Finding Elementary First Integrals for Rational Second Order Ordinary Differential Equations

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Abstract

Here we present an algorithm to find elementary first integrals of rational second order ordinary differential equations (SOODEs). In [17], we have presented the first algorithmic way to deal with SOODEs, introducing the basis for the present work. In [18], the authors used these results and developed a method to deal with SOODEs and a classification of those. Our present algorithm is based on a much more solid theoretical basis (many theorems are presented) and covers a much broader family of SOODEs than before since we do not work with restricted ansatz. Furthermore, our present approach allows for an easy integrability analysis of SOODEs and much faster actual calculations.

Keyword: Elementary first integrals, Second Order Differential Ordinary Equations

PACS: 02.30.Hq

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1 Introduction

The differential equations (DEs) are the most widespread way to formulate the evolution of any given system in many scientific areas. Therefore, for the last three centuries, much effort has been made in trying to solve them.

Broadly speaking, we may divide the approaches to solving ODEs in the ones that classify the ODE and the ones that do not (classificatory and non-classificatory methods). Up to the end of the nineteenth century, we only had many (unconnected) classificatory methods to try to deal with the solving of ODEs. Sophus Lie then introduced his method [1, 2, 3] that was meant to be general and try to solve any ODE, i.e., non-classificatory. Despite this appeal, the Lie approach had a shortcoming: namely, in order to deal with the ODE, one has to know the symmetries of the given ODE. Unfortunately, this part of the procedure was not algorithmic (mind you that the classificatory approach is algorithmic by nature). So, for many decades, the Lie method was not put to much “practical” use since to “guess” the symmetries was considered to be as hard as guessing the solution to the ODE itself. In [4, 5], an attempt was made to make this searching for the symmetries to the ODE practical and, consequently, make the Lie method more used.

So, despite all these efforts, a non-classificatory algorithmic approach was still missing. The first (semi) algorithmic approach applicable to solving first order ordinary differential equations (FOODEs) was made by M. Prelle and M. Singer [6]. The attractiveness of the PS method lies not only in the fact that it is based on a totally different theoretical point of view but, also in that, if the given FOODE has a solution in terms of elementary functions, the method guarantees that this solution will be found (though, in principle it can admittedly take an infinite amount of time to do so). The original PS method was built around a system of two autonomous FOODEs of the form \( \dot{x} = P(x, y) \), \( \dot{y} = P(x, y) \) with \( P \) and \( P \) polynomials in \( C[x, y] \) or, equivalently, the form \( y' = R(x, y) \), with \( R(x, y) \) a rational function of its arguments.

The PS approach has its limitations, for instance, it deals only with rational FOODEs. But, since it is so powerful in many respects, it has generated many extensions [7, 8, 9, 10, 11, 12, 13, 14] Nevertheless, all these extensions deal only with FOODEs. In particular, the second order ordinary differential equations (SOODEs) play a very important role, for instance, in the physical sciences. So, with this in mind, we have produced [17] a PS-type approach to deal with SOODEs. This approach dealt with SOODEs that presented elementary\(^2\) solutions (with two elementary first integrals).

Here, we present an algorithm that comes supported by several theoretical results. It proves to be very fast (in real applications) and, although being not completely general, we could not find any interesting example where it could not be used.

In section 2, we present the state of the art up to the present paper. In the following section, we introduce the algorithm to find the first integral. In section 4, we present some examples of SOODEs where the algorithm is successful. Finally, we present our conclusions and point out some directions to further our work.

\(^2\)For a formal definition of elementary function, see [15].
2 Earlier Results

In the paper [6], one can find an important result that, translated to the case of SOODEs of the form

\[ y'' = \frac{M(x, y, y')}{N(x, y, y')} = \phi(x, y, y'), \tag{1}\]

where \( M \) and \( N \) are polynomials in \((x, y, y')^3\), can be stated as:

**Theorem 1:** If the SOODE (1) has a first integral that can be written in terms of elementary functions, then it has one of the form:

\[ I = w_0 + \sum_i m_i \ln(w_i), \tag{2}\]

where \( m \) is an integer and the \( w_i \)s are algebraic functions of \((x, y, y')\).

The integrating factor for a SOODE of the form (1) is defined by:

\[ R(\phi - y'') = \frac{dI(x, y, y')}{dx} \tag{3} \]

where \( \frac{d}{dx} \) represents the total derivative with respect to \( x \).

Below we will present some results and definitions (previously presented on [17]) that we will need. First let us remember that, on the solutions, \( dI = I_x dx + I_y dy + I_y' dy' = 0 \). So, from equation (3), we have:

\[ R(\phi - y'') = I_x dx + I_y dy + I_y' dy' = dI = 0. \tag{4} \]

Since \( y' dx = dy \), we have

\[ R[(\phi + S y') dx - S dy - dy'] = dI = 0, \tag{5} \]

adding the null term \( S y' dx - S dy \), where \( S \) is a function of \((x, y, y')\). From equation (5), we have:

\[ I_x = R(\phi + Sy'), \quad I_y = -RS, \quad I_y' = -R, \tag{6} \]

that must satisfy the compatibility conditions. Thus, defining the differential operator \( D \):

\[ D \equiv \partial_x + y' \partial_y + \phi \partial_{y'}, \tag{7} \]

after a little algebra, that can be shown to be equivalent to:

\[ D[S] = -\phi y + S\phi y' + S^2, \tag{8} \]
\[ D[R] = -R(S + \phi y'), \tag{9} \]
\[ R_y = R_y' S + S_y R. \tag{10} \]

\[ ^3\text{From now on, } f' \text{ denotes } df/dx. \]
\[ ^4\text{For a formal definition of algebraic function, see [15].} \]
Combining (8) and (9) we obtain

\[ D[RS] = -R \phi_y. \]  

(11)

We will now briefly explain the algorithm proposed on [17]. In that paper, we made the conjecture that it is always possible to have an first integral \( I \) such that \( RS \) is a rational function of \( (x, y, y') \), if the corresponding SOODE presented an elementary solution. From equation (11), we then see that this conjecture implies that both \( R \) and \( S \) are also rational functions.

Using this into equation (8) and trying to solve it, we are left with the task of solving a third degree algebraic systems of equations on the desired coefficients of the polynomials defining the rational function \( S \). Supposing that this stage was overcome, we would then substitute the \( S \) just found in equation (9) and apply the a Prelle-Singer type approach to find \( R \).

Basically, that was the novelty presented on [17], as far as we know, the first semi-algorithmic approach to tackle SOODEs.

3 The Algorithm

In the present section, we will present a non-classificatory algorithm to search elementary first integrals for SOODEs. The method is based on a Darboux type procedure and, analogously to the Prelle-Singer [6] approach for FOODEs, it guarantees that, for a class of SOODEs, it will eventually find the desired first integrals.

The algorithm relies on several theoretical results that will be introduced on the next sub-section.

3.1 Theoretical Foundations

Let us start then by a corollary to theorem 1 concerning \( S \) and \( R \).

**Corollary 1:** If a SOODE of the form (1) has a first order elementary first integral then the integrating factor \( R \) for such a SOODE and the function \( S \) defined in the previous section can be written as algebraic functions of \( (x, y, y') \).

**Proof:** Using the above mentioned result by Prelle and Singer, there is always a first integral \( I = w_0 + \sum_i c_i \ln(w_i) \) for the SOODE. So we have, using equation (3),

\[ R \left( \frac{M}{N} - y'' \right) = I_x + y'I_y + y''I_{y'} \Rightarrow R = -I_{y'} \]  

(12)

where \( I_u \equiv \partial_u I \). From equation (2), we have:

\[ I_{y'} = w_0y' + \sum_i c_i \frac{w_i y'}{w_i}. \]  

(13)

Then \( I_{y'} \) is an algebraic function of \( (x, y, y') \) and, by equation (12), so is \( R \).

From equations (6), one can see that:

\[ S = \frac{I_y}{I_{y'}} = \frac{w_0y + \sum_i c_i \frac{w_i}{w_i}}{w_0y' + \sum_i c_i \frac{w_i y'}{w_i}}. \]  

(14)
Therefore, $S$ is also an algebraic function of $(x, y, y')$. □

In order to produce the further mathematical results we need, we are going to re-define the functions $R$ and $S$.

Suppose we have a SOODE of the form (1). We will then define the integrating factor as:

$$ R(M - N y'') = \frac{dI(x, y, y')}{dx} = I_x dx + I_y dy + I_{y'} dy' = 0. \quad (15) $$

This is equivalent to use the following re-definition:

$$ R = \frac{R}{N}. \quad (16) $$

Proceeding analogously to what we did with equation (4), we get:

$$ R \left[ (M + S y') dx - S dy - N dy' \right] = dI = 0, \quad (17) $$

adding the null term $S y' dx - S dy$, where $S$ is a function of $(x, y, y')$.

This is equivalent to use the following re-definition:

$$ S = S N. \quad (18) $$

Using equations (16) and (18) into equations (9), (10) and (11), we get:

$$ D \frac{D[R]}{R} = -(S + N_x + N_y y' + M_{y'}), \quad (19) $$

$$ R_y N - R N_y = N^2 (R_{y'} S + S_{y'} R) \quad (20) $$

$$ S D \frac{D[R]}{R} + D[S] = -(N M_y - M N_y) \quad (21) $$

where $D$ is defined as $D = N D$. Since $N$ is polynomial and $R$ and $S$ are algebraic, so are $R$ and $S$.

Now, we will introduce some theorems that will be the basis of our algorithm.

**Theorem 2:** Consider a SOODE of the form (1), that presents an elementary first integral $I$. If $S$ is a rational function of $(x, y, y')$, then the integrating factor $R$ for this SOODE can be written as:

$$ R = \prod_i p_i^{n_i} \quad (22) $$

where $p_i$ are irreducible polynomials in $(x, y, y')$ and $n_i$ are non-zero rational numbers.

To prove Theorem 2 we will need the following lemma:

**Lemma 1:** Consider a function $F$ of $(x, y, z)$. If the differential of $F$ can be written as $dF = A(X dx + Y dy + Z dz)$, where $X$, $Y$ and $Z$ are polynomial functions of $(x, y, z)$ and $A$ is an algebraic function of $(x, y, z)$, then the first order ordinary differential equation defined as

$$ \frac{dy}{dx} = -\frac{X}{Y} \quad (23) $$
where \( z \) is regarded as a parameter, has \( F(x, y, z) = C \) (where \( C \) is a constant) as a general solution

**Proof of Lemma 1:** Consider the first order ordinary differential equation defined by (23). If \( f \) is a function of \((x, y)\) such that

\[
(Y \partial_x - X \partial_y)[f(x, y)] = 0, \tag{24}
\]

then \( f(x, y) = k \) (where \( k \) is a constant) is a general solution of (23). Applying \((Y \partial_x - X \partial_y)\) to \( F(x, y, z) \) we get \( Y F_x - X F_y \). But, by hypothesis, \( F_x = AX \) and \( F_y = AY \) leading to

\[
(Y \partial_x - X \partial_y)[F(x, y, z)] = Y AX - X AY = 0.
\]

This implies that \( F(x, y, z) = C \) is a general solution of (23). \( \Box \)

**Proof of Theorem 2:** If the hypothesis of the theorem are satisfied, \( S \) is a rational function of \((x, y, y')\). So we can write \( S = P/Q \) where \( P \) and \( Q \) are polynomials. Substituting this into equation (17) we get

\[
\frac{R}{Q} [(M Q + P y')\,dx - P \,dy - N Q \,dy'] = dI = 0, \tag{25}
\]

By using Lemma 1 we have that \( I(x, y, y') = C \) is a general solution of the FOODE defined by

\[
\frac{dy}{dx} = \frac{M Q + P y'}{P}. \tag{26}
\]

If the polynomials \((M Q + P y')\) and \( P \) have a common factor we can write \((M Q + P y') = T_1 t_{12}, P = T_2 t_{12}\) and (26) as

\[
\frac{dy}{dx} = \frac{T_1}{T_2}. \tag{27}
\]

Since \( I(x, y, y') \) is an elementary function (by hypothesis) then, by the theorem of Prelle and Singer[6], there exists an integrating factor \( R_{12} \) for the FOODE (27) of the form

\[
R_{12} = \prod f_i^{m_i} \tag{28}
\]

where \( f_i \) are irreducible polynomials in \((x, y)\) and \( m_i \) are non-zero rational numbers. By other side, from equation (25) we have that

\[
I_x = \frac{R}{Q} t_{12} T_1; \quad I_y = \frac{R}{Q} t_{12} T_2, \tag{29}
\]

and, so, \( \frac{R}{Q} t_{12} \) is an integrating factor for (27). This implies that

\[
R_{12} = \mathcal{F}(I) \frac{R}{Q} t_{12}, \tag{30}
\]

where \( \mathcal{F}(I) \) is a function of the first integral \( I \). From (30) is easily seen that

\[
\mathcal{F}(I) R = R_{12} \frac{Q}{t_{12}} \tag{31}
\]
and we may notice (see (28)) that \( \mathcal{F}(I) \mathcal{R} \) can be expressed as \( \prod p_i^{n_i} \). But, since \( \mathcal{R} \) is an integrating factor for the SOODE (1), so is \( \mathcal{R} \equiv \mathcal{F}(I) \mathcal{R} \). Finally, the reasoning we apply to the pair of variables \((x, y)\) when we make use of Lemma 1, could be carried out analogously with the pairs \((x, y')\) and \((y, y')\) as well. So, we can conclude that there exists an integrating factor of the form (22).

**Theorem 3:** Consider a SOODE of the form (1), that presents an elementary first integral \( I \). If \( S \) (defined above) is a rational function of \((x, y, y')\) \( (S = P/Q, \) where \( P \) and \( Q \) are polynomials in \((x, y, y')\)), then \((\mathcal{R}/Q) | \mathcal{D}[\mathcal{R}/Q], \) i.e. \( \mathcal{D}[\mathcal{R}/Q]/(\mathcal{R}/Q) \) is a polynomial.

**Proof of Theorem 3:** Substituting \( S = P/Q \) in the compatibility conditions (19,21), we get:

\[
Q \frac{\mathcal{D}[\mathcal{R}]}{\mathcal{R}} = -P - Q (N_x + N_y y' + M_y), \tag{32}
\]

\[
\frac{P}{Q} \frac{\mathcal{D}[\mathcal{R}]}{\mathcal{R}} + \frac{Q \mathcal{D}[P] - P \mathcal{D}[Q]}{Q^2} = -(N M_y - M N_y) \tag{33}
\]

Multiplying (33) by \( Q \), one gets:

\[
P \frac{\mathcal{D}[\mathcal{R}]}{\mathcal{R}} + \mathcal{D}[P] - P \frac{\mathcal{D}[Q]}{Q} = -Q (N M_y - M N_y) \tag{34}
\]

and finally

\[
P \left( \frac{\mathcal{D}[\mathcal{R}]}{\mathcal{R}} - \frac{\mathcal{D}[Q]}{Q} \right) = -\mathcal{D}[P] - Q (N M_y - M N_y). \tag{35}
\]

If one adds \(-\mathcal{D}[Q]\) to both sides of (32), one gets:

\[
Q \left( \frac{\mathcal{D}[\mathcal{R}]}{\mathcal{R}} - \frac{\mathcal{D}[Q]}{Q} \right) = -\mathcal{D}[Q] - P - Q (N_x + N_y y' + M_y') \tag{36}
\]

Re-writing (36,35), we have:

\[
Q \left( \frac{\mathcal{D}[\mathcal{R}/Q]}{\mathcal{R}/Q} \right) = -\mathcal{D}[Q] - P - Q (N_x + N_y y' + M_y') \tag{37}
\]

and

\[
P \left( \frac{\mathcal{D}[\mathcal{R}/Q]}{\mathcal{R}/Q} \right) = -\mathcal{D}[P] - Q (N M_y - M N_y). \tag{38}
\]

Since, by definition, \( P, Q, M \) and \( N \) are polynomial and \( \mathcal{D} \) is a differential operator with polynomial coefficients, the right-hand side of both (37,38) are polynomial. Therefore, in principle, \( \frac{\mathcal{D}[\mathcal{R}/Q]}{\mathcal{R}/Q} \) is rational. So, let us represent it as \( A/B \), where \( A \) and \( B \) are polynomial that do not have any common factors. By doing that, one can write (37,38) as:

\[
Q \left( \frac{A}{B} \right) = \mathcal{P}_1 \tag{39}
\]

\[
P \left( \frac{A}{B} \right) = \mathcal{P}_2 \tag{40}
\]
where \( P_1 = -\mathcal{D}[Q] - P - Q (N_x + N_y y' + M_y') \) and \( P_2 = -\mathcal{D}[P] - Q (N M_y - M N_y) \).

In order to satisfy (39,40) simultaneously, since \( A \) and \( B \) do not have common factors, it would be necessary that \( B | Q \) and \( B | P \). But, again by hypothesis, \( P \) and \( Q \) do not have common factors. So, one can conclude that we have \( B = 1 \).

Therefore, \( A/B = \frac{D[R/Q]}{R/Q} = \text{polynomial}. \)

\[ \text{Corollary 2: Consider a SOODE of the form (1), that presents an elementary first integral } I. \text{ If } S \text{ is a rational function of } (x, y, y') \text{ (} S = P/Q, \text{ where } P \text{ and } Q \text{ are polynomials in } (x, y, y') \text{), then } R/Q \text{ can be written as:} \]

\[ \frac{R}{Q} = \prod_i v_i^{m_i} \quad (41) \]

where \( v_i \) are irreducible eigenpolynomials (in \( (x, y, y') \)) of the \( \mathcal{D} \) operator and \( m_i \) are non-zero rational numbers.

\[ \begin{align*}
\text{Proof of Corollary 2: From theorem 2 we have that } R &= \prod_i p_i^{n_i}, \text{ where } p_i \text{ are irreducible polynomials in } (x, y, y') \text{ and } n_i \text{ are non-zero rational numbers. Since } Q \text{ is polynomial, we have } Q &= \prod_j q_j^{k_j}, \text{ where } q_j \text{ are irreducible polynomials in } (x, y, y') \text{ and } k_j \text{ are non-zero positive integers. So,} \\
\frac{R}{Q} &= \prod_i v_i^{m_i} \quad (42) \\
\text{where } v_i \text{ are irreducible polynomials (in } (x, y, y')) \text{ and } m_i \text{ are non-zero rational numbers. Since, by theorem 3, } \mathcal{D}[R/Q]/(R/Q) \text{ is polynomial, we have:} \\
\frac{\mathcal{D}[R/Q]}{R/Q} &= \frac{\mathcal{D}[\prod_i v_i^{m_i}]}{\prod_i v_i^{m_i}} = \sum m_i \frac{\mathcal{D}[v_i]}{v_i} = \text{polynomial}. \quad (43) \\
\text{So, since the } v_i \text{'s are irreducible, } v_i | \mathcal{D}[v_i]. \Box \end{align*} \]

3.2 The Algorithm Itself

In this section, we will make use of the mathematics constructed above in order to produce a “semi-algorithm” to deal with SOODEs of the class defined in (1).

From theorem 3 we can write:

\[ \frac{R}{Q} = T = \prod_i v_i^{m_i} \Rightarrow \frac{\mathcal{D}[T]}{T} = \sum m_i \frac{\mathcal{D}[v_i]}{v_i} = \sum m_i g_i \quad (44) \]

where \( v_i \) are irreducible eigenpolynomials (in \( (x, y, y') \)) of the \( \mathcal{D} \) operator and the \( g' \text{'s} \) are the corresponding cofactors.

From equations (37,38), we can then write:

\[ Q \left( \frac{\mathcal{D}[T]}{T} \right) = Q \sum m_i g_i = -\mathcal{D}[Q] - P - Q (N_x + N_y y' + M_y') \quad (45) \]

and

\[ P \left( \frac{\mathcal{D}[T]}{T} \right) = P \sum m_i g_i = -\mathcal{D}[P] - Q (N M_y - M N_y). \quad (46) \]

Equations (45,46) will be the basis of our procedure. Let us begin discussing our procedure by talking us through the main steps before formalizing the algorithm.
As it will become clear as we go along, the first step of our procedure is the most costly one, i.e., time consuming. In the same way as in the Prelle-Singer procedure for FOODEs [6], determining the eigenpolynomials and corresponding cofactors for the \( D \) operator is a task that grows as the degree of such polynomials grow. Our present procedure start by performing this search. Assuming that this search succeeded and we found \( v_i \) and \( g_i \) for some degree \( \text{deg} \), checking (45,46), we see that it remains to be determined the following set of variables:

\[
\left\{ m_i, P, Q \right\}
\]

where \( m_i \) is a constant and \( P \) and \( Q \) are polynomials. How to determine these? Again, by inspecting (45,46), we note that if we construct a generic polynomial \( Q \) (in \((x, y, y')\)) of some degree \( \text{deg}Q \), we can infer the maximum degree of the polynomial \( P \) (\( \text{deg}P \)). So, we construct such polynomials (the most general ones for the corresponding degrees) and solve (45,46) for the coefficients of such polynomials and for \( m_i \). By doing that, we would have found \( m_i, P \) and \( Q \) and, consequently, \( T = \prod_i v_i^{m_i} \). Since we have \( R/Q = T \), we would have found \( R \), the integrating factor for the SOODE in question!

Once \( R \) and \( S^5 \) have been determined, we have all the partial first derivatives of the first order differential invariant, \( I(x, y, y') \), which is constant on the solutions (see equations (6)). This invariant can then be obtained as

\[
I(x, y, y') = \int R(\phi + Sy') \, dx - \int \left[ RS + \frac{\partial}{\partial y} \int R(\phi + Sy') \, dx \right] dy - \\
\int \left\{ R + \frac{\partial}{\partial y'} \left[ \int R(\phi + Sy') \, dx - \int \left[ RS + \frac{\partial}{\partial y} \int R(\phi + Sy') \, dx \right] dy \right] \right\} dy'.
\]

(47)

Basically, there are a few scenarios that can arise in the case of SOODEs (in regard to the first order differential invariants): One can find two pair of independent \( S \) and \( R \), leading to two independent \( I \)'s. In this case, we would have fully integrated the SOODE (by solving for \( y' \), using one of the invariants, and substituting in the other to find \( y \)). On the other hand, it is possible that we can only find one such invariant and, in that case, we can only reduce the SOODE to a FOODE.

This overview above indicates the main trust of our procedure, bellow we will present a step-by-step *modus operandi* of how to implement the above scenario.

- **Steps of the Algorithm**
  1. Set \( \text{Deg} = 1 \).
  2. For the given SOODE, we determine the eigenpolynomials and the associated cofactors, up to degree \( \text{Deg} \).
  3. Set \( \text{Deg}Q = 1 \).
  4. Construct a generic polynomial \( Q \) (in \((x, y, y')\)) of degree \( \text{Deg}Q \) and a generic polynomial \( P \) (in \((x, y, y')\)) of degree \( \text{Deg}P = \text{Deg}Q + \text{MAX}(\text{deg}M - 1, \text{deg}N) \).\(^{6}\)
  5. Try to solve equations (45,46) for the coefficients defining \( Q \) and \( P \) and for \( m_i \).
  6. If we are successful, go to step 7. In the opposite case, if \( \text{Deg}Q < 10 \text{Deg} \), we make \( \text{Deg}Q = \text{Deg}Q + 1 \) and return to step 4. If \( \text{Deg}Q = 10 \text{Deg} \), set \( \text{Deg} = \text{Deg} + 1 \) and return to step 2.

\(^{5}\)Please remember that \( S = P/Q \).

\(^{6}\)Where \( \text{deg}M \) and \( \text{deg}N \) are the degree of \( M \) and \( N \), respectively.
7. Now that we have $S$ and $R$ we, after checking if that solution satisfy equation (20), in the affirmative case, using equation (47), calculate the associated first order differential invariant. If the solution does not satisfy (20), set $DeqQ = DegQ + 1$ and return to step 4 above.

4 Examples

In the present section, we are going to show some examples of the usage of our method. In the first example, we are going to point out the improvements that have been made in comparison to the first algorithmic approach to solving SOODEs presented on [17]. Later we will apply the method to another two potentially interesting physical examples found on a very interesting paper [18] where the authors use our results [17] and construct a classification of SOODEs and present different heuristics to integrate each one of them. We will comment on that classification in the light of our present theoretical results and issuing algorithm and make some remarks about the capability of our method to analyze the integrability of the corresponding SOODEs. Finally, on a more mathematical tone, we will show that the method can solve equations previously unsolved by other powerful techniques.

4.1 First Example

A rich source of non-linear DEs in physics are the highly non-linear equations of General Relativity. Einstein’s equations are, of course, in general, partial DEs, but there exist classes of space-times where the symmetry imposed reduces these equations to ODEs in one independent variable. One such class is that of static, spherically symmetric space-times, which depend only on the radial variable, $r$. The metric for a general statically spherically space-time has two free functions, $\lambda(r)$ and $\mu(r)$ say. On imposing the condition that the matter in the spacetime is a perfect fluid, Einstein’s equations reduce to two coupled ODEs for $\lambda(r)$ and $\mu(r)$. Specifying one of these functions reduces the problem to solving an ODE (of first or second order) for the other.

Following this procedure, Buchdahl [19] obtained an exact solution for a relativistic fluid sphere by considering the so-called isotropic metric

$$ds^2 = (1 - f)^2(1 + f)^{-2}dt^2 - (1 + f)^4[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)]$$

with $f = f(r)$. The field equations for $f(r)$ reduce to

$$ff'' - 3f'^2 - r^{-1}ff' = 0.$$ 

Changing to the notation of this paper, with $y(x) = f(r)$, we get:

$$yy'' - 3y'^2 - (1/x)yy' = 0.$$ \hspace{1cm} (48)

Let us now apply our new method (section 3) to the above equation. First, we have to calculate the eigenpolynomials of degree=1 (and the associated cofactors) for the differential operator associated with equation (48) given by:

$$D = x y \partial_x + x y' \partial_y + y' (3y'x + y) \partial_{y'}.$$ \hspace{1cm} (49)
These are found to be:

\[ v_1 = x \quad g_1 = y \]  
\[ v_2 = y \quad g_2 = xy' \]  
\[ v_3 = y' \quad g_3 = 3y'x + y \]  
\[ (50) \]

Now, following the steps of the algorithm (given on section 3), we can find two independent solutions:

- **First solution**

\[ S = -3y'x \]  
\[ R = \frac{1}{x^2 y^4} \]  
\[ (53) \]
\[ (54) \]

Using equation (47), we find, for this pair of \( S \) and \( R \):

\[ I = \frac{y'}{y^3 x}. \]  
\[ (55) \]

- **Second solution**

\[ S = -\frac{x(2yx + 3y' + 3x^2 y')}{1 + x^2} \]  
\[ R = \frac{1 + x^2}{x^2 y^4} \]  
\[ (56) \]
\[ (57) \]

Using equation (47), we find, for this pair of \( S \) and \( R \):

\[ I = \frac{y' + x^2 y' + y x}{y^3 x}. \]  
\[ (58) \]

In comparison to the method we have presented on [17], we see that our present approach has some remarkable differences: It covers a much broader “universe” of SOODEs since, in the present algorithm, we do not have restrictions imposed on \( R \) (in the algorithm presented on [17], we made a conjecture that led to \( R \) being necessarily rational). The present approach is much more theoretically sound than the previous one, we produced a lot of theorems to support the algorithm. From a practical point of view, this new approach is much more efficient. For example, equation (8) will generate a system of third degree algebraic equations in the coefficients we want to determine. On the other hand, in the algorithm introduced in section 3, equations (45, 46) will mostly generate first degree algebraic equations since only the terms \( (Q \sum m_i g_i) \) and \( (P \sum m_i g_i) \) produce second degree equations on the desired coefficients.

In [18], based on our results [17], the authors produced an way of finding a second independent solution for the \( S \) and \( R \) from the first one\(^7\). As it is clear from the above, our present approach finds both pairs independently. They also classify the SOODEs they dealt with into three types. The example we presented above (that was originally treated in [17]) was also used by the authors in [18] as a representative of type I SOODEs.

In the following two sub-sections, we will introduce an example for each of the other two types.

---

\(^7\)At the time they presented their paper, they used our results from [17] where we introduced the theory and were concerned with the finding of only one pair of \( S \) and \( R \).
4.2 Second Example

Now, we are going to consider the Helmholtz oscillator with friction:

\[ y'' + c_1 y' + c_2 y - \beta y^2 = 0. \]  \hfill (59)

It is a well-known result that the above equation is integrable for the choice \( c_2 = \frac{6}{25} c_1 ^2 \) [20]. As we shall see, our method takes care of such concerns, i.e., since the algorithm translates the solving of the SOODE into solving systems of algebraic equations, if we consider the arbitrary constants appearing on the SOODE \((c_1, c_2, \beta)\) as variables (as well as the coefficients defining the eigenpolynomials and cofactors), the solution of such algebraic systems will provide the regions of integrability naturally. In order to clarify this, let us then follow our procedure in more detail than in the previous example:

In order to follow our algorithm, the first thing to do is to determine the eigenpolynomials (and corresponding cofactors) for the \( \mathcal{D} \) operator:

\[ \mathcal{D} = \partial_x + y' \partial_y + \left(-c_1 y' - c_2 y + \beta y^2\right) \partial_{y'} \]  \hfill (60)

The equation to be solved is:

\[ \mathcal{D}[v] = g v \]  \hfill (61)

where \( v \) is the eigenpolynomial and \( g \) is the cofactor.

It turns out that for degree 1 and 2 there is no solution to (61). A generic polynomial of degree 3 is given by:

\[
\begin{align*}
v &= a_1 + a_2 y^2 y' + a_3 y y'^2 + a_4 x^3 + a_5 y^3 + a_6 y'^3 + a_7 x y y' + a_8 x + \\
&\quad a_9 y + a_{10} x^2 + a_{11} y y' + a_{12} y'^2 + a_{13} x y' + a_{14} x y + a_{15} y'^2 + \\
&\quad a_{16} x^2 y' + a_{17} x^2 y + a_{18} x y^2 + a_{19} x y'^2 + a_{20} y'
\end{align*}
\]  \hfill (62)

Analyzing the \( \mathcal{D} \) operator (eq. (60)) and equation (61), one may conclude that the maximum degree for \( g \) is 1. So,

\[ g = b_1 + b_2 x + b_3 y + b_4 y'. \]  \hfill (63)

Solving equation (61) will be translated into solving the following algebraic system:

\[
\begin{align*}
sys &= (\beta a_2 - a_5 b_3 = 0, \beta a_{13} - c_2 a_7 - a_{12} b_2 - a_{18} b_1 - a_{14} b_3 = 0, \\
&\quad -a_5 b_2 - a_{18} b_3 + \beta a_7 = 0, 2 \beta a_{19} - a_{17} b_3 - a_{12} b_2 - a_{18} b_4 = 0, \\
&\quad -a_{10} b_4 - a_{16} b_1 - c_1 a_{16} - a_{13} b_2 + a_{17} = 0, -a_{19} b_2 - a_{16} b_4 = 0, \\
&\quad -a_{16} b_3 - a_{17} b_4 - a_7 b_2 = 0, \\
&\quad -a_{10} b_3 - a_{17} b_1 - a_{14} b_2 - c_2 a_{16} = 0, -a_{10} b_1 - a_8 b_2 + 3 a_4 = 0, \\
&\quad -a_4 b_2 + 0, -a_{17} b_2 - a_4 b_3 = 0, -a_1 b_2 - a_8 b_1 + 2 a_{10} = 0, \\
&\quad -a_{10} b_2 - a_4 b_1 = 0, -a_{16} b_2 - a_4 b_4 = 0, -a_6 b_3 - a_3 b_4 = 0, \\
&\quad -a_{18} b_2 + \beta a_{16} - a_{17} b_3 = 0, -a_3 b_2 - a_7 b_4 - a_{19} b_3 = 0, \\
&\quad -a_3 b_3 + 3 \beta a_6 - a_2 b_4 = 0, -a_9 b_3 - c_2 a_{11} + a_{18} - a_{12} b_1 + \beta a_{20} = 0, \\
&\quad -c_2 a_2 + \beta a_{11} - a_{12} b_3 - a_5 b_1 = 0, \\
&\quad -c_2 a_{20} + a_{14} - a_1 b_3 - a_9 b_1 = 0, -3 c_1 a_6 + a_3 - a_{15} b_4 - a_6 b_1 = 0,
\end{align*}
\]
A solution to the system above is:

\[ 2 \beta a_3 - a_2 b_3 - a_5 b_4 = 0, a_{19} - 2 c_1 a_{15} + a_{11} - a_{20} b_4 - a_{15} b_1 = 0, \]
\[-a_6 b_4 = 0, a_{13} - a_1 b_4 - a_{20} b_1 - c_1 a_{20} + a_9 = 0, \]
\[a_8 - a_1 b_1 = 0, -3 c_2 a_6 + 2 a_2 - 2 c_1 a_3 - a_{15} b_3 - a_{11} b_4 - a_3 b_1 = 0, \]
\[-2 c_2 a_{15} + 2 a_{12} - a_{20} b_3 + a_7 - a_9 b_4 - a_{11} b_1 - c_1 a_{11} = 0, \]
\[-2 c_1 a_{19} - a_{13} b_4 - a_{19} b_1 - a_{15} b_2 + a_7 = 0, \]
\[-a_{20} b_2 - a_8 b_4 - a_{13} b_1 + 2 a_{16} - c_1 a_{13} + a_{14} = 0, -a_6 b_2 - a_{19} b_4 = 0, \]
\[-2 c_2 a_{19} - a_{11} b_2 - c_1 a_7 - a_{13} b_3 + 2 a_{18} - a_{14} b_4 - a_7 b_1 = 0, \]
\[2 a_{17} - c_2 a_{13} - a_9 b_2 - a_{14} b_1 - a_8 b_3 = 0, \]
\[-a_{12} b_4 - a_2 b_1 - a_{11} b_3 - c_1 a_2 - 2 c_2 a_3 + 3 a_5 + 2 \beta a_{15} = 0 \]  
(64)

A solution to the system above is:

\[ a_{20} = 0, a_4 = -\frac{2}{3} \beta a_{11}, a_9 = \frac{4}{25} c_2^2 a_{11}, c_2 = \frac{6}{25} c_1^2, a_3 = 0, \]
\[a_{18} = 0, a_{12} = 0, a_5 = 0, a_{15} = 0, a_{16} = 0, b_3 = 0, b_4 = 0, a_{19} = 0, \]
\[a_7 = 0, a_2 = 0, a_8 = 0, b_2 = 0, a_1 = 0, b_1 = -\frac{6}{5} c_1, a_{11} = a_{11}, a_{13} = 0, \]
\[a_{10} = \frac{4}{5} c_1, a_{17} = 0, a_{14} = 0, a_6 = 0. \]  
(65)

Leading to:

\[ v = -\frac{2}{3} \beta y^3 + \frac{4}{25} c_2^2 y^2 + \frac{4}{5} c_1 y y' + y^2 \]
\[ g = -\frac{6}{5} c_1 \]
\[ c_2 = \frac{6}{25} c_1^2 \]  
(66)

As we have mentioned, the solution via our procedure takes care of the analysis of integrability of the SOODE. The above value for \( c_2 \) is an well known result for the integrability of the Helmholtz system.

So, following the remaining steps of our algorithm, solving equations (45) and (46) to find \( P \) and \( Q \), we can get two independent solutions:

- **First solution**
  \[ S = \frac{-25 \beta y^2 + 4 c_1^2 y + 10 c_1 y'}{10 c_1 y + 25 y'} \]  
(67)
  \[ R = \frac{2 c_1 y + 5 y'}{-50 \beta y^3 + 12 c_1^2 y^2 + 60 c_1 y y' + 75 y^2} \]  
(68)

Using equation (47), we find, for this pair of \( S \) and \( R \):

\[ I = 6 c_1 x + 5 \ln \left( -12 c_1^2 y^2 + 50 \beta y^3 - 60 c_1 y y' - 75 y^2 \right) \]  
(69)

- **Second solution**
  \[ S = \frac{6 c_1^2 y + 25 c_1 y' - 25 \beta y^2}{25 y'} \]
  \[ R = y' \left( -\frac{2}{3} \beta y^3 + \frac{4}{25} c_1^2 y^2 + \frac{4}{5} c_1 y y' + y^2 \right)^{-5/6} \]
(70)
(71)
Using equation (47), we find, for this pair of \( S \) and \( R \):

\[
I = \int \frac{(6 c_1^2 y + 25 c_1 y' - 25 \beta y^2)}{\left( \frac{4}{25} c_1^2 y^2 - \frac{2}{5} \beta y^3 + \frac{4}{5} c_1 y y' + y'^2 \right)^{5/6}} \, dy
\]  

(72)

What new properties of our algorithm can be emphasized by this example? As mentioned, the method is blind as for the origin of the parameters we want to determine. So, one can use this to include the integrability analysis in a very natural way. We see that, since our method is based on very sound theoretical results (see section 3) about the nature of \( R \), we are left with a truly algorithmic procedure of the PS-type, i.e., we reduce the solving of a SOODE to an algebraic problem.

### 4.3 Third Example

The next example is also physically motivated, the force free Duffing-van der Pol oscillator:

\[
y'' + \left( \alpha + \beta y^2 \right) y' - \gamma y + y^3 = 0
\]  

(73)

For this case, there are also some integrability aspects. We are not going to dwell on that now. This was already done on the last example. It is well known (see for instance, [21]) that the integrability occurs for \( \gamma = -3 \beta^{-2}, \alpha = 4 \beta^{-1} \).

Applying our method (section 3) to that equation, we first have to calculate the eigenpolynomials to the appropriate degree (and the associated cofactors) for the differential operator associated with equation (73) given by:

\[
D = \partial_x + y' \partial_y + \left( -y' \alpha - \beta y^2 y' + \gamma y - y^3 \right) \partial_{y'}
\]  

(74)

These are:

\[
v_1 = \frac{y + y' \beta}{\beta}, \quad g_1 = -\frac{3 + \beta^2 y^2}{\beta}
\]

(75)

\[
v_2 = \frac{(3 y + \beta^2 y^3 + 3 y' \beta)}{3 \beta}, \quad g_2 = -\frac{3}{\beta}
\]

(76)

Now, following the steps of the algorithm (given on section 3), we can find two independent solutions:

- **First solution**

\[
S = \frac{\beta^2 y^2 + 1}{\beta}
\]

(77)

\[
R = \frac{3 \beta}{(3 y + \beta^2 y^3 + 3 y' \beta)}
\]

(78)

Using equation (47), we find, for this pair of \( S \) and \( R \):

\[
I = \frac{3 x + \ln \left( 3 y \beta \alpha - 9 y + \beta^2 y^3 + 3 y' \beta \right)}{\beta}
\]  

(79)

- **Second solution**

\[
S = \frac{4 y' \beta + y' \beta^3 y^2 + 3 y + y^3 \beta^2}{y'}
\]

(80)

\[
R = \frac{y'}{\sqrt{y + \frac{1}{3} y^3 \beta^2 + y' \beta (y' \beta + y)}}
\]

(81)
Using equation (47), we find, for this pair of $S$ and $R$:

\[
I = - \left(3 y + y^3 \beta^2 + 3 y' \beta \right)^{2/3} \beta^{2/3} + 2 \ln \left(\sqrt[3]{3 y + y^3 \beta^2 + 3 y' \beta - y \beta^{2/3}} \right) - \\
\ln \left(\left(3 y + y^3 \beta^2 + 3 y' \beta \right)^{2/3} + \sqrt[3]{3 y + y^3 \beta^2 + 3 y' \beta} \beta^{2/3} + y^2 \beta^{4/3} \right) + \\
2 \sqrt[3]{3} \arctan \left(\frac{\sqrt[3]{3 y + y^3 \beta^2 + 3 y' \beta} + y \beta^{2/3}}{3 y^{2/3}} \right)
\]

(82)

In this example, we skipped the integrability analysis (that was already exemplified on the previous one but could be done in the same fashion here) and followed the same procedure as in the two first examples to find the functions $S$ and $R$.

### 4.4 Some Comments on the Examples Above

Before we present the fourth (and final) example, let us make some considerations on the three previous ones: In [17], we have introduced the basis for an approach to algorithmically solve SOODEs. Later on, [18] picked up some of these ideas and have developed a different approach to the task. They also used our function $S$ as one of the main ingredients of their method. Shortly, they say they use the same ansatz as we did in [17] for $S$:

\[
S = \frac{a(x, y) + b(x, y) y'}{c(x, y) + d(x, y) y'}
\]

(83)

Actually, this is a more general form than the one we considered in [17]. As we do here, we consider $S$ to be a plain rational function on $(x, y, y')$ (not rational in respect only to $y'$). Using their format (83), our case would correspond to $(a, b, c, d)$ being polynomials on $(x, y)$\(^8\). This is an important point in understanding the differences between our present algorithmic method and the approach presented on [18]. To further our discussion, let us introduce the results for $S$ and $R$, presented on [18], for the three examples above:

- For example 1 above (4.1)
  The authors of [18] have found two pairs of $S$ and $R$:

\[
S_1 = - \frac{3 y'}{x}, \quad R_1 = \frac{1}{y^4 x} \\
S_2 = - \frac{y'}{x}, \quad R_2 = \frac{1}{y^5 x}
\]

(84)

- For example 2 above (4.2)
  For this case, the authors have found:

\[
S_1 = - \beta y^2 + \frac{4 c_1^2}{2} y + \frac{2 c_1}{5} y' \\
S_2 = - \beta y^2 + \frac{6 c_1^2}{2} y + \frac{c_1}{3} y' \\
R_1 = - (y' + \frac{2 c_1 y}{5}) e^{\frac{6 c_1}{5} x} \\
R_2 = - y' e^{c_1 x}
\]

(85)

\(^8\)Please note that, in our considerations, we were not limited to numerators and denominators of degree 1 in $y'$, etc.
• For example 3 above (4.3)

In this third example, the authors of [18] have found only one pair of $S$ and $R$:

$$S_1 = \frac{1}{\beta} + \beta y^2 \quad R_1 = e^{\frac{3x}{\beta}}$$

(86)

In order to obtain the above results, the authors also used an ansatz for $R$:

$$R = A(x, y) + B(x, y) y'.$$

(87)

They use the two ansatz (for $S$ and $R$) into equations (8), (9) and (10) thus obtaining a system of partial differential equations for the unknown functions $(a, b, c, d, A, B)$ of $(x, y)$, this can be very trick and not algorithmic in essence (see example 3 above where the authors could only find one independent pair of $S$ and $R$). On the other hand, our method only consider that $S$ has to be a rational function of $(x, y, y')$ and, due to the theoretical results presented on section 3, we know the general form of $R$ and that led to an algorithmic Darboux type procedure to determine $S$ and $R$.

As can be seen from equations (85, 86), the method presented on [18] can find, in principle, non-algebraic $S$ and $R$ (our algorithm deals with rational $S$ and algebraic $R$). But, for the examples presented on [18], the authors found only rational $S$ functions and, therefore, in principle, inside the applicability of our method. As mentioned already, the point is that our approach is algorithmic (in the Prelle-Singer sense) and, computationally speaking, much faster to apply. Basically, we convert solving a SOODE to solving an algebraic system while the method presented on [18] transforms the solving of the SOODE in solving a system of partial differential equations.

The three examples above are physically motivated examples and were previously dealt with in [17] (example 1) and [18] (examples 1, 2, 3). In this latter reference, these were classified into three different types. It is worth to point out that, in our algorithm, that is not necessary, i.e., all examples are treated equally since our procedure is non-classificatory.

In order to conclude this section with examples, we will pass now to a more mathematical note and present an academic SOODE with the intention to show that the present approach is capable of solving it while many powerful techniques fail.

### 4.5 Fourth Example

Let us consider the following SOODE:

$$y'' = -\frac{y'^2}{-y - 1 + 3xy'}$$

(88)

Let us now apply our new method (section 3) to the above equation. First, we have to calculate the eigenpolynomials of degree=1 (and the associated cofactors) for the differential operator associated with equation (48) given by:

$$D = (-y - 1 + 3xy') \partial_x + (-y' y - y' + 3y'^2 x) \partial_y - y'^2 \partial_y'. $$

(89)
These are found to be (up to degree=1):

\[ v_1 = y' \quad g_1 = -y' \quad (90) \]

Now, following the steps of the algorithm (given on section 3), we can find two independent solutions:

- **First solution**
  \[
  S = -\frac{y'(-2y + 3xy' - 2)}{(-y - 1 + 3xy')} \\
  R = -y - 1 + 3xy' 
  \quad (91) (92)
  \]

Using equation (47), we find, for this pair of \( S \) and \( R \):

\[
I = 2xy^2y + 2y^2x - 2y^3x^2 - \frac{2}{3}y^2y' - \frac{4}{3}yy' - \frac{2}{3}y'. 
\quad (93)
\]

- **Second solution**
  \[
  S = \left( -\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) y' \\
  R = y'\left( -\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) 
  \quad (94) (95)
  \]

Using equation (47), we find, for this pair of \( S \) and \( R \):

\[
I = \arcsin \left( \frac{y\sqrt{3} + \sqrt{3}}{\sqrt{(6xy' - 3y - 3)^2 + \left(y\sqrt{3} + \sqrt{3}\right)^2}} \right) + \frac{\sqrt{3}}{2} \ln \left( y' \right). 
\quad (96)
\]

The great advantage of having a non-classificatory, algorithmic approach is that we can tackle every SOODE in the same manner. The above example, despite its simple appearance, could not even be reduced by the powerful Maple solver (release 9.5). In our method, the most costly part of the algorithm is to determine the eigenpolynomials (and corresponding cofactors) and, for this example, this is actually very simple.

**5 Conclusion**

In [17], we have developed a method, based on a conjecture, to deal with SOODEs that presented an elementary solution (possessing two elementary first integrals).

In that same paper, we have introduced a function \( S \) to transform the Pfaffian equation related to the particular SOODE under consideration into a 1-form proportional to the differential of the first integral. That function \( S \) was instrumental in finding the integrating factor for the SOODE.

Here, in the present paper, we introduce many theoretical results concerning that function \( S \) and present an way to calculated it and the integrating factor \( R \) via a Darboux-type procedure.

Briefly, we construct a differential operator \( D \),

\[
D = N D = N \partial_x + y'N \partial_y + M \partial_{y'}, 
\quad (97)
\]
extracted from the SOODE, and the corresponding eigenpolynomials and cofactors will be the building blocks of $S$ and $R$. Furthermore, the procedure is semi-algorithmic and, given enough time, if the solution exists, it will find it (as does the Prelle-Singer approach for FOODEs).

From an operational point of view, our method is very sound, since, as mentioned in sub-section 4.2, our algorithm converts the solving of the SOODE into solving (essentially) first degree algebraic equations.

As a consequence of this, our approach is capable of analyzing the integrability regions for the SOODE (for the case where it presents undetermined parameters). Since we translate the SOODE into an algebraic system, if we consider those parameters as variables, the method solves the system given the values for which there is an integration possible.

Our method is not general since it is limited to the cases where there exists an elementary first integral and, furthermore, to the cases where $S$ is rational. Actually, in respect to the first limitation mentioned above, it is possible that our method solves a case where the first order differential invariant is not elementary (see example 2, sub-section 4.2). The point is that, if $S$ and $R$ are of the right format (i.e., rational $S$ and $R = \prod_i p_i^{n_i}$, where $p_i$ are irreducible polynomials in $(x, y, y')$ and $n_i$ are non-zero rational numbers.), our method will work. The method is sure to work if there is an elementary first order invariant. We would like to elaborate a little bit about the second restriction mentioned above ($S$ being rational): although all of our theoretical results apply to that particular situation, we could not, so far, find any an interesting (not easily solvable) example where $S$ is an algebraic function of $(x, y, y')$ (the general case for elementary first order invariants, see Corollary 1 above). So, the restriction does not seem to be very restrictive. For instance, in [18], all cases of physical interest are covered by our method, i.e., present rational $S$.

In regard to future work, many extensions of the present paper can be pursued: We intend to further our work to include Liouvillian first order differential invariants (along the lines we followed, in dealing with FOODEs [13]). We can also include SOODEs with elementary functions (see [7, 14], for related work applied to FOODEs). As hinted above, we also intend to further analyze the capabilities of our method in the determination of integrability regions for the parameters present on the SOODE.

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