Comparing Merton model and Gram-Charlier model to capture skewness and kurtosis on bond performance

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Abstract. The existence of skewness and kurtosis in finance data distribution is as a generalization of the normal density. Non-normal skewness and kurtosis of underlying asset of bond issuer significantly contribute to the phenomenon of volatility smile. Merton jump diffusion model is one of the first beyond Black-Scholes model in the sense that it tries to capture the effect of skewness and kurtosis of the asset prices density by a simple addition of a compound Poisson jump process. Another approach to consider the effect of skewness and kurtosis in asset prices for bond valuation is the Gram-Charlier (G-C) expansion. Hermite polynomial is used to get an expansion of the probability distribution in G-C method. In this paper we compare Merton Jump Diffusion (MJD) Model and G-C model in the term of equity and default probability. The result showed that G-C model is more consistent than MJD model when the skewness and kurtosis are taken into account.

1. Introduction
Bond valuation modeling without coupons began by [1] using a formula similar to the Black-Scholes (B-S) option price [2]. The practical assumption in the Black-Scholes model is that the return on company assets is normally distributed with constant volatility. However, it is often found that the return of assets that are not normally distributed, in this case have skewness and kurtosis values that do not meet the standards. Several studies have found evidence that financial data is very sensitive to market conditions. It is due to the nature of financial data which tends to contain extreme data [3]. The extreme nature of a group of data is often identified based on the nature of heavy tail or fat tail from a data distribution.

Heavy tail or fat tail phenomenon from data distribution, namely the form of probability distribution that describes the nature of data that has a thicker shape than the normal distribution in the tail of the distribution. This heavy tail phenomenon is often also followed by the occurrence of a large frequency of occurrence in the part of the mode of data distribution which is often referred to as the occurrence of leptokurtic or excess kurtosis. It also became a motivation for researchers to develop a non-Gaussian model [4, 5].

Several theories have been put forward to solve this problem. Jarrow and Rudd [6] use the deviation from the Gaussian moment as an unknown distribution approach with Edgeworth series expansion. The Cornish-Fisher model is using α-th quantile development based on the transformation of the variable random Gaussian standard into a non-standard Gaussian random variable [7]. While [8] has model stock price returns with Gram-Charlier expansion for option valuations.
Merton models are the main paper in reference to bond valuations. This theory has developed rapidly, including the existence of Merton Jump Diffusion Model which considers the existence of skewness and excess kurtosis. One paper that discusses this theory is Matsuda [9] with the Levy process approach. In addition, the Gram-Charlier expansion as a generalization of the normal distribution to overcome the existence of skewness and kurtosis parameters is also quite widely used in this problem [10]. One approximation method used is an alternative approach to Hermite polynomials [11]. Knight and Satchell [12] have applied the Gram-Charlier theory to the measurement of option prices. Berberan Santos [13] developed the Gram-Charlier model with Hermite polynomials for any distribution and applied to the option price assessment. Chateau and Dufresne [14] use Hermite polynomials for Gram-Charlier expansion on the valuation of option prices with basic assets normally distributed. Abdurakhman and Maruddani [16] provide an analysis of the effect of skewness and kurtosis on bond valuations on corporate data bond issues in Indonesia.

Based on the background of the above problems, this paper discusses a new mathematical model for bond valuation with a period of time with asset data having skewed and fat tails (kurtosis). The valuation includes calculating the estimated equity and the chance of bankruptcy of the company that issued the bond based on the company’s assets. Modeling will be carried out based on Merton Jump Diffusion Model and Gram-Charlier expansion with Hermite polynomial approach.

2. Bond Valuation with Gram-Charlier Model

Gram-Charlier expansion is a very popular expansion to estimate normal density. In the last two decades, the use of this expansion has been introduced in the field of finance for leptokurtic return model, skewness, group volatility and so on. Hermite polynomials can be used to obtain expansion of probability functions in a series of derivatives \( n(z) \). One of the advantages of this polynomial is the fact that the density function can be formally expanded as:

\[
g(z) = c_n H_n(z) n(z) \tag{1}
\]

Where \( n(z) \) is the standard normal density function, \( H_n(z) \) is the Hermite order \( n \)th polynomial of equation (3). The coefficient \( c_n \) of (4) is derived from the Hermite Polynomial. If the equation (1) of the two segments multiplied by \( H_n(z) \), and it is integrated from \(-\infty \) to \( \infty \). Gram-Charlier expansion which is defined the normal standard density approximation until 4th moment is

\[
g(z) = n(z) \left\{ 1 + \mu_1 z + \frac{1}{2} (\mu_2 - 1)(z^2 - 1) + \left( \frac{1}{6} \mu_3 - \frac{1}{2} \mu_1 \right) (z^3 - 3z) + \left( \frac{1}{24} (\mu_4 - 6\mu_2 + 3) (z^4 - 6z^2 + 3) \right) \right\}
\]

(2)

with

\[ \mu_i = E[X - \mu]^i \]

In order to remedy the assumption of a Gaussian marginal distribution for the underlying asset returns in the classical Black-Scholes-Merton model, we use general Gaussian alternative classes for the underlying asset. Jarrow and Rudd in 1982 [6] model the distribution of stock price with an Edgeworth series expansion. Corrado and Su in 1996 [8] modeled the distribution of stock log prices with a Gram-Charlier series expansion. Those method focuses on the skewness and kurtosis deviation from normality for stock returns. If asset prices follow geometric Brownian motion with model

\[
A(t) = A(0) \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) T + \sigma W(t) \right\} \tag{3}
\]

Which \( W(t) \sim N(0, T) \) and \( \ln A(t) \sim N(\mu, \sigma^2) \), with mean \( \mu = \ln A_0 + \left( r - \frac{\sigma^2}{2} \right) T \) and the variance is \( \sigma^2 T \), then
\[ \ln A(t) = \ln A_0 + \left( r - \frac{\sigma^2}{2} \right) T + \sigma W(t) \]

\( W(t) \sim GC(0, T, \mu_3, \mu_4) \). Because of \( A(t) \) linear with \( W(t) \), we get \( \ln A_T \sim GC(0, T, \mu_3, \mu_4) \). For simplified of calculation, we use standard normal transformation

\[ Z = \frac{\ln A(t) - \mu}{\sigma \sqrt{T}} \]

For \( Z \sim GC(0,1, \mu_3, \mu_4) \). Use the expectation formula [16], we have equity expectation of the firm at maturity date is: [16]

\[ E^T_{T_0, GC} = E^T_{T_0} + \frac{\mu_3}{3!} I_1 + \frac{\mu_4 - 3}{4!} I_2 \]  (4)

With

\[ I_1 = \sigma \sqrt{T} A(0) \left( n(d_1) \left[ 2\sigma \sqrt{T} - d_1 \right] + \sigma^2 T N(d_1) \right) \]

\[ I_2 = \sigma \sqrt{T} A(0) \left[ \left( n(d_1) \left(d_1^2 - 3\sigma \sqrt{T}(d_1 - \sigma \sqrt{T}) - 1 \right) \right) + \left( \sigma \sqrt{T} \right)^3 N(d_1) \right] \]

We define \( \tau = \text{default time} \)

\[ \tau = \begin{cases} \infty & A(t) \geq K \\ T & A(t) < K \end{cases} \]

The default probabilities of the firm at maturity date [16]:

\[ P(A(t) < K) = \int_{-\infty}^{-d_2} n(z) \left\{ 1 + \frac{\mu_3}{3!} (z^3 - 3z) + \frac{\mu_4 - 3}{4!} (z^4 - 6z^2 + 3) \right\} dz \]  (5)

3. **Bond Valuation with Merton Jump Diffusion Model**

Merton Jump Diffusion (MJD) model is one of Black-Scholes theory development to capture the skewness and excess kurtosis of the underline asset density by a simple addition of a compound Poisson Process. Merton uses Levy process by adding compound Poisson process (discontinuous jump process) into Brownian motion with drift (continuous diffusion process). The probability that an asset price jumps during a small time interval \( dt \) can be written using Poisson process \( dN \), as

\[ P\{\text{an asset price jumps once in } dt\} = P\{dN(t) = 1\} \approx \lambda \ dt \]

\[ P\{\text{an asset price jumps more than once in } dt\} = P\{dN(t) \geq 2\} \approx 0 \]

\[ P\{\text{an asset price does not jumps in } dt\} = P\{dN(t) = 0\} \approx 1 - \lambda \ dt \]

Where the parameter \( \lambda \) is the intensity of the jump process which is independent of time \( t \). MJD dynamics of asset prices which is incorporates the above properties take the Stochastic Differential Equation with the form

\[ \frac{dA(t)}{A(t)} = (\alpha - \lambda k)dt + \sigma dW(t) + (y_t - 1)dN(t) \]  (6)

\( y_t \) is nonnegative random variables drawn from lognormal distribution \( \ln Y_T \sim N(\mu, \delta^2) \). Then the solve of Stochastic Differential Equation from the above equation is

\[ dA(t) = (\alpha - \lambda k)A(t)dt + \sigma A(t)dW(t) + (y_t - 1)dA(t)N(t) \]  (7)

Then the dynamics of asset prices is

\[ A(T) = A(0) \exp \left\{ \left( r - \frac{\sigma^2}{2} - \lambda k \right) T + \sigma W(T) + \Sigma_{i=1}^{N_t} Y_i \right\} \]  (8)
In this theory, we add three parameters, $\lambda$, $\mu$, and $\delta$ to the original Black-Scholes model which give us to control skewness and excess kurtosis, which equation: [9]

\[\text{Mean}[A(T)] = \alpha - \frac{\sigma^2}{2} - \lambda \left( \exp\left(\mu + \frac{\delta^2}{2}\right) - 1 \right) + \lambda \mu\]

\[\text{Variance}[A(T)] = \sigma^2 + \lambda \delta^2 + \lambda \mu^2\]

\[\text{Skewness}[A(T)] = \frac{\lambda (3 \delta^2 \mu + \mu^3)}{(\sigma^2 + \lambda \delta^2 + \lambda \mu^2)^{3/2}}\]

\[\text{Excess Kurtosis}[A(T)] = \frac{\lambda (3 \delta^4 + 6 \delta^2 \mu + \mu^3)}{(\sigma^2 + \lambda \delta^2 + \lambda \mu^2)^{3/2}}\]

Then we have equity expectation of the firm at maturity date is

\[E_{T_0,MJD}^T = \sum_{i=1}^{N_i} \exp\left(\frac{-\lambda \tau}{\delta^2}\right) V_{BS}(S_t, \tau = T - t, \sigma_i, r_i)\]  \hspace{1cm} (9)

with:

\[\lambda = \lambda (1 + k) = \lambda \exp\left(\mu + \frac{\delta^2}{2}\right)\]

\[\sigma_i^2 = \sigma^2 + \frac{i \delta^2}{\tau}\]

\[r_i = r - \lambda k + \frac{i \ln(1 + k)}{\tau} = r - \lambda \left( \exp\left(\mu + \frac{\delta^2}{2}\right) - 1 \right) + \frac{i (\mu + \frac{\delta^2}{2})}{\tau}\]

$V_{BS}$ = equity expectation of Black-Scholes model without jump. Use the formula in [17], the default probabilities of the firm at maturity date ($T$) is:

\[P(A_T < K) = \sum_{i=0}^{\infty} \frac{\exp\left(-\lambda \tau\right) (\lambda \tau)^i}{i!} N\left(\frac{\ln(K) - \ln(A_T) - (r - \sigma^2/2 - \lambda k)T - \mu \tau)}{\sqrt{\sigma^2 T + i \sigma^2}}\right)\]  \hspace{1cm} (10)

4. Data and Methods

Indonesia corporate bond data is derived from publicly available databases obtained from Indonesian Bond Pricing Agency (IBPA) 2017 on website [www.ibpa.co.id](http://www.ibpa.co.id). We use bond data issued by PT. Bank Tabungan Pensiunan Nasional Tbk on 2017 code BTPN named “Obligasi III BTPN Tahap II Tahun 2017 Seri B” which is issued on October 17th, 2107. The profile structure of this bond is given at table 1.

| Table 1. Bond Profile Structure of “Bank Tabungan Pensiunan Nasional Tbk” on 2017 |
|-----------------------------------------------|
| **Outstanding** | **Issue Term** | **Coupon Structure** |
| 900,000,000,000.00 | 3 years | Fixed 7.5% |

Total asset data of the firm is published by Indonesian Bank consists of monthly prices October 2012 until September 2017 on website [www.bi.go.id](http://www.bi.go.id). According to [18], investors’ main concern will be on the return on investment which refers to the percentage growth in the value of an asset.
5. Results and Discussion
For deriving the probability of default, equity, and liability of the bond, we have to do some steps for fulfilling the assumptions. First, we have to check whether the natural logarithm of total assets data is normally distribution or not. We do hypothesis test of normality distribution with Jarque-Bera Test. The result of the Jarque-Bera test is given at table 2. Because of the p-value (0.03892) is less than \( \alpha \) (0.05), we can conclude that this series is not normally distributed.

| Table 2. Jarque-Bera Test for Ln Return Assets |
|-----------------------------------------------|
| X-squared | df     | p-value |
|-----------|--------|---------|
| 13.4275   | 2      | 0.03892 |

Then, we have to estimate some parameters. In table 3, we give summarize for those parameters.

| Table 3. Summarize Value of the Parameters |
|-------------------------------------------|
| Parameter                          | Value                  |
|---------------------------------------|------------------------|
| Asset Value at October 2012 \( (A_o) \) | 85,932,429,000,000      |
| Interest risk free rate \( (r) \)     | 5.5 %                  |
| Mean \( (\mu) \)                      | 0.00704665             |
| Variance \( (\sigma^2) \)             | 0.00056698             |
| Volatility \( (\sigma) \)             | 1.9351 %               |
| Minimum                               | -0.03163991            |
| Maximum                               | 0.07507912             |
| Skewness                              | 0.76758620             |
| Kurtosis                              | 3.53310100             |

The kurtosis value in Table 3 is greater than 3, it can be concluded that the data on the assets of BTPN are heavy tail. This is an indication that there is extreme data on asset data. So the Merton Jump Diffusion Model and Gram-Charlier approach can be applied to this case. The results computation using equations (4),(5),(9), and (10) is given at table 4. From that table we can interpret shows that the probability of bankruptcy at maturity payments is very small on those two models, with MJD model give smaller probability. The expectation of equity also shows good performance. It can be seen from the value on those two models shows exceeds from debt value. It indicates that the default probability of PT. BTPN at maturity time is very small. The expectation value of equity in the model also far exceeds the debt of the company, so it can be said that the company has good enough performance to be able to pay bond debt until maturity.

| Table 4 Bond Valuation of Bank Tabungan Pensiunan Nasional Tbk on 2017 |
|---------------------------------------------------------------|
| Gram-Charlier Model    | Merton Jump Diffusion Model                        |
|------------------------|-----------------------------------|
| Expectation of Equity | 84,429,920,000,000                 | 84,254,970,000,000              |
| Expectation of Liability | 1,502,512,000,000           | 1,677,462,000,000               |
| Default Probability    | \( 2.932583 \times 10^{-69} \) | \( 5.634933 \times 10^{-12} \) |

6. Conclusion
From the asset data profile, it shows extreme value at some time point. So, those two methods fit to modelling this case. Both of two methods, Gram-Charlier Model and Merton Jump Diffusion Model,
give suit results for profiling the bond performance of Bank Tabungan Pensiunan Nasional Tbk on 2017. This bond has AAA rating from S&P Rating so it has good financial performance and it is appropriate with this research result that this corporation has very small probability of default.

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