ON SOME HADAMARD-TYPE INEQUALITIES FOR DIFFERENTIABLE $m$–CONVEX FUNCTIONS

*M. EMİN ÖZDEMİR, ♠AHMET OCAK AKDEMİR, AND ★MERVE AVCI

Abstract. In this paper some new inequalities are proved related to left hand side of Hermite-Hadamard inequality for the classes of functions whose derivatives of absolute values are $m$–convex. New bounds and estimations are obtained. Applications for some Theorems are given as well.

1. INTRODUCTION

Let $f : I \to \mathbb{R}$ be a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. If $f$ is a convex function then the following double inequality, which is well-known in the literature as Hermite-Hadamard inequality, holds [see [5], p. 137];

\[ f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \]  

(1.1)

For recent results, generalizations and new inequalities related to the inequality presented above see [1]-[4].

In [10], Toader defined the concept of $m$–convexity as the following;

Definition 1. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $m$–convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have

\[ f \left( tx + m(1-t)y \right) \leq tf(x) + m(1-t)f(y). \]

Denote by $K_m(b)$ the set of the $m$–convex functions on $[0, b]$ for which $f(0) \leq 0$.

Several papers have been written on $m$–convex functions on $[0, b]$ and we refer the papers [7], [8], [9], [10], [11], [12], [13], [14], [15] and [16]. In [17], Dragomir and Agarwal proved following inequality for convex functions;

Theorem 1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on $I^0$ and $a, b \in I$, where $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then the following inequality holds;

\[ \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \]

(1.2)

In [4], Pearce and Pečarić proved the following inequalities for convex functions;

2000 Mathematics Subject Classification. 26D15.

Key words and phrases. $m$–Convex, Hadamard-Type Inequalities, Hölder inequality, Power mean inequality, Favard’s inequality.

♦Corresponding author.
Theorem 2. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I \) and \( a, b \in I \), where \( a < b \). If \( |f'|^q \) is convex on \([a, b]\) for some \( q \geq 1 \), then

\[
(1.3) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}
\]

and

\[
(1.4) \quad \left| \frac{f(a + b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.
\]

In [7], Bakula et al. proved the following inequality for \( m \)-convex functions:

Theorem 3. Let \( I \) be an open real interval such that \( [0, \infty) \subset I \). Let \( f : I \to \mathbb{R} \) be a differentiable function on \( I \) such that \( f' \in L[a, b] \), where \( 0 \leq a < b < \infty \). If \( |f'|^q \) is \( m \)-convex on \([a, b]\) for some fixed \( m \in (0, 1] \) and \( q \in [1, \infty) \), then;

\[
\left| \frac{f(a + b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \min \left\{ \left( \frac{|f'(a)|^q + m |f'(b)|^q}{2} \right)^{\frac{1}{q}}, \left( \frac{m |f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.
\]

In [13], Dragomir established following inequalities of Hadamard-type similar to above.

Theorem 4. Let \( f : [0, \infty) \to \mathbb{R} \) be a \( m \)-convex function with \( m \in (0, 1] \). If \( 0 \leq a < b < \infty \) and \( f \in L_1[a, b] \), then one has the inequality:

\[
(1.5) \quad f \left( \frac{a + b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) + mf \left( \frac{x}{m} \right) dx \leq \frac{m + 1}{4} \left[ f(a) + f(b) + mf \left( \frac{a}{m} \right) + mf \left( \frac{b}{m} \right) \right].
\]

The following classical inequality is well-known in the literature as Favard’s inequality (see [18, 19, p.216]):

Theorem 5. (i) (Favard’s inequality) Let \( f \) be a non-negative concave function on \([a, b]\). If \( q \geq 1 \), then

\[
(1.6) \quad \frac{2^q}{q+1} \left( \frac{1}{b-a} \int_a^b f(x)dx \right)^q \geq \frac{1}{b-a} \int_a^b f^q(x)dx.
\]

If \( 0 < q < 1 \) the reverse inequality holds in \((1.7)\).

(ii) (Thunsdorff’s inequality) If \( f \) is a non-negative, convex function with \( f(a) = 0 \), then for \( q \geq 1 \) the reversed inequality holds in \((1.8)\).

Motivated by the above results, in this paper we consider new Hadamard-type inequalities for functions whose derivatives of absolute values are \( m \)-convex by using fairly elementary analysis and some classical inequalities like Hölder inequality, Power-mean inequality and Favard’s inequality. These new results gives new upper bounds for the Theorem 2-3. We also give some applications.
2. MAIN RESULTS

To prove our main results, we use following Lemma which was used by Alomari et al. (see [6]).

Lemma 1. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \), be a differentiable mapping on \( I \) where \( a, b \in I \), with \( a < b \). Let \( f' \in L[a, b] \), then the following equality holds:

\[
\begin{align*}
f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx &= \frac{b - a}{4} \left[ \int_0^1 t f' \left( \frac{a + b}{2} + (1 - t) a \right) \, dt + \int_0^1 (t - 1) f' \left( t b + (1 - t) \frac{a + b}{2} \right) \, dt \right].
\end{align*}
\]

Theorem 6. Let \( f : [0, \infty) \to \mathbb{R} \), be a differentiable mapping such that \( f' \in L[a, b] \). If \( |f'| \) is \( m \)-convex on \([a, b]\), where \( 0 \leq a < b < \infty \) and for some fixed \( m \in (0, 1] \), then the following inequality holds:

\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \min \{ T_1, T_2, T_3, T_4 \}
\]

where

\[
\begin{align*}
T_1 &= \frac{b - a}{12} \left[ 2 \left| f' \left( \frac{a + b}{2} \right) \right| + m \left| f' \left( \frac{a}{m} \right) + f' \left( \frac{b}{m} \right) \right| \right], \\
T_2 &= \frac{b - a}{12} \left[ \left| f' \left( \frac{a + b}{2} \right) \right| + m \left| f' \left( \frac{a + b}{2m} \right) + m \left| f' \left( \frac{b}{m} \right) \right| \right| \\
T_3 &= \frac{b - a}{12} \left[ \left| f' \left( \frac{a}{m} \right) + \left| f' \left( \frac{b}{m} \right) \right| \right| + 2m \left| f' \left( \frac{a + b}{2m} \right) \right| \\
T_4 &= \frac{b - a}{12} \left[ \left| f' \left( \frac{a + b}{2} \right) \right| + m \left| f' \left( \frac{a + b}{2m} \right) + m \left| f' \left( \frac{a}{m} \right) + f' \left( \frac{b}{m} \right) \right| \right| \right].
\end{align*}
\]

Proof. From the equality which is given in the Lemma 1 and by using the properties of modulus, we have

\[
\begin{align*}
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| &\leq \frac{b - a}{4} \left[ \int_0^1 \left| f' \left( \frac{a + b}{2} + (1 - t) a \right) \right| \, dt \\
&\quad + \int_0^1 \left| t - 1 \right| \left| f' \left( t b + (1 - t) \frac{a + b}{2} \right) \right| \, dt \right].
\end{align*}
\]

By using \( m \)-convexity of \( |f'| \) on \([a, b]\), we know that for any \( t \in [0, 1] \)

\[
\left| f' \left( \frac{t a + b}{2} + (1 - t) a \right) \right| \leq t \left| f' \left( \frac{a + b}{2} \right) \right| + m (1 - t) \left| f' \left( \frac{a}{m} \right) \right|
\]

(2.3)
and

\[ f'(tb + (1 - t) \frac{a + b}{2}) \leq (1 - t) \left| f'(\frac{a + b}{2}) \right| + mt \left| f'(\frac{b}{m}) \right|. \]

From the inequalities (2.3) and (2.4), we obtain

\[ \left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{4} \left[ \int_0^1 t \left| f'(\frac{a + b}{2}) \right| + m(1 - t) \left| f'(\frac{a}{m}) \right| \right] dt \]

\[ + \int_0^1 (1 - t) \left( (1 - t) \left| f'(\frac{a + b}{2}) \right| + mt \left| f'(\frac{b}{m}) \right| \right) dt \].

By calculating the above integrals, we get the following inequality;

\[ (2.5) \]

\[ \left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{12} \left[ 2 \left| f'(\frac{a + b}{2}) \right| + m \left( \left| f'(\frac{a}{m}) \right| + \left| f'(\frac{b}{m}) \right| \right) \right]. \]

Analogously, we obtain the following inequalities;

\[ (2.6) \]

\[ \left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{12} \left[ \left| f'(a) \right| + \left| f'(\frac{a + b}{2}) \right| + \frac{m}{2} \left| f'(\frac{b}{m}) \right| \right] \]

\[ (2.7) \]

\[ \left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{12} \left[ 2 \left| f'(\frac{a + b}{2}) \right| + m \left| f'(\frac{a + b}{2m}) \right| \right] \]

and

\[ (2.8) \]

\[ \left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{12} \left[ \left| f'(a) \right| + \left| f'(\frac{a + b}{2}) \right| + \frac{m}{2} \left| f'(\frac{b}{m}) \right| \right]. \]

From the inequalities (2.5), (2.6), (2.7) and (2.8), we get the desired result. \( \square \)

**Corollary 1.** If we choose \( m = 1 \) in (2.7), we obtain the inequality;

\[ \left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{12} \left[ 2 \left| f'(\frac{a + b}{2}) \right| + \left| f'(b) \right| \right]. \]

**Corollary 2.** Under the assumptions of Theorem 6;

i) If we choose \( m = 1 \) and \( |f'| \) is increasing in (2.7), we obtain the inequality;

\[ \left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{12} \left[ 2 \left| f'(\frac{a + b}{2}) \right| + |f'(b)| \right]. \]
Let \( \lambda > 0 \).

From Lemma 1 and by using the properties of modulus, we have

\[
\left| f\left( \frac{a + b}{2} \right) - 4 \sum_{k=0}^{\lambda - 1} f\left( \frac{a + b}{2^k} \right) \right| \leq \frac{b - a}{12} \left[ 2 f'\left( \frac{a + b}{2} \right) \right].
\]

ii) If we choose \( m = 1 \) and \( |f'| \) is decreasing in \( [2, 3] \), we obtain the inequality:

\[
\left| f\left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{12} \left[ 2 f'\left( \frac{a + b}{2} \right) \right].
\]

iii) If we choose \( m = 1 \) and \( |f'|(\frac{a + b}{2})| = 0 \) in \( [2, 3] \), we obtain the inequality:

\[
\left| f\left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{12} \left[ |f'(a)| + |f'(b)| \right].
\]

iv) If we choose \( m = 1 \) and \( |f'(a)| = |f'(b)| = 0 \) in \( [2, 3] \), we obtain the inequality:

\[
\left| f\left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{12} \left[ |f'(a)| + |f'(b)| \right].
\]

**Theorem 7.** Let \( f : [0, \infty) \rightarrow \mathbb{R} \), be a differentiable mapping such that \( f' \in L[a, b] \). If \( |f'|^{\frac{q}{q+1}} \) is \( m \)-convex on \([a, b]\), where \( 0 \leq a < b < \infty \), for some fixed \( m \in (0, 1] \) and \( p > 1 \), then the following inequality holds:

\[
(2.9) \left| f\left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4 (p + 1)^\frac{1}{p}} \left( \frac{1}{2} \right) \min \{ U_1, U_2, U_3, U_4 \}
\]

where \( \frac{1}{q} + \frac{1}{p} = 1 \) and

\[
U_1 = \left( \left| f'\left( \frac{a + b}{2} \right) \right|^q + m |f'\left( \frac{a}{m} \right)|^q \right)^{\frac{1}{q}} + \left( \left| f'\left( \frac{a + b}{2} \right) \right|^q + m |f'\left( \frac{b}{m} \right)|^q \right)^{\frac{1}{q}},
\]

\[
U_2 = \left( \left| f'(a) \right|^q + m |f'\left( \frac{a + b}{2m} \right)|^q \right)^{\frac{1}{q}} + \left( \left| f'(b) \right|^q + m |f'\left( \frac{b}{2m} \right)|^q \right)^{\frac{1}{q}},
\]

\[
U_3 = \left( \left| f'(a) \right|^q + m |f'\left( \frac{a + b}{2m} \right)|^q \right)^{\frac{1}{q}} + \left( \left| f'(b) \right|^q + m |f'\left( \frac{b}{2m} \right)|^q \right)^{\frac{1}{q}},
\]

\[
U_4 = \left( \left| f'\left( \frac{a + b}{2} \right) \right|^{q'} + m |f'\left( \frac{a}{m} \right)|^{q'} \right)^{\frac{1}{q'}} + \left( \left| f'\left( \frac{b}{2} \right) \right|^{q'} + m |f'\left( \frac{b}{2m} \right)|^{q'} \right)^{\frac{1}{q'}}.
\]

**Proof.** From Lemma 1 and by using the properties of modulus, we have

\[
(2.10) \left| f\left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4} \left[ \int_0^1 |t| f'\left( \frac{a + b}{2} + (1 - t) a \right) \, dt + \int_0^1 |t - 1| f'\left( \frac{b}{2} + (1 - t) a \right) \, dt \right].
\]
By applying the Hölder inequality to the inequality (2.11), we get

\[
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right|
\leq \frac{b-a}{4} \left[ \left( \int_0^1 t^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'(t \left( \frac{a+b}{2} \right) + (1-t) a \right) \right|^q \, dt \right]^{\frac{1}{q}}
\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left( \int_0^1 t^p \, dt \right)^{\frac{1}{p}} \left[ \left( \int_0^1 \left| f'(t \left( \frac{a+b}{2} \right) + (1-t) a \right) \right|^q \, dt \right]^{\frac{1}{q}}
+ \left( \int_0^1 (1-t)^p \, dt \right) \left( \int_0^1 \left| f'(tb + (1-t) \left( \frac{a+b}{2} \right)) \right|^q \, dt \right) \left( \int_0^1 \left| f'(tb + (1-t) \left( \frac{a+b}{2} \right)) \right|^q \, dt \right)^{\frac{1}{q}}.
\]

It is easy to see that

\[
\int_0^1 t^p \, dt = \int_0^1 (1-t)^p \, dt = \frac{1}{p+1}.
\]

Hence, by \(m\)-convexity of \(|f'|^q\) on \([a,b]\), we obtain the inequality:

\[
(2.11)\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right|
\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left( \int_0^1 t^p \, dt \right)^{\frac{1}{p}} \left[ \left( \int_0^1 \left| f'(t \left( \frac{a+b}{2} \right) + (1-t) a \right) \right|^q \, dt \right]^{\frac{1}{q}}
+ \left( \int_0^1 (1-t)^p \, dt \right) \left( \int_0^1 \left| f'(tb + (1-t) \left( \frac{a+b}{2} \right)) \right|^q \, dt \right) \left( \int_0^1 \left| f'(tb + (1-t) \left( \frac{a+b}{2} \right)) \right|^q \, dt \right)^{\frac{1}{q}}.
\]

By a similar argument to the proof of Theorem 6, analogously, we obtain the following inequalities:

\[
(2.12)\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right|
\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left( \int_0^1 t^p \, dt \right)^{\frac{1}{p}} \left[ \left( \int_0^1 \left| f'(t \left( \frac{a+b}{2} \right) + (1-t) a \right) \right|^q \, dt \right]^{\frac{1}{q}}
+ \left( \int_0^1 (1-t)^p \, dt \right) \left( \int_0^1 \left| f'(tb + (1-t) \left( \frac{a+b}{2} \right)) \right|^q \, dt \right) \left( \int_0^1 \left| f'(tb + (1-t) \left( \frac{a+b}{2} \right)) \right|^q \, dt \right)^{\frac{1}{q}}.
\]

\[
(2.13)\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right|
\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left( \int_0^1 t^p \, dt \right)^{\frac{1}{p}} \left[ \left( \int_0^1 \left| f'(t \left( \frac{a+b}{2} \right) + (1-t) a \right) \right|^q \, dt \right]^{\frac{1}{q}}
+ \left( \int_0^1 (1-t)^p \, dt \right) \left( \int_0^1 \left| f'(tb + (1-t) \left( \frac{a+b}{2} \right)) \right|^q \, dt \right) \left( \int_0^1 \left| f'(tb + (1-t) \left( \frac{a+b}{2} \right)) \right|^q \, dt \right)^{\frac{1}{q}}.
\]
Corollary 3. Under the assumptions of Theorem 7, if we choose \( m = 1 \), we obtain the inequality:

\[
(2.14) \quad \left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4 (p+1) \frac{q}{p}} \left( 1 + \frac{1}{2} \right) \left[ \left( \left| f' \left( \frac{a+b}{2} \right) \right| p + \left| f' \left( \frac{a}{m} \right) \right| q \right)^{\frac{1}{q}} + \left( \left| f' \left( \frac{a+b}{2} \right) \right| q + \left| f' \left( \frac{b}{m} \right) \right| q \right)^{\frac{1}{q}} \right].
\]

From the inequalities (2.11) - (2.14), we obtain the inequality in (2.9). The second inequality in (2.9) follows from facts that:

\[
\lim_{p \to \infty} \left( \frac{1}{1 + p} \right)^{\frac{1}{p}} = 1, \quad \lim_{p \to 1^+} \left( \frac{1}{1 + p} \right)^{\frac{1}{p}} = \frac{1}{2}
\]

and

\[
\frac{1}{2} < \left( \frac{1}{1 + p} \right)^{\frac{1}{p}} < 1.
\]

Corollary 4. Under the assumptions of Theorem 7:

i) If we choose \( m = 1 \) and \( |f'|^{\frac{p}{p-1}} \) is increasing in \( (a, b) \), we obtain the inequality:

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4 (p+1) \frac{q}{p}} \left( \left| f' \left( \frac{a+b}{2} \right) \right| q + \left| f' \left( \frac{b}{m} \right) \right| q \right)^{\frac{1}{q}}.
\]

ii) If we choose \( m = 1 \) and \( |f'|^{\frac{p}{p-1}} \) is decreasing in \( (a, b) \), we obtain the inequality:

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4 (p+1) \frac{q}{p}} \left( \left| f' \left( \frac{a+b}{2} \right) \right| q + \left| f' \left( \frac{a}{m} \right) \right| q \right)^{\frac{1}{q}}.
\]

iii) If we choose \( m = 1 \) and \( \left| f' \left( \frac{a+b}{2} \right) \right|^{\frac{p}{p-1}} = 0 \) in (2.14), we obtain the inequality:

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4 (p+1) \frac{q}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} \left( \left| f' \left( \frac{a}{m} \right) \right| + \left| f' \left( \frac{b}{m} \right) \right| \right).
\]
iv) If we choose \( m = 1 \) and \( |f'(a)|^\frac{m}{2} = |f'(b)|^\frac{m}{2} = 0 \) in (2.14), we obtain the inequality:

\[
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4(p+1)^\frac{1}{p}} \left( \frac{1}{2} \right)^\frac{1}{q} \left| f'\left(\frac{a+b}{2}\right) \right|.
\]

**Theorem 8.** Let \( f : [0, \infty) \to \mathbb{R} \), be a differentiable mapping such that \( f' \in L[a, b] \). If \( |f'|^q \) is \( m \)-convex on \([a, b]\), where \( 0 \leq a < b < \infty \), for some fixed \( m \in (0, 1) \) and \( q \geq 1 \), then the following inequality holds:

\[
(2.15) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left( \frac{1}{2} \right)^{1-q} \min\{V_1, V_2, V_3, V_4\}
\]

where

\[
V_1 = \left( \frac{1}{3} \right) \left| f'\left(\frac{a+b}{2}\right) \right|^q + \frac{m}{6} \left| f'\left(\frac{a}{m}\right) \right|^q + \left( \frac{1}{3} \right) \left| f'\left(\frac{a}{m}\right) \right|^q,
\]

\[
V_2 = \left( \frac{1}{6} \right) \left| f'(a) \right|^q + \frac{m}{3} \left| f'\left(\frac{a+b}{2m}\right) \right|^q + \left( \frac{1}{3} \right) \left| f'\left(\frac{a+b}{2m}\right) \right|^q,
\]

\[
V_3 = \left( \frac{1}{6} \right) \left| f'(a) \right|^q + \frac{m}{3} \left| f'\left(\frac{a+b}{2m}\right) \right|^q + \left( \frac{1}{6} \right) \left| f'(b) \right|^q + \frac{m}{3} \left| f'\left(\frac{a+b}{2m}\right) \right|^q,
\]

\[
V_4 = \left( \frac{1}{3} \right) \left| f'\left(\frac{a+b}{2}\right) \right|^q + \frac{m}{6} \left| f'\left(\frac{a}{m}\right) \right|^q + \left( \frac{1}{6} \right) \left| f'(b) \right|^q + \frac{m}{3} \left| f'\left(\frac{a+b}{2m}\right) \right|^q.
\]

**Proof.** From Lemma 1, we can write

\[
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left[ \int_0^1 t \left| f'\left(\frac{a+b}{2} + (1-t) \frac{a}{m}\right) \right| dt + \int_0^1 (t-1) \left| f'\left(\frac{a+b}{2} + (1-t) \frac{a}{m}\right) \right| dt \right].
\]

By applying the Power-mean inequality, we get

\[
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left[ \left( \int_0^1 t dt \right)^{1-q} \left( \int_0^1 t \left| f'\left(\frac{a+b}{2} + (1-t) \frac{a}{m}\right) \right|^q dt \right)^{\frac{1}{q}} 
\]

\[
+ \left( \int_0^1 (t-1) dt \right)^{1-q} \left( \int_0^1 (t-1) \left| f'\left(\frac{a+b}{2} + (1-t) \frac{a}{m}\right) \right|^q dt \right)^{\frac{1}{q}} \right].
\]
Now by using $m$–convexity of $|f'|^q$ on $[a, b]$ and by computing the integrals, we obtain the following inequality:

\[(2.16) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{1}{3} \left| f'\left(\frac{a+b}{2}\right)\right|^q + \frac{m}{6} \left| f'\left(\frac{a}{m}\right)\right|^q \right)^{\frac{1}{q}} \right. \]

\[+\left(\frac{1}{3} \left| f'\left(\frac{a+b}{2}\right)\right|^q + \frac{m}{6} \left| f'\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} \].

Hence, by a similar argument to the proofs of Theorem 6-7, analogously, we obtain the following inequalities:

\[(2.17) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{1}{6} \left| f'\left(\frac{a+b}{2}\right)\right|^q + \frac{m}{3} \left| f'\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} \right. \]

\[+\left(\frac{1}{6} \left| f'\left(\frac{a+b}{2}\right)\right|^q + \frac{m}{3} \left| f'\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} \],

\[(2.18) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{1}{6} \left| f'\left(\frac{a+b}{2}\right)\right|^q + \frac{m}{3} \left| f'\left(\frac{a+b}{2m}\right)\right|^q \right)^{\frac{1}{q}} \right. \]

\[+\left(\frac{1}{6} \left| f'\left(\frac{a+b}{2}\right)\right|^q + \frac{m}{3} \left| f'\left(\frac{b}{2m}\right)\right|^q \right)^{\frac{1}{q}} \],

and

\[(2.19) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{1}{3} \left| f'\left(\frac{a+b}{2}\right)\right|^q + \frac{m}{6} \left| f'\left(\frac{a}{m}\right)\right|^q \right)^{\frac{1}{q}} \right. \]

\[+\left(\frac{1}{6} \left| f'\left(\frac{a+b}{2}\right)\right|^q + \frac{m}{3} \left| f'\left(\frac{a}{2m}\right)\right|^q \right)^{\frac{1}{q}} \].

By the inequalities (2.16)–(2.19), we obtain the inequality (2.15). \qed
Corollary 5. Under the assumptions of Theorem 8, if we choose $m = 1$, we obtain the inequality:

$$
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| 
\leq \frac{b-a}{4} \left( \frac{1}{2} \right)^{1-\frac{1}{q}} \left( \frac{1}{3} \left| f' \left( \frac{a+b}{2} \right) \right|^q + \frac{1}{6} \left| f' \left( b \right) \right|^q \right)^{\frac{1}{q}}.
$$

Corollary 6. Under the assumptions of Theorem 8:

i) If we choose $m = 1$ and $|f'|^q$ is increasing in $[2.15]$, we obtain the inequality:

$$
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{1}{2} \right)^{1-\frac{1}{q}} \left( \frac{1}{3} \left| f' \left( \frac{a+b}{2} \right) \right|^q + \frac{1}{6} \left| f' \left( b \right) \right|^q \right)^{\frac{1}{q}}.
$$

ii) If we choose $m = 1$ and $|f'|^q$ is decreasing in $[2.15]$, we obtain the inequality:

$$
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{1}{2} \right)^{1-\frac{1}{q}} \left( \frac{1}{3} \left| f' \left( \frac{a+b}{2} \right) \right|^q + \frac{1}{6} \left| f' \left( a \right) \right|^q \right)^{\frac{1}{q}}.
$$

iii) If we choose $m = 1$ and $|f' (\frac{a+b}{2})|^q = 0$ in $[2.15]$, we obtain the inequality:

$$
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{8} \left( \frac{1}{3} \right)^{\frac{1}{q}} (|f'(a)| + |f'(b)|).
$$

iv) If we choose $m = 1$ and $|f'(a)|^q = |f'(b)|^q = 0$ in $[2.15]$, we obtain the inequality:

$$
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{1}{6} \right)^{1-\frac{1}{q}} \left| f' \left( \frac{a+b}{2} \right) \right|.
$$

Theorem 9. Let $f, g : [0, b] \to \mathbb{R}$, be concave and $m$–concave functions, $m \in (0, 1)$, where $0 \leq a < b < \infty$ and $q \geq 1$. Then

$$
f \left( \frac{a+b}{2} \right) g \left( \frac{a+b}{2} \right) \geq \frac{(p+1) \frac{1}{q} (q+1) \frac{1}{q}}{16} \times \left( \frac{1}{b-a} \int_a^b \left[ f(x) + mf \left( \frac{x}{m} \right) \right] \left[ g(x) + mg \left( \frac{x}{m} \right) \right] \, dx \right).
$$

where $\frac{1}{q} + \frac{1}{p} = 1$.

If $f, g$ are convex and $m$–convex functions, with $f(0) = 0$, then the reverse of the above inequality holds.

Proof. Since $f, g$ are $m$–concave, by using the inequality (1.3), we can write

$$
f \left( \frac{a+b}{2} \right) \geq \frac{1}{b-a} \int_a^b f(x) + mf \left( \frac{x}{m} \right) \, dx
$$
and
\[ g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b g(x) + mg\left(\frac{x}{m}\right) dx. \]

By using Favard’s inequality for \( p \)–th powers of both sides of inequality, we have
\[
f^p\left(\frac{a+b}{2}\right) \geq \left(\frac{1}{b-a} \int_a^b f(x) + mf\left(\frac{x}{m}\right) dx\right)^p \geq \frac{p+1}{2p} \left[ \frac{1}{b-a} \int_a^b \left( f(x) + mf\left(\frac{x}{m}\right) \right)^p dx \right]
\]
and similarly, we have
\[
g^q\left(\frac{a+b}{2}\right) \geq \frac{q+1}{2q} \left[ \frac{1}{b-a} \int_a^b \left( g(x) + mg\left(\frac{x}{m}\right) \right)^q dx \right].
\]

It follows that
\[
f\left(\frac{a+b}{2}\right) \geq \frac{(p+1)^\frac{1}{2}}{2} \left[ \frac{1}{b-a} \int_a^b \left( f(x) + mf\left(\frac{x}{m}\right) \right)^p dx \right]^\frac{1}{p}
\]
and
\[
g\left(\frac{a+b}{2}\right) \geq \frac{(q+1)^\frac{1}{2}}{2} \left[ \frac{1}{b-a} \int_a^b \left( g(x) + mg\left(\frac{x}{m}\right) \right)^q dx \right]^\frac{1}{q}.
\]

By multiplying both sides of the above inequalities, we get
\[
f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \geq \frac{(p+1)^\frac{1}{2}}{4} \left[ \frac{1}{b-a} \int_a^b \left( f(x) + mf\left(\frac{x}{m}\right) \right)^p dx \right]^\frac{1}{p} \times \left[ \frac{1}{b-a} \int_a^b \left( g(x) + mg\left(\frac{x}{m}\right) \right)^q dx \right]^\frac{1}{q}.
\]

By using Hölder inequality, we have
\[
f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \geq \frac{(p+1)^\frac{1}{2}}{16} \frac{(q+1)^\frac{1}{2}}{16} \times \left[ \frac{1}{b-a} \int_a^b \left[ f(x) + mf\left(\frac{x}{m}\right) \right] dx \right] \left[ g(x) + mg\left(\frac{x}{m}\right) \right] dx.
\]

If \( f, g \) are \( m \)–convex, then using Thunsdorff’s inequality we obtain desired result.
\[ \square \]
Corollary 7. Under the assumptions of Theorem 9, if we choose \( m = 1 \), we obtain the inequality:

\[
f \left( \frac{a + b}{2} \right) g \left( \frac{a + b}{2} \right) \geq \frac{(p + 1)^\frac{1}{p} (q + 1)^\frac{1}{q}}{4} \times \left( \frac{1}{b - a} \int_a^b f(x)g(x)dx \right).
\]

3. APPLICATIONS TO SOME SPECIAL MEANS

We now consider the applications of our Theorems to the following special means

a) The arithmetic mean:

\[
A = A(a, b) := \frac{a + b}{2}, \quad a, b \geq 0,
\]

b) The logarithmic mean:

\[
L = L(a, b) := \begin{cases} 
\frac{b - a}{\ln b - \ln a} & \text{if } a = b, \\
\frac{a - b}{\ln a - \ln b} & \text{if } a \neq b,
\end{cases} \quad a, b \geq 0,
\]

c) The \( p \)-logarithmic mean:

\[
L_p = L_p(a, b) := \begin{cases} 
\left[ \frac{p^{p+1} - q^{p+1}}{(p+1)(q-b)} \right]^{1/p} & \text{if } a \neq b, \\
\frac{1}{p} & \text{if } a = b,
\end{cases} \quad p \in \mathbb{R} \setminus \{-1, 0\}; \quad a, b > 0.
\]

We now derive some sophisticated bounds of the above means.

**Proposition 1.** Let \( a, b \in \mathbb{R}, \; 0 < a < b \) and \( n \in \mathbb{Z}, \; |n| \geq 2 \). Then, we have:

\[
|A^n(a, b) - L^n_n(a, b)| \leq \min \{K_1, K_2, K_3, K_4\}
\]

where

\[
K_1 = \frac{n(b - a)}{12} \left[ 2|A(a, b)|^{n-1} + m \left( \left| \frac{a}{m} \right|^{n-1}, \left| \frac{b}{m} \right|^{n-1} \right) \right],
\]

\[
K_2 = \frac{n(b - a)}{12} \left[ |A(a, b)|^{n-1} + m \left| \frac{A(a, b)}{m} \right|^{n-1} + A \left( |a|^{n-1}, m \left| \frac{b}{m} \right|^{n-1} \right) \right],
\]

\[
K_3 = \frac{n(b - a)}{12} \left[ A \left( |a|^{n-1} + |b|^{n-1} \right) + 2m \left| \frac{A(a, b)}{m} \right|^{n-1} \right],
\]

\[
K_4 = \frac{n(b - a)}{12} \left[ |A(a, b)|^{n-1} + m \left| \frac{A(a, b)}{m} \right|^{n-1} + A \left( m \left| \frac{a}{m} \right|^{n-1}, |b|^{n-1} \right) \right].
\]

**Proof.** The proof is immediate from Theorem 6 applied for \( f(x) = x^n \), which is an \( m \)-convex function. \( \square \)

**Proposition 2.** Let \( a, b \in \mathbb{R}, \; 0 < a < b \) and \( n \in \mathbb{Z}, \; |n| \geq 2, \; k \geq 1 \). Then, we have:

\[
|A^n(a, b) - L^n_k(a, b)| \leq \frac{b - a}{4} \left( \frac{1}{2} \right)^{1 - \frac{1}{k}} \min \{L_1, L_2, L_3, L_4\}
\]
where

\[ L_1 = \frac{n}{k} 2A \left[ \left( 2A \left( \frac{1}{3} \left| (A(a,b)) \frac{q(a-k)}{k} \right| + \frac{m}{6} \left| a \frac{q(a-k)}{k} \right| \right) \right)^{\frac{1}{3}} \right. \\
+ \left. \left( 2A \left( \frac{1}{3} \left| (A(a,b)) \frac{q(a-k)}{k} \right| + \frac{m}{6} \left| b \frac{q(a-k)}{k} \right| \right) \right)^{\frac{1}{3}} \right] \]

\[ L_2 = \frac{n}{k} 2A \left[ \left( 2A \left( \frac{1}{6} \left| a \frac{q(a-k)}{k} \right| + \frac{m}{3} \left| A(a,b) \frac{q(a-k)}{k} \right| \right) \right)^{\frac{1}{3}} \right. \\
+ \left. \left( 2A \left( \frac{1}{3} \left| (A(a,b)) \frac{q(a-k)}{k} \right| + \frac{m}{6} \left| b \frac{q(a-k)}{k} \right| \right) \right)^{\frac{1}{3}} \right] \]

\[ L_3 = \frac{n}{k} 2A \left[ \left( 2A \left( \frac{1}{6} \left| a \frac{q(a-k)}{k} \right| + \frac{m}{3} \left| A(a,b) \frac{q(a-k)}{k} \right| \right) \right)^{\frac{1}{3}} \right. \\
+ \left. \left( 2A \left( \frac{1}{6} \left| b \frac{q(a-k)}{k} \right| + \frac{m}{3} \left| A(a,b) \frac{q(a-k)}{k} \right| \right) \right)^{\frac{1}{3}} \right] \]

\[ L_4 = \frac{n}{k} 2A \left[ \left( 2A \left( \frac{1}{3} \left| (A(a,b)) \frac{q(a-k)}{k} \right| + \frac{m}{6} \left| a \frac{q(a-k)}{k} \right| \right) \right)^{\frac{1}{3}} \right. \\
+ \left. \left( 2A \left( \frac{1}{6} \left| b \frac{q(a-k)}{k} \right| + \frac{m}{3} \left| A(a,b) \frac{q(a-k)}{k} \right| \right) \right)^{\frac{1}{3}} \right] \]

Proof. The assertion follows from Theorem 8 applied to \( f(x) = x^\frac{1}{n} \), which is an \( m \)-convex function. \( \square \)

REFERENCES

[1] S.S. Dragomir, Two mappings in connection to Hadamard’s inequalities, Journal of Math. Anal. Appl., 167 (1992), 49-56.
[2] U.S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula, Appl. Math. Comput., 147 (2004), 137-146.
[3] U.S. Kirmaci and M.E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Lett., 167 (1992), 49-56.
[4] C.E.M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulæ, Appl. Math. Lett., 13(2) (2000), 51–55.
[5] J.E. Pečarić, F. Proschan and Y.L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press Inc., 1992.
[6] M.W. Alomari, M. Darus and U.S. Kirmaci, Some inequalities of Hermite-Hadamard type for s-convex functions, Acta Mathematica Scientia, (2011) 31B(4):1643–1652.
$[7]$ M.K. Bakula, M.E. Özdemir and J. Pečarić, Hadamard-type inequalities for $m$–convex and $(\alpha, m)$–convex functions, *J. Inequal. Pure and Appl. Math.*, 9, (4), (2007), Article 96.

$[8]$ M.K. Bakula, J. Pečarić and M. Ribibić, Companion inequalities to Jensen’s inequality for $m$–convex and $(\alpha, m)$–convex functions, *J. Inequal. Pure and Appl. Math.*, 7 (5) (2006), Article 194.

$[9]$ S.S. Dragomir and G. Toader, Some inequalities for $m$–convex functions, Studia University Babes Bolyai, *Mathematica*, 38 (1) (1993), 21–28.

$[10]$ G. Toader, Some generalisation of the convexity, *Proc. Colloq. Approx. Opt.*, Cluj-Napoca, (1984), 329-338.

$[11]$ M.E. Özdemir, M. Avci and E. Set, On some inequalities of Hermite-Hadamard type via $m$–convexity, *Applied Mathematics Letters*, 23 (2010), 1065-1070.

$[12]$ G. Toader, On a generalization of the convexity, *Mathematica*, 30 (53) (1988), 83-87.

$[13]$ S.S. Dragomir, On some new inequalities of Hermite-Hadamard type for $m$–convex functions, *Tamkang Journal of Mathematics*, 33 (1) (2002).

$[14]$ H. Kavurmaci, M. Avci, M.E. Özdemir, New Ostrowski type inequalities for $m$–convex functions and applications, accepted.

$[15]$ M.E. Özdemir, E. Set and M.Z. Sarıkaya, Some new Hadamard’s type inequalities for co-ordinated $m$–convex and $(\alpha, m)$–convex functions, *Hacettepe J. of Math. and Ist.*, 40, 219-229, (2011).

$[16]$ M.Z. Sarıkaya, M.E. Özdemir and E. Set, Inequalities of Hermite–Hadamard’s type for functions whose derivatives absolute values are $m$–convex, *RGMIA Res. Rep. Coll.* 13 (2010) Supplement, Article 5.

$[17]$ S.S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.*, 11 (5), 91-95, (1998).

$[18]$ N. Latif, J. Pečarić and I. Perić, Some New Results Related to Favard’s Inequality, *J. Inequal. Appl.*, 2009, Article ID 128486.

$[19]$ J. Pečarić, F. Proschan and Y.L. Tong, Convex functions, Partial Orderings and Statistical Applications, *Academic Press*, 1992.

$\ast$ *Atatürk University, K. K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey*

$\bullet$ *Ağrı İbrahim Çėcen University Faculty of Science and Letters, Department of Mathematics, 04100, Ağrı, Turkey*

*E-mail address: emos@atauni.edu.tr*

*E-mail address: ahmetakdemir@agri.edu.tr*