Private Heavy Hitters and Range Queries in the Shuffled Model

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Abstract

An exciting new development in differential privacy is the shuffled model, which makes it possible to circum-
vent the large error lower bounds that are typically incurred in the local model, while relying on much weaker
trust assumptions than in the central model. In this work, we study two basic statistical problems, namely, heavy
hitters and d-dimensional range counting queries, in the shuffled model of privacy. For both problems we devise
algorithms with polylogarithmic communication per user and polylogarithmic error; a direct consequence is an al-
gorithm for approximating the median with similar communication and error. These bounds significantly improve
on what is possible in the local model of differential privacy, where the error must provably grow polynomially
with the number of users.

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## Contents

1 Introduction
   1.1 Further related work ................................................................. 3

2 Preliminaries
   2.1 Shuffled model .............................................................................. 5
   2.2 Frequency oracle .......................................................................... 5

3 Frequency oracle and heavy hitters
   3.1 Protocol based on count-min sketch .................................................. 6
   3.2 Approximate differential privacy for insensitive functions ..................... 8
   3.3 Smoothness of binomial distribution .................................................. 9
   3.4 Privacy proof ............................................................................... 9
   3.5 Protocol based on Hadamard response ............................................. 10

4 Range queries
   4.1 Reduction to private frequency oracle via the matrix mechanism ............ 14
   4.2 Single-dimensional range queries ................................................... 18
   4.3 Multi-dimensional range queries .................................................... 20
   4.4 Guarantees for differentially private range queries .............................. 22

A Proofs of auxiliary lemmas from Section 3 ............................................. 25
1 Introduction

Consider a data set $x_1, \ldots, x_n$ distributed among $n$ users for which we wish to be able to compute or approximate some function $g(x_1, \ldots, x_n)$ without compromising the privacy of user data. Specifically, we consider the setting where $g$ comes from a family $Q$ of queries, and we wish to preprocess $x_1, \ldots, x_n$ in such a way that we can answer any query in $Q$ without further participation of the users. Let $[B] = \{1, \ldots, B\}$. We focus on two fundamental classes of queries:

- For data values $x_i \in [B]$, a frequency query $q_g(x_1, \ldots, x_n) = |\{i \mid y \in x_i\}|$ counts the number of occurrences of a particular value $y \in [B]$.

- For data values $x_i \in [B]^d$, a $d$-dimensional cube, a range counting query $q_R(x_1, \ldots, x_n) = |\{i \mid x_i \in R\}|$ determines how many users hold a vector in a given axis-aligned rectangle $R$.

We note that accurate frequency queries allow us to compute heavy hitters, i.e., the most frequent values among $x_1, \ldots, x_n$. Moreover, accurate range counting queries allow us to approximate the median of the input in the sense of $M$-estimation\[4\] i.e., finding a value $\hat{x}$ that approximately minimizes $\sum_i |x_i - \hat{x}|$.

Differential privacy is a well-studied framework for answering queries in a privacy-preserving manner. Given a family $Q$ of queries, differentially private query release [28] computes a single value $R(x_1, \ldots, x_n)$ such that for every $q \in Q$ we can compute an approximation of $q(x_1, \ldots, x_n)$ from $R(x_1, \ldots, x_n)$. A key insight in differential privacy is that it is often possible to sample $R(x_1, \ldots, x_n)$ from a distribution such that the statistical information about each user input $x_i$ that can be learned from $R(x_1, \ldots, x_n)$ is small. Specifically, the distribution of $R(x_1, \ldots, x_n)$ should remain roughly the same if we replace $x_i$ by some other value $x_i'$.

Most work in differential privacy has been done in the central model where some trusted entity curates all inputs and computes the value $R(x_1, \ldots, x_n)$. In privacy-sensitive settings, there may not exist a single entity that all users trust enough to surrender their data. For this reason, deployments of differential privacy [31, 22, 3] have been done in the so-called local model of differential privacy, where the computation of $R(x_1, \ldots, x_n)$ is distributed among users. More specifically, user $i$ computes a value $R'(x_i)$ dependent only on their own input and releases it, after which $R'(x_1), \ldots, R'(x_n)$ are combined by a central server to compute $R(x_1, \ldots, x_n)$. This means that the local computation at user $i$ determines how much information about $x_i$ is revealed. Since $R'(x_i)$ and $R'(x_i')$ need to have nearly identical distributions for any choice of $x_i'$, the output of $R'$ cannot depend too strongly on its input, which suggests the need for $R'$ to inject sufficient “noise” to obfuscate the input. For many kinds of queries, it is known that privacy is possible in the local model with additive error that grows sublinearly in $n$, often of order $O(\sqrt{n})$. Unfortunately, this tends to be significantly more than what is possible in the central model, with the result that $n$ needs to be very large to get similarly precise results.

Shuffled model. Bittau et al. [10] introduced the Encode-Shuffle-Analyze (ESA) framework as a “middle ground” between central and local differential privacy. The idea is to combine random shuffling of data with local randomness introduced by users. The shuffling of all user outputs (each user can send any number of messages to the shuffler) can be implemented in a way that does not expose user data, yet is practical, robust, and scalable for example using the cryptographic primitive of a mixnet (see the discussion in [16]). This has the effect of hiding the origin of each data item or anonymizing the outputs of local computations, which turns out to enable private protocols with error far lower than what is possible in the local model while not requiring one to place the same level of trust in the curator as in the central model. Cheu et al. [16] provided a formalization of the core components of the framework, which we refer to as the shuffled model, illustrated in Figure 1. The ESA framework can be viewed as a lightweight alternative to secure multi-party computation (MPC) protocols. We discuss these further in Section 1.1 but note for now that in contrast to the ESA model, such protocols rely on user participation for every query that has to be evaluated and generally do not protect against a series of adaptive queries revealing private information.

Our results. We present new algorithms for private frequency estimation (see Theorem 3.6) and range counting (see Theorems 4.9 and 4.10) in the shuffled model that significantly improve on what can be achieved in the local model of differential privacy. Our results are summarized in Table 1 focusing on communication requirements of

\[1\]There is no hope for a private algorithm that accurately estimates the value of the median, see [45, Section 3].
users, the size of the additive error on query answers, and the time required to answer a query (after creating a data structure based on the shuffled dataset).

To our best knowledge, the only previously-known upper bounds for these problems in the shuffled model (going beyond local differential privacy) followed via a reduction to private aggregation [16]. Using the currently best protocol for private aggregation in the shuffled model [4, 32] yields the result stated in the first row of Table 1.

Interestingly, our results are not far from the best results obtainable in the central model. For instance, the best possible error in the central model for frequency estimation is \( \Theta(\min(\log \frac{1}{\delta}, \log B)) \) [45]. For \( d \)-dimensional range counting, the best-known upper bound on error is \( (\log(B) + \log(n)^{O(d)})/\varepsilon \) [27], though we note that for \( d = 1 \), it is possible to get a much better dependence on \( B \) [12].

**Overview of techniques.** Our protocols use the privacy blanket tool of Balle et al. [5], which introduces noise by letting some users send uniformly random messages to the shuffler, forming a noise “blanket.” Informally, it suffices that each possible message output of the shuffler has a fair probability of having originated from the blanket, which ensues if the number of distinct messages is well below the blanket size.

For the frequency estimation problem with shared randomness (i.e., all users can make use of the same hash functions), our protocol is based on the count-min sketch [19], which can be efficiently implemented in the shuffled model. Indeed, all possible updates to a count-min sketch can be described using a small number of messages (indexes into the sketch) and we show that count-min frequency estimates are noise-robust and do not deteriorate when blanket noise is added to the sketch. For the frequency estimation problem without public randomness, we devise a protocol based on the Hadamard response technique [2, 1] but for the shuffled model. Compared to the local model, in which this technique was previously applied, it turns out that we can afford to include much more information in each message about a data item, again due to the privacy blanket.

Our protocol for range queries is based on a collection \( B \) of \( O(B \log^d B) \) \( d \)-dimensional rectangles in \( [B]^d \) with the property that an arbitrary rectangle can be formed as the disjoint union of \( O(\log^d B) \) rectangles from \( B \). Furthermore, each point in \( [B]^d \) is contained in \( O(\log^d B) \) rectangles from \( B \). The existence of such a collection is well-known in the literature on range reporting, e.g., one can take the Cartesian product of sets of dyadic intervals in each dimension. The idea is now to privately approximate the number of points in each rectangle in \( B \) using our frequency estimation protocol as a black box. To turn this into a protocol with small maximum communication in the shuffled model, we develop an approach analogous to the matrix mechanism [39, 40].

**Remark.** Some recent work [30, 5] has established generic results showing that shuffling provides privacy amplification of locally differentially private protocols. In particular, for an \( (\varepsilon, 0) \)-differentially private local randomizer, one can construct a protocol in the shuffled model that is \( \left( O \left( e^{\varepsilon \sqrt{\log(1/\delta)}}, \delta \right) \right) \)-differentially private and with the same level of accuracy. While one can combine this amplification result with local differentially private protocols to
obtain protocols in the shuffled model, there are two drawbacks. First, the resulting guarantees can be weaker since the amplification result holds only for relatively small values of \( \varepsilon \) (namely, \( \varepsilon = O(\log(n/\log(1/\delta))) \)) \cite{5}. Indeed, using the locally differentially private protocol RAPPOR \cite{31,2} and the privacy amplification result, one can obtain a frequency estimation protocol in the shuffled model with error \( \tilde{O}(n^{1/4}) \); in contrast, our protocol has error \( \tilde{O}(1) \). Second, the shuffled model protocols derived using privacy amplification are necessarily single-message protocols; we suspect that frequency estimation with error \( \tilde{O}(1) \) may not be possible in the single-message shuffled model.

1.1 Further related work

There have been several recent works that study differentially private algorithms in the central or local models for various statistical and learning tasks \cite{25,43,47,8,14,13,15}. More recently, the shuffled model \cite{10,16} has emerged as a middle ground between central and local differential privacy. A number of works have sought to formalize the trade-offs in the shuffled model with respect to standard local and central differential privacy \cite{30} as well as devise private schemes in this model for tasks such as secure aggregation \cite{4,16,5,32}. Our work is largely motivated by the aforementioned body of works demonstrating the power of the shuffled model, namely, its ability to enable private protocols with lower error than in the local model while placing less trust in a central server or curator.

The specific problems we consider in the shuffled model, namely, range counting, M-estimation of the median, heavy hitters, as well as tightly connected problems such as interquartile range, have been well motivated in the literature and studied in the context of differential privacy in the central and local models. Dwork and Lei \cite{23} initiated work on establishing a connection between differential privacy and robust statistics, and were able to provide differentially private estimators for median as well as interquartile range, \( \alpha \)-trimmed mean, and regression. Subsequently, Lei \cite{33} provided an approach for privately releasing a wide class of M-estimators (including for median) that are statistically consistent. While such M-estimators can also be obtained indirectly from non-interactive release of the density function \cite{47}, the aforementioned approach exhibits an improved rate of convergence. Furthermore, motivated by risk bounds under privacy constraints, Duchi et al. \cite{23} provided private versions of information-theoretic bounds for minimax risk of M-estimation of the median.

| Problem | Messages per user | Message size in bits | Error | Query time |
|---------|------------------|----------------------|-------|------------|
| Frequency estimation (private randomness) | \( B \) | \( \log B \) | \( \sqrt{(\log B) \log \frac{1}{\varepsilon}} \) (expected) | 1 |
| Frequency estimation (shared randomness) | \( \left( \frac{\log B}{\varepsilon} + \frac{\log \frac{1}{\varepsilon^2}}{\varepsilon} \right) \log B \) | \( \log n + \log \log B \) | \( \frac{\log B}{\varepsilon} + \frac{\log \frac{1}{\varepsilon^2}}{\varepsilon} \) | \( \log B \) |
| Frequency estimation (private randomness) | \( \frac{\log \frac{1}{\varepsilon^2}}{\varepsilon} \) | \( \log(n) \log B \) | \( \log B + \frac{\log(B) \log \frac{1}{\varepsilon^2}}{\varepsilon} \) | \( \frac{n(\log \frac{1}{\varepsilon^2}) \log(n) \log B}{\varepsilon^2} \) |
| \( d \)-dimensional range counting (shared) | \( \frac{\log^2(B)}{\varepsilon^2} + \frac{\frac{1}{\varepsilon^2}}{\varepsilon^2} \) \( \log B \) | \( \log n + \log \log B \) | \( \frac{\log(B) \log \frac{1}{\varepsilon^2}}{\varepsilon} \) \( \log B \) | \( \log^{d+1} B \) |
| \( d \)-dimensional range counting (private) | \( \frac{\log^2(B)}{\varepsilon^2} + \frac{\frac{1}{\varepsilon^2}}{\varepsilon^2} \) \( \log B \) | \( \log(n) \log B \) | \( \frac{\log^2(B) \log \frac{1}{\varepsilon^2}}{\varepsilon} \) | \( \frac{n \log^{d+1}(B) \log \frac{1}{\varepsilon^2}}{\varepsilon^2} \) |

Table 1: Overview of results on differentially private query release in the shuffled model. For frequency estimation each user is assumed to hold \( k = 1 \) value from \[B\]. The query time stated is the additional time to answer a query, assuming a preprocessing of the output of the shuffler that takes time linear in its length. Note that frequencies and counts are not normalized, i.e., they are integers in \( \{0, \ldots, n\} \). For simplicity, constant factors are suppressed, the bounds are stated for error probability \( \beta = B^{-O(1)} \), and the following are assumed: dimension \( d \) is a constant, \( n \) is bounded above by \( B \), and \( \delta < 1/\log B \).
Range counting queries have also been an important subject of study. Early works on differentially private histograms \cite{24,33} apply naturally to range queries, though the approach of summing up histogram entries yields large errors for longer range queries. Xiao et al. \cite{48} showed how to obtain differentially private range count queries by using Haar wavelets, while Hay et al. \cite{44} formalized the method of maintaining a hierarchical representation of data; the aforementioned two works were compared and refined by Qardaji et al. \cite{42}. Cormode et al. \cite{17} showed how to translate many of the previous ideas to the local model of differential privacy. The matrix mechanism of Li et al. \cite{39,40} also applies to the problem of range counting queries. An alternate line of work for tackling multi-dimensional range queries that relied on conceiving differentially private versions of k-d trees and quadtrees was presented by Cormode et al. \cite{20}. Finally, Bun et al. \cite{12} also considered the related problem of differentially private release for learning threshold functions.

Heavy hitters and frequency estimation, which we also consider in this work, have also been studied extensively in the standard models of differential privacy \cite{7,6,11,35,46,1}.

Secure multi-party computation. If we accept user involvement in the computation of each query \(q\), then there is a rich theory of secure multi-party computation (SMPC) that allows \(f(x_1,\ldots,x_n)\) to be computed without revealing anything about \(x_i\) except what can be inferred from \(f(x_1,\ldots,x_n)\) itself (see e.g., the book of Cramer et al. \cite{21}). Generic SMPC protocols have large overheads compared to the cost of computing \(f(x_1,\ldots,x_n)\) in a non-private setting. Kilian et al. \cite{37} studied SMPC protocols for heavy hitters, obtaining near-linear communication complexity. We are not aware of any work on optimizing SMPCs for range queries. Note that the fact that \(f(x_1,\ldots,x_n)\) is revealed means that SMPC protocols do not protect privacy in the strong sense required for differential privacy. In particular, it can be hard to quantify the privacy in situations where many SMPC computations are executed on the same data.

2 Preliminaries

For any positive integer \(B\), let \([B]\) = \(\{1,2,\ldots,B\}\). For any set \(\mathcal{Y}\), we denote by \(\mathcal{Y}^*\) the set consisting of sequences of elements of \(\mathcal{Y}\), i.e., \(\mathcal{Y}^* = \bigcup_{n\ge0} \mathcal{Y}^n\). Suppose \(S\) is a multiset whose elements are drawn from a set \(\mathcal{X}\). With a slight abuse of notation, we will write \(S \subseteq \mathcal{X}\) and for \(x \in \mathcal{X}\), we write \(m_S(x)\) to denote the multiplicity of \(x\) in \(S\). For an element \(x \in \mathcal{X}\) and a non-negative integer \(k\), let \(k \times \{x\}\) denote the multiset with \(k\) copies of \(x\) (e.g., \(3 \times \{x\}\) = \{x,x,x\}\).

We now introduce the basics of differential privacy that we will need. Fix a finite set \(\mathcal{X}\), the space of reports of users. A dataset is an element of \(\mathcal{X}^*\), namely a tuple consisting of elements of \(\mathcal{X}\). Let \(\text{hist}(X) \in \mathbb{N}[\mathcal{X}]\) be the histogram of \(X\): for any \(x \in \mathcal{X}\), the \(x\)th component of \(\text{hist}(X)\) is the number of occurrences of \(x\) in the dataset \(X\). We will consider datasets \(X, X'\) to be equivalent if they have the same histogram (i.e., the ordering of the elements \(x_1,\ldots,x_n\) does not matter). For a multiset \(S\) whose elements are in \(\mathcal{X}\), we will also write \(\text{hist}(S)\) to denote the histogram of \(S\) (so that the \(x\)th component is the number of copies of \(x\) in \(S\)).

Let \(n \in \mathbb{N}\), and consider a dataset \(X = (x_1,\ldots,x_n) \in \mathcal{X}^n\). For an element \(x \in \mathcal{X}\), let \(f_X(x) = \frac{\text{hist}(X)_x}{n}\) be the frequency of \(x\) in \(X\), namely the fraction of elements of \(X\) which are equal to \(x\). Two datasets \(X, X'\) are said to be neighboring if they differ in a single element, meaning that we can write (up to equivalence) \(X = (x_1,\ldots,x_{n-1},x_n)\) and \(X' = (x_1,\ldots,x_{n-1},x'_n)\). In this case, we write \(X \sim X'\). Let \(Z\) be a set; we now define the differential privacy of a randomized function \(P : \mathcal{X}^n \to Z\):

**Definition 2.1** (Differential privacy \cite{26,24}). A randomized algorithm \(P : \mathcal{X}^n \to Z\) is \((\varepsilon,\delta)\)-differentially private if for every pair of neighboring datasets \(X \sim X'\) and for every set \(S \subseteq Z\), we have

\[
P[P(X) \in S] \le e^\varepsilon \cdot P[P(X') \in S] + \delta,
\]

where the probabilities are taken over the randomness in \(P\). Here, \(\varepsilon \ge 0\), \(\delta \in [0,1]\).

We will use the following compositional property of differential privacy.
Lemma 2.1 (Post-processing, e.g., [29]). If P is \((\varepsilon, \delta)\)-differentially private, then for every randomized function A, the composed function \(A \circ P\) is \((\varepsilon, \delta)\)-differentially private.

For a positive real number \(a\), \(\log(a)\) denotes the logarithm base 2 of \(a\), and \(\ln(a)\) denotes the natural logarithm of \(a\).

2.1 Shuffled model

We briefly review the shuffled model of differential privacy introduced by Bittau et al. [10] and formalized by Cheu et al. [16]. The input to the model is a dataset \((x_1, \ldots, x_n) \in \mathcal{X}^n\), where the item \(x_i \in \mathcal{X}\) is held by user \(i\). A protocol in the shuffled model is the composition of three algorithms:

- The local randomizer \(R : \mathcal{X} \rightarrow \mathcal{Y}^*\) takes as input the data of one user, \(x_i \in \mathcal{X}\), and outputs a sequence \((y_{i,1}, \ldots, y_{i,m_i})\) of messages; here \(m_i\) is a positive integer.
- The shuffler \(S : \mathcal{Y}^* \rightarrow \mathcal{Y}^*\) takes as input a sequence of elements of \(\mathcal{Y}\), say \((y_1, \ldots, y_m)\), and outputs a random permutation, i.e., the sequence \((y_{\pi(1)}, \ldots, y_{\pi(m)})\), where \(\pi \in S_m\) is a uniformly random permutation on \([m]\). The input to the shuffler will be the concatenation of the outputs of the local randomizers.
- The analyzer \(A : \mathcal{Y}^* \rightarrow \mathcal{Z}\) takes as input a sequence of elements of \(\mathcal{Y}\) (which will be taken to be the output of the shuffler) and outputs an answer in \(\mathcal{Z}\) which is taken to be the output of the protocol \(P\).

We will write \(P = (R, S, A)\) to denote the protocol whose components are given by \(R, S,\) and \(A\). It is evident that the main distinction between the shuffled and local model is the introduction of the shuffler \(S\) between the local randomizer and the analyzer. Similar to the local model, in the shuffled model the analyzer is untrusted; hence privacy must be guaranteed with respect to the input to the analyzer, i.e., the output of the shuffler. Formally, we have:

Definition 2.2 (Differential privacy in the shuffled model, [16]). A protocol \(P = (R, S, A)\) is \((\varepsilon, \delta)\)-differentially private if, for any dataset \(X = (x_1, \ldots, x_n)\), the algorithm

\[
(x_1, \ldots, x_n) \rightarrow S(R(x_1), \ldots, R(x_n))
\]

is \((\varepsilon, \delta)\)-differentially private.

Notice that the output of \(S(R(x_1), \ldots, R(x_n))\) can be simulated by an algorithm that takes as input the multiset consisting of the union of the elements of \(R(x_1), \ldots, R(x_n)\) (which we denote as \(\bigcup_i R(x_i)\), with a slight abuse of notation) and outputs a uniformly random permutation of them. Thus, by Lemma 2.1, it can be assumed without loss of generality for privacy analyses that the shuffler simply outputs the multiset \(\bigcup_i R(x_i)\).

2.2 Frequency oracle

We now describe a basic data primitive that we study in this paper. Fix positive integers \(B\) and \(k \leq B\) as well as positive real numbers \(\alpha\) and \(\beta\). In the \((\alpha, \beta, k)\)-frequency oracle problem [36, 77], each user \(i \in [n]\) holds a subset \(S_i \subset [B]\) of size at most \(k\). Equivalently, user \(i\) holds the sum of the one-hot encodings of the elements of \(S_i\), i.e., the vector \(x_i \in \{0, 1\}^B\) such that \((x_i)_j = 1\) if and only if \(j \in S_i\). Note that \(\|x_i\|_1 \leq k\) for all \(i\).

The goal is to design a (possibly randomized) data structure \(\mathcal{F}O\) and a deterministic algorithm \(A\) (frequency oracle) that takes as input the data structure \(\mathcal{F}O\) and an index \(j \in [B]\), and outputs in time \(T\) an estimate \(\hat{x}_j \in \mathbb{R}\) that, with high probability, is within an additive \(\alpha\) from \(\sum_{i=1}^n (x_i)_j\). Formally:

Definition 2.3 ((\(\alpha, \beta, k\))-frequency oracle). A protocol with inputs \(x_1, \ldots, x_n \in \{0, 1\}^B\) computes an \((\alpha, \beta, k)\)-frequency oracle if it outputs a pair \((\mathcal{F}O, A)\) such that for all datasets \((x_1, \ldots, x_n)\),

\[
\mathbb{P} \left[ \forall j \in [B] : A(\mathcal{F}O, j) - \sum_{i=1}^n (x_i)_j \leq \alpha \right] \geq 1 - \beta.
\]

The probability in the above expression is over the randomness in creating the data structure \(\mathcal{F}O\).
Note that given such a frequency oracle, one can recover the $2\alpha$-heavy hitters, namely those $j$ such that $\sum_{i=1}^{n}(x_{i,j}) \geq 2\alpha$, in time $O(TB)$, by querying $A(\text{FO}, 1), \ldots, A(\text{FO}, B)$. (See the remark after Theorem 3.6.)

3 Frequency oracle and heavy hitters

Most prior work \cite{balle2017private, dwork2014our, raskhodnikova2010private} has focused on the case of computing heavy hitters when each user holds only a single element, i.e., the case $k = 1$ of the $(\alpha, \beta, k)$-frequency oracle. In this paper our focus is also primarily on the case when $k = 1$ and we do not try to optimize our algorithms for large $k$. However, in the application to computing range queries (Section 4), we will apply our protocols for a frequency oracle as a black box, and will need to deal with the case when $k$ may be larger than 1; in particular, $k$ will grow at most $\text{poly log}(n)$. When $k > 1$ and $\alpha = \Omega(n)$, the $(\alpha, \beta, k)$-frequency oracle problem has also been referred to as that of (differentially private) selection \cite{liu2015privacy}.

3.1 Protocol based on count-min sketch

In this subsection, we give a public-coin protocol for finding heavy hitters in the shuffled model with error $\text{poly log } n$ and communication per user $\text{poly log } n$ bits. Our protocol is based on combining the count-min sketch \cite{cormode2005data} and the privacy blanket \cite{balle2017private}.

Overview. We start by recalling the main idea behind the count-min sketch. Assume that each of $n$ users holds an input from $[B]$ where $n \ll B$. We hash the universe $[B]$ into $s$ buckets where $s = O(n)$. Then for each user, we increment the bucket to which its input hashes. This ensures that for every element of $[B]$, its hash bucket contains an overestimate of the number of users having that element as input. However, these bucket values are not enough to unambiguously recover the number of users holding any specific element of $[B]$—this is because on average, $B/s$ different elements hash to the same bucket. To overcome this, the count-min sketch repeats the above idea $\tau = \text{O}(\log B)$ times using independent hash functions.\footnote{It is possible to introduce a space-time trade-off here: By increasing $s$ we can decrease the number of repetitions needed. This will in turn improve privacy but increase space and communication requirements. For simplicity we present our results just for the case $s = O(n)$.} Doing so ensures that for each element $j \in [B]$, it is the case that (i) no other element $j' \in [B]$ hashes to the same buckets as $j$ for all $\tau$ repetitions, and (ii) for at least one repetition, no element of $[B]$ that is held by a user (except possibly $j$ itself) hashes to the same bucket as $j$. To make the count-min data structure differentially private, we use a privacy blanket as was done by Balle et al. \cite{balle2017private}. Specifically, we ensure that sufficient independent noise is added to each bucket of each repetition of the count-min data structure. This is done by letting each user independently increment every bucket with a very small probability. The full description of our protocol appears in Algorithm 1.

Analysis. We next state the lemmas summarizing the accuracy, efficiency, and privacy guarantees of Algorithm 1.

Lemma 3.1 (Accuracy of $P_{\text{CM}}$). Let $n$, $B$, and $\tau$ be positive integers and let $\gamma \in [0, 1]$ be a real parameter. Then, the estimate $\hat{x}_j$ produced by $Q_{\text{CM}}$ on input $j \in [B]$ and as an outcome of the shuffled-model protocol $P_{\text{CM}} = (R_{\text{CM}}, S, A_{\text{CM}})$ with input $x_1, \ldots, x_n \in \{0, 1\}^B$ ($\|x_i\|_1 \leq k$) satisfies $\hat{x}_j \geq \sum_{i=1}^{n} x_{i,j}$ and

$$\Pr \left[ \hat{x}_j - \sum_{i=1}^{n} x_{i,j} \leq O(\gamma n) \right] \geq 1 - (kn/s)^{\tau} - 2^{\Theta(\log(s \tau) - \gamma n)}.$$

Proof. We consider the entries $\{ C[t, h_t[j]] \mid t \in [\tau] \}$ of the noisy count-min data structure. We first consider the error due to the other inputs that are held by the users. Then, we consider the error due to the privacy blanket. We bound each of these two errors with high probability and then apply a union bound.

First, we note that for any element $j \in [B]$, the probability that for every repetition index $t \in [\tau]$, some element $j' \in [B]$ held by one of the users (except possibly $j$ itself) satisfies $h_t(j') = h_t(j)$, is at most $(kn/s)^{\tau}$.

It remains to show that with probability at least $1 - 2^{\Theta(\log(s \tau) - \gamma n)}$, the absolute value of the blanket noise in each of these entries is at most $O(\gamma n)$. By a union bound over all $s \tau$ pairs of bucket indices and repetition indices,
Algorithm 1: Local randomizer, analyzer and query for heavy hitter computation via count-min.

\[ R^{CM}(n, B, \tau, \gamma, s) : \]
\[ \text{Input:} \text{Subset } S \subseteq [B] \text{ specifying the user’s input set} \]
\[ \text{Parameters:} n, B, \tau, s \in \mathbb{N} \text{ and } \gamma \in [0, 1] \]
\[ \text{Public Randomness: A random hash family } \{ h_t : [B] \rightarrow [s], \forall t \in [\tau] \} \]
\[ \text{Output: A multiset } T \subseteq [\tau] \times [s] \]
\[ \text{for } j \in S \text{ do} \]
\[ \text{for } t \in [\tau] \text{ do} \]
\[ \text{Add the pair } (t, h_t(j)) \text{ to } T. \]
\[ \text{for } t \in [\tau] \text{ do} \]
\[ \text{for } \ell \in [s] \text{ do} \]
\[ \text{Sample } b_{t,\ell} \text{ from } \text{Ber}(\gamma). \]
\[ \text{if } b_{t,\ell} = 1 \text{ then} \]
\[ \text{Add the pair } (t, \ell) \text{ to } T. \]
\[ \text{return } T. \]

\[ A^{CM}(n, B, \tau, s) : \]
\[ \text{Input: Multiset } \{y_1, \ldots, y_m\} \text{ containing outputs of local randomizers} \]
\[ \text{Parameters:} n, B, \tau, s \in \mathbb{N} \]
\[ \text{Public Randomness: A random hash family } \{ h_t : [B] \rightarrow [s], \forall t \in [\tau] \} \]
\[ \text{Output: A noisy count-min data structure } C : [\tau] \times [s] \rightarrow \mathbb{N} \]
\[ \text{for } t \in [\tau] \text{ do} \]
\[ \text{for } \ell \in [s] \text{ do} \]
\[ C[t, \ell] = 0. \]
\[ \text{for } j \in [m] \text{ do} \]
\[ C[y_j] \leftarrow C[y_j] + 1. \]
\[ \text{return } C \]

\[ Q^{CM}(n, B, \tau, s) : \]
\[ \text{Input: Element } j \in [B] \]
\[ \text{Parameters:} n, B, \tau, s \in \mathbb{N} \]
\[ \text{Public Randomness: A random hash family } \{ h_t : [B] \rightarrow [s], \forall t \in [\tau] \} \]
\[ \text{Output: A non-negative real number which is an estimate of the frequency of element } j \]
\[ \text{return } \hat{x}_j := \min\{C[t, h_t(j)] : t \in [\tau]\} \]

it is enough to show that for each \( t \in [\tau] \) and each \( \ell \in [s] \), with probability at least \( 1 - 2^{-\Theta(\gamma n)} \), the absolute value of the blanket noise in \( C[t, h_t[j]] \) is at most \( O(\gamma n) \). This follows from the fact that the blanket noise in the entry \( C[t, h_t[j]] \) is the sum of \( n \) independent \( \text{Ber}(\gamma) \) random variables (one contributed by each user). The bound now follows from the multiplicative Chernoff bound.

Finally, by a union bound, we get that the overall error is at most \( O(\gamma n) \) with probability at least \( 1 - (kn/s)^\tau - 2^{\Theta(\log(s \tau) - \gamma n)} \).

Note that the protocol based on count-min never underestimates the count of an element. (However, this will not be the case for the protocol based on the Hadamard response (Section 3.5).)

Lemma 3.2 (Efficiency of \( P^{CM} \)). Let \( n, B, \tau, s \) be positive integers and \( \gamma \in [0, 1] \). Then,

1. With probability at least \( 1 - n \cdot 2^{-\Theta(s \tau)} \), the output of \( P^{CM}(n, B, \tau, \gamma, s) \) on input \( S \) consists of at most \(|S| + O(\gamma s \tau)\) messages each consisting of \( [\log_2(\tau)] + [\log_2(s)] \) bits.
2. The runtime of the analyzer $A^{\text{CM}}(n,B,\tau,s)$ on input \{y_1,\ldots,y_m\} is $O(\tau s + m)$ and the space of the data structure that it outputs is $O(\tau s \log m)$ bits.

3. The runtime of any query $Q^{\text{CM}}(n,B,\tau,s)$ is $O(\tau)$.

Proof. The second and third parts follow immediately from the operation of Algorithm \[\text{I}\] To prove the first part, we note that each user sends $|S|$ messages corresponding to its inputs along with a number of “blanket noise” terms. This number is a random variable drawn from the binomial distribution Bin$(\tau s, \gamma)$. Moreover, each of these messages is a pair consisting of a repetition index (belonging to $[\tau]$) and a bucket index (belonging to $[s]$). The proof now follows from the multiplicative Chernoff bound along with a union bound over all $n$ users. \[\square\]

Lemma 3.3 (Privacy of $P^{\text{CM}}$). Let $n$ and $B$ be positive integers. Then, for $\gamma n \geq \max \left\{ 6k^2\tau/\varepsilon, \frac{90\ln(2\tau k/\delta)}{\varepsilon^2} \right\}$, the algorithm $S \circ P^{\text{CM}}(n,B,\tau,\gamma,s)$ is $(\varepsilon,\delta)$-differentially private.

The proof of Lemma 3.3 uses some general tools linking sensitivity of vector-valued functions, smoothness of distributions and approximate differential privacy—these are given next.

3.2 Approximate differential privacy for insensitive functions

To prove Lemma 3.3 we will need some general tools linking sensitivity of vector-valued functions, smoothness of distributions, and approximate differential privacy.

Definition 3.1 (Sensitivity). The $\ell_1$-sensitivity (or sensitivity, for short) of $f : X^n \to \mathbb{Z}^m$ is given by:

$$
\Delta(f) = \max_{X \sim X'} \| f(X) - f(X') \|_1.
$$

It is well-known [26] that the mechanism given by adding independent Laplacian noise with variance $2\Delta(f)^2/\varepsilon^2$ to each coordinate of $f(X)$ is $(\varepsilon,0)$-differentially private. Laplace noise, however, is unbounded in both the positive and negative directions, and this causes issues in the shuffled model (roughly speaking, it would require each party to send infinitely many messages). In our setting we will need to ensure that the noise added to each coordinate is bounded, so to achieve differential privacy we will not be able to add Laplacian noise. As a result we will only be able to obtain $(\varepsilon,\delta)$-differential privacy for $\delta > 0$. We specify next the types of noise that we will use instead of Laplacian noise.

Definition 3.2 (Smooth distributions). Suppose $D$ is a distribution supported on $\mathbb{Z}$. For $k \in \mathbb{N}$, $\varepsilon \geq 0$ and $\delta \in [0,1]$, we say that $D$ is $(\varepsilon,\delta,k)$-smooth if for all $-k \leq k' \leq k$,$$
\mathbb{P}_{Y \sim D} \left[ \frac{\mathbb{P}_{Y' \sim D}[Y' = Y]}{\mathbb{P}_{Y' \sim D}[Y' = Y + k']} \geq e^{\|k'|\varepsilon} \right] \leq \delta.
$$

Definition 3.3 (Incremental functions). Suppose $k \in \mathbb{N}$. We define $f : X^n \to \mathbb{Z}^m$ to be $k$-incremental if for all neighboring datasets $X \sim X'$, $\| f(X) - f(X') \|_{\infty} \leq k$.

The following lemma formalizes the types of noise we can add to $f(X)$ to obtain such a privacy guarantee. Its proof appears in the Appendix.

Lemma 3.4. Suppose $f : X^n \to \mathbb{Z}^m$ is $k$-incremental (Definition 3.3) and $\Delta(f) = \Delta$. Suppose $D$ is a distribution supported on $\mathbb{Z}$ that is $(\varepsilon,\delta,k)$-smooth. Then the mechanism

$$
X \mapsto f(X) + (Y_1,\ldots,Y_m),
$$

where $Y_1,\ldots,Y_m \sim D$, i.i.d., is $(\varepsilon',\delta')$-differentially private, where $\varepsilon' = \varepsilon \cdot \Delta, \delta' = \delta \cdot \Delta$.\[\square\]
3.3 Smoothness of binomial distribution

In order to prove Lemma 3.3, we will also use the following statement about the smoothness of the binomial distribution (that we will invoke with a small value of the head probability $\gamma$). Its proof appears in the Appendix.

**Lemma 3.5** (Smoothness of Bin$(n, \gamma)$). Let $n \in \mathbb{N}$, $\gamma \in [0, 1/2]$, $0 \leq \alpha \leq 1$, and $k \leq 2\gamma n/2$. Then, the distribution Bin$(n, \gamma)$ is $(\varepsilon, \delta, k)$-smooth with $\varepsilon = \ln((1 + \alpha)/(1 - \alpha))$ and $\delta = e^{-\frac{2\gamma n}{2}} + e^{-\frac{2\gamma n}{2\varepsilon^2}}$.

3.4 Privacy proof

We are now ready to prove Lemma 3.3 thereby establishing the privacy of Algorithm 1.

**Proof of Lemma 3.3** Fix $\varepsilon, \delta$. Notice that $S \circ R_{CM}(n, B, \tau, \gamma, s)$ can be obtained as a post-processing of the noisy count-min data structure $C : [\tau] \times [s] \rightarrow \mathbb{N}$ in Algorithm 1, so it suffices to show that the algorithm bringing the players’ inputs to this count-min data structure is $(\varepsilon, \delta)$-differentially private. Consider first the count-min data structure $C : [\tau] \times [s] \rightarrow \mathbb{N}$ with no noise, so that $C[t, \ell]$ measures the number of inputs $x$ inside some user’s set $S_i$ such that $h_i(x) = \ell$. We next note that the function mapping the users’ inputs $(S_1, \ldots, S_n)$ to $C$ has sensitivity (in terms of Definition 3.1) at most $k\tau$ and is $k$-incremental (in terms of Definition 3.3). Moreover, Lemma 3.5 (with $\alpha = \varepsilon/3$) implies that the binomial distribution Bin$(n, \gamma)$ is $(\varepsilon/(\tau k), \delta/(\tau k), k)$-smooth (in terms of Definition 3.2) as long as $\delta \geq 2\tau k e^{-\frac{2\gamma n}{2\varepsilon^2}}$ and $k \leq \varepsilon \gamma n/(6\tau k)$. In particular, we need

$$\gamma n \geq \max \left\{ 6k^2 \tau / \varepsilon, \frac{90 \ln(2\tau k/\delta)}{\varepsilon^2} \right\}.$$  

By construction in Algorithm 1, $C[t, s] = C[t, s] + \text{Bin}(n, \gamma)$, where the binomial random variables are independent for each $t, s$. Applying Lemma 3.4, we get that the count-min data structure is $(\varepsilon, \delta)$-differentially private (with respect to Definition 2.2).

By combining Lemmas 3.1 and 3.3, we immediately obtain the following:

**Theorem 3.6.** There is a sufficiently large positive absolute constant $\zeta$ such that the following holds. Suppose $n, B \in \mathbb{N}$, and $0 \leq \varepsilon, \delta \leq 1$. Consider the shuffled-model protocol $P_{CM} = (R_{CM}, S, A_{CM})$ with $\tau = \log(2B/\beta)$, $s = 2kn$, and

$$\gamma = \frac{1}{n} \cdot \zeta \cdot \max \left\{ \log(kn), k^2 \tau / \varepsilon, \frac{\log(\tau k/\delta)}{\varepsilon^2} \right\}.$$  

Then $P_{CM}$ is $(\varepsilon, \delta)$-differentially private (Definition 2.2) and for inputs $x_1, \ldots, x_n \in \{0, 1\}^B (\|x_i\|_1 \leq k)$, the estimates $\hat{x}_j$ produced by $Q_{CM}$ satisfy:

$$\mathbb{P} \left[ \forall j \in [B] \quad \hat{x}_j - \sum_{i=1}^n x_{i,j} \leq O \left( \log(kn) + \frac{k^2 \tau}{\varepsilon} + \frac{\log(\tau k/\delta)}{\varepsilon^2} \right) \right] \geq 1 - \beta.$$  

**Proof.** Privacy is an immediate consequence of Lemma 3.3. To establish accuracy (i.e., (1)), note first that Lemma 3.1 guarantees that for any $j \in [B]$, $\hat{x}_j - \sum_{i=1}^n x_{i,j} \leq O(\gamma n)$ with probability at least $1 - (kn/s)^\tau - 2\zeta_0(\log(s\tau) - \gamma n)$, for some constant $\zeta_0 \leq 1$.

Now (1) follows from a union bound over all $j \in [B]$ as well as the fact that for sufficiently large $\zeta$,

$$\gamma n \geq \zeta/2 \cdot (k^2 \tau + \log(kn)) \geq \log(2kn \cdot \log(2B/\beta)) + 1/\zeta_0 \cdot \log(2B/\beta) = \log(s\tau) + 1/\zeta_0 \cdot \log(2B/\beta),$$

so that $2\zeta_0(\log(s\tau) - \gamma n) \leq \beta/(2B)$.  

\[\square\]
We remark that the error guarantee as stated in Theorem 3.6 is one-sided, i.e., \( \hat{x}_j \) is always an overestimate of the true frequency \( \sum_{i=1}^{n} x_{i,j} \). If we are willing to sacrifice the one-sided error, the third term in (1) can be decreased from \( \frac{\log((\tau k)/\delta)}{\epsilon^2} \) to \( \frac{\log((\tau k)/\delta)}{\epsilon^2} \) by adding a debiasing step in the analyzer \( A_{CM} \) in Algorithm 1.

We can immediately apply Theorem 3.6 to recover all the heavy hitters by simply iterating over all elements in \( B \), computing an estimate of each count (which would be accurate up to an additive polylogarithmic factor) and outputting the elements whose count-estimate is larger than a certain (polylogarithmic) threshold. This gives a runtime of \( O(B) \). Algorithm 1 can be combined with the prefix tree idea of Bassily et al. [6] to reduce the server decoding time (for recovering all heavy hitters and their counts up to additive polylogarithmic factors) from \( O(B) \) to \( \tilde{O}(n) \). The combined algorithm would compute \( \log(B) \) differentially private count-min data structures, one for every prefix length. Then, the efficient decoding procedure would identify the counts of heavy prefixes of each length and only examine the two children of such heavy prefixes (thereby pruning away the subtrees rooted at infrequent prefixes); this reduces the server decoding time to \( \tilde{O}(n) \).

3.5 Protocol based on Hadamard response

The protocol based on the count-min sketch of the previous section achieves nearly optimal (i.e., up to polylogarithmic factors) time and accuracy for computing a \((\alpha, \beta, k)\)-frequency oracle for \( k = 1 \); see Table 1. However, the protocol relied on the availability of public randomness, i.e., the ability of the server to send a common string of random bits to all users. In this section we address the question of what can be achieved in the absence of such common random bits. The analogous question for local differential privacy (i.e., without shuffling) was considered recently in [2, 4]. These works made use of the Hadamard response for the local randomizers in place of previous techniques which relied on public randomness. The Hadamard response was also used in [17, 18, 41] for similar applications, namely private frequency estimation.

Overview. For \( B \in \mathbb{N} \) that is a power of 2, let \( H_B \in \{ -1, 1 \}^{B \times B} \) denote the \( B \times B \) Hadamard matrix and for \( j \in [B] \), set \( \mathcal{H}_{B,j} := \{ j' \in [B] \mid H_{j,j'} = 1 \} \). By orthogonality of the rows of \( H_B \), we have that \( |\mathcal{H}_{B,j}| = B/2 \) and for all \( j \neq j' \), \( |\mathcal{H}_{B,j} \cap \mathcal{H}_{B,j'}| = B/4 \). For any \( \tau \in \mathbb{N} \), we denote the \( \tau \)-wise Cartesian product of \( \mathcal{H}_{B,j} \) by \( \mathcal{H}_{B,j}^{\tau} \subset [B]^\tau \). In the Hadamard response [2], a user whose data consists of index \( j \in [B] \) sends to the server a random index \( j' \in [B] \) that is chosen uniformly at random from \( \mathcal{H}_{B,j} \) with probability \( \frac{e^\epsilon}{1+e^\epsilon} \) and is otherwise chosen uniformly from \( [B] \setminus \mathcal{H}_{B,j} \) with probability \( \frac{1}{1+e^\epsilon} \).

In the shuffled model, much less randomization is needed to protect a user’s privacy than in the local model of differential privacy where the Hadamard response, as described above, was previously applied. In particular, we can allow the users to send more information about their data to the server, along with some “blanket noise” which helps to hide the true value of any one individual’s input. Our adaptation of the Hadamard response to the shuffled model for computing an \((\alpha, \beta, k)\)-frequency oracle proceeds as follows (Algorithm 2). Suppose the \( n \) users possess data \( S_1, \ldots, S_n \subset [B] \) such that \( |S_i| \leq k \); equivalently, they possess \( x_1, \ldots, x_n \in \{0, 1\}^B \), such that for each \( i \in [n], ||x_i||_1 \leq k \) (the nonzero indices of \( x_i \) are the elements of \( S_i \)). Given \( x_i \), the local randomizer \( R^{Had} \) augments its input by adding \( k - ||x_i||_1 \) arbitrary elements from the set \( \{ B+1, \ldots, 2B \} \) (recall that \( k \leq B \)). (Later, the analyzer will simply ignore the augmented input in \( \{ B+1, \ldots, 2B \} \) from the individual randomizers. The purpose of the augmentation is to guarantee that all sets \( S_i \) will have cardinality exactly \( k \), which facilitates the privacy analysis.) Let the augmented input be denoted \( \tilde{x}_i \), so that \( \tilde{x}_i \in \{0, 1\}^{2B} \), and \( ||\tilde{x}_i||_1 = k \). For each index \( j \) at which \( \langle \tilde{x}_i \rangle_j \neq 0 \), the local randomizer chooses \( \tau \) indices \( a_{j,1}, \ldots, a_{j,\tau} \) in \( \mathcal{H}_{2B,j}^\tau \) uniformly and independently, and sends each tuple \( (a_{j,1}, \ldots, a_{j,\tau}) \) to the shuffler. It also generates \( \rho \) tuples \( (a_{g,1}, \ldots, a_{g,\tau}) \) where each of \( a_{g,1}, \ldots, a_{g,\tau} \) is uniform over \( [2B] \), and sends these to the shuffler as well; these latter tuples constitute “blanket noise” added to guarantee differential privacy.

Given the output of the shuffler, the analyzer \( A^{Had} \) determines estimates \( \hat{x}_j \) for the frequencies of each \( j \in [B] \) by counting the number of messages \( (a_{i,1}, \ldots, a_{i,\tau}) \in [2B]^\tau \) which belong to \( \mathcal{H}_{2B,j}^\tau \). The rationale is that each user \( i \) such that \( j \in S_i \) will have sent such a message in \( \mathcal{H}_{2B,j}^\tau \). As the analyzer could have picked up some of the “blanket noise” in this count, as well as tuples sent by users holding some \( j' \neq j \), since \( \mathcal{H}_{2B,j}^\tau \cap \mathcal{H}_{2B,j'}^\tau \neq \emptyset \), it then corrects this count (Algorithm 2 Line 19) to obtain an unbiased estimate \( \hat{x}_j \) of the frequency of \( j \).
**Analysis.** We next state the theorems summarizing the accuracy, efficiency, and privacy guarantees of Algorithm $P_{\text{Had}}$.

**Theorem 3.7 (Privacy of $P_{\text{Had}}$).** Fix $n, B \in \mathbb{N}$ with $B$ a power of 2. Then with $\tau = \log n, \varepsilon \leq 1$, and $\rho = \frac{36 \ln 1/\delta}{\varepsilon^2}$ the algorithm $S \circ R_{\text{Had}}(n, B, \tau, \rho, k)$ is $(\varepsilon, \delta \cdot \exp(\varepsilon^2)/\varepsilon)$-differentially private.

**Proof.** For convenience let $P := S \circ R_{\text{Had}}(n, B, \tau, \rho, k)$ be the protocol whose $(\varepsilon, \delta)$-differential privacy we wish to establish. With slight abuse of notation, we will assume that $P$ operates on the augmented inputs $(\tilde{x}_1, \ldots, \tilde{x}_n)$ (see Algorithm $P_{\text{Had}}$ Line 4). In particular, for inputs $(x_1, \ldots, x_n)$ that lead to augmented inputs $(\tilde{x}_1, \ldots, \tilde{x}_n)$, we will let $P(\tilde{x}_1, \ldots, \tilde{x}_n)$ be the output of $P$ when given as inputs $x_1, \ldots, x_n$. Let $\mathcal{Y}$ be the set of multisets consisting of elements of $\{0, 1\}^{\log 2B \times \tau}$; notice that the output of $P$ lies in $\mathcal{Y}$.

By symmetry, it suffices to show that for any augmented inputs of the form $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_{n-1}, \tilde{x}_n)$ and $\tilde{X}' = (\tilde{x}_1, \ldots, \tilde{x}_{n-1}, \tilde{x}'_n)$, and for any subset $U \subset \mathcal{Y}$, we have that

$$
\mathbb{P}[P(\tilde{x}_1, \ldots, \tilde{x}_n) \in U] \leq e^\varepsilon \cdot \mathbb{P}[P(\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{x}'_n) \in U] + \delta. \tag{2}
$$

We first establish (2) for the special case that $\tilde{x}_n, \tilde{x}'_n$ differ by 1 on two indices, say $j, j'$, while having the same $\ell_1$ norm: in particular, we have $|\tilde{x}_n(j) - \tilde{x}'_n(j)| = 1$ and $|\tilde{x}_n(j') - \tilde{x}'_n(j')| = 1$. By symmetry, without loss of generality we may assume that $j = 0, j' = 1$ and that $(\tilde{x}_n)_j - (\tilde{x}'_n)_j = 1$ while $(\tilde{x}'_n)_j - (\tilde{x}_n)_j = 1$. To establish (2) in this case, we will in fact prove a stronger statement: for inputs $(\tilde{x}_1, \ldots, \tilde{x}_n)$, define the view of an adversary, $\text{View}_P(\tilde{x}_1, \ldots, \tilde{x}_n)$, as the tuple consisting of the following components:

- For each $i \in [n-1]$, the set $\mathcal{S}_i := \bigcup_{j:(\tilde{x}_n)_j = 1 \{a_1, \ldots, a_j, \tau\}}$ of tuples output by user $i$ corresponding to her true input $\tilde{x}_n$.
- The set $\mathcal{S}_n := \bigcup_{j:(\tilde{x}_n)_j = 1 \{a_1, \ldots, a_j, \tau\}}$ of tuples output by user $n$ corresponding to her true (augmented) input $\tilde{x}_n$, except (if applicable) the string that would be output if $(\tilde{x}_n)_1 = 1$ or $(\tilde{x}_n)_n = 0$.
- The multiset $(y_1, \ldots, y_m)$ consisting of the outputs of the $n$ users of the protocol $P$.

It then suffices to show the following:

$$
\mathbb{P}_{\text{View}_P(\tilde{x}_1, \ldots, \tilde{x}_n)} \left[ \frac{\mathbb{P}[	ext{View}_P(\tilde{x}_1, \ldots, \tilde{x}_{n-1}, \tilde{x}_n) = V]}{\mathbb{P}[	ext{View}_P(\tilde{x}_1, \ldots, \tilde{x}_{n-1}, \tilde{x}'_n) = V]} \geq e^\varepsilon \right] \leq \delta. \tag{3}
$$

(See [5] Theorem 3.1 for a similar argument.)

Notice that each of the elements $y_1, \ldots, y_m$ in the output of the protocol $P$ consists of a tuple $(a_1, \ldots, a_\tau)$, where each $a_1, \ldots, a_\tau \in \{0, 1\}$. Now we will define a joint distribution (denoted by $D$) of random variables $(W_{a_1, \ldots, a_\tau})_{a_1, \ldots, a_\tau} \in [2B]^\tau$, $Q, Q'$, where, for each $(a_1, \ldots, a_\tau) \in [2B]^\tau$, $W_{a_1, \ldots, a_\tau} \in \mathbb{Z}_{+0}$ and $Q, Q' \in [2B]^\tau$, as follows. For each tuple $(a_1, \ldots, a_\tau) \in [2B]^\tau$, let $W_{a_1, \ldots, a_\tau}$ be jointly distributed from a multinomial distribution over $[2B]^\tau$ with $\rho n$ trials. For each $(a_1, \ldots, a_\tau) \in [2B]^\tau$, let $W_{a_1, \ldots, a_\tau}$ be the random variable representing the number of tuples $(\tilde{a}_g, 1, \ldots, \tilde{a}_g, \tau)$ generated on Line 8 of Algorithm $P_{\text{Had}}$ satisfying $(\tilde{a}_g, 1, \ldots, \tilde{a}_g, \tau) = (a_1, \ldots, a_\tau)$. Notice that the joint distribution of all $W_{a_1, \ldots, a_\tau}$ is the same as the joint distribution of $W_{a_1, \ldots, a_\tau}$, for $(a_1, \ldots, a_\tau) \in [2B]^\tau$. Intuitively, $W_{a_1, \ldots, a_\tau}$ represents the “blanket noise” added by the outputs $(\tilde{a}_g, 1, \ldots, \tilde{a}_g, \tau)$ in Line 8 of Algorithm $P_{\text{Had}}$.

Also let $Q, Q' \in [2B]^\tau$ be random variables that are distributed uniformly over $\mathcal{H}_0$ and $\mathcal{H}'$, respectively. Then since the tuples $(\tilde{a}_g, 1, \ldots, \tilde{a}_g, \tau)$ are distributed independently of the tuples $(a_1, \ldots, a_j, \tau)$, (3) is equivalent to

$$
\mathbb{P}_{W_{a_1, \ldots, a_\tau}, Q, Q' \sim D} \left[ \frac{\mathbb{P}_{W_{a_1, \ldots, a_\tau}, Q, Q' \sim D} [\text{View}(a_1, \ldots, a_\tau) \in [2B]^\tau : W_{a_1, \ldots, a_\tau} + 1[Q = (a_1, \ldots, a_\tau)] = w_{a_1, \ldots, a_\tau} + 1[q = (a_1, \ldots, a_\tau)]]}{\mathbb{P}_{W_{a_1, \ldots, a_\tau}, Q, Q' \sim D} [\text{View}(a_1, \ldots, a_\tau) \in [2B]^\tau : W_{a_1, \ldots, a_\tau} + 1[Q' = (a_1, \ldots, a_\tau)] = w_{a_1, \ldots, a_\tau} + 1[q' = (a_1, \ldots, a_\tau)]} \geq e^\varepsilon \right] \leq \delta. \tag{4}
$$

Set $\bar{w}_{a_1, \ldots, a_\tau} := w_{a_1, \ldots, a_\tau} + 1[q = (a_1, \ldots, a_\tau)]$. By the definition of $D$ we have

$$
\mathbb{P}_{W_{a_1, \ldots, a_\tau}, Q, Q' \sim D} [\text{View}(a_1, \ldots, a_\tau) \in [2B]^\tau : W_{a_1, \ldots, a_\tau} + 1[Q = (a_1, \ldots, a_\tau)] = \bar{w}_{a_1, \ldots, a_\tau}]
$$

$$
= \mathbb{E}_{Q \sim D} \left[ \frac{(2B)^{-\tau \rho n} \cdot \{\bar{w}_{a_1, \ldots, a_\tau} - 1[q = (a_1, \ldots, a_\tau)]\}_{a_1, \ldots, a_\tau} \in [2B]^\tau} {(2B)^{-\tau} \cdot \sum_{a_1', \ldots, a_\tau' \in \mathcal{H}_0} \bar{w}_{a_1', \ldots, a_\tau'} \cdot \sum_{a_1', \ldots, a_\tau' \in [2B]^\tau} w_{a_1', \ldots, a_\tau'} \cdot \sum_{a_1', \ldots, a_\tau' \in \mathcal{H}_0}} \right].
$$
In the above equation, the notation such as \( \left( \prod_{a_1,\ldots,a_{\tau} \in [2B]} \tilde{w}_{a_1,\ldots,a_{\tau}} \right)^{((2B)^\tau)!} \) refers to the multinomial coefficient, equal to \( \prod_{a_1,\ldots,a_{\tau} \in [2B]} \tilde{w}_{a_1,\ldots,a_{\tau}} \). Similarly, for the denominator of the expression in (4),

\[
\mathbb{P}_{W_1,\ldots,a_{\tau},Q,Q' \sim \mathcal{D}} \left[ \forall (a_1,\ldots,a_{\tau}) \in [2B]^\tau : W_{a_1,\ldots,a_{\tau}} + 1 [Q' = (a_1,\ldots,a_{\tau})] = \tilde{w}_{a_1,\ldots,a_{\tau}} \right] = \mathbb{E}_{Q \sim \mathcal{D}} \left[ (2B)^{-\tau} \rho_n \cdot \left( \{ \tilde{w}_{a_1,\ldots,a_{\tau}} - 1 [Q' = (a_1,\ldots,a_{\tau})] \}_{(a_1,\ldots,a_{\tau}) \in [2B]^\tau} \right) \right] = \left( \frac{2}{2B} \right)^\tau \cdot (2B)^{-\tau} \rho_n \cdot \sum_{a_1',\ldots,a_{\tau}' \in \mathcal{H}_1} \tilde{w}_{a_1',\ldots,a_{\tau}'}.
\]

Thus, (4) is equivalent to

\[
\mathbb{P}_{W_1,\ldots,a_{\tau},g,Q,q' \sim \mathcal{D}} \left[ \sum_{a_1',\ldots,a_{\tau}' \in \mathcal{H}_1} \tilde{w}_{a_1',\ldots,a_{\tau}'} \geq \varepsilon \right] \leq \delta \cdot \left( \frac{\varepsilon^2 \rho}{3} \right). \tag{5}
\]

Notice that \( \sum_{a_1',\ldots,a_{\tau}' \in \mathcal{H}_{2B,0}} \tilde{w}_{a_1',\ldots,a_{\tau}'} \) is distributed as \( 1 + \text{Bin}(\rho n, 2^{-\tau}) \), since \( q \in \mathcal{H}_{2B,0} \) with probability 1 (by definition of \( \mathcal{D} \)), and each of the \( \rho n \) trials in determining the counts \( w_{a_1,\ldots,a_{\tau}} \) belongs to \( \mathcal{H}_{2B,0} \) with probability \( 2^{-\tau} \). Similarly, \( \sum_{a_1',\ldots,a_{\tau}' \in \mathcal{H}_{2B,1}} \tilde{w}_{a_1',\ldots,a_{\tau}'} \) is distributed as \( \text{Bin}(\rho n + 1, 2^{-\tau}) \); notice in particular that \( q \), which is distributed uniformly over \( \mathcal{H}_{2B,0} \), is in \( \mathcal{H}_{2B,1} \) with probability \( 2^{-\tau} \). By the multiplicative Chernoff bound, we have that, for \( \eta \leq 1 \),

\[
\mathbb{P}_{W \sim \text{Bin}(\rho n, 1/\eta)} \left[ |W - \rho| > \rho \eta \right] \leq \exp \left( -\frac{\eta^2 \rho}{3} \right). \tag{6}
\]

As long as we take \( \rho = \frac{36\ln(1/\delta)}{\varepsilon^2} \), (6) will be satisfied with \( \eta = \varepsilon/6 \), which in turn implies (5) since

\[
\frac{(1 + \varepsilon/6) \rho + 1}{\rho(1 - \varepsilon/6)} \leq \frac{e^{\varepsilon/6} \rho + 1}{e^{-\varepsilon/3} \rho} \leq \frac{e^{4\varepsilon/6} \rho}{e^{-\varepsilon/3} \rho} \leq e^\varepsilon,
\]

where the second inequality above uses \( (\rho + 1)/\rho \leq e^{\varepsilon/2} \) for our choice of \( \rho \).

We have thus established (2) for the case that \( \tilde{x}_n, \tilde{x}'_n \) differ by 1 on two indices. For the general case, consider any neighboring datasets \( X = (\tilde{x}_1,\ldots,\tilde{x}_n) \) and \( X' = (\tilde{x}_1,\ldots,\tilde{x}_{n-1},\tilde{x}'_n) \); we can find a sequence of at most \( k - 1 \) intermediate datasets \( (\tilde{x}_1,\ldots,\tilde{x}_{n-1},\tilde{x}'_n^{\mu}) \), \( 1 \leq \mu \leq k - 1 \) such that \( \tilde{x}_n^{\mu-1} \) and \( \tilde{x}_n^{\mu} \) differ by 1 on two indices. Applying (2) to each of the \( k \) neighboring pairs in this sequence, we see that for any \( \mathcal{U} \subset \mathcal{Y} \),

\[
\mathbb{P}[P(X') \in \mathcal{U}] \leq e^{\varepsilon k} \cdot \mathbb{P}[P(X') \in \mathcal{U}] + \delta \cdot (1 + e^\varepsilon + \cdots + e^{(k-1)\varepsilon}) \leq e^{\varepsilon k} \cdot \mathbb{P}[P(X') \in \mathcal{U}] + \delta \cdot \frac{2e^{k\varepsilon}}{\varepsilon},
\]

where we have used \( \varepsilon \leq 1 \) in the final inequality above.

\[\square\]

**Theorem 3.8 (Accuracy of \( P_{\text{Hadamard}} \).** Fix \( n, B \in \mathbb{N} \) with \( B \) a power of 2. Then with \( \tau, \rho \) as in Theorem 3.7, the estimate \( \hat{x} \) produced in \( A_{\text{Hadamard}} \) in the course of the shuffled-model protocol \( P_{\text{Hadamard}} = (R_{\text{Hadamard}}, S, A_{\text{Hadamard}}) \) with input \( x_1,\ldots,x_n \in \{0,1\}^B \) satisfies

\[
\mathbb{P} \left[ \left\| \hat{x} - \sum_{i=1}^n x_i \right\|_\infty \leq \sqrt{3 \ln(2B/\beta)} \cdot \max \{ 3 \ln(2B/\beta), \rho + k \} \right] \geq 1 - \beta.
\]

**Proof.** Fix any \( j \in [B] \). Let \( \zeta_j = \sum_{i=1}^n (x_i)_j \). We will upper bound the probability that \( \xi_j := \hat{x}_j - \zeta_j \) is large. Notice that the distribution of \( \hat{x}_j \) is given by

\[
\frac{1}{1 - 2^{-\tau}} \cdot (\zeta_j + \text{Bin}(\rho n + kn - \zeta_j, 2^{-\tau}) - (\rho n + kn)2^{-\tau}).
\]

This is because each of the \( \rho n \) tuples \( (\tilde{a}_{g,1},\ldots,\tilde{a}_{g,B}) \) chosen uniformly from \([2B]^\tau\) on Line 8 of Algorithm 2 has probability \( 2^{-\tau} \) of belonging to \( \mathcal{H}_{2B,\beta} \), and each of the \( kn - \zeta_j \) tuples \( (a'_j,1,\ldots,a'_{j'},\tau) \) (for \( j' \in S_i, j' \neq j, i \in [n] \))
satisfies in $A$ (Efficiency of each of the $\zeta_j$). Moreover, for all $j$, we have

$$
\Pr[j \in \mathcal{H}_{2B,j}^\tau | A] \leq 2 \exp \left( -\frac{\eta^2 c}{3} \right).
$$

For any reals $c > 0$ and $0 \leq \eta \leq 1$, by the Chernoff bound, we have

$$
\Pr[j \in \mathcal{H}_{2B,j}^\tau | A] \leq 2 \exp \left( -\frac{\eta^2 c}{3} \right).
$$

We have $2 \exp(-\eta^2 c/3) \leq \beta/B$ as long as $\eta \geq \sqrt{3 \ln(2B/\beta)} / c$. Set $c = \rho + k - \zeta_j/n$, so that $\rho \leq c \leq \rho + k$.

First suppose that $\sqrt{3 \ln(2B/\beta)} (\rho + k - \zeta_j/n) \leq 1$. Then we see that

$$
\Pr[|\hat{x}_j - \zeta_j| > \sqrt{3 \ln(2B/\beta)} (\rho + k)] \leq \beta/B.
$$

In the other case, namely $\rho + k - \zeta_j/n = c < 3 \ln(2B/\beta)$, set $\eta = 3 \ln(2B/\beta)/c$, and we see that

$$
\Pr[|\hat{x}_j - \zeta_j| > 3 \ln(2B/\beta)] \leq \beta/B.
$$

The combination of (7) and (8) with a union bound over all $j \in [B]$ completes the proof of Theorem 3.8.

Corollary 3.9. Fix $n, B$ with $B$ a power of 2. Then with $\tau, \rho$ as in Theorem 3.8 and $k = 1$, the estimate $\hat{x}$ produced in $A^{\text{Had}}$ in the course of the shuffled model protocol $P^{\text{Had}} = (R^{\text{Had}}, S, A^{\text{Had}})$ with input $x_1, \ldots, x_n \in \{0, 1\}^B$ satisfies

$$
\Pr\left[ \left\| \hat{x} - \sum_{i=1}^n x_i \right\|_\infty \leq O\left( \sqrt{\log(B/\beta) \cdot \max\{\log(B/\beta), \log(1/\delta)/\varepsilon^2\}} \right) \right] \geq 1 - \beta.
$$

Next we summarize the communication and computation settings of the shuffled model protocol $P^{\text{Had}}$.

Theorem 3.10 (Efficiency of $P^{\text{Had}}$). Let $n, B, \tau, \rho, k \in \mathbb{N}$. Then the protocol $P^{\text{Had}} = (R^{\text{Had}}(n, B, \tau, \rho, k), S, A^{\text{Had}}(n, B, \tau, \rho, k))$ satisfies the following:

1. On input $(x_1, \ldots, x_n) \in \{0, 1\}^B$, the output of the local randomizers $R^{\text{Had}}(n, B, \tau, \rho, k)$ consists of $n(k + \rho)$ messages of length $\tau \log 2B$ bits.
2. The runtime of the analyzer $A^{\text{Had}}(n, B, \tau, \rho, k)$ on input $\{y_1, \ldots, y_m\}$ is at most $O(Bm\tau)$ and its output has space $O(B \log(n(k + \rho)))$ bits. Moreover, if $\tau = \log n$ (i.e., as in Theorem 2.7), and if its input $\{y_1, \ldots, y_m\}$ is the output of the local randomizers on input $x_1, \ldots, x_n$ (so that $m = n(\rho + k)$), there is a modification of the implementation of $A^{\text{Had}}$ in Algorithm 2 that, for $\beta \in [0, 1]$, completes in time $O((\rho + k)n \log^3 B + kn + B \log 1/\beta)$ with probability $1 - \beta$.
3. There is a separate modification of $A^{\text{Had}}(n, B, \tau, \rho, k)$ that on input $\{y_1, \ldots, y_m\}$ produces an output data structure $(FO, A)$ with space $O(m\tau \log B)$ bits, such that a single query $A(FO, j)$ of some $j \in [B]$ takes time $O(m\tau \log B)$. 


Proof. The first item is immediate from the definition of $R_{\text{Had}}$ in Algorithm 2. For the second item, note first that $A_{\text{Had}}$ as written in Algorithm 2 takes time $O(Bm\log B)$: for each message $y_j = (a_{j,1}, \ldots, a_{j,\tau})$, it loops through each $j \in [B]$ to check if each $a_{i,g} \in \mathcal{H}_{2B,j}$ for $1 \leq g \leq \tau$ (determination of whether $a_{i,g} \in \mathcal{H}_{2B,j}$ takes time $O(\log B)$).

Now suppose that the messages $y_1, \ldots, y_m$ are the union of the multisets output by each of $n$ shufflers on input $x_1, \ldots, x_n$. Notice that, for messages of the form $(a_{j,1}, \ldots, a_{j,\tau}) \in \mathcal{H}_{2B,j}$ (Line 5 of Algorithm 2), the number of such messages $y_j$ such that $(a_{j,1}, \ldots, a_{j,\tau}) \in \mathcal{H}_{2B,j}$ is distributed as $1 + \text{Bin}(2B - 1, 1/n)$ (recall $\tau = \log n$). Moreover, for messages of the form $(\tilde{a}_{g,1}, \ldots, \tilde{a}_{g,\tau})$ (Line 8 of Algorithm 2), the number of $j'$ such that $(\tilde{a}_{g,1}, \ldots, \tilde{a}_{g,\tau}) \in \mathcal{H}_{2B,j'}$ is distributed as $\text{Bin}(2B, 1/n)$. Therefore, by the multiplicative Chernoff bound, for any $0 \leq \beta \leq 1$, the sum, over all $j' \in [B]$, of the number of messages $(a_{j',1}, \ldots, a_{j',\tau})$ that belong to $\mathcal{H}_{2B,j'}$ is bounded above by

$$kn + \frac{4Bn(1 + \ln(1/\beta))}{n} = kn + 4B(1 + \ln(1/\beta))$$

with probability $1 - \beta$. Next, consider any individual message $y_i = (a_{i,1}, \ldots, a_{i,\tau})$ (as on Line 16). Notice that the set of $j$ such that $\{a_{i,1}, \ldots, a_{i,\tau}\} \subset \mathcal{H}_{2B,j}$ can be described as follows: write $j = (j_1, \ldots, j_{\log_2 B}) \in \{0, 1\}^{2\log_2 B}$ to denote the binary representation of $j$, and arrange the $\log_2 B$-bit binary representations of each of $a_{i,1}, \ldots, a_{i,\tau}$ to be the rows of a $\tau \times \log_2 B$ matrix $A \in \{0, 1\}^{\tau \times \log_2 B}$. Then $\{a_{i,1}, \ldots, a_{i,\tau}\} \subset \mathcal{H}_{2B,j}$ if and only if $A_j = 0$, where arithmetic is performed over $\mathbb{F}_2$. This follows since for binary representations $i = (i_1, \ldots, i_{\log_2 B}) \in \{0, 1\}^{\log_2 B}$ and $j = (j_1, \ldots, j_{\log_2 B}) \in \{0, 1\}^{\log_2 B}$, the $(i,j)$-element of $H_B$ is $(-1)^{\sum_{t=1}^{\log_2 B} i_t j_t}$. Using Gaussian elimination one can enumerate the set of $j \in \{0, 1\}^{\log_2 B}$ in the kernel of $A$ in time proportional to the sum of $O(\log^3 B)$ and the number of $j$ in the kernel. Since the number of such $j$ is bounded above by (9), the total running time of this modification of $A_{\text{Had}}$ becomes $O((\rho + k)n \log^3 B + kn + B \log(1/\beta))$.

For the last item of the theorem, the analyzer simply outputs the collection of all tuples $y_i = (a_{i,1}, \ldots, a_{i,\tau})$; to query the frequency of some $j \in [B]$, we simply run the for loop on Line 14 of Algorithm 2 together with the debiasing step of Line 19.

4 Range queries

We recall the definition of range queries. Let $X = [B]$ and consider a dataset $X = (x_1, \ldots, x_n) \in [B]^n$. A counting query is specified by a vector $w \in \mathbb{R}^B$, and the answer to this counting query on the dataset $X$ is given by $\langle w, \text{hist}(X) \rangle$. Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product; throughout the paper, we slightly abuse notation and allow an inner product to be taken of a row vector and a column vector. A 1-dimensional range query $[j, j']$, where $1 \leq j \leq j' \leq B$, is a counting query such that $w_j = w_{j+1} = \cdots = w_{j'} = 1$, and all other entries of $w$ are 0. For $d$-dimensional range queries, the elements of $[B]$ will map to points on a $d$-dimensional grid, and a certain subset of vectors $w \in \{0, 1\}^B$ represent the $d$-dimensional range queries. In this section, we use the frequency oracle protocols in Section 3 to derive protocols for computing counting queries with per-user communication $\text{poly} \log(B)$ and additive error $\text{poly} \log(\text{max}\{n, B\})$.

In Section 4.1 we adapt the matrix mechanism of [39, 40] to use the frequency oracle protocols of Section 3 as a black-box for computation of counting queries, which include range queries as a special case. In Section 4.2 we instantiate this technique for the special case of 1-dimensional range queries, and in Section 4.3 we consider the case of multi-dimensional range queries. In Section 4.4 we collect the results from Sections 4.1 through 4.3 to formally state our guarantees on range query computation in the shuffled model, as well as the application to $M$-estimation of the median, as mentioned in the introduction.

4.1 Reduction to private frequency oracle via the matrix mechanism

Our protocol for computing range queries is a special case of a more general protocol, which is in turn inspired by the matrix mechanism of [39, 40]. We begin by introducing this more general protocol and explaining how it allows
The intuition behind the privacy of (10) is as follows: (10) can be obtained as a post-processing of the mechanism.
\[ X \mapsto M(\text{hist}(X)) + \Delta_M \cdot z, \]  
namely via multiplication by \( M^{-1} \). If we choose, for instance, each \( z_i \) to be an independent Laplacian of variance \( 2/\varepsilon \), then the algorithm \( X \mapsto M(\text{hist}(X)) + \Delta_M \cdot z \) is simply the Laplace mechanism, which is \((\varepsilon, 0)\)-differentially private \([25]\).

In our modification of the matrix mechanism, the parties will send data that allows the analyzer to directly compute the “pre-processed input” \( M(\text{hist}(X)) + \Delta_M \cdot z \). Moreover, due to limitations of the shuffled model and to reduce communication, the distribution of the noise \( z \) will be different from what has been previously used \([39],[40]\).

For our application, we will require \( M \) to satisfy the following properties:

1. For any counting query \( w \) corresponding to a \( d \)-dimensional range query, \( wM^{-1} \) has at most \( \text{poly log}(B) \) nonzero entries, and all of those nonzero entries are bounded in absolute value by some \( c > 0 \). (Here \( w \in \{0, 1\}^B \) is viewed as a row vector.)
2. \( \Delta_M \leq \text{poly log}(B) \).

By property (2) above and the fact that all entries of \( M \) are in \( \{0, 1\} \), (approximate) computation of the vector \( M(\text{hist}(X)) \) can be viewed as an instance of the frequency oracle problem where user \( i \in [n] \) holds the \( \leq \text{poly log}(B) \) nonzero entries of the vector \( M(\text{hist}(x_i)) \). This follows since \( M(\text{hist}(x_i)) \) is the \( x_i \)-th column of \( M \), \( \Delta_M \leq \text{poly log}(B) \), and \( \text{hist}(X) = \sum_{i=1}^{n} \text{hist}(x_i) \). Moreover, suppose there is some choice of local randomizer and analyzer (such as those in Section \( 3 \)) that approximately solve the frequency oracle problem, i.e., compute an approximation \( \hat{y} \) of \( M(\text{hist}(X)) \) up to an additive error of \( \text{poly log} B \), in a differentially private manner. Since \( wM^{-1} \) has at most \( \text{poly log}(B) \) nonzero entries, each of magnitude at most \( c \), it follows that

\[ \langle wM^{-1}, \hat{y} \rangle \]  
approximates the counting query \( \langle w, \text{hist}(X) \rangle \) up to an additive error of \( c \cdot \text{poly log}(B) \).

#### Algorithm 3: Local randomizer for matrix mechanism

1. \( R^{\text{matrix}}(n, B, M, R^{\text{FO}}) \):
   
   **Input:** \( x \in [B] \), parameters \( n, B \in \mathbb{N}, M \in \{0, 1\}^{B \times B}, R^{\text{FO}} : \{0, 1\}^B \rightarrow \mathcal{T}^* \)
   
   **Output:** Multiset \( S \subseteq \mathcal{T} \), where \( \mathcal{T} \) is the output set of \( R^{\text{FO}} \)

2. Let \( A_x \leftarrow \{ j \in [B] : M_{jx} \neq 0 \} /\ A_x \) is the set of nonzero entries of the \( x \)-th column of \( M \)

3. **return** \( R^{\text{FO}}(A_x) \)

Perhaps surprisingly, for any constant \( d \geq 1 \), we will be able to find a matrix \( M \) that satisfies properties (1) and (2) above for \( d \)-dimensional range queries with \( c = 1 \). This leads to the claimed \( \text{poly log}(B) \) error for computation of \( d \)-dimensional range queries, as follows: the local randomizer \( R^{\text{matrix}} \) (Algorithm 3) is parametrized by integers \( n, B \in \mathbb{N} \), a matrix \( M \in \{0, 1\}^{B \times B} \), and a local randomizer \( R^{\text{FO}} : [B] \rightarrow \mathcal{T}^* \) that can be used in a shuffled model protocol that computes a frequency oracle. (Here \( \mathcal{T} \) is an arbitrary set, and \( R^{\text{FO}} \) computes a sequence of messages in \( \mathcal{T} \).) Given input \( x \in [B] \), \( R^{\text{matrix}} \) returns the output of \( R^{\text{FO}} \) when given as input the set of nonzero entries of the \( x \)-th column of \( M \). The corresponding analyzer \( A^{\text{matrix}} \) (Algorithm 4) is parametrized by integers \( n, B \in \mathbb{N} \), a matrix \( M \in \{0, 1\}^{B \times B} \), and an analyzer \( A^{\text{FO}} \) for computation of a frequency oracle in the shuffled model. Given a multiset \( S \) consisting of the shuffled messages output by individual randomizers \( R^{\text{matrix}} \), it returns \( \text{hist} \), namely the inner product of \( wM^{-1} \) and the output of \( A^{\text{FO}} \) when given \( S \) as input. To complete the construction of a protocol for range query computation in the shuffled model, it remains to find a matrix \( M \) satisfying properties (1) and (2) above. We will do so in Sections 4.2 and 4.3. First we state here the privacy and accuracy guarantees of the shuffled protocol \( P^{\text{matrix}} = (R^{\text{matrix}}, S, A^{\text{matrix}}) \).

**Theorem 4.1 (Privacy of \( P^{\text{matrix}} \)).** Suppose \( R^{\text{FO}} \) is a local randomizer for computation of an \((\alpha, \beta, k)\)-frequency oracle with \( n \) users and universe size \( B \), which satisfies \((\varepsilon, \delta)\)-differential privacy in the shuffled model. Suppose \( M \in \{0, 1\}^B \) satisfies \( \Delta_M \leq k \). Then the shuffled protocol \( S \circ R^{\text{matrix}}(n, B, M, R^{\text{FO}}) \) is \((\varepsilon, \delta)\)-differentially private.

**Proof.** Let \( \mathcal{Y} \) be the message space of the randomizer \( R^{\text{FO}} \), and \( \mathcal{Y}' \) be the set of multisets consisting of elements of \( \mathcal{Y} \). Let \( P = S \circ R^{\text{matrix}}(n, B, M, R^{\text{FO}}) \). Consider neighboring datasets \( X = (x_1, \ldots, x_n) \in [B]^n \) and \( X' = X \)
(12) follows by the fact that \( \Pr \{ X \in T \} \leq e^3 \cdot \Pr \{ X' \in T \} + \delta \).

For \( i \in [n] \), let \( S_i = \{ j \in [B] : M_{j,x_i} \neq 0 \} \) and \( S'_n = \{ j \in [B] : M_{j,x'_n} \neq 0 \} \). Since \( \Delta_M \leq k \), we have \( |S_i| \leq k \) for \( i \in [n] \) and \( |S'_n| \leq k \). Since the output of \( R_{\text{matrix}} \) on input \( x_i \) is simply \( R_{\text{FO}}(S_i) \),

\[
P(X) = S(R_{\text{FO}}(S_1), \ldots, R_{\text{FO}}(S_n)), \quad P(X') = S(R_{\text{FO}}(S_1), \ldots, R_{\text{FO}}(S_{n-1}), R_{\text{FO}}(S'_n)).
\]

Then (12) follows by the fact that \( (S_1, \ldots, S_n) \) and \( (S_1, \ldots, S_{n-1}, S'_n) \) are neighboring datasets for the \( (\alpha, \beta, k) \)-frequency problem and \( S \circ R_{\text{FO}} \) is \( (\varepsilon, \delta) \)-differentially private.

**Theorem 4.2 (Accuracy & efficiency of \( P_{\text{matrix}} \)).** Suppose \( R_{\text{FO}}, A_{\text{FO}} \) are the local randomizer and analyzer for computation of an \( (\alpha, \beta, k) \)-frequency oracle with \( n \) users and universe size \( B \). Suppose also that \( W \subset \{0,1\}^B \) is a set of counting queries and \( M \subset \{0,1\}^B \) is such that, for any \( w \in W \), \( \|wM^{-1}\|_1 \leq a \) and \( \Delta_M \leq k \). Consider the shuffled model protocol \( P_{\text{matrix}} = (R_{\text{matrix}}(n, B, M, R_{\text{FO}}), S, A_{\text{matrix}}(n, B, M, A_{\text{FO}}, W)) \). For any dataset \( X = (x_1, \ldots, x_n) \), let the (random) estimates produced by the protocol \( P_{\text{matrix}} \) on input \( X \) be denoted by \( f_w \in [0,1] \) (\( w \in W \)). Then:

\[
\Pr \{ \forall w \in W : |f_w - \langle w, \text{hist}(X) \rangle| \leq \alpha \cdot a \} \geq 1 - \beta.
\]

Moreover, if the set of nonzero entries of \( wM^{-1} \) and their values can be computed in time \( T \), and \( A_{\text{Had}} \) releases a frequency oracle \( (\text{FO}, A) \) which takes time \( T' \) to query an index \( j \), then for any \( w \in W \), the estimate \( f_w \) can be computed in time \( O(T + a \cdot T') \) by \( A_{\text{matrix}} \).

**Proof.** For \( i \in [n] \), let \( S_i = \{ j \in [B] : M_{j,x_i} \neq 0 \} \) be the set of nonzero entries of the \( x_i \)th column of \( M \). Denote by \( (\text{FO}, A) \) the frequency oracle comprising the output \( A_{\text{FO}}(S(R_{\text{FO}}(S_1), \ldots, R_{\text{FO}}(S_n))) \). Define \( \hat{y} \in \mathbb{R}^B \) by \( \hat{y}_j = A(\text{FO}, j) \), for \( j \in [B] \). Then the output of \( P_{\text{matrix}} \), namely

\[
P_{\text{matrix}}(X) = A_{\text{matrix}}(S(R_{\text{matrix}}(x_1), \ldots, R_{\text{matrix}}(x_n))),
\]

is given by the map associating each \( w \in W \) to \( \langle wM^{-1}, \hat{y} \rangle \) (Algorithms 3 and 4).

Since \( (\text{FO}, A) \) is an \( (\alpha, \beta, k) \)-frequency oracle, we have that

\[
\Pr \{ \| \hat{y} - \text{hist}(S_1, \ldots, S_n) \|_\infty \leq \alpha \} \geq 1 - \beta.
\]

Notice that the histogram of \( S_i \) is given by the \( x_i \)th column of \( M \), which is equal to \( M \text{hist}(x_i) \). Thus \( \text{hist}(S_1, \ldots, S_n) = M \text{hist}(x_1, \ldots, x_n) \). By Holder’s inequality, it follows that with probability \( 1 - \beta \), for all \( w \in W \),

\[
|\langle wM^{-1}, \hat{y} \rangle - \langle wM^{-1}, M \text{hist}(x_1, \ldots, x_n) \rangle| \leq \alpha \cdot \|wM^{-1}\|_1 \leq \alpha \cdot a.
\]

But \( \langle wM^{-1}, M \text{hist}(x_1, \ldots, x_n) \rangle = wM^{-1} M \text{hist}(x_1, \ldots, x_n) = \langle w, \text{hist}(x_1, \ldots, x_n) \rangle \) is the answer to the counting query \( w \). This establishes (13).

The final claim involving efficiency follows directly from Line 3 of Algorithm 4.

---

**Algorithm 4: Analyzer for matrix mechanism**

1. **Input:** Multiset \( S \subset [B] \) consisting of the shuffled reports;

   Parameters \( W \subset \{0,1\}^B \) specifying a set of counting queries, \( n, B \in \mathbb{N}, M \in \{0,1\}^{B \times B} \), analyzer \( A_{\text{FO}} \) for frequency oracle computation

2. **Output:** Map associating each \( w \in W \) to \( f_w \in [0,1] \), specifying an estimate for each counting query \( w \)

3. **return** Map associating each \( w \in W \) to \( f_w := \sum_{j \in [B], (wM^{-1})_j \neq 0} (wM^{-1})_j \cdot A_{\text{FO}}(j) \) // Let \( \hat{y} \in \mathbb{R}^B \) be such that \( \hat{y}_j = A_{\text{FO}}(j) \); then this returns the map associating \( w \in W \) to \( \langle wM^{-1}, \hat{y} \rangle \).
Proof of Lemma 4.3. The first part is immediate from the definition of \( \mathcal{C}_B \). For the second part, given \( u \), we walk towards the root, continually going to the parent of the current node. The first time we arrive at a node that is the left child of its parent, we will be at a node in \( \mathcal{C}_B \); we let this node be \( u \).

Next we make two more definitions that will aid in the analysis:

Definition 4.3. For an integer \( j \in [B] \), let \( v(j) \) denote the number of steps from a node to its parent one must take starting at the leaf \( v_{\log_2 B, j} \) of the tree \( \mathcal{T}_B \) to get to a node in \( \mathcal{C}_B \). Equivalently, \( v(j) \) is the 2-adic valuation of \( j \) (i.e., the base-2 logarithm of the largest power of 2 dividing \( j \)).
Definition 4.4. For a positive integer $j$, let $c(j)$ be the number of ones in the binary representation of $j$.

By property (1) of Definition 4.2 the set of all values $y_{t,s}$, for $0 \leq t \leq \log B$, $1 \leq s \leq B/2^t$, is entirely determined by the values $z_a$; in particular, for any $v_{t,s}$, $y_{t,s}$ is the sum of all $z_a$ for which the leaf $v_{t,s}$ is a descendant of $v_{t,s}$. Conversely, given the values of $y_{t,s}$ for which $v_{t,s} \in C_B$ (equivalently, the values $y_{j,s_j}$ for $j \in [B]$), the values $z_j = y_{\log B,j}$ are determined as follows:

\[ z_j = y_{\log B,j} = y_{\log B-v(j),j/2^{v(j)}} - \sum_{t=1}^{v(j)-1} y_{\log B-v(j)+t,j/2^{v(j)-t}-1}. \]  

(14)

Graphically, we follow the path $P$ from $v_{\log B,j}$ to the root until we hit a node $v_{t,s}$ in $C_B$; then $z_j$ is the difference of $y_{t,s}$ and the sum of the variables stored at the left child of each node in the path $P$. (See Figure 2 for an example.)

It follows from the argument in the previous paragraph that the linear transformation that sends the vector $(z_1, \ldots, z_B)$ to the vector $((y_{t,s_1}, \ldots, y_{t,s_B}))$ is invertible; let $M_B \in \{0,1\}^{B \times B}$ be the matrix representing this linear transformation. By (14), which describes the linear transformation induced by $M_B^{-1}$, we have that $M_B^{-1} \in \{-1,0,1\}^{B \times B}$.

Since each leaf has $1 + \log B$ ancestors (including itself), we immediately obtain:

Lemma 4.4. The sensitivity of $M_B$ is given by $\Delta M_B = 1 + \log B$.

Next consider any range query $[j,j']$, so that $1 \leq j \leq j' \leq B$, and let $w \in \mathbb{R}^B$ be the row vector representing this range query (see Section 4.1). In particular all entries of $w$ are 0 apart from $w_j, w_{j+1}, \ldots, w_{j'}$, which are all 1.

Lemma 4.5. For a vector $w$ representing a range query $[j,j']$, the vector $wM_B^{-1}$ belongs to $\{-1,0,1\}^B$, and it has at most $c(j-1) + c(j') \leq 2 \log B$ nonzero entries. Moreover, the set of these nonzero entries (and their values) can be computed in time $O(\log B)$.

Proof of Lemma 4.5. Since $M_B$ is invertible, $wM_B^{-1}$ is the unique vector $\nu \in \mathbb{R}^B$ such that for any values of $\{y_{t,s}\}_{0 \leq t \leq B, s \in [B/2^t]}$ satisfying property (1) of Definition 4.2, we have

\[ z_j + z_{j+1} + \cdots + z_{j'} = y_{\log B,j} + \cdots + y_{\log B,j'} = \langle \nu, (y_{t,s_1}, \ldots, y_{t,s_B}) \rangle. \]

Next let $v_{t,s}$ be the first node in $C_B$ that is reached on the leaf-to-root path starting at $v_{\log B,j'}$. Recall from Definition 4.3 that $\hat{t} = \log B - v(j')$. Consider the path on the tree $T_B$ from the root $v_{1,1}$ to the node $v_{\hat{t},\hat{s}}$. Suppose the right child is taken at $h - 1$ vertices of this path; it is not hard to see that $h = c(j')$ (see Definition 4.4). For $1 \leq k \leq h$, at the $k$th vertex on this path where the right child is taken, set $v'_{t',s'}$ to be the left child of the parent vertex (so that $v'_{t',s'}$ is not on the path). By Lemma 4.3, $v_{t',s'} \in C_B$. Also set $v'_{h,s_h} = v_{\hat{t},\hat{s}}$. Then from Definition 4.2 (property (1)) we have

\[ z_1 + \cdots + z_{j'} = y_{\log B,1} + \cdots + y_{\log B,j'} = \sum_{k=1}^{h} y'_{t',s'} \]  

(15)

The same computation for $j - 1$ replacing $j'$ yields, with $\hat{h} = c(j - 1)$,

\[ z_1 + \cdots + z_{j-1} = y_{\log B,1} + \cdots + y_{\log B,j-1} = \sum_{k=1}^{\hat{h}} y_{t,k,s_k} \]  

(16)

where the pairs $(\hat{t}_k, \hat{s}_k)$ replace the pairs $(t',s')$. Taking the difference of (15) and (16) yields

\[ z_j + \cdots + z_{j'} = \sum_{k=1}^{h} y'_{t',s'} - \sum_{k=1}^{\hat{h}} y_{t,k,s_k}, \]

i.e., $z_j + \cdots + z_{j'}$ is a linear combination of at most $c(j - 1) + c(j')$ elements of $\{y_{t,s} : v_{t,s} \in C_B\}$, with coefficients in $\{-1,1\}$. The sets $\{(t'_k, s'_k)\}_{1 \leq k \leq h}$ and $\{(\hat{t}_k, \hat{s}_k)\}_{1 \leq k \leq \hat{h}}$ can be computed in $O(\log B)$ time by walking on the leaf-to-root path starting at $v_{\log B,j'}$ and $v_{\log B,j-1}$, respectively. This establishes Lemma 4.5.
Lemmas 4.4 and 4.5 establish properties (1) and (2) required of the matrix $M = M_B$ to guarantee $\text{poly} \log (B)$ accuracy and $\text{poly} \log (B)$ communication for private computation of 1-dimensional range queries. In the following section we use $M_B$ to construct a matrix which satisfies the same properties for $d$-dimensional range queries for any $d \geq 1$.

### 4.3 Multi-dimensional range queries

Fix any $d \geq 1$, and suppose the universe $\mathcal{X}$ consists of $B_0$ buckets in each dimension, i.e., $\mathcal{X} = [B_0]^d$. In this case, a range query $[j_1, j'_1] \times [j_2, j'_2] \times \cdots \times [j_d, j'_d]$ is specified by integers $j_1, j_2, \ldots, j_d, j'_1, j'_2, \ldots, j'_d \in [B_0]$ with $j_i \leq j'_i$ for all $i = 1, 2, \ldots, d$.

Throughout this section, we will consider the case that $d$ is a constant (and $B_0$ is large). Moreover suppose that $B_0$ is a power of 2 (again, this is without loss of generality since we can pad each dimension to be a power of 2 at the cost of a blowup in $|\mathcal{X}|$ by at most a factor of $2^d$). Write $B = |\mathcal{X}| = B_0^d$. Our goal is to define a matrix $M_{B,d}$ which satisfies analogues of Lemmas 4.4 and 4.5 for $w \in \{0, 1\}^B$ representing multi-dimensional range queries (when $[B]$ is identified with $[B_0]^d$).

The idea behind the construction of $M_{B,d}$ is to apply the linear transformation $M_{B_0}$ in each dimension, operating on a single-dimensional slice of the input vector $(z_{j_1, \ldots, j_d})_{j_1, \ldots, j_d \in [B_0]}$ (when viewed as a $d$-dimensional tensor) at a time. Alternatively, $M_{B,d}$ can be viewed combinatorially through the lens of range trees [9]: $M_{B,d}$ is a linear transformation that takes the vector $(z_{j_1, \ldots, j_d})$ to a $B$-dimensional vector whose components are the values stored at the nodes of a range tree defined in a similar manner to the range query tree $T_B$ for the case $d = 1$. However, we opt to proceed linear algebraically: the matrix $M_{B,d}$ is defined as follows. Fix a vector $z \in \mathbb{R}^B$. We will index the elements of $z$ with $d$-tuples of integers in $[B_0]$, i.e., we will write $z = (z_{j_1, \ldots, j_d})_{j_1, \ldots, j_d \in [B_0]}$. For $1 \leq p \leq d$, let $M_{B,d}^{\text{pre}}$ be the linear transformation that applies $M_{B_0}$ to each vector $(z_{j_1, \ldots, j_{p-1}, j_{p+1}, \ldots, j_d})$, where $j_1, \ldots, j_{p-1}, j_{p+1}, \ldots, j_d \in [B_0]$. That is, $M_{B_0}$ is applied to each slice of the vector $z$, where the slice is being taken along the $p$th dimension. Then let

$$M_{B,d} := M_{B,d}^{\text{pre}} \circ \cdots \circ M_{B,d}^{\text{pre}}(z).$$

(17)

We will also use an alternate characterization of $M_{B,d}$, which we develop next. First identify $\mathbb{R}^B$ with the $d$-wise tensor product of $\mathbb{R}^{B_0}$, in the following (standard) manner: Let $e_1, \ldots, e_{B_0} \in \mathbb{R}^{B_0}$ be the standard basis vectors in $\mathbb{R}^{B_0}$. Then the collection of all $e_{j_1} \otimes \cdots \otimes e_{j_d}$, where $j_1, \ldots, j_d \in [B_0]$, form a basis for $\mathbb{R}^{B_0} \otimes \cdots \otimes \mathbb{R}^{B_0}$. Under the identification $\mathbb{R}^B \cong (\mathbb{R}^{B_0})^\otimes d$, a vector $z = (z_{j_1, \ldots, j_d})_{j_1, \ldots, j_d \in [B_0]} \in \mathbb{R}^B$ is identified with the following linear combination of these basis vectors:

$$\sum_{j_1, \ldots, j_d \in [B_0]} z_{j_1, \ldots, j_d} e_{j_1} \otimes \cdots \otimes e_{j_d}.$$

Under this identification, the matrix $M_{B,d}$ corresponds to the following linear transformation of $(\mathbb{R}^{B_0})^\otimes d$:

$$M_{B_0} \otimes \cdots \otimes M_{B_0} : (\mathbb{R}^{B_0})^\otimes d \rightarrow (\mathbb{R}^{B_0})^\otimes d.$$

In the following lemmas, we will often abuse notation to allow $M_{B,d}$ to represent both the above linear transformation as well as the matrix in $\mathbb{R}^{B \times B}$ representing this transformation.

**Lemma 4.6.** We have that $M_{B,d} \in \{0, 1\}^{B \times B}$, and the sensitivity of $M_{B,d} : \mathbb{R}^B \rightarrow \mathbb{R}^B$ is bounded by $\Delta M_{B,d} \leq (1 + \log B_0)^d$.

**Proof of Lemma 4.6.** Notice that the entry of $M_{B,d}$ is given by the following product:

$$\prod_{p=1}^{d} (M_{B_0})_{j_p, j'_p}. $$
Since $M_{B_0} \in \{0, 1\}^{B_0 \times B_0}$, it follows immediately that $M_{B,d} \in \{0, 1\}^{B \times B}$. Moreover, to upper bound the sensitivity of $M_{B,d}$ we note that for any $(j_1', \ldots, j_d') \in [B_0]^d$,

$$
\sum_{(j_1, \ldots, j_d) \in [B_0]^d} \prod_{p=1}^{d} (M_{B_0})_{j_p,j_p'} = \prod_{p=1}^{d} \left( \sum_{j_p=1}^{B_0} (M_{B_0})_{j_p,j_p'} \right) \leq (\Delta M_{B_0})^d \leq (1 + \log B_0)^d,
$$

where the last inequality above uses Lemma 4.4.

**Lemma 4.7.** For the vector $w$ representing any range query $[j_1, j_1'] \times \cdots \times [j_d, j_d']$, the vector $wM_{B,d}^{-1}$ belongs to $\{-1, 0, 1\}^B$ and moreover it has at most

$$
\prod_{p=1}^{d} (c(j_p - 1) + c(j_p')) \leq (2 \log B_0)^d = (2 \log(B^{1/d}))^d
$$

nonzero entries.

**Proof of Lemma 4.7** The inverse $M_{B,d}^{-1}$ of $M_{B,d}$ is given by the $d$-wise tensor product $M_{B_0}^{-1} \otimes \cdots \otimes M_{B_0}^{-1}$. This can be verified by noting that this tensor product and $M_{B,d}$ multiply (i.e., compose) to the identity:

$$
(M_{B_0}^{-1} \otimes \cdots \otimes M_{B_0}^{-1}) \cdot M_{B,d} = (M_{B_0}^{-1} \otimes \cdots \otimes M_{B_0}^{-1}) \cdot (M_{B_0} \otimes \cdots \otimes M_{B_0})

= (M_{B_0}^{-1} \cdot M_{B_0}) \otimes \cdots \otimes (M_{B_0}^{-1} \cdot M_{B_0})

= I_{B_0} \otimes \cdots \otimes I_{B_0}

= I_B.
$$

Recall that the (row) vector $w$ representing the range query $[j_1, j_1'] \times \cdots \times [j_d, j_d']$ satisfies, for each $(j''_1, \ldots, j''_d) \in [B_0]^d$, $w_{j''_1,\ldots,j''_d} = 1$ if and only if $j''_p \in [j_p, j'_p]$ for all $1 \leq p \leq d$, and otherwise $w_{j''_1,\ldots,j''_d} = 0$. Therefore, we may write $w$ as the product of row vectors $w = w_1 \otimes \cdots \otimes w_d$, where for $1 \leq p \leq d$, $w_p$ is the (row) vector representing the range query $[j_p, j'_p]$. In particular, for $1 \leq j'' \leq B_0$, the $j''$th entry of $w_p$ is 1 if and only if $j'' \in [j_p, j'_p]$. It follows that

$$
wM_{B,d}^{-1} = (w_1 \otimes \cdots \otimes w_d)(M_{B_0}^{-1} \otimes \cdots \otimes M_{B_0}^{-1}) = w_1M_{B_0}^{-1} \otimes \cdots \otimes w_dM_{B_0}^{-1}. \tag{18}
$$

By Lemma 4.5, for $1 \leq p \leq d$, the vector $w_pM_{B_0}^{-1}$ has entries in $\{-1, 0, 1\}$, at most $c(j_p - 1) + c(j'_p)$ of which are nonzero. Since $wM_{B,d}^{-1}$ is the tensor product of these vectors and the set $\{-1, 0, 1\}$ is closed under multiplication, it also has entries in $\{-1, 0, 1\}$, at most $\prod_{p=1}^{d} (c(j_p - 1) + c(j'_p))$ of which are nonzero.

The following lemma allows us to bound the running time of the local randomizer (Algorithm 3) and analyzer (Algorithm 4).

**Lemma 4.8.** Given $B, d$ with $B = B_0^d$, the following can be computed in $O(\log^d B_0)$ time:

1. Given indices $(j_1, \ldots, j_d) \in [B_0]^d$, the nonzero indices of $M_{B,d}$ for the column indexed by $(j_1, \ldots, j_d)$.
2. Given a vector $w \in \mathbb{R}^B$ specifying a range query, the set of nonzero elements of $wM_{B,d}^{-1}$ and their values (which are in $\{-1, 1\}$).

**Proof of Lemma 4.8** We first deal with the case $d = 1$, i.e., the matrix $M_{B,1} = M_B$. Given $j, j' \in [B]$, the $(j', j)$-entry of $M_B$ is 1 if and only if the node $v_{j,j'}$ of the tree $T_B$ is an ancestor of the leaf $v_{\log B,j'}$. Since $t_j = \lfloor \log_2 j \rfloor$, $s_j = 2(j - 2^{t_j - 1}) - 1$, whether or not $v_{t_j,s_j}$ is an ancestor of $v_{\log B,j'}$ can be determined in $O(\log B)$ time, thus establishing (1) for the case $d = 1$. Notice that the statement of Lemma 4.5 immediately gives (2) for the case $d = 1$.

To deal with the case of general $d$, notice that $M_{B,d} = (M_{B_0})^{\otimes d}$. Therefore, for a given $(j_1, \ldots, j_d)$ the set

$$
\{(j_1', \ldots, j_d') : (M_{B,d})(j_1', \ldots, j_d'),(j_1, \ldots, j_d) = 1\}
$$

(19)
of nonzero indices in the \((j_1, \ldots, j_d)\)-th column of \(M_{B,d}\) is equal to the Cartesian product

\[
\bigotimes_{1 \leq p \leq d} \{j'_p : (M_{B_0})_{j'_p, j_p} = 1\}.
\]

Since each of the sets \(\{j'_p : (M_{B_0})_{j'_p, j_p} = 1\}\) can be computed in time \(O(\log B_0)\) (using the case \(d = 1\) solved above), and is of size \(O(\log B_0)\), the product of these sets \((19)\) can be computed in time \(O(\log^d B_0)\), thus completing the proof of item (1) in the lemma.

The proof of item (2) for general \(d\) is similar. For \(1 \leq p \leq d\), let \(w_p\) be the vector in \(R^{B_0}\) corresponding to the 1-dimensional range query \([j_p, j'_p]\). Then recall from \((18)\) we have that \(wM_{B,d}^{-1} = w_1M_{B_0}^{-1} \otimes \cdots \otimes w_dM_{B_0}^{-1}\). By item (2) for \(d = 1\), the nonzero entries of each of \(w_pM_{B_0}^{-1}\) (and their values) can be computed in time \(O(\log B_0)\); since each of these sets has size \(O(\log B_0)\), the set of nonzero entries of \(wM_{B,d}^{-1}\), which is the Cartesian product of these sets, as well as the values of these entries, can be computed in time \(O(\log^d B_0)\).

\[ \Box \]

### 4.4 Guarantees for differentially private range queries

In this section we state the guarantees of Theorems 4.1 and 4.2 on the privacy and accuracy of the protocol \(P_{\text{matrix}} = (R_{\text{matrix}}(n, B, M, R^{FO}), S, A_{\text{matrix}}(n, B, M, A^{FO}))\) for range query computation when \(M = M_{B,d}\) and the pair \((R^{FO}, A^{FO})\) is chosen to be either \((R^{CM}, A^{CM})\) (count-min sketch-based approach; Algorithm 1) or \((R^{Had}, A^{Had})\) (Hadamard response-based approach; Algorithm 2). For the count-min sketch-based frequency oracle, we obtain

**Theorem 4.9.** Suppose \(B_0, n, d \in \mathbb{N}\), \(B = B_0^d\), and \(0 \leq \varepsilon < 1\), and \(\beta, \delta \geq 0\). Consider the shuffled-model protocol \(P_{\text{matrix}} = (R_{\text{matrix}}, S, A_{\text{matrix}})\), where:

- \(R_{\text{matrix}} = R_{\text{matrix}}(n, B, M_{B,d}, R^{Had})\) is defined in Algorithm 3;
- \(A_{\text{matrix}} = A_{\text{matrix}}(n, B, M_{B,d}, A^{Had})\) is defined in Algorithm 4;
- and \(R^{CM} = R^{CM}(n, B, \log 2B/\beta, \gamma, 2kn)\) and \(A^{CM} = A^{CM}(n, B, \log 2B/\beta, 2kn)\) are defined in Algorithm 7 where

\[
\gamma = \frac{1}{n} \cdot \max \left\{ \log(kn), \frac{k^2 \log(B/\beta) / \varepsilon}{\varepsilon^2}, \frac{\log(\tau k/\beta)}{\varepsilon^2} \right\}
\]

and \(k = (\log 2B^{1/d})^d\).

Then:

- The protocol \(P_{\text{matrix}}\) is \((\varepsilon, \delta)\)-differentially private in the shuffled model (Definition 2.2).
- For any dataset \(X = (x_1, \ldots, x_n) \in ([B_0]^d)^n\), with probability at least \(1 - \beta\), the frequency estimate of \(P_{\text{matrix}}\) for each \(d\)-dimensional range query has additive error at most

\[
O \left( (2 \log B_0)^d \cdot \left( \log(n) + \frac{\log(B_0/\beta) \cdot (\log 2B_0)^{2d}}{\varepsilon} + \frac{\log(\log(B_0/\beta) \cdot (\log 2B_0)^{d}/\delta)}{\varepsilon^2} \right) \right).
\]

- With probability at least \(1 - \beta\), each local randomizer sends a total of at most

\[
\tilde{m} := O \left( (\log 2B_0)^d \cdot \log(B/\beta) \cdot \left( \log(n) + \frac{\log(B_0/\beta) \cdot (\log 2B_0)^{2d}}{\varepsilon} + \frac{\log((\log B_0/\beta)(\log 2B_0)^d/\delta)}{\varepsilon^2} \right) \right)
\]

messages, each of length \(O(\log(B/\beta) + \log((\log 2B_0)^d n))\). Moreover, in time \(O(n \tilde{m})\), the analyzer produces a data structure of size \(O(n \log(B/\beta)(\log 2B_0)^d \log(n \tilde{m}))\) bits, such that a single range query can be answered in time \(O((\log 2B_0)^d \cdot \log(B/\beta))\).

**Proof of Theorem 4.9.** Lemma 4.6 guarantees that \(\Delta_{MB,d} \leq (1 + \log B_0)^d = (\log 2B^{1/d})^d\). (Recall our notation that \(B = (B_0)^d\).) Then by Theorem 4.1 to show \((\varepsilon, \delta)\)-differential privacy of \(P_{\text{matrix}}\) it suffices to show \((\varepsilon, \delta)\)-differential privacy of the shuffled-model protocol \(P^{CM} := (R^{CM}, S, A^{CM})\). For the parameters above this follows from Theorem 3.6
Next we show accuracy of $P^{\text{matrix}}$. Lemma 4.7 guarantees that for any $w \in \{0, 1\}^B$ representing a range query, $w M_{B,d}^{-1}$ has at most $(2 \log B_0)^d$ nonzero entries, all of which are either $-1$ or $1$. Moreover, by Theorem 3.6, the shuffled model protocol $P^{\text{CM}}$ provides an $(\alpha, \beta, (\log 2 B_0)^d)$-frequency oracle with

$$\alpha \leq O \left( \log((\log 2 B_0)^d n) + \log(B/\beta) \cdot \frac{(\log 2 B_0)^{2d}}{\epsilon} + \frac{\log(\log(B/\beta) \cdot (\log 2 B_0)^d/\delta)}{\epsilon^2} \right).$$

By Theorem 4.2, it follows that with probability $1 - \beta$, the frequency estimates of $P^{\text{matrix}}$ on each $d$-dimensional range query have additive error at most

$$O \left( (2 \log B_0)^d \cdot \left( \log(n) + \log(B/\beta) \cdot \frac{(\log 2 B_0)^{2d}}{\epsilon} + \frac{\log(\log(B/\beta) \cdot (\log 2 B_0)^d/\delta))}{\epsilon^2} \right) \right).$$

This establishes the second item. The final item follows from Lemma 3.2, part (2) of Lemma 4.8 and the final sentence in the statement of Theorem 4.2.

Similarly, for the Hadamard response-based frequency oracle, we obtain the following:

**Theorem 4.10.** Suppose $B_0, n, d \in \mathbb{N}, B = B_0^d$, and $0 \leq \epsilon, \delta \leq 1$, and $\beta, \delta \geq 0$. Consider the shuffled-model protocol

$$P^{\text{matrix}} = (P^{\text{matrix}}, S, A^{\text{matrix}}),$$

where:

- $P^{\text{matrix}} = P^{\text{matrix}}(n, B, M_{B,d}, R^{\text{Had}})$ is defined in Algorithm 3;
- $A^{\text{matrix}} = A^{\text{matrix}}(n, B, M_{B,d}, A^{\text{Had}})$ is defined in Algorithm 4;
- and $R^{\text{Had}} = R^{\text{Had}}(n, B, \log n, \rho, (\log 2 B)^d)$ and $A^{\text{Had}} = A^{\text{Had}}(n, B, \log n, \rho, (\log 2 B)^d)$ are defined in Algorithm 2 and

$$\rho = \frac{36(\log 2 B)^{2d} \ln(\log 2 B)^d/(\epsilon \delta))}{\epsilon^2}. \quad (20)$$

Then:

- The protocol $P^{\text{matrix}}$ is $(\epsilon, \delta)$-differentially private in the shuffled model (Definition 2.2).
- For any dataset $X = (x_1, \ldots, x_n) \in ([B_0]^d)^n$, with probability $1 - \beta$, the frequency estimate of $P^{\text{matrix}}$ for each $d$-dimensional range query has additive accuracy at most $O(\epsilon^{-1}d/2(2 \log B_0)^d \cdot (\log(\log B)/\beta \delta))$.
- The local randomizers send a total of $O(n \cdot \rho)$ messages, each of length $O(n \log B_0)$ bits. The analyzer can either (a) produce a data structure of size $O(B \log n)$ bits such that a single range query can be answered in time $O((\log B_0)^d)$, or (b) produce a data structure of size $O(n \rho \log n)$ such that a single range query can be answered in time $O(n \rho \log n)$.

**Proof of Theorem 4.10.** Lemma 4.6 guarantees that $\Delta_{M_{B,d}} \leq (1 + \log B_0)^d = (\log 2 B_0)^d$. Then by Theorem 4.1, to show $(\epsilon, \delta)$-differential privacy of $P^{\text{matrix}}$ it suffices to show $(\epsilon, \delta)$-differential privacy of the shuffled-model protocol $P^{\text{Had}} := (R^{\text{Had}}, S, A^{\text{Had}})$. By Theorem 3.7, with $k = (\log 2 B_0)^d$, this holds with $\rho$ as in (20). (In particular, the parameter $\epsilon$ in Theorem 3.7 is set to $\epsilon/(\log 2 B_0)^d$, and the parameter $\delta$ in Theorem 3.7 is set to $\delta \cdot \epsilon/(\epsilon \cdot (\log 2 B_0)^d)$.)

Next we show accuracy of $P^{\text{matrix}}$. Lemma 4.7 guarantees that for any $w \in \{0, 1\}^B$ representing a range query, $w M_{B,d}^{-1}$ has at most $(2 \log B_0)^d$ nonzero entries, all of which are either $-1$ or $1$. Moreover, by Theorem 3.8, with $k = (\log 2 B_0)^d$ and $\rho$ as in (20), for any $1 \geq \beta \geq 0$, the shuffled model protocol $P^{\text{Had}}$ provides a $(\sqrt{3 \ln(2B_0/β)} \cdot \max\{3 \ln(2B_0/β), 2\beta\}, \beta, (\log 2 B_0)^d)$-frequency oracle. (Here we have used that $k \leq \rho$.) By Theorem 4.2, it follows that with probability $1 - \beta$, the frequency estimates of $P^{\text{matrix}}$ on each $d$-dimensional range query have additive error at most

$$\frac{(2 \log B_0)^d \cdot \sqrt{3 \ln(2B_0/β)} \cdot \max\{3 \ln(2B_0/β), 2\beta\}}{\epsilon} \leq \frac{12(2 \log B_0)^{2d} \cdot \sqrt{\log(2B_0/β) \cdot (\log((\log 2 B_0)^d + 1)/(\beta \epsilon))}}{\epsilon},$$

which establishes the claim regarding accuracy of $P^{\text{matrix}}$.

To establish the last item (regarding efficiency), notice that the claims regarding communication (the number of messages and message length) follow from Theorem 3.10 with $k = (\log 2 B_0)^d$. Part (a) of the claim regarding
efficiency of the analyzer follows from item 2 of Theorem 3.10 and the last sentence in the statement of Theorem 4.2. Part (b) of the claim regarding efficiency of the analyzer follows from item 3 of Theorem 3.10 and the last sentence in the statement of Theorem 4.2.
A  Proofs of auxiliary lemmas from Section 3

In this section, we prove Lemmas 3.4 and 3.5.

**Lemma 3.4** Suppose $f : \mathcal{X}^n \rightarrow \mathbb{Z}^m$ is k-incremental (Definition 3.3) and $\Delta(f) = \Delta$. Suppose $\mathcal{D}$ is a distribution supported on $\mathbb{Z}$ that is $(\varepsilon, \delta, k)$-smooth. Then the mechanism

$$X \mapsto f(X) + (Y_1, \ldots, Y_m),$$

where $Y_1, \ldots, Y_m \sim \mathcal{D}$, i.i.d., is $(\varepsilon', \delta')$-differentially private, where $\varepsilon' = \varepsilon \cdot \Delta, \delta' = \delta \cdot \Delta$.

**Proof of Lemma 3.4** Consider neighboring datasets $X = (x_1, \ldots, x_{n-1}, x_n)$ and $X' = (x_1, \ldots, x_{n-1}, x'_n)$. We will show

$$\mathbb{P}\left[ Y_1, \ldots, Y_m \sim \mathcal{D} \left| f(X) + (y_1, \ldots, y_m) \geq e^{\varepsilon'} \right. \right] \leq \delta'. \quad (21)$$

To see that (21) suffices to prove the Lemma 3.4, fix any subset $\mathcal{S} \subseteq \mathbb{Z}^m$, and write $P(X) = f(X) + (Y_1, \ldots, Y_m)$ to denote the randomized protocol. Let $\mathcal{T}$ denote the set of $f(X) + (y_1, \ldots, y_m) \in \mathbb{Z}^m$ such that the event in (21) does not hold; then we have $\mathbb{P}[P(X) \notin \mathcal{T}] \leq \delta'$. It follows that

$$\mathbb{P}[P(X) \in \mathcal{S}] \leq \delta + \sum_{w \in \mathcal{T} \cap \mathcal{S}} \mathbb{P}[P(X) = w]$$

$$= \delta' + \sum_{w \in \mathcal{T} \cap \mathcal{S}} \mathbb{P}_Y f(X) + (Y_1, \ldots, Y_m) = w$$

$$= \delta' + \sum_{w \in \mathcal{T} \cap \mathcal{S}} e^{\varepsilon'} \cdot \mathbb{P}[P(X') = w]$$

$$\leq \delta' + e^{\varepsilon'} \mathbb{P}[P(X') \in \mathcal{S}].$$

It then suffices to show (21). For $j \in [m]$, let $k_j = f(X)_j - f(X')_j$. Since the sensitivity of $f$ is $\Delta$, we have $\sum_{j=1}^m |k_j| \leq \Delta$. It follows that (21) is equivalent to

$$\mathbb{P}_y \left[ \prod_{j=1}^m \mathbb{P}_{Y_j \sim \mathcal{D}}[Y_j = y_j] \geq e^{\varepsilon'} \middle| \sum_{j=1}^m k_j \right] \leq \delta'. \quad (22)$$

For (22) to hold in turn suffices, by a union bound and the fact that at most $\Delta$ of the $k_j$ are nonzero, that for each $j$ with $k_j \neq 0$,

$$\mathbb{P}_{y \sim \mathcal{D}}[Y = y] \geq e^{k_j|\varepsilon'|/\Delta]} \leq \delta'/\Delta. \quad (23)$$

But (23) follows for $\varepsilon'/\Delta = \varepsilon, \delta'/\Delta = \delta$ since $\mathcal{D}$ is $(\varepsilon, \delta, k)$-smooth. This completes the proof. \hfill $\square$

**Lemma 3.5** Let $n \in \mathbb{N}$, $\gamma \in [0, 1/2]$, $0 \leq \alpha \leq 1$, and $k \leq \alpha \gamma n/2$. Then, the distribution $\text{Bin}(n, \gamma)$ is $(\varepsilon, \delta, k)$-smooth with $\varepsilon = \ln((1 + \alpha)/(1 - \alpha))$ and $\delta = e^{-\frac{\alpha^2 \gamma n}{8}} + e^{-\frac{\alpha^2 \gamma n}{8+2\alpha}}$.

**Proof of Lemma 3.5** Recall that for $Y \sim \text{Bin}(n, \gamma)$ and $0 \leq y \leq n$, we have $\mathbb{P}[Y = y] = \gamma^y (1 - \gamma)^{n-y} \binom{n}{y}$. Thus, we have that, for any $k \geq k' \geq -k$,

$$\frac{\mathbb{P}_{Y \sim \text{Bin}(n, \gamma)}[Y = y]}{\mathbb{P}_{Y \sim \text{Bin}(n, \gamma)}[Y = y + k']} = \frac{(1 - \gamma)^k}{\gamma^{k'}} \cdot \frac{(y + k')!(n - y - k')!}{y!(n - y)!}. \quad (24)$$
We define the interval \( \mathcal{E} := [(1 - \alpha)\gamma n + k', (1 + \alpha)\gamma n - k'] \) where \( \alpha \) is any positive constant smaller than 1. As long as \( k' \leq \alpha \gamma n/2 \), \( \mathcal{E} \) contains the interval \( \mathcal{E}' := [(1 - \alpha/2)\gamma n, (1 + \alpha/2)\gamma n] \). By the multiplicative Chernoff Bound, we have that
\[
P_{y \sim \text{Bin}(n, \gamma)}[y \notin \mathcal{E}] \leq e^{-\frac{\alpha^2 n}{2}} + e^{-\frac{\alpha^2 n}{8 + 2\alpha}}.
\] (25)

Note that for any \( y \in \mathcal{E} \), if \( k' \geq 0 \), it is the case that
\[
\frac{(1 - \gamma)k'}{\gamma^{k'}} \frac{(y + k')!(n - y - k')!}{y!(n - y)!} \leq \frac{(1 - \gamma)k'}{\gamma^{k'}} \frac{(y + 1) \cdots (y + k')}{(n - y) \cdots (n - y - k' + 1)} \leq (1 + \alpha)k'.
\] (26)

For \( y \in \mathcal{E} \) and if \( k' \leq 0 \), it is the case that
\[
\frac{(1 - \gamma)k'}{\gamma^{k'}} \frac{(y + k')!(n - y - k')!}{y!(n - y)!} = \frac{\gamma^{|k'|}}{(1 - \gamma)^{|k'|}} \frac{(n - y + 1) \cdots (n - y + |k'|)}{y(y - 1) \cdots (y - |k'|)} \leq \left( \frac{1 - \gamma + \gamma |k'|}{1 - \alpha(1 - \gamma)} \right)^{|k'|} \leq \left( \frac{1 + \alpha}{1 - \alpha} \right)^{|k'|},
\] (27)

where the last inequality above holds if \( \gamma \leq 1/2 \). We now proceed to show smoothness by conditioning on the event \( y \in \mathcal{E} \) as follows:
\[
P_{y \sim \text{Bin}(n, \gamma)} \left[ \frac{\mathbb{P}_{Y \sim \text{Bin}(n, \gamma)}[Y = y]}{\mathbb{P}_{Y \sim \text{Bin}(n, \gamma)}[Y = y + k']} \right] \geq e^{\frac{|k'|}{\gamma}} \]
\[
= \mathbb{P}_{y \sim \text{Bin}(n, \gamma)} \left[ \frac{\mathbb{P}_{Y \sim \text{Bin}(n, \gamma)}[Y = y]}{\mathbb{P}_{Y \sim \text{Bin}(n, \gamma)}[Y = y + k']} \right] \geq e^{\frac{|k'|}{\gamma}} \mathbb{P}[y \notin \mathcal{E}] + e^{\frac{-\alpha^2 n}{8 + \alpha}}
\] (28)

\[
= e^{-\frac{\alpha^2 n}{8} + \frac{\alpha^2 n}{8 + 2\alpha}}.
\] (29)

where (28) follows from (25) and (29) follows from (24), (26), and (27) as well as our choice of \( \varepsilon \).}

\[\square\]

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