EXTREMAL FIRST DIRICHLET EIGENVALUE OF DOUBLY CONNECTED PLANE DOMAINS AND DIHEDRAL SYMMETRY

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Abstract. We deal with the following eigenvalue optimization problem: Given a bounded domain $D \subset \mathbb{R}^2$, how to place an obstacle $B$ of fixed shape within $D$ so as to maximize or minimize the fundamental eigenvalue $\lambda_1$ of the Dirichlet Laplacian on $D \setminus B$. This means that we want to extremize the function $\rho \mapsto \lambda_1(D \setminus \rho(B))$, where $\rho$ runs over the set of rigid motions such that $\rho(B) \subset D$. We answer this problem in the case where both $D$ and $B$ are invariant under the action of a dihedral group $D_n$, $n \geq 2$, and where the distance from the origin to the boundary is monotonous as a function of the argument between two axes of symmetry. The extremal configurations correspond to the cases where the axes of symmetry of $B$ coincide with those of $D$.

1. Introduction and Statement of the Main Result

The relations between the shape of a domain and the eigenvalues of its Dirichlet or Neumann Laplacian, have been intensively investigated since the 1920’s when Faber [5] and Krahn [12] have proved independently the famous eigenvalue isoperimetric inequality first conjectured by Rayleigh (1877): the first Dirichlet eigenvalue $\lambda_1(\Omega)$ of any bounded domain $\Omega \subset \mathbb{R}^n$ satisfies

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*),$$

where $\Omega^*$ is a ball having the same volume as $\Omega$. We refer to the review papers of Ashbaugh [1, 2] and Henrot [9] for a survey of recent results on optimization problems involving eigenvalues.

The present work deals with the following eigenvalue optimization problem: Given a bounded domain $D$, we want to place an obstacle (or a hole) $B$, of fixed shape, inside $D$ so as to maximize or minimize the fundamental eigenvalue $\lambda_1$ of the Laplacian or Schrödinger operator on $D \setminus B$ with Zero Dirichlet conditions on the boundary.

In other words, the problem is to optimize the principal eigenvalue function $\rho \mapsto \lambda_1(D \setminus \rho(B))$, where $\rho$ runs over the set of rigid motions such that $\rho(B) \subset D$.

The first result obtained in this direction concerned the case where both $D$ and $B$ are disks of given radii. Indeed, it follows from Hersch’s work [10] that the maximum of $\lambda_1$ is achieved when the disks are concentric (see also [14]). This result has been extended to any dimension by several authors (Harrell, Kröger and Kurata [8], Kesavan [11], ...). Actually, Harrell, Kröger and Kurata [8] gave a more general result showing that, if the domain $D$ satisfies an interior symmetry property with respect to a hyperplane $P$ passing through the center of the spherical obstacle $B$.

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(which means that the image by the reflection with respect to \( P \) of one component of \( D \setminus P \) is contained in \( D \)), then the Dirichlet fundamental eigenvalue \( \lambda_1(D \setminus B) \) decreases when the center of \( B \) moves perpendicularly to \( P \) in the direction of the boundary of \( D \). In the particular case where both the domain \( D \) and the obstacle \( B \) are balls, this implies that the minimum of \( \lambda_1(D \setminus B) \) corresponds to the limit case where \( B \) touches the boundary of \( D \).

Notice that when the obstacle \( B \) is a disk, only translations of \( B \) may affect the \( \lambda_1 \) of \( D \setminus B \) and the optimal placement problem reduces to the choice of the center of \( B \) inside \( D \).

In the present work we investigate a kind of dual problem in the sense that we consider a nonspherical obstacle \( B \) whose center of mass is fixed inside \( D \), and seek the optimal positions while turning \( B \) around its center.

It is of course hopeless to expect a universal solution to this problem. In fact, we will restrict our investigation to a class of domains satisfying a dihedral symmetry and a monotonicity conditions.

Thus, let \( D \) be a simply-connected plane domain and assume that the following conditions are satisfied:

(i) (\( \mathbb{D}_n \)-symmetry) for an integer \( n \geq 2 \), \( D \) is invariant under the action of the dihedral group \( \mathbb{D}_n \) of order \( 2n \) generated by the rotation \( \rho_{\frac{2\pi}{n}} \) of angle \( \frac{2\pi}{n} \) and a reflection \( S \). Such a domain admits \( n \) axes of symmetry passing through the origin and such that the angle between 2 consecutive axes is \( \frac{2\pi}{n} \).

(ii) (monotonicity of the boundary) the distance \( d(O, x) \) from the origin to a point \( x \) of the boundary of \( D \) is monotonous as a function of the argument of \( x \), in a sector delimited by two consecutive symmetry axes.

Notice that assumption (i) guarantees that the center of mass of \( D \) is at the origin. Regular \( n \)-gons centered at the origin are the simplest examples of domains satisfying these assumptions. More generally, if \( g \) is any positive even \( \frac{2\pi}{n} \)-periodic continuous function that is monotonous on the interval \((0, \frac{\pi}{n})\), then the domain

\[
D = \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < g(\theta)\},
\]

satisfies assumptions (i) and (ii). Actually, up to a rigid motion, any domain satisfying assumptions (i) and (ii) can be parametrized in such a manner.

It is worth noticing that, due to the monotonicity condition, the “distance to the origin” function on the boundary of \( D \) achieves its maximum and its minimum alternatively at the intersection points of \( \partial D \) with the \( 2n \) half-axes of symmetry. The \( n \) points of \( \partial D \) at maximal (resp. minimal) distance from the origin will be called ”outer vertices” (resp. ”inner vertices”) of \( D \).

Our main result is the following

**Theorem 1.** Let \( D \) and \( B \) be two plane domains satisfying the assumptions of \( \mathbb{D}_n \)-symmetry and monotonicity (i) and (ii) above for an integer \( n \geq 2 \). Assume furthermore that \( B \) has \( C^2 \) boundary and that \( \rho(B) \subset D \) for all \( \rho \in \text{SO}(2) \). Then, the fundamental Dirichlet eigenvalue \( \lambda_1(D \setminus B) \) of \( D \setminus B \) is optimized exactly when the axes of symmetry of \( B \) coincide with those of \( D \).

The maximizing configuration corresponds to the case where the outer vertices of \( B \) and \( D \) lie on the same half-axes of symmetry (we will then say that \( B \) occupies the “ON” position in \( D \)).
The minimizing configuration corresponds to the case where the outer vertices of $B$ lie on the half-axes of symmetry passing through the inner vertices of $D$ (this is what will be called the “OFF” position).

Actually, we will prove that, except for the trivial case where $D$ or $B$ is a disk, the fundamental Dirichlet eigenvalue of $D \setminus B$ decreases gradually when $B$ switches from “ON” to “OFF”.

The main ingredients of the proof of Theorem 1 are Hadamard’s variation formula for $\lambda_1$ and the technique of domain reflection initiated by Serrin [17] in PDE’s setting.

Examples of maximal (left) and minimal (right) configurations with $n = 2, 3$ and 4 respectively

Extensions of Theorem 1 to the following situations can be obtained up to slight changes in the proof (indeed, only the Hadamard formula should be replaced by the variation formula corresponding to the new functional):

1. Soft obstacles: instead considering the Dirichlet Laplacian on $D \setminus B$, we consider the Schrödinger type operator

$$H(\alpha, B) := \Delta - \alpha \chi_B$$
acting on $H^1_0(D)$, where $\alpha > 0$ and $\chi_B$ is the indicator function of $B$. Optimization problems related to the fundamental eigenvalue of operators of this kind have been investigated in particular in [8] and [3]. Under the assumptions of Theorem 1 on $D$ and $B$, $\forall \alpha > 0$, the fundamental eigenvalue of $H(\alpha, B)$ achieves its maximum at the “ON” position and its minimum at the “OFF” position.

(2) Wells: this case corresponds to the operator $H(\alpha, B)$ with $\alpha < 0$. Under the circumstances of Theorem 1, $\forall \alpha < 0$, the first eigenvalue of $H(\alpha, B)$ achieves its maximum at the “OFF” position and its minimum at the “ON” position.

(3) Stationary problem: the problem now is to optimize the Dirichlet energy $J(D \setminus B) := \int_{D \setminus B} |\nabla u|^2 dx$ of the unique solution $u$ of the problem

$$\begin{cases}
\Delta u = -1 & \text{in } D \setminus B \\
u = 0 & \text{on } \partial(D \setminus B),
\end{cases}$$

This problem was treated in [11, Section 2] in the case where both $D$ and $B$ are balls. Under the assumptions of Theorem 1 on $D$ and $B$, one can prove that $J(D \setminus B)$ achieves its maximum when $B$ is at the “ON” position and its minimum when $B$ is at the “OFF” position.

2. Proof of the main result

Without loss of generality, we may assume that the domain $D$ and the obstacle $B$ are centered at the origin and are both symmetric with respect to the $x_1$-axis so that they can be parametrized in polar coordinates by

$$D = \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < g(\theta)\},$$
$$B = \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < f(\theta)\},$$

where $f$ and $g$ are two positive even $\frac{2\pi}{n}$-periodic functions which are nondecreasing on $(0, \frac{\pi}{n})$. To avoid technicalities, we suppose throughout that $g$ is continuous and $f$ is $C^2$. Extensions of our result to a wider class of domains would certainly be possible up to some additional technical difficulties.

The condition that the obstacle $B$ can freely rotate around his center inside $D$, that is $\rho(B) \subset D$ for all $\rho \in SO(2)$, amounts to the following:

$$f(\frac{\pi}{n}) = \max_{0 \leq \theta \leq 2\pi} f(\theta) < \min_{0 \leq \theta \leq 2\pi} g(\theta) = g(0).$$

Let us denote, for all $t \in \mathbb{R}$, by $\rho_t$ the rotation of angle $t$, that is, $\forall \zeta \in \mathbb{R}^2 \cong \mathbb{C}$, $\rho_t(\zeta) = e^{it} \zeta$, and set

$$B_t := \rho_t(B) \text{ and } \Omega(t) := D \setminus B_t.$$ 

Let $\lambda(t)$ be the fundamental eigenvalue of the Dirichlet Laplacian on $\Omega(t)$. It is well known that, since it is simple, the first Dirichlet eigenvalue $\lambda(t)$ is a differentiable function of $t$ (see [10] [15]). We denote by $u(t)$ the one parameter family of nonnegative first eigenfunctions satisfying, $\forall t \in \mathbb{R}$,

$$\begin{cases}
\Delta u(t) = -\lambda(t)u(t) & \text{in } \Omega(t) \\
u(t) = 0 & \text{on } \partial\Omega(t) \\
\int_{\Omega(t)} u^2(t) = 1.
\end{cases}$$
Theorem 1 is a consequence of the following (see [4, 6, 7, 16]):

\[ \lambda(t) = \int_{\partial B_t} \left| \frac{\partial u(t)}{\partial n} \right|^2 \eta \cdot v \, d\sigma, \]

where \( \eta \) is the inward unit normal vector field of \( \partial \Omega(t) \) (hence, along \( \partial B_t \) the vector \( \eta \) is outward with respect to \( B_t \)) and \( v \) denotes the restriction to \( \partial \Omega(t) = \partial D \cup \partial B_t \) of the deformation vector field. In our case, the vector \( v \) vanishes on \( \partial D \) and is given by \( v(\zeta) = \mathbf{k} \cdot \zeta \) for all \( \zeta \in \partial B_t \).

Since both \( \Omega \) and \( B \) are invariant by the dihedral group \( D_n \), it follows that, \( \forall t \in \mathbb{R}, \Omega(t + \frac{2\pi}{n}) = \Omega_t \). Moreover, if we denote by \( S_0 \) the reflection with respect to the \( x_1 \)-axis, then we clearly have \( S_{-1} = S_0 \circ \rho_i \circ S_0 \) which gives \( B_{-1} = S_0(B_t) \) and \( \Omega_{-1} = S_0(\Omega_t) \). Hence, as a function of \( t \), the first Dirichlet eigenvalue of \( \Omega_t \) is even and periodic of period \( \frac{2\pi}{n} \). That is, \( \forall t \in \mathbb{R}, \lambda(t + \frac{2\pi}{n}) = \lambda(t) \) and \( \lambda(-t) = \lambda(t) \).

Therefore, it suffices to investigate the variations of \( \lambda(t) \) on the interval \([0, \frac{2\pi}{n}]\) and Theorem 1 is a consequence of the following:

**Theorem 2.** Assume that neither \( D \) nor \( B \) is a disk.

(i) \( \forall t \in (0, \frac{2\pi}{n}), \lambda'(t) < 0 \). Hence, \( \lambda(t) \) is strictly decreasing on \((0, \frac{2\pi}{n})\).

(ii) \( \forall k \in \mathbb{Z}, \lambda'(k \frac{2\pi}{n}) = 0 \) and \( k \frac{2\pi}{n}, k \in \mathbb{Z}, \) are the only critical points of \( \lambda \) on \( \mathbb{R} \).

Hence, \( \lambda(t) \) achieves its maximum for \( t = 0 \mod \frac{2\pi}{n} \) which corresponds to the “ON” position, and its minimum for \( t = \frac{2\pi}{n} \mod \frac{2\pi}{n} \) which corresponds to the “OFF” position. Of course, if \( D \) or \( B \) is a disk, then the function \( \lambda(t) \) is constant.

In what follows we will denote, for any \( \alpha \in \mathbb{R} \), by \( z_\alpha \) the \( \theta = \alpha \) axis, that is \( z_\alpha := \{r e^{i\alpha} : r \in \mathbb{R}\} \), and by \( z_\alpha^+ \) the half-axis \( \{r e^{i\alpha} : r \geq 0\} \).

We start the proof with the following elementary lemma.

**Lemma 1.** Let \( K \) be a plane domain defined in polar coordinates by \( K = \{r e^{i\theta} : \theta \in [0, 2\pi), 0 \leq r < h(\theta)\} \), where \( h \) is a positive \( 2\pi \)-periodic function of classe \( C^1 \), and let \( v \) be a vector field whose restriction to \( \partial K \) is given by

\[ \eta \cdot v(\theta) := \eta(\theta) v(\theta) = \mathbf{h}(\theta) e^{i\theta} = h(\theta) e^{i\theta}. \]

We denote by \( \eta \) the unit outward normal vector field of \( \partial K \). One has, at any point \( h(\theta) e^{i\theta} \) of \( \partial K \) where \( \eta \) is defined,

(i) \( \eta(\theta) := \eta(h(\theta) e^{i\theta}) = \frac{h(\theta) e^{i\theta} - h'(\theta) e^{i\theta}}{\sqrt{h'^2(\theta) + e^{2i\theta}}} \)

(ii) \( \eta \cdot v(\theta) = \frac{\eta(\theta)}{\sqrt{h'^2(\theta) + e^{2i\theta}}} \). Hence, \( \eta \cdot v(\theta) \) has constant sign on an interval \( I \) if and only if \( h \) is monotonous in \( I \).

(iii) if for some \( \alpha > 0 \), the domain \( K \) is symmetric with respect to the axis \( z_\alpha \), then the function \( \eta \cdot v \) is antisymmetric w.r.t this axis, that is

\[ \eta \cdot v(\alpha + \theta) = -\eta \cdot v(\alpha - \theta). \]

**Proof.** Assertions (i) and (ii) are direct consequences from the definition of \( K \). The fact that \( K \) is symmetric with respect to the axis \( z_\alpha \) implies that the function \( h \) satisfies \( h(\alpha + \theta) = h(\alpha - \theta) \). Therefore, (iii) follows immediately from (ii).
We will denote by $S_\alpha$ the symmetry with respect to the axis $z_\alpha$. We will also denote, for $\alpha < \beta$, by $\gamma(\alpha, \beta)$ the sector delimited by $z_\alpha^+$ and $z_\beta^-$, that is

$$\gamma(\alpha, \beta) = \{re^{i\theta}; r > 0 \text{ and } \alpha < \theta < \beta\}.$$

**Lemma 2.** Let $D$ be as above. For all $t \in (0, \frac{\pi}{n})$, we have:

$$S_{\frac{\pi}{n} + t} \left(D \cap \gamma \left(\frac{\pi}{n} + t, \frac{2\pi}{n} + t\right)\right) \subseteq D \cap \gamma \left(t, \frac{\pi}{n} + t\right).$$

Moreover, if $D$ is not a disk, then

$$S_{\frac{\pi}{n} + t} \left(\partial D \cap \gamma \left(\frac{\pi}{n} + t, \frac{2\pi}{n} + t\right)\right) \cap D \neq \emptyset.$$

**Proof.** The action of the symmetry $S_{\frac{\pi}{n} + t}$ is given in polar coordinates by $S_{\frac{\pi}{n} + t}(re^{i\theta}) = re^{i(2\frac{\pi}{n} + t - \theta)}$. Hence,

$$S_{\frac{\pi}{n} + t} \left(D \cap \gamma \left(\frac{\pi}{n} + t, \frac{2\pi}{n} + t\right)\right) = S_{\frac{\pi}{n} + t}(D) \cap \gamma \left(t, \frac{\pi}{n} + t\right).$$

Moreover, the domain $D$ being parametrized by a positive even $\frac{2\pi}{n}$-periodic function $g(\theta)$, that is $D = \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < g(\theta)\}$, its image $S_{\frac{\pi}{n} + t}(D)$ can be parametrized in the same manner by the function $g^*(\theta) = g(\theta - 2t)$. Thus

$$S_{\frac{\pi}{n} + t}(D) \cap \gamma \left(t, \frac{\pi}{n} + t\right) = \{re^{i\theta}; \theta \in \left(t, \frac{\pi}{n} + t\right), 0 \leq r < g(\theta - 2t)\}.$$

Therefore, we need to prove that $F(\theta) = g(\theta) - g^*(\theta)$ is nonnegative for every $\theta$ in the interval $(t, \frac{\pi}{n} + t)$. This will be possible thanks to the assumptions of symmetry (that is $g$ is even and $\frac{2\pi}{n}$-periodic) and monotonicity (that is $g$ is nondecreasing on $[0, \frac{\pi}{n})$). Indeed, these properties imply that on the interval $(t, \frac{\pi}{n} + t)$,

- $g$ achieves its maximum at $\theta = \frac{\pi}{n}$,
- $g^*$ achieves its minimum at $\theta = 2t$.

Four cases must be considered separately:

- If $t < \theta \leq \min\{2t, \frac{\pi}{n}\}$, we may write, since $g$ is even, $F(\theta) = g(\theta) - g(2t - \theta)$, with $0 \leq 2t - \theta < \theta \leq \frac{\pi}{n}$. Since $g$ is nondecreasing on $[0, \frac{\pi}{n}]$, we get $F(\theta) \geq 0$.
- If $\max\{2t, \frac{\pi}{n}\} \leq \theta < \frac{\pi}{n} + t$, we may write, since $g$ is even and $\frac{2\pi}{n}$-periodic, $F(\theta) = g(2\frac{\pi}{n} - \theta) - g(\theta - 2t)$ with $0 \leq \theta - 2t < 2\frac{\pi}{n} - \theta \leq \frac{\pi}{n}$. Hence, $F(\theta) \geq 0$. 

\begin{align*}
\text{case } 2t < \frac{\pi}{n} & \quad \text{case } 2t > \frac{\pi}{n}
\end{align*}
- If $2t < \frac{\pi}{n}$ and $2t \leq \theta \leq \frac{\pi}{n}$, then $0 \leq \theta - 2t < \frac{\pi}{n}$ and, then, $F(\theta) = g(\theta) - g(\theta - 2t) \geq 0$.

- If $2t > \frac{\pi}{n}$ and $\frac{\pi}{2n} \leq \theta \leq 2t$, then $0 \leq 2t - \theta < 2\frac{\pi}{n} - \theta \leq \frac{\pi}{n}$ and, then, $F(\theta) = g(2\frac{\pi}{n} - \theta) - g(2t - \theta) \geq 0$.

Hence, $F(\theta)$ is nonnegative for all $\theta$ in $(t, \frac{\pi}{n} + t)$.

Now, if $D$ is not a disk, then $g$ is nonconstant on $[0, \frac{\pi}{n}]$. Following the arguments above, we deduce that the function $F(\theta)$ is positive somewhere on $(t, \frac{\pi}{n} + t)$ which means that $S_{\frac{\pi}{n} + t}(\partial D \cap \sigma (\frac{\pi}{n} + t, \frac{2\pi}{n} + t))$ meets the interior of $D$. \hfill $\Box$

**Proof of Theorem 2.** First, notice that the function $\lambda$ is an even and $\frac{2\pi}{n}$-periodic function of $t$, one immediately gets, $\forall k \in \mathbb{Z}$, $\lambda(k\frac{\pi}{n} - t) = \lambda(k\frac{\pi}{n} + t)$ and, then,

$$\lambda'(k\frac{\pi}{n}) = 0.$$  

Alternatively, one can deduce that $\lambda'(k\frac{\pi}{n}) = 0$ from Hadamard’s variation formula (1) after noticing that the domain $\Omega(k\frac{\pi}{n})$ is symmetric with respect to the $x_1$-axis and that the first Dirichlet eigenfunction $u(k\frac{\pi}{n})$ satisfies $u \circ S_0 = u$, where $S_0$ is the symmetry with respect to the $x_1$-axis.

Let us fix $a t \in (0, \frac{\pi}{n})$ and denote by $u$ the nonnegative first Dirichlet eigenfunction of $\Omega(t)$ satisfying $\int_{\Omega(t)} u^2 = 1$. The domain $\Omega(t)$ is clearly invariant by the rotation $\rho_{\frac{2\pi}{n}}$ of angle $\frac{2\pi}{n}$, hence $u \circ \rho_{\frac{2\pi}{n}} = u$. On the other hand, the domain $B$ being parametrized by a positive even $\frac{2\pi}{n}$-periodic function $f(\theta)$, that is $B = \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < f(\theta)\}$, one has

$$B_t = \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < h(\theta)\},$$

with $h(\theta) = f(\theta - t)$. Hence, the function $\eta_t \cdot v$ is invariant by $\rho_{\frac{2\pi}{n}}$ (Lemma 1) and we have (Hadamard formula (1))

$$\lambda'(t) = \int_{\partial B_t} \left| \frac{\partial u}{\partial \eta_t} \right|^2 \eta_t \cdot v \, d\sigma = n \int_{\partial B_t \cap \sigma(t, \frac{2\pi}{n} + t)} \left| \frac{\partial u}{\partial \eta_t} \right|^2 \eta_t \cdot v \, d\sigma.$$  

Since $B_t$ is symmetric with respect to the axis $z_{\frac{\pi}{n} + t}$, we have (Lemma 1), $\eta_t \cdot v(z_{\frac{\pi}{n} + t + \theta}) = -\eta_t \cdot v(z_{\frac{\pi}{n} + t - \theta})$, or, equivalently, $\eta_t \cdot v(x) = -\eta_t \cdot v(x^*)$, where $x^*$ denotes the symmetric of $x$ with respect to $z_{\frac{\pi}{n} + t}$. This yields

$$\lambda'(t) = n \int_{\partial B_t \cap \sigma(z_{\frac{2\pi}{n} + t}, \frac{2\pi}{n} + t)} \left( \left| \frac{\partial u}{\partial \eta_t}(x) \right|^2 - \left| \frac{\partial u}{\partial \eta_t}(x^*) \right|^2 \right) \eta_t \cdot v(x) \, d\sigma.$$  

Notice that the function $h(\theta)$ is decreasing between $\frac{\pi}{n} + t$ and $\frac{2\pi}{n} + t$ and, then, $\eta_t \cdot v$ is nonnegative on $\partial B_t \cap \sigma(z_{\frac{2\pi}{n} + t}, \frac{2\pi}{n} + t)$ (Lemma 1).

Let $H(t) := \Omega(t) \cap \sigma(z_{\frac{2\pi}{n} + t}, \frac{2\pi}{n} + t)$. Applying Lemma 2 and since $B_t$ is symmetric with respect to the axis $z_{\frac{2\pi}{n} + t}$, one gets

$$S_{\frac{\pi}{n} + t}(H(t)) \subset \Omega(t) \cap \sigma(z_{\frac{2\pi}{n} + t}).$$

Hence, the function $w(x) = u(x) - u(x^*)$ is well defined on $H(t)$ and satisfies $w(x) = 0$ for all $x$ in $\partial H(t) \cap (\partial B_t \cup z_{\frac{\pi}{n} + t} \cup z_{\frac{2\pi}{n} + t})$. Moreover, since $u$ vanishes on $\partial D$ and is positive inside $\Omega(t)$, $w(x) \leq 0$ for all $x$ in $\partial H(t) \cap \partial D$ and $w(x) < 0$ for certain $x$ in $\partial H(t) \cap \partial D$ (recall that $D$ is not a disk and apply the second part of Lemma 2).
Therefore, the nonconstant function \( w \) satisfies the following:

\[
\begin{aligned}
\Delta w &= -\lambda(t)w & \text{in } H(t) \\
w &\leq 0 & \text{on } \partial H(t).
\end{aligned}
\]

Hence, \( w \) must be nonpositive on the whole of \( H(t) \). Otherwise, a nodal domain \( V \subset H(t) \) of \( w \) would have the same first Dirichlet eigenvalue as \( \Omega(t) \). But, due to the invariance of \( \Omega(t) \) by \( \rho \), the domain \( \Omega(t) \) would contain \( n \) copies of \( V \) leading to a strong contradiction with the domain monotonicity theorem for eigenvalues.

Therefore, \( \Delta w \geq 0 \) in \( H(t) \) and \( w \) achieves its maximal value (i.e. zero) on \( \partial B_t \cap \sigma(\frac{\pi}{n} + t, \frac{2\pi}{n} + t) \subset \partial H(t) \). The Hopf maximum principle (see [13, Theorem 7, ch.2]) then implies that, at any regular point \( x \) of \( \partial B_t \cap \sigma(\frac{\pi}{n} + t, \frac{2\pi}{n} + t) \), one has

\[
\frac{\partial w}{\partial \eta}(x) = \frac{\partial u}{\partial \eta}(x) - \frac{\partial u}{\partial \eta}(x^*) < 0.
\]

It follows that \( \lambda'(t) \leq 0 \) and that the equality holds if and only if \( \eta_t \cdot v \equiv 0 \). By Lemma 1, this last equality occurs if and only if \( f \) is constant which means that \( B \) is a disk.

\[\square\]

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