Path-Additions of Graphs *

Franz J. Brandenburg, Alexander Esch, and Daniel Neuwirth

University of Passau, 94030 Passau, Germany
{brandenb, eschalex, neuwirth}@fim.uni-passau.de

Abstract. Path-addition is an operation that takes a graph and adds an internally vertex-disjoint path between two vertices together with a set of supplementary edges. Path-additions are just the opposite of taking minors.

We show that some classes of graphs are closed under path-addition, including non-planar, right angle crossing, fan-crossing free, quasi-planar, (aligned) bar 1-visibility, and interval graphs, whereas others are not closed, including all subclasses of planar graphs, bounded treewidth, \(k\)-planar, fan-planar, outer-fan planar, outer-fan-crossing free, and bar \((1,j)\)-visibility graphs.

1 Introduction

The characterization of planar graphs by forbidden minors is one of the highlights of graph theory. It is commonly known as Kuratowski’s theorem [25] and states that a graph \(G\) is planar if and only if there is no subgraph that can be obtained from \(K_5\) or \(K_{3,3}\) by subdividing edges. In other words, \(K_5\) and \(K_{3,3}\) are the forbidden topological minors of the planar graphs [19]. Equivalently, there is Wagner’s Theorem [38] which states that \(G\) is planar if and only if \(K_5\) or \(K_{3,3}\) cannot be obtained from \(G\) by edge contractions, edge deletions and the removal of isolated vertices, i.e., \(K_5\) and \(K_{3,3}\) are the minors of \(G\). An edge contraction merges the endvertices of an edge and a subdivision splits an edge into a path of length two, i.e., it places a new vertex on an edge. For more details we refer the reader to Diestel’s textbook [19].

The theorems of Kuratowski and Wagner are a prominent result in the theory of graph minors culminating in the Robertson-Seymour or Graph Minor Theorem, which states that every class of graphs \(\mathcal{G}\) closed under taking minors has a finite set of graphs, called the obstruction set or the set of forbidden minors. A

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graph $H$ belongs to $G$ if and only if $H$ does not contain a graph in the obstruction set of $G$ as a minor [5,19,32].

The path-addition operation is an extended inverse of edge contraction and combines subdivision and edge insertion. Subdivisions replace an edge by an internally vertex-disjoint path. A path-addition starts from scratch and introduces a new path. Since some classes of graphs are not hereditary, such as chordal graphs [11,19] or graphs with a strong visibility representation [17], we augment a path-addition by a set of supplementary edges to meet the requirements of a given class of graphs. A path-addition can be emulated by an edge insertion, followed by a subdivision and further edge insertions. Moreover, we require that an inserted path is long, since too short paths would violate the defining properties of some classes of graphs. For the the supplementary edges we add the restriction that they are incident to at least one internal vertex of the path.

Clearly, if a class of graphs $G$ is closed under taking minors and under path-addition, then $G$ is trivial, i.e., $G = \emptyset$, or $G$ consists of the empty graph, or $G$ is the set of all graphs. In particular, if $G$ contains a non-empty graph, then the closure of $G$ under path-addition includes any complete graph $K_k$ as a minor. To see this, first add paths until a graph with at least $k$ vertices is obtained, and then add vertex disjoint paths to obtain a subdivision of $K_k$ as a subgraph.

Path-additions of graphs were introduced by Brandenburg et al. [9]. They are helpful to distinguish classes of beyond-planar graphs. Beyond-planar is a collective term for classes of graphs that are defined by restrictions on crossings in visual representations. Particular examples are 1-planar [31] and $k$-planar graphs [30], fan-planar [3,4] and fan-crossing free graphs [14], quasi-planar graphs [1], right angle crossing (RAC) graphs [18], bar [16], bar ($1,j$) [10], and $1$-visibility graphs [7], rectangle visibility graphs [23], as well as map graphs [12,37]. Some of these classes are specialized by an alignment of the vertices, such as outerplanar, outer 1-planar [2,22], outer-fan-planar [3], outer-fan-crossing free, and aligned (or semi) bar 1-visibility [9,20] graphs.

We summarize our results on the closure of certain classes of graphs under path-addition in Table 1. The closure properties under subdivision and edge contraction are known or easy to obtain. The results show that there is no implication between path-addition, subdivision, and edge contraction.

In this work, we study the closure of classes of graphs under path-addition, where the graphs are defined by a visual representation. In Section 2, we introduce basic concepts. A positive closure is studied in Section 3 and a negative closure in Section 4. We conclude with some open problems.

2 Preliminaries

We consider simple, undirected graphs $G = (V,E)$ that are defined by a visual representation. For general graph theoretic terms we refer to [17,19]. A class of graphs is the set of all graph satisfying a particular property. A class of graphs $G$ is hereditary if $G \in G$ implies that every induced subgraph of $G$ is in $G$ [34].
An embedding or drawing $\mathcal{E}(G)$ is a mapping of a graph $G$ into the plane where each vertex is mapped to a point and each edge $e = (u, v)$ to a Jordan curve connecting the points of $u$ and $v$. There is a straight-line drawing if all curves of edges are straight lines. Two edges cross if their curves intersect. (Curves of) Edges are not allowed to pass through the points of other vertices, and edges incident to a vertex do not cross.

A generalized visibility representation represents each vertex of a graph by a horizontal segment, called a bar, which is a rectangle of small height and positive width. Each edge is represented by a vertical line of sight with $k$-visibility, such that the line of sight may traverse the bars of at most $k$ other vertices. Then an edge crosses a vertex. Sometimes, horizontal and vertical are exchanged. Planar visibility [17] with non-transparent bars is obtained if $k = 0$. In an interval representation $k$ is unbounded such that $k$ overlapping bars or intervals induce $K_k$ as a subgraph.

There are several versions of visibility including strong, $\epsilon$, and weak visibility. In the strong and $\epsilon$-versions there is an edge if and only if there is a visibility. An $\epsilon$-version requires a line of sight of width $\epsilon > 0$. This makes a subtle difference, since, in the planar case, $K_{2,3}$ is an $\epsilon$-visibility graph but is not a strong visibility graph. Furthermore, the recognition problem is $\mathcal{NP}$-hard for strong visibility graphs and solvable in linear time for $\epsilon$-visibility graphs [17]. Planar visibility graphs were characterized by Wismath [40] and Tamassia and Tollis [36]. In the weak version there is a visibility if there is an edge. Hence, edges can be omitted. A class of weak visibility graphs is hereditary, and the weak visibility graphs are exactly the induced subgraphs of the strong or $\epsilon$-visibility graphs. Also interval graphs assume the strong version of visibility [34] and are not hereditary.

Planarization is a useful tool for embeddings and generalized visibility representations. It represents the vertices and edges of the given graph and the edge-edge and edge-vertex crossings. The planarization $\mathcal{P}(\mathcal{E}(G))$ of an embed-

| graph                               | path-addition | sub-division | edge-contraction |
|-------------------------------------|---------------|--------------|------------------|
| planar                              | −             | +            | +                |
| $k$-planar                          | −             | +            | −                |
| right angle crossing (RAC)          | +             | +            | −                |
| fan-planar                          | −             | +            | −                |
| fan-crossing free                   | +             | +            | −                |
| quasi-planar                        | +             | +            | −                |
| bar 1-visibility                    | +             | +            | −                |
| bar (1, $j$)-visibility             | −             | +            | −                |
| outerplanar                         | −             | −            | −                |
| outer 1-planar                      | −             | −            | −                |
| outer fan-planar                    | −             | −            | −                |
| outer fan-crossing free             | −             | −            | −                |
| aligned bar 1-visibility (AB1V)     | +             | −            | −                |

Table 1. Closure properties of classes of graphs
Embedding $\mathcal{E}(G)$ is obtained by placing a dummy vertex of degree four at each crossing point of two edges and thereby subdividing each crossed edge. Thereafter, $\mathcal{P}(\mathcal{E}(G))$ is an ordinary planar embedding and there are vertices, edges, and faces, whose boundary consists of edges and edge segments and is determined by the vertices and crossing points on the boundary. At each vertex $v$ (including the dummy vertices) there is a rotation system describing the cycling ordering of the edges incident to $v$ or of the neighbors of $v$. An embedding is triangulated if each face of the planarization is a triangle.

Similarly, the planarization $\mathcal{P}(\mathcal{E}(G))$ of a generalized visibility representation introduces a dummy vertex at each point on the boundary of a bar or a rectangle, where a line of sight either contacts or crosses the bar. In graph drawing these points are often called ports. Thereafter, the boundary of a bar is partitioned into segments in counterclockwise order, such that the boundary of a bar consists of horizontal straight lines and of orthogonal polylines. Each line of sight is a sequence of straight vertical segments, which are either in free space or in the interior of a bar. Each such segment is an edge of the planarization, which in turn is an orthogonal drawing of a planar graph [17]. The planarization of a generalized visibility representation introduces open and closed faces, where a closed face lies in the interior of a bar and therefore is excluded for a placement of other bars or points.

### 2.1 Classes of Graphs

Embeddings and generalized visibility representations are a rich source for the definition of classes of graphs, and, in particular, for beyond-planar graphs. A graph $G$ is $k$-planar [30,31] if it admits a drawing such that each edge is crossed by at most $k$ other edges, and is $k$-quasi-planar [1] if there are no $k$ pairwise crossing edges. In a fan-planar drawing [3,4] an edge is allowed to cross two or more edges if the crossed edges are incident to a vertex and form a fan, whereas a fan-crossing free drawing [14] excludes edges that cross two edges that are incident to the same vertex. A right angle crossing graph (RAC) [18] allows crossings of edges if all edges are represented by straight lines and edges cross at a right angle. Finally, a map graph [12,37] is obtained from a planar dual. Here each vertex is represented by a region and there is an edge if and only if two regions share at least one point. This results in a complete subgraph $K_k$ if $k$ regions meet at a single point.

Accordingly, a graph $G$ is a bar $k$-visibility graph [16] if it admits a visibility representation with $k$-visibility. Then the line of sight of each edge may traverse or cross at most $k$ other bars. If $k = 1$ and each bar is passed by at most $j$ edges, we obtain bar $(1,j)$-visibility graphs [10] and 1-visibility graphs [7] if additionally $j = 1$. The two-dimensional extension with non-transparent rectangles for vertices and horizontal and vertical lines of sight for edges leads to rectangle visibility graphs [23].

Many of these concepts come with a parameter $k$, where $k = 0$ corresponds to the planar case and $k = 1$ is most commonly used. Moreover, there are...
diverse generalizations, for example from right angle to large angle drawings \cite{18}, visibility representations in 3D \cite{6}, and L-shape visibility \cite{27}, see also \cite{26}.

On the other hand, a common specialization comes with an alignment of the vertices such that they are placed on a line or are attached to a line. Then all vertices appear in the outer face. So we obtain outerplanar and outer 1-planar graphs \cite{2,22}, outer-fan-planar \cite{3} and outer-fan-crossing free graphs, as well as aligned bar 1-visibility graphs \cite{9}. In the planar case, aligned bar visibility graphs and outerplanar graphs coincide \cite{21}. Outer 1-planar graphs are planar \cite{2} and are a proper subclass of outer fan-planar, outer fan-crossing free and (weak) aligned bar 1-visibility graphs \cite{9}. Here, each vertex is represented by a vertical bar with bottom at the $x$-axis and there is an edge if there is a horizontal line of sight that traverses or crosses at most one other bar.

### 2.2 Path-Addition

A path-addition adds a path of sufficient length between two vertices and adds a set of supplementary edges which have at least one end vertex on the path. These are technical restrictions which help to preserve the properties of a particular class. The set of supplementary edges $F$ is not needed if the class of graphs is hereditary, in which case we let $F = \emptyset$. Otherwise, $F$ is computed from a representation.

**Definition 1.** For a graph $G = (V, E)$, two vertices $u, v \in V$, and an internally vertex-disjoint path $P = (u, w_1, \ldots, w_t, v)$ with $w_i \not\in V$ for $1 \leq i \leq t$ from $u$ to $v$, the path-addition results in a graph $G' = (V \cup W, E \cup Q \cup F)$ such that $W = \{w_1, \ldots, w_t\}$ is the set of internal vertices of $P$, $Q$ consists of the edges of $P$ and $F$ is a set of supplementary edges with at least one endpoint in $W$. We denote $G'$ by $G \oplus P \oplus F$.

**Definition 2.** A class of graphs $\mathcal{G}$ is closed under path-addition if for every graph $G$ in $\mathcal{G}$ and for every internally vertex-disjoint path $P$ of length at least $|G| - 1$ between two vertices $u$ and $v$ of $G$ there is a set of edges $F$ such that $G \oplus P \oplus F$ is in $\mathcal{G}$. If $\mathcal{G}$ is hereditary, then we let $F = \emptyset$.

Our definition of path-additions comes with two parameters, the length of the added path and a set of supplementary edges $F$.

We have chosen long paths to be independent of a particular representation of the graph and to avoid a situation where a too short path must violate requirements from the given class of graphs. If paths $P_1, \ldots, P_r$ are added successively to a graph $G$, then the length of the paths increases at least exponentially such that $|P_i| \geq |G| - 1 + |P_1| + \ldots + |P_{i-1}|$, and $|P_i| \geq 2^{i-1}(G - 1) + 1$.

The set of supplementary edges $F$ is added to preserve a given class. This seems necessary if the class of graphs is not closed under taking subgraphs, e.g., for chordal or triangulated graphs and for strong visibility representations. The edges are only constrained by the fact that one endvertex is from the new path.
3 Positive Closure Results

Clearly, a graph \( G \) remains non-planar if a path (and further edges) are added to \( G \). Similarly, non-planarity is preserved by subdivision, whereas a planar graph may result from a non-planar one by an edge contraction or an edge or a vertex removal.

**Corollary 1.** The class of non-planar graphs is closed under path-addition.

Many classes of graphs admit the routing of a vertex-disjoint path along a simple or shortest path between two vertices. Details are obtained from the visual representation. This technique is applicable if a given edge or vertex can be crossed by a new edge between two new vertices.

**Theorem 1.** The following classes of graphs are closed under path-addition:

- right angle crossing graphs (RAC)
- fan-crossing free graphs
- quasi-planar graphs
- bar 1-visibility graphs
- aligned bar 1-visibility graphs (AB1V)
- interval graphs

**Proof.** Consider a graph \( G \) with vertices \( u \) and \( v \) and an internally vertex-disjoint path \( P = (u_0, \ldots, u_r) \) from \( u \) to \( v \). Let \( S = (v_0, \ldots, v_s) \) be a shortest path from \( u \) to \( v \) in \( G \). Thus \( u = u_0 = v_0 \) and \( v = u_q = v_0 \).

First, we consider RAC, fan-crossing free and quasi-planar graphs, where a graph \( G \) is given by an embedding \( E(G) \). Since these classes are hereditary, the set of supplementary edges is empty. Path \( P \) must be added to the embedding, and we route \( P \) along \( S \) in \( E(G) \). Consider the \( i \)-th vertex \( v_i \) of \( S \). If \( (e_1, \ldots, e_t) \) is the rotation system at vertex \( v_i \) given by \( E(G) \), then \( S \) enters \( v_i \) via edge \( e_h \) and leaves \( v_i \) via \( e_j \). Suppose that \( h - j \leq j - h \) in the circular ordering of the rotation system. The other case is similar. Then route a section of length \( j - h - 1 \) of \( P \) around \( v_i \). We associate this section of \( P \) with \( v_i \). If vertices \( u_1, \ldots, u_q \) have been associated with the vertices \( v_0, \ldots, v_{i-1} \), then associate \( u_q + 1, \ldots, u_h = j \) to \( v_i \) and place these vertices close to \( v_i \) such that the edge \( (u_q + \nu, u_{q+\nu+1}) \) crosses \( e_{h+\nu} \) for \( \nu = 1, \ldots, h - j - 1 \), see Fig. [1]. Note that the first edge \( (u_{q+1}, u_{q+2}) \) crosses edge \( (v_{i-1}, v_i) \) of \( S \) if \( j - h < h - j \). If edge \( (v_i, v_{i+1}) \) of \( S \) crosses edge \( f \) in \( E(G) \), then edge \( (u_\mu, u_\mu+1) \) of \( P \) crosses \( f \), where \( u_\mu \) is the last vertex of \( S \) associated with \( v_i \) and \( u_{\mu+1} \) is the first vertex associated with \( v_{i+1} \). If \( E(G) \) is a RAC drawing, then the vertices associated with each \( v_i \) are placed such that the edges \( (u_q + \nu, u_{q+\nu+1}) \) and \( e_{h+\nu} \) for \( \nu = 1, \ldots, h - j - 1 \) cross at a right angle. Edge \( (u_\mu, u_{\mu+1}) \) is parallel to the edge \( (v_i, v_{i+1}) \) and if \( (v_i, v_{i+1}) \) crosses some edge \( f \) at a right angle, then so does \( (u_\mu, u_{\mu+1}) \). Clearly, the edges of \( P \) cannot introduce a fan-crossing.

If \( P \) is longer than the sum of the associated segments, then insert the remaining subpath just before \( v \).
It remains to show that a length of \( n - 1 \) suffices for the routing of \( P \) in \( E(G) \). A RAC graph has at most \( 4n - 10 \) edges \([18]\). Then \( G \) is maximal and no further edge can be added. So assume that \( G \) is maximal. If a segment of length \( \lambda \) of \( P \) is associated with vertex \( v_i \), then at least \( 4\lambda \) edges of \( G \) can be associated with the segment. Each edge \((u_{q+\nu}, u_{q+\nu+1})\) crosses an edge \( e_{h+\nu} = (v_i, w_{\nu}) \) and by the maximality of \( G \) there is an edge \((w_{\nu}, w_{\nu+1})\) which is not crossed by any edge of \( P \) since \( S \) is a shortest path. Hence, we can associate the edges incident to \( v_i \) together with the edges connecting consecutive endvertices according to the rotation system with the segment of \( P \) associated with \( v_i \), and this is a 4 : 1 relation. In consequence, a path \( P \) of length at most \((4n - 10)/4\) can be routed between vertices \( u \) and \( v \) in the RAC embedding \( E(G) \). The same argument applies to fan-crossing free graphs with at most \( 4n - 8 \) edges \([14]\).

For quasi-planar graphs, we start with a single edge \( e \) for \( P \) and follow \( S \) from \( u \) to \( v \). If there is a violation of quasi-planarity and \( e \) crosses two other crossing edges \( f \) and \( g \), then we subdivide \( e \) with a vertex \( u \) between \( e \times f \) and \( e \times g \), where \( e \times f \) is the crossing point of \( e \) and \( f \). We account \( u \) to the other endvertex of \( f \), which is not passed by \( S \) and \( P \) if \( S \) is a shortest path. Hence, \( P \) does not introduce three mutually crossing edges and a length of \( n - 1 \) suffices for \( P \).

Next, we consider generalized visibility representations. In a bar 1-visibility representation with horizontal bars shoot a vertical ray from vertex \( u \) to the top. Suppose the ray traverses bars \( b_1, \ldots, b_r \). Then introduce a small bar \( b'_{i} \) of width \( \epsilon \) just above \( b_i \) such that \( b'_{i} \) is not traversed by any other line of sight, whereas \( (b'_{i-1}, b'_{i}) \) traverses \( b_i \), where \( b_0 \) is the bar of \( u \). Proceed similarly for a path from \( v \). Introduce a new topmost bar between the rays. The length of the new path from \( u \) to \( v \) is at most \( n - 1 \) if the sum of the distances of \( u \) and \( v \) to the top is at most the distance to the bottom. Otherwise, we go to the bottom.

In case of a strong bar 1-visibility, one must add further edges to meet the requirements. These edges can be retrieved from the bar 1-visibility representation.

In an aligned bar 1-visibility representation, the bars of inner vertices of the path \( P \) are shorter than the bars of the given vertices and they are inserted in alternation with bars of vertices of \( G \), as elaborated in \([9]\).

Finally, in an interval representation, join the intervals of \( u \) and \( v \) by a sequence of overlapping intervals, and add supplementary edges to preserve interval graphs.

There are many other classes of graphs \([11, 34]\), and it seems likely that some are closed under path-addition, e.g., rectangle visibility graphs.

### 4 Negative Closure Results

As aforesaid, the closure under path-addition admits the construction of every complete graph \( K_k \) as a minor. In consequence, path-addition opposes taking minors and can be regarded as an anti-minor.
Fig. 1. Routing a new path with dotted edges at a vertex in parallel with a given path whose edges are drawn bold

Corollary 2. A minor closed class of graphs $G$ is closed under path-addition if and only if $G$ is trivial, i.e., $G$ consists of the empty set, $G$ contains only the empty graph, or $G$ is the set of all graphs.

Forthcoming, we shall exclude trivial classes of graphs.

As an immediate consequence of the construction of $K_k$ minors for arbitrary $k > 0$ we obtain that classes of graphs with bounded treewidth and all subclasses of the planar graphs are destroyed by path-addition.

Corollary 3. A class of graphs $G$ is not closed under path-addition if $G$ has bounded treewidth.

Corollary 4. If $G$ is a subclass of planar graphs, then $G$ is not closed under path-addition. In particular, the outerplanar, outer 1-planar, series-parallel, and planar graphs are not closed under path-addition.

For further non-closure results we use the following technique: Suppose there is a cycle $C$ of length $r$ separating a graph $G$ into an inner and an outer component such that the components are nonempty and $C$ can be traversed only $c \cdot k$ times, where $c$ is the length of $C$. Then we can add $c \cdot k + 1$ paths to violate this property.

There are some obstacles for the application of this technique. First, a graph in many classes of beyond-planar graphs does not necessarily have a unique embedding or visibility representation. In consequence, two vertices $u$ and $v$ may be separated by a cycle $C$ in one embedding whereas $C$ does not separate them in another embedding. Second, the removal of the cycle does not necessarily partition a graph into components, since connectivity may be preserved by crossing edges. Therefore, we consider a planarization. The removal of a vertex $v$ from the planarization induces the removal of the vertex and all edges of the given graph that are incident to $v$. In particular, if $v$ is a crossing point of two edges then both edges are removed, and similarly for an edge-vertex crossing in visibility representations where the edge and the crossed vertex are removed.
Definition 3. A graph $G$ has the separating cycle property, SCP for short, if for every planarization $P(E(G))$ of an embedding or a generalized visibility representation there is a cycle $C = (v_0, e_1, v_1, \ldots, e_r, v_r)$ of consecutive vertices and edges such that $G - C$ decomposes into $s > 1$ components $C_1, \ldots, C_s$ and there are at least two components $C_i$ and $C_j$ with $i \neq j$ containing a vertex of $G$.

If each edge or vertex of $C$ can be traversed at most $k$ times, then there is an upper bound on the number of path-additions between two vertices in distinct components.

Lemma 1. If a class of graphs $G$ contains a graph $G$ with SCP and every edge or vertex of $G$ can be crossed at most $k$ times, then $G$ is not closed under path-addition.

Proof. Consider two vertices $u$ and $v$ in different components of $G - C$, and add at least $c \cdot k + 1$ paths between $u$ and $v$, where $c$ is the length of $C$. Each path must traverse $C$ which allows at most $c \cdot k$ traversals.

In consequence, we obtain:

Corollary 5. The 1-planar graphs and the $(1,j)$ bar 1-visibility graphs for $j \leq 4$ satisfy SCP and are not closed under path-addition.

Proof. The extended wheel graphs $XW_{2k}$ are 1-planar graphs with two poles $p$ and $q$ and a cycle of $2k$ vertices such that $p$ is inside and $q$ is outside the cycle, or vice-versa [33,35]. Since each edge of the cycle can be traversed at most once, there are at most $2k$ path-additions for $p$ and $q$. In fact, an extended wheel graph does not admit any further path between its poles.

Similarly, there are bar $(1,j)$-visibility graphs for $1 \leq j \leq 4$ with a fixed bar 1-visibility representation [10] which satisfy SCP and each bar can be traversed at most $j$ times.

In passing, we note that 4-map graphs are not closed under path-addition, since they are the triangulated 1-planar graphs [8,13].

Lemma 2. For every $k \geq 0$ there is a complete bipartite graph $K_{2,q}$ which satisfies SCP for $k$-planar and bar $(1,k)$-visibility graphs.

Proof. For $k$-planarity, let $q \geq 4k + 9$ and, towards a contradiction, assume that $K_{2,q}$ does not satisfy SCP. Let $\{u_1, u_2\}$ and $\{v_1, \ldots, v_q\}$ be the sets of vertices of $K_{2,q}$. Suppose there is an initial planar quadrangle $(u_1, v_1, u_2, v_2)$, otherwise, the edges $(u_1, v_i)$ and $(u_2, v_1)$ cross for $i = 2, \ldots, q$ in the rotation system at $u_1$, which violates $k$-planarity. To violate SCP, all vertices $v_3, \ldots, v_q$ must lie in one face of the quadrangle $(u_1, v_1, u_2, v_2)$ and there is no other quadrangle as a separating cycle. Suppose all of $v_1, \ldots, v_q$ lie in the outer face. Then each pair of edges incident to $v_i$ crosses one edge of the quadrangle for $i = 3, \ldots, q$, which violates $k$-planarity for $q \geq 4k + 9$.

The same argument applies to bar $(1,k)$-visibility representations, where the edges incident to $v_i$ for $i = 3, \ldots, q$ traverse one of the bars of the vertices from the initial quadrangle $(u_1, v_1, u_2, v_2)$. \qed
Corollary 6. The \( k \)-planar graphs and the bar \((1, k)\) visibility graphs are not closed under path-addition for every \( k \geq 0 \).

It is not immediately clear that fan-planar graphs \([3, 4, 24]\) satisfy SCP, since there is no upper bound on the number of crossings per edge. However, Proposition 1 of Kaufmann and Ueckerdt \([24]\) comes close. It states that a connected planar graph can be expended to a maximal fan-planar graph preserving the planar edges. As an example consider the crossed dodecahedral graph from \([24]\).

The dodecahedral graph is a planar 3-regular graph of the dodecahedron with 20 vertices, 30 edges, and 12 pentagonal faces. In the crossed version there is a \( K_5 \) for each face.

Lemma 3. The fan-planar graphs are not closed under path-addition.

**Proof.** We claim that the dodecahedral graph with a crossed pentagram in each face has a unique fan-planar embedding up to graph automorphism \([35]\). Towards a contradiction, suppose there is another embedding. Then some \( K_5 \) subgraphs must be embedded with at least one pair of crossing edges and at least one edge \((v_i, v_{i+2})\) in the outer face, where \((v_1, \ldots, v_5)\) is the cyclic ordering of the vertices. However, there are two \( K_5 \) with vertices \( v_i, v_{i+1}, w, z_1, z_2 \) and \( v_{i+1}, v_{i+2}, w, z_3, z_4 \) such the edges \((v_i+1, z_2)\) for \( i = 1, 2, 3, 4 \) do not admit a fan-planar embedding, since they must cross \((v_i, v_{i+1})\). In addition, a crossed pentagram is impenetrable for any vertex-disjoint path, since all edges are crossed by a fan. Hence, one cannot add an internally vertex-disjoint path from a vertex of the inner face to a vertex of the outer face of the underlying planar pentagram. \( \square \)

Lemma 4. The outer-fan-planar and the outer-fan-crossing free graphs are not closed under path-addition.

**Proof.** Consider \( K_5 \), which is outer-fan-planar, and then add paths from one vertex \( v_1 \) to all other vertices. The obtained graph \( G \) is biconnected and, if outer-fan-planar, would admit a straight-line outer-fan-planar drawing with all vertices on a circle, as proved in \([3]\), Lemma 1. However, this is impossible.

Similarly, \( K_4 \) is outer-fan-crossing-free and the addition of a path for each pair of vertices violates this property. \( \square \)

At last, we consider fan-planar graphs. Binucci et al. \([4]\) have proved that there are 2-planar graphs that are not fan-planar. The graph is illustrated in Fig. 2 where each bold line is replaced by a fan-planar embedding of \( K_7 \) and each thin or line represents an edge.

Lemma 5. The fan-planar graphs are not closed under path-addition.

**Proof.** Consider graph \( G \) from Binucci et al. \([4]\) and remove the edges \((v_2, v_6)\) and \((v_3, v_9)\). The so-obtained graph is fan-planar, where each bold line represents a fan-planar embedding of \( K_7 \). Replace the edges \((v_2, v_6)\) and \((v_3, v_9)\) by vertex-disjoint paths \( P_1 \) and \( P_2 \). Binucci et al. argue that the edges \((v_2, v_6)\) and \((v_3, v_9)\) must be routed on one side of the cycle \( v_1, \ldots, v_{10} \), and this also holds for \( P_1 \) and \( P_2 \), which, inevitably, introduces a crossing of \((v_1, v_7)\) by two independent edges, and similarly for \((v_4, v_8)\). \( \square \)
Fig. 2. A non-fan-planar graph with bold edges representing a fan-planar embedding of $K_7$ from [4].

5 Conclusion

In this work we have investigated the path-addition operation which is opposite of taking minors and can be emulated by edge addition and subdivision. We have shown that some classes of graphs are closed under path-addition and others are not. It might be worthwhile to investigate the closure of classes of graphs under path-addition. For example, a graph that is obtained from a planar graph by path-addition is a RAC graph, and a graph that is obtained from a 1-planar graph is a fan-crossing free and a bar 1-visibility graph. Which graphs are RAC graphs and cannot be obtained by path-addition from planar graphs, even with a relaxed version of path-addition without a length restriction?

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