Graph polynomials and Tutte-Grothendieck invariants: an application of elementary finite Fourier analysis

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Abstract

This paper is based on a series of talks given at the Patejdlovka Enumeration Workshop held in the Czech Republic in November 2007. The topics covered are as follows. The graph polynomial, Tutte-Grothendieck invariants, an overview of relevant elementary finite Fourier analysis, the Tutte polynomial of a graph as a Hamming weight enumerator of its set of tensions (or flows), description of a family of polynomials containing the graph polynomial which yield Tutte-Grothendieck invariants in a similar way.

1 Introduction

The graph polynomial is a generalization of the Vandermonde determinant (which may be viewed as the graph polynomial of a complete graph) that was considered by Sylvester and Petersen in the nineteenth century in connection with binary quantic forms. Alon, Tarsi and Matiyasevich in more recent years have found that it contains a good deal of information about the vertex colourings of a graph. For example, the number of proper 3-colourings of a graph is a simply defined function of the coefficients of its graph polynomial. In this article we consider a family of polynomials containing the graph polynomial and ask whether other Tutte-Grothendieck invariants can be obtained in a similar way. Our results are obtained by expressing the relevant parameter as the partition function of a vertex colouring model (such as the Potts model) or, in different language, the graph parameter obtained from homomorphisms to a weighted graph.

In Section 2 we define the graph polynomial and explore its relation to proper vertex colourings. In Section 3 Tutte-Grothendieck invariants are defined and their pervasiveness noted. In Section 4 a potted account is given of Fourier analysis on finite Abelian groups which will be used to obtain our results. In Section 4.4 tensions and flows of a graph are defined and the view of the Tutte polynomial as a Hamming weight enumerator propounded. In the final Section 5 we characterize those polynomials which share with the graph polynomial the property of yielding a Tutte-Grothendieck invariant from the $\ell_2$-norm of their coefficients. More generally, the graph polynomial is seen to belong to a family of polynomials whose $\ell_2$-norm is equal to an evaluation of the complete weight enumerator of the set of tensions (or flows) of the graph.

An expanded version of Section 4 can be found in [12], and an even more fulsome presentation is given in [11]. The book [21] is recommended for an introduction to finite Fourier analysis and its wide range of applications.
2 The graph polynomial

Let $G = (V, E)$ be a graph with some fixed, arbitrary orientation of its edges, and denote its directed edge set by $\overrightarrow{E}$.

Let $Q$ be a finite set of size $q$. A proper vertex $q$-colouring using colour set $Q$ is an assignment of colours $(c_v : v \in V) \in Q^V$ such that $c_u \neq c_v$ whenever $\{u, v\} \in E$. The number of proper vertex $q$-colourings of $G$ is denoted by $P(G; q)$ (an evaluation of the chromatic polynomial of $G$ at $q$).

Let $x = (x_v : v \in V)$ be a tuple of commuting indeterminates indexed by $V$ and define the graph polynomial $F(G)$ in $\mathbb{C}[x]$ by

$$F(G; x) = \prod_{(u, v) \in \overrightarrow{E}} (x_u - x_v).$$

Given an assignment of values $c = (c_v : v \in V) \in \mathbb{C}^V$ to the indeterminates $x = (x_v : v \in V)$, the graph polynomial takes a non-zero value if and only if $c$ corresponds to a proper vertex colouring with colour set $Q = \{c_v : v \in V\}$. By restricting $c_v$ to one of the $q$ points $1, \zeta, \ldots, \zeta^{q-1}$ on the unit circle a criterion emerges for the existence of a proper vertex $q$-colouring of $G$ in terms of the polynomial $F(G; x)$.

The algebraic variety of points $\{c_v : v \in V\}$ for $c_v \in \{1, \zeta, \ldots, \zeta^{q-1}\}$ corresponds to the ideal $(x_v^q - 1 : v \in V)$ of the ring $\mathbb{C}[x]$. Denote the graph polynomial modulo the ideal generated by the polynomials $x_v^q - 1$ by

$$F(q)(G; x) = F(G; x) \mod (x_v^q - 1 : v \in V).$$

By Lagrange interpolation,

$$F(q)(G; x) = \sum_{(a_v : v \in V) \in \{0, 1, \ldots, q - 1\}^V} \prod_{v \in V} \frac{x_v - \zeta^{a_v}}{\zeta^{a_v} - \zeta^a} F(G; (\zeta^{a_v} : v \in V)),$$

where the summations are over $(a_v : v \in V) \in \{0, 1, \ldots, q - 1\}^V$ and the last line follows since $\prod_{v \neq a_v} (\zeta^{a_v} - \zeta^a) = \zeta^{(q - 1)a} \prod_{b \neq 0} (1 - \zeta^b) = \zeta^{-a_v q}$. The relationship between coefficients of the polynomial $F(q)(G; x)$ and its evaluations at points $(\zeta^{a_v} : v \in V)$ is exhibited here as a basis change between the basis of monomials $\prod_{v \in V} x_v^{a_v}$ and the basis of polynomials $\prod_{v \in V} x_v^{a_v} - 1$. The connection is the Fourier transform. This article is an elaboration of this remark.

Alon and Tarsi [3] use the “Combinatorial Nullstellensatz” [11] to prove that $F(q)(G; x) \neq 0$ if and only if $P(G; q) > 0$, and also show that more can be said.

For a polynomial $F(x) \in \mathbb{C}[x]/(x_v^q - 1 : v \in V)$, let $[x^a]F(x)$ denote the coefficient of the monomial $x^a = \prod_{v \in V} x_v^{a_v}$ in its expansion to the monomial basis. In particular, $[x^0]F(x)$ is the constant term of $F(x)$. The (squared) $\ell_2$-norm $\|F(x)\|_2^2$ is defined by

$$\|F(x)\|_2^2 = \sum_{a \in \{0, 1, \ldots, q - 1\}^V} |[x^a]F(x)|^2.$$

The graph polynomial has not yet acquired the qualification of a proper name. The “Sylvester-Petersen polynomial” might be a candidate [13,17]. Matiyasevich analyses the graph polynomial of the line graph of a cubic plane graph in order to obtain reformulations of the Four Colour Theorem [14]. Alon and Tarsi [2,20] interpret its coefficients in terms of orientations; their interpretations in terms of proper vertex colourings will be described in this section. Ellingham and Goddyn [8] call the graph polynomial the graph monomial averring that the latter has a less anonymous character than the former.
That this is a norm includes the fact that \( F(q)(G; x) \neq 0 \) if and only if \( \|F(q)(G; x)\|_2 \neq 0 \).

**Theorem 2.1.** \( \triangleright \) For each \( q \in \mathbb{N} \),

\[
\|F(q)(G; x)\|_2^2 = q^{-|V|} 4^{|E|} \sum_{e \in \{0, 1, \ldots, q-1\}^V} \prod_{uv \in E} \sin^2 \frac{\pi (c_v - c_u)}{q},
\]

the sum being over all vertex colourings of \( G \) with colours \( \{0, 1, \ldots, q-1\} \). In particular, for \( q = 3 \) this is \( 3^{|E| - |V|} P(G; 3) \).

For the next theorem we require a further definition. A \((q,1)\)-flow of \( G \) is a partial orientation of \( G \) with the property that at each vertex the number of edges directed out of \( v \) is congruent to the number of edges directed into \( v \) modulo \( q \). (A partial orientation is obtained when some edges of \( G \) are assigned an orientation while the other edges remain undirected.) By referring to the fixed orientation \( \overline{E} \) of \( G \), it is possible to use the equivalent definition as an assignment of values \( (b_e : e \in E) \) to the edges of \( G \) with the properties that \( b_e \in \{0, 1, -1\} \) and the net flow (incoming minus outgoing values) at each vertex is equal to zero modulo \( q \).

**Theorem 2.2.** \( \triangleright \) For each \( q \in \mathbb{N} \),

\[
\|F(q)(G; x)\|_2^2 = (-1)^{|E|} \sum_{(q,1)\text{-flows } b} (-2)^{|E| - |b|},
\]

where \( |b| = \#\{e \in E : b_e \neq 0\} \).

One aim of this article is to reveal the underlying relationship between Theorems 2.1 and 2.2 in a more general context. The other is to characterize those polynomials of the form

\[
\prod_{(u,v) \in \overline{E}} \sum_{a,b \in \{0,1,\ldots,q-1\}} f(a,b)x_u^a x_v^b \mod (x_v^a - 1 : v \in V)
\]

whose \( \ell_2 \)-norm is a Tutte-Grothendieck invariant (such as \( P(G; q) \)). The graph polynomial is the case \( f(1,0) = 1, f(0,1) = -1 \) and \( f(a, b) = 0 \) otherwise, and Theorem 2.1 says that for \( q = 3 \) its \( \ell_2 \)-norm is the Tutte-Grothendieck invariant \( 3^{|E| - |V|} P(G; 3) \).

### 3 Tutte-Grothendieck invariants

Let \( G = (V, E) \) be a graph, loops and parallel edges permitted, with \( k(G) \) components, rank \( r(G) = |V| - k(G) \) and nullity \( n(G) = |E| - r(G) \).

Deleting an edge \( e \in E \) gives a graph \( G \setminus e \) with one fewer edge than \( G \). Contracting \( e \) gives a graph \( G/e \) with one fewer vertex and one fewer edge than \( G \). Many graph parameters may be recursively defined via contraction-deletion recurrences.

**Definition 3.1.** A function \( F \) from (isomorphism classes of) graphs to \( \mathbb{C}[\alpha, \beta, \gamma, x, y] \) is a Tutte-Grothendieck invariant if it satisfies, for each graph \( G = (V, E) \) and any edge \( e \in E \),

\[
F(G) = \begin{cases} 
\gamma^{|V|} & E = \emptyset, \\
x F(G/e) & e \text{ a bridge,} \\
y F(G \setminus e) & e \text{ a loop,} \\
\alpha F(G/e) + \beta F(G \setminus e) & e \text{ not a bridge or loop.}
\end{cases}
\]
See for example the accounts in [22, 5, 10] for an appreciation of the ubiquity of Tutte-Grothendieck invariants. For $A \subseteq E$, the subgraph $(V, A)$ is obtained from $G$ by deleting edges not in $A$. Given $G = (V, E)$, the rank of the graph $(V, A)$ is denoted by $r(A)$. A Tutte-Grothendieck invariant is an evaluation of the Tutte polynomial, defined by

$$T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}. \quad (2)$$

The Tutte polynomial is a rescaling of the Whitney rank polynomial of $G$ (for which see for example [10]), a generating function for $(|A|, r(A))$ over all subgraphs $(V, A)$ of $G$. The coefficients of the Tutte polynomial are non-negative integers (see for example [4, 5]), a fact while not evident from its definition in equation (2) is more readily seen in its alternative formulation as a Tutte-Grothendieck invariant with $\alpha = \beta = \gamma = 1$.

**Theorem 3.2.** If $F$ is a Tutte-Grothendieck invariant satisfying the equations (1) then

$$F(G) = \gamma^k(G) \alpha^{r(G)} \beta^m(G) T(G; \frac{x}{\alpha}, \frac{y}{\beta}).$$

See [5] for how to interpret this evaluation for the case $\alpha = 0$ or $\beta = 0$.

**Example 3.3.** The monochromial $P(G) = P(G; q, y)$ (bad colouring polynomial, coboundary polynomial, partition function of the $q$-state Potts model) is defined by

$$P(G; q, y) = \sum_{c \in Q^V} \# \{(u, v) \in E : c_u = c_v\}, \quad (3)$$

where $Q$ is a set of $q$ colours (states) and $c = (c_v : v \in V)$ is a vertex colouring of $G$ using colours from $Q$. It is easily verified that the function $P$ satisfies

$$P(G) = \begin{cases} q^{|V|} & E = \emptyset, \\ (y + q - 1)P(G/e) & e \text{ a bridge}, \\ yP(G \setminus e) & e \text{ a loop}, \\ (y - 1)P(G/e) + P(G \setminus e) & e \text{ not a bridge or loop}. \end{cases}$$

By Theorem 3.2,

$$P(G; q, y) = q^k(G)(y - 1)^{r(G)} T(G; \frac{y-1+q}{y-1}, y). \quad (4)$$

In particular, the chromatic polynomial $P(G; q)$, counting the number of proper vertex $q$-colourings of $G$, is given by

$$P(G; q) = q^k(G)(-1)^{r(G)} T(G; 1 - q, 0).$$

Let $Q$ be a set of size $q$ (later in this article to be an additive Abelian group of order $q$) and $w = (w_{a,b})$ a tuple of complex numbers indexed by $(a, b) \in Q \times Q$. Assume that the edges $\{u, v\}$ of $G = (V, E)$ have been given an arbitrary, fixed orientation $(u, v)$. Denote by $\overline{E}$ the resulting set of directed edges. Consider the partition function for a vertex $Q$-colouring model that assigns a weight $w_{a,b}$ to a directed edge $(u, v)$ coloured $(a, b)$:

$$F(G; w) = \sum_{c \in Q^V} \prod_{(u, v) \in \overline{E}} w_{c_u, c_v} = \sum_{c \in Q^V} \prod_{(a, b) \in Q^E} \prod_{(u, v) \in \overline{E} : (c_u, c_v) = (a, b)} w_{a,b}. \quad (5)$$
This partition function may be interpreted as the weight of a graph homomorphism \( G \to H \), where \( H \) is the directed graph on vertex set \( Q \) and edge set \( \{(a, b) : w_{a,b} \neq 0\} \), with edge weights \( w_{a,b} \), i.e., the weighted graph \( H \) has adjacency matrix \((w_{a,b})_{a,b \in Q} \). (It is possible to also have vertex weights for \( H \) in addition to its edge weights, but this will not be considered here. See for example \([9, 19]\) and \([13, 7]\) for more on vertex colouring models and on graph homomorphisms.)

**Theorem 3.4.** The graphical invariant \( F(G; w) \) defined by equation (5) is a Tutte-Grothendieck invariant if and only if there are constants \( y, w \) such that

\[
  w_{a,b} = \begin{cases} 
    w & a \neq b, \\
    y & a = b.
  \end{cases}
\]

In this case \( F(G; w) = F(G; w, y) = q^{k(G)}w^{n(G)}(y-w)^{r(G)}T(G; \frac{y-(q-1)w}{y-w}, \frac{w}{y}) \). (If \( w = 0 \) then \( F(G; 0, y) = y^{|E|} \) and if \( w = y \) then \( F(G; y, y) = q^{|V|y^{|E|}} \).)

A sketch only of a proof of Theorem 3.4 is given\(^2\). (A fuller version will appear in a forthcoming paper.) The following lemma is the main tool.

**Lemma 3.5.** If \( u_1, \ldots, u_r, v \in \mathbb{C} \) satisfy

\[
  u_1^m + u_2^m + \cdots + u_r^m = rv^m, \tag{6}
\]

for all integers \( m \geq 0 \), then

\[
  u_1 = u_2 = \cdots = u_r = v.
\]

**Proof.** We may assume \( v \neq 0 \). Rewriting the relation (6) in terms of ordinary generating functions,

\[
  (1 - u_1 z)^{-1} + (1 - u_2 z)^{-1} + \cdots + (1 - u_r z)^{-1} = r(1 - vz)^{-1},
\]

for all \( z \in \mathbb{C} \) such that

\[
  |z| < \min\{|u_i|^{-1} : 1 \leq i \leq r, u_i \neq 0\} \cup \{|v|^{-1}\}.
\]

Multiplying out to clear fractions, this is to say that

\[
  \sum_{1 \leq i \leq r} (1 - vz) \prod_{j \neq i} (1 - u_j z) = r \prod_{1 \leq j \leq r} (1 - u_j z).
\]

Equating coefficients of \( z^k \) in this polynomial of degree \( r \) in \( z \) yields

\[
  \sum_{1 \leq i \leq r} e_k(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_r) = re_k(u_1, \ldots, u_r),
\]

where \( e_k \) is the \( k \)th elementary symmetric function. Cancelling terms just involving the \( u_i \) (which comprise altogether \( r - k \) copies of \( e_k(u_1, \ldots, u_r) \) on the left-hand side of the equation: each \( k \)-subset of \( \{u_1, \ldots, u_r\} \) occurs in \( r - k \) of the sets \( \{u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_r\} \)) and factoring out the resulting common factor of \( v \) (on the left-hand side of the equation, each \( (k-1) \)-subset of \( \{u_1, \ldots, u_r\} \) occurs in \( r - k + 1 \) terms),

\[
  (r - k + 1)v e_{k-1}(u_1, \ldots, u_r) = ke_k(u_1, \ldots, u_r).
\]

\(^2\)I am grateful to Delia Garijo for alerting me to the fact that I was assuming the truth of something that required proof, and also for her description of how she has been tackling a related, stronger result.
By this recursive formula we obtain

\[ e_k(u_1, \ldots, u_r) = \frac{r - k + 1}{k} v e_{k-1} = \binom{r}{k} v^k. \]

This implies that \( u_1, \ldots, u_r \) are uniquely determined as the roots of the polynomial \( (z - v)^r \), i.e., \( u_i = v \) for each \( 1 \leq i \leq r \).

**Proof.** (of Theorem 3.4.) In one direction, given that \( w_{a,b} = w \) for \( a \neq b \) and \( w_{a,a} = y \), the evaluation of the Tutte polynomial follows from that of the monochromial given in equation (iv) in Example 3.3 above with \( x = y/w \).

In the other direction, suppose that there are constants \( \alpha, \beta, \gamma, x, y \) such that \( F(G; w) = F(G) \) satisfies the relations (i) for a Tutte-Grothendieck invariant. By checking that this is indeed the case for the three families of graphs \( X_m, Y_m, Z_m \) (\( m \in \mathbb{N} \)) itemized below the desired conclusion is reached. Each of these families of graphs possess the following virtues:

(i) the graphs obtained by contracting or deleting an edge are of the same form or belong to one of the other families, and
(ii) it is possible to write down \( F \) as given by the partition function (v) as a succinct formula, thereupon to substitute this formula into the contraction-deletion recurrence of (i), and finally to avail oneself of Lemma 3.5 (or a variant of this lemma).

For \( m \in \mathbb{N} \) consider:

(i) \( Y_m \), the graph on one vertex with \( m \) loops. \( F(Y_1) = \sum_{a \in Q} w_{a,a} = qy \). The relation \( F(Y_m) = y F(Y_{m-1}) = qy^m \) is used to show that \( w_{a,a} = y \) for each \( a \in Q \).

(ii a) \( X_m \), the graph on two vertices connected by \( m \) parallel edges. \( F(X_1) = \sum_{a,b \in Q} w_{a,b} = qx \). The relation \( F(X_m) = \alpha F(Y_{m-1}) + \beta F(X_{m-1}) \) for \( m \geq 2 \) is used to show that \( \{w_{a,b} : a, b \in Q\} \) contains at most two distinct values \( y, w \); there is \( S \) with \( \{(a, a) : a \in Q\} \subseteq S \subseteq Q \times Q \) such that \( w_{a,b} = y \) for \((a, b) \in S \), and \( w_{a,b} = w \) otherwise.

(ii b) \( X_m^n \), the graph \( X_m \) with \( n \) edges oriented in one direction, \( m - n \) in the other. That \( F(X_m) \) is independent of any orientation of the edges of \( X_m \) (giving a graph \( X_m^n \)) is used to show that \( w_{a,b} = w_{b,a} \) for all \( a, b \in Q \), i.e., the set \( S \) defined in (ii a) is closed under the involution \((a, b) \mapsto (b, a)\).

(iii) \( Z_m \), the star graph with \( m \) edges (one vertex degree \( m \), and \( m \) vertices degree 1). The relation \( F(Z_m) = x F(Z_{m-1}) = qx^m \) is used to show that \#\( \{b \in Q : (a, b) \in S\} \) is independent of \( a \), whereby it follows from (ii b) that either \( S = \{(a, a) : a \in Q\} \) or \( S = Q \times Q \).

We now know how to recognize a Tutte-Grothendieck invariant. To aid our search amongst graph polynomials of the sort defined in Section 2 we shall use instruments from Fourier analysis, a subject to which we now turn.

### 4 Fourier analysis on finite Abelian groups

#### 4.1 The algebra \( \mathbb{C}^Q \)

Let \( Q \) be an additive Abelian group of order \( q \). In later sections \( Q = \mathbb{Z}_q \), the integers under addition modulo \( q \).
The set \( \mathbb{C}^Q \) of functions \( f : Q \rightarrow \mathbb{C} \) forms a \( q \)-dimensional Hermitian inner product space. The inner product is defined by

\[
\langle f, g \rangle = \sum_{a \in Q} f(a) \overline{g(a)},
\]

the bar denoting complex conjugation. The \( \ell_2 \)-norm is defined by \( \| f \|_2 = \langle f, f \rangle^{\frac{1}{2}} \) and defines a metric on the space \( \mathbb{C}^Q \).

The space \( \mathbb{C}^Q \) has an orthonormal basis of indicator functions \( \{ \delta_a : a \in Q \} \),

\[
\delta_a(b) = \begin{cases} 
1 & a = b, \\
0 & a \neq b.
\end{cases}
\]

There are several definitions of multiplication that make \( \mathbb{C}^Q \) an algebra:

(i) **Pointwise product**

\[
f \cdot g(a) = f(a)g(a).
\]

(ii) **Convolution**

\[
f \ast g(a) = \sum_{b \in Q} f(a)g(b - a).
\]

(iii) **Cross-correlation**

\[
f \star g(a) = \sum_{b \in Q} \overline{f(a)}g(b + a).
\]

The effect of these operations on the indicator functions is as follows:

\[
\delta_a \cdot \delta_b = \delta_a(b)\delta_a, \quad \delta_a \ast \delta_b = \delta_{a+b}, \quad \delta_a \star \delta_b = \delta_{b-a}.
\]

The Abelian group \( Q \) has **dual group** equal to the set of **characters** of \( Q \) under pointwise multiplication. For each \( c \in Q \), the character \( \chi_c : Q \rightarrow \mathbb{C}^\times \) is a group homomorphism:

\[
\chi_c(a + b) = \chi_c(a)\chi_c(b) \quad \text{for all } a, b \in Q.
\]

The multiplicative group of characters of \( Q \) is isomorphic to the additive group \( Q \). (This is only true when \( Q \) is a finite Abelian group, and, for the applications later in this article, is the reason why only finite Abelian groups are considered.)

The set \( \{ \chi_c : c \in Q \} \) forms an orthogonal basis for \( \mathbb{C}^Q \), with \( \langle \chi_a, \chi_b \rangle = q\delta_a(b) \).

In the algebra \( \mathbb{C}^Q \),

\[
\chi_a \cdot \chi_b = \chi_{a+b}, \quad \chi_a \ast \chi_b = q\delta_a(b)\chi_a = \chi_a \star \chi_b.
\]

Supposing the additive group \( Q \) has the further structure of a ring (such as \( \mathbb{Z}_q \) with addition and multiplication modulo \( q \)), a **generating character** \( \chi \) satisfies \( \chi_c(b) = \chi(ab) \) for all \( a, b \in Q \). When \( Q = \mathbb{Z}_q \), the character \( \chi \) defined by \( \chi(a) = e^{2\pi i a/q} \) (or \( e^{2\pi i ca/q} \) for any fixed \( c \) coprime with \( q \)) is a generating character.

### 4.2 The Fourier transform

The evaluation of the Fourier transform of a function at a point is the projection of the function onto a character:

\[
\hat{f}(b) = \langle f, \chi_b \rangle = \sum_{a \in Q} f(a)\chi_b(-a),
\]
i.e.,
\[ \hat{f} = \sum_{b \in Q} f(b) \chi_{-b}. \]

Orthogonality of the basis \( \{ \chi_c : c \in Q \} \) yields:

(i) the Fourier inversion formula,
\[ f(a) = q^{-1} \langle \hat{f}, \chi_{-a} \rangle = q^{-1} \sum_{b \in Q} \hat{f}(b) \chi_{b}(a), \]
i.e., the Fourier transform may be regarded as a change of basis from indicators to characters:
\[ f = \sum_{a \in Q} f(a) \delta_{a} = q^{-1} \sum_{b \in Q} \hat{f}(b) \chi_{b}. \]

(ii) Plancherel’s formula,
\[ \langle \hat{f}, \hat{g} \rangle = q \langle f, g \rangle. \]

(iii) Parseval’s formula,
\[ \| f \|_2^2 = q \| \hat{f} \|_2^2. \]

Thus the normalized Fourier transform \( f \mapsto q^{-\frac{1}{2}} \hat{f} \) is a unitary transformation, giving an isometry of the metric space \( \mathbb{C}^Q \).

The Fourier transform is an isomorphism of the algebra \( (\mathbb{C}^Q, \ast) \) with the algebra \( (\mathbb{C}^Q, \cdot) \):
\[ \hat{f} \ast g = q^{-1} \hat{f} \cdot \hat{g}, \quad \hat{f} \ast g = \hat{f} \cdot \hat{g}, \quad \hat{f} \ast g = \hat{f} \cdot \hat{g}, \]
(7)

and in particular
\[ \hat{f} \ast f = |\hat{f}|^2 \]
(the finite version of the Wiener-Khintchine formula). That the Fourier transform is an isometry carrying convolution to pointwise multiplication makes it useful in the analysis of random walks on Cayley graphs on \( Q \), where steps on the graph correspond to addition of group elements – see for example [21] and the references therein. To prove the formulae in (7) it suffices to determine the effect of the Fourier transform on basis functions and then appeal to linearity and distributivity. For example, \( \delta_{a} \ast \delta_{b} = \delta_{a-b} = \chi_{a-b} = \bar{\delta}_{a} \cdot \delta_{b} \).

For an additive subgroup \( P \) of \( Q \), the annihilator of \( P \) is defined by
\[ P^\# = \{ b \in Q : \forall a \in P \chi_{b}(a) = 1 \}. \]
and is isomorphic to the quotient group \( Q/P \).

Extend the indicator function notation from elements to subsets \( P \subseteq Q \) by setting \( \delta_P = \sum_{a \in P} \delta_{a} \).

For our purposes, a key property of the Fourier transform is that
\[ \hat{\delta}_P = |P| \delta_{P^\#}. \]

By Fourier inversion,
\[ \delta_P \ast f(b) = q^{-1} \langle \hat{\delta}_P \ast \hat{f}, \chi_{-b} \rangle, \]
giving the Poisson summation formula
\[ \sum_{a \in P} f(a + b) = |P^\#|^{-1} \sum_{a \in P^\#} \hat{f}(a) \chi_{b}(a). \]
4.3 The algebra $\mathbb{C}Q^n$ and the polynomial ring $\mathbb{C}[x]/(x_i^q - 1 : 1 \leq i \leq n)$

In this section we assume that the Abelian group $Q$ has the further structure of a commutative ring. Let $Q = Q^n$ denote the $n$-fold direct product of $Q$, which is an Abelian group of order $q^n$ and a module over $Q$. Put a ring structure on $Q^n$ by defining componentwise multiplication of $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in Q^n$,

$$ab = (a_1b_1, \ldots, a_nb_n).$$

The Hermitian inner product space $\mathbb{C}Q^n$ is the $n$-fold tensor product of $\mathbb{C}Q$: given $f_1, \ldots, f_n \in \mathbb{C}Q$ define

$$f_1 \otimes \cdots \otimes f_n(a_1, \ldots, a_n) = f_1(a_1) \cdots f_n(a_n),$$

and in particular $f^\otimes_n(a) = f(a_1) \cdots f(a_n)$.

The characters of $Q^n$ are the functions defined by $\chi_a = \chi_{a_1} \otimes \cdots \otimes \chi_{a_n}$.

Define the Euclidean (dot) product by

$$a \cdot b = a_1b_1 + \cdots + a_nb_n.$$

If $\chi$ a generating character for $Q$, then $\chi^\otimes_n$ is a generating character for $Q^n$:

$$\chi_a(b) = \chi^\otimes_n(ab) = \chi(a_1b_1) \cdots \chi(a_nb_n) = \chi(a \cdot b).$$

Given that $Q$ has a generating character, for a submodule $P$ of $Q^n$ the annihilator

$$P^\perp = \{b \in Q^n : \forall a \in P \quad \chi_a(b) = 1\}$$

is equal to the orthogonal submodule

$$P^\perp = \{b \in Q^n : \forall a \in P \quad a \cdot b = 0\}.$$

The Fourier transform on $Q^n$ is given by

$$f_1 \otimes \cdots \otimes f_n = \hat{f}_1 \otimes \cdots \otimes \hat{f}_n,$$

and in particular $\hat{f}^\otimes_n = \hat{f}^\otimes_n$.

It may be helpful to spell out the relationship between polynomials in the ring $\mathbb{C}[x]/(x_i^q - 1 : 1 \leq i \leq n)$ (where $x = (x_i : 1 \leq i \leq n)$ is an $n$-tuple of commuting indeterminates) and functions in the space $\mathbb{C}Q^n$. The aim of course is to translate statements about the reduced graph polynomial $F(q)(G; x)$, which belongs to $\mathbb{C}[x]/(x_i^q - 1 : v \in V)$, into statements about functions in $\mathbb{C}Q^n$. The latter space has now the advantage of familiarity and the accoutrements of a succinct notation.

Take $Q = \mathbb{Z}_q$, which has generating character $\chi(a) = \zeta^a$ for primitive $q$th root of unity $\zeta$.

The algebra $\mathbb{C}Q^n$ is isomorphic to $\mathbb{C}[x]/(x_i^q - 1 : 1 \leq i \leq n)$ and the following correspondences obtain:

$$\delta_a = \delta_{a_1} \otimes \cdots \otimes \delta_{a_n} \quad \text{with} \quad x^a = \prod_{1 \leq i \leq n} x_i^{a_i},$$

$$\chi_a = \chi_{a_1} \otimes \cdots \otimes \chi_{a_n} \quad \text{with} \quad \prod_{1 \leq i \leq n} \frac{x_i^q - 1}{\zeta^{-a_i} x_i - 1},$$

$$f = \sum_{a \in \mathbb{Z}_q^n} f(a) \delta_a \quad \text{with} \quad F(x) = \sum_{a \in \mathbb{Z}_q^n} f(a)x^a.$$
Finally,
\[ F(\zeta^{a_1}, \ldots, \zeta^{a_n}) = \hat{f}(a), \]
and Lagrange interpolation on points \( \{(\zeta^{a_1}, \ldots, \zeta^{a_n}) : (a_1, \ldots, a_n) \in \mathbb{Z}_q^n\} \) is the Fourier basis change:
\[
\sum_{a \in \mathbb{Z}_q^n} f(a)x^a = q^{-n} \sum_{a \in \mathbb{Z}_q^n} \hat{f}(a) \prod_{i=1}^n \frac{x_{i}^{q^{a_i}} - 1}{x_{i} - 1}.
\]

4.4 Weight enumerators and the Tutte polynomial

We finish this section on Fourier analysis with a discussion of the Tutte polynomial as a weight enumerator that gives us the opportunity at the same time to define flows and tensions of a graph, which definitions are needed for the next section.

It will be convenient to extend the domain of a function \( f \) on elements \( a \in \mathbb{Q}^n \) to subsets \( P \subseteq \mathbb{Q}^n \), setting \( f(P) = \sum_{a \in P} f(a) \).

The Hamming weight of \( a = (a_1, \ldots, a_n) \) is \(|a| = \#\{i : a_i \neq 0\}\). The Hamming weight enumerator of \( P \) is defined to be the generating function for vectors in \( P \) counted according to their number of zero entries:
\[
\sum_{a \in P} x^{n-|a|} = (x\delta_0 + \delta_{\mathbb{Q}\setminus 0})^{\otimes n}(P).
\]

The complete weight enumerator of \( P \) keeps account of the number of entries equal to a given element of \( \mathbb{Q} \):
\[
\sum_{a \in P} \prod_{a \in Q} x^{\#\{1 \leq i \leq n : a_i = a\}} = (\sum_{a \in Q} x_a \delta_a)^{\otimes E}(P).
\]

The Hamming weight enumerator is the specialization \( x_0 = x \) and \( x_a = 1 \) for \( 0 \neq a \in \mathbb{Q} \).

For a submodule \( P \) of \( \mathbb{Q}^n \),
\[ \hat{\delta}_P = |P|\delta_{P^\perp}. \]

The Poisson summation formula
\[ f(P + b) = \frac{1}{|P^\perp|} \hat{f} \cdot \chi_b(P^\perp), \]
with \( b = 0 \) and \( f = f^{\otimes n} \) gives the duality formula between the complete weight enumerator of \( P \) (with \( x_a = f(a) \)) and the complete weight enumerator of \( P^\perp \) (with \( x_a = \hat{f}(a) \)). When \( \hat{f} = x\delta_0 + \delta_{\mathbb{Q}\setminus 0} \) it yields the MacWilliams duality formula for Hamming weight enumerators.

Recall that the graph \( G = (V, E) \) has a fixed orientation of its edges, with directed edge set denoted by \( \overrightarrow{E} \). Represent this ground orientation as a matrix \( \gamma \) indexed by \( V \times E \), setting
\[
\gamma_{v,e} = \begin{cases} 
+1 & \text{if } e = (u, v) \text{ in } \overrightarrow{E}, \\
-1 & \text{if } e = (v, u) \text{ in } \overrightarrow{E}, \\
0 & \text{if } e \text{ is not incident with } v.
\end{cases}
\]

A Q-tension of \( G \) is a vector \( a \in \mathbb{Q}^E \) comprising the differences between endpoints in a vertex colouring \( c \in \mathbb{Q}^V \), i.e., if \( e = (u, v) \) then the Q-tension \( a \) associated with \( c \) is defined by
\[ a_e = \sum_{v \in V} \gamma_{v,e} c_v = c_v - c_u. \]
A $Q$-flow of $G$ is a vector $b \in Q^E$ such that, for each vertex $v$,

$$\sum_{e \in E} \gamma_{v,e}b_e = 0.$$ 

When $G$ is planar, $Q$-flows of $G$ correspond to $Q$-tensions of the planar dual graph $G^*$. In particular, when $Q = \mathbb{F}_2$, the $\mathbb{F}_2$-flows of $G$ (cycles/ Eulerian subgraphs) correspond to $\mathbb{F}_2$-tensions (cutsets) of $G^*$. 

A nowhere-zero $Q$-tension is one that takes only non-zero values, and arises from a proper vertex $Q$-colouring; similarly, a nowhere-zero $Q$-flow is a flow takes non-zero values only (and for plane graphs corresponds to a proper face $Q$-colouring of the embedded graph). 

If $P$ is the set of $Q$-tensions of $G$ (of which there are $q^{r(G)}$) then $P^\perp$ is the set of $Q$-flows of $G$ (of which there are $q^{n(G)}$). With this notation, the monochromial is given by

$$\sum_{e \in Q^V} y^{|E|-|a|} = q^{k(G)} \sum_{a \in P} y^{|E|-|a|},$$

since there are $q^{k(G)}$ vertex $Q$-colourings yielding any given $Q$-tension. Consequently, by Example 3.3, the Hamming weight enumerator of the set $P$ of $Q$-tensions of $G$ is a specialization of the Tutte polynomial:

$$\sum_{a \in P} y^{|E|-|a|} = (y-1)^{r(G)}T(G; y - 1 + q \frac{y}{y-1}, y).$$

By the Poisson summation formula (MacWilliams duality),

$$(y\delta_0 + \delta_{Q\setminus\emptyset})^{\otimes E}(P) = q^{-n(G)}[(y-1+q)\delta_0 + (y-1)\delta_{Q\setminus\emptyset}]^{\otimes E}(P^\perp).$$

Putting $x = \frac{y-1+q}{y-1}$, the Hamming weight enumerator of the set $P^\perp$ of $Q$-flows of $G$ is given by

$$\sum_{b \in P^\perp} x^{|E|-|b|} = (x-1)^{n(G)}T(G; x, \frac{x-1+q}{x-1}).$$

A corollary of Theorem 3.3 is that if an evaluation of the complete weight enumerator of $Q$-tensions (or $Q$-flows) is a Tutte-Grothendieck invariant (an evaluation of the Tutte polynomial with a certain simple type of prefactor) then it is in fact an evaluation of the Hamming weight enumerator. In fact, the proof of Theorem 3.3 says the same is true of any class of graphs that contains multiple loops on one vertex, multiple parallel edges between two vertices, and stars whose central vertex is of arbitrary degree. This notably includes the class of planar graphs.

There are nevertheless (infinite) classes of graphs for which an evaluation of the complete weight enumerator of $Q$-tensions of $G$ coincides with the value of a Tutte-Grothendieck invariant and yet is not an evaluation of the Hamming weight enumerator.

For example, if $G = (V, E)$ is the line graph of a plane cubic graph then a result ultimately due to Penrose [10] (but see [8] for a full account) is that

$$\sum_{c \in \mathbb{Z}_3^V} 0^{|\{(u,v)\in E: c_u = c_v\}|}(-1)^{|\{(u,v)\in E: c_u - c_v = -1\}|} = (-1)^{|V|}P(G; 3),$$

i.e., the complete weight enumerator of $\mathbb{Z}_3$-tensions of $G$ with $x_0 = 0, x_1 = 1, x_{-1} = -1$ is an evaluation of the Tutte polynomial. However, since the class of line graphs of plane cubic graphs is not closed under deletion or contraction, one is prevented from calling this a Tutte-Grothendieck invariant.
5 Polynomials akin to the graph polynomial

Suppose \( F^{(q)}(G; x) \in \mathbb{C}[x]/(x_q^2 - 1 : v \in V) \) is a graph polynomial of the general form

\[
F^{(q)}(G; x) = \prod_{(u,v) \in E} \sum_{(a,b) \in \mathbb{Z}_q^2} f(a,b) x_u^a x_v^b = \sum_{c \in \{\mathbb{Z}_q^2\}_E} f^{\otimes E}(c) \prod_{(u,v) \in E} x_u^{c_{u,v}} x_v^{c_{v,u}},
\]

where \( c = (c_e : e \in E) \), \( c_e = (c_{u,e}, c_{v,e}) \) for edge \( e \) directed as \( (u, v) \) in \( \tilde{E} \), and \( f^{\otimes E}(c) = \bigotimes_{e \in E} f(c_{u,e}, c_{v,e}) \). The graph polynomial of Petersen et al. introduced in Section 2 is the case \( f(1,0) = 1, f(0,1) = -1 \) and \( f(a,b) = 0 \) otherwise. (Henceforth the name “Petersen’s graph polynomial” will be used when it needs to be distinguished.)

In this section we address the following questions:

(A) When is the partition function of the vertex colouring (states) model\(^3\)

\[
\sum_{d \in \mathbb{Z}_q^V} F^{(q)}(G; (\zeta^{d_v} : v \in V)) = q^{|V| |x_q^0| F^{(q)}(G; x)}
\]

a Tutte-Grothendieck invariant (an evaluation of the Tutte polynomial)?

(B) When is the squared \( \ell_2 \)-norm

\[
\|F^{(q)}(G; x)\|_2^2 = \sum_{a \in \mathbb{Z}_q^V} |x^a| F^{(q)}(G; x)|^2
\]

a Tutte-Grothendieck invariant?

(C) What are the equivalents of Theorems 2.1 and 2.2 in this more general case?

By Parseval’s formula,

\[
\|F^{(q)}(G; x)\|_2^2 = q^{-|V|} \sum_{d \in \mathbb{Z}_q^V} |F^{(q)}(\zeta^{d_v} : v \in V)|^2,
\]

where, writing \( c \) for the vector with entries \( (c_{u,e}, c_{v,e}) = (d_u, d_v) \),

\[
|F^{(q)}(G; \zeta^{d_v} : v \in V)|^2 = |\hat{f}^{\otimes E}(c)|^2.
\]

Since \( |\tilde{f}|^2 = \hat{f} \ast \hat{f} \), this implies that the \( \ell_2 \)-norm of \( F^{(q)}(G; x) \) is equal to the constant term of the polynomial \( \tilde{F}^{(q)}(G; x) \) in \( \mathbb{C}[x]/(x_q^2 - 1 : v \in V) \) defined by

\[
\tilde{F}^{(q)}(G; x) = \prod_{(u,v) \in E} f \ast f(a,b) x_u^a x_v^b.
\]

\(^3\) The vertex colouring model assigns weight \( F^{(q)}(G; (\zeta^{d_v} : v \in V)) \) to a given vertex colouring \( d \in \mathbb{Z}_q^V \). In terms of graph homomorphisms, this vertex colouring model corresponds to considering \( d \) as a homomorphism from \( G \) to a weighted directed graph \( H \) on vertex set \( \mathbb{Z}_q \), with an edge \( (c, d) \) having weight

\[
\sum_{a,b} f(a,b) x_u^a x_v^b = \tilde{f}(c,d),
\]

The total weight of the homomorphism \( d : G \to H \) is the product of all the weights on \( (d_u, d_v) \) for edges \( (u, v) \) of \( G \), i.e., \( \tilde{f}^{\otimes E}(c) \) where \( c \in (\mathbb{Z}_q^2)^E \) is defined by \( (c_{u,e}, c_{v,e}) = (d_u, d_v) \). The partition function in question (A) is a sum over all homomorphisms \( [d, \text{ encoded by } c \in (\mathbb{Z}_q^2)^E] \) weighted in this way.
For example, the $\ell_2$-norm of Petersen’s graph polynomial

$$F^{(q)}(G; \mathbf{x}) = \prod_{(u,v) \in E} (x_u - x_v) \mod (x_v^q - 1 : v \in V)$$

is the constant term of the polynomial

$$|F^{(q)}(G; \mathbf{x})|^2 = \prod |x_u - x_v|^2 = \prod (x_u - x_v)(x_u^{-1} - x_v^{-1}),$$

$$= \prod (2 - x_u x_v^{-1} - x_u^{-1} x_v) \mod (x_v^q - 1 : v \in V).$$

[This calculation uses the correspondence of the ideal $(x_v^q - 1 : v \in V)$ with the algebraic variety of points $(\zeta^q : v \in V)$, i.e., indeterminates $x_v$ are roots of unity, for which complex conjugates are the same as multiplicative inverses.]

Let $M = \{(a, a) : a \in Q\}$ be the submodule of $Q \times Q$ comprising monochromatic pairs. The orthogonal submodule is $M^\perp = \{(a, -a) : a \in Q\}$. By Theorem 3.4, $[\mathbf{x}^0]F^{(q)}(G; \mathbf{x})$ is a Tutte-Grothendieck invariant if and only if there are constants $y, w$ such that $\hat{f} = y\delta_M + w\delta_{Q \times Q \setminus M}$. By the above remarks, $\|F^{(q)}(G; \mathbf{x})\|^2$ is a Tutte-Grothendieck invariant if and only if

$$\hat{f} * \hat{f} = y\delta_M + w\delta_{Q \times Q \setminus M}.$$

By Fourier inversion, this is the case if and only if $f * f = (y + (q-1)w)\delta_0 + (y - w)\delta_{M^\perp \setminus 0}$.

**Proposition 5.1.** The constant term of $F^{(q)}(G; \mathbf{x})$ is a Tutte-Grothendieck invariant if and only if

$$F^{(q)}(G; \mathbf{x}) = \prod_{(u,v) \in E} \left[ y + (q-1)w + (y - w)(x_u^q - 1)x_v + \cdots + x_u x_v^q - 1 \right] \mod (x_v^q - 1 : v \in V),$$

in which case

$$[\mathbf{x}^0] F^{(q)}(G; \mathbf{x}) = (qw)^{n(G)} (y - w)^{r(G)} T(G; \frac{y - (q-1)w}{y - w}, \frac{y}{w}).$$

For example, when $y = 0, w = 1$ and $q = 3$ this says that

$$\prod_{(u,v) \in E} (2 - x_u x_v^2 - x_u^2 x_v) \mod (x_v^3 - 1 : v \in V)$$

has constant term $3^n(G)(-1)^{r(G)}T(G; -2, 0) = 3^{|E| - |V|}P(G; 3)$.

That the constant term of the polynomial defined in Proposition 5.1 is a Tutte polynomial evaluation can be seen by inspection since, for $a, b \in \mathbb{Z}_q$,

$$\zeta^{(q-1)a+b} + \zeta^{(q-2)a+2b} + \cdots + \zeta^{(q-1)b} = \begin{cases} -1 & a \neq b \\ q - 1 & a = b, \end{cases}$$

so that in this case

$$q^{|V|}[\mathbf{x}^0] F^{(q)}(G; \mathbf{x}) = \sum_{a \in \mathbb{Z}_q} F^{(q)}(G; (\zeta^a v : v \in V))$$

$$= \sum_{c \in \mathbb{Z}_q} (qw)^{\# \{(u,v) \in E : c_u = c_v\}} (qw)^{\# \{(u,v) \in E : c_u \neq c_v\}}.$$
Whereas Proposition 5.1 limits the graph polynomials which have constant term equal to an evaluation of the Tutte polynomial to a single family – giving a rather dull answer to question (A) above – the possible choices for $f$ defining $F^{(q)}(G; \mathbf{x})$ so that the $\ell_2$-norm $\|F^{(q)}(G; \mathbf{x})\|_2^2$ is a Tutte-Grothendieck invariant are unlimited – making the answer to question (B) potentially equally as dull. The criterion $|f|^2 \equiv y\delta_M + u\delta_{Q \setminus Q \setminus M}$ for any complex numbers $z_{a,b}$ that satisfy $|z_{a,b}|^2 = y$ if $a = b$ and $|z_{a,b}|^2 = w$ otherwise.

Nonetheless, it seems worth describing a family of polynomials which contains Petersen’s graph polynomial as a special case and in some sense naturally generalizes it. In this family it is also possible to give a meaningful answer to question (C) asking for equivalents to Theorems 2.1 and 2.2.

### 5.1 A family of polynomials containing the graph polynomial

Suppose that $\text{supp}(f) \subseteq \{(a, b) : a + sb = t\}$ for some constants $s, t \in \mathbb{Z}_q$. Then

$$F^{(q)}(G; (\zeta^{d_v} : v \in V)) = \prod_{(u,v) \in E} \sum_{(a,b) \in \mathbb{Z}_q^2} f(a, b)\zeta^{ad_u + bd_v} = \sum_{e \in (\mathbb{Z}_q^2)^E} f^{\otimes E}(e) \prod_{(u,v) \in E} \zeta^{c_{a,v} d_u + c_{b,v} d_v}.$$ 

The equation $f(a, b) = f(t - sb, b) =: g(b)$ defines $g \in \mathbb{C}^{\mathbb{Z}_q}$ and the sum over $c \in (\mathbb{Z}_q^2)^E$ can be rewritten as a sum over $b \in \mathbb{Z}_q^E$. In particular, $s = 1$ when the polynomial $\sum_{a,b} f(a, b)x_u^{a}x_v^{b}$ is homogeneous.

Given that $a_e + sb_e = t$, we have $a_e d_u + b_e d_v = (t - sb_e) d_u + b_e d_v = b_e(d_v - sd_u) + td_u$. For $e = (u,v) \in E$, define $S : \mathbb{Z}_q^V \rightarrow \mathbb{Z}_q^E$ by

$$(Sd)_e = d_v - sd_u$$

and $T : \mathbb{Z}_q^V \rightarrow \mathbb{Z}_q^E$ by

$$(Td)_e = td_u.$$ 

For $b \in \mathbb{Z}_q^E$, the transpose $S^T$ is given by

$$(S^T b)_v = \sum_{e = (u,v) \in \overline{E}} b_e - s \sum_{e = (v,u) \in \overline{E}} b_e$$

and

$$(T^T b)_v = t \sum_{e = (v,u) \in \overline{E}} b_e.$$ 

When $s = 1$ (which is the case for Petersen’s graph polynomial) the linear transformation $S$ is the coboundary and $S^T$ the boundary. Here the submodule $\ker(S^T)$ comprises the $\mathbb{Z}_q$-flows of $G$ and $\text{im}(S)$ the $\mathbb{Z}_q$-tensions of $G$. 

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We have
\[
F(q)(G; x) = \prod_{(u,v) \in E} \sum_{b \in \mathbb{Z}_q} g(b) x_u^{f_{-b}} x_v^b
\]
\[
= \sum_{b \in \mathbb{Z}_q^E} \prod_{e=(u,v) \in E} g(b_e) x_u^{f_{-b_e}} x_v^{b_e}
\]
\[
= \sum_{b \in \mathbb{Z}_q^E} g \otimes E(b) \prod_{v \in V} x_v^{S^T b + T^T 1},
\]
where \( 1 \) is the all-one vector in \( \mathbb{Z}_q^V \). \((T^T 1 \) is \( t \) times the outdegree score of \( E \)).

The following theorem provides an answer to the question (A) posed in the previous section, and more.

**Theorem 5.2.** If \( S^T b = a - T^T 1 \) then

\[
[x^a] F(q)(G; x) = g \otimes E(\ker(S^T) + b),
\]
a complete coset weight enumerator of \( \ker(S^T) \).

In particular, the coefficient \( [x^T 1] F(q)(G; x) \) is an evaluation of the complete weight enumerator of \( \ker(S^T) \) (and of \( \im(S) \)).

For example, in Petersen’s graph polynomial, where \( g = \delta_0 - \delta_1 \),

\[
[x^T 1] \prod_{(u,v) \in E} (x_u - x_v \mod (x_v^0 - 1 : v \in V)) = \sum_{(q,1)\text{-flows } b} 0^{\#\{e \in E : b_e = -1\}} (-1)^{\#\{e \in E : b_e = 1\}},
\]

where a \((q,1)\)-flow is a \( \mathbb{Z}_q \)-flow taking values only in \( \{0, 1, -1\} \) (and here the sum need only range over those taking values in \( \{0, 1\} \)).

When \( s = 1 \) (for which \( S \) is the coboundary, \( \im(S) \) the set of \( \mathbb{Z}_q \)-tensions, \( \ker(S^T) \) the set of \( \mathbb{Z}_q \)-flows) and

\[
F(q)(G; x) = \prod_{(u,v) \in E} \sum_{b \in \mathbb{Z}_q} g(b) x_u^{f_{-b}} x_v^b,
\]

the coefficient \( [x^T 1] F(q)(G; x) \) is an evaluation of the Tutte polynomial if and only if \( g = x \delta_0 + \delta_{-1} \) (by Theorem 3.3) this is the case covered by Proposition 5.1. If \( g \) does not take this form then the coefficient \( [x^T 1] F(x) \) is not an evaluation of the Hamming weight enumerator of \( \mathbb{Z}_q \)-flows but of some other specialization of the complete weight enumerator.

To find the \( \ell_2 \)-norm, observe that, for \( d \in \mathbb{Z}_q^V \),

\[
F(q)(G; (\zeta^{d_v} : v \in V)) = \sum_{b \in \mathbb{Z}_q^E} g \otimes E(b) \zeta^{(S^T b) \cdot d + T^T 1 \cdot d}
\]
\[
= \sum_{b \in \mathbb{Z}_q^E} g \otimes E(b) \zeta^{b \cdot Sd + 1 \cdot Td}
\]
\[
= \zeta^{1 \cdot Td} \hat{g} \otimes E(-Sd),
\]
and

\[
|F(q)(G; (\zeta^{d_v} : v \in V))|^2 = |\hat{g} \otimes E(-Sd)|^2 = |\hat{g} \otimes E(Sd)|^2.
\]
By Parseval’s formula,
\[
\| F^{(q)}(G; x) \|^2_2 = q^{-|V|} \sum_{d \in \mathbb{Z}_q^V} |\hat{g} \otimes F(Gd)|^2.
\]
\[
= q^{-|V|} |\ker(S)| \sum_{b \in \text{im}(S)} (|\hat{g}|^2)^{\otimes E}(b).
\]

By the Poisson summation formula, and using $\text{im}(S)^\perp = \ker(S^\top)$, $|\ker(S)| = q^{|V|}/|\text{im}(S)|$, we deduce the following, which provides an answer to question (C).

**Theorem 5.3.** If
\[
F^{(q)}(G; x) = \prod_{(u,v) \in E} \sum_{b \in \mathbb{Z}_q} g(b)x_u^{t-b}x_v^b,
\]
then
\[
\| F^{(q)}(G; x) \|^2_2 = \frac{1}{|\text{im}(S)|} \sum_{b \in \text{im}(S)} (|\hat{g}|^2)^{\otimes E}(b)
\]
\[
= \sum_{b \in \ker(S^\top)} (g \ast g)^{\otimes E}(b),
\]
where as usual $S : \mathbb{Z}_q^V \to \mathbb{Z}_q^E$ is defined by $(Sd)_e = d_v - sd_u$ for $e = (u,v) \in E$.

**Example 5.4.** Petersen’s graph polynomial modulo $(x_v^q - 1 : v \in V)$ has $s = 1 = t$, $g = \delta_0 - \delta_1$, $g \ast g = 2\delta_0 - \delta_1 - \delta_{-1}$. The transformation $S : \mathbb{Z}_q^V \to \mathbb{Z}_q^E$ is the coboundary operator, $S^\top$ the boundary, $\ker(S^\top)$ the set of $\mathbb{Z}_q$-flows of $G$. This gives Tarsi’s result, Theorem 2.2, that the $\ell_2$-norm of Petersen’s graph polynomial modulo $(x_v^q - 1 : v \in V)$ is equal to
\[
(-1)^{|E|} \sum_{b \in \{-1,0,1\}^E \setminus \ker(S^\top)} (-2)^{\# \{e \in E : b_e = 0\}},
\]
where the sum is over $(q,1)$-flows of $G$.

**Example 5.5.** The polynomial
\[
\prod_{u,v \in E} (x_u + x_v)
\]
is a generating function for score vectors of orientations of $G$, and as such its number of non-zero coefficients turns out to be equal to $T(G; 2, 1)$, the number of forests of $G$. (See for example [10].) By Theorem 3.4 with $g = \delta_0 + \delta_1$, $g \ast g = 2\delta_0 + \delta_1 + \delta_{-1}$, when this polynomial is reduced modulo $(x_v^q - 1 : v \in V)$ it has $\ell_2$-norm equal to $T(G; 2, 4)$. Determining how many non-zero coefficients the polynomial has (its $\ell_0$-norm) when reduced modulo $(x_v^q - 1 : v \in V)$ includes as a subproblem determining whether a graph is $\mathbb{Z}_q$-connected, a notion defined in [11].

Theorem 3.4 applied to the result of Theorem 5.3 has the following consequence, answering question (B).

**Corollary 5.6.** The $\ell_2$-norm $\| F^{(q)}(G; x) \|^2_2$ of the polynomial defined in Theorem 5.3 is an evaluation of the Tutte polynomial $T(G; x, y)$ with $(x - 1)(y - 1) = q$ if and only if $s = 1$ and $g \ast g$, equivalently $|\hat{g}|^2$, is constant on $\mathbb{Z}_q \setminus 0$.  

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We finish with three examples of functions $g$ satisfying the conditions of Corollary 5.6 yielding families of polynomials that have $\ell_2$-norm equal to a Tutte-Grothendieck invariant.

A $(q, k, \ell)$-difference set in an Abelian group $Q$ is a subset $P$ of size $k$ with the property that $\#\{a, b \in P : a - b = c\} = \ell$ for each $c \in Q \setminus 0$. For example, $Q \setminus 0$ is a $(q, q - 1, q - 2)$-difference set. All non-zero $c$ have exactly $q - 2$ ways of being written as $a - b$ for $a, b \in Q \setminus 0$ since for given $a \in Q \setminus \{0, c\}$ there is a unique $b \in Q \setminus \{0, c\}$ with $a - b = c$.

Note that a function is constant on non-zero values if and only if the same is true of its Fourier transform: if $f = \delta_0 + \delta_{Q^0}$ then $\hat{f} = (t + 1)q \delta_0 + (t - 1)\delta_{Q^0}$. This fact, together with the equation $\delta_P \star \delta_P = \sum_{c \in Q} \#\{a, b \in P : a - b = c\} \delta_c$, implies that the Fourier transform $\delta_P \star \delta_P = |\hat{\delta}_P|^2$ is constant on $Q \setminus 0$ if and only if $P$ is a $(q, k, \ell)$-difference set in $Q$, i.e., $\delta_P \star \delta_P = k\delta_0 + \ell\delta_{Q^0}$.

**Example 5.7.** If $g = \delta_P$ for some $P \subseteq \mathbb{Z}_q$, or more generally $g = \delta_P + r\delta_{\mathbb{Z}_q \setminus P}$ for any constant $r$, then $|\hat{g}|^2$ is constant on $\mathbb{Z}_q \setminus 0$ if and only if $P$ is a difference set in $\mathbb{Z}_q$. When $P = \mathbb{Z}_q \setminus 0$ this is the family of polynomials described in Proposition 5.7 whose constant terms were also Tutte-Grothendieck invariants.

A $(q, k, \ell, m)$-partial difference set in $Q$ is a subset $P$ of size $k$ with the property that $\delta_P \star \delta_P = k\delta_0 + m\delta_{Q^0}$. For example, a subgroup $P$ of size $k$ is a $(q, k, k)$-partial difference set.

**Example 5.8.** If $P \subseteq \mathbb{Z}_q \setminus 0$ and $g = \delta_P - \delta_{\mathbb{Z}_q \setminus (P, 0)}$ then $|\hat{g}|^2$ is constant on $\mathbb{Z}_q \setminus 0$ if and only if $q$ is odd and $P$ is a Paley difference set or partial difference set, i.e., $|P| = (q - 1)/2$ and

$$\delta_P \star \delta_P = \begin{cases} 
\frac{q - 1}{2}\delta_0 + \frac{q - 5}{4}\delta_P + \frac{q - 1}{4}\delta_{\mathbb{Z}_q \setminus (P, 0)} \\
\frac{q - 1}{2}\delta_0 + \frac{q - 3}{4}\delta_{\mathbb{Z}_q/0}, 
\end{cases}$$

according as $q \equiv \pm 1 \pmod{4}$. (For odd prime $q$, the set of non-zero squares in $\mathbb{Z}_q$ is an example of such a $P$.)

**Example 5.9.** When $q$ is prime and $g = \sum_{a \in \mathbb{Z}_q} \psi(a) \delta_a$ for a multiplicative character $\psi$ of $\mathbb{Z}_q^\times$, then $|\hat{g}|^2 = q\delta_{\mathbb{Z}_q^0}$, i.e., the polynomial

$$F^{(q)}(G; x) = \prod_{(u, v) \in E} \sum_{b \in \mathbb{Z}_q} \psi(b)x_u^{v-b}x_v^b$$

has $\ell_2$-norm $q^{|E| - |V|} P(G; q)$. (The case $q = 3$ is Petersen’s graph polynomial reduced modulo $(x_v^3 - 1 : v \in V)$.)

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