Square Complex Orthogonal Designs with no Zero Entry for any $2^m$ Antennas

Smarajit Das, Student Member, IEEE and B. Sundar Rajan, Senior Member, IEEE

Abstract—Space-time block codes from square complex orthogonal designs (SCODs) have been extensively studied and most of the existing SCODs contain large number of zeros. The zeros in the designs result in high peak-to-average power ratio and also impose a severe constraint on hardware implementation of the code while turning off some of the transmitting antennas whenever a zero is transmitted. Recently, SCODs with no zero entry have been constructed for $2^a$ transmit antennas whenever $a+1$ is a power of 2. Though there exists codes for 4 and 16 transmit antennas with no zero entry, there is no general method of construction which gives codes for any number of transmit antennas. In this paper, we construct SCODs for any power of 2 number of transmit antennas having all its entries non-zero. Simulation results show that the codes constructed in this paper outperform the existing codes for the same number of antennas under peak power constraint while performing the same under average power constraint.

I. PRELIMINARIES

Space-Time Block Codes (STBCs) from complex orthogonal designs (CODs) have been extensively studied in [1], [2], [3]. Due to the orthogonality of the designs, the codes have linear decoding complexity, that is, they are single symbol decodable (SSD). Generally, a linear-processing complex orthogonal design (LPCOD) is a $p \times n$ matrix $G$ in $k$ complex variables $x_1, x_2, \cdots, x_k$ such that each non-zero entry of the matrix is a complex linear combinations of the variables $x_1, x_2, \cdots, x_k$ and their conjugates $x_1^*, x_2^*, \cdots, x_k^*$ satisfying $G^H G = (|x_1|^2 + |x_2|^2 + \cdots + |x_k|^2)I_n$, where $G^H$ is the complex conjugate transpose of $G$ and $I_n$ is the $n \times n$ identity matrix. An LPCOD $G$ is called complex orthogonal design (COD) if the non-zero entries of $G$ are the complex values $\pm x_1, \pm x_2, \cdots, \pm x_k$ or their complex conjugates (entries with complex linear combinations of the variables and their conjugates are not allowed).

For the construction of codes with low peak-to-average power ratio (PAPR), we relax the conditions imposed on the entries of a COD. We define $\lambda$-scaled square complex orthogonal design, for a positive integer $\lambda$, ($\lambda$-scaled COD) $G$ as a $n \times n$ matrix in $k$ complex variables $x_1, x_2, \cdots, x_k$ such that any non-zero entry of the matrix is a variable or its complex conjugate, or the negative of these and all the entries of any subset of columns of the matrix is scaled by $\frac{1}{\sqrt{\lambda}}$ satisfying the condition: $G^H G = (|x_1|^2 + \cdots + |x_k|^2)I_n$. Notice that a $\lambda$-scaled COD with no column scaled by $\frac{1}{\sqrt{\lambda}}$ is a COD (corresponds to $\lambda = 1$). In columns with scaling by $\frac{1}{\sqrt{\lambda}}$ all the variables appear exactly $\lambda$ times. In this paper, $\lambda$ is always a power of 2 and call these codes simply scaled-CODs. To construct codes with all its entries non-zero, the notion of co-ordinate interleaved complex variables is found to be useful. This type of variable is used extensively in the construction of single-symbol decodable STBCs that are not CODs [12]. Given two complex variables $x_i$ and $x_k$ where $x_i = x_iI + jx_iQ$ and $x_k = x_kI + jx_kQ$, the coordinate interleaved variables corresponding to the variables $x_i$ and $x_k$, are $x_{i,k} = x_iI + jx_kQ$ and $x_{k,i} = x_kI + jx_iQ$.

Definition 1: An LPCOD is called coordinate interleaved scaled complex orthogonal designs (CIS-COD) if any non-zero entry of the matrix is a variable or a coordinate interleaved variable, or their complex conjugates, or multiple of these by $\pm \frac{1}{\sqrt{\lambda}}$ where $\lambda$ is a power of 2. Note that any scaled-COD is a CIS-COD, but not conversely.

It is known that the maximum rate $R$ of an $n \times n$ LPCOD is $\frac{a+1}{n}$ where $n = 2^a(2b+1)$, $a$ and $b$ are positive integers [2]. Several authors have constructed LPCODs for $2^a$ antennas achieving maximal rate [2], [4], [5], [6]. In [2], the following induction method is used to construct SCODs for $2^a$ antennas, $a = 2, 3, \cdots$, starting from $G_1 = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$,

$$G_a = \begin{bmatrix} G_{a-1} & -x_{a+1}^* I_{2^a-1} \\ x_{a+1}I_{2^a-1} & G_{a-1}^H \end{bmatrix}$$

where $G_a$ is a $2^a \times 2^a$ complex matrix. Note that $G_a$ is a COD in $a+1$ complex variables $x_1, x_2, \cdots, x_{a+1}$. Moreover, each row and each column of the matrix $G_a$ contains only $a+1$ non-zero elements and all other entries in the same row or column are filled with zeros. The fraction of zeros, defined as the ratio of the number of zeros to the total number of entries in a design, for $G_a$, is

$$2^a - a - 1 \over 2^a = 1 - a + 1 \over 2^a = 1 - R.$$

For the constructions in [2], [4], [5], [6] also, the fraction of zeros is given by $R$. Reducing number of zeros in a SCOD for more than 2 transmit antennas (for two antennas, the Alamouti code does not have any zeros), is important for many reasons including improvement in Peak-to-Average Power Ratio (PAPR) and also the ease of practical implementation of these codes in wireless communication system [13].
construction \((1)\) as shown below

\[
G_3 = \begin{bmatrix}
    x_1 - x_4^* & -x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
    x_2 & x_1 & -x_3^* & 0 & -x_4 & 0 & 0 & 0 \\
    x_3 & 0 & x_1^* & x_2^* & 0 & 0 & 0 & 0 \\
    0 & x_3 - x_2 & x_1 & 0 & 0 & -x_4^* & 0 & 0 \\
    x_4 & 0 & 0 & 0 & x_1^* & x_2^* & x_3^* & 0 \\
    0 & x_4 & 0 & -x_2 & x_1 & 0 & x_3^* & 0 \\
    0 & 0 & x_4 & 0 & -x_3 & 0 & x_1 - x_2^* & 0 \\
    0 & 0 & x_4 & 0 & -x_3 & 0 & x_2 - x_1^* & 0 \\
\end{bmatrix},
\]

\(3\)

contains 50 per cent of entries zeros. But, Yuen et al, in [7], have constructed a new rate-1/2, SCOD \(G_Y\) of size 8 with no zeros in the design matrix using Amicable Complex Orthogonal Design (ACOD) [11] where \(G_Y\) is given in \((5)\).

\[
G_Y = \begin{bmatrix}
    x_1^* & x_1 & x_2 - x_2^* & x_3 & -x_4 & x_4 - x_4^* \\
    jx_1 - jx_1 & jx_2 & jx_2^* & jx_3 & jx_3^* & jx_4 & jx_4^* \\
    -x_2 & x_2 & x_1^* & x_1 & -x_4^* & -x_3 & x_3 \\
    -jx_2^* & -jx_2 & jx_1 & jx_1 & jx_4 & jx_4 - jx_3 & jx_3 \\
    -x_3 & x_3 & x_4^* & x_4^* & x_1 & x_1^* & x_2^* & x_2^* \\
    -jx_3^* & -jx_3 & -jx_2^* & -jx_4 & jx_1 & x_1^* & x_2 & jx_2 \\
    -x_4 & x_4 & x_3^* - x_3 & -x_2^* & -x_2 & x_1^* & x_1 \\
    -jx_4^* & -jx_4 & jx_3 & jx_3 & jx_2 & jx_2 & jx_1 - jx_1 \\
\end{bmatrix}
\]

\(5\)

Observe that for a fixed average power per codeword, due to the presence of zeros in \(G_3\), the peak power transmission in an antenna using \(G_3\) will be higher than that of an antenna using \(G_Y\). Hence, it is clear that the PAPR for the code \(G_Y\) is lower than that of the code \(G_3\). Hence, lower the fraction of zeros in a code, lower will be the PAPR of the code. In [8], [9], [10], another rate-1/2, 8 antenna code with no zero entry, denoted by \(G_{TWMS}\) shown at the top of this page, has been reported. Observe that \(G_{TWMS}\) has entries that are coordinated interleaved variables and hence has larger signaling complexity as explained in the following subsection.

\[\text{is not a COD and the number of zeros in (7) is zero. Notice that some of the entries of (7) can be written as}\]

\[
\begin{align*}
    (s_{11} - s_{12} - s_{12}^* - s_{22}) &= -(s_{11} - j s_{22}) = -s_{1,2}; \\
    (s_{11} - s_{22} + s_{22}^* + s_{12}) &= -(s_{21} - j s_{11}) = -s_{2,1}; \\
    (s_{11} + s_{22} + s_{22}^* + s_{12}) &= s_{21} + j s_{11} = s_{2,1}; \\
    (s_{11} + s_{22} - s_{22}^* - s_{12}) &= -(s_{11} + j s_{22}) = -s_{1,2}.
\end{align*}
\]

The code \(W_{TJC}\) reported in [1] is a NZE 4-antenna code and the NZE 4-antenna code \(W_{YGT}\) reported in [13] is

\[
\frac{1}{\sqrt{2}} \begin{bmatrix}
    s_{11} - s_{22} & s_{11}^* + s_{22}^* & s_{11}^* - s_{22}^* \\
    s_{11} + s_{22} & s_{11}^* - s_{22}^* & s_{11}^* + s_{22}^* \\
    s_{11} - s_{22} & s_{11}^* - s_{22}^* & s_{11}^* + s_{22}^* \\
    s_{11} + s_{22} & s_{11}^* + s_{22}^* & s_{11}^* - s_{22}^* \\
\end{bmatrix}. \tag{9}
\]

It is important to note that whenever the code matrix has entries with more than one complex variable like 8 of the 16 entries in (9), the number of possible transmitted values increases compared to having only one complex variable or its conjugate with or without negation. For example, if \(s_{11}\) and \(s_{22}\) take values from 16-QAM, 4 bits are needed to specify either one of them whereas 8 bits are needed to specify \(s_{11} - s_{22}\). We say that the signaling complexity in specifying \(s_{11} - s_{22}\) is more compared to specifying either \(s_{11}\) or \(s_{22}\) alone. In this sense, the signaling complexity of (9) is more than that of the code (1).

Whenever coordinate interleaving appears, as in (7), some of the entries are of the form \(s_{ij} \pm j s_{kQ}\) where \(i \neq k\). Now, suppose \(s_{11}\) and \(s_{22}\) take values from a unrotated square QAM constellation, say 16-QAM, \(\{(x, y) | x, y \in \{ \pm 1, \pm 3 \}\}\) for concreteness and illustration purposes. To specify a value taken by \(s_{11}\), one needs two look-up tables with four entries each, one to specify \(s_{11}\) and the other to specify \(s_{11}Q\). To specify a coordinate interleaved term like \(s_{11} + j s_{22}\), also one needs two look-up tables with four entries each, one to specify \(s_{11}\) and the other to specify \(s_{22}Q\). However, if one needs to rotate the 16-QAM constellation, for some purposes like guaranteeing full-diversity, then to specify the value taken by a term like \(s_{11}\) one needs a look up table with 16 entries to specify \(s_{11}\) and \(s_{11}Q\) uniquely specifies \(s_{11}Q\). This is true for coordinate interleaved terms also. Notice that a look up table with 16-entries need more memory/space than two look up tables with 4 entries each. In such cases also, we say that the signaling complexity increases. Since coordinate interleaving is a specific complex linear combination of variables as seen from (8) and designs using coordinate interleaving generally use rotated constellations for full-diversity and/or optimum coding gain, we say that designs that have entries that are linear combinations of several variables increase the signaling complexity of the design. Accordingly, the signaling complexity

\[\text{A. Signaling Complexity}\]

The code given in [1] obtained from Amicable Orthogonal Designs [11]
for 4-antennas obtained from (1). Notice that the signaling complexity of (9) is larger than that of (7), since there are 4 real variables involved in 8 entries of the matrix.

B. Contributions

Notice that by multiplying the matrix (1) with a unitary matrix the resulting matrix will continue to be a COD with lesser number of zeros and it is not difficult to locate unitary matrices that will result in a design with no zero entries. However, such a design is likely to have large signaling complexity which needs to be avoided. Obtaining a unitary matrix which reduces the number of zero entries while not increasing the signaling complexity is a nontrivial task which has been attempted in [15], [16] with partial success. It is known that there always exist codes with no zero entry for 2 antennas transmit antennas if $a + 1$ is a power of 2 [15]. For example, for 8 antennas, we have the scaled-COD $R_3$ with no zero entry

$$R_3 = \frac{1}{\sqrt{2}} \begin{bmatrix}
    x_1^+ & x_1^- & x_2^+ & x_2^- & x_3^+ & x_3^- & x_4^+ & x_4^- \\
    x_1 & -x_1 & x_2 & -x_2 & x_3 & -x_3 & x_4 & -x_4 \\
    -x_2 & x_2 & x_1 & -x_1 & -x_3 & x_3 & x_4 & -x_4 \\
    x_3 & -x_3 & x_4 & -x_4 & x_1 & -x_1 & -x_2 & x_2 \\
    -x_4 & x_4 & -x_3 & x_3 & -x_2 & x_2 & x_1 & -x_1 \\
    -x_1 & x_1 & -x_2 & x_2 & -x_3 & x_3 & -x_4 & x_4 \\
    x_2 & -x_2 & -x_1 & x_1 & x_3 & -x_3 & -x_4 & x_4 \\
    -x_3 & x_3 & x_4 & -x_4 & -x_1 & x_1 & x_2 & -x_2
\end{bmatrix}$$

In general, for $2^a$ antennas, there exists a scaled-COD (denoted by $R_a$) with fraction of zeros equal to $(1 - \frac{a+1}{2^{a+1}})$ for all $a$ [15]. It is clear that the above quantity is not equal to zero if $a + 1$ is not a power of 2. It is therefore important to construct codes with no zero entry for $2^a$ antennas when $a + 1$ is not a power of 2. However, it is known there exists a code for 4 transmit antennas with no zero entry [16] given by

$$L_2 = \begin{bmatrix}
    x_1 & -x_2 & x_1 & x_2 \\
    x_2 & -x_1 & x_2 & x_1 \\
    x_3 & x_4 & x_3 & x_4 \\
    x_4 & x_3 & x_4 & x_3
\end{bmatrix}$$

Note that the signaling complexity of the above code is slightly more than that of the code $R_3$ as all the non-zero entries in $R_3$ are variables or its conjugates (upto scaling) while some of the entries in $L_2$ contains co-ordinate interleaved variables. For 16 transmit antennas, there also exist a code with no zero entry given by (9) where $x_{i,k} = x_{i,l} + j x_{i,k} [14]$. Other than 4 and 16 antennas, no code is known for $2^a$ antennas where $a + 1$ is not a power of 2. Note that, in $L_2$, only $x_1$ and $x_2$ form co-ordinate interleaved variables denoted by $x_{1,2}, x_{2,1}$ whereas the other complex variable does not appear as coordinate interleaved with other variables. This particular observation is also valid for all the no zero entry designs constructed in this paper. We will come to this observation later when the method for the construction of such codes is described.

In this paper, we provide a general procedure to construct SCODs with no zero entries for any power of two number of antennas, with marginal increase in the signaling complexity. Our contributions are summarized as follows:

- Maximal-rate square CQDs with no zero entry for $2^a$ transmit antennas for any integer $a$.
- Our construction is based on the multiplication of the code in (1) by a suitable pre-multiplying and a post-multiplying matrix consisting of only $\pm \sqrt{\lambda}$ or 0 where $\lambda$ is a power of 2 and hence easy to construct. We give a closed form expression for these pre- and post-multiplying matrices.
- Only two variables of the design get coordinate interleaved and hence the increase in the signaling complexity compared to the one (if at all it existed) with no variables coordinate interleaved is very small.

The remaining content of the paper is organized as follows: In Section II, we prove the main result of the paper given by Theorem 1 and discuss the signaling complexity of the constructed codes. Simulation results are given in Section III and concluding remarks constitute Section IV.

II. CONSTRUCTION OF SCODS WITH NO ZERO ENTRY

In this section, we construct square CIS-CODs for any number of antennas, with no zero entry. We denote the constructed code by $L_{2^a}$.
are non-zero.

Our construction is based on the multiplication of the code given in (1) by a suitably chosen pre-multiplying and post-multiplying matrices so that the signaling complexity of the resulting code increases marginally when compared with the codes of (1). For illustration, there exists two unitary matrices \(P, Q\) of order 16 given by (11), which when multiplied with \(G_4\) give a code \(PG_4Q\) given by (12) in which none of the entries is zero. In both the matrices \(P\) and \(Q\), as well as in all the matrices throughout the paper, \(-1\) is represented by simply the minus sign.

In order to construct codes for any power of 2 number antennas with no zero entry, we introduce some notations:

Let \(\mathbb{F}_2\) be the finite field with two elements denoted by 0 and 1 with addition denoted by \(b_1 \oplus b_2\) and multiplication denoted by \(b_1b_2\) where \(b_1, b_2 \in \mathbb{F}_2\).

Let \(B\) be a finite subset of the set of natural numbers with \(c\) being its largest element and \(a\) being the smallest integer such that \(2^a > c\). We can always identify each element of \(B\) with an element of \(\mathbb{F}_2^c\) using the following correspondence: \(x \in B \rightarrow (x_a, \cdots, x_0) \in \mathbb{F}_2^c\) such that \(x = \sum_{j=0}^{c-1} x_j2^j, x_j \in \mathbb{F}_2\). The all zero vector and all one vector in \(\mathbb{F}_2^a\) are denoted by 0 and 1 respectively. For \(x \in B\), \(|x|\) denotes the Hamming weight of \(x\). Let \(x = (x_a, \cdots, x_0), y = (y_a, \cdots, y_0), x_i, y_i \in \mathbb{F}_2\) for \(i = 0, 1, \cdots, a - 1\). Let \(x \oplus y\) denote the component-wise modulo-2 addition of \(x\) and \(y\) respectively i.e.,

\[
x \oplus y = (x_a \oplus y_a, \cdots, x_0 \oplus y_0).
\]

Let \(Z_l = \{0, 1, \cdots, l - 1\}\). We identify \(Z_{2^a}\) with the set of \(a\)-tuple binary vectors \(\mathbb{F}_2^a\) in the standard way, i.e., any element of \(Z_{2^a}\) is identified with its radix-2 representation vectors (of length \(a\)). For convenience, the set \(Z_{2^a}\) is used as a collection of positive integers and sometimes as the set of vectors. For a set \(K \subset Z_{2^a}\) and \(m \in Z_{2^a}\), let \(|K|\) be the number of elements in the set \(K\) and \(m \oplus K := \{m \oplus a \mid a \in K\}\). For two sets \(A\) and \(B\), let \(A \setminus B = \{x \in A \mid x \notin B\}\). For two matrices \(A = [a_{ij}]\) and \(B\), the tensor product of \(A\) and \(B\) denoted by \(A \otimes B\), is the matrix \([a_{ij}B]\). For \(\alpha, \beta\) positive integers with \(\beta > \alpha\), define \([\alpha, \beta] := \{\alpha, \alpha + 1, \cdots, \beta\}\\mbox{.}

In the following, we construct the no zero entry code for \(2^a\) antennas in two steps: First, (i) we construct a code \(K_a\) from \(G_a\) such that the number of non-zero entries in \(K_a\) is a power of 2 and, then, (ii) we construct a code \(L_a\) with no zero entry from \(K_a\).

Define \(b = \lfloor \log_2(a) \rfloor + 1, m = 2^b - a - 1\) and \(q = a - 2^{b-1}\). It is clear that for all \(x \leq a\), we can express \(x\) as \(x = \sum_{j=0}^{b-1} x_j2^j\) with \(x_j \in \mathbb{F}_2\). Let

\[
P_a = \{0, 2^0, 2^1, \cdots, 2^{a-1}\},
\]

\[
Q_a = \begin{cases} 
\emptyset & \text{if } a + 1 \text{ is a power of } 2, \\
\{1 \oplus 2^m, 1 \oplus 2^{m+1}, \cdots, 1 \oplus 2^{a-1}\} & \text{otherwise},
\end{cases}
\]

and

\[
T_a = P_a \cup Q_a,
\]

Note that \(|T_a(i)| = 2^b\) for all \(i\). With

\[
W_a = \begin{bmatrix}
A_0 & 0 & \cdots & 0 \\
0 & A_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{2^{m-1}}
\end{bmatrix},
\]

\[
K_a = W_a G_a W_a^T,
\]

where

\[
A_i = \begin{cases} 
I_{2^{a-m}} & \text{if } ||i|| \text{ is even,} \\
I_{2^{a-m-1}} \otimes (\frac{1}{\sqrt{2}} H_2) & \text{if } ||i|| \text{ is odd}
\end{cases}
\]

and \(H_2 = \begin{bmatrix} 1 & 1 \\
-1 & 1 \end{bmatrix}\).

One nice property of the matrix \(K_a\) is that the number of non-zero entries in \(K_a\) is a power of 2. Let \(N_i^{(G_a)}, N_i^{(K_a)}\) be the set of the column indices of the non-zero entries in the \(i\)-th row of \(G_a\) and \(K_a\) respectively. It is known [15] that \(N_i^{(G_a)} = \{i\} \cup \{i \oplus 2^j \mid j = 0 \to a - 1\}\).

The following lemma describes the set \(N_i^{(K_a)}\).

**Lemma 1**: Let \(a\) be a positive integer, \(s \in Z_{2^a}\) and \(T_a(s)\) be as given by (10). Then \(N_i^{(K_a)} = T_a(s)\).

**Proof**: Let

\[
K_a = \begin{bmatrix}
K_{0,0} & K_{0,1} & \cdots & K_{0,2^{m-1}} \\
K_{1,0} & K_{1,1} & \cdots & K_{1,2^{m-1}} \\
\vdots & \vdots & \ddots & \vdots \\
K_{2^{m-1},0} & K_{2^{m-1},1} & \cdots & K_{2^{m-1},2^{m-1}}
\end{bmatrix}
\]

where \(K_{i,j}\) is a square matrix of order 2\(^{a-m}\) for \(0 \leq i, j \leq 2^{m-1}\). Similarly, we write \(G_a\) in the above form and let \((i,j)\)-th block matrix of \(G_a\) be \(G_{i,j}\). We have \(K_{i,j} = A_i G_{i,j} A_j\) and

\[
K_{i,j} = \begin{cases} 
G_{i,j} & \text{if } ||i|| \text{ even, } ||j|| \text{ even,} \\
(I_{2^{a-m-1}} \otimes H_2)G_{i,j} & \text{if } ||i|| \text{ even, } ||j|| \text{ odd,} \\
(I_{2^{a-m-1}} \otimes H_2)G_{i,j} & \text{if } ||i|| \text{ odd, } ||j|| \text{ even,} \\
(I_{2^{a-m-1}} \otimes H_2)G_{i,j} & \text{if } ||i|| \text{ odd, } ||j|| \text{ odd}
\end{cases}
\]

We now compute \(N_i^{(K_a)}\) as follows: Let \(s = 2^{a-m}r + t\). We now consider two cases: (i) \(|r|\) even and (ii) \(|r|\) odd.

Let \(a = 2^{a-m}j + \beta = 2^{a-m}j + 2^{a-m-1} - 1\). For the first case, we have

\[
N_i^{(K_a)} = (N_i^{(G_a)} \cap \alpha, \beta) \bigcup \left(\left( N_i^{(G_a)} \cup N_i^{(G_a)} \right) \cap \alpha, \beta \right)
\]

where

\[
Z = \bigcup_{j=0}^{2^{m-1}} \left(N_i^{(G_a)} \cap \alpha, \beta\right)
\]

with \(|T_a(i)| = 2^b\) for all \(i\). With

\[
W_a = \begin{bmatrix}
A_0 & 0 & \cdots & 0 \\
0 & A_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{2^{m-1}}
\end{bmatrix},
\]

where

\[
A_i = \begin{cases} 
I_{2^{a-m}} & \text{if } ||i|| \text{ is even,} \\
I_{2^{a-m-1}} \otimes (\frac{1}{\sqrt{2}} H_2) & \text{if } ||i|| \text{ is odd}
\end{cases}
\]

and \(H_2 = \begin{bmatrix} 1 & 1 \\
-1 & 1 \end{bmatrix}\).
For all \( x \), \( Two \) distinct rows of \( \tilde{A} \).

By simplifying the above expression, we have \( \sqrt{2} \).

\[ P = \frac{1}{\sqrt{2}} \]

\[ Q = \frac{1}{2} \]

\[ (11) \]

\[ \begin{bmatrix}
\sqrt{2}x_1 - \sqrt{2}x_2 & -x_3 & -x_4
\sqrt{2}x_1 - \sqrt{2}x_2 & -x_3 & -x_4
\sqrt{2}x_1 - \sqrt{2}x_2 & -x_3 & -x_4
\end{bmatrix} \]

TABLE I

\( M_a, \tilde{M}_a \) AND \( M_a' \) FOR \( a = 3, 4, \cdots, 9 \)

\[ \begin{array}{cccccccc}
| a | \{3\} & \{3\} & \{3, 4\} & \{3, 5, 6\} & \{3, 5, 6, 7\} & \{3, 5, 6, 7\} & \{3, 5, 6, 7, 9\} & \{3, 5, 6, 7\} \\
| M_a | \{3\} & \{3\} & \{3, 4\} & \{3, 5, 6\} & \{3, 5, 6, 7\} & \{3, 5, 6, 7\} & \{3, 5, 6, 7, 9\} & \{3, 5, 6, 7\} \\
| M_a' | \{3\} & \{6\} & \{3, 6\} & \{3, 5, 6\} & \{3, 5, 6, 7\} & \{6, 10, 12, 14\} & \{3, 5, 6, 7, 9\} & \{6, 10, 12, 14\} \\
| \tilde{M}_a | \{3\} & \{4\} & \{4, 6\} & \{4, 9\} & \{4, 12, 15\} & \{4, 12, 15\} & \{4, 12, 15\} & \{4, 12, 15\} \\
| x \leq a | \{2\} & \{3\} & \{3\} & \{3\} & \{3\} & \{3\} & \{3\} & \{3\} \\
\end{array} \]

\[ (12) \]

By simplifying the above expression, we have \( Z = s + 1 \oplus \{2^{a-m-1}, 2^{a-m+1}, \cdots, 2^{a-1}\} \).

Hence \( N_{\lambda/K_a}^{(s)}(K_a) = T_a^{(s)} \).

The proof for the case when \( ||s|| \) is odd is similar.

Two distinct rows of \( K_a \), say the \( s \)th and the \( t \)th are said to be \textit{non-intersecting} if \( N_{\lambda/K_a}^{(s)} \cap N_{\lambda/K_a}^{(t)} = \emptyset \).

Let \( M_a = \{0 < x \leq a \mid x \neq 2^k \text{ for any } k = 0, 1, \cdots\} \).

For all \( x \in M_a \), write \( x = \sum_{j=0}^{b-1} x_j 2^j \). Define a function \( g \) on \( M_a \) as follows:

\[ g(x) = \begin{cases} 
2x & \text{if } x_{b-1} = 0, \\
2x + 1 - 2^b & \text{if } x_{b-1} = 1.
\end{cases} \]

\[ (18) \]

\[ \tilde{M}_a = \{g(x) \mid x \in M_a\}. \]

Note that \( \tilde{M}_a = M_a \) for \( a = 2^b - 2, 2^b - 1 \). For all other values of \( a, M_a \neq \tilde{M}_a \). Let \( J_a = M_a \setminus \tilde{M}_a \) and \( H_a = \tilde{M}_a \setminus M_a \). It is shown in [17] that \( H_a \) is the smallest possible such \( \phi \).
values of \( a \), we have
\[
H_a = \{2^{\frac{a+1}{2}}, 2\left(\frac{a+1}{2}\right) + 1, \ldots, 2^{2b-1} - 1\},
\]
\[
J_a = \{2q + 3, 2q + 5, \ldots, 2^{\left\lfloor \frac{a}{2} \right\rfloor} - 1\}.
\]
Note that \( J_a \) is one-one. Now \( 2^{f(x)-1} \neq 2^{f(y)-1} \) for \( j = 0, 1, \ldots, b - 1 \) as \( f(x) \neq f(y) \) for all \( x \in M_a \). Therefore, \( \|x\| = 1 + \|x\| \) for all \( x \in M_a \) where \( \|x\| \) stands for the Hamming weight of \( x \). But \( \|x\| \geq 2 \) as \( x \) is not a power of 2, hence \( \|x\| \geq 3 \).

Similarly, \( \|x \times y\| = 2 + \|x\| + \|y\| \) for all \( x, y \in M_a \). Now \( \|x \times y\| \geq 1 \) as \( x \neq y \), which implies that \( \|x \times y\| \geq 3 \) for all \( x', y' \in M_a \). In general, \( \|y_1 \oplus y_2 \oplus \cdots \oplus y_k\| = k + \|y_1 \oplus y_2 \oplus \cdots \oplus y_k\| \) for \( k \leq a - b, y_1 \neq y_2 \neq \cdots \neq y_k \). So for all \( k \geq 3 \), \( k \leq a - b, \|y_1 \oplus y_2 \oplus \cdots \oplus y_k\| \geq 3 \). As \( \|h(x)\| = 3 \) for \( x = 3 \), the MHD of \( S \) is 3.

**Proof for Type-II**: Let \( y = \sum_{j=0}^{b-1} c_jy_j, c_j \in F_2 \), \( j \in M_a \). For all \( j \in M_a \), if \( y \neq 0, 1 \), then \( y \neq 0, 1 \) and \( y \neq 0, 1 \).

**Case (i)**: Let \( y = 1 + 2^a + 2^\alpha \in M_a \) with \( \alpha > 0 \). Then \( y = h(z) \) for some \( z \in M_a \) such that \( z_{b-1} = 1 \) and \( \|z\| = 2 \). Therefore, \( \|z\| = 2 \) for some \( \beta \in \{0, 1, \ldots, b - 2\} \). Moreover, \( \|z\| \leq a \) and hence \( \|z\| \leq a - 2^{b-1} \).

\[ \text{Now } y = h(z) = 2(2^{b-1} \oplus 2^\beta) - 1 + 2^{b-1} = 1. \]

**Case (ii)**: Let \( y = r + s' \) for some \( r', s' \in M_a \) where \( r' = h(r), s' = h(s) \) and \( r, s \in M_a \). Let \( r = \sum_{j=0}^{b-1} r_j 2^j, s = \sum_{j=0}^{b-1} s_j 2^j \). We have \( y = \alpha \) \( 2^f(r) - 1 \) \( \sum_{j=0}^{b-1} r_j 2^{b-j} \) \( \sum_{j=0}^{b-1} r_j 2^{b-j} \) \( \oplus \) \( s = \sum_{j=0}^{b-1} s_j 2^j \).

Without loss of generality, we can assume that \( s_{b-1} = 0, s_{b-1} = 1 \), therefore \( s = r \) and \( g(s) = g(r) \). Now \( f(s) = g(s) \) if \( s_{b-1} = 1 \) and \( f(r) \leq g(r) \). Therefore \( f(s) = 1 \) is not a power of 2 for all \( t \in M_a \). Therefore, \( f = 1 \) for all \( t \in S \).

**Case (iii)**: Suppose \( y = y_1 \oplus y_2 \oplus y_3 \) where \( y_i = h(y_i) \) for some \( y_i \in M_a, i = 1, 2, 3 \). As the Hamming weight of \( y \) is 3, we must have \( y = 2^{f(y_1) - 1} \oplus 2^{f(y_2) - 1} \oplus 2^{f(y_3) - 1} \). If \( y \) is of the form \( y = 1 + 2^a + 2^\alpha \), then \( f(y) = 1 \) for some \( i \in \{1, 2, 3\} \) which is not true as \( 1 \notin M_a \).
two distinct elements \( x, y \in C_j^{(a)} \) for any \( j \in \{0, 1, \cdots, 2^b - 1 \} \).

Proof: We identify the set \( Z_{2^b} \) with \( \mathbb{F}_2^b \) as before. Let \( M'_i \) be as given by (14) and \( S \) be the sub-space of \( Z_{2^b} \) spanned by the elements of \( M'_i \). We define a relation \( \sim \) on \( Z_{2^b} \) as follows: For all \( \alpha, \beta \in Z_{2^b} \), \( \alpha \sim \beta \) if \( \alpha + \beta \in S \). One can easily check that this relation is an equivalence relation. Moreover, the number of elements in any equivalence class is \( 2^{a-b} \) and hence the number of equivalence classes is \( \frac{2^b}{2^{a-b}} = 2^b \). Let \( x, y \in C_j^{(a)} \) for some \( j \in \{0, 1, \cdots, 2^b - 1 \} \) and \( x \neq y \). We show that \( T_a^{(x)} \cap T_b^{(y)} = \phi \). Now \( |T_a^{(x)} \cap T_b^{(y)}| = |x \oplus T_a^{(x)} \cap T_b^{(y)}| = |T_a^{(x)} \cap T_b^{(x)}| \). But \( x \oplus y \in S \) and \( x \oplus y \neq 0 \). By Lemma 3, \( T_a^{(x)} \cap T_b^{(y)} = \phi \).

The above lemma is used to prove the main theorem of the paper given below.

**Theorem 1:** Let \( a \) be any non-zero positive integer and \( K_a \) be the matrix given by (13). Let \( B_i \) be a \( 2^{a-b} \times 2^a \) matrix formed by the rows of \( K_a \) indexed by the elements of \( C_i^{(a)} \) for \( i = 0 \) to \( 2^b - 1 \). Let \( B_i = H B_i \) where \( H \) is a Hadamard matrix of order \( 2^a \). Define

\[
L_a = 2^{-\frac{a+1}{2}} \begin{bmatrix} \bar{B}_0 \\ \bar{B}_1 \\ \vdots \\ \bar{B}_{2^b-1} \end{bmatrix}.
\]  

The matrix \( L_a \) is a rate-\( \frac{a+1}{2a} \) code with no zero entry for \( 2^a \) transmit antennas.

Proof: Let

\[
B' = \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{2^b-1} \end{bmatrix}.
\]  

and \( \bar{H} = I_{2^b} \otimes H \). The matrix \( B' \) is related to \( K_a \) by \( B' = PK_a \) where \( P \) is a permutation matrix of size \( 2^a \times 2^a \). We have \( L_a = 2^{-\frac{a+1}{2}} \bar{H} B' = 2^{-\frac{a+1}{2}} \bar{H} PK_a \). \( L_a \) is a CISTCOD as \( 2^{-\frac{a+1}{2}} \bar{H} P \) is a unitary matrix. By Lemma 1 and Lemma 3, the number of non-zero elements in any row of \( B' \) is \( 2^b \) and the number of non-zero elements in any row of \( L_a \) is \( 2^{a-b} \cdot 2^b = 2^a \). Therefore, all the entries in \( L_a \) are non-zero.

It is clear that \( L_a = 2^{-\frac{a+1}{2}} \bar{H} PK_a = 2^{-\frac{a+1}{2}} \bar{H} PW_{a} G_{a} W_{a} \). Let \( U_{a} = 2^{-\frac{a+1}{2}} \bar{H} PW_{a} \). We have \( L_a = U_{a} G_{a} W_{a} \). For \( 2^5 \) transmit antennas, the pre-multiplying matrix \( U_5 \) and post-multiplying matrix \( W_5 \) are displayed in Fig. 5 and Fig. 6 respectively. The code \( 2L_5 \) is displayed in Fig. 7.

## III. Simulation Results

The symbol error rate performance of the code with no zero entry constructed in this paper (denoted as NZCOD) in the plots which means COD with No Zero) for 16 antennas is compared with the code with 37.5% zeros (denoted as RZCOD) and the code with 68.75% zeros (denoted as SCOD) of same order in Fig. 1 under peak power constraint. Similarly, in Fig. 2 the performance comparison of the corresponding codes under average power constraint is shown. The average power constraint performance of NZCOD matches with that of the RZCOD and SCOD, while the NZCOD performs better than the other two codes under peak power constraint as seen in Fig. 1. Similarly, for 32 antennas, the performance comparison shown in Fig. 3 and in Fig. 4 establish the fact that the NZCOD performs better than the others under peak power constraint while under average power constraint, all the codes perform identically.

## IV. Conclusion

We have constructed square complex orthogonal designs for all power of \( a \) antenas such that none of the entries in the matrix is zero. These codes have significant advantage over the existing codes in term of PAPR as the existing codes has zeros in its matrices. The only sacrifice that is made in the construction of these codes is that the signaling complexity of the these codes is marginally greater than the existing codes (with zero entries) as some of the entries in the codes of this paper consist of co-ordinate interleaved variables. An interesting direction to pursue is to investigate whether it is possible to construct codes with no zero-entry and also having lesser signaling complexity than the ones constructed in this paper. We conjecture that such codes do not exist.

## References

[1] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, “Space-time block codes from orthogonal designs,” IEEE Trans. Inform. Theory, vol. 45, pp. 1456-1467, July 1999.

[2] O. Tirkkonen and A. Hottinen, “Square matrix embeddable STBC for complex signal constellations Space-time block codes from orthogonal design,” IEEE Trans. Inform. Theory, Vol 48, no. 2, pp. 384-395, Feb. 2002.

[3] X. B. Liang “Orthogonal Designs with Maximal Rates,” IEEE Trans.Inform. Theory, Vol.49, pp. no. 10, 2468-2503, Oct. 2003.

[4] J. F. Adams, P. D. Lax, and R. S. Phillips, “On matrices whose real linear combinations are nonsingular,” Proc. Amer. Math. Soc., vol. 16, 1965, pp. 318-322.

[5] T. Jozefiak, “Realization of Hurwitz-Radon matrices,” Queen’s Papers on Pure and applied Mathematics, vol. 36, pp. 346-351.

[6] W. Wolfe, Amicable Orthogonal Designs-existence, Canadian J. Mathematics, vol.28, no.5, pp.1006-1020, 1976.

[7] X. Yuen, Y.L. Guan and T. T. Tjhung, “Orthogonal space time block code from amicable orthogonal design,” Proc. IEEE. Int. Conf. Acoustic, Speech and Signal, 2004.

[8] L. C. Tran, T. A. Wysocki, A. Mertins and J. Seberry, Complex Orthogonal Space-Time Processing in wireless communications, Springer, 2006.

[9] J. Seberry, L. C. Tran, Y. Wang, B. J. Wysocki, T. A. Wysocki, T. Xia and Y. Zhao, “New complex orthogonal space-time block codes of order eight,” in T.A.Wysocki, B.Honary and B.J.Wysocki, editors, Signal Processing for Telecommunications and Multimedia, Vol.27 of Multimedia systems and applications, pp.173-182, Springer, New York, 2004.

[10] Y. Zhao, J. Seberry, T. Xia, Y. Wang, B. J. Wysocki, T. A. Wysocki, L. C. Tran, “Amicable orthogonal designs of order 8 for complex space-time block codes,” To appear in Australian Journal of Combinatorics (AJC).

[11] A. V. Geramita and J. Seberry, Orthogonal Designs: Quadratic forms and Hadamard matrices, Vol.43, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York and Basel, 1979.

[12] Zafar Ali Khan and B. Sundar Rajan, “Single-Symbol Maximum-Likelihood Decodable Linear STBCs,” IEEE Trans. Inform. Theory, Vol.52, No.5, May 2006, pp.2062-2091.

[13] C. Yuen, Y. L. Guan and T. T. Tjhung, “Power-Balanced Orthogonal Space-Time Block Code,” To appear in IEEE Trans. Vehicular Technology.

[14] Ankita Pramanik, B. Sundar Rajan, Smarajit Das, “A Square complex Orthogonal Design with No Zero Entry for 36 Antennas,” Submitted to IEEE Trans. Inform. Theory
Fig. 1. The performance of the NZCOD, RZCOD and SCOD for 16 Transmit antennas using QAM modulation.

Fig. 2. The performance of the NZCOD, RZCOD and SCOD for 16 Transmit antennas using QAM modulation.

Fig. 3. The performance of the NZCOD, RZCOD and SCOD for 32 Transmit antennas using QAM modulation.

Fig. 4. The performance of the NZCOD, RZCOD and SCOD for 32 Transmit antennas using QAM modulation.

[15] Smarajit Das and B. Sundar Rajan, “Square Complex Orthogonal Designs with Low PAPR and Signaling Complexity,” To appear in IEEE Transactions on Wireless Communications. Also available as arXiv:0807-4128v1 [cs.IT] 25 Jul 2008.

[16] Smarajit Das and B. Sundar Rajan, “A Class of Maximal-Rate, Low-PAPR, Non-square Complex Orthogonal Designs,” Submitted to IEEE Transactions on Wireless Communication. Also available as arXiv:0808.1400v [cs.IT] 10 Aug 2008.
Fig. 5. The pre-multiplying matrix $U_5$ for 32 antennas where $s = \frac{1}{\sqrt{2}}$. 
**Fig. 6.** The post-multiplying matrix $W_5$ for 32 antennas where $s = \frac{1}{\sqrt{2}}$.
Fig. 7. The \([32,32,6]\) code \(L_5\) with no zero entry where \(y_5 = \frac{x_5}{\sqrt{2}}, y_6 = \frac{x_6}{\sqrt{2}}, \tilde{x}_1 = x_{1,2}\) and \(\tilde{x}_2 = x_{2,1}\)