Finite Larmor radius approximation for the Fokker-Planck-Landau equation

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Abstract

The subject matter of this paper concerns the derivation of the finite Larmor radius approximation, when collisions are taken into account. Several studies are performed, corresponding to different collision kernels. The main motivation consists in computing the gyroaverage of the Fokker-Planck-Landau operator, which plays a major role in plasma physics. We show that the new collision operator enjoys the usual physical properties; the averaged kernel balances the mass, momentum, kinetic energy and dissipates the entropy.

Keywords: Finite Larmor radius approximation, Boltzmann relaxation operator, Fokker-Planck-Landau equation.

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1 Introduction

Many studies in plasma physics concern the energy production through thermonuclear fusion. In particular this reaction can be achieved by magnetic confinement i.e.,

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a tokamak plasma is controlled by applying a strong magnetic field. Large magnetic fields induce high cyclotronic frequencies corresponding to the fast particle dynamics around the magnetic lines. We concentrate on the linear problem, by neglecting the self-consistent electro-magnetic field. The external electro-magnetic field is supposed to be a given smooth field

\[
E = -\nabla_x \phi, \quad B^\varepsilon = \frac{B(x)}{\varepsilon} b(x), \quad |b| = 1
\]

when \( \varepsilon > 0 \) is a small parameter, destined to converge to 0, in order to describe strong magnetic fields. The scalar function \( \phi \) stands for the electric potential, \( B(x) > 0 \) is the rescaled magnitude of the magnetic field and \( b(x) \) denotes its direction. As usual, we appeal to the kinetic description for studying the evolution of the plasma. The notation \( f^\varepsilon = f^\varepsilon(t, x, v) \geq 0 \) stands for the presence density of a population of charged particles with mass \( m \) and charge \( q \). This density satisfies

\[
\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{q}{m} (E + v \wedge B^\varepsilon) \cdot \nabla_v f^\varepsilon = Q(f^\varepsilon), \quad (t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3
\]

(1)

\[
f^\varepsilon(0, x, v) = f^{\text{in}}(x, v), \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3
\]

(2)

where \( Q \) denotes a collision kernel. The interpretation of the density \( f^\varepsilon \) is straightforward: the number of charged particles contained at time \( t \) inside the infinitesimal volume \( dx dv \) around the point \( (x, v) \) of the position-velocity phase space is given by \( f^\varepsilon(t, x, v) dx dv \). The equation (1) accounts for the fluctuation of the density \( f^\varepsilon \) due to the transport but also to the collisions. We analyze here the linear relaxation operator, but also the bilinear Fokker-Planck-Landau operator.

When neglecting the collisions the limit model as \( \varepsilon \searrow 0 \) comes by averaging with respect to the fast cyclotronic motion \([19, 24, 14, 1, 2, 3, 4, 5]\). The problem reduces to homogenization analysis and can be solved using the notion of two-scale convergence \([16, 17, 15]\).

We point out that a linearized and gyroaveraged collision operator has been written in \([25]\), but the implementation of this operator seems very hard. We refer to \([8, 9]\) for a general guiding-center bilinear Fokker-Planck collision operator. Another difficulty lies in the relaxation of the distribution function towards a local Maxwellian equilibrium. Most of the available model operators, in particular those which are linearized near a
Maxwellian, are missing this property. Very recently a set of model collision operators
has been obtained in [18], based on entropy variational principles [7].

We study here the finite Larmor radius scaling \textit{i.e.}, the typical perpendicular spatial
length is of the same order as the Larmor radius and the parallel spatial length is much
larger. We assume that the magnetic field is homogeneous and stationary

\[ B^\varepsilon = \left( 0, 0, \frac{B}{\varepsilon} \right) \]

for some constant \( B > 0 \) and therefore (1) becomes

\[ \partial_t f^\varepsilon + \frac{1}{\varepsilon} (v_1 \partial_{x_1} f^\varepsilon + v_2 \partial_{x_2} f^\varepsilon) + v_3 \partial_{x_3} f^\varepsilon + \frac{q}{m} E \cdot \nabla_v f^\varepsilon + \frac{\omega_c}{\varepsilon} (v_2 \partial_{v_1} f^\varepsilon - v_1 \partial_{v_2} f^\varepsilon) = Q(f^\varepsilon) \] (3)

where \( \omega_c = \frac{qB}{m} \) stands for the rescaled cyclotronic frequency. The density \( f^\varepsilon \) is
decomposed into a dominant density \( f \) and fluctuations of orders \( \varepsilon, \varepsilon^2, \ldots \)

\[ f^\varepsilon = f + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots \] (4)

Combining (3), (4) yields, with the notations \( \overline{x} = (x_1, x_2), \overline{v} = (v_1, v_2), \perp \overline{v} = (v_2, -v_1) \)

\[ \mathcal{T} f := \overline{v} \cdot \nabla_{\overline{x}} f + \omega_c \perp \overline{v} \cdot \nabla_v f = 0 \] (5)

\[ \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 = Q(f) \] (6)

The equation (5) appears as a divergence constraint

\[ \text{div}_{\overline{x},v} \{ f(\overline{v},0,\omega_c \perp \overline{v},0) \} = 0. \]

Equivalently, (6) says that at any time \( t \) the density \( f(t, \cdot, \cdot) \) remains constant along
the flow associated to \( \overline{v} \cdot \nabla_{\overline{x}} + \omega_c \perp \overline{v} \cdot \nabla_v \)

\[ \frac{d\overline{X}}{ds} = \nabla(s), \quad \frac{dX_3}{ds} = 0, \quad \frac{d\overline{V}}{ds} = \omega_c \perp \nabla(s), \quad \frac{dV_3}{ds} = 0 \] (7)

and therefore, at any time \( t \), the density \( f(t, \cdot, \cdot) \) depends only on the invariants of (7)

\[ f(t, x, v) = g \left( t, x_1 + \frac{v_2}{\omega_c}, x_2 - \frac{v_1}{\omega_c}, x_3, r = |\overline{v}|, v_3 \right). \]

The time evolution for \( f \) comes by eliminating \( f^1 \) in (6). For doing that, we project onto
the kernel of \( \mathcal{T} \), which is orthogonal to the range of \( \mathcal{T} \). In order to get a explicit model
for \( f \) we need a simpler representation for the orthogonal projection on \( \text{ker } T \). Actually this projection appears as the average along the characteristic flow \( (7) \). Denoting by \( \langle \cdot \rangle \) this projection, we obtain

\[
\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f \rangle = \langle Q(f) \rangle, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3.
\]

By one hand, averaging \( \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f \) leads to another transport operator

\[
\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f \rangle = \partial_t f + \langle \frac{\langle E \rangle}{B} \cdot \nabla f \rangle + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} f.
\]

The key point here is to choose as new coordinates the invariants of \( (7) \) and to observe that the partial derivatives with respect to these invariants commute with the average operator. More generally, for any smooth vector field \( \xi = (\xi_x, \xi_v) \), we obtain the following commutation formula between the divergence and average operators, cf. Proposition 3.3

\[
\langle \text{div}_{x,v} \xi \rangle = \text{div} \left\{ \left\langle \frac{\xi_x}{\omega_c} + \frac{\xi_v}{|v|} \right\rangle + \left\langle \frac{\xi_v \cdot \frac{\xi_v}{|v|}}{\omega_c |v|} \right\rangle - \left\langle \frac{\xi_v \cdot \frac{\xi_v}{|v|}}{\omega_c |v|} \right\rangle \right\} + \partial_{x_3} \langle \xi_{x_3} \rangle
\]

By the other hand we need to compute the average of the collision kernel \( Q \) which is a more complicated task. It is convenient to focus first on the relaxation Boltzmann operator [21]

\[
Q_B(f(t, x, \cdot))(v) = \frac{1}{\tau} \int_{\mathbb{R}^3} s(v, v') \{ M(v) f(t, x, v') - M(v') f(t, x, v) \} \, dv'
\]

where \( \tau > 0 \) is the relaxation time, \( s(v, v') \) is the scattering cross section and \( M \) is the Maxwellian equilibrium with temperature \( \theta \)

\[
M(v) = \frac{1}{(2\pi \theta/m)^{3/2}} e^{-m|v|^2/(2\theta)}, \quad v \in \mathbb{R}^3.
\]

We need to average functions like \( (x, v) \to \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \), where \( C(v, v') \) is a given function. Since the invariants of the flow \( (\vec{X}, \vec{V}) \) combines \( \vec{\pi} \) and \( \vec{\tau} \), we get a position-velocity integral operator cf. Proposition 4.2

\[
\left\langle \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \right\rangle (x, v) = \omega_c^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} C(|\vec{\pi}|, v_3, |\vec{\tau}'|, v_3', z) f(\vec{\pi}, x, 3, v') \, dv' dx_1' dx_2'
\]
with \( z = \omega \bar{x} + \frac{1}{\omega} \bar{v} - (\omega \bar{x} + \frac{1}{\omega} \bar{v}) \). We prove that averaging \( Q_B \) will lead to a position-velocity integral operator of the same form

\[
\langle Q_B \rangle f(x,v) = \langle Q_B(f) \rangle (x,v) = \langle \frac{\omega^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S(|\bar{v}|, v_3, |\bar{v}|, v_3', z) \{ M(v)f(x', x_3, v') - M(v')f(x, v) \} \, dv' \, dx' \, dx_2 \rangle
\]

(see Theorem 1.1 for the definition of \( S \)). Observe that \( \langle Q_B \rangle \) is global in \((\bar{x}, v)\), but remains local in \(x_3\). In particular it satisfies only a global mass balance, which comes easily by Fubini theorem and the symmetry of \( S \) cf. Remark 4.6

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle Q_B \rangle f(x,v) \, dv \, dx = 0.
\]

In the case of the relaxation operator \( Q_B \) we obtain the limit model

**Theorem 1.1** Assume that the scattering cross section satisfies \([22], [27]\) and that \( E(x) = -\nabla_x \phi(x), \phi \in W^{2,\infty}(\mathbb{R}^3) \). Let us consider \( f^{in} \geq 0, f^{in} \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^2(M^{-1}dx\, dv) \) and denote by \( f^\varepsilon \) the weak solution of \([3], [2] \) with \( Q = Q_B \) for any \( \varepsilon > 0 \). We assume that \( (f^\varepsilon)_{\varepsilon > 0} \) is bounded in \( L^\infty(\mathbb{R}_+, L^2(M^{-1}dx\, dv)) \). Then the family \( (f^\varepsilon)_{\varepsilon > 0} \) converges weakly \( \star \) in \( L^\infty(\mathbb{R}_+, L^2(M^{-1}dx\, dv)) \) to the weak solution of

\[
\partial_t f + \frac{\langle E \rangle}{B} \cdot \nabla f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} f = \langle Q_B \rangle f, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \quad (9)
\]

\[
f(0, x, v) = \langle f^{in} \rangle (x,v), \quad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \quad (10)
\]

where the averaged relaxation operator is given by

\[
\langle Q_B \rangle f(x,v) = \langle \frac{\omega^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S(|\bar{v}|, v_3, |\bar{v}|, v_3', z) \{ M(v)f(x', x_3, v') - M(v')f(x, v) \} \, dv' \, dx' \, dx_2 \rangle
\]

with \( z = \omega \bar{x} + \frac{1}{\omega} \bar{v} - (\omega \bar{x} + \frac{1}{\omega} \bar{v}) \) and the averaged scattering cross section writes

\[
S(r, v_3, r', v_3', z) = \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}) \chi(r, r', z)
\]

with

\[
\chi(r, r', z) = \frac{1}{\pi^2 \sqrt{|z|^2 - (r - r')^2}} \frac{1}{\sqrt{(r + r')^2 - |z|^2}}, \quad r, r' \in \mathbb{R}_+, v_3, v_3' \in \mathbb{R}, \quad z \in \mathbb{R}^2.
\]

The averaging technique allows us to treat many different collision operators, for example the Fokker-Planck kernel (see Appendix A for details)

\[
Q_{FP}(f) = \frac{\theta}{m\tau} \text{div}_v \left( \nabla_v f + \frac{m}{\theta} v f \right) = \frac{\theta}{m\tau} \text{div}_v \left\{ M \nabla_v \left( \frac{f}{M} \right) \right\}.
\]
Theorem 1.2 The limit model when $\varepsilon \to 0$ of (11), (12) with $Q = Q_{FP}$ is given by

$$\partial_t f + \frac{\langle \frac{\langle E \rangle}{B} \rangle}{\varepsilon} \cdot \nabla_x f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} f = \langle Q_{FP} \rangle f, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$$

(11)

$$f(0, x, v) = \langle f^{\text{in}} \rangle (x, v), \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$$

(12)

where the averaged Fokker-Planck operator and the diffusion matrix $\mathcal{L}$ write

$$\langle Q_{FP} \rangle f(x, v) = \frac{\theta}{mT} \text{div}_{\omega, x, v} \left\{ M \mathcal{L} \nabla_{\omega, x, v} \left( \frac{f}{M} \right) \right\}$$

$$\mathcal{L} = \begin{pmatrix} 2(I_3 - e_3 \otimes e_3) & -E \\ E & I_3 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Notice that the averaged Fokker-Planck operator contains no derivatives with respect to $x_3$ since the diffusion matrix $\mathcal{L}$ has only zero entries on the third line and column; averaging the Fokker-Planck operator leads to diffusion in velocity but also with respect to the perpendicular position coordinates.

Our main motivation concerns the bilinear Fokker-Planck-Landau equation, more exactly how to average kernels like

$$Q_{FPL}(f, f)(v) = \text{div}_v \left\{ \int_{\mathbb{R}^3} \sigma(|v - v'|)S(v - v')\left[f(v')\nabla_v f(v) - f(v)\nabla_{v'} f(v')\right] dv' \right\}$$

where $\sigma$ denotes the scattering cross section and $S(w) = I - \frac{w \otimes w}{|w|^2}$ is the orthogonal projection on the plane of normal $w$, cf. [20]. Recall that $Q_{FPL}$ satisfies the mass, momentum and kinetic energy balances

$$\int_{\mathbb{R}^3} Q_{FPL}(f, f) dv = 0, \quad \int_{\mathbb{R}^3} vQ_{FPL}(f, f)dv = 0, \quad \int_{\mathbb{R}^3} \frac{|v|^2}{2}Q_{FPL}(f, f)dv = 0.$$ 

Moreover it decreases the entropy $f \ln f$ since, by standard computations, we obtain

$$\int_{\mathbb{R}^3} \ln f \ Q_{FPL}(f, f) \ dv$$

$$= -\frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sigma(f(v)f(v')) \left(\nabla_v \ln f(v) - \nabla_{v'} \ln f(v')\right)^2 \frac{dv' dv}{|v - v'|^2} \leq 0.$$ 

We expect that the averaged Fokker-Planck-Landau operator satisfies the same properties. Nevertheless we will see that all of them hold true only globally in velocity and perpendicular position coordinates. Indeed, the averaged collision kernel will account
for the interactions between Larmor circles (characterized by the center $\vec{x} + \frac{1}{q} \vec{v}/\omega_c$ and the radius $|\vec{v}|/|\omega_c|$) rather than between particles. We show that the averaged Fokker-Planck-Landau kernel has the form

$$\langle Q_{FPL} \rangle (f, f) : = \langle Q_{FPL}(f, f) \rangle (x, v)$$

$$= \text{div}_{\omega_c, x, v} \left\{ \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma \chi f(\vec{x'}, x_3, v') A^+ \nabla_{\omega_c, x, v} f(x, v) \, dv' dx'_1 dx'_2 \right\}$$

$$- \text{div}_{\omega_c, x, v} \left\{ \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma \chi f(x, v) A^- \nabla_{\omega_c, x', v'} f(\vec{x'}, x_3, v') \, dv' dx'_1 dx'_2 \right\}$$

(13)

with $z = \omega_c \vec{x} + \frac{1}{q} \vec{v} - (\omega_c \vec{x} + \frac{1}{q} \vec{v})$, $\sigma = \sigma(\sqrt{|z|^2 + (v_3 - v'_3)^2})$, $\chi = \chi(|\vec{v}|, |\vec{v}'|, z)$ and

$$\sigma \chi A^+(r, v_3, v'_3, z) = \sum_{i=1}^{4} \xi'(\vec{x}, v, \vec{v}, v') \otimes \xi'(\vec{x}, v, \vec{v}, v')$$

$$\sigma \chi A^-(r, v_3, v'_3, z) = \sum_{i=1}^{4} \varepsilon_i \xi'(\vec{x}, v, \vec{v}, v') \otimes \xi'(\vec{v}, v, \vec{x}, v)$$

for some vector fields $(\xi'_i)_{1 \leq i \leq 4}$ and $\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \varepsilon_4 = 1$, see Proposition 5.9. Actually $A^+, A^-$ have only zero entries on the third line and column and therefore, averaging the Fokker-Planck-Landau kernel leads to diffusion (and convolution) with respect to velocity but also perpendicular position coordinates. To the best of our knowledge, this is the first completely explicit result on this topic. In particular, the above collisional kernel decreases the entropy $f \ln f$ since, by standard computations we obtain (see Proposition 5.10)

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \ln f \langle Q_{FPL} \rangle (f, f) \, dv dx_1 dx_2 = -\frac{\omega_c^2}{2} \sum_{i=1}^{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f f'$$

$$\times (\xi^i \cdot \nabla \ln f - \varepsilon_i (\xi^i) \cdot \nabla' \ln f')^2 \, dv' dx'_1 dx'_2 \, dv dx_1 dx_2 \leq 0, \quad x_3 \in \mathbb{R}.$$ 

Here, for any $\xi, \eta \in \mathbb{R}^6$, the notations $\xi \otimes \eta$ stands for the matrix whose entries are $(\xi \otimes \eta)_{kl} = \xi_k \eta_l, 1 \leq k, l \leq 6$. We obtain formally the following stability result

**Theorem 1.3** Let us consider $f^{in} \geq 0$, $(1 + |\ln f^{in}|) f^{in} \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and denote by $f^\varepsilon$ the solution of (3), (2) with $Q = Q_{FPL}$, for any $\varepsilon > 0$. Then the limit $f = \lim_{\varepsilon \searrow 0} f^\varepsilon$ satisfies

$$\partial_t f + \frac{1}{B} \nabla f + v_3 \partial_{x_3} f + \frac{q}{m} (E_3) \partial_{\nu} f = \langle Q_{FPL} \rangle (f, f), \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$$

(14)
\[ f(0, x, v) = \langle f^i \rangle(x, v), \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \]  

where the averaged Fokker-Planck-Landau operator is given by (13).

Our paper is organized as follows. In Section 2 we introduce the average operator along a characteristic flow. Section 3 is devoted to the commutation properties between average and first order differential operators. The average of the linear Boltzmann kernel is computed in Section 4. We establish its main properties and we prove the convergence result stated in Theorem 1.1. Section 5 is dedicated to the bilinear Fokker-Planck-Landau kernel, Theorem 1.3. We give a explicit form of its average and check the main physical properties. We prove the mass, momentum and total energy conservations for smooth solutions of the averaged Fokker-Planck-Landau equation coupled to the Poisson equation for the electric field. We also show that the mean Larmor circle center and power (with respect to the origin) are left invariant. Up to our knowledge this has not been reported yet.

2 Average operator

We recall briefly the definition and properties of the average operator corresponding to the transport operator \( \mathcal{T} \), whose definition in the \( L^2(\mathbb{R}^3 \times \mathbb{R}^3) \) setting is 

\[ \mathcal{T} u = \text{div}_{x,v}(u b), \quad b = (\tau, 0, \omega_c \frac{1}{\omega_c} \tau, 0), \quad \omega_c = \frac{qB}{m} \]

for any function \( u \) in the domain 

\[ \text{D}(\mathcal{T}) = \{ u(x, v) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) : \text{div}_{x,v}(u b) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \}. \]

We denote by \( \| \cdot \| \) the standard norm of \( L^2(\mathbb{R}^3 \times \mathbb{R}^3) \). The characteristics \((X, V)(s; x, v)\) associated to \( \tau \cdot \nabla_x + \omega_c \frac{1}{\omega_c} \tau \cdot \nabla_v \), see (7), satisfy

\[ \frac{d}{ds} \left\{ X + \frac{1}{\omega_c} V \right\} = 0, \quad \frac{dV}{ds} = \omega_c \frac{1}{\omega_c} V, \quad \frac{dX_3}{ds} = 0, \quad \frac{dV_3}{ds} = 0 \]

implying that

\[ \bar{V}(s) = R(-\omega_c s)\tau, \quad \bar{X}(s) = \tau + \frac{1}{\omega_c} \tau - \frac{1}{\omega_c} \bar{V}(s), \quad X_3(s) = x_3, \quad V_3(s) = v_3 \]
where \( R(\alpha) \) stands for the rotation of angle \( \alpha \)

\[
R(\alpha) = \begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}.
\]

All the trajectories are \( T_c = 2\pi/\omega_c \) periodic and we introduce the average operator, see \cite{2}, for any function \( u \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \)

\[
\langle u \rangle (x, v) = \frac{1}{T_c} \int_{0}^{T_c} u(X(s; x, v), V(s; x, v)) \, ds
= \frac{1}{2\pi} \int_{0}^{2\pi} u \left( \overline{x} + \frac{i}{\omega_c} \overline{v} - \frac{i}{\omega_c} \{ R(\alpha) \overline{v} \}, x_3, R(\alpha) \overline{v}, v_3 \right) \, d\alpha.
\]

It is convenient to introduce the notation \( e^{i\phi} \) for the \( \mathbb{R}^2 \) vector \((\cos \phi, \sin \phi)\). Assume that the vector \( \overline{v} \) writes \( \overline{v} = \| \overline{v} \| e^{i\phi} \). Then \( R(\alpha) \overline{v} = \| \overline{v} \| e^{i(\alpha + \phi)} \) and the expression for \( \langle u \rangle \) becomes

\[
\langle u \rangle (x, v) = \frac{1}{2\pi} \int_{0}^{2\pi} u \left( \overline{x} + \frac{i}{\omega_c} \overline{v} - \frac{i}{\omega_c} \{ \| \overline{v} \| e^{i(\alpha + \phi)} \}, x_3, \| \overline{v} \| e^{i(\alpha + \phi)}, v_3 \right) \, d\alpha.
\] (16)

Notice that \( \langle u \rangle \) depends only on the invariants \( \overline{x} + \frac{i}{\omega_c} \, \| \overline{v} \|, x_3, v_3 \) and therefore belongs to ker \( T \). The following two results are justified in \cite{3}, Propositions 2.1, 2.2. The first one states that averaging reduces to orthogonal projection onto the kernel of \( T \). The second one concerns the invertibility of \( T \) on the subspace of zero average functions and establishes a Poincaré inequality.

**Proposition 2.1** The average operator is linear continuous. Moreover it coincides with the orthogonal projection on the kernel of \( T \) i.e.,

\[
\langle u \rangle \in \text{ker} \, T \quad \text{and} \quad \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} (u - \langle u \rangle) \varphi \, dv \, dx = 0, \quad \forall \varphi \in \text{ker} \, T. \quad (17)
\]

**Remark 2.1** Notice that \( \langle X, \overline{V} \rangle \) depends only on \( s \) and \( (\overline{x}, \overline{v}) \) and thus the variational characterization in \( (17) \) holds true at any fixed \( (x_3, v_3) \in \mathbb{R}^2 \). Indeed, for any \( \varphi \in \text{ker} \, T \), \((x_3, v_3) \in \mathbb{R}^2 \) we have

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} (u \varphi)(x, v) \, d\overline{x} \, d\overline{v} = \frac{1}{T_c} \int_{0}^{T_c} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} u(x, v) \varphi(X(-s; x, v), x_3, V(-s; x, v), v_3) \, d\overline{x} \, d\overline{v} \, ds
= \frac{1}{T_c} \int_{0}^{T_c} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} u(X(s; x, v), x_3, V(s; x, v), v_3) \varphi(x, v) \, d\overline{x} \, d\overline{v} \, ds
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle u \rangle (x, v) \varphi(x, v) \, d\overline{x} \, d\overline{v}.
\]
We have the orthogonal decomposition of $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ into invariant functions along the characteristics (7) and zero average functions
\[ u = \langle u \rangle + (u - \langle u \rangle), \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u - \langle u \rangle) \langle u \rangle \mathrm{d}v \mathrm{d}x = 0. \]

Notice that $T^* = -T$ and thus the equality $\langle \cdot \rangle = \text{Proj}_{\ker T}$ implies
\[ \ker \langle \cdot \rangle = (\ker T)^\perp = (\ker T^*)^\perp = \overline{\text{Range } T}. \]

In particular $\text{Range } T \subset \ker \langle \cdot \rangle$. Actually we show that $\text{Range } T$ is closed, which will give a solvability condition for $Tu = w$ (cf. [3], Propositions 2.2).

**Proposition 2.2** The restriction of $T$ to $\ker \langle \cdot \rangle$ is one to one map onto $\ker \langle \cdot \rangle$. Its inverse belongs to $L(\ker \langle \cdot \rangle, \ker \langle \cdot \rangle)$ and we have the Poincaré inequality
\[ \|u\| \leq \frac{2\pi}{|\omega_c|} \|Tu\|, \quad \omega_c = \frac{qB}{m} \neq 0 \]
for any $u \in D(T) \cap \ker \langle \cdot \rangle$.

The natural space when dealing with the linear Boltzmann kernel $Q_B$ is $L^2(M^{-1} \mathrm{d}x \mathrm{d}v)$ rather than $L^2(\mathrm{d}x \mathrm{d}v)$. Motivated by that we introduce the operator $T_M : D(T_M) \subset L^2(M^{-1} \mathrm{d}x \mathrm{d}v) \to L^2(M^{-1} \mathrm{d}x \mathrm{d}v)$ given by $T_M u = \text{div}_{x,v}(ub)$ for any function $u$ in the domain
\[ D(T_M) = \{ u(x,v) \in L^2(M^{-1} \mathrm{d}x \mathrm{d}v) : \text{div}_{x,v}(ub) \in L^2(M^{-1} \mathrm{d}x \mathrm{d}v) \}. \]

Straightforward arguments show that $u \in D(T_M)$ iff $u/\sqrt{M} \in D(T)$ and $T_M(u) = \sqrt{M}T(u/\sqrt{M})$ for any $u \in D(T_M)$. In particular we have $\ker T_M = \sqrt{M} \ker T$. Notice that formula (16) still defines a linear bounded operator on $L^2(M^{-1} \mathrm{d}x \mathrm{d}v)$, denoted by $\langle \cdot \rangle_M$, which coincides with the orthogonal projection on the kernel of $T_M$, with respect to the scalar product of $L^2(M^{-1} \mathrm{d}x \mathrm{d}v)$. Indeed, taking into account that $M(v)$ is constant along the characteristic flow of (7), we have for any $u \in L^2(M^{-1} \mathrm{d}x \mathrm{d}v)$
\[ \langle u \rangle_M = \sqrt{M} \left\langle \frac{u}{\sqrt{M}} \right\rangle \in \sqrt{M} \ker T = \ker T_M \]
and for any $\varphi \in \ker T_M$
\[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u - \langle u \rangle_M) \varphi(x,v) \mathrm{d}x \mathrm{d}v \cdot \frac{M(v)}{M^2} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \frac{u}{\sqrt{M}} - \left\langle \frac{u}{\sqrt{M}} \right\rangle \right) \frac{\varphi}{\sqrt{M}} \mathrm{d}x \mathrm{d}v = 0. \]
The Poincaré inequality holds also true, with the same constant, since for any \( u \in D(\mathcal{T}_M) \cap \ker \langle \cdot \rangle_M \) we can write
\[
\|u\|_{L^2(M^{-1})} = \left\| \frac{u}{\sqrt{\omega_c}} \right\| \mathcal{T} \left( \frac{u}{\sqrt{M}} \right) = \frac{2\pi}{|\omega_c|} \|\mathcal{T}_M u\|_{L^2(M^{-1})} = \frac{2\pi}{|\omega_c|} \|\mathcal{T}_M u\|_{L^2(M^{-1})}.
\]

From now on, for the sake of simplicity, we will use only the notations \( \mathcal{T}, \langle \cdot \rangle \), independently of acting on \( L^2(dx dv) \) or \( L^2(M^{-1}dx dv) \).

## 3 Average and first order differential operators

We intend to average transport operators, see [3]. Moreover, in order to handle the Fokker-Planck-Landau kernel we will need to average second order differential operators. For doing that it is convenient to identify derivations which leave invariant \( \ker \mathcal{T} \).

It turns out that these derivations are those along the invariants
\[
\psi_1 = x_1 + \frac{v_2}{\omega_c}, \quad \psi_2 = x_2 - \frac{v_1}{\omega_c}, \quad \psi_3 = x_3, \quad \psi_4 = \sqrt{(v_1)^2 + (v_2)^2}, \quad \psi_5 = v_3.
\]

We introduce also \( \psi_0 = -\frac{\bar{\omega}_c}{\omega_c} \), with \( \bar{\omega}_c = |\bar{\omega}|e^{i\alpha}, \ \alpha \in [0, 2\pi[. \) Notice that \( \psi_0 \) has a jump of \( \frac{2\pi}{\omega_c} \) across \( \bar{\omega} \in \mathbb{R}_+ \times \{0\} \) but not its gradient with respect to \( \bar{\omega} \)
\[
\nabla_{\bar{\omega}} \alpha = -\frac{1}{|\bar{\omega}|^2}, \quad \nabla_{\bar{\omega}} \psi_0 = \frac{\nabla \psi_0}{\omega_c|\bar{\omega}|^2}, \quad \mathcal{T} \psi_0 = 1.
\]

The idea is to consider the fields \( (b^j)_{0 \leq i \leq 5} \) such that
\[
b^i \cdot \nabla_{x,v} \psi_j = \delta_{ij}, \quad 0 \leq i, j \leq 5.
\]

Indeed, the map \( (x, v) \rightarrow (\psi_i(x, v))_{0 \leq i \leq 5} \) defines a change of coordinates
\[
x_1 = \psi_1 + \frac{\psi_4}{\omega_c} \sin(\omega_c \psi_0), \quad x_2 = \psi_2 + \frac{\psi_4}{\omega_c} \cos(\omega_c \psi_0), \quad x_3 = \psi_3,
\]
\[
v_1 = \psi_4 \cos(\omega_c \psi_0), \quad v_2 = -\psi_4 \sin(\omega_c \psi_0), \quad v_3 = \psi_5.
\]

Therefore any function \( u = u(x, v) \) can be written \( u(x, v) = U(\psi(x, v)), \ \psi = (\psi_i)_{0 \leq i \leq 5} \) and thus, for any \( i \in \{0, 1, ..., 5\} \) we have
\[
b^i \cdot \nabla_{x,v} u = b^i \cdot \sum_{j=0}^5 \frac{\partial U}{\partial \psi_j}(\psi(x, v)) \nabla_{x,v} \psi_j = \frac{\partial U}{\partial \psi_i}(\psi(x, v)).
\]

In other words the derivations \( b^i \cdot \nabla_{x,v} \) act like \( \partial_{\psi_i}, 0 \leq i \leq 5 \). In particular if \( u \in \ker \mathcal{T} \), meaning that \( U \) does not depend on \( \psi_0 \), then \( b^i \cdot \nabla_{x,v} u = \partial_{\psi_i} U(\psi(x, v)) \) does not depend
on $\psi_0$, saying that $\ker T$ is left invariant by $b^i \cdot \nabla_{x,v}$, $0 \leq i \leq 5$. The following result comes by direct computation and is left to the reader. For any smooth vector fields $\xi, \eta$ on $\mathbb{R}^6$, the notation $[\xi, \eta]$ stands for their Poisson bracket i.e.,
\[
\[\xi, \eta\] = (\xi \cdot \nabla_{x,v})\eta - (\eta \cdot \nabla_{x,v})\xi.
\]

**Proposition 3.1** The fields $(b^i)_{0 \leq i \leq 5}$ satisfying
\[
b^i \cdot \nabla_{x,v} \psi_j = \delta^i_j, \quad 0 \leq i, j \leq 5
\]
are given by
\[
b^0 \cdot \nabla_{x,v} = v \cdot \nabla_{x} + \omega_c \cdot \nabla_{x,v} \quad b^1 \cdot \nabla_{x,v} = \partial_{x_1}, \quad b^2 \cdot \nabla_{x,v} = \partial_{x_2}, \quad b^3 \cdot \nabla_{x,v} = \partial_{x_3}
\]
\[
b^4 \cdot \nabla_{x,v} = -\frac{\omega_c \cdot \nabla_{x,v}}{|v|^2} \quad b^5 \cdot \nabla_{x,v} = \partial_{v_3}.
\]
Moreover the Poisson brackets between $(b^i)_{0 \leq i \leq 5}$ vanishes or equivalently the derivations $b^i \cdot \nabla_{x,v}$, $0 \leq i \leq 5$ are commuting.

**Remark 3.1** Notice that $(b^i)_{i \neq 4}$ are divergence free and $\text{div}_{x,v} b^4 = \frac{1}{|v|}$.

We claim that the operators $u \rightarrow \text{div}_{x,v}(ub^i)$, with domain
\[
D(\text{div}_{x,v}(\cdot, b^i)) = \{u \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) : \text{div}_{x,v}(ub^i) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)\}, \quad 0 \leq i \leq 5
\]
are commuting with the average operator. More generally we establish the following result.

**Proposition 3.2** Assume that the field $c \cdot \nabla_{x,v}$ is in involution with $b \cdot \nabla_{x,v} = v \cdot \nabla_{x} + \omega_c \cdot \nabla_{x,v}$ i.e., $[c, b] = 0$. Then the operator $\text{div}_{x,v}(\cdot, c)$ is commuting with the average operator associated to the flow of $b \cdot \nabla_{x,v}$ that is, for any function $u \in D(\text{div}_{x,v}(\cdot, c))$ its average $\langle u \rangle$ belongs to $D(\text{div}_{x,v}(\cdot, c))$ and
\[
\text{div}_{x,v}(\langle u \rangle c) = \langle \text{div}_{x,v}(uc) \rangle.
\]

**Proof.** Let us consider $u \in D(\text{div}_{x,v}(\cdot, c))$. For any $\varphi \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker T$ we have
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle \text{div}_{x,v}(uc) \rangle \varphi \, dv \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \text{div}_{x,v}(uc) \varphi \, dv \, dx = -\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} uc \cdot \nabla_{x,v} \varphi \, dv \, dx. \quad (18)
\]
But $T(c \cdot \nabla_{x,v} \varphi) = c \cdot \nabla_{x,v}(T \varphi) = 0$ saying that $c \cdot \nabla_{x,v} \varphi \in \ker T$ and thus
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} uc \cdot \nabla_{x,v} \varphi \, dv \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle u \rangle c \cdot \nabla_{x,v} \varphi \, dv \, dx. \quad (19)
\]
Combining (18), (19) we obtain for any $\varphi \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker T$
\[\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle \text{div}_{x,v}(uc) \rangle \varphi \, dv \, dx = - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle u \rangle \cdot \nabla_{x,v} \varphi \, dv \, dx. \] (20)

Actually the previous equality holds also true for smooth functions $\varphi \in \ker \langle \cdot \rangle$. Indeed, by Proposition 2.2, for any smooth function $\varphi \in \ker \langle \cdot \rangle$ there is $\psi \in D(T) \cap \ker \langle \cdot \rangle$ such that $T \psi = \varphi$ and thus $c \cdot \nabla_{x,v} \varphi = c \cdot \nabla_{x,v}(T \psi) = T(c \cdot \nabla_{x,v} \psi) \in \text{Range } T = \ker \langle \cdot \rangle$.

Using now the orthogonality between $\ker T$ and $\ker \langle \cdot \rangle$ we deduce that
\[\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle \text{div}_{x,v}(uc) \rangle \varphi \, dv \, dx = 0 = - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle u \rangle \cdot \nabla_{x,v} \varphi \, dv \, dx, \quad \varphi \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker \langle \cdot \rangle.\]

Finally (20) is verified for any smooth $\varphi$, implying that $\langle u \rangle \in D(\text{div}_{x,v}(\cdot \ c))$ and $\text{div}_{x,v}(\langle u \rangle \ c) = \langle \text{div}_{x,v}(uc) \rangle$.

\[\square\]

We want to average transport operators, which are written in conservative forms. In order to obtain averaged model still written in conservative form, it is worth to establish the following commutation formula between average and divergence. For the sake of simplicity we discard all difficulties related to the required minimal smoothness.

**Proposition 3.3** For any smooth field $\xi = (\xi_x, \xi_v) \in \mathbb{R}^6$ we have the equality
\[\langle \text{div}_{x,v} \xi \rangle = \text{div}_{x} \langle \xi_x \rangle + \text{div}_{v} \left\{ \frac{1}{|\mathbf{v}|} \partial_{x3} \langle \xi_{x3} \rangle + \frac{1}{|\mathbf{v}|} \partial_{v3} \langle \xi_{v3} \rangle \right\} + \partial_{x3} \langle \xi_{x3} \rangle.\]

In particular we have for any smooth field $\xi_x \in \mathbb{R}^3$
\[\langle \text{div}_{x,v} \xi_x \rangle = \text{div}_{x} \langle \xi_x \rangle \]

and for any smooth field $\xi_v \in \mathbb{R}^3$
\[\langle \text{div}_{x,v} \xi_v \rangle = \text{div}_{v} \left\{ \frac{1}{|\mathbf{v}|} \partial_{x3} \langle \xi_{x3} \rangle + \frac{1}{|\mathbf{v}|} \partial_{v3} \langle \xi_{v3} \rangle \right\} + \partial_{v3} \langle \xi_{v3} \rangle.\]
Proof. By construction we have \( \sum_{i=0}^{5} b^i \otimes \nabla_{x,v} \psi_i = I \) and thus

\[
\xi = \sum_{i=0}^{5} (\xi \cdot \nabla_{x,v} \psi_i) b^i.
\]

The main statement follows thanks to Proposition 3.2 since we have

\[
\langle \text{div}_{x,v} \xi \rangle = \left\langle \sum_{i=0}^{5} \text{div}_{x,v} \{ (\xi \cdot \nabla_{x,v} \psi_i) b^i \} \right\rangle = \text{div}_{x,v} \left\{ \sum_{i=0}^{5} (\xi \cdot \nabla_{x,v} \psi_i) b^i \right\}.
\]

The other statements come by considering the fields \((\xi_x, 0)\) and \((0, \xi_v)\).

A direct consequence of Proposition 3.3 is the computation of the average for the transport operator in \(\mathbb{E}\).

**Proposition 3.4** Assume that the electric field derives from a smooth potential i.e., \(E = -\nabla_x \phi\). Then for any \(f \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker T\) we have

\[
\left\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 \right\rangle = \partial_t f + \langle v_3 \partial_{x_3} f \rangle + \frac{q}{m} \langle E \rangle \partial_{v_3} f.
\]

Proof. We can write

\[
\left\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 \right\rangle = \partial_t f + \langle v_3 \partial_{x_3} f \rangle + \frac{q}{m} \langle E \cdot \nabla_v f \rangle
\]

since \(\langle \mathcal{T} f^1 \rangle = 0\) and \(\langle \partial_t f \rangle = \partial_t \langle f \rangle = \partial_t f\). The average of \(v_3 \partial_{x_3} f\) comes easily thanks to Proposition 3.2

\[
\langle v_3 \partial_{x_3} f \rangle = \langle \text{div}_{x,v} \{ f v_3 b^3 \} \rangle = \text{div}_{x,v} \{ \langle f v_3 \rangle b^3 \} = \text{div}_{x,v} \{ f v_3 b^3 \} = v_3 \partial_{x_3} f.
\]

Observe that \(\mathcal{T}(f \phi) = f \, \nabla \cdot E \phi = -f \, \nabla \cdot \mathcal{E}\) and thus \(\langle f \, \nabla \cdot E \rangle = 0\). Thanks to Proposition 3.3 one gets

\[
\langle \text{div}_{v} \{ f E \} \rangle = \text{div}_{v} \left\langle f \, \frac{\nabla \cdot E}{\omega_v} \right\rangle + \mathcal{T} \left\langle f \, \frac{\nabla \cdot E}{\omega_v |\mathcal{E}|^2} \right\rangle + \partial_{v_3} \langle f E_3 \rangle
\]

implying that

\[
\frac{q}{m} \langle \text{div}_{v} \{ f E \} \rangle = \text{div}_{v} \left\{ f \left\langle \frac{\nabla \cdot E}{\omega_v} \right\rangle \right\} + \frac{q}{m} \partial_{v_3} \{ f \langle E_3 \rangle \}.
\]

Using again Proposition 3.2 notice that

\[
\partial_{v_3} \langle E_3 \rangle = \text{div}_{x,v} \{ \langle E_3 \rangle b^5 \} = \langle \text{div}_{x,v} \{ E_3 b^5 \} \rangle = \langle \partial_{v_3} E_3 \rangle = 0
\]

and

\[
\text{div}_{v} \left\langle \frac{\nabla \cdot E}{\omega_v} \right\rangle = \langle \text{div}_{v} \frac{\nabla \cdot E}{\omega_v} \rangle = 0
\]

and our statement follows. \(\square\)
Remark 3.2 We have proved that averaging the transport operator $a \cdot \nabla_{x,v} := v_3 \partial_{x_3} + \frac{q}{m} E \cdot \nabla_v$ leads to $A \cdot \nabla_{x,v} := \langle \frac{+T}{B} \rangle \cdot \nabla_v + v_3 \partial_{x_3} + \frac{q}{m} \langle E_3 \rangle \partial_{v_3}$ which verifies

$$\langle a \cdot \nabla_{x,v} f \rangle = A \cdot \nabla_{x,v} f, \quad f \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker T.$$ 

By construction, the operator $A \cdot \nabla_{x,v}$ leaves invariant the subspace of smooth functions of $\ker T$. By antisymmetry (since $\text{div}_{x,v} A = 0$) it is easily seen that $A \cdot \nabla_{x,v}$ also leaves invariant the subspace of smooth functions in $\ker \langle \cdot \rangle$. Indeed, consider $h$ a zero average smooth function and let us prove that $\langle A \cdot \nabla_{x,v} h \rangle = 0$ For any smooth $f$ in $\ker T$ we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} A \cdot \nabla_{x,v} h f dv dx = - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} h A \cdot \nabla_{x,v} f dv dx = 0$$

by the orthogonality between $\ker \langle \cdot \rangle$ and $\ker T$, and thus $\langle A \cdot \nabla_{x,v} h \rangle = 0$. Finally $A \cdot \nabla_{x,v}$ is commuting with the average operator $\langle A \cdot \nabla_{x,v} f \rangle = A \cdot \nabla_{x,v} \langle f \rangle$ for any smooth $f$.

4 The relaxation collision operator

In this section we analyze the linear Boltzmann collision kernel [23, 22]

$$Q_B(f)(x,v) = \frac{1}{\tau} \int_{\mathbb{R}^3} s(v,v')\{M(v)f(x,v') - M(v')f(x,v)\} dv'$$

where the scattering cross section satisfies

$$s(v,v') = s(v', v), \quad 0 < s_0 \leq s(v,v') \leq S_0 < +\infty, \quad v,v' \in \mathbb{R}^3.$$ 

(22)

We recall the standard properties of this operator. Here $Q^\pm_B$ denote the gain/loss relaxation collision operators

$$Q^+_B(f)(v) = \frac{1}{\tau} \int_{\mathbb{R}^3} s(v,v')M(v)f(v') dv', \quad Q^-_B(f)(v) = \frac{1}{\tau} \int_{\mathbb{R}^3} s(v,v')M(v')f(v) dv'.$$

Proposition 4.1 Assume that the scattering cross section satisfies (22). Then

1. The gain/loss collision operators $Q^\pm_B$ are linear bounded operators on $L^2(M^{-1}dv)$, with $\|Q^\pm_B\| \leq S_0/\tau$, and symmetric with respect to the scalar product of $L^2(M^{-1}dv)$.

2. For any $f \in L^2(M^{-1}dv)$ we have

$$\int_{\mathbb{R}^3} Q_B(f)(v)f(v) \frac{dv}{M} = -\frac{1}{2\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} s(v,v')M(v)M(v') \left[ \frac{f(v)}{M(v)} - \frac{f(v')}{M(v')} \right]^2 dv' \leq 0.$$
We want to average $Q_B(f)$ for functions $f$ satisfying the constraint (5). In this section the operators $\mathcal{T}$ and $\langle \cdot \rangle$ should be understood in the $L^2(M^{-1}dxdv)$ framework. We need to compute the average of functions like $\int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv'$ where $C(v, v')$ is a given function. The corresponding result in the bidimensional framework has been announced in [6]. We will see that we only need to consider functions $C$ which are left invariant by any rotation around $e_3 = (0, 0, 1)$. Therefore we assume that for any orthogonal matrix $O \in \mathcal{M}_3(\mathbb{R})$ such that $Oe_3 = e_3$ we have

$$C(t^t O v, t^t O v') = C(v, v'), \quad v, v' \in \mathbb{R}^3. \tag{23}$$

These functions are precisely those depending only on $|\overrightarrow{v}|, v_3, |\overrightarrow{v'}|, v'_3$ and the angle between $\overrightarrow{v}$ and $\overrightarrow{v'}$

$$C(v, v') = \tilde{C}(|\overrightarrow{v}|, v_3, |\overrightarrow{v'}|, v'_3, \varphi), \quad \varphi = \arg \overrightarrow{v'} - \arg \overrightarrow{v}.$$

**Proposition 4.2** Assume that the function $C(v, v')$ satisfies (23) and belongs to the space $L^2(M^{-1}(v)M(v')dvdv')$. Then for any function $f \in \ker \mathcal{T}$ we have

$$\left( \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \right)(x, v) = \omega_c^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} C(|\overrightarrow{v}|, v_3, |\overrightarrow{v'}|, v'_3, z) f(\overrightarrow{v'}, x_3, v') \, dv' \, dx_1 \, dx_2$$

where $z = \omega_c \overrightarrow{r} + \frac{1}{2} \overrightarrow{v} - (\omega_c \overrightarrow{x'} + \frac{1}{2} \overrightarrow{v'})$

$$C(r, v_3, r', v'_3, z) = \frac{\tilde{C}(r, v_3, r', v'_3, \varphi) + \tilde{C}(r, v_3, r', v'_3, -\varphi)}{2\pi^2 \sqrt{|z|^2 - (r - r')^2} \sqrt{(r + r')^2 - |z|^2}} 1_{\{|r-r'|<|z|<r+r'\}}$$

and for any $|z| \in (|r-r'|, r+r')$, $\varphi \in (0, \pi)$ is the unique angle such that

$$|z|^2 = r^2 + (r')^2 - 2rr' \cos \varphi.$$

**Proof.** For $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ we have

$$\left( \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \right)^2 \leq \int_{\mathbb{R}^3} (C(v, v'))^2 M(v') \, dv' \int_{\mathbb{R}^3} \frac{(f(x, v'))^2}{M(v')} \, dv'$$

implying that

$$\left\| \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \right\|_{L^2(M^{-1}dxdv')} \leq \|C\|_{L^2(M^{-1}(v)M(v')dvdv')} \|f\|_{L^2(M^{-1}dxdv')}.$$
Therefore the function \( (x, v) \rightarrow \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \) belongs to \( L^2 \left( M^{-1} dx dv \right) \) and can be averaged in \( L^2 \left( M^{-1} dx dv \right) \). Consider \( (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \). By formula (16) we have

\[
I := \left\langle \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \right\rangle (x, v)
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^3} C(|\mathbf{v}| e^{i\alpha}, v_3, v') f \left( \frac{\mathbf{v}}{|\mathbf{v}|} - \frac{1}{\omega_c} \{ |\mathbf{v}| e^{i\alpha} \}, x_3, v' \right) \, dv' \, d\alpha.
\]

For any fixed \( \alpha \in [0, 2\pi) \) we use the cylindrical coordinates

\[
v' = (r' e^{i(\varphi + \alpha)}, v'_3), \quad r' \in \mathbb{R}_+, \quad \varphi \in [-\pi, \pi]
\]

and therefore

\[
I = \frac{1}{2\pi} \int_{\mathbb{R}_+} \int_0^{2\pi} \int_{-\pi}^\pi C(|\mathbf{v}| e^{i\alpha}, v_3, r' e^{i(\varphi + \alpha)}, v'_3) 
\times f \left( \frac{\mathbf{v}}{|\mathbf{v}|} - \frac{1}{\omega_c} \{ |\mathbf{v}| e^{i\alpha} \}, x_3, r' e^{i(\varphi + \alpha)}, v'_3 \right) r' \, dr' \, d\varphi \, dv' \, d\alpha.
\]

But \( f \in \ker \mathcal{T} \) and thus there is \( g \) such that

\[
f(x, v) = g \left( \frac{\mathbf{v}}{|\mathbf{v}|}, x_3, |\mathbf{v}|, v_3 \right)
\]

implying that

\[
f \left( \frac{\mathbf{v}}{|\mathbf{v}|} - \frac{1}{\omega_c} \{ |\mathbf{v}| e^{i\alpha} \}, x_3, r' e^{i(\varphi + \alpha)}, v'_3 \right) = g \left( \frac{\mathbf{v}}{|\mathbf{v}|} - \frac{1}{\omega_c} \{ |\mathbf{v}| e^{i\alpha} \} + \frac{1}{\omega_c} \{ r' e^{i(\varphi + \alpha)} \}, x_3, r', v'_3 \right).
\]

By one hand notice that \( r' e^{i(\varphi + \alpha)} - |\mathbf{v}| e^{i\alpha} = le^{i(\psi + \alpha)} \) where \( l^2 = r^2 + (r')^2 - 2rr' \cos \varphi \), \( r = |\mathbf{v}| \) and \( \psi \) depends on \( r, r' \), \( \varphi \) but not on \( \alpha \). By the other hand, since \( C \) is invariant by rotation around \( e_3 \) we deduce that

\[
C \left( re^{i\alpha}, v_3, r' e^{i(\varphi + \alpha)}, v'_3 \right) = \tilde{C} (r, v_3, r', v'_3, \varphi), \quad \varphi = \arg \mathbf{v}' - \arg \mathbf{v}.
\]

The map \( \varphi \rightarrow l(\varphi) = \sqrt{r^2 + (r')^2 - 2rr' \cos \varphi} \) defines a coordinate change between \( \varphi \in (0, \pi) \) and \( l \in (|r - r'|, r + r') \) and

\[
d\varphi = \frac{2dl}{\sqrt{l^2 - (r - r')^2} \sqrt{(r + r')^2 - l^2}}.
\]
By Fubini theorem one gets

\[ I = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} \int_{\mathbb{R}^+} \frac{2\pi}{x} \tilde{C}(r, v_3, r', v'_3, \varphi) g \left( \varphi + \frac{\pi}{\omega_c} + \frac{1}{\omega_c} \left\{ \iota e^{i(\psi + \alpha)} \right\}, x_3, r', v'_3 \right) r' \, da'd\varphi dv'_3 \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} \int_{\mathbb{R}^+} \frac{2\pi}{x} \tilde{C}(r, v_3, r', v'_3, \varphi) g \left( \varphi + \frac{\pi}{\omega_c} + \frac{1}{\omega_c} \left\{ \iota e^{i\alpha} \right\}, x_3, r', v'_3 \right) r' \, da'd\varphi dv'_3 \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} \int_{0}^{2\pi} \frac{2\pi}{x} \tilde{C}(r, v_3, r', v'_3, \varphi) + \tilde{C}(r, v_3, r', v'_3, -\varphi) \}

\times g \left( \varphi + \frac{\pi}{\omega_c} + \frac{1}{\omega_c} \left\{ \iota \varphi \right\}, x_3, r', v'_3 \right) r' \, da'd\varphi dv'_3 \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} \int_{0}^{2\pi} \frac{2\pi}{x} \tilde{C}(r, v_3, r', v'_3, \varphi(l)) + \tilde{C}(r, v_3, r', v'_3, -\varphi(l)) \}

\times g \left( \varphi + \frac{\pi}{\omega_c} + \frac{1}{\omega_c} \left\{ \iota e^{i\alpha} \right\}, x_3, r', v'_3 \right) \frac{2r' \, d\varphi d\alpha d\varphi' dv'_3}{\sqrt{l^2 - (r - r')^2 \sqrt{(r + r')^2 - l^2}}} \]

For any \( \alpha' \in (0, 2\pi) \) we have

\[ g \left( \varphi + \frac{\pi}{\omega_c} + \frac{1}{\omega_c} \left\{ \iota e^{i\alpha} \right\}, x_3, r', v'_3 \right) = f \left( \varphi + \frac{\pi}{\omega_c} + \frac{1}{\omega_c} \left\{ \iota e^{i\alpha} \right\}, x_3, r', e^{i\alpha'}, v'_3 \right) \]

and performing the change of coordinates \( v' = (r' e^{i\alpha'}, v'_3) \) leads to

\[ I = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \frac{2\pi}{x} \tilde{C}(r, v_3, r', v'_3, \varphi(l)) + \tilde{C}(r, v_3, r', v'_3, -\varphi(l)) \}

\times f \left( \varphi + \frac{\pi}{\omega_c} + \frac{1}{\omega_c} \left\{ \iota e^{i\alpha} \right\}, x_3, r', e^{i\alpha'}, v'_3 \right) \frac{1}{\sqrt{l^2 - (r - r')^2 \sqrt{(r + r')^2 - l^2}}} \]

\[ = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{2\pi}{x} \tilde{C}(r, v_3, v', v'_3, \varphi(l)) + \tilde{C}(r, v_3, v', v'_3, -\varphi(l)) \}

\times f \left( \varphi + \frac{\pi}{\omega_c} + \frac{1}{\omega_c} \left\{ \iota e^{i\alpha} \right\}, x_3, v' \right) \frac{1}{\sqrt{l^2 - (|v| - |v'|)^2 \sqrt{(|v| + |v'|)^2 - l^2}}} \]

Using the notation

\[ C(r, v_3, r', v'_3, z) = \frac{\tilde{C}(r, v_3, r', v'_3, \varphi) + \tilde{C}(r, v_3, r', v'_3, -\varphi)}{2\pi^2 \sqrt{|z|^2 - (r - r')^2 \sqrt{(r + r')^2 - |z|^2}}} 1_{|r - r'| < |z| < r + r'} \]

where for any \( |z| \in (|r - r'|, r + r') \), \( \varphi \in (0, \pi) \) is the unique angle such that

\[ |z|^2 = r^2 + (r')^2 - 2r'r' \cos \varphi \]

one gets

\[ I = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} C(|v|, v_3, |v'|, v'_3) f \left( \varphi + \frac{\pi}{\omega_c} + \frac{1}{\omega_c} \left\{ \iota e^{i\alpha} \right\}, x_3, v' \right) d\varphi dv' \]
We take as new coordinates
\[ \bar{x}^2 = \bar{x} + \frac{\bar{\nu}}{\omega_c} + \frac{\{l e^{i \alpha}\}}{\omega_c} - \frac{\bar{\nu'}}{\omega_c} \]
Observing that \( \det \frac{\partial (x'_i, x'_j)}{\partial (x_i, x_j)} = \frac{1}{\omega_c} \) we deduce that
\[ I = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} C(\bar{\nu}, v_3, |\bar{\nu}|; v'_3, (\omega_c \bar{x} + \bar{\nu}) - (\omega_c \bar{x}' + \bar{\nu}')) f(x'_1, x'_2, x_3, v') \, dv' \, dx'_1 \, dx'_2. \]

Remark 4.1 The constraint \( T \, f = 0 \) allows us to reduce the right hand side of (24) to a four dimensional integral. Indeed, thanks to (26) we obtain, using the notation \( \bar{y} = \bar{x} + \frac{\bar{\nu}}{\omega_c} \)
\[ I = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} C(\bar{\nu}, v_3, |\bar{\nu}|; v'_3, (\omega_c \bar{x} + \bar{\nu}) - (\omega_c \bar{x}' + \bar{\nu}')) g \left( \bar{x}' + \frac{\bar{\nu}'}{\omega_c}, x_3, |\bar{\nu}|, v'_3 \right) \, dv' \, dx'_1 \, dx'_2 
= \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} C(\bar{\nu}, v_3, |\bar{\nu}|; v'_3, \omega_c (\bar{y} - \bar{y}')) g(\bar{y}, x_3, |\bar{\nu}|, v'_3) \, dv' \, dy'_1 \, dy'_2 
= 2\pi \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}_+} C(\bar{\nu}, v_3, r', v'_3, \omega_c (\bar{y} - \bar{y}')) g(\bar{y}, x_3, r', v'_3) r' \, dv' \, dy'_1 \, dy'_2. \]
We prefer to keep the five dimensional integral representation since, in the sequel, we will introduce similar integral terms, but with densities \( f \) not satisfying the constraint \( T \, f = 0 \).

Remark 4.2 If the function \( \bar{C}(r, v_3, r', v'_3, \varphi) \) is odd with respect to \( \varphi \), then \( C = 0 \) and
\[ \left\langle \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \right\rangle = 0, \quad f \in \ker T. \]

Remark 4.3 Let \( \chi \) be the function
\[ \chi(r, r', z) = \frac{1_{|r - r'| < |z| < r + r'}}{\pi^2 \sqrt{|z|^2 - (r - r')^2} \sqrt{(r + r')^2 - |z|^2}} \]
for any \( r, r' \in \mathbb{R}_+, z \in \mathbb{R}^2 \). Then for every \( r, r' \in \mathbb{R}_+ \), \( \chi(r, r', z) \, dz \) is a probability measure on \( \mathbb{R}^2 \)
\[ \int_{\mathbb{R}^2} \chi(r, r', z) \, dz = 1, \quad r, r' \in \mathbb{R}_+ \]
and \( \left\langle \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \right\rangle \) appears as a convolution with respect to the invariants \( \omega_c \bar{x} + \bar{\nu} \). Indeed, using the formula
\[ f(\bar{x}', x_3, v') = g \left( \bar{x}' + \frac{\bar{\nu}'}{\omega_c}, x_3, |\bar{\nu}|, v'_3 \right) \]
we obtain
\[
\left\langle \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \right\rangle (x, v) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} C(|\vec{v}|, v_3, |\vec{v'}|, v_3' , (\omega c x + \frac{1}{\omega} \vec{v}) - (\omega c x + \frac{1}{\omega} \vec{v'})) \times g \left( \frac{\vec{v'}}{\omega c}, x, |\vec{v'}|, v_3' \right) \, d(\omega c \vec{v} + \frac{1}{\omega} \vec{v'}) \, dv'.
\]

**Remark 4.4** The conclusion of Proposition 4.2 also holds true for bounded functions \( f \) which are constant along the flow \( (7) \), provided that \( C(v, v') \in L^\infty(\mathrm{d}v; L^1(\mathrm{d}v')) \) and satisfies (23). Indeed, in this case \( f \rightarrow \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \) is bounded on \( L^\infty(\mathrm{d}x \mathrm{d}v) \)

\[
\left\| \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \right\|_{L^\infty(\mathrm{d}x \mathrm{d}v)} \leq \| C \|_{L^\infty(\mathrm{d}v; L^1(\mathrm{d}v'))} \| f \|_{L^\infty(\mathrm{d}x \mathrm{d}v)}
\]

and using the \( L^\infty \) version of the average operator, the same computations as those in the proof of Proposition 4.2 show that

\[
\left\langle \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \right\rangle (x, v) = \omega c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} C(|\vec{v}|, v_3, |\vec{v'}|, v_3' , z) f(\vec{v'}, x, v') \, dv' \, dx_1' \, dx_2'.
\]

**Corollary 4.1** Assume that the scattering cross section satisfies (22) and

\[
s(v, v') = \sigma(|v - v'|), \quad v, v' \in \mathbb{R}^3
\]

for some function \( \sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \). Then for any \( f \in \ker T \) we have

\[
\langle Q_B f \rangle (x, v) = \frac{\omega c^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S(|\vec{v}|, v_3, |\vec{v'}|, v_3' , z) \left\{ M(v) f(\vec{v'}, x, v') - M(v') f(x, v) \right\} \, dv' \, dx_1' \, dx_2'
\]

with \( z = \omega c x + \frac{1}{\omega} \vec{v} - (\omega c \vec{v} + \frac{1}{\omega} \vec{v'}) \) and

\[
S(r, v_3, r', v_3', z) = \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}) \chi(r, r', z).
\]

**Proof.** Clearly the function \( C(v, v') = \sigma(|v - v'|) M(v) \) satisfies (23), belongs to \( L^2(M^{-1}(v)M(v')dvdv') \) and we have

\[
\tilde{s}(r, v_3, r', v_3', \varphi) = \sigma(\sqrt{r^2 + (r')^2 - 2rr' \cos \varphi + (v_3 - v_3')^2})
\]

\[
S(r, v_3, r', v_3', z) = \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}) \chi(r, r', z).
\]

Thanks to Proposition 4.2 we obtain, with \( z = (\omega c x + \frac{1}{\omega} \vec{v}) - (\omega c \vec{v} + \frac{1}{\omega} \vec{v'}) \)

\[
\left\langle \int_{\mathbb{R}^3} s(v, v') M(v) f(x, v') \, dv' \right\rangle = \omega c^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} S(|\vec{v}|, v_3, |\vec{v'}|, v_3' , z) M(v) f(\vec{v'}, x, v') \, dv' \, dx_1' \, dx_2'.
\]
Since $f$ belongs to $L^2(M^{-1}dx dv)$ and remains constant along the flow $\mathbf{T}$ we have

$$\left\langle \int_{\mathbb{R}^3} s(v, v')M(v')f(x, v) \, dv' \right\rangle = f(x, v) \left\langle \int_{\mathbb{R}^3} s(v, v')M(v') \, dv' \right\rangle$$

where the first average operator should be understood in the $L^2(M^{-1}dx dv)$ setting and the second one in the $L^\infty(dx dv)$ setting. Remark 4.4 applied with $C(v, v') = s(v, v')M(v') \in L^\infty(dx; L^1(dx'))$ and the constant function $1 \in L^\infty(dx dv)$ yields

$$\left\langle \int_{\mathbb{R}^3} s(v, v')M(v')f(x, v) \, dv' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S(|\vec{v}|, v_3, |\vec{v}'|, v_3', z)M(v') \, dv' dx_1' dx_2'.$$

Therefore we obtain

$$\left\langle \int_{\mathbb{R}^3} s(v, v')M(v')f(x, v) \, dv' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S(|\vec{v}|, v_3, |\vec{v}'|, v_3', z)M(v')f(x, v) \, dv' dx_1' dx_2'$$

and our statement follows immediately.

We intend to extend the previous averaged collision operator to all densities $f$, not only those satisfying the constraint $\mathbf{T}f = 0$. Think that, when simulating numerically these models, the particle density may not satisfy exactly $\mathbf{T}f = 0$, and thus we need to construct such an extension. One possibility consists to appeal to the decomposition $f = \langle f \rangle + (f - \langle f \rangle)$ and to neglect the fluctuation $f - \langle f \rangle$, leading to the operator

$$f \rightarrow \tilde{Q}_B f := \langle Q_B \langle f \rangle \rangle$$

$$= \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S(|\vec{v}|, v_3, |\vec{v}'|, v_3', z)\{M(v)\langle f \rangle (x, v) - M(v')\langle f \rangle (x, v)\} \, dv' dx_1' dx_2'$$

for any $f \in L^2(M^{-1}dx dv)$. Clearly $\tilde{Q}_B$ coincides with $\langle Q_B f \rangle$ for any $f \in \ker \mathbf{T}$. Notice that for any $(x, v), (x_3', v_3')$ the function

$$(\vec{x}', \vec{v}') \rightarrow S(|\vec{v}|, v_3, |\vec{v}'|, v_3', \omega_c \vec{v}' + \frac{1}{2} \vec{v} - (\omega_c \vec{v}' + \frac{1}{2} \vec{v}))M(v)$$

depends only on $\omega_c \vec{v}' + \frac{1}{2} \vec{v}, |\vec{v}'|$ and therefore, thanks to Remark 3.1 we obtain a simpler form

$$\tilde{Q}_B f = \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S(|\vec{v}|, v_3, |\vec{v}'|, v_3', z)\{M(v)f(\vec{x}', x_3, v') - M(v')\langle f \rangle (x, v)\} \, dv' dx_1' dx_2'.$$

Nevertheless notice that it is not possible to remove the average in the loss part of the previous operator. Since $Q_B$ and $\langle \rangle$ are linear bounded operators on $L^2(M^{-1}dx dv)$ we deduce that $\tilde{Q}_B$ is linear bounded on $L^2(M^{-1}dx dv)$. The properties of $\tilde{Q}_B$ come
immediately from the properties of \( Q_B \), cf. Proposition 4.1. For example we have for any \( f \in L^2(M^{-1}dx dv) \)

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} Q_B f \frac{f}{M} dv dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle Q_B (f) \rangle \frac{f}{M} dv dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} Q_B (f) \frac{f}{M} dv dx
\]

\[
= -\frac{1}{2\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} s(v, v') M(v) M(v') \left[ \frac{\langle f \rangle (x, v)}{M(v)} - \frac{\langle f \rangle (x, v')}{M(v')} \right]^2 dv' dv dx \leq 0.
\]

Another possible extension is given by

\[
\langle Q_B \rangle f := \frac{\omega_0^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S(|\mathbf{v}|, v_3, |\mathbf{v}'|, v'_3, z) \{ M(v)f(\mathbf{v}, x_3, v') - M(v')f(x, v) \} dv' dx'_1 dx'_2
\]

for any \( f \in L^2(M^{-1}dx dv) \), which is very similar to the operator \( Q_B \) in (21). We keep this operator as extension for the operator in Corollary 4.1. The properties of \( \langle Q_B \rangle \) are summarized below.

**Proposition 4.3** Assume that the scattering cross section satisfies (22), (27). Then

1. The operator \( \langle Q_B \rangle \) is linear bounded on \( L^2(M^{-1}dx dv) \) and symmetric with respect to the scalar product of \( L^2(M^{-1}dx dv) \).

2. For any \( f \in L^2(M^{-1}dx dv) \) we have

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle Q_B \rangle (f) \frac{f}{M} dv dx = -\frac{\omega_0^2}{2\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S(|\mathbf{v}|, v_3, |\mathbf{v}'|, v'_3, z) M(v) M(v') \times \left[ \frac{f(x, v)}{M(v)} - \frac{f(\mathbf{v}, x_3, v')}{M(v')} \right]^2 dv' dx'_1 dx'_2 dv dx \leq 0.
\]

**Proof.** 1. The boundedness of the loss part follows easily since it is the multiplication by the bounded function (see Remark 4.3)

\[
\frac{\omega_0^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S(|\mathbf{v}|, v_3, |\mathbf{v}'|, v'_3, z) M(v') \ dv' dx'_1 dx'_2 = \frac{1}{\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} S(|\mathbf{v}|, v_3, |\mathbf{v}'|, v'_3, -z') M(v') \ dz' dv' \leq \frac{S_0}{\tau}.
\]

For the gain part we use the inequalities

\[
\omega_0^2 \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S(|\mathbf{v}|, v_3, |\mathbf{v}'|, v'_3, z) f(\mathbf{v}', x_3, v') \ dv' dx'_1 dx'_2 \right)^2 \\
\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S f^2(\mathbf{v}', x_3, v') M(v') \ dv' dx'_1 dx'_2 \quad \omega_0^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} S M(v') \ dv' dx'_1 dx'_2 \\
\leq S_0 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} S f^2(\mathbf{v}', x_3, v') M(v') \ dv' dx'_1 dx'_2.
\]
Thanks to Remark 4.3 we deduce, with $L^2_M = L^2(M^{-1}dx dv)$, that
\[
\left\| \frac{\omega^2}{\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} SM(v) f(x', x_3, v') \, dv' dx'_1 dx'_2 \right\|_{L^2_M}^2 \leq \frac{\omega^2}{\tau^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} M(v) S_0 \\
\times \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x', x_3, v')}{M(v')} \, dv' dx'_1 dx'_2 \\
= \frac{S_0}{\tau^2} \int_{\mathbb{R}^3} M(v) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x', x_3, v')}{M(v')} \, \omega^2 \int_{\mathbb{R}^3} S \, dx_1 dx_2 dv' dx'_1 dx'_2 dx_3 \\
\leq \frac{S_0^2}{\tau^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x', x_3, v')}{M(v')} \, dv' dx'_1 dx'_2 dx_3 = \frac{S_0^2}{\tau^2} \| f \|_{L^2_M}^2,
\]

2. Interchanging $(\overline{x}, v')$ with $(x, v)$ and observing that this change leaves invariant $S$, yield for any $f, g \in L^2(M^{-1}dx dv)$
\[
\langle QB \rangle f, g \rangle_{L^2_M} \\
= \frac{\omega^2}{\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} SM(v) M(v') \left[ \frac{f(x, x_3, v')}{M(v')} - \frac{f(x, v)}{M(v)} \right] \\
\times dv' dx'_1 dx'_2 \frac{g(x, v)}{M(v)} \, dv \\
= -\frac{\omega^2}{\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} SM(v) M(v') \left[ \frac{f(x, x_3, v')}{M(v')} - \frac{f(x, v)}{M(v)} \right] \\
\times dx_1 dx_2 \frac{g(x, v)}{M(v)} \\
= \frac{\omega^2}{2\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} SM(v) M(v') \left[ \frac{f(x, x_3, v')}{M(v')} - \frac{f(x, v)}{M(v)} \right] \left[ \frac{g(x, x_3, v')}{M(v')} - \frac{g(x, v)}{M(v)} \right] \\
dv' dx'_1 dx'_2 dv \\
\]

which justifies the symmetry of $\langle QB \rangle$ and its negativity.

\[\square\]

**Remark 4.5** Contrary to $Q_B$, the operator $\langle QB \rangle$ is non local in space. The value of $\langle QB \rangle f$ at the point $(x, v)$ depends on all the values of $f$ in the set
\[
A(x, v) = \{(x', x'_3, x_3, v') : S(|\overline{v}|, v_3, |\overline{v}'|, v'_3, (\omega_c \overline{x} + \overline{v}) - (\omega_c \overline{x} + \overline{v}')) > 0 \}
\]
\[
= \{(x', x'_3, x_3, v') : |\overline{v} - |\overline{v}'| < |(\omega_c \overline{x} + \overline{v}) - (\omega_c \overline{x} + \overline{v}')| < |\overline{v} + |\overline{v}'|\}.
\]

Observe that if we denote by $C_{x,v}$ the Larmor circle
\[
C_{x,v} = \{(x', x'_3, x_3) : |\omega_c \overline{v}' - (\omega_c \overline{x} + \overline{v})| = |\overline{v}'| \}
\]
then we have
\[
C_{x,v} \times \{v' : v' \in \mathbb{R}^3 \} \subset \overline{A(x, v)}
\]
where $\overline{X}$ stands for the adherence of $X$ in $\mathbb{R}^6$. In particular $\{x\} \times \mathbb{R}^3 \subset \overline{A(x, v)}$. 

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**Remark 4.6** The gain/loss parts of $\langle Q_B \rangle$ are bounded on $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and for any $f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ we have the global balance of the mass $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle Q_B \rangle f \, dv \, dx = 0$. Indeed, we have

$$\| \langle Q_B \rangle^+ f \|_{L^1} \leq \frac{\omega^2}{\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} S(\|v\|, |v'|, v''_3, z) M(v) |f(x', x_3, v')| \, dv' \, dx'_1 \, dx'_2 \, dx \leq \frac{S_0}{\tau} \| f \|_{L^1}$$

and similarly

$$\| \langle Q_B \rangle^- f \|_{L^1} \leq \frac{S_0}{\tau} \| f \|_{L^1}.$$

The global balance follows by interchanging $(\overline{x}', v')$ with $(\overline{x}, v)$.

Combining (6), Propositions 3.4, 4.1 and (28) yields the limit model in Theorem 1.1.

**Proof.** (of Theorem 1.1) Clearly $0 \leq f^\varepsilon \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ and

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^\varepsilon(t, x, v) \, dv \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{in}(x, v) \, dv \, dx, \quad t \in \mathbb{R}_+, \varepsilon > 0.$$

We consider a sequence $(\varepsilon_k)_k \subset \mathbb{R}^*_+$ converging to 0 such that $\lim_{k \to +\infty} f^{\varepsilon_k} = f$ weakly * in $L^\infty(\mathbb{R}_+, L^2(M^{-1} \, dx \, dv))$. Using the weak formulation of (3), (2) with test functions $\eta(t) \varphi(x, v)$, $\eta \in C^1_c(\mathbb{R}_+)$, $\varphi \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3)$, we deduce, after multiplication by $\varepsilon_k$ and letting $k \to \infty$, that the limit density satisfies the constraint

$$\mathcal{T} f(t) = 0, \quad t \in \mathbb{R}_+.$$

(29)

Considering test functions like $\eta(t) \varphi(x, v)$ with $\eta \in C^1_c(\mathbb{R}_+)$, $\varphi \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker \mathcal{T}$ one gets

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{\varepsilon_k} \{ \eta \varphi + \eta v_3 \partial_{x_3} \varphi + \frac{q}{m} E(x) \cdot \nabla \varphi \} \, dv \, dx \, dt + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{in} \eta(0) \varphi \, dv \, dx = -\int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \eta Q_B(f^{\varepsilon_k}) \varphi \, dv \, dx \, dt.$$

(30)
The symmetry of $Q_B$ cf. Proposition 4.1 allows us to write

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \eta Q_B(f^{\varepsilon_k}) \varphi \, dvdx \, dt = \lim_{k \to +\infty} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \eta f^{\varepsilon_k} Q_B(\varphi M) \, dvdx \, \frac{dt}{M} = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \eta Q_B(\varphi M) \, dvdx \, \frac{dt}{M} = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \eta Q_B(f) \varphi \, dvdx \, dt = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \eta \langle Q_B(f) \rangle \varphi \, dvdx \, dt
\]

(31)

since $f(t) \in \ker T$, $t \in \mathbb{R}^+$ and thus $\langle Q_B(f) \rangle = \langle Q_B \rangle (f)$. For the other terms in (30) we obtain thanks to Proposition 3.4, Remark 3.2

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{\varepsilon_k}(\partial_t + A \cdot \nabla_{x,v})(\eta \varphi) \, dvdx \, dt = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(\partial_t + A \cdot \nabla_{x,v})(\eta \varphi) \, dvdx \, dt = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(\partial_t + A \cdot \nabla_{x,v})(\eta \varphi) \, dvdx \, dt
\]

(32)

and

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{in}(x,v) \eta(0) \varphi(x,v) \, dvdx = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \eta(0) \langle f^{in} \rangle (x,v) \varphi(x,v) \, dvdx.
\]

(33)

Combining (30), (31), (32), (33) yields for any smooth test function $\eta(t) \varphi(x,v)$ with $\varphi \in \mathcal{T}$,

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(\partial_t + A \cdot \nabla_{x,v})(\eta \varphi) \, dvdx \, dt = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle f^{in} \rangle (0) \varphi(x,v) \, dvdx + \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle Q_B \rangle (f) \eta \varphi \, dvdx \, dt
\]

(34)

By Remark 3.2 we know that $A \cdot \nabla_{x,v}$ leaves invariant the subspace of zero average functions and therefore it is easily seen that (34) is trivially satisfied for any test function $\eta(t) \psi(x,v)$, with $\psi \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker \langle \cdot \rangle$. Finally (34) holds true for any smooth test function, saying that $f$ solves (9). We are done if we prove the uniqueness for the solution of (9), (10) (and in this case all the family $(f^\varepsilon)_\varepsilon$ will converge towards $f$, weakly $\ast$ in $L^\infty(\mathbb{R}^+, L^2(M^{-1}dvdx))$). Assume that $f$ solves (9) with zero initial condition. By standard arguments one gets

\[
\partial_t |f| + \frac{\langle \mathbf{1} E \rangle}{B} \cdot \nabla_{x,v}|f| + v_3 \partial_{x_3}|f| + \frac{q}{m} \langle E_3 \rangle \partial_{v_3}|f| = \langle Q_B \rangle (f) \text{ sgn} f
\]

implying that

\[
\frac{d}{dt} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} |f(t,x,v)| \, dvdx = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \langle Q_B \rangle (f) \text{ sgn} f(t,x,v) \, dvdx, \quad t \in \mathbb{R}^+.
\]
Our conclusion comes by observing that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle Q_B \rangle (f) \, \text{sgn} f \, dv \, dx = \frac{\omega^2}{\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(M(v)f(t, x, x', v') - M(v')f(t, x, v)) \, \text{sgn} f(t, x, v) \, dv' \, dx'_1 \, dx'_2 \, dvdx \\
= \frac{\omega^2}{\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(v') \{f(t, x, v) \text{sgn} f(t, x', x, v') - |f(t, x, v)|\} \, dv' \, dx'_1 \, dx'_2 \, dvdx \leq 0.
\]

Remark 4.7 It is easily seen that the integro-differential operator in (9) propagates the constraint \( T f = 0 \). We are done if we prove that \( f_s = f \) for any \( s \in \mathbb{R} \), where \( f_s(t, x, v) = f(t, X(s; x, v), V(s; x, v)) \) and \((X, V)\) is the characteristic flow (7). A direct computation shows that \( T \) and \( A \cdot \nabla_{x,v} f_s = (A \cdot \nabla_{x,v} f)_s \).

Observe also that
\[
\langle Q_B \rangle^+ f_s = (Q_B)^+ f = (\langle Q_B \rangle^+ f)_s, \quad (Q_B)^- f_s = (\langle Q_B \rangle^- f)_s
\]
and therefore \( (Q_B)^- f_s = (\langle Q_B \rangle^- f)_s \). Finally both \( f, f_s \) satisfy (9), (10) and our statement follows by the uniqueness that we have established before.

Clearly the transport equation (9) can be written in the reduced phase space \((\underline{y}, x, r) = (\underline{v}, x, v_3)\), since, by the constraint \( T f = 0 \), we know that \( f(t, x, v) = g(t, \underline{y}, x, r, v_3) \). We obtain
\[
\partial_t g + \frac{\langle 4E \rangle}{B} \cdot \nabla_{\underline{y}} g + v_3 \partial_{x_3} g + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} g = \frac{2\pi}{m} \frac{\omega^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(r, v_3, r', v_3', \omega_\varepsilon(\underline{y} - \underline{y}')) \\
\times \{M g(t, \underline{y}, x, r, v_3) - M' g(t, \underline{y}, x, r, v_3)\} r' \, dv' \, dv'_3 \, d\underline{y}'
\]
where
\[
M = \frac{1}{(2\pi \theta/m)^{3/2}} e^{-\frac{m \varepsilon (r^2 + v_3^2)}{2}}, \quad M' = \frac{1}{(2\pi \theta/m)^{3/2}} e^{-\frac{m \varepsilon (r'^2 + v_3'^2)}{2}}.
\]

Remark 4.8 The family \((f^\varepsilon)_{0 < \varepsilon \leq 1}\) remains bounded in \( L^\infty(\mathbb{R}_+, \mathcal{L}_2(M^{-1}dv)) \) for potentials of the form \( \phi(x) = \overline{\phi}(x) + \phi_3(x_3) \). Indeed, in this case observe that the energy function \( W^\varepsilon(x, v) := \frac{mv^2}{2} + q(\varepsilon \overline{\phi}(x) + \phi_3(x_3)) \) satisfies
\[
\partial_t W^\varepsilon + \frac{1}{\varepsilon} (\underline{v} \cdot \nabla_{\underline{v}} + \omega_c \frac{1}{\varepsilon} \cdot \nabla_{\underline{v}}) W^\varepsilon + v_3 \partial_{x_3} W^\varepsilon + \frac{q}{m} E(x) \cdot \nabla_{\underline{v}} W^\varepsilon = 0
\]
and therefore one gets
\[
\left\{ \partial_t + \frac{1}{\varepsilon}(\nabla \cdot \mathbf{v} + \nu \cdot \nabla \mathbf{v}) + \nu_3 \partial_{x_3} + \frac{q}{m} E(x) \cdot \nabla_v \right\} f^\varepsilon = \left( \frac{(f^\varepsilon)^2}{2M(v) \exp(-\frac{q}{\theta} \mathcal{F}(\mathbf{T}) + \phi_3(s_3))} \right) Q_B(f^\varepsilon) f^\varepsilon = \frac{Q_B(f^\varepsilon) f^\varepsilon}{M(v) \exp(-\frac{q}{\theta} \mathcal{F}(\mathbf{T}) + \phi_3(s_3))}.
\]

Integrating with respect to \((x, v)\) yields the bound
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(f^\varepsilon(t, x, v))^2}{M(v) \exp(-\frac{q}{\theta} \mathcal{F}(\mathbf{T}) + \phi_3(s_3))} \, dv \, dx \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(f^{\text{in}}(x, v))^2}{M(v) \exp(-\frac{q}{\theta} \mathcal{F}(\mathbf{T}) + \phi_3(s_3))} \, dv \, dx
\]
implies the uniform estimate
\[
\|f^\varepsilon\|_{L^\infty(\mathbb{R}^+, L^2(M^{-1} dv dx))} \leq \exp\left( \left| q \right| \frac{\|\mathcal{F}\|_{L^\infty} + \|\phi_3\|_{L^\infty}}{\theta} \right) \|f^{\text{in}}\|_{L^2(M^{-1} dv dx)}, \quad 0 < \varepsilon \leq 1.
\]

5 The Fokker-Planck-Landau operator

In this section we focus on the Fokker-Planck-Landau equation \([10, 11, 12, 13]\). The rate of change of the density \(f_s\), corresponding to a population of charged particles of specie \(s\), due to collisions with charge particles of specie \(s'\) writes

\[
Q_{FPL}(f_s, f_{s'}) = \frac{1}{m_s} \text{div}_v \int_{\mathbb{R}^3} \frac{\mu_{ss'}^2 \sigma_{ss'} (|v - v'|)|v - v'|^3}{m_\sigma} \left( \frac{1}{m_s} f_{s'}(v') (\nabla_v f_s)(v) - \frac{1}{m_{s'}} f_s(v) (\nabla_v f_{s'})(v') \right) \, dv'
\]

where \(\mu_{ss'} = \frac{m_s m_{s'}}{m_s + m_{s'}}\) is the reduced mass of the pair \(\{s, s'\}\), \(\sigma_{ss'} = \sigma_{s's} > 0\) is the scattering cross section between species \(\{s, s'\}\) and the matrix \(S(w) = \left( I - \frac{w \otimes w}{|w|^2} \right)\) corresponds to the orthogonal projection on the plane orthogonal to \(w\). As the electron mass is much smaller than the ion mass, we consider only the collisions between ions, whose distribution function is denoted by \(f^\varepsilon\) and satisfies

\[
\partial_t f^\varepsilon + \frac{1}{\varepsilon}(\nabla \cdot \mathbf{v} + \omega_e \cdot \mathbf{v}) f^\varepsilon + \nu_3 \partial_{x_3} f^\varepsilon + \frac{q}{m} E \cdot \nabla_v f^\varepsilon = Q_{FPL}(f^\varepsilon, f^\varepsilon), \quad (t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3
\]

where \(q > 0\) is the ion charge and \(m\) is the ion mass. As in the relaxation case, the limit model comes by averaging the collision kernel \(Q_{FPL}\). The treatment of the Fokker-Planck-Landau kernel is much elaborated. Therefore we content ourselves of formal computations. The main properties of the Fokker-Planck-Landau operator are summarized below.
Proposition 5.1 Consider the Fokker-Planck-Landau kernel between ions

\[ Q_{FPL}(f, f) = \frac{1}{4} \text{div}_v \int_{\mathbb{R}^3} \sigma_{ii}(|v-v'|) |v-v'|^3 S(v-v') (f(v') (\nabla_v f)(v) - f(v) (\nabla_v f)(v')) \, dv'. \]

Then the mass, momentum and kinetic energy balances hold true

\[ \int_{\mathbb{R}^3} m Q_{FPL}(f, f) \, dv = 0, \quad \int_{\mathbb{R}^3} mv Q_{FPL}(f, f) \, dv = 0, \quad \int_{\mathbb{R}^3} \frac{m|v|^2}{2} Q_{FPL}(f, f) \, dv = 0. \]

Moreover the entropy production \( D := -\int_{\mathbb{R}^3} (1 + \ln f) Q_{FPL}(f, f) \, dv \) is non negative

\[ D = \frac{1}{8} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sigma(|v-v'|) |v-v'| f(v) f(v') (v-v') \wedge (\nabla_v \ln f(v) - \nabla_v \ln f(v'))^2 \, dv' dv \geq 0. \]

Proof. All statements come easily by integration by parts, observing that \( A_{ii}(v, v') + A_{ii}(v', v) = 0 \), where \( A_{ii}(v, v') = \sigma_{ii}(|v-v'|) |v-v'|^3 S(v-v') (f(v') \nabla_v f(v) - f(v) \nabla_v f(v')) \) and \( S(v-v')(v-v') = 0 \).

With the notation \( \sigma(|v-v'|) = \frac{1}{2} \sigma_{ii}(|v-v'|) |v-v'|^3 \) the collision kernel becomes

\[ Q_{FPL}(f, f)(v) = \text{div}_v \int_{\mathbb{R}^3} \sigma(|v-v'|) S(v-v') (f(v') (\nabla_v f)(v) - f(v) (\nabla_v f)(v')) \, dv'. \]

### 5.1 Preliminary computations

The Fokker-Planck-Landau operator combines convolution and differential operators in \( v \). Therefore its average can be determined by studying the commutation properties between convolution and derivation with respect to the average. First we apply the commutation formula between divergence and average. Next we are looking for commutation between convolution and average. It is convenient to split \( Q_{FPL} \) into its gain and loss parts \( Q_{FPL}^+ \). We introduce the following notations, for any function \( g \) and vectors \( w_1, w_2 \)

\[ \langle g \rangle_{\sigma_S} := \left\langle \int_{\mathbb{R}^3} g(x, v') \sigma(|v-v'|) S(v-v') \, dv' \right\rangle \]

\[ \langle g, w_1 \rangle_{\sigma_S} := \left\langle \int_{\mathbb{R}^3} g(x, v') \sigma(|v-v'|) S(v-v') w_1 \, dv' \right\rangle \]

\[ \langle g, w_1, w_2 \rangle_{\sigma_S} := \left\langle \int_{\mathbb{R}^3} g(x, v') \sigma(|v-v'|) S(v-v') : w_1 \otimes w_2 \, dv' \right\rangle. \]

Let us establish some useful formulae based on Proposition 4.2. For any orthogonal matrix \( O \in \mathcal{M}_3(\mathbb{R}) \) we consider the application \( (v, v') \rightarrow S('Ov - 'Ov') \). It is easily seen that

\[ S('Ov - 'Ov') = 'OS(v-v')O. \]
Notice also that for any orthogonal matrix $O \in \mathcal{M}_3(\mathbb{R})$ such that $Oe_3 = e_3$ we have $(tOv, 0) = tO(v, 0)$ and $(\perp Ov, 0) = tO(\perp v, 0)$ for any $v \in \mathbb{R}^3$.

**Lemma 5.1** The following applications are left invariant by any rotation around $e_3$, that is they satisfy (23)

$$S(v - v') : (\perp 0) \otimes (\perp 0), \quad S(v - v') : (\perp 0) \otimes (\perp 0), \quad S(v - v') : (\perp 0) \otimes (\perp 0)$$

Proof. For any rotation $O$ around $e_3$ we have

$$S(tOv - tOv') : (tOv, 0) \otimes (tOv, 0) = tO \perp (v - v') \perp \perp 0) \otimes tO \perp (0)$$

$$= S(v - v') : (\perp 0) \otimes (\perp 0).$$

The other invariances follow similarly. \hfill \Box

Therefore the formula in Proposition 4.2 applies to all previous functions and we obtain

**Proposition 5.2** For any $z \in \mathbb{R}^2$ such that $|r - r'| < |z| < r + r'$ we denote by $\varphi \in (0, 2\pi)$ the angle satisfying $r^2 + (r')^2 - 2rr' \cos \varphi = |z|^2$. Then for any function $f \in \ker \mathcal{T}$ we have, with the notation $z = (\omega_c x + \perp \perp 0) - (\omega_c x' + \perp 0)$

1. 

$$\langle f, (\perp 0), (\perp 0) \rangle_{\perp S} = \omega_c^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}) f(x, x, v') \chi(|\perp 0), |\perp 0), z)$$

$$\times \left\{ r^2 - \frac{r^2 (r - r' \cos \varphi)^2}{|z|^2 + (v_3 - v_3')^2} \right\} dv' dx'_1 dx'_2$$

2. 

$$\langle f, (\perp 0), (\perp 0) \rangle_{\perp S} = \langle f, (\perp 0), (\perp 0) \rangle_{\perp S} = 0$$

3. 

$$\langle f, (\perp 0), (\perp 0) \rangle_{\perp S} = \omega_c^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}) f(x, x, v') \chi(|\perp 0), |\perp 0), z)$$

$$\times \left\{ r^2 - \frac{r^2 (r')^2 \sin^2 \varphi}{|z|^2 + (v_3 - v_3')^2} \right\} dv' dx'_1 dx'_2$$

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4. 
\[ \langle f, (\overrightarrow{v'}, 0), (\overrightarrow{v}, 0) \rangle_{\sigma_S} = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v'_3)^2}) f(x', x_3, v') \chi(|\overrightarrow{v}|, |\overrightarrow{v'}|, z) \times \left\{ r r' \cos \varphi - \frac{r r'(r \cos \varphi - r' (r - r' \cos \varphi))}{|z|^2 + (v_3 - v'_3)^2} \right\} \, dv' \, dx'_1 \, dx'_2 \]

5. 
\[ \langle f, (\overrightarrow{v'}, 0), (\overrightarrow{-v}, 0) \rangle_{\sigma_S} = \langle f, (\overrightarrow{-v'}, 0), (\overrightarrow{v}, 0) \rangle_{\sigma_S} = 0 \]

6. 
\[ \langle f, (\overrightarrow{-v'}, 0), (\overrightarrow{-v}, 0) \rangle_{\sigma_S} = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v'_3)^2}) f(x', x_3, v') \chi(|\overrightarrow{v}|, |\overrightarrow{v'}|, z) \times \left\{ r r' \cos \varphi - \frac{r^2 (r')^2 \sin^2 \varphi}{|z|^2 + (v_3 - v'_3)^2} \right\} \, dv' \, dx'_1 \, dx'_2. \]

**Proof.** We need to compute the functions \( \check{C}, C \) defined in Proposition 4.2. In each case we have

1. 
\[
C(v, v') = \sigma(|v - v'|) S(v - v') : (\overrightarrow{v}, 0) \otimes (\overrightarrow{v}, 0) = \sigma(|v - v'|) \left\{ |\overrightarrow{v}|^2 - \frac{(\overrightarrow{v} - \overrightarrow{v'}) \cdot \overrightarrow{v}|^2}{|v - v'|^2} \right\}
\]
\[
\check{C}(r, v_3, r', v'_3, \varphi) = \sigma(\sqrt{r^2 + (r')^2} - 2rr' \cos \varphi + (v_3 - v'_3)^2) \left\{ r^2 - \frac{(v^2 - r r' \cos \varphi)^2}{|z|^2 + (v_3 - v'_3)^2} \right\}
\]
\[
C(r, v_3, r', v'_3, z) = \chi(r, r', z) \sigma(\sqrt{|z|^2 + (v_3 - v'_3)^2}) \left\{ r^2 - \frac{r^2 (r' - r' \cos \varphi)^2}{|z|^2 + (v_3 - v'_3)^2} \right\}.
\]

2. 
\[
C(v, v') = \sigma(|v - v'|) S(v - v') : (\overrightarrow{v}, 0) \otimes (\overrightarrow{-v'}, 0)
\]
\[= -\sigma(|v - v'|) \frac{[(\overrightarrow{v} - \overrightarrow{v'}) \cdot \overrightarrow{v}] [(\overrightarrow{v} - \overrightarrow{v'}) \cdot \overrightarrow{-v'}]}{|v - v'|^2}
\]
\[
\check{C}(r, v_3, r', v'_3, \varphi) = -\sigma(\sqrt{|z|^2 + (v_3 - v'_3)^2}) \frac{(r^2 - r r' \cos \varphi) r' r' \sin \varphi}{|z|^2 + (v_3 - v'_3)^2}
\]
\[C = 0.
\]

3. 
\[
C(v, v') = \sigma(|v - v'|) S(v - v') : (\overrightarrow{-v}, 0) \otimes (\overrightarrow{-v}, 0) = \sigma(|v - v'|) \left\{ |\overrightarrow{v}|^2 - \frac{(v \cdot \overrightarrow{v})^2}{|v - v'|^2} \right\}
\]
We also need to compute the averages 

\[
\langle f, (\overline{\nu}, 0) \rangle_{\sigma S}, \quad \langle f, (\overline{\nu}, 0) \rangle_{\sigma S}, \quad \langle f, (\overline{\nu}, 0) \rangle_{\sigma S}, \quad \langle f, (\overline{\nu}, 0) \rangle_{\sigma S}.
\]
Notice that the functions $\sigma S(v - v')(\nabla, 0), \sigma S(v - v')(\nabla, 0), \sigma S(v - v')(\nabla, 0), \sigma S(v - v')(\nabla, 0)$ writes

$$D(v, v') = (\tilde{D} \nabla + \tilde{D'} \nabla, \tilde{D}_v v_3 + \tilde{D}'_v v'_3)$$

for some scalar functions $\tilde{D}, \tilde{D}', \tilde{D}_v, \tilde{D}'_v$ depending on $|\nabla|, v_3, |\nabla|, v'_3$ and $\varphi$, the angle between $(\nabla, 0), (\nabla, 0)$. Performing the same steps as in the proof of Proposition 4.2 we obtain (see Appendix B for proof details)

**Proposition 5.3** Consider $\tilde{D} = \tilde{D}(|\nabla|, v_3, |\nabla|, v'_3, \varphi), \tilde{D}' = \tilde{D}'(|\nabla|, v_3, |\nabla|, v'_3, \varphi)$ two functions and

$$D(v, v') = \tilde{D} \nabla + \tilde{D}' \nabla, \quad D_3(v, v') = \tilde{D} v_3 + \tilde{D}' v'_3.$$

Then for any $f \in \ker T$ we have

1. 

$$\left\langle \int_{R^3} D(v, v') f(x, v') \, dv' \right\rangle = \omega^2 \int_{R^2} \int_{R^3} D(|\nabla|, v_3, |\nabla|, v'_3, (\omega \cdot \nabla + \nabla') - (\omega \cdot \nabla' + \nabla))
	\times f(x', x_3, v') \, dv' \, dx'_1 \, dx'_2$$

where

$$D(r, v_3, r', v'_3, z) = \begin{pmatrix} z_2 & z_1 \\ -z_1 & z_2 \end{pmatrix} \frac{1}{2|z|}[\tilde{D}(r, v_3, r', v'_3, \varphi)re^{-i\psi} + \tilde{D}(r, v_3, r', v'_3, -\varphi)re^{i\psi}
+ \tilde{D}'(r, v_3, r', v'_3, \varphi)r'e^{i(\varphi - \psi)} + \tilde{D}'(r, v_3, r', v'_3, -\varphi)r'e^{-i(\varphi - \psi)}]\chi(r, r', z).$$

2. 

$$\left\langle \int_{R^3} D_3(v, v') f(x, v') \, dv' \right\rangle = \omega^2 \int_{R^2} \int_{R^3} D_3(|\nabla|, v_3, |\nabla|, v'_3, (\omega \cdot \nabla + \nabla') - (\omega \cdot \nabla' + \nabla))
	\times f(x', x_3, v') \, dv' \, dx'_1 \, dx'_2$$

where

$$D_3(r, v_3, r', v'_3, z) = \frac{1}{2}|(\tilde{D}(r, v_3, r', v'_3, \varphi) + \tilde{D}(r, v_3, r', v'_3, -\varphi))v_3
+ (\tilde{D}'(r, v_3, r', v'_3, \varphi) + \tilde{D}'(r, v_3, r', v'_3, -\varphi))v'_3|\chi(r, r', z).$$
The angles $\varphi, \psi \in (0, \pi)$ are such that $|z|^2 = r^2 + (r')^2 - 2rr' \cos \varphi$, $(r')^2 = r^2 + |z|^2 + 2r|z| \cos \psi$.

It is worth analyzing the case of even/odd coefficients $\tilde{D}, \tilde{D}'$.

**Proposition 5.4** With the same notations as in Proposition 5.3 assume that the functions $\tilde{D}, \tilde{D}'$ are even with respect to $\varphi$. Then we have

1. $\mathcal{D}(r, v_3, r', v'_3, z) = [(\tilde{D}(\varphi) + \tilde{D}'(\varphi)) r \cos \psi + |z|\tilde{D}'(\varphi)] \chi(r, r', z) \frac{z}{|z|}$

2. $\mathcal{D}_3(r, v_3, r', v'_3, z) = [(\tilde{D}(\varphi) + \tilde{D}'(\varphi)) v_3 + \tilde{D}'(\varphi)(v'_3 - v_3)] \chi(r, r', z)$

**Proof.** 1. Clearly we have

$$\frac{1}{2} [\tilde{D}(r, v_3, r', v'_3, \varphi) r e^{-i\psi} + \tilde{D}(r, v_3, r', v'_3, -\varphi) r e^{i\psi}] = \tilde{D}(r, v_3, r', v'_3, \varphi) r (\cos \psi, 0)$$

and

$$\frac{1}{2} [\tilde{D}'(r, v_3, r', v'_3, \varphi) r' e^{i(\varphi - \psi)} + \tilde{D}'(r, v_3, r', v'_3, -\varphi) r' e^{-i(\varphi - \psi)}] = \tilde{D}'(r' (\cos(\psi - \varphi), 0).$$

Consider now the triangle of vertices $O = (0, 0), A = (r, 0), A' = r'e^{i\varphi}$ in $\mathbb{R}^2$. The definitions for $\varphi, \psi$ assure that $|z| = |AA'|$ and that $\psi$ is the supplement of the angle opposite to $OA'$. Applying the cosine theorem with respect to the angle opposite to $OA$ one gets

$$r^2 = (r')^2 + |z|^2 - 2r'|z| \cos(\psi - \varphi).$$

Combining with the definition of $\psi$ yields

$$0 = 2|z|^2 - 2r'|z| \cos(\psi - \varphi) + 2r|z| \cos \psi$$

implying

$$r \cos \psi - r' \cos(\psi - \varphi) + |z| = 0. \quad (38)$$

Finally one gets

$$\mathcal{D}(r, v_3, r', v'_3, z) = [\tilde{D}(\varphi) r \cos \psi + \tilde{D}'(\varphi) r' \cos(\psi - \varphi)] \chi(r, r', z) \frac{z}{|z|}$$

$$= [(\tilde{D}(\varphi) + \tilde{D}'(\varphi)) r \cos \psi + \tilde{D}'(\varphi)(r' \cos(\psi - \varphi) - r \cos \psi)] \chi(r, r', z) \frac{z}{|z|}$$

$$= [(\tilde{D}(\varphi) + \tilde{D}'(\varphi)) r \cos \psi + |z|\tilde{D}'(\varphi)] \chi(r, r', z) \frac{z}{|z|}.$$
2. It follows immediately observing that

\[
D_3(r, v_3, r', v'_3, z) = (\bar{D}(\varphi) v_3 + \bar{D}'(\varphi) v'_3) \chi(r, r', z)
= [(\bar{D}(\varphi) + \bar{D}'(\varphi)) v_3 + \bar{D}'(\varphi) (v'_3 - v_3)] \chi(r, r', z).
\]

\[
\square
\]

**Proposition 5.5** With the same notations as in Proposition 5.3 assume that the functions \(\bar{D}, \bar{D}'\) are odd with respect to \(\varphi\). Then we have

1. \(D(r, v_3, r', v'_3, z) = -[\bar{D}(\varphi) + \bar{D}'(\varphi)] r \sin \psi \chi(r, r', z) \frac{z}{|z|}\)

2. \(D_3(r, v_3, r', v'_3, z) = 0\).

**Proof.** 1. We have

\[
\frac{1}{2} [D(r, v_3, r', v'_3, \varphi) r e^{-i\psi} + D(r, v_3, r', v'_3, -\varphi) r e^{i\psi}] = -(0, \bar{D}(r, v_3, r', v'_3, \varphi) r \sin \psi)
\]

and

\[
\frac{1}{2} [D'(r, v_3, r', v'_3, \varphi) r' e^{i(\varphi - \psi)} + D'(r, v_3, r', v'_3, -\varphi) r' e^{-i(\varphi - \psi)}] = -(0, \bar{D}' r' \sin(\psi - \varphi)).
\]

The sine theorem applied in the triangle of vertices \(O = (0, 0), A = (r, 0), A' = r' e^{i\varphi}\) implies

\[
r \sin \psi = r' \sin(\psi - \varphi).
\]

We deduce that

\[
D(r, v_3, r', v'_3, z) = -[\bar{D}(\varphi) r \sin \psi + \bar{D}'(\varphi) r' \sin(\psi - \varphi)] \chi(r, r', z) \frac{z}{|z|}
= -[\bar{D}(\varphi) + \bar{D}'(\varphi)] r \sin \psi \chi(r, r', z) \frac{z}{|z|}
\]

2. Clearly we have \(\bar{D}(\varphi) + \bar{D}(-\varphi) = \bar{D}'(\varphi) + \bar{D}'(-\varphi) = 0\) and therefore \(D_3 = 0\).

The averages in (36) come immediately appealing to Propositions 5.4, 5.5.

**Corollary 5.1** With the notations in Proposition 5.3 we have for any function \(f \in \ker T\)
1.\[ \langle f, (\overline{\nu}, 0) \rangle_S = -\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f x^2 \frac{r^2 - rr' \cos \varphi}{|z|^2 + (v_3 - v_3')^2} \left( \frac{(v_3 - v_3')^2}{|z|^2} \right) \, dv' dx_1 dx_2 \]

2.\[ \langle f, (\overline{\nu'}, 0) \rangle_S = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f x^2 \frac{(r')^2 - rr' \cos \varphi}{|z|^2 + (v_3 - v_3')^2} \left( \frac{(v_3 - v_3')^2}{|z|^2} \right) \, dv' dx_1 dx_2 \]

3.\[ \langle f, (\overline{\mathbf{n}}, 0) \rangle_S = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f x^2 \frac{r^2 - rr' \cos \varphi}{|z|^2} \, (z, 0) \, dv' dx_1 dx_2 \]

4.\[ \langle f, (\overline{\mathbf{n'}}, 0) \rangle_S = -\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f x^2 \frac{(r')^2 - rr' \cos \varphi}{|z|^2} \, (z, 0) \, dv' dx_1 dx_2. \]

**Proof.**

1. We consider the function \( D(v, v') = \sigma(|v - v'|)S(v - v')(\overline{\nu}, 0) = (\tilde{D} \overline{\nu} + \tilde{D}' \overline{\nu'}, \tilde{D}(v_3 - v_3)) \) where

\[
\tilde{D}(r, v_3, r', v_3', \varphi) = \sigma \left( 1 - \frac{r^2 - rr' \cos \varphi}{r^2 + (r')^2 - 2rr' \cos \varphi + (v_3 - v_3')^2} \right)
\]

\[
\tilde{D}'(r, v_3, r', v_3', \varphi) = \sigma \frac{r^2 - rr' \cos \varphi}{r^2 + (r')^2 - 2rr' \cos \varphi + (v_3 - v_3')^2}.
\]

Thanks to Proposition 5.4 and the identity \( r \cos \psi = -\frac{r^2 - rr' \cos \varphi}{|z|} \), we obtain

\[
\left\langle \int_{\mathbb{R}^3} (\tilde{D} \overline{\nu} + \tilde{D}' \overline{\nu'}) f(x, v') \, dv' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f(x', x'_2, x_3, v') \chi \times \left( r \cos \psi + |z| \right) \left( \frac{r^2 - rr' \cos \varphi}{|z|^2 + (v_3 - v_3')^2} \right) \frac{1}{|z|} \, dv' dx_1 dx_2
\]

\[
= -\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f(x', x'_2, x_3, v') \chi \times \left( \frac{r^2 - rr' \cos \varphi}{|z|^2 + (v_3 - v_3')^2} \right) \frac{1}{|z|} \, dv' dx_1 dx_2
\]

and also

\[
\left\langle \int_{\mathbb{R}^3} \tilde{D}(v_3' - v_3) f(x, v') \, dv' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f(x', x'_2, x_3, v') \chi \times \left( \frac{r^2 - rr' \cos \varphi}{|z|^2 + (v_3 - v_3')^2} \right) (v_3' - v_3) \, dv' dx_1 dx_2
\]

which justifies the first statement.

2. We take \( D(v, v') = \sigma(|v - v'|)S(v - v')(\overline{\nu'}, 0) = (\tilde{D} \overline{\nu'} + \tilde{D}' \overline{\nu}, \tilde{D}(v_3' - v_3)) \) where

\[
\tilde{D}(r, v_3, r', v_3', \varphi) = \sigma \frac{(r')^2 - rr' \cos \varphi}{r^2 + (r')^2 - 2rr' \cos \varphi + (v_3 - v_3')^2}
\]
By Proposition 5.5 we deduce that
\[
\tilde{D}'(r, v_3, r', v_3') = \sigma \left(1 - \frac{(r')^2 - rr' \cos \varphi}{r^2 + (r')^2 - 2rr' \cos \varphi + (v_3 - v_3')^2}\right).
\]

Notice that by (37), (38) we have
\[r \cos \psi + |z| = r' \cos(\psi - \varphi) = \frac{(r')^2 + |z|^2 - r^2}{2|z|} = \frac{(r')^2 - rr' \cos \varphi}{|z|}
\]
and in this case we obtain
\[
\left< \int_{\mathbb{R}^3} (\tilde{D} \nabla + \tilde{D}' \nabla') f(x, v') \, dv' \right> = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f(x_1', x_2', x_3, v') \chi \left[ r \cos \psi + |z| \left(1 - \frac{(r')^2 - rr' \cos \varphi}{|z|^2 + (v_3 - v_3')^2}\right) \right] \frac{1}{|z|} \, dv' \, dx_1' \, dx_2'.
\]
\[
\left< \int_{\mathbb{R}^3} \tilde{D}(v_3 - v_3') f(x, v') \, dv' \right> = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f(x_1', x_2', x_3, v') \chi \left( r \cos \psi + |z| \left(1 - \frac{(r')^2 - rr' \cos \varphi}{|z|^2 + (v_3 - v_3')^2}\right) \right) \frac{1}{|z|} \, dv' \, dx_1' \, dx_2'.
\]
justifying the second statement.

3. We take \(D(v, v') = -\sigma(|v - v'|)(\frac{v - v'}{|v - v'|}) \cdot \nabla f(x, v') \) where
\[
\tilde{D}(r, v_3, r', v_3', \varphi) = -\sigma \frac{rr' \sin \varphi}{r^2 + (r')^2 - 2rr' \cos \varphi + (v_3 - v_3')^2} = -\tilde{D}'.
\]
By Proposition 5.5 we deduce that \(\left< \int_{\mathbb{R}^3} D(v, v') f(x, v') \, dv' \right> = 0\). Therefore
\[
\left< f, (\frac{\nabla}{\varphi}, 0) \right>_{\sigma S} = \left< \int_{\mathbb{R}^3} \sigma(|v - v'|)(\frac{v - v'}{|v - v'|}) \cdot \nabla f(x, v') \, dv' \right>.
\]
Applying now Proposition 5.4 with \(\tilde{D} = \sigma, \tilde{D}' = 0\) we obtain
\[
\left< \int_{\mathbb{R}^3} \sigma(|v - v'|) f(x, v') \nabla \, dv' \right> = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f(x) r \cos \psi \frac{1}{|z|} \, dv' \, dx_1' \, dx_2'.
\]
and finally
\[
\left< f, (\frac{\nabla}{\varphi}, 0) \right>_{\sigma S} = -\omega_c^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sigma f \chi r \cos \psi \frac{1}{|z|} \, dv' \, dx_1' \, dx_2'.
\]
4. As before, by Proposition 5.5 we have
\[ \left\langle \int_{\mathbb{R}^3} \sigma(|v - v'|) f(x, v') \frac{(v - v') \otimes (v - v')}{|v - v'|^2} (\perp v', 0) \, dv' \right\rangle = 0 \]
and by Proposition 5.4 we obtain
\[ \left\langle \int_{\mathbb{R}^3} \sigma(|v - v'|) f(x, v') v' \, dv' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f(\chi(\sqrt{z^2 + (v_3 - v'_3)^2}) f(\perp v', x_3, v') \chi(|v|, |v'|, z)) S(\perp z, v_3' - v_3) \, dv' \, dx_1' \, dx_2'. \]
At the end one gets
\[ \langle f, (\perp v', 0) \rangle_{\sigma S} = -\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f(\chi(\sqrt{z^2 + (v_3 - v'_3)^2}) f(\perp v', x_3, v') \chi(|v|, |v'|, z)) S(\perp z, v_3' - v_3) \, dv' \, dx_1' \, dx_2'. \]

The last average we will need is \( \langle f \rangle_{\sigma S} = \langle \int_{\mathbb{R}^3} f(x, v') \sigma(|v - v'|) S(v - v') \, dv' \rangle \). By similar computations as those in the proofs of Propositions 4.2, 5.3 we obtain (see Appendix B for details)

**Proposition 5.6** For any function \( f \in \ker \mathcal{T} \) we have
\[ \langle f \rangle_{\sigma S} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f(\chi(\sqrt{z^2 + (v_3 - v'_3)^2}) f(\perp v', x_3, v') \chi(|v|, |v'|, z)) S(\perp z, v_3' - v_3) \, dv' \, dx_1' \, dx_2'. \]

### 5.2 The averaged Fokker-Planck-Landau operator

We are ready to determine the average of the Fokker-Planck-Landau kernel. For the sake of presentation we treat separately the gain and loss parts. Recall that the Fokker-Planck-Landau gain part appears as a velocity diffusion, where the diffusion matrix is a convolution in velocity
\[ Q_{FPL}(f, f) = \text{div}_v \left\{ \int_{\mathbb{R}^3} \sigma(|v - v'|) S(v - v') f(v') \, dv' \nabla_v f(v) \right\}. \]
The averaged Fokker-Planck-Landau kernel will keep the same structure, nevertheless diffusion and convolution have to be considered both in velocity and space perpendicular directions, as we have already observed in the relaxation case (see Remark 4.3). The proof is postponed to Appendix C.
Proposition 5.7 For any function \( f = f(x,v) \) satisfying the constraint \( T f = 0 \) we have

\[
\langle Q_{FPL}^+(f, f) \rangle = \text{div}_{\omega_3,v} \left\{ \omega_3^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}) f(x_1', x_2', x_3, v') \chi(|\overline{v}|, |\overline{v}'|, z) \times A^+ \nabla_{\omega_3,v} f(x, v) \, dv' dx'_1 dx'_2 \right\}
\]

with \( \text{div}_{\omega_3,v} = \frac{1}{\omega_3} \text{div}_x, \nabla_{\omega_3,v} = \frac{1}{\omega_3} \nabla_x \)

\[
A^+(r, v_3, r', v_3') = \frac{(r')^2 \sin^2 \varphi (v_3 - v_3')^2 \left( \frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|}, \frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|} \right) \otimes 2}{|z|^2(2r + (r')^2)} + \left[ \frac{r - r' \cos \varphi \left( \frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|}, \frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|} \right) + \left( \frac{\langle \overline{z}, 0 \rangle}{|\overline{z}|}, 0 \right) \right] \otimes 2 + \frac{(r')^2 \sin^2 \varphi \left( \frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|}, -\frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|} \right) \otimes 2}{|z|^2(2r + (r')^2)}
\]

\[
+ \left[ \frac{(r' \cos \varphi - r)(v_3 - v_3') \left( \frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|}, \frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|} \right) + \left( \frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|}, -\frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|} \right) \right] \otimes 2 + \frac{(v_3 - v_3') \left( \frac{z, 0}{|z|}, -|z|e_3 \right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}
\]

where \( z = (\omega_3 \overline{v} + \overline{v}) - (\omega_3 \overline{v} + \overline{v}) \) and for any \( r, r' \in \mathbb{R}_+, z \in \mathbb{R}^2 \) such that \( |r - r'| < |z| < r + r', \the angle \varphi \in (0, \pi) \) is given by \( |z|^2 = r^2 + (r')^2 - 2rr' \cos \varphi \).

Remark 5.1 Clearly \( A^+ \) is symmetric and positive. Notice also that the vectors \((e_3, 0)\) and \((z, 0), (\overline{\frac{z}{|z|}}, v_3 - v_3')\) are orthogonal on

\[
\left( \frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|}, \frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|} \right), \left( \frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|}, -\frac{\langle \overline{v}, 0 \rangle}{|\overline{v}|} \right), \left( \frac{\langle \overline{z}, 0 \rangle}{|\overline{z}|}, 0 \right), \frac{(v_3 - v_3') \left( \frac{z, 0}{|z|}, -|z|e_3 \right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}
\]

saying that

\[
A^+(e_3, 0) = A^+(z, 0), (\overline{\frac{z}{|z|}}, v_3 - v_3') = 0.
\]

Actually we have for any \( z \neq 0 \)

\[
\ker A^+(r, v_3, r', v_3', z) = \text{span} \{ (e_3, 0), (z, 0), (\overline{\frac{z}{|z|}}, v_3 - v_3') \}.
\]

A similar result can be carried out for the loss part \( Q_{FPL}^- \) (see Appendix \( \square \) for the proof).

Proposition 5.8 For any function \( f = f(x,v) \) satisfying the constraint \( T f = 0 \) we have

\[
\langle Q_{FPL}^-(f, f) \rangle = \text{div}_{\omega_3,v} \left\{ \omega_3^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}) f(x, v) \chi(|\overline{v}|, |\overline{v}'|, z) \times A^- \nabla_{\omega_3,v} f(x_1', x_2', x_3, v') \, dv' dx'_1 dx'_2 \right\}
\]
Consider a function $f = f(x, v)$ satisfying the constraint $T f = 0$. Then

\[
\begin{align*}
A^{-}(r, v_3, r', v_3') &= \frac{rr'\sin^2 \varphi}{|z|^2 + (v_3 - v_3')^2} \left( (\frac{v_3}{|v|}, \frac{1}{|v|}) \right) \otimes \left( (\frac{v_3'}{|v'|}, \frac{1}{|v'|}) \right) \\
&+ \left[ \frac{r - r' \cos \varphi}{|z|} \left( (\frac{v_3}{|v|}, \frac{1}{|v|}) \right) + \left( \frac{1}{|z|}, 0 \right) \right] \\
&\otimes \left[ \frac{r \cos \varphi - r'}{|z|} \left( (\frac{v_3'}{|v'|}, \frac{1}{|v'|}) \right) + \left( \frac{1}{|z|}, 0 \right) \right] \\
&+ \frac{rr'\sin^2 \varphi}{|z|^2} \left( (\frac{v_3}{|v|}, \frac{1}{|v|}) \right) \otimes \left( (\frac{v_3'}{|v'|}, \frac{1}{|v'|}) \right) \\
&+ \left[ \frac{(r' \cos \varphi - r)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \right] \left( (\frac{1}{|v|}, 0) \right) \frac{v_3' - \varphi}{|v'} \\
&\otimes \left[ \frac{(r - r' \cos \varphi)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \right] \left( (\frac{1}{|v|}, 0) \right) \frac{v_3'}{|v'} \\
&+ \left[ \frac{(v_3 - v_3')(x_0)}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \right] \left( (\frac{0}{|v|}, 0) \right) \frac{v_3'}{|v'} \\
&\otimes \left[ \frac{(v_3 - v_3')(z_0)}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \right] \left( (\frac{0}{|v|}, 0) \right) \frac{v_3'}{|v'}
\end{align*}
\]

where $z = (\omega, x + 1 v) - (\omega, x' + 1 v')$ and for any $r, r' \in \mathbb{R}_+, z \in \mathbb{R}^2$ such that $|r - r'| < |z| < r + r'$, the angle $\varphi \in (0, \pi)$ is given by $|z|^2 = r^2 + (r')^2 - 2rr' \cos \varphi$. For any $x = (x_1, x_2), x' = (x_1', x_2') \in \mathbb{R}^2, v, v' \in \mathbb{R}^3$ we introduce the fields

\[
\begin{align*}
\xi^1(x, v, x', v') &= \{\sigma \chi \}^{1/2} \frac{rr'\sin^2 \varphi}{|z|^2 + (v_3 - v_3')^2} \left( (\frac{v_3}{|v|}, \frac{1}{|v|}) \right) \\
\xi^2(x, v, x', v') &= \{\sigma \chi \}^{1/2} \left[ \frac{r - r' \cos \varphi}{|z|} \left( (\frac{v_3}{|v|}, \frac{1}{|v|}) \right) + \left( \frac{1}{|z|}, 0 \right) \right] \\
\xi^3(x, v, x', v') &= \{\sigma \chi \}^{1/2} \frac{r' \sin \varphi}{|z|} \left( (\frac{1}{|v|}, 0) \right) \\
\xi^4(x, v, x', v') &= \{\sigma \chi \}^{1/2} \frac{r' \sin \varphi}{|z|} \left( (\frac{1}{|v|}, 0) \right) \\
\xi^5(x, v, x', v') &= \{\sigma \chi \}^{1/2} \frac{r' \sin \varphi}{|z|} \left( (\frac{1}{|v|}, 0) \right) \\
\xi^6(x, v, x', v') &= \{\sigma \chi \}^{1/2} \frac{r' \sin \varphi}{|z|} \left( (\frac{1}{|v|}, 0) \right)
\end{align*}
\]

where $r = |v|, r' = |v'|, z = (\omega, x + 1 v) - (\omega, x' + 1 v'), \sigma = \sigma \chi \pi, z \in (0, \pi)$ is given by $|z|^2 = r^2 + (r')^2 - 2rr' \cos \varphi, |r - r'| < |z| < r + r'$. Thanks to Propositions 5.7, 5.8 we obtain the representation formula

**Proposition 5.9** Consider a function $f = f(x, v)$ satisfying the constraint $T f = 0$. Then
1. The averaged Fokker-Planck-Landau operator writes

\[
\omega_c^{-2} \langle Q_{FPL}(f, f) \rangle (x, v) = \\
\text{div}_{\omega,x,v} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sum_{i=1}^{4} f(x', x_3, v') \xi_i^1(\overline{x}, v, \overline{x'}, v') \otimes \xi_i^1(\overline{x}, v, \overline{x'}, v') \nabla_{\omega,x,v} f(x, v) \, dv' \, dx_1' \, dx_2' \right\}
\]

\[
- \text{div}_{\omega,x,v} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sum_{i=1}^{4} f(x, v) \xi_i^1(\overline{x}, v, \overline{x'}, v') \otimes \varepsilon_i \xi_i^1(\overline{x'}, v', \overline{x}, v) \nabla_{\omega,x',v'} f(\overline{x'}, x_3, v') \, dv' \, dx_1' \, dx_2' \right\}
\]

(41)

where \( \varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \varepsilon_4 = 1 \).

2. The following properties hold true for any fixed \( x_3 \in \mathbb{R} \)

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle Q_{FPL}(f, f) \rangle (x, v) \, dv \, dx_1 \, dx_2 = 0, \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} v \langle Q_{FPL}(f, f) \rangle (x, v) \, dv \, dx_1 \, dx_2 = 0
\]

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \frac{1}{2} \langle Q_{FPL}(f, f) \rangle (x, v) \, dv \, dx_1 \, dx_2 = 0, \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \ln f \langle Q_{FPL}(f, f) \rangle (x, v) \, dv \, dx_1 \, dx_2 \leq 0.
\]

**Proof.** 1. By Proposition 5.1 we know that

\[
\omega_c^{-2} \langle Q_{FPL}^-(f, f) \rangle (x, v) = \\
\text{div}_{\omega,x,v} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sum_{i=1}^{4} f(x', x_3, v') (\xi_i^1(\overline{x}, v, \overline{x'}, v') \otimes \nabla_{\omega,x,v} f(x, v) \, dv' \, dx_1' \, dx_2' \right\}
\]

Observe that we have \( \chi(r', r, -z) = \chi(r, r', z) \). Therefore the permutation \( (\overline{x}, v) \leftrightarrow (\overline{x'}, v') \) leads to

\[
\xi_1^1(\overline{x'}, v', \overline{x}, v) = \{\sigma \chi\}^{1/2} \frac{r \sin \varphi}{|z|} \left( \frac{(\overline{\nu}^0, 0)}{|\overline{\nu}'|} - \frac{1}{|\overline{\nu}'|} \right)
\]

\[
\xi_2^1(\overline{x'}, v', \overline{x}, v) = \{\sigma \chi\}^{1/2} \frac{r - r \cos \varphi}{|z|} \left( \frac{(\overline{\nu}^0, 0)}{|\overline{\nu}'|} - \frac{1}{|\overline{\nu}'|} \right) - \left( \frac{1}{|z|}, 0 \right)
\]

\[
\xi_3^1(\overline{x'}, v', \overline{x}, v) = \{\sigma \chi\}^{1/2} \frac{r \sin \varphi}{|z|} \left( \frac{(\overline{\nu}^0, 0)}{|\overline{\nu}'|} - \frac{1}{|\overline{\nu}'|} \right)
\]

\[
\xi_4^1(\overline{x'}, v', \overline{x}, v) = \{\sigma \chi\}^{1/2} \frac{r - r \cos \varphi}{|z|} \left( \frac{(\overline{\nu}^0, 0)}{|\overline{\nu}'|} - \frac{1}{|\overline{\nu}'|} \right) - \left( \frac{1}{|z|}, 0 \right)
\]

where \( z = (\omega_c \overline{x} + \frac{1}{2} \overline{\nu}) - (\omega_c \overline{x'} + \frac{1}{2} \overline{\nu}') \). By Proposition 5.3 one gets

\[
\omega_c^{-2} \langle Q_{FPL}^-(f, f) \rangle (x, v) = \\
\text{div}_{\omega,x,v} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sum_{i=1}^{4} f(x, v) \xi_i^1(\overline{x}, v, \overline{x'}, v') \otimes \varepsilon_i \xi_i^1(\overline{x'}, v', \overline{x}, v) \nabla_{\omega,x',v'} f(\overline{x'}, x_3, v') \, dv' \, dx_1' \, dx_2' \right\}
\]

(43)
2. The mass, third momentum component and kinetic energy balances for the averaged Fokker-Planck-Landau operator come by the corresponding properties of the Fokker-Planck-Landau kernel. Indeed, since $1, v_3, |v|^2$ belong to $\ker T$, we can write for any $x_3 \in \mathbb{R}$, thanks to Remark 2.1

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \left\{1, v_3, \frac{|v|^2}{2} \right\} \langle Q_{FPL}(f, f) \rangle \, dv \, dx_2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \left\{1, v_3, \frac{|v|^2}{2} \right\} Q_{FPL}(f, f) \, dv \, dx_2 = (0, 0, 0).$$

Similarly since $\ln f \in \ker T$ one gets

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \ln f \langle Q_{FPL}(f, f) \rangle \, dv \, dx_2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \ln f Q_{FPL}(f, f) \, dv \, dx_2 (44) = -\frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma(|v - v'|) f(x, v) f(x, v') \frac{(v - v') \wedge (\nabla_v \ln f - \nabla_{v'} \ln f)^2}{|v - v'|^2} \, dv' dv \, dx_2 \leq 0.$$

Finally observe that $\langle v_1 \rangle = \langle v_2 \rangle = 0$, $\langle Q_{FPL}(f, f) \rangle \in \ker T$ and thus trivially

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} (v_1, v_2) \langle Q_{FPL}(f, f) \rangle \, dv \, dx_2 = (0, 0).$$

We establish formally the limit model stated in Theorem 1.3.

**Proof.** (of Theorem 1.3) Plugging the Ansatz $f^\varepsilon = f + \varepsilon f^1 + \varepsilon^2 f^2 + ...$ into (35) we obtain

$$\left( \partial_t + v_3 \partial_{x_3} + \frac{q}{m} \mathbf{E} \cdot \nabla_v + \frac{1}{\varepsilon} T \right) (f + \varepsilon f^1 + ...) = Q_{FPL}(f, f) + \varepsilon (Q_{FPL}(f, f^1) + Q_{FPL}(f^1, f)) + ...$$

implying that

$$T f = 0, \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} \mathbf{E} \cdot \nabla_v f + T f^1 = Q_{FPL}(f, f).$$

Applying the average operator, we deduce by Propositions 3.4, 5.9 that $f$ satisfies

$$\partial f + \frac{\langle \frac{1}{B} \mathbf{E} \rangle}{B} \cdot \nabla f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{e_3} f = \langle Q_{FPL} \rangle (f, f).$$

\[\Box\]
As for the relaxation Boltzmann operator, we are searching for extensions of the averaged Fokker-Planck-Landau operator to the whole space of densities \( f = f(x, v) \), not necessarily in the kernel of \( T \). One possibility is to consider the extension \( \langle Q_{\text{FPL}} \rangle \) obtained thanks to (41), that is for any \( f \)

\[
\omega_c^{-2} \langle Q_{\text{FPL}} \rangle (f, f) = \\
\operatorname{div}_{\omega, x, v} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sum_{i=1}^4 f(x, x_i, v') \xi^i(\bar{x}, v, \bar{x}, v') \otimes \xi^i(\bar{x}, v, \bar{x}, v') \nabla_{\omega, x, v} f(x, v) \, dv' \, dx'_1 \, dx'_2 \right\} \\
- \operatorname{div}_{\omega, x, v} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sum_{i=1}^4 f(x, v) \xi^i(\bar{x}, v, \bar{x}, v') \otimes \varepsilon_i \xi^i(\bar{x}, v', \bar{x}, v) \nabla_{\omega, x', v'} f(\bar{x}, x, v') \, dv' \, dx'_1 \, dx'_2 \right\}.
\]

What is remarkable is that this extension still satisfies the mass, third momentum component, kinetic energy balances and decreases the entropy \( f \ln f \), globally in \((\bar{x}, v)\).

**Proposition 5.10** Consider two functions \( f = f(x, v), \varphi = \varphi(x, v) \). For any \( x_3 \in \mathbb{R} \) we have

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle Q_{\text{FPL}} \rangle (f, f) \, dv \, dx_1 \, dx_2 = -\frac{\omega_c^2}{2} \times 46
\]

\[
\sum_{i=1}^4 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^4} f' f' \xi^i \cdot \nabla \ln f - \varepsilon_i (\xi^i)' \nabla' \ln f' (\xi^i) \cdot \nabla \varphi - \varepsilon_i (\xi^i)' \nabla' \varphi' \, dv' \, dx'_1 \, dx'_2 \, dv \, dx_1 \, dx_2
\]

where

\[
f = f(x, v), \quad f' = f'(x_1', x'_2, x_3, v'), \quad \nabla \varphi = \nabla_{\omega, v, v'} \varphi(x, v), \quad \nabla' \varphi' = \nabla_{\omega, x', v'} \varphi(x_1', x'_2, x_3, v')
\]

\[
\xi^i = \xi^i(x_1, x_2, v, x'_1, x'_2, v'), \quad (\xi^i)' = \xi^i(x_1', x'_2, v', x_1, x_2, v).
\]

In particular the averaged Fokker-Planck-Landau operator satisfies the mass, third momentum component, kinetic energy balances (globally in \((x_1, x_2, v)\))

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \left( 1, v_3, \frac{|v|^2}{2} \right) \langle Q_{\text{FPL}} \rangle (f, f) \, dv \, dx_1 \, dx_2 = (0, 0, 0)
\]

and decreases the entropy \( f \ln f \) (globally in \((x_1, x_2, v)\))

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \ln f \langle Q_{\text{FPL}} \rangle (f, f) \, dv \, dx_1 \, dx_2 \leq 0.
\]
Proof. Notice that for any $1 \leq i \leq 4$ we have $\xi^i \cdot (e_3, 0) = 0$ and therefore the operator $\text{div}_{\omega, x, v}$ acts only in $(x_1, x_2, v)$. Thus, for any fixed $x_3 \in \mathbb{R}$ we can perform integration by parts with respect to $(x_1, x_2, v)$.

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle Q_{FPL} \rangle (f, f) \varphi \, dv \, dx_1 \, dx_2 = -\sum_{i=1}^{4} \omega_i^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f \, f' \times \left\{ (\xi^i \cdot \nabla \varphi)(\xi^i \cdot \nabla f) - \varepsilon_i (\xi^i \cdot \nabla \varphi)((\xi^i)' \cdot \nabla' \ln f') \right\} \, dv' \, dx'_1 \, dx'_2 \, dv \, dx_1 \, dx_2. \tag{47}
\]

Performing the change of variables $(x'_1, x'_2, v') \leftrightarrow (x_1, x_2, v)$ yields

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle Q_{FPL} \rangle (f, f) \varphi \, dv \, dx_1 \, dx_2 = -\sum_{i=1}^{4} \omega_i^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f \, f' \times \left\{ ((\xi^i)' \cdot \nabla' \varphi')( (\xi^i)' \cdot \nabla' \ln f') - \varepsilon_i((\xi^i)' \cdot \nabla' \varphi')(\xi^i \cdot \nabla \ln f) \right\} \, dv \, dx_1 \, dx_2 \, dv' \, dx'_1 \, dx'_2. \tag{48}
\]

Combining (47), (48) one gets by Fubini theorem

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle Q_{FPL} \rangle (f, f) \varphi \, dv \, dx_1 \, dx_2 = -\frac{\omega_i^2}{2} \sum_{i=1}^{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f \, f' T^i \, dv' \, dx'_1 \, dx'_2 \, dv \, dx_1 \, dx_2 \tag{49}
\]

where

\[
T^i = (\xi^i \cdot \nabla \varphi - \varepsilon_i (\xi^i)' \cdot \nabla' \varphi') (\xi^i \cdot \nabla \ln f - \varepsilon_i (\xi^i)' \cdot \nabla' \ln f'), \quad 1 \leq i \leq 4.
\]

Clearly, the divergence form of $\langle Q_{FPL} \rangle$ guarantees the mass conservation and (46) applied with $\varphi = \ln f$ ensures that the entropy $f \ln f$ is decreasing

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \ln f \, \langle Q_{FPL} \rangle (f, f) \, dv \, dx_1 \, dx_2 = -\frac{\omega_i^2}{2} \sum_{i=1}^{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} f \, f' \times (\xi^i \cdot \nabla \ln f - \varepsilon_i (\xi^i)' \cdot \nabla' \ln f')^2 \, dv' \, dx'_1 \, dx'_2 \, dv \, dx_1 \, dx_2 \leq 0, \ x_3 \in \mathbb{R}.
\]

It remains to show the kinetic energy and third momentum component balances. Thanks to formula (46) it is sufficient to check that

\[
\xi^i \cdot \nabla \frac{|v|^2}{2} - \varepsilon_i (\xi^i)' \cdot \nabla' \frac{|v'|^2}{2} = 0, \quad 1 \leq i \leq 4.
\]

The above condition is trivially satisfied for $i \in \{1, 2\}$. For $i = 3$ we have

\[
\xi^3 \cdot \nabla \frac{|v|^2}{2} - \varepsilon_3 (\xi^3)' \cdot \nabla' \frac{|v'|^2}{2} = -\left\{ \sigma \chi \right\}^{1/2} r' \sin \varphi \frac{\varphi}{|z|} r + \left\{ \sigma \chi \right\}^{1/2} r \sin \varphi \frac{r}{|z|} \varphi' = 0.
\]
Finally, when $i = 4$ we obtain
\[
\xi^4 \cdot \nabla \frac{|v|^2}{2} - \varepsilon_i(\xi^4)' \cdot \nabla' \frac{|v'|^2}{2}
= \{\sigma \chi\}^{1/2} \left\{ \frac{(r' \cos \varphi - r)(v_3 - v'_3)r + |z|^2v_3}{|z|\sqrt{|z|^2 + (v_3 - v'_3)^2}} + \frac{(r \cos \varphi - r')(v'_3 - v_3)r' + |z|^2v'_3}{|z|\sqrt{|z|^2 + (v_3 - v'_3)^2}} \right\}
= \{\sigma \chi\}^{1/2} \frac{v_3 - v'_3}{\sqrt{|z|^2 + (v_3 - v'_3)^2}} \left[ r^2 + (r')^2 - 2rr' \cos \varphi \right] = 0.
\]
Notice also that for any $i \in \{1, 2, 3, 4\}$ we have
\[
\xi^i \cdot \nabla v_3 - \varepsilon_i(\xi^i)' \cdot \nabla' v'_3 = 0
\]
saying that $\langle Q_{FPL} \rangle$ satisfies
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} v_3 \langle Q_{FPL} \rangle (f, f) \, dv \, dx_1 \, dx_2 = 0.
\]

Remark 5.2 By the formula (46) we deduce that the positive smooth functions $f$ satisfying $\langle Q_{FPL} \rangle (f, f) = 0$ are those verifying
\[
\xi^i \cdot \nabla \ln f - \varepsilon_i(\xi^i)' \cdot \nabla' \ln f' = 0, \quad i \in \{1, 2, 3, 4\}.
\]
In particular (49) holds true for any Maxwellian $f$ which belongs to $\ker T$, since in that case
\[
\langle Q_{FPL} \rangle (f, f) = \langle Q_{FPL}(f, f) \rangle = 0 = 0.
\]
We deduce that
\[
\xi^i \cdot \nabla \varphi - \varepsilon_i(\xi^i)' \cdot \nabla' \varphi' = 0, \quad i \in \{1, 2, 3, 4\}
\]
for any function $\varphi(x, v) = \alpha(x)|v|^2 + \beta(x) \cdot v + \gamma(x)$ satisfying $T \varphi = 0$, and in particular for the functions
\[
x_1 + \frac{v_2}{\omega_c}, \quad x_2 - \frac{v_1}{\omega_c}, \quad x_3, \quad v_3, \quad |\mathbf{x}|^2 + 2\mathbf{x} \cdot \frac{1}{\omega_c} \quad \xi = \mathbf{x} + \frac{1}{\omega_c} \quad |\mathbf{p}|^2 = \frac{1}{\omega_c} - \frac{|\mathbf{p}|^2}{\omega_c^2}.
\]
Notice that $\mathbf{x} + \frac{1}{\omega_c}$ is the center of the circle obtained by projecting the Larmor circle $C_{x,v}$ onto $(x'_1, x'_2)$ and $|\mathbf{x}|^2 + 2\mathbf{x} \cdot \frac{1}{\omega_c}$ is the power of the origin $(0, 0)$ with respect to the same circle. We obtain, thanks to (46) and (50), the balances
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \left( \mathbf{x} + \frac{1}{\omega_c} \right) \langle Q_{FPL} \rangle (f, f) \, dv \, dx_1 \, dx_2 = (0, 0)
\]
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \left( |\vec{\tau}|^2 + 2\vec{\tau} \cdot \frac{\vec{\tau}}{\omega_c} \right) \langle \mathcal{Q}_{FPL} \rangle(f, f) \, dv \, dx_1 \, dx_2 = 0
\]

for any smooth function \( f \).

The previous identities allow us to establish the mass, momentum, total energy, Larmor circle center and power conservations for smooth solutions of (14), (15) coupled with the Poisson equation for the electric field

\[
E = -\nabla \phi, \quad \varepsilon_0 \operatorname{div} \mathbf{E}(t, x) = q \int_{\mathbb{R}^3} f(t, x, v) \, dv =: \rho(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3. \quad (51)
\]

**Theorem 5.1** Assume that \((f, E)\) is a smooth solution for

\[
\partial_t f + \frac{\langle \frac{1}{B} \mathbf{E} \rangle}{B} \nabla_x f + v_3 \partial_{x_3} f + \frac{q}{m} (E_3) \partial_{v_3} f = \langle \mathcal{Q}_{FPL} \rangle(f, f), \quad \mathcal{T} f = 0, \quad (t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3
\]

\[
\varepsilon_0 \operatorname{div}_x E(t, x) = q \int_{\mathbb{R}^3} f(t, x, v) \, dv, \quad E = -\nabla \phi, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3.
\]

Then we have

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \, dv \, dx = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} x \, dv \, dx = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \, dv \, dx = 0
\]

\[
\frac{d}{dt} \left\{ \frac{\varepsilon_0}{2} \int_{\mathbb{R}^3} |E|^2 \, dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|v|^2}{2} f \, dv \, dx \right\} = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \frac{|\vec{\tau}|^2}{2} + \vec{\tau} \cdot \frac{\vec{\tau}}{\omega_c} \right) f \, dv \, dx = 0.
\]

**Proof.** The mass conservation comes by the conservative formulation of the Vlasov equation, which writes

\[
\partial_t f + \operatorname{div} \left\{ f \left( \frac{\langle \mathbf{E} \rangle}{B} \right) \right\} + \partial_{x_3} \{ f v_3 \} + \frac{q}{m} \partial_{v_3} \{ f \langle E_3 \rangle \} = \langle \mathcal{Q}_{FPL} \rangle(f, f). \quad (52)
\]

Multiplying (52) by \( v \) and integrating with respect to \((x, v)\) yield

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v f \, dv \, dx - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{q}{m} f \langle E_3 \rangle e_3 \, dv \, dx = 0
\]

since for functions in \( \ker \mathcal{T} \) we can write

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \langle \mathcal{Q}_{FPL} \rangle(f, f) \, dv \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v (\langle \mathcal{Q}_{FPL}(f, f) \rangle \, dv \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (v) \mathcal{Q}_{FPL}(f, f) \, dv \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (0, 0, v_3) \mathcal{Q}_{FPL}(f, f) \, dv \, dx = 0.
\]
Using the Poisson equation and the identity
\[
\text{div}_x E \cdot E = \text{div}_x (E \otimes E) - \frac{1}{2} \nabla_x |E|^2
\]  
(53)
we obtain, taking into account that \( f \in \ker \mathcal{T} \)
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{q}{m} f \langle E_3 \rangle \ dv \ dx = \frac{q}{m} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f E_3 \ dv \ dx = \frac{\varepsilon_0}{m} \int_{\mathbb{R}^3} E_3 \text{div}_x E \ dx
\]
\[
= \frac{\varepsilon_0}{m} \int_{\mathbb{R}^3} \left\{ \text{div}_x (E_3 E) - \frac{1}{2} \partial_{x_3} |E|^2 \right\} \ dx = 0
\]
and thus \( \frac{d}{dt} \int_{\mathbb{R}^3} v f \ dv \ dx = 0 \). Multiplying (52) by \( \frac{mv}{2} \) and integrating with respect to \((x,v)\) yield
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} m|v|^2 f \ dv \ dx - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} qfv_3 \langle E_3 \rangle \ dv \ dx = 0.
\]
Thanks to the continuity equation
\[
\partial_t \int_{\mathbb{R}^3} f \ dv + \text{div}_x \int_{\mathbb{R}^3} f \frac{\langle \frac{1}{|E|} \rangle}{B} \ dv + \partial_{x_3} \int_{\mathbb{R}^3} f v_3 \ dv = 0
\]
we have
\[
- q \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f v_3 \langle E_3 \rangle \ dv \ dx = - q \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f v_3 E_3 \ dv \ dx = q \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f v_3 \partial_{x_3} \phi \ dv \ dx
\]
\[
= - q \int_{\mathbb{R}^3} \phi \partial_{x_3} \int_{\mathbb{R}^3} f v_3 \ dv \ dx = \int_{\mathbb{R}^3} \phi \left\{ \partial_t \rho + q \text{div}_x \int_{\mathbb{R}^3} f \frac{\langle \frac{1}{|E|} \rangle}{B} \ dv \right\} \ dx
\]
\[
= \int_{\mathbb{R}^3} \phi \partial_t (\varepsilon_0 \text{div}_x E) \ dx - q \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_x \phi \cdot \frac{\langle \frac{1}{|E|} \rangle}{B} f \ dv \ dx
\]
\[
= - \int_{\mathbb{R}^3} \nabla_x \phi \cdot \partial_t (\varepsilon_0 E) \ dx + q \int_{\mathbb{R}^3} \langle E \rangle \cdot \frac{\langle \frac{1}{|E|} \rangle}{B} f \ dx
\]
\[
= \frac{\varepsilon_0}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |E|^2 \ dx
\]
implying
\[
\frac{d}{dt} \left\{ \frac{\varepsilon_0}{2} \int_{\mathbb{R}^3} |E|^2 \ dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} m|v|^2 f \ dv \ dx \right\} = 0.
\]
By Remark [5.2] we know that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{T} \langle Q_{FPL} \rangle (f, f) \ dv \ dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \mathcal{T} \frac{1}{\omega_c} + \frac{1}{\omega_c} \right) \langle Q_{FPL} \rangle (f, f) \ dv \ dx
\]
\[
- \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{\omega_c} \langle Q_{FPL} \rangle (f, f) \ dv \ dx = 0
\]
and therefore, multiplying (52) by \( \vec{x} \) one gets
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \vec{x} \, dv \, dx - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \left( \frac{\vec{E}}{B} \right) \, dv \, dx = 0.
\]

Appealing to the Poisson equation we have
\[
q \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \left( \frac{\vec{E}}{\mu_0} \right) \, dv \, dx = q \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \vec{E} \, dv \, dx = \int_{\mathbb{R}^3} \rho \vec{E} = \varepsilon_0 \int_{\mathbb{R}^3} \text{div}_x \vec{E} \vec{E} \, dx = \varepsilon_0 \int_{\mathbb{R}^3} \left( \text{div}_x \vec{E} \otimes \vec{E} - \frac{1}{2} \nabla |\vec{E}|^2 \right) \, dx = 0
\]
and therefore \( \frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \vec{x} \, dv \, dx = 0 \). In particular the mean Larmor circle center is left invariant
\[
\frac{d}{dt} \left\{ \frac{\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \left( \frac{\vec{x} + \frac{\vec{E}}{\omega} \cdot \omega} {\omega_c} \right) \, dv \, dx}{\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \, dv \, dx} \right\} = 0.
\]

By Remark 5.2 we know that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \omega_c^2 |\vec{x}|^2 + 2 \omega_c \vec{x} \cdot \frac{1}{\omega} \right) \langle Q_{FP} \rangle (f, f) \, dv \, dx = 0
\]
and for any \( \psi(x) \in C^1(\mathbb{R}^3) \) we can write
\[
\int_{\mathbb{R}^3} \psi(x) \text{div} \vec{x} \int_{\mathbb{R}^3} f \vec{x} \, dv \, dx - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \vec{x} \cdot \nabla \psi \, dv \, dx
\]
\[
= -\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \, \mathcal{T} \psi \, dv \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi \, \mathcal{T} f \, dv \, dx = 0
\]
saying that \( \text{div}_x \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \vec{x} \, dv \, dx = 0 \). Therefore we deduce
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \frac{|\vec{x}|^2}{2} + \vec{x} \cdot \frac{1}{\omega} \right) f \, dv \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \left( \frac{\vec{E}}{B} \right) \cdot \left( \vec{x} + \frac{1}{\omega} \right) \, dv \, dx
\]
\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \left( \frac{\vec{E}}{B} \right) \cdot \left( \vec{x} + \frac{1}{\omega} \right) \, dv \, dx = -\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \vec{x} \vec{E} \, dv \, dx + \int_{\mathbb{R}^3} \frac{1}{\omega} \vec{x} \vec{E} \, dv \, dx + \int_{\mathbb{R}^3} \vec{E} \cdot \nabla_x \int_{\mathbb{R}^3} f \vec{x} \, dv \, dx
\]
\[
= -\frac{\varepsilon_0}{qB} \int_{\mathbb{R}^3} \vec{E} \cdot \nabla_x \int_{\mathbb{R}^3} f \vec{x} \, dv \, dx = -\frac{\varepsilon_0}{qB} \int_{\mathbb{R}^3} \vec{x} \left( \text{div}_x (\vec{E} \otimes \vec{E}) - \frac{1}{2} \nabla |\vec{E}|^2 \right) \, dx
\]
\[
= -\frac{\varepsilon_0}{qB} \int_{\mathbb{R}^3} \sum_{j=1}^{3} \sum_{j=1}^{3} \left( \frac{\partial x_j}{\partial y_j} E_1 E_j - \frac{\partial y_j}{\partial y_j} E_2 E_j \right) \, dx
\]
\[
= 0
\]
implying that the mean Larmor circle power (with respect to the origin) is left invariant.
A Proof of Theorem 1.2

Proof. The Fokker-Planck kernel being a second order differential operator, we appeal twice to Proposition 3.3. For any \( f \in \ker T \), taking \( \xi = M \nabla_v \left( \frac{f}{M} \right) \) yields

\[
\left\langle \text{div}_v \left( M \nabla_v \left( \frac{f}{M} \right) \right) \right\rangle = \text{div}_{\omega, v} \left\{ \left\langle M \nabla_{\frac{|v|}{|\nabla v|}} \left( \frac{f}{M} \right) \right\rangle + \left\langle M \nabla_{\frac{|v|}{|\nabla v|}} \left( \frac{f}{M} \right) \cdot \frac{\nabla v}{|\nabla v|} \right\rangle - \left\langle M \nabla_{\frac{|v|}{|\nabla v|}} \left( \frac{f}{M} \right) \cdot \frac{\nabla v}{|\nabla v|} \right\rangle \right\} + \partial_{v_3} \left\langle M \partial_{v_3} \left( \frac{f}{M} \right) \right\rangle.
\]

Since \( \partial_{v_3} \) commutes with \( \langle \cdot \rangle \) (cf. Proposition 3.2) we deduce

\[
\partial_{v_3} \left\langle M \partial_{v_3} \left( \frac{f}{M} \right) \right\rangle = \partial_{v_3} \left\{ M \left\langle \partial_{v_3} \left( \frac{f}{M} \right) \right\rangle \right\} = \partial_{v_3} \left\{ M \partial_{v_3} \left( \frac{f}{M} \right) \right\}.
\]

It remains to compute the averages

\[
\left\langle M \nabla_{\frac{|v|}{|\nabla v|}} \left( \frac{f}{M} \right) \right\rangle, \left\langle M \nabla_{\frac{|v|}{|\nabla v|}} \left( \frac{f}{M} \right) \cdot \frac{\nabla v}{|\nabla v|} \right\rangle, \left\langle M \nabla_{\frac{|v|}{|\nabla v|}} \left( \frac{f}{M} \right) \cdot \frac{\nabla v}{|\nabla v|} \right\rangle.
\]

These averages come easily, thanks to Proposition 3.3, observing that

\[
\partial_{v_1} \left( \frac{f}{M} \right) = \text{div}_v \left( \frac{f}{M}, 0, 0 \right), \partial_{v_2} \left( \frac{f}{M} \right) = \text{div}_v \left( 0, \frac{f}{M}, 0 \right),
\]

\[
\nabla_{\frac{|v|}{|\nabla v|}} \left( \frac{f}{M} \right) \cdot \frac{-\nabla v}{|\nabla v|} = \text{div}_v \left( \frac{f}{M} \cdot \frac{-\nabla v}{|\nabla v|} \right), \nabla_{\frac{|v|}{|\nabla v|}} \left( \frac{f}{M} \right) \cdot \frac{\nabla v}{|\nabla v|} = \text{div}_v \left( \frac{f}{M} \cdot \frac{\nabla v}{|\nabla v|} \right) - 2 \frac{f}{M}.
\]

We obtain

\[
\left\langle M \nabla_{\frac{|v|}{|\nabla v|}} \left( \frac{f}{M} \right) \right\rangle = M \nabla_{\omega, v} \left( \frac{f}{M} \right),
\]

\[
\left\langle M \nabla_{\frac{|v|}{|\nabla v|}} \left( \frac{f}{M} \right) \cdot \frac{\nabla v}{|\nabla v|} \right\rangle = M \left( \frac{(\nabla v, 0)}{|\nabla v|}, \frac{1}{|\nabla v|} \right) \cdot \nabla_{\omega, x, v} \left( \frac{f}{M} \right),
\]

\[
\left\langle M \nabla_{\frac{|v|}{|\nabla v|}} \left( \frac{f}{M} \right) \cdot \frac{-\nabla v}{|\nabla v|} \right\rangle = -M \left( \frac{(\nabla v, 0)}{|\nabla v|}, \frac{-\nabla v}{|\nabla v|} \right) \cdot \nabla_{\omega, x, v} \left( \frac{f}{M} \right).
\]

Our conclusion follows by combining (54), (55), (56), (57). The diffusion matrix \( L \) is positive and for any \( \xi = (\xi_x, \xi_v) \in \mathbb{R}^6 \) we have

\[
L \xi \cdot \xi = |\xi_x|^2 + |\xi_v - \frac{1}{|\nabla v|} \xi_v|^2 + (\xi_{v_3})^2 \geq 0
\]

with equality iff \( \xi_x = \xi_v = (0, 0) \) and \( \xi_{v_3} = 0 \).
B Proofs of Propositions 5.3, 5.6

Proof. (of Proposition 5.3) We follow the same arguments as those in the proof of Proposition 4.2. The details are left to the reader.

1. Using cylindrical coordinates we obtain

\[
J := \left\langle \int_{\mathbb{R}^3} D(v, v') f(x, v') \, dv' \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \int_{\mathbb{R}^+} D(\|v\|e^{i\alpha}, v_3, r'e^{i(\varphi+\alpha)}, v'_3) \times f \left( \pi + \frac{1}{\omega_c} - \frac{\|v\|e^{i\alpha}}{\omega_c}, x_3, r'e^{i(\varphi+\alpha)}, v'_3 \right) \, r'dr'd\varphi dv'_3 d\alpha
\]

where in the last equality we have used the constraint \( f \in \ker \mathcal{T} \) i.e., there is \( g \) such that

\[
f(x, v) = g \left( \pi + \frac{1}{\omega_c}, x_3, \|v\|, v_3 \right), \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3.
\]

We have \( r'e^{i(\varphi+\alpha)} - \|v\|e^{i\alpha} = le^{i(\psi+\alpha)} \) where \( l^2 = r^2 + (r')^2 - 2rr'\cos \varphi, \ r = \|v\| \) and \((r')^2 = r^2 + l^2 + 2rl \cos \psi\). Notice that \( \psi \in (0, \pi) \) if \( \varphi \in (0, \pi) \) and \( \psi \in (-\pi, 0) \) if \( \varphi \in (-\pi, 0) \). Also \( \psi = \psi(\varphi) \) is odd with respect to \( \varphi \) that is \( \psi(-\varphi) = -\psi(\varphi) \). By hypothesis we deduce that

\[
D(\re^{i\alpha}, v_3, r'e^{i(\varphi+\alpha)}, v'_3) = \tilde{D}(r, v_3, r', v'_3, \varphi) \re^{i\alpha} + \tilde{D}'(r, v_3, r', v'_3, \varphi) \r'e^{i(\varphi+\alpha)}
\]

and thus, if we denote by \( R(\alpha) \) the rotation of angle \( \alpha \) in \( \mathbb{R}^2 \) one gets

\[
J = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \int_{\mathbb{R}^+} R(\alpha + \psi)[\tilde{D}(r, v_3, r', v'_3, \varphi) \re^{-i\psi} + \tilde{D}'(r, v_3, r', v'_3, \varphi) \r'e^{i(\varphi+\psi)}] \times g \left( \pi + \frac{1}{\omega_c} - \frac{\|l\e^{i(\alpha+\psi)}\}}{\omega_c}, x_3, r', v'_3 \right) \, r'dr'd\varphi dv'_3 d\alpha
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \int_{\mathbb{R}^+} R(\alpha)[\tilde{D}(r, v_3, r', v'_3, \varphi) \re^{-i\psi} + \tilde{D}'(r, v_3, r', v'_3, \varphi) \r'e^{i(\varphi+\psi)}] \times g \left( \pi + \frac{1}{\omega_c} - \frac{\|l\e^{i\alpha}\}}{\omega_c}, x_3, r', v'_3 \right) \, r'dr'd\varphi dv'_3 d\alpha.
\]
Using the symmetry of $\psi$ with respect to $\varphi$ and changing $\varphi$ against $l$ yield

$$J = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \int_0^\pi R(\alpha)[\tilde{D}(r, v_3, r', v'_3, \varphi) \, r e^{-i\varphi} + \tilde{D}(r, v_3, r', v'_3, -\varphi) \, r e^{i\varphi}$$

$$+ \tilde{D}'(r, v_3, r', v'_3, \varphi) \, r' e^{i(\varphi - \varphi)} + \tilde{D}'(r, v_3, r', v'_3, -\varphi) \, r' e^{-i(\varphi - \varphi)}]

$$

$$\times g(\overline{\tau} + \frac{1}{\omega_c} \{le^{ia}\}, x_3, r', v'_3) r' d\varphi' d\psi_d\alpha$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^\pi \int_0^\pi R(\alpha)[\tilde{D}(r, v_3, r', v'_3, \varphi) \, r e^{-i\varphi} + \tilde{D}(r, v_3, r', v'_3, -\varphi) \, r e^{i\varphi}$$

$$+ \tilde{D}'(r, v_3, r', v'_3, \varphi) \, r' e^{i(\varphi - \varphi)} + \tilde{D}'(r, v_3, r', v'_3, -\varphi) \, r' e^{-i(\varphi - \varphi)}]

$$

$$\times g(\overline{\tau} + \frac{1}{\omega_c} \{le^{ia}\}, x_3, r', v'_3) \frac{r' d\varphi' d\psi_d\alpha}{\sqrt{l^2 - (r - r')^2 \sqrt{(r + r')^2 - l^2}}}.$$

Notice that $R(\alpha) = e_1 \otimes e^{-ia} - e_2 \otimes \mathbb{1} e^{-ia}$ and for any $\alpha' \in [0, 2\pi)$ we have

$$g(\overline{\tau} + \frac{1}{\omega_c} \{le^{ia}\}, x_3, r', v'_3) = f(\overline{\tau} + \frac{1}{\omega_c} \{le^{ia}\}, x_3, r'e^{ia'}, v'_3).$$

Performing the change of coordinates $v' = (r'e^{ia'}, v'_3)$ and $-z = \frac{1}{2}\{le^{ia}\}$ leads to

$$J = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^\pi \int_0^\pi \int_0^\pi \{e_1 \otimes e^{-ia} - e_2 \otimes \mathbb{1} e^{-ia}\}[\tilde{D}(\varphi) \, r e^{-i\varphi} + \tilde{D}(-\varphi) \, r e^{i\varphi}$$

$$+ \tilde{D}'(\varphi) \, r' e^{i(\varphi - \varphi)} + \tilde{D}'(-\varphi) \, r' e^{-i(\varphi - \varphi)}]

$$

$$\times f(\overline{\tau} + \frac{1}{\omega_c} \{le^{ia}\} - \frac{1}{\omega_c} \{r' e^{ia'}\}, x_3, r'e^{ia'}, v'_3) \chi(r, r', -\frac{1}{2}\{le^{ia}\})r' d\varphi' d\psi_d\alpha' d\varphi_d\alpha$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{D}(|\overline{\tau}|, v_3, |v'|, v'_3, z) f(\overline{\tau} + \frac{1}{\omega_c} \{le^{ia}\} - \frac{1}{\omega_c} \{v'|\}, x_3, v') \, dv' dz$$

$$= \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{D}(|\overline{\tau}|, v_3, |v'|, v'_3, (\omega_c \overline{\tau} + \frac{1}{\omega_c} v') - (\omega_c v' + \frac{1}{\omega_c} \overline{\tau})) f(x_1', x_2', x_3, v') \, dv' dx_1' dx_2'.$$

The statement in \ref{2.}. follows similarly. \hfill \Box

**Proof.** (of Proposition \ref{5.6}) Observe that

$$\langle f \rangle_{\sigma S} = \left\langle \int_{\mathbb{R}^3} f(x, v') \sigma(|v - v'|) \, dv' \right\rangle I - \left\langle \int_{\mathbb{R}^3} f(x, v') \sigma(|v - v'|) \frac{(v - v') \otimes (v - v')}{|v - v'|^2} \, dv' \right\rangle.$$

Applying Proposition \ref{12} with $C = 1$ one gets $\langle \int_{\mathbb{R}^3} f \sigma \, dv' \rangle = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sigma f \chi \, dv' dx_1' dx_2'$.

It remains to compute the second average. Using cylindrical coordinates and the con-
For any constraint $Tf = 0$ i.e., $f(x, v) = g\left(\overline{x} + \frac{1}{\omega_c} r, x_3, |\overline{v}|, v_3\right)$ we obtain

$$K := \left\langle \int_{\mathbb{R}^3} f(x, v') \sigma(|v - v'|) \frac{(v - v') \otimes (v - v')}{|v - v'|^2} \, dv' \right\rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \int_{\mathbb{R}_+} \sigma\left(\frac{\sqrt{\overline{v}^2 + (v_3 - v'_3)^2}}{\overline{v}}\right) \left(\frac{(le^{i(\alpha + \psi)}, v_3 - v'_3)^2}{l^2 + (v_3 - v'_3)^2}\right)$$

$$\times f\left(\overline{x} + \frac{1}{\omega_c} r, x_3, r' e^{i(\varphi + \alpha)}, v_3\right) r' dr' d\varphi dv_3 \, d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \int_{\mathbb{R}_+} \sigma\left(\frac{\sqrt{\overline{v}^2 + (v_3 - v'_3)^2}}{\overline{v}}\right) \left(\frac{(le^{i(\alpha + \psi)}, v_3 - v'_3)^2}{l^2 + (v_3 - v'_3)^2}\right)$$

$$\times g\left(\overline{x} + \frac{1}{\omega_c} r, x_3, r', v_3\right) r' dr' d\varphi dv_3 \, d\alpha.$$

We introduce $l, \psi$ such that $r'e^{i(\varphi + \alpha)} - |\overline{v}| e^{i\alpha} = le^i(\psi + \alpha)$. We have the relations $l^2 = r^2 + (r')^2 - 2rr' \cos \varphi$, $r = |\overline{v}|$ and $(r')^2 = r^2 + l^2 + 2rl \cos \psi$. Since $l, \psi$ are not depending on $\alpha$ one gets

$$K = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \int_{\mathbb{R}_+} \sigma\left(\frac{\sqrt{l^2 + (v_3 - v'_3)^2}}{l^2 + (v_3 - v'_3)^2}\right)$$

$$\times g\left(\overline{x} + \frac{1}{\omega_c} r, x_3, r', v_3\right) r' dr' d\varphi dv_3 \, d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \int_{\mathbb{R}_+} \sigma\left(\frac{\sqrt{l^2 + (v_3 - v'_3)^2}}{l^2 + (v_3 - v'_3)^2}\right)$$

$$\times g\left(\overline{x} + \frac{1}{\omega_c} r, x_3, r', v_3\right) r' dr' d\varphi dv_3 \, d\alpha.$$

Since $l$ is even with respect to $\varphi$ we obtain, after changing $\varphi$ against $l$

$$K = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \int_{\mathbb{R}_+} \sigma\left(\frac{\sqrt{l^2 + (v_3 - v'_3)^2}}{l^2 + (v_3 - v'_3)^2}\right)$$

$$\times g\left(\overline{x} + \frac{1}{\omega_c} r, x_3, r', v_3\right) r' dr' d\varphi dv_3 \, d\alpha$$

$$= \frac{2}{\pi} \int_0^{2\pi} \int_{|r - r'|}^{(r + r')^2} \int_{\mathbb{R}_+} \sigma\left(\frac{\sqrt{l^2 + (v_3 - v'_3)^2}}{l^2 + (v_3 - v'_3)^2}\right)$$

$$\times g\left(\overline{x} + \frac{1}{\omega_c} r, x_3, r', v_3\right) r' dr' dl \, d\varphi dv_3 \, d\alpha$$

For any $\alpha' \in [0, 2\pi)$ we have

$$g\left(\overline{x} + \frac{1}{\omega_c} r, x_3, r', v_3\right) = f\left(\overline{x} + \frac{1}{\omega_c} r, x_3, r', e^{i\alpha'}, v_3\right).$$
Performing the change of coordinates $v' = (r'e^{iα'}, v_3')$ and $z = -{\frac{1}{2}} \{e^{iα}\}$ leads to

$$K = \int_0^{2π} \int_{\mathbb{R}_+} \int_0^{2π} \int_{\mathbb{R}_+} \frac{σ(\sqrt{r^2 + (v_3 - v_3')^2})}{r^2 + (v_3 - v_3')^2} (le^{iα}, v_3 - v_3) \otimes 2$$

$$× f \left( \frac{\frac{1}{2}}{ω_c} + \frac{\frac{1}{2}}{ω_c} \left\langle r'e^{iα'} \right\rangle, x, v, r'e^{iα'}, v_3' \right) χ(r, r', -\frac{1}{2}\{e^{iα}\})r'dr'dα'dv_3'dl'dα$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \frac{σ(\sqrt{|z|^2 + (v_3 - v_3')^2})}{|z|^2 + (v_3 - v_3')^2} \left( \frac{1}{ω_c} \frac{z}{ω_c} - \frac{1}{ω_c} \right) x, v, f$$

Finally we obtain

$$\left\langle f \right\rangle_{\mathcal{S}} = ω_c^2 \int_{\mathbb{R}^4} \int_{\mathbb{R}^3} σ(\sqrt{|z|^2 + (v_3 - v_3')^2}) f(x', x', x', v') χ S( \frac{1}{ω_c} z, v_3 - v_3 ) dv'dx'dx_2.$$

C Proofs of Propositions 5.7, 5.8

Proof. (of Proposition 5.7) Let us introduce the notation

$$ξ_v(x, v) = \int_{\mathbb{R}^3} σ(|v - v'|)S(v - v')f(x, v')\nabla_v f(x, v) dv'.$$

Thanks to Proposition 3.3 we have

$$\left\langle Q_{FP}^{\mathcal{S}}(f, f) \right\rangle = \left\langle \text{div}_v ξ_v \right\rangle = \frac{1}{ω_c} \text{div}_v \left\{ \left\langle \frac{1}{ω_c} ξ_v \right\rangle + \left\langle \frac{1}{ω_c} ξ_v \cdot \frac{1}{|v|} \right\rangle \frac{|v|}{|v|} - \left\langle \frac{1}{ω_c} ξ_v \cdot \frac{1}{|v|} \right\rangle \frac{|v|}{|v|} \right\}$$

$$+ \text{div}_v \left\{ \left\langle \frac{1}{ω_c} ξ_v \cdot \frac{1}{|v|} \right\rangle \frac{|v|}{|v|} + \left\langle \frac{1}{ω_c} ξ_v \cdot \frac{1}{|v|} \right\rangle \frac{|v|}{|v|} \right\} + \partial_{v_3} \left\langle ξ_{v_3} \right\rangle.$$

We need to compute $\left\langle ξ_v \right\rangle, \left\langle ξ_v \cdot \frac{1}{|v|} \right\rangle, \left\langle ξ_v \cdot \frac{1}{|v|} \right\rangle$. By Proposition 3.1 we know that $\sum_{i=0}^{5} b^i \otimes \nabla_{x, v} ψ_i = I$ and thus

$$\partial_{v_i} f = \sum_{i=0}^{5} \partial_{v_i} ψ_i b^i \cdot \nabla_{x, v} f = \frac{v_2}{ω_c |v|^2} b^0 \cdot \nabla_{x, v} f - \frac{1}{ω_c} b^2 \cdot \nabla_{x, v} f + \frac{v_1}{|v|} b^4 \cdot \nabla_{x, v} f.$$

Similarly we have

$$\partial_{v_2} f = \sum_{i=0}^{5} \partial_{v_2} ψ_i b^i \cdot \nabla_{x, v} f = -\frac{v_1}{ω_c |v|^2} b^0 \cdot \nabla_{x, v} f + \frac{1}{ω_c} b^1 \cdot \nabla_{x, v} f + \frac{v_2}{|v|} b^4 \cdot \nabla_{x, v} f.$$
leading to
\[ \nabla_v f = b^0 \cdot \nabla_{x,v} f \left( \frac{(\nabla f, 0)}{\omega_c |v|^2} + \left( \frac{-\nabla v f}{\omega_c}, \partial_{v_3} f \right) + b^4 \cdot \nabla_{x,v} f \left( \frac{(v, 0)}{|v|} \right). \]

Taking into account that all derivations \( b^i \cdot \nabla_{x,v}, 0 \leq i \leq 5 \) leave invariant ker \( T \), cf. Proposition 3.1, we obtain
\[
\langle \xi_v \rangle = \langle f, (\frac{(\nabla f, 0)}{|v|}) \rangle_{\sigma_S} \frac{b^0 \cdot \nabla_{x,v} f}{|v|^2} + \langle f \rangle_{\sigma_S} \left( \frac{-\nabla v f}{\omega_c}, \partial_{v_3} f \right) + \langle f, (v, 0) \rangle_{\sigma_S} \frac{b^4 \cdot \nabla_{x,v} f}{|v|} \\
= \left\{ \frac{\langle f, (\nabla f, 0) \rangle_{\sigma_S}}{|v|^2} \right\} \left( \frac{(v, 0)}{|v|}, \frac{(\nabla f, 0)}{|v|} \right) - \frac{\langle f, (v, 0) \rangle_{\sigma_S}}{|v|^2} \left( \frac{(\nabla f, 0)}{|v|}, \frac{(v, 0)}{|v|} \right) \right\} \nabla_{\omega, v} f \\
- \langle f \rangle_{\sigma_S} (E, -e_3 \otimes e_3) \nabla_{\omega, v} f
\]

where the lines of the matrix \( E \in \mathcal{M}_3(\mathbb{R}) \) are \( e_2, -e_1, 0 \). Similarly, thanks to the identities \( \langle f, (\nabla f, 0) \rangle_{\sigma_S} = \langle f, (\frac{\nabla f, 0}{|v|}) \rangle_{\sigma_S} = 0 \) we obtain
\[
\left\langle \xi_v \cdot \frac{\nabla f}{|v|} \right\rangle = \left\langle \xi_v \cdot \frac{(\nabla f, 0)}{|v|} \right\rangle \\
= \frac{\langle f, (\nabla f, 0) \rangle_{\sigma_S}}{|v|^2} \left( \frac{-\nabla v f}{\omega_c}, -\partial_{v_3} f \right) + \frac{\langle f, (v, 0) \rangle_{\sigma_S}}{|v|^2} \left( \frac{(\nabla f, 0)}{|v|}, -\frac{(v, 0)}{|v|} \right) + \left( E \frac{\langle f, (v, 0) \rangle_{\sigma_S}}{|v|^2} e_3 \otimes e_3 \right) \frac{\langle f, (v, 0) \rangle_{\sigma_S}}{|v|^2} \right\} \nabla_{\omega, v} f
\]

and
\[
\left\langle \xi_v \cdot \frac{\nabla f}{|v|} \right\rangle = \left\langle \xi_v \cdot \frac{(\nabla f, 0)}{|v|} \right\rangle \\
= \frac{\langle f, (\nabla f, 0) \rangle_{\sigma_S}}{|v|^2} \left( \frac{-\nabla v f}{\omega_c}, -\partial_{v_3} f \right) + \frac{\langle f, (v, 0) \rangle_{\sigma_S}}{|v|^2} \left( \frac{(\nabla f, 0)}{|v|}, -\frac{(v, 0)}{|v|} \right) + \left( E \frac{\langle f, (v, 0) \rangle_{\sigma_S}}{|v|^2} e_3 \otimes e_3 \right) \frac{\langle f, (v, 0) \rangle_{\sigma_S}}{|v|^2} \right\} \nabla_{\omega, v} f.
\]

In the last equality we have taken into account that
\[
e_3 \otimes e_3 \frac{\langle f, (\frac{\nabla f, 0}{|v|}) \rangle_{\sigma_S}}{|v|} = 0.
\]
Clearly $\langle Q_{FPL}(f,f)\rangle$ has the form in \(39\) with $A^+ = \begin{pmatrix} A^+_{xx} & A^+_{xe} \\ A^+_{ex} & A^+_{ee} \end{pmatrix}$ where

$$
(A^+_{xx}, A^+_{xe}) = \frac{r - r' \cos \varphi}{|z|} \frac{ (+z, 0) }{|z|} \otimes \begin{pmatrix} (\tau, 0) \\ -(\tau, 0) \end{pmatrix}
\quad + \quad \frac{v^2}{|z|} \frac{ (v_3 - v_3')^2 }{|z|^2 + (v_3 - v_3')^2} \frac{ (z, 0) }{|z|} \otimes \begin{pmatrix} -(\tau, 0) \\ (\tau, 0) \end{pmatrix}
\quad + \quad \frac{1 - (r'^2 \sin^2 \varphi)}{|z|^2 + (v_3 - v_3')^2} \frac{ (\tau, 0) }{|z|} \otimes \begin{pmatrix} (\tau, 0) \\ -(\tau, 0) \end{pmatrix}
\quad + \quad \frac{r - r' \cos \varphi}{|z|} \frac{ (+z, 0) }{|z|} \otimes \begin{pmatrix} (\tau, 0) \\ -(\tau, 0) \end{pmatrix}
\quad + \quad \frac{1 - (r'^2 \cos \varphi)}{|z|^2 + (v_3 - v_3')^2} \frac{ -(\tau, 0) }{|z|} \otimes \begin{pmatrix} (\tau, 0) \\ -(\tau, 0) \end{pmatrix}
\quad + \quad \frac{(r-r' \cos \varphi)(v_3 - v_3')}{|z| \sqrt{|z|^2 + (v_3 - v_3')^2}} e_3 \otimes \begin{pmatrix} (\tau, 0) \\ -(\tau, 0) \end{pmatrix}
\quad + \quad \frac{e_3 \otimes e_3 S( (+z, v_3' - v_3) )E}{ e_3 \otimes e_3 S( (+z, v_3' - v_3) )e_3 \otimes e_3 )}.
$$

(59)

and

$$
(A^+_{xe}, A^+_{ee}) = \begin{pmatrix} \frac{1 - (r'^2 \sin^2 \varphi)}{|z|^2 + (v_3 - v_3')^2} \frac{ (\tau, 0) }{|z|} \otimes \begin{pmatrix} (\tau, 0) \\ -(\tau, 0) \end{pmatrix}
\quad + \quad \frac{r - r' \cos \varphi}{|z|} \frac{ +(z, 0) }{|z|} \otimes \begin{pmatrix} (\tau, 0) \\ -(\tau, 0) \end{pmatrix}
\quad + \quad \frac{1 - (r'^2 \cos \varphi)}{|z|^2 + (v_3 - v_3')^2} \frac{ (\tau, 0) }{|z|} \otimes \begin{pmatrix} (\tau, 0) \\ -(\tau, 0) \end{pmatrix}
\quad + \quad \frac{(r-r' \cos \varphi)(v_3 - v_3')}{|z| \sqrt{|z|^2 + (v_3 - v_3')^2}} e_3 \otimes \begin{pmatrix} (\tau, 0) \\ -(\tau, 0) \end{pmatrix}
\quad + \quad \frac{e_3 \otimes e_3 S( +(z, v_3' - v_3) )E}{ e_3 \otimes e_3 S( -(z, v_3' - v_3) )e_3 \otimes e_3 )}.
$$

(60)
It is easily seen that the matrix $A^+$ writes

\[
A^+ = \left( 1 - \frac{(r')^2 \sin^2 \varphi}{|z|^2 + (v_3 - v_3')^2} \right) \left( \frac{v_3}{|v_3|}, \frac{v_3'}{|v_3'|} \right) \otimes \left( \frac{v_3}{|v_3|}, \frac{v_3'}{|v_3'|} \right) \otimes \left( \frac{v_3}{|v_3|}, \frac{v_3'}{|v_3'|} \right) + \left( 1 - \frac{(r - r') \cos \varphi}{|z|^2 + (v_3 - v_3')^2} \right) \left( \frac{v_3}{|v_3|}, \frac{v_3'}{|v_3'|} \right) \otimes \left( \frac{v_3}{|v_3|}, \frac{v_3'}{|v_3'|} \right) \otimes \left( \frac{v_3}{|v_3|}, \frac{v_3'}{|v_3'|} \right) + \frac{r - r' \cos \varphi}{|z|} \left( \frac{1}{|z|}, 0 \right) \otimes \left( \frac{v_3}{|v_3|}, \frac{v_3'}{|v_3'|} \right) \otimes \left( \frac{1}{|z|}, 0 \right) + \frac{r - r' \cos \varphi}{|z|} \left( \frac{v_3}{|v_3|}, 0 \right) \otimes \left( \frac{v_3}{|v_3|}, \frac{v_3'}{|v_3'|} \right) \otimes \left( 1 \right) + \left( \frac{1}{|z|}, -\frac{1}{|z|} \right) \right) \otimes \left( \frac{1}{|z|}, -\frac{1}{|z|} \right) + \frac{(r - r') \cos \varphi}{|z|} \left( \frac{v_3 - v_3'}{|z|}, 0 \right) \otimes \left( \frac{v_3}{|v_3|}, \frac{v_3'}{|v_3'|} \right) \otimes \left( \frac{v_3 - v_3'}{|z|}, 0 \right) - \frac{(r - r') \cos \varphi}{|z|} \left( v_3 - v_3' \right) \left( \frac{(v_3 - v_3')(z,0)}{|z|}, -|z|e_3 \right) \otimes \left( \frac{(v_3 - v_3')(z,0)}{|v_3|}, -|z|e_3 \right) - \frac{(r - r') \cos \varphi}{|z|} \left( v_3 - v_3' \right) \left( \frac{(v_3 - v_3')(z,0)}{|v_3|}, -|z|e_3 \right) \otimes \left( \frac{(v_3 - v_3')(z,0)}{|v_3|}, -|z|e_3 \right) + B^+ = A_1^+ + A_2^+ + A_3^+ + A_4^+ + A_5^+ + A_6^+ + B^+ \quad (61)
\]

where

\[
B^+ = \begin{pmatrix}
\iota ES(\frac{1}{z}, v_3') - v_3 \end{pmatrix} E & \begin{pmatrix} ES(\frac{1}{z}, v_3') - v_3 \end{pmatrix} e_3 \otimes e_3 \\
-e_3 \otimes e_3 S(\frac{1}{z}, v_3') - v_3 \end{pmatrix} E & e_3 \otimes e_3 S(\frac{1}{z}, v_3') - v_3 \end{pmatrix} e_3 \otimes e_3
\]

Observe that for any $z \in \mathbb{R}^2$, $v_3, v_3' \in \mathbb{R}$ the family

\[
\frac{(z,0)}{|z|}, \left( \frac{v_3 - v_3'}{|z|^2 + (v_3 - v_3')^2} \right), \left( \frac{\iota}{|z|}, \left( \frac{v_3 - v_3'}{|z|^2 + (v_3 - v_3')^2} \right) \right), \left( \frac{\iota}{|z|}, \left( \frac{v_3 - v_3'}{|z|^2 + (v_3 - v_3')^2} \right) \right)
\]

is a orthonormal basis of $\mathbb{R}^3$. Therefore we have

\[
S(\frac{1}{z}, v_3 - v_3) = \left( \frac{(z,0)}{|z|}, \left( \frac{v_3 - v_3'}{|z|^2 + (v_3 - v_3')^2} \right) \right) \otimes \left( \frac{(z,0)}{|z|}, \left( \frac{v_3 - v_3'}{|z|^2 + (v_3 - v_3')^2} \right) \right)
\]

implying that for any $\xi = (\xi_x, \xi_v) \in \mathbb{R}^6$

\[
(B^+ \xi, \xi) = S(\frac{1}{z}, v_3') : (\frac{\iota}{v_3} - \xi_v) \otimes (\frac{\iota}{v_3} - \xi_v)
\]

and thus

\[
B^+ = \left( \frac{(z,0)}{|z|}, 0 \right) \otimes \left( \frac{v_3 - v_3'}{|z|^2 + (v_3 - v_3')^2} \right) \otimes \left( \frac{z,0}{|z|}, \frac{v_3 - v_3'}{|z|^2 + (v_3 - v_3')^2} \right) \otimes \left( \frac{z,0}{|z|}, \frac{v_3 - v_3'}{|z|^2 + (v_3 - v_3')^2} \right) =: B^+_1 + B^+_2. \quad (62)
\]
Observe that
\[ A_1^+ + A_3^+ + A_4^+ + B_1^+ = \frac{(r')^2 \sin^2 \varphi}{|z|^2 \sqrt{|z|^2 + (v_3 - v_3')^2}} \left( \frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{(\overline{v}, 0)}{|\overline{v}|} \right) \right)^2 + \left( \frac{r - r' \cos \varphi}{|z|} \frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{(\overline{v}, 0)}{|\overline{v}|} \right) \right)^2 \]

since \( \frac{(r' - r \cos \varphi)^2}{|z|^2} = 1 - \frac{(r' \cos \varphi)^2}{|z|^2} \) and
\[ A_2^+ + A_5^+ + A_6^+ + B_2^+ = \frac{(r')^2 \sin^2 \varphi}{|z|^2 \sqrt{|z|^2 + (v_3 - v_3')^2}} \left( \frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{(\overline{v}, 0)}{|\overline{v}|} \right) \right)^2 + \left( \frac{(r - r' \cos \varphi)(v_3 - v_3')}{|z|} \frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{(\overline{v}, 0)}{|\overline{v}|} \right) \right)^2 \]

Our conclusion follows by combining (61), (62), (63), (64).

Proof. (of Proposition 5.8) We consider
\[ \xi_v(x, v) = \int_{\mathbb{R}^3} \sigma(|v - v'|) S(v - v') f(x, v) \nabla_{v'} f(x, v') \, dv'. \]

By Proposition 3.3 we have
\[ \langle Q_{\tilde{F}PL}(f, f) \rangle = \langle \text{div}_v \xi_v \rangle = \frac{1}{\omega_c} \text{div}_v \left\{ \langle \xi_v \rangle + \langle \xi_v \langle \frac{\overline{v}}{|\overline{v}|} \rangle \frac{\overline{v}}{|\overline{v}|} - \langle \xi_v \langle \frac{\overline{v}}{|\overline{v}|} \rangle \frac{\overline{v}}{|\overline{v}|} \rangle \right\} \]

\[ + \text{div}_v \left\{ \langle \xi_v \langle \frac{\overline{v}}{|\overline{v}|} \rangle \frac{\overline{v}}{|\overline{v}|} + \langle \xi_v \langle \frac{\overline{v}}{|\overline{v}|} \rangle \frac{\overline{v}}{|\overline{v}|} \rangle + \partial_{v_3} \langle \xi_{v_3} \rangle \right\} \]

\[ = \text{div}_{\omega_c} \langle E\xi_v \rangle + \langle \xi_v \langle \frac{\overline{v}}{|\overline{v}|} \rangle \frac{\overline{v}}{|\overline{v}|} \rangle - \langle \xi_v \langle \frac{\overline{v}}{|\overline{v}|} \rangle \frac{\overline{v}}{|\overline{v}|} \rangle \rangle \]

\[ + \text{div}_v \left\{ \langle \xi_v \langle \frac{\overline{v}}{|\overline{v}|} \rangle \frac{\overline{v}}{|\overline{v}|} + \langle \xi_v \langle \frac{\overline{v}}{|\overline{v}|} \rangle \frac{\overline{v}}{|\overline{v}|} \rangle + \langle e_3 \otimes e_3 \xi_v \rangle \right\} \]

As in the proof of Proposition 5.7, we obtain
\[ \nabla_{v'} f(x, v') = b^0 \cdot \nabla_{x, v'} f \left( \frac{\langle \frac{\overline{v'}}{\overline{v'}} \rangle |v'|^2 + \left( \frac{-\nabla f}{\omega_c}, \partial_{v_3} f \right) \right) + b^4 \cdot \nabla_{x, v'} f \left( \frac{\langle \overline{v} \rangle |v'|}{|v'|} \right) \]
and therefore

\[
\langle \xi_v \rangle = f(x,v) \left\langle \frac{b^0 \cdot \nabla_{x,v} f}{\omega_c |v'|^2}, (\frac{\perp v}{v}, 0) \right\rangle_{\sigma_S} - f(x,v) \left\langle \frac{\partial_{x_2} f}{\omega_c}, e_1 \right\rangle_{\sigma_S} + f(x,v) \left\langle \frac{\partial_{x_1} f}{\omega_c}, e_2 \right\rangle_{\sigma_S}
\]

\[+ f(x,v) \left\langle \partial_{v_3} f, e_3 \right\rangle_{\sigma_S} + f(x,v) \left\langle \frac{b^1 \cdot \nabla_{x,v} f}{|v'|}, (\frac{\perp v}{v}, 0) \right\rangle_{\sigma_S} = \]

\[= -\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \sigma f(x,v) \chi \frac{r'}{\chi} \left\langle \frac{(\frac{\perp v}{v}, 0)}{|v'|^2}, (\frac{\perp v}{v}, 0) \right\rangle_{\sigma_S} \right\rangle_{\mathbb{R}^2} \left( v_3 - v_3' \right) \left( \frac{v_3 - v_3'}{|z|^2} \right) \right\rangle_{\sigma_S}
\]

\[\right\rangle_{\mathbb{R}^2} \left( -e_3 \otimes e_3 \right) \nabla_{\omega,x,v'} f \, dv' \, dx_1' \, dx_2'.
\]

Similarly, thanks to the identities

\[\left\langle \frac{b^0 \cdot \nabla_{x,v} f}{\omega_c |v'|^2}, (\frac{\perp v}{v}, 0), (\frac{\perp \theta}{v}, 0) \right\rangle_{\sigma_S} = 0\]

we obtain

\[\langle \xi_v \cdot (\frac{\perp \theta}{v}, 0) \rangle = -f(x,v) \left\langle \frac{\partial_{x_2} f}{\omega_c |v'|}, (\frac{\perp \theta}{v}, 0) \right\rangle_{\sigma_S} \cdot e_1 + f(x,v) \left\langle \frac{\partial_{x_1} f}{\omega_c |v'|}, (\frac{\perp \theta}{v}, 0) \right\rangle_{\sigma_S} \cdot e_2
\]

\[+ f(x,v) \left\langle \partial_{v_3} f, (\frac{\perp \theta}{v}, 0) \right\rangle_{\sigma_S} \cdot e_3 + f(x,v) \left\langle \frac{b^1 \cdot \nabla_{x,v} f}{|v'|}, (\frac{\perp \theta}{v}, 0), (\frac{\perp \theta}{v}, 0) \right\rangle_{\sigma_S}
\]

\[= -\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \sigma f(x,v) \chi \left( \cos \varphi - r \right) \left( v_3 - v_3' \right) \left( \frac{v_3 - v_3'}{|z|^2} \right) \right\rangle_{\mathbb{R}^2} \left( -e_3 \otimes e_3 \right) \nabla_{\omega,x,v'} f \, dv' \, dx_1' \, dx_2'.
\]

and

\[\langle \xi_v \cdot \left( \frac{\perp \theta}{v}, 0 \right) \rangle = -f(x,v) \left\langle \frac{\partial_{x_2} f}{\omega_c |v'|}, \left( \frac{\perp \theta}{v}, 0 \right) \right\rangle_{\sigma_S} \cdot e_1 + f(x,v) \left\langle \frac{\partial_{x_1} f}{\omega_c |v'|}, \left( \frac{\perp \theta}{v}, 0 \right) \right\rangle_{\sigma_S} \cdot e_2
\]

\[+ f(x,v) \left\langle \partial_{v_3} f, \left( \frac{\perp \theta}{v}, 0 \right) \right\rangle_{\sigma_S} \cdot e_3 + f(x,v) \left\langle \frac{b^1 \cdot \nabla_{x,v} f}{|v'|^2}, \left( \frac{\perp \theta}{v}, 0 \right), \left( \frac{\perp \theta}{v}, 0 \right) \right\rangle_{\sigma_S}
\]

\[= -\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \sigma f(x,v) \chi \left( \cos \varphi - r \right) \left( \frac{\perp \theta}{v}, 0 \right) \right\rangle_{\mathbb{R}^2} \left( -e_3 \otimes e_3 \right) \nabla_{\omega,x,v'} f \, dv' \, dx_1' \, dx_2'.
\]
Obviously \( \langle Q_{FPL}(f, f) \rangle \) has the form in (40) with \( A^- = \begin{pmatrix} A^-_{xx} & A^-_{xv} \\ A^-_{vx} & A^-_{vv} \end{pmatrix} \) where

\[
(A^-_{xx}, A^-_{xv}) = \frac{r' - r \cos \varphi}{|z|} \frac{(\binom{z}{0}, 0)}{|v'|} \otimes \left( \frac{(v', 0)}{|v'|}, \frac{(-v', 0)}{|v'|} \right) \\
+ \frac{(r' - r \cos \varphi)(v_3 - v_3')^2}{|z|^2 + (v_3 - v_3')^2} \frac{(z, 0)}{|z|} \otimes \left( \frac{(-v', 0)}{|v'|}, \frac{(v', 0)}{|v'|} \right) \\
+ \left( \cos \varphi - \frac{rr' \sin^2 \varphi}{|z|^2 + (v_3 - v_3')^2} \right) \frac{(\overline{v}, 0)}{|\overline{v}|} \otimes \left( \frac{(\overline{v}', 0)}{|\overline{v}'|}, \frac{(-\overline{v}', 0)}{|\overline{v}'|} \right) \\
+ \frac{r - r' \cos \varphi}{|z|} \frac{(|v|, 0)}{|v|} \otimes \left( \frac{(+z, 0)}{|z|}, 0 \right) \\
+ \left( \cos \varphi + \frac{r - r' \cos \varphi}{|z|^2 + (v_3 - v_3')^2} \right) \frac{(|v|, 0)}{|v|} \otimes \left( \frac{(-v', 0)}{|v'|}, \frac{(-v', 0)}{|v'|} \right) \\
- \frac{(r - r' \cos \varphi)(v_3 - v_3')}{|z|^2 + (v_3 - v_3')^2} \frac{(|v|, 0)}{|v|} \otimes \left( \frac{(-v', 0)}{|v'|}, \frac{(-v', 0)}{|v'|} \right) \\
- \langle +z, v_3' - v_3 \rangle \langle E, -e_3 \otimes e_3 \rangle)
\]

and

\[
(A^-_{vx}, A^-_{vv}) = \left( \cos \varphi - \frac{r'r' \sin^2 \varphi}{|z|^2 + (v_3 - v_3')^2} \right) \frac{(\overline{v}, 0)}{|\overline{v}|} \otimes \left( \frac{(+v', 0)}{|v'|}, \frac{(+v', 0)}{|v'|} \right) \\
+ \frac{r - r' \cos \varphi}{|z|} \frac{(|v|, 0)}{|v|} \otimes \left( \frac{(+z, 0)}{|z|}, 0 \right) \\
- \left( \cos \varphi + \frac{r - r' \cos \varphi}{|z|^2 + (v_3 - v_3')^2} \right) \frac{(|v|, 0)}{|v|} \otimes \left( \frac{(-v', 0)}{|v'|}, \frac{-v', 0)}{|v'|} \right) \\
+ \frac{(r - r' \cos \varphi)(v_3 - v_3')}{|z|^2 + (v_3 - v_3')^2} \frac{(|v|, 0)}{|v|} \otimes \left( \frac{(v_3 - v_3')}{|z|}, -e_3 \right) \\
- \frac{(r' - r \cos \varphi)(v_3 - v_3')}{|z|^2 + (v_3 - v_3')^2} \frac{(|v|, 0)}{|v|} \otimes \left( \frac{(v_3 - v_3')}{|v'|}, \frac{(-v', 0)}{|v'|} \right) \\
- e_3 \otimes e_3 \langle +z, v_3' - v_3 \rangle \langle E, -e_3 \otimes e_3 \rangle.
\]
It is easily seen that the matrix $A^-$ writes

$$
A^- = \left( \cos \varphi - \frac{rr' \sin^2 \varphi}{|z|^2 + (v_3 - v_3')^2} \right) \left( \frac{(\overline{v}, 0)}{|v|}, \frac{(\overline{v}', 0)}{|v'|} \right) \otimes \left( \frac{(\overline{v}', 0)}{|v'|}, \frac{+\overline{v}', 0)}{|v'|} \right) \\
+ \left( \cos \varphi + \frac{(r - r' \cos \varphi)(r' - r \cos \varphi)}{|z|^2 + (v_3 - v_3')^2} \right) \left( \frac{+\overline{v}, 0}{|v|}, \frac{\overline{v}, 0}{|v|} \right) \otimes \left( \frac{-\overline{v}, 0}{|v|}, \frac{+\overline{v}, 0}{|v|} \right) \\
+ \frac{r' - r \cos \varphi}{|z|} \left( \frac{(\overline{v}, 0)}{|v|}, \frac{(\overline{v}', 0)}{|v'|} \right) \otimes \left( \frac{(\overline{v}', 0)}{|v'|}, \frac{-\overline{v}, 0}{|v|} \right) \\
- \frac{r - r' \cos \varphi}{|z|} \left( \frac{(\overline{v}, 0)}{|v|}, \frac{(\overline{v}', 0)}{|v'|} \right) \otimes \left( \frac{(\overline{v}', 0)}{|v'|}, \frac{-\overline{v}, 0}{|v|} \right) \\
+ \frac{(r - r' \cos \varphi)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \left( \frac{(\overline{v}, 0)}{|v|}, \frac{(\overline{v}, 0)}{|v|} \right) \otimes \left( \frac{+\overline{v}, 0}{|v|}, \frac{-\overline{v}, 0}{|v|} \right) \\
+ \frac{(r - r' \cos \varphi)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \left( \frac{(\overline{v}, 0)}{|v|}, \frac{(\overline{v}, 0)}{|v|} \right) \otimes \left( \frac{+\overline{v}, 0}{|v|}, \frac{-\overline{v}, 0}{|v|} \right)
$$

\[\text{(67)}\]

where, cf. (62)

$$
B^- = \left( \begin{array}{cc}
^{t}ES\left( (\overline{z}, v_3' - v_3) \right) E & ES\left( (\overline{z}, v_3' - v_3) \right) e_3 \otimes e_3 \\
-e_3 \otimes e_3 S\left( (\overline{z}, v_3' - v_3) \right) E & e_3 \otimes e_3 S\left( (\overline{z}, v_3' - v_3) \right) e_3 \otimes e_3
\end{array} \right)
\quad \text{(68)}
$$

Observe that

$$
A^- + A^- + A^- + A^- + B^- = \frac{rr' \sin^2 \varphi}{|z|^2 + (v_3 - v_3')^2} \left( \begin{array}{cc}
\frac{(\overline{v}, 0)}{|v|}, \frac{(\overline{v}, 0)}{|v|} \\
\frac{+\overline{v}, 0}{|v|}, \frac{+\overline{v}, 0}{|v|}
\end{array} \right) \otimes \left( \begin{array}{cc}
\frac{(\overline{v}', 0)}{|v'|}, \frac{(\overline{v}', 0)}{|v'|} \\
\frac{-\overline{v}, 0}{|v'|}, \frac{-\overline{v}, 0}{|v'|}
\end{array} \right)
$$

\[\text{(69)}\]

since

$$
\frac{rr' \sin^2 \varphi}{|z|^2 + (v_3 - v_3')^2} + \frac{(r - r' \cos \varphi)(r \cos \varphi - r')}{|z|^2} = \cos \varphi - \frac{rr' \sin^2 \varphi}{|z|^2 + (v_3 - v_3')^2}
$$

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and
\[
A_2^{-} + A_5^{-} + A_6^{-} + B_2^{'} = \frac{rr^{'} \sin^2 \varphi}{|z|^2} \left( \left( \frac{1}{v}, 0 \right), \left( \frac{\varphi}{|v|}, 0 \right) \right) \otimes \left( \left( \frac{\vec{v}^{'}, 0}{|\vec{v}'^{|}}, \left( \frac{\vec{v}^{'}, 0}{|\vec{v}'^{|}}} \right) \right)
+ \left( \frac{r - r^{'} \cos \varphi}{|z| \sqrt{|z|^2 + (v_3 - v_3^{'})^2}} \left( \left( \frac{1}{v}, 0 \right), \left( \frac{\varphi}{|v|}, 0 \right) \right) - \left( \frac{(v_3 - v_3^{'}) (z, 0)}{|z|}, -|z| e_3 \right) \right)
\otimes \left[ \frac{(r \cos \varphi - r^{'}) (v_3 - v_3^{'})}{|z| \sqrt{|z|^2 + (v_3 - v_3^{'})^2}} \left( \left( \frac{1}{v^{'}}, 0 \right), \left( \frac{\varphi}{|v^'|}, 0 \right) \right) - \left( \frac{(v_3 - v_3^{'}) (z, 0)}{|z|}, -|z| e_3 \right) \right] \right) \] (70)

since
\[
\frac{rr^{'} \sin^2 \varphi}{|z|^2} + \frac{(r^{' \cos \varphi} - r) (r' - r \cos \varphi) (v_3 - v_3^{'})^2}{|z|^2 \left| v_3^{' - v_3} \right|^2} = \cos \varphi + \frac{(r^{' \cos \varphi} - r) (r' - r \cos \varphi)}{|z|^2 + (v_3 - v_3^{'})^2}
\]

Our conclusion follows by combining (67), (68), (69), (70).

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