Abstract

Several general results for the spectral determinant of the Schrödinger operator on metric graphs are reviewed. Then, a simple derivation for the $\zeta$-regularized spectral determinant is proposed, based on the Roth trace formula. Two types of boundary conditions are studied: functions continuous at the vertices and functions whose derivative is continuous at the vertices. The $\zeta$-regularized spectral determinant of the Schrödinger operator acting on functions with the most general boundary conditions is conjectured in conclusion. The relation to the Ihara, Bass and Bartholdi formulae obtained for combinatorial graphs is also discussed.

PACS numbers: 02.70.Hm, 02.10.Ox

(Some figures in this article are in colour only in the electronic version)
Because the operator \(-\Delta + V(x)\) acts in a space of infinite dimension, the computation of the spectral determinant (1) requires in practice some regularization. Starting from the trace of Green’s function, i.e. the Laplace transform of the partition function \(Z(t) = \sum_n e^{-tE_n}\),

\[
G(\gamma) = \sum_n \frac{1}{\gamma + E_n} = \int_0^\infty dt Z(t) e^{-\gamma t},
\]

furnishes a first possible regularization, used in [1, 7, 8, 10, 32, 33, 42, 44]. Performing some integration with respect to the spectral parameter we obtain

\[
S_{GF}^{\gamma}(\gamma') \overset{\text{def}}{=} \exp \int_{\gamma_0}^{\gamma} d\gamma' G(\gamma') = S(\gamma)/S(\gamma_0).
\]

For example, consider the Laplace operator on a line \([0,L]\) with Neumann boundary conditions. The spectrum is in this case \(E_n = (n\pi/L)^2\) with \(n \in \mathbb{N}\) and we obtain \(S_{GF}^{\gamma}(\gamma) = \frac{\gamma \sinh \sqrt{\gamma L}}{\sqrt{\gamma_0 \sinh \sqrt{\gamma_0 L}}}.\) This procedure, however, leaves some arbitrary in the choice of the parameter \(\gamma_0\).

Another well-known regularization for determinants is the \(\zeta\)-regularization whose starting point is to introduce the \(\zeta\)-function

\[
\zeta(s,\gamma) \overset{\text{def}}{=} \sum_n (\gamma + E_n)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-\gamma t} Z(t)
\]

that we have expressed as a Mellin transform of the partition function, for convenience for the following discussion. Note that \(\zeta(1,\gamma) = G(\gamma).\) The \(\zeta\)-regularized determinant is then related to a derivative of the \(\zeta\)-function:

\[
S^{\zeta}(\gamma) \overset{\text{def}}{=} \exp -\frac{d\zeta}{ds}(0,\gamma).
\]

In this case no arbitrary is left within the calculation of the determinant. Returning to the simple example considered above of the Laplace operator on a line \([0,L]\) with Neumann boundary conditions, we get in this case \(S^{\zeta}(\gamma) = 2\sqrt{\gamma} \sinh \sqrt{\gamma L}\) (derived below).

An important remark is that various regularizations only differ by some \(\gamma\)-independent prefactors. This explains why the choice of the regularization has no consequence on the properties studied in [1, 6, 7, 10, 14, 32, 33, 42] since they are always related to derivatives of \(\ln S(\gamma)\), with respect to the spectral parameter \(\gamma\) or other parameters. In a recent work [44] a procedure was proposed in order to construct the spectral determinant of a graph by combinations of the determinants of subgraphs. In this case it is crucial to define precisely the prefactor of the spectral determinant.

In a recent paper [17], Friedlander derived the \(\zeta\)-regularized determinant for the Schrödinger operator on a metric graph. One should also mention that the analysis of the regularized determinant of the Laplace operator for general boundary conditions, \(\det(\gamma - \Delta)\) with the prime indicating exclusion of zero mode contribution if there is some, was the subject of the very recent work [19]. It is the purpose of this paper to propose another derivation of the \(\zeta\)-regularized determinant for the Schrödinger operator \(\det[\gamma - \gamma + V(x)]\). Our approach is based on the Roth trace formula [34] and will allow simple extension of Friedlander’s result to other choices of boundary conditions at the vertices.

In the next section we set notations. In section 3, we mostly recall some known general results obtained by Desbois in [10] and needed for the following sections. Section 4 focuses on the case of functions continuous at the vertices and section 5 on the case of functions with derivative continuous at vertices. The \(\zeta\)-regularization of the spectral determinant is provided in section 6.
2. Metric graphs and the Laplace operator

A graph is a set of vertices (here labelled with Greek letters $\alpha$, $\beta$, $\ldots$) linked by bonds (denoted as $(\alpha \beta)$, $\ldots$). Each bond $(\alpha \beta)$ is associated with two arcs (oriented bonds) $\alpha \beta$ and $\beta \alpha$ (arc will also be labelled with roman indices $a$, $b$, $\ldots$). For the arc $a = \alpha \beta$, the reversed arc is denoted by $\bar{a} = \beta \alpha$. We introduce the adjacency matrix: $a_{\alpha \beta} = 1$ if $\alpha$ is linked to $\beta$ by a bond and $a_{\alpha \beta} = 0$ otherwise. We denote by $m_\alpha = \sum_\beta a_{\alpha \beta}$ the coordination number (valency) of the vertex. The graph is said to be a metric graph (or a quantum graph) if each bond is identified with a finite interval $(0, l_\alpha) \in \mathbb{R}$, where $l_\alpha$ designates the length of the bond linking the vertices $\alpha$ and $\beta$. In this case we may consider a scalar function $\psi(x)$, characterized by a set of $B$ components $\psi_{a \beta}(x_{a \beta})$ with $x_{a \beta} \in (0, l_\alpha)$. The component is labelled by the arc in order to specify the direction along which the coordinate is measured (this redundancy of the notation implies the obvious relation $\psi_{a \beta}(x_{a \beta}) = \psi_{\bar{b} \alpha}(l_{\alpha \beta} - x_{a \beta})$).

Having introduced scalar functions living on the graph we may define the action of the Laplace operator $\Delta$ on such functions. Along a wire it acts as the usual second derivative $(\Delta \psi)(x) = \frac{1}{x^2} \psi(x)$. At the vertices, the set of functions on which $\Delta$ acts must satisfy some boundary conditions in order to ensure self-adjointness of the operator. Let us denote by $\psi(0)$ and $\psi'(0)$ the vectors of size $2B$ gathering the values taken by the components and the derivatives at the vertices $\psi_{a \beta}(0)$ and $\psi'_{a \beta}(0)$, respectively. The most general boundary conditions ensuring self-adjointness of $\Delta$ are of the form

$$C \psi(0) + D \psi'(0) = 0,$$

where the $2B \times 2B$ matrices $C$ and $D$ satisfy (i) $CD^† = DC^†$. (ii) The $2B \times 4B$ matrix $(C, D)$ has maximal rank (equal to $2B$) [24]. Note that characterizing which arc is connected to which other arc, these two matrices encode all the information on the topology of the graph.

3. General results for the spectral determinant

In this section we mostly recall some results obtained by Desbois [10] for the spectral determinant of the Schrödinger operator for general boundary conditions. We compute the spectral determinant by constructing Green’s function $G(x, y) \overset{\text{def}}{=} \{ x | \frac{1}{y - D^2 + V(x)} y \}$, where $D_x \overset{\text{def}}{=} \frac{d}{dx} - i A(x)$ is the covariant derivative. Note that the replacement of $\Delta$ by $D^2$ is motivated by physical considerations [1, 31, 39]. It might also be useful in order to study the winding properties of Brownian curves in the graph [7, 42]. It will affect calculations in a very simple manner through the introduction of additional (magnetic) phases. In the following it will be understood that the $2B$-vector $\psi'(0)$ of equation (6) gathers the covariant derivatives.

3.1. Arc determinant (1)

Let us consider two points $x$ and $y$ belonging to the two arcs $a$ and $b$, respectively. On the arc $a$ we use the set of independent solutions $f_\alpha$ and $f_\bar{b}$ of the differential equation

$$\left[ y - \frac{d^2}{dx^2} + V_y(x) \right] f(x) = 0$$

such that $f_\alpha(0) = 1$ and $f_\bar{b}(l_\alpha) = 0$ (see appendix A). Green’s function depends on two coordinates $x$ and $y$ and therefore must be specified by two indices (here $a$ and $b$) to which arcs they belong to:

$$G_{a,b}(x_a, y_b) = G_a f_a(x_a) e^{i A_a x_a} + G_{\bar{b}} f_{\bar{b}}(x_{\bar{b}}) e^{i A_{\bar{b}} x_{\bar{b}}} + \delta_{(a), (\bar{b})} \frac{e^{i A_{a,b} (x_a - y_b)}}{W_a} f_a(\max(x_a, y_a)) f_{\bar{b}}(\max(x_{\bar{b}}, y_{\bar{b}})).$$

(7)
where the matrix $M$ is the gauge transformation. In one dimension, a dependence of the vector potential in the coordinate may always be removed by a convenient gauge transformation.

One must assign a value of Green’s function $G_a$ to each arc $a$. For a continuous boundary condition at the vertex, the matrices $C^{(a)}$ and $D^{(a)}$ take the form (29). In this case one can assign to each vertex $a$ a single ‘vertex variable’ $G_a$.

where $G_a$ and $G_{\bar{a}}$ denote the values of Green’s function at the two ends of the bond $(a)$ (see figure 1(a)). We have introduced the notation $\delta_{(a),(b)} \equiv \delta_{a,b} + \delta_{\bar{a},\bar{b}}$. Note that the dependence of $G_a$ and $G_{\bar{a}}$ in the coordinate $y$ is implicit. $A_a = A(x)$ for $x \in (a)$ is the (constant) vector potential on the bond and $\theta_a = \theta_a(x_a)$ is the magnetic flux along the wire. $W_a = -f'_a(l_a)$ is the Wronski determinant of the two linearly independent solutions, equation (A.8). All the values of Green’s function at the vertices $G_a(0, y_b) = G_a$ (figure 1(a)) are gathered in the vector $G(0)$ of size $2B$ and all covariant derivatives $(D_x G)_{a,b}(0, y_b)$ in the vector $G'(0)$. We obtain

$$G'(0) = M G(0) + \begin{pmatrix}
\vdots \\
0 \\
f_b(y_b) e^{-iA_b y_b} & -b \\
0 \\
\vdots
\end{pmatrix}$$

where the matrix $M$ couples an arc $a$ to itself and to the reversed arc $\bar{a}$:

$$M_{a,b} \overset{\text{def}}{=} \delta_{a,b} f'_a(0) - \delta_{\bar{a},\bar{b}} f'_{\bar{a}}(l_{\bar{a}}) e^{i\theta_{\bar{a}}}.$$  

(9)

Noting that $f'_a(l_a) = f'_{\bar{a}}(l_{\bar{a}}) \in \mathbb{R}$ (see appendix A) and $\theta_{\bar{a}} = -\theta_a$ shows that this matrix is Hermitian. Imposing boundary conditions at vertices, $CG(0) + DG'(0) = 0$, we obtain all values, $G_a$, that could be reinjected into equation (7). Green’s function at coinciding points reads

$$G_a,a(x_a, x_a) = -[(C + DM)^{-1}D_{l_{\bar{a}}} f'_{\bar{a}}] - [(C + DM)^{-1}D_{l_a} f_a e^{i\theta_a}$$

$$- [(C + DM)^{-1}D_{l_{\bar{a}}} f_{\bar{a}} e^{i\theta_{\bar{a}}} - [(C + DM)^{-1}D_{l_a} f_a + \frac{1}{W_a} f_a f_{\bar{a}}.$$  

(10)

Integration over the graph may be decomposed as integration over the bonds:

$$\int_{\text{Graph}} dx G(x, x) = \sum_{(a)} \int_0^{l_a} dx_a G_{a,a}(x_a, x_a).$$

The integration of the product of the functions $f_a$ and $f_{\bar{a}}$ is given by equations (A.10), (A.11); hence,

$$G(\gamma) = \int_{\text{Graph}} dx G(x, x) = \text{Tr}[(C + DM)^{-1}D\partial_x M] - \partial_x \sum_{(a)} \ln[-f'_a(l_a)].$$  

(11)

In one dimension, a dependence of the vector potential in the coordinate may always be removed by a convenient gauge transformation.
where the trace runs over the $2B$ arc indices. Up to now, nothing has been assumed on the nature of the boundary conditions. An expansion of the trace provides a trace formula for $\int G(x, x) dx$, where the expansion is interpreted as the sum of contributions over cycles (orbits) in the graph.

We now suppose that the matrices $C$ and $D$ do not depend on the spectral parameter $\gamma$. Performing an integration $\int S(\gamma) = \int S(\gamma)$, we obtain the first important result of Desbois:

$$S(\gamma) = (-1)^V \prod_{(a)} f'_a(l_a) \det(C + DM), \quad (12)$$

where the product runs over the $B$ bonds. This provides a general expression of the spectral determinant in terms of $(2B \times 2B)$-matrix coupling arcs. We repeat that the matrix $M$ couples reversed arcs and contains local information related to the potential on each bound. On the other hand, the matrices $C$ and $D$, characterizing which arcs arrive and issue from the vertices, encode the information on the topology of the graph. Note that the structure $\det(C + DM)$ was also obtained in [19] in the absence of the potential where the determinant $\det(-\Delta)$ was analysed (the prime indicates exclusion of zero mode). When $V(x) = 0$ the matrix $M$ takes the form $M_{a,b} = -\sqrt{\gamma} (\delta_{a,b} \coth \sqrt{\gamma l_a} - \delta_{a,\bar{b}} \sinh \sqrt{\gamma l_a} e^{i\theta_{a}})$. The prefactor in equation (12) has been chosen arbitrarily and for convenience for the following. We will come back to the question of the prefactor in section 6.

**Limit $\gamma \to \infty$.** For a large positive spectral parameter we can ignore the potential; therefore, $f'_a(0) \simeq -\sqrt{\gamma}$ and $f'_a(L) \simeq -\sqrt{\gamma} e^{-\sqrt{\gamma} L_a}$. Hence the matrix $M$ becomes diagonal: $M_{a,b} \simeq -\delta_{a,b} \sqrt{\gamma}$, and we obtain

$$S(\gamma) \simeq (-1)^V 2^{-B} \gamma^{-B/2} e^{\sqrt{\gamma} E} \det(C - \sqrt{\gamma} D), \quad (13)$$

where $\mathcal{E} = \sum_{(a)} l_a$ is the ‘total length’ of the graph.

**Example.** Let us consider a ring of perimeter $L$ pierced by a magnetic flux $\theta$ (a graph in which the number of vertices can be reduced to $V = 1$ with $B = 1$ bond). We denote by $1$ and $\bar{1}$ the two reversed arcs. Let us consider boundary conditions described by the matrices given below by equation (29). We obtain

$$S(\gamma) = \frac{\lambda - f'_1(0) - f'_\bar{1}(0) + 2 f'_1(L) \cos \theta}{-f'_1(L)}. \quad (14)$$

It is interesting to point that this formula has found a practical physical application for a potential of the form $V(x) \propto x(L - x)$ in the context of the study of decoherence by the electron–electron interaction in a phase-coherent metallic ring [43] (hence $f_1(x)$ is a Hermite function and $f_{\bar{1}}(x) = f_1(L - x)$ due to the property $V(x) = V(L - x)$).

The case $V(x) = 0$ is analysed in section 6.

### 3.2. Arc determinant (2)

The above derivation uses the variables $G_{a}$ giving the value of Green’s function at the vertices. Another natural choice is to deal with (covariant) derivatives of Green’s function at the vertices. We write

$$G_{a,b}(x_a, y_b) = G_{a}^\prime g_{a}(x_a) e^{i A_{a} x_a} + G_{b}^\prime g_{b}(x_b) e^{i A_{b} x_b} + \delta_{(a), (b)} \frac{W_{a}^\prime}{W_{a}} g_{a}(\max(x_a, y_a)) g_{b}(\max(x_b, y_b)), \quad (15)$$
where the functions $g_a$ and $g_\bar{a}$ are solutions of the Schrödinger equation on the bond (a) for fixed values of the derivative at the boundaries $g'_a(0) = 1$ and $g'_\bar{a}(l_a) = 0$ (see appendix A). Following the lines of the previous subsection, we obtain

$$G(0) = NG'(0) - \begin{pmatrix} \vdots & \vdots & \vdots \\ 0 & g_b(y_b) e^{-iA y_b} & \leftarrow b \\ \vdots & g_b(y_b) e^{-iA y_b} & \leftarrow \bar{b} \\ \vdots & 0 & \vdots \end{pmatrix},$$

where we have introduced the arc matrix

$$N_{a,b} \overset{\text{def}}{=} \delta_{a,b} g_a(0) + \delta_{a,\bar{b}} g_\bar{a}(l_a) e^{i\theta}.$$  \hspace{1cm} (17)

Imposing the boundary conditions at the vertices we obtain the values $G'_{a}$. Reinjecting these expressions in equation (15), integration over the coordinate and summation over bonds give another general trace formula for Green’s function

$$G(\gamma) = \int_{\text{Graph}} d\gamma \frac{\det(CN + D)}{-g_a(l_a)} \ln \left[-g_a(l_a)\right].$$

For the $\gamma$-independent matrices $C$ and $D$, integration over the spectral parameter leads to

$$S(\gamma) = (-1)^{V} \prod_{(a)} \frac{-1}{g_a(l_a)} \ln \left[-g_a(l_a)\right].$$

where the $\gamma$-independent prefactor was chosen in order to match with (12). This expression can be directly related to (12) by using the relations (demonstrated in appendix A)

$$N = M^{-1} \quad \text{and} \quad \det M = \frac{1}{\det N} = \prod_{(a)} \frac{f'_a(l_a)}{g_a(l_a)}.$$ \hspace{1cm} (20)

**Example.** We consider again the very simple case of a ring, but this time we choose to consider boundary conditions of the type described below by equation (40). We obtain

$$S(\gamma) = \frac{\mu - g_1(0) - g_1(0) - 2g_1(L)\cos \theta}{-g_1(L)}.$$ \hspace{1cm} (21)

In the absence of the potential we obtain

$$S(\gamma) = \mu \sqrt{\gamma} \sinh \sqrt{\gamma} L + 2 \cosh \sqrt{\gamma} L + \cos \theta.$$ \hspace{1cm} (22)

The limit $\mu \to \infty$ gives the Neumann determinant of the bond: $S(\gamma) \to \mu \sqrt{\gamma} \sinh \sqrt{\gamma} L = \mu L \gamma \prod_{n=1}^{\infty} \left(1 + \left(\frac{\sqrt{\gamma} L}{\pi n}\right)^2\right)$.

### 3.3. Arc determinant (3), scattering matrices and the $\zeta$ function

Result (12) can be reorganized more conveniently by introducing the arc matrix

$$R = \left(\sqrt{\gamma} + M\right)/\left(\sqrt{\gamma} - M\right),$$

i.e. $M = \sqrt{\gamma}(R + I_{2B})^{-1}(R - I_{2B}) = \sqrt{\gamma}(R - I_{2B})(R + I_{2B})^{-1}$, where $I_{2B}$ is the $2B \times 2B$ identity matrix. We denote its matrix elements

$$R_{a,b} = \delta_{a,b} r_a + \delta_{a,\bar{b}} e^{i\theta}.$$ \hspace{1cm} (24)
These matrix elements have a clear physical meaning: $r_a$ and $t_a$ are the analytic continuations (to negative energies $E \to -\gamma$) of reflection and transmission probability amplitudes through the potential $V_\nu(x)$ (see appendix A): $R$ is therefore the analytic continuation of the bond scattering matrix. We also introduce the analytic continuation of the vertex scattering matrix (scattering matrix interpretation is discussed in [24, 39, 41])

$$Q = (-C + \sqrt{T}D)^{-1}(C + \sqrt{T}D),$$

(25)

(for instance we can check that $Q$ is unitary for $\sqrt{T} = -ik$, reflecting current conservation at the vertex).

We obtain another interesting result of Desbois [10]:

$$S(\gamma) = (-1)^V \prod_{(a)} \frac{-1}{f'_a(l_a)} \frac{\det(C - \sqrt{T}D)}{\det(1_{2B} + R)} \frac{\det(1_{2B} - QR)}{\det(1_{2B} + R)}.$$

(26)

Let us describe the structure of this result: the product over bonds $\prod_{(a)} f'_a(l_a)$ and the determinant $\det(1_{2B} + R)$ involve (local) information about the potential on the bonds (recall that $R$ couples an arc to itself and its reversed arc, only). This is even more clear from equation (A.17) that leads to

$$\prod_{(a)} \{-f'_a(l_a)\} \det(1_{2B} + R) = 2^B \gamma^{B/2} \prod_{(a)} I_a = (4\gamma)^{B/2} \left( \prod_a R_{a,\bar{a}} \right)^{1/2},$$

(27)

where the first product runs over the $B$ bonds and the last one over the $2B$ arcs. Now let us consider the determinant $\det(C - \sqrt{T}D)$: organizing the arcs by gathering arcs issuing from the same vertex, the matrices $C$ and $D$ take some block diagonal structure; hence, $\det(C - \sqrt{T}D)$ encodes some (local) information about the vertices. The last determinant $\det(1_{2B} - QR)$ mixes information about potential and the connection of arcs to vertices in a nontrivial way, and characterizes the (global) information about the topology.

This last part of the spectral determinant vanishes on the spectrum of $-D^2 + V(x)$, i.e. for $\gamma = -E_n$. Because $Q$ and $R$ have the meaning of scattering matrices, the equation $\det(1_{2B} - QR) = 0$ may be understood as a quantization condition in the manner of Bohr–Sommerfeld.

It is well known that the determinant $\det(1_{2B} - QR)$ can be expanded in terms of primitive cycle contributions, with the structure of a $\zeta$-function. We recall that a cycle (a periodic orbit) is the equivalence class of all ordered sets of arcs $\mathcal{C} = (a_1, a_2, \ldots, a_n)$ identical by cyclic permutations and such that $\forall i \in \{1, \ldots, n\} \; \text{end}(a_{i-1}) = \text{beginning}(a_i)$ (with $a_0 \equiv a_n$). An orbit is said primitive, and denoted by $\tilde{\mathcal{C}}$, if it cannot be decomposed as a repetition of a smaller orbit. We define the weight of the orbit $\mathcal{C}$ as $\omega(\mathcal{C}) \equiv (QR)_{a_1a_2}^\gamma (QR)_{a_2a_3}^\gamma \cdots (QR)_{a_na_1}^\gamma$. The determinant may be rewritten as an infinite product over primitive orbits as

$$\det(1_{2B} - QR) = \prod_{\tilde{\mathcal{C}}} \left( 1 - \omega(\tilde{\mathcal{C}}) \right)$$

(28)

(see for example [1, 37] and references therein). This relation emphasizes that the spectral determinant may be interpreted as a $\zeta$-function (or $L$-function) for primitive orbits of graphs. $\zeta$-functions are powerful tools of particular importance, in number theory and graph theory for example, since they play the role of generating functions for primitive elements. The most famous $\zeta$-function is the Riemann $\zeta$-function $\zeta(s)^{-1} = \prod_{\text{prime } p} (1 - p^{-s})$ giving access to the distribution of prime numbers.

2 Note that this weight can be decomposed as $\omega(\mathcal{C}) = v(\mathcal{C}) | b(\mathcal{C})|$ where we have identified the parts related to the scattering by the vertices $v(\mathcal{C}) = Q_{a_1b_1} Q_{a_2b_2} \cdots Q_{a_nb_n}$ and the scattering by the bonds $b(\mathcal{C}) = R_{b_{1a_1}} R_{b_{2a_2}} \cdots R_{b_{na_n}}$, respectively (because $R$ only couples reversed arcs we have $b_{i-1} \in \{a_i, \bar{a}_i\}$).
To close this section, let us emphasize that the representations (12,19,26) are remarkable in the sense that, despite the Laplace operator acts on a space of infinite dimension, its spectral determinant may be related to determinants of finite size matrices: all the above expressions have involved arc matrices of size $2B \times 2B$. We will now see that we can express the spectral determinant in terms of $V \times V$ vertex matrices in general smaller\(^3\).

4. Continuous boundary conditions

4.1. Roth’s trace formula

Let us consider the important case of the Laplace operator acting on functions that are continuous at the vertices: $\psi_{\alpha\beta}(0) = \psi_{\alpha}$ for all vertex $\beta$ neighbours of $\alpha$, and $\sum_\beta a_{\alpha\beta}(D_x \psi)_{\alpha\beta}(0) = \lambda_{\alpha} \psi_{\alpha}$, where the connectivity matrix ensures that the sum runs over vertex neighbours of $\alpha$. The parameter $\lambda_{\alpha}$ must be chosen real in order to ensure self-adjointness of the Schrödinger operator; it can be understood as the weight of a $\delta$ potential at the vertex (these boundary conditions are sometimes denoted as ‘$\delta$-coupling’ [12, 13]). In the appropriate basis where arcs are gathered by vertices from which they issue, the matrices $C$ and $D$ have block diagonal structures (with $V$ blocks $\alpha = 1, \ldots, V$) where the $m_\alpha \times m_\alpha$ block $\alpha$ corresponds to the vertex. A set of $\gamma$-independent matrices $C$ and $D$ corresponding to $\delta$-couplings are made of $m_\alpha \times m_\alpha$ blocks

$$C^{(\alpha)} = \begin{pmatrix} -\lambda_{\alpha} & 0 & 0 & \ldots & 0 \\ -1 & 0 & 0 & \ldots & 0 \\ -1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \ldots & 1 \end{pmatrix} \quad \text{and} \quad D^{(\alpha)} = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$  

(29)

It will be useful to remark that $\det(C^{(\alpha)} - \sqrt{\gamma}D^{(\alpha)}) = -(\lambda_{\alpha} + m_\alpha \sqrt{\gamma})$. The vertex ‘scattering matrix’ for the vertex $Q^{(\alpha)} = (-C^{(\alpha)} + \sqrt{\gamma}D^{(\alpha)})^{-1}(C^{(\alpha)} + \sqrt{\gamma}D^{(\alpha)})$ takes the form [1]

$$Q^{(\alpha)} = \frac{2}{m_\alpha + \lambda_{\alpha}/\sqrt{\gamma}} \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix} - I_{m_\alpha}.$$  

(30)

(for $\gamma = 0$ we recognize the weights introduced by Roth [34]: $\frac{2}{m_\alpha}$ for visiting the vertex $\alpha$ and $\frac{2}{m_\alpha} - 1$ for a reflection on it). Note that the limit $\lambda_{\alpha} = \infty$ corresponds to the Dirichlet boundary conditions $\psi_i(0) = 0$.

Moreover, let us consider the case $V(\chi) = 0$; then $r_a = 0$ and $t_a = e^{-\sqrt{\gamma}l_a}$ (note that this latter is the analytic continuation of the transmission probability amplitude through a free interval $t_a = e^{ikl_a}$ for $\sqrt{\gamma} = \sqrt{-k^2 - i0^+}$). Starting from (26) and using (A.17) we get

$$S(\gamma) = 2^{-B} \prod_a \left( m_u + \frac{\lambda_{\alpha}}{\sqrt{\gamma}} \right)^{\frac{\gamma u}{\sqrt{\gamma}}} e^{\gamma \tau C} \prod_{\alpha \in C} \left( 1 - \alpha(\tilde{C}) e^{-\sqrt{\gamma}(\xi(\tilde{C}) + i0^+)} \right),$$  

(31)

where the first product runs over the $V$ vertices. $\mathcal{C} = \sum_a l_a$ is the total length of the graph. The weights of the orbit $C = (a_1, a_2, \ldots, a_n)$ are now denoted by $v(C) \rightarrow \alpha(C) =$

\(^3\) The spectral determinant can only be expressed in terms of vertex matrices when boundary conditions at the vertices are such that it is possible to assign a unique variable to each vertex. This can only be done for ‘permutation invariant’ boundary conditions [10]. Two such particular cases are discussed in the following two sections.
\( Q_{\alpha\beta}, Q_{\alpha\gamma}, \ldots, Q_{\alpha\delta} \), (see footnote 2) for \( Q \) given by gathering the blocks (30). The notations \( \ell(C) \) and \( \theta(C) \) designate the length of the orbit and the magnetic flux enclosed by it, respectively. Thanks to the simple structure of the matrix \( R \) for \( V(\mathbf{x}) = 0 \), the bond scattering part of the weight of the orbit is simply \( b(C) \rightarrow e^{-\sqrt{\gamma\ell(C)} + i\theta(C)} \).

If we choose moreover vertex scattering with \( \lambda_\alpha = 0 \), the weights \( \alpha(C) \) become \( \gamma \)-independent, which allows a simple Laplace inversion of the derivative \( \frac{1}{\gamma} \ln S(\gamma) = \int_0^\infty dt Z(t) e^{-\gamma t} \). We recover the Roth’s trace formula [34]:

\[
Z(t) = \frac{L}{2\sqrt{\pi t}} + \frac{V - B}{2} + \frac{1}{2\sqrt{\pi t}} \sum_c \ell(C) \alpha(C) e^{-\ell(C)^2/4t} + i\theta(C) \]

for \( \lambda_\alpha = 0 \forall \alpha \), (32)

where the sum now runs over all periodic orbits \( C \), where \( \tilde{C} \) is the primitive orbit related to the orbit \( C \) (see [1, 34] for more details).

4.2. Vertex determinant

Another interesting expression of the spectral determinant in terms of \((V \times V)\)-matrix coupling vertices may be obtained by a construction analogous to the one given above. The assumption that the functions are continuous at the vertices allows us to deal with vertex variables (see figure 1(b)):

\[
G_{\alpha\beta,\mu\nu}(x_{\alpha\beta}, y_{\mu\nu}) = G_{\alpha}(x_{\alpha\beta}) e^{iA_{\alpha\beta}(x_{\alpha\beta})} + G_{\beta}(x_{\beta\alpha}) e^{iA_{\beta\alpha}(x_{\beta\alpha})} + \delta(\mu\nu,(\alpha\beta)) e^{iA_{\alpha\beta}(x_{\alpha\beta} - y_{\alpha\beta})} W_{\alpha\beta} f_{\alpha\beta}(\max(x_{\alpha\beta}, y_{\alpha\beta})) f_{\beta\alpha}(\max(x_{\beta\alpha}, y_{\beta\alpha})),
\]

where \( G_{\alpha} \) is the value of Green’s function at the vertex. The boundary condition at the vertex \( \mu \) takes the form

\[
\sum \nu M_{\mu\nu} G_{\nu} = \delta_{\mu\alpha} f_{\alpha\beta}(y_{\alpha\beta}) e^{-iA_{\alpha\beta}y_{\alpha\beta}} + \delta_{\mu\beta} f_{\beta\alpha}(y_{\beta\alpha}) e^{-iA_{\beta\alpha}y_{\beta\alpha}},
\]

where the \((V \times V)\)-matrix coupling vertices are given by

\[
M_{\alpha\beta} = \delta_{\alpha\beta} \left( \lambda_\alpha - \sum \mu a_{\mu\alpha} f'_{\mu\alpha}(0) \right) + a_{\alpha\beta} f'_{\beta\alpha}(l_{\alpha\beta}) e^{-i\lambda_{\alpha\beta}}.
\]

We can now obtain the value of Green’s function at the vertices and reinject these expressions in (33). Integration of

\[
G_{\alpha\beta,\alpha\beta}(x_{\alpha\beta}, x_{\alpha\beta}) = (M^{-1})_{\alpha\beta} f_{\alpha\beta}(x_{\alpha\beta})^2 + [e^{i\lambda_{\alpha\beta}} (M^{-1})_{\alpha\beta} + e^{-i\lambda_{\alpha\beta}} (M^{-1})_{\beta\alpha}] f_{\alpha\beta}(x_{\alpha\beta}) f_{\beta\alpha}(x_{\beta\alpha})
\]

and

\[
+ (M^{-1})_{\beta\alpha} f_{\beta\alpha}(x_{\beta\alpha})^2 + \frac{1}{W_{\alpha\beta}} f_{\alpha\beta}(x_{\alpha\beta}) f_{\beta\alpha}(x_{\beta\alpha})
\]

can be performed along the same lines as before. One obtains [8]

\[
S(\gamma) = \prod_{(\alpha\beta)} [-f'_{\alpha\beta}(l_{\alpha\beta})]^{-1} \det \mathcal{M}
\]

(37)

where the product runs over the bonds of the graph. Equation (37) is less general than equation (12) since it corresponds to a particular choice of boundary conditions at the vertices; however, it involves the information about the graph in a more compact manner in the sense that the \( V \times V \) matrix \( \mathcal{M} \) is smaller than the \( 2B \times 2B \) matrix \( C + DM \).
In the absence of a potential, $V(x) = 0$, we recover the Pascaud and Montambaux result [33]

$$S(\gamma) = \prod_{(\alpha\beta)} \sinh \frac{\sqrt{\gamma} l_{\alpha\beta}}{\sqrt{\gamma}} \det \mathcal{M},$$

(38)

for

$$\mathcal{M}_{\alpha\beta} = \delta_{\alpha\beta} \left( \lambda_{\alpha} + \sqrt{\gamma} \sum_{\mu} a_{\alpha\mu} \coth \sqrt{\gamma} l_{\alpha\mu} - a_{\alpha\beta} \frac{e^{-i \theta_{\alpha\beta}}}{\sinh \sqrt{\gamma} l_{\alpha\beta}} \right).$$

(39)

Note the alternative derivation of (38) and (37) with the path integral [1] and [9]. It is worth pointing that the full spectrum of the Laplace operator is not always given by $\det \mathcal{M} = 0$: it may occur in some particular cases that eigenstates of the Schrödinger operator vanish at all vertices $\psi_{\alpha} = 0$ for $\psi(x) \neq 0$. Some examples are discussed in detail in [40] (this is related to the phenomenon known as ‘bound state in the continuum’ in the scattering situation, for graphs with some infinitely long wires [38, 45]).

Finally, note that the $\gamma$-independent prefactor in (37) is chosen in order to match with formulae (12) and (26): one can easily check that the $\gamma \to \infty$ behaviours coincide. Note that, in [1], a direct relation between (38) and (31) was established, without a posteriori matching procedure.

5. Continuous derivative boundary conditions

5.1. Trace formula

Another interesting case worth to be discussed is the case of the Schrödinger operator acting on functions whose derivative is continuous at the vertices $(D_x \psi)_{\alpha\beta}(0) = \psi'_{\alpha}$ for all vertex $\beta$ neighbours of $\alpha$, and $\sum_{\beta} a_{\alpha\beta} \psi_{\alpha\beta}(0) = \mu_{\alpha} \psi_{\alpha}$. This choice is denoted as ‘$\delta'$-coupling’ in [12, 13]. A set of $\gamma$-independent boundary matrices at the vertex $\alpha$ is in this case

$$C^{(\alpha)} = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

and

$$D^{(\alpha)} = \begin{pmatrix} -\mu_{\alpha} & 0 & \ldots & 0 \\ -1 & 1 & 0 & \ldots & 0 \\ -1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

(40)

that gives $\det(C^{(\alpha)} - \sqrt{\gamma} D^{(\alpha)}) = (-1)^{m_{\alpha} + 1/2} (\mu_{\alpha} + m_{\alpha} \sqrt{\gamma})$. From (25), we get the vertex ‘scattering matrix’ for the vertex $\alpha$:

$$Q^{(\alpha)} = \frac{-2}{m_{\alpha} + \mu_{\alpha} \sqrt{\gamma}} \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix} + I_m;$$

(41)

therefore, the limit $\mu_{\alpha} = \infty$ corresponds to the Neumann boundary conditions $\psi'_{\alpha}(0) = 0$.

Starting again from equation (26), a little bit of algebra gives the spectral determinant in the absence of the potential

$$S(\gamma) = 2^{-\beta} \prod_{\alpha} (m_{\alpha} + \sqrt{\gamma} \mu_{\alpha}) e^{\frac{\mu_{\alpha}}{\sqrt{\gamma}}} e^{\gamma \sum_{\alpha} \log (1 - \beta (\tilde{C})) e^{-\sqrt{\gamma} (\log (\tilde{C}))^2}},$$

(42)
where the weights \( v(C) \rightarrow \beta(C) \) are computed with the matrix elements of \((41)\). It is interesting to point out that these weights may be simply related to the weights \( \alpha(C) \) of the previous section thanks to the substitution \( \lambda_\alpha \rightarrow \gamma \mu_\alpha \) and adding a sign \((-1)^{\#\text{arcs of } C}\).

If we, moreover, set \( \mu_\alpha = 0 \) we may obtain the trace formula for the partition function

\[
Z(t) = \frac{\mathcal{L}}{2\sqrt{\pi t}} + \frac{B - V}{2} + \frac{1}{2\sqrt{\pi t}} \sum_C \ell(\tilde{C}) \beta(C) e^{-\frac{\ell^2}{4t} + i\theta(C)}
\]

for \( \mu_\alpha = 0 \ \forall \alpha \) (43) that generalizes the result of Roth to the case of the Laplace operator acting on functions whose derivative is continuous at the vertices. Compared to the case of functions continuous at the vertices discussed in the previous section, equation (32), only the constant term and the weights of the orbits have changed in sign.

5.2. Vertex determinant

The expression of the spectral determinant may also be obtained by using vertex variables \([44]\):

\[
S(\gamma) = \prod_{(\alpha\beta)} (-g_{\alpha\beta}(l_{\alpha\beta}))^{-1/2} \text{det} N
\]

with

\[
N_{\alpha\beta} = \delta_{\alpha\beta} \left( \mu_\alpha - \sum_\nu a_{\alpha\nu} g_{\alpha\nu}(0) - a_{\alpha\beta} g_{\alpha\beta}(l_{\alpha\beta}) e^{-i\theta_{\alpha\beta}} \right),
\]

where the functions \( g_\alpha \) were introduced above (and in appendix A). In the absence of the potential, \( V(x) = 0 \), we obtain

\[
S(\gamma) = \prod_{(\alpha\beta)} \sqrt{\gamma} \sinh \sqrt{\gamma} l_{\alpha\beta} \text{det} N,
\]

where \( N_{\alpha\beta} = \delta_{\alpha\beta} \left( \mu_\alpha + \frac{1}{\sqrt{\gamma}} \sum_\nu a_{\alpha\nu} \coth \sqrt{\gamma} l_{\alpha\nu} \right) + a_{\alpha\beta} \frac{e^{-\gamma}_{\alpha\beta}}{\sqrt{\gamma} \sinh \sqrt{\gamma} l_{\alpha\beta}}.\) (46)

6. From the trace formula to the \( \zeta \)-regularized determinant

The question of the prefactor of the spectral determinant is the subject of this section. It is important to fix the \( \gamma \)-independent prefactor: in all the expressions given in the previous sections, \((12), (19), (26), (31), (38), (42)\) and \((44)\), the prefactor has been chosen arbitrarily, nevertheless in a way that all expressions match when varying continuously boundary conditions. However, this choice is a priori not related to a particular regularization and has only been made for convenience.

6.1. Functions continuous at the vertices

Let us first consider the simplest case of the Laplace operator acting on functions continuous at the vertices in the absence of the potential. Moreover, we set the parameters \( \lambda_\alpha = 0 \), when Roth’s trace formula for the partition function holds. Mellin transform of Roth’s partition function (32) reads

\[
\xi(s, \gamma) = -\mathcal{L} \frac{\Gamma(s - 1/2)}{\Gamma(-1/2) \Gamma(s)} \gamma^{-s+1/2} + \frac{V - B}{2} \gamma^{-s} + \frac{1}{\sqrt{\pi} \Gamma(s)} \gamma^{-s+1/2} \sum_C \ell(\tilde{C}) \alpha(C) e^{i\theta(C)} \left( \frac{2}{\sqrt{\gamma} \ell(\tilde{C})} \right)^{-s+1/2} K_{1/2}(\sqrt{\gamma} \ell(\tilde{C})), (47)
\]
where $K_\nu(z)$ is the MacDonald function (modified Bessel function) \cite{18}. The first and third terms are of the form $\prod_{i=1}^n h(s)$ for a function $h(s)$ regular for $s \to 0$; therefore, we can use the limit $h_{ss} \to 0 \left[ \frac{1}{\sqrt{s}} h(s) \right]$ = $h(0)$. The analysis of the third term requires the formula
\[
\left[ \frac{d K_{\nu}(z)}{dz} \right]_{i=1/2} = -\frac{\sqrt{\pi}}{2 z} e^z \text{Ei}(-2z) \quad \text{\cite{18}}.
\]
Using $K_{1/2}(z) = \frac{\sqrt{\pi}}{2 z} e^{-z}$ we get
\[
d\zeta(0, \gamma) = -\sqrt{\gamma L} - \frac{V}{2} \ln \gamma + \sum_{c} \frac{\ell(\zeta) a(\zeta)}{\ell(\zeta)} e^{-\sqrt{\gamma L} \cdot v(\zeta)}.
\]
(48)

Sum over orbits can be decomposed as a sum over primitive orbits and their repetitions $\sum_{c} = \sum_{c} \sum_{n=1}^{\infty}$. Using $\ell(c) = n \ell(\zeta(\zeta))$, $\theta(c) = n \theta(\zeta)$ and $a(c) = a'(\zeta)$. Therefore
\[
d\zeta(0, \gamma) = -\sqrt{\gamma L} - \frac{V}{2} \ln \gamma - \sum_{c} \ln(1 - a'(\zeta)) e^{-\sqrt{\gamma L} \cdot v(\zeta)},
\]
(49)

that is
\[
S_{\gamma}(\rho) = \gamma^{\frac{1}{n}} e^{\sqrt{\gamma L}} \prod_{c} \frac{1}{\sqrt{\gamma}} (1 - a'(\zeta)) e^{-\sqrt{\gamma L} \cdot v(\zeta))} = \gamma^{\frac{1}{n}} e^{\sqrt{\gamma L}} \det(1_{2B} - Q R).
\]
(50)

We recognize part of expression (31) (see also equation (65) of \cite{1}). We have related equation (31) to another representation in terms of a vertex matrix, equation (38); hence,
\[
S_{\gamma}(\rho) = \prod_{(ab)} \frac{2 \sinh \sqrt{\gamma L} \theta_{ab}}{\sqrt{\gamma}} \det M = \left( \prod_{a} m_a \right) S(\gamma) \quad \text{for} \quad V(x) = 0 \quad \text{and} \quad \lambda_a = 0 \forall a.
\]
(51)

The previous formula may easily be generalized to the case of the Schrödinger equation for a continuous boundary condition with arbitrary parameters $\lambda_a$. Using the observation that various regularizations only differ by a $\gamma$-independent prefactor, we conclude that $S_{\gamma}(\rho) = 2^{\gamma} / \left( \prod_{a} m_a \right) S(\gamma)$ also holds in the presence of the potential
\[
S_{\gamma}(\rho) = \prod_{(ab)} \frac{-2 \theta_{ab}}{\sqrt{\gamma L} \theta_{ab}} \det M = \left( \prod_{a} m_a \right) S(\gamma)
\]
(52)

This is precisely Friedlander’s result \cite{17} (cf.\cite{44}), where the correspondence of notations is discussed). The product $\prod_{(ab)} \left[ -2 / f'_{ab}(\theta_{ab}) \right]$ is the Dirichlet determinant of the graph (determinant when Dirichlet boundary conditions are chosen at all vertices that make all wires independent).

Example 1: wire. Let us consider a wire of length $L$ ($B = 1$ bond and $V = 2$ vertices) with no potential $V(a) = 0$. The matrices $C$ and $D$ for boundary conditions of type \cite{29} are
\[
C = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
(53)

where $\lambda_1$ and $\lambda_2$ describe the boundary conditions at the two vertices. Equation (61) gives
\[
S_{\gamma}(\rho) = 2 \sqrt{\gamma} \sinh \sqrt{\gamma L} + (\lambda_1 + \lambda_2) / 2 \cosh \sqrt{\gamma L} + \lambda_1 \lambda_2 / 2 \sinh \sqrt{\gamma L}.
\]
(54)

This result can be obtained straightforwardly from (52).

For $\lambda_1 = \lambda_2 = 0$ we recover the Neumann determinant $S_{\gamma}(\rho) = 2 \sqrt{\gamma} \sinh \sqrt{\gamma L} = 2 \gamma L \prod_{n=1}^{\infty} (1 + \frac{\gamma}{n})$ where $E_n = \left( \frac{\gamma}{2} \right)^2$ with $n \in \mathbb{N}$ are the eigenvalues of the Laplace operator on $(0, L)$ for Neumann boundary conditions.
Note that the second term $2 \cosh \sqrt{\gamma L}$ corresponds to the Neumann/Dirichlet determinant (Neumann at $x = 0$ and Dirichlet at $x = L$ or vice versa) retained for $\lambda_1 = 0$ and $\lambda_2 = \infty$ (with $\lambda_1 \lambda_2 = 0$).

The last term $\frac{2 \sinh \sqrt{\gamma L}}{\sqrt{\gamma}}$ is the Dirichlet determinant ($\lambda_1 = \lambda_2 = \infty$).

**Example 2: ring.** We consider a ring of perimeter $L$ ($B = 1$ bond and $V = 1$ vertex) with no potential $V(x) = 0$ and pierced by a flux $\theta$. The matrices $C$ and $D$ for boundary conditions of type (29) are

$$C = \begin{pmatrix} -\lambda & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix};$$  \hspace{1cm} (55)

therefore, (61) gives

$$S^\zeta_{\text{ring}}(\gamma) = 2(\cosh \sqrt{\gamma L} - \cos \theta) + \lambda \frac{\sinh \sqrt{\gamma L}}{\sqrt{\gamma}}.$$  \hspace{1cm} (56)

It may also be obtained from (52) knowing how a loop contributes to $M$ (see [1]).

For $\lambda = 0$ we obtain $S^\zeta_{\text{ring}}(\gamma) = 2(\cosh \sqrt{\gamma L} - \cos \theta) = 4 \sin^2 \frac{\theta}{2} \prod_{n \in \mathbb{Z}} (1 + \frac{\gamma}{E_n})$ where $E_n = \left(\frac{2\pi n}{L}\right)^2$ with $n \in \mathbb{Z}$ being the spectrum of the ring.

**Example 3: star graph.** We consider a star graph with $B$ arms (then $V = B + 1$) with no potential $V(x) = 0$. We introduce a parameter $\lambda$ at the central vertex and set all other boundary parameters equal to zero, $\lambda_\alpha = 0$. In this case it is more easy to use (51). The $B$ wires are ‘dead arms’ and may be taken into account very simply through the following rule [32]: a dead arm issuing from the vertex $\alpha$ gives a contribution $\sqrt{\gamma} \tanh \sqrt{\gamma \ell_a}$ that should be added to the matrix element $M_{\alpha\alpha}^a$ corresponding to the graph from which the dead arm is removed; in the product over bonds of equation (51) we replace $\sinh \sqrt{\gamma \ell_a}$ by $\cosh \sqrt{\gamma \ell_a}$.

This simple rule can easily be recovered using the path integral formalism introduced in [1]. It allows us to treat the star graph as a graph with one vertex and $B$ dead arms. Therefore $M = \lambda + \sqrt{\gamma} \sum_{a=1}^B \tanh \sqrt{\gamma \ell_a}$ and the spectral determinant is given by

$$S^\zeta_{\text{star}}(\gamma) = \frac{2B}{B} \prod_{a=1}^B \cosh \sqrt{\gamma \ell_a} \left(\lambda + \sqrt{\gamma} \sum_{a=1}^B \tanh \sqrt{\gamma \ell_a}\right).$$  \hspace{1cm} (57)

If we consider the case $\lambda = 0$ and take the limit $\gamma \to 0$ we recover the result of [19]:

$$S^\zeta_{\text{star}}(\gamma) \simeq \frac{\gamma B \ell}{B} \gamma = \gamma \det'(-\Delta),$$  \hspace{1cm} (58)

where $\det'(-\Delta)$ is the determinant when the zero mode contribution has been excluded.

### 6.2. Functions with derivative continuous at the vertices

The reader has remarked that there is very little difference between the Roth trace formula (32) and its generalization (43): the second term proportional to $V-B$ is changed in sign and the definition of the weights are changed: when $\lambda_\alpha = 0$ and $\mu_\alpha = 0$ we have simply $\beta(\tilde{C}) = (-1)^{\# \text{ars(C)}} \alpha(\tilde{C})$. We can therefore perform a similar calculation and get in the case $V(x) = 0$ and $\mu_\alpha = 0$

$$S^\zeta(\gamma) = \gamma \frac{2B}{B} \prod_{\tilde{C}} (1 - \beta(\tilde{C}) e^{-\sqrt{\gamma \ell(\tilde{C}) + i\theta(\tilde{C})}}).$$  \hspace{1cm} (59)
By a similar continuity argument as in the previous subsection we get for $V(x) \neq 0$ and $\mu_\alpha \neq 0$

$$S^I_\gamma (\gamma) = \prod_{(\alpha \beta)} \frac{-2}{g_{\alpha \beta}(l_{\alpha \beta})} \frac{\det N}{\prod_\alpha m_\alpha},$$

the first term (product over the bonds) now coincides with the Neumann determinant of the graph (product of spectral determinants for all disconnected wires with Neumann boundary conditions). This latter expression hence provides some generalization of Friedlander’s result to another kind of boundary conditions.

7. Conclusion

Using Roth’s trace formula for the partition function and its generalization, we have obtained the expression of the $\gamma$-regularized spectral determinant of the Schrödinger operator acting on functions continuous at the vertices, equation (52), or functions whose derivative is continuous, equation (60). It is worth emphasizing that in these formulae, the $\gamma$-independent prefactor is now fully determined by some properties of the graph, i.e. the $\gamma$-regularization fixes a priori the prefactor of the spectral determinant, whereas in formulae (12), (19), (26), (31), (38), (42) and (44) it was chosen arbitrarily a posteriori.

We have summarized the relations between the various expressions of the spectral determinant given in the paper in figure 2.

The work of Desbois [10] on the case of general boundary conditions, in particular equations (12), (19) and (26), allows us in principle to go continuously from one to the other.
of these two situations by modifying continuously the boundary condition matrices $C$ and $D$. Therefore, by continuity this leads us to conjecture that the \( \zeta \)-regularized spectral determinant of the Schrödinger operator for the most general boundary conditions is given by

\[
S_\zeta(\gamma) = \left(-1\right)^V \prod_{\alpha} m_\alpha \prod_{(a)} -2 \det(C + DM) = \frac{\det(\gamma^{-1/4}C - \gamma^{1/4}D)}{\prod_{\alpha} (-m_\alpha) \sqrt{\prod_{a,\bar{a}} R_{a,\bar{a}}} \det(1_{2B} - QR)}.
\]  

(61)

where \( \prod_{\alpha} \) runs over the \( V \) vertices, \( \prod_{(a)} \) over the \( B \) bonds and \( \prod_{a} \) over the \( 2B \) arcs. It would be interesting to provide a direct proof of this formula.

**Acknowledgment**

It is my pleasure to acknowledge useful discussions with Jean Desbois.

**Appendix A. Bond scattering**

Let us consider the Schrödinger equation on the bond \((a)\) for an energy \( -\gamma = E = +k^2 \):

\[
\left[ -\gamma - \frac{d^2}{dx_a^2} + V_a(x_a) \right] \psi(x_a) = 0 \quad \text{for} \quad x_a \in (0, l_a).
\]  

(A.1)

We introduce three useful bases of solutions of this differential equation.

**Basis 1.** We denote by \( \phi_a(x_a) \) the scattering state with an incoming wave from the left, with \( \phi_a(x_a < 0) = e^{ik_a x_a} + r_a e^{-ik_a x_a} \) and \( \phi_a(x_a > l_a) = t_a e^{ik(x_a - l_a)} = t_a e^{-ik l_a} \), where \( r_a, t_a \) are the reflection and transmission amplitudes through the potential:

\[
\phi_a(0) = 1 + r_a \quad \text{and} \quad \phi_a(l_a) = t_a.
\]  

(A.2)

The scattering state incoming from the right is naturally denoted by \( \phi_{\bar{a}}(x_{\bar{a}}) \).

A well-known sum rule relating integral of the square modulus of the wavefunction (i.e. local density of states in the scattering region) to the scattering matrix is the Krein–Friedel sum rule [16, 27, 36]. Generalizations of this sum rule in the context of metric graphs have been obtained in [40, 41]. For \( E = k^2 = -\gamma > 0 \), let us change notation and introduce \( \Psi^{(L)}(x) = \frac{1}{\sqrt{2\pi}} \phi_a(x) \) and \( \Psi^{(R)}(x) = \frac{1}{\sqrt{2\pi}} \phi_{\bar{a}}(l_a - x) \), the left and right stationary scattering states, respectively. We can deduce from the local version of the Krein–Friedel sum rule demonstrated in [41] for graphs that

\[
\int_0^{l_a} dx |\Psi^{(\alpha)}(x)|^2 \Psi^{(\beta)*}(x) = \frac{1}{2i\pi} \left( J \frac{dJ}{dE} + \mathcal{J} - \mathcal{J}^\dagger \right)_{\alpha\beta} \quad \text{with} \quad \alpha, \beta \in \{L, R\},
\]  

(A.3)

where \( \mathcal{J} \) is the \( 2 \times 2 \) scattering matrix describing the scattering by the potential on the bond

\[
\mathcal{J} = \begin{pmatrix} r_a & t_a \\ t_{\bar{a}} & r_{\bar{a}} \end{pmatrix}.
\]  

(A.4)

We obtain

\[
\int_0^{l_a} dx |\phi_a(x)|^2 = -2i\sqrt{E} \left( r_a \frac{dr_a}{dE} + t_a \frac{dt_a}{dE} + \frac{r_a - r_{\bar{a}}^*}{4E} \right)
\]  

(A.5)

\[
\int_0^{l_a} dx \phi_a(x)^* \phi_{\bar{a}}(l_a - x) = -2i\sqrt{E} \left( r_{\bar{a}} \frac{dr_{\bar{a}}}{dE} + t_{\bar{a}} \frac{dt_{\bar{a}}}{dE} + \frac{t_{\bar{a}} - t_a^*}{4E} \right).
\]  

(A.6)
Basis 2. Let us introduce the solution $f_\alpha(x_\alpha)$ satisfying the boundary conditions

\[ f_\alpha(0) = 1 \quad \text{and} \quad f_\alpha(l_\alpha) = 0. \]

It follows that $f_\alpha(x)$ is a real function for $\gamma \in \mathbb{R}$. For example, when $V(x) = 0$, the function reads $f_\alpha(x) = \frac{\sinh \sqrt{\mathcal{V}}(x-x_\alpha)}{\sinh \sqrt{\mathcal{V}}(l_\alpha-x_\alpha)}$. Another independent solution of this differential equation is naturally denoted by $f_\alpha(l_\alpha - x_\alpha) = f_\alpha(x_\alpha)$ and takes the values 0 for $x_\alpha = 0$ and 1 for $x_\alpha = l_\alpha$ (do not confuse the two functions $f_\alpha$ and $\bar{f}_\alpha$ with the components of a scalar function). The Wronskian of the two solutions is

\[ W_\alpha = W[f_\alpha(x) + f_\alpha(l_\alpha - x)] = -f_\alpha'(l_\alpha) = -f_\alpha'(l_\alpha). \]  

(A.7)

Sum rules can be obtained as follows [8]: we remark that $\partial_\gamma f_\alpha(x)$ is a solution of the differential equation $\left[ \gamma - \frac{d^2}{dx^2} + V_\alpha(x) \right] \partial_\gamma f_\alpha(x) = -f_\alpha(x)$ for the boundary conditions $\partial_\gamma f_\alpha(0) = \partial_\gamma f_\alpha(l_\alpha) = 0$. Integration straightforwardly gives

\[ \partial_\gamma f_\alpha(x) = -\frac{1}{W_\alpha} \left[ f_\alpha(x) \int_0^x dx' f_\alpha(x' - x') + f_\alpha(x - x') \int_x^{l_\alpha} dx' f_\alpha^2(x') \right]. \]  

(A.9)

We deduce

\[ \int_0^{l_\alpha} dx f_\alpha(x)^2 = -\partial_\gamma f_\alpha'(0) \]  

(A.10)

\[ \int_0^{l_\alpha} dx f_\alpha(x) f_\alpha(l_\alpha - x) = \partial_\gamma f_\alpha'(l_\alpha). \]  

(A.11)

They replace the Krein–Friedel sum rule like formulæ given above for scattering states.

One can establish the relation between the two bases of solutions $\{\phi_\alpha(x_\alpha), \phi_\alpha(x_\beta)\}$ and $\{f_\alpha(x_\alpha), f_\alpha(x_\beta)\}$ [39]:

\[ f_\alpha(x_\alpha) = \frac{(1 + r_\alpha)\phi_\alpha(x_\alpha) - t_\alpha\phi_\alpha(x_\beta)}{(1 + r_\alpha)(1 + r_\beta) - t_\alpha t_\beta}. \]  

(A.12)

Two relations follow:

\[ f_\alpha'(0) = -\sqrt{\gamma} \frac{(1 - r_\alpha)(1 + r_\beta) + t_\alpha t_\beta}{(1 + r_\alpha)(1 + r_\beta) - t_\alpha t_\beta} \]  

(A.13)

\[ f_\alpha'(l_\alpha) = -\sqrt{\gamma} \frac{2t_\alpha}{(1 + r_\alpha)(1 + r_\beta) - t_\alpha t_\beta}, \]  

(A.14)

where we have performed some analytic continuation to negative energies $\gamma = -k^2 - i0^+$. Let us introduce the $2 \times 2$ block in the matrix $M$ related to the two arcs $a$ and $\bar{a}$:

\[ M_{(a)} = \begin{pmatrix} f_\alpha'(0) & -f_\alpha'(l_\alpha) e^{i\delta_a} \\ -f_\alpha'(l_\alpha) e^{i\delta_a} & f_\alpha'(0) \end{pmatrix}. \]  

(A.15)

Relations (A.13), (A.14) with (23) immediately show that the $2 \times 2$ block in the matrix $R$ related to the two arcs $a$ and $\bar{a}$ indeed encodes the reflection and transmission amplitudes:

\[ R_{(a)} = \begin{pmatrix} r_\alpha & t_\alpha e^{i\delta_a} \\ t_\alpha e^{i\delta_a} & r_\alpha \end{pmatrix}, \]  

(A.16)

which demonstrates that $R$ is the ‘bond scattering matrix’ (or more precisely its analytic continuation to energy $E \to -\gamma$). We obtain the useful relation

\[ -f_\alpha'(l_\alpha) \det(1 + R_{(a)}) = 2\sqrt{\gamma} t_\alpha \]  

(A.17)

(we recall that $t_\alpha = t_\alpha$ follows from the one-dimensional character of the wire: it can be demonstrated by computing the Wronskian determinant of the two scattering states at the two edges of the interval).
Basis 3. Another useful set of solutions is the functions $g_a(x_a)$ and $g_{\bar{a}}(x_{\bar{a}})$ whose derivative takes values 1 and 0 at the two ends of the interval:

$$g_a'(0) = 1 \quad \text{and} \quad g_a'(l_a) = 0. \quad (A.18)$$

For example, when $V(x) = 0$, the function reads $g_a(x) = -\frac{\cosh \sqrt{\gamma} (l_a - x)}{\sqrt{\gamma} \sinh \sqrt{\gamma} l_a}$. The Wronski determinant is given by

$$W_{g_a} = W[\{g_a(x), g_{\bar{a}}(l_a - x)\}] = -g_a(l_a) = -g_{\bar{a}}(l_a). \quad (A.19)$$

Following the method of [8] recalled above, one can construct the derivative of the function with respect to the spectral parameter $\partial_\gamma g_a(x) = -\frac{1}{W_{g_a}} \int_0^{l_a} d x' g_a(x') g_{\bar{a}}(l_a - x') + g_{\bar{a}}(l_a - x) f_a^{l_a} dx' g_a^2(x')$. Therefore

$$\int_0^{l_a} d x g_a(x)^2 = \partial_\gamma g_a(0) \quad (A.20)$$

$$\int_0^{l_a} d x g_a(x) g_{\bar{a}}(l_a - x) = \partial_\gamma g_a(l_a). \quad (A.21)$$

It is also interesting to relate this basis of solutions to the two other bases:

$$g_a(x_a) = -\frac{1}{\sqrt{\gamma}} \frac{(1 - r_a) \phi_a(x_a) + l_a \phi_{\bar{a}}(x_0)}{(1 - r_a)(1 - r_{\bar{a}}) - t_a l_{\bar{a}}} \quad (A.22)$$

from which we obtain

$$g_a(0) = -\frac{1}{\sqrt{\gamma}} \frac{(1 + r_a)(1 - r_{\bar{a}}) + l_a l_{\bar{a}}}{(1 - r_a)(1 - r_{\bar{a}}) - t_a l_{\bar{a}}} \quad (A.23)$$

$$g_a(l_a) = -\frac{1}{\sqrt{\gamma}} \frac{2l_a}{(1 - r_a)(1 - r_{\bar{a}}) - t_a l_{\bar{a}}} \quad (A.24)$$

We also easily demonstrate that

$$g_a(x_a) = \frac{f_a^{l_a}(0) f_a(x_a) + f_a'(l_a) f_{\bar{a}}(x_0)}{f_a^{l_a}(0) f_a(0) - f_a'(l_a) f_{\bar{a}}'(l_a)} \quad (A.25)$$

from which

$$g_a(0) = \frac{f_a^{l_a}(0)}{f_a^{l_a}(0) f_a(0) - f_a'(l_a) f_{\bar{a}}'(l_a)} \quad (A.26)$$

$$g_a(l_a) = \frac{f_{\bar{a}}(l_a)}{f_a^{l_a}(0) f_a(0) - f_a'(l_a) f_{\bar{a}}'(l_a)}. \quad (A.27)$$

The two matrices introduced above are hence simply related by

$$N(a) = \begin{pmatrix} g_a(0) & g_a(l_a) e^{i\theta_a} \\ g_{\bar{a}}(l_a) e^{i\theta_{\bar{a}}} & g_{\bar{a}}(0) \end{pmatrix} = {M(a)}^{-1}. \quad (A.28)$$

Note also that $\det M(a) = f_a^{l_a}(l_a) / g_a(l_a)$. 

17
Appendix B. $\zeta$-functions for combinatorial graphs

The study of $\zeta$-functions (also denoted as $L$-functions in the mathematical literature) and trace formulae in combinatorial graphs has attracted a lot of interest [2, 3, 20–22, 37] (they found recently some application in the context of quantum chaos [29, 30]); a brief recent review on mathematical aspects may be found in the introduction of [28]. We show in this appendix that the general trace formula for combinatorial graphs (Bartholdi’s formula [2] generalizing Bass [3] and Ihara [22] trace formulae) may be deduced from the trace formula obtained in [1] for metric graphs. This latter is itself a particular case of the general trace formula obtained by Desbois [10] recalled in section 3.

We consider the trace formula for the Laplace operator acting on functions continuous at the vertices. Our starting point is the equality, equations (26) and (37),

$$(-1)^V \frac{\det(C - \sqrt{\gamma}D)}{\det(1_{2B} + R)} \det(1_{2B} - QR) = \det M. \tag{B.1}$$

We recall that in this case $(-1)^V \det(C - \sqrt{\gamma}D) = \prod_{\alpha}(\lambda_\alpha + m_\alpha \sqrt{\gamma})$.

Note that (B.1) is related to the spectral determinant of the metric graph when the boundary condition matrices $C$ and $D$ are $\gamma$ independent. However, having established the relation between the vertex determinant and the arc determinant, equation (B.1) also holds when the parameters $\lambda_\alpha$ depend on $\gamma$ (the case considered below). In this case, equation (B.1) is however not anymore related to the spectral determinant of the metric graph.

Let us consider the case with no potential $V(x) = 0$ and set all the lengths of the wires equal: $l_a = lV a$. For convenience, we introduce the notation

$$e^{-\sqrt{\gamma}l} = uw, \tag{B.2}$$

where $u$ and $w$ are two real or complex numbers. The bond scattering matrix elements are $R_{ij} = uw\delta_{i,j}$ and therefore $\det(1_{2B} + R) = [1 - (uw)^2]^B$. We consider the case where the boundary condition parameters $\lambda_\alpha$ are related to the valencies of the vertex by the relation

$$m_\alpha + \frac{\lambda_\alpha}{\sqrt{\gamma}} = 2w. \tag{B.3}$$

The matrix $QR$ has elements $(QR)_{ij} = u$ if the arc $j$ terminates at the vertex from which issues the arc $i$ (with $j \neq i$); $(QR)_{ii} = u(1 - w)$; all other matrix elements are zero. We write (notations reminiscent of those of [2, 28])

$$QR = u(\mathcal{B} - w\mathcal{F}), \tag{B.4}$$

where $\mathcal{B}_{ij} = 1$ if end($j$) =beginning($i$), $\mathcal{B}_{ij} = 0$ otherwise; the matrix $\mathcal{F}$ couples the reversed arcs $\mathcal{F}_{ij} = \delta_{i,j}$.

On the other hand, from (39) we find

$$M_{a\beta} = \frac{2\sqrt{\gamma}}{1 - (uw)^2} \{\delta_{a\beta}[w + (m_\alpha - w)(uw)^2] - uwa_{a\beta}\}. \tag{B.5}$$

Using that $(-1)^V \det(C - \sqrt{\gamma}D) = (2u\sqrt{\gamma})^V$ we deduce Bartholdi’s formula for the zeta function [2, 10, 28, 29]

$$Z(u, w)^{-1} = \prod_{\mathcal{C}}(1 - (1 - w)^{n_{\mathcal{C}}} u^{\mathcal{C}}) \tag{B.6}$$

$$= \det(1_{2B} - u(\mathcal{B} - w\mathcal{F}))$$

$$= (1 - (uw)^2)^{B_{\mathcal{V}}} \det(1_{\mathcal{V}}(1 - (uw)^2) - uA + uw^2\mathcal{V}), \tag{B.7}$$
where \( \ell(C) \) is the length of the orbit (number of wires visited) and \( n_R(C) \) is the number of reflections of the orbit on vertices. \( A \) denotes the adjacency matrix with matrix elements \( a_{\alpha\beta} \).

The matrix \( Y \) is the diagonal matrix gathering the valencies: \( Y_{\alpha\beta} = m_\alpha \delta_{\alpha\beta} \). This formula was used in [7, 10] as a generating function for counting of the orbits with finite number of reflections. It has also found some interesting application in the context of quantum chaos in combinatorial graphs [29, 30].

Note that combinatorial graphs are related to metric graphs when all lengths are equal and for permutation invariant boundary conditions at the vertices. It follows that only two weights can be attributed to an orbit passing at a vertex (\( u \) for no reflection and \( u(1 - w) \) for a reflection) and therefore (B.6) is the most general trace formula for combinatorial graphs (up to the straightforward addition of magnetic fluxes in matrices \( A \) and \( B \)). As a consequence all the trace formulae obtained for metric graphs with permutation invariant boundary conditions at the vertices (we have considered the particular case of continuous boundary conditions here) would lead to the Bartholdi formula when lengths are taken to be equal and the potential vanishes.

For \( w = 1 \) we recover the Bass formula [3, 37] (this was also discussed in [7, 10])

\[
Z(u, 1)^{-1} = \prod_{\tilde{C}_B}(1 - u^{\ell(C_B)}) \quad (B.8)
\]

\[
= \det(1_{2B} - u(\mathcal{B} - \mathcal{F})) = (1 - u^2)^{B-V} \det(1_Y - uA + u^2(Y' - 1_Y)) \quad (B.9)
\]

for the \( \zeta \)-function over the backtrack-less orbits \( \tilde{C}_B \) (orbits with no reflection). The Bass formula generalizes the Ihara formula [22] demonstrated for regular graphs.

**References**

[1] Akkermans E, Comtet A, Desbois J, Montambaux G and Texier C 2000 On the spectral determinant of quantum graphs Ann. Phys., NY 284 10–51
[2] Bartholdi L 1999 Counting paths in graphs L’Enseignement Math. 45 83–131
[3] Bass H 1992 The Ihara–Selberg zeta function of a tree lattice Int. J. Math. 3 717
[4] Berkolaiko G and Keating J P 1999 Two-point spectral correlations for star graphs J. Phys. A: Math. Gen. 32 7827
[5] Chekhov L O 1999 A spectral problem on graphs and \( L \)-functions Russ. Math. Surv. 54 1197
[6] Comtet A, Desbois J and Majumdar S N 2002 The local time distribution of a particle diffusing on a graph J. Phys. A: Math. Gen. 35 L687
[7] Comtet A, Desbois J and Texier C 2005 Functionals of the Brownian motion, localization and metric graphs J. Phys. A: Math. Gen. 38 R341–83
[8] Desbois J 2000 Spectral determinant of Schrödinger operators on graphs J. Phys. A: Math. Gen. 33 L63
[9] Desbois J 2000 Time-dependent harmonic oscillator and spectral determinant on graphs Eur. Phys. J. B 15 201
[10] Desbois J 2001 Spectral determinant on graphs with generalized boundary conditions Eur. Phys. J. B 24 261
[11] Desbois J 2002 Occupation times distribution for Brownian motion on graphs J. Phys. A: Math. Gen. 35 L673
[12] Exner P 1995 Lattice Kronig–Penney Models Phys. Rev. Lett. 74 3503
[13] Exner P 1997 A duality between Schrödinger operators on graphs and certain Jacobi matrices Ann. Inst. Henri Poincaré A 66 359
[14] Ferrier M, Angers L, Rowe A C H, Guéron S, Bouchiat H, Texier C, Montambaux G and Mailly D 2004 Direct measurement of the phase coherence length in a GaAs/GaAlAs square network Phys. Rev. Lett. 93 246804
[15] Forman R 1993 Determinants of Laplacians on graphs Topology 32 35
[16] Friedel J 1952 The distribution of electrons round impurities in monovalent metals Phil. Mag. 43 153
[17] Friedlander L 2006 Determinant of the Schrödinger operator on a metric graph Contemp. Math. 415 151
[18] Gradshteyn I S and Ryzhik I M 1994 Table of Integrals, Series and Products 5th edn (New York: Academic)
[19] Harrison J M and Kirsten K 2009 Zeta functions of quantum graphs arXiv:0911.2509
[20] Hashimoto K I 1989 Zeta functions of finite graphs and representations of \( p \)-adic groups Adv. Stud. Pure Math. 15 211
[21] Hashimoto K I 1990 On zeta and L-functions of finite graphs Int. J. Math. 1 381
[22] Ihara Y 1966 On discrete subgroup of the two by two projective linear group over p-adic field J. Math. Soc. Japan 18 219
[23] Keating J P, Marklof J and Winn B 2003 Value distribution of the eigenfunctions and spectral determinants of quantum star graphs Commun. Math. Phys. 241 421
[24] Kostrykin V and Schrader R 1999 Kirchhoff’s rule for quantum wires J. Phys. A: Math. Gen. 32 595
[25] Kostrykin V and Schrader R 2007 Heat kernel on metric graphs and a trace formula Contemp. Math. 447 175
[26] Kottos T and Smilansky U 1999 Periodic orbit theory and spectral statistics for quantum graphs Ann. Phys., NY 274 76
[27] Krein M G 1953 Trace formulas in perturbation theory Matem. Sbornik 33 597
[28] Mizuno H and Sato I 2005 A new proof of Bartholdi’s theorem J. Algebra Comb. 22 259–71
[29] Oren I, Godel A and Smilansky U 2009 Trace formulae and spectral statistics for discrete Laplacians on regular graphs (I) J. Phys. A: Math. Theor. 42 415101
[30] Oren I and Smilansky U 2010 Trace formulae and spectral statistics for discrete Laplacians on regular graphs (II) arXiv:1003.1445

[31] Pascaud M and Montambaux G 1998 Interference effects in mesoscopic rings and wires Phys.—Usp. 41 182
[32] Pascaud M 1998 Magnétisme orbital de conducteurs mésoscopiques désordonnés et propriétés spectrales de fermions en interaction PhD Thesis Université Paris 11
[33] Pascaud M and Montambaux G 1999 Persistent currents on networks Phys. Rev. Lett. 82 4512
[34] Roth J-P 1983 Spectre du Laplacien sur un graphe C. R. Acad. Sci. Paris 296 793
[35] Roth J-P 1983 Le spectre du Laplacien sur un graphe Colloque de Théorie du potentiel—Jacques Deny Orsay p 521

[36] Smith F T 1960 Lifetime matrix in collision theory Phys. Rev. 118 349
[37] Stark H M and Terras A A 1996 Zeta functions of finite graphs and coverings Adv. Math. 121 124
[38] Stillinger F H and Herrick D R 1975 Bound states in the continuum Phys. Rev. A 11 446
[39] Texier C and Montambaux G 2001 Scattering theory on graphs J. Phys. A: Math. Gen. 34 10307–26
[40] Texier C 2002 Scattering theory on graphs (2): the Friedel sum rule J. Phys. A: Math. Gen. 35 3389–407
[41] Texier C and Büttiker M 2003 Local Friedel sum rule in graphs Phys. Rev. B 67 245410
[42] Texier C and Montambaux G 2005 Quantum oscillations in mesoscopic rings and anomalous diffusion J. Phys. A: Math. Gen. 38 3455–71
[43] Texier C and Montambaux G 2005 Dephasing due to electron–electron interaction in a diffusive ring Phys. Rev. B 72 115327

Texier C and Montambaux G 2006 Phys. Rev. B 74 209902 (Erratum)

[44] Texier C 2008 On the spectrum of the Laplace operator of metric graphs attached at a vertex—spectral determinant approach J. Phys. A: Math. Theor. 41 085207
[45] von Neumann J and Wigner E 1929 Über merkwürdige diskrete Eigenwerte Phys. Z. 30 465