Widths of weighted Sobolev classes with weights that are functions of distance to some $h$-set: some limiting cases

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1 Introduction

Let $X$, $Y$ be sets, and let $f_1, f_2 : X \times Y \to \mathbb{R}_+$. We write $f_1(x, y) \lesssim y f_2(x, y)$ (or $f_2(x, y) \gtrsim y f_1(x, y)$) if for any $y \in Y$ there exists $c(y) > 0$ such that $f_1(x, y) \leq c(y) f_2(x, y)$ for any $x \in X$; $f_1(x, y) \asymp y f_2(x, y)$ if $f_1(x, y) \lesssim y f_2(x, y)$ and $f_2(x, y) \lesssim y f_1(x, y)$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain (i.e., a bounded open connected set), and let $g, v : \Omega \to (0, \infty)$ be measurable functions. For each measurable vector-value $d$-function $\psi : \Omega \to \mathbb{R}^l$, $\psi = (\psi_k)_{1 \leq k \leq l}$, and for any $p \in [1, \infty)$ we set

$$\|\psi\|_{L^p(\Omega)} = \left( \max_{1 \leq k \leq l} |\psi_k(x)|^p \ dx \right)^{1/p}.$$

Let $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_+^d := (\mathbb{N} \cup \{0\})^d$, $|\beta| = \beta_1 + \ldots + \beta_d$. For any distribution $f$ defined on $\Omega$ we write $\nabla^\beta f = \left( \frac{\partial^\beta f}{\partial x^\beta} \right)_{|\beta|=r}$ (here partial derivatives are taken in the sense of distributions), and denote by $l_{r,d}$ the number of components of the vector-valued distribution $\nabla^\beta f$. We set

$$W_{p,q}(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \exists \psi : \Omega \to \mathbb{R}^{l_{r,d}} : \|\psi\|_{L^p(\Omega)} \leq 1, \nabla^\beta f = g \cdot \psi \} \quad \text{(we denote the corresponding function } \psi \text{ by } \frac{\nabla^\beta f}{g},)$$

$$\|f\|_{L^q(\Omega)} = \|f\|_{q,v} = \|f v\|_{L^q(\Omega)}, \quad L_{q,v}(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \|f\|_{q,v} < \infty \}.$$

We call the set $W_{p,q}(\Omega)$ a weighted Sobolev class. Observe that $W_{p,1}(\Omega) = W_p^r(\Omega)$ is a non-weighted Sobolev class. For properties of weighted Sobolev spaces and their generalizations, we refer the reader to the books [13, 14, 21, 27, 36, 37] and the survey paper [20].
Let \((X, \| \cdot \|_X)\) be a normed space, let \(X^*\) be its dual, and let \(L_n(X), n \in \mathbb{Z}_+\), be the family of subspaces of \(X\) of dimension at most \(n\). Denote by \(L(X, Y)\) the space of continuous linear operators from \(X\) into a normed space \(Y\). Also, by \(\text{rk} A\) denote the dimension of the image of an operator \(A \in L(X, Y)\), and by \(\|A\|_{X \rightarrow Y}\), its norm.

By the Kolmogorov n-width of a set \(M \subset X\) in the space \(X\), we mean the quantity
\[
d_n(M, X) = \inf_{L \in L_n(X)} \sup_{x \in M} \inf_{y \in L} \|x - y\|_X,
\]
by the linear n-width, the quantity
\[
\lambda_n(M, X) = \inf_{A \in L(L(X, X)), \text{rk} A \leq n} \sup_{x \in M} \|x - Ax\|_X,
\]
and by the Gelfand n-width, the quantity
\[
d^n(M, X) = \inf_{x_1, \ldots, x_n \in X^*} \sup_{x \in M} \{\|x\| : x_j(x) = 0, 1 \leq j \leq n\} = \inf_{A \in L(L(X, \mathbb{R}^n))} \sup_{x \in M \cap \ker A}\{\|x\| : x \in M \cap \ker A\}.
\]

In estimating Kolmogorov, linear, and Gelfand widths we set, respectively, \(\vartheta_1(M, X) = d_1(M, X)\) and \(q = q\), \(\vartheta_1(M, X) = \lambda_1(M, X)\) and \(\hat{q} = \min\{q, p\}\), \(\lambda_1(M, X) = d^1(M, X)\) and \(\hat{q} = p\).

In the 1960-1980s problems concerning the values of the widths of function classes in \(L_q\) and of finite-dimensional balls \(B^n_p\) in \(l^n_q\) were intensively studied. Here \(l^n_q\) (1 \(\leq q \leq \infty\)) is the space \(\mathbb{R}^n\) with the norm
\[
\|(x_1, \ldots, x_n)\|_q = \|(x_1, \ldots, x_n)\|_{l^n_q} = \begin{cases} (|x_1|^q + \cdots + |x_n|^q)^{1/q}, & \text{if } q < \infty; \\ \max\{|x_1|, \ldots, |x_n|\}, & \text{if } q = \infty, \end{cases}
\]
\(B^n_p\) is the unit ball in \(l^n_q\). For more details, see [29, 33, 34].

Let us formulate the result on widths of non-weighted Sobolev classes on a cube in the space \(L_q\). We set
\[
\vartheta_{p,q,r,d}(x_1, \ldots, x_n) = \begin{cases} \frac{\delta}{d} + \left(\frac{1}{q} - \frac{1}{p}\right)_+, & \text{if } p \geq q \text{ or } \hat{q} \leq 2; \\ \min \left\{\frac{\delta}{d} + \min\left\{\frac{1}{p} - \frac{1}{q}, \frac{1}{2} - \frac{1}{q}\right\}, \frac{\delta}{2d}\right\}, & \text{if } p < q, \hat{q} > 2. \end{cases}
\]

**Theorem A.** (see, e.g., [7,12,19,35]). Let \(r \in \mathbb{N}, 1 \leq p, q \leq \infty, \frac{r}{d} + \frac{1}{q} - \frac{1}{p} > 0\). In addition, we suppose that
\[
\frac{\delta}{d} + \min \left\{\frac{1}{p} - \frac{1}{q}, \frac{1}{2} - \frac{1}{q}\right\} \neq \frac{\hat{q}\delta}{2d}
\]
in the case \(p < q \) and \(\hat{q} > 2\). Then
\[
\vartheta_n(W^r_p([0, 1]^d), L_q([0, 1]^d)) \asymp n^{-\vartheta_{p,q,r,d}}.
\]
The problem concerning estimates of widths of weighted Sobolev classes in weighted \(L_q\)-space was studied by Birman and Solomyak \cite{7}, El Kolli \cite{15}, Triebel \cite{36,38}, Mynbaev and Otelbaev \cite{27}, Boykov \cite{8,9}, Lizorkin and Otelbaev \cite{26,28}, Aitenova and Kusainova \cite{1,2}. For details, see, e.g., \cite{46}.

Let \(|\cdot|\) be a norm on \(\mathbb{R}^d\), and let \(E, E' \subset \mathbb{R}^d, x \in \mathbb{R}^d\). We set

\[
\operatorname{diam}_{\cdot} E = \sup \{|y - z| : y, z \in E\}, \quad \operatorname{dist}_{\cdot}(x, E) = \inf \{|x - y| : y \in E\}.
\]

**Definition 1.** Let \(\Omega \subset \mathbb{R}^d\) be a bounded domain, and let \(a > 0\). We say that \(\Omega \in \text{FC}(a)\) if there exists a point \(x^* \in \Omega\) such that, for any \(x \in \Omega\), there exist \(T(x) > 0\) and a curve \(\gamma_x : [0, T(x)] \to \Omega\) with the following properties:

1. \(\gamma_x \in \text{AC}[0, T(x)], \left| \frac{d\gamma_x(t)}{dt} \right| = 1\) a.e.,
2. \(\gamma_x(0) = x, \gamma_x(T(x)) = x^*\),
3. \(\text{Bat}(\gamma_x(t)) \subset \Omega\) for any \(t \in [0, T(x)]\).

**Definition 2.** We say that \(\Omega\) satisfies the John condition (and call \(\Omega\) a John domain) if \(\Omega \in \text{FC}(a)\) for some \(a > 0\).

For a bounded domain the John condition coincides with the flexible cone condition (see definition in \cite{2}). Reshetnyak \cite{30,31} found an integral representation for functions on a John domain \(\Omega\) in terms of their derivatives of order \(r\). This representation yields that for \(r d - \left(\frac{1}{p} - \frac{1}{q}\right)_+ > 0\) (for \(r d - \left(\frac{1}{p} - \frac{1}{q}\right)_+ > 0\), respectively) the class \(W^r_p(\Omega)\) can be continuously (respectively, compactly) imbedded into \(L_q(\Omega)\) (i.e., the conditions of continuous and compact imbeddings are the same as for \(\Omega = [0, 1]^d\)). Moreover, in \cite{4,39} it was proved that if \(\Omega\) is a John domain and \(p, q, r, d\) are such as in Theorem \(A\) then widths have the same orders as for \(\Omega = [0, 1]^d\).

Throughout we suppose that \(\Omega \subset (-\frac{1}{2}, \frac{1}{2})^d\) (here \(\Omega\) is the closure of \(\Omega\)).

Denote by \(H\) the set of all non-decreasing functions defined on \((0, 1]\).

**Definition 3.** (see \cite{10}). Let \(\Gamma \subset \partial \Omega\) be a nonempty compact set and \(h \in H\). We say that \(\Gamma\) is an \(h\)-set if there are a constant \(c_* > 1\) and a finite countably additive measure \(\mu\) on \(\mathbb{R}^d\) such that \(\sup \mu = \Gamma\) and

\[
c_*^{-1} h(t) \leq \mu(B_t(x)) \leq c_* h(t)
\]

for any \(x \in \Gamma\) and \(t \in (0, 1]\).

Throughout we suppose that \(1 < p \leq \infty, 1 \leq q < \infty, r \in \mathbb{N}, \delta := r + \frac{d}{q} - \frac{4}{p} > 0\).

We denote \(\log x := \log_2 x\).

Let \(\Gamma \subset \partial \Omega\) be an \(h\)-set,

\[
g(x) = \varphi_g(\operatorname{dist}_{\cdot}(x, \Gamma)), \quad v(x) = \varphi_v(\operatorname{dist}_{\cdot}(x, \Gamma)),
\]

for any \(x \in \Gamma\) and \(t \in (0, 1]\).
where $\varphi_g, \varphi_v : (0, \infty) \to (0, \infty)$. Suppose that in some neighborhood of zero
\[ h(t) = t^\theta \log |t|^\gamma \tau (|\log t|), \quad 0 < \theta < d, \tag{5} \]
\[ \varphi_g(t) = t^{-\beta_g} \log |t|^{-\alpha_g} \rho_g(|\log t|), \quad \varphi_v(t) = t^{-\beta_v} \log |t|^{-\alpha_v} \rho_v(|\log t|), \tag{6} \]
where $\rho_g, \rho_v, \tau$ are absolutely continuous functions,
\[ \lim_{y \to +\infty} \frac{y \tau'(y)}{\tau(y)} = \lim_{y \to +\infty} \frac{y \rho_g'(y)}{\rho_g(y)} = \lim_{y \to +\infty} \frac{y \rho_v'(y)}{\rho_v(y)} = 0. \tag{7} \]

For $\beta_v < \frac{d - \theta}{q}$, in [40, 41, 43] there were obtained sufficient conditions for embedding of $W_{p,g}^r(\Omega)$ into $L_{q,v}^\gamma(\Omega)$, and order estimates of Kolmogorov, Gelfand and linear widths were found. Here we consider the limiting case
\[ \beta_v = \frac{d - \theta}{q}, \quad \alpha_v > 1 - \frac{\gamma}{q}. \tag{8} \]

We set $\beta = \beta_g + \beta_v, \alpha = \alpha_g + \alpha_v, \rho(y) = \rho_g(y)\rho_v(y), \mathcal{Z} = (p, q, r, d, a, c*, h, g, v), \mathcal{Z}_* = (\mathcal{Z}, \text{diam} \Omega)$.

**Theorem 1.** There exists $n_0 = n_0(\mathcal{Z})$ such that for any $n \geq n_0$ the following assertion holds.

1. Let $\beta - \delta + \theta \left(\frac{1}{q} - \frac{1}{p}\right)_+ < 0$. We set
\[ \sigma_*(n) = (\log n)^{-\alpha + \frac{1}{q} + \frac{(\beta - \delta)r}{p} \rho(\log n)\tau^\frac{\beta - \delta}{q} (\log n)}. \]

- Let $p \geq q$ or $p < q, \hat{q} \leq 2$. We set
\[ \theta_1 = \frac{\delta}{d} - \left(\frac{1}{q} - \frac{1}{p}\right)_+, \quad \theta_2 = \frac{\delta - \beta}{\theta} - \left(\frac{1}{q} - \frac{1}{p}\right)_+, \tag{9} \]
\[ \sigma_1(n) = 1, \quad \sigma_2(n) = \sigma_*(n). \tag{10} \]

Suppose that $\theta_1 \neq \theta_2, j_* \in \{1, 2\}$,
\[ \theta_{j_*} = \min\{\theta_1, \theta_2\}. \]

Then
\[ \vartheta_n(W_{p,g}^r(\Omega), L_{q,v}^\gamma(\Omega)) \asymp n^{-\theta_{j_*}} \sigma_*(n). \]
Let \( p < q, \hat{q} > 2 \). We set

\[
\theta_1 = \frac{\delta}{d} + \min \left\{ \frac{1}{p} - \frac{1}{q}, \frac{1}{2} - \frac{1}{\hat{q}} \right\}, \quad \theta_2 = \frac{\hat{q}\delta}{2d},
\]

\[
\theta_3 = \frac{\delta - \beta}{\theta} + \min \left\{ \frac{1}{p} - \frac{1}{q}, \frac{1}{2} - \frac{1}{\hat{q}} \right\}, \quad \theta_4 = \frac{\hat{q}(\delta - \beta)}{2\theta},
\]

\[
\sigma_1(n) = \sigma_2(n) = 1, \quad \sigma_3(n) = \sigma_4(n) = \sigma_*(n).
\]

Suppose that there exists \( j_* \in \{1, 2, 3, 4\} \) such that

\[
\theta_{j_*} < \min_{j \neq j_*} \theta_j.
\]

Then

\[
\vartheta_n(W_{p,q}^r(\Omega), L_{q,v}(\Omega)) \asymp n^{-\theta_{j_*}} \sigma_{j_*}(n).
\]

2. Let \( \beta - \delta + \theta \left( \frac{1}{q} - \frac{1}{p} \right) \geq 0 \). In addition, we suppose that \( \alpha_0 := \alpha - \frac{1}{q} > 0 \) for \( p < q \) and \( \alpha_0 := \alpha - 1 - (1 - \gamma) \left( \frac{1}{q} - \frac{1}{p} \right) > 0 \) for \( p \geq q \). Then

\[
\vartheta_n(W_{p,q}^r(\Omega), L_{q,v}(\Omega)) \asymp (\log n)^{-\alpha_0} \rho(\log n)^{-\left(\frac{1}{q} - \frac{1}{p}\right)} (\log n).
\]

Remark 1. From Theorem A it follows that for \( \frac{\delta - \beta}{\theta} > \frac{\delta}{d} \) the order estimates are the same as in the non-weighted case.

Remark 2. Formulas in Theorem 1 differ from formulas in [43] by the power of the logarithmic factor.

The upper estimates follow from the general result about the estimate of widths of function classes on sets with tree-like structure. Problems on estimating widths and entropy numbers for embedding operators of weighted function classes on trees were studied in papers of Evans, Harris, Lang, Solomyak, Lifshits and Linde [16, 23–25, 32].

Without loss of generality, as \(| \cdot |\) we may take \(|(x_1, \ldots, x_d)| = \max_{1 \leq i \leq d} |x_i|\).

## 2 Proof of the upper estimate

In this section, we obtain upper estimates for widths in Theorem 1.

The following lemma was proved in [44] (see inequalities (60)).

\[
\]
Lemma 1. Let \( \Lambda_\ast : (0, \infty) \to (0, \infty) \) be an absolutely continuous function such that \( \lim_{y \to +\infty} \frac{\Lambda_\ast(y)}{\Lambda_\ast(0)} = 0 \). Then for any \( \varepsilon > 0 \)

\[
t^{-\varepsilon} \leq \frac{\Lambda_\ast(ty)}{\Lambda_\ast(y)} \leq t^\varepsilon, \quad 1 \leq y < \infty, \quad 1 \leq t < \infty.
\]  \((15)\)

Let \( c_\ast \geq 1 \) be the constant from the definition of an \( h \)-set. From \((5), (6), (7)\) and Lemma \( \text{II} \) it follows that there exists \( c_0 = c_0(3) \geq c_\ast \) such that

\[
\frac{h(t)}{h(s)} \leq c_0, \quad \frac{\varphi_\ast(t)}{\varphi_\ast(s)} \leq c_0, \quad \frac{\varphi_v(t)}{\varphi_v(s)} \leq c_0, \quad j \in \mathbb{N}, \quad t, s \in [2^{-j-1}, 2^{-j+1}].
\]  \((16)\)

Let \( (\Omega, \Sigma, \nu) \) be a measure space. We say that sets \( A, B \subset \Omega \) do not overlap if \( \nu(A \cap B) = 0 \). Let \( m \in \mathbb{N} \cup \{\infty\} \), \( E, E_1, \ldots, E_m \subset \Omega \) be measurable sets. We say that \( \{E_i\}_{i=1}^m \) is a partition of \( E \) if the sets \( E_i \) do not overlap pairwise and \( \nu[\bigcup_{i=1}^m E_i \triangle E] = 0 \).

Let \( (T, \xi_0) \) be a tree rooted at \( \xi_0 \). We introduce a partial order on \( V(T) \) as follows: we say that \( \xi' > \xi \) if there exists a simple path \( (\xi_0, \xi_1, \ldots, \xi_n, \xi') \) such that \( \xi = \xi_k \) for some \( k \in \mathbb{N} \). In this case, we set \( \rho_T(\xi, \xi') = \rho_T(\xi', \xi) = n + 1 - k \). In addition, we denote \( \rho_T(\xi, \xi) = 0 \). If \( \xi' > \xi \) or \( \xi' = \xi \), then we write \( \xi' \geq \xi \) and denote \( [\xi, \xi'] := \{\xi'' \in V(T) : \xi \leq \xi'' \leq \xi'\} \). This partial order on \( T \) induces a partial order on its subtree.

Given \( j \in \mathbb{Z}_+, \xi \in V(T) \), we set

\[
V_j(\xi) := V^T_j(\xi) := \{\xi' \geq \xi : \rho_T(\xi, \xi') = j\}.
\]

For each vertex \( \xi \in V(T) \) we denote by \( T_\xi = (T_\xi, \xi) \) a subtree in \( T \) with vertex set \( \{\xi' \in V(T) : \xi' \geq \xi\} \).

In \([10, 11]\) a tree \( (A, \eta_{j,1}) \) with vertex set \( \{\eta_{j,i}\}_{j \geq j_\ast, i \in I_j} \) was constructed, as well as the partition of \( \Omega \) into subdomains \( \Omega[\xi], \xi \in V(A) \). Moreover, \( V^A_{j-j_\ast}(\eta_{j,1}) = \{\eta_{j,i}\}_{i \in I_j} \) and there exists a number \( \overline{s} = \overline{s}(a, d) \in \mathbb{N} \) such that

\[
diam_{\Omega[\eta_{j,i}]} > 2^{-\overline{s}j}, \quad \text{dist}_{\Omega[\xi]}(x, \Gamma) > 2^{-\overline{s}j}, \quad x \in \Omega[\eta_{j,i}],
\]

\[
\text{card } V^A_{j-j_\ast}(\eta_{j,i}) \lesssim \frac{h(2^{-\overline{s}j})}{h(2^{-\overline{s}j})}, \quad j' \geq j \geq j_\ast.
\]

In particular,

\[
\text{card } V^A_{1}(\eta_{j,i}) \lesssim 1, \quad j \geq j_\ast.
\]  \((17)\)

We set

\[
u(\eta_{j,i}) = u_j = \varphi_g(2^{-\overline{s}j}) \cdot 2^{-(r - \frac{d}{r})\overline{s}j}, \quad w(\eta_{j,i}) = w_j = \varphi_v(2^{-\overline{s}j}) \cdot 2^{-(s_0 - \frac{d}{r})/r}.
\]  \((18)\)

Given a subtree \( D \subset A \), we denote \( \Omega[D] = \cup_{\xi \in V(D)} \Omega[\xi] \).

In \([15]\) sufficient conditions for embedding \( W^r_{p,g}(\Omega) \) into \( L_{q,v}(\Omega) \) were obtained; here \((16)\) holds and the functions \( g, v \) satisfy \((4)\). Let us formulate these results.
Theorem B. Let \( u, w \) be defined by (18), \( 1 < p < q < \infty \). Suppose that there exist \( l_0 \in \mathbb{N} \) and \( \lambda \in (0, 1) \) such that

\[
\left( \sum_{i=j+l_0}^{\infty} \frac{h(2^{-\frac{p}{p-q} i)}}{h(2^{-\frac{p}{p-q}} j)} w_i^q \right)^{1/q} \leq \lambda, \quad j \geq j_*.
\] (19)

Let \( \sup_{j \geq j_*} u_j \left( \sum_{i=j}^{\infty} \frac{h(2^{-\frac{p}{p-q} i})}{h(2^{-\frac{p}{p-q}} j)} w_i^q \right)^{1/q} < \infty \). Then \( W_{p,q}(\Omega) \subset L_{q,v}(\Omega) \) and for any \( k \geq j_* \), \( \xi_* \in \mathbf{V}^A_{k-j_*}(n_{j_*,1}) \) there exists a linear continuous operator \( P : L_{q,v}(\Omega) \to \mathcal{P}_{r-1}(\Omega) \) such that for any subtree \( \mathcal{D} \subset \mathcal{A} \) rooted at \( \xi_* \) and for any function \( f \in W_{p,q}^r(\Omega) \)

\[
\|f - Pf\|_{L_{q,v}(\mathcal{D})} \leq \sup_j u_j \left( \sum_{i=j}^{\infty} \frac{h(2^{-\frac{p}{p-q} i})}{h(2^{-\frac{p}{p-q}} j)} w_i^q \right)^{1/q} \left\| \nabla^r f \right\|_{L_p(\mathcal{D})}.
\]

Theorem C. Let \( p \geq q, \xi_* \in \mathbf{V}^A_{k-j_*}(n_{j_*,1}) \), and let the functions \( u, w \) on \( \mathbf{V}(\mathcal{A}) \) be defined by (18). We set \( \hat{w}_j = w_j \cdot \left( \frac{h(2^{-\frac{p}{p-q} i})}{h(2^{-\frac{p}{p-q}} j)} \right)^{\frac{1}{p}} \), \( \hat{u}_j = u_j \cdot \left( \frac{h(2^{-\frac{p}{p-q} i})}{h(2^{-\frac{p}{p-q}} j)} \right)^{\frac{1}{q}} \), \( k \leq j < \infty \). Let

\[
M_{\hat{u}, \hat{w}}(k) := \sup_{k \leq j < \infty} \left( \sum_{i=j}^{\infty} \hat{w}_i^q \right)^{\frac{1}{q}} \left( \sum_{i=k}^{\infty} \hat{w}_i^p \right)^{\frac{1}{p}} < \infty, \quad 1 < p = q < \infty, \quad (20)
\]

\[
M_{\hat{u}, \hat{w}}(k) := \left( \sum_{j=k}^{\infty} \left( \sum_{i=j}^{\infty} \hat{w}_i^q \right)^{\frac{1}{q}} \left( \sum_{i=k}^{\infty} \hat{w}_i^p \right)^{\frac{1}{p}} \right)^{\frac{p}{p-q}} \hat{w}_j^q \ < \infty, \quad q < p. \quad (21)
\]

Then \( \mathbf{W}^{r}_{p,q}(\Omega[\mathcal{A}_{\xi_*}]) \subset L_{q,v}(\Omega[\mathbf{V}_{\xi_*}]) \) and there exists a linear continuous operator \( P : L_{q,v}(\Omega) \to \mathcal{P}_{r-1}(\Omega) \) such that for any subtree \( \mathcal{D} \subset \mathcal{A}_{\xi_*} \) rooted at \( \xi_* \) and for any function \( f \in \mathbf{W}^{r}_{p,q}(\Omega) \)

\[
\|f - Pf\|_{L_{q,v}(\mathcal{D})} \leq M_{\hat{u}, \hat{w}}(k) \left\| \nabla^r f \right\|_{L_p(\mathcal{D})}.
\]

Suppose that (5), (6), (7), (8) hold.
From (6), (8) and (18) it follows that

\[
u(\eta_{j,i}) = u_j = 2^{\varphi_j} \left( \frac{p \rho_\varphi}{\rho_\varphi} \right) (s_j)^{-\alpha_\varphi} \rho_\varphi(s_j), \quad w(\eta_{j,i}) = w_j = 2^{-\frac{\pi}{q}} (s_j)^{-\alpha_\varphi} \rho_\varphi(s_j). \quad (22)
\]

Recall that \( \delta = r + \frac{4}{q} - \frac{4}{p} \).
Corollary 1. Let $1 < p < q < \infty$, $r \in \mathbb{N}$, $\delta > 0$, and let the conditions (5), (6), (7), (8) hold. In addition, we suppose that

$$\text{either } \beta - \delta < 0 \quad \text{or} \quad \beta - \delta = 0, \quad \alpha > \frac{1}{q}. \tag{23}$$

Then $W_{p,g}^r(\Omega) \subset L_{q,v}(\Omega)$ and for any $k \geq j_*$, $\xi_* \in V^{A}_{k-j_*}(\eta_{j_*})$ there exists a linear continuous operator $P : L_{q,v}(\Omega) \to P_{r-1}(\Omega)$ such that for any subtree $D \subset A$ rooted at $\xi_*$ and for any function $f \in W_{p,g}^r(\Omega)$

$$\|f - Pf\|_{L_{q,v}(\Omega[D])} \lesssim 2^{-(\delta - \beta)\|k\| - \alpha + \frac{1}{q} \rho(\|k\|)} \left\| \frac{\nabla f}{g} \right\|_{L_p(\Omega[D])}. \tag{19}$$

Proof. From (5) and (22) it follows that

$$\sum_{i,j} h(2^{-k_i}) u_i^j = \sum_{i,j} 2^{-\beta \|i\|} - \alpha q \rho^q(\|i\|) \cdot 2^p \gamma(\|j\|) \cdot \frac{\gamma(\|j\|)}{2^p \gamma(\|j\|)} \gtrsim \frac{1}{3} \tag{24}$$

This together with Lemma 1 implies (19). Further,

$$\sup_{j \geq k} \left( \sum_{i,j} h(2^{-k_i}) u_i^j \right)^{\frac{1}{q}} \gtrsim \frac{1}{3} \tag{25}$$

It remains to apply Theorem 1.

Let us consider the case $p \geq q$. We apply Theorem 1. For $j \geq k$, we have

$$\hat{u}_j = 2^{2\gamma(\beta_0 - \delta - \frac{1}{p})} \gamma(\|j\|) - \alpha q \rho^q(\|j\|) \cdot 2^{-\frac{\beta_0 - \delta - \frac{1}{p}}{p} \gamma(\|j\|)} \cdot \frac{k^\gamma(\|j\|)}{k^\gamma(\|j\|)} \tag{26}$$

Corollary 2. Let $1 < p \leq \infty$, $1 \leq q < \infty$, $p \geq q$, $r \in \mathbb{N}$, $\delta > 0$ and let conditions (5), (6), (7), (8) hold. Suppose that either $\beta - \delta + \theta \left( \frac{1}{q} - \frac{1}{q} \right) < 0$ or $\beta - \delta + \theta \left( \frac{1}{q} - \frac{1}{p} \right) > 0$. Then $W_{p,g}^r(\Omega) \subset L_{q,v}(\Omega)$ and for any $k \geq j_*$, $\xi_* \in V^{A}_{k-j_*}(\eta_{j_*})$ there exists a linear continuous operator $P : L_{q,v}(\Omega) \to P_{r-1}(\Omega)$ such that for any subtree $D \subset A$ rooted at $\xi_*$ and for any function $f \in W_{p,g}^r(\Omega)$

$$\|f - Pf\|_{L_{q,v}(\Omega[D])} \lesssim 2^{-(\beta - \delta)\|k\| - \alpha + \frac{1}{q} \rho(\|k\|)} \left\| \frac{\nabla f}{g} \right\|_{L_p(\Omega[D])} \tag{19}$$

in the case $\beta - \delta + \theta \left( \frac{1}{q} - \frac{1}{q} \right) < 0$, and

$$\|f - Pf\|_{L_{q,v}(\Omega[D])} \lesssim 2^{-\theta \left( \frac{1}{q} - \frac{1}{p} \right) \|k\| - \alpha + 1 + \frac{1}{q} - \frac{1}{p} \rho(\|k\|)} \left\| \frac{\nabla f}{g} \right\|_{L_p(\Omega[D])} \tag{19}$$

in the case $\beta - \delta + \theta \left( \frac{1}{q} - \frac{1}{p} \right) = 0$. 

\[8\]
Proof. Let \( p = q \). Applying (25) and (20) and taking into account that \( \alpha_v > \frac{1 - \gamma}{q} \) and \( \beta_y - r + \frac{d}{p} - \frac{\theta}{p} < 0 \), we get

\[
M_{\hat{u}, \hat{w}}(k) \lesssim \sup_{l \geq k} (\hat{s}l)^{-\alpha_v + \frac{1 - \gamma}{q}} \rho_v(\hat{s}l)^{\frac{1}{q}} (\hat{s}l) \left( \sum_{j=k}^{\infty} 2^{\rho(\beta - \delta, \overline{j})} (\hat{s}j) \rho'(\alpha_v, \overline{j}) \rho'_v(\overline{j}) \tau_{\hat{s}j}^{\frac{1}{p'}} \right)^{\frac{1}{p}}.
\]

If \( \beta - \delta < 0 \), then by Lemma II

\[
M_{\hat{u}, \hat{w}}(k) \lesssim 2^{(\beta - \delta, \overline{k})} (\overline{k})^{-\alpha_v + \frac{1}{q}} \rho(\overline{k}).
\]

Let \( \beta - \delta = 0 \). We may assume that \( -\alpha_v + \frac{\gamma}{p} + \frac{1}{p'} > 0 \) (otherwise, we multiply \( \hat{u}_j \) by \( \frac{g}{k'} \) with some \( c > 0 \)). Then

\[
M_{\hat{u}, \hat{w}}(k) \lesssim (\overline{k})^{-\alpha_v + 1} \rho(\overline{k}).
\]

Let \( p > q \). Applying (25) and (21) and taking into account that \( \alpha_v > \frac{1 - \gamma}{q} \) and \( \beta_y - r + \frac{d}{p} - \frac{\theta}{p} = \beta - \delta \), we get

\[
M_{\hat{u}, \hat{w}}(k) \lesssim 2^{-\overline{k}} (\overline{k})^\gamma (\frac{1}{q} - \frac{1}{p}) \tau_{\overline{k}}^\frac{1}{p} (\overline{k}) \times
\]

\[
\left( \sum_{j=k}^{\infty} (\hat{s}j)^{\frac{1}{p'}} (\rho_v(\hat{s}j)^{\frac{1}{q}}) (\rho'_v(\hat{s}j)^{\frac{1}{p'}} (\hat{s}j)^{\frac{1}{p'}} \sigma(\hat{s}j)^{\frac{1}{p'}}) \right)^{\frac{1}{p'}}.
\]

where

\[
\sigma(\hat{s}j) = \left( \sum_{i=k}^{\infty} 2^{\rho(\beta - \delta, \overline{j})} (\rho'_v(\overline{j}))^{\frac{1}{p'}} (\tau_{\overline{j}})^{\frac{1}{p'}} (\overline{j})^{\frac{1}{p'}} \right)^{\frac{1}{p'}}.
\]

If \( \beta - \delta + \theta \left( \frac{1}{q} - \frac{1}{p} \right) < 0 \), then

\[
\sigma(\hat{s}j) \lesssim 2^{-\overline{k}} (\overline{k})^\gamma (\frac{1}{q} - \frac{1}{p}) \rho(\overline{k}) \tau_{\overline{k}}^\frac{1}{p} (\overline{k}),
\]

and by the second relation in (8) we have

\[
M_{\hat{u}, \hat{w}}(k) \lesssim 2^{(\beta - \delta, \overline{k})} (\overline{k})^{-\alpha_v + \frac{1}{q}} \rho(\overline{k}).
\]

If \( \beta - \delta + \theta \left( \frac{1}{q} - \frac{1}{p} \right) = 0 \) and \( \alpha > 1 + (1 - \gamma) \left( \frac{1}{q} - \frac{1}{p} \right) \), then we may assume that \( -\alpha_v + \frac{\gamma}{p} + \frac{1}{p'} > 0 \). We have

\[
M_{\hat{u}, \hat{w}}(k) \lesssim 2^{-\theta \left( \frac{1}{q} - \frac{1}{p} \right)} (\overline{k})^{-\alpha_v + \frac{1}{q}} \rho(\overline{k}).
\]

This completes the proof. \( \square \)
Remark 3. Notice that in order to prove Theorems B and C we use estimates for norms of summation operators on trees, which are obtained in [42]. If \( \beta - \delta + \theta \left( \frac{1}{q} - \frac{1}{p} \right) > 0 \), then these estimates can be proved easier (we argue similarly as in [41], Lemma 5.1).

Applying Corollaries 1 and 2 and arguing similarly as in [43] Theorem 1], we obtain the desired upper estimate of widths.

3 Proof of the lower estimate

In this section, we obtain the lower estimates of widths in Theorem 1.

If \( \frac{\delta - \beta}{\theta} > \frac{4}{3} \), then by Theorem A (see also Remark 1) and by the upper estimate of \( \vartheta_n(W_p^r(\Omega), L_{q,v}(\Omega)) \), which is already obtained, we have \( \vartheta_n(W_p^r(\Omega), L_{q,v}(\Omega)) \lesssim \vartheta_n(W_p^r([0, 1]^d), L_q([0, 1]^d)) \). On the other hand, there is a cube \( \Delta \subset \Omega \) with side length \( l(\Delta) \gtrsim 1 \) such that \( g(x) \gtrsim 1 \), \( v(x) \gtrsim 1 \) for any \( x \in \Delta \) (see [43]). Hence, \( \vartheta_n(W_p^r(\Omega), L_{q,v}(\Omega)) \gtrsim \vartheta_n(W_p^r([0, 1]^d), L_q([0, 1]^d)) \). Thus, we obtained the order estimates of widths in the case \( \frac{\delta - \beta}{\theta} > \frac{4}{3} \).

Consider the case \( \frac{\delta - \beta}{\theta} \leq \frac{4}{3} \). In order to obtain the lower estimates we argue similarly as in [43]. It is sufficient to prove the following assertions.

Proposition 1. Let \( \frac{\delta - \beta}{\theta} \leq \frac{4}{3} \). Suppose that one of the following conditions holds: 1) \( \beta - \delta + \theta \left( \frac{1}{q} - \frac{1}{p} \right) < 0 \) or 2) \( \beta = \delta \), \( p < q \). Then there exist \( t_0 = t_0(3_s) \in \mathbb{N} \) and \( k = k(3_s) \in \mathbb{N} \) such that for any \( t \in \mathbb{N} \), \( t \geq t_0 \) there exist functions \( \psi_{j,t} \in C_0^\infty(\mathbb{R}^d) \) \( (1 \leq j \leq j_t) \) with pairwise non-overlapping supports such that

\[
\vartheta_{j,t} \gtrsim 2^{\beta k t} (\hat{k} t)^{-\gamma - 1} (\hat{k} t),
\]

\[
\left\| \frac{\nabla \psi_{j,t}}{g} \right\|_{L_p(\Omega)} = 1, \quad \| \psi_{j,t} \|_{L_{q,v}(\Omega)} \gtrsim 2^{(\beta - \gamma) k t} (\hat{k} t)^{-\alpha + \frac{1}{q} + \frac{1}{p}} \rho(\hat{k} t).
\]

Proposition 2. Let \( \beta - \delta + \theta \left( \frac{1}{q} - \frac{1}{p} \right) = 0 \), \( p \gtrsim q \). Then there exist \( t_0 = t_0(3_s) \in \mathbb{N} \) and \( k = k(3_s) \in \mathbb{N} \) such that for any \( t \in \mathbb{N} \), \( t \geq t_0 \) there exist functions \( \psi_{j,t} \in C_0^\infty(\mathbb{R}^d) \) \( (1 \leq j \leq j_t) \) with pairwise non-overlapping supports such that

\[
\vartheta_{j,t} \gtrsim 2^{\beta k t} (\hat{k} t)^{-\gamma - 1} (\hat{k} t),
\]

\[
\left\| \frac{\nabla \psi_{j,t}}{g} \right\|_{L_p(\Omega)} = 1, \quad \| \psi_{j,t} \|_{L_{q,v}(\Omega)} \gtrsim 2^{-\beta k t} (\hat{k} t)^{-\gamma + \frac{1}{q} + 1 + \frac{1}{p}} \rho(\hat{k} t).
\]
First we formulate the Vitali covering theorem \[22, \text{p. 408}\].

**Theorem D.** Denote by $B(x, t)$ the open or closed ball of radius $t$ with respect to some norm on $\mathbb{R}^d$ centered in $x$. Let $E \subset \mathbb{R}^d$ be a finite union of balls $B(x_i, r_i)$, $1 \leq i \leq l$. Then there exists a subset $I \subset \{1, \ldots, l\}$ such that the balls $\{B(x_i, r_i)\}_{i \in I}$ are pairwise non-overlapping and $E \subset \bigcup_{i \in I} B(x_i, 3r_i)$.

Let $K$ be a family of closed cubes in $\mathbb{R}^d$ with axes parallel to coordinate axes. Given a cube $K \in K$ and $s \in \mathbb{Z}_+$, we denote by $\Xi_s(K)$ the partition of $K$ into $2^{sd}$ closed non-overlapping cubes of the same size, and we set $\Xi(K) := \bigcup_{s \in \mathbb{Z}_+} \Xi_s(K)$.

Given a cube $\Delta \in \Xi(-1/2, 1/2)^d$ such that $\Delta \cap \Gamma \neq \emptyset$, we define the cubes $Q_\Delta$, $\hat{Q}_\Delta$ and the points $x_\Delta$, $\hat{x}_\Delta$ as follows.

Let $m \in \mathbb{N}$, $\Delta \in \Xi_1(-1/2, 1/2)^d$, $\Delta \cap \Gamma \neq \emptyset$. We choose $x_\Delta \in \Delta \cap \Gamma$ and a cube $Q_\Delta$ such that $\Delta \in \Xi_1(Q_\Delta)$, \[ \text{dist}_1(x_\Delta, \partial Q_\Delta) \geq 2^{-m-1}. \] (34)

Denote by $\hat{x}_\Delta$ the center of $Q_\Delta$. Then \[ Q_\Delta = \hat{x}_\Delta + 2^{-m+1} \left[ -\frac{1}{2}, \frac{1}{2} \right]^d. \] (35)

We set \[ \hat{Q}_\Delta = \hat{x}_\Delta + 3 \cdot 2^{-m} \left[ -\frac{1}{2}, \frac{1}{2} \right]^d, \quad \hat{Q}_\Delta = \hat{x}_\Delta + 2^{-m+2} \left[ -\frac{1}{2}, \frac{1}{2} \right]^d. \] (36)

Recall that the norm $| \cdot |$ is defined by $|(x_1, \ldots, x_d)| = \max_{1 \leq i \leq d} |x_i|$. Let $k \in \mathbb{N}$ (it will be chosen later). For each $l \in \mathbb{Z}_+$ we set \[ \hat{E}_l(\Delta) = \{ x \in \hat{Q}_\Delta : \text{dist}_1(x, \Gamma) \leq 2^{-m-kl+2} \}, \quad E_l(\Delta) = \hat{E}_l(\Delta) \cap Q_\Delta \cap \Omega. \] (37)

Notice that \[ \hat{Q}_\Delta = \hat{E}_0(\Delta). \] (38)

Denote by $\text{mes} A$ the Lebesgue measure of the measurable set $A \subset \mathbb{R}^d$.

**Lemma 2.** The following estimate holds: \[ \text{mes} \hat{E}_l(\Delta) \leq \frac{2^{-md-(d-\theta)kl} m^\gamma \tau(m)}{(m + kl)^\gamma \tau(m + kl)^\gamma}, \] (39)

In addition, there exists $m_0 = m_0(3\gamma)$ such that for $m \geq m_0$ \[ \text{mes} E_l(\Delta) \geq \frac{2^{-md-(d-\theta)kl} m^\gamma \tau(m)}{(m + kl)^\gamma \tau(m + kl)^\gamma}. \] (40)
Proof. Let us prove (39). Consider the covering of the set $\hat{E}_i(\Delta)$ by cubes $x + K$, $x \in \Gamma \cap \hat{Q}_\Delta$, $K = (-2^{-m-kl+3}, 2^{-m-kl+3})$. We take a finite subcovering; applying Theorem D (the balls are taken with respect to $| \cdot |$), we get a family of pairwise non-intersecting balls $\{x_i + K\}_{i=1}^N$ such that $\{x_i + 3K\}_{i=1}^N$ is a covering of $\hat{E}_i(\Delta)$. Since $\cup_{i=1}^N (x_i + K)$ is contained in a ball $B$ of radius $R \gtrsim 2^{-m}$, we have

$$\sum_{i=1}^N \mu(x_i + K) \leq \mu(B) \overset{39}{\lesssim} h(2^{-m});$$

since $x_i \in \Gamma$, we get $\mu(x_i + K) \overset{42}{\gtrsim} h(2^{-m-kl})$ and $N \lesssim \frac{h(2^{-m})}{h(2^{-m-kl})}$. Finally,

$$\mes \hat{E}_i(\Delta) \lesssim \sum_{i=1}^N \mes (x_i + 3K) \lesssim 2^{-m-kl+\delta} \frac{h(2^{-m})}{h(2^{-m-kl})}.$$

It remains to apply (42).

Let us prove (40). Denote by $Q^*_\Delta$ the homothetic transform of the cube $Q_\Delta$ with respect to its center with the coefficient $1 - 2^{-kl-3}$. We set

$$\{\Delta^L_{i=1} = \{\Delta' \in \Xi_{m+kl+3}([-1/2, 1/2]^d) : \Delta' \subset Q^*_\Delta, \Delta' \cap \Gamma \neq \emptyset\}.$$  

It can be proved similarly as formula (4.20) in [10] that $L \lesssim \frac{h(2^{-m})}{h(2^{-m-kl})}$. Since $\Delta, \cap \Gamma \neq \emptyset$, it follows from the definition of $\Delta_i$ and $Q^*_\Delta$ that $\cup_{i=1}^L \Delta_i \subset \hat{E}_i(\Delta) \cap Q_\Delta$. Finally, for any $j \in \{1, \ldots, L\}$

$$\text{card} \{i \in \Gamma : \mes (Q_{\Delta_i} \cap Q_{\Delta_j}) > 0\} \lesssim 1.$$

Therefore, it is sufficient to prove that $\mes (Q_{\Delta_i} \cap \Omega) \lesssim 2^{-(m+kl)d}$.

Let $x \in Q_{\Delta_i} \cap \Omega$, $|x - x_{\Delta_i}| \leq 2^{-m-kl-5}$. This point exists since $x_{\Delta_i} \in \Gamma \subset \partial \Omega$ and (33) holds with $m + kl + 3$ instead of $m$; moreover, $\text{dist}_{\Gamma}(x, \partial Q_{\Delta_i}) \gtrsim 2^{-m-kl}$. Let $x_*$ and $\gamma_x(\cdot) : [0, T(x)] \to \Omega$ be such as in Definition 11. There exists $m_0 = m_0(3_*)$ such that $x_* \notin Q_{\Delta_i}$ for $m \geq m_0$. Let $\gamma_x(t_*) \in \partial Q_{\Delta_i}$. Then $t_* \gtrsim 2^{-m-kl}$. By Definition 11 the ball $B_{\gamma_x(t_*)}$ is contained in $\Omega$. It remains to observe that $\mes (B_{\gamma_x(t_*)} \cap Q_{\Delta_i}) \gtrsim 2^{-(m+kl)d}$. \hfill \Box

Remark 4. From (39) it follows that $\mes (\hat{Q}_\Delta \cap \Gamma) = 0$.

Suppose that $m \geq m_0(3_*)$.
Choose $k = k(3_*)$ such that for any $l \in \mathbb{Z}_+$

$$\mes \left(\hat{E}_i(\Delta) \setminus \hat{E}_{i+1}(\Delta)\right) \lesssim \frac{2^{-md-(d-3)kl} m^d \tau(m)}{(m + kl) \tau(m + kl)} \quad (41)$$

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(it is possible by (15), (39) and (40)).

Let \( \psi \in C_0^\infty(\mathbb{R}^d) \), supp \( \psi \subset \left[-\frac{1}{2}, \frac{1}{2}\right]^d, \psi\big|_{\left[-\frac{1}{4}, \frac{1}{4}\right]^d} = 1 \), \( \psi(x) \in [0, 1] \) for any \( x \in \mathbb{R}^d \).

We set
\[
\psi_\Delta(x) = \psi(2^{m-2}(x - \hat{x}_\Delta)).
\]
\( \psi_\Delta \) satisfies (42).

Then
\[
\text{supp } \psi_\Delta \subset \hat{Q}_\Delta, \quad \psi_\Delta|_{\hat{Q}_\Delta} = 1,
\]
\( \psi_\Delta \) satisfies (43).

\[
\left| \frac{\nabla^r \psi_\Delta(x)}{g(x)} \right| \lesssim \frac{1}{5^r} 2^{-\beta_g(m+\hat{k}l)}(m + \hat{k}l)^{\alpha_g} \rho_g^{-1}(m + \hat{k}l) \cdot 2^{rm}, \quad x \in \hat{E}_l(\Delta) \setminus \hat{E}_{l+1}(\Delta).
\]
\( \left| \frac{\nabla^r \psi_\Delta(x)}{g(x)} \right| \lesssim \frac{1}{5^r} 2^{-\beta_g(m+\hat{k}l)}(m + \hat{k}l)^{\alpha_g} \rho_g^{-1}(m + \hat{k}l) \cdot 2^{rm}, \quad x \in \hat{E}_l(\Delta) \setminus \hat{E}_{l+1}(\Delta).
\]
\( \left| \frac{\nabla^r \psi_\Delta(x)}{g(x)} \right| \lesssim \frac{1}{5^r} 2^{-\beta_g(m+\hat{k}l)}(m + \hat{k}l)^{\alpha_g} \rho_g^{-1}(m + \hat{k}l) \cdot 2^{rm}, \quad x \in \hat{E}_l(\Delta) \setminus \hat{E}_{l+1}(\Delta).
\]

We set \( c_\Delta = \left| \frac{\nabla^r \psi_\Delta}{g} \right|_{L_p(\hat{Q}_\Delta)}^{-1} > 0 \).

**Lemma 3.** The following estimates hold:
\[
c_\Delta \geq \frac{2(\beta_g - r + \frac{\theta}{q})m - \alpha_g \rho_g(m)}{5^r}, \quad c_\Delta \left\| \psi_\Delta \right\|_{L_{q,v}(\Omega)} \geq \frac{2(\beta - \delta)p_m - \alpha_g \rho_g(m)}{5^r}.
\]

**Proof.** We estimate the value \( \left\| \frac{\nabla^r \psi_\Delta}{g} \right\|_{L_p(\hat{Q}_\Delta)} \) from above. First we notice that from the conditions \( \frac{\delta - \beta}{\delta} \leq \frac{\delta}{d}, \theta < d \) and \( \beta_v \geq \frac{d - \theta}{q} \) it follows that
\[
\beta_g + \frac{d - \theta}{p} > 0.
\]
Hence, by Remark 3
\[
\left\| \frac{\nabla^r \psi_\Delta}{g} \right\|_{L_p(\hat{Q}_\Delta)}^p \lesssim \sum_{l \in \mathbb{Z}_+} \left\| \frac{\nabla^r \psi_\Delta}{g} \right\|_{L_p(\hat{E}_l(\Delta) \setminus \hat{E}_{l+1}(\Delta))}^p \lesssim \sum_{l \in \mathbb{Z}_+} 2^{-p\beta_g(m+\hat{k}l)}(m + \hat{k}l)^{p\alpha_g / 2}(m + \hat{k}l) \cdot 2^{rm} \cdot 2^{-d(m-\theta)l} \cdot 2^{-d-m} \cdot \frac{m^\gamma \tau(m)}{(m + \hat{k}l)^\gamma \tau(m + \hat{k}l)} \lesssim \frac{2 p^{\left(\beta_g + r - \frac{\theta}{2}\right)m - \alpha_g \rho_g(m)}}{5^r}
\]
This implies the first inequality in (15). Let us prove the second inequality. Taking into account that \( \psi_\Delta|_{\hat{Q}_\Delta} = 1 \) and \( \beta_v = \frac{d - \theta}{q} \), we get
\[
\left\| \psi_\Delta \right\|_{L_{q,v}(\Omega)}^q \geq \sum_{l \in \mathbb{Z}_+} \left\| \nabla^r \psi_\Delta \right\|_{L_{q}(\hat{E}_l(\Delta) \setminus \hat{E}_{l+1}(\Delta))}^q \lesssim \frac{1}{5^r}.
\]
\[
\sum_{l \in \mathbb{Z}_+} 2^{\beta_q (m+kl)} (m+kl)^{-\alpha_q \rho^q_v(m+kl)} \cdot 2^{-md-(d-\delta)kl} m^{\gamma \tau(m)} (m+kl)^{\gamma \tau(m+kl)} \geq 2^{(\beta_q - d)m - \delta \rho^q_v(m)}.
\]

It remains to apply the first inequality in (45). \(\square\)

**Proof of Proposition 1.** Let

\[
\{ \Delta_{\nu} \}_{\nu \in \mathcal{N}} = \left\{ \Delta \in \Xi_{\hat{k}t} \left( \left[ -\frac{1}{2}, \frac{1}{2} \right] \right) : \Delta \cap \Gamma \neq \emptyset \right\}.
\]

Then \(\{ \hat{Q}_{\Delta_{\nu}} \}_{\nu \in \mathcal{N}}\) is a covering of \(\Gamma\). Denote by \(Q^*_\Delta\) the homothetic transform of \(\hat{Q}_\Delta\) with respect to its center with coefficient 3. Applying Theorem D, we get that there exists a subset \(\mathcal{N}' \subset \mathcal{N}\) such that \(\{ \hat{Q}_{\Delta_{\nu}} \}_{\nu \in \mathcal{N}'}\) are pairwise non-overlapping and \(\{ Q^*_\Delta \}_{\nu \in \mathcal{N}'}\) is a covering of \(\Gamma\). We claim that

\[
\text{card} \mathcal{N}' \geq \frac{Z}{3^*} 2^{\hat{k}t} (\hat{k}t)^{-\gamma \tau^{-1}(\hat{k}t)}. \tag{48}
\]

Indeed,

\[
\text{card} \mathcal{N}' \cdot 2^{-\hat{k}t} (\hat{k}t)^{\gamma \tau(\hat{k}t)} \geq \text{card} \mathcal{N}' \cdot h(2^{-\hat{k}t}) \geq \sum_{\nu \in \mathcal{N}'} \mu(Q^*_\Delta) \geq \mu(\Gamma) \approx 1.
\]

We take \(\{ c_{\Delta_{\nu}} \psi_{\Delta_{\nu}} \}_{\nu \in \mathcal{N}'}\) as the desired function set. It remains to apply Lemma 3 with \(m = \hat{k}t\) and (48). \(\square\)

Let us prove Proposition 2. Since \(\beta - \delta + \theta \left( \frac{1}{q} - \frac{1}{p} \right) = 0\) and \(\beta_v = \frac{d - \delta}{q}\), then

\[
\beta_g = r - \frac{d}{p} + \frac{\theta}{p}. \tag{49}
\]

Let \(t \in \mathbb{N}\) be sufficiently large, and let \(\Delta \in \Xi_{\hat{k}t} \left( \left[ -\frac{1}{2}, \frac{1}{2} \right] \right)\), \(\Delta \cap \Gamma \neq \emptyset\). For each \(s \in \mathbb{Z}_+\) we set

\[
\{ \Delta_{s,i} \}_{i \in J_s} = \left\{ \Delta' \in \Xi_{\hat{k}(t+s)} \left( \left[ -\frac{1}{2}, \frac{1}{2} \right] \right) : \Delta' \subset \hat{Q}_\Delta, \ \Delta' \cap \Gamma \neq \emptyset \right\}. \tag{50}
\]

Let

\[
f_{\Delta}(x) = \sum_{s=0}^t \sum_{i \in J_s} \psi_{\Delta_{s,i}}(x),
\]

where functions \(\psi_{\Delta_{s,i}}\) are defined by formula similar to (42).

There are a number \(t_0 = t_0(3^*)\) and a cube \(\Delta_0 \in \Xi_{\hat{k}(t-t_0)} \left( \left[ -\frac{1}{2}, \frac{1}{2} \right] \right)\) such that \(\Delta \subset \Delta_0\), \(\Gamma \cap \Delta_0 \neq \emptyset\) and \(\text{supp} f_{\Delta} \subset \hat{Q}_{\Delta_0}\).
Let \( l \in \mathbb{Z}_+ \), \( x \in \hat{E}_l(\Delta_0) \setminus \hat{E}_{l+1}(\Delta_0) \) (see (37) with \( m = \hat{k}(t-t_0) \)). Then \( \text{dist}_{|1|}(x, \Gamma) \gtrsim 2^{-k(t+l)} \). We estimate \( \left| \frac{\nabla^r f_\Delta(x)}{g(x)} \right| \) from above. If \( x \in \text{supp } \psi_{\Delta,s,i} \) for some \( i \in J_s \), then

\[
\left| \frac{\nabla^r f_\Delta(x)}{g(x)} \right| \lesssim \frac{1}{3_s} \left( \frac{1}{5} \right) \sum_{s=0}^{l+s_0} 2^{-\beta_\vartheta k(t+l)} (\hat{k}(t+l))^\alpha \rho_g^{-1}(\hat{k}(t+l)) \cdot 2^{rk(t+s)}.
\]

Moreover, by (50) we get \( s \leq l + s_0 \) with \( s_0 = s_0(3_s) \). Since \( \text{supp } \psi_{\Delta,s,i} \subseteq \hat{Q}_{\Delta,s,i} \), by the definition of \( \hat{Q}_{\Delta,s,i} \), it follows that for any \( x \in \hat{Q}_{\Delta_0} \) the inequality \( \{i \in J_s : x \in \text{supp } \psi_{\Delta,s,i}\} \lesssim 1 \) holds. Hence, for \( l \leq t - s_0 \)

\[
\left| \frac{\nabla^r f_\Delta(x)}{g(x)} \right| \lesssim \frac{1}{3_s} \left( \frac{1}{5} \right) \sum_{s=0}^{l+s_0} 2^{-\beta_\vartheta k(t+l)} (\hat{k}(t+l))^\alpha \rho_g^{-1}(\hat{k}t),
\]

and for \( l > t - s_0 \)

\[
\left| \frac{\nabla^r f_\Delta(x)}{g(x)} \right| \lesssim \frac{1}{3_s} \left( \frac{1}{5} \right) \sum_{s=0}^{l} 2^{-\beta_\vartheta k(t+l)} (\hat{k}(t+l))^\alpha \rho_g^{-1}(\hat{k}(t+l)) \cdot 2^{rk(t+s)} \lesssim 2^{-\beta_\vartheta k(t+l)} (\hat{k}(t+l))^\alpha \rho_g^{-1}(\hat{k}(t+l)) \cdot 2^{rk t}.
\]

This yields that

\[
\left\| \frac{\nabla^r f_\Delta}{g} \right\|_{L_p(\Omega)} \lesssim \sum_{l=0}^{\infty} \left\| \frac{\nabla^r f_\Delta}{g} \right\|_{L_p(\hat{E}_l(\Delta_0) \setminus \hat{E}_{l+1}(\Delta_0))} \lesssim 3_s
\]

\[
\leq \sum_{l=0}^{t-s_0} 2^{p(r-\beta_\vartheta)(\hat{k}(t+l))^\alpha \rho_g^{-p}(\hat{k}t)} \cdot 2^{-ktd-(d-\theta)\hat{k}t} (\hat{k})^\gamma \tau(\hat{k}t) + (\hat{k}(t+l))^\gamma \tau(\hat{k}(t+l)) \lesssim 2^{-\hat{k}t \theta} (\hat{k})^\alpha \rho_g^{-1}(\hat{k}t).
\]

Thus,

\[
\left\| \frac{\nabla^r f_\Delta}{g} \right\|_{L_p(\Omega)} \lesssim 2^{\frac{kdt}{\rho} \rho_g^{-1}(\hat{k}t)}. \quad (51)
\]

Let us estimate \( \| f_\Delta \|_{L_{q,p}(\Omega)} \) from below. Let \( x \in E_l(\Delta) \setminus \hat{E}_{l+1}(\Delta) \). Then \( \text{dist}_{|1|}(x, \Gamma) \lesssim 3_s \), \( 2^{-k(t+l)} \) and there exists \( l_0 = l_0(3_s) \) such that for \( 0 \leq s \leq l - l_0 \) there exists \( i_s \in J_s \).
such that \( x \in \hat{Q}_{\Delta_{s_{i_s}}} \). (Indeed, since \( x \in Q_\Delta \) by (57), there exists a point \( y \in \Gamma \cap \hat{Q}_{\Delta} \) such that \( |x - y| \geq 2^{-k(t+s)}\). We choose a cube \( \Delta_{s_{i_s}} \) that contains the point \( y \). By the definition of the cube \( Q_{\Delta_{s_{i_s}}} \), we have \( \hat{x}_{\Delta_{s_{i_s}}} \in \Delta_{s_{i_s}} \); hence, \( |y - \hat{x}_{\Delta_{s_{i_s}}}| \leq 2^{-k(t+s)} \).

Therefore, \( |x - \hat{x}_{\Delta_{s_{i_s}}}| \leq |x - y| + |y - \hat{x}_{\Delta_{s_{i_s}}}| \leq c(3_s)2^{-k(t+s)} + 2^{-k(t+s)} \) for some \( c(3_s) > 0 \). It remains to apply (36), (50) and the inequality \( s \leq l - l_0 \). Hence, for \( \frac{1}{2} \leq l \leq t \) we have \( |f_\Delta(x)| \geq \frac{t}{3} \). Consequently,

\[
\|f_\Delta\|_{L_{q,v}(\Omega)}^q \geq \sum_{t/2 \leq l \leq t} \|f_\Delta\|_{L_{q,v}(E_l(\Delta) \setminus E_{l+1}(\Delta))}^q \gtrsim \frac{t}{3}.
\]

**Proof of Proposition 2.** Let the set of cubes \( \{\Delta_{v}\}_{v \in \mathcal{N}} \) be defined by formula (57), and let \( F_{\Delta_v} = c_{\Delta_v} f_{\Delta_v} \), with \( c_{\Delta_v} \) such that \( \|\nabla^r F_{\Delta_v}\|_{L_p(\Omega)} = 1 \). From (51) and (52) it follows that

\[
\|F_{\Delta_v}\|_{L_{q,v}(\Omega)} \gtrsim \frac{t}{3} \cdot 2^{\beta_0 qk(t+l)} \cdot 2^{-k(l - d - \theta)q} \cdot 2^{-k(l - d - \theta)q} \cdot 2^{-\alpha v \rho_v(kt)\gamma(kt)} \cdot \frac{(kt)^\gamma(kt)}{(k(t + l))^{\gamma(kt + l)}} \gtrsim \frac{t}{3}.
\]

\[\|f_\Delta\|_{L_{q,v}(\Omega)} \gtrsim 2^{-\alpha v q + \frac{1}{2} v} \rho_v(kt);\]
i.e.,

\[\|f_\Delta\|_{L_{q,v}(\Omega)} \gtrsim 2^{-\alpha v q + \frac{1}{2} v} \rho_v(kt).\]

**Remark 5.** Let \( \beta_v = \frac{d - \theta}{q} \), \( \beta - \delta + \theta \left( \frac{1}{q} - \frac{1}{p} \right)_+ = 0 \). In addition, let \( \alpha < \frac{1}{q} \) in the case \( 1 < p < q < \infty \), and let \( \alpha < 1 + (1 - \gamma) \left( \frac{1}{q} - \frac{1}{p} \right) \) in the case \( p \geq q \). Then Propositions 1 and 2 hold; it implies that \( \vartheta_n(W_{p,q}^r(\Omega), L_{q,v}(\Omega)) = \infty \) for any \( n \in \mathbb{Z}_+ \).

In particular, if we take \( \vartheta_n = d_n \), then we get that the deviation of \( W_{p,q}^r(\Omega) \) from any finite-dimensional subspace is infinite.

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