GLOBAL ERADICATION FOR SPATIALLY STRUCTURED POPULATIONS BY REGIONAL CONTROL

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Abstract. This paper concerns problems for the eradication of a population by acting on a subregion $\omega$. The dynamics is described by a general reaction-diffusion system, including one or more populations, subject to a vital dynamics with either local logistic or nonlocal logistic terms. For the one population case, a necessary condition and a sufficient condition for eradicability (zero-stabilizability) are obtained, in terms of the sign of the principal eigenvalue of a suitable elliptic operator acting on the domain $\Omega \setminus \omega$. A feedback harvesting-like control with a large constant harvesting rate realizes eradication of the population. The problem of eradication is then reformulated in a more convenient way, by taking into account the total cost of the damages produced by a pest population and the costs related to the choice of the relevant subregion, and approximated by a regional optimal control problem with a finite horizon. A conceptual iterative algorithm is formulated for the simulation of the proposed optimal control problem. Numerical tests are given to illustrate the effectiveness of the results. Relevant regional control problems for two populations reaction-diffusion models, such as prey-predator system, and an SIR epidemic system with spatial structure and local/nonlocal force of infection, have been analyzed too.

1. Introduction and setting of the problems. Consider a biological population that is free to move in a habitat $\Omega \subset \mathbb{R}^2$ and is subject to a control acting in a subset $\omega$ compactly embedded in $\Omega$. If there is no population flux through the boundary of

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the habitat \( \partial \Omega \) then the population dynamics may be described by models having the following general structure

\[
\begin{align*}
\partial_t y(x, t) - d \Delta y(x, t) &= \eta(x) y(x, t) - y(x, t) F((By(\cdot, t))(x)) + \chi_\omega(x) u(x, t), \quad (x, t) \in Q \\
\partial_n y(x, t) &= 0, \quad (x, t) \in \Sigma \\
y(x, 0) &= y_0(x), \quad x \in \Omega
\end{align*}
\]

(1)

for the the population density \( y(x, t) \) at position \( x \), and time \( t \). Here \( \Omega \) is a bounded domain (open and connected) with a sufficiently smooth boundary \( \partial \Omega \), \( \bar{T} \in (0, +\infty], ~ Q = \Omega \times (0, \bar{T}), \Sigma = \partial \Omega \times (0, \bar{T}) \).

\( \eta(x) \) is the growth rate at position \( x \): \( y(x, t) F((By(\cdot, t))(x)) \) is a logistic term, and \( B \in L(L^2(\Omega)); d \in (0, +\infty) \) is the diffusion coefficient. The homogeneous Neumann boundary condition describes the no flux of the population through the boundary of the habitat (the habitat is isolated). \( y_0(x) \) denotes the initial population density at position \( x \). If not stated otherwise \( \bar{T} = +\infty \) is taken.

The control \( u \) acts in the open subset \( \omega \) which has a sufficiently smooth boundary; \( \chi_\omega \) is the characteristic function of \( \omega \).

Here are the hypotheses we are going to use in what follows:

(H1) \( \eta, y_0 \in L^\infty(\Omega), y_0(x) \geq 0 \) a.e. in \( \Omega \);

(H2) \( B \in L(L^2(\Omega)) \cap L(L^\infty(\Omega)), (Bz)(x) \geq 0 \) a.e. in \( \Omega \), for any \( z \in L^2(\Omega) \) such that \( z(x) \geq 0 \) a.e. in \( \Omega \);

(H3) \( F : [0, +\infty) \to [0, +\infty) \) is a continuously differentiable function, \( F(0) = 0, ~ F'(s) \geq 0 \) for any \( s \geq 0 \).

Two cases are of particular interest.

Case 1. If we take \((Bz)(x) = \rho(x)z(x)\) for \( z \in L^2(\Omega), \) where \( \rho \in L^\infty(\Omega), \rho(x) \geq 0 \) a.e. in \( \Omega \), and \( F(s) = s^\theta \) for \( s \in [0, +\infty), \) where \( \theta \in (1, +\infty) \) is a constant, then (1) describes a Fisher-like pest population dynamics with a local logistic term.

Case 2. If we take \((Bz)(x) = \int_\Omega k(x, x')z(x')dx'\) for \( z \in L^2(\Omega), \) where \( k \in L^\infty(\Omega \times \Omega), k(x, x') \geq 0 \) a.e. in \( \Omega \times \Omega \), and \( F(s) = s^\theta \) for \( s \in [0, +\infty), \) where \( \theta \in (1, +\infty) \) is a constant, then (1) describes a population dynamics with nonlocal logistic term.

For other population models with nonlocal terms we refer to [3]–[7], [23].

In either cases, for \( F \equiv 0 \), we get a linear model.

When dealing with a pest population, the problem of controlling the population for its eradication becomes relevant; for \( \bar{T} = +\infty \), we are interested to know if there exists a control \( u \) acting in \( \omega \) such that the solution \( y^u \) to (1) satisfies \( y^u(x, t) \geq 0 \) in \( Q \) and \( \lim_{t \to +\infty} y^u(\cdot, t) = 0 \) in a certain functional space.

We shall prove firstly some basic properties of the solution to (1). Next we continue with an investigation of the zero-stabilizability (eradicability) for (1).

**Definition 1.1.** The population is internally zero-stabilizable (eradicable) if for any \( y_0 \) satisfying (H1) there exists a control \( u \in L^\infty_{\text{loc}}(\bar{\omega} \times [0, +\infty)) \) such that the solution \( y^u \) to (1) satisfies

\[
0 \leq y^u(x, t) \quad \text{a.e. in } Q, \quad (2)
\]

and

\[
\lim_{t \to +\infty} y^u(\cdot, t) = 0 \quad \text{in } L^\infty(\Omega). \quad (3)
\]
If for any $y_0$ satisfying (H1) such a control $u$ exists, we say that (1) is internally zero-stabilizable (eradicable).

We are dealing here with zero-stabilizability (eradicability) and zero-stabilization (eradication) with state constraints (the state of the system must be nonnegative).

We shall prove that the zero-stabilizability is deeply related to the magnitude of the principal eigenvalue $\lambda_1(\omega)$ for
\begin{equation}
\begin{cases}
-d\Delta \psi(x) - \eta(x)\psi(x) = \lambda \psi(x), & x \in \Omega \setminus \overline{\omega} \\
\psi(x) = 0, & x \in \partial \omega \\
\partial \nu \psi(x) = 0, & x \in \partial \Omega,
\end{cases}
\end{equation}
and to the magnitude of the principal eigenvalue $\lambda_{1,\gamma}(\omega)$ for
\begin{equation}
\begin{cases}
-d\Delta \psi(x) - \eta(x)\psi(x) + \gamma \chi_\omega(x)\psi(x) = \lambda \psi(x), & x \in \Omega \\
\partial \nu \psi(x) = 0, & x \in \partial \Omega
\end{cases}
\end{equation}
(where $\gamma \in [0, +\infty)$). The existence and the basic properties of these principal eigenvalues follow via Rayleigh’s principle. We will see that the mapping $\gamma \mapsto \lambda_{1,\gamma}(\omega)$ is strictly increasing and that
\[\lim_{\gamma \to +\infty} \lambda_{1,\gamma}(\omega) = \lambda_1(\omega).\]

As far as the zero-stabilizability is concerned, the following result shall be proved in Section 3:

**Theorem 1.2.** If (1) is internally zero-stabilizable, then $\lambda_1(\omega) \geq 0$.

Conversely, if $\lambda_1(\omega) > 0$, then (1) is internally zero-stabilizable and for sufficiently large $\gamma > 0$ the feedback control $u := -\gamma y$ realizes (2) and (3) for any $y_0$ satisfying (H1).

**Remark 1.** In the particular case $F \equiv 0$, the zero-stabilizability is actually equivalent to $\lambda_1(\omega) > 0$.

Moreover, if we consider the solution $y$ to (1) corresponding to a stabilizing feedback control $u := -\gamma y$, i.e., $y$ satisfies
\begin{equation}
\begin{cases}
\partial_y y(x, t) - d\Delta y(x, t) = \eta(x) y(x, t) - y(x, t)F((By(\cdot, t))(x)) \\
-\gamma \chi_\omega(x) y(x, t), & (x, t) \in Q \\
\partial_y y(x, t) = 0, & (x, t) \in \Sigma \\
y(x, 0) = y_0(x), & x \in \Omega
\end{cases}
\end{equation}
(note that using Banach’s fixed point result we may conclude that problem (6) has a unique and nonnegative solution; see Section 2), then
\[\lim_{t \to +\infty} y(\cdot, t) = 0 \quad \text{in } L^\infty(\Omega),\]
at the rate of $\exp\{-\lambda_1(\omega)t\}$. Here $\gamma > 0$ is a constant affordable harvesting effort.

Since $y$, the solution to (6), behaves as $\exp\{-\lambda_1(\omega)t\}$ as $t \to +\infty$, it is important to find a subset $\omega$ such that $\lambda_1(\omega)$ to be as large as possible in order to diminish as much as possible the damages produced by the pest population.

From a measure theoretic point of view the geometry of $\omega$ can be described by its Minkowski functionals. In two dimensions there are three such functionals and they are proportional to more commonly known quantities such as the area, the perimeter and the Euler-Poincaré characteristic (see [24], [25]-p. 30). In the present paper we shall refer to two such functionals, the area and the perimeter.
Indeed, in the simplest case, for the cost to be paid in order to act in $\omega$ we may assume the form $\alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega)$ ($\alpha$ and $\beta$ are given positive constants). We may interpret this cost as due to the installation of harvesting devices; usually it depends upon the number of installed devices, which, for a uniform distribution, is proportional to the area of $\omega$, and on costs due to the spatial spread of the installation sites, which are proportional to the elongation of $\omega$, i.e. to the length of $\partial \omega$. In synthesis a reasonable goal is to

\[
\text{(P0) Minimize} \left\{ \frac{\zeta}{\lambda_1(\omega)} + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega) \right\},
\]

subject to $\omega$. Here $\zeta$ is a positive constant depending on $y_0$.

An alternative formulation of the above minimization problem follows. For a given constant harvesting effort $\gamma > 0$ such that $\lambda_1(\omega) \gamma > 0$ we consider $y$, the solution corresponding to the feedback control $u := -\gamma y$, i.e. $y$ satisfies (6). Since the damages produced by the pest population are proportional to the quantity

\[
\int_0^\infty \int_\Omega y(x,t) dx \ dt,
\]

while the cost related to the choice of the subregion $\omega$ is $\alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega)$, we may require to solve the following minimization problem

\[
\text{(P) Minimize} \left\{ \theta \int_0^\infty \int_\Omega \tilde{y}(x,t) dx \ dt + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega) \right\},
\]

subject to $\omega$, where $\theta$ is a positive constant.

The relationship between problems (P0) and (P) shall be discussed later. In what follows we shall however focus on problem (P).

Problems (P0) and (P) may be treated as shape optimization problems. A convenient tool for handling the shape of $\omega \subset \mathbb{R}^2$ is the implicit representation interface according to which $\partial \omega$ is the zero-isocontour of some function $\varphi : \overline{\Omega} \to \mathbb{R}$ called its implicit function as $\omega = \{ x \in \Omega ; \varphi(x) > 0 \}$, while $\partial \omega = \{ x \in \overline{\Omega} ; \varphi(x) = 0 \}$ (the level set method).

If $\varphi$ is an implicit function of $\omega$, and taking into account that

\[
\text{length}(\partial \omega) = \int_\Omega \delta(\varphi(x)) |\nabla \varphi(x)| dx
\]

(De Giorgi’s formula), we may rewrite problem (P) as

\[
\text{(P) Minimize} \left\{ \theta \int_0^\infty \int_\Omega \tilde{y}^\varphi(x,t) dx \ dt + \alpha \int_\Omega H(\varphi(x)) dx + \beta \int_\Omega \delta(\varphi(x)) |\nabla \varphi(x)| dx \right\},
\]

subject to $\varphi$, where $\tilde{y}^\varphi$ is the solution to

\[
\begin{cases}
\partial_t y(x,t) - d \Delta y(x,t) = \eta(x)y(x,t) - y(x,t)F((B y(\cdot,t))(x)) - \gamma H(\varphi(x)) y(x,t), & (x,t) \in Q \\
\partial_\nu y(x,t) = 0, & (x,t) \in \Sigma \\
y(x,0) = y_0(x), & x \in \Omega.
\end{cases}
\]

Here $H$ is the Heaviside function and $\delta(z)$ is the Dirac mass at $z$.

From now on we will not distinguish between the subregion $\omega$ and its implicit function $\varphi$. 
An extension of Problem (1) to reaction-diffusion systems modelling the interaction of two populations, such as a prey-predator model, and an SIR epidemic system with local/nonlocal force of infection, will be investigated too. The problem of eradication of a specialized predator (pest) population or an infection, respectively, via a regional control, will be treated following the same protocol.

Let us notice that the present paper concerns harvested population dynamics with constant harvesting effort. There is however a huge literature devoted to optimal harvesting problems. We mention only a few of the most recent papers and monographs: [1], [2], [8], [13], [14], [18], [19], [21], [27], [31], [32]. We are dealing in the present paper with a different control problem: the stabilization with regional control. For other control problems in mathematical biology see [29], [30] and the references therein.

The stabilization for population dynamics with space structure is also a problem of great importance; some results have been obtained by the authors for reaction-diffusion systems in [3]-[7]. For basic results and methods in shape optimization we refer to [15], [17], [20], [24], [25], [33], [34], [36], [37]. We mention that some basic properties for the solution to population dynamics with nonlocal terms have been obtained in [23].

Here is the plan of the paper. In Section 2 we derive some basic results concerning the solution to (1) and to a slight generalization of it. In Section 3 we obtain a necessary condition and a sufficient condition for the zero-stabilizability of (1) in terms of the sign of $\lambda$. Section 4 concerns the relationship between problems (P0) and (P). Section 5 is devoted to the regional control for an approximation of problem (P). In Section 6 we consider an iterative algorithm to decrease at each iteration the total cost by changing the subregion where the control acts. Some numerical results are given. In Section 7 we discuss the eradicability (zero-stabilizability) of a specialized pest population. In Section 8, the regional eradication of a specialized pest population is investigated as well.

2. Basic results concerning the solution to (1). Consider the following linear parabolic equation with homogeneous Neumann/Dirichlet boundary conditions and initial condition

$$\begin{align*}
\frac{\partial v(x,t)}{\partial t} - d\Delta v(x,t) &= \tilde{\eta}(x,t)v(x,t) + f(x,t), & (x,t) \in Q \\
\frac{\partial v(x,t)}{\partial \nu} &= 0, & (x,t) \in \Sigma_0 \\
v(x,t) &= 0, & (x,t) \in \Sigma \setminus \Sigma_0 \\
v(x,0) &= v_0(x), & x \in \Omega,
\end{align*}$$

where $\Sigma_0 = \Gamma \times (0,\bar{T})$, and $\Gamma \subset \partial \Omega$, $\Gamma$ and $\partial \Omega \setminus \Gamma$ are sufficiently smooth.

**Definition 2.1.** We say that $v \in C([0,T];L^2(\Omega)) \cap AC([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^2_{loc}(0,T;H^2(\Omega))$ for any real $T$, $0 < T \leq \bar{T}$, is a (weak) solution to (7) if

$$\begin{align*}
\frac{\partial v(x,t)}{\partial t} - d\Delta v(x,t) &= \tilde{\eta}(x,t)v(x,t) + f(x,t) & \text{a.e. in } Q \\
\frac{\partial v(x,t)}{\partial \nu} &= 0 & \text{a.e. in } \Sigma_0 \\
v(x,t) &= 0 & \text{a.e. in } \Sigma \setminus \Sigma_0 \\
v(x,0) &= v_0(x) & \text{a.e. in } \Omega.
\end{align*}$$

It is well known that for any $v_0 \in L^2(\Omega)$ and $\tilde{\eta} \in L^\infty_{loc}(\overline{\Gamma}); f \in L^2_{loc}(\overline{\Gamma})$ problem (7) has a unique (weak) solution (see [1], [12]). Moreover, if in addition $v_0(x) \geq 0$ a.e. in $\Omega$ and if $\tilde{\eta}_1, \tilde{\eta}_2 \in L^\infty_{loc}(\overline{\Gamma})$, $f_1, f_2 \in L^2_{loc}(\overline{\Gamma})$ and $\tilde{\eta}_1(x,t) \leq \tilde{\eta}_2(x,t)$, $0 \leq f_1(x,t) \leq f_2(x,t)$ a.e. in $\Omega$ and if $\tilde{\eta}_1, \tilde{\eta}_2 \in L^\infty_{loc}(\overline{\Gamma})$
Concerning the solutions to (7) we get via Banach’s fixed point theorem that for any $v_1(x, t) \leq v_2(x, t)$ a.e. in $Q$, then

$$0 \leq v_1(x, t) \leq v_2(x, t) \quad \text{a.e. in } Q,$$

where $v_j$ is the solution to (7) corresponding to $\eta := \eta_j$, $f := f_j$ (see [1], [12], [22], [35]).

If $v_0 \in L^\infty(\Omega)$ a.e. in $\Omega$ and if $f \in L^1_{\text{loc}}(\partial \Omega)$, then the solution to (7) satisfies $v \in L^1_{\text{loc}}(\Gamma)$. Now, we consider a slightly more general problem than (1):

$$\begin{cases}
\partial_t y(x, t) - \Delta y(x, t) = \tilde{\eta}(x, t) y(x, t) - y(x, t) F((By(\cdot, t))(x)) \\
\quad + \chi_{\omega}(x) u(x, t), & (x, t) \in Q \\
\partial_{\nu} y(x, t) = 0, & (x, t) \in \Sigma_0 \\
y(x, t) = 0, & (x, t) \in \Sigma \setminus \Sigma_0 \\
y(x, 0) = y_0(x), & x \in \Omega.
\end{cases}
$$

(8)

**Definition 2.2.** We say that $v \in C([0, T]; L^2(\Omega)) \cap AC([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap L^2_{\text{loc}}((0, T); H^2(\Omega))$ for any real $T$, $0 < T \leq T$, is a (weak) solution to (8) if

$$\begin{cases}
\partial_t y(x, t) - \Delta y(x, t) = \tilde{\eta}(x, t) y(x, t) - y(x, t) F((By(\cdot, t))(x)) \\
\quad + \chi_{\omega}(x) u(x, t), & \text{a.e. in } Q \\
\partial_{\nu} y(x, t) = 0 & \text{a.e. in } \Sigma_0 \\
y(x, t) = 0 & \text{a.e. in } \Sigma \setminus \Sigma_0 \\
y(x, 0) = y_0(x) & \text{a.e. in } \Omega.
\end{cases}
$$

If $u \equiv 0$ or if $u(x, t) := h(x, t)y(x, t)$, where $h \in L^1_{\text{loc}}(\Gamma)$, then using the results concerning the solutions to (7) we get via Banach’s fixed point theorem that for any $\tilde{\eta} \in L^1_{\text{loc}}(\Gamma)$, (8) has a unique solution $y^u$. This solution belongs to $L^1_{\text{loc}}(\Gamma)$. The comparison result for (7) allows us to conclude that the solution to (8) satisfies

$$0 \leq y^u(x, t) \leq \tilde{y}(x, t) \quad \text{a.e. in } Q,$$

where $\tilde{y}$ is the solution corresponding to (8) corresponding to $F \equiv 0$.

Remark that if $u \in L^1_{\text{loc}}(\Gamma \times [0, \tilde{T}))$, then (1) has at most one solution $y^u$ (which must belong to $L^1_{\text{loc}}(\Gamma)$). Moreover, if such a solution exists then

$$0 \leq \overline{y}(x, t) \leq y^u(x, t) \quad \text{a.e. in } (\Omega \setminus \Gamma) \times (0, \tilde{T}),$$

where $\overline{y}$ is the solution to

$$\begin{cases}
\partial_t y(x, t) - \Delta y(x, t) = \eta(x)y(x, t) \\
\quad - y(x, t) F((By^u(\cdot, t))(x)), & \text{in } (\Omega \setminus \Gamma) \times (0, \tilde{T}) \\
\partial_{\nu} y(x, t) = 0 & \text{in } \Sigma \\
y(x, t) = 0 & \text{in } \partial\omega \times (0, \tilde{T}) \\
y(x, 0) = y_0(x) & \text{in } \Omega \setminus \Gamma
\end{cases}
$$

(if $z \in L^2(\Omega \setminus \Gamma)$, then we denote by the same symbol $z$ its extension by 0 on $\Gamma$).

3. **Proof of Theorem 1.2.**

**Proof.** We shall prove firstly that the mapping $\gamma \mapsto \lambda_{1, \gamma}(\omega)$ is strictly increasing and that

$$\lim_{\gamma \to +\infty} \lambda_{1, \gamma}(\omega) = \lambda_1(\omega).$$

Indeed, by Rayleigh’s principle we get that
\[ \lambda_1(\omega) = \min \left\{ \frac{\int_{\Omega\setminus\overline{\omega}} (d|\nabla \psi(x)|^2 - \eta(x)|\psi(x)|^2) dx}{\int_{\Omega\setminus\overline{\omega}} |\psi(x)|^2 dx} : \psi \in H^1(\Omega \setminus \overline{\omega}), \psi(x) = 0 \text{ in } \partial \omega, \psi \neq 0_{L^2(\Omega\setminus\overline{\omega})} \right\} \]

and this minimum is reached at a \( \psi \) if and only if \( \psi \) is an eigenfunction to (4) corresponding to \( \lambda := \lambda_1(\omega) \). Moreover, there exists at least one eigenfunction \( \psi \) satisfying \( \psi(x) \geq 0 \) in \( \Omega \setminus \overline{\omega} \).

By Rayleigh’s principle we also get that
\[ \lambda_{1\gamma}(\omega) = \min \left\{ \frac{\int_{\Omega}(d|\nabla \psi(x)|^2 - \eta(x)|\psi(x)|^2) dx + \gamma \int_{\omega} |\psi(x)|^2 dx}{\int_{\Omega} |\psi(x)|^2 dx} : \psi \in H^1(\Omega), \psi \neq 0_{L^2(\Omega)} \right\} \]

and this minimum is reached at a \( \psi^{\gamma} \) if and only if \( \psi^{\gamma} \) is an eigenfunction to (5) corresponding to \( \lambda := \lambda_{1\gamma}(\omega) \). Moreover, the set of all eigenfunctions corresponding to \( \lambda_{1\gamma}(\omega) \) is a vector space of dimension 1 and there exists at least one eigenfunction \( \psi^{\gamma} \) satisfying \( \psi^{\gamma}(x) \geq 0 \) in \( \Omega \). Actually, the regularity results for elliptic equations implies that \( \psi^{\gamma} \in C(\overline{\Omega}) \) and using the maximum and comparison principles it is possible to show that \( \psi^{\gamma}(x) > 0 \) in \( \overline{\Omega} \). It follows that there exists a constant \( \zeta_\gamma > 0 \) such that \( \psi^{\gamma}(x) \geq \zeta_\gamma \) for any \( x \in \overline{\Omega} \).

It follows immediately that the mapping \( \gamma \mapsto \lambda_{1\gamma}(\omega) \) is strictly increasing. Let \( \psi^{\gamma} \) be the unique eigenfunction to (5) corresponding to \( \lambda := \lambda_{1\gamma}(\omega) \) and satisfying \( \|\psi^{\gamma}\|_{L^2(\Omega)} = 1 \) and \( \psi^{\gamma}(x) \geq \zeta_\gamma > 0 \) in \( \overline{\Omega} \), where \( \zeta_\gamma \) is a constant, and let \( \psi_1 \) be an eigenfunction to (4) corresponding to \( \lambda := \lambda_1(\omega) \) and satisfying \( \|\psi_1\|_{L^2(\Omega\setminus\overline{\omega})} = 1 \) and \( \psi_1(x) \geq 0 \) a.e. in \( \Omega \setminus \overline{\omega} \). This yields
\[ \lambda_1(\omega) = \int_{\Omega\setminus\overline{\omega}} |d|\nabla \psi_1(x)|^2 - \eta(x)|\psi_1(x)|^2| dx \]
\[ = \int_{\Omega} |d|\nabla \tilde{\psi}_1(x)|^2 - \eta(x)|\tilde{\psi}_1(x)|^2| dx + \gamma \int_{\omega} |\tilde{\psi}_1(x)|^2 dx \geq \lambda_{1\gamma}(\omega), \]

where \( \tilde{\psi}_1 \) is the extension of \( \psi_1 \) by 0 in \( \omega \). Hence, \( \lambda_{1\gamma}(\omega) \leq \lambda_1(\omega) \) for any \( \gamma > 0 \), and there exists \( \lim_{\gamma \to +\infty} \lambda_{1\gamma}(\omega) = \lambda_1(\omega) \).

Since
\[ \lambda_1(\omega) \geq \lambda_{1\gamma}(\omega) = \int_{\Omega} |d|\nabla \psi^{\gamma}(x)|^2 - \eta(x)|\psi^{\gamma}(x)|^2| dx + \gamma \int_{\omega} |\psi^{\gamma}(x)|^2 dx, \]

for any \( \gamma > 0 \), we may infer that \( \{\psi^{\gamma}\} \) is bounded in \( H^1(\Omega) \) and \( \gamma \int_{\omega} |\psi^{\gamma}(x)|^2 dx \) is bounded in \( \mathbb{R} \). It follows that there exists a subsequence, also denoted by \( \{\psi^{\gamma}\} \), such that
\[ \psi^{\gamma} \rightharpoonup \psi \quad \text{in } H^1(\Omega), \]
\[ \psi^{\gamma} \to \psi \quad \text{in } L^2(\Omega), \quad \psi^{\gamma} \to 0 \quad \text{in } L^2(\omega) \]
as \( \gamma \to +\infty \), where \( \psi \in H^1(\Omega) \). It follows immediately that \( \psi \in H^1(\Omega\setminus\overline{\omega}) \), \( \psi(x) = 0 \) in \( \partial \omega \), and \( \|\psi\|_{L^2(\Omega\setminus\overline{\omega})} = 1 \).

By (10) we obtain that
\[ \lambda_1(\omega) \geq \limsup_{\gamma \to +\infty} \lambda_{1\gamma}(\omega) \]
\[ \geq \liminf_{\gamma \to +\infty} \lambda_{1\gamma}(\omega) \geq \liminf_{\gamma \to +\infty} \int_{\Omega\setminus\overline{\omega}} |d|\nabla \psi^{\gamma}(x)|^2 - \eta(x)|\psi^{\gamma}(x)|^2| dx \]
\[
\geq \int_{\Omega \setminus \overline{\omega}} \left[ d|\nabla \psi(x)|^2 - \eta(x)|\psi(x)|^2 \right] dx \geq \lambda_1(\omega),
\]
and so we get the conclusion.

Here we consider \( T = +\infty \). Assume that (1) is internally zero-stabilizable. For any \( y_0 \) satisfying (H1) there exists \( u \in L^\infty_{\text{loc}}(\overline{\omega} \times [0, +\infty)) \) such that (2) and (3) are satisfied. Since \( \lim_{t \to +\infty} y^u(\cdot, t) = 0 \) in \( L^\infty(\Omega) \), we get that for any \( \varepsilon > 0 \), there exists \( T_\varepsilon > 0 \) such that
\[
0 \leq F((By^u(\cdot, t))(x)) \leq \varepsilon
\]
a.e. \((x, t) \in \Omega \times [T_\varepsilon, +\infty)\). Using the comparison result in Section 2 we may conclude that
\[
0 \leq q(x, t) \leq y^u(x, t),
\]
a.e. in \((\Omega \setminus \overline{\omega}) \times [0, +\infty)\), where \( q \) is the solution to
\[
\left\{ \begin{array}{ll}
\partial_t q(x, t) - d\Delta q(x, t) = \eta(x)q(x, t) \\
\partial_\nu q(x, t) = 0, \\
q(x, t) = 0, \\
q(x, 0) = y_0(x),
\end{array} \right.
\]
\((x, t) \in (\Omega \setminus \overline{\omega}) \times (0, +\infty)\),
and  \( \omega \) is such that \( \lambda_1(\omega) \geq 0 \).

Let us remark firstly that for \( y_0 \) not identically zero on \( \Omega \setminus \overline{\omega} \) it follows via the backward uniqueness for linear parabolic equations that \( q(x, T_\varepsilon) \) is not identically zero on \( \Omega \setminus \overline{\omega} \), and consequently \( y^u(x, T_\varepsilon) \) is not identically zero on \( \Omega \setminus \overline{\omega} \).

Using again the comparison result in Section 2 we have that
\[
0 \leq \tilde{q}(x, t) \leq y^u(x, t),
\]
a.e. in \((\Omega \setminus \overline{\omega}) \times [T_\varepsilon, +\infty)\), where \( \tilde{q} \) is the solution to
\[
\left\{ \begin{array}{ll}
\partial_t q(x, t) - d\Delta q(x, t) = \eta(x)q(x, t) - \varepsilon q(x, t) \\
\partial_\nu q(x, t) = 0, \\
q(x, t) = 0, \\
q(x, T_\varepsilon) = y^u(x, T_\varepsilon),
\end{array} \right.
\]
\((x, t) \in (\Omega \setminus \overline{\omega}) \times (T_\varepsilon, +\infty)\),
and consequently \( \lambda_1(\omega) > \varepsilon > 0 \). Since this inequality holds for any positive \( \varepsilon \), we may infer that \( \lambda_1(\omega) > 0 \).

Conversely, if \( \lambda_1(\omega) > 0 \), then we know that \( \lim_{\gamma \to +\infty} \lambda_{1, \gamma}(\omega) = \lambda_1(\omega) \), which implies that for a sufficiently large \( \gamma > 0 \) we have that \( \lambda_{1, \gamma}(\omega) > 0 \). Consider the feedback control \( u := -\gamma y \). We shall prove that this feedback control realizes (2) and (3). With this feedback control system (1) becomes (6). System (6) has a unique solution \( y \) which is nonnegative. Using the comparison result in Section 2 we get that
\[
0 \leq y(x, t) \leq \tilde{y}(x, t)
\]
a.e. in \( Q \), where \( \tilde{y} \) is the solution to
\[
\left\{ \begin{array}{ll}
\partial_t y(x, t) - d\Delta y(x, t) = \eta(x)y(x, t) - \gamma\chi_\omega(x)y(x, t), \\
\partial_\nu y(x, t) = 0, \\
y(x, 0) = \tau\psi^\gamma(x),
\end{array} \right.
\]
\((x, t) \in Q\),
where \( \tau > 0 \) is a constant such that \( y_0(x) \leq \tau\psi^\gamma(x) \) a.e. \( x \in \Omega \). Since \( \lambda_{1, \gamma}(\omega) > 0 \) and \( \tilde{y}(x, t) = \tau\psi^\gamma(x) \exp\{-\lambda_{1, \gamma}(\omega)t\} \) for any \( (x, t) \in Q \), we may conclude that
\[
\lim_{t \to +\infty} \|y(\cdot, t)\|_{L^\infty(\Omega)} = 0
\]
at the rate of \( \exp\{-\lambda_{1\gamma}(\omega)t\}\).

4. **Relationship between (P0) and (P).** In spite of the fact that (P0) and (P) are not identical, we remark that if \( F \equiv 0 \) and if \( y_0(x) = \tau \psi_1(x) \) a.e. \( x \in \Omega \), where \( \tau > 0 \) and \( \psi_1 \) is the eigenfunction corresponding to the eigenvalue \( \lambda_{1\gamma}(\omega) \) and to (5) and satisfying

\[
\psi_1(x) > 0, \quad \forall x \in \overline{\Omega}, \quad \|\psi_1\|_{L^1(\Omega)} = 1,
\]

(it is also well known that \( \psi_1 \in C(\overline{\Omega}) \) as well) then the solution corresponding to (6) (and to \( F \equiv 0 \)) is given by

\[
y(x, t) = \tau \psi_1(x) \exp\{-\lambda_{1\gamma}(\omega)t\}, \quad x \in \Omega, \ t > 0,
\]

and consequently

\[
\int_0^\infty \int_\Omega y(x, t)dx \ dt = \tau \int_\Omega \psi_1(x)dx \int_0^\infty \exp\{-\lambda_{1\gamma}(\omega)t\}dt = \frac{\tau}{\lambda_{1\gamma}(\omega)}.
\]

Hence (P) may be rewritten as

\[
\text{Minimize} \left\{ \frac{\theta \tau}{\lambda_{1\gamma}(\omega)} + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega) \right\},
\]

subject to \( \omega \), and it follows that for \( \zeta = \theta \tau \) we have that (P0) \( \equiv \) (P).

However, problem (P) gives a more accurate formulation of our goals. For the problem with logistic term this is indeed obvious.

**Remark 2.** If problem (P) corresponding to \( F \equiv 0 \) has an optimal \( \omega \), then \( \int_0^\infty \int_\Omega y(x, t)dx \ dt < +\infty \), and since the solution \( y \) to (6) behaves as \( \exp\{-\lambda_{1\gamma}(\omega)t\} \), as \( t \to +\infty \), we may conclude that \( \lambda_{1\gamma}(\omega) > 0 \).

If \( F \equiv 0 \) and if \( \tau_1 \psi_1(x) \leq y_0(x) \leq \tau_2 \psi_1(x) \) a.e. \( x \in \Omega \), where \( 0 < \tau_1 \leq \tau_2 \), then

\[
\tau_1 \psi_1(x) \exp\{-\lambda_{1\gamma}(\omega)t\} \leq y(x, t) \leq \tau_2 \psi_1(x) \exp\{-\lambda_{1\gamma}(\omega)t\}, \quad x \in \Omega, \ t > 0,
\]

and consequently

\[
\tau_1 \int_\Omega \psi_1(x)dx \int_0^\infty \exp\{-\lambda_{1\gamma}(\omega)t\}dt = \frac{\tau_1}{\lambda_{1\gamma}(\omega)}
\]

\[
\leq \int_0^\infty \int_\Omega y(x, t)dx \ dt \leq \tau_2 \int_\Omega \psi_1(x)dx \int_0^\infty \exp\{-\lambda_{1\gamma}(\omega)t\}dt = \frac{\tau_2}{\lambda_{1\gamma}(\omega)}
\]

a.e. \((x, t) \in Q\). This implies that

\[
\frac{\theta \tau_1}{\lambda_{1\gamma}(\omega)} + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega)
\]

\[
\leq \theta \int_0^\infty \int_\Omega y(x, t)dx \ dt + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega)
\]

\[
\leq \frac{\theta \tau_2}{\lambda_{1\gamma}(\omega)} + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega).
\]

For a general \( F \) we have that

\[
\theta \int_0^\infty \int_\Omega y(x, t)dx \ dt + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega)
\]

\[
\leq \theta \int_0^\infty \int_\Omega y^0(x, t)dx \ dt + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega),
\]

where \( y \) is the solution to (6) and \( y^0 \) is the solution to (6) corresponding to \( F \equiv 0 \).
5. Localizing the subregion $\omega$. Without loss of generality we can assume that $\theta = 1$. We shall use the level set approach (see also [15], [17], [20], [24], [25], [33], [34], [36], [37]) to treat problem (P), where $\gamma > 0$ is fixed.

We shall approximate problem (P) by the following one where the Heaviside function is substituted by its mollified version $H_\varepsilon(s) = \frac{1}{2}(1 + \frac{2}{\pi} \arctan \frac{s}{\varepsilon})$, its derivative by its mollified version $\delta_\varepsilon(s) = \frac{1}{\pi \varepsilon}(\frac{1}{1 + \frac{s^2}{\varepsilon^2}} - \frac{1}{\varepsilon})$. For a small but fixed $\varepsilon > 0$ and for a large but fixed $T > 0$, we shall deal with the following approximating regional control problem with a finite horizon:

\[(\text{RC}) \quad \text{Minimize } J(\varphi),\]

where $\varphi : \bar{\Omega} \to \mathbb{R}$ is a smooth function,

$$J(\varphi) = \int_0^T \int_\Omega y(\varphi(x),t)dx \, dt + \alpha \int_\Omega \varphi(x) dx + \beta \int_\Omega \delta_\varepsilon(\varphi(x)) |\nabla \varphi(x)| dx,$$

and $y(\cdot)$ is the solution to

\[
\begin{align*}
\partial_t y(x,t) - d\Delta y(x,t) &= \eta(x)y(x,t) - y(x,t)F((By(\cdot,t))(x)) - \gamma H_\varepsilon(\varphi(x))y(x,t), \\
\partial_r y(x,t) &= 0, \\
y(x,0) &= y_0(x),
\end{align*}
\]

where $T := T$.

The next result gives the directional derivative of $J$:

**Theorem 5.1.** For any smooth functions $\varphi, \psi : \bar{\Omega} \to \mathbb{R}$ we have that

\[
dJ(\varphi)(\psi) = \int_\Omega \delta_\varepsilon(\varphi(x))\psi(x)[\gamma \int_0^T r^\varphi(x,t)y^\varphi(x,t)dt + \alpha \\
- \beta \ \text{div} \left( \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|} \right) dx + \beta \int_\Omega \delta_\varepsilon(\varphi(x)) \partial_r \varphi(x) \psi(x) d\sigma,
\]

where $r^\varphi$ is the solution to

\[
\begin{align*}
\partial_r r + d\Delta r &= -\eta(x)r + r(x,t)F((By^\varphi(\cdot,t))(x)) \\
+ \{B^* [y^\varphi(\cdot,t)r(\cdot,t)F((By^\varphi(\cdot,t))(\cdot))](x) + \gamma H_\varepsilon(\varphi(x))r + 1, \\
\partial_r r(x,t) &= 0, \\
r(x,T) &= 0,
\end{align*}
\]

It follows in a standard way that the linear parabolic problem (13) has a unique solution $r^\varphi$ (see [12], [22]).

**Proof.** As in the proof of Lemma 3 in [8] it is possible to prove that for any smooth functions $\varphi, \psi : \bar{\Omega} \to \mathbb{R}$ we have

$$\frac{1}{\xi} [y^\varphi + \xi \psi - y^\varphi] \to z \quad \text{in } C([0,T]; L^\infty(\Omega)),$$

as $\xi \to 0$, where $z$ is the solution to the problem.
If we multiply the first equation in (14) by \( r \) and after an easy calculation we infer that
\[
\partial_t z - d\Delta z = \eta(x)z - z(x,t) F\big((By^\varphi(\cdot,t))(x)\big) \\
- y^\varphi(x,t) F'\big((By^\varphi(\cdot,t))(x)\big) (Bz(\cdot,t))(x) - \gamma \delta_\xi(\varphi(x)) y^\varphi(\cdot,t) z, \\
\partial_\nu z(x,t) = 0, \\
z(x,0) = 0, \\
(x,t) \in Q, \\
(x,t) \in \Sigma, \\
x \in \Omega. 
\] 

The linear parabolic problem (14) has a unique solution \( z \) (see [12], [22]).

For any smooth functions \( \varphi, \psi \) we have that
\[
\lim_{\xi \to 0} \frac{1}{\xi} [J(\varphi + \xi \psi) - J(\varphi)] = \int_0^T \int_\Omega \rho(x,t) dx \, dt + \alpha \int_\Omega \partial_\xi(\varphi(x)) \psi(x) dx \\
+ \beta \int_\Omega \delta_\xi(\varphi(x)) \psi(x) dx + \beta \int_\Omega \delta_\xi(\varphi(x)) \frac{\nabla \varphi(x) \cdot \nabla \psi(x)}{|\nabla \varphi(x)|} dx \\
= \int_0^T \int_\Omega z(x,t) dx \, dt + \alpha \int_\Omega \delta_\xi(\varphi(x)) \psi(x) dx \\
+ \beta \int_\Omega \text{div} \delta_\xi(\varphi(x)) \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|} \psi(x) dx - \beta \int_\Omega \delta_\xi(\varphi(x)) \psi(x) \text{div} \left( \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|} \right) dx. 
\]

After applying the Gauss-Ostrogradski’s formula (see [12, p. 13]) we obtain
\[
dJ(\varphi)(\psi) = \int_0^T \int_\Omega \rho(x,t) dx \, dt + \alpha \int_\Omega \delta_\xi(\varphi(x)) \psi(x) dx \\
- \beta \int_\Omega \delta_\xi(\varphi(x)) \text{div} \left( \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|} \right) \psi(x) dx \\
+ \beta \int_\Omega \delta_\xi(\varphi(x)) \partial_\nu \varphi(x) \psi(x) \sigma. 
\] 

If we multiply the first equation in (14) by \( r^\varphi \) and integrate over \( Q \) we obtain that
\[
\int_0^T \int_\Omega r^\varphi(x,t) [\partial_t z(x,t) - d\Delta z(x,t)] dx \, dt = \int_0^T \int_\Omega r^\varphi(x,t) \eta(x) z(x,t) dx \, dt \\
+ \int_0^T \int_\Omega r^\varphi(x,t) [-z(x,t) F\big((By^\varphi(\cdot,t))(x)\big) \\
- y^\varphi(x,t) F'\big((By^\varphi(\cdot,t))(x)\big) (Bz(\cdot,t))(x) \big] dx \, dt \\
+ \int_0^T \int_\Omega r^\varphi(x,t) [-\gamma \delta_\xi(\varphi(x)) y^\varphi(\cdot,t) \psi(x) - \gamma H_\xi(\varphi(x)) z(x,t) \big] dx \, dt, 
\]
and after an easy calculation we infer that
\[
\int_0^T \int_\Omega z(x,t) [\partial_t r^\varphi(x,t) + d r^\varphi(x,t) \big] dx \, dt = \int_0^T \int_\Omega r^\varphi(x,t) \eta(x) z(x,t) dx \, dt \\
+ \int_0^T \int_\Omega r^\varphi(x,t) [-z(x,t) F\big((By^\varphi(\cdot,t))(x)\big) \big] dx \, dt \\
- \int_0^T \int_\Omega [B^* \gamma^\varphi(\cdot,t) \gamma^\varphi(\cdot,t) F'\big((By^\varphi(\cdot,t))(\cdot)\big)](x) z(x,t) dx \, dt \\
+ \int_0^T \int_\Omega r^\varphi(x,t) [-\gamma \delta_\xi(\varphi(x)) y^\varphi(\cdot,t) \psi(x) - \gamma H_\xi(\varphi(x)) z(x,t) \big] dx \, dt. 
\]

Since \( r^\varphi \) satisfies the first equation in (13) we immediately get that
\[
\int_0^T \int_\Omega z(x,t) dx \, dt = \gamma \int_0^T \int_\Omega r^\varphi(x,t) y^\varphi(x,t) \delta_\xi(\varphi(x)) \psi(x) dx \, dt. 
\]
By (16) and (15) we get (12) (the formula for the derivative of $J$).

Remark 3. From (12) in Theorem (5.1) we get that the gradient descent with respect to $\varphi$ (see [24]) is

$$\begin{cases}
\partial_t \varphi(x, \xi) = \delta_\epsilon(\varphi(x, \xi))-\gamma \int_0^T r^2(x, t)y^2(x, t)dt - \alpha \\
\quad + \beta \text{div} \left( \frac{\nabla \varphi(x, \theta)}{|\nabla \varphi(x, \theta)|} \right) & x \in \Omega, \ \xi > 0 \\
\delta_\epsilon(\varphi(x, \xi)) & x \in \partial \Omega, \ \xi > 0
\end{cases}$$

(17)

Remark 4. If there is an optimal $\varphi^*$, then we get by Theorem (5.1) that $\varphi^*$ is a steady-state of (17).

Remark 5. If we take $(Bz)(x) = \rho(x)z(x)$ for $z \in L^2(\Omega)$, where $\rho \in L^\infty(\Omega)$, $\rho(x) \geq 0$ a.e. in $\Omega$, and $F(s) = s$ for $s \in [0, +\infty)$, then (13) becomes

$$\begin{cases}
\partial_t r + d\Delta r = -\eta(x)r + 2\rho(x)r(x, t)y^2(x, t) + \gamma H_\varepsilon(\varphi(x))r + 1, & (x, t) \in Q \\
\partial_t r(x, t) = 0, & (x, t) \in \Sigma \\
r(x, T) = 0, & x \in \Omega.
\end{cases}$$

Remark 6. If we take $(Bz)(x) = \int_\Omega k(x, x')z(x')dx'$ for $z \in L^2(\Omega)$, where $k \in L^\infty(\Omega \times \Omega)$, $k(x, x') \geq 0$ a.e. in $\Omega \times \Omega$, and $F(s) = s$ for $s \in [0, +\infty)$, then (13) becomes

$$\begin{cases}
\partial_t r + d\Delta r = -\eta(x)r + r(x, t) \int_\Omega k(x, x')y^2(x', t)dx' \\
\quad + \int_\Omega k(x', x)y^2(x', t)r(x', t)dx' + \gamma H_\varepsilon(\varphi(x))r + 1, & (x, t) \in Q \\
\partial_t r(x, t) = 0, & (x, t) \in \Sigma \\
r(x, T) = 0, & x \in \Omega.
\end{cases}$$

6. **Numerical implementation.** Theorem (5.1) allows us to derive a conceptual iterative algorithm to improve at each step the function $\varphi$ (the subregion where the control acts) in order to obtain a smaller value for $J$ (for gradient algorithms see also [10]).

**Step 0.** set $k := 0$, $J^{(0)}$ a large value and $\xi_0 > 0$ a small constant initialize $\varphi^{(0)} = \varphi^{(0)}(x, 0)$.

**Step 1.** compute $y^{(k+1)}$ the solution of (11) corresponding to $\varphi = \varphi^{(k)}(\cdot, 0)$

compute $J^{(k+1)} = \int_0^T \int_\Omega y^{(k+1)}(x, t)dx \ dt$

$$+ \alpha \int_\Omega H_\varepsilon(\varphi^{(k)}(x, 0))dx \ + \beta \int_\Omega \delta_\epsilon(\varphi^{(k)}(x, 0)) |\nabla \varphi^{(k)}(x, 0)| dx.$$

**Step 2.** if $|J^{(k+1)} - J^{(k)})| < \varepsilon_1$ or $J^{(k+1)} \geq J^{(k)}$ then STOP

else go to Step 3.

**Step 3.** compute $r^{(k+1)}$ the solution of problem (13) corresponding to $\varphi^{(k+1)}(\cdot, 0)$ and $y^{(k+1)}$.

**Step 4.** compute $\varphi^{(k+1)}$ using (17) and the initial condition $\varphi^{(k+1)}(x, 0) = \varphi^{(k)}(x, \xi_0)$ and a semi-implicit timestep scheme

**Step 5.** if $||\varphi^{(k+1)} - \varphi^{(k)}||_{L^2} < \varepsilon_2$ then STOP

else $k := k + 1$
parameters. For details about the gradient methods see [10, §2.3].

Numerical tests

In the numerical tests, we consider $\Omega = (0, 1) \times (0, 1)$. The domain $\Omega$ is approximated by a grid of $(N + 1) \times (N + 1)$ equidistant nodes, namely

$$\{(x_1^i, x_2^j) : x_1^i = (i-1)\Delta x_1, x_2^j = (j-1)\Delta x_2, i, j = 1, 2, ..., N+1, \Delta x_1 = \Delta x_2 = 1/N\}$$

$((x_1, x_2) \in \Omega$ is the space variable). We also discretize the time interval $[0, T]$ using $(M+1)$ equidistant nodes, $t^m = (m-1)\Delta t$, $m = 1, 2, ..., M+1$, $\Delta t = T/M$. We take $M$ and $N$ to be even. Using the backward Euler implicit scheme, we approximate the parabolic system from Step 1 by a finite difference method, ascending with respect to time levels. The parabolic system from Step 3 is approximated by a finite difference method too, now descending with respect to time levels. The obtained algebraic linear systems are solved by Gaussian elimination. Integrals from Step 1 are numerically computed using Simpson’s rule corresponding to the approximating grid. In Step 4, we use the semi-implicit Gauss-Seidel iterative method to approximate the solution at the current iteration (see [24]). Therefore, we can use the new values of $\varphi$ immediately after they are computed.

In each figure of the following tests, the white area represents the subregion $\omega$ that provides a smaller value for $J$.

We set the diffusion coefficient to be $d = 1$, the final time $T = 1$, $\gamma = 1$ and $F = 0$. We take the space discretization step $\Delta x_1 = \Delta x_2 = 0.05$, and the time discretization step $\Delta t = 0.025$. For the convergence parameters we consider $\varepsilon_1 = \varepsilon_2 = 0.0001$ and for the regularization parameter $\varepsilon = 1$.

**Test 1.** We consider the initial population density $y_0(x_1, x_2) = x_1 + 2x_2$, $(x_1, x_2) \in \Omega$, and a constant natural growth rate of the population, e.g. $\eta(x_1, x_2) = 3$, $(x_1, x_2) \in \Omega$. To start with, we take $\varphi^{(0)}(x_1, x_2, 0) = 1$, $(x_1, x_2) \in \Omega$. This means that the control is acting in the whole domain $\Omega$. We set $\beta = 0$ (no penalization for the length of $\partial \omega$), and the penalization $\alpha \in \{0.5, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ for the area of $\omega$. From the numerical tests, we observe that for $\alpha \in \{0.5, 1, 2, 3, 4\}$ the final iteration of $\omega$ is the same as the initial one (due to the fact that the cost $\alpha$ to be paid per unit of area is small and there is no cost per unit of perimeter). For the other considered values of $\alpha$, there is a fragmentation into small parts of the initial region $\omega$, as it can be seen in Figure 1.

We point out that, the region where the control is acting is decreasing with the increase in $\alpha$. The same result is obtained for $\alpha \in \{11, 12, 13, 14\}$. If we increase $\alpha$ much more (e.g., $\alpha = 20$), the white zone dissipates. This means that the cost to be paid in order to act in $\omega$ becomes too big, so the harvesting is unprofitable and it will not be achieved (the level set function returned by the computer program is negative in each point of the grid). This means that the best thing to do is to let the system uncontrolled ($\omega = \emptyset$).

**Test 2.** Consider the same $\eta$, and $y_0(x_1, x_2) = x_1 + x_2$, $(x_1, x_2) \in \Omega$. If the initialization of $\varphi$ is made by a circle of radius 0.25 and center $(0.5, 0.5)$, $\varphi^{(0)}(x_1, x_2, 0) = 0.25 - \sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2}$, $(x_1, x_2) \in \Omega$, then the level set function returned by the computer program is positive in each point of the grid,
that control will act on the entire domain Ω.

Let’s notice a symmetry of the final iteration of ω is decreasing (e.g., for α = 0.6, the last iteration of ϕ is negative in each point of the grid).

Test 3. Consider again η(x₁, x₂) = 3, (x₁, x₂) ∈ Ω, and a uniformly distributed initial population density y₀(x₁, x₂) = 1, (x₁, x₂) ∈ Ω. We take now ϕ(0)(x₁, x₂, 0) = sin(7πx₁)sin(7πx₂), (x₁, x₂) ∈ Ω, a function that produces a checkerboard shape. For α ∈ {0.5, 1, 2.5} and β = 0.001, the obtained results can be seen in Figure 3. Let’s notice a symmetry of the final iteration of ω with respect the the center of the whole domain Ω.

In all these numerical tests, the convergence was obtained by the tests in STEP 2 (in all figures, the numbers in the brackets represent the numbers of iterations needed to fulfill the tests in STEP 2). The approximate solutions obtained by the program corresponding to the above algorithm are non-negative. By refining the time and the space stepsizes, we obtain similar numerical results.

7. Extension to reaction-diffusion systems. Let us now consider the following extension of Equation (1) to reaction-diffusion systems

\[
\begin{cases}
\partial_t h(x,t) - d_1 \Delta h(x,t) = r(x)h(x,t) - \rho(x)h(x,t)^2 \\
- f(h(x,t))(B_0p(x,t))(x), \\
\partial_t p(x,t) - d_0 \Delta p(x,t) = -a(x)p(x,t) + c_0(B_0^*f(h(\cdot,t)))(x)p(x,t) + \chi_\omega(x)u(x,t), \\
\partial_x h(x,t) = \partial_x p(x,t) = 0, \\
h(x,0) = h_0(x), \\
p(x,0) = p_0(x),
\end{cases} \quad (x,t) \in Q
\]

For β = 0.3, and α varying in the set \{0, 0.1, 0.2, 0.3, 0.4, 0.5\}. Thus, it is assumed that control will act on the entire domain Ω.

Taking now the natural growth rate of the population η(x₁, x₂) = x₂sin(x₁), (x₁, x₂) ∈ Ω, we get the results from Figure 2. It can be seen that, for bigger α, the area of ω is decreasing (e.g., for α = 0.6, the last iteration of ϕ is negative in each point of the grid).

The representation of final iteration of ω for α ∈ \{5, 6, 7, 8, 9, 10\} and β = 0.
It includes spatially structured prey-predator models for a specialized predator (a pest) as well as a class of epidemic models.

For a prey-predator system in which the pest population is the predator population, we have the following situation.

Here $d_1, d_2$ are positive constants, $h(x, t)$ is the spatial density at time $t$ of a prey population distributed over the habitat $\Omega$, and $p(x, t)$ is the spatial density...
at time $t$ of a predator population distributed over the habitat $\Omega$. $r(x)$ is the growth rate and $\rho(x)h(x,t)^2$ is a local logistic term for the prey population, while $a(x)$ is the decreasing rate of the predator population. $f(h(x,t))(B_0p(.,t))(x)$ gives the density of captured prey population; the prey captured is transformed into biomass via a constant conversion rate $c_0 > 0$, yielding a numerical response to predation $c_0(B_0^*f(h(.,t))(x)p(x,t)$, where $B_0 \in L(L^2(\Omega))$ and $B_0^*$ is its adjoint. In applications we have in mind several forms of the functional response to predation, such as $f(h) = ch$, $c > 0$ (Lotka-Volterra), or $f(h) = \frac{c_1h}{1+c_2h^s}$, $c_1,c_2 > 0$, $s \in \mathbb{N}$ (Holling type $s+1$).

The homogeneous Neumann boundary condition shows that there is no flux of population through the boundary of the habitat (the domain is isolated), and $h_0, p_0$ are the initial densities of the prey, predator populations, respectively. The control $u$ acts in the open subset $\omega$.

Two cases are of particular interest.

**Case 1’.** If we take $(B_0z)(x) = z(x)$ for $z \in L^2(\Omega)$, then the numerical response to predation in (18) is $c_0f(h(x,t))(p(x,t)$.

**Case 2’.** If we take $(B_0z)(x) = \int_\Omega k(x,x')z(x')dx'$, for $z \in L^2(\Omega)$, $k \in L^\infty(\Omega \times \Omega)$, $k(x,x') \geq 0$ a.e. in $\Omega \times \Omega$, then the numerical response to predation in (18) is $c_0\int_\Omega k(x,x')f(h(x',t))(x)p(x,t)$ and the kernel $k$ represents a redistribution rate. Remark that if the predators come back to their initial position and produce offsprings, then the numerical response to predation is $c_0(B_0^*f(h(.,t))(x)p(x,t)$. Actually, the adjoint of $B_0$ is given in this case by $(B_0^*z)(x) = \int_\Omega k(x',x)z(x')dx'$, for any $z \in L^2(\Omega)$.

If the prey is captured at position $x$ by predators comming from any position $x'$, and if the predators arrived at position $x$ stays at this new position and produce offsprings (the predators follow the prey), then the prey-predator dynamics is described by

$$
\begin{aligned}
\partial_t h(x,t) - d_1\Delta h(x,t) &= r(x)h(x,t) - \rho(x)h(x,t)^2 - f(h(x,t))(B_0p(.,t))(x), \\
\partial_t p(x,t) - d_2\Delta p(x,t) &= -a(x)p(x,t) + c_0f(h(x,t))(B_0p(.,t))(x) + \chi_\omega(x)u(x,t), \\
\partial_t h(x,t) &= \partial_x p(x,t) = 0, \\
h(x,0) &= h_0(x), \quad p(x,0) = p_0(x), \\
x \in \Omega.
\end{aligned}
$$

A spatially structured SIR epidemic model with nonlocal infection rate was proposed by D.G. Kendall in [28], (it has been widely studied in literature, see e.g. [11], [26, p. 64], [16, p. 150]). Here we present it in the following form.

$$
\begin{aligned}
\partial_t s(x,t) - d_1\Delta s(x,t) &= b(x)s(x,t) - \mu(x)s(x,t) - \rho(x)s(x,t)^2 - (B_0i(.,t))(x)s(x,t), \\
\partial_t i(x,t) - d_2\Delta i(x,t) &= -(\mu(x) + \bar{\mu})(x)i(x,t) + (B_0i(.,t))(x)s(x,t) + \chi_\omega(x)u(x,t), \\
\partial_t s(x,t) &= \partial_x i(x,t) = 0, \\
s(x,0) &= s_0(x), \quad i(x,0) = i_0(x), \\
x \in \Omega.
\end{aligned}
$$

(20)

where $s(x,t)$ and $i(x,t)$ denote the spatial density of the susceptible population, and the infective population respectively.
According to Kendall’s proposal, the rate of contagion may be nonlocal as follows

\[(B_0i(\cdot,t))(x) = \int_{\Omega} k(x,x')i(x',t)dx',\]

for a suitably chosen kernel. We get the local case by choosing \(k(x,x') = \tilde{c}\delta(x-x')\), where \(\tilde{c}\) is a positive constant.

In (20) the vital dynamics of the susceptible population is described by

\[b(x)s(x,t) - \mu(x)s(x,t) - \rho(x)s(x,t)^2,\]

considering that only the susceptible population reproduces; \(b(x)\) is the birth rate; \(\mu(x)\) is the death rate; \(\rho(x)s(x,t)^2\) is a logistic term.

As far as the infective population is concerned, \(\mu(x)\) is the natural death rate, while \(\tilde{\mu}(x)\) includes the natural healing rate (with a solid immunity) and the extra death rate due to the disease. The control term \(\chi_\omega(x)u(x,t)\) may represent a regional intervention by treatment and/or isolation.

It is obvious that model (20) may be viewed as a particular case of (19). If we consider the local contagion case, then (20) is a particular case of (18).

**Eradicability (zero-stabilizability) of a specialized predator (pest) population**

In what follows we will focus on the eradication of a specialized predator (pest) species, in a prey-predator system.

Here are the hypotheses we are going to use in what follows:

1. (H1') \(\rho, \alpha, h_0, p_0 \in L^\infty(\Omega), r(x) \geq r_0 > 0, \rho(x) \geq \rho_0 > 0, h_0(x) \geq 0, p_0(x) \geq 0\) a.e. in \(\Omega\);
2. (H2') \(B_0 \in L(L^2(\Omega)) \cap L(L^\infty(\Omega)), (B_0z)(x) \geq 0\) a.e. in \(\Omega\), for any \(z \in L^2(\Omega)\) such that \(z(x) \geq 0\) a.e. in \(\Omega\);
3. (H3') \(f : [0, +\infty) \to [0, +\infty)\) is a continuously differentiable function satisfying

   \[f(0) = 0, 0 \leq f'(h) \leq L_0,\]

   for any \(h \in [0, +\infty)\), where \(L_0 > 0\) is a constant.

Since we are dealing with a pest population, the problem of controlling the pest population for its eradication becomes relevant; for \(T = +\infty\), we are interested to know if there exists a control \(u\) acting in \(\omega\) such that the solution \((h^u, p^u)\) to (18) satisfies \(h^u(x,t) \geq 0, p^u(x,t) \geq 0\) in \(Q\) and \(\lim_{t \to +\infty} p^\nu(\cdot,t) = 0\) in a certain functional space.

We shall prove firstly some basic properties of the solution to (18). Next we continue with an investigation of the zero-stabilizability (eradicability) of the predator population for (18).

**Definition 7.1.** We say that \((h,p)\) is a (weak) solution to (18) if \(h, p \in C([0,T];L^2(\Omega)) \cap AC([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L_{loc}^2([0,T];H^2(\Omega))\) for any real \(T, 0 < T \leq \tilde{T}\), and if all relations in (18) are satisfied a.e.

If \(u \equiv 0\) or if \(u(x,t) := w(x,t)p(x,t)\), where \(w \in L^\infty_{loc}(\overline{Q})\), then using the results concerning the solutions to (7) we get via Banach’s fixed point theorem that (18) has a unique solution \((h^u, p^u)\). This solution belongs to \(L^\infty_{loc}(\overline{Q}) \times L^\infty_{loc}(\overline{Q})\) and has nonnegative components.

**Definition 7.2.** The predator population is internally zero-stabilizable (eradicable) if for any \(h_0, p_0\) satisfying (H1') there exists a control \(u \in L^\infty_{loc}(\overline{\omega} \times [0, +\infty))\) such that the solution \((h^u, p^u)\) to (18) satisfies

\[0 \leq h^u(x,t), \quad 0 \leq p^u(x,t)\quad\text{a.e. in } Q,\]

(21)
\begin{align}
\lim_{t \to +\infty} p^n(\cdot, t) &= 0 \quad \text{in } L^\infty(\Omega). \tag{22}
\end{align}

We are dealing here with zero-stabilizability and zero-stabilization with state constraints (the components of the state of the system must be nonnegative).

We shall see that the zero-stabilizability is deeply related to the magnitude of the principal eigenvalue $\lambda_1^\infty$ for
\begin{equation}
\begin{cases}
-d_2 \Delta \psi(x) + a(x) \psi(x) - c_0(B_0^\ast f(K(\cdot)))(x) \psi(x) = \lambda \psi(x), & x \in \Omega \setminus \Sigma, \\
\psi(x) = 0, & x \in \partial \omega, \\
\partial_\nu \psi(x) = 0, & x \in \partial \Omega,
\end{cases}
\tag{23}
\end{equation}
and to the magnitude of the principal eigenvalue $\lambda_{1\gamma}^\infty$ for
\begin{equation}
\begin{cases}
-d_2 \Delta \psi(x) + a(x) \psi(x) + c_0(B_0^\ast f(K(\cdot)))(x) \psi(x) + \gamma \chi(x) \psi(x) = \lambda \psi(x), & x \in \Omega, \\
\partial_\nu \psi(x) = 0, & x \in \partial \Omega
\end{cases}
\tag{24}
\end{equation}
($\gamma > 0$). Here $K$ is the maximal nonnegative solution to
\begin{equation}
\begin{cases}
-d_1 \Delta K(x) = r(x) K(x) - \rho(x) K(x)^2, & x \in \Omega, \\
\partial_\nu K(x) = 0, & x \in \partial \Omega
\end{cases}
\end{equation}
(this problem has exactly two nonnegative solutions, one being trivial; actually $K \in C(\overline{\Omega}), K(x) > 0, \forall x \in \overline{\Omega}$ - see [9]).

The existence and the basic properties of the principal eigenvalues to (23) and (24) follow via Rayleigh’s principle. Let us notice (see Section 3) that the mapping $\gamma \mapsto \lambda_{1\gamma}^\infty$ is increasing and that $\lim_{\gamma \to +\infty} \lambda_{1\gamma}^\infty = \lambda_1^\infty$.

As far as the zero-stabilizability is concerned, the following result shall be proved in the present section:

**Theorem 7.3.** If the predator population in (18) is internally zero-stabilizable, then $\lambda_1^\infty \geq 0$.

Conversely, if $\lambda_1^\infty > 0$, then the pest population is internally zero-stabilizable, and for sufficiently large $\gamma > 0$, the feedback control $u := -\gamma p$ realizes (21) and (22) for any $h_0, p_0$ satisfying (H1').

**Proof.** Assume that the predator population is eradicable. For any $h_0, p_0$ satisfying (H1'), $h_0(x) > 0, p_0(x) > 0$ a.e. in $\Omega$, and for any stabilizing control $u$ we have that for any sufficiently small $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that
\begin{equation}
0 \leq (B_0 p^u(\cdot, t)))(x) \leq \varepsilon
\end{equation}
a.e. $\Omega \times [T(\varepsilon), +\infty)$. It follows that
\begin{equation}
0 \leq \tilde{h}(x,t) \leq h^0(x,t)
\end{equation}
a.e. $\Omega \times [T(\varepsilon), +\infty)$, where $\tilde{h}$ is the solution to
\begin{equation}
\begin{cases}
\partial_t h(x,t) - d_1 \Delta h(x,t) = r(x) h(x,t) - \rho(x) h(x,t)^2, & (x,t) \in \Omega \times (T(\varepsilon), +\infty), \\
\partial_\nu h(x,t) = 0, & (x,t) \in \partial \Omega \times (T(\varepsilon), +\infty), \\
h(x,T(\varepsilon)) = h^0(x, T(\varepsilon)), & x \in \Omega.
\end{cases}
\end{equation}
Since
\begin{equation}
\lim_{t \to +\infty} \|\tilde{h}(\cdot, t) - K(\cdot)\|_{L^\infty(\Omega)} = 0,
\end{equation}
Remark 7. If $h_0(x) \leq K(x)$, then it follows that $p(x, t) \leq p^0(x, t)$ a.e. in $Q$, and consequently
\[
\lim_{t \to +\infty} \|p(\cdot, t)\|_{L^\infty(\Omega)} = 0
\]
Remark 8. Since, usually the density of the prey population is less or equal than $K(x)$ even when there is no predator in the habitat, the hypothesis $h_0(x) \leq K(x)$ is not a very restrictive one.

We see how important is to maximize $\lambda_{\gamma}^\omega$ with respect to $\omega$.

Remark 8. A similar result to Theorem (7.3) may be obtained for the eradicability (zero-stabilizable) of the predator population described by (19), if we consider $\lambda_{\gamma}^\omega$ the principal eigenvalue to

$$
\begin{cases}
-d_2 \Delta \psi(x) + a(x)\psi(x) - c_0 f(K(x))(B_0p(\cdot))(x) = \lambda \psi(x), & x \in \Omega \setminus \omega \\
\psi(x) = 0, & x \in \partial \omega \\
\partial_n \psi(x) = 0, & x \in \partial \Omega.
\end{cases}
$$

8. Regional eradication of a specialized (pest) predator. Let $\gamma > 0$ be a constant affordable harvesting effort and consider $(h, p)$ the solution to (18) corresponding to the feedback control $u := -\gamma p$.

Let us formulate the problem of eradicating the pest population at a minimum cost. The damages produced by the pest population are proportional to the quantity (the total captured prey individuals)

$$
\int_0^\infty \int_\Omega f(h(x, t))(B_0p(\cdot, t))(x)dx \, dt,
$$

while the cost related to the choice of the subregion $\omega$ is

$$
\alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega);
$$

hence we may ask to

(P')

Minimize \( \{ \theta \int_0^\infty \int_\Omega f(h(x, t))(B_0p(\cdot, t))(x)dx \, dt + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega) \} \)

subject to $\omega$, where $\theta$ is a positive constant.

For $h_0(x) \leq K(x)$ a.e. in $\Omega$, where $K$ has been defined in the previous section, we conclude that $h(x, t) \leq K(x)$ a.e. in $Q$, and consequently we get that

$$
0 \leq f(h(x, t))(B_0p(\cdot, t))(x) \leq f(K(x))(B_0p(\cdot, t))(x) \leq f(K(x))(B_0y(\cdot, t))(x)
$$

a.e. in $Q$, where $y$ is the solution to

$$
\begin{cases}
\partial_t v(x, t) - d_2 \Delta v(x, t) = -a(x)v(x, t) + c_0(B_0^\gamma f(K(\cdot)))(x)v(x, t) \\
-\gamma \chi(x)v(x, t), & (x, t) \in Q \\
\partial_n v(x, t) = 0, & (x, t) \in \partial \Sigma \\
v(x, 0) = p_0(x), & x \in \Omega.
\end{cases}
$$

This yields

$$
\theta \int_0^\infty \int_\Omega f(h(x, t))(B_0p(\cdot, t))(x)dx \, dt + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega)
$$

$$
\leq \theta \int_0^\infty \int_\Omega f(K(x))(B_0y(\cdot, t))(x)dx \, dt + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega).
$$

Without restricting the generality we may assume that $\theta = 1$. Problem (P') may be treated by using the level set method as in Section 5. Another possibility is to treat a related weaker problem for a single population species

(RP)

Minimize \( \{ \int_0^\infty \int_\Omega f(K(x))(B_0y(\cdot, t))(x)dx \, dt + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial \omega) \} \),
subject to \( \omega \). If for a certain \( \omega \) we get a small value for \((RP)\), then for the same \( \omega \) we get even a smaller value for \((P')\) (due to the fact that \( y(x,t) \geq p(x,t) \) a.e. in \( Q \)).

Problem \((RP)\) may be rewritten as

\[
\text{Minimize } \left\{ \int_0^\infty \int_\Omega f(K(x))(B_0y(\cdot,t))(x)dx \, dt + \alpha \int_\Omega H(\varphi(x))dx + \beta \int_\Omega \delta(\varphi(x))|\nabla \varphi(x)|dx \right\}
\]

subject to \( \varphi \), where \( y \) is the solution to

\[
\begin{aligned}
\partial_t y(x,t) - d_2 \Delta y(x,t) &= -a(x)y(x,t) + c_0(B_0^*f(K(\cdot)))(x)y(x,t) \\
-\gamma H(\varphi(x))y(x,t), & \quad (x,t) \in Q \\
\partial_\nu y(x,t) &= 0, & \quad (x,t) \in \partial \Sigma \\
y(x,0) &= p_0(x), & \quad x \in \Omega.
\end{aligned}
\]

Problem \((RP)\) may be approximated by

\[
\text{Minimize } \tilde{J}(\varphi),
\]

where \( \varphi : \overline{\Omega} \to \mathbb{R} \) is a smooth function,

\[
\tilde{J}(\varphi) = \int_0^T \int_\Omega f(K(x))(B_0y(\cdot,t))(x)dx \, dt + \alpha \int_\Omega H(\varphi(x))dx + \beta \int_\Omega \delta(\varphi(x))|\nabla \varphi(x)|dx,
\]

and \( y^\varepsilon \) is the solution to

\[
\begin{aligned}
\partial_t y(x,t) - d_2 \Delta y(x,t) &= -a(x)y(x,t) + c_0(B_0^*f(K(\cdot)))(x)y(x,t) \\
-\gamma H(\varphi(x))y(x,t), & \quad (x,t) \in Q \\
\partial_\nu y(x,t) &= 0, & \quad (x,t) \in \partial \Sigma \\
y(x,0) &= p_0(x), & \quad x \in \Omega.
\end{aligned}
\]

Here \( \varepsilon \) is a small positive number and \( T > 0 \) is a large positive number, and \( \tilde{T} := T \).

The next result gives the directional derivative of \( \tilde{J} \):

**Theorem 8.1.** For any smooth functions \( \varphi, \psi : \overline{\Omega} \to \mathbb{R} \) we have that

\[
ad\tilde{J}(\varphi)(\psi) = \int_\Omega \delta(\varphi(x))\psi(x)[\gamma \int_0^T r^\varphi(x,t)y^\varphi(x,t) \, dt + \alpha \int_\Omega H(\varphi(x))dx + \beta \int_\Omega \delta(\varphi(x))|\nabla \varphi(x)|dx]
\]

where \( y^\varphi \) is the solution to \((25)\), and \( r^\varphi \) is the solution to

\[
\begin{aligned}
\partial_t r(x,t) + d_2 \Delta r(x,t) &= a(x)r(x,t) - c_0(B_0^*f(K(\cdot)))(x)r(x,t) \\
+\gamma H(\varphi(x))r(x,t) + (B_0^*f(K(\cdot)))(x), & \quad (x,t) \in Q \\
\partial_\nu r(x,t) &= 0, & \quad (x,t) \in \Sigma \\
r(x,T) &= 0, & \quad x \in \Omega.
\end{aligned}
\]

The proof follows in the same manner as the proof of Theorem \((5.1)\).

**Remark 9.** From \((26)\) in Theorem \((8.1)\) we get that the gradient descent with respect to \( \varphi \) is given by \((17)\), where \( r^\varphi \) is the solution to \((27)\).
Remark 10. A similar result may be obtained for the regional eradication of the predator population related to (19).

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