On the stability of boundary-layer flow over a rotating cone in an enforced axial free stream

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August 2021

Abstract. Studies into the boundary-layer stability of rotating cones in an axial free stream have considered both cases of broad-angled cones and slender-angled ones. This distinction is made due to the findings that the dominant instability in the quiescent-fluid case results from different physical mechanisms for broad and slender cones. These investigations made use of an Orr-Sommerfeld-type analysis derived from the assumption of a small perturbation to a similarity-type solution for the mean fluid flow around the cone. In this work, we investigate the efficacy of a combination of sophisticated numerical techniques to find the basic flow, using ideas both from automatic differentiation and spectral collocation methods, to maximize the accuracy of the obtained solution. We inspect the solutions obtained for the Orr-Sommerfeld system in this setting and compare them to the results from previous studies.

Keywords: boundary layers, rotating flow, stability

1. Introduction

Details of the stability properties of boundary layer flows over rotating cones are of central importance to a number of industrial and aerospace applications. These problems can largely be separated into two families: those in which the cone’s rotation occurs within a quiescent external fluid, and those involving an externally-imposed mean flow which interacts with the rotating cone. An example of an important problem belonging to the latter family is that of the transition of a flow to turbulence as it passes over the nose cone of a turbofan engine’s air intake. Understanding the influence of geometrical parameters governing the flow’s behaviour, in particular the spatio-temporal stability of the boundary layer, allows in principle the designing of machinery and engine components to ameliorate the well-known costs of turbulence development.

The geometry of the general dimensional problem is shown in Figure 1, and consists of a cone rotating about its medial axis, with an additional (possibly vanishing) axis-aligned external mean flow impinging on the tip of the cone. In this work, all quantities marked with an asterisk are to be understood as being dimensional, while the corresponding symbols lacking an asterisk are non-dimensional. Our coordinate system comprises along-surface (streamwise) and surface-perpendicular (normal) directions, denoted $x^*$ and $z^*$ respectively, and an angle around the cone’s
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axis, denoted $\theta$. The half angle of the cone is represented by $\psi$, and the angular frequency with which the cone rotates is given by $\Omega^*$. The strength of the external slip velocity resulting from the oncoming axial flow is represented by $U^*$, and is given by a power law (see [1]).

\[ U^* = C^* x^{**m} \]

\[ V_w^* = \Omega^* x^* \sin \psi \]

**Figure 1.** In the dimensional problem formulation we use along-surface and surface-perpendicular coordinates, $x^*$ and $z^*$ respectively, as well as an angle of rotation about the cone’s axis, $\theta$. The cone’s angular frequency about this axis is denoted $\Omega^*$, and its half angle is given by $\psi$.

1.1. Governing system for the boundary-layer flow

Non-dimensionalising length quantities on a characteristic along-surface distance, $l^*$, and performing the usual analysis of near-boundary viscous fluid flow, we introduce a non-dimensional boundary-layer coordinate $\zeta = \frac{Re^2}{x^*}$, where $Re$ is the Reynolds number. Non-dimensionalising fluid velocities using the local swirl velocity, $x^* \Omega^* \sin \psi$, induced by the cone’s rotation, and using a simple similarity variable leads to non-dimensional governing equations for the local velocities $U = (U(\zeta), V(\zeta), Re^2 W(\zeta))$, given by

\[
W' + 2U = 0, \quad (1a)
\]

\[
WU' + U^2 - V^2 = mT_s^2 + U'', \quad (1b)
\]

\[
WV'' + 2UV' = V'', \quad (1c)
\]

\[
V^2 \cot \psi = P', \quad (1d)
\]

where $P$ is the non-dimensional boundary-layer pressure and

\[
T_s = \frac{C^* x^{**m}}{x^* \Omega^* \sin \psi} \quad (2)
\]

is a parameter representing the relative strength of the oncoming axial flow to the local swirl. Note that $m$ is the exponent in the power law that defines the external slip velocity driven by the axial flow. The governing system (1) are subject to the boundary conditions

\[
U = W = 0, \quad V = 1, \quad \text{on } \zeta = 0, \quad (3a)
\]

\[
U \to T_s, \quad V \to 0, \quad \text{as } \zeta \to \infty. \quad (3b)
\]
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This formulation is capable of describing the dynamics for cases in which axial flow is not present (i.e. a cone rotating in an otherwise quiescent fluid), or in which axial flow is present but the cone’s half-angle is precisely $\psi = \pi/2$ (so that $m = 1$), which is equivalent to an axial flow impinging upon a rotating disk. These cases have been studied previously (for instance see [2] for an investigation of the rotating disk boundary layer and [3, 4] for cones both experiencing an oncoming axial flow and rotating within a quiescent fluid), and much progress has been made in characterising the flow’s behaviour and stability properties. When we wish to investigate the boundary-layer stability of an axially-driven flow near a cone of half-angle $\psi < \pi/2$ we encounter a problem in the above formulation, in that the flow strength parameter $T_s$ contains a dependence on the stream wise coordinate. In its definition (2) we clearly see that stream wise-coordinate independence is only achieved if $\psi = \pi/2$, representing the absence of an axial flow, or $m = 1$, so that the cone degenerates to a disk.

2. Boundary layer flow for a cone with $\psi < \pi/2$ experiencing an oncoming axial flow

It is possible to make progress by performing a Mangler transformation on the governing equations (giving newly transformed dimensional along-surface and normal coordinate directions $\mathbf{x}^*$ and $\mathbf{z}^*$ respectively), and writing the transformed velocity components in terms of a streamfunction, $f$ (see [5]). We introduce two new parameters

$$s = \left(\frac{V_\infty}{U_e}\right)^2,$$

$$\eta_1 = \mathbf{z}^* \left(\frac{m + 3}{6} \frac{U_e}{\nu^* \mathbf{x}^*}\right)^{1/2},$$

which define the coordinates for the streamfunction formulation of the governing system. The $s$ coordinate is a measure of the relative strength of the local swirl velocity at some stream wise location to the external slip velocity corresponding to that location, and satisfies $s^{-1/2} = T_s$ in terms of our earlier flow strength parameter (2). The $\eta_1$ coordinate represents the new surface-normal direction scaled on a boundary-layer displacement thickness according to the new velocity scales.

Writing the along-surface and azimuthal dimensional velocity components as

$$U^* = U_e \left(\frac{\partial f}{\partial \eta_1}(\eta_1, s) + 1\right), \quad V^* = V_\infty g(\eta_1, s),$$

we may derive the streamfunction formulation for the boundary-layer system. Denoting derivatives with respect to $\eta_1$ by primes, we have

$$f''' + (f + \eta_1) f'' + \frac{2m}{m + 3} \left(1 - (f' + 1)^2\right)$$

$$+ \frac{2s}{m + 3} \left[g^2 + 2(1 - m) \left(f'' \frac{\partial f}{\partial s} + (f' + 1) \frac{\partial f'}{\partial s}\right)\right] = 0,$$

$$(6a)$$

$$g'' + (f + \eta_1) g' - \frac{4}{m + 3} (f' + 1) g + \frac{4(1 - m)s}{m + 3} \left(g \frac{\partial f}{\partial s} - (f' + 1) \frac{\partial g}{\partial s}\right) = 0,$$

$$(6b)$$
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subject to the boundary conditions

\begin{align}
  f &= 0, \quad f' = -1, \quad g = 1 \quad \text{on } \eta_1 = 0, \quad (7a) \\
  f' &\to 0, \quad g \to 0 \quad \text{as } \eta_1 \to \infty. \quad (7b)
\end{align}

Our choice to non-dimensionalise velocities on the local swirl strength thus leads to the non-dimensional velocity components being given in terms of the streamfunction by

\begin{equation}
  U = s^{-\frac{1}{2}} \left( \frac{\partial f}{\partial \eta_1} (\eta_1, s) + 1 \right), \quad V = g(\eta_1, s),
\end{equation}

which allows us to recover the local dimensional velocity values at a given streamwise location. The formulation above differs slightly from previous works, most notably [5], due to the fact that our chosen numerical method requires decaying conditions in the far field. The details of the numerical procedure will be discussed in §3.

2.1. Perturbed boundary-layer system for the cone

As outlined in [4], it is possible to define a local Reynolds number by considering the velocity scale as a function of the streamwise coordinate along the cone. In this problem, we denote this local Reynolds number at a location \( x_s^* \) by

\begin{equation}
  Re_L = x_s^* \Omega^* \delta^* \sin \psi \nu^*,
\end{equation}

where \( \delta^* \) is the boundary-layer thickness and \( \nu^* \) is the fluid’s kinematic viscosity.

To determine the stability of the boundary layer flow at some non-dimensional location \( x_s \) corresponding to local Reynolds number \( Re_L \), we perturb the mean flow from the previous section with disturbances given by

\begin{equation}
  (\hat{u}, \hat{v}, \hat{w}, \hat{p}) = (u(\zeta), v(\zeta), w(\zeta), p(\zeta)) \exp(i(\alpha x + \beta R_L \theta - \omega t)),
\end{equation}

where \( \zeta \) is the unscaled non-dimensional boundary-normal coordinate within the boundary layer, \( \alpha \) is the disturbance wavenumber in the streamwise direction, \( \beta \) is the wavenumber in the azimuthal direction, and \( \gamma \) is the frequency. The two different boundary-normal coordinates are related by

\begin{equation}
  \eta_1 = \zeta \left( \frac{m + 3}{2s^2} \sin \psi \right)^{\frac{1}{2}}.
\end{equation}

In general, the wavenumbers and frequencies are permitted to be complex as appropriate for the particular form of instability being investigated in the spatio-temporal analysis. However, geometrical considerations imply that the \( \theta \)-wavenumber must satisfy \( \beta Re_L = n \in \mathbb{N} \setminus \{0\} \), which corresponds to the existence of an integer number of vortices. This is required to enforce a periodicity constraint.

Substituting these disturbance ansätze into the original Navier–Stokes system leads to a large system of equations (omitted for brevity) which form a quadratic eigenvalue problem in \( \alpha \) and \( \beta \). Noting that the assumption of a unidirectional viscous baseflow allows the derivation of a simpler perturbed system, namely the Orr–Sommerfeld equation (see, for instance, [6]), we may also perform the same operation with the full disturbance system to obtain an Orr–Sommerfeld analogue.
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for the rotating cone system. This is also achieved by neglecting terms obtained as a result of streamline-curvature effects in the full equations.

The cone system’s Orr-Sommerfeld equation is given simply by

\[ \left[ i(D^2 - k^2)^2 + R_L(\alpha U + \beta V - \gamma)(D^2 - \gamma^2) - R_L(\alpha D^2 U + \beta D^2 V) \right] w = 0, \]  

(12)

where \( D \) represents a derivative with respect to \( \zeta \) and \( k = \sqrt{\alpha^2 + \beta^2} \) is the effective wavenumber of the disturbance.

Solving the full eigenvalue problem, or the Orr-Sommerfeld system, for certain combinations of values for \( \alpha, \beta, \) and \( \gamma \) at each local Reynolds number \( Re_L \) for a given half-angle \( \psi \) yields a dispersion relation for the disturbances. This will allow us to study the occurrence of convective instabilities in the boundary-layer flow.

3. Numerical method

3.1. Spectral collocation method

In this work, the numerical results described have been obtained by utilising a (pseudo)-spectral collocation method based on Chebyshev polynomials, allowing for much higher precision in the solution than a corresponding finite difference or finite volume method (see [7]). For instance, compared to simpler finite-difference approaches to solving the base flow, for which the accuracy is formally second-order in the grid size, it is possible to achieve supergeometric accuracy using a spectral method based on Chebyshev polynomials. In essence, this method is predicated on the representation of all dependent flow variables in the form of a set of coefficients which appear in their expansions in terms of the Chebyshev polynomial basis. For this discussion, we consider a single dependent variable, \( f \), valid on some mapped domain defined by an appropriate object with \( N \) nodes (and \( N \) Chebyshev basis functions denoted \( T_n \) for \( n \in \{0, 1, \ldots, N - 1\} \). Denote the coordinate mapping by a smooth bijective function \( x : S \to X \) for the canonical interval \( S = [-1, 1] \). Let the mapped Chebyshev-Lobatto points be \( x_k = x(s_k) \) for \( k \in \{0, 1, \ldots, N - 1\} \). Then under that mapping, we can define functions \( U_n(x) = T_n(s(x)) \), and so our dependent variable may be written as a sum

\[ f(x) = \sum_{n=0}^{N-1} \alpha_n U_n(x) = \sum_{n=0}^{N-1} \alpha_n T_n(s(x)). \]  

(13)

This gives us a handle to be able to transfer derivatives of dependent variables directly onto derivatives of the corresponding linear combinations of Chebyshev functions. One point to note is that the resulting linear algebra problems that must be solved when representing a set of differential equations this way exhibit a much greater degree of fill-in compared to the equivalent problems posed in finite-difference formulations. This can lead to a significant trade-off of accuracy for solution speed and memory requirements for higher-dimensional problems. For this particular problem of solving one-dimensional base-flow profiles, the advantages of having a far more accurate solution outweigh the marginal increase in computation time.

In particular, the processes for solving for the baseflow profiles as well as the resulting eigenvalue system(s) were performed on the same spectral representations of variables from the outset, which minimises the introduction of truncation and discretisation errors. Previous approaches to investigating the stability of this
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boundary-layer flow computed base flow profiles using a finite-difference approach, followed by an interpolation onto a new mesh designed for use with the aforementioned spectral method. Given the potential for vastly different node distributions between these two domain representations, there is the possibility of introducing errors into the basic flow profiles that could impact a stability computation if following such a process.

3.2. Automatic differentiation for nonlinear terms

Additionally, a method for automatically differentiating non-linear expressions in differential equations allows the numerical method to evaluate them with machine precision (see [8, 9]), so that the computed basic solutions and perturbed quantities do not suffer from the feed-through effect of truncation errors during the evaluation of numerical Jacobians.

As a brief introduction to the mode of operation of the automatic differentiation process, we consider one particular definition of so-called 'dual numbers'. Denote a dual number (with a single dual component) as \( \hat{x} = x + x' \epsilon \), where \( x \) is the leading-order part and \( x' \) is the dual component with \( \epsilon \) an infinitesimal. The dual number follows all usual rules of real or complex arithmetic, as appropriate, with the condition that \( \epsilon^2 = 0 \), i.e. the dual unit is nilpotent. The result of this arithmetic on dual numbers yields, for example

\[
\hat{x} + \hat{y} = (x + x' \epsilon) + (y + y' \epsilon) = (x + y) + (x' + y') \epsilon,
\]

and

\[
\hat{x} \hat{y} = (x + x' \epsilon)(y + y' \epsilon) = xy + (x'y + y'x) \epsilon.
\]

For a polynomial \( P(\xi) = \sum_{k=0}^{n} a_k \xi^k \), we have

\[
P(\hat{x}) = \sum_{k=0}^{n} a_k (x + x' \epsilon)^k = a_0 + \sum_{k=1}^{n} a_k \left(x^k + kx^{k-1}x' \epsilon\right) = \sum_{k=0}^{n} a_k x^k + x' \left(\sum_{k=1}^{n} a_k kx^{k-1}\right) \epsilon = P(x) + x' P'(x) \epsilon.
\]

From these examples we can see the essential idea that the dual components of the dual numbers transform as would first derivatives of the leading-order expressions, satisfying product and chain rules under arithmetic and application of (smooth) functions. By insisting that the dual component always transforms according to the arithmetic of first derivatives, we obtain a system by which derivatives of expressions are automatically evaluated along with the expression itself. In order for this system
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to work, we must provide rules for the application of elementary functions to dual numbers, for instance
\[ \sin(\tilde{x}) = \sin(x + x'\epsilon) = \sin(x) + x'\cos(x)\epsilon. \]

Once this is done, in combination with an implementation of all arithmetic manipulations on dual numbers, dual components are calculated using the chain rule, product rule, and basic arithmetic rules for derivatives. The calculated values are then accurate to machine precision and not subject to truncation error as would be the case for finite differences, for example. For this system to work we must seed the derivatives by setting a unity dual component on the variable with respect to which we wish to differentiate an expression.

This is particularly important for iterated problems, such as solving the base-flow (6) spatially in \( \eta_1 \) while marching through \( s \). A by-product of using automatic differentiation techniques is that it was possible to design a system by which non-linear terms are automatically (and exactly) linearised for inclusion in an algorithmically-generated Jacobian.

4. Results

4.1. Investigating the convergence properties of the spatial solver

In order to assess the performance of our numerical scheme, in particular its behaviour for the spatial solution of the base-flow profiles, we restrict our attention to the basic flow components for the case \( s = 0 \). We shall use the Grid Convergence Index (GCI) method to evaluate the computations, by using three choices of mesh refinement for the Chebyshev basis set introduced in the previous section. Focussing on the profiles for \( f' \), we consider meshes composed of Chebyshev basis sets containing 12, 18, and 27 modes and sample the values obtained at twenty locations in the region of greatest change of \( f' \). This gives us a constant refinement factor of \( r_{21} = r_{32} = 1.5 \), leading to local apparent convergence orders with maximum value \( p_{\text{max}} = 12.42 \) and minimum value \( p_{\text{min}} = 1.33 \). Figure 2 shows this result in the rapid convergence of the \( f' \) profiles to the one obtained via Richardson extrapolation. The average apparent order is \( p_{\text{ave}} = 5.07 \).

Restricting our attention to the region \( \eta_1 < 3 \), for which the value of \( f' \) is not close to its limiting value, we can obtain an average GCI value of 0.2%. Figure 3 shows that the errors for this computation are insignificant, and only play a role when the function values are settling to their far-field limiting values, as one would expect. This analysis shows that using this spectral method yields significantly better-converged solutions than are obtainable via an approach using central differencing, for example.

4.2. Baseflow profiles for varying \( s \)

Figures 4 and 5 show the non-dimensional streamwise and azimuthal velocity components for the steady boundary-layer flow for a range of values of \( s \). While the dependence of the basic flow profiles on the half-angle of the cone, \( \psi \), still exists in the governing equations through the parameter \( m \), the magnitude of this influence is far smaller than that caused by variations in \( s \) as \( \psi \) changes. An angle of \( \psi = 70^\circ \) was used to generate the basic profiles seen in Figure 4 and Figure 5, but choosing other angles yields sets of profiles which are almost indistinguishable. This observation makes sense
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Figure 2. Profiles for $f'$ for differing mesh resolutions $N \in \{12, 18, 27\}$, along with the Richardson extrapolation for the curves following the GCI method. The solution is qualitatively well-converged and the closer view shows this more clearly.

Figure 3. The extrapolated profile for $f'$ with error bars corresponding to the local GCI measure applied. The larger view shows that the error bars are compact, and a closer view reveals that there are only few locations at which errors are appreciably non-zero.

since our non-dimensionalisation of the base velocity components involved a spatially-dependent term which takes account of the cone’s increasing surface velocity (in which the half-angle of the cone plays a critical role) as we move down in the streamwise direction.

We see from the definition of $s$ that as $s$ increases, the effect of the cone’s rotation on the mean flow field dominates that of the incoming axial flow. This results in the streamwise velocity components beginning to take on points of inflection, and at sufficiently large values of $s$ finding a local maximum in the fluid domain before dropping down to the inviscid solution at the edge of the boundary layer.

4.3. Neutral curves for the Orr–Sommerfeld stability problem

To demonstrate the differences between solving the full stability problem and the simplified Orr–Sommerfeld system we choose to investigate a convective instability in line with previous analytical studies [10]. In particular, it is known that rotating broad cones (of large half angle) in an otherwise still fluid are susceptible to a crossflow instability mode, and this mode arises from the inflectional nature of the streamwise velocity component. In order to find neutral stability curves for this instability mode, we consider stationary disturbances with $\gamma_i = 0$ and $\frac{d\gamma}{d\beta} = 1$, as this forces the vortices
to rotate with the surface of the cone. Solving the dispersion relation for each local Reynolds number then yields a pair \((\alpha, \beta)\) corresponding to a point on the neutral curve for this instability mode. Noting that the combined effect of the streamwise and azimuthal wavenumbers is to define an effective spiral direction for the instability mode, we define \(k_\delta = \sqrt{\alpha^2 + \beta^2} \sin \psi\) to be the modified wavenumber and \(\epsilon = \arctan \frac{\beta \sin \psi}{\alpha r}\) to be the modified waveangle of the disturbances. Figure 6 shows the neutral stability curves obtained as a result of solving the eigenvalue problems corresponding to the simplified Orr–Sommerfeld approximation for the axial flow case for a range of \(s\) values. These exhibit close agreement with results seen previously for this stability approach to the axial flow problem.

The natural next step is to graduate to solving the full perturbation equations for the axial flow case, which is complicated by more inherently two-dimensional base flow details. Some insight can be found in [10] for the corresponding perturbation solutions for the case of a rotating cone in an otherwise still fluid. One clear feature of these neutral curves is that the Orr–Sommerfeld approximation finds a lower critical Reynolds number for the instability, and fails to find the second lobe of the neutral curve which is obtained by solving the full system. This can be explained by the fact that in deriving the Orr–Sommerfeld system, we necessarily neglected streamline-curvature effects of the flow. These streamline-curvature effects are the source of a centrifugally-driven instability which is not dominant for broad cones, hence the appearance of the second, smaller lobe at a higher Reynolds number than the primary lobe. In both cases, whether considering the Orr–Sommerfeld or full stability equations, the neutral curves agree with the asymptotic results of [10] in the limit of large Reynolds numbers.

5. Concluding remarks

When investigating the convective stability of the boundary-layer flow over a rotating cone (in an otherwise still fluid), it can be seen that numerical formulations accurately model the instabilities found via previous asymptotic analyses. A simplifying Orr–Sommerfeld-type approach to the stability problem, utilising the assumption of a parallel base flow and the subsequent neglect of streamline-curvature effects, yields
neutral stability curves that broadly follow those obtained by solving the more costly full problem. However, this simpler approach has two drawbacks: the critical local Reynolds number for the onset of this convective instability is underestimated slightly, and the loss of streamline-curvature effects eliminates the existence of a second lobe in the neutral stability curve found by solving the full perturbed system.

It is our aim to extend the solution of the full stability problem to cases of axial flow and gain a clear picture of the interplay between the dominant sources of convective instability as the cone’s half angle is changed. As mentioned earlier, the dominant physical basis for convective instability depends on the cone’s half angle [10, 11], so the numerical approach of relying on an Orr–Sommerfeld approximation will be deficient in identifying the neutral stability curve for the instability as streamline-curvature effects will be ignored.

As discussed previously, the combination of a highly-accurate spectral method with automatic differentiation makes it possible to greatly reduce a number of sources of error in our numerical investigations. Using this numerical method, along with solving the full stability problem and resolving both branches of the neutral curves for axial flow, we will have a clearer picture of the stabilising or destabilising effect of varying the level of axial flow for any given cone angle. Obtaining these results will additionally allow us to quantify the effect of crossflow or centrifugal instabilities in the rotating cone geometry, which is valuable in being able to find parameter regimes for which a boundary layer’s development can be accurately predicted.
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