Doubly excited states in a chiral waveguide-QED system: description and properties

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Abstract. In modern quantum optics chiral waveguide quantum-electrodynamical (wQED) systems are attracting a lot of attention from the perspective of fundamental science, and possible interesting applications. In our work we theoretically analyze the eigenstates in a two-excitation domain of an ensemble of two-level atoms that are periodically spaced, and asymmetrically coupled to a guided mode. We found that in a regime when all atoms emit photons in-phase, most eigenstates in such a system can be well-approximated and described through the eigenstates from a single excitation domain, while the rest present a superposition of bound states with two strongly attracting excitations, and states, for which the excitations strongly repel from each other occupying the opposite edges of the system.

1. Introduction

During last several years the topic of chiral quantum systems \cite{1} became of high interest in a quantum optical community. The main reason for this is that emitters that are asymmetrically coupled to propagating photonic modes have already demonstrated in experiments such interesting, and unique properties as, for example, a strong coherent Bragg reflection \cite{2}, and optical isolation \cite{3}. From a theory perspective, strongly correlated dissipation-immune photon bound states arising in chiral waveguide-QED systems are of big interest \cite{4, 5} with a recent experimental demonstration \cite{6}.

In this work we analyze the eigenstates of a periodic chain of two-level systems that asymmetrically interact with a single guided mode (and in-phase with each other) in a two-excitation domain. We demonstrate that most states in this case can be qualitatively characterized within the rank-1 or rank-2 approximation that corresponds to the existence of two independent excitations in the system. We also show that there are states that can not be described within this framework, and which have a form of a superposition made of two strongly attracting excitations in a bound state, and an edge state, for which excitations prefer to occupy opposite edges of the system.

2. Theoretical framework

We consider a periodic 1D chain of identical $N$ two-level atoms (TLAs) with interatomic spacing $d$ that are in the vicinity of a photonic waveguide, and which are coupled asymmetrically to
δ-Lindblad superoperator accounting for the description of the dissipative dynamics, \( \omega \), where \( \Delta \) on a reduced density matrix for such an ensemble has the following form [7]:

\[
\dot{\rho}_s(t) = i \sum_k \Delta \omega_k \left[ \hat{\sigma}_k^- \rho_s(t) \hat{\sigma}_k^+ \right] + \hat{L} \rho_s(t),
\]

\[
\hat{L} \rho_s(t) = \sum_{k \neq l} \frac{\gamma_{kl}}{2} \left( 2 \hat{\sigma}_k^- \hat{\sigma}_l^+ \rho_s(t) - \hat{\sigma}_k^- \rho_s(t) \hat{\sigma}_l^+ - \hat{\sigma}_k^+ \rho_s(t) \hat{\sigma}_l^- - \hat{\sigma}_k^+ \hat{\sigma}_l^- \rho_s(t) \right) + \sum_{k,l} \delta_{kl} \left( -i \left[ \hat{\sigma}_l^- \rho_s(t), \hat{\sigma}_l^+ \right] + i \left[ \hat{\sigma}_l^- \rho_s(t), \hat{\sigma}_l^+ \right] \right),
\]

where \( \Delta \omega_k = -\omega_{0,k} + \delta_{kk}, \) with \( \omega_{0,k} \) - transition frequency for atom labeled \( k, \) \( \hat{L} \) is the Lindblad superoperator accounting for the description of the dissipative dynamics, \( \delta_{kl} = 4\pi k_{0,d}^2 \text{Re} \left[ |d_k|^2 \mathbf{G}(\mathbf{r}_k, \mathbf{r}_l, \omega_{0,l}) d_l^* / \hbar \right], \) \( \gamma_{kl} = 8\pi k_{0,l}^2 \text{Im} \left[ d_k^* \mathbf{G}(\mathbf{r}_k, \mathbf{r}_l, \omega_{0,l}) d_l / \hbar \right], \) and the spontaneous emission rates \( \omega_{0,k} \) are responsible for the coherent, and dissipative coupling of atoms, and are described through the classical electromagnetic Green’s tensor of the photonic environment \( \mathbf{G}(\mathbf{r}_k, \mathbf{r}_l, \omega_{0,l}). \)

We want to find the probability that a pair of atoms with numbers \( k, \) and \( l, k \neq l \) that are excited at time \( t, \) given by the respective component of the density matrix \( \langle e_k e_l | \hat{\rho}_s(t) | e_k e_l \rangle = \rho_{kl,kl}(t), \) where \( |e_k e_l\rangle = |e_k\rangle |e_l\rangle \) \( \otimes N^{-2}. \) Therefore, one can obtain:

\[
\hat{\rho}_{kl,kl}(t) = -\left( \gamma_{kk} + \gamma_{ll} \right) \rho_{kl,kl}(t) + \sum_{m \neq k,l} \left( \rho_{km,kl}(t) G_{l,m} + \rho_{kl,km}(t) G^*_{l,m} + \rho_{lm,kl}(t) G_{k,m} + \rho_{kl,lm}(t) G^*_{k,m} \right),
\]

with \( G_{k,l} = i \delta_{kl} - \frac{\gamma_{kl}}{2}, \) and by further substituting \( \rho_{ij,kl}(t) = C_{i,j}(t) \cdot C_{k,l}(t), \) one arrives at:

\[
\hat{\rho}_{k,l}(t) = -\frac{\gamma_{kk} + \gamma_{ll}}{2} \rho_{kl,kl}(t) + \sum_{m=\\text{like}} \left( G_{l,m} C_{k,m}(t) + G_{k,m} C_{l,m}(t) \right),
\]

where \( C_{k,l}(t) \) is a probability amplitude that atoms \( k,l \) are excited (assumes \( k \neq l, \) and \( k > (or <) l). \) As we consider a set of two-level atoms that are close to a waveguide, and positioned periodically along it, and at the same distance from it’s surface, we can use the Green’s tensor from [8], which gives \( G_{k,l} = -\gamma^{(l)}(\gamma^{(b)}) e^{ik_{l}^{(b)}} |r_{l}|d, \) where the spontaneous emission rate into forward-propagating mode \( \gamma^{(l)} \) is chosen when \( k > l, \) and \( \gamma^{(b)} \) otherwise. Moreover, from now on we take \( d = 2\pi / k_{l}^{(b)} m, m \in Z, \) which corresponds to the case when all atoms emit photons in-phase as this will simplify the further analysis significantly.

3. Results
We begin with the analysis of the spectrum, which mathematically corresponds to the eigenfrequencies of normal modes of Eq. (3), and as there are 2 excited atoms out of \( N, \)
then we have \( N(N - 1)/2 \) eigenstates in general. As one can see from Fig. 2 (a), among all of the eigenstates there are the ones with a noticeable higher values of the Inverse Participation Ratio (IPR) that is given by

\[
\text{IPR} = \frac{\sum_{n \neq m} |C_{n,m}(j)|^4}{(\sum_{n \neq m} |C_{n,m}(j)|^2)^2}
\]

- a common measure of a degree of localization. The probability distributions of these states can be described as a linear superposition of a bound state, when two excitations are strongly localized near each other \((|C_{n,m}(j)| \sim e^{-|n-m|})\), and states, for which they are located at different chain edges \(|C_{n,m}| \sim e^{[n-m]-N}\) (see label (1) in Fig. 2 (a), and Fig. 2 (b)). However, the states with larger frequency shifts from a bare atomic resonance \(\Delta \omega_j = \omega_j - \omega_0\) have a lower IPR (more delocalized, see label (2) in Fig. 2 (a)), and for them both excitations tend to be localized on the right edge (Fig. 2 (c)).

![Figure 2](image-url)

**Figure 2.** (a) Frequency shifts \((\Delta \omega_j/\gamma(j))\), and emission rates \((\gamma_j/\gamma(j))\) for 2-excitation domain (colored circles). Green squares - single particle spectrum, red circles - half-sums of all combinations of single-particle energies. The color identifies the Inverse Participation Ratio. (b)-(d) Logarithms of the probabilities for atoms \(n\), and \(m\) to be excited for the states labeled (1)-(3) in (a). (e) - rank-2 approximation of (d) formed from the corresponding eigenstates from a single excitation domain. The parameters are \(N = 30\), \(\gamma(j) = 1\), \(\gamma(0) = 10^{-2}\).

Most of other eigenenergies can be well described either by a single excitation energies \(\epsilon_i\) (green squares in Fig. 2 (a)) or as an average of all possible unordered pairs of thereof \((\epsilon_i + \epsilon_j)/2\) \((i \neq j\), red circles in Fig. 2 (a)). This means that the eigenstate in this case can be approximated, roughly speaking, by the following expression

\[
|\psi(k)\rangle \sim |\psi(i)\rangle \langle \psi(j)| + |\psi(j)\rangle \langle \psi(i)|,
\]

which is an eigenstate of the effective Hamiltonian \(\hat{H}_{\text{eff}}^{(1)} \otimes \hat{I} + \hat{I} \otimes \hat{H}_{\text{eff}}^{(1)}\) with \(\hat{H}_{\text{eff}}^{(1)}\) being the Hamiltonian for a single-excitation domain, which is an effective Hamiltonian describing two independent excitations in a system. However, due to the Pauli principle, one has to multiply the state \(|\psi(k)\rangle\) by a term \(1 - \delta_{n,m}\) as two excitations can not occupy the same atom as a consequence of a photon blockade \((P_{n,n} = 0\) for any state). This corresponds to rank-2(1) approximation of a 2D representation of a given state \(k\) once \(i \neq (=) j\), and as seen by comparison of (d) with (e) in Fig. 2 (state labeled (3) in Fig. 2 (a)), this approximation is qualitatively valid.

**4. Conclusions**

In this work we have analyzed the doubly excited eigenstates in an ensemble of periodically spaced two-level atoms asymmetrically interacting with a single guided mode. We found that if the system period is a multiple of a wavelength of a guided mode, then most eigenstates
can be qualitatively well described by making use of the eigenstates from a single excitation domain, and this can be done by constructing either rank-1 or rank-2 approximate states. We also show that some states cannot be described within this framework, and, depending on their frequency shift from a bare atomic resonance $\omega_0$, they gradually morph from a superposition state of strongly attracting (bound states), and repelling (edge states) excitation pairs to states, where both excitations lie at one edge of the system due to the coupling asymmetry.

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