In the last article of this series (see [10]) we will first explain how Artin’s reciprocity law for unramified abelian extensions can be formulated with the help of power residue symbols, and then show that, in this case, Artin’s reciprocity law was already stated by Bernstein [3] in the case where the base field contains the roots of unity necessary for realizing the Hilbert class field as a Kummer extension. Bernstein’s article appeared in 1904, almost 20 years before Artin conjectured his version of the reciprocity law, and seems to have been overlooked completely.

Let me also remark that although we will present Bernstein’s reciprocity law only for unramified extensions (Takagi created the general class field theory dealing with ramified abelian extensions long after Bernstein’s work), the generalization to arbitrary abelian extensions of number fields is straightforward.

With hindsight, the basic idea is this: the Artin isomorphism is a decomposition law for abelian extensions $K/k$. By adjoining suitable roots of unity to $k$, the extension $K/k$ will lift to a Kummer extension. In Kummer extensions, the decomposition of prime ideals is governed by power residue symbols. We may therefore harbor some hope of being able to express the content of Artin’s reciprocity law using power residue symbols. Such a description is easy to give in the case when the base field contains the roots of unity that are necessary for writing the Hilbert class field $K$ as a Kummer extension.

Bernstein’s reciprocity law apparently did not play any role at all in the development of Artin’s reciprocity law because it has not been noticed at all. In Sect. 12 we will give (a corrected version of) Bernstein’s reciprocity law, and in Sect. 13 we will show that this reciprocity law contains several classical observations on the quadratic and cubic power residue characters of quadratic units.

### 12. Bernstein’s Reciprocity Law

Let us now see how to formulate Artin’s reciprocity law with the help of power residue symbols. Before we do so we have to recall a few basic properties of singular numbers and power residue symbols.

**Power Residue Symbols.** Let $h \geq 2$ be an integer, and let $k$ be a number field containing a primitive $h$-th root of unity $\zeta = \zeta_h$. For $\alpha \in k^\times$ and prime ideals $p \nmid h\alpha$ we define the $h$-th power residue symbol $(\alpha/p)_h$ by demanding that its values are $h$-th roots of unity satisfying the congruence

$$
\left( \frac{\alpha}{p} \right)_h \equiv \alpha^{(Np-1)/h} \mod p.
$$

Observe that $Np \equiv 1 \mod h$ since $\mathbb{Q}(\zeta) \subseteq k$. 

An element $\alpha \in k^\times$ is called singular if $(\alpha) = a^h$ is an $h$-th ideal power of some fractional ideal $a$. It is called primitive if $k(\sqrt[h]{\alpha})/k$ is unramified at all primes dividing $h$. If $\alpha$ is singular and primitive, then $k(\sqrt[h]{\alpha})/k$ is unramified at all finite primes (if $h > 2$, the extension is automatically unramified at the infinite primes because $k$, as an extension of $\mathbb{Q}(\zeta)$, is totally complex).

The main observation we will need is the following classical result:

**Lemma 12.1.** If $\alpha \in k^\times$ is singular and primitive, then the power residue symbol $(\alpha/p)_{h}$ is well defined for all prime ideals $p \mid h$.

In fact, fix a prime ideal $p \mid h$. If $p \mid (\alpha)$, then write $(\alpha) = a^h$. Choose an ideal $b$ in the ideal class $[a]$ generated by $a$ such that $b$ is coprime to $p$. Then $a = \gamma b$. Then $\alpha = \gamma^h \beta$ for some $\beta \in k^\times$ with $(\beta) = b^h$. Clearly $(\alpha/q)_{h} = (\beta/q)_{h}$ for all prime ideals $q \mid \alpha \beta$, and $(\beta/p)_{h}$ is defined since $p \nmid \beta$ by construction.

**Bernstein’s Reciprocity Law.** Now we will formulate Bernstein’s version of the reciprocity law. We will distinguish two cases.

**Case I.** $\zeta_{h} \in k$.

Consider an algebraic number field $k$ whose class group is cyclic of order $h$, and assume that $k$ contains a primitive $h$-th root of unity $\zeta$. Then the Hilbert class field of $K$ has the form $K = k(\sqrt[h]{\omega})$ for some $\omega \in k^\times$.

**Theorem 12.2.** Let $k$ be a number field with cyclic class group $\text{Cl}(k) = \langle c \rangle$ of order $h$, and assume that $k$ contains the $h$-th roots of unity $\mu_{h} = \langle \zeta \rangle$. Then we can choose $\omega \in k^\times$ in such a way that $K = k(\sqrt[h]{\omega})$ is the Hilbert class field of $k$, and that

$$
\left(\frac{\omega}{p}\right) = \zeta^{e} \iff [p] = c^{e}
$$

(12.1)

for all prime ideals $p \mid h$, where $(\frac{\omega}{p})$ denotes the $h$-th power residue symbol in $k$.

**Proof.** Let $K = k(\sqrt[h]{\omega})$ be the Hilbert class field of $k$. Let $p$ denote a prime ideal in $k$, and let $\sigma$ to $\alpha = \sqrt[h]{\omega}$ we find $\alpha^{\sigma} \equiv \alpha^{Np} = \alpha^{Np-1} \alpha = \omega^{(Np-1)/h} \alpha \equiv (\frac{\alpha}{\beta}) \alpha \mod p$. Artin’s reciprocity law induces an exact sequence

$$
1 \rightarrow P_{k} \rightarrow I_{k} \rightarrow \text{Gal}(K/k) \rightarrow 1,
$$

where the map $I_{k} \rightarrow \text{Gal}(K/k)$ is induced by sending a prime ideal $p$ to its Frobenius automorphism. Since the Artin map only depends on the ideal class of $p$ we get an isomorphism $\text{Cl}(k) \rightarrow \text{Gal}(K/k)$.

Let $q$ be a prime ideal in the ideal class $c$; then $(\frac{\omega}{q}) = \zeta^{a}$ for some $a$ coprime to $h$. Write $ab \equiv 1 \mod h$; replacing $\omega$ by $\omega^{b}$ we find that $K = k(\sqrt[h]{\omega})$ and $(\frac{\omega}{q}) = \zeta$. The multiplicativity of the Artin map then guarantees that $(\frac{\omega}{p}) \equiv \zeta^{e}$ if and only if $[p] \in c^{e}$.

Clearly, Thm. 12.2 is equivalent to Artin’s reciprocity law in unramified abelian extensions of number fields $k$ with cyclic class group of order $h$ and $\zeta_{h} \in k$: if (12.1) holds, then the map $p \rightarrow (\frac{\omega}{p})$ induces a canonical isomorphism $\rho : \text{Cl}(k) \rightarrow \mu_{h}$ between the ideal class group of $k$ and the group $\mu_{h}$ of $h$-th roots of unity. Similarly, mapping $\sigma \in \text{Gal}(K/k)$ to the root of unity $\chi(\sigma) = \sqrt[h]{\omega^{-1}}$ defines a canonical isomorphism $\chi : \text{Gal}(K/k) \rightarrow \mu_{h}$. Composing $\rho$ with $\chi^{-1}$ then provides us with
a canonical isomorphism $\rho$ between the class group $\text{Cl}(k)$ and the Galois group $\text{Gal}(K/k)$ of the Hilbert class field $K$ of $k$.

The restriction of Thm. 12.2 to subextensions of the Hilbert class field can be proved similarly:

**Theorem 12.3.** Let $\ell$ be a prime number and $k$ a number field whose $\ell$-class group is cyclic of order $m = \ell^n$. Assume that $k$ contains the $m$-th roots of unity, and let $K/k$ be the cyclic unramified extension of degree $m$.

Let $c$ be an ideal class with order $m$ in $\text{Cl}(k)$. Then we can choose $\omega \in k^\times$ in such a way that $K = k(\sqrt[m]{\omega})$ and

$$\left(\frac{\omega}{p}\right) = \zeta^e \iff \left[p\right]^{h/m} = c^e \quad (12.2)$$

for all prime ideals $p \nmid h$, where $(\cdot)_{p}$ denotes the $m$-th power residue symbol in $k$.

The proof proceeds exactly as above.

**Case II.** $\zeta_h \notin k$.

If $k$ does not contain the roots of unity necessary for defining the $h$-th power residue symbol, the situation is slightly more involved. In this case, assume that $\text{Cl}(k)$ is cyclic of order $h$ and set $k' = k(\zeta)$ for some primitive $h$-th root of unity. The Hilbert class field $k^1/k$ becomes a Kummer extension over $k'$, and we can write $k^1k^3 = k'(\sqrt[m]{\omega})$ for a suitable $\omega \in k'$. Observe that the degree $(k'(\sqrt[m]{\omega}) : k)$ divides the class number $h$; if a subextension of $k^1/k$ is unramified, this degree will be strictly smaller than $h$.

In fact, set $\nu = (k^1 \cap k' : k)$; class field theory predicts that the group $N_{k'/k}\text{Cl}(k')$ has index $\nu$ in the class group $\text{Cl}(k)$. We will see below that the description via power residue symbols will allow us to characterize only those ideal classes that belong to this subgroup $N_{k'/k}\text{Cl}(k')$ of index $\nu$ in $\text{Cl}(k)$.

Recall that $k^1/k$ is a class field with conductor 1 for the ideal group $P_k$ of principal ideals in $k$. By the Translation Theorem of class field theory, the extension $k^1k'/k'$ is also a class field with conductor 1 for the ideal group $H_{k'} = \{a' \in I_{k'} : N_{k'/k}a' \in P_k\}$ consisting of all ideals whose norms down to $k$ are principal.

For formulating (a piece of) Artin’s reciprocity law using power residue symbols we use the well known transfer formula:

**Lemma 12.4.** Let $L/K$ be a finite abelian extension of number fields. If $F/K$ is a finite extension, then $FL/F$ is abelian, and we have, for all prime ideals $p$ unramified in $F$,

$$\left(\frac{FL/F}{q}\right)_{L} = \left(\frac{L/K}{N_{F/K}q}\right) = \left(\frac{L/K}{p}\right)^f,$$

where $q$ is a prime ideal in $F$ above $p$, and where $f = f(q|p)$ is the inertia degree of $q$. If, in particular, $F/K$ is a subextension of $L/K$, then

$$\left(\frac{L/F}{q}\right) = \left(\frac{L/K}{N_{F/K}q}\right) = \left(\frac{L/K}{p}\right)^f.$$
In our case, (12.3) says that

\[
\left(\frac{k^{1}k' / k'}{a}\right)_{k} = \left(\frac{k^{1}/k}{N_{k'/k}a}\right).
\]

Let \(c\) be an ideal class generating the cyclic group \(Cl(k)\); then \(c^e\) is a norm of some class \(C \in Cl(k')\), and \(c^e\) generates the subgroup \(N_{k'/k}Cl(k')\) of \(Cl(k)\). Let \(p'\) be a prime ideal in \(C\); as in Case I we can choose \(\omega \in k'\) in such a way that \(\left(\frac{\omega}{p'}\right) = \zeta^e\), where \(\zeta\) is a primitive \(h\)-th root of unity. Thus we get

**Theorem 12.5 (Bernstein’s Reciprocity Law).** Let \(k\) be a number field whose class group \(Cl(k) = \langle \zeta \rangle\) is cyclic of order \(h\), and let \(k' = k(\zeta)\), where \(\zeta\) is a primitive \(h\)-th root of unity. Then we can choose \(\omega \in k'\) in such a way that the Hilbert class field \(K\) of \(k\) lifts to the Kummer extension \(Kk' = k'(\sqrt[\zeta]{\omega})\), and that for prime ideals \(p'\) in \(k'\) we have

\[
\left(\frac{\omega}{p'}\right) = \zeta^e \iff N_{k'/k}p' \in c^e
\]

for all prime ideals \(p' \nmid h\), where \(\left(\frac{\cdot}{\cdot}\right)\) denotes the \(h\)-th power residue symbol in \(k'\).

The extension of Bernstein’s reciprocity law to number fields whose class groups are not necessarily cyclic is purely formal:

**Theorem 12.6 (Bernstein’s Reciprocity Law).** Let \(k\) be a number field whose class group \(Cl(k) = \langle c_1, \ldots, c_q \rangle\) is the direct sum of groups \(C_j = \langle c_j \rangle\) of prime power order \(h_j\) (then the class number \(h\) of \(k\) is given by \(h = h_1 \cdots h_q\)). Let \(\zeta_j\) denote a primitive \(h_j\)-th root of unity, and set \(k' = k(\zeta_1, \ldots, \zeta_q)\).

Then there exist elements \(\omega_1, \ldots, \omega_q \in k'\) such that \(k'(\sqrt[\zeta_1]{\omega_1}, \ldots, \sqrt[\zeta_q]{\omega_q})\) is the compositum of \(k'\) and the Hilbert class field of \(k\). Moreover, the \(\omega_j\) can be chosen in such a way that

\[
\left(\frac{\omega_j}{p'}\right) = \zeta_j^{e_j} \text{ for } j = 1, \ldots, q \text{ if and only if } [N_{k'/k}p'] = c_1^{e_1} \cdots c_q^{e_q}
\]

for all prime ideals \(p' \nmid h\), where \(\left(\frac{\cdot}{\cdot}\right)\) denotes the \(h\)-th power residue symbol in \(k'\).

The connection between explicit reciprocity laws for power residue symbols and Artin’s reciprocity law was used by Artin himself to prove special cases of his reciprocity law in [1]. Artin was able to prove his reciprocity law for general abelian extensions only after Chebotarev provided the key idea in his proof what became known as Chebotarev’s density theorem.

Bernstein’s formulation of his reciprocity law in [3] is only correct for number fields containing the appropriate roots of unity:

**Bernstein’s Formulation.** Let \(k\) be a number field with class number \(h\). Decompose the class group \(Cl(k) = C_1 \oplus \cdots \oplus C_q\) into groups \(C_j\) of prime power order \(h_j\) (then \(h = h_1 \cdots h_q\)), and pick generators \(c_j\) of the \(C_j\). Let \(\zeta_j\) be a primitive \(h_j\)-th root of unity, let \(k' = k(\zeta_1, \ldots, \zeta_q)\), and put \(r = (k' : k)\).

Then there exist elements \(\omega_1, \ldots, \omega_q \in k'\) such that \(k'(\sqrt[\zeta_1]{\omega_1}, \ldots, \sqrt[\zeta_q]{\omega_q})\) is the compositum of \(k'\) and the Hilbert class field of \(k\). Moreover, the \(\omega_j\) can be chosen in such a way that the relations

\[
\left(\frac{\omega_j^{(j)}}{p'}\right) = \zeta_1^{e_j}, \ldots, \left(\frac{\omega_j^{(j)}}{p'}\right) = \zeta_q^{e_j}
\]

\(^1\)Observe that \(\omega\) is a \(\nu\)-th power, so we only get \(h/\nu\)-th roots of unity on both sides.
are the necessary and sufficient conditions for

\[ [N_{k'/k}'] = e_1 \cdots e_q, \]

where

\[ e_1 = \sum_{\rho=1}^r e_{1,\rho}, \ldots, e_q = \sum_{\rho=1}^r e_{q,\rho}. \]

12.1. Power Residue Symbols. It is clear that the right hand side in the correspondence (12.4) does not change if we replace \( p' \) by one of its conjugates. Let us therefore convince ourselves that the left hand side is also invariant.

To this end, assume that \((\omega/p') = \zeta^e\) and let \( \sigma \) denote an automorphism of \( K_{k'/k} \). Then \( k'(\sqrt[p']{\omega}) = k'(\sqrt[p']{a}) \) implies that \( \omega^\sigma = \alpha^h \omega^a \) for \( \alpha \in k' \) and some exponent \( a \) coprime to \( h \). Since \( k'/k \) is abelian, it follows from Kummer theory (see e.g. [7, Satz 147] or [11, Lemma 14.7]) that we also must have \( \zeta^\sigma = \zeta^a \) for the same value of \( a \).

The basic formalism of power residue symbols (see [9, Ch. 4]) now shows

\[ (\omega/p')^\sigma = (\omega^\sigma/p^\sigma) = (\omega/a^\sigma)^a = (\omega/p)^{\sigma}, \]

hence

\[ (\omega/p^\sigma) = (\omega/p^\sigma)^\sigma \]

(12.5) as expected.

Now let \( \tau = \sigma^{-1} \); then

\[ (\omega/p')^\sigma = (\omega/p^\tau)^\sigma = (\omega^\tau). \]

This implies that if \((\omega/p') = \zeta^e\), then

\[ \prod_{\sigma \in \text{Gal}(K_{k'/k})} (\omega/a^\sigma)^\sigma = \prod_{\sigma \in \text{Gal}(k'/k)} (\omega/p^\sigma)^\sigma = [N_{k'/k}] \zeta^e = 1. \]

This means that in the special case where \( \text{Cl}(k) \) is cyclic or prime order \( p \) and where \( k \) does not contain the \( p \)-th roots of unity, the product of the power residue symbols in Bernstein’s formulation is always trivial, that is, we have \( e_1 = 0 \). This shows that Bernstein’s formulation of his reciprocity law is incorrect in the case where roots of unity have to be adjoined.

13. Power Residue Characters of Quadratic Units

The corrected formulation of Bernstein’s reciprocity law is much more than a twisted version of Artin’s reciprocity law; in fact, Bernstein’s reciprocity law contains many results on the power residue characters of quadratic units obtained by Dirichlet, Kronecker, Scholz, Aigner and a host of other mathematicians in the wake of Emma Lehmer’s work on these topics in the 1970s.

In this section we will be content with giving a few examples that show how to apply Bernstein’s reciprocity law to such problems. This approach can be found in various articles on the power residue characters of units, for example in Halter-Koch [6].

Before we begin let us recall the relevant notation. Let \( k \) be a number field containing the \( m \)-th roots of unity, and let \( \varepsilon \) be a unit in the ring of integers \( \mathcal{O}_k \).

Define the \( m \)-th power residue symbol \((\alpha/p)_m\) for prime ideals \( p \) in \( \mathcal{O}_k \) coprime to
ma by \((\alpha/p)_m = \zeta_m^r\) if \(\alpha^{(Np-1)/m} \equiv \zeta_m^r \mod p\) (observe that \(Np \equiv 1 \mod m\) since \(k\) contains the \(m\)-th roots of unity). If the symbol \((\varepsilon/p)_m\) does not depend on the choice of the prime ideal \(p\) above \(p\), then we set \((\varepsilon/p)_m = (\varepsilon/p)_m\).

13.1. **Dirichlet.** The following example due to Dirichlet shows that Bernstein’s reciprocity law is weaker than Artin’s if \(\nu \neq 1\).

Let \(p \equiv 1 \mod 8\) be prime, write \(p = a^2 + 4b^2\) and let \(\varepsilon\) denote the fundamental unit of \(Q(\sqrt{p})\). The 2-class group of \(K = Q(\sqrt{-p})\) is cyclic of order divisible by 4. The unique cyclic quartic unramified extension \(K/k\) contains the quadratic subextension \(k' = k(i)\); this implies \(K' = K\), hence the extension \(K'/k'\) is quadratic, and we have \(\nu = 2\) in this case.

We will see below that \(K = k(\sqrt{\varepsilon})\), and that \(K/k\) is a cyclic quartic extension. The extension \(L/Q'\) with \(Q' = Q(i)\), on the other hand, is a biquadratic extension since \(L = Q'(\sqrt{a + 2bi}, \sqrt{a - 2bi})\).

Applying Bernstein’s reciprocity law (more exactly the special case Thm. [12.3]) to this situation we find

\[
\left(\frac{\varepsilon}{q}\right) = \begin{cases} +1 & \text{if } q^{\frac{1}{4}} = x^2 + py^2, \\ -1 & \text{if } q^{\frac{1}{4}} = 2x^2 + 2xy + \frac{p+1}{2}y^2. \end{cases}
\]

**Proof.** Recall (see [3] or [9, Ex. 5.10]) that \((a + 2bi)\varepsilon\) is a square in \(Q(i, \sqrt{p})\); this shows that \(K = k'(\sqrt{\varepsilon}) = k'(\sqrt{a + 2bi})\). Since \(b\) is even, the extension \(K/k'\) is unramified at the primes above 2, and the fact that \(k'(\sqrt{a + 2bi}) = k'(\sqrt{a - 2bi})\) shows that primes above \(p\) cannot ramify either. Thus \(K'/k\) is unramified everywhere.

The 2-class group \(Cl_2(k)\) is cyclic by Gauss’s genus theory. The unique element of order 2 in the ideal class group of \(k\) is represented by the ideal \(2 = (2, 1 + \sqrt{-p})\). If \(q\) is a prime ideal with odd prime norm \(q\) in the class \([2]\), then \(2p\) is principal, and taking norms shows that \(2q = w^2 + py^2\). Since \(q\) and \(y\) must be odd, we can set \(w = 2x + y\) and \(y\) and find, after cancelling the common factor 2 from both sides, that \(q = 2x^2 + 2xy + \frac{p+1}{4}y^2\).

For applying Bernstein’s reciprocity law observe that \(K = k'(\sqrt{\varepsilon^2})\), and that the fourth power residue symbols \((\varepsilon^2/p)_4\) are just quadratic residue symbols \((\varepsilon/p)\). \(\square\)

**Example.** Consider the number field \(k = Q(\sqrt{17})\). Its class group \(Cl(k)\) is cyclic of order 4 and is generated by the class of the prime ideal \(q = (3, 1 + \sqrt{17}); \). Observe that \(\varepsilon = 4 + \sqrt{17} = \eta^2\) for \(\eta = \frac{\sqrt{17 + 4\sqrt{17}}}{1 + \sqrt{17}}\).

Scholz’s reciprocity law (see [9, Ch. 5]) shows that \((\varepsilon/q)_4 = (q/17)_4(17/q)_4\) for primes \(q \equiv 1 \mod 4\) with \((\frac{17}{q}) = 1\), hence \((\varepsilon/q)_4 = +1\) if \(q = x^2 + 17y^2\), and
\((\varepsilon/q) = -1\) if \(q = 2x^2 + 2xy + 9y^2\). This result is due to Dirichlet \([5, \S 4]\) and was rediscovered by Brandler (see \([4]\) and \([9, \text{Ch. 5}]\)).

13.2. **Kronecker.** Let \(a\) be an odd positive integer, and assume that \(m = 3a^2 \pm 4\) is squarefree. Then

\[
\varepsilon = \frac{1}{3} \left( \frac{3a + \sqrt{3m}}{2} \right)^2 = \pm 1 + 3a + \frac{\sqrt{3m}}{2}
\]

is a unit in the ring of integers of \(k = \mathbb{Q}(\sqrt{3m})\). If \(a = 3b\) and \(m = 27b^2 \pm 4\), then any unit \(\varepsilon\) is a cube in \(k\) if and only if it is a cube in \(k\), which in turn is equivalent to \(3(3a + \sqrt{3m})\) being a cube in \(\mathcal{O}_k = \mathbb{Z}[1 + \sqrt{3m}/2]\). The equation \(9a + 3\sqrt{3m} = (1 + 9\sqrt{3}m)^3\) leads to

\[
9a = r(r^2 + 9rs^2m) \quad \text{and} \quad 3 = 3s(3r^2 + ms^2),
\]

which is easily seen to be impossible.

The cyclic unramified extension \(K'/k\) descends to \(F = \mathbb{Q}(\sqrt{-m})\) in the sense that the abelian extension \(K'/F\) contains a cyclic cubic unramified subextension \(L/F\). We can construct this extension explicitly by setting \(\theta = \sqrt[3]{\varepsilon} + 3\sqrt[3]{\varepsilon'}\); in fact we find

\[
\theta^3 = \varepsilon + \varepsilon' + 3\sqrt[3]{\varepsilon\varepsilon'}(\sqrt[3]{\varepsilon} + \sqrt[3]{\varepsilon'}) = 27b^2 + 2 + 3\theta,
\]

hence \(L\) is the compositum of \(F\) and the cubic extension generated by a root \(\theta\) of the polynomial \(x^3 - 3x - 27b^2 + 2\).

Here are a few small examples of odd values of \(b\) for which \(m = 27b^2 - 4\) is squarefree; in the table below, \(h\) denotes the class number of the quadratic number field \(\mathbb{Q}(\sqrt{-m})\).

| \(b\) | \(-m\) | \(h\) |
|------|------|------|
| 1    | -23  | 3    |
| 3    | -239 | 15   |
| 5    | -671 | 30   |
| 7    | -1319| 45   |
| 9    | -2183| 42   |

Here is a similar table for \(m = 27b^2 + 4\):

| \(b\) | \(-m\) | \(h\) |
|------|------|------|
| 1    | -31  | 3    |
| 3    | -247 | 6    |
| 5    | -679 | 18   |
| 7    | -1327| 15   |
| 9    | -2191| 30   |

Applying Bernstein’s reciprocity law to the two examples \(b = 1\) with class number 3 we find

**Proposition 13.2.** Let \(\varepsilon = \frac{25 + 3\sqrt{69}}{2}\) be the fundamental unit of \(k = \mathbb{Q}(\sqrt{69})\). Then

\[
\left(\frac{\varepsilon}{p}\right)_3 = 1 \iff p = x^2 + xy + 6y^2
\]

for primes \(p\) with \((\frac{-3}{p}) = (\frac{-23}{p}) = 1\).

Similarly, let \(\eta = \frac{29 + 3\sqrt{93}}{2}\) be the fundamental unit of \(k = \mathbb{Q}(\sqrt{93})\). Then

\[
\left(\frac{\varepsilon}{p}\right)_3 = 1 \iff p = x^2 + xy + 8y^2
\]

for primes \(p\) with \((\frac{-3}{p}) = (\frac{-31}{p}) = 1\).
The last example was essentially discovered by Kronecker, who considered in [8] the splitting field $L$ of the polynomial $f(x) = (x^3 - 10x)^2 + 31(x^2 - 1)^2$. This number field $L$ has degree 6 and Galois group $S_3$; it is the compositum of the complex quadratic number field $\mathbb{Q}(\sqrt{-31})$ and the cubic field with discriminant $-31$ generated by a root of $x^3 + 11x^2 + 38x + 31$. Write $\omega = \frac{-1 + \sqrt{-3}}{2}$ and $\overline{\omega} = \frac{-1 + \sqrt{-31}}{2}$; then $\eta = 1 - \omega + 3\omega = \frac{1}{2}(3\sqrt{-3} + \sqrt{-31})$ is a unit in $L' = L(\sqrt{-3})$, and $\eta^2 = \frac{1}{4}(-29 + \sqrt{93})$ actually shows that $\eta$ is the fundamental unit in $\mathbb{Q}(\sqrt{-3}, \sqrt{-31})$.

Now prime ideals $\mathfrak{p}$ in $k$ above primes $p$ with $(\frac{-31}{p}) = +1$ split completely in $\mathbb{Q}(\sqrt{-31})$ and the Hilbert class field of $k$ if and only if $\mathfrak{p}$ is principal, which is equivalent to $p$ being represented by the principal form $x^2 + xy + 8y^2$. Since $L' = k'(\sqrt{\varepsilon})$, Bernstein’s reciprocity law tells us that

$$\left(\frac{\varepsilon}{p}\right)_4 = 1 \iff p = x^2 + xy + 8y^2$$

for primes $p$ with $(\frac{-31}{p}) = (\frac{-3}{p}) = +1$.

A general result containing Kronecker’s example as a special case can be found e.g. in Weinberger [12].

13.3. Quartic Character of Certain Quadratic Units. Let $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{4}$ be primes such that $(\frac{p}{q}) = +1$. The 2-class group of the complex quadratic number field $k = \mathbb{Q}(\sqrt{-pq})$ is cyclic of order divisible by 4, hence $k$ admits a cyclic quartic unramified extension $K/k$. Over $K' = k'(\sqrt{\varepsilon})$, this extension can be realized as a Kummer extension:

**Proposition 13.3.** Let $\varepsilon$ denote the fundamental unit of $F = \mathbb{Q}(\sqrt{pq})$, where $p$ and $q$ are as above. Then $\varepsilon \equiv s \pmod{4}$ for $s \in \{\pm 1\}$, and $K' = k'(\sqrt{\varepsilon})$.

**Proof.** Write $\varepsilon = T + U\sqrt{pq}$; from $T^2 - pqU^2 = 1$ it follows that $T$ is odd and $U \equiv 0 \pmod{4}$, hence $\varepsilon \equiv \pm 1 \pmod{4}$. By a routine calculation one verifies that $k'(\sqrt{\varepsilon})/k$ is a cyclic quartic unramified extension, from which the claim follows since $k$ has a cyclic 2-class group. \hfill $\Box$

The 2-class group $\text{Cl}_2(k)$ of $k = \mathbb{Q}(\sqrt{-pq})$, where $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{4}$ are prime, is cyclic of order divisible by 4. The principal class of in the form class group with discriminant $-pq$ is represented by $Q_0(x,y) = x^2 + xy + my^2$ with $m = \frac{pq-1}{4}$, and the unique class of order 2 by $Q_1(x,y) = qx^2 + qxy + ny^2$ with $n = \frac{pq+3}{4}$. Bernstein’s reciprocity law applied to $k$ now immediately gives the following

**Proposition 13.4.** Let $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{4}$ be primes such that $(\frac{p}{q}) = +1$, and let $\varepsilon$ denote the fundamental unit of $F = \mathbb{Q}(\sqrt{pq})$. Choose $s \in \{\pm 1\}$ such that $s\varepsilon \equiv 1 \pmod{4}$, and set $pq = 4m - 1$ and $p + q = 4n$. Then

$$\left(\frac{\varepsilon s}{\ell}\right)_4 = \begin{cases} +1 & \text{if } \ell^{h/4} = x^2 + xy + my^2, \\ -1 & \text{if } \ell^{h/4} = qx^2 + qxy + ny^2 \end{cases}$$

for all primes $\ell$ that split in $F' = \mathbb{Q}(i, \sqrt{pq})$, where $h$ denotes the class number of $k = \mathbb{Q}(\sqrt{-pq})$. 


From $\ell \equiv 1 \pmod{4}$ we see that $x$ must be odd and $y = 2z$ must be even; thus
\[x^2 + xy + my^2 = x^2 + 2xz + 4mz^2 = (x + z)^2 + pqz^2,\]
\[qxy + qy + ny^2 = qx^2 + 2qy(x + z) + 4nz^2 = q(x + z)^2 + pz^2,\]
hence our result can also be stated in the form
\[
\left(\frac{s \varepsilon}{\ell}\right)_4 = \left\{ \begin{array}{ll}
+1 & \text{if } \ell^h/4 = x^2 + p q y^2, \\
-1 & \text{if } \ell^h/4 = q x^2 + y^2.
\end{array} \right.
\]

**Example 1.** In the simplest example $p = 13$, $q = 3$ we find $\varepsilon = 25 + 4\sqrt{39}$, $s = 1$ and $h = 4$, hence
\[
\left(\frac{\varepsilon}{\ell}\right)_4 = \left\{ \begin{array}{ll}
+1 & \text{if } \ell = x^2 + 39y^2, \\
-1 & \text{if } \ell = 3x^2 + 13y^2.
\end{array} \right.
\]
Since $\ell \equiv 1 \pmod{4}$ we see that $2 \mid y$ in the first and $2 \mid x$ in the second case.

The first few primes $\ell \equiv 1 \pmod{4}$ represented by $Q_1(x, y) = x^2 + 39y^2$ and $Q_2(x, y) = 3x^2 + 13y^2$ are

| $\ell$ | 61 | 157 | 181 | 277 | 313 | 337 |
|-------|----|-----|-----|-----|-----|-----|
| $Q$   | $Q_2(4, 1)$ | $Q_1(1, 2)$ | $Q_1(5, 4)$ | $Q_1(11, 2)$ | $Q_2(10, 1)$ | $Q_2(2, 5)$ |
| $(\varepsilon/\ell)_4$ | -1 | +1 | +1 | +1 | -1 | -1 |

**Example 2.** If $p = 37$ and $q = 3$, then $\varepsilon = 295 + 28\sqrt{11}$, hence $s = -1$. Here we have $h = 8$, and we find the following results:
\[
\left(\frac{-\varepsilon}{\ell}\right)_4 = \left\{ \begin{array}{ll}
+1 & \text{if } \ell^2 = x^2 + 11y^2, \\
-1 & \text{if } \ell^2 = 3x^2 + 37y^2.
\end{array} \right.
\]
Composition of forms shows that this result is equivalent to
\[
\left(\frac{-\varepsilon}{\ell}\right)_4 = \left\{ \begin{array}{ll}
+1 & \text{if } \ell = x^2 + 11y^2, 3x^2 + 37y^2 \\
-1 & \text{if } \ell^2 = 4x^2 \pm xy + 7y^2.
\end{array} \right.
\]
The first few primes $\ell \equiv 1 \pmod{4}$ represented by $Q_1(x, y) = x^2 + 37y^2$, $Q_2(x, y) = 3x^2 \pm xy + 7y^2$ and $Q_4 = 4x^2 - xy + 7y^2$ are

| $\ell$ | 73 | 157 | 181 | 229 | 337 |
|-------|----|-----|-----|-----|-----|
| $Q$   | $Q_4(2, 3)$ | $Q_4(6, 1)$ | $Q_4(2, 5)$ | $Q_2(8, 1)$ | $Q_2(10, 1)$ |
| $(-\varepsilon/\ell)_4$ | -1 | -1 | -1 | +1 | +1 |

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