Fractional Calculus of Variations of Several Independent Variables

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Abstract. We prove multidimensional integration by parts formulas for generalized fractional derivatives and integrals. The new results allow us to obtain optimality conditions for multidimensional fractional variational problems with Lagrangians depending on generalized partial integrals and derivatives. A generalized fractional Noether’s theorem, a formulation of Dirichlet’s principle and an uniqueness result are given.

1 Introduction

The research field concerned with extremal values of functionals is called the calculus of variations [12, 34]. Often, variational functionals are given in the form of an integral that involves an unknown function and its derivatives. In the simplest case, one thinks of single variable integration. However, results can be further extended to the multi-time calculus. Variational problems are particularly attractive because of their manyfold applications. For example, in physics, engineering, and economics, the variational integral may represent some action, energy, or cost functional [9, 35]. The calculus of variations possesses also important connections with other fields of mathematics. Here we are interested in connections with fractional calculus, which is a generalization of the standard calculus that considers integrals and derivatives of noninteger (real or complex) order [13, 27, 30]. The first question linking the two areas was brought up in the XIXth century by Niels Heinrik Abel (1802–1829). Abel’s mechanical problem asks about a curve, lying in a vertical plane, for which the time taken by a material point sliding without friction from the highest point to the lowest one, is destined function of time [1]. The problem is a generalization of the tautochrone problem, which is part of the calculus of variations. Nevertheless, only in 1996–1997, with the works of Riewe [28, 29], the fractional calculus of variations became an important research field per se [4, 6, 7, 10, 11, 15, 17, 22, 25]. It is nowadays of strong interest, with many authors contributing to its theory and applications. For the state of the art we refer the reader to the recent book [20].

Our goal is to develop a theory of the fractional calculus of variations by considering multidimensional fractional variational problems with Lagrangians depending

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on generalized partial fractional operators. Moreover, applications to physics are discussed (see Example 1, Sections 4.2 and 4.3). Our results generalize the fractional calculus of variations for functionals involving multiple integrals studied in [3,26,31], as well as previous works about extremizers of single variable integral functionals with generalized fractional operators [2,23,24].

The text is organized as follows. In Section 2 we give definitions and basic properties for the generalized ordinary and partial fractional operators. Main results are then proved and discussed in Sections 3 and 4: multidimensional fractional integration by parts formulas are given in Section 3 (Theorems 2 and 3); while in Section 4 we obtain fractional partial differential equations of the Euler–Lagrange type for multi-time variational problems with a Lagrangian depending on generalized partial fractional operators (Theorems 4 and 5), we prove a generalized fractional Dirichlet’s principle (Theorems 6 and 7), and a fractional Noether’s symmetry theorem (Theorem 8). We end with Section 5 of conclusion.

2 Generalized Fractional Operators

In this section we give definitions of generalized ordinary and partial fractional operators. By the choice of an appropriate kernel, these operators can be reduced to the standard fractional integrals and derivatives. For more on the subject we refer the reader to [2,14,23,24].

Definition 1 (Generalized fractional integral) Let \( f : [a,b] \rightarrow \mathbb{R} \). The operator \( K_P^\alpha \) is defined by

\[
K_P^\alpha f(t) := p \int_a^t k_\alpha(t,\tau)f(\tau)d\tau + q \int_t^b k_\alpha(\tau,t)f(\tau)d\tau,
\]

where \( P = (a,t,b,p,q) \) is the parameter set (p-set for brevity), \( t \in (a,b) \), \( p,q \) are real numbers, and \( k_\alpha(t,\tau) \) is a kernel which may depend on \( \alpha \). The operator \( K_P^\alpha \) is referred as the operator \( K \) (K-op for simplicity) of order \( \alpha \) and p-set \( P \).

Theorem 1 (cf. Theorem 2.3 of [23]) Let \( k_\alpha \) be a difference kernel, i.e., \( k_\alpha(t,\tau) = k_\alpha(t-\tau) \) and \( k_\alpha \in L_1((0,b-a);\mathbb{R}) \). Then \( K_P^\alpha : L_1((a,b);\mathbb{R}) \rightarrow L_1((a,b);\mathbb{R}) \) is well defined, bounded and linear operator.

Definition 2 (Generalized Riemann–Liouville fractional derivative) Let \( P \) be a given parameter set. The operator \( A_P^\alpha \), \( 0 < \alpha < 1 \), is defined by

\[
A_P^\alpha := D \circ K_P^{1-\alpha},
\]

where \( D \) denotes the standard derivative. We refer to \( A_P^\alpha \) as operator \( A \) (A-op) of order \( \alpha \) and p-set \( P \).

Definition 3 (Generalized Caputo fractional derivative) Let \( P \) be a given parameter set. The operator \( B_P^\alpha \), \( \alpha \in (0,1) \), is defined by

\[
B_P^\alpha := K_P^{1-\alpha} \circ D
\]

and is referred as the operator \( B \) (B-op) of order \( \alpha \) and p-set \( P \).
Operators $A$, $B$ and $K$ reduce to the classical fractional integrals and derivatives for suitably chosen kernels and $p$-sets (see [23, 24]). The notation was introduced in [2] and is now standard [16, 25].

Now, we shall define generalized partial fractional operators. For $n \in \mathbb{N}$, let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n) \in \mathbb{R}^n$ with $0 < \alpha_i < 1$, $i = 1, \ldots, n$, and $\Delta_n = (a_1, b_1) \times \cdots \times (a_n, b_n) \subset \mathbb{R}^n$, $t = (t_1, \ldots, t_n) \in \Delta_n$. Generalized partial fractional integrals and derivatives are natural generalizations of the corresponding one-dimensional generalized fractional integrals and derivatives, being taken with respect to one or several variables.

**Definition 4 (Generalized partial fractional integral)** Let $f = f(t_1, \ldots, t_n): \Delta_n \to \mathbb{R}$. The generalized partial Riemann–Liouville fractional integral of order $\alpha_i$ with respect to the $i$th variable $t_i$ is given by

$$K_{P_i}^{\alpha_i} f(t) := p_i \int_{a_i}^{t_i} k_{\alpha_i}(t_i, \tau)f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)d\tau$$

$$+ q_i \int_{t_i}^{b_i} k_{\alpha_i}(\tau, t_i)f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)d\tau,$$

where $P_i = \langle a_i, t_i, b_i, p_i, q_i \rangle$. We refer to $K_{P_i}^{\alpha_i}$ as the partial operator $K$ (partial $K$-op) of order $\alpha_i$ and $p$-set $P_i$.

**Definition 5 (Generalized partial Riemann–Liouville fractional derivative)** Let $P_i = \langle a_i, t_i, b_i, p_i, q_i \rangle$. The generalized partial Riemann–Liouville fractional derivative of order $\alpha_i$ with respect to the $i$th variable $t_i$ is given by

$$A_{P_i}^{\alpha_i} f(t) := \frac{\partial}{\partial t_i} \left( p_i \int_{a_i}^{t_i} k_{1-\alpha_i}(t_i, \tau)f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)d\tau \right)$$

$$+ q_i \int_{t_i}^{b_i} k_{1-\alpha_i}(\tau, t_i)f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)d\tau \right) = \left( \frac{\partial}{\partial t_i} K_{P_i}^{1-\alpha_i} f \right)(t).$$

The operator $A_{P_i}^{\alpha_i}$ is referred as the partial operator $A$ (partial $A$-op) of order $\alpha_i$ and $p$-set $P_i$.

**Definition 6 (Generalized partial Caputo fractional derivative)** Let $P_i = \langle a_i, t_i, b_i, p_i, q_i \rangle$. The generalized partial Caputo fractional derivative of order $\alpha_i$ with respect to the $i$th variable $t_i$ is given by

$$B_{P_i}^{\alpha_i} f(t) := p_i \int_{a_i}^{t_i} k_{1-\alpha_i}(t_i, \tau)\frac{\partial}{\partial \tau} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)d\tau$$

$$+ q_i \int_{t_i}^{b_i} k_{1-\alpha_i}(\tau, t_i)\frac{\partial}{\partial \tau} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)d\tau = \left( K_{P_i}^{1-\alpha_i} \frac{\partial}{\partial \tau} f \right)(t)$$

and is referred as the partial operator $B$ (partial $B$-op) of order $\alpha_i$ and $p$-set $P_i$. 

Similarly to the one-dimension case, partial operators $K$, $A$ and $B$ reduce to 
the standard partial fractional integrals and derivatives. The left- and right-sided 
Riemann–Liouville partial fractional integrals with respect to the $i$th variable $t_i$ are 
obtained by choosing the kernel $k_{\alpha_i}(t_i, \tau) = \frac{1}{\Gamma(\alpha_i)}(t_i - \tau)^{\alpha_i - 1}$ and $p$-sets $P_{t_i} = (a_i, t_i, b_i, 1, 0)$ and $P_{t_i} = (a_i, t_i, b_i, 0, 1)$, respectively:

\[
K_{P_{t_i}}^{\alpha_i} f(t) = \frac{1}{\Gamma(\alpha_i)} \int_{a_i}^{t_i} (t_i - \tau)^{\alpha_i - 1} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau =: a_i K_{t_i}^{\alpha_i} f(t),
\]
\[
K_{P_{t_i}}^{\alpha_i} f(t) = \frac{1}{\Gamma(\alpha_i)} \int_{t_i}^{b_i} (\tau - t_i)^{\alpha_i - 1} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau =: t_i K_{b_i}^{\alpha_i} f(t).
\]

The standard left- and right-sided Riemann–Liouville and Caputo partial fractional 
operators reduce to the computation of one-variable generalized fractional operators.

Similarly to the one-dimension case, partial operators $K$, $A$ and $B$ reduce to 
the standard partial fractional integrals and derivatives. The left- and right-sided 
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obtained by choosing the kernel $k_{\alpha_i}(t_i, \tau) = \frac{1}{\Gamma(\alpha_i)}(t_i - \tau)^{\alpha_i - 1}$ and $p$-sets $P_{t_i} = (a_i, t_i, b_i, 1, 0)$ and $P_{t_i} = (a_i, t_i, b_i, 0, 1)$, respectively:

\[
P_{t_i}^{\alpha_i} f(t) = \frac{1}{\Gamma(1 - \alpha_i)} \frac{\partial}{\partial t_i} \int_{a_i}^{t_i} (t_i - \tau)^{-\alpha_i} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau =: a_i P_{t_i}^{\alpha_i} f(t),
\]
\[
P_{t_i}^{\alpha_i} f(t) = \frac{1}{\Gamma(1 - \alpha_i)} \frac{\partial}{\partial t_i} \int_{t_i}^{b_i} (\tau - t_i)^{-\alpha_i} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau =: t_i P_{b_i}^{\alpha_i} f(t);
\]

if $P_{t_i} = (a_i, t_i, b_i, 0, 1)$, then

\[
-A_{P_{t_i}}^{\alpha_i} f(t) = \frac{1}{\Gamma(1 - \alpha_i)} \frac{\partial}{\partial t_i} \int_{a_i}^{t_i} (t_i - \tau)^{-\alpha_i} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau =: t_i P_{f_i}^{\alpha_i} f(t),
\]
\[
-B_{P_{t_i}}^{\alpha_i} f(t) = \frac{1}{\Gamma(1 - \alpha_i)} \frac{\partial}{\partial t_i} \int_{t_i}^{b_i} (\tau - t_i)^{-\alpha_i} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau =: t_i P_{b_i}^{\alpha_i} f(t).
\]

**Remark 1** In Definitions [1] [5] and [6] all the variables except $t_i$ are kept fixed. That 
choice of fixed values determines a function $f_{t_1, t_2, \ldots, t_i-1, t_{i+1}, \ldots, t_n}(t_i) = f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n)$. By Definitions [1] [2] [3] and [1] [5] [6] we have

\[
K_{P_{t_i}}^{\alpha_i} f_{t_1, t_2, \ldots, t_i-1, t_{i+1}, \ldots, t_n}(t_i) = K_{P_{t_i}}^{\alpha_i} f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n),
\]
\[
A_{P_{t_i}}^{\alpha_i} f_{t_1, t_2, \ldots, t_i-1, t_{i+1}, \ldots, t_n}(t_i) = A_{P_{t_i}}^{\alpha_i} f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n),
\]
\[
B_{P_{t_i}}^{\alpha_i} f_{t_1, t_2, \ldots, t_i-1, t_{i+1}, \ldots, t_n}(t_i) = B_{P_{t_i}}^{\alpha_i} f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n).
\]

Therefore, as in the integer-order case, computation of partial generalized fractional 
operators reduces to the computation of one-variable generalized fractional operators.
3 Generalized Fractional Integration by Parts for Functions of Several Variables

The integration by parts formula plays a crucial role in the principle of virtual works. In this section, it is of our interest to obtain such formula for generalized fractional operators. Throughout the section, $i \in \{1, \ldots, n\}$ is arbitrary but fixed.

**Definition 7 (Dual $p$-set)** For a given $p$-set $P_i = \langle a_i, t_i, b_i, p_i, q_i \rangle$ we denote by $P_i^*$ the $p$-set $P_i^* = \langle a_i, t_i, b_i, q_i, p_i \rangle$. We say that $P_i^*$ is the dual of $P_i$.

**Theorem 2** Let $\alpha_i \in (0,1)$ and $P_i = \langle a_i, t_i, b_i, p_i, q_i \rangle$ be a parameter set. Moreover, let $k_{\alpha_i}$ be a difference kernel, i.e., $k_{\alpha_i}(t_i, \tau) = k_{\alpha_i}(t_i - \tau)$ such that $k_{\alpha_i} \in L_1((0, b_i - a_i); \mathbb{R})$. If $f : \mathbb{R}^n \to \mathbb{R}$ and $\eta : \mathbb{R}^n \to \mathbb{R}$, $f, \eta \in C (\Delta_n; \mathbb{R})$, then the generalized partial fractional integrals satisfy the identity

$$
\int_{\Delta_n} f : K_{P_i}^{\alpha_i} \eta \, dt_n \ldots dt_1 = \int_{\Delta_n} \eta : K_{P_i^*}^{\alpha_i} f \, dt_n \ldots dt_1,
$$

where $P_i^*$ is the dual of $P_i$.

**Proof** Let $\alpha_i \in (0,1)$, $P_i = \langle a_i, t_i, b_i, p_i, q_i \rangle$ and $f, \eta \in C (\Delta_n; \mathbb{R})$. Define

$$
F(t, \tau) := \begin{cases} 
|p_i k_{\alpha_i}(t_i - \tau)| \cdot |f(t)| \cdot |\eta(t_1, \ldots, t_i-1, \tau, t_{i+1}, \ldots, t_n)| & \text{if } \tau < t_i \\
|q_i k_{\alpha_i}(\tau - t_i)| \cdot |f(t)| \cdot |\eta(t_1, \ldots, t_i-1, \tau, t_{i+1}, \ldots, t_n)| & \text{if } \tau > t_i 
\end{cases}
$$

for all $(t, \tau) \in \Delta_n \times (a_i, b_i)$. Since $f$ and $\eta$ are continuous functions on $\Delta_n$, they are bounded on $\Delta_n$, i.e., there exist real numbers $C, D > 0$ such that $|f(t)| \leq C$ and $|\eta(t)| \leq D$ for all $t \in \Delta_n$. Therefore,

\begin{align*}
\int_{\Delta_n} \left( \int_{a_i}^{b_i} F(t, \tau) \, d\tau \right) \, dt_n \ldots dt_1 \\
= \int_{\Delta_n} \left( \int_{a_i}^{t_i} |p_i k_{\alpha_i}(t_i - \tau)| \cdot |f(t)| \cdot |\eta(t_1, \ldots, t_i-1, \tau, t_{i+1}, \ldots, t_n)| \, d\tau \\
+ \int_{t_i}^{b_i} |q_i k_{\alpha_i}(\tau - t_i)| \cdot |f(t)| \cdot |\eta(t_1, \ldots, t_i-1, \tau, t_{i+1}, \ldots, t_n)| \, d\tau \right) \, dt_n \ldots dt_1 \\
\leq C \cdot D \int_{\Delta_n} \left( \int_{a_i}^{t_i} |p_i k_{\alpha_i}(t_i - \tau)| \, d\tau + \int_{t_i}^{b_i} |q_i k_{\alpha_i}(\tau - t_i)| \, d\tau \right) \, dt_n \ldots dt_1 \\
\leq C \cdot D \left( |q_i| + |p_i| \right) \|k_{\alpha_i}\| \prod_{i=1}^{n} (b_i - a_i) \\
< \infty.
\end{align*}
Theorem 3 (Generalized fractional integration by parts) Let $\alpha_1 \in (0,1)$ and $P_i = \{a_i, t_i, b_i, p_i, q_i\}$ be a parameter set and $f, \eta \in C^1(\Delta_n; \mathbb{R})$. Moreover, let $k_{\alpha_i}$ be a difference kernel such that $k_{1-\alpha_i} \in L_1((0, b_i - a_i); \mathbb{R})$ and $K_{P_i}^{1-\alpha_i} f \in C^1(\Delta_n; \mathbb{R})$.

Then

$$\int_{\Delta_n} f \cdot B_{P_i}^{\alpha_i} \eta \, dt_n \ldots dt_1 = \int_{\partial \Delta_n} \eta \cdot K_{P_i}^{1-\alpha_i} f \cdot \nu^i \, d(\partial \Delta_n) - \int_{\Delta_n} \eta \cdot A_{P_i}^{\alpha_i} f \, dt_n \ldots dt_1,$$

where $\nu^i$ is the outward pointing unit normal to $\partial \Delta_n$.

**Proof** By definition of the generalized partial Caputo fractional derivative, Theorem 2 and the standard integration by parts formula (see, e.g., [8]), one has

$$\int_{\Delta_n} f \cdot B_{P_i}^{\alpha_i} \eta \, dt_n \ldots dt_1 = \int_{\Delta_n} f \cdot K_{P_i}^{1-\alpha_i} \eta \, dt_n \ldots dt_1 = \int_{\Delta_n} \eta \cdot K_{P_i}^{1-\alpha_i} f \, dt_n \ldots dt_1$$

$$= \int_{\partial \Delta_n} \eta \cdot K_{P_i}^{1-\alpha_i} f \cdot \nu^i \, d(\partial \Delta_n) - \int_{\Delta_n} \eta \cdot A_{P_i}^{\alpha_i} f \, dt_n \ldots dt_1.$$
4 The Generalized Fractional Calculus of Variations of Several Independent Variables

Variational problems with functionals depending on several independent variables arise, for example, in mechanics, for systems with infinite number of degrees of freedom, like a vibrating elastic solid. Fractional variational problems involving multiple integrals have been already studied in different contexts. We can mention here [3,5,7,20], where the multidimensional fractional Euler–Lagrange equations for the field are obtained, or [18,19], where a first and a second fractional Noether-type theorem are proved. In this section we present a more general approach to the subject by considering functionals depending on generalized fractional operators. In the sequel we use the notion of generalized fractional gradient.

Definition 8 (The generalized fractional gradient operator) Let \( n \in \mathbb{N} \), \( P = (P_{1}, \ldots, P_{n}) \), and \( \alpha \in (0, 1)^{n} \). We define the generalized fractional gradient of a function \( f : \mathbb{R}^{n} \to \mathbb{R} \) with respect to a generalized fractional operator \( T \) by

\[
\nabla_{T}^{\alpha} f := \sum_{i=1}^{n} c_{i} T_{P_{i}}^{\alpha_{i}} f,
\]

where \( \{c_{i} : i = 1, \ldots, n\} \) denotes the standard basis in \( \mathbb{R}^{n} \). Additionally, we define \( \nabla_{T}^{\alpha} f \) for a vector function \( f : \mathbb{R}^{n} \to \mathbb{R}^{N} \) by

\[
\nabla_{T}^{\alpha} f := [\nabla_{T}^{\alpha_{1}} f_{1}, \ldots, \nabla_{T}^{\alpha_{n}} f_{N}].
\]

4.1 The Fundamental Problem

Let \( \alpha = (\alpha_{1}, \ldots, \alpha_{n}) \) and \( \beta = (\beta_{1}, \ldots, \beta_{n}) \) be such that \( \alpha_{i}, \beta_{i} \in (0, 1) \), and \( P_{j} = (P_{i}^{j}, \ldots, P_{i}^{n}) \), where \( P_{i}^{j} = \langle a_{i}, t_{i}, b_{i}, p_{i}^{j}, q_{i}^{j} \rangle \), \( i = 1, \ldots, n \), \( j = 1, 2 \). Consider the problem of finding an extremizer \( u : \Delta_{n} \to \mathbb{R}^{N} \) of the functional

\[
J[u] = \int_{\Delta_{n}} F(t, u(t), \nabla_{B_{1}}^{\alpha_{1}} u(t), \nabla_{K_{2}}^{\beta_{2}} u(t)) \, dt_{n} \ldots dt_{1}
\]

subject to the boundary condition

\[
u(t)|_{\partial \Delta_{n}} \equiv \psi(t),
\]

where \( \psi : \partial \Delta_{n} \to \mathbb{R}^{N} \) is a given function. For simplicity of notation we write

\[
\{u\}_{P_{1}, P_{2}}^{\alpha, \beta} (t) = \left( t, u(t), \nabla_{B_{1}}^{\alpha_{1}} u(t), \nabla_{K_{2}}^{\beta_{2}} u(t) \right)
\]

and \( dt = dt_{n} \ldots dt_{1} \). As usually, we denote by \( \partial_{i} F \), \( i = 1, \ldots, M \) (\( M \in \mathbb{N} \)), the partial derivative of function \( F : \mathbb{R}^{M} \to \mathbb{R} \) with respect to its \( i \)th argument. We assume that \( F \in C^{2}(\Delta_{n} \times \mathbb{R}^{N} \times \mathbb{R}^{2nN} ; \mathbb{R}) \), \( t \mapsto \partial_{N+k_{n}+i} F (u)_{P_{1}, P_{2}}^{\alpha_{1}, \beta_{2}} (t) \) has continuously differentiable partial integral \( K_{1, \alpha_{i}}^{1, -\alpha_{i}} \) and continuous partial derivative \( A_{P_{1}}^{\alpha_{i}} \); and \( t \mapsto \partial_{N+k_{n}+i} F (u)_{P_{1}, P_{2}}^{\alpha_{1}, \beta_{2}} (t) \) has continuous partial integral \( K_{2, \beta_{2}}^{1, \beta_{2}} \), where \( i = 1, \ldots, n \) and \( k = 1, \ldots, N \). Moreover, we suppose that \( k_{\alpha} \) and \( k_{\beta} \) are difference kernels such that \( k_{1-\alpha_{i}}, k_{\beta_{2}} \in L_{1}((0, b_{i} - a_{i}); \mathbb{R}) \), \( i = 1, \ldots, n \).
**Definition 9** A continuously differentiable function $u \in C^1(\bar{\Delta}_n; \mathbb{R}^n)$ is said to be admissible for the variational problem (1)–(2) if, for all $i \in \{1, \ldots, n\}$, $B^\alpha_{P^1_i}$ and $K^\beta_{P^2_i}$ exist and are continuous on $\bar{\Delta}_n$ and $u$ satisfies the boundary condition (2).

**Theorem 4** Let $u$ be a solution to problem (1)–(2). Then, $u$ satisfies the following system of fractional partial differential equations of Euler–Lagrange type:

$$
\sum_{i=1}^{n} \left[ -A^\alpha_{P^1_i} \partial_{N+n+i} F \{ u \}^\alpha_{P^1_i, P^2_i} (t) + K^\beta_{P^2_i} \partial_{n+N(k+n)+i} F \{ u \}^\alpha_{P^1_i, P^2_i} (t) \right] \\
+ \partial_{n+k} F \{ u \}^\alpha_{P^1_i, P^2_i} (t) = 0, \quad (3)
$$

$k = 1, \ldots, N$, for all $t \in \Delta_n$.

**Proof** Suppose that $u$ is an extremizer of $\mathcal{J}$. For $\eta \in C^1(\bar{\Delta}_n; \mathbb{R}^n)$ such that $B^\alpha_{P^1_i}$ and $K^\beta_{P^2_i}$ are continuous for all $i \in \{1, \ldots, n\}$, and $\eta(t)|_{\partial \Delta_n} = 0$, $\varepsilon \in \mathbb{R}$, the function $\hat{u}(t) = u(t) + \varepsilon \eta(t)$ is still admissible. Define

$$
J(\varepsilon) := \mathcal{J}[\hat{u}] = \int_{\Delta_n} F \left( t, \hat{u}(t), \nabla^\alpha_{P^1_i} \hat{u}(t), \nabla^\beta_{P^2_i} \hat{u}(t) \right) dt.
$$

Then, a necessary condition for $u$ to be an extremizer of $J$ is given by $\frac{dJ}{d\varepsilon}|_{\varepsilon=0} = 0$, that is,

$$
\int_{\Delta_n} \sum_{k=1}^{N} \left( \partial_{n+k} F \{ u \}^\alpha_{P^1_i, P^2_i} (t) \cdot \eta_k(t) + \sum_{i=1}^{n} \left[ \partial_{N+n+i} F \{ u \}^\alpha_{P^1_i, P^2_i} (t) B^\alpha_{P^1_i} \eta_k(t) \\
+ \partial_{n+N(k+n)+i} F \{ u \}^\alpha_{P^1_i, P^2_i} (t) K^\beta_{P^2_i} \eta_k(t) \right] \right) dt = 0.
$$

By integration by parts formulas (Theorems 2 and 3) and since $\eta(t)|_{\partial \Delta_n} \equiv 0$, one has

$$
\int_{\Delta_n} \left( \partial_{n+N(k+n)+i} F \cdot B^\alpha_{P^1_i} \eta_k \right) dt = - \int_{\Delta_n} \eta_k \cdot A^\alpha_{P^1_i} \partial_{N+n+i} F dt,
$$

$$
\int_{\Delta_n} \left( \partial_{n+N(k+n)+i} F \cdot K^\beta_{P^2_i} \eta_k \right) dt = \int_{\Delta_n} \eta_k \cdot K^\beta_{P^2_i} \partial_{n+N(k+n)+i} F dt,
$$

where $i = 1, \ldots, n$ and $k = 1, \ldots, N$. Therefore,

$$
\int_{\Delta_n} \sum_{k=1}^{N} \eta_k(t) \left( \partial_{n+k} F \{ u \}^\alpha_{P^1_i, P^2_i} (t) + \sum_{i=1}^{n} \left[ -A^\alpha_{P^1_i} \partial_{N+n+i} F \{ u \}^\alpha_{P^1_i, P^2_i} (t) \\
+ K^\beta_{P^2_i} \partial_{n+N(k+n)+i} F \{ u \}^\alpha_{P^1_i, P^2_i} (t) \right] \right) dt = 0.
$$

Finally, by the fundamental lemma of the calculus of variations, we arrive to (3).
**Definition 10** We say that an admissible function $u$ is an extremal for problem 11–12 if it satisfies the system of fractional partial differential equations 13.

Using similar techniques as in the proof of Theorem 14 one can prove the following theorem.

**Theorem 5** Let $u: \Delta_n \to \mathbb{R}^N$ be an extremizer of

$$J[u] = \int_{\Delta_n} F \left( t, u(t), \nabla_{\partial\Delta_n} u(t), \nabla u(t) \right) dt_n \ldots dt_1$$

subject to the boundary condition $u(t)|_{\partial\Delta_n} = \psi(t)$, where $\psi: \partial\Delta_n \to \mathbb{R}^N$ is a given function. Then, $u$ satisfies the system of multidimensional generalized Euler–Lagrange equations

$$\sum_{i=1}^n A_{p_i,1}^{\alpha} \partial_{n+k_1+i} F \left[ u \right]_{p_1,p_2}^\alpha \beta \left( t \right) + \frac{\partial}{\partial t_i} \partial_{n+N(k+n)+i} F \left[ u \right]_{p_1,p_2}^\alpha \beta \left( t \right) = \partial_{n+k} F \left[ u \right]_{p_1,p_2}^\alpha \beta \left( t \right),$$

$k = 1, \ldots, N$, for all $t \in \Delta_n$.

**Example 1** Consider a medium motion whose displacement may be described by a scalar function $u(t,x)$, where $x = (x_1, x_2)$. For example, this function may represent the transverse displacement of a membrane. Suppose that the kinetic energy $T$ and the potential energy $V$ of the medium are given by $T \left( \frac{\dot{u}}{\rho} \right) = \frac{1}{2} \int \rho \left( \frac{\dot{u}}{2} \right)^2 dx$ and $V(u) = \frac{1}{2} \int k|\nabla u|^2 dx$, respectively, where $\rho(x)$ is the mass density and $k(x)$ is the stiffness, both assumed positive. Then, the classical action functional is $J(u) = \frac{1}{2} \int \left( \rho \left( \frac{\dot{u}}{2} \right)^2 - k|\nabla u|^2 \right) dx dt$. We shall illustrate what are the Euler–Lagrange equations when the Lagrangian density depends on generalized fractional derivatives. When we have the Lagrangian with the kinetic term depending on the operator $B_{\alpha}^\gamma$, then the fractional action functional has the form

$$J(u) = \frac{1}{2} \int_{\Delta_n} \left[ \rho \left( B_{\alpha}^\gamma u \right)^2 - k|\nabla u|^2 \right] dx dt. \quad (4)$$

The fractional Euler–Lagrange equation satisfied by an extremizer function of $J$ is

$$\rho A_{p_1,1} B_{p_2}^\gamma u - \nabla \cdot (k\nabla u) = 0.$$ 

If $\rho$ and $k$ are constants, then the equation $\rho A_{p_1,1} B_{p_2}^\gamma u - c^2 \Delta u = 0$, $c^2 = k/\rho$, can be called the generalized time-fractional wave equation. Now assume that the kinetic and the potential energy depend on operators $B_{\alpha}^\gamma$ and $B_{\beta}^\alpha$, $P = (P_{x_1}, P_{x_2})$, $\beta = (\beta_1, \beta_2)$, respectively. Then the action functional for the system has the form

$$J(u) = \frac{1}{2} \int_{\Delta_n} \left[ \rho \left( B_{\alpha}^\gamma u \right)^2 - k|\nabla_{B_{\beta}^\alpha} u|^2 \right] dx dt. \quad (5)$$

The fractional Euler–Lagrange equation satisfied by an extremizer of $J$ is

$$\rho A_{p_1,1} B_{p_2}^\gamma u - \sum_{i=1}^2 A_{p_{1,i}}^{\beta_i} (k B_{p_{2,i}}^{\beta_i} u) = 0.$$ 

If $\rho$ and $k$ are constants, then $A_{p_1,1} B_{p_2}^\gamma u - c^2 \left( \sum_{i=1}^2 A_{p_{1,i}}^{\beta_i} B_{p_{2,i}}^{\beta_i} u \right) = 0$ can be called the generalized space- and time-fractional wave equation.
4.2 Dirichlet’s Principle

One of the most important variational principles for a PDE is Dirichlet’s principle for the Laplace equation. We shall present its generalized fractional counterpart. In this section we assume that \( N = 1 \). We show that the solution of the generalized fractional boundary value problem

\[
\begin{align*}
\sum_{i=1}^{n} A_{P_i}^{\alpha_i} \left( B_{P_i}^{\alpha_i} u \right) &= 0 \quad \text{in} \quad \Delta_n, \\
u &= \psi \quad \text{on} \quad \partial\Delta_n,
\end{align*}
\]

(6)  (7)

can be characterized as a minimizer of the energy functional

\[
J[u] = \int_{\Delta_n} |\nabla_{B_p}^\alpha u|^2 \, dt
\]

(8)
on the set

\[ A = \left\{ u \in C^1(\Delta_n; \mathbb{R}) : B_{P_i}^{\alpha_i} u \in C^1(\Delta_n; \mathbb{R}), u|_{\partial\Delta_n} = \psi \right\}, \]

where \( \alpha \in (0, 1)^n \), \( P = (P_1, \ldots, P_n) \), \( P^* = (P_t^1, \ldots, P_t^n) \), and \( k_{1-\alpha_i} \) is a difference kernel such that \( k_{1-\alpha_i} \in L_1((0, b_i - a_i); \mathbb{R}) \), \( i = 1, \ldots, n \).

Remark 2 In the following we assume that both problems, (6)–(7) and minimization of (8) on the set \( A \), have solutions.

Theorem 6 (Generalized fractional Dirichlet’s principle) Let \( \alpha \in (0, 1)^n \) and \( u \in A \). Then \( u \) solves the boundary value problem (6)–(7) if and only if \( u \) satisfies

\[
J[u] = \min_{w \in A} J[w].
\]

(9)

Proof Multiply the equation (6) by any \( v \in C^1(\Delta_n; \mathbb{R}) \) such that \( v|_{\partial\Delta_n} = 0 \) and \( B_{P_i}^{\alpha_i} v \) is continuously differentiable on the rectangle \( \Delta_n \). Then, after integration,

\[
\int_{\Delta_n} v \sum_{i=1}^{n} A_{P_i}^{\alpha_i} \left( B_{P_i}^{\alpha_i} u \right) \, dt = 0.
\]

The generalized integration by parts formula in Theorem 3 yields

\[
\int_{\Delta_n} \nabla_{B_p}^\alpha u \cdot \nabla_{B_p}^\alpha v \, dt = 0,
\]

(10)
as there is no boundary term since \( v|_{\partial\Delta_n} = 0 \). By (10) and properties of the scalar product, one has

\[
\int_{\Delta_n} |\nabla_{B_p}^\alpha (u + v)|^2 \, dt = \int_{\Delta_n} |\nabla_{B_p}^\alpha u|^2 \, dt + 2 \int_{\Delta_n} \nabla_{B_p}^\alpha u \cdot \nabla_{B_p}^\alpha v \, dt + \int_{\Delta_n} |\nabla_{B_p}^\alpha v|^2 \, dt
\]

\[
\geq \int_{\Delta_n} |\nabla_{B_p}^\alpha u|^2 \, dt.
\]

Conversely, if \( u \) satisfies (9), then, by Theorem 4 \( u \) is a solution to (6)–(7).
Theorem 7 There exists at most one solution $u \in \mathcal{A}$ to problem (6)–(7).

**Proof** Let $u \in \mathcal{A}$ be a solution to problem (6)–(7). Assume that $\hat{u}$ is another solution to problem (6)–(7). Then $w = u - \hat{u} \neq 0$ and

$$\int_{\Delta_n} w \sum_{i=1}^{n} A_{\alpha_i}^{\alpha_i} (B_{\beta_i}^{\alpha_i} w) \, dt = 0.$$ 

By the generalized integration by parts formula (Theorem 3), and since $w|_{\partial \Delta_n} = 0$, one has

$$\int_{\Delta_n} \sum_{i=1}^{n} (B_{\beta_i}^{\alpha_i} w)^2 \, dt = \int_{\Delta_n} |\nabla_{B_{\beta_i}} w|^2 \, dt = 0.$$ 

Note that $|\nabla_{B_{\beta_i}} w|^2$ is a non-negative definite quantity. The volume integral of a non-negative definite quantity is equal to zero only in the case when this quantity is zero itself throughout the volume. Thus, $\nabla_{B_{\beta_i}} w = 0$. Since $w$ is continuously differentiable and $k_{1-\alpha_i} \in L_1 ((0, b_i - a_i); \mathbb{R})$, we have

$$\frac{\partial}{\partial t_i} w(t) \equiv 0, \quad i = 1, \ldots, n,$$

that is, $\nabla w = 0$. Because $w = 0$ on $\partial \Delta_n$, we deduce that $w = 0$. In other words, $u = \hat{u}$.

4.3 The Multidimensional Generalized Fractional Noether’s Theorem

Emmy Noether’s theorem [21] states that conservation laws in classical mechanics follow whenever the Lagrangian function is invariant under a one-parameter continuous group that transforms dependent and/or independent variables [32, 33]. In this section we prove a Noether-type theorem for variational problems that depend on continuous group that transforms dependent and/or independent variables [32, 33]. In this section we prove a Noether-type theorem for variational problems that depend on continuous group that transforms dependent and/or independent variables.

**Definition 11** Functional (11) is said to be invariant under an $\varepsilon$-parameter family of infinitesimal transformations

$$\hat{u}(t) = u(t) + \varepsilon \xi(t, u(t)) + o(\varepsilon) \quad (11)$$

with $\xi \in C^1 (\tilde{\Delta}_n; \mathbb{R}^N)$ such that $B_{\beta_i}^{\alpha_i} \xi$ and $K_{\beta_i}^{\alpha_i} \xi$ exist and are continuous on $\tilde{\Delta}_n$, $i \in \{1, \ldots, n\}$, if

$$\int_{\Delta_n} F \left( t, u(t), \nabla_{B_{\beta_i}} \hat{u}(t), \nabla_{K_{\beta_i}} \hat{u}(t) \right) \, dt = \int_{\Delta_n} F \left( t, \hat{u}(t), \nabla_{B_{\beta_i}} \hat{u}(t), \nabla_{K_{\beta_i}} \hat{u}(t) \right) \, dt$$

for any $\Delta_n^* \subseteq \Delta_n$.

The following result provides a necessary condition of invariance.

**Lemma 1** If functional (11) is invariant under an $\varepsilon$-parameter family of infinitesimal transformations (11), then

$$\sum_{k=1}^{N} \left( \partial_{n+k} F \{ u \}_{\beta_i}^{\alpha_i} (t) \cdot \xi_k(t, u) + \sum_{i=1}^{n} \left[ \partial_{N+k+1+i} F \{ u \}_{\beta_i}^{\alpha_i} (t) B_{\beta_i}^{\alpha_i} \xi_k(t, u) \right. 

+ \partial_{n+1+k(n+i)+1} F \{ u \}_{\beta_i}^{\alpha_i} (t) K_{\beta_i}^{\alpha_i} \xi_k(t, u) \right] \right) = 0. \quad (12)$$
Proof By Definition 11, invariance of functional (1) under transformations (10) is equivalent to
\[ F\left(t, u, \nabla_{B,t}^{\alpha} u, \nabla_{K,p}^{\beta} u\right) = F\left(t, \bar{u}, \nabla_{B,t}^{\alpha} \bar{u}, \nabla_{K,p}^{\beta} \bar{u}\right). \] (13)

Let us differentiate (13) with respect to \( \varepsilon \):
\[ \frac{d}{d\varepsilon} F\left(t, u(t) + \varepsilon \xi(t, u(t)) + o(\varepsilon), \nabla_{B,t}^{\alpha} (u(t) + \varepsilon \xi(t, u(t)) + o(\varepsilon))\right). \]
\[ \nabla_{K,p}^{\beta} (u(t) + \varepsilon \xi(t, u(t)) + o(\varepsilon)) = 0. \]

Putting \( \varepsilon = 0 \) and applying definitions and properties of partial generalized fractional operators, we obtain (12).

In order to state Noether’s theorem in a compact form, we follow [10]. More precisely, we introduce two bilinear operators.

**Definition 12** Let \( f, g \in C^{1}(\Delta_{n}; \mathbb{R}) \) for \( K_{P_i}^{\beta-\alpha^i} g \in C^{1}(\Delta_{n}; \mathbb{R}) \). We define the following bilinear operators:
\[ D_{P_i}^{P_1} [f, g] := fA_{P_i}^{\alpha}, g + gB_{P_i}^{\alpha}, f, \]
\[ I_{P_i}^{P_2} [f, g] := -fK_{P_i}^{\beta}, g + gK_{P_i}^{\beta}, f, \]
\( i = 1, \ldots, n. \)

Now we are ready to state our generalized fractional Noether’s theorem.

**Theorem 8 (Multidimensional generalized fractional Noether’s theorem)**
If functional (1) is invariant, in the sense of Definition 11, then
\[ \sum_{k=1}^{N} \sum_{i=1}^{n} \left[D_{P_i}^{P_1} [\xi_k(t, u(t))], \partial_{N+n+1} F \{ u \}^{\alpha, \beta}_{P_i, P_2} (t) \right] + I_{P_i}^{P_2} [\xi_k(t, u(t)), \partial_{n+N(k+n)+1} F \{ u \}^{\alpha, \beta}_{P_i, P_2} (t)] = 0 \] (14)
along any extremal of (1).

Proof By equations (3) we have
\[ \partial_{n+k} F \{ u \}^{\alpha, \beta}_{P_i, P_2} (t) = \sum_{i=1}^{n} \left[A_{P_i}^{\alpha}, \partial_{N+n+i} F \{ u \}^{\alpha, \beta}_{P_i, P_2} (t) \right] - K_{P_i}^{\beta}, \partial_{n+N(k+n)+1} F \{ u \}^{\alpha, \beta}_{P_i, P_2} (t), \quad k = 1, \ldots, N. \] (15)

Putting (15) into (12), we obtain that
\[ \sum_{k=1}^{N} \sum_{i=1}^{n} \left[\xi_k(t, u(t))A_{P_i}^{\alpha}, \partial_{N+n+i} F \{ u \}^{\alpha, \beta}_{P_i, P_2} (t) \right] - \xi_k(t, u(t))K_{P_i}^{\beta}, \partial_{n+N(k+n)+1} F \{ u \}^{\alpha, \beta}_{P_i, P_2} (t) + \partial_{N+n+i} F \{ u \}^{\alpha, \beta}_{P_i, P_2} (t)B_{P_i}^{\alpha}, \xi_k(t, u(t)) \]
\[ + \partial_{n+N(k+n)+1} F \{ u \}^{\alpha, \beta}_{P_i, P_2} (t)K_{P_i}^{\beta}, \xi_k(t, u(t)) \] = 0.
Finally, we arrive to (14) by Definition 12.

**Example 2** Let \( N = 1, \alpha, \beta \in (0,1)^n, \ c \in \mathbb{R} \) and \( P = (P_1, \ldots, P_n) \) with \( P_i = (a_i, t_i, b_i, p_i, q_i), \ i = 1, \ldots, n. \) Consider the \( \varepsilon \)-parameter family of infinitesimal transformations

\[
\bar{u}(t) = u(t) + \varepsilon c + o(\varepsilon)
\]

and the functional

\[
\mathcal{J}[u] = \int_{\Delta_n} F \left( t, \nabla_B^\alpha P u(t) \right) dt.
\]

Then, for any \( \Delta_n^* \subseteq \Delta_n, \) we have

\[
\int_{\Delta_n^*} F \left( t, \nabla_B^\alpha \bar{u}(t) \right) dt = \int_{\Delta_n^*} F \left( t, \nabla_B^\alpha u(t) \right) dt.
\]

Hence, \( \mathcal{J}[u] \) is invariant under transformations and Theorem 8 asserts that

\[
\sum_{i=1}^{n} D_{P_i}^\alpha \left[ c, \partial_{n+i} F \left( t, \nabla_B^\alpha u(t) \right) \right] = 0.
\]

**5 Conclusion**

Partial fractional integrals and derivatives can be defined in different ways and, consequently, in each case one must consider different variational problems. In this paper we unify and extend previous results of the multidimensional calculus of variations by considering more general operators that reduce to the standard fractional integrals and derivatives by an appropriate choice of kernels and \( p \)-sets. After proving generalized integration by parts formulas, we obtained Euler–Lagrange equations, a generalized fractional Dirichlet’s principle, and a fractional Noether’s theorem. As an example, we obtained a generalized space- and time-fractional wave equation.

This paper marks the born of the generalized multidimensional fractional calculus of variations. Much remains to be done. For example, if boundary conditions are not imposed at the initial problem, then Theorem 4 needs to be complemented with transversality conditions. Problems subject to constraints can also be considered.

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