Brane russian doll

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Abstract

It is shown that an \((n - 1)\)-dimensional inflating brane world instantonically created from nothing can exist in the region beyond the Rindler horizon of the Lorentzian spacetime associated with another inflating brane world in \(n\) dimensions which is also instantonically created from nothing. Generalizing this construction we obtain an unbounded from above tower of successive brane worlds inside brane worlds, each having one more dimension than the one which it nests and one less dimension than the one which nests it.
The large recent influx of papers on brane-world cosmology (for a review see [1]) clearly shows the stir produced in the physical community by the two seminal papers by Randall and Sundrum [2,3] in 1999 on the physics of extra dimensions. These two papers were preceded by related ideas put forwards by Rubakov and Shaposhnikov [4], Visser [5] and Arkani-Hamed, Dimopoulos and Dvali [6]. In the spirit of the cosmological Garriga-Sasaki model [7], the present paper deals with the proposal of the instantonic creation of an inflating brane world inside the unobservable Lorentzian region of another larger brane. This new proposal would allow us to consider a novel mechanism for dimensional reduction and, when trivially extended to larger (even infinite) towers of branes inside branes, our model may accommodate a gradually induced resolution of the hierarchy problem [2,3].

Let us start by considering a four-dimensional brane-world embedded in an anti-de Sitter (AdS) bulk whose six-dimensional Euclidean metric can be written in the form

$$ds^2 = dr^2 + \rho_0^2 \sinh^2(r/\rho_0) \left( d\chi^2 + \sin^2\chi d\Omega_4^2 \right),$$

where $d\Omega_4^2$ is the metric on the unit four-sphere, and $\rho_0 = (-10/\Lambda_6)^{1/2}$, with $\Lambda_6$ the six-dimensional cosmological constant. If we were going to consider a compact brane-world in five-dimensional spacetime, this could then be described by means of an instanton constructed following e.g. the cutting-pasting procedure introduced by Garriga and Sasaki [7], that is by excising the spacetime region defined by the radial extra coordinate $r$ at values larger than a given arbitrary value $r_0$, and gluing two copies of the remaining spacetime along the five-sphere at the given $r = r_0$. A four-brane can then be introduced on the hypersurface at $r_0$ if we provide that brane with a tension

$$\sigma = \frac{3}{4\pi G_6 \rho_0} \coth \left( \frac{r_0}{\rho_0} \right),$$

so that the Israel’s matching conditions [8] are satisfied. The resulting instantonic
configuration can be interpreted as a semiclassical path for the creation of a five-dimensional universe from nothing. Clearly this is a mathematical nice model without direct application to the observable universe which we live in, and exactly corresponds merely to the five-dimensional extension of the Garriga-Sasaki instanton. Similarly, one can cut our five-dimensional instanton in half to get a solution that interpolates between nothing, at the south pole, and a four-dimensional spherical brane of radius $H^{-1} = \rho_0 \sinh (r_0/\rho_0)$, at the equator. This is the five-dimensional de Sitter instanton with the inside of the brane filled with an AdS bulk.

On the other hand, one can also consider the evolution of the resulting brane after its creation as being described by the analytical continuation of metric (1) such that

\[ 7 \]
\[ \chi \rightarrow iHt + \pi/2. \]

This gives

\[ ds^2 = dr^2 + \rho_0^2 \sinh^2 (r/\rho_0) \left[ -H^2 dt^2 + \cosh^2 (Ht) d\Omega_4^2 \right] . \]  

(3)

On the brane at $r = r_0$, metric (3) reduces to $ds^2 = -dt^2 + H^{-2} \cosh^2 (Ht) d\Omega_4^2$, which represents an inflating five-dimensional de Sitter space. It does not cover the whole of the Lorentzian spacetime, but only the exterior region of the Rindler horizon at $r = 0$. Covering also the interior region to complete description of the whole spacetime requires the complexification of the coordinates $r$ and $t$: $r \rightarrow it_c$, $Ht \rightarrow r_c - i\pi/2$, with which the metric becomes

\[ ds^2 = -dt_c^2 + \rho^2 \sin^2 (t_c/\rho_0) \left[ dr_c^2 + \sinh^2 r_c d\Omega_4^2 \right] . \]  

(4)

This is the line element that describes a six-dimensional open FRW universe with the coordinate $r_c$ running from 0 to $\pi\rho_0$, at which surface there is a Cauchy horizon connecting to a new spacetime. However, as it was also noted by Garriga and Sasaki in the five-dimensional case [7], in the present six-dimensional framework, even though we were living in a five-dimensional universe, the extended solution (4) would
be nothing but a mathematical idealization, describing a spacetime which is physically unreachable from the assumed physical region external to the Rindler horizon that contains the brane. Nevertheless, the usual four-dimensional instantonic picture of the birth from nothing of an inflating de Sitter universe, according to the no boundary condition, can also be recovered from metric (4). In fact, starting with our six-dimensional spacetime (1), it is always possible to construct a compact 3-brane-world instanton inside the continuation \((r \leq r_0 \rightarrow it_c, \text{with } t_c \leq t_{c0})\) of the above higher-dimensional brane instanton by excising the spacetime regions at \(t_c \geq t_{c0}\) and at \(r_c \geq r_{c0}\) (with \(t_{c0}\) and \(r_{c0}\) being given arbitrary values of coordinates \(t_c\) and \(r_c\)), and gluing two copies of the remaining spacetime along the four-sphere at \(t_c = t_{c0}, r_c = r_{c0}\). This can be readily seen by noting that, on the hypersurface at \(t_c = t_{c0}\), metric (4) reduces to

\[
    ds^2 = dz^2 + \tilde{\rho}_0^2 \sinh^2 (z/\tilde{\rho}_0) d\Omega_4^2,
\]

(5)

where \(z = \tilde{\rho}_0 r_c\), with

\[
    0 \leq \tilde{\rho}_0 = \rho_0 \sin (t_{c0}/\rho_0) \leq \rho_0.
\]

(6)

This is the line element of a five-dimensional Euclidean AdS space. On this hypersurface, one can repeat the construction of a brane-world, this time a four-dimensional world with a three-brane, by simply exising the spacetime region \(z > z_0\), and gluing two copies of the remaining spacetime. On \(z = z_0\) a brane with tension \(\sigma_5 = \frac{3}{4\pi G_6 \rho_0} \coth (z_0/\tilde{\rho}_0)\) can finally be introduced. The associated instanton would represent the path for the creation of a four-universe from nothing in the region beyond the Rindler horizon corresponding to a brane-world with one more dimension. Together with the fact that the final brane-world is filled with an AdS bulk, this distinguishes the present model from the original Garriga-Sasaki scenario. Again the Lorentzian evolution of the observable brane after creation should be given by the analytical con-
tinuation of one of the spherical coordinates of the unit metric $d\Omega_4^2$. Taking this to be expressed as $d\psi^2 + \sin^2 \psi d\Omega_3^2$, one can continue coordinate $\psi$ such that $\psi \to i\tilde{H}\tilde{t} + \pi/2$, in which $\tilde{H}^{-1} = \tilde{\rho}_0 \sinh (z_0/\tilde{\rho}_0)$, to induce the following metric

$$
\begin{align*}
    ds^2 &= dz^2 + \tilde{\rho}_0^2 \sinh^2 (z/\tilde{\rho}_0) \left[ -\tilde{H}^2 d\tilde{t}^2 + \cosh^2 (\tilde{H}\tilde{t}) d\Omega_3^2 \right].
\end{align*}
$$

(7)

The conformal diagram of the Lorentzian spacetime corresponding to the observable brane-world spacetime, a finite region of which (but not all) is described by this metric, is given in Fig.1. Such a brane world at $z = z_0$ is an inflating de Sitter space with Hubble rate $\tilde{H}^{-1}$:

$$
\begin{align*}
    ds^2 &= -d\tilde{t}^2 + \tilde{H}^{-2} \cosh^2 (\tilde{H}\tilde{t}) d\Omega_3^2.
\end{align*}
$$

(8)

This resulting observable brane has the same bound state for gravity and the same spectrum of Kaluza-Klein excitations as those obtained starting with a five-dimensional
Euclidean AdS spacetime [7], with an energy gap given now by \((3/2)\tilde{H}\). We have thus been able to construct a four-dimensional de Sitter brane-world instanton starting with a six-dimensional AdS instanton, which is localized in the region beyond the event horizon of the five-dimensional Lorentzian brane. What matters now is to compute the action of the four-dimensional brane-world instanton within the realm of the five-brane instanton. This can be accomplished by summing the Euclidean actions corresponding to the brane-worlds in the six- and five-dimensional AdS spaces, starting with the nonlinear generalization of the zero mode for the respective brane [2,3]. Thus, for the Euclidean six-dimensional AdS space (with topology \(R^1 \times S^5\)) we shall use the metrical ansatz

\[
ds^2 = g_{ab}dx^a dx^b = dr^2 + a(r)^2 \gamma_{\mu\nu}dx^\mu dx^\nu,\tag{9}\]

with \(\gamma_{\mu\nu}\) the five-dimensional metric and \(a(r) = \rho_0 \sinh(r/\rho_0)\). In order to avoid the singular character of the one-brane solution and prepare the system to accommodate a cosmological scenario, we shall use a configuration made of by two concentric branes which would cosmologically evolve independently of each other. To the brane at \(r = r_0\), we thus add a new brane with a negative tension given by reversing sign of Eq. (2) at some radius \(r_1 < r_0\), excising the spacetime region \(r > r_1\) and identifying the edges of the two copies of AdS space bulk along the five-sphere at \(r = r_1\) [7]. Then, expressing the surface term \(\int d^5x \sqrt{\mathcal{H}} \text{Tr} K\) in terms of the brane’s tension, \(\sigma\), i.e. \(\text{Tr} K = 32\pi G_6 \rho_0 \sigma/3\), we have for the Euclidean action of the system formed by two concentric branes with tensions \(\sigma_i (i = 0, 1)\), respectively located at \(r_i\) and separated by a region of AdS bulk described by the generalized metric (9)

\[
S_E^{(6)} = \frac{1}{16\pi G_6} \int d^6x a^5 (2\Lambda_6 - R) \sqrt{\gamma^{(5)}} + \frac{4}{3} \sum_i \sigma_i a^5 V_{5}^{(\gamma)},\tag{10}\]
where $V^{(5)}_5 = \int d^5x \sqrt{\gamma^{(5)}}$ is the five-volume. Introducing an energy-momentum tensor

$$T_{ab} = -\frac{g_{ab}\Lambda_6}{8\pi G_6} - \sum_i a^2 \sigma_i \gamma^{(5)}_{ab} \delta(r - r_i),$$

(11)
in which $\gamma^{(5)}_{ab} = 0$ provided that $a$ or $b$ equals $r$, eliminating the scalar curvature $R$ from the trace of the Einstein equation, $R = 3\Lambda_6 + 20\pi G_6 \sum_i \sigma_i$, and expliciting the six-dimensional cosmological constant as $\Lambda_6 = -10/\rho_0^2$, we finally obtain, after performing the integral,

$$S^{(6)}_E = -\frac{1}{3\pi G_N H^2_i} \sum_{i=0}^{1} (-1)^i \cosh \left(\frac{r_i}{\rho_0}\right) \left[\frac{1}{\sinh^2 \left(\frac{r_i}{\rho_0}\right)} + \frac{15}{32} \sinh^2 \left(\frac{r_i}{\rho_0}\right) - \frac{1}{2 \coth^2 \left(\frac{r_i}{\rho_0}\right)}\right] S^{(6)i}_E,$$

(12)
where

$$S^{(6)i}_E = -\frac{2V^{(5)}_5}{3\pi G_N H^2_i},$$

(13)
with $G_N = G_6/\rho_0^2$ the Newtonian gravitational constant, and $H_i = a_i^{-1} = 1/ (\rho_0 \sinh (r_i/\rho_0))$.

The calculation of the Euclidean action for the generalized five-dimensional AdS instanton follows a similar pattern, to finally produce:

$$S^{(5)}_E = -\frac{1}{15} \sum_{j=0}^{1} (-1)^j \left[\frac{z}{\rho_0 \sinh^2 \left(\frac{z}{\rho_0}\right)} - \coth \left(\frac{z}{\rho_0}\right) + \frac{4}{15} \cosh \left(\frac{z}{\rho_0}\right) \sinh \left(\frac{z}{\rho_0}\right)\right] S^{(5)j}_E,$$

(14)
where

$$S^{(5)j}_E = -\frac{5V^{(4)}_4}{8\pi G_N \rho_0 H^2_j},$$

(15)
with $V^{(4)}_4 = \int d^4x \sqrt{\gamma^{(4)}}$ the four-volume, $\gamma_{\mu\nu}$ the four-dimensional metric entering the ansatz for the generalized line element $ds^2 = dz^2 + a^2(z)\gamma_{\mu\nu} dx^\mu dx^\nu$, $G_N = G_6/\rho_0^2$, and $H_j = 1/ (\rho_0 \sinh (z/\rho_0))$. The probability for the creation of a five-brane instanton containing a four-brane instanton itself created in the interior region of the Lorentzian continuation of the first instanton will be then proportional to $\exp(S_E)$, with $S_E =$
\( S_E^{(6)} + S_E^{(5)} \). It can be checked that such a probability will closely depend on the combined valued for the fixed coordinates \( r \) and \( z \) at the brane locations.

The above instantonic construction can be readily extended to encompass any \( n \)-dimensional starting Euclidean metric, \( ds^2 = dr^2 + \rho_0^2 \sinh^2(r/\rho_0) \left( d\chi^2 + \sin^2 \chi d\Omega_{n-2}^2 \right) \), which, if we follow such a construction for every \( p \)-dimensional Lorentzian interior region (with \( n \leq p \leq 5 \)) beyond the event horizon of the successive branes, will lead to a russiandoll-like tower of inflating brane worlds. In such a construct, the innest brane should be chosen to be the lowest dimensional and the outest brane would be \((n-1)\)-dimensional, and each brane would be fully unobservable for the rest of branes in the whole tower. Inflating brane-worlds could thus exist inside the unobservable region of higher dimensional brane-worlds which would also be inflating. Our own universe could in this way be just one (the lowest dimensional or not) among a large (perhaps infinite) tower of universes with different number of physical dimensions, which would be geometrically unobservable for us. The creation of each of these universes should be from nothing only in the sense of absence of the spacetime of its own observable region, but other even large regions outside the given universe’s horizon would be allowed to exist at its origin. In any event, while the brane scenario presented in this paper offers the opportunity: of establishing a new and perhaps advantageous mechanism for dimensional reduction or increase, to allow a gradual solution of the hierarchy problem, and shed new light at the concept of creation from nothing in cosmology, the application of the holographic idea [9] to the creation of de Sitter universes by the procedure described in this work leads to the encoding on the cosmological horizon of only a finite number of degrees of freedom [10], in contradiction [11] with string and M theories. One would then adhere to a pure higher dimensional general-relativity.
interpretation of the brane worlds rather than adcribing a string- or M-theory origin for such worlds.

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References

1 C. Barceló and M. Visser, JHEP 0010 (2000) 019.

2 L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370.

3 L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690.

4 V. Rubakov and M. Shaposhnikov, Phys. Lett. B125 (1983) 136.

5 M. Visser, Phys. Lett. B159 (1985) 22.

6 N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429 (1998) 263.

7 J. Garriga and M. Sasaki, Phys. rev. D62 (2000) 043523.

8 W. Israel, N. Cimento B44 (1966) 1; B48 (1966) 463.

9 R. Bousso, JHEP 0011 (2000) 038.

10 R. Bousso, Bekenstein bounds in de Sitter and flat space, hep-th/0012052.

11 W. Fischler, A. Kashani-Poor, R. McNees and S. Paban, The Acceleration of the Universe, a Challenge for String Theory, hep-th/0104181.