Lieb-Robinson bounds and the simulation of time evolution
of local observables in lattice systems

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Abstract
This is an introductory text reviewing Lieb-Robinson bounds for open and
closed quantum many-body systems. We introduce the Heisenberg picture for
time-dependent local Liouvillians and state a Lieb-Robinson bound. Finally we
discuss a number of important consequences in quantum many-body theory.

1 Introduction

In lattice systems one might expect that, due to the locality of the interaction, there
is some limit to the speed with which correlations can propagate. Similar to the light
cone in special relativity, there should be a space time cone, outside of which a local
perturbation of such a system should not be able to influence any measurement in a
significant way. That this intuition can indeed be made rigorous was first shown by
Elliott H. Lieb and Derek W. Robinson in a seminal work [36] in 1972.

Today, the term Lieb-Robinson bound generally refers to upper bounds on the speed
of propagation of some measure of correlation. Outside the space time cone defined by
this speed, any signal is typically exponentially suppressed in the distance. The results
of Lieb and Robinson, originally derived in the setting of translation invariant 1D spin
systems with short range, or exponentially decaying interactions [36] have since been
tightened [26, 46, 38] and extended to more general graphs [30, 44] and to interactions
decaying only polynomially with the distance, both, for spin systems [43] and
fermionic systems [30] (see also Ref. [47] for a review). Lieb-Robinson bounds have
been proven for Liouvillian dynamics first in Ref. [55], where Liouvillian dynamics is
a generalization of Hamiltonian dynamics that can also capture the effect of a certain
type of noise. The bounds have recently been strengthened for a specific subclass of
Liouvillians in Ref. [14] and have been generalized to time-dependent Liouvillian dyna-
mics in Refs. [48, 7]. Indeed, Lieb-Robinson bounds provide the basis for a wealth of
statements in quantum many-body theory, mostly as a mathematical proof tool, but also
as an argument justifying numerical techniques. We will touch upon these implications
and discuss the simulation of time evolution in more detail.

To keep the presentation both self-contained and concise, we mainly focus on Li-
ouvillian dynamics as presented in Ref. [7]. The chapter is structured as follows: In
the beginning, we introduce the setting and the necessary notation in Section 2. This
includes in particular an introduction to Liouvillian dynamics in both the Schrödinger
and Heisenberg picture and a discussion of the relevant measures for approximation

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errors that are needed later to state the Lieb-Robinson bound and their physical interpretation. In the last part of Section 2 we explain the setting of spin lattice systems. Next, we state a general Lieb-Robinson bound in Section 3 and mention various consequences. In particular, we explain the locality and simulability of time evolution in more detail in Section 4. Finally, in Section 5, we state the Lieb-Robinson bound for fermions and introduce the Jordan-Wigner transform, which is a mapping between spin systems and fermionic systems.

2 Setting and notation

In this section we introduce the necessary formalism to describe the dynamics of spin lattice systems evolving under local Liouvillian dynamics, including local Hamiltonian dynamics as a special case. While Hamiltonian time evolution describes the dynamics of closed systems, Liouvillian dynamics also captures the case of so-called open quantum systems \[^{39}\], which are systems coupled to memoryless “baths”. Such couplings can be used to model Markovian “noise” perturbing the evolution of the system.

2.1 Schrödinger and Heisenberg picture for time-dependent Liouvillians

We start by introducing some notation and some basic mathematical facts. For some Hilbert space $\mathcal{H}$ of finite dimension $\text{dim}(\mathcal{H})$ let us denote the space of linear operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. Together with the Hilbert-Schmidt inner product, defined by $\langle A, B \rangle := \text{Tr}(A^\dagger B)$ for $A, B \in \mathcal{B}(\mathcal{H})$, the operators $\mathcal{B}(\mathcal{H})$ are also a Hilbert space. Importantly, this defines the Hilbert Schmidt adjoint of a superoperator. A superoperator is a linear map $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, i.e., $T \in \mathcal{B}(\mathcal{H})$ and its (Hilbert Schmidt) adjoint $T^\dagger \in \mathcal{B}(\mathcal{H})$ is defined via $\langle X, T^\dagger(Y) \rangle := \langle T(X), Y \rangle$ for all $X, Y \in \mathcal{B}(\mathcal{H})$. The subspace of observables $\mathcal{A}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ are the Hermitian, i.e. self-adjoint operators and the set of states $\mathcal{S}(\mathcal{H})$ (also called density operators) are positive Hermitian operators with unit trace. Given an observable $A \in \mathcal{A}(\mathcal{H})$ and a state $\rho \in \mathcal{S}(\mathcal{H})$ the expectation value is

$$\langle A \rangle_\rho := \text{Tr}(\rho A).$$ (1)

When considering time evolution one is confronted with the following scenario: At some time $s$ the system is in some initial state $\rho$ and at a later time $t \geq s$ (throughout this chapter)

$$t \geq s$$ (2)

one measures some observable $A$ that gives rise to an expectation value $\langle A \rangle_\rho(s, t)$. The time evolution can be described either in the Schrödinger picture or the Heisenberg picture. In the Schrödinger picture, one evolves the initial state $\rho$, given at time $s$, forward in time until time $t$ is reached at which the measurement is performed. In the Heisenberg picture, in turn, one evolves the observable $A$ backwards in time from $t$ to the time $s$ at which the initial state is given.

In the Schrödinger picture one considers the states to be time-dependent. In the case of a closed quantum system evolving under a Hamiltonian $H$, the system state at time $t$ is the solution of the linear initial value problem

$$\frac{d}{dt}\rho_s(t) = -i[H(t), \rho_s(t)], \quad \rho_s(s) = \rho,$$ (3)
where the solutions of the dynamical equations carry the initial time \( s \) as a label for reasons that become clear once we switch to the Heisenberg picture. In case of an open quantum system, so if a system is coupled to further degrees of freedom giving rise to decoherence and dissipation, time evolution is still reflected by a completely positive and trace preserving map, but is no longer governed by the Schrödinger equation. Still, in many instances, e.g., for many physically relevant situations with weak coupling, one can still approximate the dynamics very well by the solution of the linear initial value problem
\[
\frac{d}{dt}\rho_s(t) = L^\dagger_s(\rho_s(t)), \quad \rho_s(s) = \rho, \tag{4}
\]
where \( L^\dagger : \mathbb{R} \to \mathcal{B} (\mathcal{H}) \) is called the Liouvillian\(^1\), where the time dependence is given by the input \( t \in \mathbb{R} \). Throughout this chapter we only consider time dependencies that are piecewise continuous. For an equation of motion of this form, the only constraint is that the time evolution maps states to states, i.e., is completely positive and trace preserving. This is equivalent \(^{70}\) to the Liouvillian \( L^\dagger_1 \) having a Lindblad representation \(^{39}\), i.e. it must be of the form
\[
L^\dagger (\rho) = -i[H, \rho] + \sum_{\mu=1}^{\dim(\mathcal{H})^2} (2L^\dagger_{\mu} \rho L_{\mu} - L_{\mu}^\dagger L_{\mu} \rho - \rho L_{\mu}^\dagger L_{\mu} ), \tag{5}
\]
for some time-dependent operators \( H : \mathbb{R} \to \mathcal{A}(\mathcal{H}) \) and \( L_{\mu} : \mathbb{R} \to \mathcal{B}(\mathcal{H}). \)

Liouvillian dynamics is ubiquitous in many contexts in physics. It has recently been studied particularly intensely in the context of cold atoms in optical lattices \(^{16, 15, 53, 4}\), trapped ions \(^{6, 57}\), driven dissipative Rydberg gases \(^{23}\), and macroscopic atomic ensembles \(^{35}\). Also dissipative state preparation \(^{69}\), dissipative phase transitions \(^{16}\), noise-driven criticality \(^{20}\) and nonequilibrium topological phase transitions \(^{5}\) have been considered.

The initial value problem (4) defines the propagator (also called dynamical map)
\[
T_{L^\dagger_1}(t, s) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \via
\]
\[
T_{L^\dagger_1}(t, s)(\rho) := \rho_s(t), \tag{6}
\]
which is also the unique solution of the initial value problem
\[
\frac{d}{dt}T_{L^\dagger_1}(t, s) = L^\dagger_1 T_{L^\dagger_1}(t, s), \quad T_{L^\dagger_1}(s, s) = \text{id}. \tag{7}
\]
The expectation value at time \( t \) then is
\[
\langle A \rangle_{\rho}(s, t) = \text{Tr} [T_{L^\dagger_1}(t, s)(\rho) A]. \tag{8}
\]
If the Liouvillian \( L^\dagger_1 \) is time-independent, a state satisfying \( L^\dagger_1(\rho) = 0 \) is called stationary state. The role played by stationary states is reminiscent of the role of ground states of Hamiltonians. For the case of a unique stationary state the spectral gap of the Liouvillian is a measure of the speed of convergence \(^{33}\) towards this stationary state.

Evolving some state \( \rho \) from \( s \) to \( r \geq s \) and then from \( r \) to \( t \geq r \) also yields \( \rho_s(t) \) and hence the propagator has the composition property \( T_{L^\dagger_1}(t, r)T_{L^\dagger_1}(r, s) = T_{L^\dagger_1}(t, s) \) for all \( t \geq r \geq s \). It is a good exercise to derive the differential equation
\[
\frac{d}{ds}T_{L^\dagger_1}(t, s) = -T_{L^\dagger_1}(t, s)L^\dagger_1, \tag{9}
\]
\(^1\) As we will later mostly work in the Heisenberg picture it is convenient to denote the Liouvillian in the Schrödinger picture by \( L^\dagger_1 \) rather than \( L \).
from this property.

We are now ready to introduce the Heisenberg picture, in which the states are constant and the observables are defined as solutions of a dynamical equation. Of course, both pictures must yield the same expectation values, i.e.,

\[ \langle A \rangle_\rho (s, t) = \text{Tr} (\rho \tau_\mathcal{L} (t, s)(A)), \]

where

\[ \tau_\mathcal{L} (s, t) = T_\mathcal{L}^\dagger (t, s) \]

is the adjoint of \( T_\mathcal{L} (t, s) \) in the Hilbert-Schmidt inner product. \( \tau_\mathcal{L} \) is the propagator in the Heisenberg picture. Using Eq. (9), it is not hard to see that it is the unique solution of

\[ \frac{d}{ds} \tau_\mathcal{L} (s, t) = -\mathcal{L}_s \tau_\mathcal{L} (s, t), \quad \tau_\mathcal{L} (t, t) = \text{id}, \]

which coincides with the largest singular values of \( \mathcal{B} \). If \( \mathcal{B} \) is an observable, then its norm is its largest eigenvalue in magnitude and thus a bound on the range of values one can obtain when \( \mathcal{B} \) is measured. Let us state this more explicitly by using the well-known formula \[ \| \mathcal{B} \| = \sup_{\rho \in \mathcal{S} (\mathcal{H})} | \text{Tr} (\rho \mathcal{B}) |. \]
Figure 1: An example for an interaction hypergraph. The dots denote the vertices and the blue frames the hyperedges. The maximum number of nearest neighbors is $Z = 4$ (the hyperedge containing 3 vertices in the lower right corner).

Considering the case where $B = A - A'$ is the difference of two observables $A, A' \in \mathcal{A}(\mathcal{H})$ this means that the operator norm is the physically relevant norm to measure closeness of the observables. If $\|A - A'\|$ is small, then $A$ and $A'$ will have almost the same expectation value on all states. In the same way as the Hilbert space norm induces the operator norm, the operator norm gives rise to the norm

$$\|T\| := \sup_{\|A\|=1} \|T(A)\|$$

(18)

for superoperators $T \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$, where the supremum is taken over all observables $A \in \mathcal{A}$ of unit norm. So for $T, T' \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$

$$\|T - T'\| = \sup_{\|A\|=1} \|T(A) - T'(A)\|$$

(19)

and hence if $\|T - T'\|$ is small, then $T$ and $T'$ will map any observable to observables that yield almost the same expectation values for all states. Consequently, this norm is the physically relevant norm for measuring closeness of super-operators. When applied to propagators, it can hence be used to quantify the closeness of the corresponding time evolutions.

### 2.3 Lattice systems and local Liouvillians

Quantum lattice systems are formally described by a set of (spatial) sites that are considered to be the vertices and the interactions between them the edges of a (hyper)graph (see also Fig. 1). In this section we explain this setting for spin systems in detail and consider fermionic systems in Section 5.

Let us assume that the set of sites $V$ is finite, i.e., there are only finitely many sites, and that each site $x \in V$ is associated with a finite dimensional Hilbert space $\mathcal{H}_x$. The Hilbert space of some subsystem $X \subset V$ is denoted by $\mathcal{H}_X := \bigotimes_{x \in X} \mathcal{H}_x$ and $\mathcal{H} := \mathcal{H}_V$. For an operator $A \in \mathcal{B}(\mathcal{H})$ we define its support $\text{supp}(A)$ to be the smallest subset $X \subset V$ such that it acts as the identity outside of $X$, i.e., $A_X = A \otimes 1_{V \setminus X}$. The set of operators supported on $X$ is denoted by $\mathcal{B}_X(\mathcal{H}) := \{ A \in \mathcal{B}(\mathcal{H}) : \text{supp}(A) \subset X \}$ and the subspace of observables by $\mathcal{A}_X(\mathcal{H}) \subset \mathcal{B}_X(\mathcal{H})$. For a Liouvillian $\mathcal{L}$ on $\mathcal{B}(\mathcal{H})$ we define its support to be

$$\text{supp}(\mathcal{L}) := \bigcup \{ X \subset V : \mathcal{A}_{V \setminus X}(\mathcal{H}) \subset \ker(\mathcal{L}) \}$$

(20)
i.e., the part of the system where $L$ corresponds to a non-trivial time evolution. The set of Liouvillians supported on $X$ is denoted by $L_X(H)$. Often we omit the Hilbert space and write, e.g., $A_X$ instead of $A_X(H)$.

We are interested in time evolution under Liouvillians that are sums of local Liouvillians and hence of the form

$$L = \sum_{X \subset V} L_X, \quad L_X \in L_X.$$  \hfill (21)

In many physically relevant situations many of the local terms $L_X$, in particular those belonging to large sets $X$, will be zero. This structure reflects interactions and dissipation processes that are finite-ranged. The interaction graph $E$ of the Liouvillian is the set of all subsets of $V$ for which the Liouvillian contains a non zero term, i.e.,

$$E := \{X \subset V : L_X \neq 0\}.$$  \hfill (22)

As an example, consider the case of a 1D system with nearest neighbor interactions and open boundary conditions. If the sites are $V = \{1, \ldots, N\}$, the interaction graph is $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{N - 1, N\}\}$ in that case.

The interaction (hyper)graph $E$ defines the distance $d(X, Y)$ between any two sets $X, Y \subset V$ of vertices. The distance $d(X, Y)$ is equal to 0 if and only if $X \cap Y \neq \emptyset$ and otherwise equal to the length of the shortest path connecting $X$ and $Y$, and $\infty$ if there is no connecting path. A path between two sets $X, Y \subset V$ is a sequence of elements of $E$, such that the first element contains a vertex in $X$, each element of the path shares at least one vertex with the following element and the last element contains a vertex in $Y$. Note that $d$ is a degenerate metric on subsets of $V$. In the above 1D example the graph distance of the two sets $\{j\}, \{k\} \subset V$ would simply be $d(\{j\}, \{k\}) = |j - k|$, as one would expect.

3 A Lieb-Robinson bound

In this section we state and explain a very general Lieb-Robinson bound for the speed of propagation of correlations in spin systems under arbitrary time-dependent Liouvillian dynamics. Our goal is to make statements about local time evolution, i.e., time evolution of local observables arising from local interactions and local noise. In order to make this precise, let us impose some technical constraints on some possibly time-dependent Liouvillian $L_s \in L_V$, which we consider to be fixed from now on. Local time evolution is captured by a Liouvillian $L$ that is a sum of strictly local terms $L_X$, each of which is bounded in norm by $b$, and a maximum number of nearest neighbors $Z$. In more detail, we define

$$L = \sum_{X \in E} L_X, \quad L_X : \mathbb{R} \to L_X(H), \text{ piecewise continuous},$$  \hfill (23)

$$b := \sup_{s, X} \|L_X(s)\|,$$  \hfill (24)

$$Z := \max_{X \in E} |\{Y \in E : Y \cap X \neq \emptyset\}|. \hfill (25)$$

The parameters $b$ and $Z$ will determine the Lieb-Robinson speed and also the final results about the spatial truncation

$$L_{\mid V'} := \sum_{X \subset V'} L_X$$  \hfill (26)
of the Liouvillian $\mathcal{L}$ to some region $V' \subset V$. Now we are ready to state the Lieb-Robinson bound for this setting. Similar results on Liouvillians can be found in Refs. \[48, 55\]. The theorem is quite general and it might not be immediately obvious how statements about propagation of information are implied. But this will become clear in the next section.

**Theorem 1 (Lieb-Robinson Bound \[7\])**. Let $\mathcal{L} : \mathbb{R} \to \mathbb{L}(\mathcal{H})$ be a local Liouvillian as specified in Eqn. (23)–(25) and $X, Y \subset V$. Then, for every $\mathcal{K}_Y \in \mathbb{L}_Y(\mathcal{H})$, $A_X \in \mathbb{B}_X(\mathcal{H})$ and $s \leq t$

$$\| \mathcal{K}_Y \tau_{\mathcal{L}}(s, t)(A_X) \| \leq C \| \mathcal{K}_Y \| \| A_X \| \, e^{v(t-s)-d(X,Y)},$$

where $v = \exp(1)bZ$ and $C$ is some constant depending polynomially on the size of the smaller of the two sets $X$ and $Y$.

Remembering that the Liouvillian maps an observable to its time derivative, the theorem tells us that an evolved observable $\tau_{\mathcal{L}}(s, t)(A_X)$ remains basically unchanged when evolved with respect to a Liouvillian $\mathcal{K}_Y$ that is supported on a region a distance much larger than $v(t-s)$ away from $X$, i.e., that $\tau_{\mathcal{L}}(s, t)(A_X)$ is almost the identity outside the corresponding space-time cone. More intuitively, the Lieb-Robinson bound tells us that information travels with a velocity bounded by the *Lieb-Robinson speed* $v$ of the considered lattice system. In the special case $\mathcal{K}_Y = i[B_Y, \cdot]$ for some $B_Y \in \mathbb{A}_Y(\mathcal{H})$, Eq. (27) yields a Lieb-Robinson bound in the more common form of an upper bound on $\| [B_Y, \tau_{\mathcal{L}}(s, t)(A_X)] \|$ (compare Refs. \[48, 55\]).

If a system is mixing in the sense that all states are driven towards a steady state then information encoded in the initial state gets lost at some point. This puts an upper bound on the distance over which information can propagate. Therefore, one might expect that there is some effective Lieb-Robinson speed that decreases in time. This is indeed true for certain systems with fluctuating disorder \[10\] and for a class of Liouvillian dynamics \[14\]. Very recently, a Lieb-Robinson bound with Gaussian decay in the distance $d(X, Y)$ has been derived for Hamiltonian dynamics \[38\].

Finally, note that the restriction to finite-dimensional subsystems is not merely for simplicity of notation: For infinite-dimensional systems the situation can be quite different. For some anharmonic lattices \[44\], and other instances of strongly correlated models \[60\] Lieb-Robinson bounds can still be found, as well as for commutator-bounded operators \[56\]. Still, counterexamples to Lieb-Robinson bounds are known for models with infinite-dimensional constituents \[18\].

## 4 Consequences of Lieb-Robinson bounds

Lieb-Robinson bounds are fundamental for a plethora of statements concerning various properties of locally interacting systems. We first discuss immediate consequences as far as the dynamics of such systems is concerned. Next, we turn to implications for the classical simulation of time evolution. Finally, we discuss static properties that can be derived from Lieb-Robinson bounds.

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Footnote: In Ref. \[7\] the bound is given for an arbitrary metric on the vertex set and the Liouvillians are allowed to have interaction range $\alpha$ in that metric. Our interaction graph distance $d$ is induced by a metric on $V$ for which $\alpha = 1$. 

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4.1 Quasi-locality of quantum dynamics

The result of the last section suggests that the terms of the Liouvillian whose support is sufficiently far away from the support of an observable are irrelevant for the time evolution. More precisely, one should be able to spatially truncate the Liouvillian \( \mathcal{L} \) to some region \( V' \subset V \). If \( X \) is sufficiently far from the boundary of \( V' \), i.e., if \( d(X, V \setminus V') \) is larger than the radius \( v \cdot (t - s) \) of the space time cone of \( \tau_{\mathcal{L}}(s, t)(A_X) \), then the dynamics of \( A_X \) under the truncated Liouvillian \( \mathcal{L}_{V'} \) and the original Liouvillian \( \mathcal{L} \) should be very similar. In the next theorem we will see that this is indeed the case if the underlying interaction graph is of finite spacial dimension, which we define first. Let us denote the “sphere” around some subsystem \( X \in E \) with radius \( n \) by

\[
S_X(n) := \{ Y \in E : d(Y, X) = n \}. \tag{28}
\]

Then we say that an interaction graph \( E \) is of spatial dimension \( \mu \) if there is a constant \( M > 0 \) that only depends on local properties of the interaction graph such that for all \( X \in E \)

\[
|S_X(n)| \leq M n^{\mu-1}. \tag{29}
\]

For example, the interaction graph of next-neighbor Liouvillians on a -dimensional cubic lattice has dimension \( k \).

**Theorem 2** (Quasi-locality of local Liouvillian dynamics [7]). Let \( \mathcal{L} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) be a local Liouvillian as specified in Eqn. (23) – (25) and let its interaction graph be of spatial dimension \( \mu \) with the constant \( M \) as defined in Eq. (29). Then, for all \( X \subset V' \subset V \) with \( D := d(X, V \setminus V') \geq 2\mu - 1 \), \( A_X \in B_X(\mathcal{H}) \), and \( s \leq t \)

\[
\left\| \tau_{\mathcal{L}_{V'}}(s, t)(A_X) - \tau_{\mathcal{L}}(s, t)(A_X) \right\| \leq \frac{2M}{Z} D^{\mu-1} e^{v(t-s) - D} \| A_X \|, \tag{30}
\]

where \( v = \exp(1)h \mathcal{Z} \) is the Lieb-Robinson speed.

So, colloquially speaking, the full dynamics of local observables can be approximated with exponential accuracy by the dynamics of a sufficiently large subsystem. Of course, the size of the subsystem depends on the desired time span of the evolution. In particular the locality result makes an extension of time evolution to infinitely large lattices possible, i.e., it can be employed to rigorously define the thermodynamic limit. Theorem 2 has two immediate physical consequences which can hence be seen as an interpretation of the result. For the rest of this section consider a lattice system with \( V = \{1, \ldots, N\} \) and let \( \rho \) be a product state, i.e., \( \rho = \bigotimes_{j=1}^{N} \rho_j \) where \( \rho_j \in \mathcal{S}(\mathcal{H}_{\langle j \rangle}) \) for all \( j \) and moreover, let \( X, Y \subset V \) such that \( X \cap Y = \emptyset \).

**Suppression of correlation functions:** Consider two observables \( A_X \in A_X \) and \( B_Y \in A_Y \). Their correlation coefficient in state \( \rho \in \mathcal{S}(\mathcal{H}) \) is proportional to the covariance

\[
\text{cov}_\rho(A_X, B_Y) := \langle A_X B_Y \rangle_\rho - \langle A_X \rangle_\rho \langle B_Y \rangle_\rho. \tag{31}
\]

As \( \rho \) is a product state, \( \text{cov}_\rho(A_X, B_Y) = 0 \). Now, Theorem 2 tells us that as long as \( v \cdot (t - s) \ll d(X, Y)/2 \) the correlation coefficient of the time evolved observables will remain very small. More precisely, \( \text{cov}_\rho(\tau(s, t)(A_X), \tau(s, t)(B_Y)) \) is upper bounded by \( \exp(v \cdot (t - s) - d(X, Y)/2) \) up to a constant factor. The measurement statistics of the two observables can show correlations only after the dynamics of the system has correlated the two regions \( X \) and \( Y \) (see Ref. [43] for a similar discussion in the context of Hamiltonian dynamics).
Figure 2: The space time cone of an observable $A_X$ in one spatial dimension. The truncation error scales exponentially in the distance $D - v \cdot (t - s)$.

**Speed of information propagation:** The lattice system can also be thought of as a channel that one might want to use to send information from $X$ to $Y$ in the following way: One party encodes a message by preparing at time $s$ the part of the initial state $\rho$ in the region $X$ in a particular way, the other party tries to retrieve the message by measuring on region $Y$ at time $t$. Lieb-Robinson bounds can be used to show that the amount of information that can be transferred in this way in a time span $t - s$ is exponentially suppressed if $d(X,Y)$ is larger than $v \cdot (t - s)$. This can be made precise in the sense that the classical information capacity is exponentially small outside the cone, if the quantum many-body systems is used as a quantum channel\cite{9}. In Ref.\cite{11}, the ballistic propagation of excitations and information propagation constrained by Lieb-Robinson bounds has been experimentally explored in systems of cold atoms.

### 4.2 Classical simulation of quantum dynamics

By classical simulation of quantum dynamics we mean the calculation of expectation values of local observables $\langle A_X \rangle_\rho(s,t)$, so that one could, for instance, plot them over time. If one tries to do that naively, i.e., by calculating the full propagator $\tau_L(s,t)$ on a classical computer, one quickly runs into problems even with just having enough memory to store the propagator. For instance, if one has $N$ subsystems with a local Hilbert space dimension of 2, then to completely specify the propagator in a naive way, one needs $2^{4N}$ complex numbers. Therefore, if one aims at classically simulating local observables one needs to come up with a smart simulation scheme that only deals with the information relevant for the simulation. We sketch two such schemes here:

**Time evolution as (unitary) circuits:** Here the quasi-locality Theorem\cite{2} is of great help, since it already tells us that one can truncate the dynamics to a set $V'$ containing the space time cone of the observable instead of considering the full system $V$. The arising error is exponentially small in the distance between the space time cone and the truncation. So the simulation cost does clearly not depend on the system size and the dynamics can hence be implemented efficiently in that. Of course, implementing the full simulation naively on $V'$ is still by far not optimal. Famously, one can decompose the propagator $\tau_L(s,t)$ into products over short time steps and more local propagators, which is often called Trotter-decomposition\cite{65}. At the heart of this approach is the following product formula that can be used to bound the error one makes by de-
composing the propagator of a Liouvillian that is a sum of two Liouvillians \( \mathcal{L} \) and \( \mathcal{K} \) into the product of the propagators of these Liouvillians.

**Theorem 3** (Trotter product formula \([34, 7]\)). Let \( K \) be the number of elements of the interaction graph of \( \mathcal{L} \), then there exist constants \( b, c \) that depend only on local properties of the Liouvillians, and the size of the largest set in the interaction graphs of \( \mathcal{L} \) and \( \mathcal{K} \), but are in particular independent of the number of sites, such that

\[
\| \tau_{\mathcal{L}+\mathcal{K}}(s, t) - \tau_{\mathcal{L}}(s, t) \tau_{\mathcal{K}}(s, t) \| \leq c(t - s)^2 e^{b(t - s) K}.
\]

(32)

One can now decompose the time span \( t - s \) into short time steps \( [s_{j+1}, s_j] \) and in each of these intervals approximate the propagator by a product of the local propagators \( \tau_{\mathcal{L}_X}(s_{j+1}, s_j) \) for each edge \( X \) in the interaction graph of the Liouvillian. In other words, the full propagator can be approximated by a “circuit” of local propagators.

The number of time steps needed to reach a simulation with total error upper bounded by \( \epsilon \) is proportional to \( (t - s)^2 K^2 / \epsilon [34] \). Of course, the above covers Hamiltonian dynamics as a special case. However, there one would rather apply similar ideas to the time evolution operator \( \exp(-i (t - s) H) \) rather than the propagator. In a variant of this circuit description for Hamiltonian dynamics in 1D, the time evolution operator can be approximated by a circuit of constant depth and time-dependent gates \([50] \).

**Time-dependent density-matrix renormalization group methods:** A similar mindset is also fundamental for the simulation of time evolution using so-called tensor network states. The situation is particularly clear in 1D systems with sites \( V = \{1, \ldots, N\} \) in pure states undergoing local Hamiltonian dynamics. If the initial state has a strong decay of initial correlations, then the time evolution can for short times be efficiently grasped in terms of matrix-product states (MPS) \([22, 59, 51] \). These are variational state vectors that are described by \( O(dN D^2) \) variational parameters, where \( D \in \mathbb{N} \) is a refinement parameter and \( d \) the dimension of the local Hilbert space. There are several variants of this approach, based on either a Trotter-decomposition \([65] \) or a time-dependent variational principle \([24] \). Such schemes are subsumed under the term *time-dependent density matrix renormalization group method* (t-DMRG). At the heart of the functioning of t-DMRG is the insight that states generated by short time local Hamiltonian dynamics will have low entanglement. This can be formalized \([19] \) in terms of so-called area laws \([29, 3, 1, 17] \) that arise as a consequence of a Lieb-Robinson bound.

An area law is an upper bound on the entanglement of a state. More precisely, we say that a pure state satisfies an area law if for any region \( R \subset V \) the (Rényi) entropy of the reduced state on \( R \) can be bounded by the size of the boundary of \( R \), up to a constant. States of 1D systems satisfying an area law can be provably well approximated by matrix product states \([57] \). Indeed, t-DMRG simulates time evolution for short times to essentially machine precision. For long times, the entropy will in general grow too much, as then sites are in the space time cone of too many sites of the lattice, and an efficient simulation in terms of matrix-product states is hence \([61] \) no longer possible \([62, 12] \). That is, the power of the t-DMRG approach can be rigorously grasped in terms of Lieb-Robinson bounds. For 1D local Liouvillian dynamics, variants of t-DMRG have also been proposed \([68, 72] \), usually as variational principles over matrix-product operators, the mixed state analogues of matrix-product states, or by means of suitable sampling employing classical stochastic processes in Hilbert space \([53] \).
4.3 Static properties derived from Lieb-Robinson bounds

Among the most important applications of Lieb-Robinson bounds are proof techniques related to static properties of quantum lattice systems, and not time-dependent ones. Here we briefly mention some of them:

**Clustering of correlations in Hamiltonian systems:** One of the most relevant applications concerns the decay of correlations in the ground state of a local Hamiltonian with a spectral gap \( \Delta E > 0 \), first shown in Refs. [26, 30] and further generalized in Ref. [45]. The basic intuition underlying this intricate insight is that the spectral gap \( \Delta E \) essentially defines a time scale in the system, which in turn can be related to a length scale.

**Theorem 4** (Clustering of correlations in unique ground states [26, 46]). Let \( H \in \mathcal{A}(\mathcal{H}) \) be a local Hamiltonian with a unique ground state \( \psi \) and a spectral gap \( \Delta E > 0 \) and \( X, Y \subseteq V \). Then, for every \( A_X \in \mathcal{B}_X(\mathcal{H}) \) and \( B_Y \in \mathcal{B}_Y(\mathcal{H}) \)

\[
|\text{cov}_\psi(A_X, B_Y)| \leq C \|A_X\| \|B_Y\| e^{-\mu d(X,Y)}. \tag{33}
\]

\( C \) and \( \mu \) are constants both depending on \( \Delta E \). Moreover, \( C \) depends on the lattice geometry and the smaller of the surface areas of \( X \) and \( Y \), and \( \mu \) depends on the Lieb-Robinson speed.

The proof of this statement confirmed a long-standing conjecture in condensed-matter physics, that gapped Hamiltonian systems have exponentially clustering correlations in the ground state.

**Clustering of correlations in Liouvillian systems:** A similar intuition actually holds true for Liouvillian systems, where the role of the ground state of Hamiltonian systems is taken over by the stationary state. Clustering of correlations in local Liouvillian systems has first been considered in Ref. [55] and made rigorous and largely generalized in Ref. [33]: If a local Liouvillian is primitive (that is, if its stationary state has full rank) and has a spectral gap which is independent of the system size, then correlation functions between local observables again decay exponentially as a function of the distance between their supports.

**Area laws of ground states of gapped Hamiltonians:** It has been shown using Lieb-Robinson bounds that ground states of 1D local Hamiltonian systems with spectral gap \( \Delta E > 0 \) always satisfy an area law for the Rényi entropies (for a review, see Ref. [17]). This result has since been tightened [3] and area laws have also been shown for some instances of gapped higher-dimensional Hamiltonian systems [52]. For local Liouvilians, general area laws (in terms of entropic measures suitable for mixed states) can be derived for stationary states [33], again using Lieb-Robinson bounds.

**Approximating 1D ground states of gapped Hamiltonians with MPS:** Since ground states of any 1D local Hamiltonian with a spectral gap \( \Delta E > 0 \) satisfy an area law for Rényi entropies they can be approximated [67] by matrix product states (MPS). This is used by the static density-matrix renormalization group method (DMRG) [58] for simulating ground state properties [59], which has led to a wealth of novel insights in condensed matter physics.

**Higher dimensional Lieb-Schultz-Mattis theorems:** The Lieb-Schultz-Mattis theorem [37, 2] is an upper bound on the spectral gap of certain one-dimensional quantum spin systems. Using Lieb-Robinson bounds, a higher-dimensional Lieb-Schultz-Mattis theorem has been proven in Refs. [26, 28].

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\( \Delta E \) is the difference between the ground state energy and the energy of the first exited state.
Stability and further properties of ground states: Lieb-Robinson bounds are one of the pillars of the formalism grasping the stability of ground states of a class of Hamiltonians (frustration-free Hamiltonians satisfying certain topological order conditions) under local perturbations. This has developed into a field of research in its own right, and we merely touch upon the topic here. Starting point is the concept of quasi-adiabatic continuation \[31\], which is a tool to connect dynamical properties of a Hamiltonian to static ones and relies on Lieb-Robinson bounds. Importantly, quasi-adiabatic continuation is a cornerstone of the proof of the stability of topological order under local perturbations \[8\] and related proofs of the stability of the spectral gap, of frustration-free Hamiltonians under general, quasi-local perturbations \[42\]. With similar tools, the stability of the area law for the entanglement entropy of the ground state can be proven \[41, 1\].

Stability of stationary states: Inspired by the stability results on Hamiltonian ground states, Lieb-Robinson bounds have also been used to prove the stability of stationary states of certain local Liouvillians \[13, 33\].

Structure of elementary excited states: The structure of elementary excited has been explored using Lieb-Robinson bounds, which can be approximated by superimposing ground states to which local operators have been applied \[25\].

5 Fermionic Hamiltonians

While Lieb-Robinson bounds are usually stated for spin lattice system, they also hold for systems of fermions on a lattice. The situation is particularly simple for 1D systems with nearest neighbor coupling only, since in that case the Jordan-Wigner transform can be applied. In this section we first state a fermionic Lieb-Robinson bound and then introduce the Jordan-Wigner transform.

Again, as with spin lattice systems, we have an interaction (hyper)graph \((V, E)\) but now work in the picture of second quantization, i.e., operators are given in terms of the fermionic creation and annihilation operators \(f_j\) and \(f_k^{\dagger}\) for \(j, k \in V\). These fermionic operators satisfy

\[
\{f_j, f_k^{\dagger}\} = \delta_{j,k},
\]

where \(\{A, B\} := AB + BA\) is the anti-commutator. According to the fermion number parity superselection rule only observables that are even polynomials in the fermionic operators can occur in nature. A polynomial of fermionic operators is called even if it can be written as a linear combination of monomials, where each monomial is a product of an even number of fermionic operators from \(f_j\) and \(f_k^{\dagger}\). Correspondingly, we denote the algebra of the parity preserving observables acting on a region \(X \subset V\) by \(\mathcal{G}_X\) for short. Now one can prove a fermionic Lieb-Robinson bound in the same way as Theorem 1 is proven:

Theorem 5 (Fermionic Lieb-Robinson bound). Let

\[
H = \sum_{X \in E} H_X
\]

be a local time-dependent Hamiltonian with \(H_X : \mathbb{R} \to \mathcal{G}_X\) and \(\|H_X(r)\| \leq b\) for all \(X \in E\) and \(r \in \mathbb{R}\), \(\tau\) its propagator, and \(Z\) the maximum number of nearest neighbors as defined in Eq. (25). Then, for every \(A_X \in \mathcal{G}_X\), \(B_Y \in \mathcal{G}_Y\) and \(s, t \in \mathbb{R}\),

\[
\|[B_Y, \tau(s, t)(A_X)]\| \leq C \|B_Y\| \|A_X\| e^{\nu|t-s|-d(X,Y)},
\]

(36)
where \( v = \exp(1) b \mathbb{Z} \) and \( C \) is some constant depending polynomially on the size of the smaller of the two sets \( X \) and \( Y \).

For the unphysical case where \( B_Y \) and \( A_X \) are observables that are odd polynomials in the fermionic operators one can still prove a similar Lieb-Robinson bound for the anti-commutator, providing a relevant proof-tool \[27\].

For the case of 1D systems with nearest neighbor interactions only, the analogy between fermionic and spin systems is even stronger in the sense that such systems can be mapped to each other by the Jordan-Wigner transform \[32\]. Note that a higher dimensional variant has also been developed \[66\].

Consider a one-dimensional lattice with vertices \( V = \{1, \ldots, N\} \). The Hilbert space of the spin-1/2 model on \( V \) is given by \( \mathcal{H} := \bigotimes_{j \in V} \mathcal{H}_j \) with \( \mathcal{H}_j \cong \mathbb{C}^2 \). We denote by \( X_j, Y_j, Z_j \in \mathcal{A}_{\{j\}} \) the Pauli operators acting on site \( j \) of the spin chain. Then the Jordan-Wigner-Transformation is given by

\[
    f_j + f_j^\dagger = w_{2j-1} := X_j \prod_{j' < j} Z_{j'},
\]

\[
    if_j - if_j^\dagger = w_{2j} := Y_j \prod_{j' < j} Z_{j'},
\]

where the \( \{w_j\}_{j=1}^{2N} \) are called Majorana operators. The Majorana operators satisfy the anti-commutation relation \( \{w_j, w_k\} = 2 \delta_{j,k} \). It can be verified with elementary calculations that

\[
    f_j = \frac{1}{2} (w_{2j-1} - iw_{2j}),
\]

\[
    f_j^\dagger f_j = \frac{1}{2} (1 - iw_{2j-1} w_{2j}),
\]

as well as

\[
    Z_j = -iw_{2j-1} w_{2j} = 2f_j^\dagger f_j - 1, \tag{41}
\]

\[
    X_j = w_{2j-1} \prod_{j' < j} Z_{j'}, \tag{42}
\]

\[
    Y_j = w_{2j} \prod_{j' < j} Z_{j'}, \tag{43}
\]

and

\[
    \forall j \leq k : \quad f_j^\dagger f_k = \frac{1}{4} S_j^+ ( \prod_{j \leq j' < k} Z_{j'} ) S_k^-, \quad \text{where} \quad S_j^\pm := X_j \pm iY_j. \tag{44}
\]

Most importantly, as can be seen from Eq. \(44\), the Jordan-Wigner-Transformation preserves locality in the sense that a one-dimensional fermionic Hamiltonian with nearest neighbor or short range hopping and short range density-density interactions is mapped to a spin chain Hamiltonian with only short range interactions.

6 Conclusion

We have reviewed the Heisenberg picture for time-dependent Liouvillian dynamics in spin lattice systems. For this setting we have stated a Lieb-Robinson bound. Such
bounds give rise to a plethora of statements about locally interacting systems which we have reviewed subsequently. Finally, we have explained the relevance for fermionic systems. We hope that this text serves as an introduction to Liouvillian dynamics on spin lattice systems and provides an overview of important consequences of Lieb-Robinson bounds.

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