On rime Ansatz

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Abstract

The ice Ansatz on matrix solutions of the Yang–Baxter equation is weakened to a condition which we call rime. Generic rime solutions of the Yang–Baxter equation are described. We prove that the rime non-unitary (respectively, unitary) R-matrix is equivalent to the Cremmer–Gervais (respectively, boundary Cremmer–Gervais) solution. Generic rime classical r-matices satisfy the (non-)homogeneous associative classical Yang-Baxter equation.

Let $V$ be a vector space. Among solutions $\hat{R} \in \text{End} (V \otimes V)$ of the Yang-Baxter equation

$$
(\hat{R} \otimes \mathbb{1})(\mathbb{1} \otimes \hat{R})(\hat{R} \otimes \mathbb{1}) = (\mathbb{1} \otimes \hat{R})(\hat{R} \otimes \mathbb{1})(\mathbb{1} \otimes \hat{R}),
$$

(1)

there is a class characterized by the so called ice condition (see lectures [1] for details) which says that $\hat{R}_{ij}^{kl}$ can be different from zero only if the sets of upper and lower indices coincide,

$$
\hat{R}_{kl}^{ij} \neq 0 \quad \Rightarrow \quad \{i, j\} \equiv \{k, l\}.
$$

We introduce the rime Ansatz, relaxing the ice condition: the entry $\hat{R}_{ij}^{kl}$ can be different from zero if the set of the lower indices is a subset of the set of the upper indices,

$$
\hat{R}_{kl}^{ij} \neq 0 \quad \Rightarrow \quad \{k, l\} \subset \{i, j\}.
$$

Matrices for which it holds will be referred to as rime matrices. A rime matrix has the structure (to avoid redundancy, we fix $\beta_{ii} = 0, \gamma_{ii} = 0 = \gamma'_{ii}$)

$$
\hat{R}_{kl}^{ij} = \alpha_{ij}\delta_i^k\delta_j^l + \beta_{ij}\delta_k^i\delta_l^j + \gamma_{ij}\delta_k^i\delta_l^j + \gamma'_{ij}\delta_k^i\delta_l^j \quad \text{(no summation)}.
$$

(2)

We call a rime matrix “strict” if $\alpha_{ij}\gamma_{ij} \neq 0 \forall i$ and $j, i \neq j$. Strict rime matrices are necessarily not ice.

All propositions hereafter are given without proofs, for further details see [2].

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Proposition 1. The Yang-Baxter equation implies that the quantity $\beta := \beta_{ij} + \beta_{ji}$ does not depend on $i$ and $j$. A strict rime Yang-Baxter solution \cite{2} has, up to a change of basis, the form

$$
\hat{R}_{kl}^{ij}(\beta_{ij}) = (1 - \beta_{ji})\delta_{i}^{j}\delta_{k}^{l} + \beta_{ij}\delta_{k}^{i}\delta_{l}^{j} - \beta_{ij}\delta_{i}^{l}\delta_{k}^{j} + \beta_{ji}\delta_{i}^{j}\delta_{k}^{i}.
$$

(4)

Lemma 1. The rime Yang-Baxter solution \cite{4} is of Hecke type,

$$
\hat{R}^2(\beta_{ij}) = \beta \hat{R}(\beta_{ij}) + (1 - \beta) I , \quad I := 1 \otimes 1.
$$

(5)

When $\beta \neq 2$, $\hat{R}(\beta_{ij})$ is of $\mathfrak{gl}$-type: multiplicities of the eigenvalues of 1 and $\beta - 1$ are, respectively, $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$.

Proposition 2. The non-unitary ($\beta \neq 0$) strict rime Yang-Baxter solutions \cite{4} are parameterized by a point $\phi \in \mathbb{P}^n\mathbb{C}$ in a projective space, $\phi = (\phi_1 : \phi_2 : \ldots : \phi_n)$, such that $\phi_i \neq 0$ and $\phi_i \neq \phi_j$, $i \neq j$. The $\beta_{ij}$’s of the solution $\hat{R}(\beta_{ij})$ are given by

$$
\beta_{ij} = \frac{\beta \phi_i}{\phi_i - \phi_j}, \quad \hat{R} := \hat{R}(\frac{\beta \phi_i}{\phi_i - \phi_j}).
$$

(6)

If one and only one $\phi_i = 0$, then $\hat{R}$ is again rime Yang-Baxter solution (it is not strict).

Proposition 3 The unitary ($\beta = 0$) rime $R$-matrices $\hat{R}(\beta_{ij}^{(0)})$ are parameterized by a vector $(\mu_1, \ldots, \mu_n)$ such that $\mu_i \neq \mu_j$,

$$
\beta_{ij}^{(0)} = \frac{1}{\mu_i - \mu_j}, \quad \hat{R}_0 := \hat{R}(\beta_{ij}^{(0)}).
$$

(7)

The Cremmer-Gervais solution \cite{3} of the Yang-Baxter equation in its general two-parametric form reads (we use a rescaled matrix with eigenvalues 1 and $-q^{-2}$)

$$
(\hat{R}_{CG,p})_{kl}^{ij} = q^{-2\theta_{ij}}p^{i-j}\delta_{k}^{i}\delta_{l}^{j} + (1 - q^{-2}) \sum_{i \leq s < j} p^{i-s}\delta_{s}^{i}\delta_{l}^{i+j-s} - (1 - q^{-2}) \sum_{j < s < i} p^{i-s}\delta_{s}^{i}\delta_{l}^{i+j-s},
$$

(8)

where $\theta_{ij}$ is the step function ($\theta_{ij} = 1$ when $i > j$ and $\theta_{ij} = 0$ when $i \leq j$).

Proposition 4 The rime $R$-matrix $\hat{R}$ \cite{5} transforms to the Cremmer-Gervais matrix

$$
\hat{R} = (X(\phi) \otimes X(\phi)) \hat{R}_{CG,1} (X^{-1}(\phi) \otimes X^{-1}(\phi)), \quad \beta = 1 - q^{-2},
$$

(9)

by a change of basis with the matrix $X(\phi)$ whose entries are the elementary symmetric polynomials $e_i(x_1, \ldots, x_N) = \sum_{s_1 < \ldots < s_i} x_{s_1}x_{s_2} \ldots x_{s_i}$,

$$
X_{j}^{k}(\phi) := e_{j}(\phi_1, \ldots , \hat{\phi}_k, \ldots , \phi_n), \quad \det X = \prod_{j<k}(\phi_j - \phi_k).
$$

(10)

We now consider solutions of the classical Yang-Baxter (cYB) equation (i.e., classical $r$-matrices), $[r_{12}, r_{23}] + [r_{12}, r_{13}] + [r_{13}, r_{23}] = 0$ which are (quasi-)classical limits of the rime $R$-matrices above. Denote by $P$ the permutation operator, $P x \otimes y = y \otimes x$.
Corollary 1 The non-unitary rime R-matrix \( R = P\hat{R} \) is linear in the parameter \( \beta \),
\[
R = I + \beta r , \quad \text{where} \quad r = \sum_{i,j:i\neq j} \frac{1}{\phi_i - \phi_j} (\phi_i e^i_j - \phi_j e^j_i) \otimes (e^i_j - e^j_i) .
\] (11)

\( R \) is a quantization of the non-skew-symmetric rime \( r \)-matrix, \( r + r_{21} = P - I \). Similarly \( R_{CG,1} := P\hat{R}_{CG,1} = I + \beta r_{CG} \), thus the matrix \( r \) is equivalent to the matrix
\[
r_{CG} = \sum_{i<j} \sum_{s=1}^{j-i} (e^i_{i+s-1} \otimes e^i_{j-s+1} - e^i_{i+s-1} \otimes e^i_{j-s+1}) , \quad r = \text{Ad}_{X(\phi)} \otimes \text{Ad}_{X(\phi)}(r_{CG}) .
\] (12)

The non-skew-symmetric cYB solution such as \( r \) and \( r_{CG} \) are classified by Belavin-Drinfeld triples [4]. Gerstenhaber and Giaquinto introduced the notion of boundary skew-symmetric cYB solution: a solution which lies on the boundary of the space of skew-symmetric solutions of the modified classical Yang-Baxter equation [5] (these solutions are in turn into one-to-one correspondence with non-skew-symmetric solutions of cYB).

Proposition 5 The unitary rime R-matrix \( R_0 = P\hat{R}_0 \) is the quantization\(^1\) of the skew-symmetric rime matrix \( r_0 \)
\[
R_0 = 1 + r_0 , \quad r_0 = \sum_{i,j:i<j} \beta_{ij}^{(0)} Z^i_j \wedge Z^j_i ,
\] (13)

where \( Z^i_j := e^i_j - e^j_i \) generate a subalgebra of the matrix algebra and \( x \wedge y := x \otimes y - y \otimes x \). The cYB solution \( r_0 \) is equivalent to the Cremmer-Gervais boundary solution [5, 6].

\[
b = \sum_{i<j} \sum_{k=1}^{j-i} e^{i+k}_i \wedge e^{j-k+1}_j , \quad r_0 = \text{Ad}_{X(\mu)} \otimes \text{Ad}_{X(\mu)}(b) .
\] (14)

By a sophisticated construction [5] one can show that \( b \) is the boundary solution attached to \( r_{CG} \) [6]. In the rime basis, the cYB solutions \( b \) and \( r_{CG} \) are transformed to the \( r \)-matrices \( r_0 \) and \( r \), respectively. One of advantages of the rime basis is that the unitary limit is explicit: expand \( \phi_i = 1 + \beta \mu_i + o(\beta) \) and take the limit
\[
\beta_{ij} = \frac{\beta (1 + \beta \mu_i + o(\beta))}{\beta \mu_i - \beta \mu_j + o(\beta)} \xrightarrow{\beta \to 0} \beta_{ij}^{(0)} = \frac{1}{\mu_i - \mu_j} .
\]

Hence \( R \xrightarrow{\beta \to 0} R_0 \) and \( \beta r \xrightarrow{\beta \to 0} r_0 \) (but the transform \( X(\phi) \) becomes singular in the limit).

On the matrix level one can write \( R_0 \) as an exponential due to the nilpotency of \( r_0 \) [6]
\[
r_0^2 = 0 \quad \Rightarrow \quad R_0 = e^{r_0} = I + r_0 .
\]
The idempotency of \( r \) yields a similar exponential formula for \( R \)
\[
r^2 = -r \quad \Rightarrow \quad R = e^{hr} = I + (1 - e^{-h}) r ,
\]
so the quasi-classical approximation is exact in the renormalized parameter \( \beta = \beta(h) = 1 - e^{-h} \).

\(^1\)The small parameter can be absorbed in the \( \mu \)'s
Consider the splitting of the cYB equation: \( A'(r) - A(r) = 0, \)
\[
A(r) := r_{13} r_{12} - r_{12} r_{23} + r_{23} r_{13}, \quad A'(r) := r_{12} r_{13} - r_{23} r_{12} + r_{13} r_{23}.
\]

It was shown in [7] that a solution of the associative cYB (acYB) equation \( A(r) = 0 \) defines the Newtonian coalgebra structure of an infinitesimal bialgebra [8]. The boundary \( r \)-matrices \( b \) and \( r_0 \) satisfy the acYB equations \( A(b) = 0, \ A(r_0) = 0. \)

The non-skew-symmetric \( r \)-matrices \( r_{CG} \) and \( r \) satisfy the non-homogeneous acYB equation
\[
A(r) = -r_{13}.
\]
and \( r + r_{21} = P - I \) (thus \( A(\tilde{r}) = \frac{1}{4} I \otimes I \otimes I, \ \tilde{r} + \tilde{r}_{21} = P \); where \( \tilde{r} = r + \frac{1}{2} I \)).

In the associative algebra generated by \( r_{13}, r_{23}, r_{12} \) having as relations the non-homogeneous acYB equations \( A(r) = \xi r_{13}, \ A'(r) = \xi r_{13} \) and the Hecke condition \( r_{ij}^2 = \xi r_{ij} + \eta, \ \{ij\} = \{13\}, \{23\}, \{12\} \) (\( \xi \) and \( \eta \) are arbitrary constants) the following identities hold [2]
\[
r_{12} r_{23} r_{12} = r_{23} r_{12} r_{23}, \quad r_{12} r_{13} r_{23} = r_{23} r_{13} r_{12}.
\]
i.e., the matrix \( r \) satisfies both forms of the “quantum” Yang-Baxter equation.

The acYB (respectively, non-homogeneous acYB) solutions are related to the Rota–Baxter operators of zero (respectively, non-zero) weight. In the example of integration and summation operators, the deformation from zero to non-zero weight is given by the Euler-Maclaurin formula [9].

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