Discrete phase-space structure of $n$-qubit mutually unbiased bases

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Abstract

We work out the phase-space structure for a system of $n$ qubits. We replace the field of real numbers that label the axes of the continuous phase space by the finite field $\text{GF}(2^n)$ and investigate the geometrical structures compatible with the notion of unbiasedness. These consist of bundles of discrete curves intersecting only at the origin and satisfying certain additional properties. We provide a simple classification of such curves and study in detail the four- and eight-dimensional cases, analyzing also the effect of local transformations. In this way, we provide a comprehensive phase-space approach to the construction of mutually unbiased bases for $n$ qubits.

Key words: Mutually unbiased bases; discrete phase space; Galois fields

1. Introduction

Quantum mechanics describes physical systems through the density operator $\hat{\rho}$. For continuous systems, this operator lives in an infinite-dimensional complex Hilbert space and its relations with the physical properties of the system are far from obvious. To overcome these conceptual difficulties, a number of phase-space methods have been devised, which result in a striking formal similarity with classical mechanics [1].
Textbook examples of this subject are usually presented in terms of continuous variables, typically position and momentum. However, there are many quantum systems that can be appropriately described in a finite-dimensional Hilbert space. These include, among other, spins, multilevel atoms, optical fields with a fixed number of photons, electrons or molecules with a finite number of sites, etc. An elegant way of approaching these systems was proposed by Weyl [2] in his description of quantum kinematics as an Abelian group of ray rotations. Similar results were also obtained by Schwinger [3–5], who showed that a set of unitary operators (defined through cyclic permutations of state vectors) can be constructed such that they are the generators of a complete operator basis, in terms of which all possible quantities related to the physical system can be built.

The structure of the phase space associated to a $d$-dimensional system (a qudit, in the modern parlance of quantum information) has been addressed by a number of authors. A possible approach was taken by Hannay and Berry [6], considering a grid constrained to admit only periodic probability distributions, which implies that it is effectively a $2d \times 2d$-dimensional torus. The same strategy was adopted by Leonhardt [7, 8] and used to deal with different aspects of quantum information [9–11]. This method offers a way for treating even-dimensional systems, since the grid has both integer and half-odd coordinates.

However, the mainstream of research has focused on a phase space pictured as a $d \times d$ lattice. This line was started by Buot [12], who introduced a discrete Weyl transform that generates a Wigner function on the toroidal lattice $\mathbb{Z}_d$ (with $d$ odd). More recently, these ideas have been developed further by other authors [13–24]. In particular, when the dimension is a power of a prime, one can label the points in the $d \times d$ grid with elements of the finite Galois field $\mathbb{GF}(d)$. At first sight, the use of elements of $\mathbb{GF}(d)$ as coordinates could be seen as an unnecessary complication, but it turns out to be an essential step: only by doing this we can endow the phase space with similar geometrical properties as the ordinary plane. Note also that though the restriction to powers of primes rules out many quantum systems, this formulation is ideally suited for the outstanding case of $n$ qubits we deal in this paper.

In these finite descriptions, the Wigner function, being the Weyl representative of the density operator, naturally emerges as a function that takes values only at the points defining the discrete mesh of the phase space (while preserving some properties that make it a special object in quantum mechanics). A remarkable feature is that one can sum the Wigner function along different axes (including skew ones) to obtain correct probability distributions for observables associated with those axes. Although the axis observables cannot be complementary in the usual sense (their commutator cannot be proportional to the identity operator), they will have a closely related property: every eigenstate of either one of them is a state of maximum uncertainty with respect to the other.

This makes a deep connection with the notion of mutually unbiased bases (MUBs) [25, 26], which were introduced as a central tool for quantum state reconstruction [27]. They also play a relevant role in a proper understanding of complementarity [28–31], in cryptographic protocols [32, 33], and in quantum error correction codes [34,35]. Recently, they have also found uses in quantum game theory, in particular to provide a convenient tool for solving the so-called mean king problem [36–43].

It has been shown [44] that the maximum number of MUBs can be at most $d+1$. Actually, it is known that if $d$ is prime or power of a prime the maximal number of MUBs can be achieved [45]. Different explicit constructions of MUBs in prime power dimensions have been suggested in a number of recent papers [46–53]. Remarkably though, there is no known answer for any other values of $d$, although there are some attempts to find a solution in some simple cases, such as $d =$
or when \(d\) is a nonprime integer squared [54–57]. Recent work has suggested that the answer to this question may well be related with the nonexistence of finite projective planes of certain orders [58, 59] or with the problem of mutually orthogonal Latin squares in combinatorics [60, 61].

The construction of MUBs is closely related to the possibility of finding \(d+1\) disjoint classes, each one having \(d-1\) commuting operators [which proves useful to arrange in a table with \((d-1) \times (d+1)\) entries], so the corresponding eigenstates form sets of MUBs [62]. Nevertheless, these MUB operators can be organized in several different nontrivial tables, leading to different factorization properties of the MUB [63]. It has been recently noticed [64] that such arrangements are related with special types of curves. We have previously analyzed [65] those curves in the four-dimensional case (corresponding to two qubits) and have shown that they can be obtained through local transformations from rays (straight lines passing though the origin), so that the six possible \(3 \times 5\) tables of operators lead to the same (and unique) factorization of the MUBs.

In the present paper we go further and analyze the general situation of \(n\) qubits. In particular, we classify the admissible curves in specific bundles and show how the properties of these curves can be used to determine nontrivial sets of MUBs.

2. Mutually unbiased bases and discrete phase space

2.1. Constructing mutually unbiased bases in prime dimensions

We start by considering a system living in a Hilbert space \(\mathcal{H}_d\), whose dimension \(d\) is assumed for now to be a prime number \(p\). The different outcomes of a maximal test constitute an orthogonal basis of \(\mathcal{H}_d\) [66]. One can also look for orthogonal bases that, in addition, are “as different as possible”. This is the idea behind MUBs and can be formally stated as follows: two bases \(\{|u_i\rangle\}\) and \(\{|v_j\rangle\}\) are mutually unbiased when

\[
|\langle u_i | v_j \rangle|^2 = \frac{1}{d}.
\]

Unbiasedness also applies to measurements: two nondegenerate tests are mutually unbiased if the bases formed by their eigenstates are MUBs. Therefore, the measurements of the components of a spin 1/2 along \(x, y,\) and \(z\) axes are all unbiased. It is also obvious that for these finite quantum systems unbiasedness is tantamount of complementarity.

It is useful to choose a computational basis \(|n\rangle\) \((n = 0, \ldots, d-1)\) in \(\mathcal{H}_d\) and introduce the basic operators

\[
X|n\rangle = |n+1\rangle, \quad Z|n\rangle = \omega^n|n\rangle,
\]

where \(\omega = \exp(2\pi i/d)\) is a \(d\)th root of the unity and addition and multiplication must be understood modulo \(d\). These operators \(X\) and \(Z\), which are generalizations of the Pauli matrices, were studied long ago by Weyl [2] and have been used recently by many authors in a variety of applications [67,68]. They generate a group under multiplication known as the generalized Pauli group and obey

\[
ZX = \omega XZ,
\]

which is the finite-dimensional version of the Weyl form of the commutation relations.
As anticipated in the Introduction, one can construct MUBs by finding \( d + 1 \) disjoint classes (each one having \( d - 1 \) commuting operators), such that the corresponding eigenstates form sets of MUBs. We follow the explicit construction in reference [52], which starts with the following sets of operators:

\[
\{X^k\}, \quad \{Z^k X^{mk}\},
\]

with \( k = 1, \ldots, d - 1 \) and \( m = 0, \ldots, d - 1 \). One can easily check that

\[
\text{Tr}(X^k X^{k'}) = d \delta_{k,k'}, \quad \text{Tr}(Z^k Z^{k'}) = d \delta_{k,k'},
\]

\[
\text{Tr}[(Z^k X^{mk})(Z^{k'} X^{m'k'})^\dagger] = d \delta_{k,k'} \delta_{m,m'}.
\]

These pairwise orthogonality relations indicate that, for every value of \( m \), we generate a maximal set of \( d - 1 \) commuting operators and that all these classes are disjoint. In addition, the common eigenstates of each class \( m \) form different sets of MUBs.

2.2. Mutually unbiased bases for \( n \) qubits

When the space dimension \( d = p^n \) is a power of a prime it is natural to view the system as composed of \( n \) constituents (particles), each of dimension \( p \). We adapt the previous construction to this case, although with an eye on the particular case of \( n \) qubits (in the Appendix we summarize the basic notions of finite fields needed for our purposes in this paper).

The main idea consists in labeling the states with elements of the finite field \( \text{GF}(2^n) \), instead of natural numbers. We denote by \( |\alpha\rangle \), with \( \alpha \in \text{GF}(2^n) \), an orthonormal basis in the Hilbert space of the system. Operationally, the elements of the basis can be labeled by powers of a primitive element, and the basis reads

\[
\{|0\rangle, |\sigma\rangle, \ldots, |\sigma^{2^n-1} = 1\rangle\}.
\]

These vectors are eigenvectors of the operators \( Z_\beta \) belonging to the generalized Pauli group, whose generators are now defined as

\[
Z_\beta = \sum_{\alpha \in \text{GF}(2^n)} \chi(\alpha \beta) |\alpha\rangle \langle \alpha|, \quad X_\beta = \sum_{\alpha \in \text{GF}(2^n)} |\alpha + \beta\rangle \langle \alpha|,
\]

so that

\[
Z_\alpha X_\beta = \chi(\alpha \beta) X_\beta Z_\alpha,
\]

where \( \chi \) is an additive character defined in (A.5).

The operators (2.7) can be factorized into tensor products of powers of single-particle Pauli operators \( \sigma_z \) and \( \sigma_x \), whose expression in the standard basis of the two-dimensional Hilbert space is

\[
\sigma_z = |1\rangle \langle 1| - |0\rangle \langle 0|, \quad \sigma_x = |0\rangle \langle 1| + |1\rangle \langle 0|.
\]

This factorization can be carried out by mapping each element of \( \text{GF}(2^n) \) onto an ordered set of natural numbers as in equation (A.7). A convenient choice for this is the selfdual basis, since the
finite Fourier transform factorizes then into a product of single-particle Fourier operators, which leads to

\[ Z_\alpha = \sigma_z^{a_1} \otimes \ldots \otimes \sigma_z^{a_n}, \quad X_\beta = \sigma_x^{b_1} \otimes \ldots \otimes \sigma_x^{b_n} \]  

(2.10)

where \((a_1, \ldots, a_n)\) and \((b_1, \ldots, b_n)\) are the corresponding coefficients.

The simplest geometrical structures in the discrete phase space are straight lines, i.e., collections of points \((\alpha, \beta) \in \mathbb{GF}(2^n) \times \mathbb{GF}(2^n)\) satisfying the relation

\[ \zeta \alpha + \eta \beta = \vartheta, \]  

(2.11)

where \(\zeta, \eta\) and \(\vartheta\) are fixed elements of \(\mathbb{GF}(2^n)\). Two lines \(\zeta \alpha + \eta \beta = \vartheta, \zeta' \alpha + \eta' \beta = \vartheta'\) are parallel if they have no common points, which implies that \(\eta \zeta' = \zeta \eta'\). If the lines (2.12) are not parallel they cross each other. A ray is a line passing through the origin, so its equation is

\[ \alpha = 0, \quad \text{or} \quad \beta = \lambda \alpha, \]  

(2.13)

or, in parametric form,

\[ \alpha(\kappa) = \mu \kappa, \quad \beta(\kappa) = \nu \kappa, \]  

(2.14)

where \(\kappa\) is the parameter running through the field. The rays are the simplest nonsingular (i.e., with no selfintersection) additive structures in phase space, in the sense that

\[ \alpha(\kappa + \kappa') = \alpha(\kappa) + \alpha(\kappa'), \quad \beta(\kappa + \kappa') = \beta(\kappa) + \beta(\kappa'). \]  

(2.15)

This means that by summing the coordinates of the origin and of any point in a ray we obtain another point on the same ray. In particular, this opens the possibility of introducing operators that generate “translations” along these rays [22].

The rays have a very remarkable property: the monomials \(Z_\alpha X_\beta\) (labeled by phase-space points) belonging to the same ray commute

\[ Z_{\alpha_1} X_{\beta_1} = \lambda_{\alpha_1} X_{\beta_1}, \quad Z_{\alpha_2} X_{\beta_2} = \lambda_{\alpha_2} X_{\beta_2} \quad Z_{\alpha_1} X_{\beta_1} = \lambda_{\alpha_1} X_{\beta_1}, \]  

(2.16)

and thus, have a common system of eigenvectors \(\{ | \psi_{\nu, \lambda} \rangle \}\), with \(\lambda, \nu \in \mathbb{GF}(2^n)\):

\[ Z_\alpha X_{\lambda \alpha} | \psi_{\nu, \lambda} \rangle = \exp(i \xi_{\nu, \lambda}) | \psi_{\nu, \lambda} \rangle, \]  

(2.17)

where \(\lambda\) is fixed and \(\exp(i \xi_{\nu, \lambda})\) is the corresponding eigenvalue, so \(| \psi_{\nu, 0} \rangle = | \nu \rangle\) are eigenstates of \(Z_\alpha\) (displacement operators labeled by the ray \(\beta = 0\), which we take as the horizontal axis). Indeed, we have that

\[ | \langle \psi_{\nu, \lambda} | \psi_{\nu', \lambda'} \rangle |^2 = \delta_{\lambda, \lambda'} \delta_{\nu, \nu'} + \frac{1}{d} (1 - \delta_{\lambda, \lambda'}), \]  

(2.18)

and, in consequence, they are MUBs [27]. Since each ray defines a set of \(2^n - 1\) commuting operators, if we introduce \(2^n + 1\) sets of commuting operators (which from now on will be called displacement operators) as

\[ \{ X_\beta \}, \quad \{ Z_\alpha X_{\beta = \lambda \alpha} \}, \]  

(2.19)

then we have a whole bundle of \(2^n + 1\) rays (which is obtained by varying the “slope” \(\lambda\)) that allows us to construct a complete set of MUB operators arranged in a \((2^n - 1) \times (2^n + 1)\) table.
We wish to emphasize that in our approach we do not assign a quantum state to each line in phase space (as in reference [22]), but rather we use them to label Pauli displacement operators.

3. Curves in discrete phase space

3.1. Additive curves and the commutativity condition

The rays are not the only additive structures that exist in the discrete phase space. One can check that the parametric curves (passing through the origin)

\[
\alpha(\kappa) = \sum_{m=0}^{n-1} \alpha_m \kappa^{2^m}, \quad \beta(\kappa) = \sum_{m=0}^{n-1} \beta_m \kappa^{2^m}, \quad (3.1)
\]

satisfy the condition (2.15) too. If we also require the displacement operators labeled with the points of (3.1) to commute with each other, we must impose

\[
\text{tr}(\alpha' \beta') = \text{tr}(\alpha' \beta), \quad (3.2)
\]

where \(\alpha' = \alpha(\kappa')\) and \(\beta' = \beta(\kappa')\). Then, the coefficients \(\alpha_m\) and \(\beta_m\) fulfill the following restrictions (the indices must be understood modulus \(n\))

\[
\sum_{m=0}^{n-1} \alpha_{n-m} \beta_{n-m+q}^2 = \sum_{m=0}^{n-1} \beta_{n-m} \alpha_{n-m+q}^2, \quad q = 1, \ldots, n - 1. \quad (3.3)
\]

This can be rewritten in an invariant form by summing up all Frobenius automorphisms of (3.3):

\[
\sum_{m \neq k} \text{tr}(\alpha_m \beta_k) = 0. \quad (3.4)
\]

Whenever the condition (3.4) holds true, we can associate to each curve (3.1), with given coefficients \(\vec{\alpha} = (\alpha_0, \ldots, \alpha_{n-1})\) and \(\vec{\beta} = (\beta_0, \ldots, \beta_{n-1})\), a state \(|\psi_{\vec{\alpha}, \vec{\beta}}\rangle\).

The curves fulfilling equation (3.4) will be called additive commutative curves. When such a curve contains \(2^n - 1\) different points (apart from the origin), the monomials \(Z_{\alpha}X_{\beta}\) form also a set of commuting operators (as happened for the rays).

In fact, consider a subset \(Z_{\alpha(\kappa)}X_{\beta(\kappa)}\) such that

\[
[Z_{\alpha(\kappa)}X_{\beta(\kappa)}, Z_{\alpha(\kappa')}X_{\beta(\kappa')}] = 0, \quad \text{Tr}(Z_{\alpha(\kappa)}X_{\beta(\kappa')}Z_{\alpha(\kappa')}^\dagger X_{\beta(\kappa')}^\dagger) = 0, \quad (3.5)
\]

for any \(\kappa, \kappa' \in \text{GF}(p^n)\), i.e. a disjoint set of \(d\) mutually commuting monomials, including \(Z_{\alpha(0)}X_{\beta(0)} = 1\). Then, the eigenstates of any two disjoint sets of these mutually commuting monomials form MUBs.

To prove this important result, we first note that any set of commuting monomials can be obtained by applying Clifford operations \(U_\lambda\) to the simplest set \(\{Z_{\kappa}\}\):

\[
U_\lambda Z_{\kappa} U_\lambda^\dagger = \phi(\lambda, \kappa) D_{\lambda, \kappa}, \quad (3.6)
\]

where \(\phi(\lambda, \kappa)\) is an unessential phase factor \([\phi(\lambda, 0) = \phi(0, \kappa) = 1]\) and \(D_{\lambda, \kappa} = Z_{\alpha(\kappa)}X_{\beta(\kappa)}\) fulfill

\[
[D_{\lambda, \kappa}, D_{\lambda', \kappa'}] = 0, \quad \text{Tr}(D_{\lambda, \kappa}D_{\lambda', \kappa'}^\dagger) = d \delta_{\kappa, \kappa'}, \quad (3.7)
\]
since the transformations $U_{\lambda}$ are nondegenerate. For two disjoint set of commuting monomials \( \{D_{\lambda,\kappa}\} \) and \( \{D_{\lambda',\kappa'}\} \) we have then

\[
\text{Tr}(D_{\lambda,\kappa}D_{\lambda',\kappa'}^\dagger) = \begin{cases} d\,\delta_{\kappa,\kappa'}, & \lambda = \lambda', \\ d\,\delta_{\kappa,0}\delta_{\kappa',0}, & \lambda \neq \lambda'. \end{cases} \tag{3.8}
\]

Let \( \{|v\rangle\} \) be the basis of eigenstates of $Z_{\kappa}$. It is worth noting the following expansion

\[
|v\rangle\langle v| = \frac{1}{d} \sum_{\kappa} \chi(-v\kappa)Z_{\kappa} = \frac{1}{d} \sum_{\kappa} \chi(v\kappa)Z_{\kappa}^\dagger. \tag{3.9}
\]

Now, if we define the states \( |\psi_{\nu,\lambda}\rangle = U_{\lambda}|v\rangle \) and \( |\psi_{\nu',\lambda'}\rangle = U_{\lambda'}|v'\rangle \), where $U_{\lambda}$ is as in equation (3.6), then a direct calculation using (3.9) shows that (2.18) holds true for them and so they are indeed MUBs, as announced.

Finding sets of MUBs can be thus reduced to the problem of arranging additive curves in $2^n + 1$ bundles of mutually nonintersecting curves.

Due to the condition (2.15), points of a curve form a finitely generated Abelian group, which allows us to determine all the curve from any $n$ points and, in particular, from the “first” $n$ consecutive points. For instance, taking the parameter $\kappa$ polynomially ordered (that is, $\kappa = \sigma, \sigma^2, \ldots, \sigma^{2^n - 1}$), we have that $\alpha(\sigma^k) + \alpha(\sigma^{k+1}) = \alpha(\sigma^k + \sigma^{k+1}) = \alpha(\sigma^k(1 + \sigma)) = \alpha(\sigma^{k+L(1)})$, where $L(\lambda)$ is the Jacobi logarithm [69]. Given a primitive polynomial, sometimes is possible to evaluate $L(1)$ in a simple form.

For GF(2) the only irreducible polynomial is $x^2 + x + 1 = 0$, and this immediately leads to $L(1) = L(\sigma^3) = 2$, in such a way that $\alpha(\sigma^k) + \alpha(\sigma^{k+1}) = \alpha(\sigma^{k+2})$ and two points are enough to determine any additive curve. In the case of GF(2), if we use the irreducible polynomial $x^3 + x + 1 = 0$, we obtain that $L(1) = L(\sigma^7) = 3$, so that $\alpha(\sigma^k) + \alpha(\sigma^{k+1}) = \alpha(\sigma^{k+3})$ and we need three points to generate the curve.

### 3.2. Nonsingularity condition

To classify all the additive commutative curves, we first discuss the condition of nonsingularity (i.e., nonselfintersection), which means that there are no $\kappa' \neq \kappa$ such that

\[
\alpha(\kappa) = \alpha(\kappa'), \quad \beta(\kappa) = \beta(\kappa'). \tag{3.10}
\]

If one of these equations would be fulfilled, then $G^m[\varepsilon(\kappa)] = 0$ (with $\varepsilon = \alpha$ or $\beta$) for $m = 0, \ldots, n - 1$.

Let us introduce the following matrix that will play a relevant role in our subsequent analysis:

\[
W_\varepsilon = \begin{pmatrix}
\varepsilon_0 & \varepsilon_1 & \varepsilon_2 & \cdots \\
\varepsilon^2_{n-1} & \varepsilon^2_0 & \varepsilon^2_1 & \cdots \\
\varepsilon^4_{n-2} & \varepsilon^4_{n-1} & \varepsilon^4_0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \tag{3.11}
\]

where the rows are determined by the coefficients in (3.1) and the corresponding expansions of $G^m[\varepsilon(\kappa)]$ (with $\varepsilon = \alpha, \beta$ in our case). If $\det W_\varepsilon$ and/or $\det W_\beta$ do not vanish simultaneously, the curve (3.11) has no selfintersection.
The condition
\[ \det \mathbf{W}_\alpha = \det \mathbf{W}_\beta = 0, \] (3.12)
indicates that the ranks of the matrices $\mathbf{W}_\alpha$ and $\mathbf{W}_\beta$ are smaller than the dimension of the system, but it does not guarantee that there exist $\kappa' \neq \kappa$ satisfying (3.10), because the solutions of $\alpha(\kappa) = \alpha(\kappa')$ and $\beta(\kappa'') = \beta(\kappa''')$ can form disjoint sets. Consequently, (3.12) is necessary but not sufficient to determine if a curve is singular. Another necessary but not sufficient condition of singularity is
\[ \det(\mathbf{W}_\alpha + \mathbf{W}_\beta) = 0. \]

A curve that fulfills $\det \mathbf{W}_\alpha \neq 0$ and/or $\det \mathbf{W}_\beta \neq 0$ will be called a regular curve. For such curves the coordinate $\alpha$ (when $\det \mathbf{W}_\alpha \neq 0$) or $\beta$ (when $\det \mathbf{W}_\beta \neq 0$) take all the values in the field.

Nonsingular curves satisfying (3.12) will be called exceptional curves. The conditions (3.12) mean that $S_m(\alpha)$ and $S_m(\beta)$ are not linearly independent (for $m = 0, \ldots, n - 1$), so neither $\alpha$ or $\beta$ run through the whole field (in other words, the values of $\alpha$ and $\beta$ are degenerate). The number of linearly independent powers of $\alpha$ (respectively $\beta$) is equal to the rank of the matrix $\mathbf{W}_\alpha$ (respectively $\mathbf{W}_\beta$) and the quantities $n - \text{rank} \mathbf{W}_\alpha$ (respectively $n - \text{rank} \mathbf{W}_\beta$) determine the degree of degeneration of every allowed value of $\alpha$ (respectively $\beta$).

It is interesting to note that the determinant of the matrix (3.11) takes only the values zero and one; i.e., $\det \mathbf{W}_\varepsilon \in \mathbb{Z}_2$, which can be easily seen by observing that $(\det \mathbf{W}_\varepsilon)^2 = \det \mathbf{W}_\varepsilon$.

4. Regular curves

4.1. Explicit forms

Given a regular curve, we can invert one of the relations (3.1) and, by substituting into the other one, we find an explicit equation of the curve. When $\det \mathbf{W}_\alpha \neq 0$ and $\det \mathbf{W}_\beta \neq 0$, the coordinates $\alpha$ and $\beta$ are nondegenerate and the curve can be written either as
\[ \beta = \sum_{m=0}^{n-1} \phi_m \alpha^{2^m}; \] (4.1)

or
\[ \alpha = \sum_{m=0}^{n-1} \psi_m \beta^{2^m}. \] (4.2)

However, when $\det \mathbf{W}_\alpha \neq 0$ but $\det \mathbf{W}_\beta = 0$, the coordinate $\beta$ is degenerate and the curve cannot be expressed in the form (4.2). We will refer to the corresponding curve as $\alpha$-curve. Similarly, if $\det \mathbf{W}_\beta \neq 0$ but $\det \mathbf{W}_\alpha = 0$, the coordinate $\alpha$ is degenerate and the curve cannot be expressed in the form (4.1): we will call it a $\beta$-curve.

4.2. Commutativity conditions

The commutativity condition (3.4) can be further simplified for regular curves. When $\det \mathbf{W}_\alpha \neq 0$ (or $\det \mathbf{W}_\beta \neq 0$) we obtain, by direct substitution of the explicit forms (4.1) or (4.2) into (3.2), the following restrictions on the coefficients $\phi_m$ (or $\psi_m$)
\[ \phi_j = \phi_{n-j}^{2^j}, \quad \psi_j = \psi_{n-j}^{2^j}, \quad j = 1, \ldots, [(n-1)/2], \quad (4.3) \]

where \([\ ]\) denotes the integer part. For even values of \(n\), the additional requirements \(\phi_{n/2} = \phi_{n/2}^{2^{n/2}}, \psi_{n/2} = \psi_{n/2}^{2^{n/2}}\) should be fulfilled, which basically implies that \(\phi_{n/2}, \psi_{n/2}\) belong to the subfield \(GF(2^{n/2})\).

Because the regular curves are nonsingular per definition, we do not have to carry out the whole analysis involving the parametric forms of curves and the properties of the corresponding \(W_\alpha\) and \(W_\beta\), but just to write down explicit expressions using directly (4.3).

Two regular curves defined explicitly as \(\beta = f(\alpha, \vec{\phi})\) and \(\beta = f(\alpha, \vec{\phi}')\), with \(\vec{\phi} = (\phi_0, \ldots, \phi_{n-1})\) [or \(\alpha = g(\beta, \vec{\psi})\) and \(\alpha = g(\beta, \vec{\psi}')\), with \(\vec{\psi} = (\psi_0, \ldots, \psi_{n-1})\)] are not mutually intersecting (except at the origin), if

\[ \det(W_\alpha + W_\phi) \neq 0, \quad \det(W_\psi + W_\psi') \neq 0, \quad (4.4) \]

for \(\alpha\)- and \(\beta\)-curves, respectively, and the matrices \(W\) have been defined in (3.11).

An \(\alpha\)-curve \(\beta = f(\alpha, \vec{\phi})\) intersects with a \(\beta\)-curve \(\alpha = g(\beta, \vec{\psi})\) when the polynomial \(\beta = f[g(\beta, \vec{\psi})], \vec{\phi}']\) (or \(\alpha = f[\alpha, \vec{\phi}], \vec{\psi}'\)) has at least one nonzero root.

It follows from (4.4) that the regular curves

\[ \beta = \phi_0\alpha + \sum_{m=1}^{n-1} \phi_m \alpha^{2^m}, \quad (4.5) \]

where \(\phi_m\) \((m = 1, \ldots, n-1)\) are fixed and fulfill (4.3), and \(\phi_0\) runs through the whole field, belong to a bundle of nonintersecting curves, since the matrices \(W_\alpha + W_\phi\) take now the diagonal form

\[ W_\alpha + W_\phi = \text{diag} \left[ \phi_0 + \phi_0', (\phi_0 + \phi_0')^2, \ldots, (\phi_0 + \phi_0')^{2^{n-1}} \right], \quad (4.6) \]

so that

\[ \det(W_\alpha + W_\phi) = (\phi_0 + \phi_0')^{2^n-1}, \quad (4.7) \]

and thus, \(\det(W_\alpha + W_\phi) = 1\) if \(\phi_0 \neq \phi_0'\). To complete the bundle to \(n + 1\) curves we add the ray \(\alpha = 0\), which obviously has no common points with (4.5).

Similarly, the curves

\[ \alpha = \psi_0\beta + \sum_{m=1}^{n-1} \psi_m \beta^{2^m} \quad (4.8) \]

form bundles of nonintersecting curves, except that now we have to add the ray \(\beta = 0\) to complete the bundle.

4.3. Examples

4.3.1. Regular curve

Let us consider the following parametric curve in \(GF(2^4)\)

\[ \alpha = \sigma^2 \kappa + \kappa^2 + \sigma^4 \kappa^4, \quad \beta = \sigma^3 \kappa + \sigma^6 \kappa^2 + \sigma^6 \kappa^4, \quad (4.9) \]
where $\sigma$ is the primitive element. The associated matrices are

\[
W_\vec{\alpha} = \begin{pmatrix}
\sigma^2 & 1 & \sigma^4 \\
\sigma & \sigma^4 & 1 \\
1 & \sigma^2 & \sigma
\end{pmatrix}, \quad W_\vec{\beta} = \begin{pmatrix}
\sigma^3 & \sigma^6 & \sigma^6 \\
\sigma^5 & \sigma^6 & \sigma^5 \\
\sigma^3 & \sigma^3 & \sigma^5
\end{pmatrix}.
\] (4.10)

One can check that $\det W_\vec{\alpha} = \det W_\vec{\beta} = 1$, and the explicit forms of the curve are

\[
\beta = \sigma^6 \alpha + \sigma^3 \alpha^2 + \sigma^5 \alpha^4, \quad \text{or} \quad \alpha = \sigma^6 \beta + \sigma^3 \beta^2 + \sigma^5 \beta^4,
\] (4.11)

whose coefficients satisfy again the condition (4.3). The set of commuting operators corresponding to this curve is

\[
\{Z_{\sigma^6} X_{\sigma^6}, Z_{\sigma^3} X_{\sigma^3}, Z_{\sigma^5} X_{\sigma^5}, Z_{\sigma^7} X_{\sigma^7}, Z_{\sigma^2} X_{\sigma^4}, Z_{\sigma^3} X_{\sigma^2}\}.
\] (4.12)

4.3.2. $\alpha$-curve

To the parametric curve

\[
\alpha = \sigma^2 \kappa^4, \quad \beta = \sigma^2 \kappa + \kappa^2 + \sigma^4, \] (4.13)

it corresponds the matrices

\[
W_\vec{\alpha} = \begin{pmatrix}
0 & 0 & \sigma^2 \\
\sigma^4 & 0 & 0 \\
0 & \sigma & 0
\end{pmatrix}, \quad W_\vec{\beta} = \begin{pmatrix}
\sigma^2 & 1 & \sigma \\
\sigma^4 & \sigma^4 & 1 \\
1 & \sigma^4 & \sigma
\end{pmatrix}.
\] (4.14)

Now we have $\det W_\vec{\alpha} = 1$ and $\det W_\vec{\beta} = 0$, which leads to the following explicit form of the $\alpha$-curve

\[
\beta = \sigma^6 \alpha + \sigma^3 \alpha^2 + \sigma^6 \alpha^4,
\] (4.15)

whose coefficients satisfy again the condition (4.3). The corresponding $W_\vec{\phi}$ matrix is degenerate in this case. The set of commuting operators is

\[
\{Z_{\sigma^6} X_{\sigma^6}, Z_{\sigma^3} X_{\sigma^3}, Z_{\sigma^5} X_{\sigma^5}, Z_{\sigma^7} X_{\sigma^7}, Z_{\sigma^2} X_{\sigma^4}, Z_{\sigma^3} X_{\sigma^2}\},
\] (4.16)

and the curve belongs to the bundle $\beta = \psi_0 \alpha + \sigma^5 \alpha^2 + \sigma^6 \alpha^4$.

5. Exceptional curves

The analysis of exceptional curves is considerably more involved. As we have stressed above, the points on the curve do not take all the values in the field and their admissible values are fixed by the structural equations

\[
\sum_{m=0}^{r_\alpha} v_m \alpha^{2m} = 0, \quad \sum_{m=0}^{r_\beta} \tau_m \beta^{2m} = 0,
\] (5.1)
where \( r_\alpha = \text{rank} W_\alpha \leq n - 1 \) and \( r_\beta = \text{rank} W_\beta \leq n - 1 \), which are a consequence of the linear dependence of \( \alpha^{2^m} \) and \( \beta^{2^m} \). The coordinates \( \alpha \) and \( \beta \) of an exceptional curve are \( \deg \alpha = 2^{n-r_\alpha} \) and \( \deg \beta = 2^{n-r_\beta} \) times degenerate, respectively. In other words, if \((\alpha_j, \beta_j)\) is a point of an exceptional curve, for each \( \alpha_j \) there are \( 2^{n-r_\alpha} \) values of \( \beta \), such that the points \((\alpha_j, \beta_k)\) \((k = 1, \ldots, 2^{n-r_\beta})\) belong to the same curve and, conversely, for each \( \beta_j \) there are \( 2^{n-r_\beta} \) values of \( \alpha \), such that the points \((\alpha_k, \beta_j)\) also belong to the same curve.

Due to the nonsingularity condition, there are \( 2^n \) different pairs of points \((\alpha, \beta)\) belonging to the curve, so the condition \( r_\alpha + r_\beta \geq n \) is satisfied. For instance, for GF\((2^2)\) the only possibility is \( r_\alpha = r_\beta = 1 \), and the only type of degeneration is \( \deg \alpha = \deg \beta = 2 \). For GF\((2^3)\) there are three possibilities: \( r_\alpha = r_\beta = 2 \) (\( \deg \alpha = \deg \beta = 2 \)), \( r_\alpha = 1, r_\beta = 2 \) (\( \deg \alpha = 4, \deg \beta = 2 \)), and \( r_\alpha = 2, r_\beta = 1 \) (\( \deg \alpha = 2, \deg \beta = 4 \)).

When a curve equation can be found [i.e., a relation of the type \( F(\alpha) = G(\beta) \), where \( F(\alpha) \) and \( G(\beta) \) are polynomials of degrees \( 2^{(r_\alpha-1)} \) and \( 2^{(r_\beta-1)} \)], it establishes a direct correspondence between the roots of \((5.1)\). To define uniquely the curve, the equation \( F(\alpha) = G(\beta) \) should be supplemented with the structural equations. Nevertheless, such a relation exists only when the conditions \( r_{\alpha,\beta} \geq (n+1)/2 \) hold.

There are two ways of approaching the classification of exceptional curves. The first is a direct analysis of the parametric form \((3.1)\) [whose coefficients satisfy the commutativity relation \((3.4)\) and the corresponding determinants vanish]. We have to determine the rank of the matrices \( W_\alpha \) and \( W_\beta \), find the structural relations \((5.1)\), and check the nonsingularity condition. The main difficulty with this approach is the complicated form of \((3.4)\), which is related to that fact that there is no one-to-one correspondence between the parametric form of a curve and points in the discrete phase space, in the sense that the same curve can be defined by several different parametric equations.

We shall take an alternative route and construct all the possible exceptional curves by imposing \textit{ab initio} the nonsingularity and commutativity conditions. An important ingredient in this construction is the existence of \textit{a priori} information about the degree of degeneration in the \( \alpha \) and \( \beta \) directions. We shall outline the main idea and study in detail only the eight-dimensional case.

### 5.1. Constructing exceptional curves

Let us consider a nonsingular curve with degenerations \( 2^{n-r_\alpha} \) and \( 2^{n-r_\beta} \) along the \( \alpha \) and \( \beta \) axes, respectively. The structural equations \((5.1)\) can be represented as

\[
\alpha \prod_{j=1}^{2^{\alpha-1}} (\alpha + \alpha_j) = 0, \quad \beta \prod_{j=1}^{2^{\beta-1}} (\beta + \beta_j) = 0,
\]

with all the roots \( \alpha_j \) and \( \beta_j \) different. Since only the powers \( \alpha^{2^m} \) and \( \beta^{2^m} \) \((m = 0, \ldots, r_{\alpha,\beta})\) can appear in \((5.2)\), we obtain the following restriction on the roots \( \alpha_j \) and \( \beta_j \)

\[
S_k(\alpha_j) = 0, \quad S_k(\beta_j) = 0, \quad k \neq 2^{r_{\alpha,\beta}} - 2^m
\]

where

\[
S_k(\xi) = \sum_{i_1 < i_2 < \ldots < i_k} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k},
\]
are symmetrical functions of the roots. This restriction implies that only \( r_\alpha \) \( (r_\beta) \) roots \( \alpha_j \) \( (\beta_j) \) are linearly independent.

Condition (5.2) implies that, given a degree of degeneration and once one of the structural equations is fixed, the other structural equation is not arbitrary. In other words, having determined the admissible points along one of the axis, all the admissible points of an additive commutative curve along the other axis are uniquely defined.

The simplest situation corresponds to the case when degeneration along both axes is the same: \( \deg \alpha = \deg \beta = g = 2^{n-r} \) \((r \geq n/2)\), i.e., the ranks of the corresponding matrices (3.11) are \( r = r_\alpha = r_\beta \). Suppose that the structural equation for \( \alpha \) is fixed. Then, the commutativity condition is equivalent to the following set of equations

\[
\text{tr}(\alpha_k \beta_j) = 0, \quad \left\{ \begin{array}{l}
j = 1, \ldots, n-r, \quad k = 1, \ldots, r, \\
j = 1, \ldots, r, \quad k = 1, \ldots, n-r,
\end{array} \right.
\]

(5.5)

where \( (\alpha_k, \beta_k) \) are linearly independent roots and the points \((\alpha_j, 0)\) and \((0, \beta_j)\), with \( j = 1, \ldots, 2^{n-r} \), belong to the curve.

Equations (5.5) mean that (5.2) can be always written as \( \text{tr}(\alpha \xi) = 0, \text{tr}(\beta \zeta) = 0 \), with \( \xi \) and \( \zeta \) being fixed elements of \( GF(2^n) \).

In the doubly degenerate case, \( g = 2 \), the value of \( \beta_1 \) is uniquely determined from the first condition in (5.5) and the curve can be represented as a disjoint union of two straight lines

\[
\beta^{(1)} = \frac{\beta_1}{\alpha_1} \alpha, \quad \beta^{(2)} = \frac{\beta_1}{\alpha_1} \alpha + \beta_1,
\]

(5.6)

with \( \alpha = \alpha_1, \ldots, \alpha_{2r-1} \). It is worth noting that (5.5) in this case is just a structural equation, so that

\[
\beta_1 = \frac{v_1}{v_0} = \frac{S_{2r-2}}{S_{2r-1}},
\]

(5.7)

where \( v_k \) are the coefficients in (5.1) and \( S_r \) are the symmetrical functions (5.3) of arguments \( \alpha_k \) \((k = 1, \ldots, 2^r - 1)\). Each different ordered set \( \{\alpha_1, \ldots, \alpha_r\} \), with \( \alpha_j \neq \alpha_k \neq 0 \) determines thus an exceptional doubly degenerate curve.

For higher degenerations, the curve is a disjoint union of \( g \) straight lines

\[
\beta^{(1)} = \lambda \alpha, \quad \beta^{(2)} = \lambda \alpha + \beta_1, \ldots, \beta^{(g)} = \lambda \alpha + \beta_{g-1},
\]

(5.8)

where \( \beta_m \) \((m = r + 1, \ldots, g - 1)\) are obtained as the possible different linear combinations of \( \beta_j \) \((j = 1, \ldots, r)\) and

\[
\lambda = \frac{\beta_1}{\alpha_1} = \ldots = \frac{\beta_{r}}{\alpha_r}.
\]

(5.9)

The above ordering indicates that the points \((\alpha_j, \beta_j)\) belong to the same straight line. Then, \( \beta_k = \beta_1 \alpha_k / \alpha_1 \) \((k = 2, \ldots, r)\) and \( \beta_1 \) are uniquely expressed in terms of admissible values of \( \alpha_j \) from (5.5), which is convenient to rewrite in terms of the parameter \( \lambda \): \( \text{tr}(\lambda \alpha_j \alpha_k) = 0 \), for \( j, k = 1, \ldots, r \). It is clear that the exceptional curves constructed using the equations (5.6) to (5.8) are nonsingular.

If the degeneration along \( \alpha \) and \( \beta \) axes are different (say, \( r_\alpha > r_\beta \)), then the curves can be represented as a collection of nonintersecting “parallel” curves

\[
\beta^{(1)} = f(\alpha), \quad \beta^{(2)} = f(\alpha) + \delta_1, \ldots, \beta^{(2^{n-r_\alpha})} = f(\alpha) + \delta_{2^{n-r_\alpha}-1},
\]

(5.10)
where \( f(\alpha) \) is the function

\[
f(\alpha) = \sum_{k=0}^{n-r-1} f_k \alpha^k,
\]

and the commutativity condition leads to the restrictions \( \text{tr}(\delta_i \alpha_j) = 0 \), which fix the values of \( \delta_i \).

The intersection problem can be studied using the same criterion as for the regular curves, taking into account that those conditions should be satisfied only at the admissible points of the curve.

5.2. Examples

5.2.1. Four-dimensional case

In the case of \( \text{GF}(2^2) \) the only exceptional curves are doubly degenerate, \( r_\alpha = r_\beta = 1 \). Besides, the structural equation is of second order: \( \alpha(\alpha + \alpha_1) = 0 \), so that any one of the three possible exceptional curves can be represented as a union of straight lines:

\[
\beta^{(1)} = \alpha_1^{-2} \alpha, \quad \beta^{(2)} = \alpha_1^{-2} \alpha + \alpha_1^{-1},
\]

with \( \alpha = 0, \alpha_1 \). More explicitly, for a given value of \( \alpha_1 \) we have the following curve:

\[
(0,0), (\alpha_1, \alpha_1^{-1}), (0, \alpha_1^{-1}), (\alpha_1, 0).
\]

In this case it is impossible to write down an equation that relates \( \alpha \) and \( \beta \).

As an example, consider the curve

\[
(0,0), (\sigma, \sigma^2), (0, \sigma^2), (\sigma, 0),
\]

where \( \alpha_1 = \sigma \). The structural equations are then

\[
\alpha^2 = \sigma \alpha, \quad \beta^2 = \sigma^2 \beta.
\]

The implication of the curve type on the factorization of the basis will be discussed in section 7.

5.2.2. Eight-dimensional case

Two types of exceptional curves exist in the case of \( \text{GF}(2^3) \): (i) doubly degenerate in both directions, which corresponds to \( r_\alpha = r_\beta = 2 \); (ii) doubly degenerate in one direction and quadruply degenerate in the other, which corresponds to \( r_\alpha = 2, r_\beta = 1 \) or \( r_\alpha = 1, r_\beta = 2 \).

In the case (i), any exceptional curve can be represented as a union of two lines

\[
\beta^{(1)} = \beta_1 \alpha_1^{-1} \alpha, \quad \beta^{(2)} = \beta_1 (\alpha_1^{-1} \alpha + 1),
\]

where, according to equation 5.14,

\[
\beta_1 = \frac{1}{\alpha_1 + \alpha_2} + \frac{1}{\alpha_1} + \frac{1}{\alpha_2},
\]

and the admissible values of \( \alpha \) are \( 0, \alpha_1, \alpha_2, \) and \( \alpha_1 + \alpha_2 \). For a fixed set \( \{\alpha_1, \alpha_2\} \) the following exceptional curve is defined
6. Local transformations

Local transformations induce nontrivial transformations in the curve, although they preserve the factorization properties (in a given basis). We recall that in any selfdual basis we can represent the monomial $Z_{\alpha}X_{\beta}$ in the following way

$$Z_{\alpha}X_{\beta} = \bigotimes_{j=1}^{n} \sigma^{a_j} \sigma^{b_j} = \bigotimes_{j=1}^{n} (a_j, b_j),$$

(6.1)

From the above equation we find that there are 21 exceptional curves due to the permutational symmetry between $\alpha_2$ and $\alpha_1 + \alpha_2$.

As an example, consider

$$(0, 0), (\alpha_1, 0), (\alpha_1, \beta_1), (\alpha_2, \beta_1(\alpha_1^{-1}\alpha_2 + 1)), (\alpha_2, \beta_1\alpha_1^{-1}\alpha_2),$$

$$(\alpha_1 + \alpha_2, \beta_1\alpha_1^{-1}\alpha_2), (\alpha_1 + \alpha_2, \beta_1(\alpha_1^{-1}\alpha_2 + 1)), (0, \beta_1).$$

(5.18)

where $\alpha_1 = \sigma^4, \alpha_2 = \sigma^3$. The structural equations are

$$\sigma^6\alpha + \sigma^4\alpha^2 + \alpha^4 = 0, \quad \sigma^2\beta + \sigma^6\beta^2 + \beta^4 = 0,$$

(5.20)

and the curve has the form

$$\beta^2 + \sigma^5\beta = \sigma^6\alpha + \sigma^2\alpha^2.$$

(5.21)

In the case (ii), one of coordinates (say, for instance, $\alpha$) is still doubly degenerate, while the other one is quadruply degenerate. Then, the coordinate $\beta$ takes only two values: 0 and $\delta$, while the allowed values of $\alpha$ are 0, $\alpha_1$, $\alpha_2$, and $\alpha_1 + \alpha_2$, so such a curve has the form

$$\beta^{(1)} = f(\alpha), \quad \beta^{(2)} = f(\alpha) + \delta,$$

(5.22)

where

$$f(\alpha) = \frac{\delta}{\alpha_2(\alpha_1 + \alpha_2)(\alpha_1\alpha + \alpha^2)},$$

(5.23)

and $\delta$ satisfies $\text{tr}(\delta \alpha_{1,2}) = 0$, which leads to $\delta = \beta_1$.

Explicitly, the points of such a curve are

$$(0, 0), (\alpha_1, 0), (\alpha_2, 0), (\alpha_1 + \alpha_2, 0), (\alpha_1, \delta), (\alpha_2, \delta), (\alpha_1 + \alpha_2, \delta), (0, \delta),$$

(5.24)

so there are seven different curves of this type due to permutational symmetry between $\alpha_1, \alpha_2$, and $\alpha_1 + \alpha_2$.

As an example of such a degenerate curve, consider the points

$$(0, 0), (\sigma^3, 0), (\sigma^5, 0), (\sigma^2, 0), (\sigma^3, \sigma^6), (\sigma^5, \sigma^6), (\sigma^2, \sigma^6), (0, \sigma^6),$$

(5.25)

where $\alpha_1 = \sigma^3, \alpha_2 = \sigma^5$. The corresponding structural equations are

$$\sigma^3\alpha + \sigma^2\alpha^2 + \alpha^4 = 0, \quad \sigma^6\beta + \beta^2 = 0.$$

(5.26)

6. Local transformations

Local transformations induce nontrivial transformations in the curve, although they preserve the factorization properties (in a given basis). We recall that in any selfdual basis we can represent the monomial $Z_{\alpha}X_{\beta}$ in the following way

$$Z_{\alpha}X_{\beta} = \bigotimes_{j=1}^{n} \sigma^{a_j} \sigma^{b_j} = \bigotimes_{j=1}^{n} (a_j, b_j),$$

(6.1)
where \( \otimes \prod_j \) denotes the tensor product over the index \( j \). Under local transformations (rotations of angle \( \pi/2 \) around \( z, x, \) and \( y \) axes, which we call \( z-, x- \) and \( y- \)rotations) applied to the \( j \)th particle, it transforms as

\[
\begin{align*}
    z : (a_j, b_j) &\mapsto (a_j + b_j, b_j), \\
x : (a_j, b_j) &\mapsto (a_j, b_j + a_j), \\
y : (a_j, b_j) &\mapsto (a_j + a_j + b_j, b_j + a_j + b_j) = (b_j, a_j).
\end{align*}
\]

(6.2)

To give a concrete example, suppose we consider a \( z \)-rotation. The operator \( \sigma_z \), corresponding to \((a_j = 1, b_j = 0)\), is transformed into \((a_j = 1 + 0 = 1, b_j = 0)\); i.e., into itself, while the operator \( \sigma_x \), corresponding to \((a_j = 0, b_j = 1)\), is mapped onto \((a_j = 0 + 1 = 1, b_j = 1)\), which coincides with \( \sigma_y \). In the same way \( \sigma_y \) is mapped onto \( \sigma_x \), while the identity \((a_j = 0, b_j = 0)\) is mapped onto itself.

In terms of the field elements it is equivalent to

\[
\begin{align*}
    z : &\quad \alpha \mapsto \alpha' = \alpha + \sum_k \theta_k \text{tr}(\beta \theta_k), \\
    \beta \mapsto \beta' = \beta, \\
    x : &\quad \alpha \mapsto \alpha' = \alpha, \\
    \beta \mapsto \beta' = \beta + \sum_k \theta_k \text{tr}(\alpha \theta_k), \\
    y : &\quad \alpha \mapsto \alpha' = \alpha + \sum_k \theta_k \text{tr}[\alpha + \beta] \theta_k], \\
    \beta \mapsto \beta' = \beta + \sum_k \theta_k \text{tr}[\alpha + \beta] \theta_k],
\end{align*}
\]

(6.3)

where \( \theta \) is the selfdual basis.

These transformations are nonlinear in the field elements: starting with a standard set of MUB operators related to rays, we obtain another set of MUB operators parametrized with points of curves, but leading to the same factorization structure. Indeed, consider a ray as in equation (2.14). Then, under \( z-, x- \), and \( y- \)rotations we have

\[
\begin{align*}
    z : &\quad \alpha \mapsto \alpha' = \mu \kappa + \sum_{m=0}^{n-1} \kappa^{2m} \nu^m \sum_k \theta_k^{2m+1}, \\
    \beta \mapsto \beta' = \nu \kappa, \\
    x : &\quad \alpha \mapsto \alpha' = \mu \kappa, \\
    \beta \mapsto \beta' = \nu \kappa + \sum_{m=0}^{n-1} \kappa^{2m} \mu^2 \sum_k \theta_k^{2m+1}, \\
    y : &\quad \alpha \mapsto \alpha' = \mu \kappa + \sum_{m=0}^{n-1} \kappa^{2m} (\nu + \mu)^2 \sum_k \theta_k^{2m+1} \\
    \beta \mapsto \beta' = \nu \kappa + \sum_{m=0}^{n-1} \kappa^{2m} (\nu + \mu)^2 \sum_k \theta_k^{2m+1}
\end{align*}
\]

(6.4)
Table 1
Curves generated by applying the rotations indicated in the left column to the ray $\beta = 0$ in the case of two qubits.

| Rotation | Curves |
|----------|--------|
| $x \otimes 1$ | $\beta = \sigma \alpha + \alpha^2$, $\beta^2 = \sigma^2 \beta$ |
| $y \otimes 1$ | $\sigma \beta + \beta^2 = \sigma^2 \alpha + \alpha^2$, $\alpha^2 = \sigma^2 \alpha$, $\beta^2 = \sigma \beta$ |
| $1 \otimes x$ | $\beta = \sigma^2 \alpha + \alpha^2$, $\beta^2 = \sigma \beta$ |
| $1 \otimes y$ | $\sigma^2 \beta + \beta^2 = \sigma \alpha + \alpha^2$, $\alpha^2 = \sigma \alpha$, $\beta^2 = \sigma^2 \beta$ |
| $x \otimes x$ | $\beta = \alpha$ |
| $y \otimes y$ | $\beta = \sigma^2 \alpha + \alpha^2$, $\beta^2 = \sigma \beta$ |
| $x \otimes y$ | $\sigma \beta + \sigma^2 \beta = \alpha + \sigma^2 \alpha$, $\alpha^2 = \sigma \alpha$, $\beta^2 = \sigma \beta$ |
| $y \otimes x$ | $\alpha = \sigma \beta + \beta^2$, $\alpha^2 = \sigma^2 \alpha$ |

Note that the $z$- and $x$-rotations produce regular curves

$$z : \alpha = \mu \nu^{-1} \beta \mapsto \alpha' = \mu \nu^{-1} \beta + \sum_{m=0}^{n-1} \beta^2 \sum_{k} \theta^2_{k+m+1}, \quad (6.5)$$

$$x : \beta = \mu^{-1} \nu \alpha \mapsto \beta' = \mu^{-1} \nu \alpha + \sum_{m=0}^{n-1} \alpha^2 \sum_{k} \theta^2_{k+m+1}. \quad (6.6)$$

Meanwhile, the $y$-rotation may lead to exceptional curves. In this case we always have $\kappa = (\mu + \nu)^{-1}(\alpha + \beta)$, and thus the explicit equation of that curve is either

$$\alpha = \mu (\nu + \mu)^{-1}(\alpha + \beta) + \sum_{m=0}^{n-1} (\alpha + \beta)^2 \sum_{k} \theta^2_{k+m+1}, \quad (6.6)$$

or

$$\beta = \nu (\nu + \mu)^{-1}(\alpha + \beta) + \sum_{m=0}^{n-1} (\alpha + \beta)^2 \sum_{k} \theta^2_{k+m+1}. \quad (6.7)$$

For instance, in the two-qubit case, starting from the ray $\beta = 0$, we can generate all the curves shown in table 1. In particular, it can be proven that in there are only two equivalence classes of curves [65].

7. Factorization structure and curve bundles

In this section we discuss bundles leading to MUBs with different factorization structures. As we have stated before, given a basis in the field, any operator, labeled by a point of a curve, is factorized into a product of one-particle Pauli operators. For qubit systems the selfdual is the most appropriate, for the Fourier operator is factorized, and thus, the factorization of $Z_\alpha$ and $X_\alpha$ is straightforward. Now, let us divide each monomial $Z_\alpha X_\beta$, into two parts, so that the first part contains $k$ Pauli operators and the second part $n - k$ operators. If any first “block” of the set of $d - 1$ commuting generalized Pauli operators commutes with all the other “blocks”, we will say that the corresponding curve is factorized into two sets. Obviously, the second blocks would
then also commute between them. Moreover, inside the first or second blocks may exist some “sub-blocks” that commute with corresponding sub-blocks, etc. In the end, we can represent any curve \( \Gamma \in \text{GF}(2^n) \) in the following factorized form:

\[
\Gamma = \{m_1, m_2, \ldots, m_N\}, \quad 0 < m_1 \leq m_2 \leq \ldots \leq m_N, \quad \sum_i m_i = n, \tag{7.1}
\]

where \( m_i \in \mathbb{N} \) is the number of particles in the \( i \)-th block that cannot be factorized anymore. It is clear that \( \{m_1, m_2, \ldots, m_N\} \) is just a partition of the integer \( n \), so the maximum number of terms is \( n \), which corresponds to a completely factorized curve, \( \Gamma = \underbrace{1, 1, \ldots, 1}_n \), and the minimum number of terms is one, corresponding to a completely nonfactorized curve, \( \Gamma = \{n\} \).

One can construct bundles that contain only regular curves, as it was shown in section 4. A systematic construction of bundles containing both regular and exceptional curves is a more involved task, which can be carried out numerically for low dimensions.

As an example, consider the ray \( \beta = 0 \) over \( \text{GF}(2^2) \), so the corresponding set of operators is \( \{Z_\sigma, Z_{\sigma^2}, Z_{\sigma^3}\} \). In the selfdual basis \( (\sigma, \sigma^2) \) this set is factorized into \( (\sigma z_1, 11z_2, \sigma z_2) \). Then, the curve \( \beta = 0 \) is represented as \( \Gamma = \{1, 1, 1\} \), i.e., both particles are factorized. The ray \( \beta = \sigma \alpha \), whose points label the set \( \{Z_\sigma X_{\sigma^2}, Z_{\sigma^2} X_{\sigma^3}, Z_{\sigma^3} X_\sigma\} \), has the following factorization \( (\sigma z_2 \sigma x_2, \sigma z_2 \sigma x_2, \sigma y \sigma z_x) \), so that it can be represented as \( \Gamma = \{2\} \), which means that there are no factorized blocks. In the case of three qubits the possible partitions are \( \{1, 1, 1\} \) (e.g., the ray \( \beta = 0 \)), \( \{1, 2\} \) (e.g., the regular curve \( \beta = \sigma^3 \alpha^2 + \sigma^4 \alpha^3 \)), and \( \{3\} \) (e.g., the ray \( \beta = \sigma^3 \alpha \)).

The representation \( \{7.1\} \) is invariant under local transformations. The corresponding basis preserves the factorization of the operator set, and local transformations preserve the factorization structure of the curve. This means that all the completely factorized curves can be obtained from a single factorized ray, say \( \beta = 0 \). Nevertheless, the curves with the same factorization structure are not necessarily equivalent under local transformations (except in the trivial two-qubit case \([65]\)).

A bundle may contain curves with different factorizations. We can characterize different bundles with a set of numbers that indicate the number of completely factorized curves \( \underbrace{n_1, 1, 1, \ldots, 1}_n \) (structure), completely factorized except a single two-particle block (curve of the type \( \underbrace{n, 1, 1, \ldots, 1, 1, 2\}}_{n-2} \)), etc., until completely nonfactorized curves \( \{n\} \). In other words, we assign to the bundle the set of numbers

\[
(k_1, k_2, \ldots, k_{p(n)}), \quad \sum_j k_j = 2^n + 1, \tag{7.2}
\]

which indicate the number of curves factorized in \( n \) one-dimensional blocks, \( k_1 \); the number of curves factorized in \( n - 2 \) one-dimensional blocks and one two-dimensional block, \( k_2 \), etc, and \( p(n) \) is the number of partitions of an integer \( n \).

7.1. Curves over \( \text{GF}(2^2) \)

As we have discussed, an additive commutative curve over \( \text{GF}(2^2) \) can be expressed as

\[
\alpha(\kappa) = \alpha_0 \kappa + \alpha_1 \kappa^2, \quad \beta(\kappa) = \beta_0 \kappa + \beta_1 \kappa^2, \tag{7.3}
\]
where the commutativity condition (3.3) impose the following restrictions on the coefficients \( \alpha_j \) and \( \beta_j \)

\[
\alpha_1 \beta_0 + (\alpha_1 \beta_0)^2 = \alpha_0 \beta_1 + (\alpha_0 \beta_1)^2. 
\] (7.4)

In this simple case, the whole analysis can be carried out from the parametric form. Nevertheless, it is more convenient to separate types of curves on regular and exceptional, according to our discussion in sections 4 and 5.

All the possible additive commutative structures can be divided into two types:

a) 12 regular curves, which can be constructed according to the general rule (4.3), among which there are four rays

\[
\beta = \lambda \alpha, \quad \alpha = 0, 
\] (7.5)

and 8 curves

\[
\alpha\text{-curves}: \beta = \eta \alpha + \alpha^2, \quad \beta\text{-curves}: \alpha = \eta \beta + \beta^2. 
\] (7.6)

b) 3 exceptional curves that can be represented as a union of two parallel lines (5.12) or in the parametric form

\[
\alpha = \mu (\kappa + \kappa^2), \quad \beta = \mu^2 (\sigma \kappa + \sigma^2 \kappa^2). 
\] (7.7)

Every point of these exceptional curves is doubly degenerate and the admissible values of \( \alpha \) and \( \beta \) are \( \{0, \mu\} \) and \( \{0, \mu^2\} \), respectively.

It is important to stress that it is possible to obtain all the curves of form (7.5), (7.6), and (7.7) from the rays after some (nonlinear) operations, corresponding to local transformations of operators [65]. The families of such transformations are the following: 8 curves (the rays \( \beta = \alpha \) and \( \beta = 0 \) among them) can be obtained from the single ray \( \alpha = 0 \) (corresponding to the vertical axis) and the other 5 curves (the ray \( \beta = \sigma^2 \alpha \) among them) from the ray \( \beta = \sigma \alpha \).

The simplest curve bundle contains just rays. There are three rays (\( \beta = \alpha, \beta = 0, \) and \( \alpha = 0 \)) with completely factorizable structure \( \{1, 1\} \) and two rays (\( \beta = \sigma \alpha \) and \( \beta = \sigma^2 \alpha \) with nonfactorizable structure \( \{2\} \). Since any other bundle can be obtained from the ray bundle by applying some local transformations, the only bundle structure is \( (3, 2) \), i.e., in any bundle there are three factorizable curves and two nonfactorizable (having EPR-states as basis states).

7.2. Curves over GF(2^3)

A generic additive commutative curve over GF(2^3) is given by

\[
\alpha = \alpha_0 \kappa + \alpha_1 \kappa^2 + \alpha_2 \kappa^4, \quad \beta = \beta_0 \kappa + \beta_1 \kappa^2 + \beta_2 \kappa^4. 
\] (7.8)

The commutative condition in this case is much more complicated than for GF(2^2), so a full analysis of all the possible curves becomes cumbersome if we start with (7.8). Instead, we can follow the procedure of sections 4 and 5.

A generic regular curves has always one of the following forms

\[
\beta = \phi_0 \alpha + \phi^2 \alpha^2 + \phi \alpha^4, \quad \alpha = \psi_0 \beta + \psi^2 \beta^2 + \psi \beta^4. 
\] (7.9)

for \( \alpha \)- and \( \beta \)-curves, respectively. All in all, we get 100 different regular curves. There are 21 doubly degenerate exceptional curves of the form (5.18) and 14 exceptional curves (5.24), which are quadruply degenerate in one direction and doubly degenerate in the other.
The simplest way of forming bundles of commutative curves is given in (4.5). Then, four bundles of nine curves each
\[
\beta = \phi_0 \alpha + \phi^2 \alpha^2 + \phi \alpha^4, \quad \alpha = 0,
\] (7.10)
where \( \phi_0 \in \text{GF}(2^3) \) and \( \text{tr}(\phi) = 0 \), have the factorization structure \((3, 0, 6)\). The choice \( \phi = 0 \) leads to the ray structure (2.13). Among them, only three rays \( \beta = 0, \alpha = 0, \beta = \alpha \) have the structure \( \{1, 1, 1\} \), while the other six rays have the structure \( \{3\} \).

All the other bundles with \( \text{tr}(\phi) = 0 \) (i.e., \( \phi \) takes the values \( \sigma, \sigma^2, \) and \( \sigma^4 \)) can be generated from the bundle with \( \phi = 0 \) by applying local transformations. In particular, \( \phi = \sigma \) is generated by an \( x \)-rotation of the first qubit, \( \phi = \sigma^2 \) by an \( x \)-rotation of the second qubit, and \( \phi = \sigma^4 \) by an \( x \)-rotation of the first and second qubits.

Another four bundles of nine curves
\[
\beta = \phi_0 \alpha + \phi^2 \alpha^2 + \phi \alpha^4, \quad \alpha = 0,
\] (7.11)
where \( \phi_0 \in \text{GF}(2^3) \) and \( \text{tr}(\phi) = 1 \) generate all the structures \((1, 6, 2)\). For instance, in the bundle with \( \phi = 1 \) the curves with \( \phi_0 = 0, \sigma^7 = 1 \) have the factorization \( \{3\} \) and all the other values of \( \phi_0 \in \text{GF}(2^3) \) generate the curves with the factorization \( \{1, 2\} \). The ray \( \alpha = 0 \) has the factorization \( \{1, 1, 1\} \).

As in the previous case, all the bundles with \( \text{tr}(\phi) = 1 \) (\( \phi = \sigma^3, \sigma^5, \sigma^6 \)) can be obtained form the bundle with \( \phi = 1 \) by some local transformations: \( \phi = \sigma^k \alpha \) by an \( x \)-rotation of the first qubit, \( \phi = \sigma^0 \) by an \( x \)-rotation of the second qubit and \( \phi = \sigma^k \) by an \( x \)-rotation of the third qubits.

The aforementioned local rotations can be also viewed as a relabeling of the points of the curves \( \beta = \phi_0 \alpha + \phi^2 \alpha^2 + \phi \alpha^4 \) according to \( \alpha \mapsto \sigma^k \alpha \) and \( \beta \mapsto \sigma^{-k} \beta \), with \( k = 2, 4, 1 \), respectively. On the other hand, \( k = 3, 5, 6 \) correspond to nonlocal transformations and lead to the bundles of curves corresponding to the \((3, 0, 6)\) structure.

It is possible to obtain one more type of bundles with different factorization structure and constituted only by regular curves. To this end we recall that the nonintersection condition between two regular curves has the form
\[
(\phi_0 + \phi_0')^7 + \text{tr}[(\phi_0 + \phi_0')(\phi + \phi')] = 1,
\] (7.12)
so \( \phi_0 \) and \( \phi_0' \) never coincide. Now, let us take three nonintersecting regular curves satisfying (7.12)
\[
\text{tr}[(\phi_0 + \phi_0')(\phi + \phi')] = 0,
\]
\[
\text{tr}[(\phi_0 + \phi_0')(\phi' + \phi'')] = 0,
\]
\[
\text{tr}[(\phi_0 + \phi_0')(\phi + \phi'')] = 0,
\] (7.13)
and construct a set of curves according to
\[
\beta = (\phi_0^a + \phi_0^b) \alpha + (\phi^a + \phi^b)^2 \alpha^2 + (\phi^a + \phi^b) \alpha^4,
\] (7.14)
where \( \phi_0^a, \phi_0^b \) and \( \phi^a, \phi^b \) are coefficients of the previously defined curves. In this way we generate five additional new curves:
\[
\beta = (\phi_0 + \phi_0') \alpha + (\phi + \phi')^2 \alpha^2 + (\phi + \phi') \alpha^4,
\]
\[
\beta = (\phi_0 + \phi_0'') \alpha + (\phi + \phi'')^2 \alpha^2 + (\phi + \phi'') \alpha^4,
\]
\( \beta = (\phi' + \phi'')\alpha + (\phi' + \phi'')^2\alpha^2 + (\phi' + \phi'')\alpha^4, \) \hfill (7.15)

\( \beta = (\phi_0 + \phi'_0 + \phi''_0)\alpha + (\phi + \phi' + \phi'')^2\alpha^2 + (\phi + \phi' + \phi'')\alpha^4, \)

\( \beta = 0, \)

and one has to add the last curve \( \alpha = 0 \) to complete the set of nine curves.

We observe that the three “initial” curves can be obtained in the same way using some of the curves in (7.15). This implies that any curve constructed in this way should satisfy the condition

\[ \text{tr}(\phi_0 \phi) = 0. \] \hfill (7.16)

All these sets of curves lead to the factorization structure \((2, 3, 4)\). One example of such type of bundle is

\[
\begin{align*}
\{1, 1, 1\} &\mapsto \alpha = 0, \beta = 0 \\
\{1, 2\} &\mapsto \\
&\quad \beta = \sigma^6\alpha + \sigma^3\alpha^2 + \sigma^5\alpha^4, \\
&\quad \beta = \sigma^2\alpha + \sigma^5\alpha^2 + \sigma^6\alpha^4, \\
&\quad \beta = \sigma^4\alpha + \sigma^3\alpha^2 + \sigma^5\alpha^4 \\
&\quad \beta = \sigma^3\alpha, \\
\{3\} &\mapsto \\
&\quad \beta = \sigma^5\alpha + \sigma^5\alpha^2 + \sigma^6\alpha^4, \\
&\quad \beta = \sigma\alpha + \sigma^2\alpha^2 + \sigma\alpha^4, \\
&\quad \beta = \alpha + \sigma^2\alpha^2 + \sigma\alpha^4.
\end{align*}
\] \hfill (7.17)

There is one more type of bundles with the structure \((0, 9, 0)\), i.e., which contains only curves with the factorization \(\{1, 2\}\). Such bundles always contain exceptional curves. One example of those bundles is

a) Regular curves

\[
\begin{align*}
\alpha &= \sigma^2\beta + \sigma^3\beta^2 + \sigma^5\beta^4, \quad \alpha = \sigma^6\beta + \sigma^3\beta^2 + \sigma^5\beta^4, \\
\beta &= \sigma^2\alpha + \sigma^3\alpha^2 + \sigma^5\alpha^4, \quad \beta = \sigma^6\alpha + \sigma^3\alpha^4, \\
\alpha &= \beta + \sigma^6\beta^2 + \sigma^3\beta^4, \quad \beta = \alpha + \sigma^3\alpha^2 + \sigma^5\alpha^4, \\
\alpha &= \sigma^3\beta^2 + \sigma^5\beta^4.
\end{align*}
\] \hfill (7.18)

b) Exceptional curves

\[
\begin{align*}
\beta^2 + \sigma^5\beta &= \sigma^2\alpha^2 + \sigma^6\alpha, \quad \text{tr}(\sigma^4\beta) = 0, \quad \text{tr}(\sigma^5\alpha) = 0; \\
\beta^2 + \sigma^2\beta &= \sigma^6\alpha^2 + \sigma^5\alpha, \quad \text{tr}(\sigma^6\beta) = 0, \quad \text{tr}(\sigma^2\alpha) = 0.
\end{align*}
\] \hfill (7.19)

where the structural equations are written in the trace form.

The rays with \(\{1, 2\}\) factorization structure cannot be obtained from rays by local transformations. Moreover, not all the curves with factorization \(\{3\}\) can be obtained from the rays of the same type.
All these structures can be obtained from each other by nonlocal transformations, which can be always reduced to a combination of CNOT gates and local transformations. To each bundle with a given factorization structure corresponds a set of nonlocal transformations preserving such structure. Such a symmetry can be used to determine the optimum tomographic procedure and will be discussed elsewhere.

The phase-space approach presented here also provides an alternative to the graph-state classification of all the possible stabilizer states for \( n \)-qubit systems. In fact, each additive curve represents a basis in the \( 2^n \) dimensional Hilbert space, so that the stabilizer state is one element of such basis. Not each curve can be directly associated with a graph state [70], but it can be reduced to an appropriate graph state through local Clifford transformations [71]. While the classification of the graph states represents a formidable task for large numbers of qubits, the phase-space approach allows working with algebraic structures. Although the local equivalence is still an open problem, at least we can determine some elements of equivalence classes under local Clifford transformations in a relatively simple form. Besides, several nonlocal qubit operations as SWAP or CNOT gates can be nicely represented in terms of curves transformations. The above-mentioned problems will be analyzed in future work.

8. Conclusions

It has relatively recently been realized that several different types of MUBs, with respect to their factorization properties, exist for a system of \( n \) qubits. Such bases are related to different arrangements of generalized Pauli monomials into sets of commuting operators. The construction of a whole set of MUBs is an involved problem, especially for large number of qubits. The simplest MUB construction was discovered by Wootters and it is related to straight lines in the discrete phase space. We have shown that all the other MUBs are connected with a special type of discrete curves. Although, in principle, we can classify all the possible additive commutative curves and even determine which are related through local transformations, arranging them in bundles of nonintersecting curves is still an involved problem. Nevertheless, we can find some of such bundles according to a “recipe” [specifically equations (4.5) and (4.8)], which represents an essential progress in this field.

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Appendix A. Galois fields

In this appendix we briefly recall the minimum background needed in this paper. The reader interested in more mathematical details is referred, e.g., to the excellent monograph by Lidl and Niederreiter [69].
A commutative ring is a set $R$ equipped with two binary operations, called addition and multiplication, such that it is an Abelian group with respect to the addition, and the multiplication is associative. Perhaps, the motivating example is the ring of integers $\mathbb{Z}$ with the standard sum and multiplication. On the other hand, the simplest example of a finite ring is the set $\mathbb{Z}_n$ of integers modulo $n$, which has exactly $n$ elements.

A field $F$ is a commutative ring with division, that is, such that 0 does not equal 1 and all elements of $F$ except 0 have a multiplicative inverse (note that 0 and 1 here stand for the identity elements for the addition and multiplication, respectively, which may differ from the familiar real numbers 0 and 1). Elements of a field form Abelian groups with respect to addition and multiplication (in this latter case, the zero element is excluded).

The characteristic of a finite field is the smallest integer $p$ such that

$$p^1 = 1 + 1 + \ldots + 1 = 0$$ (A.1)

and it is always a prime number. Any finite field contains a prime subfield $\mathbb{Z}_p$ and has $d = p^n$ elements, where $n$ is a natural number. Moreover, the finite field containing $p^n$ elements is unique and is called the Galois field $\text{GF}(p^n)$.

Let us denote as $\mathbb{Z}_p[x]$ the ring of polynomials with coefficients in $\mathbb{Z}_p$. Let $P(x)$ be an irreducible polynomial of degree $n$ (i.e., one that cannot be factorized over $\mathbb{Z}_p$). Then, the quotient space $\mathbb{Z}_p[x]/P(x)$ provides an adequate representation of $\text{GF}(p^n)$. Its elements can be written as polynomials that are defined modulo the irreducible polynomial $P(x)$. The multiplicative group of $\text{GF}(p^n)$ is cyclic and its generator is called a primitive element of the field.

As a simple example of a nonprime field, we consider the polynomial $x^2 + x + 1 = 0$, which is irreducible in $\mathbb{Z}_2$. If $\sigma$ is a root of this polynomial, the elements $\{0, 1, \sigma, \sigma^2 = \sigma + 1 = \sigma^{-1}\}$ form the finite field $\text{GF}(2^2)$ and $\sigma$ is a primitive element.

The map $\alpha \mapsto \alpha^p$, where $\alpha \in \text{GF}(p^n)$ is a linear automorphism of $\text{GF}(p^n)$: $(\alpha + \beta)^n = \alpha^n + \beta^n$, and $(\alpha \beta)^n = \alpha^n \beta^n$. It is called the Frobenius automorphism and will be represented in the form

$$S^k(\alpha) = \alpha^{p^k}.$$ (A.2)

The elements of the prime field are invariant under action of the this automorphism.

Another basic map is the trace

$$\text{tr}(\alpha) = \alpha + \alpha^2 + \ldots + \alpha^{p^n-1} = \sum_{k=0}^{n-1} S^k(\alpha),$$ (A.3)

which satisfies

$$\text{tr}(\alpha + \beta) = \text{tr}(\alpha) + \text{tr}(\beta),$$ (A.4)

and also leaves the prime field invariant. In terms of it we define the additive characters as

$$\chi(\alpha) = \exp \left[ \frac{2\pi i}{p} \text{tr}(\alpha) \right],$$ (A.5)

and posses two important properties:

$$\chi(\alpha + \beta) = \chi(\alpha) \chi(\beta),$$

$$\sum_{\alpha \in \text{GF}(p^n)} \chi(\alpha \beta) = p^n \delta_{0,\beta},$$ (A.6)
Any finite field $\text{GF}(p^n)$ can be also considered as an $n$-dimensional linear vector space. Given a basis $\{\theta_k\}$, $(k = 1, \ldots, n)$ in this vector space, any field element can be represented as

$$\alpha = \sum_{k=1}^{n} a_k \theta_k,$$
(A.7)

with $a_k \in \mathbb{Z}_p$. In this way, we map each element of $\text{GF}(p^n)$ onto an ordered set of natural numbers $\alpha \Leftrightarrow (a_1, \ldots, a_n)$.

Two bases $\{\theta_1, \ldots, \theta_n\}$ and $\{\theta'_1, \ldots, \theta'_n\}$ are dual when

$$\text{tr}(\theta_k \theta'_l) = \delta_{k,l}.$$  
(A.8)

A basis that is dual to itself is called selfdual.

There are several natural bases in $\text{GF}(p^n)$. One is the polynomial basis, defined as

$$\{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}\},$$
(A.9)

where $\sigma$ is a primitive element. An alternative is the normal basis, constituted of

$$\{\sigma, \sigma^p, \ldots, \sigma^{p^{n-1}}\}.$$ 
(A.10)

The choice of the appropriate basis depends on the specific problem at hand. For example, in $\text{GF}(2^2)$ the elements $\{\sigma, \sigma^2\}$ are both roots of the irreducible polynomial. The polynomial basis is $\{1, \sigma\}$ and its dual is $\{\sigma^2, 1\}$, while the normal basis $\{\sigma, \sigma^2\}$ is selfdual.

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