Boundary singularities of $N$-harmonic functions

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1 Introduction

Let $\Omega$ be a domain in $\mathbb{R}^N$ ($N \geq 2$) with a $C^2$ compact boundary $\partial \Omega$. A function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is $p$-harmonic if

$$\int_{\Omega} |Du|^{p-2} \langle Du, D\phi \rangle \, dx = 0$$

for any $\phi \in C^1_0(\Omega)$. Such functions are locally $C^{1,\alpha}$ for some $\alpha \in (0,1)$. In the case $p = N$, the function $u$ is called $N$-harmonic. The $N$-harmonic functions play an important role as a natural extension of classical harmonic functions. They also appear in the theory of bounded distortion mappings [8]. One of the main properties of the class of $N$-harmonic functions is its invariance by conformal transformations of the space $\mathbb{R}^N$. This article is devoted to the study of $N$-harmonic functions which admit an isolated boundary singularity. More precisely, let $a \in \partial \Omega$ and $u \in W^{1,N}_{\text{loc}}(\Omega) \cap C(\Omega \setminus \{a\})$ be a $N$-harmonic function vanishing on $\partial \Omega \setminus \{a\}$, then $u$ may develop a singularity at the point $a$. Our goal is to show the existence of such singular solutions, and then to classify all the positive $N$-harmonic functions with a boundary isolated singularity. We denote by $n_a$ the outward normal unit vector to $\Omega$ at $a$. The main result we prove are presented below:

There exists a unique positive $N$-harmonic function $u = u_{1,a}$ in $\Omega$, vanishing on $\partial \Omega \setminus \{a\}$ such that

$$\lim_{x \to a} \frac{|x - a|}{|x - a|^\sigma} u(x) = -(\sigma, n_a)$$

uniformly on $S^{N-1} \cap \overline{\Omega} = \{ \sigma \in S^{N-1} : \langle \sigma, n_a \rangle < 0 \}$.

The functions $u_{1,a}$ plays a fundamental role in the description of all the positive singular $N$-harmonic functions since we the next result holds

Let $u$ be a positive $N$-harmonic function in $\Omega$, vanishing on $\partial \Omega \setminus \{a\}$. Then there exists $k \geq 0$ such that

$$u = ku_{1,a}.$$

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When $u$ is no longer assumed to be positive we obtain some classification results provided its growth is limited as shows the following

Let $u$ be a $N$-harmonic function in $\Omega$, vanishing on $\partial\Omega \setminus \{a\}$ and verifying

$$|u| \leq Mu_{1,a},$$

for some $M \geq 0$. Then there exists $k \in \mathbb{R}$ such that

$$u = ku_{1,a}. \quad (1.4)$$

In the last section we give a process to construct $p$-harmonic regular functions ($p > 1$) or $N$-harmonic singular functions as product of one variable functions. Starting from the existence of $p$-harmonic functions in the plane under the form $u(x) = u(r, \sigma) = r^\beta \omega(\theta)$ (see [5]), our method, by induction on $N$, allows us to produce separable solutions of the spherical $p$-harmonic spectral equation

$$-\text{div}_x \left( \left( \beta^2 v^2 + |\nabla\sigma v|^2 \right)^{(p-2)/2} \nabla\sigma v \right) = \lambda_{N,\beta} \left( \beta^2 v^2 + |\nabla\sigma v|^2 \right)^{(p-2)/2} v. \quad (1.5)$$

on $S^{N-1}$, where $\lambda_{N,\beta} = \beta (N - 1 + (\beta - 1)(p - 1))$. This equation equation is naturally associated to the existence of $p$-harmonic functions under the form $u(x) = |x|^\beta v(x/|x|)$. As a consequence, we express $p$-harmonic functions under the form of a product of $N$-explicit functions of one real variable. If we represent $\mathbb{R}^N$ as the set of $\{x = (x_1, ..., x_N)\}$ where $x_1 = r \sin \theta_{N-1} \sin \theta_{N-2}... \sin \theta_2 \sin \theta_1$, $x_2 = r \sin \theta_{N-1} \sin \theta_{N-2}... \sin \theta_2 \cos \theta_1$, ..., $x_{N-1} = r \sin \theta_{N-1} \cos \theta_{N-2}$ and $x_N = r \cos \theta_{N-1}$ with $\theta_1 \in [0, 2\pi]$ and $\theta_k \in [0, \pi]$, for $k = 2, ..., N-1$, then, for any integer $k$ the function

$$u(x) = (r \sin \theta_{N-1} \sin \theta_{N-2}... \sin \theta_2)^{\beta_k} \omega_k(\theta_1) \quad (1.6)$$

is $p$-harmonic in $\mathbb{R}^N$, in which expression $\beta_k > 1$ is an algebraic number depending on $k$ and $\omega_k$ is a $\pi/k$-antiperiodic solutions of a completely integrable homogeneous differential equation. Moreover $N$-harmonic singular functions are also obtained under the form

$$u(x) = r^{-\beta_k} (\sin \theta_{N-1} \sin \theta_{N-2}... \sin \theta_2)^{\beta_k} \omega_k(\theta_1). \quad (1.7)$$

Our paper is organized as follows: 1- Introduction. 2- Construction of fundamental singular $N$-harmonic functions. 3- The classification theorem. 4- Separable solutions of the $p$-harmonic spectral problem.

2 Construction of fundamental singular $N$-harmonic functions

We denote by $\mathcal{H}_N$ the group of conformal transformations in $\mathbb{R}^N$. This group is generated by homotheties, inversion and isometries. Our first result is classical, but we repeat the proof for the sake on completeness.

**Proposition 2.1** Let $u$ be a $N$-harmonic function in a domain $G \subset \mathbb{R}^N$ and $h \in \mathcal{H}_N$. Then $u_h = u \circ h$ is $N$-harmonic in $h^{-1}(G)$. 

2
Proof. Because for any \( p > 1 \) the class of \( p \)-harmonic functions is invariant by homotheties and isometries, it is sufficient to prove the result if \( h \) is the inversion \( \mathcal{I}_0^1 \) with center the origin in \( \mathbb{R}^N \) and power 1. We set \( y = \mathcal{I}_0^1(x) \) and \( v(y) = u(x) \). For any \( i = 1, ..., N \)

\[
u_i(x) = \sum_j \left( \delta_{ij} |x|^{-2} - 2 |x|^{-4} x_ix_j \right) v_{y_i}(y).
\]

Then

\[
|Du|^2(x) = |x|^{-4} |Dv|^2(y) = |y|^4 |Dv|^2(y).
\]

If \( \phi \) is a test function, we denote similarly \( \psi(y) = \phi(x) \), thus

\[
\langle Du, D\phi \rangle = |x|^{-4} \langle Dv, D\psi \rangle = |y|^4 \langle Dv, D\psi \rangle,
\]

and

\[
\int_G |Du|^N - 2 \langle Du, D\phi \rangle dx = \int_{\mathcal{I}_0^1(G)} |y|^{2N} |Dv|^{N-2} \langle Dv, D\psi \rangle |D\mathcal{I}_0^1| dy
\]

Because \( |D\mathcal{I}_0^1| = |\det(\partial x_i/\partial y_j)| = |y|^{-2N} \), the result follows. \( \square \)

**Proposition 2.2** Let \( N \geq 2 \), \( B = B_1(0) \) and \( a \in \partial B \). Then there exists a unique positive \( N \)-harmonic function \( U^i \) in \( B \) which vanishes on \( \partial B \setminus \{a\} \) and satisfies

\[
U^i(x) = \frac{1 - |x|^2}{|x - a|^2} (1 + o(1)) \quad \text{as} \quad x \to a. \tag{2.1}
\]

**Proof.** We first observe that the coordinates functions \( x_i \) are \( N \)-harmonic and positive in the half-space \( H_1 = \{ x \in \mathbb{R}^N : x_i > 0 \} \) and vanishes on \( \partial H_1 \). Therefore, the functions \( \chi_i(x) = x_i/|x|^{2} \) are also \( N \)-harmonic and singular at 0. Without loss of generality we can assume that \( a \) is the origin of coordinates, and that \( B \) is the ball with radius 1 and center \((-1, 0, ..., 0)\). Let \( \omega \) be the point with coordinates \((-2, 0, ..., 0)\). By the inversion \( \mathcal{I}_0^1 \), \( a \) is invariant and \( B \) is transformed into the half space \( H_1 \). Since \( \chi_1 \) is \( N \)-harmonic in \( H_1 \), the function

\[
x \mapsto \chi_1 \circ \mathcal{I}_0^1(x) = \frac{-|x|^2 + 2x_1}{2|x|^2}
\]

is \( N \)-harmonic and positive in \( B = \{ x : |x|^2 + 2x_1 < 0 \} \), vanishes on \( \partial B \) and is singular at \( x = 0 \). If we set \( x'_i = x_i + 1, \ x'_i = x_i \) for \( i = 2, ..., N \), \( N \) and \( U^i(x') = \chi_1 \circ \mathcal{I}_0^1(x) \), then the \( x' \) coordinates of \( a \) are \((1, 0, ..., 0)\) and

\[
U^i(x') = \frac{1 - |x'|^2}{2|x'|^2} = \frac{1 - |x|'}{|x' - a|^2} (1 + o(1)) \quad \text{as} \quad x' \to a.
\]

Let \( \tilde{U}^i \) be another positive \( N \)-harmonic function in \( B \) which verifies (2.1) and vanishes on \( \partial B \setminus \{a\} \). Thus, for any \( \delta > 0 \), \( (1 + \delta)\tilde{U}^i \), is positive, \( N \)-harmonic, and \( U^i - (1 + \delta)\tilde{U}^i \) is negative near \( a \). By the maximum principle, \( U^i \leq (1 + \delta)\tilde{U}^i \). Letting \( \delta \to 0 \), and permuting \( U^i \) and \( \tilde{U}^i \) yields \( \tilde{U}^i = U^i \). \( \square \)

By performing the inversion \( \mathcal{I}_0^1 \), we derive the dual result
Proposition 2.3 Let $N \geq 2$, $G = B^i(0)$ and $a \in \partial B$. Then there exists a unique positive $N$-harmonic function $U^c$ in $G$ which vanishes on $\partial B \setminus \{a\}$ and satisfies

$$U^c(x) = o(\ln|x|) \quad \text{as} \quad |x| \to \infty,$$ (2.2)

and

$$U^c(x) = \frac{|x| - 1}{|x - a|^2}(1 + o(1)) \quad \text{as} \quad x \to a.$$ (2.3)

Proof. The assumption (2.2) implies that the function $U = U^c \circ I_0$, which is $N$-harmonic in $B \setminus \{0\}$ verifies

$$U(x) = o(\ln(1/|x|)) \quad \text{near} \quad 0.$$ (2.4)

By (2.2), 0 is a removable singularity and thus $U$ can be extended as a positive $N$-harmonic function in $B$ which satisfies (2.1). This implies the claim. $\square$

We denote by $\rho(x)$ the signed distance from $x$ to $\partial \Omega$. Since $\partial \Omega$ is $C^2$, there exists $\beta_0 > 0$ such that if $x \in \mathbb{R}^N$ verifies $-\beta_0 \leq \rho(x) \leq \beta_0$, there exists a unique $\xi_x \in \partial \Omega$ such that

$$|x - \xi_x| = |\rho(x)|.$$ (2.5)

Furthermore, if $\nu_{\xi_x}$ is the outward unit vector to $\partial \Omega$ at $\xi_x$, $x = \xi_x - \rho(x)\nu_{\xi_x}$. In particular $\xi_x - \rho(x)\nu_{\xi_x}$ and $\xi_x + \rho(x)\nu_{\xi_x}$ have the same orthogonal projection $\xi_x$ onto $\partial \Omega$.

Let $T_{\beta_0}(\Omega) = \{x \in \mathbb{R}^N : -\beta_0 \leq \rho(x) \leq \beta_0\}$, then the mapping $\Pi : [-\beta_0, \beta_0] \times \partial \Omega \to T_{\beta_0}(\Omega)$ defined by $\Pi(\rho, \xi) = \xi - \rho \nu(\xi)$ is a $C^2$ diffeomorphism. Moreover $D\Pi(0, \xi)(1, e) = e - \nu_\xi$ for any $e$ belonging to the tangent space $T_\xi(\partial \Omega)$ to $\partial \Omega$ at $\xi$. If $x \in T_{\beta_0}(\Omega)$, we define the reflection of $x$ through $\partial \Omega$ by $\psi(x) = \xi_x + \rho(x)\nu_{\xi_x}$. Clearly $\psi$ is an involutive diffeomorphism from $\Omega \cap T_{\beta_0}(\Omega)$ to $\Omega^c \cap T_{\beta_0}(\Omega)$, and $D\psi(x) = I$ for any $x \in \partial \Omega$. If a function $v$ is defined in $\Omega \cap T_{\beta_0}(\Omega)$, we define $\tilde{v}$ in $T_{\beta_0}(\Omega)$ by

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \Omega \cap T_{\beta_0}(\Omega) \\ -v \circ \psi(x) & \text{if } x \in \Omega^c \cap T_{\beta_0}(\Omega). \end{cases}$$ (2.4)

Lemma 2.4 Assume that $0 \in \partial \Omega$. Let $v \in C^{1,\alpha}(\Omega \cap T_{\beta_0}(\Omega) \setminus \{0\})$ be a solution of (1.4) in $\Omega \cap T_{\beta_0}(\Omega)$ vanishing on $\partial \Omega \setminus \{0\}$. Then $\tilde{v} \in C^{1,\alpha}(T_\beta(\Omega) \setminus \{0\})$ is solution of a quasilinear equation

$$\sum_j \frac{\partial}{\partial x_j} \tilde{A}_j(x, D\tilde{v}) = 0$$ (2.5)

in $T_\beta(\Omega) \setminus \{0\}$ where the $\tilde{A}_j$ are $C^1$ functions defined in $T_\beta(\Omega)$ where they verify

$$\begin{cases} (i) \quad \tilde{A}_j(x, 0) = 0 \\ (ii) \quad \sum_{i,j} \frac{\partial^{\tilde{A}}_j}{\partial \eta_i}(x, \eta) \xi_i \xi_j \geq \Gamma |\eta|^{p-2} |\xi|^2 \\ (iii) \quad \sum_{i,j} \frac{\partial \tilde{A}_j}{\partial \eta_i}(x, \eta) \leq \Gamma |\eta|^{p-2} \end{cases}$$ (2.6)

for all $x \in T_\beta(\Omega) \setminus \{0\}$ for some $\beta \in (0, \beta_0]$, $\eta \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$ and some $\Gamma > 0$. 

4
Proof. The assumptions \((2.6)\) implies that weak solutions of \((2.5)\) are \(C^{1,\alpha}\), for some \(\alpha > 0\) and satisfy the standard a priori estimates. As it is defined the function \(\tilde{v}\) is clearly \(C^2\) in \(T_{\beta_0}(\Omega) \setminus \{0\}\). Writing \(Dv(x) = -D(\tilde{v} \circ \psi(x)) = -D\psi(x)(D\tilde{v}(\psi(x)))\) and \(x = \psi(x) = \psi^{-1}(x)\)

\[
\int_{\Omega \cap T_\beta(\Omega)} |Dv|^{p-2} Dv \, D\zeta \, dx = \int_{\Omega \cap T_\beta(\Omega)} |D\psi(D\tilde{v})|^{p-2} D\psi(D\tilde{v}) \, D\psi(D\zeta) \, |D\psi| \, d\tilde{x}.
\]

But

\[
D\psi(D\tilde{v}) \cdot D\psi(D\zeta) = \sum_k \left( \sum \frac{\partial \psi_i}{\partial x_k} \frac{\partial \tilde{v}}{\partial x_i} \right) \left( \sum \frac{\partial \psi_i}{\partial x_k} \frac{\partial \zeta}{\partial x_i} \right) = \sum_j \left( \sum \frac{\partial \psi_i}{\partial x_k} \frac{\partial \psi_j}{\partial x_i} \frac{\partial \tilde{v}}{\partial x_i} \right) \frac{\partial \zeta}{\partial x_j}.
\]

We set \(b(x) = |D\psi|\),

\[
A_j(x, \eta) = |D\psi| \cdot |D\psi(\eta)|^{p-2} \sum_i \left( \sum \frac{\partial \psi_i}{\partial x_k} \frac{\partial \psi_j}{\partial x_i} \right) \eta_i,
\]

and

\[
A(x, \eta) = (A_1(x, \eta), ..., A_N(x, \eta)) = |D\psi| \cdot |D\psi(\eta)|^{p-2} (D\psi)^t D\psi(\eta).
\]

For any \(\xi \in \partial \Omega\), the mapping \(D\psi(\partial \Omega)(\xi)\) is the symmetry with respect to the hyperplane \(T_\xi(\partial \Omega)\) tangent to \(\partial \Omega\) at \(\xi\), so \(|D\psi(\xi)| = 1\). Inasmuch \(D\psi\) is continuous, a lengthy but standard computation leads to the existence of some \(\beta \in (0, \beta_0)\) such that \((2.6)\) holds in \(T_{\beta}(\Omega) \cap \Omega\) and \(A\) on \(T_{\beta}(\Omega) \cap \Omega\), then inequalities \((2.6)\) are satisfied in \(T_{\beta}(\Omega)\).

These three results allows us to prove our main result

**Theorem 2.5** Let \(\Omega\) be an open subset of \(\mathbb{R}^N\) with a compact \(C^2\) boundary, \(\rho(x) = \text{dist} (x, \partial \Omega)\) and \(a \in \partial \Omega\). Then there exists one and only one positive \(N\)-harmonic function \(u\) in \(\Omega\), vanishing on \(\partial \Omega \setminus \{a\}\) verifying

\[
\lim_{x \to a} \frac{|x - a|}{|x - a|} u(x) = -\langle \sigma, n_a \rangle \quad (2.9)
\]

uniformly on \(S^{N-1} \cap \Omega\) and

\[
u(x) = o(\ln |x|) \quad \text{as} \quad |x| \to \infty, \quad (2.10)
\]

if \(\Omega\) is not bounded.

Proof. Uniqueness follows from \((2.9)\) by the same technique as in the previous propositions. **Step 1 (Existence).** If \(\Omega\) is not bounded, we perform an inversion \(T_m^{|m-a|^2}\) with center some \(m \in \Omega\). Because of \((2.10)\), the new function \(u \circ T_m^{|m-a|^2}\) is \(N\)-harmonic in \(\Omega' = T_m^{|m-a|^2}(\Omega)\) and satisfies \((2.9)\). Thus we are reduced to the case were \(\Omega\) is bounded. Since \(\Omega\) is \(C^2\), it satisfies the interior and exterior sphere condition at \(a\). By dilating \(\Omega\), we can assume that the exterior and interior tangent spheres at \(a\) have radius 1. We denote them by
$B_1(\omega^i)$ and $B_1(\omega^-)$, their respective centers being $\omega^i = a - n_a$ and $\omega^- = a + n_a$. We set $V^i(x) = U^i(x - \omega^i)$ and $V^-(x) = U^-(x - \omega^-)$ where $U^i$ and $U^-$ are the two singular $N$-harmonic functions described in Proposition 2.2 and Proposition 2.3 respectively in $B_1(\omega^i)$ and $B_1(\omega^-)$, with singularity at point $a$. For $\epsilon > 0$, we put $\Omega_\epsilon = \Omega \cup B_\epsilon(a)$, $\Sigma_\epsilon = \Sigma \cap \partial B_\epsilon(a)$ and $\partial^*\Omega_\epsilon = \partial\Omega \cap B_\epsilon'(a)$. Let $u_\epsilon$ be the solution of

$$
\begin{cases}
\text{div}(|Du_\epsilon|^{N-2} Du_\epsilon) = 0 \quad \text{in } \Omega_\epsilon \\
u_\epsilon = 0 \quad \text{on } \partial^*\Omega_\epsilon \\
u_\epsilon = V^\epsilon \quad \text{on } \Sigma_\epsilon.
\end{cases}
$$

(2.11)

This solution is obtained classically by minimisation of a convex functional over a class of functions with prescribed boundary value on $\partial\Omega_\epsilon$. For any $x \in B_1(\omega^i)$, there holds

$$\text{dist} \ (x, \partial B_1(\omega^-)) = |x - \omega^-| - 1 \geq \text{dist} \ (x, \partial\Omega) \geq \text{dist} \ (x, \partial B_1(\omega^i)) = 1 - |x - \omega^i|.$$ 

Thus

$$V^i(x) \leq V^\epsilon(x) \quad \forall x \in B_1(\omega^i),$$

by using (2.11), (2.3) and the maximum principle. Therefore

$$V^i(x) \leq u_\epsilon(x) \leq V^\epsilon(x) \quad \forall x \in B_1(\omega^i) \cap \Omega_\epsilon$$

and

$$u_\epsilon(x) \leq V^\epsilon(x) \quad \forall x \in \Omega_\epsilon.$$ 

Finally, for $0 < \epsilon' < \epsilon$, $u_{\epsilon'}|_{\Sigma_{\epsilon'}} \leq V^\epsilon|_{\Sigma_{\epsilon'}} = u_\epsilon|_{\Sigma_{\epsilon'}}$. Thus

$$u_{\epsilon'}(x) \leq u_\epsilon(x) \quad \forall x \in \Omega_{\epsilon'}.$$ 

The sequence $\{u_\epsilon\}$ is increasing with $\epsilon$. By classical a priori estimates concerning quasilinear equations, it converges to some positive $N$-harmonic function $u$ in $\Omega$ which vanishes on $\partial\Omega \setminus \{a\}$ and verifies

$$V^i(x) \leq u(x) \quad \forall x \in B_1(\omega^i),$$

and

$$u(x) \leq U^\epsilon(x) \quad \forall x \in \Omega.$$ 

This implies

$$\frac{1 - |x - \omega|^2}{2 |x - a|^2} \leq u(x) \quad \forall x \in B_1(\omega^i),$$

(2.12)

and

$$u(x) \leq \frac{|x - \omega|}{2 |x - a|^2} \quad \forall x \in \Omega,$$

(2.13)

By scaling we can prove the following estimate

$$u(x) \leq C \frac{\rho(x)}{|x - a|^2} \quad \forall x \in \Omega,$$

(2.14)

for some $C > 0$: for simplicity we can assume that $a$ is the origin of coordinates and, for $r > 0$ set $u_r(y) = u(ry)$. Clearly $u_r$ is $N$-harmonic in $\Omega/r$ and

$$\max\{|Du_r(y)| : y \in \Omega/r \cap (B_{3/2} \setminus B_{1/2})\} \leq C \max\{|u_r(z)| : z \in \Omega/r \cap (B_2 \setminus B_{1/2})\},$$

where $C$, which depends on the curvature of $\partial\Omega/r$, remains bounded as long as $r \leq 1$. Since $Du_r(y) = rDu(ry)$, we obtain by taking $ry = x$, $|y| = 1$ and using (2.13) with general $a$,
\[ |Du(x)| \leq C |x - a|^{-2}. \] By the mean value theorem, since \( u \) vanishes on \( \partial \Omega \setminus \{ a \} \), holds.

**Step 2.** In order to give a simple proof of the estimate \( 2.9 \), we fix the origin of coordinates at \( a = 0 \) and the normal outward unit vector at \( a \) to be \( -e_N \). If \( \tilde{u} \) is the extension of \( u \) by reflection through \( \partial \Omega \), it satisfies \( 2.5 \) in \( T_{\beta}(\Omega) \setminus \{0\} \) (see lemma \( 2.4 \)). For \( r > 0 \), set \( \tilde{u}^r(x) = r\tilde{u}(rx) \). Then \( \tilde{u}^r \) is solution of

\[
\sum_j \frac{\partial}{\partial x_j} \tilde{A}_j(rx, Du^r) = 0 \tag{2.15}
\]

in \( T_{\beta/r}(\Omega/r) \setminus \{0\} \). By the construction of \( \tilde{A}_j(x, \eta) \), we can note that

\[
\lim_{r \to 0} \tilde{A}_j(rx, \eta) = |\eta|^{p-2} \eta_j, \quad \forall \eta \in \mathbb{R}^N.
\]

Furthermore, for any \( x \in T_{\beta}(\Omega) \setminus \{0\} \), \( \rho(x) = \rho(\psi(x)) \) and \( c |x| \leq |\psi(x)| \leq c^{-1} |x| \) for some \( c > 0 \), the estimate \( 2.14 \) holds if \( u \) is replaced by \( \tilde{u}^r \), \( \Omega \) by \( T_{\beta/r}(\Omega/r) \) and \( \rho(x) \) by \( \rho_r(x) := \text{dist}(x, \Omega/r) \) i.e.

\[
|\tilde{u}^r(x)| \leq C|x|^{-2} \rho_r(x) \quad \forall x \in T_{\beta/r}(\Omega/r).
\]

For \( 0 < a < b \) fixed and for some \( 0 < r_0 \leq 1 \) the spherical shell \( \Gamma_{a,b} = \{ x \in \mathbb{R}^N : a \leq |x| \leq b \} \) is included into \( T_{\beta/r}(\Omega/r) \) for all \( 0 < r \leq r_0 \). By the classical regularity theory for quasilinear equations \( 12 \) and lemma \( 2.4 \), there holds

\[
\| Du^r \|_{C^0(\Gamma_{2/3,2})} \leq C_r \| \tilde{u}^r \|_{L^\infty(\Gamma_{1/2,2})},
\]

where \( C_r \) remains bounded because \( r \leq 1 \). By Ascoli’s theorem, \( 2.12 \) and \( 2.4 \), \( \tilde{u}^r(x) \) converges to \( x_N |x|^{-2} \) in the \( C^1(\Gamma_{2/3,2}) \)-topology. This implies in particular that \( r^2 D\tilde{u}(rx) \) converges uniformly in \( \Gamma_{2/3,2} \) to \( -2x_N |x|^{-4}x + |x|^{-2} e_N \). Using the expression of \( D\tilde{u} \) in spherical coordinates we obtain

\[
r^2 \tilde{u}_\phi i - r \tilde{u}_\phi e + \frac{r}{\sin \phi} \nabla_{\sigma'} \tilde{u} \to -2\sigma_N i + e_N \text{ uniformly on } S^{N-1} \text{ as } r \to 0,
\]

where \( \cos \phi = x_N |x|^{-1}, i = x/|x|, e \) is derived from \( x/|x| \) by a rotation with angle \( \pi/2 \) in the plane \( 0, x, N \) ( \( N \) being the North pole), and \( \nabla_{\sigma'} \) is the covariant gradient on \( S^{N-2} \). Inasmuch \( i, e \) and \( \nabla_{\sigma'} \) are orthogonal, the components of \( e_N \) are \( \cos \phi, \sin \phi \) and \( 0 \), thus

\[
\tilde{u}_\phi (r, \sigma', \phi) \to -\sin \phi \text{ as } r \to 0.
\]

Since

\[
\tilde{u}(r, \sigma', \phi) = \int_{\pi/2}^\phi \tilde{u}_\phi (r, \sigma', \theta) d\theta,
\]

the previous convergence estimate establishes \( 2.9 \).

**Definition 2.6** We shall denote by \( u_{1,a} \) the unique positive \( N \)-harmonic function satisfying \( 2.9 \), and call it the fundamental solution with a point singularity at \( a \).
3 The classification theorem

In this section we characterize all the positive \( N \)-harmonic functions vanishing on the boundary of a domain except one point. The next statement is an immediate consequence of Theorem 2.5 and [2 Th. 2.11].

**Theorem 3.1.** Let \( \Omega \) be a bounded domain with a \( C^2 \) boundary and \( a \in \partial \Omega \). If \( u \) is a positive \( N \)-harmonic function in \( \Omega \) vanishing on \( \partial \Omega \setminus \{ a \} \), there exists \( M \geq 0 \) such that

\[
u(\omega) = \max_{\omega} \frac{\partial u}{\partial \nu}(x) < M \quad \forall x \in \Omega
\]  

In the next theorem, which extends [2 Th. 2.13], we characterize all the signed \( N \)-harmonic functions with a moderate growth near the singular point.

**Theorem 3.2.** Let \( \Omega \) be a bounded domain with a \( C^2 \) boundary and \( a \in \partial \Omega \). Assume that \( u_{1,a} \) has only a finite number of critical points in \( \Omega \). If \( u \) is a \( N \)-harmonic function in \( \Omega \) vanishing on \( \partial \Omega \setminus \{ a \} \) verifying \( |u(x)| \leq Mu_{1,a}(x) \) for some \( M > 0 \) and any \( x \in \Omega \), there exists \( k \in [-M,M] \) such that \( u = ku_{1,a} \).

**Proof.** We define \( k \) as the minimum of the \( \ell \) such that \( u \leq \ell u_{1,a} \) in \( \Omega \). Without any loss of generality we can assume \( k > 0 \). Then either the tangency of the graphs of the functions \( u \) and \( ku_{1,a} \) is achieved in \( \Omega \) \setminus \{ a \} \), or it is achieved asymptotically at the singular point \( a \). In the first case we considered two sub-cases:

(i) The coincidence set \( G \) of \( u \) and \( ku_{1,a} \) has a connected component \( \omega \) isolated in \( \Omega \). In this case there exists a smooth domain \( U \) such that \( \overline{\omega} \subset U \) and \( \delta > 0 \) such that \( ku_{1,a} - u \geq \delta \) on \( \partial U \). The maximum principle implies that \( ku_{1,a} - u \geq \delta \) in \( U \), a contradiction.

(ii) In the second sub-case any connected component \( \omega \) of the coincidence set touches \( \partial \Omega \setminus \{ a \} \), or the two graphs admits a tangency point on \( \partial \Omega \setminus \{ a \} \). If \( m \in \omega \cap \partial \Omega \setminus \{ a \} \) or is such a tangency point, the regularity theory implies \( \partial u(m) = ku_{1,a}(m) \). By Hopf boundary lemma, \( u_{1,a}(m) < \partial m \). By the mean value theorem, the function \( w = ku_{1,a} - u \) satisfies an equation

\[
Lw = 0
\]

which is elliptic and non degenerate near \( m \) (see [3, 4]), it follows that \( w \) vanishes in a neighborhood of \( m \) and the two graphs cannot be tangent only on \( \partial \Omega \setminus \{ a \} \). Assuming that \( \omega \neq \Omega \), let \( x_0 \in \partial \Omega \setminus \omega \) such that dist \((x_0,\omega) < r_0 = \rho(x_0) = \rho(x_0,\partial \Omega)\) and let \( y_0 \in \omega \) be such that \( |x_0 - y_0| = r_0 \). Since \( u_{1,a} \) has at most a finite number of critical points, we can choose \( x_0 \) such that \( y_0 \) is not one of these critical points. By assumption \( w = ku_{1,a} - u \) is positive in \( B_{r_0}(x_0) \) and vanishes at a boundary point \( y_0 \). Since the equations are not degenerate at \( y_0 \) there holds

\[
k\partial u_{1,a}(y_0)/\partial \nu - \partial u(y_0)/\partial \nu < 0
\]

where \( \nu = (y_0 - x_0)/r_0 \), which contradicts the fact that the two graphs are tangent at \( y_0 \).

Next we are reduced to the case where the graphs of \( u \) and \( ku_{1,a} \) are separated in \( \Omega \) and asymptotically tangent at the singular point \( a \). There exists a sequence \( \{ \xi_n \} \subset \Omega \) such that \( \lim_{n \to \infty} u(\xi_n)/u_{1,a}(\xi_n) = k \). We set \( x_n - a = r_n, u_n(y) = r_n u(a + r_n y) \) and \( v_n(y) = r_n u_{1,a}(a + r_n y) \). Both \( u_n \) and \( v_n \) are \( N \)-harmonic in \( \Omega_n = (\Omega - a)/r_n \). The functions \( u_n \) and \( v_n \) are locally uniformly bounded in \( \Omega_n \setminus \{ 0 \} \). It follows, by using classical regularity results, that, there exists sub-sequences, such that \( \{ u_{n_k} \} \) and \( \{ v_{n_k} \} \) converge respectively to \( U \) and \( V \) in the \( C_{l_{loc}}^1 \)-topology of \( \Omega_{n_k} \setminus \{ 0 \} \). The functions \( U \) and \( V \) are \( N \)-harmonic
in $H \approx \mathbb{R}^N_+ = \{x = (x_1, x_2, ..., x_N) : x_N > 0\}$ and vanish on $\partial H \setminus \{0\}$. Since it can be assumed that $(\zeta_n - a)/r_n \to \xi$, there holds $U \leq kV$ in $H$, $U(\xi) = kV(\xi)$, if $\xi \in H$, and $\partial U/\partial x_N = k\partial V(\xi)/\partial x_N > 0$, if $\xi \in \partial H$ (notice that $|\xi| = 1$). If $\xi \in \partial H$, Hopf lemma applies to $V$ at $\xi$ and, using the same linearization with the linear operator $L$ as in the previous proof, it yields to $U = kV$. If $\xi \in H$, we use the fact that $|Du_{1,a}(x)| \geq \beta > 0$ for $|x - a| \leq \alpha$ for some $\beta, \alpha > 0$. Thus $|Dv_{\omega}(\xi)| \geq \beta$. The non-degeneracy of $V$ and the strong maximum principle lead again to $U = kV$. Whatever is the position of $\xi$, the equality between $U$ and $kV$ and the convergence in $C^1_{loc}$ leads to the fact that for any $\epsilon > 0$ there exists $n_{\epsilon} \in \mathbb{N}$ such that $n \geq n_{\epsilon}$ implies

$$(k - \epsilon)u_{1,a}(x) \leq u(x) \leq (k + \epsilon)u_{1,a}(x) \quad \forall x \in \Omega \cap \partial B_{r_n}(a).$$

By the comparison principle between $N$-harmonic functions this inequality holds true in $\Omega \setminus \partial B_{r_n}(a)$. Since $r_n \to 0$ and $\epsilon$ is arbitrary, this ends the proof.

Remark. The assumption that $u_{1,a}$ has only isolated critical points in $\Omega$ is clearly satisfied in the case of a ball, a half-space or the complementary of a ball where no critical point exists. It is likely that this assumption always holds but we cannot prove it. However the Hopf maximum principle for $p$-harmonic functions (see [1]) implies that $u_{1,a}$ cannot have local extremum in $\Omega$.

4 Separable solutions of the $p$-harmonic spectral problem

In this section we present a technique for constructing signed $N$-harmonic functions, regular or singular, as a product of functions depending only on one real variable. Some of the results were sketched in [16]. The starting point is the result of Krol [5] dealing with the existence of 2-dimensional separable $p$-harmonic functions (the construction of singular separable $p$-harmonic functions was performed in [4]).

Theorem 4.1 (Krol) Let $p > 1$. For any positive integer $k$ there exists a unique $\beta_k > 0$ and $\omega_k : \mathbb{R} \to \mathbb{R}$, with least antiperiod $\pi/k$, such that

$$u_k(x) = |x|^\beta_k \omega_k(x/|x|)$$

is $p$-harmonic in $\mathbb{R}^2$; $\beta_k$ is the unique root $\geq 1$ of

$$(2k - 1)X^2 - \frac{pk^2 + (p - 2)(2k - 1)}{p - 1}X + k^2 = 0. \quad (4.2)$$

$(\beta_k, \omega_k)$ is unique up to translation and homothety over $\omega_k$.

This result is obtained by solving the homogeneous differential equation satisfied by $\omega_k = \omega$:

$$-\left( (\beta^2 \omega^2 + \omega_0^2)^{(p-2)/2} \omega \right)_{\theta} = \beta (1 + (\beta - 1)(p - 1)) (\beta^2 \omega^2 + \omega_0^2)^{(p-2)/2} \omega. \quad (4.3)$$

In the particular case $k = 1$, then $\beta_1 = 1$ and $\omega_1(\theta) = \sin \theta$. For the other values of $k$ the $\beta_k$ are algebraic numbers and the $\omega_k$ are not trigonometric functions, except if $p = 2$. More generally, if one looks for $p$-harmonic functions in $\mathbb{R}^N \setminus \{0\}$ under the form $u(x) = u(r, \sigma) = r^\beta v(\sigma)$, $r = |x| > 0$, $\sigma = x/|x| \in S^{N-1}$, one obtains that $v$ verifies

$$-div_{\sigma} \left( (\beta^2 v^2 + |\nabla_{\sigma} v|^2)^{(p-2)/2} \nabla_{\sigma} v \right) = \lambda_{N, \beta} (\beta^2 v^2 + |\nabla_{\sigma} v|^2)^{(p-2)/2} v \quad (4.4)$$
on $S^{N-1}$, where $\lambda_{N,\beta} = \beta (N - 1 + (\beta - 1)(p - 1))$ and $\text{div}_\sigma$ and $\nabla_\sigma$ are respectively the divergence and the gradient operators on $S^{N-1}$ (endowed with the Riemannian structure induced by the imbedding of the sphere into $\mathbb{R}^N$). This equation, called the spherical $p$-harmonic spectral problem, is the natural generalization of the spectral problem of the Laplace-Beltrami operator on $S^{N-1}$. His methods combined ODE estimates and shooting arguments. Later on, Tolksdorf \[\text{[11]}\] introduced an entirely new method for proving the existence of solutions on any $C^2$ smooth manifold $S$, with Dirichlet boundary conditions. Only the case $\beta > 0$ was treated in \[\text{[11]}\], and, by a small adaptation of Tolksdorf approach, the case $\beta > 0$ was considered in \[\text{[16]}\]. We develop below a method which allows to express solutions as product of explicit one variable functions.

### 4.1 The 3-D case

Let $(r, \theta, \phi) \in (0, \infty) \times [0, 2\pi] \times [0, \pi]$ be the spherical coordinates in $\mathbb{R}^3$

\[
\begin{align*}
    x_1 &= r \cos \theta \sin \phi \\
    x_2 &= r \sin \theta \sin \phi \\
    x_3 &= r \cos \phi
\end{align*}
\]

Then \[\text{(4.4)}\] turns into

\[
- \frac{1}{\sin \phi} \left[ \sin \phi \left( \beta^2 v^2 + v^2_\phi + \frac{v^2_\theta}{\sin^2 \phi} \right) \right]^{(p-2)/2} = \frac{1}{\sin^2 \phi} \left[ \left( \beta^2 v^2 + v^2_\phi + \frac{v^2_\theta}{\sin^2 \phi} \right) \right]^{(p-2)/2}_\theta
\]

\[
= \beta (2 + (\beta - 1)(p - 1)) \left( \beta^2 v^2 + v^2_\phi + \frac{v^2_\theta}{\sin^2 \phi} \right)^{(p-2)/2} v
\]

(4.5)

We look for a function $v$ under the form

\[
v(\theta, \phi) = (\sin \phi)^\beta \omega(\theta)
\]

(4.6)

then

\[
\beta^2 v^2 + v^2_\phi + \frac{v^2_\theta}{\sin^2 \phi} = (\sin \phi)^{2\beta - 2}(\beta^2 \omega^2 + \omega^2_\theta),
\]

\[
\frac{1}{\sin^2 \phi} \left[ \left( \beta^2 v^2 + v^2_\phi + \frac{v^2_\theta}{\sin^2 \phi} \right) \right]^{(p-2)/2}_\theta = (\sin \phi)^{(\beta -1)(p-1)-1} \left( (\beta^2 \omega^2 + \omega^2_\theta)^{(p-2)/2} \omega \right)_\theta
\]

\[
\left( \beta^2 v^2 + v^2_\phi + \frac{v^2_\theta}{\sin^2 \phi} \right)^{(p-2)/2} v = (\sin \phi)^{(\beta -1)(p-1)+1} (\beta^2 \omega^2 + \omega^2_\theta)^{(p-2)/2} \omega,
\]

and

\[
\frac{1}{\sin \phi} \left[ \sin \phi \left( \beta^2 v^2 + v^2_\phi + \frac{v^2_\theta}{\sin^2 \phi} \right) \right]^{(p-2)/2}_\phi
\]

\[
= \beta (\sin \phi)^{(\beta -1)(p-1)-1} \left[ ((\beta - 1)(p - 1) + 1) - \sin^2 \phi ((\beta - 1)(p - 1) + 2) \right] (\beta^2 \omega^2 + \omega^2_\theta)^{(p-2)/2} \omega.
\]
Proof. We first recall (see [17] for details) that the $\beta_p$ where $\beta = \frac{p}{n}$ more, if $\beta > 1$.

It follows that $\omega$ satisfies the same equation (4.3). The next result follows immediately from Theorem 4.1.

**Theorem 4.2** Assume $N = 3$ and $p > 1$. Then for any positive integer $k$ there exists a $p$-harmonic function $u$ in $\mathbb{R}^3$ under the form

$$u(x) = u(r, \theta, \phi) = r^{\beta_k} (\sin \phi)^{\beta_k} \omega_k(\theta) \quad (4.7)$$

where $\beta_k$ and $\omega_k$ are as in Theorem 4.1.

In the case $p = 3$ we can use the conformal invariance of the 3-harmonic equation in $\mathbb{R}^3$ to derive

**Theorem 4.3** Assume $p = N = 3$. Then for any positive integer $k$ there exists a $p$-harmonic function $u$ in $\mathbb{R}^3 \setminus \{0\}$ under the form

$$u(x) = u(r, \theta, \phi) = r^{-\beta_k} (\sin \phi)^{\beta_k} \omega_k(\theta) \quad (4.8)$$

where $\beta_k$ and $\omega_k$ are as in Theorem 4.1.

As a consequence of Theorem 4.3 we obtain signed 3-harmonic functions under the form (4.7) in the half space $\mathbb{R}^3_+ = \{x : x_2 > 0\}$, vanishing on $\partial \mathbb{R}^3_+ \setminus \{0\}$, with a singularity at $x = 0$. They correspond to even integers $k$. The extension to general smooth domains $\Omega$ is a deep challenge. In the particular case $k = 1$, we have seen that $\beta_1 = 1$ and $\omega_1(\theta) = \sin \theta = x_2$, that we already know.

### 4.2 The general case

We assume that $N > 3$ and write the spherical coordinates in $\mathbb{R}^N$ under the form

$$x = \{(r, \sigma) \in (0, \infty) \times S^{N-1} = (r, \sin \phi \sigma', \cos \phi) : \sigma' \in S^{N-2}, \phi \in [0, \pi]\} \quad (4.9)$$

The main result concerning separable $p$-harmonic functions is the following.

**Theorem 4.4** Let $N > 3$ and $p > 1$. For any positive integer $k$ there exists $p$-harmonic functions in $\mathbb{R}^N$ under the form

$$u(x) = u(r, \sigma', \phi) = (r \sin \phi)^{\beta_k} w(\sigma') \quad (4.10)$$

where $\beta_k$ is the unique root $\geq 1$ of (4.2) and $w$ is solution of (4.12) with $\beta = \beta_k$. Furthermore, if $p = N$ there exists a singular $N$-harmonic function under the form

$$u(x) = u(r, \sigma', \phi) = r^{-\beta_k} (\sin \phi)^{\beta_k} w(\sigma') \quad (4.11)$$

Proof. We first recall (see [17] for details) that the $SO(N)$-invariant unit measure on $S^{N-1}$ is $d\sigma = a_N \sin^{N-2} \theta d\sigma'$ for some $a_N > 0$, and

$$\nabla_\sigma v = -v_\phi e + \frac{1}{\sin \phi} \nabla_{\sigma'} v.$$

where $e$ is derived from $x/|x|$ by the rotation of center 0 angle $\pi/2$ in the plane going thru 0, $x/|x|$ and the north pole. The weak formulation of (4.4) expresses as

$$\int_0^\pi \int_{S^{N-1}} \left( \beta^2 v^2 + v_\phi^2 + \frac{1}{\sin^2 \phi} |\nabla_\sigma' v|^2 \right)^{(p-2)/2} \left( v_\phi \zeta_\phi + \frac{1}{\sin^2 \phi} \nabla_\sigma' v. \nabla_\sigma \zeta \right) \sin^{N-2} \phi d\sigma' d\phi$$

$$= \lambda_{N, \beta} \int_0^\pi \int_{S^{N-1}} \left( \beta^2 v^2 + v_\phi^2 + \frac{1}{\sin^2 \phi} |\nabla_\sigma' v|^2 \right)^{(p-2)/2} \zeta v \sin^{N-2} \phi d\sigma' d\phi \quad (4.12)$$
or, equivalently
\[
- \frac{1}{\sin^{N-2} \phi} \left[ \sin^{N-2} \phi \left( \beta^2 v^2 + v_\phi^2 + \frac{1}{\sin^2 \phi} |\nabla_\sigma v|^2 \right)^{(p-2)/2} \phi \right] \\
- \frac{1}{\sin^2 \phi} \text{div}_{\sigma'} \left[ \left( \beta^2 v^2 + v_\phi^2 + \frac{1}{\sin^2 \phi} |\nabla_\sigma v|^2 \right)^{(p-2)/2} \nabla_{\sigma'} v \right]
\]
\[\text{(4.13)}\]

where \( \text{div}_{\sigma'} \) is the divergence operator acting on vector fields on \( S^{N-2} \). We look again for \( p \)-harmonic functions under the form
\[
u(r, \sigma) = u(r, \sigma', \phi) = r^\beta v(\sigma', \phi) = r^\beta \sin^\beta \phi w(\sigma').
\]
\[\text{(4.14)}\]

Then
\[
\left( \beta^2 v^2 + v_\phi^2 + \frac{1}{\sin^2 \phi} |\nabla_\sigma v|^2 \right)^{(p-2)/2} = (\sin \phi)^{(\beta-1)(p-2)} \left( \beta^2 w^2 + |\nabla_{\sigma'} w|^2 \right)^{(p-2)/2},
\]
thus
\[
\frac{1}{\sin^{N-2} \phi} \left[ \sin^{N-2} \phi \left( \beta^2 v^2 + v_\phi^2 + \frac{1}{\sin^2 \phi} |\nabla_\sigma v|^2 \right)^{(p-2)/2} \phi \right]
\]
\[= \beta(\sin \phi)^{(\beta-1)(p-1)-1} \left( (N - 2 + (\beta - 1)(p - 1)) - (N - 1 + (\beta - 1)(p - 1)) \sin^2 \phi \right)
\times \left( \beta^2 w^2 + |\nabla_{\sigma'} w|^2 \right)^{(p-2)/2} w,
\]
and
\[
\frac{1}{\sin^2 \phi} \text{div}_{\sigma'} \left[ \left( \beta^2 v^2 + v_\phi^2 + \frac{1}{\sin^2 \phi} |\nabla_\sigma v|^2 \right)^{(p-2)/2} \nabla_{\sigma'} v \right]
\]
\[= (\sin \phi)^{(\beta-1)(p-1)-1} \text{div}_{\sigma'} \left[ \left( \beta^2 w^2 + |\nabla_{\sigma'} w|^2 \right)^{(p-2)/2} \nabla_{\sigma'} w \right]
\]
Finally \( w \) satisfies
\[
- \text{div}_{\sigma'} \left[ \left( \beta^2 w^2 + |\nabla_{\sigma'} w|^2 \right)^{(p-2)/2} \nabla_{\sigma'} w \right] = \lambda_{N-1,\beta} \left( \beta^2 w^2 + |\nabla_{\sigma'} w|^2 \right)^{(p-2)/2} w
\]
\[\text{(4.15)}\]
on \( S^{N-2} \), which is the desired induction. \( \square \)

In order to be more precise, we can completely represent the preceding solutions by introducing the generalized Euler angles in \( \mathbb{R}^N = \{ x = (x_1, \ldots, x_N) \} \)
\[
\begin{cases}
  x_1 = r \sin \theta_{N-1} \sin \theta_{N-2} \ldots \sin \theta_2 \sin \theta_1 \\
  x_2 = r \sin \theta_{N-1} \sin \theta_{N-2} \ldots \sin \theta_2 \cos \theta_1 \\
  \vdots \\
  x_{N-1} = r \sin \theta_{N-1} \cos \theta_{N-2} \\
  x_N = r \cos \theta_{N-1}
\end{cases}
\]
\[\text{(4.16)}\]
where $\theta_1 \in [0, 2\pi]$ and $\theta_k \in [0, \pi]$, for $k = 2, ..., N - 1$. Notice that $\theta_{N-1}$ is the variable $\phi$ in the representation (4.14). The above theorem combined with the induction process yields to the following.

**Theorem 4.5** Let $N > 3$ and $p > 1$. For any positive integer $k$ there exists $p$-harmonic functions in $\mathbb{R}^N$ under the form

$$u(x) = (r \sin \theta_{N-1} \sin \theta_{N-2} \ldots \sin \theta_2)^{\beta_k} \omega_k(\theta_1)$$  \hspace{1cm} (4.17)

where $(\beta_k, \omega_k)$ are obtained in Theorem 4.4. Furthermore, if $p = N$ there exists a singular $N$-harmonic function under the form

$$u(x) = r^{-\beta_k} (\sin \theta_{N-1} \sin \theta_{N-2} \ldots \sin \theta_2)^{\beta_k} \omega_k(\theta_1).$$  \hspace{1cm} (4.18)

**References**

[1] Borghol R., *Singularités au bord de solutions d'équations quasilineaires*, Thèse de Doctorat, Univ. Tours (in preparation).

[2] Bidaut-Véron M. F., Borghol R. & Véron L., *Boundary Harnack inequalities and a priori estimates of singular solutions of quasilinear equations*, Calc. Var. and P. D. E., to appear.

[3] Friedman A., & Véron L., *Singular solutions of some quasilinear elliptic equations*, Arch. Rat. Mech. Anal. **96**, 359-387 (1986).

[4] Kichenassamy S. & Véron L., *Singular solutions of the p-Laplace equation*, Math. Ann. **275**, 599-615 (1986).

[5] Krol I. N., *The behavior of the solutions of a certain quasilinear equation near zero cusps of the boundary*, Proc. Steklov Inst. Math. **125**, 130-136 (1973).

[6] Libermann G, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12**, 1203-1219 (1988).

[7] Manfredi J. & Weitsman A., *On the Fatou Theorem for p-Harmonic Functions*, Comm. P. D. E. **13**, 651-668 (1988).

[8] Rešetnjak, Ju. *Spatial mappings with bounded distortion* (Russian), Sibirsk. Mat. Ž. **8**, 629-658 (1967).

[9] Serrin J., *Local behaviour of solutions of quasilinear equations*, Acta Math. **111**, 247-302 (1964).

[10] Serrin J. & Zou H., *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math. **189**, 79-142 (2002).

[11] Tolksdorff P., *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Part. Diff. Equ. **8**, 773-817 (1983).

[12] Tolksdorff P., *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Equ. **51**, 126-140 (1984).

[13] Trudinger N., *On Harnack type inequalities and their applications to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20**, 721-747 (1967).
[14] Véron L., *Some existence and uniqueness results for solution of some quasilinear elliptic equations on compact Riemannian manifolds*, Colloquia Mathematica Societatis János Bolyai 62, 317-352 (1991).

[15] Véron L., *Singularities of solutions of second order quasilinear elliptic equations*, Pitman Research Notes in Math. 353, Addison-Wesley- Longman (1996).

[16] Véron L., *Singularities of some quasilinear equations*, Nonlinear diffusion equations and their equilibrium states, II (Berkeley, CA, 1986), 333-365, Math. Sci. Res. Inst. Publ., 13, Springer, New York (1988).

[17] Vilenkin N. *Fonctions spéciales et théorie de la représentation des groupes*, Dunod, Paris (1969).

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