ON WEIGHTED COMPACTNESS OF COMMUTATORS OF SCHRÖDINGER OPERATORS

QIANJUN HE AND PENGTAO LI

Abstract. Let $L = -\Delta + V(x)$ be a Schrödinger operator, where $\Delta$ is the Laplacian operator on $\mathbb{R}^d$ ($d \geq 3$), while the nonnegative potential $V(x)$ belongs to the reverse Hölder class $B_q$, $q > d/2$. In this paper, we study weighted compactness of commutators of some Schrödinger operators, which include Riesz transforms, standard Calderón-Zygmund operators and Littlewood-Paley functions. These results generalize substantially some well-known results.

1. Introduction

Let us consider the Schrödinger differential operators $L = -\Delta + V(x)$ on $\mathbb{R}^d$ with $d \geq 3$, where $\Delta$ is the Laplace operator on $\mathbb{R}^d$, and the potential $V \geq 0$ is a function satisfying the reverse Hölder inequality (called the reverse Hölder class $B_q$) for some $q > d/2$

$$\left( \frac{1}{|B|} \int_B V(y)^q \right)^{1/q} \leq C \frac{1}{|B|} \int_B V(y)dy$$

(1.1)

for any ball $B \subset \mathbb{R}^d$. The general theory of semigroup, in particular Yosida's generating theorem [54], implies that $L$ is the infinitesimal generator of semigroups, formally denoted by $T_t = e^{-tL}$, that solves the diffusion problem

$$\begin{cases}
\frac{\partial}{\partial t} u(x, t) = Lu(x, t), & (x, t) \in \mathbb{R}^d \times [0, \infty); \\
u(x, 0) = f(x), & x \in \mathbb{R}^d,
\end{cases}$$

by setting $u(x, t) = e^{-tL} f(x)$.

In this paper, we study weighted compactness of commutators of some classical operators of harmonic analysis associated with Schrödinger operators. In functional analysis, an important branch is the theory of compact operators. Let $L$ be a linear operator from a Banach space $X$ to another Banach space $Y$. We call $L$ a compact operator if the image under $L$ of any bounded subset of $X$ is a relatively compact subset of $Y$. One of classical examples of compact operators is the compact imbedding of Sobolev spaces. By such imbedding, it can be converted an elliptic boundary value problem into a Fredholm integral equation. For further information on compact operators, we refer the reader to Conway [18], Folland-Stein [24] and Kutateladze [35].

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In 1978, Uchiyama [48] first studied the compactness of commutators of a singular integral operator with the kernel \( \Omega \in \operatorname{Lip}_1(\mathbb{R}^{d-1}) \) defined by

\[
T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^d} \frac{\Omega(y/|y|)}{|y|^d} f(x - y) dy.
\]

He obtained that the commutator \([b, T_\Omega]\) is compact on \(L^p(\mathbb{R}^d)\), \(1 < p < \infty\), if and only if \(b \in \operatorname{CMO}(\mathbb{R}^d)\), where \(\operatorname{CMO}(\mathbb{R}^d)\) denotes the closure of \(C_c^\infty(\mathbb{R}^d)\) in the topology of \(\operatorname{BMO}(\mathbb{R}^d)\). In 1984, Janson and Peetre [32] established the theory of paracommutators \(T_b\) defined by

\[
(T_b^{s,t} f)(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \hat{b}(\xi - \eta) A(\xi, \eta) |\xi|^s |\eta|^t \hat{f}(\eta) d\eta.
\]

Under some assumptions of \(A(\cdot, \cdot)\), Janson and Peetre proved that if \(b \in \operatorname{CMO}(\mathbb{R}^d)\), then \(T_b^{0,0}\) is compact, see [32, Theorems 13.2 and 13.3]. The commutators and higher commutators of convolution singular integrals are special cases of \(T_b^{0,0}\). Then the result of [32] is a generalization of that in [48].

Since then, the study on the compactness of commutators of different operators has attracted much more attention. For examples, the compactness of commutators of linear Fourier multipliers and pseudodifferential operators was considered by Cordes [19]. Peng [40] gave the the compactness of paracommutators \(T_b\). Beatrous and Li [1] studied the boundedness and compactness of the commutators of Hankel type operators. Krantz and Li [33, 34] applied the compactness characterization of the commutator \([b, T_\Omega]\) to study Hankel type operators on Bergman spaces. Wang [49] showed that the commutators of fractional integral operators are compact from \(L^p(\mathbb{R}^d)\) to \(L^q(\mathbb{R}^d)\). In 2009, Chen and Ding [15] proved that the commutator of singular integrals with variable kernels is compact on \(\mathbb{R}^d\) if and only if \(b \in \operatorname{CMO}(\mathbb{R}^d)\). In [15], the authors also established the compactness of Littlewood-Paley square functions in [16]. After that, Chen, Ding and Wang [17] obtained the compactness of commutators for Marcinkiewicz integrals on Morrey spaces. Liu and Tang [38] studied the compactness for higher order commutators of oscillatory singular integral operators. Li and Peng [36] investigated compact commutators of Riesz transforms associated to Schrödinger operators. Li, Mo and Zhang [37] established a compactness criterion with applications to the commutators associated with Schrödinger operators. For more information about the compactness problems of commutators, see also [2–4, 9–14, 20, 21, 26, 29–31, 46, 47, 50–53] and the references therein.

The study of Schrödinger operator \(\mathcal{L} = -\Delta + V\) recently attracted much attention, see [5–8, 22, 23, 27, 42, 55]. In particular, Shen [42] considered \(L^p\) estimates for Schrödinger operators \(\mathcal{L}\) with certain potentials which include Schrödinger Riesz transforms

\[
R_j^\mathcal{L} = \frac{\partial}{\partial x_j} \mathcal{L}^{-1/2}, \quad j = 1, \ldots, n.
\]

Shen also proved that the Schrödinger type operators: \(\nabla(-\Delta + V)^{-1/2} \nabla\), \(\nabla(-\Delta + V)^{-1/2} \nabla\), \((-\Delta + V)^{-1/2} \nabla\) with \(V \in B_d\), and \((-\Delta + V)^\gamma\) with \(\gamma \in \mathbb{R}\) and \(V \in B_{d/2}\), are standard Calderón-Zygmund operators.

Recently, Bongioanni, Harboure and Salinas [5] proved \(L^p(\mathbb{R}^d)\) \((1 < p < \infty)\) boundedness for commutators of Riesz transforms associated with Schrödinger operators with \(\operatorname{BMO}(\rho)\) functions which include the class of \(\operatorname{BMO}\) functions. In [6],
Bongioanni et al. established the weighted boundedness for Riesz transforms, fractional integrals and Littlewood-Paley functions associated to Schrödinger operators with weight $\mathcal{A}_p$ class which includes the Muckenhoupt weight class. Tang and his collaborators [43–45] have established weighted norm inequalities for some Schrödinger type operators, which include commutators of Riesz transforms, fractional integrals, and Littlewood-Paley functions related to Schrödinger operators (see also [7, 8]).

Naturally, it will be a very interesting problem to ask whether we can establish the weighted compactness of commutators of some Schrödinger type operators with $\text{CMO}(\rho)$ functions and weight $\mathcal{A}_p$ class. In this paper, we give a positive answer.

To obtain the conclusion, we will utilize a new estimate of kernels and new weighted inequalities for new maximal operators. It is worth pointing out that our method is applicable to more general Schrödinger type operators, and generalizes the results obtained in [36, 37]. The paper is organized as follows. In Section 2, we give some notation and several basic results which will play a crucial role in the sequel. In Section 3, we establish the weighted compactness of commutators of Riesz transforms, standard Calderón-Zygmund operators and Littlewood-Paley functions associated with Schrödinger operators.

Throughout this paper, we let $C$ denote constants that are independent of the main parameters involved but whose value may differ from line to line. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$. By $A \lesssim B$, we mean that there exists a constant $C > 0$ such that $A \leq CB$. 

2. Some notation and basic results

We first recall some notation. Given $B = B(x, r)$ and $\lambda > 0$, we will write $\lambda B$ for the $\lambda$-dilate ball, which is the ball centered at $x$ and with radius $\lambda r$. Given a Lebesgue measurable set $E$ and a weight $w$, the symbol $|E|$ denotes the Lebesgue measure of $E$, and $w(E) := \int_E w(x)dx$.

For $0 < p < \infty$,

$$
\|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^d} |f(y)|^p w(y)dy \right)^{1/p}.
$$

The following auxiliary function $m_V(x)$ was first introduced by Shen [42] and is widely used in the research of Schrödinger operators:

$$
\rho(x) = \frac{1}{m_V(x)} = \sup_{r > 0} \left\{ r : \frac{1}{p(d-2)} \int_{B(x, r)} V(y)dy \leq 1 \right\}.
$$

Obviously, $0 < m_V(x) < \infty$ if $V \neq 0$. In particular, $m_V(x) = 1$ with $V = 1$ and $m_V(x) \sim (1 + |x|)$ with $V = |x|^2$.

For different $x$ and $y$, Shen [42] gave the following important inequality.

Lemma 2.1. ([42, Lemmas 1.4 and 1.8]) Assume that $V \in \mathcal{B}_q$ for $q > d/2$.

(i) There exist constants $k_0 > 0$, $C_0 > 1$ and $C > 0$ such that

$$
\frac{1}{C_0} (1 + |x - y|m_V(x))^{-k_0} \leq \frac{m_V(x)}{m_V(y)} \leq C_0 (1 + |x - y|m_V(x))^{k_0/(k_0 + 1)}. \tag{2.1}
$$

In particular, $m_V(x) \sim m_V(y)$ if $|x - y| \leq C/m_V(x)$. 

(ii) For $0 < r < R < \infty$,
\[
\frac{1}{r^{d-2}} \int_{B(x,r)} V(y)dy \leq C(R/r)^{d/q-2} \frac{1}{R^{d-2}} \int_{B(x,R)} V(y)dy. \tag{2.2}
\]

By $0 < m_V(x) < \infty$ and (2.2), Guo et al. [27] got the following result.

**Lemma 2.2.** ([27, Lemma 1]) Suppose that $V \in B_q$ for some $q > d/2$ and let $K > \log_2 C_0 + 1$, where $C_0$ is the constant in (2.1). Then for any $x \in \mathbb{R}^d$ and $R > 0$, we have
\[
\frac{1}{(1+m_V(x)R)^K} \int_{B(x,R)} V(y)dy \leq CR^{d-2}. \tag{2.3}
\]

For a number $\theta > 0$ and a ball $B = B(x_0, r)$ with center at $x_0$ and radius $r$, we denote $\Psi_\theta(B) = (1 + r/\rho(x_0))^{\theta}$.

A weight will always mean a nonnegative locally integrable function. As in [6], we say that a weight $w$ belongs to the class $A_p^\theta$, $1 < p < \infty$, if there is a constant $C$ such that for all balls $B = B(x, r)$,
\[
\left( \frac{1}{\Psi_\theta(B)|B|} \int_B w(y)dy \right) \left( \frac{1}{\Psi_\theta(B)|B|} \int_B w^{-1/(p-1)}(y)dy \right)^{p-1} \leq C.
\]

We also say that a nonnegative function $w$ satisfies the $A_1^\theta$ condition if there exists a constant $C$ such that for all balls $B$,
\[
M_V^\theta(w)(x) \leq Cw(x) \quad a.e. \ x \in \mathbb{R}^d,
\]

where
\[
M_V^\theta f(x) = \sup_{B \ni x} \frac{1}{\Psi_\theta(B)|B|} \int_B |f(y)|dy.
\]

Since $\Psi_\theta(B) \geq 1$, obviously, $A_p \subset A_p^\theta$ for $1 < p < \infty$, where $A_p$ denote the classical Muckenhoupt weights ([25, 39]).

Since
\[
\Psi_\theta(B) \leq \Psi_\theta(2B) \leq 2^\theta \Psi_\theta(B),
\]

we remark that balls can be replaced by cubes in the definitions of $A_p^\theta$ for $p \geq 1$ and $M_V^\theta$. For convenience, in the rest of this paper, for fixed $\theta > 0$, we use the notation $\Psi(B)$ and $A_p^\theta$ instead of $\Psi_\theta(B)$ and $A_p^\theta$, respectively.

The next lemma follows from the definition of $A_p^\theta$ ($1 \leq p < \infty$):

**Lemma 2.3.** ([43]) Let $1 < p < \infty$. Then the following assertions hold.

(i) If $1 < p_1 < p_2 < \infty$, then $A_{p_1}^\theta \subset A_{p_2}^\theta$.

(ii) $w \in A_p^\theta$ if and only if $w^{-1/(p-1)} \in A_p$, where $1/p + 1/p' = 1$.

(iii) If $w \in A_p^\theta$ for $1 \leq p < \infty$, then
\[
\frac{1}{\Psi(\Theta)|\Theta|} \int_{\Theta} |f(y)|dy \leq C \left( \frac{1}{w(5\Theta)} \int_{\Theta} |f|^p w(y)dy \right)^{1/p},
\]

where $w(E) = \int_E w(x)dx$. In particular, letting $f = \chi_E$ for any measurable set $E \subset \Theta$, we have
\[
\frac{|E|}{\Psi(\Theta)|\Theta|} \leq C \left( \frac{w(E)}{w(5\Theta)} \right)^{1/p}. \tag{2.4}
\]
In [5], Bongioanni et al. introduced a new space $\text{BMO}(\rho)$ defined by

$$\|f\|_{\text{BMO}(\rho)} = \sup_{B \subset \mathbb{R}^d} \frac{1}{\Psi(B)|B|} \int_B |f(x) - f_B| dx < \infty,$$

where $f_B = \frac{1}{|B|} \int_B f(y) dy$, $\Psi(B) = (1 + r(\rho(x_0))^\theta$, $B = B(x_0, r)$ and $\theta > 0$. We denote by $\text{CMO}(\rho)$ the closure of $C_c^\infty$ in the topology of $\text{BMO}(\rho)$, where $C_c^\infty$ is the set of all smooth functions on $\mathbb{R}^d$ with compact supports.

To prove the weighted boundedness for the area functions related with Schrödinger operators, Tang et al. [45] consider the following variant of maximal operator $M_{V,\eta}, 0 < \eta < \infty,$ defined as

$$M_{V,\eta}f(x) := \sup_{B \ni x} \frac{1}{(|\Psi(B)|) \eta |B|} \int_B |f(y)| dy.$$

One of the main results obtained in [45] is the weighted $L^p$-boundedness of $M_{V,\eta}, 0 < \eta < \infty.$ Precisely,

**Lemma 2.4.** Let $1 < p < \infty$ and $p' = p/(p - 1)$. If $w \in A_p$, then there exists a constant $C > 0$ such that

$$\|M_{V,p'}\|_{L^p(w)} \leq C \|f\|_{L^p(w)}.$$

**Remark 2.5.** If $\eta = 1$, Lemma 2.4 holds for $1 < p_0 < p < \infty$.

There exists many operators related with $-\Delta + V$ are standard Calderón-Zygmund operators ([42]), for instance,

$$\begin{align*}
\nabla(-\Delta + V)^{-1}\nabla, & \quad V \in B_n, \\
\nabla(-\Delta + V)^{-1/2}, & \quad V \in B_n, \\
(-\Delta + V)^{-1/2}\nabla, & \quad V \in B_n, \\
(-\Delta + V)^{\gamma}, & \quad \gamma \in \mathbb{R} \& \quad V \in B_{n/2},
\end{align*}$$
and $\nabla^2(-\Delta + V)^{-1}\nabla$ with $V$ being a nonnegative polynomial. In particular, the kernels $K$ of the operators mentioned above all satisfy the following conditions: for some $\delta_0 > 0$ and any $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, there exists a constant $C_l$ such that

$$|K(x, y)| \leq \frac{C_l}{(1 + |x - y|)(m_V(x) + m_V(y))^{l+1}} \frac{1}{|x - y|^d}$$

(2.5)

and

$$|K(x + h, y) - K(x, y)| + |K(x, y + h) - K(x, y)| \leq \frac{C_l}{(1 + |x - y|)(m_V(x) + m_V(y))} \frac{|h|^{\delta_0}}{|x - y|^{d+\delta_0}}$$

(2.6)

whenever $x, y, h \in \mathbb{R}^d$ and $|h| < |x - y|/2$.

Next we give a result of maximal Calderon-Zygmund operators associated with Schrödinger type.

**Lemma 2.6.** Let $1 < p < \infty$. If $w \in A_p$, then there exists a constant $C > 0$ such that

$$\|T^* f\|_{L^p(w)} \leq C \|f\|_{L^p(w)},$$

where the maximal operator $T^*$ is defined by

$$T^* f(x) := \sup_{\epsilon > 0} |T_\epsilon f(x)| = \sup_{\epsilon > 0} \left| \int_{|y - x| > \epsilon} K(x, y) f(y) dy \right|.$$
We remark that the maximal operator can be controlled by $M_{V,q}$ in $L^p(w)$, and it is proved in [43]. Thus, using Lemma 2.4, it implies that Lemma 2.6 holds.

The following result about the weighted $L^p$ boundedness of commutator $[b, T]$ which can be found in [43].

**Lemma 2.7.** Let $1 < p < \infty$, $b \in BMO(\rho)$ and $w \in A^p_\rho$. Then there exists a constant $C_p > 0$ such that

$$
||[b, T]||_{L^p(w)} \leq C_p \|b\|_{BMO(\rho)} \|f\|_{L^p(w)}.
$$

Next we consider another class $V \in B_q$ for $q \geq d/2$ for Riesz transforms associated with Schrödinger operators. Let

$$
T_1 = (-\Delta + V)^{-1} V, \quad T_2 = (-\Delta + V)^{-1/2} V^{1/2} \quad \text{and} \quad T_3 = (-\Delta + V)^{-1/2} \nabla.
$$

Tang considered the weighted estimates for the operators $T_i$, $i = 1, 2, 3$, in [43].

**Lemma 2.8.** Suppose that $V \in B_q$ and $q \geq d/2$. Then the following three statements hold.

(i) If $q' \leq p < \infty$ and $w \in A^p_{p/q'}$, then $||T_1 f||_{L^p(w)} \leq C \|f\|_{L^p(w)}$.

(ii) If $(2q)' \leq p < \infty$ and $w \in A^p_{p/(2q)'}$, then $||T_2 f||_{L^p(w)} \leq C \|f\|_{L^p(w)}$.

(iii) If $p_0 \leq p < \infty$ and $w \in A^p_{p/p_0}$, where $1/p_0 = 1/q - 1/d$ and $d/2 \leq q < d$, then

$$
||T_3 f||_{L^p(w)} \leq C \|f\|_{L^p(w)}.
$$

In [43], using Lemma 2.8 and a pointwise estimate of the kernels of $T_i$, $i = 1, 2, 3$, the author got the weighted $L^p$ boundedness of commutator $[b, T_i]$, $i = 1, 2, 3$, with $b \in BMO(\rho)$.

**Lemma 2.9.** Suppose that $V \in B_q$, $q \geq d/2$. Let $b \in BMO(\rho)$. Then the following three statements hold.

(i) If $q' \leq p < \infty$ and $w \in A^p_{p/q'}$, then

$$
||[b, T_1] f||_{L^p(w)} \leq C \|b\|_{BMO(\rho)} \|f\|_{L^p(w)}.
$$

(ii) If $(2q)' \leq p < \infty$ and $w \in A^p_{p/(2q)'}$, then

$$
||[b, T_2] f||_{L^p(w)} \leq C \|b\|_{BMO(\rho)} \|f\|_{L^p(w)}.
$$

(iii) If $p_0 \leq p < \infty$ and $w \in A^p_{p/p_0}$, where $1/p_0 = 1/q - 1/d$ and $d/2 \leq q < d$, then

$$
||[b, T_3] f||_{L^p(w)} \leq C \|b\|_{BMO(\rho)} \|f\|_{L^p(w)}.
$$

We list some estimates of the kernel $K_i$ of operator $T_i$, $i = 1, 2, 3$, and refer the reader to Guo-Li-Peng [27] and Shen [42].

**Lemma 2.10.** Suppose $V \in B_q$ for some $q > n/2$. Then there exist constants $\delta > 0$ and $C_l$ such that for $0 < |h| < |x - y|/16$ and $l > 0$,

$$
|K_1(x, y)| \leq \frac{C_l}{(1 + |x - y|m_V(x))^l} \frac{1}{|x - y|^{d-2}} V(y), \quad (2.7)
$$

$$
|K_1(x + h, y) - K_1(x, y)| \leq \frac{C_l}{(1 + |x - y|m_V(x))^l} \frac{|h|^\delta}{|x - y|^{d-2+\delta}} V(y), \quad (2.8)
$$

$$
|K_2(x, y)| \leq \frac{C_l}{(1 + |x - y|m_V(x))^l} \frac{1}{|x - y|^{d-1}} V^{1/2}(y) \quad (2.9)
$$
If $B$ be decomposed into finite disjoint cubes $\{Q_i\}_{i=1,\ldots,m}$ such that
\[
B \subset \bigcup_{i=1}^{m} Q_i \subset 2^{d/2}B
\]
and
\[
r_i/2 \leq \frac{1}{m_{V}(x)} \leq 2^{d/2}C_0 r_i
\]
for some $x \in Q_i = Q(x_i,r_i)$, where $C_0$ is the same as in (2.1) of Lemma 2.1.

Lemma 2.13. Suppose that $0 < \eta < \infty$ and $V \in B_q,q \geq d/2$. For any ball $B = B(x_0,r)$, we have for $x \in B$
\[
\frac{1}{|B|} \int_B |f(y)|dy \leq (2^{d})^d M_{V,\eta} f(x).
\]
Proof. It is sufficient to consider two cases.

Case 1: \( r < 1/m_V(x_0) \). Since \( r < 1/m_V(x_0) \) implies \( \Psi(B) \sim 1 \), this case is easy to handle and we omit the details.

Case 2: \( r \geq 1/m_V(x_0) \). Using Lemma 2.13, there exist finite disjoint cubes \( Q_i(x_i, r_i), i = 1, \ldots, m, \) such that

\[
\begin{aligned}
&\int_B |f(y)|dy \leq \sum_{i=1}^m \int_{Q_i} |f(y)|dy, \\
&|B| \leq \sum_{i=1}^m |Q_i| \leq (2\sqrt{d})^d |B|, \\
&r_i/2 \leq 1/m_V(x_i) \leq 2\sqrt{dC_0 r_i}.
\end{aligned}
\]

Note that \( r_i < C/m_V(x_i) \) implies \( \Psi(Q_i) \sim 1 \). For \( x \in B \), we then have

\[
\int_B |f(y)|dy \leq \sum_{i=1}^m |Q_i| M_{V,\eta}f(x) \leq (2\sqrt{d})^d |B| M_{V,\eta}f(x).
\]

This finished the proof. \( \square \)

Proof of Theorem 2.11. We first prove (i). Take

\[
T_{1, r} f(x) = \int_{|x-y|>r} K_1(x, y) f(y) dy.
\]

For \( B = B(x, r/16) \), we divide \( f \) as \( f = f_1 + f_2 \), where \( f_1 := f\chi_{16B} \). It follows from Lemma 2.13 and Lemma 2.8 (i) with \( w = 1 \) that

\[
\begin{aligned}
|T_{1, r} f(x)| &= \frac{1}{|B|} \int_B |T_{1, r} f(x)|dy \\
&\leq \frac{1}{|B|} \int_B |T_1 f(y)|dy + \frac{1}{|B|} \int_B |T_1 f_1(y)|dy + \frac{1}{|B|} \int_B |T_1 f_2(y) - T_{1, r} f(x)|dy \\
&\leq M_{V,\eta}(T_1 f)(x) + \frac{1}{|B|^{1/q'}} \|T_1 f_1\|_{L^{q'}} + \frac{1}{|B|} \int_B |T_1 f_2(y) - T_{1, r} f(x)|dy \\
&\leq M_{V,\eta}(T_1 f)(x) + C \left( \frac{1}{|B|} \int_B |f(y)|^{q'} dy \right)^{1/q'} + \frac{1}{|B|} \int_B |T_1 f_2(y) - T_{1, r} f(x)|dy \\
&\leq M_{V,\eta}(T_1 f)(x) + C \left( M_{V,\eta}(|f|^{q'}) \right)^{1/q'} + \frac{1}{|B|} \int_B |T_1 f_2(y) - T_{1, r} f(x)|dy.
\end{aligned}
\]

For the third term in the last inequality, we have

\[
\begin{aligned}
\frac{1}{|B|} \int_B |T_1 f_2(y) - T_{1, r} f(x)|dy \\
= \frac{1}{|B|} \int_B \left| \int_{|y|>r} K_1(y, \xi) f(\xi) d\xi - \int_{|x-\xi|>r} K_1(x, \xi) f(\xi) d\xi \right| dy \\
\leq \frac{1}{|B|} \int_B \left( \int_{|x-\xi|>r} |K_1(y, \xi) - K_1(x, \xi)||f(\xi)| d\xi \right) dy \\
=: \frac{1}{|B|} \int_B I_1(y) dy.
\end{aligned}
\]
Now, set $h = |y - x|$. Since $|y - x| < r/16 < |x - \xi|/16$ for $y \in B$, by (2.8) in Lemma 2.10, we can obtain that for $l = \theta \eta/q' + K$,

$$I_1(y) \leq \sum_{k=0}^{\infty} \int_{2^k r < |x - \xi| \leq 2^{k+1} r} \frac{C_l}{(1 + |x - \xi| m V(x))^l} \frac{|y - x|^\delta}{|x - \xi|^d + \delta} V(\xi) |f(\xi)| d\xi$$

$$\leq \sum_{k=0}^{\infty} \frac{C_l}{(1 + m V(x) 2^kr)^l} \left( \int_{|x - \xi| \leq 2^{k+1} r} |f(\xi)| d\xi \right)^{1/q'} |f(\xi)|^{\frac{\delta}{q'}} d\xi$$

$$\leq C \sum_{k=0}^{\infty} \frac{C_l(M V, \eta(|f|^\prime)(x))^{1/q'}}{(1 + m V(x) 2^kr)^l} \left( \int_{B(x, 2^k r)} V(\xi) d\xi \right)$$

$$\leq C(M V, \eta(|f|^\prime)(x))^{1/q'} \sum_{k=0}^{\infty} \frac{\rho^\delta}{(2^kr)^{d-2+\delta}} (2^kr)^d$$

$$\leq C(M V, \eta(|f|^\prime)(x))^{1/q'}.$$

Here we have used (2.2) for $R = 2^{k+1} r$, and (2.3) in Lemma 2.2 for $R = 2^k r$. The estimate for $I_1(y)$ implies that

$$\frac{1}{|B|} \int_B I_1(y) dy \leq C(M V, \eta(|f|^\prime)(x))^{1/q'}$$

and

$$T_{1,\text{Max}} f(x) \leq M V, \eta(T_1 f)(x) + C \left( M V, \eta(|f|^\prime) \right)^{1/q'}.$$

Hence, using Lemma 2.3 (i), Lemma 2.4 and Lemma 2.8 (i), we have

$$\|T_{1,\text{Max}}\|_{L^p(w)} \leq \|M V, \eta(T_1 f)\|_{L^p(w)} + C \|M V, \eta(|f|^\prime)(x))^{1/q'}\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$$

for $p > q' > 1$. This finishes the proof of (i).

Similar to (i), (ii) can be obtained easily. We omit the details.

It remains to handle the maximal operator $T_{3,\text{Max}}$. For any $x$, let

$$T_{3,r} = \int_{|x-y| > r} K_3(x, y) f(y) dy.$$

For $B = B(x, r/16)$, we split $f = f_1 + f_2$, where $f_1 = f \chi_{16B}$. Similarly, we can obtain

$$T_{3,r} f(x) \leq M V, \eta(T_3 f)(x) + C \left( M V, \eta(|f|^\prime) \right)^{1/q'} + \frac{1}{|B|} \int_B I_3(y) dy,$$

where $I_3(y)$ denotes the following integral:

$$I_3(y) = \int_{|x - \xi| > r} |K_3(y, \xi) - K_3(x, \xi)| |f(\xi)| d\xi.$$

Since $y \in B$ and $h = |y - x| < r/16 < |x - \xi|$, we deduce from (2.12) that $I_3(y) \lesssim I_{3,1}(x) + I_{3,2}(x)$, where

$$I_{3,1}(x) := \int_{|x - \xi| > r} \frac{|r|^\delta}{|x - \xi|^{d+\delta}} V(\xi) \left( \int_{B(\xi, |x - \xi|)} \frac{1}{|\xi - u|^{d-1}} \right) |f(\xi)| d\xi,$$

$$I_{3,2}(x) := \int_{|x - \xi| > r} \frac{|r|^\delta}{|x - \xi|^{d+\delta}} |f(\xi)| d\xi.$$
For $I_{3,2}(x)$, we have
\[
I_{3,2}(x) \lesssim r^\delta \sum_{k=0}^\infty \int_{2^k r < |x-\xi| < 2^{k+1} r} \frac{1}{(1 + m V(x) 2^k r)^{\delta/d_0}} \frac{1}{(2^k r)^{d+\delta}} d\xi
\]
\[
\lesssim r^\delta \sum_{k=0}^\infty \frac{1}{(2^k r)^{\delta}} (M_{V,q}(|f|^{p_0'})(x))^{1/p_0'}
\]
\[
\lesssim (M_{V,q}(|f|^{p_0'})(x))^{1/p_0'}.
\]
Since $|u - \xi| < |x - \xi|$ yields $|x - u| \leq |x - \xi| + |\xi - u| < 2|x - \xi|$, we can apply Hölder’s inequality and the Hardy-Littlewood-Sobolev inequality with $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{d}$ to obtain
\[
I_{3,1}(x) \leq \sum_{k=0}^\infty \int_{2^k r < |x-\xi| < 2^{k+1} r} C_{I \frac{r^{\delta}}{(2^k r)^{d+\delta}}} \left( \int_{B(x,2^k r)} \frac{V(u)}{|x-u|^{d-1}} \right) |f(\xi)| d\xi
\]
\[
\leq \sum_{k=0}^\infty \frac{C_{I \frac{r^{\delta}}{(2^k r)^{d+\delta}}}}{(1 + m V(x) 2^k r)^{\delta/d_0}} \left( \int_{B(x,2^k r)} V(\xi) d\xi \right)^{1/q} (2^{k+1} r)^{d/p_0'}
\]
\[
\leq C \sum_{k=0}^\infty \frac{(M_{V,q}(|f|^{p_0'})(x))^{1/p_0'}}{(1 + m V(x) 2^k r)^{\delta}} \left( \int_{B(x,2^k r)} V(\xi) d\xi \right)^{1/q} (2^{k+1} r)^{d/p_0'}
\]
\[
\leq C \sum_{k=0}^\infty \frac{(M_{V,q}(|f|^{p_0'})(x))^{1/p_0'}}{(1 + m V(x) 2^k r)^{\delta}} \left( \int_{B(x,2^k r)} V(\xi) d\xi \right)^{1/q} (2^{k+1} r)^{d/p_0'}
\]
\[
\leq C(M_{V,q}(|f|^{p_0'})(x))^{1/p_0'} \sum_{k=0}^\infty \frac{r^{\delta}}{(2^k r)^{d-1+\delta}} (2^{k+1} r)^{d/q - d + (d/p_0)+d-2}
\]
\[
\leq C(M_{V,q}(|f|^{p_0'})(x))^{1/p_0'}.
\]
Here we have used the fact that $d/q - d + (d/p_0)+d-2 = d-1$ and $1/p_0 = 1/q - 1/d$. Thus, by a similar manner as the case (i), we obtain the desired result. This completes the proof of Lemma 2.11. \hfill \Box

Finally, we continue to investigate the Littlewood-Paley functions related to Schrödinger operators. We first introduce some notations. For $(x, t) \in \mathbb{R}^{d+1} = \mathbb{R}^d \times (0, \infty)$, let $T_s = e^{-s\mathcal{L}}$ and
\[
(Q_t f)(x) = t^2 \left( \frac{dT_s}{ds} \right)_{s=t^2} f(x).
\]
The Littlewood-Paley $g$-function $g_Q$ and the area function $S_Q$ related to Schrödinger operators (cf. [3, 22, 44, 45]) are defined by
\[
g_Q(f)(x) := \left( \int_0^\infty |Q_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2}
\]
(2.13)
and
\[
S_Q(f)(x) := \left( \int_0^\infty \int_{|x-y|<t} |Q_t(f)(y)|^2 \frac{dydt}{t^{d+1}} \right)^{1/2}.
\]
(2.14)
In [6, 45], the authors proved that the weighted boundedness of $g_Q$ and $S_Q$, respectively.
Lemma 2.14. Let $1 < p < \infty$. If $w \in A^p_p$, then there exists a constant $C$ such that 
\[ \|g_Q(f)\|_{L^p(w)} \leq C \|f\|_{L^p(w)} \quad \text{and} \quad \|S_Q(f)\|_{L^p(w)} \leq C \|f\|_{L^p(w)}. \]

The commutators of $g_Q$ and $S_Q$ with $b \in \text{BMO}(\rho)$ are defined by 
\[ g_{Q,b}(f)(x) = \left( \int_0^\infty |Q_t((b(x) - b(\cdot))f)(x)|^2 \frac{dt}{t} \right)^{1/2} \]
and 
\[ S_{Q,b}(f)(x) = \left( \int_0^\infty \int_{|x-y|<t} |Q_t((b(x) - b(\cdot))f)(y)|^2 \frac{dydt}{t^{d+1}} \right)^{1/2}. \]

The following lemma contains weighted norm inequalities for the commutators $g_{Q,b}$ and $S_{Q,b}$. This result can be found in [44, 45].

Lemma 2.15. Let $b \in \text{BMO}(\rho)$ and $1 < p < \infty$. If $w \in A^p_p$, then there exists a constant $C$ such that 
\[ \|g_{Q,b}(f)\|_{L^p(w)} \leq C \|b\|_{\text{BMO}(\rho)} \|f\|_{L^p(w)} \quad \text{and} \quad \|S_{Q,b}(f)\|_{L^p(w)} \leq C \|b\|_{\text{BMO}(\rho)} \|f\|_{L^p(w)}. \]

In [22], the authors introduce some properties for the integral kernel $Q_t(x,y)$ of the operators $Q_t$ in (2.13) and (2.14).

Lemma 2.16. There exist positive constants $c$ and $\delta_0 \leq 1$ such that for any $l \geq 0$ there is a constant $C_l$ so that the following inequalities hold:
(i) $|Q_t(x,y)| \leq C_l t^{-\delta} \left( 1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)} \right)^{-l} \exp \left( -c|x-y|^2/t^2 \right)$,
(ii) $|Q_t(x+h,y) - Q_t(x,y)| \leq C_l \left( |h|/|t| \right)^{\delta_0} t^{-\delta} \left( 1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)} \right)^{-l} \exp \left( -c|x-y|^2/t^2 \right)$
for all $|h| \leq t$.

We define the space $B = L^2(\mathbb{R}^{d+1}, dydt/t^d)$ to be the set of all measurable functions $a : \mathbb{R}^{d+1} \to \mathbb{C}$ endowed the norm 
\[ |a|_B = \left( \int_{\mathbb{R}^{d+1}} |a(y,t)|^2 \frac{dydt}{t^d} \right)^{1/2} < \infty. \]

Let $\varphi$ be a nonnegative infinitely differentiable function on $\mathbb{R}^+$ such that $\varphi(s) = 1$ for $0 < s < 1$ and $\varphi(x) = 0$ for $s \geq 2$. Then the function $\varphi_t(x,y) := \frac{1}{t^d} \varphi(\frac{|x-y|}{t})$ satisfies 
\[ |\varphi_t(x,y) - \varphi(x',y)| \leq \frac{|x-x'|}{t^2} \chi_{[0,2]} \left( \min\{|x-y|,|x'-y|\} \right) \]
for $|x-y| > 2|x-x'|$.

Denote by $\widetilde{K}(\cdot,\cdot)$ the kernel defined as follows:
\[ \widetilde{K}(x,z) := \left\{ t^{1/2} \varphi_t(x,y)Q_t(y,z) \right\}_{(y,z) \in \mathbb{R}^{d+1}}. \]

It is easy to see that 
\[ \left( \int_0^\infty \int_{|x-y|<t} |Q_t(y,z)|^2 \frac{dydt}{t^{d+1}} \right)^{1/2} \leq |\widetilde{K}(x,z)|_B. \tag{2.15} \]

The following estimate of $\widetilde{K}(\cdot,\cdot)$ was obtained by Tang et al. [45].
Lemma 2.17. Let $\delta_0$ as same as in Lemma 2.16. Then for any $l$ we have

$$|\tilde{K}(x, z)| \lesssim \frac{C_l}{(1 + |x - z|(m_V(x) + m_V(y)))^l} \frac{1}{|x - z|^{d'}}.$$  

We end this section by a general weighted version of the Frechet-Kolmogrov theorem, which was proved by Xue, Yabuta and Yan [53].

Lemma 2.18. Let $w$ be a weight on $\mathbb{R}^d$. Assume that $w^{-1/(p_0 - 1)}$ is also a weight on $\mathbb{R}^d$ for some $p_0 > 1$. Let $0 < p < \infty$ and $\mathcal{F}$ be a subset in $L^p(w)$, then $\mathcal{F}$ is sequentially compact in $L^p(w)$ if the following three conditions are satisfied:

(i) $\mathcal{F}$ is bounded, i.e., $\sup_{f \in \mathcal{F}} \|f\|_{L^p(w)} < \infty$;

(ii) $\mathcal{F}$ uniformly vanishes at infinity, i.e.,

$$\lim_{N \to \infty} \sup_{f \in \mathcal{F}} \int_{|x| > N} |f(x)|^p w(x) dx = 0;$$

(iii) $\mathcal{F}$ is uniformly equicontinuous, i.e.,

$$\lim_{|h| \to 0} \sup_{f \in \mathcal{F}} \int_{\mathbb{R}^d} |f(x + h) - f(x)|^p w(x) dx = 0.$$

Note that a operator $T: V \to Y$ is said to be a compact operator if $T$ is continuous and maps bounded subsets into sequentially compact subsets.

3. Compactness of commutators of Schrödinger type operators

In this section, we will establish the weighted compactness of commutators of Riesz transforms, standard Calderón-Zygmund operators and Littlewood-Paley functions associated with Schrödinger operators.

3.1. The weighted compactness of $[b, T]$.

First of all, we consider the weighted compactness of $[b, T]$.

Theorem 3.1. Let $1 < p < \infty$, $b \in \text{CMO}(\rho)$ and $w \in A^p_\rho$. Then $[b, T]$ is a compact operator from $L^p(\mathbb{R}^d)$ to itself.

Proof. Since

$$|[b_1, T]f(x) - [b_2, T]f(x)| \leq ||[b_1 - b_2, T]f(x)||$$

and $b \in \text{CMO}(\rho) \subset \text{BMO}(\rho)$, then by Lemma 2.7, the commutator $[b, T]$ is continuous on $L^p(w)$. Hence, for any bounded set $F \subset L^p(w)$, where $f \in F$ with $\|f\|_{L^p(w)} \lesssim 1$, it suffices to prove that

$$\mathcal{F} = \{[b, T]f : f \in F, b \in \text{CMO}(\rho)\}$$

is a sequentially compact subset. According to a density argument, if $b \in \text{CMO}(\rho)$, then there exists a sequence of functions $b_\epsilon \in C^\infty_c(\mathbb{R}^d)$ such that

$$\|b - b_\epsilon\|_{\text{BMO}(\rho)} < \epsilon.$$  

Thus, by Lemma 2.7, we show that

$$\|[b, T] - [b_\epsilon, T]\|_{L^p(w) \to L^p(w)} \leq \|[b - b_\epsilon, T]\|_{L^p(w) \to L^p(w)} \lesssim \epsilon.$$
Therefore, it is enough to prove that \( \mathcal{F} \) is sequentially compact. Without loss of generalization, we will verify \( \mathcal{F} \) satisfies the conditions (i) – (iii) of Lemma 2.18 for \( b \in C_0^\infty(\mathbb{R}^d) \). The proof is divided into three steps.

**Step I:** \( \mathcal{F} \) satisfies the condition (i). First, by Lemma 2.7, we have

\[
\sup_{f \in F} \| [b, T] f \|_{L^p(w)} \leq C \| b \|_{\text{BMO}(\rho)} \| f \|_{L^p(w)} < \infty,
\]

which yields the fact that the set \( \mathcal{F} \) is bounded.

**Step II:** \( \mathcal{F} \) satisfies the condition (ii). We adapt the method used in [36] to verify the condition (ii) of Lemma 2.18. Assume that \( b \in C_0^\infty(\mathbb{R}^d) \) and \( \text{supp} \, b \subset B(0, R) \), where \( B(0, R) \) is a ball of radius \( R \) and centered at origin in \( \mathbb{R}^d \). For \( \nu > 2 \), set \( B^c = \{ x \in \mathbb{R}^d : |x| > \nu R \} \). Then we have

\[
\int_{|x| > \nu R} |[b, T] f(x)|^p w(x) \, dx \leq \int_{|x| > \nu R} \left( \int_{|y| < \rho(0)} |K(x, y)||b(y)||f(y)| \, dy \right)^p w(x) \, dx.
\]

It can be deduced from (2.1) and the scaling technique directly that for any \( x, y \in \mathbb{R}^d \) and \( c \in (0, 1] \),

\[
\frac{1}{C_0(1 + |x - y|m_V(y))^{k_0 + 1}} \leq \frac{1}{C_0} \frac{1}{c(1 + |x - y|m_V(y))^{1/(1 + k_0)}},
\]

where the constants \( k_0 \) and \( C_0 \) is as same as in (2.1) of Lemma 2.1.

Since \( |x| > \nu R \) implies \( |x - y| > (1 - 1/\nu)|x| \) with \( \nu > 2 \), applying (2.5) and Hölder’s inequality, we have

\[
\int_{|y| < R} |K(x, y)||b(y)||f(y)| \, dy
\]

\[
\leq \int_{|y| < R} \frac{C_l}{1 + |x - y|(m_V(x) + m_V(y))} \frac{1}{|x - y|^d} |b(y)||f(y)| \, dy
\]

\[
\leq \int_{|y| < R} \frac{C_l}{(1 - 1/\nu)^d|x|^d (1 + (1 - 1/\nu)|x|m_V(x))^d} \frac{1}{(1 - 1/\nu)^d} |b(y)||f(y)| \, dy
\]

\[
\leq \frac{C_l\|b\|_{L^\infty(\mathbb{R}^d)}}{(1 - 1/\nu)^d|x|^d (1 + (1 - 1/\nu)|x|m_V(x))^d} \left( \int_{|y| < R} |f(y)| \, dy \right)^{1/p} \left( \int_{|y| < R} w^{-1/(p-1)}(y) \, dy \right)^{1-1/p}.
\]

Thus, by using (3.1), it follows that

\[
\left( \int_{|x| > \nu R} |[b, T] f(x)|^p w(x) \, dx \right)^{1/p}
\]

\[
\leq C\|b\|_{L^\infty(\mathbb{R}^d)} \| f \|_{L^p(w)} \sum_{j=0}^\infty \frac{1}{(1 - 1/\nu)^d(2^j \nu R)^d (1 + (1 - 1/\nu)(2^j \nu R)m_V(0))^{d/(k_0 + 1)}}
\]

\[
\times \left( \int_{2^{j+1} \nu R < |x| < 2^{j+1} \nu R} w(x) \, dx \right)^{1/p} \left( \int_{|y| < R} w^{-1/(p-1)}(y) \, dy \right)^{1-1/p}.
\]
Next, we will divide the discussion on the convergence of the above series into two cases.

Case I: $R > \rho(0)$. Since $R > \rho(0)$ implies $1/(1 + (2^j \nu R)/\rho(0)) < 1/(1 + 2^j \nu) \leq 1/2^j \nu$, if $l > 2^j(k_0 + 1)$, it holds
\[
\sum_{j=0}^{\infty} \frac{(1 + 2^j \nu R/\rho(0))^{2^g}}{(1 + (2^j \nu R)/\rho(0))^{l/(k_0 + 1)}} \leq \sum_{j=0}^{\infty} \frac{1}{(2^j \nu)^{l/(k_0 + 1) - 2^g}} \leq \frac{C}{\nu^{l/(k_0 + 1) - 2^g}}.
\]

Case II: $R \leq \rho(0)$. Note that $R$ and $\rho$ are finite, there exists finite integer $N \geq \lceil \log_2(\rho(0)/R) \rceil + 1$ such that $2^N R > \rho(0)$. Hence, this case goes back to Case I and we get
\[
\sum_{j=0}^{\infty} \frac{(1 + 2^j \nu R/\rho(0))^{2^g}}{(1 + (2^j \nu R)/\rho(0))^{l/(k_0 + 1)}} \leq \sum_{j=0}^{\infty} \frac{1}{(2^j \nu)^{l/(k_0 + 1) - 2^g}} \leq \frac{C2^N(l/(k_0 + 1) - 2^g)}{\nu^{l/(k_0 + 1) - 2^g}}.
\]

By the above argument, we obtain
\[
\left( \int_{|x| > \nu R} |[b, T]f(x)|^p w(x) dx \right)^{1/p} \leq C \left( \frac{\|b\|_{L^\infty(\mathbb{R}^d)} \|f\|_{L^p(w)} \|w\|_{A_p^f}}{(1 - 1/\nu)^{d+1/(k_0+1)} \nu^{l/(k_0+1) - 2^g}} \right) \max\{2^N(l/(k_0+1) - 2^g), 1\},
\]
which implies that for any \( p > 1 \) and \( l > 2\theta(k_0 + 1) \),

\[
\lim_{\nu \to \infty} \int_{|x| > \nu R} |[b, T]f(z)|^p w(x)dx = 0
\]

holds whenever \( f \in \mathcal{F} \).

**Step III:** \( \mathcal{F} \) satisfies condition (iii). It remains to show that the set \( \mathcal{F} \) is uniformly equicontinuous. It suffices to verify that for any \( \epsilon > 0 \), if \( |h| \) is sufficiently small and only depends on \( \epsilon \), then

\[
||[b, T]f(h + \cdot) - [b, T]f(\cdot)||_{L^p(w)} \leq C\epsilon
\]

holds uniformly for \( f \in \mathcal{F} \).

For any \( x \in \mathbb{R}^d \), we divide \( [b, T]f(x + h) - [b, T]f(x) = \sum_{i=1}^4 I_i(x) \), where

\[
I_1(x) := \int_{|x-y| > |a|h} K(x, y)(b(x + h) - b(x))f(y)dy;
\]

\[
I_2(x) := \int_{|x-y| > |a|h} (K(x + h, y) - K(x, y))(b(x + h) - b(y))f(y)dy;
\]

\[
I_3(x) := \int_{|x-y| \leq |a|h} K(x, y)(b(x) - b(y))f(y)dy;
\]

\[
I_4(x) := \int_{|x-y| \leq |a|h} K(x + h, y)(b(x + h) - b(y))f(y)dy.
\]

Clearly, by the definition of \( T^* \) and \( b \in C^\infty_c(\mathbb{R}^d) \), we have

\[
|I_1(x)| \leq |h|\|\nabla b\|_{L^\infty(\mathbb{R}^d)} T^* f(x),
\]

which, together with Lemma 2.6, indicates that

\[
\|I_1\|_{L^p(w)} \leq |h|\|T^* f\|_{L^p(w)} \leq C|h|\|f\|_{L^p(w)}.
\]

For \( I_2(x) \), take \( a > 2 \). Using (2.6) and \( \|b\|_{L^\infty(\mathbb{R}^d)} \leq C \), we have

\[
|I_2(x)| \lesssim \sum_{k=0}^\infty \int_{2^ka|h| < |x-y| \leq 2^{k+1}a|h|} \frac{1}{(1 + |x-y| m_V(x))^{l}} \frac{|h|^{\delta_0}}{|x-y|^{d+\delta_0}} |f(y)|dy
\]

\[
\lesssim \sum_{k=0}^\infty \int_{2^ka|h| < |x-y| \leq 2^{k+1}a|h|} \frac{1}{(1 + |x-y| m_V(x))^{l}} \frac{|h|^{\delta_0}}{|x-y|^{d+\delta_0}} |f(y)|dy
\]

\[
\lesssim \sum_{k=0}^\infty \frac{1}{(1 + (2^ka|h|) m_V(x))^{l}} \frac{|h|^{\delta_0}}{(2^ka|h|)^{d+\delta_0}} \int_{B(x, 2^{k+1}a|h|)} |f(y)|dy
\]

\[
\lesssim M_{V, \eta}f(x) \sum_{k=0}^\infty \frac{|h|^{\delta_0}}{(2^ka|h|)^{\delta_0}}
\]

\[
\lesssim a^{-\delta_0} M_{V, \eta}f(x),
\]

where we have used the constant \( l = \theta \eta \). Hence, it follows from Lemma 2.4 that for \( \eta = p' \),

\[
\|I_2\|_{L^p(w)} \leq C\alpha^{-\delta_0} \|M_{V, \eta}f\|_{L^p(w)} \leq C\alpha^{-\delta_0} \|f\|_{L^p(w)}.
\]
For $I_3(x)$, applying (2.5) and $b \in C_0^\infty(\mathbb{R}^d)$, we obtain

$$|I_3(x)| \leq \|\nabla b\|_{L^\infty(\mathbb{R}^d)} \int_{|x-y| \leq a|h|} \frac{C_I}{(1 + |x-y|m(y)x)^d} \frac{1}{|x-y|^{d-1}} |f(y)| dy$$

$$\lesssim \sum_{j=-\infty}^{0} \int_{2^{j-1}a|h| < |x-y| \leq 2^j a|h|} \frac{1}{(1 + (2^{j-1}a|h|)m(y)x)^d} \frac{1}{(2^{j-1}a|h|)^{d-1}} |f(y)| dy$$

$$\lesssim \sum_{j=-\infty}^{0} \frac{1}{(1 + (2^{j-1}a|h|)m(y)x)^d} \frac{1}{(2^{j-1}a|h|)^{d-1}} \int_{B(x,2^ja|h|)} |f(y)| dy$$

$$\lesssim M_{V,n}f(x) \sum_{j=-\infty}^{0} 2^{j-1}a|h|$$

where we have used the constant $l = \theta \eta$. Hence, we use Lemma 2.4 to deduce that

$$\|I_3\|_{L^p(w)} \leq Ca|h||M_{V,n}f\|_{L^p(w)} \leq Ca|h||f\|_{L^p(w)}$$

for $\eta = p'$.

The estimate of $I_4(x)$ is similar to that of $I_3(x)$. Since $|x-y| \leq a|h|$, we have $|x+h-y| \leq (a+1)|h|$. Using the kernel property of $T$ in (2.5) and $b \in C_0^\infty(\mathbb{R}^d)$, we have

$$|I_4(x)| \lesssim \int_{|x+h-y| \leq (a+1)|h|} \frac{C_I}{(1 + |x+h-y|m(y)x)^d} \frac{1}{|x+h-y|^{d-1}} |f(y)| dy$$

$$\lesssim \sum_{j=-\infty}^{0} \frac{1}{(1 + (2^{j-1}(a+1)|h|)^d} \frac{1}{(2^{j-1}(a+1)|h|)^{d-1}} \int_{B(x+h,2^{j}(a+1)|h|)} |f(y)| dy$$

$$\lesssim M_{V,n}f(x) \sum_{j=-\infty}^{0} 2^{j-1}(a+1)|h|$$

$$\lesssim (a+1)|h|M_{V,n}f(x),$$

where we have used the constant $l = \theta \eta$. Thus,

$$\|I_4\|_{L^p(w)} \leq C(a+1)|h||M_{V,n}f\|_{L^p(w)} \leq C(a+1)|h||f\|_{L^p(w)}.$$

Combining with the estimations of $I_1(x)$, $I_2(x)$, $I_3(x)$ and $I_4(x)$, it implies that

$$\|[b,T]f(\cdot + h) - [b,T]f(\cdot)|_{L^p(w)} \leq \sum_{i=1}^{4} \|I_i\|_{L^p(w)}$$

$$\leq C(|h| + a^{-\delta_0} + a|h| + (a+1)|h|)||f||_{L^p(w)}.$$

Consequently, for any $\varepsilon > 0$, we can choose $a$ large enough such that

$$\max\{1/a^2, 1/(a+1)^2, 1/a^{\delta_0}\} < \varepsilon,$$

and set $|h|$ being sufficiently small satisfying $|h| < \min\{1/a^2, 1/(a+1)^2\}$. Letting $a \to \infty$, we can see that $\mathcal{F}$ is uniformly equicontinuous (condition (iii)). This completes the proof of Theorem 3.1. \qed
3.2. The weighted compactness of $g_b$ and $S_{Q,b}$.

**Theorem 3.2.** Let $1 < p < \infty$, $b \in \text{CMO}(\rho)$ and $w \in A_p^\rho$. Then $g_b$ and $S_{Q,b}$ are compact operators from $L^p(w)$ to itself.

**Proof.** We first prove that $S_{Q,b}$ is a compact operator from $L^p(w)$ to itself. Since

$$|S_{Q,b}f(x) - S_{Q,b}f(x)| \leq |S_{Q,b_1-b_2}f(x)|,$$

by the argument in the proof of Theorem 3.1 and Lemma 2.15, we only need to prove that for $b \in C_c^\infty(\mathbb{R}^d)$, $\mathcal{S} = \{[b, T_1]f : f \in F, b \in \text{BMO}(\rho)\}$ satisfies the conditions (ii) – (iii) of Lemma 2.18. We divide the proof into two steps.

**Step I:** $\mathcal{S}$ satisfies the condition (ii).

Suppose $\text{supp } b \subset \{ z : |z| < R \}$ and choose $\nu > 2$. For $|x| > \nu R > 2R$, we have $b(x) = 0$. Therefore, by the Minkowski inequality, we have

$$|S_{Q,b}f(x)| = \left( \int_0^\infty \int_{|x-y| < t} \left| \int_{\mathbb{R}^d} Q_t(y, z)(b(x) - b(z))f(z)dz \right|^2 \frac{dydt}{t^{d+1}} \right)^{\frac{1}{2}} \leq \int_{|x| < R} \left( \int_0^\infty \left| \int_{|x-y| < t} |Q_t(y, z)|^2 \frac{dydt}{t^{d+1}} \right)^{\frac{1}{2}} |b(z)||f(z)|dz.$$

Using (2.15), we show that

$$|S_{Q,b}f(x)| \leq \int_{|x| < R} \left| K(x, z) \right|_{\mathcal{B}} |b(z)||f(z)|dz.$$

Notice that $|K(x, z)|_{\mathcal{B}}$ and $|K(x, y)|$ are both dominated by

$$C_l \frac{1}{(1 + |x-z|(m_\nu(x) + m_\nu(y)))^l |x-z|^n}.$$

By Lemma 2.17, following the argument in Step II of the proof of Theorem 3.1, we obtain

$$\left( \int_{|x| > \nu R} |S_{Q,b}f(x)|^p w(x)dx \right)^{\frac{1}{p}} \leq C \frac{\|b\|_{L^\infty(\mathbb{R}^d)} \|f\|_{L^p(w)}}{\|w\|_{A_p^\rho}} \left( 1 - 1/\nu^d + \frac{1}{\nu^{d+1}} \right) \max\{2N(\frac{1}{\nu^{d+1}} - 2\theta), 1\}.$$

For $f \in F$, letting $\nu \to \infty$ reaches

$$\int_{|x| > \nu R} |S_{Q,b}f(x)|^p w(x)dx \to 0,$$

i.e., $\mathcal{S}$ satisfies the condition (ii) in Lemma 2.18.

**Step II:** $\mathcal{S}$ satisfies the condition (iii). It suffices to prove that for $1 < p < \infty$ and $w \in A_p^\rho$,

$$\lim_{|h| \to 0} \|S_{Q,b}f(\cdot + h) - S_{Q,b}f(\cdot)\|_{L^p(w)} = 0.$$

Note that

$$|S_{Q,b}f(x + h) - S_{Q,b}f(x)| \leq \left( \int_0^\infty \int_{|x-y| < t} |D(x, y, t)|^2 \frac{dydt}{t^{d+1}} \right)^{\frac{1}{2}},$$

where

$$D(x, y, t) = \int_{\mathbb{R}^d} Q_t(x+h, z)(b(x+h) - b(z))f(z)dz - \int_{\mathbb{R}^d} Q_t(y, z)(b(x) - b(z))f(z)dz.$$
For any $x \in \mathbb{R}^d$, choose $a > 0$, and write $D(x, y, y) = D_1 + D_2 + D_3 + D_4$, where

$$
\begin{align*}
D_1(x) &:= \int_{|x-z| > a|h|} Q_t(y, z)(b(x + h) - b(z))f(z)dz; \\
D_2(x) &:= \int_{|x-z| > a|h|} (Q_t(y + h, z) - Q_t(y, z))(b(x + h) - b(z))f(z)dz; \\
D_3(x) &:= \int_{|x-z| \leq a|h|} Q_t(y, z)(b(x) - b(z))f(z)dz; \\
D_4(x) &:= \int_{|x-z| \leq a|h|} Q_t(y + h, z)(b(x + h) - b(z))f(z)dz.
\end{align*}
$$

For $D_3$ and $D_4$, it can be deduced from (2.15) and Minkowski's inequality that

$$
\left(\int_0^\infty \int_{|x-y| < t} |D_3|^2 \frac{dydt}{t^{d+1}}\right)^{\frac{1}{2}} \leq \int_{|x-z| \leq a|h|} |\tilde{K}(x, z)|_B |b(x) - b(z)||f(z)|dz
$$

and

$$
\left(\int_0^\infty \int_{|x-y| < t} |D_4|^2 \frac{dydt}{t^{d+1}}\right)^{\frac{1}{2}} \leq \int_{|x-z| \leq a|h|} |\tilde{K}(x + h, z)|_B |b(x + h) - b(z)||f(z)|dz.
$$

Since $|\tilde{K}(x, z)|_B$ and $|\tilde{K}(x, y)|$ are both dominated by

$$
\frac{C_l}{(1 + |x-z|((m_V(x) + m_V(y)))^l |x-z|^{n}},
$$

the argument in the proof of Theorem 3.1, we use Lemma 2.17 to obtain

$$
\left\|\left(\int_0^\infty \int_{|x-y| < t} |D_3|^2 \frac{dydt}{t^{d+1}}\right)^{\frac{1}{2}}\right\|_{L^p(w)} \leq C_a|h||f||\|_{L^p(w)}
$$

and

$$
\left\|\left(\int_0^\infty \int_{|x-y| < t} |D_4|^2 \frac{dydt}{t^{d+1}}\right)^{\frac{1}{2}}\right\|_{L^p(w)} \leq C(a + 1)|h||f||\|_{L^p(w)}.
$$

For $D_2$, by Minkowski’s inequality, we have

$$
\left(\int_0^\infty \int_{|x-y| < t} |D_2|^2 \frac{dydt}{t^{d+1}}\right)^{\frac{1}{2}} \leq \int_{|x-z| > a|h|} K_Q(x, z)|b(x + h) - b(z)||f(z)|dz,
$$

where

$$
K_Q(x, z) = \left(\int_0^\infty \int_{|x-y| < t} |Q_t(y + h, z) - Q_t(y, z)|^2 \frac{dydt}{t^{d+1}}\right)^{\frac{1}{2}}.
$$

We claim that for $|x - z| > 2h$

$$
|K_Q(x, z)| \leq \frac{C_l}{(1 + |x-z|((m_V(x) + m_V(z)))^l |x-z|^{n + \delta_0}},
$$

where $\delta_0$ be as in Lemma 2.16. In fact, consider $|x - z| > 2h$ and define $E = \{y : |y - x| \geq |x - z|/2\}$. Hence, we can apply (ii) of Lemma 2.16 to obtain

$$
|K_Q(x, z)|^2 \leq \int_{\mathbb{R}^d} \int_{|y-x|/2} |Q_t(y + h, z) - Q_t(y, z)|^2 \frac{dtdy}{t^{d+1}}.
$$
\[
\begin{align*}
&\leq C|h|^{\delta_0} \int_{\mathbb{R}^d} \int_{|y-x|/2}^{\infty} \frac{\rho(x)^{2l}}{t^{3d+1+2\delta_0+2l}} (t+|z-y|)^{2N} \, dtdy \\
&\leq C|h|^{\delta_0} \int_E \int_{|y-x|/2}^{\infty} \frac{\rho(x)^{2l}}{t^{2N}} \, dtdy \\
&\quad + C|h|^{\delta_0} \int_{E^c} \int_{|y-x|/2}^{\infty} \frac{\rho(x)^{2l}}{t^{3d+1+2\delta_0+2l}} (t+|z-y|)^{2N} \, dtdy \\
&=: III_1 + III_2.
\end{align*}
\]

For \(III_1\), we then have
\[
III_1 \leq C|h|^{\delta_0} \int_E \frac{\rho(x)^{2l}}{|y-x|^{2d+2\delta_0}} \, dy \leq C \left( \frac{|x-z|}{\rho(x)} \right)^{-2l} |h|^{2\delta_0}. 
\]

If \(y \in E^c\), then \(|y-x| < |x-z|/2 < |y-z| < 2|x-z|\), and hence
\[
III_2 \leq |h|^{2\delta_0} \int_{E^c} \left( \int_{|y-x|/2}^{\infty} \frac{\rho(x)^{2l}}{t^{3d+1+2\delta_0+2l}} (t+|z-y|)^{2N} \, dtdy \right)
\]
\[
=: III_{2a} + III_{2b}.
\]

For \(III_{2a}\) and \(III_{2b}\), letting \(d + \delta_0 < N - l < (3d + 2\delta_0)/2\), we can get
\[
III_{2a} \leq C \frac{|h|^{2\delta_0}}{|x-z|^{2N}} \int_{E^c} \int_{|y-x|/2}^{\infty} \frac{\rho(x)^{2l}}{t^{3d+1+2N+2l}} \, dtdy \\
\leq C \frac{|h|^{2\delta_0}}{|x-z|^{2N}} \int_{|y-x|<|x-z|/2} \frac{\rho(x)^{2l}}{t^{3d+2\delta_0+2N+2l}} \, dtdy \\
\leq C \left( \frac{|x-z|}{\rho(x)} \right)^{-2l} |h|^{2\delta_0}.
\]

and
\[
III_{2b} \leq C|h|^{\delta_0} \int_{E^c} \int_{|y-x|/2}^{\infty} \frac{\rho(x)^{2l}}{t^{3d+1+2\delta_0+2l}} \, dtdy \\
\leq C \frac{\rho(x)^{2l}|h|^{2\delta_0}}{|x-z|^{3d+2\delta_0+2l}} \int_{|y-x|<|x-z|/2} \, dy \\
\leq C \left( \frac{|x-z|}{\rho(x)} \right)^{-2l} |h|^{2\delta_0}.
\]

Combining the above inequalities, we obtain the desired inequality (3.5).

Taking \(a > 2\), similar to estimate \(I_2(x)\) in Step III of proof of Theorem 3.1, we obtain
\[
\left\| \left( \int_0^{\infty} \int_{|x-y|<t} |D_2|^2 \frac{dydt}{t^{d+1}} \right)^{1/2} \right\|_{L^p(w)} \leq Ca^{-\delta_0} \|f\|_{L^p(w)}. \tag{3.6}
\]

It remains to estimate \(D_1\). Since
\[
\left( \int_0^{\infty} \int_{|x-y|<t} |D_1|^2 \frac{dydt}{t^{d+1}} \right)^{1/2}
\]
\begin{align*}
\leq |b(x + h) - b(x)| \left( \int_0^\infty \int_{|x-y| < t} \int_{|x-z| > a|h|} Q_t(y, z) f(z) dz \right)^{2 \frac{dydt}{t^{d+1}}} \right)^{\frac{1}{2}} \\
= |b(x + h) - b(x)| S_{Q,a,h} f(x),
\end{align*}
we claim that
\begin{equation}
S_{Q,a,h} f(x) \lesssim M_{V,\eta}(S_Q f)(x) + (M_{V,\eta}(|f|^{q_0})(x))^{\frac{1}{q_0}} + M_{V,\eta} f(x), \tag{3.7}
\end{equation}
where $1 < q_0 < p$. Let $B$ denote the ball centered at $x$ and with radius $r = a|h|/2$. Furthermore, let $f_1 = f \chi_{2B}$ and $f_2 = f - f_1$. Thus, by Lemma 2.13, we get
\begin{align*}
S_{Q,a,h} f(x) &= \frac{1}{|B|} \int_B S_{Q,a,h} f(x) d\xi \\
&\leq \frac{1}{|B|} \int_B S_Q f(\xi) d\xi + \frac{1}{|B|} \int_B S_Q f_1(\xi) d\xi \\
&\quad + \frac{1}{|B|} \int_B |S_Q f_2(\xi) - S_{Q,a,h} f(x)| d\xi \\
&\leq M_{V,\eta}(S_Q f)(x) + L_1 + L_2.
\end{align*}
By Lemma 2.13, Lemma 2.14 with $w = 1$ and Hölder’s inequality, for any $1 < q_0 < p$, we show that
\begin{align*}
L_1 &\leq \frac{1}{|B|^{1/q_0}} \left( \int_B |S_Q f_1(\xi)|^{q_0} d\xi \right)^{\frac{1}{q_0}} \\
&\leq C \frac{1}{|B|^{1/q_0}} \left( \int_B |f_1(\xi)|^{q_0} d\xi \right)^{\frac{1}{q_0}} \leq C(M_{V,\eta}(|f|^{q_0})(x))^{\frac{1}{q_0}}.
\end{align*}
Now, let us estimate $L_2$. By the Minkowski inequality, we have
\begin{align*}
|S_Q f_2(\xi) - S_{Q,a,h} f(x)| &\leq \left( \int_0^\infty \int_{|x-y| < t} \int_{|x-z| > a|h|} Q_t(y + \xi - x, z) f(z) dz \right)^{2 \frac{dydt}{t^{d+1}}} \right)^{\frac{1}{2}} \\
&\quad - \left( \int_0^\infty \int_{|x-y| < t} \int_{|x-z| > a|h|} Q_t(y, z) f(z) dz \right)^{2 \frac{dydt}{t^{d+1}}} \right)^{\frac{1}{2}} \\
&\leq \left( \int_0^\infty \int_{|x-y| < t} \int_{|x-z| > a|h|} |Q_t(y + \xi - x, z) - Q_t(y, z)| f(z) dz \right)^{2 \frac{dydt}{t^{d+1}}} \right)^{\frac{1}{2}} \\
&\leq \int_{|x-z| > a|h|} \left( \int_0^\infty \int_{|x-y| < t} \int_{|x-z| > a|h|} |Q_t(y + \xi - x, z) - Q_t(y, z)| \frac{dydt}{t^{d+1}} \right)^{\frac{1}{2}} |f(z)| dz.
\end{align*}
Since $a > 2$ and $|\xi - x| < a|h|/2$ implies $|x - z| > 2|\xi - x|$, it holds
\begin{align*}
\left( \int_0^\infty \int_{|x-y| < t} \int_{|x-z| > a|h|} |Q_t(y + \xi - x, z) - Q_t(y, z)| \frac{dydt}{t^{d+1}} \right)^{\frac{1}{2}}
\end{align*}
It follows from (3.7) that

\[
|S_Qf_2(\xi) - S_{Q,a,h}f(x)| \\
\leq \sum_{j=0}^{\infty} \int_{|x-z|<2^j a |h|} (1 + (2^j - 1) a |h|) |f(z)|dz \\
\leq CM_{V,q} |f(x)| \sum_{j=0}^{\infty} \frac{1}{2^j a |h|} \\
\leq CM_{V,q} |f(x)|. 
\]

Hence, we obtain \( L_2 \leq CM_{V,q} |f(x)| \). Combining \( L_1 \) and \( L_2 \), the claim (3.7) holds.

Now we are ready to give the estimates of \( x \).

Consequently, for any \( \epsilon > 0 \), we can choose a large enough such that

\[
\max\{1/a^2, 1/(a+1)^2, 1/a^{\delta_0}\} < \epsilon, 
\]

and set \( |h| \) being small enough such that \( |h| < \min\{1/a^2, 1/(a+1)^2\} \).

Letting \( a \to \infty \), we have the uniformly equicontinuous (condition (iii)) of \( S \).

Next we will prove that \( g_0 \) is a compact operator from \( L^p(w) \) to itself. In order to this result, we need the following claim:

**Claim 3.3.** Let \( \delta_0 \) as same as in Lemma 2.16. Then for any \( l \) we have

\[
\left( \int_0^\infty |Q_l(x,y)|^\frac{2}{l} \frac{dt}{t} \right)^\frac{1}{2} \leq \frac{C_l}{(1+|x-y|(m_V(x)+m_V(y)))^\frac{1}{l}|x-y|^d} \\
\]

and

\[
\left( \int_0^\infty |Q_l(x,y) - Q_l(\xi,y)|^\frac{2}{l} \frac{dt}{t} \right)^\frac{1}{2} \leq \frac{C_l}{(1+|x-y|(m_V(x)+m_V(y)))^\frac{1}{l}|x-y|^d+\delta_0}  \\
\]

for \( |x-y| > 2|x-\xi| \).
If Claim 3.3 holds, then similar to the proof of $S_{Q,b}$, we can easily obtain $g_b$ is a compact operator on $L^p(w)$. Now we proceed to prove Claim 3.3. Using inequality (a) in Lemma 2.16, we obtain

$$
\int_0^\infty |Q_t(x,y)|^{2l} \frac{dt}{t} \leq \int_0^\infty \frac{\rho(x)^{2l}}{t^{2d+1+2l}} \frac{t^{2N}}{(t+|x-y|)^{2N}} dt
$$

$$
\leq \left( \int_0^{1-|x-y|} + \int_{1-|x-y|}^\infty \right) \frac{\rho(x)^{2l}}{t^{2d+1+2l}} \frac{t^{2N}}{(t+|x-y|)^{2N}} dt =: W_1 + W_2.
$$

For $W_1$ and $W_2$, let $d+l < N < (2d+2l+1)/2$. We have the following estimates:

$$
W_1 \leq \frac{\rho(x)^{2l}}{|x-y|^{2N}} \int_0^{1-|x-y|} \frac{1}{t^{2d+1+2l-2N}} dt \leq C \left( \frac{|x-y|}{\rho(x)} \right)^{-2l} \frac{1}{|x-y|^{2d}}
$$

and

$$
W_2 \leq \int_{1-|x-y|}^\infty \frac{\rho(x)^{2l}}{2d+1+2l} \leq C \left( \frac{|x-y|}{\rho(x)} \right)^{-2l} \frac{1}{|x-y|^{2d}}.
$$

Combining the above inequalities, we get the first inequality of Claim 3.3. The second inequality of Claim 3.3 is similarly to be proved, we omit the details.

By the above arguments, we finish the proof of Theorem 3.2. \qed

3.3. The weighted compactness of $[b,T_i], i = 1, 2, 3$.

Next, we discuss the weighted compactness of $[b,T_i], i = 1, 2, 3$, on $L^p(w)$.

**Theorem 3.4.** Suppose that $V \in B_q, q > d/2$. Let $b \in \text{CMO}(\rho)$. Then the following three statements hold.

(i) If $q' \leq p < \infty$ and $w \in A_{p/q'}^\rho, [b,T_1]$ is a compact operator from $L^p(w)$ to itself.

(ii) If $(2q)' < p < \infty$ and $w \in A_{p/(2q')}^\rho, [b,T_2]$ is a compact operator from $L^p(w)$ to itself.

(iii) If $p_0 < p < \infty$ and $w \in A^\rho_{p/p_0},$ where $1/p_0 = 1/q - 1/d$ and $d/2 < q < d$, $[b,T_3]$ is a compact operator from $L^p(w)$ to itself.

**Proof.** Similar to the proof of Theorem 3.1, and following the process of the proofs of Theorem 2.1, Theorem 2.5 and Theorem 2.7 in [36], we can easily obtain the desired results. Hence, we omit the details. \qed

Let

$$
T_1^* = V(-\Delta + V)^{-1}, \quad T_2^* = V^{1/2}(-\Delta + V)^{1/2} \quad \text{and} \quad T_3^* = \nabla(-\Delta + V)^{-1/2}.
$$

By duality, the following weighted $L^p$-compactness of $T_i^*$, $i = 1, 2, 3$ can be deduced from Theorem 3.4 immediately.

**Corollary 3.5.** Suppose that $V \in B_q$ and $q > d/2$. Let $b \in \text{CMO}(\rho)$. Then the following three statements hold.

(i) If $1 < p < q$ and $w^{-\frac{1}{p-1}} \in A_{p'/q'}^\rho, [b,T_1^*]$ is a compact operator from $L^p(w)$ to itself.

(ii) If $1 < p < 2q$ and $w^{-\frac{1}{p-1}} \in A_{p/(2q')}^\rho, [b,T_2^*]$ is a compact operator from $L^p(w)$ to itself.

(iii) If $1 < p < p_0$ and $w^{-\frac{1}{p_0-1}} \in A_{p'/p_0}^\rho,$ where $1/p_0 = 1/q - 1/d$ and $d/2 < q < d$, $[b,T_3^*]$ is a compact operator from $L^p(w)$ to itself.
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