Tropical limit of log-inflection points for planar curves

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Abstract. This paper describes the behaviour of log-inflection points (that is, points of inflection with respect to the parallelization of \((\mathbb{C}^\times)^2\) given by the multiplicative group law) of curves in \((\mathbb{C}^\times)^2\) under passage to the tropical limit. Assuming that the limiting tropical curve is smooth, we show that log-inflection points accumulate by pairs at the midpoints of bounded edges of it.

Bibliography: 11 titles.

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§ 1. Introduction

Denote by \(\mathbb{C}^\times = \mathbb{C} \setminus \{0\}\) the complex torus and let \(f: (\mathbb{C}^\times)^2 \to \mathbb{C}\) be a Laurent polynomial in two variables with the Newton polygon \(\Delta\). The zero locus

\[ V_f = \{(z, w) \in (\mathbb{C}^\times)^2 \mid f(z, w) = 0\} \subset (\mathbb{C}^\times)^2 \]

is a noncompact complex curve. For nonsingular \(V_f\) the logarithmic Gauss map (see [5])

\[ \gamma_f: V_f \to \mathbb{C}P^1 \]

is defined by \(\gamma_f(z, w) = (z \frac{\partial f}{\partial z} : w \frac{\partial f}{\partial w})\). This map depends only on \(V_f\) and not on \(f\) itself. It is the map that takes a point of \(V_f\) to the slope of its tangent plane with respect to the parallelization of \((\mathbb{C}^\times)^2\) given by the multiplicative translations \((z, w) \mapsto (\alpha z, \beta w), \alpha, \beta \in \mathbb{C}^\times\).

Definition 1. Critical points of \(\gamma_f\) are called log-inflection points of \(V_f\).

Denote the set of critical points by \(\rho V_f \subset V_f\). If all coefficients of the Laurent polynomial \(f\) are real we define

\[ \mathbb{R}V_f = \{(z, w) \in (\mathbb{R}^\times)^2 \mid f(z, w) = 0\} \subset (\mathbb{R}^\times)^2. \]

We have

\[ \gamma_f|_{\mathbb{R}V_f}: \mathbb{R}V_f \to \mathbb{R}P^1, \]

and we denote by \(\rho \mathbb{R}V_f \subset \mathbb{R}V_f\) the set of critical points of \(\gamma_f|_{\mathbb{R}V_f}\) (called real log-inflection points).

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Recall the definition of tropical limits (see, for instance, [3], [1], as well as [10]). Let
\[ f_t = \sum_{j,k \in \mathbb{Z}} a_{jk,t} z^j w^k \]
be a family of Laurent polynomials parametrized by an (infinite) set \( A \). Suppose that the degree of \( f_t \) is universally bounded in \( z, z^{-1}, w \), and \( w^{-1} \). Let \( \alpha: A \to \mathbb{R} \) be a map whose image is unbounded above. The pair \((f_t, \alpha)\) is called a scaled sequence of polynomials \( f_t \) (\( \alpha \) is the scaling).

The scaled sequence of polynomials is said to converge tropically if the limit
\[ a_{jk}^{trop} = \lim_{t \in A} \log_{\alpha(t)} |a_{jk,t}| \in \mathbb{R} \cup \{-\infty\} \]
exists for all \( j, k \in \mathbb{Z} \). Here the limit is taken with respect to the directed set \( A \), where the order is given by \( \alpha \). Define the limiting Newton polygon
\[ \Delta = \text{ConvexHull}\{(j, k) \in \mathbb{Z}^2 | a_{jk}^{trop} \neq -\infty\} \subset \mathbb{R}^2. \]
It is a bounded convex lattice polygon. The set \( \mathbb{R} \cup \{-\infty\} \) is called the set of tropical numbers and is denoted by \( \mathbb{T} \). Denote by \( \text{Log}: (\mathbb{C}^\times)^2 \to \mathbb{R}^2 \) the map defined by
\[ \text{Log}(z, w) = (\log |z|, \log |w|). \]
If a scaled sequence of polynomials \((f_t, \alpha)\) converges tropically, then the limit
\[ C = \lim_{t \in A} \text{Log}_{\alpha(t)}(V_t) \]
exists in the sense of Hausdorff, that is, as a limit of closed subsets of \( \mathbb{R}^2 \) with the topology given by the Hausdorff distance. It is called the tropical limit of \( \{V_t \subset (\mathbb{C}^\times)^2\}_{t \in A} \).

More generally, given a scaled sequence \((V_t, \alpha)\) of curves \( V_t \subset (\mathbb{C}^\times)^n \) we say that it converges tropically to \( C \subset \mathbb{R}^n \) if \( C \) is the limit of \( \text{Log}_{\alpha(t)}(V_t) \) in the Hausdorff sense.

It can be proved (see [9], and see [3] for a more general statement) that the limit \( C \) is a rectilinear balanced graph in \( \mathbb{R}^n \). The edges \( E \) of \( C \) are straight intervals with rational slopes, and can be prescribed integer weights \( w(E) \) (coming from the \( V_t \) for large \( \alpha(t) \)) so that the balancing condition
\[ \sum_{E \ni v} w(E)u(E) = 0 \]
holds for every vertex \( v \in C \), where \( u(E) \in \mathbb{Z}^n \) is the primitive vector parallel to \( E \) in the direction away from \( v \).

A rectilinear graph \( C \subset \mathbb{R}^n \) with these properties is called a tropical curve. If \( n = 2 \) then \( C \) defines a lattice subdivision of the Newton polygon \( \Delta \). If each polygon of the subdivision is a triangle of area \( 1/2 \) (the minimal possible area for a lattice polygon) then \( C \) is called smooth. For details we refer to [4] and [1].

**Definition 2.** Let \( C \) be a smooth plane tropical curve. The tropical parabolic locus of \( C \) is the set \( \rho C \subset C \) formed by the midpoints of all bounded edges of \( C \).
The main result of this paper is the following.

**Theorem 3.** Let $C \subset \mathbb{R}^2$ be a smooth tropical curve, and $\{V_t \subset (\mathbb{C}^*)^2\}_{t \in A}$ be a family of complex curves parametrized by a scaling sequence $\alpha: A \to \mathbb{R}$. Suppose that $V_t$ converges tropically to $C$ and that the Newton polygon $\Delta$ for $C$ coincides with the Newton polygons of $V_t$ for large $\alpha(t)$. Then $\text{Log}_{\alpha(t)}(\rho V_t)$ converges to $\rho C$ in the Hausdorff metric as $\alpha(t) \to +\infty$. Furthermore, a small neighbourhood of an arbitrary point in $\rho C$ contains exactly two points of $\text{Log}_{\alpha(t)}(\rho V_t)$ for large $\alpha(t)$.

In other words, $2\rho C$ (that is, the tropical parabolic locus $\rho C$ taken with multiplicity two) is the tropical limit of $\rho V_t$.

**Remark 4.** Theorem 3 provides a 2–1 covering map $\Phi_t: \rho V_t \to \rho C$ for large $\alpha(t)$. If the family $\{V_t\}_{t \in A}$ is defined over $\mathbb{R}$, then $\rho V_t$ is invariant under the involution $\text{conj}$ of complex conjugation in $(\mathbb{C}^*)^2$. In particular, the pair $\Phi_t^{-1}(p)$ is $\text{conj}$-invariant for each $p \in \rho C$.

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§ 2. The case of real curves

Let $\{V_t \subset (\mathbb{C}^*)^2\}_{t \in A}$ be a family of real curves (that is, curves invariant with respect to the involution of complex conjugation) tropically convergent to a smooth tropical curve $C \subset \mathbb{R}^2$ with respect to a scaling $\alpha: A \to \mathbb{R}$. Assume that the Newton polygon for $C$ coincides with the Newton polygons of $V_t$ for large $\alpha(t)$.

Let $E$ be a bounded edge of $C$. Denote by $p \in E$ the midpoint of $E$, and take $t \in A$ with $\alpha(t)$ sufficiently large so that the 2–1 covering map $\Phi_t: \rho V_t \to \rho C$ is defined.

**Definition 5.** The edge $E$ is called $V_t$-twisted if $\text{conj}$ preserves $\Phi_t^{-1}(p)$ pointwise. Otherwise $\text{conj}$ interchanges the points in $\Phi_t^{-1}(p)$, and we call $E$ $V_t$-untwisted.

Clearly $E$ is $V_t$-twisted if $\Phi_t^{-1}(p) \subset \mathbb{R}V_t$, and $V_t$-untwisted if $\Phi_t^{-1}(p)$ forms a complex-conjugated pair of points.

**Definition 6** (cf. [1]). A subset $T$ of the set of bounded edges of a smooth tropical curve $C \subset \mathbb{R}^2$ is called twist-admissible if it satisfies the following condition: for any cycle $\gamma$ of $C$ (considered as a graph), with edges $E_1, \ldots, E_k$ from $T$ and maybe some edges not from $T$,

$$\sum_{i=1}^{k} u(E_i) = (0, 0) \mod 2, \quad (2)$$

where $u(E_i) \in \mathbb{Z}^2$ is the primitive integer vector in the direction of $E_i$.

Recall that a patchworking polynomial may be defined by

$$f_t(z_1, z_2) = \sum_{j \in B} \beta_j(t)(\alpha(t))^{a_j}z^j.$$
Here $j = (j_1, j_2)$, $z^j = (z_1^{j_1}, z_2^{j_2})$, $j \mapsto a_j$ is a strictly convex function from a finite subset $B \subset \mathbb{Z}^2$ to $\mathbb{R}$, and $\{\beta_j(t)\}_{t \in A} \subset \mathbb{R}$ is a family converging (with respect to the directed set $A$) to a nonzero real number. Denote

$$\sigma_j = \text{sign}\left(\lim_{t \to A} \beta_j(t)\right) \in \{-1, +1\}.$$ 

The family $(V_t)_{t \in A}$ converges tropically to a tropical curve $C$ defined by the tropical polynomial with the coefficients $a_j$ (see [4], for example). According to Viro’s patchworking theorem (see [2] and [11]), if the tropical curve $C$ is smooth then the rigid isotopy class of the curve $V_t = \{(x, y) \in (\mathbb{R}^\times)^2 \mid f_t(x, y) = 0\}$ for large values of $\alpha(t)$ is determined by $C$ together with the signs $\sigma_j$ (for example, see [1]).

**Proposition 7.** Let $V_t$ be a family of real curves given by a patchworking polynomial as above, and let $C$ be its tropical limit. Assume that $C$ is smooth. For each bounded edge $E$ of $C$, denote by $v_1^E$ and $v_2^E$ the two vertices of the segment $\Delta_E$ dual to $E$ in the subdivision of $\Delta$ defined by $C$. The segment $\Delta_E$ is adjacent to two other triangles of the subdivision of $\Delta$. Denote by $v_3^E$ and $v_4^E$ vertices of these two triangles different from $v_1^E$ and $v_2^E$. Then $E$ is $V_t$-twisted if and only if

- $\sigma_{v_1^E} \sigma_{v_2^E} \sigma_{v_3^E} \sigma_{v_4^E} > 0$ provided that the coordinates modulo 2 of $v_3^E$ and $v_4^E$ are distinct;
- $\sigma_{v_1^E} \sigma_{v_2^E} < 0$ provided that the coordinates modulo 2 of $v_3^E$ and $v_4^E$ are the same.

The following corollary follows easily from Proposition 7 and known facts about Viro’s patchworking (see [1]).

**Corollary 8.** The set of $V_t$-twisted edges of $C$ is twist-admissible. Conversely, for any twist-admissible subset $T$ of bounded edges of a smooth tropical curve $C$ there exists a family $\{V_t \subset (\mathbb{C}^\times)^2\}_{t \in A}$ of real curves such that $T$ is the set of $V_t$-twisted edges.

§ 3. **Tropical limit of the logarithmic Gauss map**

Let $\{V_t\}_{t \in A} = \{f_t = 0\}_{t \in A} \subset (\mathbb{C}^\times)^2$ be a family of complex curves parametrized by the scaled sequence $\alpha : A \to \mathbb{R}$. Let $\widetilde{V}_t$, $t \in A$, be the graph of the logarithmic Gauss map $\gamma_{f_t}$ (which we denote by $\gamma_t$ for brevity) given by

$$\widetilde{V}_t = \{(z_t, \gamma_t(z_t)) \mid z_t \in V_t\} \subset (\mathbb{C}^\times)^2 \times \mathbb{CP}^1.$$

Denote by $\pi_1 : \widetilde{V}_t \to V_t \subset (\mathbb{C}^\times)^2$ the projection onto the first factor and by $\pi_2 : \widetilde{V}_t \to \mathbb{CP}^1$ the projection onto the second factor. The map $\pi_1$ is an isomorphism and the map $\pi_2$ is of degree $2 \text{Area}(\Delta)$ by the Bernstein-Kushnirenko formula (see [6] and also [7], Lemma 2).

Denote by $\mathbb{TP}^n$ the tropical projective space of dimension $n$ (see [1], for example). Denote by $\pi_1^{\text{trop}} : \mathbb{R}^2 \times \mathbb{TP}^1 \to \mathbb{R}^2$ the projection onto the first factor and by $\pi_2^{\text{trop}} : \mathbb{R}^2 \times \mathbb{TP}^1 \to \mathbb{TP}^1$ the projection onto the second factor.

**Proposition 9.** There exists a subfamily $A' \subset A$ with unbounded $\alpha(A') \subset \mathbb{R}$ such that the subsequence $(\widetilde{V}_t)_{t \in A'}$ converges tropically to a tropical curve $\widetilde{C}$ in $\mathbb{R}^2 \times \mathbb{TP}^1$. Furthermore, $\pi_1^{\text{trop}}(\widetilde{C}) = C$. 
Proof. The proposition is a special case of Theorem 39 in [3]. We apply this theorem to the projective curves $V_t, t \in A$, obtained as the closures of $\tilde{V}_t \cap (\mathbb{C}^\times)^3$ in $\mathbb{CP}^3$. We set $\tilde{C}$ to be the closure in $\mathbb{R}^2 \times TP^1$ of $\bar{C} \cap \mathbb{R}^3$, where $\bar{C} \subset TP^3$ is the tropical limit of the subfamily from Theorem 39 in [3]. By the uniqueness of tropical limits we have $C = \pi_1^{trop}(\tilde{C})$.

Definition 10. We refer to the map $\pi_2^{trop}|_{\bar{C}} : \tilde{C} \to TP^1$ as the tropical limit of the family $\gamma_t$; we call it the tropical Gauss map.

Remark 11. This construction extends to the case of hypersurfaces of higher dimensions. However in this text, we will focus on the case of curves.

Remark 12. A priori the tropical curve $\tilde{C}$ does not depend on $C$ alone but also on an approximation $(V_t)_{t \in A}$ of $C$, and even on the choice of the subfamily $A' \subset A$. Nevertheless, the tropical limit of the ramification locus of $\gamma_t$ is a well-defined subset of $C$ and does not depend on these choices as shown in this paper.

We now define the degree of the projection $\pi : \Gamma \subset \mathbb{R}^n \mapsto \mathbb{R}^k$, where $\Gamma$ is a tropical curve, and where we identify $\mathbb{R}^n$ with $\mathbb{R}^k \times \mathbb{R}^{n-k}$. If the image $\pi(\Gamma)$ is a point, we say that $\pi$ is of degree 0. Assume that $\pi(\Gamma)$ does not reduce to a point. If $x \in \pi(\Gamma)$ is a generic point, then the fibre $\pi^{-1}(x)$ intersects $\Gamma$ transversally (meaning that any point of intersection lies in the relative interior of an edge of $\Gamma$) at finitely many points. Denote by $e$ the edge of $\pi(\Gamma)$ containing $x$ and denote by $Z_e$ the sublattice of $\mathbb{Z}^k \times \mathbb{Z}^{n-k}$ generated by $0 \times \mathbb{Z}^{n-k}$ and $(\vec{v}, 0)$, where $\vec{v}$ is the primitive vector in the direction of $e$. Let $z \in \Gamma \cap \pi^{-1}(x)$, and denote by $\bar{e}$ the edge of $\Gamma$ containing $z$. Define the local tropical intersection number $\nu_z(\Gamma, \pi^{-1}(x))$ as the index of the sublattice in $Z_e$ generated by $\mathbb{Z}^{n-k}$ and a primitive vector in the direction of $\bar{e}$ (multiplied by the weight of $\bar{e}$). The degree of $\pi$ is defined as the sum

$$\deg(\pi) := \sum_{z \in \pi^{-1}(x) \cap \Gamma} \nu_z(\Gamma, \pi^{-1}(x)).$$

It follows from the balancing condition that the degree is independent of the point $x$ (see also Lemma 40 in [3]).

The following proposition follows directly from the proof of Proposition 42 in [3] (because before taking the tropical limit the degree of $\pi_1$ is 1 and the degree of $\pi_2$ is $2 \text{Area}(\Delta)$).

Proposition 13. One has $\deg(\pi_1^{trop}) = 1$ and $\deg(\pi_2^{trop}) = 2 \text{Area}(\Delta)$.

One can define similarly the local degree of $\pi$. Let $s \in \Gamma_1$ and let $U_s$ be a small neighbourhood of $s$ only containing points on edges adjacent to $s$. If $\pi$ maps the neighbourhood $U_s$ to a point, we say that $\pi$ is of local degree 0 at $s$. If not, let $x \in \pi(\Gamma) \cap \pi(U_s)$ be a generic point. Define the local degree of $\pi$ at $s$ as the sum of the local tropical indices of intersection $\nu_z(\Gamma, \pi^{-1}(x))$ over all $z \in U_s$.

§ 4. First properties of the tropical Gauss map

Proposition 14. Let $\{V_t \subset (\mathbb{C}^\times)^2\}_{t \in A}$ be a family of complex curves parameterized by the scaling sequence $\alpha : A \to \mathbb{R}$. Assume that $\{V_t \subset (\mathbb{C}^\times)^2\}_{t \in A}$ converges
tropically to a smooth plane tropical curve $C$. Let $e$ be an edge of $C$ of slope $p \in \mathbb{Q}P^1$, and let $x$ belong to the relative interior of $e$. If a scaled sequence $x_t \in V_t$ satisfies

$$\lim_{t \in A} \log_{\alpha(t)}(x_t) = x,$$

then

$$\lim_{t \in A} \gamma_t(x_t) = p.$$

**Proof.** Denote by $\{f_t = \sum_{j,k} a_{j,k,t} z^j w^k \}_t \in A$ a family of polynomials defining the family of complex curves $(V_t)_{t \in A}$. Consider the tropical curve $C_x$ obtained by translating the tropical curve $C$ along the vector $-x \in \mathbb{R}^2$. Denote by $\{f_{t,x} \}_{t \in A}$ the family of polynomials given by

$$f_{t,x}(z, w) = \alpha(t)^{-b} f_t(\alpha(t)x_1 z, \alpha(t)x_2 w) = \alpha(t)^{-b} \sum a_{j,k,t} z^j w^k \alpha(t)^{b_{j,k}},$$

where $x = (x_1, x_2)$ and $b_{j,k} = jx_1 + kx_2$, and

$$b = b_{j_1,k_1} + a^{\text{trop}}_{j_1,k_1} = b_{j_2,k_2} + a^{\text{trop}}_{j_2,k_2},$$

where the $a^{\text{trop}}_{j,k}$ are the corresponding coefficients of the tropical curve $C$ and $(j_1, k_1)$, and $(j_2, k_2)$ are the integer points on the edge (of the subdivision of $\Delta$) dual to the edge $e$.

The family of complex curves $\{V_{t,x} \}_{t \in A} = \{f_{t,x} = 0 \}_{t \in A}$ converges tropically to $C_x$. Furthermore, denoting by $\Phi_x$ the isomorphism

$$V_{t,x} \xrightarrow{\Phi_x} V_t,$$

$$(z, w) \mapsto (\alpha(t)x_1 z, \alpha(t)x_2 w),$$

one has the following commutative diagram

$$\begin{array}{ccc}
V_{t,x} & \xrightarrow{\Phi_x} & V_t \\
\downarrow{\gamma_{t,x}} & & \downarrow{\gamma_t} \\
\mathbb{CP}^1 & \xrightarrow{\text{Id}} & \mathbb{CP}^1
\end{array}$$

It is then enough to prove the proposition for the point $(0, 0) \in C_x$ and the family $V_{t,x}$. Since $b > b_{j,k} + a^{\text{trop}}_{j,k}$ for all $(j, k) \in (\Delta \cap \mathbb{Z}^2) \setminus \{(j_1, k_1), (j_2, k_2)\}$, there exist real numbers $c_{j,k}$ satisfying $c_{j_1,k_1} = c_{j_2,k_2} = 0$ and $c_{j,k} > 0$ otherwise such that

$$f_{t,x}(z, w) = \sum_{(j,k) \in \Delta \cap \mathbb{Z}^2} a_{j,k,t} \alpha(t)^{-a^{\text{trop}}_{j,k} - c_{j,k}} z^j w^k.$$

Since the tropical curve $C$ is smooth, there exists a $\mathbb{Z}$-affine automorphism $G$ of the plane such that $G(j_1, k_1) = (0, 0)$ and $G(j_2, k_2) = (1, 0)$. Write $G = M + (v_1, v_2)$, where $M \in \text{GL}_2(\mathbb{Z})$ and $(v_1, v_2) \in \mathbb{Z}^2$. The map $M$ gives rise to a multiplicative automorphism $\Phi_M$ of $(\mathbb{C}^*)^2$. Define

$$g_t(z, w) = z^{v_1} w^{v_2} f_{t,x}(\Phi_M(z, w))$$

and

$$W_t = \{g_t = 0\}.$$
One has $\Phi_M(W_t) = V_{t,x}$ and the family of complex curves $(W_t)_{t \in A}$ converges tropically to the tropical curve $D$ which is the image of $C$ under $M^{-1}$. The edge of slope $p$ containing $(0,0)$ is then mapped to an edge of slope $\infty$. Furthermore, one has the following commutative diagram:

$$
\begin{array}{ccc}
W_t & \xrightarrow{\Phi_M} & V_{t,x} \\
\gamma_{gt} \downarrow & & \downarrow \gamma_{t,x} \\
\mathbb{CP}^1 & \overset{M^{-1}}{\longrightarrow} & \mathbb{CP}^1
\end{array}
$$

One deduces that it is enough to prove the statement for the point $x = (0,0)$ on the vertical edge $e$. One has

$$g_t(z, w) = a_{j_1k_1,t}x(t)^{-a_{j_1k_1}} + a_{j_2k_2,t}x(t)^{-a_{j_2k_2}} + R_t(z, w),$$

where $R_t(z, w)$ is a Laurent polynomial such that its coefficients $r_{jk,t}$ satisfy $r_{jk,t} = O(\alpha(t)^c)$, where $c < 0$ is some real number.

We divide $g_t$ by $a_{j_1k_1,t}x(t)^{-a_{j_1k_1}}$ and make the change of variables

$$z \mapsto a_{j_2k_2,t}^{-1}x(t)^{a_{j_2k_2}}z.$$

This does not affect the limiting tropical curve since the tropical limit of $a_{j_2k_2,t}x(t)^{-a_{j_2k_2}}$ is zero. We end with a polynomial $h_t$ of the form

$$h_t(z, w) = 1 + z + S_t(z, w),$$

where, as above, $S_t(z, w)$ is a Laurent polynomial such that the coefficients $s_{jk,t}$ satisfy $s_{jk,t} = O(\alpha(t)^c)$. Then

$$\gamma_{h_t}(z_t, w_t) = \left[ z_t \left( 1 + \frac{\partial S_t}{\partial z} \right); w_t \frac{\partial S_t}{\partial w} \right].$$

Let $x_t = (z_t, w_t) \in \{ h_t = 0 \}$ be points such that $\lim_{t \to +\infty} \log_t(x_t) = (0,0)$. Since $z_t = -1 - S_t(z_t, w_t)$, we obtain $\lim_{t \to +\infty} z_t = -1$. One also has

$$\lim_{t \to +\infty} \left( z_t \frac{\partial S_t}{\partial z} \right) = \lim_{t \to +\infty} \left( w_t \frac{\partial S_t}{\partial w} \right) = 0,$$

which proves the proposition.

Taking the logarithm with base $\alpha(t)$ we obtain the following corollary, where $1_T = 0$ is the neutral element for tropical multiplication.

**Corollary 15.** Let $\{ V_t \subseteq (\mathbb{C}^\times)^2 \}_{t \in A}$ be a family of complex curves parametrized by the scaled sequence $\alpha: A \to \mathbb{R}$ which converges tropically to a smooth plane tropical curve $C$. Let $e$ be an edge of $C$ of slope $p \in \mathbb{Q}P^1$, $x$ be a point of the relative interior of $e$, and $x_t \in V_t$ be a sequence such that $\lim_{t \in A} \log_{\alpha(t)}(x_t) = x$. Then, up to passing to a subsequence,

- if $p \neq 0$ or $\infty$, then $\lim_{t \in A} \log_{\alpha(t)}(\gamma_t(x_t)) = [1_T : 1_T] \in \mathbb{TP}^1$;
\begin{itemize}
\item if \( p = 0 \), then \( \lim_{t \in A} \log_{\alpha(t)}(\gamma_t(x_t)) = [a : 1T] \in \mathbb{TP}^1 \), for some \( a \in [-\infty, 0] \);
\item if \( p = \infty \), then \( \lim_{t \in A} \log_{\alpha(t)}(\gamma_t(x_t)) = [1T : a] \in \mathbb{TP}^1 \), for some \( a \in [-\infty, 0] \).
\end{itemize}

Since the tropical map \( \pi_{1}^{\text{trop}} \) is of degree 1, one has a continuous inclusion \( i_C : C \hookrightarrow \widetilde{C} \) such that \( \pi_{1}^{\text{trop}} \circ i_C = \text{id} \).

**Proposition 16.** We have \( \pi_{2}^{\text{trop}}(s) = [1T : 1T] \in \mathbb{TP}^1 \) for any \( s \in i_C(\text{Vert}(C)) \).

**Proof.** At least one of the edges adjacent to \( s \) must have slope different from 0 and \( \infty \). Then the proposition follows immediately from Corollary 15.

**Proposition 17.** For any \( s \in i_C(\text{Vert}(C)) \) the tropical map \( \pi_{2}^{\text{trop}} \) is of local degree 1 at \( s \).

**Proof.** Since the logarithmic Gauss map commutes with multiplicative automorphisms, one can assume that \( s = i_C(0,0) \). We perform a multiplicative automorphism \( \Phi_M \) as in the proof of Proposition 14 and denote by \( (W_t)_{t \in A} \) the family of complex curves such that \( \Phi_M(W_t) = V_t \). Denote by \( \widetilde{W}_t \) the graph of the logarithmic Gauss map on \( W_t \) and by \( \Phi := (\Phi_M, t^{-1}M^{-1}) \mapsto \widetilde{W}_t \) to \( \widetilde{V}_t \). Denote by \( \widetilde{D} \) the tropical limit (up to passing to a subsequence) of \( \widetilde{W}_t \) and set \( D = \pi_{1}^{\text{trop}}(\widetilde{D}) \) and \( s' = i_D(0,0) \). By definition, the map \( \Phi_M \) sends \( \log_{\alpha(t)}^{-1}(0,0) \subset W_t \) to \( \log_{\alpha(t)}^{-1}(0,0) \subset V_t \). Since \( \widetilde{W}_t \) (\( \widetilde{V}_t \) respectively) is a graph over \( W_t \) (\( V_t \)), respectively, there exist a small neighbourhood \( U_{s'} \) of \( s' \) and a small neighbourhood \( U_s \) of \( s \) such that \( \Phi(\log_{\alpha(t)}^{-1}(U_{s})) \subset \log_{\alpha(t)}^{-1}(U_s) \).

Now let \( y \in \pi_{2}^{\text{trop}}(U_s) \) be a generic point, and let \( (y_t)_{t \in A} \in \mathbb{CP}^1 \) be a sequence such that \( \lim_{t \in A} \log_{\alpha(t)}(y_t) = y \). It follows from Proposition 43 in [3] that the local degree of \( \pi_{2}^{\text{trop}} \) at \( s \) is equal to the index of intersection of \( \widetilde{V}_t \) and \( \pi_{2}^{-1}(y_t) \) in \( \log_{\alpha(t)}^{-1}(U_s) \), for sufficiently large \( \alpha(t) \). Applying the map \( \Phi \), one deduces that it is enough to prove the statement for the family of complex curves \( (W_t)_{t \in A} \) given by a family of polynomials

\[ h_t(z, w) = 1 + z + w + S_t(z, w), \]

where \( S_t(z, w) \) is a Laurent polynomial such that its coefficients \( s_{j,k,t} \) satisfy \( s_{j,k,t} = O(\alpha(t)^c) \). The graph of the logarithmic Gauss map over \( W_t \) is then given by

\[ \widetilde{W}_t = \left\{ z, w, \left[ z \left( 1 + \frac{\partial S_t}{\partial z} \right) : w \left( 1 + \frac{\partial S_t}{\partial w} \right) \right] \mid (z, w) \in W_t \right\}, \]

and the graph of the logarithmic Gauss map on the line \( \mathscr{L} = \{1 + z + w = 0\} \) is given by

\[ \mathscr{L} = \{z, w, [z : w] \mid (z, w) \in \mathscr{L} \}. \]

Denote by \( \tilde{L} \) the tropical limit of \( \mathscr{L} \). It follows from the description of \( \widetilde{W}_t \) and \( \mathscr{L} \) that there exists a small neighbourhood \( U \) of the point \( (0,0,0) \) in \( \mathbb{R}^3 \) such that \( \widetilde{D} \cap U = \tilde{L} \cap U \). But the map \( \pi_{2}^{\text{trop}} : \tilde{L} \to \mathbb{TP}^1 \) is of degree one, which proves the proposition.

**Example 18.** In the left-hand side of Figure 1, we draw the tropical limit (when \( t \to +\infty \)) of \( \widetilde{V}_t \), for \( V_t = \{1 + x + y + t^{-1}xy\} \). We draw also the image of this tropical limit under the first projection \( \pi_{1}^{\text{trop}} \). In the right-hand side of Figure 1 we do the same for \( V_t = \{1 + x + y + t^{-1}x^2\} \).
Figure 1. The tropical limit of $\tilde{V}_t$ for $V_t = \{1 + x + y + t^{-1}xy\}$ and for $V_t = \{1 + x + y + t^{-1}x^2\}$.

§ 5. Proofs of Theorem 3, Proposition 7 and Corollary 8

The tropical Gauss map $\pi_2^{trop}$ contracts a priori some edges of $i_2(C)$. It is then not immediately clear how to locate the tropical limit of the logarithmic inflection points on $C$. To overcome this difficulty, we embed $\tilde{C}$ and $\mathbb{TP}^1$ in higher-dimensional spaces so that the new tropical Gauss map does not contract any edges coming from $C$ (see Proposition 19). Then we prove that the tropical Gauss map is of local degree 2 at each point coming from a midpoint of $C$ (see Proposition 20).

Denote by $m_1, \ldots, m_l$ the midpoints of bounded edges of $C$, and by $x_1^t, \ldots, x_l^t$ some points in $V_t$ such that $\lim_{t \in A} \log_{\alpha(t)}(x_i^t) = m_i$, for $1 \leq i \leq l$. Put $\gamma_t(x_i^t) = [u_i^t : v_i^t]$, for $1 \leq i \leq l$, and let

$$
\Phi_t: \mathbb{CP}^1 \to \mathbb{CP}^{l+1},
$$

$$
[u : v] \mapsto [u : v : \varphi_1^t(u, v) : \ldots : \varphi_l^t(u, v)],
$$

where $\varphi_i^t(u, v) = uv_i^t - vu_i^t$ are linear maps.

Consider the line $\mathcal{L}_t = \Phi_t(\mathbb{CP}^1)$ and denote by $L$ its tropical limit in $\mathbb{TP}^{l+1}$ (after passing to a subsequence), which exists by the compactness theorem (see [3], § 3.4). Also set

$$
\tilde{V}_t = \{(z_t, \Phi_t \circ \gamma_t(z_t)) \mid z_t \in V_t\} \subset (\mathbb{C}^\times)^2 \times \mathbb{CP}^{l+1},
$$

and denote by $\tilde{C} \subset \mathbb{R}^2 \times \mathbb{TP}^{l+1}$ its tropical limit (after passing to a subsequence). One has isomorphisms $\tilde{V}_t \simeq \tilde{V}_t$ and $\mathcal{L}_t \simeq \mathbb{CP}^1$ given by projections. The projection onto the second factor $\mathbb{R}^2 \times \mathbb{TP}^{l+1} \to \mathbb{TP}^{l+1}$ restricts to a projection $\pi^{trop}: \tilde{C} \to L$ of degree 2 $\text{Area}(\Delta)$, the projection $\mathbb{R}^2 \times \mathbb{TP}^{l+1} \to \mathbb{R}^2 \times \mathbb{TP}^1$ restricts to a projection $p: \tilde{C} \to \tilde{C}$ of degree 1 and the projection $\mathbb{TP}^{l+1} \to \mathbb{TP}^1$ restricts to a projection
\( q : L \to \mathbb{TP}^1 \) of degree 1. Furthermore, the following diagram commutes:

\[
\begin{array}{ccc}
\hat{C} & \xrightarrow{\pi_{1}\text{trop}} & L \\
p & \downarrow & q \\
\hat{C} & \xrightarrow{\pi_2\text{trop}} & \mathbb{TP}^1 \\
& \downarrow & \pi_{1}\text{trop} \\
& & C
\end{array}
\]

Denote by \( i_p : \hat{C} \to \hat{C} \) the continuous inclusion that is right-inverse to the projection \( \pi_{1}\text{trop} \circ p \), and denote by \( s_L \) the image in \( L \) of \([1_T : 1_T] \in \mathbb{TP}^1\) by the continuous inclusion \( \mathbb{TP}^1 \to L \) that is right-inverse to \( q \).

It follows from Proposition 17 that for any \( s \in \text{Vert}(C) \) one has \( \pi_{1}\text{trop}(i_p(s)) = s_L \). In fact, if \( \pi_{1}\text{trop}(i_p(s)) \neq s_L \), a small neighbourhood of \( i_C(s) \) would be contracted to \([1_T : 1_T] \) by \( \pi_2\text{trop} \), contradicting Proposition 17. Moreover, since \( \pi_{2}\text{trop} \) is locally of degree 1 at \( i_C(s) \) and since the projections \( p \) and \( q \) are both of degree 1, we deduce that the tropical map \( \pi_{1}\text{trop} \) is locally of degree 1 at \( i_p(s) \).

**Proposition 19.** Let \( s \) be a vertex of \( C \) and let \( \hat{e} \) be a bounded edge of \( i_p(C) \) adjacent to \( i_p(s) \). Then \( \hat{e} \) is not contracted by \( \pi_{1}\text{trop} \).

**Proof.** Assume that \( \hat{e} \) is contracted by \( \pi_{1}\text{trop} \), and denote by \( e \) the edge of \( C \) containing \( \pi_{1}\text{trop} \circ p(\hat{e}) \). First we prove that \( i_p(e) = \hat{e} \).

Assume that \( \hat{e} \nsubseteq i_p(e) \). Then \( \hat{e} \) is adjacent to a vertex \( s' \in \text{Vert}(\hat{C}) \setminus i_p(\text{Vert}(C)) \).

Since \( \hat{e} \) is contracted by \( \pi_{1}\text{trop} \) by assumption, one has \( \pi_{1}\text{trop}(s') = \pi_{1}\text{trop}(s) = s_L \). Since \( s' \) is a vertex, there exists an edge incident to \( s' \) but not belonging to \( i_p(\text{Edge}(C)) \). Such an edge is contracted by \( \pi_{1}\text{trop} \circ p \) and so is not contracted by \( \pi_{1}\text{trop} \). Then the local degree of \( \pi_{1}\text{trop} \) at \( s' \) is at least 1. Since every vertex in \( i_p(\text{Vert}(C)) \) already contributes 1 to the degree of \( \pi_{1}\text{trop} \) and since \#(\text{Vert}(C)) = 2\text{Area}(\Delta) \), we conclude that the degree of \( \pi_{1}\text{trop} \) is at least \( 2\text{Area}(\Delta) + 1 \), which is impossible. Thus one has \( i_p(e) = \hat{e} \).

It follows from the construction of \( \hat{C} \) that for any bounded edge \( e_C \) of \( C \) the set \( i_p(e_C) \) consists of at least two edges (if \( m_C \) is the midpoint of \( e_C \), there is a point of \( \hat{C} \) in the boundary divisor of \( \mathbb{TP}^{d+1} \) projecting to \( m_C \) ). Then \( i_p(e_C) \neq \hat{e} \) and \( \hat{e} \) cannot be contracted by \( \pi_{1}\text{trop} \).

The proposition is proved.

We call a tropical map \( f \) **finite** over a subset \( F \) of the target domain if each point in \( F \) has a finite preimage.

**Proposition 20.** Let \( e \) be a bounded edge of \( C \), and let \( m \) be the midpoint of \( e \). The map \( \pi_{1}\text{trop} \) is finite over \( i_p(e) \) and is of local degree 1 at any point of \( i_p(e) \setminus \{i_p(m)\} \). Its local degree at \( i_p(m) \) is 2.

**Proof.** Denote by \( T \) the image of \( i_p(e) \) by the map \( \pi_{1}\text{trop} \). The set \( T \) is a tree since \( T \subseteq L \). We show first that \( s_L \) is a vertex of valence 1 of \( T \). Assume that the valence of \( s_L \) is at least 2 and denote by \( a_1 \) and \( a_2 \) two edges of \( T \) adjacent to \( s_L \). Since
T is a tree, the sets $a_1 \setminus s_L$ and $a_2 \setminus s_L$ lie in two different connected components of $T \setminus s_L$. Since $i_p(e)$ is connected, this means that there are at least 3 vertices of $i_p(e)$ mapped to $s_L$. As in the proof of Proposition 19 this gives a contradiction with the degree of $\pi_{\text{trop}}$.

Denote by $e_L$ the edge of $T$ adjacent to $s_L$. Assume first that the edges $e^1_1$ and $e^2_1$ of $i_p(e)$ adjacent to the vertices over $s_L$ join over the edge $e_L$ (see Figure 2). In this case $i_p(e) = e^1_1 \cup e^2_1$. Since the local degree of $\pi_{\text{trop}}$ at the two vertices in $i_p(\text{Vert}(C))$ adjacent to $i_p(e)$ is equal to 1, the local degree of $\pi_{\text{trop}}$ on the edges $e^1_1$ and $e^2_1$ is equal to 1. Then the vertex adjacent to $e^1_1$ and $e^2_1$ is $i_p(m)$ (the lengths of $e^1_1$ and $e^2_1$ are equal).

![Figure 2. The two edges $e^1_1$ and $e^2_1$ join over the edge $e_L$.](image)

Moreover, the map $\pi_{\text{trop}}$ is of local degree 2 at $i_p(m)$. In fact, if the local degree were greater than 2, then by the balancing condition $s_L$ would have at least 3 preimages under $\pi_{\text{trop}}$ over the set $(\pi_{\text{trop}} \circ p)^{-1}(e)$, which is impossible.

Assume now that the two edges $e^1_1$ and $e^2_1$ do not join over $e_L$, and denote by $s_1$ the other vertex of $T$ adjacent to $e_L$ (see Figure 3).

![Figure 3. The two edges $e^1_1$ and $e^2_1$ do not join over the edge $e_L$.](image)

In this case the preimage of $s_1$ consists of two vertices at which the map $\pi_{\text{trop}}$ is of local degree 1. As in the proof of Proposition 19 we see that no edge of $i_e(C)$ incident to $i_e(s_1)$ is contracted by $\pi_{\text{trop}}$. 
Now one concludes easily by induction.

**Proof of Theorem 3.** Let $e$ be a bounded edge of $C$, and denote by $v_1$ and $v_2$ the vertices adjacent to $e$. Consider the fibration $\lambda_t: V_t \to C$ as defined in [8]. It follows from Proposition 17 that there exists a small neighbourhood $\mathcal{N}_i$ of $v_i$, $i = 1, 2$, in $C$ such that for $t$ big enough the map $\gamma_t|_{\lambda_t^{-1}(\mathcal{N}_i)}$ is of degree 1 on its image (see Figure 4).

![Figure 4. The sets $\lambda_t^{-1}(\mathcal{N}_i)$ inside $V_t$.](image)

It follows from Proposition 14 that the image under $\gamma_t$ of a boundary component of $\lambda_t^{-1}(\mathcal{N}_i)$ converges to the slope of the edge supporting this boundary component. Consider a small circle $\Gamma \subset \mathbb{CP}^1$ centred at the slope of $e$ and of radius $\varepsilon$. The image under $\gamma_t$ of the boundary component of $\lambda_t^{-1}(\mathcal{N}_i)$ corresponding to $e$ is contained in the same connected component of $\mathbb{CP}^1 \setminus \Gamma$ as the slope $s_e$ of $e$. Then for $t$ big enough, $\Gamma$ is contained in $\gamma_t(\lambda_t^{-1}(\mathcal{N}_i))$, and the set $\gamma_t^{-1}|_{\lambda_t^{-1}(\mathcal{N}_i)}(\Gamma)$ is a circle $\Gamma_i \subset \lambda_t^{-1}(\mathcal{N}_i)$ (see Figure 5).

![Figure 5. The circles $\Gamma_i$ inside the sets $\lambda_t^{-1}(\mathcal{N}_i)$.](image)

Note that one of the two connected components of $\lambda_t^{-1}(\mathcal{N}_i) \setminus \Gamma_i$ contains only the boundary component of $\lambda_t^{-1}(\mathcal{N}_i)$ corresponding to $e$. If not, then there would exist a path in $V_t \setminus \Gamma_i$ with endpoints $a_t$ and $b_t$ such that the tropical limit of $a_t$ lies in the interior of $e$ and the tropical limit of $b_t$ lies in the interior of another edge incident to $v_i$ (see Figure 6). But it follows from Proposition 14 that for $t$ big enough such a path should intersect $\Gamma_i$. 
Figure 6. The path with endpoints $a_t$ and $b_t$.

Figure 7. The nonconvex tropical limit of a branch of $\mathbb{R}V_t$ corresponding to an edge $E$.

Figure 8. The convex tropical limit of a branch of $\mathbb{R}V_t$ corresponding to an edge $E$.

The restriction of $\gamma_t$ to the annulus $\mathcal{A}$ with boundary $\Gamma_1 \cup \Gamma_2$ gives us a map of degree 2 to the disc with boundary $\Gamma$ which contains $s_e$. In fact, the preimage of $\Gamma$ in $\mathcal{A}$ under $\gamma_t$ consists exactly of $\Gamma_1 \cup \Gamma_2$ since the map $\gamma_t$ is of degree $2 \text{Area}(\Delta)$ and since by Proposition 17 the other vertices of $C$ each contribute 1 to the degree. By the Riemann-Hurwitz formula, such a map has two critical points in $\mathcal{A}$. Their tropical limit must be a point with local degree of $\pi_{\text{trop}}$ greater than one.

Proposition 20 now implies the theorem as the points in $\rho V_t$ are the critical points of $\gamma_t$.

Proof of Proposition 7. Suppose that $\sigma_{v_1 E} \sigma_{v_4 E} = -1$. Then there is a branch of $\mathbb{R}V_t$ corresponding to $E$ in the positive quadrant. Its tropical limit is nonconvex (see Figure 7) if and only if $\sigma_{v_3 E} \sigma_{v_4 E} = -1$. Thus both log-inflection points corresponding to $E$ must be real (and lie in different quadrants).

If $\sigma_{v_3 E} \sigma_{v_4 E} = 1$ then the tropical limits of both the corresponding branches are convex (see Figure 8). Suppose that $\mathcal{B}$, one of the branches of $\mathbb{R}V_t$ corresponding to $E$, contains two real log-inflection points. Consider the second branch $\mathcal{B}'$ of $\mathbb{R}V_t$ corresponding to $E$. If $\mathcal{B}'$ is also logarithmically nonconvex then we have more
than two inflection points for $E$, which contradicts Theorem 3. If the branch $B'$ is logarithmically convex then we can find a tangent line to $B'$ parallel to the tangent at an inflection point of $B$ (in the logarithmic coordinates). This gives us at least three points (counted with multiplicity) in the preimage of some point under the logarithmic Gauss map $\gamma_t$. It follows from Proposition 17 that there also are $2 \text{Area}(\Delta) - 2$ vertices of $C$ in this preimage. We arrive at a contradiction with $\deg \gamma_t = 2 \text{Area}(\Delta)$.

This finishes the proof of the proposition in the case $\sigma_{v_1}^E \sigma_{v_2}^E = -1$.

The case $\sigma_{v_1}^E \sigma_{v_2}^E = 1$ is reduced to the above by applying the coordinate change $z \mapsto -z$ and/or $w \mapsto -w$.

Bibliography

[1] E. Brugallé, I. Itenberg, G. Mikhalkin and K. Shaw, “Brief introduction to tropical geometry”, Proceedings of the Gökova Geometry–Topology conference 2014, Gökova Geometry/Topology Conference (GGT), Gökova 2015, 1–75 pp.
[2] I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky, Discriminants, resultants, and multidimensional determinants, Math. Theory Appl., Birkhäuser Boston, Inc., Boston, MA 1994, x+523 pp.
[3] I. Itenberg, L. Katzarkov, G. Mikhalkin and I. Zharkov, Tropical homology, arXiv: 1604.01838.
[4] I. Itenberg, G. Mikhalkin and E. Shustin, Tropical algebraic geometry, 2nd ed., Oberwolfach Semin., vol. 35, Birkhäuser Verlag, Basel 2009, x+104 pp.
[5] M.M. Kapranov, “A characterization of $A$-discriminantal hypersurfaces in terms of the logarithmic Gauss map”, Math. Ann. 290:2 (1991), 277–285.
[6] A.G. Kushnirenko, “Newton polytopes and the Bezout theorem”, Funktsional. Anal. i Prilozhen. 10:3 (1976), 82–83; English transl. in Funct. Anal. Appl. 10:3 (1976), 233–235.
[7] G. Mikhalkin, “Real algebraic curves, the moment map and amoebas”, Ann. of Math. (2) 151:1 (2000), 309–326.
[8] G. Mikhalkin, “Decomposition into pairs-of-pants for complex algebraic hypersurfaces”, Topology 43:5 (2004), 1035–1065.
[9] G. Mikhalkin, “Enumerative tropical algebraic geometry in $\mathbb{R}^2$”, J. Amer. Math. Soc. 18:2 (2005), 313–377.
[10] G. Mikhalkin, “Tropical geometry and its applications”, International congress of mathematicians, vol. II, Eur. Math. Soc., Zürich 2006, pp. 827–852.
[11] O. Viro, “Dequantization of real algebraic geometry on logarithmic paper”, European congress of mathematics, vol. I (Barcelona 2000), Progr. Math., vol. 201, Birkhäuser, Basel 2001, 135–146 pp.