A Low-Rank Rounding Heuristic for Semidefinite Relaxation of Hydro Unit Commitment Problems

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Abstract—Hydro unit commitment is the problem of maximizing water use efficiency while minimizing start-up costs in the daily operation of multiple hydro plants, subject to constraints on short-term reservoir operation, and long-term goals. A low-rank rounding heuristic is presented for the semidefinite relaxation of the mixed-integer quadratic-constrained formulation of this problem. In addition to limits on reservoir and generator operation, transmission constraints are represented by an approximate AC power flow model. In our proposed method, the mathematical program is equivalently formulated as a QCQP problem solved by convex relaxation based on semidefinite programming, followed by a MILP solution of undefined unit commitment schedules. Finally, a rank reduction procedure is applied. Effectiveness of the proposed heuristic is compared to branch-and-bound solutions for numerical case studies of varying sizes of the generation and transmission systems.

Index Terms—Hydroelectric power generation, power generation dispatch, unit commitment, quadratic programming, relaxation methods, heuristic algorithms.

I. INTRODUCTION

The economic operation of hydro-dominated power systems is implicitly associated with efficient use of water resources for power generation. The nonlinear characteristics of hydropower production models and transmission constraints pose computational challenges. Most often such challenges are overcome by simplification of either or both models, as evidenced in the literature. For instance, in [1] a formulation with AC power flow equations is presented at the expense of a linear representation of hydro production functions, whereas DC power flow formulations are more commonly proposed for the solution of large-scale instances of short-term hydro-thermal coordination problems, e.g. [2]. Because such linear representations of the production function invariably ignore head variation effects on productivity, linear piece-wise formulations have been presented over the years [3]. Furthermore, hydro-electric unit commitment adds complexity to the problem due to the combinatorial nature of unit configuration optimization. Lagrangian relaxation and dynamic programming [4] were once common strategies in the literature for problem resolution.

Although it has been shown [5] that load demand can be formulated as a non-anticipative variable, in this paper a deterministic model is considered. Moreover, water inflows could also be modeled as stochastic variables, although such formulation is dependent on the availability of reliable data in the considered time resolution.

In the literature, short-term operation scheduling of hydrothermal power systems is commonly decomposed into two properly coordinated subproblems. In [6], the hydro unit commitment (HUC) problem is solved by means of a network flow model using priority-list-based dynamic programming with reservoir aggregation. The strategy in [7] is based on mixed-integer linear programming (MILP) as an alternative to the so-called “curse of dimensionality” assailing dynamic programming methods.

A quadratic formulation with hydro unit efficiency functions is presented in [8] for individual representation of hydropower stations in a Lagrangian relaxation framework with a packing technique. The HUC problem is then transformed into a linear master problem with construction of a nonlinear integer subproblem and consideration of forbidden operation zones. This particular decomposition helps to reduce the duality gap in every iteration if compared to classical Lagrangian relaxation. Subproblems are solved by a Quasi-Newton algorithm, implementing sequential quadratic programming (SQP). Lagrangian-based heuristics [9] can also approximate the solution of the HUC problem by linearizing the hydropower model with inner minimization of thermal subproblems.

In [10] a nonlinear HUC model is presented, where the objective is a nonlinear function of water release with linear constraints. Integer variables are used to represent model characteristics related to forbidden zones and unit startup and shutdown. Quadratic programming (QP) and barrier methods are used to solve the problem.

In this paper a quadratic model [11] of the hydroelectric production function is used, such that water discharge can be formulated as a function of active power generation, i.e:

\[
\hat{q}_{t,h}(.) = \sum_{u=1}^{N_{u,t,h}} x_{t,h,u} \left( \alpha_{h,u} p_{t,h,u}^2 + \beta_{h,u} p_{t,h,u} + \gamma_{h,u} \right), \tag{1}
\]

where \( r \) represents water discharge, and \( \alpha, \beta, \gamma \) are coefficients of the water discharge curve for given active power output \( p \) and unit configuration \( u \) at hydro plant \( h \) and hour \( t \). The choice of unit configurations is determined by discrete
variables:
\[ x_{t,h,u} \in \{0, 1\}, \quad \forall (t, h, u) \] (2)

subject to hourly configuration uniqueness constraints:
\[ \sum_{u=1}^{N_{u,t,h}} x_{t,h,u} = 1 \]
and unit startup availability between successive hours :
\[ \sum_{u=1}^{N_{u,t,h}} (u \cdot x_{t,h,u}) - \sum_{u=1}^{N_{u,t-1,h,u}} (u \cdot x_{t-1,h,u}) \leq y_{t,h} \] (4)

where \( N_{u,t,h} \) is the number of unit configurations available for commitment, and \( y_{t,h} \geq 0 \) is the number of units scheduled for startup. In order to solve this mixed-integer, quadratically-constrained quadratic program (MIQCQP) in a computationally efficient manner, we initially solve a semidefinite programming-based (SDP) convex relaxation [12] of the problem with AC power flow formulation, as originally proposed in [11]. This convex SDP relaxation has been previously shown to guarantee valid lower bounds with relatively small gaps. In this paper we propose a rounding heuristic that, after the initial relaxation, solves a MILP subproblem to determine the optimal unit configuration of the remaining undefined discrete variables for the active power generation dispatches obtained from the solution to the SDP relaxation. Once the active power dispatches and unit commitments are obtained, a re-optimization of the AC power flow using SDP is performed, followed by a rank-reduction procedure for improved solution.

II. SOLUTION METHODOLOGY

A. Problem formulation

It is based on the quadratic relaxation of discrete variables of the day-ahead HUC problem with hourly discretization [11], such that:
\[
\begin{align*}
\min_x & \quad x^T C x \\
\text{subject to} & \quad x^T A_i x = a_i, \quad i = 1, \ldots, m \\
& \quad x^T B_j x \leq b_j, \quad j = 1, \ldots, n \\
& \quad x \succeq 0
\end{align*}
\]

where \( C \) represents water-use and unit startup costs, \( A_i \) represents active and reactive power balance, long-term reservoir operation goals, unit configuration uniqueness constraints, and fixed voltage at the slack bus. Limits on active and reactive power generation, reservoir storage, as well as power flow and bus voltages are represented in \( B_j \). Problem variables are represented by
\[
x = (1, x, y, p, q, e, f)^	op
\]

where, in addition to unit configuration \( x \) and startup control variables \( y \), active and reactive power, as well as real and imaginary parts of bus voltages in Cartesian coordinates, represented by \( p, q, e, \) and \( f \), respectively.

The next sections describe the solution methodology proposed in this paper as depicted in Algorithm 1.

Algorithm 1 Solution methodology

1. Solve SDP (P1) to obtain \( \tilde{p}, \tilde{x} \)
2. \( \Omega_1(\tilde{x}) \leftarrow \{(t, h, u) : \tilde{x}_{t,h,u} \notin \{0, 1\}\} \)
3. \( \Omega_2(\tilde{x}) \leftarrow \{(t, h, u) : \tilde{x}_{t,h,u} = 1\} \)
4. Let \( \tilde{p}_{t,h} \leftarrow \sum_{u=1}^{N_{u,t,h}} \tilde{p}_{t,h,u} \cdot \tilde{x}_{t,h,u} \)
5. Solve MILP (P2) for \( \tilde{p}, \tilde{x} \in \Omega_2(\tilde{x}) \) to obtain \( x^* \in \{0, 1\} \)
6. Solve the OPF (22) for \( x^* \) to obtain \( \tilde{p}, \tilde{q}, \tilde{e}, \tilde{f} \)
7. Apply rank reduction procedure to obtain \( p^*, q^*, e^*, f^* \)
8. Return \( p^*, x^*, q^*, y^*, e^*, f^* \)

B. Semidefinite relaxation

The SDP relaxation [13] of the QCQP (5) model is given by:
\[
\begin{align*}
(P1) \quad \min_{x, y} & \quad C \cdot x + e^T y \\
\text{s. t.} & \quad p = A(X) + F_p(V); \quad 1 = A(X) \\
& \quad q = Q(X) + F_q(V); \quad g = Q(X) \\
& \quad 0 \geq \tilde{P}(X); \quad 0 \geq \tilde{Q}(X) \\
& \quad r \geq \tilde{R}(X); \quad s = \tilde{S}(V) \\
& \quad \bar{f} \geq \tilde{F}(V); \quad \bar{v} \geq \tilde{F}(V) \\
& \quad y \geq \xi(X) \quad 0 \leq \bar{v} \leq X, V \\
& \quad 1 = \text{rank}(X) = \text{rank}(V) \quad 0 \leq y
\end{align*}
\]

where the rank constraints on \( X \) and \( V \) are relaxed, and the Frobenius product is represented by the symbol \( \bullet \).

\[
x_{t,h,u} = [\Delta P_{t,h,u}, x_{t,h,u}, \Delta Q_{t,h,u}]^T; \quad y = \{y_{t,h,u}\} \quad \forall t, h, u
\]

\[
X = \sum_{(t,h,u)} \xi_{t,h,u} \xi_{t,h,u}^T \otimes x_{t,h,u} x_{t,h,u}^T; \quad V = \sum_{t=1}^{T} \xi_t \xi_t^T \otimes \nu_t \nu_t^T
\]

\[
\nu_t = [e_{t,1}, \ldots, e_{t,N_B}, f_{t,1}, \ldots, f_{t,N_B}]^T \quad \forall t = 1, \ldots, T
\]

where \( \xi_{t,h,u} \) is a vector on the Euclidean space with appropriate dimensions (in this case \( \mathbb{R}^{2\cdot N_B} \)), with value 1 in the corresponding \((t, h, u)\) position. Matrix \( C \) represents water cost coefficients:
\[
C = \sum_{t,h,u} \lambda_{h} \cdot \tilde{C}_{t,h,u}
\]

\[
\tilde{C}_{t,h,u} = \xi_{t,h,u} \xi_{t,h,u}^T \otimes \left[ \begin{array}{ccc} \alpha_{h} & \tilde{p}_{h,u} & 0 \\ \tilde{p}_{h,u} & \tilde{q}_{h,u} & 0 \\ 0 & 0 & \tilde{q}_{h,u} \end{array} \right] \quad \forall t, h, u
\]

Startup costs are in vector \( c \in \mathbb{R}^{\dim(y)} \), with elements \( c_{t,h,u} = \delta_h \); \( \forall t, h, u \). Active power generation constraints are represented by linear mapping \( P(\cdot) \), where, for each \( P_{t,h} \in \mathbb{R}^{\dim(x) \times \dim(x)} \quad \forall i = 1, \ldots, N_B \) and \( t = 1, \ldots, T \), we have:
\[
P_{t,h} = \sum_{h \in \Psi_t} \sum_{u=1}^{N_{u,h}} \xi_{t,h,u} \xi_{t,h,u}^T \otimes \left[ \begin{array}{ccc} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \forall h, u
\]

where \( P_{h,u} \) is the minimum active power generation, Linear mapping \( Q(\cdot) \) is similarly constructed. In order to represent the active and reactive power injection on every bus we
use $F_p(\cdot)$ and $F_q(\cdot)$, with elements $F_{p_{i,t}}$ and $F_{q_{i,t}} \in \mathbb{R}^{2N_x \times 2N_p}$.

$$F_{p_{i,t}} = - \sum_{j \in \Omega_i} \xi_{i,j}^T \otimes Y_{i,j} ; F_{q_{i,t}} = - \sum_{j \in \Omega_i} \xi_{i,j}^T \otimes Y_{i,j} \tag{9}$$

$$Y_{i,j} = \frac{1}{2} \begin{bmatrix} G_{i,j} + G_{i,j}^T & B_{i,j} - B_{i,j}^T \\ B_{i,j} - B_{i,j}^T & G_{i,j} + G_{i,j}^T \end{bmatrix} \tag{10}$$

$$Y_{i,j} = \frac{1}{2} \begin{bmatrix} B_{i,j} + B_{i,j}^T & G_{i,j} - G_{i,j}^T \\ G_{i,j} - G_{i,j}^T & B_{i,j} + B_{i,j}^T \end{bmatrix} \tag{11}$$

where $g_{i,j}$ and $b_{i,j}$ are conductance and susceptance, and vectors $p$ and $q$ contain active and reactive power loads. The value of the module of the slack-bus voltage $V_{slack}$ is represented by $S(\cdot)$, with matrices $S_i \in \mathbb{R}^{2N_p \times 2N_p}$.

$$S_i = \xi_{i,lack} \otimes \xi_{i,lack}^T : S_{T+t} = \xi_{N_p+i,slack} \otimes \xi_{N_p+i,slack}^T \tag{12}$$

and vector $s \in \mathbb{R}^{2T}$, with elements $s_t = V_{slack}, s_{T+t} = 0, \forall t = 1, \ldots, T$. Active power flow is limited using linear mapping $F(\cdot)$, where $F_{l,t} \in \mathbb{R}^{2N_p \times 2N_p}$ is given by:

$$F_{l,t} = -F_{N,L+t} = \xi_{i,l} \otimes Y_{i,j} \tag{13}$$

Vector $\bar{\mathbf{t}} \in \mathbb{R}^{2T \times N_L}$ of power flow limits has elements $\bar{\mathbf{t}}_{l,t} = \bar{\mathbf{t}}_{N_L+t} = \bar{\mathbf{t}}_{l} \forall \forall t = 1, \ldots, T$, such that $(i, j) \in \mathcal{L}$. Voltage variables at bus $i$ and time $t$ are limited in linear mapping $\mathcal{J}_{l,t}(\cdot)$ where matrices $\mathcal{J}_{l,t} \in \mathbb{R}^{2N_p \times 2N_p}$ are given by:

$$\mathcal{J}_{l,t} = -\mathcal{J}_{N_B+t} = \xi_{i,l} \xi_{i,l}^T \otimes (\xi_{i,l} + \xi_{N_B+t,\xi}) \tag{14}$$

and voltage limits ($\bar{\mathbf{v}}_i$ and $\bar{\mathbf{v}}_i$) are obtained from $\bar{\mathbf{v}} \in \mathbb{R}^{2N_p}$.

$$\bar{\mathbf{v}}_{i,t} = \bar{\mathbf{v}}_i \vdash \bar{\mathbf{v}}_{N_B+t} = \bar{\mathbf{v}}_i \forall t = 1, \ldots, T ; i = \{1, \ldots, N_B\} \setminus i\text{slack}. \tag{15}$$

Active power generation target constraint at hydro plant $h$ is represented in linear mapping $\mathcal{G}(\cdot)$, where matrices $\mathcal{G}_h \in \mathbb{R}^{(\text{dim}(x)) \times \text{dim}(x)}$ are given by:

$$\mathcal{G}_h = \sum_{t=1}^{T} \sum_{t=1}^{N_{u_{t,h}}} \xi_{u_{t,h},u} \xi_{u_{t,h},u}^T \otimes \begin{bmatrix} 0 & \frac{1}{2} \bar{P}_{h,u}^T \\ \frac{1}{2} \bar{P}_{h,u} & 0 \end{bmatrix} \tag{16}$$

$\forall h \in \{1, \ldots, N_H\} \setminus h\text{slack}$, where $g$ contains active power generation targets obtained from longer-term hydropower scheduling models. Active and reactive power generation limits are represented by linear mappings $\mathcal{P}(\cdot)$ contain matrices $\mathcal{P}_{t,h,u} \in \mathbb{R}^{(\text{dim}(x)) \times \text{dim}(x)}$.

$$\mathcal{P}_{t,h,u} = \xi_{t,h,u} \xi_{t,h,u}^T \otimes \begin{bmatrix} 0 & \frac{1}{2} \bar{P}_{h,u} \\ \frac{1}{2} \bar{P}_{h,u} & 0 \end{bmatrix} \tag{17}$$

This is similar for linear mapping $\mathcal{Q}(\cdot)$. Reservoir dynamics at hydro plant $h$ and time $t$ is given in $\mathcal{R}(\cdot)$ containing matrices $\mathcal{R}_{h,t} \in \mathbb{R}^{(\text{dim}(x)) \times \text{dim}(x)}$, and $\mathcal{R}_{h,t} = -\mathcal{R}_{N_H+h,t}$ such that

$$\mathcal{R}_{h,t} = \frac{1}{2} \sum_{t=1}^{T} \left( \sum_{u=1}^{N_{u_{t,h}}} \mathcal{C}_{t,h,u} + \sum_{u=1}^{N_{u_{t,h}}} \mathcal{C}_{t-h,u} \right) \tag{18}$$

and the vector $\mathbf{r} \in \mathbb{R}^{2T \times N_H}$ with elements:

$$r_{h,t} = r_{h,t} - r_{h,t} - \forall t = 1, \ldots, T \text{ and } h = 1, \ldots, N_H \text{, where } a_{h,t}, b_{h,t}, c_{h,t} \text{ represent given water inflow, initial reservoir volume and spillage, respectively.}$$

Unit configuration status variables and their respective uniqueness constraints are represented in $\mathcal{A}(\cdot)$ where matrices $\mathcal{A}_{h,t} \in \mathbb{R}^{\text{dim}(x)}$ are given by:

$$\mathcal{A}_{h,t} = \sum_{u=1}^{N_{u_{t,h}}} \xi_{t,h,u} \xi_{t,h,u}^T \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{19}$$

$\forall t = 1, \ldots, T \text{ and } h = 1, \ldots, N_H$. Startup constraints uses $\mathcal{E}(\cdot)$ with $\mathcal{E}_{t,h} \in \mathbb{R}^{\text{dim}(x)}$, such that:

$$\mathcal{E}_{t,h} = \sum_{u=1}^{N_{u_{t,h}}} \left( \xi_{t,h,u} \xi_{t,h,u}^T - \xi_{t-1,h,u} \xi_{t-1,h,u}^T \right) \otimes \begin{bmatrix} 0 & 0 \\ u & 0 \\ 0 & 0 \end{bmatrix} \tag{20}$$

$\forall t = 1, \ldots, T \text{ and } h = 1, \ldots, N_H$.

C. Solving undefined unit commitment schedules

A feasible active power dispatch $\hat{\mathbf{p}}$ is obtained from the solution to the SDP relaxation in (P1), which may contain undefined unit commitment schedules represented by the set:

$$\Omega_1(\mathbf{x}) = \{(t,h,u) : \bar{x}_{t,h,u} \notin \{0, 1\}\} \tag{20}$$

A MILP problem is formulated at domain $x \in \Omega_1(\mathbf{x})$ with:

$$P_{t,h}^* = \sum_{u=1}^{N_{u_{t,h}}} \mathbf{1}_{(t,h,u) \in \Omega_1(\mathbf{x})} \cdot (\Delta P_{t,h}^* + P_{h} \cdot x_{t,h,u}) \tag{21}$$

such that:

$$\text{(P2)} \min_{x \in \Omega_1(\mathbf{x}), y} \sum_{(t,h) \in \Omega_1(\mathbf{x})} w_{(t,h)} |x_{t,h}| \leq 1 \tag{22}$$

subject to:

$$\sum_{u=1}^{N_{u_{t,h}}} \mathbf{1}_{(t,h,u) \in \Omega_1(\mathbf{x})} x_{t,h,u} = 1 \tag{23}$$

$$\sum_{u=1}^{N_{u_{t,h}}} \left( \mathbf{1}_{(t,h,u) \in \Omega_1(\mathbf{x})} x_{t,h,u} + \mathbf{1}_{(t,h,u) \in \Omega_2(\mathbf{x})} \right)$$

for $t = 1, \ldots, N_T$ and $h = 1, \ldots, N_H$, where $\mathbf{1}_{w \in A} = \{1, 0\}$, if $w \in A$ and 0 otherwise. Water use and unit start-up costs are represented by $wc(\cdot)$ and $\delta_h(\cdot)$ respectively. A feasible unit commitment schedule $x^*$ is obtained from solving (P2).
D. AC power flow re-optimization

Since active power generation dispatches \( \hat{p} \) remain feasibly unchanged after solving (P2) for the undefined unit commitment schedules, it is necessary to guarantee feasibility of the updated water discharge values resulting from \( z^* \) whilst minimizing costs with water use in a stripped down formulation of (6). This is accomplished by solving the following SDP formulation of the optimal AC power flow problem to find \( \tilde{q}, \hat{q}, \hat{\epsilon}, \) and \( f \):

\[
\min \frac{\tilde{C}}{Z} \quad \text{s.t.} \quad \tilde{A}(Z) = \hat{a}, \quad \tilde{B}(Z) \leq \tilde{b}, \quad Z \succeq 0 \tag{22}
\]

where \( Z = z z^T \) is the outer product of \( z = (1, p, q, e, f)^T \), water use costs are represented by \( C \). \( A(\cdot) \) represents active and reactive power balance, long-term reservoir operation goals, and fixed voltage at the slack bus. Limits on active and reactive power generation, reservoir storage, as well as power flow and bus voltages are defined by \( B(\cdot) \). Analogously to the initial relaxation, the rank-1 constraint on \( Z \) is relaxed.

E. Rank reduction procedure

In our proposed rounding heuristic we seek to further improve \( Z^* \) resulting from solving (22) by applying rank reduction in an iterative procedure [14], however, by updating matrix \( Z_r^* \):

\[
Z^*_{r+1} \leftarrow Z^*_r + \omega_r \cdot D_r \tag{23}
\]

Also for the current OPF SDP formulation the solution’s rank is upper-bounded by the tree width of the power network plus one as demonstrated in [16]. The rank reduction procedure listed in Algorithm 2 maintains feasibility as explained in the remainder of this section, and also converges up to a theoretical limit of the rank in the final solution [15]. Moreover, because \( Z \) does not appear in the objective function, new solutions resulting from the rank reduction procedure maintain the value of the objective function. A central idea to this procedure is to calculate \( D_r \) such that at every iteration it decreases the rank of \( Z_r^* \) maintaining its feasibility, i.e. \( D_r \) is be subject to:

\[
\begin{align*}
D_r \cdot Y_i &= 0, \quad i = 1, \ldots, N_B \tag{24} \\
D_r \cdot Y_{ij} &= 0, \quad \forall (i, j) \in L \tag{25} \\
D_r \cdot E_l &= 0, \quad l = 1, 2, \ldots, 2N_B \tag{26}
\end{align*}
\]

where \( D_r \) is subject to:

\[
\begin{align*}
D_r \cdot Y_i &= 0, \quad i = 1, \ldots, N_B \tag{24} \\
D_r \cdot Y_{ij} &= 0, \quad \forall (i, j) \in L \tag{25} \\
D_r \cdot E_l &= 0, \quad l = 1, 2, \ldots, 2N_B \tag{26}
\end{align*}
\]

where \( D_r = R_r S_r R_r^\top \) and matrix \( R_r \) are obtained from the Cholesky decomposition of \( Z^*_r = R_r R_r^\top \). By the distributive property of the Frobenius product we obtain the following relationship:

\[
D_r \cdot Y_i = R_r S_r R_r^\top \cdot Y_i = S_r \cdot R_r^\top Y_i R_r = 0. \tag{27}
\]

Similarly for the other constraints, we obtain a linear system:

\[
\begin{align*}
\text{svec}(R_r^\top Y_i R_r)^\top \text{svec}(S_r) &= 0, \quad i = 1, \ldots, N_B \tag{28} \\
\text{svec}(R_r^\top Y_{ij} R_r)^\top \text{svec}(S_r) &= 0, \quad (i, j) \in L \tag{29} \\
\text{svec}(R_r^\top E_l R_r)^\top \text{svec}(S_r) &= 0, \quad l = 1, 2, \ldots, 2N_B \tag{30}
\end{align*}
\]

Algorithm 2 Rank reduction procedure.

**Require:** \( Z^* = \{Y_i \forall i \in \{1, \ldots, N_B\}\}, \{Y_{ij} \forall (i, j) \in L\}, \{E_l \forall l \in \{1, \ldots, 2N_B\}\} \) and \( Z^*_r = Z^* \)

1: for \( r \leftarrow 1, \ldots, \text{rank}(Z^*) + \epsilon - 1 \) do
2: Obtain Cholesky decomposition \( Z^*_r = R_r R_r^\top \)
3: Obtain \( \epsilon_r \) by get the basis of (28)-(30).
4: if \( \text{nullity}(A) \) do
5: Calculate \( S_r = \text{findSDP}(S_r, \text{svec}(\epsilon_r)) \)
6: end for
7: \( D_r \leftarrow -R_r S_r R_r^\top \)
8: \( Z^*_{r+1} \leftarrow Z^*_r + \lambda_{\max}(S_r) \cdot D_r \)
9: if rank\( (Z^*_{r+1}) = 1 \) or \( S_r = 0 \) then
10: break
11: end if
12: end for

where \( \text{svec}(\cdot) : \mathbb{R}^{n \times n} \to \mathbb{R}^{n(n+1)/2} \) is a linear transformation of a symmetric matrix into a vector.

We can represent the linear system in (28), (29) and (30) as \( A s = 0 \), which is equivalent to say that \( s \) is in the null space of \( A \). In order to solve this linear system SVD decomposition of matrix \( A = U \Sigma L^\top \) is used. If matrix \( A \) is rank-deficient, then the null space is generated by the columns of \( L \) corresponding to the zero singular values on \( \Sigma \). Then \( \text{svec}(S_r) \) is generated by the basis of the null space of \( A \). Finally, it is necessary to enforce positive semidefiniteness on \( S_r \) as the convex combination of the basis vectors of the null space of \( A \):

\[
S_r = \sum_{i=1}^{\text{nullity}(A)} \text{svec}(\epsilon_r) \sigma_i \tag{31}
\]

where \( \epsilon_r \) is a basis of the null space of \( A \). The search for the semidefinite convex combination uses the algorithm in [17] to determine if is feasible to obtain a semidefinite matrix from a convex combination of two matrices. This algorithm is therefore identified as findSDP(\( \cdot, \cdot \) : \( \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} \)).

Then from (23) we obtain:

\[
Z^*_r + \omega_r \cdot D_r = R_r (I + \omega_r S_r) R_r^\top. \tag{32}
\]

From (32) we conclude that the necessary condition for positive semidefiniteness of the new solution is given by:

\[
(Z^*_r + \omega_r \cdot D_r) \succeq 0 \iff (I + \omega_r S_r) \succeq 0. \tag{33}
\]

If \( S_r \) is indefinite, then the algorithm is unable to reduce its rank at the \( r \)-th iteration. It should be noted, however, that in theory it should still be possible to reduce its rank in further iterations, given that \( S_r \neq S_{r+1} \) because of the added perturbations, and also because the SDP formulation of the network constraints guarantees that at least one rank-1 solution exists. In that sense, additional \( \epsilon \) iterations are provisioned. Parameter \( \omega_r \), on the other hand, is a scale of perturbation matrix \( D_r \) that induces rank reduction of \( Z^*_r \) in at least one degree by reducing the rank of \( (I + \omega_r S_r) \). This is achieved by obtaining the maximum eigenvalue of \( S_r \): \( \omega_r = -\lambda_{\max}(S_r) \).
TABLE I

| Case   | Obj. Func. ($) | Relative Gap ($) |
|--------|----------------|------------------|
| SDP    | B&B            | RH               |
| 3-GEN  | 1,598,129      | 1,598,999        | 1,598,974       | 840 | 845 |
| IEEE-14| 2,711,258      | 2,711,718        | 2,711,823       | 459 | 565 |
| IEEE-30| 5,587,150      | 5,587,329        | 5,588,811       | 178 | 1,661 |
| IEEE-57| 13,221,266     | 13,221,609       | 13,222,070      | 342 | 803  |
| IEEE-118| 29,659,377    | ?                | 29,703,639      | ?   | 44,262 |

TABLE II

| Case    | Time (s) |
|---------|----------|
| SDP     | B&B      |
| 3-GEN   | 0.78     | 45.64   | 22.13 |
| IEEE-14 | 1.83     | 46.72   | 45.16 |
| IEEE-30 | 10.00    | 560.594 | 20.32 |
| IEEE-57 | 33.03    | 10,475.46 | 64.55 |
| IEEE-118| 270.15   | ?       | 1,009.97 |

III. Numerical Results

The proposed solution methodology was implemented in MATLAB® R2016. The SDPA [18] solver was used for solving SDP problems. The HUC MILP subproblem is solved using IBM® CPLEX® 10.2. A total of five numerical cases studies were analyzed for an increasing size of the test systems. Hydro plant data were obtained from the Brazilian power system operator for actual plants for a typical day. Network data were obtained from modified versions of IEEE test systems. Objective function results and gaps relative to the initial SDP relaxation are listed on Table I for the initial semidefinite relaxation (SDP), the B&B algorithm solving a SDP relaxation in every node (this method is an exact method developed on [11]), as well as the rounding heuristic (RH). Table II lists the computer time required by each of the procedures.

It can be observed that the solutions obtained by the B&B and rounding heuristic algorithms were equivalent up to very small error margins. In the case of the IEEE 118-bus problem, the B&B algorithm does not converge. Results in Table I suggest that the first relaxation is very close to the optimal solution obtained by B&B, and consequently that it is possible to approximate the objective function with a small error margin, and, in the case of the proposed rounding heuristic, with reduced computing time. In all the cases the final unit commitment give us different result for B&B and RH methods.

IV. Conclusion

For the study cases presented the proposed rounding heuristic efficiently approximates the optimal solutions of different HUC problems. This is most likely due to the fact that the initial semidefinite relaxation already finds near-optimal solutions, as evidenced by the results obtained by the B&B algorithm. Furthermore, our rank reduction heuristic enhances the method presented in [14] by adding a procedure to identify the possibility to find a perturbation matrix that reduces in at least I the rank of the solution per iteration. It is possible to employ a more conservative rounding heuristic, as long as the tradeoff between computational burden and relative gap is considered. However, we note that the observed performance of the proposed rounding heuristic is compatible with the results presented in [19], in which it is demonstrated that a maximum of 13.8% gap is guaranteed for problems of max-cut type, and applicable to QCQP problems as the one formulated in this paper.

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