A STRENGTHENING OF FREIMAN’S $3k - 4$ THEOREM

BÉLA BOLLOBÁS, IMRE LEADER, AND MARIUS TIBA

Abstract. In its usual form, Freiman’s $3k - 4$ theorem states that if $A$ and $B$ are subsets of $\mathbb{Z}$ of size $k$ with small sumset (of size close to $2k$) then they are very close to arithmetic progressions. Our aim in this paper is to strengthen this by allowing only a bounded number of possible summands from one of the sets. We show that if $A$ and $B$ are subsets of $\mathbb{Z}$ of size $k$ such that for any four-element subset $X$ of $B$ the sumset $A + X$ has size not much more than $2k$ then already this implies that $A$ and $B$ are very close to arithmetic progressions.

1. Introduction

Starting with the classical Cauchy-Davenport theorem [3–5], the sizes of sumsets in Abelian groups have been studied in a host of papers. A new direction in this study was introduced in [1]: given sets of integers $A$ and $B$, say, does $B$ have a small subset $B'$ such that $|A + B'|$ is large, perhaps even comparable to $|A + B|$? In particular, in [1] the following result was proved.

Theorem 1. Let $A$ and $B$ be finite non-empty subsets of $\mathbb{Z}$ with $|A| \geq |B|$. Then there exist elements $b_1, b_2, b_3 \in B$ such that

$$|A + \{b_1, b_2, b_3\}| \geq |A| + |B| - 1.$$ 

Freiman’s $3k - 4$ theorem (see Freiman [6,7] and also Lev and Smeliansky [10] and Stan chesscu [12]), states the following, in the case where the sets have the same size.

Let $A$ and $B$ be finite subsets of $\mathbb{Z}$ with $|A| = |B| = k$ such that $|A + B| = 2k - 1 + r$, where $r \leq k - 3$. Then there are arithmetic progressions $P$ and $Q$ of the same common difference, containing $A$ and $B$ respectively, such that each has size $k + r$.

A result of this kind is often called an ‘inverse’ theorem, as it describes what happens when an inequality is close to being tight. (These are also often known as ‘stability’ results.)

What about a ‘bounded sumset’ version of Freiman’s $3k - 4$ theorem? Our aim in this paper is to show that indeed we can weaken the condition that $A + B$ is small to just a condition that all sumsets of $A$ with a bounded number of terms of $B$ are small. This is really rather surprising. One could view this as a kind of inverse result to Theorem 1.

Theorem 2. There are constants $c, \varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ the following holds. Let $A$ and $B$ be finite non-empty subsets of $\mathbb{Z}$ with $n = |A| = |B|$. Suppose that for any four elements $b_1, b_2, b_3, b_4 \in B$ we have

$$|A + \{b_1, b_2, b_3, b_4\}| \leq (2 + \varepsilon)n - 1.$$ 

Then there are arithmetic progressions $P, Q$ in $\mathbb{Z}$ of size $(1 + \varepsilon + c\varepsilon^2)n$ with the same common difference such that $B \subset Q$ and

$$|A \Delta P| \leq c\varepsilon n.$$ 

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The key point in this result is that the sizes of the progressions \( P \) and \( Q \) are ‘about’ \((1 + \varepsilon)n\): the precise form of the error term in this, as quadratic in \( \varepsilon \), is not so important. Note that the dependence of the error terms (the sizes of \(|A\Delta P|\) and \(|B\Delta Q|\)), as linear functions of \( \varepsilon \), is best possible. This may be seen by taking \( B \) to be an interval of length \( n \) and \( A \) to consist of of an interval of length \((1 - 2\varepsilon)n\) together with, on either side of it, random half-sized subsets of the adjacent intervals of length \( 2\varepsilon n \).

We remark that the condition that \( A \) has small symmetric difference with \( P \) cannot be strengthened to insist that \( A \) lies inside \( P \). This is because one may always have a few ‘rogue’ points in \( A \), far from the rest of \( A \). The same would not apply to \( B \), because adding a few faraway points to \( B \) is a great help in selecting the four points.

It would be extremely interesting to decide whether this result remains valid if we consider three elements instead of four. To elaborate on this, the way that Theorem 1 is proved is by taking the three points of \( B \) as follows: we first take the smallest and largest points of \( B \) (the reader will see that we have to take those points, as may be seen from the case when the two sets are intervals), and then choosing the third point at random from the remaining points of \( B \). It is very surprising that these two different ingredients mesh together so well, to give exactly the lower bound required. However, and this is the key point, this approach cannot be used to prove a three-element version of Theorem 2. Indeed, this may be seen by taking \( B = [0, n] \) and \( A = X \cup (X + n) \) where \( X \) is a random set of density \( 1/2 \) in an interval of length \( n \).

The plan of the paper is as follows. In Section 2 we mention some background results from \([1]\) that we will need. We do give a brief summary of the proof of Theorem 1 to help make the paper self-contained and also because the methods there form a good ‘toy case’ of the arguments in our main result. Indeed, those background results from \([1]\) that we do not prove here may be proved by methods very similar to those in the proof of Theorem 1. Then in Section 3 we present our main result.

Our notation is standard. To make our paper more readable, we often omit integer-part signs when these do not affect the argument.

Sometimes we write ‘\( x \mod d \)’ as shorthand for the infinite arithmetic progression \( \{ y \in \mathbb{Z} : y \equiv x \mod d \} \), and refer to it as a fibre mod \( d \). When \( S \) is a subset of \( \mathbb{Z} \) we often write \( S^x \) for the intersection of this fibre with \( S \) – when the value of \( d \) is clear. (We sometimes write \( S^x \) as \( S^x_d \) when we want to stress the value of \( d \).) Thus \( S^x = S \cap \pi^{-1}(x) \), where \( \pi = \pi_d \) denotes the natural projection from \( \mathbb{Z} \) to \( \mathbb{Z}_d \). We also write \( \tilde{S} \) for \( \pi_d(S) \).

When we write a probability or an expectation over a finite set, we always assume that the elements of the set are being sampled uniformly. Thus, for example, for a finite set \( X \subseteq \mathbb{Z} \) we denote the expectation and probability when we sample uniformly over all \( x \in X \) by respectively \( \mathbb{E}_{x \in X} \) and \( \mathbb{P}_{x \in X} \).

For more general background on sumsets, see the survey of Breuillard, Green and Tao \([2]\) or the books of Nathanson \([11]\) or Tao and Vu \([13]\). For extensions of Freiman’s \( 3k - 4 \) theorem to other settings, such as \( \mathbb{Z}_p \) in particular, see Grynkiewicz \([8]\).

To end the Introduction, we give a brief overview of the proof of Theorem 2. With \( B \) having first element 0 and last element \( m \), we start by showing that (unless we are done)
A must have a large projection onto \( \mathbb{Z}_m \). This is accomplished by a careful analysis of what happens when we take our four points of \( B \) to be \( 0, m \) and two random points: it is the interaction of the two random points that is critical here. From this we find that \( B \) is contained in a quite short arithmetic progression, and in turn this will imply that \( A \) is also relatively close to an arithmetic progression. However, these progressions are not small enough to give Theorem 2, and so we need an argument that ‘boosts’ this. This is a very delicate analysis of, again, how the two random translates interact with each other and with the two fixed translates. In fact, it is rather surprising that this boosting argument actually works: in length it is most of the proof. (It is Theorem 9 below).

2. Prerequisites

We start by giving an indication of the proof of Theorem 1. As explained above, the main idea is to fix two elements as the first and last elements of \( B \) and then to select the remaining element at random.

Let \( B \) have first element 0 and last element \( m \). Then \( A + \{0, m\} = A \cup (A + m) \) satisfies \( \pi_m(A) = \pi_m(A \cup (A + m)) = \tilde{A} \), and

\[
|A \cup (A + m)| \geq |A| + |\tilde{A}|.
\]

(1)

To see this, note that \((A + m)^x = A^x + m\) for every \( x \in \mathbb{Z} \). Hence, if \( x \in \tilde{A} = \pi_m(A) \) then \((A + m)^x = A^x + m \neq A^x\), so

\[
\left[ A \cup (A + m) \right]^x = A^x \cup (A + m)^x \geq |A^x| + 1.
\]

Consequently,

\[
|A \cup (A + m)| \geq \sum_{x \in \tilde{A}} \left| \left[ A \cup (A + m) \right]^x \right| \geq \sum_{x \in \tilde{A}} |A^x| + 1 = |A| + |\tilde{A}|,
\]

completing the proof of (1).

We mention that the way that the proof of Theorem 1 in [1] now proceeds is to consider the excess contribution to \( |A \cup (A + m)| \) that comes when we form the union with \( A + b \), where \( b \) is chosen uniformly at random from the remaining points of \( B \). One finds that this has expectation at least \( |A| \) times \( 1 - |\tilde{A}|/|B| \). If this is negative then the bound (1) is already enough to finish the proof, while if it is positive then it is easy to check that together with (1) it gives the required bound.

This actually gives the following stronger version of Theorem 1. Let \( A \) and \( B \) be finite non-empty subsets of integers with \( |A| \geq |B| \) and \( B \) having smallest element 0 and greatest element \( m \). Then we have

\[
\max_{b_1, b_2, b_3 \in B} |A + \{b_1, b_2, b_3\}| \geq \mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| \geq |A| + |B| - 1.
\]

(2)

We now pass to some related results from [1]. These are proved along broadly similar lines to the above results. The first one is about the case of equality.

**Theorem 3.** Let \( A \) and \( B \) be finite non-empty subsets of \( \mathbb{Z} \) with \( |A| = |B| \), with \( \min(B) = 0 \) and \( \max(B) = m \). Suppose that when we choose an element \( b \) of \( B \setminus \{0, m\} \) uniformly at
random we have
\[ \mathbb{E}_{b \in B \setminus \{0, m\}} \left| A + \{0, b, m\} \right| \leq |A| + |B| - 1. \] (3)

Then \( A \) and \( B \) are arithmetic progressions with the same common difference.

In particular, suppose that when we choose any three elements \( b_1, b_2, b_3 \) of \( B \) we have
\[ \left| A + \{b_1, b_2, b_3\} \right| \leq |A| + |B| - 1. \]

Then \( A \) and \( B \) are arithmetic progressions with the same common difference.

Then we need another version of Theorem 4.

**Theorem 4.** Let \( A \) and \( B \) be finite non-empty subsets of \( \mathbb{Z} \), with \( \min(B) = 0 \) and \( \max(B) = m \). Then
\[ \mathbb{E}_{b \in B \setminus \{0\}} \left| A + \{0, b, m\} \right| \geq |A| + |\pi_m(A)| + |A| \max \left( 0, \frac{|B| - 1 - |\pi_m(A)|}{|B| - 1} \right). \]

We note also a simple variant of the ideas above.

**Lemma 5.** Let \( A, B \) and \( A_1 \) be finite non-empty subsets of \( \mathbb{Z} \), with \( \min(B) = 0 \) and \( \max(B) = m \). Then with \( \tilde{A} = \pi_m(A) \) and \( \tilde{B} = \pi_m(B) \) we have
\[ \mathbb{E}_{b \in B \setminus \{0\}} \left| (A_1 + b) \setminus \pi_m^{-1}(\tilde{A}) \right| \geq |A_1| \max \left( 0, \frac{\tilde{B} - |\tilde{A}|}{\tilde{B}} \right). \]

The next result follows easily from Lemma 5.

**Lemma 6.** Let \( A \) and \( B \) be finite non-empty subsets of \( \mathbb{Z} \), with \( \min(B) = 0 \) and \( \max(B) = m \). Suppose that \( |B| - 1 = |B| \geq 8 |\tilde{A}| \), where \( \tilde{A} = \pi_m(A) \) and \( \tilde{B} = \pi_m(B) \). Then
\[ \mathbb{E}_{b_2, b_3 \in B \setminus \{0\}} \left| A + \{0, b_2, b_3\} \right| \geq 2.5 |A|. \]

The last result from [1] that we need is a strengthening of Theorem 4.

**Theorem 7.** Let \( A \) and \( B \) be finite non-empty subsets of \( \mathbb{Z} \) with \( |A| = |B| \), with \( \min(B) = 0 \) and \( \max(B) = m \). Then
\[ \mathbb{E}_{b \in B \setminus \{0\}} \left| A + \{0, b, m\} \right| \geq |A| + |B| - 1 + \max \left( 0, \frac{2|\pi_m(A)| - m(m - (|B| - 1)) - 1}{|B| - 1} \right). \]

3. The Main Result

In this section we prove Theorem 2, our stability version of Theorem 1. As we mentioned earlier, the dependence of the error terms on \( \varepsilon \), being linear, is of best possible order. It will turn out that a key ingredient is actually the following weaker version of Theorem 2.

**Theorem 8.** For any \( \delta > 0 \) there exists \( \varepsilon > 0 \) such that the following holds. Let \( A \) and \( B \) be finite non-empty subsets of \( \mathbb{Z} \) with \( n = |A| = |B| \). Suppose that
\[ \max_{b_1, b_2, b_3, b_4 \in B} \left| A + \{b_1, b_2, b_3, b_4\} \right| \leq 2n - 1 + \varepsilon n. \]
Then there exist arithmetic progressions $P$ and $Q$ in $\mathbb{Z}$ with the same common difference and size $\lfloor(1 + \delta)n\rfloor$ such that $|A \setminus P| \leq \delta n$ and $B \subset Q$.

We will also need the following technical result.

**Theorem 9.** For all sufficiently small $\alpha, \beta, \varepsilon > 0$, taking $\nu = (\varepsilon + \varepsilon^2(1-\alpha)^{-1})(1-4\alpha+\beta)^{-1}$ and $\mu = 2^{15}(\varepsilon + \beta)$ the following holds. Let $A$ and $B$ be finite non-empty subsets of $\mathbb{Z}$ with $n = |A| = |B|$. Suppose that there exist integer arithmetic progressions $P$ and $Q$ with the same common difference $d$ and sizes $\lfloor(1 + \alpha)n\rfloor$ and $\lfloor(1 + \beta)n\rfloor$ such that

$$|A \setminus P| \leq \alpha n \text{ and } B \subset Q.$$  \hfill (5)

Then there exist integer arithmetic progressions $P'$ and $Q'$ with common difference $d$ and sizes $\lfloor(1 + \mu)n\rfloor$ and $\lfloor(1 + \nu)n\rfloor$ such that $|A \setminus P'| \leq \mu n$ and $B \subset Q'$.

**Proof.** Fix $\alpha, \beta, \varepsilon > 0$ sufficiently small. By Theorem 3 we may assume that

$$\varepsilon n \geq 1. \hfill (7)$$

Set $R = d\mathbb{Z}$, and assume without loss of generality that $P, Q \subset R$. Form the partition $A = A_1 \cup A_2$ given by $A_1 = A \cap R$ and $A_2 = A \cap R^c$ and write $n_1 = |A_1|$ and $n_2 = |A_2|$. By (5) we have

$$n \geq n_1 \geq (1 - \alpha)n \geq 9n_2. \hfill (8)$$

Assume that $\min(B) = 0$ and $\max(B) = md$. Take any subset $B_1 \subset B$ of size $n_1 = |B_1|$ such that $\min(B_1) = 0$ and $\max(B_1) = md$. On the one hand, by construction we have $(A_1 + B_1) \cap (A_2 + B_1) = \emptyset$. On the other hand, by (6) we have

$$\mathbb{E}_{b_1, b_3 \in B_1 \setminus \{md\}}|A + \{0, b_2, b_3, md\}| \leq 2n - 1 + \varepsilon n.$$  \hfill (9)

The last two relations imply

$$\mathbb{E}_{b_2 \in B_1 \setminus \{md\}}|A_1 + \{0, b_2, md\}| + \mathbb{E}_{b_2, b_3 \in B_1 \setminus \{md\}}|A_2 + \{0, b_2, b_3\}| \leq 2n - 1 + \varepsilon n.$$  \hfill (10)

By Theorem 4 we have

$$\mathbb{E}_{b_2 \in B_1 \setminus \{md\}}|A_1 + \{0, b_2, md\}| \geq 2n_1 - 1.$$  \hfill (11)

Lemma 6 and (8) tell us that

$$\mathbb{E}_{b_2, b_3 \in B_1 \setminus \{md\}}|A_2 + \{0, b_2, b_3\}| \geq 2.5n_2.$$  \hfill (12)

The last three inequalities imply

$$\mathbb{E}_{b_2 \in B_1 \setminus \{md\}}|A_1 + \{0, b_2, md\}| \leq 2n_1 - 1 + \varepsilon n_1.$$  \hfill (13)

and also

$$n_2 \leq 2\varepsilon n \text{ i.e. } n_1 \geq (1 - 2\varepsilon)n. \hfill (14)$$
By (6) and (10) we have
\[ \max_{b_2,b_3 \in B_1 \setminus \{md\}} \left| A_1 + \{0, b_2, b_3, md\} \right| \leq 2n_1 - 1 + 8\varepsilon n_1. \]  \hspace{1cm} (11)

For the rest of the proof we focus on the sets \( A_1 \) and \( B_1 \) and thus we may assume without loss of generality that \( d = 1 \).

**The construction of \( Q' \).** By Theorem 7 we have
\[ \mathbb{E}_{b_2 \in B_1 \setminus \{md\}} \left| A_1 + \{0, b_2, m\} \right| \geq 2n_1 - 1 + \frac{(2|\pi_m(A_1)| - m)(m - (n_1 - 1)) - 1}{n_1 - 1}. \]  \hspace{1cm} (12)

By (5), we have \(|\pi_m(A_1)| \geq n - 2\alpha n\) and also \( m \leq 1 + \beta n \); consequently,
\[ 2|\pi_m(A_1)| - m \geq (1 - 4\alpha - \beta)n \geq (1 - 4\alpha - \beta)n_1. \]  \hspace{1cm} (13)

From (9), (12) and (13) we have
\[ \frac{(1 - 4\alpha - \beta)n_1(m + 1 - n_1) - 1}{n_1 - 1} \leq \varepsilon n_1, \]  \hspace{1cm} (14)

and so our bounds on the parameters imply
\[ m + 1 \leq n_1 + (1 - 4\alpha - \beta)^{-1}(\varepsilon(n_1 - 1) + \frac{1}{n_1}) \leq n + (1 - 4\alpha - \beta)^{-1}(\varepsilon n + \varepsilon^2(1 - \alpha)^{-1}n) = (1 + (1 - 4\alpha - \beta)^{-1}(\varepsilon + \varepsilon^2(1 - \alpha)^{-1}))n. \]

Finally, we conclude the arithmetic progression \( Q' = \{0, d, \ldots, md\} \) has common difference \( d \) and size at most \([1 + \nu]n\) and contains \( B \).

**The construction of \( P' \).** By translating \( A_1 \), we may assume that the partition \( A_1 = A_l \sqcup A_c \sqcup A_r \) given by
\[ A_l = (-\infty, -m) \cap A_1, A_c = \{-m, 0\} \cap A_1, A_r = [0, \infty) \cap A_1. \]
satisfies
\[ |A_r| = k, |A_l| = k - \tau, |A_c| = n_1 - 2k + \tau, \]  \hspace{1cm} (15)

where \( \tau \in \{0, 1\} \). By (9) and (N) we have \( k \leq 2^{-10}n_1 \), and by (7) and (8) we have \( 2\varepsilon n_1 \geq 1 \). However, we may assume that
\[ 2^{-10}n_1 \geq k \geq 2^{10}(\beta + \varepsilon)n_1 \geq 2^9, \]  \hspace{1cm} (16)

and so Theorem 4 implies
\[ \mathbb{E}_{b_2 \in B_1 \setminus \{m\}} \left| A_c + \{0, b_2, m\} \right| \geq 2(n_1 - 2k + \tau) + (n_1 - 2k + \tau)\frac{2k - \tau - 1}{n_1 - 1}. \]

In particular, there exist \( b_2 \in B_1 \setminus \{m\} \) and there exists a partition \([-m, m] = Y \sqcup Y^c \) such that
\[ |Y| = 2m - \left[ 2(n_1 - 2k + \tau) + (n_1 - 2k + \tau)\frac{2k - \tau - 1}{n_1 - 1} \right] \geq 2k - \tau - 1, \]
In particular, there exists

\[ |Y^c| = 2(n_1 - 2k + \tau) + (n_1 - 2k + \tau) \frac{2k - \tau - 1}{n_1 - 1} \]  \tag{17}

\[ \geq 2n_1 - 1 - 2k + \tau - \frac{(2k - \tau - 1)^2}{n_1 - 1}, \]  \tag{18}

and

\[ Y^c \subset A_c \cup (A_c + m) \cup (A_c + b_2). \]

We now break the argument into two cases.

**Case 1:**

\[ \left| [-m - k, -m) \cap A_1 \right| \geq k/2 \text{ and } \left| [0, k) \cap A_1 \right| \geq k/2. \]  \tag{19}

In this case, by (17) we get

\[ \left| [-m, 0) \cap Y \right| \geq k - 1 \text{ or } \left| [0, m) \cap Y \right| \geq k - 1. \]

Let \( Z = Y \cap [-m, 0). \) We may assume that \( |Z| \geq k - 1. \) By (16) we have

\[ |Z| \geq k/2. \]  \tag{20}

**Claim A.**

\[ \mathbb{P}_{b_3 \in [0,4k) \cap B_1} \left( A_1 + b_3 \right) \cap Z \geq k/32. \]

In particular, there exists \( b_3 \in B_1 \) such that

\[ \left| (A_1 + b_3) \cap Z \right| \geq k/32. \]

**Proof.** By (5), (10) and (16) we have

\[ \left| [0, 4k) \setminus B_1 \right| \leq \left| [0, m) \setminus B \right| + |B_2| \leq \beta n + n_2 \leq 4(\beta + \varepsilon)n_1 \leq k/4. \]  \tag{21}

For \( x \in [-m + 3k, 0), \) by (5), (10), (15) and (16) we have

\[ \left| (x - 4k, x] \setminus A_1 \right| \leq \left| (x - 3k, x] \setminus A_1 \right| + k \leq \left| [-m, 0) \setminus A_1 \right| + k \]
\[ \leq m - (n_1 - 2k + \tau) + k \leq \beta n + n_2 + 3k \]  \tag{22}
\[ \leq 4(\beta + \varepsilon)n_1 + 3k \leq 7k/2. \]  \tag{23}

For \( x \in [-m, -m + 3k), \) by (19) we get

\[ \left| (x - 4k, x] \setminus A_1 \right| \leq 7k/2. \]  \tag{24}

For \( z \in [-m, 0), \) by (21), (22), (25) we get

\[ \mathbb{P}_{b_3 \in [0,4k) \cap B_1} \left( z \in A_1 + b_3 \right) \geq 1/16. \]  \tag{26}
Therefore, by (20) and (26) we conclude
\[
\mathbb{E}_{b_3 \in [0,4k)} \left| (A_1 + b_3) \cap Z \right| = \sum_{z \in Z} \mathbb{P}_{b_3 \in [0,4k)} \left( z \in A_1 + b_3 \right) \geq k/32.
\]

This concludes the proof of Claim A. \(\square\)

We now turn to the second case.

**Case 2:**
\[
\left| [-m - k, -m) \cap A_1 \right| \leq k/2 \quad \text{or} \quad \left| [0, k) \cap A_1 \right| \leq k/2.
\]

In this case we may assume
\[
\left| [-m - k, -m) \cap A_1 \right| \leq k/2 \quad \text{and} \quad \left| [-\infty, -m - k) \cap A_1 \right| \geq k/4. \tag{27}
\]

Put \( Z = (-\infty, -m) \setminus A_1 \).

**Claim B.**
\[
\mathbb{E}_{b_3 \in [0,k) \cap B_1} \left| (A_1 + b_3) \cap Z \right| \geq k/32.
\]

In particular, there exists \( b_3 \in B_1 \) such that \( (A_1 + b_3) \cap Z \geq k/32 \).

**Proof.** Let \( S \) be set of the greatest \( k/4 \) elements of \( [-\infty, -m - k) \cap A_1 \).

For \( x \in S \), by (27) we have
\[
\left| [x, x + k) \cap Z \right| \geq k/4. \tag{28}
\]

By (5), (10) and (16) we get
\[
\left| [0,k) \setminus B_1 \right| \geq k - \left| [0, m) \setminus B \right| - |B_2| \geq k - \beta n - n_2 \geq k - 4(\beta + \varepsilon)n_1 \geq 7k/8. \tag{29}
\]

For \( x \in S \), by (28) and (29) we have \( \mathbb{P}_{b_3 \in [0,k) \cap B_1} \left( x + b_3 \in Z \right) \geq 1/8 \), so
\[
\mathbb{E}_{b_3 \in [0,k) \cap B_1} \left| (A_1 + b_3) \cap Z \right| = \sum_{x \in S} \mathbb{P}_{b_3 \in [0,k) \cap B_1} \left( x + b_3 \in Z \right) \geq k/32.
\]

This ends the proof of Claim B. \(\square\)

We now return to the proof of Theorem 9. By construction, we have
\[
A_1 + \{0, b_2, b_3, m\} \supset A_1 \cup (A_r + m) \cup \left( [A_c \cup (A_c + b_2) \cup (A_c + m)] \cap Y^c \right) \cup \left( (A_1 + b_3) \cap Z \right).
\]
So by Claim A and Claim B, together with (11), (15), (16) and (17), we obtain
\[
2n_1 - 1 + 8\varepsilon n_1 \geq k + (k - \tau) + (2n_1 - 1 - 2k + \tau - \frac{(2k - \tau - 1)^2}{n_1 - 1}) + \frac{k}{32} \\
\geq 2n_1 - 1 + \frac{k}{32} - \frac{(2k - \tau - 1)^2}{n_1 - 1} \geq 2n_1 - 1 + \frac{k}{32} - \frac{4k^2}{n_1} \\
\geq 2n_1 - 1 + \frac{k}{64}.
\]
Therefore we have \( k \leq 2^9 \varepsilon n_1 \).

Finally, we conclude the arithmetic progression \( P' = \{0, d, \ldots, md\} \) with common difference \( d \) and size at most \( \lfloor (1 + \beta)n \rfloor \) satisfies \( |A \setminus P'| \leq n_2 + 2k \leq 2^{11} \varepsilon n \leq \mu n \).

This finishes the proof of Theorem 9. \( \square \)

We now prove Theorem 8.

Proof of Theorem 8 Fix \( \delta < 2^{-10} \) and choose \( \varepsilon = 2^{-40} \delta^8 \). By Theorem 3, we may assume that \( \varepsilon n \geq 1 \). In particular, we have \( \varepsilon^{1/2} n \geq 2^{20} \) and \( \varepsilon^{1/8} \leq 2^{-10} \). We may also assume that \( \min(B) = 0 \) and \( \max(B) = m \).

The proof will proceed via the next three lemmas.

Lemma 10.
\[
|\pi_m(A)| \geq (1 - 4\varepsilon^{1/4})n.
\]

Proof. The proof of this lemma is based on two claims. Suppose for a contradiction that
\[
|\tilde{A}| \leq (1 - 4\varepsilon^{1/4})n.
\]

Claim A. There exists \( b_1 \in B \) such that
\[
|A + \{0, b_1, m\}| \geq 2n - 1 - 2^{-1} \varepsilon^{1/2} n.
\]
and
\[
|\pi_m[A + \{0, b_1, m\}]| \leq (1 - \varepsilon^{1/2}) n.
\]

Proof. By Theorem 7 we have
\[
\mathbb{E}_{b \in B \setminus \{m\}} |A + \{0, b, m\}| \geq 2n - 1.
\]
By hypothesis we have
\[
\max_{b \in B} |A + \{0, b, m\}| \leq 2n - 1 + \varepsilon n.
\]
By Markov’s inequality, it follows that
\[
\mathbb{P}_{b \in B \setminus \{m\}} \left( |A + \{0, b, m\}| \geq 2n - 1 - 2^{-1} \varepsilon^{1/2} n \right) > 1 - 4 \varepsilon^{1/2}.
\]
Now form the partition $A = A_1 \cup A_2$ such that

$$A_1^x = \begin{cases} A^x & \text{if } |A^x| \geq 2, \\ \emptyset & \text{if } |A^x| \leq 1. \end{cases} \quad \text{and} \quad A_2^x = \begin{cases} A^x & \text{if } |A^x| \leq 1, \\ \emptyset & \text{if } |A^x| \geq 2. \end{cases} \quad (31)$$

From $|\tilde{A}| \leq (1 - 4\varepsilon^{1/4})n$ we get $|A_1| \geq 4\varepsilon^{1/4}n$. Lemma 5 gives

$$\mathbb{E}_{b \in B \setminus \{m\}} \left| (A_1 + b) \setminus \pi^{-1}(\tilde{A}) \right| \geq |A_1| \max \left( 0, \frac{|\tilde{B} - |\tilde{A}||}{|\tilde{B}|} \right) \geq 4\varepsilon^{1/4}n \frac{n - 1 - (1 - 4\varepsilon^{1/4})n}{n - 1} \geq 8\varepsilon^{1/2}n.$$ 

Trivially, we have

$$\max_{b \in B \setminus \{m\}} \left| (A_1 + b) \setminus \pi^{-1}(\tilde{A}) \right| \leq n.$$ 

By Markov’s inequality, we deduce

$$\mathbb{P}_{b \in B \setminus \{m\}} \left( \left| (A_1 + b) \setminus \pi^{-1}(\tilde{A}) \right| \geq 4\varepsilon^{1/2}n \right) > 4\varepsilon^{1/2}. \quad (32)$$

By (30) and (32), it follows that there exists $b_1 \in B \setminus \{m\}$ such that

$$\left| A + \{0, b_1, m\} \right| \geq 2n - 1 - 2^{-1}\varepsilon^{1/2}n$$

and

$$\left| (A_1 + b_1) \setminus \pi^{-1}(\tilde{A}) \right| \geq 4\varepsilon^{1/2}n. \quad (33)$$

By (1), together with (1), (31) and (33), it follows that

$$2n - 1 + \varepsilon n \geq \left| A + \{0, b_1, m\} \right| \geq \left| A \cup (A + m) \right| + \left| A + b_1 \setminus [A \cup (A + m)] \right| \geq |A| + |\tilde{A}| + \left| (A + b_1) \setminus \pi^{-1}_m(\tilde{A}) \right| \geq |A| + |\tilde{A}| + \left| (\tilde{A} + b_1) \setminus \tilde{A} \right| + \frac{1}{2} \left| (A_1 + b_1) \setminus \pi^{-1}_m(\tilde{A}) \right| \geq (1 + 2\varepsilon^{1/2})n + \left| \tilde{A} \cup (\tilde{A} + b_1) \right|.$$ 

Hence

$$\left| \pi_m[A + \{0, b_1, m\}] \right| = |\tilde{A} \cup (\tilde{A} + b_1)| \leq (1 - \varepsilon^{1/2})n.$$ 

This establishes Claim A. \hfill \Box

**Claim B.**

$$\mathbb{E}_{b_2 \in B \setminus \{m\}} \left| A + \{0, b_1, b_2, m\} \right| > 2n - 1 + \varepsilon n.$$
Proof. From the first part of Claim A we get

\[ |A + \{0, b_1, m\}| \geq 2n - 1 - 2^{-1} \varepsilon^{1/2} n \]

From the second part of Claim A and Lemma 5 we get

\[ \mathbb{E}_{b_2 \in B \setminus \{m\}} \left| (A + b_2) \setminus (A + \{0, b_1, m\}) \right| \geq \mathbb{E}_{b_2 \in B \setminus \{m\}} \left| (A + b_2) \setminus (A + b_1) \right| \]

\[ \geq |A| \max \left( 0, \frac{|B| - |A \cup (A + b_1)|}{|B|} \right) \]

\[ \geq n \max \left( 0, \frac{n - 1 - (1 - \varepsilon^{1/2})^n}{n - 1} \right) \geq \frac{3}{4} \varepsilon^{1/2} n. \]

Combining the inequalities above, we obtain

\[ \mathbb{E}_{b_2 \in B \setminus \{m\}} \left| A + \{0, b_1, b_2, m\} \right| \geq 2n - 1 + \frac{1}{4} \varepsilon^{1/2} n > 2n - 1 + \varepsilon n. \]

This proves Claim B. \( \square \)

This completes the proof of Lemma 10 as we see that we have our desired contradiction. \( \square \)

We now move on to our second lemma.

Lemma 11. There exists an arithmetic progression Q of size \((1 + 32\varepsilon^{1/4})n\) such that \(B \subset Q\).

Proof. The proof of this lemma is based on the following two claims.

Claim A. For every \(\tilde{c} \in \tilde{B} + \tilde{B}\) there exists a set \(\tilde{A}' \subset \tilde{A}\) such that

\( \tilde{A}' + \tilde{c} \subset \tilde{A} \) and \( |\tilde{A}'| \geq (1 - 10\varepsilon^{1/4})n \).

Proof. Let \(b_1, b_2 \in B\) such that \(\tilde{c} = \tilde{b}_1 + \tilde{b}_2\), and fix \(b \in \{b_1, b_2\}\). By (1), (1) and Lemma 10 we have

\[ 2n - 1 + \varepsilon n \geq |A + \{0, b, m\}| \geq |A \cup (A + m)| + |(A + b) \setminus (A \cup (A + m))| \]

\[ \geq |A| + |\tilde{A}| + |(\tilde{A} + \tilde{b}) \setminus \tilde{A}| \geq (2 - 4\varepsilon^{1/4})n + |(\tilde{A} + \tilde{b}) \setminus \tilde{A}|. \]

Therefore \(|(\tilde{A} + \tilde{b}) \setminus \tilde{A}| \leq 5\varepsilon^{1/4} n\), and in particular, we have \(|(\tilde{A} + \tilde{c}) \setminus \tilde{A}| \leq 10\varepsilon^{1/4} n\). We conclude that the set \(\tilde{A}' = \{\tilde{a} \in \tilde{A} : \tilde{a} + \tilde{c} \in \tilde{A}\}\) has size \(|\tilde{A}'| \geq (1 - 10\varepsilon^{1/4})n\), so Claim A is proved. \( \square \)

Claim B.

\[ |\tilde{B} + \tilde{B}| \leq (1 + 32\varepsilon^{1/4})|\tilde{B}|. \]
Proof. Recall that $|\tilde{A}| \leq n$ and $|\tilde{B}| = n - 1$. Consider the set

$$E(\tilde{A}) = \{(a_1, a_2, a_3, a_4) \in \tilde{A}^4 : a_1 - a_2 = a_3 - a_4\}.$$ 

Trivially, $|E(\tilde{A})| \leq n^3$. On the other hand, by the previous claim we have $|E(\tilde{A})| \geq (1 - 10\varepsilon^{1/4})^2 n^2 |\tilde{B} + \tilde{B}|$. This completes the proof of Claim B. □

Returning to the proof of Lemma 11, by Claim B we have $|\tilde{B} + \tilde{B}| \leq (1 + 32\varepsilon^{1/4})|\tilde{B}|$. Kneser’s inequality [9] now implies that $\tilde{B} + \tilde{B}$ is a union of cosets of a subgroup $H$ of $\mathbb{Z}_m$ with $|H| \geq (1 - 32\varepsilon^{1/4})|\tilde{B}|$. Because $0 \in \tilde{B}$, it follows that $\tilde{B} + \tilde{B} = \tilde{H}$ where $|\tilde{H}| \leq (1 + 32\varepsilon^{1/4})|\tilde{B}|$ and $|\tilde{B} \setminus \tilde{H}| = 0$. If we let $Q$ be the unique arithmetic progression in $\mathbb{Z}$ satisfying $\{0, m\} \subset Q \subset [0, m]$ and $\pi_m(Q) = \tilde{H}$ then we can conclude that $|Q| = 1 + |\tilde{H}| \leq (1 + 32\varepsilon^{1/4})n$ and $B \subset Q$. This finishes the proof of Lemma 11. □

Our final lemma is as follows.

**Lemma 12.** There exists a translate $P$ of $Q$ such that $|A \setminus P| \leq 16\varepsilon^{1/8}n$.

Proof. Let $X$ be the set of the first $\varepsilon^{1/8}n$ elements of $A$, and assume that

$$0 = \min(B) = \min(Q) \text{ and } m = \max(B) = \max(Q).$$

Suppose for a contradiction that for every $x \in X$ we have $|(x + Q) \cap A| \leq (1 - 16\varepsilon^{1/8})n$, so, in particular, $|(x + B) \cap A| \leq (1 - 16\varepsilon^{1/8})n$. On the one hand, from this assumption we get

$$\mathbb{E}_{b \in B} \left( |X + b| \setminus |A \cup (A + m)| \right) = \sum_{x \in X} \mathbb{P}_{b \in B} \left( x + b \not\in A \cup (A + m) \right)$$

$$= \sum_{x \in X} \mathbb{P}_{b \in B} \left( x + b \not\in A \cup (X + m) \right) = \sum_{x \in X} \frac{|x + B| \setminus |A \cup (X + m)|}{|B|}$$

$$\geq \sum_{x \in X} \max \left( 0, \frac{|B| - |(x + B) \cap A| - |X|}{|B|} \right)$$

$$\geq \varepsilon^{1/8} n \max \left( 0, \frac{n - (1 - 16\varepsilon^{1/8})n - \varepsilon^{1/8}n}{n} \right) = 15\varepsilon^{1/4}n.$$ 

On the other hand, from (11), (10) and Lemma 10 we have

$$2n - 1 + \varepsilon n \geq \max_{b \in B} \left| A + \{0, b, m\} \right| \geq \left| A \cup (A + m) \right| + \max_{b \in B} \left| A + b \setminus |A \cup (A + m)| \right|$$

$$\geq |A| + |\tilde{A}| + \max_{b \in B} \left| A + b \setminus |A \cup (A + m)| \right|$$

$$\geq (2 - 4\varepsilon^{1/4})n + \max_{b \in B} \left| A + b \setminus |A \cup (A + m)| \right|. $$

This is a contradiction, as desired, completing our proof. □

With this, we have proved Theorem 8 since it is implied by Lemma 11 and Lemma 12. □
To end this section, we prove Theorem 2.

**Proof of Theorem 2.** The strategy is to first apply Theorem 8, and then apply Theorem 9 twice. Consider the function $\delta(\varepsilon)$ given by Theorem 8, and the functions $\nu(\alpha, \beta, \varepsilon), \mu(\alpha, \beta, \varepsilon)$ given by Theorem 9. It is easy to check that $\delta \to 0$ as $\varepsilon \to 0$, and $\mu, \nu \to 0$ as $\alpha, \beta, \varepsilon \to 0$.

It is also a simple computational task to find $c > 0$ such that

$$
\mu_2 \leq c\varepsilon \text{ and } \nu_2 \leq \varepsilon + c\varepsilon^2 \text{ as } \varepsilon \to 0.
$$

The result follows from these estimates. \qed

To end the paper, we give one of the many possible conjectures.

Our inverse theorem about $\mathbb{Z}$, Theorem 2, involves taking four elements from $B$ instead of three elements. We believe that a similar result should hold with three elements. We state this as a conjecture in the weaker form corresponding to Theorem 8 instead of Theorem 2, although we do believe that the stronger form should be true as well.

**Conjecture 13.** For every $\delta > 0$ there exists $\varepsilon > 0$ such that the following is true. Suppose that $A$ and $B$ are finite subsets of $\mathbb{Z}$ of equal size $n$ such that for any elements $b_1, b_2, b_3 \in B$ we have

$$
|A + \{b_1, b_2, b_3\}| \leq (2 + \varepsilon)n - 1.
$$

Then there exist arithmetic progressions $P, Q$ in $\mathbb{Z}$ with the same common difference such that

$$
B \subset Q \text{ and } |A \Delta P|, |B \Delta Q| \leq \delta n.
$$

We remark that if this conjecture is proved then, by virtue of Theorem 7, it would actually yield a strengthening of itself in which $|B \Delta Q| \leq (1 + o(1))\varepsilon n$. However, as we remarked earlier, it is important to note, for Conjecture 3, that the selection of three elements cannot be done as in Theorem 11 by taking the maximum and minimum elements of $B$ plus one more element.

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Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge, CB3 0WA, UK, and Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA

Email address: b.bollobas@dpmms.cam.ac.uk

Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge, CB3 0WA, UK

Email address: i.leader@dpmms.cam.ac.uk

Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge, CB3 0WA, UK

Email address: mt576@cam.ac.uk