Long-Time Asymptotic Behavior for the Discrete Defocusing mKdV Equation

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Abstract
In this article, we apply Deift–Zhou nonlinear steepest descent method to analyze the long-time asymptotic behavior of the solution for the discrete defocusing mKdV equation

\[ \dot{q}_n = \left(1 - q_n^2\right) (q_{n+1} - q_{n-1}) \]

with decay initial value

\[ q_n(t = 0) = q_n(0), \]

where \( n = 0, \pm 1, \pm 2, \ldots \) is a discrete variable and \( t \) is continuous time variable. This equation was proposed by Ablowitz and Ladik.

Keywords Discrete defocusing mKdV equation · Lax pair · Riemann–Hilbert problem · Deift–Zhou steepest descent method · Long-time asymptotic behavior

Mathematics Subject Classification 35B40 · 35Q58 · 35Q15

1 Introduction

Since Deift and Zhou developed nonlinear steepest descent method in 1993 (Deift and Zhou 2017), it has been used to analyze the long-time asymptotic behavior of a wide variety of continuous integrable systems, such as the mKdV equation, the NLS equation, the sine–Gordon equation, the KdV equation, and the Camassa–Holm...
equation (Deift et al. 1993; Cheng et al. 1999; Vartanian 2000; Grunert and Teschl 2009; De Monvel et al. 2009). However, there still has been little work on the long-time behavior of the discrete integrable systems, except for the Toda lattice and discrete NLS equation (Krüger and Teschl 2009; Yamane 2014).

In this article, we consider the following discrete defocusing mKdV equation

\[
\dot{q}_n = \left( 1 - q_n^2 \right) (q_{n+1} - q_{n-1})
\]

with initial value

\[
q_n(t = 0) = q_n(0),
\]

where \(\dot{q}_n = dq_n(t)/dt\), \(n = 0, \pm 1, \pm 2, \ldots\) and \(t\) is continuous time variable. Equation (1) was first proposed by Ablowitz (1977). They further showed that Eq. (1) is the semi-discrete version of the classical mKdV equation (Ablowitz 1977)

\[
\tau(x, \tau) + 6u^2u_x(x, \tau) - u_{xxx}(x, \tau) = 0.
\]

Narita found two kinds of Miura transformations

\[
q_n = \frac{(u_{n-1} + u_{n+1})u_n - 2u_{n-1}u_{n+1}}{(u_{n-1} - u_{n+1})u_n},
\]

\[
q_n = \frac{u_{n-1} - 2u_n + u_{n+1}}{u_{n-1} - u_{n+1}}
\]

between Eq. (1) and Sokolov–Shabat equation (Narita 1997)

\[
\dot{u}_n = 4(u_{n-1} - u_n)(u_n - u_{n+1})/(u_{n-1} - u_{n+1}).
\]

Some kinds of exact solutions of Eq. (1) were obtained by using homotopy analysis method and exp-function method (Wang et al. 2008; Zhu 2007). With Darboux transformation, Wen and Gao (2010) obtained the explicit solutions for Eq. (1) on the discrete spectral of Lax pair. However, the solutions of Eq. (1) with initial-boundary condition on the continuous spectral have been still unknown by using inverse scattering transformation or Riemann–Hilbert approach. So in this paper, we would like to apply Riemann–Hilbert approach/Deift–Zhou nonlinear steepest descent method to investigate the long-time behavior of the solution for the initial value problem of discrete defocusing mKdV Eqs. (1)–(2).

The organization of this paper is as follows. In Sect. 2, we introduce appropriate eigenfunctions and spectral functions to reformulate initial value problem of discrete mKdV Eqs. (1)–(2) as a Riemann–Hilbert problem(RHP). From Sects. 3–8, we transform the RHP to a model one by using a series of deformations and decompositions. In Sect. 9, we show the existence and boundedness of the operator as solving the RHP. At the last section, we obtain the asymptotic behavior of the solution of Eq. (1).
2 Riemann–Hilbert Problem

In this section, we first investigate the solvability of the initial value problem (1)–(2) equation and then transform it into a RHP. Moreover, we further express $q_n$ with a solution of the obtained RHP.

Similar to Proposition 2.1 in Yamane (2014), we can show the following proposition.

Proposition 2.1 Let $s$ be a nonnegative integer. If the initial condition (2) satisfies

$$\|q_n(0)\|_{1,s} = \sum_{n=-\infty}^{\infty} (1 + |n|)^s |q_n(0)| < \infty,$$

(3)

$$\|q_n(0)\|_{\infty} = \sup_n |q_n(0)| < 1,$$

(4)

then Eq. (1) admits an unique solution in the space

$$l^{1,s} = \left\{ \{c_n\}_{n=-\infty}^{\infty} : \sum_{n=-\infty}^{\infty} (1 + |n|)^s |c_n| < \infty, 0 \leq t < \infty \right\}.$$

Proof We define

$$c_{-\infty} = \prod_{n=-\infty}^{\infty} \left( 1 - |q_n|^2 \right), \quad \rho_0 = (1 - c_{-\infty})^{1/2},$$

then we have

$$\sup_n |q_n(0)| \leq \rho_0.$$  

(5)

It is easy to verify that $c_{-\infty}$ and $\rho_0$ are both conserved quantities. Therefore, we consider Eq. (1) as an ordinary differential equation with respect to $t$, whose solution belongs to the Banach space $l^{1,s} \subset l^\infty$ under the condition (3) and (4).

Given the ball in $l^\infty$

$$B := \left\{ \{q_n(t)\} \in l^\infty, |q_n(t) - q_n(0)| < \rho_0 \right\},$$

we show that Eq. (1) admits a solution in the space $B$. Since the right-hand side of (1) is Lipschitz continuous and bounded, there exits a $t_1$ such that Eq. (1) admits solution in $B$ for $t \in (0, t_1]$. By the standard argument about ordinary differential equations in Lang (2012), we get that $t_1$ depends on $\rho_0$ so as we have (5). Because we have known that $\rho_0$ is conserved, we have

$$\sup_n |q_n(t_1)| \leq \rho_0.$$  

(6)
Similarly, we could extend the solution of Eq. (1) to \( t \in (t_1, 2t_1] \). Repeating the procedure above, we then can extend the solution to \( t \in [0, \infty) \) in the space \( l^\infty \). Therefore, for \( 0 \leq t < \infty \), we have

\[
\sup_n |q_n(t)| \leq \rho_0.
\]

Based on the fact that there is a solution for (1)–(2) in \( l^\infty \) shown above, we further verify that the solution belongs to the Banach space \( l^{1,s} \). From Eq. (1), we get

\[
\|\dot{q}_n(t)\|_{1,s} \leq \text{Const.} \|q_n(t)\|_{1,s}.
\]

By integrating \( \dot{q}_n(t) \) with respect to \( t \), we get

\[
\|q_n(t)\|_{1,s} \leq \|q_n(0)\|_{1,s} + \text{Const.} \int_0^t \|q_n(\tau)\|_{1,s} \, d\tau.
\]

By virtue of the Gronwall inequality, it follows that \( \|q_n(t)\|_{1,s} \) grows at most exponentially and does not blow up.

For the uniqueness, if we set

\[
q_n(0) = 0,
\]

then the problem (1)–(2) only admits a zero solution. \( \square \)

In the following, we transform the initial problem (1)–(2) into a RHP. It is known that the discrete mKdV equation admits a Lax pair (Ablowitz 1977)

\[
\begin{align*}
X_{n+1} &= z^{\sigma_3} X_n + Q_n X_n, \\
\dot{X}_n &= z^{2\sigma_3} X_n + B_n X_n,
\end{align*}
\tag{7}
\tag{8}
\]

where \( X_n \) is a \( 2 \times 2 \) matrix, and

\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z^{\sigma_3} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \quad Q_n = \begin{pmatrix} 0 & q_n \\ q_n & 0 \end{pmatrix},
\]

\[
B_n = \begin{pmatrix} -q_{n-1}q_n & q_nz + q_{n-1}z^{-1} \\ q_nz^{-1} + q_{n-1}z & -q_{n-1}q_n \end{pmatrix}.
\]

The Lax pair (7)–(8) admit the following asymptotic Jost solutions

\[
X_n(z, t) \sim z^{n \sigma_3} \begin{pmatrix} e^{z^2 t} & 0 \\ 0 & e^{-z^2 t} \end{pmatrix}, \quad n \to \pm \infty.
\]
Making transformation

\[ Y_n = z^{-n\sigma_3} \begin{pmatrix} e^{-z^2t} & 0 \\ 0 & e^{-z^{-2}t} \end{pmatrix} X_n, \]

then we have

\[ Y_{n+1} - Y_n = z^{-\sigma_3} \tilde{Q}_n Y_n, \]  

where

\[ \tilde{Q}_n = z^{-n\hat{\sigma}_3} e^{-\frac{t}{2}(z^2 - z^{-2})\hat{\sigma}_3} Q_n \]

\[ = \begin{pmatrix} 0 & q_n z^{-2n} e^{-t(z^2 - z^{-2})} \\ q_n z^{2n} e^{t(z^2 - z^{-2})} & 0 \end{pmatrix}, \]  

and for a 2 \times 2 matrix \( A \), the symbol \( e^{\hat{\sigma}_3 A} \) is defined by

\[ e^{\hat{\sigma}_3 A} \equiv e^{\sigma_3 A} e^{-\sigma_3}. \]

Denoting \( Y_n^{(\pm)} \) as the 2 \times 2 eigenfunctions of (12) such that

\[ Y_n^{(-)} \to I \text{ as } n \to -\infty, \]

\[ Y_n^{(+)} \to I \text{ as } n \to +\infty, \]

then we have

\[ Y_n^{(-)} = I + \sum_{k=-\infty}^{n-1} z^{-\sigma_3} \tilde{Q}_k Y_k^{(-)}, \]

\[ Y_n^{(+)} = I - \sum_{k=n}^{\infty} z^{-\sigma_3} \tilde{Q}_k Y_k^{(+)}. \]

By WKB expansion method, it follows that

\[ Y_n^{(-)} = z^{-n\hat{\sigma}_3} e^{\frac{t}{2}(z^2 - z^{-2})\hat{\sigma}_3} \]

\[ \times \begin{pmatrix} 1 + O(z^{-2}, \text{ even}) & zq_{n-1} + O(z^3, \text{ odd}) \\ z^{-1}q_{n-1} + O(z^{-3}, \text{ odd}) & 1 + O(z^2, \text{ even}) \end{pmatrix}, \]

\[ Y_n^{(+)} = z^{-n\hat{\sigma}_3} e^{\frac{t}{2}(z^2 - z^{-2})\hat{\sigma}_3} \]

\[ \times \begin{pmatrix} c_n^{-1} + O(z^2, \text{ even}) & -c_n^{-1}z^{-1}q_n + O(z^{-3}, \text{ odd}) \\ -c_n^{-1}zq_n + O(z^3, \text{ odd}) & c_n^{-1} + O(z^{-2}, \text{ even}) \end{pmatrix}, \]
where \( O(z^{\pm 3}, \text{odd}) \) (\( O(z^{\pm 2}, \text{even}) \)) means the remaining part containing \( z^{\pm 3}, z^{\pm 5}, \ldots (z^{\pm 2}, z^{\pm 4}, \ldots, \) respectively) and

\[
c_n = \prod_{k=n}^{\infty} \left( 1 - |q_k|^2 \right).
\]

Rewriting

\[
Y_{n}^{(-)} = \begin{pmatrix} Y_{n,1}^{(-)} & Y_{n,2}^{(-)} \end{pmatrix}, \quad Y_{n}^{(+)} = \begin{pmatrix} Y_{n,1}^{(+)} & Y_{n,2}^{(+)} \end{pmatrix},
\]

where \( Y_{n,j} \) is the \( j \)th column of \( Y_{n}^{(\pm)} \), and letting

\[
X_{n}^{(\pm)} = z^{n \sigma_3} \begin{pmatrix} e^{z^2 t} 0 \\ 0 e^{-z^2 t} \end{pmatrix} Y_{n}^{(\pm)}, \quad (17)
\]

from (11) and (12), we find that (7) admits matrix solutions

\[
X_{n}^{(-)} = \begin{pmatrix} X_{n,1}^{(-)} & X_{n,2}^{(-)} \end{pmatrix}, \quad X_{n}^{(+)} = \begin{pmatrix} X_{n,1}^{(+)} & X_{n,2}^{(+)} \end{pmatrix}
\]

such that

\[
X_{n,1}^{(-)} \to z^n e^{z^2 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_{n,2}^{(-)} \to z^{-n} e^{-z^2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{as} \quad n \to -\infty, \quad (18)
\]

\[
X_{n,1}^{(+)} \to z^n e^{z^2 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_{n,2}^{(+)} \to z^{-n} e^{-z^2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{as} \quad n \to \infty. \quad (19)
\]

From spectral problem (7), we know that

\[
\det(X_{n}^{(\pm)}) \neq 0,
\]

which implies that \( X_{n}^{(\pm)} \) are invertible. Thus, by the linearity of the eigenfunction, there exist four functions \( a(z), b(z), a^*(z) \) and \( b^*(z) \), such that

\[
X_{n,1}^{(-)}(z, t) = a(z) X_{n,1}^{(+)}(z, t) + b(z) X_{n,2}^{(+)}(z, t), \quad (20)
\]

\[
X_{n,2}^{(-)}(z, t) = b^*(z) X_{n,1}^{(+)}(z, t) + a^*(z) X_{n,2}^{(+)}(z, t), \quad (21)
\]

which combining with (7), we can get the symmetry

\[
a^*(z) = \overline{a(z^{-1})}, \quad b^*(z) = \overline{b(z^{-1})}.
\]

By using (17) and (20), we find that

\[
a(z) = \frac{\det(X_{n,1}^{(-)}, X_{n,2}^{(+)}))}{\det X_{n}^{(+)}} = \frac{\det(Y_{n,1}^{(-)}, Y_{n,2}^{(+)}))}{\det Y_{n}^{(+)}}.
\]
\[ b(z) = \frac{\det(X_{n,1}^{(+)}, X_{n,1}^{(-)})}{\det X_{n}^{(+)}} = \frac{\det(Y_{n,1}^{(+)}, Y_{n,1}^{(-)})}{\det Y_{n}^{(+)}}. \]

From (10), (13) and (14), it is worthy of notice that both \( Y_{n,1}^{(-)} \) and \( Y_{n,2}^{(+)} \) belong to \( C[|z| \geq 1] \cup \{\infty\} \) and they are both analytic outside the unit circle. While both \( Y_{n,2}^{(-)} \) and \( Y_{n,1}^{(+)} \) are continuous on the closed unit disk and analytic inside the unit circle. Thus, using (15)–(16), we know that \( a(z) \) is analytic outside the unit circle, moreover \( a \to 1 \) as \( z \to \infty \).

Besides, \( b \) is continuous on the unit circle, and
\[ a^*(0) = 1. \] (22)

By calculating the determinant of \( X_{n}^{(-)} \) and \( X_{n}^{(+)} \), we obtain that
\[ |a(z)|^2 - |b(z)|^2 = c_{-\infty} > 0, \quad \text{for } |z| = 1, \]
which implies \( a \neq 0 \) on the unit circle. Let \( r = b/a, \bar{r} = b^*/a^* \), we then have
\[ \bar{r}(z) = r(\bar{z}^{-1}), \quad 0 \leq |r(z)| < 1. \]

We now construct the RHP for Eq. (1). Defining a 2 × 2 analytic matrix function on \( \mathbb{C} \setminus \{|z| = 1\} \)
\[ m(z; n, t) = \begin{cases} 
1 & 0 \\
0 & c_n 
\end{cases} e^{-\frac{1}{2}(z^2 - z^{-2})\sigma_3} \frac{Y_{n,1}^{(-)}}{a}, Y_{n,2}^{(+)} \quad |z| > 1, \\
1 & 0 \\
0 & c_n 
\end{cases} e^{-\frac{1}{2}(z^2 - z^{-2})\sigma_3} \frac{Y_{n,1}^{(+)} Y_{n,2}^{(-)}}{a^*} \quad |z| < 1, \] (23)
from (15)–(16), we can get the expansion of \( m \)
\[ m = I + m_{12} z^{-1} + \cdots, \quad z \to \infty. \] (24)

Moreover, with (15), (22) and (23), we find that
\[ q_n = \frac{m_{12}}{z} \bigg|_{z=0} = \frac{d}{dz} m_{12} \bigg|_{z=0}, \] (25)
where \( m_{12} \) is the (1,2)-entry of the matrix \( m \).

By using (20) and (21), we can derive that
\[ m^+ = m^- v, \quad z \in \Sigma, \] (26)
where $\Sigma = \{z : |z| = 1\}$ is called jump curve (see Fig. 1), and

$$v(z; n, t) = z^n \hat{\sigma}_3 e^{-\frac{i}{2} (z^2 - z^{-2}) \hat{\sigma}_3} \begin{pmatrix} 1 - |r(z)|^2 & -\bar{r}(z) \\ r(z) & 1 \end{pmatrix},$$

is called the jump matrix.

In summary, we have got the RHP of $m$

$$\begin{cases} 
    \text{m analytic on } \mathbb{C}\backslash \{|z| = 1\}, \\
    \text{m}^+ = m^-v, \quad z \in \Sigma, \\
    m(z; n, t) \to I \quad \text{as } z \to \infty. 
\end{cases}$$

Letting

$$\varphi = \frac{i}{2} (z^2 - z^{-2}) - n \log z,$$

we obtain that

$$v(z; n, t) = e^{-\varphi \hat{\sigma}_3} \begin{pmatrix} 1 - |r(z)|^2 & -\bar{r}(z) \\ r(z) & 1 \end{pmatrix}.$$  \hfill (28)

### 3 The First RHP Deformation

In this section, we change the RHP of $m$ into a new equivalent one, such that its jump matrix admits a helpful lower/upper triangular factorization. We take a real constant number $0 < V_0 < 2$, we will discuss the asymptotic behavior under assumption $|n| \leq V_0 t, \ t \to \infty.$
From (27), the function $\varphi$ has four first-order stationary phase points

$$S_1 = A, \quad S_2 = \bar{A}, \quad S_3 = -A, \quad S_4 = -\bar{A},$$

where

$$A = 2^{-1} \left( \sqrt{2 + \frac{n}{t}} - i \sqrt{2 - \frac{n}{t}} \right).$$

Let

$$z = |z| e^{i\theta}.$$ 

From (27), we have

$$\text{Re} \varphi = \frac{t}{2} \cos(2\theta)(|z|^2 - |z|^{-2}) - n \log |z|,$$

which leads to the sign figure of $\text{Re} \varphi$ as shown in Fig. 2.

We denote $S_j S_{j+1}$ as the arc on unit circle from $S_j$ to $S_{j+1}$ such that the central angle of the arc is less than $\pi$ ($S_5 = S_1$). Define $\delta$ as an analytic function on $\mathbb{C} \setminus \{|z| = 1\}$ satisfying an scalar RHP

$$\begin{cases}
\delta_+(z) = \delta_-(z) \left( 1 - |r|^2 \right) & \text{on } S_1 S_2 \cup S_3 S_4, \\
\delta_+(z) = \delta_-(z) & \text{on } S_2 S_3 \cup S_4 S_1, \\
\delta \to 1 & \text{as } z \to \infty,
\end{cases}$$

where

$$r = \sqrt{2 + \frac{n}{t}}.$$
which admits a unique solution
\[ \delta(z) = e^{-\frac{1}{2\pi i} \left( \int_{S_1}^{S_2} + \int_{S_3}^{S_4} \right) \log(1-|r(\tau)|^2) \frac{d\tau}{\tau-z}}, \] (29)

where \( \int_{S_j}^{S_{j+1}} \), \( j = 1, 3 \) denote the integral on \( S_j S_{j+1} \). Noticing that
\[ \sup_{|z|=1} |r(z)| < 1, \]
it can be shown that both \( \delta \) and \( \delta^{-1} \) are bounded.

We introduce a transformation on \( \mathbb{C}\{|z|=1\} \)
\[ m^{(1)} = m\delta^{-\sigma_3}, \] (30)

where the jump curve still is \( \Sigma^{(1)} = \Sigma \), but its orientation is clockwise on \( S_1 S_2 \cup S_3 S_4 \) and counterclockwise on \( S_2 S_3 \cup S_4 S_1 \) as shown in Fig. 3.

On \( \Sigma \), we find that
\[ (m\delta^{-\sigma_3})_+ = m_- v\delta^{-\sigma_3}_+ = (m\delta^{-\sigma_3})_- \delta^\sigma_3 v\delta^{-\sigma_3}_-, \] (31)

where
\[
\delta^\sigma_3 v\delta^{-\sigma_3}_- = e^{-\psi_3} \begin{pmatrix}
1 - |r(z)|^2 & \delta_- \delta^{-1}_+ & -\overline{r(z)} \delta_+ \\
r(z) \delta^{-1}_- \delta^{-1}_+ & \delta^{-1}_- \delta_+ & 1
\end{pmatrix}
\]
\[
e^{-\psi_3} \begin{pmatrix}
1 - |r(z)|^2 & -\overline{r(z)} \delta^2_+ \\
r(z) \delta^{-2}_- & 1
\end{pmatrix},
\]
e^{-\psi_3} \begin{pmatrix}
1 & -\overline{r(z)} \delta^2_+ \\
r(z) \delta^{-2}_- & 1 - |r(z)|^2
\end{pmatrix}
\[
\begin{cases}
\begin{pmatrix}
1 & -\bar{r}(z)\delta^2_+ e^{-2\varphi} \\
0 & 1 \\
r\delta^{-2}_- e^{2\varphi} & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
r(z)\delta^{-2}_- e^{2\varphi} & 1
\end{pmatrix}
on \overline{S_2S_3} \cup \overline{S_4S_1}, \\
\begin{pmatrix}
1 & 0 \\
r\delta^{-2}_- e^{2\varphi} & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1-|r|^2 & 1
\end{pmatrix}
on \overline{S_1S_2} \cup \overline{S_3S_4}.
\end{cases}
\]

Also, we verify that
\[
m^{(1)} = m^{\delta^{-\sigma_3}} \to I \quad \text{as} \quad z \to \infty.
\]

So if setting
\[
v^{(1)} = \begin{cases}
\delta^\sigma_3 v^{\delta^{-\sigma_3}} & \text{on} \quad \overline{S_1S_2} \cup \overline{S_3S_4}, \\
\delta^\sigma_3 v^{\delta^{-\sigma_3}} & \text{on} \quad \overline{S_2S_3} \cup \overline{S_4S_1},
\end{cases}
\]
then we get RHP for \( m^{(1)} \) as follows
\[
m^{(1)}_+ = m^{(1)_-} v^{(1)}, \quad z \in \Sigma^{(1)}
\]

To express the jump matrix more concise, we set
\[
\rho(z) = \begin{cases}
-\bar{r} & \text{on} \quad \overline{S_1S_2} \cup \overline{S_3S_4}, \\
\bar{r} & \text{on} \quad \overline{S_2S_3} \cup \overline{S_4S_1},
\end{cases}
\]
and
\[
\bar{\rho}(z) = \rho(\bar{z}^{-1}),
\]
then we have
\[
v^{(1)} = b^{-1}_- b_+,
\]
where
\[
b_- = \begin{pmatrix}
1 & 0 \\
\bar{\rho}\delta^{-2}_- e^{2\varphi} & 1
\end{pmatrix}, \quad b_+ = \begin{pmatrix}
1 & \rho\delta^2_+ e^{-2\varphi} \\
0 & 1
\end{pmatrix}.
\]

\section{The Second RHP Deformation}

In this section, we would decompose \( \rho \) and \( \bar{\rho} \) into the rational part and two small parts, respectively, so that we can make the second RHP transformation with augmented jump contour (see Fig. 4).
Denoting
\[ d = 2^{-1} \min \left( \sqrt{2 + n/t}, \sqrt{2 - n/t} \right), \]
and making the decomposition
\[ \rho = R + h_I + h_{II}, \]
\[ \bar{\rho} = \bar{R} + \bar{h}_I + \bar{h}_{II}, \]
we then can obtain the following estimates
\[ |e^{-2\varphi} h_I| \leq Ct^{1/2-l} \text{ on } \Sigma, \]
\[ |e^{-2\varphi} h_{II}| \leq Ct^{-q/2} \text{ on } L, \]
\[ |e^{-2\varphi} R(z)| \leq Ce^{-Ct\epsilon^2} \text{ on } L^\epsilon, \]
\[ |e^{2\varphi} \bar{h}_I| \leq Ct^{1/2-l} \text{ on } \Sigma, \]
\[ |e^{2\varphi} \bar{h}_{II}| \leq Ct^{-q/2} \text{ on } \bar{L}, \]
\[ |e^{2\varphi} \bar{R}(z)| \leq Ce^{-Ct\epsilon^2} \text{ on } \bar{L}^\epsilon, \]
where we can see \( L, \bar{L} \) in Fig. 4 and for a positive real number \( \epsilon (< d/2) \)
\[ L^\epsilon = \{ z \in L : |z - S_j| \geq \epsilon \text{ for } j = 1, 2, 3, 4 \}, \]
\[ \bar{L}^\epsilon = \{ z \in \bar{L} : |z - S_j| \geq \epsilon \text{ for } j = 1, 2, 3, 4 \}. \]
In the following, we first get down to the decomposition of $\rho$ on the one side of $\bar{S}_1 S_2$; then we shall study the decomposition of $\bar{\rho}$ on the other side of $\bar{S}_1 S_2$. At last, in the similar way, we would investigate the decomposition on the other curves, like $\bar{S}_2 S_3$, $\bar{S}_3 S_4$ and $\bar{S}_4 S_1$.

4.1 Decomposition of $\rho$ on $\bar{S}_1 S_2$

We consider $\rho$ as a function of $\theta$. If we set $\theta_0 = -\arg(A)$, then the region of $\rho$ is $(-\theta_0, \theta_0)$, which is a symmetric interval about $\theta = 0$. Thus, we could decompose $\rho$ as the sum of an odd function and an even function

$$\rho(\theta) = H_e(\theta^2) + \theta H_o(\theta^2),$$

$H_o$ and $H_e$ can be approximated by Taylor’s expansion at the point $\theta^2 = \theta_0^2$

$$H_o(\theta^2) = \mu_o^0 + \cdots + \mu_o^k \left( \theta^2 - \theta_0^2 \right)^k + O(\theta^2 - \theta_0^2)^{k+1},$$

$$H_e(\theta^2) = \mu_e^0 + \cdots + \mu_e^k \left( \theta^2 - \theta_0^2 \right)^k + O(\theta^2 - \theta_0^2)^{k+1}.$$

Define

$$R(\theta) = \sum_{l=0}^{k} \left( \mu_e^l (\theta^2 - \theta_0^2)^l + \theta \mu_o^l (\theta^2 - \theta_0^2)^l \right),$$

$$h(\theta) = \rho(\theta) - R(\theta),$$

$$\alpha(z) = (z - S_1)^q (z - S_2)^q,$$

where $k$ and $q$ are two fixed positive integers and have the relationship

$$k = 4q + 1.$$

Notice that

$$R(\pm \theta_0) = \rho(\pm \theta_0), \quad \theta = -i \log z, \quad \text{on} \quad \bar{S}_1 S_2,$$

we consider $R$ as a function of complex number $z$, which could be extended analytically to a fairly large neighborhood of $\bar{S}_1 S_2$.

We consider a function

$$\psi = \varphi / (it),$$
then (27) implies that $\psi$ strictly decrease on $(-\theta_0, \theta_0)$. Thus, we can consider $\theta$ as a function of $\psi$ and $h/\alpha$, which are defined by

$$(h/\alpha)(\psi) = \begin{cases} 
  h \left( \theta(\psi) \right) / \alpha \left( \theta(\psi) \right), & \psi(0) \leq \psi \leq (-\theta_0), \\
  0, & \text{otherwise on the real line.}
\end{cases}$$

We verify that

$$(h/\alpha)(\theta) = O((\theta \pm \theta_0)^{k+1-q}), \quad \theta \to \pm \theta_0.$$ 

Since both $S_1$ and $S_2$ are the first-order stationary phase points, one gets

$$\frac{d\theta}{d\psi} = \left( \frac{d\varphi}{itd\theta} \right)^{-1} = O((\theta \pm \theta_0)^{-1}), \quad \theta \to \pm \theta_0.$$ 

For any integer $1 \leq l \leq \frac{3q+2}{2}$, we deduce that

$h/\alpha \in H^l_\psi,$  \hspace{1cm} (44)

where $H^l_\psi$ norm is uniformly bounded with respect to $(n, t)$ as long as $|n| \leq V_0 t$.

We apply Fourier transform to $(h/\alpha)(\theta)$ to decompose $h$ into two parts

$$h = h_I + h_{II},$$

where

$$h_I(\theta) = \alpha(\theta) \int_t^\infty e^{i\psi(\theta)} \widehat{(h/\alpha)}(s) ds,$$  \hspace{1cm} (45)

$$h_{II}(\theta) = \alpha(\theta) \int_{-\infty}^t e^{i\psi(\theta)} \widehat{(h/\alpha)}(s) ds,$$  \hspace{1cm} (46)

and

$$\left( \frac{\hat{h}}{\alpha} \right)(s) = \int_{-\infty}^\infty e^{-is\psi} \left( \frac{h}{\alpha} \right)(s) ds.$$ 

Thus, we have

$$\rho = R + h_I + h_{II}.$$ 

By using (44)–(45) and Schwartz inequality, we have

$$\left| e^{-2\psi} h_I \right| \leq Ct^{\frac{1}{2}-l} \text{ on } S_1S_2.$$  \hspace{1cm} (47)
Let
\[ p = \sqrt{d^2 + 2d + \frac{1}{2}} - \sqrt{\frac{1}{2}}, \]
and introduce a curve
\[ L_{12} = l_{12} \cup l_{21} \cup \tilde{L}_{12}, \] (48)
where
\[ l_{12} = \{ S_1 + S_1 e^{i\pi} u : 0 \leq u \leq p \}, \]
\[ l_{21} = \{ S_2 + S_2 e^{-i\pi} (p - u) : 0 \leq u \leq p \}, \]
\[ \tilde{L}_{12} = \text{arc} \left( S_1 + S_1 e^{i\pi} p, S_2 + S_2 e^{-i\pi} p \right). \]

From (46), it is shown that \( h_{II} \) can be analytically extend to \( \{ \text{Re} \phi > 0 \} \), and
\[ \left| e^{-2\phi h_{II}} \right| \leq C e^{-t R e(i \psi)} |\alpha|. \] (49)

Since that both \( S_1 \) and \( S_2 \) are first-order saddle points and \( \tilde{L}_{12} \) is a closed curve in the inner of \( \{ \text{Re} \phi > 0 \} \), we have
\[ \text{Re} i \psi \geq \begin{cases} 
Cu^2 & \text{on } l_{12}, \\
C(p - u)^2 & \text{on } l_{21}, \\
C & \text{on } \tilde{L}_{12}.
\end{cases} \] (50)

It is easy to check that
\[ |\alpha| = \begin{cases} 
O(u^q) & \text{on } l_{12}, \\
O((p - u)^q) & \text{on } l_{21}, \\
O(1) & \text{on } \tilde{L}_{12}.
\end{cases} \] (51)

With (50) and (51), we obtain
\[ \left| e^{-2\phi h_{II}} \right| \leq Ct^{-q/2} \text{ on } L_{12}. \] (52)

For sufficiently small positive number \( \epsilon \) (\( \epsilon < d/2 \)), we define
\[ I_{12}^\epsilon = \{ z \in l_{12} : |z - S_1| \geq \epsilon \}, \]
\[ I_{21}^\epsilon = \{ z \in l_{21} : |z - S_2| \geq \epsilon \}, \]
\[ L_{12}^\epsilon = I_{12}^\epsilon \cup I_{21}^\epsilon \cup \tilde{L}_{12}. \]
Since \( \text{dist}(L_{12}^\epsilon, \{S_1, S_2\}) \geq \epsilon \), it follows that

\[
\text{Re}(i \psi) \geq C \epsilon^2,
\]

and with the boundedness of \( R(z) \) on \( L_{12}^\epsilon \), we get

\[
\left| e^{-2\psi} R(z) \right| \leq C e^{-C \epsilon^2} \quad \text{on} \quad L_{12}^\epsilon.
\]  (53)

### 4.2 Decomposition of \( \tilde{\rho} \) on \( S_1 S_2 \)

For the one side of \( S_1 S_2 \), we have decomposed \( \rho \) into three parts

\[
\rho = R + h_I + h_{II}.
\]

Similar result can be obtained for \( \tilde{\rho} \) on \( S_1 S_2 \) and set

\[
\tilde{p} = \frac{\sqrt{2(2 - \frac{n}{\tau})}}{\sqrt{2 + \frac{n}{\tau}} + \sqrt{2 - \frac{n}{\tau}}}.
\]

Consider the contour \( \tilde{L}_{12} = \tilde{l}_{12} \cup \tilde{l}_{21} \), where

\[
\tilde{l}_{12} = \{S_1 - S_1 e^{-\frac{i\pi}{4}} u : 0 \leq u \leq \tilde{p}\},
\]

\[
\tilde{l}_{21} =: \{S_2 - S_2 e^{\frac{i\pi}{4}} (\tilde{p} - u) : 0 \leq u \leq \tilde{p}\}.
\]

Noticing decomposition

\[
\tilde{\rho} = \tilde{R} + \tilde{h}_I + \tilde{h}_{II},
\]

where

\[
\tilde{R}(z) = \overline{R(\tilde{z}^{-1})},
\]

then in a similar way to the derivation of \( h_I \) and \( h_{II} \), we can get the estimates

\[
\left| e^{2\psi} \tilde{h}_I \right| \leq C t^{\frac{3}{2} - \frac{q}{4}} \quad \text{on} \quad \overline{S_1 S_2},
\]  (54)

\[
\left| e^{2\psi} \tilde{h}_{II} \right| \leq C t^{-\frac{q}{4}} \quad \text{on} \quad \tilde{L}_{12}.
\]  (55)

Let

\[
\tilde{l}_{12}^\epsilon = \{z \in \tilde{l}_{12} : |z - S_1| \geq \epsilon\},
\]

\[
\tilde{l}_{21}^\epsilon = \{z \in \tilde{l}_{21} : |z - S_2| \geq \epsilon\},
\]
\[ \bar{L}_1^\varepsilon = I_1^\varepsilon \cup I_2^\varepsilon, \]
then for a fixed \( \varepsilon \), we get
\[
\left| e^{2\varphi} \bar{R}(z) \right| \leq C e^{-Ct\varepsilon^2} \quad \text{on} \quad \bar{L}_1^\varepsilon. \tag{56}
\]

**Remark 4.1** We have decomposed \( \rho \) and \( \bar{\rho} \) on \( S_1S_2 \); and we could get (36)–(37) on \( S_2S_3, S_3S_4 \) and \( S_4S_1 \): \( \rho = R + h_I + h_{II} \), similarly. What’s more, we could get (38)–(43) in the same way.

### 4.3 The Second RHP Deformation

According to the decomposition for \( \rho \) and \( \bar{\rho} \), we would make the second deformation to get a new RHP equivalent to the original one. Set
\[
b_{\pm} = b_{\pm}^e + b_{\pm}^o,
\]
where
\[
b_{+}^e = \begin{pmatrix} 1 & (R + h_{II})\delta^2 e^{-2\varphi} \\ 0 & 1 \end{pmatrix}, \quad b_{+}^o = \begin{pmatrix} 1 & h_I\delta^2 e^{-2\varphi} \\ 0 & 1 \end{pmatrix},
\]
\[
b_{-}^e = \begin{pmatrix} 1 & (\bar{R} + \bar{h}_{II})\delta^{-2} e^{2\varphi} \\ 0 & 1 \end{pmatrix}, \quad b_{-}^o = \begin{pmatrix} 1 & \bar{h}_I\delta^{-2} e^{2\varphi} \\ 0 & 1 \end{pmatrix}.
\]
Then, for (34)–(37), we have
\[
v^{(1)} = (b_{-}^o b_{-}^e)^{-1} b_{+}^o b_{+}^e \quad \text{on} \quad \Sigma^{(1)}.
\]
Since both \( h_{II} \) and \( R \) is analytic on \( \Omega_5 \) and \( \Omega_6 \), \( b_{+}^e \) is analytic on \( \Omega_5 \) and \( \Omega_6 \); similarly, \( b_{-}^e \) is analytic on \( \Omega_3 \) and \( \Omega_4 \). Thus, we define a new holomorphic function on \( \mathbb{C} \setminus \Sigma^{(2)} \) as shown in Fig. 4:

\[
m^{(2)} = \begin{cases} m^{(1)} & \text{on} \quad \Omega_1 \quad \text{and} \quad \Omega_2, \\
m^{(1)} (b_{-}^e)^{-1} & \text{on} \quad \Omega_3 \quad \text{and} \quad \Omega_4, \\
m^{(1)} (b_{+}^e)^{-1} & \text{on} \quad \Omega_5 \quad \text{and} \quad \Omega_6, \end{cases} \tag{57}
\]

and it is easy to see that
\[
\lim_{z \to \infty} m^{(2)} = \lim_{z \to \infty} m^{(1)} = I.
\]

From (32) and (57), it follows that
\[
m^{(2)}_+ = m^{(2)}_+ v^{(2)} \quad \text{on} \quad \Sigma^{(2)}, \tag{58}
\]
where

\[ v^{(2)} = (b_+^{(2)})^{-1} b_-^{(2)} \]  \hspace{1cm} \tag{59} \]

and

\[ b_+^{(2)} = \begin{cases} 
  b_+ & \text{on } L, \\
  I & \text{on } \bar{L}, \\
  b'_+ & \text{on } \Sigma,
\end{cases} \quad b_-^{(2)} = \begin{cases} 
  I & \text{on } L, \\
  b'_- & \text{on } \bar{L}, \\
  b'_- & \text{on } \Sigma
\end{cases} \]

with

\[ w^{(2)} = \pm (b^{(2)}_+ - I), \quad w^{(2)} = w^{(2)}_+ + w^{(2)}_. \]

We have completed the second RHP transformation.

5 Reduction of the RHP

In this section, we would like to reduce the previous RHP to a model RHP. We firstly apply an integral operator introduced in Beals and Coifman (1984) to write \( m^{(2)} \) in the integral form. Then, we reduce this integral to the sum of the decaying part and the integral related to the leading order.

5.1 Partition of Matrices

Recall that \( m^{(2)} \) can be written in the integral form

\[
m^{(2)}(z) = I + \frac{1}{2\pi i} \int_{\Sigma^{(2)}} \frac{((I - C_{w^{(2)}})^{-1} I)(\tau)w^{(2)}(\tau)}{\tau - z} \, d\tau, \hspace{1cm} \tag{60}\]

where \( C_{w^{(2)}} = C_+(\cdot w^{(2)}_-) + C_- (\cdot w^{(2)}_+) \), and \( C_{\pm} \) are the Cauchy operators defined by

\[
C_{\pm}(f)(z) = \frac{1}{2\pi i} \int_{\Sigma^{(2)}} \frac{f(\tau)d\tau}{\tau - z_{\pm}}, \quad f \in L^2(\Sigma^{(2)}), \quad z \in \Sigma^{(2)}.
\]

By (24), (25), (30) and (57), we then get

\[
q_n = \frac{d}{dz} \left( (m^{(2)})_12 \delta^{-1} \right) \bigg|_{z=0} = \delta(0)^{-1} \frac{d}{dz} \left( m^{(2)} \right)_12 \bigg|_{z=0} = \delta(0)^{-1} \frac{1}{2\pi i} \int_{\Sigma^{(2)}} z^{-2} \left[ ((I - C_{w^{(2)}})^{-1} I)(z)w^{(2)}(z) \right]_12 \, dz.
\hspace{1cm} \tag{61}\]

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Now we would like to decompose $w^{(2)}_{\pm}$ into two parts

$$w^{(2)}_{\pm} = w'_{\pm} + w^c_{\pm},$$

where

$$w^c_{\pm} = w'^c_{\pm} + w^b_{\pm},$$

$$w'^c_{\pm} :$$

- $w'^c_{\pm} = w^{(2)}_{\pm}$ on $|z| = 1$,
- $w'^a_{\pm}$ is equal to the contribution to $w^{(2)}_{\pm}$ from terms of type $h_{II}$ and $\bar{h}_{II}$,

$$w^b_{\pm} :$$

- $w^b_{\pm} = w^{(2)}_{\pm} - w'^c_{\pm}$ on $L^\epsilon \cup \bar{L}^\epsilon$,
- $w^b_{\pm} = 0$ on $\Sigma^{(2)} \setminus (L^\epsilon \cup \bar{L}^\epsilon)$.

So we set a contour consisting of four crosses

$$\Sigma' = \Sigma^{(2)} \setminus (\Sigma \cup L^\epsilon \cup \bar{L}^\epsilon) = \bigcup_{j=1}^4 \Sigma_j,$$

where $\Sigma_j$ is the small cross connected to $S_j$ for $j = 1, 2, 3, 4$. Note that the orientation of $\Sigma'$ and $\Sigma_j$ are determined by that of $\Sigma^{(2)}$ (see Fig. 5).

**Lemma 5.1** There exists a constant $C > 0$, such that

$$\left\| w'^a_{\pm} \right\|_{L^2(\Sigma')} \leq Ct^{-1} \quad \text{on} \quad \Sigma^{(2)},$$

$$\left\| w'^b_{\pm} \right\|_{L^2(\Sigma')} \leq Ce^{-\gamma \epsilon t} \quad \text{on} \quad \Sigma^{(2)}. \quad (63)$$

Moreover, we have

$$\left\| w'_{\pm} \right\|_{L^2(\Sigma')} \leq Ct^{-1/4}, \quad (64)$$

$$\left\| w'_{\pm} \right\|_{L^1(\Sigma')} \leq Ct^{-1/2}. \quad (65)$$
Proof Since $\delta$ and $\delta^{-1}$ is bounded, from the estimate (38)–(43), we get (62) and (63), and from (50), we get

$$|\delta^2 \text{Re}^{-2\varphi}| \leq \text{Const} e^{-Ct|z-S_1|^2} \quad \text{on} \quad l_{12} \cap \Sigma',$$

which leads to

$$\int_{l_{12} \cap \Sigma'} |\delta^2 \text{Re}^{-2\varphi}| dz \leq \text{Const} t^{-1/2},$$

$$\int_{l_{12} \cap \Sigma'} |\delta^2 \text{Re}^{-2\varphi}|^2 dz \leq \text{Const} t^{-1/2}.$$

Since all $S_j$ are the first-order stationary phase point of $\varphi$, we obtain that

$$\int_{L \cap \Sigma'} |\delta^2 \text{Re}^{-2\varphi}| dz \leq \text{Const} t^{-1/2},$$

$$\int_{L \cap \Sigma'} |\delta^2 \text{Re}^{-2\varphi}|^2 dz \leq \text{Const} t^{-1/2}.$$

For $\delta^{-2} \text{Re}^{2\varphi}$ on $\bar{L} \cap \Sigma'$, we also have

$$\int_{\bar{L} \cap \Sigma'} |\delta^{-2} \text{Re}^{2\varphi}| dz \leq \text{Const} t^{-1/2},$$

$$\int_{\bar{L} \cap \Sigma'} |\delta^{-2} \text{Re}^{2\varphi}|^2 dz \leq \text{Const} t^{-1/2}.$$

Therefore, by simply analyzing the entry of $w'_{\pm}$, we deduce (64) and (65). \qed

5.2 Some Resolvent and Estimates

We shall decompose

$$\int_{\Sigma^{(2)}} z^{-2}((I - C_{w^{(2)}})^{-1} I)(z) w^{(2)}(z) dz,$$

into the principal part and the decaying part by the resolvent identity.

Under the assumption that both $(I - C_{w^{(2)}})^{-1}$ and $(I - C_{w'})^{-1}$ exist and are bounded, by the second resolvent identity, we have

$$\int_{\Sigma^{(2)}} z^{-2}(I - C_{w^{(2)}})^{-1} I w^{(2)}$$

$$= \int z^{-2}(I - C_{w'})^{-1} I w' + \int z^{-2} w' + \int z^{-2}(I - C_{w'})^{-1} (C_{w'} I) w^{(2)}$$

$$+ \int z^{-2}(I - C_{w'})^{-1} (C_{w'} I) w' + \int z^{-2}(I - C_{w'})^{-1} C_{w'}.$$
× (I − C_{w^{(2)}})^{-1}(C_{w^{(2)}}I)w^{(2)}\\
= \int z^{-2}(I − C_w)^{-1}I w' + I + III + IV.

(66)

Since the length of \( \Sigma^{(2)} \) is finite, by (62) and (63), we have

\[ \| w^c_+ \|_{L^s(\Sigma^{(2)})}, \| w^c_- \|_{L^s(\Sigma^{(2)})} \leq Ct^{-1} \quad \text{for } s = 1, 2. \]

(67)

For \( \emptyset \notin \Sigma^{(2)} \), \( z^{-2} \) is bounded on \( \Sigma^{(2)} \), so we obtain

\[ |I| \leq Ct^{-1}. \]

Because the Cauchy integral operator is bounded on \( L^2(\Sigma^{(2)}) \) and

\[ \| C_{w^c I} \|_{L^2 \to L^2} \leq 1, \]

we can get that

\[ \| C_{w^c I} \|_{L^2(\Sigma^{(2)})} \leq \| C_{w^c} \|_{L^2(\Sigma^{(2)})} \| w(2) \|_{L^2} \]

\[ \leq C \| w^c_+ \|_{L^2(\Sigma^{(2)})} + \| w^c_- \|_{L^2(\Sigma^{(2)})} \leq Ct^{-1}. \]

(68)

Similarly, from (64), we have

\[ \| C_{w' I} \|_{L^2(\Sigma^{(2)})} = O(t^{-1/4}) \quad \text{as } t \to \infty. \]

(69)

From (62)–(63) and \( \| C_{\pm} \|_{L^2 \to L^2} \leq 1 \), for \( f \in L^2(\Sigma^{(2)}) \), we can show that

\[ \| C_{w^c} f \|_{L^2(\Sigma^{(2)})} \leq \| C_{w^c} \|_{L^2(\Sigma^{(2)})} \| f \|_{L^2(\Sigma^{(2)})} \]

\[ \leq C \| f \|_{L^2(\Sigma^{(2)})} \left( \sup_{z \in \Sigma^{(2)}} |w^c_+(z)| + \sup_{z \in \Sigma^{(2)}} |w^c_-(z)| \right) \]

\[ \leq Ct^{-1} \| f \|_{L^2(\Sigma^{(2)})}, \]

(70)

by which, we obtain that

\[ |II| \leq C \| C_{w^c} I \|_{L^2 \to L^2} \| w(2) \|_{L^2} \]

\[ \leq C \| C_{w^c} I \|_{L^2} (\| w' \|_{L^2} + \| w^c \|_{L^2}) \]

\[ \leq Ct^{-1}, \]

\[ |III| \leq C \| C_{w^c} I \|_{L^2 \to L^2} \| w^c \|_{L^2} \]

\[ \leq Ct^{-1}, \]

\[ |IV| \leq C \| C_{w^c} \|_{L^2 \to L^2} \| C_{w^{(2)}} I \|_{L^2 \to L^2} \| w(2) \|_{L^2} \]

\[ \leq C \| C_{w^c} \|_{L^2 \to L^2} (\| C_{w^c} I \|_{L^2} + \| C_{w^c I} \|_{L^2})(\| w' \|_{L^2} + \| w^c \|_{L^2}) \]

\[ \leq Ct^{-1}. \]
From these estimates, combining (61) and (66), we obtain the following equation

\[ q_n = \frac{\delta(0)^{-1}}{2\pi i} \int_{\Sigma(z)} z^{-2} \left[ (I - C_w')^{-1} Iw' \right]_{12} (z) dz + O(t^{-1}). \]  

(71)

Let

\[ C_w' = C_+ (\cdot \times w'_{\Sigma'}) + C_- (\cdot \times w'_{\Sigma'}), \]

which is an operator on \( 2 \times 2 \) Hilbert space \( L^2(\Sigma') \). Since \( w' \) vanishes on \( \Sigma^{(2)} \setminus \Sigma' \), by (2.58) in Deift and Zhou (2017), we could write (71) as

\[ q_n = \frac{\delta(0)^{-1}}{2\pi i} \int_{\Sigma'} z^{-2} \left[ (I - C_w'^\Sigma')^{-1} Iw' \right]_{12} (z) dz + O(t^{-1}), \]

(72)

where \( C_{w'} \) in (71) is an operator on \( L^2(\Sigma^{(2)}) \), whose kernel is \( w' \).

**Remark 5.2** We have assumed in this subsection that both \( (I - C_{w^{(2)}})^{-1} \) and \( (I - C_{w'})^{-1} \) exist and are bounded. In fact, in Sect. 9, we will prove that both \( (I - C_{w^{(2)}})^{-1} \) and \( (I - C_{w'})^{-1} \) exist and are bounded uniformly as \( t \to \infty \).

### 6 Four Crosses

In this section, we may decompose \( w' \) into four parts according to the cross. Moreover, we will make some estimates such that the principal part is more accurate for \( q_n \).

Define matrix functions on \( \Sigma' \)

\[ w^j = \begin{cases} w' & \text{on } \Sigma_j, \\ 0 & \text{on } \Sigma' \setminus \Sigma_j, \end{cases} \quad j = 1, 2, 3, 4, \]

(73)

then we define integral operators on \( L^2(\Sigma') \) with kernels in (73)

\[ A_j = C_{w^j}'', \quad j = 1, 2, 3, 4. \]

(74)

For \( A_j \), we have the following result.

**Proposition 6.1** Given \( j, k = 1, 2, 3, 4 \) \( (j \neq k) \), we have

\[ \| A_j A_k \|_{L^2 \to L^2} \leq C t^{-\frac{1}{2}}, \]

(75)

\[ \| A_j A_k \|_{L^\infty \to L^2} \leq C t^{-\frac{3}{4}}. \]

(76)

The proof of this proposition is similar to Lemma 3.5 in Deift and Zhou (2017). Referring to (64) and (65), we only have to replace \( A', B' \) in Lemma 3.5 of Deift and Zhou (2017) with \( A_j, A_k \), respectively; then, Proposition 6.1 follows.
Assuming that as \( t \to \infty \), \((1 - A_j)^{-1}\) exists and is uniformly bounded which would be proven in Sect. 9 because \( C_w^{\Sigma'} = \sum_j A_j \), by direct calculation, we get

\[
1 - \sum_{j \neq k} A_j A_k (1 - A_k)^{-1} = (1 - C_w^{\Sigma'})^{-1} \left[ 1 + \sum_j A_j (1 - A_j)^{-1} \right].
\]

We verify that for any \( f \in L^{\Sigma'} \),

\[
\| A_j (f) \|_{L^2} \leq \| C_+ (f w_+^j) \|_{L^2} + \| C_- (f w_-^j) \|_{L^2} \\
\leq (\| w_+^j \|_{L^\infty} + \| w_-^j \|_{L^\infty}) \| f \|_{L^2} \\
\leq 2 \| w_\pm^j \|_{L^\infty} \| f \|_{L^2} \leq C t^{-1} \| f \|_{L^2},
\]

which implies

\[
\| A_j \|_{L^2} = O(t^{-1}).
\]

With Proposition 6.1, as \( t \to \infty \), we have

\[
(1 - C_w^{\Sigma'})^{-1} = \left[ 1 + \sum_j A_j (1 - A_j)^{-1} \right] \left[ 1 - \sum_{j \neq k} A_j A_k (1 - A_k)^{-1} \right]^{-1},
\]

then we obtain that

\[
\int z^{-2} \left[ (1 - C_w^{\Sigma'})^{-1} I \right] (z) w'(z) \\
= \int z^{-2} \left[ I + \sum_j A_j (1 - A_j)^{-1} I \right] (z) w'(z) dz + \int z^{-2} \left\{ \left[ 1 + \sum_j A_j (1 - A_j)^{-1} \right] \\
\times \left[ 1 - \sum_{j \neq k} A_j A_k (1 - A_k)^{-1} \right]^{-1} \sum_{j \neq k} A_j A_k (1 - A_k)^{-1} I \right\} (z) w'(z) dz.
\]

and

\[
A_j A_k (1 - A_k)^{-1} = A_j A_k + A_j A_k (1 - A_k)^{-1} A_k.
\]

By the definition of \( A_k \), we learn that \( A_k I \in L^2(\Sigma') \) and

\[
\| A_k I \|_{L^2} \leq C \| w^k \|_{L^2} \leq C \| w' \|_{L^2} \leq C t^{-1/4}.
\]
From the uniform boundedness of the operator \((1 - A_k)^{-1}\), we obtain \((1 - A_k)^{-1} A_k I\) belong to \(L^2(\Sigma')\) and

\[
\left\| (1 - A_k)^{-1} A_k I \right\|_{L^2} \leq Ct^{-1/4}.
\]

Combining (75), (76), (80) and (81), we get that

\[
\left[ \sum_{j \neq k} A_j A_k (1 - A_k)^{-1} \right] I \in L^2(\Sigma'),
\]

and

\[
\left\| \left[ \sum_{j \neq k} A_j A_k (1 - A_k)^{-1} \right] I \right\|_{L^2(\Sigma')} \leq Ct^{-3/4}.
\]

If we can verify that

\[
\left\| \left[ 1 + \sum_j A_j (1 - A_j)^{-1} \right] \left[ 1 - \sum_{j \neq k} A_j A_k (1 - A_k)^{-1} \right]^{-1} \right\|_{L^2 \to L^2}
\]

is bounded uniformly as \(t \to \infty\), by Hölder’s inequality, we can write (79) in the form

\[
\int z^{-2} [(1 - C^w)'I] (z) w'(z)
\]

\[
= \int z^{-2} \left[ I + \sum_j A_j (1 - A_j)^{-1} I \right] (z) w'(z) dz + O(t^{-1}).
\]

With Eqs. (64)–(65), we obtain that

\[
\left[ 1 - \sum_{j \neq k} A_j A_k (1 - A_k)^{-1} \right]^{-1}, \quad t \to \infty
\]

exists and is a bounded operator on \(L^2(\Sigma')\). Also, by (77), we know that

\[
1 + \sum_j A_j (1 - A_j)^{-1}
\]

exists and is bounded on \(L^2(\Sigma')\). Thus, we have proven the uniform boundedness of (82).

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With (79), if we prove that for any pair of different numbers in \( \{1, 2, 3, 4\} \) \((j \neq k)\) and some constant \(C\)
\[
\left| \int_{\Sigma'} z^{-2} [A_j (1 - A_j)^{-1} I](z)w^k(z) \right| \leq Ct^{-1},
\]
(84)
and have the following theorem.

**Theorem 6.2** The potential function \( q_n \) admits the asymptotic estimate
\[
q_n = \frac{\delta(0)^{-1}}{2\pi i} \sum_j \int_{\Sigma_j} z^{-2} \left\{ [(1 - A_j)^{-1} I](z)w^j(z) \right\}_{12} \, dz + O(t^{-1}), \quad t \to \infty
\]
(85)

**Proof** By using (72), (83) and (84), we could complete the proof of (85) by consider the following relation
\[
A_j(1 - A_j)^{-1} = (1 - A_j)^{-1} - 1.
\]
\(\square\)

To prove (84), considering
\[
[A_j (1 - A_j)^{-1} I]w^k = A_j (1 - A_j)^{-1} A_j I w^k + A_j I w^k,
\]
we obtain the estimate
\[
\int_{\Sigma_k} |A_j I w^k| = \int_{\Sigma_k} \left| \left( \int_{\Sigma_j} \frac{w^j(\eta)d\eta}{\eta - \zeta} \right) w^k(\zeta) \right| d\zeta \\
\leq C \| w^j \|_{L^1(\Sigma_j)} \| w^k \|_{L^1(\Sigma_k)} \leq Ct^{-1},
\]
\[
\int_{\Sigma_k} |A_j (1 - A_j)^{-1} A_j I w^k| = \int_{\Sigma_k} \left| \left( \int_{\Sigma_j} \frac{(1 - A_j)(1 - A_j)^{-1} A_j I(\eta)w^j(\eta)d\eta}{\eta - \zeta} \right) w^k(\zeta) \right| d\zeta \\
\leq C \int_{\Sigma_j} |(1 - A_j)^{-1} A_j I(\eta)w^j(\eta)|d\eta \| w^k \|_{L^1(\Sigma_k)} \\
\leq C \| (1 - A_j)^{-1} A_j I \|_{L^2(\Sigma_j)} \| w^j \|_{L^1(\Sigma_j)} \| w^k \|_{L^1(\Sigma_k)} \\
\leq C \| A_j I \|_{L^2(\Sigma_j)} \| w^j \|_{L^2(\Sigma_j)} \| w^k \|_{L^1(\Sigma_k)} \\
\leq C \| A_j I \|_{L^2(\Sigma_j)} \| w' \|_{L^2(\Sigma_j)}^2 \leq Ct^{-1}.
\]
Then (84) follows.

With Theorem 6.2, the original RHP can be reduced to four RHPs on four separated crosses and the leading order is only about these four RHPs.
7 Rotation and Scaling

In this section, we would introduce scaling operators concerning each stationary phase point. Before getting down to the scaling operator, we shall first make some preparations. From (27), we show that for any \( j = 1, 2, 3, 4 \),

\[
\phi''(S_j) = (-1)^j 2i S_j^{-2} \sqrt{4t^2 - n^2}. \tag{86}
\]

Let \( T_1 = T_2 = 1 \) and \( T_3 = T_4 = -1 \), we define the following quantities for \( j = 1, 2, 3, 4 \)

\[
\delta_j(z) = e^{\frac{(-1)^j}{2\pi} \int_{T_j}^{S_j} \log(1 - |r(\tau)|^2) \frac{d\tau}{\tau - z}}, \tag{87}
\]

\[
\nu_j = -\frac{1}{2\pi} \log(1 - |r(S_j)|^2), \quad l_j(z) = \int_{T_j}^{S_j} \frac{d\tau}{\tau - z}, \tag{88}
\]

\[
\chi_j(z) = \frac{1}{2\pi i} \int_{T_j}^{S_j} \log \frac{1 - |r(\tau)|^2}{1 - |r(S_j)|^2} \frac{d\tau}{\tau - z}, \tag{89}
\]

where the symbol \( \int_{T_j}^{S_j} \) stands for the integral on the arc \( T_j S_j \) from \( T_j \) to \( S_j \). These quantities have the relationship

\[
\delta_j(z) = e^{(-1)^j \frac{1}{2\pi} \int_{T_j}^{S_j} \log(1 - |r(\tau)|^2) \frac{d\tau}{\tau - z}} \cdot e^{(-1)^j \nu_j l_j(z)} \]

\[
= \left( \frac{S_j - z}{T_j - z} \right)^{(-1)^j \nu_j} \cdot e^{(-1)^j \chi_j(z)}. \tag{90}
\]

We verify that \( \delta(z) = \prod_{j=1}^{4} \delta_j(z) \) by making product directly and define

\[
\hat{\delta}_j(z) = \frac{\delta(z)}{\delta_j(z)}.
\]

Since each \( S_j \) is first-order stationary phase point, we have

\[
\phi(z) = \phi(S_j) + \phi''(S_j)(z - S_j)^2 + \phi_j(z), \tag{91}
\]

where

\[
\phi_j(z) = O(|z - S_j|^3).
\]

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We extend the four small crosses to four infinite ones

\[
\begin{align*}
\Sigma(S_j) &= (S_j + S_je^{i\pi/4}\mathbb{R}) \cup (S_j + S_je^{-i\pi/4}\mathbb{R}), \\
&\quad \text{oriented inward like } \Sigma_j \text{ for } j = 1, 3, \\
\Sigma(S_j) &= (S_j + S_je^{i\pi/4}\mathbb{R}) \cup (S_j + S_je^{-i\pi/4}\mathbb{R}), \\
&\quad \text{oriented outward like } \Sigma_j \text{ for } j = 2, 4,
\end{align*}
\]

and define the contours

\[
\Sigma_j(0) = \begin{cases} 
(e^{i\pi/4}\mathbb{R}) \cup (e^{-i\pi/4}\mathbb{R}), & \text{oriented inward like } \Sigma_j, \quad j = 1, 3, \\
(e^{i\pi/4}\mathbb{R}) \cup (e^{-i\pi/4}\mathbb{R}), & \text{oriented outward like } \Sigma_j, \quad j = 2, 4.
\end{cases}
\]

In fact, we have a mapping from $\Sigma_j(0)$ to $\Sigma(S_j)$ (See Fig. 6)

\[
M_j : \Sigma_j(0) \to \Sigma(S_j), \\
z \mapsto \beta_j z + S_j,
\]

where

\[
\beta_j = \frac{1}{2}(4t^2 - n^2)^{-1/4}iS_j(-1)^j. \tag{92}
\]

Directly calculating we find that

\[
\varphi''(S_j)\beta_j^2 = (-1)^{j+1}\frac{i}{2}.
\]

We introduce the scaling operator

\[
N_j : (C^0 \cup L^2)(\Sigma(S_j)) \to (C^0 \cup L^2)(\Sigma_j(0)), \\
f(z) \mapsto (N_j f)(z) = f((\beta_j z + S_j),
\]

which is the pull-back of $M_j$. 
By Eqs. (90) and (91), we obtain

\[ N_j(e^{-\varphi})(z) = e^{-\varphi(S_j) - \psi''(S_j) \beta_j z^2 - N_j \varphi_j(z)} \]

\[ = S_j^n e^{-\varphi(S_j) - \psi''(S_j) \beta_j z^2 - N_j \varphi_j(z)}, \tag{93} \]

\[ N_j \delta_j(z) = \delta_j(\beta_j z + S_j) + \left( \frac{\beta_j}{\beta_j z + S_j - T_j} \right)^{(1)} \delta_j(z), \]

\[ \times e^{(1) - (1)\chi_j(\beta_j z + S_j)}, \tag{94} \]

Further from (93) and (94), we get

\[ N_j(\delta_j e^{-\varphi})(z) = \delta_j^0 \delta_j^1(z), \tag{95} \]

where

\[ \delta_j^0 = S_j \left( \frac{\beta_j}{\beta_j z + S_j - T_j} \right)^{(1)} \delta_j(z), \tag{96} \]

\[ \delta_j^1(z) = \left( \frac{S_j - T_j}{\beta_j z + S_j - T_j} \right)^{(1)} \delta_j(z), \tag{97} \]

Note that \( \Sigma(S_j) \) is extension of \( \Sigma_j \). Let \( \hat{w}^j_\pm \) be the zero extension of \( w^j_\pm |_{\Sigma_j} \) on \( \Sigma(S_j) \); then, the related operator on \( L^2(\Sigma(S_j)) \) is denoted as \( \hat{A}_j = C_{\hat{w}^j} \) with kernel

\[ \hat{w}^j = \hat{w}^j_+ + \hat{w}^j_-, \quad j = 1, 2, 3, 4. \]

Define a \( 2 \times 2 \) matrix \( \Delta^0_j = (\sigma^0_j)_{ij} \) and the related operator \( \tilde{\Delta}^0_j; \tilde{\Delta}^0_j \phi = \phi \Delta^0_j \) for \( 2 \times 2 \) matrix \( \phi \); then \( \tilde{\Delta}^0_j \) and its inverse are bounded. Letting

\[ \hat{w}^j = (\Delta^0_j)^{-1}(N_j \hat{w}^j_\pm) \Delta^0_j, \quad \tilde{w}^j = \hat{w}^j_+ + \hat{w}^j_-, \]

and \( \alpha_j = C_{\hat{w}^j} : L^2(\Sigma_j(0)) \rightarrow L^2(\Sigma_j(0)) \), direct calculation shows that

\[ \alpha_j = \tilde{\Delta}^0_j N_j \hat{A}_j N_j^{-1}(\tilde{\Delta}^0_j)^{-1}, \tag{98} \]

\[ \hat{A}_j = N_j^{-1}(\tilde{\Delta}^0_j)^{-1} \alpha_j \tilde{\Delta}^0_j N_j. \tag{99} \]

Noticing that the support of \( N_j \hat{w}^j \) belongs to \( M_j^{-1} \Sigma_j \), by using (115), we obtain

\[ (\Delta^0_j)^{-1}(N_j \hat{w}^j_\pm) \Delta^0_j = (\Delta^0_j)^{-1}(N_j \hat{w}^j_\pm) \Delta^0_j = \begin{pmatrix} 0 & R(\beta_j z + S_j)\delta_j(z)^2 \\ 0 & 0 \end{pmatrix}. \tag{100} \]
Similarly, on $M_j^{-1}(\Sigma_j \cap \bar{L}) \setminus \{0\}$, we have

$$(\Delta_j^0)^{-1}(N_j \hat{w}^j) \Delta_j^0 = (\Delta_j^0)^{-1}(N_j \hat{w}^j) \Delta_j^0 = \begin{pmatrix} 0 & 0 \\ -\tilde{R}(\beta_j z + S_j) \delta_j^1(z)^{-2} & 0 \end{pmatrix},$$

(101)

**8 Convergence**

Noticing that the principal part of $q_n$ consists of four integrals, respectively, on four separate crosses: $\Sigma_j$ $(j = 1, 2, 3, 4)$, we consider the convergence of the principal part in this section. That is, we estimate four terms

$$\int_{\Sigma_j} z^{-2}[(1 - A_j)^{-1}I](z)w^j(z)dz, \quad j = 1, 2, 3, 4.$$  

By (99), we have the following equalities

$$\int_{\Sigma_j} z^{-2}[(1 - A_j)^{-1}I](z)w^j(z)dz = \beta_j \int_{\Sigma_j(0)} (\beta_j z + S_j)^{-2}[(1 - \alpha_j)^{-1}(\Delta_j^0)](z)(\Delta_j^0)^{-1}N_j \hat{w}^j(z)dz$$

$$= \beta_j(\delta_j^0)^2 \int_{\Sigma_j(0)} [(1 - \alpha_j)^{-1}I](z)(\Delta_j^0)^{-1}N_j(\cdot \times \hat{w}^j)(z)\Delta_j^0 dz. \quad (102)$$

We estimate the limit of formula (102). For this purpose, we investigate the convergence of the following functions

$$(\beta_j z + S_j)^{-2}N_j R(z)\delta_j^1(z)^2, \quad N_j R(z)\delta_j^1(z)^2,$$

$$(\beta_j z + S_j)^{-2}N_j \tilde{R}(z)\delta_j^1(z)^2, \quad N_j \tilde{R}(z)\delta_j^1(z)^2.$$

**Proposition 8.1** For any arbitrary fixed constant $\gamma \ (0 < 2\gamma < 1)$, on $M_j^{-1}(\Sigma_j \cap L) \cap \{z : \pm z/e^{i\pi/4} > 0\}$, respectively, we have

$$\left| (\beta_j z + S_j)^{-2}N_j R(z)\delta_j^1(z)^2 - S_j^{-2}R(S_j \pm) e^{-iz^2/2}z^{2iv_j} \right| \leq C e^{-\frac{1}{2}\gamma z^2 - t^{-\frac{1}{4}} \log t},$$

(103)

$$\left| N_j R(z)\delta_j^1(z)^2 - R(S_j \pm) e^{-iz^2/2}z^{2iv_j} \right| \leq C e^{-\frac{1}{2}\gamma z^2 - t^{-\frac{1}{4}} \log t},$$

(104)

where $R(S_j^+) = \tilde{r}(S_j)$ on $M_j^{-1}(\Sigma_j \cap L) \cap \{z : z/e^{i\pi/4} > 0\}$ and $R(S_j^-) = -\frac{\tilde{r}(S_j)}{1-|\tilde{r}(S_j)|^2}$ on $M_j^{-1}(\Sigma_j \cap L) \cap \{z : z/e^{i\pi/4} < 0\}$.  

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This proposition can be proved in a similar way to proposition 10.1 in Yamane (2014), we omit it here. Similarly, we also have the following proposition.

**Proposition 8.2** For any arbitrary fixed constant $\gamma (0 < 2\gamma < 1)$, on $M_j^{-1}(\Sigma_j \cap \tilde{L}) \cap \{z : \pm ze^{-i\pi/4} > 0\}$, respectively, we have

$$
\left| (\beta_j z + S_j)^{-2} N_j \tilde{R}(z) \delta_j^1(z)^{-2} - S_j^{-2} \tilde{R}(S_j \pm) e^{iz^2/2z^{-2iv_j}} \right| \leq Ce^{i\gamma z^2} t^{-\frac{1}{2}} \log t, \quad (105)
$$

$$
\left| N_j \tilde{R}(z) \delta_j^1(z)^{-2} - \tilde{R}(S_j \pm) e^{iz^2/2z^{-2iv_j}} \right| \leq Ce^{i\gamma z^2} t^{-\frac{1}{2}} \log t, \quad j = 1, 3, \quad (106)
$$

where $\tilde{R}(S_j+) = r(S_j)$ on $M_j^{-1}(\Sigma_j \cap \tilde{L}) \cap \{z : z e^{i\pi/4} > 0\}$ and $\tilde{R}(S_j-) = -\frac{r(S_j)}{1 - |r(S_j)|^2}$ on $M_j^{-1}(\Sigma_j \cap \tilde{L}) \cap \{z : z e^{i\pi/4} < 0\}$.

**Remark 8.3** For even number $j$, in the same way, we can get the following result

\[
(\beta_j z + S_j)^{-2} N_j R(z) \delta_j^1(z)^2 = S_j^{-2} R(S_j \pm) e^{iz^2/2z^{-2iv_j}}
\]

\[
+ O(e^{-\frac{i\gamma z^2}{2}t^{-\frac{1}{2}} \log t}), \quad \text{on } M_j^{-1}(\Sigma_j \cap L) \cap \{z : \pm ze^{i\pi/4} > 0\},
\]

\[
N_j R(z) \delta_j^1(z)^2 = R(S_j \pm) e^{iz^2/2z^{-2iv_j}}
\]

\[
+ O(e^{-\frac{i\gamma z^2}{2}t^{-\frac{1}{2}} \log t}), \quad \text{on } M_j^{-1}(\Sigma_j \cap L) \cap \{z : \pm ze^{i\pi/4} > 0\},
\]

\[
\beta_j z + S_j)^{-2} N_j \tilde{R}(z) \delta_j^1(z)^{-2} = S_j^{-2} \tilde{R}(S_j \pm) e^{iz^2/2z^{-2iv_j}}
\]

\[
+ O(e^{\frac{i\gamma z^2}{2}t^{-\frac{1}{2}} \log t}), \quad \text{on } M_j^{-1}(\Sigma_j \cap L) \cap \{z : \pm z/e^{i\pi/4} > 0\},
\]

\[
N_j \tilde{R}(z) \delta_j^1(z)^{-2} = \tilde{R}(S_j \pm) e^{iz^2/2z^{-2iv_j}}
\]

\[
+ O(e^{\frac{i\gamma z^2}{2}t^{-\frac{1}{2}} \log t}), \quad \text{on } M_j^{-1}(\Sigma_j \cap L) \cap \{z : \pm z/e^{i\pi/4} > 0\}.
\]

We consider some new matrices and operators that are limits of those on $\Sigma_j(0)$ which is a union of four parts

$$
\Sigma_j(0) = \bigcup_{k=1}^{k} \Sigma_j^k(0),
$$

where

$$
\Sigma_j^k(0) = e^{i(2k-1)\pi/4} \mathbb{R}^+.
$$

For $j = 1, 2, 3, 4$, we define

$$
w_{\pm}^{j, \infty} = \lim_{t \to \infty} \tilde{w}_{\pm}^{j, \infty}, \quad w^{j, \infty} = w_{+}^{j, \infty} + w_{-}^{j, \infty}.
$$
Defining operators
\[
\alpha_j^\infty = C_{w,j,\infty} : L^2(\Sigma_j(0)) \to L^2(\Sigma_j(0)),
\]
then by (104), (106) and Remark 8.3, we have
\[
\| \tilde{w}_j^j - w_{j,\infty}^j \|_{L^2} \leq Ct^{-1} \log t,
\]
\[
\| (C_{w,j} - C_{w,j,\infty}) I \|_{L^2} \leq Ct^{-1} \log t.
\]

Direct calculation shows that
\[
\| (1 - \alpha_j^\infty)^{-1} I - (1 - \alpha_j)^{-1} I \|_{L^2} \leq Ct^{-1} \log t.
\]

In summary, by using (85), (102), (103), (105), (108), Remark 8.3, we get that
\[
q_n = \frac{\delta(0)^{-1}}{2\pi i} \sum_{j=1}^{4} \beta_j S_j^{-2}(\delta_j^0)^2 \left\{ \int_{\Sigma_j(0)} [(1 - \alpha_j^\infty)^{-1} I](z) w_{j,\infty}(z) dz \right\}_{12}
+ O(t^{-1} \log t).
\]

For the branch cut of \( z^{ij} \), we first consider that of \( \delta_j^1 \). From (97), it is reasonable to consider \( M_j^{-1} S_j^{-1} T_j \cup (\beta_j^{-1}(T_j - S_j) + \mathbb{R}^-) \) as the branch cut of \( \delta_j^1 \). As \( t \to \infty \), we find that this branch cut becomes \( \mathbb{R}^- = \{ x \in \mathbb{R} \} \). Therefore, we consider \( \mathbb{R}^- \) as the branch cut of \( z^{ij} \) in the remaining of this article.

9 The Boundedness of Operators

In this section, we would investigate the existence and boundedness of the following operators
\[
(I - C_{w,0})^{-1}, \quad (I - C_{w'})^{-1}, \quad (1 - A_j)^{-1}, \quad (1 - \hat{A}_j)^{-1},
\]
\[
(1 - \alpha_j)^{-1}, \quad (1 - \alpha_j^\infty)^{-1} \quad (j = 1, 2, 3, 4).
\]

Remark 9.1 Assume that operator \( (1 - \alpha_j^\infty)^{-1} \) exists and is bounded on \( L^2(\Sigma_j(0)) \) for all \( j \). With Propositions 8.1, 8.2 and Remark 8.3, we deduce that \( (1 - \alpha_j)^{-1} \) exists and is uniformly bounded as \( t \to \infty \). Because \( M_j \) is homeomorphic from \( \Sigma_j(0) \) to \( \Sigma(S_j) \), \( N_j \) is invertible and \( \| N_j \|_{L^2(\Sigma(S_j) \to L^2(\Sigma_j(0)))} \) is bounded as well as \( \| N_j^{-1} \|_{L^2(\Sigma(S_j) \to L^2(\Sigma_j(0)))} \). Moreover, noticing that \( \Delta_j^0 \) and its inverse are invertible, we get the existence and boundedness of \( (1 - \hat{A}_j)^{-1} \) from those of \( (1 - \alpha_j)^{-1} \). More precisely, by Eq. (98), we have
\[
(1 - \hat{A}_j)^{-1} = N_j^{-1}(\Delta_j^0)^{-1}(1 - \alpha_j)^{-1}\Delta_j^0 N_j.
\]
By (2.58) in Deift and Zhou (2017), we derive the existence and boundedness of $(1 - A_j)^{-1}$ from those of $(1 - \hat{A}_j)^{-1}$; thus, $(1 - C_{w^j})^{-1}$ exists and is uniformly bounded with Eq. (78). Moreover, by (2.59) in Deift and Zhou (2017), $(I - C_{w^j})^{-1}$ exists and is uniformly bounded.

Because of (62)–(63) and boundedness of $\Sigma^{(2)}$, we get that

$$\|C_{w^{(2)}} - C_{w^j}\|_{L^2(\Sigma^{(2)}) \to L^2(\Sigma^{(2)})} \to 0.$$ 

Thus, by the second resolvent identity, we obtain that for sufficiently large $t$, the existence and boundedness of $(I - C_{w^j})^{-1}$ implies those of $(I - C_{w^{(2)}})^{-1}$.

According to Remark 9.1, the remaining work in this section is to prove the existence and boundedness of $(1 - \alpha_j^\infty)^{-1}$. We discuss this problem in two cases when $j$ is odd and $j$ is even.

If $j$ is odd, we change the orientation of $\Sigma_j^1(0)$ and $\Sigma_j^4(0)$ in $\Sigma_j(0)$. In fact, this change does not affect $\alpha_j^\infty$ as an operator on $L^2(\Sigma_j(0))$. So we could add the real line with orientation marked in Fig. 7, we define matrices

$$w_{j,e}^{i,e} = \begin{cases} w_{j,\infty}^{i,e} & \text{on } \Sigma_j^2(0) \cup \Sigma_j^3(0), \\ -w_{j,\infty}^{i,e} & \text{on } \Sigma_j^1(0) \cup \Sigma_j^4(0), \\ 0 & \text{on } \mathbb{R}, \end{cases}$$

$$w_j^{i,e} = w_+^{j,e} + w_-^{j,e}, \quad \alpha_j^e = C_{w,i,e},$$

where

$$w_+^{j,e} = \begin{pmatrix} 0 & 0 \\ r(S_j)e^{iz^2/2}z^{-2iv_j} & 0 \\ 0 & \frac{r(S_j)}{1 - |r(S_j)|^2}e^{-iz^2/2}z^{2iv_j} \\ 0 & 0 \end{pmatrix}, \quad z \in e^{i\pi/4}\mathbb{R}_+,$$

$$w_-^{j,e} = \begin{pmatrix} 0 & 0 \\ -r(S_j)^{-1}e^{-iz^2/2}z^{-2iv_j} & 0 \\ 0 & \frac{r(S_j)}{1 - |r(S_j)|^2}e^{iz^2/2}z^{2iv_j} \\ 0 & 0 \end{pmatrix}, \quad z \in e^{3i\pi/4}\mathbb{R}_+,$$

otherwise on $\Sigma^e$. 

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\[ v^{j,e} = (b^{j,e}_-)^{-1} b^{j,e}_+ = (1 - w^{j,e}_-)^{-1} (1 + w^{j,e}_+), \]

and define a meromorphic function \( \sigma \)

\[
\sigma = \begin{cases} 
    z^{i v_j} \sigma_3 & \text{on } \Omega_2^e \cup \Omega_5^e, \\
    b^{j,e}_+ z^{i v_j} \sigma_3 & \text{on } \Omega_1^e \cup \Omega_5^e, \\
    b^{j,e}_- z^{i v_j} \sigma_3 & \text{on } \Omega_4^e \cup \Omega_6^e. 
\end{cases} \quad (110)
\]

We simply denote \( v^e = v^{j,e} \), also \( \text{det } \sigma = 1 \) implies that \( \sigma \) is invertible. Let

\[
v^{e, \sigma} = \sigma_+^{-1} v^e \sigma_+, \quad \text{on } \Sigma_1^e, \quad (111)
\]

then we find that

\[
v^{e, \sigma} = \begin{cases} 
    v^0 := e^{-\frac{i z^2}{2} \sigma_3} \begin{pmatrix} 1 & \tilde{r}(S_j) \\ -r(S_j) & 1 - |r(S_j)|^2 \end{pmatrix} & \text{on } \mathbb{R}, \\
    I & \text{on } \Sigma_1^e \setminus \mathbb{R},
\end{cases} \quad (112)
\]

where

\[
v^{e, \sigma} = (b^{e, \sigma}_-)^{-1} b^{e, \sigma}_+ = (1 - w^{e, \sigma}_-)^{-1} (1 + w^{e, \sigma}_+)
\]

\[
= \begin{pmatrix} 1 & 0 \\ -r(S_j) e^{i z^2/2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{r}(S_j) e^{-i z^2/2} \\ 0 & 1 \end{pmatrix}. \quad (113)
\]

Let

\[
w^{e, \sigma} = w^{e, \sigma}_+ + w^{e, \sigma}_-.
\]

Since the operator \( C_{w^{e, \sigma}|\mathbb{R}} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) satisfies

\[
\| C_{w^{e, \sigma}|\mathbb{R}} \| \leq \sup_{z \in \mathbb{R}} |e^{-i z^2/2} \tilde{r}(S_j)| \leq \| r \|_{L^\infty(\mathbb{R})} < 1,
\]
by Lemma 2.56 in Deift and Zhou (2017), we have \( \| C_{w,e,\sigma} \| < 1 \), where \( C_{w,e,\sigma} \) is an operator on \( L^2(\Sigma^e) \). Thus \( (1 - C_{w,e,\sigma})^{-1} \) exists and is bounded. Like the step 5 in section 3 of Deift and Zhou (2017), we can deduce the existence and boundedness of \( (1 - \alpha_j^e)^{-1} \). So the boundedness of \( (1 - \alpha_j^e)^{-1} \) follows.

As for \( j \) is even, we could go through the process once again as \( j \) is odd, but we simply complete it by making conjugate transformation to \( w_{j,e}^{\pm} \). Notice that for \( j \) is even, \( \Sigma^e_j \) admits the orientation that is different from the case of \( j \) is odd. Exactly, each line of \( \Sigma^e_j \) admits different orientation.

Let \( w_{j,e}^{j,e} = w_{j,e}^+ + w_{j,e}^- \), where \( w_{j,e}^{j,e} \) are given by

\[
w_{j,e}^{j,e} = \begin{cases} w_{j,e}^{j,e,+} & \text{on } \Sigma^e_j(0) \cup \Sigma^3_j(0), \\ w_{j,e}^{j,e,-} & \text{on } \Sigma^1_j(0) \cup \Sigma^4_j(0), \\ 0 & \text{on } \mathbb{R}. \end{cases}
\]

Further,

\[
w_{+}^{j,e} = \begin{cases} r(S_j)e^{-iz^2/2}z^{2ivj} & z \in e^{-i\pi/4}\mathbb{R}_+, \\ \bar{r}(S_j)e^{iz^2/2}z^{-2ivj} & z \in e^{-3i/4}\mathbb{R}_+, \\ 0 & \text{otherwise on } \Sigma^e, \end{cases}
\]

\[
w_{-}^{j,e} = \begin{cases} 0 & z \in e^{i\pi/4}\mathbb{R}_+, \\ \bar{r}(S_j)e^{iz^2/2}z^{-2ivj} & z \in e^{3i\pi/4}\mathbb{R}_+, \\ 0 & \text{otherwise on } \Sigma^e. \end{cases}
\]

Define a conjugate operator \( t: \)

\[ T \tilde{f}(z) = \overline{\tilde{f}(\bar{z})}. \]

We could find that \( Tw_{j,e}^{j,e} \) is almost the same as \( w_{j,e}^{j,e} \) on \( \Sigma^e \) by replacing \( \bar{r}(S_j) \) to \( r(S_{j-1}) \). Noticing that

\[ \| \bar{r} \|_{L^\infty} = \| r \|_{L^\infty} < 1, \]

the proof of even number \( j \) follows as the proof of odd number \( j \).
10 The Asymptotic of Discrete mKdV Equation

In this section, we give the asymptotic behavior of $q_n$ shown in Eq. (127).

By (109), we only have to estimate

$$\left\{ \int_{\Sigma_j(0)} [(1 - \alpha_j^{-\infty})^{-1} I](z) w^{j,\infty}(z) dz \right\}_{12}.$$  \hspace{1cm} (114)

We expand the $m^j$ in the form

$$m^j(z) = I + \frac{1}{2\pi i} \int_{\Sigma_j(0)} \frac{[(1 - \alpha_j^{-\infty})^{-1} I](\tau) w^{j,\infty}(\tau) d\tau}{\tau - z}$$

$$= I - z^{-1} m^1_j(z) + \cdots,$$ \hspace{1cm} (115)

where

$$m^1_j(z) = \int_{\Sigma_j(0)} [(1 - \alpha_j^{-\infty})^{-1} I](z) w^{j,\infty}(z) \frac{dz}{2\pi i}.$$ \hspace{1cm} (116)

And $m^j$ is analytic on $\mathbb{C} \setminus \Sigma_j(0)$ and satisfy

$$m^j_+ = m^j_- v^{j,\infty} \quad \text{on} \quad \Sigma_j(0),$$ \hspace{1cm} (117)

where

$$v^{j,\infty} = (1 - w^{j,\infty}_-)^{-1} (1 + w^{j,\infty}_+).$$

For odd number $j$, we set

$$\Sigma^{\infty}_j = \Sigma^e_j,$$

and the orientation of $\Sigma^{\infty}$ is different from $\Sigma^e$ only on $\mathbb{R}$ (See Fig. 8). Further we denote $\tilde{v}^{j,\infty}_\pm$ as the zero extension of $w^{j,\infty}_\pm$ on $\Sigma^\infty_j$. 

Fig. 8 $\Sigma^{\infty}_j$ for $j$ is odd
Define $\Phi(z) = m^j(z)\sigma(z)$ on $\mathbb{C}\setminus\Sigma^\infty$, then we would find that

$$
\Phi_+ = \begin{cases} 
\Phi_- & \text{on } (e^{i\pi/4}\mathbb{R}) \cup (e^{-i\pi/4}\mathbb{R}) \setminus \{0\}, \\
\Phi_-(v^0)^{-1} & \text{on } \mathbb{R}.
\end{cases}
$$

(118)

From the asymptotic behavior of $m^j$ as $z \to \infty$, we have

$$
\Phi(z)z^{-iv\sigma^3} = I - z^{-1}m_1^j + \cdots.
$$

(119)

which implies that $\Phi$ is analytic on $\mathbb{C}\setminus\mathbb{R}$. Letting $\hat{\Phi} = \Phi z^{-i\nu_j^2/\sigma^3}$ on $\mathbb{C}\setminus\mathbb{R}$, we then get a model RHP

$$
\begin{aligned}
\hat{\Phi}_+ &= \hat{\Phi}_- 
\begin{pmatrix} 1 - |r(S_j)|^2 & -\overline{r(S_j)} \\
r(S_j) & 1 \end{pmatrix}
\quad \text{on } \mathbb{R}, \\
\hat{\Phi}z^{-iv\sigma^3}e^{i\nu_j^2/\sigma^3} &\to I \quad \text{as } z \to \infty.
\end{aligned}
$$

By using the result (110) in Deift and Zhou (1994), we could get that

$$
(m_1^j)_{12} = \frac{i(2\pi)^{1/2}e^{i\pi/4}e^{-\pi\nu_j^2/2}}{r(S_j)\Gamma(-i\nu_j)} , \quad j = 1, 3.
$$

(120)

For the case when $j$ is even, we can write $m^j$ as

$$
m^j = I + \int_{\Sigma_j(0)} [(1 - C_{w_j,e}^{-1})I](\tau)w_j,e(\tau)d\tau \\
= I - z^{-1}m_1^j(z) + \cdots,
$$

(121)

where

$$
m_1^j = \int_{\Sigma_j(0)} [(1 - C_{w_j,e}^{-1})I](z)w_j,e(z)dz.
$$

(122)

Because $C_{w_j,e} = T C_{T w_j,e} T$ and $T^2 = 1$, we obtain

$$
C_{T w_j,e} = T C_{w_j,e} T
$$

(123)

and

$$
\int_{\Sigma_j(0)} [(1 - C_{T w_j,e}^{-1})I](z)T w_j,e(z)dz \\
= \int_{\Sigma_j(0)} [T(1 - C_{w_j,e}^{-1})TI](z)T w_j,e(z)dz
$$
\[
= \int_{\Sigma_j(0)} T[(1 - C_{w^{j,e}})^{-1} I \times w^{j,e}(z)]dz
= \int_{\Sigma_j(0)} [(1 - C_{w^{j,e}})^{-1} I](z)w^{j,e}(z)dz.
\] (124)

Therefore, in a similar way to the case when \( j \) is odd, we have

\[
\left\{ \int_{\Sigma_j(0)} [(1 - C_{w^{j,e}})^{-1} I](z)T w^{j,e}(z)dz \right\}_{12} = \frac{i(2\pi)^{1/2}e^{i\pi/4}e^{-\pi v_j/2}}{r(S_j)\Gamma(-iv_j)}.
\] (125)

Combining (124) and (125) gives

\[
(m_j^1)_{12} = -\frac{i(2\pi)^{1/2}e^{-i\pi/4}e^{-\pi v_j/2}}{r(S_j)\Gamma(iv_j)}, \quad j = 2, 4,
\] (126)

where \( v_j = -\frac{1}{2\pi} \log(1 - |r(S_j)|^2) \).

Finally, combining (109), (115), (116), (120) and (126) gives the asymptotic behavior of the discrete mKdV equation

\[
q_n = \delta(0)^{-1} \sum_{j=1}^{4} \beta_j S_j^{-2}(\delta_j^0)^{2}(m_j^1)_{12} + O(t^{-1} \log t), \quad \text{for } |n| \leq V_0t,
\] (127)

where \( S_j, \quad j = 1, 2, 3, 4 \) are stationary points; \( \beta_j \) and \( \delta_j^0 \) are given by (92) and (96), respectively, and

\[
(m_j^1)_{12} = (-1)^{j-1}\frac{i(2\pi)^{1/2}e^{-i\pi/4}e^{-\pi v_j/2}}{r(S_j)\Gamma((-1)^jiv_j)}, \quad j = 1, 2, 3, 4.
\]

The asymptotic formula (127) is composed of a decaying term and a leading term coming from four stationary phase points \( S_j, \quad j = 1, 2, 3, 4 \). In the leading term \( \delta_j^0 \) contains three oscillatory factors: \( S_j^n, e^{-\frac{1}{2}(S_j^2 - S_j^{-2})} \) and \( \beta_j^{(-1)^{j-1}iv_j} \). Since \( n/t \) is fixed, as \( t \) tends to the infinity, \( n \) also tends to the infinity. If we set \( \theta_j = \arg S_j \) and \( \kappa \) the imaginary part of \( S_j^2 \), then we can write the three oscillatory terms in the form

\[
S_j^n e^{-\frac{1}{2}(S_j^2 - S_j^{-2})} \beta_j^{(-1)^{j-1}iv_j} = e^{\frac{i}{2}(2n\theta_j - 2\kappa + (-1)^jv_j \log t)} \psi_j(n/t),
\]

where \( \psi_j \) is a function about \( n/t \). Then, \( \beta_j(\delta_j^0)^2 \) behaves like \( \text{const.}t^{-1/2}e^{ip_jt+iq_j \log t}, \)

\( p_j \in \mathbb{R} \) and \( q_j \in \mathbb{R} \).

In this article, we have got the long-time asymptotic formula (127) for the solutions of the initial value problem for the discrete defocusing mKdV Eqs. (1)–(2) by the Deift–Zhou steepest descent method. To our knowledge, with exception to Grunert, Teschl and Yamane’s recent work (Grunert and Teschl 2009; Yamane 2014, 2015, 2019a, b), there has been little work on asymptotic behavior for discrete integrable systems by the
Deift–Zhou steepest descent method. There is almost no work on asymptotic behavior for discrete integrable systems with nonzero boundary conditions.

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