CONSTANTS FOR ARTIN-LIKE PROBLEMS IN KUMMER AND DIVISION FIELDS

AMIR AKBARY AND MILAD FAKHARI

ABSTRACT. We apply the character sums method of Lenstra, Moree, and Stevenhagen to explicitly compute the constants in the Titchmarsh divisor problem for Kummer fields and division fields of Serre curves. We derive our results as special cases of a general result on the product expressions for the sums in the form

\[ \sum_{n=1}^{\infty} \frac{g(n)}{\#G(n)} \]

in which \( g(n) \) is a multiplicative arithmetic function and \( \{G(n)\} \) is a certain family of Galois groups. Our results extend the application of the character sums method to the evaluation of constants, such as the Titchmarsh divisor constants, that are not density constants.

1. INTRODUCTION

Let \( a \) be a non-zero integer that is not \( \pm 1 \). Let \( h \) be the largest integer for which \( a \) is a perfect \( h \)-th power. In 1927, Emil Artin proposed a conjecture for the density of primes \( q \) for which a given integer \( a \) is a primitive root modulo \( q \). More precisely, Artin conjectured that the density is

\[
A_a = \prod_{p \text{ prime}} \left( 1 - \frac{1}{[\mathbb{Q}(\zeta_p, a^{1/p}) : \mathbb{Q}]} \right) = \prod_{p \text{ prime}} \left( 1 - \frac{1}{p-1} \right) \prod_{p \text{ prime}} \left( 1 - \frac{1}{p(p-1)} \right).
\]

Here \( \zeta_p \) is a primitive \( p \)-th root of unity in a fixed algebraic closure of \( \mathbb{Q} \). Observe that \( A_a = 0 \) if \( a \) is a perfect square as \( [\mathbb{Q}(\zeta_2, a^{1/2}) : \mathbb{Q}] = 1 \) for such \( a \).

In 1957, computer calculations of the approximate density for various values of \( a \) by D. H. Lehmer and E. Lehmer revealed some discrepancies from the conjectured value \( A_a \) (see \[21\] Section 2)). The reason for these inconsistencies is the dependency between the splitting conditions in Kummer fields \( \mathbb{Q}(\zeta_p, a^{1/p}) \).

To deal with these dependencies, Artin suggested an entanglement correction factor that appears when \( a_{sf} \equiv 1 \pmod{4} \), where \( a = a_{sf} \cdot b^2 \) in which \( b \) is the largest integer such that \( b^2 \) divides \( a \) (see the preface to Artin’s collected works \[3\]). More precisely, the corrected conjectured density \( \delta_a \) is

\[
\delta_a = \begin{cases} A_a & \text{if } a_{sf} \not\equiv 1 \pmod{4}, \\ E_a \cdot A_a & \text{if } a_{sf} \equiv 1 \pmod{4}, \end{cases}
\]

where

\[
E_a = 1 - \mu(|a_{sf}|) \prod_{p \mid b} \frac{1}{p-2} \prod_{p \mid b^2} \frac{1}{p^2 - p - 1}.
\]

Date: April 9, 2024.

2020 Mathematics Subject Classification. 11N37, 11A07.

Key words and phrases. Generalized Artin problem, character sums, Titchmarsh divisor problems in the family of number fields.
Here, $\mu(.)$ is the Möbius function. Hooley [10] proved the modified conjecture in 1967 under the assumption of the Generalized Riemann Hypothesis (GRH) for the Kummer fields $K_n = \mathbb{Q}(\zeta_n, a^{1/n})$ for square-free values of $n$. For any $n$, let $G(n)$ be the Galois group of $K_n/\mathbb{Q}$. More precisely, Hooley proved, under the GRH, that the primitive root density is

$$
\sum_{n=1}^{\infty} \frac{\mu(n)}{\#G(n)},
$$

and then showed that the above sum equals the corrected conjectured density $\delta_{a}$ in (1.2).

In [14], Lenstra, Moree, and Stevenhagen introduced a method which allows one to find product expressions for densities in Artin-like problems. Their method directly studies the primes that do not split completely in a Kummer family attached to $a$, without considering the summation expressions such as (1.4) for the densities. In [14, Theorem 4.2], they express the correction factor (1.3), when $a$ is non-square and the discriminant $d$ of $K_2 = \mathbb{Q}(a^{1/2})$ is odd (equivalently $a$ is non-square and $a_{sf} \equiv 1 \pmod{4}$), as

$$
E_a = 1 + \prod_{p|a} \frac{-1}{\#G(p) - 1}.
$$

The authors of [14] achieve this by constructing a quadratic character $\chi = \prod_p \chi_p$ of a certain profinite group $A = \prod_p A_p$ such that $\ker \chi = \text{Gal}(K_\infty/\mathbb{Q})$, where $K_\infty = \bigcup_{n \geq 1} K_n$ (see Section 2 for details). They derive (1.2) as a special case of the following general theorem ([14, Theorem 3.3]) in the context of profinite groups.

**Theorem 1.1** (Lenstra-Moree-Stevenhagen). Let $A = \prod_p A_p$, with Haar measure $\nu = \prod_p \nu_p$, and the quadratic character $\chi = \prod_p \chi_p : A \rightarrow \{\pm 1\}$ a non-trivial character obtained from a family of continuous quadratic characters $\chi_p : A_p \rightarrow \{\pm 1\}$, with $\chi_p$ trivial for almost all primes $p$. Then for $G = \ker \chi$ and $S = \prod_p S_p$, a product of $\nu_p$-measurable subsets $S_p \subset A_p$ with $\nu_p(S_p) > 0$, we have

$$
\frac{\nu(G \cap S)}{\nu(G)} = \left(1 + \prod_p \frac{1}{\nu_p(S_p)} \int_{S_p} \chi_p d\nu_p \right) \cdot \frac{\nu(S)}{\nu(A)}.
$$

The above theorem shows that if $\frac{\nu(G \cap S)}{\nu(G)} \neq \frac{\nu(S)}{\nu(A)}$, then the density of $G \cap S$ in $G$ can be expressed as the density of $S$ in $A$ multiplied by a correction factor that can be written explicitly in terms of the average of local characters $\chi_p$ over $S_p$. Moreover, since $\nu = \prod_p \nu_p$, $S = \prod_p S_p$, and $A = \prod_p A_p$, the quotient $\nu(S)/\nu(A)$ can be written as a product over primes $p$.

Our goals in this paper are two-fold. In one direction, in Theorem 1.6 and Corollary 4.1, we will show how the character sums method of [14] can be adapted to directly deal with the sum obtained by replacing $\mu(n)$ in (1.4) by a general multiplicative function $g(n)$. This is an approach different from the one given in Theorem 1.1 in which a density given as a product, i.e., $\nu(S)/\nu(A)$, is corrected to another density, i.e., $\nu(G \cap S)/\nu(G)$, which is not explicitly given as an infinite sum. In another direction, we describe how the method of [14] can be adapted to derive product expressions for the general sums similar to (1.4) in which $\mu(n)$ is replaced by a multiplicative arithmetic function that could be supported on non-square free integers (all the examples given in [14] are dealing with arithmetical functions supported on square-free integers). Such arithmetic sums appear naturally on many Artin-like problems (i.e., problems related to distributions of functions of the residual indices of integers modulo primes or subsets of primes). In addition, some of them, such as Titchmarsh divisor problems for families of number fields, are not problems related to the natural density of subsets of integers. In this direction, our Theorem 1.2 provides a product formula for the constant appearing in the Generalized Artin Problem for multiplicative functions $f$ (see Problem 1.5) in full generality.
We continue with our general setup. Let \( a = \pm a_0^2 \), where \( e \) is the largest positive integer such that \(|a|\) is a perfect \( e \)-th power, and \( \text{sign}(a_0) = \text{sign}(a) \). (Note that the exponent \( e \) can differ from the exponent \( h \) defined at the beginning of the introduction. For example, if \( a = -3^2 \), then \( e = 2 \), however, \( h = 1 \).) In our arguments, the integer \( a \) is fixed; so we suppress the dependency on \( a \) in most of our notations. We fix a solution of the equation \( x^2 - a_0 = 0 \) and denote it by \( a_0^{1/2} \). The quadratic field \( K = \mathbb{Q}(a_0^{1/2}) \), the so-called entanglement field in \([14] \text{ p. 495}\), plays an important role in our arguments (see Section 2 for the justification of the terminology). We denote the discriminant of \( K \) by \( D \). Observe that for an integer \( a \neq 0, \pm 1 \), we have three different cases based on the parity of the exponent \( e \) and the sign of \( a \):

(i) Odd exponent case, in which \( e \) is odd;
(ii) Square case, in which \( e \) is even and \( a > 0 \);
(iii) Twisted case, in which \( e \) is even and \( a < 0 \).

We refer to cases (i) and (ii) as untwisted cases. Note that for odd exponent case \( K = K_2 \), for square case \( K_2 = \mathbb{Q} \) and \( K \neq K_2 \), and for twisted case \( K_2 = \mathbb{Q}(i) \) and \( K \neq K_2 \).

For a Kummer family \( \{K_n\} \), the Galois elements in \( G(n) = \text{Gal}(K_n/\mathbb{Q}) \) are determined by their actions on the \( n \)-th roots of \( a \) and the \( n \)-th roots of unity. Thus, any Galois automorphism can be realized as a group automorphism of the multiplicative group

\[
R_n = \{ \alpha \in \mathbb{Q}^\times; \; \alpha^n \in \langle a \rangle \},
\]

the group of \( n \)-radicals of \( a \). This yields the injective homomorphisms

\[
(1.6) \quad r_n : G(n) \rightarrow A(n) := \text{Aut}_{\mathbb{Q}^\times} A_n(R_n),
\]

where \( A(n) \) is the group of automorphisms of \( R_n \) fixing elements of \( \mathbb{Q}^\times \). For \( n = \prod p^k \in \mathbb{N} \), we have \( A(n) \cong \prod p^k \in A(p^k) \). Let \( \nu_p(e) \) denote the multiplicity of \( p \) in \( e \). Let \( \Phi(n) \) be the Euler totient function. For odd \( p \),

\[
\#A(p^k) = p^{k - \min\{k, \nu_p(e)\}} \Phi(p^k),
\]

and for \( p = 2 \),

\[
\#A(2^k) = \begin{cases} 
2^{k - \min\{k, s-1\}} \Phi(2^k) & \text{if } e \text{ is odd or } a > 0, \\
2^{k - \min\{k, s-1\}} \Phi(2^{k+1}) & \text{if } e \text{ is even and } a < 0,
\end{cases}
\]

where

\[
(1.7) \quad s = \begin{cases} 
\nu_2(e) + 1 & \text{if } e \text{ is odd or } a > 0, \\
\nu_2(e) + 2 & \text{if } e \text{ is even and } a < 0.
\end{cases}
\]

In particular, \( \#A(p) = \#G(p) \). (See Proposition 3.1 for a proof of these claims.)

The following theorem, related to the family of Kummer fields \( K_n \), gives us the product expressions of a large family of summations involving the orders of the Galois groups of \( K_n/\mathbb{Q} \).

**Theorem 1.2.** Let \( a = \pm a_0^2 \), where \( a_0 \) and \( e \) are defined as above, \( K = \mathbb{Q}(a_0^{1/2}) \), and let \( D \) be the discriminant of \( K \). Let \( g \) be a multiplicative arithmetic function such that

\[
\sum_{n \geq 1} \frac{|g(n)|}{\#G(n)} < \infty,
\]

where \( G(n) = \text{Gal}(\mathbb{Q}(\zeta_n, a^{1/n})/\mathbb{Q}) \) for all \( n \geq 1 \). Let \( A(n) \) be as defined above. Then,

\[
(1.8) \quad \sum_{n=1}^{\infty} \frac{g(n)}{\#G(n)} = \prod_p \sum_{k \geq 0} \frac{g(p^k)}{\#A(p^k)} + \prod_p \sum_{k \geq \ell(p)} \frac{g(p^k)}{\#A(p^k)},
\]
where

\[
\ell(p) = \begin{cases} 
0 & \text{if } p \text{ is odd and } p \nmid D, \\
1 & \text{if } p \text{ is odd and } p \mid D, \\
s & \text{if } p = 2 \text{ and } D \text{ is odd}, \\
\max\{2, s\} & \text{if } p = 2 \text{ and } 4 \nmid D, \\
2 & \text{if } p = 2, 8 \mid D, \text{ and } (\nu_2(e) = 1 \text{ and } a < 0), \\
\max\{3, s\} & \text{if } p = 2, 8 \mid D, \text{ and } (\nu_2(e) \neq 1 \text{ or } a > 0). 
\end{cases}
\]

Remarks 1.3. (i) In the summation (1.4) appearing in Artin’s primitive root conjecture, we have \( g(n) = \mu(n) \). In this case, formula (1.8) for \( g(n) = \mu(n) \) provides a unified way of expressing the constant in Artin’s primitive root conjecture as a sum of products over primes. Note that if \( e \) is even and \( a > 0 \), i.e., \( a \) is a perfect square, we have

\[
\sum_{k \geq 0} \frac{\mu(2^k)}{#A(2^k)} = 0 \quad \text{and} \quad \sum_{k \geq \ell(2)} \frac{\mu(2^k)}{#A(2^k)} = 0,
\]

where the first sum is zero since \( #A(2) = 1 \), and the second sum is zero since \( \ell(2) \geq 2 \). Hence, \((1.8)\) vanishes. Also, if \( e \) is even and \( a < 0 \), then \( \ell(2) \geq 2 \). Thus, \((1.8)\) reduces to \((1.1)\). If \( e \) is odd and \( D \) is even, then again \( \ell(2) \geq 2 \) and \((1.8)\) reduces to \((1.1)\). The only remaining case is when \( e \) is odd and \( D \) is odd (equivalently \( e \) odd and \( a_{sf} = 1 \text{ (mod } 4) \)), where \((1.8)\) reduces to \( E_a \cdot A_a \) given in \((1.2)\).

(ii) As a consequence of Theorem 1.1, we can derive necessary and sufficient conditions for the vanishing of

\[
\sum_{n=1}^{\infty} \frac{g(n)}{#G(n)}.
\]

More precisely, \((1.9)\) vanishes if and only if one of the following holds:

(a) For a prime \( p \nmid 2D \), we have \( \sum_{k \geq 0} \frac{g(p^k)}{#A(p^k)} = 0 \).

(b) We have

\[
\prod_{p \mid 2D} \sum_{k \geq 0} \frac{g(p^k)}{#A(p^k)} + \prod_{p \mid 2D, k \geq \ell(p)} \frac{g(p^k)}{#A(p^k)} = 0.
\]

In the case of Artin’s conjecture, (a) is never satisfied and (b) holds if and only if \( a \) is a perfect square.

(iii) If \( #G(n) \) were a multiplicative function, then the sum in \((1.8)\) would have been equal to the product \( \prod_{p} \frac{g(p^k)}{#G(p^k)} \). However, this is not the case for the Kummer family, and thus, the sum in \((1.8)\) may differ from the above naive product. If the sum and the product are not equal, then a complex number \( E_{a,g} \) is called a correction factor if

\[
\sum_{n=1}^{\infty} \frac{g(n)}{#G(n)} = E_{a,g} \prod_{p} \sum_{k \geq 0} \frac{g(p^k)}{#G(p^k)}.
\]

The expression \((1.8)\) provides precise information on the correction factor \( E_{a,g} \). In fact, if \( \sum_{k \geq 0} \frac{g(p^k)}{#G(p^k)} \neq 0 \) for all primes \( p \mid 2D \), we have

\[
\sum_{n=1}^{\infty} \frac{g(n)}{#G(n)} = \left( \prod_{p \mid 2D} \sum_{k \geq 0} \frac{g(p^k)}{#A(p^k)} + \prod_{p \mid 2D, k \geq \ell(p)} \frac{g(p^k)}{#A(p^k)} \right) \prod_{p} \sum_{k \geq 0} \frac{g(p^k)}{#G(p^k)}.
\]
On the other hand, if \( \sum_{k \geq 0} \frac{g(p^k)}{#G(p^k)} = 0 \) for some prime \( p \mid 2D \), and \( \sum_{n \geq 1} \frac{g(n)}{#G(n)} \neq 0 \), then the product \( \prod_p \sum_{k \geq 0} \frac{g(p^k)}{#G(p^k)} \) cannot be corrected.

(iv) For integer \( a \neq 0, \pm 1 \), let \( n_a = \prod_{p|2D} p^{\ell(p)} \), where \( D \) and \( \ell(p) \) are as in Theorem [1.2]. Then, by taking \( g(n) = 1/n^2 \), for \( \Re(z) > 0 \), in Theorem [1.2] and comparing the coefficients of \( 1/n^2 \) in both sides of (1.8), we get

\[
[Q(\zeta_n, a^{1/n}) : Q] = \begin{cases} 
\#A(n) & \text{if } n_a \nmid n, \\
\frac{1}{2}\#A(n) & \text{if } n_a \mid n.
\end{cases}
\]

The formula (1.8) can be used to study the constants in many Artin-like problems. We next apply this formula in the computation of the average value of a specific arithmetic function attached to a Kummer family. More precisely, for \( \{K_n := Q(\zeta_n, a^{1/n})\}_{n \geq 1} \), we define

\[
\tau_a(p) = \# \{ n \in \mathbb{N}; \ p \text{ splits completely in } K_n/Q \}.
\]

The Titchmarsh divisor problem attached to a Kummer family concerns the behaviour of \( \sum_{p \leq x} \tau_a(p) \) as \( x \to \infty \) (see [1] for the motivation behind this problem and its relation with the classical Titchmarsh divisor problem on the average value of the number of divisors of shifted primes). Under the assumption of the GRH for the Dedekind zeta function of \( K_n/Q \) for \( n \geq 1 \), Felix and Murty [8 Theorem 1.6] proved that

\[
(1.10) \quad \sum_{p \leq x} \tau_a(p) \sim \left( \sum_{n \geq 1} \frac{1}{[K_n : Q]} \right) \cdot \text{li}(x),
\]

as \( x \to \infty \), where \( \text{li}(x) = \int_2^x \frac{1}{\log t} dt \). They do not provide an Euler product expression for the constant appearing in the main term of (1.10). As a direct consequence of Theorem [1.2] with \( g(n) = 1 \), we readily find an explicit product formula for the constant appearing in (1.10).

**Proposition 1.4.** Let \( a = \pm a_0^e \) with \( e = \prod_p p^{\nu_p(e)} \), and let \( D \) be the discriminant of \( K = Q(a_0^{1/2}) \). Then, if \( e \) is odd or \( a > 0 \),

\[
(1.11) \quad \sum_{n \geq 1} \frac{1}{[K_n : Q]} = \left( 1 + \frac{c_0}{3 \cdot 2^{\nu_2(e)} - 2} \prod_{p \mid D} \frac{p^{\nu_p(e)+2} + p^{\nu_p(e)+1} - p^2}{p^{\nu_p(e)+3} + p^{\nu_p(e)} - p^2} \right) 
\times \prod_p \left( 1 + \frac{p^{\nu_p(e)+2} + p^{\nu_p(e)+1} - p^2}{p^{\nu_p(e)}(p-1)(p^2-1)} \right),
\]

where

\[
c_0 = \begin{cases} 
1/4 & \text{if } 4 \parallel D \text{ and } \nu_2(e) = 0, \text{ or if } 8 \parallel D \text{ and } \nu_2(e) = 1, \\
1/16 & \text{if } 8 \parallel D \text{ and } \nu_2(e) = 0, \\
1 & \text{otherwise.}
\end{cases}
\]

If \( e \) is even and \( a < 0 \) (i.e., the twisted case)

\[
(1.12) \quad \sum_{n \geq 1} \frac{1}{[K_n : Q]} = \left( 1 + \frac{c_0}{3 \cdot 2^{\nu_2(e)+2} - 2} \prod_{p \parallel D} \frac{p^{\nu_p(e)+2} + p^{\nu_p(e)+1} - p^2}{p^{\nu_p(e)+3} + p^{\nu_p(e)} - p^2} \right) 
\times \left( 1 + \frac{2^{\nu_2(e)+2} - 2^{\nu_2(e)}}{3 \cdot 2^{\nu_2(e)}} \right) \prod_{p \parallel 2} \left( 1 + \frac{p^{\nu_p(e)+2} + p^{\nu_p(e)+1} - p^2}{p^{\nu_p(e)}(p-1)(p^2-1)} \right),
\]
where
\[ c_0 = \begin{cases} 
4 & \text{if } 8 \nmid D \text{ and } \nu_2(e) = 1, \\
1 & \text{otherwise}.
\end{cases} \]

Let \( c_a \) denote the constant given in (1.11) and (1.12). We can write \( c_a = q_a \cdot u \), where \( q_a \) is a rational number depending on \( a \), and \( u \) is the universal constant
\[(1.13) \quad \sum_{n=1}^{\infty} \frac{1}{n \Phi(n)} = \prod_p \left(1 + \frac{p}{(p-1)(p^2-1)} \right) = 2.203856 \ldots,\]
where \( \Phi(n) \) is the Euler totient function. Note that \( n \Phi(n) \) is the “generic/expected” degree of the extension \( \mathbb{Q}(\zeta, a^{1/n}) / \mathbb{Q} \); however, this may not be the case due to entanglement phenomena. Moreover, observe that if \( \nu_p(e) = 0 \) for all \( p \), the expressions for products over all primes \( p \) given in (1.11) and (1.12) reduce to (1.13). This is in accordance with [2, Theorem 1.4] in which (1.13) appears as the average constant while varying \( a \). Thus, on average over \( a \) the universal constant appears. The product expressions of Proposition 1.4 provide a convenient way of computing the numerical value of \( c_a \) for a given value of \( a \). For example \( c_2 = c_{-2} = 2.258 \ldots \).

The classical Artin conjecture and the Titchmarsh divisor problem for a Kummer family are instances of a more general problem that we now describe. For an integer \( a \neq 0, \pm 1 \) and a prime \( p \nmid a \), the residual index of \( a \mod p \), denoted by \( i_a(p) \), is the index of the subgroup \( \langle a \rangle \) in the multiplicative group \( (\mathbb{Z}/p\mathbb{Z})^\times \). There is a vast amount of literature on the study of asymptotics of functions of \( i_a(p) \) as \( p \) varies over primes. In [16, p. 377], the following problem is proposed.

**Problem 1.5** (Generalized Artin Problem). Determine integers \( a \) and arithmetic functions \( f(n) \) for which the asymptotic formula
\[ \sum_{p \leq x} f(i_a(p)) \sim c_{f,a} \text{li}(x), \]
as \( x \to \infty \), hold, where
\[(1.14) \quad c_{f,a} := \sum_{n \geq 1} \frac{g(n)}{[K_n : \mathbb{Q}]}. \]
Here \( g(n) = \sum_{d|n} \mu(d)f(n/d) \) is the Möbius inverse of \( f(n) \), where \( \mu(n) \) is the Möbius function.

Note that by setting \( f(n) \) as the characteristic function of the set \( S = \{1\} \), hence \( g(n) = \mu(n) \), in Problem 1.5 we get the Artin conjecture, and \( f(n) = d(n) \) (the divisor function), hence \( g(n) = 1 \), gives the Titchmarsh divisor problem for a Kummer family, this is true since \( \tau_a(p) = d(i_a(p)) \) (see [8, Lemma 2.1] for details). Also, a conjecture of Laxton from 1969 (see [13] and [20, p. 313]) predicts that for \( f(n) = 1/n \), the generalized Artin problem determines the density of primes in the sequence given by the recurrence \( w_{n+2} = (a + 1)w_{n+1} - aw_n \), where \( a > 1 \) is a fixed integer. Another instance of Problem 1.5 appears in a conjecture of Bach, Lukes, Shallit, and Williams [4] in which the constant \( c_{f,2} \) for \( f(n) = \log n \) appears in the main term of the asymptotic formula for \( \log P_2(x) \), where \( P_2(x) \) is the smallest \( x \)-pseudopower of the base 2.

A notable result on the Generalized Artin Problem, due to Felix and Murty [8, Theorem 1.7], establishes, under the assumption of GRH, the asymptotic
\[(1.15) \quad \sum_{p \leq x} f(i_a(p)) = c_{f,a} \text{li}(x) + O_{\alpha} \left( \frac{x}{(\log x)^{2-\epsilon-\alpha}} \right), \]
for \( \epsilon > 0 \). Here \( f(n) \) is an arithmetic function whose Möbius inverse \( g(n) \) satisfies
\[ |g(n)| \ll d_k(n)^{r}(\log n)^{\alpha}, \]
with \( k, r \in \mathbb{N} \) and \( 0 < \alpha < 1 \) all fixed, where \( d_k(n) \) denotes the number of representations of \( n \) as product of \( k \) positive integers. Observe that the identity (1.8) in Theorem 1.2 conveniently furnishes a product formula in full generality for the constant \( c_{f,a} \) in (1.15) when \( f \) (equivalently \( g \)) is a multiplicative function. This product formula is valuable for studying the vanishing criteria for \( c_{f,a} \) and their numerical evaluations for different \( f \).

We now comment on the proof of Theorem 1.2. Observe that the Kummer family \( \{K_n\} \), we can consider the inverse systems \((G(n))_{n \in \mathbb{N}}, (i_{n_1,n_2})_{n_1,n_2}\) and \((A(n))_{n \in \mathbb{N}}, (j_{n_1,n_2})_{n_1,n_2}\) ordered by divisibility relation on \( \mathbb{N} \), where \( G(n) \) and \( A(n) \) are as defined before and \( i_{n_1,n_2} : G(n_2) \to G(n_1) \) and \( j_{n_1,n_2} : A(n_2) \to A(n_1) \), for \( n_1 | n_2 \), are restriction maps. By taking the inverse limits on both sides of (1.6) we have the injective continuous homomorphism

\[
r : G = \lim_{\leftarrow} G(n) \to A = \lim_{\leftarrow} A(n)
\]

of profinite groups, where \( G = \text{Gal}(K_X/\mathbb{Q}) \) and \( A = \text{Aut}_{\mathbb{Q}^\times \cap R_X}(R_X) \) with \( K_X = \bigcup_{n \geq 1} K_n \) and \( R_X = \bigcup_{n \geq 1} R_n \). As profinite groups, both \( G \) and \( A \) are endowed with topologies that make \( G \) and \( A \) into compact topological spaces, and thus, they can be equipped by Haar measures. We will show that Theorem 1.2 is a corollary of the following theorem attached to a general setting of profinite groups \( G \) and \( A \).

**Theorem 1.6.** Let \( ((G(n))_{n \in \mathbb{N}}, (i_{n_1,n_2})_{n_1,n_2}) \) and \( ((A(n))_{n \in \mathbb{N}}, (j_{n_1,n_2})_{n_1,n_2}) \) be surjective inverse systems of finite groups ordered by divisibility relation on \( \mathbb{N} \). Moreover, for \( n \geq 1 \), assume that there are injective maps \( r_n : G(n) \to A(n) \) compatible with surjective transition maps \( i_{n_1,n_2} \) and \( j_{n_1,n_2} \), i.e., for \( n_1 | n_2 \), the diagram

\[
\begin{array}{ccc}
G(n_2) & \xrightarrow{r_{n_2}} & A(n_2) \\
\downarrow{i_{n_1,n_2}} & & \downarrow{j_{n_1,n_2}} \\
G(n_1) & \xrightarrow{r_{n_1}} & A(n_1)
\end{array}
\]

commutes. Let \( r : G = \lim_{\leftarrow} G(n) \to A = \lim_{\leftarrow} A(n) \) be the resulting injective continuous homomorphism of profinite groups. Let \( \mu_m \) be the multiplicative group of \( m \)-th roots of unity for a fixed \( m \). Suppose there exists an exact sequence

\[
1 \to G \xrightarrow{r} A \xrightarrow{\chi} \mu_m \to 1,
\]

where \( \chi \) is a continuous homomorphism. Let \( g \) be an arithmetic function such that

\[
\sum_{n \geq 1} \frac{|g(n)|}{#G(n)} < \infty.
\]

Consider the natural projections \( \pi_{A,n} : A \to A(n) \) and let

\[
\tilde{g} = \sum_{n \geq 1} g(n)\chi_{\ker \pi_{A,n}},
\]

where \( \chi_{\ker \pi_{A,n}} \) denotes the characteristic function of \( \ker \pi_{A,n} \). Let \( \nu_A \) be the normalized Haar measure attached to \( A \). Then, \( \tilde{g} \in L^1(\nu_A) \) (the space of \( \nu_A \)-integrable functions) and

\[
\sum_{n \geq 1} \frac{g(n)}{#G(n)} = \sum_{i=0}^{m-1} \int_A \tilde{g} \chi^i d\nu_A.
\]

Observe that Theorem 1.6 is quite general and can be applied in the evaluation of sums of the form \( \sum_{n \geq 1} g(n)/#G(n) \) for any family \( \{G(n)\} \) of finite groups satisfying the assumptions of the theorem. The family of Galois groups of a Kummer family is an instance of such families. Another example is the family of Galois groups of the division fields attached to a *Serre elliptic curve* \( E \) (see
Section 7 for the definition). Following [14, Section 8], in Section 7 we show that the family of Galois groups of the division fields \( \{ \mathbb{Q}(E[n]) \} \) attached to a Serre curve \( E \) satisfies the conditions of Theorem 1.6 and we deduce the following proposition.

**Proposition 1.7.** Let \( \mathbb{Q}(E[n]) \) denote the \( n \)-division field of a Serre elliptic curve defined over \( \mathbb{Q} \). Let \( \Delta \) be the discriminant of any Weierstrass model for \( E \). Let \( D \) be the discriminant of the quadratic field \( K = \mathbb{Q}(\Delta^{1/2}) \). Let \( g(n) \) be a multiplicative arithmetic function such that

\[
\sum_{n \geq 1} \frac{|g(n)|}{[\mathbb{Q}(E[n]) : \mathbb{Q}]} < \infty.
\]

Then,

\[
\sum_{n=1}^\infty \frac{g(n)}{[\mathbb{Q}(E[n]) : \mathbb{Q}]} = \prod_p \left( 1 + \sum_{k \geq 0} \frac{g(p^k)}{\# \text{Aut}(E[p^k])} \right) + \prod_p \sum_{k \geq 0} \frac{g(p^k)}{\# \text{Aut}(E[p^k])},
\]

where

\[
\ell(p) = \begin{cases} 
0 & \text{if } p \text{ is odd and } p \nmid D, \\
1 & \text{if } p \text{ is odd and } p \mid D, \\
1 & \text{if } p = 2 \text{ and } D \text{ is odd}, \\
2 & \text{if } p = 2 \text{ and } 4 \mid D, \\
3 & \text{if } p = 2 \text{ and } 8 \mid D,
\end{cases}
\]

and

\[
\# \text{Aut}(E[p^k]) = \begin{cases} 
p^{4k-3}(p^2 - 1)(p - 1) & \text{if } k \geq 1, \\
1 & \text{if } k = 0.
\end{cases}
\]

Note that for a Serre curve, \( D \neq 1 \) (see [14, p. 510]) and thus \( K \) is a quadratic field. Also, observe that the above proposition for \( g(n) = 1 \) reduces to the product expression of the Titchmarsh divisor problem for the family of division fields attached to a Serre curve \( E \). We note that the product expression for this constant and two other constants corresponding to different \( g(n) \)'s for such families are given in [5] Theorem 5 by determining the value of \([\mathbb{Q}(E[n]) : \mathbb{Q}]\) for a Serre curve \( E \) (see [5] Proposition 17 (iv)) and employing [12, Lemma 3.12]. It is worth mentioning that a similar approach in finding the expression (1.11) using the exact formulas for \([K_n : \mathbb{Q}]\) as given in [22] Proposition 4.1 will result in the tedious case by case computations that does not appear to be straightforward. Especially when \( a < 0 \), this approach seems to be intractable. The method of [14] as described above provides an elegant approach to establishing identities similar to (1.11) and (1.12).

The structure of the paper is as follows. We describe our adaptation of the character sums method of [14] in Sections 2 and 3 and prove Proposition 3.3 that plays a crucial role in our explicit computation of the constants in the Kummer case. Section 4 is dedicated to a proof of Theorem 1.6. The proofs of Theorem 1.2 and its consequence, Proposition 1.4 are given respectively in Sections 5 and 6. Finally, a brief discussion on Serre curves and the proof of Proposition 1.7 are provided in Section 7.

**Notations 1.8.** The following notations are used throughout the paper. The letter \( p \) denotes a prime number, \( k \) denotes a non-negative integer, the letter \( n \) denotes a positive integer, the multiplicity of the prime \( p \) in the prime factorization of \( n \) is denoted by \( \nu_p(n) \), the cardinality of a finite set \( S \) is denoted by \( \# S \), \( 1_S \) is the characteristic function of a set \( S \), \( \overline{\mathbb{Q}} \) is an algebraic closure of \( \mathbb{Q} \), \( \zeta_n \) denotes a primitive root of unity in \( \overline{\mathbb{Q}} \), and \( \Phi(n) \) is the Euler totient function. In Sections 4, 5 and 6, \( a = \pm a_0 \), with \( \text{sign}(a_0) = \text{sign}(a) \), is a non-zero integer other than \( \pm 1 \), the collection \( \{ K_n = \mathbb{Q}(\zeta_n, a^{1/n}) \}_{n \in \mathbb{N}} \) is the family of Kummer fields and \( K = \mathbb{Q}(a_0^{1/2}) \) is the entanglement field attached to this family. \( D \) is the discriminant of \( K \), the Galois group of \( K_n \) over \( \mathbb{Q} \) is denoted by
G(n), the inverse limit of the directed family \( \{G(n)\} \) is denoted by \( G, \mu_\infty \subseteq \overline{Q} \) denotes the group of all roots of unity, and \( Q_{ab} = Q(\mu_\infty) \) is the maximal abelian extension of \( Q \). The group of \( n \)-radicals of the integer \( a = \pm a_0^n \) is denoted by \( R_n \) and \( R_\infty = \bigcup_{n \geq 1} R_n \). The group of automorphisms of \( R_n \) (respectively \( R_\infty \)) that fix \( Q^\times \) is denoted by \( A(n) \) (respectively \( A \)). The inverse limit of the system \( \{A(p^k)\}_{k \geq 1} \) is denoted by \( A_p \). The map \( \pi_{A,n} \) (respectively \( \varphi_{p^k} \)) is the projection map from \( A \) (respectively \( G \) and \( A_p \)) to \( A(n) \) (respectively \( G(n) \) and \( A(p^k) \)). The profinite completion of \( Z \) is denoted by \( \hat{Z} \) and \( Z_n \) is the ring of \( p \)-adic integers. The normalized Haar measures on \( G, A \), and \( A_p \) are denoted respectively by \( \nu_G, \nu_A \), and \( \nu_{A_p} \). The space of \( \nu \)-integrable functions is denoted by \( L^1(\nu) \). In Section 4, \( G(n) \), \( A(n), A(p^k) \), \( G, \hat{A}, A_p, \pi_{A,n}, \varphi_{p^k}, \nu_G, \nu_A \), and \( \nu_{A_p} \) are used in the general setting of profinite groups. Finally, in Section 7, \( E[n] \) denotes the group of \( n \)-division points over \( \overline{Q} \) of an elliptic curve \( E \) defined over \( \mathbb{Q} \) given by a Weierstrass equation with discriminant \( \Delta \), and \( K = Q(\Delta^{1/2}) \) of discriminant \( D \) is the entanglement field attached to the family of division fields of a Serre elliptic curve. We denote the group of automorphisms of \( E[n] \) by \( \text{Aut}(E[n]) \) and the multiplicative group of \( 2 \times 2 \) matrices with entries in \( \mathbb{Z}/n\mathbb{Z} \) by \( \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \).

2. The Associated Character to a Kummer Family

Recall that for an integer \( a \neq 0, \pm 1 \), we set \( a = \pm a_0^n \) where \( \text{sign}(a) = \text{sign}(a_0) \) and \( e \) is the largest such integer. We have the group embedding \( Q \cong \langle a_0 \rangle \subset \overline{Q}^\times \) defined by sending 1 to \( a_0 \). Since \( \overline{Q}^\times \) is a divisible group, we can extend this embedding to an embedding \( Q \to \overline{Q}^\times \). For \( a_0 \), we fix such embedding \( q \to a_0^n \) and write \( a_0^n \) for the image of this embedding in \( \overline{Q}^\times \). We denote the image of 1/2 in this embedding by \( a_0^{1/2} \).

We next define a quadratic character which describes the entanglements within a given Kummer family \( \{K_n\} \). Let \( \mu_\infty = \bigcup_{n \geq 1} \mu_n(\overline{Q}) \) be the group of all roots of unity in \( \overline{Q} \). Then, \( \mu_\infty \) is contained in \( K_\infty = \bigcup_{n \geq 1} K_n \). In addition, the infinite extension \( K_\infty/Q \) is the compositum of \( Q(a_0^n) \) and \( Q_{ab} \) (the maximal abelian extension of \( Q \)), where

\[
\text{Hom}(a_0^n, Q_{ab}) = \mathbb{Q}(a_0^{1/2})
\]

(see [14, Lemma 2.5] and the discussion on the last paragraph of [14, p. 494]). The equality (2.1) demonstrates the entanglement of two fields \( \mathbb{Q}(a_0^n) \) and \( Q_{ab} \) since their intersection is a non-trivial extension of \( \mathbb{Q} \) (i.e., the field \( \mathbb{Q}(a_0^{1/2}) \)). This justifies calling \( K = \mathbb{Q}(a_0^{1/2}) \) the entanglement field. In [14, p. 494] it is proved that

\[
A = \text{Aut}_{Q_\infty \times R_\infty}(R_\infty) \cong \text{Hom}(a_0^n/a_0^2, \mu_\infty) \rtimes \text{Aut}(\mu_\infty),
\]

where \( a_0^n = \{a_0^n \mid b \in \mathbb{Z}\} \), and for \( (\phi_1, \sigma_1), (\phi_2, \sigma_2) \in A \) we have

\[
(\phi_1, \sigma_1)(\phi_2, \sigma_2) = (\phi_1 \cdot (\sigma_1 \circ \phi_2), \sigma_1 \circ \sigma_2).
\]

Note that \( G = \text{Gal}(K_\infty/Q) \) can be embedded in \( A \). Thus, if \( (\phi, \sigma) \in \text{Hom}(a_0^n/a_0^2, \mu_\infty) \rtimes \text{Aut}(\mu_\infty) \) is an element of \( G \), then, by (2.1), the action of \( \phi \) and \( \sigma \) on \( a_0^{1/2} \) must be the same. One can show that \( (\phi, \sigma) \in A \) is in \( A \) if and only if \( \phi \) and \( \sigma \) act in a compatible way on \( a_0^{1/2} \), i.e.,

\[
\phi(a_0^{1/2}) = \frac{\sigma(a_0^{1/2})}{a_0^{1/2}} \in \mu_2
\]

(see [14, p. 494]). (For simplicity, we used \( \phi(a_0^{1/2}) \) instead of \( \phi(a_0^{1/2})a_0^2 \)). We elaborate on (2.3) by considering two distinct quadratic characters \( \psi_K \) and \( \chi_D \) on \( A \) which are related to the entanglement field \( K = \mathbb{Q}(a_0^{1/2}) \) of discriminant \( D \). The quadratic character \( \psi_K : A \to \mu_2 \) corresponds to the
action of \( \phi \)-component of \((\phi, \sigma) \in A \) on \( a_0^{1/2} \), i.e.,

\[
\psi_K(\phi, \sigma) = \phi(a_0^{1/2}) \in \mu_2.
\]

This is a non-cyclotomic character, i.e., \( \psi_K \) does not factor via the natural map \( A \to \text{Aut}(\mu_\infty) \) (see [14, p. 495]). The other quadratic character,

\[
\chi_D : A \to \text{Aut}(\mu_\infty) \cong \hat{\mathbb{Z}}^\times \to \mu_2,
\]

corresponds to the action of the cyclotomic component \( \text{Aut}(\mu_\infty) \) of \( A \) on \( K = \mathbb{Q}(a_0^{1/2}) \) of discriminant \( D \), i.e.,

\[
\chi_D(\phi, \sigma) = \frac{\sigma(a_0^{1/2})}{a_0^{1/2}} \in \mu_2.
\]

Hence, by [6, Proposition 5.16 and Corollary 5.17], \( \chi_D \) factors via the lift of the Kronecker symbol \( (D/p) \) to \( \text{Aut}(\mu_\infty) \cong \hat{\mathbb{Z}}^\times \).

The characters \( \chi_D \) and \( \psi_K \) are not the same on \( A \) since one is cyclotomic, and the other is not. Moreover, by (2.3), an element \( x \in A \) is in \( G \) if and only if \( \psi_K(x) = \chi_D(x) \). Thus, if \( r : G \to A \) is the natural embedding defined in [14, p. 493], the image of \( r \) is the kernel of the non-trivial quadratic character \( \chi = \psi_K \cdot \chi_D : A \to \mu_2 \). In other words, the sequence

\[
1 \to G \xrightarrow{r} A \xrightarrow{\chi=\psi_K \cdot \chi_D} \mu_2 \to 1
\]

is an exact sequence (see [14, Theorem 2.9]).

Let \( A(p^k) = \text{Aut}_{\mathbb{Q}(\sqrt{D})}(R_{p^k}) \) and \( A_p = \varprojlim A(p^k) \). Since an element of \( A \) can be determined by its action on prime power radicals, we have that \( A \cong \prod_p A_p \) (see [14 formula (2.10), p. 495] and [15, p. 20]). The character \( \chi_D \) is the lift of the Kronecker symbol \( (D/p) \) to \( A \via \) the maps

\[
A \cong \left( \prod_p A_p \right)^{\text{proj}} \xrightarrow{\text{proj}} \text{Aut}(\mu_\infty) \left( \cong \prod_p \mathbb{Z}_p^\times \right) \xrightarrow{\text{proj}} (\mathbb{Z}/|D|\mathbb{Z})^\times,
\]

where the first projection comes via (2.2). Since \( D \) is a fundamental discriminant, \( \chi_D = \prod_{p|D} \chi_{D,p} \) where \( \chi_{D,p} \) is the lift of the Legendre symbol modulo \( p \) to \( A_p \) for odd \( p \), and \( \chi_{D,2} \) is the lift of one of the Dirichlet characters mod 8 to \( A_2 \) (see [7, Chapter 5]). More precisely, if \( D \) is odd, then \( \chi_{D,2} = 1 \); if \( 4 \parallel D \), then \( \chi_{D,2} \) is the lift to \( A_2 \) of \( (\frac{\hat{a}}{2}) \), the unique Dirichlet character mod 8 of conductor 4; and if \( 8 \parallel D \), then \( \chi_{D,2} \) is the lift to \( A_2 \) of \( (\frac{\hat{a}}{8}) \), one of the two Dirichlet characters mod 8 of conductor 8. For the case \( 8 \parallel D = \pm 2^a \prod_{i=1}^k p_i \), if \( D > 0 \) and the number of \( 1 \leq i \leq k \) with \( p_i \equiv 3 \pmod{4} \) is even, or \( D < 0 \) and the number of \( 1 \leq i \leq k \) with \( p_i \equiv 3 \pmod{4} \) is odd, then \( \chi_{D,2} \) is the lift to \( A_2 \) of \( (\frac{\hat{a}}{2}) \). Otherwise, \( \chi_{D,2} \) is the lift to \( A_2 \) of \( (\frac{\hat{a}}{8}) \).

Next, we show that \( \chi \) can be written as a product of local characters \( \chi_p : A_p \to \mu_2 \). Note that \( \psi_K \) factors via \( A_2 \). Let \( \psi_{K,2} : A_2 \to \mu_2 \) be the corresponding homomorphism obtained from factorization of \( \psi_K \) via \( A_2 \). For odd primes \( p \nmid D \), set \( \chi_p = 1 \). Let \( \chi_p = \chi_{D,p} \) for odd primes \( p \mid D \) and for prime 2 let \( \chi_2 = \chi_{D,2} : \psi_{K,2} \). Therefore, by the above construction of \( \chi \), we have the decomposition \( \chi = \prod_p \chi_p \).

3. The local characters \( \chi_p \)

In this section, we find the smallest values of \( k \), as a function of \( p \) and \( a \), for which the local character \( \chi_p \) factors via \( A(p^k) \). In other words, we will determine the values of \( k \) for which \( \chi_p \) is trivial on \( \ker \varphi_{p^k} \) and it is nontrivial on \( \ker \varphi_{p^{k-1}} \), where \( \varphi_{p^k} \) is the projection map from \( A_p \) to \( A(p^k) \). The values are recorded in the statements of Theorem 1.2 and Proposition 3.3. We start by giving a concrete description of the groups \( A(p^k) \), for positive integers \( k \), as subgroups of matrices \( \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \).
where \( b \in \mathbb{Z}/p^k\mathbb{Z} \) and \( d \in \left( \mathbb{Z}/p^k\mathbb{Z} \right)^\times \). We achieve this by choosing a certain compatible system of generators for the groups \( R_{p^k} \), where \( k \geq 1 \).

**Proposition 3.1.** (i) If \( e \) is even and \( a < 0 \), by choosing a suitable set of generators for \( R_{2^e} \), we have that
\[
A(2^k) \cong \left\{ \begin{array}{ll}
1 & b \in \mathbb{Z}/2^k\mathbb{Z}, d \in \left( \mathbb{Z}/2^k\mathbb{Z} \right)^\times, \text{ and } 2b + 1 \equiv d \pmod{2^{\min(k,\nu_2(e)+1)}}
\end{array} \right\}.
\]

(ii) If \( p \) is odd, or \( p = 2 \) and \( e \) is odd, or \( p = 2 \) and \( a > 0 \), by choosing a suitable set of generators for \( R_{p^k} \), we have that
\[
A(p^k) \cong \left\{ \begin{array}{ll}
1 & b \in \mathbb{Z}/p^k\mathbb{Z}, d \in \left( \mathbb{Z}/p^k\mathbb{Z} \right)^\times, \text{ and } 2b + 1 \equiv d \pmod{p^{\min(k,\nu_p(e)+1)}}
\end{array} \right\}.
\]

(iii) Let \( \Phi(n) \) be the Euler totient function and \( s \) be as defined in (1.7). For odd \( p \),
\[
\#A(p^k) = p^{k-\min(k,\nu_p(e)+1)}\Phi(p^k),
\]
and for \( p = 2 \),
\[
\#A(2^k) = \begin{cases}
2^{k-\min(k,s-1)}\Phi(2^k) & \text{if } e \text{ is odd or } a > 0, \\
2^{k-\min(k,s-1)}\Phi(2^{k+1}) & \text{if } e \text{ is even and } a < 0.
\end{cases}
\]

**Proof.** (i) Let \( a = -a_0^e \) as before and \( e = 2^{\nu_2(e)}e_1 \), where \( \nu_2(e) \geq 1 \) and \( e_1 \) is odd. For \( q \in \mathbb{Q} \), let \( q \rightarrow a_0^e \) be the fixed embedding \( \mathbb{Q} \rightarrow \mathbb{Q}^\times \) defined at the beginning of Section 2. We also fix a collection \( \{\zeta_n; n \in \mathbb{N}\} \) of primitive roots of unity for which \( (\zeta_{mn})^m = \zeta_n \) for \( m, n \in \mathbb{N} \).

Recall that \( R_{2^k} \) is the group of \( 2^k \)-radicals. We have
\[
R_{2^k} = \langle \zeta_{2^{k+1}}(a_0^e)^{1/2^{k-\nu_2(e)}}, \zeta_{2^k} \rangle = \langle \beta, \zeta_{2^k} \rangle.
\]
An automorphism \( \tau \in A(2^k) \) is determined by its action on these generators of \( R_{2^k} \), i.e., \( \beta \) and \( \zeta_{2^k} \). We have \( \tau(\beta) = \beta\zeta_{2^k}^{d(\tau)} \) and \( \tau(\zeta_{2^k}) = \zeta_{2^k}^{d(\tau)} \), where \( b(\tau) \in \mathbb{Z}/2^k\mathbb{Z} \) and \( d(\tau) \in \left( \mathbb{Z}/2^k\mathbb{Z} \right)^\times \). We consider two cases.

Case 1: \( k \geq \nu_2(e) + 1 \). We have
\[
a_0^e \tau(\zeta_{2^{k+1}}^{2^{k-\nu_2(e)}}) = (\zeta_{2^{k+1}}^{2^{k-\nu_2(e)}})(\beta\zeta_{2^k}^{d(\tau)})^{2^{k-\nu_2(e)}} = a_0^e \zeta_{2^{k+1}}^{2^{k-\nu_2(e)}} \zeta_{2^k}^{2^{k-\nu_2(e)}}.
\]
From here we get
\[
\zeta_{2^k}^{d(\tau)} = \zeta_{2^k}^{2^{2k-\nu_2(e)-1}+b(\tau)2^{k-\nu_2(e)}}.
\]
Since \( k \geq \nu_2(e) + 1 \), this is equivalent to \( 2b(\tau) + 1 \equiv d(\tau) \pmod{2^{\nu_2(e)+1}} \).

Case 2: \( k < \nu_2(e) + 1 \). We have
\[
a_0^e \tau(\zeta_{2^{k+1}}^{2^{k-\nu_2(e)}}) = (\zeta_{2^{k+1}}^{2^{k-\nu_2(e)}})(\beta\zeta_{2^k}^{d(\tau)})^{2^{k-\nu_2(e)}} = a_0^e \zeta_{2^{k+1}}^{2^{k-\nu_2(e)}} \zeta_{2^k}^{2^{k-\nu_2(e)}}.
\]
From here we get
\[
\zeta_{2^k}^{d(\tau)} = \zeta_{2^k}^{2^{k-\nu_2(e)}+2b(\tau)}.
\]
This is equivalent to \( 2b(\tau) + 1 \equiv d(\tau) \pmod{2^k} \).

So any \( \tau \in A(2^k) \) injects to a matrix \( \begin{pmatrix} 1 & b(\tau) \\ 0 & d(\tau) \end{pmatrix} \) in the group of matrices given in part (i) of the proposition. Thus, \( A(2^k) \) is isomorphic to a subgroup of the given group of matrices. We claim that \( A(2^k) \) is, in fact, isomorphic to the whole of this group of matrices.

To prove the claimed isomorphism, it is enough to show that for any \( \tau \in A(2^k) \), the congruence
\[
2b(\tau) + 1 \equiv d(\tau) \pmod{2^{\min(k,\nu_2(e)+1)}}
\]
is the only relation between \( b(\tau) \) and \( d(\tau) \) appearing in \( \tau(\beta) = \beta\zeta_{2^k}^{d(\tau)} \) and \( \tau(\zeta_{2^k}) = \zeta_{2^k}^{d(\tau)} \). Recall that \( R_{2^k} \) is generated by \( \beta \) and \( \zeta_{2^k} \) and the elements of \( A(2^k) \) (automorphisms of \( R_{2^k} \) that fix \( \mathbb{Q}^\times \))
are determined by their actions on $\beta$ and $\zeta_{2^k}$. Also, such automorphisms should fix the rational elements of $R_{2^k}$ and preserve any relation between $\beta$ and $\zeta_{2^k}$. Since $R_{2^k}$ is a multiplicative abelian group, any such relationship should be in the form

$$(3.2) \quad \beta^m \zeta^n_{2^k} = r \in \mathbb{Q}^\times.$$ 

To study (3.2) we consider two cases.

Case (a): Let $k \geq \nu_2(e) + 1$ and $\beta^m \zeta^n_{2^k} = r \in \mathbb{Q}^\times$. Then, $|a_0^{m,n/2^k-\nu_2(e)}| = |r| \in \mathbb{Q}^\times$. Hence, $m = m_1(2^k-\nu_2(e))$, for $m_1 \in \mathbb{Z}$. Replacing this in (3.2) yields $\zeta_{2^k}^{m_1(2^k-\nu_2(e)-1)}n \in \mathbb{Q}^\times$. Hence, $n = -m_1(2^k-\nu_2(e)-1) + \ell \Phi(2^k)$, for $\ell \in \mathbb{Z}$. Thus, the relations (3.2) are in the form

$$(3.3) \quad \beta^{m_1(2^k-\nu_2(e))} \zeta_{2^k}^{-m_1(2^k-\nu_2(e)-1)} = (a_0^{e_1})^{m_1}$$

for $m_1 \in \mathbb{Z}$. (Note that $\beta^{2^k-\nu_2(e)} \zeta_{2^k}^{-2^k-\nu_2(e)-1} = a_0^{e_1}$ and thus the relations $\beta^{m_1(2^k-\nu_2(e))} \zeta_{2^k}^{-m_1(2^k-\nu_2(e)-1)} = -(a_0^{e_1})^{m_1}$ cannot happen.) Now if

$$\beta^{m_1(2^k-\nu_2(e))} \zeta_{2^k}^{-m_1(2^k-\nu_2(e)-1)} = (a_0^{e_1})^{m_1}$$

applying $\tau \in A(2^k)$ on both sides of this identity and following a computation similar to Case 1 above, we conclude that

$$(3.4) \quad 2b(\tau) + 1 \equiv d(\tau) \pmod{2^{\nu_2(e)+1-\nu_2(m_1)}}$$

for $0 \leq \nu_2(m_1) \leq \nu_2(e)$, and no condition if $\nu_2(m_1) > \nu_2(e)$. Since $m_1$ can be any arbitrary integer, then

$$(3.5) \quad 2b(\tau) + 1 \equiv d(\tau) \pmod{2^{\nu_2(e)+1}}$$

implies all the congruences (3.4).

Case (b): Let $k < \nu_2(e) + 1$ and $\beta^m \zeta^n_{2^k} \in \mathbb{Q}^\times$. Then, $\zeta_{2^k}^m \zeta_{2^k}^n \in \mathbb{Q}^\times$, which implies $\zeta_{2^k}^{m+n} = \pm 1$. Hence, $m = 2m_1$ for $m_1 \in \mathbb{Z}$, and $n = -m_1 + \ell \Phi(2^k)$ for $\ell \in \mathbb{Z}$. Thus, the relations (3.2) are in the form

$$\beta^{2m_1} \zeta_{2^k}^{-m_1} = (a_0^{e_1})^{2(2^k-\nu_2(e))}$$

for $m_1 \in \mathbb{Z}$. (Note that $\beta^{2}\zeta_{2^k}^{-1} = (a_0^{e_1})^{2/2^k-\nu_2(e)}$ and thus the relations $\beta^{2m_1} \zeta_{2^k}^{-m_1} = -(a_0^{e_1})^{2(2^k-\nu_2(e))}$ cannot happen.) Applying $\tau \in A(2^k)$ on both sides of this identity and following a computation similar to Case 2 above, we conclude that

$$(3.6) \quad 2b(\tau) + 1 \equiv d(\tau) \pmod{2^{k-\nu_2(m_1)}}$$

for $0 \leq \nu_2(m_1) \leq k - 1$, and no relation if $\nu_2(m_1) > k - 1$. Since $m_1$ can be any arbitrary integer, then

$$(3.7) \quad 2b(\tau) + 1 \equiv d(\tau) \pmod{2^k}$$

implies all these relations.

In conclusion, from (3.4), (3.5), and (3.6), we get that the congruence (3.1) is the only existing relation among the entries of matrices $\begin{pmatrix} \beta & d(\tau) \\ b(\tau) & 1 \end{pmatrix}$. Hence, the injection $\tau \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & d(\tau) \end{pmatrix}$ establishes the claimed isomorphism.

(ii) The proof is analogous to the proof of (ii) by considering compatible systems of generators for the groups $R_{p^k}$’s. More precisely, for $p = 2$, $R_{2^k} = \langle \zeta_{2^k}, (a_0^{e_1})^{1/2^k-\nu_2(e)}, \zeta_{2^k} \rangle$, where $\gcd(e_1, 2) = 1$, and, for odd $p$, $R_{p^k} = \langle \zeta_{p^k}, (\pm a_0^{e_1})^{1/p^k-\nu_p(e)}, \zeta_{p^k} \rangle$, where $\gcd(e_1, p) = 1$.

(iii) This is a consequence of parts (i) and (ii), since the sizes of the groups of matrices in parts (i) and (ii) are the same as the claimed sizes for $A(p^k)$. 

The following proposition indicates the significance of the integer $s$ defined in (1.7).
Proposition 3.2. The number $s$ defined in (1.7) is the smallest integer $k$ for which $\psi_{K,2}$ factors via $A(2^k)$.

Proof. For integers $k \geq 0$, let $\varphi_{p^k} : A_p \to A(p^k)$ be the projection map. It is enough to show that $\psi_{K,2}$ is non-trivial on $\ker \varphi_{2^{e_1}}$ and is trivial on $\ker \varphi_{2^{e_2}}$. We write the proof for the twisted case, where $s = \nu_2(e) + 2$. The proof for the untwisted case is similar.

Assume that $a = -(a_0^{e_1})^{2^{e_2}}$ as in part (i) of Proposition 3.1 and assume the compatibility conditions for roots of $a_0$ and roots of unity described there. Considering

$R_{2^{e_2}} = \langle \zeta_{2^{e_2}}^{1/4}, \zeta_{2^{e_2}}^{2} \rangle$, 

let $\alpha \in A_2$ be such that

$\tau_2 = \varphi_{2^{e_2}+2}(\alpha) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 + 2^{e_2+1} \end{array}\right) \in A(2^{e_2+2})$.

Observe that $\alpha \in \ker \varphi_{2^{e_2}+1}$ and we have

(3.7) $\tau_2(\zeta_{2^{e_2}+3}(a_0^{e_1})) = \zeta_{2^{e_2}+3}(a_0^{e_1})^{1/4}$.

Raising both sides of (3.7) to power 2 and observing that $\zeta_{2^{e_2}+2}$ and $(a_0^{e_1})^{1/2}$ are in $R_{2^{e_2}+2}$, we get

(3.8) $\tau_2(\zeta_{2^{e_2}+2})\tau_2((a_0^{e_1})^{1/2}) = \zeta_{2^{e_2}+2}(a_0^{e_1})^{1/2}$.

Now since $\tau_2(\zeta_{2^{e_2}+2}) = \zeta_{2^{e_2}+2}^{1+2^{e_2}+1} = -\zeta_{2^{e_2}+2}$, the equation (3.8) implies that

$\tau_2(a_0^{e_1/2}) = -a_0^{e_1/2}$.

Hence,

$\tau_2(a_0^{1/2}a_0^{-1/2}) = -a_0^{-1/2}a_0^{-1/2}$.

Thus for $\alpha \in \ker \varphi_{2^{e_2}+1}$, we have $\psi_{K,2}(\alpha) = -1$. Hence, $\psi_{K,2}$ is non-trivial on $\ker \varphi_{2^{e_2}+1}$.

Next, let $\alpha \in A_2$ be such that $\alpha \in \ker \varphi_{2^{e_2}+2}$. Hence,

$\tau_3 = \varphi_{2^{e_2}+3}(\alpha) = \left(\begin{array}{cc} 1 & 0 \\ b & d \end{array}\right) \in A(2^{e_2+3})$

and

(3.9) $\left(\begin{array}{cc} 1 & 0 \\ b & d \end{array}\right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$ in $A(2^{e_2+2})$.

Hence, $b = 2^{e_2+2}b_1$ for some integer $b_1$. We have

$\tau_3(\zeta_{2^{e_2}+4}(a_0^{e_1})) = \zeta_{2^{e_2}+4}(a_0^{e_1})^{1/8}$ $\zeta_{2^{e_2}+2}^{2^{e_2+2}+b_1}$.

Squaring both sides of this identity yields

$\tau_3(\zeta_{2^{e_2}+3})\tau_3((a_0^{e_1})^{1/4}) = \zeta_{2^{e_2}+3}(a_0^{e_1})^{1/4}$.

This implies

(3.10) $\tau_3((a_0^{e_1})^{1/4}) = \frac{\zeta_{2^{e_2}+3}(a_0^{e_1})^{1/4}}{\zeta_{2^{e_2}+3}^{2^{e_2}+3}}$.

Now observe, from (3.9), that

(3.11) $d = 1 + 2^{e_2+2}d_1$

for some integer $d_1$. Raising both sides of (3.10) to power 2 and employing (3.11) yield

$\tau_3((a_0^{e_1})^{1/2}) = (a_0^{e_1})^{1/2}$. 
Hence,
\[ \tau_3(a_0^{1/2}e_1^{1-1}/2) = a_0^{1/2}e_1^{1-1}/2. \]
Thus, \( \psi_{K,2} \) is trivial on \( \ker \varphi_{2^{e_2+1}} \).

\[ \square \]

The following proposition is essential in proving Theorem 1.2.

**Proposition 3.3.** Let \( \ell(p) \) be the smallest integer \( k \) for which \( \chi_p \) factors via \( A(p^k) \). Then

\[ \ell(p) = \begin{cases} 
0 & \text{if } p \text{ is odd and } p \nmid D, \\
1 & \text{if } p \text{ is odd and } p \mid D, \\
s & \text{if } p = 2 \text{ and } D \text{ is odd}, \\
\max\{2, s\} & \text{if } p = 2 \text{ and } 4 \mid D, \\
2 & \text{if } p = 2, 8 \nmid D, \text{ and } (\nu_2(e) = 1 \text{ and } a < 0), \\
\max\{3, s\} & \text{if } p = 2, 8 \mid D, \text{ and } (\nu_2(e) \neq 1 \text{ or } a > 0). 
\end{cases} \]

**Proof.** If \( p \nmid 2D \), by the definition of \( \chi_p \), we have that \( \chi_p \) is constantly equal to 1. Thus, the assertion holds.

If \( p \) is an odd integer dividing \( D \), then \( \chi_p \) is the Legendre symbol mod \( p \), so the result follows.

If \( p = 2 \) and \( D \) is odd, then \( \chi_2 = \psi_{K,2} \). Thus, the result follows from Proposition [3.2].

If \( p = 2 \) and \( 4 \mid D \), then \( \chi_2 = \psi_{K,2} \chi_{D,2} \), where \( \chi_{D,2} \) is the Dirichlet character mod 8 of conductor 4. We are looking for a positive integer \( k \) such that \( \psi_{K,2}(\alpha) \neq \chi_{D,2}(\alpha) \) for an element \( \alpha \in \ker \varphi_{2^k-1} \), and \( \psi_{K,2}(\alpha) = \chi_{D,2}(\alpha) \) for all \( \alpha \in \ker \varphi_{2^k} \). Note that 2 is the smallest value of \( k \) for which \( \chi_{D,2} \) factors via \( A(2^k) \), and, by Proposition [3.2], \( s \) is the smallest value of \( k \) for which \( \psi_{K,2} \) factors via \( A(2^k) \). Thus, \( \chi_{D,2} \) is trivial on \( \ker \varphi_{2^k} \) for \( k \geq 2 \) and is nontrivial on \( \ker \varphi_{2^k} \) for \( 0 \leq k \leq 1 \). Also, \( \psi_{K,2} \) is trivial on \( \ker \varphi_{2^k} \) for \( k \geq s \) and is nontrivial on \( \ker \varphi_{2^k} \) for \( 0 \leq k < s \). Using these facts and a case-by-case analysis in terms of the values of \( \nu_2(e) \) and for the untwisted and twisted cases, we can see that the claimed assertion, in this case, holds. More precisely, if \( \nu_2(e) = 0 \), then \( \chi_2 \) factors via \( A(2^2) \). Otherwise, \( \chi_2 \) factors via \( A(2^s) \). The only case that needs special attention is when \( s = 2 \), i.e., \( a \) is an exact perfect square (i.e., \( a > 0 \) and \( \nu_2(e) = 1 \)). In this case, \( \max\{2, s\} = 2 \) and both \( \psi_{K,2} \) and \( \chi_{D,2} \) are trivial on \( \ker \varphi_{2^2} \), hence \( \chi_2 \) is trivial on \( \ker \varphi_{2^2} \). Let \( \alpha \in \ker \varphi_{2^2} \) be such that \( \varphi_{2^2}(\alpha) = (1 0 \ 0 \ 1) \in A(2^2) \). Note that \( 0 + 1 = 3 \) (mod 2), so by Proposition [3.1(ii)] such \( \alpha \) exists. We have \( \chi_2(\alpha) = \psi_{K,2}(\alpha) \chi_{D,2}(\alpha) = (1)(1) = 1 = -1 \). Thus, \( \chi_2 \) is non-trivial on \( \ker \varphi_{2^2} \). Hence, \( \chi_2 \) factors via \( A(2^2) = A(2^s) = A(2^{\max\{2, s\}}) \) but not via \( A(2) \).

If \( p = 2 \) and \( 8 \mid D \), similar to part (iv), a case-by-case analysis in terms of the values of \( \nu_2(e) \), and for the untwisted and twisted cases, we can verify the result. (Note that in this case 3 is the smallest values of \( k \) for which \( \chi_{D,2} \) factors via \( A(2^k) \).) More precisely, if \( \nu_2(e) = 0 \) or 1, and \( -a \) is not a perfect square, then \( \chi_2 \) factors via \( A(2^3) \). Also, if \( \nu_2(e) = 1 \) and \( a < 0 \), then \( \chi_2 \) factors via \( A(2^5) \). Otherwise, \( \chi_2 \) factors via \( A(2^s) \). Two cases need special attention.

Case 1: The number \( a \) is negative of an exact perfect square (i.e., \( a < 0 \) and \( \nu_2(e) = 1 \)). In this case, \( \max\{3, s\} = 3 \) and both \( \psi_{K,2} \) and \( \chi_{D,2} \) are trivial on \( \ker \varphi_{2^3} \), hence \( \chi_2 \) is trivial on \( \ker \varphi_{2^3} \). Thus, \( \chi_2 \) acts through \( A(2^3) \). Let \( \alpha \in \ker \varphi_{2^3} \). Then \( \varphi_{2^3}(\alpha) = (1 0 \ 0 \ 1) \in A(2^3) \) is such that \( (1 0 \ 0 \ 1) \equiv (1 0 \ 0 \ 1) \) (mod 2^2). Hence,

\[
\left( \begin{array}{cc} 1 & 0 \\ b & d \end{array} \right) \in \left\{ \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & 5 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 4 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 4 & 5 \end{array} \right) \right\} \subset A(3). 
\]

Since for each \( \alpha \) corresponding to the above matrices we have \( \chi_2(\alpha) = \psi_{K,2}(\alpha) \chi_{D,2}(\alpha) = 1 \), we conclude that \( \chi_2 \) is trivial on \( \ker \varphi_{2^3} \). Now let \( \alpha \in \ker \varphi_{2^3} \) be such that \( \varphi_{2^3}(\alpha) = (1 0 \ 0 \ 1) \in A(2^3) \). Note that \( 2(6) + 1 = 1 \) (mod 2^2), so by Proposition [3.1(ii)] such \( \alpha \) exists. We have \( \chi_2(\alpha) = \psi_{K,2}(\alpha) \chi_{D,2}(\alpha) = (1)(1) = 1 \). Thus, \( \chi_2 \) is non-trivial on \( \ker \varphi_{2^3} \). Hence, \( \chi_2 \) factors via \( A(2^3) \) as claimed.
Case 2: The number $a$ is an exact perfect fourth power (i.e., $a > 0$ and $\nu_2(a) = 2$). In this case, \( \max\{3, s\} = 3 \) and both $\psi_{K,2}$ and $\chi_{D,2}$ are trivial on $\ker \varphi_{2^3}$, hence $\chi_D$ is trivial on $\ker \varphi_{2^3}$. Let $\alpha \in \ker \varphi_{2^3}$ be such that $\varphi_{2^3}(\alpha) = \left(\frac{1}{p}, \frac{0}{3}\right) \in A(2^3)$. Note that $0 + 1 \equiv 5 \pmod{2^2}$, so by Proposition 3.1(ii) such $\alpha$ exists. We have $\chi_2(\alpha) = \psi_{K,2}(\alpha)\chi_{D,2}(\alpha) = (1)(-1) = -1$. Thus, $\chi_2$ is non-trivial on $\ker \varphi_{2^3}$. Hence, $\chi_2$ factors via $A(2^3) = A(2^s) = A(2^{\max\{3, s\}})$.

\[ \square \]

4. PROOF OF THEOREM 1.6

Proof of Theorem 1.6 Let $\nu_G$ be the normalized Haar measure on the profinite group $G$, and $\nu_A$ be the normalized Haar measure on the profinite group $A$. We start by writing the summation

\[ \sum_{n \geq 1} \frac{g(n)}{\#G(n)} \]

in terms of measures of certain measurable subgroups of $G$. For this purpose, let $\pi_{G,n} : G \to G(n)$ be the projection map for each $n \geq 1$. Then, $G/\ker \pi_{G,n} \cong G(n)$ and $[G : \ker \pi_{G,n}] = \#G(n)$. Hence, since $\ker \pi_{G,n}$ is a closed subgroup of $G$, we have $\nu_G(\ker \pi_{G,n}) = 1/\#G(n)$ (see [9, Lemma 18.1.1.(a)]). Thus,

\[ (4.1) \quad \sum_{n \geq 1} \frac{g(n)}{\#G(n)} = \sum_{n \geq 1} g(n)\nu_G(\ker \pi_{G,n}). \]

Observe that

\[ (4.2) \quad [A : r(\ker \pi_{G,n})] = [A : r(G)][r(G) : r(\ker \pi_{G,n})]. \]

Also, since $\chi$ is continuous and $\ker \chi = r(G)$, then $r(G)$ is a closed subgroup of $A$ and hence it is $\nu_A$-measurable. Similarly, since $r(\ker \pi_{G,n})$ is a closed subgroup of $r(G)$ and $r(G)$ is a closed subgroup of $A$, then $r(\ker \pi_{G,n})$ is a closed subgroup of $A$ and hence it is $\nu_A$-measurable. Thus, from (4.2), we have

\[ (4.3) \quad \nu_G(\ker \pi_{G,n}) = \frac{\nu_A(r(\ker \pi_{G,n}))}{\nu_A(r(G))}. \]

Now, since

\[ (4.4) \quad 1 \to G \xrightarrow{r} A \xrightarrow{\chi} \mu_m \to 1 \]

is an exact sequence, by (4.3), we have

\[ (4.5) \quad \sum_{n \geq 1} g(n)\nu_G(\ker \pi_{G,n}) = \sum_{n \geq 1} g(n)\frac{\nu_A(r(\ker \pi_{G,n}))}{\nu_A(r(G))} = \frac{1}{\nu_A(\ker \chi)} \sum_{n \geq 1} g(n)\nu_A(r(\ker \pi_{G,n})). \]

Next, we show that $r(\ker \pi_{G,n}) = \ker(\pi_{A,n}) \cap \ker \chi$, where $\pi_{A,n} : A \to A(n)$ is the projection map for each $n \geq 1$. To prove this claim, we note that the diagram

\[ (4.6) \quad \begin{array}{ccc}
G & \xrightarrow{\pi_{G,n}} & G(n) \\
\downarrow r & & \downarrow r_n \\
A & \xrightarrow{\pi_{A,n}} & A(n)
\end{array} \]

commutes. For a group $H$, let $e_H$ denote its identity element. Note that if $\sigma \in \ker \pi_{G,n}$, then $r_n(\pi_{G,n}(\sigma)) = r_n(e_{G(n)}) = e_{A(n)}$. Hence, by the commutative diagram (4.6), we have $r(\sigma) \in \ker \chi$.\[ \square \]
Proposition 4, we have $r(\sigma) \in r(G) = \ker \chi$. Therefore, 
(4.7) $r(\ker \pi_{G,n}) \subset \ker \pi_{A,n} \cap \ker \chi$.

On the other hand, if $\alpha \in \ker \pi_{A,n} \cap \ker \chi \subset \ker \chi = r(G)$, then there exists a $\sigma \in G$ such that $r(\sigma) = \alpha$. Moreover, $r(\sigma) \in \ker \pi_{A,n}$ means $\pi_A(r(\sigma)) = e_{A(n)}$. Hence, $r_n(\pi_{G,n}(\sigma)) = e_{A(n)}$ as (4.6) is commutative. Thus, $\sigma \in \ker \pi_{G,n}$, since $r_n$ is injective. This shows that 
(4.8) $\ker \pi_{A,n} \cap \ker \chi \subset r(\ker \pi_{G,n})$.

Therefore, from (4.7) and (4.8), we have 
(4.9) $r(\ker \pi_{G,n}) = \ker \pi_{A,n} \cap \ker \chi$.

From (4.9), we have 
(4.10) 
\[
\sum_{n \geq 1} g(n)\nu_A(r(\ker \pi_{G,n})) = \sum_{n \geq 1} g(n)\nu_A(\ker \pi_{A,n} \cap \ker \chi) \\
= \sum_{n \geq 1} g(n) \int_A 1_{\ker \pi_{A,n} \cap \ker \chi} d\nu_A \\
= \int_A \left( \sum_{n \geq 1} g(n)1_{\ker \pi_{A,n}} \right) 1_{\ker \chi} d\nu_A.
\]

To justify the interchange of the summation and the integral in the last equality, observe that 
\[
\left| \sum_{n=1}^m g(n)1_{\ker \pi_{A,n} \cap \ker \chi} \right| \leq \sum_{n \geq 1} |g(n)|1_{\ker \pi_{A,n} \cap \ker \chi}.
\]

Since by the assumption, $\sum_{n \geq 1} |g(n)|/\#G(n)$ converges, then, by [17, Theorem 1.27], $\sum_{n \geq 1} |g(n)|1_{\ker \pi_{A,n} \cap \ker \chi}$ is integrable. Thus, by Lebesgue’s dominated convergence theorem (see [17, Theorem 1.34]), the interchange of the summation and the integral in (4.10) is justified. Also, since $\#A(n) \geq \#G(n)$, then $\sum_{n \geq 1} |g(n)|/\#A(n) < \infty$. Hence, by [17, Theorem 1.38], 
\[
\tilde{g} = \sum_{n \geq 1} g(n)1_{\ker \pi_{A,n}} \in L^1(\nu_A).
\]

Now from (4.1), (4.5), and (4.10), we have 
(4.11) 
\[
\sum_{n \geq 1} \frac{g(n)}{\#G(n)} = \int_A \tilde{g}1_{\ker \chi} d\nu_A - \int_A \sum_{n \geq 1} g(n)1_{\ker \pi_{A,n}} 1_{\ker \chi} d\nu_A.
\]

Note that the character $\chi : A \to \mu_m$ in (4.4) induces the character $\chi' : A/{r(G)} \to \mu_m$ by $\chi'(\tilde{\alpha}) = \chi(\alpha)$, where $\alpha \in A$ and $\tilde{\alpha}$ is the coset associated to $\alpha$ in $A/{r(G)}$. In other words, $\chi$ is the lift of $\chi'$ to $A$. Since $\chi'$ sends a generator of $A/{r(G)}$ to a generator of $\mu_m$, then $\chi'$ is a generator of the group of characters of $A/{r(G)}$ denoted by $A/{r(G)}$. Thus, for $\tilde{\alpha} \in A/{r(G)}$, by [18, Chapter VI, Proposition 4], we have 
\[
\sum_{\epsilon \in A/{r(G)}} \epsilon(\tilde{\alpha}) = \sum_{i=0}^{m-1} (\chi')^i(\tilde{\alpha}) = \begin{cases} 
  m & \text{if } \tilde{\alpha} = e_{A/{r(G)}}, \\
  0 & \text{if } \tilde{\alpha} \neq e_{A/{r(G)}}.
\end{cases}
\]

Therefore, since $\tilde{\alpha} = e_{A/{r(G)}}$ means $\alpha \in \ker \chi$, we have 
\[
\sum_{i=0}^{m-1} \chi'^i(\alpha) = \begin{cases} 
  m & \text{if } \alpha \in \ker \chi, \\
  0 & \text{if } \alpha \notin \ker \chi.
\end{cases}
\]
This implies \( \sum_{i=0}^{m-1} \chi^i(\alpha) = m \cdot 1_{\ker \chi}(\alpha) \). Thus,

\[
\int_A \tilde{g} 1_{\ker \chi} \, d\nu_A = \int_A \tilde{g} \sum_{i=0}^{m-1} \chi^i \, d\nu_A.
\]

Furthermore, by (4.4), we have \( [A : \ker \chi] = [A : r(G)] = m \). Hence, \( \nu_A(\ker \chi) = 1/m \). Thus, the desired result follows from (4.11) and (4.12).

The following corollary considers a special case of Theorem 1.6.

**Corollary 4.1.** In Theorem 1.6 suppose that \( g \) is a multiplicative arithmetic function. Assume that \((h_n)_{n \in \mathbb{N}}\) is a family of isomorphisms \( h_n : A(n) \to \prod_{p \mid n} A(p^k) \), compatible with transition maps \((j_{n1,n2})_{n1,n2} \), that results in a topological isomorphism \( A \cong \prod_p A_p \), where \( A_p = \varprojlim A(p^i) \). In addition, let \( \chi = \prod_p \chi_p \), where the continuous characters \( \chi_p : A_p \to \mathbb{C} \) are trivial except for finitely many \( p \)'s. Then,

\[
\tilde{g}_p = \sum_{k \geq 0} g(p^k) 1_{\ker \varphi_p} \in L^1(\nu_{A_p})
\]

and

\[
\sum_{n \geq 1} \frac{g(n)}{\#G(n)} = \sum_{i=0}^{m-1} \int_A \tilde{g} \chi^i \, d\nu_A,
\]

where \( \nu_{A_p} \) is the normalized Haar measure on \( A_p \).

**Proof.** By Theorem 1.6, we have

\[
\sum_{n \geq 1} \frac{g(n)}{\#G(n)} = \sum_{i=0}^{m-1} \int_A \tilde{g} \chi^i \, d\nu_A.
\]

Since \( g(n) \) is multiplicative, \( A \cong \prod_p A_p \), \( \nu_A = \prod_p \nu_{A_p} \), \( \chi = \prod_p \chi_p \), and \( \tilde{g} = \prod_p \tilde{g}_p \), then (4.14) yields (4.13). Note that, since \( \#A(p^k) \geq \#G(p^k) \), then \( \sum_{n \geq 1} |g(p^k)|/\#A(p^k) < \infty \). Hence, by [17] Theorem 1.38, \( \tilde{g}_p \in L^1(\nu_{A_p}) \). Thus, the integrals in (4.13) are finite.

5. PROOF OF THEOREM 1.2

**Proof of Theorem 1.2.** Let the profinite group \( A \) and the character \( \chi \) be as defined in Section 2. We employ Corollary 4.1 and compute \( \int_{A_p} \tilde{g}_p d\nu_{A_p} \) and \( \int_{A_p} \tilde{g}_p \chi_p d\nu_{A_p} \) for primes \( p \). Since \( \ker \varphi_p \) is a closed subgroup of \( A_p \), we have \( \nu_{A_p}(\ker \varphi_p) = 1/[A_p : \ker \varphi_p] = 1/\#A(p^k) \). Observe that

\[
\int_{A_p} \tilde{g}_p d\nu_{A_p} = \int_{A_p} \sum_{k \geq 0} g(p^k) 1_{\ker \varphi_p} \, d\nu_{A_p} = \sum_{k \geq 0} g(p^k) \nu_{A_p}(\ker \varphi_p).
\]

(5.1)

Observe that if \( S \subset A_p \) is \( \nu_{A_p} \)-measurable, then \( \int_S \chi_p(\alpha x) d\nu_{A_p}(x) = \int_S \chi_p(x) d\nu_{A_p}(x) \) for any \( \alpha \in S \). From here we conclude that if \( \chi_p \) is non-trivial on \( S \), then \( \int_S \chi_p d\nu_{A_p} = 0 \). Hence, by
Proposition 3.3

\[ \int_{A_p} \tilde{g}_p \chi_p d\nu_{A_p} = \int_{A_p} \left( 1_{A_p} \chi_p + g(p)1_{\ker \varphi_p} \chi_p + \cdots + g(p^k)1_{\ker \varphi_{p^k}} \chi_p + \cdots \right) d\nu_{A_p} = 0 + \sum_{k \geq \ell(p)} g(p^k) \nu_{A_p}(\ker \varphi_{p^k}) = \sum_{k \geq \ell(p)} \frac{g(p^k)}{\#A(p^k)}. \]

(5.2)

Thus, by Corollary 4.1 with \( m = 2 \), (5.1), and (5.2), we get (1.8). \( \square \)

6. PROOF OF PROPOSITION 1.4

Proof of Proposition 1.4. For integer \( k \geq 1 \) and odd prime \( p \), let

\[ k' = \begin{cases} 
0 & \text{if } k \leq \nu_p(e), \\
-k - \nu_p(e) & \text{if } k > \nu_p(e), 
\end{cases} \]

and for \( k \geq 1 \) and \( p = 2 \), let

\[ k' = \begin{cases} 
0 & \text{if } k \leq \nu_2(e) \text{ and } (a > 0 \text{ or } e \text{ is odd}), \\
1 & \text{if } k \leq \nu_2(e) \text{ and } (a < 0 \text{ and } e \text{ is even}), \\
-k - \nu_2(e) & \text{if } k > \nu_2(e). 
\end{cases} \]

Then, from Proposition 3.1(iii), we have

\[ \#A(p^k) = \begin{cases} 
p^{k+k'-1}(p-1) & \text{if } k \geq 1, \\
1 & \text{if } k = 0. 
\end{cases} \]

(6.3)

Now by employing (6.3) in (1.8) we get

\[ \sum_{n=1}^{\infty} \frac{g(n)}{\#G(n)} = \prod_p \left( 1 + \sum_{k \geq 1} \frac{g(p^k)}{p^{k+k'-1}(p-1)} \right) + \prod_p \sum_{k \geq \ell(p)} \frac{g(p^k)}{\#A(p^k)}. \]

(6.4)

We set \( g = 1 \) in (6.4) to get the product expression for the constant in the conjectured asymptotic formula in the Titchmarsh Divisor Problem for a given Kummer family. Therefore, by (6.4),

\[ \sum_{n \geq 1} \frac{1}{[K_n : \mathbb{Q}]} = \left( 1 + \prod_{p \mid 2D} \frac{C_p}{1 + B_p} \right) \prod_p \left( 1 + B_p \right), \]

(6.5)

for the following values for \( B_p \) and \( C_p \).

If \( p \) is odd, we have

\[ B_p = \sum_{k \geq 1} \frac{1}{p^{k+k'-1}(p-1)} = \frac{p^\nu_p(e) + 2 + p^\nu_p(e)+1 - p^2}{p^\nu_p(e)(p-1)(p^2 - 1)}, \]

(6.6)

and \( C_p = B_p \), where \( k' \) is given by (6.1).

For \( p = 2 \), we have the following cases for \( B_2 \) and \( C_2 \) with \( k' \) as given by (6.2).

Case (i). Let \( e \) be odd or \( a > 0 \). Hence, \( s = \nu_2(e) + 1 \). Then \( B_2 \) is the same as (6.6) with \( p = 2 \). Now, if \( D \) is odd; or \( 4 \mid D \) and \( s \geq 2 \); or \( 8 \mid D \) and \( s \geq 3 \), then

\[ C_2 = \sum_{k \geq \ell(2)} \frac{1}{2^{k+k'-1}} = \frac{2}{2^\nu_2(e)(2^2 - 1)}. \]
Otherwise,
\[ C_2 = \sum_{k \geq \ell(2)} \frac{1}{2^{k+k'-1}} = \frac{2^{\nu_2(e)+1}}{2^\beta (2^2 - 1)}, \]
where \( \beta = 2 \) if \( 4 \| D \) and \( s = 1 \); and \( \beta = 4 \) if \( 8 \| D \) and \( s \in \{1, 2\} \).

Case (ii). Let \( e \) be even and \( a < 0 \). Then
\[ B_2 = \sum_{k \geq 1} \frac{1}{2^{k+k'-1}} = \frac{2^{\nu_2(e)+2} - 2^{\nu_2(e)} - 1}{2^{\nu_2(e)}(2^2 - 1)}. \]
If \( 8 \| D \) and \( \nu_2(e) = 1 \), we have \( \ell(2) = 2 \). Hence,
\[ C_2 = \sum_{k \geq \ell(2)} \frac{1}{2^{k+k'-1}} = \frac{1}{2^2 - 1}. \]
Otherwise, we have \( \ell(2) = s = \nu_2(e) + 2 \) and thus
\[ C_2 = \sum_{k \geq \ell(2)} \frac{1}{2^{k+k'-1}} = \frac{1}{2^{\nu_2(e)+1}(2^2 - 1)}. \]

By applying the above expressions in (6.5) and by case-by-case simplifying, we get (1.11).

7. Serre Curves

Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \). Let \( \mathbb{Q}(E[n]) \) be the \( n \)-division field of \( E \). By taking the inverse limit of the natural injective maps
\[ r_n : \text{Gal}(\mathbb{Q}(E[n])/\mathbb{Q}) \to \text{Aut}(E[n]) \cong \text{GL}_2(\mathbb{Z}/n\mathbb{Z}), \]
over all \( n \geq 1 \), we have an injective profinite homomorphism
\[ r : \text{Gal}(\mathbb{Q}(E[x])/\mathbb{Q}) \to \text{Aut}(E[x]) \cong \text{GL}_2(\widehat{\mathbb{Z}}), \]
where \( E[x] = \bigcup_{n=1}^{\infty} E[n] \). Let \( \Delta \) be the discriminant of any Weierstrass model for \( E \). Set \( K = \mathbb{Q}(\Delta^{1/2}) \) and let \( D \) be the discriminant of \( K \). In anticipation of applying Theorem 1.6, let \( \det \) be the determinant map \( \text{det} : \text{GL}_2(\widehat{\mathbb{Z}}) \to \widehat{\mathbb{Z}}^\times \) and
\[ \chi_D : \text{GL}_2(\widehat{\mathbb{Z}}) \xrightarrow{\text{det}} \widehat{\mathbb{Z}}^\times \xrightarrow{(B)} \mu_2 \]
be the composition of \( \det \) with the lift to \( \widehat{\mathbb{Z}}^\times \) of the Kronecker symbol attached to \( D \). We note that \( \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3 \), where \( S_3 \) is the symmetric group on three letters. Let
\[ \psi : \text{GL}_2(\widehat{\mathbb{Z}}) \to \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3 \xrightarrow{\text{sign}} \mu_2 \]
be the composition of the projection map from \( \text{GL}_2(\widehat{\mathbb{Z}}) \) to \( \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \) with the signature character on \( S_3 \). Let \( G = \text{Gal}(\mathbb{Q}(E[x])/\mathbb{Q}) \). For \( \eta \in G \) we can show that the image of \( r(\eta) \) under \( \psi \) is the same as \( \chi_D(r(\eta)) = \eta(\Delta^{1/2})/\Delta^{1/2} \) (see (26) in [5] and discussion before it). We now set \( \chi = \chi_D \circ \psi \).

The above construction of the character \( \chi \) is described by J.-P. Serre in [19]. In addition, in [19, Section 5.5], Serre shows that the character \( \chi \) constructed above is non-trivial and \( r(G) \) is contained in \( \ker \chi \), hence \( [\text{GL}_2(\widehat{\mathbb{Z}}) : r(G)] \geq 2 \). We name \( E \) a Serre curve if \( [\text{GL}_2(\widehat{\mathbb{Z}}) : r(G)] = 2 \). This is equivalent to saying that \( r(G) = \ker \chi \). Thus, letting \( A = \text{GL}_2(\widehat{\mathbb{Z}}) \), for Serre curve \( E \), the sequence
\[(7.1)\]
\[ 1 \longrightarrow G \xrightarrow{r} A \xrightarrow{\chi} \mu_2 \longrightarrow 1 \]
is an exact sequence. In addition for a Serre curve \( K = \mathbb{Q}(\Delta^{1/2}) \) is a quadratic field (see the first paragraph of [14 p. 510] for explanation).
The quadratic character \( \chi : \text{GL}_2(\hat{\mathbb{Z}})(\cong \prod_p \text{GL}_2(\mathbb{Z}_p)) \to \mu_2 \) can be written as a product of local characters \( \chi_p : \text{GL}_2(\mathbb{Z}_p) \to \mu_2 \). Observe that since \( \psi \) factors via \( \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \), then it factors via \( \text{GL}_2(\mathbb{Z}_2) \). Let \( \psi : \text{GL}_2(\mathbb{Z}_2) \to \mu_2 \) be the corresponding homomorphism obtained from factorization of \( \psi \) via \( \text{GL}_2(\mathbb{Z}_2) \). For primes \( p \mid 2D \), let \( \chi_p \) be constantly equal to 1. For odd primes \( p \mid D \), let \( \chi_p = \chi_{D,p} \) be the lift of the Legendre symbol mod \( p \) to \( \mathbb{Z}_p^\times \), i.e.,

\[
\chi_p : \text{GL}_2(\mathbb{Z}_p^\times) \overset{\text{det}}{\longrightarrow} \mathbb{Z}_p^\times \longrightarrow \mu_2
\]

where the last map is the composition of projection map to \( \mathbb{Z}/p\mathbb{Z} \) and the Legendre symbol mod \( p \). For prime 2, let \( \chi_2 = \chi_{D,2} : \psi_2 \), where \( \chi_{D,2} \), similarly to the Kummer case, is the lift of one of the Dirichlet characters mod 8 to \( \mathbb{Z}_2^\times \) (if \( D \) is odd, then \( \chi_{D,2} \) is trivial). Therefore, by the above construction of \( \chi \), we have the decomposition \( \chi = \prod_p \chi_p^\prime \).

Let \( A_p = \text{GL}_2(\mathbb{Z}_p) \) and \( A(p^k) = \text{GL}_2(\mathbb{Z}/p^k\mathbb{Z}) \). The following is an analogous of Proposition 3.3 for Serre curves.

**Proposition 7.1.** For a Serre curve \( E \), assume the above notations. Let \( \ell(p) \) be the smallest integer \( k \) for which \( \chi_p \) factors via \( A(p^k) \). Then

\[
\ell(p) = \begin{cases} 
0 & \text{if } p \text{ is odd and } p \nmid D, \\
1 & \text{if } p \text{ is odd and } p \mid D, \\
1 & \text{if } p = 2 \text{ and } D \text{ is odd,} \\
2 & \text{if } p = 2 \text{ and } 4 \mid D, \\
3 & \text{if } p = 2 \text{ and } 8 \mid D.
\end{cases}
\]

**Proof.** If \( p \nmid 2D \), then \( \chi_p \) is constantly equal to 1. Hence, it factors via \( A(1) \). If \( p \) is odd and \( p \mid D \), then \( \chi_p \) is the Legendre symbol mod \( p \), and so it factors via \( A(p) \), and since it is non-trivial, it does not factor via \( A(1) \). The result for \( p = 2 \) follows from the construction of \( \chi_2 \) described above, noting that the smallest integer \( k \) for which \( \psi_2 \) factors via \( A(p^k) \) is \( k = 1 \), for \( 4 \mid D \) the smallest such \( k \) is \( k = 2 \), and for \( 8 \parallel D \) the smallest such \( k \) is \( k = 3 \).

We are now ready to prove our last remaining assertion.

**Proof of Proposition LZ** Following steps similar to the proof of Theorem 1.2 and by employing Corollary 4.1 with \( m = 2 \), Proposition 7.1 and, for integer \( n \geq 1 \),

\[
\# \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) = \prod_{p^i \mid n} p^{4e-3}(p^2 - 1)(p - 1)
\]

(see [11, p. 231]) we have the stated product expression.

**Acknowledgements.** The authors thank the reviewers for their valuable comments and suggestions. The authors thank David Basil and Solaleh Bolvardizadeh for help computing the explicit constants \( c_n \) of Proposition 1.4 for certain values of \( a \).

**Declarations**

**Funding.** This research is partially supported by NSERC.

**Data availability statement.** Data sharing does not apply to this article as no datasets were generated or analyzed during the current study.

**Conflict of interest.** The authors have no conflicts of interest to declare relevant to this article’s content.
REFERENCES

[1] Amir Akbary and Dragos Ghioca, A geometric variant of Titchmarsh divisor problem, Int. J. Number Theory 8 (2012), no. 1, 53–69, DOI 10.1142/S1793042121500030. MR2887882
[2] Amir Akbary and Adam Tyler Felix, On the average value of a function of the residual index, Springer Proc. Math. Stat. 251 (2018), 19–37. MR3880381
[3] Emil Artin, The collected papers of Emil Artin, Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1965. Edited by Serge Lang and John T. Tate. MR0176888
[4] Eric Bach, Richard Lukes, Jeffrey Shallit, and H. C. Williams, Results and estimates on pseudopowers, Math. Comp. 65 (1996), no. 216, 1737–1747. MR1355005
[5] Renee Bell, Clifford Blakestad, Alina Carmen Cojocaru, Alexander Cowan, Nathan Jones, Vlad Matei, Geoffrey Smith, and Isabel Vogt, Constants in Titchmarsh divisor problems for elliptic curves, Res. Number Theory 6 (2020), no. 1, Paper No. 1, 24, DOI 10.1007/s40993-019-0175-9. MR4041152
[6] David A. Cox, Primes of the form \(x^2 + ny^2\), 2nd ed., Pure and Applied Mathematics (Hoboken), John Wiley & Sons, Inc., Hoboken, NJ, 2013. Fermat, class field theory, and complex multiplication. MR3236783
[7] Harold Davenport, Multiplicative number theory, 3rd ed., Graduate Texts in Mathematics, vol. 74, Springer-Verlag, New York, 2000. Revised and with a preface by Hugh L. Montgomery. MR1790423
[8] Adam Tyler Felix and M. Ram Murty, A problem of Fomenko’s related to Artin’s conjecture, Int. J. Number Theory 8 (2012), no. 7, 1687–1723, DOI 10.1142/S1793042112500984. MR2968946
[9] Michael D. Fried and Moshe Jarden, Field arithmetic, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 11, Springer-Verlag, Berlin, 2008. Revised by Jarden. MR2445111
[10] Christopher Hooley, On Artin’s conjecture, J. Reine Angew. Math. 225 (1967), 209–220, DOI 10.1515/crll.1967.225.209. MR207630
[11] Neal Koblitz, Introduction to elliptic curves and modular forms, 2nd ed., Graduate Texts in Mathematics, vol. 97, Springer-Verlag, New York, 1993. MR1216136
[12] E. Kowalski, Analytic problems for elliptic curves, J. Ramanujan Math. Soc. 21 (2006), no. 1, 19–114. MR2226355
[13] Peter Stevenhagen, The correction factor in Artin’s primitive root conjecture, J. Théor. Nombres Bordeaux 15 (2003), no. 1, 383–391 (English, with English and French summaries). Les XXIèmes Journées Arithmetiques (Lille, 2001).
[14] Samuel S. Wagstaff Jr., Pseudoprimes and a generalization of Artin’s conjecture, Acta Arith. 41 (1982), no. 2, 141–150, DOI 10.4064/aa-41-2-141-150. MR674829

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF LETHBRIDGE, LETHBRIDGE, ALBERTA T1K 3M4, CANADA

Email address: amir.akbary@uleth.ca

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF LETHBRIDGE, LETHBRIDGE, ALBERTA T1K 3M4, CANADA

Email address: milad.fakhari@uleth.ca