Linear Matrix Inequality Design of Exponentially Stabilizing Observer-Based State Feedback Port-Hamiltonian Controllers

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Abstract—The design of an observer-based state feedback controller with guaranteed passivity properties for port-Hamiltonian systems (PHS) is addressed using linear matrix inequalities (LMIs). The observer gain is freely chosen and the LMIs conditions such that the state feedback is equivalent to control by interconnection with an input strictly passive and/or an output strictly passive and zero-state detectable port-Hamiltonian controller are established. It is shown that the proposed controller exponentially stabilizes a class of infinite-dimensional PHS and asymptotically stabilizes a class of finite-dimensional nonlinear PHS. A Timoshenko beam model and a microelectromechanical system are used to illustrate the proposed approach.

Index Terms—Boundary control systems, linear matrix inequality, Luenberger observer, nonlinear systems, port-Hamiltonian systems, PHS, state feedback.

I. INTRODUCTION

PORT-HAMILTONIAN systems (PHS) have been introduced in [1] and have shown to be well suited for the modeling and control of multiphysical systems [2], [3]. They have been widely studied for finite-dimensional systems in [2], [4], [5], and [6] and generalized to infinite-dimensional systems in [7], [8], and [9]. The main feature of PHS is to describe physical systems in terms of energy and to express the energy exchanges between the different internal components of these systems and their environment through an appropriate geometric structure.

Control of PHS using interconnection and damping assignment (IDA) has been proposed in [4] and [5] and extensively developed for linear systems in [6], in which linear matrix inequalities (LMIs) have been employed to solve the IDA control problem. The LMIs proposed in [6] allow to design a static-state feedback that guarantees a desired closed-loop behavior. It can be seen as an alternative to traditional approaches, such as pole-placement, $L\infty$-control, or $H\infty$-control. Similar LMIs conditions have been used for the dual problem of observer design in [10]. Further works on observer design for linear and nonlinear PHS have been reported in [11], [12], [13], and [14], in which the passive properties of PHS are used to ensure the convergence of the observer. Nevertheless, few results are reported regarding observer-based state feedback (OBSF) control design for PHS.

In [10], the design of an OBSF controller that allows to achieve desired closed-loop performances and guarantees closed-loop stability when it is applied to a finite-dimensional linear system is proposed using a dual observer-based compensator. In [15], an OBSF design is proposed such that the equivalent controller is a PHS. Similarly, in [16], a similar controller for infinite-dimensional PHS with distributed actuation is proposed. However, in the last two references, stability only is considered and the closed-loop performances can only be modified through damping injection. Recently in [17], an OBSF controller has been proposed based on the resolution of a modified algebraic Riccati equation (ARE) such that: (1) the design is based on a discretized model of an infinite-dimensional PHS (as the ones presented in [7] and [8]), and (2) the asymptotic closed-loop stability is guaranteed when applying the finite-dimensional OBSF controller to a class of boundary controlled PHS (BC-PHS) [8], [9]. The design procedure consists in assuming a given state feedback gain and obtain the observer gain by solving an ARE to guarantee that the dynamic controller is a PHS [17].

In this work, we propose an extension of the preliminary results proposed in [18]. Using an early lumping approach as in [17], an OBSF design method based on LMIs, which guarantees that the dynamic controller is input strictly passive (ISP) and/or output strictly passive (OSP) and zero-state detectable (ZSD), is proposed. The ISP, OSP, and ZSD properties of the controller allow to guarantee exponential stability of the closed-loop dynamics when applied to a class of BC-PHS [8], [9] and the asymptotic stability when applied to a class of nonlinear PHS [2]. In both cases, the design of the controller is performed on a finite-dimensional linear approximation of the open-loop systems, and it is also possible to specify (up to a certain degree) the performance of the closed-loop finite-dimensional linear system, whereas the closed-loop stability of the infinite-dimensional/nonlinear system is guaranteed. An important improvement with respect to [17] and [18] is that the proposed controller ensures exponential stability when applied to BC-PHS, overcoming the limitation that only asymptotic stability could be achieved. The exponential stability is possible thanks to a direct feedback term, whereas the observer-based feedback allows to modify and improve the closed-loop response. Moreover, the LMI-based design method is based on a set of explicit design parameters. An important extension of the proposed control method is that it can be equally applied to finite-dimensional nonlinear PHS. This would also be the case for the method proposed in [17] provided that the

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designed controller is OSP and ZSD. In this article, these conditions are formalized.

The rest of this article is organized as follows. Section II presents the design procedure in terms of LMI s of the proposed OBSF controller and the stability results. Section III presents two examples, an infinite-dimensional Timoshenko beam model on a 1-D spatial domain and a finite-dimensional nonlinear microelectromechanical system (MEMS). The design procedure and the closed-loop performances by means of numerical simulations are illustrated. Finally, Section IV concludes this article.

II. OBSERVER-BASED STATE FEEDBACK DESIGN

Consider the following linear PHS

\[
\begin{align*}
P : \quad \dot{x}(t) &= (J - R)Qx(t) + Bu(t), \quad x(0) = x_0 \\
y(t) &= B^TQx(t)
\end{align*}
\]  

(1)

where \(x(t) \in \mathbb{R}^n\) is defined for all \(t \geq 0\), \(x_0 \in \mathbb{R}^n\) is the unknown initial condition, \(u(t) \in \mathbb{R}^m\) is the input, and \(y(t) \in \mathbb{R}^m\) is the power conjugated output of \(u(t)\), which in this work is considered to be measurable. \(J = -J^T\), \(R = R^T \geq 0\), and \(Q = Q^T > 0\) are all known real matrices of size \(n \times n\) and \(B \in \mathbb{R}^{n \times m}\). We assume that \(J - R\) and \(R\) are full rank, i.e., rank\((J - R)\) = rank\((R)\) = \(m\). The Hamiltonian function of (1) is \(H(t) = \frac{1}{2}x^T(t)Qx(t)\), and for models of physical systems corresponds to the stored energy. For simplicity, we refer to the system (1) as the system \((A, B, C)\), with \(A = (J - R)\) and \(C = B^TQ\), and we assume that it is controllable and observable.

Define the following Lunenburg observer:

\[
\begin{align*}
\dot{x}(t) &= A_D\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)) + BD_Dy(t) \\
\dot{\hat{y}}(t) &= B^TQ\hat{x}(t), \quad \hat{x}(0) = \hat{x}_0
\end{align*}
\]  

(2)

for the PHS (1), where \(\hat{x} \in \mathbb{R}^n\) is the estimation of \(x\), \(\hat{x}_0\) is a known initial condition, \(A_D = A - BD_DC\) with \(0 < D_C \in \mathbb{R}^{n \times n}\) is a damping matrix, and \(L \in \mathbb{R}^{n \times m}\) is the observer gain to be designed. Note that (2) can be interpreted as a damped version of a Lunenburg observer since the design of \(L\) is not based on the open-loop matrix \(A\) but on the closed-loop matrix \(A_D\). Then, we design the feedback gain \(K\) such that the following OBSF control law

\[
\begin{align*}
u(t) &= r(t) - K\hat{x}(t) - D_Dy(t) \\
r(t) &\in \mathbb{R}^{n \times m}, \quad K \in \mathbb{R}^{n \times n}
\end{align*}
\]  

(3)

is equivalent to the control by interconnection with an ISP or/and OSP and ZSD PHS controller. Hence, if the considered linear system is the discretization of a linear BC-PHS [8], [9] or the approximation of a nonlinear PHS [2], the closed-loop stability is also guaranteed if (2) and (3) are applied to:

1) the original BC-PHS;
2) the original nonlinear PHS.

In the following section, we recall the LMI design approach proposed in [6], which is instrumental in our developments.

A. Observer Design by LMIs

Define the error of the state as \(\epsilon(t) := x(t) - \hat{x}(t)\). The error dynamics is obtained from (1) and (2)

\[
\begin{align*}
\dot{\epsilon}(t) &= (A_D - LC)\epsilon(t), \quad \epsilon(0) := \epsilon_0 := x_0 - \hat{x}_0
\end{align*}
\]  

(4)

where \(\epsilon_0\) is an unknown initial condition. We recall the following proposition from [6], which is used for the design of the matrix \(L\) such that \(A_D - LC\) is Hurwitz.

**Proposition 1** ([6]): Denote by \(B_1\) a full rank \((n - m) \times n\) matrix that annihilates \(B\), i.e., \(B_1B = 0\). Let us also denote \(E_1 = B_1A_D\). There exist matrices \(J_d = -J_d^T\), \(R_d = R_d^T \geq 0\), \(Q_d = Q_d^T > 0\), and \(F\) such that \((J_d - R_d)Q_d = A_D + BF\) if and only if there exists a solution \(X = X^T \in \mathbb{R}^{n \times n}\) to the LMIs

\[
X > 0
\]

- \([E_1XB^T_1 + B_1XE^T_1] \geq 0\).  

(5)

Given such an \(X\), compute \(S_d\) as follows:

\[
S_d = \begin{pmatrix}
B_1 \\
B^T
\end{pmatrix}^{-1} \begin{pmatrix}
E_1X \\
-B^TXE^T_1(B_1B^T_1)^{-1}B_1
\end{pmatrix}
\]  

(6)

then the following matrices:

\[
J_d = \frac{1}{2}(S_d - S^T_d), \quad R_d = -\frac{1}{2}(S_d + S^T_d)
\]

\[
Q_d = X^{-1}, F = (B^T B)^{-1}B^T(S_dX^{-1} - A_D)
\]

(7)

satisfy \(J_d = -J_d^T\), \(R_d = R_d^T \geq 0\), \(Q_d = Q_d^T > 0\), and \((J_d - R_d)Q_d = A_D + BF\).

**Remark 1:** The dual problem, for the design of the matrix \(L\) for the error system (4), consists in following Proposition 1, but replacing \(A_1\) by \(A_D\), \(B\) by \(B^T\), and \(F\) by \(-F^T\). The reader can also refer to [10, Prop. 1].

The performances obtained using Proposition 1 are in terms of \(Q_d\) (energy matrix) and \(R_d\) (dissipation matrix). As it is mentioned in [6], the LMI (5) can be slightly modified in order to keep the energy matrix in a desired interval and to have sufficient but not excessive damping. This is formalized in the following corollary.

**Corollary 1:** Under the same statements of Proposition 1, if the following LMIs:

\[
\begin{align*}
\Lambda_1^{-1} - X &< 0 \\
\Lambda_1^{-1} + X &< 0 \\
-\Xi_1 - E_1XB^T_1 + B_1XE^T_1 &\leq 0 \\
-\Xi_2 - E_1XB^T_1 - B_1XE^T_1 &\leq 0
\end{align*}
\]  

(8)

have a solution \(X = X^T\) for some symmetric matrices \(\Lambda_1, \Lambda_2 \in \mathbb{R}^{n \times n}\) and \(\Xi_1, \Xi_2 \in \mathbb{R}^{(n - m) \times (n - m)}\), such that \(0 < \Lambda_1 < \Lambda_2\) and \(0 \leq \Xi_1 < \Xi_2\), then \(A_1 < Q_d < A_2\). Moreover, choosing

\[
S_d = \begin{pmatrix}
B_1 \\
B^T
\end{pmatrix}^{-1} \begin{pmatrix}
E_1X \\
-B^TXE^T_1(B_1B^T_1)^{-1}B_1 - \gamma G^T
\end{pmatrix}
\]  

(9)

for some scalar \(\gamma > 0\), and the matrices \(J_d, R_d, F\) as in (7), then \(A_D + BF = (J_d - R_d)Q_d\) with \(R_d > 0\), which ensures exponential stability of (4).

**Proof:** The proof of Corollary 1 is a direct application of [6, Prop. 7 and Remark 8]. See also [10, Prop. 1].

**Remark 2:** Matrices \(\Lambda_1\) and \(\Lambda_2\) allow to fix the lowest and highest eigenvalues of \(Q_d\), respectively. Matrices \(\Xi_1\) and \(\Xi_2\) bound the damping term, whereas the scalar \(\gamma > 0\) implies \(R_d > 0\) [6], and then, since \(Q_d > 0\), exponential behavior is ensured.

In the following section, we consider the Lunenburg observer (2) already designed by Corollary 1 using the dual problem, i.e., replacing \(A_1\) by \(A_D\), \(B\) by \(B^T\), and \(L = -F^T\). Then, we design the matrix \(K\) of the control law (3) such that the closed-loop system is equivalent.
Δ = αI satisfy the following:

is verified by replacing (13) by their expression with respect to J and \( R \) such that the LMIs in (13) are satisfied. Indeed, the OBSF (2) and (3) is equivalent to the interconnection defined by

\[
\begin{align*}
\dot{y}(t) &= (J_x - R_c)Q\hat{x}(t) + B_cu_c(t) + Br(t) \\
\hat{P}(t) &= \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} = B_c^TQ\hat{x}(t) + D_cu_c(t) \\
\end{align*}
\]

with \( H_c = \frac{1}{2}x^TQ\hat{x} \), if \( K = B_c^TQ_c, \ B_c = L \), and the following matching equation:

\[
A_D - LC - BK = (J_x - R_c)Q_c, \quad (12)
\]

is satisfied for some matrices \( J_x = J_x^T, R_c, R_c^T \geq 0, Q_c = Q_c^T > 0 \) of appropriate dimensions.

**Proof:** Consider in a first instance only the feedthrough term \( D_cu_c(t) \) in (11). The interconnection (10), with \( r(t) = 0 \), produces the following closed-loop PHS:

\[
\hat{x} = (A - BD_cB_c^T)Qx = (J - (R + BD_cB_c^T))Qx
\]

where \((R + BD_cB_c^T)^\top = (J - (R + BD_cB_c^T))^\top \geq 0\), since \( D_c = D_c^T > 0 \). The closed-loop system is then \( \hat{x} = (J - (R + BD_cB_c^T))Qx \) where \( (J - (R + BD_cB_c^T))^\top \) is Hurwitz [2]. Hence, the direct feedthrough term in (11) guarantees the exponential decay of the solutions of (1). This result also holds in the case of BC-PHs defined on 1-D spatial domains [8], [9] provided that the boundary controller is ISP [19]. In the case of lumped nonlinear system, the use of a negative output feedback renders the system asymptotically stable provided that La Salle’s invariance principle is satisfied at the equilibrium [2].

**Corollary 2:** Consider \((A, B, C), 0 < D_c \in \mathbb{R}^{m \times m}\) and a matrix \( L \in \mathbb{R}^{n \times n}\) such that \( A_L := A_D - LC \) is Hurwitz with \( A_D = A - BD_cC \). The OBSF controller (2) and (3) is equivalent to the PHS (11) by computing a solution \( X = X^\top \) to the LMIs

\[
\begin{align*}
2\Gamma_1 - BL^T - LB^T - A_LX - XA_L^T \leq 0 \\
-2\Gamma_2 + BL^T + LB^T - A_LX - XA_L^T \leq 0
\end{align*}
\]

for some \( n \times n \) symmetric matrices \( \Gamma_1, \Gamma_2, \Delta_1, \text{ and } \Delta_2 \) such that \( 0 \leq \Gamma_1 < \Gamma_2 \) and \( 0 < \Delta_1 < \Delta_2 \). Then, with \( S_c = X - BL^T, \) one can obtain \( J_x = \frac{1}{2}(S_c - S_c^T), R_c = -\frac{1}{2}(S_c + S_c^T), Q_c = X^\top, A_c = L, \) and \( K = B_c^TQ_c \). Furthermore, the following results hold.

i) Matrices \( R_c \) and \( Q_c \) satisfy the following:

1. \( \Gamma_1 \leq R_c \leq \Gamma_2 \);
2. \( \Delta_1 \leq Q_c \leq \Delta_2 \).

ii) If \( \Delta_1 > 0 \), then the observer estimation error converges exponentially to zero and (11) is ISP, OSP, and ZSD with respect to the input/output pair \( u_c, y_c \).

**Proof:** The matching equation (12) is satisfied by the solution of (13) and \( J_x = -J_x^T, R_c = R_c^T, Q_c = Q_c^T \). To conclude that the proposed solution leads to a PHS, it must be verified that \( R_c \geq 0 \) and \( Q_c > 0 \). From (13)

\[
\begin{align*}
2\Gamma_1 & \leq BL^T - BL^T - A_LX - XA_L^T \leq 2\Gamma_2 \\
\Delta_2^{-1} & \leq X \leq \Delta_1^{-1}
\end{align*}
\]

Replacing \( X, A_LX - BL^T \) by their expression with respect to \( S_c \) and \( Q_c \), and inverting the second inequality, we obtain

\[
\begin{align*}
2\Gamma_1 & \leq -(S_c + S_c^T) \leq 2\Gamma_2 \\
\Delta_1 & \leq Q_c \leq \Delta_2.
\end{align*}
\]

Since \( R_c = -\frac{1}{2}(S_c + S_c^T) \), we conclude that \( Q_c > 0 \) and \( R_c \geq 0 \) since \( \Delta_1 \geq 0 \) and \( \Gamma_1 \geq 0 \). Implication (i) is verified by replacing \( R_c = -\frac{1}{2}(S_c + S_c^T) \) in (14). Implication (ii) is the exponential convergence to the observer follows since \( R_c \geq 0 \) and \( Q_c > 0 \). The ISP, OSP, and ZSD properties are directly verified since \( R_c > 0 \) and \( D_c > 0 \) [20, Lemma 2]. It is always possible to find some positive definite matrices \( \Gamma_1, \Gamma_2, \Delta_1, \text{ and } \Delta_2 \) such that the LMIs in (13) are satisfied. Indeed, since \( A_L \) is assumed to be Hurwitz, for any matrix \( 0 < Q_c \in \mathbb{R}^{n \times n} \), there exists a unique \( X > 0 \) such that the following Lyapunov equation \( A_LX + XA_L^T + Q_c = 0 \) is satisfied. Then, given some \( Q_c > 0 \), the LMIs in (13) become \( 2\Gamma_1 \leq BL^T - LB^T \leq 2\Gamma_2 \) and \( \Delta_2^{-1} \leq X \leq \Delta_1^{-1} \). By choosing \( Q_c = -\alphaI \leq BL^T - LB^T + \alpha\Gamma_1 \), with \( \alpha > 0 \) a scalar that guarantees \( Q_c > 0 \), the LMIs in (13) become \( 2\Gamma_1 \leq \alpha\Gamma_1 \leq 2\Gamma_2 \) and \( \Delta_2^{-1} \leq X \leq \Delta_1^{-1} \) for some \( \alpha > 0 \) and with \( X > 0 \). For these two last inequalities, we can always find some positive definite upper and lower bounds. A simple choice for the matrices \( \Gamma_1, \Gamma_2, \Delta_1, \text{ and } \Delta_2 \) is the identity modulated by a constant.

The main result of this article is given in the following theorem.

**Theorem 3:** Let (1) be the finite-dimensional linear approximation of the following:

i) a linear 1-D BC-PHS, as defined by (17)–(20), or

ii) an OSP and ZSD finite-dimensional nonlinear PHS, as defined by (21).

The OBSF (2) and (3) with gains \( K \) and \( L \) designed using Corollary 2 exponentially stabilizes 3 or asymptotically stabilizes 3, if \( \Gamma_1 > 0 \).

**Proof:** By Proposition 2, the OBSF (2) and (3) is equivalent to (10) and (11). By Corollary 2, (11) is ISP, OSP, and ZSD if \( \Gamma_1 > 0 \). Hence, the proof of (i) follows by direct application of [19, Th. IV.2], which states that the power-preserving interconnection of a BC-PHS defined on a 1-D spatial domain and an ISP finite-dimensional system is exponentially stable, and the proof of (ii) follows by direct application of [2, Prop. 4.3.1], which states that the power preserving interconnection between an OSP and a ZSD systems is asymptotically stable. 

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**Fig. 1.** Power preserving interconnection.
Remark 3: The ISP, OSP, and ZSD properties of the controller are fundamental for guaranteeing the closed-loop exponential stability when dealing with the control of infinite-dimensional linear systems or the asymptotic stability when dealing with finite-dimensional nonlinear systems. The PHS allow to verify these properties by matrix conditions that can be satisfied by solving the LMIs of Corollary 2. The matrices $R$, and $Q$, define, respectively, the parameters of the dissipation and the energy function of the controller, which in turn define the time constants of the controller. Hence, the tuning matrices $\Gamma$ and $\Delta$ can be interpreted as bounds for the controller’s time constants. This shall be illustrated in the examples in the following section.

III. EXAMPLES

In this section, we illustrate the design approach on an infinite-dimensional Timoshenko flexible beam model and on a nonlinear model of a microelectromechanical optical switch.

A. Boundary Control of a Flexible Beam

The Timoshenko beam model describes the behavior of a thick beam in a 1-D spatial domain. It admits the BC-PHS formulation (18)–(20) with matrices

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$H(\zeta) = \begin{bmatrix} T(\zeta) & 0 & 0 & 0 \\ 0 & \rho(\zeta)^{-1} & 0 & 0 \\ 0 & 0 & E(\zeta) & 0 \\ 0 & 0 & 0 & I_p(\zeta)^{-1} \end{bmatrix}$$

and state variables $z = (z_1, z_2, z_3, z_4)^T$, defined as the shear displacement $z_1(\zeta, t) = w_1(\zeta, t) - \phi(\zeta, t)$, the transverse momentum distribution $z_2(\zeta, t) = \rho(\zeta)w_1(\zeta, t)$, the angular displacement $z_3(\zeta, t) = \phi(\zeta, t)$, and the angular momentum distribution $z_4(\zeta, t) = I_p(\zeta)\phi(\zeta, t)$. $w(\zeta, t)$ and $\phi(\zeta, t)$ are, respectively, the transverse displacement of the beam and the rotation angle of neutral fiber of the beam. We use lower indexes $\zeta$ and $t$ to refer to the partial derivative with respect to that index. $T(\zeta)$ is the shear modulus, $\rho(\zeta)$ is the mass per unit length, $E(\zeta)$ is the Young’s modulus of elasticity $E$ multiplied by the moment of inertia of a cross section $I$, and $I_p(\zeta)$ is the rotational momentum of inertia of a cross section. Note that $T(\zeta)z_1(\zeta, t)$ is the shear force, $\rho(\zeta)^{-1}z_2(\zeta, t)$ is the longitudinal velocity, $E(\zeta)z_3(\zeta, t)$ is the torque, and $I_p(\zeta)^{-1}z_4(\zeta, t)$ is the angular velocity.

We consider the beam clamped at the left-hand side, i.e., $\rho(\cdot)^{-1}z_2(a, t) = 0$ and $I_p(\cdot)^{-1}z_4(a, t) = 0$, and we define the following inputs and outputs:

$$u(t) = \begin{bmatrix} T(b)z_1(b, t) \\ E(\zeta)z_3(b, t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} \rho(b)^{-1}z_2(b, t) \\ I_p(b)^{-1}z_4(b, t) \end{bmatrix}.$$

The energy balance is given by $\dot{H}(t) = u(t)^T \cdot y(t)$. In this case, we have force and torque actuators at the right-hand side of the beam and collocated transverse and angular velocity sensors. The reader is referred to [21] for more details on the model, to [8] and [9] for the well posedness of this class of systems, and to [19] and [22] for exponential stability analysis. For simplicity, we use the following parameters of the model $T = 1 \text{ Pa}$, $\rho = 1 \text{ kg m}^{-3}$, $EI = 1 \text{ Pa m}^4$, $I_p = 1 \text{ Kg m}^2$, $a = 0 \text{ m}$, and $b = 1 \text{ m}$.

A finite-dimensional approximation of the system using the finite-differences discretization scheme on staggered grids proposed in [23] is

$$J = \begin{bmatrix} 0 & D & 0 & -F \\ -D^T & 0 & 0 & 0 \\ 0 & 0 & D & F^T \\ F & 0 & -D^T & 0 \end{bmatrix}, \quad R = 0$$

$$Q = \begin{bmatrix} hQ_1 & 0 & 0 & 0 \\ 0 & hQ_2 & 0 & 0 \\ 0 & 0 & hQ_3 & 0 \\ 0 & 0 & 0 & hQ_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

where

$$D = \frac{1}{h^2} \begin{bmatrix} 1 & 0 & \ldots & 0 \\ -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}, \quad F = \frac{1}{2h} \begin{bmatrix} 1 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 1 \end{bmatrix}.$$
from both designs, by tuning the design parameters of the state feedback properly, the closed-loop responses become very similar.

Fig. 4 shows the influence of the design parameter $\Delta_1$ in the temporal response of the end-tip position of the beam. Increasing the value of $\Delta_1$ reduces the settling time until the controller becomes overdamped.

By Theorem 3, the reduced-order OBSF controller exponentially stabilizes the BC-PHS.

B. Microelectromechanical Optical Switch

MEMS are microrobots with an electronic actuation part. Due to the miniaturization of technology, MEMS are an important tool in the microrobotic industry. In optics, for instance [24], using tiny mirrors MEMS allow to connect two optical devices without converting continuous signals into electronic ones. A dynamical model of this system can be found in [24] and its port-Hamiltonian representation in [25]

$$
\begin{align*}
\dot{y} &= \frac{1}{r} \frac{\partial H}{\partial Q} = \frac{\varepsilon A}{q_{\text{max}} - q} \left( \frac{\partial H}{\partial \varphi} \right) + \left( \frac{0}{0} \right) u \\
H &= \frac{p^2}{2m} + \frac{1}{2} k_1 q^2 + \frac{1}{4} k_2 q^4 + \frac{Q^2}{2C(q)}
\end{align*}
$$

where $q(t)$, $p(t)$, and $Q(t)$ are, respectively, the position, the momentum, and the charge in the capacitor, $k_1$ and $k_2$ are the spring coefficients, $m$ is the mass of the moving part, $C(q)$ is the nonlinear capacitance that depends on the gap of the MEMS, $b > 0$ and $r > 0$ are the damping and resistance constant parameters, respectively, $\varepsilon$ is the dielectric constant, $A_s$ is the surface of the MEMS, and $q_{\text{max}}$ is such that $q < q_{\text{max}}$. The input of the system $u(t)$ is the input voltage and $y(t)$ is the supplied current. The balance equation of the Hamiltonian is

$$
H(t) = -b \left( \frac{\partial u}{\partial t} \right)^2 - ry(t)^2 + g(t)u(t),
$$

which implies that the system is OSP. Under realistic operation conditions, we can assume that the state space of the system is such that $Q(t) > 0$ for all $t > 0$, allowing to
conclude that the system is ZSD. The parameters of the plant are given in Table III. The linearization of (15) around an equilibrium point is given by

\[
A = \begin{pmatrix}
0 & 1 \\
-m & -k_1 - 3k_2q^* - b - \frac{k_2}{A_e e_r} & -\frac{Q^*}{A_e e_r} & 0 \\
Q^* & 0 & \frac{Q^*}{A_e e_r} & \frac{Q^* - q_{\max}}{A_e e_r} \\
\end{pmatrix}
\]

with the * symbol representing the equilibrium point used for the linearization (see the values in Table III).

For the OBSF design, we choose \( D_c = 10^4 \). Using this value, the eigenvalues of the matrix \( A_D = A - B D_c C \) are close to the eigenvalues of the matrix \( A \). In a second step, the matrix \( L \) is designed such that \( A_D - L C \) is a Hurwitz matrix. Similarly, as in the previous example, we use the LMI and the LQE methods using Corollary 1 by minimizing the cost function \( J = \int_0^T (\dot{x}_1^T \bar{Q} \dot{x}_1 + y_1^T \bar{R} y_1 + 2 \dot{x}_1^T \bar{N} y_1) dt \), respectively.

The design parameters are given in Table IV where the matrices \( C_q, C_p, \) and \( C_Q \) are defined as

\[
C_q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C_p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_Q = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

For the simulation, time \( t = [0, 10 \text{ ms}] \) is used with a step time \( \delta t = 1 \mu s \). The initial conditions are set equal to \( q(0) = q^*, \dot{q}(0) = \dot{q}^* \), and \( Q(0) = 0.9Q^* \) for the nonlinear system, whereas for the observer, all initial conditions are set exactly at the equilibrium point. Figs. 5–7 show the temporal responses of the position, momentum, and the electric charge of the closed-loop system using the feedthrough term only (\( u = -D_c y \)). In this case, the LQE method has been used to speed-up the observed position and momentum in exchange of a bigger overshoot.

Now, for both observers a state feedback matrix \( K \) is designed using Corollary 2 with the design parameters given in Table V. Fig. 8 shows the dynamical responses for the position \( q(t) \) (for the sake of space...
only the position is shown. For both observers, one can achieve similar closed-loop behaviors by tuning the state feedback properly.

Finally, Fig. 9 shows the influence of $\Delta_2$ in the temporal response of the position. One can see that increasing the value of $\Delta_1$ the settling time is reduced.

Theorem 3 guarantees that the OBSF controller asymptotically stabilizes the nonlinear system (15). In general, the design procedure can be summarized as follows.

1) Approximate the infinite-dimensional/nonlinear model by a linear finite-dimensional model.
2) Assign the observer behavior to the finite-dimensional model. To this end, different strategies can be used, namely LMI, LQE, or pole-placement, for instance.
3) Tune the matrices $\Gamma_1$, $\Gamma_2$, $\Delta_1$, and $\Delta_2$ for the design of $K$.
4) Apply the OBSF controller to the original system.

IV. CONCLUSION

An OBSF LMI-based design is proposed for linear PHS. The feedback consists of a Luenberger observer and a negative feedback on the observed state. The novelty and main contribution of this article is a constructive design method for the OBSF gains based on (i) a discretized model of BC-PHS, or (ii) a linearized model of a nonlinear PHS. The observer gain is designed freely and the state feedback gain is designed such that the OBSF controller is ISP and/or OSP, and ZSD. The closed-loop performances can then be modified by tuning some explicit controller parameters while guaranteeing the closed-loop exponential/asymptotic stability when the OBSF is applied to the original system. An infinite-dimensional Timoshenko beam model and a finite-dimensional nonlinear model of a microelectromechanical actuator have been used to illustrate the effectiveness of the proposed approach.

APPENDIX

Boundary controlled PH system on 1-D domain

In this section, the definition of BC-PHS is given. The reader is referred to [8] and [9] for further details and definitions. A BC-PHS is a dynamical system governed by the following partial differential equation:

$$\frac{\partial z}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta}(H(\zeta)z(\zeta, t)) + P_0 H(\zeta)z(\zeta, t)$$

$$z(\zeta, 0) = z_0(\zeta)$$

$$W_S \left( \begin{array}{c} f_0(t) \\ e_0(t) \end{array} \right) = u(t)$$

$$y(t) = W_C \left( \begin{array}{c} f_0(t) \\ e_0(t) \end{array} \right)$$

where the initial condition is given by (18), the boundary input by (19), and the boundary output by (20). Here, $z(\zeta, t) \in \mathbb{R}^n$ is the state variable with initial condition $z_0(\zeta)$. $\zeta \in [a, b]$ is the 1-D domain and $t \geq 0$ is the time. $P_1 = PT_1 \in \mathbb{R}^{n \times n}$ is a nonsingular matrix, $P_0 = -P_1^T \in \mathbb{R}^{n \times n}$, $H(\zeta)$ is a bounded and continuously differentiable matrix-valued function satisfying for all $\zeta \in [a, b]$ and $H(\zeta) = \hat{H}^T(\zeta)$ and $mI < \hat{H}(\zeta) < MI$ with $0 < m < M$ both scalars independent on $\zeta$. The Hamiltonian energy function of (17) is given by $H(t) = \frac{1}{2} \int_a^b \hat{H}(\zeta(t), t^2) \hat{H}(\zeta)z(\zeta, t) d\zeta$. The boundary port variables defined as

$$\left( \begin{array}{c} f_0(t) \\ e_0(t) \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} P_1 & -P_1 \\ I & I \end{array} \right) \left( \begin{array}{c} H(b)z(b, t) \\ H(a)z(a, t) \end{array} \right).$$

$W_S$, $W_C \in \mathbb{R}^{n \times 2n}$ are two matrices such that if $W_S \Sigma W_C^T = W_C^T \Sigma W_S^T = 0$ and $W_C \Sigma W_C^T = I$, with $\Sigma = \left[ \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right]$, then $H(t) = u(t)^T y(t)$.

ISP, OSP, and ZSD (non)linear control system

In this section, the definitions of ISP, OSP, and ZSD (non)linear systems are given. The reader is referred to [2] for further details and definitions. Consider a control system

$$\dot{x} = f(x, u), \quad y = h(x, u)$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, and $f(\cdot)$ and $h(\cdot)$ sufficiently smooth differentiable mappings, then (21) is as follows:

1) ISP if there exists $\delta > 0$ such that it is dissipative with respect to the supply rate $s(u, y) = u^T y - \delta|u|^2$;
2) OSP if there exists $\epsilon > 0$ such that it is dissipative with respect to the supply rate $s(u, y) = u^T y - \epsilon|y|^2$;
3) ZSD if $u(t) = 0, y(t) = 0 \forall t \geq 0$, implies $\lim_{t \to \infty} x(t) = 0$.

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