ELLIPTIC BOUNDARY VALUE PROBLEMS FOR THE STATIONARY VACUUM SPACETIMES

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Abstract. We develop a general method of proving the ellipticity of boundary value problems for the stationary vacuum space time, by showing that the stationary vacuum field equations are elliptic subjected to a geometrically natural collection of boundary conditions in the projection formalism. Using this we prove that the moduli space of stationary vacuum spacetimes admits Banach manifold structure.

1. Introduction

A stationary spacetime \((V^{(4)}, g^{(4)})\) is a 4-manifold with a smooth Lorentzian metric \(g^{(4)}\) of signature \((-,+,+,+),\) which has a time-like Killing vector field. A trivial example is the flat Minkowski space \((\mathbb{R}^{1,3}, g_{\text{Min}})\), where

\[ g_{\text{Min}} = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2. \]

Stationary vacuum spacetimes are stationary spacetimes \((V^{(4)}, g^{(4)})\) that solve the vacuum Einstein field equations

\[ \text{Ric}_{g^{(4)}} = 0. \]

They are important and much studied in general relativity. There are two famous nontrivial examples: the Schwarzschild metric,

\[ ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2); \]

and the Kerr metric,

\[ ds^2 = -(1 - \frac{2Mr}{\Sigma})dt^2 - \frac{4Marsin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + (r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma})\sin^2\theta d\phi^2, \]

cf.\cite{W}.

Throughout this paper, we assume that the spacetime \((V^{(4)}, g^{(4)})\) is \textit{globally hyperbolic}, i.e. it admits a Cauchy surface \(\Sigma\). The topology of a globally hyperbolic spacetime \(V^{(4)}\) is necessarily \(\Sigma \times \mathbb{R}\). In the following, we recall two well-known formulations of the stationary vacuum field equations — the hypersurface formalism and the projection formalism.

In the hypersurface formalism, one can define a global time function \(t\) on \(V^{(4)}\) so that \(\partial_t\) is the Killing field. The metric \(g^{(4)}\) may then be written globally in the form

\[ g^{(4)} = -N^2 dt^2 + (g_{\text{M}})_{ij}(dx^i + Y^i dt)(dx^j + Y^j dt), \]
where \( \{x^i\} (i = 1, 2, 3) \) are local coordinates of the space-like hypersurface \( M = \{t = 0\} \) in \( V^{(4)} \). The lapse function \( N \), the shift vector \( Y = Y^i \partial x_i \), and the induced metric \( g_M = (g_M)_{ij} dx^i dx^j \) are all independent of the time \( t \).

The stationary spacetime \((V^{(4)}, g^{(4)})\) is vacuum if and only if the following stationary vacuum field equations hold on \( M \), cf. [M],

\[
\begin{aligned}
2NK - LYg &= 0, \\
\text{Ric}_{g_M} + (\text{tr} K)K - 2K^2 - \frac{1}{6} D^2 N + \frac{1}{4} L_Y K &= 0, \\
s_{g_M} + (\text{tr} K)^2 - |K|^2 &= 0, \\
\delta K + d(\text{tr} K) &= 0.
\end{aligned}
\]

(1.3)

Here \( D^2 N \) denotes the Hessian of function \( N \); \( s_{g_M} \) denotes the scalar curvature of the metric \( g_M \) on \( M \); and \( K \) is the second fundamental form of the hypersurface \( M \subset (V^{(4)}, g^{(4)}) \).

It is known and easy to see that when the hypersurface \( M \) is a closed 3-manifold, there are no non-flat stationary vacuum solutions to the field equations (1.3). Lichnerowicz (cf. [L]) proved that a geodesically complete stationary vacuum spacetime is necessarily flat Minkowski space \( g_{\text{Min}} \) when the 3-manifold \( M \) is complete and asymptotically flat, cf. also [A2] for a generalization with no asymptotic condition.

Thus nontrivial solutions of (1.3) only exist on 3-manifolds with nonempty boundary. The standard case is when \( M \) is diffeomorphic to \( B^3 \) (the interior case) or \( \mathbb{R}^3 \setminus B^3 \) (the exterior case), where \( B^3 \) is the unit 3-ball. This paper concerns the exterior case with asymptotical flatness assumption, cf. § 2.1.

Since the boundary \( \partial M \) is nonempty, the issue of boundary conditions arises. In [B1], through a Hamiltonian analysis of the vacuum Einstein equation, Bartnik proposed a collection of boundary data given by

\[
(g_M^T, H_{g_M}, K(n)^T, tr^T K),
\]

which consists of the tangential restriction \( g_M^T \) of the metric \( g_M \) to the boundary \( \partial M \), the mean curvature \( H_{g_M} \) of \( \partial M \subset (M, g_M) \), the mixed (perpendicular and tangential) part \( K(n)^T \) of \( K \) on \( \partial M \), and the tangential trace \( tr^T K \) of the second fundamental form \( K \) along \( \partial M \).

In [B1], Bartnik conjectured that among all the admissible asymptotically flat extensions of such boundary data to \( \mathbb{R}^3 \setminus B^3 \), the infimum of the ADM mass is realized by one arising from a stationary vacuum spacetime, that is a set of data \((g, N, Y)\) on \( M \) satisfying the stationary vacuum Einstein field equations. A natural question raised in [B1] is whether the stationary vacuum Einstein field equations equipped with the Bartnik boundary conditions are elliptic. This is one of the main motivations for the present paper.

In fact, to check the ellipticity of boundary problems for the stationary vacuum equations, it is essential to find the right gauge terms and compute the principal symbols. However, it is very complicated to carry out this work in the hypersurface formalism. So alternatively, we use the projection formalism in this paper.

In the projection formalism, we use \( S \) to denote the collection of all trajectories of the Killing field \( \partial_t \) in \((V^{(4)}, g^{(4)})\), i.e. \( S \) is the orbit space of the action of 1-parameter group \( \mathbb{R} \) generated by \( \partial_t \). Since the spacetime is globally hyperbolic, the quotient space \( S \) is a smooth 3-manifold and the metric \( g^{(4)} \) restricted to the horizontal distribution — the orthogonal complement of \( \text{span}\{\partial_t\} \) in \( TV^{(4)} \) — induces a well-defined Riemannian metric \( g_S \) on \( S \). Let \( \pi : V^{(4)} \to S \) denote the projection...
Elliptic boundary value problems for the stationary vacuum spacetimes

map, then metric $g^{(4)}$ is globally of the form

\begin{equation}
(1.4) \quad g^{(4)} = -e^{2u}(dt + \theta)^2 + \pi^*g_S.
\end{equation}

Here $\theta$ is a 1-form on $S$ so that the dual of the Killing vector field $\partial_t$ is $\xi = -e^{2u}(dt + \theta)$. The twist tensor $\omega$ is defined as

\begin{equation}
(1.5) \quad \omega = \frac{1}{2} \star_{g^{(4)}} (\xi \wedge d\xi),
\end{equation}

where $\star_{g^{(4)}}$ denotes the hodge star operator of the metric $g^{(4)}$. The twist tensor provides a measurement of the integrability of the horizontal distribution $TS$ in $V^{(4)}$. It actually lives on the quotient manifold $S$, because equation (1.5) is equivalent to

\[
\omega = \frac{1}{2} e^{3u} \star g_S \, d\theta.
\]

It is easy to observe that under the reparametrization of time

\[
t' = t + f,
\]

where $f$ is a function on $S$, the formula (1.4) becomes

\[
g^{(4)} = -e^{2u}(dt' + \theta') + \pi^*g_S,
\]

with $\theta' = \theta - df$. The twist tensor $\omega$ remains invariant under this gauge transformation. Therefore, a stationary spacetime $(V^{(4)}, g^{(4)})$ corresponds uniquely to a collection of data $(g_S, u, d\theta)$ or $(g_S, u, \omega)$ on the quotient manifold $S$. We refer to [K] and [CH] for more details of the projection formalism.

Notice that the restriction $(\pi|_M : M \rightarrow S)$ of the projection $\pi$, gives a diffeomorphism between the hypersurface $M$ and the quotient manifold $S$. Thus boundary value problems in the setting $\{M, (g_M, N, Y)\}$ can be transferred to equivalent boundary value problems for $\{S, (g_S, u, \omega)\}$ via this diffeomorphism and vice versa. In certain respects, the projection formalism is more canonical, since there are many distinct hypersurfaces giving rise to the same stationary solution on the $4$-manifold, but the projection data is unique.

The stationary vacuum field equations in the projection formalism, which are equivalent to (1.3) in the hypersurface formalism, are given by, cf.[H1],[H2],

\[
\begin{cases}
Ric_{g_S} - D^2u - (du)^2 - 2e^{-4u}(\omega \otimes \omega - |\omega|^2g_S) = 0, \\
\Delta_{g_S}u - |du|^2 - 2e^{-4u}|\omega|^2 = 0, \\
\delta\omega + 3\langle du, \omega \rangle = 0, \\
d\omega = 0.
\end{cases}
\]

Here $\Delta_{g_S}$ denotes the geometric Laplacian operator of the metric $g_S$, i.e. $\Delta_{g_S}u = -\text{tr}_{g_S} D^2g_Su$. The last equation indicates that $\omega$ is exact. In the case $S \cong \mathbb{R}^3 \setminus B^3$, we can assume $\omega = d\phi$ for some function $\phi$ on $S$. Thus the system above can be expressed equivalently as,

\[
(1.6) \begin{cases}
Ric_{g_S} - D^2u - (du)^2 - 2e^{-4u}(d\phi \otimes d\phi - |d\phi|^2g_S) = 0, \\
\Delta_{g_S}u - |du|^2 - 2e^{-4u}|d\phi|^2 = 0, \\
\Delta_{g_S}\phi + 3\langle du, d\phi \rangle = 0.
\end{cases}
\]

Compared with the system (1.3), the work of choosing proper gauge terms and dealing with the principal symbols turns out to be much easier in (the conformal transformation of) the system above. Thus it is of interest to study the ellipticity
of the system (1.6) with geometrically natural prescribed boundary conditions on the quotient manifold $S$.

Rather than transforming the Bartnik boundary conditions from the slice $M$ to the quotient space $S$, here we analyze some much simpler boundary conditions arising naturally from the projection formalism. In view of the Bartnik conditions, we choose $(g_T^S, H_{gS})$ — the tangential restriction $g_T^S$ of the metric $g_S$ to the boundary $\partial S$ and the mean curvature $H_{gS}$ of the boundary $\partial S \subset (S, g_S)$ — to be prescribed. In addition, we pose a restriction on the twist tensor $\omega$, by requiring $\omega(n) = n(\phi)$ fixed on the boundary $\partial S$, where $n$ is the unit normal vector of the boundary pointing outwards. Actually the collection of boundary data,

$$(1.7) \quad \{g_T^S, H, n(\phi)\},$$

also arises naturally from the boundary terms in the variation of a functional on $S$, which is the reduction of the Einstein Hilbert action from $V^{(4)}$ to $S$, c.f. §3.

The first main theorem we will prove is the ellipticity of the boundary data (1.7).

**Theorem 1.1.** The stationary vacuum field equations (1.6) and boundary conditions (1.7) form an elliptic boundary value problem, modulo gauge transformations.

To prove this theorem, we first present in §2 the conformal transformation of the vacuum field equations, which gives a differential operator with simpler symbols. For the purpose of ellipticity, we modify the equations using certain gauge terms. After that, in §3 ellipticity for (the conformal transformation of) the boundary conditions (1.7) is proved with respect to different choices of gauge terms.

**Remark.** The method we use to prove ellipticity in this paper is not only valid for the boundary conditions (1.7). It can also be applied to more general boundary value problems for the stationary vacuum field equations.

In §4 we prove a manifold structure theorem for the moduli space $\mathcal{E}_C = \mathcal{E}_C^{m,\alpha}$ of stationary vacuum spacetimes. The space $\mathcal{E}_C$ is basically the space of all $C^{m,\alpha}$ asymptotically flat stationary vacuum solutions to the system (1.6) on $S$ modulo the action of the group $\mathcal{D}_0^{m+1,\alpha}(S)$ of diffeomorphisms on $S$ that are equal to the identity map when restricted on $\partial S$. In addition, based on the boundary conditions (1.7), we have a natural map $\Pi$, from the moduli space $\mathcal{E}_C$ to the space of boundary data defined as follows,

$$(1.8) \quad \Pi(g_S, u, \phi) = (g_T^S, H, n(\phi)).$$

Here $\text{Met}^{m,\alpha}(\partial S)$ is the space of $C^{m,\alpha}$ metrics on $\partial S$; $C^{m-1,\alpha}(\partial S)$ is the space of $C^{m-1,\alpha}$ functions on $\partial S$. By applying the ellipticity result, we will prove the following theorem.

**Theorem 1.2.** The moduli space $\mathcal{E}_C$ is an infinite dimensional $C^\infty$ Banach manifold, and the map $\Pi$ is $C^\infty$ smooth and Fredholm, of Fredholm index $0$.

The theorems we prove in this paper are generalizations of the results proved in [AK], where spacetimes are static. Related work can be found in [J],[M1-3],[R] and elsewhere. In a sequel to this work, we plan to discuss ellipticity of the more complicated Bartnik boundary conditions.

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2. Background Discussion

2.1. Asymptotic flatness. Throughout this paper, the quotient manifold $S$ is assumed to be diffeomorphic to $\mathbb{R}^3 \setminus B^3$, with $B^3$ the unit 3-ball. When it goes to the infinite end of $S$, we assume that the data $(g_S, u, \phi)$ is asymptotically flat, in the sense that

$$g - g_F \to 0, \quad u \to 0, \quad \phi \to 0, \quad \text{as} \quad r \to \infty,$$

where $g_F$ is the flat metric on $\mathbb{R}^3 \setminus B^3$ and $r$ is the pull back to $S$ of the radius function on $\mathbb{R}^3 \setminus B^3$ under a fixed diffeomorphism. To describe rigorously the decay behavior above, we use the weighted Holder spaces defined as follows, cf. [B2], [LP].

**Definition 2.1.** We define several Banach spaces for $m \in \mathbb{N}$ and $\alpha, \delta \in \mathbb{R}$ on a general Riemannian manifold $M \cong \mathbb{R}^3 \setminus B^3$:

- $C^0_m(M) = \{\text{functions } v \text{ on } M : ||v||_{C^0_m} = \Sigma_{k=0}^m \sup_{x} r^{k+\delta} |\nabla^k v| < \infty\}$,
- $C^m_{\delta, \alpha}(M) = \{\text{functions } v \text{ on } M : ||v||_{C^m_{\delta, \alpha}} = \sup_{x,y} [\min(r(x), r(y))]^{m+\alpha+\delta} \frac{|\nabla^m v(x) - \nabla^m v(y)|}{|x - y|^\alpha} < \infty\}$,
- $Met_{\delta, \alpha}^m(M) = \{\text{metrics } g \text{ on } M : (g_{ij} - \delta_{ij}) \in C^m_{\delta, \alpha}(M)\}$,
- $(T_p^m)_{\delta, \alpha}(M) = \{\text{p - tensors } \tau \text{ on } M : \tau_{ij} \in C^m_{\delta, \alpha}(M)\}$,
- $(\wedge_p^m)_{\delta, \alpha}(M) = \{\text{p - forms } \sigma \text{ on } M : \sigma_{i_1 i_2 \ldots i_p} \in C^m_{\delta, \alpha}(M)\}$.

**Definition 2.2.** The data $(g_S, u, \phi)$ is called asymptotically flat of order $\delta$ if

$$(g_S, u, \phi) \in [Met_{\delta, \alpha}^m \times C^m_{\delta, \alpha} \times C^m_{\delta, \alpha}](S),$$

for some $m, \alpha$ and $\delta$.

Throughout the following, the orders $m, \alpha$ and the decay rate $\delta$ are fixed, and chosen to satisfy,

$$m \geq 2, \quad 0 < \alpha < 1, \quad \frac{1}{2} < \delta < 1.$$

**Remark.** In the previous section, we introduced the diffeomorphism $\pi|M : M \to S$ between the hypersurface $M$ and the quotient space $S$. In fact, under this diffeomorphism, the asymptotic flatness condition (2.1) of $(g_S, u, \phi)$ in $S$ is equivalent to the asymptotic condition in $M$:

$$(g_M, u, Y) \in [Met_{\delta, \alpha}^m \times C^m_{\delta, \alpha} \times (\wedge_{\delta})^m_{\delta, \alpha}](M),$$

which, furthermore, is equivalent to the decay behavior as in Bartnik’s work.

2.2. Conformal transformation. To simplify the symbols of the stationary field equations (1.6), we first apply a conformal transformation on the quotient manifold $S$:

$$g = e^{2u} g_S.$$  

Under such a transformation, the data $(g_S, u, \phi)$ is in 1-1 correspondence to the triple $(g, u, \phi)$; and if $(g_S, u, \phi)$ is asymptotically flat as described in §2.1, it also holds that the data $(g, u, \phi)$ is asymptotically flat, i.e.

$$(g, u, \phi) \in [Met_{\delta, \alpha}^m \times C^m_{\delta, \alpha} \times C^m_{\delta, \alpha}](S).$$
Furthermore, the stationary vacuum field equations (1.6), which are expressed in terms of \((g_S, u, \phi)\), can be simplified equivalently to the following system for \((g, u, \phi)\), cf. [K],

\[
\begin{align*}
(I) \quad & \begin{cases}
Ric_g - 2du \otimes du - 2e^{-4u}d\phi \otimes d\phi = 0, \\
\Delta_g u - 2e^{-4u}|d\phi|^2 = 0, \\
\Delta_g \phi + 4\langle du, d\phi \rangle = 0.
\end{cases}
\end{align*}
\]

Here we point it out that the field equations above can be expressed in an equivalent way, where the Ricci tensor in \(Ric_g\) is replaced by the Einstein tensor \(Ein_g = Ric_g - \frac{1}{2}s_g g\). In fact, the trace of the first equation is given by

\[
s_g - 2|du|^2 - 2e^{-4u}|d\phi|^2.
\]

Let \(T_g\) be the term

\[
T_g = \frac{1}{2}(s_g - 2|du|^2 - 2e^{-4u}|d\phi|^2)g.
\]

Then it is easy to see that, system \((I)\) is equivalent to the following system \((II)\) by inserting \(T_g\) into the first equation,

\[
(II) \quad \begin{cases}
Ric_g - 2du \otimes du - 2e^{-4u}d\phi \otimes d\phi - T_g = 0, \\
\Delta_g u - 2e^{-4u}|d\phi|^2 = 0, \\
\Delta_g \phi + 4\langle du, d\phi \rangle = 0.
\end{cases}
\]

Notice that by rearranging the terms, the first equation in \((II)\) can be expressed as

\[
(2.3) \quad (Ric_g - \frac{1}{2}s_g g) - 2du \otimes du - 2e^{-4u}d\phi \otimes d\phi + (|du|^2 + e^{-4u}|d\phi|^2)g = 0,
\]

where the leading term — the term with highest order of derivative with respect to the data \((g, u, \phi)\) — is exactly the Einstein tensor \((Ric_g - \frac{1}{2}s_g g)\).

Observe that the system \((I)\) (or \((II)\)) is not elliptic, because the full system is invariant under diffeomorphisms, i.e. if \((g, u, \omega)\) is a solution of the Einstein field equations, then the pull back data \(\Psi^*(g, u, \omega)\) under some diffeomorphism \(\Psi\) on \(S\), is also a solution. So to ensure ellipticity, as is usual we modify the system using gauge terms, and obtain

\[
(III) \quad \begin{cases}
Ric_g - 2du \otimes du - 2e^{-4u}d\phi \otimes d\phi + T + \delta^*G = 0, \\
\Delta_g u - 2e^{-4u}|d\phi|^2 = 0, \\
\Delta_g \phi + 4\langle du, d\phi \rangle = 0.
\end{cases}
\]

The pair \((T, G)\) can be

\[
\begin{cases}
T_1 = 0, \\
G_1 = \beta_\tilde{g}(g),
\end{cases}
\]

where \(\tilde{g}\) is a reference metric near \(g\), and \(\beta\) is the Bianchi operator:

\[
\beta_\tilde{g}g = \delta_\tilde{g}g + \frac{1}{2}dtr_\tilde{g}g.
\]

This corresponds to inserting \(G_1 = \beta_\tilde{g}(g)\) (the Bianchi gauge) into the system \((II)\).

Alternately, one can set \((T, G)\) to be

\[
\begin{cases}
T_2 = -T_g, \\
G_2 = \delta_\tilde{g}(g),
\end{cases}
\]
which corresponds to inserting $G_2 = \delta_3(g)$ (the divergence gauge) into (II).

We will be concerned with both distinct choices of gauge terms here. In the case $(T, G) = (T_1, G_1)$, the principal symbols of the system (III) are simple and ellipticity can be proved by straightforward computation. However, such a system is not self-adjoint, which makes it not suitable for the proof of the manifold theorem in §4. On the other hand, the system (III) with $(T, G) = (T_2, G_2)$ is formally self-adjoint, whereas its principal symbols are much more complicated. We will use the ellipticity result of the case $(T, G) = (T_1, G_1)$ to prove the ellipticity for the gauge choice $(T_2, G_2)$. We refer to §3 for more details.

Since the boundary $\partial S$ is not empty, it is necessary to include a boundary condition for the gauge term $G$. A convenient choice is

$$(2.4) \quad G = 0 \quad \text{on} \ \partial S.$$  

Next we will prove that, equipped with this boundary restriction, solutions to the gauged system (III) when $(T, G) = (T_2, G_2)$, correspond to solutions to the stationary vacuum system (II) modulo diffeomorphisms.

2.3. Moduli space of stationary vacuum spacetimes. We begin by defining the following subsets of the space of stationary vacuum solutions.

**Definition 2.3.**

$$E_C = \{(g,u,\phi) \in [\text{Met}^m_s \times C^m_r \times C^m_r](S) : \text{solutions of (II) with } \delta_3 g = 0 \text{ on } \partial S\};$$

$$Z_C = \{(g,u,\phi) \in [\text{Met}^m_s \times C^m_r \times C^m_r](S) : \text{solutions of (III) where } (T, G) = (T_2, G_2) \text{ and boundary condition (2.4)}\}.$$  

The following lemma shows that $Z_C \subset E_C$.

**Lemma 2.4.** Elements in $Z_C$ are elements in $E_C$ which satisfy the divergence gauge condition $\delta_3 g = 0$ in $S$.

**Proof.** It suffices to prove the gauge term $G_2$ is zero in (III) under the boundary condition (2.4). From the equation (2.3) we know that, if $(T, G) = (T_2, G_2)$ then leading term of the first equation in (III) is the Einstein tensor $Ein_g$, where we have the Bianchi identity, $\delta_3 Ein_g = 0$. Thus, taking the divergence (with respect to $g$) of the first equation in (III), we obtain

$$\delta_3 \{Ein_g - 2du \otimes du - 2e^{-4u}d\phi \otimes d\phi + (|du|^2 + e^{-4u}|d\phi|^2)g + \delta^* G_2 \} = 0,$$

which gives,

$$\delta_3 \{-2du \otimes du - 2e^{-4u}d\phi \otimes d\phi + (|du|^2 + e^{-4u}|d\phi|^2)g\} + \delta \delta^* G_2 = 0.$$

Basic computation gives

$$\delta \{-2du \otimes du - 2e^{-4u}d\phi \otimes d\phi + (|du|^2 + e^{-4u}|d\phi|^2)g\} = -2(\Delta_3 u)du - 8e^{-4u}(du, d\phi)d\phi - 2e^{-4u}(\Delta_3 \phi)d\phi + 4e^{-4u}du|d\phi|^2.$$

Together with the equations: $\Delta_3 u - 2e^{-4u}|d\phi|^2 = 0$ and $\Delta_3 \phi + 4(du, d\phi) = 0$ from (III), it is easy to see that the expression above is equal to zero, and consequently

$$\delta \delta^* G_2 = 0.$$
Thus we obtain the following system for $G_2$,
\[
\begin{cases}
\delta \delta^* G_2 = 0 \text{ on } S, \\
G_2 = 0 \text{ on } \partial S.
\end{cases}
\]
Integration by parts gives:
\[
0 = \int_S (\delta \delta^* G_2, G_2) = \int_S |\delta^* G_2|^2 - \int_{\partial S} \delta^* G_2(n, G_2) - \int_{\partial S} \delta^* G_2(n, G_2).
\]
Here the finite boundary term must vanish because $G_2 = 0$ on $\partial S$. Basic computation shows that the term $\delta^* G_2(n, G_2)$ decays at the rate $r^{-2\delta-2}$, so the boundary term at infinity is also zero. It follows that $\delta^* G_2 = 0$ in $S$. Thus, $G_2$ is a Killing field vanishing on $\partial S$, and hence $G_2 = 0$.

\[\square\]

Remark. The lemma above shows that adding the divergence gauge to the system (II) preserves the stationary vacuum property of the solutions. In contrast, it is unknown in general whether adding the Bianchi gauge to system (II) will work in the same way. In the case $(T, G) = (T_1, G_1)$, the leading term in the first equation of (III) is the Ricci tensor $Ric_g$. Thus instead of taking divergence as in the lemma 2.4, one needs to apply the Bianchi operator to the first equation, which yields $\beta \delta^* G_1 = 0$. The operator $\beta \delta^*$ is not positive in general, so the argument above does not apply to the Bianchi gauge.

Next, we will show $E_C \subset Z_C$ in the sense of moduli space.

First define a Banach space $X^m_\delta(S)$ of asymptotically flat vector fields vanishing on $\partial S$:
\[
X^m_\delta(S) = \{\text{vector fields } X \text{ on } S : X^i \in C^m_\delta(S) \text{ and } X = 0 \text{ on } \partial S\}.
\]
Then the following lemma holds for the space $X^m_\delta(S)$.

**Lemma 2.5.** The map $\delta \delta^* : X^m_\delta(S) \rightarrow (\Lambda^1)^{m-2,\alpha}_\delta(S)$ is an isomorphism.

**Proof.** From the proof of the previous lemma, one sees that kernel of $\delta \delta^* : X^m_\delta(S) \rightarrow (\Lambda^1)^{m-2,\alpha}_\delta(S)$ is zero. On the other hand, $\delta \delta^*$ is an elliptic operator with Fredholm index 0, thus it is an isomorphism.

\[\square\]

Next, let $D_0^{m+1,\alpha}(S)$ be the group of $C^{m+1,\alpha}_\delta$ diffeomorphisms of $S$ which equal to the identity map on $\partial S$. These are diffeomorphisms decaying asymptotically to the identity at the rate $r^{-\delta}$. The group $D_0^{m+1,\alpha}(S)$ acts freely and continuously on $Met^m_\delta(S)$ by pull back and one has the following local result.

**Theorem 2.6.** Given any $g \in Met^m_\delta(S)$ near $\tilde{g}$, there is a unique diffeomorphism $\Psi \in D_0^{m+1,\alpha}(S)$ near the identity map $Id$ such that the pull back metric $\Psi^* g$ satisfies the divergence gauge condition $\delta_\delta(\Psi^* g) = 0$.

**Proof.** Define a map $F$ as follows,
\[
F : [D_0^{m+1,\alpha} \times Met^m_\delta](S) \rightarrow (\Lambda^1)^{m-1,\alpha}_\delta(S),
\]
\[
F(\Psi, g) = \delta_\delta(\Psi^*(g)).
\]
Linearization of $F$ at $(Id, \tilde{g})$ with respect to $(X, h)$ is given by
\[
D_0 F(X, h) = \delta \delta^* X + \delta(h).
\]
Here $X \in X^{m+1,\alpha}_i(S)$, and hence $\delta\delta^*$ is an isomorphism by the previous lemma. According to the inverse function theorem, for any $g$ in a neighbourhood of $\tilde{g}$, there exists a unique $\Psi$ near $Id$ such that $F(\Psi, g) = 0$, which proves the theorem. 

\[\Box\]

Now define the moduli space $\mathcal{E}_C = \mathcal{E}^{m,\alpha}_C$ to be the quotient of the space $\mathcal{E}_C$ by the diffeomorphism group:

$$\mathcal{E}_C = \mathcal{E}_C / \mathcal{D}^{m+1,\alpha}_0(S).$$

By Lemma 2.4, any element of $Z_C$ is in one of the equivalence classes in $\mathcal{E}_C$. Conversely, given any stationary vacuum data $(g, u, \phi)$ near $\tilde{g}$, according to the theorem above, one can choose a unique diffeomorphism $\Psi \in \mathcal{D}^{m+1,\alpha}_0(S)$ near $Id$ so that $\delta\tilde{g}(\Psi^*g) = 0$, i.e. the pull back data $\Psi^*(g, u, \phi)$ belongs to $Z_C$. Therefore, locally elements in the set $Z_C$ near $\tilde{g}$ are in 1-1 correspondence to equivalence classes in the moduli space $\mathcal{E}_C$ near $[\tilde{g}]$.

2.4. Boundary Conditions. As in the Introduction, we pose a geometrically natural collection of boundary conditions on $\partial S$:

$$\begin{cases}
g_T = \gamma, \\
H_g = \lambda, \\
n_g(\phi) = f,
\end{cases}$$

where $\gamma \in Met^{m,\alpha}(\partial S)$ is a fixed metric of the surface $\partial S$; and $\lambda, f \in C^{m-1,\alpha}(\partial S)$ are prescribed functions on $\partial S$. Under the conformal transformation (2.2), these tensors become

$$g_T = e^{-2u}g^T, \quad H_g = e^{\alpha}(H_g - 2n_g(u)), \quad n_g = e^\alpha n_g.$$ 

Thus one can translate the boundary conditions (2.5) to the following for the data $(g, u, \omega)$,

$$\begin{cases}
e^{-2u}g_T = \gamma, \\
H_g - 2n_g(u) = -ue^{-u}, \\
n_g(\phi) = e^{-u}f,
\end{cases}$$

on $\partial S$.

Pairing these boundary conditions with the gauged field equations (III), we obtain the following boundary value problem,

$$\begin{cases}
Ric_g - 2du \otimes du - 2e^{-4u}d\phi \otimes d\phi + T + \delta^* G = 0 \\
\Delta_g u - 2e^{-4u}d\phi^2 = 0 \\
\Delta_g \phi + 4 < du, d\phi > = 0
\end{cases}$$

on $S$, and

$$\begin{cases}
G = 0 \\
e^{-2u}g_T = \gamma \\
H - 2n(u) = -ue^{-u} \\
n(\phi) = e^{-u}f
\end{cases}$$

on $\partial S$. 

The main step to prove Theorem 1.1 is verifying that the boundary value problem above is elliptic. To do this, we define a differential operator \( P = (L, B) \) based on it, where \( L \) denotes the interior operator and \( B \) the boundary operator. The interior operator \( L \), mapping the data \((g, u, \phi)\) to the interior equations of (2.7), is defined as follows,

\[
L: [Met^{m,\alpha}_\delta \times C^m_{\delta} \times C^2_{\delta}](S) \to [(S_2)^{m-2,\alpha}_{\delta+2} \times C^m_{\delta+2} \times C^2_{\delta+2}](S),
\]

\[
L(g, u, \phi) = \{ 2(Ric_g - 2du \otimes du - 2e^{-4u}d\phi \otimes d\phi + \delta^* G + T),
8(\Delta_g u - 2e^{-4u}|d\phi|^2),
8e^{-4u}(\Delta_g \phi + 4(du, d\phi)) \}.
\]

Here \( S_2 \) denotes the symmetric 2-tensors and \((S_2)^{m,\alpha}_\delta\) is the space of \( C^{m,\alpha} \) asymptotically flat symmetric 2-tensors which is defined similarly as the spaces of tensor fields in Definition 2.1. The extra scalar factors 2, 8 and function \( 8e^{-4u} \) are for later use when proving self-adjointness below. They do not affect the ellipticity.

The boundary operator \( B \), mapping the data \((g, u, \phi)\) to the boundary terms in (2.7), is given by,

\[
B: [Met^{m,\alpha}_\delta \times C^m_{\delta} \times C^2_{\delta}](S) \to [(T^{m-1,\alpha}_1 \times C^{m-1,\alpha}) \times S^{m,\alpha}_2 \times C^{m-1,\alpha} \times C^{m-1,\alpha}](\partial S),
\]

\[
B(g, u, \phi) = \{ G,
 e^{-2u} \tilde{g}^T - \gamma,
 H - 2n(u) - e^{-u} \lambda,
n(\phi) - e^{-u} f \},
\]

where we write \( G \in [T^{m-1,\alpha}_1 \times C^{m-1,\alpha}](\partial S) \) because the gauge term \( G \) is a 1-tensor on \( S \), and when restricted to \( \partial S \), it induces a tangential 1-tensor \( G^T \) and a \( C^{m-1,\alpha} \) function \( G(n) \) on \( \partial S \). For simplicity of notation, we will use \( B^{m,\alpha}(S) \) to denote the target space of \( B \), i.e.

\[
B^{m,\alpha}(S) = [(T^{m-1,\alpha}_1 \times C^{m-1,\alpha}) \times S^{m,\alpha}_2 \times C^{m-1,\alpha} \times C^{m-1,\alpha}](\partial S).
\]

Throughout this paper, \( P \) will be written as \( P = (L_1, B_1) \) if the gauge terms in \( L, B \) correspond to the Bianchi gauge, and \( P = (L_2, B_2) \) if the divergence gauge is applied.

Let \((g, u, \phi)\) be a fixed element in the zero set \( P^{-1}(0) \), and choose the reference metric \( \tilde{g} = g \) in the gauge term \( G \). The linearization of \( P \) at \((g, u, \phi)\) is given by

\[
DP(h, v, \sigma) = (DL(h, v, \sigma), \ DB(h, v, \sigma)),
\]

where \((h, v, \sigma)\) is an infinitesimal deformation of the data \((g, u, \phi)\), and \( DL, DB \) are linearizations of the operators \( L \) and \( B \), expressed as follows,

\[
DL: [(S_2)^{m,\alpha}_\delta \times C^m_{\delta} \times C^2_{\delta}](S) \to [(S_2)^{m-2,\alpha}_{\delta+2} \times C^m_{\delta+2} \times C^2_{\delta+2}](S),
\]

\[
DL(h, v, \sigma) = \{ D^* Dh - Z(h) + O_1,
8\Delta_g v + O_1,
8e^{-4u}(\Delta_g \sigma + 4(du, d\sigma)) + O_0 \}.
\]
and

\[ DB : \mathbb{[C}^{m,\alpha}(S) \times C^m(S) \times C^m(S)] \rightarrow B^m(S), \]

\[ DB(h, v, \sigma) = \{ G'_h, e^{-2u(-2vg + h)}|_{\partial S}, H'_h - 2n(v) + O_0, n(\sigma) + O_0 \}. \]

In the expression above, \( D^* Dh = -\nabla^i \nabla_i h \). The terms \( Z \) and \( G'_h \) depend on the choice of gauge terms. They are of the form (2.8)

\[ \begin{cases} Z_1(h) = 0, \\ (G_1)'_h = \beta \tilde{g} h, \end{cases} \]

when the Bianchi gauge is chosen, and

\[ \begin{cases} Z_2(h) = D^2(trh) + \Delta_g(trh)g + (\delta \delta h)g, \\ (G_2)'_h = \delta \tilde{g} h, \end{cases} \]

when the divergence gauge is used.

The expressions \( O_1 \) and \( O_0 \) stand for terms which involve the derivative of \((h, v, \sigma)\) with order not higher than 1 and 0. In the linearization \( DB \), the term \( H'_h \) denotes the variation of the mean curvature. We refer to \( \S 5 \) for the details of the calculation.

Since ellipticity only depends on the principal part of the operator, we can remove the lower order terms \( O_1 \) and \( O_0 \) in \( DP \) and study the simplified operator \( P(h, v, \sigma) = (L(h, v, \sigma), B(h, v, \sigma)) \), where \( L \) and \( B \) are as follows:

\[ L : \mathbb{[C}^{m,\alpha}(S) \times C^m(S) \times C^m(S)] \rightarrow \mathbb{[C}^{m,\alpha}(S) \times C^m(S) \times C^m(S)], \]

\[ L(h, v, \sigma) = \{ D^* Dh - Z(h), 8\Delta_g v, 8e^{-4u}(\Delta_g \sigma + 4(du, d\sigma)) \}. \]

and

\[ B : \mathbb{[C}^{m,\alpha}(S) \times C^m(S) \times C^m(S)] \rightarrow B^m(S), \]

\[ B(h, v, \sigma) = \{ G'_h, e^{-2u(-2vg + h)}|_{\partial S}, H'_h - 2n(v), n(\sigma) \}. \]

In the last component of \( L \), we keep the lower order term \( 4(du, d\sigma) \) for the purpose of self-adjointness discussed later; again this does not affect the ellipticity.

In the following section, if the pair \((Z, G'_h)\) takes the values in (2.8), the operator \( P \) will be denoted by \( P_1 = (L_1, B_1) \); and if equipped with the divergence gauge (2.9), \( P \) will be written as \( P_2 = (L_2, B_2) \). We will prove the ellipticity of both operators \( P_1 \) and \( P_2 \). As a consequence, the boundary value problem (2.7) is elliptic with respect to both choices of gauges.
3. Ellipticity

Throughout this section, we use $\xi$ to denote a generic 1–form on $S$, $\eta$ to denote a nonzero 1–form tangent to the boundary $\partial S$, i.e. $\eta(n) = 0$, and $\mu$ a nonzero 1–form normal to the boundary $\partial S$, i.e. $\mu^T = 0$.

To check ellipticity, we will follow [ADN]: Let $P = (L, B)$ be a differential operator consisting of an interior operator $L$ and a boundary operator $B$. Denote the matrix of principal symbols of the interior operator at $\xi$ as $L(\xi)$ and the matrix of principal symbols of the boundary operator as $B(\xi)$. The operator $P$ forms an elliptic boundary value problem if and only if the following two conditions hold:

(A) (properly elliptic condition): determinant $l(\xi)$ of $L(\xi)$ has no nontrivial real root;

(B) (complementing boundary condition): Take the adjoint matrix $L^*(\xi)$ of $L(\xi)$. Let $\xi = (\eta + z\mu)$. The rows of $B \cdot L^*(\eta + z\mu)$ are linearly independent modulo $t^+(z)$, where $t^+(z) = \prod(z - z_k)$ and $\{z_k\}$ are the roots of $l(\eta + z\mu) = 0$ having positive imaginary parts.

3.1. Ellipticity with the Bianchi gauge.

Theorem 3.1. $P_1$ is an elliptic operator.

Proof. It is easy to observe that the matrix of principal symbols for $L_1$ at $\xi$ is given by

$$L_1(\xi) = \begin{bmatrix} \xi^2 I_{6 \times 6} & 0 & 0 \\ 0 & 8|\xi|^2 & 0 \\ 0 & 0 & 8e^{-4u}|\xi|^2 \end{bmatrix}.$$ 

The adjoint matrix of $L(\xi)$ is then given by

$$L_1^*(\xi) = \begin{bmatrix} 64e^{-4u}|\xi|^{14} I_{6 \times 6} & 0 & 0 \\ 0 & 8e^{-4u}|\xi|^{14} & 0 \\ 0 & 0 & 8|\xi|^{14} \end{bmatrix}.$$ 

The determinant of $L_1(\xi)$ is $l(\xi) = 64e^{-4u}|\xi|^{16}$. So it is obvious that the interior operator is properly elliptic.

The root of $l(\eta + z\mu)$ with positive imaginary part is $z = \pm i|\eta|$, and this implies

$$t^+(z) = (z - i|\eta|)^8.$$

Let $C$ be a generic vector in $\mathbb{C}^8$. The complementing boundary condition holds if the equation below has no nontrivial solution in $\mathbb{C}^8$:

$$C \cdot B_1(\eta + z\mu) \cdot L_1^*(\eta + z\mu) = 0 \quad (\text{mod } t^+(z)).$$

One sees easily that equation (3.1) is equivalent to the following,

$$(z - i|\eta|) \cdot \begin{bmatrix} 64e^{-4u} I_{6 \times 6} & 0 & 0 \\ 0 & 8e^{-4u} & 0 \\ 0 & 0 & 8 \end{bmatrix} = 0.$$

And furthermore, this holds if and only if the following is true,

$$C \cdot B_1(\eta + i|\eta|\mu) \cdot \begin{bmatrix} 64e^{-4u} I_{6 \times 6} & 0 & 0 \\ 0 & 8e^{-4u} & 0 \\ 0 & 0 & 8 \end{bmatrix} = 0.$$

So to prove the complementing boundary condition (B), it suffices to prove that the matrix of principal symbol $B_1(\xi)$, has trivial kernel, where $\xi = \eta + i|\eta|\mu$. In
the following, the subscript 0 represents the direction normal to \( \partial S \), while indices 
1, 2 represent the tangential directions on \( \partial S \). Write the nonzero tangential 1-form \( \eta = (\eta_1, \eta_2) \), then \( \tilde{\xi} = (i|\eta|, \eta_1, \eta_2) \). Basic computation (cf. §5.2) shows that the
principal symbols of the boundary operator \( B_1 \) at \( \tilde{\xi} \) are given by,

\[
\begin{align*}
|\eta|h_{00} - i\eta_1h_{10} - i\eta_2h_{20} - \frac{1}{2}|\eta|(h_{00} + h_{11} + h_{22}) &= 0 \\
|\eta|h_{01} - i\eta_1h_{11} - i\eta_2h_{21} + \frac{1}{2}i\eta_1(h_{00} + h_{11} + h_{22}) &= 0 \\
|\eta|h_{02} - i\eta_1h_{12} - i\eta_2h_{22} + \frac{1}{2}i\eta_2(h_{00} + h_{11} + h_{22}) &= 0 \\
-2v + h_{11} &= 0 \\
-2v + h_{22} &= 0 \\
-\frac{1}{2}|\eta|(h_{11} + h_{22}) - i\eta_1h_{10} - i\eta_2h_{20} + 2|\eta|v &= 0 \\
-|\eta|\sigma &= 0
\end{align*}
\]

According to equations (3.5), (3.6) and (3.7), we can replace \( h_{11} \) and \( h_{22} \) by \( 2v \)
and \( h_{21} \) by 0. Then equation (3.8) gives

\[2|\eta|v - (i\eta_1h_{10} + i\eta_2h_{20}) - 2|\eta|v = 0,\]
i.e.

\[(i\eta_1h_{10} + i\eta_2h_{20}) = 0.\]

Equation (3.2) gives:

\[\frac{1}{2}|\eta|h_{00} - (i\eta_1h_{10} + i\eta_2h_{20}) - 2|\eta|v = 0.\]

It follows that

\[h_{00} = 4v.\]

Multiplying (3.3) by \((i\eta_1)\) and (3.4) by \((i\eta_2)\), and then summing them, we obtain,

\[2|\eta|^2v + 4|\eta|^2v = 0.\]

Thus \( v = 0 \) and consequently \( h_{ij} = 0 \) for all \( 0 \leq i, j \leq 2.\)

Finally, it’s obvious from equation (3.9) that \( \sigma = 0. \) This completes the proof.

\[\square\]

\textbf{Remark.} One can see from the proof above that principal symbols of the operator
\( P_1 \) are simple so that ellipticity follows from a direct verification of the conditions
\((A)\) and \((B)\). However, in the divergence-gauge case, principal symbols of the
operator \( P_2 \) are too complicated for us to carry out the same computation as above.
In the following, we will use an intermediate operator which has Bianchi gauge term
\( G_1 \) in the interior operator and divergence gauge \( G_2 \) in the boundary operator, to
approach the ellipticity of the operator \( P_2. \)

3.2. ellipticity for the \( \delta \) gauge. We begin with the following lemma:

\textbf{Lemma 3.2.} If we replace \((G_1')_h\) by \((G_2')_h\) in the boundary part \( B_1 \) of \( P_1 \), the
operator is still elliptic.
Proof. This can be proved by a slight modification of the previous proof. After changing \( \beta(h) \) to \( \delta(h) \) (they only differ by a trace term) in the boundary operator, the new principal symbols of the boundary operator at \( \xi = (i\|\eta\|, \eta_1, \eta_2) \) become

\[
\begin{align*}
|\eta|h_{00} - i\eta_1 h_{10} - i\eta_2 h_{20} &= 0 \\
|\eta|h_{01} - i\eta_1 h_{11} - i\eta_2 h_{21} &= 0 \\
|\eta|h_{02} - i\eta_1 h_{12} - i\eta_2 h_{22} &= 0 \\
-2v + h_{11} &= 0 \\
h_{12} &= 0 \\
-2v + h_{22} &= 0 \\
\frac{1}{2}|\eta|(h_{11} + h_{22}) - i\eta_1 h_{10} - i\eta_2 h_{20} + 2|\eta|v &= 0 \\
-|\eta|\sigma &= 0
\end{align*}
\]

By equations (3.13), (3.14) and (3.15), we can replace \( h_{11} \) and \( h_{22} \) by \( 2v \) and \( h_{21} \) by 0. Then (3.16) gives

\[
2|\eta|v - (i\eta_1 h_{10} + i\eta_2 h_{20}) - 2|\eta|v = 0,
\]
i.e.

\[
(i\eta_1 h_{10} + i\eta_2 h_{20}) = 0.
\]

Consequently, equation (3.10) yields \( h_{00} = 0 \).
Multiplying (3.11) by \( (i\eta_1) \) and (3.12) by \( (i\eta_2) \), and then summing, we obtain

\[
2|\eta|^2v = 0.
\]

Thus \( v = 0 \) and consequently \( h_{ij} = 0 \) for all \( 0 \leq i, j \leq 2 \).

Next, we use the method exploit in \([AK]\) to prove ellipticity for the operator \( P_2 \).

Theorem 3.3. The operator \( P_2 \) is elliptic.

Proof. The ellipticity of a general operator \( P = (L, B) \) is equivalent to the existence of a uniform estimate:

\[
|||(h, v, \sigma)|||_{C^{m,\alpha}} \leq C(|||L(h, v, \sigma)|||_{C^{m-k,\alpha}} + |||B(h, v, \sigma)|||_{C^{m-j,\alpha}} + |||(h, v, \sigma)|||_{C^0}),
\]

together with such an estimate for the adjoint operator. In the expression above, \( k \) and \( j \) denote the order of derivative in the principal parts of the operators \( L \) and \( B \).

The operator \( P_2 \) is then elliptic as a consequence of the following two facts, which are proved in Lemma 3.4 and Proposition 3.5 below.

(1) The inequality (3.18) holds for \( P_2 \);
(2) The operator \( P_2 \) is formally self-adjoint.

Lemma 3.4. Inequality (3.18) holds for \( P_2 \), i.e.

\[
|||(h, v, \sigma)|||_{C^{m,\alpha}} \leq C(|||L_2(h, v, \sigma)|||_{C^{m-2,\alpha}} + |||B_2(h, v, \sigma)|||_{C^{m-j,\alpha}} + |||(h, v, \sigma)|||_{C^0}).
\]
Proof. By Lemma 3.2, the inequality (3.19) must hold if $L_2$ is replaced by $L_1$, i.e.

$$|| (\hat{h}, v, \sigma) ||_{C^m otimes C^s} \leq C (|| L_1 (h, v, \sigma) ||_{C^{m-2, \alpha}} + || B_2 (h, v, \sigma) ||_{C^{m-2, \alpha}} + || (h, v, \sigma) ||_{C^0}).$$

Observe that $L_1 (h, v, \sigma) = L_2 (h, v, \sigma) + (D^2 (\text{tr} h) + \Delta_g (\text{tr} h)) g + 2 \delta \delta \delta \delta (h, v, \sigma).$ So by the interpolation inequality, $|| h ||_{C^{m-1, \alpha}} \leq \epsilon || h ||_{C^m} + \epsilon^{-1} || h ||_{C^s}$, it suffices to prove (3.20) $|| \delta \delta (h) ||_{C^{m-1, \alpha}} \leq C (|| L_2 (h, v, \sigma) ||_{C^{m-2, \alpha}} + || B_2 (h, v, \sigma) ||_{C^{m-2, \alpha}} + || (h, v, \sigma) ||_{C^0},$

and

(3.21)

$$|| D^2 \text{tr} h ||_{C^{m-2, \alpha}} \leq C (|| L_2 (h, v, \sigma) ||_{C^{m-2, \alpha}} + || B_2 (h, v, \sigma) ||_{C^{m-2, \alpha}} + || (h, v, \sigma) ||_{C^0}).$$

First notice that, $L_2 (h)$ is the 2nd-order part of the linearization of the map:

$$\Phi (g) = \text{Ric} g + \delta^* G_2 + T_2 - 2 du \otimes du - 2 e^{-4u} d\phi \otimes d\phi.$$

From the proof of Lemma 2.4, one sees that

$$\delta \Phi (g) + 2 (\Delta_g u - 2 e^{-4u} |d\phi|^2) du + 2 e^{-4u} (\Delta_g \phi + 4 \langle du, d\phi \rangle) d\phi = \delta \delta^* \delta g.$$

Assume $g$ is a zero of $\Phi$. Linearizing the above equation at $\hat{g} = g$ with respect to $h$ gives

$$\delta D \Phi (h) + O_0 = \delta \delta^* \delta (h),$$

where $O_0$ denotes terms of 0-derivative order with respect to $h$. It is of derivative order 3 on the right hand side of the equation above, so the left hand side must be also of order 3, hence we obtain,

$$\delta L_2 (h) = \delta \delta^* \delta (h).$$

The operator $\delta \delta^*$ is elliptic with respect to Dirichlet boundary conditions, and $\delta h$ is included in the boundary operator $B_2$. Thus inequality (3.20) holds.

To prove inequality (3.21), we use the Gauss equation at $\partial S$:

$$|A_g|^2 - H_g + s_g^T = s_g - 2 \text{Ric}_g (n, n),$$

where $A_g$ is the second fundamental form of $\partial M \subset (M, g)$ and $s_g^T$ is the scalar curvature of the metric $g^T$ on $\partial M$. It follows that

$$|A_g|^2 - H_g + s_g^T|^h_\partial = - L_2 (n, n) + 2 \delta \delta (h) + O_1,$$

where $O_1$ denotes terms of derivative order no higher than 1 with respect to $h$.

Observe that $s_g'_{\partial T} = \Delta_g (\text{tr} h) + \delta \delta (h^T) + O_1$ and the terms $A_g', H_g'$ only involve first order derivatives in $h$ so they can be ignored according to the interpolation inequality. Writing $h^T = B_{2,0} + 2 v g^T$, where $B_{2,0} = h^T - 2 v g^T$ is one of the boundary conditions for $P_2$, it following that

$$s_g'_{\partial T} = \Delta_g (\text{tr} h^T - 2 v) + \delta \delta (B_{2,0}) + O_1.$$

Therefore, by the ellipticity of the Laplace operator on $\partial S$, and together with (3.20) being true, we obtain the estimate for $(\text{tr} h^T - 2 v)$ along $\partial S$:

(3.22)

$$|| (\text{tr} h^T - 2 v) ||_{\partial S} \leq C (|| L_2 (h, v, \sigma) ||_{C^{m-2, \alpha}} + || B_2 (h, v, \sigma) ||_{C^{m-2, \alpha}} + || h ||_{C^0}).$$

Since the term $(h^T - 2 v g^T)$ is included in the boundary operator, $\text{tr} (h^T - 2 v g^T) = (\text{tr} h^T - 4 v) ||_{\partial S}$ is also controlled. Comparing with (3.22), we obtain the control for $v$ on $\partial S$:

$$|| v ||_{\partial S} \leq C (|| L_2 (h, v, \sigma) ||_{C^{m-2, \alpha}} + || B_2 (h, v, \sigma) ||_{C^{m-2, \alpha}} + || h ||_{C^0}).$$
In addition, $\Delta_g v$ is one of the components of the interior operator $L_2$, so from the ellipticity of Laplace operator with Dirichlet boundary condition, we obtain the uniform estimate for $v$ over $S$,

$$||v||_{C^{m,\alpha}} \leq C(||L_2(h, v, \sigma)||_{C^{m-2,\alpha}} + ||B_2(h, v, \sigma)||_{C^{m-1,\alpha}} + ||(h, v, \sigma)||_{C^0}).$$

Furthermore, observe that $\Delta_g \sigma$ is the last component of the interior operator $L_2$ and $\mathbf{n}(\sigma)$ is one of the boundary terms in $B_2$. Thus, based on the ellipticity of Laplace operator with Neumann boundary condition, we also have the uniform estimate for $\sigma$ over $S$:

$$||\sigma||_{C^{m,\alpha}} \leq C(||L_2(h, v, \sigma)||_{C^{m-2,\alpha}} + ||B_2(h, v, \sigma)||_{C^{m-1,\alpha}} + ||(h, v, \sigma)||_{C^0}).$$

Now with $v, \sigma$ being well controlled, inequality (3.21) is equivalent to the following inequality,

$$||D^2 tr h||_{C^{m-2,\alpha}} \leq C(||L_2(h)||_{C^{m-2,\alpha}} + ||\delta(h)||_{AS}||_{C^{m-1,\alpha}} + ||h^T||_{AS}||_{C^{m,\alpha}} + ||H^T_k||_{AS}||_{C^{m-1,\alpha}} + ||h||_{C^0}).$$

This estimate is proved in Lemma 3.2 of [AK], and this completes the proof of the uniform estimate.

$\square$

**Proposition 3.5.** Let $\mathcal{M}_2$ be the space of data $(h, v, \sigma)$ which is in the kernel of the boundary operator $B_2$, i.e.

$$\mathcal{M}_2 = \{ (h, v, \sigma) \in (S^2)_{\delta}^{m,\alpha} \times C^\alpha_{\delta} \times C^\alpha_{\delta} | (S) :$$

$$\begin{cases}
\delta_g(h) = 0, \\
h^T - 2vg^T = 0, \\
H^T_k - 2\mathbf{n}(v) = 0, & \text{on } \partial S \\
\mathbf{n}(\sigma) = 0,
\end{cases}$$

(3.23)

Then the operator $L_2 : \mathcal{M}_2 \rightarrow (S^2)_{\delta}^{m-2,\alpha} \times C^\alpha_{\delta} \times C^\alpha_{\delta} | (S)$, given by

$L_2(h, v, w) = \{ D^* Dh - Z_2(h), 8\Delta_g v, 8e^{-4u}[\Delta_g \sigma + 4(du, d\sigma)] \}$

$$= \{ D^* Dh - D^2 (tr h) - \Delta_g (tr h)g - (\delta \delta h) g, 8\Delta_g v, 8e^{-4u}[\Delta_g \sigma + 4(du, d\sigma)] \},$$

is formally self-adjoint.

**Proof.** We will prove this proposition by showing that $L_2$ arises as the 2nd variation of a natural variational problem on the data $(g, u, \phi)$.

To begin, the Einstein equation $Ein_{g^{(4)}} = 0$ is the functional derivative of the Einstein-Hilbert action

$$I_{EH} = \int_{V^{(4)}} R^{(4)} g^2 dvol_{g^{(4)}}.$$

Reducing this action from 4-dimensional spacetime $V^{(4)}$ to the 3-dimensional quotient space $S$, one obtains the following functional on the data $(g, u, \phi)$, of which the Euler-Lagrange equations are exactly the field equations (II) in §2,

$$I_{eff} = \int_S s - 2|du|^2 - 2e^{-4u}|d\phi|^2 dvol_g.$$

We refer to [H1][H2] for further discussion of the action $I_{eff}$. 
Since the boundary \( \partial S \) is nonempty, as is well known it is necessary to add boundary terms to the action such as Gibbons-Hawking boundary terms, cf.\cite{GH}. The proper action in our case is given by

\[
I = \int_S s - 2|du|^2 - 2e^{-4u}|d\phi|^2 dvol_g + 2\int_{\partial S} H_g dvol_g + 16\pi m_{ADM}(g).
\]

Next, let \( (E,F,H) \) denote the expressions in the system (II), i.e.

\[
E[(g,u,\phi)] = \frac{1}{2} (s - 2|du|^2 - 2e^{-4u}|d\phi|^2)g - Ric_g + 2(du)^2 + 2e^{-4u}(d\phi)^2,
\]

(3.24)  
\[
F[(g,u,\phi)] = -4\Delta_g u + 8e^{-4u}|d\phi|^2,
\]

(3.25)  
\[
H[(g,u,\phi)] = -4e^{-4u}(\Delta_g \phi + 4(du,d\phi)).
\]

Then the variation of \( I \) with respect to \( g \) is given by

\[
I'_g(h) = \int_S (E,h) dvol_g + \int_{\partial S} -\langle A,h \rangle + Htrh^T dvol_g + \int_{\partial S} n(trh) + \delta h(n)dvol_{\partial S} + 16\pi(m_{ADM}(g))' h.
\]

(3.26)  
We refer to §5.3 for the details of the computation. To abbreviate notation, we shall omit the volume form in the following.

Notice that the terms in the second line of the equation (3.26) can be removed, because we have

\[
\int_{\partial S\infty} n(trh) + \delta h(n) = 16\pi(m_{ADM}(g))' h,
\]

based on the definition of ADM mass and its variation, cf.\cite{RT},\cite{B1}.

Basic computation shows the variations of \( I \) with respect to \( u \) and \( \phi \) are of the form,

(3.27)  
\[
I'_u(v) = \int_S (F,v) + \int_{\partial S} -4n(u)v + \int_{\partial S\infty} -4n(u)v,
\]

and

\[
I'_\phi(\sigma) = \int_S (H,\sigma) + \int_{\partial S} -4e^{-4u}n(\phi)\sigma + \int_{\partial S\infty} -4e^{-4u}n(\phi)\sigma.
\]

By simply checking the decay rate, one sees easily that the boundary terms at infinity in the expressions above are both zero.

Now let \( (g,u,\phi) \) be a triple such that \( (E,F,H)[(g,u,\phi)] = 0 \), and take a 2-parameter variation of data \( (g_{st}, u_{st}, \phi_{st}) = (g,u,\phi) + s(h,v,\sigma) + t(k,w,\zeta) \), with infinitesimal deformations \( (h,v,\sigma), (k,w,\zeta) \in M_2 \).

Based on the boundary conditions in the expression (3.23), we have \( h^T = 2vg^T \). The equation (3.25) then becomes:

(3.28)  
\[
I'_g(h) = \int_S (E,h) + \int_{\partial S} 2vh.
\]

Take one more variation of the equation (3.28) with respect to \( k \), and we obtain

(3.29)  
\[
I''_g(h,k) = \int_S (E_k,h) + \int_{\partial S} 2vh_k + 4vwH.
\]
Similar operation of the equations (3.26) and (3.27) yields,

\[(3.30)\]
\[I''_\alpha(v, w) = \int_S \langle \mathbf{F}'_w, v \rangle + \int_{\partial S} -4\mathbf{n}(w)v,\]

and

\[(3.31)\]
\[I''(\sigma, \zeta) = \int_S \langle \mathbf{H}'_\zeta, \sigma \rangle + \int_{\partial S} -4e^{-4\mathbf{n}(\zeta)}\sigma.\]

From the symmetry of second variation, we know that \(I''(h, k) = I''(k, h),\)
\(I''(w, v) = I''(v, w)\) and \(I''(\sigma, \zeta) = I''(\zeta, \sigma).\) The equations (3.29 – 31) then imply that:

\[
\int_S [(\mathbf{E}'_k, h) + (\mathbf{F}'_w, v) + (\mathbf{H}'_\zeta, \sigma)] + \int_{\partial S} [2vH_k' + 4vwH - 4\mathbf{n}(w)v - 4e^{-4\mathbf{n}(\zeta)}\sigma] \\
= \int_S [(\mathbf{E}'_h, k) + (\mathbf{F}'_v, w) + (\mathbf{H}'_\sigma, \zeta)] + \int_{\partial S} [2wH_k' + 4vwH - 4\mathbf{n}(v)w - 4e^{-4\mathbf{n}(\sigma)}\zeta].
\]

By the boundary condition in (3.23), \(H_k' - 2\mathbf{n}(v) = 0\ \mathbf{n}(\sigma) = 0\ on \ \partial S,\) and the same for \((k, w, \zeta).\) Thus we can remove the boundary terms above and obtain

\[(3.32)\]
\[\int_S [(\mathbf{E}'_k, h) + (\mathbf{F}'_w, v) + (\mathbf{H}'_\zeta, \sigma)] = \int_S [(\mathbf{E}'_h, k) + (\mathbf{F}'_v, w) + (\mathbf{H}'_\sigma, \zeta)].\]

On the other hand, from the boundary condition \(\delta h = \delta k = 0\ on \ \partial S,\) it follows that,

\[(3.33)\]
\[\int_S (\delta^* \delta k, h) = \int_S (\delta k, \delta h) = \int_S (\delta^* \delta h, k).\]

Combining equations (3.32) and (3.33), we obtain,

\[(3.34)\]
\[\int_S (\langle \mathbf{E}'_k - \delta^* \delta k, \mathbf{F}'_w, \mathbf{H}'_\zeta \rangle, (h, v, \sigma)) = \int_S (\langle \mathbf{E}'_h - \delta^* \delta h, \mathbf{F}'_v, \mathbf{H}'_\sigma \rangle, (k, w, \zeta)).\]

Notice that the terms of second order and first order derivative in \((\mathbf{E}'_h - \delta^* \delta h, \mathbf{F}'_v, \mathbf{H}'_\sigma)\) are the same as in the operator \(-\frac{1}{2}L_2;\) and the zero order terms in the equation (3.34) can be removed because of symmetry. Therefore it follows that

\[
\int_S \langle L_2(k, w, \zeta), (h, v, \sigma) \rangle = \int_S \langle L_2(h, v, \sigma), (k, w, \zeta) \rangle,
\]

which proves the formal self-adjointness of the operator \(P_2.\)

\[\Box\]

Ellipticity of the operator \(P_2\) implies that the boundary value problem (2.7) with the divergence gauge is elliptic. Together with the local equivalence between the sets \(Z_C\) and \(E_C\) in §2.3, we conclude that the collection of boundary conditions (2.6) is elliptic for the stationary vacuum field equations (II) modulo diffeomorphisms in \(D_0^{n+1,\alpha}(S).\)
3.3. **Back to** $g_S$.

It is now basically trivial to prove the ellipticity of the system (1.6) equipped with boundary conditions (1.7), using the result we have obtained.

First observe that, by combining the first and second equations in (1.6), the system is equivalent to the following one,

\[
\begin{aligned}
&\begin{cases}
Ric_g - D^2 u - (du)^2 - 2e^{-4u} (d\phi \otimes d\phi - |d\phi|^2 g_S) \\
(\Delta_{g_S} u - |du|^2 - 2e^{-4u} |d\phi|^2) g_S = 0,
\end{cases}
\end{aligned}
\]

(3.35)

Denote $T_S$ as the trace term

\[
T_S = \frac{1}{2}(s_{g_S} + 4\Delta_{g_S} u - 4|du|^2 - 2e^{-4u} |d\phi|^2) g_S,
\]

and let $G_S$ be the pull back by conformal transformation of the divergence gauge term $\delta g$, i.e.

\[
G_S = \delta g. \bar{g}_S (e^2 u g_S),
\]

where $\bar{g}_S$ is a reference metric near $g_S$.

Inserting $(T_S + \delta^*_{g_S} g_S)$ to the first equation in system (3.35), we obtain

\[
\begin{aligned}
&\begin{cases}
Ric_g - D^2 u - (du)^2 - 2e^{-4u} (d\phi \otimes d\phi - |d\phi|^2 g_S) \\
(\Delta_{g_S} u - |du|^2 - 2e^{-4u} |d\phi|^2) g_S + T_S + \delta^*_{g_S} G_S = 0,
\end{cases}
\end{aligned}
\]

(3.36)

According to the system above, we define a differential operator $P_S = (L_S, B_S)$, which consists of the interior operator $L_S$, mapping the data $(g_S, u, \phi)$ to the interior expressions in (3.36), given by

\[
L_S : |Met^m_{\delta} \times C^m_{\delta} \times C^m_{\delta}|(S) \rightarrow |(S_2)^{m-2,2}_{\delta+2} \times C^m_{\delta+2} \times C^m_{\delta+2}|(S)
\]

\[
L_S (g_S, u, \phi) = \begin{cases}
2[Ric_g - D^2 u - (du)^2 - 2e^{-4u} (d\phi \otimes d\phi - |d\phi|^2 g_S) \\
(\Delta_{g_S} u - |du|^2 - 2e^{-4u} |d\phi|^2) g_S + T_S + \delta^*_{g_S} G_S],
\end{cases}
\]

\[
\begin{aligned}
8e^{-2u}(\Delta_{g_S} u - |du|^2 - 2e^{-4u}|d\phi|^2),
8e^{-6u}(\Delta_{g_S} \phi + 3\langle du, d\phi \rangle).
\end{aligned}
\]

and the boundary operator $B_S$, mapping the data $(g_S, u, \phi)$ to boundary data including the gauge term $G_S$ and the terms in (1.7), given by

\[
B_S : |Met^m_{\delta} \times C^m_{\delta} \times C^m_{\delta}|(S) \rightarrow |(T_{1}^{m-1,1} \times C^{m-1,1}) \times g^2_{S} \times C^{m-1,1} \times C^{m-1,1}|(\partial S),
\]

\[
B(g, u, \omega) = \{ G_S, \bar{g}_S |_{\partial S} - \gamma, \ H_\gamma - \lambda, \ n_{g_S}(\phi) - f \}.
\]

In addition, define an operator $Q$ as the conformal transformation in (2.2),

\[
Q : |Met^m_{\delta} \times C^m_{\delta} \times C^m_{\delta}|(S) \rightarrow |Met^m_{\delta} \times C^m_{\delta} \times C^m_{\delta}|(S)
\]

\[
Q(g_S, u, \phi) = (e^{2u} g_S, u, \phi),
\]
It is easy to see, by elementary computation, that the operator $\mathcal{P}_S$ is exactly the composition of $\mathcal{P}_2$ in §2 and $\mathcal{Q}$, i.e.

$$\mathcal{P}_S = \mathcal{P}_2 \circ \mathcal{Q}.$$ 

The operator $\mathcal{P}_2$ has already been proved to be elliptic and $\mathcal{Q}$ is obviously an isomorphism. As a consequence, the operator $\mathcal{P}_S$ is also elliptic. This gives the proof of Theorem 1.1.

In the following section, we will apply the ellipticity of $\mathcal{P}_2$ to prove the manifold theorem for the moduli space $\mathcal{E}_C$ of stationary vacuum spacetimes.

4. Manifold Theorem

Throughout this section, $(\tilde{g}, \tilde{u}, \tilde{\phi})$ denotes a collection of the conformal data which solves (II). We start by defining the following Banach spaces.

**Definition 4.1.**

$$\mathcal{M}_S = \{ (g, u, \phi) \in [\text{Met}^{m,\alpha}_S \times C^{m,\alpha}_S \times C^{m,\alpha}_S](S) : \begin{cases} \delta g = 0, \\ e^{-2u} g T |_{\partial S} = \gamma \\ H - 2n(u) = e^{-u} \lambda \\ n(\phi) = e^{2uf} \end{cases} \text{ on } \partial S, \text{ for some fixed } \gamma, \lambda \text{ and } f \} ;$$

$$\mathcal{M}_C = \{ (g, u, \phi) \in [\text{Met}^{m,\alpha}_S \times C^{m,\alpha}_S \times C^{m,\alpha}_S](S) : \delta g = 0 \text{ on } \partial S \}.$$

In the definition of $\mathcal{M}_S$, we modify the previous boundary condition $n(\phi) = e^{-uf}$ into $n(\phi) = e^{2uf}$, to ensure that the operator $D\hat{\Phi}$ below is formally self-adjoint on the tangent space $T\mathcal{M}_S$. This does not affect the elliptic property of the operator.

Define a map:

$$\Phi : \mathcal{M}_C \rightarrow [\text{Met}^{m,\alpha}_S \times C^{m,\alpha}_S \times C^{m,\alpha}_S](S)$$

$$\Phi(g, u, \phi) = (E - \delta^* \delta g, F, H)$$

where the terms $E, F, H$ are defined as in (3.24). Thus, the zero set $\Phi^{-1}(0)$ consists of stationary vacuum data $(g, u, \phi)$ satisfying $\delta g = 0$ on $S$, i.e.

$$\Phi^{-1}(0) = \mathcal{Z}_C,$$

where $\mathcal{Z}_C$ is as in Definition 2.3. Henceforth, based on the analysis in §2.3, to prove the moduli space $\mathcal{E}_C$ has the structure of a Banach manifold, it suffices to prove the zero set $\Phi^{-1}(0)$ is a smooth Banach manifold. The main step of that is the following theorem.

**Theorem 4.2.** At the point $(\tilde{g}, \tilde{u}, \tilde{\phi}) \in \Phi^{-1}(0)$, the linearization $D\Phi$ is surjective and its kernel splits in $T\mathcal{M}_C$.

4.1. Proof of Theorem 4.2. Surjectivity can be proved in a similar way as in [A1] and [AK]. Let $D\hat{\Phi}$ be the restriction of $D\Phi$ to the subspace $T\mathcal{M}_S \subset T\mathcal{M}_C$, i.e.

$$D\hat{\Phi} = D\Phi|_{T\mathcal{M}_S}.$$
Then the operator $D\hat{\Phi}$ is elliptic by Theorem 3.3. So $\text{Im}(D\hat{\Phi})$ is closed in $\left((S_2)^{m-2,\alpha}_0 \times C^{m-2,\alpha}_{\delta+2}\right)(S)$ and has finite dimensional cokernel $K$. If $K$ is trivial, then $D\hat{\Phi}$ is surjective, and hence so is $D\Phi$.

If $K$ is nontrivial, then from the self-adjoint property of $D\hat{\Phi}$ (cf. §5.4), it follows that,

$$K^\perp = \text{Im}D\hat{\Phi}.$$  

Thus for any element $(k, w, \zeta) \in K$ and an arbitrary element $(h, u, \sigma) \in T\mathcal{M}_S$, the following equation holds,

$$\int_S \langle D\hat{\Phi}(h, v, \sigma), (k, w, \zeta) \rangle = 0.$$

To prove surjectivity of $D\Phi$, it suffices to prove that for any triple $(k, w, \zeta) \in K$, there exists an element $(h, v, \sigma) \in T\mathcal{M}_C$ such that $\int_S \langle D\Phi(h, v, \sigma), (k, w, \zeta) \rangle \neq 0$.

Assume this is not true, i.e. there exists an element $(k, w, \zeta) \in K$ such that,

$$\int_S \langle D\Phi(h, v, \sigma), (k, v, \zeta) \rangle = 0, \quad \forall (h, v, \sigma) \in T\mathcal{M}_C.$$  

First choose $(h, v, \sigma) = (\delta^* X, L_X u, L_X \phi)$, for some vector field $X$ which vanishes on $\partial S$. Thus we are varying the data using diffeomorphisms in $D^{m+1,\alpha}_0(S)$. In this case, since the stationary vacuum field equations (II) are invariant under diffeomorphisms, it follows that

$$D\Phi(\delta^* X, L_X u, L_X \phi) = (\delta^* Y, 0, 0) \quad \text{at} \quad (\tilde{g}, \tilde{u}, \tilde{\phi}),$$

where $Y = \delta \delta^* X$. Note that Lemma 2.5 shows the operator $\delta \delta^*$ is surjective, so $Y$ can be arbitrarily prescribed. Moreover, the fact $(h, v, \sigma) \in T\mathcal{M}_C$ implies that $\delta h = 0$ on $\partial S$, so that $Y = 0$ on $\partial S$. It follows from the equation (4.1) that,

$$0 = \int_S \langle \delta^* Y, k \rangle = \int_S \langle Y, \delta k \rangle + \int_{\partial S} k(Y, n) = \int_S \langle Y, \delta k \rangle,$$

and thus,

$$\int_S \langle D\Phi(k, w, \zeta), (h, v, \sigma) \rangle = 0. \quad \text{at} \quad (\tilde{g}, \tilde{u}, \tilde{\phi}),$$  

Next applying integration by parts to (4.1), we obtain

$$\int_S \langle D\Phi(k, w, \zeta), (h, v, \sigma) \rangle + \int_{\partial S} \tilde{B}[(h, v, \sigma), (k, w, \zeta)] = 0.$$

This holds for any $(h, v, \sigma) \in T\mathcal{M}_C$, thus it implies that

$$D\Phi(k, w, \zeta) = 0 \quad \text{on} \quad S,$$

and

$$\int_{\partial S} \tilde{B}[(h, v, \sigma), (k, w, \zeta)] = 0, \quad \forall (h, v, \sigma) \in T\mathcal{M}_C.$$  

Here the bilinear form $\tilde{B}$ is as follows

$$\tilde{B}[(h, v, \sigma), (k, w, \zeta)] = B[(h, v, \sigma), (k, w, \zeta)] - B[(k, w, \zeta), (h, v, \sigma)].$$
where
\[ B[(h, v, \sigma), (k, w, \zeta)] = -k(\delta h, n) + \frac{1}{2} \{-\langle \nabla_n h, k \rangle - k(n, dtr h) + trk[n(trh + \delta h(n)] \} \]
\[ + [4n(w) - 4k(n, d\bar{u}) + 2trkn(\bar{u})]v \]
\[ + 4e^{-4v}\sigma[n(\zeta) - k(n, d\bar{\phi}) + \frac{1}{2}trkn(\bar{\phi}) - 4wn(\bar{\phi})]. \]

On the other hand, since the operator \( D\hat{\Phi} \) is formally self-adjoint in the space \( T_M S \), the cokernel \( K \) of \( D\hat{\Phi} \) is the same as the kernel of \( D\hat{\Phi} \) in \( T_M S \). Therefore, the element \((k, w, \zeta)\) must satisfy the following boundary conditions,

\[
\begin{align*}
\delta k &= 0 \\
k^T - 2w\bar{y}^T &= 0 \\
H'_k - 2n(w) - 2n'(\bar{u}) + w(H - 2n(\bar{u})) &= 0 \\
n(\zeta) + n'_k(\bar{\phi}) - 2wn(\bar{\phi}) &= 0 \\
\end{align*}
\]  

on \( \partial S \).

Based on the first equation \( \delta k = 0 \), together with the fact that \( h \in TM_C \) implies \( \delta h = 0 \) on \( \partial S \), the bilinear form \( B \) can be simplified by removing the divergence terms and becomes,

\[
B[(h, v, \sigma), (k, w, \zeta)] = \frac{1}{2} \{-\langle \nabla_n h, k \rangle - k(n, dtr h) + trk[n(trh)] \} \\
+ [4n(w) - 4k(n, d\bar{u}) + 2trkn(\bar{u})]v \\
+ 4e^{-4v}\sigma[n(\zeta) - k(n, d\bar{\phi}) + \frac{1}{2}trkn(\bar{\phi}) - 4wn(\bar{\phi})].
\]

Taking a triple \((h, v, \sigma)\) such that \( h = 0 \), \( \nabla_n h = 0 \) and \( \sigma = v = 0 \) on \( \partial S \), and inserting it into equation (4.4), we obtain,

\[
\int_{\partial S} 4n(v)w + 4e^{-4v}\zeta n(\sigma) = 0.
\]

The terms \( n(v) \) and \( n(\sigma) \) can be chosen to be arbitrary functions along \( \partial S \), so this implies that,

\[
w = \zeta = 0 \text{ on } \partial S.
\]

Consequently, based on the second equation in (4.5), we obtain

\[
k^T = w\bar{y}^T = 0 \text{ on } \partial S;
\]

and, according to the last equation in (4.5),

\[
n(\zeta) + n'_k(\bar{\phi}) = 4wn(\bar{\phi}) = 0 \text{ on } \partial S.
\]

Since \( k^T = 0 \) on \( \partial S \), the trace of \( k \) is \( trk = k(n, n) \). Thus in the last line of equation (4.6), the term \[n(\zeta) - k(n, d\bar{\phi}) + \frac{1}{2}trkn(\bar{\phi}) - 4wn(\bar{\phi}) \] vanishes on \( \partial S \) because of the following computation,

\[
n(\zeta) - k(n, d\bar{\phi}) + \frac{1}{2}trkn(\bar{\phi}) - 4wn(\bar{\phi}) \\
= n(\zeta) - k(n, d\bar{\phi}) + \frac{1}{2}k(n, n)n(\bar{\phi}) - 4wn(\bar{\phi}) \\
= n(\zeta) + n'_k(\bar{\phi}) - 4wn(\bar{\phi}) \\
= 0.
\]
Here the second equality is based on the formula of the linearization of the unit normal vector \( n \), cf.\$5:
\[
n' = -k(n) + \frac{1}{2}k(n, n)n.
\]
In addition, we have \( \zeta = 0 \) from equation (4.7). Therefore, the form \( B \) can be simplified further by removing the last line in (4.6) and becomes,
\[
B[(h, v, \sigma), (k, w, \zeta)] = \frac{1}{2} \left\{ -\langle \nabla_n h, k \rangle - k(n, dtrh) + trk(n(trh)) \right\}
+ [4n(w) - 4k(n, d\tilde{u}) + 2trk(n(\tilde{u}))]v
\]
Choose a triple \((h, v, \sigma)\) so that \( h = 0 \) and \( \nabla_n h = 0 \) on \( \partial S \) for equation (4.4). It follows that,
\[
\int_{\partial S} [4n(w) - 4k(n, d\tilde{u}) + 2trk(n(\tilde{u}))]v = 0,
\]
Since the term \( v \) can be arbitrarily prescribed on \( \partial S \), one obtains
\[
(4.11) \quad 4n(w) - 4k(n, d\tilde{u}) + 2trk(n(\tilde{u})) = 0 \quad \text{on} \ \partial S,
\]
which is equivalent to the following equation since \( trk = k(n, n) \),
\[
(4.12) \quad n(v) + n'(\tilde{u}) = 0 \quad \text{on} \ \partial S.
\]
Combining this with the third equation in (4.5), one obtains
\[
(4.13) \quad H' = 0 \quad \text{on} \ \partial S.
\]
Based on equation (4.11), we can simplify the form \( B \) further into the following expression,
\[
(4.14) \quad B[(h, v, \sigma), (k, w, \zeta)] = \frac{1}{2} \left\{ -\langle \nabla_n h, k \rangle - k(n, dtrh) + trk(n(trh)) \right\}
\]
Consequently (4.4) implies that the following equation holds for any \( h \in TM_C \),
\[
(4.15) \quad \int_{\partial S} \left\{ -\langle \nabla_n h, k \rangle - k(n, dtrh) + trk(n(trh)) \right\} - \left\{ -\langle \nabla_n k, h \rangle - h(n, dtrk) + trh(n(trk)) \right\} = 0.
\]
It follows from equations (4.13) and (4.15) that, c.f.[AK],
\[
(4.16) \quad (A'_{k})^T = 0 \quad \text{on} \ \partial S.
\]
Combining the equations (4.2), (4.3), (4.7), (4.8), (4.9), (4.12), and (4.16), we conclude that the element \((k, w, \zeta)\) must satisfy the following system,
\[
(4.17) \begin{cases}
D\Phi(k, w, \zeta) = 0, & \text{on} \ S, \\
\delta k = 0, & \text{on} \ S, \\
\end{cases}
\]
\[
(4.18) \begin{cases}
k^T = 0, \\
(A'_{k})^T = 0, \\
w = \zeta = 0, & \text{on} \ \partial S, \\
n(v) + n'(\tilde{u}) = 0, \\
n(\zeta) + n'(\tilde{\phi}) = 0.
\end{cases}
\]
**Remark** The first equation in (4.17) implies that the variation of \( (E - \delta^*\delta g, F, H) \) with respect to the deformation \((k, w, \zeta)\) vanishes, i.e.
\[
D(E - \delta^*\delta g, F, H)_{\{g, \tilde{u}, \tilde{\phi}\}}(k, w, \zeta) = 0,
\]
Together with the second equation in (4.17), we observe that \((k, w, \zeta)\) is in fact a vacuum deformation, i.e. it makes the linearization of \((E,F,H)\) vanish,

\[(4.19) \quad D(E,F,H)_{(\tilde{g}, \tilde{u}, \tilde{\phi})}(k, w, \zeta) = 0.\]

Translating to normal geodesic gauge

\[
 k \rightarrow \hat{k} = k + \delta^* V,
\]

where \(V\) is a vector field such that \(V = 0\ \kappa_0 = 0\) on \(\partial S\), the boundary conditions in (4.18) imply that the Cauchy data for \((k, w, \zeta)\) vanishes on \(\partial S\). Thus, to complete the proof, we will use the following unique continuation result, which is proved in §4.2.

**Proposition 4.3.** The boundary value problem formed by equations (4.17) and (4.18) has only the trivial solution \(k = w = \zeta = 0\).

As a consequence, \(D\Phi\) is surjective. In such circumstances, it is a standard fact that the kernel of \(D\Phi\) splits, cf. [A1]. This completes the proof of Theorem 4.2.

### 4.2. Proof of Proposition 4.3.

The proof below is a generalization of the unique continuation result of [AH] from Riemannian Einstein metrics to stationary Lorentzian Einstein metrics. We first formulate a local result as follows.

In the manifold \((S, \tilde{g})\), take an embedded cylinder \(C \cong I \times B^2 \subset \mathbb{R}^3\), where \(I = [0, 1]\) and \(B^2\) is the unit disk, in such a manner that the horizontal boundary \(\partial_0 C = \{0\} \times B^2\) is embedded in \(\partial S\), and the vertical boundary \(\partial C = I \times S^1\) is located in the interior of \(S\). Equip \(C\) with the induced metric \(\tilde{g}\), and choose H-harmonic coordinates \(\{\tau, x^i\}(\tau \geq 0, i = 1, 2)\), such that level set \(\{\tau = 0\}\) coincides with the horizontal boundary \(\partial_0 C\), cf. [AH] for the definition of H-harmonic coordinates.

Notice that under the H-harmonic coordinate system, a generic metric \(g\) in \(C\) can be written in the form,

\[
g = z d\tau^2 + \gamma(\psi^i d\tau + x^i)(\psi^j d\tau + x^j).
\]

Here \(\gamma\) is the induced metric on the level sets of \(\tau\) function, \(z\) is called the lapse function and \(\psi\) is the shift vector. In addition, by expressing the Ricci tensor \(Ric_g\) in these coordinates, one can obtain the following equations on every \(\tau\)-level set \(\{\tau = \text{constant}\}\) :

\[
(\partial^2 + z^2 \Delta - 2\psi^k \partial_k \partial \tau + \psi^k \psi^l \partial^2_{kl})\gamma_{ij} = -2z^2 (Ric_g)_{ij} - 2z \nabla_i \nabla_j z + Q_{ij}(\gamma, \partial \gamma),
\]

\[
(4.20)
\]

\[
\Delta z + |A_\gamma|^2 z + z Ric_g(N, N) - \psi(H_\gamma) = 0,
\]

\[
(4.21)
\]

\[
\Delta \psi^i + 2z(D^2 x^i, A_\gamma) + z \partial_\tau H_\gamma + 2[(A_\gamma)^{ij} \nabla_j z - \frac{1}{2} H \nabla^i z] + 2z Ric_0^g = 0,
\]

\[
(4.22)
\]

where the Laplacian operator \(\Delta\) and covariant derivative \(\nabla\) are with respect to the induced metric \(\gamma\) on the level surface. In the equations above, \(N\) denotes the normal vector of the surface \(\{\tau = \text{constant}\} \subset C\), which is equal to

\[
(4.23)
\]

\[
N = \frac{1}{z}(\partial_\tau - \psi),
\]

and thus the second fundamental form is given by

\[
(4.24)
\]

\[
A_\gamma = \frac{1}{2} L_N \gamma.
\]
In the equation (4.20), $Q_{ij}(\gamma, \partial \gamma)$ is a term which involves at most first order derivatives of $(\gamma, z, \psi)$ in all directions and the 2nd tangential derivatives of $\psi$.

In addition, on the vertical boundary $\partial C$, we have the following conditions in H-harmonic coordinates:

$$
(4.25) \quad z|_{\partial C} \equiv 1, \psi|_{\partial C} \equiv 0.
$$

Without loss of generality, we can assume the cylinder $C$ is sufficiently small so that $\tilde{g}$ is $C^{m, \alpha}$ close to the standard product metric on the cylinder.

**Proposition 4.4.** Let data $(\tilde{g}, \tilde{u}, \tilde{\phi})$ be a stationary vacuum solution, $\Phi(\tilde{g}, \tilde{u}, \tilde{\phi}) = 0$ in $C$. If $(k, w, \zeta)$ is an infinitesimal deformation of $(\tilde{g}, \tilde{u}, \tilde{\phi})$ such that it satisfies the equations (4.17) in $C$ and the boundary conditions (4.18) on $\partial_0 C$, then there exists a vector field $X$ with $X = 0$ on $\partial_0 C$, such that

$$
(4.26) \quad k = \delta^* X, \quad w = L_X \tilde{u}, \quad \text{and} \quad \zeta = L_X \tilde{\phi}.
$$

**Proof.** First we define a Banach space $\mathcal{M}^*$ as follows,

$$
\mathcal{M}^* = \{ (g, u, \phi) \in [M_{44}^{m, \alpha} \times C_4^{m, \alpha} \times C_4^{m, \alpha}](C) : \delta^* g = 0 \text{ on } \partial C \text{ and,} \}
$$

$$
(4.26) \quad g^T, A, u, \phi, n(u) \text{ and } n(\phi) \text{ are all fixed on } \partial_0 C\}.
$$

Observe that the deformation $(k, w, \zeta)$, by the hypothesis, is tangent to the space $\mathcal{M}^*$, i.e. $(k, w, \zeta) \in T \mathcal{M}^*$. Thus, we can assume $(k, w, \zeta)$ is the infinitesimal deformation of a smooth curve $(\tilde{g}_t, u_t, \phi_t)$ at $t = 0$, where $(\tilde{g}_t, u_t, \phi_t) \in \mathcal{M}^*$ for $t \in (-\epsilon, \epsilon)$, with some $\epsilon > 0$, and $(\tilde{g}_0, u_0, \phi_0) = (\tilde{g}, \tilde{u}, \tilde{\phi})$.

According to [AH], there exists a smooth curve of $C^{m+1, \alpha}$ diffeomorphisms $\Psi_t$ of $C$, which equal to $Id_{\partial_0 C}$ on $\partial_0 C$ for all $t \in (-\epsilon, \epsilon)$ and $\Psi_0 = Id$ in $C$, so that $\Psi_t^* (\tilde{g}_t)$ share the same H-harmonic coordinates. We denote the infinitesimal variation of the new curve $(\Psi_t^* (\tilde{g}_t), \Psi_t^* (u_t), \Psi_t^* (\phi_t))$ at $t = 0$ as $(g', u', \phi')$. It is given by

$$
(g', u', \phi') = (k + \delta_7^* X, w + L_X \tilde{u}, \zeta + L_X \tilde{\phi}),
$$

for some vector field $X$, with $X = 0$ on $\partial_0 C$. Therefore, to prove the proposition, it suffices to prove that $g' = u' = \phi' = 0$.

For simplicity of notation, the normalized curve $(\Psi_t^* (\tilde{g}_t), \Psi_t^* (u_t), \Psi_t^* (\phi_t))$ will still be denoted as $(\tilde{g}_t, u_t, \phi_t)$ in the following argument. Since the infinitesimal deformation $(g', u', \phi')$ is the sum of a vacuum deformation $(k, w, \zeta)$, cf.(4.19), and a diffeomorphism deformation $\frac{d}{dt} \Psi_t^*$, it must preserve the stationary vacuum property, i.e. it satisfies the following equation:

$$
\frac{d}{dt}|_{t=0}(E, F, H)[(\tilde{g}_t, u_t, \phi_t)] = 0 \quad \text{in} \quad C,
$$

which furthermore implies that,

$$
(4.27) \quad \zeta'_{\tilde{g}_t} = (2|dt|^2 + 2e^{-4u_t}|d\phi_t|^2)',
$$

$$
(4.28) \quad Ric'_{\tilde{g}_t} = (2du_t \otimes du_t + 2e^{-4u_t}d\phi_t \otimes d\phi_t)',
$$

$$
(4.29) \quad (\Delta u_t - 2e^{-4u_t}|d\phi_t|')' = 0,
$$

$$
(4.30) \quad (\Delta \phi_t + 4(du_t, d\phi_t))' = 0,
$$

where the prime superscript $'$ means $\frac{d}{dt}|_{t=0}$. Let $\{\tau, x^i\}(i = 1, 2)$ denote the common H-harmonic coordinates for $g_t$, with the lapse function denoted as $z_t$ and the shift vector $\psi_t$. Thus the metric $g_t$ is in the
form,
\[ g_\tau = z_\tau d\tau^2 + \gamma_t(\psi_\tau^i d\tau + x^i)(\psi_\tau^j d\tau + x^j). \]

Write \((\gamma', z', \psi', u', \phi')\) as the infinitesimal variation of the curve \((\gamma_t, z_t, \psi_t, u_t, \phi_t)\) at \(t = 0\), then by the boundary conditions in \((4.26)\), we obtain the following equations
\[
\begin{align*}
(4.31) \\
\gamma' &= 0 \\
(A')^T &= 0 \\
u' &= 0 \\
\phi' &= 0 \\
\mathbf{n}(u') + \mathbf{n}'(u_0) &= 0 \\
\mathbf{n}(\phi') + \mathbf{n}'(\phi_0) &= 0,
\end{align*}
\]
on the boundary surface \(\partial_0 C = \{\tau = 0\}\).

Notice that equations \((4.20 - 22)\) hold for all \((\gamma_t, z_t, \psi_t, u_t, \phi_t), \ t \in (-\epsilon, \epsilon)\). Linearization of the equation \((4.21)\) at \(t = 0\) gives
\[
\begin{align*}
0 &= \Delta z' + |A|^2 z' + z' Ric_{g_0}(\mathbf{n}, \mathbf{n}) - \psi'(H) \\
&\quad + \Delta' z + (|A|_t^2) z' + z'[Ric_{g_0}(\mathbf{n}, \mathbf{n})]' - \psi(H_t').
\end{align*}
\]

Here the terms \(\Delta', (|A|_t^2)', \) and \(H_t'\) only involve the tangential variation of \(\gamma\) and \(A\), thus they are all zero on the boundary surface \(\partial_0 C\) according to \((4.31)\). In addition, for the term \([Ric_{g_0}(\mathbf{n}, \mathbf{n})]'\), we have the following equation:
\[
\begin{align*}
[Ric_{g_0}(\mathbf{n}, \mathbf{n})]' &= Ric_{g_0}^\prime(\mathbf{n}, \mathbf{n}) + 2Ric_{g_0}(\mathbf{n}', \mathbf{n}) \\
&= (2du_t \otimes du_t + 2e^{-4u_0} d\phi_t \otimes d\phi_t)'(\mathbf{n}, \mathbf{n}) \\
&\quad + 2(2du_0 \otimes du_0 + 2e^{-4u_0} d\phi_0 \otimes d\phi_0)(\mathbf{n}', \mathbf{n}) \\
&\quad + [(2du \otimes du + 2e^{-4u} d\phi \otimes d\phi)(\mathbf{n}, \mathbf{n})]'.
\end{align*}
\]
The second equality above is based on the equation \((4.28)\) and the fact that \((g_0, u_0, \phi_0)\) is a vacuum solution. Equation \((4.33)\) shows that \([Ric_{g_0}(\mathbf{n}, \mathbf{n})]'\) only involves the variation of \(u, \mathbf{n}(u)\) and \(\mathbf{n}(\phi)\). Thus it is also zero on \(\partial_0 C\) according to the boundary condition \((4.31)\).

Henceforth, by restricting the equation \((4.32)\) to \(\partial_0 C\), we can remove those terms on the second line and obtain
\[
\Delta z' + |A|^2 z' + z' Ric_{g_0}(\mathbf{n}, \mathbf{n}) - \psi'(H) = 0 \quad \text{on} \ \partial_0 C.
\]

For the same reason, it follows from the linearization of the equation \((4.22)\) that
\[
\Delta(\psi') + 2z'(D^2 x^i, A) + z' \partial_i H + 2[(A)^i_j \nabla^j z - \frac{1}{2} H \nabla^i z'] + 2z' Ric_{g_0}^{\mu i} = 0 \quad \text{on} \ \partial_0 C.
\]

Furthermore, on the boundary of the surface \(\partial_0 C\), we have the Dirichlet condition for \(z'\) and \(\psi'\), because linearization of the boundary condition \((4.25)\) gives,
\[ z'|_{\partial C} = 0, \psi'|_{\partial C} = 0. \]

Since \(g_0\) is assumed to be \(C^{m, \alpha}\) close to the standard cylinder metric, equations \((4.34)\) and \((4.35)\) together with the Dirichlet boundary condition, imply that
\[
(4.36) \quad z' = \psi' = 0 \quad \text{on} \ \partial_0 C.
\]
Based on (4.23) and (4.24), we have

\[ A_{r_1}^{2} = \frac{1}{2} \left[ C \left( \frac{\partial}{\partial x} - \psi_1 \right) \gamma_i \right]^2. \]

On the boundary \( \partial \Omega C \), \( (A_{r_1}^{2})^T = 0 \) \( \gamma' = \psi' = 0 \) by (4.31) and (4.36). Hence it follows from the equation above that,

\[ (4.37) \quad \partial_r \gamma_{ij} = 0 \quad \text{on} \quad \partial \Omega C. \]

Observe that the unit normal vector \( N \) of \( \partial \Omega C \) and the normal vector \( n \) of \( \partial S \) are related by

\[ N = -n \]

on the boundary \( \partial \Omega C \). From (4.23) and (4.36), it follows that,

\[ n'(u_0) = -\left[ \frac{1}{z_t} (\partial_r - \psi_1) \right]'(u_0) = 0 \quad \text{on} \quad \partial \Omega C. \]

Therefore, according to the boundary conditions in (4.31), we obtain

\[ (4.38) \quad \partial_r u' = 0 \quad \text{on} \quad \partial \Omega C. \]

Similarly, one can derive that,

\[ (4.39) \quad \partial_r \phi' = 0 \quad \text{on} \quad \partial \Omega C. \]

By the conditions in (4.31) and (4.37–39), the triple \((\gamma', u', \phi')\) has trivial Cauchy data on the boundary \( \partial \Omega C \). In the interior of \( C \), linearization of the equation (4.20) shows

\[ (4.40) \quad (\partial^2_r + w^2 \gamma^k \partial^2_{kl} - 2 \sigma^k \partial_r \sigma^l + \sigma^k \sigma^l \partial^2_{kl}) \gamma_{ij}' = O(\gamma', w', \sigma', u', \phi'), \]

where \( O(\gamma', w', \sigma', u', \phi') \) is used to denote a term when it only depends on the tangential derivatives (at most 2nd order) of \( w', \sigma' \) and derivatives (at most 1st order) of \( \gamma', u', \phi' \). Equations (4.29) and (4.30) gives:

\[ (4.41) \quad g^{\alpha \beta} \partial^2_{\alpha \beta} u' = O(\gamma', w', \sigma', u', \phi'), \]

\[ (4.42) \quad g^{\alpha \beta} \partial^2_{\alpha \beta} \phi' = O(\gamma', w', \sigma', u', \phi'), \]

which are equivalent to the following equations,

\[ (4.43) \quad [\partial^2_r - 2 \sigma^i \partial^2_{0i} + (w^{-2} \gamma^{ij} + \sigma^i \sigma^j) \partial^2_{ij}] u' = O(\gamma', w', \sigma', u', \phi'), \]

\[ (4.44) \quad [\partial^2_r - 2 \sigma^i \partial^2_{0i} + (w^{-2} \gamma^{ij} + \sigma^i \sigma^j) \partial^2_{ij}] \phi' = O(\gamma', w', \sigma', u', \phi'), \]

since \( g^{00} = w^{-2}, g^{\beta i} = -w^{-2} \sigma^i, \) and \( g^{kl} = \gamma^{kl} + w^{-2} \sigma^k \sigma^l \), where index 0 denotes the \( \partial_r \) direction.

Observe that equations (4.40) and (4.43–44) have the same principal operator on \((\gamma', u', \phi')\). We denote it as \( P \),

\[ P = [\partial^2_r - 2 \sigma^i \partial^2_{0i} + (w^{-2} \gamma^{ij} + \sigma^i \sigma^j) \partial^2_{ij}]. \]

It is the same as the operator in \([AH]\), and hence, as is shown there, \( \gamma' = u' = \phi' = 0 \) on condition that the Cauchy data of \((\gamma', u', \phi')\) vanishes on the boundary \( \partial \Omega C \). This completes the proof.

\[ \square \]
Proposition 4.4 implies that there exists a vector field \( Z \), which is zero on \( \partial S \), such that \( k = \delta^* Z, w = L_Z \tilde{u}, \zeta = L_Z \tilde{\phi} \) in a neighborhood \( U \) of \( \partial S \). From \([A1]\), \( Z \) can be uniquely extended to \( S \cong \mathbb{R}^3 - B \) so that \( k = \delta^* Z \) holds globally. From the second equation in (4.17), it follows that,
\[
\delta \delta^* Z = \delta k = 0.
\]

For a fixed radius \( R > 1 \), let \( B_R \subset S \) denote the pull back of a closed ball of radius \( R \) under a chosen diffeomorphism \( S \cong \mathbb{R}^3 - B^3 \), and \( A_\epsilon \) denote the annulus between \( B_{R-\epsilon} \) and \( B_R \). Take a cutoff function \( f \in C^{m+1,\alpha}(S) \) such that \( f|_{B_{R-\epsilon}} \equiv 1 \) and \( f|_{S\setminus B_R} \equiv 0 \). Let \( W \) be the compactly supported vector field \( W = \tilde{f} Z \). Since \( Z \) is bounded in \( B_R \), we can take \( \epsilon \) small enough such that,
\[
\begin{align*}
\int_S (W, Z) &= \int_{B_{R-\epsilon}} \int_{A_\epsilon} |Z, Z|^2 + \int_{A_\epsilon} \langle f Z, Z \rangle \\
&\geq \frac{1}{2} \int_{B_{R/2}} |Z|^2.
\end{align*}
\]

According to Lemma 2.5, the map \( \delta \delta^* \) is surjective, therefore there exist a vector field \( Y \), which is asymptotically zero of decay rate \( (4 + \delta) \) and \( Y|_{\partial S} = 0 \), such that
\[
\delta \delta^* Y = W.
\]

Notice that \( \delta^* Z \) has the decay rate \( \delta \), since \( \delta^* Z = k \). From this one can derive that \( Z \) can blow up no faster than \( r^{2-\delta} \) (cf. §5.5). Therefore, applying integration by parts, one can obtain
\[
\begin{align*}
\int_S \langle W, Z \rangle &= \int_S \langle \delta \delta^* Y, Z \rangle \\
&= \int_S \langle Y, \delta \delta^* Z \rangle = 0.
\end{align*}
\]

From equations (4.45) and (4.46), it is easy to derive that \( Z = 0 \) in \( B_{R/2} \), thus \( k, w, \) and \( \zeta \) vanish in \( B_{R/2} \), which further implies that they are vanishing globally because of ellipticity. This finishes the proof of Proposition 4.3.

In conclusion, we obtain the following result:

**Theorem 4.5.** The moduli space \( \mathcal{E}_C \) is an infinite dimensional \( C^\infty \) Banach manifold, with tangent space
\[
T_{[\tilde{g}, \tilde{u}, \tilde{\phi}]} \mathcal{E}_C \cong \text{Ker}(D\Phi_{(\tilde{g}, \tilde{u}, \tilde{\phi})}).
\]

**Proof.** This is an immediate consequence of Theorem 4.2, the fact from §2.3 that \( \Phi^{-1}(0) = \mathcal{E}_C \) (locally), and the implicit function theorem in Banach spaces.

Moreover, from the ellipticity results in §3, it follows that,

**Theorem 4.6.** The boundary map
\[
\Pi : \mathcal{E}_C \to \mathcal{E}_2 \equiv C^{m-1,\alpha}_{S^2 \times C^{m-1,\alpha}}, (\partial S)
\]
\[
\Pi[(g, u, \phi)] = (e^{-2u} g^T, e^n(H_g - 2n_g(u)), e^n n_g(\phi))
\]
is a \( C^\infty \) Fredholm map, of Fredholm index 0.

**Proof.** The fact from §3.2 that the operator \( P_2 \) is elliptic implies that the boundary map \( \Pi \)
\[
\Pi : \mathcal{Z}_C \equiv C^{m-1,\alpha}_{S^2 \times C^{m-1,\alpha}}, (\partial S),
\]
\[
\Pi[(g, u, \phi)] = (e^{-2u} g^T, e^n(H_g - 2n_g(u)), e^n n_g(\phi))
\]
is smooth and Fredholm. It is of Fredholm index 0 because $P_2$ is formally self-adjoint. Moreover, since we show in §2.3 that locally $\mathcal{E}_C = \mathbb{Z}_C$, it follows that $\Pi$ is also a smooth Fredholm map and of index 0.

Now translating the results above from conformal data $(g, u, \phi)$ back to $(g_S, u, \phi)$ via the isomorphism $Q$ as in §3.3, proves Theorem 1.2.

□

Remark. All the methods and results in this paper can be applied equally well to the interior problem where $S \cong B^3$.

5. Appendix

In this section, we provide the details of the computation of the linearization of operator $\mathcal{P}$, the linearization of reduced Hilbert-Einstein functional $I$, and some other basic results used in this paper. We refer to [Be] for elementary formulas of the differentials of various geometric tensors.

5.1. Linearization of the interior operator $\mathcal{L}$. Let $h$ be the infinitesimal deformation of the metric $g$ on $S$. The variation of Ricci tensor is given by,

\[
2Ric'_h = D^* Dh - 2\delta^* \delta h - D^2 (tr h) + O_0.
\]

(5.1)

The variation of scalar curvature is given by,

\[
s'_h = \Delta_g (tr h) + \delta \delta h + O_0.
\]

(5.2)

Linearization of the gauge term $\delta^* G$ in operator $\mathcal{L}$ are as follows. For the Bianchi gauge, we have,

\[
2[\delta^* G_1]'_h = 2[\delta^* \beta g (g)]'_h
\]

(5.3)

\[
= L_{\beta g h} + 2\delta^* \beta g h
\]

\[
= L_{\delta g} h + 2\delta^* \delta g h + 2\delta^2 (tr h);
\]

and for the divergence gauge, we have,

\[
2[\delta^* G_2]'_h = 2[\delta^* \delta g g]'_h = L_{\delta g} h + 2\delta^* \delta g h.
\]

(5.4)

Combining equations (5.1) and (5.3), one can derive the linearization for $\mathcal{L}$, with the Bianchi gauge, at $\hat{g} = g$:

\[
L_1(h) = D^* Dh + O_0.
\]

Combining equations (5.1 − 2) and (5.4), one can derive the linearization for $\mathcal{L}$, with the divergence gauge, at $\hat{g} = g$:

\[
L_2(h) = D^* Dh - D^2 (tr h) - (\Delta_g (tr h) + \delta \delta h) g + O_0.
\]
5.2. Linearization of the boundary operator $B$. We know that the normal vector $n$ of $\partial S$ satisfies the following equations,
\[
\begin{aligned}
g(n, n) &= 1, \\
g(n, T) &= 0,
\end{aligned}
\]
where $T$ is a tangential vector. Let $n'_h$ denote the variation of $n$ with respect to deformation $h$. Then linearization of the above equations gives,
\[
\begin{aligned}
2g(n, n'_h) + h(n, n) &= 0, \\
g(n'_h, T) + h(n, T) &= 0,
\end{aligned}
\]
from which, one can solve for the term $n'_h$ as,
\[
(5.5) \quad n'_h = -\frac{1}{2}h(n, n)n - h(n)n + \frac{1}{2}h(n, n)n.
\]

The variation $H'_h$ of mean curvature $H_g$ is given by
\[
(5.6) \quad 2H'_h = 2(trA)'_h = 2trA'_h - 2(A_g, h),
\]
where $A_g$ is the second fundamental form of $\partial S \subset (S, g)$, defined by $A_g = \frac{1}{2}L_n g$. Linearization of $A$ is as follows,
\[
2A'_h = (L_n h + L_{n'_h} g) = \nabla_n h + 2h \circ \nabla n + L_{n'_h} g.
\]
Taking the trace of the equation above, we obtain,
\[
2trA'_h = \nabla_n trh + 2(h, A) - 2\delta(n'_h).
\]
Plugging the expression (5.5) for $n'_h$ into the equation above, we obtain,
\[
(5.7) \quad 2trA'_h = \nabla_n trh + 2(h, A) + 2\delta^T(h(n)^T - n(h(n, n))) + O_0
\]
\[
= \nabla_n trh + 2(h, A) + 2\delta^T(h(n)^T) + O_0.
\]
Combining equations (5.6) and (5.7) gives,
\[
H'_h = \frac{1}{2}\nabla_0 (h_{11} + h_{22}) - \Sigma_{k=1}^2 \nabla^k (h_{0k}) + O_0,
\]
which is the same as used in the symbol computation in §3.1.

5.3. Variation of the functional $I$. First, we define a functional $\tilde{I}$ as,
\[
\tilde{I} = \int_S s_g - 2|du|^2 - 2e^{-4u}|d\phi|^2 dvol_g.
\]
Since the variation of scalar curvature $s_g$ is given by,
\[
s'_h = \Delta_g(trh) + \delta\delta h - (Ric_g, h),
\]
Elliptic boundary value problems for the stationary vacuum spacetimes

linearization of \( I \) with respect to the metric is as follows,

\[
\tilde{I}'(h) = \int_S [\Delta_g(trh) + \delta \delta h - \langle Ric, h \rangle + 2h(du, du) + 2e^{-4u}h(d\phi, d\phi) \\
+ \frac{1}{2} trh(s_g - 2|du|^2 - 2e^{-4u}|d\phi|^2)] \\
= \int_S [\Delta_g(trh) + \delta \delta h \\
+ \langle -Ric + 2du \otimes du + 2e^{-4u}d\phi \otimes d\phi + \frac{1}{2}(s_g - 2|du|^2 - 2e^{-4u}|d\phi|^2)g, h \rangle] \\
= \int_S [\Delta_g(trh) + \delta \delta h + \langle E, h \rangle].
\]

The term \( \int_S [\Delta_g(trh) + \delta \delta h] \) in the expression above can be converted to a boundary term as,

\[
\int_S [\Delta_g(trh) + \delta \delta h] = -\int_{\partial S} [n(trh) + \delta h(n)] - \int_{\partial S_{\infty}} [n(trh) + \delta h(n)] \\
= -\int_{\partial S} [n(trh) - n(h_{00}) + \langle h, A \rangle] - \int_{\partial S_{\infty}} [n(trh) + \delta h(n)].
\]

To balance the finite boundary term, we add an extra term \( I_B = \int_{\partial S} 2H_g \) to the functional \( I \). Notice that the first variation of \( I_B \) with respect to the metric is given by,

\[
I'_B(h) = \int_{\partial S} [2H'_h + tr^T h H_g] \\
= \int_{\partial S} [n(trh) - 2\delta(n'_h) + tr^T h H_g].
\]

For a generic vector field \( V \) on \( \partial S \), we have \( \delta V = \delta^T V^T - n(V_0) \). Thus, we can simplify the term \( \delta(n'_h) \) in the boundary term above and obtain,

\[
I'_B(h) = \int_{\partial S} [n(trh) - n(h_{00}) + tr^T h H_g].
\]

Combining equations (5.8) and (5.9) we can obtain the formulae of variation for the functional \( \tilde{I} + I_B \):

\[
(\tilde{I} + I_B)'(h) = \int_S \langle E, h \rangle + \int_{\partial S} [\langle -A, h \rangle + tr^T h H_g] - \int_{\partial S_{\infty}} [n(trh) + \delta h(n)].
\]

To remove the boundary term at infinity, we use the mass \( m_{ADM}(g) \), as shown in equation (3.25).

5.4. Formal self-adjointness of \( D\Phi \). To prove the self-adjointness for \( D\Phi \), we will use the functional \( I \), as defined in §3,

\[
I = \int_S s_g - 2|du|^2 - 2e^{-4u}|d\phi|^2 dvol_g + 2 \int_{\partial S} H dvol_g + 16\pi m_{ADM}(g).
\]
Recall from §3, the first variation of $I$ is given by,

$$
I'_{(g,\tilde{u},\phi)}(h, v, \sigma) = \int_S \langle (E, F, H), (h, v, \sigma) \rangle + \int_{\partial S} \left[ -\langle A_{\tilde{g}}, h \rangle + H_{\tilde{g}} tr h^T - 4n(u)v - 4e^{-4u}n(\phi) \right].
$$

(5.10)

Let $(h, v, \sigma)$ be in the tangent space $T_M$, which is defined by

$$
T_M = \{(h, v, \sigma) \in [(\delta, \alpha)_{\delta} \times C^m_{\alpha} \times C^m_{\alpha}] (S) : \delta h = 0, \quad h^T - 2v\tilde{g}^T = 0, \quad [e^u(H_{\tilde{g}} - 2n(u))]_{(h, v)} = 0, \quad [e^{-2u}n(\phi)]_{(h, v, \sigma)} = 0, \quad \text{on } \partial S \}.
$$

(5.11)

Applying the boundary conditions in (5.13) to the equation (5.10), we obtain,

$$
I'_{(g,\tilde{u},\phi)}(h, v, \sigma) = \int_S \langle (E, F, H), (h, v, \sigma) \rangle + \int_{\partial S} \left[ -\langle A_{\tilde{g}}, 2v\tilde{g} \rangle + 2vH_{\tilde{g}} tr \tilde{g}^T - 4n(u)v - 4e^{-4u}n(\phi) \right]
$$

$$
= \int_S \langle (E, F, H), (h, v, \sigma) \rangle + \int_{\partial S} [2v(H_{\tilde{g}} - 2n(u)) - 4e^{-4u}n(\phi)].
$$

Taking the variation of $I'$ with respect a deformation $(k, w, \zeta) \in T_M$, we obtain,

$$
I''_{(g,\tilde{u},\phi)}[(h, v, \sigma), (k, w, \zeta)] = \int_S \langle (E', F', H'), (k, w, \zeta) \rangle (h, v, \sigma) \rangle + \int_{\partial S} \{2v[H_{\tilde{g}} - 2n(u)]_{(k, w)} - \sigma[4e^{-4u}n(\phi)]_{(k, w, \zeta)} \}
$$

$$
+ \int_{\partial S} \left\{ \frac{1}{2} tr k^T [2v(H_{\tilde{g}} - 2n(u)) - 4e^{-4u}n(\phi)] \right\}.
$$

According to (5.11), we have $k^T = 2u\tilde{g}^T$ and $2v[H_{\tilde{g}} - 2n(u)]_{(k, w)} = -2vw(H_{\tilde{g}} - 2n(u))$. Thus the boundary terms in the expression above can be simplifies as

$$
I''_{(g,\tilde{u},\phi)}[(h, v, \sigma), (k, w, \zeta)]
$$

$$
= \int_S \langle (E', F', H'), (k, w, \zeta) \rangle (h, v, \sigma) \rangle + \int_{\partial S} 2vw[H_{\tilde{g}} - 2n(u)]_{(k, w)}
$$

$$
+ \int_{\partial S} \{-\sigma[4e^{-4u}n(\phi)]_{(k, w, \zeta)} - 8we^{-4u}n(\phi) \}
$$

$$
= \int_S \langle (E', F', H'), (k, w, \zeta) \rangle (h, v, \sigma) \rangle + \int_{\partial S} 2vw[H_{\tilde{g}} - 2n(u)]_{(k, w)}
$$

$$
+ \int_{\partial S} 4e^{-4u}\sigma[2wn(\phi) - (\phi)]_{(k, \zeta)}.
$$

Here the last boundary term vanishes, because

$$
4e^{-4u}\sigma[2wn(\phi) - (\phi)]_{(k, \zeta)} = 0.
$$
Thus we obtain,

\[
\int_S \langle (E', F', H')_{(k, w, \zeta)}, (h, v, \sigma) \rangle = \int_S \langle (E', F', H')_{(h, v, \sigma)}, (k, w, \zeta) \rangle.
\]

In addition, it is easy to derive that

\[
\int_S \langle \delta^* \delta k, h \rangle = \int_S \langle \delta^* \delta h, k \rangle \quad \text{for } k, h \in T\mathcal{M}_S.
\]

Combining equations (5.12) and (5.13), we obtain the formal self-adjointness for \(D\Phi\), i.e.

\[
\int_S \langle D\Phi[(k, w, \zeta)], (h, v, \sigma) \rangle = \int_S \langle D\Phi[(h, v, \sigma)], (k, w, \zeta) \rangle, \quad \forall (k, w, \zeta), (h, v, \sigma) \in T\mathcal{M}_S.
\]

### 5.5. The blow-up rate of  \( Z \)

Given that \( \delta^* Z \) has the decay rate \( \delta \) (denoted as \( \delta^* Z \sim r^{-\delta} \)), we will find an upper bound for the blow-up rate of \( Z \) in the following.

For a radius \( r \) large enough, let \( S_r \subset S \) be the pull back of the sphere of radius \( r \) under the chosen diffeomorphism \( S \cong \mathbb{R}^3 \setminus B^3 \). Let \( N_r \) denote the unit normal vector of the sphere pointing outwards, then we have

\[
\delta^* Z(N_r, N_r) = N_r[g(Z, N_r)] + \hat{g}(Z, \nabla N_r, N_r).
\]

One can extend \( N_r \) in a way such that \( \nabla N_r, N_r = 0 \), and thus

\[
N_r[g(Z, N_r)] \sim r^{-\delta}.
\]

Therefore, \( g(Z, N_r) \) blows up no faster than \( r^{1-\delta} \).

Let \( Z^T \) denote the tangential component of \( Z \) along \( S_r \), then \( Z = Z^T + g(Z, N_r)N_r \).

Basic calculation shows

\[
2\delta^* Z(N_r, Z^T) = \hat{g}(\nabla N_r, Z, Z^T) + \hat{g}(\nabla Z^T, Z, N_r)
\]

\[
= \hat{g}(\nabla N_r, (Z^T + g(Z, N_r)N_r), Z^T) + Z^T[\hat{g}(Z, N_r)] - \hat{g}(Z, \nabla Z^T, N_r)
\]

\[
= \hat{g}(\nabla N_r, Z^T, Z^T) + Z^T[\hat{g}(Z, N_r)] - A(Z^T, Z^T)
\]

\[
= |Z^T|N_r(|Z^T|) + Z^T[\hat{g}(Z, N_r)] - A(Z^T, Z^T),
\]

where \( A \) denotes the second fundamental form of the hypersurface \( S_r \subset (S, \tilde{g}) \).

Thus we obtain

\[
N_r(|Z^T|) - A(Z^T, \frac{Z^T}{|Z^T|}) = 2\delta^* Z(N_r, \frac{Z^T}{|Z^T|}) = 2\delta^* Z(N_r, \frac{Z^T}{|Z^T|}) - Z^T[\hat{g}(Z, N_r)].
\]

This implies that \( \partial_r(|Z^T|) - \frac{1}{2}|Z^T| \) blows up no faster than \( r^{1-\delta} \), and therefore the increasing rate of \( |Z^T| \) is at most \( r^{2-\delta} \).

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