One-way unlocalizable information deficit

Xue-Na Zhu\(^1\) and Shao-Ming Fei\(^2,3\)

\(^1\) Department of Mathematics, South China University of Technology, Guangzhou 510640, People’s Republic of China
\(^2\) School of Mathematical Sciences, Capital Normal University, Beijing 100048, People’s Republic of China
\(^3\) Max-Planck-Institute for Mathematics in the Sciences, D-04103 Leipzig, Germany

E-mail: jing_feng1986@126.com and feishm@cnu.edu.cn

Received 6 March 2013, in final form 21 May 2013
Published 24 July 2013
Online at stacks.iop.org/JPhysA/46/325303

Abstract

We introduce a one-way unlocalizable information deficit with respect to the one-way information deficit, similar to the definition of one-way unlocalizable quantum discord with respect to one-way quantum discord. The properties of the one-way unlocalizable information deficit and the relations among one-way unlocalizable information deficit, one-way unlocalizable quantum discord, one-way quantum discord, one-way information deficit and other quantum correlations are investigated. Analytical formulas of the one-way unlocalizable information deficit are given with detailed examples.

PACS numbers: 03.65.Ud, 03.67.Mn

1. Introduction

Quantum entanglement [1] is of special importance in quantum information processing, such as quantum teleportation, dense coding and remote state preparation. To quantify quantum entanglement, various entanglement measures have been suggested [2]. Considerable efforts have been made to estimate these entanglement measures and investigate the roles played by these measures in different information processing. The one-way unlocalizable entanglement in terms of entanglement of assistance has been presented in [3]. It is shown that the polygamous nature of distributed quantum entanglement in multipartite systems is strongly related to this unlocalizable character.

Quantum correlations other than quantum entanglement have also been extensively explored recently. It has been shown that some quantum information processing could be carried out without quantum entanglement. For instance, quantum discord [4] plays an important role in some quantum information processing such as assisted optimal state discrimination, in which only one side discord is required in the optimal process of assisted state discrimination, while another side discord and entanglement are not necessary [5].
With respect to one-way quantum discord, one-way unlocalizable quantum discord has been introduced and studied in [8]. In this paper, with respect to the one-way information deficit, we introduce and study the one-way unlocalizable information deficit. Systematic relations among one-way unlocalizable information deficit, one-way unlocalizable quantum discord, one-way quantum discord, one-way information deficit and other quantum correlations are presented. Tradeoff relations are discussed.

2. One-way unlocalizable information deficit

2.1. Definition

Quantum correlation is emerging as a primitive notion in physics following an essential extension of the classical Shannon information theory into the quantum domain. There have been many different definitions of measures for quantum correlations.

Let $H_A$ and $H_B$ be the $m$- and $n$-dimensional ($m \leq n$) vector spaces, respectively. The quantum discord of a bipartite quantum state $\rho^{AB} \in H_A \otimes H_B$ is defined by [6, 7]

$$\delta^- (\rho^{AB}) = S(\rho^B) - S(\rho^{AB}) + \min_{\{\Pi_i^B\}} \sum_i p_i S(\rho_i^A),$$

where $S(\rho) = -\text{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy, $\rho^B = \text{Tr}_A(\rho^{AB})$ is the reduced density matrix of the system $B$, $p_i = \text{Tr}(I_A \otimes \Pi_i^B)\rho^{AB}(I_A \otimes \Pi_i)$ with $I_A$ being the identity operator on the subsystem $A$, $\rho_i^A = \text{Tr}_B(I_A \otimes \Pi_i^B\rho^{AB})/p_i$ is the state of the subsystem $A$ after the measurement on $B$. $\Pi_i^B = |i\rangle \langle i|$ is the von Neumann measurement on $B$ satisfying $\sum_i \Pi_i^B = I_B$, with $I_B$ the identity operator on $B$, $|i\rangle$, $i = 1, \ldots, n$, is the computational basis.

Inspired by the definition of unlocalizable entanglement in [3], the authors in [8] provided the quantity one-way unlocalizable quantum discord

$$\delta^-_u (\rho^{AB}) = S(\rho^B) - S(\rho^{AB}) + \max_{\{\Pi_i^B\}} \sum_i p_i S(\rho_i^A).$$

Closely related to the one-way quantum discord, the one-way information deficit [9, 10] is defined as the minimal increase of entropy after a von Neumann measurement on $B$ [9]:

$$\Delta^- (\rho^{AB}) = \min_{\{\Pi_i^B\}} S\left(\sum_i \Pi_i^B \rho^{AB} \Pi_i^B\right) - S(\rho^{AB}).$$

Similar to the quantum discord (1) and the one-way unlocalizable quantum discord (2), corresponding to the one-way information deficit (3), we define the one-way unlocalizable information deficit

$$\Delta^-_u (\rho^{AB}) = \max_{\{\Pi_i^B\}} S\left(\sum_i \Pi_i^B \rho^{AB} \Pi_i^B\right) - S(\rho^{AB}).$$

Dual to the one-way information deficit $\Delta^- (\rho^{AB})$, $\Delta^-_u (\rho^{AB})$ is the maximum distance of relative entropy from the state $\rho^{AB}$ to the set $S^-$, which can be created reversibly under one-way communications.

If the maximum in (4) is taken over all the von Neumann measurements $\{\Pi_i^B\}$ which do not disturb reduced states $\rho^B = \text{Tr}_A(\rho^{AB})$ locally, then the one-way unlocalizable information deficit $\Delta^-_u (\rho^{AB})$ is just the relative entropy of nonlocality [11]:

$$N_{RE}^- (\rho^{AB}) = \max_{\{\Pi_i^B\}} [S(\rho^{AB}) - S(\rho^{AB})],$$

where $\rho^{AB} = \sum_i (I_A \otimes \Pi_i^B)\rho^{AB}(I_A \otimes \Pi_i^B)$. Hence, the one-way unlocalizable information deficit $\Delta^-_u (\rho^{AB})$ is equal to the relative entropy of nonlocality $N_{RE} (\rho^{AB})$ for a set of special states $\rho^{AB}$ such that $\rho^B$ is invariant under all von Neumann measurements $\{\Pi_i^B\}$. \[2\]
2.2. Polygamy inequality

Similar to the relations between $\delta^- (\rho^{AB})$ and $\delta^- (\rho^{AB})$ investigated in [3], in the following we study the relations between $\Delta^- (\rho^{AB})$ and $\Delta^- (\rho^{AB})$, the relations among the quantum discord, the one-way unlocalizable quantum discord, the one-way information deficit and the one-way unlocalizable information deficit, as well as tradeoff relations.

We first present some properties of the one-way unlocalizable information deficit. From [12, 13], any partial von Neumann measurement on $B$ can be modeled as an indirect measurement with an apparatus $Q$ initialized in a fixed pure state $|0\rangle^M$, $\rho_1 = |0\rangle^M \otimes \rho_{AB}$, and applying a unitary operator $U$ on the whole state, $\rho_2 = U \rho_1 U^\dagger$, where $U = I_A \otimes U_{AB}$ and $\text{Tr}_Q[U(|0\rangle^M \otimes \rho_{AB} U^\dagger] = \sum_i \Pi_i^B \rho^{AB} \Pi_i^B$. As an important measure of quantum entanglement, the distillable entanglement $E_D (\rho^{AB})$ defined in [14, 15] is the asymptotic number of standard singlets that can be prepared from a system in the state $\rho^{AB}$ by local operations. From [9], one has $E_D^{M|AB} (U \rho_1 U^\dagger) = S(\sum_i \Pi_i^B \rho^{AB} \Pi_i^B) - S(\rho^{AB})$. Hence, if a bipartite state $\rho^{AB}$ has nonzero quantum discord $\delta^- > 0$, any von Neumann measurement on $B$ creates distillable entanglement between the measurement apparatus and the total system $AB$. The maximal distillable entanglement created in a von Neumann measurement on $B$ is equal to the one-way unlocalizable information deficit: $\Delta^- (\rho^{AB}) = \max_{MU} E_D^{M|AB} (U \rho_1 U^\dagger)$. Therefore, the one-way unlocalizable information deficit has the following properties.

(i) $\Delta^- (\rho^{AB})$ does not increase under arbitrary quantum operations $\Lambda_B$ on $B$

$$\Delta^- (\Lambda_B (\rho^{AB})) \leq \Delta^- (\rho^{AB}),$$

as $E_D^{M|AB}$ does not increase under local operations and classical communications.

(ii) $\Delta^- (\rho^{AB})$ does not increase on average under stochastic local operations and classical communications (SLOCC):

$$\sum_i q_i \Delta^- (\sigma_i^{AB}) \leq \Delta^- (\rho^{AB}),$$

where $q_i = \text{Tr}[V_i \rho^{AB} V_i^\dagger]$, $\sigma_i = V_i \rho^{AB} V_i^\dagger / q_i$ and $V_i$ are Kraus operators characterizing a local quantum operation on $B$ with $\sum_i V_i^\dagger V_i = I$. Inequality (7) is due to the fact that the distillable entanglement does not increase on average under SLOCC: $\sum_i q_i E (\sigma_i^{AB}) \leq E (\rho^{AB})$.

From definitions (1)–(4), we have the following lower bounds of the one-way unlocalizable information deficit.

**Theorem 1.** For any bipartite quantum state $\rho^{AB}$, we have

$$\begin{align*}
\Delta^- (\rho^{AB}) &\geq \Delta^- (\rho^{AB}); \\
\Delta^- (\rho^{AB}) &\geq \delta^- (\rho^{AB}); \\
\Delta^- (\rho^{AB}) &\geq \Delta^- (\rho^{AB}) - \delta^- (\rho^{AB}).
\end{align*}$$

**Proof.** The first inequality is easily obtained from the definitions of $\Delta^- (\rho^{AB})$ and $\Delta^- (\rho^{AB})$. For $p_i = \text{Tr}[\Pi_i^B \rho^{AB} \Pi_i^B]$ and $\rho_i^A = \Pi_i^B \rho^{AB} \Pi_i^B / p_i$, one has the following equality [9]:

$$\sum_i p_i S (\rho_i^A) = S \left( \sum_i \Pi_i^B \rho^{AB} \Pi_i^B \right) - S \left( \sum_i \Pi_i^B \rho^{AB} \Pi_i^B \right).$$
According to the definition of one-way unlocalizable quantum discord, we obtain

\[
\delta^- (\rho^{AB}) = S(\rho^B) - S(\rho^{AB}) + \max_{\{\rho_i^B\}} \left\{ -S \left( \sum_i \Pi_i^B \rho^{AB} \Pi_i^B \right) \right\}
\]

\[
\in S(\rho^B) - S(\rho^{AB}) + \max_{\{\rho_i^B\}} \left\{ -S \left( \sum_i \Pi_i^B \rho^{AB} \Pi_i^B \right) \right\}
\]

\[
= \Delta^- (\rho^{AB}) - \Delta^- (\rho^B) = \Delta^- (\rho^{AB}).
\]

The last inequality in (10) can be proved similarly

\[
\delta^- (\rho^{AB}) = S(\rho^B) - S(\rho^{AB}) + \min_{\{\rho_i^B\}} \left\{ -S \left( \sum_i \Pi_i^B \rho^{AB} \Pi_i^B \right) \right\}
\]

\[
\geq S(\rho^B) - S(\rho^{AB}) + \min_{\{\rho_i^B\}} \left\{ -S \left( \sum_i \Pi_i^B \rho^{AB} \Pi_i^B \right) \right\}
\]

\[
= \Delta^- (\rho^{AB}) - \Delta^- (\rho^B).
\]

Here, in fact, \( \Delta^- (\rho^B) = \log_2 n - S(\rho^B). \)

The relationship between \( \delta^- \) (defined on single copies) and \( \Delta^- \) was shown in [16]. The one-way information deficit is non-negative and zero only for states with zero quantum discord. To compare the one-way unlocalizable deficit with other measures of quantum correlation, one-way unlocalizable quantum discord, we consider the Bell-diagonal states

\[
\rho_{ABm}^{AB} = \frac{1}{4} \left( I^A \otimes I^B + \sum_{i=1}^3 c_i \sigma_i \otimes \sigma_i \right),
\]

where \( c_i \) are real numbers and \( \sigma_i \) are Pauli matrices.

For the state \( \rho_{ABm}^{AB} \), one has \( \Delta^- (\rho_{ABm}^{AB}) = \delta^- (\rho_{ABm}^{AB}) \). It can also be proven that \( \Delta^- (\rho_{ABm}^{AB}) = \delta^- (\rho_{ABm}^{AB}) \). A von Neumann measurement \( \{\Pi_{0i}^B, \Pi_i^B\} \) of a two-qubit system can be characterized by a unit vector \( n = (n_1, n_2, n_3)^T \) on the Bloch sphere [17]

\[
\Pi_i^B = \frac{1}{2} \left( I^B + \sum_{j=1}^3 n_j \sigma_j^B \right), \quad \Pi_i^B = \frac{1}{2} \left( I^B - \sum_{j=1}^3 n_j \sigma_j^B \right).
\]

From equation (12), we have \( \rho_{ABm}^{B} = \text{Tr}_A(\rho_{ABm}^{AB}) = I^B/2 \). Set \( n_1 = \cos(x/2) \sin(y/2), n_2 = \cos(x/2) \cos(y/2) \) and \( n_3 = \sin(x/2) \). We have \( S(\sum_i \Pi_i^B \rho_{ABm}^{AB}) = S(I^B/2) = S(\rho_{ABm}^{B}). \)

Moreover, from [11] we have

\[
\Delta^- (\rho_{ABm}^{AB}) = \delta^- (\rho_{ABm}^{AB}) = f(c_{\min}) - f(c_1, c_2, c_3),
\]

where

\[
f(x) = -\frac{1 + x}{2} \log_2 \left( \frac{1 + x}{2} \right) - \frac{1 - x}{2} \log_2 \left( \frac{1 - x}{2} \right),
\]

\[
c_{\min} = \min\{|c_1|, |c_2|, |c_3|\}
\]

\[
f(c_1, c_2, c_3) = \left( \frac{1 - c_1 - c_2 - c_3}{4} \right) \log_2 \left( \frac{1 - c_1 - c_2 - c_3}{4} \right) + \left( \frac{1 - c_1 + c_2 + c_3}{4} \right) \log_2 \left( \frac{1 - c_1 + c_2 + c_3}{4} \right) + \left( \frac{1 + c_1 - c_2 + c_3}{4} \right) \log_2 \left( \frac{1 + c_1 - c_2 + c_3}{4} \right) + \left( \frac{1 + c_1 + c_2 - c_3}{4} \right) \log_2 \left( \frac{1 + c_1 + c_2 - c_3}{4} \right) - 1.
\]
Generally, one has the following conclusion.

**Theorem 2.** For any bipartite quantum state $\rho^{AB}$, the one-way unlocalizable information deficit is zero if and only if the one-way unlocalizable quantum discord is zero.

**Proof.** If the one-way unlocalizable information deficit is zero: $\Delta^-_\mu (\rho^{AB}) = 0$, from the first inequality of theorem 1, we obtain that $\Delta^- (\rho^{AB}) = 0$. Then, from equations (3) and (4), we have $\max_{|\Pi^B_\mu|} S\left( \sum_i \Pi^B_i \rho^{AB} \Pi^B_i \right) = S(\rho^{AB})$ and $\min_{|\Pi^B_\mu|} S\left( \sum_i \Pi^B_i \rho^{AB} \Pi^B_i \right) = S(\rho^{AB})$. Therefore for all $\{\Pi^B_i\}$, formula $S\left( \sum_i \Pi^B_i \rho^{AB} \Pi^B_i \right) = S(\rho^{AB})$ is true. By using equation (9), one proves the result.

If the one-way unlocalizable quantum discord is zero: $\delta^-_\mu (\rho^{AB}) = 0$, since $0 \leq \delta^- (\rho^{AB}) \leq \delta^-_\mu (\rho^{AB})$, then we have $\delta^- (\rho^{AB}) = 0$. Hence, $\rho^{AB}$ has the form $\rho^{AB} = \sum_i p_i \rho^A_i \otimes |i\rangle \langle i|$, and $\delta^-_\mu (\rho^{AB}) = 0$. $\square$

2.3. Discussions on tradeoff relations

In the following, we study the tradeoff relations. The one-way unlocalizable quantum entanglement is defined by [8]

$$S^-_\chi (\rho^{AB}) = \min_{|\Pi^B|} \left[ S(\rho^A) - \sum_i p_i S(\rho^A_i) \right].$$

(14)

For a tripartite pure state $|\psi\rangle^{ABC}$, in terms of the Buscemi Gour–Kim equation [8], a tradeoff relation between the one-way unlocalizable quantum discord for $A$ and $B$ with the measurement on $B$ and the one-way unlocalizable quantum entanglement for $C$ and $B$ has been obtained in [8]

$$\delta^-_\mu (\rho^{AB}) = S(\rho^B) - S^-_\chi (\rho^{BC}).$$

(15)

For states that all the measurements $|\Pi^B_i|$ do not disturb $\rho^B$ locally, there is a tradeoff relation between the one-way unlocalizable deficit for $A$ and $B$ with the measurement on $B$ and the one-way unlocalizable quantum entanglement for $C$ and $B$ [11]

$$\Delta^-_\mu (\rho^{AB}) = S(\rho^B) - S^-_\chi (\rho^{BC}).$$

(16)

According to the tradeoff relations (15) and (16), if $\rho^{AB}$ satisfies the particular condition that all the von Neumann measurements $|\Pi^B_i|$ do not disturb $\rho^B$ locally, for instance, the state $\rho^{AB}_m$ in (12), we have

$$\Delta^-_\mu (\rho^{AB}) = \delta^-_\mu (\rho^{AB}).$$

(17)

3. Conclusion

We have introduced the one-way unlocalizable information deficit. The essential properties of the one-way unlocalizable information deficit and some foundational relations among the one-way unlocalizable information deficit, the one-way unlocalizable quantum discord, the one-way quantum discord, the one-way information deficit and other quantum correlations have been investigated. As different kinds of quantum correlations play different roles in different quantum information processing, these properties and relations might allow us to highlight the deep relations among deep processing and correlations.
Acknowledgment

This work is supported by the NSFC under number 11275131.

References

[1] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Rev. Mod. Phys. 81 865
[2] Streltsov A, Kampermann H and Bruß D 2010 New J. Phys. 12 123004
[3] Buscemi F, Gour G and Kim J S 2009 Phys. Rev. A 80 012324
[4] Modi K, Brodutch A, Cable H, Paterek T and Vedral V 2012 Rev. Mod. Phys. 84 1655
[5] Roa L, Retamal J C and Alid-Vaccarezza M 2011 Phys. Rev. Lett. 107 080401
   Li B, Fei S M, Wang Z X and Fan H 2012 Phys. Rev. A 85 022328
[6] Zurek W H 2000 Ann. Phys., Lpz. 9 855
[7] Ollivier H and Zurek W H 2001 Phys. Rev. Lett. 88 017901
[8] Xi Z J, Fan H and Li Y M 2012 Phys. Rev. A 85 052102
[9] Streltsov A, Kampermann H and Bruß D 2011 Phys. Rev. L 106 160401
[10] Horodecki M, Horodecki P, Horodecki R, Oppenheim J, Sen A, Sen U and Synak-Radtke B 2005 Phys. Rev. A 71 062307
[11] Xi Z J, Wang X G and Li Y M 2012 Phys. Rev. A 85 042325
[12] Schlosshauer M 2005 Rev. Mod. Phys. 76 1267
[13] Ozawa M 1984 J. Math. Phys. 25 79
[14] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Phys. Rev. A 54 3824
[15] Plenio M B and Virmani S 2007 Quantum Inform. Comput. 7 1
[16] Zurek W H 2003 Phys. Rev. A 67 012320
[17] Liu X M, Ma J, Xi Z J and Wang X G 2011 Phys. Rev. A 83 012327