String fluids and membrane media

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Abstract

String (membrane) theory could be considered as degenerate case of relativistic continuous media theory. The paper presents models of media, which are continuous distributions of interacting membranes, strings or particles.

1 Introduction (Kinematics)

To describe the motion of elastic medium one has to specify the motion of every element of medium. In the present paper elements of medium are particles, strings or membranes. To specify their motion one has to describe their world lines or world surfaces.

Here we suppose, that world surfaces (lines) do not intersect each other, and every point of space-time \( M \) belongs to some world surface (line).

Let world surfaces are numerated by parameters \( \phi = (\phi^1, \ldots, \phi^n) \) (coordinates across world surface). \( V_\phi \) is world surface number \( \phi \). Let points of world surfaces are numerated by parameters \( \xi = (\xi^1, \ldots, \xi^{D-n}) \) (coordinates along world surface).

To describe the motion of elastic medium one can use explicit or implicit form

\[
V_\phi = \{ X \in M | X = x(\xi, \phi) \}, \quad \text{or} \quad V_\phi = \{ X \in M | \phi(X) = \phi \}.
\]

I.e. space-time coordinates are functions \( x = (x^0, \ldots, x^{D-1}) \) of \( \xi \) and \( \phi \), or world surface \( V_\phi \) is intersection of level surfaces for \( n \) space-time scalar fields \( \phi = (\phi^1, \ldots, \phi^n) \).

Explicit form is standard in relativistic string (membrane) theory \([1–4]\).

Implicit form was used in some papers on relativistic string (membrane) theory \([5, 6, 7, 9, 10, 11]\), which deal with continuous distributions of strings (membranes). The aim of the present paper is to relate this field theory approach to relativistic elastic media mechanics.

Implicit form allows to build differential form \( J = f(\phi)d\phi^1 \wedge \ldots \wedge d\phi^n \), which represents density of membranes. Some theories use other parametrizations \( J = dI \) of form \( J \) (see \([8]\) and references in these papers). These theories are discussed in Section 5 of present paper.

Kinematics of elastic media is described (in implicit form) by map \( \varphi : M \rightarrow F \), where \( M \) is pseudorimannian manifold (space-time), \( \dim M = D \), \( X^M \) are coordinates on \( M \) (Euler coordinates), \( g_{MN} \) is metric; \( F \) is manifold, \( \dim F = n < D \), \( \phi^n \) are coordinates on \( F \) (Lagrange coordinates).

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World surface $V_{\phi}$ is defined by means of inverse image $V_{\phi} = \varphi^{-1}(\phi)$.

Points of $\textbf{F}$ represents elements of elastic medium. In further discussion we refer space $\textbf{F}$ as medium. Now we do not introduce any additional structure at manifold $\textbf{F}$, but later we have to add some structure on $\textbf{F}$ to build appropriate action.

In the kinematics we construct here gradients $d\varphi^\alpha$ have to be space- or light-like. To study a causal structure of the theory one has to investigate the properties of energy-momentum tensor.

2 Action

To construct action for elastic medium we impose restrictions 1) action does not involve higher derivatives of $\varphi$, 2) field $\varphi$ is minimally coupled to metric, 3) action does not involve other fields at $\textbf{M}$, 4) action is invariant under change of coordinates on $\textbf{M}$ (Euler coordinates), 5) action is invariant under change of coordinates on $\textbf{F}$ (Lagrange coordinates).

Items 1, 2 and 3 force us to write action in the following form

$$S_{\phi} = \int d^D X \sqrt{|g|} L_{\phi}(\varphi^\alpha, \partial_M \varphi^\alpha, g_{MN}).$$  \hspace{1cm} (1)

Item 4 require Lagrangian $L_{\phi}$ to be scalar with respect to change of coordinates on $\textbf{M}$. Under coordinate transformation on $\textbf{M}$ fields $\varphi^\alpha$ are scalars, metric $g_{MN}$ is tensor, and gradients $\partial_M \varphi^\alpha$ are covectors. Any scalar constructed from these fields is function of $\varphi^\alpha$ and scalar products of covectors $\partial_M \varphi^\alpha$

$$G^{\alpha\beta} = g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta.$$  \hspace{1cm} (2)

So, $L_{\phi} = L_{\phi}(\varphi^\alpha, G^{\alpha\beta})$.

Item 5 require Lagrangian $L_{\phi}$ to be scalar with respect to change of coordinates on $\textbf{F}$. With respect to change of coordinates on $\textbf{F}$ matrix $G^{\alpha\beta}$ is tensor (induced inverse metric at $\textbf{F}$). We are unable to build any nontrivial scalar from $G^{\alpha\beta}$, so some additional parameters, which behave under change of coordinates on $\textbf{F}$ as tensor fields, are necessary. These additional tensor fields specify the structure at $\textbf{F}$, which was mentioned in previous section. From the point of view of field theory these tensors are functional parameters of action, from the point of view of elastic media mechanics they describes the medium (just in previous section we assign name “medium” to auxiliary space $\textbf{F}$).

Let matrix $G^{\alpha\beta}$ is nondegenerate, i.e. $\det G^{\alpha\beta} \neq 0$ (rank $\partial_M \varphi^\alpha = n$). Nondegeneracy condition means, that inverse image $\varphi^{-1}(\phi)$ of point $\phi \in \textbf{F}$ is (locally) a surface of dimension $D - n$. Medium we consider consist of such membranes, numerated by points of space $\textbf{F}$.

One can introduce inverse matrix $G_{\alpha\beta}$ (metric) and volume form $\omega_{\textbf{F}}$

$$G^{\alpha\beta} G_{\beta\gamma} = \delta^\alpha_\gamma, \quad G = \det G^{\alpha\beta} = \left(\det G^{\alpha\beta}\right)^{-1}, \quad \omega_{\textbf{F}} = \sqrt{|G|} d\phi^1 \wedge \ldots \wedge d\phi^n.$$  \hspace{1cm} (3)

Tensors $G_{\alpha\beta}$ and $G^{\alpha\beta}$ are assumed to be used for raising and lowering of indices at space $\textbf{F}$ (Greek indices).

Coordinate system with $D - n$ coordinates specified by relation $x^\alpha = \varphi^\alpha$ is attendant coordinate system. Energy density in attendant coordinates is $\rho = -L$. 

2


3 Energy-momentum and stress tensors

By variation of metric, or variation of $\partial_K \varphi^\alpha$ one can find energy-momentum tensor (i.e. general relativity definition of energy-momentum tensor and mechanical definition produce the same expression)

$$T_{MN} = g_{MN}L_\phi - 2 \frac{\partial L_\phi}{\partial G_{\alpha\beta}} \partial_M \varphi^\alpha \partial_N \varphi^\beta = g_{MK} \left( \delta^K_N L_\phi - \frac{\partial L_\phi}{\partial (\partial_K \varphi^\alpha)} \partial_N \varphi^\alpha \right).$$

(4)

$$\nabla_K T^K_N = \partial_N \varphi^\alpha \left( \frac{\partial L_\phi}{\partial \varphi^\alpha} - \nabla_K \frac{\partial L_\phi}{\partial (\partial_K \varphi^\alpha)} \right) = \partial_N \varphi^\alpha \left( \frac{1}{\sqrt{|g|}} \delta S_\phi \right).$$

(5)

So, if rank $\partial_N \varphi^\alpha = n$ then field equations $\delta S_\phi \delta \varphi^\alpha = 0$ are equivalent to energy-momentum conservation $\nabla_M T^M_N = 0$.

Contravariant energy-momentum tensor $T^{MN}$ could be projected by $\varphi_*$ from space-time $M$ to medium $F$. New medium tensor $S^{\mu\nu}$ we refer as stress tensor

$$S^{\mu\nu} = T^{MN} \partial_M \varphi^\mu \partial_N \varphi^\nu = G^{\mu\nu} L_\phi - 2 \frac{\partial L_\phi}{\partial G_{\alpha\beta}} G^{\alpha\mu} G^{\beta\nu}.$$  

(6)

It looks like energy-momentum tensor formula with $g^{MN}$ replaced by $G^{\alpha\beta}$.

4 Perfect membrane fluid

It was stated in section 2 that to build Lagrangian, which is scalar with respect to $M$ and $F$, one has to specify at $F$ some tensor fields, which are parameters of action describing the medium. The simplest choice is to introduce at $F$ a volume form $\Omega_F = f(\phi) d\phi^1 \wedge \ldots \wedge d\phi^n$. Using volume forms $\Omega_F$ and $\omega_F$ one can build the scalar, which is ratio of volumes $\frac{1}{n!} \Omega_F^{\alpha_1 \ldots \alpha_n} \omega_F^{\alpha_1 \ldots \alpha_n} = f(\varphi)/\sqrt{|G|}$.

Lagrangian $L_\phi$ is a function $L_{\text{fluid}}$ of $\varphi$ and scalar $f(\varphi)/\sqrt{|G|}$.

$$S_{\text{fluid}} = \int d^D x \sqrt{|g|} L_{\text{fluid}} \left( \varphi, f(\varphi) \sqrt{\det G^{\alpha\beta}} \right)$$

(7)

Action (7) with $n = D - 1$ describes fluid or gas (with no dissipation). If $n < D - 1$ one has analogue of fluid, which consists of membranes with nonintersecting world surfaces.

This action was considered in paper [11].

Equations of motion for simple media are consequence of energy-momentum conservation, so to check correspondence of model (7) to fluid one can compare just energy-momentum tensors

$$T_{MN} = P_{MN} L_{\text{fluid}} + \tilde{P}_{MN} P_{\text{fluid}}.$$  

(8)

Here $P_{MN}$ is projector to world surface (surface $\varphi = \text{const}$), $\tilde{P}_{MN} = G_{\alpha\beta} \partial_M \varphi^\alpha \partial_N \varphi^\beta$ is projector to directions orthogonal to world surface. Pressure (in attendant coordinate system) is derived from Lagrangian by following “Legendre transformation” (prime means derivative by $f(\varphi)/\sqrt{|G|}$)

$$P_{\text{fluid}} = L_{\text{fluid}} - L'_{\text{fluid}} f \sqrt{\det G^{\alpha\beta}}.$$  

(9)

1Under $(-, +, +, +)$ space-time signature we use here, tensor (6) has opposite sign with respect to standard stress tensor introduced in mechanics.
If pressure \( P_{\text{fluid}} \) vanish (linear \( L_{\text{fluid}} \)), then one has gas of membranes with zero pressure, i.e. each world surface \( \varphi = \text{const} \) behaves as free membrane. (Equivalence of equations of motion in this case with equations of motion of free membrane was demonstrated in papers [9].)

**Example:** fluid with linear equation of state \( P_{\text{lin, fluid}} = k \rho_{\text{lin, fluid}} \) is described by Lagrangian \( L_{\text{lin, fluid}} = -\left( \det G^{\alpha \beta} \right)^{1/2} \).

## 5 Membrane fluids and nonlinear electrodynamics-type theories

One can write action (7) in terms of closed form \( J \) (“current”)

\[
J = \varphi^{\ast} \Omega_F = f(\varphi) \, d\varphi^1 \wedge \ldots \wedge d\varphi^n. \quad (10)
\]

\[
||J|| = \sqrt{1/n! J_{M_1 \ldots M_n} J^{M_1 \ldots M_n} = f(\varphi) \sqrt{\det G^{\alpha \beta}}.}
\]

\[
S_{\text{fluid}}[\varphi] = \int d^D x \sqrt{|g|} \, L_{\text{fluid}}(\varphi, ||J||).
\]

If fluid is homogeneous, then Lagrangian is independent on \( \varphi \), i.e. \( L_{\text{h, fluid}}(||J||) \).

So,

\[
S_{\text{h, fluid}}[\varphi] = \int d^D x \sqrt{|g|} \, L_{\text{h, fluid}}(||J||). \quad (11)
\]

One can use other definition of \( J \)

\[
J = dI. \quad (12)
\]

Any \( J \) defined by (10) could be written in form (12).

Definition (12) allows to build nonlinear electrodynamics-type theory with action

\[
S_{\text{nonlin}}[I] = \int d^D x \sqrt{|g|} \, L_{\text{h, fluid}}(||J||). \quad (13)
\]

Theories of this type are also called “string fluids” [8].

For both actions (11), (13) the only physical field is \( J \). E.g. both energy-momentum tensors are written in terms of field \( J \) only by the same formula. Fields \( I \) and \( \varphi \) play the role of “potentials”, i.e. they are used to derive field equations, which could be written in terms of \( J \) only.\(^2\)

So, both theories look very similar, but nonlinear electrodynamics-type theory is different from membrane fluid theory (even if the function \( L_{\text{h, fluid}} \) is the same).

If field \( J \) could be represented in form (10) (i.e. in both forms (10) and (12)), then

\[
\frac{\delta S_{\text{h, fluid}}}{\delta \varphi^\alpha} = \frac{1}{(n - 1)!} \frac{\delta S_{\text{nonlin}}}{\delta J_{M_1 \ldots M_n}} \epsilon^{\alpha \alpha_1 \ldots \alpha_{n-1}} f(\varphi) \partial_{M_1} \varphi^{\alpha_1} \ldots \partial_{M_{n-1}} \varphi^{\alpha_{n-1}}, \quad (14)
\]

\(^2\)Field equations for (11) has the following two forms (see (11))

\[
\frac{\delta S_{\text{h, fluid}}}{\delta \varphi^\alpha} = 0 \iff \frac{\delta S_{\text{h, fluid}}}{\delta \varphi^\alpha} \partial_{M} \varphi^{\alpha} = 0.
\]

The last one could be written in terms of \( J \) only.
or (the other form of same relation)

\[
\frac{\delta S_{\text{h, fluid}}}{\delta \varphi^\alpha} \partial_M \varphi^\alpha = \frac{1}{(n-1)!} \frac{\delta S_{\text{nonlin}}}{\delta I_{M_1 \ldots M_{n-1}}} J_{M_1 \ldots M_{n-1}},
\]

here

\[
\frac{\delta S_{\text{nonlin}}}{\delta I_{M_1 \ldots M_{n-1}}} = -\partial_M \left( \sqrt{|g|} \frac{\partial L_{\text{h, fluid}}}{\partial J_{M_1 \ldots M_{n-1}}} \right).
\]

(15)

I.e. definition (12) admits wider set of fields \( J \), and generates stricter field equations.

Fields \( J \), which could be represented in form (10) and satisfy field equations \( \frac{\delta S_{\text{nonlin}}}{\delta I_{M_1 \ldots M_{n-1}}} = 0 \) are solutions for both theories. Some fields \( J \), which could be represented in form (10) and satisfy field equations \( \frac{\delta S_{\text{nonlin}}}{\delta I_{M_1 \ldots M_{n-1}}} = 0 \) are not solutions of field equations \( \frac{\delta S_{\text{nonlin}}}{\delta I_{M_1 \ldots M_{n-1}}} = 0 \). Similarly some fields \( J \), which satisfy field equations \( \frac{\delta S_{\text{nonlin}}}{\delta I_{M_1 \ldots M_{n-1}}} = 0 \) could not be represented in form (10) (these fields do not represent membrane fluids).

6 Elastic membrane media

Elastic medium in contrast to fluid tends to preserve not only volume, but also form, i.e. distance between close points, which determined by metric. In classical mechanics it means existence of certain “nondeformed metric”, deviation from which is deformation. So, at space \( \Phi \) one has to define metric \( G_{\alpha\beta} \), which does not depend on map \( \varphi \) and describes medium itself. Metric \( G_{\alpha\beta} \) specifies nondeformed state.

Nondeformed state is defined as state with stress tensor \( S^{\mu\nu} \) vanishing and zero external forces (including gravity).

In the framework of general relativity defining of such state looks impossible, but stress tensor \( S^{\mu\nu} \) is defined (6) in terms of internal geometry of space \( \Phi \). “Non-deformed state” we are looking for is metric at \( \Phi \), so one can formulate the following problem

\[
S^{\mu\nu} |_{g_{\alpha\beta} = G_{\alpha\beta}^0} = 0.
\]

(16)

Metric \( G_{\alpha\beta}^0 \), which is solution of (16) is nondeformed state.

One could refer medium as elastic medium iff equations (16) possess discrete set of solutions.

Strain tensor for elastic medium is defined as \( \epsilon_{\mu\nu} = \frac{1}{2} \left( G_{\mu\nu} - G_{\mu\nu}^0 \right) \).

6.1 Isotropic elastic membrane media

Isotropic medium is characterised by metric \( g_{\Phi \alpha\beta} \) (metric \( g_{\Phi \alpha\beta} \) could be different from \( G_{\alpha\beta}^0 \)) and some scalar fields at \( \Phi \).

Let us introduce the following notation \( \tilde{G}_{\alpha\gamma} = G_{\alpha\beta} g_{\beta\gamma} \).

By using two \( n \)-dimensional metrics \( (g_{\Phi \alpha\beta} \) and \( G_{\alpha\beta} \) one can build \( n \) independent scalars. E.g. one can consider roots or coefficients of secular equation

\[
\det(\tilde{G}_{\alpha\gamma} - \lambda \delta_{\alpha\gamma}) = \sum_{k=0}^{n} (-\lambda)^{n-k} f_k = 0, \quad f_k = \tilde{G}_{[\alpha_1 \ldots \alpha_k]} \]  (17)
\[ f_0 = 1, \quad f_1 = \tilde{G}^\alpha_\alpha = \text{tr}\, \tilde{G}, \quad f_2 = \frac{1}{2} \left( (\text{tr}\, \tilde{G})^2 - \text{tr}(\tilde{G}^2) \right), \quad f_n = \det \tilde{G}^\alpha_\gamma = \frac{g_F}{G}. \]

Lagrangian of isotropic elastic membrane medium is function of \( f_k \) and some scalars at \( F \)

\[ S_{\text{iso}} = \int d^2x \sqrt{\left| g \right|} L_{\text{iso}}(\varphi, f_1, \ldots, f_n). \tag{18} \]

If \( L_{\text{iso}} \) depends on \( \varphi \) and \( f_n \) only, then it describes again perfect membrane fluid.

### 6.2 Linear isotropic membrane media

To consider equations of motion and energy-momentum tensor in linear order on strain tensor \( \epsilon_{\mu\nu} \), one need scalars \( f_k \) up to second order on \( \epsilon_{\mu\nu} \). Under this accuracy any two scalars \( f_k \) could be used to express all other scalars \( f_k \). So, to construct action (under this level of accuracy) one can choose any two scalars \( f_{k_1} \) and \( f_{k_2} \). It is just the matter of convenience.

It is convenient to use \( f_n = \det \tilde{G} \), to simplify comparison with perfect fluid. Another scalar, which is natural to use is \( f_1 = \text{tr} \tilde{G} = g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta \, g_{\alpha\beta} \). Its form reproduces standard \( \sigma \)-model Lagrangian for set of scalar fields.

In linear approximation isotropic elastic membrane medium is described by two elastic moduli \( \mu, \lambda \), density in nondeformed state \( \rho_0 \) and metric of non-deformed state \( G^{0}_{\alpha\beta} = g_{\alpha\beta} \). E.g. the possible Lagrangian is

\[ L_{\text{lin.}} = -\frac{(\rho_0 - \mu)^2}{\rho_0 + \lambda} \left( f_n^{1\frac{2(\rho_0 + \lambda)}{\rho_0 - \mu}} - 1 \right) - \frac{\mu}{2} (f_1 - n) - \rho_0. \tag{19} \]

### 6.3 Perturbations of linear membrane media

Lagrangian \( L_{\text{lin.}} \) requires gradients \( d\varphi^\alpha \) to be space-like. It is natural to study propagation of perturbations representing fields \( \varphi^\alpha \) in the form \( \varphi^\alpha = \zeta^\alpha + \chi^\alpha \). Here \( \zeta^\alpha \) is a solution of equations of motion generated by Lagrangian \( L_{\text{lin.}} \), and \( \chi^\alpha \) is small perturbation (displacement vector). Gradients \( d\chi^\alpha \) are assumed to be time- or light-like. Expansion of Lagrangian \( L_{\text{lin.}} \) in series on \( \chi^\alpha \) allows to build perturbation theory.

Let space-time is direct product of longitudinal space \( V \) with metric \( \gamma_{mn} \) and medium space \( F \), i.e. \( ds^2 = \gamma_{mn} dz^m dz^n + g_{\alpha\beta} dx^\alpha dx^\beta \). Metric coefficients \( \gamma_{mn} \), \( g_{\alpha\beta} \), unperturbed density \( \rho_0 \) and elastic moduli \( \lambda \) and \( \mu \) are independent on coordinates. To lower and raise Greek and Latin indices we use matrices \( g_{\alpha\beta} \) and \( \gamma_{mn} \) in inverse matrices.

Fields \( \zeta^\alpha = x^\alpha \) are solution of equations of motion, because energy-momentum tensor \( T^{(0)}_{MN} = -\rho_0 P_{MN} \) (here \( P_{MN} \) is projector to surface \( \zeta = \text{const} \) obviously satisfies conservation law.

Up to quadratic terms one has

\[ L_{\text{lin.}} \approx -\rho_0 \left( 1 + \partial_\alpha \chi^\alpha + \frac{1}{2} \partial_m \chi^\alpha \partial^m \chi_\alpha \right) - \frac{\rho_0 + \lambda}{2} (\partial_\alpha \chi^\alpha)^2 - \frac{\mu}{2} \partial_\alpha \chi^\gamma \partial^\alpha \chi_\gamma + \frac{\rho_0 - \mu}{2} \partial_\alpha \chi^\gamma \partial_\gamma \chi^\alpha. \tag{20} \]

Linearized equations of motion have the following form

\[ \frac{\delta S_{\text{lin.}}}{\delta \chi^\alpha} \approx -\rho_0 \partial_m \partial^m \chi_\alpha + \mu \partial_\mu \partial^\mu \chi_\alpha + (\lambda + \mu) \partial_\alpha (\partial_\gamma \chi^\gamma). \tag{21} \]
It is obvious that this equation admit transverse and longitudinal waves, which propagate along coordinates $x$ with velocities $C_{\perp}$ and $C_{\parallel}$ and perturbations, which propagate along coordinates $z$ (iff $\dim V > 1$, i.e. iff the medium consists of strings or membranes, but not of particles) with speed of light

$$C_{\perp} = \sqrt{\frac{\mu}{\rho_0}} \quad C_{\parallel} = \sqrt{\frac{\lambda + 2\mu}{\rho_0}} \quad C_{\perp}' = 1. \quad (22)$$

In the case $\dim V = 1$, one has the standard equations for displacement vector in homogeneous isotropic elastic medium.

7 Conclusion

Consideration of linearized theory is first step of quantizing of initial model (19). In the linearized theory quantization is just introducing of creation-annihilation operators for phonons. The next stage is perturbation theory, which have to take into account self-interaction of fields $\chi$, originated from high order term.

The models considered admits various generalisations, e.g. one can easily introduce interactions with other fields. (Lagrangian for membrane media itself could be used as interaction term for scalar fields $\varphi^a$.)

Existence of volume form $\Omega_F$ at space $F$ allows to introduce current $J = \varphi^*\Omega_F$ at space $M$. Current $J$ satisfies “kinematic conservation law” $dJ = 0 \quad (9)$. E.g. if $\dim M = \dim F + 1$, then vector $*J$ is standard $D$-dimensional current density.

To describe interaction one has just to include in action standard current-field interaction term, e.g. interaction with electromagnetic field described by $\int_M J \wedge A$.

The considered models of “membrane media” reproduces some properties of strings and membranes, which are elements of medium. It reveals the close relation between modern string theory and relativistic continuous media theory. This relation has to become fruitful for both theories.

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