Excluded Volume Model for Quarkyonic Matter II: Three-flavor Shell-like Distribution of Baryons in Phase Space

Dyana C. Duarte,1 Saul Hernandez-Ortiz,1 and Kie Sang Jeong1

1Institute for Nuclear Theory, University of Washington, Seattle, WA 98195, USA

(Dated: July 28, 2020)

We extend the excluded volume model of iso-spin symmetric two-flavor dense Quarkyonic matter II including strange baryons and quarks and address its implications for neutron stars. The effective size of baryons are defined from the diverging hard-core potentials in the short interdistance regime. Around the hard-core density, the repulsive core between baryons at short-distances leads to a saturation in the number density of baryons and generates the perturbative quarks from the lower phase space which leads to the shell-like distribution of baryons by Pauli’s exclusion principle. The strange quark Fermi sea always appears in the high densities but Λ hyperon shell only appears when the effective size of Λ hyperon is smaller than the effective size of nucleons. We find that the pressure of strange quarkyonic matter can be large enough to support neutron stars with two-time solar mass and can have a large sound speed \(c_s^2 \approx 0.7\). The fraction of the baryon number carried by perturbative quarks is about 30% at the inner core of most massive neutron stars.
I. INTRODUCTION

Observation of GW170817 [2, 3] provided important information for understanding dense nuclear matter. The possible range of tidal deformability is confined in 90% confidence level [2, 3] and the subsequent analyses constrained the corresponding radius $R_{1.4} \leq 13.5$ km [4–11]. Recent GW190425 observation [12] constrained the possible range $R_{M \gtrsim 1.4} \leq 15$ km including higher mass states. Meanwhile, it has been required for the hard enough equation of state (EoS) to support two-times solar mass ($M_{\odot}$) states, which usually leads to larger radius state [13–15]. To reconcile these observations, the EoS should be soft enough for the low density regime and hard enough for the higher density regime, so that the strong pressure of the inner core in higher densities can support larger mass state and the weaker pressure of the outer core in lower densities can satisfy $R_{1.4} \leq 13.5$ km constraint. Then, the expected soft-hard evolution of EoS should accompany the sound velocity $c_s^2 > 0.3$ around few times of the normal nuclear density ($\rho_0$) [16–24]. Beyond the density regime of inner core where the hard EoS is supported, the softened EoS is expected under the causality and conformal limit constraints [5, 6, 18–25].

However, it is hard to reconcile both constraints from fundamental principles. If one considers mean-field potentials between baryons, certain universal repulsive contributions are expected for EoS [26, 27] at high densities as the newly generated degrees of freedom lead to the soft EoS through various decay channels into the low energy states [28–31]. Even if the stiff evolution is obtained by some kind of model, it is hard to explain the expected softening evolution at the high density limit by these same first principles. Some kind of phase transition to quark matter can be introduced. A phase transition to quark matter attenuates the hard nature of the EoS. There is much literature that debates about the signals of such a hypothetical phase transition [16–18, 25, 32–36]. As an alternative candidate for a solution, it is worthwhile to consider Quarkyonic-like model [1, 37–41] which naturally generates the hard-soft evolution of EoS.

The Quarkyonic matter concept is based on large $N_c$ quantum chromodynamics (QCD) [42, 43]. If one supposes a large Fermi sphere ($T \to 0$) in the large $N_c$ limit, the quasi-quark state around the Fermi surface whose momenta are distributed in the range of confinement ($|\vec{k}Q_i - \vec{k}Q_j| < \Lambda_{\text{QCD}}$, $|\vec{k}Q_i| \approx k_F^Q$ where $i, j \in \{1, \cdots N_c\}$) will be confined in the baryon-like state. The confinement mechanism is expected to be similar to the mechanism for the baryon state in vacuum as the Debye screening due to the quark loop is suppressed by $1/N_c$ [37]. In this circumstance, as the confined quark momenta are correlated within $|\vec{k}Q_i - \vec{k}Q_j| < \Lambda_{\text{QCD}}$, the baryon-like state has the minimum momentum $k_F^Q \approx N_c k_F^Q$. Thus, one can expect the shell-like distribution of the baryons and the almost free quarks occupying the lower phase sphere. The transition from the ordinary nuclear matter to Quarkyonic matter may occur at few times of $\rho_0$ where the soft-hard evolution of EoS is expected. Whence $k_F^Q \sim O(\Lambda_{\text{QCD}})$, the lower phase sphere will be saturated by the free quarks. Then, by Pauli’s principle, the momenta of confined quarks should become larger than the saturated momenta $k_F^Q$, which leads to the sudden enhancement of chemical potential of the baryon-like state ($k_F^Q \sim O(\Lambda_{\text{QCD}}) \to N_c k_F^Q$). This is not a usual first-order phase transition as the pressure is not fixed but suddenly and smoothly enhanced by the enhanced chemical potential and there is no discontinuity for the increment of energy density and the baryon number density [37]. From this point, most of baryon number increment is taken by the saturated quarks and eventually, the shell-like baryon distribution will disappear and the perturbative QCD matter will appear at extreme density limit ($k_F^Q \sim O(\sqrt{N_c} \Lambda_{\text{QCD}})$), as the Debye screening begins to block the confinement process ($r_{\text{Debye}} \sim O(N_c^{-1})$).

This concept was introduced to describe the hard-soft evolution of EoS in the previous literature [11, 39, 41]. The model construction with the explicit shell-like distribution reproduced the plausible result satisfying the aforementioned constraints [39]. The 2-flavor generalization of the aforementioned model [39] is studied under the $\beta$-equilibrium condition [40]. In the phenomenological model construction, one can consider the hard-core repulsive interaction whose scale can be regarded as the effective size of the baryon [44, 45]. In the single flavor excluded volume model [1], the repulsive core dynamically generates the shell-like phase structure of baryon which reproduces the stiff evolution of EoS with $c_s^2 \approx 0.7$ as analyzed in the literatures [1, 11, 16, 18, 24]. In the previous work [41], the excluded volume model was extended to the 3-flavor mixture of the baryon and quarks where the scale of repulsive core is adopted from the first principle studies [62–64]. It was argued for the dynamical role of the multi-flavor hard-core repulsion in Ref. [11], but as the shell-like distribution was omitted, the resulting EoS was not hard enough to satisfy the physical constraints.

In this paper, we present the 3-flavor excluded volume model with explicit shell-like distribution of baryons appearing after the saturation momentum of the quark Fermi sea. The paper is organized as follows. In Sec. [11] we present a brief introduction of excluded volume model and the possible structure of the shell-like distribution. In Sec. [111] we explain the physical configurations, EoS, and corresponding mass-radius relations obtained under the physical constraints including the electromagnetic charge neutrality and the equilibrium constraints from weak interactions. Finally, in Sec. [1V] we summarize our results and discuss possible developments for the future work.
II. EXPLICIT STRUCTURE OF SHELL-LIKE DISTRIBUTION OF BARYONS

In dense Quarkyonic matter \cite{37}, the quark wave functions distributed around the quark Fermi surface are clearly confined in baryon-like states because Debye screening is suppressed in large $N_c$ limit \cite{42,43}. The matter looks like the normal nuclear matter in the low density regime as the momentum of a quark is distributed in the confinement range. However, when the matter density reaches few times of $\rho_0$ where $k_F^b \sim O(\Lambda_{\text{QCD}})$ so that lowest momentum states become distributed away from the clear confinement range, the quark Fermi sea is formed from the low momentum phase space. Whence the Fermi sea is saturated, the confined quark should take larger momentum than the fully occupied lower phase by Pauli’s exclusion principle (shell-like momentum distribution of baryons) \cite{1,37,39}. Around the onset moment, the pressure of the system will be continuously and stiffly increasing as the chemical potential should

\begin{equation}
\sum_{n,p,\Lambda} \frac{1}{n} \left( 1 - \frac{\tilde{n}_b}{n_0} \right) \int_0^{K_F^b} \frac{d^3k}{k^3} k^2 \left( n_k^p + n_k^\Lambda + (1 + \alpha)n_\Lambda \right),
\end{equation}

where $K_F^b$ represents the enhanced Fermi momentum due to the reduced available volume and $\alpha$ determines the strength of hard core repulsive interaction between surrounding baryon and $\Lambda$ hyperon in range of $|\alpha| < 0.2$. In the context of the presumed effective size of particle, this approach could be understood as the cold-dense limit of the Van der Waals (VdW) EoS in Fermi-Dirac statistics \cite{49,63}. As a simple example, $K_F^b$ can be obtained from $\mu^* = \mu^{id}(n_F^{\text{ex}}, T \to 0)$ without attraction term if one derives the intensive number from the VdW EoS in Fermi-Dirac statistics \cite{65}. However, as we focus on the high density regime where the interdistance of particles becomes order of the hard-core radius, $n_0 > 0.65$ fm$^{-3} \sim 4\rho_0$ will be considered, which is a different order of magnitude from the size used in Refs. \cite{55,63}. One may adopt a well constructed model \cite{18,20,24,26,27,61} and use a Maxwell construction to accommodate the low density properties of nuclear matter. The variation range $|\alpha| < 0.2$ for the hard-core size of $\Lambda$ is supposed by considering the possible error band of $\Lambda N$ potential from lattice QCD \cite{65,68}, which is relatively small in comparison with the variation range studied in Refs. \cite{55,60}. If the SU(3) flavor symmetry breaking term is non-negligible or kaon condensation plays a significant role \cite{72,74}, the effective size of $\Lambda$ can be different from current variation range.

The energy density of the corresponding system can be described as the one of non-ideal free fermions having effective size \cite{1,41}:

\begin{equation}
\varepsilon_b = \left( 1 - \frac{\tilde{n}_b}{n_0} \right) \frac{1}{\pi^2} \sum_{\text{n.p.}\Lambda} \int_0^{K_F^b} dk k^2 \left( n_k^p + n_k^\Lambda + (1 + \alpha)n_\Lambda \right) + \frac{(3\pi^2)^{\frac{5}{3}}}{4\pi^2} - n_b^\frac{4}{3} + \cdots,
\end{equation}

where the electron mass is suppressed. If one takes non-relativistic limit, baryon chemical potential can be obtained as follows:

\begin{equation}
\mu_i \simeq m_i + \frac{(3\pi^2)^{\frac{5}{3}}}{10\pi^2 m_i} n_b^\frac{4}{3} + \omega_i \sum_{j} \frac{(3\pi^2)^{\frac{5}{3}}}{10\pi^2 m_j} \frac{2}{3n_0} n_b^\frac{4}{3} + \cdots,
\end{equation}

A. Brief summary of excluded volume model for Quarkyonic-like model

As introduced in the previous literature \cite{1,41}, one can simplify the baryon-baryon central potential whose strong repulsive core is expected at high density regime \cite{65,69} by supposing the infinite-well shaped potential whose hard-core radius is around $r_c \simeq 0.6$ fm scale. Among the low-lying baryon octet, nucleons and $\Lambda$ hyperon would be the lightest particles which have the strong repulsive core at short interdistance according to the lattice QCD calculation \cite{65–69}. The phenomenological configuration strongly depends on how one defines the lower boundary of the distribution because the dynamical equilibrium constraints are related through the shell-like distribution. In this section, we will briefly introduce the excluded volume model approach, and present the explicit structure of the shell-like baryon distribution formed by the dynamically saturated quark Fermi sea. We will use the following abbreviations to denote the baryons and quarks: $B$ represents the total baryons including quarks, $b$ represents the baryon (hadron) and the $Q$ represents the saturated quarks. The abbreviations appearing with the lowcase romans, $b_i$ represents the baryon flavor flavors \{$n, p, \Lambda$\} and $Q_i$ represents the quark flavors \{$u, d, s$\}.\n
\begin{equation}
\sum_{i} n_{b_i} = \frac{n_b}{1 - \tilde{n}_b/n_0} = \frac{2}{(2\pi)^3} \int_0^{K_{F_b}^b} d^3k,
\end{equation}
\begin{equation}
\tilde{n}_b = n_n + n_p + (1 + \alpha)n_\Lambda,
\end{equation}

where $K_{F_b}^b$ represents the enhanced Fermi momentum due to the reduced available volume and $\alpha$ determines the strength of hard core repulsive interaction between surrounding baryon and $\Lambda$ hyperon in range of $|\alpha| < 0.2$. In the context of the presumed effective size of particle, this approach could be understood as the cold-dense limit of the Van der Waals (VdW) EoS in Fermi-Dirac statistics \cite{49,63}. As a simple example, $K_{F_b}^b$ can be obtained from $\mu^* = \mu^{id}(n_F^{\text{ex}}, T \to 0)$ without attraction term if one derives the intensive number from the VdW EoS in Fermi-Dirac statistics \cite{65}. However, as we focus on the high density regime where the interdistance of particles becomes order of the hard-core radius, $n_0 > 0.65$ fm$^{-3} \sim 4\rho_0$ will be considered, which is a different order of magnitude from the size used in Refs. \cite{55,63}. One may adopt a well constructed model \cite{18,20,24,26,27,61} and use a Maxwell construction to accommodate the low density properties of nuclear matter. The variation range $|\alpha| < 0.2$ for the hard-core size of $\Lambda$ is supposed by considering the possible error band of $\Lambda N$ potential from lattice QCD \cite{65,68}, which is relatively small in comparison with the variation range studied in Refs. \cite{55,60}. If the SU(3) flavor symmetry breaking term is non-negligible or kaon condensation plays a significant role \cite{72,74}, the effective size of $\Lambda$ can be different from current variation range.

The energy density of the corresponding system can be described as the one of non-ideal free fermions having effective size \cite{1,41}:

\begin{equation}
\varepsilon_b = \left( 1 - \frac{\tilde{n}_b}{n_0} \right) \frac{1}{\pi^2} \sum_{\text{n.p.}\Lambda} \int_0^{K_{F_b}^b} dk k^2 \left( n_k^p + n_k^\Lambda + (1 + \alpha)n_\Lambda \right) + \frac{(3\pi^2)^{\frac{5}{3}}}{4\pi^2} - n_b^\frac{4}{3} + \cdots,
\end{equation}

where the electron mass is suppressed. If one takes non-relativistic limit, baryon chemical potential can be obtained as follows:

\begin{equation}
\mu_i \simeq m_i + \frac{(3\pi^2)^{\frac{5}{3}}}{10\pi^2 m_i} n_b^\frac{4}{3} + \omega_i \sum_{j} \frac{(3\pi^2)^{\frac{5}{3}}}{10\pi^2 m_j} \frac{2}{3n_0} n_b^\frac{4}{3} + \cdots,
\end{equation}
where $\omega_i = \partial \tilde{n}_b / \partial n_i$ ($\omega_{n,p} = 1$, $\omega_\Lambda = 1 + \alpha$). As one can find from the third term, the chemical potential of a specific flavor [1] can be enhanced without having large $n_{k_F}^x$ if the some part of system volume is occupied by the other finite size particles. Thus, to accommodate a heavier baryon (denote flavor $h$), its effective size should be small so that the contribution from the third term be suppressed ($\omega_h \ll 1$). Therefore, it is unlikely to have the higher mass state such as $\Delta(1232)$ if the particle has the effective size in similar order to $n_0$. Due to the intrinsic divergence around $n_B \sim n_0$, this system cannot accommodate $n_B > n_0$ and contains unphysical configuration ($n_s^2 \gg 1$).

If new degrees of freedom are considered as is done in the Hagedorn model [73], the dynamically generated quark degrees freedom lead to a physically plausible explanation in accordance with Quarkyonic matter concept [1][41]. Once the quark Fermi sea is saturated, the baryons should have the shell-like distribution in the momentum space as a consequence of Pauli’s exclusion principle: the quarks confined in the baryon should have the momentum larger than the saturated quark Fermi momentum. If one assumes the iso-spin symmetric quark configuration as discussed in Ref. [1], the lower boundary of the distribution is simply obtained as $k_F^b = N_c k_F^Q$. Even if the asymmetric configuration is considered, the scale can be estimated around $k_F^b \sim N_c \max.\ {k_F^u, k_F^d, k_F^s}$ as the quarks confined in a baryon should have a common scale of momentum. The detailed argument for the $k_F^b$ in the iso-spin asymmetric configuration will be given in next subsection. The baryon number in excluded volume density within the explicit shell-like structure can be written from $k_F^b$ as

$$\tilde{n}_{b_i} = \frac{n_b}{1 - \tilde{n}_b/n_0} = \frac{2}{(2\pi)^3} \int_{k_F^b}^{[k_F + \Delta]_{b_i}} d^3k, \quad \text{(5)}$$

where the upper boundary of the baryon distribution has been defined by assuming the fully occupied phase space:

$$[k_F + \Delta]_{b_i} = \left(3\pi^2 \tilde{n}_{b_i} + k_F^{b_i} \right)^{\frac{2}{3}} , \quad \text{(6)}$$

where the $\Delta$ is the width of the baryon distribution [1].

In Quarkyonic matter, the quark Fermi sea would be continuously saturated without any signature of a first-order phase transition according to large $N_c$ gauge dynamics [1][37][41]. Thus, smooth interpolation of a quark’s energy should be possible in both directions around the Fermi surface. In this excluded volume model approach, the energy interpolation is continuous by analytic definition but an unphysical divergence appears at the onset moment of saturation. Huge energy enhancement due to the sudden formation of the shell-like baryon distribution leads to the unphysical energy dispersion relation corresponding to $\partial n_B / \partial n_0 \gg 1, \partial n_B / \partial n_0 \ll 0$ ($n_B = n_b + n_Q$). To attenuate the unphysical divergence, an enhanced phase measure $\mathcal{M}_i(k^2)$ for the saturated quarks can be introduced. The modified measure $\mathcal{M}_i(k^2) > k^2$ effectively enhances the free quark density around the saturation moment of free quarks and converges to the ideal gas limit ($\mathcal{M}_i(k^2) \rightarrow k^2$) at high density regime ($k_F^Q \gg \Lambda_{\text{QCD}}$). Then the quark number density can be written in baryon number unit as follows:

$$n_{Q_i} = \frac{1}{\pi^2} \int_0^{k_F^Q} dk \mathcal{M}_i(k^2), \quad \text{(7)}$$

where the tilde in the subscript denotes the number density in baryon unit. The relatively rapid growth of quark density at the onset moment ($n_b \approx n_0$) makes an effective barrier for $\delta n_Q$ in the variation of the baryon number density ($n_b < n_0 - \delta n_Q$), which prevents the unphysical divergence and leads to the gradual formation of the shell-like distribution. The energy density with explicit shell-like baryon distribution can be written as follows:

$$\varepsilon_{\text{QY}} = 2 \left(1 - \frac{n_b}{n_0}\right) \sum_i \int_{k_F^b}^{[k_F + \Delta]_{b_i}} d^3k \left(2\pi\right)^3 \frac{1}{(2\pi)^3} \left(k^2 + m_{b_i}^2\right)^{\frac{1}{2}} + \frac{N_c}{\pi^2} \sum_j \int_0^{k_F^Q} dk \mathcal{M}_j(k^2) \left(k^2 + m_{Q_j}^2\right)^{\frac{1}{2}} + \frac{(3\pi^2)^{\frac{1}{2}}}{4\pi^2} n_s^4. \quad \text{(8)}$$
Corresponding baryon (n, p, and Λ) chemical potential can be obtained as

\[
\mu_b = \frac{\partial \varepsilon_{\text{av.}}}{\partial n_b} = \left(1 - \frac{n_b}{n_0}\right) \left\{ \frac{[k_F + \Delta]_{b_1}^2}{\pi^2} \left(\frac{[k_F + \Delta]_{b_1}^2 + m_{b_1}^2}{\pi^2} \right)^{\frac{1}{2}} \frac{\partial [k_F + \Delta]_{b_1}}{\partial n_b} \right. \\
+ \left. \sum_{j \neq i} \frac{\omega_j}{n_0} \left(\frac{[k_F + \Delta]_{b_j}^2}{\pi^2} \left(\frac{[k_F + \Delta]_{b_j}^2 + m_{b_j}^2}{\pi^2} \right)^{\frac{1}{2}} \frac{\partial [k_F + \Delta]_{b_j}}{\partial n_b} \right) \right\}
\]

\[\omega_j = \frac{\omega_i^{\text{(n.p.A)}}}{\sum_{j \neq i} \omega_j^{\text{(n.p.A)}}} \left(\frac{[k_F + \Delta]_{b_j}^2}{\pi^2} \left(\frac{[k_F + \Delta]_{b_j}^2 + m_{b_j}^2}{\pi^2} \right)^{\frac{1}{2}} \frac{\partial [k_F + \Delta]_{b_j}}{\partial n_b} \right) \right\}, \tag{9}
\]

where the partial derivatives are calculated as

\[
\frac{\partial [k_F + \Delta]_{b_i}}{\partial n_{b_i}} = \frac{\pi^2}{[k_F + \Delta]_{b_i}^2} \left(\frac{1}{1 - \tilde{n}_b - \omega_i n_b} \right) \left(\frac{1 - \tilde{n}_b - \omega_i n_b}{n_0} \right), \tag{10}
\]

\[
\frac{\partial [k_F + \Delta]_{b_i}}{\partial n_{b_i}} = \frac{\pi^2}{[k_F + \Delta]_{b_i}^2} \left(\frac{1}{1 - \tilde{n}_b - \omega_i n_b} \right) \left(\frac{\omega_i n_b}{n_0} \right), \tag{11}
\]

with \(\omega_{n,p} = 1\), \(\omega_\Lambda = 1 + \alpha\). Again, the characteristic feature of excluded volume model can be found from the \(\omega_i\) dependent terms of chemical potential [9]. Even if there exist only few numbers of a specific flavor of baryon, the corresponding chemical potential can be enhanced if the space is taken by the other baryons. By the same reason, we only consider 3-flavors for the baryon side as n, p, and Λ are expected to have similar order of \(n_0\) [1]. The quark chemical potential in baryon units can be obtained in a similar way:

\[
\mu_{\tilde{Q}_i} = \frac{\partial \varepsilon_{\text{av.}}}{\partial n_{\tilde{Q}_i}}
\]

\[
= \left(1 - \frac{n_b}{n_0}\right) \sum_k \left\{ \frac{[k_F + \Delta]_{b_k}^2}{\pi^2} \left(\frac{[k_F + \Delta]_{b_k}^2 + m_{b_k}^2}{\pi^2} \right)^{\frac{1}{2}} \frac{\partial [k_F + \Delta]_{b_k}}{\partial n_{\tilde{Q}_i}} \right. \\
+ \left. \sum_{j \neq i} \frac{\partial k_{Q_i}^{b_j}}{\partial k_{Q_i}^{b_j}} \left(\frac{[k_F + \Delta]_{b_j}^2 + m_{b_j}^2}{\pi^2} \right)^{\frac{1}{2}} \frac{\partial [k_F + \Delta]_{b_j}}{\partial n_{\tilde{Q}_i}} \right\} + N_c \left(\frac{k_{Q_i}^{b_k}}{n_0} \right)^2, \tag{12}
\]

where the partial derivatives are calculated as

\[
\frac{\partial [k_F + \Delta]_{b_k}}{\partial n_{\tilde{Q}_i}} = \frac{\pi^2}{[k_F + \Delta]_{b_k}^2} \left(\frac{k_{b_k}^{b_k}}{[k_F + \Delta]_{b_k}^2} \right)^{\frac{1}{2}} \frac{\partial k_{b_k}^{b_k}}{\partial n_{\tilde{Q}_i}}, \tag{13}
\]

\[
\frac{\partial k_{Q_i}^{b_j}}{\partial n_{\tilde{Q}_i}} = \frac{\pi^2}{M_i \left(k_{Q_i}^{b_j} \right)^2}. \tag{14}
\]

As a consequence of Pauli’s exclusion principle, the chemical potential of saturated quark gets contributions from the baryon distribution as well because \(k_{Q_i}^{b_j}\) emerges as a consequence of the saturated quark Fermi sea. Also, one

1 A detailed argument for the possible emergence of \(\Delta(1232)\) is given in Appendix A.
can anticipate another singularity possibly arising in the iso-spin asymmetric configuration. Suppose the shell-like distribution \( k^b_F > 0 \) formed by ahead saturation of \( d \) quark Fermi sea \( (k^d_F > 0) \) and \( u \) Fermi quark sea about to appear \( (k^u_F \simeq 0) \). Then, the derivatives (13)-(14) may diverge if \( \mathcal{M}_u (k^u_F^2) \to 0 \) in \( k^u_F \to 0 \) limit. To avoid the singularity problem, an infrared regulator can be simply introduced as \( \mathcal{M}_i (k^2) = k^2 + \Lambda^2_{Q_i} \) which leads to following configurations:

\[
\begin{align*}
n_{\bar{Q}_i} &= \frac{1}{\pi^2} \int_{0}^{k^2_{F_{\bar{Q}_i}}} dk \left( k^2 + \Lambda^2_{Q_i} \right) = \frac{k^2_{F_{\bar{Q}_i}}}{2\pi^2} \left( 1 + 3 \left( \Lambda_{Q_i}/k^2_{F_{\bar{Q}_i}} \right)^2 \right), \\
\varepsilon_{Qy} &= 2 \left( 1 - \bar{n}_b/n_0 \right) \sum_{i} \int_{k^2_{F_{\bar{Q}_i}}}^{k^2_{F_{\bar{Q}_i}} + \Delta k_{b_i}} \frac{d^3k}{(2\pi)^3} \left( k^2 + m^2_{b_i} \right)^{\frac{1}{2}} + N_c \frac{\pi^2}{\Lambda^2_{Q_i}} \sum_{j} \int_{0}^{k^2_{F_{Q_j}}} dk \left( k^2 + \Lambda^2_{Q_j} \right) \left( k^2 + m^2_{Q_j} \right)^{\frac{1}{2}} + \frac{(3\pi^2)^2}{4\pi^2} n^4_e,
\end{align*}
\]

where the criteria for \( \mathcal{M}_i (k^2) \) are satisfied in the both limits. The regulator \( \Lambda_{Q_i} \) could be understood as an a priori non-perturbative contribution remaining on the saturated quark Fermi surface.

**B. Explicit structure of shell-like distribution of baryon**

The iso-spin asymmetry naturally appears under consideration of electro-weak interactions and subsequent equilibrium conditions. This asymmetric configuration can arise in either of the baryons and quarks. The details will be strongly dependent on \( k^b_F \) as the physical constraints between the baryons and quarks are related through the shell-like baryon distribution. \( k^b_F \) can be supposed differently depending on the assumption about the confined quark state distributed slightly above the saturated quark Fermi surface: if the confined quark momenta around the saturated quark Fermi surface are strongly correlated, \( k^b_F \) should show weak dependence on the flavor asymmetry while it can depend strongly on the asymmetry if the confined quark momenta are weakly correlated. Following we propose two phenomenological assumptions for the two different scenarios.
1. Assumption I: strongly correlated momentum of confined quark

In the large $N_c$ limit, one may assume a strong correlation between the momenta of confined quarks as the confinement mechanism should be very similar to the one of hadron state in vacuum. This clear confinement should occur even for the quarks whose momentum is distributed just above the Fermi surface, where the occupation number is almost one. In a simplest guess for the constituent quarks of baryon, one can imagine the confined quarks sharing a same size of momentum $k_F^q = k_F^q/3$ balanced by the internal interaction. If all the quark momenta are strongly correlated as appearing in this simple guess, the difference between the momenta of two confined quarks should be minimal, even though the flavor asymmetry of saturated quarks becomes large ($|k_F^{u,F} - k_F^{d,F}| \ll \Lambda_{QCD}$, $k_F^{u,F} > k_F^{d,F}$, and $|k_F^{d,F} - k_F^{s,F}| > \Lambda_{QCD}$ where $i,j$ denotes the quark flavor). For example, one may imagine iso-spin asymmetric configuration where $d$ quarks are saturated first as total baryon density increases and $u$ quarks follow after (Fig. 1(a)). Then, the momentum of $u$ quark confined in the lower boundary of the baryon shell should be closely distributed around $k_F^u$ as follows:

$$k_{conf.}^u = k_F^u + r_{qq}^u w_q (k_F^u - k_F^d),$$

(17)

where $r_{qq}^u$ determines the correlation strength and the strong correlation weight function $w_q(x)$ is assumed to slowly converge to 1 in the $x = |k_F^u - k_F^d| > \Lambda_{QCD}$ limit:

$$w_q(x) = 1 - \exp\left(-|x|^2/\delta^2\right),$$

(18)

where $\delta$ determines the non-trivial range ($w_q(x < \Lambda_{QCD}) < 1$). If one assigns a small negative $r_{qq}^u$, the minimal difference $|k_{conf.}^u - k_{conf.}^d| \ll \Lambda_{QCD}$ will be guaranteed. As illustrated in Fig. 2(a), $k_{conf.}^u \to k_F^u + r_{qq}^u$ in $|k_F^d - k_F^u| > \Lambda_{QCD}$ limit and $k_{conf.}$ slowly reduces even when a large $|r_{qq}^u|$ is assigned, which implies $k_{conf.}^F \approx N_{max.} [k_F^u, k_F^d]$. Under this assumption, one can anticipate the minimal flavor asymmetry in quark Fermi sea: populating a specific flavor of quark Fermi sea is saturated first as total baryon density increases and the flavor asymmetry of saturated quarks becomes large ($|r_{qq}^u|$). In the strong correlation assumption, the lower boundary of baryon distribution can be defined as follows:

$$k_F^u = \Theta(k_F^d - k_F^u) (3k_F^d + r_{qq}^u w_q (k_F^d - k_F^u)) + \Theta(k_F^u - k_F^d) (3k_F^u + 2r_{qq}^u w_q (k_F^u - k_F^d)),$$

(19)

$$k_F^d = \Theta(k_F^d - k_F^u) (3k_F^d + 2r_{qq}^u w_q (k_F^d - k_F^u)) + \Theta(k_F^u - k_F^d) (3k_F^u + r_{qq}^u w_q (k_F^u - k_F^d)),$$

(20)

$$k_F^s = \Theta(k_F^d - k_F^u) (3k_F^d + r_{qq}^u w_q (k_F^d - k_F^u) + r_{qq}^u w_q (k_F^u - k_F^d)) + \Theta(k_F^u - k_F^d) (3k_F^u + r_{qq}^u w_q (k_F^u - k_F^d)),$$

(21)

where $\Theta(x)$ represents the unit step function and $r_{qq}^u$ determines the correlation strength between the light and strange quark momentum. Around the saturation moment of quark Fermi sea, $k_F^u$ will be determined by the largest quark Fermi momentum as $k_F^u \approx 3k_F^d$. Also, $k_F^u \geq k_F^d$, $k_F^u \geq k_F^d$ conditions in the high density regime are understood from the weak decay channel of $d$ and $s$ quarks.

2. Assumption II: weakly correlated momentum of confined quark

On the other hand, one can imagine the weakly correlated momenta of the confined quarks in the confinement range $|k_{conf.}^u - k_{conf.}^d| \lesssim \Lambda_{QCD}$. If one considers the non-zero chiral condensate in the confined baryon phase and the symmetry restoration at high density regime [77–83], the confinement mechanism of the quarks distributed slightly above the saturated Fermi surface would be quite different from the one of vacuum case where the symmetry is broken. The confined state would rather look like the correlated state of three non-perturbative quarks whose ground energy scale is $m_b \approx 1$ GeV, than the clearly distinguishable baryon state. In this weakly correlated assumption, if the confined quark momentum of a specific flavor becomes enhanced by saturation ($k_{conf.}^d > k_{conf.}^u$) so that the flavor asymmetry becomes large, the other confined quarks can take some lower unoccupied phase space ($k_F^u > k_{conf.}^u > k_F^d - \Lambda_{QCD}$) to minimize the ground state energy (Fig. 1(b)). If we assume the similar condition where $d$ quark Fermi sea is saturated first, then confined $u$ quark momentum can be suggested as

$$k_{conf.}^u = k_F^u + r_{qq} w_q (k_F^u - k_F^d),$$

(22)

where $r_{qq}$ determines the weak correlation strength and the weak correlation weight function $w_q(x)$ is assumed to rapidly converge to 1 in the $x = |k_F^u - k_F^d| > \Lambda_{QCD}$ limit:

$$w_q(x) = \text{erf}(x/\delta),$$

(23)
As summarized in Ref. [41], the weak interactions leads to following constraints:

\[
\text{ration should be constrained by the baryon number conservation, charge neutrality, and possible weak interactions.}
\]

The Fermi surface are allowed to decay onto the other Fermi surface of different flavor. Under the baryon number conservation, within \( k_F^u > k_F^d \) condition. \textbf{Left (a):} \( k_F^u \) under the strong correlation assumption \([17]\) is plotted with black solid (red dashed) line and \( r_{qq}^u = -30 \text{ MeV} \) (\( r_{qq}^u = -60 \text{ MeV} \)). The confined quark momenta are closely located around \( k_F^u \) and weakly dependent on \( |k_F^u - k_F^d| \). \textbf{Right (b):} \( k_F^u \) under the weak correlation assumption \([22]\) is plotted with black solid (red dashed) line and \( r_{qq}^u = -100 \text{ MeV} \) (\( r_{qq}^u = -140 \text{ MeV} \)). The confined quark momenta rapidly deviate away from \( k_F^u \) as \( |k_F^u - k_F^d| \) becomes large.

\[
\text{where \( \text{erf}(x) \) denotes the error function and \( \delta \) has the same role given in Eq. \([17]\). With a large negative \( r_{qq}^w \), the non-negligible difference \( |k_{\text{conf.}}^u - k_{\text{conf.}}^d| < \Lambda_{\text{QCD}} \) can be obtained. As one can find in Fig. 2(b), the error function makes relatively fast reduction of \( k_{\text{conf.}}^u \), even at small \( |k_F^u - k_F^d| \). Comparing to the case of strongly correlated assumption, relatively larger flavor asymmetry is anticipated among the saturated quarks: if \( k_{\text{conf.}}^u \) is distributed away from \( k_F^u \), \( k_{\text{conf.}}^u \) can have smaller magnitude than \( k_{\text{conf.}}^u \).} \]

\[
\text{Under the same conditions, } k_F^u \text{ can be defined as follows:}
\]

\[
k_F^u = \Theta(k_F^d - k_F^u) \left( 3k_F^d + r_{qq}^w w \left( k_F^d - k_F^u \right) \right) + \Theta(k_F^u - k_F^d) \left( 3k_F^u + 2r_{qq}^w w \left( k_F^u - k_F^d \right) \right),
\]

\[
k_F^d = \Theta(k_F^u - k_F^d) \left( 3k_F^u + 2r_{qq}^w w \left( k_F^u - k_F^d \right) \right) + \Theta(k_F^d - k_F^u) \left( 3k_F^d + r_{qq}^w w \left( k_F^d - k_F^u \right) \right),
\]

\[
k_F^s = \Theta(k_F^d - k_F^s) \left( 3k_F^d + r_{qq}^w w \left( k_F^d - k_F^s \right) \right) + \Theta(k_F^s - k_F^d) \left( 3k_F^s + r_{qq}^w w \left( k_F^s - k_F^d \right) \right) + \Theta(k_F^u - k_F^s) \left( 3k_F^u + r_{qq}^w w \left( k_F^u - k_F^s \right) \right) + \Theta(k_F^s - k_F^u) \left( 3k_F^s + r_{qq}^w w \left( k_F^s - k_F^u \right) \right) + \Theta(k_F^d - k_F^s) \left( 3k_F^d + r_{qq}^w w \left( k_F^d - k_F^s \right) \right) + \Theta(k_F^s - k_F^d) \left( 3k_F^s + r_{qq}^w w \left( k_F^s - k_F^d \right) \right),
\]

\[
\text{where } r_{qq}^s \text{ determines the correlation strength between the light and strange quark momentum.}
\]

\[
\text{III. EQUATION OF STATE FOR THE QUARKYONIC-LIKE MATTER}
\]

\textbf{Equilibrium constraints and parameter set:} in the 3-flavor system with electron clouds, the physical configuration should be constrained by the baryon number conservation, charge neutrality, and possible weak interactions. As summarized in Ref. [41], the weak interactions leads to following constraints:

\[
\mu_n = \mu_p + \mu_e, \quad \mu_d = \mu_{\bar{u}} + N_c \mu_e, \quad \mu_s = \mu_{\bar{d}} \text{ (when } n_A \neq 0, n_A = 0 \text{ if } \mu_A < m_A), \quad \mu_{\bar{s}} = \mu_{\bar{u}} \text{ (when } n_{\bar{z}} \neq 0, n_{\bar{z}} = 0 \text{ if } \mu_{\bar{z}} < N_c m_{\bar{s}}),
\]

\[
\text{where } \mu_{\bar{Q}} = N_c \mu_Q \text{ denotes the quark chemical potential in the unit of baryon number. The saturated quark on the Fermi surface are allowed to decay onto the other Fermi surface of different flavor. Under the baryon number}
\]

FIG. 2. Illustration of \( k_{\text{conf.}}^u \) within \( k_F^u > k_F^d \) condition. \textbf{Left (a):} \( k_F^u \) under the strong correlation assumption \([17]\) is plotted with black solid (red dashed) line and \( r_{qq}^u = -30 \text{ MeV} \) (\( r_{qq}^u = -60 \text{ MeV} \)). The confined quark momenta are closely located around \( k_F^u \) and weakly dependent on \( |k_F^u - k_F^d| \). \textbf{Right (b):} \( k_F^u \) under the weak correlation assumption \([22]\) is plotted with black solid (red dashed) line and \( r_{qq}^u = -100 \text{ MeV} \) (\( r_{qq}^u = -140 \text{ MeV} \)). The confined quark momenta rapidly deviate away from \( k_F^u \) as \( |k_F^u - k_F^d| \) becomes large.
conservation and charge neutrality, these constraints leads to the dynamical equilibrium condition:

\[
\begin{align*}
& \text{if } n_\Lambda = 0, \, n_s = 0, \, \mu_n = N_c \mu_d - \mu_e = \mu_p + \mu_e, \\
& \text{if } n_\Lambda \neq 0, \, n_s \neq 0, \, \mu_n = N_c \mu_d - \mu_e = \mu_\Lambda = \mu_p + \mu_e = N_c \mu_s - \mu_e, \\
& \text{if } n_\Lambda = 0, \, n_s \neq 0, \, \mu_n = N_c \mu_d - \mu_e = \mu_\Lambda = \mu_p + \mu_e, \\
& \text{if } n_\Lambda \neq 0, \, n_s = 0, \, \mu_n = N_c \mu_d - \mu_e = \mu_p + \mu_e = N_c \mu_s - \mu_e,
\end{align*}
\]

where $\mu_n = N_c \mu_d - \mu_e$ is the generalization of the dynamical equilibrium constraint $\mu_N = N_c \mu_Q$ in the iso-spin symmetric configuration $[1]$. Hereafter, we will calculate all the physical quantities under the constraints. Following numbers will be used as the representative parameter set: $N_c = 3$ for the number of colors, $\{m_{n,p} = 1 \text{ GeV}, m_\Lambda = 1.2 \text{ GeV}, m_{q, s} = 0.333 \text{ GeV}, m_s = 0.533 \text{ GeV}\}$ for the fermion masses, $\{r_{qs}^u = -30 \text{ MeV and } r_{qs}^w = -60 \text{ MeV}\}$ for the strong correlation assumption, $\{r_{qq}^u = -100 \text{ MeV and } r_{qq}^w = -140 \text{ MeV}\}$ for the weak correlation assumption, $n_0 = 6\rho_0$ for the hard-core density ($r_c \simeq 0.6 \text{ fm}$) and corresponding regulator $\Lambda_Q = 180 \text{ MeV}$ which attenuate the unphysical noise.

### A. Density profile of particles in the excluded volume model with shell-like baryon distribution

We present the density profile of particles to understand the complicated dynamical properties from the shell-like distribution. The density profiles of particles are plotted in Fig. 3 under $n_0 = 6\rho_0$, and $\Lambda_Q = 180 \text{ MeV}$ conditions. As one can find in the profile (a) and (c) of Fig. 3, the stronger repulsive core ($\alpha = 0.2$) for $\Lambda$ hyperon suppresses the emergence of $\Lambda$ degree of freedom even in the high density regime while the weaker repulsive core ($\alpha = -0.2$) allows $n_\Lambda > 0$ at the same regime as shown in the profile (b) and (d). One can find the reason from the $\omega_\Lambda$ dependent terms of the baryon chemical potential $[9]$, it becomes hard to satisfy the equilibrium constraint $[29]$ with the other constraints in simultaneous way because $\mu_\Lambda$ is enhanced by $\omega_\Lambda > 1$.

Meanwhile, when $\Lambda$ degree of freedom is suppressed, $s$ quark takes relatively large portion than the one of the cases where $n_\Lambda > 0$ with $\alpha = -0.2$ ((b) and (d) of Fig. 3). As can be found in Eq. $[12]$, the quark chemical potential has the contribution from the shell-like distribution if the quark Fermi momentum is related to the confined quark momentum via Pauli’s principle. Because there is no $\Lambda$ shell-like distribution in the $\mu_s$ for $\alpha = 0.2$ case, relatively large $n_c$ can be accommodated satisfying the constraint $[30]$ where $\mu_c$ has the contribution from the $n, p$ shell. By the same reason, one can understand the difference between the profiles from the strong and weak correlation assumptions. Under the strong correlation of the confined quark momenta, the large iso-spin asymmetry in the quark Fermi sea enhances the lower boundary of nucleon momentum as $k_F^{n,p} \simeq 3k_F^d$, by which the nucleons in the shell obtain huge energy enhancement. Thus, it is dynamically favored for the iso-spin symmetric configuration of the light quark Fermi sea by the constraint $[33]$ (Fig. 3(a)). If there is the $\Lambda$ shell ($n_\Lambda > 0$), all the quark Fermi sea becomes almost symmetric and the asymmetric configuration only appears in the shell-like distribution of the baryon side by the constraint $[32]$ in the high density regime (Fig. 3(b)). However, if the confined quark momenta are weakly correlated, it is allowed to have large flavor asymmetry for the quark Fermi sea in the same density regime (see the profiles (c) and (d) in Fig. 3) as $k_F^{p,n} \leq 3k_F^d$ and $k_F^\Lambda \leq 3k_F^s$. Therefore, the constraints $[32]$ and $[33]$ can be satisfied in that asymmetric configuration.

In all the cases, the quark Fermi sea is saturated in order of $d, u, s$ quark flavor. After the saturation, ($n_B \geq 5\rho_0$), each baryon density profile looks converging to the asymptotic number and the quark Fermi sea takes all the increment of the baryon number density ($dn_B \simeq dn_Q$). By definition, this model does not contain the essential attractive and repulsive potentials required to reproduce the low density properties of nuclear matter. The saturation moment of $d$ quark Fermi sea can differ by the proper modifications to acquire the low density properties. The expected possible configurations at the low density regime is denoted as shaded area in Fig. 3. The qualitative behavior of the density profile does not change when different hard-core density $n_0 = 5\rho_0$ is assigned.

### B. Equation of state and speed of sound

In the zero-temperature limit, the pressure and corresponding sound velocity can be found as

\[
\begin{align*}
p_{qy} &= -\varepsilon_{qy} + \mu_B n_B, \\
\varepsilon_s^2 &= \left(\frac{\partial p_{qy}}{\partial \varepsilon_{qy}}\right)_s = \frac{n_B}{\mu_B} \frac{\partial m_B}{\partial n_B}.
\end{align*}
\]
FIG. 3. Density profiles of fermions ($n_0 = 6\rho_0$, and $\Lambda Q = 0.18$ GeV). The profiles in the upper (a, b) and lower (c, d) sides are obtained under the strong and weak correlation assumption, respectively. The profiles in the left (a, c) and right (b, d) sides are obtained under $\alpha = 0.2$ and $\alpha = -0.2$ condition, respectively. The shaded band denotes the possible deviation depending on the saturation moment of $d$ quark Fermi sea which may rely on the proper EoS which covers the low density regime. As can be found in the EoS plots (Figs. 4(a, c)), the EoS stiffly increases around the saturation moment of the quark Fermi sea. The evolution becomes more stiff when the stronger repulsive core ($n_0 = 5\rho_0$) is considered. Because each quark flavor can be separately saturated in this system, there are several stiffly increasing segments in the evolution curve of EoS. This tendency does not appear when all the quark flavors simultaneously saturate [140]. This evolution look similar to the results presented in Refs. [34, 35] where the first-order phase transition is implied via the hybrid quark-meson-meson model [32–35]. However, as one can find in the sound velocity plots (Figs. 4(b, d)), $c_s^2 > 0$ in the stiffly increasing segment, which means that our model does not present the first-order phase transition around the onset moment of quark sea saturation. In the high density regime, the stiffness becomes moderated and looks converging to the relativistic ideal limit ($c_s^2 = 1/3$) as one can anticipate from the definition of the model [16]. Although the stiffness converges to the ideal limit, the weaker repulsive core ($n_0 = 6\rho_0$) leads to harder EoS in the high density regime because the system can accommodate more baryons under the weaker repulsion. The details of the stiff increments of EoS can be understood from the corresponding sound velocity plots and the density profiles. Under the strong correlation assumption for the confined quark momenta, one can read the overall stiffness of EoS from the the peak value of sound velocity as max.$[c_s^2] \simeq 0.6$ while its peak becomes max.$[c_s^2] \simeq 0.7$ under the weak correlation assumption. The multiple peaks and the scale of sound velocity is compatible with the results from Ref. [34, 35, 61]. As discussed in Sec. III A, the flavor asymmetry of the quark Fermi sea evidently appears under the weak correlation assumption, which can make relatively large energy enhancement to the baryon shell side. In comparison of the peaks in the sound velocity, the scale of the final bump after the $s$ quark saturation
is determined by the existence of the Λ shell-like distribution. After the saturation of all the quark Fermi sea, the shell-like distributions of the baryons rapidly expand as the saturated quarks take almost all of the total baryon number density increment \( \partial n_B / \partial n_{\tilde{Q}} \simeq 1 \), so that \( k_F^3 \simeq N_c k_F^Q \). If \( n_\Lambda > 0 (\alpha = -0.2) \), the expanding Λ shell makes the bump slightly larger than the one of the no Λ shell (\( \alpha = 0.2 \)) circumstance. The locations of early appearing peak can be altered by the phenomenological modification of EoS for the low density regime as the saturation moment of \( d \) quark Fermi sea may depend on the modification.

C. Mass-Radius relation of Quarkyonic neutron star

Now we can explore the possible Quarkyonic configuration in the compact stellar state by solving Tolman-Oppenheimer-Volkof (TOV) equations [84, 85]:

\[
\frac{dp(r)}{dr} = \frac{G(\epsilon(r) + p(r))(M(r) + 4\pi r^3 p(r))}{r(r - 2GM(r))}, \tag{37}
\]

\[
\frac{dM(r)}{dr} = 4\pi r^2 \epsilon(r), \tag{38}
\]
FIG. 5. Maxwell constructions of EoS (left) and mass-radius relations from the solution of TOV equations (right). Abbreviations (a) and (w) represent the strong and weak correlation assumptions for the confined quark momenta, respectively. The crust EoS \cite{86,87} is used for the lower density regime of \(0 \leq \rho_B \leq 0.5\rho_0\). In the density range of \(0.5\rho_0 < \rho_B \leq \rho_M\), two different parameter sets \cite{27} are used for the nuclear EoS: {Urbana IX force: \(\tilde{a} = -28.3\) MeV, \(\tilde{b} = 10.7\) MeV} for the plots in (a, b) and \({V_{2n}^{PW} + V_{4n}^{R} = 150}\): \(\tilde{a} = -29.8\) MeV, \(\tilde{b} = 13.6\) MeV} for the plots in (c, d). The EoS of the excluded volume model is used for the higher density regime beyond \(\rho_M\). Stiffer evolution with \(n_0 = 5\rho_0\) (a, b): \(M_{\text{max}} = 2.03M_\odot\) and \(R_{1.4} = 12.5\) km. Softer evolution with \(n_0 = 6\rho_0\) (c, d): \(M_{\text{max}} = 1.8M_\odot\) and \(R_{1.4} = 11.5\) km. Gray shaded inner (outer) band represents the 68.3% (95.4%) credence range of \(M_{\text{max}}\) estimated from Ref. \cite{15}.

where \(G\) is the gravitational constant and the boundary conditions \(p(R_{\text{star}}) = 0\) and \(M(R_{\text{star}}) = M_{\text{star}}\) is assumed. To obtain physically reasonable mass-radius relation, the low density part of our model needs modification as it does not contain the essential attractive and repulsive contribution required to describe the low density nuclear matter properties. The EoS of our model will be kept from the intermediate regime to high density limit because we focus on the role of the hard-core repulsive interaction and the dynamically generated shell-like distribution of baryons at the high density regime. Below a critical density (say \(n_B \leq \rho_M\)), some proper EoS can be adopted instead of introducing additional mean-field potentials. At the extremely low density regime \((0 \leq n_B \leq 0.5\rho_0)\), the EoS of outer crust \cite{86,87} will be used. Since the low density configuration of our model can be simply regarded as the neutron matter (see the profiles in Fig. 3), the nuclear EoS developed for neutron rich matter \cite{27} can be used for the intermediate density regime \((0.5\rho_0 \leq n_B \leq \rho_M)\) as

\[
E/A = \left(p_F^2 + m_n^2\right)^{\frac{1}{2}} - m_n + \tilde{a} \left(\frac{n_n}{\rho_0}\right) + \tilde{b} \left(\frac{n_n}{\rho_0}\right)^2,
\]  

(39)
where $p_F^0 = (3\pi^2 n_n)^{1/3}$ represents the neutron Fermi momentum in the ideal gas limit. The attractive ($\tilde{a}$) and repulsive ($\tilde{b}$) coefficients have been determined by the possible 3-body nucleon forces. The parameter sets (Urbana IX force: $\tilde{a} = -28.3$ MeV, $\tilde{b} = 10.7$ MeV) and \{$V_{2\pi}^{PW} + V_{\mu=150}^{R}$: $\tilde{a} = -29.8$ MeV, $\tilde{b} = 13.6$ MeV\} are used in the stiffer and softer nuclear EoS, respectively. The stiffer (softer) nuclear EoS is interpolated with our high density EoS with $n_0 = 5\rho_0$ ($n_0 = 6\rho_0$), requiring minimal Maxwell construction interval ($P_{\text{nucl.}}(\mu_M) = P_{\text{quark}}(\mu_M)$ where $\mu_M \equiv \mu_B(\mu_M)$).

As one can find in the Maxwell constructions of the EoS obtained under $n_0 = 5\rho_0$ condition (Figs. 5(a, b)), the moment of interpolation differs by the assumptions on the high density EoS. Among the interpolated curves plotted in Fig. 5(a), the $\alpha$ = 0.2 cases look smoothly interpolated and it minimally appears for the interval which looks like first-order transition while the $\alpha$ = -0.2 cases show the non-negligible interval. Although the saturation moments of the light quark Fermi sea can be included in the non-negligible interval with the $\alpha$ = -0.2 cases, one can still regard the interpolated EoS as the effective Quarkyonic-like model because the enhancement of the saturated quark number density is quite small in the interval. Corresponding mass radius relations are presented in Fig. 5(b). The low mass stage is governed by the nuclear EoS with Urbana IX model and the high mass tails are determined by the Quarkyonic-like excluded volume model. The highest mass state is appearing as $\{M_{\text{max.}} = 2.03M_\odot, R_{M_{\text{max.}}} = 11.4\text{ km}\}$ where the weaker repulsive core of $\Lambda$ ($\alpha = -0.2$) and the weakly correlated confined quark momenta are assumed. The other curves are barely located in the possible range estimated from the recent observation $\{M_{\text{max.}} = 2.14_{-0.13}^{+0.20}M_\odot\}$. The constraints from the GW observations, $R_{1.4} = 12.5\text{ km} < 13.5\text{ km}$ [2, 8] and $R_{1.8} = 12.2\text{ km} < 15\text{ km}$ [12], are satisfied.

If the $n_0 = 6\rho_0$ condition is considered (Figs. 5(c, d)), the interpolation interval minimally appears in all the cases and the Maxwell construction is done before the saturation of d quark Fermi sea. The low mass stage is governed by the nuclear EoS with \{$V_{2\pi}^{P} + V_{\mu=150}^{R}$\} potential. While the interpolated EoS can be regarded as the Quarkyonic-like model, $M_{\text{max.}} = 2.14_{-0.13}^{+0.20}M_\odot$ cannot be reproduced from the EoS. The highest mass state appears as $\{M_{\text{max.}} = 1.8M_\odot, R_{M_{\text{max.}}} = 10.4\text{ km}\}$ in both cases of the weaker repulsive core of $\Lambda$ ($\alpha = -0.2$). The other curves present the maximal mass around $M_{\text{max.}} \sim 1.75M_\odot$ and the corresponding radius in the range of $10\text{ km} \leq R_{M_{\text{max.}}} \leq 10.5\text{ km}$.

In comparison with the previously reported work [41], one can find that the formation of the shell-like distribution of the baryon state by Pauli’s principle can enhance EoS hard enough to support the large mass state of neutron star. Although it is necessary to adopt the nuclear EoS for the lower density regime, the higher mass state evolution are determined by the EoS of the Quarkyonic-like excluded volume model. As mentioned above, the most cases of Maxwell construction are finished before the saturation moment of the quark Fermi sea. In the curves plotted in Fig. 5(b), the deviation point of higher mass tail appears at $\{M_{\text{star}} = 1.8M_\odot, R_{M_{\text{star}}} = 12\text{ km}\}$ from that moment, the saturated quarks begin to take most of the total baryon density increment ($\partial n_B/\partial n_\tilde{Q} \simeq 1$). The portion of the saturated quarks at the neutron star core can be estimated from the evolution of EoS and corresponding density profiles. For $M_{\text{max.}} = 2.03M_\odot$ state (on the curve of $\alpha = -0.2$ in Fig. 5(b)), the portions at the neutron star core can be found as $n_\tilde{Q} \simeq 0.26n_B$ and $\varepsilon_\tilde{Q} \simeq 0.27\varepsilon_{\text{quark}}$. For $M_{\text{max.}} = 2.01M_\odot$ state (on the curve of $\alpha = 0.2$ in Fig. 5(b)), they can be found as $n_\tilde{Q} \simeq 0.33n_B$ and $\varepsilon_\tilde{Q} \simeq 0.35\varepsilon_{\text{quark}}$. The scale of sound velocity appears as max.$[c_s^2] \simeq 0.65$ in the both cases. The resulting stellar mass number and evolution tendency are comparable with the results of Refs. [25, 34, 35, 61], although the fundamental physical principle is different from the Quarkyonic matter concept.

IV. SUMMARY AND DISCUSSION

In this paper, the single flavor excluded volume model [1] is extended to the three-flavor model under consideration of Pauli’s exclusion principle which leads to the shell-like distribution of baryons. In perspective of the presumed finite effective size of the particles, the baryon part of this excluded volume model could be understood as the zero-size, and the Maxwell construction is done before the saturation of quark Fermi sea. The low mass stage is governed by the nuclear EoS with $V_{2\pi}^{P} + V_{\mu=150}^{R}$ potential. While the interpolated EoS can be regarded as the Quarkyonic-like model, $M_{\text{max.}} = 2.14_{-0.13}^{+0.20}M_\odot$ cannot be reproduced from the EoS. The highest mass state appears as $\{M_{\text{max.}} = 1.8M_\odot, R_{M_{\text{max.}}} = 10.4\text{ km}\}$ in both cases of the weaker repulsive core of $\Lambda$ ($\alpha = -0.2$). The other curves present the maximal mass around $M_{\text{max.}} \sim 1.75M_\odot$ and the corresponding radius in the range of $10\text{ km} \leq R_{M_{\text{max.}}} \leq 10.5\text{ km}$.

In comparison with the previously reported work [41], one can find that the formation of the shell-like distribution of the baryon state by Pauli’s principle can enhance EoS hard enough to support the large mass state of neutron star. Although it is necessary to adopt the nuclear EoS for the lower density regime, the higher mass state evolution are determined by the EoS of the Quarkyonic-like excluded volume model. As mentioned above, the most cases of Maxwell construction are finished before the saturation moment of the quark Fermi sea. In the curves plotted in Fig. 5(b), the deviation point of higher mass tail appears at $\{M_{\text{star}} = 1.8M_\odot, R_{M_{\text{star}}} = 12\text{ km}\}$ from that moment, the saturated quarks begin to take most of the total baryon density increment ($\partial n_B/\partial n_\tilde{Q} \simeq 1$). The portion of the saturated quarks at the neutron star core can be estimated from the evolution of EoS and corresponding density profiles. For $M_{\text{max.}} = 2.03M_\odot$ state (on the curve of $\alpha = -0.2$ in Fig. 5(b)), the portions at the neutron star core can be found as $n_\tilde{Q} \simeq 0.26n_B$ and $\varepsilon_\tilde{Q} \simeq 0.27\varepsilon_{\text{quark}}$. For $M_{\text{max.}} = 2.01M_\odot$ state (on the curve of $\alpha = 0.2$ in Fig. 5(b)), they can be found as $n_\tilde{Q} \simeq 0.33n_B$ and $\varepsilon_\tilde{Q} \simeq 0.35\varepsilon_{\text{quark}}$. The scale of sound velocity appears as max.$[c_s^2] \simeq 0.65$ in the both cases. The resulting stellar mass number and evolution tendency are comparable with the results of Refs. [25, 34, 35, 61], although the fundamental physical principle is different from the Quarkyonic matter concept.
the strong correlation case.

As a consequence of the saturation of the quark Fermi sea, the pressure increases stiffly by two or three steps with emergence of the shell-like distributions, which is a different feature from the result of other literatures where all the quark degrees of freedom appears simultaneously \cite{1, 10}. The corresponding sound velocity shows its peak value as \( \max|c^2_s| \approx 0.6 \) (\( \max|c^2_s| \approx 0.7 \)) for the strong (weak) correlation assumption for the confined quark momenta. Although the Quarkyonic matter would not have the evident first-order phase transition nature, the EoS looks similar to the result of the previously reported works where the phase transition is implied \cite{34, 35, 61}. Beyond the saturation moments of the quark Fermi sea, the stiffness of EoS becomes moderate and converges to the conformal looks similar to the result of the previously reported works where the phase transition is implied \cite{34, 35, 61}. Beyond momenta. Although the Quarkyonic matter would not have the evident first-order phase transition nature, the EoS peak value as \( \max|c^2_s| \approx 1 \)

\[ \tilde{Q} \]

and \( \epsilon \) comparable numbers with the results of Refs. \cite{25, 34, 35, 61}.

In comparison with the previous study \cite{41} where the stiff evolution was not evident enough, this excluded volume model approach reproduced the required stiff evolution of EoS even for the 3-flavor circumstance. It would be rather required for the existence of a degree of freedom to support 2\( M_\odot \) state in the high densities. At least, we demonstrated that the repulsive hard-core of baryons and the dynamically generated Quarkyonic-like configuration can be an alternative approach for understanding dense nuclear matter via fundamental principle. However, it would need more improvements as the current model cannot cover all the possible range of the massive neutron star \cite{15} and accommodate the matter properties at low densities. If one keeps the physical scale of the hard-core radius \cite{64, 69}, various types of potential \cite{25, 27} can be referred to for the low density regime and the model can be refined to satisfy the low density matter constraints. Meanwhile, one can improve the current model in the VdW EoS framework \cite{44, 63}. For example, the baryon part of the current model can be improved by following the treatment of Carnahan-Starling modification \cite{46, 59, 62, 63} where the additional repulsive contribution is reflected in the larger repulsive core size than the estimated scale in Refs. \cite{64, 69}. In either approaches, the required soft nature for the low densities and stiffer nature for the high densities can be achieved by the additional attractive and repulsive contributions to the EoS.

In microscopic aspect, there would be debates about the baryon-like state located on the lower boundary of the shell-like distribution. In this model, the baryon like state is clearly distinguished from the saturated quark states and the non-perturbative regulator \( \Lambda_Q \) is introduced for the quark phase measure. However, depending on the chiral symmetry restoration \cite{77, 83} and the quark confinement mechanism around the quark Fermi sea \cite{93, 94}, the baryon like-state can be differently understood. It would be still the baryon state with restored chiral symmetry or the correlated state of quarks under non-perturbative dynamics. The similarity and discrepancy between the Quarkyonic matter concept and the other approaches which involve the quark degrees of freedom would be understood via further studies about the possible baryon-like states since the phase transition nature should also be related with the strong correlation patterns of quarks on the surface.

ACKNOWLEDGMENTS

Authors acknowledge useful discussions with Larry McLerran and Sanjay Reddy during development of this work. Authors also thank Toru Kojo and Gerald Miller for inspiring discussions. Authors acknowledge the support of the Simons Foundation under the Multifarious Minds Program grant 557037. The work of Dyana C. Duarte, Saul Hernandez-Ortiz and Kie Sang Jeong was supported by the U.S. DOE under Grant No. DE-FG02-00ER41132

Appendix A: Possible emergence of \( \Delta(1232) \) isobar

The \( \Delta(1232) \) isobar may emerge via the energetic collisions or in the dense neutron rich matter. In this work, the low density configuration \( (n_B \leq 3\rho_0) \) appears as the neutron rich matter (see the profiles plotted in Fig. 3). If one assumes similar size of the repulsive core for the baryons \( (\omega_{n,p} = \omega_\Delta = 1, n_0 = 5\rho_0) \), the chemical potentials of
where $m_n = 1 \text{ GeV}$, $m_{\Delta} = 1.3 \text{ GeV}$, and the neutron rich circumstance ($n_B \simeq n_n$, $n_p, n_{\Delta} \simeq 0$) is understood. Firstly, $\mu_{\Delta} > \mu_n$ in all the relevant density regime ($n_B \leq 3\rho_0$). If one considers the possible emergence via the scattering $nn \rightarrow p\Delta^-$, the energy relation is satisfied in the relevant densities ($n_B \simeq 2.5\rho_0$). However, this scattering barely happens as the momentum conservation is not always matched. Another possibility can be imagined in our model as $n + d \rightarrow \Delta^- + u$ after the saturation of $d$ quark Fermi sea. In this scenario, the liberated $u$ quark falls down to the lower phases space but the emerging $\Delta^-$ should fill the phase space from the lower shell boundary ($k_F^\Delta \simeq 3k_F^d$).

Under the simplified configuration where $n_B \simeq n_n + n_{\tilde{d}}$, the baryon (9) and quark (9) chemical potentials can be written as follows:

$$\mu_n = \left( \frac{n_0}{n_0 - n_n} \right) \left( k_F + \Delta \right)_n^2 + m_n^2 \right)^{\frac{1}{2}} - \frac{1}{\pi^2 n_0} \int_{K_F^\Delta}^{k_F^\Delta} dkk^2 (k^2 + m_n^2)^{\frac{1}{2}},$$

$$\mu_{\Delta} = \left( k_F^\Delta + m_{\Delta}^2 \right)^{\frac{1}{2}} + \frac{1}{n_0} \left\{ \tilde{n}_{ex} \left( \left| k_F + \Delta \right)_n^2 + m_n^2 \right)^{\frac{1}{2}} - \frac{1}{\pi^2} \int_{k_F^\Delta}^{[k_F + \Delta]_n} dkk^2 (k^2 + m_n^2)^{\frac{1}{2}} \right\},$$

$$\mu_d = 2 \left( 1 - \frac{n_n}{n_0} \right) \frac{k_F^d}{k_F^\Delta} + \frac{\Lambda_d^2}{2} \left\{ \left( \left| k_F + \Delta \right)_n^2 + m_n^2 \right)^{\frac{1}{2}} - \left( k_F^d + m_d^2 \right)^{\frac{1}{2}} \right\} + \left( k_F^d + m_d^2 \right)^{\frac{1}{2}},$$

$$\mu_u = \left( 1 - \frac{n_n}{n_0} \right) \frac{k_F^u}{\Lambda_u^2} \left\{ \left( \left| k_F + \Delta \right)_n^2 + m_n^2 \right)^{\frac{1}{2}} - \left( k_F^u + m_u^2 \right)^{\frac{1}{2}} \right\} + m_u,$$

where $k_F^d = k_F^\Delta = 3k_F^d$ is assumed in small $k_F^d$ limit. In this case, $\mu_n + \mu_d < \mu_{\Delta} + \mu_u$ around the expected saturation moments of the quark Fermi sea ($3\rho_0 \leq n_B \leq 5\rho_0$) but the energy relation can be satisfied when the iso-spin asymmetry of the saturated quarks is large. However, it is unlikely to accommodate $\Delta$ isobar degrees of freedom in the Quarkyonic-like system as that large flavor asymmetry of saturated quark is not appearing under the physical contraints.

[1] K. S. Jeong, L. McLerran and S. Sen, Phys. Rev. C 101, no.3, 035201 (2020) doi:10.1103/PhysRevC.101.035201 [arXiv:1908.04799 [nucl-th]].
[2] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 119, no. 16, 161101 (2017) doi:10.1103/PhysRevLett.119.161101 [arXiv:1710.05832 [gr-qc]].
[3] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 121, no. 16, 161101 (2018) doi:10.1103/PhysRevLett.121.161101 [arXiv:1805.11581 [gr-qc]].
[4] F. J. Fattoyev, J. Piekarewicz and C. J. Horowitz, Phys. Rev. Lett. 120, no. 17, 172702 (2018) doi:10.1103/PhysRevLett.120.172702 [arXiv:1805.06115 [nucl-th]].
[5] E. Annala, T. Gorda, A. Kurkela and A. Vuorinen, Phys. Rev. Lett. 120, no. 17, 172703 (2018) doi:10.1103/PhysRevLett.120.172703 [arXiv:1711.02644 [astro-ph.HE]].
[6] A. Vuorinen, Nucl. Phys. A 982, 36 (2019) doi:10.1016/j.nuclphysa.2018.10.011 [arXiv:1807.04480 [nucl-th]].
[7] C. Raithel, F. zel and D. Psaltis, Astrophys. J. 857, no. 2, L23 (2018) doi:10.3847/2041-8213/aabd7f [arXiv:1803.07687 [astro-ph.HE]].
[8] E. R. Most, L. R. Weih, L. Rezzolla and J. Schaffner-Bielich, Phys. Rev. Lett. 121, no. 26, 261103 (2018) doi:10.1103/PhysRevLett.121.261103 [arXiv:1803.00549 [gr-qc]].
[9] I. Tews, J. Margueron and S. Reddy, Eur. Phys. J. A 55, no. 6, 97 (2019) doi:10.1140/epja/i2019-12774-6 [arXiv:1901.09874 [nucl-th]].
[10] I. Tews, J. Margueron and S. Reddy, AIP Conf. Proc. 2127, no. 1, 020009 (2019) doi:10.1063/1.5117799 [arXiv:1905.11212 [nucl-th]].
[11] C. D. Capano, I. Tews, S. M. Brown, B. Margalit, S. De, S. Kumar, D. A. Brown, B. Krishnan and S. Reddy, Nat Astron. 4, 625632 (2020) doi:10.1038/s41550-020-1014-6 [arXiv:1908.10352 [astro-ph.HE]].
