On Riedtmann’s Lie algebra of the gentle one-cycle algebra \( \Lambda(n - 1, 1, 1) \)

Hui Chen\(^a\) and Dong Yang\(^a\)

\(^a\)Department of Mathematics, Nanjing University, Nanjing, P. R. China; \(^b\)School of Biomedical Engineering and Informatics, Nanjing Medical University, Nanjing, P. R. China

ABSTRACT
An extended version of Riedtmann’s Lie algebra of the gentle one-cycle algebra \( \Lambda(n - 1, 1, 1) \) is computed and is shown to admit a Cartan decomposition by the positive roots of the root system of type \( BC_n \).

ARTICLE HISTORY
Received 21 February 2023
Revised 13 July 2023
Communicated by K. Misra

KEYWORDS
Euler form; Riedtmann Lie algebra; Root system of type \( BC \)

2020 MATHEMATICS SUBJECT CLASSIFICATION
16G20; 17B37

1. Introduction
The gentle one-cycle algebra \( \Lambda(n - 1, 1, 1) \) is defined as the quotient of the path algebra of

\[
1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n \circlearrowleft \alpha
\]

by the ideal generated by \( \alpha^2 \). This algebra is known to be derived-discrete [3, 17], and in recent years it has attracted much attention mainly because it appears in the theory of cluster tubes [6, 7, 16, 19, 20]. It is of particular interest that on the one hand due to [2], the support \( \tau \)-tilting theory of this algebra can be used to model the cluster combinatorics of type \( B_n \), and on the other hand, a Caldero-Chapoton map is defined in [6, 7, 20] to realise the cluster algebra of type \( C_n \). Moreover, the algebra is Iwanaga–Gorenstein of dimension 1 and is used to realise the positive part of the simple complex Lie algebra of type \( C_n \), as an example of the general theory on constructing non-simply-laced Lie algebras in [9].

This paper is concerned with the structure of Riedtmann’s Lie algebra \( L(A) \) of \( A = \Lambda(n - 1, 1, 1) \), which is the free \( \mathbb{Z} \)-module with basis the isomorphism classes of indecomposable \( A \)-modules endowed with the Lie bracket with structure constants given by the Euler characteristics of suitable varieties of submodules. The algebra \( A \) has only finitely many isomorphism classes of indecomposable \( A \)-modules and a classification is given in [4]. This helps us to obtain a complete description of the Lie bracket of \( L(A) \) (Proposition 4.2). Moreover, we introduce a symmetric bilinear form on the Grothendieck group of \( A \) in terms of the Cartan matrix of \( A \) (Definition 3.2). It turns out to be a modified symmetric Euler form\(^1\) (Theorem A1). This form allows us to establish a Gabriel’s theorem for \( A \) (Theorem 3.3): taking dimension vectors defines a surjective map from the set of isomorphism classes of indecomposable \( A \)-modules to the set of positive roots of the root system of type \( BC_n \), which sends the symmetric Euler form to the natural bilinear form on roots. We also use this form to extend the Riedtmann Lie algebra \( L(A) \) by adding an abelian Lie algebra given by the Grothendieck group of \( A \), following [13, 18].

CONTACT Hui Chen huichen@njmu.edu.cn Department of Mathematics, Nanjing University, Nanjing 210093, P. R. China.
Dedicated to Jie Xiao on the occasion of his 60th birthday.
\(^1\)Note that the usual Euler form is not well-defined since \( A \) has infinite global dimension.

\( © \ 2023 \) Taylor & Francis Group, LLC
The next step is to describe the extended Riedtmann’s Lie algebra $\tilde{L}(A)$ by generators and relations. We introduce a complex Lie algebra $g$, which is generated by $x_1, x_2, \ldots, x_{n-1}, x_n, x'_n, h_1, \ldots, h_n$ subject to the following relations:

- \[ \{x_1, \ldots, x_{n-1}, x_n, h_1, \ldots, h_{n-1}, h_n\} \text{ satisfy the Serre relations for the Borel subalgebra } b_B \text{ of the simple complex Lie algebra of type } B; \]
- \[ \{x_1, \ldots, x_{n-1}, x'_n, h_1, \ldots, h_{n-1}, \frac{1}{2} h_n\} \text{ satisfy the Serre relations for the Borel subalgebra } b_C \text{ of the simple complex Lie algebra of type } C; \]
- \[ [x_n, x'_n] = 0 \text{ and } [[x_{n-1}, x_n], x'_n] = 0. \]

Then $g = b_B + b_C$. Moreover, the space $h = \text{span}\{h_1, \ldots, h_n\}$ is a Cartan subalgebra of $g$ and the corresponding Cartan decomposition of $g$ is given by the positive roots of the root system of type $BC_n$ (Theorem 2.4).

Let $S_1, \ldots, S_n$ be the simple $A$-modules supported on the vertices $1, \ldots, n$, respectively, and let $S'_n$ be the unique 2-dimensional indecomposable $A$-module supported on the vertex $n$. Let $h_{S_1}, \ldots, h_{S_n}, h_{S'_n}$ be the image of $S_1, \ldots, S_n, S'_n$ in the Grothendieck group, respectively. The following theorem is our main result (see Theorem 4.1).

**Theorem 1.1.** The assignment $h_i \mapsto h_{S_i}(1 \leq i \leq n-1), \frac{1}{2} h_n \mapsto h_{S_n}, x_i \mapsto S_i(1 \leq i \leq n), x'_n \mapsto S'_n$ extends to an isomorphism $g \longrightarrow \tilde{L}(A) \otimes \mathbb{Z} \mathbb{C}$ of complex Lie algebras.

Therefore, the Lie subalgebra of $\tilde{L}(A)$ generated by $S_1, \ldots, S_{n-1}, S_n, h_{S_1}, \ldots, h_{S_{n-1}}, 2h_n(=h_{S_n})$ is isomorphic to $b_B$ and the Lie subalgebra generated by $S_1, \ldots, S_{n-1}, S'_n, h_{S_1}, \ldots, h_{S_{n-1}}, \frac{1}{2} h_n(=h_{S_n})$ is isomorphic to $b_C$. We remark that the Lie subalgebra generated by $S_1, \ldots, S_{n-1}, S'_n$ is exactly the Lie algebra $\mathcal{P}(\Lambda)$ in [9].

The structure of the paper is as follows. In Section 2 we recall the root system of type $BC$, introduce the Lie algebra $g$ and study its Cartan decomposition. In Section 3 we introduce a symmetric bilinear form for the algebra $\Lambda(n-1,1,1)$ and establish a Gabriel’s theorem. In Section 4 we describe the Lie bracket for the extended Riedtmann’s Lie algebra $\tilde{L}(A)$ and prove the isomorphism between $g$ and $\tilde{L}(A)$. In the appendix we show that the symmetric bilinear form introduced in Section 3 is a modified symmetric Euler form.

### 2. A Lie algebra of type $BC^+$

In this section we introduce a complex Lie algebra, which admits a Cartan decomposition by the positive roots of the root system of type $BC$.

#### 2.1. The root system of type $BC$

Let $\mathbb{R}^n$ be the $n$-dimensional real space with the usual metric $(-, -)$, and $e_1, \ldots, e_n$ be the natural basis. In classical Lie theory, the root systems of type $B_n$, $C_n$ and $BC_n$ are (see Bourbaki [5, Chapter VI, Sections 4.5, 4.6, and 4.14])

- $\Phi_B = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}$,
- $\Phi_C = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid 1 \leq i \leq n\}$,
- $\Phi_{BC} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\} \cup \{\pm 2e_i \mid i = 1, \ldots, n\}$.

Clearly $\Phi_{BC} = \Phi_B \cup \Phi_C$. The corresponding simple roots are

- $\Delta_B = \{e_i - e_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{e_n\}$,
- $\Delta_C = \{e_i - e_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{2e_n\}$,
- $\Delta_{BC} = \{e_i - e_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{e_n\}$. 

The corresponding positive roots are
\[ \Phi^+_B = \{ \epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n \} \cup \{ \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n \} \cup \{ \epsilon_i \mid 1 \leq i \leq n \}, \]
\[ \Phi^+_C = \{ \epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n \} \cup \{ \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n \} \cup \{2\epsilon_i \mid 1 \leq i \leq n \}, \]
\[ \Phi^+_{BC} = \{ \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n \} \cup \{ \epsilon_i \mid 1 \leq i \leq n \} \cup \{2\epsilon_i \mid 1 \leq i \leq n \}. \]

In the rest of this paper we will use \( \Phi, \Phi^+ \) and \( \Delta \) to denote \( \Phi_{BC}, \Phi^+_{BC} \) and \( \Delta_{BC} \) for simplicity.

### 2.2. The Borel subalgebras of the Lie algebras of type B and type C

In this section, we review some basic results on the Borel subalgebras of the simple complex Lie algebras of type \( B_n \) and type \( C_n \). Most of the results in this subsection are from J. E. Humphreys [10] and V. Kac [11], but we will change some notations.

#### 2.2.1. The Borel subalgebra of the Lie algebra of type B

Let \( b_B \) be the Borel subalgebra of the simple complex Lie algebra of type \( B_n \), i.e., it is the complex Lie algebra generated by \( \{ x_i, h_i \mid 1 \leq i \leq n \} \) with relations:

(B1) \[ [h_i, h_j] = 0, 1 \leq i, j \leq n. \]

(B2) \[ [h_i, x_j] = \begin{cases} 2x_j, & 1 \leq i = j \leq n; \\ -x_j, & |i - j| = 1 \text{ but } i \neq n; \\ -2x_j, & j = n - 1 \text{ and } i = n; \\ 0, & \text{otherwise}. \end{cases} \]

(B3) When \( i \neq j, \)

\[ \begin{cases} [x_i, [x_i, x_j]] = 0, & |i - j| = 1 \text{ but } i \neq n; \\ [x_i, [x_i, [x_i, x_j]]] = 0, & j = n - 1 \text{ and } i = n; \\ [x_i, x_j] = 0, & \text{otherwise}. \end{cases} \]

Let \( h \) be the subspace of \( b_B \) spanned by \( h_1, \ldots, h_n \) and \( g_B^+ \) be the Lie subalgebra generated by \( x_1, \ldots, x_n \). Then \( h \) is a maximal abelian subalgebra of \( b_B, b_B = h \oplus g_B^+ \) and \( g_B^+ \) admits the decomposition \( g_B^+ = \bigoplus_{\mu \in \Phi^+_B} g_B(\mu), \)

where \( g_B(\mu) = \{ x \in b_B \mid [h, x] = \mu(h)x, \forall h \in h \}, \)

and the matrix \( (\varepsilon_i(h_i))_{i,j} \)

\[
\begin{pmatrix}
 1 \\
-1 & 1 \\
& & \ddots & \ddots \\
& & & -1 & 1 \\
& & & & & -1 & 2
\end{pmatrix}
\]

Precisely, putting \( x_{i,i} = x_i \) for \( 1 \leq i \leq n \) and \( x_{i,j} = [x_{i,j-1}, x_j] \) for \( 1 \leq i < j \leq n \), we have:

1. \( g_B(\varepsilon_i - \varepsilon_j)(1 \leq i < j \leq n) \) is 1-dimensional with basis \( x_{i,j-1} \).
2. \( g_B(\varepsilon_i)(1 \leq i \leq n) \) is 1-dimensional with basis \( x_{i,n} \).
3. \( g_B(\varepsilon_i + \varepsilon_j)(1 \leq i < j \leq n) \) is 1-dimensional with basis \( x_{i,n}, x_{j,n} \).

#### 2.2.2. The Borel subalgebra of the Lie algebra of type C

Let \( b_C \) be the Borel subalgebra of the simple complex Lie algebra of type \( C_n \), i.e., it is the complex Lie algebra generated by \( \{ x'_i, h'_i \mid 1 \leq i \leq n \} \) with relations:

(C1) \[ [h'_i, h'_j] = 0, 1 \leq i, j \leq n. \]
Precisely, putting

\[ [h_i', x_j'] = \begin{cases} 
2x_j', & 1 \leq i = j \leq n; \\
-x_j', & |i-j|=1 \text{ but } j \neq n; \\
-2x_j', & j=n \text{ and } i=n-1; \\
0, & \text{otherwise.}
\]

(C3) When \( i \neq j \),

\[
\begin{cases}
[x_i', [x_i', x_j']] = 0, & |i-j|=1 \text{ but } j \neq n; \\
[x_i', [x_i', x_j']] = 0, & j = n \text{ and } i = n-1; \\
[x_i', x_j'] = 0, & \text{otherwise.}
\end{cases}
\]

Let \( h' \) be the subspace of \( b_C \) spanned by \( h_1', \ldots, h_n' \) and \( g_C^+ \) be the Lie subalgebra generated by \( x'_1, \ldots, x'_n \). Then \( h' \) is a maximal abelian subalgebra of \( b_C \), \( b_C = h \oplus g_C^+ \) and \( g_C^+ \) admits the decomposition \( b_C^+ = \bigoplus_{\mu \in \Phi_C^+} g_C(\mu) \), where

\[ g_C(\mu) = \{ x \in b_C | [h', x] = \mu(h')x, \forall h' \in h' \}, \]

and the matrix \((\varepsilon_j(h'))_{ij}\) is

\[
\begin{pmatrix}
1 & 1 & & & \\
-1 & 0 & & & \\
& & \ddots & & \\
& & & -1 & 1 \end{pmatrix}
\]

Precisely, putting \( x'_{ij} = x'_{ij} \) for \( 1 \leq i \leq n \) and \( x'_{ij} = [x'_{ij-1}, x'_{ij}] \) for \( 1 \leq i < j \leq n \), we have

1. \( g_C(\varepsilon_i - \varepsilon_j)(1 \leq i < j \leq n) \) is 1-dimensional with basis \( x'_{ij-1} \).
2. \( g_C(2\varepsilon_i)(1 \leq i < n) \) is 1-dimensional with basis \( [x'_{i,n-1}, x'_{i,n}] \) and \( g_C(2\varepsilon_n) \) is 1-dimensional with basis \( x'_{n,n} \).
3. \( g_C(\varepsilon_i + \varepsilon_j)(1 \leq i < j < n) \) is 1-dimensional with basis \( [x'_{j,n-1}, x'_{j,n}] \) and \( g_C(\varepsilon_i + \varepsilon_n)(1 \leq i < n) \) is 1-dimensional with basis \( x'_{i,n} \).

### 2.3. A Lie algebra of type BC

**Definition 2.1.** Let \( g \) be the complex Lie algebra generated by \( x_1, x_2, \ldots, x_{n-1}, x_n, x_n', \) and \( h_1, \ldots, h_n \) satisfying the following relations:

\[
\begin{align*}
(C1) \quad & \{x_1, \ldots, x_n, h_1, \ldots, h_n\} \ \text{satisfy the relations (B1), (B2) and (B3) as in Section 2.2.1;} \\
(C2) \quad & \{x'_1, \ldots, x'_n, h'_1, \ldots, h'_n\} \ \text{satisfy the relations (C1), (C2) and (C3) as in Section 2.2.2, where } x'_i = x_i, \\
& \text{and } h'_i = h_i \text{ for } 1 \leq i \leq n - 1, \text{ and } h'_n = \frac{1}{2} h_n; \\
(C3) \quad & [x_n, x'_n] = 0 \text{ and } [[x_{n-1}, x_n], x'_n] = 0.
\end{align*}
\]

It is clear that \( b_B \) and \( b_C \) are Lie subalgebras of \( g \), and that \( h = \text{span}(h_1, \ldots, h_n) \) is a maximal abelian subalgebra of \( g \). Denote by \( g^+ \) the Lie subalgebra of \( g \) generated by \( x_1, \ldots, x_n, x_n' \). Then \( g_B^+ \) and \( g_C^+ \) are Lie subalgebras of \( g^+ \). We keep the notation in Sections 2.2.1 and 2.2.2 for basis elements of \( g_B^+ \) and \( g_C^+ \).

**Lemma 2.2.** \( g = h \oplus g^+ \), and \( g^+ = g_B^+ + g_C^+ \).
Proof. We need to prove \([g_B^+, x_i^n] \subseteq g_B^+ + g_C^-\) and \([g_C^+, x_i^n] \subseteq g_B^+ + g_C^-\). We only show the first inclusion since the second one is similar. Precisely, we show \([x, x_i^n] \in g_C^-\) for basis elements \(x\) of \(g_B^+\).

Case 1: \(x = x_{i,j-1}\) with \(1 \leq i < j \leq n\). The element \([x_{i,j-1}, x_i^n]\) is generated by \(x_1, \ldots, x_{n-1}, x_i^n\), and hence belongs to \(g_C^-\).

Case 2: \(x = x_{i,n}\) with \(1 \leq i \leq n\). We claim that \([x, x_i^n] = 0\). For \(i = n - 1\) and \(i = n\) this is the condition (BC3). For \(1 \leq i \leq n - 2\) we have

\[
[x_{i,n}, x_i^n] = \left[[[x_{i,n-2}, x_{i,n-1}], x_i^n], x_i^n\right] = \left[[[x_{i,n-2}, [x_{i,n-1}, x_i^n]], x_i^n\right] + \left[[x_{i,n-1}, x_i^n], x_i^n\right] = \left[[x_{i,n-2}, [x_{i,n-1}, x_i^n]], x_i^n\right] + \left[[x_{i,n-1}, x_i^n], [x_i^n, x_{i,n-2}]\right]
\]

For the third equality notice that \([x_{i,n}, x_{i,n-2}] \in g_B^- (\epsilon_n + (\epsilon_i - \epsilon_{n-1})) = 0\). The last equality holds because (BC3) holds and \([x_i^n, x_{i,n-2}] \in g_C^-(2\epsilon_n + (\epsilon_i - \epsilon_{n-1})) = 0\).

Case 3: \(x = [x_{i,n}, x_{j,n}]\) with \(1 \leq i < j \leq n\). Then \([x, x_i^n] = 0\), due to Case 2.

The following result is a consequence of Lemma 2.2.

**Proposition 2.3.** The Lie algebra \(g\) has a basis

\[
\begin{align*}
(g1) & \quad x_{i,j}(1 \leq i \leq j \leq n), [x_i^n, x_{i,n}], (1 \leq i < j \leq n), x_i^n, (1 \leq i \leq n), [x_{i,n}, x_{i,n}], (1 \leq i \leq n - 1); \\
(g2) & \quad h_1, \ldots, h_n.
\end{align*}
\]

Consequently, the dimension of \(g\) is \(3n^2 + 3n\), and \(g = b_B + b_C\).

It follows from Lemma 2.2 that \(h\) is a Cartan subalgebra of \(g\). Then by Proposition 2.3 and the description of the root spaces in Section 2.2, we obtain the following Cartan decomposition of \(g\).

**Theorem 2.4.** \(g\) has the decomposition \(g = \bigoplus_{\mu \in \Phi^+ \cup \{0\}} g(\mu)\), where \(g(\mu) = \{x \in g \mid [h, x] = \mu(h)x, \forall h \in h\}\) satisfies

- \(g(0)\),
- \(g(\epsilon_i - \epsilon_j)(1 \leq i < j \leq n)\) is 1-dimensional with basis \(x_{i,j-1}\);
- \(g(\epsilon_i)(1 \leq i \leq n)\) is 1-dimensional with basis \(x_{i,n}\);
- \(g(2\epsilon_i)(1 \leq i \leq n)\) is 1-dimensional with basis \([x_{i,n-1}, x_{i,n}]\) and \(g(2\epsilon_n)\) is 1-dimensional with basis \(x_i^n, x_{i,n}\);
- \(g(\epsilon_i + \epsilon_j)(1 \leq i < j < n)\) is 2-dimensional with basis \([x_{i,n}, x_{j,n}]\) and \([x_{i,n-1}, x_{i,n}]\); and \(g(\epsilon_i + \epsilon_n)(1 \leq i < n)\) is 2-dimensional with basis \([x_{i,n}, x_{n}, x_i^n]\) and \(x_i^n\).

**Corollary 2.5.** The quotient of \(g\) by the Lie ideal generated by \(x_i^n\) (respectively, by \(x_n\)) is isomorphic to \(b_B\) (respectively, \(b_C\)).

**Proof.** By the proof of Lemma 2.2, the Lie ideal generated by \(x_i^n\) has a basis \(x_i^n, \ldots, x_{i,n} (1 \leq i \leq n)\), \([x_{i,n-1}, x_{i,n}]\) \((1 \leq i \leq j \leq n - 1)\). Similarly, the Lie ideal generated by \(x_n\) has a basis \(x_{i,n}, \ldots, x_{n} (1 \leq i \leq n)\), \([x_{i,n}, x_{j,n}]\) \((1 \leq i < j \leq n)\). The desired result follows immediately.

3. **A Gabriel’s theorem for the algebra \(\Lambda(n - 1, 1, 1)\)**

In this section we establish a Gabriel’s theorem for the gentle one-cycle algebra \(\Lambda(n - 1, 1, 1)\).

Let \(K = C\). Consider the bound quiver \((Q, I)\)

\[
1 \to 2 \to \cdots \to n \to a, \quad a^2.
\]

The path algebra of this bound quiver is exactly the derived-discrete algebra \(\Lambda(n - 1, 1, 1)\) in the notation of [3]. In the rest of this paper we will denote this algebra by \(A\) and we will identify an \(A\)-module with a
representation of \((Q, I)\), i.e., a representation of \(Q\) satisfying the relation \(\alpha^2 = 0\). Let \(\text{rep}_K(Q, I)\) denote the category of finite-dimensional representations of \((Q, I)\). Consider the following representations:

- **\(U_{ij}\) for \(1 \leq j \leq i \leq n\):**

\[
\begin{array}{c}
0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow K \rightarrow 1 \rightarrow \cdots \rightarrow 1 \rightarrow K^2 \rightarrow \alpha \\
\downarrow j \hspace{1cm} \downarrow i \hspace{1cm} \downarrow n \hspace{1cm} \downarrow \alpha \nexists \end{array}
\]

- **\(U_{ij}\) for \(1 \leq i < j \leq n\):**

\[
\begin{array}{c}
0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow K \rightarrow 1 \rightarrow \cdots \rightarrow 1 \rightarrow K^2 \rightarrow \alpha \\
\downarrow i \hspace{1cm} \downarrow j \hspace{1cm} \downarrow n \hspace{1cm} \downarrow \alpha \nexists \end{array}
\]

- **\(V_i\) for \(1 \leq i \leq n\):**

\[
\begin{array}{c}
0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow K \rightarrow 1 \rightarrow \cdots \rightarrow 1 \rightarrow K \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \\
\downarrow i \hspace{1cm} \downarrow \alpha \nexists \end{array}
\]

- **\(W_{ij}\) for \(1 \leq i \leq j < n\):**

\[
\begin{array}{c}
0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow K \rightarrow 1 \rightarrow \cdots \rightarrow 1 \rightarrow K \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \\
\downarrow i \hspace{1cm} \downarrow j \hspace{1cm} \downarrow \alpha \nexists \end{array}
\]

Here, \(e_1 = \left(\begin{array}{c}1 \\ 0 \end{array}\right)\), \(e_2 = \left(\begin{array}{c}0 \\ 1 \end{array}\right)\), \(I = \left(\begin{array}{cc}1 & 0 \\ 0 & 1 \end{array}\right)\), and \(\alpha = \left(\begin{array}{cc}0 & 0 \\ 0 & 1 \end{array}\right)\).

**Theorem 3.1.** [4, Theorem 3.3] The representations \(U_{ij}\), \(V_i\) and \(W_{ij}\) form a complete set of representatives of the isoclasses of indecomposable objects in \(\text{rep}_K(Q, I)\).

The simple representations in \(\text{rep}_K(Q, I)\) are \(\{S_i = W_{ij}| 1 \leq i \leq n - 1\} \cup \{S_n = V_n\}\) and \(P_1 = U_{n,1}, \ldots, P_n = U_{n,n}\) form a complete set of pairwise non-isomorphic indecomposable projective representations of \(\text{rep}_K(Q, I)\). Thus the Cartan matrix of the algebra \(A\) in the sense of [1, Definition 3.7] is the \(n \times n\)-matrix

\[
C_A = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
1 & 1 & \cdots & 1 & 0 \\
2 & 2 & \cdots & 2 & 2 \\
\end{pmatrix}
\]

Recall that the **dimension vector** of a representation \(M\) of the quiver \(Q\) is defined to be the vector \(\dim M = (\dim_K M_1, \ldots, \dim_K M_n)\). Below we define a symmetric bilinear form on the Grothendieck group \(K_0(\text{rep}_K(Q, I))\) with integer values. In the appendix we will explain that it is a modified symmetric Euler form.
Definition 3.2. For any $M$ and $N$ in $\mathrm{rep}_K(Q, I)$, define
\[
(M, N)_A = (\dim M)(C_A^{-1} + C_A^{-T})(\dim N)^T.
\]

Then we obtain a similar result as Gabriel’s theorem [8]:

Theorem 3.3. (a) Let $D$ be the set of dimension vectors of indecomposable representations in $\mathrm{rep}_K(Q, I)$, and let $\alpha_i = \dim S_i$. There is a bijection: $D \rightarrow \Phi^+$ which takes $\alpha_i$ to $\varepsilon_i - \varepsilon_{i+1} (1 \leq i < n)$, $\alpha_n$ to $\varepsilon_n$ and which preserves addition. From now on, we identify $D$ with $\Phi^+$.

(b) The fiber of $\alpha \in \Phi^+$ under $\dim$ is:
\[
\begin{align*}
W_{ij-1}, & \quad \text{if } \alpha = \varepsilon_i - \varepsilon_j (1 \leq i < j \leq n); \\
V_i, & \quad \text{if } \alpha = \varepsilon_i (1 \leq i \leq n); \\
U_{ii}, & \quad \text{if } \alpha = 2\varepsilon_i (1 \leq i \leq n); \\
U_{ij} \text{ and } U_{ji}, & \quad \text{if } \alpha = \varepsilon_i + \varepsilon_j (1 \leq i < j \leq n).
\end{align*}
\]

(c) For any $M$ and $N$ in $\mathrm{rep}_K(Q, I)$, we have $(M, N)_A = (\dim M, \dim N)$.

Proof. We only need to prove (c). This is because the matrix
\[
C_A^{-1} + C_A^{-T} = \begin{pmatrix}
2 & -1 \\
-1 & 2 & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & 2 & -1 \\
& & & -1 & 1
\end{pmatrix}
\]
is exactly the matrix of the bilinear form $(-, -)$ with respect to the basis $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. \qed

4. Riedtmann’s Lie algebra for $\Lambda(n-1, 1, 1)$

In this section we compute an extended version of Riedtmann’s Lie algebra of $\Lambda(n-1, 1, 1)$ and show that it is isomorphic to the Lie algebra $g$ introduced in Section 2.3.

4.1. The extended Riedtmann’s Lie algebra

We adopt the notation in Section 3. Let $P(A)$ be a set of representatives of all isomorphism classes of objects of $\mathrm{rep}_K(Q, I)$. Let $\mathcal{H}(A)$ be the free $\mathbb{Z}$-module with basis $\{u_X \mid X \in P(A)\}$. According to [14] (see also [12, 15]), the following multiplication makes $\mathcal{H}(A)$ a $\mathbb{Z}$-algebra with identity $u_0$:
\[
u_X \cdot u_Y = \sum_{Z \in P(A)} \chi(V(X, Y; Z))u_Z,
\]
where $V(X, Y; Z) = \{0 \leq Z_1 \subseteq Z \mid Z_1 \cong Y, Z/Z_1 \cong X\}^2$ and $\chi(V(X, Y; Z))$ is the Euler characteristic of $V(X, Y; Z)$. This sum is finite because $\chi(V(X, Y; Z)) \neq 0$ implies that there is a short exact sequence $0 \rightarrow X \rightarrow Z \rightarrow Y \rightarrow 0$, and hence $\dim Z = \dim X + \dim Y$.

Let $I(A) = \{U_{ij} \mid 1 \leq j \leq i \leq n\} \cup \{W_{ij} \mid 1 \leq i < j \leq n\} \cup \{V_i \mid 1 \leq i \leq n\} \cup \{W_{ij} \mid 1 \leq i \leq j < n\}$, and let $L(A)$ be the $\mathbb{Z}$-submodule of $\mathcal{H}(A)$ spanned by $\{u_X \mid X \in I(A)\}$. Then according to [14, Corollary 2.3], $L(A)$ is a Lie subalgebra of $\mathcal{H}(A)$ with respect to the commutator $[u_X, u_Y] = u_X \cdot u_Y - u_Y \cdot u_X$. In the rest, for $X \in \mathrm{rep}_K(Q, I)$, we use $X$ instead of $u_X$ for simplicity.

Recall that there is a symmetric bilinear form $(-, -)_A$ on the Grothendieck group $K_0(\mathrm{rep}_K(Q, I))$ with integer values. For a representation $M \in \mathrm{rep}_K(Q, I)$ denote its image in $K_0(\mathrm{rep}_K(Q, I))$ by $h_M$. Let

\[\text{This is opposite to Riedtmann’s multiplication.}\]
4.2. The main result

Let \( n \) be the \( \mathbb{Z} \)-submodule of \( g \) spanned by the basis elements in Proposition 2.3 and \( n^+ \) the one spanned by the elements in (g1). They are Lie subalgebras of \( g, n \otimes \mathbb{Z} \mathbb{C} = g \) and \( n^+ \otimes \mathbb{Z} \mathbb{C} = g^+ \). The following is the main result of this paper. Let \( S_1 = W_{1,1}, \ldots, S_{n-1} = W_{n-1,n-1}, S_n = V_n, S_n = U_{n,n} \).

Theorem 4.1. (a) There is an injective homomorphism \( n^+ \rightarrow L(A) \) of Lie algebras over \( \mathbb{Z} \) which sends \( x_1, \ldots, x_{n-1}, x_n x_i^n \) to \( S_i, \ldots, S_{n-1}, S_n, S_n \). By extending the scalars, it becomes an isomorphism \( n^+ \otimes \mathbb{Z} \mathbb{C} \rightarrow L(A) \otimes \mathbb{C} \).

(b) The homomorphism \( n^+ \rightarrow L(A) \) in (a) extends to an injective homomorphism \( U(n^+) \rightarrow \mathcal{H}(A) \) of \( \mathbb{Z} \)-algebras. By extending the scalars, it becomes an isomorphism \( U(n^+) \otimes \mathbb{C} \rightarrow \mathcal{H}(A) \otimes \mathbb{C} \) of \( \mathbb{C} \)-algebras.

(c) The homomorphism \( n^+ \rightarrow L(A) \) in (a) extends to an injective homomorphism \( \varphi: n \rightarrow \tilde{L}(A) \) of Lie algebras over \( \mathbb{Z} \), by \( h_i \mapsto h_{n_i}(1 \leq i \leq n-1) \) and \( h_n \mapsto 2h_{n_n} \). By extending the scalars, it becomes an isomorphism \( n \otimes \mathbb{Z} \mathbb{C}[\frac{1}{2}] \rightarrow \tilde{L}(A) \otimes \mathbb{C}[\frac{1}{2}] \) of Lie algebras over \( \mathbb{C}[\frac{1}{2}] \).

To prove Theorem 4.1 we need some preparation. First we describe the Lie brackets of the basis elements of \( L(A) \).

Proposition 4.2. The following equalities hold in \( L(A) \):

(a) \([W_{ij}, W_{lm}] = \delta_{j+1,l}W_{ij,m} - \delta_{m+1,l}W_{ij,l} \) for \( 1 \leq i \leq j \leq n-1 \) and \( 1 \leq l \leq m \leq n-1 \),

(b) \([W_{ij}, V_l] = \delta_{j+1,l}V_{ij,l} \) for \( 1 \leq i \leq j \leq n-1 \) and \( 1 \leq l \leq n \),

(c) \([W_{ij}, U_{lm}] = \delta_{j+1,l}U_{ij,m} + \delta_{j+1,l}U_{ij,l} \) for \( 1 \leq i \leq j \leq n \) and \( 1 \leq l \leq m \leq n \),

(d) \([V_l, V_j] = U_{ij} - U_{ij,l} \) for \( 1 \leq i, j \leq n \),

where \( \delta_{ij} \) is the Kronecker symbol. For other pairs of indecomposable representations, the Lie bracket of them equals 0.

Proof. We only prove (c) and the proof of the other cases is similar. It is clear that \( \dim W_{ij} + \dim U_{lm} = \epsilon_i - \epsilon_{j+1} + \epsilon_i + \epsilon_m \) belongs to \( \Phi^+ \) if and only if \( l = j+1 \) or \( m = j+1 \). So \([W_{ij}, U_{lm}] = 0 \) unless \( l = j+1 \) or \( m = j+1 \), by Theorem 3.3(a).

Case 1: \( l = m = j+1 \). Then \( \dim W_{ij} + \dim U_{j+1,j+1} = \epsilon_i + \epsilon_{j+1}, \) so by Theorem 3.3(b) the element \([W_{ij}, U_{j+1,j+1}] \) is a linear combination of \( U_{ij+1} \) and \( U_{j+1,j} \). Now it is easy to see from the structures of \( W_{ij}, U_{j+1,j+1}, U_{ij+1} \) and \( U_{j+1,j} \) in Section 3 that neither \( U_{ij+1} \) nor \( U_{j+1,j} \) has a submodule isomorphic to \( W_{ij} \), and both \( U_{ij+1} \) and \( U_{j+1,j} \) have a unique submodule isomorphic to \( U_{j+1,j+1} \) with quotient isomorphic to \( W_{ij} \). So \([W_{ij}, U_{j+1,j+1}] = U_{ij+1} + U_{j+1,j} \).

Case 2: \( l = j+1 \neq m \). Then \( \dim W_{ij} + \dim U_{j+1,m} = \epsilon_i + \epsilon_m, \) so by Theorem 3.3(b) the element \([W_{ij}, U_{j+1,m}] \) is a linear combination of \( U_{ij} \) and \( U_{ij,m} \). Now it is easy to see from the structures of \( W_{ij}, U_{j+1,m}, U_{ij,m} \) and \( U_{ij,m} \) in Section 3 that neither \( U_{ij,m} \) nor \( U_{ij,m} \) has a submodule isomorphic to \( W_{ij} \). If \( i \neq m \), \( U_{ij,m} \) has a unique submodule isomorphic to \( U_{j+1,m} \) with quotient isomorphic to \( W_{ij} \), and \( U_{ij,m} \) has no submodule isomorphic to \( U_{j+1,m} \); if \( i = m \), \( V(W_{ij}, U_{j+1,i}; U_{ii}) = \mathbb{C} \) and \( \chi(V(W_{ij}, U_{j+1,i}; U_{ii})) = 1 \). So \([W_{ij}, U_{j+1,m}] = U_{ij,m} \).

Case 3: \( l \neq m = j+1 \). This is similar to Case 2. \( \square \)

Corollary 4.3. \( \{S_1, \ldots, S_{n-1}, S_n, S_n'\} \) is a set of generators of \( L(A) \otimes \mathbb{Z}[\frac{1}{2}] \).
Proof. We need to prove that any indecomposable representation $M$ is generated by $W_{1,1}, W_{2,2}, \ldots, W_{n-1,n-1}, V_n, U_{n,n}$.

Case 1: $M = W_{i,j}$, $1 \leq i \leq j \leq n - 1$. This follows from $[W_{i,i}, W_{i+1,j}] \overset{4.2(a)}{=} W_{i,j}$ by induction on $j$.

Case 2: $M = V_i$, $1 \leq i \leq n$. This follows from $[W_{i,i}, V_{i+1}] \overset{4.2(b)}{=} V_i$ by decreasing induction on $i$.

Case 3: $M = U_{i,j}$, $1 \leq i, j \leq n$. We first assume $i = n$. Then $[W_{n,n-1}, U_{n,n}] \overset{4.2(c)}{=} U_{n,i} + U_{j,n}$. Moreover, $[V_j, V_n] \overset{4.2(d)}{=} U_{n,j} - U_{j,n}$. So $U_{i,n} = \frac{[W_{n,n-1}, U_{n,n}] - [V_j, V_n]}{2}$ and $U_{n,j} = \frac{[W_{n,n-1}, U_{n,n}] + [V_j, V_n]}{2}$. This also solves the problem for $j = n$. Next we assume $i \neq n$ and $j \neq n$. Then $U_{i,j} \overset{4.2(c)}{=} [W_{i,n-1}, U_{n,j}]$. The proof finishes. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. (a) By Proposition 4.2, the assignment $x_1 \mapsto S_1, \ldots, x_{n-1} \mapsto S_{n-1}, x_n \mapsto S_n, x'_n \mapsto S'_n$ extends to a homomorphism $n^+ \to L(A)$ of Lie algebras over $\mathbb{Z}$. By extending the scalars, we obtain a homomorphism $n^+ \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right] \to L(A) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right]$ of Lie algebras over $\mathbb{Z}\left[\frac{1}{2}\right]$. By Corollary 4.3, this homomorphism is surjective, and hence bijective because both sides are free over $\mathbb{Z}\left[\frac{1}{2}\right]$ of rank $\frac{3n^2+n}{2}$. This implies that the homomorphism $n^+ \to L(A)$ is injective.

(b) We have the following chain of homomorphisms of $\mathbb{Z}$.-algebras:

$$U(n^+) \longrightarrow U(L(A)) \longrightarrow H(A).$$

The first homomorphism is injective by (a), and the second one is injective by [14, Proposition 3.1]. So the composition is injective. By extending the scalars we obtain a chain of injective homomorphisms of $\mathbb{Q}$.-algebras:

$$U(n^+) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow U(L(A)) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H(A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The first homomorphism is bijective by (a) and the second one is bijective by the proof of [14, Proposition 3.1].

(c) It is enough to prove $\varphi([h_i, x_j]) = [\varphi(h_i), \varphi(x_j)]$ and $\varphi([h_i, x'_n]) = [\varphi(h_i), \varphi(x'_n)]$, more precisely,

- $\varphi([h_i, x_j]) = [h_i, [x_j, S_{j}]] = [S_i, S_j]A_S$ for $1 \leq i \leq n - 1$ and $1 \leq j \leq n$;
- $\varphi([h_i, x'_n]) = [h_i, [x'_n, S'_n]] = [S_i, S'_n]A_{S'_n}$ for $1 \leq i \leq n - 1$;
- $\varphi([h_n, x_j]) = [2h_n, S_j] = 2(S_n, S_j)_A$ for $1 \leq j \leq n$;
- $\varphi([h_n, x'_n]) = [2h_n, S'_n] = 2(S_n, S'_n)A_{S'_n}$.

This is straightforward. □

Remark 4.4. On basis elements, the homomorphism in Theorem 4.1(a) is given by

$$x_{i,j-1} \mapsto W_{i,j-1} \quad (1 \leq i < j \leq n)$$
$$x_{i,n} \mapsto V_i \quad (1 \leq i \leq n)$$
$$[x_{i,n}, x_{j,n}] \mapsto U_{j,i} - U_{i,j} \quad (1 \leq i < j \leq n)$$
$$x'_n \mapsto U_{n,n}$$
$$x'_{i,n} \mapsto U_{i,n} + U_{n,i} \quad (1 \leq i \leq n - 1)$$
$$[x_{j,n-1}, x'_{i,n}] \mapsto U_{i,j} + U_{j,i} \quad (1 \leq i \leq j \leq n - 1).$$

We have the following corollary of Theorem 4.1, Corollary 2.5, and Remark 4.4.

Corollary 4.5. The subspace of $\tilde{L}(A) \otimes_{\mathbb{Z}} \mathbb{C}$ spanned by $U_{i,j} + U_{j,i} \quad (1 \leq i < j \leq n)$ is a Lie ideal and the corresponding quotient is isomorphic to $b_B$. The subspace of $\tilde{L}(A) \otimes_{\mathbb{Z}} \mathbb{C}$ spanned by $V_i \quad (1 \leq i \leq n)$, $U_{i,j} - U_{j,i} \quad (1 \leq i < j \leq n)$ is a Lie ideal and the corresponding quotient is isomorphic to $b_C$. 
Appendix

In this appendix we show that the bilinear form \((-,-)_A\) introduced in Section 3 is a modified symmetric Euler form for \(A\).

Since the global dimension of the algebra \(A\) is infinite, the usual Euler form \((M,N) = \sum_{p=0}^{\infty} (-1)^p \dim_K \Ext_A^p(M,N)\) is not well-defined. We modify it as follows. First define
\[
\langle M,N \rangle_t = \sum_{p=0}^{\infty} \dim_K \Ext_A^p(M,N)(-t)^p.
\]

This form is not additive but additive with respect to split short exact sequences.

**Theorem A1.** (a) For any \(M\) and \(N\) in \(\rep_K(Q,I)\), the power series \(\langle M,N \rangle_t\) is a rational function in \(t\) and \(t = 1\) is not a pole.
(b) The form \((-,-)_1\) is additive.
(c) The matrix of the form \((-,-)_1\) with respect to the basis \(\{S_1, \ldots, S_n\}\) is \(C^{-t}_A\).

By Theorem A1(b), the form \((-,-)_1\) can be considered as a bilinear form on \(K_0(\rep_K(Q,I))\). Restricted to the subgroup generated by the classes of \(S_1, \ldots, S_{n-1}, U_{n,0}\), it is exactly the Euler form \((-,-)_H\) in [9, Section 3.3] because these modules have projective dimension 1. Recall from Definition 3.2 that there is a bilinear form \((-,-)_A\).

**Corollary A2.** For any \(M\) and \(N\) in \(\rep_K(Q,I)\) we have
\[
(M,N)_A = \langle M,N \rangle_1 + \langle N,M \rangle_1
\]

**Proof.** By Theorem A1(c), the matrix of the symmetric form associated to \((-,-)_1\) is exactly \(C^{-1}_A + C^{-t}_A\).

We start to prove Theorem A1. We first make a table for \(\langle M,N \rangle_t\), where \(M\) and \(N\) are indecomposable. This is obtained by a direct computation on \(\Ext_A^p(M,N)\), which we omit.

**Lemma A3.** The following Table A1 gives the values of \(\langle M,N \rangle_t\) with column \(M\) and row \(N\).

Notice that in the table of Lemma A3 all the entries are polynomials in \(t\) but two, which are \(\sum_{p=0}^{\infty} (-t)^p = \frac{1}{1+t}\) and \(\sum_{p=1}^{\infty} (-t)^p = -\frac{t}{1+t}\). So Theorem A1(a) is proved and the form \(\langle M,N \rangle_1\) is well-defined.

**Corollary A4.** The following Table A2 gives the values of \(\langle M,N \rangle_1\) with column \(M\) and row \(N\).

The following proposition is Theorem A1(b).

**Proposition A5.** The form \((-,-)_1\) is additive, i.e., \(\forall M\) and \(N\) in \(\rep_K(Q,I)\),
\[
\langle M,N \rangle_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} (\dim_K M_i)(\dim_K N_j)\langle S_i, S_j \rangle_1.
\]

**Proof.** We may assume that \(M\) and \(N\) are indecomposable. We prove the equality for some typical cases by explicitly computing the values of both sides.

Case 1. \(M = U_{l,k}\) and \(N = U_{i,j}\) for \(i \leq k \leq l < j\). By Corollary A4 we have
\[
\text{LHS} = \langle U_{l,k}, U_{i,j} \rangle_1 = 0,
\]
and by direct computation,

\[
\text{RHS} = \sum_{a=k}^{n-1} \sum_{a'=l}^{n} \langle S_a, S_a' \rangle_1 + \sum_{a=k}^{n-1} \sum_{a'=l}^{n} \langle S_a, S_{a+1} \rangle_1 + \sum_{a=k}^{n} \sum_{a'=l}^{n-1} \langle S_a, S_{a'} \rangle_1 + \sum_{a=l}^{n} \sum_{a'=l}^{n} \langle S_a, S_{a'} \rangle_1
\]

\[
= \sum_{a=k}^{n} \langle S_a, S_a \rangle + \sum_{a=k}^{n-1} \langle S_a, S_{a+1} \rangle_1 + \sum_{a=j}^{n} \langle S_a, S_a \rangle_1 + \sum_{a=j}^{n} \langle S_{a-1}, S_a \rangle_1
\]

\[
+ \sum_{a=l}^{n} \langle S_a, S_a \rangle_1 + \sum_{a=0}^{n-1} \langle S_a, S_{a+1} \rangle_1 + \sum_{a=j}^{n} \langle S_a, S_a \rangle_1 + \sum_{a=j}^{n} \langle S_{a-1}, S_a \rangle_1
\]

\[
= [(n-k) \times 1 + \frac{1}{2}] + (n-k) \times (-1) + [(n-j) \times 1 + \frac{1}{2}] + (n-j+1) \times (-1)
\]

\[
+ [(n-l) \times 1 + \frac{1}{2}] + (n-l) \times (-1) + [(n-j) \times 1 + \frac{1}{2}] + (n-j+1) \times (-1)
\]

\[
= \frac{1}{2} + \frac{1}{2} - 1 + \frac{1}{2} + \frac{1}{2} - 1 = 0.
\]
Case 2. \( M = V_j \) and \( N = V_i \) for \( l < i \). By Corollary A4

\[
LHS = \langle V_j, V_i \rangle_1 = -\frac{1}{2}
\]

and on the other hand,

\[
RHS = \sum_{a=1}^{n} \sum_{a'=i}^{n} \langle S_a, S_{a'} \rangle_1 = \sum_{a=i}^{n} \langle S_a, S_a \rangle_1 + \sum_{a=i}^{n} \langle S_{a-1}, S_a \rangle_1 = [(n-i) \times 1 + \frac{1}{2}) + (n-i+1) \times (-1)] = \frac{1}{2} - 1 = -\frac{1}{2}.
\]

Case 3. \( M = W_{lk} \) and \( N = W_{ij} \) for \( l < i \leq k + 1 \leq j < n \). By Corollary A4

\[
LHS = \langle W_{lk}, W_{ij} \rangle = -1,
\]
and by direct computation,

$$\text{RHS} = \sum_{a=l}^{k} \sum_{a'=l}^{j} \langle S_a, S_{a'} \rangle_1 = \sum_{a=l}^{k} (S_a)_1 + \sum_{a=l}^{k} \langle S_{a-1}, S_a \rangle_1 + \langle S_k, S_{k+1} \rangle_1 \quad (\text{note: } k + 1 < n)$$

$$= (k - i + 1) \times 1 + (k - i + 1) \times (-1) + (-1) = -1.$$

The proof of the other cases is similar and we obtain the additivity.

Finally, by Corollary A4 the matrix of the form $\langle M, N \rangle_1$ with respect to the basis $\{S_1, \ldots, S_n\}$ is:

$$\begin{pmatrix}
1 & -1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & -1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & -1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & -1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 1/2
\end{pmatrix},$$

which is exactly $C_{-1}^{-t}$. This is Theorem A1(c).

Acknowledgments

The authors thank Bangming Deng and Changjian Fu for answering their questions. They thank Changjian Fu for reading a preliminary version.

Funding

The second author acknowledges support by the National Natural Science Foundation of China No.12031007.

References

[1] Assem, I., Simson, D., Skowroński, A. (2006). *Elements of the Representation Theory of Associative Algebras*. Vol. 1. London Mathematical Society Student Texts, Vol. 65. Cambridge: Cambridge University Press. Techniques of Representation Theory.

[2] Bakke Buan, A., Marsh, R. J., Vatne, D. F. (2010). Cluster structures from 2-Calabi–Yau categories with loops. *Math. Z.* 265(4):951–970.

[3] Bobiński, G., Geiß, C., Skowroński, A. (2004). Classification of discrete derived categories. *Cent. Eur. J. Math.* 2(1):19–49.

[4] Boos, M., Reineke, M. (2012). B-orbits of 2-nilpotent matrices and generalizations. In: Joseph, A., Melenikov, A., Penkov, I., eds. *Highlights in Lie Algebraic Methods*. Progress in Mathematics, Vol. 295. New York: Birkhäuser/Springer, pp. 147–166.

[5] Bourbaki, N. (1968). *Eléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1337, Hermann, Paris.

[6] Fu, C., Geng, S., Liu, P. (2020). Cluster algebras arising from cluster tubes ii: the caldero-chapoton map. *J. Algebra* 544:228–261.

[7] Fu, C., Geng, S., Liu, P. (2021). Cluster algebras arising from cluster tubes i: integer vectors. *Math. Z.* 297:1793–1824.

[8] Gabriel, P. (1972). Unzerlegbare Darstellungen I. *Manuscripta Math.* 6:71–103.

[9] Geiß, C., Leclerc, B., Schröer, J. (2016). Quivers with relations for symmetrizable Cartan matrices III: Convolution algebras. *Represent. Theory* 20:375–413.

[10] Humphreys, J. E. (1972). *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics, Vol. 9. New York: Springer–Verlag.
[11] Kac, V. G. (1990). Infinite-Dimensional Lie Algebras, 3rd ed. Cambridge: Cambridge University Press.
[12] Lusztig, G. (1991). Intersection cohomology methods in representation theory. In: Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990). Tokyo: Math. Soc. Japan, 1991, pp. 155–174.
[13] Peng, L., Xiao, J. (1997). Root categories and simple Lie algebras. J. Algebra 198(1):19–56.
[14] Riedtmann, C. (1994). Lie algebras generated by indecomposables. J. Algebra 170(2):526–546.
[15] Schofield, A. Notes on constructing Lie algebras from finite-dimensional algebras, manuscript.
[16] Vatne, D. F. (2011). Endomorphism rings of maximal rigid objects in cluster tubes. Colloq. Math. 123:63–93.
[17] Vossieck, D. (2001). The algebras with discrete derived category. J. Algebra 243(1):168–176.
[18] Xiao, J. (1997). Drinfeld double and ringel–green theory of Hall algebras. J. Algebra 190:100–144.
[19] Yang, D. (2012). Endomorphism algebras of maximal rigid objects in cluster tubes. Commun. Algebra 40: 4347–4371.
[20] Zhou, Y., Zhu, B. (2014). Cluster algebras arising from cluster tubes. J. London Math. Soc. (2) 89:703–723.