Minimum Spanning Tree Cycle Intersection Problem

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Abstract—Consider a connected graph $G$ and let $T$ be a spanning tree of $G$. Every edge $e \in G - T$ induces a cycle in $T \cup \{e\}$. The intersection of two distinct such cycles is the set of edges of $T$ that belong to both cycles. We consider the problem of finding a spanning tree that has the least number of such non-empty intersections.

1. Introduction

In this article we present what we believe is a novel problem in graph theory, namely the Minimum Spanning Tree Cycle Intersection (MSTCI) problem.

The problem can be expressed as follows. Let $G$ be a graph and $T$ a spanning tree of $G$. Every edge $e \in G - T$ induces a cycle in $T \cup \{e\}$. The intersection of two distinct such cycles is the set of edges of $T$ that belong to both cycles. Consider the problem of finding a spanning tree that has the least number of such pairwise non-empty intersections.

The problem arose while investigating a (yet unpublished) method for mesh deformation in the area of digital geometry processing, see [Botsch, 2010]. The method requires to solve a linear system and the sparsity of the matrix is related to the solution of this problem.

The remaining of this section is dedicated to express the MSTCI problem in the context of well established theories to motivate other points of view. Section 2 sets some notation and convenient definitions. In Section 3 the complete graph case is analyzed. Section 4 presents a variety of interesting properties, and a conjecture in the slightly general case of a graph (not necessarily complete) that admits a star spanning tree. Section 5 explores systematically the space of spanning trees to provide evidence that the conjecture is well posed. Section 6 collects the conclusions of the article.

1.1. Intersection graph theory

Intersection graph theory is a longstanding and central area of graph theory covered in important textbooks [McKee and McMorris, 1999]. It is concerned with the study of intersection graphs.

Let $\mathcal{F} = \{S_1, \ldots, S_k\}$ be any family of sets. The intersection graph denoted $\Omega(\mathcal{F})$ is the graph having $\mathcal{F}$ as vertex set with $S_i$ adjacent $S_j$ if and only if $i \neq j$ and $S_i \cap S_j \neq \emptyset$.

Let $G = (V, E)$ be a graph and $\mathcal{F} = \{S_1, \ldots, S_k\}$ be the family of edge sets corresponding to the cycles of $G$. Then $\Omega(\mathcal{F})$ has as vertex set the cycles of $G$ and two vertices are adjacent if the corresponding cycles share at least one edge. In this setting, let $T$ be a spanning tree of $G$ and $G - T \subset E$ the set of complementary edges of $T$. An edge $e \in G - T$ induces a cycle in $T \cup \{e\}$ which obviously is a cycle of $G$. So $T$ has a canonical mapping to some subgraph of $\Omega(\mathcal{F})$. The MSTCI problem can be expressed in the following terms: find the spanning tree $T$ such that it maps to the sparsest possible subgraph in $\Omega(\mathcal{F})$.

1.2. Matroid theory

The classical matroid theory as developed by Tutte in [Tutte, 1965] is a fundamental theoretical toolbox with very deep insights in graph theory.

The family of sets described in the previous subsection closely resembles the polygon matroid. An interesting formulation of the MSTCI problem can be expressed in terms of its dual matroid $B(G)$, namely the bond matroid.

The bond matroid can be defined as follows. Let $G = (V, E)$ be a connected graph. The atoms of $B(G)$ are the edge subsets $A \subset E$ such that $G - A$ determine two connected components $G_1$ and $G_2$ and such that every edge of $A$ has one end in each component.

A dendroid $D$ of a matroid $M$ is a set that intersects all the atoms and is minimal, meaning that if we delete an element of $D$ then there exists an atom $A \in M$ such that $A \cap D = \emptyset$. The dendroids of $B(G)$ are the sets of edges of the spanning trees of $G$.

Let $T$ be a spanning tree of $G$. So the edges of $T$ define a dendroid $D$ of $B(G)$. Note that for every edge $e \in T$, $T - \{e\}$ determine two subtrees that span two connected components $G_1$ and $G_2$. So there is a natural injective map between the edges of $T$ and the atoms of $B(G)$:

$$\phi_T : E(T) \rightarrow B(G)$$
where $A$ is the atom corresponding to the set of edges linking $G_1$ and $G_2$. As a remark, note that $e \in A$. In the language of matroid theory this set of $|V| - 1$ atoms (ie. the image of $\partial T$) of $B(G)$ is called the dendroid basis determined by $D$.

To formulate the MSTCII problem in this framework we have to be precise about the pairwise intersection of cycles. In this sense let $T$ be a spanning tree of a connected graph $G = (V,E)$. Let $S$ be the dendroid basis determined by the edges of $T$ and $A \in S$ an atom. Clearly there exist a unique edge $e \in T$ such that $\phi_T(e) = A$. Note that each edge $e' \in A - \{e\}$ determine a cycle $c' \in T \cup \{e'\}$ and that $e \in c'$. So two such cycles have non-empty intersection. It is not difficult to realize that every non-empty pairwise intersection is of this form. If we manage to count the set of this pair of edges of all the atoms of $S$ we could express the MSTCII problem as an alternative minimization problem: find the spanning tree such that its corresponding dendroid basis has the least number of such pair of edges.

### 1.3. Homology theory

The importance of homological methods [Cartan, 1956] in topology and geometry cannot be overemphasized. These methods constitute fundamental algebraic tools that enable the computation of invariant quantities of spaces. In its original form the main invariant was the number of “holes” of a space known as its Betti numbers.

We introduce some elementary notions based these notes [Dewan, 2010]. Let $R$ be a commutative ring, and suppose we have a sequence of $R$-modules $M_i$ and homomorphisms $d_i$

$$
\ldots \xrightarrow{d_3} M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \ldots
$$

such that $\forall i d_i d_{i+1} = 0$. Such a sequence is called a chain complex of $R$-modules, and denoted $M_*$. Because the composition of adjacent homomorphisms is trivial, $im(d_{i+1}) \subseteq ker(d_i)$. Therefore, for each module in the chain there is a quotient

$$
H_i(M_*) = \frac{ker(d_i)}{im(d_{i+1})}
$$

called the $i$-th homology group of $M_*$. In graph theory these definitions become very concrete. Let $R$ be a principal ideal domain (ie. $\mathbb{Z}$) and let $G = (V,E)$ be a graph. Consider an arbitrary orientation of the edges that maps every edge $e \in E$ to a triple: $e \mapsto (e, s, t)$, where $s, t \in V$ are the source and target ends of the orientation of $e$. Let $M_0$ be the free $R$-module over $V$ (formal linear combinations of the vertices of $G$). And $M_1$ the $R$-module generated by the oriented edges subject to the relations $(e, s, t) + (e, t, s) = 0$ (where $(e, t, s)$ expresses de traversal of the edge $(e, s, t)$ in the opposite direction). Now consider the boundary homomorphism $\partial : M_1 \to M_0$

$\partial : M_1 \to M_0$

defined on the generators as $\partial(e, s, t) = t - s$. Not surprisingly $ker(\partial)$ is denoted the cycle space of $G, \mathcal{C}(G)$. Since if we consider in $M_1$ the linear combination representing a cycle $c$ of $G$ then $\partial(c) = 0$. It is not difficult to check that $rank(\mathcal{C}(G)) = |E| - |V| + c$ where $c$ is the number of connected components of $G$. Now we can define the following chain complex $G_*$

$$
\ldots \xrightarrow{d_3} 0 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} 0 \xrightarrow{d_{-1}} \ldots
$$

As $im(d_2) = 0$ then $H_1(G_*) = ker(\partial) = \mathcal{C}(G)$.

An interesting result is the following. Let $T$ be a spanning tree of $G$. Note that if $e \in G - T$ then $T \cup e$ has only one cycle: $c_e$. The set of those cycles generate $\mathcal{C}(G)$. Symbolically:

$$
\{c_e \forall e \in G - T\} = \mathcal{C}(G) = H_1(G_*)
$$

In other words: each spanning tree determines some basis of $H_1(G_*)$. In particular if $T$ is a spanning tree that is a solution of the MSTCII problem then the tree intersection number (defined in the next section) of $T$ could be a finer invariant of $G$.

### 2. Preliminaries

#### 2.1. Overview

In the first part of this section we present some of the terms that are used in the article. Then we define the notion of closest-point and closest-point-set. Finally we show a convenient cycle partition.

#### 2.2. Notation

Let $G = (V,E)$ be a graph and $T$ a spanning tree of $G$, then we will refer to the edges $e \in T$ as tree-edges and to the edges $e \in G - T$ as cycle-edges.

Every cycle-edge $e$ induces a cycle in $T \cup \{e\}$, we will call such a cycle a tree-cycle. And we shall call $C_T$ to the set of tree-cycles of $T$.

The intersection of two tree-cycles is the set of edges of $T$ that belong to both cycles. We will define three functions concerning the intersection of tree-cycles.

The first is $\cap_T(\cdot, \cdot) : C_T \times C_T \to \{0, 1\}$

$$
\cap_T(c_i, c_j) = \begin{cases} 1 & c_i \cap c_j \neq \emptyset, c_i \neq c_j \\ 0 & \text{otherwise} \end{cases}
$$

Note that the trivial case $c_i = c_j$ is excluded. This arbitrary decision will simplify future computations.

The second is $\cap_T(\cdot) : C_T \to \mathbb{N}$

$$
\cap_T(c_i) := \sum_{c_j \in C_T} \cap_T(c_i, c_j)
$$

We will call $\cap_T(c)$ the cycle intersection number of $c$. Given a tree-cycle $c$ we will denote $\cap_T(c)$ as the set of
tree-cycles that have non-empty intersection with $c$. More precisely:
\[ \cap_{T,c} \equiv \{ c' \in C_T : \cap_T(c, c') = 1 \} \]

Note that $| \cap_{T,c} | = \cap_T(c)$.

To define the third function consider $\mathcal{T}_G$ to be the set of spanning trees of $G$, so the definition will be as follows:
\[ \cap_G : \mathcal{T}_G \to \mathbb{N} \]
\[ \cap_G(T) := \frac{1}{2} \sum_{c \in C_T} \cap_T(c) \]

We will call $\cap_G(T)$ the tree intersection number of $T$. Clearly the set $\min \cap_G(T)$ is the set of solutions of the MSTCI problem. If the graph is clear from the context we could drop the subindex and simply write: $\cap(T)$.

We shall call star spanning tree to a spanning tree that has one vertex that connects to all other vertices. And $K_n$ to the complete graph on $n$ nodes. If $G = (V, E)$ we will say that $|V| = n$ is the number of vertices of $G$, $|c| = k$ is the length of the cycle $c$ and $|p|$ is the length of the path $p$. Also $uTv$ will denote the unique path between $u, v \in V$ in the spanning tree $T$: $d_T(v)$ will be the degree of $v \in V$ relative to the spanning tree $T$. We will denote $N(v)$ to the set of neighbor nodes of $v \in V$.

Finally we use the terms “node” and “vertex” interchangeably.

2.3. Closest point

In this section we prove the following simple fact: if $G = (V, E)$ is a connected graph, $T$ a spanning tree of $G$ and $c \in C_T$ a tree-cycle, then for every node $v \in V$ there exists a unique node $w \in c$ that minimizes the distance to $v$ in $T$. We shall denote that node closest-point $(v, c)$.

**Lemma 1.** Let $G = (V, E)$ be a connected graph, $T$ a spanning tree of $G$ and $c \in C_T$ a tree-cycle then for every node $v \in V$ there exists a unique node $w \in c$ such that
\[ |vTw| \leq |vTu| \quad \forall u \in c \]

Proof. The proof proceeds by contradiction. If $v \in c$ it is obviously its own unique closest point. Suppose that $v \notin c$ and that there are two distinct nodes $w, w' \in c$ such that $|vTw| = |vTw'| \leq |vTu| \quad \forall u \in c$. Obviously $w' \notin vTw$ and $w \notin vTw'$, we conclude that $vTw \cup wTw'w'Tw'$ determine a cycle in $T$ which contradicts the fact that $T$ is a tree. $\square$

The uniqueness of the closest-point $(v, c)$ leads to the following definition.

**Definition 2.** Let $G = (V, E)$ be a connected graph, $T$ a spanning tree of $G$ and $c \in C_T$ a tree-cycle, then the set of closest points to a node $w \in c$ is defined as follows
\[ \text{closest-point}(u, c) := \{ v \in V - c : \text{closest-point}(v, c) = u \} \]

2.4. Tree cycle intersection partition

Now we define a partition of the set $\cap_{T,c}$. More precisely, let $G$ be a connected graph, $T$ a spanning tree of $G$ and $c \in C_T$ a tree-cycle. As defined above the set $\cap_{T,c}$ is the set of tree-cycles that have non-empty intersection with $c$.

Let us consider any tree-cycle $c' \in \cap_{T,c}$ induced by a cycle-edge $e = (v, w)$. In this setting we can define the following partition:

- **Internal tree-cycles:** $c'$ is internal if $v, w \in c$.
- **External tree-cycles:** $c'$ is external if $v \notin c$ and $w \in c$.
- **Transit tree-cycles:** $c'$ is transit if $v, w \notin c$.

Let us denote them $\cap_{T,c}^i, \cap_{T,c}^e, \cap_{T,c}^t$, respectively. This partition will be convenient to simplify the computation of the intersection number of $c$.

3. Tree cycles of complete graphs

3.1. Overview

In this section we analyze the complete graph case $G = K_n$. First we deduce a formula to compute the cycle intersection number. Then we prove that the tree-cycles of a star spanning tree achieve the minimum cycle intersection number. Finally we conclude that the star spanning trees are the unique solutions of the MSTCI problem.

3.2. Cycle intersection number formula

In this subsection we consider the problem of finding a formula to count tree-cycle intersections. More precisely, let $G = K_n$, $T$ a spanning tree of $G$ and $c$ a tree-cycle, we intend to derive a formula to calculate $\cap_T(c)$.

The idea behind the formula is to consider the partition of $\cap_{T,c}$, defined in the previous section. And then by combinatorial arguments compute the number of elements in each class.

We shall analyze in turn the three classes: $\cap_{T,c}^i, \cap_{T,c}^e, \cap_{T,c}^t$. In this section we will consider $c' \in \cap_T(c)$ to be a tree-cycle induced by a cycle-edge $e = (v, w)$.

The simplest case is the internal tree-cycles class: $\cap_{T,c}^i$. Let $c'$ be an internal tree-cycle. By definition the nodes $v$ and $w$ belong to $c$, so the following holds: $(c' \cap T) \subset c$ because there is a unique path from $v$ to $w$ in $T$. So basically counting the number of internal tree-cycles reduces to count the pairings of the nodes of $c$ excluding some obvious cases. The cases that should be excluded are: the pairing of a node with itself and with its neighbors in $c$. Then the number of internal tree-cycles is:
\[ | \cap_{T,c}^i | = \frac{(k - 3)k}{2} \]
where $k$ is, as before, equal to $|c|$. The quotient is obviously due to the fact that every cycle is counted twice.
Next we consider the class of external tree-cycles. Now let \( c' \) be an external tree-cycle. In this case exactly one of the extremal nodes \( (v \text{ or } w) \) belong to \( c \). Without loss of generality (as we are considering undirected edges), suppose that \( v \notin c \) and \( w \in c \). Clearly \( w \neq \text{closest} - \text{point} (v, c) \) because in that case \( c' \cap c = \emptyset \) and consequently \( c' \notin \cap c \) which contradicts our hypothesis. As that is the only particular case that should be excluded, the number of external tree-cycles is:

\[
|\cap T_c | = (n-k)(k-1),
\]

where \( n = |V| \) is the number of vertices of \( G \) and \( k = |c| \) is the length of \( c \).

Now we consider the class of transit tree-cycles. In this case the key observation depends on the closest - point - set definition of the previous section. Let's define two classes of cycle-edges:

1. A cycle-edge \( e = (v, w) \) is called intraset cycle-edge if both \( v, w \in \text{closest} - \text{point} - \text{set} (u, e) \) for some \( u \in c \).
2. A cycle-edge \( e = (v, w) \) is called interset cycle-edge if \( v \in \text{closest} - \text{point} - \text{set} (u, c) \) and \( w \in \text{closest} - \text{point} - \text{set} (u, c) \) where \( u_i, u_j \in c \) and \( u_i \neq u_j \).

Then:

- Every intraset cycle-edge induce a tree-cycle \( c' \) such that \( c' \cap c = \emptyset \).
- Every interset cycle-edge induce a tree-cycle \( c' \) such that \( c' \cap c \neq \emptyset \).

So we should consider interset cycle-edges or equivalently, the pairing of the nodes that are in different sets. Let \( q_i = |\text{closest} - \text{point} - \text{set} (u_i, c) | \) be defined for all \( w_i \in c \), then the number of transit tree-cycles is:

\[
|\cap T_c | = \sum_{i<j} q_i q_j = \frac{1}{2} \sum_{i=1}^{k} q_i (n-k-q_i)
\]

Finally, the intersection number formula is the aggregation of the three classes:

\[
\cap T(c) = |\cap c | = |\cap T_c | + |\cap T_c | + |\cap T_c | = \frac{(k-3)k}{2} + (n-k)(k-1) + \frac{1}{2} \sum_{i=1}^{k} q_i(n-k-q_i),
\]

where \( n \) is the number of vertices of \( G \), \( k = |c| \) and \( q_i = |\text{closest} - \text{point} - \text{set} (w_i, c) | \), for \( w_i \in c \).

### 3.3. Main result

In this subsection we start by defining transiteless tree-cycles. Then we prove two lemmas. The first shows that for every cycle \( c \in G = K_n \) we can build a spanning tree \( T \) such that \( c \) is a tree-cycle of \( T \) and the intersection number \( \cap T(c) \) is minimal. And the second calculates the intersection number of tree-cycles of star spanning trees. Finally we prove the main result of this section, namely that star spanning trees minimize \( \cap (\cdot) \) in the case of complete graphs.

**Definition 3.** Let \( G = (V, E) \) be a connected graph, \( T \) a spanning tree of \( G \) and \( c \in C_T \) a tree-cycle, we call \( c \) a transitless tree-cycle if \( |\cap T(c) | = 0 \).

As an important remark, note that the number of elements in the internal and external classes of \( c \) are independent of the spanning tree because they depend exclusively on the numbers \( n = |V| \) and \( k = |c| \). So two spanning trees \( T_1 \) and \( T_2 \) that have \( c \) as a tree-cycle induce an intersection number (for \( c \)) that only differ in the number of elements in their transit classes. We conclude that transitless tree-cycles have minimal intersection number \( \cap T(c) \).

**Lemma 4.** Let \( G = K_n \) and let \( c \) be a cycle of \( G \) then the following procedure lead to a spanning tree \( T \) that minimizes the intersection number of \( c \):

1. Remove exactly one edge \( e \in c \).
2. Choose a vertex \( v \in c \).
3. Define \( T \) as follows:
   - The edges \( e - c \)
   - A star centered at \( v \) of the vertices \( (G - c) \cup \{v\} \).

**Proof.** Note that \( T \) is a spanning tree of \( G \) and \( c \) is a tree-cycle of \( T \). So if we prove that \( |\cap T(c) | = 0 \), then the intersection number \( \cap T(c) \) is minimal. This is the case, because by construction:

- \( |\text{closest} - \text{point} - \text{set} (u, c) | = 0 \) for all \( u \in c, u \neq v \).
- \( |\text{closest} - \text{point} - \text{set} (v, c) | = n - k \).

So \( |\cap T(c) | = \sum_{i<j} q_i q_j = 0 \).

**Lemma 5.** Let \( G = K_n \) and let \( T_s \) be a star spanning tree of \( G \). Then the following property holds:

\[
\cap T_s (c) = 2(n-3)
\]

for any tree-cycle \( c \) of \( T_s \).

**Proof.** Clearly the tree-cycles in \( T_s \) have the same intersection number (by symmetry). Let \( c \) be a tree-cycle of \( T_s \). Note that \( c \) is a triangle (\( |c| = 3 \)), so the corresponding internal tree-cycles class is empty: \( \cap T_s (c) = 0 \). Also note that \( c \) is a transitless tree-cycle because its nodes are: the central node and two leaf nodes of \( T_s \). So the external tree-cycle class is the only non-empty class:

\[
\cap T_s (c) = |\cap T_s (c) | = 2(n-3)
\]

**Proposition 6.** Let \( G = K_n \) and let \( T_s \) be a star spanning tree of \( G \). Then the following property holds:

\[
\cap T_s (c) \leq \cap T (c)
\]

where \( T \) is any spanning tree of \( G \).

**Proof.** We shall prove the proposition by contradiction. Suppose that a spanning tree \( T \) and a tree-cycle \( c \) of \( T \) exist such that:

\[
\cap T(c) < \cap T_s (c) = 2(n-3)
\]
We can assume that \( c \) is transitless because, if it's not the case, by lemma \( 4 \) we can build a spanning tree \( T' \) such that \( \cap_{T'}(c) < \cap_T(c) \). In this context the inequality can be expressed as

\[
\cap_T(c) = |\cap_{T,c}'| + |\cap_{T,c}''| = \frac{(k - 3)k}{2} + (n - k)(k - 1) < 2(n - 3)
\]

Expanding and simplifying the expression we have

\[
-\frac{1}{2}k^2 + \frac{1}{2}k - 3n + 6 < 0
\]

The roots of this quadratic polynomial are: \( r_1 = 3 \) and \( r_2 = 2(n - 2) \). We should consider two cases depending on the relation of the roots:

1) \( r_1 < r_2 \)

2) \( r_1 > r_2 \)

The case \( r_1 = r_2 \) can be discarded because it leads to a fractional number of nodes \((n = \frac{3}{2})\). In the first case the inequality holds for \( k < r_1 = 3 \) or \( k > r_2 = 2(n - 2) \). The case \( k < 3 \) is an obvious contradiction since the size of the cycle must be \( |c| = k \geq 3 \). The case \( k > 2(n - 2) \) combined with the fact that \( k \leq n \) induces the following inequality

\[
r_1 = 3 < r_2 = 2(n - 2) < k \leq n
\]

which implies a contradiction: \( 3 < n < 4 \), since \( n \) is a positive integer.

The second case \((r_1 = 3 > r_2 = 2(n - 2))\) imply \( n < \frac{3}{2} \). So the only case that should be considered is \( n = 3 \) since \( k \leq n \). But, this inequality is false because \( k = 3 \) is a root of the quadratic polynomial. □

**Corollary 7.** Let \( G = K_n \) and let \( T_c \) be a star spanning tree of \( G \). Then the following property holds

\[
\cap(T_c) = \cap(T),
\]

where \( T \) is any spanning tree of \( G \).

Proof. As expressed by proposition \( 6 \) a tree-cycle of a star spanning tree has the minimal intersection number among all tree-cycles. Since any tree-cycles of a star spanning tree has the same intersection number, we conclude that the tree intersection number of a star spanning tree \( \cap(T_c) \) is minimal among all spanning trees. □

This corollary can be further improved to a strict inequality. In other words: star spanning trees are the unique minimizers of \( \cap(T) \).

**Corollary 8.** Let \( G = (V, E) \) be a solution of the \( MSTCI \) problem for complete graphs. In this subsection we present two formulas for graphs \( G = (V, E) \) that admits a star spanning tree \( T_c \). Let us denote \( v \in V \) to the central node of \( T_c \).

The first formula corresponds to the cycle intersection number of a tree-cycle \( c = (u, v, w) \in C_{T_c} \) namely \( \cap(T_c, c) \). Recall from the previous section that \( c \) does not intersect neither transit nor internal tree-cycles: \( \cap(T_c, c) = 0 \) and \( \cap(T, c) = 0 \). So its non-empty intersections are the tree-cycles in the set \( \cap(T, c) \). Note that the remaining incident edges to \( u \) and \( w \), are the only source of tree-cycles that have non-empty intersection with \( c \). So the formula is straightforward:

\[
\cap(T_c, c) = d(u) - 2 + d(w) - 2,
\]

where \( d(u) \) and \( d(w) \) are the degrees of \( u \) and \( w \), resp.

Now we shall deduce a formula for the tree intersection number \( \cap(T_c) \). Based on the preceding observations and the fact that each node \( u \in V - \{v\} \) is involved in

4. Further generalization

4.1. Overview

Now we explore some aspects of a slightly more general case, namely: the \( MSTCI \) problem in the context of a graph (not necessarily complete) \( G = (V, E) \) that admits a star spanning tree \( T_c \). In the first part we present a formula to calculate \( \cap(T_c) \). In the second part we show that \( \cap(T_c) \) is a local minimum in the domain of what we refer to as the “spanning tree graph”. In the third part we prove a result that suggests a general observation: the fact that a spanning tree of a graph \( G \) being a solution of the \( MSTCI \) problem doesn’t depend on an intrinsic property of \( T \) but on the particular embedding of \( T \) in \( G \). Finally we conjecture a generalization of Corollary \( 2 \) \( \cap(T_c) \leq \cap(T) \) for every spanning tree \( T \) of \( G \).

4.2. Formulas for star spanning trees

In this subsection we present two formulas for graphs \( G = (V, E) \) that admit a star spanning tree \( T_c \). Let us denote \( v \in V \) to the central node of \( T_c \).

The first formula corresponds to the cycle intersection number of a tree-cycle \( c = (u, v, w) \in C_{T_c} \), namely \( \cap(T_c, c) \). Recall from the previous section that \( c \) does not intersect neither transit nor internal tree-cycles: \( \cap(T_c, c) = 0 \) and \( \cap(T, c) = 0 \). So its non-empty intersections are the tree-cycles in the set \( \cap(T, c) \). Note that the remaining incident edges to \( u \) and \( w \) are the only source of tree-cycles that have non-empty intersection with \( c \). So the formula is straightforward:

\[
\cap(T_c, c) = d(u) - 2 + d(w) - 2,
\]

where \( d(u) \) and \( d(w) \) are the degrees of \( u \) and \( w \), resp.
$d(u) - 1$ tree-cycles and for each of those tree-cycles it produces $d(u) - 2$ non-empty intersections (note that the pairwise intersections are counted twice). The formula is as follows:

$$\cap(T_s) = \frac{1}{2} \sum_{u \in V - \{v\}} (d(u) - 1)(d(u) - 2) - \frac{1}{2} \sum_{u \in V - \{v\}} d(u)^2 - 3d(u) + 2$$

If we denote $d$ as the degree vector of $G$, that is: a vector that has in the $i$-th component the degree of the $i$-th vertex. And taking into account that $\sum_{u \in V} d(u) = 2m$ where $m = |E|$, the formula can be expressed as:

$$\cap(T_s) = \frac{1}{2} \|[d]^2 - 6m - (n - 1)(n - 6)\]$$

### 4.3. Star spanning tree as a local minimum

In this subsection we prove that a star spanning tree is a local minimum with respect to the tree intersection number in the domain of the spanning tree graph. We start by defining this second order graph of the original graph $G = (V, E)$. Then we analyze the structure of the neighbors of a star spanning tree $T_s$. Finally we demonstrate the result by a bijection between tree-cycles to conclude that $\cap(T_s)$ is a local minimum.

#### Definition 9.
Let $G = (V, E)$ be a graph, and $S$ a subgraph of $G$. We denote as $e \leftrightarrow e'$ to the operation of replacing the edge $e \in S$ with the edge $e' \in G - S$.

We call this operation edge replacement on $S$.

#### Definition 10.
Let $G = (V, E)$ be a graph. We denote $SP_G$ to the graph that has one node for every spanning tree of $G$ and an edge between two nodes if the corresponding spanning trees differ in exactly one edge replacement. We call this graph the spanning tree graph of $G$.

#### 4.3.1. Neighborhood of $T_s$.
Let $G = (V, E)$ be a graph that admits a star spanning tree $T_s$ with $v \in V$ as its center. Let $\alpha_x$ be the node corresponding to $T_s$ in $SP_G$, and let $\alpha_T$ (with corresponding spanning tree $T$) be any neighbor of $\alpha_x$. By definition $T_s$ and $T$ differ in exactly one edge replacement $e \leftrightarrow e'$ where $e = (v, w) \in T_s$ and $e' = (u, w) \in T$. Note that $T$ is exactly the same as $T_s$ except that the node $w$ is no longer connected to the central node $v$ but is connected to the intermediate node $u$. This similar structure has direct consequences in the intersection numbers of both trees.

Now we prove the result of this section.

#### Theorem 11.
Let $G = (V, E)$ be a graph that admits a star spanning tree $T_s$ with $v \in V$ as its center. Then, $T_s$ is a local minimum respect to the tree intersection number in the domain of $SP_G$.

Proof. Let $T$ be a spanning tree corresponding to a neighbor of $T_s$ in $SP_G$. Then we want to prove that $\cap(T_s) \leq \cap(T)$. We shall proceed by defining a bijection between the tree-cycles of both trees $\{c \leftrightarrow d : c \in C_{T_s} \land d \in C_T\}$ such that $\cap(c) \leq \cap(d)$, this strategy clearly implies the thesis since by definition:

$$\cap(T_s) = \frac{1}{2} \sum_{c} \cap_{T_s}(c) \leq \frac{1}{2} \sum_{d} \cap_{T}(d) = \cap(T)$$

Let $e_{T_s} \leftrightarrow e_T$ with $e_{T_s} = (v, w) \in T_s$ and $e_T = (u, w) \in T$ be the edge replacement in $SP_G$. Consider the following simple facts:

- $e_{T_s}$ is a cycle-edge in $T_s$, with corresponding tree-cycle $c$
- $e_T$ is a cycle-edge in $T$, with corresponding tree-cycle $d$
- Except for $e_{T_s}$ and $e_T$, $T_s$ and $T$ have the same set of cycle-edges. For every $e \in E - T_s - T$ we denote $e_{T_s}$ and $e_T$ to the corresponding tree-cycles in $T_s$ and $T$, resp.

According to this naming convention, we can define the following “natural” bijection between tree-cycles:

$$\{c \leftrightarrow d\} \cup \{c_e \leftrightarrow d_e : e \in E - T_s - T\}$$

In order to compare the intersection numbers of the bijected pairs it is convenient to distinguish the following partition:

- Case 1: the pair induced by the edge replacement, $\{c \leftrightarrow d\}$
- Case 2: pairs induced by cycle-edges non-incident to $u$ nor to $w$, $\{c_e \leftrightarrow d_e : e \in E - T_s - T \land u \notin e \land w \notin e\}$
- Case 3: pairs induced by cycle-edges incident to $u$ or $w$, $\{c_e \leftrightarrow d_e : e \in E - T_s - T \land (u \in e \lor v \in e)\}$

Case 1 is the easiest: note that $c$ and $d$ are the same tree-cycle $(u, v, w)$, which is a transitless triangle, so its intersection number is determined by its external intersections:

$$\cap(c) = d(u) - 2 + d(w) - 2 = \cap(d)$$

Case 2 is similar, let $e = (h, k)$ be a cycle-edge non-incident to $u$ or to $w$ and $c_e \leftrightarrow d_e$ its corresponding pair of bijected tree-cycles. Clearly $e$ determines the transitless triangle $(h, v, k)$ both in $T_s$ and $T$, and as $d_{T_s}(h) = d_T(h)$, $d_{T_s}(k) = d_T(k) = 1$, then every other edge incident to $h$ or $k$ induces a tree-cycle that intersects $(h, v, k)$. We conclude that:

$$\cap(c_e) = d(h) - 2 + d(k) - 2 = \cap(d_e)$$

Case 3 is the one that should be analyzed more carefully. As we already know how to calculate intersection numbers of tree-cycles in $T_s$, we will focus on the tree-cycles of $T$. We will further divide this partition in two subpartitions:

- Case 3.1: pairs induced by cycle-edges incident to $u$, $\{c_e \leftrightarrow d_e : e \in E - T_s - T \land u \in e\}$
- Case 3.2: pairs induced by cycle-edges incident to $w$, $\{c_e \leftrightarrow d_e : e \in E - T_s - T \land w \in e\}$
In case 3.1 the situation is as follows: the cycle-edge $e = (u, k)$ defines the tree-cycle $c_e = d_e = (u, v, h)$ (both in $T$ and $T_h$). The important details are:

- $d_T(u) = 2$: $u$ induces $d(u) - 3$ intersections
- $d_T(k) = 1$: $k$ induces $d(k) - 2$ intersections
- $d_T(w) = 1$: $w$ induces $d(w) - 1$ intersections
- $d(w) \geq 2$ since it is connected at least to $u$ and $v$ in $G$
- $w$ may have an incident cycle-edge connecting it to $k$, so we should avoid counting twice that intersection

Now we claim that

$$\cap_T(d_e) \geq d(u) - 3 + d(k) - 2 + d(w) - 1 - \epsilon(w, k)$$

where

$$\epsilon(w, k) = \begin{cases} 
1 & (w, k) \in E \\
0 & \text{otherwise}
\end{cases}$$

The inequality follows since $d(w) - 1 - \epsilon(w, k) \geq 1$.

In case 2 the situation is as follows: the cycle-edge $e = (w, h)$ defines the tree-cycle $d_e = (w, u, v, h)$ in $T$ and $c_e = (w, v, h)$ in $T_h$. The important details are:

- $d_T(u) = 2$: $u$ induces $d(u) - 2$ intersections
- $d_T(h) = 1$: $h$ induces $d(h) - 2$ intersections
- $d_T(w) = 1$: $w$ induces $d(w) - 2$ intersections
- $u$ may have an incident cycle-edge connecting it to $h$, so we should avoid counting twice that intersection

And we claim that

$$\cap_T(d_e) \geq d(w) - 2 + d(h) - 2 + d(u) - 2 - \epsilon(u, h)$$

The inequality follows since $d(u) - 2 - \epsilon(u, h) \geq 0$.

4.4. Intrinsic tree invariants

In this subsection we consider the following question: is there any correlation between an intrinsic tree invariant and the tree intersection number of the spanning trees for every graph? If so we could formulate an alternative characterization of the MSTCI problem expressed in terms of the invariant.

By intrinsic tree invariant we denote a map $f : \mathcal{T} \rightarrow \mathbb{R}$ on the set of all trees. Of particular interest are the degree-based topological indices [Gutman, 2013]. The topological index that motivated our question is the atom-bond connectivity (ABC) index [Estrada, 1998]. As shown by [Furtula, 2009] the star trees are maximal among all trees respect to the ABC index. In the previous section we proved that in the complete graph the star spanning trees are minimal respect to the tree intersection number. Consequently we can formulate a natural question: is there a negative correlation between the ABC index of the spanning trees and their corresponding intersection numbers?

We will prove that the answer to our question is negative. Without loss of generality we will consider positive correlation (negative correlation is analogous). The underlying idea of the proof is as follows: suppose that there exists an intrinsic tree invariant $f : \mathcal{T} \rightarrow \mathbb{R}$ such that for every graph $G$ the intersection number $\cap_T()$ is positively correlated with $f$, this can be expressed as:

$$f(T_1) \leq f(T_2) \iff \cap_G(T_1) \leq \cap_G(T_2), \forall G, T_1, T_2$$

According to this property if we consider two trees $T_1$ and $T_2$ and two graphs $G$ and $H$ such that $T_1, T_2 \in \mathcal{T}_G$ and $T_1, T_2 \in \mathcal{T}_H$, then this equivalence follows:

$$\cap_G(T_1) \leq \cap_G(T_2) \iff \cap_H(T_1) \leq \cap_H(T_2)$$

So it suffices to show that there exist $T_1, T_2, G$ and $H$ such that the equivalence is not satisfied to answer the question negatively.

First we prove a simple lemma regarding the tree intersection number of a spanning tree $T$ under the removal of a cycle-edge. Namely if a cycle-edge $e$ is removed from $G$ then the tree intersection number of $T$ decreases exactly in the intersection number of its corresponding tree-cycle.

**Lemma 12.** Let $G = (V, E)$ be a graph, $T \in \mathcal{T}_G$ a spanning tree, $e \in G - T$ a cycle-edge, and $c$ the corresponding tree-cycle, then the following holds:

$$\cap_G(T) - e(T) = \cap_G(T) - \cap(c)$$

Proof. As the spanning tree $T$ is the same in both graphs: $G$ and $G - e$, then the remaining cycle-edges define the same tree-cycles so their pairwise intersection relations are identical. As $c$ is not a cycle in $G - e$ then the equality follows. □

**Theorem 13.** There is no intrinsic tree invariant $f : \mathcal{T} \rightarrow \mathbb{R}$ positively correlated with the intersection number $\cap_G()$ for every graph $G$.

Proof. We will proceed by contradiction: let $f$ be such an intrinsic tree invariant. Then by definition for arbitrary graphs $G$ and $H$ the following equivalences hold

$$f(T_1) \leq f(T_2) \iff \cap_G(T_1) \leq \cap_G(T_2)$$

$$f(T_1) \leq f(T_2) \iff \cap_H(T_1) \leq \cap_H(T_2)$$

Where $T_1, T_2 \in \mathcal{T}_G$ and $T_1, T_2 \in \mathcal{T}_H$. This in turn imply that

$$\cap_G(T_1) \leq \cap_G(T_2) \iff \cap_H(T_1) \leq \cap_H(T_2)$$

The proof will be based on showing two graphs and two spanning trees such that the latter equivalence is not valid.

- Let $G$ be the complete graph $K_n$
Let $H$ be the graph $K_n - \{e_{i,1}, \ldots, e_{i,n-3}\}$ where the edges $e_{i,1}, \ldots, e_{i,n-3}$ are $n - 3$ edges incident to some arbitrary node $v_i$. We will refer to $v_i$ as the almost disconnected node of $H$. Note that $d(v_i) = 2$.

Let $T_1$ be the star spanning tree $T_s$.

Let $T_2$ be the spanning tree defined as $T_s - \{e_i\} \cup \{e_{i,j}\}$, where $e_i$ is the edge that connects some arbitrary node $v_i$ (in $H$ this role will be played by the almost disconnected node) to the center of the star and $e_{i,j}$ is an edge that connects $v_i$ to a different node $v_j$.

It is easy to check that $T_1$ and $T_2$ are spanning trees of both $G$ and $H$. If we also suppose that $|V| = n > 4$ then by corollary 8

$$\cap_G(T_1) < \cap_G(T_2)$$

By the previous equivalence it is expected that $\cap_H(T_1) < \cap_H(T_2)$ as well. But we will show that this is not the case.

By a suitable labelling of the nodes of $H$ we can refer to: the center of the star spanning tree as $v_1$, the almost disconnected node of $H$ as $v_2$ and the other neighbor of $v_2$ as $v_3$. By lemma 12 arises that

$$\cap_H(T_1) = \cap_{H-e_{2,3}}(T_1) + \cap_{T_1}(e_{2,3})$$

$$\cap_H(T_2) = \cap_{H-e_{1,2}}(T_2) + \cap_{T_2}(e_{1,2})$$

Where $e_{2,3}$ and $e_{1,2}$ are the tree-cycles induced by $e_{2,3}$ and $e_{1,2}$ in $T_1$ and $T_2$, resp. Since the remaining tree-cycles corresponding to both trees are the same then

$$\cap_{H-e_{2,3}}(T_1) = \cap_{H-e_{1,2}}(T_2)$$

And this imply the following

$$\cap_H(T_1) - \cap_H(T_2) = \cap_{T_1}(e_{2,3}) - \cap_{T_2}(e_{1,2})$$

It is an easy exercise to check that

$$\cap_{T_1}(e_{2,3}) = \cap_{T_2}(e_{1,2}) = d(v_3) - 2 = n - 3$$

At this point we can conclude that

$$\cap_H(T_1) = \cap_H(T_2)$$

Contradicting the fact that $f$ is positively correlated with the tree intersection number for every graph. □

The underlying key fact of this result is that a spanning tree $T$ that solves the the MSTC1 problem for a graph $G$ does not depend on intrinsic properties of $T$ but on the embedding of $T$ in $G$.

Note that as an interesting side effect this demonstration shows that a star spanning tree is not necessarily a strict local minimum in the spanning tree graph (see previous subsection).

4.5. Intersection number conjecture

In this subsection we present the conjecture $\cap(T_s) \leq \cap(T)$ for every spanning tree $T$ generalizing theorem 11. Then we explore two ideas to simplify a hypothetical counterexample of the conjecture. The first is based on the notion of interbranch cycle-edge. We show that if a non-star spanning tree $T$ exists such that $\cap(T) < \cap(T_s)$, then the inequality must hold if we remove the interbranch cycle-edges. The second is based on the notion of principal subtree. In this case we show that the inequality must hold for some principal subtree of $T$. This ideas will be of practical use in the next section.

4.5.1. The conjecture statement. We present below the conjecture that generalizes the case of complete graphs.

**Conjecture 14.** Let $G = (V, E)$ be a graph that admits a star spanning tree $T_s$, then

$$\cap(T_s) \leq \cap(T)$$

for every spanning tree $T \in \mathcal{R}_G$.

As an important remark, a demonstration of this result seems difficult if approached by a local-to-global strategy as in the complete graph case exposed previously.

4.5.2. Counterexample simplification. In this part we consider some ideas to simplify a hypothetical counterexample of conjecture 14.

Below we define the notion of interbranch cycle-edge.

**Definition 15.** Let $G = (V, E)$ be a graph that admits a star spanning tree $T_s$ and let $v \in V$ be the center of $T_s$. Let $T \in \mathcal{R}_G$ be a spanning tree. We call interbranch cycle-edge of $T$ to any cycle-edge of $T$, $e = (u, w)$, such that closest point $(v, e) \neq u, w$, where $e$ is the induced tree-cycle of $e$ in $T$.

The intuition behind this definition is that the paths $vTu$ and $vTw$ belong to different branches. Or equivalently, $u$ and $w$ are not collinear with respect to $v$ in $T$. The following lemma shows that if we can find a counterexample to the conjecture 14 (i.e.: $\cap(T) < \cap(T_s)$) then we can build a simpler counterexample removing from $G$ the interbranch cycle-edges of $T$.

**Lemma 16.** Let $G = (V, E)$ be a graph that admits a star spanning tree $T_s$ with $v \in V$ as its center. Let $T \in \mathcal{R}_G$ be a spanning tree such that $\cap_G(T) < \cap_G(T_s)$ and let $\Delta_T$ be the set of interbranch cycle-edges of $T$, then $\cap_{G-\Delta_T}(T) < \cap_{G-\Delta_T}(T_s)$

Proof. Let $e = (u, w) \in \Delta_T$. Note that $e$ is also a cycle-edge in $T_s$ since $v \neq u, w$ by definition of interbranch cycle-edge. So $e$ determines the tree-cycle $c$ in $T_s$ and the tree-cycle $c'$ in $T$. By the intersection number formula it arises that $\cap_{T_s}(e) = d(u) - 2 + d(w) - 2$. On the other hand, since the other neighbors of $u$ and
w are connected to v, they belong to distinct tree-cycles with non-trivial intersection with respect to \( c' \) in \( T \). We conclude that

\[
\cap_w(c') \geq d(u) - 2 + d(w) - 2 = \cap_w(c).
\]

Hence by lemma\(^\text{[12]}\)

\[
\cap_{G-w}(T) = \cap_G(T) - \cap_T(c') < \cap_G(T_s) - \cap_T(c) = \cap_{G-w}(T_s)
\]

Applying the same procedure for every edge in \( \Delta_T \), the claimed inequality follows. □

**Definition 17.** Let \( T = (V, E) \) be a rooted tree graph with root \( v \in V \). Let \( w \in N(v) \) then we call principal subtree respect to \( w \) to the subtree spanned by \( v \) and the nodes \( w \in V \) such that \( w \in vT_u \).

The next lemma expresses the intersection number of a spanning tree (without interbranch cycle-edges) as the sum of the intersection number of its principal subtrees.

**Lemma 18.** Let \( G = (V, E) \) be a graph that admits a star spanning tree \( T_s \) with \( v \in V \) as its center. Let \( T \) be a spanning tree of \( G \) without interbranch cycle-edges (ie: \( \Delta_T = \emptyset \)), then the following holds

\[
\cap_G(T) = \sum_{w \in N(v)} \cap_{G_w}(T_w)
\]

where \( T_w \) is the principal subtree of \( w \in N(v) \) considering \( T \) as a rooted tree with \( v \) as its root. And \( G_w \) is the subgraph spanned by \( T_w \).

Proof. As \( \Delta_T = \emptyset \) there are no cycle-edges connecting any two such principal subtrees. This implies that the nonempty intersections between tree-cycles of \( T \) must occur inside each subtree. This determines a partition of \( C_T \) and the claimed expression follows. □

The following corollary in line with lemma\(^\text{[16]}\) further simplifies a hypothetical counterexample of conjecture\(^\text{[14]}\).

**Corollary 19.** Let \( G = (V, E) \) be a graph that admits a star spanning tree \( T_s \) with \( v \in V \) as its center. Let \( T \) be a spanning tree of \( G \) without interbranch cycle-edges (ie: \( \Delta_T = \emptyset \)) such that \( \cap(T) < \cap(T_s) \) then

\[
\cap(T_w) < \cap(G_w \wedge T_s)
\]

for some \( G_w \). Where \( T_w \) is the principal subtree of \( w \in N(v) \) considering \( T \) as a rooted tree with \( v \) as its root; \( G_w \) is the subgraph of \( G \) spanned by \( T_w \); \( G_w \wedge T_s \) is the subtree of \( T_s \) restricted to \( G_w \) namely the intersection between \( G_w \) and \( T_s \).

Proof. First note that the \( G_w \)’s are edge disjoint since \( \Delta_T = \emptyset \). This partition of the edges of \( G \) also determines a partition of \( T_s \) such that \( \cap(T_s) = \sum_{w \in N(v)} \cap(G_w \wedge T_s) \). As the parts are in a natural bijective relation since they are the subtrees of \( T \) and \( T_s \) restricted to each \( G_w \), we can express the intersection number of \( T \) and \( T_s \) as follows

\[
\cap(T) = \sum_{w \in N(v)} \cap(T_w) \leq \sum_{w \in N(v)} \cap(G_w \wedge T_s) = \cap(T_s)
\]

And from the bijection we can deduce that \( \cap(T_w) < \cap(G_w \wedge T_s) \) for some \( G_w \). □

5. Programmatic exploration

5.1. Overview

In this section we present some experimental results to reinforce conjecture\(^\text{[14]}\). We proceed by trying to find a counterexample based on our preceding observations. In the first part we focus on the complete analysis of small graphs, ie: graphs of at most 9 nodes. In the second part we analyze larger families of graphs by random sampling instances.

5.2. General remarks

In the previous section we showed that the space of candidate counterexamples of conjecture\(^\text{[14]}\) can be reduced. The general picture is as follows:

- Let \( G = (V, E) \) be a graph that admits a star spanning tree \( T_s \) with \( v \in V \) as its center
- In the case that we can find some non-star spanning tree \( T \) of \( G \) such that \( \cap(T) < \cap(T_s) \)
- Then we can “simplify” the instance by removing the interbranch cycle-edges with respect to \( T \) in \( G \) without affecting the inequality (see lemma\(^\text{[16]}\))
- We can further reduce the instance by focusing on the case where \( d_T(v) = 1 \), that is: the degree of \( v \) restricted to \( T \) is 1 (see corollary\(^\text{[19]}\))

This considerations can be used to implement algorithms to explore the space of spanning trees more efficiently. Since the algorithms will generate instances in this ‘reduced’ form instead of a brute force approach.

5.3. Complete analysis of small graphs

In this subsection we present an algorithm to explore the spanning tree space. The algorithm proceed by exhaustively analyzing all the reduced graphs of a given number of nodes. The size of the space increases exponentially with respect to the number of nodes, so it has a major limitation: it can be used to analyze only small graphs. The main part is sketched in Algorithm\(^\text{[4]}\).

The details of the algorithm are the following:

- The input parameter \( n \) is the number of nodes of the graphs to explore
- \( \text{GenerateAllTrees}(n-1) \) is a function that returns the list of all trees of \( n-1 \) nodes
- \( \text{GenerateGraph}(w, T') \) is a function that builds a graph \( G \). Based on the tree \( T' \), it adds a new node (v) that will play the role of the central node of a star spanning tree, then adds the edge (v, w) to define our candidate tree counterexample \( T \). Finally adds all the other edges that link v to the rest of the nodes to obtain \( G \). It returns the graph \( G \) and \( (\Delta) \) the set of “possible” non-interbranch cycle edges.
- \( \text{IntersectionNumber}(\phi, G) \) is a function that calculates the intersection number of \( T \) in \( G \cup \phi \), where \( \phi \subset \Delta \) is a subset of supplementary edges of \( G \).
we used the package \textit{nauty} in the experiments. The column of the intersection conjecture. Table 1 shows the size of the graph family, i.e.: $|V|$. The column \textit{instances} is the number of instances processed.

### 5.4. Random sampling of large graphs

In this section we present another algorithm to explore the spanning tree space. The strategy in this case is to sample reduced graphs of a given number of nodes. The main part is sketched in Algorithm 2.

The details of the algorithm are the following:

- The input parameters are: $n$ the number of nodes of the graphs and $k$ the size of the sample
- $\text{GenerateRandomTree}(n)$ is a function that returns a random tree $T$ of $n$ nodes, where the node $v$ that will play the role of center of the star has degree 1 restricted to $T$.
- $\text{GenerateGraph}(T)$ is a function that builds a reduced graph $G$. Based on the tree $T$, adds all the edges that link $v$ to the rest of the nodes to obtain $G$. It returns the graph $G$ and $(\Delta)$ a random set of non-interbranch cycle edges.

- $\text{IntersectionNumber}(\phi, G)$ same as algorithm \ref{StarIntersection}
- $\text{StarIntersectionFormula}(\phi, G)$ same as algorithm \ref{StarIntersection}
- The algorithm finds a counterexample of the conjecture if: $\text{IntersectionNumber}(\phi, G) < \text{StarIntersectionFormula}(\phi, G)$

We used a uniformly distributed random number generator. To generate trees we used a simple algorithm that randomly connects a new node to an already connected tree. The non-interbranch cycle-edge set is built by associating a Bernoulli trial to each such possible edge. To achieve some diversity for each tree we built three different sets to obtain a sparse, medium and dense sets based on corresponding probabilities \(0.1, 0.5, 0.9\).

The proposed algorithm did not find a counterexample of the intersection conjecture. Table 2 shows the size of the experiments. The column \textit{nodes} is the number of nodes of the graph family, i.e: $|V|$. The column \textit{instances} is the number of instances processed.

### 6. Conclusion

In this article we introduced the Minimum Spanning Tree Cycle Intersection (MSTCI) problem.

We proved by enumerative arguments that the star spanning trees are the unique solutions of the problem in the context of complete graphs.

We conjectured a generalization to the case of graphs (not necessarily complete) that admit a star spanning tree. In this sense we showed that the star spanning tree is a local minimum in the domain of the spanning tree graph. We deduced a closed formula for the tree intersection number of star spanning trees in this setting. We proposed two ideas to attempt to find a counterexample of the conjecture. Those ideas were the basis of two strategies to programatically explore the space of solutions in the

| Table 1. SMALL instances results |
|---|
| nodes | instances (approx.) |
| 4 | 5 |
| 5 | 33 |
| 6 | 251 |
| 7 | 4200 |
| 8 | 125000 |
| 9 | 790000 |

| Table 2. RANDOM instances results |
|---|
| nodes | instances |
| 25 | 300000 |
| 50 | 300000 |
| 100 | 300000 |
| 200 | 150000 |
| 400 | 300 |

\begin{algorithm}
\begin{algorithmic}
\STATE $\mathcal{T} \leftarrow \text{GenerateAllTrees}(n-1)$
\FOR {each tree $T' \in \mathcal{T}$}
\FOR {each node $w \in T'$}
\STATE $G, \Delta \leftarrow \text{GenerateGraph}(w, T')$
\FOR {each subset $\phi \subset \Delta$}
\STATE $\text{check} (\text{IntersectionNumber}(\phi, G) < \text{StarIntersectionFormula}(\phi, G))$
\ENDFOR
\ENDFOR
\ENDFOR
\end{algorithmic}
\end{algorithm}

\begin{algorithm}
\begin{algorithmic}
\FOR {$i := 1..k$}
\STATE $T \leftarrow \text{GenerateRandomTree}(n)$
\STATE $G, \Delta \leftarrow \text{GenerateRandomGraph}(T)$
\STATE $\text{check} (\text{IntersectionNumber}(\phi, G) < \text{StarIntersectionFormula}(\phi, G))$
\ENDFOR
\end{algorithmic}
\end{algorithm}

\textit{Algorithm 1} CounterexampleSearch($n$)

\textit{Algorithm 2} CounterexampleRandomSearch($n, k$)

Note that the analyzed graphs are reduced in the sense explained previously. The cycle-edges are non-interbranch by construction and $d_T(v) = 1$ since $v$ is only connected to $w$ in $T$ (ie. there is a single principal subtree). As the algorithm iterates over all possible spanning subtrees $T'$ and all the combinations of possible non-interbranch cycle-edges, every instance is guaranteed to be explored at least once.

- \textit{StarIntersectionFormula}(\phi, G) is a function that calculates the intersection number of the star spanning tree in $G \cup \phi$
- The algorithm finds a counterexample of the conjecture if: $\text{IntersectionNumber}(\phi, G) < \text{StarIntersectionFormula}(\phi, G)$

To generate all non-isomorphic trees of $|V| - 1$ nodes we used the package \textit{nauty} \cite{McKay and Piperno, 2014}.

The proposed algorithm did not find a counterexample of the intersection conjecture. Table 1 shows the size of the experiments. The column \textit{nodes} is the number of nodes of the graph family, i.e: $|V|$. The column \textit{instances} is the number of instances processed.

\begin{algorithm}
\begin{algorithmic}
\STATE $\mathcal{T} \leftarrow \text{GenerateAllTrees}(n-1)$
\FOR {each tree $T' \in \mathcal{T}$}
\FOR {each node $w \in T'$}
\STATE $G, \Delta \leftarrow \text{GenerateGraph}(w, T')$
\FOR {each subset $\phi \subset \Delta$}
\STATE $\text{check} (\text{IntersectionNumber}(\phi, G) < \text{StarIntersectionFormula}(\phi, G))$
\ENDFOR
\ENDFOR
\ENDFOR
\end{algorithmic}
\end{algorithm}
pursue of a counterexample. The negative result of the experiments suggest that the conjecture is well posed. Unlike the complete graph context, in this slightly more general case, star spanning trees are not unique; there are other spanning trees $T$ such that $\cap(T_s) = \cap(T)$.

We proved a general result that shows that spanning trees that solve the MSTCI problem don’t depend on some intrinsic property but on their particular embedding in the ambient graph.

An interesting direction of research is to consider the MSTCI problem for other families of graphs, i.e.: graphs that do not admit a star spanning tree. Of particular interest is the class of triangular meshes, i.e.: graphs that model the immersion of compact surfaces in the 3D euclidean space.

Another interesting direction of research is related to proving to which complexity class the MSTCI problem belongs to. In case of belonging to the NP-hard class, it will be important to find approximate, probabilistic and heuristic algorithms.

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