Parametrisations of elements of spinor and orthogonal groups using exterior exponents

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Abstract

We present new parametrizations of elements of spinor and orthogonal groups of dimension 4 using Grassmann exterior algebra. Theory of spinor groups is an important tool in theoretical and mathematical physics namely in the Dirac equation for an electron.

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An exterior algebra, invented by G. Grassmann in the year 1844 [1], has many applications in different fields of mathematics and physics. Here we present a new application of Grassmann algebra to the theory of spinor and orthogonal groups.

Clifford algebras. Let $p, q, n$ be nonnegative integer numbers and $n = p+q$. And let $\mathcal{C}(p, q)$ be a real Clifford algebra [2] of the signature $(p, q)$ with generators $e^1, \ldots, e^n$ such that
\[ e^a e^b + e^b e^a = 2\eta^{ab} e, \quad a, b = 1, \ldots, n, \]
where $e$ is the identity element of Clifford algebra and $\eta^{ab}$ are elements of the diagonal matrix of dimension $n$

$$\eta = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$$

with $p$ pieces of 1 and $q$ pieces of $-1$ on the diagonal. The Clifford algebra $\mathcal{Cl}(p, q)$ can be considered as $2^n$-dimensional vector space with basis elements

$$e, e^a, e^{a_1a_2}, \ldots, e^{a_1\ldots a_{n-1}}, e^{12\ldots n}, \quad 0 \leq a_1 < \ldots < a_k \leq n \quad (1)$$

numbered by ordered multi-indices of lengths from 0 to $n$. Any element of Clifford algebra $\mathcal{Cl}(p, q)$ can be written in the form of decomposition w.r.t. the basis (1)

$$U = ue + u_a e^a + \sum_{a_1 < a_2} u_{a_1a_2} e^{a_1a_2} + \ldots + u_{1\ldots n} e^{1\ldots n} \quad (2)$$

with real coefficients $u, u_a, u_{a_1a_2}, \ldots, u_{1\ldots n}$. Elements of the form

$$U = \sum_{a_1 < \ldots < a_k} u_{a_1\ldots a_k} e^{a_1\ldots a_k}$$

are called elements of rank $k$. Denote by $\mathcal{Cl}_k(p, q)$ the subspace of rank $k$ elements. We have

$$\mathcal{Cl}(p, q) = \mathcal{Cl}_0(p, q) \oplus \ldots \oplus \mathcal{Cl}_n(p, q).$$

An element $U \in \mathcal{Cl}(p, q)$ is called even (odd) if this element is a sum of elements of even (odd) ranks. Hence

$$\mathcal{Cl}_{\text{Even}}(p, q) = \mathcal{Cl}_0(p, q) \oplus \mathcal{Cl}_2(p, q) \oplus \ldots,$$

$$\mathcal{Cl}_{\text{Odd}}(p, q) = \mathcal{Cl}_1(p, q) \oplus \mathcal{Cl}_3(p, q) \oplus \ldots,$$

$$\mathcal{Cl}(p, q) = \mathcal{Cl}_{\text{Even}}(p, q) \oplus \mathcal{Cl}_{\text{Odd}}(p, q).$$

**Reverse operation.** Let us define a linear reverse operation $\sim: \mathcal{Cl}(p, q) \to \mathcal{Cl}(p, q)$ with the aid of the following rules:

$$e^\sim = e, \quad (e^a)^\sim = e^a, \quad (UV)^\sim = V^\sim U^\sim, \quad \forall U, V \in \mathcal{Cl}(p, q).$$

In particular,

$$(e^{a_1} \ldots e^{a_k})^\sim = e^{a_k} \ldots e^{a_1}.$$
Spinor groups. Let $n = p + q \leq 5$. Consider the following set of even elements of the Clifford algebra $\mathcal{C}(p, q)$:

$$\text{Spin}^+(p, q) = \{ S \in \mathcal{C}_{\text{Even}}(p, q) : S^* S = e \}.$$ 

This set is closed w.r.t. Clifford product and contains the identity element $e$. Elements of this set are invertible. Therefore $\text{Spin}^+(p, q)$ can be considered as a group (Lie group) w.r.t. the Clifford product. This group is called spinor group.

The set of second rank elements $\mathcal{C}_2(p, q)$ with the commutator $[A, B] = AB - BA$ is the Lie algebra of the Lie group $\text{Spin}^+(p, q)$ [3].

Let us define an exponent of Clifford algebra elements $\exp : \mathcal{C}(p, q) \rightarrow \mathcal{C}(p, q)$ by the formula

$$\exp A = e + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \ldots.$$ 

Spinor groups in cases $p + q = 4$. The sign $\simeq$ denotes group isomorphisms.

It is known [3] that

- $\text{Spin}^+(4, 0) \simeq \text{Spin}^+(0, 4) \simeq \text{Spin}(4)$.
- $\text{Spin}^+(1, 3) \simeq \text{Spin}^+(3, 1)$.
- $\forall S \in \text{Spin}(4)$ there exists $B \in \mathcal{C}_2(4, 0)$ such that $S = \exp B$.
- $\forall S \in \text{Spin}^+(1, 3)$ there exists $B \in \mathcal{C}_2(1, 3)$ such that $S = \exp B$, or $S = -\exp B$. Note that the set of exponents of rank 2 elements do not form a group.
- There exist elements $S \in \text{Spin}^+(2, 2)$ such that these elements can’t be represented in the form $\pm \exp B$, where $B \in \mathcal{C}_2(2, 2)$.

Exterior (Grassmann) multiplication of Clifford algebra elements.

Let us define the associative and distributive operation of exterior multiplication of Clifford algebra elements (denoted by $\wedge$)

$$e^{a_1} \wedge e^{a_2} \wedge \ldots \wedge e^{a_k} = e^{[a_1 e^{a_2} \ldots e^{a_k}]}.$$
where square brackets denote the operation of alternation of indices. The Clifford algebra $\mathcal{C}(p, q)$, considered with the exterior product, can be identified with the Grassmann algebra of dimension $n$.

**Exterior exponent.** Consider the exterior exponent $\widehat{\exp} : \mathcal{C}(p, q) \to \mathcal{C}(p, q)$

$$\widehat{\exp}(B) = e + B + \frac{1}{2!} B \wedge B + \frac{1}{3!} B \wedge B \wedge B + \ldots,$$

where $B \in \mathcal{C}(p, q)$. We interested in exterior exponent of second rank elements. In this case in the right hand part of (3) there are finite number of nonzero summands. In particular, for the case $n = 4$ there are only three summands

$$\widehat{\exp}(B) = e + B + \frac{1}{2} B \wedge B,$$

where $B \in \mathcal{C}_2(p, q)$, $p + q = 4$. We see that

$$(\widehat{\exp}(B))^{\sim} = \widehat{\exp}(-B) = e - B + \frac{1}{2} B \wedge B.$$ 

It is not hard to prove that

$$(\widehat{\exp}(B))^{\sim} \widehat{\exp}(B) = \lambda e,$$

where $\lambda = \lambda(B)$ is a scalar that depends on coefficients of the element $B$.

**Main theorem.** If $U \in \mathcal{C}(p, q)$ is written in the form (2), then we denote $\text{Tr} U = u$. Also denote $\ell = e^1 e^2 e^3 e^4$ and $\epsilon = \text{Tr}(\ell^2)$.

**Theorem 1** Let $n = p + q = 4$. Any element $S$ of the group $\text{Spin}_+(p, q)$ can be represented in one of two following forms:

- If $\text{Tr} S \neq 0$, then there exists $B \in \mathcal{C}_2(p, q)$ such that $\lambda = \lambda(B) > 0$ and

  $$S = \pm \frac{1}{\sqrt{\lambda}} \widehat{\exp} B.$$  

  The sign (plus or minus) at the right hand part is equal to the sign of the number $\text{Tr} S$.

- If $\text{Tr} S = 0$, then there exists $B \in \mathcal{C}_2(p, q)$ such that $B \wedge B = 0$, $\epsilon(1 + \beta) \geq 0$, where $\beta = \text{Tr}(B^2)$ and

  $$S = B \pm \ell \sqrt{\epsilon(1 + \beta)}.$$  

  The sign (plus or minus) at the right hand part is equal to the sign of the number $\text{Tr}(\ell^{-1} S)$. 

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Proof. Let $S \in \text{Spin}_+(p, q)$ be such that $\text{Tr} \, S = \alpha \neq 0$. We write $S$ and $S^\sim$ in the form

$$S = \alpha e + U + F, \quad S^\sim = \alpha e - U + F,$$

where $U \in \mathcal{C}_2(p, q)$, $F \in \mathcal{C}_4(p, q)$. For $n = 4$ it follows that

$$U^2 = U \wedge U + \gamma e, \quad \alpha^2 e - F^2 - \gamma^2 e \in \mathcal{C}_0(p, q), \quad 2\alpha F - U \wedge U \in \mathcal{C}_4(p, q).$$

The identity

$$S^\sim S = (\alpha^2 e - F^2 - \gamma^2 e) + (2\alpha F - U \wedge U) = e$$

gives us

$$2\alpha F - U \wedge U = 0 \Rightarrow F = \frac{1}{2\alpha} U \wedge U.$$

That means

$$S = \alpha e + U + \frac{1}{2\alpha} U \wedge U = \alpha \exp\left(\frac{1}{\alpha} U\right).$$

Denoting

$$B = \frac{1}{\alpha} U, \quad \alpha = \pm \frac{1}{\sqrt{\lambda}},$$

we get

$$S = \pm \frac{1}{\sqrt{\lambda}} \exp B,$$

where

$$\lambda e = \exp(B) \exp(-B). \quad (6)$$

It is easy to prove that if an element $S \in \text{Spin}_+(p, q)$ is such that $\text{Tr} \, S = 0$, then $S$ can be represented in the form (5). This completes the proof. ■

If $S \in \text{Spin}_+(p, q)$ has the form (4), then we say that $S$ is given with the aid of semi-polynomial parametrisation. That means, coefficients of $S$ in the basis (1) are polynomials of second degree of coefficients $b_{ij}$ multiplied on one and the same factor $1/\sqrt{\lambda}$, where $\lambda = \lambda(B)$ is the polynomial of second degree of coefficients $b_{ij}$.

If $S \in \text{Spin}_+(p, q)$ has the form (5), then we say that $S$ is given with the aid of adjoint semi-polynomial parametrisation.

Let us write down explicit form of elements $S \in \text{Spin}_+(p, q)$, $p + q = 4$ using real coefficients of $b_{ij} \in B \in \mathcal{C}_2(p, q)$

$$B = b_{12} \, e_{12} + b_{13} \, e_{13} + b_{14} \, e_{14} + b_{23} \, e_{23} + b_{24} \, e_{24} + b_{34} \, e_{34}.$$
We have
\[ \overline{\exp}(B) = e + b_{12}e^{12} + b_{13}e^{13} + b_{14}e^{14} + b_{23}e^{23} + b_{24}e^{24} + b_{34}e^{34} + (b_{14}b_{23} - b_{13}b_{24} + b_{12}b_{34})e^{1234} \]

Expressions for the scalar \( \lambda = \lambda(B) \) that satisfy (6) and for \( \epsilon(1 + \beta) \) are depend on a signature \((p, q)\).

For \((p, q) = (0, 4), (4, 0)\)
\[
\lambda = 1 + b_{12}^2 + b_{13}^2 + b_{14}^2 + b_{23}^2 + b_{14}^2b_{23}^2 - 2b_{13}b_{14}b_{23}b_{24} + b_{24}^2 + b_{13}b_{24}^2 + 2b_{12}b_{14}b_{23}b_{34} - 2b_{13}b_{14}b_{24}b_{34} + b_{34}^2 + b_{12}^2b_{34}^2, \\
\epsilon(1 + \beta) = 1 - b_{12}^2 - b_{13}^2 - b_{14}^2 - b_{23}^2 - b_{24}^2 - b_{34}^2.
\]

For \((p, q) = (1, 3)\)
\[
\lambda = 1 - b_{12}^2 - b_{13}^2 - b_{14}^2 + b_{23}^2 - b_{14}^2b_{23}^2 + 2b_{13}b_{14}b_{23}b_{24} + b_{24}^2 - b_{13}b_{24}^2 - 2b_{12}b_{14}b_{23}b_{34} + 2b_{12}b_{13}b_{24}b_{34} + b_{34}^2 - b_{12}^2b_{34}^2, \\
\epsilon(1 + \beta) = -1 - b_{12}^2 - b_{13}^2 - b_{14}^2 + b_{23}^2 + b_{24}^2 + b_{34}^2.
\]

For \((p, q) = (2, 2)\)
\[
\lambda = 1 + b_{12}^2 - b_{13}^2 - b_{14}^2 - b_{23}^2 + b_{14}^2b_{23}^2 - 2b_{13}b_{14}b_{23}b_{24} - b_{24}^2 + b_{13}b_{24}^2 + 2b_{12}b_{14}b_{23}b_{34} - 2b_{12}b_{13}b_{24}b_{34} + b_{34}^2 + b_{12}^2b_{34}^2, \\
\epsilon(1 + \beta) = 1 - b_{12}^2 + b_{13}^2 + b_{14}^2 + b_{23}^2 + b_{24}^2 - b_{34}^2.
\]

For \((p, q) = (3, 1)\)
\[
\lambda = 1 + b_{12}^2 + b_{13}^2 - b_{14}^2 + b_{23}^2 - b_{14}^2b_{23}^2 + 2b_{13}b_{14}b_{23}b_{24} - b_{24}^2 - b_{13}b_{24}^2 - 2b_{12}b_{14}b_{23}b_{34} + 2b_{12}b_{13}b_{24}b_{34} - b_{34}^2 - b_{12}^2b_{34}^2, \\
\epsilon(1 + \beta) = -1 + b_{12}^2 + b_{13}^2 - b_{14}^2 + b_{23}^2 - b_{24}^2 - b_{34}^2.
\]

With the aid of these formulas we get the general form (4), (5) of elements \( S \in \text{Spin}_+(p, q) \).

The proof of the following known theorem is straightforward.

**Theorem 2** Let \( n = p + q = 2, 3 \). Any element \( S \in \text{Spin}_+(p, q) \) can be represented in the following form
\[ S = \pm e\sqrt{1 + \beta} + B, \tag{7} \]
where \( B \in \mathcal{O}_2(p, q) \) is such that \( \beta = \text{Tr}(B^2) \geq -1 \). The sign (plus or minus) at the right hand part of (7) is equal to the sign of the number \( \text{Tr} S \). ■
Orthogonal groups. Consider the Lie groups of special orthogonal matrices of dimension $n = p + q$

$$SO(p, q) = \{ P \in GL(n, \mathbb{R}) : P^T \eta P = \eta, \ \det P = 1 \},$$

where $P^T$ is the transposed matrix. For $pq \neq 0$ the groups $SO(p, q)$ have two disconnected components \[^{[1]}\]. The component of the group $SO(p, q)$ that contains the identity matrix is a subgroup $SO_+(p, q)$. In cases $pq = 0$ we have $SO(0, n) = SO(n, 0) = SO(n)$ and this group has only one component, i.e. $SO_+(n) = SO(n)$.

For $p + q = 4$ the following propositions are valid \[^{[3]}\]:

- $SO_+(4, 0) \simeq SO_+(0, 4) \simeq SO(4)$.
- $SO_+(1, 3) \simeq SO_+(3, 1)$.
- $\forall P \in SO(4)$ there exists an anti-Hermitian matrix of fourth order ($A^T = -A$) such that $P = \exp A$.
- $\forall P \in SO_+(1, 3)$ there exists a matrix $A$ of fourth order such that $\eta A^T \eta = -A$, ($\eta = diag(1, -1, -1, -1)$) and $P = \exp A$.
- There exist matrices from $SO_+(2, 2)$ that can’t be represented in the form $\pm \exp A$, where $\eta A^T \eta = -A$, ($\eta = diag(1, 1, -1, -1)$).

Connection between spinor and orthogonal groups. It is known \[^{[3]}\] that the spinor group $Spin_+(p, q)$ double cover the orthogonal group $SO_+(p, q)$. This connection can be expressed by the formula

$$S^a e^a S = p_b^a e^b. \tag{8}$$

In this formula a pair of elements $\pm S$ of spinor group are connected with the matrix $P = \|p_b^a\|$ from the orthogonal group.

Consider in more details the case ($p, q = (1, 3)$, which is important for physics. Let us take expressions for $S \in Spin_+(1, 3)$ from Theorem 1 and, using \[^{[3]}\], calculate corresponding elements of the matrix $P = \|p_b^a\|$. Then we get the following formulas.
If $S$ has the form (4) and $\lambda = \lambda(B) > 0$, then elements of the matrix $T = \lambda P$ have the form

$$
\begin{align*}
\tau_1^1 &= 1 + b_{12}^2 + b_{13}^2 + b_{14}^2 + b_{23}^2 + b_{14}^2 b_{23}^2 - 2 b_{13} b_{14} b_{23} b_{24} + b_{24}^2 + b_{13}^2 b_{24}^2 + 2 b_{12} b_{14} b_{23} b_{34} - 2 b_{12} b_{13} b_{24} b_{34} + b_{34}^2 + b_{12}^2 b_{34}^2, \\
\tau_2^2 &= 2 b_{12} + 2 b_{13} b_{23} + 2 b_{14} b_{24} + 2 b_{14} b_{23} b_{34} - 2 b_{13} b_{24} b_{34} + 2 b_{12} b_{34}^2, \\
\tau_3^1 &= 2 b_{13} - 2 b_{12} b_{23} - 2 b_{14} b_{24} + 2 b_{13} b_{24}^2 + 2 b_{14} b_{34} - 2 b_{12} b_{24} b_{34}, \\
\tau_4^1 &= 2 b_{14} + 2 b_{14} b_{23}^2 - 2 b_{12} b_{24} - 2 b_{13} b_{23} b_{24} - 2 b_{13} b_{34} + 2 b_{12} b_{23} b_{34}, \\
\tau_2^2 &= 2 b_{12} - 2 b_{13} b_{23} - 2 b_{14} b_{24} + 2 b_{14} b_{23} b_{34} - 2 b_{13} b_{24} b_{34} + 2 b_{12} b_{34}^2, \\
\tau_2^3 &= 2 b_{12} b_{13} - 2 b_{23} + 2 b_{14}^2 b_{23} - 2 b_{13} b_{14} b_{24} + 2 b_{12} b_{14} b_{34} - 2 b_{24} b_{34}, \\
\tau_4^3 &= 2 b_{12} b_{14} - 2 b_{13} b_{14} b_{23} - 2 b_{24} + 2 b_{13}^2 b_{24} - 2 b_{12} b_{13} b_{34} + 2 b_{23} b_{34}, \\
\tau_3^2 &= 2 b_{13} + 2 b_{12} b_{23} - 2 b_{14} b_{24} + 2 b_{13} b_{24}^2 - 2 b_{14} b_{34} - 2 b_{12} b_{24} b_{34}, \\
\tau_4^3 &= 2 b_{12} b_{13} + 2 b_{23} - 2 b_{14}^2 b_{23} + 2 b_{13} b_{14} b_{24} - 2 b_{12} b_{14} b_{34} - 2 b_{24} b_{34}, \\
\tau_3^3 &= 1 - b_{12}^2 + b_{13}^2 - b_{14}^2 - b_{23}^2 + b_{14}^2 b_{23}^2 - 2 b_{13} b_{14} b_{23} b_{24} + b_{24}^2 + b_{13}^2 b_{24}^2 + 2 b_{12} b_{14} b_{23} b_{34} - 2 b_{12} b_{13} b_{24} b_{34} - b_{34}^2 + b_{12}^2 b_{34}^2, \\
\tau_4^4 &= 2 b_{13} b_{14} + 2 b_{12} b_{14} b_{23} - 2 b_{12} b_{13} b_{24} - 2 b_{23} b_{24} - 2 b_{34} + 2 b_{12}^2 b_{34}, \\
\tau_4^3 &= 2 b_{14} + 2 b_{14} b_{23}^2 + 2 b_{12} b_{24} - 2 b_{13} b_{23} b_{24} + 2 b_{13} b_{34} + 2 b_{12} b_{23} b_{34}, \\
\tau_4^4 &= 2 b_{12} b_{14} + 2 b_{13} b_{14} b_{23} + 2 b_{24} - 2 b_{13} b_{24} b_{24} + 2 b_{12} b_{13} b_{34} + 2 b_{23} b_{34}, \\
\tau_2^4 &= 2 b_{13} b_{14} - 2 b_{12} b_{14} b_{23} + 2 b_{12} b_{13} b_{24} - 2 b_{23} b_{24} + 2 b_{34} - 2 b_{12}^2 b_{34}, \\
\tau_4^4 &= 1 - b_{12}^2 - b_{13}^2 + b_{14}^2 + b_{23}^2 + b_{14}^2 b_{23}^2 - 2 b_{13} b_{14} b_{23} b_{24} - b_{24}^2 + b_{13}^2 b_{24}^2 + 2 b_{12} b_{14} b_{23} b_{34} - 2 b_{12} b_{13} b_{24} b_{34} - b_{34}^2 + b_{12}^2 b_{34}^2.
\end{align*}
$$

If $S$ has the form (5), $\epsilon(1 + \beta) \geq 0$ and $B \wedge B = 0$, then elements of the
matrix $P$ have the form

\[
\begin{align*}
p_1^1 &= -1 + 2b_{23}^2 + 2b_{24}^2 + 2b_{34}^2, \\
p_1^2 &= 2b_{13}b_{23} + 2b_{14}b_{24} + 2b_{34}\sqrt{\rho}, \\
p_1^3 &= -2b_{13}b_{23} + 2b_{14}b_{34} - 2b_{24}\sqrt{\rho}, \\
p_1^4 &= -2b_{12}b_{24} - 2b_{13}b_{34} + 2b_{23}\sqrt{\rho}, \\
p_2^1 &= -2b_{13}b_{23} - 2b_{14}b_{24} + 2b_{34}\sqrt{\rho}, \\
p_2^2 &= -1 - 2b_{13}^2 - 2b_{14}^2 + 2b_{34}^2, \\
p_2^3 &= 2b_{12}b_{13} - 2b_{24}b_{34} + 2b_{14}\sqrt{\rho}, \\
p_2^4 &= 2b_{12}b_{14} + 2b_{23}b_{34} - 2b_{13}\sqrt{\rho}, \\
p_3^1 &= 2b_{12}b_{14} + 2b_{23}b_{34} - 2b_{24}\sqrt{\rho}, \\
p_3^2 &= -1 - 2b_{12}^2 - 2b_{14}^2 + 2b_{24}^2, \\
p_3^3 &= 2b_{13}b_{14} - 2b_{23}b_{24} + 2b_{12}\sqrt{\rho}, \\
p_3^4 &= 2b_{13}b_{14} - 2b_{23}b_{24} + 2b_{12}\sqrt{\rho}, \\
p_4^1 &= -1 - 2b_{12}^2 - 2b_{13}^2 + 2b_{23}^2, \\
p_4^2 &= -1 - 2b_{12}^2 - 2b_{13}^2 - b_{14}^2 + b_{23}^2 + b_{24}^2 + b_{34}^2 \geq 0.
\end{align*}
\]

where

\[\rho = \epsilon(1 + \beta) = -1 - b_{12}^2 - b_{13}^2 - b_{14}^2 + b_{23}^2 + b_{24}^2 + b_{34}^2 \geq 0.\]

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