigwedge^{\geq 0} \mathfrak{g}$ is conformal of central charge $-2\dim \mathfrak{g}$. We have

$$\bigwedge^{\geq 0} \mathfrak{g} = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} \bigwedge^{\geq 0} (\mathfrak{g})_{\Delta}, \quad \bigwedge^{\geq 0} (\mathfrak{g}) = \bigwedge^{\geq 0} (\mathfrak{g})_0 \cong \bigwedge^* (\mathfrak{g}^*).$$

Set

$$Q = \sum_{i=1}^{\dim \mathfrak{g}} \mu_V(x_i) \otimes \psi^*_i - 1 \otimes \frac{1}{2} C_{ij} \psi^*_i \psi^*_j \psi_k \in C(\widehat{\mathfrak{g}}_{-\kappa_{\mathfrak{g}}}, V).$$

Then $(Q(0))^2 = 0$, and $(C(\widehat{\mathfrak{g}}_{-\kappa_{\mathfrak{g}}}, V), Q(0))$ is a differential graded vertex algebra, whose cohomological grading is defined by $\deg a = \deg V$, $\deg \partial a = 1$, $\deg \partial^* a = 0$. This is called the chiral quantum moment map.
deg \partial^n \psi_i = -1, n \geq 0. The corresponding cohomology \( H^{\hat{g}+\bullet}_c(\g, V) \) is the semi-infinite \( \hat{g} \)-cohomology with coefficient in \( V \), and is naturally a vertex superalgebra. If \( V \) is conformal with central charge \( c_V \), then \( H^{\hat{g}+\bullet}_c(\g, V) \) is conformal with central charge \( c_V + 2 \dim \g \).

More generally, if \( M \) is an object in \( \text{KL}_{-\g} \), then \( C(\g, M) = M \otimes (\hat{g}^{\bullet+})^\circ \) is a \( C(\g, V^{-\g}(\g)) \)-module. Hence, \( (C(\g, M), Q(0)) \) is naturally a differential graded vertex algebra module over \( C(\g, V^{-\g}(\g)) \). The corresponding cohomology \( H^{\hat{g}+\bullet}_c(\g, M) \) is naturally a module over \( H^{\hat{g}+\bullet}_c(\g, V^{-\g}(\g)) \).

Set \( \kappa^* = -\kappa - \kappa_0 \). For \( M \in \text{KL}_{-\g}, N \in \text{KL}_{\kappa^*}, M \otimes N \) is an object of \( \text{KL}_{-2\kappa} \) with respect to the diagonal action of \( \hat{g}_{-2\kappa} \).

**Theorem 3.1.** Let \( V \) be a vertex algebra object in \( \text{KL}_{\kappa} \), which is a strict quantization of \( X \). Assume that (1) there exists a closed subscheme \( S \) of \( X \) such that the action map \( G \times S \to X, (g, s) \to gs \), is an isomorphism of \( G \)-schemes, (2) the chiral moment map \( \text{gr} \rho^*_V : J_\infty X \to J_\infty g^* \) is flat. Then,

\[
\text{gr} H^{\hat{g}+\bullet}_c(\g, \text{op}(\g, M)) \cong (\text{gr}(J_\infty S) \otimes J_\infty g^* \otimes \text{gr} M) \otimes H^\bullet(\g, \mathbb{C})
\]

for any \( M \in \text{KL}_{\kappa}^{\text{op}} \). If \( W \) is a vertex algebra object \( W \in \text{KL}_{\kappa^*} \), the vertex algebra \( H^{\hat{g}+\bullet}_c(\g, V^{\text{op}} \otimes W) \) is good, and we have

\[
\text{SS}(H^{\hat{g}+\bullet}_c(\g, V^{\text{op}} \otimes W)) \cong J_\infty S \times J_\infty g^* \text{SS}(W)
\]

and

\[
\hat{X}_{H^{\hat{g}+\bullet}_c(\g, V^{\text{op}} \otimes W)} \cong S \times g^* \hat{X}_W.
\]

**Proof.** Note that \( (C(\g, V^{\text{op}} \otimes W), Q(0)) \) is a direct sum of finite-dimensional subcomplexes since \( V^{\text{op}} \otimes W \) is a direct sum of objects in \( \text{KL}_{\kappa}^{\text{op}} \). Note also that \( \text{gr} C(\g, V^{\text{op}} \otimes M) \cong C(\g^{\bullet+}, \text{gr} V^{\text{op}} \otimes \text{gr} M) \cong C(\g^{\bullet+}, \text{gr}(J_\infty X) \otimes \text{gr} M) \). Hence there is a spectral sequence \( E_r \Rightarrow H^{\hat{g}+\bullet}(\g, V^{\text{op}} \otimes M) \) such that

\[
E_1^{n-q, q} = H^{\hat{g}+n}(\g^{\bullet+}, \text{gr}(J_\infty X) \otimes \text{gr} M) \cong (\text{gr}(J_\infty S) \otimes J_\infty g^* \otimes \text{gr} M) \otimes H^n(\g, \mathbb{C}).
\]

Because elements of \( \text{gr}(J_\infty S) \otimes J_\infty g^* \otimes \text{gr} M \) are \( g \)-invariant, it follows that \( d_r \) is identically zero for all \( r \geq 1 \), and we get that

\[
\text{gr} H^{\hat{g}+\bullet}(\g, V^{\text{op}} \otimes M) \cong (\text{gr}(J_\infty S) \otimes J_\infty g^* \otimes \text{gr} M) \otimes H^\bullet(\g, \mathbb{C}).
\]

This completes the proof. \( \square \)

For \( M \in \text{KL}_{-\kappa} \), \( C(\g, M) \) is a \( V^0(\g) \)-module by the action \( x_i \mapsto \hat{x}_i := Q(0)\psi_i, i = 1, \ldots, \dim \g \). Set

\[
C(\g, M) = \{ c \in C(\g, M) \mid (\hat{x}_i)(0)c = (\psi_i)(0)c = 0, \forall i = 1, \ldots, \dim \g \}.
\]

Then \( C(\g, M) \) is a subcomplex of \( C(\g, M) \), and the cohomology of the complex \( (C(\g, M), Q(0)) \) is the relative cohomology \( H^{\hat{g}+\bullet}(\g, g, M) \) with coefficient in \( M \). If \( V \) is a vertex algebra object in \( \text{KL}_{-\kappa} \), \( H^{\hat{g}+\bullet}(\g, g, V) \) is naturally a vertex superalgebra.

**Proposition 3.2.** For \( M \in \text{KL}_{-\kappa} \),

\[
H^{\hat{g}+\bullet}(\g, g, M) \cong H^{\hat{g}+\bullet}(\g, g, M) \otimes H^\bullet(\g, \mathbb{C}).
\]
Proof. We may assume that $M$ is finitely generated. Consider the Hochschild-Serre spectral sequence $E_r \Rightarrow H^{\infty/2+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, M)$ for the subalgebra $\mathfrak{g} \subset \widehat{\mathfrak{g}}_{-\kappa_g}$. In the first term we have

$$E_1^{p,q} \cong C^p(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, M) \otimes H^q(\mathfrak{g}, \mathbb{C}).$$

In the second term we have

$$(15) \quad E_2^{p,q} \cong H^p(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, M) \otimes H^q(\mathfrak{g}, \mathbb{C}).$$

We can therefore represent classes in $E_2^{p,q}$ as sums of tensor products $\omega_1 \otimes \omega_2$ of a cocycle $\omega_1$ in $C^p(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, M)$ representing a class in $H^p(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, M)$ and a cocycle $\omega_2$ representing a class in $H^q(\mathfrak{g}, \mathbb{C})$. Applying the differential to this class, we find that it is identically equal to zero because $\omega_1$ is $\mathfrak{g}$-invariant. Therefore all the classes in $E_2$ survive. Moreover, all of the vectors of the two factors in the decomposition (15) lift canonically to the cohomology $H^{\infty/2+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, M)$, and so we obtain the desired isomorphism. \hfill \Box

The following assertion follows immediately from Theorem 3.1 and Proposition 3.2.

**Theorem 3.3.** Let $V, S, M$, be as in Theorem 3.1. Then we have

$$\text{gr } H^{\infty/2+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, V^{op} \otimes M) \cong \begin{cases} O(J_\infty S) \otimes J_\infty \mathfrak{g}^* & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$SS(H^{\infty/2+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, V^{op} \otimes W)) \cong J_\infty S \times J_\infty \mathfrak{g}^* \times SS(W),$$

and

$$\tilde{X}_{H^{\infty/2+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, V^{op} \otimes W))} \cong S \times \mathfrak{g}^* \times \tilde{X}_W,$$

for a vertex algebra object $W$ in $\text{KL}_{\kappa_\mathfrak{g}}$.

**Proposition 3.4.** Let $M \in \text{KL}_{-\kappa_g}$. Suppose that $M$ is free over $U(t^{-1} \mathfrak{g}[t^{-1}])$ and cofree over $U(t \mathfrak{g}[t])$. Then $H^{\infty/2+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, M) = 0$ for $i \neq 0$.

Proof, Since $M$ is free over $U(t^{-1} \mathfrak{g}[t^{-1}])$, we obtain that $H^{\infty/2+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, M) = 0$ for $i < 0$ by considering the Hochschild-Serre spectral sequence for the subalgebra $t^{-1} \mathfrak{g}[t^{-1}] \subset \widehat{\mathfrak{g}}_{-\kappa_g}$, see [Vor, Theorem 2.3]. Hence by Proposition 3.2

$H^{\infty/2+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, M) = 0$ for $i < 0$. Next, consider the Hochschild-Serre spectral sequence $E_r \Rightarrow H^{\infty/2+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, M)$ for the subalgebra $\mathfrak{g}[t] \subset \widehat{\mathfrak{g}}_{-\kappa_g}$ as in [Vor, Theorem 2.2]. By definition, we have $E_1^{p,q} = H^q(\mathfrak{g}[t], M \otimes \bigwedge^- p(\widehat{\mathfrak{g}}/\mathfrak{g}[t])$. Since $M$ is cofree over $U(t \mathfrak{g}[t])$, so is $M \otimes \bigwedge^- p(\widehat{\mathfrak{g}}/\mathfrak{g}[t])$. Hence

$$(16) \quad E_1^{p,q} \cong (M \otimes \bigwedge^- p(\widehat{\mathfrak{g}}/\mathfrak{g}[t])) \otimes H^q(\mathfrak{g}, \mathbb{C}).$$

It follows that $E_1^{p,q} = 0$ for $p > \dim G$, and therefore $H^{\infty/2+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, M) = 0$ for $i > \dim G$. Therefore, by Proposition 3.2 we have $H^{\infty/2+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, M) = 0$ for $i > 0$. \hfill \Box

The form $\kappa_c = -\frac{1}{2} \kappa_\mathfrak{g}$ is called the critical level for $\mathfrak{g}$. For $M, N \in \text{KL}_{\kappa_c}$, we have $M \otimes N \in \text{KL}_{-\kappa_g}$. We set

$$(17) \quad M \circ N = H^{\infty/2+i}(\widehat{\mathfrak{g}}_{-\kappa_g}, \mathfrak{g}, M \otimes N).$$
4. Feigin-Frenkel center

Let
\[ j(\mathfrak{g}) = Z(V^{\infty}(\mathfrak{g})) = V^{\infty}(\mathfrak{g})^0 \]
the Feigin-Frenkel center of the vertex algebra \( V^{\infty}(\mathfrak{g}) \) (\cite{FF2}). We have isomorphisms (\cite{EF, Pre1})
\[ \hat{X}(j(\mathfrak{g})) \cong \mathfrak{g}^* / G, \quad \text{Spec}(\text{gr } j(\mathfrak{g})) \cong J_\infty(\mathfrak{g}^* / G), \quad \text{Zhu}(j(\mathfrak{g})) \cong Z(\mathfrak{g}), \]
where \( Z(\mathfrak{g}) \) denotes the center of \( U(\mathfrak{g}) \). The grading of \( V^{\infty}(\mathfrak{g}) \) induces a grading of \( j(\mathfrak{g}) \):
\[ j(\mathfrak{g}) = \bigoplus_{\Delta \in Z_{\geq 0}} j(\mathfrak{g})_\Delta, \quad j(\mathfrak{g})_\Delta = j(\mathfrak{g}) \cap V^{\infty}(\mathfrak{g})_\Delta. \]
Let \( d_1, \ldots, d_{rk \mathfrak{g}} \) be the exponents of \( \mathfrak{g} \), where \( rk \mathfrak{g} \) is the rank of \( \mathfrak{g} \). Choose homogeneous strong generators \( P_1, \ldots, P_{rk \mathfrak{g}} \in j(\mathfrak{g}) \) with \( P_i \in j(\mathfrak{g})_{d_i + 1} \). Their images form homogeneous generators of \( \text{R}_j(\mathfrak{g}) = \mathcal{O}(\mathfrak{g}^*)^G, \quad \text{gr } j(\mathfrak{g}) = \mathcal{O}(J_\infty \mathfrak{g}^*)^{J_\infty G}, \quad \text{Zhu}(j(\mathfrak{g})) = Z(\mathfrak{g}), \)
respectively. We use the notation
\[ P_i(z) = \sum_{n \in \mathbb{Z}} P_{i,n} z^{-n-1} = \sum_{n \in \mathbb{Z}} P_{i,n} z^{-n-d_i-1}, \]
so that the operator \( P_{i,n} \) has degree \(-n\) on \( V^{\infty}(\mathfrak{g}) \).

Let \( M \in KL_{\infty} \). Then \( j(\mathfrak{g}) \) naturally acts on \( M \), and hence, \( M \) can be regarded as a module of the polynomial ring
\[ Z = \mathbb{C}[P_{i,n}; i = 1, \ldots, rk \mathfrak{g}, n \in \mathbb{Z}] \]
Set
\[ Z_{(\geq 0)} = \mathbb{C}[P_{i,n}; i = 1, \ldots, rk \mathfrak{g}, n \geq 0], \quad Z_{(<0)} = \mathbb{C}[P_{i,n}; i = 1, \ldots, rk \mathfrak{g}, n < 0]; \]
\[ Z_{>0} = \mathbb{C}[P_{i,n}; i = 1, \ldots, rk \mathfrak{g}, n > 0], \quad Z_{<0} = \mathbb{C}[P_{i,n}; i = 1, \ldots, rk \mathfrak{g}, n < 0]; \]
\[ Z_0 = \mathbb{C}[P_{i,0}; i = 1, \ldots, rk \mathfrak{g}] \]
Then \( Z_{(<0)} \subset Z_{<0}, Z_{(\geq 0)} \supset Z_{\geq 0}, \) and \( Z_{(\geq 0)} \cong j(\mathfrak{g}) \). We have the isomorphism
\[ o : Z_0 \cong \text{Zhu}(j(\mathfrak{g})) \cong Z(\mathfrak{g}). \]
For \( \lambda \in P_+ \), set \( V_\lambda = V^{\infty}_\lambda \). We regard \( V_\lambda \) a \( Z_{\geq 0} \)-graded \( \mathfrak{g}_\lambda \)-module by giving the degree zero to the highest weight vector. Let \( L_\lambda \) be the unique simple graded quotient of \( V_\lambda \). Let \( \chi_\lambda : Z \to \mathbb{C} \) be the evaluation at \( L_\lambda \). Since \( L_\lambda \) is graded,
\[ \chi_\lambda(P_{i,n}) = \left\{ \begin{array}{ll} \gamma_\lambda(o(P_{i,n})) & \text{for } n = 0 \\ 0 & \text{for } n \neq 0, \end{array} \right. \]
where \( \gamma_\lambda : Z(\mathfrak{g}) \to \mathbb{C} \) is the evaluation at \( \mathfrak{g} \).
\( V_\lambda \). Let \( KL[\lambda] \) be the full subcategory of \( KL \) consisting of objects \( M \) such that \( \chi_\lambda(P_{i,0})m = \gamma_\lambda(o(P_{i,0}))m \) for all \( i \). Then
\[ KL = \bigoplus_{\lambda \in P_+} KL[\lambda]. \]
For \( M \in KL \), let \( M = \bigoplus_{\lambda \in P_+} M[\lambda], \quad M[\lambda] \in KL[\lambda] \), be the corresponding decomposition. Clearly, \( V_\lambda, L_\lambda \in KL[\lambda] \) and any simple object in \( KL[\lambda] \) is isomorphic to \( L_\lambda[d] \) for some \( d \in \mathbb{C} \), where \( L_\lambda[d] \) denotes the \( \mathfrak{g}_\lambda \)-module \( L_\lambda \) whose grading is shifted as \( (L_\lambda[d])_{d'} = (L_\lambda)_d ), \)
We denote by \( Z\text{-Mod} \) the category of positive energy representations of the vertex algebra \( j(\mathfrak{g}) \), that is, the category of \( Z \)-modules \( M \) that admits a grading \( M = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} M_d, \quad h \in \mathbb{C} \), such that \( P_{i,n}M_d \subset M_{d-n} \) for all \( i \) and \( n \). For \( M \in Z\text{-Mod}, \) we set \( ch M = \sum_{d \in \mathbb{C}} q^d \dim M_d \) if \( \dim M_d < \infty \) for all \( d \). Let
$\cal Z\text{-Mod}^{[\lambda]}$ be the full subcategory of $\cal Z\text{-Mod}$ consisting of objects $M$ on which $P_{i,0}$ acts as $\chi_{\lambda}(P_{i,0})$ for all $i = 1, \ldots, \text{rk}\, \mathfrak{g}$, and set $\cal Z\text{-Mod}_{\text{reg}} = \bigoplus_{\lambda \in P_{+}} \cal Z\text{-Mod}^{[\lambda]}$.

5. Chiral Differential Operators on $G$

There are two commuting Hamiltonian actions on the cotangent bundle $T^*G = G \times \mathfrak{g}^*$ of $G$ given by $g.(h, x) = (hg^{-1}, gx)$ and $g.(h, x) = (gh, x)$. The corresponding moment maps are

$$
\mu_L : G \times \mathfrak{g}^* \to \mathfrak{g}^*, \quad (g, x) \mapsto x,
$$

and

$$
\mu_R : G \times \mathfrak{g}^* \to \mathfrak{g}^*, \quad (g, x) \mapsto g.x,
$$

respectively. The algebra $G$ on operators $\cal D$ identity. There exists a unique vertex algebra structure on $\cal Z\text{-Mod}$.

By (11), we have $H_{\cal Z}\otimes^+ \mathfrak{g} = \mathfrak{g}$ and (12), $H_{\cal Z}\otimes^+ \mathfrak{g} = \mathfrak{g}$.

Proof. By (11), we have $H_{\cal Z}\otimes^+ \mathfrak{g} = \mathfrak{g}$ and (12), $H_{\cal Z}\otimes^+ \mathfrak{g} = \mathfrak{g}$.

Lemma 5.1. (i) For any Poisson algebra object $R$ in $\text{Coh}^G(\mathfrak{g}^*)$, we have

$$
H_{\cal Z}\otimes^+ \mathfrak{g}, \mathcal{O}(T^*G) \otimes R \cong R.
$$

(ii) For any vertex Poisson algebra object $V$ in $\text{Coh}^G(\mathfrak{g}^*)$, we have

$$
H_{\cal Z}\otimes^+ \mathfrak{g}, \mathcal{O}(J^\infty(T^*G) \otimes V) \cong V.
$$

Proof. Let $\kappa$ be an invariant symmetric bilinear form of $\mathfrak{g}$ as before. Let

$$
\mathcal{D}^{ch}_{G,\kappa} = U(\mathfrak{g}[t]) \otimes U(\mathfrak{g}[t]) \otimes \mathfrak{c}_1 \otimes \mathcal{O}(J^\infty(G)),
$$

where $\mathfrak{g}[t] = J^\infty \mathfrak{g}$ acts on $\mathcal{O}(J^\infty(G))$ by $x \mapsto x_L := (J^\infty \mu_L)^*(x)$ and $1$ acts as the identity. There exists a unique vertex algebra structure on $\mathcal{D}^{ch}_{G,\kappa}$ such that

$$
\pi_L : V^\kappa(\mathfrak{g}) \hookrightarrow \mathcal{D}^{ch}_{G,\kappa}, \quad u[0] \mapsto u \otimes 1_{J^\infty(G)} (u \in U(\mathfrak{g}[t])),
$$

$$
\mathcal{O}(J^\infty(G)) \hookrightarrow \mathcal{D}^{ch}_{G,\kappa}, \quad f \mapsto 1 \otimes f,
$$

are homomorphisms of vertex algebras, and

$$
x(z)f(w) \sim \frac{1}{z-w}(x_L f)(w) \quad (x \in \mathfrak{g} \subset V^\kappa(\mathfrak{g}), \quad f \in \mathcal{O}(G) \subset \mathcal{O}(J^\infty(G))
$$

(BMS2 AG). The vertex algebra $\mathcal{D}^{ch}_{G,\kappa}$ is called the algebra of chiral differential operators (cdo) on $G$ at level $\kappa$, which is the special case of the chiral differential operators on a smooth algebraic variety introduced independently by Malikov, Schechtman and Vaintrob MSV and Beilinson and Drinfeld BD. The cdo $\mathcal{D}^{ch}_{G,\kappa}$ is a strict chiral quantization of the cotangent bundle $T^*G$ to $G$. In particular, $\mathcal{D}^{ch}_{G,\kappa}$ is simple for any $\kappa$ by Theorem 2.2.

We have $\mathcal{D}^{ch}_{G,\kappa} \to (\mathcal{D}^{ch}_{G,\kappa})^{op}$, $x \mapsto -x$, $x \in \mathfrak{g}$, $f \mapsto f$, $f \in \mathcal{O}(G)$, and so we do not distinguish between $\mathcal{D}^{ch}_{G,\kappa}$ and $(\mathcal{D}^{ch}_{G,\kappa})^{op}$.

According to GMS2 AG, there is a vertex algebra embedding

$$
\pi_R : V^\kappa(\mathfrak{g}) \hookrightarrow \mathcal{D}^{ch}_{G,\kappa},
$$
and \( V^\kappa(g) \) and \( V^{\kappa*}(g) \) form a dual pair in \( D^g_{G,\kappa} \), i.e.,

\[
(20) \quad V^\kappa(g) \cong \text{Com}(V^{\kappa*}(g), D^g_{G,\kappa}) = (D^g_{G,\kappa})^{\pi_R(g[t])},
\]

\[
(21) \quad V^{\kappa*}(g) = \text{Com}(V^\kappa(g), D^g_{G,\kappa}) = (D^g_{G,\kappa})^{\pi_L(g[t])}.
\]

Suppose that \( \kappa \neq \kappa_c \), and let \( \omega_L \) and \( \omega_R \) be the Sugawara conformal vector of \( V^\kappa(g) \) and \( V^{\kappa*}(g) \), respectively. Then

\[
\omega_{D^g_{G,\kappa}} = \omega_L + \omega_R
\]
gives a conformal vector of \( D^g_{G,\kappa} \) of central charge \( 2 \dim G \). In fact, the left-hand side makes sense even at the critical level \( \kappa = \kappa_c \), and \( \omega_{D^g_{G,\kappa}} \) gives a conformal vector of \( D^g_{G,\kappa} \) of central charge \( 2 \dim G \) for all \( \kappa \) (GMS1). We have

\[
D_{G,\kappa} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (D_{G,\kappa})^\Delta, \quad (D_{G,\kappa})_0 \cong O(G) \cong \bigoplus_{\lambda \in P_+} V_\lambda \otimes V_\Lambda,
\]

where \( \lambda^* = -w_0(\lambda) \) and \( w_0 \) is the longest element of the Weyl group \( W \) of \( g \).

**Proposition 5.2.** The cdo \( D^g_{G,\kappa} \) is a direct sum of objects in \( KL^{\kappa,\text{ord}} \) (resp. \( KL^{\kappa*}_{\text{ord}} \)).

**Proof.** First, suppose that \( \kappa = \kappa_c \). Consider the decomposition

\[
D^g_{G,\kappa_c} = \bigoplus_{\lambda \in P_+} (D^g_{G,\kappa_c})[\lambda]
\]

with respect to the action \( \pi_R \). Clearly, each direct summand \( (D^g_{G,\kappa_c})[\lambda] \) is closed under the action of both \( \pi_L \) and \( \pi_R \) of \( \mathfrak{g}_{\kappa_c} \). Thus, it is enough to show that \( (D^g_{G,\kappa_c})[\lambda],\Delta = (D^g_{G,\kappa_c})[\lambda] \cap (D^g_{G,\kappa_c})^\Delta \) is finite-dimensional for all \( \lambda, \Delta \). By the definition (19), \( D^g_{G,\kappa_c} \) is cofree over \( U(tg[t]) \), and we have

\[
(D^g_{G,\kappa_c})_t g[t] \cong U(\mathfrak{g}_{\kappa_c}) \otimes U(g[t] \oplus C_1) O(G) \cong \bigoplus_{\lambda \in P_+} V_\lambda \otimes V_\Lambda.
\]

Hence the direct summand \( (D^g_{G,\kappa_c})[\lambda] \) is also cofree over \( U(tg[t]) \) and we have

\[
(D^g_{G,\kappa_c})[\lambda] \cong V_\lambda \otimes V_\Lambda \quad \text{as graded vector spaces}
\]

(not as \( \mathfrak{g}_{\kappa_c}, \bar{\mathfrak{g}}_{\kappa_c} \)-modules), and we are done.

Next suppose that \( \kappa \neq \kappa_c \). Let \( L^L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \) and \( L^R(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \) be the fields corresponding to \( \omega_L \) and \( \omega_R \), respectively. Set \( D^g_{G,\kappa}[d_1, d_2] = \{ v \in D^g_{G,\kappa}, L^L_0 v = d_1 v, L^R_0 v = d_2 v \} \), so that \( (D_{G,\kappa})^\Delta = \bigoplus_{d_1 + d_2 = \Delta} D^g_{G,\kappa}[d_1, d_2] \). We have

\[
D^g_{G,\kappa} = \bigoplus_{d \in \mathbb{C}} D^g_{G,\kappa}[\bullet, d] = \bigoplus_{d \in \mathbb{C}} D^g_{G,\kappa}[d, \bullet],
\]

where \( D^g_{G,\kappa}[\bullet, d] = \bigoplus_{d' \in \mathbb{C}} D^g_{G,\kappa}[d', d], \ D^g_{G,\kappa}[d, \bullet] = \bigoplus_{d' \in \mathbb{C}} D^g_{G,\kappa}[d, d'] \). Therefore, it is sufficient to show that \( D^g_{G,\kappa}[\bullet, d] \) and \( D^g_{G,\kappa}[d, \bullet] \) are objects of \( KL^{\kappa,\text{ord}} \) and \( KL^{\kappa*,\text{ord}} \), respectively. This is equivalent to that \( D^g_{G,\kappa}[d_1, d_2] \) is finite-dimensional for all \( d_1, d_2 \in \mathbb{C} \). Since the argument is symmetric, we may assume that \( (\kappa - \kappa_c) / \kappa_g \notin \mathbb{Q}_{\leq 0} \). By [Zhu2], there exists an increasing filtration

\[
0 = M_1 \subset M_1 \subset \ldots, \quad D^g_{G,\kappa} = \bigcup_p M_p,
\]
such that
\[ \bigoplus_p M_p/M_{p-1} \cong \bigoplus_{\lambda \in P_+} V^\kappa_\lambda \otimes D(V^\kappa_\lambda), \]
where \( D(V^\kappa_\lambda) \) is the contradirectional dual of \( V^\kappa_\lambda \). However, for a given \( d_1 \), there exists only finitely many \( \lambda \in P_+ \) such that \( (V^\kappa_\lambda)_{d_1} \neq 0 \), where \( (V^\kappa_\lambda)_{d_1} \) denotes the \( L^\kappa_\lambda \)-eigenspace of \( V^\kappa_\lambda \) with eigenvalue \( d_1 \). It follows that \( \mathcal{D}^{ch}_{G,\kappa}[d_1,d_2] \) is finite-dimensional for all \( d_1,d_2 \in \mathbb{C} \).

By Proposition \[5.2\], \( \mathcal{D}^{ch}_{G,\kappa} \) is a conformal vertex algebra object in \( KL_\kappa \) (resp. \( KL_{\kappa'} \)) with the chiral quantum moment map \( \pi_L \) (resp. \( \pi_R \)).

Let \( M \in KL_{\kappa'} \). According to [AG] Theorem 6.5, we have the canonical isomorphism
\[ \mathcal{D}^{ch}_{G,\kappa} \otimes M \cong \mathcal{D}^{ch}_{G,\kappa-\kappa'} \otimes M \]
for \( M \in KL_{\kappa'} \).

In terms of the notation \([17]\) we have
\[ \mathcal{D}^{ch}_{G,\kappa} \circ M \cong M \cong M \circ \mathcal{D}^{ch}_{G,\kappa} \]
for \( M \in KL_{\kappa'} \).

We have the canonical isomorphism \([ACM]\)
\[ \text{Zhu} \mathcal{D}^{ch}_{G,\kappa} \cong \mathcal{D}_G, \]
where \( \mathcal{D}_G \) denotes the algebra of global differential operators on \( G \). Under this identification, the filtration of \( \text{Zhu} \mathcal{D}^{ch}_{G,\kappa} \) coincides with the natural filtration on \( \mathcal{D}_G \) and the map \([3]\) for \( V^\kappa(g) \) recovers the well-known isomorphism \( \mathcal{O}(T^*G) \rightarrow \mathfrak{g} \mathcal{D}_G \).

The algebra homomorphism between \( \text{Zhu} \)'s algebras induced by \( \pi_L \) and \( \pi_R \) are the embeddings \( U(\mathfrak{g}) \hookrightarrow \mathcal{D}_G \) defined by \( \mathfrak{g} \ni x \mapsto x_L \) and \( \mathfrak{g} \ni x \mapsto x_R \), respectively.

For a \( \mathcal{D}_G \)-module \( M \), define the \( \mathcal{D}^{ch}_{G,\kappa} \)-module \( \text{Ind}^{\mathcal{D}^{ch}_{G,\kappa}}_{\mathcal{D}_G} M \) by
\[ \text{Ind}^{\mathcal{D}^{ch}_{G,\kappa}}_{\mathcal{D}_G} M = U(\hat{\mathfrak{g}}_\kappa) \otimes U(\mathfrak{g}[t]) \otimes \mathbb{C}1 \left( \mathcal{O}(J_\infty G) \otimes \mathcal{O}(G)M \right), \]
where \( \mathfrak{g}[t] \oplus \mathbb{C}1 \) acts on \( \mathcal{O}(J_\infty G) \otimes \mathcal{O}(G)M \) by \( (u(t))(g \otimes m) = (u(t)g) \otimes m + g \otimes (u(0)m) \) for \( u \in \mathfrak{g}[t] \), \( f \in \mathcal{O}(J_\infty G) \), \( m \in M \).

Let \( \mathcal{D}^{ch}_{G,\kappa} \text{-Mod} \) be the category of positive energy representations of \( \mathcal{D}^{ch}_{G,\kappa} \), and let \( \mathcal{D}_G \text{-Mod} \) be the category of \( \mathcal{D}_G \)-modules.

**Theorem 5.3** ([ACM] Theorem 5.2). The functor \( \text{Ind}^{\mathcal{D}^{ch}_{G,\kappa}}_{\mathcal{D}_G} : \mathcal{D}_G \text{-Mod} \rightarrow \mathcal{D}^{ch}_{G,\kappa} \text{-Mod} \) gives an equivalence of categories.

**Corollary 5.4.** Any \( M \in \mathcal{D}^{ch}_{G,\kappa} \text{-Mod} \) is free over \( U(t^{-1}\mathfrak{g}[t^{-1}]) \) and cofree over \( U(t\mathfrak{g}[t]) \).
6. QUANTUM DRINFELD-SOKOLOV REDUCTION AND EQUIVARIANT AFFINE $W$-ALGEBRAS

Let $f$ be a nilpotent element of $\mathfrak{g}$, $\{e, h, f\}$ an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ associated with $f$,

$$S_f = f + \mathfrak{g}^e \subset \mathfrak{g} \cong \mathfrak{g}^*$$

the Slodowy slice at $f$, where $\mathfrak{g}^e$ is the centralizer of $e$ in $\mathfrak{g}$. It is known that $S_f$ is a Poisson variety, where the Poisson structure of $S_f$ is described as follows ([GG]).

Set

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [h, x] = 2jx\},$$

so that $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$. Choose a Lagrangian subspace $I$ of $\mathfrak{g}_{1/2}$ with respect to the symplectic form $\mathfrak{g}_{1/2} \times \mathfrak{g}_{1/2} \to \mathbb{C}$, $(x, y) \mapsto (f[x, y])$. Then $\mathfrak{m} := I \oplus \bigoplus_{j \geq 1} \mathfrak{g}_j$ is a nilpotent Lie subalgebra of $\mathfrak{g}$. Let $M$ be the unipotent subgroup of $G$ corresponding to $\mathfrak{m}$. Then the projection $\mu : \mathfrak{g}^* \to \mathfrak{m}^*$ is the moment map for the $M$-action. Let $\chi : \mathfrak{m} \to \mathbb{C}$ be the character defined by $\chi(x) = (x, f)$. Then $\chi$ is a regular value of $\mu$. Moreover, we have the isomorphism of the affine varieties

$$M \times S_f \to \mu^{-1}(\chi) = \chi + \mathfrak{m}^\perp, \quad (g, s) \mapsto g.s$$

Hence,

$$S_f \cong \mu^{-1}(\chi)/M = \mathfrak{g}^*/\chi M$$

has the structure of a reduced Poisson variety.

According to [Gin], this construction can be generalized as follows. Let $R$ be a Poisson algebra object in $\text{QCoh}^G(\mathfrak{g}^*)$, $X = \text{Spec} R$. Then the moment map $\mu : X \to \mathfrak{g}^* \to \mathfrak{m}^*$ for the $M$-action is flat. We get the reduced Poisson scheme

$$X/\chi M = \mu^{-1}(\chi)/M = S_f \times \mathfrak{g}^* X.$$

By applying this to $X = T^*G = G \times \mathfrak{g}^*$ for the moment map $\mu_L$, we obtain the equivariant Slodowy slice [Los].

$$S_{G, f} = G \times M (f + \mathfrak{m}^+)$$

at $f$. The Poisson variety $S_{G, f}$ is an example of a twisted contangent bundle over $G/M$ ([GG 1.5]). In particular, $S_{G, f}$ is a smooth symplectic variety. We have $S^\text{op}_{G, f} \cong S_{G, f}$ since $(T^*G)^\text{op} \cong T^*G$. We identify $S_{G, f}$ with $G \times S_f$ via the $G$-equivariant isomorphism

$$\nu : S_{G, f} = G \times M (\chi + \mathfrak{m}^+) \to G \times S_f, \quad (g, (n, s)) \mapsto (gn, s),$$

($g \in G, n \in M, s \in S_f$). The action of $G$ on $S_{G, f}$ is Hamiltonian with the moment map

$$\mu : S_{G, f} = G \times S_f \to \mathfrak{g}^*, \quad (g, s) \mapsto g.s,$$

which is flat ([Slo]). Hence, by [Lo], (28), we have

$$H^*_{\mu^*}(g, \mathcal{O}(S_{G, f}) \otimes R) \cong \mathcal{O}(S_f \times \mathfrak{g}^* \text{Spec } R) \otimes H^*_{\mu^*}(g, \mathbb{C})$$

for a Poisson algebra object $R$ in $\text{QCoh}^G(\mathfrak{g}^*)$.

The same argument as in [Slo] shows that

$$J_n \mu : J_n S_{G, f} = J_n G \times J_n S_f \to J_n \mathfrak{g}^*$$
is flat for all \( n \in \mathbb{Z}_{>0} \), and hence the same is true for \( n = \infty \). Thus, by (12), we have
\[
H^{+\bullet}(\mathfrak{g}^\ell, \mathcal{O}(J_{\infty}S_{G,f}) \otimes V) \cong \mathcal{O}(J_{\infty}S_f \times_{\mathfrak{g}^*} SS(V)) \otimes H^\bullet(\mathfrak{g}, \mathbb{C}).
\]
for any Poisson vertex algebra object \( V \) in \( \text{Coh}^{f=0}(J_{\infty}\mathfrak{g}^*) \).

For \( M \in \text{KL}_\kappa \), let \( H^0_{DS,f}(M) \) be the BRST cohomology of the quantized Drinfeld-Sokolov reduction associated with \( f \) (and with the Dynkin grading) with coefficients in \( M \) (FF1, KRW), which is defined as follows. Set \( \mathfrak{g}_{\geq 1} = \bigoplus_{j \geq 1} \mathfrak{g}_j \subset \mathfrak{g}_{>0} = \bigoplus_{j > 0} \mathfrak{g}_j \). Then \( \tilde{\chi} : \mathfrak{g}_{\geq 1}[t, t^{-1}] \to \mathbb{C}, x t^n \mapsto \delta_{n,-1}(f|x) \), defines a character. Let \( F_{\tilde{\chi}} = U(\mathfrak{g}_{>0}[t, t^{-1}]) \otimes U(\mathfrak{g}_{>0}[t]+\mathfrak{g}_{\geq 1}[t^{-1}]) \mathbb{C}_{\tilde{\chi}} \), where \( \mathbb{C}_{\tilde{\chi}} \) is the one-dimensional representation of \( \mathfrak{g}_{>0}[t] + \mathfrak{g}_{\geq 1}[t, t^{-1}] \) on which \( \mathfrak{g}_{\geq 1}[t] \) acts by the character \( \tilde{\chi} \) and \( \mathfrak{g}_{>0}[t] \) acts trivially. We have
\[
H^0_{DS,f}(M) = H^{+\bullet}(\mathfrak{g}_{>0}[t, t^{-1}], M \otimes F_{\tilde{\chi}}),
\]
where \( H^{+\bullet}(\mathfrak{g}_{>0}[t, t^{-1}], N) \) is the semi-infinite \( \mathfrak{g}_{>0}[t, t^{-1}] \)-cohomology [Fel] with coefficients in a \( \mathfrak{g}_{>0}[t, t^{-1}] \)-module \( N \).

The (affine) \( W \)-algebra associated with \( (\mathfrak{g}, f) \) at level \( \kappa \) is by definition [FF1, KRW] the vertex algebra
\[
\mathcal{W}^\kappa(\mathfrak{g}, f) = H^0_{DS,f}(V^\kappa(\mathfrak{g})).
\]
The vertex algebra \( \mathcal{W}^\kappa(\mathfrak{g}, f) \) is a strict chiral quantization of the Slodowy slice \( S_f \) ([DSK, A4]).

More generally, if \( V \) is a vertex algebra object in \( \text{KL}_\kappa \), then \( H^0_{DS,f}(V) \) is a vertex algebra, and \( \mu_V : V^\kappa(\mathfrak{g}) \to V \) induces a vertex algebra homomorphism \( \mathcal{W}^\kappa(\mathfrak{g}, f) \to H^0_{DS,f}(V) \). If \( V \) is a conformal vertex algebra object in \( \text{KL}_\kappa \) with central charge \( c_V \), then it follows from [KRW, §2.2] that \( H^0_{DS,f}(V) \) is conformal with central charge
\[
c_V - \dim \mathcal{O}_f - \frac{3}{2} \dim \mathfrak{g}_{1/2} + 12(\rho|\rho) - 3(k + h^\vee)|h|^2,
\]
where \( \mathcal{O}_f = G.f \) and \( k = 2(h^\vee)^2/\kappa \). Note that \( \dim \mathcal{O}_f = \dim \mathfrak{g} - \dim \mathfrak{g}_0 - \dim \mathfrak{g}_{1/2} \).

**Theorem 6.1.**

(i) ([A4]) For \( M \in \text{KL}_\kappa \) we have \( H^0_{DS,f}(M) = 0 \) for \( i \neq 0 \). Therefore the functor \( \text{KL}_\kappa \to \mathcal{W}^\kappa(\mathfrak{g}, f)\text{-Mod}, M \mapsto H^0_{DS,f}(M) \), is exact.

(ii) ([A4]) For \( M \in \text{KL}_{\kappa^d} \), we have
\[
\text{gr} H^0_{DS,f}(M) \cong \mathcal{O}(J_{\infty}S_f) \otimes_{\mathcal{O}(J_{\infty}\mathfrak{g}^*)} \text{gr} M.
\]
Hence, if \( V \) is a vertex algebra object in \( \text{KL}_\kappa \), then the vertex algebra \( H^0_{DS,f}(V) \) is good, and \( \text{gr} H^0_{DS,f}(V) \cong \mathcal{O}(J_{\infty}S_f) \otimes_{\mathcal{O}(J_{\infty}\mathfrak{g}^*)} (\text{gr} V) \). In particular,
\[
\tilde{X} H^0_{DS,f}(V) \cong \tilde{X} V \times_{\mathfrak{g}^*} S_f.
\]

(iii) ([A3]) For any vertex algebra object \( V \) in \( \text{KL}_\kappa \), we have
\[
\text{Zhu}(H^0_{DS,f}(V)) \cong H^0_{DS}(\text{Zhu}(V)),
\]
where in the right-hand side \( H^0_{DS}(? \right) \) is the finite-dimensional analogue of the quantized Drinfeld-Sokolov reduction as described in [A3].
Corollary 6.2. For a vertex algebra object $V$ in $KL_\kappa$, the vertex algebra $H^0_{DS,f}(V)$ is a chiral quantization of $X_V \times g^* S_f$. If in addition $V$ is a strict chiral quantization of $X = \tilde{X}_V$, then $H^0_{DS}(V)$ is a strict chiralization of $X \times g^* S_f$.

Define the equivariant affine $W$-algebra associated with $(G, f)$ at level $\kappa$ by

$$W^\kappa_{G,f} := H^0_{DS}(D^\kappa_{G,\kappa}).$$

By Corollary 6.2, $W^\kappa_{G,f}$ is a strict chiralization of

$$\tilde{X} W^\kappa_{G,f} = X_{D^\kappa_{G,\kappa}} \times g^* S_f \cong S_{G,f}$$

Since $S_{G,f}$ is a smooth symplectic variety, $W^\kappa_{G,f}$ is simple by Theorem 2.2. Because $D^\kappa_{G,\kappa}$ is conformal with central charge 2, $W^\kappa_{G,f}$ is conformal with central charge

$$\dim g + \dim g_0 - \frac{1}{2} \dim g_{1/2} + 12(\rho | h) - 3(k + h^\vee) | h |^2.$$  

(33)

We have $W^\kappa_{G,f} = \bigoplus_{\Delta \in \Delta^g} (W^\kappa_{G,f})_{\Delta}$ and the conformal weight of $W^\kappa_{G,f}$ is not bounded from the below.

We have $W^\kappa_{G,f} \cong (W_{G,f})^{op}$ since $D^\kappa_{G,\kappa} \cong (D^\kappa_{G,\kappa})^{op}$, and so we do not distinguish between them.

By Corollary 6.2, the vertex algebra homomorphism $\pi_L : V^\kappa(g) \to D^\kappa_{G,\kappa}$ induces the embedding

$$W^\kappa(g, f) \hookrightarrow W^\kappa_{G,f},$$

which we also denote by $\pi_L$. The map $\text{gr} \pi_L : \text{gr} W^\kappa(g, f) = \mathcal{O}(J_\infty S_f) \to \text{gr} W^\kappa_{G,f} = \mathcal{O}(J_\infty S_{G,f})$ is induced by the projection $J_\infty S_{G,f} \cong J_\infty (G \times S_f) \cong J_\infty G \times J_\infty S_f \to J_\infty S_f$. On the other hand, the vertex algebra homomorphism $\pi_R : V^\kappa(g) \to D^\kappa_{G,\kappa}$ induces the vertex algebra homomorphism

$$V^\kappa(g, f) \to W^\kappa_{G,f},$$

which we denote also by $\pi_R$. In particular, $W^\kappa_{G,f}$ is a $\hat{g}_\kappa$-$\kappa$-module. The map $\text{gr} \pi_R : \text{gr} V^\kappa(g) = \mathcal{O}(J_\infty g^*) \to \text{gr} W^\kappa_{G,f} = \mathcal{O}(J_\infty S_{G,f})$ is the dual to the chiral moment map

$$J_\infty \mu : J_\infty S_{G,f} = J_\infty G \times J_\infty S_f \to J_\infty g^*, \quad (g, f) \to g f.$$  

(34)

Proposition 6.3. The $\hat{g}_\kappa$-$\kappa$-module $W^\kappa_{G,f}$ is free over $U(g[t^{-1}]t^{-1})$. Therefore, there exists a filtration $0 = M_0 \subset M_1 \subset M_2 \subset \ldots$, $\text{gr} W^\kappa_{G,f} = \bigcup_p M_p$, as a $\hat{g}_\kappa$-$\kappa$-module such that each successive quotient is isomorphic to $V^\kappa_\lambda$ for some $\lambda \in P_+.$

Proof. Since $W^\kappa_{G,f}$ is an object of $KL_\kappa$, it is sufficient to show that we have $H_i(g[t^{-1}]t^{-1}, W^\kappa_{G,f}) = 0$ for $i > 0$. We consider the spectral sequence $E_r \Rightarrow H^*_{t^{-1} g[t^{-1}]}(W^\kappa_{G,f})$ such that the $E_1$-term is isomorphic to $H^1_{t^{-1} g[t^{-1}], \text{gr} W^\kappa_{G,f}} \cong H^1_{t^{-1} g[t^{-1}], \mathcal{O}(J_\infty S_{G,f})}$, that is, the Koszul homology of $\mathcal{O}(J_\infty S_{G,f})$ as a $\mathcal{O}(J_\infty g^*)$-module. Since $J_\infty \mu$ is a flat, we have that $H_i(g[t^{-1}]t^{-1}, \mathcal{O}(J_\infty S_{G,f})) = 0$ for $i > 0$. Hence the spectral sequence collapses at $E_1 = E_\infty$, and we get that $H_i(g[t^{-1}]t^{-1}, W^\kappa_{G,f}) = 0$ for $i > 0$ as required.

By Proposition 6.3, it follows that the vertex algebra homomorphism $\pi_R : V^\kappa(g) \to W^\kappa_{G,f}$ is injective.
Proposition 6.4. The $\hat{g}_\kappa$-module $W_{G,f}^\kappa$ is cofree over $U(\mathfrak{g}[t])$ and we have

$$(W_{G,f}^\kappa)^{\mathfrak{g}[t]_i} \cong \bigoplus_{\lambda \in P_+} H_{D,S,f}^0(\mathcal{V}_\lambda^{\kappa}) \otimes V_\lambda,$$

as $W^\kappa(\mathfrak{g}, f) \otimes U(\mathfrak{g})$-module. In particular,

$W^\kappa(\mathfrak{g}, f) \cong \text{Com}(V^{\kappa^*}(\mathfrak{g}), W_{G,f}^\kappa) = (W_{G,f}^\kappa)^{\mathfrak{g}[t]}$

as vertex algebras.

Proof. Consider the spectral sequence $E_r \Rightarrow H^*(\mathfrak{g}[t], W_{G,f}^\kappa)$ such that the $E_1$-term is isomorphic to $H^*(\mathfrak{g}[t], \text{gr} W_{G,f}^\kappa)$. Since $\text{gr} W_{G,f}^\kappa \cong O(S_{G,f})$, we have $H^*(\mathfrak{g}[t], \text{gr} W_{G,f}^\kappa) \cong \delta_{i,0}O(J_{\infty}S_f) \otimes O(G)$. It follows that the spectral sequence collapses at $E_1 = E_\infty$, and we get that

$$(35) \; \text{gr} H^*(\mathfrak{g}[t], W_{G,f}^\kappa) \cong \delta_{i,0}O(J_{\infty}S_f) \otimes O(G).$$

In particular, $W_{G,f}^\kappa$ is cofree over $U(\mathfrak{g}[t])$.

On the other hand, by Theorem 5.11 the embedding

$$(D_{G,K}^{ch})^{\mathfrak{g}[t]} \cong \bigoplus_{\lambda \in P_+} \mathcal{V}_\lambda^{\kappa} \otimes V_\lambda \hookrightarrow D_{G,K}^{ch}$$

induces the embedding

$$H_{D,S,f}^0((D_{G,K}^{ch})^{\mathfrak{g}[t]}) \cong \bigoplus_{\lambda \in P_+} H_{D,S,f}^0(\mathcal{V}_\lambda^{\kappa}) \otimes V_\lambda \hookrightarrow W_{G,f}^\kappa,$$

and the image is contained in $(W_{G,f}^\kappa)^{\mathfrak{g}[t]}$. Since $\text{gr} H_{D,S,f}^0(\mathcal{V}_\lambda^{\kappa}) \cong O(J_{\infty}S_f) \otimes V_\lambda$, we have $\text{gr} \left( \bigoplus_{\lambda \in P_+} H_{D,S,f}^0(\mathcal{V}_\lambda^{\kappa}) \otimes V_\lambda \right) \cong O(J_{\infty}S_f) \otimes O(G)$. Therefore, by (35), we get that $(W_{G,f}^\kappa)^{\mathfrak{g}[t]} \cong \bigoplus_{\lambda \in P_+} H_{D,S,f}^0(\mathcal{V}_\lambda^{\kappa}) \otimes V_\lambda$.

Proposition 6.5. The $\hat{g}_\kappa$-module $W_{G,f}^\kappa$ is a direct sum of objects in $\text{KL}^{ord}_{\kappa^*}.$

Proof. First, suppose that $\kappa = \kappa_c$. Consider the decomposition

$$W_{G,f}^\kappa = \bigoplus_{\lambda \in P_+} W_{G,f}^\kappa \big|_{[\lambda]}.$$

Then $(W_{G,f}^\kappa)_{[\lambda]} = H_{D,S,f}^0((D_{G,K}^{ch})_{[\lambda]})$. In the same way as in the proof of Proposition 5.2, using Proposition 6.4, we find that $(W_{G,f}^\kappa)_{[\lambda]} \cong H_{D,S,f}(\mathcal{V}_\lambda^{\kappa}) \otimes \mathcal{V}_\lambda^{*}$ as graded vector spaces and we are done.

So suppose that $\kappa \neq \kappa_c$. Let $\omega$ be the conformal vector of $W_{G,f}^\kappa$. Then $\omega = \omega_\mathfrak{g} + \omega_\mathfrak{g}$, where $\omega_\mathfrak{g}$ and $\omega_\mathfrak{g}$ are conformal vectors of $W_{\kappa}^{\mathfrak{g}}(\mathfrak{g}, f)$ and $V^{\kappa^*}(\mathfrak{g})$, respectively. Let $L^\mathfrak{g}(z) = \sum_{n \in \mathbb{Z}} L_n^\mathfrak{g} z^{-n-2}$ and $L^\mathfrak{g}(z) = \sum_{n \in \mathbb{Z}} L_n^\mathfrak{g} z^{-n-2}$ be the fields corresponding to $\omega_\mathfrak{g}$ and $\omega_\mathfrak{g}$, respectively. Set $W_{G,f}^\kappa[d_1, d_2] = \{ v \in W_{G,f}^\kappa \mid L^\mathfrak{g}_v = d_1 v, L^\mathfrak{g}_v = d_2 v \}$, so that $(W_{G,f}^\kappa)_{[\lambda]} = \bigoplus_{d_1 + d_2 = \Delta} W_{G,f}^\kappa[d_1, d_2]$. We have

$$W_{G,f}^\kappa = \bigoplus_{d \in \mathbb{C}} W_{G,f}[d, \bullet].$$
as a $\mathfrak{g}_{\kappa}$-module, where $W_{G,f}^\kappa[d, \bullet] = \bigoplus_{d' \in \mathbb{C}} W_{G,f}^\kappa[d, d']$. So it is sufficient to show that $W_{G,f}^\kappa[d, \bullet]$ is an object of $KL_{\kappa, d}^{ord}$, or equivalently, $W_{G,f}^\kappa[d_1, d_2]$ is finite-dimensional for all $d_1, d_2 \in \mathbb{C}$. If $(\kappa - \kappa_\ast)/\kappa_\ast \not\in \mathbb{Q}_{\geq 0}$, set $N_p = H^0_{DS,f}(M_p)$, where $M_p$ is as in [23]. By Theorem 5.1, $N_\bullet$ defines a filtration of $W_{G,f}^\kappa$ such that

$$\bigoplus_{o} N_p/N_{p-1} \cong \bigoplus_{\lambda \in P_+} H^0_{DS,f}(V_\lambda^\kappa) \otimes D(V_\lambda^\kappa).$$

We have $H^0_{DS,f}(V_\lambda^\kappa) = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0_{DS,f}(V_\lambda^\kappa)_{d+h\lambda}$, and $\dim H^0_{DS,f}(V_\lambda^\kappa)_{d} < \infty$ for all $d$, where $H^0_{DS,f}(V_\lambda^\kappa)_{d}$ denotes the $L_0^\kappa$-eigenspace of $H^0_{DS,f}(V_\lambda^\kappa)$ with eigenvalue $d$, and

$$h_\lambda = \frac{(\lambda + 2\rho | \lambda) - \frac{1}{2}\lambda(h) - \frac{1}{2}(\lambda - \frac{k + h'}{2}h + \rho)^2 - |\rho|^2}{2(k + h')} + \frac{1}{2}(\rho|h) - \frac{(k + h')}{8}h^2. $$

It follows that, for a given $d_1$, there exists only finitely many $\lambda \in P_+$ such that $H^0_{DS,f}(V_\lambda^\kappa)_{d_1} \neq 0$, and therefore $W_{G,f}^\kappa[d_1, d_2]$ is finite-dimensional for all $d_1, d_2 \in \mathbb{C}$. If $(\kappa - \kappa_\ast)/\kappa_\ast \not\in \mathbb{Q}_{\geq 0}$, there exists [Zhu2] a decreasing filtration

$$D^\kappa_{G,f} = M_1 \supset M_1 \supset \ldots, \quad \bigcap_{p} M_p = 0,$$ 

such that

$$\bigoplus_{o} M_p/M_{p+1} \cong \bigoplus_{\lambda \in P_+} V_\lambda^\kappa \otimes D(V_\lambda^\kappa).$$

Setting $N_p = H^0_{DS,f}(M_p)$, we get a filtration of $W_{G,f}^\kappa$ such that

$$\bigoplus_{o} N_p/N_{p+1} \cong \bigoplus_{\lambda \in P_+} H^0_{DS,f}(V_\lambda^\kappa) \otimes D(V_\lambda^\kappa).$$

Since there exist only finitely many $\lambda \in P_+$ such that $D(V_\lambda^\kappa)_{d_1} \neq 0$ for a given $d_1$, it follows that $W_{G,f}^\kappa[d_1, d_2]$ is finite-dimensional for all $d_1, d_2 \in \mathbb{C}$. This completes the proof. 

By Proposition 6.3, $W_{G,f}^\kappa$ is a conformal vertex algebra object in $KL_{\kappa, d}^{ord}$ with the chiral quantum moment map $\pi_R$.

By Theorem 5.3, we have the following assertion.

**Theorem 6.6.** Let $M \in KL_{\kappa}$. Then $H^{\ast + i}_{BS}(\hat{\mathfrak{g}}_{-\kappa_\ast}, \mathfrak{g}, W_{G,f}^\kappa \otimes M) = 0$ for $i \neq 0$, where the BRST cohomology is taken with respect to the action of $\hat{\mathfrak{g}}_{-\kappa_\ast}$ on $W_{G,f}^\kappa \otimes M$ given by $x(w \otimes m) = \pi_R(x)w \otimes m + w \otimes xm$. If $M \in KL_{\kappa, d}^{ord}$ we have

$$\text{gr} H^{\ast + 0}(\hat{\mathfrak{g}}_{-\kappa_\ast}, \mathfrak{g}, W_{G,f}^\kappa \otimes M) \cong O(J_{\infty S_f}) \otimes O(J_{\infty \mathfrak{g}^\ast}) \text{ gr } M.$$

**Theorem 6.7.** For $M \in KL_{\kappa}$, we have the isomorphism

$$H^\bullet_{DS,f}(M) \cong H^{\ast + \bullet}(\hat{\mathfrak{g}}_{-\kappa_\ast}, \mathfrak{g}, W_{G,f}^\kappa \otimes M)$$

as $W^\kappa(\mathfrak{g}, f)$-modules, where $W^\kappa(\mathfrak{g}, f)$ acts on the first factor $W_{G,f}^\kappa$ on the right-hand side. If $M$ is a vertex algebra object in $KL_{\kappa}$, this is an isomorphism of vertex algebras.
Proof. We may assume that $M \in \text{KLs}^\text{ord}$ since the cohomology functor commutes with injective limits. We may also assume that $M$ is $\mathbb{Z}_{\geq 0}$-graded: $M = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} M_d$: $(xt^n) M_d \subset M_{d-n}$ for $x \in \mathfrak{g}$, $n \in \mathbb{Z}$. Note that

$$(37) \quad \text{gr} H^*_{\text{KLs},f}(M) \cong \text{gr} H^*_{\text{KLs}}(\hat{\mathfrak{g}}-\kappa, \mathfrak{g}, W_{G,f} \otimes M)$$

by Theorem 0.1 and Theorem 0.6.

Let

$$C = \bigoplus_{i \in \mathbb{Z}} C^i, \quad C^i = \bigoplus_{p+q=i} C^{p,q},$$

$$C^{p,q} = M \otimes D_{G,\kappa}^{ch} \otimes \bigwedge^{\infty/2+p} (\mathfrak{g}) \otimes \bigwedge^{\infty/2+q} (\mathfrak{g}_{>0}),$$

where $F_i$ is the vertex algebra generated by fields $\Psi_\alpha(z), \alpha \in \Delta_{1/2}$, with the OPEs $\Psi_\alpha(z) \Psi_\beta(w) \sim (f[x_\alpha, x_\beta])/(z-w)$, and $\bigwedge^{\infty/2+q} (\mathfrak{g}_{>0})$ is the vertex superalgebra generated by odd fields $\psi_\alpha(z), \psi_\alpha^*(z), \alpha \in \Delta_{>0}$, with the OPEs $\psi_\alpha(z) \psi_\beta^*(w) \sim \delta_{\alpha, \beta}(z-w)$. Here, $x_\alpha$ is the root vector of $\mathfrak{g}$ corresponding to the root $\alpha$, $\Delta_j = \{ \alpha \in \Delta \mid x_\alpha \in \mathfrak{g}_1 \}$, $\Delta_{>0} = \bigcup_{j>0} \Delta_j$, and we have taken the Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}_0$ and $\Delta$ is the corresponding set of roots of $\mathfrak{g}$.

Set $Q_{(0)} = (Q_{\hat{\mathfrak{g}}}(0)) + (Q_{DS}(0))$, where

$$Q_{\hat{\mathfrak{g}}}(z) = \sum_{i=1}^{\dim \mathfrak{g}} \left( (\pi_M(x_i(z)) + \pi_R(x_i(z))) \psi_i^*(z) - \frac{1}{2} \sum_{i,j,k} c_{ij}^k : \psi_i^*(z) \psi_j^*(z) \psi_k(z) ; \right)$$

$$Q_{DS}(z) = \sum_{\alpha \in \Delta_{>0}} \left( \pi_L(x_\alpha(z)) + \Phi_\alpha(z) \psi_\alpha^*(z) - \frac{1}{2} \sum_{\alpha, \beta, \gamma \in \Delta_{>0}} c_{\alpha, \beta, \gamma}^{(2)} : \psi_\alpha^*(z) \psi_\beta^*(z) \psi_\gamma(z) ; \right),$$

where $\pi_M$ is the action of $\hat{\mathfrak{g}}_\kappa$ on $M$, $\{ x_i \}$ is a basis of $\mathfrak{g}$, $c_{ij}^k$ is the corresponding structure constant, $c_{\alpha, \beta, \gamma}^{(2)}$ is the structure constant of $\mathfrak{g}_{>0}$ with respect to the basis $\{ x_\alpha \}$, and we have set $\Phi_\alpha(z) = (f[x_\alpha])$ for $\alpha \in \Delta_j$, $j \geq 1$, and omitted the tensor product symbol. Then $(Q_{\hat{\mathfrak{g}}}(0))^2 = (Q_{DS}(0))^2 = 0, \{ Q_{\hat{\mathfrak{g}}}(0), Q_{DS}(0) \} = 0, C$ has the structure of a double complex.

Let $(E_r, d_r)$ be the spectral sequence for the total cohomology $H^*(C, Q_{(0)})$ such that $d_0 = (Q_{DS}(0))$ and $d_1 = (Q_{\hat{\mathfrak{g}}}(0))$. We claim that this spectral sequence converges to $H^*(C, Q_{(0)})$. Indeed, first suppose that $\kappa = \kappa_\text{c}$. Then by (22), $(D_{G,\kappa}^{ch}) |_{[\lambda]}$ admits a decomposition $(D_{G,\kappa}^{ch}) |_{[\lambda]} = \bigoplus_{d_1, d_2 \in \mathbb{Z}_{\geq 0}} (D_{G,\kappa}^{ch}) |_{[\lambda]}[d_1, d_2]$ such that $\pi_L(x^n)(D_{G,\kappa}^{ch}) |_{[\lambda]}[d_1, d_2] \subset (D_{G,\kappa}^{ch}) |_{[\lambda]}[d_1 - n, d_2], \pi_R(x^n)(D_{G,\kappa}^{ch}) |_{[\lambda]}[d_1, d_2] \subset (D_{G,\kappa}^{ch}) |_{[\lambda]}[d_1 - n, d_2]$ for $x \in \mathfrak{g}, n \in \mathbb{Z}$. It follows that $C$ is a direct sum of subcomplexes

$$C_{\lambda, D} = \bigoplus_{D = d+\Delta} M \otimes (D_{G,\kappa}^{ch}) |_{[\lambda]}[d] \otimes \bigwedge^{\infty/2+p} (\mathfrak{g}) \otimes (C^p \otimes (\mathfrak{g}) \otimes \bigwedge^{\infty/2+q} (\mathfrak{g}_{>0}) \otimes (M_{\Delta})), \lambda \in P_+, D \in \mathbb{Z}_{\geq 0},$$

$\lambda \in P_+, D \in \mathbb{Z}_{\geq 0}$, and we have $C_{\lambda, D} \cap C^p = 0$ for a sufficiently large $p$. Next suppose that $\kappa \neq \kappa_\text{c}$. The increasing filtration $M_\bullet$ of $D_{G,\kappa}^{ch}$ in (23) induces a filtration $C_\kappa$ of the complex $C$. As above, we find that the spectral sequence converges on each $C^p$. Since the cohomology functor commutes with the injective limits, we get a vector space isomorphism $H^*(C) \cong \lim H^*(C_p) \cong \lim E_{\infty}^{p,q} \cong E_{\infty}$, where $E_{\infty}^{p,q}$ is the spectral sequence for $H^*(C_p)$. 
By definition, we have

\[ E_1^{p,q} = M \otimes H^{0}_{DS,f}(D_{G,\kappa}^{ch}) \otimes \bigwedge^{\infty/2+p}(g) \cong \delta_{q,0} M \otimes W_{G,f}^{\kappa} \otimes \bigwedge^{\infty/2+p}(g), \]

and

\[ E_2^{p,q} \cong \delta_{q,0} H_{-\kappa,\kappa}^{\infty/0}(g, W_{G,f}^{\kappa} \otimes M) \otimes H^{0}(g, C). \]

Since elements of \( H_{-\kappa,\kappa}^{\infty/0}(g, W_{G,f}^{\kappa} \otimes M) \) are \( g \)-invariant, it follows that \( d_r = 0 \) for all \( r \geq 2 \). Therefore, the spectral sequence collapses at \( E_2 = E_\infty \), and we obtain the isomorphism

\[ H^i(C, Q(0)) \cong H^{\infty/0}_{-\kappa,\kappa}(g, W_{G,f}^{\kappa} \otimes M) \otimes H^i(g, C). \]

On the other hand, by (24), we have the canonical isomorphism

\[ H^*(C, Q(0)) \cong H^*(C', Q'_{(0)}), \]

where

\[ C' = \bigoplus_{p,q} (C')^{p,q}, \quad (C')^{p,q} = M \otimes D_{G,0}^{ch} \otimes \bigwedge^{\infty/2+p}(g) \otimes F_{\chi} \otimes \bigwedge^{\infty/2+q}(n), \]

\[ Q'(z) = Q_{\delta}(z) + Q_{DS}(z), \]

\[ Q'_{\delta}(z) = \sum_{i=1}^{\dim g} \pi_R(x_i)(z) \psi_i^*(z) - \frac{1}{2} \sum_{i,j,k} c_{i,j}^k : \psi_i^*(z) \psi_j^*(z) \psi_k(z) : , \]

\[ Q'_{DS}(z) = \sum_{\alpha \in \Delta_{>0}} (\pi_M(x_\alpha)(z) + \pi_L(x_\alpha(z)) + \Phi_\alpha(z)) \psi_\alpha^*(z) + \frac{1}{2} \sum_{\alpha, \beta, \gamma \in \Delta_{>0}} c_{\alpha, \beta}^\gamma : \psi_\alpha^*(z) \psi_\beta^*(z) \psi_\gamma(z) : . \]

Consider the map

\[ H^0_{DS,f}(M) \to H^0(C', Q'_{(0)}), \quad [c] \mapsto [c \otimes 1_{D_{G,\kappa}^{ch}} \otimes 1_{\Lambda^{\infty/2+*}(g)}], \]

where \( 1_{D_{G,\kappa}^{ch}} \) and \( 1_{\Lambda^{\infty/2+*}(g)} \) denotes the vacuum vectors of \( D_{G,\kappa}^{ch} \) and \( \Lambda^{\infty/2+*}(g) \), respectively. We claim that this is an isomorphism. Indeed, this respects the filtration on each side, and gives rise to a homomorphism

\[ \text{gr} H^0_{DS,f}(M) \to \text{gr} H^0(C', Q'_{(0)}) \cong \text{gr} H^0(C, Q(0)) \cong \text{gr} H_{-\kappa,\kappa}^{\infty/0}(g, W_{G,f}^{\kappa} \otimes M), \]

which is an isomorphism by (37). Therefore (39) is an isomorphism as well. This completes the proof.

In terms of the notation (17), we have

\[ W_{G,f}^{\kappa} \circ M \cong H^0_{DS,f}(M) \cong M \circ W_{G,f}^{\kappa} \]

for \( M \in \text{KL}_{\kappa,\kappa} \).

Let \( U_{G,f} \) denote the equivariant finite \( W \)-algebra introduced by Losev [Los].

**Theorem 6.8.** We have \( \text{Zhu}(W_{G,f}^{\kappa}) \cong U_{G,f} \) for any \( \kappa \).
Proof. The statement follows from [Los, Remark 3.1.4], Theorem 6.1 (3) and the description of the finite-dimensional analogue of the quantized Drinfeld-Sokolov reduction given in [A5].

7. Chiral universal centralizer

Let \( f, f' \) be nilpotent elements of \( \mathfrak{g} \). Let

\[
I_{G,f,f'} = H^0_{DS,f}(W^\kappa_{G,f}),
\]

where the Drinfeld-Sokolov reduction of \( W^\kappa_{G,f} \) is taken with respect to the action \( \pi_R \). By Corollary 6.2, \( Z^\kappa_{G,f,f'} \) is a conformal vertex algebra that is a strict chiral-ization of

\[
S_{G,f} \times_{\mathfrak{g}^*} S_f = (G \times S_f) \times_{\mathfrak{g}^*} S_f',
\]

where \( S_{G,f} \to \mathfrak{g}^* \) is given by the moment map (29). If \( f = f' \), the central charge of \( I_{G,f,f} \) is independent of \( \kappa \) and is given by

\[
2 \dim \mathfrak{g}_0 - \dim \mathfrak{g}_{1/2} + 24(\rho|\rho).
\]

Set

\[
I^\kappa_G := I_{G,f,prin,f,prin},
\]

where \( f_{prin} \) is a principal nilpotent element of \( \mathfrak{g} \). Then \( I^\kappa_G \) is a strict chiral quantization of the universal centralizer

\[
I^\kappa_G = (G \times S_{f,prin}) \times_{\mathfrak{g}^*} S_{f,prin}.
\]

The central charge of \( I^\kappa_G \) is given by

\[
2 \mathrm{rk} \mathfrak{g} + 48(\rho|\rho').
\]

By [BFM, BF], Theorem 6.1 (3), and the description of the finite-dimensional analogue of the quantized Drinfeld-Sokolov reduction given in [A5], we have the isomorphisms

\[
R_{I^\kappa_G} \cong H^*_C(\mathrm{Gr}_G), \quad \text{Zhu}(I^\kappa_G) \cong H^*_C(\mathrm{Gr}_G).
\]

The vertex algebra \( I^\kappa_G \) is called the chiral universal centralizer associated with \( G \) at level \( \kappa \).

For \( G = SL_2 \) and a generic \( \kappa \), we expect that the chiral universal centralizer \( I^\kappa_G \) coincides with the modified regular representations of the Virasoro algebra constructed by I. Frenkel and M. Zhu [FMZ].

8. More on cdo on \( G \) at the critical level

For the rest of this article we restrict to ourselves the case that \( \kappa = \kappa_c \). Set

\[
D^{ch}_G := D^{ch}_{G,\kappa_c}.
\]

Since \( \kappa_c^* = \kappa_c \), we have a vertex algebra homomorphism

\[
\pi_L \otimes \pi_R : V^{\kappa_c}(\mathfrak{g}) \otimes V^{\kappa_c}(\mathfrak{g}) \to D^{ch}_G.
\]

Note that

\[
\mathrm{Com}((\pi_L \otimes \pi_R)(V^{\kappa_c}(\mathfrak{g}) \otimes V^{\kappa_c}(\mathfrak{g})), D^{ch}_G) = (D^{ch}_G)^{\mathfrak{g}[t] \otimes \mathfrak{g}[t]}.
\]
Lemma 8.1. We have
\[ \pi_L(V^{\kappa_e}(\mathfrak{g})) \cap \pi_R(V^{\kappa_e}(\mathfrak{g})) = \pi_L(\mathfrak{z}(\mathfrak{g})) = \pi_R(\mathfrak{z}(\mathfrak{g})) = (D^h_G)^{\mathfrak{g}[t] \times \mathfrak{g}[t]}. \]
In particular, both \( \pi_L \) and \( \pi_R \) give the isomorphism
\[ \mathfrak{z}(\mathfrak{g}) \rightarrow (D^h_G)^{\mathfrak{g}[t] \times \mathfrak{g}[t]} \]
of vertex algebras.

Proof. By (20) and (21), \( \pi_L(\mathfrak{z}(\mathfrak{g})) = \pi_L(V^{\kappa_e}(\mathfrak{g})) \cap (D^h_{G,\kappa_e})^\pi_L(\mathfrak{g}[t]) = \pi_L(V^{\kappa_e}(\mathfrak{g})) \cap \pi_R(V^{\kappa_e}(\mathfrak{g})). \) Similarly, \( \pi_R(\mathfrak{z}(\mathfrak{g})) = \pi_L(V^{\kappa_e}(\mathfrak{g})) \cap \pi_R(V^{\kappa_e}(\mathfrak{g})). \) Also, \( (D^h_G)^{\mathfrak{g}[t] \times \mathfrak{g}[t]} = \pi_L(V^{\kappa_e}(\mathfrak{g})) \cap \pi_R(V^{\kappa_e}(\mathfrak{g})). \) This completes the proof. \( \square \)

Let \( S : \mathcal{O}(G) \rightarrow \mathcal{O}(G) \) be the antipode of the Hopf algebra \( \mathcal{O}(G). \) The map \( S \) induces the antipode \( J_\infty S : \mathcal{O}(J_\infty G) \rightarrow \mathcal{O}(J_\infty G) \) of the Hopf algebra \( \mathcal{O}(J_\infty G). \) Since \( J_\infty S \circ \pi_L(x) = \pi_R(x) \circ J_\infty S, \) \( J_\infty S \circ \pi_R(x) = \pi_L(x) \circ J_\infty S \) for \( x \in J_\infty \mathfrak{g}, \) \( J_\infty S \) extends to the vertex algebra homomorphism
\[ \tilde{\tau} : D^h_G \rightarrow D^h_G \]
such that \( \tilde{\tau}(f) = S(f), f \in \mathcal{O}(G), \tilde{\tau} \circ \pi_L = \pi_R \circ \tilde{\tau}, \tilde{\tau} \circ \pi_R = \pi_L \circ \tilde{\tau}. \) Thus, \( \tilde{\tau} \) restricts to the involutive automorphism \( \tau \) of the vertex subalgebra \( \mathfrak{z}(\mathfrak{g}) = (D^h_G)^{\mathfrak{g}[t] \times \mathfrak{g}[t]}, \) and we have
\[ \tau(\pi_L(z)) = \pi_R(z), \quad \tau(\pi_R(z)) = \pi_L(z) \]
for \( z \in \mathfrak{z}(\mathfrak{g}). \) It follows that (42) factors through the vertex algebra homomorphism
\[ V^{\kappa_e}(\mathfrak{g}) \otimes \mathfrak{z}(\mathfrak{g}) V^{\kappa_e}(\mathfrak{g}) \rightarrow D^h_G, \]
where \( \mathfrak{z}(\mathfrak{g}) \) acts on the first factor by the natural inclusion \( \mathfrak{z}(\mathfrak{g}) \rightarrow V^{\kappa_e}(\mathfrak{g}), \) and on the second by twisting the action by \( \tau. \)

The following assertion can be regarded as an affine analogue of [Ner, Theorem 2].

Proposition 8.2. The vertex algebra homomorphism (44) is injective.

Proof. The homomorphism between the associated graded vertex Poisson algebras induced by (44) is dual to the morphism
\[ J_\infty G \times J_\infty \mathfrak{g}^* \rightarrow J_\infty \mathfrak{g}^* \times J_\infty (\mathfrak{g}^* / G) J_\infty \mathfrak{g}^*, \quad (g, x) \mapsto (x, gx). \]
On the other hand, the morphism
\[ G \times \mathfrak{g}_{\text{reg}}^* \rightarrow \mathfrak{g}_{\text{reg}}^* \times \mathfrak{g}^* / G \mathfrak{g}_{\text{reg}}^*, \quad (g, x) \mapsto (x, gx). \]
is smooth and surjective ([Ric]), where \( \mathfrak{g}_{\text{reg}}^* \) is the open dense subset of \( \mathfrak{g}^* \) consisting of regular elements. Therefore, the morphism
\[ J_\infty G \times J_\infty \mathfrak{g}_{\text{reg}}^* \rightarrow J_\infty \mathfrak{g}_{\text{reg}}^* \times J_\infty (\mathfrak{g}_{\text{reg}}^* / G) J_\infty \mathfrak{g}_{\text{reg}}^*, \quad (g, x) \mapsto (x, gx) \]
is formally smooth and surjective ([?, Remark 2.10]). Hence (45) is dominant. \( \square \)
9. **Drinfeld-Sokolov reduction at the Critical level**

We have

\[ \text{ch} \mathcal{V}_\lambda = \sum_{w \in W} \epsilon(w) e^{w \circ \lambda} \prod_{\alpha \in \Delta^+} \left(1 - q^{\lambda + \rho, \alpha^\vee}\right) \prod_{j=1}^\infty \left(1 - q^j\right)^{r_k g}, \]

where \( W \) is the Weyl group of \( g \), \( \Delta^+ \) is the set of positive real roots of \( \hat{g}_{\kappa_c} \). Also, we have

\[ \text{ch} \mathbb{L}_\lambda = \sum_{w \in W} \epsilon(w) e^{w \circ \lambda} \prod_{\alpha \in \Delta^+} \left(1 - q^{\lambda + \rho, \alpha^\vee}\right) \prod_{\alpha \in \hat{\Delta}^+} \left(1 - e^{-\alpha}\right) \]

\[ \prod_{j=1}^\infty \left(1 - q^j\right)^{r_k g}, \]

(46)

Fix a principal nilpotent element \( f = f_{\text{prin}} \) of \( g \), and set \( H^0_{DS}(?) = H^0_{DS,f}(?) \).

By [FF2], we have the isomorphism

\[ \mathfrak{z}(\hat{g}) \cong W_{\kappa_c}(g, f) \]

of vertex algebras. Thus, for \( M \in KL \), \( H^0_{DS}(M) \) is naturally a module over \( \mathcal{Z} \).

By Theorem 6.1, we have the exact functor

\[ KL[\lambda] \rightarrow \mathcal{Z}\text{-Mod}[\lambda], \ M \mapsto H^0_{DS}(M). \]

For \( \lambda \in P_+ \), define the \( \mathcal{Z} \)-module

\[ \mathfrak{z}_\lambda := H^0_{DS}(\mathcal{V}_\lambda). \]

Note that \( \mathcal{Z}_{>0} \) trivially acts on \( \mathfrak{z}_\lambda \). The grading of \( \mathcal{V}_\lambda \) induces a grading on \( \mathfrak{z}_\lambda \) such that

\[ \mathfrak{z}_\lambda = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} (\mathfrak{z}_\lambda)_{-\lambda(\rho^\vee)+d}, \quad (\mathfrak{z}_\lambda)_{-\lambda(\rho^\vee)} = \mathbb{C} \]

see [AI] 4.6 for the details. By [AI] and the Euler-Poincaré principle, we have

\[ \text{ch} \mathfrak{z}_\lambda = \frac{q^{-\lambda(\rho^\vee)} \prod_{\alpha \in \Delta^+} (1 - q^{\lambda + \rho, \alpha^\vee}) \prod_{j=1}^\infty (1 - q^j)^{r_k g}}{r_k g}, \]

that is,

\[ \text{ch} \mathcal{V}_\lambda = \text{ch} \mathfrak{z}_\lambda \text{ ch } \mathbb{L}_\lambda. \]

Consider Li’s canonical filtration \( F^p \mathfrak{z}_\lambda \) of \( \mathfrak{z}_\lambda \). Since \( \mathfrak{z}(\hat{g}) \) is commutative, \( P_i(n) F^p \mathfrak{z}_\lambda \subset F^p \mathfrak{z}_\lambda \) for all \( i, p \) and \( n \geq 0 \). Hence, each graded subspace \( F^p \mathfrak{z}_\lambda / F^{p+1} \mathfrak{z}_\lambda \) of \( \text{gr} \mathfrak{z}_\lambda \) is naturally a module over \( \mathcal{Z} \). In particular,

\[ \overline{\mathfrak{z}_\lambda} := \mathfrak{z}_\lambda / F^1 \mathfrak{z}_\lambda = \mathfrak{z}_\lambda / \mathcal{Z}_{<0} \mathfrak{z}_\lambda \]

is a \( \mathcal{Z} \)-module.

Since \( \chi = (f|?) \in g^* \) is regular, there is a surjective algebra homomorphism

\[ \mathcal{Z} \twoheadrightarrow A_\chi \subset U(g), \]

constructed by Rybnikov [Ryb], where \( A_\chi \) denotes the Mishchenko-Fomenko subalgebra of \( U(g) \) associated with \( \chi \).

**Proposition 9.1.** Let \( \lambda \in P_+ \).

(i) \( \mathfrak{z}_\lambda \) is a free \( \mathcal{Z}_{<0} \)-module.
(ii) There is an isomorphism \( \bar{\mathfrak{z}}_\lambda \cong V_\lambda \) of \( \mathbb{Z} \)-modules, where \( \mathbb{Z} \) acts on \( V_\lambda \) via the map (51).

(iii) The \( \mathbb{Z} \)-module \( \mathfrak{z}_\lambda \) is cyclic, and is generated by the image of the highest weight vector of \( V_\lambda \).

(iv) The \( \mathbb{Z} \)-module \( \bar{\mathfrak{z}}_\lambda \) has a unique socle spanned by a homogeneous vector of maximal degree \( \lambda(\rho^\vee) \).

Proof. We have \( \text{gr} \, V_\lambda = \mathcal{O}(J_{\infty}g^*) \otimes V_\lambda \), and by (52),

\[
\text{gr} \, \mathfrak{z}_\lambda \cong (\mathcal{O}(J_{\infty}(\chi + n^\perp)) \otimes V_\lambda)^{J_{\infty}N},
\]

where \( N \) is the unipotent subgroup of \( G \) corresponding to the maximal nilpotent subalgebra \( n := g > 0 \) of \( g \). Since \( J_{\infty}(\chi + n^\perp) \) is a free \( J_{\infty}N \)-module, we have

\[
\text{gr} \, \mathfrak{z}_\lambda \cong (\mathcal{O}(J_{\infty}(\chi + n^\perp))^{J_{\infty}N} \otimes V_\lambda \cong \mathcal{O}(J_{\infty}\mathcal{S}_f) \otimes V_\lambda
\]
as \( \mathcal{O}(J_{\infty}\mathcal{S}_f) \)-modules. Hence, \( \text{gr} \, \mathfrak{z}_\lambda \) is free over \( \mathbb{Z} \langle \rho_\lambda \rangle \), and therefore, \( \mathfrak{z}_\lambda \) is free over \( \mathbb{Z} \langle \rho_\lambda \rangle \). Also, we get that

\[
\bar{\mathfrak{z}}_\lambda \cong \text{gr} \, \mathfrak{z}_\lambda / F^1 \text{gr} \, \mathfrak{z}_\lambda \cong V_\lambda
\]
as vector spaces. Therefore, \( \mathbb{Z} \) acts on \( V_\lambda \) via (52), and one finds that this action is identical to the action defined by (51), see [FF1, §2.1]. Since \( V_\lambda \) is a cyclic \( \mathcal{A}_\lambda \)-module generated by the highest weight vector (FFR2), it follows that \( \mathfrak{z}_\lambda \) is a cyclic \( \mathbb{Z}_{\leq 0} \)-module generated by the image of the highest weight vector of \( V_\lambda \).

According to [FFTL], the \( \mathcal{A}_\lambda \)-module \( V_\lambda \) has a unique socle, and hence, so does \( \bar{\mathfrak{z}}_\lambda \). As \( \bar{\mathfrak{z}}_\lambda \) is cyclic, the socle must be spanned by a homogeneous vector of maximal degree, which is \(- (w_0 \lambda)(\rho^\vee) = \lambda(\rho^\vee) \).

Let \( v_\lambda \in \mathfrak{z}_\lambda \) be the image of the highest weight vector of \( V_\lambda \). By Proposition 9.1, the map \( \mathbb{Z}_{\leq 0} \to \mathfrak{z}_\lambda \), \( z \mapsto z v_\lambda \), is surjective, and thus,

\[
\mathfrak{z}_\lambda \cong \mathbb{Z}_{\leq 0} / I_\lambda,
\]

where \( I_\lambda \) is the kernel of the above map. In particular, \( \mathfrak{z}_\lambda \) has the structure of a unital commutative algebra. In [FG, FFT] it was shown that

\[
\mathfrak{z}_\lambda \cong \text{End}_{\mathbb{Z}_{\geq 0}}(V_\lambda).
\]

By (52) and Proposition 9.1 we have

\[
\text{ch} \, \bar{\mathfrak{z}}_\lambda = \frac{q^{-\lambda(\rho^\vee)} \prod_{\alpha \in \Delta_+} (1 - q^{\lambda(\rho^\vee) + \alpha})}{\prod_{i=1}^q \prod_{j=1}^{d_i} (1 - q^i)} = q^{-\lambda(\rho^\vee)} \prod_{\alpha \in \Delta_+} (1 - q^{\lambda(\rho^\vee) + \alpha}) (1 - q^{\alpha + \alpha}),
\]

which can also be obtained by the standard method [Kac1, 10.9].

Let \( C_{[\lambda]} \) be the unique simple graded quotient of \( \mathfrak{z}_\lambda \), that is, the one-dimensional graded \( \mathbb{Z} \)-module corresponding to \( \chi_\lambda \), which is concentrated in degree \(- \lambda(\rho^\vee) \).

**Proposition 9.2.** For \( \lambda \in P_+ \), we have \( H^0_{DS}(L_\lambda) \cong C_{[\lambda]} \).

Proof. The surjection \( V_\lambda \to L_\lambda \) induces a surjection \( \mathfrak{z}_\lambda \to H^0_{DS}(L_\lambda) \). Hence \( H^0_{DS}(L_\lambda) \) is cyclic by Proposition 9.1. Since \( \ker \chi_\lambda \) annihilates it, \( H^0_{DS}(L_\lambda) \) must be one-dimensional.

**Proposition 9.3.** For \( \lambda \in P_+ \), the functor \( \text{KL}^{[\lambda]} \to \mathbb{Z} \text{-Mod}^{[\lambda]} \) is faithful. For \( M \in \text{KL}^{[\lambda]} \), we have

\[
\text{ch} \, M = q^{\lambda(\rho^\vee)} \text{ch} \, L_\lambda \cdot \text{ch} \, H^0_{DS}(M).
\]
Proof. The first statement follows from the exactness of the functor and Proposition 9.2. To see the second assertion, note that we have
\[ \text{ch } M = \sum_{d \in \mathbb{C}} [M : \mathbb{L}_\lambda[-d]] q^d \text{ch } \mathbb{L}_\lambda, \]
where \([M : \mathbb{L}_\lambda[-d]]\) is the multiplicity of \(\mathbb{L}_\lambda[-d]\) as a graded representation. Since \(H^0_D(M)\) is exact, we have
\[ \text{ch } H^0_D(M) = \sum_{d \in \mathbb{C}} [M : \mathbb{L}_\lambda[-d]] q^d \text{ch } H^0_D(\mathbb{L}_\lambda) = q^{-\lambda(\rho^\vee)} \sum_{d \in \mathbb{C}} [M : \mathbb{L}_\lambda[-d]] q^d. \]
This completes the proof. \(\square\)

For a \(\mathbb{Z}\)-module \(M\), we set
\[ M^* := \text{Hom}_{\mathbb{Z}_{\leq 0}}(M, \mathbb{Z}_{\leq 0}), \]
and consider \(M^*\) as a \(\mathbb{Z}\)-module by the action \((z f)(m) = f(zm)\). Also, let \(\tau^* M\) denote the \(\mathbb{Z}\)-module obtained by twisting the action of \(\mathbb{Z}\) on \(M\) as \(z m = \tau(z) m\).

**Proposition 9.4.** For \(\lambda \in P_+\), we have the following:

(i) \(\tilde{\mathfrak{g}}_\lambda \cong \mathfrak{g}_\lambda\),

(ii) \(\tau(\tilde{\mathfrak{g}}_\lambda) \cong \mathfrak{g}_\lambda^\tau\).

**Proof.** (i) Clearly, \(I_\lambda\) annihilates \(\tilde{\mathfrak{g}}_\lambda\). Hence it is enough to show that \(\tilde{\mathfrak{g}}_\lambda\) is a cyclic module over \(\mathbb{Z}_{\leq 0}\). Since \(\mathfrak{g}_\lambda\) is free over \(\mathbb{Z}_{\leq 0}\), \(\tilde{\mathfrak{g}}_\lambda\) is free over \(\mathbb{Z}_{\leq 0}\) as well. We have
\[ \tilde{\mathfrak{g}}_\lambda \cong \text{Hom}_\mathbb{C}(\tilde{\mathfrak{g}}_\lambda, \mathbb{C}) \]
as \(\mathbb{Z}_{\leq 0}/\mathbb{Z}_{=0}\)-modules, and since \(\tilde{\mathfrak{g}}_\lambda\) has a unique socle by Proposition 9.1, \(\tilde{\mathfrak{g}}_\lambda\) is cyclic. Therefore, \(\tilde{\mathfrak{g}}_\lambda\) is a cyclic module over \(\mathbb{Z}_{<0}\) as required. (ii) The \(\tilde{\mathfrak{g}}_{\kappa_i,}\)-submodule generated by \(V_{\kappa_i} \otimes V_{\lambda_i} \subset \mathcal{O}(G)\) by the action \(\pi_{\kappa_i}\) (resp. \(\pi_{\lambda_i}\)) is isomorphic to \(V_{\lambda_i} \otimes V_{\lambda_i}\) (resp. \(V_{\kappa_i} \otimes V_{\lambda_i}\)). Hence, \(V_{\lambda_i} \otimes V_{\lambda_i} \subset \mathcal{O}(G)\) is annihilated by \(\pi_{\kappa_i}(I_{\lambda_i})\) and \(\pi_{\lambda_i}(I_{\lambda_i})\), and we get that \(\tau(I_{\lambda_i}) = I_{\lambda_i}^\tau\). \(\square\)

For \(b \geq 1\), define the vertex algebra
\[ V(\mathfrak{g})_b := V^{\kappa_i}(\mathfrak{g}) \otimes_{\lambda_i(\tilde{\mathfrak{g}})} V^{\kappa_i}(\mathfrak{g}) \otimes_{\lambda_i(\tilde{\mathfrak{g}})} \cdots \otimes_{\lambda_i(\tilde{\mathfrak{g}})} V^{\kappa_i}(\mathfrak{g}), \]
where the tensor product \(\otimes_{\lambda_i(\tilde{\mathfrak{g}})}\) is taken with respect to the action \(z \otimes 1 - 1 \otimes \tau(z)\) for \(z \in \lambda_i(\tilde{\mathfrak{g}})\). Let \(\iota_i : V^{\kappa_i}(\mathfrak{g}) \mapsto V(\mathfrak{g})_b\) be the vertex algebra embedding that sends \(u \in V^{\kappa_i}(\mathfrak{g})\) to the \(i\)-th factor of \(V^{\kappa_i}(\mathfrak{g})\). This induces a Lie algebra homomorphism \(\tilde{\mathfrak{g}}_{\kappa_i} \mapsto \text{End}_C(M)\) for any \(V(\mathfrak{g})_b\)-module \(M\), which we denote also by \(\iota_i\).

Let \(KL_b\) be the category of \(V(\mathfrak{g})_b\)-modules consisting of objects \(M\) that belongs to \(KL\) as \(\iota_i(\tilde{\mathfrak{g}}_{\kappa_i})\)-module for all \(i = 1, \ldots, b\).

By Proposition 8.2, we have a vertex algebra embedding \(V(\mathfrak{g})_{b=2} \mapsto \mathcal{D}^{ch}_G\). In particular, \(\mathcal{D}^{ch}_{G_{b=2}}\) is an object of \(KL_{b=2}\).

Set
\[ \mathbb{V}_{\lambda,b} := \mathbb{V}_{\lambda} \otimes \mathbb{V}_{\lambda} \otimes \cdots \otimes \mathbb{V}_{\lambda'} = \mathbb{V}_{\lambda} \otimes_{\lambda} \mathbb{V}_{\lambda} \otimes_{\lambda} \cdots \otimes_{\lambda} \mathbb{V}_{\lambda'}, \mathbb{V}_{\lambda'} \in KL_b, \]
where \(\lambda' = \lambda\) if \(b\) is odd and \(\lambda' = \lambda^*\) of \(b\) is even.
Proposition 9.5. For \( b \geq 2 \), we have \( H^0_{DS}(\mathcal{V}_\lambda, b) \cong \mathcal{V}_{\lambda^*, b-1} \) (resp. \( \cong \mathcal{V}_{\lambda^*, b-1} \)), where the Drinfeld-Sokolov reduction is taken with respect to the action \( \iota_1 \) (resp. \( \iota_r \)) of \( \hat{\mathfrak{g}}_{\kappa, r} \).

Proof. We only show the assertion for the action \( \iota_1 \). We have \( \mathcal{V}_{\lambda, b} = \mathcal{V}_\lambda \otimes_{\mathcal{V}_\lambda^*, b-1} \mathcal{V}_{\lambda^*, b-1} \).

Since \( H^0_{DS}(\mathcal{V}_\lambda) \cong \mathcal{V}_\lambda \) is obviously free over \( \mathcal{V}_\lambda \), \( H^0_{DS}(\mathcal{V}_{\lambda, b}) \cong H^0_{DS}(\mathcal{V}_\lambda) \otimes_{\mathcal{V}_\lambda} \mathcal{V}_{\lambda^*, b-1} \cong \mathcal{V}_\lambda \otimes_{\mathcal{V}_\lambda} \mathcal{V}_{\lambda^*, b-1} \cong \mathcal{V}_{\lambda^*, b-1} \) by Künneth’s theorem. \( \square \)

Let \( \mathcal{K}_b^\Delta \) be the full subcategory of \( \mathcal{K}_b \) consisting of objects \( M \) that admits an increasing filtration \( 0 = M_0 \subset M_1 \subset \ldots, M = \bigcup_p M_p \), such that each successive quotient \( M_p/M_{p-1} \) is isomorphic to \( \mathcal{V}_{\lambda, b} \) for some \( \lambda \in P_+ \). Dually, let \( \mathcal{K}_b^\nabla \) be the full subcategory of \( \mathcal{K}_b \) consisting of objects \( M \) that admits a decreasing filtration \( M = M_0 \supset M_1 \supset \ldots, \bigcap_p M_p = 0 \), such that each successive quotient \( M_p/M_{p+1} \) is isomorphic to the contragradient dual \( D(\mathcal{V}_{\lambda, b}) \) of \( \mathcal{V}_{\lambda, b} \) for some \( \lambda \in P_+ \). Let \( \mathcal{K}_b^\Delta \cap \mathcal{K}_b^\nabla \) be the full subcategory of \( \mathcal{K}_b \) consisting of objects \( M \) that belongs to both \( \mathcal{K}_b^\Delta \) and \( \mathcal{K}_b^\nabla \).

10. The BRST Cohomology \( H^{\frac{\Delta}{2}+\bullet}(Z, ?) \)

Let \( \bigwedge^{\infty/2+\bullet}(\mathfrak{g}) \) be the charged free Fermions generated by odd fields \( \psi_1(z), \ldots, \psi_{\text{rk} \mathfrak{g}}(z), \psi^*_1(z), \ldots, \psi^*_{\text{rk} \mathfrak{g}}(z) \), with OPEs

\[
\psi_i(z)\psi^*_j(z) \sim \frac{\delta_{ij}}{z - w}, \quad \psi_i(z)\psi_j(z) \sim \psi^*_i(z)\psi^*_j(z) \sim 0.
\]

Define

\[
T^{gh}(z) = \sum_{n \in \mathbb{Z}} L_n^{gh} z^{-n-2} := \sum_{i=1}^{\text{rk} \mathfrak{g}} d_i : \psi^*_i(z) \partial_z \psi_i(z) : - \sum_{i=1}^{\text{rk} \mathfrak{g}} (d_i + 1) : \psi_i(z) \partial \psi^*_i(z) :
\]

\[
= \sum_{i=1}^{\text{rk} \mathfrak{g}} (\partial \psi^*_i(z)) \psi^*_i(z) - \sum_{i=1}^{\text{rk} \mathfrak{g}} (d_i + 1) : \partial (\psi_i(z) \psi^*_i(z)) :.
\]

Then \( T^{gh}(z) \) defines a conformal vector of \( \bigwedge^{\infty/2+\bullet}(\mathfrak{g}) \) with central charge

\[-2(\text{rk} \mathfrak{g} + 24(\rho, \rho^0)).\]

The conformal weight of \( \psi_i \) is \( d_i + 1 \) are \( d_i + 1 \) and \(-d_i \), respectively. The conformal weight of \( \bigwedge^{\infty/2+\bullet}(\mathfrak{g}) \) is bounded from below and each homogeneous space \( \bigwedge^{\infty/2+\bullet}(\mathfrak{g}) \Delta \) is finite-dimensional.

Define the odd field

\[
Q^Z(z) = \sum_{n \in \mathbb{Z}} Q^Z(n) z^{-n-1} := \sum_{i=1}^{\text{rk} \mathfrak{g}} P_i(z) \otimes \psi^*_i(z).
\]

on the vertex algebra \( \mathfrak{g} \otimes \bigwedge^{\infty/2+\bullet}(\mathfrak{g}) \). So

\[
Q^Z(z) = \sum_{i=1}^{\text{rk} \mathfrak{g}} \sum_{n \in \mathbb{Z}} (p_i)(-n) \otimes (\psi^*_i)(n-1).
\]

Since \( \mathfrak{g} \) is commutative, we have \( Q^Z(z)Q^Z(w) \sim 0 \), and thus, \( (Q^Z(z))^2 = 0 \). Hence, for \( M \in Z\text{-Mod}, \mathcal{M} \otimes \bigwedge^{\infty/2+\bullet}(\mathfrak{g}), Q^Z(z) \) is a cochain complex, where the
cohomological degree is defined by \( \deg(\psi_i) = -1 \), \( \deg(\psi_i^*(x)) = 1 \), \( \deg m \otimes |0\rangle = 0 \) for \( m \in M \). Denote by \( H^{\infty/2+\bullet}(Z, M) \) the corresponding cohomology. If \( V \) is a vertex algebra object in KL and \( M \) is a \( V \)-module, then \( H^{\infty/2+\bullet}(Z, V) \) is a vertex algebra and \( H^{\infty/2+\bullet}(Z, M) \) is a \( H^{\infty/2+\bullet}(Z, V) \)-module.

Lemma 10.1. Let \( M \in Z \)-Mod and suppose that \( M \) is free as a \( Z(\leq 0) \)-module. Then \( H^{\infty/2+i}(Z, M) = 0 \) for \( i < 0 \).

Proof. Immediate from [Vor] Theorem 2.3, see the proof of Lemma 10.2 below. \( \square \)

In the followings, for \( M, N \in Z \)-Mod we consider \( M \otimes N \) as a \( Z \)-module by the action

\[
P_i(z) \mapsto P_i(z) \otimes 1 - 1 \otimes \tau(P_i)(z)
\]

unless otherwise stated.

Lemma 10.2. Let \( M, N \in Z \)-Mod and suppose that \( M \) is free of finite rank as a \( Z(\leq 0) \)-module. Then \( H^{\infty/2+i}(Z, M \otimes N) = 0 \) for \( i < 0 \) and

\[
H^{\infty/2+0}(Z, M \otimes N) \cong \text{Hom}_Z(M^*, \tau^*N).
\]

Proof. Consider the Hochschild-Serre spectral sequence \( E_2 \Rightarrow H^{\infty/2+\bullet}(Z, M \otimes N) \) for the subalgebra \( Z(\leq 0) \subset Z \) as in [Vor] Theorem 2.3. By definition, the \( E_1 \)-term is the opposite Koszul homology \( H^{op}_i(Z(\leq 0), M \otimes N) \) of the \( Z(\leq 0) \)-module \( M \otimes N \). Since \( M \) is free over \( Z(\leq 0) \), \( M \) is also free over \( Z(\leq 0) \), and so is \( M \otimes N \). Hence \( H^{op}_i(Z(\leq 0), M \otimes N) = 0 \) for \( i \neq 0 \) and \( H^{op}_0(Z(\leq 0), M \otimes N) \cong M \otimes Z(\leq 0) \). It follows that we can write \( E_2^{p,q} = \delta_{q,0} \chi P(Z(\geq 0), M \otimes Z(\leq 0), N) \), the Lie algebra cohomology of \( M \) as a module over the commutative Lie algebra \( \bigoplus_{i=1}^{k} \bigoplus_{n \geq 0} Z_{i(n)} \) with respect to the action \( (P_i)(n) \mapsto (P_i)(n) \otimes 1 - 1 \otimes \tau(P_i)(n) \). The spectral sequence collapses at \( E_2 = E_\infty \), and we have

\[
H^{\infty/2+i}(Z, M \otimes N) \cong \begin{cases} H^i(Z(\geq 0), M \otimes Z(\leq 0), N) & \text{for } i \geq 0 \\ 0 & \text{for } i < 0. \end{cases}
\]

Now since \( M \) is free as a \( Z(\leq 0) \)-module, we have

\[
M \otimes Z(\leq 0) \cong \text{Hom}_Z(M, \tau^*N).
\]

It follows from the definition that

\[
H^{\infty/2+0}(Z, M \otimes N) \cong \text{Hom}_Z(M, \tau^*N) \subset \text{Hom}_Z(Z(\leq 0), M, \tau^*) N)
\]

\( \square \)

For \( M, N \in Z \)-Mod, the complex \( C(Z, M \otimes N) = M \otimes N \otimes \bigwedge^{\infty/2+\bullet}(Z) \) is a direct sum of subcomplexes \( C(Z, M \otimes N)_d = \bigoplus_{d_1 + d_2 + \Delta = d} M_{d_1} \otimes N_{d_2} \otimes \bigwedge^{\infty/2+\bullet}(Z) \), \( \Delta \in \mathbb{C} \), and so \( H^{\infty/2+\bullet}(Z, M \otimes N) \) is graded:

\[
H^{\infty/2+\bullet}(Z, M \otimes N) = \bigoplus_{d \in \mathbb{C}} H^{\infty/2+\bullet}(Z, M \otimes N)_d.
\]

Set \( \text{ch}_q H^{\infty/2+\bullet}(Z, M \otimes N) = \sum_{d \in \mathbb{C}} q^d \dim H^{\infty/2+\bullet}(Z, M \otimes N)_d \) when it is well-defined.

Proposition 10.3. Let \( \lambda \in P_+ \), \( M \in Z \)-Mod. Then \( H^{\infty/2+0}(Z, \delta_\lambda \otimes M) \cong \text{Hom}_Z(\delta_\lambda, \tau^*M) \). If \( \tau^*M \) is a quotient of \( \delta_\lambda \), we have

\[
\text{ch} H^{\infty/2+0}(Z, \delta_\lambda \otimes M) \cong q^{\lambda(\rho^*)} \text{ch} M.
\]
Proof. By Proposition \[9.4\] \( H^\mp+0(Z, \mathfrak{z} \otimes M) \cong \text{Hom}_Z(\mathfrak{z}_\lambda, \tau^* M) \), which is isomorphic to \( \tau^* M \) if \( \tau^* M \) is quotient of \( \mathfrak{z}_\lambda \) as a \( Z \)-module. Since under the isomorphism \( \mathfrak{z}_\lambda \cong \mathfrak{z}_\lambda \) in Lemma \[10.2\] the generator \( v_\lambda \) of \( \mathfrak{z}_\lambda \) corresponds to the dual element of \( \mathfrak{z} \) of conformal weight \( \lambda(\rho^\vee) \), we get the assertion. \( \square \)

Set \( W = W_G = W_{G, f_{\text{prin}}}^c, \) \( S = S_G = S_{G, f_{\text{prin}}}, \) \( \mathcal{S} = S_{f_{\text{prin}}} \).

By \( \text{Lemma 10.4} \), \( W \) is conformal with central charge

\( \dim g + \text{rk} g + 24(\rho^\vee). \)

Let \( \omega_W \) be the conformal vector of \( W \), \( L^W(z) = \sum_{n \in \mathbb{Z}} L_n^W z^{-n-2} \) be the corresponding field.

Lemma 10.4. The subspace \( \mathfrak{z}(\mathfrak{g}) \subset W \) is preserved by the action of \( L_n^W \) with \( n \geq -1 \).

Proof. Clearly, \( \mathfrak{z}(\mathfrak{g}) \) is preserved by the action of the translation operator \( L_n^W \).

Let \( z \in \mathfrak{z}(\mathfrak{g}) = \mathfrak{w}^{[0]} \). Since \( [L_n^W, x_n] = (m - n)x_{m+n} \) for \( x \in g \), \( x_n L_n^W z = -(n-m)x_{m+n} z = 0 \) for \( m, n \geq 0 \). Hence \( L_n^W z \in \mathfrak{z}(\mathfrak{g}) \) for \( n \geq 0 \).

For \( N \in Z\text{-Mod}_{\text{reg}}, H^\mp+0(Z, W \otimes N) \) is a graded \( V_{\mathcal{K}}(\mathfrak{g}) \)-module, where \( V_{\mathcal{K}}(\mathfrak{g}) \) acts on the first factor \( W \). Hence we have a functor

\( Z\text{-Mod}_{\text{reg}} \to KL, \quad N \mapsto H^\mp+0(Z, W \otimes N). \)

Lemma 10.5. We have \( H^\mp+i(Z, W \otimes N) = 0 \) for \( i < 0, N \in Z\text{-Mod} \). Therefore, the functor \( \text{Lemma 10.4} \) is left exact.

Proof. Since \( SS(W) \cong J_{\mathcal{S}} S \cong J_{\mathcal{S}} S \times J_{\mathcal{S}} G \), \( g_{\mathcal{S}} W \) is free over \( O(J_{\mathcal{S}} S) \). Hence, \( W \) is free over \( \mathfrak{z}(\mathfrak{g}) \) in the usual associative sense and so is \( W \otimes N \). Hence the assertion follows from Lemma \[10.4 \].

Proposition 10.6. For \( N \in Z\text{-Mod}_{\text{reg}}, \) we have

\( H^\mp+0(Z, W \otimes N)^{[0]} \cong H^\mp+0(Z, W^{[0]} \otimes N) \cong \bigoplus_{\lambda \in P^+} V_\lambda \otimes \text{Hom}_Z(\mathfrak{z}_\lambda, N) \)

as \( g \)-modules.

Proof. Set

\[ C = \bigoplus_{i \in \mathbb{Z}} C^i, \quad C^i = \bigoplus_{p+q = i} C^{p,q}, \quad C^{p,q} = W \otimes N \otimes \bigwedge_{p+q}^{\infty/2+p} (\mathfrak{z}(\mathfrak{g})) \otimes \bigwedge_q (\mathfrak{t} \otimes [\mathfrak{g}])^* \]

Let \( Q_{[0]} \) be the differential of the Chevalley complex \( W \otimes \bigwedge^* ((\mathfrak{t} \otimes [\mathfrak{g}])^*) \) for the computation of the Lie algebra cohomology \( \text{Lie}^* (\mathfrak{t} \otimes [\mathfrak{g}], W) \). We extend \( Q_{[0]} \) to a differential of \( C \) by letting it trivially act on the factor \( N \otimes \bigwedge_{p+q}^{\infty/2+p} (\mathfrak{z}(\mathfrak{g})) \). Let \( Q_3 \) be the differential of the complex \( W \otimes N \otimes \bigwedge_{p+q}^{\infty/2+p} (\mathfrak{z}(\mathfrak{g})) \) which defines \( H^\mp+*(Z, W \otimes N) \). We extend \( Q_3 \) to a differential of \( C \) by letting it trivially act on the factor \( \bigwedge^* ((\mathfrak{t} \otimes [\mathfrak{g}])^*) \). Since \( \{Q_{[0]}, Q_3\} = 0, C \) is equipped with the structure of a double complex.
Consider the spectral sequence $E_r \Rightarrow H^\bullet_{\text{tot}}(C)$ such that $d_0 = Q_{\mathfrak{g}[t]}$ and $d_1 = Q_+$.
Since $W$ is coffee over $U(\mathfrak{g}[t])$, we have
\[ E_1^{p,q} \cong H^q(\mathfrak{g}[t], W) \otimes N \otimes \bigwedge^{\infty/2+p}(\mathfrak{g}(\mathfrak{g})) \cong \delta_{q,0} W^{\mathfrak{g}[t]} \otimes N \otimes \bigwedge^{\infty/2+p}(\mathfrak{g}(\mathfrak{g})) \]
\[ \cong \delta_{q,0} \bigoplus_{\lambda \in P_+} (\mathfrak{z}(\lambda) \otimes \mathfrak{V}_{\lambda}) \otimes N \otimes \bigwedge^{\infty/2+p}(\mathfrak{g}(\mathfrak{g})), \]
\[ E_2^{p,q} \cong \delta_{q,0} \bigoplus_{\lambda \in P_+} \mathfrak{V}_{\lambda} \otimes H_{\mathfrak{z}+p}(\mathfrak{Z}, \mathfrak{z}(\lambda) \otimes N). \]
Therefore, the spectral sequence collapses at $E_2 = E_\infty$, and we obtain the isomorphism $H^{i}_{\text{tot}}(C) \cong \bigoplus_{\lambda \in P_+} \mathfrak{V}_{\lambda} \otimes H_{\mathfrak{z}+i}(\mathfrak{Z}, \mathfrak{z}(\lambda) \otimes N)$. In particular, by Lemma \[10.2\] we have
\[ H^0_{\text{tot}}(C) \cong \bigoplus_{\lambda \in P_+} \mathfrak{V}_{\lambda} \otimes \operatorname{Hom}_{\mathfrak{Z}}(\mathfrak{z}(\lambda), N). \]
\[ (60) \]
Next, let $E'_r \Rightarrow H^\bullet_{\text{tot}}(C)$ be the spectral sequence such that $d_0 = Q_0$ and $d_1 = Q_{\mathfrak{g}[t]}$. We have
\[ (E'_1)^{p,q} \cong H^{\infty/2+q}(\mathfrak{Z}, W \otimes N) \otimes \bigwedge^p((\mathfrak{g}[t])^*), \]
\[ (E'_2)^{p,q} \cong H^p(\mathfrak{g}[t], H_{\mathfrak{z}+q}(\mathfrak{Z}, W \otimes N)). \]
By Lemma \[10.3\] $(E'_2)^{p,q} = 0$ for $p < 0$ or $q < 0$. It follows that $H^0_{\text{tot}}(C) \cong H_{\mathfrak{z}+0}(\mathfrak{Z}, W \otimes N)^{\mathfrak{g}[t]}$. Hence, by \[60\], we obtain that
\[ H_{\mathfrak{z}+0}(\mathfrak{Z}, W \otimes N)^{\mathfrak{g}[t]} \cong \bigoplus_{\lambda \in P_+} \mathfrak{V}_{\lambda} \otimes \operatorname{Hom}(\mathfrak{z}(\lambda), N). \]

\[ \square \]

**Proposition 10.7.** For $\lambda \in P_+$, we have
\[ H_{\mathfrak{z}+0}(\mathfrak{Z}, W \otimes C_{[\lambda]}) \cong L_\lambda \]
as graded $\mathfrak{g}_{\kappa_+}$-modules.

**Proof.** Note that $z \in \mathfrak{z}$ acts as the constant $\chi_\lambda(z)$ on $H_{\mathfrak{z}+0}(\mathfrak{Z}, W \otimes C_{[\lambda]})$. Therefore, $H_{\mathfrak{z}+0}(\mathfrak{Z}, W \otimes C_{[\lambda]})$ is a direct sum of $L_\lambda$ as $\mathfrak{g}_{\kappa_+}$-modules (FG). On the other hand, by Proposition \[10.6\] $H_{\mathfrak{z}+0}(\mathfrak{Z}, W \otimes C_{[\lambda]})^{\mathfrak{g}[t]} \cong \mathfrak{V}_\lambda$, which is concentrated in degree 0, hence the assertion.

\[ \square \]

**Proposition 10.8.** For $N \in \mathfrak{Z} \text{-Mod}_{[\lambda]}$,
\[ \operatorname{ch} H_{\mathfrak{z}+0}(\mathfrak{Z}, W \otimes N) \leq \operatorname{ch} N \operatorname{ch} L_\lambda, \]
that is, the dimension of each weight space of $\operatorname{ch} H_{\mathfrak{z}+0}(\mathfrak{Z}, W \otimes N)$ is equal to or smaller than that of $N \otimes L_\lambda$.

**Proof.** We may assume that $N \in \mathfrak{Z} \text{-Mod}_{[\lambda]}$ for some $\lambda \in P_+$. First, consider the case $N$ is a quotient of $\mathfrak{z}_\lambda$. Then there exists a decreasing filtration $N = N_0 \supset N_1 \supset N_2 \supset \ldots, \cap_{p} N_p = 0$, of $\mathfrak{Z}$-modules such that each successive quotient is a direct sum of $C_{[\lambda]}$. Consider the induced filtration of the complex and the corresponding spectral sequence $E_r \Rightarrow H_{\mathfrak{z}+\bullet}(\mathfrak{Z}, W \otimes N)$. As the $E_1$-term is
\[ \bigoplus_i H_{\mathfrak{z}+i}(\mathfrak{Z}, W \otimes N_i/N_{i+1}) \cong N \otimes H_{\mathfrak{z}+i}(\mathfrak{Z}, W \otimes C_{[\lambda]}), \]
Lemma \[10.5\] gives that
$E^{p,q}_1 = 0$ for $p + q < 0$. As a result, as graded vector spaces, $H^* Z(\mathcal{V}, \mathcal{W} \otimes N)$ is a quotient of $N \otimes H^* Z(\mathcal{V}, \mathcal{W} \otimes C_{\lambda}) \cong N \otimes \mathbb{L}_{\lambda}$ by Proposition 10.7.

Next, let $N$ be an arbitrary object in $Z\text{-Mod}_{\text{reg}}$. Then there exists a filtration $0 = N_0 \subset N_1 \subset N_2 \subset \ldots \subset N = \bigcup_i N_i$ such that each successive quotient is a quotient of $\mathcal{H}_\lambda$ for some $\lambda \in P_+$. By Lemma 10.8, we get that $\text{ch} H^* Z(\mathcal{V}, \mathcal{W} \otimes N) \leq \sum_i \text{ch} H^* Z(\mathcal{V}, \mathcal{W} \otimes N_i/N_{i+1}) \leq \text{ch} N \circ \mathbb{L}_{\lambda}$. □

Define the vertex algebra $\mathcal{V}_2^S$ by

$$
\mathcal{V}_2^S := H^* (\mathcal{Z}, \mathcal{W} \otimes \mathcal{W}).
$$

**Theorem 10.9.** We have

$$
\mathcal{V}_2^S \cong \mathcal{D}_G^{ch}
$$

as vertex algebras.

**Proof.** The vertex algebra embedding $V^{\kappa_\gamma}(g) \hookrightarrow \mathcal{W}$ to the first factor (resp. the second factor) of $\mathcal{W} \otimes \mathcal{W}$ induces the vertex algebra homomorphism

$$
\pi_L : V^{\kappa_\gamma}(g) \rightarrow \mathcal{V}_2^S
$$

(resp. $\pi_R : V^{\kappa_\gamma}(g) \rightarrow \mathcal{V}_2^S$).

Consider the decomposition $\mathcal{V}_2^S = \bigoplus_{\lambda \in P_+} (\mathcal{V}_2^S)_{[\lambda]}$ with respect to the action of $\mathfrak{g}_\kappa_\gamma$ on the first factor of $\mathcal{W} \otimes \mathcal{W}$. Then $(\mathcal{V}_2^S)_{[\lambda]} = H^* (\mathcal{Z}, \mathcal{W}_{[\lambda]} \otimes \mathcal{W}_{[\lambda']})).$ By Proposition 10.8, we have

$$
\text{ch}(\mathcal{V}_2^S)_{[\lambda]} \leq \text{ch} \mathcal{W}_{[\lambda']} \circ \mathcal{L}_\lambda \leq \text{ch}(\mathcal{D}_G^{ch})_{[\lambda]}
$$

for $\lambda \in P_+$. Therefore, it is sufficient to show that there is a non-trivial vertex algebra homomorphism $\mathcal{D}_G^{ch} \rightarrow \mathcal{W}_2$ since $\mathcal{D}_G^{ch}$ is simple. Note that since $\mathcal{D}_G^{ch}$ is $\mathbb{Z}_{\geq 0}$-graded, so is $\mathcal{V}_2^S$ by (64).

By using Proposition 10.6 twice, we obtain that

$$
(\mathcal{V}_2^S)_{[\lambda]} \circ \mathcal{L}_\lambda \cong \mathcal{V}_{\lambda} \otimes \mathcal{L}_\lambda \mathcal{V}_{\lambda'}.
$$

(65)

In particular,

$$
\mathcal{V}_2^S_{[\lambda]} \mathcal{L}_\lambda = \mathcal{V}_{\lambda} \otimes \mathcal{V}_{\lambda'}.\n$$

(66)

It follows from (64) that

$$
\mathcal{V}_2^S_{[\lambda]} \mathcal{L}_\lambda \cong \bigoplus_{\lambda \in P_+} \mathcal{V}_{\lambda} \otimes \mathcal{L}_\lambda \mathcal{V}_{\lambda'}.
$$

(67)

as $\mathfrak{g} \times \mathfrak{g}$-modules.

Since $\mathcal{V}_2^S$ is $\mathbb{Z}_{\geq 0}$-graded, $(\mathcal{V}_2^S)_{[\lambda]} \mathcal{L}_\lambda \cong \mathcal{O}(G)$.

(68)

as $\mathfrak{g} \times \mathfrak{g}$-modules.

Since $\mathcal{V}_2^S$ is $\mathbb{Z}_{\geq 0}$-graded, $(\mathcal{V}_2^S)_{[\lambda]} \mathcal{L}_\lambda \cong \mathcal{O}(G)$.

(69)

as $\mathfrak{g} \times \mathfrak{g}$-modules.

Since $\mathcal{V}_2^S$ is $\mathbb{Z}_{\geq 0}$-graded, $(\mathcal{V}_2^S)_{[\lambda]} \mathcal{L}_\lambda \cong \mathcal{O}(G)$.
of the commutative algebra $(V_S^z)_0$ such that $(V_S^z)_0(0) = V_0 \otimes V_0 = \mathbb{C}$. It follows that the projection $(V_S^z)_0 \to (V_S^z)_0(0) = \mathbb{C}$ is an algebra homomorphism.

**Lemma 10.10.** Let $A$ be a unital commutative $G \times G$-algebra, that is, unital commutative $\mathbb{C}$-algebra equipped with an action of $G \times G$ on $A$ such that the multiplication map $A \otimes A \to A$ is a $G \times G$-module homomorphism, where $G \times G$ diagonally acts on $A \otimes A$. Suppose that $A \cong A \otimes \mathbb{C}$ is an algebra homomorphism. Then $A \cong \mathcal{O}(G)$ as commutative $G \times G$-algebras.

**Proof.** For a $G$-module $M$, let $\mu_M : M \to \mathcal{O}(G) \otimes M$ be the comodule map. Thus, $\mu_M(m) = \sum_i f_i \otimes m_i$ if $g.m = \sum_i f_i(g)m_i$ for all $g \in G$. Then $\bar{\mu}_M : \mathcal{O}(G) \otimes M \to \mathcal{O}(G) \otimes M$, $f \otimes m \mapsto f \mu(m)$, gives a linear isomorphism such that $\bar{\mu}_M \circ (g \otimes 1) = (g \otimes 1) \circ \mu_M$ for all $g \in G$. Set $\nu_M = (\bar{\mu}_M)^{-1}$, so that $(g \otimes 1) \circ \nu_M = \nu_M \circ (g \otimes 1)$. Define $\nu_M : M \to \mathcal{O}(G) \otimes M$ by $\nu_M(m) = \sum_i f_i \otimes m_i$ if $g^{-1}.m = \sum_i f_i(g)m_i$ for all $g \in G$. We have the linear isomorphism

$$M \cong \mathcal{O}(G) \otimes M \rightarrow (\mathcal{O}(G) \otimes M)^{\Delta(G)}, \quad m \mapsto \nu_M(m).$$

We claim that if $R$ is a commutative $G$-algebra, this is an isomorphism of algebras. Indeed, consider the algebra homomorphism $\epsilon \otimes 1 : \mathcal{O}(G) \otimes R \to \mathbb{C} \otimes R = R$, where $\epsilon$ is a counit. We have $(\epsilon \otimes 1)(\nu_R(r)) = r$, and so, $(\epsilon \otimes 1)(\nu_R(r)\nu_R(r') - \nu_R(r'r')) = (\epsilon \otimes 1)(\nu_R(r))(\epsilon \otimes 1)(\nu_R(r')) - (\epsilon \otimes 1)(\nu_R(r'r')) = r'r' - rr' = 0$. Because $\nu_R(r)\nu_R(r') - \nu_R(r'r') \in (\mathcal{O}(G) \otimes R)^{\Delta(G)}$ and the restriction of $(\epsilon \otimes 1)$ to $(\mathcal{O}(G) \otimes R)^{\Delta(G)}$ is an injection, it follows that that $\nu_R(r)\nu_R(r') = \nu_R(r'r')$.

Let $\Phi : A \to \mathcal{O}(G)$ be an isomorphism as $G \times G$-modules. For a $G$-module $M$, we have the linear isomorphism

$$M \cong (A \otimes M)^{\Delta(G)}, \quad m \mapsto (\Phi \otimes 1)(\nu_M(m)).$$

If $R$ is a commutative $G$-algebra, this is an isomorphism of algebras by the same argument as above.

We conclude that there are isomorphisms

$$A \cong (\mathcal{O}(G) \otimes A)^{\Delta(G)} \cong (A \otimes \mathcal{O}(G))^{\Delta(G)} \cong \mathcal{O}(G).$$

This completes the proof. □

By Lemma 10.10, we can identify $(V_S^z)_0$ with $\mathcal{O}(G)$ as commutative $G \times G$-algebras. Since $x(z)f(w) \sim \frac{1}{z-w}(xf)(w)$ for $x \in \mathfrak{g}$ and $f \in (V_S^z)_0$, we have a non-trivial vertex algebra homomorphism $D_G^h \to V_S^z$ as required. □

**Theorem 10.11.** Let $V$ be a vertex algebra object in KL.

(i) We have $V \cong H^{\varphi + \varepsilon}_0(Z, W \otimes H^0_D(S)(V))$ as vertex algebras.

(ii) Let KL($V$) be the category of $V$-module objects on KL. Then $M \mapsto H^{\varphi + \varepsilon}_0(Z, W \otimes H^0_D(S)(M))$ gives an identity functor in KL($V$).

**Proof.** (i) Set

$$C = \bigoplus_{i \in \mathbb{Z}} C^i, \quad C^i = \bigoplus_{p+q=i} C^{p,q}, \quad C^{p,q} = W \otimes W \otimes V \otimes \bigoplus_{j=0}^{\infty/2+q} (\mathfrak{g}) \otimes \bigoplus_{j=0}^{\infty/2+q} (\mathfrak{g}).$$

Let $d_j$ be the differential of the complex $W \otimes W \otimes \bigoplus_{j=0}^{\infty/2+q} (\mathfrak{g})$ which defines $H^{\infty/2+q}(Z, W \otimes W)$. We regard $d_j$ as a differential of $C$ which trivially acts on the
factors $V$ and $\Lambda^\infty/2+\bullet(\hat{g})$. Let $d_\hat{g}$ be the differential of the complex $W \otimes V \otimes \Lambda^\infty/2+\bullet(\hat{g})$ which defines $H^{\infty/2+\bullet}(\hat{g}, W \otimes V)$. We consider $d_\hat{g}$ as a differential of $C$ which trivially acts on the first factor $W$ and $\Lambda^\infty/2+i(\hat{g}))$. Since $\{d_3, d_\hat{g}\} = 0$, $C$ is equipped with the structure of a double complex.

Let $H^{\bullet}_{\text{tot}}(C)$ be the total cohomology of $C$. Note that $C$ is a direct sum of finite-dimensional subcomplexes. In fact, we have $C = \bigoplus_{\lambda, \mu, \nu \in P_+} C_{\lambda, \mu, \nu}$ as complexes, where $C_{\lambda, \mu, \nu} = W[\tilde{\lambda}] \otimes W[\tilde{\mu}] \otimes V[\tilde{\nu}] \otimes \Lambda^\infty/2+\bullet(\hat{g}) \otimes \Lambda^\infty/2+\bullet(\hat{g})$, and each $C_{\lambda, \mu, \nu}$ decomposes into a direct sum of finite-dimensional eigenspaces of the total action of the Hamiltonian, which are preserved by $d_\text{tot} = d_3 + d_\hat{g}$.

Let $E_r \Rightarrow H^{\bullet}_{\text{tot}}(C)$ be the spectral sequence such that $d_0 = d_\hat{g}$ and $d_1 = d_3$. We have

$$E_1^{p,q} \cong H^{\infty/2+p}(Z, W \otimes W) \otimes \Lambda^\infty/2+p(\hat{g}),$$

$$E_2^{p,q} \cong H^{\infty/2+p}(\hat{g}, H^{\infty/2+p}(Z, W \otimes W) \otimes V).$$

By Theorem 10.11, $H^{\infty/2+p}(Z, W \otimes W)$ is a module over $H^{\infty/2+p}(Z, W \otimes W) \cong D_G^{ch}$. Hence $H^{\infty/2+p}(Z, W \otimes W)$ is free over $U(\hat{g}[t^{-1}])$ by Corollary 5.4, and hence $E_2^{p,q} = 0$ if $p < 0$ or $q < 0$. It follows that

$$H^{0}_{\text{tot}}(C) \cong E_2^{0,0} = H^{\infty/2+0}(\hat{g}, H^{\infty/2+0}(Z, W \otimes W) \otimes V) \cong H^{\infty/2+0}(\hat{g}, D_G^{ch} \otimes V) \cong V$$

as vertex algebras.

Next, consider the spectral sequence $E'_r \Rightarrow H^{\bullet}_{\text{tot}}(C)$ such that $d_0 = d_\hat{g}$ and $d_1 = d_3$. We have

$$(E'_1)^{p,q} \cong W \otimes H^0_{DS}(V) \otimes \Lambda^\infty/2+p(\hat{g}) \otimes \Lambda^q(\hat{g}, C),$$

$$(E'_2)^{p,q} \cong H^{\infty/2+p}(Z, W \otimes H^0_{DS}(V)) \otimes \Lambda^q(\hat{g}, C).$$

Since $(E'_2)^{p,q} = 0$ for $p < 0$ or $q < 0$, we obtain the vertex algebra isomorphism

$$H^{0}_{\text{tot}}(C) \cong H^{\infty/2+0}(Z, W \otimes H^0_{DS}(V)).$$

The assertion now immediately follows from (68) and (69). (ii) The same proof as (1) applies.

Let $\text{KL}_0$ be the full subcategory of $Z$-$\text{Mod}_{\text{reg}}$ consisting of objects $M$ that is isomorphic to $H^0_{DS}(\hat{M})$ for some $\hat{M} \in \text{KL}$.

**Proposition 10.12.** For $M \in \text{KL}_0$, we have

$$M \cong H^0_{DS}(H^{\infty/2+0}(Z, W \otimes M)).$$

**Proof.** Let $\hat{M} \in \text{KL}$ and set $M \cong H^0_{DS}(\hat{M}) \in \text{KL}_0$. By Theorem 10.11, $\hat{M} \cong H^{\infty/2+0}(Z, W \otimes M)$. Hence $M \cong H^0_{DS}(H^{\infty/2+0}(Z, W \otimes M))$. □

**Proposition 10.13.** Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence on $Z$-$\text{Mod}_{\text{reg}}$.

(i) Suppose that $M_1, M_2 \in \text{KL}_0$. Then $M_3 \in \text{KL}_0$ and (70) induces the exact sequence $0 \to H^{\infty/2+0}(Z, M_1) \to H^{\infty/2+0}(Z, M_2) \to H^{\infty/2+0}(Z, M_3) \to 0$. 
(ii) Suppose that \( M_2, M_3 \in \text{KL}_0 \). Then \( M_1 \in \text{KL}_0 \) and (70) induces the exact sequence \( 0 \to H_{\bullet}^{+0}(Z, M_1) \to H_{\bullet}^{+0}(Z, M_2) \to H_{\bullet}^{+0}(Z, M_3) \to 0 \).

Proof. By Lemma 10.5, (70) induces the exact sequence \( 0 \to H_{\bullet}^{+0}(Z, M_1) \to H_{\bullet}^{+0}(Z, M_2) \to H_{\bullet}^{+0}(Z, M_3) \).

(i) Let \( N = \text{Im} \psi \). Applying the exact functor \( H_{DS}^{0}(?) \) to the exact sequence \( 0 \to H_{\bullet}^{+0}(Z, M_1) \to H_{\bullet}^{+0}(Z, M_2) \to N \to 0 \), we obtain the exact sequence \( 0 \to M_1 \to M_2 \to H_{DS}^{0}(N) \to 0 \) by Lemma 10.12. Thus, \( H_{DS}^{0}(N) \cong M_2/M_1 \cong M_3 \), and so \( M_3 \in \text{KL}_0 \). Moreover, \( N \cong H_{\bullet}^{+0}(Z, W \otimes M_3) \) by Theorem 10.11, whence the second assertion.

(ii) Let \( N \) denote the cokernel of \( \psi \). Applying \( H_{DS}^{0}(?) \) to the exact sequence \( H_{\bullet}^{+0}(Z, M_2) \to H_{\bullet}^{+0}(Z, M_3) \to N \to 0 \), we obtain the exact sequence \( M_2 \to M_3 \to H_{DS}^{0}(N) \to 0 \). Since the first map is surjective, \( H_{DS}^{0}(N) = 0 \), and thus \( N = 0 \) by Proposition 9.3. The exact sequence \( 0 \to H_{\bullet}^{+0}(Z, M_1) \to H_{\bullet}^{+0}(Z, M_2) \to H_{\bullet}^{+0}(Z, M_3) \to 0 \) gives an exact sequence \( 0 \to H_{DS}^{0}(H(Z, W \otimes N_1)) \to N_2 \to N_3 \to 0 \), and thus, \( N_1 \cong H_{DS}^{0}(H(Z, W \otimes N_1)) \).

Proposition 10.14. Let \( M \in \text{KL} \). Suppose that \( N = H_{DS}^{0}(M) \) admits a filtration \( 0 = N_0 \subset N_1 \subset N_2 \subset \ldots \subset N = \bigcup_{p} N_p \) such that each successive quotient is an object of \( \text{KL}_0 \). Then \( M_i = H_{\bullet}^{+0}(Z, W \otimes N_i) \) defines an increasing filtration of \( M \) such that each successive quotient \( M_i/M_{i-1} \) is isomorphic to \( H_{\bullet}^{+0}(Z, W \otimes N_i/N_{i-1}) \).

Proof. First, we show by induction on \( i \geq 0 \) that \( N_i \in \text{KL}_0 \). There is nothing to show for \( i = 0 \). So let \( i > 0 \). By the induction hypothesis and Proposition 10.13, \( N_i/N_{i-1} \in \text{KL}_0 \). Hence \( N_i \in \text{KL}_0 \). This again by Proposition 10.13, \( N_{i+1} \in \text{KL}_0 \). Since \( N_{i+1} \in \text{KL}_0 \) for all \( i \geq 0 \), we have \( M_i/M_{i-1} = H_{\bullet}^{+0}(Z, W \otimes (N_i/N_{i-1})) \) and this completes the proof.

11. Genus zero chiral algebras of class S

For \( b \geq 2 \), set \( C_b = C(\bigoplus_{i=1}^{b-1} Z^{i,i+1}, W^b) := W^b \otimes \bigg( \bigwedge_{\bullet}^{+1} \left( \hat{\mathfrak{s}}(\hat{\mathfrak{g}}) \right) \bigg)^{\otimes b-1} \),

and let

\[
Q^{(b)}(z) = \sum_{i=1}^{b-1} Q^{(i,i+1)}(z),
\]

where

\[
Q^{(i,i+1)}(z) = \sum_{j=1}^{r \cdot g} (\rho_i(P_j(z)) - \rho_{i+1}(\tau(P_j(z))) \rho_{gh,i}(\psi_j^*(z)),
\]

where \( \rho_i \) denote the action on the \( i \)-th factor of \( W \otimes^r \) and \( \rho_{gh,i} \) denotes the action on the \( i \)-th factor of \( \bigwedge_{\bullet}^{+1} \left( \hat{\mathfrak{s}}(\hat{\mathfrak{g}}) \right) \)^{\otimes b-1}. Clearly \( Q^{(b)}(z) Q^{(b)}(w) \sim 0 \). \( (C_r, Q^{(b)}_{(0)}) \) is the differential graded vertex algebra, and the corresponding cohomology is denoted
by \( H^{\mathfrak{m}+\bullet}(\bigoplus_{i=1}^{b-1} Z^{i,i+1}, W^{\otimes b}) \). The vertex algebra \( H^{\mathfrak{m}+\bullet}(\bigoplus_{i=1}^{b-1} Z^{i,i+1}, W^{\otimes b}) \) is \( \frac{1}{2} \mathbb{Z} \)-graded by the Hamiltonian

\[
H = \sum_{i=1}^{b} \rho_i(L_0^W) + \sum_{i=1}^{b-1} \rho_{gb_i}(L_0^h).
\]

We are now in a position to define the main objects of this article. Let

\[
V_{G,1}^S = W = W_G,
\]

and let

\[
V_{G,b}^S := H^{\mathfrak{m}+0}(\bigoplus_{i=1}^{b-1} Z^{i,i+1}, W^{\otimes b}_G)
\]

for \( b \geq 1 \). Note that by Theorem \[10.9\] we have

\[
V_{G,2}^S \cong D_G^b.
\]

The embedding \( V^{\kappa}(g) \to W_G \) induces the vertex algebra homomorphism

\[
V^{\kappa}(g)^{\otimes b} \to V_{G,b}^S
\]

By definition,

\[
\rho_i(z) = \rho_{i+1}(\tau(z))
\]

on \( H^{\mathfrak{m}+0}(\bigoplus_{i=1}^{b-1} Z^{i,i+1}, W^{\otimes b}_G) \) for \( z \in Z, 1 \leq i < b \). Therefore, \( \rho_i \) factors through the vertex algebra homomorphism

\[
V(g)_b \to V_{G,b}^S.
\]

In particular, \( V_{G,b}^S \) belongs to \( KL_b \).

**Lemma 11.1.** For each \( b \geq 2 \), we have the following.

(i) \( H^{\mathfrak{m}+i}(\bigoplus_{i=1}^{b-1} Z^{i,i+1}, W^{\otimes b}_G) = 0 \) for \( i < 0 \).

(ii) \( V_{G,b}^S \cong H^{\mathfrak{m}+0}(Z, W_G \otimes V_{G,b-1}^S) \cong H^{\mathfrak{m}+0}(Z, V_{G,b-1}^S \otimes W_G) \).

**Proof.** We prove (i) and (ii) by induction on \( b \geq 2 \). For \( b = 2 \), (i) has been proved in Lemma \[11.4\] and there is nothing to show for (ii). Let \( b > 2 \). Consider the spectral sequence \( E_r \Rightarrow H^{\mathfrak{m}+\bullet}(\bigoplus_{i=1}^{b-1} Z^{i,i+1}, W^{\otimes b}_G) \) such that

\[
E_1^{p,q} = W_G \otimes H^{\mathfrak{m}+q}(\bigoplus_{i=1}^{b-2} Z^{i,i+1}, W^{\otimes b-1}_G) \otimes \bigwedge^\infty/2+p(\mathfrak{g}(\hat{g}))
\]

and \( E_2^{p,q} = H^{\mathfrak{m}+p}(Z, W_G \otimes H^{\mathfrak{m}+q}(\bigoplus_{i=1}^{b-2} Z^{i,i+1}, W^{\otimes b-1}_G)) \). By Lemma \[10.5\] and the induction hypothesis, \( E_2^{p,q} = 0 \) if \( q < 0 \) or \( p < 0 \). Therefore, we get that

\[
V_{G,b}^S \cong E_2^{0,0} \cong H^{\mathfrak{m}+0}(Z, W_G \otimes V_{G,b-1}^S).
\]

\[\square\]

**Proposition 11.2.** For any \( b, b' \geq 1 \), we have \( H^{\mathfrak{m}+i}(Z, V_{G,b}^S \otimes V_{G,b'}^S) = 0 \) for \( i < 0 \) and \( H^{\mathfrak{m}+0}(Z, V_{G,b}^S \otimes V_{G,b'}^S) \cong V_{G,b+b'}^S \) as vertex algebras.
Proof. The first statement follows from Lemma 11.1 and Proposition 11.9. We prove the second statement by induction on \( b \geq 1 \). For \( b = 1 \), the statement has been proved in Lemma 11.1. So let \( b \geq 2 \), and consider the cohomology 
\[
H^\varpi_+ (Z \oplus 2, V_{G,b-1}^S \otimes W_G \otimes V_{G,b}^S)
\]
of the complex 
\[
(V_{G,b-1}^S \otimes W_G \otimes V_{G,b}^S \otimes \bigwedge^\varpi_+ (\mathfrak{g}(\mathfrak{g})) \otimes \bigwedge^\varpi_+ (\mathfrak{g}(\mathfrak{g})), Q_{(0)}),
\]
with \( Q_{(0)} = Q^{(1)}_{(0)} + Q^{(2)}_{(0)} \), where \( Q^{(1)}_{(0)} \) is the differential of the complex for the cohomology \( H^\varpi_+ (Z, V_{G,b-1}^S \otimes W_G) \) that trivially acts on the factor \( V_{G,b}^S \) and the second factor of \( \bigwedge^\varpi_+ (\mathfrak{g}(\mathfrak{g}))^{\otimes 2} \), and \( Q^{(1)}_{(0)} \) is the differential of the complex for the cohomology \( H^\varpi_+ (Z, W_G \otimes V_{G,b}^S) \) that trivially acts on the factor \( V_{G,b-1}^S \) and the first factor of \( \bigwedge^\varpi_+ (\mathfrak{g}(\mathfrak{g}))^{\otimes 2} \). There is a spectral sequence
\[
E_r \Rightarrow H^\varpi_+ (Z \oplus 2, V_{G,b-1}^S \otimes W_G \otimes V_{G,b}^S)
\]
such that
\[
E_2^{p,q} = H^\varpi^p (Z, H^\varpi^q (Z, V_{G,b-1}^S \otimes W_G) \otimes V_{G,b}^S).
\]
By Lemma 11.1 and Proposition 11.9, \( E_2^{p,q} = 0 \) for \( p < 0 \) or \( q < 0 \). Therefore,
\[
H^\varpi_+^0 (Z \oplus 2, V_{G,b-1}^S \otimes W_G \otimes V_{G,b}^S) \cong E_2^{0,0} \cong H^\varpi_+^0 (Z, V_{G,b}^S \otimes V_{G,b}^S).
\]
There is another spectral sequence
\[
E'_r \Rightarrow H^\varpi_+ (Z \oplus 2, V_{G,b-1}^S \otimes W_G \otimes V_{G,b}^S)
\]
such that
\[
(E'_2)^{p,q} = H^\varpi^p (Z, V_{G,b-1}^S \otimes H^\varpi^q (Z, W_G \otimes V_{G,b}^S)).
\]
Again by Lemma 11.1 and Proposition 11.9, \((E'_2)^{p,q} = 0 \) for \( p < 0 \) or \( q < 0 \). It follows that
\[
H^\varpi_+^0 (Z \oplus 2, V_{G,b-1}^S \otimes W_G \otimes V_{G,b}^S) \cong (E'_2)^{0,0} \cong H^\varpi_+^0 (Z, V_{G,b}^S \otimes V_{G,b}^S) \cong V_{G,b}^S
\]
by the induction hypothesis, whence the statement. \( \square \)

Proposition 11.3. We have \( H^0_{DS} (V_{G,b+1}^S) \cong V_{G,b}^S \) for \( b \geq 1 \).

Proof. We have nothing to prove for \( b = 1 \). Let \( b > 1 \). We have \( H^0_{DS} (V_{G,b+1}^S) \cong H^\varpi_+^0 (\mathfrak{g} - \mathfrak{k}_b, W_G \otimes V_{G,b+1}^S) \cong H^\varpi_+^0 (\mathfrak{g} - \mathfrak{k}_b, W_G \otimes H^\varpi_+^0 (Z, V_{G,b}^S \otimes W_G)). \) As in the proof of Theorem 10.11, we find that there is an isomorphism
\[
H^\varpi_+^0 (\mathfrak{g} - \mathfrak{k}_b, W_G \otimes H^\varpi_+^0 (Z, V_{G,b}^S \otimes W_G)) \cong H^\varpi_+^0 (Z, H^\varpi_+^0 (\mathfrak{g} - \mathfrak{k}_b, W_G \otimes V_{G,b}^S) \otimes W_G).
\]
But the latter is isomorphic to \( H^\varpi_+^0 (Z, V_{G,b-1}^S \otimes W_G) \cong V_{G,b}^S \) by the induction hypothesis. \( \square \)

Theorem 11.4. The vertex algebra \( V_{G,r}^S \) is simple for all \( b \geq 1 \).
Proof. For \( b = 1, 2 \), we already know that \( V^S_{G,b} \) is simple. So let \( b > 2 \) and let \( I \) be a nonzero submodule of \( V^S_{G,b} \). The embedding \( I \hookrightarrow V^S_{G,b} \) induces the embedding \( H^0_{DS}(I) \hookrightarrow H^0_{DS}(V^S_{G,b}) = V^S_{G,b-1} \). Therefore, \( H^0_{DS}(I) \) is a submodule of \( V^S_{G,b-1} \), which is nonzero since \( I \cong H^0_{DS} \mathbb{Z} \otimes H^0_{DS}(I) \) by Theorem 10.11. Because \( V^S_{G,b-1} \) is simple by induction hypothesis, we get that \( H^0_{DS}(I) = V^S_{G,b-1} \). But then \( I = H^0_{DS} \mathbb{Z} \otimes H^0_{DS}(I) = H^0_{DS}(I) = V^S_{G,b-1} \). This completes the proof.

Consider the decomposition \( V^S_{G,b} = \bigoplus_{\lambda \in P_+} (V^S_{G,b})_{|\lambda} \) with respect to the action of \( Z_0 \).

**Proposition 11.5.** For \( r \geq 3 \), the vertex algebra \( V^S_{G,r} \) is conical. We have for \( \lambda \in P_+, (z_1, \ldots, z_b) \in T_b \) that

\[
\text{tr}_{V^S_{G,b}}(q^L z_1 z_2 \ldots z_b) = \left( q^{(\lambda, \rho')} \prod_{j=1}^{b-2} (1 - q^j)^{\text{rk} g} \right) \prod_{k=1}^{b} \text{tr}_{V^S_{G,b}}(q^{-D} z_k),
\]

where \( D \) is the standard degree operator of the affine Kac-Moody algebra, and so

\[
\text{tr}_{V^S_{G,b}}(q^{-D} z) = \frac{\sum_{w \in W} \epsilon(w) e^{w_{\alpha_\lambda}(z)}}{\prod_{j=1}^{b} (1 - q^j)^{\text{rk} g} \prod_{\alpha \in \Delta_+} \prod_{j=1}^{b} (1 - q^j e^{-\alpha}(z))(1 - q^j e^\alpha(z)).}
\]

**Proof.** The first assertion follows from the second assertion since \( \lambda(\rho') \geq 0 \) and \( \lambda(\rho') = 0 \) if and only if \( \lambda = 0 \) for \( \lambda \in P_+ \). By Proposition 9.3 and Proposition 11.5 we have for \( z \in T, w \in T^{b-1} \),

\[
\text{tr}_{V^S_{G,b}}(q^L z w) = q^{\lambda(\rho')} \text{tr}_{V^S_{G,b}}(q^{-D} z) \text{tr}_{V^S_{G,b-1}}(q^{-D} w)
\]

which equals to

\[
\left( q^{(\lambda, \rho')} \prod_{j=1}^{b-2} (1 - q^j)^{\text{rk} g} \right) \prod_{k=1}^{b} \text{tr}_{V^S_{G,b}}(q^{-D} z) \text{tr}_{V^S_{G,b-1}}(q^{-D} w)
\]

by (50). On the other hand for \( b = 2 \), we know that

\[
\text{tr}_D(q^L z w) = \text{tr}_{V^S_{G,b}}(q^{-D} z) \text{tr}_{V^S_{G,b}}(q^{-D} w)
\]

by (22). Hence the assertion follows inductively.

**Corollary 11.6.** The vertex algebra \( V^S_{G,b} \) is good for all \( b \geq 1 \).

**Proof.** We already know the statement for \( b = 1, 2 \) and the statement for \( b \geq 3 \) follows from Proposition 11.5.

**Proposition 11.7.** The vertex algebra \( V^S_{G,b} \) is conformal with central charge

\[
b \dim g - (b - 2) \text{rk} g - 24(b - 2)(\rho')^2.
\]
Proof. Set

\[ \omega_b = \sum_{i=1}^{b} (\omega_{(i)}) + \sum_{i=1}^{b-1} (\omega_{\rho_{\psi}}), \]

Then \( \omega_b \) defines a conformal vector of the complex \( C_b \) of central charge \( b \dim g - (b - 2) \text{rk} g - 24(b - 2)(\rho | \rho') \).

By Lemma \[10.4\] we have

\[
T(z)P_1(w) \sim \frac{1}{z-w} \partial P_1(w) + \frac{d_1 + 1}{(z-w)^2} P_1(w) + \sum_{j=2}^{d_1+2} \frac{\rho(1)^j i^j}{(z-w)^{j+1}} q_j^{(i)}(w),
\]

where \( q_j^{(i)} \) is some homogeneous vector of \( \mathfrak{g} \) of conformal weight \( d_i - j + 2 \). Hence,

\[
Q(\omega) \omega_b = \sum_{s=1}^{b-1} \sum_{i=1}^{\text{rk} g} \sum_{j=2}^{d_i+1} \partial^j \left( (\pi_s q_j^{(i)})^\ast - \pi_{s+1} (\tau q_j^{(i)}) (\psi_s^i) \right).
\]

By definition of \( Q(\omega) \), the element \( \sum_{s=1}^{b-1} \left( \pi_s q_j^{(i)} \otimes 1 - 1 \otimes \pi_s \tau q_j^{(i)} \right) (\psi_s^i) \) is a coboundary, and so that is, there exists \( z \in \mathfrak{g} \) such that \( \omega_b = \omega_b \) is annihilated by the action of \( \mathfrak{g} \) and so is \( \omega_b \). Therefore, \( \omega_b \) has to belong to \( C = V_0 \otimes V_0 \subset \mathcal{O}(G) = (V)_{G,b}^0 \).

Finally, we will show by induction on \( b \geq 2 \) that the central charge of \( \omega_b \) is the same as that of \( \omega_b \). Let \( b = 2 \). It is enough to show that \( \omega_{D_{G,b}} \) is the unique conformal vector of \( D_{G,b} \) such that \( \omega_{(1)} = (\omega_{D_{G,b}})_{(1)} \) and \( \omega \in (D_{G,b}^0 \otimes \theta). \) Since \( (D_{G,b}^0 \otimes \theta) \) is stable under the action of the \( sL \)-triple \( \{L-1, L_0, L_1\} \) associated with \( \omega_{D_{G,b}} \) and \( (D_{G,b}^0 \otimes \theta) = C \), it follows that \( \theta \otimes L-1 \) is injective on \( (D_{G,b}^0 \otimes \theta) \). Hence one can apply \[ Mor \] Lemma 4.1] twice to obtain that \( \omega - \omega_{D_{G,b}} = \partial \theta a \) for some \( a \in (D_{G,b}^0 \otimes \theta)^0 \). Since \( a \) is a constant multiplication of the vacuum vector, this gives that \( \omega = \omega_{D_{G,b}} \). So let \( b \geq 3 \). Note that since \( V_{G,b}^0 \) is conic, the same argument using \[ Mor \] Lemma 4.1] implies that \( \omega_b \) is the unique conformal vector of \( V_{G,b}^0 \) such that \( \omega_{(1)} \) coincides with the Hamiltonian \( (72) \). Let \( c \) be the central charge of \( \omega_b \). By Proposition \[11.3\] \( \omega_b \) gives rise to a conformal vector of \( V_{G,b-1}^0 \) with central charge \( c - \dim g + \text{rk} g + 24(\rho | \rho') \), which is annihilated by \( \mathfrak{g} \). Hence the uniqueness of the conformal vector of \( V_{G,b-1}^0 \) and the induction hypothesis gives the required result.

\[ \square \]

Remark 11.8. In type ADE the central charge of \( V_{G,b}^0 \) can be expressed as

\[
(b - 2(b - 2)|h') \dim g - (b - 2) \text{rk} g
\]

using the strange formula \( \dim g = 12|\rho|^2/h' \).
Proposition 11.9. For any \( r \geq 1 \), \( V_{G,b}^S \in KL^b \cap KL^S \). In particular, \( V_{G,b}^S \) is free over \( U(\mathfrak{g}[t^{-1}]) \) and cofree over \( U(t\mathfrak{g}[t]) \) by the action \( \iota_\nu \), \( \nu = 1, \ldots, b \).

Proof. We prove the statement by induction on \( b \geq 1 \).

For \( r = 1 \), \( V_{G,1}^S = W_G \) is free over \( U(\mathfrak{g}[t^{-1}]) \) and cofree over \( U(t\mathfrak{g}[t]) \). Hence, \( V_{G,1}^S \in KL^1 \cap KL^S \).

For \( b \geq 2 \), \( V_{G,b}^S \) is non-negatively graded and self-dual. Indeed, for \( b = 2 \) it was proved in [Zhu] that \( V_{G,2}^S = D_G^b \) is self-dual. For \( b \geq 3 \), \( V_{G,b}^S \) is conical by Proposition 11.5. Hence [NH] the Shapovalov form is defined on \( V_{G,b}^S \). By Theorem 11.4, this form is non-degenerate, and therefore \( V_{G,b}^S \) is self-dual. Thus, this is sufficient to show that \( V_{G,b}^S \in KL^b \). By the induction hypothesis and Proposition 11.3, \( H_{DS}^1(V_{G,b}^S) = V_{G,b-1}^S \) belongs to \( KL^b \). Therefore Proposition 10.14 and Proposition 10.15 give that \( V_{G,b}^S \in KL^b \).

\( \square \)

Theorem 11.10. Let \( b, b' \geq 1 \) such that \( b + b' \geq 3 \). We have

\[
H^{\mp+p}(\mathfrak{g}, \mathfrak{g}, V_{G,b}^S \otimes V_{G,b'}^S) = 0 \quad \text{for } i \neq 0,
\]

and

\[
V_{G,b}^S \circ V_{G,b'}^S \cong V_{G,b+b'-2}^S
\]
as vertex algebras.

Proof. The first statement follows from Proposition 11.9. For \( b = 1 \) or \( b' = 1 \), the second statement has been proved in Proposition 11.3 since \( V_{G,1}^S \circ V_{G,b}^S \cong H_{DS}^b(V_{G,b}^S) \) by (10). So suppose that \( b \geq 2 \). Set

\[
C = V_{G,b-1}^S \otimes W_G \otimes V_{G,b'}^S \otimes \bigwedge^{\mp+*}(\mathfrak{g}) \otimes \bigwedge^{\mp+*}(\mathfrak{g}).
\]

Let \( Q_3 \) be the differential of the complex \( V_{G,b-1}^S \otimes W_G \otimes \bigwedge^{\mp+*}(\mathfrak{g}) \) for the cohomology \( H^{\mp+p}(\mathfrak{z}, V_{G,b-1}^S \otimes W_G) \). We consider \( Q_3 \) as a differential on \( C \) that trivially acts on the factors \( V_{G,b}^S \) and \( \bigwedge^{\mp+*}(\mathfrak{g}) \). Let \( Q_3 \) be the differential of the complex \( W_G \otimes V_{G,b}^S \otimes \bigwedge^{\mp+*}(\mathfrak{g}) \) for \( H^{\mp+*}(\mathfrak{g}, W_G \otimes V_{G,b}^S) \). We consider \( Q_3 \) as a differential on \( C \) that trivially acts on the factors \( V_{G,b}^S \) and \( \bigwedge^{\mp+*}(\mathfrak{g}) \). Then \( C \) has the structure of a double complex. Let \( H_{\text{tot}}^*(C) \) be the total cohomology of \( C \) with the differential \( Q = Q_3 + Q_3 \). As in the proof of Theorem 10.11 we find that the total complex \( C \) is a direct sum of finite-dimensional subcomplexes.

Let \( E_r \Rightarrow H_{\text{tot}}^*(C) \) be the spectral sequence such that \( d_0 = Q_3 \) and \( d_1 = Q_3 \). We have

\[
E_2^{p,q} \cong H^{\mp+q}(\mathfrak{g}, H^{\mp+p}(\mathfrak{g}, V_{G,b-1}^S \otimes W_G) \otimes V_{G,b'}^S).
\]

Then \( E_2^{p,q} = 0 \) if \( p < 0 \) or \( q < 0 \) by Lemma 10.3 and Proposition 11.9 and thus, we obtain the vertex algebra isomorphism

\[
H_{\text{tot}}^0(C) \cong E_2^{0,0} \cong H^{\mp+0}(\mathfrak{g}, V_{G,b}^S \otimes V_{G,b'}^S) = V_{G,b}^S \circ V_{G,b'}^S.
\]

Next, consider the spectral sequence \( E'_r \Rightarrow H_{\text{tot}}^*(C) \) such that \( d_0 = Q_3 \) and \( d_0 = Q_3 \). We have

\[
(E'_2)^p,q \cong H^{\mp+p}(\mathfrak{g}, V_{G,b-1}^S \otimes H^{\mp+q}(\mathfrak{g}, W_G \otimes V_{G,b'}^S)).
\]
Then \((E'_2)^{p,q} = 0\) if \(p < 0\) or \(q < 0\), and we obtain the isomorphism
\[
H^0_{\text{tot}}(C) \cong (E_2)^{0,0} = H^{\mu,\nu}(Z, V_{G,b-1}^S \otimes V_{G,b'}^S) \cong V_{G,b+b'}^S
\]
by Proposition \ref{prop:11.12}. By \((77)\) and \((78)\), we obtain the required isomorphism. \(\Box\)

**Remark 11.11.** \(V_{G,b}^S\) is the unique vertex algebra object in \(\text{KL}\) (with respect to the first or the last action of \(\tilde{g}_{nc}\)) such that \(V_{G,1}^S \circ V_{G,b}^S \cong V_{G,b-1}^S\). Indeed, let \(V\) be a vertex algebra object in \(\text{KL}\) such that \(V_{G,1}^S \circ V_{G,b}^S \cong H^0_{DS}(V_{G,b}^S) \cong V_{G,b-1}^S\). Then by Theorem \ref{thm:11.11} we have \(V \cong H^{\mu,\nu}(Z, W \otimes V_{G,b-1}^S) \cong V_{G,b}^S\).

**Remark 11.12.** If there exists a vertex algebra object \(V \in \text{KL}\) with vertex algebra embeddings
\[
V \hookrightarrow V_{G,r}^S \quad \text{and} \quad V_{G,r}^S \rightarrow H^0_{DS}(V),
\]
then \(V_{G,b}^S \cong V\). Indeed, the embedding \(V \hookrightarrow V_{G,b}^S\) induces the embedding \(H^0_{DS}(V) \hookrightarrow H^0_{DS}(V_{G,b}^S) \cong V_{G,b-1}^S\). The existence of the embedding \(V_{G,b-1}^S \hookrightarrow H^0_{DS}(V)\) then forces that \(H^0_{DS}(V) \cong V_{G,b-1}^S\). It follows that \(V \cong V_{G,b}^S\) by Theorem \ref{thm:11.11}.

**Theorem 11.13.** The associated variety \(X_{V_{G,b}^S}\) of the vertex algebra \(V_{G,b}^S\) is isomorphic to \(W_b^G\) for all \(b \geq 1\).

**Proof.** Set \(X_b = X_{V_{G,b}^S}, R_b = R(V_{G,b}^S)/\sqrt{10} = \mathcal{O}(X_b), A_b = \mathcal{O}(W_b^G).\) We wish to show that \(R_b \cong A_b\). We regard both \(R_b\) and \(A_b\) as Poisson algebra objects of \(\text{QCoh}^G(\mathfrak{g}^*)\) by the moment maps \(\mu : X_b \rightarrow \mathfrak{g}^*\) and \(\mu : W_b^G \rightarrow \mathfrak{g}^*\) with respect to the action of the first copy of \(G\). Since preimages of \(\mathfrak{g}_{reg}^*\) by the moment maps are open and dense in \(X_b\) and \(W_b^G\), the restriction maps \(R_b \rightarrow R_b|_{\mathfrak{g}_{reg}^*}\) and \(A_b \rightarrow A_b|_{\mathfrak{g}_{reg}^*}\) are injective.

We prove by induction on \(b \geq 2\) that \(R_b|_{\mathfrak{g}_{reg}^*} \in \text{Coh}^G(\mathfrak{g}_{reg}^*)\) is a free \(\mathcal{O}(\mathfrak{g}_{reg}^*)\)-module and coincides with \(A_b|_{\mathfrak{g}_{reg}^*}\). Since the complement of \(\mathfrak{g}_{reg}^*\) has codimension 3 in \(\mathfrak{g}^*\) \((\text{[V2]}\)), this shows that
\[
R_b \cong R_b|_{\mathfrak{g}_{reg}^*} \cong A_b|_{\mathfrak{g}_{reg}^*} \cong A_b.
\]

We have
\[
\kappa(R_b) \cong R_{b-1}, \quad \kappa(A_b) \cong A_{b-1},
\]
where \(\kappa\) is the Kostant-Whittaker reduction, that is, \(\kappa(M) = M|_b = M \otimes \mathcal{O}(\mathfrak{g}^*), \mathcal{O}(S)\). On the other hand, it was shown in \([Ric]\) using the equivariant descent theory that the Kostant-Whittaker reduction gives an equivalence
\[
\text{QCoh}^G(\mathfrak{g}_{reg}^*) \rightarrow \text{Rep}(I_G^I), \quad M \mapsto \kappa(M),
\]
where \(\text{Rep}(I_G^I)\) is the category of representations of the group scheme \(I_G^I\) over \(S\), which are quasi-coherent over \(S\). Moreover, from the proof it follows that the above equivalence restricts to the equivalence between the category of the Poisson algebra objects in \(\text{QCoh}^G(\mathfrak{g}_{reg}^*)\) and that of the Poisson algebra objects \(\text{Rep}(I_G^I)\). Here, a Poisson algebra object in \(\text{Rep}(I_G^I)\) is a Poisson algebra \(R\) equipped with a Poisson algebra homomorphism \(\nu : \mathcal{O}(S) \rightarrow R\), and a \(\text{Rep}(I_G^I)\)-module structure is compatible with the \(\mathcal{O}(S)\)-module structure given by \(\nu\).

Hence, by \((80)\) and the induction hypothesis, it follows that \(R_b|_{\mathfrak{g}_{reg}^*} \cong A_b|_{\mathfrak{g}_{reg}^*}\). Moreover, since it is a free \(\mathcal{O}(\mathfrak{g}_{reg}^*)\)-module by the induction hypothesis, \(R_{b-1}\) is
free as $\mathcal{O}(\delta)$-module. Thus, the above equivalence show that $R_b|_{\mathfrak{g}_{reg}} = A_b|_{\mathfrak{g}_{reg}}$ is a free $\mathcal{O}_{\mathfrak{g}_{reg}}$-module. This completes the proof.

Conjecture 1. The vertex algebra $V_{G,b}^S$ is quasi-lisse \cite{AK} for all $b \geq 1$, that is, $X_{V_{G,b}^S} = W_G^b$ has finitely many symplectic leaves.

Appendix A. Examples

Let $G = SL_2$. We identify the space of the symmetric invariant bilinear forms on $\mathfrak{g}$ with $\mathbb{C}$ by the correspondence $\mathbb{C} \ni k \mapsto k\kappa_\mathfrak{g}/2h^\vee = k\kappa_\mathfrak{g}/4$. Let $W_G^k$ be the vertex algebra generated by fields $S(z), a(z), b(z), c(z), d(z)$, subjected to the OPEs

\[
\begin{align*}
  a(z)a(w) &\sim 0, \quad c(z)c(w) \sim 0, \quad a(z)c(w) \sim 0, \\
  a(z)b(w) &\sim \frac{1}{2(z-w)} :a(w)^2 :, \quad a(z)d(w) \sim \frac{1}{2(z-w)} :a(w)c(w) :, \\
  b(z)c(w) &\sim -\frac{1}{2(z-w)} :a(w)c(w) :, \quad c(z)d(w) \sim \frac{1}{2(z-w)} :c(w)^2 :, \\
  b(z)b(w) &\sim \frac{2k+3}{4} \left( \frac{1}{z-w} :a'(w)a(w) : + \frac{1}{(z-w)^2} :a(w)^2 : \right), \\
  d(z)d(w) &\sim \frac{2k+3}{4} \left( \frac{1}{z-w} :c'(w)c(w) : + \frac{1}{(z-w)^2} :c(w)^2 : \right), \\
  b(z)d(w) &\sim \frac{1}{z-w} \left( \frac{1}{2} :a(z)d(z) :: - b(z)c(z) : + \frac{2k+1}{4} :a'(z)c(z) : \right) \\
  &\quad + \frac{2k+3}{4(z-w)^2} :a(w)c(w) :, \\
  S(z)S(w) &\sim -\frac{k+2}{z-w} \partial S(w) - \frac{2(k+2)}{(z-w)^2} S(w) + \frac{(k+2)^2 c_k}{2(z-w)^4}, \quad c_k = 1 - \frac{6(k+1)^2}{k+2}, \\
  S(z)a(w) &\sim \frac{1}{z-w} b(w) + \frac{2k+1}{4(z-w)^2} a(w), \\
  S(z)b(w) &\sim -\frac{1}{z-w} :S(w)a(w) :: - \frac{2k+7}{4(z-w)^2} b(w) - \frac{(k+2)(2k+1)}{2(x-w)^4} a(w), \\
  S(z)c(w) &\sim \frac{1}{z-w} d(w) + \frac{2k+1}{4(z-w)^2} c(w), \\
  S(z)d(w) &\sim -\frac{1}{z-w} :S(w)c(w) :: - \frac{2k+7}{4(z-w)^2} d(w) - \frac{(k+2)(2k+1)}{2(x-w)^4} c(w).
\end{align*}
\]

The equivariant affine $W$-algebra $W_G^k$ is the quotient of $\bar{W}_G^k$ by the submodule generated by the singular vector $a(z)d(z) - b(z)c(z) :: - a'(z)c(z) : = -1$.

The conformal vector (the stress tensor) of $W_G^k$ is given by

\[
T(z) = :S(z)(a(z)c'(z) - a'(z)c(z)) :: + : (b(z)d'(z) - b'(z)d(z)) :: \\
+ \frac{2k+7}{2} : a'(z)d'(z) :: - b'(z)c'(z) :: \\
- \frac{6k+17}{24} (3 : a''(z)c'(z) :: + : a'''(z)c(z) :).
\]
We have
\[ T(z)S(w) \sim \frac{1}{z - w} \partial S(w) + \frac{1}{(z - w)^2} 2S(w) + \frac{(2k + 1)(3k + 4)}{2(z - w)^4}, \]
\[ T(z)a(w) \sim \frac{1}{z - w} \partial a(w) - \frac{1}{2(z - w)^2} a(w), \]
\[ T(z)b(w) \sim \frac{1}{z - w} \partial b(w) + \frac{1}{2(z - w)^2} b(w) + \frac{2k + 1}{2(z - w)^3} a(w), \]
\[ T(z)c(w) \sim \frac{1}{z - w} \partial c(w) - \frac{1}{2(z - w)^2} c(w), \]
\[ T(z)d(w) \sim \frac{1}{z - w} \partial d(w) + \frac{1}{2(z - w)^2} b(w) + \frac{2k + 1}{2(z - w)^3} c(w), \]
and
\[ T(z)T(w) \sim \frac{1}{z - w} \partial T(w) + \frac{1}{(z - w)^2} 2T(w) + \frac{2(2 - 3k)}{2(z - w)^4}. \]

Thus, the central charge of \( W^k_G \) is \( 2(2 - 3k) \), and
\[ \Delta_a = \Delta_c = -\frac{1}{2}, \quad \Delta_b = \Delta_d = \frac{1}{2}, \quad \Delta_S = 2. \]

We have an embedding
\[ W^k(\mathfrak{sl}_2) \hookrightarrow W^k_G, \]
and the image of \( W^k(\mathfrak{sl}_2) \) is generated by \( S(z) \). Indeed, for \( k \neq -2, -S(z)/(k+2) \) is a conformal vector of central charge \( c_k = 1 - \frac{6(k+1)^2}{k+2} \), and for \( k = 2, S(z) \) generates a commutative vertex subalgebra.

The embedding
\[ \mathcal{V}^k(\mathfrak{sl}_2) \hookrightarrow W^k_G, \quad k^* = -k - 4, \]
is given as follows:
\[ e(z) \mapsto S(z)c(z)^2 : + : d(z)^2 : - \frac{2k + 7}{2} : c(z)d'(z) : - : c'(z)d(z) : \\
+ \frac{2k + 7}{4} : c'(z)^2 : - \frac{2k + 3}{8} : c''(z)c(z) :, \]
\[ b(z) \mapsto 2 : S(z)a(z)c(z) : + 2 : b(z)d(z) : \\
- \frac{2k + 7}{2} : a(z)d'(z) + : b'(z)c(z) : - : a'(z)d(z) : - : b(z)c'(z) : \\
+ \frac{2k + 7}{2} : a'(z)c'(z) : - \frac{2k + 3}{4} : a''(z)c(z) :, \]
\[ f(z) \mapsto -S(z)a(z)^2 : - : b(z)^2 : + \frac{2k + 7}{2} : a(z)b'(z) - a'(z)b(z) : \\
- \frac{2k + 7}{4} : a'(z)^2 : + \frac{2k + 3}{8} : a''(z)a(z) :. \]

The vertex algebras \( W^k(\mathfrak{sl}_2) \) and \( V^{k^*}(\mathfrak{sl}_2) \) from a dual pair inside \( W^k_G \).
We have
\[ e(z)a(w) \sim -\frac{1}{z-w}c(w), \quad e(z)b(w) \sim -\frac{1}{z-w}d(w), \]
\[ h(z)a(w) \sim -\frac{1}{z-w}a(w), \quad h(z)b(w) \sim -\frac{1}{z-w}b(w), \]
\[ h(z)c(w) \sim \frac{1}{z-w}c(w), \quad h(z)c(w) \sim \frac{1}{z-w}c(w), \]
\[ f(z)c(w) \sim -\frac{1}{z-w}a(w), \quad f(z)d(w) \sim -\frac{1}{z-w}b(w), \]
\[ e(z)c(w) \sim e(z)d(w) \sim 0, \quad f(z)a(w) \sim f(z)b(w) \sim 0. \]

In \( R_{W_G} = O[S_G] = O[G] \otimes O[S] \), the images of \( a, b, c \), \( d \) generate \( O[G] \) and the image of \( S \) generates \( O[S] \).

As in Section\(^{10}\) we set \( W_G = W_G^{\kappa = -2} \). The complex \( C_G \) equals to \( W_G \otimes W_G \otimes \wedge^*(\mathfrak{g}^\vee) \).

For \( u \in W_G \) we set \( u_1 = u \otimes 1, u_2 = 1 \otimes u_2 \in W_G \otimes W_G \). The field \( Q^{(2)}(z) \) in (71) is given by
\[ Q^{(2)}(z) = S_1(z)\psi^*(z) - S_2(z)\psi^*(z). \]

The isomorphism
\[ D^{ch}_G \cong V_{G,2}^S = H^{\hat{\mathfrak{g}} + 0}(Z, W_G \otimes W_G) \]
is given by
\[
\begin{align*}
  u(z) & \mapsto u_1(z) \quad (u \in \mathfrak{g}), \\
  x_{11}(z) & \mapsto c_1(z)b_2(z) : + : d_1(z)a_2(z) : + : c_1(z)a_2(z)\psi(z)\psi^*(z) :, \\
  x_{12}(z) & \mapsto - : a_1(z)b_2(z) : - : b_1(z)a_2(z) : - : a_1(z)a_2(z)\psi(z)\psi^*(z) :, \\
  x_{21}(z) & \mapsto - : c_1(z)d_2(z) : - : d_1(z)c_2(z) : - : c_1(z)c_2(z)\psi(z)\psi^*(z) :, \\
  x_{22}(z) & \mapsto - : a_1(z)d_2(z) : - : b_1(z)c_2(z) : - : a_1(z)c_2(z)\psi(z)\psi^*(z) : .
\end{align*}
\]

Here \( x_{ij}(z), 1 \leq i, j \leq 2 \), are generators of the commutative vertex algebra \( O(J_{\infty}G) \) satisfying the relation \( x_{11}(z)x_{22}(z) - x_{12}(z)x_{21}(z) = 1 \).

**Theorem A.1.** We have \( V_{G = S L_2, b = 3}^S \cong SB((\mathbb{C}^2)^{\otimes 3}) \).

**Proof.** By Remark\(^{11,12}\) it is enough to construct vertex algebra homomorphisms
\[ SB((\mathbb{C}^2)^{\otimes 3}) \rightarrow H^{\hat{\mathfrak{g}} + 0}(Z, W_G \otimes D^{ch}_G) \cong V_{G,3}^S, \]
\[ V_{G,2}^S \cong D^{ch}_G \rightarrow H^{0}_{DS}(SB((\mathbb{C}^2)^{\otimes 3})), \]
which can be done directly. \( \square \)

**Theorem A.2.** (BLL\(^*\) Conjecture 1). We have \( V_{G = S L_2, b = 4}^S \cong L_{-2}(D_4) \).

**Proof.** By Remark\(^{11,12}\) it is enough to show that there are nonzero vertex algebra homomorphisms
\[ L_{-2}(D_4) \rightarrow V_{G,4}^S \quad \text{and} \quad V_{G,3}^S \rightarrow H^{0}_{DS}(L_{-2}(D_4)). \]

Both maps can be constructed directly. We explain the details only for the first map.
We have
\[ V^S_{G,4} \cong V^S_{G,3} \circ V^S_{G,3} = H^{\mp+0}(\g-4, \g, V^S_{G,3} \otimes V^S_{G,3}) \cong H^{\mp+0}(\g-4, \g, SB((\g^2)^{\otimes 3}) \otimes SB((\g^2)^{\otimes 3})). \]

Let \( U = SB((\g^2)^{\otimes 3}) \otimes SB((\g^2)^{\otimes 3}) \), where the \( \g[t] \) acts diagonally. As easily seen there is a non-trivial vertex algebra homomorphism \( \psi : V^{-2}(D_4) \to U \). Composing it with the vertex algebra homomorphism
\[ U \to H^{\mp+0}(\g-4, \g, SB((\g^2)^{\otimes 3}) \otimes SB((\g^2)^{\otimes 3})), \quad u \mapsto [u \otimes 1], \]
we get a vertex algebra homomorphism
\[ \psi : V^{-2}(D_4) \to H^{\mp+0}(\g-4, \g, SB((\g^2)^{\otimes 3}) \otimes SB((\g^2)^{\otimes 3})). \]
This is nonzero, because \( \psi \) sends the vacuum vector to the vacuum vector. By [AMor1], \( L_{-2}(D_4) \) is a quotient of \( V^{-2}(D_4) \) by a submodule generated by three singular vectors of weight 2. The map \( \psi \) sends two of the three singular vectors to zero, but it does not kill the remaining singular vector, say \( w \). Nevertheless, one finds that \( \psi(w) \) is a codomain, that is, \( \psi(w) = 0 \). Hence \( \psi \) induces a vertex algebra embedding
\[ L_{-2}(D_4) \hookrightarrow H^{\mp+0}(\g-4, \g, SB((\g^2)^{\otimes 3}) \otimes SB((\g^2)^{\otimes 3})) \cong V^S_{G,4} \]
as required.

\[ \square \]

References

[A1] Tomoyuki Arakawa. Representation theory of \( W \)-algebras. \textit{Invent. Math.}, 169(2):219–320, 2007.

[A2] Tomoyuki Arakawa. Characters of representations of affine Kac-Moody Lie algebras at the critical level. \textit{arXiv:0706.1817 [math.QA]}.

[A3] Tomoyuki Arakawa. A remark on the \( C_2 \) cofiniteness condition on vertex algebras. \textit{Math. Z.}, 270(1-2):559–575, 2012.

[A4] Tomoyuki Arakawa. Associated varieties of modules over Kac-Moody algebras and \( C_2 \)-cofiniteness of \( W \)-algebras. \textit{Int. Math. Res. Not.}, 2015:11605–11666, 2015.

[A5] Tomoyuki Arakawa. Rationality of \( W \)-algebras: principal nilpotent cases. \textit{Ann. Math.}, 182(2):565–694, 2015.

[A6] Tomoyuki Arakawa. Associated varieties and Higgs branches (a survey). In \textit{Vertex algebras and geometry}, volume 711 of \textit{Contemp. Math.}, pages 37–44. Amer. Math. Soc., Providence, RI, 2018.

[ACM] Tomoyuki Arakawa, Dimitar Chebotarov, and Fyodor Malikov. Algebras of twisted chiral differential operators and affine localization of \( \g \)-modules. \textit{Sel. Math. New Ser.}, 17(1):1–46, 2011.

[AK] Tomoyuki Arakawa and Kazuya Kawasetsu. \textit{Quasi-lisse vertex algebras and modular linear differential equations}, volume to appear in Kostant Memorial Volume. Birkhauser.

[AMal] Tomoyuki Arakawa and Fyodor Malikov. A chiral Borel-Weil-Bott theorem. \textit{Adv. Math.}, 229(5):2908–2949, 2012.

[AMor1] Tomoyuki Arakawa and Anne Moreau. Joseph ideals and Lisse minimal \( W \)-algebras. \textit{J. Inst. Math. Jussieu}, 17(2):397–417, 2018.

[AMor2] Tomoyuki Arakawa and Anne Moreau. Arc spaces and chiral symplectic cores. \textit{arXiv:1802.06533 [math.RT]}.

[AG] Sergey Arkhipov and Dennis Gaitsgory. Differential operators on the loop group via chiral algebras. \textit{Int. Math. Res. Not.}, (4):165–210, 2002.

[BLL+] Christopher Beem, Madalena Lemos, Pedro Liendo, Wlodek Pochaers, Leonardo Rastelli, and Balt C. van Rees. Infinite chiral symmetry in four dimensions. \textit{Comm. Math. Phys.}, 336(3):1359–1433, 2015.
[BPRvR] Christopher Beem, Wolger Peelaers, Leonardo Rastelli, and Balt C. van Rees. Chiral algebras of class S. J. High Energy Phys., (5):020, front matter+67, 2015.

[BR] Christopher Beem and Leonardo Rastelli. Vertex operator algebras, Higgs branches, and modular differential equations. preprint. [arXiv:1707.07679[hep-th]]

[BD] Alexander Beilinson and Vladimir Drinfeld. Chiral algebras, volume 51 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004.

[BF] Roman Bezrukavnikov and Michael Finkelberg. Equivariant Satake category and Kostant-Whittaker reduction. Mosc. Math. J., 8(1):39–72, 183, 2008.

[BFM] Roman Bezrukavnikov, Michael Finkelberg, and Ivan Mirković. Equivariant homology and K-theory of affine Grassmannians and Toda lattices. Compos. Math., 141(3):746–768, 2005.

[BFN1] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. Coulomb branches of 3d n = 4 quiver gauge theories and slices in the affine Grassmannian. arXiv:1604.03625 [math.RT].

[BFN2] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. Ring objects in the equivariant derived category arising from coulomb branches. arXiv:1706.02112[math.RT].

[Bor] Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. Proc. Nat. Acad. Sci. U.S.A., 83(10):3068–3071, 1986.

[CG] Neil Chriss and Victor Ginzburg. Representation theory and complex geometry. Birkhäuser Boston, Inc., Boston, MA, 1997.

[DSK] Alberto De Sole and Victor G. Kac. Finite vs affine W-algebras. Japan. J. Math., 1(1):137–261, 2006.

[EF] David Eisenbud and Edward Frenkel. Appendix to [?], 2001.

[Fel] Boris Feigin. Semi-infinite homology of Lie, Kac-Moody and Virasoro algebras. Uspekhi Mat. Nauk, 39(2(236)):195–196, 1984.

[FF1] Boris Feigin and Edward Frenkel. Quantization of the Drinfeld’’d-Sokolov reduction. Phys. Lett. B, 246(1-2):75–81, 1990.

[FF2] Boris Feigin and Edward Frenkel. Affine Kac-Moody algebras at the critical level and Gel’fand-Dikii algebras. In Infinite analysis, Part A, B (Kyoto, 1991), volume 16 of Adv. Ser. Math. Phys., pages 197–215. World Sci. Publ., River Edge, NJ, 1992.

[FF1] Boris Feigin, Edward Frenkel, and Leonid Rybnikov. On the endomorphisms of Weyl modules over affine Kac-Moody algebras at the critical level. Lett. Math. Phys., 88(1-3):163–173, 2009.

[FFR2] Boris Feigin, Edward Frenkel, and Leonid Rybnikov. Oper with irregular singularity and spectra of the shift of argument subalgebra. Duke Math. J., 155(2):337–363, 2010.

[FMTL] Boris Feigin, Edward Frenkel, and Valerio Toledano Laredo. Gaudin models with irregular singularities. Adv. Math., 233(3):873–948, 2010.

[Fre1] Edward Frenkel. Wakimoto modules, opers and the center at the critical level. Adv. Math., 195(2):297–404, 2005.

[Fre2] Edward Frenkel. Langlands correspondence for loop groups, volume 103 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2007.

[FBZ] Edward Frenkel and David Ben-Zvi. Vertex algebras and algebraic curves, volume 88 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, second edition, 2004.

[FG] Edward Frenkel and Dennis Gaitsgory. D-modules on the affine Grassmannian and representations of affine Kac-Moody algebras. Duke Math. J., 125(2):279–327, 2004.

[FHL] Igor B. Frenkel, Yi-Zhi Huang, and James Lepowsky. On axiomatic approaches to vertex operator algebras and modules. Mem. Amer. Math. Soc., 104(494):viii+64, 1993.

[FYZ] Igor B. Frenkel and Yongchang Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. Duke Math. J., 66(1):123–168, 1992.

[FMZ] Igor Frenkel and Minxian Zhu. Vertex algebras associated to modified regular representations of the Virasoro algebra. Adv. Math., 229(6):3468–3507, 2012.

[Gai] Davide Gaiotto. N = 2 dualities. J. High Energy Phys., (8):034, front matter + 57, 2012.

[GMN] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. Wall-crossing, Hitchin systems, and the WKB approximation. Adv. Math., 234:239–403, 2013.
[GG] Wee Liang Gan and Victor Ginzburg. Quantization of Slodowy slices. *Int. Math. Res. Not.*, (5):243–255, 2002.

[Gin] Victor Ginzburg. Harish-Chandra bimodules for quantized Slodowy slices. *Represent. Theory*, 13:236–271, 2009.

[GK] Victor Ginzburg and David Kazhdan. Construction of symplectic varieties arising in `Sicilian theories'. *in preparation*.

[GMS1] Vassily Gorbounov, Fyodor Malikov, and Vadim Schechtman. Gerbes of chiral differential operators. *Math. Res. Lett.*, 7(1):55–66, 2000.

[GMS2] Vassily Gorbounov, Fyodor Malikov, and Vadim Schechtman. On chiral differential operators over homogeneous spaces. *Int. J. Math. Math. Sci.*, 26(2):83–106, 2001.

[Kac1] Victor G. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.

[Kac2] Victor Kac. *Vertex algebras for beginners*, volume 10 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 1998.

[Kac3] Victor Kac. Introduction to vertex algebras, poisson vertex algebras, and integrable Hamiltonian PDE. In Caselli F. De Concini C. De Sole A. Callegaro F., Carnevalo G., editor, *Perspectives in Lie Theory*, volume 19 of *Springer INdAM Series* 19, pages 3–72. Springer, 2017.

[KRW] Victor Kac, Shi-Shyr Roan, and Minoru Wakimoto. Quantum reduction for affine superalgebras. *Comm. Math. Phys.*, 241(2-3):307–342, 2003.

[KW] Victor Kac and Minoru Wakimoto. On characers of irreducible highest weight modules of negative integer level over affine lie algebras. arXiv:1706.08387 [math.RT].

[KT] Masaki Kashiwara and Toshiyuki Tanisaki. Characters of irreducible modules with non-critical highest weights over affine Lie algebras. In *Representations and quantizations (Shanghai, 1998)*, pages 275–296. China High. Educ. Press, Beijing, 2000.

[KS] Bertrand Kostant and Shlomo Sternberg. Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. *Ann. Physics*, 176(1):49–113, 1987.

[Kuw] Toshiro Kuwabara. BRST cohomologies for symplectic reflection algebras and quantizations of hypertoric varieties. *Transform. Groups*, 20(2):437–461, 2015.

[Li] Haisheng Li. Abelianizing vertex algebras. *Comm. Math. Phys.*, 259(2):391–411, 2005.

[Los] Ivan Losev. Quantized symplectic actions and W-algebras. *J. Amer. Math. Soc.*, 23(1):35–59, 2010.

[MSV] Fyodor Malikov, Vadim Schechtman, and Arkady Vaintrob. Chiral de Rham complex. *Comm. Math. Phys.*, 204(2):439–473, 1999.

[MT] Gregory W. Moore and Yuji Tachikawa. On 2d TQFTs whose values are holomorphic symplectic varieties. In *String-Math 2011*, volume 85 of *Proc. Sympos. Pure Math.*, pages 191–207. Amer. Math. Soc., Providence, RI, 2012.

[Mor] Yuto Moriwaki. On classification of conformal vectors in vertex operator algebra and the vertex algebra automorphism group. arXiv:1810.04899 [math.QA].

[Mus] Mircea Mustaţă. Jet schemes of locally complete intersection canonical singularities. *Invent. Math.*, 145(3):397–424, 2001. With an appendix by David Eisenbud and Edward Frenkel.

[MV] Ivan Mirković and Kari Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007.

[Ner] Ju. A. Neretin. Differential operators on Lie groups. *Uspekhi Mat. Nauk*, 36(2(218)):195–196, 1981.

[Ric] Simon Riche. Kostant section, universal centralizer, and a modular derived Satake equivalence. *Math. Z.*, 286(1-2):223–261, 2017.

[Ryb] Leonid G. Rybnikov. The shift of invariants method and the Gaudin model. *Funktsional. Anal. i Prilozhen.*, 40(3):30–43, 96, 2006.

[Slo] Peter Slodowy. *Simple singularities and simple algebraic groups*, volume 815 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Tac1] Yuji Tachikawa. *On some conjectures on VOAs*. [http://member.ipmu.jp/yuji.tachikawa/transp/voaconj2.pdf](http://member.ipmu.jp/yuji.tachikawa/transp/voaconj2.pdf)

[Tac2] Yuji Tachikawa. On “categories” of quantum field theories. *Proc. Int. Cong. of Math. Rio de Janeiro*, 2:2695–2718, 2018.

[Vel] Ferdinand D. Veldkamp. The center of the universal enveloping algebra of a Lie algebra in characteristic p. *Ann. Sci. École Norm. Sup. (4)*, 5:217–240, 1972.
[Vor] Alexander A. Voronov. Semi-infinite homological algebra. *Invent. Math.*, 113(1):103–146, 1993.

[Zhu1] Minxian Zhu. Vertex operator algebras associated to modified regular representations of affine Lie algebras. *Adv. Math.*, 219(5):1513–1547, 2008.

[Zhu2] Minxian Zhu. A Weyl module filtration for the vertex algebra of differential operators. *Represent. Theory*, 15:370–384, 2011.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: arakawa@kurims.kyoto-u.ac.jp