Gravitational radiospectrometer

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Abstract

Gravitational lensing is predicted by general relativity and is found in observations. When a gravitating body is surrounded by a plasma, the lensing angle depends on a frequency of the electromagnetic wave due to refraction properties, and the dispersion properties of the light propagation in plasma. The last effect leads to dependence, even in the uniform plasma, of the lensing angle on the frequency, what resembles the properties of the refractive prism spectrometer. The strongest action of this spectrometer is for the frequencies slightly exceeding the plasma frequency, what corresponds to very long radiowaves.

1 Introduction

An ordinary theory of the gravitational lensing is developed for the light propagation in the vacuum. Gravitational lensing in the vacuum is achromatic because the deflection angle for the photon does not depend on the frequency of the photon [1]. In the limit of a weak lensing in vacuum by a body with a mass $M$ the deflection angle for the photon (Einstein angle) is $\hat{\alpha} = \frac{4GM}{c^2b} = \frac{2r_g}{b}$, under condition $b \gg r_g$, where $b$ is impact parameter, $r_g$ is the Schwarzschild radius [1].

Propagation of the light in the medium at presence of the gravity field was considered by many authors [2], [3], [4], [5] and references there. If we consider inhomogeneous medium (without gravity) the light rays move along the curved trajectories in this medium. In the papers concerning gravitational deflection the inhomogeneous medium was considered. The deflection due to the gravitation, and the deflection due to the inhomogenity of the medium had been considered separately, without an account of the influence of the dispersion in plasma on the light propagation in the gravitational field. In this work we show that even in the homogeneous medium the dispersion in the plasma leads to dependence of the light deflection angle on the wavelength, what is different from the constant deflection angle in the vacuum.

It have been shown [1], [5], [6], [7], [8], that light propagation in the gravitational field in the vacuum may be formulated as its propagation in a inhomogeneous medium with an effective refraction index $n_g$, depending on metric. It have been shown also, that in presence of the medium in the gravitation field, such analogy is valid too. In this case we should use the refraction index, which is a multiplication of the effective gravitational refraction index $n_g$ and usual refraction index $n$, determined by the physical properties of the medium: $n_{\text{eff}} = n_g n$ [5], [4]. In the case when both $n_g$ and $n$ are close to unity, the combined deviation of the refraction index from unity is reduced to the sum of both separate effects [2], [3], [5].

In this work we consider the gravitational lensing in a homogeneous plasma. Plasma is a dispersive medium, where the refraction index depends on the frequency of the photon. Therefore in the plasma the photons with different frequencies move with different velocities, namely the photons with smaller frequency (or bigger wavelength) move with smaller group velocity of the
light signal. We obtain here, that in a homogeneous plasma, in the presence of gravity, the deflection angle of the photon depends on the frequency of the photon, and discuss observational effects of this phenomenon.

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2 Light propagation in an inhomogeneous plasma in a week gravitational field

Let us consider a static space-time with the metric

\[
d s^2 = g_{ik} \, dx^i \, dx^k = g_{\alpha\beta} \, dx^\alpha \, dx^\beta + g_{00} \, (dx^0)^2, \quad i, k = 0, 1, 2, 3, \quad \alpha, \beta = 1, 2, 3. \tag{1}
\]

Here \( g_{ik} \) do not depend on the time. Let us assume that the gravitational field is week, so that

\[
g_{ik} = \eta_{ik} + h_{ik}, \quad h_{ik} \ll 1, \quad h_{ik} \to 0 \text{ under } x^\alpha \to \infty. \tag{2}
\]

Here \( \eta_{ik} \) is the metric of a flat space \((-1, 1, 1, 1)\), and \( h_{ik} \) is a small perturbation. Note [7], that

\[
g^{ik} = \eta^{ik} - h^{ik}, \quad \eta^{ik} = \eta_{ik}, \quad h^{ik} = h_{ik}. \tag{3}
\]

Let us consider in this gravitational field a static inhomogeneous plasma with the refraction index \( n \), which depends on the space location \( x^\alpha \) and the frequency of the photon \( \omega(x^\alpha) \):

\[
n^2 = 1 - \frac{\omega^2}{[\omega(x^\alpha)]^2}, \quad \omega^2 = \frac{4\pi e^2 N(x^\alpha)}{m}. \tag{4}
\]

Here \( \omega(x^\alpha) \) is the frequency of the photon, which depends on space coordinates \( x^1, x^2, x^3 \) due to presence of gravitational field (gravitational red shift). We denote \( \omega(\infty) = \omega \), and \( e \) is the charge of the electron, \( m \) is the mass of the electron, \( N(x^\alpha) \) is the electron concentration in the inhomogeneous plasma, \( \omega_e \) is the electron plasma frequency in this plasma. To consider a general case let us assume

\[
N(x^\alpha) = N_0 + N_1(x^\alpha), \quad N_0 = \text{const}, \quad N_1(\infty) = 0. \tag{5}
\]

Here \( N_1 \) is not supposed to be small compared to \( N_0 \), let us denote:

\[
\omega^2 = \omega_0^2 + \omega_1^2, \quad \omega_0^2 = K_e N_0, \quad \omega_1^2 = K_e N_1, \quad K_e = \frac{4\pi e^2}{m}. \tag{6}
\]

The gravitational optic in a medium in a curved space-time, was investigated in [9]. Consider a set of three-dimensional surfaces in a static four-D space-time, which are characterized by monotonically increasing phase angle. These 3-D surfaces are called 3-D waves, or phase waves. It was found in [9], the connection between the phase velocity \( u \), and a 4-vector of the photon momentum \( p^i \), written with using the refraction index of medium \( n \), \( n = c/u \), \( c \) is the light velocity in a vacuum, as

\[
\frac{c^2}{u^2} = n^2 = 1 + \frac{p^i p^i}{(p^0 \sqrt{-g_{00}})^2}. \tag{7}
\]
The refraction index $n$, defined for plasma in (4), is in a general case the function of $x^i$ and $\omega(x^\alpha)$, which are determined by the properties of the medium, and the photon frequency. In the case of the vacuum ($n = 1$) we can obtain from (7) the usual relation for the square of the photon 4-vector: $p_ip^i = 0$. In the medium, the square of the photon 4-vector is not equal to zero. For the medium in a flat space-time we have,

$$g_{00} = -1, \quad g_{\alpha\alpha} = 1, \quad p^0 = -p_0, \quad p^\alpha = p_\alpha \quad n^2 = 1 + \frac{-(p^0)^2 + (p^\alpha)^2}{(p^0)^2},$$

and obtain the usual relation between the space and time components of the 4-vector of the photon

$$p^\alpha = n^2(p^0)^2. \quad (9)$$

The time component of the photon 4-vector is its energy, therefore $p^0$ is proportional to the frequency of the photon $\omega$ [8]. We have in the flat space-time

$$p^0 = C\omega, \quad C = \text{const} > 0, \quad (10)$$

and in a space-time with gravity we have [8], [9],

$$p^0\sqrt{-g_{00}} = C\omega(x^\alpha), \quad (11)$$

what physically determines the gravitational red shift. The coordinate $x^\alpha$ and the momentum $p^\alpha$ are connected by the relation $dx^\alpha/d\lambda = p^\alpha$, where $\lambda$ is a parameter changing along the photon trajectory (see below). Consider a photon moving along $z$-axis, with the frequency at infinity equal to $\omega$. Without the gravity and medium inhomogeneity its unperturbed trajectory is a straight line along $z$ axis. The photon 4-vector for the unperturbed trajectory is $p^i = (p^0, 0, 0, p^3)$, and the relation between the space and time components, using the flat metric $\eta_{ik}$, is

$$(p^3)^2 = n_0^2(p^0)^2. \quad (12)$$

Here we denote [6]

$$n_0 = n(\infty) = \sqrt{1 - \frac{\omega_0^2}{\omega^2}}. \quad (13)$$

It is convenient to use the coordinate $z$ as the parameter $\lambda$. Then we have the components of the photon 4-vector in a simple form:

$$p^i = (1/n_0, 0, 0, 1), \quad p_i = (-1/n_0, 0, 0, 1). \quad (14)$$

Thus the photon momentum in plasma is a time-like 4-vector

$$p^i_\alpha p_i = 1 - \frac{1}{n_0} = 1 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)^{-1} < 0, \quad (15)$$

While the phase velocity $u = \frac{c}{m}$ in plasma is larger than the light velocity in vacuum $c$, the group velocity $v_{gr}$ is less than $c$, so that the larger frequency corresponds to the larger group velocity, tending to $c$ in the limit. For the group velocity we have the relation [10], [9]

$$\frac{c}{v_{gr}} = \frac{\partial}{\partial \omega}(n_0\omega) = \frac{1}{\sqrt{1 - \omega_0^2/\omega^2}}, \quad v_{gr} = c\sqrt{1 - \omega_0^2/\omega^2} = cn_0 < c, \quad (16)$$

so that $v_{gr}u = c^2$. To find a constant $C$ in (11), we consider this relation at infinity, where $\omega(x^\alpha) = \omega$, and $p^0$ is defined by (14). We have then

$$p^0 = C\omega, \quad C = \frac{1}{n_0\omega}. \quad (17)$$
The trajectories of the photon in presence of the gravitational field may be found from the variational principle \[ (18) \]

\[ \delta \left( \int p_i \, dx^i \right) = 0, \]

with the restriction \[ (7) \], which may be written in the form

\[ W(x^i, p_i) = \frac{1}{2} \left[ g^{ij} p_i p_j - (n^2 - 1) \left( p_0 \sqrt{-g^{00}} \right)^2 \right] = 0. \] \[ (19) \]

Here we define the scalar function \( W(x^i, p_i) \) of \( x^i \) and \( p_i \). The variational principle \[ (18) \], with the restriction condition \( W(x^i, p_i) = 0 \), leads to the system of differential equations \[ (9) \]:

\[ \frac{dx^i}{d\lambda} = \frac{\partial W}{\partial p_i}, \quad \frac{dp_i}{d\lambda} = -\frac{\partial W}{\partial x^i}, \] \[ (20) \]

with the parameter \( \lambda \) changing along the light trajectory. Let us introduce a variable

\[ \chi = p_0 \sqrt{-g^{00}} = -p_0 \sqrt{-g_{00}} = -C \omega (x^\alpha) = -\frac{1}{n_0 \omega} \omega (x^\alpha), \] \[ (21) \]

by using of which we can transform \( W(x^i, p_i) \) to a simpler form

\[ W(x^i, p_i) = \frac{1}{2} \left[ g^{00} p_0 p_0 + g^{0\alpha} p_0 p_\alpha - (n^2 - 1) p_0^2 (-g^{00}) \right] = \frac{1}{2} \left[ g^{0\alpha} p_0 p_\alpha - n^2 \chi^2 \right]. \] \[ (22) \]

From \[ (20) \] we obtain the system of equations for the space components \( p_\alpha \):

\[ \frac{dx^\alpha}{d\lambda} = g^{0\alpha} p_0, \quad \frac{dp_\alpha}{d\lambda} = \frac{1}{2} g^{0\beta} p_\beta p_\gamma + \frac{1}{2} \left( n^2 \chi^2 \right)_\alpha. \] \[ (23) \]

For the inhomogeneous plasma \( n = n(\chi, x^\alpha) \), and we have, using \[ (14), (25), (21) \] the relation

\[ \frac{1}{2} \left( n^2 \chi^2 \right)_\alpha = \frac{1}{2} \left[ \left( 1 - \frac{K_\epsilon N_0}{|\omega(x^\alpha)|^2} - \frac{K_\epsilon N_1(x^\alpha)}{|\omega(x^\alpha)|^2} \right) \chi^2 \right]_\alpha = \frac{1}{2} \left[ \chi^2 - \frac{1}{n_0^2 \omega^2} K_\epsilon N_0 - \frac{1}{n_0^2 \omega^2} K_\epsilon N_1(x^\alpha) \right]_\alpha = \chi \frac{\partial \chi}{\partial x^\alpha} - \frac{1}{2} \frac{K_\epsilon}{n_0^2 \omega^2} \frac{\partial N_1(x^\alpha)}{\partial x^\alpha}. \] \[ (24) \]

As follows from \[ (20) \], the variable \( p_0 \) is constant along the trajectory, so from \[ (21) \] we have

\[ \frac{\partial \chi}{\partial x^\alpha} = p_0 \left( \sqrt{-g^{00}} \right)_\alpha, \] \[ (25) \]

and obtain finally

\[ \frac{1}{2} \left( n^2 \chi^2 \right)_\alpha = \chi p_0 \left( \sqrt{-g^{00}} \right)_\alpha - \frac{1}{2} \frac{1}{n_0^2 \omega^2} K_\epsilon \frac{\partial N_1(x^\alpha)}{\partial x^\alpha} = \frac{1}{2} p_0^2 (-g^{00})_\alpha - \frac{1}{2} \frac{1}{\omega^2 - \omega_0^2} K_\epsilon \frac{\partial N_1(x^\alpha)}{\partial x^\alpha}. \] \[ (26) \]

Using \[ (20) \], we reduce equations \[ (20) \] to the form

\[ \frac{dx^\alpha}{d\lambda} = g^{0\beta} p_\beta, \quad \frac{dp_\alpha}{d\lambda} = -\frac{1}{2} g^{0\beta} p_\beta p_\gamma + \frac{1}{2} \left( n^2 \chi^2 \right)_\alpha - \frac{1}{2} \frac{1}{\omega^2 - \omega_0^2} K_\epsilon \frac{\partial N_1(x^\alpha)}{\partial x^\alpha}. \] \[ (27) \]

The system \[ (27) \] describes the light propagation in an inhomogeneous plasma, with account the dispersion, in the presence of a gravitational field. Let us consider the photon moving along \( z \)-axis in a flat space with a homogeneous plasma, and use the coordinate \( z \) as the parameter \( \lambda \). In approximation of small perturbation \( h_{ik} \) and week inhomogeneity, \( N_1 \ll N_0 \), one can integrate equations, calculating right-hand side of equations \[ (27) \] by using the unperturbed trajectory of the photon, with \( p_i \) from \[ (14) \], in the right hand side. At this simplification, the first and second terms in the second equation \[ (27) \] can be written as
\[- \frac{1}{2} g^{\beta\gamma}_{\alpha} p_{\beta} p_{\gamma} + \frac{1}{2} p_{0}^{2} (-g^{00})_{,\alpha} = - \frac{1}{2} g^{\beta\gamma}_{\alpha} p_{\beta} + \frac{1}{2} p_{0}^{2} (-g^{00})_{,\alpha}. \]

In the weak gravitational field we can write, using (2), (3) and (14)
\[- \frac{1}{2} g^{33}_{\alpha} p_{3} + \frac{1}{2} p_{0}^{2} (-g^{00})_{,\alpha} = \frac{1}{2} (h_{33})_{,\alpha} + \frac{1}{2} \frac{1}{n_{0}^{2}} (h_{00})_{,\alpha} = \frac{1}{2} \left( h_{33} + \frac{\omega^{2} h_{00}}{\omega^{2} - \omega_{0}^{2}} \right)_{,\alpha}. \]

Finally we obtain the equation describing the light propagation in a weakly inhomogeneous plasma, in the presence of a small gravitational field, as
\[\frac{dp_{\alpha}}{dz} = \frac{1}{2} \left( h_{33} + \frac{\omega^{2} h_{00}}{\omega^{2} - \omega_{0}^{2}} \right)_{,\alpha} - \frac{1}{2} \frac{K_{e}}{\omega^{2} - \omega_{0}^{2}} \frac{\partial N_{1}(x^{\alpha})}{\partial x^{\alpha}}. \]

The deflection angle in the plane, perpendicular to the unperturbed light trajectory, is determined as
\[\hat{\alpha}_{\beta} = [p_{\beta}(+\infty) - p_{\beta}(-\infty)]/p, \quad p = \sqrt{p_{3}^{2} + p_{2}^{2} + p_{0}^{2}} = |p_{3}| = 1, \quad \beta = 1, 2. \]

Here \( p \) is defined by the unperturbed trajectory, and \( \beta = 1, 2 \) are related to \( x, y \) axes. After integration, we have the following expression for the deflection angle of the photon with the unperturbed trajectory along \( z \) axis
\[\hat{\alpha}_{b} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial h_{33}}{\partial b} + \frac{\omega^{2}}{\omega^{2} - \omega_{0}^{2}} \frac{\partial h_{00}}{\partial b} - \frac{K_{e}}{\omega^{2} - \omega_{0}^{2}} \frac{\partial N_{1}(r)}{\partial b} \right) \right) dz. \]

If the problem is axially symmetric, it is convenient to introduce the impact parameter \( b \). Let us consider the photon moving along the axis \( z \) in the plasma with the impact parameter \( b \) relative to the point mass, and the plasma has a spherically-symmetric distribution of a concentration around this point mass, \( N = N(r) = N_{0} + N_{1}(r) \). The situation is axially symmetric, so the position of the photon is given by \( b \) and \( z \), and the absolute value of the 3-radius-vector is \( r = \sqrt{b^{2} + z^{2}} \), instead of \( r = \sqrt{x_{1}^{2} + x_{2}^{2} + z^{2}} \). We have the following expression for the deflection angle in the plane perpendicular to direction of the unperturbed photon:
\[\hat{\alpha}_{b} = \frac{1}{2} \left( \frac{\partial h_{33}}{\partial b} + \frac{\omega^{2}}{\omega^{2} - \omega_{0}^{2}} \frac{\partial h_{00}}{\partial b} - \frac{K_{e}}{\omega^{2} - \omega_{0}^{2}} \frac{\partial N_{1}(r)}{\partial b} \right) \right) dz. \]

Note that \( \hat{\alpha}_{b} < 0 \) corresponds to bending of the light trajectory to the direction of the gravitation center, and \( \hat{\alpha}_{b} > 0 \) corresponds to the opposite deflection.

## 3 Particular cases

### 3.1 Vacuum, and a homogeneous medium without dispersion

Let us consider a homogeneous medium without dispersion, with the refraction index \( n = \text{const} \geq 1 \) not depending on the frequency \( \omega \). The equations describing the light propagation follow from (23):
\[\frac{dx^{\alpha}}{d\lambda} = g^{\alpha\beta} p_{\beta}, \quad \frac{dp_{\alpha}}{d\lambda} = -\frac{1}{2} g^{\beta\gamma}_{\alpha} p_{\beta} p_{\gamma} + \frac{n^{2}}{2} (\chi^{2})_{,\alpha} \]

The right-hand side of equation (23) for the \( p_{\alpha} \) is transformed, using (24), (25), and for determination of the deflection angle \( \hat{\alpha}_{b} \) of the photon, moving with the impact parameter \( b \) in the spherically symmetric gravitational field, we have
\[- \frac{1}{2} g^{\beta\gamma}_{\alpha} p_{\beta} p_{\gamma} + \frac{n^{2}}{2} (\chi^{2})_{,\alpha} = -\frac{1}{2} g^{\beta\gamma}_{\alpha} p_{\beta} p_{\gamma} + \frac{n^{2}}{2} p_{0}^{2} (-g^{00})_{,\alpha} = -\frac{1}{2} g^{33}_{\alpha} p_{3}^{2} + \frac{n^{2}}{2} p_{0}^{2} (-g^{00})_{,\alpha} = (35)\]
between 3-vector \( \mathbf{r} \)

where \( \mathbf{r} \) is the 3-radius-vector.

Substituting the limits of integration, we obtain the following expression for the deflection angle with the well known formula \( [13], [14], [15] \) for the vacuum case, derived from the geodesic equation,

\[
\hat{\alpha}_\beta = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial}{\partial x^\beta} (h_{33} + h_{00}) \, dz.
\]

We see here, that in case of the non-dispersive medium, the constant index of the refraction is canceled, and the photon trajectory is the same as in the vacuum, in presence of the gravitational field, in spite of lower velocity of the light propagation in the medium. Note that the motion of photons in 4-space, in the medium, is not described by the geodesic equation (neither massive not zero), because the light propagation in the medium with the refraction is determined not only by the gravitational field, but also by the medium.

### 3.2 Homogeneous plasma in a weak gravitational field

Here we represent the main new result of this work: the dependence of the deflection angle on the photon frequency in the homogeneous plasma, due to account of the dispersion. In the homogeneous plasma from \([33]\), we have expression for the deflection angle as

\[
\hat{\alpha}_b = \int_{0}^{\infty} \frac{\partial}{\partial b} \left( h_{33} + \frac{1}{1 - \omega_0^2/\omega^2} h_{00} \right) \, dz.
\]

Let us calculate the deflection angle for the photon moving in the homogeneous plasma in the Schwarzschild metric \([7]\) of the point mass \( M \)

\[
ds^2 = -(1 - r_g/r) \, dt^2 + \frac{dr^2}{1 - r_g/r} + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2),
\]

where \( r_g = \frac{2GM}{c^2} \) is the Schwarzschild gravitational radius. In the weak field approximation this metric is written as \([7]\)

\[
ds^2 = ds_0^2 + \frac{r_g}{r} (c^2 dt^2 + dr^2),
\]

where \( ds_0^2 \) is the flat part of metric \( \left( ds_0^2 = \eta_{ik} dx^i dx^k \right) \). The components \( h_{ik} \) are written in the Cartesian frame as \([7]\)

\[
h_{00} = \frac{r_g}{r}, \quad h_{\alpha\beta} = \frac{r_g}{r} n_\alpha n_\beta, \quad h_{33} = \frac{r_g}{r} \cos^2 \theta.
\]

Here \( n_\alpha \) is the unit vector of 3-radius-vector \( r_\alpha = (x_1, x_2, x_3) \), the angle \( \theta \) is the polar angle between 3-vector \( r^\alpha \) and \( z \)-axis, and \( \cos \theta = z/r = z/\sqrt{b^2 + z^2} \). The integrals in \([33]\) are taken analytically:

\[
\int \frac{\partial}{\partial b} h_{00} \, dz = \int \frac{\partial}{\partial b} \frac{r_g}{\sqrt{b^2 + z^2}} \, dz = -r_g \int \frac{b}{(b^2 + z^2)^{3/2}} \, dz = -r_g \frac{z}{b \sqrt{b^2 + z^2}} + \text{const},
\]

\[
\int \frac{\partial}{\partial b} h_{33} \, dz = \int \frac{\partial}{\partial b} \frac{z^2}{b^2 + z^2} \, dz = -r_g \int \frac{3z^2b}{(b^2 + z^2)^{3/2}} \, dz = -r_g \frac{z^3}{b (b^2 + z^2)^{3/2}} + \text{const}.
\]

Substituting the limits of integration, we obtain the following expression for the deflection angle in the homogeneous plasma

\[
\hat{\alpha}_b = \int_{0}^{\infty} \frac{\partial}{\partial b} \left( h_{33} + \frac{1}{1 - \omega_0^2/\omega^2} h_{00} \right) \, dz = -\frac{r_g}{b} \left( 1 + \frac{1}{1 - \omega_0^2/\omega^2} \right).
\]
Here $\alpha_b < 0$ for $\omega > \omega_0$, what means that the light ray is bent to the direction of gravitation center, as it occurs in the vacuum. The formula (43) is valid only for $\omega > \omega_0$, because the waves with $\omega < \omega_0$ do not propagate in the plasma [12]. In the theory of the gravitational lensing the deflection angle is usually defined as the difference between the initial and the final ray directions $\alpha = e_{in} - e_{out}$, where $e$ is the unit tangent vector of a ray [1]. Therefore, if we use this definition, we will have the expression with the opposite sign:

$$\hat{\alpha} = \frac{rg}{b} \left( 1 + \frac{1}{1 - \frac{\omega_0^2}{\omega^2}} \right),$$

(44)

which turns into the deflection angle [1] for vacuum $2rg/b$, when $\omega \to \infty$. For lower frequencies the deflection angle may be much larger than in the vacuum, and the image of the point source will be represented by the line, or the ring (see Fig.1,2), on which the frequency is decreasing with increasing of the distance from the source on the plane of the view. Such effect may happen only for photons with radio frequencies, because optical frequencies are much higher than the plasma frequency $\omega_0$, so the effect should be negligible. We see therefore, that the gravitational lens in plasma is acting as the gravitational radiospectrometer, see Fig.1,2.

3.3 Inhomogeneous plasma with gravity

Let us consider inhomogeneous plasma in presence of a weak gravitational field. Let us consider a case when plasma concentration $N(\infty) = 0$. In our notations we have $N_0 = 0$, $\omega_0 = 0$, $N = N_1$. In this case we have from (33) the following expression for the deflection angle, valid when $|n-1| \ll 1$,

$$\alpha_b = \frac{1}{2} \int_{-\infty}^{\infty} \left( \frac{\partial h_{33}}{\partial b} + \frac{\partial h_{00}}{\partial b} - \frac{1}{\omega^2} K_e \frac{\partial N_1}{\partial b} \right) dz.$$ 

(45)

The case of the inhomogeneous plasma with gravity was considered in [2], [3], [5]. Let us calculate the deflection angle for the photon, moving in the Schwarzschild metric, with the concentration of plasma in the form [5]

$$N(r) = N_m \left( \frac{R}{r} \right)^h, \quad N_m = \text{const}, \quad R = \text{const}, \quad h > 0.$$ 

(46)

It is evident from (45) that in the week gravitational field, in presence of plasma which implies a small perturbation in the photon propagation ($\omega^2 \gg \omega_e^2$), both effects: the deflection due to gravity and the deflection due to inhomogeneity of medium (non-relativistic effect) can be considered separately. The first and the second terms in (45) lead to usual Einstein angle of deflection, following from (44) at $\omega_0^2 = 0$. Let us calculate deflection due to the inhomogeneity of plasma, in the third term in (45) for the density distribution (46):

$$\frac{1}{2} \int_{-\infty}^{\infty} \left( -\frac{1}{\omega^2} K_e \frac{\partial N_1}{\partial b} \right) dz = -\frac{1}{\omega^2} K_e N_m R^h \int_0^{\infty} \frac{\partial}{\partial b} \left( \frac{1}{r^h} \right) dz.$$ 

(47)

After differentiation

$$\frac{\partial}{\partial b} \left( \frac{1}{r^h} \right) = \frac{\partial}{\partial b} \left( \frac{1}{(z^2 + b^2)^{h/2}} \right) = -\frac{hb}{(z^2 + b^2)^{h/2+1}}$$

(48)

we perform the integration using [16], and properties of the $\Gamma$-function:

$$\int_0^{\infty} \frac{dz}{(z^2 + b^2)^{h/2+1}} = \frac{1}{hb^{h+1}} \frac{\sqrt{\pi} \Gamma \left( \frac{b}{2} + \frac{1}{2} \right)}{\Gamma \left( \frac{h}{2} \right)}, \quad \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$ 

(49)

So we obtain

$$\int_0^{\infty} \frac{\partial}{\partial b} \left( \frac{1}{z^2 + b^2} \right) dz = -\frac{1}{b^h} \sqrt{\pi} \Gamma \left( \frac{b}{2} + \frac{1}{2} \right) \frac{\Gamma \left( \frac{h}{2} \right)}{\Gamma \left( \frac{h}{2} + \frac{1}{2} \right)},$$

(50)
and for the third term in \([15]\) we have
\[
\frac{1}{2} \int_{-\infty}^{\infty} \left( -\frac{1}{\omega^2} K_c \frac{\partial N_1(r)}{\partial b} \right) dz = \frac{1}{\omega^2} \frac{4\pi e^2}{m} N_m \left( \frac{R}{b} \right)^\hbar \frac{\sqrt{\pi} \Gamma \left( \frac{b}{2} + \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} \right)}.
\] (51)

Finally, for the deflection angle near the Schwarzschild metric with plasma concentration \([16]\) we obtain the expression
\[
\hat{\alpha}_o = -\frac{r_g}{b} + \frac{1}{\omega^2} \frac{4\pi e^2}{m} N_m \left( \frac{R}{b} \right)^\hbar \frac{\sqrt{\pi} \Gamma \left( \frac{b}{2} + \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} \right)}.
\] (52)

which is the same, as in \([5]\), where this formula is written in terms of the wave length instead of the frequency.

4 Discussion

Let us consider the case of a weekly inhomogeneous medium without gravity, with the refraction index \(n = n_0 + n_1, \quad n_0 = \text{const}, \quad n_1 \ll n_0\). The system of equation for the trajectory of the photon in this case follows from equations \([23]\) with the flat metric \(g_{ik} = \eta_{ik}\), see \([10]\),
\[
\frac{dx^\alpha}{dz} = p^\alpha, \quad \frac{dp^\alpha}{dz} = -\frac{1}{2} \eta^{\alpha\gamma} p_\beta p_\gamma + \frac{1}{2} \left( n^2 \chi^2 \right)_\alpha = \frac{1}{2n_0^2} \frac{\partial n^2}{\partial x^\alpha} \approx \frac{1}{n_0} \frac{\partial n}{\partial x^\alpha}.
\] (53)

The light propagation in a week gravitational field may be considered, as a propagation in a flat space with the "gravitational" refraction index, which for the Schwarzschild metric is written as \([1, 2, 3, 4, 5, 6]\)
\[
n_g = 1 + \frac{r_g}{r}, \quad \frac{r_g}{r} \ll 1.
\] (54)

The total effective refraction index \(n_{eff}\), in presence of plasma with the proper refraction index \(n\), is determined \([4, 5]\) as \(n_{eff} = n n_g\). When plasma in a week gravitational field has a refraction index close to unity, it follows from the definition of \(n_{eff}\), that in the linear approximation the effective refraction index is obtained as a sum of two different additions to the unity \([2, 3, 5]\):
\[
n_{eff} = 1 + \frac{r_g}{r}, \quad \frac{r_g}{r} \ll 1, \quad \frac{\omega^2(r)}{\omega^2} \ll 1.
\] (55)

From \([53]\) we obtain the deflection angle \([52]\). Note that when \(n^2 - 1\) is not small, \(n_{eff}\) cannot be represented in the form \([53]\), and such case was considered in the subsection 3.2.

The main new result of this work is obtaining of the dependence, of the lensing angle on the frequency in a homogenous plasma in the gravitational field. This effect has a relativistic nature, and is connected with the dispersive properties of plasma. It is interesting, that, as shown in the subsection 3.1, in the medium without dispersion, the trajectories of photons with different frequencies (energies) are exactly the same as in the vacuum, while their velocities are less that the vacuum light velocity \(c\).

Observational effect of such frequency dependence is easy to explain on the example of lensing by the Schwarzschild point-mass lens. This lens gives two images of source, on the opposite sides from lens. Angular positions of images depend on the Schwarzschild radius of lens and positions of source, lens and observer. The dependence of the deflection angle on the frequency in plasma lead to the phenomenon, that instead of two concentrated images with complicated spectra, we will have two line images, formed by the photons with different frequencies, which are deflected by different angles (Fig.1,2).

The description of the mass distribution as a point mass (Schwarzschild lens) is rarely sufficient for gravitational lensing considerations \([1]\). In reality the gravitational lenses have more a complicated structure, and position of images different from that of the point-mass lens. But all standard models of gravitational lenses are based on the same Einstein deflection angle \(2r_g/b\), which should be modified for sufficiently long waves, according to our formula \([14]\), in the presence of plasma. Note also, that taking into account of plasma effects on the gravitational lensing may influence the
spectrum of the microwave background radiation, leading to the dependence of power spectrum of the fluctuations on the photon wavelength.

The light signals are propagated with the group velocity. It follows from (44), that the smaller group velocity (smaller frequency and bigger wavelength) corresponds to a larger deflection angle. Hence the effect of difference in the gravitational deflection angles is significant for larger wavelengths, when $\omega$ is approaching $\omega_e$, what is possible only for the radio waves. Therefore, the gravitating center in plasma is acting as a radiospectrometer. The longest radiowaves are registered in the band $\sim 3 \cdot 10^3$ cm, corresponding to $\nu \simeq 10^7$ Hz = 10 MHz and $\omega = 2\pi\nu \simeq 6 \cdot 10^7$ sec$^{-1}$. The spectroscopic effects of lensing will be important when $N_e \geq 3 \cdot 10^5$ cm$^{-3}$, corresponding to 10 % difference in the lensing angles. Such electron densities may be expected around the supermassive black holes, or during lensing at earlier stages of the universe expansion at $z \geq 10^3$.

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Figure 2: Axis on lensing by the Schwarzschild point-mass lens. The case of the Einstein ring. Instead of a thin ring corresponding to the vacuum lensing (the inner circle of the ring) we have a thick ring, formed by the photons of different frequencies.
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