GENERALIZATION OF THE “STARK UNIT” FOR
ABELIAN L-FUNCTIONS WITH MULTIPLE ZEROS

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Let $K/k$ be a normal extension of number fields with abelian Galois
group $G$. Suppose there is an infinite place $v$ of $k$ such that the set
of nontrivial characters $\chi : G \to \mathbb{C}^\times$ which are ramified at all infinite
places except $v$ is nonempty. Then the corresponding Artin L-functions
$L(\chi, s)$ have zeros of the first order at $s = 0$, and if we fix any embed-
ding of the bigger field $K$ to $\mathbb{C}$ which extends $v$ then Stark’s conjecture
predicts existence of a “unit” $\varepsilon \in \mathbb{Q} \otimes O_K^\times$ such that for every such $\chi$

$$L'(\chi, 0) = \sum_{g \in G} \chi(g) \log |g\varepsilon|.$$  

We refer to the book [1] concerning the formulation and consisten-
ty of the conjecture. It is stated there for a character of an arbitrary
Galois representation, but the case of abelian character would imply
the general one.

In this note we fix a nonnegative integer $r \geq 0$ and consider those $\chi$
for which the L-function has zero of order exactly $r$ at $s = 0$. Let

$$L(\chi, s) = c(\chi)s^r + o(s^r), \quad s \to 0$$

with $c(\chi) \neq 0$. First we show that the leading Laurent coefficients
c($\chi$) for such $\chi$ can be all described by means of a single element
$\varepsilon_r \in \mathbb{R} \bigwedge^r \mathbb{Z}[G] O_K^\times$ of the $\mathbb{R}$-span of the $r$-th exterior power of the module
of units $O_K^\times$ taken in the category of $\mathbb{Z}[G]$-modules. Further, we prove
that the Stark conjecture is equivalent to the statement that

$$\varepsilon_r \in \mathbb{Q} \bigwedge^r \mathbb{Z}[G] O_K^\times.$$  

Our main result is Theorem 2. Following [1], we consider here slightly
more general L-functions $L_S(\chi, s)$ where $S$ is a set of places of $k$ containing
all the infinite places $S_{\infty} \subseteq S$. If one puts $S = S_{\infty}$ then these are
the ordinary Artin L-functions as above, namely $L(\chi, s) = L_{S_{\infty}}(\chi, s)$.  

In general $\varepsilon_r$ belongs to $\mathbb{R} \bigwedge^r \mathbb{Z}[G] U$, where $U \subset K^\times$ is the module of
$S$-units of the bigger field $K$. If $S = S_{\infty}$ then $U = O_K^\times$.  

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We should mention that such elements already appeared in [2], but under very special assumptions on the set of places $S$ (Hypotheses 2.1 of [2]). Although these assumptions allow to refine the Stark conjecture, we find it convenient from the computational point of view to keep the set $S$ as small as possible. Indeed, when we increase $S$ then the order of zero of $L_S(\chi, s)$ at $s = 0$ either stays constant or increases. The module $U$ of $S$-units also grows. Hence if one really wants to verify the Stark conjecture numerically, the most simple choice will be $S = S_\infty$. We consider one numerical example below. In view of the above-said, the present note may be considered as a naive version of [2], where we do everything what can be done simply by linear algebra.

1. The conjecture

We start with a reformulation of the abelian Stark conjecture. Let $K/k$ be a normal extension of number fields with an abelian Galois group $G$, $S$ be a set of places of $k$ containing all the infinite places $S_\infty \subseteq S$ and $S_K$ be the set of all places of $K$ lying above places in $S$. Let $Y$ be the free abelian group on $S_K$ and

$$X = \left\{ \sum_{w \in S_K} x_w w \mid x_w \in \mathbb{Z}, \sum x_w = 0 \right\} \subseteq Y.$$ 

Let

$$U = \left\{ x \in K^\times \mid \text{ord}_w(x) = 0 \quad \forall w \notin S_K \right\}$$

be the group of $S$-units in $K$. We have a $G$-module isomorphism

$$\log : \mathbb{R}U \longrightarrow \mathbb{R}X$$

$$\log(x) = \sum_{w \in S_K} \log |x|_w w.$$ 

Also $\mathbb{Q}X \cong \mathbb{Q}U$ as $G$-modules (see [1], p.26), but there is no canonical choice for this isomorphism.

The modified Artin L-function attached to a character $\chi : G \longrightarrow \mathbb{C}^\times$ is defined for $\Re(s) > 1$ as

$$(1) \quad L_S(\chi, s) = \prod_{v \notin S, \chi|_{I_v} \equiv 1} (1 - \chi(\sigma_v) Nv^{1-s})^{-1},$$

where for every finite place $v$ of $k$ we denote by $G_v, I_v$ and $\sigma_v \in G_v/I_v$ the decomposition group, the inertia group and the Frobenius element correspondingly. The order of zero at $s = 0$ of this L-function is

$$r(\chi) = \# \left\{ v \in S \mid \chi|_{G_v} \equiv 1 \right\}$$.
for $\chi \neq 1$ and $r(1) = \#S - 1$ (Proposition II.3.4 in [1]). Here $1$ is the trivial character. Let $c(\chi)$ be the leading Laurent coefficient of the $L$-function $L_S(\chi, s)$ at $s = 0$:

$$L_S(\chi, s) = c(\chi)s^{r(\chi)} + o(s^{r(\chi)}), \quad s \to 0.$$ 

Let us fix a nonnegative integer $r$ such that there are characters $\chi$ with $r(\chi) = r$. In particular, should be $r \leq \#S$. Let $C$ be a nonempty set of characters satisfying the two conditions:

(i) if $\chi \in C$ then $\chi^\alpha = \alpha \circ \chi \in C$ for every $\alpha \in \text{Aut}(\mathbb{C}/\mathbb{Q})$;
(ii) $r(\chi) = r$ for all $\chi \in C$.

For any $\chi$ the element $E_\chi = \frac{1}{\#G} \sum_{g \in G} \chi(g)g$ is an idempotent in $\mathbb{C}[G]$. Let

$$E_C = \sum_{\chi \in C} E_\chi.$$ 

It is an idempotent since $E_\chi E_{\chi'} = 0$ when $\chi \neq \chi'$, and it belongs to $\mathbb{Q}[G]$ due to the property (i) of the set $C$. Consider also the generalized “Stickelberger element”

$$\Theta_C = \sum_{\chi \in C} c(\overline{\chi}) E_\chi.$$ 

It obviously belongs to the subspace $E_C \mathbb{C}[G]$, but since $c(\overline{\chi}) = c(\chi)$ we have $\Theta_C \in E_C \mathbb{R}[G]$ in fact.

Let $\log^{(r)} : \mathbb{R} \wedge^r U \to \mathbb{R} \wedge^r X$ be the $G$-module isomorphism induced by $\log : \mathbb{R}U \to \mathbb{R}X$, where exterior products are always taken in the category of modules over the commutative ring $\mathbb{Z}[G]$. If $r = 0$ this means $\wedge^0 U = \wedge^0 X = \mathbb{Z}[G]$ and $\log^{(0)} = 1$.

**Conjecture 1.** Let $C$ be a nonempty set of characters satisfying the conditions (i) and (ii) from above. Then

$$\Theta_C \mathbb{Q} \wedge^r X = \mathbb{Q} \log^{(r)} \left( E_C \wedge^r U \right).$$

Notice that both $\Theta_C \wedge^r X$ and $\log^{(r)} (E_C \wedge^r U)$ are maximal discrete lattices in the real vector space $E_C \mathbb{R} \wedge^r X$ of dimension $\#C$. Indeed, $E_C \wedge^r X$ is obviously a maximal lattice in $E_C \mathbb{R} \wedge^r X$, and $\Theta_C$ is an invertible transformation of this space since $\det(\Theta_C) = \prod_{\chi \in C} c(\chi) \neq 0$. Analogously, $E_C \wedge^r U$ modulo torsion is maximal in $E_C \mathbb{R} \wedge^r U$, and $\log^{(r)} : E_C \mathbb{R} \wedge^r U \to E_C \mathbb{R} \wedge^r X$ is invertible.

\[\text{1If } r = 0 \text{ the conjecture states } \Theta_C \in E_C \mathbb{Q}[G]. \text{ The same claims the Stark conjecture for all } \chi \in C, \text{ which is true (Theorem III.1.2 of [1]). We assume } r > 0 \text{ further in this note.}\]
The conjecture claims these lattices are commensurable.

**Theorem 1.** Conjecture 1 is equivalent to the Stark conjecture “over \( \mathbb{Q} \)” (Conjecture I.5.1 of [1]) for all characters \( \chi \in C \).

Let us choose any isomorphism \( f : \mathbb{Q}X \longrightarrow \mathbb{Q}U \). Then the Stark regulator for each \( \chi \) is defined as \( R(\chi, f) = \det(\log \circ f|_{\text{Hom}_G(\chi, CX)}) \). Put \( A(\chi, f) = \frac{c(\chi)}{R(\chi, f)} \). Stark’s conjecture claims that \( A(\chi^\alpha, f) = A(\chi, f)^\alpha \) for every \( \alpha \in \text{Aut}(\mathbb{C}/\mathbb{Q}) \). We can equivalently state it as

\[
A = \sum_{\chi \in C} A(\chi, f) \mathbb{E}_\chi \in \mathbb{Q}[G].
\]

Let us introduce also \( R = \sum_{\chi \in C} R(\chi, f) \mathbb{E}_\chi \). Then \( \Theta_C = AR \).

**Proof of Theorem 1.** For any \( x_1, \ldots, x_r \in X \)

\[
R(\chi, f) \mathbb{E}_\chi x_1 \wedge \cdots \wedge x_r = \mathbb{E}_\chi \log \circ f(x_1) \wedge \cdots \wedge \log \circ f(x_r)
\]

by the definition of \( R(\chi, f) \). Indeed, \( \text{Hom}(\chi, CX) \cong \mathbb{E}_\chi CX \) and dimension of this space is \( r \). Therefore

\[
R \bigwedge^r X = \log^{(r)} \left( \mathbb{E}_CX \bigwedge^r f(X) \right),
\]

and \( \mathbb{Q} \log^{(r)} (\mathbb{E}_C \bigwedge^r U) = R \frac{\mathbb{Q}}{ \bigwedge^r X} \). Also \( \Theta_C \frac{\mathbb{Q}}{ \bigwedge^r X} = R (A Q \bigwedge^r X) \). Since multiplication by \( R \) is an invertible transformation of \( \mathbb{E}_C \mathbb{R} \bigwedge^r X \), Conjecture 1 is now equivalent to the statement \( \mathbb{E}_C \mathbb{Q} \bigwedge^r X = A \mathbb{Q} \bigwedge^r X \). This is in turn equivalent to \( A \in \mathbb{E}_C \mathbb{Q}[G] \). Recall that we reduced the Stark conjecture to the same statement (2). \( \square \)

We have already remarked that Conjecture 1 claims commensurability of the two lattices \( \Theta_C \bigwedge^r X \) and \( \log^{(r)} (\mathbb{E}_C \bigwedge^r U) \). Let us show that the ratio of covolumes of these lattices is a rational number. Let \( f : \mathbb{Q}X \longrightarrow \mathbb{Q}U \) be the isomorphism from above. Then \( \log^{(r)} (\mathbb{E}_C \bigwedge^r f(X)) \) is commensurable with \( \log^{(r)} (\mathbb{E}_C \bigwedge^r U) \), and

\[
\Theta_C \bigwedge^r X = AR \bigwedge^r X = A \log^{(r)} \left( \mathbb{E}_C \bigwedge^r f(X) \right)
\]

due to (3). The ratio of covolumes of \( \Theta_C \bigwedge^r X \) and \( \log^{(r)} (\mathbb{E}_C \bigwedge^r f(X)) \) is therefore \( \det(A) = \prod_{\chi \in C} A(\chi, f) \). For a character \( \psi : G \longrightarrow \mathbb{C} \) of an arbitrary representation of \( G \) one can define the Artin L-function \( L_S(\psi, s) \) and introduce the numbers \( c(\psi), R(\psi, f) \) and \( A(\psi, f) = \frac{c(\psi)}{R(\psi, f)} \).
Since the latter numbers satisfy \( A(\psi_1 + \psi_2) = A(\psi_1)A(\psi_2) \),
\[
\det(A) = \prod_{\chi \in \mathcal{C}} A(\chi, f) = A \left( \sum_{\chi \in \mathcal{C}} \chi, f \right).
\]

Due to the property (i) of \( C \) the character \( \psi = \sum_{\chi \in \mathcal{C}} \chi \) takes rational values on \( G \). For such characters the Stark conjecture is known to be true (Corrolary II.7.4 in \([1]\)).

It means \( A(\psi, f) \in \mathbb{Q} \), hence our claim is proved.

2. **Generalization of the "Stark unit"**

Let \( S = \{v_1, \ldots, v_n\} \) and choose some \( w_i \in S_K \) above each \( v_i \in S \). For \( w \in S_K \) let \( w^* \in \text{Hom}_G(Y, \mathbb{Z}[G]) \) be the map
\[
w^* \left( \sum x_w x' \right) = \frac{1}{\#G_{v_i}} \sum x_{gw} g.
\]

Then we can consider \( w_{i_1}^* \wedge \cdots \wedge w_{i_r}^* \in \text{Hom}_G(\bigwedge^r G, \mathbb{Z}[G]) \) for any \( i_1 < \cdots < i_r \). This homomorphism maps \( y_1 \wedge \cdots \wedge y_r \) to \( \det(w_{i_k}^*(y_{i_k}))_{k=1}^r \).

Let \( E_r = \sum_{\chi: r(\chi) = r} E_{\chi} \). It is an idempotent in the group algebra \( \mathbb{Q}[G] \).

Recall that \( U \subset K^\times \) is the group of \( S \)-units of \( K \).

**Theorem 2.**

There is a unique element \( \varepsilon_r \in E_r \mathbb{R} \bigwedge^r U \) such that for every \( \chi \neq 1 \) with \( r(\chi) = r \)
\[
c(\chi) = \chi \left( \prod_{i=1}^r \left( \log^{(r)}(\varepsilon_r) \right) \right)
\]
where \( \{i_1 < \cdots < i_r\} = \{i : \chi|_{G_{v_i}} \equiv 1\} \), and (in case \( r = n - 1 \))
\[
c(1) = 1 \left( (-1)^i w_i^* \wedge \cdots \wedge w_{i-1}^* \wedge w_{i+1}^* \wedge \cdots \wedge w_n^* \left( \log^{(r)}(\varepsilon_r) \right) \right)
\]
with an arbitrary \( i \).

**Conjecture**

for the set of characters \( C_r = \{\chi : r(\chi) = r\} \) is true if and only if \( \varepsilon_r \in E_r \mathbb{Q} \bigwedge^r U \).

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2We refer here to the truth of the Stark conjecture for a rational-valued character, although one can find simple arguments in this case. In fact our \( \psi \) is a sum of permutation characters. A permutation character is the one induced from the trivial character on a subgroup. Since \( A(\text{Ind} \chi, f) = A(\chi, f) \), the Stark conjecture is true for \( \psi \) because it is true for a trivial character.

3Let us write for the moment \( \varepsilon_r^K, w_i^K \) and \( U_K \) to indicate the relation to \( K/k \). Let us take some intermediate extension \( k \subset F \subset K \) and choose for the places \( w_i^F \) exactly the ones under \( w_i^K \). Then one can prove that \( \varepsilon_r^F = \mathcal{N}^{(r)}(\varepsilon_r^K) \), where \( \mathcal{N}^{(r)} : \mathbb{R} \bigwedge^r U_K \rightarrow \mathbb{R} \bigwedge^r U_F \) is the map induced by the norm \( \mathcal{N} : U_K \rightarrow U_F \).
Proof. We want to construct a certain $G$-module isomorphism

$$\Omega : E_r \mathbb{Q} \wedge^r X \overset{\sim}{\longrightarrow} E_r \mathbb{Q}[G].$$

Suppose first $r \neq n - 1$, so that all $\chi \in C_r$ are nontrivial. Let us check that

$$\Omega = \sum_{i_1 < \cdots < i_r} w^*_{i_1} \wedge \cdots \wedge w^*_{i_r} \big|_{E_r \wedge^r X}$$

is such an isomorphism. Since its image obviously belongs to $E_r \mathbb{Q}[G]$, it is enough to prove that $\Omega$ gives an invertible map from $E_r \mathbb{Q} \wedge^r X$ to $E_r \mathbb{Q}[G]$ or, equivalently, from each $E_\chi \mathbb{C} \wedge^r X$ to $E_\chi \mathbb{C}[G]$. The last space is of dimension 1. Notice that if $\chi|_{G_{v_1}}$ is nontrivial then $E_\chi w_1 = 0$ in $\mathbb{CY}$. If $\chi|_{G_{v_1}} \equiv 1$ then $E_\chi w_1 \in E_\chi \mathbb{C}X$. Let $\{i_1 < \cdots < i_r\} = \{i : \chi|_{G_{v_1}} \equiv 1\}$. Then $E_\chi w_{i_1}, \ldots, E_\chi w_{i_r}$ is a basis of $E_\chi \mathbb{C}X$, and therefore the space $E_\chi \mathbb{C} \wedge^r X = \mathbb{C} E_\chi w_{i_1} \wedge \cdots \wedge w_{i_r}$ has dimension 1. Since $\Omega(E_\chi w_{i_1} \wedge \cdots \wedge w_{i_r}) = E_\chi$ we are done.

Let now $r = n - 1$ and

$$\Omega = (E_r - E_1) \sum_{i_1 < \cdots < i_r} w^*_{i_1} \wedge \cdots \wedge w^*_{i_r} + E_1(-1)^i \tilde{w}_i^* \bigg|_{E_r \wedge^r X}$$

where we denote $w^*_{i_1} \wedge \cdots \wedge w^*_{i_{r-1}} \wedge w^*_{i_{r+1}} \wedge \cdots \wedge w^*_{i_r}$ by $\tilde{w}_i^*$. We see that this map is independent of $i$ and it is an isomorphism from $E_r \mathbb{Q} \wedge^r X$ to $E_r \mathbb{Q}[G]$. We can tensor with $\mathbb{C}$ again, and if $\chi \neq 1$ then $\Omega$ is an isomorphism from each $E_\chi \mathbb{C} \wedge^r X$ to $E_\chi \mathbb{C}[G]$. $E_1 \mathbb{C} \wedge^r X$ is generated over $\mathbb{C}$ by $E_1(w_2 - w_1) \wedge \cdots \wedge (w_n - w_1) = E_1 \sum_{j} (-1)^{j-1} \tilde{w}_j$, where $\tilde{w}_j = w_1 \wedge \cdots \wedge w_{j-1} \wedge w_{j+1} \wedge \cdots \wedge w_n$. Since $E_1 \tilde{w}_i^*(\tilde{w}_j) = E_1$ when $i = j$ and 0 otherwise, we are done.

With $\Omega$ constructed above we have an invertible map

$$\Omega \circ \log^{(r)} : E_r \mathbb{R} \wedge^r U \overset{\sim}{\longrightarrow} E_r \mathbb{R}[G].$$

Notice that the property of $\varepsilon_r$ can be reformulated as

$$c(\chi) = \overline{\chi}(\Omega \circ \log^{(r)}(\varepsilon_r))$$

for all $\chi$ with $r(\chi) = r$. Since $\Theta_r = \sum_{r(\chi) = r} c(\chi) E_\chi$ is a unique element in $E_r \mathbb{R}[G]$ with the property $c(\chi) = \overline{\chi}(\Theta_r)$ for all such $\chi$, then $\varepsilon_r = (\Omega \circ \log^{(r)})^{-1} \Theta_r$ is a unique element satisfying the conditions.

Since $\Omega : E_r \mathbb{R} \wedge^r X \longrightarrow E_r \mathbb{R}[G]$ is an isomorphism and it maps $E_r \mathbb{Q} \wedge^r X$ to $E_r \mathbb{Q}[G]$, it also maps $\Theta_r \mathbb{Q} \wedge^r X$ to $\Theta_r \mathbb{Q}[G]$. Hence Conjecture $\text{[H]}$ is equivalent to the statement that the isomorphism $\text{[H]}$ maps $E_r \mathbb{Q} \wedge^r U$ to $\Theta_r \mathbb{Q}[G]$. If the conjecture is true then $\varepsilon_r \in E_r \mathbb{Q} \wedge^r U$. Suppose conversely that $\varepsilon_r \in E_r \mathbb{Q} \wedge^r U$. Then $(\Omega \circ \log^{(r)})^{-1}(\Theta_r \mathbb{Q}[G]) = \cdots$
\(\mathbb{Q}[G] \varepsilon_r \subset E_r \mathbb{Q} \wedge^r U\). In fact \(\mathbb{Q}[G] \varepsilon_r = E_r \mathbb{Q} \wedge^r U\) since these \(\mathbb{Q}\)-vector spaces have the same dimension \(\#C_r\). Thus the conjecture is true. \(\Box\)

We consider this element \(\varepsilon_r\) (conjecturally belonging to \(E_r \mathbb{Q} \wedge^r U\)) as a generalization of the “Stark unit”. Similarly to the latter, \(\varepsilon_r\) is defined via units modulo torsion and it is unique as soon as some place of the bigger field \(K\) over each place of the base field \(k\) is fixed.

There is a (certainly non-unique) way to write \(c(\chi)\) as a single determinant. Let us show it for \(\chi \neq 1\). Consider \(E = \sum_{\alpha \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} E_{\chi^\alpha}\) and let \(\{i_1 < \cdots < i_r\} = \{i : \chi|_{G_{v_i}} \equiv 1\}\). Since \(E \mathbb{Q} \wedge^r X\) is generated over \(\mathbb{Q}[G]\) by the element \(E w_{i_1} \wedge \cdots \wedge E w_{i_r}\), each element \(x \in E \mathbb{Q} \wedge^r X\) can be represented (non-uniquely) in the form \(x = x_1 \wedge \cdots \wedge x_r\) with some \(x_i \in E \mathbb{Q} X\). Since \(QU \cong \mathbb{Q} X\) as \(G\)-modules, we can represent

\[
E \varepsilon_r = \epsilon_1 \wedge \cdots \wedge \epsilon_r
\]

with some (non-unique) \(\epsilon_i \in E QU\). Then

\[
c(\chi) = \frac{1}{\prod_{j=1}^r \#G_{v_{i_j}}} \det \left( \sum_{g \in G} \chi(g) \log |g \epsilon_k|_{w_{i_j}} \right)_{1 \leq k, j \leq r}.
\]

3. Example with the Hilbert class field of \(\mathbb{Q}(\sqrt{229})\).

The field \(k = \mathbb{Q}(\sqrt{229})\) has class number \(h = 3\). To construct its Hilbert class field together with some units in it let us consider the polynomial

\[
x^3 - 4x + 1.
\]

Let \(F\) be its splitting field. It can have degree 3 or 6 over \(\mathbb{Q}\). Since this equation has discriminant \(d = 229\), the splitting field should contain \(k\). Hence \(K\) is of degree 6 with \(\text{Gal}(K/\mathbb{Q}) \cong S_3\), and \([K : k] = 3\).

We claim that \(K\) is the Hilbert class field of \(k\). Let us show \(K/k\) is an unramified extension. Let \(F = \mathbb{Q}(x^3 - 4x + 1)\). This cubic field is ramified at 229 only, so \(K\) is also ramified at 229 only since it is the compositum of \(k\) and \(F\). Further,

\[
x^3 - 4x + 1 = (x - 29)^2(x - 171) \pmod{229}
\]

implies that the decomposition of (229) into primes in \(O_F\) has form \(p^2b\). Then \(pO_K = \mathfrak{p}\) should be prime and \(bO_K = \mathfrak{b}^2\) for a prime \(\mathfrak{b}\). This is because \(K\) is a Galois field, so in the decomposition of (229) = \(\mathfrak{p}^2\mathfrak{b}^2\) all primes should have the same power. This power is the ramification index \(e_K(229) = 2\). We have \(e_K(229) = e_k(229) = 2\), therefore \(K/k\) is unramified.
Let $U = O_\kappa^x$ and $x \in U$ be the root of [5]. Notice that $x - 2$ is also a unit then. Indeed, $(x - 2)^3 + 6(x - 2)^2 + 8(x - 2) + 1 = x^3 - 4x + 1 = 0$. We could also take some unit $\epsilon \in O_\kappa^x$, and it is easy to check that $\epsilon$ together with the conjugates of $x$ and $x - 2$ under $G = \text{Gal}(K/k)$ generate the subgroup of finite index in $U$. Let $E_1 = \frac{1}{3} \sum_{g \in G} g$ and $E_2 = 1 - E_1$. Then $\epsilon \in E_1 U$ and $x, x - 2 \in E_2 U$. Thus the element
\[
x \land (x - 2)
\]
generates $E_2 \mathbb{Q} \land ^2 U$ over $E_2 \mathbb{Q}[G]$.

We also need the Stickelberger element $\Theta_2$. We compute it using the Kronecker limit formula for real quadratic fields invented by Hecke. Expansion at $s = 0$ of the partial zeta function $\zeta(g, s)$ (for $g \in G$) starts as
\[
\zeta(g, s) = -\frac{\log(\epsilon)}{2} s + \rho(g)s^2 + o(s^2), \quad s \to 0
\]
where $\epsilon = 15.066372975210...$ is the fundamental unit of $k$ and the coefficient $\rho(g)$ can be computed as follows. We take a quadratic form $Q$ of discriminant 229 which represents the class of ideals mapped to $g \in G$ by the Artin map from the class field theory. Let $x > x'$ be its roots. Then
\[
\rho(g) = -\frac{1}{4} \int_{-\log(\epsilon)}^{\log(\epsilon)} \log \left( y(v) | \eta(z(v)) \right) dv + \text{const}
\]
where $z(v) = \frac{xe^{\pi i/2} - x'e^{-\pi i/2}}{e^{\pi i/2} - e^{-\pi i/2}}$, $y(v)$ is imaginary part of $z(v)$ and the constant is independent of $g$. This follows from the functional equation and formula for expansion near $s = 1$ given with similar integrals in [3] (page 90) or [4] (page 162). The unity $1 \in G$ can be represented by the form $Q_0 = x^2 - 15x - 1$, and the other elements (each of which generates $G$) by $Q_1 = 3x^2 - 11x - 9$ and $Q_2 = 9x^2 - 11x + 3$. The integral $\int_{-\log(\epsilon)}^{\log(\epsilon)} \log(\eta(z)^4) dv$ equals $-16.230647798060277...$ for $Q_0$. Since $\zeta(g, s) = \zeta(g^{-1}, s)$, the value of this integral for both $Q_1$ and $Q_2$ is the same number $-6.808829434682019...$ Therefore we find
\[
\Theta_2 = E_2 \sum_{g \in G} \rho(g)g = 1.570303060563043... \cdot (1 - g/2 - g^2/2)
\]
where $g \in G$ is any of the two generators in the last expression.

Let us enumerate the roots of [5] as $x_0 = -2.1149075414...$, $x_1 = 0.2541016883...$, $x_2 = 1.8608058531...$. Let $w_{1,2} : K \to \mathbb{R}$ be the embeddings in which $x$ takes the value $x_0$ and $\sqrt{229}$ is positive and negative correspondingly. Let $g$ be a generator of $G$ such that $x$ take the value $x_1$ under $gw_1$. Since $\text{Gal}(K/\mathbb{Q})$ acts as the permutation group $S_3$ on
the set \((x_0, x_1, x_2)\) and \(G\) is the subgroup of even permutations (cyclic shifts in this case), we see that \(x\) will take the values \(x_2, x_1\) and \(x_0\) under \(g^2w_1, gw_2\) and \(g^2w_2\) correspondingly. Hence

\[
\log^{(2)}(x \wedge (x - 2)) = \det \begin{pmatrix}
\sum_{i=0}^{2} \log |x_i| g^i & \sum_{i=0}^{2} \log |x_i| g^{-i} \\
\sum_{i=0}^{2} \log |x_i + 2| g^i & \sum_{i=0}^{2} \log |x_i + 2| g^{-i}
\end{pmatrix} w_1 \wedge w_2
\]

\[
= 7.066363772533693\ldots \cdot (g - g^2) w_1 \wedge w_2
\]

Let \(\Omega = w_1^* \wedge w_2^*\) and

\[
\Omega \left( \log^{(2)}(x \wedge (x - 2)) \right) = 7.066363772533693\ldots \cdot (g - g^2).
\]

Thus \(\Theta_2 = -1.570303060563043\ldots \cdot \frac{(g - g^2)^2}{2} = \frac{1}{9}(g^2 - g) \Omega \left( \log^{(2)}(x \wedge (x - 2)) \right)\) and

\[
\varepsilon_2 = \frac{1}{9}(g^2 - g) x \wedge (x - 2) = \frac{1}{9} \left( \frac{x^9}{x^g} \right) \wedge (x - 2).
\]

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