The linear and nonlinear Jaynes-Cummings model for the multiphoton transition

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With the Jaynes-Cummings model, we have studied the atom and light field quantum entanglement of multiphoton transition, and researched the effect of initial state superposition coefficient $C_1$, the transition photon number $N$, the quantum discord $\delta$ and the nonlinear coefficient $\chi$ on the quantum entanglement degrees. We have given the quantum entanglement degrees curves with time evolution, and obtained some results, which should have been used in quantum computing and quantum information.

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1. Introduction

The interaction between a two-level system and a harmonic oscillator is ubiquitous in different physical setups, ranging from quantum optics to condensed matter and applications to quantum information. Typically, due to the parameter accessibility of most experiments, the rotating-wave approximation (RWA) can be applied producing a solvable dynamics called the Jaynes-Cummings model (JCM) [1], this model has been widely generalized to treat various interactions between atoms and photons. These include, e.g., multilevel atoms interact with multimode quantized fields, and various multiphoton processes in quantum optics [2]. It gives arise to many quantum phenomena that can not be explained in classical terms, such as the collapses and revivals of the atomic population inversion [3], squeezing of the field [4], and atom-cavity entanglement [5]. Recent experiments with Rydberg atoms and microwave photons in a superconducting cavity have turned the JCM from a theoretical curiosity to a useful and testable enterprise [6]. Such a system is also suitable for quantum state engineering and quantum information processing.

Entanglement plays a central role in quantum information, quantum computation and communication, and quantum cryptography. A lot of schemes are proposed for many-particle entanglement generation. The simplest scheme to investigate the atom-field entanglement is the Jaynes-Cummings model (JCM) describing an interaction of a two-level atom with a singlemode quantized radiation field. This model is of fundamental importance for quantum optics [7, 8] and is realizable to a very good approximation in experiments with Rydberg atoms in high-Q superconducting cavities, trapped ions, superconducting circuits etc. [9, 10]. The model predicts a variety of interesting phenomena. The atom-field entanglement is among them. An investigation of the atom-field entanglement for JCM has been initiated by Phoenix and Knight [11] and Gea-Banacloche [12]. Gea-Banacloche has derived an asymptotic result for the JCM state vector which is valid when the field is initially in a coherent state with a large mean photon number.

The atomic systems have found new application in quantum information processing [13]. The main requirement in quantum information processing task is the quantum entanglement. The interaction of atoms with cavity field has been shown to be an efficient source of atom-atom, atom-cavity, and cavity-cavity entanglement [14-16]. Appreciable number of studies related to the study of influence of atom-field interaction on the entanglement between atom and field has been done. The reason behind the study of quantum entanglement in such system is due to its fundamental nature and its applicability. However as discussed in previous chapter quantum entanglement do not exhausts the complete quantum correlations

The theory outlined in has been generalized for two-photon JCM [17], two-photon JCM with nondegenerate two-photon and Raman transitions [18, 19], two-atom JCM [20], two-atom one-mode Raman coupled model

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Two-photon processes are known to play a very important role in atomic systems due to high degree of correlation between emitted photons. An interest for investigation of the two-photon JCM is stimulated by the experimental realization of a two-photon one-atom micromaser on Rydberg transitions in a microwave cavity. JCM with nondegenerate two-photon transitions have attracted a great deal of attention. The foregoing model have been considered in terms of atomic population dynamics research, field statistics research, field and atom squeezing analysis, atom and field entropy and entanglement examining. The two-atom two-photon JCM for initial two-mode coherent cavity field has been investigated for nondegenerate two-photon transitions in.

The single photon and double photon Jaynes-Cummings model had been studied largely. In this paper, we have studied the atom and light field quantum entanglement of multiphoton transition, and researched the effect of initial state superposition coefficient $C_1$, the initial photon number $n$, the transition photon number $N$ ($N = 1, 2, 3, 4, 5, 6$), the quantum discord $\delta$ and nonlinear coupling constant $\chi$ on the quantum entanglement degrees. We have given the quantum entanglement degrees curves with time evolution, and obtained some results. When the transition photon number $N$ increases, the entanglement degrees oscillation get faster. When the initial state superposition coefficient $C_1 = 0$, with the quantum discord $\delta$ increase, the entanglement degrees oscillation get slowly. When the initial state superposition coefficient $C_1 = 0.76$, the the quantum discord increases, the entanglement degrees oscillation get quickly. when the nonlinear coefficient $\alpha > 0$, the entanglement degrees oscillation get quickly. When the nonlinear coefficient $\alpha < 0$, the entanglement degrees oscillation get slow. It is benefit to the atom and light field entanglement obviously. These results have important significance in the quantum communication and quantum information.

2. The multiphoton Jaynes-Cummings model and entanglement degrees

Let us consider the N-photon Jaynes-Cummings model, the Hamiltonian is

$$H = \omega a^+ a + \frac{1}{2} \omega_0 \sigma_z + g(a^+ a \sigma_+ + a^N \sigma_+) + \chi a^+ a^2, \quad (\hbar = 1)$$

(1)

where $a(a^+)$ is the annihilation (creation) operator for a photon in an electromagnetic mode of frequency, $\omega_0$ is the radiative field mode frequency, $\sigma_z = | a > < a | - | b > < b |$, $\sigma_+ = | a > < b |$, $\sigma_- = | b > < a |$. The nonlinear part $\chi a^+ a^2$ may be thought of as having been obtained from a quartic potential through the rotating-wave approximation, $\chi$ denotes the third-order susceptibility of Kerr medium, it is the nonlinear coupling constant of light field and nonlinear medium. This Hamiltonian, simple as it may seem, appears to lie in the center of many theoretical investigations. Drummond and Walls used it to examine the optical bistability in nonlinear medium. Mittelmann and Walls and then Risken et al. used it to examine the optical bistability in nonlinear medium.

The initial state is

$$| \psi(0) > = c_1(0) | a, n > + c_2(0) | b, n + N >$$

(2)

where $|c_1(0)|^2 + |c_2(0)|^2 = 1$, the state $| b >$ is atom ground state, state $| a >$ is atom excited state, and the wave function at any time is

$$| \psi(t) > = c_1(t) | a, n > + c_2(t) | b, n + N >,$$

(3)

substituting Eqs. (1) and (3) into Schrodinger equation

$$i \frac{\partial}{\partial t} | \psi(t) > = H | \psi(t) >,$$

(4)
we obtain
\[
i\frac{\partial}{\partial t}(c_1(t) \mid a, n > c_2(t) \mid b, n + N >) = (\omega a_1 + \frac{1}{2}\omega_0\sigma_1 + g(a^n\sigma_- + a^N\sigma_+) + \chi a^n\sigma_2)(c_1(t) \mid a, n > c_2(t) \mid b, n + N >)
\]
\[
= (\omega n c_1(t) + \frac{1}{2}\omega_0 c_1(t) + gc_1(t)\sqrt{(n + 1)(n + 2) \cdots (n + N) + \chi(n - 1)c_1(t)}) \mid n, a >
\]
\[
+ (\omega(n + N)c_2(t) - \frac{1}{2}\omega_0 c_2(t) + gc_2(t)\sqrt{(n + 1)(n + 2) \cdots (n + N)}
\]
\[
+ \chi(n + N)(n + N - 1)c_2(t)) \mid n + N, b >,
\]
comparing the both sides of Eq. (5), we have
\[
i\frac{\partial}{\partial t}c_1(t) = \omega n c_1(t) + \frac{1}{2}\omega_0 c_1(t) + gc_1(t)\sqrt{(n + 1)(n + 2) \cdots (n + N) + \chi(n - 1)c_1(t)},
\]
\[
i\frac{\partial}{\partial t}c_2(t) = \omega(n + N)c_2(t) - \frac{1}{2}\omega_0 c_2(t) + gc_2(t)\sqrt{(n + 1)(n + 2) \cdots (n + N) + \chi(n + N)(n + N - 1)c_2(t)},
\]
using Laplace transforms to the both sides of Eqs. (6) and (7), we get
\[
ip\mathcal{L}_1(p) - \frac{1}{2}\omega_0 L_1(p) + g\sqrt{(n + 1)(n + 2) \cdots (n + N)}L_2(p) + \chi(n - 1)L_1(p),
\]
\[
ip\mathcal{L}_2(p) - \frac{1}{2}\omega_0 L_2(p) + g\sqrt{(n + 1)(n + 2) \cdots (n + N)}L_1(p)
\]
\[
+ \chi(n + N)(n + N - 1)L_2(p),
\]
We obtain
\[
L_1(p) = \frac{c_1}{p + i\omega n + \frac{1}{2}\omega_0 + i\chi n(n - 1)} - i\frac{g\sqrt{(n + 1)(n + 2) \cdots (n + N)}}{p + i\omega n + \frac{1}{2}\omega_0 + i\chi n(n - 1)} \left( \frac{D}{p + A} + \frac{E}{p + B} \right),
\]
\[
L_2(p) = \frac{D}{p + A} + \frac{E}{p + B},
\]
where \(L_1(p) = \mathcal{L}[c_1(t)]\) and \(L_2(p) = \mathcal{L}[c_2(t)]\) are Laplace transform of functions \(c_1(t), c_2(t),\) and
\[
A = i\omega(2n + N) + i\chi[2n^2 + 2(N - 1)n + N^2 - N] + i\sqrt{\omega_1^2 + \omega_2^2},
\]
\[
B = i\omega(2n + N) + i\chi[2n^2 + 2(N - 1)n + N^2 - N] - i\sqrt{\omega_1^2 + \omega_2^2},
\]
\[
D = \frac{(\sqrt{\omega_1^2 + \omega_2^2} - \omega_2)c_2 + \omega_1 c_1}{2\sqrt{\omega_1^2 + \omega_2^2}},
\]
\[
E = \frac{(\sqrt{\omega_1^2 + \omega_2^2} - \omega_2)c_2 - \omega_1 c_1}{2\sqrt{\omega_1^2 + \omega_2^2}},
\]
Using Laplace retransforms to Eqs. (10) and (11), we have
\[ c_1(t) = e^{-\frac{i}{2}(2n+N)t}e^{-\frac{i}{2}x(2n^2+2(N-1)n+N^2-N)t}[c_1 \cos\left(\frac{\sqrt{\omega_1^2 + \omega_2^2}}{2} t\right) - \iota \frac{c_1 \omega_1 + c_2 \omega_2}{\sqrt{\omega_1^2 + \omega_2^2}} \sin\left(\frac{\sqrt{\omega_1^2 + \omega_2^2}}{2} t\right)] \]  

\[ c_2(t) = e^{-\frac{i}{2}(2n+N)t}e^{-\frac{i}{2}x(2n^2+2(N-1)n+N^2-N)t}[c_2 \cos\left(\frac{\sqrt{\omega_1^2 + \omega_2^2}}{2} t\right) - \iota \frac{c_1 \omega_1 - c_2 \omega_2}{\sqrt{\omega_1^2 + \omega_2^2}} \sin\left(\frac{\sqrt{\omega_1^2 + \omega_2^2}}{2} t\right)] \]  

where \( \delta = \omega_0 - N\omega, \omega_1 = 2g\sqrt{(n+1)(n+2) \cdots (n+N)}, \omega_2 = \delta - N\chi(2n+N-1) \). 

With the state (2), we can obtain the density operator of atom-photon system
\[ \hat{\rho}(t) = |\psi(t)\rangle \langle \psi(t)| + |c_1(t)|^2|n, a > < n, a| + |c_1(t)c_2(t)|n, a > < b, n+2| 
+c_2(t)c_1(t)|n+2, b > < a, n| + |c_2(t)|^2|n+2, b > < b, n+2|, \]

the reduce density operator of atom \( A \) is
\[ \hat{\rho}_A(t) = tr_{(\alpha)} \hat{\rho}(t) 
= |c_1(t)|^2|a > < a| + |c_2(t)|^2|b > < b|, \]

the matrix form of \( \hat{\rho}_A(t) \) at basis vectors \(|a > \) and \(|b >\)
\[ \hat{\rho}_A(t) = \begin{pmatrix} |c_1(t)|^2 & 0 \\
0 & |c_2(t)|^2 \end{pmatrix}, \]

the quantum system entanglement degrees is
\[ E(t) = -tr(\hat{\rho}_A(t) \log_2 \hat{\rho}_A(t)) 
= -tr\left( \begin{pmatrix} |c_1(t)|^2 & 0 \\
0 & |c_2(t)|^2 \end{pmatrix} \cdot \begin{pmatrix} \log_2 |c_1(t)|^2 & 0 \\
0 & \log_2 |c_2(t)|^2 \end{pmatrix} \right), \]
\[ = -(|c_1(t)|^2 \log_2 |c_1(t)|^2 + |c_2(t)|^2 \log_2 |c_2(t)|^2), \]  

3. Numerical result

In this section, we shall calculate the quantum entanglement degrees with Eqs. (15), (16) and (20). The entanglement degree \( E \) is in the range of \( 0 \approx 1 \). From FIGs. 1 to 5, we calculate the entanglement degree with linear Jaynes-Cummings model, i.e., \( \chi = 0 \). In FIGs. 1 and 2, the initial state superposition coefficient \( C_1 = 0 \), the entanglement degree \( E = 0 \), i.e., the atom and light field is not in entangled state at initial state, the initial photon number \( n = 0 \). With the time evolution, the atom and light field should be in the entangled state \( E \neq 0 \). In FIG. 1 (a)-(f), the quantum discord \( \delta^2 = 0 \), the transition photon numbers \( N \) are 1, 2, 3, 4, 5 and 6, respectively. In FIG. 1 (a)-(f), the entanglement degree \( 0 \leq E \leq 1 \). When the numbers of photons \( N \) increases, the entanglement degrees oscillation get faster. When \( N = 1, 2, 3 \), the evolution curves of entanglement degrees change slowly with time \( t \), and stay the time of entanglement degrees \( E \approx 1 \) more long. When \( N = 4, 5, 6 \), the entanglement degree oscillate quickly, and they oscillate more quickly with \( N \) increase. In FIG. 2, the quantum discord \( \delta^2 = 4g^2 \), and other parameters are the same as FIG. 1. Comparing FIG. 2 with FIG. 1, When the quantum discord \( \delta \) increase, the entanglement degrees oscillation get slowly. In FIG. 3, the initial state superposition coefficient \( C_1 = 0.96 \), and other parameters are the same as FIG. 1. The atom and light field is in entangled state at initial state. With the time evolution, the atom and light field should be in the entangled state. When \( N \) increase, the entangled degrees increase for \( N = 1, 2, 3, 4, 5, 6 \). In FIGs. 4 and 5, the initial state superposition coefficient \( C_1 = 0.76 \), the atom and light
field is in maximum entangled state at initial state. In FIG. 4, the quantum discord \( \delta^2 = 0 \). With the time evolution, the atom and light field should be in the maximum entangled state for \( N = 1, 2, 3, 4, 5, 6 \). In FIG. 5, the quantum discord \( \delta^2 = 8g^2 \). With the quantum discord \( \delta \) increase, when \( N = 1, 2, 3 \), the entanglement degree oscillate quickly, when \( N = 4, 5, 6 \), they keep the maximum entangled state.

From FIGs. 6 to 9, we calculate the entanglement degree with nonlinear Jaynes-Cummings model, i.e., \( \chi \neq 0 \). The initial state superposition coefficient \( C_1 = 0 \), the initial photon number \( n = 0 \), the quantum discord \( \delta^2 = 4g^2 \), and they corresponding to nonlinear coefficient are \( \alpha = \frac{\chi}{g} = 0.2, 0.8, -0.2 \) and \(-0.8 \). Comparing FIGs. 6 with 7, we find when the nonlinear coefficient increase (\( \alpha > 0 \)), the entanglement degrees oscillation get quickly. Comparing FIGs. 8 with 9, we find when the nonlinear coefficient decrease \( \alpha < 0 \), the entanglement degrees oscillation get quickly. Comparing FIG. 6 with 2, we find when the nonlinear coefficient \( \alpha > 0 \), the entanglement degrees oscillation get quickly, to the disadvantage of the atom and light field entanglement. Comparing FIG. 8 with 2, we find when the nonlinear coefficient \( \alpha < 0 \), the entanglement degrees oscillation get slow. It is benefit to the atom and light field entanglement obviously.

4. Conclusion

In this paper, we have studied the atom and light field quantum entanglement of multiphoton transition, and researched the effect of initial state superposition coefficient \( C_1 \), the initial photon number \( n \), the transition photon number \( N (N = 1, 2, 3, 4, 5, 6) \), the quantum discord \( \delta \) and nonlinear coupling constant \( \chi \) on the quantum entanglement degrees. We have given the quantum entanglement degrees curves with time evolution, and obtained some results. When the transition photon number \( N \) increases, the entanglement degrees oscillation get faster. When the initial state superposition coefficient \( C_1 = 0 \), with the quantum discord \( \delta \) increase, the entanglement degrees oscillation get slowly. When the initial state superposition coefficient \( C_1 = 0.76 \), the the quantum discord increases, the entanglement degrees oscillation get quickly, when the nonlinear coefficient \( \alpha > 0 \), the entanglement degrees oscillation get quickly. When the nonlinear coefficient \( \alpha < 0 \), the entanglement degrees oscillation get slow. It is benefit to the atom and light field entanglement obviously. These results have important significance in the quantum communication and quantum information.

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FIG. 1: The curves of the atom and light field entanglement degrees with time evolution, the initial state superposition coefficient $C_1 = 0$, the initial photon number $n = 0$, the quantum discord $\delta^2 = 0$, the nonlinear coefficient $\alpha = \frac{\chi}{g} = 0$.

FIG. 2: The curves of the atom and light field entanglement degrees with time evolution, the initial state superposition coefficient $C_1 = 0$, the initial photon number $n = 0$, the quantum discord $\delta^2 = 4g^2$, the nonlinear coefficient $\alpha = \frac{\chi}{g} = 0$. 
FIG. 3: The curves of the atom and light field entanglement degrees with time evolution, the initial state superposition coefficient $C_1 = 0.96$, the initial photon number $n = 0$, the quantum discord $\delta^2 = 0$, the nonlinear coefficient $\alpha = \chi / g = 0$.

FIG. 4: The curves of the atom and light field entanglement degrees with time evolution, the initial state superposition coefficient $C_1 = 0.76$, the initial photon number $n = 0$, the quantum discord $\delta^2 = 0$, the nonlinear coefficient $\alpha = \chi / g = 0$. 
FIG. 5: The curves of the atom and light field entanglement degrees with time evolution, the initial state superposition coefficient $C_1 = 0.76$, the initial photon number $n = 0$, the quantum discord $\delta^2 = 8g^2$, the nonlinear coefficient $\alpha = \frac{\chi}{g} = 0$.

FIG. 6: The curves of the atom and light field entanglement degrees with time evolution, the initial state superposition coefficient $C_1 = 0$, the initial photon number $n = 0$, the quantum discord $\delta^2 = 4g^2$, the nonlinear coefficient $\alpha = \frac{\chi}{g} = 0.2$. 
FIG. 7: The curves of the atom and light field entanglement degrees with time evolution, the initial state superposition coefficient $C_1 = 0$, the initial photon number $n = 0$, the quantum discord $\delta^2 = 4g^2$, the nonlinear coefficient $\alpha = \frac{\chi}{g} = 0.8$.

FIG. 8: The curves of the atom and light field entanglement degrees with time evolution, the initial state superposition coefficient $C_1 = 0$, the initial photon number $n = 0$, the quantum discord $\delta^2 = 4g^2$, the nonlinear coefficient $\alpha = \frac{\chi}{g} = -0.2$. 
FIG. 9: The curves of the atom and light field entanglement degrees with time evolution, the initial state superposition coefficient $C_1 = 0$, the initial photon number $n = 0$, the quantum discord $\delta^2 = 4g^2$, the nonlinear coefficient $\alpha = \frac{\chi}{g} = -0.8$. 