CONVEX ORDER FOR CONVOLUTION POLYNOMIALS OF BOREL MEASURES

ANDRZEJ KOMISARSKI AND TERESA RAJBA

Abstract. We give necessary and sufficient conditions for Borel measures to satisfy the inequality introduced by Komisarski, Rajba (2018). This inequality is a generalization of the convex order inequality for binomial distributions, which was proved by Mrowiec, Rajba, Wąsowicz (2017), as a probabilistic version of the inequality for convex functions, that was conjectured as an old open problem by I. Raşa. We present also further generalizations using convex order inequalities between convolution polynomials of finite Borel measures. We generalize recent results obtained by B. Gavrea (2018) in the discrete case to general case. We give solutions to his open problems and also formulate new problems.

1. Introduction

Let $\mu$ and $\nu$ be two finite Borel measures (e.g. probability distributions) on $\mathbb{R}$ with finite first moments (i.e. $\int |x| \mu(dx) < \infty$ and the same for $\nu$). We say that $\mu$ is \textit{smaller than} $\nu$ in the convex order (denoted as $\mu \leq_{cx} \nu$) if

$$
\int_{\mathbb{R}} \varphi(x) \mu(dx) \leq \int_{\mathbb{R}} \varphi(x) \nu(dx)
$$

for all convex functions $\varphi : \mathbb{R} \to \mathbb{R}$.

Note that both integrals always exist (finite or infinite).

Let $P$ and $Q$ be two real polynomials of $m$ variables. They can be treated as convolution polynomials of finite Borel measures $\mu_1, \ldots, \mu_m$ (product of variables corresponds to convolution of measures). We are interested, when the relation $P(\mu_1, \ldots, \mu_m) \leq_{cx} Q(\mu_1, \ldots, \mu_m)$ holds.

Our investigation is motivated by the recent result of J. Mrowiec, T. Rajba and S. Wąsowicz [11] who proved the following convex ordering relation for convolutions of binomial distributions $B(n, x)$ and $B(n, y)$ ($n \in \mathbb{N}$, $x, y \in [0, 1]$):

$$
B(n, x) \ast B(n, y) \leq_{cx} \frac{1}{2}(B(n, x) \ast B(n, x) + B(n, y) \ast B(n, y)),
$$

which is a probabilistic version of the inequality involving Bernstein polynomials and convex functions, that was conjectured as an open problem by I. Raşa [12] (see also [1], [2], [8], [5] for further results on the I. Raşa problem).

In [7], we gave a generalization of the inequality (1.1). We introduced and studied the following convex ordering relation

$$
\mu \ast \nu \leq_{cx} \frac{1}{2}(\mu \ast \mu + \nu \ast \nu),
$$

where $\mu$ and $\nu$ are two probability distributions on $\mathbb{R}$. The inequality (1.2) can be regarded as the Raşa type inequality. In [7], we proved Theorem 2.3 providing a very useful sufficient condition for verification that $\mu$ and $\nu$ satisfy (1.2). We applied Theorem 2.3 for $\mu$ and $\nu$ from various families of probability distributions. In particular, we obtained a new proof for binomial distributions, which is significantly simpler and shorter than that given in [11]. By (1.2), we can also obtain inequalities related to some approximation operators associated with $\mu$ and $\nu$. (such as Bernstein-Schnabl operators, Mirakyan-Szász operators, Baskakov operators and others, cf. [7]).

2010 Mathematics Subject Classification. Primary 26D15; Secondary 60E15, 39B62.

Key words and phrases. Bernstein polynomials, stochastic order, stochastic convex order, convex functions, functional inequalities related to convexity, Muirhead inequality.
sequences (2.1) $$\mu \leq \mu$$ probabilistic methods. Instead, he used complex analysis. the following inequality given in [5]. discrete case. In our considerations, $$\mu$$ finite Borel measures on $$\mathbb{R}$$. In the particular case of discrete probability distributions, our assumptions on the sequences $$(a_k)$$ and $$(b_k)$$ are weaker than those given in [3].

In Section 3, we consider a generalization of (1.2) for more than two measures. As a generalization of results from [4], we present the Raşa type inequalities for convex orders for convolution polynomials of finite Borel measures on $$\mathbb{R}$$. In Section 4, we give solutions to B. Gavrea’s problems (presented in [5]) and list some new problems.

2. THE BASIC CASE OF TWO MEASURES

In the sequel we adapt some notation from theory of probability and stochastic orders (see [14]). Let $$\mu$$ be a finite Borel measure (e.g. a probability distribution) on $$\mathbb{R}$$. For $$x \in \mathbb{R}$$ the delta symbol $$\delta_x$$ denotes the one-point probability distribution satisfying $$\delta_x(\{x\}) = 1$$. Function $$F(x) = F_\mu(x) = \mu((\infty, x])$$ is the cumulative distribution function of $$\mu$$ (simply the distribution function). The complementary cumulative distribution function, or simply the tail distribution, is defined as $$\overline{F}(x) = \mu(\mathbb{R}) - F(x) = \mu((x, \infty))$$. If $$\mu$$ and $$\nu$$ are finite Borel measures such that $$\mu(\mathbb{R}) = \nu(\mathbb{R})$$ and $$F_\mu(x) \geq F_\nu(x)$$ for all $$x \in \mathbb{R}$$, then $$\mu$$ is said to be smaller than $$\nu$$ in the usual stochastic order (denoted by $$\mu \leq_{st} \nu$$). An important characterization of the usual stochastic order for probability distributions is given in the following theorem.

**Theorem 2.1** ([14], p. 5). Two probability distributions $$\mu$$ and $$\nu$$ satisfy $$\mu \leq_{st} \nu$$ if, and only if, there exist two random variables $$X$$ and $$Y$$ defined on the same probability space, such that the distribution of $$X$$ is $$\mu$$, the distribution of $$Y$$ is $$\nu$$ and $$P(X \leq Y) = 1$$. In [7], we gave a very useful sufficient condition, that can be used for the verification of the inequality (1.2).

**Theorem 2.2** ([7]). Let $$\mu$$ and $$\nu$$ be two probability distributions with finite first moments, such that $$\mu \leq_{st} \nu$$ or $$\nu \leq_{st} \mu$$. Then

$$\mu * \nu \leq_{st} \frac{1}{2}(\mu * \mu + \nu * \nu).$$

As an application of Theorem 2.2 we obtain that (2.1) holds for $$\mu$$ and $$\nu$$ from various families of probability distributions: binomial, Poisson, negative binomial, beta, gamma and Gaussian distributions.
The condition presented in Theorem 2.2 is sufficient but it is not necessary. In the following theorem we give a necessary and sufficient condition for finite Borel measures \( \mu \) and \( \nu \) to satisfy the inequality (2.1).

**Theorem 2.3.** Let \( \mu \) and \( \nu \) be two finite Borel measures on \( \mathbb{R} \) with finite first moments. Let \( F \) and \( G \) be the distribution functions corresponding to \( \mu \) and \( \nu \), respectively. Then the following conditions are equivalent:

1. \( \mu(\mathbb{R}) = \nu(\mathbb{R}) \) and \( (F - G) \ast (F - G) \geq 0 \),
2. \( \mu \ast \nu \leq_{\text{ex}} \frac{1}{2}(\mu \ast \mu + \nu \ast \nu) \).

**Proof.** First we show that (2) implies \( \mu(\mathbb{R}) = \nu(\mathbb{R}) \). For the convex function \( \varphi(x) = 1 \) \( (x \in \mathbb{R}) \) we have:

\[
2\mu(\mathbb{R})\nu(\mathbb{R}) = \int_{-\infty}^{\infty} 1 \ (2\mu \ast \nu)(dx) \leq \int_{-\infty}^{\infty} (\mu \ast \mu + \nu \ast \nu)(dx) = (\mu(\mathbb{R}))^2 + (\nu(\mathbb{R}))^2.
\]

In turn, taking the convex function \( \varphi(x) = -1 \) \( (x \in \mathbb{R}) \) we obtain:

\[
-2\mu(\mathbb{R})\nu(\mathbb{R}) = \int_{-\infty}^{\infty} (-1) \ (2\mu \ast \nu)(dx) \leq \int_{-\infty}^{\infty} (-1) \ (\mu \ast \mu + \nu \ast \nu)(dx) = -(\mu(\mathbb{R}))^2 - (\nu(\mathbb{R}))^2.
\]

Consequently, we have \( 2\mu(\mathbb{R})\nu(\mathbb{R}) = (\mu(\mathbb{R}))^2 + (\nu(\mathbb{R}))^2 \), which implies \( (\mu(\mathbb{R}) - \nu(\mathbb{R}))^2 = 0 \). It follows that \( \mu(\mathbb{R}) = \nu(\mathbb{R}) \). It remains to show that if \( \mu(\mathbb{R}) = \nu(\mathbb{R}) \), then (2) is equivalent to \( (F - G) \ast (F - G) \geq 0 \).

The relation (2) is equivalent to fulfilling the following inequality

\[
(\varphi(x))(\mu \ast \nu)(dx) = \int_{-\infty}^{\infty} \varphi(x)(\mu \ast \nu)(dx) \geq \int_{-\infty}^{\infty} \varphi(x)(2\mu \ast \nu)(dx)
\]

for all convex functions \( \varphi : \mathbb{R} \to \mathbb{R} \). Note that every convex function \( \varphi \) is a pointwise limit of an increasing sequence \( (\varphi_n) \) of convex, piecewise linear functions. Therefore (due to monotone convergence theorem for integrals) (2.2) is valid for convex functions if, and only if, it is valid for convex, piecewise linear functions. On the other hand, every convex, piecewise linear function is a linear combination with non-negative coefficients of a linear (affine) function and functions of the form (2.3) (see below). If \( \varphi(x) = ax + b \) is a linear function, then we have equality in (2.2) (both sides of (2.2) are equal to \( 2b(\mu(\mathbb{R}))^2 + 2a\mu(\mathbb{R}) (\int x\mu(dx) + \int x\nu(dx)) \)). It follows that (2.2) is valid for convex functions if, and only if, it is valid for functions of the form

\[
(\varphi(x) = (x - A)_+ = \max(x - A, 0),
\]

where \( A \in \mathbb{R} \).

In the following computation symbols \( \overline{F} \) and \( \overline{G} \) stand for the tail distributions of \( \mu \) and \( \nu \), respectively. We use the Fubini Theorem several times. We assume that all the definite integrals are integrals on open intervals. Let \( \lambda \) denote the Lebesgue measure on the real line \( \mathbb{R} \). Besides the positive measures \( \mu \) and \( \nu \), we also study the signed measure \( \mu - \nu \). Integrability of all considered functions and applicability of the Fubini Theorem follows from our assumption that \( \mu \) and \( \nu \) have finite first moments (e.g., \( \mu(\mathbb{R}) = \nu(\mathbb{R}) \)) implies \( F - G = \overline{G} - \overline{F} \), hence \( \int_{-\infty}^{\infty} |F(t) - G(t)| \ \lambda(dt) \leq \int_{-\infty}^{0} (F(t) + G(t)) \ \lambda(dt) + \int_{0}^{\infty} (\overline{F}(t) + \overline{G}(t)) \ \lambda(dt) = \int_{-\infty}^{\infty} |x| \ \mu(dx) + \int_{-\infty}^{\infty} |x| \ \nu(dx) < \infty \).
Let $A \in \mathbb{R}$. Then we have
\[
\int_{-\infty}^{\infty} (x - A) \nu(x + \nu - 2\mu \nu)(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1 \lambda(dx)(\mu - \nu)^2(dx) = [A < z < x] =
\]
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} 1 (\mu - \nu)^2(dx)\lambda(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1 (\mu - \nu)(dv)(\mu - \nu)(dv)\lambda(dx) =
\]
\[
[A < z < u + v] = \int_{\mathbb{R}} \int_{\mathbb{R}} 1 (\mu - \nu)(dv)(\mu - \nu)(dv) =
\]
\[
\text{we substitute } t = z - u = \int_{\mathbb{R}} \int_{\mathbb{R}} 1 (\mu - \nu)(dv)(\mu - \nu)(dv) = [A < u + t < u + v] =
\]
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} 1 (\mu - \nu)(dv)(\mu - \nu)(dv)dt = \int_{\mathbb{R}} (\mathcal{F}(t) - \mathcal{G}(t))(\mathcal{F}(A - t) - \mathcal{G}(A - t))\lambda(dt) =
\]
\[
((\mathcal{F} - \mathcal{G})(\mathcal{F} - \mathcal{G}))(A) = ((F - G)(F - G))(A).
\]

The above identity completes the proof of the theorem.

**Remark 2.4.** Theorem 2.3 is a generalization of Theorem 2.1. Indeed, if $\mu \leq_{st} \nu$ or $\nu \leq_{st} \mu$, then obviously the condition (1) in Theorem 2.1 is satisfied.

In the following proposition we give more precise estimation of the difference of integrals given in (2.2).

**Proposition 2.5.** Let $\mu$ and $\nu$ be two finite Borel measures on $\mathbb{R}$ with finite first and second moments. Let $\mathcal{P} = \int_{\mathbb{R}} x \mu(dx)$ and $\mathcal{V} = \int_{\mathbb{R}} x \nu(dx)$. Assume, that $\mu \ast \nu \leq_{cx} \frac{1}{2}(\mu \ast \mu \ast \nu \ast \nu)$ and $\varphi$ is a twice differentiable convex function. If both sides of (2.2) are finite, then
\[
\inf_x \varphi''(x) \cdot (\mathcal{P} - \mathcal{V})^2 \leq \int_{\mathbb{R}} \varphi(x)(\mu \ast \mu \ast \nu \ast \nu - 2\mu \ast \nu)(dx) \leq \sup_x \varphi''(x) \cdot (\mathcal{P} - \mathcal{V})^2
\]
(we set $\infty \cdot 0 = \infty$ and $-\infty \cdot 0 = -\infty$).

**Proof.** The proofs of both inequalities are similar, therefore we will prove only the left one. If $\inf_x \varphi''(x) = -\infty$, then the inequality is obvious (the left side is $-\infty$). Assume that $m := \inf_x \varphi''(x)$ is finite. Then $x \mapsto \varphi(x) - \frac{m}{2} \cdot x^2$ is a convex function. We have
\[
\int_{\mathbb{R}} \left( \varphi(x) - \frac{m}{2} \cdot x^2 \right)(\mu \ast \mu \ast \nu \ast \nu - 2\mu \ast \nu)(dx) \geq 0,
\]
which implies
\[
\int_{\mathbb{R}} \varphi(x)(\mu \ast \mu \ast \nu \ast \nu - 2\mu \ast \nu)(dx) \geq \frac{m}{2} \cdot \int_{\mathbb{R}} x^2(\mu \ast \mu \ast \nu \ast \nu - 2\mu \ast \nu)(dx) = m \cdot (\mathcal{P} - \mathcal{V})^2.
\]

We need to justify the last equality. If $\mu = \nu = 0$, then there is nothing to do. If $a := \mu(\mathbb{R}) = \nu(\mathbb{R}) > 0$, then $\frac{\mu}{a}$ and $\frac{\nu}{a}$ are probability distributions. We consider independent random variables $X_1, X_2, Y_1, Y_2$ such that the distribution of $X_1$ and $X_2$ is $\frac{\mu}{a}$ and the distribution of $Y_1$ and $Y_2$ is $\frac{\nu}{a}$. Then we have
\[
\int_{\mathbb{R}} x^2(\mu \ast \mu \ast \nu \ast \nu - 2\mu \ast \nu)(dx) = a^2 \cdot \int_{\mathbb{R}} x^2 \left( \frac{\mu}{a} \ast \frac{\mu}{a} \ast \frac{\nu}{a} \ast \frac{\nu}{a} - 2\frac{\mu}{a} \ast \frac{\nu}{a} \right)(dx) =
\]
\[
a^2 \left( \mathbb{E}(X_1 + X_2)^2 + \mathbb{E}(Y_1 + Y_2)^2 - 2\mathbb{E}(X_1 + Y_1)^2 \right) =
\]
\[
a^2 \left( \mathbb{E}X_1^2 + \mathbb{E}X_2^2 + 2\mathbb{E}X_1\mathbb{E}X_2 + \mathbb{E}Y_1^2 + \mathbb{E}Y_2^2 + 2\mathbb{E}Y_1\mathbb{E}Y_2 - 2\mathbb{E}X_1^2 - 2\mathbb{E}Y_1^2 - 4\mathbb{E}X_1\mathbb{E}Y_1 \right) =
\]
\[
2a^2(\mathbb{E}X_1 - \mathbb{E}Y_1)^2 = 2(\mathcal{P} - \mathcal{V})^2.
\]

In the proof of the right inequality we use the convexity of the function $x \mapsto \frac{m}{2} \cdot x^2 - \varphi(x)$, where $M = \sup_x \varphi''(x)$. \qed
Consider now the discrete probability distribution \( \mu \) concentrated on the set of non-negative integers \( \{0, 1, 2, \ldots \} \), with \( a_k = \mu(\{k\}) \) \( (k = 0, 1, 2, \ldots) \). Then the probability generating function corresponding to \( \mu \) is given by the formula

\[
f(z) = \sum_{k=0}^{\infty} a_k z^k.
\]

**Theorem 2.6.** Let \( \mu \) and \( \nu \) be discrete probability distributions concentrated on the set of non-negative integers \( \{0, 1, 2, \ldots \} \), with \( a_k = \mu(\{k\}) \) and \( b_k = \nu(\{k\}) \) \( (k = 0, 1, 2, \ldots) \). Assume that \( \mu \) and \( \nu \) have finite first moments. Let \( F, f \) and \( G, g \) be the distribution functions and the generating functions corresponding to \( \mu \) and \( \nu \), respectively. Then the following conditions are equivalent:

1. For all convex functions \( \varphi : \mathbb{R} \to \mathbb{R} \)

\[
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (a_i a_j + b_i b_j) \varphi(i + j) \geq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} 2a_i b_j \varphi(i + j).
\]

2. \( (F - G) \ast (F - G) \geq 0 \).

3. \( \frac{d^k}{dz^k} \left( \frac{f(z) - g(z)}{z - 1} \right)^2 \bigg|_{z=0} \geq 0 \) for \( k = 0, 1, \ldots \).

**Proof.** Since (2.4) is equivalent to the relation \( \mu \ast \nu \leq_{cx} \frac{1}{2}(\mu \ast \mu + \nu \ast \nu) \), the equivalence of (1) and (2) clearly follows from Theorem 2.3. It suffices to prove the equivalence of (2) and (3).

In the following calculations, we use the existence and finiteness of the first moments of the probability distributions \( \mu \) and \( \nu \), which implies that all the following series are absolutely convergent for \( z \in [-1, 1] \) and we can change the summation order. By the equality \( \sum_{k=0}^{\infty} a_k = 1 \), we have

\[
\frac{f(z) - 1}{z - 1} = \sum_{k=0}^{\infty} a_k \frac{z^k - 1}{z - 1} = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} a_k z^i = \sum_{i=0}^{\infty} \sum_{k=i+1}^{\infty} a_k z^i = \sum_{i=0}^{\infty} \sum_{k=i+1}^{\infty} z^i = \sum_{i=0}^{\infty} F(i) z^i
\]

for every \( z \in [-1, 1] \), where \( F \) is the tail distribution of \( \mu \). Note that \( \sum_{i=0}^{\infty} F(i) = \int_{\mathbb{R}} x \mu(dx) < \infty \). Similarly, we obtain

\[
\frac{g(z) - 1}{z - 1} = \sum_{i=0}^{\infty} G(i) z^i,
\]

where \( G \) is the tail distribution of \( \nu \). Therefore for every \( z \in [-1, 1] \) we have

\[
\left( \frac{f(z) - g(z)}{z - 1} \right)^2 = \left( \sum_{i=0}^{\infty} (F - G)(i) z^i \right)^2 = \left( \sum_{i=0}^{\infty} (G - F)(i) z^i \right)^2 = \left( \sum_{i=0}^{\infty} ((G - F) \hat{\ast} (G - F))(i) z^i \right)^2.
\]

Here \( \hat{\ast} \) denotes the discrete convolution (the Euler product) of sequences. Condition (3) is equivalent to the non-negativity of all the coefficients in the above series. Note that if \( i \in \mathbb{Z} \) and \( x \in [i, i+1] \), then \( ((G - F) \ast (G - F))(x) = (x - i)((G - F) \hat{\ast} (G - F))(i) + (i + 1 - x)((G - F) \hat{\ast} (G - F))(i-1) \) (here we put \( ((G - F) \hat{\ast} (G - F))(i) = 0 \) for \( i < 0 \)). Therefore the non-negativity of all the terms \( ((G - F) \hat{\ast} (G - F))(i) \) is equivalent to (2). The theorem is proved.

**Remark 2.7.** B. Gavrea [5] also gave the condition (3) as necessary and sufficient to satisfy the condition (1), but assuming that the radii of convergence of functions \( f \) and \( g \) are greater then 1 (in particular, there exist all the moments of \( \mu \) and \( \nu \)). The assumption in Theorem 2.6 is weaker, we assume only the existence of the first moments of \( \mu \) and \( \nu \). Furthermore, in Theorem 2.3 we give necessary and sufficient condition for all distributions, not just for discrete.
3. The case of \( m \) measures

In this section we consider \( m \) finite Borel measures on \( \mathbb{R} \), \( \mu_1, \ldots, \mu_m \), with finite first moments. We denote \( F_i(x) = \mu_i((\infty, x]) \) and \( F'_i(x) = \mu_i((x, \infty)) \) for \( i = 1, \ldots, m \) and \( x \in \mathbb{R} \).

Let \( P \) and \( Q \) be two polynomials of \( m \) variables. They can be treated as convolution polynomials of the measures \( \mu_1, \ldots, \mu_m \) (product of variables corresponds to convolution of measures). We are interested, when the relation \( P(\mu_1, \ldots, \mu_m) \leq_{\text{cx}} Q(\mu_1, \ldots, \mu_m) \) holds. Since \( \leq_{\text{cx}} \) is defined for non-negative measures, we generally assume that the polynomials have non-negative coefficients, although in the proofs we consider also differences of such polynomials.

**Proposition 3.1.** Let \( \mu_1, \ldots, \mu_m \) be finite Borel measures on \( \mathbb{R} \) with finite first moments. For \( i = 1, \ldots, m \), we denote \( a_i = \mu_i(\mathbb{R}) \) and \( b_i = \int x \, d\mu_i(dx) \). Let \( P, Q \) be polynomials of \( m \) variables with non-negative coefficients, such that \( P(\mu_1, \ldots, \mu_m) \leq_{\text{cx}} Q(\mu_1, \ldots, \mu_m) \). Then \( P(a_1, \ldots, a_m) = Q(a_1, \ldots, a_m) \) and \( \frac{\partial P}{\partial(x_1, \ldots, x_m)}(a_1, \ldots, a_m) = \frac{\partial Q}{\partial(x_1, \ldots, x_m)}(a_1, \ldots, a_m) \) (the directional derivatives along the vector \( (b_1, \ldots, b_m) \) at the point \( (a_1, \ldots, a_m) \)).

**Proof.** Let \( P = \sum_j p_j \prod_{i=1}^m x_i^{k_{ji}} \) and \( Q = \sum_j q_j \prod_{i=1}^m x_i^{l_{ji}} \). Considering the convex functions \( \varphi(x) = 1 \) and \( \varphi(x) = -1 \) we obtain \( \int 1 \, dP(\mu_1, \ldots, \mu_m) = \int 1 \, dQ(\mu_1, \ldots, \mu_m) \), which implies \( P(a_1, \ldots, a_m) = Q(a_1, \ldots, a_m) \). Taking the convex functions \( \varphi(x) = x \) and \( \varphi(x) = -x \) we get \( \int x \, dP(\mu_1, \ldots, \mu_m) = \int x \, dQ(\mu_1, \ldots, \mu_m) \), which is equivalent to \( \sum_j p_j \sum_{i=1}^m k_{ji} a_i \Pi_{i=1}^m a_i^{k_{ji}} = \sum_j q_j \sum_{i=1}^m l_{ji} b_i \Pi_{i=1}^m a_i^{l_{ji}} \). Consequently we obtain \( \frac{\partial P}{\partial(x_1, \ldots, x_m)}(a_1, \ldots, a_m) = \frac{\partial Q}{\partial(x_1, \ldots, x_m)}(a_1, \ldots, a_m) \). \( \square \)

**Theorem 3.2.** Let \( \mu_1, \ldots, \mu_m \) be finite Borel measures on \( \mathbb{R} \) with finite first moments. We assume that \( \mu_1(\mathbb{R}) = \cdots = \mu_m(\mathbb{R}) \) and \( (F_i - F_j) \ast (F_i - F_j) \geq 0 \) for all \( i, j = 1, \ldots, m \). Let \( P \) and \( Q \) be polynomials of variables \( x_1, \ldots, x_m \) with non-negative coefficients, such that \( (Q - P)(x_1, \ldots, x_m) = \sum_{i \neq j} R_{i,j}(x_1, \ldots, x_m)(x_i - x_j)^2 \), where \( R_{i,j} \) are polynomials of variables \( x_1, \ldots, x_m \) with non-negative coefficients. Then \( P(\mu_1, \ldots, \mu_m) \leq_{\text{cx}} Q(\mu_1, \ldots, \mu_m) \).

**Proof.** Since \( R_{i,j} \) are polynomials with non-negative coefficients, it follows that \( R_{i,j}(\mu_1, \ldots, \mu_m) \) are finite non-negative measures. Let \( \varphi \) be a function which is affine or of form \( \varphi(x) = (x - A)_+ \), where \( A \in \mathbb{R} \). Then \( \varphi \) is integrable with respect to every polynomial of measures \( \mu_1, \ldots, \mu_m \) and we have

\[
\int \varphi(x) \, (Q - P)(\mu_1, \ldots, \mu_m)(dx) = \sum_{i \neq j} \int \varphi(x) \, (R_{i,j}(\mu_1, \ldots, \mu_m) \ast (\mu_i - \mu_j)^2)(dx) = \sum_{i \neq j} \int \varphi(u + v) \, (\mu_i - \mu_j)^2(du)R_{i,j}(\mu_1, \ldots, \mu_m)(dv) \geq 0,
\]

because, by Theorem 2, the internal integral in each component of the above sum is non-negative.

Since other convex functions are limits of convex combinations with non-negative coefficients of functions considered above, we obtain \( P(\mu_1, \ldots, \mu_m) \leq_{\text{cx}} Q(\mu_1, \ldots, \mu_m) \). \( \square \)

**Remark 3.3.** Let \( \mu_1, \ldots, \mu_k \) be finite Borel measures on \( \mathbb{R} \) with finite first moments. If \( \mu_1, \ldots, \mu_k \) are pairwise comparable in the usual stochastic order (for each \( 1 \leq i, j \leq k \) we have \( \mu_i \leq_{\text{st}} \mu_j \) or \( \mu_i \geq_{\text{st}} \mu_j \)), then the inequalities \( (F_i - F_j) \ast (F_i - F_j) \geq 0 \) are satisfied for all \( i, j = 1, \ldots, m \).

Before we state the next theorem, we need to present two definitions.

In the set of all the \( m \)-tuples \( p = (p_1, \ldots, p_m) \) of non-negative integers we consider the following quasiorder.

**Definition 3.4.** We say that \( q \) majorizes \( p \) (denoted by \( p < q \) or \( q \succ p \)) if

1. \( \sum_{i=1}^m \hat{p}_i = \sum_{i=1}^m \hat{q}_i \),

where \( \hat{p}_i \) and \( \hat{q}_i \) are the numbers of \( p_i \) and \( q_i \), respectively.
Definition 3.6. Let \( m \in \mathbb{N} \) and let \( \Pi \) be the set of all permutations of the set \( \{1, \ldots, m\} \). For every \( m \)-tuple \( p = (p_1, \ldots, p_m) \) of non-negative integers, we define the following polynomial:

\[
W^p(x_1, \ldots, x_m) := \frac{1}{m!} \sum_{\pi \in \Pi} \prod_{i=1}^{m} x_{\pi(i)}^{p_i}.
\]

Clearly \( W^p \) is a symmetric polynomial with non-negative coefficients. If \( q \) is a permutation of \( p \), then \( W^q = W^p \). In particular \( W^\Pi = W^p \).

Theorem 3.7. Let \( m \in \mathbb{N} \) and let \( \mu_1, \ldots, \mu_m \) be finite Borel measures on \( \mathbb{R} \) satisfying \( \mu_1(\mathbb{R}) = \cdots = \mu_m(\mathbb{R}) \) and \( f(x) \mu_1(dx) \leq \infty \) for \( f \in \mathbb{R} \). Let \( F_1(x) = \mu_1((\infty, x]) \) for \( x \in \mathbb{R} \). Assume that \( (F_1, F_2) \) satisfies the condition (S). Then \( W^\Pi(\mu_1, \ldots, \mu_m) \leq \sup_{x} W^q(\mu_1, \ldots, \mu_m) \).

Proof. In view of Lemma 3.5 and transitivity of \( \leq_{cx} \), it is enough to consider the case when the pair \( p \prec q \) satisfies condition (S). Let \( l_1 < l_2 \) be the indices given in condition (S). For every \( \pi \in \Pi \) we define \( \pi' \in \Pi \) by \( \pi'(l_1) = \pi(l_2), \pi'(l_2) = \pi(l_1) \) and \( \pi'(l) = \pi(l) \) for \( l \notin \{l_1, l_2\} \). We have

\[
(W^q - W^p)(x_1, \ldots, x_m) = \frac{1}{m!} \sum_{\pi \in \Pi} \prod_{l=1}^{m} x_{\pi(l)}^{p_l} - \prod_{l=1}^{m} x_{\pi(l)}^{q_l} = \frac{1}{m!} \sum_{1 \leq u < v \leq m} \sum_{\pi \in \Pi} \prod_{l=1}^{m} x_{\pi(l)}^{q_l} - \prod_{l=1}^{m} x_{\pi(l)}^{q_l} = \frac{1}{m!} \sum_{1 \leq u < v \leq m} \prod_{l \neq l_1, l_2} x_{\pi(l)}^{q_l} \left( x_{u}^{\tilde{p}_1 + 1} x_{v}^{\tilde{q}_2} + x_{v}^{\tilde{p}_1 + 1} x_{u}^{\tilde{q}_2} - x_{u}^{\tilde{p}_1} x_{v}^{\tilde{q}_2} + x_{u}^{\tilde{p}_1} x_{v}^{\tilde{q}_2} \right)
\]

Note that

\[
x_{u}^{\tilde{p}_1 + 1} x_{v}^{\tilde{q}_2} + x_{v}^{\tilde{p}_1 + 1} x_{u}^{\tilde{q}_2} - x_{u}^{\tilde{p}_1} x_{v}^{\tilde{q}_2} + x_{u}^{\tilde{p}_1} x_{v}^{\tilde{q}_2} = \left( x_{u} - x_{v} \right) \left( x_{u}^{\tilde{p}_1} x_{v}^{\tilde{q}_2} - x_{u}^{\tilde{p}_1} x_{v}^{\tilde{q}_2} \right) = \left( x_{u} - x_{v} \right)^2 \sum_{j=0}^{\tilde{p}_1 - 1} \sum_{l \neq l_1, l_2} x_{u}^{\tilde{p}_1 + j} x_{v}^{\tilde{q}_2 - j}.
\]

It follows that

\[
(W^q - W^p)(x_1, \ldots, x_m) = \sum_{1 \leq u < v \leq m} \left( x_{u} - x_{v} \right)^2 \sum_{\pi \in \Pi} \prod_{l=1}^{m} x_{\pi(l)}^{q_l} = \sum_{\pi \in \Pi} \prod_{l=1}^{m} x_{\pi(l)}^{q_l} - \sum_{\pi \in \Pi} \prod_{l=1}^{m} x_{\pi(l)}^{q_l} = \sum_{\pi \in \Pi} \prod_{l=1}^{m} x_{\pi(l)}^{q_l}.
\]

By Theorem 3.2 we obtain \( W^p(\mu_1, \ldots, \mu_m) \leq \sup_{x} W^q(\mu_1, \ldots, \mu_m) \).
Remark 3.8. Theorem 3.7 is an analogue of Muirhead’s inequality (see [6], Theorem 45 or [10], Section 3G) with positive numbers replaced by Borel measures on \( \mathbb{R} \), multiplication replaced by convolution, and \( \leq \) replaced by \( \leq_{\text{cx}} \). Moreover, if \( x_1, \ldots, x_m > 0 \), then applying Theorem 3.7 with \( \mu_l = \delta_{l,n} \) (for \( l = 1, \ldots, k \)) and the convex function \( \varphi(x) = e^x \), we obtain the classical Muirhead inequality with integer exponents: If \( p < q \) and \( x_1, \ldots, x_m > 0 \), then \( W^p(x_1, \ldots, x_m) \leq W^q(x_1, \ldots, x_m) \).

If we apply Theorem 3.7 with \((p) = (1, \ldots, 1)\) and \((q) = (m, 0, \ldots, 0)\), we get the following corollary, which generalizes Raşa type inequalities proved in [11], [7] and [3].

Corollary 3.9. If \( \mu_1, \ldots, \mu_m \) are finite Borel measures on \( \mathbb{R} \) satisfying assumptions of Theorem 3.7, then

\[
\mu_1 \ast \cdots \ast \mu_m \leq_{\text{cx}} \left[ (\mu_1)^{*m} + \cdots + (\mu_m)^{*m} \right].
\]

In particular

\[
\sum_{i_1, \ldots, i_m=0}^{n} \left( b_{n,i_1}(x_1) \cdots b_{n,i_m}(x_m) + \cdots + b_{n,i_1}(x_m) \cdots b_{n,i_m}(x_m) - mb_{n,i_1}(x_1) \cdots b_{n,i_m}(x_m) \right) \varphi \left( \frac{i_1+\cdots+i_m}{m} \right) \geq 0,
\]

in the case of \( \mu_i = \delta(n,x_i) \) \((x_i \in [0,1])\) for \( i = 1, \ldots, m \).

One might expect that every polynomial inequality valid for non-negative real numbers has its counterpart for finite Borel measures and convex orders. The following example shows that it is very far from true.

Example 3.10. Let \( P(x,y) = \frac{1}{4} x^3 y + \frac{1}{4} x y^3 \) and \( Q(x,y) = \frac{1}{3} x^4 + \frac{2}{3} x^2 y^2 + \frac{1}{3} y^4 \). The polynomials \( P \) and \( Q \) are symmetric and homogeneous polynomials of degree 4. We have \( Q(x,y) - P(x,y) = \frac{1}{3} (x-y)^4 \geq 0 \) for every \( x, y \in \mathbb{R} \). Both \( P \) and \( Q \) have non-negative coefficients and \( P(1,1) = Q(1,1) = 1 \). It follows that \( P(\mu, \nu) \) and \( Q(\mu, \nu) \) are probability distributions whenever \( \mu \) and \( \nu \) are probability distributions. If the expected values (means) \( \mathbb{E}_\mu \) and \( \mathbb{E}_\nu \) are finite, then \( \mathbb{E} P(\mu, \nu) = 2(\mathbb{E}_\mu + \mathbb{E}_\nu) = \mathbb{E} Q(\mu, \nu) \). Despite all this regularity the inequality \( P(\mu, \nu) \leq_{\text{cx}} Q(\mu, \nu) \) is not valid for \( \mu = \delta_0 \) and \( \nu = \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1 \) (note that \( F - G \geq 0 \), hence \( (F - G)^* (F - G) \geq 0 \), cf. assumptions of Theorem 1 and Theorem 3.7). Indeed, \( P(\mu, \nu) = \frac{5}{16} \delta_0 + \frac{7}{16} \delta_1 + \frac{3}{16} \delta_2 + \frac{1}{16} \delta_3 \) and \( Q(\mu, \nu) = \frac{41}{128} \delta_0 + \frac{32}{128} \delta_1 + \frac{30}{128} \delta_2 + \frac{4}{128} \delta_3 + \frac{128}{128} \delta_4 \) and for the convex function \( \varphi(x) = \max(0,x-2) \) we have \( \int \varphi(x) P(\mu, \nu)(dx) = \frac{41}{16} > \frac{6}{128} = \int \varphi(x) Q(\mu, \nu)(dx) \), hence \( P(\mu, \nu) \not\leq_{\text{cx}} Q(\mu, \nu) \).

4. Open problems

For \( n \in \mathbb{N} \) the classical Bernstein operators \( B_n : C([0,1]) \to C([0,1]) \), defined by

\[
(B_n f)(x) = \sum_{i=0}^{n} b_{n,i}(x) f \left( \frac{i}{n} \right) \quad \text{for } x \in [0,1],
\]

with the Bernstein basic polynomials

\[
b_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i} \quad \text{for } i = 0, 1, \ldots, n, \ x \in [0,1],
\]

are the most prominent positive linear approximation operators (see [3]).

The inequality (1.1) is the probabilistic version of the following inequality involving Bernstein polynomials and convex functions, that was conjectured as an open problem by I. Raşa in [12]:

\[
(4.1) \quad \sum_{i,j=0}^{n} \left( b_{n,i}(x)b_{n,j}(x) + b_{n,i}(y)b_{n,j}(y) - 2b_{n,i}(x)b_{n,j}(y) \right) f \left( \frac{i+j}{2n} \right) \geq 0
\]

for each convex function \( f \in C([0,1]) \) and for all \( x, y \in [0,1] \).
Raşa [13] remarked, that (4.1) is equivalent to
\[(4.2) \quad (B_{2n,f})(x) + (B_{2n,f})(y) \geq 2 \sum_{i=0}^{n} \sum_{j=0}^{n} b_{n,i}(x)b_{n,j}(y) f \left( \frac{i+j}{2n} \right). \]

B. Gavrea [4] presented the following generalization of the problem of I. Raşa [12].

**Problem 1.** Let \( D = [0,1] \times [0,1] \), \( g \in C(D) \) and \( n \in \mathbb{N} \). The Bernstein operator is then defined by
\[(4.3) \quad (B_{n,g})(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n} b_{n,i}(x)b_{n,j}(y) g \left( \frac{i+j}{2n} \right) \quad \text{for} \quad (x,y) \in D. \]

Give a characterization of the class of convex functions \( g \) defined on \( D \), satisfying
\[(4.4) \quad (B_{n,g})(x,x) + (B_{n,g})(y,y) - 2(B_{n,g})(x,y) \geq 0 \]
for all \((x,y) \in D\).

Remark 4.1 [5]. We note that, if \( \varphi \in C([0,1]) \) is a convex function, and
\[ g(x,y) = \varphi \left( \frac{x+y}{2} \right) \quad \text{for} \quad (x,y) \in D, \]
then (4.1) coincides with the Raşa inequality (4.1).

Remark 4.2. Note that if (4.3) is satisfied for all \((x,y) \in D\), then also
\[(4.5) \quad (B_{n,g})(x,x) + (B_{n,g})(y,y) - 2(B_{n,g})(x,y) - (B_{n,g})(x,y) \geq 0 \]
for all \((x,y) \in D\). Adding inequalities (4.3) and (4.4), we obtain
\[(4.6) \quad (B_{n,g})(x,x) + (B_{n,g})(y,y) - (B_{n,g})(x,y) - (B_{n,g})(x,y) \geq 0. \]

Taking into account (4.5), we consider a modification of Problem 1.

**Problem 1’.** Let \( D = [0,1] \times [0,1] \), \( g \in C(D) \) and \( n \in \mathbb{N} \). Give a characterization of the class of convex functions \( g \) defined on \( D \), satisfying
\[(4.7) \quad (B_{n,g})(x,x) + (B_{n,g})(y,y) - (B_{n,g})(x,y) - (B_{n,g})(x,y) \geq 0 \]
for all \((x,y) \in D\).

Remark 4.3. The inequality (4.7) has the probabilistic interpretation. It is equivalent to the following inequality
\[(4.8) \quad E g \left( \frac{X}{n}, \frac{Y}{n} \right) + E g \left( \frac{Y}{n}, \frac{X}{n} \right) \leq E g \left( \frac{X_1}{n}, \frac{X_2}{n} \right) + E g \left( \frac{Y_1}{n}, \frac{Y_2}{n} \right), \]
where \((X,Y), (X_1,X_2), (Y_1,Y_2)\) are pairs of independent random variables such that \( X, X_1, X_2 \sim B(n,x) \) and \( Y, Y_1, Y_2 \sim B(n,y) \).

We use the following notation: \( X \sim \mu \) means that \( \mu \) is the probability distribution of a random variable \( X \).

The inequality (4.7) is not satisfied for all convex functions \( g \in C(D) \). Let us take \( g(x,y) = |x - y| \).
Then \( g \) is convex, \( g(0,0) = g(1,1) = 0 \) and \( g(0,1) = g(1,0) = 1 \). Let \( X, X_1, X_2 \sim B(n,0) = \delta_0 \), \( Y, Y_1, Y_2 \sim B(n,1) = \delta_1 \) be independent random variables. We obtain
\[ E g \left( \frac{X}{n}, \frac{Y}{n} \right) + E g \left( \frac{Y}{n}, \frac{X}{n} \right) = 1 + 1 > 0 + 0 = E g \left( \frac{X_1}{n}, \frac{X_2}{n} \right) + E g \left( \frac{Y_1}{n}, \frac{Y_2}{n} \right), \]
consequently the inequality (4.7) is not fulfilled.
Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be a convex function. We consider the Jensen gap corresponding to \( g \) that is given by
\[
\mathcal{J}(g; (x_1, x_2), (y_1, y_2)) = \frac{g(x_1, x_2) + g(y_1, y_2)}{2} - g \left( \frac{x_1 + x_2 + y_1 + y_2}{2} \right)
\]
for all \( x_1, x_2, y_1, y_2 \in \mathbb{R} \). Since \( g \) is convex,
\[
\mathcal{J}(g; (x_1, x_2), (y_1, y_2)) \geq 0 \quad \text{and} \quad \mathcal{J}(g; (x_1, y_2), (y_1, x_2)) \geq 0
\]
for all \( x_1, x_2, y_1, y_2 \in \mathbb{R} \).

We will consider convex functions \( g \) satisfying the inequality
\[
\mathcal{J}(g; (x_1, x_2), (y_1, y_2)) \geq \mathcal{J}(g; (x_1, y_2), (y_1, x_2))
\]
for all \( x_1, x_2, y_1, y_2 \in \mathbb{R} \) such that
\[
(y_1 - x_1)(y_2 - x_2) > 0.
\]
By (4.10), we can write (4.9) in the form
\[
\frac{g(x_1, x_2) + g(y_1, y_2)}{2} - g \left( \frac{x_1 + x_2 + y_1 + y_2}{2} \right) \geq \frac{g(x_1, y_2) + g(y_1, x_2)}{2} - g \left( \frac{x_1 + y_2 + y_1 + x_2}{2} \right),
\]
or equivalently
\[
g(x_1, x_2) + g(y_1, y_2) \geq g(x_1, y_2) + g(y_1, x_2).
\]

The following theorem says, that if the convex function \( g \) satisfies the inequality (4.11), which is equivalent to the inequality (4.9), and the random variables \( X \) and \( Y \) (which are not necessary binomially distributed) are chosen to satisfy some sufficient condition, then the inequality (4.7) is satisfied (up to natural number \( n \)).

**Theorem 4.4.** Let \( X \) and \( Y \) be two independent random variables with finite first moments, such that
\[
X \preceq_{st} Y \quad \text{or} \quad Y \preceq_{st} X.
\]
Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be a convex function satisfying
\[
g(x_1, x_2) + g(y_1, y_2) \geq g(x_1, y_2) + g(y_1, x_2)
\]
for all \( x_1, x_2, y_1, y_2 \in \mathbb{R} \) such that
\[
(y_1 - x_1)(y_2 - x_2) > 0.
\]
Then
\[
\mathbb{E} g(X_1, X_2) + \mathbb{E} g(Y_1, Y_2) \geq \mathbb{E} g(X, Y) + \mathbb{E} g(Y, X),
\]
where \( X_1, X_2 \) and \( Y_1, Y_2 \) are independent random variables such that \( X_1, X_2 \sim X \) and \( Y_1, Y_2 \sim Y \).

**Proof.** Without loss of generality we may assume that \( X \preceq_{st} Y \). By Theorem 2.1 there exist two independent random vectors \( (X_1, Y_1) \) and \( (X_2, Y_2) \) such that
\[
X_1, X_2 \sim X, \quad Y_1, Y_2 \sim Y, \quad P(X_1 \leq Y_1) = 1 \quad \text{and} \quad P(X_2 \leq Y_2) = 1.
\]
By (4.10) and (4.13) we obtain
\[
P \left( g(X_1, X_2) + g(Y_1, Y_2) \geq g(X_1, Y_2) + g(Y_1, X_2) \right) = 1
\]
which implies
\[
\mathbb{E} g(X_1, X_2) + \mathbb{E} g(Y_1, Y_2) \geq \mathbb{E} g(X_1, Y_2) + \mathbb{E} g(Y_1, X_2).
\]
Since (4.17) is equivalent to (4.15), the theorem is proved. \( \Box \)
Remark 4.7. We note that, if
\[(4.18)\]
then the inequality
\[(4.18)\]
Bernstein type operator
for functions \(g\) that appear in Problem 1’.

Corollary 4.5. Let \(g : [0, 1]^2 \to \mathbb{R}\) be a convex function satisfying
\[g(x_1, x_2) + g(y_1, y_2) \geq g(x_1, y_2) + g(y_1, x_2)\]
for all \(x_1, x_2, y_1, y_2 \in \mathbb{R}\) such that \((y_1 - x_1)(y_2 - x_2) > 0\). Then
\[(B_{n,n}g)(x, x) + (B_{n,n}g)(y, y) - (B_{n,n}g)(x, y) - (B_{n,n}g)(y, x) \geq 0.\]

We present a new open problem, which is a generalization of Problem 1 [5].

Problem 3. Let \(k \in \mathbb{N}\), \(n_i \in \mathbb{N}\) for \(i = 1, \ldots, k\), \(\sum_{i=1}^{k} n_i = m\), \(D = [0, 1]^k\) and \(g \in C(D)\). The Bernstein type operator \(B_{n_1, \ldots, n_k}\) is defined by
\[(B_{n_1, \ldots, n_k}g)(x_1, \ldots, x_k) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_k=0}^{n_k} b_{n_1, i_1}(x_1) \cdots b_{n_k, i_k}(x_k) g\left(\frac{i_1}{n_1}, \ldots, \frac{i_k}{n_k}\right)\]
for \((x_1, \ldots, x_k) \in D\).

Give a characterization of the class of convex functions \(g\) defined on \(D\) and satisfying
\[(4.18)\]
for all \((x_1, \ldots, x_k) \in D\).

In [3], we proved the following generalization of the Raşa inequality [12].

Theorem 4.6 ([3, Theorem 2.2, (2.5)]). Let \(k \in \mathbb{N}\), \(n_i \in \mathbb{N}\) for \(i = 1, \ldots, k\) and \(\sum_{i=1}^{k} n_i = m\). Then
\[(4.19)\]
for all convex functions \(\varphi \in C([0, 1])\) and \(x_1, \ldots, x_k \in [0, 1]\).

Remark 4.7. We note that, if \(\varphi \in C([0, 1])\) is a convex function, and
\[g(x_1, \ldots, x_k) = \varphi\left(\frac{n_1}{m}x_1 + \cdots + \frac{n_k}{m}x_k\right)\]
for \((x_1, \ldots, x_k) \in D\), then the inequality \[(4.18)\] coincides with \[(4.19)\], which was proved in [3].

Remark 4.8. If the inequality \[(4.18)\] is satisfied for all \((x_1, \ldots, x_k) \in D\), then
\[(4.20)\]
for all \((x_1, \ldots, x_k) \in D\).
Problem 3'. With assumptions such as in Problem 3., give a characterization of the class of convex functions $g$ defined on $D$ and satisfying (4.21) for all $(x_1, \ldots, x_k) \in D$.

B. Gavrea [3] presented also the following open problem.

Problem 4. ([3], Problem 2.) If $a_{n,k}(x) = {n+k \choose k} (1-x)^{n+1} x^k$ and if $\varphi$ is a convex continuous function on $[0, 1]$, prove or disprove the following inequality:

(4.21) $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (a_{n,i}(x) a_{n,j}(x) + a_{n,i}(y) a_{n,j}(y) - 2a_{n,i}(x) a_{n,j}(y)) \varphi \left( \frac{i+j}{2n+i+j} \right) \geq 0.$

We will show that (4.21) is not valid in general. Let $\varphi(u) = u$ and let $x \neq y$. Let $\mu$ be the negative binomial probability distribution with parameters $n+1$ and $x$. Then $\mu$ is concentrated on the set of non-negative integers and it satisfies $\mu\{(k)\} = \binom{n+k}{k} (1-x)^{n+1} x^k = a_{n,k}(x)$ for $k = 0, 1, \ldots$. Similarly, let $\nu$ be the negative binomial probability distribution with parameters $n+1$ and $y$. Let $F$ and $G$ be the cumulative distribution functions of $\mu$ and $\nu$, respectively. If $x < y$, then $F(u) < G(u)$ for $u \geq 0$. If $x > y$, then $F(u) > G(u)$ for $u \geq 0$ (see proof of Lemma 2.5.c in [7]). In both cases we have $F(u) = G(u) = 0$ for $u < 0$, thus $(F-G)*(F-G)(u) > 0$ for $u > 0$. Observe that for every $u \geq 0$ we have

$$\frac{u}{2n+u} - \int_0^u \frac{u}{2n+u} (u-y) + \lambda(dy).$$

Consequently,

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (a_{n,i}(x) a_{n,j}(x) + a_{n,i}(y) a_{n,j}(y) - 2a_{n,i}(x) a_{n,j}(y)) \frac{i+j}{2n+i+j} = \int_{-\infty}^{\infty} \frac{u}{2n+u} (\mu*\nu*\nu - 2\mu*\nu)(du) =$$

$$\int_{-\infty}^{\infty} \frac{u}{2n} (\mu*\nu*\nu - 2\mu*\nu)(du) - \int_{-\infty}^{\infty} \frac{4n}{(2n+u)^2} (u-y)+\lambda(dy)(\mu*\nu*\nu - 2\mu*\nu)(du) =$$

$$0 - \int_0^{\infty} \frac{4n}{(2n+u)^2} \int_{-\infty}^{\infty} (u-y)+\lambda(dy)(\mu*\nu*\nu - 2\mu*\nu)(du)\lambda(dy) =$$

$$- \int_0^{\infty} \frac{4n}{(2n+u)^2} ((F-G)*(F-G))(y)\lambda(dy) < 0.$$
[10] A. Marshall, I. Olkin, B. Arnold, *Inequalities: Theory of Majorization and Its Applications*, Second Edition, Springer, 2011.

[11] J. Mrowiec, T. Rajba, S. Wąsowicz, *A solution to the problem of Raşa connected with Bernstein polynomials*, J. Math. Anal. Appl., 446 (2017), 864-878.

[12] I. Raşa, 2. Problem, p. 164. In: Report of Meeting Conference on Ulam’s Type Stability, Rytro, Poland, June 2–6, 2014, Ann. Univ. Paedagog. Crac. Stud. Math. 13 (2014), 139–169. DOI: 10.2478/aupcsm-2014-0011

[13] I. Raşa, *Bernstein polynomials and convexity: recent probabilistic and analytic proofs*, The Workshop "Numerical Analysis, Approximation and Modeling", T. Popoviciu Institute of Numerical Analysis, Cluj-Napoca, June 14, 2017, http://ictp.acad.ro/zileleacademice-clusiene-2017/.

[14] M. Shaked, J. G. Shanthikumar, *Stochastic orders*, Springer, 2007.

**Andrzej Komisarski, Department of Probability Theory and Statistics, Faculty of Mathematics and Computer Science, University of Łódź, ul. Banacha 22, 90-238 Łódź, Poland**

*E-mail address: andkom@math.uni.lodz.pl*

**Teresa Rajba, University of Bielsko-Biała, Department of Mathematics, ul. Willowa 2, 43-309 Bielsko-Biała, Poland**

*E-mail address: trajba@ath.bielsko.pl*