Electromechanical power flowcharts in systems of electrical circuits

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Abstract
We present an original undergraduate level compilation for the physics of electromechanical systems with special attention to power flow. An approach based on energy considerations is presented that is specially suited to compute the mechanical and electrical actions of electromagnetic fields and to draw power flowcharts that clarify the path taken by energy in typical devices. The procedure guarantees energy conservation and provides a consistent way for auditing the power flow.

1 Introduction

Classical textbooks on electromagnetics currently taught in undergraduate engineering courses often place great emphasis on electromagnetic induction. It is not surprising, since it plays a paramount role in energy transport and electromechanical conversion in today’s technology. Yet, in our perspective, a more thorough analysis of the interplay of electrical, mechanical and magnetic variables in a typical machine would also fit into an undergraduate engineering course on electromagnetics. It is our intention to sketch the theory
of such a systems placing a special focus on power flow tracking and charting. On the other hand, texts on electromechanical systems, often appeared in a Mechatronics context\[2, 5\] make use of analytical mechanics to provide the mathematical framework for a generalized formulation of the subject. We propose a way in between the raw Lorentz force and the analytical mechanics approach, which is better suited for students with no previous course on theoretical mechanics. It is bases on power balances which, in turn, rely on the conservative character of the energy in electromagnetic systems in the low frequency regime, under which most electromechanical systems operate.

We consider electromagnetic currents at low frequencies, meaning that signals propagating at the speed of light do not experience a significant change of phase when they come through the electric machine. We will further assume that conduction currents flow in closed paths (circuits). They may move, typically rotate about a fixed axis, and accordingly, dynamical actions generate a mechanical power flow. Faraday’s law, on its turn, induces electromotive forces in circuits, so that there is also electrical power.

This lesson aims at providing a systematic way for computing mechanical and electromotive forces and power flows and check that they are consistent with energy conservation. We do it both analytically and graphically.

2 Electromagnetic potential energy

Considering the volume density of power delivered by the electromagnetic field to the conduction currents

\[ p = \vec{J} \cdot \vec{E} \]  

(1)

it is possible to write an overall power balance

\[ \iiint p \, d\tau = - \iiint \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \, d\tau - \iiint \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \, d\tau - \iiint (\vec{E} \times \vec{H}) \cdot d\vec{\sigma} \]  

(2)

where, assuming constitutive relations \( \vec{D} = \vec{D}(\vec{E}) \), \( \vec{B} = \vec{B}(\vec{H}) \)\[2\] and defining

\[
\begin{align*}
  u_e & := \int_0^\tau \vec{D} \cdot d\vec{D} \\
  u_m & := \int_0^\tau \vec{B} \cdot d\vec{B} \\
  \vec{S} & := \vec{E} \times \vec{H}
\end{align*}
\]

(3)

it follows that

\[ \iiint p \, d\tau = - \iiint \frac{\partial (u_e + u_m)}{\partial t} \, d\tau - \iiint \vec{S} \cdot d\vec{\sigma} \]  

(4)

that renders physical interpretations of \( u_e, u_m, \vec{S} \) as electric and magnetic energy densities and current respectively.

\(^*\)it is further assumed that \( \frac{\partial B_i}{\partial H_j} = \frac{\partial B_j}{\partial H_i} \), \( \frac{\partial D_i}{\partial E_j} = \frac{\partial D_j}{\partial E_i} \)

\(^1\)Vector \( \vec{S} \) is known as \textit{Poynting Vector}
Slowly varying electric charges and currents yield electric and magnetic fields that decay at most as $1/r^2$, so that the flux of the Poynting vector $\vec{S}$ becomes negligible if the integration is taken over a big sphere. That allows us to define a potential energy $U_{em}$ from which conduction currents take or deliver power.

$$U_{em} := \iiint_{\text{all space}} (u_e + u_m) \, d\tau$$  \hspace{2cm} (5)

In linear and isotropic materials $\vec{D} = \epsilon \vec{E}$, $\vec{B} = \mu \vec{H}$, where $\epsilon, \mu$ do not depend on the fields. Then,

$$\begin{align*}
  u_e &= \frac{1}{2} \epsilon |\vec{E}|^2 \\
  u_m &= \frac{1}{2 \mu} |\vec{B}|^2 \\
  U_{em} &= \frac{1}{2} \iiint_{\text{all space}} (\epsilon |\vec{E}|^2 + \frac{1}{\mu} |\vec{B}|^2) \, d\tau
\end{align*}$$  \hspace{2cm} (6)

3 Electromotive and mechanical forces on circuits

Next, we set the scenario of electromechanics with currents. A set of $n$ electric circuits, each one being traversed by a magnetic flux $\Phi_j$, $j = 1, \ldots, n$, is positioned by a collection of $m$ generalized geometrical coordinates $q_1, \ldots, q_m$. Each electric circuit is given energy at a rate $-I_j \dot{\Phi}_j$ and every mechanical degree of freedom receives a power $F_k \dot{q}_k$, where $F_k$ is the generalized force corresponding to the $k$-th generalized coordinate $q_k$. As the action of the electromagnetic field on conduction currents is conservative, it follows that

$$\frac{dU_{em}}{dt} = \sum_{j=1}^{n} I_j \dot{\Phi}_j - \sum_{k=1}^{m} F_k \dot{q}_k$$  \hspace{2cm} (7)

whence, if energy $U$ is expressed as a function of the fluxes and coordinates $U = F(\Phi_j, q_k)$, we readily arrive at

$$\begin{align*}
  I_j &= \frac{\partial F}{\partial \Phi_j} \\
  F_k &= -\frac{\partial F}{\partial q_k}
\end{align*}$$  \hspace{2cm} (8)

Eq(7) implies a way to compute $U_{em}$ by raising the set of currents from zero to their final values keeping the positions constant

$$U(I, q) = \sum_{j=1}^{n} \int_{0}^{\Phi_j} I_j \, d\Phi_j(I, q) = \sum_{j=1}^{n} I_j \Phi_j(I, q) - \sum_{j=1}^{n} \int_{0}^{I_j} \Phi_j(I, q) \, dI_j$$  \hspace{2cm} (9)

where the sets of $I_1, \ldots, I_n$ currents and $q_1, \ldots, q_m$ coordinates are denoted generically as $I, q$. It is often useful to perform a Legendre transformation and define the co-energy $W$ as

$$W := \sum_{j=1}^{n} I_j \Phi_j - U = \sum_{j=1}^{n} \int_{0}^{I_j} \Phi_j(I, q) \, dI_j$$  \hspace{2cm} (10)
and express it as a function of the currents and the coordinates \( W = G(I, q_k) \), so that
\[
\begin{align*}
\Phi_j &= \frac{\partial G}{\partial I_j} \\
F_k &= \frac{\partial G}{\partial q_k}
\end{align*}
\]
(11)

When \( F(\Phi_j, q_k) \) is a quadratic function of the fluxes, energy and co-energy are equal \( U = W \). This is the case in linear systems.

The electromotive force on circuit \( j \) is always given by Faraday’s law
\[
E_j = -\frac{d\Phi_j}{dt}
\]
(12)

4 Generalized forces on circuits from external magnetic fluxes

In this section we consider the mechanical actions on a set \( C_1 \) of \( n_1 \) electric circuits originated by fields created by other set \( C_2 \) of \( n_2 \) currents which are not exactly known or we are not interested in. It is assumed that the magnetic flux through the \( \ell \)-th circuit in \( C_1 \) can be split into two contributions: \( \Gamma_{\ell j}, \Psi_{\ell j} \), from \( C_1, C_2 \), respectively. All we need is an expression for the fluxes \( \Psi_{\ell j}(q_k, t) \); \( \ell = 1, \ldots, n_1; j = 1, \ldots, n_2 \) through the \( \ell \)-th circuit in \( C_1 \) caused by the \( j \)-th current in \( C_2 \). From Eq.8,
\[
F_k = -\frac{dF}{dq_k} = \frac{1}{2} \sum_{j \in C_1 \cup C_2} \int_0^\Phi I_j(\phi, q_k) d\phi_j = \sum_{j \in C_1 \cup C_2} \Phi_j I_j(\Phi, q_k) - \sum_{j \in C_1 \cup C_2} \Phi_j I_j(\Phi, q_k) + \sum_{j \in C_1 \cup C_2} \Phi_j I_j(\Phi, q_k) - \sum_{j \in C_1 \cup C_2} \Phi_j I_j(\Phi, q_k)
\]
(13)

whose partial derivative yields
\[
F_k = -\frac{dF}{dq_k} = \sum_{j \in C_1 \cup C_2} \int_0^{l(\Phi, q_k)} \frac{\partial \Phi_j(i, q_k)}{\partial q_k} di_j
\]
(14)

If we consider mechanical actions on a coordinate \( q_k \) that positions circuits in \( C_1 \) from fields created by circuits in \( C_2 \), the currents taken into account are only those in \( C_1 \) and the fluxes \( \Psi_j \) from \( C_2 \) do not depend on said currents. Thus,
\[
F_k = \sum_{\ell=1}^{n_1} \int_0^{l_\ell(\Phi, q_k)} \frac{\partial \Psi_{\ell j}(q_k, t)}{\partial q_k} dl_\ell = \sum_{\ell=1}^{n_1} l_\ell \frac{\partial \Psi_{\ell j}(q_k, t)}{\partial q_k}
\]
(15)

and the mechanical power delivered to circuits in \( C_1 \) is
\[
P_{m,C_1} = \sum_{k=1}^{n_1} \sum_{\ell=1}^{n_1} l_\ell \frac{\partial \Phi_{\ell j}(q_k, t)}{\partial q_k} \dot{q}_k = \sum_{k=1}^{n_1} \sum_{\ell=1}^{n_1} l_\ell \frac{\partial \Psi_{\ell j}(q_k, t)}{\partial q_k} \dot{q}_k
\]
(17)
whereas the electrical power evaluates to

$$P_{e,C_1} = \sum_{k=1}^{m} \sum_{\ell=1}^{n_1} I_\ell \frac{d\Phi_\ell(q_k,t)}{dt} = \sum_{k=1}^{m} \sum_{\ell=1}^{n_1} I_\ell \frac{d[\Psi_\ell(q_k,t) + \Gamma_\ell(q_k,t)]}{dt}$$  \hspace{1cm} (18)$$

When all $\Psi_j(q_k,t)$ do not depend on time, their contribution to the sum of Eqs.17,18 cancels, meaning that they do not contribute any power to $C_1$, and they act just as catalyst for electro-mechanical energy conversion.

It is possible to define a partial potential energy for the circuits in $C_1$.

$$U_p := \sum_{j=1}^{n_1} \int I_j d\Gamma_j = \sum_{j=1}^{n_1} I_j \Gamma_j(I,q) - \sum_{j=1}^{n_1} \int_0^I \Gamma_j(q,i)d\hat{i}_j$$  \hspace{1cm} (19)$$

whose time derivative is

$$\frac{dU_p}{dt} = \sum_{j=1}^{n_1} \frac{dI_j}{dt} \Gamma_j(I,q) + \sum_{j=1}^{n_1} I_j \frac{d\Gamma_j(I,q)}{dt} - \sum_{j=1}^{n_1} \Gamma_j(q,I) \frac{dI_j}{dt} \sum_{j=1,k=1}^{n_1,n_1} \int_0^I \frac{\partial \Gamma_j(q,i)}{\partial q_k} d\hat{i}_j \hat{q}_k$$  \hspace{1cm} (20)$$

which, combined with Eqs.17,18 yields

$$\frac{dU_p}{dt} + P_{m,C_1} + P_{e,C_1} = 0$$  \hspace{1cm} (21)$$

so that, when external fluxes are constant, electrical and mechanical powers come only at the expense of the partial potential $U_p$.

Finally, as a particularization of Eq.16, it is worth noting that when $q_k$ determines the position the $j$-th circuit the mechanical action of an external field whose magnetic flux through the circuit is $\Psi(q_k)$ reads

$$F_k = I_j \frac{\partial \Psi_j(q_k,t)}{\partial q_k}$$  \hspace{1cm} (22)$$

5 Power flowchart

In this section we write down expressions for energy flows in electromechanical systems. Energy is conserved and accordingly positive and negative fluxes should balance out. The rate of decrease of the potential energy $U_{em}$ should match the power delivered by the electromagnetic field to the electric circuits and the mechanical degrees of freedom. We next check that this is the case. The $j$-th electric circuit receives a power from the electromagnetic field $P_{e,j}$ given by

$$P_{e,j} = I_j E_j = -I_j \frac{d\Phi_j}{dt}$$  \hspace{1cm} (23)$$

The power supplied to the $k-$th mechanical degree of freedom $P_{m,k}$ is

$$P_{m,k} = F_k \dot{q}_k = \frac{\partial G}{\partial q_k} \dot{q}_k = \sum_{j=1}^{n} I_j \frac{d\Phi_j}{dt} - \sum_{j=1,k=1}^{n,n} \int_0^I \frac{\partial \Phi_j(I,q)}{\partial q_k} d\hat{i}_j \hat{q}_k$$  \hspace{1cm} (24)$$

Deriving Eq.16 one obtains

$$\frac{dU}{dt} = \sum_{j=1}^{n_1} \frac{dI_j}{dt} \frac{d\Phi_j}{dt} + \sum_{j=1}^{n_1} I_j \frac{d\Phi_j}{dt} - \sum_{j=1}^{n_1} \frac{dI_j}{dt} - \sum_{j=1,k=1}^{n_1,n_1} \int_0^I \frac{\partial \Phi_j(I,q)}{\partial q_k} d\hat{i}_j \hat{q}_k$$  \hspace{1cm} (25)$$
which, combined with Eqs.23,24 yields

\[ \frac{dU_{em}}{dt} + \sum_{j=1}^{n} P_{e,j} + \sum_{k=1}^{m} P_{m,k} = 0 \]  

(26)

that is another statement of Eq.7.

An equation in which an addition of powers equals zero is a power balance equation. It can be drawn as node in a graph if the edges represent the powers. Fig.1 depicts the power balance Eq.26 in the case \( n = 2, m = 2 \).

Figure 1: Graph representation of the power balance equation \( \frac{dU_{em}}{dt} + \sum_{j=1}^{2} P_{e,j} + \sum_{k=1}^{2} P_{m,k} = 0 \) that provides a visual picture of energy conservation.

An equation in which an addition of powers equals zero is a power balance equation. It can be drawn as node in a graph if the edges represent the powers. Fig.1 depicts the power balance Eq.26 in the case \( n = 2, m = 2 \).

Figure 2: Flowchart of the path followed by the electrical and mechanical powers power \( P_{e,1}, P_{m,1} \) coming from Eq.26. \( P_{e,1} \) combines with power from an electrical source \( P_{S,1} \) to be dissipated as Joule power \( P_{J,1} \) by a resistor, whereas the mechanical power is used to overcome a mechanical friction \( P_{r,1} \) and increase a kinetic energy \( T \).

All mechanical and electrical powers can be traced further to the mechanical or electrical systems they are fed to or drawn from. Fig.2 amplifies the scope
of the flowchart to include information about the next stages of the power. For example, the electrical power delivered by the field to the first circuit $P_{e,1}$ enters a new balance in which it is added to the power coming from an electric source $P_{S,1}$ to feed a resistor where $P_{J,1}$ is dissipated. The mechanical power that flows into the first geometrical degree of freedom also enters a mechanical power balance that splits it into a power dissipated by mechanical friction $P_{r,1}$ and the time derivative of the kinetic energy or a moving part.

![Figure 3](image-url)  
**Figure 3:** Possible complete flowchart of power in a $n = 2, m = 2$ electromechanical system.

Fig.3 includes the analysis of all power coming from the electromagnetic field. New equations can be obtained by adding those of any number of nodes. Internal power flows are canceled out. Fig.4 represents the equation coming from all the nodes inside the closed red curve, which yields a global power audit of the system.

![Figure 4](image-url)  
**Figure 4:** By selecting a group of nodes in the graph a new equation is obtained adding the powers represented by all the external edges. If all the nodes are selected, we get a global energy audit of the system.

When there is an external source of constant magnetic fields, we have proved
in Section 4 that there is another balance equation, represented by Eq. 21. Besides, electric and mechanical powers from the constant magnetic field cancel out. It is represented in Fig. 5.

Figure 5: When a system is under constant magnetic fields generated elsewhere, two balances emerge. One is for the electrical and mechanical powers induced by the local magnetic fields and the other accounts for the cancellation of the electrical and mechanical powers combined coming from the external sources.
6 Examples

6.1 Variable reluctance circuit

As a first example we may consider the system of Fig. 6. An electric source supplies an electric current $I$ which is wound $N$ times around the ferromagnetic core of a magnetic circuit. Part of the circuit may rotate by an angle $\alpha$, thus making the reluctance $R(\alpha)$ variable. It can be approximated by

$$ R(\alpha) = \frac{2e}{\mu_0 ha(2\beta - \alpha)} $$  \hspace{1cm} (27)

where $a, e, \beta$ are geometrical parameters read from Fig. 6 and $h$ is the height (unseen dimension) of the system.

The electromotive force, from Faraday’s law results

$$ E = -N^2 \frac{dI}{dt} R^{-1} = -N^2 R^{-1} \frac{dI}{dt} - N^2 I \frac{\partial R^{-1}}{\partial \alpha} \dot{\alpha} $$  \hspace{1cm} (28)

so that the electrical power supplied to the circuit by the field reads

$$ P_{e,1} = -N^2 R^{-1} I \frac{dI}{dt} = N^2 I^2 \frac{\partial R^{-1}}{\partial \alpha} \dot{\alpha} $$  \hspace{1cm} (29)

The electromagnetic energy stored in the field is

$$ U_{em} = F = \frac{N^2 I^2}{2R(\alpha)} $$  \hspace{1cm} (30)
The mechanical torque results

\[ F_\alpha = \frac{\partial F}{\partial \alpha} = \frac{N^2 I^2}{2} \frac{\partial R^{-1}(\alpha)}{\partial \alpha} \]  

(31)

giving a mechanical power

\[ P_{m,1} = F_\alpha \dot{\alpha} = \frac{N^2 I^2}{2} \frac{\partial R^{-1}(\alpha)}{\partial \alpha} \]  

(32)

It is now straightforward to check that Eqs. 29, 32 and the time derivative of Eq. 30 add to zero, which indicates that the energy balance matches.

Then electrical and mechanical powers may enter further balances. Typically, there is a power furnished by the electrical source \( V_s I \) and a Joule dissipation in the resistance of the wire \( R I^2 \). There may also be a gain in kinetic energy \( \dot{T} \) of the rotary part and a mechanical viscous friction dissipation \( b \dot{\alpha}^2 \), as represented in the flowchart of Fig. 7.

Figure 7: Power flowchart for the variable reluctance system of Fig. 6. Power from the source is traced down to the increase of kinetic energy, mechanical friction, Joule dissipation and energy storage in the field.
6.2 Electromagnetic clutch

We may next consider the electromechanical system of Fig.8 made by two electric circuits $C_1, C_2$ which may rotate about a fixed axis $z$, through angles $\alpha, \beta$ respectively. Both circuits may have voltage sources $V_{S1}, V_{S2}$, resistances $R_1, R_2$ and are traversed by electric currents $I_1, I_2$, respectively. The mechanical rotations encounter viscous friction torques $-b\dot{\alpha}, -b\dot{\beta}$, and may increase the kinetic energies $T_1, T_2$. The mutual induction matrix is

$$
\mathbf{M} = \begin{pmatrix}
L & M \cos(\beta - \alpha) \\
M \cos(\beta - \alpha) & L
\end{pmatrix}
$$

Now we can evaluate the electromotive forces and electrical powers for both circuits, using Faraday's law

$$
\begin{pmatrix}
\mathcal{E}_1 \\
\mathcal{E}_2
\end{pmatrix} = -\frac{d}{dt} \begin{pmatrix}
LI_1 + MI_2 \cos(\beta - \alpha) \\
LI_2 + MI_1 \cos(\beta - \alpha)
\end{pmatrix}
$$

$$
\begin{pmatrix}
\mathcal{E}_1 \\
\mathcal{E}_2
\end{pmatrix} = \begin{pmatrix}
-L\dot{I}_1 + MI_2 \sin(\beta - \alpha)(\dot{\beta} - \dot{\alpha}) - M\dot{I}_2 \cos(\beta - \alpha) \\
-L\dot{I}_2 + MI_1 \sin(\beta - \alpha)(\dot{\beta} - \dot{\alpha}) - M\dot{I}_1 \cos(\beta - \alpha)
\end{pmatrix}
$$

$$
\begin{pmatrix}
P_{e,1} \\
P_{e,2}
\end{pmatrix} = \begin{pmatrix}
-MI_1 I_2 \sin(\beta - \alpha)(\dot{\beta} - \dot{\alpha}) - M\dot{I}_2 \cos(\beta - \alpha) \\
-MI_2 \dot{I}_2 + MI_1 I_2 \sin(\beta - \alpha)(\dot{\beta} - \dot{\alpha}) - M\dot{I}_1 \cos(\beta - \alpha)
\end{pmatrix}
$$

The mechanical torques and powers, according to Eq[11], are

$$
\begin{pmatrix}
F_1 \\
F_2
\end{pmatrix} = \begin{pmatrix}
MI_1 I_2 \sin(\beta - \alpha) \\
-MI_1 I_2 \sin(\beta - \alpha)
\end{pmatrix}
$$

$$
\begin{pmatrix}
P_{m,1} \\
P_{m,2}
\end{pmatrix} = \begin{pmatrix}
MI_1 I_2 \sin(\beta - \alpha)\dot{\alpha} \\
-MI_1 I_2 \sin(\beta - \alpha)\dot{\beta}
\end{pmatrix}
$$

\[\text{In this problem energy and coenergy are the same, because the system is linear}\]
The electromagnetic energy stored in the field is

\[ U_{em} = \frac{1}{2} (LI_1^2 + 2MI_1I_2 \cos(\beta - \alpha) + LI_2^2) \]  

(39)

and its time derivative evaluates to

\[ \frac{dU_{em}}{dt} = LI_1 \dot{I}_1 - MI_1 \dot{I}_2 \sin(\beta - \alpha)(\dot{\beta} - \dot{\alpha}) + M \dot{I}_1 \dot{I}_2 \cos(\beta - \alpha) + MI_1 \dot{I}_2 \cos(\beta - \alpha) + LI_2 \dot{I}_2 \]  

(40)

It is now straightforward to check Eq.7 adding the powers of Eqs.36,38 and 40 to zero. The electrical and mechanical powers delivered by the field enter further power balances. Electrical powers may be added to those supplied by other sources and dissipate in resistors. Mechanical powers may increase kinetic energies and dissipate by frictional forces. A complete power flowchart for the system is represented in Fig.9. We can select all nodes to write down the equation:

\[ I_1 V_{S1} + I_2 V_{S2} = \dot{U} + \dot{T}_1 + \dot{I}_1^2 R_1 + I_2^2 R_2 + b\dot{\alpha}^2 + b\dot{\beta}^2 \]  

(41)

Figure 9: Power flowchart for the system of Fig.8. The power delivered (or absorbed) by the electric sources may be traced down to dissipative losses in resistors and viscous frictions and kinetic energy gains in the rotating masses.

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