MINIMAL SURFACES IN $\mathbb{R}^3$ PROPERLY PROJECTING INTO $\mathbb{R}^2$

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ABSTRACT. For all open Riemann surface $\mathcal{N}$ and real number $\theta \in (0, \pi/4)$, we construct a conformal minimal immersion $X = (X_1, X_2, X_3) : \mathcal{N} \to \mathbb{R}^3$ such that $X_3 + \tan(\theta)|X_1| : \mathcal{N} \to \mathbb{R}$ is positive and proper. Furthermore, $X$ can be chosen with arbitrarily prescribed flux map.

Moreover, we produce properly immersed hyperbolic minimal surfaces with non-empty boundary in $\mathbb{R}^3$ lying above a negative sublinear graph.

1. INTRODUCTION

The conformal structure of a complete minimal surface plays a fundamental role in its global properties. It is then important to determine the conformal type of a given minimal surface. An open Riemann surface is said to be hyperbolic if and only if it carries a negative non-constant sub-harmonic function. Otherwise, it is said to be parabolic. Compact Riemann surfaces with empty boundary are said to be elliptic.

Complete minimal surfaces with finite total curvature or complete embedded minimal surfaces with finite topology in $\mathbb{R}^3$ are properly immersed and have parabolic conformal type (for further information, see [Os, JM, CM, MPR, MP2]). On the other hand, there exists properly immersed hyperbolic minimal surfaces in $\mathbb{R}^3$ with arbitrary non-compact topology [Mo, AFM, FMM].

It is then interesting to elucidate how properness and completeness influence the conformal geometry of minimal surfaces. In [Lo2] it is shown that any open Riemann surface admits a conformal complete minimal immersion in $\mathbb{R}^3$, even with arbitrarily prescribed flux map. In this paper we extend this result to the family of proper minimal immersions, proving considerably more (see Theorem 4.4):

**Theorem I.** For all open Riemann surface $\mathcal{N}$, group morphism $p : \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^3$ and real number $\theta \in (0, \pi/4)$, there exists a conformal minimal immersion $X = (X_1, X_2, X_3) : \mathcal{N} \to \mathbb{R}^3$ satisfying that:

- $X_3 + \tan(\theta)|X_1| : \mathcal{N} \to \mathbb{R}$ is positive and proper, and
- $\int_{\gamma} \partial X = ip(\gamma)$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$, where $\partial$ is the complex differential operator.

This result is sharp in the sense that the angle $\theta$ cannot be zero. Indeed, by the Strong Half Space Theorem [HM] properly immersed minimal surfaces in a half space are planes. Contrariwise, Theorem I shows that any wedge of angle greater than $\pi$ in $\mathbb{R}^3$ contains minimal surfaces properly immersed in $\mathbb{R}^3$, even of hyperbolic type. In particular, neither open wedges nor closed wedges of angle greater than $\pi$ are universal regions for surfaces (see [MP1] for a good setting). Other Picard conditions for properly immersed minimal surfaces in $\mathbb{R}^3$ guaranteeing parabolicity can be found in [Lo1].

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From Theorem I follow some remarkable results concerning not only minimal surfaces. We are going to mention three of them related to proper harmonic maps into $C$, proper holomorphic null curves in $C^3$ and maximal surfaces in the Lorentz-Minkowski space $R^3_1$.

Schoen and Yau conjectured that there are no proper harmonic maps from $D$ to $C$ with flat metrics, and connected this question with the existence of hyperbolic minimal surfaces in $R^3$ properly projecting into $R^2$ [SY, p. 18]. A counterexample to this conjecture follows from the results in [DF], which imply the existence of proper harmonic maps from any finite bounded Riemann surface (that is to say, a compact Riemann surface minus a finite collection of pairwise disjoint closed discs) into $R^2$. It remains open whether or not a hyperbolic minimal surface in $R^3$ can be properly projected into $R^2$. Theorem I answers positively this question for minimal surfaces with arbitrary open conformal structure (just notice that the harmonic map $(X_1, X_3) : N \to R^2$ is proper).

It is well known that any open Riemann surface properly holomorphically embeds in $C^3$ and immerses in $C^2$ [Bi, Nar, Re]. Moreover, there are proper null immersions in $C^3$ of the unit disc $[Mo]$, and of any open parabolic Riemann surface of finite topology [Pi, Lo2]. Theorem I also shows that any open Riemann surface admits a proper null immersion in $C^3$, and a holomorphic immersion in $C^2$ properly projecting into $R^2$. Indeed, choosing $p = 0$ in Theorem I and labeling $X^+ = (X_1^+, X_2^+, X_3^+)$ as the conjugate minimal immersion of $X$, the map $X + iX^+ = (F_1, F_2, F_3) : N \to C^3$ is a proper holomorphic null immersion, and $(F_1, F_3) : N \to C^2$ is a holomorphic immersion which properly projects into $R^2$.

Finally, from Theorem I follows the existence of proper Lorentzian null holomorphic immersions in $C^3$ (see [UY]) and proper conformal maximal immersions in the Lorentz-Minkowski space, with singularities and arbitrary conformal structure. See [AI] for the hyperbolic simply connected case.

The last part of the paper is devoted to properly immersed minimal surfaces in $R^3$ with non empty boundary. A Riemann surface $M$ with non empty boundary is said to be parabolic if bounded harmonic functions on $M$ are determined by their boundary values, or equivalently, if the harmonic measure of $M$ respect to a point $P \in M \setminus \partial(M)$ is full on $\partial(M)$. Otherwise, the surface is said to be hyperbolic (see [AS, Pe] for a good setting). For instance, $D - \{1\}$ is parabolic whereas $D_+ := D \cap \{z \in C \mid \text{Im}(z) > 0\}$ is hyperbolic. Properly immersed minimal surfaces with non empty boundary lying in a half space of $R^3$ are parabolic [CKMR], and the same result holds for proper minimal graphs in $R^3$ [Ne]. It is also known that any properly immersed minimal surface in $R^3$ with non empty boundary lying over a negative sublinear graph in $R^3$ and whose Gaussian image is contained in a hyperbolic domain of the Riemann sphere is parabolic [LP]. We prove the following complementary result (see Theorem 5.1), which also shows that the condition about the size of the Gauss map in [LP] plays an important role:

**Theorem II.** There exists a conformal minimal immersion $X = (X_1, X_2, X_3) : D_+ \to R^3$ such that $(X_1, X_3) : D_+ \to R^2$ is proper and $\lim_{n \to \infty} \min \frac{X_1(p_n)}{|X_1(p_n)| + 1} = 0$ for all divergent sequence $\{p_n\}_{n \in \mathbb{N}}$ in $D_+$.

Theorem II contributes to the understanding of Meeks’ conjecture about parabolicity of minimal surfaces with boundary. This conjecture asserts that any properly immersed minimal surface lying above a negative half catenoid is parabolic.

The techniques developed in this paper may be applied to a wide range of problems on minimal surfaces theory. In the paper [AFL] complete minimal surfaces in $R^3$ with arbitrary conformal structure and whose Gauss map misses two points are constructed. Our tools come from deep results on approximation theory by meromorphic functions [Sc1, Sc2, Ro]. The most useful one is the Approximation Lemma in Section 2, where accurate use of Runge’s approximation theorem and classical theory of Riemann surfaces [AS, FK] is made. In this way, we can refine the classical
construction methods of complete minimal surfaces (see, among others, \[JX, \text{Nad, LMM, MUY}\] for a good setting).

The paper is laid out as follows. In Section 2 we introduce the necessary background on Riemann surfaces and approximation theory, and the required notations. Furthermore, we prove the Approximation Lemma. Section 3 is devoted to some preliminaries on minimal surfaces in \( \mathbb{R}^3 \). Finally, Theorems I and II are proved in Sections 4 and 5, respectively.

2. Riemann Surfaces and Approximation Results

Given a topological surface \( W, \partial(W) \) will denote the one dimensional topological manifold determined by the boundary points of \( W \). Given \( S \subset W \), call by \( S^0 \) and \( S \) the interior and the closure of \( S \) in \( W \), respectively.

A Riemann surface \( M \) is said to be open if it is non-compact and \( \partial(M) = \emptyset \). As usual, \( \overline{C} = C \cup \{ \infty \} \) will denote the Riemann sphere. We denote \( \partial \) as the global complex operator given by \( \partial |_U = \frac{\partial f}{\partial z}dz \) for any conformal chart \((U, z)\) on \( M \).

Remark 2.1. Throughout this paper \( N \) will denote a fixed but arbitrary open Riemann surface.

Let \( S \) denote a subset of \( N, S \neq N \). We denote by \( \mathcal{F}_0(S) \), respectively \( \mathcal{F}(S) \), as the space of continuous functions \( f : S \to \mathbb{C} \), respectively \( f : S \to \overline{C} \), which are holomorphic, respectively meromorphic, on an open neighborhood of \( S \) in \( N \), and \( f^{-1}(\infty) \subset S^0 \). Likewise, \( \mathcal{F}_0(S) \), respectively \( \mathcal{F}^*(S) \), will denote the space of continuous functions \( f : S \to \mathbb{C} \), respectively \( f : S \to \overline{C} \), being holomorphic, respectively meromorphic, on \( S^0 \) and satisfying that \( f^{-1}(\infty) \subset S^0 \).

As usual, a 1-form \( \theta \) on \( S \) is said to be of type \( (1, 0) \) if for any conformal chart \((U, z)\) in \( N, \theta |_{U \cap S} = h(z)dz \) for some function \( h : U \cap S \to \overline{C} \). We denote by \( \Omega_0(S) \), respectively \( \Omega(S) \), as the space of holomorphic, respectively meromorphic, 1-forms on an open neighborhood of \( S \) in \( N \), and without poles on \( S - S^0 \). We call \( \Omega^*(S) \) as the space of 1-forms \( \theta \) of type \( (1, 0) \) on \( S \) such that \((\theta |_U)/dz \in \mathcal{F}^*(S \cap U)\) for any conformal chart \((U, z)\) on \( N \). Likewise we define \( \Omega_0^*(S) \).

Let \( \mathfrak{Div}(S) \) denote the free commutative group of divisors of \( S \) with multiplicative notation. If \( D = \prod_{i=1}^{n} Q_n^i \in \mathfrak{Div}(S) \), where \( n_i \in \mathbb{Z} - \{0\} \) for all \( i \), the set \( \{Q_1, \ldots, Q_n\} \) is said to be the support of \( D \), written \( \text{supp}(D) \). A divisor \( D \in \mathfrak{Div}(S) \) is said to be integral if \( D = \prod_{i=1}^{n} Q_n^i \) and \( n_i \geq 0 \) for all \( i \). Given \( D_1, D_2 \in \mathfrak{Div}(S) \), \( D_1 \geq D_2 \) if and only if \( D_1 - D_2 \) is integral. For any \( \theta \in \mathcal{F}(S) \) we denote by \((\theta)_0 \) and \((\theta)_\infty \) its associated integral divisors of zeroes and poles in \( S \), respectively, and label \((f) = \frac{(f)_0}{(f)_\infty} \) as the divisor associated to \( f \) on \( S \). Likewise we define \((\theta)_0, (\theta)_\infty \) for any \( \theta \in \Omega(S) \), and call \( (\theta) = \frac{(\theta)_0}{(\theta)_\infty} \) as the divisor of \( \theta \) on \( S \).

In the sequel we will assume that \( S \) is a compact subset of \( N \) and \( W \subset N \) an open subset containing \( S \).

By definition, a connected component \( V \) of \( W - S \) is said to be bounded in \( W \) if \( \partial(V) \cap W \) is compact, where \( \partial(V) \) is the closure of \( V \) in \( N \).

Definition 2.2. A compact subset \( S \subset W \) is said to be admissible in \( W \) if and only if:

- \( W - S \) has no bounded components in \( W \),
- \( M_S := S^o \) consists of a finite collection of pairwise disjoint compact regions in \( W \) with \( C^0 \) boundary,
- \( C_S := S - M_S \) consists of a finite collection of pairwise disjoint analytical Jordan arcs, and
- any component \( a \) of \( C_S \) with an endpoint \( P \in M_S \) admits an analytical extension \( \beta \) in \( W \) such that the unique component of \( \beta - a \) with endpoint \( P \) lies in \( M_S \).
Observe that if $S$ is admissible in $\mathcal{N}$ then it is admissible in $W$ as well, but the contrary is in general false.

We shall say that a function $f \in F^*(S)$, respectively $f \in F_0^*(S)$, can be uniformly approximated on $S$ by functions in $F(W)$, respectively in $F_0(W)$, if there exist $\{f_n\}_{n \in \mathbb{N}} \subset F(W)$, respectively $\{f_n\}_{n \in \mathbb{N}} \subset F_0(W)$, such that $\{|f_n - f|\}_{n \in \mathbb{N}} \to 0$ uniformly on $S$. We also say that $\{f_n|_S\}_{n \in \mathbb{N}} \to f$ in the $\omega$-topology. In particular all $f_n$ have the same set $\mathcal{P}_f$ of poles on $S^\circ$. A 1-form $\theta \in \Omega^*(S)$, respectively $\theta \in \Omega_0^*(S)$, can be uniformly approximated on $S$ by 1-forms in $\Omega(W)$, respectively in $\Omega_0(W)$, if there exists $\{\theta_n\}_{n \in \mathbb{N}} \subset \Omega(W)$, respectively $\{\theta_n\}_{n \in \mathbb{N}} \subset \Omega_0(W)$, such that $\{\frac{\theta_n - \theta}{\theta_n} - f\}_{n \in \mathbb{N}} \to 0$ uniformly on $S \cap U$, for any conformal closed disc $(U, dz)$ on $W$. In particular all $\theta_n$ have the same set of poles $\mathcal{P}_\theta$ on $S^\circ$. As above, we say that $\{\theta_n|_S\}_{n \in \mathbb{N}} \to \theta$ in the $\omega$-topology.

Recall that a compact Jordan arc in $\mathcal{N}$ is said to be analytical if it is contained in an open analytical Jordan arc in $\mathcal{N}$.

Given an admissible compact set $S \subset W$, a function $f : S \to \mathbb{C}^n$, $n \in \mathbb{N}$, is said to be smooth if $f|_{M_S}$ admits a smooth extension $f_\beta$ to an open domain $V$ containing $M_S$, and for any component $\alpha$ of $C_S$ and any open analytical Jordan arc $\beta$ in $W$ containing $\alpha$, $f$ admits a smooth extension $f_\beta$ to $\beta$ satisfying that $f_\beta|_{V \cap \beta} = f_\alpha|_{V \cap \beta}$. A function $f \in F^*(S)$ is said to be smooth if $f|_{S - V}$ is smooth, where $V$ is any open neighbourhood of $f^{-1}(\infty)$ such that $S - V$ is admissible in $W - f^{-1}(\infty)$. Analogously, a 1-form $\theta \in \Omega^*(S)$ is said to be smooth if $(\theta|_{S \cap U})/dz \in F^*(S \cap U)$ is smooth for any closed conformal disc $(U, z)$ on $W$ such that $S \cap U$ is an admissible set. Given a smooth $f \in F^*(S)$, we set $df \in \Omega^*(S)$ as the smooth 1-form given by $df|_{M_S} = d(f|_{M_S})$ and $df|_{a \cap U} = (f \circ a)'(z)dz|_{a \cap U}$, where $(U, z = x + iy)$ is a conformal chart on $W$ such that $a \cap U = z^{-1}(\mathbb{R} \cap z(U))$. Obviously, $df|_\gamma(t) = (f \circ \alpha)'(t)dt$ for any component $\alpha$ of $C_S$, where $t$ is any smooth parameter along $\alpha$.

A smooth 1-form $\theta \in \Omega^*(S)$ is said to be exact if $\theta = df$ for some smooth $f \in F^*(S)$, or equivalently if $f|_\gamma \theta = 0$ for all $\gamma \in \mathcal{H}_1(S, \mathbb{Z})$.

Several extensions of classical Runge’s Theorem can be found in [Ro, Sc1, Sc2]. For our purposes, we need only the following compilation result:

**Theorem 2.3.** Let $S \subset W$ be a not necessarily connected admissible compact subset in $W$.

Then any function $f \in F^*(S)$ can be uniformly approximated on $S$ by functions in $F(W) \cap F_0(W - \mathcal{P}_f)$, where $\mathcal{P}_f = f^{-1}(\infty)$. Furthermore, if $D \in \mathcal{D}_n(S)$ is an integral divisor satisfying that $\text{supp}(D) \subset S^\circ$, then the approximation sequence $\{f_n\}_{n \in \mathbb{N}}$ in $F(W)$ can be chosen satisfying that $(f - f_n|_S)0 \geq D$.

### 2.1. The Approximation Lemmas

Throughout this section, $W \subset \mathcal{N}$ will denote an open connected subset of finite topology, and $S$ an admissible compact subset in $W$.

**Lemma 2.4.** Consider $f \in F^*(S)$ such that $f$ never vanishes on $S - S^\circ$. Then $f$ can be uniformly approximated on $S$ by functions $\{f_n\}_{n \in \mathbb{N}} \subset F(W)$ satisfying that $(f_n) = (f)$ on $W$ for all $n$. In particular, $f_n$ is holomorphic and never vanishing on $W - S$ for all $n$.

**Proof.** Let $\mu$ and $b$ denote the genus of $W$ and the number of topological ends of $W - \text{supp}((f))$. It is well known (see [FK]) that there exist $2\mu + b - 1$ cohomologically independent 1-forms in $\Omega(W) \cap \Omega_0(W - \text{supp}((f)))$ generating the first holomorphic De Rham cohomology group $\mathcal{H}^1_{\text{hol}}(W - \text{supp}((f)))$. Furthermore, the 1-forms can be chosen having at most single poles at points of $\text{supp}((f))$. Thus, the map $\mathcal{H}^1_{\text{hol}}(W - \text{supp}((f))) \to \mathbb{C}^{2\mu + b - 1}, \tau \mapsto (\int_{c} \tau)_{c \in B_0}$, where $B_0$ is any homology basis of $W - \text{supp}((f))$, is a linear isomorphism, and there exists $\tau \in \Omega(W) \cap$
\(\Omega_0(W - \text{supp}(f))\) with at most single poles at points of \(\text{supp}(f)\) such that \(\frac{1}{2\pi i} \int_{1_{\gamma}} \tau \in \mathbb{Z}\) for all \(\gamma \in \mathcal{H}_1(W - \text{supp}(f), \mathbb{Z})\) and \(df/f - \tau \in \Omega_0^1(S)\) is exact. Set \(f_0 = fe^{-\int \tau} \in F^*(S)\).

It is clear that \(\log(f_0) \in F_0^1(S)\), and so \(f_0\) is holomorphic and never vanishing on \(S\). By Theorem 2.3, there exists a sequence \(\{h_n\}_{n \in \mathbb{N}} \subset F_0^1(W)\) converging to \(\log(f_0)\) in the \(\omega\)-topology on \(S\).

The sequence \(f_n = e^{h_n + \int \tau}, n \in \mathbb{N}\), solves the lemma. \(\square\)

**Lemma 2.5.** Consider \(\theta \in \Omega^*(S)\) such that \(\theta\) never vanishes on \(S - S^0\).

Then \(\theta\) can be uniformly approximated on \(S\) by 1-forms \(\{\theta_n\}_{n \in \mathbb{N}}\) in \(\Omega(W)\) satisfying that \((\theta_n) = (\theta)\) on \(W\). In particular, \(\theta_n\) is holomorphic and never vanishing on \(W - S\) for all \(n\).

**Proof.** Let \(\tau\) be a non zero 1-form in \(\Omega_0(W)\) with finitely many zeroes, none of them lying in \(S - S^0\). Label \(f = \theta/\tau \in F_\infty(S)\). By Lemma 2.4, \(f\) can be approximated in the \(\omega\)-topology on \(S\) by a sequence \(\{f_n\}_{n \in \mathbb{N}}\) in \(F(W)\) satisfying that \((f_n) = (f)\) on \(W\) for all \(n\). It suffices to take \(\theta_n := f_n\tau, n \in \mathbb{N}\). \(\square\)

**Lemma 2.6 (The Approximation Lemma).** Let \(\Phi = (\Phi_j)_{j=1,2,3}\) be a smooth triple in \(\Omega_0^1(S)^3\) such that \(\sum_{j=1}^3 \Phi_j^2 = 0\), \(\sum_{j=1}^3 |\Phi_j|^2\) never vanishes on \(S\), and \(\Phi|_{M^*_S} \in \Omega_0^1(M_S)^3\). Then \(\Phi\) can be uniformly approximated on \(S\) by a sequence \(\{\Phi_n = (\Phi_{n,j})_{j=1,2,3}\}_{n \in \mathbb{N}} \subset \Omega_0(W)^3\) satisfying that:

(i) \(\sum_{j=1}^3 \Phi_{n,j}^2 = 0\) and \(\sum_{j=1}^3 |\Phi_{n,j}|^2\) never vanishes on \(W\),

(ii) \(\Phi_n - \Phi\) is exact on \(S\) for all \(n\).

**Proof.** Label \(g = \frac{\Phi_1}{\Theta_1 - \Theta_2}, \eta_1 = \frac{1}{2} \Phi_1 = \Phi_1 - i \Phi_2\) and \(\eta_2 = g\Phi_3 = -\Phi_1 - i \Phi_2 \in \Omega_0^1(S)\). We start with the following claim:

**Claim 2.7.** Without loss of generality, we can assume that \(g|_{M_S}\) is not constant.

**Proof.** Suppose for a moment that \(g|_{M_S}\) is constant, and up to replacing \(\Phi\) by \(\Phi \cdot A\) for a suitable orthogonal matrix \(A \in O(3, \mathbb{R})\), assume that \(g \neq \infty\). For each \(h \in F_0(W)\), set \(\eta_2(h) = (g + h)^2 \eta_1\) and \(\Phi_3(h) = \eta_1(g + h)\). Let \(B\) be a homology basis of \(\mathcal{H}_1(M_S, \mathbb{Z})\), label \(\mu\) as its cardinal number and consider the holomorphic map \(T : F_0(W) \to \mathbb{C}^2, T(h) = (f_1((\eta_2(h) - \eta_2, \Phi_3(h) - \Phi_3))_{i \in B}\).

Note that the analytical subset \(T^{-1}(0)\) is conical, that is to say, if \(T(h) = 0\) then \(T(\lambda h) = 0\) for all \(\lambda \in \mathbb{C}\). Furthermore, since \(J_0(W)\) has infinite dimension we can choose a non constant \(h \in T^{-1}(0)\). Take \(\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{C}\) converging to zero, set \(h_n := \lambda_n h \in T^{-1}(0)\) for all \(n\), and notice that \(\{h_n\}_{n \in \mathbb{N}} \to 0\) in the \(\omega\)-topology on \(S\).

Set \(\Psi_n := (\psi_{1,n}, \psi_{2,n}, \psi_{3,n}) := \left(\frac{1}{2}(\eta_1 - \eta_2(h_n)), \frac{1}{2}(\eta_1 + \eta_2(h_n)), \Phi_3(h_n)\right) \in \Omega_0^1(S) \cap \Omega_0(0)(M_S)^3\), and observe that \(\sum_{j=1}^3 \Phi_{n,j}^2 = 0\) and \(\sum_{j=1}^3 |\psi_{n,j}|^2\) never vanishes on \(S\) and \(g_n := \frac{\Phi_{n,1}}{\psi_{1,n}} - i \Phi_{n,2}\) is holomorphic and non constant on \(M_S\). It is clear that \(\Psi_n - \Phi\) is exact on \(M_S\), \(n \in \mathbb{N}\). Furthermore, we can slightly deform \(\Psi_n|_{M_S}\) so that \(\Psi_n - \Phi\) is exact on \(S\) as well, \(n \in \mathbb{N}\) (we leave the details to the reader).

If the lemma holds for \(\Psi_n\) for all \(n\), we can construct a sequence \(\{\Psi_{n,m}\}_{m \in \mathbb{N}} \subset \Omega_0^1(S)^3\) converging to \(\Psi_n\) in the \(\omega\)-topology on \(S\) and satisfying that \(\Psi_{n,m} - \Psi_n\) is exact on \(S\) for all \(n\). A standard diagonal argument proves the claim. \(\square\)

**Claim 2.8.** Without loss of generality, we can assume that \(g, 1/g\) and \(dg\) never vanish on \(\partial(M_S) \cup C_S\) (hence the same holds for \(\eta_i, i = 1, 2, 3\), and \(\Phi, j = 1, 2, 3\)). In particular, \(g \in F^*(S)\) and \(dg \in \Omega^*(S)\).

**Proof.** Take a sequence \(M_1 \supset M_2 \supset \ldots\) of tubular neighborhoods of \(M_S\) in \(W\) such that \(M_n \subset M^o_{n-1}\), for any \(n\), \(\cap_{n \in \mathbb{N}} M_n = M_S\), \(\Phi\) (and so \(g\)) meromorphically extends (with the same name) to \(M_1\), \(\sum_{j=1}^3 |\Phi_j|^2 \neq 0\) on \(M_1\), and \(g, 1/g\), and \(dg\) never vanish on \(\partial(M_n)\) for all \(n\) (take into account...
Claim 2.7). Choose $M_n$ in such a way that $S_n := M_n \cup C_S$ is an admissible subset of $W$ and $\gamma - M_n^0$ is a (non-empty) Jordan arc for any component $\gamma$ of $C_S$. In particular, $C_{S_n} = C_S - M_n^0$, $n \in \mathbb{N}$.

Let $(h_n, \psi_{3,n}) \in F^*(S_n) \times \Omega_0^3(S_n)$ be smooth data such that
\begin{itemize}
  \item $(h_n, \psi_{3,n})|_{M_{n}} = (g, \phi_3)|_{M_{n}}$ and $\sum_{i=1}^3 |\psi_{j,n}|^2$ never vanishes on $S_n$, where $\Psi_n = (\psi_{j,n})_{j=1,2,3} = (\frac{1}{h_n - h}, \frac{1}{h_n + h}, 1) \psi_{3,n} \in \Omega_0^3(S_n)$, $n \in \mathbb{N}$,
  \item $h_n, 1/h_n$ and $dh_n$ never vanish on $\partial(M_{n}) \cup C_{S_n}$,
  \item $\Psi_n|_{S} - \Phi$ is exact on $S$, and
  \item the sequence $\{\Psi_n|_{S}\}_{n \in \mathbb{N}} \subset \Omega_0^3(S)$ converges to $\Phi$ in the $\omega$-topology on $S$.
\end{itemize}

The construction of these data is standard, we omit the details.

Label $T \subset \Omega_0(W)^3$ as the subspace of data $\Psi$ formally satisfying $(i)$ and $(ii)$. If the Lemma held for any of the data in $\{\Psi_n | n \in \mathbb{N}\}$, $\Psi_n$ would lie in the closure of $T$ in $\Omega_0^3(S_n)$ with respect to the $\omega$-topology on $S_n$ for all $n \in \mathbb{N}$, hence the same would occur for $\Phi$ and we are done. \qed

Let $B_S$ be a homology basis of $H_1(S, \mathbb{Z})$ and label $n$ its cardinal number. Endow $F_0^3(S)$ with the maximum norm, and consider the Fréchet differentiable map:
\[P : F_0^3(S) \times F_0^3(S) \rightarrow C^{3\nu}, \quad P((h_1, h_2)) = \left( \int_c (e^{\mu \eta_1 - h_1} - 1) \eta_1, (e^{h_2 + h} - 1) \eta_2, (e^{h_2} - 1) \phi_3 \right)_{c \in B_S} \]

The meromorphic data inside the integrals arise from multiplying $g$ by $e^{\mu \eta_1}$ and $\phi_3$ by $e^{h_2}$. Label $A_0 : F_0^3(S) \times F_0^3(S) \rightarrow C^{3\nu}$ as the Fréchet derivative of $P$ at $(0,0)$.

**Claim 2.9.** $A_0|_{F_0^3(W) \times F_0^3(W)}$ is surjective.

**Proof.** Reason by contradiction and assume that $A_0(F_0^3(W) \times F_0^3(W))$ lies in a complex subspace $U = \{( \langle x_c, y_c, z_c \rangle )_{c \in B_S} \in C^{3\nu} | \sum_{c \in B_S} \left( A_c x_c + B_c y_c + D_c z_c \right) = 0 \}$, where $A_c, B_c$ and $D_c \in \mathbb{C}$ for all $c \in B_S$ and $\sum_{c \in B_S} \left( |A_c| + |B_c| + |D_c| \right) \neq 0$. This simply means that:
\[(2.1) \quad - \int_{\Gamma_1} h \eta_1 + \int_{\Gamma_2} h \eta_2 = \int_{\Gamma_1} h \eta_1 + \int_{\Gamma_2} h \eta_2 + \int_{\Gamma_3} h \phi_3 = 0\]
for all $h \in F_0(W)$, where $\Gamma_1 = \sum_{c \in B_S} A_c \mathcal{C}, \Gamma_2 = \sum_{c \in B_S} B_c \mathcal{C}$ and $\Gamma_3 = \sum_{c \in B_S} D_c \mathcal{C}$.

Label $\Sigma_0 = \{ f \in F_0(W) | ( f ) \geq (\phi_3)^2 \}$. By Theorem 2.3, the function $h = df/\phi_3 \in F_0^3(S)$ lies in the closure of $F_0^3(W)$ in the $\omega$-topology on $F_0^3(S)$ for any $f \in \Sigma_0$. Therefore, equation (2.1) can be applied formally to $h = df/\phi_3$, getting that $\int_{\Gamma_1} h df = \int_{\Gamma_2} g df = 0$ for all $f \in \Sigma_0$. Integrating by parts,
\[(2.2) \quad \int_{\Gamma_1} \frac{f dg}{g^2} = \int_{\Gamma_2} f dg = 0\]
for all $f \in \Sigma_0$.

Let us show that $\Gamma_1 = 0$.

Let $\mu$ and $b$ denote the genus of $W$ and the number of ends of $W$. It is well known (see [FK]) that there exist $2\mu + b - 1$ cohomologically independent 1-forms in $\Omega_0(W)$ generating the first holomorphic De Rham cohomology group $H_{\text{hol}}^1(W)$ of $W$. Thus, the map $H_{\text{hol}}^1(W) \rightarrow C^{2\mu + b - 1}$, $\tau \mapsto \left( \int_c \tau \right)_{c \in B_0}$, where $B_0$ is any homology basis of $W$, is a linear isomorphism. Assume that $\Gamma_1 \neq 0$ and take $[\tau] \in H_{\text{hol}}^1(W)$ such that $\int_{\Gamma_1} \tau \neq 0$. Since $W$ is an open surface, $F_0(W)$ has infinite dimension and we can find $F \in F_0(W)$ such that $(\tau + dF) \geq (d\theta) (g) [\tau] (\phi_3)^2$. Set $h := (\tau + dF)(\phi_3)^2$ and note that $(h) \geq (\phi_3)^2$. By Theorem 2.3, $h$ lies in the closure of $\Sigma_0$ in $F_0^3(S)$ with respect to the
Indeed, like in the proof of Claim 2.8 consider a sequence \( \{ \partial \} \) of Proof.

By a similar argument \( \Gamma_2 = 0 \) and equation (2.1) becomes:

\[
(2.3) \\
\int_{\Gamma_3} h \phi_3 = 0
\]

for all \( h \in \mathcal{F}_0(W) \).

Since \( \sum_{i \in B_3} (|A_i| + |B_i| + |D_i|) \neq 0 \), then \( \Gamma_3 \neq 0 \). Reason as above and choose \( |\tau| \in H^1_{\text{hol}}(W) \) and \( F \in \mathcal{F}_0(W) \) such that \( \int_{\Gamma_3} \tau \neq 0 \) and \( (\tau + dF)_0 \geq (\phi_3) \). Set \( h := \frac{\tau + dF}{\phi_3} \) and note that \( h \in \mathcal{F}_0^+(S) \).

By Theorem 2.3, \( h \) lies in the closure of \( \mathcal{F}_0(W) \) in \( \mathcal{F}_0^+(S) \) with respect to the \( \omega \)-topology, and equation (2.3) gives that \( \int_{\Gamma_3} \tau + dF = \int_{\Gamma_3} \tau = 0 \), a contradiction. This proves the claim. 

Let \( \{ e_1, \ldots, e_{3^N} \} \) be a basis of \( \mathbb{C}^{n+\nu} \), fix \( H_i = (h_{1,i}, h_{2,i}) \in A_0^{-1}(e_i) \cap (\mathcal{F}_0(W) \times \mathcal{F}_0(W)) \) for all \( i \), and set \( Q_0 : \mathbb{C}^{n+\nu} \to \mathbb{C}^{n+\nu} \) as the analytical map given by

\[
Q_0(z_i)_{i=1,\ldots,3^N} = \mathcal{P}(\sum_{i=1,\ldots,3^N} z_i H_i).
\]

By Claim 2.9 \( d(Q_0) \) is an isomorphism, so there exists a closed Euclidean ball \( U \subset \mathbb{C}^{n+\nu} \) centered at the origin such that \( Q_0 : U \to Q_0(U) \) is an analytical diffeomorphism. Furthermore, notice that \( 0 = Q_0(0) \in Q_0(U) \) is an interior point of \( Q_0(U) \).

On the other hand, by Lemmas 2.4 and 2.5 there exists a sequence \( \{(f_n, \psi_n)\}_{n \in \mathbb{N}} \subset (\mathcal{F}(W) \times \Omega_0(W)) \) such that \( (f_n) = (g) \) and \( (\psi_n) = (\phi_3) \) for all \( n \), and \( (f_n, \psi_n)_{n \in \mathbb{N}} \to (g, \phi_3) \) in the \( \omega \)-topology on \( \mathcal{F}^+(S) \times \Omega_0^+(S) \).

Label \( \mathcal{P}_n : \mathcal{F}_0^+(S) \times \mathcal{F}_0^+(S) \to \mathbb{C}^{n+\nu} \) as the Fréchet differentiable map

\[
\mathcal{P}_n(h_1, h_2) = \left( \int_{\mathbb{C}} (e^{h_2 - h_1} \eta_{1,n} - \eta_1, e^{h_2 + h_1} \eta_{2,n} - \eta_2, e^{h_3} \psi_n - \phi_3) \right)_{c \in \mathcal{B}_3},
\]

where \( \eta_{1,n} = \frac{1}{2} \psi_n(1/f_n - f_n) \) and \( \eta_{2,n} = \frac{1}{2} \psi_n(1/f_n + f_n) \), and call \( \mathcal{Q}_n : \mathbb{C}^{n+\nu} \to \mathbb{C}^{n+\nu} \) as the analytical map \( \mathcal{Q}_n(z_{i=1,\ldots,3^N}) = \mathcal{P}_n(\eta_{i=1,\ldots,3^N} H_i) \) for all \( n \in \mathbb{N} \). Since \( \{ \mathcal{Q}_n \}_{n \in \mathbb{N}} \to \mathcal{Q}_0 \) uniformly on compacts subsets of \( \mathbb{C}^{n+\nu} \) without loss of generality we can suppose that \( \mathcal{Q}_n : U \to \mathcal{Q}_n(U) \) is an analytical diffeomorphism and \( 0 \in \mathcal{Q}_n(U) \) for all \( n \).

Label \( y_n = (y_{1,n}, \ldots, y_{3^N,n}) \) as the unique point in \( U \) such that \( \mathcal{Q}_n(y_n) = 0 \) and note that \( \{ y_n \}_{n \in \mathbb{N}} \to 0 \). Setting

\[
g_n = e^{\sum_{i=1}^{3^N} y_{i,n} h_i}, \quad \phi_{3,n} = e^{\sum_{i=1}^{3^N} y_{i,n} h_i} \psi_n
\]

for all \( n \in \mathbb{N} \), the sequence \( \{ (g_n, \phi_{3,n}) \}_{n \in \mathbb{N}} \) solves the lemma. 

**Corollary 2.10.** In the previous lemma we can choose \( \phi_{3,n} = \phi_3 \) for all \( n \in \mathbb{N} \), provided that \( \phi_3 \) extends holomorphically to \( W \) and \( \phi_3 \) never vanishes on \( C_S \).

**Proof.** Without loss of generality, we can suppose that \( g \), \( 1/g \) and \( dg \) never vanish on \( \partial(M_S) \cup C_S \).

Indeed, like in the proof of Claim 2.8 consider a sequence \( \{ M_n \}_{n \in \mathbb{N}} \) of tubular neighborhoods of \( M_S \) in \( W \) such that \( \cap_{n=1}^{\infty} M_n = M_S \), \( S_n := M_n \cup C_S \) is admissible in \( W \), \( 1/g \) and \( dg \) never vanishes on \( \partial(M_n) \) and \( \gamma : M_n^0 \) is a (non-empty) Jordan arc for any component \( \gamma \) of \( C_S \), for all \( n \in \mathbb{N} \). Let \( h_n \in \mathcal{F}^+(S_n) \) be a smooth datum such that:

- \( h_n(M_S) = (1/h_n - h_n), 1/2(1/h_n + h_n), 1) \phi_3 \in \Omega_0^+(S_n)^3, \ n \in \mathbb{N}, \)
- \( h_n \), \( 1/h_n \) and \( dh_n \) never vanish on \( \partial(M_{S_n}) \cup C_{S_n}, \)
- \( \Psi_n | S - \Phi \) is exact on \( S \), and
- the sequence \( \{ \Psi_n | S \}_{n \in \mathbb{N}} \subset \Omega_0^+(S)^3 \) converges to \( \Phi \) in the \( \omega \)-topology on \( S \).
Reasoning as in the proof of Claim 2.8, if the Lemma held for any of the data in \{Ψ_n \mid n ∈ N\} the same would occur for \(Φ\) and we are done.

Reasoning like in the proof of Lemma 2.6, we can prove that \(\tilde{A}_0 : F_0(W) → C^v\) is surjective, where \(\tilde{A}_0\) is the Fréchet derivative of \(\tilde{P} : F_0^n(S) → C^{2v}\), \(\tilde{P}(h) := P(h, 0)\). Then take \(\tilde{H}_i ∈ \tilde{A}_0^{-1}(e_i) \cap F_0(W)\) for all \(i\), where \(\{e_1, ..., e_{2v}\}\) is a basis of \(C^{2v}\), and define \(\tilde{Q}_0 : C^{2v} → C^{2v}\) by \(\tilde{Q}_0((z_i)_{i=1,...,2v}) = \tilde{P}(\sum_{i=1,...,2v} z_i \tilde{H}_i)\).

Use Riemann-Roch Theorem to find a holomorphic function \(H ∈ F_0(W)\) such that \((H) = (φ_3)|_{W - S}\), and then Lemma 2.4 to get \(\{f_n\}_{n ∈ N} ⊂ F(W)\) such that \((f_n) = (g|_S)\) for all \(n\) and \(\{f_n\}_{n ∈ N} → g/H in the ω-topology on F^3(S)\).

Set \(\tilde{P}_n : F_0^n(S) → C^{2v}\) by \(\tilde{P}_n(h) = (\int_c (e^{-h}η_{1,n} - η_{1,c}e^h η_{2,n} - η_{2})_c ∈ B^c\), where \(η_{1,n} = \frac{1}{2}φ_3(\frac{1}{j_2h} - f_n H)\) and \(η_{2,n} = \frac{1}{2}φ_3(\frac{1}{j_2h} + f_n H)\), and call \(\tilde{Q}_n : C^{2v} → C^{2v}\) as the analytical map \(\tilde{Q}_n((z_i)_{i=1,...,2v}) = \tilde{P}_n(\sum_{i=1,...,2v}(z_i \tilde{H}_i))\) for all \(n ∈ N\). To finish, reason as in the proof of Lemma 2.6.

3. Weierstrass Representation and Flux Map of Minimal Surfaces

Let \(W\) be an open Riemann surface and let \(X = (X_1, X_2, X_3) : W → \mathbb{R}^3\) be a conformal minimal immersion. Denote by \(φ_j = \partial X_j\), \(j = 1, 2, 3\), and \(Φ = \partial X\). The 1-forms \(φ_k\) are holomorphic, have no real periods and satisfy that \(∑_{k=1}^3 φ_k^2 = 0\). Furthermore, the intrinsic metric in \(W\) is given by \(ds^2 = ∑_{k=1}^3 |φ_k|^2\), hence \(φ_k, k = 1, 2, 3\), have no common zeroes.

Conversely, a vectorial holomorphic 1-form \(Φ = (φ_1, φ_2, φ_3)\) on \(W\) without real periods and satisfying that \(∑_{k=1}^3 φ_k^2 = 0\) and \(∑_{k=1}^3 |φ_k(P)|^2 ≠ 0\) for all \(P ∈ W\), determines a conformal minimal immersion \(X : W → \mathbb{R}^3\) by the expression:

\[
X = \text{Re} \int Φ.
\]

By definition, the triple \(Φ\) is said to be the Weierstrass representation of \(X\). The meromorphic function \(g = \frac{φ_3}{φ_1 - φ_2}\) corresponds to the Gauss map of \(X\) up to the stereographic projection (see [Os]).

We need the following definition:

**Definition 3.1.** Given a proper region \(M \subset N\) with possibly non empty boundary, we denote by \(M(M)\) the space of maps \(X : M → \mathbb{R}^3\) extending as a conformal minimal immersion to an open neighborhood of \(M\) in \(N\). This space will be equipped with the \(C^0\) topology of the uniform convergence on compact subsets of \(M\).

Given \(X ∈ M(M)\) and an arclength parameterized curve \(γ(s)\) in \(M\), the conormal vector of \(X\) at \(γ(s)\) is the unique unitary tangent vector \(μ(s)\) of \(X\) at \(γ(s)\) such that \(\{dX(γ'(s)), μ(s)\}\) is a positive basis. If in addition \(γ\) is closed, the flux \(p_X(γ)\) along \(γ\) is given by \(\int_γ μ(s)ds\), and it is easy to check that \(p_X(γ) = \text{Im} \int_γ \partial X\). The flux map \(p_X : H_1(M, Z) → \mathbb{R}^3\) is a group morphism.

Let \(S \subset N\) be a compact admissible subset. We denote by \(J(S)\) the space of smooth maps \(X : S → \mathbb{R}^3\) such that \(X|_{M_S} ∈ M(M_S)\) and \(X|_C\), is regular, that is to say, \(X|_a\) is a regular curve for all \(a ∈ C^S\). It is clear that \(Y|_S ∈ J(S)\) for all \(Y ∈ M(N)\).

Consider \(X ∈ J(S)\) and let \(ω : C^S → \mathbb{R}^3\) be an smooth normal field along \(C^S\) with respect to \(X\). This simply means that for any analytical arc \(a ∈ C^S\) and any smooth parameter \(t\) on \(X|_a\), \(ω(α(t))\) is smooth, unitary and orthogonal to \((X|_a)'(t)\), \(ω\) extends smoothly to any open analytical arc \(β\) in \(W\) containing \(a\) and \(ω\) is tangent to \(X\) on \(β ∩ S\). The normal field \(ω\) is said to be orientable respect to
X if for any component \( a \subset C_S \) having endpoints \( P_i, P_2 \) lying in \( \partial(M_S) \), and for any arclength parameter \( s \) along \( X|_s \), the basis \( B_i = \{ (X|_s)'(s_i), \alpha(s_i) \} \) of the tangent plane of \( X|_{M_S} \) at \( P_i, i = 1, 2 \), are both positive or negative, where \( s_j \) is the value of \( s \) for which \( \alpha(s_j) = P_i, i = 1, 2 \).

**Definition 3.2.** We call \( \mathcal{M}^*(S) \) as the space of marked immersions \( X_{\alpha^\circ} := (X, \alpha) \), where \( X \in \mathcal{J}(S) \) and \( \alpha \) is an orientable smooth normal field along \( C_S \) respect to \( X \), endowed with the \( C^0 \) topology of the uniform convergence of maps and normal fields.

Given \( X_{\alpha^\circ} \in \mathcal{M}^*(S) \), let \( \partial X_{\alpha^\circ} = (\hat{\phi}_j)_{j=1,2,3} \) be the complex vectorial “1-form” on \( S \) given by \( \partial X_{\alpha^\circ}|_{M_S} = \partial(X|_{M_S}), \partial X_{\alpha^\circ}(\alpha'(s)) = dX(\alpha'(s)) + i\alpha(s) \), where \( a \) is a component of \( C_S \) and \( s \) is the arclength parameter of \( X|_a \) for which \( \{dX(\alpha'(s_i)), \alpha(s_i)\} \) are positive, where as above \( s_1 \) and \( s_2 \) are the values of \( s \) for which \( \alpha(s) \in \partial(M_S) \). If \( (U, \zeta = x + iy) \) is an arclength chart on \( N \) such that \( a \cap U = z^{-1}(R \cap z(U)) \), it is clear that \( \partial X_{\alpha^\circ}|_{a \subset U} = [dX(\alpha'(s)) + i\alpha(s)]s'(x)dz|_{a \subset U} \), hence \( \partial X_{\alpha^\circ} \in \Omega^1(S)_3 \). Furthermore, \( \hat{g} = \hat{\phi}_1/(\hat{\phi}_1 - i\hat{\phi}_2) \in \mathcal{F}^*(S) \) provided that \( \hat{g}^{-1}(\infty) \subset S^0 \).

It is clear that \( \hat{\phi}_i \) is smooth on \( S, j = 1, 2, 3 \), and the same occurs for \( \hat{g} \) on \( S - \hat{g}^{-1}(\infty) \). Notice that \( \sum_{j=1}^3 \hat{\phi}_1^2 = 0, \sum_{j=1}^3 \hat{\phi}_1^2 \) never vanishes on \( S \) and Real(\( \hat{\phi}_i \)) is an “exact” real 1-form on \( S, j = 1, 2, 3 \), hence we also have \( X(P) = X(Q) + \text{Real}(\int_{Q}^{P}(\hat{\phi}_j))_{j=1,2,3}, P, Q \in S \). For these reasons, \((\hat{g}, \hat{\phi}_3)\) will be called the generalized “Weierstrass data” of \( X_{\alpha^\circ} \). As \( X|_{M_S} \in \mathcal{M}(M_S) \), then \( (\hat{\phi}_j)_{j=1,2,3} := (\hat{\phi}_j|_{M_S})_{j=1,2,3} \) and \( \hat{g} := \hat{g}|_{M_S} \) are the Weierstrass data and the meromorphic Gauss map of \( X|_{M_S} \), respectively.

The group homomorphism \( p_{X_{\alpha^\circ}} : \mathcal{H}_1(S, \mathbb{Z}) \to \mathbb{R}^3, \ p_{X_{\alpha^\circ}}(\gamma) = \text{Im} \int_{\gamma} \partial X_{\alpha^\circ}, \) is said to be the generalized flux map of \( X_{\alpha^\circ} \). Obviously, \( p_{X_{\alpha^\circ}} = p_{Y}|_{\mathcal{H}_1(S)} \) provided that \( X = Y|_S \) and \( \alpha_Y \) is the conormal field of \( Y \) along any curve in \( C_S \).

### 4. Properness and Conformal Structure of Minimal Surfaces

From now on, we label \( x_5 : \mathbb{R}^3 \to \mathbb{R} \) as the \( k \)-th coordinate function, \( k = 1, 2, 3 \). Given a compact \( M \subset \mathcal{N} \) and a map \( X : M \to \mathbb{R}^3 \), we denote \( ||X|| := \max_M \{ (\sum_{j=1}^3 x_j \circ X)^2 \}^{1/2} \) as the maximum norm. For each \( \varphi, a \subset \mathbb{R} \), we call

\[
\Pi_a(q) = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 + \tan(q)x_1 \leq a \}, \quad \Pi^*_a(q) = \mathbb{R}^3 - \Pi_a(q).
\]

The following lemma concentrates most of the technical computations required in the proof of the main theorem of this section.

**Lemma 4.1.** Let \( M, V \subset \mathcal{N} \) be two compact regions with analytical boundary such that \( M \subset V^\circ \) and \( \mathcal{N} - M \) has no bounded components in \( \mathcal{N} \).

Consider \( X \in \mathcal{M}(M) \) and let \( p : \mathcal{H}_1(V, \mathbb{Z}) \to \mathbb{R} \) be any morphism extension of \( p_X \). Suppose there are \( \theta \in (0, \pi/4) \) and \( \delta \in (0, +\infty) \) such that \( X(\partial(M)) \subset \Pi^*_\delta(\theta) \cup \Pi^*_{\delta+1}(\theta) \).

Then, for any \( \varepsilon > 0 \), there exists \( Y \in \mathcal{M}(V) \) such that \( p_Y = p, ||Y - X|| < \varepsilon \) on \( M, Y(\partial(V)) \subset \Pi^*_\delta(\theta) \cup \Pi^*_{\delta+1}(\theta) \) and \( Y(V - M) \subset \Pi^*_{\delta}(\theta) \cup \Pi^*_\delta(\theta) \).

**Proof.** We start with the following claim.

**Claim 4.2.** The lemma holds when the Euler characteristic \( \chi(V - M^\circ) \) vanishes.
Proof. Since \( M \subset V^o \) and \( V^o - M \) has no bounded components in \( V^o \), then \( V - M^o = \bigcup_{j=1}^k A_j \), where \( A_1, \ldots, A_k \) are pairwise disjoint compact annuli. Write \( \partial(A_j) = \alpha_j \cup \beta_j \), where \( \alpha_j \subset \partial(M) \) and \( \beta_j \subset \partial(V) \) for all \( j \).

Since \( \chi(\partial(M)) \subset \Pi^\omega_r(\theta) \cup \Pi^\omega_r(-\theta) \), each \( \alpha_j \) can be divided into finitely many Jordan arcs \( a^i_j \), \( i = 1, \ldots, n_j \geq 2 \), laid end to end and satisfying that either \( \chi(a^i_j) \subset \Pi^\omega_r(\theta) \) or \( \chi(a^i_j) \subset \Pi^\omega_r(-\theta) \) for all \( i \). Up to refining the partitions, we can assume that \( n_j = m \in \mathbb{N} \) for all \( j \).

An arc \( a^i_j \) is said to be positive if \( \chi(a^i_j) \subset \Pi^\omega_r(\theta) \), and negative otherwise. Notice that \( \chi(a^i_j) \subset \Pi^\omega_r(-\theta) \) for any negative (and possibly for some positive) \( a^i_j \). We also label \( Q^i_j \) and \( Q^{i+1}_j \) as the endpoints of \( a^i_j \), in such a way that \( Q^{i+1}_j = a^i_j \cap a^{i+1}_j \), \( i = 1, \ldots, m \), (obviously, \( Q^{m+1}_j = Q^1_j \)).

Let \( \{ r^i_j \mid i = 1, \ldots, m \} \) be a collection of pairwise disjoint analytical Jordan arcs in \( A_j \) such that \( r^i_j \) has initial point \( Q^i_j \in \alpha_j \), final point \( P^i_j \in \beta_j \), \( r^i_j \) is otherwise disjoint from \( \partial(A_j) \), and \( r^i_j \) meets transversally \( a^i_j \) at \( Q^i_j \), for all \( i \) and \( j \).

Let \( W \) be a tubular neighborhood of \( V \) in \( N \), that is to say, an open connected subset of \( N \) such that \( V \subset W \), \( W - V \) consists of a finite collection of open annuli, and the closure of any annuli in \( W - V \) intersects \( \partial(V) \).

Set \( M_0 = M \cup (\bigcup_{j=1}^k r^i_j) \), and notice that \( M_0 \) is an admissible subset in \( N \), and so in \( W \). Call \( \Omega^i_j \) as the closed disc in \( A_j \) bounded by \( a^i_j \cup r^i_j \cup r^{i+1}_j \) and a piece, named \( \beta^i_j \), of \( \beta_j \) connecting \( P^i_j \) and \( P^{i+1}_j \). Obviously \( \Omega^i_j \cap \Omega^{i+1}_j = r^{i+1}_j \), \( i < m \), \( \Omega^m_j \cap \Omega^1_j = r^1_j \), and \( A_j = \bigcup_{i=1}^m \Omega^i_j \). The domain \( \Omega^i_j \) is said to be positive (respectively, negative) if \( a^i_j \) is positive (respectively, negative).

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The annulus \( A_j \).}
\end{figure}

Take \( X_\alpha \in \mathcal{M}^+(M_0) \) such that \( X_\alpha |_M = X \), and
\begin{equation}
\text{dist}(\hat{X}(P^i_j), \Pi^\omega_r(\theta) \cup \Pi^\omega_r(-\theta)) > 1,
\end{equation}
for all \( i \) and \( j \). In addition, we choose \( X_\alpha \) in such a way that:

1. If \( \Omega^i_j \) is positive then \( \hat{X}(r^i_j \cup r^{i+1}_j) \subset \Pi^\omega_r(\theta) \).
2. If \( \Omega^i_j \) is negative then \( \hat{X}(r^i_j \cup r^{i+1}_j) \subset \Pi^\omega_r(-\theta) \).
The existence of such a $\hat{X}_0$ is standard, we omit the details.

We label $\hat{\phi}_k = \partial(x_k \circ \hat{X})$, $k = 1, 2, 3$, and recall that $\sum_{k=1}^{3} \hat{\phi}_k^2 = 0$ and $\sum_{k=1}^{3} |\hat{\phi}_k|^2$ never vanishes on $M_0$. Applying Lemma 2.6 to $\Phi := (\hat{\phi}_k)_{k=1,2,3}$ on $M_0$ and integrating the resulting Weierstrass data on $V$, we can find $Y_0 \in \mathcal{M}(V)$ such that

\[(4.2)\quad p_{Y_0} = p_{\hat{X}_0} = p_X = p\]

and

\[(4.3)\quad \|Y_0 - \hat{X}\| < \varepsilon/3 \quad \text{on} \quad M_0.\]

Furthermore, without loss of generality we can assume that

\[(4.4)\quad \text{dist} (Y_0(P_i^j), \Pi_\delta(\theta) \cup \Pi_\delta(-\theta)) > 1,\]

for all $j, i$, and

(2+) if $\Omega_j^i$ is positive then $Y_0(r_j^i \cup a_j^i \cup r_j^{i+1}) \subset \Pi_\delta^*(\theta)$.

(2-) if $\Omega_j^i$ is negative then $Y_0(r_j^i \cup a_j^i \cup r_j^{i+1}) \subset \Pi_\delta^*(-\theta)$.

This choice is possible thanks to the properties (4.1), (1+) and (1-).

Let $L^+ : \mathbb{R}^3 \to \mathbb{R}^3$ (respectively, $L^- : \mathbb{R}^3 \to \mathbb{R}^3$) denote the rotation of angle $\theta$ (respectively, $-\theta$) around the straight line parallel to the $x_2$-axis and containing the point $(0, 0, \delta)$. Note that $L^+(\Pi_\delta(\theta)) = \Pi_\delta(0)$, $L^+(\Pi_\delta(-\theta)) = \Pi_\delta(2\theta)$, $L^-(\Pi_\delta(\theta)) = \Pi_\delta(-\theta)$ and $L^-(\Pi_\delta(-\theta)) = \Pi_\delta(0)$.

Call $Y^+ = L^+ \circ Y_0$ and $Y^- = L^- \circ Y_0 \in \mathcal{M}(V)$. Set $I^+ = \{(i, j) \mid \Omega_j^i$ is positive\}$ and $I^- = \{(i, j) \mid \Omega_j^i$ is negative\}.

In a first step, we are going to deform the immersion $Y^+$. Roughly speaking, this deformation will be strong on $\bigcup_{(i, j) \in I^+_+} \Omega_j^i$ and hardly modify $Y^+$ on $V - \bigcup_{(i, j) \in I^-} \Omega_j^i$.

Label $\Psi = (\psi_k)_{k=1,2,3}$ as the Weierstrass data of $Y^+$ on $V$. For any $(i, j) \in I^+$ let $K_j^i$ and $\gamma_j^i$ be a closed disc with analytical boundary and a compact analytical Jordan arc in $\Omega_j^i$ satisfying that:

(a1) $K_j^i \cap \partial(\Omega_j^i) \neq \emptyset$ and consists of an arc in $\beta_j^i - \{p_j^i, p_j^{i+1}\}$, $Y^+(\Omega_j^i - K_j^i) \subset \Pi_\delta^*(0)$, and

$$\text{dist} (Y^+(\beta_j^i - K_j^i), \Pi_\delta(0) \cup \Pi_\delta(-2\theta)) > 1.$$

This choice is possible from (2-) and (4.4) provided that $K_j^i$ is chosen large enough in $\Omega_j^i$.

(a2) The endpoints $S_j^i$ and $T_j^i$ of $\gamma_j^i$ lie in $a_j^i - \{Q_j^i, Q_j^{i+1}\}$ and $\partial(K_j^i) - \beta_j^i$, respectively, and $\gamma_j^i$ is otherwise disjoint from $K_j^i \cup \partial(\Omega_j^i)$. Moreover, $\gamma_j^i$ and $a_j^i$ (resp., $\partial(K_j^i)$) meet transversally at $S_j^i$ (resp., $T_j^i$) and $\psi_3$ never vanishes on $\gamma_j^i$. See Figure 1.

Set $S_+ = (V - \bigcup_{(i, j) \in I^+} \Omega_j^i) \cup (\bigcup_{(i, j) \in I^+} (K_j^i \cup \gamma_j^i))$, and notice that $S_+$ is admissible in $\mathcal{N}$, and so in $W$. Consider $\lambda^+ > 0$ such that

\[(4.5)\quad \text{dist} ((-\lambda^+, 0, 0) + Y^+(\bigcup_{(i, j) \in I^+} K_j^i), \Pi_\delta(-2\theta)) > 1.\]

This choice is possible because $\theta \in (0, \pi/4)$.

**Claim 4.3.** There exist smooth data $\hat{\Psi} = (\hat{\psi}_k)_{k=1,2,3} \in \Omega_0(S^3)$ such that:

(i) $\hat{\psi}_3 = \psi_3$ on $S_+ - \bigcup_{(i, j) \in I^+} \gamma_j^i$ for $k = 1, 2$, and $\hat{\psi}_3 = (\psi_3)|_{S^+}$.

(ii) $\sum_{j=1}^{3} |\hat{\psi}_k|^2 = 0$, and $\sum_{j=1}^{3} |\hat{\psi}_k|^2$ never vanishes on $S_+$. 

(iii) \( \text{Re}(\int_{S_1}^{T_1} \hat{\psi}_1) = \text{Re}(\int_{S_1}^{T_1} \psi_1) - \lambda^+, \) where the integrals are computed along \( \gamma^+_t \), \((i, j) \in \mathcal{I}_+ .\)

Proof. For each \((i, j) \in \mathcal{I}_+\) consider an open conformal disc \((U, \mathbf{z})\) on \(\mathcal{N}\) (obviously depending on \((i, j)\)) so that \(\gamma^+_t \subset U\) and \(z(\gamma^+_t) = [0, 1] \subset \mathbb{R}\). Write \(\psi_3(\mathbf{z}) = f_3(\mathbf{z})d\mathbf{z}\) on \(U\) and call \(g_+\) as the meromorphic Gauss map of \(Y^+\).

Let \(\{\rho_t : \mathbb{R} \to \mathbb{C}\}_{t \in \mathbb{R}}\) be a smooth family of never vanishing smooth functions satisfying that:

- \(\rho_t(x) = 1\) if \(x \in ]-\infty, 1/3[ \cup [2/3, +\infty[\).
- If \(|t| \geq 1\) then \(\rho_t(x) = \frac{1}{|t|^{|1/2| / 3|}}\) for all \(x \in [\frac{1}{2} + \frac{1}{4} / t^2, \frac{2}{3} - \frac{1}{4} / t^2].\)
- \(A_0 \leq |\rho_t(x)| \leq A_1|t| + A_2\) for all \(t, x \in \mathbb{R}\), where \(A_0, A_1\) and \(A_2\) are suitable positive constants not depending on \(t\).

Set \(g_t = g \cdot (\rho_t \circ \mathbf{z})\) on \(\gamma^+_t\) for each \((i, j) \in \mathcal{I}_+\), and put \(g_t = g_+\) on \(S_+ - \bigcup_{(i,j) \in \mathcal{I}_+} \gamma^+_t\). Consider the generalized Weierstrass data \(\Psi_t = (\hat{\psi}_{k,t})_{k=1,2,3} \in \Omega_0^3(S_+)^3\) induced by the couple \((g_t, \psi_3)\), and notice that \(\Psi_t\) satisfies (i) and (ii).

Since \(\lim_{t \to +\infty} \text{Re}(\int_{S_1}^{T_1} \hat{\psi}_{1,t}) = -\lim_{t \to -\infty} \text{Re}(\int_{S_1}^{T_1} \hat{\psi}_{1,t}) = +\infty\), an intermediate value argument gives that \(\text{Re}(\int_{S_1}^{T_1} \hat{\psi}_{1,t}) = \text{Re}(\int_{S_1}^{T_1} \psi_1) - \lambda^+\) for some \(t_0 \in \mathbb{R}\). It suffices to set \(\Psi = \hat{\Psi}_{t_0}\). \(\square\)

Fix \(P_0 \in \mathcal{M}\).

By Corollary 2.10, there exists \(\{(\hat{\psi}_{1,n}, \hat{\psi}_{2,n})\}_{n \in \mathbb{N}} \subset \Omega_0^3(V)^2\) converging to \((\hat{\psi}_1, \hat{\psi}_2)\) uniformly on \(S_+\) and such that:

- \(\sum_{k=1}^3 |\hat{\psi}_{k,n}|^2 = 0\) and \(\sum_{k=1}^3 |\hat{\psi}_{k,n}|^2\) never vanishes on \(V\), where \(\hat{\psi}_{3,n} = \hat{\psi}_3, \forall n \in \mathbb{N}\).
- The minimal immersion \(Z_+^n \subset \mathcal{M}(V)\) with initial condition \(Z_0^n(P_0) = Y^+(P_0)\) and Weierstrass data \(\psi_n = (\hat{\psi}_{k,n})_{k=1,2,3}\) is well defined and satisfies

\[
4.6 \quad p_{Z_+^n} = p_{Y^+}, \quad \forall n \in \mathbb{N}.
\]

Notice that \(\{Z_+^n\}_{n \in \mathbb{N}}\) uniformly converges to \(Y^+\) on \(V - \bigcup_{(i,j) \in \mathcal{I}_+} \Omega^+_j\), \(\{x_1 \circ Z_+^n\}_{n \in \mathbb{N}}\) uniformly converges to \(x_1 \circ Y^+ - \lambda^+\) on \(\bigcup_{(i,j) \in \mathcal{I}_+} K_j\) and \(x_3 \circ Z_+^n = x_3 \circ Y^+, \forall n \in \mathbb{N}\) (see Claim 4.3). These facts let us find \(n_0 \in \mathbb{N}\) such that the following properties hold:

- \(\|Z^+_0 - Y^+\| < \epsilon / 3\) on \(M\).
- \(Z^+_0 \cup \bigcup_{(i,j) \in \mathcal{I}_+} (\Omega^+_j - K_j) \subset \Pi^+_2(0)\) and \(Z^+_0 \cup \bigcup_{(i,j) \in \mathcal{I}_+} (\beta^+_j - K_j) \subset \Pi^+_{2+1}(0)\). See (a1).
- \(\text{dist}(Z^+_0(\cup_{(i,j) \in \mathcal{I}_+} K_j)), \Pi^+_2(2\theta)) > 1.\) Take into account (4.5).
- \(\text{dist}(Z^+_0(P_0)\cup\Pi^+_2(-2\theta)) \cup \Pi^+_2(0)) > 1,\) for any \((i,j) \in \mathcal{I}_-.\) Use (4.4).
- \(Z^+_0(r^+_j \cup k_j^+ \cup r_j^{+1}) \subset \Pi^+_2(0),\) for any \((i,j) \in \mathcal{I}_-\). It follows from (2-...).

Denote by \(F^- := L^- \circ L^- \circ Z^+_0 \in \mathcal{M}(V)\). The above properties can be rewritten in terms of \(F^-\) as follows:

(A1) \(\|F^- - Y^-\| < \epsilon / 3\) on \(M\).
(A2) \(F^-(\cup_{(i,j) \in \mathcal{I}_+} (\Omega^+_j - K_j)) \subset \Pi^+_2(2\theta)\) and \(\text{dist}(F^-(\cup_{(i,j) \in \mathcal{I}_+} (\beta^+_j - K_j))), \Pi^+_2(2\theta)) > 1\).
(A3) \(F^- \cup \bigcup_{(i,j) \in \mathcal{I}_+} K_j \subset \Pi^+_{2+1}(0)\).
(A4) \(\text{dist}(F^-(P_0)\cup\Pi^+_2(2\theta)) > 1,\) for any \((i,j) \in \mathcal{I}_-\).
(A5) \(F^- (r^+_j \cup k_j^+ \cup r_j^{+1}) \subset \Pi^+_2(0),\) for any \((i,j) \in \mathcal{I}_-\).
Label $Y = (\phi_k)_{k=1,2,3}$ as the Weierstrass data of $F^-$ on $V$. Now, we are going to deform the immersion $F^-$ in an analogous way. For any $(i, j) \in \mathcal{I}_-$ let $K'_i$ and $\gamma'_i$ be a closed disc and an analytical Jordan arc in $\Omega'_i$ satisfying that:

\begin{itemize}
  \item (b1) $K'_i \cap \partial(\Omega'_i) \neq \emptyset$ and consists of a connected arc in $\beta'_i$, $F^-(\overline{\Omega'_i - K'_i}) \subset \Pi'_i(0)$, and
  \begin{align*}
  \text{dist} \left( F^- (\beta'_i - K'_i), \Pi_\delta(0) \cup \Pi_\delta(2\theta) \right) > 1.
  \end{align*}
  This choice is possible from (A5) and (A4).
  \item (b2) The endpoints $S'_i$ and $T'_i$ of $\gamma'_i$ lie in $a'_i$ and $\partial(K'_i) - \beta'_i$, respectively, and $\gamma'_i$ is otherwise disjoint from $K'_i \cup \partial(\Omega'_i)$. Moreover, $\gamma'_i$ and $a'_i$ (resp., $\partial(K'_i)$) meet transversally at $S'_i$ (resp., $T'_i$) and $\varphi_3$ never vanishes on $\gamma'_i$.
\end{itemize}

Set $S_- = (V - \bigcup_{(i,j) \in \mathcal{I}_-} \Omega'_j) \cup (\cup_{(i,j) \in \mathcal{I}_-} (K'_j \cup \gamma'_j))$. Consider $\lambda^- > 0$ such that

\begin{equation}
\text{dist} \left( (\lambda^-,0,0) + F^- (\cup_{(i,j) \in \mathcal{I}_-} K'_i), \Pi_\delta(2\theta) \right) > 1.
\end{equation}

Reasoning as in Claim 4.3, there exist smooth data $\tilde{Y} = (\tilde{\phi}_k)_{k=1,2,3} \in \Omega_0^3(S_-)$ satisfying that:

\begin{itemize}
  \item $\tilde{\phi}_3 = \phi_3$ on $S_- - (\cup_{(i,j) \in \mathcal{I}_-} \gamma'_j)$, $k = 1,2$, and $\tilde{\varphi}_3 = (\varphi_3)_{|S_-}$.
  \item \begin{align*}
  \sum_{k=1}^3 (4\tilde{\phi}_k)^2 = 0, \text{ and } \sum_{k=1}^3 (\tilde{\phi}_k)^2 \text{ never vanishes on } S_-.
  \end{align*}
  \item \begin{align*}
  \text{Re} \left( \frac{1}{2} \int_{S'_i} \tilde{\varphi}_i \right) = \text{Re}(\frac{1}{2} \int_{S'_i} \varphi_i) + \lambda^-,
  \end{align*}
  where as above we integrate along $\gamma'_j$ (i, j) $\in \mathcal{I}_-$.
\end{itemize}

Again by Corollary 2.10, there exist \{$(\varphi_{1,n}, \varphi_{2,n})$\}_{n \in \mathbb{N}} $\subset \Omega^2_0(V)$ converging to $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ uniformly on $\overline{S_-}$ and satisfying that:

\begin{itemize}
  \item \begin{align*}
  \sum_{k=1}^3 (\varphi_{k,n})^2 = 0, \text{ and } \sum_{k=1}^3 (\varphi_{k,n})^2 \text{ never vanishes on } V, \text{ where } \varphi_{3,n} = \varphi_3, \forall n \in \mathbb{N}.
  \end{align*}
  \item The immersion $G^-_{n} \in \mathcal{M}(V)$ with initial condition $G^-_{n}(P_0) = F^-(P_0)$ and Weierstrass data $\tilde{Y}_n = (\varphi_{k,n})_{k=1,2,3}$ is well defined and satisfies
\end{itemize}

\begin{equation}
\text{dist} \left( (\lambda^-,0,0) + F^- (\cup_{(i,j) \in \mathcal{I}_-} K'_i), \Pi_\delta(2\theta) \right) > 1.
\end{equation}

Arguing as above, we can find $n_1 \in \mathbb{N}$ such that the following properties hold:

\begin{itemize}
  \item (B1) $\|G_{n_1} - F^-\| < \varepsilon/3$ on $M$.
  \item (B2) $G^-_{n_1} \left( \cup_{(i,j) \in \mathcal{I}_-} \overline{\Omega'_i - K'_i} \right) \subset \Pi'_i(0)$ and $G^-_{n_1} \left( \cup_{(i,j) \in \mathcal{I}_-} (\beta'_i - K'_i) \right) \subset \Pi'_{i+1}(0)$. Use that $x_3 \circ G^-_{n_1} = x_3 \circ F^-$ on $V$ and (b1).
  \item (B3) $\text{dist} \left( G^-_{n_1} \left( \cup_{(i,j) \in \mathcal{I}_-} K'_i \right), \Pi_\delta(2\theta) \right) > 1$. Take into account (4.7).
  \item (B4) $G^-_{n_1} \left( \cup_{(i,j) \in \mathcal{I}_-} \overline{\Omega'_i - K'_i} \right) \subset \Pi'_i(2\theta)$. See (A2).
  \item (B5) $\text{dist} \left( G^-_{n_1} \left( \cup_{(i,j) \in \mathcal{I}_-} (\beta'_i - K'_i) \right), \Pi_\delta(2\theta) \right) > 1$. It is implied by (A2).
  \item (B6) $G^-_{n_1} \left( \cup_{(i,j) \in \mathcal{I}_-} K'_i \right) \subset \Pi'_{i+1}(0)$. It follows from (A3).
\end{itemize}

Summarizing, $Y := L^+ \circ G_{n_1} \in \mathcal{M}(V)$ satisfies the following properties:

\begin{itemize}
  \item $\|Y - X\| \leq \|G_{n_1} - F^-\| + \|F^- - Y\| + \|Y_0 - \tilde{X}\| < \varepsilon$ on $M$. Use (A1), (B1) and (4.3).
  \item $Y \left( \cup_{(i,j) \in \mathcal{I}_-} \overline{\Omega'_i - K'_i} \right) \subset \Pi'_i(\theta)$ and $Y \left( \cup_{(i,j) \in \mathcal{I}_-} \overline{\Omega'_i - K'_i} \right) \subset \Pi'_{i+1}(\theta)$. See (B2) and (B4).
  \item $\text{dist} \left( Y \left( \cup_{(i,j) \in \mathcal{I}_-} (\beta'_i - K'_i) \right), \Pi_\delta(\theta) \right) > 1$ and $\text{dist} \left( Y \left( \cup_{(i,j) \in \mathcal{I}_-} (\beta'_i - K'_i) \right), \Pi_\delta(\theta) \right) > 1$. See (B2) and (B5).
  \item $\text{dist} \left( Y \left( \cup_{(i,j) \in \mathcal{I}_-} K'_i \right), \Pi_\delta(\theta) \right) > 1$ and $\text{dist} \left( Y \left( \cup_{(i,j) \in \mathcal{I}_-} K'_i \right), \Pi_\delta(\theta) \right) > 1$. Use (B3) and (B6).
\end{itemize}
Let $p : H_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^3$ and $\theta$ be a group morphism and a real number in $(0, \pi/4)$, respectively. Let $M \subset \mathcal{N}$ be a compact region of $\mathcal{N}$ such that $\mathcal{N} - M$ has no bounded components, and consider a non flat $Y \in \mathcal{M}(M)$ satisfying that $p_Y = p|_{H_1(M, \mathbb{Z})}$ and $(x_3 + \tan(\theta)|x_1|) \circ Y > 1$.

Then for any $\epsilon > 0$ there exists a conformal minimal immersion $X : \mathcal{N} \to \mathbb{R}^3$ satisfying the following properties:

- $p_X = p$,
- $(x_3 + \tan(\theta)|x_1|) \circ X : \mathcal{N} \to \mathbb{R}$ is a positive proper function, and
- $\|X - Y\| < \epsilon$ on $M$.

**Proof.** Without loss of generality, we will assume that $\epsilon < 1$.

Let $\{M_n | n \in \mathbb{N}\}$ be an exhaustion of $\mathcal{N}$ by compact regions with analytical boundary satisfying that $M_1 = M, M_n \subset M_{n+1}$ and $\mathcal{N} - M_n$ has no bounded components in $\mathcal{N}$ for all $n \in \mathbb{N}$.

Label $Y_1 = Y$, and by Lemma 4.1 and an inductive process, construct a sequence $\{Y_n\}_{n \in \mathbb{N}}$ of minimal immersions satisfying that

(a) $Y_n \in \mathcal{M}(M_n)$ for all $n \in \mathbb{N}$,
(b) $\|Y_n - Y_{n-1}\| < \epsilon/2^n$ on $M_{n-1}$ for all $n \geq 2$,
(c) $p_{Y_n} = p|_{H_1(M_n, \mathbb{Z})}$ for all $n \in \mathbb{N}$, and
(d) $Y_n(\partial(M_n)) \subset \Pi_n^\theta(\theta) \cup \Pi_n^\theta(-\theta)$ and $Y_n(M_n - M_{n-1}) \subset \Pi_n^{\theta-1}(\theta) \cup \Pi_n^{\theta-1}(-\theta)$ for all $n \geq 2$.

By items (a) and (b) and Harnack’s theorem, $\{Y_n\}_{n \in \mathbb{N}}$ uniformly converges on compact subsets of $\mathcal{N}$ to a conformal minimal (possibly branched) immersion $X : \mathcal{N} \to \mathbb{R}^3$. Furthermore, $\|X - Y_n\| \leq \epsilon/2^n$ on $M_{n+1}$ for all $n$. In particular from (d) we have that $(x_3 + \tan(\theta)|x_1|) \circ X \geq n - 1 - \epsilon/2^{n-1}$ on $M_n - M_{n-1}$, for all $n \geq 2$. On $M_1$ we also have that $(x_3 + \tan(\theta)|x_1|) \circ X \geq 1 - \epsilon > 0$, and so $(x_3 + \tan(\theta)|x_1|) \circ X$ is a positive proper function on $\mathcal{N}$.

Since the Weierstrass data of $Y_n$ converge in the $\omega$-topology to the ones of $X$, Hurwitz’s theorem gives that either $X(\mathcal{N})$ is a point or $X$ has no branch points. However, the condition $\|X - Y_1\| < \epsilon$ implies that $X(\mathcal{N})$ can not be a point provided that $\epsilon$ is small enough.

Finally, item (c) gives that $p_X = p$ and we are done. \qed
5. Proper Minimal Surfaces in Regions with Sublinear Boundary

The main goal of this section is to prove the existence of proper hyperbolic minimal surfaces with non-empty boundary in $\mathbb{R}^3$ contained in the region above a negative sublinear graph.

Throughout this section $\mathcal{N}$ will be the complex plane $\mathbb{C}$.

**Theorem 5.1.** Let $C$ denote the set $[-1, 1] \times (0, 1] \subset \mathbb{R}^2 \equiv \mathbb{C}$ endowed with the conformal structure induced by $\mathbb{C}$.\(^1\)

Then there exists $X \in \mathcal{M}(C)$ satisfying that:

- $(x_1, x_3) \circ X : C \to \mathbb{R}^2$ is proper.
- $x_3 \circ X > 0$ on $(x_1 \circ X)^{-1}((-\infty, 0])$, and $\lim_{n \to \infty} f(p_n) = 0$ for any divergent sequence $\{p_n\}_{n \in \mathbb{N}}$ in $C$, where $f : C \to (-\infty, 0]$ is the function $f = \min\{\frac{1}{|x_1|}, 0\}$.

**Proof.** Let $D_n$ denote the rectangle $[-2, 2] \times [\frac{1}{n+1}, 2] \subset \mathbb{R}^2 \equiv \mathbb{C}$, $n \in \mathbb{N}$. Label also $D = [-2, 2] \times [0, 2]$.

**Lemma 5.2.** For any $\epsilon \in (0, 1)$ there exists a sequence of non-flat $X_k \in \mathcal{M}(D), k \in \mathbb{N}$, such that

- $(i)$ $\|(X_k - X_{k-1})\| < \epsilon/2^{k-1}$ on $D_{k-1}$ for all $k \geq 2$.
- $(ii)$ $X_k([-2, 2] \times \{\frac{1}{n+1}\}) \subset \Pi_k^*(\epsilon)$.
- $(iii)$ $X_k(D - D_{k-1}) \subset \Pi_k^*(\frac{1}{n+1}) \cup \Pi_k^*(\frac{1}{n+1})$ for all $k \geq 2$.
- $(iv)$ If $p \in D_k$ and $(x_1 \circ X_k)(p) < 0$, then $(x_3 \circ X_k)(p) > 1 - \frac{\epsilon}{\frac{1}{n+1} + 2} > 0$.

**Proof.** We are going to construct the sequence inductively. Take any non-flat $X_1 \in \mathcal{M}(D)$ satisfying $X_1(D_k) \subset \Pi_k^*(1)$. Notice that $Y_1$ fulfills (ii) and (iv). Assume there exists a non-flat immersion $X_{n-1}$ satisfying (i), (ii), (iii) and (iv), and let us construct $X_n$.

Label $L : \mathbb{R}^3 \to \mathbb{R}^3$ as the rotation of angle $\frac{1}{n-1}$ around the straight line parallel to the $x_2$-axis and containing the point $(0, 0, n-1)$. Notice that $L(\Pi_{n-1}(\frac{1}{n-1})) = \Pi_{n-1}(0)$ and $L(\Pi_n(\frac{1}{n})) = \Pi_\zeta(-\frac{1}{n(n-1)})$, where $\zeta = n - 1 + \cos(1/n)\sec(1/(n^2 - n))$.

Call $Y = L \circ X_{n-1} \in \mathcal{M}(D)$. From (ii), we have

$$Y([-2, 2] \times \{1/n\}) \subset \Pi_{n-1}^*(0).$$

By continuity and equation (5.1), there exists $\mu \in (\frac{1}{n+1}, \frac{1}{n})$, close enough to $1/n$ so that

$$Y(\Theta) \subset \Pi_{n-1}^*(0), \quad \text{where } \Theta := [-2, 2] \times [\mu, 1/n].$$

Denote $\Delta := [-2, 2] \times [0, \mu]$. Notice that $D_n = D_{n-1} \subset \Delta \cup \Theta$, and $\emptyset = D_{n-1} \cap \Delta = D_n^o \cap \Theta^o = \Theta^o \cap \Delta^c$ (see figure 2).

Fix $\lambda > 0$ such that

$$(-\lambda, 0, 0) + Y(\Delta) \subset \Pi_k^*(-\frac{1}{n(n-1)}).$$

Denote $\Phi = (\phi_k)_{k=1,2,3}$ as the Weierstrass data of $Y$ on $D$. Consider $\gamma$ an analytic Jordan arc on $\Theta$ with endpoints $P \in \partial(D_{n-1})$ and $Q \in \partial(\Delta)$ and otherwise disjoint from $\partial(\Theta)$, and meeting transversally $D_{n-1}$ and $\Delta$. Moreover, we choose $\gamma$ so that $\phi_3$ never vanishes on $\gamma$. Label $\Lambda$ as the admissible subset $D_{n-1} \cup \gamma \cup \Lambda$ in $\mathbb{C}$ and consider $\Phi = (\phi_k)_{k=1,2,3} \in \Omega_0^*(\Lambda)^3$ satisfying

\(^1\)By Carathéodory’s Theorem, $C$ is biholomorphic to the half disc $\overline{D}_+$. 
The existence of \( \hat{\Phi} \) follows by similar arguments to those used in Claim 4.3.

Let \( W \subset \mathbb{C} \) be an open topological disc containing \( D \), and without loss of generality, suppose that \( \Phi_3 \) extends holomorphically to \( W \). We can apply Corollary 2.10 to the data \( W, S = \Lambda \) and \( \Phi \in \Omega_0^3(\Lambda)^3 \), to obtain a sequence \( \{ \Phi_m = (\Phi_{k,m})_{k=1,2,3} \}_{m \in \mathbb{N}} \subset \Omega_0(D)^3 \) converging to \( \Phi \) uniformly on \( \Lambda \) and such that \( \hat{\Phi}_{3,m} = \Phi_3, \sum_{k=1}^3 \partial_{\bar{z}}^2 \Phi_{k,m} = 0 \) and \( \sum_{k=1}^3 |\hat{\Phi}_{k,m}|^2 \) never vanishes, for all \( m \in \mathbb{N} \). Fix \( P_0 \in D_{n-1} \), and consider the minimal immersion \( Y_m \in \mathcal{M}(D) \) with initial condition \( Y_m(P_0) = Y(P_0) \) and Weierstrass data \( \Phi_m \). Notice that \( \{ Y_m \}_{m \in \mathbb{N}} \) uniformly converges to \( Y \) on \( D_{n-1} \), \( \{ x_1 \circ Y_m \}_{m \in \mathbb{N}} \) uniformly converges to \( x_1 \circ Y - \lambda \) on \( \Lambda \), and \( x_3 \circ Y_m = x_3 \circ Y, \forall m \in \mathbb{N} \). As a consequence we can find \( m_0 \in \mathbb{N} \) such that:

- \( \|Y_{m_0} - Y\| < \varepsilon / 2^n - 1 \) on \( D_{n-1} \).
- \( Y_{m_0}(\Theta) \subset \Pi_n^{*}(0) \) (take into account (5.2)).
- \( Y_{m_0}(\Delta) \subset \Pi_n^* \left( \varepsilon \frac{1}{n(n-1)} \right) \). Use (5.3).

Define \( X_n := L^{-1} \circ Y_{m_0} \in \mathcal{M}(D) \). From the above properties we get

(a) \( \|X_n - X_{n-1}\| < \varepsilon / 2^n - 1 \) on \( D_{n-1} \).
(b) \( X_n(\Theta) \subset \Pi_n^{*}(\varepsilon \frac{1}{n-1}) \).
(c) \( X_n(\Delta) \subset \Pi_n^* (\varepsilon \frac{1}{n}) \).

Property (a) directly gives (i). Since \([-2, 2] \times \{ \pm 1 \} \subset \Delta \), (c) implies (ii). Taking into account that \( D_n - D_{n-1} \subset \Theta \cup \Delta \), (iii) follows from (b) and (c). Finally, property (iv) is an elementary consequence of (a), (b) and (c). \( \square \)

From (i) and Harnack’s theorem, the sequence \( \{ X_n \}_{n \in \mathbb{N}} \) uniformly converges on compact sets of \((-2, 2) \times (0, 2)\) to a conformal minimal (possibly branched) immersion \( \hat{X} : (-2, 2) \times (0, 2) \to \mathbb{R}^3 \). Since the Weierstrass data of \( X_n \) uniformly converge on compact sets of \((-2, 2) \times (0, 2)\) to the ones of \( \hat{X} \), Hurwitz’s theorem assures that either \( X \) has no branch points or \( X((-2, 2) \times (0, 2)) \) degenerates in a point. However, \( \|X - X_1\| < \varepsilon \) on \( D_1 \), and so \( \hat{X} \) does not degenerate provided that \( \varepsilon \) is small enough. Hence \( \hat{X} \) is not branched and \( X := \hat{X}|_{C} \in \mathcal{M}(\mathbb{C}) \). Denote by \( C_n = \)
which converges to 0 as $\epsilon \to 0$. Therefore (iv) gives that $x_3 + \tan(1)|x_1| \circ X \geq n - 1 - 2\epsilon$ on $C_n - C_{n-1}$, for all $n \geq 2$. Therefore, $(x_1, x_3) \circ X : C \to \mathbb{R}^2$ is proper.

Consider $P \in C$ such that $(x_1 \circ X)(P) < 0$. For $n$ large enough, $P \in C_n \subset D_n$ and $(x_1 \circ X_n)(P) < 0$ as well. Therefore (iv) gives $(x_3 \circ X)(P) \geq 1 - \epsilon > 0$ and $f(P) = 0$. Finally, consider a divergent sequence $\{P_n\}_{n \in \mathbb{N}}$ in $C$. For any $n \in \mathbb{N}$ we label $k(n) \in \mathbb{N}$ as the natural number such that $P_n \in C_{k(n)} - C_{k(n)-1}$ and note that $\{k(n)\}_{n \in \mathbb{N}}$ is divergent. From (i) and (iii),

$$0 \geq f(X(P_n)) \geq \min\left\{ \frac{k(n) - 1 - 2\epsilon}{|x_1(X(P_n))| + 1} - \tan\left(\frac{1}{k(n) - 1}\right), 0 \right\} \geq -\tan\left(\frac{1}{k(n) - 1}\right),$$

which converges to 0 as $n$ goes to $\infty$.

The proof is done. \qed

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