Entropy Spectrum of Lyapunov Exponents for Nonhyperbolic Step Skew-Products and Elliptic Cocycles

L. J. Diaz, K. Gelfert, M. Rams

1 Departamento de Matemática, PUC-Rio, Marquês de São Vicente 225, Gávea, Rio de Janeiro 22451-900, Brazil. E-mail: lodiaz@mat.puc-rio.br
2 Instituto de Matemática, Universidade Federal do Rio de Janeiro, Av. Athos da Silveira Ramos 149, Cidade Universitária - Ilha do Fundão Rio de Janeiro, 21945-909 Brazil. E-mail: gelfert@im.ufrj.br
3 Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszaw, Poland. E-mail: rams@impan.pl

Received: 18 October 2017 / Accepted: 29 August 2018
Published online: 28 March 2019 – © Springer-Verlag GmbH Germany, part of Springer Nature 2019

Abstract: We study the fiber Lyapunov exponents of step skew-product maps over a complete shift of \( N, N \geq 2 \), symbols and with \( C^1 \) diffeomorphisms of the circle as fiber maps. The systems we study are transitive and genuinely nonhyperbolic, exhibiting simultaneously ergodic measures with positive, negative, and zero exponents. Examples of such systems arise from the projective action of \( 2 \times 2 \) matrix cocycles and our results apply to an open and dense subset of elliptic \( SL(2, \mathbb{R}) \) cocycles. We derive a multifractal analysis for the topological entropy of the level sets of Lyapunov exponent. The results are formulated in terms of Legendre–Fenchel transforms of restricted variational pressures, considering hyperbolic ergodic measures only, as well as in terms of restricted variational principles of entropies of ergodic measures with a given exponent. We show that the entropy of the level sets is a continuous function of the Lyapunov exponent. The level set of the zero exponent has positive, but not maximal, topological entropy. Under the additional assumption of proximality, as for example for skew-products arising from certain matrix cocycles, there exist two unique ergodic measures of maximal entropy, one with negative and one with positive fiber Lyapunov exponent.

Contents

1. Introduction ........................................ 352
   1.1 Step skew-products with circle fibers .......... 353
   1.2 Application to \( GL^+(2, \mathbb{R}) \) and \( SL(2, \mathbb{R}) \) cocycles .......... 355

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. This research has been supported [in part] by CNE-Faperj, CNPq-Grants (Brazil), EU Marie-Curie IRSES “Brazilian–European partnership in Dynamical Systems” (FP7-PEOPLE-2012-IRSES 318999 BREUDS), and National Science Centre Grant 2014/13/B/ST1/01033 (Poland). The authors acknowledge the hospitality of IMPAN, IM-UFRJ, and PUC-Rio and thank Anton Gorodetski, Yali Liang, and Silvius Klein for their comments. They are very thankful to two anonymous referees for their useful comments.
1. Introduction

We will study the entropy spectrum of Lyapunov exponents, that is, the topological entropy of level sets of points with a common given Lyapunov exponent. This subject forms part of the multifractal analysis which, in general, studies thermodynamical quantities and objects (such as, for example, equilibrium states, entropies, Lyapunov exponents, Birkhoff averages, and recurrence rates) and their relations with geometrical properties (for example, fractal dimensions). Those properties are often encoded, and we follow this
approach, by the topological pressure and its Legendre–Fenchel transform. The novelty of this paper is that we consider transitive systems which are genuinely nonhyperbolic in the sense that their Lyapunov spectra contain zero in its interior (this property continues to hold also for perturbations) and that we provide a description of the full spectrum.

The systems that we investigate are step skew-products with circle fibers. These systems provide quite easily describable examples in which (robust) nonhyperbolicity can be studied. At the same time, they serve as models for robustly transitive and (nonhyperbolic) partially hyperbolic diffeomorphisms in a setting motivated by [10,49], for further details see [21, Section 8.3]. Let us observe that they also appear quite naturally as limit systems (using the terminology in [31]) in some non-local bifurcations and fit into the theory rigorously initiated in [32]. From another point of view, they can also be considered as actions of a group of diffeomorphisms on the circle as well as random dynamical systems [42]. An important class of examples that fit into our setting is the one of step skew-products on the circle which are induced by the projective action of a linear cocycle of $2 \times 2$-matrices. Indeed, there are a kind of paradigmatic examples which admit a fairly simple description where our results can be applied (for the complete general setting and precise results see Sect. 2).

In Sects. 1.1 and 1.2 we will skip all major technicalities and announce in a quick way our main result and its application to the study of cocycles of matrices in $\text{SL}(2, \mathbb{R})$, while in Sect. 2 we announce our results in their full generality. We point out that we always work in the lowest possible regularity and consider $C^1$ circle diffeomorphisms as fiber maps.

1.1. Step skew-products with circle fibers. Consider a finite family $f_i : S^1 \to S^1, i = 0, \ldots, N - 1$ for $N \geq 2$, of $C^1$ diffeomorphisms and the associated step skew-product

$$F : \Sigma_N \times S^1 \to \Sigma_N \times S^1, \quad F(\xi, x) = (\sigma(\xi), f_{\xi_0}(x)), \quad (1.1)$$

where $\Sigma_N = \{0, \ldots, N - 1\}^Z$. We consider the class $\text{SP}^1_{\text{shyp}}(\Sigma_N \times S^1)$ of such maps which are topologically transitive and “nonhyperbolic in a nontrivial way” in the sense that there are some “expanding region” and some “contracting region” (relative to the fiber direction) and that any of those can be reached from anywhere in the ambient space under forward/backward iterations as follows:

- Some hyperbolicity: There is a “forward blending” interval $J^+ \subset S^1$ such that for every sufficiently small interval $H$ with $H \cap J^+ \neq \emptyset$ there are $\ell \sim |\log |H||$ and a finite sequence $(\xi_0, \ldots, \xi_\ell)$ such that $f_{\xi_\ell} \circ \cdots \circ f_{\xi_0}(H)$ covers $J^+$ in an uniformly expanding way. Similarly, there is a “backward expanding blending” interval $J^-$.  

- Transitions in finite time to/from blending intervals: There exists $M \geq 1$ such that for every $x \in S^1$ there are finite sequences $(\theta_{-r}, \ldots, \theta_{-1})$ and $(\beta_0, \ldots, \beta_s)$, $s, r \leq M$, such that $f_{\beta_s} \circ \cdots \circ f_{\beta_0}(x)$ covers $J^+$ and $f_{\beta_r}^{-1} \circ \cdots \circ f_{\beta_0}^{-1}(x) \in J^+$. Similarly, there are transitions to/from the “backward blending” interval $J^-$. 

Remark 1.1. The simplest setting where the two properties above can be verified are skew-product maps defined on $\Sigma_2 \times S^1$ whose fiber maps are a Morse–Smale diffeomorphism $f_0$ with one attracting and one repelling fixed point and an irrational rotation $f_1$. The study of such maps was initiated in [33] (see also references given in [33]). Observe that small perturbations of these maps also satisfy the hypotheses, see [21, Proposition 8.8]. Indeed, this type of example plays a specially important role in this paper as they satisfy the following property of proximality:
• Proximality: For every \( x, y \in S^1 \) there exists one bi-infinite sequence \( \xi \in \Sigma_N \) such that \( |f_{\xi_n} \circ \cdots \circ f_{\xi_0}(x) - f_{\xi_n} \circ \cdots \circ f_{\xi_0}(y)| \to 0 \) and \( |f_{\xi_{-n}}^{-1} \circ \cdots \circ f_{\xi_{-1}}^{-1}(x) - f_{\xi_{-n}}^{-1} \circ \cdots \circ f_{\xi_{-1}}^{-1}(y)| \to 0 \) as \( n \to 0 \).

In our context, proximality is equivalent to the so-called synchronization property (see, for example, [40, Section 2.3]). In the context of matrix cocycles further studied below, this was first investigated in [27] (studying random products of matrices on projective spaces). Further, examples of a quite different nature can be found in [21, Section 8.1], where also the motivation for the term “blending” is discussed. Let us observe that the above properties hold open and densely among \( C^1 \) transitive nonhyperbolic step skew-products (see [21, Proposition 8.9] for details).

Given \( X = (\xi, x) \in \Sigma_N \times S^1 \), consider the (fiber) Lyapunov exponent of \( X \)
\[
\chi(X) \overset{\text{def}}{=} \lim_{n \to \pm \infty} \frac{1}{n} \log |(f^n_{\xi})'(x)|,
\]
(1.2)
(where \( f_{\xi_{-n}}^{-1} \overset{\text{def}}{=} f_{\xi_{-n}}^{-1} \circ \cdots \circ f_{\xi_{-1}}^{-1} \) and \( f_{\xi}^{n} \overset{\text{def}}{=} f_{\xi_{n-1}}^{-1} \circ \cdots \circ f_{\xi_{0}}^{-1} \)) where we assume that both limits \( n \to \pm \infty \) exist and coincide. We will analyze the topological entropy of the following level sets of Lyapunov exponents: given \( \alpha \in \mathbb{R} \) let
\[
\mathcal{L}(\alpha) \overset{\text{def}}{=} \{ X \in \Sigma_N \times S^1 : \chi(X) = \alpha \}.
\]
Here we will rely on the general concept of topological entropy \( h_{\text{top}} \) introduced by Bowen [11] (see Appendix A). Given an \( F \)-ergodic measure \( \mu \), denote by \( \chi(\mu) \) the Lyapunov exponent of \( \mu \) defined by
\[
\chi(\mu) \overset{\text{def}}{=} \int \log |(f_{\xi_0})'(x)| \, d\mu(\xi, x).
\]

**Theorem A.** For every \( N \geq 2 \) and for every \( F \in \text{SP}^1_{\text{shyp}}(\Sigma_N \times S^1) \) we have \( \mathcal{L}(\alpha) \neq \emptyset \) if and only if \( \alpha \in [\alpha_{\text{min}}, \alpha_{\text{max}}] \) for some numbers \( \alpha_{\text{min}} < 0 < \alpha_{\text{max}} \). Moreover, the map \( \alpha \mapsto h_{\text{top}}(\mathcal{L}(\alpha)) \) is continuous and concave on each interval \([\alpha_{\text{min}}, 0]\) and \([0, \alpha_{\text{max}}]\) and for all \( \alpha \in [\alpha_{\text{min}}, 0] \cup (0, \alpha_{\text{max}}] \) we have
\[
h_{\text{top}}(\mathcal{L}(\alpha)) = \sup \{ h(\mu) : \mu \in M_{\text{erg}}(\Sigma_N \times S^1), \chi(\mu) = \alpha \}.
\]
Moreover, assuming proximality, there exist unique ergodic \( F \)-invariant probability measures \( \mu_- \) and \( \mu_+ \) of maximal entropy \( h(\mu_{\pm}) = \log N \), respectively, and satisfying
\[
\alpha_{\text{min}} \leq \alpha_- \overset{\text{def}}{=} \chi(\mu_-) < 0 < \alpha_+ \overset{\text{def}}{=} \chi(\mu_+) \leq \alpha_{\text{max}}
\]
and for all \( \alpha \in (\alpha_{\text{min}}, \alpha_{\text{max}}) \setminus \{\alpha_-, \alpha_+\} \) we have
\[
0 < h_{\text{top}}(\mathcal{L}(\alpha)) < \log N.
\]

Theorem A will be a consequence of the more elaborate version stated in Theorems 1, 2, and 4, see Sect. 2 and compare Fig. 1.
1.2. Application to $\text{GL}^+(2,\mathbb{R})$ and $\text{SL}(2,\mathbb{R})$ cocycles. Consider first the group $\text{GL}^+(2,\mathbb{R})$ of all $2 \times 2$ matrices with real coefficients and positive determinant. Given $N \geq 2$, a continuous map $A: \Sigma_N \to \text{GL}^+(2,\mathbb{R})$ is called a $2 \times 2$ matrix cocycle. If $A$ is piecewise constant and depends only on the zeroth coordinate of the sequences $\xi \in \Sigma_N$, that is $A(\xi) = A_{\xi_0}$ where $A \equiv \{A_0, \ldots, A_{N-1}\} \in \text{GL}^+(2,\mathbb{R})^N$, then we refer to it as the one-step cocycle generated by $A$ or simply as the one-step cocycle $A$. One-step matrix cocycles are an object of intensive study in several branches of mathematics and also have serious physical applications. See for example [17, Sections 2 and 3], [23, Section 7.2], and [54].

Note that the projective line $\mathbb{P}^1$ is topologically the circle $S^1$ and the action of any $\text{GL}^+(2,\mathbb{R})$ matrix on $\mathbb{P}^1$ is a diffeomorphism. We will continue to take this point of view, given a matrix $A \in \text{GL}^+(2,\mathbb{R})$, define $f_A: \mathbb{P}^1 \to \mathbb{P}^1$ by

$$f_A(v) \overset{\text{def}}{=} \frac{Av}{\|Av\|}. \quad (1.3)$$

Given a one-step $2 \times 2$ matrix cocycle $A$, we denote by $F_A$ the associated step skew-product generated by the family of maps $f_{A_0}, \ldots, f_{A_{N-1}}$ as in (1.1).

To simplify our study of the Lyapunov exponents of the cocycle, we consider the one-sided one-step cocycle $A: \Sigma_N^+ \to \text{GL}^+(2,\mathbb{R})$, where $\Sigma_N^+ = \{0, \ldots, N-1\}^{\mathbb{N}_0}$. We denote

$$A^n(\xi^+) \overset{\text{def}}{=} A_{\xi_{n-1}} \circ \cdots \circ A_{\xi_1} \circ A_{\xi_0}, \quad \xi^+ \in \Sigma_N^+, \ n \geq 0.$$ 

The Lyapunov exponents of the cocycle $A$ at $\xi^+ \in \Sigma_N^+$ are the limits

$$\lambda_1(A, \xi^+) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \log \|A^n(\xi^+\|) \quad \text{and} \quad \lambda_2(A, \xi^+) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \log \|(A^n(\xi^+))^{-1}\|^{-1},$$

where $\|L\|$ denotes the norm of the matrix $L$, whenever they exist. Given $\alpha \in \mathbb{R}$, we consider the level set

$$\mathcal{L}_A^+ (\alpha) \overset{\text{def}}{=} \{ \xi^+ \in \Sigma_N^+ : \lambda_1(A, \xi^+) = \alpha \}. \quad (1.4)$$

We now consider the subgroup $\text{SL}(2,\mathbb{R}) \subset \text{GL}^+(2,\mathbb{R})$ of $2 \times 2$ matrices with real coefficients and determinant one. The space $\text{SL}(2,\mathbb{R})^N$ can be roughly divided into two subsets: elliptic and uniformly hyperbolic ones (denoted by $\mathcal{E}_N$ and $\mathcal{H}_N$, respectively).
Both $\mathcal{E}_N$ and $\mathcal{F}_N$ are open and their union is dense in $\text{SL}(2, \mathbb{R})^N$, see [57, Proposition 6]. Hyperbolic cocycles are quite well understood, for their characterization see [2]. However, much less is known about elliptic cocycles. Here we will introduce a subset of elliptic cocycles having “some hyperbolicity”, denoted by $\mathcal{E}_{N, \text{shyp}}$, which forms an open and dense subset of $\mathcal{E}_N$. For any $A \in \mathcal{E}_{N, \text{shyp}}$ we will provide a detailed description of the spectrum of its Lyapunov exponents.

To be more precise, denote by $\langle A \rangle$ the semigroup generated by $A$. Recall that an element $R \in \text{SL}(2, \mathbb{R})$ is elliptic if the absolute value of its trace is strictly less than 2; in such a case the matrix $R$ is conjugate to a rotation by some angle, called its rotation number and denoted by $\varrho(R)$. An element $A \in \text{SL}(2, \mathbb{R})$ is hyperbolic if the absolute value of its trace is strictly larger than 2, which is equivalent to the fact that the matrix $A$ has one eigenvalue with absolute value bigger than one and one smaller than one. The set $\mathcal{E}_N$ of elliptic cocycles is the set of cocycles $A \in \text{SL}(2, \mathbb{R})^N$ such that $\langle A \rangle$ contains an elliptic element.

If a matrix $R$ is elliptic then $f_R$ is conjugate to a rotation by an angle $\varrho(R)$. Note that in the case when $\varrho(R)$ is irrational mod $2\pi$ then $f_R$ is of the same type as (differentially conjugate to) the map $f_1$ in Remark 1.1. If a matrix $A$ is hyperbolic then $f_A$ has one attracting and one repelling fixed point. Note that then $f_A$ is a very specific case of a Morse–Smale diffeomorphism of $\mathbb{P}^1$ of the same type as $f_0$ in Remark 1.1.

We consider a subset $\mathcal{E}_{N, \text{shyp}}$ of $\mathcal{E}_N$ consisting of cocycles $A \in \text{SL}(2, \mathbb{R})^N$ having the following properties (the precise definition is provided in Sect. 11.7):

- Some hyperbolicity: There exists $A \in \langle A \rangle$ which is hyperbolic.
- Transitions in finite time: There exists $B \in \langle A \rangle$ which is sufficiently “close” to an irrational rotation.

Note that these properties are just a translation of the properties of systems in $\text{SP}_1^{\text{shyp}}(\Sigma_N \times \mathbb{S}^1)$ for the induced fiber maps $f_A$ and $f_B$ arising from matrix cocycles in the spirit of Remark 1.1. Indeed, it is easy to check that each such a cocycle automatically also satisfies the property Proximality. Following [2], we will show that $\mathcal{E}_{N, \text{shyp}}$ is an open and dense subset of $\mathcal{E}_N$ (see Proposition 11.23).

Given $\nu$ an ergodic measure on $\Sigma_N$ (with respect to $\sigma^+: \Sigma_N \to \Sigma_N^+$), denote

$$\lambda_1(A, \nu) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \log \|A^n(\xi^+)\| \ d\nu$$

and note that this number is the Lyapunov exponent $\lambda_1(A, \xi^+)$ for any $\nu$-typical $\xi^+$. Denote by $h(\nu)$ the metric entropy of $\nu$.

For the following two results, note that for the standard metric the Hausdorff dimension of any set in $\Sigma_N^+$ is equal to its topological entropy relative to $\sigma^+$. Thus, they can be read also in terms of Hausdorff dimension. The function $\alpha \mapsto h_{\text{top}}(\mathcal{L}_A^+(\alpha))$ is also called (Lyapunov) multifractal spectrum and among the first references where it was studied one cites [25, 26].

**Theorem B.** For every $N \geq 2$ the set $\mathcal{E}_{N, \text{shyp}}$ is open and dense in $\mathcal{E}_N$ and has the following property: For every $A$ in $\mathcal{E}_{N, \text{shyp}}$ there are numbers $0 < \alpha_+ < \alpha_{\max}$ such that the map $\alpha \mapsto h_{\text{top}}(\mathcal{L}_A^+(\alpha))$ is continuous and concave on $[0, \alpha_{\max}]$, having a unique maximum at $\alpha_+$ and

$$h_{\text{top}}(\mathcal{L}_A^+(\alpha_+)) = \log N,$$

we have $0 < h_{\text{top}}(\mathcal{L}_A^+(0)) < \log N$, and for every $\alpha \in (0, \alpha_{\max}]$ we have

$$h_{\text{top}}(\mathcal{L}_A^+(\alpha)) = \sup\{h(\nu) : \nu \in \mathcal{M}_{\text{erg}}(\Sigma_N^+), \lambda_1(A, \nu) = \alpha\}.$$
A fundamental step to prove the above theorem is to study the relations between the Lyapunov exponents of the cocycle and the ones of the associated step skew-product. With these results at hand, we can invoke the results about skew-products and prove Theorem B. More precisely, by Remark 3.1, for every $A \in \mathcal{E}_N, \text{shyp}$ the associated step skew-product $F_A$ satisfies the hypothesis of Theorems 2 and 4. Theorem 5 then translates the Lyapunov spectrum of the skew-product to the one of the cocycle. In Sect. 11.7 we prove that the set $\mathcal{E}_N, \text{shyp}$ is open and dense in $\mathcal{E}_N$. This all together hence proves Theorem B.

Note that the existence of a zero Lyapunov exponent for $\text{SL}^+(2, \mathbb{R})$ cocycles immediately “translates” to the condition of having two equal exponents for $\text{GL}^+(2, \mathbb{R})$ cocycles, just by considering the normalization $A \mapsto A/\sqrt{\det(A)}$. Thus, for such cocycles the above result can be read as follows.

**Corollary B.1.** For every $N \geq 2$ there is an open and dense subset $S \subset \text{GL}^+(2, \mathbb{R})^N$ such that for every $A \in S$ one of the two possibilities is true:

- we have $0 < h_{\text{top}}([\xi^+ \in \Sigma_N^+: \lambda_1(A, \xi^+) = \lambda_2(A, \xi^+))] < \log N$,
- the cocycle is hyperbolic.

The left inequality in Corollary B.1 also follows from [8, Theorem 3], where a different approach is used.

The study of level sets of Lyapunov exponents within the context of cocycles fits within the analysis of the simplicity of the Lyapunov spectrum. There is an intensive line of research, perhaps initiated by [37], where a measure in the base space is fixed and varying the cocycle one aims to establish conditions which guarantee that the integrated top Lyapunov is (or is not) positive, see for instance [1,3]. In a slightly different context, the designated measure in the base is the Riemannian volume, the fiber dynamics is the derivative cocycle of a volume preserving diffeomorphism, where more precise results about the spectrum of Lyapunov exponents are obtained (a dichotomy between all exponents being equal to zero versus the existence of a dominated Oseledets splitting), see for instance [5,9].

In contrast to these works, here our cocycle is fixed (within an open and dense set of elliptic cocycles), and *a priori* no base measure is designated, and we study the orbitwise Lyapunov exponents. Our measurement of the level sets will be in terms of topological entropy. For that we will establish restricted variational principles and develop a multifractal analysis in a nonhyperbolic setting, which we will now discuss.

### 1.3. Multifractal context

For uniformly hyperbolic dynamics multifractal analysis is understood in great depth and has found already far reaching applications. There is an enormous literature on this subject. To highlight a collection of results in the field at different stages of development, we refer, for example, to [50] (analyticity of pressure and its consequences), [43,44] (multifractal analysis for conformal expanding maps and Smale’s horseshoes), and [4] (mixed spectra and restricted variational principles). In many of those references, particular attention is drawn to the so-called geometric potentials because of their close relation to Lyapunov exponents, entropy, and Sinai-Ruelle-Bowen measures. One key property of uniformly hyperbolic systems, under which the classical context of multifractal analysis was developed so far, is the specification property (studied for example in [24,45,53]). Note that it is also essential (compare [12]) to guarantee the
uniqueness of equilibrium states which is another key property to study multifractal analysis. The specification property implies many further strong properties, for example that the set of all invariant probability measures is a Poulsen simplex ([51]) with a hence very rich topological structure.

The multifractal analysis theory extends also to “one-sided” nonuniformly hyperbolic systems, that is, for example to nonuniformly expanding maps where the presence of a nonpositive Lyapunov exponent is the only obstruction to hyperbolicity, that is, the spectrum of Lyapunov exponents covers a range of hyperbolicity and the zero exponent bounds this range from one side, see for example [29] (expansive Markov maps of the interval) and [35,46] (multimodal interval maps). So far, there is not much understanding of a multifractal analysis for more complicated types of nonhyperbolic systems. It is difficult to describe all the situations that can happen in general; one natural class of systems to focus on could be the systems with a designated line field (associated with the Oseledets decomposition) for which the Lyapunov exponent takes both positive and negative values arbitrarily close to zero (and also zero). Naturally, we assume topological transitivity, hence the system in question cannot split into “one-sided” nonuniformly hyperbolic parts.

Probably, the simplest setting of such “two-sided” nonhyperbolic dynamical systems (that is, with zero Lyapunov exponent in the interior of the spectrum) can be found in step skew-products with a hyperbolic horseshoe map in its base and circle diffeomorphisms in its fibers. The nonuniform hyperbolicity arises from the coexistence of contracting and expanding regions (in the fibers) which are blended by the dynamics. These properties are exemplified by the hypotheses “some hyperbolicity” and “transitions in finite time” stated in Sect. 1.1. The considered dynamics is topologically transitive and simultaneously has “horseshoes” which are contracting and “horseshoes” which are expanding in the fiber direction. These horseshoes are intermingled and there coexist dense sets of periodic points with negative and positive fiber Lyapunov exponents. As a consequence, the system exhibits ergodic hyperbolic measures with positive entropy. An important feature is the occurrence of ergodic nonhyperbolic measures (i.e., with zero Lyapunov exponent) with positive entropy, see [6]. A natural question is what type of behavior (hyperbolic or nonhyperbolic one) prevails, for example in terms of entropy. Another important question is how the degree of hyperbolicity measured in terms of exponents varies, for example, how the entropy of the corresponding level sets changes.

In [21] we provide a conceptual framework for the prototypical dynamics which present the features in the above paragraph (see also Sects. 3 and 4). Moreover, we see that the mentioned topological and ergodic properties hold even for perturbations of these systems.1 The works [21,22] contain results about the topology of the space of invariant measures which laid the basis for the multifractal analysis of the entropy of the level sets of fiber Lyapunov exponents.

1.4. Tools of multifractal analysis. In the classical approach for multifractal analysis one expresses the entropy of a level set simultaneously

- in terms of a restricted variational principle and
- in terms of the Legendre–Fenchel transform of a topological pressure function.

---

1 Indeed, as explained in [21, Section 8.3], if \( \mathcal{S} \) denotes the set of step skew-product maps \( F \) as in (1.1) which are robustly transitive and have periodic points of different indices, then there is a \( C^1 \)-open and dense subset of \( \mathcal{S} \) consisting of maps with satisfy the axioms stated in Sect. 3.
It is important to point out that in our setting the dynamical system as a whole does not satisfy the specification property and none of the classical approaches apply. Here, we follow a thermodynamic approach having as one key ingredient restricted variational principles, but combining them with restricted pressure functions. The philosophy is that in order to obtain relevant multifractal information about the respective classes of exponents one should not consider the whole variational-topological pressure, but instead its restrictions to ergodic measures with corresponding exponents, so-called restricted variational pressures, and to derive the information about entropy on level sets from so-called exhausting families. As the difficulty in our setting comes from the coexistence of negative, zero, and positive fiber Lyapunov exponents and as zero exponent measures are notoriously difficult to analyze, a natural solution is to separately consider the restricted pressures defined on the ergodic measures with negative and positive exponents, respectively.

To make the link between restricted variational pressures and the multifractal information which they carry for the relevant subsystems, we follow a somewhat general principle. While we do not have specification on the whole space, we are able to find certain families of subsets (basic sets) on which we do have this property. First, we recall the general restricted variational principle for topological entropy in [11] which provides a natural lower bound for $h_{\text{top}}(L(\alpha))$, see Sect. 5.1. In Sect. 6.1, we are going to present a general theory of restricted pressures which allows us to obtain dynamical properties of the full system knowing the properties of subsystems. Our key-concept is the existence of so-called exhausting families on which each restricted variational pressure can be approached gradually. In Sect. 6.2 we show the existence of exhausting families in our setting, treating negative and positive exponents separately. For that we will strongly use the fact that for any pair of uniformly hyperbolic sets with negative (positive) fiber exponents there exists a larger one containing them both. We show that the entropy spectrum of fiber Lyapunov exponents is described in terms of the Legendre–Fenchel transforms of the respective restricted variational pressure functions and is simultaneously given in terms of a restricted variational principle. This applies to negative/positive exponents only. As a consequence, in our setting, we show that for each $\alpha \in [\alpha_{\text{min}}, 0) \cup (0, \alpha_{\text{max}}]$ the level set $L(\alpha)$ of points with fiber exponent equal to $\alpha$ is nonempty and its topological entropy changes continuously with $\alpha$ (see Fig. 1).

We proved in [21] that any nonhyperbolic ergodic measure can be approached by hyperbolic ones (weak* and in metric entropy) and that then the difficulties arising from zero exponents can be somewhat circumvented. This provides a tool to deal with zero exponent (nonhyperbolic) ergodic measures, enables us to consider exhausting families “approaching nonhyperbolicity”, and to “glue” the two parts of the spectrum, which would be completely unrelated otherwise.

To extend our results to a description of the level set of zero exponent, we then combine a thermodynamical and an orbitwise approach. On the one hand, we study the restricted variational pressure functions and extract properties from its shape. This approach gives us convexity for free, which turns out to be a surprisingly useful property. On the other hand, in our approach we put our hands on the orbits of the level sets (the amount of their entropy provides explicit information about them), using natural recurrence properties of the systems (which is guided by the concept of so-called blending intervals in Sect. 1.1), and follow an “orbit-gluing approach”.

---

2 The use of restricted (sometimes also called hidden) pressures was initiated in [39] (for rational maps of the Riemann sphere) and subsequently used, for example, in [29] (for non-exceptional rational maps) and [46] (for multimodal interval maps).
While for exponents $\alpha \in [\alpha_{\text{min}}, 0) \cup (0, \alpha_{\text{max}}]$ we can give the full description of the Lyapunov exponent level sets, including the restricted variational principle and the exact formula for their entropy, there are very restricted tools for studying the level set $L(0)$. We are able to describe its entropy, but the restricted variational principle for $h_{\text{top}}(L(0))$ cannot be obtained by our methods. Let us observe that the fact that $L(0)$ has positive topological entropy was obtained in a similar context in [6] by proving the existence of ergodic measures with positive entropy and zero exponents.\footnote{Indeed, [6] shows the existence of a compact and invariant set with positive topological entropy consisting of points with zero Lyapunov exponent.} In this paper, this property is also obtained as a surprising consequence of the shape of the pressure map. Though positive, we also show that in our setting and assuming proximality the topological entropy of $L(0)$ is strictly smaller than the maximal, that is, the topological entropy of the system.

The systems we study always have (at least) two hyperbolic ergodic measures of maximal entropy, one with negative and one with positive fiber Lyapunov exponent. Indeed, this is an immediate consequence of [16], obtained from a different point of view of our system as a random dynamical system, that is, as a product of independent and identically distributed circle diffeomorphisms, also observing the fundamental fact that our hypotheses exclude the case that our system is a rotation extension of a Bernoulli shift. This is a particular case of a result in a more general setting [49], stated for accessible partially hyperbolic diffeomorphisms having compact center leaves, see also [52] where higher regularity is required. Under the additional assumption of proximality, with [40] we even can conclude uniqueness of the ergodic measure of maximal entropy with negative and positive exponents, respectively.

1.5. Structure of the paper. In Sect. 2 we precisely state our main results with all details from which we deduce the simplified versions Theorems A and B. In Sect. 3 we describe the setting in which we derive our results and in Sect. 4 recall some key results about ergodic approximations. In Sect. 5 we give some basic information about the thermodynamical formalism (entropy and pressure function and its convex conjugate). In Section 6.1 we introduce (in an abstract setting) the restricted pressures and exhausting families, then in Section 6.2 verify their existence in the setting of our paper. Our main result Theorem 1 is proved in Sects. 7 and 8. Theorem 2 and Corollary 3 are proved in Sect. 9. Theorem 4 is shown in Sect. 10. To apply the above results to matrix cocycles, in Sect. 11 we develop several general tools to relate Lyapunov exponents of cocycles with the ones of the induced skew-products. The main result there is Theorem 11.1 which implies Theorem 5. In Sect. 11.7 we provide some more details about the space of elliptic cocycles and describe precisely the set $E_{N, \text{shyp}}$. Appendix A recalls the definition of topological entropy of general sets.

2. Precise Statements of the Results

Let $\sigma : \Sigma_N \to \Sigma_N$, $N \geq 2$, be the usual shift map on the space $\Sigma_N \overset{\text{def}}{=} \{0, \ldots, N-1\}^\mathbb{Z}$ of two-sided sequences. We equip the shift space $\Sigma_N$ with the standard metric $d_1(\xi, \eta) \overset{\text{def}}{=} 2^{-n(\xi, \eta)}$, where $n(\xi, \eta) \overset{\text{def}}{=} \sup\{\ell i : \xi_i = \eta_i \text{ for } i = -\ell, \ldots, \ell\}$. We equip $\Sigma_N \times S^1$ with the metric $d((\xi, x), (\eta, y)) \overset{\text{def}}{=} \sup\{d_1(\xi, \eta), \min\{|x - y|, 1 - |x - y|\}\}$, where $S^1$ is viewed as the unit interval with end points identified.
We will require the step skew-product
\[ F : \Sigma_N \times \mathbb{S}^1 \rightarrow \Sigma_N \times \mathbb{S}^1, \quad F(\xi, x) = (\sigma(\xi), f_{\xi_0}(x)) \] (2.1)
to satisfy Axioms CEC± and Acc± (see Sect. 3). Sometimes, we will also take another point of view and study the underlying iterated function system (IFS) generated by the family of maps \( \{f_i\}_{i=0}^{N-1} \). We will denote by \( \pi : \Sigma_N \times \mathbb{S}^1 \rightarrow \Sigma_N \) the natural projection \( \pi(\xi, x) \overset{\text{def}}{=} \xi \).

Let \( \mathcal{M} \) be the space of \( F \)-invariant probability measures supported in \( \Sigma_N \times \mathbb{S}^1 \), equip \( \mathcal{M} \) with the weak* topology, and denote by \( \mathcal{M}_{\text{erg}} \subset \mathcal{M} \) the subset of ergodic measures. To characterize nonhyperbolicity, given \( \mu \in \mathcal{M} \) denote by \( \chi(\mu) \) its (fiber) Lyapunov exponent which is given by
\[ \chi(\mu) \overset{\text{def}}{=} \int \log |(f_{\xi_0})'(x)| \, d\mu(\xi, x). \]

An ergodic measure \( \mu \) is nonhyperbolic if \( \chi(\mu) = 0 \). Otherwise the measure is hyperbolic. In our setting, any hyperbolic ergodic measure has either a negative or a positive exponent. Accordingly, we divide the set of all ergodic measures and consider the decomposition
\[ \mathcal{M}_{\text{erg}} = \mathcal{M}_{\text{erg}, < 0} \cup \mathcal{M}_{\text{erg}, 0} \cup \mathcal{M}_{\text{erg}, > 0} \] (2.2)
into measures with negative, zero, and positive fiber Lyapunov exponent, respectively.

In our setting, each component is nonempty. In general, it is very difficult to determine which type of hyperbolicity “prevails”. For that we will study the spectrum of possible exponents and will perform a multifractal analysis of the topological entropy of level sets of equal (fiber) Lyapunov exponent.

To be more precise, a sequence \( \xi = (\ldots \xi_{-1}, \xi_0, \xi_1, \ldots) \in \Sigma_N \) can be written as \( \xi = \xi^+ \xi^- \), where \( \xi^+ \in \Sigma_N^+ \overset{\text{def}}{=} \{0, \ldots, N-1\}^N = \{0, \ldots, N-1\} \). Given finite sequences \( (\xi_0, \ldots, \xi_n) \) and \( (\xi_m, \ldots, \xi_{-1}) \), we let
\[ f_{[\xi_0, \ldots, \xi_n]} \overset{\text{def}}{=} f_{\xi_n} \circ \cdots \circ f_{\xi_0} \quad \text{and} \quad f_{[\xi_m, \ldots, \xi_{-1}]} \overset{\text{def}}{=} (f_{\xi_{m-1}} \cdots f_{\xi_{-1}})^{-1}. \]

For \( n \geq 0 \) denote also
\[ f^N_{\xi} \overset{\text{def}}{=} f_{[\xi_0, \ldots, \xi_{n-1}]} \quad \text{and} \quad f^{-n}_{\xi} \overset{\text{def}}{=} f_{[\xi_{-n}, \ldots, \xi_{-1}]}^{-1}. \]

As usual, we use the following notation for cylinder sets \( [\xi_0, \ldots, \xi_n] \overset{\text{def}}{=} \{ \eta \in \Sigma_N : \eta_0 = \xi_0, \ldots, \eta_n = \xi_n \} \) and \( [\xi_{-n}, \ldots, \xi_0] \overset{\text{def}}{=} \{ \eta \in \Sigma_N : \eta_{-1} = \xi_{-1}, \ldots, \eta_{-n} = \xi_{-n} \} \). Given \( X = (\xi, x) \in \Sigma_N \times \mathbb{S}^1 \) consider the (fiber) Lyapunov exponent of \( X \)
\[ \chi(X) \overset{\text{def}}{=} \lim_{n \to \pm \infty} \frac{1}{n} \log |(f^N_{\xi})'(x)|, \]
where we assume that both limits \( n \to \pm \infty \) exist and coincide. Note that in our context the exponent is nothing but the Birkhoff average of the continuous function (also called potential) \( \varphi : \Sigma_N \times \mathbb{S}^1 \rightarrow \mathbb{R} \) defined for \( X = (\xi, x) \) by
\[ \varphi(X) \overset{\text{def}}{=} \log |(f_{\xi_0})'(x)|. \] (2.3)
We will analyze the topological entropy of the following level sets of Lyapunov exponents: given $\alpha \in \mathbb{R}$ let
\[
\mathcal{L}(\alpha) \overset{\text{def}}{=} \{ X \in \Sigma_N \times \mathbb{S}^1 : \chi(X) = \alpha \},
\]
assuming that the Lyapunov exponent at $X$ is well defined and equal to $\alpha$. Note that each level set is invariant but, in general, noncompact. Hence we will rely on the general concept of topological entropy $h_{\text{top}}$ introduced by Bowen [11] (see Appendix A). Denoting by $\mathcal{L}_{\text{irr}}$ the set of points where the fiber Lyapunov exponent is not well-defined (either one of the limits does not exist or both limits exist but they do not coincide), we obtain the following multifractal decomposition
\[
\Sigma_N \times \mathbb{S}^1 = \bigcup_{\alpha \in \mathbb{R}} \mathcal{L}(\alpha) \cup \mathcal{L}_{\text{irr}}.
\]
Note that $\mathcal{L}(\alpha)$ will be nonempty in some range of $\alpha$, only. Under our axioms this range decomposes into three natural nonempty parts
\[
\{ \alpha : \mathcal{L}(\alpha) \neq \emptyset \} = [\alpha_{\text{min}}, 0) \cup \{0\} \cup (0, \alpha_{\text{max}}],
\]
where
\[
\alpha_{\text{max}} \overset{\text{def}}{=} \sup \{ \alpha : \mathcal{L}(\alpha) \neq \emptyset \}, \quad \alpha_{\text{min}} \overset{\text{def}}{=} \inf \{ \alpha : \mathcal{L}(\alpha) \neq \emptyset \}.
\]
We have that inf and sup are indeed attained, justifying the notation.

To state our main results, we need the following thermodynamical quantities. Denote by $h(\mu)$ the entropy of a measure $\mu$ and consider the following pressures and their convex conjugates (see Sect. 5 for details)
\[
P_{\ast}(q \varphi) \overset{\text{def}}{=} \sup_{\mu \in \mathcal{M}_{\text{erg}, \ast}} \left( h(\mu) - q \chi(\mu) \right), \quad \mathcal{E}_{\ast}(\alpha) \overset{\text{def}}{=} \inf_{q \in \mathbb{R}} \left( P_{\ast}(q \varphi) - q \alpha \right),
\]
where $\ast$ should be replaced by $< 0$ and $> 0$, respectively (recall (2.2)). In the terminology of [47], this would be called \textit{(positive/negative) variational hyperbolic pressure}, we call them simply \textit{pressures}. For simplicity we will use the notation
\[
P_{\ast}(q) \overset{\text{def}}{=} P_{\ast}(q \varphi),
\]
as $\{q \varphi\}_{q \in \mathbb{R}}$ is the only family of potentials whose pressure we are going to consider. Similarly, we define
\[
P_0(q) \overset{\text{def}}{=} \sup_{\mu \in \mathcal{M}_{\text{erg}, 0}} h(\mu).
\]
Clearly,
\[
\max\{P_{<0}(q), P_0(q), P_{>0}(q)\} = P_{\text{top}}(q \varphi)
\]
is the classical \textit{topological pressure} of $q \varphi$ with respect to $F$ (see [55, Chapter 7]). We will also write $\mathcal{E}$ for both $\mathcal{E}_{>0}$ and $\mathcal{E}_{<0}$, because the domains of those two functions are disjoint.
Theorem 1. Consider a transitive step skew-product map \( F \) as in (2.1) whose fiber maps are \( C^1 \). Assume that \( F \) satisfies Axioms CEC\( ^\pm \) and Acc\( \pm \).

Then there are numbers \( \alpha_{\min} < 0 < \alpha_{\max} \) such that \( \alpha \in [\alpha_{\min}, \alpha_{\max}] \) if and only if \( \mathcal{L}(\alpha) \neq \emptyset \). Moreover,

(a) for every \( \alpha \in [\alpha_{\min}, 0) \) we have

\[
h_{\text{top}}(\mathcal{L}(\alpha)) = \sup \{ h(\mu) : \mu \in \mathcal{M}_{\text{erg}}, \chi(\mu) = \alpha \} = \mathcal{E}_{<0}(\alpha),
\]

(b) for every \( \alpha \in (0, \alpha_{\max}] \) we have

\[
h_{\text{top}}(\mathcal{L}(\alpha)) = \sup \{ h(\mu) : \mu \in \mathcal{M}_{\text{erg}}, \chi(\mu) = \alpha \} = \mathcal{E}_{>0}(\alpha),
\]

(c) for every \( \alpha \in \{\alpha_{\min}, 0, \alpha_{\max}\} \) we have

\[
\lim_{\beta \to \alpha} h_{\text{top}}(\mathcal{L}(\beta)) = h_{\text{top}}(\mathcal{L}(\alpha)),
\]

(d) for every \( \alpha \in (\alpha_{\min}, \alpha_{\max}) \) we have \( h_{\text{top}}(\mathcal{L}(\alpha)) > 0 \).

Moreover, there exist (finitely many) ergodic measures \( \mu_+, \mu_- \) of maximal entropy \( h(\mu_+) = \log N \) and with \( \chi(\mu_-) < 0 < \chi(\mu_+) \).

To prove uniqueness of the measures \( \mu_\pm \) of maximal entropy, we require the additional assumption (see Sect. 9.1 for discussion). We say that the iterated function system (IFS) generated by the family of fiber maps \( \{f_i\}_{i=0}^{N-1} \) of the step skew-product map \( F \) is proximal\(^4\) if for every \( x, y \in S^1 \) there exists at least one sequence \( \xi \in \Sigma_N \) such that \( |f^n_\xi(x) - f^n_\xi(y)| \to 0 \) as \( |n| \to \infty \). By some abuse of notation, in this case we also say that the skew-product is proximal.

Remark 2.1. It is easy to see that the step skew-product is proximal if, for example, there exists one Morse–Smale fiber map with exactly one attracting and one repelling fixed point (North pole-South pole map) and the step skew-product satisfies Axioms CEC\( \pm \) and Acc\( \pm \).

Theorem 2. Assume the hypothesis of Theorem 1 and also proximality of the step skew-product. Then there exist unique ergodic \( F \)-invariant probability measures \( \mu_- \) and \( \mu_+ \) of maximal entropy \( h(\mu_\pm) = \log N \), respectively, and satisfying

\[
\alpha_- \overset{\text{def}}{=} \chi(\mu_-) < 0 < \alpha_+ \overset{\text{def}}{=} \chi(\mu_+).
\]

We have

\[
h_{\text{top}}(\mathcal{L}(\alpha_-)) = h_{\text{top}}(\mathcal{L}(\alpha_+)) = \log N
\]

and for all \( \alpha \notin \{\alpha_- , \alpha_+\} \) we have

\[
h_{\text{top}}(\mathcal{L}(\alpha)) < \log N.
\]

Under the hypothesis of Theorem 1, some possible shapes of the graph of the corresponding function \( \alpha \to h_{\text{top}}(\mathcal{L}(\alpha)) = \mathcal{E}(\alpha) \) are as in Fig. 3 and under the hypotheses of Theorem 2 as in Fig. 3 (left figure).

\(^4\) We borrow this terminology from [40].
Similar phenomenon as in Theorem 2 (the entropy achieving its maximum away from zero exponent) in a slightly different setting (for ergodic measures on $C^2$ systems) was observed in [52]. We note that somewhat related questions about the topology of the space of measures are considered in [7,22,28,34].

In the following, when referring to weak* and in entropy convergence we mean that a sequence of measures converges in the weak* topology and their entropies converge to the entropy of the limit measure.

**Corollary 3.** Under the hypothesis of Theorem 2, no measure which is a nontrivial convex combination of the two ergodic measures of maximal entropy is a weak* and in entropy limit of ergodic measures.

The results in [52] and our results suggest the following conjecture (which is indeed true for maximal entropy measures, by Corollary 3).

**Conjecture.** For every pair of hyperbolic ergodic measures $\mu_1$ and $\mu_2$ with $\chi(\mu_1) < 0 < \chi(\mu_2)$ every nontrivial convex combination of $\mu_1$ and $\mu_2$ cannot be approximated (weak* and in entropy) by ergodic measures.

We finally summarize the properties of (restricted) pressure functions, its Legendre–Fenchel transform, and of the entropy spectrum of Lyapunov exponents in the following theorem (compare Figs. 2 and 3).

**Theorem 4.** Under the assumptions of Theorem 1, we have the following:
(a) $P_{<0}$ and $P_{>0}$ are nonincreasing and nondecreasing convex functions, respectively,
(b) (Plateaus) There are numbers $D_{\pm}$ and $h_{\pm} > 0$ such that
$$P_{<0}(q) = h_- \text{ for all } q \geq D_- \quad \text{and} \quad P_{>0}(q) = h_+ \text{ for all } q \leq D_+.$$
(c) $h_- = h_+ = h_{\text{top}}(\mathcal{L}(0))$. 

**Fig. 2.** Some possible shapes of pressures. Left figure: Under the hypothesis of Theorem 2

**Fig. 3.** Some possible shapes of convex conjugates. Left figure: Under the hypothesis of Theorem 2
For $\alpha < 0$ the function $\alpha \mapsto \mathcal{E}(\alpha)$ is a Legendre–Fenchel transform of $q \mapsto \mathcal{P}_{\leq 0}(q)$. Similarly, for $\alpha > 0$ the function $\alpha \mapsto \mathcal{E}(\alpha)$ is a Legendre–Fenchel transform of $q \mapsto \mathcal{P}_{> 0}(q)$. In particular, $\alpha \mapsto \mathcal{E}(\alpha)$ is a concave function on the domains $\alpha < 0$ and $\alpha > 0$, respectively.

For $\alpha < 0$ the function $\alpha \mapsto \mathcal{L}(\alpha)$ achieves its maximum value $\log N$ at some points

$$\alpha_- < 0 \quad \text{and} \quad \alpha_+ > 0.$$ 

Moreover, under the assumptions of Theorem 2 we have additional properties

(k) $\mathcal{P}_{> 0}$ and $\mathcal{P}_{\leq 0}$ are differentiable at $q = 0$

and in items (d) and (i) we have strict inequalities:

$$D_+ < 0 < D_- \quad \text{and} \quad D_L \mathcal{E}(0) < 0 < D_R \mathcal{E}(0),$$

and the points $\alpha_-, \alpha_+$ in item (f) are the unique numbers $\alpha$ for which $h_{\text{top}}(\mathcal{L}(\alpha)) = \log N$.

**Remark 2.2** (Open questions). The following questions remain open. The restricted pressures can be differentiable or nondifferentiable at the beginning of the plateaus in Theorem 4 item b). The nondifferentiability of, for example, $\mathcal{P}_{> 0}$ at $D_-$ would mean that $\mathcal{E}(\alpha)$ is linear on some interval $[0, q]$. Further regularity properties (smoothness, analyticity) of the restricted pressure functions (excluding the ends of plateaus) and of the spectrum are unknown.

The asymptote of $\mathcal{P}_{> 0}$ at $q \to \infty$ is some line \(P = \alpha_{\text{max}}q + h_{\text{max}}\), similarly $\mathcal{P}_{\leq 0}$ is asymptotic to \(P = \alpha_{\text{min}}q + h_{\text{min}}\), and, in general, we do not know whether $h_{\text{max}}$ and $h_{\text{min}}$ are equal to zero (which would mean that $h_{\text{top}}(\mathcal{L}(\alpha_{\text{max}})) = h_{\text{top}}(\mathcal{L}(\alpha_{\text{min}})) = 0$; this type of problem is also studied within the field of ergodic optimization, see for example [36]). Of course, $h_{\text{max}}$ and $h_{\text{min}}$ may be positive in the case when some generators of the IFS are duplicated, for example, when $f_0 = f_1$ and $f_2 = f_3$. Note also that [8] provides sufficient conditions for $h_{\text{max}} = 0 = h_{\text{min}}$. We are also not able to exclude the possibility that $h_{\text{top}}(\mathcal{L}(\alpha_{\text{max}})) = \log N$ or $h_{\text{top}}(\mathcal{L}(\alpha_{\text{min}})) = \log N$; even under the additional assumption of proximality, we do not know if the cases $\alpha_- = \alpha_{\text{min}}$ and/or $\alpha_+ = \alpha_{\text{max}}$ do not occur.

Finally, even though we know that there do exist ergodic measures with Lyapunov exponent zero and with positive entropy (this follows from [6]), we do not know if there exist such measures with entropy arbitrarily close to $h_{\text{top}}(\mathcal{L}(0))$ (which would mean that we have the restricted variational principle also for exponent zero).

Our final result deals with cocycles. Recall, given a cocycle $A \in SL(2, \mathbb{R})^N$, the definitions of the associated skew-product $F_A: \Sigma_N \times \mathbb{P}^1 \to \Sigma_N \times \mathbb{P}^1$ with fiber maps $f_A$ as in (1.3) and the level sets $\mathcal{L}_A$ in (1.4) and $\mathcal{L}(\alpha)$ as defined in (2.4) for $F_A$. Recall also the existence of the open and dense subset $\mathcal{E}_{N, \text{shyp}} \subset \mathcal{E}_N$ in Theorem B and Sect. 11.7.

**Theorem 5.** For every $N \geq 2$ and every $A \in \mathcal{E}_{N, \text{shyp}}$ we have the following: There are numbers $0 < \alpha_+ \leq \alpha_{\text{max}}$ such that for every $\alpha \in [0, \alpha_{\text{max}}]$ we have $\mathcal{L}_A(\alpha) \neq \emptyset$. Moreover,
(a) for every $\alpha \in [0, \alpha_{\max}]$ we have

$$h_{\text{top}}(L^+_A(\frac{\alpha}{2})) = h_{\text{top}}(L(\alpha)) = h_{\text{top}}(L(-\alpha)).$$

In particular, the function $\alpha \mapsto h_{\text{top}}(L(\alpha))$ is even.

(b) For all $\alpha \in [0, \alpha_{\max}) \setminus \{\alpha_s\}$ we have

$$0 < h_{\text{top}}(L^+_A(\alpha)) < \log N = h_{\text{top}}(L^+_A(\alpha_s)).$$

The proof of Theorem 5 has two parts. First, we apply Theorems 1 and 2 to the skew-product $F_A$ whose hypotheses are satisfied for every $A \in \mathcal{C}_{N, \text{shyp}}$. Second, we perform a careful analysis of the relation between the spectra of exponents of the cocycle and the ones of the associated skew-product (see Theorem 11.1); this is done in Sect. 11.

3. Setting

We recall the precise setting of our Axioms CEC± and Acc± and their main consequences, established in [21]. The step skew-product structure of $F$ allows us to reduce the study of its dynamics to the study of the IFS generated by the fiber maps $\{f_i\}_{i=0}^{N-1}$. In what follows we always assume that $F$ is transitive.

Given a point $x \in S^1$, consider and define its forward and backward orbits by

$$O^+(x) \overset{\text{def}}{=} \bigcup_{n \geq 1} \bigcup_{(\theta_0, \ldots, \theta_{n-1})} f[\theta_0, \ldots, \theta_{n-1}](x)$$

and

$$O^-(x) \overset{\text{def}}{=} \bigcup_{m \leq 1} \bigcup_{(\theta_{-m}, \ldots, \theta_{-1})} f[\theta_{-m}, \ldots, \theta_{-1}](x),$$

respectively. Consider also the full orbit of $x$

$$O(x) \overset{\text{def}}{=} O^+(x) \cup O^-(x).$$

Similarly, we define the orbits $O^+(J)$, $O^-(J)$, and $O(J)$ for any subset $J \subset S^1$.

In requiring that the step skew-product $F$ with fiber maps $\{f_i\}_{i=0}^{N-1}$ satisfies the Axioms CEC± and Acc± we mean that there are so-called (closed) forward and backward blending intervals $J^+, J^- \subset S^1$ such that the following properties hold.

**CEC+(J+)** (Controlled Expanding forward Covering relative to $J^+$). There exist positive constants $K_1, \ldots, K_5$ such that for every interval $H \subset S^1$ intersecting $J^+$ and satisfying $|H| < K_1$ we have

- (controlled covering) there exists a finite sequence $(\eta_0 \ldots \eta_{\ell-1})$ for some positive integer $\ell \leq K_2 |\log |H|| + K_3$ such that

  $$f[\eta_0, \ldots, \eta_{\ell-1}](H) \supset B(J^+, K_4),$$

  where $B(J^+, \delta)$ is the $\delta$-neighborhood of the set $J^+$.

- (controlled expansion) for every $x \in H$ we have

  $$\log |(f[\eta_0, \ldots, \eta_{\ell-1}])'(x)| \geq \ell K_5.$$

**CEC−(J−)** (Controlled Expanding backward Covering relative to $J^-$). The step skew-product $F^{-1}$ satisfies the Axiom CEC+(J−).

**Acc+(J+)** (forward Accessibility relative to $J^+$). $O^+(\text{int } J^+) = S^1$.

**Acc−(J−)** (backward Accessibility relative to $J^-$). $O^-(\text{int } J^-) = S^1$.

When the step skew-product $F$ is transitive then there is a common interval $J \subset S^1$ satisfying CEC±(J) and Acc±(J) (see Lemma 4.5 and detailed discussion in [21, Section 2.2]).
Remark 3.1 (Remark 1.1 continued). Consider an IFS \( \{ f_i \}_{i=0}^{N-1} \) of diffeomorphisms \( f_0, \ldots, f_{N-1} : S^1 \to S^1 \) and assume that there are finite sequences \((\xi_0 \ldots \xi_r)\) and \((\zeta_0 \ldots \zeta_t)\) such that \( f[\xi_0 \ldots \xi_r] \) is Morse–Smale with exactly one attracting fixed point and one repelling fixed point and \( f[\zeta_0 \ldots \zeta_t] \) is an irrational rotation. Then by [21, Proposition 8.8], every \( C^1 \)-small perturbation of this IFS satisfies Axioms CEC± and Acc±. Moreover, the system is proximal (recall Remark 2.1).

Also note that it is enough to assume that \( f[\zeta_0 \ldots \zeta_t] \) is only \( C^2 \) conjugate to an irrational rotation (this avoids Denjoy-like counterexamples guaranteeing that every orbit is dense).

4. Ergodic Approximations

We recall some technical results from [21]. The first one claims that any nonhyperbolic ergodic measure \( \mu \) (that is, with exponent \( \chi(\mu) = 0 \)) is weak* and in entropy approximated by hyperbolic ergodic measures.

**Lemma 4.1** (Rephrasing partially [21, Theorem 1]). For every ergodic measure \( \mu \) with zero Lyapunov exponent \( \chi(\mu) = 0 \) there is a sequence of ergodic measures \( \nu_i \) with Lyapunov exponents \( \chi(\nu_i) = \beta_i \) such that \( \beta_i > 0, \beta_i \to 0, \lim_{i \to \infty} \nu_i = \mu \) in the weak* topology, and

\[
\lim_{i \to \infty} h(\nu_i) = h(\mu).
\]

The same holds true with ergodic measures \( \nu_i \) satisfying \( \chi(\nu_i) = \alpha_i \) such that \( \alpha_i < 0 \) and \( \alpha_i \to 0 \).

A further result claims that given an ergodic measure \( \mu \) with exponent \( \chi(\mu) = \alpha > 0 \) and entropy \( h(\mu) > 0 \), for every small \( \beta < 0 \) there are ergodic measures with exponents close to \( \beta \) and positive entropy, but in this construction some entropy is lost. [21, Theorem 5] bounds the amount of lost entropy that is related to the size of \( \alpha + |\beta| \). A specially interesting case occurs when the exponent \( \beta \) is taken arbitrarily close to \( 0^- \). The estimates are summarized in the next lemma.

**Lemma 4.2** (Rephrasing partially [21, Theorem 5]). There exists \( c > 0 \) such that for every ergodic measure \( \mu \) with nonzero Lyapunov exponent \( \chi(\mu) = \alpha \neq 0 \) there is a sequence of ergodic measures \( \nu_i \) with Lyapunov exponents \( \chi(\nu_i) = \beta_i \), \( \sgn \alpha \neq \sgn \beta_i \), such that \( \beta_i \to 0 \) and

\[
\lim_{i \to \infty} h(\nu_i) \geq \frac{h(\mu)}{1 + c|\alpha|}.
\]

This result also implies the following.

**Corollary 4.3.** There exist ergodic measures with negative/positive exponents arbitrarily close to 0.

The systems considered in this paper satisfy the so-called skeleton property which implies the existence of orbit pieces that allow to approximate entropy and Lyapunov exponent, see [21, Section 4] for details. The skeleton property is referred to some blending interval and to quantifiers corresponding to the entropy and a level set for the Lyapunov exponent. An important property is that if \( \mathcal{L}(\alpha) \neq 0 \) then the skeleton property holds relative to \( h = h_{\text{top}}(\mathcal{L}(\alpha)) \) and \( \alpha \).
Given a compact $F$-invariant set $\Gamma \subset \Sigma_N \times S^1$, we say that $\Gamma$ has \textit{uniform fiber expansion (contraction)} if every ergodic measure $\mu \in \mathcal{M}_{\text{erg}}(\Gamma)$ has a positive (a negative) Lyapunov exponent. It is \textit{hyperbolic} if it either has uniform fiber expansion or uniform fiber contraction. We say that a set is \textit{basic} (with respect to $F$) if it is compact, $F$-invariant, locally maximal, topologically transitive, and hyperbolic.\footnote{This definition mimics the usual definition of a basic set in a differentiable setting.}

### Proposition 4.4 ([21, Theorems 4.3 and 4.4 and Proposition 4.8]).

Given $\alpha \leq 0$ such that $\mathcal{L}(\alpha) \neq \emptyset$ and $h = h_{\text{top}}(\mathcal{L}(\alpha)) > 0$, for every $\gamma \in (0, h)$ and every small $\lambda > 0$ there is a basic set $\Gamma \subset \Sigma_N \times S^1$ such that

1. $h_{\text{top}}(\Gamma) \in [h - \gamma, h + \gamma]$ and
2. every $\nu \in \mathcal{M}_{\text{erg}}(\Gamma)$ satisfies $\chi(\nu) \in (\alpha - \lambda, \alpha + \lambda) \cap \mathbb{R}_-$.

The analogous result holds for any $\alpha \geq 0$.

A further consequence of the Axioms CEC± and Acc± is that the IFS $\{f_i\}$ is forward and backward minimal. [21, Lemma 2.2] states a quantitative version of this minimality. We also will use the following results which are simple consequences of these axioms.

### Lemma 4.5 ([21, Lemmas 2.2 and 2.3]).

Every nontrivial interval $I \subset S^1$ contains a subinterval $J \subset I$ such that $F$ satisfies Axioms CEC±($J$) and Acc±($J$). Moreover, there is a number $M = M(I) \geq 1$ such that for any point $x \in S^1$ there are finite sequences $((\theta_1, \ldots, \theta_r))$ and $((\beta_1, \ldots, \beta_s))$ with $r, s \leq M$ such that

$$f_{[\theta_1, \ldots, \theta_r]}(x) \in I \quad \text{and} \quad f_{[\beta_1, \ldots, \beta_s]}(x) \in I.$$

### Lemma 4.6 ([21, Lemma 2.4]).

For every interval $I \subset S^1$ there exist $\delta = \delta(I) > 0$ and $M = M(I) \geq 1$ such that for any interval $J \subset S^1$, $|J| < \delta$, there exists a finite sequence $(\tau_1, \ldots, \tau_m)$, $m \leq M$, such that $f_{[\tau_1, \ldots, \tau_m]}(J) \subset I$.

We finish this section with one further conclusion which we will use in Sects. 7.1 and 9.1.

### Lemma 4.7.

There does not exist a Borel probability measure $m$ on $S^1$ which is $f_i$-invariant for every $i = 0, \ldots, N - 1$.

**Proof.** By contradiction, assume that there is a Borel probability measure $m$ on $S^1$ which is simultaneously $f_i$-invariant for all $i$. Let $J \subset S^1$ be a blending interval and consider two closed disjoint small sub-intervals $J_1, J_2 \subset J$. By Axiom CEC+(J), there is some sequence $(\eta_0, \ldots, \eta_{\ell-1})$ such that $f_{[\eta_0, \ldots, \eta_{\ell-1}]}(J_1) \supset J$. From this we can conclude that $m(J \setminus J_1) = 0$. Similarly, $m(J \setminus J_2) = 0$. This implies $m(J) = 0$. Hence, by Acc±($J$) we have that $m(S^1) = 0$. But this is a contradiction. \hfill $\square$

## 5. Entropy, Pressures, and Variational Principles

In this section, we collect some general facts about entropy and pressure. We consider a general setting of a compact metric space $(X, d)$, a continuous map $F : X \to X$, and a continuous function $\varphi : X \to \mathbb{R}$.\footnotetext{This definition mimics the usual definition of a basic set in a differentiable setting.}
5.1. Entropy: restricted variational principles. Given $\alpha \in \mathbb{R}$ consider the level sets

$$\mathcal{L}(\alpha) \overset{\text{def}}{=} \{ x \in X : \varphi(x) = \alpha \},$$

where $\varphi(x) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(F^k(x)),$

whenever this limit exists. We study the topological entropy of $F$ on the set $\mathcal{L}(\alpha)$ and consider the function

$$\alpha \mapsto h_{\text{top}}(\mathcal{L}(\alpha)).$$

We will now recall some results which are known for such general setting. An upper bound for the entropy $h_{\text{top}}(\mathcal{L}(\alpha))$ (which, in fact, is sharp in many cases) is easily derived applying a general result by Bowen [11]. Denote by $\mathcal{M}(X)$ the set of all $F$-invariant probability measures and by $\mathcal{M}_{\text{erg}}(X) \subset \mathcal{M}(X)$ the subset of ergodic measures. We equip this space with the weak* topology. Given $x \in X,$ let $V_F(x) \subset \mathcal{M}(X)$ be the set of $(F$-invariant) measures which are weak* limit points as $n \to \infty$ of the empirical measures $\mu_{x,n}$

$$\mu_{x,n} \overset{\text{def}}{=} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{F^k(x)},$$

where $\delta_x$ is the Dirac measure supported on the point $x.$ Given $\mu \in \mathcal{M}(X),$ denote by $G(\mu)$ the set of $\mu$-generic points

$$G(\mu) \overset{\text{def}}{=} \{ x : \lim_{n \to \infty} \mu_{x,n} = \{ \mu \} \}.$$

Given $c \geq 0,$ define the set of its “quasi regular” points by

$$QR(c) \overset{\text{def}}{=} \{ y \in X : \text{there exists } \mu \in V_F(y) \text{ with } h(\mu) \leq c \}.$$

**Proposition 5.1.** (i) $h_{\text{top}}(QR(c)) \leq c$ ([11, Theorem 2]).

(ii) For $\mu$ ergodic we have $h(\mu) = h_{\text{top}}(G(\mu))$ ([11, Theorem 3]).

(iii) If $F$ satisfies the specification property, then for every $\mu \in \mathcal{M}(X)$ we have $h(\mu) = h_{\text{top}}(G(\mu))$ ([45, Theorem 1.2] or [24, Theorem 1.1]).

We have the following simple consequence. Let

$$\varphi(\mu) \overset{\text{def}}{=} \int \varphi \, d\mu.$$ 

**Lemma 5.2.** For every $\alpha$ such that $\mathcal{L}(\alpha) \neq \emptyset$ we have

$$\sup \{ h(\mu) : \mu \in \mathcal{M}_{\text{erg}}(X), \varphi(\mu) = \alpha \} \leq h_{\text{top}}(\mathcal{L}(\alpha)) \leq \sup \{ h(\mu) : \mu \in \mathcal{M}(X), \varphi(\mu) = \alpha \}.$$ 

Moreover, for $\alpha = \sup \{ \varphi(\mu) : \mu \in \mathcal{M}_{\text{erg}}(X) \}$ we have

$$h_{\text{top}}(\mathcal{L}(\alpha)) = \sup \{ h(\mu) : \mu \in \mathcal{M}_{\text{erg}}, \varphi(\mu) = \alpha \}.$$ 

Analogously for $\alpha = \inf \{ \varphi(\mu) : \mu \in \mathcal{M}_{\text{erg}}(X) \}.$

---

6 Note that, in fact, this result holds true for any map which has the so-called $g$-almost product property which is implied by the specification property (see [45, Proposition 2.1]). The specification property is satisfied for example for every basic set (see [51]). We emphasize that the skew-product systems we study in this paper do not satisfy the specification property.
Proof. To prove the first inequality, observe that for \( \mu \) ergodic with \( \varphi(\mu) = \alpha \) we have \( G(\mu) \subset \mathcal{L}(\alpha) \) and by Proposition 5.1 (ii) and monotonicity of topological entropy with respect to inclusion we obtain \( h(\mu) = h_{\text{top}}(G(\mu)) \leq h_{\text{top}}(\mathcal{L}(\alpha)) \).

To prove the second inequality, denote \( H(\alpha) \overset{\text{def}}{=} \sup \{ h(\mu) : \mu \in \mathcal{M}(X), \varphi(\mu) = \alpha \} \).

Note that for every \( x \in \mathcal{L}(\alpha) \) we have \( \varphi(x) = \alpha \) and hence for every \( \mu \in \mathcal{V}_F(x) \) we have \( \varphi(\mu) = \alpha \) and thus \( h(\mu) \leq H(\alpha) \). Hence, \( \mathcal{L}(\alpha) \subset \mathcal{Q}(H(\alpha)) \) and again by monotonicity and Proposition 5.1 (i) we obtain

\[
h_{\text{top}}(\mathcal{L}(\alpha)) \leq h_{\text{top}}(\mathcal{Q}(H(\alpha))) \leq H(\alpha),
\]

proving the first part of the lemma.

It remains to consider the extremal value \( \alpha = \sup \{ \varphi(\mu) : \mu \in \mathcal{M}_{\text{erg}}(X) \} \). By the ergodic decomposition, any invariant measure with extremal value \( \alpha \) has almost surely only ergodic measures with that value in its decomposition. Hence we have

\[
h_{\text{top}}(\mathcal{L}(\alpha)) \leq \sup \{ h(\mu) : \mu \in \mathcal{M}_{\text{erg}}, \varphi(\mu) = \alpha \},
\]

ending the proof. \( \square \)

We recall the following classical restricted variational principle strengthening the above lemma which will play a central role in our arguments. We point out that it requires \( \varphi \) to be continuous, only.

**Proposition 5.3** ([45, Theorem 6.1 and Proposition 7.1] or [24, Theorem 1.3] and [51]). If \( F : X \to X \) satisfies the specification property then for every \( \alpha \) such that \( \mathcal{L}(\alpha) \neq \emptyset \) we have

\[
h_{\text{top}}(\mathcal{L}(\alpha)) = \sup \{ h(\mu) : \mu \in \mathcal{M}(X), \varphi(\mu) = \alpha \}.
\]

Moreover, \( \{ \varphi(\mu) : \mu \in \mathcal{M}_{\text{erg}}(X) \} \) is an interval.

### 5.2. Pressure functions.

For a measure \( \mu \in \mathcal{M}(X) \) we define the affine functional \( P(\cdot, \mu) \) on the space of continuous functions by

\[
P(\varphi, \mu) \overset{\text{def}}{=} h(\mu) + \int \varphi \, d\mu.
\]

Given an \( F \)-invariant compact subset \( Y \subset X \), we define the topological pressure of \( \varphi \) with respect to \( F|_Y \) by

\[
P_{F|_Y}(\varphi) \overset{\text{def}}{=} \sup_{\mu \in \mathcal{M}(Y)} P(\varphi, \mu) = \sup_{\mu \in \mathcal{M}_{\text{erg}}(Y)} P(\varphi, \mu)
\]

and we simply write \( P(\varphi) = P_{F|_X}(\varphi) \) if \( Y = X \) and \( F|_X \) is clear from the context. Note that definition and equality in (5.1) are nothing but the variational principle of the topological pressure (see [55, Chapter 9] for a proof and a purely topological and equivalent definition of pressure). A measure \( \mu \in \mathcal{M}(Y) \) is an equilibrium state for \( \varphi \)
(with respect to \( F|_{Y} \)) if it realizes the supremum in (5.1). Note that \( h_{\text{top}}(Y) = P_{F|_{Y}}(0) \) is the topological entropy of \( F \) on \( Y \).

We now continue by considering a decomposition of the set of ergodic measures and studying corresponding pressure functions. Given a subset \( N \subseteq \mathcal{M}(X) \), define

\[
P(\varphi, N) \overset{\text{def}}{=} \sup_{\mu \in N} P(\varphi, \mu).
\]

Given \( N \subseteq \mathcal{M}(X) \), consider its closed convex hull \( \overline{\text{conv}}(N) \), defined as the smallest closed convex set containing \( N \). It is an immediate consequence of the affinity of \( \mu \mapsto P(\varphi, \mu) \) that

\[
P(\varphi, N) = P(\varphi, \overline{\text{conv}}(N)).
\]

A particular consequence of this equality and the ergodic decomposition of non-ergodic measures is the fact that for \( N = \mathcal{M}_{\text{erg}}(X) \) and hence \( \overline{\text{conv}}(N) = \mathcal{M}(X) \) in (5.1) it is irrelevant if we take the supremum over all measures in \( \mathcal{M}(X) \) or over the ergodic measures only (used to show the equality in (5.1)). The case of a general subset \( N \) of \( \mathcal{M}(X) \), however, will be quite different and is precisely our focus of interest.

We now analyze the pressure function for a subset of ergodic measures \( N \subseteq \mathcal{M}_{\text{erg}}(X) \).

Let \( q \in \mathbb{R} \) and consider the parametrized family \( q\varphi : X \to \mathbb{R} \) and the function

\[
\mathcal{P}_{N}(q) \overset{\text{def}}{=} P(q\varphi, N).
\]

For each \( \mu \in N \) we simply write \( \mathcal{P}_{\mu}(q) = \mathcal{P}_{[\mu]}(q) = P(q\varphi, \{\mu\}) \). We call \( \mu \in \mathcal{M}(X) \) an equilibrium state for \( q\varphi, q \in \mathbb{R} \), (with respect to \( N \)) if \( \mathcal{P}_{N}(q) = \mathcal{P}_{\mu}(q) \). Let also

\[
\varphi(N) \overset{\text{def}}{=} \left\{ \int \varphi \, d\mu : \mu \in N \right\}, \quad \underline{\varphi}_{N} \overset{\text{def}}{=} \inf \varphi(N), \quad \overline{\varphi}_{N} \overset{\text{def}}{=} \sup \varphi(N). \tag{5.2}
\]

We list the following general properties which are easy to verify (most of these properties and the ideas behind their proofs can be found in [55, Chapter 9]).

(P1) The function \( \mathcal{P}_{\mu} \) is affine and satisfies \( \mathcal{P}_{\mu} \leq \mathcal{P}_{N} \) and \( \mathcal{P}_{\mu}(0) = h(\mu) \).

(P2) Given a subset \( N' \subseteq N \), then \( \mathcal{P}_{N'} \leq \mathcal{P}_{N} \).

(P3) \( \mathcal{P}_{N}(0) = \sup\{h(\mu) : \mu \in N\} \).

(P4) The function \( \varphi \mapsto P(\varphi, N) \) is continuous and \( q \mapsto P(q\varphi, N) \) is uniformly Lipschitz continuous.

(P5) The function \( \mathcal{P}_{N} \) is convex. Consequently, \( \mathcal{P}_{N} \) is differentiable at all but at most countably many \( q \)'s and the left and right derivatives \( D_{L}\mathcal{P}_{N}(q) \) and \( D_{R}\mathcal{P}_{N}(q) \) are defined for all \( q \in \mathbb{R} \).

(P6) We have

\[
\underline{\varphi}_{N} = \lim_{q \to -\infty} \frac{\mathcal{P}_{N}(q)}{q} = \lim_{q \to -\infty} D_{L}\mathcal{P}_{N}(q) = \lim_{q \to -\infty} D_{R}\mathcal{P}_{N}(q),
\]

\[
\overline{\varphi}_{N} = \lim_{q \to \infty} \frac{\mathcal{P}_{N}(q)}{q} = \lim_{q \to \infty} D_{L}\mathcal{P}_{N}(q) = \lim_{q \to \infty} D_{R}\mathcal{P}_{N}(q).
\]

---

7 Note that in the context of the rest of the paper, skew-product maps with one-dimensional fibers, such equilibrium states indeed exist by [19, Corollary 1.5] (see also [15]). However, in a slightly different skew-product setting, they are not unique in general, see for instance the examples in [20, 38].

8 In the rest of this paper, we will study the decomposition (2.2) and have in mind the particular subset of measures \( \mathcal{M}_{\text{erg},><0} \) and \( \mathcal{M}_{\text{erg},<0} \).
(P7) The graph of $\mathcal{P}_N$ has a supporting straight line of slope $\varphi(\mu)$ for every $\mu \in \mathcal{N}$. Thus, for any $\alpha \in (\varphi_N, \overline{\varphi}_N)$ it has a supporting straight line of slope $\alpha$.

(P8) If the entropy map $\mu \mapsto h(\mu)$ is upper semi-continuous on $\mathcal{M}(\mathcal{X})$, then for any number $\alpha \in (\varphi_N, \overline{\varphi}_N)$ there is a measure $\mu_\alpha \in \mathcal{M}(\mathcal{X})$ (not necessarily ergodic and not necessarily in $\mathcal{N}$) such that $\varphi(\mu_\alpha) = \alpha$ and $q \mapsto \mathcal{P}_\mu(q)$ is a supporting straight line for $\mathcal{P}_N$.

(P9) If $\mu \in \mathcal{M}(\mathcal{X})$ is an equilibrium state for $q\varphi$ for some $q \in \mathbb{R}$ (with respect to $\mathcal{N}$), then $D_L\mathcal{P}_N(q) \leq \varphi(\mu) \leq D_R\mathcal{P}_N(q)$. Moreover, the graph of $\mathcal{P}_\mu$ is a supporting straight line for the graph of $\mathcal{P}_N$ at $(q, \mathcal{P}_N(q))$.

(P10) If the entropy map $\mu \mapsto h(\mu)$ is upper semi-continuous, then for any $q$ there are equilibrium states $\mu_{L,q}$ and $\mu_{R,q}$ for $q\varphi$ (with respect to $\mathcal{N}$) such that $\varphi(\mu_{L,q}) = D_L\mathcal{P}_N(q)$ and $\varphi(\mu_{R,q}) = D_R\mathcal{P}_N(q)$. Moreover, $\mu_{L,q}$ and $\mu_{R,q}$ can be chosen to be ergodic (but not necessarily in $\mathcal{N}$).

(P11) $\mathcal{P}_N$ is differentiable at $q$ if and only if all equilibrium states for $q\varphi$ (with respect to $\mathcal{N}$) have the same averaged value and this value is $\mathcal{P}_N(q)$. In particular, if there is a unique equilibrium state for $q\varphi$ (with respect to $\mathcal{N}$) then $\mathcal{P}_N$ is differentiable at $q$.

(P12) If $\mu \in \overline{\text{conv}}(\mathcal{N})$ is not ergodic and $\mathcal{P}_\mu(q) = \mathcal{P}_N(q)$ for some $q$, then almost all measures in the ergodic decomposition of $\mu$ are equilibrium states for $q\varphi$ (with respect to $\mathcal{N}$).

5.3. The convex conjugates of pressure functions. One of our goals is to express the topological entropy $h_{\text{top}}(\mathcal{L}(\alpha))$ of each level set $\mathcal{L}(\alpha)$ in terms of a restricted variational principle and in terms of a Legendre–Fenchel transform of an appropriate pressure function. Let us hence recall some simple facts about such transforms.

Given a subset of ergodic measures $\mathcal{N} \subset \mathcal{M}_{\text{erg}}(\mathcal{X})$, we define

$$\mathcal{E}_N(\alpha) \overset{\text{def}}{=} \inf_{q \in \mathbb{R}} \{\mathcal{P}_N(q) - q\alpha\}$$

on its domain

$$D(\mathcal{E}_N) \overset{\text{def}}{=} \{\alpha \in \mathbb{R} : \inf_{q \in \mathbb{R}} (\mathcal{P}_N(q) - q\alpha) > -\infty\}.$$ 

Observe that $(\mathcal{P}_N, \mathcal{E}_N)$ forms a Legendre–Fenchel pair.\(^9\) We list the following general properties.

\(^9\) The Legendre–Fenchel transform of a convex function $\beta : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\beta^*(\alpha) \overset{\text{def}}{=} \sup_{q \in \mathbb{R}} (\alpha q - \beta(q)).$$

and is convex on its domain $D(\beta^*) = \{\alpha \in \mathbb{R} : \beta^*(\alpha) < \infty\}$. In particular, the convex function $\beta$ is differentiable at all but at most countably many points and

$$\beta^*(\alpha) = \beta'(q)q - \beta(q) \quad \text{for} \quad \alpha = \beta'(q).$$

On the set of strictly convex functions the transform is involutive $\beta^{**} = \beta$. Formally, it is the function $\alpha \mapsto -\mathcal{E}_N(-\alpha)$ which is the Legendre–Fenchel transform of $\mathcal{P}_N(q)$, but it is common practice in the context of this paper (that we will also follow) to address $\mathcal{E}_N$ by this name.
(E1) The function $E_N$ is concave (and hence continuous). Consequently, it is differentiable at all but at most countably many $\alpha$, and the left and right derivatives are defined for all $\alpha \in D(E_N)$.

(E2) We have
\[ D(E_N) \supset (\varphi_N, \overline{\varphi}_N). \]

(E3) If $\mu$ is an equilibrium state for $q \varphi$ for some $q \in \mathbb{R}$ (with respect to $N$) and $\alpha = \varphi(\mu)$, then $h(\mu) = E_N(\alpha)$.

(E4) We have
\[ \max_{\alpha \in D(E_N)} E_N(\alpha) = \mathcal{P}_N(0). \]

Moreover, this maximum is attained at exactly one value of $\alpha$ if, and only if, $\mathcal{P}_N$ is differentiable at 0.

(E5) For every $\alpha \in D(E_N)$ we have
\[ E_N(\alpha) \geq \sup \{ h(\mu) : \mu \in N, \varphi(\mu) = \alpha \} . \]

For completeness, we give the short proof of (E5).

Proof of property (E5). Let $\alpha \in \text{int} \, D(E_N)$. Fix any $q \in \mathbb{R}$. Observe that
\[
\sup \{ h(\mu) : \mu \in N, \varphi(\mu) = \alpha \} = \sup \{ h(\mu) + q \varphi(\mu) : \mu \in N, \varphi(\mu) = \alpha \} - q \alpha \\
\leq \sup \{ h(\mu) + q \varphi(\mu) : \mu \in N \} - q \alpha \\
= \mathcal{P}_N(q) - q \alpha .
\]

Since $q$ was arbitrary, we can conclude
\[
\sup \{ h(\mu) : \mu \in N, \varphi(\mu) = \alpha \} \leq \inf_{q \in \mathbb{R}} (\mathcal{P}_N(q) - q \alpha) = E_N(\alpha)
\]
proving the property. $\square$

**Proposition 5.4.** Assume that $X$ is a basic set of the skew-product map $F : \Sigma_N \times \mathbb{S}^1 \to \Sigma_N \times \mathbb{S}^1$. Let $\varphi : X \to \mathbb{R}$ be a continuous potential. Then for $N = M_{\text{erg}}(X)$ and every $\alpha \in \text{int} \, D(E_N)$ we have
\[
\sup \{ h(\mu) : \mu \in N, \varphi(\mu) = \alpha \} = \sup \{ h(\mu) : \mu \in M(X), \varphi(\mu) = \alpha \} = E_N(\alpha).
\]

Note that to show the inequality $\leq$ in the proposition we, in fact, do not need the hypothesis of a basic set.

**Proof.** Let $\alpha \in \text{int} \, D(E_N)$. Note that $N \subset M(X)$, the above proof of (E5), and the fact that for $N = M_{\text{erg}}(X)$ we have $\mathcal{P}_N(q) = \sup \{ h(\mu) + q \varphi(\mu) : \mu \in M(X) \}$ (see [55, Corollary 9.10.1 i]) immediately implies the inequalities $\leq$.

It remains to prove the inequality $E_N(\alpha) \leq \sup \{ h(\mu) : \mu \in N, \varphi(\mu) = \alpha \}$ and hence the proposition. First recall [13] that for any Hölder continuous potential $\tilde{\varphi} : X \to \mathbb{R}$ and $\tilde{q} \in \mathbb{R}$ there is a unique equilibrium state for $\tilde{q} \tilde{\varphi}$ for a basic set of a diffeomorphism. Note that this hypothesis naturally translates to our skew-product setting. By property (P8) applied to $X$ and $N$, there is a measure $\mu_\alpha \in M(X)$ (not necessarily ergodic) such that $\varphi(\mu_\alpha) = \alpha$ and $q \mapsto \mathcal{P}_{\mu_\alpha}(q)$ is a supporting straight line for $\mathcal{P}_N$. Hence, there is $q = q(\alpha)$ such that $\mathcal{P}_N(q) = h(\mu_\alpha) + q h(\mu_\alpha)$. If $\mu_\alpha$ was already ergodic then we are
done. Otherwise, note that we can find \( \tilde{\varphi} : X \to \mathbb{R} \) Hölder continuous and arbitrarily close to the continuous potential \( \varphi : X \to \mathbb{R} \) and \( \tilde{q} \) arbitrarily close to \( q \) and an ergodic equilibrium state \( \tilde{\nu} \in \mathcal{N} \) for \( \tilde{q} \tilde{\varphi} \) such that \( \varphi(\tilde{\nu}) = \alpha \). By (P4) we have that \( P(\tilde{q} \tilde{\varphi}, \mathcal{N}) \) is arbitrarily close to \( P(q \varphi, \mathcal{N}) \). Hence, for such \( \tilde{\nu} \) we have
\[
\begin{align*}
    h(\tilde{\nu}) = P(\tilde{q} \tilde{\varphi}, \mathcal{N}) - \tilde{q} \alpha &= (P(q \varphi, \mathcal{N}) - q \alpha) + (P(\tilde{q} \tilde{\varphi}, \mathcal{N}) - P(q \varphi, \mathcal{N})) + (q \alpha - \tilde{q} \alpha).
\end{align*}
\]
Thus, we can conclude
\[
\begin{align*}
    \sup \{ h(\nu) : \nu \in \mathcal{N}, \varphi(\nu) = \alpha \} \geq (P(q \varphi, \mathcal{N}) - q \alpha).
\end{align*}
\]
Taking the infimum over all \( q \in \mathbb{R} \) we obtain
\[
\begin{align*}
    \sup \{ h(\nu) : \nu \in \mathcal{N}, \varphi(\nu) = \alpha \} \geq \inf_{q \in \mathbb{R}} (\mathcal{P}_\mathcal{N}(q) - q \alpha) = \mathcal{E}_\mathcal{N}(\alpha).
\end{align*}
\]
This finishes the proof of the lemma. \( \square \)

6. Exhausting Families

In this section, we present a general principle to perform a multifractal analysis. It was already used in several contexts having some hyperbolicity (see, for example, [30] for Markov maps on the interval, [29] for non-exceptional rational maps of the Riemann sphere, or [14] for geodesic flows of rank one surfaces). As the system as a whole does not satisfy the specification property, we consider certain families of subsets (basic sets, see Sect. 6.2.1) on which we do have specification. The general theory of restricted pressures presented here allows us to obtain dynamical properties of the full system knowing the properties of those subsets.

6.1. General framework. Let \( (X, d) \) be a compact metric space, \( F : X \to X \) a continuous map, and \( \varphi : X \to \mathbb{R} \) a continuous potential. Fix a set of ergodic measures \( \mathcal{N} \subset \mathcal{M}_{\text{erg}}(X) \). Recall that we defined for \( \alpha \in \text{int} D(\mathcal{E}_\mathcal{N}) \)
\[
\mathcal{E}_\mathcal{N}(\alpha) \triangleq \inf_{q \in \mathbb{R}} (\mathcal{P}_\mathcal{N}(q) - q \alpha).
\]
A sequence of compact \( F \)-invariant sets \( X_1, X_2, \ldots \subset X \) is said to be \( (X, \varphi, \mathcal{N}) \)-exhausting if the following holds: for every \( i \geq 1 \) we have

(exh1) \( \mathcal{M}_{\text{erg}}(X_i) \subset \mathcal{N} \),
(exh2) \( F|_{X_i} \) has the specification property,
(exh3) Given \( \mathcal{M}_i = \mathcal{M}_{\text{erg}}(X_i) \) let \( \mathcal{P}_i = \mathcal{P}_{\mathcal{M}_i} \) and
\[
\mathcal{E}_i(\alpha) \triangleq \inf_{q \in \mathbb{R}} (\mathcal{P}_i(q) - q \alpha).
\]
Then for every \( \alpha \in \text{int} D(\mathcal{E}_i) \) the restricted variational principle holds
\[
\mathcal{E}_i(\alpha) = \sup \{ h(\mu) : \mu \in \mathcal{M}_i, \varphi(\mu) = \alpha \}.
\]
(exh4) For every \( q \in \mathbb{R} \) we have
\[
\lim_{i \to \infty} P_{F|X_i}(q \varphi) = \mathcal{P}_\mathcal{N}(q).
\]
Let $\varphi_N$ and $\overline{\varphi}_N$ be as in (5.2), then

$$\varphi_N = \lim_{i \to \infty} \varphi_{M_i}, \quad \overline{\varphi}_N = \lim_{i \to \infty} \overline{\varphi}_{M_i}.$$  

Note that $(P_i, E_i)$ forms a Legendre–Fenchel pair for every $i \geq 1$.

The exhausting property for appropriate $N$ is the essential step to relate the lower bound in the restricted variational principle (5.2) to the Legendre–Fenchel transform of the restricted pressure function $P_N$. This is the requirement (exh3).

**Lemma 6.1.** It holds $\lim_{i \to \infty} E_i(\alpha) = E_N(\alpha)$. In particular, $\int D(E_N) = (\varphi_N, \overline{\varphi}_N)$.

**Proof.** Note that property (exh4) of pointwise convergence of convex functions of pressures $P_i$ to the convex function of pressure $P_N$ and the fact that $E_i$ and $E_N$ are their Legendre–Fenchel transforms imply the claim, see for instance [56].

The following result will be the main step in establishing the lower bounds for entropy in Theorem 1. We derive it in the general setting of this subsection.

**Proposition 6.2.** Assume that there exists an increasing family of sets $(X_i)_i \subset X$ which is $(X, \varphi, N)$-exhausting. Then

- we have

  $$(\varphi_N, \overline{\varphi}_N) \subset \varphi(N) \subset [\varphi_N, \overline{\varphi}_N].$$

  In particular, $\varphi(N)$ is an interval.

- For every $\alpha \in (\varphi_N, \overline{\varphi}_N)$ we have $\mathcal{L}(\alpha) \neq \emptyset$ and

  $$h_{\text{top}}(\mathcal{L}(\alpha)) \geq E_N(\alpha)$$

  $$= \lim_{i \to \infty} \sup \{ h(\mu) : \mu \in M_i, \varphi(\mu) = \alpha \}.$$  

**Proof.** By condition (exh4) and the property of pointwise convergence of convex functions to a convex function (see (P5)), we can conclude that for every $i$

$$P_{F|X_i}(q \varphi) \geq P_N(q) - \frac{1}{i}$$

for all $q \in [-i, i]$ and some sequence $(n(i))_i$. For simplicity, allowing a change of indices, we will assume that $n(i) = i$.

A particular consequence of specification of $F|X_i$ is that by Proposition 5.3 the set $\varphi(M_i)$ is an interval. Together with (exh5) this implies that $\varphi(N)$ is an interval and we have

$$(\varphi_N, \overline{\varphi}_N) \subset \varphi(N) = \bigcup_{i \geq 1} \varphi(M_i) \subset [\varphi_N, \overline{\varphi}_N],$$

(6.1)

proving the first item.

Let $\alpha \in (\varphi_N, \overline{\varphi}_N)$. For every index $i$, by Proposition 5.3, we have

$$h_{\text{top}}(\mathcal{L}(\alpha) \cap X_i) = \sup \{ h(\mu) : \mu \in M_i, \varphi(\mu) = \alpha \} \leq h_{\text{top}}(\mathcal{L}(\alpha)),$$

where for the inequality we use monotonicity of entropy. By (6.1), there is $i = i(\alpha) \geq 1$ such that $\alpha \in \varphi(M_i)$ and, in particular, we have $\mathcal{L}(\alpha) \neq \emptyset$. By (exh3), for every $\alpha \in (\varphi_N, \overline{\varphi}_N)$ and $i$ sufficiently big, we have

$$E_i(\alpha) = \sup \{ h(\mu) : \mu \in M_i, \varphi(\mu) = \alpha \}. $$
By Lemma 6.1 we have \( \lim_{i \to \infty} E_i(\alpha) = E_N(\alpha) \), concluding the proof of the proposition. \( \Box \)

6.2. Existence of exhausting families in our setting. In this section, we return to consider a transitive step skew-product map \( F \) as in (2.1) whose fiber maps are \( C^1 \) and satisfies Axioms CEC± and Acc±. Recall that the map \( F \) has ergodic measures with negative/positive exponents arbitrarily close to 0, see Corollary 4.3. The goal of this section is to prove the following proposition.

**Proposition 6.3.** Consider the set of ergodic measures \( N = \mathcal{M}_{\text{erg}, <0} \) and the potential \( \varphi: \Sigma_N \times S^1 \to \mathbb{R} \) in (2.3). Then there is a \((\Sigma_N \times S^1, \varphi, N)\)-exhausting family consisting of nested basic sets and \( \varphi(N) = [\alpha_{\min}, 0) \).

The analogous statement is true for \( N = \mathcal{M}_{\text{erg}, >0} \) with \( \varphi(N) = (0, \alpha_{\max}] \).

6.2.1. Homoclinic relations. We say that a periodic point of \( F \) is hyperbolic or a saddle if its (fiber) Lyapunov exponent is nonzero. In our partially hyperbolic setting with one-dimensional central bundle, there are only two possibilities: a saddle has either a negative or positive (fiber) Lyapunov exponent. We say that two saddles are of the same type if either both have negative exponents or both have positive exponents. Note that all saddles in a basic set are of the same (contracting/expanding) type. We say that two basic sets are of the same type if their saddles are of the same type.

Given a saddle \( P \) we define the stable and unstable sets of its orbit \( O(P) \) by

\[
W^s(O(P)) \overset{\text{def}}{=} \{ X : \lim_{n \to \infty} d(F^n(X), O(P)) = 0 \},
\]

and

\[
W^u(O(P)) \overset{\text{def}}{=} \{ X : \lim_{n \to \infty} d(F^{-n}(X), O(P)) = 0 \},
\]

respectively.

We say that a point \( X \) is a homoclinic point of \( P \) if \( X \in W^s(O(P)) \cap W^u(O(P)) \). Two saddles \( P \) and \( Q \) of the same index are homoclinically related if the stable and unstable sets of their orbits intersect cyclically, that is, if

\[
W^s(O(P)) \cap W^u(O(Q)) \neq \emptyset \neq W^s(O(Q)) \cap W^u(O(P)).
\]

In our context, homoclinic intersections behave the same as transverse homoclinic intersections in the differentiable setting. As in the differentiable case, to be homoclinically related defines an equivalence relation on the set of saddles of \( F \). The homoclinic class of a saddle \( P \), denoted by \( H(P, F) \), is the closure of the set of saddles which are homoclinically related to \( P \). A homoclinic class can be also defined as the closure of the homoclinic points of \( P \). As in the differentiable case, a homoclinic class is an \( F \)-invariant and transitive set.\( ^{10} \)

\( ^{10} \) These assertions are folklore ones, details can be found, for instance, in [18, Section 3]. Note that in our skew-product context the standard transverse intersection condition between the invariant sets of the saddles in the definition of a homoclinic relation is not required and does not make sense. However, since the dynamics in the central direction is non-critical (the fiber maps are diffeomorphisms and hence have no critical points) the intersections between invariant sets of saddles of the same type behave as “transverse” ones and the arguments in the differentiable setting can be translated to the skew-product setting (here the fact that the fiber direction is one-dimensional is essential).
Lemma 6.4. Any pair of saddles $P, Q \in \Sigma_N \times \mathbb{S}^1$ of the same type are homoclinically related.

Proof. Let us assume that $P$ and $Q$ both have negative exponents. The proof of the other case is analogous and omitted. Let $P = (\xi, p)$ and $Q = (\eta, q)$, where $\xi = (\xi_0 \ldots \xi_{n-1})^\mathbb{Z}$ and $\eta = (\eta_0 \ldots \eta_{m-1})^\mathbb{Z}$. By hyperbolicity, there is $\delta > 0$ such that

$$f^{n}_\xi([p - \delta, p + \delta]) \subset (p - \delta, p + \delta) \quad \text{and} \quad f^{m}_\eta([q - \delta, q + \delta]) \subset (q - \delta, q + \delta)$$

and such that those maps are uniformly contracting on those intervals. This immediately implies that

$$[(\xi_0 \ldots \xi_{n-1})^\mathbb{N}] \times [p - \delta, p + \delta] \subset W^s(\Sigma_P(P)),
[(\eta_0 \ldots \eta_{m-1})^\mathbb{N}] \times [q - \delta, q + \delta] \subset W^s(\Sigma_Q(Q)).$$

Similarly we get

$$[(\xi_0 \ldots \xi_{n-1})^{-\mathbb{N}}] \times \{p\} \subset W^u(\Sigma_P(P)), \quad [(\eta_0 \ldots \eta_{m-1})^{-\mathbb{N}}] \times \{q\} \subset W^u(\Sigma_Q(Q)).$$

By Lemma 4.5 there are $(\beta_0 \ldots \beta_s)$ and $(\gamma_0 \ldots \gamma_r)$ such that

$$f_{[\beta_0 \ldots \beta_s]}(p) \in (p - \delta, p + \delta) \quad \text{and} \quad f_{[\gamma_0 \ldots \gamma_r]}(q) \in (q - \delta, q + \delta).$$

By construction, this implies that

$$((\eta_0 \ldots \eta_{m-1})^{-\mathbb{N}}.\beta_0 \ldots \beta_s(\xi_0 \ldots \xi_{n-1})^\mathbb{N}, q) \in W^u(\Sigma_Q(Q)) \cap W^s(\Sigma_P(P)),
((\xi_0 \ldots \xi_{n-1})^{-\mathbb{N}}.\gamma_0 \ldots \gamma_r(\eta_0 \ldots \eta_{m-1})^\mathbb{N}, p) \in W^s(\Sigma_P(P)) \cap W^u(\Sigma_Q(Q)).$$

This proves that $P$ and $Q$ are homoclinically related. □

6.2.2. Existence of exhausting families: Proof of Proposition 6.3 We recall the following well-known fact about homoclinically related basic sets. For a proof we refer to [48, Section 7.4.2], where the hypothesis of a basic set of a diffeomorphism naturally translates to our skew-product setting.

Lemma 6.5 (Bridging). Consider two basic sets $\Gamma_1, \Gamma_2 \subset \Sigma_N \times \mathbb{S}^1$ of $F$ which are homoclinically related. Then there is a basic set $\Gamma$ of $F$ containing $\Gamma_1 \cup \Gamma_2$. In particular, for every continuous potential $\varphi$, we have

$$\max\left\{P_{F|\Gamma_1}(\varphi), P_{F|\Gamma_2}(\varphi)\right\} \leq P_{F|\Gamma}(\varphi).$$

We will base our arguments also on the following result that translates results of from Pesin–Katok theory to our setting.

Lemma 6.6. Let $\mu \in \mathcal{M}_{\text{erg.} < 0}$ with $h = h(\mu) > 0$ and $\alpha = \chi(\mu) < 0$.

Then for every $\gamma \in (0, h)$ and every $\lambda \in (0, \alpha)$ there exists a basic set $\Gamma = \Gamma(\gamma, \lambda) \subset \Sigma_N \times \mathbb{S}^1$ such that for all $q \in \mathbb{R}$ we have

$$P_{F|\Gamma}(q \varphi) \geq h(\mu) + q \int \varphi \, d\mu - \gamma - q \lambda.$$

The analogous statement is true for $\mathcal{M}_{\text{erg.} > 0}$.  


Proof. By Proposition 4.4, there exists a basic set \( \Gamma \) such that \( h_{\text{top}}(\Gamma) \geq h - \gamma \) and that for every \( \nu \in \mathcal{M}_{\text{erg}}(\Gamma) \) we have \( \chi(\nu) \in (\alpha - \lambda, \alpha + \lambda) \). The variational principle (5.1) immediately implies the lemma. \( \square \)

We are now prepared to prove Proposition 6.3.

Proof of Proposition 6.3. We first construct an exhausting family for \( \mathcal{N} = \mathcal{M}_{\text{erg}, \prec 0} \). Given \( i \geq 1 \), let us first construct a basic set \( X_i \) of contracting type such that

\[
P_{F|X_i}(q\varphi) \geq \mathcal{P}_\mathcal{N}(q) - \frac{1}{i}
\]

(6.2)

for all \( q \in [-i, i] \). By Lipschitz continuity property (P4) of pressure, there are a Lipschitz constant \( \text{Lip} \) and a finite subset \( q_1, \ldots, q_\ell \) of \( [-i, i] \) such that for every \( q \in [-i, i] \) there is \( q_k \) with

\[
\text{Lip} |q_k - q| \|\varphi\| < \frac{1}{4i}.
\]

To prove (6.2), given \( q_k \), by Lemma 6.6 there is a basic set \( X_{i,k} \) such that

\[
P_{F|X_{i,k}}(q_k\varphi) \geq \mathcal{P}_\mathcal{N}(q_k) - \frac{1}{4i}.
\]

Applying Lemma 6.5 consecutively to the finitely many basic sets \( X_{i,1}, \ldots, X_{i,\ell} \), we obtain a basic set \( X_i \) containing all these sets and satisfying (6.2). This shows (exh4) and (exh5).

By construction, all basic sets are of contracting type and hence all ergodic measures have negative Lyapunov exponent and we have (exh1). Each of them clearly satisfies (exh2) (basic sets have the specification property [51]). By Proposition 5.4 we have the restricted variational principle (exh3) on each of them.

What remains to prove is that \( \varphi(\mathcal{N}) = [\alpha_{\text{min}}, 0) \). By Corollary 4.3, the Lyapunov exponents of ergodic measures extend all the way to 0, that is, \( \overline{\varphi}_\mathcal{N} = 0 \). On the other hand, note that by (P5) we can choose an increasing sequence \( (q_j)_j \) tending to \( -\infty \) such that \( \mathcal{P}_\mathcal{N} \) is differentiable at all such \( q_j \). By (P11) and (P12) for every \( j \) there is an ergodic equilibrium state \( \mu_j \) for \( q_j \varphi \) and \( \varphi(\mu_j) \to \varphi_\mathcal{N} \). Taking \( \mu' \) which is a weak* limit of \( (\mu_j)_j \) as \( j \to \infty \), then there is an ergodic measure \( \mu'' \) in its ergodic decomposition such that \( \varphi(\mu'') = \varphi_\mathcal{N} \). In particular, we can conclude \( \mathcal{L}(\varphi_\mathcal{N}) \neq \emptyset \) and \( \alpha_{\text{min}} = \varphi_\mathcal{N} \). This concludes the proof that \( \varphi(\mathcal{N}) = [\alpha_{\text{min}}, 0) \).

The statement for \( \mathcal{N} = \mathcal{M}_{\text{erg}, \succ 0} \) is proved analogously.

The proof of the proposition is now complete. \( \square \)

7. Entropy of the Level Sets: Proof of Theorem 1

In this section, we collect the ingredients required to prove Theorem 1. Section 7.1 deals with the measures of maximal entropy. Section 7.2 provides upper bounds for the entropy of level sets with exponents of the interior of the spectrum. Section 7.5 deals with lower bounds. Sections 7.3 and 7.4 deal with the boundary of the spectrum and with exponent zero. Here the main technical result is Theorem 7.6 whose proof will be postponed to Sect. 8. The proof of Theorem 1 is concluded in Section 7.6.
7.1. Measure(s) of maximal entropy. Note that any measure of maximal entropy projects to the \((1/N, \ldots, 1/N)\)-Bernoulli measure in the base. Hence, we can use the known results about the behavior of Bernoulli measures for random dynamical systems. By [16, Theorem 8.6] (stated for products of independently and identically distributed (i.i.d.) diffeomorphisms on a compact manifold) for every Bernoulli measure \(b \in \mathcal{M}(\Sigma_N)\) there exists a (at least one) \(F\)-ergodic measure \(\mu^b_+\) with positive exponent and a (at least one) \(F\)-ergodic measure \(\mu^b_-\) with negative exponent, both projecting to \(b = \pi_* \mu^b_\pm\). Indeed, note that our axioms rule out the possibility of a measure being simultaneously preserved by all the fiber maps, see Lemma 4.7. When \(b\) is the \((1/N, \ldots, 1/N)\)-Bernoulli measure we simply write \(\mu^b_\pm\).

There are various ways to prove that there are only finitely many hyperbolic ergodic \(F\)-invariant measures projecting to the same Bernoulli measure. For example, in our setting it is a consequence of [49, Theorem 1].

7.2. Negative/positive exponents in the interior of the spectrum. We will analyze the negative part of the spectrum, the analysis of the positive part is analogous and it will be omitted.

By Proposition 6.3 there is a \((\Sigma_N \times S^1, \varphi, \mathcal{M}_{\text{erg}, <0})\)-exhausting family \(\{X_i\}_i\). Hence, in particular, for every \(\alpha \in (\alpha_{\text{min}}, 0)\) we have \(\mathcal{L}(\alpha) \neq \emptyset\) and together with Proposition 6.2 and writing \(\varphi(\mu) = \chi(\mu)\) we have

\[
h_{\text{top}}(\mathcal{L}(\alpha)) \geq \mathcal{E}_{<0}(\alpha) = \lim_{i \to \infty} \sup \left\{ h(\mu) : \mu \in \mathcal{M}(X_i), \chi(\mu) = \alpha \right\}.
\]

By Lemma 5.2, we have

\[
h_{\text{top}}(\mathcal{L}(\alpha)) \geq \sup \left\{ h(\mu) : \mu \in \mathcal{M}_{\text{erg}}, \chi(\mu) = \alpha \right\}.
\]

**Lemma 7.1.** For every \(\alpha \in (\alpha_{\text{min}}, 0)\) we have \(h_{\text{top}}(\mathcal{L}(\alpha)) = \mathcal{E}_{<0}(\alpha)\).

**Proof.** By Proposition 6.2, we already have \(h_{\text{top}}(\mathcal{L}(\alpha)) \geq \mathcal{E}_{<0}(\alpha)\) and it is hence enough to prove the other inequality. Recall that by (E5) for every \(\alpha < 0\) we have

\[
\mathcal{E}_{<0}(\alpha) \geq \sup \left\{ h(\mu) : \mu \in \mathcal{M}_{\text{erg}, <0}, \chi(\mu) = \alpha \right\}.
\]  

(7.1)

Arguing by contradiction, let us assume that there are \(\alpha \in (\alpha_{\text{min}}, 0)\) and \(\delta > 0\) so that

\[
h_{\text{top}}(\mathcal{L}(\alpha)) \geq \mathcal{E}_{<0}(\alpha) + 2\delta.
\]

Then, by continuity of \(\mathcal{E}_{<0}(\cdot)\), property (E1), there exists \(\varepsilon > 0\) such that for every \(\alpha' \in (\alpha - 2\varepsilon, \alpha + 2\varepsilon)\) we have

\[
h_{\text{top}}(\mathcal{L}(\alpha)) \geq \mathcal{E}_{<0}(\alpha') + \delta.
\]

By Proposition 4.4, there exists a basic set \(\Gamma \subset \Sigma_N \times S^1\) such that

\[
h_{\text{top}}(\Gamma) > h_{\text{top}}(\mathcal{L}(\alpha)) - \delta,
\]

and that for every \(\nu \in \mathcal{M}_{\text{erg}}(\Gamma)\) we have \(\chi(\nu) \in (\alpha - \varepsilon, \alpha + \varepsilon)\). Taking the measure of maximal entropy \(\nu \in \mathcal{M}_{\text{erg}}(\Gamma)\), with the above, for every \(\alpha' \in (\alpha - 2\varepsilon, \alpha + 2\varepsilon)\) we have

\[
h(\nu) = h_{\text{top}}(\Gamma) \geq \mathcal{E}_{>0}(\alpha') + \delta.
\]

However, \(\alpha' = \chi(\nu) \in (\alpha - \varepsilon, \alpha + \varepsilon)\) would then contradict (7.1). This proves the lemma. \(\square\)
Lemma 7.2. For every $\alpha \in (\alpha_{\text{min}}, 0)$ we have

$$h_{\text{top}}(\mathcal{L}(\alpha)) = \mathcal{E}_{<0}(\alpha) = \sup\{h(\mu) : \mu \in \mathcal{M}\text{erg}, \chi(\mu) = \alpha\}.$$ 

Proof. By Lemma 7.1 we have $h_{\text{top}}(\mathcal{L}(\alpha)) = \mathcal{E}_{<0}(\alpha)$. With Lemma 5.2, what remains to show is that

$$\mathcal{E}_{<0}(\alpha) \leq \sup\{h(\mu) : \mu \in \mathcal{M}\text{erg}, \chi(\mu) = \alpha\}.$$ 

By contradiction, assume that there is $\alpha$ such that $\mathcal{E}_{<0}(\alpha) \geq \sup\{h(\mu) : \mu \in \mathcal{M}\text{erg}, \chi(\mu) = \alpha\}$ and let $\delta > 0$ such that for every $\mu \in \mathcal{M}\text{erg}$ with $\chi(\mu) = \alpha$ we have $\mathcal{E}_{<0}(\alpha) - 3\delta > h(\mu)$. By property (exh5) of the exhausting family $\{X_i\}_i$, there exists $i_0 \geq 1$ such that for every $i \geq i_0$ we have that $\alpha \in (\varphi_{M_i}, \varphi_{M_i})$. With Lemma 6.1, we also can assume that for every $i \geq i_0$ we have $\mathcal{E}_i(\alpha) \geq \mathcal{E}_{<0}(\alpha) - \delta$. Applying Proposition 5.4 to a basic set $X_i$, there exists $\mu \in M_i \subset \mathcal{M}\text{erg}$ with $\chi(\mu) = \alpha$ satisfying $h(\mu) \geq \mathcal{E}_i(\alpha) - \delta$ and hence $h(\mu) \geq \mathcal{E}_{<0}(\alpha) - 2\delta$, a contradiction. \(\square\)

7.3. Coincidence of one-sided limits of the spectrum at zero.

Lemma 7.3. $h_0^{\pm} \equiv \lim_{\alpha \to 0^\pm} h_{\text{top}}(\mathcal{L}(\alpha)) = \lim_{\alpha \to 0} h_{\text{top}}(\mathcal{L}(\alpha))$.

Proof. By Lemma 7.1, for $\alpha \in (\alpha_{\text{min}}, 0)$ we have $h_{\text{top}}(\mathcal{L}(\alpha)) = \mathcal{E}_{<0}(\alpha)$. Hence, by (E1), this is a concave function in $\alpha$. Similarly for $\alpha \in (0, \alpha_{\text{max}})$. So we can define the numbers $h_0^+ = \lim_{\alpha \to 0^+} h_{\text{top}}(\mathcal{L}(\alpha))$.

By the restricted variational principle in Lemma 7.2, for every sequence $\alpha_k \not\to 0$ there is a sequence of ergodic measures $(\mu_k)_{k \geq 0}$ such that $\chi(\mu_k) = \alpha_k$ and $h(\mu_k) \to h_0^-$. As a consequence of Lemma 4.2 there is a corresponding sequence $(v_k)_{k \geq 1}$ with $\chi(v_k) \searrow 0$ and $h(v_k) \to h_0^-$. This implies that $h_0^- \leq h_0^+$. Reversing the roles of the negative and positive exponents we get $h_0^- \leq h_0^+$ and hence $h_0^- = h_0^+$, proving the lemma. \(\square\)

7.4. Zero and extremal exponents: upper bounds.

Lemma 7.4. $h_{\text{top}}(\mathcal{L}(0)) \leq h_0$.

Proof. As a consequence of Proposition 4.4 together with Lemma 7.3, for every $\gamma > 0$ and $\lambda > 0$ there exists $\alpha \in (-\lambda, 0)$ such that $h_{\text{top}}(\mathcal{L}(\alpha)) \geq h_{\text{top}}(\mathcal{L}(0)) - \gamma$. The assertion then follows. \(\square\)

Lemma 7.5. For $\alpha \in [\alpha_{\text{min}}, \alpha_{\text{max}}]$ we have $h_{\text{top}}(\mathcal{L}(\alpha)) \leq \lim_{\beta \to \alpha} h_{\text{top}}(\mathcal{L}(\beta))$ and

$$h_{\text{top}}(\mathcal{L}(\alpha)) = \sup\{h(\mu) : \mu \in \mathcal{M}\text{erg}, \chi(\mu) = \alpha\}.$$ 

Proof. We consider $\alpha = \alpha_{\text{min}}$, the other case is analogous. By Lemma 7.1, for every $\beta \in (\alpha_{\text{min}}, 0)$ we have already $h_{\text{top}}(\mathcal{L}(\beta)) = \mathcal{E}_{<0}(\beta)$. By the second part in Lemma 5.2, we have

$$h_{\text{top}}(\mathcal{L}(\alpha_{\text{min}})) = \sup \{h(\mu) : \mu \in \mathcal{M}\text{erg}, \varphi(\mu) = \alpha_{\text{min}}\},$$

proving the second assertion in the lemma. Hence, for every $q \in \mathbb{R}$ we have

$$\mathcal{P}_{<0}(q) = \sup\{h(\mu) + q\chi(\mu) : \mu \in \mathcal{M}\text{erg}, <0\} \geq h_{\text{top}}(\mathcal{L}(\alpha_{\text{min}})) + q\alpha_{\text{min}}.$$
which implies
\[
\mathcal{E}_{<0}(\alpha_{\min}) = \inf_{q \in \mathbb{R}} \left( P_{<0}(q) - q \alpha_{\min} \right) \geq h_{\text{top}}(\mathcal{L}(\alpha_{\min})).
\]

By the definition of \( \mathcal{E}_{<0} \) in (2.5) and [56], the left hand side is not larger than \( \lim_{\beta \to \alpha_{\min}} \mathcal{E}_{<0}(\beta) \), proving the first assertion. \( \square \)

### 7.5. Whole spectrum: lower bounds.

The following result is the final step needed to complete the proof of Theorem 1. We postpone its proof to Sect. 8.

**Theorem 7.6.** For every \( \alpha \in [\alpha_{\min}, \alpha_{\max}] \) we have
\[
\limsup_{\beta \to \alpha} \mathcal{E}(\beta) \leq h_{\text{top}}(\mathcal{L}(\alpha)).
\]

**Remark 7.7.** For \( \alpha \in (\alpha_{\min}, \alpha_{\max}) \setminus \{0\} \) the result of the above theorem follows already from Sect. 7.2. For \( \alpha \in \{\alpha_{\min}, \alpha_{\max}\} \) this result can be easily obtained by the following arguments: Take the weak* limit \( \mu \) of a sequence of measures \( (\mu_k)_k \) converging in exponent to \( \alpha \) and in entropy to \( h = \limsup_{\beta \to \alpha} \mathcal{E}(\beta) \). Indeed, such sequences exist by the already obtained description in the interior of the spectrum. The ergodic decomposition of \( \mu \) contains an ergodic measure \( \mu' \) with exponent \( \alpha \) and entropy at least \( h \). The set of \( \mu' \)-generic points is contained in \( \mathcal{L}(\alpha) \) which will imply the assertion. So, we only need to prove Theorem 7.6 for \( \alpha = 0 \). However, the proof is completely general.

### 7.6. Proof of Theorem 1.

By the arguments in Sect. 7.2, for every \( \alpha \in (\alpha_{\min}, 0) \) we have \( \mathcal{L}(\alpha) \neq \emptyset \), analogously for \( \alpha \in (0, \alpha_{\max}) \). The fact that \( \mathcal{L}(0) \neq \emptyset \) is a consequence of [6]. Note that by Theorem 4 item j) we have \( \lim_{\alpha \to 0} \mathcal{E}(\alpha) > 0 \) and hence, by Theorem 7.6, we obtain \( h_{\text{top}}(\mathcal{L}(0)) > 0 \) and, in particular, \( \mathcal{L}(0) \neq \emptyset \). By Theorem 4 item g), \( \alpha \mapsto \mathcal{E}(\alpha) \) is a concave function, and hence, by Lemma 7.1, we obtain \( h_{\text{top}}(\mathcal{L}(\alpha)) > 0 \) for every \( \alpha \in (\alpha_{\min}, 0) \cup (0, \alpha_{\max}) \). This already proves item d) of the theorem. For any \( \alpha \in \{\alpha_{\min}, \alpha_{\max}\} \) take the weak* limit \( \mu \) of a sequence of measures \( (\mu_k)_k \) converging in exponent to \( \alpha \). Indeed, such sequences exist by the already obtained description in the interior of the spectrum. The ergodic decomposition of \( \mu \) contains an ergodic measure \( \mu' \) with exponent \( \alpha \). This implies that \( \mathcal{L}(\alpha) \neq \emptyset \).

Item a) (and analogously item b)) follows from Lemmas 7.1 and 7.2 for the interior of the spectrum. The restricted variational principle for \( \alpha_{\min} \) and \( \alpha_{\max} \) follows from Lemma 5.2.

To prove item c), for \( \alpha \in \{\alpha_{\min}, 0, \alpha_{\max}\} \), again by Theorem 7.6 we have
\[
\limsup_{\beta \to \alpha} \mathcal{E}(\beta) = \limsup_{\beta \to \alpha} h_{\text{top}}(\mathcal{L}(\beta)) \leq h_{\text{top}}(\mathcal{L}(\alpha)).
\]

Now if \( \alpha = 0 \), the existence of the limit and the equality is a consequence of Lemmas 7.3 and 7.4. If \( \alpha \in \{\alpha_{\min}, \alpha_{\max}\} \), then we apply Lemma 7.5 instead proving item c).

Finally, the existence of finitely many ergodic measures of maximal entropy \( \log N \) follows by the arguments in Sect. 7.1.
8. Orbitwise Approach and Bridging Measures: Proof of Theorem 7.6

In this section, we prove Theorem 7.6 which goes as follows. In Sect. 8.1 we identify a set of orbits with appropriate properties (“cardinality” and “finite-time Lyapunov exponents” in Proposition 8.1). Based on those orbits, we construct a subset of \( \pi(\mathcal{L}(\alpha)) \) in Sect. 8.2. After some preliminary estimates in Sect. 8.3, in Sect. 8.4 we show that this subset is “large” by estimating its entropy following the approach of “bridging measures”. The proof of the theorem will be completed in Sect. 8.5.

8.1. Orbitwise approximation of ergodic measures. We will consider sets \( \Xi^+(n) \) of finite sequences of length \( n \) whose cardinalities grow exponentially fast. The following proposition provides precise estimates of how fast initial sequences are “branching out” to form sequences in \( \Xi^+(n) \). Given \( 1 \leq \ell \leq n \) we say \( (\xi_0 \ldots \xi_{\ell-1}) \) is an initial sequence of length \( \ell \) of \( \Xi^+(n) \) if there is \( (\xi_0 \ldots \xi_{n-1}) \) such that \( (\xi_0 \ldots \xi_{\ell-1} \xi_{\ell} \ldots \xi_{n-1}) \in \Xi^+(n) \).

Proposition 8.1. Given any \( \mu \in \mathcal{M}_{\text{erg}} \), for every small \( \varepsilon > 0 \) there exists a constant \( K = K(\mu, \varepsilon) > 1 \) such that for every \( n \geq 1 \) there exists a set \( \Xi^+(n) \subset \{0, \ldots, N-1\}^n \) of finite sequences of length \( n \) satisfying:

1) (Cardinality) The set \( \Xi^+(n) \) has cardinality 
\[
K^{-1} e^{n(h(\mu) - \varepsilon)} \leq \text{card}(\Xi^+(n)) \leq Ke^{n(h(\mu) + \varepsilon)}
\]

such that for every \( j \in \{0, \ldots, n\} \) and \( \ell \in \{j, \ldots, n\} \) there exist at least
\[
\frac{1}{3} K^{-2} e^{-2(\ell-j)\varepsilon} e^{(h(\mu) - \varepsilon)}
\]
initial sequences of length \( j \) of \( \Xi^+(n) \) such that each of them has between
\[
\frac{1}{3} K^{-2} e^{-2j\varepsilon} e^{(\ell-j)(h(\mu) - \varepsilon)} \text{ and } Ke^{(\ell-j)(h(\mu) + \varepsilon)}
\]
continuations to an initial sequence of length \( \ell \) of \( \Xi^+(n) \).

2) (Finite-time Lyapunov exponents) There exists an interval \( I = I(\mu, \varepsilon) \subset \mathbb{S}^1 \) such that for every \( n \geq 1 \) for each \( (\rho_0 \ldots \rho_{n-1}) \in \Xi^+(n) \) and each point \( x \in I \) for every \( j \in \{1, \ldots, n\} \) we have
\[
K^{-1} e^{j(\chi(\mu) - \varepsilon)} \leq |(f|_{\rho_0 \ldots \rho_{j-1}})'(x)| \leq Ke^{j(\chi(\mu) + \varepsilon)}.
\] (8.1)

Proof. The results are consequences of ergodicity, the definition of the Lyapunov exponent and the Brin-Katok, the Birkhoff ergodic, the Shannon-McMillan-Breiman, and the Egorov theorems (see also [21, Proposition 3.1]). For completeness we provide the details.

Given \( \varepsilon_E \in (0, \chi(\mu)/2), \varepsilon_H > 0, \) and \( \kappa \in (0, 1/6) \), there are a constant \( K > 1 \) and a set \( \Lambda \subset \Sigma_N \times \mathbb{S}^1 \) such that \( \mu(\Lambda) > 1 - \kappa \) and for every \( X = (\rho, x) \in \Lambda \) for every \( n \geq 1 \) we have
\[
K^{-1} e^{n(\chi(\mu) - \varepsilon_E)/2} \leq |(f|_{\rho_0 \ldots \rho_{n-1}})'(x)| \leq Ke^{n(\chi(\mu) + \varepsilon_E)/2}.
\] (8.2)

Let \( S \overset{\text{def}}{=} \pi(\Lambda) \subset \Sigma_N \) and \( \nu \overset{\text{def}}{=} \pi_* \mu \), where \( \pi : \Sigma_N \times \mathbb{S}^1 \to \Sigma_N \) denotes the natural projection. Note that \( \nu \) is \( \sigma \)-invariant and ergodic, \( \nu(S) \geq 1 - \kappa \), and we can also assume that for every \( \pi(\rho) \in S \) and every \( n \geq 1 \)
\[
K^{-1} e^{-n(h(\mu) + \varepsilon_H)} \leq \nu([\rho_0 \ldots \rho_{n-1}]) \leq Ke^{-n(h(\mu) - \varepsilon_H)}.
\] (8.3)
For every $n \geq 1$, define

$$\Xi^+(n) \overset{\text{def}}{=} \{(\rho_0 \ldots \rho_{n-1}) \in [0, \ldots, N-1]^n : [\rho_0 \ldots \rho_{n-1}] \cap S \neq \emptyset\}.$$  

Note that by (8.3) (notice that those cylinders are pairwise disjoint and cover a set of measure at least $1 - \kappa$) this set has cardinality $M_n$ bounded by

$$(1 - \kappa) \cdot K^{-1} e^{(h(\mu) - \varepsilon_H)} \leq M_n \leq K e^{(h(\mu) + \varepsilon_H)},$$

which proves the first assertion of item (1).

To prove the second assertion of item (1), fix positive integers $j$, $\ell$ such that $j \leq \ell \leq n$. Let $S(j) \overset{\text{def}}{=} S \cap \sigma^{-j}(S)$. Note that $v(S) > 1 - \kappa$ implies $1 - 2\kappa < v(S(j)) \leq 1$. Observe that $S(j)$ consists of sequences $\xi = (\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots)$ which by (8.3) have the property that the initial $j$-cylinder $[\xi_0 \ldots \xi_{j-1}]$ satisfies

$$K^{-1} e^{-j(h(\mu) + \varepsilon_H)} \leq v([\xi_0 \ldots \xi_{j-1}]) \leq K e^{-j(h(\mu) - \varepsilon_H)}$$

and that (because $\sigma^j(\xi) \in S$) has the property that its $(\ell - j)$-cylinder $[\xi_j \ldots \xi_{\ell-1}]$ satisfies

$$K^{-1} e^{-(\ell-j)(h(\mu) + \varepsilon_H)} \leq v([\xi_j \ldots \xi_{\ell-1}]) \leq K e^{-(\ell-j)(h(\mu) - \varepsilon_H)}.$$  

These estimates have the consequence that we can choose a subset of $S(j)$ of sequences which is $(\ell - j, 1)$-separated (with respect to $\sigma$) and whose cardinality $M_{\ell-j}$ is bounded between

$$(1 - 2\kappa) \cdot K^{-1} e^{-(\ell-j)(h(\mu) - \varepsilon_H)} \leq M_{\ell-j} \leq K e^{-(\ell-j)(h(\mu) + \varepsilon_H)}. \quad (8.4)$$

Arguing in the same way for $j$ instead of $\ell - j$, we get a subset of $S(j)$ of sequences which is $(j, 1)$-separated and whose cardinality $M_j$ is bounded between

$$(1 - 2\kappa) \cdot K^{-1} e^{j(h(\mu) - \varepsilon_H)} \leq M_j \leq K e^{j(h(\mu) + \varepsilon_H)}.$$  

Note that to any such sequence with a fixed initial $j$-cylinder $[\xi_0 \ldots \xi_{j-1}] \in S(j)$ there are at most $M_{\ell-j}$ continuations to a sequence with a corresponding $\ell$-cylinder $[\xi_0 \ldots \xi_{j-1}\xi_j \ldots \xi_{\ell-1}]$ which has a continuation to some sequence in $S(j)$.

Assume now, by contradiction, that among those sequences with initial $j$-cylinder there are only less than

$$M'_j \overset{\text{def}}{=} \frac{1}{3} K^{-2} e^{-2(\ell-j)\varepsilon_H} e^{j(h(\mu) - \varepsilon_H)}$$

of them which have more than $\frac{1}{3} K^{-2} e^{-2j\varepsilon_H} e^{(\ell-j)(h(\mu) - \varepsilon_H)}$ continuations to a $\ell$-cylinder. Note that $M_j$ gives a simple estimate from above for the number of the remaining initial cylinders. Hence, the total number of $(\ell, 1)$-separated sequences in $S(j)$ would be bounded from above by

$$M_j \cdot \frac{1}{3} K^{-2} e^{-2j\varepsilon_H} e^{(\ell-j)(h(\mu) - \varepsilon_H)} + M_j' \cdot M_{\ell-j} \leq K e^{j(h(\mu) + \varepsilon_H)} \cdot \frac{1}{3} K^{-2} e^{-2j\varepsilon_H} e^{(\ell-j)(h(\mu) - \varepsilon_H)}$$

$$+ \frac{1}{3} K^{-2} e^{-2(\ell-j)\varepsilon_H} e^{j(h(\mu) - \varepsilon_H)} \cdot K e^{(\ell-j)(h(\mu) + \varepsilon_H)}$$

$$= \frac{2}{3} K^{-1} e^{(\ell(j)(h(\mu) - \varepsilon_H)}.$$
However, from (8.4) with \( j = 0 \), there are at least \( M_l \geq (1 - 2\kappa)K^{-1}e^{\ell(h(\mu) - \varepsilon H)} \) such sequences in \( S(j) \), which leads us to a contradiction. Together with the upper estimate in (8.4), this ends the proof of item (1).

To show item (2), what remains is to prove that (8.1) holds for any point in \( I \) and not only for a point \( x \) such that \((\rho, x) \in \Lambda\) as in (8.2). As any \( X = (\rho, x) \in \Lambda \) is a point with uniform contraction (using that \( \chi(\mu) < 0 \)), we will obtain distortion control on some small neighborhood whose size depends on the constant \( K \). For that we use the following.

**Claim 8.2** ([21, Proposition 3.4]). Given \( \varepsilon_D > 0 \), let \( \delta_0 > 0 \) be such that

\[
\max_{i=0,\ldots,N-1} \max_{x,y \in \mathbb{S}^1, |y-x| \leq 2\delta_0} \log \frac{|f_i'(y)|}{|f_i'(x)|} \leq \varepsilon_D.
\]

Suppose that \( x \in \mathbb{S}^1, r > 0, n \geq 1 \) are such that for every \( \ell = 0, \ldots, n-1 \) we have

\[
|\langle f_\ell^\rho \rangle'(x)| < \frac{1}{r} \delta_0 e^{-\ell \varepsilon_D},
\]

then for every \( \ell = 0, \ldots, n-1 \) we have

\[
\sup_{x,y: |y-x| \leq r} \frac{|\langle f_\ell^\rho \rangle'(y)|}{|\langle f_\ell^\rho \rangle'(x)|} \leq e^{\ell \varepsilon_D}.
\]

Fixing some

\[
\varepsilon_D < \frac{1}{2} \min\{\varepsilon_E, |\chi(\mu) - \varepsilon_E|\},
\]

let \( \delta_0 > 0 \) be as in Claim 8.2 and choose also \( r > 0 \) such that \( K < \delta_0/r \). Thus, for every \( X = (\rho, x) \in \Lambda \) and every \( y \in (x-r, x+r) \) for every \( \ell = 0, \ldots, n-1 \) with (8.2) we obtain

\[
Ke^{\ell(\chi(\mu) - \varepsilon_E/2)} e^{-\ell \varepsilon_D} \leq |\langle f_{[\rho_0,\ldots,\rho_{\ell-1}]^\rho \rangle'(y)| \leq Ke^{\ell(\chi(\mu) + \varepsilon_E/2)} e^{\ell \varepsilon_D}.
\]

Dividing now \( \mathbb{S}^1 \) into intervals of length \( r \), at least one interval of them, denoted \( I \), must contain at least \( K^{-1}re^{\alpha(h(\mu) - \varepsilon)} \) starting points \( x \) of \((n, 1)\)-separated trajectories corresponding to \( \Sigma^+(n) \). Note that this way we perhaps disregard some of the elements in \( \Sigma^+(n) \), but we continue to denote the remaining set by \( \Sigma^+(n) \) and the proposition follows exchanging now \( K^{-1}r \) for \( K \). \( \square \)

### 8.2. Large subset of the level set.

In this section, we will construct a large subset \( \Xi \subset \pi(L(\alpha)) \). We start by fixing some quantifiers.

**Choice of quantifiers.** Given

\[
h = \limsup_{\beta \to \alpha} \mathcal{E}(\alpha),
\]

there is a sequence of ergodic measures \((\mu_k)_{k \geq 0}\) with Lyapunov exponents converging to \( \alpha \) and with the upper limit of entropies equal to \( h \). We aim to prove that \( h_{\operatorname{top}}(L(\alpha)) \geq h \). Without weakening of assumptions, by passing to a subsequence, we can assume that all the measures \( \mu_k \) have exponents of the same sign and that their entropies converge to \( h \), recall the results in Sect. 4. In the following, we will assume that \( \alpha \leq 0 \) and that all
the measures $\mu_k$ have negative exponents. The other case can be obtained by studying the map $F^{-1}$ instead of $F$.

Fix a sequence $(\varepsilon_k)_{k \geq 0}$ with $\varepsilon_k \searrow 0$ and apply Proposition 8.1 item (1) to each measure $\mu_k$: we get constants $K_k = K_k(\mu_k, \varepsilon_k)$ and for every $n \geq 1$ a set $\mathcal{E}_k^+(n)$ of finite sequences of length $n$. As, by our choice of sequences, $h(\mu_k) \to h$, we can assume that our constants $K_k$ and $\varepsilon_k$ are such that for every $k \geq 0$ and for every $\ell \geq 1$ we have

$$K_k^{-1}e^{\ell(h-\varepsilon_k)} \leq \text{card } \mathcal{E}_k^+(\ell) \leq K_k e^{\ell(h+\varepsilon_k)}. \tag{8.6}$$

We will choose a sequence $(n_k)_{k \geq 0}$ with $n_k \nearrow \infty$, which will be further specified below. By Proposition 8.1 item (1), there is $\mathcal{E}_k^+(n_k)$ such that for every $j \in \{0, \ldots, n_k\}$ and $\ell \in \{j, \ldots, n_k\}$ there exist at least

$$\frac{1}{3}K_k^{-2}e^{-2(\ell-j)\varepsilon_k}e^{j(h-\varepsilon_k)} \tag{8.7}$$

initial sequences of length $j$ of $\mathcal{E}_k^+(n_k)$, each of which has between

$$\frac{1}{3}K_k^{-2}e^{-2j\varepsilon_k}e^{j(h-\varepsilon_k)} \quad \text{and} \quad K_k e^{(\ell-j)(h+\varepsilon_k)} \tag{8.8}$$

continuations to an initial sequence of length $\ell$ of $\mathcal{E}_k^+(n_k)$.

The same arguments but applied to $\sigma^{-1}$ instead of $\sigma$ provide sets $\mathcal{E}_k^-(n_k)$ of finite sequences with the very same properties.

Finally, by Proposition 8.1 item (2) applied to $\mu_k$ and $\sigma$, there exist intervals $I_k = I(\mu_k, \varepsilon_k)$ such that for every $x \in I_k$ we have

$$K_k^{-1}e^{\ell(\chi(\mu_k)-\varepsilon_k)} \leq |(f_{[\rho_1, \ldots, \rho_l]}'(x))| \leq K_k e^{\ell(\chi(\mu_k)+\varepsilon_k)}. \tag{8.9}$$

To each interval $I_k$ we associate numbers $\delta_k > 0$ and $M_k > 0$ provided by Lemma 4.6. This ends the choice of quantifiers.

**Construction of $\mathcal{E}$.** We now are prepared to construct a subset $\mathcal{E} \subset \Sigma^N$ in the projection of $L(\alpha)$. First, we construct forward orbits on which the (forward) Lyapunov exponent is $\alpha$.

As the chosen orbit pieces, and their chosen neighborhoods, are uniformly contracting, we can fix a sequence of sufficiently fast increasing natural numbers $(n_k)_k$ such that for each $k$ for every $(\rho_1 \ldots \rho_{n_k}) \in \mathcal{E}_k^+(n_k)$ we have that

$$|f_{[\rho_1, \ldots, \rho_{n_k}]}(I_k)| < \delta_{k+1}.$$ 

Note that $n_k$ can be chosen arbitrarily large. We will further specify the choice of this sequence in Sect. 8.3. Hence, by Lemma 4.6, to each $(\rho_1 \ldots \rho_{n_k})$ we associate one (there may be several choices, we just pick one) finite sequence $(\tau_1 \ldots \tau_m)$, $m \leq M_{k+1}$, such that

$$(f_{[\tau_1, \ldots, \tau_m]} \circ f_{[\rho_1, \ldots, \rho_{n_k}]})(I_k) \subset I_{k+1}.$$ 

We point out that the sequence $(\tau_1 \ldots \tau_m)$ depends on the initial sequence $(\rho_1 \ldots \rho_{n_k})$, which is not reflected by the notation to simplify the exposition.

We consider now the set of all such concatenated finite sequences defined by

$$\mathcal{E}_k^f \overset{\text{def}}{=} \{(\rho_1 \ldots \rho_{n_k} \tau_1 \ldots \tau_m) : (\rho_1 \ldots \rho_{n_k}) \in \mathcal{E}_k^+(n_k)\}. \tag{8.9}$$
We write \((\rho_1 \ldots \rho_n) = \varrho\) and \((\tau_1 \ldots \tau_m) = \vartheta\) and say that \(\varrho\) is a main sequence and that \(\vartheta\) is a connecting sequence. Finally, we consider the set \(\Xi^+\) of all one-sided infinite sequences

\[
\Xi^+ \overset{\text{def}}{=} \{\varrho_1 \vartheta_2 \varrho_2 \ldots \varrho_k \vartheta_k \ldots : \varrho_k \vartheta_k \in \Sigma'\}.
\]

Note that by construction, for every \(k \geq 1\) we have

\[
(f_{[\varrho \vartheta \ell]} \circ \ldots \circ f_{[\varrho \vartheta 1]})(I_1) \subset I_{\ell + 1}, \quad \ell = 1, \ldots, k.
\]

By our choice of quantifiers, for every \(x \in I_1\) and every \(\xi \in \Xi^+\) we have

\[
\lim_{n \to \infty} \frac{1}{n} \log |(f^n_x)'(x)| = \alpha.
\]

(8.10)

Until now we proved, that for some interval \(I_1\) there exists a large set of forward-infinite symbolic sequences \(\xi^+\) such that for every point \(x \in I_1\) the forward orbit of \((x, \xi^+)\) satisfies (8.10). Now we can do the construction in the other (time) direction using the sets of finite sequences \(\Xi_k^- (n)\) instead of \(\Xi_k^+ (n)\). It is somewhat analogous: we take the sequence \((\mu_k)_{k \geq 1}\) of measures defined by \(\mu_1 = \mu_1, \mu_2 = \mu_2, \ldots\). For every \(k \geq 1\) we apply Proposition 8.1 item (1) to the inverse map \(F^{-1}\) and the \(F^{-1}\)-invariant measure \(\mu_{-k}\) and obtain a set of sequences that we will denote by \(\Xi_k^- (n_k)\). Then we connect them using Lemma 4.6. Note that constructed backward itineraries are expanding, thus for a backward itinerary we just get one point following it instead of an interval of points as in the forward itinerary. Note also that for each finite concatenation we get a closed interval of starting points and that these intervals form a nested sequence, the point is given by the intersection of these intervals. In this way we obtain a set \(\Xi^-\) of backward-infinite symbolic sequences such that for each of them there exists a corresponding backward orbit \((y, \xi^-)\) satisfying

\[
\lim_{n \to -\infty} \frac{1}{n} \log |(f^n_y)'(y)| = \alpha.
\]

By Lemma 4.6 we can make each of those backward orbits end at some point in \(I_1\). Hence, each of those trajectories can be prolonged into the future by any \(\xi \in \Xi^+\).

To summarize, we obtained a set of two-sided infinite sequences

\[
\Xi \overset{\text{def}}{=} \Xi^- \cdot \Xi^+ = \{\xi^- \cdot \xi^+: \xi^\pm \in \Xi^\pm\}
\]

such that \(\Xi \subset \pi(\mathcal{L}(\alpha))\), where \(\pi : \Sigma \times S^1 \to \Sigma\) denoted the natural projection. Note that \(\Xi\) depends on the choice of the quantifiers \(\Xi = \Xi((\varepsilon_k), (n_k)_k)\). The sequence \((n_k)_k\), and hence the set \(\Xi\), will be further specified in Sect. 8.3.

8.3. Specifying the set \(\Xi = \Xi((\varepsilon_k), (n_k)_k)\). We need the following technical lemma which estimates how rapidly sequences in \(\Xi = \Xi^- \cdot \Xi^+\) are “branching out”. However, for technical reasons (see the proof of Lemma 8.6), we are only interested in some specific type of branching (sequences at length \(\iota\) branch out to sequences of length \(N\), where \(\iota\) satisfies \(N - \iota \leq \iota\)). This will be used in Sect. 8.4 to construct a special probability measure.

Recall the quantifiers fixed in Sect. 8.2.
Lemma 8.3. There are a sequence \((n_k)_k\), a constant \(K > 1\), and a function \(\delta : \mathbb{N} \to (0, 1)\) with
\[
n_k > 2^k \text{ and } \lim_{N \to \infty} \delta(N) = 0 \quad (8.11)
\]
such that for every \(N \geq 1\) there exist between \(K^{-1} \cdot e^{N(h - \delta(N))}\) and \(K \cdot e^{N(h + \delta(N))}\) different sequences of length \(N\) which admit continuations to a one-sided infinite in \(\Xi^+\). Moreover,
\[
\text{for every } \iota \in \{1, \ldots, N\} \text{ satisfying } N - \iota \leq \iota
\]
there exist at least \(K^{-1} \cdot e^{\iota h - \iota \delta(\iota)}\) different finite sequences of length \(\iota\) such that each of them has at least \(K^{-1} \cdot e^{(N - \iota)h - N \delta(\iota)}\) and at most \(K \cdot e^{(N - \iota)h + N \delta(\iota)}\) different continuations to a sequence in \(\Xi^+\).

The same statement also holds for \(\Xi^-\), modulo time reversal.

Proof. We will present the proof for \(\Xi^+\), the case of \(\Xi^-\) is analogous. For the sequences \((\varepsilon_k)_k\), \((K_k)_k\), and \((M_k)_k\) chosen in Sect. 8.2 we choose a sequence \((n_k)_k\) such that
\[
\prod_{i=1}^{k+1} K_i \leq e^{\varepsilon_k - 1}, \quad (8.12)
\]
\[
\sum_{i=1}^{k-1} n_i \varepsilon_i \leq n_k \varepsilon_k \quad \text{or, equivalently,} \quad \sum_{i=1}^k n_i \varepsilon_i \leq 2n_k \varepsilon_k, \quad (8.13)
\]
and
\[
M_1 + \sum_{i=1}^{k-1} (n_i + M_{i+1}) < n_k \quad \text{and} \quad e^{M_{k+1} h} \leq e^{n_k - 1}. \quad (8.14)
\]

Note that these conditions hold simultaneously if \((n_k)_k\) grows sufficiently fast.

By the construction in Sect. 8.2, any sequence in \(\xi \in \Xi^+\) is a concatenation of main and connecting finite sequences \(\varrho_1 \theta_1 \varrho_2 \theta_2 \ldots \varrho_k \theta_k \ldots\) such that \(\varrho_k \in \Xi^+_k(n_k)\) and \(|\theta_k| \leq M_{k+1}\). To shorten notation, we say that an index \(\iota \in \varrho_k\) provided that it enumerates a symbol in the block \(\varrho_k\) within the sequence \(\xi = \varrho_1 \theta_1 \varrho_2 \theta_2 \ldots \varrho_k \theta_k \ldots\), similar for any of the other blocks. Denote by
\[
m_1 = m_1(\xi) \overset{\text{def}}{=} 0, \quad m_k = m_k(\xi) \overset{\text{def}}{=} \sum_{i=1}^{k-1} (n_i + |\theta_i|)
\]
the position of the beginning of the block \(\varrho_k\).

To show the first claim in the lemma, let \(N \geq 1\). Let \(k \geq 1\) be such that \(m_k < N \leq m_{k+1}\) and write \(N = m_k + \ell, \ell \geq 1\). Using (8.6) we have that the number of sequences of length \(N\) is given by
\[
\text{card } \Xi^+_1(n_1) \cdot \ldots \cdot \text{card } \Xi^+_{k-1}(n_{k-1}) \cdot \text{card } \Xi^+_k(\ell) \leq \prod_{i=1}^{k-1} K_i \cdot e^{Nh} \cdot e^{\sum_{i=1}^{k-1} n_i \varepsilon_i} \cdot e^{\ell \varepsilon_k}.
\]
By (8.12) and (8.13) the latter can be estimated from above by
\[ e^{n_{k-1}e_k} \cdot e^{N h} \cdot e^{2n_{k-1}e_{k-1}} e^{\ell e_k} \leq e^{N h} e^{3n_{k-1}e_{k-1}+\ell e_k} \leq e^{N(h+3\epsilon_{k-1})}, \]
with an analogous lower bound proving the claimed property.

To show the second claim in the lemma, let now \( N \geq 1 \) and \( i \in \{1, \ldots, N\} \) satisfying \( N - \epsilon \leq \ell \). First, note that either \( i \in \varrho_k \) or \( i \in \vartheta_k \) for appropriate \( k \). We will only discuss the case \( i \in \varrho_k \), the other one is simpler and hence omitted. Note that \( i \in \varrho_k \) implies
\[ m_{k-1} \leq \epsilon \leq \sum_{i=1}^{k} (n_i + |\vartheta_i|) \leq \sum_{i=1}^{k} (n_i + M_{i+1}) < n_{k+1}, \quad (8.15) \]
where the last inequality follows from (8.14). Now note that (8.15) and our hypothesis \( N \leq 2\epsilon \) imply that either \( N \in \varrho_k \) or \( N \in \vartheta_k \) or \( N \in \varrho_{k+1} \). This leads to the three cases studied below. In the course, we will be implicitly defining constants \( c^{(i)}(i, N), C^{(i)}(i, N), i = 1, 2, 3, \) needed at the end of the proof.

**Case 1:** \( i \in \varrho_k \) and \( N \in \varrho_k \). Write \( \epsilon = m_k + j \) and \( N = m_k + \ell \) and note that \( N - \epsilon = j - \ell \). Compare Fig. 4. By our choice of the quantifiers (see (8.7) in Sect. 8.2 applied to \( E_k^i(\ell) \)), there exist at least
\[ \frac{1}{3} K_k^{-2} e^{-2(\ell-j)e_k} e^{i(h-\epsilon_k)} = \frac{1}{3} K_k^{-2} e^{-2(N-i)e_k} e^{i(h-\epsilon_k)} = \frac{1}{3} K_k^{-2} e^{-(j+2(N-i))e_k} e^{i\ell} = H(\epsilon, N) \cdot e^{i\ell}, \quad (8.16) \]
where
\[ H(\epsilon, N) \overset{\text{def}}{=} \frac{1}{3} K_k^{-2} e^{-(j+2(N-i))e_k}, \quad (8.17) \]
initial sequences within the block \( \varrho_k \) each of which has between (see (8.8))
\[ \frac{1}{3} K_k^{-2} e^{-2j e_k} e^{(\ell-j)(h-\epsilon_k)} = \frac{1}{3} K_k^{-2} e^{-2j e_k} e^{(N-i)(h-\epsilon_k)} = \frac{1}{3} K_k^{-2} e^{-(2j+N-i)e_k} e^{(N-i)\ell} \overset{\text{def}}{=} c^{(1)}(i, N) \cdot e^{(N-i)\ell} \quad (8.18) \]
and

\[ K_k e^{(\ell-j)(h+\varepsilon_k)} = K_k e^{(N-\ell)(h+\varepsilon_k)} = K_k e^{(N-\ell)\varepsilon_k} \cdot e^{(N-\ell)h} \]

\[ \overset{\text{def}}{=} C^{(1)}(t, N) \cdot e^{(N-\ell)h} \] (8.19)

continuations to a sequence of length \( \ell \) within this block.

**Case 2:** \( t \in \Omega_k \) and \( N \in \partial_k \). Write \( t = m_k + j \) and \( N = m_k + n_k + \ell \) and note that \( N - t = n_k - j + \ell \). By our choice of the constants (see (8.7) in Sect. 8.2 now applied to \( \Sigma^+_k(n_k) \) with \( \ell = n_k \)) and also using \( n_k - j \leq N - t \), there exist at least

\[ \frac{1}{3} K_k^{-2} e^{-2(n_k-j)\varepsilon_k} e^{j(h-\varepsilon_k)} \geq \frac{1}{3} K_k^{-2} e^{-2(N-\ell)\varepsilon_k} e^{j(h-\varepsilon_k)} \]

\[ = \frac{1}{3} K_k^{-2} e^{-(j+2(N-\ell))\varepsilon_k} \cdot e^{jh} = H(t, N) \cdot e^{jh} \] (8.20)

initial sequences of length \( j \) within the block \( \Omega_k \). By (8.8) each of them has between

\[ \frac{1}{3} K_k^{-2} e^{-2j\varepsilon_k} e^{(n_k-j)(h-\varepsilon_k)} \] (8.21)

and

\[ K_k e^{(n_k-j)(h+\varepsilon_k)} \] (8.22)

continuations to a sequence of length \( n_k \) within this block. Each of those sequences can then be continued (without further branching) to a sequence of length \( N - m_k \) by some connecting sequence whose length is between 0 and \( M_k \). We now estimate the terms in (8.21) and (8.22). Using that \( n_k - j \leq N - t \) and \( \ell \leq M_{k+1} \) we get

Eq. (8.21) = \[ \frac{1}{3} K_k^{-2} e^{-2j\varepsilon_k} e^{-(n_k-j)\varepsilon_k} e^{-\ell h} e^{(n_k+\ell-j)h} \]

\[ \geq \frac{1}{3} K_k^{-2} e^{-(j+N-\ell)\varepsilon_k} e^{-M_{k+1} h} \cdot e^{(N-\ell)h} \overset{\text{def}}{=} C^{(2)}(t, N) \cdot e^{(N-\ell)h}, \] (8.23)

Eq. (8.22) = \[ K_k e^{(n_k-j)\varepsilon_k} e^{-\ell h} e^{(n_k+\ell-j)h} \]

\[ < K_k e^{(N-\ell)\varepsilon_k} \cdot e^{(N-\ell)h} \overset{\text{def}}{=} C^{(2)}(t, N) \cdot e^{(N-\ell)h}. \]

**Case 3:** \( t \in \Omega_k \) and \( N \in \Omega_{k+1} \). Write \( t = m_k + j \) and \( N = m_k + n_k + |\partial_k| + \ell = m_{k+1} + \ell \). First, as in Case 2, there exist initial sequences of length \( j \) within \( \Omega_k \) which can be estimated as in (8.20) and each of them has between (8.21) and (8.22) continuations to a sequence of length \( n_k \) within this block. Then each of them is continued (without further branching) by some connecting sequence of length \( |\partial_k| \) up to length \( m_{k+1} \) at the beginning of block \( \Omega_{k+1} \). Compare Fig. 5. Then, by (8.6) in Sect. 8.2 applied to \( \Sigma^+_k(\ell) \), for each such sequence there exist between

\[ K_{k+1}^{-1} e^{(h-\varepsilon_{k+1})} = K_{k+1}^{-1} e^{(N-m_{k+1})(h-\varepsilon_{k+1})} \quad \text{and} \quad K_{k+1} e^{(N-m_{k+1})(h+\varepsilon_{k+1})} \] (8.24)

continuations to a sequence of length \( N - m_k \). Thus, as for (8.20) and using \( n_k - j \leq N - t \), there exist at least

\[ \frac{1}{3} K_k^{-2} e^{-2(n_k-j)\varepsilon_k} e^{j(h-\varepsilon_k)} \geq \frac{1}{3} K_k^{-2} e^{-2(N-\ell)\varepsilon_k} e^{j(h-\varepsilon_k)} \]

\[ = \frac{1}{3} K_k^{-2} e^{-(j+2(N-\ell))\varepsilon_k} \cdot e^{jh} = H(t, N) \cdot e^{jh} \] (8.25)
initial sequences of length \( j \) within \( \varrho_k \). Moreover, multiplying the terms in (8.24) with (8.21) and (8.22), respectively, each of them has between

\[
\frac{1}{3} K_k^{-2} e^{-2j\varepsilon_k} e^{(n_k-j)(h-\varepsilon_k)} \cdot K_{k+1}^{-1} e^{(N-m_{k+1})(h-\varepsilon_{k+1})} = \frac{1}{3} K_k^{-2} K_{k+1}^{-1} e^{-2j\varepsilon_k-(n_k-j)\varepsilon_k-(N-m_{k+1})\varepsilon_{k+1}} e^{(n_k-j+N-m_{k+1})h} \geq \frac{1}{3} K_k^{-2} K_{k+1}^{-1} e^{-(2j+N-\varepsilon_k)} \cdot e^{-M_{k+1}h} e^{(N-h)\varepsilon_k} \cdot e^{(N-h)h} \text{ def } = c(3)(t, N) \cdot e^{(N-h)h}
\]

and

\[
K_k e^{(n_k-j)(h+\varepsilon_k)} \cdot K_{k+1} e^{(N-m_{k+1})(h+\varepsilon_{k+1})} = K_k K_{k+1} e^{(n_k-j+\varepsilon_k+(N-m_{k+1})\varepsilon_{k+1})} e^{(n_k-j+N-m_{k+1})h} \leq K_k K_{k+1} e^{(N-h)\varepsilon_k} \cdot e^{(N-h)h} \text{ def } = C(3)(t, N) \cdot e^{(N-h)h}
\]

continuations each of which can then be continued to an infinite sequence on \( \Xi^+ \). This ends the estimates in Case 3.

We are now ready to estimate the cardinality of the set of finite compound sequences \( \varrho_1 \vartheta_1 \ldots \varrho_{k-1} \vartheta_{k-1} \) which is given by

\[
\text{card } \Xi_1^+(n_1) \cdot \ldots \cdot \text{card } \Xi_{k-1}^+(n_{k-1}).
\]

Joining the above Cases 1, 2, and 3, let us now estimate all possible continuations to a sequence \( \varrho_1 \vartheta_1 \ldots \varrho_{k-1} \vartheta_{k-1} \varepsilon_{m_k} \ldots \varepsilon_t \) of length \( t \) which then has continuations to an infinite sequence \( \varepsilon_t \in \Xi^+ \). Writing \( t = m_k + j \), there exist (using (8.16) in Case 1, (8.20) in Case 2, and (8.25) in Case 3) at least

\[
L = \text{card } \Xi_1^+(n_1) \cdot \ldots \cdot \text{card } \Xi_{k-1}^+(n_{k-1}) \cdot H(t, N) \cdot e^{j\varepsilon_k}
\]

initial finite sequences of length \( t \) such that each of them has at least (using (8.18), (8.21), and (8.26))

\[
\min_{r=1,2,3} c^{(r)}(t, N) \cdot e^{(N-t)(h-\varepsilon_k)}
\]

and at most (using (8.19), (8.22), and (8.27))

\[
\max_{r=1,2,3} C^{(r)}(t, N) \cdot e^{(N-t)(h-\varepsilon_k)}
\]
continuations to an infinite sequence in $\Xi^+$. To conclude the proof of the lemma, we finally estimate the above terms. Recalling (see (8.9)) that $\text{card } \Xi' = \text{card } \Xi^+(n_\ell)$, with (8.6) and (8.28) we have

$$L \geq \prod_{i=1}^{k-1} \left( K_i^{-1} e^{n_i (h - \varepsilon_i)} \right) \cdot H(t, N) \cdot e^{jh}$$

$$= \prod_{i=1}^{k-1} K_i^{-1} \cdot e^{-\sum_{i=1}^{k-1} n_i \varepsilon_i} \cdot H(t, N) \cdot e^{jh} \overset{\text{def}}{=} \tilde{H}(t, N) \cdot e^{jh}.$$

We can conclude, recalling the definition of $H(t, N)$ in (8.17)

$$\tilde{H}(t, N) = \prod_{i=1}^{k-1} K_i^{-1} \cdot e^{-\sum_{i=1}^{k-1} n_i \varepsilon_i} \cdot \frac{1}{3} K_k^{-2} \cdot e^{-(j+2(N-t))\varepsilon_k}$$

(24.3)

(24.4)

(24.5)

Moreover, using the definitions of $c^{(r)}(t, N)$ in (8.18), (8.23), and (8.26), we have

$$\min_{r=1,2,3} c^{(r)}(t, N) = c^{(3)}(t, N) = \frac{1}{3} K_k^{-2} K_{k+1}^{-1} \cdot e^{-(2j+(N-t))\varepsilon_k} \cdot e^{-M_{k+1}h}$$

(24.6)

(24.7)

Finally, using (8.19), (8.23), and (8.27), we have

$$\max_{r=1,2,3} C^{(r)}(t, N) = C^{(3)}(t, N) = K_k K_{k+1} \cdot e^{(N-t)\varepsilon_k}$$

(24.8)

(24.9)

Now, choosing the function $\delta$ appropriately, this implies the claimed estimate, where the constant $K$ in the lemma takes also care of the remaining (not considered) cases. \hfill \Box

### 8.4. Estimate of the entropy of $\Xi$: bridging measures

In this section, we show that $\Xi$ has large entropy, bigger than $h$ defined in (8.5). The heart of the argument is the construction of certain Borel probability measures $v^{\pm}$ and the application of the Mass distribution principle that we recall below (see [41]). Here we follow a general principle of so-called bridging measures already used in other contexts. See for example [30, Section 5.2] and [29, Section 5.2] (where those measures are called $w$-measures). We will use Frostman’s lemma again in Sect. 11.5.
Lemma 8.4 (Mass distribution principle or Frostman’s lemma). Consider a compact metric space \((X, d)\) and a subset \(\Xi \subset X\). Let \(\nu\) be a finite Borel measure such that \(\nu(\Xi) > 0\). Suppose that there exists \(D > 0\) such that for every \(x \in \Xi\) it holds
\[
\liminf_{\varepsilon \to 0} \frac{\log \nu(B(x, \varepsilon))}{\log \varepsilon} \geq D.
\]
Then \(\text{HD}(\Xi) \geq D\), where \(\text{HD}\) denotes the Hausdorff dimension.

Remark 8.5. Consider the natural projections \(\pi^\pm: \Sigma_N^\pm \to \Sigma_N^\pm\) and the shift maps \(\sigma^\pm: \Sigma_N^\pm \to \Sigma_N^\pm\). Recall that, by construction, \(\pi^\pm(\Xi) = \Xi^\pm\). Note that \(h_{\text{top}}(\sigma^\pm, \Xi^\pm) = h_{\top}(\sigma^\pm(\Xi)) \leq h_{\top}(\sigma, \Xi)\).

Note that for the standard metric the Hausdorff dimension of any set in \(\Sigma_N^+\) is equal to its topological entropy relative to \(\sigma^+\).

We will apply Lemma 8.4 and Remark 8.5 to estimate the topological entropy of the sets \(\Xi^\pm\) relative to \(\sigma^\pm\) and hence the topological entropy of \(\Xi\) relative to \(\sigma\).

Lemma 8.6. We have \(h_{\top}(\sigma^+, \Xi^+) \geq h\) and \(h_{\top}(\sigma^-, \Xi^-) \geq h\).

Proof. Consider the sequence \((r_k)_{k \geq 0}\) given by \(r_k = 2^k\). Note that by (8.11) in Lemma 8.3 the sequence \((n_k)_{k \geq 0}\) grows faster than the sequence \((r_k)_{k \geq 0}\).

We define a probability measure \(\nu^+\) depending only on the (forward) one-sided sequences \(\Sigma_N^+\) and a measure \(\nu^-\) depending only on the (backward) one-sided sequences \(\Sigma_N^-\). The measure \(\nu^+\) is constructed as follows (the measure \(\nu^-\) is analogously defined): for every \(k \geq 1\)

- for every (“parent”) cylinder of level \(r_k\) intersecting \(\Xi^+\), \(\nu^+\) is uniformly subdistributed on its (“child”) subcylinders of level \(r_{k+1}\) intersecting \(\Xi^+\),
- as the family of cylinders of levels \(r_k\), \(k \geq 1\), generate the Borel \(\sigma\)-algebra of \(\Sigma_N^+\), we obtain a Borel probability measure on this \(\sigma\)-algebra.

Given \(L \geq 1\) and \(\xi^+ \in \Xi^+\), denote by \(\Delta^+_L(\xi^+) \overset{\text{def}}{=} [\xi_0 \ldots \xi_{L-1}]\) the cylinder of length \(L\) containing \(\xi^+\).

Claim 8.7. For every \(\xi^+ \in \Xi^+\) we have \(\lim_{L \to \infty} -\frac{1}{L} \log \nu^+(\Delta^+_L(\xi^+)) \geq h\).

Proof. First, let
\[
C \overset{\text{def}}{=} \max_{\xi^+ \in \Xi^+} \Delta^+_2(\xi^+) > 0.
\]
In what follows we fix \(\xi^+\) and omit in the notation the dependence of \(\Delta^+_L\) on \(\xi^+\). Given \(L \geq 1\), consider some cylinder \(\Delta^+_L\) of length \(L\) which has nonempty intersection with \(\Xi^+\). We are going to estimate \(\nu^+(\Delta^+_L)\). There exists a unique index \(k \geq 1\) such that \(r_k < L \leq r_{k+1}\). We have
\[
\nu^+(\Delta^+_L) = \nu^+(\Delta^+_r) \cdot \left(\frac{\nu^+(\Delta^+_{r_1})}{\nu^+(\Delta^+_r)} \cdots \frac{\nu^+(\Delta^+_{r_{k-1}})}{\nu^+(\Delta^+_{r_{k-1}})} \right) \cdot \frac{\nu^+(\Delta^+_L)}{\nu^+(\Delta^+_{r_k})} = T_1 \cdot T_2 \cdot T_3,
\]
where \(\Delta^+_r\) is the corresponding parent \(r_i\)-cylinder of \(\Delta^+_{r_{i+1}}, i = 0, \ldots, k - 1\). Let us now estimate the three terms \(T_1, T_2,\) and \(T_3\) from above. Compare Fig. 6.

First note that, since \(r_0 = 2\) is fixed, \(T_1 = \nu^+(\Delta^+_2) \leq C\) does not depend on \(L\).
For every $i = 1, \ldots, k$, by the second part of Lemma 8.3 applied to $N = r_i$ and $\tau = r_{i-1}$ (note that the hypothesis $N - \tau = i$ is satisfied), we obtain that each cylinder $\Delta_{r_{i}}^+$ contains at least

$$K^{-1} \cdot e^{(r_i - r_{i-1})h} e^{-r_i \delta(r_{i-1})}$$

subcylinders of length $r_{i+1}$ intersecting $\Xi^+$ and hence (using $r_i = 2r_{i-1}$) we have

$$T_2 \leq \prod_{i=1}^{k} \left( K \cdot e^{-(r_i - r_{i-1})h} e^{r_i \delta(r_{i-1})} \right)$$

$$= K^k \prod_{i=1}^{k} e^{r_i \delta(r_{i-1})} \cdot e^{(r_0 - r_k)h} = K^k \cdot e^{\sum_{i=1}^{k} r_i \delta(r_{i-1})} e^{-(r_k - r_0)h} = e^{-r_k h} S(k),$$

where

$$S(k) \overset{\text{def}}{=} K^k e^{\sum_{i=1}^{k} r_i \delta(r_{i-1})} e^{r_0 h}.$$

Recall the particular choice of $r_k = 2^k$. For any $r > 0$ let

$$d(r) \overset{\text{def}}{=} \max\{i : \delta(r_{i-1}) \geq r\}.$$

Let $D = \max_i \delta(r_i)$. Then for any $k$ sufficiently large so that $d(r) < k$ we have

$$\frac{1}{r_k} \sum_{i=1}^{k} r_i \delta(r_{i-1}) < \frac{1}{r_k} \sum_{i=1}^{d(r)} r_i D + \frac{1}{r_k} \sum_{i=d(r)+1}^{k} r_i r \leq 2^{d(r)+1-k} D + 2r.$$

The latter term we can make arbitrarily small by choosing a small $r$ and the former summand tends to zero with $k \to \infty$. Hence, it follows that

$$\lim_{k \to \infty} \frac{1}{r_k} \log S(k) = 0. \quad (8.29)$$

The estimate of $T_3$ is done in two steps. First, since $r_k < L \leq r_{k+1}$, there is a cylinder $\Delta_{r_k}^+$ which contains $\Delta_{L}^+$. Let $\ell$ be the number of cylinders of level $r_{k+1}$ intersecting $\Xi^+$ and subdividing $\Delta_{r_k}^+$. Note that each such cylinder, by construction of $\nu^+$, has measure

$$\frac{1}{\ell} \nu^+(\Delta_{r_k}^+).$$

By Lemma 8.3 applied to $N = r_{k+1}$ and $\tau = r_k$, we have

$$\ell \geq K^{-1} \cdot e^{(r_{k+1} - r_k)h} e^{-r_{k+1} \delta(r_k)} = K^{-1} \cdot e^{(r_{k+1} - r_k)h} e^{-2r_k \delta(r_k)}.$$
Let \( \ell' \leq \ell \) be the number of cylinders of level \( r_{k+1} \) intersecting \( \Xi^+ \) which are contained in \( \Delta^+_L \). Again by Lemma 8.3 applied now \( N = r_{k+1} \) and \( t = L \) we get

\[
\ell' \leq K \cdot e^{(r_{k+1} - L)h} e^{r_{k+1}\delta(L)}.
\]

Hence

\[
T_3 = v^+(\Delta^+_L) \cdot \frac{1}{v^+(\Delta^+_r_k)} = \ell' \frac{1}{\ell} v^+(\Delta^+_r_k) \cdot \frac{1}{v^+(\Delta^+_r_k)} \leq \frac{K}{K^{-1}} \cdot e^{(r_{k+1} - L)h} e^{r_{k+1}\delta(L)} = e^{-(L-r_k)h} R(k),
\]

where

\[
R(k) \overset{\text{def}}{=} K^2 e^{2r_k(\delta(L)+\delta(r_k))}.
\]

In this case, it follows easily that

\[
\lim_{k \to \infty} \frac{1}{r_k} \log R(k) = 0.
\]  \hspace{1cm} (8.30)

This concludes the estimates of \( T_1, T_2, T_3 \). Putting them together, we get

\[
v^+(\Delta^+_L) \leq e^{-Lh} Q(k), \quad \text{where} \quad Q(k) = K^{-1} e^{-r_0(h+\delta(r_0))} \cdot e^{-r_k h} S(k) \cdot e^{r_k h} R(k).
\]

Using (8.29) and (8.30), it follows immediately that

\[
\lim_{k \to \infty} \frac{1}{r_k} \log Q(k) = 0.
\]

As the cylinder \( \Delta^+_L \) which intersects \( \Xi^+ \) was arbitrary, we get that

\[
\lim_{L \to \infty} -\frac{1}{L} \log v^+(\Delta^+_L(\xi^+)) \geq h,
\]

proving the claim.  \( \square \)

Analogous arguments apply to \( v^- \).

Note that, by construction, we have \( v^+(\Xi^+) = v^-(\Xi^-) = 1 \). By Lemma 8.4 applied to the probability measure \( v^+ \) on the space \( \Sigma^+_N \), we obtain \( h_{\text{top}}(\sigma^+, \Xi^+) \geq h \). The same arguments give \( h_{\text{top}}(\sigma^-, \Xi^-) \geq h \) using \( v^- \) instead of \( v^+ \). This concludes the proof of the lemma.  \( \square \)

8.5. End of the proof of Theorem 7.6. Recall that so far we worked under the hypothesis that the sign of the exponents of the measures \( \mu_k \) was negative. To conclude the proof in this case, note that \( \Xi \subset \pi(\mathcal{L}(\alpha)) \) and the projections \( \pi \) and \( \pi^+ \) do not increase entropy, hence

\[
h_{\text{top}}(F, \mathcal{L}(\alpha)) \geq h_{\text{top}}(\sigma, \pi(\mathcal{L}(\alpha))) \geq h_{\text{top}}(\sigma, \Xi) \geq h_{\text{top}}(\sigma^+, \Xi^+) \geq h.
\]

This ends the proof of Theorem 7.6 in this case.

In the other case, when the exponents of the measures \( \mu_k \) are positive, we can perform the same construction for \( F^{-1} \) instead of \( F \) and construct a set \( \Xi = \Xi(F^{-1}) \). However, since we want to determine the topological entropy with respect to \( F \), we have to consider the “inverse” set of \( \Xi \), that is, the set of sequences \( \Xi' \overset{\text{def}}{=} \{(\xi^+, \xi^-) : (\xi^-, \xi^+) \in \Xi \} \). To conclude, note that \( h_{\text{top}}(\sigma^+, \pi^+(\Xi')) = h_{\text{top}}(\sigma^-, \pi^-(\Xi)) \) and apply the second assertion in Lemma 8.6.
9. Measures of Maximal Entropy: Proof of Theorem 2

As explained in Sect. 7.1, any measure of maximal entropy (with respect to \( F \)) projects to the \((1/N, \ldots, 1/N)\)-Bernoulli measure \( b \). In this section, we study its properties and conclude the proof of Theorem 2.

9.1. Synchronization. We call a Bernoulli measure \( b = (b_1, \ldots, b_N) \) nondegenerate if all weights \( b_i \), \( i = 1, \ldots, N \), are positive. By Lemma 4.7 (there is no measure which is simultaneously \( f_i \)-invariant for every \( i \)) the assumptions in [16, Theorem 8.6] are satisfied. Hence for the Bernoulli measure \( b \) above there exists a (at least one) \( F \)-ergodic measure with positive exponent and a (at least one) \( F \)-ergodic measure with negative exponent, both projecting to \( b \). In what follows, in our context, we will show that those measures are unique and will be denoted by \( \mu_0^b \).

Following for example [40], given a Bernoulli measure \( b \), we say that an IFS \( \{f_i\}_{i=0}^{N-1} \) with probabilities \( b \) is forward synchronizing if for every \( x, y \in S^1 \) for \( b \)-almost every one-sided sequence \( \xi \in \Sigma^+_N \) we have

\[
|f^n_\xi(x) - f^n_\xi(y)| \to 0. \tag{9.1}
\]

Backward synchronization is defined in the same way, but for the IFS \( \{f_i^{-1}\}_{i=0}^{N-1} \). We say that it is synchronizing if it is both backward and forward synchronizing. The IFS \( \{f_i\}_{i=0}^{N-1} \) is (forward) proximal if for every \( x, y \in S^1 \) there exists at least one sequence \( \xi \in \Sigma_N \) such that (9.1) holds, backward proximality is defined analogously. By [40, Theorem E], forward proximality of the IFS implies that every nondegenerate Bernoulli measure satisfies forward synchronization. Similarly, backward proximality implies backward synchronization.11

Lemma 9.1. For every Bernoulli measure \( b \) satisfying synchronization there are ergodic measures \( \mu^b_+ \) with positive exponent and \( \mu^b_- \) with negative exponent such that \( b = \pi^* \mu^b_+ \).

Moreover, for every ergodic measure \( \mu \) with \( b = \pi^* \mu \) we have that \( \mu = \mu^b_+ \) or \( \mu = \mu^b_- \).

Proof. Consider the set

\[
B \overset{\text{def}}{=} \{(x, y, \xi) \in S^1 \times S^1 \times \Sigma^+_N : \limsup_{n \to \infty} |f^n_\xi(x) - f^n_\xi(y)| > 0\}
\]

and write

\[
B_{(x, y)} \overset{\text{def}}{=} \{(x, y)\} \times \Sigma^+_N \cap B.
\]

Note that forward synchronization means that for every \( (x, y) \in S^1 \times S^1 \) it holds that \( b(B_{(x, y)}) = 0 \).

Given \( \xi \in \Sigma^+_N \), divide \( S^1 \) into equivalence classes by the relation

\[
x \sim_\xi y \iff \lim_{n \to \infty} |f^n_\xi(x) - f^n_\xi(y)| = 0.
\]

One can easily check that it is indeed an equivalence relation. As the fiber maps are homeomorphisms, those equivalence classes are simply connected, i.e. intervals or points.

11 In fact, [40, Theorem E] (stated for groups of circle homeomorphisms) shows even that we have exponential synchronization, that is, for a given Bernoulli measure convergence in (9.1) is exponential. However, we will not make use of this fact.
Note that in principle for $\xi$ there may exist uncountably many classes. However, there can exist only countably many classes which are nontrivial intervals. Let us denote these classes by $C_i(\xi)$ with $i \in I(\xi)$.

Let us denote by $\text{Leb}$ the Lebesgue measure.

**Claim 9.2.** For $b$-almost every $\xi$ we have $I(\xi) = \{1\}$ and $\text{Leb}(C_1(\xi)) = 1$.

**Proof.** Note that if $(x, y, \xi) \notin B$, $x \neq y$, then there is an index $i = i(x, y, \xi)$ such that $x, y \in C_i(\xi)$. As the diagonal $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1; x = y$ has $\text{Leb} \times \text{Leb}$ measure zero, we have

$$\frac{\text{Leb} \times \text{Leb} \times b(B^c)}{\text{Leb}} = \int_{\Sigma_N^+} \left( \sum_{i \in I(\xi)} \text{Leb}(C_i(\xi)) \right)^2 \, db(\xi). \quad (9.2)$$

By the comments above, synchronization, and Fubini’s Theorem we have that $\frac{\text{Leb} \times \text{Leb} \times b(B^c)}{\text{Leb}} = 1$, hence the integrand in (9.2) is 1 almost everywhere. As $\sum \text{Leb}(C_i(\xi))^2 = 1$ can happen if, and only if, $I(\xi) = \{1\}$ and $\text{Leb}(C_1(\xi)) = 1$. Therefore, for $b$-almost every $\xi$ the index set $I(\xi)$ consists of exactly one element and there is exactly one class with full Lebesgue measure. This class is the whole circle except one point. \qed

Using Claim 9.2, for $b$-almost every $\xi$, let $\{z^+(\xi)\} \overset{\text{def}}{=} \mathbb{S}^1 \setminus C_1(\xi)$ be this missing point and denote $C^+(\xi) \overset{\text{def}}{=} C_1(\xi)$.

Clearly, the synchronization in the set $C^+(\xi)$ implies that the disintegration of any positive exponent invariant ergodic measure projecting to $b$ is supported on $z^+(\xi)$ for almost all $\xi$. We will denote by $\mu^b_+$ this measure.

The same arguments applied to the IFS $\{f_i^{-1}\}$ imply that for $b$-almost every $\xi$ there is exactly one class with full Lebesgue measure, called $C^-(\xi)$, where the points synchronized as $n \to -\infty$. Analogously, $\mathbb{S}^1 = C^-(\xi) \cup \{z^-(\xi)\}$. Arguing as above we prove that $\mu^b_-$ is the only ergodic measure with negative exponent that projects to $b$ and that its disintegration is supported on the points $z^-(\xi)$.

It remains to exclude the case that there is a third ergodic measure $\mu$ (necessarily with zero exponent) projecting to $b$. Note that we have obtained a set $\Sigma^\pm$ of full measure $b$ for which the points $z^+(\xi)$ and $z^-(\xi)$ are well defined and different. Consider now the disintegration $\mu_\xi$ of $\mu$ for $\xi \in \Sigma^\pm$. The forward synchronization with the point $z^-(\xi)$ implies that the measure $\mu$ has negative exponent in those fibers. This leads to a contradiction. \qed

**9.2. End of the proof of Theorem 2.** We can now conclude the proof of Theorem 2. By Lemma 9.1 applied to the $(1/N, \ldots, 1/N)$-Bernoulli measure $b$ it follows that there is exactly a pair of ergodic measures of maximal entropy $\log N$ with positive and negative exponent $\mu_+ = \mu^b_+$ and $\mu_- = \mu^b_-$, respectively.

Assume that the second conclusion in the theorem is not true, that is, that there exists $\alpha \neq \alpha_\pm = \chi(\mu_\pm)$ such that $h_{\text{top}}(\mathcal{L}(\alpha)) = \log N$. Let us assume that $\alpha \in [0, \alpha_+]$, the proof of the other case is analogous. By property (E1) in Sect. 5.3 we have $\log N = \mathcal{E}_N(\alpha')$ with $N = M_{\text{erg}, >0}$ for all $\alpha'$ between $\alpha$ and $\alpha_+$. Hence, by (E4) the function $\mathcal{P}_N$ is not differentiable at 0. Recall that a measure of maximal entropy is an ergodic equilibrium states for $q = 0$. By property (P10) in Sect. 5.2, there exist two ergodic measures of maximal entropy (with respect to $N$) with exponents given by the (different) left and
right derivatives $D_{L/R} \mathcal{P}_N(0)$, these derivatives being nonnegative by the choice of $N$. Hence, there would exist two ergodic measures of maximal entropy with two distinct nonnegative Lyapunov exponents, contradicting Lemma 9.1. □

9.3. Proof of Corollary 3. By Lemma 9.1, applied to the (Bernoulli) measure of maximal entropy we have that there are exactly two ergodic measures of maximal entropy. Arguing by contradiction, suppose that there is another invariant measure $\mu$ of maximal entropy which is the weak* and in entropy limit of a sequence of ergodic measures, which must be nonergodic. Then almost every measure in its (nontrivial) ergodic decomposition has maximal entropy. Hence this measure is a (nontrivial) linear combination of $\mu_+$ and $\mu_-\mu$ and, in particular, $\alpha = \chi(\mu) \in (\alpha_-, \alpha_+)$. Without weakening of assumptions suppose that $\alpha \geq 0$. Recall that the function $E_{>0}$ is continuous and has a unique global maximum at $\alpha_+$. Hence for $\delta$ small $E_{>0}(\alpha') < \log N - \delta$ for all $\alpha'$ in a small neighborhood of $\alpha$. If there would exist such a sequence of ergodic measures weak* (and hence in Lyapunov exponent) and in entropy converging to $\mu$ then eventually the Lyapunov exponents of the measures would be arbitrarily close to $\alpha$ and their entropies arbitrarily close to $\log N$. This provides a contradiction with the above inequality. □

10. Shapes of Pressure and Lyapunov Spectrum: Proof of Theorem 4

Recall the properties of pressure, Legendre–Fenchel transform, and convex functions given in Sect. 5. For convenience, the items are proved in the following order: (a), (f), (g), (h), (i), (b), (c), (d), (e), (j), (k).

Property (a): Convexity follows from basic properties of pressure. The (one-sided) derivative(s) of the pressure function is equal to the Lyapunov exponent of a corresponding equilibrium state by the definition of Legendre–Fenchel transform. For the pressure $\mathcal{P}_{>0}$ all the equilibrium states have nonnegative Lyapunov exponent, for the pressure $\mathcal{P}_{<0}$ all the equilibrium states have nonpositive Lyapunov exponent. Hence property (a) follows from property (P9) in Sect. 5.2.

Properties (f), (g), and (h) are formulated in Theorem 1.

We need the following lemma.

Lemma 10.1. There is $c > 0$ such that $\frac{E(\alpha) - E(0)}{\alpha} \leq cE(0)$.

Proof. An immediate consequence of Lemma 4.2 and property (h) is that there exists $c > 0$ such that

$$E(0) \geq \frac{E(\alpha)}{1 + c|\alpha|}$$

for all $\alpha \neq 0$. Hence,

$$\frac{E(\alpha) - E(0)}{|\alpha|} \leq cE(0).$$

Taking the limit $\alpha \to 0$, we hence get

$$\max\{D_R E(0), -D_L E(0)\} \leq cE(0),$$

which proves the finiteness of the derivatives $D_L E(0)$, $D_R E(0)$. The other inequality follows from convexity of $E_{<0}$ and $E_{>0}$ and property (f). □
Property (i) follows now from Lemma 10.1 and Property (a).

Property (b) follows from the fact that the sets $M_{\text{erg}, >0}$ and $M_{\text{erg}, <0}$ contain measures with arbitrarily small Lyapunov exponent (Corollary 4.3) and hence the limit derivative of both $P_{>0}$ (as $q \to -\infty$) and $P_{<0}$ (as $q \to \infty$) is zero. The fact that those are indeed plateaus, not asymptotic behaviour, follows from property (i) proved below. Indeed, by the definition of Legendre–Fenchel transform, $D_+ = D_R E(0)$ and $D_- = D_L E(0)$.

Property (c) follows from Theorem 1 item (c). Indeed, by the definition of Legendre–Fenchel transform, $h_+ = \lim_{\alpha \to 0} E(\alpha)$ and $h_- = \lim_{\alpha \nearrow 0} E(\alpha)$.

Property (d) follows from Theorem 1 and property (a). Indeed, a concave function with a maximum in the interior of the domain is nonincreasing to the right of the maximum and nondecreasing to the left of the maximum.

Property (e) follows immediately from Theorem 1, because by the basic properties of entropy $P_{>0}(0)$ is the supremum of entropies of ergodic measures with positive Lyapunov exponents (and similarly $P_{<0}(0)$ - negative Lyapunov exponents) and those classes of measures both contain a measure of maximal entropy.

Property (j) was proved in the course of the proof of Property (i). Indeed, by contradiction, assume that $E(0) = 0$. Then Eq. (10.1) would imply that $D_L E(0) = D_R E(0) = 0$. However, this would imply that $E$ attains its maximum at zero and hence $E(0) = \log N$, a contradiction. Together with Theorem 7.6, we have $h_{\text{top}}(\mathcal{L}(0)) > 0$.

Property (k): assume now that we are under assumptions of Theorem 2 and that hence there are exactly two maxima of the entropy spectrum $E(\alpha)$, achieved at points $\alpha_- < 0$ and $\alpha_+ > 0$. As the concave function with a unique maximum in the interior of the domain has negative derivative (or one-sided derivatives in case that the derivative is not defined) to the right of the maximum and positive derivative to the left of the maximum, the required changes in items (d) and (i) follow immediately. Property (k) follows from (local) uniqueness of the maximum by (P11) is Sect. 5.2.

11. One-Step 2 × 2 Matrix Cocycles: Proof of Theorem 5

The goal of this section is to prove Theorem 5. For that, using the notation in Sect. 1.2, for every $A \in \text{SL}(2, \mathbb{R})^N$ we study the one-sided cocycle $A: \Sigma_N^+ \to \text{SL}(2, \mathbb{R})$ and consider the associated one-sided step skew-product $F_A : \Sigma_N^+ \times \mathbb{P}^1 \to \Sigma_N^+ \times \mathbb{P}^1$ defined in (1.3). Recall that the Lyapunov exponents of the cocycle $A$ at $\xi^+ \in \Sigma_N^+$ are the limits

$$\lambda_1(A, \xi^+) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \log \|A^n(\xi^+)\|,$$

$$\lambda_2(A, \xi^+) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \log \|A^n(\xi^+)\|^{-1},$$

whenever they exist. Otherwise, we denote by $\underline{\lambda}_1$ and $\bar{\lambda}_1$ and $\underline{\lambda}_2$ and $\bar{\lambda}_2$ the lower and upper limits, respectively. We analyze the spectrum of Lyapunov exponents of one-step cocycles. Note that for every $\xi^+$ we have $\lambda_2(A, \xi^+) = -\lambda_1(A, \xi^+)$ whenever those exponents are well defined. The Lyapunov exponent $\underline{\lambda}_1$ (and hence $\lambda_2$) of $A$ are intimately related to the (one-sided) Lyapunov exponent $\chi^+$ of the step skew-product $F_A$ defined as in (1.2) taking only the limit $n \to \infty$, as we will see below. Given $\alpha \geq 0$, similarly to the level sets in (2.4) we will analyze the level sets of Lyapunov exponents.
\[ \mathcal{L}^*_A(\alpha) \overset{\text{def}}{=} \{ \xi^+ \in \Sigma^+_N : \lambda_1(A, \xi^+) = \alpha \}. \]

The following is our main translation step from skew-products to cocycles.

**Theorem 11.1.** For every \( A \in \text{SL}(2, \mathbb{R})^N \) we have the following:

1. For every \( \alpha > 0 \) it holds

\[ \{ \xi^+ \in \Sigma^+_N : \lambda_1(A, \xi^+) = \alpha \} = \{ \xi^+ \in \Sigma^+_N : \chi^+(\xi^+, v) = -2\alpha \text{ for some } v \in \mathbb{P}^1 \} \]
\[ = \{ \xi^+ \in \Sigma^+_N : \chi^+(\xi^+, v) = 2\alpha \text{ for some } v \in \mathbb{P}^1 \}. \]

2. For \( \alpha = 0 \) it holds

\[ \{ \xi^+ \in \Sigma^+_N : \lambda_1(A, \xi^+) = 0 \} \subset \{ \xi^+ \in \Sigma^+_N : \chi^+(\xi^+, v) = 0 \text{ for some } v \in \mathbb{P}^1 \} \]
\[ \subset \{ \xi^+ \in \Sigma^+_N : \lambda_1(A, \xi^+) = 0 \} \]

and those three sets have the same topological entropy.

We will prove this theorem in Sect. 11.5 and conclude the proof of Theorem 5 in Sect. 11.6. In Sects. 11.1–11.4 we collect preparatory results.

### 11.1. Preliminary steps.

We first collect a series of results (see also Caveat 11.3) relating the exponents of cocycles and skew-products.

In what follows, we use the standard metric generated by \( \theta \mapsto (\cos(\theta \pi), \sin(\theta \pi)) \) mapping \( \mathbb{R}/\mathbb{Z} \) to \( \mathbb{P}^1 \), we denote by \( \text{Leb} \) the corresponding image of the Lebesgue measure. Let us start from elementary linear algebra:

**Lemma 11.2.** For every \( A \in \text{SL}(2, \mathbb{R}) \) we have \( \| A^{-1} \| = \| A \| \) and

(i) \( \min_v |f'_A(v)| = \| A \|^{-2} \) and \( \max_v |f'_A(v)| = \| A \|^2 \),

(ii) for any \( \delta > 0 \) the points \( v \in \mathbb{P}^1 \) which satisfy

\[ |f'_A(v)| \leq \frac{(1 + \delta^2)\| A \|^2}{1 + \delta^2 \| A \|^4} \]

form an interval of length \( 1 - \arctan \delta \).

Let us write

\[ f_i \overset{\text{def}}{=} f_{A_i} \text{ for every } i = 0, \ldots, N - 1. \]

Let

\[ M \overset{\text{def}}{=} \max_{i=0,\ldots,N-1} \| A_i \|. \]  \hspace{1cm} (11.1)

**Caveat 11.3.** Note that, given \( \xi^+ \) and \( \ell \), unless \( f_{[\xi_0, \xi_{\ell-1}]} \) is an isometry, the function \( |f'_{[\xi_0, \xi_{\ell-1}]}| \) attains its unique maximum and minimum at some \( v_+(\xi^+, \ell) \in \mathbb{P}^1 \) and \( v_-(\xi^+, \ell) \in \mathbb{P}^1 \), respectively, and is monotone between them. This hypothesis is explicitly stated in Lemma 11.4 and is a consequence of the hypotheses of the other lemmas in Sects. 11.2–11.4 (excluding the first part of Proposition 11.5).
Lemma 11.4. For every $\xi^+$ and $\ell \geq 1$ so that $f_{[\xi_0, \ldots, \xi_{\ell-1}]}$ and $f_{[\xi_0, \ldots, \xi_\ell]}$ are not isometries we have

$$|v_+(\xi^+, \ell) - v_+(\xi^+, \ell + 1)| \leq \arctan \left( \frac{1 - M^{-4}}{M^{-4}A^{\ell+1}(\xi^+)^4 - 1} \right)^{1/2}.$$  

Proof. Note that the hypothesis implies that $v_+(\xi^+, \ell)$ and $v_+(\xi^+, \ell + 1)$ are well defined. By Lemma 11.2(i)

$$M^2 |f'_{[\xi_0, \ldots, \xi_\ell]}(v_+(\xi^+, \ell))| \geq |f'_{[\xi_0, \ldots, \xi_{\ell-1}]}(v_+(\xi^+, \ell))| = \|A^\ell(\xi^+)\|^2 \geq M^{-2}\|A^{\ell+1}(\xi^+)\|^2.$$  

Applying Lemma 11.2(ii) to $A = A^{\ell+1}(\xi^+)$ and $v = v_+(\xi^+, \ell)$, we determine $\delta \overset{\text{def}}{=} |v_+(\xi^+, \ell) - v_+(\xi^+, \ell + 1)|$ by solving the inequality in item (ii) for $\delta$ and obtain

$$\delta^2 \leq \frac{1 - |f'_A(v)|/\|A\|^2}{|f'_A(v)|\|A\|^2 - 1} \leq \frac{1 - M^{-4}}{M^{-4}\|A\|^4 - 1},$$

where we applied the above estimates. Using Lemma 11.2, the definitions of $v_+(\xi^+, \ell)$ and $v_+(\xi^+, \ell + 1)$ and the choice of $\delta$ imply the assertion. □

11.2. Regular one-sided sequences. Using the notation in Sect. 11.1, define

$$v_0(\xi^+) \overset{\text{def}}{=} \lim_{\ell \to \infty} v_+ (\xi^+, \ell)$$

whenever this limit exists. We derive the following result about one-sided sequences $\xi^+ \in \Sigma^+_N$ which are regular for the cocycle, that is, for which $\lambda_1(A, \xi^+)$ is well-defined. Throughout this section, denote

$$a_\ell \overset{\text{def}}{=} \frac{1}{\ell} \log \|A^\ell(\xi^+)\|. \quad (11.2)$$

Proposition 11.5. Assume $\xi^+$ satisfies $\lambda_1(A, \xi^+) = \alpha$.

- If $\alpha = 0$, then $\chi^+(\xi^+, v) = 0$ for all $v \in \mathbb{P}^1$.
- If $\alpha > 0$, then $v_0(\xi^+)$ is well-defined and we have
  $$\chi^+(\xi^+, v_0(\xi^+)) = 2\alpha \quad \text{and} \quad \chi^+(\xi^+, v) = -2\alpha \text{ for all } v \neq v_0(\xi^+).$$

Proof. The case $\alpha = 0$ follows immediately from Lemma 11.2(i). So let us assume that $\alpha > 0$. Let

$$\mathcal{L}(-2\alpha, \xi^+) \overset{\text{def}}{=} \{v \in \mathbb{P}^1 : \chi^+(\xi^+, v) = -2\alpha\}.$$

Note that, by definition of $\lambda_1(A, \xi^+)$, we have $\alpha = \lim_{\ell \to \infty} a_\ell$.

Let $(\varepsilon_\ell)_\ell$ be a sequence of positive numbers $\varepsilon_\ell < 2a_\ell$ converging to 0 such that

$$\sum_{\ell \geq 1} \arctan e^{-\varepsilon_\ell} < \infty.$$  

By Lemma 11.2(i), for all $v \in \mathbb{P}^1$ for every $\ell \geq 1$ we have that

$$e^{-2\ell a_\ell} \leq |f'_{[\xi_0, \ldots, \xi_{\ell-1}]}(v)|. \quad (11.3)$$
By Lemma 11.2 (ii), for any \( \ell \) there is an interval \( I_\ell \subset \mathbb{P}^1 \) of length 1\( - \arctan e^{-\ell \varepsilon} \) such that for every point \( v \in I_\ell \)

\[
|f'_{[\xi_0...\xi_{\ell-1}]}(v)| \leq \frac{(1 + e^{-2\ell \varepsilon})e^{2\alpha \ell}}{1 + e^{2\ell(2\alpha - \varepsilon)}}
\]

(note that the right hand side is approximately \( e^{-2\ell(\varepsilon_0 - \varepsilon)} \)). By the Borel-Cantelli Lemma, almost every \( v \in \mathbb{P}^1 \) belongs to infinitely many intervals \( I_\ell \). Together with (11.3) we hence have \( \chi^+(\xi^+, v) = -2\alpha \). Thus, \( \mathcal{L}(-2\alpha, \xi^+) \) has full Lebesgue measure. What remains to prove is that \( \mathcal{L}(-2\alpha, \xi^+) \) is the whole set \( \mathbb{P}^1 \) minus one point which will turn out to be the point \( v_0(\xi^+) \).

Let \( v_1, v_2 \in \mathcal{L}(-2\alpha, \xi^+) \). Given \( \delta > 0 \), let \( L \geq 1 \) be such that for all \( \ell > L \) we have that \( |\alpha_\ell - \alpha| < \delta \) and for \( i = 1, 2 \) (for the second inequality again using (11.3))

\[
-2(\alpha + \delta) \leq -2\ell \alpha_\ell \leq \frac{1}{\ell} \log |f'_{[\xi_0...\xi_{\ell-1}]}(v_i)| \leq -2(\alpha - \delta).
\]

The points \( v_1, v_2 \) divide \( \mathbb{P}^1 \) into two intervals. Note that by definition

\[
|f'_{[\xi_0...\xi_{\ell-1}]}(v_i(\xi^+, \ell))| = \|A^\ell(\xi^+)\| \geq 1
\]

and hence \( v_i(\xi^+, \ell) \neq v_i \), \( i = 1, 2 \). Thus, let us denote by \( J_1(\ell) = J_1(\ell, v_1, v_2) \) the interval which contains \( v_i(\xi^+, \ell) \) and by \( J_2(\ell) = J_2(\ell, v_1, v_2) \) the other one. Note that by monotonicity of the derivative, we have

\[
|f'_{[\xi_0...\xi_{\ell-1}]}(v)| \leq e^{-2\ell(\alpha - \delta)} \quad \text{for every} \quad v \in J_2(\ell).
\]

**Claim 11.6.** For every \( \ell \) large enough it holds \( J_1(\ell) = J_1(\ell + 1) \).

**Proof.** By Lemma 11.2 (i) and the second estimate in (11.4) we have

\[
|f'_{[\xi_0...\xi_{\ell-1}]}(v_i(\xi^+, \ell))| \geq |f'_{[\xi_0...\xi_{\ell-1}]}(v_i(\xi^+, \ell))| \cdot M^{-2} \geq e^{2\ell(\alpha - \delta)} \cdot M^{-2} \geq 1.
\]

By (11.5) the interval \( J_1(\ell) \) must contain \( v_i(\xi^+, \ell + 1) \). \( \Box \)

**Claim 11.7.** \( v_0(\xi^+) \) is well-defined.

**Proof.** By contradiction, otherwise \( (v_i(\xi^+, \ell))_\ell \) would have at least two accumulation points which would divide \( \mathbb{P}^1 \) into two connected components. Since by the above \( \mathcal{L}(-2\alpha, \xi^+) \) has full measure and hence is dense, there would exist points \( u_1, u_2 \in \mathcal{L}(-2\alpha, \xi^+) \), one in each of them. Hence, each of the correspondingly defined intervals \( J_1(\ell, u_1, u_2) \) and \( J_2(\ell, u_1, u_2) \) would contain one of the accumulation points and hence eventually the accumulating points \( v_i(\xi^+, \ell) \), for infinitely many times. This would contradict that \( J_1(\ell, u_1, u_2) = J_1(\ell + 1, u_1, u_2) \) for all large enough \( \ell \) as in Claim 11.6. \( \Box \)

**Claim 11.8.** For every \( v \neq v_0(\xi^+) \) we have \( -2\alpha \leq \chi^+(\xi^+, v) \leq -2(\alpha - \delta) \).

**Proof.** The first inequality follows from (11.3). For the second, by the above \( \mathcal{L}(-2\alpha, \xi^+) \) has full Lebesgue measure and hence is dense in \( \mathbb{P}^1 \). Hence, for any \( v \in \mathbb{P}^1 \setminus \{v_0(\xi^+)\} \) we can find \( u_1, u_2 \in \mathcal{L}(-2\alpha, \xi^+) \) such that the points \( v \) and \( v_0(\xi^+) \) are in different components of \( \mathcal{L}(-2\alpha, \xi^+) \setminus \{u_1, u_2\} \). Thus, \( v_0(\xi^+) \in J_1(\ell, u_1, u_2) \) and hence \( v \in J_2(\ell, u_1, u_2) \) for \( \ell \) large enough. Now (11.5) implies the claim. \( \Box \)
By Claim 11.8, as $\delta > 0$ was arbitrary, we conclude that for every $v \neq v_0(\xi^+)$ we have $\chi^+(\xi^+, v) = -2\alpha$.

The only thing that remains to prove is that the Lyapunov exponent at $v_0(\xi^+)$ is $2\alpha$. For that we invoke the following lemma whose proof we postpone.

In what follows we use the Banach–Landau notation. 12

**Lemma 11.9.** For every $\xi^+$, if $(\ell_i)_i$ is a sequence of positive integers such that

a) the limit $\alpha \overset{\text{def}}{=} \lim_{i \to \infty} \frac{1}{\ell_i} \log \| A^{\ell_i}(\xi^+) \| > 0$ exists,

b) the limit $v_0(\xi^+) \overset{\text{def}}{=} \lim_{\ell \to \infty} v_+(\xi^+, \ell_i)$ exists, and

c) for every $\delta > 0$ and every $i$ we have $|v_+(\xi^+, \ell_i) - v_0(\xi^+)| = O(e^{-2\ell_i(\alpha-\delta)})$.

Then

$$\lim_{i \to \infty} \frac{1}{\ell_i} \log |f^i_{[\xi_0, \xi_{\ell_i-1}]}(v_0(\xi^+))| = 2\alpha.$$  

Let us show that the hypotheses of Lemma 11.9 are verified.

Since we consider the case $\lambda_1(A, \xi^+) = \alpha > 0$, we have hypothesis a) in Lemma 11.9. By Claim 11.7, $v_0(\xi^+) = \lim_{\ell \to \infty} v_+(\xi^+, \ell)$ is well defined and hence we have hypothesis b) in Lemma 11.9. What remains to verify is hypothesis c) in this lemma. Observe that $\alpha > 0$ implies that for every $\ell$ large enough the map $f_{[\xi_0, \xi_{\ell}]}$ is not an isometry. Hence by Lemma 11.4 with (11.2) we get

$$|v_+(\xi^+, \ell) - v_+(\xi^+, \ell + 1)| \leq \arctan \left( \frac{1 - M^{-4}}{M^{-4} e^{4(\ell+1) a_{\ell+1}} - 1} \right)^{1/2}$$

$$\leq \left( \frac{1 - M^{-4}}{M^{-4} e^{4(\ell+1) a_{\ell+1}} - 1} \right)^{1/2},$$

where the latter holds for $\ell$ sufficiently large. Hence, given $\delta > 0$ for every $\ell$ large we get (after some simple approximation steps)

$$|v_+(\xi^+, \ell) - v_+(\xi^+, \ell + 1)| \leq e^{-2(\ell+1)(\alpha_{\ell+1} - \delta)} \leq e^{-2(\ell+1)(\alpha - 2\delta)}, \quad (11.6)$$

where we also used that $\alpha = \lim_{\ell} a_\ell > 0$. Thus, for $\ell$ large enough we get

$$|v_0(\xi^+) - v_+(\xi^+, \ell)| \leq \sum_{k \geq \ell} |v_+(\xi^+, k) - v_+(\xi^+, k + 1)| \leq e^{-2(\ell+1)(\alpha - 3\delta)}.$$

This shows that for any $\delta > 0$ we have $|v_0(\xi^+) - v_+(\xi^+, \ell)| = O(e^{-2\ell(\alpha - \delta)})$. Hence we have hypothesis c) in Lemma 11.9.

We can now apply Lemma 11.9 to get $\chi^+(\xi^+, v_0(\xi^+)) = 2\alpha$.

What remains is to give the postponed proof.

**Proof of Lemma 11.9.** Clearly, with Lemma 11.2 (i), we get

$$\lim_{i \to \infty} \frac{1}{\ell_i} \log |f^i_{[\xi_0, \xi_{\ell_i-1}]}(v_0)| \leq 2\alpha.$$  

12 Recall that for two real valued functions $f, g : \mathbb{R} \to \mathbb{R}$ we have $f = O(g)$ if and only if there exists a positive number $C$ and $x_0$ such that $|f(x)| \leq C|g(x)|$ for every $x \geq x_0$. 
What remains to show is the other inequality. Observe that $\alpha > 0$ implies that for every $\ell_i$ large enough the map $f_{[\xi_0, \ldots, \xi_{\ell_i}]}$ is not an isometry. Note that by the definition of $v_+(\xi^+, \ell)$ and by Lemma 11.2 (i) we have

$$|f'_{[\xi_0, \ldots, \xi_{\ell_i-1}]}(v_+(\xi^+, \ell))| = \max_v |f'_{[\xi_0, \ldots, \xi_{\ell_i-1}]}(v)| = ||A^\ell(\xi^+)||^2.$$  

Hence, by Lemma 11.2 (ii) applied to $A = A^\ell(\xi^+)$, for every $v$ with $|v - v_+(\xi^+, \ell)| \leq \frac{1}{2} \arctan \delta'$ we have

$$|f'_{[\xi_0, \ldots, \xi_{\ell_i-1}]}(v)| \geq \frac{(1 + (\delta')^2)||A^\ell(\xi^+)||^2}{1 + (\delta')^2||A^\ell(\xi^+)||^4}.$$  

(note that the interval in Lemma 11.2 is the complement of an concentric interval centered at $v_+(\xi^+, \ell)$). Applying the above for $v = v_0$ and $\delta' = \delta_i = 2 \tan \delta_i$, where $\delta_i \equiv |v_+(\xi^+, \ell_i) - v_0|$, we have

$$|f'_{[\xi_0, \ldots, \xi_{\ell_i-1}]}(v_0)| \geq \frac{(1 + (\delta_i')^2)||A^\ell_i(\xi^+)||^2}{1 + (\delta_i')^2||A^\ell_i(\xi^+)||^4}.$$  

By hypothesis, for any $\delta > 0$ we have $\delta_i = O(e^{-2\ell_i(\alpha - \delta)})$ and hence

$$(\delta_i')^2 = O(\delta_i^2) = O(\delta_i^2) = O(e^{-4\ell_i(\alpha - \delta)}).$$

Recalling that $\alpha = \lim_i \frac{1}{\ell_i} \log ||A^{\ell_i}(\xi^+)|| > 0$, we conclude

$$\lim_{i \to \infty} \frac{1}{\ell_i} \log |f'_{[\xi_0, \ldots, \xi_{\ell_i-1}]}(v_0)| \geq \lim_{i \to \infty} \frac{1}{\ell_i} \log \left( \frac{(1 + (\delta_i')^2)||A^{\ell_i}(\xi^+)||^2}{1 + (\delta_i')^2||A^{\ell_i}(\xi^+)||^4} \right)$$

$$= -\lim_{i \to \infty} \frac{1}{\ell_i} \log ((\delta_i')^2||A^{\ell_i}(\xi^+)||^2)$$

$$\geq 4(\alpha - \delta) - 2\alpha = 2\alpha - 4\delta.$$

As $\delta$ was arbitrary, this shows the inequality $\geq$ and hence proves the lemma. $\square$

This finishes the proof of the proposition. $\square$

11.3. Nonregular one-sided sequences. We now study one-sided sequences $\xi^+ \in \Sigma_N^+$ which are nonregular for the cocycle, that is, for which $\lambda_1(A, \xi^+) < \lambda_1(A, \xi^+)$.  

**Lemma 11.10.** Assume $\xi^+$ satisfies $\lambda_1(A, \xi^+) = 0$. Then there exists a sequence $(n_i)_i$ such that for every point $v \in \mathbb{P}^1$ it holds

$$\lim_{i \to \infty} \frac{1}{n_i} \log |f'_{[\xi_0, \ldots, \xi_{n_i-1}]}(v)| = 0.$$  

**Proof.** There exists a subsequence $(n_i)_i$ so that by Lemma 11.2 (i) and with the notation in (11.2) we have $\lim_i a_{n_i} = 0$ and

$$-2a_{n_i} \leq \frac{1}{n_i} \log |f'_{[\xi_0, \ldots, \xi_{n_i-1}]}(v)| \leq 2a_{n_i} \quad \text{for every} \quad v \in \mathbb{P}^1.$$  

Since $\lambda_1(A, \xi^+) = 0$ this immediately implies the lemma. $\square$
Lemma 11.11. Assume $\xi^+$ satisfies $\lambda_1(A, \xi^+) = \alpha_1$, $\overline{\lambda}_1(A, \xi^+) = \alpha_2$ for some $0 < \alpha_1 < \alpha_2$. Then for every $\alpha \in [\alpha_1, \alpha_2]$ there exists a sequence $(m_i)_i$ such that

$$\lim_{i \to \infty} \frac{1}{m_i} \log \|A^{m_i}(\xi^+)\| = \alpha.$$ 

Moreover, for any such a sequence $(m_i)_i$ there are two cases:

(a) either the limit $v_0 \overset{\text{def}}{=} \lim_i v^+(\xi^+, m_i)$ exists and

$$\lim_{i \to \infty} \frac{1}{m_i} \log |f'_{[\xi_0 \ldots \xi_{m-1}]}(v)| = -2\alpha \quad \text{for all} \quad v \neq v_0, \quad (11.7)$$

and for all $\alpha \in (\alpha_1, \alpha_0)$, where

$$\alpha_0 \overset{\text{def}}{=} \alpha_1 \frac{\log M + \alpha_2}{\log M + \alpha_1},$$

there exists a subsequence $(n_i)_i$ of $(m_i)_i$ such that

$$\lim_{i \to \infty} \frac{1}{n_i} \log |f'_{[\xi_0 \ldots \xi_{n-1}]}(v_0)| = 2\alpha.$$ 

(b) or for all $v \in \mathbb{P}^1$ there exists a subsequence of $(m_i)_i$ for which $(11.7)$ holds.

Proof. Let $\alpha \in [\alpha_1, \alpha_2]$. We note the following simple fact.

Claim 11.12. Consider $\beta$ and $\gamma$ with $\beta > \gamma$.

- If $a_m \geq \beta$ and $a_{m+k} < \gamma$, then $m + k > m \frac{\log M + \beta}{\log M + \gamma}$.
- If $a_m \leq \gamma$ and $a_{m+k} > \beta$, then $m + k > m \frac{\log M - \gamma}{\log M - \beta}$.

Remark 11.13. Claim 11.12 implies that there exists $(m_i)_i$ so that $\lim_i a_{m_i} = \alpha$. Indeed, by hypothesis this holds if $\alpha = \alpha_2$. Otherwise, if $\alpha < \alpha_2$ and $\delta$ small, it is enough to observe that if $a_m > \alpha_2 - \delta \geq \alpha + \delta$ then $a_{m+k} < \alpha - \delta$ only if

$$m + k > m \frac{\log M + (\alpha + \delta)}{\log M + (\alpha - \delta)},$$

and hence $k \geq 2$ when $m$ is large.

Arguing as in the proof of Proposition 11.5, we get that the set

$$\mathcal{L}'(-2\alpha, \xi^+, (m_i)_i) \overset{\text{def}}{=} \left\{ v \in \mathbb{P}^1 : \lim_{i \to \infty} \frac{1}{m_i} \log |f'_{[\xi_0 \ldots \xi_{m_i-1}]}(v)| = -2\alpha \right\}$$

has full Lebesgue measure. Now choose some $v_1, v_2 \in \mathcal{L}'(-2\alpha, \xi^+, (m_i)_i)$ and as in the proof of Proposition 11.5 define the intervals $J_1(m_i) = J_1(m_i, v_1, v_2)$ and $J_2(m_i, v_1, v_2)$. Note that, as we consider a sequence for which in general $m_{i+1} \neq m_i + 1$, we cannot proceed directly to get $J_1(m_i) = J_1(m_i+1)$ as in Claim 11.6, however we can argue as follows.

Either we do have $J_1(m_i) = J_1(m_i+1)$ for all $i \geq 1$ large enough, in which case we continue exactly as in the proof of Proposition 11.5 and obtain the existence of the
limit $v_0 \overset{\text{def}}{=} \lim_i v_+(\xi^+, m_i)$ and that for every $v \neq v_0$ and for this very sequence the assertion (11.7) holds true. This proves the first part of the claim in Case (a).

Or we have $J_1(m_i) = J_1(m_{i+j})$ for infinitely many $j \geq 1$ and $J_2(m_i) = J_1(m_{i+\ell})$ for infinitely many $\ell$. Hence, taking any $v \in \mathbb{P}^1$, there is some subsequence $(m_{i_k})_k$ such that $v \in J_1(m_{i_k})$ and hence for this subsequence the assertion (11.7) holds true, proving Case (b).

It remains to prove the remaining part of Case (a) where $\alpha \in (\alpha_1, \alpha_0)$. For that we will apply Lemma 11.9. Note that hypotheses a) and b) are satisfied by the hypotheses of the Case (a) we consider. It remains to check hypothesis c) of that lemma.

**Claim 11.14.** There exists a subsequence $(n_i)_i$ of $(m_i)_i$ such that for any sufficiently small $\delta > 0$ and for every $i$ we have

$$|v_+(\xi^+, n_i) - v_0| = O(e^{-2n_i(\alpha - \delta)}).$$

With this claim at hand all hypotheses in Lemma 11.9 are satisfied and hence we have

$$\lim_{i \to \infty} \frac{1}{n_i} \log |f_{[\xi_0, .., \xi_{n_i-1}]}(v_0)| = 2\alpha$$

which concludes the proof of the lemma.

**Proof of Claim 11.14.** First observe that the definition of $\alpha_0$ implies that $\alpha_1 < \alpha_0 \leq \alpha_2$. Consider $\delta > 0$ sufficiently small (we will specify this further) such that

$$\delta < \min \{(\alpha_2 - \alpha)/2, (\alpha - \alpha_1)/2\}.$$  

By hypothesis $\overline{\lambda}_1(A, \xi^+) = \alpha_2$, there is a sequence $(r_i)_i$ for which $a_{r_i} \geq \alpha_2 - \delta$ for all $i$. To define the subsequence $(n_i)_i$ of $(m_i)_i$ we consider an auxiliary strictly increasing sequence $(t_i)_i$ given by the positive integers such that $a_{t_i} \geq \alpha_2 - \delta$ for all $i \geq 1$. For every $i \geq 1$ let

$$n_i \overset{\text{def}}{=} \max \left\{ n \in \{m_j\}, n < t_i, a_n < \alpha + \delta \right\},$$

that is, $a_{n_i}$ is the last sequence element which was below $\alpha + \delta$ before approaching values close to $\alpha_2$. Let

$$L_i'(\delta) \overset{\text{def}}{=} n_i \frac{\log M + (\alpha_2 - \delta)}{\log M + (\alpha + \delta)} \cdot \frac{\log M - (\alpha + \delta)}{\log M - (\alpha_2 - \delta)},$$

$$L_i''(\delta) \overset{\text{def}}{=} n_i \frac{\log M + (\alpha - \delta)}{\log M + (\alpha + \delta)} \cdot \frac{\log M + (\alpha_2 - \delta)}{\log M + (\alpha_1 + \delta)} \cdot \frac{\log M - (\alpha + \delta)}{\log M - (\alpha_2 - \delta)} = \frac{\log M + (\alpha - \delta)}{\log M + (\alpha_1 + \delta)} \cdot L_i'(\delta).$$

Clearly $L_i''(\delta) > L_i'(\delta) > n_i$. Let

$$\ell_i'(\delta) \overset{\text{def}}{=} \lceil L_i'(\delta) \rceil \quad \text{and} \quad \ell_i''(\delta) \overset{\text{def}}{=} \lceil L_i''(\delta) \rceil.$$  

The above implies that there are an increasing function $\tau(\delta)$, $\tau(\delta) \to 1$ as $\delta \to 0$ such that (for sufficiently large $i$)

$$\ell_i''(\delta) > n_i \tau(\delta) \frac{\log M + \alpha_2}{\log M + \alpha_1} > n_i \tau(\delta) \frac{\log M - \alpha}{\log M - \alpha_2}.$$
where for the last inequality we use that $\alpha_2 > \alpha$. Multiplying both sides by $\alpha_1 - \delta$ and recalling the definition of $\alpha_0$, we get

$$\ell_i''(\delta)(\alpha_1 - \delta) > n_i \left( \tau(\delta)\alpha_0 - \tau(\delta) \frac{\log M + \alpha_2}{\log M + \alpha_1} \delta \right).$$

Specifying now $\delta$, it will be enough that for given $\alpha$ we have that $\delta > 0$ is sufficiently small such that

$$\tau(\delta)\alpha_0 - \tau(\delta) \frac{\log M + \alpha_2}{\log M + \alpha_1} \delta > \alpha.$$

Hence, for $i$ large we obtain

$$n_i(\alpha - \delta) < \ell_i''(\delta)(\alpha_1 - \delta). \quad (11.8)$$

In what follows, we take $\delta$ in this way and further-on omit the dependence on $\delta$.

Given $n_i$, let

$$n_i' \overset{\text{def}}{=} \min\{n : n > t_i, a_n < \alpha + \delta\}$$

be the smallest index $n > t_i$ at which $a_n$ drops below $\alpha + \delta$ again and let

$$n_i'' \overset{\text{def}}{=} \min\{n : n > n_i', a_n < \alpha_1 + \delta\}$$

be the first $n > n_i'$ at which $a_n$ approaches the other accumulated exponent $\alpha_1$. Compare Fig. 7. Note that by Claim 11.12 applied twice (to the pairs of times $(n_i, t_i)$ and $(t_i, n_i')$) we have $n_i' \geq L_i' \geq \ell_i'$. In particular, invoking Remark 11.13, for any $n = n_i, \ldots, n_i'$ and hence in particular for $n = \ell_i'$, we have

$$a_n \geq \alpha - \delta.$$

Consequently, for any $n \geq \ell_i' + 1$ we have

$$n a_n \geq \ell_i'(\alpha - \delta) - (n - \ell_i') \log M.$$
Putting this together and noting also that we can assume that \( a_n > \alpha_1 - \delta \) for all \( n \), by construction, we have

\[
\begin{align*}
  a_n &\geq \alpha - \delta & \text{for all } n \in I_1 \overset{\text{def}}{=} \{n_i + 1, \ldots, \ell'\} \\
  na_n &\geq \ell'_i(\alpha - \delta) - (n - \ell'_i) \log M & \text{for all } n \in I_2 \overset{\text{def}}{=} \{\ell'_i + 1, \ldots, \ell''_i\} \\
  a_n &\geq \alpha_1 - \delta & \text{for all } n \in I_3 \overset{\text{def}}{=} \{\ell''_i + 1, \ldots\}.
\end{align*}
\] (11.9)

Recall again that the limit \( v_0 \overset{\text{def}}{=} \lim_{i \to \infty} v_+(\xi^+, n_i) \) exists since we choose the terms \( n_i \) in the sequence \( (m_j)_j \) satisfying the hypothesis of Case (a). Thus, applying Lemma 11.4 to a telescoping sum we have (for \( n_i \) large enough)

\[
|v_+(\xi^+, n_i) - v_0| \leq \sum_{\ell=0}^{\infty} |v_+(\xi, n_i + \ell) - v_+(\xi, n_i + \ell + 1)| \\
\leq \sum_{\ell=0}^{\infty} \arctan \left( \frac{1 - M^{-4}}{M^{-4} \|A^{n_i+\ell+1}(\xi^+)\|^4 - 1} \right)^{1/2} \leq \sum_{n=n_i+1}^{\infty} e^{-2n(a_n-\delta)},
\]

where the last inequality follows after some simple approximation steps as in (11.6). To finish the proof of the claim, we now divide the latter sum into three subsums over the index sets \( I_1, I_2, \) and \( I_3 \) defined in (11.9) and estimate these sums.

- **First sum:**
  \[
  \sum_{n \in I_1} e^{-2n(a_n-\delta)} \leq \sum_{n \geq n_i+1} e^{-2n\alpha} \leq e^{-2n_i\alpha} \frac{e^{-2\alpha}}{1 - e^{-2\alpha}} < e^{-2n_i(\alpha-\delta)} \frac{e^{-2\alpha}}{1 - e^{-2\alpha}}.
  \]

- **Second sum:** with (11.9) we have
  \[
  \sum_{n \in I_2} e^{-2\ell'_i(\alpha-\delta)} \leq \sum_{n = \ell'_i+1}^{\ell''_i} e^{-2\ell'_i(\log M + (\alpha-\delta)) + 2n(\log M+\delta)}
  \leq e^{-2\ell'_i(\log M + (\alpha-\delta))} \frac{e^{2(\ell''_i+1)(\log M+\delta)} - e^{2(\ell'_i+1)(\log M+\delta)}}{e^{2(\log M+\delta)} - 1}
  \]

  (with (11.8))

  \[
  = e^{-2\ell''_i(\log M + (\alpha_1+\delta))} \frac{e^{2(\log M+\delta)} - e^{-2(\ell''_i-\ell'_i-1)(\log M+\delta)}}{e^{2(\log M+\delta)} - 1}
  \leq e^{-2\ell''_i \alpha_1} \frac{e^{2(\log M+\delta)}}{e^{2(\log M+\delta)} - 1}.
  \]

Note that in the but last step, by a slight abuse of notation, we apply (11.8) for \( \ell'_i \) and \( \ell''_i \) which asymptotically coincide with \( L'_i \) and \( L''_i \), respectively.

- **Third sum:** with (11.9) we have
  \[
  \sum_{n \in I_3} e^{-2na_n} \leq \sum_{n \geq \ell''_i+1} e^{-2n(\alpha_1-\delta)} \leq e^{-2\ell''_i(\alpha_1-\delta)} \frac{e^{-2(\alpha_1-\delta)}}{1 - e^{-2(\alpha_1-\delta)}}.
  \]

Finally, recalling that (11.8), we have \( n_i(\alpha - \delta) < \ell''_i(\alpha_1 - \delta) < \ell''_i \alpha_1 \) for every sufficiently large \( i \). Hence the first sum dominates the second and the third ones and hence, for any \( \delta > 0 \) small enough we have \( |v_+(\xi^+, n_i) - v_0| = O(e^{-2n_i(\alpha-\delta)}) \). This proves the claim. \( \Box \)

The proof of the lemma is now complete. \( \Box \)
11.4. Relations between exponents of cocycles and skew-products. We have the following consequences of Lemmas 11.10 and 11.11.

**Corollary 11.15.** Assume that $\xi^+$ satisfies $\lambda_1(A, \xi^+) = \alpha_1$, $\overline{\lambda}_1(A, \xi^+) = \alpha_2$ for some $\alpha_1 < \alpha_2$.

1. If $\alpha_1 = 0$, then if $\chi^+(\xi^+, v)$ is well defined (i.e., the limit exists) then it is equal to zero.
2. If $\alpha_1 > 0$, then there is no $v \in \mathbb{P}^1$ for which the Lyapunov exponent $\chi^+(\xi^+, v)$ is well defined.

**Proof.** Item 1 follows immediately from Lemma 11.10. To show Item 2 observe that by Lemma 11.11 for any $\alpha \in (\alpha_1, \alpha_0)$ and for every $v \in \mathbb{P}^1$ there is a sequence $(m_i)_i$ for which

$$\lim_{i \to \infty} \frac{1}{m_i} \log |f'_{[\xi_0 \ldots \xi_{m_i-1}]}(v)| \in \{-2\alpha, 2\alpha\}.$$

This concludes the proof. $\square$

**Corollary 11.16.** Assume that $(\xi^+, v) \in \Sigma^+_N \times \mathbb{P}^1$ is such that $\chi^+(\xi^+, v)$ is well defined and nonzero, then $\lambda_1(A, \xi^+) = 2|\chi^+(\xi^+, v)|$.

**Proof.** By Corollary 11.15 we obtain $\lambda_1(A, \xi^+) = \overline{\lambda}_1(A, \xi^+) = \lambda_1(A, \xi^+) > 0$ and by Proposition 11.5 $\lambda_1(A, \xi^+) = 2|\chi^+(\xi^+, v)|$. $\square$

11.5. Entropy spectrum: Proof of Theorem 11.1. We finally study the topological entropy of several level sets.

**Proposition 11.17.** The sets

$$S_0 = \{\xi^+, \in \Sigma^+_N: \lambda_1(A, \xi^+) = 0\} \text{ and } S_1 = \{\xi^+, \in \Sigma^+_N: \overline{\lambda}_1(A, \xi^+) = 0\}$$

have the same topological entropy.

**Proof.** We clearly have $S_0 \subset S_1$ and hence $h_{\text{top}}(S_0) \leq h_{\text{top}}(S_1)$.

It only remains to prove the other inequality, for which we will invoke again Frostman’s Lemma 8.4. Let $h = h_{\text{top}}(S_1)$. For any $\varepsilon > 0$ consider the sets

$$X_{n, \varepsilon} \overset{\text{def}}{=} \left\{ \xi^+ \in \Sigma^+_N: \left| -\frac{1}{n} \log \|A_{\xi_0} \circ \cdots \circ A_{\xi_{n-1}}\| \right| \leq \varepsilon \right\}.$$

Note that for any $N \geq 1$ we obtain the corresponding cover by open (cylinder) sets

$$S_1 \subset \bigcup_{n \geq N} \bigcup_{\xi^+ \in X_{n, \varepsilon}} [\xi_0 \ldots \xi_{n-1}].$$

Recalling Appendix A, fix $\mathcal{A} = \{[i]: i = 0, \ldots, N-1\}$ the cover by cylinders of level 1. Note that there exists $N = N(\varepsilon) \geq 1$ such that for any open cover $\mathcal{U}$ of $S_1$, which satisfies $n_{\mathcal{A}}(U) \geq N$ for every $U \in \mathcal{U}$, we have

$$\sum_{U \in \mathcal{U}} e^{-(h-\varepsilon)n_{\mathcal{A}}(U)} > C(\varepsilon) \overset{\text{def}}{=} \frac{1}{1 - e^{-\varepsilon}}.$$

Note that $n_{\mathcal{A}}(U)$ for a cylinder set $U = [\xi_0 \ldots \xi_{n-1}]$ is just its length $n$. 


Claim 11.18. There exist $n \geq N$ and $\hat{U} = \{[\xi_0 \ldots \xi_{n-1}]: \xi^+ \in X_{n, \epsilon}\}$ a family of cylinders of equal length $n \geq N$ each one intersecting $S_1$ such that
\[
\text{card}(\hat{U}) > e^{(h-2\epsilon)n}.
\]

Proof. By the above, there exists a cover $\mathcal{U}$ of $S_1$ by cylinders
\[
\mathcal{U} = \{[\xi_0 \ldots \xi_{n-1}]: n \geq N, \xi^+ \in X_{n, \epsilon}\}
\]
such that
\[
\sum_{U \in \mathcal{U}} e^{-(h-\epsilon)n,U} > C(\epsilon).
\]
We will show that we can choose $\hat{\mathcal{U}}$ being a subfamily of $\mathcal{U}$. Indeed, by contradiction, suppose that for every $n \geq N$, denoting by $\mathcal{U}(n) \subset \mathcal{U}$ the subfamily of cylinders of length $n$, we would have $\text{card}(\mathcal{U}(n)) \leq e^{(h-2\epsilon)n}$. This would imply that
\[
\sum_{U \in \mathcal{U}} e^{-(h-\epsilon)n,U} = \sum_{n \geq N} \sum_{U \in \mathcal{U}(n)} e^{-(h-\epsilon)n,U} \leq \sum_{n \geq N} e^{(h-2\epsilon)n} e^{-(h-\epsilon)n} < C(\epsilon),
\]
contradiction. □

Now we take a sequence $(\epsilon_i)_i$ decreasing to zero and apply the above to each $\epsilon_i$. This provides a sequence $n_i = n(\epsilon_i) \geq 1$ and families $\hat{\mathcal{U}}_i$ of cylinder sets of equal length $n_i$ each satisfying
\[
\text{card}(\hat{\mathcal{U}}_i) > e^{(h-2\epsilon_i)n_i}.
\]
Note that by the standard property of cylinders, the elements in $\hat{\mathcal{U}}_i$ are pairwise disjoint.

Now, given $m \geq 1$, for each $i$ we consider the family of cylinders of length $m n_i$ which are formed by all possible cylinders which are $m$ concatenated elements from the family $\hat{\mathcal{U}}_i$, denote this family by $\hat{\mathcal{U}}_i^m$. Again, this is a family of pairwise disjoint cylinders.

Choose now a fast growing sequence $(m_i)_i$ satisfying
\[
\lim_{k \to \infty} \frac{n_{k+1}}{\sum_{i=1}^k m_i n_i} = 0 \quad \text{and} \quad \max_{i=1,\ldots,k} \frac{m_i n_i}{m_k n_k} < \frac{1}{k^2}. \quad (11.10)
\]
Let $X$ be the set of one-sided infinite sequences of the form
\[
X = \{\xi^+ = \varrho_1 \varrho_2 \ldots: \varrho_i \in \hat{\mathcal{U}}_i^{m_i}, i = 1, 2, \ldots\}.
\]

Claim 11.19. $X \subset S_0$.

Proof. Each $\ell \geq 1$ we can write as $\ell = \sum_{i=1}^k m_i n_i + j n_{k+1} + r$ for some $j \in \{0, \ldots, m_{k+1} - 1\}$ and $r \in \{0, \ldots, n_{k+1} - 1\}$. Hence, recalling (11.1), from (11.10) we obtain
\[
\left| \frac{1}{\ell} \log \|A^\ell(\xi^+)\| \right| \leq \frac{\sum_{i=1}^k m_i n_i \epsilon_i + j n_{k+1} \epsilon_{k+1} + r \log M}{\sum_{i=1}^k m_i n_i + j n_{k+1} + r} \leq k \frac{1}{k^2} + \epsilon_{k+1} + \frac{n_{k+1}}{\sum_{i=1}^k m_i n_i} \log M \to 0,
\]
as $\ell \to \infty$ (and hence $k \to \infty$). □
Claim 11.20. $h_{\text{top}}(X) \geq h$.

Proof. The construction of the set $X$ can be described as the intersection of an infinite nested family $X(\ell)$, each being a finite union of cylinders. For any $\ell_k \overset{\text{def}}{=} \sum_{i=1}^{k} m_i n_i$ denote by $X(\ell_k)$ the union of all $\ell_k$th level cylinders which intersect $X$. By construction of $X$, each cylinder in $X(\ell_k)$ contains at least $e^{(h - 2\varepsilon_{k+1})n_{k+1}}$ cylinders from $X(\ell_{k+1})$.

We will equidistribute on $X$ a probability measure $\nu$, estimate its local dimension and apply Frostman’s Lemma 8.4. For every cylinder in $X(\ell_k)$ the measure $\nu$ is to be equidistributed on its subcylinder from $X(\ell_{k+1})$. Denote by $\Delta^{+}(\xi^{+})$ the cylinder of length $\ell$ containing $\xi^{+}$. By induction, we can prove that for every $\ell_k$th level cylinder $C \in X(\ell_k)$ we have

$$\nu(C) \leq e^{-\sum_{i=1}^{k} m_i n_i (h - 2\varepsilon_{i})}.$$  

Hence, for every $\xi^{+} \in X$ we have

$$\liminf_{k \to \infty} - \frac{1}{\ell_k} \log \nu(\Delta^{+}(\xi^{+})) \geq \liminf_{k \to \infty} \frac{\sum_{i=1}^{k} m_i n_i (h - 2\varepsilon_{i})}{\sum_{i=1}^{k} m_i n_i} = h,$$

where we used (11.10). Now Frostman’s Lemma 8.4 implies $h_{\text{top}}(X) \geq h$. Since $S_0 \supset X$ and entropy is monotone, this proves the claim. □

This proves $h_{\text{top}}(S_0) \geq h_{\text{top}}(S_1)$ and finishes the proof of the proposition. □

We can now conclude the Proof of Theorem 11.1.

Proof of Theorem 11.1. Consider first the case of $\alpha > 0$. By Proposition 11.5 we have

$$\{\xi^{+} \in \Sigma^+_N : \lambda_1(A, \xi^{+}) = \alpha\} \subset \{\xi^{+} \in \Sigma^+_N : \chi^+(\xi^{+}, v) = 2\alpha \text{ for all } v \in \mathbb{P}^1\}.$$  

To obtain the other inclusion, consider $\xi^{+} \in \Sigma^+_N$, such that there is a Lyapunov regular point $(\xi^{+}, v)$ for some $v \in \mathbb{P}^1$ with exponent $\chi^+(\xi^{+}, v) = 2\alpha$ in the fiber. By Corollary 11.16, $\lambda_1(A, \xi^{+})$ is well defined. Again applying Proposition 11.5, we obtain that $\alpha = \lambda_1(A, \xi^{+})$, proving the other inclusion. The case $-2\alpha$ is analogous and hence omitted.

Now consider the case $\alpha = 0$. Again by Proposition 11.5 we have

$$\{\xi^{+} \in \Sigma^+_N : \lambda_1(A, \xi^{+}) = 0\} \subset \{\xi^{+} \in \Sigma^+_N : \chi^+(\xi^{+}, v) = 0 \text{ for all } v \in \mathbb{P}^1\}.$$  

To show the second inclusion, assume that $\chi^+(\xi^{+}, v) = 0$ for all $v$. Then either $\lambda_1(A, \xi^{+})$ exists and hence by the first claim in Proposition 11.5 must be equal to 0. Or $\alpha_1 = \lambda_1(A, \xi^{+}) < \lambda_1(A, \xi^{+})$ and then by Case (b) in Lemma 11.11 we can exclude that $\alpha_1 > 0$, hence proving $\lambda_1(A, \xi^{+}) = 0$ and thus the other inclusion. The assertion about the entropy is just Proposition 11.17. This proves the theorem. □
11.6. Entropy spectrum: Proof of Theorem 5. Now we are ready to prove Theorem 5. Note the differences in some statements in which we jump from studying the one-sided shift space $\Sigma_N^+$ to the two-sided one $\Sigma_N$.

Proof of Theorem 5. After Theorem 11.1, it remains to see the properties of the level sets of two-sided sequences. Note that by Corollary 11.16 for every $\alpha \neq 0$ we have

$$\pi(\mathcal{L}_A^+(\alpha)) \subset \mathcal{L}_{A_1}^+(|\alpha|/2),$$

where

$$\mathcal{L}_A^+(\alpha) \overset{\text{def}}{=} \{ (\xi^+, v) \in \Sigma_N^+ \times \mathbb{S}^1 : \chi^+(\xi^+, v) = \alpha \}$$

and $\pi$ denotes the natural projection. By Proposition 11.5 for every $\alpha$

$$\mathcal{L}_A^+(|\alpha|/2) \subset \pi(\mathcal{L}_A^+(\alpha)).$$

Hence, for every $\alpha > 0$ we have

$$\pi(\mathcal{L}_A^+(\alpha)) = \mathcal{L}_A^+(\alpha/2) = \pi(\mathcal{L}_A^+(\alpha)).$$

Finally, recalling that for every set $\Theta \subset \Sigma_N \times \mathbb{P}^1$ we have $h_{\text{top}}(F_A, \Theta) = h_{\text{top}}(\sigma, \pi(\Theta))$, see for instance [21, Lemma 4.9], for every $\alpha > 0$ we have

$$h_{\text{top}}(\mathcal{L}_A^+(\alpha)) = h_{\text{top}}(\mathcal{L}_A^+(\alpha/2)) = h_{\text{top}}(\mathcal{L}_A^+(\alpha)).$$

By the same argument, applying Theorem 11.1 item 2., we obtain

$$h_{\text{top}}(\mathcal{L}_A^+(0)) = h_{\text{top}}(\pi(\mathcal{L}_A^+(0))) = h_{\text{top}}(\mathcal{L}_A^+(0)).$$

We now relate the one-sided and the two-sided spectra. Note that any $(\xi, v)$ with $\chi(\xi, v) = \alpha$ satisfies $\chi^+(\xi^+, v) = \alpha$. This immediately implies that for all $\alpha$ we have $h_{\text{top}}(\mathcal{L}(\alpha)) \leq h_{\text{top}}(\mathcal{L}_A^+(\alpha))$.

Claim 11.21. For every $\alpha \geq 0$ it holds $h_{\text{top}}(\mathcal{L}_A^+(\alpha/2)) \leq h_{\text{top}}(\mathcal{L}(\alpha))$.

Proof. The case $\alpha = 0$ follows immediately from Proposition 11.5.

Consider now the case $\alpha > 0$. Again by Proposition 11.5, for every $\xi^+ \in \mathcal{L}_A^+(\alpha/2)$ the vector $v_0(\xi^+) \in \mathbb{P}^1$ is well defined and $\chi^+(\xi^+, v_0(\xi^+)) = \alpha$. We need to prove the following statement: for every $\xi^+ \in \mathcal{L}_A^+(\alpha/2)$ there exists $\eta^-$ such that for $\xi = \eta^- \cdot \xi^+$ we have $\chi(\xi, v_0(\xi^+)) = \alpha$. This will imply that $h_{\text{top}}(\mathcal{L}_A^+(\alpha/2)) \leq h_{\text{top}}(\mathcal{L}(\alpha))$.

Let us first consider the special case when the vector $v_0(\xi^+)$ is simultaneously fixed by all the maps $f_i$. In this case, this vector is also fixed by all the maps $f_i^{-1}$, and we can choose $\xi = \xi_0^+ \cdot \xi_1^+ = (\ldots \xi_0, \xi_0, \xi_1, \ldots)$. Indeed,

$$(f_{\xi_0}^{-n})'(v_0(\xi^+)) = \prod_{i=0}^{n-1} (f_{\xi_i}^{-1})'(v_0(\xi^+)) = \prod_{i=0}^{n-1} (f_{\xi_i}^{-1}(v_0(\xi^+)))^{-1} = (f_{\xi}^{-n})'(v_0(\xi^+))^{-1}$$

and hence $\chi^+(\xi^+, v_0(\xi^+)) = \alpha$ implies $\chi(\xi, v_0(\xi^+)) = \alpha$.

Assume now that there exists some $f_i$, such that $f_i(v_0(\xi^+)) \neq v_0(\xi^+)$. Since we consider $\text{SL}(2, \mathbb{R})$ cocycles, the Lyapunov spectra for the cocycle generated by $A$ and the one for the cocycle generated by $A^{-1} = \{ A_i^{-1} : A_i \in A \}$ coincide. Hence there is $\eta^+ \in \mathcal{L}_A^+(\alpha/2)$. We now apply Proposition 11.5 to $A^{-1}$. Either $v_0(\xi^+) = v_0(\xi^+, A)$
does not coincide with the vector \( v_0(\eta^+, A^{-1}) \) defined with respect to \( A^{-1} \) and \( \eta^+ \) and hence \( \chi^+(\eta^+, v_0(?^+, F_{A^{-1}})) = -\alpha \). In this case, we have \( \chi(\xi, v_0(\xi^+)) = \alpha \) for the concatenated two-sided sequences \( \xi = \eta^+, \xi^+ = (\ldots \eta_1\eta_0\xi_0\xi_1 \ldots) \). Or, \( v_0(\xi^+, A) = v_0(\xi^+, A^{-1}) \). Then for some map \( f_j \), \( v \equiv f_j(v_0(\xi^+)) \neq v_0(\xi^+) \), in which case we have \( \chi(\xi, v) = \alpha \) with \( \xi = (\ldots \eta_1\eta_0i.\xi_0\xi_1 \ldots) \), ending the proof of the claim. \( \square \)

The symmetry of the entropy spectrum of the skew-product now follows from the next claim.

**Claim 11.22.** For every \( \alpha > 0 \) it holds \( h_{\text{top}}(\mathcal{L}(\alpha)) = h_{\text{top}}(\mathcal{L}(-\alpha)) \).

**Proof.** The proof of this claim is analogous to the previous one. Let \((\xi^+, v)\) such \( \chi^+(\xi^+, v) = -\alpha \). By Corollary 11.16 we have \( \lambda_1(A, \xi^+) = 2\alpha \). Now by Proposition 11.5, we in fact have \( \chi^+(\xi^+, w) = -\alpha \) for all \( w \neq v_0(\xi^+) \), the latter being well defined. Again by Proposition 11.5 now applied to \( A^{-1} \) there is \( \eta^+ \in \mathcal{L}^+_{A^{-1}}(\alpha/2) \) so that: either \( v_0(\xi^+) = v_0(\xi^+, A) \) does not coincide with the vector \( v_0(\eta^+, A^{-1}) \) in which case we would have \( \chi(\xi, v_0(\eta^+, A^{-1})) = -\alpha \), where \( \xi = (\ldots \eta_1\eta_0\xi_0\xi_1 \ldots) \). Or \( v_0(\xi^+) = v_0(\xi^+, A) = v_0(\eta^+, A^{-1}) \). In this case, if \( f_j(v_0(\eta^+, A^{-1})) = v_0(\xi^+) \) for every \( j \) then we argue as above and let \( \xi = (\ldots \xi_1\xi_0\xi_1 \ldots) \). Otherwise, there is some map \( f_j \) such that \( w = f_j(v_0(\eta^+, A^{-1})) \neq v_0(\xi^+) \), in which case we have \( \chi(\xi, w) = \alpha \) for \( \xi = (\ldots \eta_1\eta_0i.\xi_0\xi_1 \ldots) \). This proves \( h_{\text{top}}(\mathcal{L}(\alpha/2)) \leq h_{\text{top}}(\mathcal{L}(-\alpha)) \). \( \square \)

The proof of the theorem is now complete. \( \square \)

### 11.7. The set \( \mathcal{E}_{N, \text{shyp}} \) of elliptic cocycles.

In this section, we define \( \mathcal{E}_{N, \text{shyp}} \) and prove that this set is open and dense (in \( \mathcal{E}_N \)).

First observe that, according to Remark 3.1, given \( F : \Sigma_2 \times S^1 \rightarrow \Sigma_2 \times S^1 \) with fiber maps \( f_0, f_1 \) which satisfies Axioms CEC\( \pm \) and Acc\( \pm \) there exists \( \epsilon = \epsilon(f_0, f_1) > 0 \) such that every skew-product \( G : \Sigma_2 \times S^1 \rightarrow \Sigma_2 \times S^1 \) with fiber maps \( g_0, g_1 \) which are \( \epsilon \)-close to \( f_0, f_1 \), respectively, also satisfies those axioms. Also observe that for every \( N > 2 \), every skew-product \( H : \Sigma_{N} \times S^1 \rightarrow \Sigma_{N} \times S^1 \) with fiber maps \( h_0, \ldots, h_{N-1} \) such that \( h_0 = f_0 \) and \( h_1 = f_1 \) also satisfies the axioms.

Second, take \( F \) such that \( f_0 \) is a Morse–Smale diffeomorphism with exactly two fixed points (a global attractor and a global repeller) and \( f_1 \) is an irrational rotation. By Remark 1.1 the skew-product \( F \) satisfies the axioms and hence we can define an \( \epsilon(f_0, f_1) \) as above.

Let us denote by \( \mathcal{S}_H \subset \text{SL}(2, \mathbb{R}) \) the subset of hyperbolic matrices and by \( \mathcal{I}_1 \subset \text{SL}(2, \mathbb{R}) \) the one of “irrational rotations”

\[
\mathcal{I}_1 \overset{\text{def}}{=} \left\{ \begin{pmatrix} \cos 2\pi \theta & \sin 2\pi \theta \\ -\sin 2\pi \theta & \cos 2\pi \theta \end{pmatrix} : \theta \in [0, 1), \theta \notin \mathbb{Q} \right\}.
\]

Note that if \( A \in \mathcal{S}_H \) and \( B \in \mathcal{I}_1 \), then with \( A = \{A, B\} \) the skew-product \( F_A \) satisfies the axioms and we can define an \( \epsilon(f_A, f_B) \) as above (recall (1.3)). Now let

\[
\mathcal{E}_{N, \text{shyp}} \overset{\text{def}}{=} \bigcup_{A \in \mathcal{S}_H, B \in \mathcal{I}_1} \left\{ A \in \mathcal{E}_N : \text{there exist } A', B' \in \{A\}, C \in \text{SL}(2, \mathbb{R}) \text{ so that } f_{C^{-1}A'C}, f_{C^{-1}B'C} \text{ are } \epsilon(f_A, f_B)\text{-close to } f_A, f_B, \text{ respectively} \right\}.
\]

Note that there is a natural identification of \( \text{SL}(2, \mathbb{R})^N \) with a subset of \( \mathbb{R}^{4N} \).
**Proposition 11.23.** The set \( \mathcal{E}_{N,shyp} \subset \mathcal{E}_N \) is open and dense (in \( \mathcal{E}_N \)). Moreover, for every one-step \( 2 \times 2 \) matrix cocycle \( A \in \mathcal{E}_{N,shyp} \) its induced step skew-product \( F_A \) satisfies Axioms \( \text{CEC}^\pm \) and \( \text{Acc}^\pm \) and is proximal.

Certainly the existence of a dense (and automatically open) subset of \( \mathcal{E}_N \) of cocycles with a hyperbolic element is well known, but let us sketch a proof for completeness.

**Proof.** By definition, the set \( \mathcal{E}_{N,shyp} \) is open. It remains to show its density (in \( \mathcal{E}_N \)). For that we consider the subset of \( \mathcal{E}_{N,shyp} \),

\[
\mathcal{E}_{N,shyp}^\prime \overset{\text{def}}{=} \bigcup_{A \in \mathcal{S}_1, B \in \mathcal{I}_1} \left\{ A \in \mathcal{E}_N : \text{there exist } A', B' \in \langle A \rangle, C \in \text{SL}(2, \mathbb{R}) \text{ so that } C^{-1}A' C = A, C^{-1}B' C = B \right\}.
\]

**Lemma 11.24.** \( \mathcal{E}_{N,shyp}^\prime \) is dense in \( \mathcal{E}_N \).

**Proof.** Let \( A \in \mathcal{E}_N \), that is, assume that \( \langle A \rangle \) contains elliptic matrices. Note that if \( A \in \text{SL}(2, \mathbb{R}) \) is hyperbolic then for every \( C \in \text{SL}(2, \mathbb{R}) \) we have that \( C^{-1}AC \) is also hyperbolic.

**Claim 11.25.** There is an arbitrarily small perturbation of \( A \) containing a hyperbolic element.

**Proof.** Recall that \( A \in \text{SL}(2, \mathbb{R})^N \) is called parabolic if \( |\text{trace } A| = 2 \). Notice that every parabolic \( A \) can be arbitrarily approximated by hyperbolic ones. There are three cases: (1) \( A \) contains a hyperbolic element; (2) \( A \) contains a parabolic element; and (3) \( A \) contains only elliptic elements. Observe that in the first two cases we are done. We now consider the third case.

Pick \( A_1, A_2 \in A \) and the vector \( v = (1, 0) \). Possibly after a small perturbation, we can assume that \( A_1 \) has irrational rotation number and hence, after possibly a new perturbation, we can assume that \( A(v) = v \), where \( A = A_1^k \circ A_2 \) for some (large) \( k \). If \( v \) is a hyperbolic fixed point for \( f_A \), we are done. Otherwise we consider the perturbations of \( A_2 \) given by

\[
A_{2,t} = A_2 \circ \begin{pmatrix} 1+t & 0 \\ 0 & 1/(1+t) \end{pmatrix}
\]

and observe that \( A_t(v) = v \), where \( A_t = A_1^t \circ A_{2,t} \) and that \( v \) is necessarily hyperbolic for \( f_{A_t} \). Recalling that \( \mathcal{E}_N \) is open, we can assume that this perturbation was sufficiently small such that \( A_t = \{ A_1, A_{2,t}, A_3, \ldots, A_N \} \) is elliptic and that \( \langle A_t \rangle \) contains a hyperbolic element. \( \square \)

Observe now that the above achieved hyperbolic elements will not be destroyed by sufficiently small further perturbations. It remains to obtain one further perturbation to get an element which is matrix-conjugate to an irrational rotation.

Consider an elliptic matrix \( A' = A_{ik} \circ \cdots \circ A_{ii} \). If its rotation number is already irrational we are done. Otherwise we consider rotation matrices \( R_r \), small \( r \geq 0 \), the elliptic matrix \( A'_r = A_{ik} \circ R_r \circ \cdots \circ A_{ii} \circ R_r \), and the map \( F(r) = \text{trace}(A_r) = 2 \cos(q(A_r)) \). By [2, Lemma A.4], \( F'(0) > 0 \). This immediately implies that there are arbitrarily small perturbations of \( A \) with irrational rotation number. This proves the density of \( \mathcal{E}_{N,shyp}^\prime \) (in \( \mathcal{E}_N \)) and finishes the proof of the lemma. \( \square \)
Note that the second part of the proposition just rephrases Remark 3.1, that asserts that for every $A$ in $\mathcal{E}_{N, \text{shyp}}$ the corresponding skew-product map $F_A$ satisfies the axioms and proximality. □

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Appendix A. Entropy

Let $X$ be a compact metric space. Consider a continuous map $f : X \to X$, a set $Y \subset X$, and a finite open cover $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ of $X$. Given $U \subset X$ we write $U \prec \mathcal{A}$ if there is an index $j$ so that $U \subset A_j$, and $U \nprec \mathcal{A}$ otherwise. Taking $U \subset X$ we define

$$n_{f, \mathcal{A}}(U) := \begin{cases} 0 & \text{if } U \nprec \mathcal{A}, \\ \infty & \text{if } f^k(U) \prec \mathcal{A} \text{ for all } k \in \mathbb{N}, \\ \ell & \text{if } f^k(U) \prec \mathcal{A} \text{ for all } k \in \{0, \ldots, \ell - 1\} \text{ and } f^\ell(U) \nprec \mathcal{A}. \end{cases}$$

If $\mathcal{U}$ is a countable collection of open sets, given $d > 0$ let

$$m(\mathcal{A}, d, \mathcal{U}) := \sum_{U \in \mathcal{U}} e^{-d n_{f, \mathcal{A}}(U)}.$$ 

Given a set $Y \subset X$, let

$$m_{\mathcal{A}, d}(Y) := \lim_{\rho \to 0} \inf \left\{ m(\mathcal{A}, d, \mathcal{U}) : Y \subset U, \sum_{U \in \mathcal{U}} e^{-d n_{f, \mathcal{A}}(U)} < \rho \text{ for every } U \in \mathcal{U} \right\}.$$ 

Analogously to what happens for the Hausdorff measure, $d \mapsto m_{\mathcal{A}, d}(Y)$ jumps from $\infty$ to $0$ at a unique critical point and we define

$$h_{\mathcal{A}}(f, Y) := \inf \left\{ d : m_{\mathcal{A}, d}(Y) = 0 \right\} = \sup \left\{ d : m_{\mathcal{A}, d}(Y) = \infty \right\}.$$ 

The topological entropy of $f$ on the set $Y$ is defined by

$$h_{\text{top}}(f, Y) := \sup_{\mathcal{A}} h_{\mathcal{A}}(f, Y),$$

When $Y = X$, we simply write $h_{\text{top}}(X) = h_{\text{top}}(f, X)$.

By [11, Proposition 1], in the case of $Y$ compact this definition is equivalent to the canonical definition of topological entropy (see, for example, [55, Chapter 7]).

References

1. Avila, A.: Density of positive Lyapunov exponents for SL(2, $\mathbb{R}$)-cocycles. J. Am. Math. Soc. 24(4), 999–1014 (2011)
2. Avila, A., Bochi, J., Yoccoz, J.-C.: Uniformly hyperbolic finite-valued SL(2, $\mathbb{R}$)-cocycles. Comment. Math. Helv. 85(4), 813–884 (2010)
3. Avila, A., Viana, M.: Simplicity of Lyapunov spectra: a sufficient criterion. Port. Math. (N.S.) 64(3), 311–376 (2007)
4. Barreira, L., Saussol, B.: Variational principles and mixed multifractal spectra. Trans. Am. Math. Soc. 353(10), 3919–3944 (2001)
5. Bochi, J.:Genericity of zero Lyapunov exponents. Ergod. Theory Dyn. Syst. 22(6), 1667–1696 (2002)
6. Bochi, J., Bonatti, C., Díaz, L.J.: Robust criterion for the existence of nonhyperbolic ergodic measures. Commun. Math. Phys. 344(3), 751–795 (2016)
7. Bochi, J., Bonatti, C., Gelfert, K.: Dominated Pesin theory: convex sum of hyperbolic measures. Isr. J. Math. 226(1), 387–417 (2018)
8. Bochi, J., Rams, M.: The entropy of Lyapunov-optimizing measures of some matrix cocycles. J. Mod. Dyn. 10, 255–286 (2016)
9. Bochi, J., Viana, M.: Uniform (projective) hyperbolicity or no hyperbolicity: a dichotomy for generic conservative maps. Ann. Inst. Henri Poincaré Anal. Non Linéaire 19(1), 113–123 (2002)
10. Bonatti, C., Díaz, L.J, Ures, R.: Minimality of strong stable and unstable foliations for partially hyperbolic diffeomorphisms. J. Inst. Math. Jussieu 1(4), 513–541 (2002)
11. Bowen, R.: Topological entropy for noncompact sets. Trans. Am. Math. Soc. 184, 125–136 (1973)
12. Bowen, R.: Some systems with unique equilibrium states. Math. Syst. Theory 8(3), 193–202 (1974/1975)
13. Bowen, R.: Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, volume 470 of Lecture Notes in Mathematics, revised edition. Springer, Berlin (2008). With a preface by David Ruelle, Edited by Jean-René Chazottes
14. Burns, K., Gelfert, K.: Lyapunov spectrum for geodesic flows of rank 1 surfaces. Discrete Contin. Dyn. Syst. 34(5), 1841–1872 (2014)
15. Cowieson, W., Young, L.-S.: SRB measures as zero-noise limits. Ergod. Theory Dyn. Syst. 25(4), 1115–1138 (2005)
16. Crauel, H.: Extremal exponents of random dynamical systems do not vanish. J. Dyn. Differ. Equ. 2(3), 245–291 (1990)
17. Damanik, D.: Schrödinger operators with dynamically defined potentials. Ergod. Theory Dyn. Syst. 29(5), 1745–1767 (2018)
18. Díaz, L.J., Esteves, S., Rocha, J.: Skew product cycles with rich dynamics: from totally non-hyperbolic dynamics to fully prevalent hyperbolicity. Dyn. Syst. 31(1), 1–40 (2016)
19. Díaz, L.J., Fisher, T.: Symbolic extensions and partially hyperbolic diffeomorphisms. Discrete Contin. Dyn. Syst. 29(4), 1419–1441 (2011)
20. Díaz, L.J., Gelfert, K.: Porcupine-like horseshoes: transitivity, Lyapunov spectrum, and phase transitions. Fund. Math. 216(1), 55–100 (2012)
21. Díaz, L.J., Gelfert, K., Rams, M.: Nonhyperbolic step skew-products: ergodic approximation. Ann. Inst. Henri Poincaré Anal. Non Linéaire 34(6), 1561–1598 (2017)
22. Díaz, L.J., Gelfert, K., Rams, M.: Topological and ergodic aspects of partially hyperbolic diffeomorphisms and nonhyperbolic step skew products. Proc. Steklov Inst. Math. 297(1), 98–115 (2017)
23. Duarte, P., Klein, S.: Lyapunov Exponents of Linear Cocycles: Continuity via Large Deviations, volume 3 of Atlantis Studies in Dynamical Systems. Atlantis Press, Paris (2016)
24. Fan, A., Liao, L., Peyrière, J.: Generic points in systems of specification and Banach valued Birkhoff ergodic average. Discrete Contin. Dyn. Syst. 21(4), 1103–1128 (2008)
25. Feng, D.-J.: Lyapunov exponents for products of matrices and multifractal analysis. I. Positive matrices. Isr. J. Math. 138, 353–376 (2003)
26. Feng, D.-J., Lau, K.-S.: The pressure function for products of non-negative matrices. Math. Res. Lett. 9(2-3), 363–378 (2002)
27. Furstenberg, H.: Noncommuting random products. Trans. Am. Math. Soc. 108, 377–428 (1963)
28. Gelfert, K., Kwietniak, D.: On density of ergodic measures and generic points. Ergod. Theory Dyn. Syst. 38(5), 1745–1767 (2018)
29. Gelfert, K., Przytycki, F., Rams, M.: On the Lyapunov spectrum for rational maps. Math. Ann. 348(4), 965–1004 (2010)
30. Gelfert, K., Rams, M.: The Lyapunov spectrum of some parabolic systems. Ergod. Theory Dyn. Syst. 29(3), 919–940 (2009)
31. Gorodetski, A., Il’yashenko, Y.S.: Some new robust properties of invariant sets and attractors of dynamical systems. Funktsional. Anal. i Prilozhen. 33(2), 16–30, 95 (1999)
32. Gorodetski, A., Il’yasenko, Y.S.: Some properties of skew products over a horseshoe and a solenoid. Tr. Mat. Inst. Steklova 231(Din. Sist., Avtom. i Beskonn. Gruppy), 96–118 (2000)
33. Gorodetski, A., Il’yasenko, Y.S., Kleptsyn, V., Nal’skií, M.B.: Nonremovability of zero Lyapunov exponents. Funktsional. Anal. i Prilozhen. 39(1), 27–38, 95 (2005)
34. Gorodetski, A., Pesin, Y.: Path connectedness and entropy density of the space of hyperbolic ergodic measures. In: Katok, A., Pesin, Y., Rodríguez Hertz, F. (eds.) Modern Theory of Dynamical Systems, volume 692 of Contemporary Mathematics, pp. 111–121. American Mathematical Society, Providence (2017)
35. Iommi, G., Todd, M.: Dimension theory for multimodal maps. Ann. Henri Poincaré 4(3), 591–620 (2011)
36. Jenkinson, O.: Ergodic optimization. Discrete Contin. Dyn. Syst. 15(1), 197–224 (2006)
37. Knill, O.: The upper Lyapunov exponent of SL(2, R) cocycles: discontinuity and the problem of positivity. In: Arnold, L., Crauel, H., Eckmann, J.-P. (eds.) Lyapunov Exponents (Oberwolfach, 1990), volume 1486 of Lecture Notes in Mathematics, pp. 86–97. Springer, Berlin (1991)
38. Leplaideur, R., Oliveira, K., Rios, I.: Equilibrium states for partially hyperbolic horseshoes. Ergod. Theory Dyn. Syst. 31(1), 179–195 (2011)
39. Makarov, N., Smirnov, S.: On “thermodynamics” of rational maps. I. Negative spectrum. Commun. Math. Phys. 211(3), 705–743 (2000)
40. Malicet, D.: Random walks on Homeo(S^1). Commun. Math. Phys. 356(3), 1083–1116 (2017)
41. Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability, volume 44 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (1995)
42. Navas, A.: Groups of Circle Diffeomorphisms. Chicago Lectures in Mathematics, Spanish edition. University of Chicago Press, Chicago (2011)
43. Olsen, L.: A multifractal formalism. Adv. Math. 116(1), 82–196 (1995)
44. Pesin, Y., Weiss, H.: The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples. Chaos 7(1), 89–106 (1997)
45. Pfister, C.-E., Sullivan, W.G.: On the topological entropy of saturated sets. Ergod. Theory Dyn. Syst. 27(3), 929–956 (2007)
46. Przytycki, F., Rivera-Letelier, J.: Geometric pressure for multimodal maps of the interval. Preprint arXiv:1405.2443v1. To appear in Memoirs of the American Mathematical Society
47. Przytycki, F., Rivera-Letelier, J., Smirnov, S.: Equality of pressures for rational functions. Ergod. Theory Dyn. Syst. 24(3), 891–914 (2004)
48. Robinson, C.: Dynamical Systems: Stability, Symbolic Dynamics, and Chaos. Studies in Advanced Mathematics. CRC Press, Boca Raton (1995)
49. Rodriguez Hertz, F., Rodriguez Hertz, M., Tahzibi, A., Ures, R.: Maximizing measures for partially hyperbolic systems with compact center leaves. Ergod. Theory Dyn. Syst. 32(2), 825–839 (2012)
50. Ruelle, D.: Thermodynamic Formalism: The Mathematical Structures of Equilibrium Statistical Mechanics. Cambridge Mathematical Library, 2nd edn. Cambridge University Press, Cambridge (2004)
51. Sigmund, K.: On dynamical systems with the specification property. Trans. Am. Math. Soc. 190, 285–299 (1974)
52. Tahzibi, A., Yang, J.: Invariance principle and rigidity of high entropy measures. Trans. Amer. Math. Soc. 371(2), 1231–1251 (2019)
53. Takens, F., Verbitskiy, E.: On the variational principle for the topological entropy of certain non-compact sets. Ergod. Theory Dyn. Syst. 23(1), 317–348 (2003)
54. Viana, M.: Lectures on Lyapunov Exponents, volume 145 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (2014)
55. Walters, P.: An Introduction to Ergodic Theory, volume 79 of Graduate Texts in Mathematics. Springer, New York (1982)
56. Wijsman, R.A.: Convergence of sequences of convex sets, cones and functions. II. Trans. Am. Math. Soc. 123, 32–45 (1966)
57. Yoccoz, J.-C.: Some questions and remarks about SL(2, R) cocycles. In: Brin, M., Hasselblatt, B., Pesin, Y. (eds.) Modern Dynamical Systems and Applications, pp. 447–458. Cambridge University Press, Cambridge (2004)

Communicated by C. Liverani