Dependence of Poynting vector on state of polarization

Xiao-Lu You and Chun-Fang Li

Department of Physics, Shanghai University,
99 Shangda Road, 200444 Shanghai, China

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Abstract

The dependence of the Poynting vector on the state of polarization is theoretically investigated by making use of the non-paraxial superposition of four plane waves. It is found that the Poynting vector in such a field depends on a constant unit vector, called the Stratton vector, in addition to the well-known Jones vector. It is shown that the Stratton vector is a degree of freedom to determine the polarization bases for the non-paraxial field. Only when combined with the Stratton vector can the Jones vector be able to completely describe the state of its polarization.
The Poynting vector is the intensity of energy flow in the electromagnetic field \[1\]. The cycle-averaged value of the Poynting vector in a monochromatic field takes the form,

\[
\mathbf{g} = \frac{1}{2} \Re(\mathbf{E} \times \mathbf{H}^*),
\]

where \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic fields, respectively. About two decades ago, Katsenelenbaum \[2\] studied the direction of the Poynting vector in an electromagnetic field that is formed by the superposition of four plane monochromatic waves of linear polarization. The four waves propagate in the directions that make the same acute angle with the \( z \) axis. He found that in the transverse plane, the Poynting vector in some domains is oriented in the opposite direction to the propagation of the plane waves. That is to say, the \( z \)-component of the Poynting vector in those domains is negative. That phenomenon was recently utilized \[3\] to explain the inverse energy flow in a sharp focus. We will show that such a phenomenon lies with the choice of the concrete states of linear polarization of the constituent plane waves. When the states of linear polarization are appropriately adjusted, the \( z \)-component of the resultant Poynting vector can be non-negative. The purpose of this Letter is to make use of the four-plane-wave model to investigate the dependence of the Poynting vector on the state of polarization.

To this end, we first consider a situation that is similar to what Katsenelenbaum discussed. The wavevectors of the four plane waves in free space are as follows,

\[
\begin{align*}
\mathbf{k}_1 &= k(\hat{z}\cos\vartheta + \hat{x}\sin\vartheta), \\
\mathbf{k}_2 &= k(\hat{z}\cos\vartheta + \hat{y}\sin\vartheta), \\
\mathbf{k}_3 &= k(\hat{z}\cos\vartheta - \hat{x}\sin\vartheta), \\
\mathbf{k}_4 &= k(\hat{z}\cos\vartheta - \hat{y}\sin\vartheta),
\end{align*}
\]

which are schematically indicated in Fig. 1 by the green arrows, where \( k \) is the wavenumber, \( \hat{x}, \hat{y}, \) and \( \hat{z} \) denote the unit vectors along the corresponding axes, and \( \vartheta \) is the angle that the wavevectors make with the \( z \) axis. Their electric fields are chosen to be

\[
\mathbf{E}_i^+ = \frac{\mathbf{a}_i^+}{\sqrt{2\varepsilon_0}} \exp(i\mathbf{k}_i \cdot \mathbf{x}), \quad i = 1, 2, 3, 4,
\]
where the real unit vectors
\[ a_1^\perp = \hat{x} \cos \vartheta - \hat{z} \sin \vartheta, \]
\[ a_2^\perp = \hat{x}, \]
\[ a_3^\perp = \hat{x} \cos \vartheta + \hat{z} \sin \vartheta, \]
\[ a_4^\perp = \hat{x}, \]

are the polarization vectors of the respective electric fields which are linearly-polarized and are schematically indicated in Fig. 1 by the red arrows, the meaning of the superscript ∥ will be clear later, and the factor \( \frac{1}{\sqrt{2\varepsilon_0}} \) is introduced for convenience. The electric field of the superposition of the four plane waves takes the form
\[ E^\perp = \sqrt{\frac{2}{\varepsilon_0}} [\hat{x}(\cos \vartheta \cos k_\perp x + \cos k_\perp y) - i\hat{z} \sin \vartheta \sin k_\perp x] \exp(ik_\parallel z), \]

the transverse component of which is polarized only in the \( x \) direction, where \( k_\perp = k \sin \vartheta \) and \( k_\parallel = k \cos \vartheta \). According to Maxwell’s equations, the corresponding magnetic field reads
\[ H^\perp = \sqrt{\frac{2}{\mu_0}} [\hat{y}(\cos \vartheta \cos k_\perp y + \cos k_\perp x) - i\hat{z} \sin \vartheta \sin k_\perp y] \exp(ik_\parallel z). \]

Its transverse component is polarized only in the \( y \) direction. Both electric field (3) and magnetic field (4) have \( z \)-polarized longitudinal components.

According to expression (1) for the Poynting vector, it is straightforward to make use of the electric and magnetic fields (3)-(4) to obtain the \( z \)-component of the Poynting vector,
which assumes

\[ g_z^\perp = \cos \vartheta (\cos k_\perp x + \cos k_\perp y)^2 + (1 - \cos \vartheta)^2 \cos k_\perp x \cos k_\perp y, \tag{5} \]

where the factor of the speed of light, \( c = 1/\sqrt{\varepsilon_0 \mu_0} \), is omitted. It is clear that in the transverse plane, there are domains in which \( g_z^\perp \) is less than zero. After all, there always exist such points at which one has \( \cos (k_\perp x) = -\cos (k_\perp y) \neq 0 \) and therefore \( g_z^\perp < 0 \). It is seen that only the second term on the right side of Eq. (5) can be negative. Since the first non-negative term is proportional to \( \cos \vartheta \), so under the extremely non-paraxial condition in which \( \vartheta = \pi/2 \), Eq. (5) reduces to \( g_z^\perp = \cos k_x \cos k_y \). The maximum of its negative value is equal to the maximum of its positive value. In this case, electric field (3) and magnetic field (4) reduce to

\[ \mathbf{E}^\perp = \sqrt{\frac{2}{\varepsilon_0}} (\hat{x} \cos k_y - i \hat{z} \sin k_x), \quad \mathbf{H}^\perp = \sqrt{\frac{2}{\mu_0}} (\hat{y} \sin k_x - i \hat{z} \cos k_y), \]

respectively. The amplitude of their “longitudinal” components is the same as that of their “transverse” components. On the other hand, in the zeroth-order paraxial approximation in which \( \sin \vartheta \approx 0 \), one has \( g_z^\perp \approx 4 \), which is positive. This is because in such an approximation, expressions (3) and (4) tend to the electric and magnetic fields of a linearly-polarized plane wave,

\[ \mathbf{E}^\perp \approx 2 \sqrt{\frac{2}{\varepsilon_0}} \hat{x} \exp(ikz), \quad \mathbf{H}^\perp \approx 2 \sqrt{\frac{2}{\mu_0}} \hat{y} \exp(ikz), \]

respectively. Their longitudinal components all vanish. As a matter of fact, in the first-order paraxial approximation in which \( \sin \vartheta \approx \vartheta \) and \( \cos \vartheta \approx 1 \), one has \( g_z^\perp \approx (\cos k\vartheta x + \cos k\vartheta y)^2 \), which is non-negative. This indicates that the negative value of \( g_z^\perp \) comes from higher terms. Therefore, the larger the angle \( \vartheta \) is, the bigger the maximum of the negative value of \( g_z^\perp \) is.

Expression (5) can also be written as

\[ g_z^\perp = \cos \vartheta (\cos k_\perp x - \cos k_\perp y)^2 + (1 + \cos \vartheta)^2 \cos k_\perp x \cos k_\perp y, \]

showing that \( g_z^\perp \geq 0 \) on the bisectrices \( x = \pm y \). A typical distribution of normalized \( g_z^\perp \) in the transverse plane is illustrated in Fig. 2 where \( \vartheta = 2\pi/5 \).

The \( g_z^\perp \) in (5) being negative in some domains is similar to what Katsenelenbaum found in Ref. [2]. To demonstrate that this phenomenon lies with the concrete states of linear polarization expressed by polarization vectors (2), we only adjust the polarization vectors
FIG. 2. Distribution of normalized $g^\perp_z$ in the transverse plane, where $\vartheta = 2\pi/5$, $|k_x| \leq 2.5\pi$, and $|k_y| \leq 2.5\pi$.

of the four linearly-polarized plane waves and write their electric fields as follows,

$$E_i^\parallel = \frac{a_i^\parallel}{\sqrt{2\varepsilon_0}} \exp(ik_i \cdot x),$$

where the polarization vectors $a_i^\parallel$ are given by

$$a_1^\parallel = -\hat{z}\sin \vartheta + \hat{x}\cos \vartheta,$$

$$a_2^\parallel = -\hat{z}\sin \vartheta + \hat{y}\cos \vartheta,$$

$$a_3^\parallel = -\hat{z}\sin \vartheta - \hat{x}\cos \vartheta,$$

$$a_4^\parallel = -\hat{z}\sin \vartheta - \hat{y}\cos \vartheta,$$

and the meaning of the superscript $\parallel$ will be clear shortly. The same as polarization vectors (2), all the polarization vectors in (6) stand for states of linear polarization. In this situation, the electric field of the superposition of the four plane waves becomes

$$E^\parallel = i \sqrt{\frac{2}{\varepsilon_0}} [(\hat{x}\sin k_x + \hat{y}\sin k_y) \cos \vartheta + i\hat{z}(\cos k_x + \cos k_y) \sin \vartheta] \exp(ik_z),$$

which has a $y$-polarized transverse component in addition to the $x$-polarized one. Accordingly, the corresponding magnetic field becomes

$$H^\parallel = i \sqrt{\frac{2}{\mu_0}} (\hat{y}\sin k_x - \hat{x}\sin k_y) \exp(ik_z).$$

The same as electric field (7), it has both $x$- and $y$-polarized transverse components. But its $z$-polarized component vanishes. As a consequence, the $z$-component of the Poynting vector
FIG. 3. Distribution of normalized $g^\parallel_z$ in the transverse plane, where the value of $\vartheta$ is the same as in Fig. 2 $|k_{\perp x}| \leq 2.5\pi$, and $|k_{\perp y}| \leq 2.5\pi$.

takes the form,

$$g^\parallel_z = \cos \vartheta (\sin^2 k_{\perp x} + \sin^2 k_{\perp y}),$$  \hspace{1cm} (9)

where the constant factor $c$ is omitted as before. In contrast with $g^\perp_z$, $g^\parallel_z$ is non-negative. It vanishes at points $k_{\perp x} = m\pi$ and $k_{\perp y} = n\pi$, where $m$ and $n$ are integers. And it is maximum at points $k_{\perp x} = (m + 1/2)\pi$ and $k_{\perp y} = (n + 1/2)\pi$. A typical distribution of normalized $g^\parallel_z$ in the transverse plane is illustrated in Fig. 3 where the value of $\vartheta$ is the same as in Fig. 2.

It is interesting to note that whether under the extremely non-paraxial condition or under the zeroth-order paraxial condition, $g^\parallel_z$ completely vanishes. In the former case, $\vartheta = \pi/2$, electric field (7) has only a $z$-polarized component,

$$\mathbf{E}^\parallel = -\sqrt{\frac{2}{\varepsilon_0}} \hat{z} (\cos kx + \cos ky);$$

and magnetic field (8) reduces to

$$\mathbf{H}^\parallel = i\sqrt{\frac{2}{\mu_0}} (\hat{y} \sin kx - \hat{x} \sin ky).$$

So certainly the $z$-component of the Poynting vector in such a field is equal to zero by virtue of Eq. (11). Whereas in the latter case, $\sin \vartheta = 0$, both the electric and magnetic fields vanish.

It is seen that even though the constituent plane waves in both situations are linearly polarized, electric fields (3) and (7) are different from each other. In particular, the former
tends to the electric field of a linearly-polarized plane wave in the zeroth-order paraxial approximation; whereas the latter vanishes completely in the same approximation. To explain how they are different, we resort to the constant unit vector that was first introduced by Stratton [1] and later by others [4–6] in the representation of electromagnetic fields, called the Stratton vector. According to Ref. [7], whether electric field (3) or electric field (7) can be written as follows,

\[
E = \sum_{i=1}^{4} \frac{a_i}{\sqrt{2\varepsilon_0}} \exp(ik_i \cdot x),
\]

where the polarization vector \(a_i\) of the constituent plane wave is expressed as

\[
a_i = \alpha_1 u_i + \alpha_2 v_i,
\]

\(u_i\) and \(v_i\) are the polarization bases of the plane wave that are orthogonal to each other and are determined by the Stratton vector, denoted by \(I\), in the following way,

\[
u_i = v_i \times \frac{k_i}{k}, \quad v_i = \frac{I \times k_i}{|I \times k_i|},
\]

and complex constants \(\alpha_1\) and \(\alpha_2\) satisfying \(|\alpha_1|^2 + |\alpha_2|^2 = 1\) make up the Jones vector, \(\alpha \equiv \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}\), for all the four plane waves. But it is easily checked that the Jones vector in either situation is the same, \(\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\). The polarization bases are not, however. In the first situation, the polarization bases are determined by the Stratton vector \(I^\perp = -\hat{x}\) that is perpendicular to the \(z\) axis. As a result, both electric field (3) and magnetic field (4) have non-vanishing longitudinal components. The transverse component of the electric field is polarized in the \(x\) direction and the transverse component of the magnetic field is polarized in the \(y\) direction. In the second situation, the polarization bases are determined by the Stratton vector \(I^\parallel = \hat{z}\), which is parallel to the \(z\) axis. In this situation, the longitudinal component of magnetic field (8) vanishes. But whether the electric field or the magnetic field has non-vanishing \(x\)- and \(y\)-components. From these discussions it is concluded that when the Jones vector is fixed, the polarization state of electric field (10), as well as of the corresponding magnetic field, depends on the choice of the Stratton vector. The difference between expressions (5) and (9) just reflects this dependence.

Nevertheless, it is pointed out that when the Stratton vector is chosen to be either \(I^\perp\) or \(I^\parallel\), the \(z\)-component of the Poynting vector in non-paraxial field (10) does not depend on
the Jones vector. This does not mean that the Poynting vector itself has nothing to do with the Jones vector. The point here is that the transverse component of the Poynting vector depends on the Jones vector. Indeed, when $\mathbf{I} = \mathbf{I}^\perp$, electric field (10) for any particular Jones vector takes the form

$$E^\perp = \frac{1}{\sqrt{2\varepsilon_0}}(\alpha_1 U^\perp + \alpha_2 V^\perp),$$  \hspace{1cm} (12)

where

\begin{align*}
U^\perp &= \sum_{i=1}^{4} u_i^\perp \exp(i\mathbf{k}_i \cdot \mathbf{x}) = 2[\hat{x}(\cos \vartheta \cos k_{\perp} x + \cos k_{\perp} y) - i\hat{z} \sin \vartheta \sin k_{\perp} x] \exp(ik_{\parallel} z), \\
V^\perp &= \sum_{i=1}^{4} v_i^\perp \exp(i\mathbf{k}_i \cdot \mathbf{x}) = 2[\hat{y}(\cos \vartheta \cos k_{\perp} y + \cos k_{\perp} x) - i\hat{z} \sin \vartheta \sin k_{\perp} y] \exp(ik_{\parallel} z),
\end{align*}

and $u_i^\perp$ and $v_i^\perp$ are obtained from Eqs. (11) with $\mathbf{I}$ being replaced with $\mathbf{I}^\perp$. If $\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, electric field (12) goes back to electric field (3). Accordingly, the magnetic field reads

$$H^\perp = \frac{1}{\sqrt{2\mu_0}}(\alpha_1 V^\perp - \alpha_2 U^\perp).$$

The Poynting vector in this field is obtained from Eq. (1) to be

$$g^\perp = \hat{z}g_x^\perp + \hat{x}\sigma \sin \vartheta(\cos k_{\perp} x + \cos \vartheta \cos k_{\perp} y) \sin k_{\perp} y$$

$$\quad + \hat{y}\sigma \sin \vartheta(\cos k_{\perp} y + \cos \vartheta \cos k_{\perp} x) \sin k_{\perp} x,$$

where the constant factor $c$ is omitted as before, $g_x^\perp$ is given by Eq. (5), and $\sigma = -i(\alpha_1^*\alpha_2 - \alpha_2^*\alpha_1)$ is the ellipticity of polarization. Both the $x$- and $y$-components depend on the Jones vector $\alpha$ through $\sigma$ though the $z$-component does not. Whereas when $\mathbf{I} = \mathbf{I}^\parallel$, electric field (14) for any particular Jones vector assumes

$$E^\parallel = \frac{1}{\sqrt{2\varepsilon_0}}(\alpha_1 U^\parallel + \alpha_2 V^\parallel),$$  \hspace{1cm} (14)

where

\begin{align*}
U^\parallel &= \sum_{i=1}^{4} u_i^\parallel \exp(i\mathbf{k}_i \cdot \mathbf{x}), \\
V^\parallel &= \sum_{i=1}^{4} v_i^\parallel \exp(i\mathbf{k}_i \cdot \mathbf{x}),
\end{align*}

and $u_i^\parallel$ and $v_i^\parallel$ are obtained from Eqs. (11) with $\mathbf{I}$ being replaced with $\mathbf{I}^\parallel$. If $\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, electric field (14) goes back to electric field (7). The same procedure gives for the resultant Poynting vector,

$$g^\parallel = \hat{z}g_z^\parallel + \frac{\sigma}{2}(\hat{x} \sin k_{\perp} y - \hat{y} \sin k_{\perp} x)(\cos k_{\perp} x + \cos k_{\perp} y) \sin 2\vartheta,$$  \hspace{1cm} (15)
where $g_\parallel$ is given by Eq. (9). Its $x$- and $y$-components also depend on the Jones vector through $\sigma$.

It is noticed that whether Poynting vector (13) or (15) does not depend on the two elements of the Jones vector $\alpha$ individually. They depend on the Jones vector through $\sigma$.

To explain this, we rewrite electric field (10) as

$$E = \frac{1}{\sqrt{2\varepsilon_0}}(\alpha_1 U + \alpha_2 V), \quad (16)$$

where

$$U = \sum_{i=1}^{4} u_i \exp(i k_i \cdot x), \quad V = \sum_{i=1}^{4} v_i \exp(i k_i \cdot x), \quad (17)$$

and $u_i$ and $v_i$ are given by Eqs. (11). Being the Jones vector of all the constituent plane waves of the electric field (10), $\alpha$ should be viewed as the Jones vector of the entire electric field (16). By this it is meant that the complex-valued vector functions $U$ and $V$ play the role of the polarization bases for it. Such polarization bases are different from those for the electric field of a plane wave in that they usually do not obey $U^* \cdot V = 0$ as can be readily checked with $U^\perp$ and $V^\perp$ in Eq. (12). Nevertheless, with the help of Eqs. (11), it is not difficult to prove that they are orthogonal to each other in the following sense,

$$\iiint U^* \cdot V \, dx \, dy \, dz = 0.$$ 

From this property it follows that $\alpha_1$ and $\alpha_2$ in electric field (16) are indeed the two elements of its Jones vector. They are proportional to the projections of electric field (16) onto $U$ and $V$,

$$\alpha_1 \propto \iiint U^* \cdot E \, dx \, dy \, dz, \quad \alpha_2 \propto \iiint V^* \cdot E \, dx \, dy \, dz,$$

respectively.

It is important to emphasize that for the same Jones vector $\alpha$, electric fields (12) and (14) are different in polarization. That is to say, the Jones vector is not able to completely describe the state of polarization of electric field (16). This is understandable. Equations (17) together with Eqs. (11) tell that the Stratton vector shows up as a degree of freedom to determine the polarization bases for the entire electric field (16). Only when combined with the Stratton vector can the Jones vector be able to completely describe the state of its polarization. As a result, when the Jones vector is fixed, the state of polarization of electric field (16) depends on the choice of the Stratton vector. A comparison of expressions...
and (15) indicates that given the Jones vector, the Stratton vector affects not only the $z$-component of the Poynting vector but also the transverse component.

In summary, we made use of the non-paraxial superposition of four plane waves to discuss the dependence of the Poynting vector on the state of polarization. It is known that the state of polarization of a plane wave is described by its Jones vector. Here we showed that for non-paraxial electric field (16), the Jones vector $\alpha$ is not able to completely describe the state of its polarization. In addition to the Jones vector, the Stratton vector $I$ turns out to be another degree of freedom. It plays the role of determining the polarization bases $U$ and $V$ through Eqs. (17) and (11). Therefore, it combines the Jones vector to completely describe the state of polarization of the entire electric field (16). We found that when the Jones vector is fixed, the Poynting vector depends on the choice of the Stratton vector. Furthermore, when the Stratton vector is chosen to be either $I_{\perp}$ or $I_{\parallel}$, the transverse component of the Poynting vector depends on the Jones vector though the $z$-component does not.

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