Evolution of coupled fermions under the influence of an external axial-vector field

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(Dated: December 8, 2018)

The evolution of coupled fermions interacting with external axial-vector fields is described with help of the classical field theory. We formulate the initial conditions problem for the system of two coupled fermions in (3+1)-dimensional space-time. This problem is solved using the perturbation theory. We obtain in the explicit form the expressions for the leading and next to the leading order terms in the expansion over the strength of external fields. It is shown that in the relativistic limit the intensity of the fermion field coincides with the transition probability in the two neutrinos system interacting with moving and polarized matter.

PACS numbers: 14.60.Pq, 03.50.-z
Keywords: classical field theory, particle mixing, neutrino flavor oscillations, moving and polarized matter

I. INTRODUCTION

The description of the mixed fermions evolution attracts considerable attention after the experimental confirmation of solar neutrino oscillations [1, 2]. The majority of the neutrino oscillations studies involve the quantum mechanical approach to the description of the neutrino wave function evolution (see especially Ref. [3]). Despite of the fact that quantum mechanics allows one to establish the main properties of the neutrino oscillations process this method of the neutrino oscillations treatment has several disadvantages. Neutrinos are usually supposed to be scalar particles without reference to the multicomponent single neutrino wave function. The famous Pontecorvo formula (see Ref. [4]), which is in use in many theoretical and experimental studies of neutrino oscillations, is valid only for ultrarelativistic particles. However, the exact theory of the considered process must be applicable for neutrinos with arbitrary energies. Quantum mechanical approach also does not make it clear if mass or flavor eigenstates bear more physical meaning. Therefore one can see that a theoretical model of neutrinooscillations process. For example, it was discovered in Refs. [13, 11] that the transition probability can achieve great values if a neutrino interacts with background matter by means of weak currents. Thus we should develop now not only the appropriate theory of neutrino oscillations in vacuum, but also include in out treatment possible effects of neutrino interactions with external fields. During the last three decades the approaches for the theoretical substantiation of the Mikheyev-Smirnov-Wolfenstein (MSW) effect were developed. Among them we can distinguish Refs. [12, 13] in which the neutrino relativistic wave equations in dense matter were analyzed. The $S$-matrix approach was used Ref. [14] to account for the MSW effect. The influence of moving and polarized matter was described in Refs. [15, 16].

The purpose of the present work is to provide deeper understanding of the neutrino flavor oscillations phenomenon. The approach developed in our paper can not only reproduce the Pontecorvo formula for the transition probability, but also give clear physical explanation of the corrections to this expression, which are widely discussed now (see, e.g., review [17] and references therein). The analysis used in this article is based on classical field theory methods. We study the evolution of coupled classical fermions under the influence of external axial-vector fields. A classical fermion is regarded as a first quantized field because the Dirac equation,

$$\frac{1}{c} \frac{\partial \psi}{\partial t} + \alpha \nabla \psi + \frac{imc}{\hbar} \beta \psi = 0,$$

in which we use the common notations for the gamma matrices $\alpha = \gamma^0 \gamma$ and $\beta = \gamma^0$, does contain Plank constant $\hbar$, however the wave function $\psi$ is supposed to be the non-operator object in our approach. Therefore we do not involve the second quantization in the present work. Note that the discussion of the first quantized neutrino fields is also presented in Ref. [18].

In Sec. II we start from the flavor neutrino Lagrangian...
which accounts for the interaction with external axial-vector fields. Then we derive the basic integro-differential equations for the “mass eigenstates” which exactly take into account both Lorentz invariance and the interaction with external fields. These equations are also valid in (3+1) dimensions. The perturbation theory is used for the analysis of the obtained equations. In Sec. III we discuss the neutrino fields distributions and obtain the zero order term in their expansion over the external fields strength. This result corresponds to vacuum neutrino oscillations. In Sec. IV we get the first order correction to the neutrino field intensity in vacuum and show that our formula is identical to the expression for the neutrino transition probability obtained in the quantum mechanical approach for ultrarelativistic neutrinos. This case corresponds to neutrino flavor oscillations in moving and polarized matter. Finally we discuss our results in Sec. V.

II. GENERAL FORMALISM

Without losing generality we can discuss the evolution of the two coupled fermions system \((\nu_1, \nu_2)\). These fermions are taken to interact with the external axial-vector fields \(f_{\mu}^{1,2}\). The Lagrangian for this system is expressed in the following form

\[
\mathcal{L}(\nu_1, \nu_2) = \sum_{\mu=1,2} \mathcal{L}_0(\nu_k) + g \bar{\nu}_2 \nu_1 + g^* \bar{\nu}_1 \nu_2 - \sum_{\mu=1,2} \bar{\nu}_k \gamma_\mu \nu_k f_\mu^\mu,
\]

where \(g\) is the coupling constant (\(g^*\) is the complex conjugate value), \(\gamma_\mu = \gamma_\mu (1 + \gamma^5)/2\) \((\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3)\), and

\[
\mathcal{L}_0(\nu_k) = \bar{\nu}_k (i\gamma_\mu \partial_\mu - m_k) \nu_k.
\]

is the free fermion Lagrangian, \(m_k\) are the masses of the corresponding fermions \(\nu_k\).

One of the possible examples of the fermions \(\nu_k\) is the system of neutrinos belonging to different flavor states. In this case we can identify the first fermion in Eq. (2.1) with a muon neutrino \(\nu_\mu\) or a \(\tau\)-neutrino \(\nu_\tau\) and the second one with an electron neutrino \(\nu_e\). These neutrino types are known to interact with matter composed of electrons, protons and neutrons by means of the electroweak interactions. Note that an electron neutrino interacts with background fermions via both charged and neutral weak currents whereas a muon or a \(\tau\)-neutrino are involved only in the interaction through weak neutral currents. Thus the external axial-vector fields \(f_{\mu}^{1,2}\) can be expressed in terms of the hydrodynamical currents \(j_\mu^f\) and the polarizations \(\kappa_\mu^f\) of different fermions in matter (see, e.g., Refs. [12, 13, 21]),

\[
\mathcal{C}_{1,2} = \sqrt{2} G_F \sum_{f=e,p,n} \left( j_\mu^f \rho_\mu^{1,2} + \lambda_\mu^f \kappa_\mu^{1,2} \right),
\]

where \(G_F\) is the Fermi constant and

\[
\begin{align*}
\rho_\mu^{1} &= \left( f_{3L}^{(f)} - 2Q(f) \sin^2 \theta_W + \delta_\mu \right) F_\mu^\mu, \\
\rho_\mu^{2} &= \left( f_{3L}^{(f)} - 2Q(f) \sin^2 \theta_W \right), \\
\kappa_\mu^{1} &= -f_{2L}^{(f)} + \delta_\mu, \\
\kappa_\mu^{2} &= -f_{2L}^{(f)},
\end{align*}
\]

\[
\delta_\mu = \begin{cases} 
1, & f = e; \\
0, & f = n, p.
\end{cases}
\]

Here \(f_{3L}^{(f)}\) is the third isospin component of the matter fermion \(f\), \(Q(f)\) is its electric charge and \(\theta_W\) is the Weinberg angle. The hydrodynamical currents and the polarizations are related to the fermions velocities \(v_f\) and the spin vectors \(\zeta_f\) by means of the following formulas

\[
\begin{align*}
\lambda_\mu^f &= \left( v_f(n_f, \bar{n}_f v_f) \right) \kappa_\mu^f, \\
\lambda_\mu^f &= \left( n_f(\zeta_f v_f), n_f \zeta_f \sqrt{1 - v_f^2} + n_f v_f(\zeta_f v_f) \right) \kappa_\mu^f,
\end{align*}
\]

The detailed derivation of Eqs. (2.2) is presented in Refs. [12, 13, 21].

Following the results of our previous work [9] we will study the evolution of the system (2.1) by solving the Cauchy problem. Let us choose the initial conditions in the form

\[
\begin{align*}
\nu_1(r, 0) &= 0, \\
\nu_2(r, 0) &= \xi(r),
\end{align*}
\]

where \(\xi(r)\) is the known function. If one considers the fermions \(\nu_k\) as flavor neutrinos, the initial conditions in Eq. (2.5) correspond to the common situation in a neutrino oscillations experiment, i.e. \(\nu_{\mu,\tau}\) are absent initially and \(\nu_e\) has some known field distribution. We will be interested in searching for the fields distributions \(\nu_k(r, t)\) for \(t > 0\).

In order to solve the Cauchy problem we introduce the new set of the fermions \((\psi_1, \psi_2)\) by means of the matrix transformation

\[
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.
\]

The mixing matrix (which is parameterized with help of the one angle \(\theta\) in Eq. (2.6)) is chosen in the way to eliminate the second and the third terms in Eq. (2.1). If we had studied the evolution of our system without the external fields \(f_{\mu}^{2}\), i.e. in vacuum, the fermions \(\psi_k\) would have been called mass eigenstates because they would have diagonalized the Lagrangian and therefore \(\psi_k\) would have had definite masses. In our case \(f_{\mu}^{2} \neq 0\), if we simplify the vacuum mixing terms, the matter term becomes more complicated compared to Eq. (2.1). The Lagrangian rewritten in terms of the fields \(\psi_k\) can be
expressed in the following way,
\[
\mathcal{L}(\psi_1, \psi_2) = \sum_{k=1,2} \mathcal{L}_0(\psi_k) - [\bar{\psi}_1 \gamma^\mu_1 \psi_1 (c^2 f_1^\mu + s^2 f_2^\mu) \\
+ \bar{\psi}_2 \gamma^\mu_2 \psi_2 (c^2 f_2^\mu + s^2 f_1^\mu) \\
+ sc(\bar{\psi}_1 \gamma^\mu_1 \psi_2 + \bar{\psi}_2 \gamma^\mu_2 \psi_1)(f_1^\mu - f_2^\mu)],
\] (2.7)
where
\[
\mathcal{L}_0(\psi_k) = \bar{\psi}_k (i\gamma^\mu \partial_\mu - m_k) \psi_k.
\]

It should be noted that the masses \(m_k\) of the fermions \(\psi_k\) are related to the masses \(m_k\) by the formula,
\[
m_1 = c^2 m_1 + s^2 m_2, \quad m_2 = c^2 m_2 + s^2 m_1,
\]
In Eq. (2.7) we use the notations \(c = \cos \theta\) and \(s = \sin \theta\).

Eq. (2.7) has some advantages in comparison with Eq. (2.11) despite of the more complicated matter interaction term. The terms \(g \bar{\psi} \nu_1 \nu_2\) and \(g^* \bar{\psi} \nu_1 \nu_2\) in Eq. (2.11), which are responsible for the vacuum oscillations cannot be treated within the perturbation theory. In order to describe the flavor changing processes one has to take into account these terms exactly. In Eq. (2.7) we have the additional interaction terms which can be analyzed in the common way with help of the perturbation theory if the strength of the fields \(f_{1,2}^\mu\) is supposed to be weak. The criterion of the external fields weakness is discussed in details in Sec. [V]. It is also possible to assume the convenient time dependence (sometimes it is necessary to consider the adiabatic switching-on of the interaction) of the fields \(f_{1,2}^\mu\) in Eq. (2.7). On the contrary we cannot "switch-on" or "switch-off" the constant \(g\) in Eq. (2.11) at some moments of time because it is related to the properties of a theoretical scheme underlying the phenomenological model of neutrino oscillations.

Now let us discuss the evolution of the system (2.7) with the initial conditions
\[
\psi_1(\mathbf{r}, 0) = -s \xi(\mathbf{r}), \quad \psi_2(\mathbf{r}, 0) = c \xi(\mathbf{r}),
\]
which follow from Eqs. (2.5) and (2.6). We also suppose that the external fields \(f_{1,2}^\mu\) are weak and one is able to take them into account in the lowest order of the perturbation theory. If we had solved this problem using the quantum field theory, we would have calculated the contributions of the four Feynman diagrams shown on Fig. 1.

We can always rewrite the Dirac equations which result from Eq. (2.7) in the form,
\[
i \dot{\psi}_1 = (H_1 + V_1)\psi_1 + V\psi_2, \\
i \dot{\psi}_2 = (H_2 + V_2)\psi_2 + V\psi_1,
\] (2.8)
where \(V_1 = \beta \gamma^\mu \left(c^2 f_1^\mu + s^2 f_2^\mu\right), \quad V_2 = \beta \gamma^\mu \left(c^2 f_2^\mu + s^2 f_1^\mu\right), \quad V = sc\gamma^\mu (f_1^\mu - f_2^\mu)
\]
and \(H_1 = -i\alpha \vec{\nabla} + \beta m_k\) are the free fields Hamiltonians. We are searching for the solutions of Eqs. (2.8) in the following way,
\[
\psi_k(\mathbf{r}, t) = \int \frac{d^3 p}{(2\pi)^{3/2}} [a_k(\mathbf{p}, t) \Psi^+_{k\mathbf{p}}(x) + b_k(\mathbf{p}, t) \Psi^-_{k\mathbf{p}}(x)],
\] (2.9)
where \(\Psi^+_{k\mathbf{p}}(x) = u_k(\mathbf{p}) e^{-ip_k x}\) and \(\Psi^-_{k\mathbf{p}}(x) = v_k(\mathbf{p}) e^{ip_k x}\) are the basis spinors, \(x^\mu = (t, \mathbf{r})\), \(p_k^\mu = (E_k, \mathbf{p})\) and \(E_k = \sqrt{\mathbf{p}^2 + m_k^2}\) is the energy of the fermion \(\psi_k\).

The coefficients \(a_k(\mathbf{p}, t)\) and \(b_k(\mathbf{p}, t)\) are not the creation and annihilation operators since we are using here the classical field theory. The values of these functions should be chosen in the way to satisfy the initial conditions (2.6).

Note that there is the additional time dependence of these functions in contrast to the case of the flavor changing process in vacuum (see Ref. [3]).

Using the orthonormality condition of the basis spinors \(\Psi^+_{k\mathbf{p}}(x) = u_k(\mathbf{p}) e^{-ip_k x}\) and \(\Psi^-_{k\mathbf{p}}(x) = v_k(\mathbf{p}) e^{ip_k x}\) we obtain from Eqs. (2.8) the new integro-differential equations for the functions \(a_k(\mathbf{p}, t)\) and \(b_k(\mathbf{p}, t)\),
\[
i \dot{a}_1(\mathbf{p}, t) = \frac{1}{(2\pi)^{3/2}} \left( \int d^3 r \Psi^+_{1\mathbf{p}}(x) V_1 \psi_1(\mathbf{r}, t) + \int d^3 r \Psi^+_{1\mathbf{p}}(x) V \psi_2(\mathbf{r}, t) \right),
\]
\[
i \dot{b}_1(\mathbf{p}, t) = \frac{1}{(2\pi)^{3/2}} \left( \int d^3 r \Psi^+_{1\mathbf{p}}(x) V_1 \psi_1(\mathbf{r}, t) + \int d^3 r \Psi^+_{1\mathbf{p}}(x) V \psi_2(\mathbf{r}, t) \right),
\]
\[
i \dot{a}_2(\mathbf{p}, t) = \frac{1}{(2\pi)^{3/2}} \left( \int d^3 r \Psi^+_{2\mathbf{p}}(x) V_2 \psi_2(\mathbf{r}, t) + \int d^3 r \Psi^+_{2\mathbf{p}}(x) V \psi_1(\mathbf{r}, t) \right),
\]
\[
i \dot{b}_2(\mathbf{p}, t) = \frac{1}{(2\pi)^{3/2}} \left( \int d^3 r \Psi^+_{2\mathbf{p}}(x) V_2 \psi_2(\mathbf{r}, t) + \int d^3 r \Psi^+_{2\mathbf{p}}(x) V \psi_1(\mathbf{r}, t) \right).
\]
These equations correctly take into account both the Lorentz invariance and the interaction with the external fields $f_{1,2}^{\mu}$. However if we suppose that these external fields are rather weak, it is possible to look for the solutions of Eqs. (2.10) in the form of the series, where the functions $\psi_k(r,t)$, which correspond to the function $\psi_k(r,t)$, do not depend on time. The functions $\psi_k(r,t)$ are responsible for the evolution of the considered system in vacuum, i.e. at $f_{1,2}^{\mu} = 0$. These functions in two dimensional space-time have been found in our previous work where the analogous Cauchy problem has been solved in the explicit form in (1+1)-dimensional case. However to describe the evolution of our system with the non-zero external fields in the (3+1)-dimensional space-time we should study the vacuum case also in (3+1) dimensions.

### III. EVOLUTION OF THE SYSTEM IN VACUUM

To study the behavior of $\psi_k(r,t)$, i.e. the evolution of our system in vacuum, we use the results of Ref. [9] where it was revealed that the fields distributions for the given initial conditions have the form,

$$\psi_k(r,t) = -s \int d^3r' S_k(r' - r, t)(-i\beta)\xi(r'),$$

$$\psi_2(r,t) = c \int d^3r' S_2(r' - r, t)(-i\beta)\xi(r'),$$

(3.1)

where $S_k(r,t)$ is the Pauli-Jordan function for the fermion $\psi_k$ (see, e.g., Ref. [21]). It should be noted that Eqs. (3.1) are the most general ones and valid in (3+1) dimensions. Contrary to the approach of Ref. [9] here we use the momentum representation because the integrations in Eqs. (3.1) are rather cumbersome in the coordinate representation. Thus one rewrites these expressions using the Fourier transform of the initial conditions.

Eqs. (3.1) now take the form,

$$\psi_1(r,t) = -s \int \frac{d^3p}{(2\pi)^3} e^{ipr} S_1(-p, t)(-i\beta)\xi(p),$$

$$\psi_2(r,t) = c \int \frac{d^3p}{(2\pi)^3} e^{ipr} S_2(-p, t)(-i\beta)\xi(p),$$

(3.2)

where

$$S_k(r, t) = \left[ \cos \xi_k t + i \frac{\sin \xi_k t}{\xi_k} (\alpha p + \beta m_k) \right] (i\beta),$$

and

$$\xi(p) = \int d^3pe^{-ipr}\xi(r),$$

are the Fourier transforms of the functions $S_k(r,t)$ and $\xi(r)$.

Now let us choose the initial condition. We suppose that the initial field distribution of $\nu_2$ is the plane wave, i.e. $\xi(r) = e^{i\omega r}\xi_0$, where $\xi_0$ is the normalization spinor. The Fourier transform of this function can be simply computed, $\xi(p) = (2\pi)^3\delta^3(\omega - p)\xi_0$. Using Eqs. (3.2) we get the fields distributions in the (3+1)-dimensional space-time for the plane wave initial condition,

$$\psi_1(r,t) = -se^{i\omega r} \left[ \cos \xi_1(\omega)t \right. - i \sin \xi_1(\omega)t \left. \left( \alpha \omega + \beta m_1 \right) \right] \xi_0,$$

$$\psi_2(r,t) = se^{i\omega r} \left[ \cos \xi_2(\omega)t \right. - i \sin \xi_2(\omega)t \left. \left( \alpha \omega + \beta m_2 \right) \right] \xi_0,$$

(3.4)

where $\xi_k(\omega) = \sqrt{\omega^2 + m_k^2}$.

In the following we discuss the case of the rapidly oscillating initial conditions, i.e. $\omega \gg m_{1,2}$. One obtains from Eqs. (3.4) the fields distributions for $\omega \gg m_{1,2}$ in the following form,

$$\psi_1(r,t) = -se^{i\omega r} \left[ \cos \xi_1(\omega)t \right. - i \sin \xi_1(\omega)t \left. \left( \alpha \omega + \beta m_1 \right) \right] \xi_0,$$

$$\psi_2(r,t) = se^{i\omega r} \left[ \cos \xi_2(\omega)t \right. - i \sin \xi_2(\omega)t \left. \left( \alpha \omega + \beta m_2 \right) \right] \xi_0,$$

(3.5)

where $n = \omega / \omega$ is the unit vector in the direction of the initial field distribution momentum. The fermion $\nu_1$ is absent at $t = 0$. Therefore it would be interesting to examine the field distribution $\nu_1^{(0)}(r,t)$ at $t > 0$. Using Eqs. (2.10) and (3.5) we obtain,

$$\nu_1(r,t) = c\nu_1^{(0)} + s\nu_2^{(0)} = \sin 2\theta \sin[\Delta(\omega)t] \times \left[ \sin[\sigma(\omega)t] + i(\alpha n) \cos[\sigma(\omega)t] \right] e^{i\omega t}\xi_0,$$

(3.6)

where

$$\sigma(\omega) = \frac{\xi_1(\omega) + \xi_2(\omega)}{2} \to \omega + \frac{m_1^2 + m_2^2}{4\omega},$$

$$\Delta(\omega) = \frac{\xi_1(\omega) - \xi_2(\omega)}{2} \to \frac{m_1^2 - m_2^2}{4\omega} = \frac{\Delta m^2}{4\omega}.$$
The measurable quantity of a classical spinor field is the intensity. With the help of Eq. (3.2) one gets the intensity of the fermion $\nu_1^{(0)}$ in the following form,

$$I_1^{(0)}(t) = |\nu_1^{(0)}(r, t)|^2 = \sin^2(2\theta) \sin^2[\Delta(t)]$$

$$\times \xi_0^2 |\sin[\sigma(t)] + i(\alpha \mu_0 \cos[\sigma(t)]|^2 \xi_0$$

$$= \sin^2(2\theta) \sin^2 \left( \frac{\Delta m^2}{4\omega} t \right), \quad (3.7)$$

### IV. INTERACTION OF THE SYSTEM WITH AN EXTERNAL FIELD

In order to proceed in our study of the two neutrino system evolution under the influence of the external fields $f_{1,2}^\mu$, we discuss the further correction to the vacuum case. The first order corrections to Eqs. (3.2) can be derived from Eqs. (2.9) and (2.10) and have the form,

$$\psi_{1,2}^{(1)}(r, t) = i \int \frac{d^3 p}{(2\pi)^3} e^{ipr} \left\{ E_1 \left[ sV_1 (\mathbf{G}_1^+ e^{-i\mathbf{E}_1 t} + \mathbf{G}_1^- e^{+i\mathbf{E}_1 t}) - cV (\mathbf{G}_2^+ e^{-i\mathbf{E}_2 t} + \mathbf{G}_2^- e^{+i\mathbf{E}_2 t}) \right] \right. \right.$$  

$$+ (\alpha \mu_1 + \beta \nu_2) \left[ sV_2 (\mathbf{G}_2^+ e^{-i\mathbf{E}_2 t} + \mathbf{G}_2^- e^{+i\mathbf{E}_2 t}) - cV (\mathbf{G}_2^+ e^{-i\mathbf{E}_2 t} + \mathbf{G}_2^- e^{+i\mathbf{E}_2 t}) \right] \left. \right\} (4.1)$$

where

$$\mathbf{G}_{1,2}^\pm = \int_0^t e^{i\mathbf{E}_{1,2} t} S_2(-p, t) dt, \quad \mathbf{E}_{1,2}^\pm = \int_0^t e^{i\mathbf{E}_{1,2} t} S_1(-p, t) dt, \quad \mathbf{E}_{1,2}^\pm = \int_0^t e^{i\mathbf{E}_{1,2} t} S_1(-p, t) dt. \quad (4.2)$$

When we derive Eqs. (4.1), we suppose that $a_{1,2}^{(1)}(p, 0) = b_{1,2}^{(1)}(p, 0) = 0$. It means that at $t = 0$ the fields distributions are determined by Eqs. (3.2). One can find the explicit form of the functions given in Eqs. (4.2) using Eq. (3.3),

$$\mathbf{E}_{1,2} = \frac{1}{2} \left\{ e^{+i\mathbf{E}_1 t} t \right\} + i \left\{ \frac{1}{2i\mathbf{E}_1,2} \right\} \left[ \begin{array}{c} \alpha \mu_1 + \beta \nu_2 \\ \mathbf{E}_1,2 \end{array} \right]$$

where

$$\sigma = \frac{\mathbf{E}_1 + \mathbf{E}_2}{2}, \quad \Delta = \frac{\mathbf{E}_1 - \mathbf{E}_2}{2}.$$

On the basis of Eqs. (4.1) and (4.3) we obtain the expressions for the first order corrections to the vacuum case in the following form,
\[ -sV \left\{ \frac{\sin \sigma}{\sigma} \cos \Delta t + \frac{\sin \Delta t}{\Delta} \cos \sigma t - i \frac{\alpha p + \beta m_1}{\xi_1} \sin \sigma t \sin \Delta t \left( \frac{1}{\sigma} + \frac{1}{\Delta} \right) \right\} \\
+ (\alpha p + \beta m_2) \left\{ cV_2 \left\{ -it \sin \xi_2 t + \frac{\alpha p + \beta m_2}{\xi_2} \left( t \cos \xi_2 t - \frac{\sin \xi_2 t}{\xi_2} \right) \right\} \\
- sV \left\{ i \sin \sigma t \sin \Delta t \left( \frac{1}{\sigma} - \frac{1}{\Delta} \right) + \frac{\alpha p + \beta m_1}{\xi_1} \left( \frac{\sin \Delta t}{\Delta} \cos \sigma t - \frac{\sin \sigma t}{\sigma} \cos \Delta t \right) \right\} \right\} \xi(p). \] (4.4)

Note that these expressions are valid for the arbitrary initial conditions \(\xi(p)\) and exactly take into account the Lorentz invariance. It should also be mentioned that along with the harmonic functions there are several terms in the integrands which linearly depend on time. Therefore for Eqs. (4.4) to be meaning-bearing one has to assume that the potentials \(V_{1,2}\) and \(V\) are rather weak, i.e. we study the interaction of our system with weak external fields (see Eq. (4.9) below).

The integrations in Eqs. (4.4) are rather complicated for the arbitrary initial conditions. That is why we choose the function \(\xi(p)\) analogously to Sec. III, i.e. we again suppose that \(\xi(p) = (2\pi)^3 \delta^3(\omega - p) \xi_0\). The integrations over momenta are eliminated with help of the \(\delta\)-functions. We also consider high frequency approximation, i.e. \(\omega \gg m_{1,2}\). Finally we obtain the following expressions for the fields distributions of the fermions \(\psi\),

\[
\psi_1^{(1)}(r, t) = i e^{i \omega r} \left\{ \frac{1}{2} \left\{ s[tV_1 + (\alpha n) V_1(\alpha n)](\cos[\xi_1(\omega)t] - i(\alpha n)\sin[\xi_1(\omega)t]) \right\} \xi_0, \\
- c \frac{\sin[\Delta(\omega)t]}{\Delta(\omega)} \left\{ V + (\alpha n) V(\alpha n) \right\} \left\{ \cos[\sigma(\omega)t] - i(\alpha n)\sin[\sigma(\omega)t] \right\} \right\} \xi_0, \\
\psi_2^{(1)}(r, t) = -i e^{i \omega r} \left\{ \frac{1}{2} \left\{ c[tV_2 + (\alpha n) V_2(\alpha n)](\cos[\xi_2(\omega)t] - i(\alpha n)\sin[\xi_2(\omega)t]) \right\} \xi_0. \\
- s \frac{\sin[\Delta(\omega)t]}{\Delta(\omega)} \left\{ V + (\alpha n) V(\alpha n) \right\} \left\{ \cos[\sigma(\omega)t] - i(\alpha n)\sin[\sigma(\omega)t] \right\} \right\} \xi_0. \] (4.5)

On the basis of Eqs. (4.6) and (4.5) we can derive the first order correction to the field distribution of the fermion \(\nu_1\) in the form,

\[
\nu_1^{(1)}(r, t) = -\sin 2\theta e^{i \omega r} \left\{ \frac{1}{4} \left\{ \cos 2\theta \sin[\Delta(\omega)t]/\Delta(\omega) \right\} \left\{ F_1 - F_2(\cos[\sigma(\omega)t] - i(\alpha n)\sin[\sigma(\omega)t]) \right\} \\
- t \left\{ F_1 \left\{ c^2(\cos[\xi_1(\omega)t] - i(\alpha n)\sin[\xi_1(\omega)t]) - s^2(\cos[\xi_2(\omega)t] - i(\alpha n)\sin[\xi_2(\omega)t]) \right\} \\
- F_2 \left\{ c^2(\cos[\xi_2(\omega)t] - i(\alpha n)\sin[\xi_2(\omega)t]) - s^2(\cos[\xi_1(\omega)t] - i(\alpha n)\sin[\xi_1(\omega)t]) \right\} \right\} \right\} \xi_0. \] (4.6)

where \(F_{1,2} = [f_0^2 - (f_{1,2} n)(\Sigma n)](1 + \gamma_5)\) and \(\Sigma = -\gamma^5\alpha\).

To calculate the intensity of the field \(\nu_1\) one should take into account that the final expression for the intensity must contain only terms linear in external fields. Therefore the first order correction to the intensity should be calculated with help of the formula

\[
I_1^{(1)}(t) = \nu_1^{(0)}(t), \quad I_1^{(1)}(t) = \nu_1^{(1)}(t). 
\]

Using Eqs. (4.6) and (4.6) we get the expression for \(I_1^{(1)}\),

\[
I_1^{(1)}(t) = \sin^2(2\theta) \cos 2\theta \sin[\Delta(\omega)t] \times \left\{ \left( \frac{\sin[\Delta(\omega)t]}{\Delta(\omega)} - t \cos[\Delta(\omega)t] \right) \times \left( \left( f_0^2 - (f_0 n) - f_1 n \right) \right) \right\} (1 + \gamma_5). \] (4.7)

In Eq. (4.7) we use the notation \(\langle \ldots \rangle = \xi_0^t(\ldots)\xi_0^s\). To compute the mean value with help of the normalization spinor \(\xi_0\) we can suppose that \(\xi(r) = \exp(-i E_{\nu_1} t) \xi_0\). Then we notice that for spinors corresponding to high energies one has the obvious identities \((1 + \gamma_5) \xi_0 \approx 2 \xi_0\) and \(\xi_0^t(\alpha n) \xi_0 \approx 1\). Putting together Eqs. (4.7) and (4.7) we obtain the final expression for the intensity of the fermion \(\nu_1\),

\[
I_1(t) = I_1^{(0)}(t) + I_1^{(1)}(t) = \sin^2(2\theta) \times \left\{ \left( \frac{\sin[\Delta(\omega)t]}{\Delta(\omega)} - t \cos[\Delta(\omega)t] \right) \right\} \times \left( \left( f_0^2 - (f_0 n) - f_1 n \right) \right) \right\} \right\}. \] (4.8)

Using Eq. (4.8) it is possible to define the scope of the applied method, i.e. we can evaluate the strength of external fields necessary for the perturbative approach to be valid. With help of Eq. (4.8) one obtains the inequal-
\begin{align}
A \cos 2\theta \ll \Delta(\omega), \quad A \cos 2\theta \ll 1,
\end{align}

where \( A = |[f_2^0 - (f_2 n)] - [f_1^0 - (f_1 n)]| \). If Eq. (4.9) is satisfied, the contribution of external axial-vector fields to neutrino flavor oscillations is small compared to the vacuum term. It should be noted that Eq. (4.9) is valid in the ultrarelativistic case. For neutrinos with \( \mathcal{E}_k(\omega) \sim m_k \) we should rely on Eqs. (4.14) rather than on Eqs. (4.15). In this case the condition of our method applicability will be different from Eq. (4.9).

Now let us compare Eq. (4.8) with the neutrino transition probability formula. Flavor neutrinos are considered to interact with external axial-vector fields as it is described in Eqs. (2.2)-(2.4). Then the probability to find muon or \( \tau \)-neutrinos in the electron neutrinos beam in presence of moving and polarized matter is expressed in the following way (see, e.g., Refs. [15, 16])

\begin{align}
P_{\nu_e \rightarrow \nu_{\mu, \tau}}(t) = \sin^2(2\theta_{\text{eff}}) \sin^2 \left( \frac{\pi t}{L_{\text{eff}}} \right),
\end{align}

where

\begin{align}
\sin^2(2\theta_{\text{eff}}) &= \frac{\Delta^2(\omega) \sin^2(2\theta)}{[\Delta(\omega) \cos 2\theta - A/2]^2 + \Delta^2(\omega) \sin^2(2\theta)},
\end{align}

is the definition of the effective mixing angle and

\begin{align}
\frac{\pi}{L_{\text{eff}}} = \sqrt{[\Delta(\omega) \cos 2\theta - A/2]^2 + \Delta^2(\omega) \sin^2(2\theta)},
\end{align}

is the definition of the effective oscillation length.

To discuss the weak external field limit in Eqs. (4.10)-(4.12) we should expand Eqs. (4.11) and (4.12) over small parameter \( A \) [see Eq. (4.9)]. As a result one gets

\begin{align}
\sin^2(2\theta_{\text{eff}}) &\approx \sin^2(2\theta) \left( 1 + \frac{A \cos 2\theta}{\Delta(\omega)} \right), \\
\frac{\pi}{L_{\text{eff}}} &\approx \Delta(\omega) - (A \cos 2\theta)/2.
\end{align}

It is also necessary to expand the time dependent factor in Eq. (4.10),

\begin{align}
\sin^2 \left( \frac{\pi t}{L_{\text{eff}}} \right) \approx \sin^2[2\Delta(\omega) t] - A \cos 2\theta \sin[\Delta(\omega) t] \cos[\Delta(\omega) t].
\end{align}

Note that Eq. (4.14) is valid while \( A \cos 2\theta \ll 1 \), that coincides with the second inequality in Eq. (4.9). With help of Eqs. (4.13) and (4.14) the neutrino transition probability is reduced to the form,

\begin{align}
P_{\nu_e \rightarrow \nu_{\mu, \tau}}(t) = \sin^2(2\theta) \left\{ \sin^2[\Delta(\omega) t] + \cos 2\theta \sin[\Delta(\omega) t] \right. \\
\times \left( \frac{\sin[\Delta(\omega) t]}{\Delta(\omega)} - t \cos[\Delta(\omega) t] \right) \\
\times ([f_2^0 - (f_2 n)] - [f_1^0 - (f_1 n)]),
\end{align}

which coincides with Eq. (1.8). This comparison shows that neutrino flavor oscillations in weak axial-vector fields (e.g., if a neutrino propagates in moving and polarized matter) can be treated with help of the classical field theory approach.

V. CONCLUSION

In conclusion we mention that the evolution of coupled classical fermions under the influence of external axial-vector fields has been studied in the present paper. We have discussed the particular case of two coupled fermions and formulated the Cauchy problem for this system. If the initial conditions were chosen in the appropriate way, as it has been shown in Sec. II the described system might serve as a theoretical model of neutrino flavor oscillations in matter. The initial conditions problem has been solved with help of the perturbation theory. We have found the zero and the first order terms in the fields distributions expansions over the external fields strength. It should be noted that the obtained results exactly take into account the Lorentz invariance as well as they are valid in (3+1)-dimensional space-time. The intensity of the zero order term corresponds to the case of the neutrino flavor oscillations in vacuum. Therefore we have generalized our previous calculations performed in (1+1) dimensions in Ref. [9]. The first order correction is responsible for the neutrino interaction with moving and polarized matter. We have obtained this intensity of the fermion field at great oscillations frequencies of the initial field distribution, that corresponds to ultrarelativistic neutrinos. Note that we have compared our results with the transition probability formula for neutrino flavor oscillation in moving and polarized matter derived in Refs. [15, 16] and revealed an agreement in the case of weak external fields. This comparison proves the validity of the method elaborated in our work. Finally it has been demonstrated that neutrino flavor oscillations in moving and polarized matter could be described with help of the classical field theory.

It is interesting to notice that along with the usual neutrino oscillations phase equal to \( \Delta m^2/(4\omega) \) the classical field theory approach also reveals rapid harmonic oscillations on the frequency [see, e.g., Eqs. (4.9) and (4.14)]

\begin{align}
\omega_{\text{rapid}} = \frac{\mathcal{E}_1(\omega) + \mathcal{E}_2(\omega)}{2} \rightarrow \omega + \frac{m_1^2 + m_2^2}{4\omega}.
\end{align}

However these terms are suppressed by the ratios \( m_k/\omega \) which are small for great values of \( \omega \). This case corresponds to ultrarelativistic neutrinos. The analogous terms were discussed in many previous publications devoted to neutrino flavor oscillations (see, e.g., Refs. [4, 5]). In our work it has been demonstrated that such contributions to the neutrino transition probability appear even in the classical field theory approach. These terms arise from the accurate account of the Lorentz invariance.
Acknowledgments

This research was supported by grant of Russian Science Support Foundation. The author is indebted to Alexander Grigoriev and Alexander Studenikin (MSU) for helpful discussions as well as to Takuya Morozumi for the invitation to visit Hiroshima University.

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