GENERALIZED SPRINGER THEORY
AND WEIGHT FUNCTIONS

G. Lusztig

INTRODUCTION

0.1. The generalized Springer correspondence [L1] is a bijection between, on the
one hand, the set of pairs consisting of a unipotent class in a connected reductive
group $G$ and an irreducible $G$-equivariant local system on it and, on the other hand,
the union of the sets of irreducible representations of a collection of Weyl groups
associated to $G$. (The classical case involves only some irreducible local systems
and only one Weyl group.) In this paper we show that each Weyl group appearing
in the collection has a natural weight function (see 0.2). We also show how to
extend each of these weight functions to an affine Weyl group; in fact, we describe
two such extensions, one in terms of $G$ and one in terms of the dual group $G^*$. The one in terms of $G^*$ has a surprising representation theoretic interpretation,
see 3.3.

0.2. Notation. Let $G$ be a connected reductive group over $\mathbb{C}$. We fix a prime
number $l$. By local system we mean a $\mathbb{Q}_l$-local system. The centralizer of an
element $x$ of a group $\Gamma$ is denoted by $Z_{\Gamma}(x)$. The identity component of an
algebraic group $H$ is denoted by $H^0$. For an algebraic group $H$ let $Z_H$ be the
centre of $H$. For a connected affine algebraic group $H$ let $U_H$ be the
unipotent radical of $H$. If $(W, S)$ is a Coxeter group with length function $l$ we say that $L : W \to \mathbb{N}$ is a weight function if $L(ww') = L(w) + L(w')$ whenever $w, w'$ in $W$
satisfy $l(ww') = l(w) + l(w')$.

1. A weighted Weyl group

1.1. Induction data. An induction datum for $G$ is a triple $(L, \mathcal{O}, \mathcal{E})$ where $L$ is a
Levi subgroup of a parabolic subgroup of $G$, $\mathcal{O}$ is a unipotent conjugacy class of $L$
and $\mathcal{E}$ is an irreducible $L$-equivariant local system on $\mathcal{O}$ (up to isomorphism) which
is cuspidal (in a sense that will be made precise in 1.3). To an induction datum
$(L, \mathcal{O}, \mathcal{E})$ we will associate a complex of sheaves $K$ on $G$ as follows. We choose a
parabolic subgroup $P$ for which $L$ is a Levi subgroup; let $pr : Z_L^0 \mathcal{O}U_P \to \mathcal{O}$ be

Supported in part by National Science Foundation grant 1303060.

Typeset by A\LaTeX\-iT\TeX
the projection (we identify \( Z_L^0 \mathcal{O} U_P \), a subvariety of \( P \), with \( Z_L^0 \times \mathcal{O} \times U_P \)). We have a diagram
\[
Z_L^0 \times \mathcal{O} \overset{a}{\leftarrow} \mathfrak{P} \overset{b}{\rightarrow} \mathfrak{P} \overset{c}{\rightarrow} G
\]
where
\[
\mathfrak{P} = \{(h, g) \in G \times G; h^{-1}gh \in Z_L^0 \mathcal{O} U_P \},
\]
\[
\mathfrak{P} = \{(hP, g) \in G/P \times G; h^{-1}gh \in Z_L^0 \mathcal{O} U_P \},
\]
\[
a(h, g) = pr(h^{-1}gh), b(h, g) = (hP, g), c(hP, g) = g.
\]
We have \( a^*(Q_l \boxtimes \mathcal{E}) = b^*\mathcal{E} \) where \( \mathcal{E} \) is a well defined local system on \( \mathfrak{P} \). Thus, \( K = c_!\mathcal{E} \) is well defined. According to [L1], \( K \) is an intersection cohomology complex on \( G \) whose support is \( \cup_{h \in G} hZ_L^0 \mathcal{O} U_P h^{-1}; \mathcal{O} \) is the closure of \( \mathcal{O} \).

Let \( X_G \) be the (finite) set consisting of all pairs \((C, S)\) where \( S \) is a unipotent conjugacy class in \( G \) and \( S \) is an irreducible \( G \)-equivariant local system on \( C \) (up to isomorphism). Let \([L, \mathcal{O}, \mathcal{E}]\) be the set of all \((C, S) \in X_G \) such that \( S \) is a direct summand of the local system on \( C \) obtained by restricting some cohomology sheaf of \( K|_C \). Note that subset \([L, \mathcal{O}, \mathcal{E}]\) depends only on the \( G \)-conjugacy class of \((L, \mathcal{O}, \mathcal{E})\).

1.2. For example, if \( L \) is a maximal torus of \( G \) (so that \( P \) is a Borel subgroup, \( \mathcal{O} = \{1\} \) and \( \mathcal{E} = Q_l \)), we have \( \mathfrak{P} = \{(hP, g) \in G/P \times G; h^{-1}gh \in P \} \) and \( c: \mathfrak{P} \rightarrow G \) is the Springer resolution; in this case, \( K = c_!Q_l \).

1.3. Blocks of \( X_G \). Following [L1] we define a partition of \( X_G \) into subsets called blocks. If \((C, S) \in X_G \) we say that \( S \) is cuspidal if \([\{C, S\}]\) is a block by itself said to be a cuspidal block. The definition of blocks is by induction on \( \dim \mathcal{O} \). If \( G = \{1\} \), then \( X_G \) has a single element; it forms a block. For general \( G \), the non-cuspidal blocks of \( X_G \) are exactly the subsets of \( X_G \) of the form \([L, \mathcal{O}, \mathcal{E}]\), where \((L, \mathcal{O}, \mathcal{E})\) is an induction datum for \( G \) with \( L \neq G \). (Note that the notion of cuspidality of \( \mathcal{E} \) is known from the induction hypothesis since \( \dim L < \dim G \).) The cuspidal blocks of \( X_G \) are the one element subsets of \( X_G \) which are not contained in any non-cuspidal block. The correspondence \((L, \mathcal{O}, \mathcal{E}) \mapsto [L, \mathcal{O}, \mathcal{E}]\) defines a bijection between the set of induction data of \( G \) (up to conjugation) and the set of blocks of \( X_G \), see [L1].

1.4. Let \( L, \mathcal{O}, \mathcal{E}, P, c: \mathfrak{P} \rightarrow G \) be as in 1.1 and let \( x \in \mathcal{O} \). Let \( \mathfrak{P}_x = c^{-1}(x) \). Thus, \( \mathfrak{P}_x = \{hP \in G/P; h^{-1}xh \in OU_P \} \). In [L3, §11], \( \mathfrak{P}_x \) is called a generalized flag manifold. This is justified by the following result in [L3, 11.2] in which \( U = U_{Z_G^0}(x) \).

(a) The conjugation action of \( Z_G^0(x) \) on \( \mathfrak{P}_x \) is transitive. If \( hP \in \mathfrak{P}_x \) then \( \beta_P := (hPh^{-1} \cap Z_G^0(x))U \) is a Borel subgroup of \( Z_G^0(x) \). The map \( hP \rightarrow \beta_P \) from \( \mathfrak{P}_x \) to the variety of Borel subgroups of \( Z_G^0(x) \) is a fibration. The fibres are exactly the orbits of the conjugation action of \( U \) on \( \mathfrak{P}_x \) hence are affine spaces.
We have the following result.

(b) \( \dim \mathfrak{P}_x = (\dim Z_G^0(x) - \dim Z_L^0(x))/2 \).
From [L1, 2.9] we see that the right hand side of (b) is equal to the dimension of
the $Z_G(x)$-orbit of $P$ in $G/P$ and that this orbit is connected so that, by (a), it equals $\mathfrak{P}_x$. This proves (b).

Let $W$ be the Weyl group of $G$, a finite Coxeter group, and let $S_0$ be the set of simple reflections of $W$. For any $J \subset S_0$ let $W_J$ be the subgroup of $W$ generated by $J$ and let $w_J$ be the longest element of $W_J$.

Now $P$ is a parabolic subgroup of type $I$ for a well defined subset $I$ of $S_0$. Let $W$ be the set of all $w \in W$ such that $wW_J = W_Iw$ and $w$ has minimal length in $wW_I = W_Iw$. This is a subgroup of $W$. For any $s \in S_0 - I$ we have $w_J^{l_Js}w_J = w_J^{l_Js}$ hence $\sigma_s = w_J^{l_Js}w_J = w_J^{l_Js}$ satisfies $\sigma_s^2 = 1$. Moreover we have $\sigma_s \in W$.

Let $x \in \mathcal{O}$. Let $b$ be the dimension of the variety of Borel subgroups of $P$ that contain $x$. For any $s \in S_0 - I$ let $P_s$ be the unique parabolic subgroup of type $I \cup s$ that contains $P$ and let

$$\mathfrak{P}_{s,x} = \{hP \in P_s/P; h^{-1}xh \in \mathcal{O}P\}.$$  

This is the analogue of $\mathfrak{P}_x$ when $G$ is replaced by $P_s/U_P$ hence is again a generalized flag manifold. We set

$$\mathcal{L}_0(s) = \dim \mathfrak{P}_{s,x}.$$  

One can verify that

(c) $W$ is a Weyl group with Coxeter generators $\{\sigma_s; s \in S_0 - I\}$ (see [L1]) and

(d) $\sigma_s \mapsto \mathcal{L}_0(s)$ is the restriction to $\{\sigma_s; s \in S_0 - I\}$ of a weight function $\hat{\mathcal{L}}_0$ on $\mathcal{W}$.

To verify (d), we note that $\mathcal{L}_0(s)$ can be computed explicitly in each case using (b) for $P_s/U_{P_s}$ instead of $G$. (See the next section.)

1.5. We now assume that $G$ is almost simple, simply connected. We describe in each case where $L$ is not a maximal torus, the assignment $(G, L, \mathcal{O}, \mathcal{E}) \mapsto \mathcal{W}$ and the values of the function $\mathcal{L}_0$; we will write $(G, L)$ instead of $(G, L, \mathcal{O}, \mathcal{E})$ and will specify $G, L$ by the type of $G, L/Z_L$. The notation for Weyl groups is the usual one, with the convention that a Weyl group of type $A_0$ is $\{1\}$.

(a) $(A_{k-1}, A_{n-1}^k) \mapsto A_{k-1}, \quad n \geq 2, k \geq 1; \mathcal{L}_0 = n, n, \ldots, n;

(b) (C_{2t^2+t+k}, C_{2t^2+t}) \mapsto C_k, \quad t \geq 1, k \geq 0; \mathcal{L}_0 = 1, 1, \ldots, 1, 2t + 1;

(c) (C_{2t^2+3t+k+1}, C_{2t^2+3t+1}) \mapsto C_k, \quad t \geq 0, k \geq 0; \mathcal{L}_0 = 1, 1, \ldots, 1, 2t + 2;

(d) (B_{2t^2+2t+k}, B_{2t^2+2t}) \mapsto B_k, \quad t \geq 1, k \geq 0; \mathcal{L}_0 = 1, 1, \ldots, 1, 2t + 1;$
(e) \((B_{4t^2+3t+2k}, B_{4t^2+3t} \times A_1^k) \hookrightarrow C_k, \quad t \geq 1, k \geq 0; L_0 = 2, 2, \ldots, 2, 4t + 2;\)

(f) \((B_{4t^2+5t+2k+1}, B_{4t^2+5t+1} \times A_1^k) \hookrightarrow C_k, \quad t \geq 0, k \geq 0; L_0 = 2, 2, \ldots, 2, 4t + 1;\)

(g) \((D_{2t^2+k}, D_{2t^2}) \hookrightarrow B_k, \quad t \geq 1, k \geq 0; L_0 = 1, 1, \ldots, 1, 2t;\)

(h) \((D_{4t^2+t+2k}, D_{4t^2+t} \times A_1^k) \hookrightarrow C_k, \quad t \geq 1, k \geq 0; L_0 = 2, 2, \ldots, 2, 4t - 1;\)

(i) \((D_{4t^2-t+2k}, D_{4t^2-t} \times A_1^k) \hookrightarrow C_k, \quad t \geq 1, k \geq 0; L_0 = 2, 2, \ldots, 2, 4t;\)

(j) \((E_6, A_2^2) \hookrightarrow G_2; L_0 = 1, 3;\)

(k) \((E_7, A_1^3) \hookrightarrow F_4; L_0 = 1, 1, 2;\)

(l) \((E_8, E_8) \hookrightarrow A_0;\)

(m) \((F_4, F_4) \hookrightarrow A_0;\)

(n) \((G_2, G_2) \hookrightarrow A_0.\)

(In the case where \(W\) is of type \(B_k = C_k\) the name we have chosen is such that it agrees with the type of the affine Weyl group \(\hat{W}\) in 1.5.)

In the case where \(L\) is a maximal torus that is, \((L, O, E)\) is as in 1.2, we have \(W = W\); the function \(L_0\) is constant equal to 1.

1.6. Let \(L, O, E, P\) be as in 1.1 and let \(x \in O\). Let \(\Omega\) be the set of \(P\)-orbits on \(G/P\) (under the action by left translation). For \(\omega \in \Omega\) we set \(\mathfrak{P}_x^\omega = \mathfrak{P}_x \cap \omega\) so that we have a partition \(\mathfrak{P}_x = \sqcup \mathfrak{P}_x^\omega\) where each \(\mathfrak{P}_x^\omega\) is locally closed in \(\mathfrak{P}_x\). Let \(NL\) be the normalizer of \(L\) in \(G\). We can identify \(NL/L\) with a subset of \(\Omega\) by \(nL \mapsto P - \text{orbit of } nP\) where \(n \in NL\). We can also identify \(NL/L = \mathcal{W}\) canonically so that we can identify \(\mathcal{W}\) with a subset of \(\Omega\). One can show that

(a) If \(w \in \mathcal{W}\) then \(\mathfrak{P}_x^w\) is an affine space of dimension \(\tilde{L}_0(w)\).

Let \(w_0\) be the longest element of \(\mathcal{W}\). Since \(\mathfrak{P}_x^{w_0}\) is open in \(\mathfrak{P}_x\) we deduce that

(b) \(\dim \mathfrak{P}_x = \tilde{L}_0(w_0).\)
2. A weighted affine Weyl group

2.1. In this subsection we describe an affine analogue of the generalized Springer theory. We assume that \( G \) is almost simple, simply connected and that \((L, O, \mathcal{E})\) are as in 1.1. Let \( \hat{G} = G(\mathbb{C}((\epsilon))) \) where \( \epsilon \) is an indeterminate. We can find a parahoric subgroup \( \hat{P} \) of \( \hat{G} \) whose prounipotent radical \( U_{\hat{P}} \) satisfies \( \hat{P} = U_{\hat{P}}L \), \( U_{\hat{P}} \cap L = \{1\} \). Let \( \hat{W} \) be the affine Weyl group defined by \( \hat{G} \). It is a Coxeter group with set of simple reflections \( \hat{S}_0 \). We have \( \hat{S}_0 \subset \hat{S}_0 \) naturally and the subgroup of \( \hat{W} \) generated by \( \hat{S}_0 \) can be identified with \( W \). In particular the subset \( I \subset \hat{S}_0 \) can be viewed as a subset of \( \hat{S}_0 \). Let \( \hat{S}^0_0 \) be the set of \( s \in \hat{S}_0 \) such that \( I \cup s \) generate a finite subgroup of \( \hat{W} \); this set contains \( \hat{S}_0 - I \). Let \( \hat{W} \) be the subgroup of \( \hat{W} \) defined in terms of \( \hat{W}, W, u = 1 \) as in \([L4, 25.1]\). This is a Coxeter group (in fact an affine Weyl group) with generators \( \{\sigma_s; s \in \hat{S}^0_0\} \). It contains \( W \) as the subgroup generated by \( \hat{S}_0 - I \).

For any \( g \in \hat{G} \) let \( \hat{P}_g \) be the subset of \( \hat{G}/\hat{P} \) such that \( h^{-1}gh \in Z^0_LOU_{\hat{P}} \). If \( g \in \hat{G} \) is regular semisimple, then \( \hat{P}_g \) can be viewed as an increasing union of algebraic varieties of bounded dimension. Moreover, \( \mathcal{E} \) gives rise to a local system \( \hat{\mathcal{E}} \) on \( \hat{P}_g \) in the same way as \( \mathcal{E} \) gives rise to a a local system \( \mathcal{E} \) on \( \mathcal{P} \) in 1.1. Then the homology groups \( H_i(\hat{P}_g, \hat{\mathcal{E}}) \) are defined; they are (possibly infinite dimensional) \( \mathbb{Q}_l \)-vector spaces. Using the method in \([L5]\) (patching together various generalized Springer representations for groups of rank 2 considered in \([L4]\)) we see that \( \hat{W} \) acts naturally on \( H_i(\hat{P}_g, \hat{\mathcal{E}}) \).

We now describe the type of the affine Weyl group \( \hat{W} \).

In 1.5(a), \( \hat{W} \) has type \( \tilde{A}_{k-1} \).
In 1.5(b), \( \hat{W} \) has type \( \tilde{C}_k \).
In 1.5(c), \( \hat{W} \) has type \( \tilde{C}_k \).
In 1.5(d), \( \hat{W} \) has type \( \tilde{B}_k \).
In 1.5(e), \( \hat{W} \) has type \( \tilde{C}_k \).
In 1.5(f), \( \hat{W} \) has type \( \tilde{C}_k \).
In 1.5(g), \( \hat{W} \) has type \( \tilde{B}_k \).
In 1.5(h), \( \hat{W} \) has type \( \tilde{C}_k \).
In 1.5(i), \( \hat{W} \) has type \( \tilde{C}_k \).
In 1.5(j), \( \hat{W} \) has type \( \tilde{G}_2 \).
In 1.5(k), \( \hat{W} \) has type \( \tilde{F}_4 \).
In 1.5(l), \( \hat{W} \) has type \( \tilde{A}_0 \).

In \([L2, 2.6]\) it is shown that the Weyl group \( W \) can be identified with the Weyl group of \( Z^0_G(x)/U_{Z^0_G(x)} \) where \( x \in \mathcal{O} \). The results above show that \( \hat{W} \) can be identified with the affine Weyl group associated with \( Z^0_G(x)/U_{Z^0_G(x)} \).

2.2. For any \( s \in \hat{S}_0' \) let \( \hat{P}_s \) be a parahoric subgroup of type \( I \cup \{s\} \) containing \( \hat{P} \) and let \( U_{\hat{P}_s} \) the prounipotent radical of \( \hat{P}_s \). Then \((L, O, \mathcal{E})\) can be viewed
2.3. Let \( \omega \) be the set of \( (\omega; s \in \hat{S}_0) \) of a weight function \( \hat{L} \) on the Coxeter group \( \hat{W} \).

2.3. Let \( x \in \mathcal{O} \subset L \subset \hat{G} \). We say that \( \hat{\mathfrak{P}}_x = \{ h \hat{P} \in \hat{G}/\hat{P}; h^{-1}xh \in \mathcal{O}U_{\hat{P}} \} \) is a generalized affine flag manifold. Let \( \hat{\Omega} \) be the set of \( \hat{P} \)-orbits on \( \hat{G}/\hat{P} \) (under the action by left translation). For \( \omega \in \hat{\Omega} \) we set \( \hat{\mathfrak{P}}_x^\omega = \hat{\mathfrak{P}}_x \cap \omega \) so that we have a partition \( \hat{\mathfrak{P}}_x = \bigsqcup_{\omega} \hat{\mathfrak{P}}_x^\omega \) where each \( \hat{\mathfrak{P}}_x^\omega \) is an algebraic variety. In analogy with 1.6, we can identify \( \hat{\mathcal{W}} \) with a subset of \( \hat{\Omega} \). It is likely that the following affine analogue of 1.6(a) holds.

(a) If \( w \in \hat{\mathcal{W}} \) then \( \hat{\mathfrak{P}}_x^w \) is an affine space of dimension \( \hat{\mathcal{L}}_0(w) \).

3. Another weighted affine Weyl group

3.1. We again assume that \( G \) is almost simple, simply connected. We denote by \( G^\ast \) a simple adjoint group over \( C \) of type dual to that of \( G \). Let \( (L, \mathcal{O}, \mathcal{E}) \) be an induction datum for \( G \). Let \( G^\ast \) (resp. \( L^\ast \)) be a connected reductive group over \( C \) of type dual to that of \( G \) (resp. \( L \)) we can regard \( L^\ast \) as the Levi subgroup of a parabolic subgroup of \( G^\ast \). Let \( \mathcal{E} = \imath(j(Q_l \boxtimes \mathcal{E})) \) where \( j : Z_L^0 \times \mathcal{O} = Z_L^0 \mathcal{O} \to L \) is the obvious imbedding. Then \( \mathcal{E}[d] \) (where \( d = \dim(Z_L^0 \mathcal{O}) \)) is a character sheaf on \( L \). The classification of character sheaves of \( L \) associates to \( L \) a triple \((s, C, c)\) where \( s \) is a semisimple element of finite order of \( L^\ast \), \( C \) is a connected component of \( H = Z_L^0(s) \) and \( c \) is a two-sided cell of the Weyl group \( W' \) of \( H^0 \) which is stable under the conjugation by any element of \( C \). (The triple \((s, C, c)\) is defined up to \( L^\ast \)-conjugacy.) Let \( W'^{\alpha} \) be the affine Weyl group associated to \( (Z_L^0(s)/centre)(C(\{e\})) \). Then \( W' \) can be viewed as a finite (standard) parabolic subgroup of \( W'^{\alpha} \). Note that conjugation by an element of \( C \) induces a Coxeter group automorphism \( \gamma : W'^{\alpha} \to W'^{\alpha} \) which leaves \( W' \) stable.

We describe in each case where \( L \) is not a maximal torus, the assignment \( (G, L, \mathcal{O}, \mathcal{E}) \to (W'^{\alpha}, W') \); we will write \( (G, L) \) instead of \( (G, L, \mathcal{O}, \mathcal{E}) \) and will specify \( G, L \) by the type of \( G, L/\mathcal{Z}_L \). The notation for Weyl groups and affine Weyl groups is the usual one, with the convention that a Weyl group or affine Weyl group of type \( A_0, B_0, C_0, D_0, D_1 \) is \( \{1\} \). The cases (a)-(n) below correspond to the cases (a)-(n) in 1.5.

(a) \( (A_{kn-1}, A_{n-1}^k) \to (A_{k-1}^n, A_0) \), \( n \geq 2, k \geq 1 \);

(b) \( (C_{2^t+2+t+k} ; C_{2^t+2}) \to (\tilde{B}_{2^t+2+t+k} \times \tilde{D}_{2^t}, \tilde{B}_{2^t+2} \times D_{2^t}) \), \( t \geq 1, k \geq 0 \);

(c) \( (C_{2^t+3t+k+1}; C_{2^t+3t+1}) \to (\tilde{D}_{2^t+2t+k+1} \times \tilde{B}_{2^t+2t+1} \times D_{2^t+2t+1} \times B_{2^t+2t+1}) \), \( t \geq 0, k \geq 0 \);
We set $n_t = 1$ if $t$ is even, $n_t = 2$ if $t$ is odd. In (a) with $k \geq 1$, $\gamma$ has order $n$; it permutes cyclically the $n$ copies of $A_{k-1}$; in (a) with $k = 1$, we have $\gamma = 1$. In (b) with $t \geq 2$, $\gamma$ has order $n_t$; it acts only on the $\tilde{D}$-factor; in (b) with $t = 1$, we have $\gamma = 1$. In (c) with $(t, k) \neq (0, 0)$, $\gamma$ has order $n_{t+1}$; it acts only on the $\tilde{D}$-factor; in (c) with $(t, k) = (0, 0)$, we have $\gamma = 1$. In (d) we have $\gamma = 1$. In (e), $\gamma$ has
order 2; it interchanges the two $\tilde{C}$-factors and acts nontrivially on the $\tilde{A}$-factor. In (f) with $(t, k) \neq (0, 0)$, $\gamma$ has order 2; it interchanges the two $\tilde{C}$-factors and acts nontrivially on the $\tilde{A}$-factor; in (f) with $(t, k) = (0, 0)$, we have $\gamma = 1$. In (g) with $(t, k) \neq (1, 0)$, $\gamma$ has order $n_t$; it acts on the $\tilde{D}_{12+k}$-factor. In (g) with $(t, k) = (1, 0)$ we have $\gamma = 1$. In (h) with $(t, k) \neq (1, 0)$, $\gamma$ has order $2n_t$; it interchanges the two $\tilde{D}$ factors. In (h) with $(t, k) = (1, 0)$, $\gamma$ has order 2. In (i) with $(t, k) \neq (1, 0)$, $\gamma$ has order $2n_t$; it interchanges the two $\tilde{D}$ factors. In (i) with $(t, k) = (1, 0)$, we have $\gamma = 1$. In (j), $\gamma$ has order 3; in (k), $\gamma$ has order 2. In (l),(m),(n), we have $\gamma = 1$.

We now describe in each case the two-sided cell $c$ of $W'$. If $W' = \{1\}$ then $c = \{1\}$. If $W' \neq \{1\}$, we write $W' = W'_1 \times \ldots \times W'_m$ where $W'_i$ are irreducible Weyl groups and $c = c_1 \times \ldots \times c_m$ where $c_i$ is a two-sided cell in $W'_i$. For any $i$ such that $W'_i$ is of type $A_r$, $r \geq 1$, we have $r + 1 = (h^2 + h)/2$ for some $h$ and $c_i$ is the two-sided cell associated to a unipotent cuspidal representation of a nonsplit group of type $A_r$ over $\mathbf{F}_q$. For any $i$ such that $W'_i$ is of type $B_r$ or $C_r$ with $r \geq 2$, we have $r = h^2 + h$ for some $h$ and $c_i$ is the two-sided cell associated to a unipotent cuspidal representation of a group of type $B_r$ or $C_r$ over $\mathbf{F}_q$. For any $i$ such that $W'_i$ is of type $D_r$ with $r \geq 4$, we have $r = h^2$ for some $h$ and $c_i$ is the two-sided cell associated to a unipotent cuspidal representation of a group of type $D_r$ over $\mathbf{F}_q$ (which is split if $h$ is even, nonsplit if $h$ is odd). If $W'$ is of type $E_8, F_4$ or $G_2$, $c$ is the two-sided cell associated to a unipotent cuspidal representation of a group of type $E_8, F_4$ or $G_2$ over $\mathbf{F}_q$.

3.2. We associate to an induction datum $(L, \mathcal{O}, \mathcal{E})$ of $G$ an affine Weyl group $W^a$. We define $W^a$ in terms of $(W^a, W', \gamma)$ as in [L4, 25.1]. In more detail, let $S$ be the set of all simple reflections of $W^a$. For any subset $J$ of $S$ let $W^a_J$ be the subgroup of $W^a$ generated by $J$; when $W^a_J$ is finite let $w^a_J$ be the longest element of $W^a_J$. If $J'$ is a set of simple reflections of $W'$. Let $\tilde{W}^a$ be the set of all $w \in W^a$ such that $wW^a_{J'} = W^a_{J'} w$ and $w$ has minimal length in $wW^a_{J'} = W^a_{J'} w$ and $W^a$ be the fixed point set of $\gamma : W^a \to \tilde{W}^a$. Note that $\tilde{W}^a, W^a$ are subgroup of $W^a$.

Let $K$ be the set of all $\gamma$-orbits $k$ on $S - J'$ such that $W^a_{J' \cup k}$ is finite. In each case (a)-(n), for any $k \in K$ we have $w^a_{J' \cup k} w^a_0 = w^a_{J'} w^a_0 w^a_{J' \cup k}$ hence $\tau_k = w^a_{J' \cup k} w^a_0 = w^a_0 w^a_{J' \cup k}$ satisfies $\tau_k^2 = 1$. Moreover we have $\tau_k \in W_a$. Let $a : W^a \to \mathbf{N}$ be the $a$-function of the Coxeter group $W^a$ (with standard length function), see [L4, §13]. We define $\mathcal{L} : K \to \mathbf{N}$ by $\mathcal{L}(k) = a(c\tau_k) - a(c)$ where $a(c\tau_k), a(c)$ denotes the (constant) value of the $a$-function on $c\tau_k, c$ (see [L4, 9.13]). One can verify that $W^a$ is an affine Weyl group with Coxeter generators $\{\tau_k; k \in K\}$ and that $\tau_k \mapsto \mathcal{L}(k)$ is the restriction to $\{\tau_k; k \in K\}$ of a weight function on $W^a$.

We describe below the type of the affine Weyl group $W^a$ and the values of the weight function $\mathcal{L}$ on $K$.

In 3.1(a), $W^a$ has type $\tilde{A}_{k-1}$, $\mathcal{L} = n, n, \ldots, n$.

In 3.1(b), $W^a$ has type $\tilde{B}_k$, $\mathcal{L} = 1, 1, \ldots, 1, 2t + 1$.

In 3.1(c), $W^a$ has type $\tilde{B}_k$, $\mathcal{L} = 1, 1, \ldots, 1, 2t + 2$. 

G. LUSZTIG
In 3.1(d), \(W^a\) has type \(\tilde{C}_k, \mathcal{L} = 1, 1, \ldots, 1, 2t + 1\).
In 3.1(e), \(W^a\) has type \(\tilde{C}_k, \mathcal{L} = 2, 2, \ldots, 2, 4t + 2\).
In 3.1(f), \(W^a\) has type \(\tilde{C}_k, \mathcal{L} = 1, 2, 2, \ldots, 2, 4t + 1\).
In 3.1(g), \(W^a\) has type \(\tilde{B}_k, \mathcal{L} = 1, 1, \ldots, 1, 2t\).
In 3.1(h), \(W^a\) has type \(\tilde{C}_k, \mathcal{L} = 1, 2, 2, \ldots, 2, 4t - 1\).
In 3.1(i), \(W^a\) has type \(\tilde{B}_k, \mathcal{L} = 2, 2, \ldots, 2, 4t\).
In 3.1(j), \(W^a\) has type \(\tilde{G}_2, \mathcal{L} = 1, 1, 3\).
In 3.1(k), \(W^a\) has type \(\tilde{A}_0\).
In 3.1(l),(m),(n), \(W^a\) has type \(\tilde{A}_0\).

In the case where \(L\) is a maximal torus that is, \((L, \mathcal{O}, \mathcal{E})\) is as in 1.2, we have \(s = 1\), \(W^a\) is an affine Weyl group of type dual to that of \(G\), \(W' = \{1\}, c = 1\), and \(\gamma = 1\); \(W^a = W^a\); the function \(\mathcal{L}\) is constant equal to 1.

We see that \(W\) in 1.4 is naturally imbedded (as a Coxeter group) in \(W^a\) so that \(W^a\) is an affine Weyl group associated to \(W\) and that \(\mathcal{L}_0\) in 1.4 is the restriction of \(\mathcal{L}\).

3.3. Let \(\overline{\mathbf{F}_q}\) be an algebraic closure of the finite field \(\mathbf{F}_q\). The pair \(Z^0_{G^*}(s) \supset Z^0_{L^*}(s)\) has a version \(\mathcal{G}' \supset \mathcal{G}_0'\) with \(\mathcal{G}', \mathcal{G}_0'\) being connected reductive groups over \(\overline{\mathbf{F}_q}\) of the same type as \((Z^0_{G^*}(s), Z^0_{L^*}(s))\). Let \(\mathcal{G} \supset \mathcal{G}_0\) be obtained from \(\mathcal{G}' \supset \mathcal{G}_0'\) by dividing by the centre of \(\mathcal{G}'\). Let \(F : \mathcal{G} \to \mathcal{G}\) be the Frobenius map for an \(\mathbf{F}_q\)-rational structure on \(\mathcal{G}\) which induces on the Weyl group of \(\mathcal{G}\) the same automorphism as \(\gamma\) in 3.1. We can then form the corresponding group \(\mathcal{G}(\mathbf{F}_q((\epsilon)))\) where \(\epsilon\) is an indeterminate and its subgroup \(\mathcal{G}_0(\mathbf{F}_q)\). This subgroup can be regarded as the reductive quotient of a parahoric subgroup \(\mathcal{P}\) of \(\mathcal{G}(\mathbf{F}_q((\epsilon)))\); moreover this subgroup carries a unipotent cuspidal representation as in the last paragraph of 3.1. We can induce this representation from \(\mathcal{P}\) to \(\mathcal{G}(\mathbf{F}_q((\epsilon)))\). The endomorphism algebra of this induced representation is known to be an extended affine Hecke algebra with explicitly known (possibly unequal) parameters. An examination of the cases (a)-(n) in 3.2 shows that these parameters are exactly those described by the function \(\mathcal{L}\) in 3.2.

References

[L1] G.Lusztig, *Intersection cohomology complexes on a reductive group*, Inv.Math. **75** (1984), 205-272.
[L2] G.Lusztig, *Cuspidal local systems and graded Hecke algebras I*, Publications Math. IHES **67** (1988), 145-202.
[L3] G.Lusztig, *Cuspidal local systems and graded Hecke algebras II*, Representations of groups, ed. B.Allison et al., Canad. Math. Soc. Conf. Proc., vol. 16, Amer. Math. Soc., 1995, pp. 217-275.
[L4] G.Lusztig, *Hecke algebras with unequal parameters*, CRM Monograph Ser., vol. 18, Amer. Math. Soc., 2003.
[L5] G.Lusztig, *Unipotent almost characters of simple p-adic groups*, De la Géometrie Algébrique aux Formes Automorphes, Astérisque, vol. 369-370, Soc. Math. France, 2015.

Department of Mathematics, M.I.T., Cambridge, MA 02139