Axisymmetric model of the sealing cylinder in service: analytical solutions

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ABSTRACT

The stress state analysis of the sealing cylinder is of great significance for the safe operation of the sealing system. In this study, we probe a sealing system that can be simplified as an axisymmetric problem. In service, the rubber around the central pipe contacts the casing under the action of the axial pressure, and thus the sealing function is realized. The analytical solution of the stress at the sealing interface is derived based on the Love strain function in the axisymmetric configuration. Then, the relationship between the axial pressure and the gap (between the sealing cylinder and the casing) is presented, and the contact pressure at the contact interface is also given. The numerical simulation is performed, which is in agreement with the analytical solutions in the two deformation stages. The obtained results in the current work have also been comprehensively compared with the previous results, to give suggestions for engineering selection. These findings are beneficial to obtain a deep understanding on the mechanism of the sealing process, and provide some inspirations on the new types of sealing tools for mechanical engineering, chemical engineering, petroleum engineering, etc.

KEYWORDS: sealing cylinder, Love function, analytical solution, contact pressure

1. INTRODUCTION

Packer is a kind of downhole tool in oil exploitation, and its sealing performance directly affects the safety and efficiency of many mechanical structures in service. The sealing cylinder is one kernel component of the packer sealing system, which is used to seal the gap between the central pipe and casing when an external pressure is exerted. In practice, such factors as the working environment, temperature, pressure, cylinder structure and material properties will significantly affect the sealing performance. How to properly design the sealing structure by considering these problems leaves us a big challenge. This demand urges us to fully understand the mechanism of the sealing process.

One tractable problem we must treat is the sealing failure, which is inevitable under terrible working conditions [1–5]. Many studies have been carried out to survey the failure behaviors of the sealing device. For instance, Dong et al. [6] investigated the effect of high temperature on the sealing behavior based on thermal aging experiments. It is also noted that, with the increase of well depth in petroleum exploitation, not only the temperature changes, but also the downhole pressure increases. Dong et al. [7] explored the effect of stress relaxation on the fabric rubber sealing performance under different pressures. Besides these, the CO2 corrosion is the main cause of rubber cylinder failure in the sealing system, as the mechanical properties of rubber will deteriorate. As a result, significant efforts have been made to study the influence of CO2 corrosion on the sealing failure [8–11]. The sealing failure may also be induced by improper structural designs, and thus the structural optimization of the sealing system is another task of engineers. For example, Wang et al. [12] studied the influence of inner groove shape and shoulder protection structure on the sealing property, and they designed a dual-channel packer with excellent performances. Liu et al. [13] discussed the impact of the cylinder size on the sealing performance, and one retrievable packer was invented after the optimization of structure sizes. Evidently, the sealing failure is closely related to the types of rubber material [14]. Ahmed et al. [15] studied the effect of material and chemical swelling on the sealing performance via numerical and experimental techniques. It was claimed that the NBR rubber can generate higher contact pressures than the EPDM rubber, especially when it is exposed to surfactant that results in volumetric swelling.

Although there are many factors affecting the sealing performance, the central topic on proper evaluation of the sealing performance may be ascribed to the mechanics analyses, which is beneficial to probe the sealing mechanism [16–20]. In most engineering areas, the contact pressure at the rubber–metal interface is often adopted as an index to evaluate the sealing performance. Zheng et al. [21] utilized the finite-element method (FEM) to calculate the contact stress in sealing, and several cylinders with different structures were analyzed. In their calculation, there is only one certain pressure applied on the rubber, and the whole sealing process has not been demonstrated. On this basis, Hu et al. [22] studied the influence of different axial loads on the contact pressure and compression deformation amplitude through experiments and numerical simulations. They found that the compression deformation amplitude and contact...
Figure 1 The sealing cylinder under the uniformly distributed pressure and its inner wall is constrained by displacement.

pressure difference both increase with the increase of the axial force, but the increasing speeds are different. The unresolved work is that the dependence relationship between the external load and contact pressure has not been quantitatively given in this study. During compression of the rubber cylinder, another failure mode is its instability that often occurs due to the shell structure. As a typical example, Liu [23] analyzed the stability of rubber cylinder in the sealing system when the outer boundary is constrained, and the axial and circumferential buckling has been analyzed. Moreover, Song et al. [24] derived the analytical expression of contact stress under the axial load based on the neo-Hookean constitutive relationship, which is different from the previous result [25,26]. Furthermore, Liu et al. [27] carried out the FEM simulation on the sealing process of fracturing packers under corrosion conditions, but only the qualitative relationship between the axial load and the gap was given.

Based on the previous studies, it is judged that the sealing mechanism has not been thoroughly described, and one challenging issue is that there is a lack of explicit solutions. This situation may be attributed to the physical complexity of the problem, where the rubber is a nonlinear material, and the contact behavior exists at the interface. As a consequence, the current exploration is directed toward a comprehensive study on analytical expressions between the axial load and gap, and that between the axial load and contact stress at the interface. The goal in this study is to give an efficient and convenient formula to evaluate the sealing performance. Without loss of generality, the analysis framework is built on the basis of linear elasticity, as the practical deformation of rubber is within the range of infinitesimal deformation.

The outline of this article is organized as follows. In Section 2, considering the axisymmetric boundary conditions, a Love strain function is proposed to deduce the displacement, strain and stress fields. In Section 3, the analytical expression of the gap between the rubber and casing is derived in the first deformation stage, and the FEM simulation has been made to verify the analytical result. In Section 4, the analytical expressions on the gap and contact pressure are both given, and the numerical results are also provided for comparison. In Section 5, the existing results from different literatures are compared with our analytical solution and numerical result. Finally, critical conclusions follow in Section 6.

2. MODEL FORMULATION

Let us consider the mechanical model of the sealing system, which is schematized in Fig. 1. In the three-sleeve sealing structure, the inner one is a central pipe, the middle one is a sealing cylinder and the outer one is a casing, so the whole system can be viewed as an axisymmetric model. Refer to the Cartesian coordinate system O-xyz and the polar coordinate system O-rtz. The z-axis is along the axis of the sealing cylinder, and the origin is built at the center of the central pipe. The length of the sealing cylinder is 2l, and the symbols r1, r2 and r3 represent the inner radius of the casing, respectively. Considering the infinitesimal deformation of the system, and without loss of generality, the material of the sealing cylinder is viewed as the perfectly elastic material, which obeys the generalized Hooke’s law. The in-service condition tells us that initially the sealing cylinder is in close contact with the central pipe, and there is a gap between the outer surface of the cylinder and the casing, which is denoted by δ. Under the action of a uniformly distributed pressure p in the z direction, the sealing cylinder starts to deform and then contacts the inner surface of the casing to realize the sealing function.

It is easily understood that the sealing structure experiences two stages to achieve the sealing effect in service. In the first stage, which is schematized in Fig. 2a, the outer wall of the sealing cylinder gradually approaches the casing under the action of the axial uniform load, and there is no interaction between them. In the second stage, which is schematized in Fig. 2b, when the axial uniform load increases and reaches a certain value, the sealing cylinder begins to contact the casing. There must be stress generated at the contact interface when the cylinder starts to contact the casing, which is a key quantity in the design of sealing structures. It should be pointed out that in the first deformation stage, the axial uniform load p is denoted by p_1. Herein, we define the
maximum uniform load in the first stage as $p_1$. In the second deformation stage, the value of $p$ is the sum of $p_2$ and $p_2$, where $p_2$ is the axial uniform load added in the second stage based on the level of zero value. This decomposition satisfies the principle of superposition within the scope of infinitesimal deformation.

For the convenience of modeling, the rigid body motions of the central pipe and the casing are both constrained, and they are both viewed as rigid bodies. The equilibrium equation of the sealing cylinder is expressed as

$$
\begin{align*}
\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} &= 0, \\
\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \tau_{rz} &= 0,
\end{align*}
$$

where $\sigma_r, \sigma_\theta$ and $\sigma_z$ are normal stress components in the radial, circumferential and axial directions, respectively, and $\tau_{rz}$ is the shear stress. Considering the real working conditions, the volumetric forces are omitted in the model.

The geometric equations of the sealing cylinder read

$$
\begin{align*}
\varepsilon_r &= \frac{\partial r}{\partial r}, \\
\varepsilon_\theta &= \frac{u_r}{r}, \\
\varepsilon_z &= \frac{\partial z}{\partial z}, \\
\gamma_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r},
\end{align*}
$$

where $\varepsilon_r, \varepsilon_\theta, \varepsilon_z$ and $\gamma_{rz}$ are the radial strain, circumferential strain, axial strain and shear strain, respectively, and $u_r$ and $w$ represent the radial displacement and axial displacement, respectively.

The constitutive relationship of the sealing cylinder, i.e. the generalized Hooke’s law, is given as

$$
\begin{align*}
\sigma_r &= \frac{E}{1+\mu} \left( \frac{\mu}{1-2\mu} \Theta + \varepsilon_r \right), \\
\sigma_\theta &= \frac{E}{1+\mu} \left( \frac{\mu}{1-2\mu} \Theta + \varepsilon_\theta \right), \\
\sigma_z &= \frac{E}{1+\mu} \left( \frac{\mu}{1-2\mu} \Theta + \varepsilon_z \right), \\
\tau_{rz} &= \frac{E}{2(1+\mu)} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right),
\end{align*}
$$

where $\Theta = \varepsilon_r + \varepsilon_\theta + \varepsilon_z$ is the volume strain, $E$ is the Young’s modulus and $\mu$ is the Poisson’s ratio.

Substituting Eqs (2) and (3) into (1), one can obtain the Lamé–Navier equation in terms of displacement:

$$
\begin{align*}
\frac{1}{1-2\mu} \frac{\partial \sigma_r}{\partial r} + \nabla^2 u_r - \frac{u_r}{r} &= 0, \\
\frac{1}{1-2\mu} \frac{\partial \sigma_\theta}{\partial \theta} + \nabla^2 w &= 0,
\end{align*}
$$

where $\nabla^2$ is the Laplacian operator and $\nabla^2 = (\partial/\partial r^2) + (1/r)(\partial/\partial r) + (\partial^2/\partial z^2)$.

To solve the spatial axisymmetric problem, we introduce the Love function as follows [28–30]:

$$
\xi (r, z) = A_1 z^3 + A_2 z r^2 + A_3 z \ln r,
$$

and the radial displacement $u_r$ and axial displacement $w$ can be written as

$$
\begin{align*}
u_r &= -\frac{1}{2G} \frac{\partial \xi}{\partial z}, \\
w &= \frac{1}{2G} \left[ 2(1-\mu) \nabla^2 - \frac{\partial^2 \xi}{\partial z^2} \right],
\end{align*}
$$

where $G$ is the shear modulus and $G = E/[2(1+\mu)]$.

It can be seen that the Love function satisfies the biharmonic equation $\nabla^4 \xi = 0$ naturally. Then, inserting Eq. (5) into (6), one obtains the displacement components as follows:

$$
\begin{align*}
u_r &= -\frac{1}{2G} \left( 2A_2 r + \frac{A_3}{r} \right), \\
w &= \frac{1}{6} \left[ 3 (1-2\mu) A_1 z + 4 (1-\mu) A_2 z \right].
\end{align*}
$$

Next, the strain components can be derived as

$$
\begin{align*}
\varepsilon_r &= -\frac{1}{2(1+\mu)} \left( 2A_2 + \frac{A_3}{r} \right), \\
\varepsilon_\theta &= -\frac{1}{2(1+\mu)} (2A_2 + \frac{A_3}{r}), \\
\varepsilon_z &= \frac{1}{3} \left[ 3 (1-2\mu) A_1 z + 4 (1-\mu) A_2 z \right], \\
\gamma_{rz} &= 0.
\end{align*}
$$

As a result, the stress components are acquired as

$$
\begin{align*}
\sigma_r &= 6\mu A_1 + 2 (2\mu - 1) A_2 + \frac{A_3}{r}, \\
\sigma_\theta &= 6\mu A_1 + 2 (2\mu - 1) A_2 - \frac{A_3}{r}, \\
\sigma_z &= 6 (1-\mu) A_1 + 4 (2-\mu) A_2, \\
\tau_{rz} &= 0.
\end{align*}
$$
3. ANALYTICAL AND NUMERICAL SOLUTIONS IN STAGE I

3.1 Analytical solution
At the first deformation stage, i.e. Stage I, the initial distance between the sealing cylinder and the casing is \( r_1 - r_2 \), and the range of the parameter \( r \) is \( r_1 \leq r \leq r_2 \). Under the action of the axial uniform load, the relationship between the applied pressure \( p_1 \) and the gap \( \delta \) can be given. Considering that the outer surface of the sealing cylinder is free, the boundary conditions are prescribed as

\[
\begin{align*}
  & z = l, \quad \sigma_z = -p_1, \\
  & z = 0, \quad w = 0, \\
  & r = r_1, \quad u_r = 0, \\
  & r = r_2, \quad \sigma_r = 0.
\end{align*}
\]

Substituting the above given expressions of displacement, stress and strain into the boundary conditions in Eq. (10), one can get

\[
\begin{align*}
  & 6(1 - \mu)A_1 + 4(2 - \mu)A_2 = -p_1, \\
  & -\frac{1}{2\mu} \left( 2A_2r_1 + \frac{A_1}{r_1} \right) = 0, \\
  & 6\mu A_1 + 2(2\mu - 1)A_2 + \frac{A_1}{r_1^2} = 0,
\end{align*}
\]

in which

\[
\begin{align*}
  & A_1 = \frac{(2\mu-1)r_2^2-r_1^2)\mu}{6[(1-\mu)r_1^2+(1+\mu)r_2^2]}p_1, \\
  & A_2 = \frac{1}{6}[(1-\mu)r_1^2+(1+\mu)r_2^2], \\
  & A_3 = \frac{\mu}{1+\mu}r_2^2.
\end{align*}
\]

Introducing the Love strain function

\[
\zeta (r, z) = \frac{1}{6} \frac{(2\mu-1)r_2^2-r_1^2)\mu}{[(1-\mu)r_1^2+(1+\mu)r_2^2]}p_1r^3
\]

\[
= -\frac{\mu \rho_1 r^2}{2(1-\mu)r_1^2+(1+\mu)r_2^2}z^2 r^2
\]

\[
+ \frac{\mu \rho_1 r_1^2 r_2^2}{(1-\mu)r_1^2+(1+\mu)r_2^2} z \ln r
\]

and the displacement components can then be deduced as

\[
\begin{align*}
  & u_r = \frac{\mu r_1 r_2^2}{2(1-\mu)r_1^2+(1+\mu)r_2^2}z, \\
  & w = \frac{\mu r_1 r_2^2}{2(1-\mu)r_1^2+(1+\mu)r_2^2}z.
\end{align*}
\]

As a consequence, the stress components in Stage I are obtained as

\[
\begin{align*}
  & \sigma_r = \frac{\mu r_1 r_2^2}{[(1-\mu)r_1^2+(1+\mu)r_2^2]}z, \\
  & \sigma_\theta = \frac{-\mu r_1 r_2^2}{[(1-\mu)r_1^2+(1+\mu)r_2^2]}z, \\
  & \sigma_z = -p_1, \\
  & \tau_{r\theta} = 0.
\end{align*}
\]

Figure 3 The relationship between \( r \) and \( u_r \) under different axial uniform loads.

According to the above solution results, the relationships between \( u_r \) and \( r \), between \( w \) and \( z \), and between \( \sigma_r \) and \( r \) under different uniformly distributed pressures can be obtained, respectively. To exemplify these relations, we select the parameter values according to the previous work [31]: the radius \( r_1 = 57.5 \) mm, radius \( r_2 = 73.5 \) mm, Young’s modulus \( E = 10.98 \) MPa and Poisson’s ratio \( \mu = 0.5 \).

First, as shown in Fig. 3, when the uniform pressure is fixed, the radial displacement increases with the increase of the radius and reaches the maximum value on the outer surface of the sealing cylinder, and the relationship between them is close to a linear one. The scaling law is given as \( u_r \propto r - r_1^2/r \). In particular, when \( r = r_1 \), \( u_r = 0 \), and the function \( r - r_1^2/r \) = 0. When \( r = r_2 \), the radial displacement \( u_r \propto r - r_1^2/r - r_2^2/r \). It is also shown that the greater the axial uniform pressure, the greater the radial displacement at the same position of the cylinder, and this is consistent with our experience.

Next, according to Eq. (14), the axial displacement \( w \) is dependent upon the parameter \( z \) and their dependence relationship is displayed in Fig. 4 under different axial pressures. These curves are definitely linear according to Eq. (14), i.e. \( w \propto z \), and when \( z = 0 \), the axial displacement is zero. When \( z = l \), i.e. at the upper or bottom surfaces, the axial displacement arrives at the maximum value, where the negative sign means the cylinder is in compression. Moreover, the axial displacement increases with the increase of the applied pressure, which matches the dimensional analysis.

In addition, when sealing, the contact pressure mainly comes from the radial stress, and its relationship with respect to the parameter \( r \) can be depicted in Fig. 5. It is seen that the radial stress decreases with the increase of the parameter \( r \) nonlinearly. It can be judged from Eq. (15) that there is the scaling law \( \sigma_r \propto r^{-2} \). On the outer surface of the cylinder, the free boundary leads to the result that the radial stress is zero. It is also noticed that, at the same position, the radial stress increases with the increase of the external pressure.
Altogether, in Stage I, the gap value can be expressed as

$$\delta = r_3 - r_2 - u_r (r_2)$$

$$= r_3 - r_2 - \frac{\mu r_2^2 (r_2^2 - r_1^2)}{2G[(1 - \mu) r_1^2 + (1 + \mu) r_2^2]} r_2 p_1.$$  

(16)

Evidently, it decreases with the increase of the external pressure. At the critical state, i.e. $\delta = 0$, the corresponding value of the external pressure is

$$p_{cr} = \frac{2G[(1 - \mu) r_1^2 + (1 + \mu) r_2^2]}{\mu r_2^2 (r_2^2 - r_1^2)} (r_3 - r_2).$$  

(17)

As one typical example, when we substitute the above material parameters $G$ and $\mu$ into Eq. (17), one can get the critical pressure $p_{cr} = 2.5033$ MPa.

3.2 Numerical solution

In order to verify the above analytical solution, the finite-element software ABAQUS is used to analyze the first sealing stage [32, 33]. Considering the symmetric property of the problem, the 2D axisymmetric model is chosen for modeling. The mesh type of the cylinder is selected as the CAX4RH element, and in the other parts the default elements are used. The Mooney–Rivlin constitutive model, which is a perfect model to formulate rubber, is selected in the simulation. The material of the casing is 45 steel and the central pipe is 35CrMo. The elastic moduli of the casing and the central pipe are 2.1 $\times$ 10^5 MPa, and the Poisson's ratios are both 0.3. The central pipe and the casing are fixed. Consequently, the relationship between the external pressure $p$ and the gap $\delta$ can be obtained. Four sets of displacement cloud diagrams are given through numerical simulations, as shown in Fig. 6. Obviously, as the external uniform load increases, the gap between the sealing cylinder and the casing gradually decreases, and when its value reaches zero, the sealing function can be realized.

The analytical solution and numerical result are compared in Fig. 7. It can be seen that the two curves are very close, and this fact can validate the accuracy of the analytical solution. Before the critical state appears, the gap decreases with the increase of the external pressure, and after this point, the gap becomes zero, meaning that the sealing has started. The difference between the two curves I and II lies in the critical value of the external pressure. It can be clearly seen that the variation trend of the relationship under the two solutions is the same, and the maximum error between them is 4%, which is within the engineering requirement. This again shows that the proposed Love strain function can be used to solve the radial displacement efficiently.

4. ANALYTICAL AND NUMERICAL SOLUTIONS IN STAGE II

4.1 Analytical solution

When the outer boundary of the sealing cylinder contacts the casing, the deformation comes into the second stage. In this case, both the inner and outer walls of the sealing cylinder are constrained via displacement. The contact pressure will be generated when the sealing cylinder and the casing are in contact. Discussing the relationship between the axial uniform load and the contact pressure is the key to the sealing effect for the sealing cylinder.

As mentioned earlier, the pressure value of $p_2$ is the total applied axial uniform load $p$ minus the maximum uniform load $p_{cr}$ in the first stage, which represents the axial uniform load that causes the contact pressure in the second stage. The boundary conditions are

$$\begin{align*}
    z &= 1, & \sigma_z &= -p_2, \\
    z &= 0, & w &= 0, \\
    r &= r_1, & u_r &= 0, \\
    r &= r_3, & u_r &= 0.
\end{align*}$$  

(18)
Figure 6 Numerical results on the radial displacement of the sealing cylinder under different axial loads.

Substituting the stress and displacement expressions into Eq. (18), one has

\[
\begin{cases}
6 (1 - \mu) A_1 + 4 (2 - \mu) A_2 = -p_2, \\
-\frac{1}{2G} \left( 2A_2 r_1 + A_3 \frac{1}{r_1} \right) = 0, \\
-\frac{1}{2G} \left( 2A_2 r_3 + A_3 \frac{1}{r_3} \right) = 0.
\end{cases}
\]

Then, one can further get \( A_1 = p_2/[6(\mu - 1)], A_2 = A_3 = 0 \). The corresponding expressions can be obtained as

\[
\zeta (r, z) = \frac{P^2}{6(\mu - 1)} z^3,
\]

\[
\begin{cases}
u_r = 0, \\
w = \frac{(1-2\mu)p_2}{2G(\mu-1)} z,
\end{cases}
\]

The scaling law tells us that the radial stress \( \sigma_r \) is proportional to the external pressure, which is in agreement with the curve in Fig. 8 based on Eq. (22). It indicates that the radial stress \( \sigma_r \) is independent of the parameter \( r \), which is a constant when the Poisson's ratio and external pressure are fixed. Evidently, its value is also equal to that on the outer boundary of the sealing cylinder, which is defined as the contact stress \( P_C = \sigma_r (r_3) \). It is also observed that the contact stress becomes larger as the axial uniform load increases.

According to the calculation formula of the total contact force, and inserting the values of corresponding parameters, the value
of the contact force between the sealing cylinder and the casing with respect to the external load can be obtained. For clarity, the symbol \( w_2 \) represents the axial displacement in the second stage, and \( w_{cr} \) is the absolute value of the maximum axial displacement at the end of the first stage, i.e. \( w_{cr} = \frac{z p_c}{2G[(1 - \mu) r_1^2 + (1 + \mu) r_2^2]} \). Therefore, the resultant force of the radial stress, i.e. the contact force at the cylinder–casing interface, is expressed as

\[
N = \int_S \sigma \, dS = 2\pi \sigma r_3 (1 - w_{cr} - w_2)
\]

\[
= \frac{2\pi \mu p_2 r_3}{\mu - 1} \left[ 1 - \frac{z p_c}{2G[(1 - \mu) r_1^2 + (1 + \mu) r_2^2]} - \frac{(1 - 2\mu) p_2}{2G(\mu - 1)} z \right].
\]

As an example, inserting the parameters given above into Eq. (23), one can get \( N = 48 \, 490.4931 \, p_2 \).

4.2 Numerical solution

The numerical simulation during Stage II can also be performed, and the cloud diagrams of the contact pressure between the sealing cylinder and the casing under different external uniform loads are displayed in Fig. 9a. Figure 9b shows the variation of the contact pressure with respect to the axial coordinate under four groups of uniformly distributed loads. The basic knowledge for us is that the maximum contact pressure increases with the increase of the axial uniform load. In addition, the maximum contact pressure appears at both ends of the sealing cylinder. On the contrary, the closer to the middle part, the lower the contact pressure. These results are consistent with those in previous studies [34–36].

The dependence relationship between the resultant contact force and the axial pressure is shown in Fig. 10, where the analytical solution and numerical result are compared. From the variation trend of the two sets of data, it can be seen that the solution obtained by the Love strain function is very close to the solutions obtained by numerical methods. This consistency indicates that the analytical solution proves effective to treat the sealing structure.

5. FURTHER RESULTS AND DISCUSSION

For the quantitative analysis of the contact stress at the interface, there are various results in the literature, and they are not consistent. The selection of these models should be quantitatively discussed, which is very beneficial for engineers. In what follows, we consider four results of the contact pressure, which are denoted by \( P_{C1} \), \( P_{C2} \), \( P_{C3} \), \( P_{C4} \).

First, Zhang et al. [37] gave the following contact stress expression based on the hyperelastic model of rubber:

\[
P_{C1} = k_1 p, \quad \text{Eq. (24)}
\]

where \( k_1 = \tanh(\lambda/2) \sinh[\lambda(z/H - 1)] + \cosh[\lambda(z/H - 1)], \)

\( H/h = 1 - 2r_3 (r_3 - r_1) / r_3^2, \)

\( K = E / [3(1 - 2\mu)], \)

\( \lambda = H \sqrt{8G[\ln r_3 - \ln r_1] / [K[(r_3^2 + r_1^2) / (\ln r_3 - \ln r_1) - (r_3^2 - r_1^2)]]} \)

and \( K \) is the bulk modulus. We select the value of \( \mu = 0.4995 \) considering the nearly incompressibility of rubber, and substitute the corresponding parameters mentioned above into Eq. (24) to get the value of the contact stress, where \( 0.825 \leq k_1 \leq 1.154 \).

Next, in the sealing engineering, one classical formula that is widely adopted to calculate the contact pressure [25, 26] is

\[
P_{C2} = \sigma_r = \frac{\mu}{1 - \mu} \left[ \frac{F_0}{A} - G \left( \frac{1}{\eta} - \eta^2 \right) \right]. \quad \text{Eq. (25)}
\]

where \( F_0 \) is the resultant load in the axial direction and \( A \) is the cross-sectional area of the sealing cylinder after deformation. Among them, \( p = F_0 / A \) and \( \eta = H / h \), where \( \eta \) represents the longitudinal deformation, and \( H \) and \( h \) are the heights of the sealing cylinder in the deformed state and the original state, respectively. In this theory, the volume before and after deformation is assumed incompressible; that is, the volume of the sealing cylinder is kept as a constant in the deformation process. Therefore, the parameter \( \eta \) can be expressed as

\[
\eta = \frac{H}{h} = \frac{r_2^2 - r_1^2}{r_3^2 - r_1^2}. \quad \text{Eq. (26)}
\]

Inserting Eq. (26) into (25), one can obtain the following relationships:

\[
\begin{align*}
P_{C2} = p - 2k_2 G, \\
k_2 = \frac{r_1^2 - r_2^2}{2(r_3^2 - r_1^2) - (r_3^2 - r_1^2)}.
\end{align*}
\]

However, in a recent work, Song et al. [24] gave the relationship between the axial stress and the contact pressure as follows:

\[
\begin{align*}
P_{C3} = p - 2k_3 G, \\
k_3 = \frac{a^{\eta - 1} - \eta}{4\eta} - \frac{1}{4(\rho^2 - 1)} \ln \frac{1 + \rho^2(\eta - 1)}{\eta}.
\end{align*}
\]

Figure 7 Relationships between the gap and the external pressure based on the analytical result and numerical simulation.
Figure 8 The relationship between the contact pressure and radial stress under different axial uniform loads.

Figure 9 Numerical result of contact stress between the cylinder and casing under different axial uniform loads: (a) the contact pressure $p = 3$ MPa; (b) the contact pressure takes different values.

Figure 10 Relationship between the resultant contact force and axial load based on the analytical result and numerical simulation.

where $\rho_0 = r_1/r_2$. By comparing the results with previous studies [25, 26], such as that in Eq. (25), they found that the error is even more than 200% in case of small deformation.

According to Eq. (22), the contact pressure in the current work is expressed as

$$P_{C4} = \sigma_r (r_3) = \frac{\mu}{1-\mu} p_2$$

$$= \frac{\mu}{1-\mu} (p - p_{cr}). \quad (29)$$

Equation (29) can be recast as

$$\begin{cases} P_{C4} = p - 2k_4 G, \\ k_4 = \frac{[(1-\mu)r_1^2 + (1+\mu)r_2^2](r_3-r_2)r_2}{\mu r_2^2 (r_3^2 - r_1^2)}. \end{cases} \quad (30)$$

It can be seen that the difference of above four results on the contact pressure is due to different values of $k_1$, $k_2$, $k_3$ and $k_4$. 

where $\rho_0 = r_1/r_2$. By comparing the results with previous studies [25, 26], such as that in Eq. (25), they found that the error is even more than 200% in case of small deformation.
and they are compared in Fig. 11, where the numerical result is also provided. If the above given parameters are inserted, the following three analytical results are calculated as $P_{C2} = p - 1.79$, $P_{C3} = p - 2.62$ and $P_{C4} = p - 2.5$, respectively. It is noticed that the three values are all constants when the external pressure is given, i.e. they are irrelevant of the longitudinal coordinate $z$, and this characteristic is different from that of Zhang et al. [37] and the numerical result. However, the analytical solution in the current work matches the result of Song et al. [24] very well. For example, when the uniform load is 7 MPa, the error between the results of $P_{C3}$ and $P_{C4}$ is only 2.7%. Although the numerical result on the contact pressure is not a constant, its value does not fluctuate away from $P_{C3}$ and $P_{C4}$ too much, i.e. its average value is close to these two values. Especially, the maximum error between $P_{C4}$ and the numerical result is 6% when the uniform pressure is 4 MPa. The error analysis shows that with the increase of the external pressure, the error between them decreases gradually. When the uniform load is 7 MPa, the relative error between $P_{C4}$ and the numerical result becomes 4.8%. As this fluctuation around the average value of $P_{C4}$ is small, the integration of the numerical result is close to the analytical solution on the resultant contact force. In addition, the results of Zhang et al. [37] are different from those of other cases, which may be due to the consideration of shear stress for modeling. It can be seen that the maximum error between $P_{C1}$ and $P_{C4}$ can be as high as 157%.

There is also some difference between the value of $P_{C2}$ and the other three results. For example, when $p = 7$ MPa, the relative error between $P_{C2}$ and $P_{C4}$ is 15.8%. This phenomenon may be due to the fact that, in calculation of $P_{C2}$, the incompressible volume of rubber is adopted. However, in calculation of $P_{C4}$, the volume of rubber is not a constant. Although there is a slight difference, the value of $P_{C4}$ is much closer to that of $P_{C2}$ compared with $P_{C1}$. Based on the comparison with the numerical result, it indicates that the current analytical solution on $P_{C4}$ is sufficiently accurate, and it can be put to use in predicting mechanical behaviors of the sealing structure. Another advantage is that the current solution on $P_{C4}$ is based on the linear elastic material with infinitesimal deformation, and it is convenient to derive and be used.

6. CONCLUSION

In this study, based upon the Love strain function, the relationships between the axial uniform loads and the gap, the uniform loads and the contact pressure under different axial uniform loads are obtained by using analytical and numerical methods. The following conclusions can be made.

First, through the FEM solution verification, the proposed Love strain function can be used to analyze the force of a three-cylinder sealing structure with a central pipe inside, a sealing cylinder in the middle under axial uniform loads and a casing.
outer. The derivation process of the analytical solution is simple, but it decreases the complexity of the computation. It can provide a certain and practical reference for the stress analysis of the sealing cylinder. Next, the relationship between the axial uniform loads and the gap is given, and the relationship between the axial uniform loads and the contact pressure at the interface is also given. The results show that the contact pressure at the upper and lower ends of the sealing cylinder is relatively large, and the influence of the contact pressure at both ends should be considered when designing the sealing cylinder. Third, the analytical solution and FEM simulation are used to analyze and discuss the two stages of force and deformation of the sealing cylinder. The results on the contact pressure based on five models are compared, which are beneficial for selection of proper models in design.

These findings are beneficial to obtain a deep understanding on the mechanism of the sealing process, and provide some inspirations on the new types of sealing tools for mechanical engineering, chemical engineering, petroleum engineering, etc.

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