On Diffusion Processes with Drift in a Morrey Class Containing $L_{d+2}$

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Abstract
We present new conditions on the drift of the Morrey type with mixed norms allowing us to obtain Aleksandrov type estimates of potentials of time inhomogeneous diffusion processes in spaces with mixed norms and, for instance, in $L_{d_0+1}$ with $d_0 < d$.

Keywords Diffusion processes · Singular drift · Aleksandrov estimates

Mathematics Subject Classification 60H10 · 35K10

1 Introduction and Main Results
Let $\mathbb{R}^d$ be a Euclidean space of points $x = (x^1, \ldots, x^d)$, $d \geq 2$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, let $\mathcal{F}_t$, $t \geq 0$, be an increasing family of complete $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$, and let $w_t$ be an $\mathbb{R}^d$-valued Wiener process relative to $\mathcal{F}_t$. Fix $\delta \in (0, 1)$ and denote by $S_\delta$ the set of $d \times d$ symmetric matrices whose eigenvalues are between $\delta$ and $\delta^{-1}$.

Assumption 1.1 On $\mathbb{R}^{d+1}$ we are given a smooth $S_\delta$-valued function $\sigma(t, x)$ and a smooth $\mathbb{R}^d$-valued function $b(t, x)$ with compact support.

Under this assumption the solutions of the system

$$x_s = x + \int_0^s \sigma(t_r, x_r) \, dw_r + \int_0^s b(t_r, x_r) \, dr, \quad t_s = t + s$$

form a strong Markov process $(t_s, x_s)$. Our goal in this article is to find conditions on the drift of the Morrey type with mixed norms still allowing us to obtain Aleksandrov type estimates of potentials of $(t_s, x_s)$ in spaces with mixed norms and, for instance, in $L_{d_0+1}$ with $d_0 < d$. 

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We take and fix in the whole article two sets of numbers satisfying
\[ p_0, q_0 \in [1, \infty], \quad \frac{d_0}{p_0} + \frac{1}{q_0} = 1, \quad (1.2) \]
\[ p, q \in [1, \infty], \quad \frac{d_0}{p} + \frac{1}{q} = 1, \quad (1.3) \]
where \( d_0 = d_0(d, \delta) \in (d/2, d) \) is defined in Sect. 2.

Introduce
\[ B_R = \{ x \in \mathbb{R}^d : |x| < R \}, \quad B_R(x) = x + B_R, \quad C_{T,R} = [0, T] \times B_R, \]
\[ C_{T,R}(t, x) = (t, x) + C_{T,R}, \quad C_R(t, x) = C_{R^2,R}(t, x), \quad C = C_{R(0, 0)}, \]
and let \( C_R \) be the collection of cylinders \( C_R(t, x), \quad (t, x) \in \mathbb{R}^{d+1}, \quad C = \{ C_R : R > 0 \} \).

Introduce \( b \) as a constant such that, for any \( R \in (0, \infty) \) and \( C \in C_R \),
\[ \|b I_C\|_{L_{p_0,q_0}} \leq \hat{b} R^{d/p_0 + q_0 - 1}, \quad (1.4) \]
where the norm \( \| \cdot \|_{L_{p,q}} \) is introduced as follows.

For \( p, q \in [1, \infty) \) we introduce the space \( L_{p,q} \) as the space of Borel functions on \( \mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d \) such that
\[ \| f \|^q_{L_{p,q}} := \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |f(t, x)|^p dx \right)^{q/p} dt < \infty \]
if \( p \geq q \) or
\[ \| f \|^p_{L_{p,q}} := \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}} |f(t, x)|^q dt \right)^{p/q} dx < \infty \]
if \( p \leq q \) with natural interpretation of these definitions if \( p = \infty \) or \( q = \infty \). We write \( f \in L_{p,q}(Q) \) to mean that \( f I_Q \in L_{p,q} \). Observe that \( p \) is associated with \( x \) and \( q \) with \( t \) and the interior integral is always elevated to the power \( \leq 1 \). If \( p = q \) we abbreviate \( L_{p,p} \) to \( L_p \), which is \( L_p(\mathbb{R}^{d+1}) \). We use the same symbol \( L_p \) for \( L_p(\mathbb{R}^{d+1}) \) and hope that its meaning will be clear from the context. The necessity to define \( L_{p,q} \)-norms differently for \( p \geq q \) and \( p \leq q \) is dictated by the form in which we present our version of the parabolic Aleksandrov estimate in Theorem 1.1.

Since \( b \) is bounded and has compact support, such a \( \hat{b} < \infty \) satisfying (1.4) does exist.

We are now ready to present two main results of the article in which by \( \tau_R \) we mean the first exit time of \( x_s \) from \( B_R \). These theorems are proved in Sects. 3 and 4.

**Theorem 1.1** Under Assumption 1.1 there is a constant \( \hat{b} < \infty \) depending only on \( d, \delta \), such that if (1.4) holds for any \( R \in (0, \infty) \) and \( C \in C_R \), then for any \( R \in (0, \infty), \quad x \in \mathbb{R}^d \) and Borel \( f \geq 0 \)
\[ I := E_{0,x} \int_0^{\tau_R} f(t, x_t) dt \leq \hat{N} R^{2 - d/p - 2/q} \| f \|_{L_{p,q}}, \quad (1.5) \]
where \( \hat{N} \) depends only on \( d, \delta \).

It turns out that the global condition (1.4) can be replaced with a local one at the expense of losing good control on \( \hat{N} \) in the estimate.
Theorem 1.2 Let $R_0 \in (0, \infty)$. Under Assumption 1.1 there is a constant $\hat{b} < \infty$ depending only on $d$, $\delta$, such that if (1.4) holds for any $R \in (0, R_0]$ and $C \in C_R$, then for any $R \in (0, \infty)$, $x \in \mathbb{R}^d$ and Borel $f \geq 0$

$$E_{0,x} \int_0^{\tau_R} f(t, x_t) \, dt \leq \hat{N} \|f\|_{L_{p,q}},$$

(1.6)

where $\hat{N}$ depends only on $d$, $\delta$, $R$, and $R_0$.

Remark 1.1 By shifting the origin in $\mathbb{R}^{d+1}$ one obtains estimates similar to (1.5) and (1.6) for the process starting from any point like

$$E_{t,x} \int_0^{\tau_R(v)} f(s, x_s) \, ds \leq \hat{N} \|f\|_{L_{p,q}},$$

where $\tau_R(y)$ is the first exit time of $x_s$ from $B_R(y)$.

The above result have some implications on relaxing the integrability requirement on $b$ in elliptic and parabolic Aleksandrov’s estimates.

In this part of the section Assumption 1.1 is replaced with the following.

Assumption 1.2 On $\mathbb{R}^{d+1}$ we are given a Borel $S_{\delta}$-valued function $\sigma(t, x)$ and a Borel $\mathbb{R}^d$-valued function $b(t, x)$ such that, for an $R_0 \in (0, \infty)$, estimate (1.4) holds with $\hat{b} = \hat{b}(d, \delta)$ from Theorem 1.1 for any $R \in (0, R_0]$ and $C \in C_R$.

Introduce

$$D_t = \frac{\partial}{\partial x^t}, \quad D_{ij} = D_i D_j, \quad \partial_t = \frac{\partial}{\partial t},$$

$$a = (1/2)\sigma^2, \quad Lu = a^{ij} D_{ij} u + b^i D_i u.$$ 

Recall that for a domain $Q \subset \mathbb{R}^{d+1}$ one denotes by $\partial' Q$ its parabolic boundary defined as the set of all points on $\partial Q$ which are endpoints of continuous curves of type $(t, x_t)$, $t \in [a, b]$, which start in $Q$ and belong to $Q$ for all $t < b$. Define $W^{1,2}_{p,q,loc}(Q)$ as the set of functions such that $u, Du, D^2 u, \partial_t u \in L_{p,q,loc}(Q)$. Here is a quantitative maximum principle that is a parabolic Aleksandrov estimate.

Theorem 1.3 Under Assumption 1.2 let $Q \subset \mathbb{R} \times B_R$ be bounded and $u \in W^{1,2}_{p,q,loc}(Q) \cap C(Q)$. Take a function $c \geq 0$ on $Q$. Then on $Q$

$$u \leq N \|I_{Q,u>0}(\partial_t u + Lu - cu)_-\|_{L_{p,q}} + \sup_{\partial' Q} u_+,$$

(1.7)

where $N$ depends only on $d$, $\delta$, $R_0$, and $R$.

In particular (the maximum principle), if $\partial_t u + Lu - cu \geq 0$ in $Q$ and $u \leq 0$ on $\partial' Q$, then $u \leq 0$ in $Q$.

The proof of this theorem coincides with that of Theorem 5.1 of [21] apart from the point that after reducing the general case to the one in which $u \in W^{1,2}_{p,q}(Q)$ and $b$ is bounded, here, we approximate $\sigma$, $b$, and $u$ with smooth functions by using mollifiers. Of course, we observe that the assumption about $b$ is preserved under this operation. After that, as in [21], Itô’s formula and Theorem 1.1 allow us to finish the proof.

An adaptation of Theorem 1.3 to elliptic operators yields an elliptic Aleksandrov estimate and shows advantages of having mixed norm estimates for parabolic operators. Let us use the symbol $L_p$ to mean either $L_p(\mathbb{R}^d)$ or $L_p(\mathbb{R}^{d+1})$. What is the meaning in each concrete case will be quite clear from the context. For instance, in the following theorem $L_{d_0} = L_{d_0}(\mathbb{R}^d)$.
Theorem 1.4 Under Assumption 1.2 let $\Omega$ be a bounded domain in $\mathbb{R}^d$, $u \in W^2_{d_0,\text{loc}}(\Omega) \cap C(\bar{\Omega})$. Also assume that $a$ and $b$ are independent of $t$ and $p = p_0 = d_0$ ($q = q_0 = \infty$). Take a function $c \geq 0$. Then on $\Omega$

$$u \leq N\|I_{\Omega, u > 0}(Lu - cu)_-\|_{L^d_{d_0}} + \sup_{\partial \Omega} u_+,$$

where $N$ depends only on $d$, $\delta$, $R_0$, and the diameter of $\Omega$.

Proof Take $\varepsilon > 0$, let $v(t, x) = e^{-\varepsilon t}u(x)$ and apply (1.7) to $v$ and $(0, T) \times \Omega$ in place of $u$ and $Q$, respectively. Then we get that on $\Omega$

$$u \leq N\|I_{\Omega, u > 0}(-\varepsilon u + Lu - cu)_-\|_{L^d_{d_0}} + \sup_{\partial \Omega} u_+ + e^{-T} \sup_{\partial \Omega} u_+.$$

Letting $T \to \infty$ eliminates the last term. After that letting $\varepsilon \downarrow 0$ and observing that $(-\varepsilon u + Lu - cu)_- \leq \varepsilon \sup |u| + (Lu - cu)_-$ yields (1.8). The theorem is proved.

In case when $d_0$ in Theorem 1.4 is replaced by $d$ and $b \in L_d$, the result belongs to A.D. Aleksandrov (1960), see Theorem 8 in [1,2]. The proofs are given in 1963 in [3,4]. Our Theorem 1.4 extends Aleksandrov’s result in reducing the power of summability of both: the drift and the free terms. However, we treat only the uniformly nondegenerate case.

There was a considerable interest in reducing $L_d$-norm of the free term to $L_{d_0}$-norm with $d_0 < d$. This was achieved by Cabré [6] for bounded $b$ and by Fok in [10] for $b \in L_{d+\varepsilon}$. In [18] the author allowed $b \in L_d$ and the free term in $L_{d_0}$ with $d_0 < d$. This made it possible to develop in [24] a $W^2_d$-solvability theory for linear equations with $b \in L_d$ and $p < d$.

Applied to fully nonlinear equations we can now treat $W^2_{d_0}$-solvability with “the coefficients” of the first order terms in $L_d$ (see [19]). In Theorem 1.1 of [8] our Theorem 1.4 is proved, loosely speaking, when $p = p_0 = d_0$, but we only treat true solutions. Our Theorem 1.3 is not of class $L_{d,\text{loc}}$.

The results like Theorem 1.3 are indispensable in the theory of controlled diffusion processes (see, for instance, [15]). First such result with bounded $b$ and $d$ in place of $d_0$ and $L_{d+1}$ in place of $L_{p,q}$ was published in [13,14]. It was extended by A.I. Nazarov and N.N. Uralt’seva [26] to allow $b \in L_{d+1}$. The author in [16] developed a general approach to such estimates and slightly improved the result of [26]. By using this approach A.I. Nazarov in [25] for the first time proved estimates in $L_{p,q}$-spaces with $b \in L_{p_0,q_0}$ when $d_0$ in (1.2) and (1.3) is replaced by $d$ but no further restrictions like (1.4) are imposed.

In [7] by extending some earlier results by Wang the authors prove Theorem 1.3 for $L_p$-viscosity solutions with $p = q < d + 1$ when $b$ is bounded. In our situation we have some freedom in choosing $p$, $q$ and $b \in L_{p_0,q_0}$, but we only treat true solutions. Our Theorem 1.3 covers Theorem 2.4 of [7] on the account of having mixed norms and $b \in L_{p_0,q_0}$.

The author in [21] gave a version of the result of [25] by allowing $f \in L_{p,q}$ with $p$, $q$ satisfying (1.3) and $b \in L_{p_0,q_0}$ but $p_0$, $q_0$ are supposed to satisfy (1.2) with $d$ in place of $d_0$ and (1.4) is imposed. Relaxing the assumptions on $f$ further in [22] allowed the author to investigate fine properties of the corresponding diffusion processes and lead in [23] to proving the solvability in Sobolev space $W^{1,2}_p$, where $p < d + 1$, of the equation

$$\partial_t u + \Delta u + b^i D_i u = f$$

when $f \in L_q$ with $q$ large enough, $b \in L_{d+1}$, and (1.4) is satisfied with $p_0 = q_0 = d + 1$. In this case it turns out that $b^i D_i u \in L_p$.  

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In this article we relax the restrictions on $b$ from [21] allowing $p_0, q_0$ to be as in (1.2). Observe that the function $b$ with

$$|b(t, x)| = I_{C_1}(t, x) \frac{1}{|x|} \left( \frac{|x|}{\sqrt{t}} \right)^{2/(d+1)}$$

satisfies (1.4) with finite $\hat{b}$ and $p_0 = q_0 = d_0 + 1$ but does not belong to $L_{d+1}$.

Condition (1.4), however, does not allow us to derive from the results presented here even the estimates from [26] and [16]. The point is that, if we have $b \in L_r$ and ask ourselves what $r$ should be in order to have $b \in L_{p_0,q_0}$ satisfying (1.4), then the answer is: $r \geq d + 2$ (thus explaining the title of the paper). So $b \in L_{d+1}$ are not good enough. This is somewhat discouraging but, most likely $b \in L_{d+1}$ is not good enough to have solvability in any $W^{1,2}_r$ of (1.9). In any case, by imposing (1.4), say with $p_0 = q_0 = d_0 + 1$, we allow stronger local singularities of $b$, as compared to $b \in L_{d+1}$, spread sufficiently far apart.

This article have some similarity to [21], in particular, we borrow from there quite a few results. The main difference is that in [21] we start by considering general Itô processes, which led to the requirement $q_0 < \infty$, and here we start by considering Markov diffusion processes with regular drift and diffusion coefficients. Therefore we narrow the class of processes under investigation but gain obtaining better estimates by exploiting the fact that we have solutions of equations coefficients of which we can change (see Sect. 3). Our results could possibly be relevant in investigations described in [5, 27, 28] and the references therein, where the authors present very strong results on fine properties of diffusion processes under some regularity assumptions on $\sigma$ and $b \in L_{p,q}$ with $d/p + 2/q \leq 1$. We add some information about Green’s functions of such processes.

The rest of the article is organized as follows. Section 2 contains some auxiliary results part of which is borrowed from [21]. The other part of the section is devoted to proving better summability of Green’s functions and mixed norm estimates. In Sect. 3 we prove Theorem 1.1. Theorem 1.2 is proved in Sect. 4. Section 5 is an “Appendix” where we prove a version of Gehring’s lemma used in Sect. 2.

We finish the section by some notation. If $\Gamma$ is a measurable subset of $\mathbb{R}^{d+1}$ we denote by $|\Gamma|$ its Lebesgue measure. The same notation is used for measurable subsets of $\mathbb{R}^d$ with $d$-dimensional Lebesgue measure. We hope that it will be clear from the context which Lebesgue measure is used. If $\Gamma$ is a measurable subset of $\mathbb{R}^{d+1}$ and $f$ is a function on $\Gamma$ we denote

$$\int_{\Gamma} f \, dx \, dt = \frac{1}{|\Gamma|} \int_{\Gamma} f \, dx \, dt.$$ 

In case $f$ is a function on a measurable subset $\Gamma$ of $\mathbb{R}^d$ we set

$$\int_{\Gamma} f \, dx = \frac{1}{|\Gamma|} \int_{\Gamma} f \, dx.$$ 

2 Auxiliary Results

In this section we do not suppose that any version of (1.4) holds.

For $C \subseteq \mathbb{R}$ denote by $\tau_C$ the first exit time of $(t_s, x_s)$ from $C$, but if $C = C_R$, we use the notations $\tau_R$ instead of $\tau_{C_R}$ with the hope that no confusion will be created.

Here is a combination of Lemmas 4.1 and 4.2 of [20].
Theorem 2.1 For any \((t, x) \in \mathbb{R}^{d+1}\), Borel \(f(s, y), g(y) \geq 0\) and stopping time \(\gamma\)
\[
E_{t,x} \int_0^\gamma f(t_s, x_s) \, dt \leq N(d, \delta)(A + B^2)^{d/(2d+2)} \|f\|_{L_{d+1}},
\]
\[
E_{t,x} \int_0^\gamma g(x_t) \, dt \leq N(d, \delta)(A + B^2)^{1/2} \|g\|_{L_d},
\]
where \(A = E_{t,x} \gamma\) and
\[
B = E_{t,x} \int_0^\gamma |b(t_s, x_s)| \, ds.
\]

Observe that if \(\gamma = \tau_C\) with \(C \in \mathcal{C}_R\) in Theorem 2.1, then obviously \(\gamma \leq R^2\) and \(A \leq R^2\). Since \(b\) is bounded, \(B \leq KR^2\), where \(K\) is independent of \(C\). This shows that
\[
\bar{b}_R := \sup_{\rho \leq R} \frac{1}{\rho} \sup_{C \in \mathcal{C}_\rho} E_{t,x} \int_0^{\tau_C} |b(t_s, x_s)| \, ds
\]
is finite for any \(R \in (0, \infty)\).

Remark 2.1 Usual way to deal with additive functionals shows that for any \(n = 1, 2, \ldots, R \in (0, \infty), (t, x) \in \mathbb{R}^{d+1}\), and \(C \in \mathcal{C}_R\)
\[
E_{t,x} \left( \int_0^{\tau_C} |b(t_s, x_s)| \, ds \right)^n \leq n! \bar{b}_R^n R^n.
\]

Since \(B \leq \bar{b}_R R\) if \(\gamma = \tau_C\) and \(C \in \mathcal{C}_R\), we obtain the following.

Lemma 2.2 For any Borel \(f \geq 0, R > 0, C \in \mathcal{C}_R\), and \((t, x) \in \mathbb{R}^{d+1}\) we have
\[
E_{t,x} \int_0^{\tau_C} f(t_s, x_s) \, ds \leq N(d, \delta)(1 + \bar{b}_R)^{d/(d+1)} R^{d/(d+1)} \|f\|_{L_{d+1}}.
\]

By plugging in \(f = |b|IC\) and using the fact that \(b\) is bounded and has compact support, we get
\[
E_{t,x} \int_0^{\tau_C} |b(t_s, x_s)| \, ds
\]
\[
\leq K(1 + \bar{b}_R)^{d/(d+1)} R^{d/(d+1)} \|f\|_{L_{d+1}} \leq K(1 + \bar{b}_R)^{d/(d+1)} R,
\]
where the constants \(K\) are independent of \(R > 0, C \in \mathcal{C}_R\), and \((t, x) \in \mathbb{R}^{d+1}\). It follows by definition that
\[
\bar{b}_R \leq K(1 + \bar{b}_R)^{d/(d+1)},
\]
which shows that \(\bar{b}_\infty := \lim_{R \to \infty} \bar{b}_R\) is finite.

For \(T, R \in (0, \infty)\) introduce
\[
\tau_{T,R} = \inf\{s \geq 0 : (t_s, x_s) \notin CT, R\} \quad (\tau_R = \tau_{R^2,R}).
\]

Observe that owing to (2.3) with \(f = IC\)
\[
E_{0,x} \tau_R \leq N(d, \delta)(1 + \bar{b}_\infty)^{d/(d+1)} R^2.
\]

Estimate (2.4) says that \(\tau_R\) is of order not more than \(R^2\). A very important fact which is implied by Corollary 2.6 is that \(\tau_R\) is of order not less than \(R^2\). To show this we need an additional assumption appearing after the following result, in which
\[
\tau'_R = \inf\{t \geq 0 : x_t \notin B_R\}, \quad \gamma_R = \inf\{t \geq 0 : x_t \in \tilde{B}_R\}.
\]
Theorem 2.3 There are constants \( \bar{\xi} = \bar{\xi}(d, \delta) \in (0, 1) \) and \( \bar{N} = \bar{N}(d, \delta) \) continuously depending on \( \delta \) such that if, for an \( R \in (0, \infty) \), we have

\[
\bar{N} \bar{b}_R \leq 1,
\]

then for any \( x \)

\[
P_{0,x}(\tau_R = R^2) \leq 1 - \bar{\xi}, \quad P_{0,0}(\tau_R = R^2) \geq \bar{\xi}.
\]

Moreover for \( n = 1, 2, \ldots \)

\[
P_{0,x}(\tau_R' \geq nR^2) = P_{0,x}(\tau_{nR^2,R}(x) = nR^2) \leq (1 - \bar{\xi})^n,
\]

so that

\[
E_{0,x} \tau_R' \leq N(d, \delta)R^2,
\]

and

\[
I := E_{0,x} \int_0^{\tau_R} |b(t, x_t)| \, dt \leq N(d, \delta)\bar{b}_R R.
\]

Furthermore, the probability starting from a point in \( \bar{B}_{9R/16} \) to reach the ball \( \bar{B}_{R/16} \) before exiting from \( B_R \) is bigger than \( \bar{\xi} \): for any \( x \) with \( |x| \leq 9R/16 \)

\[
P_{0,x}(\tau_R' > \gamma R/16) \geq \bar{\xi}.
\]

Proof This theorem is similar to Theorem 2.3 of [21]. The most significant difference is that in Theorem 2.3 of [21] to estimate quantities like

\[
E_{0,0} \int_0^{\tau_R} |b(s, x_s)| \, ds
\]

an estimate similar to (2.3) is used and this lead to the assumption that \( q_0 < \infty \). In our situation quantity (2.12) is less than \( \bar{b}_R R \) by definition. Taking this into account to prove the theorem one can just repeat the proof of Theorem 2.3 of [21].

We show an example of how to do that proving (2.10). By using the strong Markov property we obtain

\[
I = \sum_{n=1}^{\infty} E_{0,x} I_{\tau_{(n-1)R^2,R} > \tau_R} E \left( \int_{\tau_{(n-1)R^2,R}}^{\tau_{nR^2,R}} |b(t, x_t)| \, dt \mid \mathcal{F}_{\tau_{(n-1)R^2,R}} \right)
\]

\[
\leq J \sum_{n=1}^{\infty} P_{0,x}(\tau_{(n-1)R^2,R} = (n - 1)R^2) \leq J \sum_{n=1}^{\infty} (1 - \bar{\xi})^{n-1},
\]

where \( J = R\bar{b}_R \). This yields (2.10). The theorem is proved.

In light of (2.9) and (2.10) estimate (2.2) implies the following.

Corollary 2.4 If (2.6) holds, then for any Borel \( g \geq 0 \) and \( x \in \mathbb{R}^d \) we have

\[
E_{0,x} \int_0^{\tau_R} g(x_s) \, ds \leq N(d, \delta)(1 + \bar{b}_R)R\|g\|_{L_1}.
\]

Assumption 2.1 It holds that

\[
\bar{N}(d, \delta)\bar{b}_\infty < 1.
\]
This assumption (as well as Assumption 1.1) is supposed to hold throughout the section. We need a few more results given here without proofs because their proofs are obtained just by repeating the corresponding proofs from [21].

Here are particular cases of Theorem 2.6 and Corollary 2.7 of [21].

**Theorem 2.5** For any $\lambda, R > 0$, we have
\[
E_{0,0} e^{-\lambda \tau_R} \leq e^{\xi/2} e^{-\sqrt{\lambda} \xi/2}.
\] (2.15)

In particular, for any $t, R > 0$ we have
\[
P_{0,0}(\tau_R \leq t) \leq e^{\xi/2} \exp\left(-\frac{\xi^2 R^2}{16 t}\right).
\] (2.16)

**Corollary 2.6** There is a constant $N = N(\tilde{\xi})$ such that for any $R \in (0, \infty)$
\[
NE_{0,0} \tau_R \geq R^2.
\] (2.17)

A few more general results are related to going through a long “sausage”.

**Theorem 2.7** Let $R \in (0, \infty)$, $x, y \in \mathbb{R}^d$ and $16|x - y| \geq 3R$. For $r > 0$ denote by $S_r(x, y)$ the open convex hull of $B_r(x) \cup B_r(y)$. Then there exist $T_0, T_1$, depending only on $\tilde{\xi}$, such that $0 < T_0 < T_1 < \infty$ and the $P_{0,x}$-probability $\pi$ that $x_t$ will reach $\bar{B}_R/16(y)$ before exiting from $S_R(x, y)$ and this will happen on the time interval $[nT_0 R^2, nT_1 R^2]$ is greater than $\pi_0$, where
\[
n = \left\lfloor \frac{16|x - y| + R}{4R} \right\rfloor
\] and $\pi_0 = \frac{\tilde{\xi}}{3}$.

This is a particular case of Theorem 2.9 of [21].

**Remark 2.2** Observe that, for any fixed $x, y$, the interval $[nT_0 R^2, nT_1 R^2]$ is as close to zero as we wish if we choose $R$ small enough. Then, of course, the corresponding probability will be quite small but $> 0$.

Next follow analogs of Corollaries 2.10 and 2.11 of [21]

**Corollary 2.8** Let $\kappa \in [0, 1)$, and $|x| \leq \kappa R$. Then for any $T > 0$
\[
N P_{0,x}(\tau_R' > T) \geq e^{-v T / [(1-\kappa)R]^2},
\] (2.18)

where $N$ and $v > 0$ depend only on $\tilde{\xi}$.

**Corollary 2.9** Let $R \in (0, \infty)$. Then there exists a constant $N$, depending only on $\tilde{\xi}$, such that, for any $T > 0$, $P_{0,0}(\tau_R' > T) \leq N e^{-T/(NR^2)}$.

It is well known that, in light of Assumption 1.1, the process $(t, x_t)$ has a Green’s function, which means that we can introduce a function $G(t, x, s, y) \geq 0$ so that for all nonnegative Borel $f$ and $(t, x) \in \mathbb{R}^{d+1}$
\[
E_{t,x} \int_0^\infty f(t_r, x_r) \, dr = \int_0^\infty \int_{\mathbb{R}^d} G(t, x, s, y) f(s, y) \, dy \, ds,
\]
\[
G(s, y) = G(0, 0, s, y).
\]
Theorem 2.10 There exist $d_0 \in (1, d)$, depending only on $\delta$, $d$, and a constant $N = N(\delta, d)$ such that for any $R \in (0, \infty)$, $C \in C_R$, and $p \geq d_0 + 1$, we have
\[
\left( \int_C G^{p/(p-1)}(s, y) \, dy \, ds \right)^{(p-1)/p} \leq NR^{-d}.
\] (2.19)

Proof We basically follow the idea in [9]. Introduce $C_+$ as the set of cylinders $C_R(t, x)$, $R > 0$, $t \geq 0$, $x \in \mathbb{R}^d$. For $C = C_R(t, x) \in C_+$ let $2C = C_{2R}(t, x)$. If $C = C_R(t, x)$ set $R_C = R$.

Take $C \in C_+$ and define recursively
\[
\gamma^1 = \inf\{s \geq 0 : (t_s, x_s) \in \bar{C}\}, \quad \tau^1 = \inf\{s \geq \gamma^1 : (t_s, x_s) \notin 2C\},
\]
\[
\gamma^{n+1} = \inf\{s \geq \tau^n : (t_s, x_s) \notin \bar{C}\}, \quad \tau^{n+1} = \inf\{s \geq \gamma^{n+1} : (t_s, x_s) \notin 2C\}.
\]

Then for any nonnegative Borel $f$ vanishing outside $C$ with $\|f\|_{L_{d+1}(C)} = 1$ we have
\[
\int_C f(s, y)G(s, y) \, dy \, ds = E_{0,0} \int_0^\infty f(t_s, x_s) \, ds = \sum_{n=1}^\infty E_{0,0} E\left( \int_{\gamma^n}^{\tau^n} f(t_s, x_s) \, ds \mid \mathcal{F}_{\gamma^n} \right).
\]

By the strong Markov property and (2.3) (a.s.) ($\beta_\infty$ is estimated by using (2.14))
\[
E\left( \int_{\gamma^n}^{\tau^n} f(t_s, x_s) \, ds \mid \mathcal{F}_{\gamma^n} \right) \leq N(d, \delta)R_C^{d/(d+1)}I_{\gamma^n < \tau^n},
\]

Next, on the set $\{\gamma^n < \tau^n\}$ we have $(t_{\gamma^n}, x_{\gamma^n}) \in \bar{C}$ and by Corollary 2.6 on average it will take at least $R_C^2/N$ time for the process to exit from $2C$, that is
\[
E(\tau^n - \gamma^n \mid \mathcal{F}_{\gamma^n}) \geq R_C^2/N
\]
on $\{\gamma^n < \tau^n\}$. We conclude that
\[
\int_C f(s, y)G(s, y) \, dy \, ds \leq NR_C^{-(d+2)/(d+1)} \sum_{n=1}^\infty E_{0,0}(\tau^n - \gamma^n)
\]
\[
\leq NR_C^{-(d+2)/(d+1)}E_{0,0} \int_0^{\tau_{C_0}} I_{2C}(t_s, x_s) \, ds
\]
\[
= NR_C^{-(d+2)/(d+1)} \int_{2C} G(s, y) \, dy \, ds.
\]

The arbitrariness of $f$ yields
\[
\left( \int_C G^{(d+1)/d} \, dy \, ds \right)^{d/(d+1)} \leq N \int_{2C} G \, dy \, ds.
\]

Now, by Theorem 5.1 there exists $d_0 = d_0(d, \delta) < d$ such that
\[
\left( \int_C G^{(d_0+1)/d_0} \, dy \, ds \right)^{d_0/(d_0+1)} \leq N \int_{2C} G \, dy \, ds.
\]
for any \( C \in \mathcal{C}_+ \). Hölder’s inequality shows that this estimate also holds for \( p \geq d_0 \) in place of \( d_0 \). After that it only remains to recall that the integral of \( G \) over \([0, R^2]\times \mathbb{R}^d\) is \( R^2 \) and hence

\[
\int_{2C} G \, dx \, dt \leq NR_C^{-d}.
\]

The theorem is proved.

**Theorem 2.11** For any \( R \in (0, \infty), x \in \mathbb{R}^d \) and Borel \( f \geq 0 \)

\[
E_{0,x} \int_0^{\tau^R} f(t, x_t) \, dt \leq \hat{N} R^{(2d_0-d)/(d_0+1)} \| f \|_{L_{d_0+1}}^d, \quad (2.20)
\]

where \( \hat{N} \) depends only on \( d, \delta \).

**Proof** Observe that in light of Theorem 2.10 and the Markov property

\[
E_{0,x} \int_0^{\tau^R} f(s, x_s) \, ds = \sum_{n=0}^{\infty} E_{0,x} I_{\tau^R \geq nR^2} E \left( \int_{nR^2}^{((n+1)R^2) \land \tau^R} f(t, x_t) \, ds \mid \mathcal{F}_{nR^2} \right)
\]

\[
\leq N \| f \|_{L_{d_0+1}} R^{(d+2)d_0/(d_0+1)-d} \sum_{n=0}^{\infty} P_{0,x} (\tau^R \geq nR^2).
\]

It only remains to use Corollary 2.9. The theorem is proved.

Introduce the Green’s function \( G_R(x, y) \) of \( x \) by means of the formula

\[
E_{0,x} \int_0^{\tau^R} f(x_t) \, dt = \int_{B_R} f(y) G_R(x, y) \, dy.
\]

In this “elliptic” setting one can use Corollary 2.4 in place of (2.3) and then by mimicking the proof of Theorem 2.10 one sees that there is a constant \( N = N(d, \delta) \) such that for any \( R \in (0, \infty) \) and any ball \( B \) such that \( 4B \subset B_R \) it holds that

\[
\left( \int_B G_R^{d/(d-1)}(x, y) \, dy \right)^{(d-1)/d} \leq N \int_{2B} G_R(x, y) \, dy.
\]

Then by following the arguments in [11] (or following our arguments in Sect. 5) one can see that there exists \( d_0 = d_0(d, \delta) < d \) such that

\[
\left( \int_B G_R^{p/(p-1)}(x, y) \, dy \right)^{(p-1)/p} \leq N \int_{2B} G_R(x, y) \, dy, \quad (2.21)
\]

for any \( p \geq d_0 \) and \( B \) such that \( 4B \subset B_R \). Of course, we can take the \( d_0 \)’s to be the same.

Next (2.21) implies that

\[
\left( \int_{B_{R/4}} G_R^{p/(p-1)}(x, y) \, dy \right)^{(p-1)/p} \leq N \int_{B_R} G_R(x, y) \, dy \leq NR^{2-d},
\]

where the second inequality follows from (2.9). In the inequality between the extreme terms one can replace \( G_R \) by a smaller quantity \( G_{R/4} \). Then by using the arbitrariness of \( R \) we arrive at the following.

**Theorem 2.12** For \( p \geq d_0 \) and any \( R \in (0, \infty) \) and \( x \in \mathbb{R}^d \)

\[
\left( \int_{B_R} G_R^{p/(p-1)}(x, y) \, dy \right)^{(p-1)/p} \leq N(d, \delta) R^{2-d}.
\]
Remark 2.3 If in Theorem 2.12 we take $x_I$ to be just a Wiener process, then we will see that $d_0 > d/2$.

In the following theorem we use the interpolation technique suggested by A.I. Nazarov in [25].

Theorem 2.13 There exists a constant $\hat{N} = \hat{N}(d, \delta)$ such that for any $R \in (0, \infty)$, $x \in \mathbb{R}^d$ and Borel $f \geq 0$ estimate (1.5) holds.

Proof If $p = d_0 + 1$, then $q = d_0 + 1$ and (1.5) follows from Theorem 2.11 since $(2d_0 - d)/(d_0 + 1) = 2 - d/(d_0 + 1) - 2/(d_0 + 1)$.

If $p = d_0$ and $q = \infty$ estimate (1.5) follows from Theorem 2.12 since

$$ I \leq E_{0,x} \int_0^{t_R} \sup_{s \geq 0} f(s, x_t) \, dt = \int_{B_R} G_R(x, y) \sup_{s \geq 0} f(s, y) \, dy $$

$$ \leq \| f \|_{L_{d_0, \infty}} \| G_R \|_{L_{d_0/(d_0 - 1)}, (B_R)} \leq NR^{2 - d/d_0} \| f \|_{L_{d_0, \infty}}. $$

If $p = \infty$ and $q = 1$, estimate (1.5) holds because

$$ I \leq E_{0,x} \int_0^{\infty} \sup_{x \in \mathbb{R}^d} f(t, y) \, dt = \| f \|_{L_{p, \infty}}. $$

In case $\infty > p > d_0 + 1$ we have $p > q$ and set $\beta = p/(d_0 + 1)$ and $\alpha = \beta/(\beta - 1)$. Take a nonnegative $g(t)$ such that $(f(t, x)g(t))/g(t) = f(t, x) (0/0 = 0)$ and use Hölder’s inequality to conclude that $I \leq I_1 I_2$, where

$$ I_1 = \left( \int_0^{\infty} g^{-\alpha}(t) \, dt \right)^{1/\alpha}, $$

$$ I_2 = \left( E_{0,x} \int_0^{t_R} g^\beta(t) f^\beta(t, x_t) \, dt \right)^{1/\beta} \leq NR^{(2d_0 - d)/p} \left( \int_0^{\infty} g^{(d_0 + 1)\beta}(t) \left( \int_{\mathbb{R}^d} f^{(d_0 + 1)\beta}(t, y) \, dy \right) dt \right)^{1/(d_0 \beta + \beta)}. $$

For $g$ found from

$$ g^{-\alpha}(t) = g^{(d_0 + 1)\beta}(t) \int_{\mathbb{R}^d} f^{(d_0 + 1)\beta}(t, y) \, dy $$

we get (1.5) and this takes care of the case that $\infty > p > d_0 + 1$.

If $\infty > q > d_0 + 1$ we have $p < q$ and set $\beta = q/(q - d_0 - 1)$ and $\alpha = \beta/(\beta - 1)$. Take a nonnegative $g(y)$ such that $(f(t, y)g(y))/g(y) = f(t, y) (0/0 = 0)$ and use Hölder’s inequality to conclude that $I \leq I_1 I_2$, where

$$ I_1 = \left( E_{0,x} \int_0^{t_R} g^{-\beta}(x_t) \, dt \right)^{1/\beta} \leq NR^{(2d_0 - d)/\beta} \left( \int_{\mathbb{R}^d} g^{-d_0\beta}(y) \, dy \right)^{1/(d_0 \beta)}, $$

$$ I_2 = \left( E_{0,x} \int_0^{t_R} g^\alpha(x_t) f^\alpha(t, x_t) \, dt \right)^{1/\alpha} \leq NR^{(2d_0 - d)/(\alpha d_0 + \alpha)} \left( \int_{\mathbb{R}^d} g^{(d_0 + 1)\alpha}(y) \left( \int_0^{\infty} f^{(d_0 + 1)\alpha}(t, y) \, dt \right) dy \right)^{1/(\alpha d_0 + \alpha)}. $$

For $g$ found from

$$ g^{-d_0\beta}(y) = g^{(d_0 + 1)\alpha}(y) \int_0^{\infty} f^{(d_0 + 1)\alpha}(t, y) \, dt $$
we get \((1.5)\) after simple manipulations and this proves the theorem.

Here is a key to proving Theorem 1.1.

Corollary 2.14 Assume that there exists a constant \(\hat{b} \in (0, \infty)\) such that, for any \(C \in \mathcal{C}\)

\[
\|b\|_{L_p,q(C)} \leq \hat{b} R^{d/p+2/q-1}.
\]

(2.22)

Then \(\bar{b}_\infty \leq \hat{N} \hat{b}\).

This follows immediately from \((1.5)\) with \(f = |b|\) and the fact that a natural modification of \((1.6)\) holds for any starting point.

Finally, we need the following.

Lemma 2.15 For any \(\varepsilon > 0\) there exists \(\alpha = \alpha(\varepsilon) > 1\) such that, for any \(R > 0\) and \(x \in \mathbb{R}^d\)

\[
\left( E_{0,x} \left( \int_0^{\tau_R} |b(t,x_t)| \, dt \right)^{\alpha} \right)^{1/\alpha} \leq (1 + \varepsilon) \bar{b}_\infty R.
\]

Proof We claim that if \(\xi \geq 0, E\xi \leq A, \) and \(E\xi^2 \leq 2A^2,\) then for any \(\varepsilon > 0\) there exists \(\alpha = \alpha(\varepsilon) > 1\) such that

\[
E\xi^\alpha \leq (1 + \varepsilon) A^\alpha.
\]

Indeed, by normalizing \(\xi\) we may assume that \(A = 1.\) Then \(E\xi^2 \leq 2\) and for \(\alpha \in [1, 3/2]\)

\[
\frac{d}{d\alpha} E\xi^\alpha \leq N,
\]

where \(N\) is an absolute constant. This proves the claim. This also proves the lemma after setting

\[
\xi = \int_0^{\tau_R} |b(t,x_t)| \, dt
\]

and using Remark 2.1. The lemma is proved.

3 Proof of Theorem 1.1

Suppose that \((1.4)\) holds for any \(R \in (0, \infty)\) and \(C \in \mathcal{C}_R\) with \(\hat{b}\) satisfying

\[
\hat{N} \hat{b} \leq (2\hat{N})^{-1},
\]

where \(\hat{N}\) is taken from Theorem 2.3 and \(\hat{N}\) is taken from Theorem 2.13. For \(\lambda \in [0, \infty)\) denote by \(x_\lambda^t\) the diffusion process corresponding to \(\lambda b\) in place of \(b\) and use the superscript \(\lambda\) for other objects related to \(x_\lambda^t.\) Call a \(\lambda\) “good” if (cf. Assumption 2.1)

\[
\bar{N} \hat{b}_\infty^\lambda < 1,
\]

so that, for \(x_\lambda^t\) in place of \(x_t,\) the assertions of Theorem 2.13 and, hence, \((1.5)\) hold true. Let \(\Lambda\) be the set of good \(\lambda\)’s. Our claim is that \(1 \in \Lambda.\) Observe that \(0 \in \Lambda.\)

We are going to use the method of continuity proving, first, that \(\Lambda \cap [0, 1]\) is closed and, second, that \(\Lambda\) is open to the right (and therefore contains points even beyond 1).
If $\lambda_0 \in \Lambda \cap [0, 1], n = 1, 2, \ldots$, converge to $\lambda_0$, then by Corollary 2.14 we have $b_{\lambda_0}^{\lambda_n} \leq \hat{N}\dot{b}$, that is

$$E_{t,x} \int_0^{\tau_C^{\lambda_n}} |b(t, x_{s}^{\lambda_n})| \, ds \leq \hat{N}\dot{b} R \tag{3.1}$$

for any $(t, x) \in \mathbb{R}^{d+1}$, $R > 0$, and $C \in C_R$, where $\tau_C^{\lambda_n}$ is the first exit time of $(t, x_{s}^{\lambda_n})$ from $C$. By using Girsanov’s theorem and Fatou’s lemma one easily shows that (3.1) is also true for $n = 0$. But in that case, $\hat{N}\dot{b}_{\lambda_0}^{\lambda_n} \leq \hat{N}\dot{b} \leq 1/2 < 1$ so that, indeed, $\Lambda \cap [0, 1]$ is closed.

To prove that $\Lambda$ is open to the right, first take $\lambda = 0, \varepsilon > 0$, $(t, x) \in \mathbb{R}^{d+1}$, $R > 0$, and $C \in C_R$ and observe that for any $\beta > 0$, by Corollary 2.14 is less than $\hat{N}\dot{b}_{\lambda_0}^{\lambda_n} \leq \hat{N}\dot{b} \leq 1/2 < 1$ so that, indeed, $\Lambda \cap [0, 1]$ is closed.

Hence, for $\varepsilon$ small enough we have $\hat{N}\dot{b}_{\lambda_0}^{\lambda_n} < 1$, so that all small $\varepsilon$’s are good. Next, take a $\lambda \in \Lambda \cap (0, 1], \varepsilon > 0$, $(t, x) \in \mathbb{R}^{d+1}$, $R > 0$, and $C \in C_R$ and use Girsanov’s theorem to see that

$$E_{t,x} \int_0^{\tau_C^{\lambda_n + \varepsilon}} |(\lambda + \varepsilon)b(t, x_{s}^{\lambda_n + \varepsilon})| \, ds = E_{t,x}e^{\phi(\varepsilon)} \int_0^{\tau_C^{\lambda_n}} |(\lambda + \varepsilon)b(t, x_{s}^{\lambda_n})| \, ds, \tag{3.2}$$

where

$$\phi(\varepsilon) = \varepsilon \int_0^{\infty} \sigma^{-1}b(t, x_s) \, dw_x - (\varepsilon^2/2) \int_0^{\infty} |\sigma^{-1}b(t, x_s)|^2 \, ds.$$

Recall that $E_{t,x}e^{\phi(\beta\varepsilon)} = 1$ for any $\beta$ and observe that for any $\beta > 1$

$$E_{t,x}e^{\phi(\varepsilon)} = E_{t,x}e^{\phi(\beta\varepsilon)} \exp \left( (\varepsilon^2/2)(\beta^2 - 1) \int_0^{\infty} |\sigma^{-1}b(t, x_s)|^2 \, ds \right) \leq e^{\varepsilon^2\beta^2 K}$$

since $b$ is bounded and the range of $t$ such that $b(t, \cdot) \neq 0$ is bounded, where $K$ is a constant independent of $t, x$. We use this and Hölder’s inequality to obtain from (3.2) that

$$E_{t,x} \int_0^{\tau_C^{\lambda_n + \varepsilon}} |(\lambda + \varepsilon)b(t, x_{s}^{\lambda_n + \varepsilon})| \, ds \leq e^{\varepsilon^2\beta^2 K} \left( E_{t,x} \left( \int_0^{\tau_C^{\lambda_n}} |(\lambda + \varepsilon)b(t, x_{s}^{\lambda_n})| \, ds \right)^{\alpha} \right)^{1/\alpha}, \tag{3.3}$$

where $\alpha = \beta/(\beta - 1)$.

Recall that $\lambda$ is good, so that, for any $\varepsilon > 0$ according to Lemma 2.15, for an appropriate choice of $\beta$, the second factor on the right in (3.3) is less than $(1 + \varepsilon_1)(1 + \varepsilon/\lambda)b_{\lambda_0}^{\lambda_n} R_C$, which by Corollary 2.14 is less than $(1 + \varepsilon_1)(1 + \varepsilon/\lambda)\hat{N}\dot{b} R_C$. Since we can choose $\varepsilon$ and $\varepsilon_1$ arbitrarily, we can make the left-hand side less than $(3/2)\hat{N}\dot{b} R_C$. This shows that $\hat{b}_{\lambda_0}^{\lambda_n + \varepsilon} \leq (3/2)\hat{N}\dot{b}$. Now the condition $\hat{N}\dot{b} \leq (2\hat{N})^{-1}$ implies that $\hat{N}\dot{b}_{\lambda_0}^{\lambda_n + \varepsilon} < 1$, so that $\lambda + \varepsilon$ is good for all small enough $\varepsilon > 0$ and this brings the proof of the theorem to an end.

### 4 Proof of Theorem 1.2

We take $\dot{b} = \dot{b}(d, \delta)$ from Theorem 1.1 and split the proof into two steps.
Step 1. First we want to prove that (1.5) holds if $R \leq R_0/2$. To do that take a smooth $\xi(x)$ such that $\xi = 1$ on $B_R$, $\xi = 0$ outside $B_{2R}$, and $0 \leq \xi \leq 1$ everywhere. Observe that for any $\rho \in (0, \infty)$ and $C \in C_\rho$,

$$
\|\xi b\|_{L_{p_0-q_0}}(C) \leq \hat{b}\rho^{d/p_0+2/q_0-1}.
$$

(4.1)

Indeed, if $\rho \leq R_0$, this follows from the assumption of the theorem. However, if $\rho \geq R_0$, then $\rho \geq 2R$ and (4.1) follows from

$$
\|\xi b\|_{L_{p_0-q_0}}(C) \leq \|b\|_{L_{p_0-q_0}(C_{2R})} \leq \hat{b}(2R)^{d/p_0+2/q_0-1}.
$$

After that let $\hat{\bar{x}}_t$ be the process with drift $\xi b$. For this process (1.5) holds for all $R \in (0, \infty)$. Since the coefficients of $(t_s, x_s)$ in $\mathbb{R} \times B_R$ coincide with the coefficients of $(t_s, \hat{x}_t)$ and the coefficients are smooth, the distributions of this processes coincide before they exit from $\mathbb{R} \times B_R$. Therefore, the left-hand side of (1.5) does not change if we replace there $x_t$ with $\hat{x}_t$ and we are done with the first step.

Step 2. general $R > R_0/2$. Applying the same argument, based on the fact that on the small scale $x_t$ behaves like a process with small $\hat{b}_\infty$, and using Theorem 2.7 we see that with strict positive probability, depending only on $d, \delta$, and $R_0$, the process starting at a point $(t_0, x_0)$ will reach $\Gamma := \{t_0 + S_1, t_0 + S_2\} \times B_{R_0/10}(x_0 + R_0\epsilon_1/4)$, where $\epsilon_1$ is the first basis vector, $0 < S_1 < S_2 < \infty$ and the $S_i$’s depend only on $d, \delta$. Repeating this argument after the process reaches $\Gamma$ and taking into account that $R < \infty$, we see that with probability $\pi > 0$ depending only on $d, \delta, R_0$, and $R$, starting from any point in $\mathbb{R} \times B_R$ the process will leave $\mathbb{R} \times B_R$ before time $T$, where $T$ depends only on $d, \delta, R_0$, and $R$, that is

$$
P_{t, x}(\tau_R > T) \leq 1 - \pi.
$$

Iterating this inequality we obtain $P_{t, x}(\tau_R > nT) \leq (1 - \pi)^n$ for $n = 1, 2, \ldots$. This shows, as in the proof of Theorem 2.11, that to prove the current theorem it suffices to prove that for any $R \in (0, \infty), x \in \mathbb{R}^d$ and Borel $f \geq 0$ (notice $\tau_R$

$$
E_{0, x} \int_0^{\tau_R} f(t, x_t) \, dt \leq \hat{\tilde{N}} \|f\|_{L_{p, q}},
$$

(4.2)

where $\hat{\tilde{N}}$ depends only on $d, \delta, R_0$, and $R$.

Observe that it suffices to prove (4.2) only for smooth $f$. Fix $\lambda = \lambda(d, \delta) > 0$ such that the right-hand side of (2.15) is less than $1/2$ and introduce

$$
u(t, x) = E_{t, x} \int_0^{\tau_R} e^{-\lambda s} f(t_s, x_s) \, ds.
$$

Then $\nu$ is a continuous (smooth) nonnegative function on $\bar{C}_R$ vanishing on $\bar{C}_R \setminus (0 \times B_R)$ and hence attains $\max_{\bar{C}_R} \nu =: M$ at a point $(t_0, x_0) \in \bar{C}_R$. Let $\gamma$ be the first exit time of $(t_s, x_s)$ from $\bar{C}_{R_0/2}(t_0, x_0)$. By the strong Markov property

$$
M = u(t_0, x_0) = E_{t_0, x_0} e^{-\lambda(\tau_R \wedge \gamma)} u(t(\tau_R \wedge \gamma), x(\tau_R \wedge \gamma))
$$

$$
+ E_{t_0, x_0} \int_0^{\tau_R \wedge \gamma} e^{-\lambda s} f(t_s, x_s) \, ds.
$$

(4.3)

Here the second term admits estimating like in (1.5) by the first step. The first term is less than

$$
ME_{t_0, x_0} e^{-\lambda \gamma} I_{\gamma < \tau_R} \leq ME_{t_0, x_0} e^{-\lambda \gamma} \leq (1/2)M.
$$
Thus, (4.3) implies that
\[ M \leq N \| f \|_{L^p} + (1/2)M, \quad M \leq N, \]
and to finish the proof it only remains to observe that
\[ E_{0,x} \int_0^{t_R} f(t, x_t) \, dt \leq e^{\lambda R^2} u(0, x). \]

The theorem is proved.

5 Appendix: A Version of Gehring’s Lemma

Here we prove the parabolic version of the famous Gehring’s lemma stated as Proposition 1.3 in [12] without proof with the only hint that the proof is similar to the one given in the elliptic case in [11]. The author found it quite hard to make constructions in parabolic case “similar” to the elliptic ones given in [11] and decided to give a complete proof having a strong probabilistic flavor. One might think that the only difference between elliptic and parabolic cases is different scaling. However, in the elliptic case the doubled cubes strictly contain the original ones and in the parabolic case this is not so. Our proof is based on the ideas from [11] but the organization of the proof is different. In particular, this allows us to easily track down the dependence of constants on \( A \) and show that \( q \) is a decreasing function of \( A \), which was never done before to the best of the author’s knowledge. If \( C = C_R(t, x) \) and \( \mu > 0 \) by \( \mu C \) we mean \( C_{\mu R}(t, x) \).

**Theorem 5.1** Let in \( C_R \) be given a measurable \( f(t, x) \geq 0 \) such that, for some fixed \( p, A, B, \mu \in (1, \infty) \) satisfying \( A \leq B \) and for all \( C \in C \) such that \( \mu C \subset C_R \) we have
\[
\left( \int_C f^p \, dz \right)^{1/p} \leq A \int_{\mu C} f \, dz.
\]

Then there exists \( q = q(d, p, B) > p \) such that
\[
\left( \int_{C_R/4} f^q \, dz \right)^{1/q} \leq N(d, p, \mu) A \int_{C_R/2} f \, dz.
\]

**Proof** It is convenient to work with parabolic boxes instead of cylinders. For \( n = 0, 1, \ldots, 2^{2(n+1)} - 1 \), \( k = -2^n, -2^n + 1, \ldots, 2^n - 1 \), for \( i \geq 1 \), introduce \( D_{k_0, \ldots, k_d} \) as
\[
[k_0 2^{-2n}, (k_0 + 1) 2^{-2n}) \times [k_1 2^{-n}, (k_1 + 1) 2^{-n}) \times \cdots \times [k_d 2^{-n}, (k_d + 1) 2^{-n}).
\]
We call \( 2^{-n} \) the size of \( D_{k_0, \ldots, k_d} \). These are dyadic parabolic boxes, subsets of \( D_0 := [0, 4) \times [-1, 1]^d \). Set \( D_1 = [0, 1) \times [-1/2, 1/2]^d \) and for any box \( D = [S, S + T] \times Q \), where \( Q \) is a cube in \( \mathbb{R}^d \), denote \( 2D = [S, S + 4T] \times 2Q \), where \( 2Q \) is the concentric cube with twice the side length of \( Q \).

Routine arguments show that to prove the theorem, it suffices to show that there exists \( q = q(d, p, B) > p \) such that
\[
\left( \int_{D_1} f^q \, dz \right)^{1/q} \leq N(d, p) A \int_{2D_0} f \, dz,
\]
(5.1)
provided that a nonnegative $f$ is defined in $2D_0$ and
\[
\left( \int_D f^p \, dz \right)^{1/p} \leq A \int_{2D} f \, dz, \tag{5.2}
\]
for any $D = D_{k_0, \ldots, k_d}(n)$ such that $D \subset D_0$.

To proceed in so modified setting, for $n \geq 0$ introduce $\Sigma_n$ as the collection of $D_{k_0, \ldots, k_d}(n)$. To be consistent with probability language we add to $\Sigma_n$ the empty set. Then in the terminology from [17] the family $\{ \Sigma_n \}$ is a filtration of partitions of $D_0$. Observe that for each $n \geq 0$ and $(t, x) \in D_0$ there is only one element of $\Sigma_n$ containing $(t, x)$. We denote it by $\Gamma_n(t, x)$.

Then for each $(t, x) \in D$ define $\gamma(t, x)$ as the least $n \geq 0$ such that $3\Gamma_n(t, x) \subset D$. Clearly, if $\gamma(t, x) = n$ and $(s, y) \in \Gamma_n(t, x)$, then $\gamma(s, y) = n$. Therefore, the set $\{(t, x) : \gamma(t, x) = n\}$ is the union of some disjoint elements of $\Sigma_n$. In the terminology from [17] this means that $\gamma$ is a stopping time relative to the filtration $\{ \Sigma_n \}$.

For each $n \geq 0$ and measurable function $g \geq 0$ on $D_0$ one defines the function $g_{|\Gamma_n}$ which on each $\Gamma \in \Sigma_n$ equals its average over $\Gamma$.

Then for a fixed $\lambda > 0$ and $(t, x) \in D$ we define
\[
\tau_\lambda(t, x) = \inf\{ m \geq \gamma(t, x) : g_{|\Gamma_n}(t, x) > \lambda \}, \quad (\inf \emptyset := \infty).
\]
The set $\{ \tau_\lambda < \infty \}$ is similar to what one usually gets by applying the Riesz-Calderón-Zygmund decomposition. However, we are watching the averages of $g$ only on dyadic boxes where $\gamma$ is constant. Otherwise we continue in the usual way.

Observe that $D_0 \cap \{ g > \lambda \} \subset D_0 \cap \{ \tau_\lambda < \infty \}$ (a.e.) because of the Lebesgue differentiation theorem.

Next, assume that, for a constant $\bar{g}$, we have $g|_{\tau} \leq \bar{g}$ and take $\lambda > \bar{g}$ so that $\tau > \gamma$. Then note that the set $D_0 \cap \{ \tau_\lambda < \infty \}$ is either empty or is the disjoint union of some nonempty $\Gamma_i \subset \Sigma_{m_i}, i = 1, 2, \ldots$, on each of which $\tau_\lambda = m_i$. Trivially,
\[
\int_{\Gamma_i} g \, dz = \int_{\Gamma_i} g_{|\Gamma_i} \, dz = \int_{\Gamma_i} g_{|\tau_\lambda} \, dz,
\]
which implies that
\[
\int_{D_0} gI_{\tau_\lambda < \infty} \, dz = \int_{D_0} g_{|\tau_\lambda} I_{\tau_\lambda < \infty} \, dz.
\]
Furthermore, on the set $D_0 \cap \{ \tau_\lambda < \infty \}$ we have $g_{|\tau_\lambda} \geq \lambda, g_{|\tau_\lambda - 1} \leq \lambda$ and, since $g_{|\Gamma} \leq 2^{d+2}g_{m-1}$, we have $g_{|\tau_\lambda} \leq \nu^{-1}\lambda$, where $\nu = 2^{-d-2}$. It follows that
\[
\nu\lambda^{-1} \int_{D_0} g_{|\tau_\lambda} \, dz \leq \nu\lambda^{-1} \int_{D_0} g_{|\tau_\lambda - 1} \, dz = \nu\lambda^{-1} \int_{D_0} g_{|\tau_\lambda} I_{\tau_\lambda < \infty} \, dz
\]
\[
\leq |D_0 \cap \{ \tau_\lambda < \infty \}|. \tag{5.3}
\]
We apply this to $g = \phi f^p$, where $\phi(t, x) = [(1 - t)^{1/2} \wedge (1 - |x|)]^{d+2}$. As is easy to see on $D$ we have
\[
(\phi f^p)|_{\gamma} \leq N(d) \int_{D_0} f^p \, dz =: \bar{g}, \tag{5.4}
\]
Also observe that
\[
D_0 \cap \{ \tau_\lambda < \infty \} = \bigcup_{m \geq 0} D_0 \cap \{ \tau_\lambda = m \} =: \bigcup_{m \geq 0} D^{\lambda, m}.
\]
and for any $m$ the set $D^{\lambda,m}$ (even if empty) is the disjoint union of a family of $\Gamma \in \bigcup_n \Sigma_n$. Choose $\Gamma^1, \Gamma^2, \ldots$ from this family in such a way that $2\Gamma^1, 2\Gamma^2, \ldots$ are mutually disjoint and

$$|D^{\lambda,m}| \leq N \sum_i |\Gamma^i|,$$

where and below by $N$ we denote various constants depending only on $d$ and $p$. Also note that, since $\tau > \gamma$, each of those $\Gamma^i$ is a parabolic dyadic box of size $2^{-m}$ which is the subset of a parabolic dyadic box, say $\hat{\Gamma}^i$, of size $2^{-k}$, where $k \leq m$ is the value of $\gamma$ on $\Gamma^i$. It follows by construction that $3\hat{\Gamma}^i \subset D_0$. In particular, $3\Gamma^i \subset D_0$. Also the ratio $\phi(z_1)/\phi(z_2)$ is bounded by a constant $N$ as long as $z_1, z_2 \in 2\Gamma^i$. Therefore,

$$\lambda|\Gamma^i|^p \leq |\Gamma^i|^p \int_{\Gamma^i} \phi f^p \, dz \leq N|\Gamma^i|^p \max_{\Gamma^i} \phi \int_{\Gamma^i} f^p \, dz$$

$$\leq NA^p \min_{2\Gamma^i} \phi \left( \int_{2\Gamma^i} f^p \, dz \right)^p \leq NA^p \left( \int_{2\Gamma^i} \phi^{1/p} f \, dz \right)^p,$$

$$|\Gamma^i| \leq N_1 \frac{A}{\lambda^{1/p}} \int_{2\Gamma^i} \phi^{1/p} f \, dz.$$

One of inconveniences of the last estimate is that we do not have control of $f$ on $2\Gamma^i$. In a similar situation Gehring suggested to sacrifice some part of what is on the right to be absorbed by the left-hand side but restrict values of $f$. So following him we dominate the right-hand side by

$$N_1 \frac{A}{\lambda^{1/p}} \int_{2\Gamma^i} I_{\phi f^p > \lambda} \phi^{1/p} f \, dz + N_1 \frac{A_s^{1/p}}{\lambda^{1/p}} |2\Gamma^i|,$$

where $s > 0$ is arbitrary. For $s = N_2^{-p} A^{-p} \lambda$, where $N_2 = N_1 2^{d+2}$, we get

$$|\Gamma^i| \leq N \frac{A}{\lambda^{1/p}} \int_{2\Gamma^i} I_{\phi f^p > \lambda} \phi^{1/p} f \, dz$$

and hence, coming back to (5.3), for any $\lambda > \bar{g}$,

$$\nu \lambda^{-1} \int_{D_0} \phi f^p \, I_{\phi f^p \geq \lambda} \, dz \leq N A \lambda^{-1/p} \int_{D_0} \phi^{1/p} f \, I_{\phi f^p > N_2^{-p} A^{-p} \lambda} \, dz.$$

Multiply both sides by $\lambda^\alpha$, $\alpha \in (0, 1]$, and integrate between $\bar{g}$ and an arbitrary finite $\Lambda > \bar{g}$ to get

$$\alpha^{-1} \int_{D_0} \phi f^p \, (\phi f^p \land \Lambda)^\alpha \, dz - \alpha^{-1} \int_{D_0} \phi f^p \, (\phi f^p \land \bar{g})^\alpha \, dz$$

$$\leq N(\alpha + 1 - 1/p)^{-1} A \int_{D_0} \phi^{1/p} f \, ((N_2 A \phi^{1/p} f)^p \land \Lambda)^{\alpha + 1 - 1/p} \, dz.$$
We conclude that
\[
\int_{D_0} \phi f^p (\phi f^p \wedge \Lambda)^\alpha \, dz \leq N\left( \int_{D_0} f^p \, dz \right)^{1+\alpha} + N_3 \alpha (\alpha + 1 - 1/p)^{-1} A^{p(\alpha+1)} \int_{D_0} \phi f^p (\phi f^p \wedge \Lambda)^\alpha \, dz.
\]

Now choose \( \alpha \leq 1 \) so that
\[
N_3 \alpha (\alpha + 1 - 1/p)^{-1} B^{2p} \leq 1/2.
\]
Then we obtain
\[
\int_{D_0} \phi f^p (\phi f^p \wedge \Lambda)^\alpha \, dz \leq N\left( \int_{D_0} f^p \, dz \right)^{1+\alpha},
\]
which after sending \( \Lambda \to \infty \) and using (5.2) yields the result with \( q = p(1 + \alpha) \). The theorem is proved.

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**Conflict of interest** The author declares that he has no conflict of interest.

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