 Thermal noise of whispering gallery resonators

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By direct application of the fluctuation-dissipation theorem, we numerically calculate the fundamental dimensional fluctuations of crystalline CaF$_2$ whispering gallery resonators in the case of structural damping, and the limit that this noise imposes on the frequency stability of such resonators at both room and cryogenic temperatures. We analyze elasto-optic noise - the effect of Brownian dimensional fluctuation on frequency via the strain-dependence of the refractive index - a noise term that has so far not been considered for whispering-gallery resonators. We find that dimensional fluctuation sets a lower limit of $10^{-16}$ to the Allan deviation for a 10-millimeter-radius sphere at 5 K, predominantly via induced fluctuation of the refractive index.

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I. INTRODUCTION

The properties of a body in equilibrium at a finite temperature undergo continuous fluctuation about their mean values. The fluctuation-dissipation theorem of Callen and Welton \cite{1} established the relationship between these fluctuations and the dissipative properties of the system. This fundamental relationship is of great practical importance in precision measurements, such as the measurement and stabilization of the frequency of laser light using high-quality-factor optical cavities. Thermal noise sets a limit on the frequency stability of optical cavities, and hence on their performance in such applications.

The thermal noise of conventional Gaussian-mode optical cavities has been analyzed in detail \cite{2, 3, 4}, providing guidance in the design of such cavities. A relatively new type of high-finesse optical cavity is the whispering-gallery resonator (WGR) \cite{6}, in which light is confined inside an axially-symmetric dielectric object by continuous-total internal reflection. Solid dielectric optical WGRs were quickly recognized as potentially useful \cite{7} due to their high quality factors and small mode volumes, resulting in an intense confined coherent optical field. The lowest cavity loss demonstrated so far resulted in a linewidth of about 1 kHz in pure crystalline CaF$_2$ \cite{8}, making these resonators very attractive candidates for precision laser frequency stabilization. Understanding thermal noise in such resonators is critical for this application. For a WGR, both the fluctuation of the radius and of the refractive index must be considered. We present a calculation, by direct application of the fluctuation-dissipation theorem, of the frequency instability due to fundamental thermal fluctuation of the dimensions of crystalline CaF$_2$ resonators.

Initial estimates of thermal noise in WGRs have been made \cite{9, 10}, in particular for thermorefractive noise, the fluctuation of refractive index due to fundamental temperature fluctuations. The thermorefractive noise has recently been measured for crystalline resonators at room temperature \cite{11}. The thermal noise due to fundamental dimensional fluctuations at a fixed temperature has been less investigated. For laser frequency stabilization, the fluctuations on timescales of 0.01 s to hundreds of seconds are of interest, for applications such as clock lasers for optical clocks \cite{12} or tests of fundamental symmetries such as Lorentz invariance \cite{13}. We find that at low temperatures the thermal noise due to dimensional fluctuation dominates over the thermal noise due to fundamental temperature fluctuation. We find that the dominant component of the dimensional-fluctuation thermal noise is due to induced changes of the refractive index, via the elasto-optical effect.

To calculate the thermal noise we follow the direct approach introduced by Levin \cite{2} for standard two-mirror resonators, adapting it to WGRs. We perform the calculations numerically, by finite-element analysis, for sphere- and disk-shaped resonators.

II. CALCULATION

A. Temperature fluctuation

Our calculations reported in this paper are focused on the fundamental dimensional fluctuations that occur at a fixed temperature. However, to determine the expected dominant thermal noise terms in different regimes and to make contact with existing thermal noise results for WGRs, we compare our calculated noise levels to thermal noise arising from fundamental temperature fluctuations at a given mean temperature. These cause WGM frequency fluctuation via the dependence of refractive index on temperature (thermorefractive noise, TR) and thermal expansion of the resonator (thermoelastic noise). We evaluate the thermorefractive noise of a sphere using the expression of Gorodetsky and Grudinin \cite{10, 14},

$$S_{d_f} = \frac{8k_BT^2\Gamma(R,f,T)}{\pi^2\gamma(T)R} \left(\frac{1}{n}\frac{dn}{dT}\right)^2$$

(1)
FIG. 1. (a) Simulated mode profile of a CaF$_2$ disk, as shown by the electric field intensity (arbitrary units). The radius ($R$), surface radius of curvature ($S$) and thickness are all 1 mm. (b) Simulated strain energy density distribution in the same disk. (c) Geometries used for disk simulation.

$$\sigma_{TR}(\tau) = \frac{2T}{\pi} \sqrt{\frac{k_B \Gamma(R)}{\gamma(T)R\tau}} \left( \ln \frac{1}{n dT} \right)$$

where $\Gamma$ evaluates to 0.847 for a 1-millimetre-radius CaF$_2$ sphere at low frequency and for the entire temperature range of interest, and $\gamma$ is the thermal conductivity.

B. Brownian boundary noise

The fluctuation-dissipation theorem says that the power spectral density of fluctuations of coordinate $x$ of a body, at frequency $f$ is

$$S_x(f) = \frac{k_B T}{\pi f^2} |\text{Re}[Y(f)]|$$

where $Y(f) = 2\pi i f x(f)/F(f)$ is the susceptibility for displacement $x(f)$ under action of the force conjugate to it, $F(f)$. Levin [3] noted that

$$|\text{Re}[Y(f)]| = \frac{2W_{\text{diss}}(f)}{F^2}$$

where $W_{\text{diss}}$ is the dissipation when an oscillating force of magnitude $F$ conjugate to the desired displacement and of frequency $f$ is applied. The cycle-averaged dissipation can be written as

$$W_{\text{diss}}(f, T) = 2\pi f U(f) \phi(f, T)$$

where $U$ is the strain energy in the body at the temporal maximum of the applied force and $\phi(f, T)$ is the loss angle of the material. We assume that $\phi$ is constant with respect to frequency (structural damping). The constant loss-angle model is supported by experimental measurements on ultra-low-expansion material (ULE) Fabry-Perot cavities in the frequency range 1 Hz down to 3 mHz [2]. We thus have

$$S_x(f) = \frac{4 k_B T U(f) \phi(T)}{\pi f F^2}$$

$$\rightarrow S_{dR/R}(f) = \frac{4 k_B T U \phi(T)}{R^2 f F^2}$$

where in (7) we have specialized to the case $x = R$ and divided by the radius squared to get the spectral density of relative radius fluctuations, and the $f$-dependence of $U$ has been dropped (static approximation). Since the relative change of the WGR eigenmode frequency $\nu$ with mode radius $R$ is $d\nu/\nu = dR/R [15]$, the Allan deviation of the relative resonance frequency fluctuations for this $1/f$ spectrum is independent of integration time,

$$\sigma_{BB} = \left[ \frac{8\ln(2)}{\pi} \right]^{1/2} \frac{\sqrt{k_B T U \phi(T)}}{RF}$$

We refer to this type of thermal noise as “Brownian boundary” (BB) noise.

Thus, to calculate the fluctuation of the mode path length, we apply an oscillating force conjugate to a stretching of this path. This is a force along the circumferential optical path in the interior of the resonator, with a transverse spatial distribution identical to the mode intensity profile. A circumferential (ring) load is elastically equivalent to a radial one [15]. As the mode is very close to the surface of the resonator, the load can be closely approximated by a surface radial stress distribution with the profile of the 1-dimensional (radial) integral of the mode profile. This is the load that we apply.

The calculation is carried out in two steps. First, the electromagnetic field of a fundamental whispering gallery mode (chosen to be at $\sim 1.9152 \times 10^{14}$ Hz (1565 nm) for concreteness) is simulated using the PDE module of the finite-element package COMSOL [17], using weak-form partial differential equations following the procedure of Oxborrow [18, 19]. The field intensity along the vertical midline of a 2D cross-section of the mode is then fit by a gaussian. The field intensity distribution in the radial direction is also fitted, for use in the elasto-optic calculation described in the next subsection. These fitted profiles are the desired information from the electromagnetic simulation.

The second step is to calculate the strain energy stored in the resonator when a load with the same profile as
the electromagnetic mode intensity is applied to the resonator surface. We do this for a resonator with its axis along the [111] crystal direction, and in the static approximation (i.e. material acceleration is neglected), accurate for low-frequency noise. The calculation is carried out using COMSOL’s 3D structural mechanics module. The strain energy is integrated over the body to give \( F \), indicating that the force magnitudes used were sufficient.

In the strain energy simulation, deformation patterns observed corresponded well to the applied load profiles, indicating that the force magnitudes used were sufficient. Numerical values of CaF\(_2\) properties used in the calculation are listed in Appendix A.

C. Elasto-optic noise

The dimensional fluctuation of the resonator also affects the mode frequency indirectly via the the elasto-optic effect, the dependence of the refractive index on strain. Relative fluctuations of different points in the resonator material correspond to fluctuating strains, and we have used the approximation (i.e. material acceleration is neglected), accurate for low-frequency noise. The calculation is carried out using COMSOL’s 3D structural mechanics module. The strain energy is integrated over the body to give \( F \), indicating that the force magnitudes used were sufficient.

in this estimate the radius fluctuations will be averaged over the entire spherical surface rather than over only the mode region. For a sphere subject to a constant pressure on its surface. We expect that this will yield a dimensional fluctuation of the resonator also affecting the mode frequency indirectly via the the elasto-optic effect, the dependence of the refractive index on strain. Relative fluctuations of different points in the resonator material correspond to fluctuating strains, and we have used the approximation (i.e. material acceleration is neglected), accurate for low-frequency noise. The calculation is carried out using COMSOL’s 3D structural mechanics module. The strain energy is integrated over the body to give \( F \), indicating that the force magnitudes used were sufficient.

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applied force is \( F = 4\pi R^2 P \), thus we estimate

\[
\tilde{S}_{dR/R}(f) = \frac{1}{3\pi^2} \frac{k_B T \phi(T)}{\kappa R^3}
\]

(16)

\[
\sigma_{BB}(\tau) = \left( \frac{2(3)\ln(2)}{\pi} \right)^{1/2} \sqrt{\frac{k_B T \phi(T)}{\kappa R^3}}
\]

(17)

For a 1-millimetre-radius sphere at 5.5 K, with \( \phi(5K) = 2 \times 10^{-8} \) \( \text{[24]} \), this gives \( \sigma_{BB} = 3 \times 10^{-17} \) for CaF\(_2\) (\( \kappa = 90 \) GPa), a factor of 2.5 smaller than the value determined by more detailed calculations below.

**Elasto-optic noise** - We can model the WGM volume as a tube of length \( 2\pi R \) and radius \( r \). We estimate \( r \) from \( V_m = 2\pi R r^2 \), where the mode volume \( V_m \) is given by \( V_m \approx 3.4\pi^{3/2}(\lambda/n)^{7/6} R^{11/6} \), where \( \lambda \) is the optical wavelength,

\[
r \approx 0.335(\lambda/n)^{7/12} R^{5/12}
\]

(18)

We calculate the strain energy that results from a uniform pressure \( P \) applied to the cylindrical surface of the tube, approximating the WGR material as isotropic. Applying the plane-strain condition, the non-zero stresses in the tube are \( \sigma_{rr} = \sigma_{\theta\theta} = -P, \sigma_{zz} = -2\mu P \), where \( \mu \) is Poisson’s ratio, and the resulting strain energy is

\[
U = \pi^2 r^2 R P^2 \left[ \frac{3}{3K + G} \right]
\]

(19)

where \( G \) is the shear modulus. The applied force is \( F = (2\pi)^2 r R P \), and we get

\[
\tilde{S}_{dR/R}(\tau) = \frac{\sqrt{6(\ln 2)}}{2\pi^{7/2}} \left[ \frac{k_B T \phi}{r^2 R (3K + G)} \right]^{1/2}
\]

\[
= 0.55 \left( \frac{n}{\lambda} \right)^{7/12} R^{-11/12} \left( \frac{k_B T \phi}{3K + G} \right)^{1/2}
\]

(20)

\[
\sigma_{EO}(\tau) \approx 0.55 \left( \frac{n^{31/12}}{\lambda^{11/12} R^{11}} \right) \left( \frac{k_B T \phi}{3K + G} \right)^{1/2} \frac{1}{n} \frac{P}{2\pi R^2}
\]

(21)

For a 1-mm-radius resonator, at \( \lambda = 1.56 \mu m \) and with \( G = 45 \) GPa, this numerically evaluates at 5.5 K to \( 7 \times 10^{-16} \), close to the more accurately calculated result below.

**IV. RESULTS**

We perform the calculation for sphere and disk radii ranging from 0.1 mm to 10 mm, and for disk surface radii of curvature ranging from 0.1 mm to 10 mm. The disk shapes used are shown in Fig. \( \text{[11]} \). To numerically evaluate the Allan deviation with equations \( \text{[8]} \) and \( \text{[14]} \), we must know the material loss angle \( \phi \). Nawrodt et. al. \( \text{[24]} \) measured the \( Q \) of mechanical modes of a pure crystalline CaF\(_2\) cylinder as a function of temperature. In the cylinder’s fundamental drum mode, material motion is concentrated near the cylinder axis and away from the experimental support. We therefore take the reported \( Q \) of the cylinder’s fundamental drum mode as representative of the bulk material loss and therefore as the limiting value achievable for losses. The \( Q \) of this mode ranged from \( 10^7 \) to \( 5 \times 10^8 \) between 5 K and 300 K. Using this loss angle \( \phi = 1/Q = (2 \times 10^{-8}, 5 \times 10^{-8}) \) (5 K, 300 K values), we obtain the Allan deviation values shown in Table \( \text{[1]} \). Uncertainty in \( \phi \) is the principal uncertainty in our calculation. For different values of \( \phi \) the BB and EO noise terms scale as \( \phi^{1/2} \).

**TABLE I. Calculated Brownian boundary and elasto-optic noise for CaF\(_2\) spheres and disks at 300 K and 5 K.**

| R, S (mm) | \( \sigma_{BB} \) | \( \sigma_{EO} \) | \( \sigma_{BB} \) | \( \sigma_{EO} \) |
|----------|----------------|----------------|----------------|----------------|
| **Spheres** | | | | |
| 0.1 | \( 2 \times 10^{-14} \) | \( 9 \times 10^{-14} \) | \( 2 \times 10^{-15} \) | \( 8 \times 10^{-15} \) |
| 1.0 | \( 8 \times 10^{-16} \) | \( 1 \times 10^{-14} \) | \( 7 \times 10^{-17} \) | \( 1 \times 10^{-15} \) |
| 10 | \( 3 \times 10^{-17} \) | \( 2 \times 10^{-15} \) | \( 3 \times 10^{-18} \) | \( 1 \times 10^{-16} \) |
| **Disks** | | | | |
| 0.1, 0.15 | \( 2 \times 10^{-14} \) | \( 9 \times 10^{-14} \) | \( 2 \times 10^{-15} \) | \( 8 \times 10^{-15} \) |
| 1.0, 0.15 | \( 9 \times 10^{-16} \) | \( 1 \times 10^{-14} \) | \( 8 \times 10^{-17} \) | \( 1 \times 10^{-15} \) |
| 10, 0.15 | \( 5 \times 10^{-17} \) | \( 2 \times 10^{-15} \) | \( 4 \times 10^{-18} \) | \( 2 \times 10^{-16} \) |
| 1.0, 0.1 | \( 9 \times 10^{-16} \) | \( 1 \times 10^{-14} \) | \( 8 \times 10^{-17} \) | \( 1 \times 10^{-15} \) |
| 1.0, 1.0 | \( 9 \times 10^{-16} \) | \( 1 \times 10^{-14} \) | \( 8 \times 10^{-17} \) | \( 1 \times 10^{-15} \) |
| 1.0, 10 | \( 9 \times 10^{-16} \) | \( 1 \times 10^{-14} \) | \( 7 \times 10^{-17} \) | \( 1 \times 10^{-15} \) |
extrapolated noise levels at the $10^{-17}$ level for a 10-mm-radius sphere at 25 mK and for a 10-cm-radius sphere at 1.7 K, where we have used the 5 K value of $\phi$.

V. CONCLUSION

We have calculated the thermal noise limit to the frequency stability of crystalline CaF$_2$ whispering-gallery resonators set by dimensional fluctuations, by direct application of the fluctuation-dissipation theorem and using the widely-applied structural damping model. We have identified a new source of thermal noise in whispering gallery resonators, due to the elasto-optic effect. This noise is smaller than the thermorefractive noise at room temperature, so it has not been observed in previous room-temperature experiments. At temperatures sufficiently below room temperature (e.g. 50 K), the elasto-optic noise dominates. The total thermal noise is reduced by increasing the resonator size and reducing the temperature. WGRs have already been operated at cryogenic temperatures,$^{23,24}$ and therefore an experimental study of the noise at low temperatures should be feasible. One important result of such studies will be a determination of the loss angle $\phi$, which can at present only be estimated. For achieving extremely low total noise, it would be favourable to operate large resonators ($R \geq 50$ mm) at temperatures $\leq 100$ mK, where a noise level $\leq 4 \times 10^{-17}$ is estimated from the present work.

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Appendix A: Material properties

| Property | Value | Ref. |
|----------|-------|------|
| Refractive index $(n)$ | 1.43 | $^{[27]}$ |
| Elastic constants: | | |
| $C_{11}$ | 164 GPa | $^{[28]}$ |
| $C_{12}$ | 53.0 GPa | |
| $C_{44}$ | 33.7 GPa | |
| Thermal conductivity ($\gamma$): | | |
| 300 K | 9.71 Wm$^{-1}$K$^{-1}$ | $^{[29]}$ |
| 5.5 K | 1200 Wm$^{-1}$K$^{-1}$ | |
| Thermorefractive index $|\langle(1/n)(dn/dT)\rangle|$: | | |
| 5.5 K | 1.5 x $10^{-9}$ | $^{[27]}$ |
| Loss angle ($\phi$): | | |
| 300 K | 5 x $10^{-8}$ | $^{[24]}$ |
| 5.5 K | 2 x $10^{-8}$ | |
| Elasto-optic constants: | | |
| $p_{11}$ | 0.039 | $^{[30]}$ |
| $p_{12}$ | 0.223 | |
| $p_{44}$ | 0.051 | |
| Thermal expansion coefficient ($\alpha$): | | |
| 295 K | 1.89 x $10^{-3}$ K$^{-1}$ | $^{[31]}$ |

$^a$ Extrapolation from 20 K data using $T^3$ dependence.

Appendix B: Calculation results

We tabulate here more detailed results of the numerical calculation.

a. Electromagnetic modeling
TABLE III. Modeled fundamental electromagnetic modes: sphere

| Sphere radius (mm) | Mode azimuthal index | Mode frequency (10^{14} Hz) | Mode polar intensity 1/e^2 half-width (μm) |
|-------------------|----------------------|-----------------------------|------------------------------------------|
| 0.1               | 559                  | 1.9164                      | 4.25                                     |
| 1                 | 5706                 | 1.9152                      | 13.5                                     |
| 10                | 1.9149               | 42.0                        |                                          |

TABLE IV. Modeled fundamental electromagnetic modes: disk

| Disk radius of curvature (mm) | Mode azimuthal index | Mode frequency (10^{14} Hz) | Mode polar intensity 1/e^2 half-width (μm) |
|-------------------------------|----------------------|-----------------------------|------------------------------------------|
| 0.1                          | 0.15                 | 559                         | 1.9156                                   |
| 1                            | 0.15                 | 5706                        | 1.9152                                   |
| 10                           | 0.10                 | 57320                       | 1.9151                                   |
| 1                            | 1                    | 5707                        | 1.9151                                   |
| 1                            | 10                   | 5708                        | 1.9155                                   |

b. Strain energy calculation

TABLE V. BB noise calculation for CaF₂ spheres. \( R \): sphere radius, \( w_s \): mode radial 1/e^2 intensity half-width, \( U \): strain energy \( 10^6 \) N/m² peak load, \( F \): total force, \( \sigma_{dR/R} \): Allan deviation for \( \phi = 2 \times 10^{-8} \).

| \( R \) (mm) | \( w_s \) (μm) | \( U \) (J) | \( F \) (N) | \( \sigma_{dR/R} \) at 5.5 K |
|-------------|----------------|-------------|-------------|-----------------------------|
| 0.1         | 4.25           | 3.2×10^{-14}| 0.0047     | 2.0×10^{-16}               |
| 1           | 13.5           | 4.4×10^{-11}| 0.15        | 7.2×10^{-17}               |
| 10          | 42.0           | 5.3×10^{-9} | 4.7         | 2.6×10^{-18}               |

TABLE VI. BB noise calculation for CaF₂ disks. \( R \): disk radius, \( S \): surface radius of curvature.

| \( R, S \) (mm) | \( w_s \) (μm) | \( U \) (J) | \( F \) (N) | \( \sigma_{dR/R} \) at 5.5 K |
|----------------|----------------|-------------|-------------|-----------------------------|
| 0.1, 0.15      | 4.6            | 3.8×10^{-14}| 0.0051     | 2.0×10^{-16}               |
| 1, 0.15        | 8.2            | 1.9×10^{-11}| 0.091      | 7.7×10^{-17}               |
| 10, 0.15       | 14.1           | 1.8×10^{-9} | 1.6        | 4.5×10^{-18}               |
| 1, 0.1         | 6.7            | 1.3×10^{-11}| 0.075      | 7.9×10^{-17}               |
| 1, 1           | 11.3           | 3.3×10^{-11}| 0.13       | 7.5×10^{-17}               |
| 1, 10          | 13.0           | 4.2×10^{-11}| 0.14       | 7.3×10^{-17}               |

TABLE VII. EO noise calculation for CaF₂ spheres. \( w_p \): mode radial 1/e^2 intensity half-width, \( \rho_0 \): radial position of mode center.

| \( R \) (mm) | \( w_p \) (μm) | \( \rho_0 \) (mm) | \( U \) (J) | \( F \) (N) | \( \sigma_{dR/R} \) at 5.5 K |
|-------------|----------------|-------------------|-------------|-------------|-----------------------------|
| 0.1         | 4.25           | 1.1               | 0.0986      | 7.1×10^{-14}| 0.0088                     |
| 1           | 13.5           | 2.5               | 0.996       | 4.6×10^{-11}| 0.65                      |
| 10          | 42.0           | 5.0               | 9.99        | 2.3×10^{-8} | 41                        |

TABLE VIII. EO noise calculation for CaF₂ disks.

| \( R, S \) (mm) | \( w_p \) (μm) | \( \rho_0 \) (mm) | \( U \) (J) | \( F \) (N) | \( \sigma_{dR/R} \) at 5.5 K |
|----------------|----------------|-------------------|-------------|-------------|-----------------------------|
| 0.1, 0.15      | 4.6            | 1.1               | 0.986       | 8.8×10^{-14}| 0.0094                     |
| 1, 0.15        | 8.2            | 2.5               | 0.997       | 1.2×10^{-11}| 0.39                      |
| 10, 0.15       | 14.1           | 5.3               | 0.999       | 1.5×10^{-9} | 14                         |
| 1, 0.1         | 6.7            | 2.5               | 0.997       | 6.7×10^{-12}| 0.31                      |
| 1, 1           | 11.3           | 2.5               | 0.997       | 2.9×10^{-11}| 0.53                      |
| 1, 10          | 13.0           | 2.5               | 0.997       | 4.2×10^{-10}| 0.61                      |

TABLE VIII. EO noise calculation for CaF₂ disks.
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