The Structure of Maximum Subsets of \( \{1, \ldots, n\} \) with No Solutions to \( a + b = kc \)

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Abstract

If \( k \) is a positive integer, we say that a set \( A \) of positive integers is \( k \)-sum-free if there do not exist \( a, b, c \) in \( A \) such that \( a + b = kc \). In particular we give a precise characterization of the structure of maximum sized \( k \)-sum-free sets in \( \{1, \ldots, n\} \) for \( k \geq 4 \) and \( n \) large.

1 Introduction

A set of positive integers is called \( k \)-sum-free if it does not contain elements \( a, b, c \) such that

\[ a + b = kc, \]

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where $k$ is a positive integer. Denote by $f(n, k)$ the maximum cardinality of a $k$-sum-free set in $\{1, \ldots, n\}$. For $k = 1$ these extremal sets are well-known: Deshoulliers, Freiman, Sós, and Temkin [1] proved in particular that the maximum 1–sum-free sets in $\{1, \ldots, n\}$ are precisely the set of odd numbers and the “top half” $\left\{\left\lceil \frac{n+1}{2} \right\rceil, \ldots, n \right\}$. For $n > 8$ even $\left\{\frac{n}{2}, \ldots, n-1 \right\}$ forms the only additional extremal set. The famous theorem of Roth [4] gives $f(n, 2) = o(n)$. Chung and Goldwasser [2] solved the case $k = 3$ by showing that the set of odd integers is the unique extremal set for $n > 22$. For $k \geq 4$ the y give an example of a $k$-sum-free set [3] of cardinality $\frac{k(k-2)}{k^2-2}n + \frac{8(k-2)}{k(k^2-2)(k^4-2k^2+4)}n + O(1)$, which implies

$$\lim_{n \to \infty} \frac{f(n, k)}{n} \geq \frac{k(k-2)}{k^2-2} + \frac{8(k-2)}{k(k^2-2)(k^4-2k^2+4)},$$

and they conjectured that this lower bound is the actual value. Moreover they conjectured that extremal $k$-sum-free sets consist of three intervals of consecutive integers with slight modifications at the end-points if $n$ is large.

In this paper we prove that the first conjecture is true, and we expose a structural result which is very close to the second. Our proof is elementary. In fact it is based on two simple observations:

Suppose we are given a $k$-sum-free set $A$. Then

- $kx - y \notin A$ for all $x, y \in A$
  (Otherwise we could satisfy the equation $kx = (kx - y) + y$ in $A$.)

- for all $y \in A$ any interval centered around $\frac{ky}{2}$ cannot share more than half of its elements with $A$.
  (Otherwise we would find a pair $\left\lceil \frac{ky}{2} \right\rceil - d, \left\lceil \frac{ky}{2} \right\rceil + d$ in $A$, giving $\left(\left\lceil \frac{ky}{2} \right\rceil - d\right) + \left(\left\lceil \frac{ky}{2} \right\rceil + d\right) = ky$.)

## 2 Preparations

Let $n \in \mathbb{N}$ be large and let $k \in \mathbb{N}_{\geq 4}$. We start by agreeing on some notations.

### Notations

Let $A \subseteq \{1, \ldots, n\}$ be a set of positive integers. Denote by

$$s_A := \min A$$

and

$$m_A := \max A$$

the smallest and the largest elements of $A$ respectively.

For $l, r \in \mathbb{R}$ let

$$[l, r] := \{x \in \mathbb{N} \mid l \leq x \leq r\}$$

$$[l, r) := \{x \in \mathbb{N} \mid l \leq x < r\}$$

$$(l, r) := \{x \in \mathbb{N} \mid l < x < r\}$$

$$[l, r) := \{x \in \mathbb{N} \mid l \leq x < r\}$$

$$[l, r] := \{x \in \mathbb{N} \mid l \leq x \leq r\}$$
abbreviate intervals of integers. Continuous intervals will be indicated by the subscript \( R \).

Furthermore for any \( y \in \mathbb{N} \) and \( d \in \mathbb{N}_0(= \mathbb{N} \cup \{0\}) \) put

\[
I_y^d := \left[ \frac{k y - 1}{2} - d, \frac{k y + 1}{2} + d \right].
\]

Note that if \( k y \) is even then \( I_y^d = \{ \frac{k y}{2} - d, \frac{k y}{2} - d + 1, \ldots, \frac{k y}{2} + d \} \) and \( |I_y^d| = 2d + 1 \), while if \( k y \) is odd we have \( I_y^d = \{ \frac{k y - 1}{2} - d, \ldots, \frac{k y + 1}{2} + d \} \) and \( |I_y^d| = 2d + 2 \).

The first Lemma restates our introductory observations.

**Lemma 1** Let \( A \subseteq [1, n] \) be a \( k \)-sum-free set. If \( x, y \in A \) then \( k x - y \notin A \). If \( y \in A \) and \( d \in \mathbb{N}_0 \) then \( |I_y^d| \geq d + 1 \).

Suppose \( A' \) is a \( k \)-sum-free set consisting of intervals \( (l_i, r_i] \). The interval \( (l_i, r_i] \) is \( k \)-sum-free if \( l_i \geq \frac{2 r_i}{k} \). Moreover we observe that reasonably large consecutive intervals \( (l_{i+1}, r_{i+1}], (l_i, r_i] \) (where we assume \( r_{i+1} < l_i \)) should satisfy \( k r_{i+1} \leq l_i + s_A \). This leads to the following definition, describing a successive transformation of an arbitrary \( k \)-sum-free set \( A \) into a \( k \)-sum-free set of intervals.

**Definition 1** Let \( n \in \mathbb{N} \) and let \( A \subseteq [1, n] \) be \( k \)-sum-free with smallest element \( s := s_A \). Define sequences \( (r_i), (l_i), (A_i) \) by:

\[
A_0 := A, \quad r_1 := n,
\]

\[
l_i := \left\lfloor \frac{2 r_i}{k} \right\rfloor, \quad r_{i+1} := \left\lfloor \frac{l_i + s}{k} \right\rfloor,
\]

\[
A_i := (A_{i-1} \setminus (r_{i+1}, l_i]) \cup (l_i, r_i] \cap [s, n] \text{ for } i \geq 1.
\]

The letter \( t = t_A \) will be reserved to denote the least integer such that \( r_{t+1} < s \). Observe that, for all \( i \geq t \),

\[
A_i = A_t = [\alpha, r_t] \cup \left( \bigcup_{j=1}^{t-1} (l_j, r_j] \right),
\]

where \( \alpha = \alpha_A := \max\{l_t + 1, s\} \).

## 3 The structure of maximum \( k \)-sum-free sets

To obtain the structural result we consider the successive transformation of an arbitrary \( k \)-sum-free set \( A \) into a set \( A_i \) of intervals as in (1). Our plan is to show that each member of the transformation sequence \( (A_i) \) is \( k \)-sum-free and has size greater than or equal to \( |A| \). For \( n \) sufficiently large, depending on \( k \), and a maximum sized \( k \)-sum-free subset \( A \) of \( [1, n] \), it will turn out that \( A_i \) consists of three intervals only, i.e.: that \( t = 3 \). This observation will do to determine \( f(n, k) \), and we conclude our proof by showing that \( A \)
could be enlarged if it did not contain (nearly) the whole interval \((l_3, r_3]\) and consequently almost all elements from \((l_2, r_2]\) and \((l_1, r_1]\), so that in fact almost nothing happens during the transformation of an extremal set.

**Lemma 2** Let \(A \subseteq [1, n] \) be \(k\)-sum-free. Let \(i \in \mathbb{N}\).

a) \(A_i\) is \(k\)-sum-free.

b) \(|A_i| \geq |A_{i-1}|\).

**Proof.** a) Clearly, it is enough to prove the claim for \(i \leq t\), so we may assume that \(s \leq r_i\). Suppose there are \(a, b, c \in A_i\) with \(a + b = kc\). \(A_i\) is of the form

\[
A_i = A_{i-1} \cap [s, r_{i+1}] \cup (l_i, r_i] \cap [s, n] \cup (l_{i-1}, r_{i-1}] \cup \ldots \cup (l_1, r_1].
\]

If \(c \in (l_1, r_1]\), then \(kc > 2n\), which is impossible. If \(c \in (l_j, r_j]\) for some \(j \in [2, i]\), then \(kc \in (2r_j, l_j-1 + s]\) and the larger one of \(a, b\) must be in \((r_j, l_{j-1}]\). But \((r_{j+1}, l_j] \cap A_i = \emptyset\) by construction. Hence \(c \in A_{i-1} \cap [s, r_{i+1}]\). Now, \(kc \leq kr_{i+1} \leq l_i + s\). Since \((r_{i+1}, l_i] \cap A_i = \emptyset\), both \(a\) and \(b\) have to be in \(A_{i-1} \cap [s, r_{i+1}] = A \cap [s, r_{i+1}]\), but \(A\) is \(k\)-sum-free, a contradiction.

b) The inequality is trivial for \(i \geq t\). For \(1 \leq i < t\) we have that \(l_i \geq s\) and hence

\[
A_i = (A_{i-1} \cap [1, r_{i+1}]) \cup (l_i, r_i] \cup \left( \bigcup_{j=1}^{i-1} (l_j, r_j) \right).
\]

Thus it suffices to prove that

\[
|A_{i-1} \cap [1, r_i]| \leq |A_{i-1} \cap [1, r_{i+1}]| + \left\lceil \frac{(k-2)r_i}{k} \right\rceil.
\]

Clearly, then, it suffices to prove the inequality for \(i = 1\), i.e.: to prove that, for any \(n > 0\), and any \(k\)-sum-free subset \(A\) of \([1, n]\) with smallest element \(s_A\), we have

\[
|A| \leq |A \cap [1, r_{2,A}]| + \left\lceil \frac{(k-2)n}{k} \right\rceil,
\]

(2)

where

\[
r_{2,A} := \left\lfloor \frac{2n/k + s_A}{k} \right\rfloor.
\]

The proof is by induction on \(n\). The result is trivial for \(n = 1\). So suppose it holds for all \(1 \leq m < n\) and let \(A\) be a \(k\)-sum-free subset of \([1, n]\). Note that the result is again trivial if \(s_A > 2n/k\), so we may assume that \(s_A \leq 2n/k\), which implies that \(r_{2,A} \leq n/k\), since \(k \geq 4\).

First suppose that there exists \(x \in A \cap (n/k, 2n/k]\). Then \(1 \leq kx - n \leq n\) and the
map \( f : y \mapsto kx - y \) is a 1-1 mapping from the interval \([kx - n, n]\) to itself. For each \( y \) in this interval, at most one of the numbers \( y \) and \( f(y) \) can lie in \( A \), since \( A \) is \( k \)-sum-free.

To simplify notation, put \( w := kx - n - 1 \). Then our conclusion is that

\[
|A \cap (w, n]| \leq \frac{1}{2}(n - w). \tag{3}
\]

If \( w = 0 \) or if \( A \cap [1, w] = \emptyset \), then we are done (since \( k \geq 4 \)). Put \( B := A \cap [1, w] \). Then we may assume \( B \neq \emptyset \), hence \( s_B = s_A \). Applying the induction hypothesis to \( B \), we find that

\[
|B| = |A \cap [1, w]| \leq |B \cap [1, r_{2,B}]| + \left\lceil \frac{(k - 2)w}{k} \right\rceil. \tag{4}
\]

But \( s_B = s_A \) implies that \( r_{2,B} \leq r_{2,A} \), hence that \( B \cap [1, r_{2,B}] \subseteq A \cap [1, r_{2,A}] \). Thus (3) and (4) yield the inequality

\[
|A| \leq |A \cap [1, r_{2,A}]| + \left\lceil \frac{(k - 2)w}{k} \right\rceil + \frac{1}{2}(n - w),
\]

which in turn implies (2), since \(|A|\) is an integer. Thus we are reduced to completing the induction under the assumption that \( A \cap (n/k, 2n/k] = \emptyset \). Suppose \( x \in A \cap (r_{2,A}, n/k] \).

Then \([2n/k] + s_A < kx \leq n \) and \( kx - s_A \notin A \). In other words, we can pair off elements in \( A \cap (r_{2,A}, 2n/k] \) with elements in \((2n/k, n] \setminus A \). This immediately implies (2), and the proof of Lemma 2 is complete. \( \square \)

We have seen so far that any \( k \)-sum-free set \( A \) can be turned into a \( k \)-sum-free set \( A_t \) having overall size at least \(|A|\). The set \( A_t \) is a union of intervals, as given by (1), though note that the final interval \([\alpha, r_t] \) may consist of a single point, since \( r_t = s \) is possible. The proof of the following Lemma uses a fact shown in [3] by Chung and Goldwasser, to prove that \( t \) must be equal to three if \(|A|\) is maximum.

**Lemma 3** Let \( A \) be a maximum \( k \)-sum-free subset of \([1, n] \), where \( n > n_0(k) \) is sufficiently large. Let \( s := s_A \) and let \( t := \max\{i \in \mathbb{N} \mid r_i \geq s\} \). Then \( t = 3 \).

**Proof.** Let \( A_t \) be the set of positive integers given by (1). In a similar manner we now define a \( k \)-sum-free subset \( A'_t \) of \((0, 1]_\mathbb{R} \).

Put \( c := s/n \) and, for \( i = 1, \ldots, t \) define real numbers \( R_i, L_i \) as follows:

\[
R_1 := 1, \quad L_i := \frac{2R_i}{k}, \quad R_{i+1} := \frac{L_i + c}{k}.
\]

Then we put

\[
A'_t := [\alpha', R_t]_\mathbb{R} \cup \left( \bigcup_{j=1}^{t-1} [L_j, R_j]_\mathbb{R} \right),
\]

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where $\alpha' := \max\{L_t, c\}$. That $A'_t$ is $k$-sum-free is shown in [3]. One sees easily that

$$|A_t| \leq n \cdot \mu(A'_t) + t,$$

(5)

where $\mu$ denotes the Lebesgue-measure. Now suppose that $t \neq 3$. It is shown in [3] that there exists a constant $c_k > 0$, depending only on $k$, such that in this case

$$|\mu(A'_t)| \leq \frac{k(k - 2)}{k^2 - 2} + \frac{8(k - 2)}{k(k^2 - 2)(k^4 - 2k^2 - 4)} - c_k.$$

(6)

In fact, in the notation of page 8 of [3], an explicit value for $c_k$ (which we will use later) is given by

$$c_k = \frac{2}{k} (R(3) - R(4)),$$

which by definition of $R$ amounts to

$$c_k = \frac{8(4 - 4k^2 - 4)(k - 2)}{(k^4 - 4k^2 - 8)(k^4 - 2k^2 - 4)}.$$

(7)

Now (5) and (6) would imply that

$$|A| \leq \frac{k(k - 2)}{k^2 - 2} n + \frac{8(k - 2)}{k(k^2 - 2)(k^4 - 2k^2 - 4)} n - c_k n + t.$$ But we have seen in the introduction that $|A| \geq \frac{k(k - 2)}{k^2 - 2} n + \frac{8(k - 2)}{k(k^2 - 2)(k^4 - 2k^2 - 4)} n + O(1)$ and, since $t = O(\log_k n)$, we thus have a contradiction for sufficiently large $n$. Hence $t$ must equal three, for large enough $n$, as required.

Now we are nearly in a position to determine $f(n, k)$. We want to calculate the cardinality of an extremal $k$-sum-free set $A$ via computing $|A_3|$. Since $|A_3|$ depends on $s_A$, the following lemma will be helpful:

**Lemma 4** Let $n > n_0(k)$ be sufficiently large. If $A$ is a maximal $k$-sum-free subset of $[1, n]$, then $S - 2k \leq s_A \leq S + 3$, where $S := \frac{8n}{k^2 - 2k - 4}$.\[\Box\]

**Proof.** Set $s := s_A$. By Lemma 3, for $n > n_0(k)$ we have $r_4 < s$. Since $A$ is maximal we have $|A| = |A_3|$. Now, for a fixed $n$, the cardinality of $A_3$ is a function of $s \in [1, n]$ only. So we need to show that $|A_3(s)|$ attains its maximum value only for some $s \in [S - 2k, S + 3]$. Define

$$s' := \min\{s \in [1, n] : l_3(s) < s\}.$$ A tedious computation (see the Appendix below) yields that $s' = S + 1$ if $k$ is even and $s' = S$ or $S + 1$ if $k$ is odd. Hence

$$s' \in [S, S + 1].$$

(8)
Clearly,

\[ |A_3(s)| = \begin{cases} 
\left\lceil \frac{(k-2)n}{k} \right\rceil + r_2(s) - l_2(s) + r_3(s) - s + 1, & \text{if } s \geq s', \\
\left\lceil \frac{(k-2)n}{k} \right\rceil + r_2(s) - l_2(s) + r_3(s) - l_3(s), & \text{if } s < s'. 
\end{cases} \] (9)

How does \( |A_3(s)| \) change (ignoring its maximality for a while) if we alter \( s \)?

First suppose \( s \geq s' \). If \( s \) increases by one, then \( |A_3| \) will decrease by one unless either \( r_2 \) or \( r_3 \) increases. Now \( r_2 \) can only increase (by one) once in \( k \geq 4 \) times. Almost the same is true of \( r_3 \), though its dependence on \( l_2 \) makes things a little more complicated. However, it is not hard to see that we encounter an irreversible decrease in the cardinality of \( |A_3| \) after at most 3 steps of increment of \( s \). Hence \( |A_3(s)| < |A_3(s')| \) if \( s \geq s' + 3 \).

Next suppose \( s < s' \). If we decrease \( s \), then \( |A_3| \) cannot increase at all, since \( l_i \) will not decrease unless \( r_i \) does. Moreover, \( |A_3| \) will become smaller if the size of any interval is diminished. So we can focus our attention on \((l_2, r_2)\). While \( r_2 \) decreases once in \( k \) times, \( l_2 \) does so no more than once in \( k \left\lceil \frac{k}{2} \right\rceil \geq 2k \) times. Thus \( |A_3(s)| < |A_3(s' - 1)| \) if \( s \leq s' - 1 - 2k \).

We have now shown that, as a function of \( s \in [1, n] \), the cardinality of \( A_3 \) attains its maximum only for some \( s \in [s' - 2k, s' + 2] \). This, together with (8), completes the proof of the lemma.

Now we can prove the first conjecture of Chung and Goldwasser.

**Theorem 1**

\[ \lim_{n \to \infty} \frac{f(n, k)}{n} = \frac{k(k - 2)}{k^2 - 2} + \frac{8(k - 2)}{k(k^2 - 2)(k^4 - 2k^2 - 4)}. \]

**Proof.** Let \( A \) be a maximum \( k \)-sum-free set in \([1, n]\), with \( n \) sufficiently large. From Lemma 4 we have

\[ S_n = n \frac{S^*}{k^2 - 2k + 8} + o(1), \]

where \( S^* = \frac{8n}{k^2 - 2k + 8} \). Thus we can estimate

\[ \frac{f(n, k)}{n} = \frac{|A_3|}{n} = \frac{r_1 - l_1 + r_2 - l_2 + r_3 - S^* + 1}{n} + o(1) \]

\[ = \frac{1}{n} \left( \frac{n - 2n}{k^2 - 2k + 8} + 2n + kS^* \right) + 4n + 2kS^* + k^3S^* - S^* + o(1) \]

\[ = \frac{k^4 - 2k^3 + 2k^2 - 4k + 4}{k^4} + \frac{S^*}{nk^3} (2k^2 - 2k + 2 - k^3) + o(1) \]

\[ = \frac{k^4 - 2k^3 + 2k^2 - 4k + 4}{k^4} + \frac{8(2k^2 - 2k + 2 - k^3)}{(k^5 - 2k^3 - 4k)k^3} + o(1) \]

\[ = \frac{k^5 - 2k^4 - 4k + 8}{(k^4 - 2k^2 - 4)k} + o(1) \]

\[ = \frac{k(k - 2)}{k^2 - 2} + \frac{8(k - 2)}{k(k^2 - 2)(k^4 - 2k^2 - 4)} + o(1), \]

and the claim follows by taking the limit. \( \square \)

We can now show the main result.
Theorem 2 Let \( k \in \mathbb{N}_{\geq 4} \) and \( n > n_1(k) \). Let \( S \) and \( s' \) be as in Lemma 4. Let \( A \subseteq \{1, \ldots, n\} \) be a \( k \)-sum-free set of maximum cardinality, with smallest element \( s = s_A \).
Then \( s \in [S, S+3] \) and \( A = \mathcal{I}_3 \cup \mathcal{I}_2 \cup \mathcal{I}_1 \), where

\[
\mathcal{I}_3 = \begin{cases} \{[s, r_3], [s, r_3+1]\}, & \text{if } s \geq s' \\ \{[s, r_3], [s, r_3 \setminus \{r_3-1]\}, & \text{if } s < s' \end{cases}
\]

\[
\mathcal{I}_2 = \begin{cases} \{(l_2 + 2, r_2), [l_2 + 2, r_2 + 1]\}, & \text{if } r_3 + 1 \in A \\ \{(l_2, r_2), (l_2, r_2 + 1), [l_2, r_2), [l_2, r_2 \setminus \{r_2 - 1\}\}, & \text{if } r_3 + 1 \notin A, \end{cases}
\]

\[
\mathcal{I}_1 = \begin{cases} \{[l_1 + 2, n]\}, & \text{if } r_2 + 1 \in A \\ \{(l_1, n), (l_1, n), [l_1, n \setminus \{n - 1\}], & \text{if } r_2 + 1 \notin A, \end{cases}
\]

If \( k \) is even, then \( \mathcal{I}_i \neq \{l_i, r_i\} \setminus \{r_i - 1\} \) for \( 1 \leq i \leq 3 \).

Remark. Note that Theorem 2 does not precisely determine the \( k \)-sum-free subsets of \( \{1, \ldots, n\} \) of maximum size, for every \( n > n_1(k) \). With \( n \) and \( k \) fixed, one first needs to determine for which value(s) of \( s \in [S, S+3] \) the quantity \( |A_3(s)| \), as given by (9), is maximized. The result will depend on \( n \) and \( k \). Even then, for a fixed \( s \), not all the possibilities for \( \mathcal{I}_3 \cup \mathcal{I}_2 \cup \mathcal{I}_1 \) need be \( k \)-sum-free. See Section 4 below for further discussion.

Proof. We have already seen that \( |A_3| = |A| \). Our first aim is to show by comparing \( A_3 \) with \( A_2 \) that almost the whole interval \( (l_3, r_3] \) must be in \( A \). Having achieved this, we infer by Lemma 1 that \( (r_3, l_2] \cap A \) is nearly empty. Comparing \( A_2 \) with \( A_1 \) will then reveal that most of \( [l_2, r_2] \) is contained in \( A \). Again Lemma 1 will help us to see that \( A \) cannot share many elements with \( (r_2, l_1] \) and a final comparison of \( A_1 \) with \( A \) will conclude the proof.

(I) The first aim is easily reached if \( s := s_A \geq l_3 + 1 \). Simply note that

\[
A_2 = (A \cap [s, r_3]) \cup (l_2, r_2] \cup (l_1, r_1] \subseteq [s, r_3] \cup (l_2, r_2] \cup (l_1, r_1] = A_3.
\]

The maximality of \( |A_2| \) gives \( A_2 = A_3 \) and hence \( [s, r_3] \subseteq A \). Observe that \( s > l_3 \) together with Lemma 4 and (8) give \( S \leq s \leq S + 3 \).

Assume now that \( s \leq l_3 \). We want to show that in this case \( s = l_3 \). Suppose \( s < l_3 \) and let \( B = [S - 2k, l_3] \cap A \). Define

\[
C := I^1_{s_B} \cup \bigcup_{b \in B \setminus \{s_B\}} I^0_b.
\]

Clearly \( C \subseteq (l_3, r_3] \) for all \( n \gg 0 \). Then since \( C \) is the union of disjoint intervals, Lemma 1 gives that \( |C \setminus A| > |B| \). Hence we get the contradiction \( |A_3| = |(A_2 \setminus B) \cup (l_3, r_3]| > |A_2| - |B| + |B| = |A_2| \). Therefore we are left with \( s = l_3 \), and this implies

\[
|A_2| = |A_3| \iff |A \cap [s, r_3]| = |(l_3, r_3] \cap [s, r_3]| = |(s, r_3)|.
\]
If $r_3 \not\in A$ we can infer from (10) that

$$A \cap [s, r_3] = [s, r_3 - 1] = [l_3, r_3 - 1].$$

If $r_3 \in A$, Lemma 1 gives $kl_3 - r_3 \not\in A$, so $-k + 1 \leq kl_3 - 2r_3 \leq -1$. If $kl_3 - 2r_3 \leq -2$ we get $I_{r_3}^1 \subseteq (l_3, r_3]$ and $|I_{r_3}^1 \setminus A| \geq 2$, which is impossible since this would imply $|A_3| > |A_2|$. Hence $kl_3 - 2r_3 = -1$ and $k$ is odd. Using (10) one obtains

$$A \cap [s, r_3] = [l_3, r_3] \setminus \{r_3 - 1\}.$$ 

Suppose now that $s = l_3$ and $r_3 + 1 \in A$. Then $kl_3 - (r_3 + 1) \not\in A$ and

$$r_3 - k \leq kl_3 - (r_3 + 1) \leq r_3 - 1.$$ 

This contradicts that $[s, r_3 - 2] \subseteq A$ unless $kl_3 - (r_3 + 1) = r_3 - 1$, but then $r_3 \not\in A$ and $|A \cap [s, r_3]| = |A \cap [s, r_3 - 2]|$ which contradicts (10). Hence $r_3 + 1 \not\in A$ if $s = l_3$.

Finally note that, if $s = l_3$ and $kl_3 \geq 2r_3 - 1$, the latter being a requirement for either of the two possibilities for $I_3$ to be $k$-sum-free, then another computation similar to the one in the Appendix yields that $s \geq S$. Again, using Lemma 4 we obtain

$$S \leq s \leq S + 3,$$

as claimed in the statement of the theorem. This completes the first part of our proof.

(II) For the second part note that we have just shown

$$s \geq l_3.$$ 

Plugging (11) into the definition of $l_3$ yields (after a further tedious computation similar to that in the Appendix)

$$S - 1 \leq l_3 \leq S + 1,$$

which implies in view of (12) and (11)

$$l_3 \leq s \leq l_3 + 4.$$ 

Moreover we have observed that $[s, r_3 - 2] \subseteq A$. Let $\xi_1, \ldots, \xi_5 \in \{0, \ldots, k - 1\}$ be constants such that

$$kl_1 = 2r_1 - \xi_1$$

$$kr_2 = l_1 + s - \xi_2$$

$$kl_2 = 2r_2 - \xi_3$$

$$kr_3 = l_2 + s - \xi_4$$

$$kl_3 = 2r_3 - \xi_5.$$
We suppose that $n$ is sufficiently large, so we can be sure that

$$[ks - (r_3 - 2), k(r_3 - 2) - s] \cap A = \emptyset.$$ 

By (14) we can infer that

$$\emptyset = [k(l_3 + 4) - (r_3 - 2), k(r_3 - 2) - s] \cap A$$

$$= [r_3 - \xi_5 + 4k + 2, l_2 - \xi_4 - 2k] \cap A.$$ 

Let $J = [r_3 + 2, r_3 - \xi_5 + 4k + 1] \cap A$ and $K = \bigcup_{x \in J} \{kx - (s + 2), kx - (s + 1), kx - s\}.$ Then $K \cap A = \emptyset,$ $|K| = 3|J|$ and by (18) and (19) we have

$$K \subseteq [l_2 - \xi_4 + 2k - 2, l_2 - \xi_4 - k\xi_5 + 4k^2 + k] \subseteq \langle l_2 + k - 2, l_2 + 4k^2 + k \rangle \subseteq \langle l_2 + 2, r_2 \rangle,$$

if $n \gg 0.$ Let $B = [l_2 - \xi_4 - 2k + 1, l_2] \cap A.$ If $B \cup J \subseteq \{l_2\}$ then $A \cap [r_3 + 2, l_2 - 1] = \emptyset.$ Otherwise, with $C$ as in part (I) if $|B| > 1$ we can verify that $C \subseteq [r_2 - \frac{3k^2 - k + 2}{2}, r_2] \subseteq (l_2 + 1, r_2),$ for $n \gg 0,$ and $|C \setminus A| > |B|.$ Put $C := \emptyset$ if $|B| \leq 1.$ For large $n,$ $K$ and $C$ are disjoint. Hence $|B \cup J| < |(C \setminus A) \cup K|$ and we get

$$|A_2| = |A_1 \setminus (J \cup B \cup \{r_3 + 1\})| \cup (l_2, r_2) > |A_1 \setminus \{r_3 + 1\}|.$$ 

Thus if $r_3 + 1 \notin A$ we get $|A_2| > |A_1|$ so suppose $r_3 + 1 \in A.$ Then neither $l_2$ nor $l_2 + 1$ can be in $A_1.$ Otherwise, since $(s - \xi_4 + k), s - \xi_4 + k - 1 \in [s, s + k] \subseteq [s, r_3 - 2] \subseteq A$ we get

$$k(r_3 + 1) = l_2 + (s - \xi_4 + k) = (l_2 + 1) + (s - \xi_4 + k - 1),$$

which is impossible. But $l_2 + 1 \in A_2,$ so also in this case it follows that $|A_2| > |A_1|,$ since $l_2 + 1 \notin K \cup C$ for large $n.$ Again we conclude that $A \cap [r_3 + 2, l_2 - 1] = \emptyset.$ Consequently,

$$|A_2| = |A_1| \Leftrightarrow |A \cap ([l_2, r_2] \cup \{r_3 + 1\})| = |(l_2, r_2)|,$$

which gives $A \cap [l_2, r_2] = [l_2 + 2, r_2]$ if $r_3 + 1 \in A.$ If $r_3 + 1 \notin A$ and either $l_2 \notin A$ or $r_2 \notin A,$ we get $A \cap [l_2, r_2] = (l_2, r_2)$ or $A \cap [l_2, r_2] = [l_2, r_2),$ respectively. In case $r_3 + 1 \notin A$ and both $l_2, r_2 \in A,$ we see that $kl_2 - r_2 = r_2 - \xi_3 \notin A.$ If $\xi_3 \geq 2$ then $I_3^1 \subseteq (l_2, r_2)$ and $l_2$ could be profitably replaced. Hence $\xi_3 = 1,$ $A \cap [l_2, r_2] = [l_2, r_2] \setminus \{r_2 - 1\}$ and $k$ is odd.

(III) For the final interval $(l_1, r_1)$ we use Lemma 1 to conclude from

$$[s, r_3 - 2] \subseteq A$$

in view of (16) and (17) that, for $n \gg 0,$

$$\emptyset = A \cap [k(l_2 + 2) - (r_2 - 2), k(r_2 - 2) - (l_2 + 2)]$$

$$= A \cap [r_2 - \xi_3 + 2k + 2, l_1 + s - \xi_2 - 2k - l_2 - 2],$$

and

$$\emptyset = A \cap [k(l_2 + 2) - (r_3 - 2), k(r_2 - 2) - s]$$

$$= A \cap [2r_2 - \xi_3 + 2k - r_3 + 2, l_1 - \xi_2 - 2k].$$
Let \( J = [r_2 + 2, r_2 - \xi_3 + 2k + 1] \cap A \) and \( K = \cup_{x \in J} \{kx - s, kx - (s + 1), kx - (s + 2)\} \).
From (14) we have
\[
K \subseteq [l_1 - \xi_2 + 2k - 2, l_1 - \xi_2 - k\xi_3 + 2k^2 + k] \subseteq (l_1 + k - 2, r_1], \text{ if } n \gg 0.
\]
Let \( B = [l_1 - \xi_2 - 2k + 1, l_1] \cap A \). If \( s_B < l_1 \) with \( C \) as in (I) we can verify that, for sufficiently large \( n \),
\[
C \subseteq \left[ \frac{2r_1 - \xi_1 - k\xi_2 - 2k^2 + k - 5}{2}, r_1 \right] \subseteq (l_1, r_1],
\]
\(|C \setminus A| > |B|\) and \( \max K < s_C \). By analogy with part (II) we get \( A \cap [r_2 + 2, l_1 - 1] = \emptyset \)
and the rest of the claim follows as before.

\[
4 \text{ Estimates and Periodicity}
\]
We first want to estimate values of \( n_i(k), i = 0, 1, \) for which Lemmas 3 and 4, and Theorem 2 respectively are valid. The estimates we shall arrive at can probably be improved upon. The example of a \( k \)-sum-free set \( A \) in [3], referred to in the proof of Lemma 3, satisfies
\[
|A| > \frac{k(k - 2)}{k^2 - 2} n + \frac{8(k - 2)}{k(k^2 - 2)(k^4 - 2k^2 - 4)} n - 3.
\]
Hence the proof of Lemma 3 goes through provided \( n \) is sufficiently large so that
\[
c_k n - t_0 \geq 3,
\]
where \( t_0 = t_0(n, k) \) is the largest possible value for \( t \) in Definition 1. Now from Definition 1 we easily deduce that, if \( i < t \), then \( r_{i+1} \leq \left( \frac{4}{k^2} \right) r_i \), and hence that \( r_t \leq \left( \frac{4}{k^2} \right)^{t-1} n \). Since \( r_t \geq 1 \) a priori, we can thus estimate
\[
t_0 \leq \frac{1}{2} \log_{k/2} n + 1.
\]
Since, by (7), \( c_k = O(\frac{1}{k^c}) \), we thus deduce from (18) and (19) that one can take \( n_0(k) = O(k^6) \). It is then an easy and tedious exercise to go through the proof of Theorem 2 and check that one can also take \( n_1(k) = O(k^6) \).

Next, we explain what we mean by the word ‘periodicity’ in the title of this section. If \( k \geq 4 \) is even then, for \( n > 0 \), we have \( s' = S + 1 = \lfloor \frac{8n}{k^2 - 2k^3 - 4k} \rfloor + 1 \). Hence for a fixed \( k \), if we regard \( s' \) as a function of \( n \), then \( s'(n) + 1 = s'(n + p_k) \), where \( p_k := \frac{k^5 - 2k^3 - 4k}{8} \). For odd \( k \), we define \( p_k := k^5 - 2k^3 - 4k \) and in this case, a little more care is required to check that \( s'(n) + 8 = s'(n + p_k) \).
Now for any \( k \) and \( n \), let \( \mathcal{F}(k, n) \) denote the family of maximal \( k \)-sum-free subsets of \( \{1, ..., n\} \). Then for \( n \) sufficiently large, as estimated above, and \( k \) even (resp. \( k \) odd), the map \( s \mapsto s + 1 \) (resp. \( s \mapsto s + 8 \)) clearly induces a 1-1 correspondence between the sets in \( \mathcal{F}(k, n) \) and \( \mathcal{F}(k, n + p_k) \). This is what we mean by ‘periodicity’. This observation clearly reduces, for any fixed \( k \), the full classification of all \( k \)-sum-free subsets of \( \{1, ..., n\} \), for all \( n \), to a finite computation.

As an example, we now look at \( k = 4 \). By (7) we compute \( c_4 = \frac{47}{48290} \). Then Lemma 3 is valid at least for all \( n \) satisfying

\[
c_4 n - \frac{1}{2} \log_2 n - 1 \geq 3,
\]

which reduces to \( n \geq 11008 \). One can then check that the proof of Theorem 2 also goes through for all such \( n \). We have \( p_4 = 110 \). We now present the full classification of all 4-sum-free subsets of \( \{1, ..., n\} \), valid (at least) for all \( n \geq 11008 \). This was obtained with the help of a computer.

For each \( s, n \in \mathbb{N} \) we define the sets \( J_x(s) \), \( 1 \leq x \leq 13 \), as follows (the \( l_i \) and \( r_i \) are functions of \( s \) and \( n \) as in Definition 1):

\[
\begin{align*}
J_1 &= [S, r_3 - 1] \cup [l_2, r_2 - 1] \cup [l_1, n - 1], \\
J_2 &= [S, r_3 - 1] \cup [l_2, r_2 - 1] \cup [l_1 + 1, n], \\
J_3 &= [S, r_3 - 1] \cup [l_2 + 1, r_2] \cup [l_1, n - 1], \\
J_4 &= [S, r_3 - 1] \cup [l_2 + 1, r_2] \cup [l_1 + 1, n], \\
J_5 &= [S, r_3 - 1] \cup [l_2 + 1, r_2 + 1] \cup [l_1 + 2, n], \\
J_6(s) &= [s, r_3] \cup [l_2, r_2 - 1] \cup [l_1, n - 1], \\
J_7(s) &= [s, r_3] \cup [l_2, r_2 - 1] \cup [l_1 + 1, n], \\
J_8(s) &= [s, r_3] \cup [l_2 + 1, r_2] \cup [l_1, n - 1], \\
J_9(s) &= [s, r_3] \cup [l_2 + 1, r_2] \cup [l_1 + 1, n], \\
J_{10}(s) &= [s, r_3] \cup [l_2 + 1, r_2 + 1] \cup [l_1 + 2, n], \\
J_{11}(s) &= [s, r_3 + 1] \cup [l_2 + 2, r_2] \cup [l_1, n - 1], \\
J_{12}(s) &= [s, r_3 + 1] \cup [l_2 + 2, r_2] \cup [l_1 + 1, n], \\
J_{13}(s) &= [s, r_3 + 1] \cup [l_2 + 2, r_2 + 1] \cup [l_1 + 2, n].
\end{align*}
\]

Note that, by Theorem 2, for a given \( n \geq 11008 \), every maximal 4-sum-free subset of \( \{1, ..., n\} \) is one of the sets \( J_x(s) \), for some \( s \in [S, S + 3] = [s', - 1, s' + 2] \). By the remarks above, for each \( i \in \{0, ..., 109\} \), there are natural 1-1 correspondences between the sets in the families \( \mathcal{F}(4, n) \) for all \( n \equiv i \) (mod 110). By slight abuse of notation, we denote any such family simply by \( \mathcal{F}_i \). Our computer program yielded the following result:

If \( |\mathcal{F}_i| = 1 \), then \( i = 6, 7, 22, 23, 46, 47, 49, 51, 54, 55, 57, 59, 61, 70, 71, 73, 75, 77, 86, 87, 89 \).
or 91 and

\[ \mathcal{F}_i = \{ J_9(s') \}, \]

or \( i = 36, 37, 100 \) or 101 and

\[ \mathcal{F}_i = \{ J_9(s' + 1) \}. \]

If \( |\mathcal{F}_i| = 2 \), then \( \mathcal{F}_i \) is

\[
\begin{align*}
\{ J_9(s'), J_9(s' + 1) \} & \quad \text{if} \quad i = 93, 103, 105, 107, \\
\{ J_4, J_9(s') \} & \quad \text{if} \quad i = 9, 11, 13, 25, 27, \\
\{ J_8(s'), J_9(s') \} & \quad \text{if} \quad i = 48, 50, 56, 58, 60, 72, 74, 76, 88, 90 \\
\{ J_7(s'), J_9(s') \} & \quad \text{if} \quad i = 63, 65, 67, 79, 81.
\end{align*}
\]

If \( |\mathcal{F}_i| = 3: \)

\[
\begin{align*}
\mathcal{F}_8 = \mathcal{F}_{24} & = \{ J_4, J_8(s'), J_9(s') \}, \\
\mathcal{F}_{15} & = \{ J_4, J_7(s'), J_9(s') \}, \\
\mathcal{F}_{29} & = \{ J_4, J_9(s'), J_9(s' + 1) \}, \\
\mathcal{F}_{39} & = \{ J_9(s'), J_9(s' + 1) \}, \\
\mathcal{F}_{62} = \mathcal{F}_{78} & = \{ J_6(s'), J_7(s'), J_9(s') \}, \\
\mathcal{F}_{53} & = \{ J_9(s'), J_{10}(s'), J_9(s' + 1) \}, \\
\mathcal{F}_{83} & = \{ J_7(s'), J_9(s'), J_9(s' + 2) \}, \\
\mathcal{F}_{92} & = \{ J_8(s'), J_9(s'), J_9(s' + 1) \}, \\
\mathcal{F}_{95} = \mathcal{F}_{97} & = \{ J_7(s'), J_9(s'), J_9(s' + 1) \}, \\
\mathcal{F}_{102} & = \{ J_9(s'), J_8(s' + 1), J_9(s' + 1) \}, \\
\mathcal{F}_{109} & = \{ J_9(s'), J_7(s' + 1), J_9(s' + 1) \}.
\end{align*}
\]

If \( |\mathcal{F}_i| = 4: \)

\[
\begin{align*}
\mathcal{F}_1 = \mathcal{F}_3 = \mathcal{F}_{17} & = \{ J_2, J_4, J_7(s'), J_9(s') \}, \\
\mathcal{F}_{10} = \mathcal{F}_{12} = \mathcal{F}_{26} & = \{ J_3, J_4, J_8(s'), J_9(s') \}, \\
\mathcal{F}_{38} & = \{ J_9(s'), J_{12}(s'), J_9(s' + 1) \}, \\
\mathcal{F}_{41} = \mathcal{F}_{43} & = \{ J_9(s'), J_{12}(s'), J_9(s' + 1) \}, \\
\mathcal{F}_{52} & = \{ J_9(s'), J_9(s'), J_{10}(s'), J_9(s' + 1) \}, \\
\mathcal{F}_{64} = \mathcal{F}_{66} = \mathcal{F}_{80} & = \{ J_6(s'), J_7(s'), J_9(s'), J_9(s') \}, \\
\mathcal{F}_{104} = \mathcal{F}_{106} & = \{ J_8(s'), J_9(s'), J_8(s' + 1), J_9(s' + 1) \}, \\
\mathcal{F}_{69} & = \{ J_7(s'), J_9(s'), J_{10}(s'), J_9(s' + 1) \}.
\end{align*}
\]
If $|\mathcal{F}_i| = 5$:

$$\mathcal{F}_{14} = \{J_3, J_4, J_6(s'), J_7(s'), J_9(s')\},$$
$$\mathcal{F}_{19} = \{J_2, J_4, J_7(s'), J_9(s'), J_9(s' + 2)\},$$
$$\mathcal{F}_{28} = \{J_3, J_4, J_8(s'), J_9(s'), J_9(s' + 1)\},$$
$$\mathcal{F}_{31} = \{J_4, J_7(s'), J_9(s'), J_{12}(s'), J_9(s' + 1)\},$$
$$\mathcal{F}_{82} = \{J_6(s'), J_7(s'), J_8(s'), J_9(s'), J_9(s' + 2)\},$$
$$\mathcal{F}_{94} = \{J_6(s'), J_7(s'), J_9(s'), J_9(s' + 1), J_9(s' + 1)\},$$
$$\mathcal{F}_{99} = \{J_7(s'), J_9(s'), J_9(s' + 1), J_{10}(s' + 1), J_9(s' + 2)\},$$
$$\mathcal{F}_{108} = \{J_8(s'), J_9(s'), J_9(s' + 1), J_7(s' + 1), J_9(s' + 1)\}.$$

If $|\mathcal{F}_i| = 6$:

$$\mathcal{F}_5 = \{J_2, J_4, J_7(s'), J_9(s'), J_{10}(s'), J_9(s' + 1)\},$$
$$\mathcal{F}_{33} = \{J_2, J_4, J_7(s'), J_9(s'), J_{12}(s'), J_9(s' + 1)\},$$
$$\mathcal{F}_{45} = \{J_4, J_9(s'), J_{12}(s'), J_{13}(s'), J_7(s' + 1), J_9(s' + 1)\},$$
$$\mathcal{F}_{68} = \{J_6(s'), J_7(s'), J_9(s'), J_9(s'), J_{10}(s'), J_9(s' + 1)\},$$
$$\mathcal{F}_{85} = \{J_7(s'), J_9(s'), J_{10}(s'), J_9(s' + 1), J_{12}(s' + 1), J_9(s' + 2)\},$$
$$\mathcal{F}_{96} = \{J_6(s'), J_7(s'), J_9(s'), J_9(s'), J_9(s' + 1), J_9(s' + 1)\}.$$

If $|\mathcal{F}_i| = 7$:

$$\mathcal{F}_0 = \mathcal{F}_{16} = \{J_1, J_2, J_4, J_6(s'), J_7(s'), J_9(s'), J_9(s')\},$$
$$\mathcal{F}_{40} = \{J_4, J_8(s'), J_9(s'), J_{11}(s'), J_{12}(s'), J_8(s' + 1), J_9(s' + 1)\}.$$

If $|\mathcal{F}_i| = 8$:

$$\mathcal{F}_2 = \{J_1, J_2, J_3, J_4, J_6(s'), J_7(s'), J_8(s'), J_9(s')\},$$
$$\mathcal{F}_{21} = \{J_2, J_4, J_7(s'), J_9(s'), J_{10}(s'), J_9(s' + 1), J_{12}(s' + 1), J_9(s' + 2)\},$$
$$\mathcal{F}_{30} = \{J_3, J_4, J_6(s'), J_7(s'), J_9(s'), J_{12}(s'), J_8(s' + 1), J_9(s' + 1)\},$$
$$\mathcal{F}_{35} = \{J_2, J_4, J_7(s'), J_9(s'), J_{12}(s'), J_9(s' + 1), J_{10}(s' + 1), J_9(s' + 2)\},$$
$$\mathcal{F}_{42} = \{J_3, J_4, J_6(s'), J_9(s'), J_{11}(s'), J_{12}(s'), J_8(s' + 1), J_9(s' + 1)\},$$
$$\mathcal{F}_{98} = \{J_6(s'), J_7(s'), J_8(s'), J_9(s'), J_6(s' + 1), J_9(s' + 1), J_{10}(s' + 1), J_9(s' + 2)\}.$$

If $|\mathcal{F}_i| = 9$:

$$\mathcal{F}_{18} = \{J_1, J_2, J_3, J_4, J_6(s'), J_7(s'), J_8(s'), J_9(s'), J_9(s' + 2)\},$$
$$\mathcal{F}_{84} = \{J_6(s'), J_7(s'), J_8(s'), J_9(s'), J_{10}(s'), J_9(s' + 1), J_{12}(s' + 1), J_8(s' + 2), J_9(s' + 2)\}.$$

If $|\mathcal{F}_i| = 10$:

$$\mathcal{F}_4 = \{J_1, J_2, J_3, J_4, J_6(s'), J_7(s'), J_8(s'), J_9(s'), J_9(s' + 1)\},$$
$$\mathcal{F}_{44} = \{J_3, J_4, J_6(s'), J_9(s'), J_{11}(s'), J_{12}(s'), J_{13}(s'), J_6(s' + 1), J_7(s' + 1), J_9(s' + 1)\}.$$
If $|\mathcal{F}_i| = 11, 13$ or $14$, we get precisely one family for each size:

$\mathcal{F}_{32} = \{ J_1, J_2, J_4, J_6(s'), J_7(s'), J_8(s'), J_9(s'), J_{11}(s'), J_{12}(s'), J_8(s' + 1), J_9(s' + 1) \}$,

$\mathcal{F}_{20} = \{ J_1, J_2, J_3, J_4, J_6(s'), J_7(s'), J_8(s'), J_9(s'), J_{10}(s'), J_9(s' + 1), J_9(s' + 2), J_9(s' + 2) \}$,

$\mathcal{F}_{34} = \{ J_1, J_2, J_3, J_4, J_6(s'), J_7(s'), J_8(s'), J_9(s'), J_{11}(s'), J_{12}(s'), J_8(s' + 1), J_9(s' + 1), J_{10}(s' + 1), J_9(s' + 2) \}$.

Note, in particular, that $|\mathcal{F}(4, n)| \leq 14$ for all sufficiently large $n$. Computer simulations suggest the same may be true for any even $k$, with a similar result for odd $k$, but we leave the investigation of this possibility to a subsequent paper.

**Appendix**

As a prototype for a type of calculation which appears in several places in the paper, we now show, in the notation of Lemma 4, that $s' = S + 1$ when $k$ is even.

We must investigate the condition $l_3(s) < s$. By definition of $l_3$ this is just

\[
\left( \frac{2r_3}{k} \right) < s \iff \frac{2r_3}{k} < s \iff r_3 < \frac{ks}{2} \iff \frac{l_2 + s}{k} < \frac{ks}{2} \iff \frac{l_2 + s}{k} < \frac{ks}{2} \\
\iff l_2 < \left( \frac{k^2}{2} - 1 \right) s \iff \frac{2r_2}{k} < \left( \frac{k^2}{2} - 1 \right) s \iff r_2 < \left( \frac{k^3}{4} - \frac{k}{2} \right) s \\
\iff \frac{l_1 + s}{k} < \left( \frac{k^3}{4} - \frac{k}{2} \right) s \iff l_1 < \left( \frac{k^4}{4} - \frac{k^2}{2} - 1 \right) s \\
\iff \frac{2n}{k} < \left( \frac{k^4}{4} - \frac{k^2}{2} - 1 \right) s \iff n < \left( \frac{k^5}{8} - \frac{k^3}{4} - \frac{k}{2} \right) s \iff s > \frac{8n}{k^5 - 2k^3 - 4k} \\
\iff s > S.
\]

Thus $s' = S + 1$, as required.

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