1 Introduction

This article is a successor to our previous paper [3] and continues the theme of generalizing the Yamabe problem to various classes of singular spaces. In that earlier paper we considered this problem on ‘almost smooth’ metric-measure spaces which satisfy a small set of additional structural hypotheses. As part of this, we defined the local Yamabe invariant \( Y_\ell(M,[g]) \), which is a generalization of the quantity \( Y(S^n) \) which plays a key role in the standard Yamabe problem, and then established solvability of the Yamabe problem for any metric \( g \) on the smooth locus of one of these spaces provided it satisfies \(-\infty < Y(M,[g]) < Y_\ell(M,[g]) \). As the main application there, we find Yamabe minimizers on certain stratified spaces with iterated edge metrics.

In the present article we consider this problem in a more general setting, on the class of Dirichlet spaces which satisfy a few additional structural properties. Our main results here again concern the generalized Aubin inequality, in particular its role in establishing existence of minimizing solutions for the Yamabe energy, and we also consider the regularity for (not necessarily minimizing) critical points of this energy.

Let us begin by recalling the standard Yamabe problem. Consider the functional
\[
E(g) := \frac{\int_M \text{Scal}_g \, d\mu_g}{\text{Vol}_g(M)^{(n-2)/n}}
\]

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on the space $\mathcal{M}(M)$ of all Riemannian metrics on the compact (smooth) manifold $M^n$ with $n \geq 3$. Here, $\text{Scal}_g, d\mu_g$ and $\text{Vol}_g(M)$ are the scalar curvature, volume form and volume of $(M, g)$. This is called the (normalized) Einstein-Hilbert functional, and its critical points are the Einstein metrics on $M$.

This functional is unbounded both above and below, so it is reasonable to search for critical points using a max-min scheme. Consider the quantity

$$ Y(M, C) := \inf_{\tilde{g} \in C} E(\tilde{g}), $$

the infimum of $E$ on any conformal class $C = [g] := \{e^{2f} \cdot g \mid f \in C^\infty(M)\}$. This is called the Yamabe invariant (or Yamabe constant or conformal Yamabe invariant) of $C$. We then define

$$ Y(M) := \sup_{C \in \mathcal{C}(M)} \inf_{g \in C} E(g) = \sup_{C \in \mathcal{C}(M)} Y(M, C), $$

where $\mathcal{C}(M)$ is the space of all conformal classes on $M$. This is called the Yamabe invariant (or $\sigma$-invariant or smooth Yamabe invariant) of $M$, see [19], [31].

The Yamabe problem concerns the first part of this, namely whether it is possible to find a metric which minimizes $E$ in a given conformal class $C$. Such a metric has constant scalar curvature, and conversely, any constant scalar curvature metric is at least a critical point for $E$ in its conformal class. The second step, showing that one can find a metric $g$ which attains the max-min, so that $g$ is Einstein (and $E(g) = Y(M)$) is significantly more difficult. We refer to [23], [5], [4] for some significant progress here.

It is now well known, through successive work of Yamabe, Trudinger, Aubin and Schoen, see [24], [7] for details, that each conformal class $C$ contains a minimizer $\tilde{g}$ of $E$ restricted to that conformal class, called the Yamabe metric of that class, and

$$ \text{Scal}_{\tilde{g}} = Y(M, C) \cdot \text{Vol}_{\tilde{g}}(M)^{-2/n}. $$

When studying sequences of Yamabe metrics $g_j$ satisfying certain geometric non-collapsing assumptions, and with $E(g_j) \to Y(M)$, one is led to consider limit spaces which are Riemannian orbifolds (or Riemannian multi-folds, manifolds with conic singularities, simple edge spaces, and more general iterated edge spaces). This makes it natural to study the Yamabe problem directly on these and more general singular spaces, cf. [1], [2], [3], [20], [37].

In our previous paper [3] we consider the Yamabe problem on a compact metric-measure space $(M, d, \mu)$ which has a compatible smooth Riemannian metric $g$ on an open dense subset; we call this an almost smooth metric-measure space. Assuming also that this space is Ahlfors $n$-regular, satisfies a Sobolev inequality, and with certain growth conditions on $\text{Scal}_g$, but without specific information
about the singular set of \( M \), we define the local Yamabe invariant \( Y_\ell(M, d, \mu) \). Roughly speaking, this is the infimum of the Yamabe invariants of each of the tangent cones to \( M \). When \( M \) is smooth, \( Y_\ell(M, [g]) \) equals the Euclidean Yamabe invariant \( Y(\mathbb{R}^n) \), or equivalently, the Yamabe invariant \( Y(S^n, [g_0]) \) of the round sphere \( (S^n, g_0) \). Aubin’s inequality [6] states that \( Y(S^n, [g_0]) \) is the supremum of the set of values of the Yamabe invariants over all compact smooth conformal \( n \)-manifolds:

\[
Y(M, C) \leq Y(S^n, [g_0]) \quad \text{for every} \quad (M, C).
\]

As in the smooth case, we can define the Yamabe invariant \( Y(M, d, \mu) \) of a compact metric-measure space \((M, d, \mu)\), and it is not hard to show that the analogous Aubin-type inequality

\[
Y(M, d, \mu) \leq Y_\ell(M, d, \mu)
\]  

(1.1)

is still valid. This local Yamabe invariant contains much information about the metric near the singular points of \( M \). In [3], we showed that if \((M, d, \mu)\) is almost smooth and satisfies the extra conditions noted above, and if \( Y(M, d, \mu) < Y_\ell(M, d, \mu) \), then the energy \( E \) attains its minimum in that conformal class. We also proved that solutions of the Yamabe equation on \((M, d, \mu)\) are bounded and uniformly positive.

We generalize this yet further here and consider a Yamabe-type problem on a so-called Dirichlet space \((M, \mu, \mathcal{E})\), i.e. a finite measure space \((M, \mu)\) equipped with a Dirichlet form \( \mathcal{E} \) on \( L^2(M, \mu) \), with the scalar curvature replaced by a potential \( V \). Assuming a few other conditions on the space and potential, we define a Yamabe invariant \( Y(V) \) of \((M, \mu, \mathcal{E}, V)\), and then consider the corresponding Yamabe-type problem. After proving the generalization of (1.1), we show that if this inequality is strict, then (again under certain additional assumptions), this generalized Yamabe problem admits a minimizer. We also prove the boundedness, uniform positivity and Hölder continuity of more general solutions of the associated Yamabe equation.

This paper is organized as follows: §2 reviews the necessary terminology and defines the generalized Yamabe problem on a Dirichlet space; in §3 we establish the Aubin inequality and prove existence of minimizers of the generalized Yamabe problem; §4 contains proofs of the regularity results for solutions of the Yamabe type equation; finally, in §4, we present some examples of this generalized Yamabe problem.

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2 A generalized Yamabe problem

We begin by presenting some terminology which allows us to pose the generalized
Yamabe problem on a Dirichlet space.

2.1 Dirichlet spaces

We first review some classical facts about Dirichlet spaces; \[15\] is a comprehensive
reference for this material, but see also \[28, page 209\] or \[9\], which is sufficient
for what we do here.

Let \((M, \mu)\) be a finite measure space, and consider a nonnegative closed sym-
metric bilinear form \(E\) defined on a dense subspace \(D(E) \subset L^2(M, \mu)\); thus
\(E: D(E) \times D(E) \rightarrow \mathbb{R}^+\). We refer to this simply as a
closed symmetric form
on \(L^2(M, \mu)\), and identify \(E\) with the corresponding quadratic form
\(E(\phi, \phi)\). Because
this quadratic form is semibounded, the Friedrichs extension procedure determines
a selfadjoint operator \(L: D(L) \rightarrow L^2(M, \mu)\), with domain \(D(L)\) consisting of all
functions \(v \in D(E)\) such that

\(|E(v, \phi)| \leq C \|\phi\|_{L^2} \quad \text{for every } \phi \in D(E)\).

for some constant \(C\) (depending on \(v\) and \(E\), but not \(\phi\)). This is the generator of \(E\).
A closed symmetric form on \(L^2(M, \mu)\) is called a Dirichlet form if its generator \(L\)
is \(\text{subMarkovian}\), i.e. provided the semigroup \(e^{-tL}\) satisfies

\[0 \leq v \leq 1 \Rightarrow 0 \leq e^{-tL}v \leq 1 .\]

According to the Beurling-Deny criteria, this is equivalent to the following:

i) \(v \in D(E) \Rightarrow |v| \in D(E)\) and \(E(|v|) \leq E(v)\)

ii) \(v \in D(E)\) and \(v \geq 0 \Rightarrow v_1 := \inf \{v, 1\} \in D(E)\) and \(E(v_1) \leq E(v)\).

A triple \((M, \mu, E)\) with all these properties is called a Dirichlet space.

2.2 The Sobolev inequality

Suppose that \((M, \mu, E)\) is a Dirichlet space for which a Sobolev inequality holds.
This means that there exist \(\nu > 2\) and \(A, B > 0\) such that

\[A \|v\|^2_{L^{2\mu/2}} \leq A\mathcal{E}(v) + B \|v\|^2_{L^2} \quad \text{for } v \in D(E).\] (2.1)

Following Nash \[27\], the heat semi-group \(\{e^{-tL}\}_{t \geq 0}\) then necessarily satisfies an
ultracontractive estimate: there exists a constant \(C\) such

\[\|e^{-tL}\|_{L^1 \rightarrow L^\infty} \leq C \frac{t^{\mu/2}}, \quad 0 < t < 1.\] (2.2)
It is now known, through work of Varopoulos [36], that (2.1) and (2.2) are in fact equivalent.

It is straightforward to show that (2.1) implies the following compactness result, see [3, Proposition 1.6]:

**Proposition 2.1.** If (2.1) holds, then the inclusion
\[ D(E) \to L^{\frac{2p}{p-2}}(M) \]
is compact for any \( p \in (\nu, \infty] \).

### 2.3 Schrödinger operators

A nonnegative measurable function \( W \) is said to be *relatively form bounded* with respect to \( E \) if there exists some constant \( D > 0 \) such that
\[
\int_M W v^2 d\mu \leq D(\mathcal{E}(v) + ||v||_{L^2}^2) \quad \text{for } v \in D(E);
\]
similarly, \( W \) is *infinitesimally form bounded* with respect to \( E \) if for any \( \varepsilon > 0 \) there exists \( c(\varepsilon) \) such that :
\[
\int_M W v^2 d\mu \leq \varepsilon \mathcal{E}(v) + c(\varepsilon) \int_M v^2 d\mu \quad \text{for } v \in D(E).
\]

Since \( D(E) \hookrightarrow L^2(M) \) is compact, \( W \) is infinitesimally form bounded with respect to \( E \) if and only if the operator \((L + 1)^{-\frac{1}{2}} W (L + 1)^{-\frac{1}{2}}\) is compact on \( L^2 \).

If \( V \) is a real-valued integrable function on \( M \) and its nonpositive part \( V_- := \sup\{0, -V\} \) is relatively form bounded with respect to \( E \), we define the quadratic form
\[
\mathcal{E}_V(v) = \mathcal{E}(v) + \int_M V v^2 d\mu
\]
on the domain \( D(\mathcal{E}_V) = \{v \in D(E) : \int_M V v^2 d\mu < \infty\} \). As before, \( \mathcal{E}_V \) is densely defined, closed and semibounded, so we can define the self-adjoint operator \( L + V \) by the Friedrichs procedure.

### 2.4 The generalized Yamabe problem

Let \( V \) be integrable and suppose that \( V_- \) is relatively form bounded; suppose too that the Dirichlet space \((M, \mu, \mathcal{E})\) satisfies the Sobolev inequality (2.1). We then define the *Yamabe invariant* associate to the operator \( L + V \):
\[
Y(V) = \inf \left\{ \mathcal{E}_V(v) : v \in D(E) \text{ and } ||v||_{L^{\frac{2p}{p-2}}} = 1 \right\}.
\]
Note that (2.1) implies immediately that

\[ Y(V) \geq -D \max\{A, B\} . \]

We wish to whether there exists \( u \in \mathcal{D}(\mathcal{E}_V) \) such that

\[ \mathcal{E}_V(u) = Y(V) \quad \text{and} \quad \|u\|_{L^{2\nu}} = 1 . \]

Since

\[ \mathcal{E}_V(|u|) \leq \mathcal{E}_V(u) , \]

we can always assume that any such minimizer must be nonnegative. This minimizer must satisfy the Euler-Lagrange equation

\[ \mathcal{E}_V(u, \varphi) = Y(V) \int_M u^{\frac{\nu+2}{\nu-2}} \varphi \, d\mu \quad \text{for every} \quad \varphi \in \mathcal{D}(\mathcal{E}) . \] (2.3)

Note that by the Sobolev and H"older inequalities, the right hand side is finite.

3 Existence of minimizers

3.1 Existence theorem

Theorem 3.1. Let \((M, \mu, \mathcal{E})\) be a Dirichlet space with Sobolev inequality (2.1) for some \( \nu > 2 \) and positive constants \( A, B \). Let \( V \) be an integrable function whose nonpositive part \( V_- \) is infinitesimally form bounded with respect to \( \mathcal{E} \). Assume that

\[ Y(V) < \frac{1}{A} . \] (3.1)

Then there exists \( v \in \mathcal{D}(\mathcal{E}) \) such that

\[ \mathcal{E}_V(v) = Y(V) \quad \text{and} \quad \|v\|_{L^{2\nu}} = 1 . \]

Remark 3.2. The hypothesis (3.1) can be rephrased in terms of the optimal Sobolev constant \( A_{\text{opt}} \). This is, by definition, the smallest constant such that for every \( A > A_{\text{opt}} \) there exists \( B > 0 \) such that the Sobolev inequality (2.1) holds with that choice of \( A \) and \( B \). We also write

\[ A_{\text{opt}} = \frac{1}{\alpha(\mathcal{E})} , \]

where

\[ \alpha(\mathcal{E}) = \lim_{t \to \infty} Y(t) , \]
i.e. the limit of the Yamabe invariants associated to the constant potentials \( V \equiv t \).

Another characterization is that

\[
A_{\text{opt}} = \lim_{t \to +\infty} \left\| \left( \sqrt{L} + t \right)^{-\frac{1}{2}} \right\|_{L^2}^2.
\]

**Proof.** Using the infinitesimal form boundedness of \( V \), we see that if \( \hat{A} > A \) then there exists a positive constant \( \hat{B} \) such that:

\[
\|v\|^2_{L^{\frac{2\nu}{\nu-2}}} \leq \hat{A} \mathcal{E}_V(v) + \hat{B} \|v\|^2_{L^2} \quad \text{for all } v \in \mathcal{D}(\mathcal{E}). \tag{3.2}
\]

Choose \( \hat{A} > A \) so that \( \hat{A}Y(V) < 1 \). Since the embedding \( \mathcal{D}(\mathcal{E}_V) \to L^2 \) is compact, we can find a minimizing sequence \( u_\ell \in \mathcal{D}(\mathcal{E}_V) \) and \( u \in \mathcal{D}(\mathcal{E}_V) \) such that

a) \( u_\ell \rightharpoonup u \) weakly in \( \mathcal{D}(\mathcal{E}_V) \);

b) \( u_\ell \to u \) strongly in \( L^2 \);

c) \( u_\ell \to u \) a.e.;

d) \( \|u_\ell\|^2_{L^{\frac{2\nu}{\nu-2}}} = 1 \).

By d),

\[
\mathcal{E}_V(u_\ell - u) = \mathcal{E}_V(u_\ell) - \mathcal{E}_V(u) + \varepsilon_\ell, \quad \text{where } \lim_{\ell \to \infty} \varepsilon_\ell = 0.
\]

We now appeal to a very useful result of Brezis and Lieb [11] (we are grateful to E. Hebey for pointing us to this), which gives

\[
\lim_{\ell} \left( \|u_\ell\|^2_{L^{\frac{2\nu}{\nu-2}}} - \|u_\ell - u\|^2_{L^{\frac{2\nu}{\nu-2}}} \right) = \|u\|^2_{L^{\frac{2\nu}{\nu-2}}}. \nonumber
\]

Hence, setting \( I = \|u\|^2_{L^{\frac{2\nu}{\nu-2}}} \), then

\[
\lim_{\ell} \|u_\ell - u\|^2_{L^{\frac{2\nu}{\nu-2}}} = 1 - I.
\]

Now apply the Sobolev inequality (3.2) to \( u_\ell - u \) and pass to the limit \( \ell \to \infty \) to get

\[
(1 - I)^{1 - \frac{2}{\nu}} \leq \hat{A}Y(V) - \hat{A}\mathcal{E}_V(u).
\]
On the other hand, by definition,
\[ \mathcal{E}_V(u) \geq Y(V)I^{1 - \frac{2}{\nu}}, \]
so putting these together and recalling the choice of \( \hat{A} \) yields
\[ (1 - I)^{1 - \frac{2}{\nu}} + \hat{A}Y(V)I^{1 - \frac{2}{\nu}} \leq \hat{A}Y(V) < 1. \]
This forces \( I = 1 \), hence \( u \neq 0 \), and since \( Y(V) \geq \mathcal{E}_V(u) \), we conclude that \( u \) is a minimizer for \( \mathcal{E}_V \).

### 3.2 On the optimal Sobolev constant

We now turn to a more careful discussion of the optimal Sobolev constant \( A_{opt} \) introduced in Remark 3.2. We assume henceforth that \( M \) is a compact topological space and \( \mu \) is a Radon measure, and moreover, that the Dirichlet space is regular and strongly local. These last two conditions are:

- **(Regularity)** \( \mathcal{D}(\mathcal{E}) \cap C^0(M) \) is dense in both \( \mathcal{D}(\mathcal{E}) \) with \( \mathcal{E}_1 \)-norm and \( C^0(M) \) with uniform norm;
- **(Strong locality)** if \( u, v \in \mathcal{D}(\mathcal{E}) \) and if \( u \) is constant in a neighborhood of \( \text{supp}(v) \), then \( \mathcal{E}(u, v) = 0 \).

These conditions guarantee the existence of a bilinear form \( d\gamma \), the so-called *the energy measure*, from \( \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \) to the set of Radon measures on \( M \), such that
\[ \mathcal{E}(u, v) = \int_M d\gamma(u, v) \quad \text{for} \quad u, v \in \mathcal{D}(\mathcal{E}). \]

If the energy measure is absolutely continuous with respect to \( d\mu \), Bakry and Emery [8] call this bilinear form the *carré du champ*. The energy measure is determined by the identity
\[ \mathcal{E}(\phi u, u) - \frac{1}{2} \mathcal{E}(\phi, u^2) = \int_M \phi d\gamma(u, u) \quad \text{for} \quad u \in \mathcal{D}(\mathcal{E}) \text{ and } \phi \in \mathcal{D}(\mathcal{E}) \cap C^0(M). \]

The energy measure satisfies the Leibniz and chain rules:
\[ d\gamma(uv, w) = ud\gamma(v, w) + vd\gamma(u, w) \quad \text{for} \quad u, v, w \in \mathcal{D}(\mathcal{E}) \]
\[ d\gamma(f(u), v) = f'(u)d\gamma(u, v) \quad \text{for} \quad u, v \in \mathcal{D}(\mathcal{E}), \text{ and } f \in \text{Lip}(\mathbb{R}). \]
A regular, strongly local Dirichlet space \((M, \mu, \mathcal{E})\) has an intrinsic pseudo-distance defined by
\[ d(x, y) = \sup \{ u(x) - u(y) : u \in \mathcal{D}(\mathcal{E}) \cap C^0(M) \text{ and } d_\gamma(u, u) \leq d\mu \}; \]
the comparison \(d_\gamma(u, u) \leq d\mu\) here means that there exists a function \(f \leq 1\) such that \(d_\gamma(u, u) = fd\mu\).

If this pseudo-distance is compatible with the topology of \(M\), then for any \(y \in M\), the function \(r_y = d(y, \cdot)\) satisfies \(d_\gamma(r_y, r_y) \leq d\mu\) [3]. If \(U\) is open in \(M\), we define
\[
S(U) = \inf \left\{ \mathcal{E}(u) : \|u\|_{L^\infty} = 1 \text{ and } \text{supp } u \subset U \right\},
\]
\[
Y(U) = \inf \left\{ \mathcal{E}_V(u) : \|u\|_{L^\infty} = 1 \text{ and } \text{supp } u \subset U \right\}.
\]
We now adapt the proof of [3] Proposition 1.4, using cutoffs of these distance functions, to obtain

**Proposition 3.3.** Let \((M, \mu, \mathcal{E})\) be a regular, strongly local Dirichlet form with intrinsic distance compatible with the topology of \(M\). Then
\[
A_{\text{opt}} = \sup_{x \in M} \lim_{r \downarrow 0} A_{\text{opt}}(B(x, r)),
\]
where \(B(x, r)\) denotes the metric ball of radius \(r\) centered at \(p\). If \(A_{\text{opt}}\) is finite, then
\[
A_{\text{opt}} = \frac{1}{S_\ell}, \quad \text{where } S_\ell := \inf_{x \in M} \lim_{r \downarrow 0} S(B(x, r)).
\]
Moreover, if \(|V|\) is infinitesimally form bounded with respect to \(\mathcal{E}\), then
\[
S_\ell = Y_\ell := \inf_{x \in M} \lim_{r \downarrow 0} Y(B(x, r)).
\]

4 Regularity of solutions

In this section, we now prove various facts about regularity of solutions (which are not necessarily minimizers) of this generalized Yamabe equation. Note that this equation can be rewritten as
\[
Lu = Wu, \quad \text{where } W = -V + Y(V)u^{\frac{4}{n-2}}.
\]
Some of our results will follow from regularity results for solutions of this linear equation.
4.1 Boundedness

4.1.1 General results

**Proposition 4.1.** Let \((M, \mu, \mathcal{E})\) be a Dirichlet space with Sobolev inequality \((2.1)\). Let \(W\) be a nonnegative measurable function with \(W \in L^q\) for some \(q > \nu/2\). Assume that \(u \in \mathcal{D}(\mathcal{E})\) is a nonnegative function satisfying

\[
Lu \leq W u. \tag{4.2}
\]

Then \(u \in L^\infty\), and moreover,

\[
\|u\|_\infty \leq C \|u\|_2,
\]

where the constant \(C\) depends only on \(\|W\|_{L^q}\), \(n\), \(q\) and the constants \(A, B\).

This follows from the Gagliardo-Nirenberg inequality \([13]\); the proof is in \([12]\), but for the sake of completeness, we sketch the proof here as well.

**Proof.** The inequality \((4.2)\) means that for all nonnegative \(\varphi \in \mathcal{D}(\mathcal{E})\),

\[
q(u, \varphi) \leq \int_M W u \varphi \, d\mu.
\]

The Sobolev inequality implies that

\[
\left\| e^{-t L} \right\|_{L^1 \to L^\infty} \leq \frac{C}{t^{\nu/2}} \quad \text{for } t \in (0, 1),
\]

and hence by interpolation, if \(1 \leq r < s\), then

\[
\left\| e^{-t L} \right\|_{L^r \to L^s} \leq \frac{C}{t^{\frac{\nu}{2} \left(\frac{1}{r} - \frac{1}{s}\right)}} \quad \text{for } t \in (0, 1). \tag{4.3}
\]

Clearly,

\[
Le^{-tL} u = e^{-tL} Lu \leq e^{-tL} W u,
\]

and hence

\[
u = e^{-L} u + \int_0^1 e^{-tL} Lu \, dt \leq e^{-L} u + \int_0^1 e^{-tL} W u \, dt.
\]

Now introduce

\[
T(f) = e^{-L} f + \int_0^1 e^{-tL} W f \, dt.
\]
From (2.2) and (4.3), it follows that if $f \in L^s$ with $\frac{1}{s} < \frac{2}{\nu} - \frac{1}{q}$, then $Tf \in L^\infty$.

Indeed, if $r$ is determined by $r^{-1} = s^{-1} + q^{-1}$, then

$$
\|Tf\|_{L^\infty} \leq C\|f\|_{L^s} + \int_0^1 \|e^{-t\mathbb{L}}\|_{L^r \to L^\infty} \|W\|_{L^q} \|f\|_{L^s} \, dt \\
\leq C\|f\|_{L^s} + \int_0^1 Ct^{-\frac{s}{r} + \frac{q}{s}} \|W\|_{L^q} \|f\|_{L^s} \, dt \\
\leq C(1 + \|W\|_{L^q})\|f\|_{L^s}.
$$

A similar argument shows that if $f \in L^s$ with $\frac{1}{s} > \frac{2}{\nu} - \frac{1}{q}$, then $Tf \in L^r$ for $r \geq 1$ and $\frac{1}{r} > \frac{1}{s} + \frac{1}{q} - \frac{2}{\nu}$.

Hence, from $u \in L^2$, we obtain that $u \in L^\infty$ in a finite number of steps. \qed

**Remark 4.2.** It is easy to show using (2.1) that if $W \in L^{\frac{\nu}{2}}$, then $|W|$ is relatively form bounded with respect to $\mathcal{E}$. Indeed, if $v \in \mathcal{D}(\mathcal{E})$, then

$$
\int_M |W|v^2 \, d\mu \leq \|W\|_{L^{\frac{\nu}{2}}} \|v^2\|_{L^{\frac{4}{\nu}}} \\
\leq \|W\|_{L^{\frac{\nu}{2}}} \left[ A\mathcal{E}(v) + B\|v\|_{L^2}^2 \right].
$$

Moreover, decomposing $|W| = \inf\{|W|, \lambda\} + W^\lambda$, then for every $v \in \mathcal{D}(\mathcal{E})$,

$$
\int_M |W|v^2 \, d\mu \leq A\|W^\lambda\|_{L^{\frac{\nu}{2}}} \mathcal{E}(v) + \left[ \lambda + B\|W^\lambda\|_{L^{\frac{\nu}{2}}} \right]\|v\|_{L^2}^2.
$$

This proves the infinitesimal form boundedness since $\lim_{\lambda \to \infty} \|W^\lambda\|_{L^{\frac{\nu}{2}}} = 0$.

Another result of the same nature, which is proved exactly as in [3], requires less about $W$ but more regularity on the Dirichlet space.

**Proposition 4.3.** Let $(M, \mu, \mathcal{E})$ be a regular, strongly local Dirichlet space with intrinsic distance compatible with the topology of $M$, and with Sobolev inequality. Suppose too that the measure $\mu$ is Alfors $\nu$-regular, i.e., there exist constants $0 < c < C$ such that

$$
cr^{\nu} \leq \mu(B(x, r)) \leq Cr^{\nu} \quad \text{for all } x \in M \text{ and } r < \text{diam}(M).
$$
Suppose that $W \in L^q$ for some $q > 1$ and moreover, for all $x \in M$ and $r < \text{diam}(M)$,
\[ \int_{B(x,r)} |W|^q \, d\mu \leq \Lambda r^{q-\alpha} \quad (4.4) \]
for some constants $\Lambda$ and $\alpha \in [0, 2)$. If $u \in \mathcal{D}(\mathcal{E})$, $u \geq 0$ and
\[ Lu \leq W u , \]
then $u \in L^\infty$.

It is proved in [3] that the Morrey estimate (4.4) implies that $|W|$ is infinitesimally form bounded with respect to $\mathcal{E}$. In addition, the Gaussian estimate
\[ e^{-tL}(x, y) \leq C t^{\nu/2} e^{-\frac{d(x,y)^2}{t}} \text{ for } x, y \in M, \ t \in (0, 1) . \]
is also valid under this hypothesis.

### 4.1.2 Boundedness of solutions of the Yamabe equation

To apply the results above, we must show that the potential $W$ in (4.1) satisfies one of these hypotheses. In fact, any solution to this equation lies in a better $L^p$ space, cf. [35], [16], [26] :

**Proposition 4.4.** Let $(M, \mu, \mathcal{E})$ be a regular, strongly local Dirichlet space with Sobolev inequality. Suppose that $W$ is integrable and $W_+$ is infinitesimally form bounded with respect to $\mathcal{E}$. If $u \in \mathcal{D}(\mathcal{E})$ is a nonnegative solution to $Lu = W u$, then $u \in L^q$ for all $q \geq 2$.

**Proof.** By assumption on $W_+$, for every $\beta \geq 0$, there are positive constants $A_\beta$ and $B_\beta$ such that
\[ \|v\|_{L^{\frac{2}{q}}}^2 \leq A_\beta \mathcal{E}_{-\beta} W_+ (v) + B_\beta \|v\|_{L^2}^2 \text{ for every } v \in \mathcal{D}(\mathcal{E}) . \quad (4.5) \]

Define, for $\alpha \geq 1$,
\[ f_\alpha(x) = \begin{cases} x^\alpha & \text{if } 0 \leq x \leq \alpha^{-\frac{1}{\alpha-1}} , \\ x + (\alpha^{-\frac{\alpha}{\alpha-1}} - \alpha^{-\frac{1}{\alpha-1}}) & \text{if } \alpha^{-\frac{1}{\alpha-1}} \leq x . \end{cases} \quad (4.6) \]

This function is $C^1$ and convex. Next, for $L \geq 1$, set
\[ \phi_{\alpha,L}(x) = L^\alpha f_\alpha \left( \frac{x}{L} \right) ; \]
thus $\phi_{\alpha,L}(x) = x^{\alpha}$ on $[0, \alpha^{-1} L]$. If we finally set

$$G_{\alpha,L}(x) = \int_0^x \phi'_{\alpha,L}(t)^2 \, dt,$$

then a laborious computation gives

$$\phi_{\alpha,L}(x) \leq x^{\alpha} \quad \text{and} \quad xG_{\alpha,L}(x) \leq \frac{\alpha^2}{2\alpha - 1} (\phi_{\alpha,L}(x))^2, \quad x \geq 0. \quad (4.7)$$

By the chain rule, with $\varphi = \phi_{\alpha,L}(u)$,

$$E(\varphi) = \int_M \phi'_{\alpha,L}(u)^2 \, d\gamma(u, u) = E(G_{\alpha,L}(u), u) = \int_M G_{\alpha,L}(u) W u \, d\mu \leq \int_M W_+ u G_{\alpha,L}(u) \, d\mu \leq \frac{\alpha^2}{2\alpha - 1} \int_M W_+ \varphi^2 \, d\mu.$$

Using (4.5) with $\beta = \frac{\alpha^2}{2\alpha - 1}$ gives

$$\|\phi_{\alpha,L}(u)\|_{L^{\frac{2\alpha}{2\alpha - 1}}}^2 \leq B \frac{\alpha^2}{2\alpha - 1} \|\phi_{\alpha,L}(u)\|_{L^2}^2,$$

so that, letting $L \to \infty$, we conclude

$$u \in L^{2\alpha} \Rightarrow u \in L^{2\frac{\nu}{\nu - 2}\alpha} \quad \text{for all} \quad \alpha \geq 1.$$

This completes the proof. \[\square\]

This all leads to the

**Proposition 4.5.** Let $(M, \mu, E)$ be a regular, strongly local Dirichlet space with Sobolev inequality. Let $V$ be an integrable function with nonpositive part $V_-$ infinitesimally form bounded with respect to $E$. If $u \in \mathcal{D}(E_V)$ is a nonnegative solution to

$$Lu + Vu = Y(V)u^{\frac{\nu + 2}{\nu - 2}},$$

then for every $p \geq 2$,

$$\int_M u^p \, d\mu < \infty.$$

Indeed, the assumption that $u \in \mathcal{D}(E_V)$ and (2.1) give that $u^{\frac{4}{\nu - 2}} \in L^{\frac{\nu}{\nu - 2}}$. According to Remark 4.2, $u^{\frac{4}{\nu - 2}}$ is infinitesimally form bounded with respect to $E$. We can thus apply Proposition 4.4 with $W = -V + Y(V)u^{\frac{4}{\nu - 2}}$. 

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4.2 Positivity of solutions

The argument of [3], see also [13], can be applied verbatim to our Yamabe equation when $(M, \mu, \mathcal{E})$ is a regular, strongly local Dirichlet space with intrinsic distance compatible with the topology of $M$. Thus any nonnegative solution of this equation which is strictly positive on some ball is strictly positive everywhere, provided that (2.1) holds and $|V|$ satisfies a Morrey type estimate. However, the Harnack inequality need not hold in this generality. In the next subsection, we give a criterion which ensures Hölder continuity of solutions to the linear equation $Lu = f$, and this implies that if $u \not\equiv 0$, then it is strictly positive on some ball.

4.3 Higher regularity of solutions

We now turn to questions concerning the modulus of continuity of solutions of the equation $Lu = f$. As usual, let $(M, \mu, \mathcal{E})$ be a regular, strongly local Dirichlet space with intrinsic distance compatible with the topology of $M$. We assume that the measure $\mu$ is Ahlfors $\nu$-regular and a uniform Poincaré inequality holds. This means that if $r \leq \frac{1}{4} \text{diam } M$, then

$$
\|v - v_B\|^2_{L^2(B)} \leq Cr^2 \int_{B(x,2r)} d\gamma(v,v) \quad \text{for every } v \in D(\mathcal{E}),
$$

where $B = B(x,r)$ and $v_B = \frac{1}{\mu(B)} \int_B v \, d\mu$. For a nice review on the Dirichlet space satisfying these assumptions, see [29] and also the paper [17] for recent results.

These assumptions imply that the heat kernel of $L$ exists and satisfies Gaussian upper bounds, and also that the Sobolev inequality (2.1) holds. They also guarantee the elliptic and parabolic Harnack inequality. In particular, if $h$ is a positive harmonic function on $2B := B(x,2r)$ (so $Lh = 0$ on $2B$), then

$$
\sup_{z \in B} h(z) \leq C_H \inf_{z \in B} h(z).
$$

The Harnack constant $C_H$ depends only on the constants in the Ahlfors regularity condition and the Poincaré inequality. From this, one obtains Hölder continuity of harmonic functions.

**Lemma 4.6.** Let $h \in L^\infty(2B)$ be a solution of the equation $Lh = 0$ on a ball $2B$. Then, for all $p, q \in B(x,r)$,

$$
|h(p) - h(q)| \leq C \left( \frac{d(p,q)}{r} \right)^\beta \sup_{z \in 2B} |h(z)|.
$$

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In fact, $\beta = \log_2 \left( \frac{C_{n+1}}{C_n} \right)$ and $C = 2 \left( \frac{C_{n+1}}{C_n} \right)$. The Green function of $L$ is a symmetric function $G \in C^0((M \times M) \setminus \text{Diag})$ such that $G(x, \cdot) \in L^1(M)$ for any $x \in M$, and in addition, if $f \in L^2$, then

$$D(E) \ni u(x) = \int_M G(x, y) f(y) \, d\mu(y) \quad \text{for } \mu\text{-a.e. } x \in M$$

satisfies $Lu = f - f_M$, where $f_M = \frac{1}{\mu(M)} \int_M f \, d\mu$. Clearly, if $Lu = f$ and $\int_M u \, d\mu = 0$, then $f_M = 0$ and

$$u(x) = \int_M G(x, y) f(y) \, d\mu(y) \quad \text{for } \mu\text{-a.e. } x \in M.$$ 

**Proposition 4.7.** The Green kernel $G$ satisfies

$$|G(x, y)| \leq C \frac{d}{d(x, y)^{\nu - 2}} \quad \text{for all } x, y \in M$$

and if $p, q, y \in M$ with $d(p, q) \leq \frac{1}{8} d(p, y)$, then

$$|G(p, q) - G(q, y)| \leq C \left( \frac{d(p, q)}{d(p, y)} \right)^{\beta} \frac{1}{d(q, y)^{\nu - 2}}.$$

**Theorem 4.8.** With all the assumptions as above, suppose that $f$ satisfies the Morrey estimate

$$\int_{B(x, r)} |f| \, d\mu \leq A r^{\nu - \alpha} \quad \text{for all } x \in M, \ r \leq \frac{1}{2} \text{diam } M,$$

for some $A > 0$ and $\alpha \in [0, 2)$. If $u \in D(E)$ solves $Lu = f$, then $u$ is Hölder continuous of order $\mu = \min \{ \beta, 2 - \alpha \}$ when $\beta \neq 2\alpha$; if $\beta = 2\alpha$, then $\mu$ need only satisfy $0 < \mu < \min \{ \beta, 2 - \alpha \}$.

**Proof.** For each $x \in M$, introduce the nondecreasing function

$$\nu_x(r) = \int_{B(x, r)} |f| \, d\mu.$$

If $p, q \in M$ and $\rho := d(p, q) \leq \frac{1}{8} \text{diam } M$, then

$$|u(p) - u(q)| \leq \int_{B(p, 4\rho)} |G(p, y)||f(y)| \, d\mu(y) + \int_{B(q, 4\rho)} |G(q, y)||f(y)| \, d\mu(y)$$

$$+ \int_{M \setminus B(p, 4\rho)} |G(p, y) - G(q, y)| \, |f(y)| \, d\mu(y).$$
Using the estimates on $G$ and integrating by parts,

\[
\int_{B(p,4\rho)} |G(p,y)||f(y)| \, d\mu(y) \leq C \left( \frac{\nu_p(4\rho)}{4\nu-2\rho^2} + (\nu - 2) \int_0^{4\rho} \frac{\nu_p(r)}{r^{\nu-1}} \, dr \right) \]

so by the Morrey estimate,

\[
\int_{B(p,4\rho)} |G(p,y)||f(y)| \, d\mu(y) \leq C \rho^{2-\alpha}.
\]

The integral over $B(q,4\rho)$ is bounded by $C \rho^{2-\alpha}$ too.

For the final term,

\[
\int_{M \setminus B(p,4\rho)} |G(p,y) - G(q,y)| |f(y)| \, d\mu(y) 
\leq C \rho^{\beta} \left[ \frac{\nu_p(\text{diam } M)}{(\text{diam } M)^{\nu-2+\beta}} + (\nu - 2 + \beta) \int_{4\rho}^{\text{diam } M} \frac{\nu_p(r)}{r^{\nu-2+\beta}} \, dr \right].
\]

Since

\[
\int_{4\rho}^{\text{diam } M} \frac{\nu_p(r)}{r^{\nu-2+\beta}} \, dr \leq \Lambda \int_{4\rho}^{\text{diam } M} \frac{1}{r^{\alpha-2+\beta}} \, dr
\]

and

\[
\int_{4\rho}^{\text{diam } M} \frac{1}{r^{\alpha-1+\beta}} \, dr \leq \begin{cases} \frac{1}{\beta-2+\alpha} \frac{1}{(4\rho)^{\alpha-2+\beta}} & \text{if } 2-\alpha < \beta, \\ \frac{1}{\alpha+2-\beta} \frac{1}{(\text{diam } M)^{\alpha-2+\beta}} & \text{if } \beta < 2-\alpha, \\ \log(\text{diam } M/(4\rho)) & \text{if } \beta = 2-\alpha, \end{cases}
\]

we conclude that $u$ is Hölder continuous of order $\mu$. \qed

4.4 Conclusion

**Theorem 4.9.** Let $(M, \mu, \mathcal{E})$ be a regular, strongly local Dirichlet space with Sobolev inequality.

i) If $V_- \in L^q$ for some $q > \nu/2$ and if $Y(V) < 1/A$, then there exists a nonnegative bounded function $u \in D(\mathcal{E})$ such that

\[
\mathcal{E}_V(u) = Y(V) \quad \text{and} \quad \|u\|_{L^{\nu/2}} = 1.
\]
ii) The conclusion of i) holds if the intrinsic distance is compatible with the topology of \( M \), if \( \mu \) is Ahlfors \( \nu \)-regular, and if \( V \) satisfies the Morrey estimate (4.4).

iii) Suppose that the intrinsic distance is compatible with the topology of \( M \), \( \mu \) is Ahlfors \( \nu \)-regular, and \( |V| \) satisfies the Morrey inequality (4.4). (Note that this holds if \( V \in L^p \) for some \( p > \nu/2 \).) Assume also that \((M, \mu, E)\) satisfies the Poincaré inequality on any ball \( B = B(x, r) \subset M \), \( r \leq \frac{1}{4}\) diam \( M \), and that \( Y(V) < 1/A \). Then the solution in either case i) or case ii) is strictly positive and Hölder continuous.

5 Examples

We now explain how the general results above simplify the original proof of the usual Yamabe problem and then yield a generalization of the CR Yamabe problem.

5.1 The Riemannian Yamabe problem

We have already discussed the classical Yamabe problem when \( M^n \) is a compact smooth Riemannian manifold, \( n \geq 3 \). The metric \( \tilde{g} = v^{\frac{4}{n-2}}g \) has constant scalar curvature if and only if \( v \) is a critical point of the Yamabe functional

\[
Q_g(f) := \frac{\int_M \left[ \frac{4(n-1)}{n-2} |df|^2_g + \text{Scal}_g f^2 \right] d\mu_g}{\left( \int_M f^{\frac{2n}{n-2}} d\mu_g \right)^{1-\frac{2}{n}}}
\]

\[
= \text{Vol}_{\tilde{g}}(M)^{\frac{2}{n}-1} \int_M \text{Scal}_{\tilde{g}} d\mu_{\tilde{g}} \quad \text{for} \quad \tilde{g} = f^{\frac{4}{n-2}}g.
\]

The minimizer for this problem always exists. For this case,

- the pair of \((M, d\mu_g)\) and \( E(v) = \frac{4(n-1)}{n-2} \int_M |dv|^2_g d\mu_g \), \( v \in W^{1,2}(M, d\mu_g) \), determine the Dirichlet space;

- \( L + V = -\frac{4(n-1)}{n-2} \Delta_g + \text{Scal}_g \);

- \( \nu = n \) and \( Y_\epsilon = Y(S^n, [g_0]) \).

(Here \( \Delta_g = \text{div}\nabla \).)

The key result, due to Aubin [6] and Schoen [30], [31], [32], states that if \((M, [g])\) is not conformal to \((S^n, [g_0])\), then \( Y(M, [g]) < Y(S^n, [g_0]) \), so the existence proof above may be applied.
5.2 The contact Riemannian Yamabe problem

The second application of our results is to the Yamabe problem on contact Riemannian manifold. This problem was initially posed for CR manifolds by Jerison and Lee, [21], [22], and solved by them for manifolds not CR equivalent to the standard sphere $S^{2m+1}$. The remaining case was completed by Z. Li [25]. This problem does not seem to have been treated for non-integrable almost complex structures, but we are able to work in that more general context here.

5.2.1 The setting:

Recall from [14], [34] that a contact manifold is an odd dimensional manifold $M^{2m+1}$ with a totally non-integrable hyperplane subbundle $H \subset TM$. Thus, for each $x \in M$, there is a nondegenerate bilinear form $H_x \times H_x \to T_x M/H_x$, $(X, Y) \mapsto [X, Y] \mod H_x$.

When $M$ and $H$ are oriented, one may choose a contact form $\theta \in \Omega^1(M)$ with $H = \ker \theta$; in terms of this form, nondegeneracy of $H$ is equivalent to $\theta \wedge (d\theta)^m \neq 0$ everywhere on $M$.

A choice of $\theta$ uniquely defines the Reeb vector field $\xi \in \mathcal{X}(M)$; this is associated to $\theta$ by the conditions

$$\theta(\xi) = 1 \quad \text{and} \quad \mathcal{L}_\xi \theta = 0,$$

(here $\mathcal{L}_\xi$ is the Lie derivative of $\xi$). Thus

$$TM = H \oplus \mathbb{R} \xi.$$  \hspace{2cm} (5.1)

A contact Riemannian manifold $(\theta, g_H, J)$ is a triple, consisting of a contact form $\theta$, a Riemannian metric $g_H$ on $H$, and a compatible almost complex structure $J$ on $H$, i.e. such that

$$g_H(X, Y) = d\theta(JX, Y) \quad \text{for every} \ x \in M, \ X, Y \in H_x.$$

For any contact form $\theta$, there always exists a compatible pair $(g_H, J)$, see [10]. We can then define the Webster metric $g_\theta$ on $M$, which is Riemannian, by

$$g_\theta = \pi_H^* g_H + \theta^2,$$

where $\pi_H : TM \to H$ is the projection associated with the decomposition (5.1).

By definition, $\xi \perp H$ with respect to $g_\theta$. We also define the Tanaka-Webster scalar curvature $\text{Scal}_{g_H}$ by

$$\text{Scal}_{g_H} = \text{Scal}_{g_\theta} - \text{Ric}_{g_\theta}(\xi, \xi) + 4m.$$
The structure \((\sigma \theta, \sigma g_H, J)\), where \(\sigma \in C^\infty(M), \sigma > 0\), is said to be conformally related to \((\theta, g_H, J)\), and the conformal class \([\theta, g_H, J]\) is the set of all such conformally related structures. Associated to \((\sigma \theta, \sigma g_H, J)\) are its Reeb vector field

\[
\xi_{\sigma \theta} = \frac{1}{\sigma} \left( \xi_\theta + \frac{1}{2\sigma} J \pi_H(\nabla g^\theta \sigma) \right)
\]

and Webster metric

\[
g_{\sigma \theta} = \sigma (\pi_H^* g_H) + \sigma^2 \theta^2.
\]

The contact Riemannian analogue of the conformal Laplacian is the operator

\[
-b_m \Delta_H u + \text{Scal}_{g_H} u, \quad b_m = \frac{4(m + 1)}{m},
\]

where

\[
\Delta_H = \Delta_{g_\theta} - \xi \circ \xi
\]

is the horizontal Laplacian, which is defined as follows. For any function \(v\), consider the restriction of \(dv\) to \(H\). This has squared \(g_H\)-length

\[
|d_H v|^2(x) = \sup_{X \in H_x, g_H(X,X) \leq 1} |dv(X)|^2 = |dv|^2_{g_\theta}(x) - |\xi v|^2(x).
\]

Integrating this against the volume form \(\theta \wedge (d\theta)^m\) gives a quadratic form, and \(\Delta_H\) is then determined by

\[
\int_M |d_H v|^2 \theta \wedge (d\theta)^m = -\int_M v \Delta_H v \theta \wedge (d\theta)^m, \quad v \in C^\infty(M).
\]

The distance function \(\rho\) associated to this quadratic form is the sub-Riemannian distance

\[
\rho(x, y) = \inf \left\{ \int_0^1 |\dot{c}|^2_{g_H} \, dt : c \in C^1([0,1], M), \quad c(0) = x, c(1) = y, \quad \dot{c}(t) \in H_{c(t)} \text{ for all } t \right\},
\]

If \(d\gamma\) is the energy measure associated to the quadratic form, then

\[
\rho(x, y) = \sup \{ u(x) - u(y) : u \in \text{Lip}(M), \quad d\gamma(u, u) \leq \theta \wedge (d\theta)^m \}.
\]

Note that \(\rho\) is compatible with the geodesic distance for \(d_{g_\theta}\) in the sense that

\[
d_{g_\theta} \leq \rho \leq C \sqrt{d_{g_\theta}}
\]
for some $C > 0$.

Just as in Riemannian geometry, there is a simple conformal transformation rule for the Tanaka-Webster scalar curvature. If \( \sigma = u^{\frac{4}{\alpha-2}} \), \( \alpha = 2m + 2 \), then writing \( \hat{\theta} = \sigma \theta \), \( \hat{g}_H = \sigma g_H \), we have

\[-b_m \Delta_H u + \text{Scal}_{g_H} u = \text{Scal}_{\hat{g}_H} u^{\frac{\alpha+2}{\alpha-2}}.\] (5.2)

Noting that

\[
\int_M \text{Scal}_{\hat{g}_H} \hat{\theta} \wedge (d\hat{\theta})^m \quad \text{and} \quad \int_M \hat{\theta} \wedge (d\hat{\theta})^m = \int_M [b_m |d_H u|^2 + \text{Scal}_{g_H} u^2] \theta \wedge (d\theta)^m
\]

we define the contact Yamabe invariant of \((M, [\theta, g_H])\) by

\[
Y(M, [\theta, g_H]) = \inf_{u > 0} \frac{\int_M [b_m |d_H u|^2 + \text{Scal}_{g_H} u^2] \theta \wedge (d\theta)^m}{\left( \int_M |u|^{\frac{2\alpha}{\alpha-2}} \theta \wedge (d\theta)^m \right)^{1-\frac{2}{\alpha}}}.
\]

If this infimum is attained by some \((\hat{\theta}, \hat{g}_H)\), then the Euler-Lagrange equation for this functional shows that \((M, \hat{\theta}, \hat{g}_H)\) has constant Tanaka-Webster scalar curvature.

5.2.2 The Heisenberg group:

The basic model contact Riemannian manifold is the Heisenberg group

\[ h_m = (\mathbb{R}^{2m+1}, \theta_0) := \{(x, y, t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}, \theta_0 = dt - \sum y_j dx_j \} \]

with metric \( g_0 = |dx|^2 + |dy|^2 \) on the horizontal distribution

\[ H = \ker \theta_0 = \text{span}\left\{ \frac{\partial}{\partial x_j} - y_j \frac{\partial}{\partial t}, \frac{\partial}{\partial y_j}, \; j = 1, \ldots, m \right\}. \]

It is simple to check that \( \text{Scal}_{g_0} = 0 \), while

\[ Y(h_m, [\theta_0, g_0]) > 0. \]

(Since \( h_m \) is noncompact, this invariant is the infimum over compactly supported smooth nonnegative functions.)
The Heisenberg group satisfies a uniform Poincaré-Wirtinger inequality: there exists a constant $C > 0$ such that on any $\rho$-ball $B_r$ of radius $r > 0$, 
\[
\int_{B_r} u^2 \theta_0 \wedge (d\theta_0)^m \leq C r^2 \int_{B_r} |d_H u|^2 \theta_0 \wedge (d\theta_0)^m
\]
for all $u \in L^1(B_r)$ with 
\[
|d_H u|^2 \in L^1(B_r) \quad \text{and} \quad \int_{B_r} u \theta_0 \wedge (d\theta_0)^m = 0.
\]

5.2.3 The local Yamabe invariant:

Jerison and Lee \cite{21} posed the problem of showing that on a given CR manifold, the local Yamabe invariant equals the contact Yamabe invariant of the Heisenberg group in normal CR-coordinates. It turns out that if one uses Darboux coordinates instead, then this is not difficult. Indeed, let $(M^{2m+1}, \theta, g_H)$ be a compact contact Riemannian manifold. For each $p \in M$, there exists a diffeomorphism 
\[
\varphi: U \to \mathbb{B}(1) = \{(x, y, t) \in \mathbb{R}^{2m+1} : \rho((x, y, t), 0) < 1\},
\]
where $U$ is a neighbourhood of $p$, such that 
\[
\varphi^* \theta_0 = \theta, \quad (\varphi^* g_0)_p = (g_H)_p \quad \text{and} \quad \varphi^* g_0 - g_H = O(\sqrt{\varepsilon}) \text{ on } U_\varepsilon := \varphi^{-1}(\mathbb{B}(\varepsilon)).
\]
Note that the Tanaka-Webster scalar curvature is assumed to be bounded, hence the local Yamabe invariant is equal to the local Sobolev invariant. The Heisenberg group has vanishing Tanaka-Webster scalar curvature and the Sobolev constant varies continuously when the metric varies in the space of continuous metrics, hence, we have 
\[
Y(U_\varepsilon, [\theta, g_H]) = Y(\mathbb{B}(\varepsilon), [\theta_0, g_0]) \left(1 + O(\sqrt{\varepsilon})\right) = Y(h_m, [\theta_0, g_0]) \left(1 + O(\sqrt{\varepsilon})\right).
\]

Since the Heisenberg group satisfies the Poincaré-Wirtinger inequality, it is easy to show that any compact contact Riemannian manifold $(M^{2n+1}, \theta, g_H)$ satisfies a local Poincaré-Wirtinger inequality. There exist positive constants $C, R > 0$ such that on any ball $B_r \subset M$ of radius $r \in (0, R)$, 
\[
\int_{B_r} u^2 \theta \wedge (d\theta)^m \leq C r^2 \int_{B_{2r}} |d_H u|^2 \theta \wedge (d\theta)^m
\]
for any $u \in L^1(M)$ with 
\[
|d_H u|^2 \in L^1(B_{2r}) \quad \text{and} \quad \int_{B_r} u \theta \wedge (d\theta)^m = 0.
\]
5.2.4 The contact Riemannian Yamabe problem:

The second application of our general result is to the contact Riemannian Yamabe problem. This is a generalization of the first main result of Jerison and Lee.

**Theorem 5.1.** Let $(M^{2m+1}, \theta, g_H)$ be a compact contact Riemannian manifold. Assume that

$$Y(M, [\theta, g_H]) < Y(h_m, [\theta_0, g_0]).$$

Then there exists a positive function $u \in C^\infty(M)$ such that the Tanaka-Webster scalar curvature of $(u^{\frac{m}{m}}, \theta, u^{\frac{m}{m}}g_H)$ is constant.

In this setting,

- $d\mu = \theta \wedge (d\theta)^m$,
- $E(v) = \frac{4(m+1)}{m} \int_M |d_Hv|^2 \theta \wedge (d\theta)^m$,
- $v \in D(E) = \{v \in L^2(M) : |d_Hv|^2 \in L^1(M)\}$,
- $L + V = -\frac{4(m+1)}{m}\Delta_H + \text{Scal}_{g_H}$,
- $\nu = 2m + 2$,
- $Y_\ell = Y(h_m, [\theta_0, g_0])$.

Although our result gives only a positive bounded solution $u \in D(E)$, the hypoelliptic properties of $\Delta_H$ directly show that $u \in C^\infty$.

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