A CONCISE PROOF OF KRUSKAL’S THEOREM ON TENSOR DECOMPOSITION

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Abstract. A theorem of J. Kruskal from 1977, motivated by a latent-class statistical model, established that under certain explicit conditions the expression of a 3-dimensional tensor as the sum of rank-1 tensors is essentially unique. We give a new proof of this fundamental result, which is substantially shorter than both the original one and recent versions along the original lines.

1. Introduction

In [10], J. Kruskal proved that, under certain explicit conditions, the expression of a 3-dimensional tensor (i.e., a 3-way array) of rank $r$ as a sum of $r$ tensors of rank 1 is unique, up to permutation of the summands. (See also [8, 9].) This result contrasts sharply with the well-known non-uniqueness of expressions of matrices of rank at least 2 as sums of rank-1 matrices. The uniqueness of this tensor decomposition is moreover of fundamental interest for a number of applications, ranging from Kruskal’s original motivation by latent-class models used in psychometrics, to chemistry and signal processing, as mentioned in [11] and its references. In these fields, the expression of a tensor as a sum of rank-1 tensors is often referred to as the Candecomp or Parafac decomposition. Recently, Kruskal’s theorem has been used as a general tool for investigating the identifiability of a wide variety of statistical models with hidden variables [1, 2].

As noted in [11], Kruskal’s original proof was “rather inaccessible,” leading a number of authors to work toward a shorter and more intuitive presentation. This thread, which continued to follow the basic outline of Kruskal’s approach in which his ‘Permutation Lemma’ plays a key role, culminated in the proof given in [11]. In this paper, we present a new and more concise proof of Kruskal’s theorem, Theorem 3 below, that follows an entirely different approach. While the resulting theorem is identical, the alternative argument given here offers a new perspective on the role of Kruskal’s explicit condition ensuring uniqueness.

While Kruskal’s theorem gives a sufficient condition for uniqueness of a decomposition, the condition is known in general not to be necessary. Of particular note are recent independent works of De Lathauwer [4] and Jiang and Sidiropoulos [6], which give a different, though in some ways more narrow, criterion that can ensure uniqueness. See also [12] for the connection between these works.

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It would, of course, be highly desirable to obtain conditions (more involved than Kruskal’s) that would ensure the essential uniqueness of the expression of a rank $r$ tensor as a sum of rank-1 tensors under a wider range of assumptions on the size and rank of the tensor. Note that both Kruskal’s condition and that of [4, 6] can be phrased algebraically, in terms of the non-vanishing of certain polynomials in the variables of a natural parameterization of rank $r$ tensors. This algebraic formulation allows one to conclude that generic rank $r$ tensors of certain sizes have unique decompositions. Having explicit understanding of these polynomial conditions is essential for certain applications, such as in [1]. The general problem of determining for which sizes and ranks of generic tensors the decomposition is essentially unique, and what explicit algebraic conditions can ensure uniqueness, remains open.

2. Notation

Throughout, we work over an arbitrary field.

For a matrix such as $M_k$, we use $m_j^k$ to denote the $j$th column, $m_i^k$ to denote the $i$th row, and $m_{ij}^k$ the $(i,j)$th entry. We use $(S)$ to denote the span of a set of vectors $S$. With $[r] = \{1, 2, 3, \ldots, r\}$, we denote by $\mathfrak{S}_r$ the symmetric group on $[r]$.

Given matrices $M_i$ of size $s_i \times r$, the matrix triple product $[M_1, M_2, M_3]$ is an $s_1 \times s_2 \times s_3$ tensor defined as a sum of $r$ rank-1 tensors by

$$[M_1, M_2, M_3] = \sum_{i=1}^{r} m_i^1 \otimes m_i^2 \otimes m_i^3,$$

so

$$[M_1, M_2, M_3](j, k, l) = \sum_{i=1}^{r} m_i^1 m_i^2 m_i^3.$$

A matrix $A$ of size $t \times s_1$ acts on an $s_1 \times s_2 \times s_3$ tensor $T$ ‘in the $l$th coordinate.’ For example, with $l = 1$

$$(A \ast_1 T)(i, j, k) = \sum_{n=1}^{s_1} a_{in} T(n, j, k),$$

so that $A \ast_1 T$ is of size $t \times s_2 \times s_3$. One then easily checks that

$$A \ast_1 [M_1, M_2, M_3] = [AM_1, M_2, M_3],$$

with similar formulas applying for actions in other coordinates.

**Definition.** The Kruskal rank, or $K$-rank, of a matrix is the largest number $j$ such that every set of $j$ columns is independent.

**Definition.** We say a triple of matrices $(M_1, M_2, M_3)$ is of type $(r; a_1, a_2, a_3)$ if each $M_i$ has $r$ columns and the K-rank of $M_i$ is at least $r - a_i$.

In a slight abuse of notation, we will say a product $[M_1, M_2, M_3]$ is of type $(r; a_1, a_2, a_3)$ when the triple $(M_1, M_2, M_3)$ is of that type.

Note that with this definition, type $(r; a_1, a_2, a_3)$ implies type $(r; b_1, b_2, b_3)$ as long as $a_i \leq b_i$ for each $i$. Thus $a_i$ is a bound on the gap between the K-rank of the matrix $M_i$ and the number $r$ of its columns. Intuitively, when the $a_i$ are small it should be easier to identify the $M_i$ from the product $[M_1, M_2, M_3]$.

We will not need to be explicit about the number of rows in any of the $M_i$, though type $(r; a_1, a_2, a_3)$ of course implies $M_i$ has at least $r - a_i$ rows.
3. The proof

We begin by establishing a lemma that generalizes a basic insight that has been rediscovered many times over the years, in which matrix diagonalizations arising from matrix slices of a 3-dimensional tensor are used to understand the tensor decomposition. A few such instances of the appearance of this idea include [3,7], and other such references are mentioned in [5] where the idea is exploited for computational purposes.

Lemma 1. Suppose $(M_1, M_2, M_3)$ is of type $(r; 0, 0, r - 1)$; $N_1, N_2, N_3$ are matrices with $r$ columns; and $[M_1, M_2, M_3] = [N_1, N_2, N_3]$. Then there is some permutation $\sigma \in S_r$ such that the following holds:

Let $I \subseteq [r]$ be any maximal subset (with respect to inclusion) of indices with the property that $\langle \{m_{\sigma(i)}^\alpha\}_{i \in I} \rangle$ is 1-dimensional. Then

1. $\langle \{m_{\sigma(i)}^\alpha\}_{i \in I} \rangle = \langle \{n_{\sigma(i)}^\alpha\}_{i \in I} \rangle$, for $j = 1, 2, 3$ and
2. $\sigma$ is also maximal for the property that $\langle \{n_{\sigma(i)}^\alpha\}_{i \in I} \rangle$ is 1-dimensional.

Proof. That $(M_1, M_2, M_3)$ is of type $(r; 0, 0, r - 1)$ means $M_1, M_2$ have full column rank, and $M_3$ has no zero columns.

Choose some vector $c$ that is not orthogonal to any of the columns of $M_3$, so that $c^T M_3$ has no zero entries. Then

$$A = c^T \ast_3 [M_1, M_2, M_3] = [M_1, M_2, c^T M_3] = M_1 \text{ diag}(c^T M_3) M_2^T$$

is a matrix of rank $r$. Since

$$A = c^T \ast_3 [N_1, N_2, N_3] = [N_1, N_2, c^TN_3] = N_1 \text{ diag}(c^TN_3) N_2^T,$$

$N_1$ and $N_2$ must also have rank $r$, and $c^TN_3$ has no zero entries. These two expressions for $A$ also show that the span of the columns of $M_j$ is the same as that of the columns of $N_j$ for $j = 1, 2$. Expressing the columns of $M_j$ and $N_j$ in terms of a basis given by the columns of $M_j$, we may henceforth assume $M_1 = M_2 = I_r$, the $r \times r$ identity, and $N_1, N_2$ are invertible. Thus $A = \text{ diag}(c^T M_3)$.

Now let $S_i$ denote the slice of $[M_1, M_2, M_3] = [N_1, N_2, N_3]$ with fixed third coordinate $i$, so $S_i$ is an $r \times r$ matrix. Recalling that $\tilde{m}_i$ and $\tilde{n}_i$ denote the $i$th rows of $M_j$ and $N_j$, we have

$$S_i = \text{ diag}(\tilde{m}_i) = N_1 \text{ diag}(\tilde{n}_i) N_2^T.$$

Note the matrices

$$S_i A^{-1} = \text{ diag}(\tilde{m}_i) \text{ diag}(c^T M_3)^{-1} = N_1 \text{ diag}(\tilde{n}_i) \text{ diag}(c^T N_3)^{-1} N_1^{-1},$$

for various choices of $i$, commute. Thus their (right) simultaneous eigenspaces are determined. But from the two expressions for $S_i A^{-1}$ we see its $\alpha$-eigenspace is spanned by the set

$$\{e_j = m_{ij}^3 | m_{i,j}^3/(c^T m_{i}^3) = \alpha\},$$

and also by the set

$$\{n_{i}^3 | n_{i,j}^3/(c^T n_{i}^3) = \alpha\}.$$
This condition is equivalent to $\mathbf{m}_j^3$ and $\mathbf{n}_j^3$ being scalar multiples of one another. Such a set $\mathcal{I}$ is therefore exactly of the sort described in the statement of the lemma. As the simultaneous eigenspaces are also spanned by similar sets defined in terms of the columns of $N_1$, one may choose a permutation $\sigma$ so that claim 2 holds, as well as claim 1 for $j = 1$.

The case $j = 2$ of claim 1 is similarly proved using the transposes of $A$ and the $S_i$. As the needed permutation of the columns of the $N_j$ in the two cases of $j = 1, 2$ is dependent only on the maximal sets $\mathcal{I}$, a common $\sigma$ may be chosen. Finally, the case $j = 3$ follows from equating eigenvalues in the two expressions giving diagonalizations for $S_i A^{-1}$, to see that for all $i$

$$m_{3,j}^3 / c^T m_j^3 = n_{i,\sigma(j)}^3 / c^T n_{\sigma(j)}^3,$$

so $\mathbf{m}_j^3$ and $\mathbf{n}_{\sigma(j)}^3$ are scalar multiples of one another. □

This lemma quickly yields a special case of Kruskal’s theorem, when two of the matrices in the product are assumed to have full column rank.

**Corollary 2.** Suppose $(M_1, M_2, M_3)$ is of type $(r; 0, 0, r - 2)$; $N_1, N_2, N_3$ are matrices with $r$ columns; and $[M_1, M_2, M_3] = [N_1, N_2, N_3]$. Then there exists some permutation matrix $P$ and invertible diagonal matrices $D_i$ with $D_1 D_2 D_3 = I_r$ such that $N_i = M_i D_i P$.

**Proof.** Since $(M_1, M_2, M_3)$ is also of type $(r; 0, 0, r - 1)$, we may apply Lemma 1. As in the proof of that lemma, we may also assume $M_1 = M_2 = I_r$. But $M_3$ has K-rank at least 2, so every pair of columns is independent. Thus the maximal sets of indices in Lemma 1 are all singletons. Thus with $P$ acting to permute columns by $\sigma$, the one-dimensionality of all eigenspaces shows there is a permutation $P$ and invertible diagonal matrices $D_1, D_2$ with $N_i = M_i D_i P = D_i P$ for $j = 1, 2$.

Thus $[M_1, M_2, M_3] = [N_1, N_2, N_3]$ implies

$$[I_r, I_r, M_3] = [D_1 P, D_2 P, N_3] = [D_1, D_2, N_3 P^T] = [I_r, I_r, N_3 P^T D_1 D_2],$$

which shows $M_3 = N_3 P^T D_1 D_2$. Setting $D_3 = (D_1 D_2)^{-1}$, we find $N_3 = M_3 D_3 P$. □

We now use the lemma to give a new proof of Kruskal’s Theorem in its full generality. Note that the condition on the $a_i$ stated in the following theorem is equivalent to Kruskal’s condition in [10] that $(r - a_1) + (r - a_2) + (r - a_3) \geq 2r + 2$.

**Theorem 3** (Kruskal, [10]). Suppose $(M_1, M_2, M_3)$ is of type $(r; a_1, a_2, a_3)$ with $a_1 + a_2 + a_3 \leq r - 2$; $N_1, N_2, N_3$ are matrices with $r$ columns, and $[M_1, M_2, M_3] = [N_1, N_2, N_3]$. Then there exists some permutation matrix $P$ and invertible diagonal matrices $D_i$ with $D_1 D_2 D_3 = I_r$ such that $N_i = M_i D_i P$.

**Proof.** We need only consider $a_1 + a_2 + a_3 = r - 2$. We proceed by induction on $r$, with the case $r = 2$ (and 3) already established by Corollary 2. We may also assume $a_1 \leq a_2 \leq a_3$. We may furthermore restrict to $a_2 \geq 1$, since the case $a_1 = a_2 = 0$ is established by Corollary 2.

We first claim that it will be enough to show that, for some $1 \leq i \leq 3$, there is some set of indices $\mathcal{J} \subset [r]$, $1 \leq |\mathcal{J}| \leq r - a_i - 2$, and a permutation $\sigma \in \mathfrak{S}_r$ such that

$$\langle \{ \mathbf{m}_j^i \}_{j \in \mathcal{J}} \rangle = \langle \{ \mathbf{n}_{\sigma(j)}^i \}_{j \in \mathcal{J}} \rangle.$$


To see this, if there is such a set \( \mathcal{J} \), assume for convenience \( i = 1 \) (the cases \( i = 2, 3 \) are similar), and the columns of \( M_i, N_i \) have been reordered so that \( \sigma = id \) and \( \mathcal{J} = [s] \). Let \( \Pi \) be a matrix with nullspace the span described in equation (11). Then
\[
[\Pi M_1, M_2, M_3] = \Pi \ast_1 [M_1, M_2, M_3] = \Pi \ast_1 [N_1, N_2, N_3] = [\Pi N_1, N_2, N_3].
\]

But since the first \( s \) columns of \( \Pi M_1 \) and \( \Pi N_1 \) are zero, these triple products can be expressed as triple products of matrices with only \( r - s \) columns. That is, using the symbol \( \sim \) to denote deletion of the first \( s \) columns,
\[
[\Pi M_1, \tilde{M}_2, \tilde{M}_3] = [\Pi \tilde{N}_1, \tilde{N}_2, \tilde{N}_3].
\]

For \( i = 2, 3 \), since \( M_i \) has K-rank \( \geq r - a_i \), the matrix \( \tilde{M}_i \) has K-rank \( \geq \min(r - a_i, r - s) \). Since the nullspace of \( \Pi \) is spanned by the first \( s \) columns of \( M_1 \), and \( M_1 \) has K-rank \( \geq r - a_1 \), ones sees that \( \Pi \tilde{M}_i \) has K-rank \( \geq r - s - a_1 \), as follows:

For any set of \( r - s - a_1 \) columns of \( \Pi \tilde{M}_1 \), consider the corresponding columns of \( M_1 \), together with the first \( s \) columns. This set of \( r - a_1 \) columns of \( M_1 \) is therefore independent, so the span of its image under \( \Pi \) is of dimension \( r - s - a_1 \). This span must then have as a basis the chosen set of \( r - s - a_1 \) columns of \( \Pi \tilde{M}_1 \), which are therefore independent. Thus \( [\Pi M_1, \tilde{M}_2, \tilde{M}_3] \) is of type \( (r - s; a_1, b_2, b_3) \), where \( b_i = \max(0, a_i - s) \) for \( i = 2, 3 \). Note also that \( s \leq r - a_1 - 2 \) implies \( a_1 + b_2 + b_3 \leq r - s - 2 \).

We may thus apply the inductive hypothesis to \( [\Pi \tilde{M}_1, \tilde{M}_2, \tilde{M}_3] = [\Pi \tilde{N}_1, \tilde{N}_2, \tilde{N}_3] \), and, after an allowed permutation and scalar multiplication of the columns of the \( N_i \), conclude that \( \tilde{M}_i = \tilde{N}_i \) for \( i = 2, 3 \). But this means we can now take the set \( \mathcal{J} \) described in equation (11) to be a singleton set \( \{j\} \), with \( j > s \), and \( i = 2 \).

Again applying the argument developed thus far implies that, allowing for a possible permutation and rescaling, all but the \( j \)th columns of \( M_3 \) and \( N_3 \) are identical. As \( m_j^3 = n_j^3 \), this shows \( M_3 = N_3 \). Applying this argument yet again, with \( i = 3 \), and varying choices of \( j \), then shows \( M_1 = N_1 \) and \( M_2 = N_2 \), up to the allowed permutation and rescaling. The claim is thus established.

We next argue that some set of columns of some \( M_i, N_i \) meets the hypotheses of the above claim.

Let \( \Pi_1 \) be any matrix with nullspace \( \langle \{n_i^3\}_{1 \leq i \leq a_1 + a_2} \rangle \), spanned by the first \( a_1 + a_2 \) columns of \( N_3 \). Let \( Z \) be the set of indices of all zero columns of \( \Pi_3 M_3 \). Since every set of \( r - a_3 = a_1 + a_2 + 2 \) columns of \( M_3 \) is independent, \( |Z| \leq a_1 + a_2 \). Note also that at least 2 columns of \( \Pi_3 M_3 \) are independent, since the span of any \( a_1 + a_2 + 2 \) columns of \( \Pi_3 M_3 \) is at least 2 dimensional.

Let \( S_1, S_2 \) be any disjoint subsets of \( [r] \) such that \( |S_1| = a_2, |S_2| = a_1 \), \( Z \subseteq S_1 \cup S_2 = S \), and \( S \) excludes at least two indices of independent columns of \( \Pi_3 M_3 \). Let \( \Pi_1 = \Pi_1(S_1) \) be any matrix with nullspace \( \langle \{m_i^1\}_{i \in S_1} \rangle \), and let \( \Pi_2 = \Pi_2(S_2) \) be any matrix with nullspace \( \langle \{m_i^2\}_{i \in S_2} \rangle \).

Now consider
\[
[\Pi_1 M_1, \Pi_2 M_2, \Pi_3 M_3] = \Pi_3 \ast_3 (\Pi_2 \ast_2 (\Pi_1 \ast_1 [M_1, M_2, M_3]))
\]
\[
= \Pi_3 \ast_3 (\Pi_2 \ast_2 (\Pi_1 \ast_1 [N_1, N_2, N_3])) = [\Pi_1 N_1, \Pi_2 N_2, \Pi_3 N_3].
\]

By the specification of the nullspace of \( \Pi_3 \), the columns of all \( N_i \) with indices in \( [a_1 + a_2] \) can be deleted in this last product. In the first product, one can similarly delete the columns of the \( M_i \) with indices in \( S \), due to the specifications of the
nullspaces of $\Pi_1$ and $\Pi_2$. Using $\sim$ to denote the deletion of these columns, we have
\begin{equation}
[\Pi_1 \tilde{M}_1, \Pi_2 \tilde{M}_2, \Pi_3 \tilde{M}_3] = [\Pi_1 \tilde{N}_1, \Pi_2 \tilde{N}_2, \Pi_3 \tilde{N}_3],
\end{equation}
where these products involve matrix factors with $r - a_1 - a_2 = a_3 + 2$ columns.

The matrix $\Pi_1 \tilde{M}_1$ in fact has full column rank. To see this, note that it can also be obtained from $M_1$ by (a) first deleting columns with indices in $S_2$, then (b) multiplying on the left by $\Pi_1$, and finally (c) deleting the columns arising from those in $M_1$ with indices in $S_1$. Since $M_1$ has K-rank at least $r - a_1$, step (a) produces a matrix with $r - a_1$ columns, and full column rank. Since the nullspace of $\Pi_1$ is spanned by certain of the columns of this matrix, step (b) produces a matrix whose non-zero columns are independent. Step (c) then deletes all zero columns to give a matrix of full column rank. Similarly, the matrix $\Pi_2 \tilde{M}_2$ has full column rank.

Noting that $\Pi_3 \tilde{M}_3$ has no zero columns since $Z \subseteq S$, we may thus apply Lemma \[\Pi\] to the products of equation \[\Pi\]. In particular, we find that there is some $\sigma \in \mathcal{S}_r$ with $\sigma([r] \setminus S) = [r] \setminus [a_1 + a_2]$ such that if $\mathcal{I}$ is a maximal subset of $[r] \setminus S$ with respect to the property that $\langle \{\Pi_3 m^j_i\}_{i \in \mathcal{I}} \rangle$ is 1-dimensional, then
\begin{equation}
\langle \{\Pi_j m^j_i\}_{i \in \mathcal{I}} \rangle = \langle \{\Pi_{\sigma(i)} m^j_i\}_{i \in \mathcal{I}} \rangle
\end{equation}
for $j = 1, 2, 3$.

Since we chose $S$ to exclude indices of two independent columns of $\Pi_3 M_3$, there will be such a maximal subset $\mathcal{I}$ of $[r] \setminus S$ that contains at most half the indices. We thus pick such an $\mathcal{I}$ with $|\mathcal{I}| \leq (r - a_1 - a_2)/2 = a_3/2 + 1$, and consider two cases:

**Case** $a_1 = 0$: Then $S_2 = \emptyset$, and $\Pi_2$ has trivial nullspace and thus may be taken to be the identity. Since $a_3 \geq a_2 \geq 1$, this implies $|\mathcal{I}| \leq a_3 = r - a_2 - 2$. The sets $\{m^j_i\}_{i \in \mathcal{I}}$ and $\{n^2_{\sigma(i)}\}_{i \in \mathcal{I}}$ therefore satisfy the hypotheses of the claim.

**Case** $a_1 \geq 1$: Note that $|\mathcal{I}| + a_2 + 1 \leq a_3/2 + a_2 + 2 < a_2 + a_3 + 2 = r - a_1$, so for any index $k$, the columns of $M_1$ indexed by $\mathcal{I} \cup S_1 \cup \{k\}$ are independent. This then implies that for $j = 1$ the spanning set on the left of equation \[\Pi\] is independent, so the spanning set on the right is as well. Thus the set $\{n^j_{\sigma(i)}\}_{i \in \mathcal{I}}$ is also independent. Note next that equation \[\Pi\] implies that, for $i \in \mathcal{I}$, there are scalars $b^j_i, c^j_i$ such that
\begin{equation}
n^1_{\sigma(i)} - \sum_{j \in \mathcal{I}} b^j_i m^j_i = \sum_{k \in S_1} c^j_k m^j_k.
\end{equation}
Now for any $p \in S_1$, $q \in S_2$, let
\[S'_1 = (S_1 \setminus \{p\}) \cup \{q\}, \quad S'_2 = (S_2 \setminus \{q\}) \cup \{p\}.
\]
Choosing $\Pi'_1$ and $\Pi'_2$ to have nullspaces determined as above by the index sets $S'_1$ and $S'_2$, and applying Lemma \[\Pi\] to $[\Pi'_1 M_1, \Pi'_2 M_2, \Pi_3 M_3] = [\Pi'_1 N_1, \Pi'_2 N_2, \Pi_3 N_3]$, similarly shows that for some permutation $\sigma'$ and any $i' \in \mathcal{I}$ there are scalars $d^j_k, f^j_k$ such that
\begin{equation}
n^1_{\sigma'(i')} - \sum_{j \in \mathcal{I}} d^j_k m^j_i = \sum_{i' \in S'_1} f^j_{i'} m^j_i.
\end{equation}
Note that since the same $\Pi_3$ was used, the set $\mathcal{I}$ is unchanged here, and $\sigma$ and $\sigma'$ must have the same image on $\mathcal{I}$. Picking $i' \in \mathcal{I}$ so that $\sigma'(i') = \sigma(i)$, and
subtracting equation (4) from (5) shows
\[ \sum_{j \in \mathcal{I}} (b^i_j - d^i_j) m^j = \sum_{k \in \mathcal{S}_1 \setminus \{p\}} (f^i_k - c^i_p) m^k + f^i_q m^q - c^i_p m^p. \]

But since the columns of \( M \) appearing in this equation are independent, we see that \( f^i_q = c^i_p = 0 \). By varying \( p \), we conclude that \( n^1_{\sigma(i)} \in \langle \{m^1_i\}_{i \in \mathcal{I}} \rangle \). Thus \( \langle \{n^1_{\sigma(i)}\}_{i \in \mathcal{I}} \rangle \subseteq \langle \{m^1_i\}_{i \in \mathcal{I}} \rangle \). Since both of these spanning sets are independent, and of the same cardinality, their spans must be equal. Since \( |\mathcal{I}| \leq r - a_1 - 2 \), the set \( \mathcal{I} \) satisfies the hypotheses of the claim.

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