Partition function of \( N \) composite bosons

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The partition function of composite bosons ("cobosons" for short) is calculated in the canonical ensemble, with the Pauli exclusion principle between their fermionic components included in an exact way through the finite temperature many-body formalism for composite quantum particles we recently developed. To physically understand the very compact result we obtain, we first present a diagrammatic approach to the partition function of \( N \) elementary bosons. We then show how to extend this approach to cobosons with Pauli blocking and interaction between their fermions. These diagrams provide deep insights on the structure of a coboson condensate, paving the way toward the determination of the critical parameters for their quantum condensation.

I. INTRODUCTION

A century ago, Albert Einstein suggested that as temperature decreases, non-interacting elementary bosons must undergo a phase transition with a macroscopic number of these bosons "condensed" into the system ground state. Such a condensation occurs below a critical temperature which decreases with the boson number \( N \) as \( N^{2/3} \). Interest in Bose-Einstein condensation (BEC) has been revived a decade ago by its first experimental realization thanks to advanced cooling and gas trapping techniques.\(^1\) These techniques now allow the study of condensation in geometrically different or low-dimensional potential wells in which a fixed number of bosons are trapped. In addition, highly controllable Feshbach resonances\(^2\) opened the route to the study of the BEC-BCS crossover in atomic systems.\(^3\)

As the effect of interaction between particles decreases with particle density, a condensation similar to the condensation of non-interacting elementary bosons predicted by Einstein should in principle occur in a dilute gas of bosonic particles, i.e., composite particles made of an even number of fermions. And indeed, such a phase transition is now commonly produced in ultracold atomic vapors.\(^4\) Yet, Bose-Einstein condensation in the case of semiconductor excitons has been searched for decades\(^7\)–\(^13\), even though these particles were for a long time considered as the most promising candidate to evidence this remarkable macroscopic quantum effect: due to their very light effective mass, the exciton quantum degeneracy at density easy to experimentally achieve should occur below a few kelvins while temperatures as low as micro-kelvins are required for atoms. By contrast, evidence of condensation in exciton-polaritons\(^14\) has been demonstrated in semiconductor quantum well embedded inside a microcavity\(^15\)–\(^17\) and more clearly in a trap\(^18\).

One reason for such a long time search could be that, due to their internal degrees of freedom, semiconductor excitons exist in bright and dark states, i.e., excitons coupled or not coupled to light. This coupling goes along with an increase of the bright exciton energy, leaving dark excitons in the lowest-energy state. So, the Bose-Einstein condensate of excitons must be dark, i.e., not coupled to light.\(^19\)–\(^21\) Another reason could be that, in addition to Coulomb interaction between carriers, excitons also interact in a non-standard way through carrier exchanges induced by the Pauli exclusion principle between electrons and between holes. We may wonder if the Pauli exclusion principle at density necessary for condensation does not substantially affect the quantum condensation of a coboson gas. In relation to this question, we wish to mention that, although the BCS wave function ansatz with all Cooper pairs condensed into the same state successfully explains the physical properties observed in conventional superconductors, this Pauli exclusion principle still makes the exact wave function for \( N \) Cooper pairs, as deduced from the Richardson-Gaudin procedure, quite different from the BCS wave function ansatz.\(^22\)

Although quite successful in treating systems of interacting elementary particles, either bosonic and fermionic, conventional many-body formalism is inadequate when it comes to cobosons like the excitons: first, conventional many-body theory such as the Green’s function formalism is constructed in the grand canonical ensemble whereas \( \alpha \)\( \times \)size excitons dissociate through a Mott transition when their number reaches \( L^3/\alpha \), which is the maximum number a sample volume \( L^3 \) can accommodate. Secondly, conventional many-body theory presumes some kind of Hamiltonian which normally consists of a part for the particle kinetic energy and a part for interaction between particles. But, attempts to construct energy-like effective scatterings between cobosons through a “bosonization procedure” fail, by nature, to allow exchanges between the particle fermionic components because their fermions must be frozen into a fixed configuration: the problem comes from the fact that fermion exchanges are dimensionless; so, they cannot lead to energy-like scatterings in order to possibly appear in the Hamiltonian. These two reasons led us to seek for a new many-body formalism in which the number of cobosons is fixed.

A zero temperature formalism for composite quantum particles which allows handling fermion exchanges induced by the Pauli exclusion principle in an exact way
was proposed by Combescot et al. We then extended this coboson formalism to finite temperature, paving the way to solving a large variety of coboson many-body effects. The goal of this work is to derive the partition function in the canonical ensemble based on this finite temperature formalism. Through it, all statistical thermodynamic properties, including the critical temperature for quantum condensation, should be possible to obtain.

To start, we reconsider the partition function of non-interacting elementary bosons. The one commonly known is in the grand canonical ensemble. From it, we can mathematically extract the partition function in the canonical ensemble; in practice, however, its numerical implementation is quite tricky. Here, we instead propose a direct derivation of this canonical partition function based on a recursion relation. Through this recursion relation we are directly led to the well-known compact form for the canonical partition function of non-interacting elementary bosons given in Eq. (10). Its diagrammatic representation has the great advantage to allow easy identification of the fully uncondensed, partially condensed and fully condensed contributions.

To show the power of our diagrammatic approach, next we consider interacting elementary bosons. We show how to perform a many-body expansion of the canonical partition function through a recursion relation similar to the one used for non-interacting bosons. Interestingly, we find that the partition function for interacting elementary bosons maintains the same recursion relation—and the same compact form—as for ideal elementary bosons provided that we add interactions in each n-particle entangled configuration. While this is reminiscent of cluster expansion for quantum systems, here we do not need to assume the property that the partition functions can be divided into groups of “connected” particles. They automatically show up.

We then turn to the canonical partition function of N cobosons made of two fermions, like the excitons. After recalling the key commutators of the coboson many-body formalism, we first calculate the recursion relation of this partition function at first order in fermion exchange in the absence of interaction scatterings between these cobosons. Although this can be done through a brute-force use of commutators, we have here chosen to present a physically intuitive way in getting this partition function through the extension of the diagrammatic approach we used for non-interacting elementary bosons. Surprisingly, we find that the coboson partition function can be cast in the same compact form as for non-interacting elementary bosons provided that we take into account the possibility that cobosons exchange their fermionic components due to the indistinguishability in each n-particle entangled configuration. Since fermion exchange does not lead to a normal particle-particle potential, this canonical partition function is fundamentally different from the one of non-interacting elementary bosons. Our result evidences that cobosons do not all condense into the same state, as non-interacting elementary bosons do in a BEC condensate. The similar structure of the elementary boson and coboson partition functions may help us build possible links between condensate wave functions and critical parameters for the BEC’s of elementary bosons and excitons. Moreover, the statistical entropy derived from the partition function enables us to study the relation between quantum entanglement in quantum information language and the composite particle bosonic nature.

The present paper is organized as follows: In Sec. II, we briefly introduce the compact form for the canonical partition function of non-interacting elementary bosons. Next we present the diagrammatic approach to derive the recursion relation between canonical partition functions. Then we extend this diagrammatic approach to interacting elementary bosons. In Sec. III we first briefly discuss complexities intrinsic in the coboson systems. We then introduce the interaction expansion which allows us to split the coboson partition function into a non-interacting part and an interacting part. Finally, we use a diagrammatic approach to calculate the partition function at zeroth order and also at first order in interaction scattering with Pauli exchange treated at first order. Consequences and significances of our results are discussed in the end.

II. ELEMENTARY BOSONS

A. Ideal(non-interacting) Bose gas

We consider a gas of non-interacting elementary bosons with kinetic energy \( e_k = \frac{\hbar^2 k^2}{2m} \). Since these bosons do not interact, the energy of each \( k \) state occupied by \( N_k \) bosons simply is \( N_k e_k \); so, the partition function for this ideal Bose gas in the canonical ensemble reads, for \( \beta = 1/k_B T \), as

\[
\tilde{Z}_N^{(0)} = \sum_{\{N_k\}_N} e^{-\beta \sum_k N_k e_k},
\]

the sum being taken over all possible boson numbers subject to \( \sum_k N_k = N \).

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Then, taking into account interaction between the fermionic components of the cobosons becomes rather straightforward due to similarities between interacting elementary bosons and interacting cobosons, differences coming from additional Pauli exchange processes.

The key result of this work is the recursion relation given in Eq. (12) for the canonical partition functions of N cobosons. This recursion relation leads to the partition function in the same compact form as the one of non-interacting elementary bosons. Our result evidences that cobosons do not all condense into the same state, as non-interacting elementary bosons do in a BEC condensate. The similar structure of the elementary boson and coboson partition functions may help us build possible links between condensate wave functions and critical parameters for the BEC’s of elementary bosons and excitons. Moreover, the statistical entropy derived from the partition function enables us to study the relation between quantum entanglement in quantum information language and the composite particle bosonic nature.

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1. Canonical partition function starting from grand canonical ensemble

To lift the constraint in the sum of Eq. 11, one commonly turns to the grand partition function with $\mu$ fixed instead of $N$, defined as

$$\bar{Z}^{(GC)} = \sum_{N=0}^{\infty} e^{\beta \mu N} \bar{Z}^{(0)}_N.$$  \hspace{1cm} (2)

A compact form for $\bar{Z}^{(GC)}$ is easy to obtain by noting that it also reads

$$\bar{Z}^{(GC)} = \sum_{N=0}^{\infty} \prod_{N_k=0}^{\infty} e^{-\beta N_k (\varepsilon_k - \mu)} = \prod_k \frac{1}{1 - e^{-\beta (\varepsilon_k - \mu)}}.$$  \hspace{1cm} (3)

The chemical potential $\mu$ is ultimately adjusted for the mean value of the particle number in the grand canonical ensemble to equal the number of bosons at hand.

Equation (2) shows that the partition function in the canonical ensemble, $\bar{Z}^{(0)}_N$, is just the prefactor of $e^{\beta \mu N}$ in $\bar{Z}^{(GC)}$. This prefactor can be obtained from the $N$th derivative of $\bar{Z}^{(GC)}$ with respect to $e^{\beta \mu}$. It has been shown that this yields a compact form to the canonical partition function which reads as $^{30,31}$

$$\bar{Z}^{(0)}_N = \sum_{\{p_k\}} \frac{1}{p_1! (\frac{z(\beta)}{1})^{p_1} \cdots (\frac{z(N\beta)}{N})^{p_N}}.$$  \hspace{1cm} (4)

The $p_k$'s are a set of non-negative integers such that

$$N = p_1 + 2p_2 + \cdots + Np_N,$$  \hspace{1cm} (5)

while $z(n\beta)$ is defined as

$$z(n\beta) = \sum_k e^{-n\beta \varepsilon_k}.$$  \hspace{1cm} (6)

2. Direct approach to the canonical partition function

The above derivation of the canonical partition function, based on derivatives of the partition function in the grand canonical ensemble, is smart but completely formal. It moreover presupposes the knowledge of the partition function in the grand canonical ensemble. We here present a direct derivation of the canonical partition function for a boson number $N$. This derivation is not only useful for possible extension to cobosons, but, through its diagrammatic representation, it provides a physical understanding of the various terms as coming from the fully uncondensed, partially condensed and fully condensed configurations.

Let $|\tilde{\psi}(N_k)_N\rangle$ be normalized $N$-particle eigenstate of the system Hamiltonian $\tilde{H}_0$ with $N_k$ bosons having an energy $\varepsilon_k$. The canonical partition function given in Eq. 11 can be rewritten as

$$\bar{Z}^{(0)}_N = \sum_{\{N_k\}_N} \langle \tilde{\psi}(N_k)_N | e^{-\beta \tilde{H}_0} | \tilde{\psi}(N_k)_N \rangle.$$  \hspace{1cm} (7)

We can circumvent the difficulty coming from the restriction, $\sum_k N_k = N$, in the sum over all possible configurations $\{N_k\}_N$ by using the closure relation in the $N$-elementary boson subspace written in terms of single boson operators $\tilde{B}_k$. These operators are such that $(\tilde{H}_0 - \varepsilon_k)\tilde{B}_k^\dagger |v\rangle = 0$ where $|v\rangle$ denotes the vacuum state, with a commutation relation given by

$$[\tilde{B}_k, \tilde{B}_k^\dagger] = \delta_{kk'}.$$  \hspace{1cm} (8)

This closure relation reads as

$$\tilde{I}_N = \frac{1}{N!} \sum_{\{k\}} \tilde{B}_{k_1}^\dagger \cdots \tilde{B}_{k_N}^\dagger |v\rangle \langle \tilde{B}_{k_N} \cdots \tilde{B}_{k_2} \tilde{B}_{k_1}|v\rangle,$$  \hspace{1cm} (9)

as can be checked from $\tilde{I}_2^2 = \tilde{I}_2$ and to generalize to $\tilde{I}_N$. Since the $|\tilde{\psi}(N_k)_N\rangle$'s are eigenstates of $\tilde{H}_0$, a closure relation also exists for normalized $|\tilde{\psi}(N_k)_N\rangle$'s, reading as

$$\tilde{I}_N = \sum_{\{N_k\}_N} |\tilde{\psi}(N_k)_N\rangle \langle \tilde{\psi}(N_k)_N|.$$  \hspace{1cm} (10)

By injecting Eq. 9 in front of $|\tilde{\psi}(N_k)_N\rangle$ in Eq. 11 and by getting rid of the $|\tilde{\psi}(N_k)_N\rangle$ states through Eq. 10, we can rewrite $\bar{Z}^{(0)}_N$ as

$$\bar{Z}^{(0)}_N = \frac{1}{N!} \sum_{\{k\}} \langle v | \tilde{B}_{k_1}^\dagger \cdots \tilde{B}_{k_N} e^{-\beta \tilde{H}_0} \tilde{B}_{k_N}^\dagger \cdots \tilde{B}_{k_2}^\dagger \tilde{B}_{k_1} |v\rangle.$$  \hspace{1cm} (11)

The Hamiltonian $\tilde{H}_0$ for non-interacting elementary bosons reads as $\tilde{H}_0 = \sum_k \varepsilon_k \tilde{B}_k^\dagger \tilde{B}_k$; so, the above canonical partition function readily reduces to

$$\bar{Z}^{(0)}_N = \frac{1}{N!} \sum_{\{k\}} e^{-\beta (\varepsilon_{k_1} + \cdots + \varepsilon_{k_N})} \langle v | \tilde{B}_{k_1}^\dagger \cdots \tilde{B}_{k_N} \tilde{B}_{k_N}^\dagger \cdots \tilde{B}_{k_2}^\dagger \tilde{B}_{k_1}^\dagger |v\rangle.$$  \hspace{1cm} (12)

Note that (i) the $k$'s in the sum now take all possible values without restriction. (ii) a given $\{N_k\}_N$ configuration appears once only in Eq. 11, while it appears many times in Eq. 12, which explains the presence of the $1/N!$ prefactor.
3. Recursion relation for $\bar{Z}_N^{(0)}$

The scalar product in the above equation can be calculated using the commutator $[,]$. It allows us to replace $\bar{B}_{k_N}^{1\dagger} \bar{B}_{k_N}$ by $\delta_{k_N,k_N} + \bar{B}_{k_N}^{1\dagger} \bar{B}_{k_N}$. The $\delta_{k_N,k_N}$ term, when inserted into Eq. (12), readily gives

$$\frac{1}{N!} z^2(\beta) \left[ (N-1)! \bar{Z}_{N-1}^{(0)} \right].$$

(13)

To evaluate the $\bar{B}_{k_N}^{1\dagger} \bar{B}_{k_N}$ term, we push the operator $\bar{B}_{k_N}$ to the right according to the commutator $[,]$. This yields $\delta_{k_N,k_N-1} \langle v | \bar{B}_{k_1} \cdots \bar{B}_{k_N-1} \bar{B}_{k_N}^{1\dagger} \bar{B}_{k_N-1} \cdots \bar{B}_{k_1} | v \rangle$ which simply reads as

$$\delta_{k_N,k_N-1} \langle v | \bar{B}_{k_1} \cdots \bar{B}_{k_N-1} \bar{B}_{k_N}^{1\dagger} \bar{B}_{k_N-1} \cdots \bar{B}_{k_1} | v \rangle$$

(14)

which are equivalent when inserted into Eq. (12) through a relabeling of the dummy indices $k_i$'s. Repeating the same procedure as above, we replace $\bar{B}_{k_N-1}^{1\dagger} \bar{B}_{k_N}$ by $\delta_{k_N-1,k_N} + \bar{B}_{k_N}^{1\dagger} \bar{B}_{k_N-1}$. The term in $\delta_{k_N-1,k_N}$, when inserted into Eq. (12), readily gives

$$\frac{1}{N!} (N-1) z(2\beta) \left[ (N-2)! \bar{Z}_{N-2}^{(0)} \right].$$

(15)

The term in $\bar{B}_{k_N}^{1\dagger} \bar{B}_{k_N-1}$, calculated by pushing $\bar{B}_{k_N-1}$ to the right, yields $(N-2)$ equivalent terms; and so on...

So, we end with a nicely compact recursion relation which simply reads as

$$\bar{Z}_N^{(0)} = \frac{1}{N} \left[ z(\beta) \bar{Z}_{N-1}^{(0)} + z(2\beta) \bar{Z}_{N-2}^{(0)} + \cdots + z(N\beta) \right]$$

$$= \frac{1}{N} \sum_{p=1}^{N} z(p\beta) \bar{Z}_{N-p}^{(0)},$$

(16)

with $\bar{Z}_0^{(0)}$ taken as 1. Using this recursion relation, it is easy to recover the expression of the canonical partition function obtained from the grand canonical ensemble $\bar{Z}_N$, as given in Eq. (4). As illustration, we give the lowest few $\bar{Z}_N^{(0)}$'s in Appendix 1.

4. Diagrammatic procedure

It is possible to recover the recursion relation (16) between the canonical partition functions using diagrams. The diagram of Fig. 2 represents the scalar product of $N$ elementary bosons $(k_1, \cdots, k_N)$. We can set up reduction rules to relate this scalar product to those of lower number of bosons. As depicted in Fig. 2 this is done by connecting $k_N$ on the left to one of the $k_i$'s on the right; this $k_i$ can be either $k_N$ as in Fig. 2(a) (leaving behind a scalar product of $N-1$ bosons) or any other $k_i$'s like $k_{N-1}$ as in Fig. 2(b), which leads to $(N-1)$ similar terms once summation over dummy $k$ indices is performed. In the diagram of Fig. 2(b), we can connect $k_{N-1}$ on the left either to $k_N$ as in Fig. 2(c) (leaving behind a scalar product of $N-2$ bosons), or to any other $k_i$'s like $k_{N-2}$ as in Fig. 2(d), which leads to $(N-2)$ similar terms once summation over dummy $k$ is performed; and so on...

We then readily find that the process of Fig. 2(a) gives a contribution equal to $(1/N!) z(\beta)(N-1)! \bar{Z}_{N-1}^{(0)}$. The process of Fig. 2(c), which imposes a relabeling of the dummy indices $k_i$'s, being the partition function for a condensate made of $p$ elementary bosons, all in the same state.

We are going to show that the partition function for $N$ cobosons obeys a similar recursion relation, provided that we take into account fermion exchanges and interaction scatterings between the composite particles entangled in a condensate. However, before turning to cobosons, let us go one step further by considering interacting elementary bosons. We are going to show that a recursion relation exists provided that we replace $z(n\beta)$ for a non-interacting $n$-boson condensate by a modified $\tilde{Z}(n\beta)$ which contains interaction between bosons.

B. Interacting Bose gas

We now consider interacting elementary bosons. Their Hamiltonian reads

$$\tilde{H} = \tilde{H}_0 + \tilde{V}$$

$$= \sum_k \epsilon_k \bar{B}_k^{1\dagger} \bar{B}_k + \frac{1}{2} \sum_{kk'q} V_q \bar{B}_k^{1\dagger} \bar{B}_{k+q}^{1\dagger} \bar{B}_{k'}^{1\dagger} \bar{B}_k,$$

(17)

the operators $\bar{B}_k^{1\dagger}$ still obeying the commutation relation $[,]$. The canonical partition function reads in terms of the $N$-boson eigenstates of the system, $(\tilde{H} - \tilde{E}_{N,\xi}) |\psi_{N,\xi}\rangle = 0$, as

$$\tilde{Z}_N = \sum_{\xi} e^{-\beta \tilde{E}_{N,\xi}} = \sum_{\xi} \langle \tilde{\psi}_{N,\xi} | e^{-\beta \tilde{H}} | \psi_{N,\xi} \rangle.$$  

(18)
To get rid of these unknown eigenstates, we follow the same procedure as in Sec. II A 2 . We insert the closure relation (4) for N elementary bosons in front of $|\bar{\psi}_{N,\xi}\rangle$ in Eq. (13) and use the fact that $\bar{1}_N = \sum |\bar{\psi}_{N,\xi}\rangle\langle\bar{\psi}_{N,\xi}|$. The canonical partition function then reads as

$$Z_N = \frac{1}{N!} \sum_{\{k\}} \langle v | \bar{B}_{k_1} \cdots \bar{B}_{k_N} e^{-\beta \bar{H}} \bar{B}_{k_1}^\dagger \cdots \bar{B}_{k_N}^\dagger | v \rangle. \quad (19)$$

Next, we perform a many-body expansion of $Z_N$. We first rewrite $e^{-\beta \bar{H}}$ using the Cauchy integral formula as

$$e^{-\beta \bar{H}} = \int_{C} \frac{dz}{2\pi i} e^{-\beta z}, \quad (20)$$

where the integration path C is a circle of finite radius centered at the $\bar{H}$ value on the complex plane. (For simplicity, we omit this subscript C in the following.) The operator $1/(z - \bar{H})$ is expanded for $\bar{H} = \bar{H}_0 + \bar{V}$ through

$$\frac{1}{z - \bar{H}} = \frac{1}{z - \bar{H}_0} + \frac{1}{z - \bar{H}_0} \bar{V} \frac{1}{z - \bar{H}}. \quad (21)$$

This leads us to split the partition function as

$$Z_N = Z_N^{(0)} + Z_N^{(1)} + \cdots \quad (22)$$

The zeroth-order $Z_N^{(0)}$ in interaction reads as in Eq. (11) while the first order is given by

$$Z_N^{(1)} = \frac{1}{N!} \sum_{\{k\}} \int \frac{dz}{2\pi i} e^{-\beta z}$$

$$\times \langle v | \bar{B}_{k_1} \cdots \bar{B}_{k_N} \frac{1}{z - \bar{H}_0} \bar{V} \frac{1}{z - \bar{H}_0} \bar{B}_{k_N}^\dagger \cdots \bar{B}_{k_1}^\dagger | v \rangle. \quad (23)$$

$\bar{H}_0$ acting on $\bar{B}_{k_N}^\dagger \cdots \bar{B}_{k_1}^\dagger | v \rangle$ gives $\varepsilon_{k_1} + \cdots + \varepsilon_{k_N}$ while

$$\int \frac{dz}{2\pi i} (z - \bar{H}_0)^2 = -\beta e^{-\beta \varepsilon}; \quad (24)$$

so, $Z_N^{(1)}$ appears as

$$Z_N^{(1)} = -\beta \frac{1}{N!} \sum_{\{k\}} e^{-\beta (\varepsilon_{k_1} + \cdots + \varepsilon_{k_N})}$$

$$\times \langle v | \bar{B}_{k_1} \cdots \bar{B}_{k_N} \bar{V} \bar{B}_{k_N}^\dagger \cdots \bar{B}_{k_1}^\dagger | v \rangle. \quad (25)$$

A convenient way to calculate the above matrix element is to introduce commutators

$$[\bar{V}, \bar{B}_{p}^\dagger]_\varepsilon = \sum_{kq} V_q \bar{B}_{p+q}^\dagger \bar{B}_{k-q} = \bar{V}_{p}^\dagger, \quad (26)$$

$$[\bar{V}_{p} \bar{B}_{p}^\dagger, \bar{B}_{p'}^\dagger]_\varepsilon = \sum_{q} V_q \bar{B}_{p+q}^\dagger \bar{B}_{p'-q}. \quad (27)$$

By pushing $\bar{V}$ in Eq. (25) to the right using these commutators, we get $(N - 1) + \cdots + 1 = N(N - 1)/2$ terms

which contribute equally to $Z_N^{(1)}$ through a relabeling of the dummy indices $k_{\alpha}$'s. By symmetrizing the process, i.e., by also pushing $\bar{V}$ to the left, we end with the first-order term in interaction reading as

$$Z_N^{(1)} = -\beta \frac{1}{N!} C_{N}^{2} \sum_{\{k\}} e^{-\beta (\varepsilon_{k_1} + \cdots + \varepsilon_{k_N})} \sum_{q} V_q$$

$$\times \frac{1}{2} \langle v | \bar{B}_{k_1} \cdots \bar{B}_{k_N} \bar{B}_{k_{N+1}+q}^\dagger \bar{B}_{k_{N+1}-q}^\dagger \cdots \bar{B}_{k_1}^\dagger | v \rangle + c.c. \rangle. \quad (28)$$

This matrix element is shown in the diagram of Fig. 3(a).

For $N = 2$, we readily get, since $\bar{V} \bar{B}_{k_2}^\dagger \bar{B}_{k_1}^\dagger | v \rangle = \sum_{q} V_q \bar{B}_{k_2+q}^\dagger \bar{B}_{k_1-q}^\dagger | v \rangle$,

$$Z_2^{(1)} = -\frac{\beta}{2} \bar{V}(\bar{\beta}, \bar{\beta}) \quad (29)$$

with $\bar{V}(\bar{\beta}, \bar{\beta})$ defined through

$$\bar{V}(\bar{n}_1, \bar{n}_2 \beta) = \sum_{k_1, k_2} e^{-\beta (\bar{n}_1 \varepsilon_{k_1} + \bar{n}_2 \varepsilon_{k_2})} (V_{0} + V_{k_1-k_2}). \quad (30)$$

$\bar{V}(\bar{\beta}, \bar{\beta})$ corresponds to the two processes shown in Fig. 3(b), indicated by two columns of $k$ vectors ($k_1, k_2$) and ($k_2, k_1$) separated by a dashed line on the left of the diagram. To understand how the result develops for large $N$, we have explicitly derived $Z_N^{(1)}$ for $N = 4$ in Appendix. For arbitrary $N$, we isolate $Z_{N-2}^{(0)} Z_{N-3}^{(0)} \cdots$ from the diagram of Fig. 3(a) in the same way as for ideal elementary bosons. The prefactor of $Z_{N-2}^{(0)}$ is made of the processes involving $(k_N, k_{N-1})$ shown in Fig. 3(a). Their contribution to $Z_N^{(1)}$ reads as

$$-\beta \frac{1}{N!} C_{N}^{2} \bar{V}(\bar{\beta}, \bar{\beta}) [(N - 2)! Z_{N-2}^{(0)}] = -\frac{\beta}{2} \bar{V}(\bar{\beta}, \bar{\beta}) Z_{N-2}^{(0)}. \quad (31)$$

The prefactor of $Z_{N-3}^{(0)}$ is made of processes involving $(k_N, k_{N-1})$ and one of the $(k_1, \cdots, k_{N-2})$, let say $k_{N-2}$. As shown in Fig. 3(b) there are four such entangled processes indicated by the four columns of $k$ vectors separated by dashed lines on the left of the diagram. In Fig. 3(b), $k_{N-2}$ "condenses" either with $k_{N-1}$ or with $k_N$. Since there are $C_{N-2}^{2}$ ways to choose this $k_{N-2}$ boson among ($k_1, \cdots, k_{N-2}$), the contribution of such processes to $Z_N^{(1)}$ reads as

$$-\frac{1}{N!} C_{N}^{2} C_{N-2} \left(\bar{V}(\bar{\beta}, \bar{2\beta}) + \bar{V}(\bar{2\beta}, \bar{\beta})\right) [(N - 3)! Z_{N-3}^{(0)}]$$

$$= -\frac{\beta}{2} \bar{V}(\bar{\beta}, \bar{2\beta}) + \bar{V}(\bar{2\beta}, \bar{\beta}) Z_{N-3}^{(0)}. \quad (32)$$
To get the prefactor of $\tilde{Z}_{N-4}^{(0)}$, we isolate two $k$'s out of $(k_1, \ldots, k_{N-2})$, let say $(k_{N-2}, k_{N-3})$. Since there are $C_N^{N-2}$ ways to choose these two $k$’s, the contribution to $\tilde{Z}_{N}^{(1)}$ of the entangled processes between $(k_N, k_{N-1}, k_{N-2}, k_{N-3})$ reads as

$$-rac{1}{N!} C_2^{N-2} \left(2\bar{V}(\beta, 3\beta) + 2\bar{V}(2\beta, 2\beta) + 2\bar{V}(3\beta, 1\beta)\right) \times \left(\frac{1}{N-4}!\tilde{Z}_{N-4}^{(0)}\right)$$

$$= -\frac{\beta}{2} \left(\bar{V}(\beta, 3\beta) + \bar{V}(2\beta, 2\beta) + \bar{V}(3\beta, 1\beta)\right) \tilde{Z}_{N-4}^{(0)}. \tag{33}$$

The three terms in the parentheses originate from the 12 processes shown in Fig. 4(c). They correspond to all possible permutations of $(k_N, k_{N-1}, k_{N-2}, k_{N-3})$ on the left which make the same four $k$’s on the right entangled, i.e., the $(k_{N-2}, k_{N-3})$ must not “condense” with themselves; and so on...

So, we finally get

$$\tilde{Z}_{N}^{(1)} = -\frac{\beta}{2} \sum_{n=1}^{N} \hat{V}(n\beta) \tilde{Z}_{N-n}^{(0)} \tag{34}$$

with

$$\hat{V}(n\beta) = \sum_{p=1}^{n-1} \bar{V}(p\beta, (n-p)\beta). \tag{35}$$

By using the recursion relation between the $\tilde{Z}_{N}^{(0)}$’s given in Eq. (16), we get the partition function of $N$ interacting elementary bosons at first order in interaction as

$$\tilde{Z}_{N} \simeq \frac{1}{N} \sum_{n=1}^{N} \left[z(n\beta) - \frac{\beta n}{2} \hat{V}(n\beta)\right] \tilde{Z}_{N-n}^{(0)}. \tag{36}$$

Note that the second term in the brackets depends on density $N/L^3$ since the $V_0$ scattering depends on sample volume as $1/L^3$, which is physically reasonable for many-body effects.

It actually is possible to write $\tilde{Z}_{N}$ in a compact form like Eq. (11). For that, we must transform Eq. (36) into a recursion relation between the $\tilde{Z}_{N}$’s similar to Eq. (16). To do it, we rewrite $\tilde{Z}_{N-n}^{(0)}$ on the right-hand side of Eq. (36) in terms of $\tilde{Z}_{N-n}$ using Eq. (22). Equation (36) then becomes

$$\tilde{Z}_{N} \simeq \frac{1}{N} \sum_{n=1}^{N} z(n\beta) \left[\tilde{Z}_{N-n} - \tilde{Z}_{N-n}^{(1)}\right] + \tilde{Z}_{N}^{(1)}. \tag{37}$$

Next, we note that, due to Eq. (34),

$$-\frac{1}{N} \sum_{n=1}^{N} z(n\beta) \tilde{Z}_{N-n}^{(1)} = \frac{1}{N} \beta \sum_{n=1}^{N} z(n\beta) \sum_{m=1}^{N-m} \hat{V}(m\beta) \tilde{Z}_{N-n-m}^{(0)} \tag{38}$$

As the right-hand side also reads

$$\frac{1}{N} \beta \sum_{n=1}^{N} \hat{V}(n\beta) (N-n) \tilde{Z}_{N-n}^{(0)}$$

$$= -\tilde{Z}_{N}^{(1)} - \frac{\beta}{N} \sum_{n=1}^{N} n \hat{V}(n\beta) \tilde{Z}_{N-n}^{(0)}, \tag{39}$$

we end with $\tilde{Z}_{N}$ correct up to first order in interaction reading as

$$\tilde{Z}_{N} \simeq \frac{1}{N} \sum_{n=1}^{N} z(n\beta) \tilde{Z}_{N-n} \tag{40}$$

with, for $n \geq 2$,

$$\hat{z}(n\beta) = z(n\beta) - \frac{\beta n}{2} \hat{V}(n\beta). \tag{41}$$

It is then straightforward to transform Eq. (40) into a compact form like Eq. (11) with $z(n\beta)$ replaced by $\hat{z}(n\beta)$.

We have demonstrated that, up to first order in interaction, the canonical partition function for interacting elementary bosons takes the same compact form as for non-interacting elementary bosons provided that we replace $z(n\beta)$ by $\hat{z}(n\beta)$ of Eq. (41). For the perturbative regime to be valid, $N\beta V_0$ must be smaller than 1. Since $V_0$ scales as $1/L^3$, this imposes $NV/L^3 \ll k_BT$. Higher orders in interaction are obtained in the same way using Eq. (21). We then rewrite $\tilde{Z}_{N-n}$’s in terms of $\tilde{Z}_{N-n}$’s to obtain a recursion relation similar to Eq. (41).

III. COMPOSITE BOSONS

A. Intrinsic difficulties with cobosons

We now consider cobosons made of two fermions like the excitons. Some difficulties immediately arise when compared to the ideal Bose gas we previously considered.

\begin{itemize}
  \item \end{itemize}
Interaction in the absence of fermion exchange follows dashed-lines to represent the Pauli scattering $\Lambda^{(n\ j)}$ to the same exchange processes. For simplicity, in the $\alpha\Lambda$ states ($D_{\alpha}$ where $a_{\alpha}$ brings an interaction between cobosons.

In addition to this interaction, cobosons also feel each other through the Pauli exclusion principle between their fermionic components. This “Pauli interaction” in fact dominates most coboson many-body effects. As a result, it is impossible to avoid considering interaction between bosons once we have decided to take into account their composite nature. To properly handle many-body effects between cobosons with creation operators $B^\dagger$, we adopt the commutation formalism introduced in Ref. 23:

\begin{equation}
B_i^\dagger = \sum_{k\alpha,\beta} a^\dagger_{k\alpha} b^\dagger_{k\beta} \langle k\beta, k\alpha | i \rangle,
\end{equation}

where $a^\dagger_{k\alpha}$ and $b^\dagger_{k\beta}$ are creation operators of their fermionic components, we adopt the commutation formalism introduced in Ref. 23:

(i) Fermion exchanges in the absence of fermion-fermion interaction follow from

\begin{align}
[H, B]^\dagger_i &= E_i B_i^\dagger + V_i^\dagger, \\
[D_{mi}, B]^\dagger_j &= \sum_n \Lambda^{(n\ j)} B_n^\dagger,
\end{align}

\begin{equation}
D_{mi} \text{ being such that } D_{mi}|v\rangle = 0. \text{ The Pauli scattering } \Lambda^{(n\ j)} \text{ associated with fermion exchange is shown in Fig. 5(a). It corresponds to an exchange of fermion } \alpha \text{ or } \beta \text{ between cobosons in states } (i, j), \text{ which then end in states } (m, n). \text{ Note that } \Lambda^{(n\ j)} \text{ and } \Lambda^{(m\ i)} \text{ correspond to the same exchange processes. For simplicity, in the following, we shall use the first diagram with crossing dashed-lines to represent the Pauli scattering } \Lambda^{(n\ j)}. \end{equation}

(ii) Interaction in the absence of fermion exchange follows

\begin{equation}
\eta = N \left(\frac{a_X}{L}\right)^3,
\end{equation}

this parameter appearing as $\eta^{n-1}$ in processes in which $n$ excitons are involved.

\section{Formal expression of the canonical partition function for cobosons}

The canonical partition function of $N$ cobosons is defined in terms of $N$-pair eigenstate energies, $(H - \xi_{N,\xi})|\psi_{N,\xi}\rangle = 0$, as

\begin{equation}
Z_N = \sum_{\xi} e^{-\beta \xi_{N,\xi}} = \sum_{\xi} \langle \psi_{N,\xi}|e^{-\beta H}|\psi_{N,\xi}\rangle.
\end{equation}

We can get rid of these unknown eigenstates by inserting the closure relation for $N$ cobosons made of two fermions. Instead of Eq. (9), this closure relation has been shown to read as

\begin{equation}
I_N = \left(\frac{1}{N!}\right)^2 \sum_{\{i\}} B_{i_1}^\dagger B_{i_2}^\dagger \cdots B_{i_N}^\dagger |v\rangle \langle v| B_{i_N} \cdots B_{i_2} B_{i_1}.
\end{equation}

The fact that these cobosons are made of two fermions appears through the prefactor $(1/N!)^2$ instead of $1/N!$.

By inserting Eq. (19) in front of $|\psi_{N,\xi}\rangle$ in Eq. (18) and by using the closure relation $I_N = \sum_{\xi} \langle \psi_{N,\xi}|\psi_{N,\xi}\rangle$ for the $N$-pair eigenstates, we can rewrite Eq. (18) as

\begin{equation}
Z_N = \left(\frac{1}{N!}\right)^2 \sum_{\{i\}} \langle v| B_{i_1} \cdots B_{i_N} e^{-\beta H} B_{i_N}^\dagger \cdots B_{i_2}^\dagger |v\rangle.
\end{equation}

We wish to stress that difference with the canonical partition function for elementary bosons given in Eq. (13) is not so much the prefactor change from $1/N!$ to $(1/N!)^2$ as the fact that the coboson operators $B_{i}^\dagger$’s now commute in a different way from the elementary boson operators.
In addition, since these cobosons interact, the Hamiltonian $H$ in $e^{-\beta H}$ cannot be simply replaced by the sum of individual boson energies as in Eq. 12.

To calculate the scalar product of Eq. [51], we use the commutators for coboson operators given in Eqs. [49]-[48]. As for interacting elementary bosons, we first use the Cauchy integral formula [20] to rewrite $e^{-\beta H}$ in order to possibly perform an interaction expansion. This interaction expansion follows from

$$\frac{1}{z - H} B_i^\dagger = B_i^\dagger \frac{1}{z - H - E_i} + \frac{1}{z - H} V_i^\dagger \frac{1}{z - H - E_i},$$

as easy to check using Eq. [45]. So,

$$e^{-\beta H} B_i^\dagger = B_i^\dagger e^{-\beta(H+E_i)} + \int \frac{dz}{2\pi i} \frac{e^{-\beta z}}{z - H - E_i} V_i^\dagger \frac{1}{z - H - E_i}.$$  

By symmetrizing the expansion procedure, as necessary since we are going to truncate the interaction expansion, as usual in many-body problems, we are led to split $Z_N$ as

$$Z_N = \sum_{i_N} e^{-\beta E_{i_N}} [\Gamma_N(i_N) + I_N(i_N)].$$  

The $I_N(i_N)$ part, which comes from the second term of Eq. [62], is given by

$$I_N(i_N) = \frac{1}{2} \left( \frac{1}{N} \right)^2 \sum_{i_1 \cdots i_{N-1}} \int \frac{dz}{2\pi i} e^{-\beta(z - E_{i_N})} \times \left[ \langle v | B_{i_1} \cdots B_{i_N} \frac{1}{z - H} V_{i_N}^\dagger \frac{1}{z - H - E_{i_N}} B_{i_{N-1}}^\dagger \cdots B_{i_1}^\dagger | v \rangle + c.c. \right]$$

To obtain $I_N(i_N)$ at first order in $V_{i_1}^\dagger$, we can push the operator $1/(z - H - E_{i_{i_N}})$ to the right by only keeping the first term in Eq. [32]. This leads to replacing $H_{i_N}$ on the right of the above matrix element by $E_{i_{N-1}} + \cdots + E_{i_{i_1}}$ and $H$ on the left by $E_{i_{i_N}} + \cdots + E_{i_1}$. Since

$$\int \frac{dz}{2\pi i} \frac{e^{-\beta z}}{(z - E_{i_1} - \cdots - E_{i_{i_N}})^2} = -\beta e^{-\beta(E_{i_1} + \cdots + E_{i_{i_N}})},$$

$I_N(i_N)$ appears as

$$I_N(i_N) \approx -\beta \left( \frac{1}{N} \right)^2 \sum_{i_1 \cdots i_{N-1}} e^{-\beta(E_{i_1} + \cdots + E_{i_{i_N-1}})} \times \frac{1}{2} \left[ \langle v | B_{i_1} \cdots B_{i_N} V_{i_N}^\dagger B_{i_{N-2}}^\dagger \cdots B_{i_1}^\dagger | v \rangle + c.c. \right].$$

To go further, we use Eq. [40] to push $V_{i_1}^\dagger$ to the right. By noting that $V_{i_N}^\dagger B_{i_{N-1}}^\dagger \cdots B_{i_1}^\dagger | v \rangle$ gives $(N - 1)$ terms like

$$\sum_{m} \xi \left( \frac{n}{m} \frac{i_N}{i_{i_N}} \right) B_m^\dagger B_m^\dagger B_m^\dagger B_{i_{N-2}}^\dagger \cdots B_{i_1}^\dagger | v \rangle$$

which give equal contribution to $I_N(i_N)$ when relabeling the dummy indices $i_n$’s, we end with

$$I_N(i_N) \approx -\beta \left( \frac{1}{N} \right)^2 \sum_{i_1 \cdots i_{N-1}} e^{-\beta(E_{i_1} + \cdots + E_{i_{i_N-1}})} \times \left[ \langle v | B_{i_1} \cdots B_{i_N} B_{m_{i_1}} B_{m_{i_{N-2}}} \cdots B_{m_{i_1}} | v \rangle \xi \left( \frac{n}{m} \frac{i_N}{i_{i_N}} \right) + c.c. \right].$$

This term physically corresponds to the diagram of Fig. 6 in which two out of the $N$ cobosons interact before possibly exchanging their fermions with the other cobosons. We now consider the $I_N(i_N)$ term of $Z_N$ which comes from the first term of Eq. [62]. It reads

$$I_N(i_N) = \frac{1}{2} \left( \frac{1}{N} \right)^2 \sum_{i_1 \cdots i_{N-1}} \langle v | B_{i_1} \cdots B_{i_N} B_{i_{N-1}}^\dagger \cdots B_{i_1}^\dagger | v \rangle.$$

The same equation [52] leads us to split $I_N(i_N)$ as

$$\langle v | B_{i_1} \cdots B_{i_N} B_{i_{N-1}}^\dagger \cdots B_{i_1}^\dagger | v \rangle \approx -\beta e^{-\beta(E_{i_1} + \cdots + E_{i_{i_N-1}})}.$$  

which is similar to the scalar product appearing in Eq. [54] except that we now have $B_{i_N}$ on the left. Its lowest order in $V_i^\dagger$ is obtained by replacing the right H operator by $E_{i_{i_N-2}} + \cdots + E_{i_1}$ and the left H operator by $-E_{i_N} + E_{i_{i_N-2}} + \cdots + E_{i_1}$. Integration over $z$ in Eq. [52] again gives $-\beta e^{-\beta(E_{i_1} + \cdots + E_{i_{i_N-1}})}$. So, by symmetrizing the above process, we get

$$I_N(i_N, i_{i_N-1}) \approx -\beta \left( \frac{1}{N} \right)^2 \sum_{i_1 \cdots i_{N-2}} e^{-\beta(E_{i_1} + \cdots + E_{i_{i_N-2}})} \times \frac{1}{2} \left[ \langle v | B_{i_1} \cdots B_{i_N} V_{i_{i_N-1}}^\dagger B_{i_{i_N-2}}^\dagger \cdots B_{i_1}^\dagger | v \rangle + c.c. \right].$$

To go further, we again use Eq. [40] $V_{i_{N-1}}^\dagger B_{i_{N-2}}^\dagger \cdots B_{i_1}^\dagger | v \rangle$ then leads to $(N - 2)$ terms similar to

$$\sum_{m} \xi \left( \frac{n}{m} \frac{i_{i_N-1}}{i_{i_{i_N-1}}} \right) B_m^\dagger B_m^\dagger B_m^\dagger B_{i_{N-3}}^\dagger \cdots B_{i_1}^\dagger | v \rangle.$$
which ultimately gives \( I_N(i_N, i_{N-1}) \) as
\[
I_N(i_N, i_{N-1}) \simeq \frac{\beta(N-2)}{2 (N!)^2} \sum_{i_1, \ldots, i_{N-2}} e^{-\beta(E_{i_1} + \cdots + E_{i_{N-2}})} \times \left[ \sum_{m_n} \langle v|B_{i_1} \cdots B_{i_N} B_{i_{N-1}}^\dagger B_{i_{N-2}}^\dagger \cdots B_{i_1}^\dagger |v \rangle \xi \left( \frac{n}{m_{i_{N-1}}} \right) + \text{c.c.} \right].
\]  

(64)

To calculate \( \Gamma_N(i_N, i_{N-1}) \), we proceed in the same way, namely we push \( e^{-\beta H} \) in the scalar product to the right using Eq. (52), and so on... After summing over \( i_N \) and \( i_{N-1} \), the \( I_N(i_N, i_{N-1}) \) and \( I_N(i_N, i_{N-1}) \) terms actually give equal contribution through a relabeling of the \( i \)'s. So, by considering all equivalent terms, namely \( (N-1) + (N-2) + \cdots + 1 = N(N-1)/2 \), we end with
\[
Z_N \simeq Z_N^{(0)} + Z_N^{(1)} = \frac{1}{N!} \left[ \tilde{Z}_N^{(0)} + \tilde{Z}_N^{(1)} \right],
\]  

(65)

where the zeroth-order term in interaction scattering is
\[
\tilde{Z}_N^{(0)} = \frac{1}{N!} \sum_{\{v\}} e^{-\beta(E_{i_1} + \cdots + E_{i_N})} \langle v|B_{i_1} \cdots B_{i_N} B_{i_{N-1}}^\dagger B_{i_{N-2}}^\dagger \cdots B_{i_1}^\dagger |v \rangle,
\]

(66)

while the first-order term in \( \xi \) reads as
\[
\tilde{Z}_N^{(1)} = -\frac{\beta}{2} \frac{1}{N!} C^2_N \sum_{\{v\}} e^{-\beta(E_{i_1} + \cdots + E_{i_N})} \times \left[ \sum_{m_n} \langle v|B_{i_1} \cdots B_{i_N} B_{i_{N-1}}^\dagger B_{i_{N-2}}^\dagger \cdots B_{i_1}^\dagger |v \rangle \xi \left( \frac{n}{m_{i_{N-1}}} \right) + \text{c.c.} \right].
\]  

(67)

Note that in Eq. (65), we have turned from \( Z_N \) to \( \tilde{Z}_N \) in order to better see the consequences of the boson composite nature, \( \tilde{Z}_N^{(0)} \) in Eq. (66) and \( \tilde{Z}_N^{(0)} \) in Eq. (12) then being formally identical: their unique but major difference lies in the commuting relations these \( B_{i_k}^\dagger \) and \( B_{i_k}^\dagger \) operators have.

The canonical partition function \( Z_N \) in Eq. (45) appears as an expansion in interaction scattering \( \xi \). In the case of electrons and holes bound into excitons through Coulomb processes, \( \xi \) scales as the exciton Rydberg \( R_X \) multiplied by the exciton volume \( a_X^3 \) and divided by the sample volume \( L^3 \). So, for \( N \) excitons, \( \tilde{Z}_N^{(1)}/\tilde{Z}_N^{(0)} \) scales as \( N \beta \xi \approx \beta R_X \eta \) where \( \eta \) is the dimensionless many-body parameter defined in Eq. (47). The many-body interaction expansion we perform is thus valid for \( \beta R_X \eta \ll 1 \), i.e., \( \eta \ll k_B T/R_X \). This ratio is small compared to 1 if the lowest relative motion exciton state only is populated. Note that \( \eta \) actively controls the exciton physics, because, for \( \eta > 1 \), excitons dissociate into an electron-hole plasma through a Mott transition.

C. Partition function at zeroth order in \( \xi \)

To grasp how the Pauli exclusion principle affects the canonical partition function of \( N \) cobosons, let us concentrate on its zeroth-order term in interaction scattering given in Eq. (45). The calculation of the scalar product appearing in \( \tilde{Z}_N^{(0)} \) can be done through a brute-force use of Eqs. (43) and (44). However, as for elementary bosons, calculating this scalar product diagrammatically greatly helps the understanding of the physical processes this part of the partition function contains. This is why here we present a diagrammatic derivation of the recursion relation existing between the \( \tilde{Z}_N^{(0)} \)'s, which is similar to the one we gave for elementary bosons. For readers not at ease with diagrams, we also give in Appendix III the brute-force calculation of \( \tilde{Z}_N^{(0)} \) for low \( N \).

1. Diagrammatic derivation of recursion relation for \( \tilde{Z}_N^{(0)} \)

The scalar product appearing in \( \tilde{Z}_N^{(0)} \) looks very much like the scalar product of \( N \) elementary bosons shown in Fig. 4 except that the \( k_n \) lines are now replaced by \( i_n \) double-lines representing the fermions \( \alpha_n \) and \( \beta_n \) of the coboson \( i_n \). As for elementary bosons, we can connect the \( i_N \) double-line on the left to the \( i_N \) double-line on the right, leaving the \( (N-1) \) other cobosons unaffected, in the same way as in Fig. 4(a). This process readily leads to a contribution to \( \tilde{Z}_N^{(0)} \) given by
\[
\frac{1}{N!} \delta(\beta) \left[ (N-1)! \tilde{Z}_N^{(0)} \right].
\]  

(68)

We can also connect the \( i_N \) double-line on the left to one of the \( (N-1) \) other double-lines on the right, let say \( i_{N-1} \). The \( i_{N-1} \) double-line on the left then has to be connected to one of the \( i_{N-1} \)'s on the right; this can be either to \( i_{N-1} \) or to one of the \( (N-2) \) double-lines like \( i_{N-2} \). The first process leads to the diagrams (a,b) of Fig. 7 since the cobosons \( i_N \) and \( i_{N-1} \) can exchange their fermions, these two cobosons appear either as in Fig. 4(a) or as in Fig. 4(b). The physical processes corresponding to these two diagrams bring a contribution to \( \tilde{Z}_N^{(0)} \) given by
\[
(N-1) \frac{1}{N!} \delta(2\beta) \left[ (N-2)! \tilde{Z}_{N-2}^{(0)} \right].
\]  

(69)
where \( \tilde{z}(2\beta) = z(2\beta) - L(\beta, \beta) \), the fermion exchange part \( L(\beta, \beta) \) being defined through

\[
L(n_1\beta, n_2\beta) = \sum_{i_1, i_2} e^{-\beta(i_1 E_{i_1} + i_2 E_{i_2})} \left( \frac{\alpha_{i_1 i_2}}{\beta} \right). \tag{70}
\]

FIG. 8: Diagrams corresponding to the prefactor of \( \tilde{Z}_{N-4}^{(0)} \).

We now consider processes in which \( i_{N-1} \) on the left is connected to \( i_{N-2} \) on the right (in the same way as in Fig. 2(d)). We can then connect \( i_{N-2} \) on the left to \( i_N \) or to any of the other \((N-3)\) cobosons like \( i_{N-3} \) on the right. The first possibility leads to the diagrams shown in Figs. 4(c,d), in which the three cobosons \((i_N, i_{N-1}, i_{N-2})\) possibly exchange their fermions. If we restrict to one fermion exchange only, we get the three processes shown in Fig. 3(d) in which two cobosons are in the same state, while in the process of Fig. 3(c) the three cobosons are condensed into the same state. So, the processes of Figs. 4(c,d) bring a contribution to \( \tilde{Z}_{N}^{(0)} \) given by

\[
(N - 1)(N - 2) \frac{1}{N!} \tilde{z}(3\beta) \left[ (N - 3)! \tilde{Z}_{N-3}^{(0)} \right]. \tag{71}
\]

where \( \tilde{z}(3\beta) \) at first order in fermion exchange is equal to \( z(3\beta) - 3L(2\beta, \beta) \). To go one step further, we isolate the cobosons \((i_1, \ldots, i_{N-4})\), while the cobosons \((i_N, i_{N-1}, i_{N-2}, i_{N-3})\) form a condensate in which they possibly exchange their fermions as shown in Fig. 5. If we restrict to one fermion exchange only, we must connect any two double-lines by exchange, leaving unaffected the other two double-lines, these lines imposing their cobosons to be in the same state. This brings a contribution to \( \tilde{Z}_{N}^{(0)} \) given by

\[
(N - 1)(N - 2)(N - 3) \frac{1}{N!} \tilde{z}(4\beta) \left[ (N - 4)! \tilde{Z}_{N-4}^{(0)} \right]. \tag{72}
\]

where \( \tilde{z}(4\beta) \) at first order in fermion exchange is equal to \( z(4\beta) - 4L(3\beta, \beta) - 2L(2\beta, 2\beta) \). The \( z(4\beta) \) term comes from diagram (a), the four \( L(3\beta, \beta) \) term come from diagrams (b,d,e,g) while the two \( L(2\beta, 2\beta) \) term come from diagrams (c,f).

Using the same procedure, we end with the following recursion relation between the \( \tilde{Z}_{N}^{(0)} \)'s

\[
\tilde{Z}_{N}^{(0)} = \frac{1}{N} \sum_{n=1}^{N} \tilde{z}(n\beta) \tilde{Z}_{N-n}^{(0)}. \tag{73}
\]

This is just the one for elementary bosons \( \tilde{Z}_{N}^{(0)} \) but with \( z(n\beta) \) replaced by \( \tilde{z}(n\beta) \): \( \tilde{z}(n\beta) = z(\beta) \), while, for \( n \geq 2 \), \( \tilde{z}(n\beta) \), reads, at lowest order in fermion exchange,

\[
\tilde{z}(n\beta) \simeq z(n\beta) - \frac{n}{2} \sum_{m=1}^{n-1} L(m\beta, (n-m)\beta) \tag{74}
\]

with \( L(n_1\beta, n_2\beta) = L(n_2\beta, n_1\beta) \), as seen from Eq. (70).

The recursion relation (73) allows us to write \( \tilde{Z}_{N}^{(0)} \) in the same form as \( Z_N \) in Eq. (4) with \( z(n\beta) \) simply replaced by \( \tilde{z}(n\beta) \). We must however note that, in order to get \( \tilde{Z}_{N}^{(0)} \) at first order only in fermion exchange, we have to keep one \( \tilde{z}(n\beta) \) only, while taking the other \( p \)-boson condensates as \( z(p\beta) \).

2. Partition function of a \( n \)-coboson condensate at zeroth order in \( \xi \)

\( \tilde{z}(n\beta) \) appears as the partition function of a \( n \)-coboson condensate with fermion exchange between their fermionic components. The diagrammatic representation of the partition function for a \( n \)-elementary boson condensate is shown in Fig. 9(a) with the double-lines replaced by single lines. This diagram indeed imposes \( i_n = i_{n-1} = \cdots = i_1 \). As these \( n \) bosons have the same energy, their partition function is given by

\[
\sum_{i=1}^{N} e^{-n\beta E_i} = z(n\beta). \]

To get the partition function of a \( n \)-coboson condensate, we must add fermion exchange to this diagram. At first order, this corresponds to processes like the one of Fig. 9(b) with one fermion exchange between any two double-lines. The cobosons unaffected by this exchange imposes \( i_{p+r-1} = i_{p+r-2} = \cdots = i_p \) and

\[
\text{FIG. 9: Diagrams contributing to } \tilde{z}(n\beta).\]
\(i_p = \cdots = i_1 = i_n = i_{n-1} = \cdots = i_{p+r}\). So, the diagram (b) brings an exchange term equal to \(L(r\beta, (n-r)\beta)\) to the partition function of the \(n\)-coboson condensate. Due to the various ways \(p\) can be chosen and the fact that \(L(r\beta, (n-r)\beta) = L((n-r)\beta, r\beta)\), such an exchange leads to a contribution to the partition function of a \(n\)-coboson condensate given by \((n/2)[L(r\beta, (n-r)\beta) + L((n-r)\beta, r\beta)]\). Note that, as scatterings involving \(n\) cobosons bring a factor \((a_N^N/L^3)(n-1)\), keeping fermion exchange between two cobosons corresponds to performing a many-body expansion at lowest order in density.

D. Partition function at first order in \(\xi\)

We now turn to the contribution at first order in interaction scattering to the canonical partition function of \(N\) cobosons, as given in Eq. (1). It is fundamentally similar to the canonical partition function of \(N\) interacting elementary bosons given in Eq. \((28)\). One just has to include fermion exchanges in the processes considered in our previous calculations.

Let us first consider it for \(N = 2\). It reads

\[
\hat{Z}_2^{(1)} = -\frac{\beta}{2!} \sum_{i_1i_2} e^{-\beta(E_{i_1} + E_{i_2})} \left\{ \langle v|B_{i_1}B_{i_2}B_{i_1}^\dagger B_{i_2}^\dagger |v \rangle \xi \left( \frac{m}{n_{i_1i_2}} \right) + \text{c.c.} \right\} .
\]  

(75)

Using the commutators \((43, 44)\), we find that the scalar product in the above relation reads as \(\delta_{i_1i_2} \delta_{i_2i_1} - \Lambda (i_1^2i_2^2)\); so, \(\hat{Z}_2^{(1)}\) is equal to

\[
\hat{Z}_2^{(1)} = -\frac{\beta}{2} \xi(\beta, \beta),
\]  

(76)

where \(\xi(\beta, \beta)\) follows from

\[
\xi(n_1\beta, n_2\beta) = \sum_{i_1i_2} e^{-\beta(n_1E_{i_1} + n_2E_{i_2})} \xi(i_1, i_2).
\]  

(77)

The scattering \(\hat{\xi}(i_1, i_2)\) corresponds to all possible direct and exchange interaction processes between incoming cobosons \((i_1, i_2)\) ending in states \((i_1i_2)\). It precisely reads

\[
\hat{\xi}(i_1, i_2) = \xi(i_2i_1) + \xi(i_1i_2) - \xi^\text{in}(i_1^2i_2^2) - \xi^\text{in}(i_2^2i_1^2).
\]  

(78)

Precise definition of the exchange scattering \(\xi^\text{in}\) can be found in Ref. \(23\).

\(\hat{Z}_N^{(1)}\) for arbitrary \(N\) is calculated by writing it as a sum of terms proportional to \(\hat{Z}_N^{(0)}\). This can be done through a brute-force calculation using the key commutators of the coboson many-body formalism. In [Appendix IV] we show the calculation for \(N = 3\). Instead, we here give a more enlightening derivation based on diagrams. The scalar product appearing in \(\hat{Z}_N^{(1)}\) is shown in Fig. 4. The prefactor of \(\hat{Z}_N^{(0)}\) is made of \((i_N, i_{N-1})\) cobosons only (see Fig. 10(a)). It just corresponds to the four direct and exchange interaction processes appearing in \(\hat{Z}_2^{(1)}\). We readily get their contribution to \(\hat{Z}_N^{(1)}\) as

\[
-\frac{\beta}{N!} C_2^N C_1^N \xi(\beta, \beta) \left[(N-2)!\hat{Z}_N^{(0)}(2)\right] = -\frac{\beta}{2} \xi(\beta, \beta) \hat{Z}_N^{(0)}/1.
\]  

(79)

To get the prefactor of \(\hat{Z}_N^{(0)}\), we isolate one more cobosons out of \((N-2)\), let say \(i_{N-2}\), and we draw all entangled processes. This imposes \(i_{N-2}\) not to be connected with itself, as in the diagram of Fig. 10(b). By noting that \(\xi(2\beta, \beta) = \xi(2\beta, \beta)\), these two processes lead to

\[
-\frac{\beta}{N!} C_2^N C_1^N \xi(\beta, \beta) \left[(N-3)!\hat{Z}_N^{(0)}(3)\right] = -\frac{\beta}{2} \xi(\beta, \beta) \hat{Z}_N^{(0)}/1.
\]  

(80)

Note that we can also have exchange processes like the ones of Fig. 10(c) which connect three cobosons. The associated scatterings, however, are \((a_N^N/L^3)\) smaller than diagram (b). So, the dominant prefactor of \(\hat{Z}_N^{(0)}\) is the one given in Eq. \((80)\).

As for interacting elementary bosons (see Eq. \((33)\)), the prefactor of \(\hat{Z}_N^{(0)}\) in \(\hat{Z}_N^{(1)}\) is obtained by isolating two cobosons out of \((i_1, \cdots, i_{N-2})\), let say \((i_{N-2}, i_{N-3})\), and by drawing all entangled processes between \((i_N, i_{N-1}, i_{N-2}, i_{N-3})\), like in the diagrams of Fig. 4(c). This brings a contribution to \(\hat{Z}_N^{(1)}\) given by

\[
-\frac{\beta}{N!} C_2^N C_1^N (2\beta, 3\beta) + 2\xi(2\beta, 2\beta) + 2\xi(3\beta, 1\beta)
\]  

\[\times [(N-4)!\hat{Z}_N^{(0)}(4)].\]  

(81)

So, we end with an expansion of \(\hat{Z}_N^{(1)}\) similar to the one for interacting elementary bosons, \(\hat{Z}_N^{(1)}\), namely

\[
\hat{Z}_N^{(1)} = -\frac{\beta}{2} \sum_{n=2}^N \xi(n\beta) \hat{Z}_N^{(0)}/n.
\]  

(82)
with

\[ \tilde{\xi}(n\beta) = \sum_{p=1}^{n-1} \tilde{\xi}(p\beta, (n-p)\beta). \] (83)

By adding the Pauli part \( \tilde{Z}_N^{(0)} \) of the \( N \)-coboson partition function given in Eq. (83), we find that the canonical partition function of these composite quantum particles is given at lowest order in \( (a_X^2/L^3) \) by

\[ \tilde{Z}_N \simeq \frac{1}{N} \sum_{n=1}^{N} \left[ \tilde{z}(n\beta) - \frac{\beta N}{2} \tilde{\xi}(n\beta) \right] \tilde{Z}_N^{(0)}. \] (84)

We can go further and transform the above equation into a recursion relation between the \( \tilde{Z}_N \)'s by following the procedure we have used for interacting elementary bosons. We then end with \( \tilde{Z}_N \) correct up to first order in both, Pauli exchange and interaction scattering, as

\[ \tilde{Z}_N \simeq \frac{1}{N} \sum_{n=1}^{N} \tilde{z}(n\beta) \tilde{Z}_N^{(0)}. \] (85)

where the partition function for a \( n \)-coboson condensate is given, for \( n \geq 2 \), by

\[ \tilde{z}(n\beta) = z(n\beta) - \frac{\beta n}{2} \tilde{\xi}(n\beta). \] (86)

It is then straightforward to show that Eq. (86) leads to a compact form for \( \tilde{Z}_N \) similar to Eq. (11) with \( z(n\beta) \) replaced by \( \tilde{z}(n\beta) \). A similar compact form for the canonical partition function of cobosons to all orders in interaction and fermion exchange appears to us as conceptually obvious, although beyond the scope of the present work.

\section*{IV. CONCLUSIONS}

We propose a diagrammatic approach to the canonical partition function of \( N \) cobosons. In addition to the usual diagrams representing the condensation processes existing for elementary bosons, the Pauli exclusion principle generates new diagrams for fermion exchanges between the fermionic components of cobosons. The partition function we obtain provides grounds for the study of coboson quantum condensation. Here, we calculate in details the canonical partition functions of non-interacting elementary bosons as well as interacting elementary bosons and interacting composite bosons at first order in interaction and fermion exchange. In all cases, the partition function takes the same compact form as the one of non-interacting elementary bosons provided that we include interaction and fermion exchange in the partition function \( z(n\beta) \) of the \( n \)-particle condensate.

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\section*{Appendix I. \( \tilde{Z}_N^{(0)} \) FOR LOW \( N \)'S}

For \( N = 1 \), the canonical partition function reduces to

\[ \tilde{Z}_1^{(0)} = z(\beta). \] (A.1)

For \( N = 2 \), the recursion relation (11) gives

\[ \tilde{Z}_2^{(0)} = \frac{1}{2!} [z^2(\beta) + z(2\beta)] \] (A.2)

in agreement with Eq. (11) taken for \((p_1 = 2)\) or \((p_2 = 1)\).

This \( \tilde{Z}_2^{(0)} \) taken in the recursion relation for \( \tilde{Z}_3^{(0)} \) gives

\[ \tilde{Z}_3^{(0)} = \frac{1}{3!} [z^3(\beta) + 3z(\beta)z(2\beta) + 2z(3\beta)], \] (A.3)

which agrees with Eq. (11) taken for \((p_1 = 3)\), \((p_1 = 1, p_2 = 1)\) or \((p_2 = 1)\).

These \( \tilde{Z}_1^{(0)}, \tilde{Z}_2^{(0)} \) and \( \tilde{Z}_3^{(0)} \) taken in the recursion relation (11) for \( N = 4 \) give

\[ \tilde{Z}_4^{(0)} = \frac{1}{4!} \left[ z^4(\beta) + 6z^2(\beta)z(2\beta) + 8z(\beta)z(3\beta) + 6z(4\beta) + 3z^2(2\beta) \right] \] (A.4)

in agreement with Eq. (11) taken for \((p_1 = 4)\), \((p_1 = 2, p_2 = 1)\), \((p_1 = p_3 = 1)\), \((p_4 = 1)\) or \((p_2 = 2)\). We note that the sum of prefactors in these partition functions, e.g., \((1 + 6 + 8 + 6 + 3)/4!\) in the case of 4 bosons, is equal to 1. So, these prefactors physically correspond to the probability of the condensation process at hand.
and we can connect \( k \) processes of diagram 11(c) are going to appear twice. Their contribution to two processes lead to the diagrams shown in Fig. 12(a).

In diagrams 11(a) or (b), we can also connect \( k \) processes, shown in Figs. 12(b,c), will appear twice.

Finally, from diagram 11(c), we can connect \( k \), \( k \), which gives equivalent contribution; so, these processes, shown in Figs. 12(b,c), will appear twice.

(ii) To go further, we consider diagrams 12(b,d), and we connect \( k \) to \( k \), as shown in Figs. 13(a,b).

Collecting all the terms and noting that \( \mathcal{V}(n_1\beta,n_2\beta) = \mathcal{V}(n_2\beta,n_1\beta) \), we end with

\[
\tilde{Z}_4^{(1)} = -\frac{\beta}{2} \mathcal{V}(\beta,\beta) \tilde{Z}_4^{(0)} + \mathcal{V}(\beta,2\beta) \tilde{Z}_4^{(0)} + \mathcal{V}(2\beta,\beta) \tilde{Z}_4^{(0)}.
\]

\[
\tilde{Z}_4^{(0)} = \frac{1}{N!} \sum_{\{i\}} e^{-\beta(E_{i_1}+\cdots+E_{i_N})} \langle v|B_{i_1} \cdots B_{i_N} B_{i_1}^\dagger \cdots B_{i_N}^\dagger |v \rangle.
\]
To understand how the recursion relation for the $\tilde{Z}_N^{(0)}$ given in Eq. (73) develops, let us explicitly calculate $\tilde{Z}_N^{(0)}$ for $N = 2$ and $N = 3$.

A. Two cobosons

Equation (43) allows us to write the scalar product of two cobosons shown in Fig. 13(a) as

$$\langle v|B_i B_i B_i B_i |v\rangle = \langle v|B_i (\delta_{i1} - D_{i2} + B_i B_i) B_i |v\rangle$$

(C.2)

By inserting the term in $\delta_{i12}$ into $\tilde{Z}_2^{(0)}$, we readily get its contribution to $\tilde{Z}_2^{(0)}$ as

$$\frac{1}{2!} z(\beta) \tilde{Z}_1^{(0)}.$$ (C.3)

The corresponding diagram is shown in Fig. 13(b). Using Eq. (44) for the term in $D_{i12}$, we get

$$- \sum_m \langle v|B_i B_m |v\rangle \Lambda (m_{i1} i_2) = - \Lambda (i_1 i_2).$$ (C.4)

The corresponding diagram is shown in Fig. 13(c). When inserted into $\tilde{Z}_2^{(0)}$, this term leads to $-L(\beta, \beta)$.

Finally, the term in $B_i B_{i12}$ gives $\langle v|B_i B_{i12} |v\rangle \delta_{i12}$ as shown in Fig. 13(d). This imposes $i_1 = i_2$ and yields $z(2\beta)$. So, we end with

$$\tilde{Z}_2^{(0)} = \frac{1}{2!} \left[ z(\beta) \tilde{Z}_1^{(0)} + z(2\beta) - L(\beta, \beta) \right].$$ (C.5)

with $\tilde{Z}_1^{(0)} = z(\beta)$. We can rewrite this expression as Eq. (73), with $\tilde{z}(2\beta) = z(2\beta) - L(\beta, \beta)$ in agreement with Eq. (74) taken for $N = 2$.

B. Three cobosons

Equation (43) gives the scalar product of three cobosons as

$$\langle v|B_i B_i B_i B_i B_i B_i B_i B_i |v\rangle$$

(C.6)

$$= \langle v|B_i B_i (\delta_{ii3} - D_{i3i3} + B_i B_i) B_i |v\rangle.$$ (C.7)

The term in $\delta_{i13}$, when inserted into Eq. (73) taken for $N = 3$, readily yields a contribution to $\tilde{Z}_3^{(0)}$ given by

$$\frac{1}{3!} z(\beta) \left[ 2\tilde{Z}_2^{(0)} \right],$$

which corresponds to the diagram of Fig. 13(a).

For the term in $D_{i13}$ of Eq. (C.6), we use Eq. (44) to replace $D_{i13} B_i$ by $\sum_m B_m \Lambda (m_{i1} i_3) + B_i B_i D_{i3}$ and we use again Eq. (44) for $D_{i3i3} B_i$. This leads to

$$- \sum_m \langle v|B_i B_i B_i B_i |v\rangle \Lambda (m_{i1} i_3) + \Lambda (m_{i1} i_3) B_i |v\rangle.$$ (C.8)

When inserted into Eq. (C.10), these two terms contribute equally through a relabeling of $(i_1, i_2)$. So, the term in $D_{i13}$ gives a contribution to $\tilde{Z}_3^{(0)}$ given by

$$- \frac{2}{3!} \sum e^{-\beta(E_{i1} + E_{i2} + E_{i3})} \Lambda (m_{i1} i_3) \langle v|B_i B_i B_i B_i B_i B_i |v\rangle.$$ (C.9)

This term is shown in Fig. 13(b). The scalar product in the above equation gives two delta terms, namely $\delta_{miz}$ and $\delta_{mzi}$, plus one exchange term in $\Lambda (m_{i1} m)$ that we can neglect if we only want the first-order correction to $\tilde{Z}_3^{(0)}$. The two delta terms shown in Figs. 13(c) and (d) give $-2(1/3!)L(\beta, \beta)\tilde{Z}_1^{(0)}$ and $-2(1/3!)L(2\beta, \beta)$ respectively.

Finally, the term in $B_i B_i B_i$ of Eq. (C.6) is calculated by pushing $B_i$ to the right according to Eq. (43),

$$\langle v|B_i B_i B_i B_i |v\rangle = \delta_{i13} - D_{i3i3} + B_i B_i B_i |v\rangle.$$ (C.10)

The scalar product in the first term of the above equation, shown in Fig. 13(a), is calculated by replacing $B_i B_i$ with $\delta_{i13} - D_{i3i3} + B_i B_i B_i$ according to Eq. (43). Since $D_{i3i3} B_i |v\rangle$ gives $\sum_m B_m |v\rangle \Lambda (m_{i1} i_3)$, these three terms shown in Fig. 13(b) ultimately yield a contribution to $\tilde{Z}_3^{(0)}$ given by

$$\frac{1}{3!} z(\beta) \tilde{Z}_1^{(0)} - L(2\beta, \beta) + z(3\beta).$$ (C.11)

$$\frac{1}{3!} z(\beta) \tilde{Z}_1^{(0)} - L(2\beta, \beta) + z(3\beta).$$ (C.12)
In the second term of Eq. (C.11), shown in Fig. 16(b), we just have to replace the scalar product by $\delta_{i_{m}i_{1}}\delta_{i_{2}i_{3}}$ if we want this term at first order in fermion exchange only. These two terms, shown in Fig. 16(c), yield $-2(1/3)L(2\beta, \beta)$.

Collecting all the above terms, we end with

$$
\hat{Z}_{3}^{(0)} = \frac{1}{3} [z(\beta)Z_{2}^{(0)} + (z(2\beta) - L(\beta, \beta))\hat{Z}_{1}^{(0)} + (z(3\beta) - 3L(2\beta, \beta))]
$$

in agreement with Eqs. (E.13).

**Appendix IV. CALCULATION OF $\hat{Z}_{3}^{(1)}$**

We here calculate the partition function at first order in interaction scattering, $\hat{Z}_{3}^{(1)}$, given in Eq. (67) for three cobosons, namely

$$
\hat{Z}_{3}^{(1)} = -\frac{\beta}{4} \sum e^{-\beta(E_{i_{1}} + E_{i_{2}} + E_{i_{3}})}
$$

$$
\times \left[ \langle \nu | B_{1} B_{2} B_{3} B_{m} B_{n} B_{1} B_{1} \rangle \xi \left( \frac{n_{i_{2}}}{m_{i_{2}}} \right) + \text{c.c.} \right].
$$

(E.1)

The above scalar product is calculated by first replacing $B_{i_{1}} B_{i_{1}1}$ with $\delta_{i_{1}i_{1}} - D_{i_{1}i_{1}} + B_{i_{1}} B_{a}$ according to Eq. (13).

(i) The $\delta_{i_{3}i_{1}}$ term leads to a contribution to $\hat{Z}_{3}^{(1)}$ given by

$$
-\frac{\beta}{4} \sum e^{-\beta(2\beta_{1} + E_{i_{3}})} \left[ \langle \nu | B_{1} B_{2} B_{3} B_{m} B_{n} B_{1} B_{1} \rangle \xi \left( \frac{n_{i_{2}}}{m_{i_{2}}} \right) + \text{c.c.} \right].
$$

(E.2)

As $\langle \nu | B_{1} B_{2} B_{3} B_{m} B_{n} B_{1} B_{1} \rangle = \delta_{i_{3}m} \delta_{i_{2}m} + \delta_{i_{3}i_{n}} \delta_{i_{2}m} - \Lambda_{i_{3}i_{1}i_{1}}$, we ultimately get this contribution to $\hat{Z}_{3}^{(1)}$ as

$$
-2\frac{\beta}{4} \zeta(\beta, 2\beta)
$$

(E.3)

with $\zeta(\nu_{j}, \nu_{j})$ defined in Eqs. (17,18), the factor of 2 coming from the c.c. part.

(ii) The term in $D_{i_{3}i_{1}}$, inserted into Eq. (C.11), leads to

$$
\frac{\beta}{4} \sum e^{-\beta(E_{i_{1}} + E_{i_{2}} + E_{i_{3}})}
$$

$$
\times \left[ \langle \nu | B_{1} B_{2} B_{3} D_{i_{3}i_{1}} B_{m} B_{n} B_{1} B_{1} \rangle \xi \left( \frac{n_{i_{2}}}{m_{i_{2}}} \right) + \text{c.c.} \right].
$$

(E.4)

Using Eq. (C.14), we get $D_{i_{3}i_{1}} B_{m} B_{n} B_{1} B_{1}$ as

$$
\sum_{p} \Lambda \left( \frac{p_{i_{3}i_{1}}}{m_{i_{3}i_{2}}} \right) B_{m} B_{n} B_{1} B_{1} (\nu) + (\nu \rightarrow n).
$$

(E.5)

So, Eq. (E.14) gives

$$
\frac{\beta}{2} \sum e^{-\beta(E_{i_{1}} + E_{i_{2}} + E_{i_{3}})}
$$

$$
\times \left[ \langle \nu | B_{1} B_{2} B_{3} B_{m} B_{n} B_{1} B_{1} \rangle \xi \left( \frac{n_{i_{2}}}{m_{i_{2}}} \right) + \text{c.c.} \right].
$$

(E.6)

The sum over $m$ corresponds to a scattering represented by a diagram similar to the one of Fig. 16(c). As it involves three cobosons, this term leads to a contribution to $Z_{3}^{(1)}$ of the order of $(a_{1}/L^{3})^{2}$ which can be neglected in a first-order calculation.

(iii) The term in $B_{i_{1}i_{3}} B_{1}$ leads to a contribution to $Z_{3}^{(1)}$, given by

$$
\frac{\beta}{4} \sum e^{-\beta(E_{i_{1}} + E_{i_{2}} + E_{i_{3}})}
$$

$$
\times \left[ \langle \nu | B_{1} B_{2} B_{3} B_{m} B_{n} B_{1} B_{1} \rangle \xi \left( \frac{n_{i_{2}}}{m_{i_{2}}} \right) + \text{c.c.} \right].
$$

(E.7)

To get it, we replace $B_{i_{2}i_{1}}$ by $\delta_{i_{2}i_{1}} - D_{i_{2}i_{1}} + B_{i_{2}} B_{i_{3}}$: The term in $\delta_{i_{2}i_{1}}$ is equivalent to the one of Eq. (E.2) if we interchange $i_{2}$ and $i_{3}$; so, it gives a contribution equal to

$$
-2\frac{\beta}{4} \zeta(2\beta, \beta).
$$

(E.8)

The term in $D_{i_{2}i_{1}}$ leads to

$$
\frac{\beta}{4} \sum e^{-\beta(E_{i_{1}} + E_{i_{2}} + E_{i_{3}})}
$$

$$
\times \left[ \langle \nu | B_{1} D_{i_{2}i_{1}} B_{2} B_{3} B_{m} B_{n} B_{1} B_{1} \rangle \xi \left( \frac{n_{i_{2}}}{m_{i_{2}}} \right) + \text{c.c.} \right].
$$

(E.9)

Since the above scalar product already contains one fermion exchange associated with $D_{i_{2}i_{1}}$, we can reduce $B_{i_{2}i_{1}} B_{m} B_{n} B_{1} B_{1}$ to $D_{i_{2}i_{1}} B_{m} B_{1} B_{1} + \delta_{i_{3}i_{3}} B_{m} B_{n} B_{1} B_{1}$ at lowest order in $a_{1}/L^{3}$. When inserted into Eq. (E.9), we get

$$
\frac{\beta}{4} \sum e^{-\beta(E_{i_{1}} + E_{i_{2}} + E_{i_{3}})} \left[ \langle \nu | B_{1} D_{i_{2}i_{1}} B_{2} B_{3} B_{m} B_{n} B_{1} B_{1} \rangle \xi \left( \frac{n_{i_{2}}}{m_{i_{2}}} \right) + \text{c.c.} \right].
$$

(E.10)

As $\langle \nu | B_{1} D_{i_{2}i_{1}} B_{m} B_{n} B_{1} B_{1} \rangle$ reduces to $A_{i_{2}i_{1}}$, the term in $D_{i_{2}i_{1}}$ leads to a scattering involving three cobosons; so, it gives a contribution of the order $(a_{1}/L^{3})^{2}$ which can be neglected at lowest order. The term in $B_{i_{1}i_{3}} B_{2} B_{1}$ gives

$$
\frac{\beta}{4} \sum e^{-\beta(E_{i_{1}} + E_{i_{2}} + E_{i_{3}})}
$$

$$
\times \left[ \langle \nu | B_{1} B_{2} B_{3} B_{m} B_{n} B_{1} B_{1} \rangle \xi \left( \frac{n_{i_{2}}}{m_{i_{2}}} \right) + \text{c.c.} \right].
$$

(E.10)
As $\langle v | B_{i_1} B_{i_1}^\dagger | v \rangle = \langle v | \delta_{i_1 i_1} | v \rangle$, the above contribution reduces to

$$
\begin{align}
&z(\beta) - \frac{\beta}{4} \sum e^{-\beta (E_{i_3} + E_{i_4})} \left[ \langle v | B_{i_2} B_{i_3} B_{i_4}^\dagger B_{i_5}^\dagger | v \rangle \xi \left( \frac{n}{m_{i_2}} \right) + c.c. \right] \\
&= z(\beta) \bar{Z}_2^{(1)} = -\frac{\beta}{2} \bar{\xi}(\beta, \beta) \tilde{Z}_1^{(0)}. \\
&(E.11)
\end{align}
$$

All these terms combine to yield, with $\tilde{\xi}(n, \beta)$ defined in Eq. (83),

$$
\bar{Z}_3^{(1)} = -\frac{\beta}{2} \left[ \bar{\xi}(2, \beta) \bar{Z}_1^{(0)} + \bar{\xi}(3, \beta) \right],
$$

(E.12)
in agreement with Eq. (82).

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