Distributions of countable models of disjoint unions of Ehrenfeucht theories

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Abstract

We describe Rudin–Keisler preorders and distribution functions of numbers of limit models for disjoint unions of Ehrenfeucht theories. Decomposition formulas for these distributions are found.

Keywords: disjoint union of theories, Ehrenfeucht theory, distribution of countable models, decomposition formula.

In [1], a description is obtained for distributions of countable models of quite o-minimal Ehrenfeucht theories in terms of Rudin–Keisler preorders and distribution functions of numbers of limit models. In the present paper, using a general theory of classification of countable models of complete theories [2, 3] as well as the description [1] of specificity for quite o-minimal theories, we describe distributions of countable models of disjoint unions of Ehrenfeucht theories in terms of Rudin–Keisler preorders and distribution functions of numbers of limit models. Besides, we derive decomposition formulas for these distributions.

1 Preliminaries

In this section we give the necessary information from [2, 3].

Recall that the number of pairwise non-isomorphic models of theory $T$ and of cardinality $\lambda$ is denoted by $I(T, \lambda)$.

Definition 1.1 [4] A theory $T$ is called Ehrenfeucht if $1 < I(T, \omega) < \omega$.

Definition 1.2 [5] Type $p(\bar{x}) \in S(T)$ is said to be powerful in a theory $T$ if every model $\mathcal{M}$ of $T$ realizing $p$ also realizes every type $q \in S(T)$, that is, $\mathcal{M} \models S(T)$.

Since for any type $p \in S(T)$ there exists a countable model $\mathcal{M}$ of $T$, realizing $p$, and the model $\mathcal{M}$ realizes exactly countably many types, the availability of a powerful type implies that $T$ is small, that is, the set $S(T)$ is countable. Hence for any type $q \in S(T)$ and its realization $\bar{a}$, there exists a prime model $\mathcal{M}(\bar{a})$ over $\bar{a}$, i. e., a model of $T$ containing $\bar{a}$ with $\mathcal{M}(\bar{a}) \models q(\bar{a})$ and such that $\mathcal{M}(\bar{a})$ is elementarily embeddable to any model realizing the type $q$. Since all prime models over realizations of $q$ are isomorphic, we denote these models by $\mathcal{M}_q$. Models $\mathcal{M}_q$ are called almost prime or q-prime.

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Definition 1.3 [2 6 7] Let $p$ and $q$ be types in $S(T)$. We say that the type $p$ is dominated by a type $q$, or $p$ does not exceed $q$ under the Rudin–Keisler preorder (written $p \leq_{RK} q$), if $M_q = p$, that is, $M_p$ is an elementary submodel of $M_q$ (written $M_p \leq M_q$). Besides, we say that a model $M_p$ is dominated by a model $M_q$, or $M_p$ does not exceed $M_q$ under the Rudin–Keisler preorder, and write $M_p \leq_{RK} M_q$.

Syntactically, the condition $p \leq_{RK} q$ (and hence also $M_p \leq_{RK} M_q$) is expressed thus: there exists a formula $\varphi(\bar{x}, \bar{y})$ such that the set $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\}$ is consistent and $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\} \models p(\bar{x})$. Since we deal with a small theory (there are only countably many types over any tuple $\bar{a}$ and so any consistent formula with parameters in $\bar{a}$ is deducible from a principal formula with parameters in $\bar{a}$), $\varphi(\bar{x}, \bar{y})$ can be chosen so that for any formula $\psi(\bar{x}, \bar{y})$, the set $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}$ being consistent implies that $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\} \models \psi(\bar{x}, \bar{y})$. In this event the formula $\varphi(\bar{x}, \bar{y})$ is said to be $(p, q)$-principal.

Definition 1.4 [2 6 7] Types $p$ and $q$ are said to be domination-equivalent, realization-equivalent, Rudin–Keisler equivalent, or $RK$-equivalent (written $p \sim_{RK} q$) if $p \leq_{RK} q$ and $q \leq_{RK} p$. Models $M_p$ and $M_q$ are said to be domination-equivalent, Rudin–Keisler equivalent, or $RK$-equivalent (written $M_p \sim_{RK} M_q$).

As in [8], types $p$ and $q$ are said to be strongly domination-equivalent, strongly realization-equivalent, strongly Rudin–Keisler equivalent, or strongly $RK$-equivalent (written $p \equiv_{RK} q$) if for some realizations $\bar{a}$ and $\bar{b}$ of $p$ and $q$ respectively, both $tp(\bar{b}/\bar{a})$ and $tp(\bar{a}/\bar{b})$ are principal. Models $M_p$ and $M_q$ are said to be strongly domination-equivalent, strongly Rudin–Keisler equivalent, or strongly $RK$-equivalent (written $M_p \equiv_{RK} M_q$).

Clearly, domination relations form preorders, and (strong) domination-equivalence relations are equivalence relations. Here, $M_p \equiv_{RK} M_q$ implies $M_p \sim_{RK} M_q$.

If $M_p$ and $M_q$ are not domination-equivalent then they are non-isomorphic. Moreover, non-isomorphic models may be found among domination-equivalent ones.

In Ehrenfeucht examples, models $M_{p_0}, \ldots, M_{p_{n-3}}$ are domination-equivalent but pairwise non-isomorphic.

A syntactic characterization for the model isomorphism between $M_p$ and $M_q$ is given by the following proposition. It asserts that the existence of an isomorphism between $M_p$ and $M_q$ is equivalent to the strong domination-equivalence of these models.

Proposition 1.5 [2 7] For any types $p(\bar{x})$ and $q(\bar{y})$ of a small theory $T$, the following conditions are equivalent:

1. the models $M_p$ and $M_q$ are isomorphic;
2. the models $M_p$ and $M_q$ are strongly domination-equivalent;
3. there exist $(p, q)$- and $(q, p)$-principal formulas $\varphi_{p,q}(\bar{y}, \bar{x})$ and $\varphi_{q,p}(\bar{x}, \bar{y})$ respectively, such that the set

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi_{p,q}(\bar{y}, \bar{x}), \varphi_{q,p}(\bar{x}, \bar{y})\}$$

is consistent;
4. there exists a $(p, q)$- and $(q, p)$-principal formula $\varphi(\bar{x}, \bar{y})$, such that the set

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\}$$

is consistent.
Definition 1.6 \[2,7\] Denote by $\text{RK}(T)$ the set $\text{PM}$ of isomorphism types of models $\mathcal{M}_p$, $p \in S(T)$, on which the relation of domination is induced by $\leq_{\text{RK}}$, a relation deciding domination among $\mathcal{M}_p$, that is, $\text{RK}(T) = (\text{PM}; \leq_{\text{RK}})$. We say that isomorphism types $\tilde{M}_1, \tilde{M}_2 \in \text{PM}$ are domination-equivalent (written $\tilde{M}_1 \sim_{\text{RK}} \tilde{M}_2$) if so are their representatives.

Clearly, the preordered set $\text{RK}(T)$ has a least element, which is an isomorphism type of a prime model.

Proposition 1.7 \[2,7\] If $I(T, \omega) < \omega$ then $\text{RK}(T)$ is a finite preordered set whose factor set $\text{RK}(T)/\sim_{\text{RK}}$, with respect to domination-equivalence $\sim_{\text{RK}}$, forms a partially ordered set with a greatest element.

Definition 1.8 \[2,3,7,9\] A model $\mathcal{M}$ of a theory $T$ is called limit if $\mathcal{M}$ is not prime over tuples and $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_n$ for some elementary chain $(\mathcal{M}_n)_{n \in \omega}$ of prime models of $T$ over tuples. In this case the model $\mathcal{M}$ is said to be limit over a sequence $\mathbf{q}$ of types or $\mathbf{q}$-limit, where $\mathbf{q} = (q_n)_{n \in \omega}$, $\mathcal{M}_n = \mathcal{M}_{q_n}$, $n \in \omega$. If the sequence $\mathbf{q}$ contains unique type $q$ then the $\mathbf{q}$-limit model is called limit over the type $q$.

Denote by $I_p(T, \omega)$ the number of pairwise non-isomorphic countable models of the theory $T$, each of which is prime over a tuple, by $I_l(T)$ the number of limit models of $T$, and by $I_l(T, q)$ the number of limit models over a type $q \in S(T)$.

Definition 1.9 \[3,9\] A theory $T$ is called $p$-categorical (respectively, $l$-categorical, $p$-Ehrenfeucht, and $l$-Ehrenfeucht) if $I_p(T, \omega) = 1$ (respectively, $I_l(T) = 1$, $1 < I_p(T, \omega) < \omega$, $1 < I_l(T) < \omega$).

Clearly, a small theory $T$ is $p$-categorical if and only if $T$ countably categorical, and if and only if $I_l(T) = 0$; $T$ is $p$-Ehrenfeucht if and only if the structure $\text{RK}(T)$ finite and has at least two elements; and $T$ is $p$-Ehrenfeucht with $I_l(T) < \omega$ if and only if $T$ is Ehrenfeucht.

Let $\tilde{\mathcal{M}} \in \text{RK}(T)/\sim_{\text{RK}}$ be the class consisting of isomorphism types of domination-equivalent models $\mathcal{M}_{p_1}, \ldots, \mathcal{M}_{p_n}$. Denote by $\text{IL}(\tilde{\mathcal{M}})$ the number of equivalence classes of models each of which is limit over some type $p_i$.

Theorem 1.10 \[2,7\] For any countable complete theory $T$, the following conditions are equivalent:

1. $I(T, \omega) < \omega$;
2. $T$ is small, $|\text{RK}(T)| < \omega$ and $\text{IL}(\tilde{\mathcal{M}}) < \omega$ for any $\tilde{\mathcal{M}} \in \text{RK}(T)/\sim_{\text{RK}}$.

If (1) or (2) holds then $T$ possesses the following properties:

(a) $\text{RK}(T)$ has a least element $\mathcal{M}_0$ (an isomorphism type of a prime model) and $\text{IL}(\tilde{\mathcal{M}}_0) = 0$;
(b) $\text{RK}(T)$ has a greatest $\sim_{\text{RK}}$-class $\tilde{\mathcal{M}}_1$ (a class of isomorphism types of all prime models over realizations of powerful types) and $|\text{RK}(T)| > 1$ implies $\text{IL}(\tilde{\mathcal{M}}_1) \geq 1$;
(c) if $|\tilde{\mathcal{M}}| > 1$ then $\text{IL}(\tilde{\mathcal{M}}) \geq 1$.

Moreover, the following decomposition formula holds:

$$I(T, \omega) = |\text{RK}(T)| + \sum_{i=0}^{(|\text{RK}(T)/\sim_{\text{RK}}| - 1)} \text{IL}(\tilde{\mathcal{M}}_i),$$

(1)

where $\tilde{\mathcal{M}}_0, \ldots, \tilde{\mathcal{M}}_{|\text{RK}(T)/\sim_{\text{RK}}| - 1}$ are all elements of the partially ordered set $\text{RK}(T)/\sim_{\text{RK}}$.  

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In Figure 1, a and b, possible variants for Hasse diagrams of Rudin–Keisler preorders ≤_{RK} and values of distribution functions IL of numbers of limit models on ∼_{RK}-equivalence classes are represented for the cases I(T, ω) = 3 and I(T, ω) = 4. In Figure 2, corresponding configurations for I(T, ω) = 5 are shown.

**Definition 1.11** [10] The disjoint union \( \bigsqcup_{n \in \omega} M_n \) of pairwise disjoint structures \( M_n \) for pairwise disjoint predicate languages \( \Sigma_n, n \in \omega \), is the structure of language \( \bigsqcup_{n \in \omega} \Sigma_n \cup \{ P_n^{(1)} \mid n \in \omega \} \) with the universe \( \bigsqcup_{n \in \omega} M_n \), \( P_n = M_n \), and interpretations of predicate symbols in \( \Sigma_n \) coinciding with their interpretations in \( M_n \), \( n \in \omega \). The disjoint union of theories \( T_n \) for pairwise disjoint languages \( \Sigma_n \) accordingly, \( n \in \omega \), is the theory

\[
\bigcup_{n \in \omega} T_n \models \text{Th} \left( \bigcup_{n \in \omega} M_n \right),
\]

where \( M_n \models T_n \), \( n \in \omega \).

Clearly, the theory \( T_1 \sqcup T_2 \) does not depend on choice of disjoint union \( M_1 \sqcup M_2 \) of models \( M_1 \models T_1 \) and \( M_2 \models T_2 \). Besides, the cardinality of RK\( (T_1 \sqcup T_2) \) is equal to the product of
cardinalities for $\text{RK}(T_1)$ and $\text{RK}(T_2)$, and the relation $\leq_{\text{RK}}$ on $\text{RK}(T_1 \sqcup T_2)$ equals the Pareto relation [11] defined by preorders in $\text{RK}(T_1)$ and $\text{RK}(T_2)$. Indeed, each type $p(\bar{x})$ of $T_1 \sqcup T_2$ is isolated by set consisting of some types $p_1(\bar{x}^1)$ and $p_2(\bar{x}^2)$ of theories $T_1$ and $T_2$ respectively, as well as of formulas $P^1(x^1_i)$ and $P^2(x^2_i)$ for all coordinates in tuples $\bar{x}^1$ and $\bar{x}^2$. For types $p(\bar{x})$ and $p'(\bar{y})$ of $T_1 \sqcup T_2$, we have $p(\bar{x}) \leq_{\text{RK}} p'(\bar{y})$ if and only if $p_1(\bar{x}^1) \leq_{\text{RK}} p'_1(\bar{y}^1)$ (in $T_1$) and $p_2(\bar{x}^2) \leq_{\text{RK}} p'_2(\bar{y}^2)$ (in $T_2$).

Thus, the following proposition holds.

**Proposition 1.12** [3] [12] For any small theories $T_1$ and $T_2$ of disjoint predicate languages $\Sigma_1$ and $\Sigma_2$ respectively, the theory $T_1 \sqcup T_2$ is mutually $\text{RK}$-coordinated with respect to its restrictions to $\Sigma_1$ and $\Sigma_2$. The cardinality of $\text{RK}(T_1 \sqcup T_2)$ is equal to the product of cardinalities for $\text{RK}(T_1)$ and $\text{RK}(T_2)$, i.e.,

$$I_p(T_1 \sqcup T_2, \omega) = I_p(T_1, \omega) \cdot I_p(T_2, \omega),$$

and the relation $\leq_{\text{RK}}$ on $\text{RK}(T_1 \sqcup T_2)$ equals the Pareto relation defined by preorders in $\text{RK}(T_1)$ and $\text{RK}(T_2)$.

**Remark 1.13** [3] [12] An isomorphism of limit models of theory $T_1 \sqcup T_2$ is defined by isomorphisms of restrictions of these models to the sets $P_1$ and $P_2$. In this case, a countable model is limit if and only if some its restriction (to $P_1$ or to $P_2$) is limit and the following equality holds:

$$I(T_1 \sqcup T_2, \omega) = I(T_1, \omega) \cdot I(T_2, \omega).$$

Thus, the operation $\sqcup$ preserves both $p$-Ehrenfeucht and $l$-Ehrenfeucht (if components are $p$-Ehrenfeucht), and, by [3], we obtain the equality

$$I_l(T_1 \sqcup T_2) = I_l(T_1) \cdot I_p(T_2, \omega) + I_p(T_1, \omega) \cdot I_l(T_2) + I_l(T_1) \cdot I_l(T_2).$$

### 2. Distributions of countable models

In this section, using Theorem [1.10] and Proposition [1.12] we give a description of Rudin–Keisler preorders and distribution functions of numbers of limit models for disjoint unions $T_1 \sqcup T_2$ of Ehrenfeucht theories $T_1$ and $T_2$, as well as propose representations of this distributions, based on the decomposition formula [11].

Using the formulas (1)–(4) we obtain the following equalities:

$$I(T_1, \omega) \cdot I(T_2, \omega) = I(T_1 \sqcup T_2, \omega) = I_p(T_1 \sqcup T_2, \omega) + I_l(T_1 \sqcup T_2) =$$

$$= I_p(T_1, \omega) \cdot I_p(T_2, \omega) + I_l(T_1 \sqcup T_2) = I_l(T_1) \cdot I_p(T_2, \omega) + I_p(T_1, \omega) \cdot I_l(T_2) + I_l(T_1) \cdot I_l(T_2)$$

implying

$$I(T_1, \omega) \cdot I(T_2, \omega) = I_p(T_1, \omega) \cdot I_p(T_2, \omega) + I_l(T_1) \cdot I_p(T_2, \omega) + I_p(T_1, \omega) \cdot I_l(T_2) + I_l(T_1) \cdot I_l(T_2).$$

In view of Proposition [1.12] the Hasse diagrams for distributions of countable models for disjoint unions $T_1 \sqcup T_2$ of Ehrenfeucht theories $T_1$ and $T_2$ are constructed as images of Pareto relations for these theories $T_1$ and $T_2$. Here, $\sim_{\text{RK}}$-equivalent vertices for $T_1$ and $T_2$ are transformed to $\sim_{\text{RK}}$-equivalent pairs for $T_1 \sqcup T_2$. Hence, each $\sim_{\text{RK}}$-class for $T_1$, consisting of
$k$ vertices, united with a $\sim_{\text{RK}}$-class for $T_2$, consisting of $m$ vertices, produces a $\sim_{\text{RK}}$-class $\tilde{z}$ for $T_1 \sqcup T_2$, consisting of $km$ vertices. Thus, in the formula (1), the value $I_p(T_1, \omega) \cdot I_p(T_2, \omega)$ is represented as a sum of products $|\tilde{x}| \cdot |\tilde{y}|$ for each equivalence class $\tilde{x}$ in $\text{RK}(T_1)$ and each equivalence class $\tilde{y}$ in $\text{RK}(T_2)$:

$$I_p(T_1 \sqcup T_2, \omega) = I_p(T_1, \omega) \cdot I_p(T_2, \omega) = \sum_{\tilde{x} \in \text{RK}(T_1), \tilde{y} \in \text{RK}(T_2)} |\tilde{x}| \cdot |\tilde{y}|. \quad (7)$$

Following the formula (1), each $\sim_{\text{RK}}$-class $\tilde{z}$ has some number $I_l(\tilde{z})$ of limit models over types defining that class. This number is expressed by the numbers $I_l(\tilde{x})$ and $I_l(\tilde{y})$ of limit models for the $\sim_{\text{RK}}$-class $\tilde{x}$ in $\text{RK}(T_1)$ and the $\sim_{\text{RK}}$-class $\tilde{y}$ in $\text{RK}(T_2)$, generating $\tilde{z}$, by the following formula:

$$I_l(\tilde{z}) = I_l(\tilde{x}) \cdot |\tilde{y}| + |\tilde{x}| \cdot I_l(\tilde{y}) + I_l(\tilde{x}) \cdot I_l(\tilde{y}). \quad (8)$$

By (7) and (5), for the theory $T_1 \sqcup T_2$, the decomposition formula (1) has the following form:

$$I(T_1 \sqcup T_2, \omega) = \sum_{\tilde{x}, \tilde{y}} |\tilde{x}| \cdot |\tilde{y}| + \sum_{\tilde{x}, \tilde{y}} (I_l(\tilde{x}) \cdot |\tilde{y}| + |\tilde{x}| \cdot I_l(\tilde{y}) + I_l(\tilde{x}) \cdot I_l(\tilde{y})). \quad (9)$$

Notice that the graph $\Gamma$ for the Pareto relation correspondent to the Rudin–Keisler preorder of the theory $T_1 \sqcup T_2$ is represented as the product of the graphs $\Gamma_1$ and $\Gamma_2$ for the Rudin–Keisler preorders of the theories $T_1$ and $T_2$, and $\Gamma_1 \times \Gamma_2$ is a (Boolean) lattice if and only if $\Gamma_1$ and $\Gamma_2$ are (Boolean) lattices.

Thus, the following theorem holds, generalizing Theorem 24 in [1].

**Theorem 2.1** For any Ehrenfeucht theories $T_1$ and $T_2$ with graphs $\Gamma_1$ and $\Gamma_2$ of Rudin–Keisler preorders the theory $T_1 \sqcup T_2$ has the Rudin–Keisler preorder, represented by the product $\Gamma_1 \times \Gamma_2$, and the decomposition formula of the form (1). The structure $\Gamma_1 \times \Gamma_2$ for $T_1 \sqcup T_2$ forms a (Boolean) lattice if and only if $\Gamma_1$ and $\Gamma_2$ form (Boolean) lattices.

The following proposition shows that the function $I_l(\cdot)$ is monotone with respect to disjoint unions of Ehrenfeucht theories.

**Proposition 2.2** The functions $I_l(\tilde{x})$ and $I_l(\tilde{y})$ of numbers of limit models for $\sim_{\text{RK}}$-classes $\tilde{x}$ in $\text{RK}(T_1)$ and $\tilde{y}$ in $\text{RK}(T_2)$ monotonically increase (do not decrease) with respect to Rudin–Keisler preorders, having monotonically increasing (non-decreasing) cardinalities $|\tilde{x}|$ and $|\tilde{y}|$, if and only if the function $I_l(\tilde{z})$ of numbers of limit models for $\sim_{\text{RK}}$-classes $\tilde{z}$ in $\text{RK}(T_1 \sqcup T_2)$ monotonically increase (do not decrease) with respect to Rudin–Keisler preorder, having monotonically increasing (non-decreasing) cardinalities $|\tilde{z}|$.

Proof. Assume that the cardinalities $|\tilde{x}|$ and $|\tilde{y}|$ monotonically increase (do not decrease) with respect to Rudin–Keisler preorders. If the functions $I_l(\tilde{x})$ and $I_l(\tilde{y})$ monotonically increase (do not decrease) with respect to Rudin–Keisler preorders and $\tilde{z}_1 <_{\text{RK}} \tilde{z}_2$ ($\tilde{z}_1 \leq_{\text{RK}} \tilde{z}_2$), then $I_l(\tilde{z}_1) < I_l(\tilde{z}_2)$ ($I_l(\tilde{z}_1) \leq I_l(\tilde{z}_2)$) in view of the formula (5).

The reverse implication takes place, since $\text{RK}(T_1)$ and $\text{RK}(T_2)$ are isomorphic to substructures of the structure $\text{RK}(T_1 \sqcup T_2)$. □

In addition to the examples of Hasse diagrams given in [1], we have a series of new examples. Below we describe some of them.
Example 2.3 Consider the disjoint union of theory $T_1$ with the Hasse diagram shown in Fig. 1a and of theory $T_2$ with the first Hasse diagram shown in Fig. 1b. By Theorem 2.1 we have the theory $T_1 \sqcup T_2$ with $3 \cdot 4 = 12$ countable models, having the Boolean lattice with $2 \cdot 2 = 4$ prime models over finite sets and with 8 limit models. The decomposition formula (9) has the following form:

$$3 \cdot 4 = 4 + 8 = 2 \cdot 2 + (0 + 1 + 2 + (1 \cdot 1 + 1 \cdot 2 + 1 \cdot 2)).$$

The Hasse diagram for the theory $T_1 \sqcup T_2$ is shown in Fig. 3.

Replacing, respectively, 1 and 2 limit models of the theories $T_1$ and $T_2$ by $k > 0$ and $m > 0$ the equation (10) is transformed to the following:

$$(k + 2)(m + 2) = 2 \cdot 2 + (0 + k + m + (k + m + km)).$$

Example 2.4 Consider the disjoint union of theory $T_1 \sqcup T_2$ in Example 2.3 and of theory $T_3$ with the first Hasse diagram shown in Fig. 2. By Theorem 2.1 we have the theory $T_1 \sqcup T_2 \sqcup T_3$ with $3 \cdot 4 \cdot 5 = 60$ countable models, having the Boolean lattice with $2 \cdot 2 \cdot 2 = 8$ prime models over finite sets and with 52 limit models. The decomposition formula (9) has the following form:

$$3 \cdot 4 \cdot 5 = 8 + 52 = 2 \cdot 2 \cdot 2 + (0 + 1 + 2 + 3 + 5 + 7 + 11 + 23).$$

The Hasse diagram for the theory $T_1 \sqcup T_2 \sqcup T_3$ is shown in Fig. 4.

Replacing, respectively, 1, 2, 3 limit models of the theories $T_1$, $T_2$, $T_3$ by $k > 0$, $m > 0$, $n > 0$ the equation (11) is transformed to the following:

$$(k + 2)(m + 2)(n + 2) = 2 \cdot 2 \cdot 2 + (0 + k + m + n + (k + m + km) + (k + n + kn) + (m + n + mn) +
+(k + m + n + km + kn + mn + kmn)).$$

Example 2.5 Consider the disjoint union of theory $T_1$ with the third Hasse diagram shown in Fig. 1b and the theory $T_2$ with the second Hasse diagram shown in Fig. 2. By Theorem 2.1 we have the theory $T_1 \sqcup T_2$ with $4 \cdot 5 = 20$ countable models, having the lattice with
$3 \cdot 3 = 9$ prime models over finite sets and with 11 limit models. The decomposition formula \( (9) \) has the following form:

$$4 \cdot 5 = 9 + 11 = 3 \cdot 3 + (0 + 1 + 1 + 1 + 1 + 3 + 3).$$

The Hasse diagram for the theory $T_1 \sqcup T_2$ is shown in Fig. 5.

Example 2.6 Consider the disjoint union of theory $T_1 \sqcup T_2$ in Example 2.5 and the theory $T_3$ with the third Hasse diagram shown in Fig. 2. By Theorem 2.1 we have the theory $T_1 \sqcup T_2 \sqcup T_3$ with $4 \cdot 5 \cdot 5 = 100$ countable models, having the lattice with $3 \cdot 3 \cdot 3 = 27$ prime models over finite sets and with 73 limit models. The decomposition formula \( (9) \) has the following form:

$$4 \cdot 5 \cdot 5 = 27 + 73 = 3 \cdot 3 \cdot 3 + (1 \cdot 10 + 2 \cdot 2 + 3 \cdot 4 + 5 \cdot 5 + 11 \cdot 2).$$

The Hasse diagram for the theory $T_1 \sqcup T_2 \sqcup T_3$ is shown in Fig. 6.

Example 2.7 Consider the disjoint union of theory $T_1$ with the first Hasse diagram shown in Fig. 1 b and the theory $T_2$ with the second Hasse diagram shown in Fig. 2. By Theorem 2.1 we have the theory $T_1 \sqcup T_2$ with $4 \cdot 5 = 20$ countable models, having the lattice with $2 \cdot 3 = 6$ prime models over finite sets and with 14 limit models. The decomposition formula \( (9) \) has the following form:

$$4 \cdot 5 = 6 + 14 = 2 \cdot 3 + (0 + 1 + 1 + 2 + 5 + 5).$$

The Hasse diagram for the theory $T_1 \sqcup T_2$ is shown in Fig. 7.

Example 2.8 Consider the disjoint union of theory $T_1 \sqcup T_2$ in Example 2.7 and of the theory $T_3$ with the first Hasse diagram shown in Fig. 2. By Theorem 2.1 we have the theory $T_1 \sqcup T_2 \sqcup T_3$ with $4 \cdot 5 \cdot 5 = 100$ countable models, having the lattice with $2 \cdot 2 \cdot 3 = 12$ prime
models over finite sets and with 88 limit models. The decomposition formula (9) has the following form:

\[ 4 \cdot 5 \cdot 5 = 12 + 88 = 2 \cdot 2 \cdot 3 + (1 \cdot 2 + 2 \cdot 1 + 3 \cdot 1 + 5 \cdot 2 + 7 \cdot 2 + 11 \cdot 1 + 23 \cdot 2). \]

The Hasse diagram for the theory \( T_1 \sqcup T_2 \sqcup T_3 \) is shown in Fig. 8.

**Example 2.9** Consider the disjoint union of theory \( T_1 \) with the first Hasse diagram shown in Fig. 1 and of the theory \( T_2 \) with the third Hasse diagram shown in Fig. 2. By Theorem 2.1 we have the theory \( T_1 \sqcup T_2 \) with \( 4 \cdot 5 = 20 \) countable models, having the lattice with \( 2 \cdot 3 = 6 \) prime models over finite sets and with 14 limit models. The decomposition formula (9) has the following form:

\[ 4 \cdot 5 = 6 + 14 = 2 \cdot 3 + (2 \cdot 3 + 8). \]

The Hasse diagram for the theory \( T_1 \sqcup T_2 \) is shown in Fig. 9.

**Example 2.10** Consider the disjoint union of theory \( T_1 \sqcup T_2 \) in Example 2.9 and of the theory \( T_3 \) with the first Hasse diagram shown in Fig. 1. By Theorem 2.1 we have the theory \( T_1 \sqcup T_2 \sqcup T_3 \) with \( 4 \cdot 5 \cdot 5 = 100 \) countable models, having the lattice with \( 2 \cdot 2 \cdot 3 = 12 \) prime models over finite sets and with 88 limit models. The decomposition formula (9) has the following form:

\[ 4 \cdot 5 \cdot 5 = 12 + 88 = 2 \cdot 2 \cdot 3 + (2 \cdot 3 + 3 \cdot 2 + 11 \cdot 3 + 35). \]

The Hasse diagram for the theory \( T_1 \sqcup T_2 \sqcup T_3 \) is shown in Fig. 10.

**Example 2.11** Consider the disjoint union of theory \( T_1 \sqcup T_2 \) in Example 2.5 and of the theory \( T_3 \) with the first Hasse diagram shown in Fig. 1. By Theorem 2.1 we have the theory \( T_1 \sqcup T_2 \sqcup T_3 \) with \( 4 \cdot 4 \cdot 5 = 80 \) countable models, having the lattice with \( 3 \cdot 3 \cdot 2 = 18 \) prime models over finite sets and with 88 limit models.
Figure 9: Figure 10:

prime models over finite sets and with 62 limit models. The decomposition formula (9) has the following form:

\[ 4 \cdot 4 \cdot 5 = 18 + 62 = 3 \cdot 3 \cdot 2 + (1 \cdot 5 + 2 \cdot 2 + 3 \cdot 2 + 5 \cdot 5 + 11 \cdot 2). \]

The Hasse diagram for the theory \( T_1 \sqcup T_2 \sqcup T_3 \) is shown in Fig. 11.

Example 2.12 Consider the disjoint union \( T_1 \sqcup T_2 \sqcup T_3 \) of theory \( T_1 \) with the third Hasse diagram shown in Fig. 11, of theory \( T_2 \) with the first Hasse diagram shown in Fig. 2, and of theory \( T_3 \) with the third Hasse diagram shown in Fig. 2. By Theorem 2.1 we have the theory \( T_1 \sqcup T_2 \sqcup T_3 \) with 4 \cdot 5 \cdot 5 = 100 countable models, having the lattice with 3 \cdot 3 \cdot 2 = 18 prime models over finite sets and with 82 limit models. The decomposition formula (9) has the following form:

\[ 4 \cdot 4 \cdot 5 = 18 + 62 = 3 \cdot 3 \cdot 2 + (1 \cdot 2 + 2 \cdot 2 + 3 \cdot 2 + 5 \cdot 1 + 7 \cdot 2 + 11 \cdot 2 + 23 \cdot 1). \]

The Hasse diagram for the theory \( T_1 \sqcup T_2 \sqcup T_3 \) is shown in Fig. 12.

Example 2.13 Consider the disjoint union of theory \( T_1 \) with the second Hasse diagram shown in Fig. 11, b, where one limit model is replaced by \( k \) \( > \) 0 ones, and of theory \( T_2 \) with the Hasse diagram shown in Fig. 11, a, where one limit model is replaced by \( m \) \( > \) 0 ones. By Theorem 2.1 we have the theory \( T_1 \sqcup T_2 \sqcup T_3 \) with \( (k + 3)(m + 2) \) countable models, having the Hasse diagram with 3 \cdot 2 = 6 prime models over finite sets and with \( k + m + (k + 2m + km) \) limit models. The decomposition formula (9) has the following form:

\[ (k + 3)(m + 2) = 3 \cdot 2 + (k + m + (k + 2m + km)). \]

The Hasse diagram for the theory \( T_1 \sqcup T_2 \) is shown in Fig. 13.

Example 2.14 Consider the disjoint union of theory \( T_1 \) with the second Hasse diagram shown in Fig. 11, b, where one limit model is replaced by \( k \) \( > \) 0 ones, and of theory \( T_2 \) with
similar Hasse diagram, where one limit model is replaced by \( m > 0 \) ones. By Theorem 2.1 we have the theory \( T_1 \sqcup T_2 \) with \((k + 3)(m + 3)\) countable models, having the Hasse diagram with \( 3 \cdot 3 = 9 \) prime models over finite sets and with \( k + m + (2k + 2m + km) \) limit models. The decomposition formula (9) has the following form:

\[
(k + 3)(m + 3) = 3 \cdot 3 + (k + m + (2k + 2m + km)).
\]

The Hasse diagram for the theory \( T_1 \sqcup T_2 \) is shown in Fig. 14.

In the latter two examples quotients with respect to \( \sim_{RK} \) produce Boolean lattices with four elements.

References

[1] B.Sh. Kulpeshov, S.V. Sudoplatov, Distributions of countable models of quite o-minimal Ehrenfeucht theories // [arXiv:1802.08078v1 [math.LO]]. — 2018.
[2] S.V. Sudoplatov, Classification of countable models of complete theories, Part 1. — Novosibirsk : Novosibirsk State Technical University Publishing House, 2014.

[3] S.V. Sudoplatov, Classification of countable models of complete theories, Part 2. — Novosibirsk : Novosibirsk State Technical University Publishing House, 2014.

[4] T.S. Millar, Decidable Ehrenfeucht theories // Proc. Sympos. Pure Math., 1985, vol. 42, pp. 311–321.

[5] M. Benda, Remarks on countable models // Fund. Math., 1974, vol. 81, No. 2, pp. 107–119.

[6] D. Lascar, Ordre de Rudin–Keisler et poids dans les theories stables // Z. Math. Logic Grundlagen Math., 1982, vol. 28, pp. 413–430.

[7] S.V. Sudoplatov, Complete theories with finitely many countable models. I // Algebra and Logic, 2004, vol. 43, No. 1, pp. 62–69.

[8] P. Tanović, Theories with constants and three countable models // Archive for Math. Logic, 2007, vol. 46, No. 5–6, pp. 517–527.

[9] S.V. Sudoplatov, Hypergraphs of prime models and distributions of countable models of small theories // J. Math. Sciences, 2010, vol. 169, No. 5, pp. 680–695.

[10] R.E. Woodrow, Theories with a finite number of countable models and a small language. Ph. D. Thesis. — Simon Fraser University, 1976. — 99 p.

[11] S.V. Sudoplatov, E.V. Ovchinnikova, Discrete Mathematics: Textbook. — Moscow : Uralt, 2016–2018. — 280 p.

[12] S.V. Sudoplatov, Inessential combinations of small theories // Reports of Irkutsk State University. Series “Mathematics”, 2009, vol. 2, No. 2, pp. 158–169.