SECOND ORDER NON-AUTONOMOUS LATTICE SYSTEMS
AND THEIR UNIFORM ATTRACTORS

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Abstract. The existence of the uniform global attractor for a second order non-autonomous lattice dynamical system (LDS) with almost periodic symbols has been carefully studied. Considering the nonlinear operators \( \{ f_{1i}(u_j | j \in I_{q1}) \}_{i \in \mathbb{Z}^n} \) and \( \{ f_{2i}(u_j | j \in I_{q2}) \}_{i \in \mathbb{Z}^n} \) of this LDS, up to our knowledge it is the first time to investigate the existence of uniform global attractors for such second order LDSs. In fact there are some previous studies for first order autonomous and non-autonomous LDSs with similar nonlinear parts, cf. \([3, 24]\). Moreover, the LDS under consideration covers a wide range of second order LDSs. In fact, for specific choices of the nonlinear functions \( f_{1i} \) and \( f_{2i} \) we get the autonomous and non-autonomous second order systems given by \([1, 25, 26]\).

1. Introduction. Lattice dynamical systems (LDSs) have attracted much attention in the literature. They arise naturally in a wide variety of applications, for instance, in propagation of nerve pulses in myelinated axons, electrical engineering, pattern recognition, image processing, chemical reaction theory, etc. In each case, they have their own form, but in some cases, they appear as spatial discretizations of continuous partial differential equations \([9, 13]\).

In this work we study the existence of the uniform global attractor for the following second order non-autonomous LDS:

\[
\ddot{u}_i + (A_1 u)_i + (A_2 u)_i + f_{1i}(u_j | j \in I_{q1}) + f_{2i}(u_j | j \in I_{q2}) = g_i(t), i \in \mathbb{Z}^n, t > \tau, \tau \in \mathbb{R},
\]

with initial data

\[
u_i(\tau) = u_{0i,\tau}, \quad \dot{u}_i(\tau) = u_{1i,\tau}, \quad i \in \mathbb{Z}^n,
\]

where \( \mathbb{Z}^n \) is the product of \( n \) integer sets, \( A_1, A_2 \) are linear operators, \( f_{1i}, f_{2i} \) are nonlinear functions, and \( g_i \) is an external term, to be determined later. Considering the nonlinear functions \( f_{1i} \) and \( f_{2i} \) of (1), one can see that such a non-autonomous LDS covers a wide range of second order autonomous and non-autonomous second order LDSs. In fact for particular choices of the nonlinear part, we get the autonomous and non-autonomous second order systems given by \([1, 25, 26]\).

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Recently, the existence of global attractor, uniform attractor, pullback attractor, and random attractor for different types of autonomous, non-autonomous, and stochastic LDSs in standard and weighted spaces of infinite lattices have been carefully investigated \[1, 2, 3, 4, 5, 7, 8, 14, 15, 17, 18, 20, 21, 22, 23, 25, 26\].

Since LDSs are infinite systems of ordinary differential equations or of difference equations, indexed by points in a lattice such as the unbounded \(n\)-dimensional integer lattice \(\mathbb{Z}^n\), proving the asymptotic compactness of the solution operators is a crucial step towards proving the existence of attractors for the system. Such compactness property in autonomous systems is obtained by the uniform estimates on the tails of solutions with a bounded set of initial data when \(\infty\). In the present case, the LDS (1)-(2) is non-autonomous with almost periodic symbols and therefore it defines a family of processes, instead of a semigroup in the autonomous case. In such a case, the estimates on the tails of the solutions must be uniform with respect to initial data from a bounded set as well as all translations of the time symbol \(g \in \mathcal{H}(g_0)\) in the system.

This paper is organized as follows. In section 2, the non-autonomous LDS (6)-(7) with almost periodic symbols is introduced, where abstract assumptions on the linear and nonlinear parts of the system and some introductory results are presented. In section 3, the non-autonomous LDS (6)-(7) is written in the abstract form (40)-(41) in a suitable extended phase space, where the well-posedness of the system (40)-(41) is established, a family of processes associated with this system is defined, and the existence of a uniform absorbing set for this family of processes is verified. In section 4, the uniform estimates on the tails of solutions with respect to initial data from the uniform absorbing set and the time symbol are introduced, where such estimates are needed to obtain the asymptotic compactness of the solution semigroup, then by the semigroup theory, the uniform global attractor is presented.

2. Preliminaries. For a positive integer \(n\), we consider the Hilbert space

\[
\ell^2 = \left\{ u = (u_i)_{i \in \mathbb{Z}^n} : i = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n, u_i \in \mathbb{R}, \sum_{i \in \mathbb{Z}^n} u_i^2 < \infty \right\},
\]

whose inner product and norm are given by:

\[
\langle u, v \rangle = \sum_{i \in \mathbb{Z}^n} u_i v_i, \|u\| = \langle u, u \rangle^{1/2}, \quad u = (u_i)_{i \in \mathbb{Z}^n}, v = (v_i)_{i \in \mathbb{Z}^n} \in \ell^2.
\]

In the Hilbert space \(H = \ell^2 \times \ell^2\), we consider the inner product and norm,

\[
\langle \varphi_1, \varphi_2 \rangle_H = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle, \|\varphi_1\|_H = \langle \varphi_1, \varphi_1 \rangle_H^{1/2}, \quad \varphi_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \varphi_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in H.
\]

For \(m = 1, 2\), \(i = (i_1, \ldots, i_n) \in \mathbb{Z}^n\), and a nonnegative integer \(q_m\), the set \(I_{iq_m}\) is defined as follows:

\[
I_{iq_m} = \left\{ j = (j_1, \ldots, j_n) \in \mathbb{Z}^n : \max_{1 \leq l \leq n} |j_l - i_l| \leq q_m \right\}.
\]

Let \(g_0 : \mathbb{R} \to \ell^2\) with \(g_0(t) = (g_{0i}(t))_{i \in \mathbb{Z}^n}\), be an almost periodic function. From [16], we know that such a function is bounded and uniformly continuous on
\(R\), therefore \(g_0 \in C_b (R, l^2)\), where \(C_b (R, l^2)\) is the space of bounded continuous functions on \(R\) with the norm
\[
\|g\|_{C_b} = \sup_{t \in R} \|g (t)\|_2 , \quad g \in C_b (R, l^2) .
\]
Moreover, since \(g_0 : R \to l^2\) is an almost periodic function, by Bohner’s criterion [16], the set of translations \(\{g_0 (\cdot + h) : h \in R\}\) is precompact in \(C_b (R, l^2)\). Let \(H (g_0) = \{g_0 (\cdot + h) : h \in R\}\) in \(C_b (R, l^2)\). Then for any \(g \in H (g_0)\), \(g\) is almost periodic and \(H (g) = H (g_0)\). Here \(H (g)\) is called the hull of the function \(g\).

In the extended phase space \(E = H \times H (g_0)\), we consider the norm:
\[
\||\phi||_E = \left(\|\varphi\|_H^2 + \|g\|_{C_b}^2\right)^{1/2} , \quad \phi = (\varphi, g) \in E . \tag{5}
\]

Considering the symbol space \(H (g_0)\) and the LDS (1)-(2), we shall study the existence of the uniform global attractor (with respect to \(g \in H (g_0)\)) for the following second order non-autonomous LDS in \(H\):
\[
\ddot{u} + A_1 \dot{u} + A_2 u + f_1 (\dot{u}) + f_2 (u) = g (t) , \quad t > 0, \tau \in R, g \in H (g_0) , \tag{6}
\]
\[
u (\tau) = (w_{0i}, \tau)_{i \in Z^n} = w_0 \tau , \quad \ddot{u} (\tau) = (u_{1i}, \tau)_{i \in Z^n} = u_{1\tau} , \tag{7}
\]
where \(u = (u_i)_{i \in Z^n}, A_m u = ((A_m u)_i)_{i \in Z^n}, f_m (u) = (f_m (u_j | j \in I_{q_m}))_{i \in Z^n}\), for \(m = 1, 2, g (t) = (g_i (t))_{i \in Z^n}\).

Up to our knowledge it is the first time to study the existence of the uniform global attractor for such a second order non-autonomous LDS with nonlinear part of the form
\[
f_1 (\dot{u}) + f_2 (u) - g (t) = (f_{1\tau} (\dot{u}_j | j \in I_{q_1}) + f_{2\tau} (u_j | j \in I_{q_2}) - g_i (t))_{i \in Z^n} .
\]
In fact there are some previous studies for first order autonomous and non-autonomous LDSs with similar nonlinear part, cf. [3, 24].

Within this work, we assume that:
\(\text{(A0)}\) \(g \in H (g_0)\), where \(g_0 : R \to l^2\) is an almost periodic function.
\(\text{(A1)}\) For \(m = 1, 2, A_m : l^2 \to l^2\) is a positive self-adjoint bounded linear operator which can be represented in the following form
\[
A_m = A_{m1} + A_{m2} + \cdots + A_{mn} ,
\]
such that for \(k = 1, \cdots, n\), there exists a bounded linear operator \(D_m : l^2 \to l^2\) with
\[
(D_m u)_i = \sum_{l=-l_m}^{l_m} d_{mk,i} u_{i,k,l} , \quad u = (u_i)_{i \in Z^n} \in l^2 , \tag{8}
\]
\(l_m\) is a positive integer, \(d_{mk,i} \in R, l = -l_m, \ldots, l_m, \) not all of them are zeros,
\[
i_{k,l} = (i_1, i_2, \cdots, i_{k-1}, i_k + l, i_{k+1}, \cdots, i_n) \in Z^n ,
\]
and
\[
A_m = D_m^* D_m = D_m D_m^* , \quad \|D_m\|_\infty \leq C_m , \tag{9}
\]
where \(C_m\) is a positive constant, \(\|\cdot\|_\infty\) is the norm of a bounded linear operator in \(l^2\), and \(D_m^*\) is the adjoint operator of \(D_m\). That is, for \(u = (u_i)_{i \in Z^n}, v = (v_i)_{i \in Z^n} \in l^2\),
\[
(D_m^* u)_i = \sum_{l=-l_m}^{l_m} d_{mk,-l} u_{i,k,l} , \quad (D_m u, v) = (u, D_m^* v) . \tag{10}
\]
Since \( \|D_{mk}\|_O \leq C_m \), it is clear that
\[
|a_{mk,l}| \leq C_m, \quad l = -l_m, \ldots, l_m. \tag{11}
\]

(A2) For \( m = 1, 2 \), considering the nonlinear function \( f_{mi} : \mathbb{R}^{(2q_m+1)} \to \mathbb{R} \), there exist positive constants \( c_m, r_2 \) and a positive integer \( I_m \) such that for \( (u_i)_{i \in \mathbb{Z}^n} \in \ell^2 \) and \( i \in \mathbb{Z}^n \),
\[
f_{mi}(u_j = 0 \mid j \in I_{iq_m}) = 0, \tag{12}
\]
\[
f_{i1}(u_j \mid j \in I_{iq_1}) u_i \geq 0, \text{ for } u_j \in \mathbb{R}, \|i\|_0 \leq I_1, \tag{13}
\]
\[
f_{2i}(u_j \mid j \in I_{iq_2}) u_i \geq 0, \text{ for } \sqrt{\sum_{j \in I_{iq_2}} u_j^2} > r_2, \|i\|_0 \leq I_2, \tag{14}
\]
\[
f_{mi}(u_j \mid j \in I_{iq_m}) u_i \geq c_m u_i^2, \text{ for } u_j \in \mathbb{R}, \|i\|_0 > I_m, \tag{15}
\]
where
\[
\|i\|_0 = \max_{1 \leq l \leq n} |i_l|, \quad i = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n.
\]

(A3) There exists a positive constant \( c_3 \) such that for \( (u_i)_{i \in \mathbb{Z}^n}, (v_i)_{i \in \mathbb{Z}^n} \in \ell^2 \) and \( i \in \mathbb{Z}^n \),
\[
|f_{i1}(u_j \mid j \in I_{iq_1}) - f_{i1}(v_j \mid j \in I_{iq_1})| \leq c_3 \sum_{j \in I_{iq_1}} |u_j - v_j|. \tag{16}
\]

(A4) There exists a positive continuous increasing function \( M_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) such that for \( (u_i)_{i \in \mathbb{Z}^n} \in \ell^2 \), \( R > 0 \), and \( i \in \mathbb{Z}^n \),
\[
|f'_{2i,k}(u_j \mid j \in I_{iq_2})| \leq M_2(R), \text{ for } k \in I_{iq_2}, \sqrt{\sum_{j \in I_{iq_2}} u_j^2} \leq R, \tag{17}
\]
where \( f'_{2i,k}(u_j \mid j \in I_{iq_2}) \) is the derivative of \( f_{2i}(u_j \mid j \in I_{iq_2}) \) with respect to \( k \in I_{iq_2} \). There exists \( \delta_2 > 0 \) such that for \( (u_i)_{i \in \mathbb{Z}^n} \in \ell^2 \) and \( i \in \mathbb{Z}^n \),
\[
|F_{2i,k}(u_j \mid j \in I_{iq_2})| \leq \delta_2 |u_i|, \quad k \in I_{iq_2} \setminus \{i\}, \tag{18}
\]
where
\[
F_{2i}(u_j \mid j \in I_{iq_2}) = \int_0^{u_i} \tilde{f}_{2i}(r, u_j \mid j \in I_{iq_2} \setminus \{i\}) \, dr, \tag{19}
\]
\( \tilde{f}_{2i} \) is the function \( f_{2i} \), but \( u_i \) in \( f_{2i} \) is replaced by \( r \) in \( \tilde{f}_{2i} \). The positive constant \( \delta_2 \) is sufficiently small such that there exists \( 0 < \mu < 1 \) with
\[
\frac{\beta_2}{8} - \delta_2 > 0, \quad \mu \beta_2 - \frac{\beta_2^2}{2\beta_2} (2q_2 + 1)^{2n} - \mu^2 \left( 1 + \frac{c^2}{\beta_2} (2q_1 + 1)^{2n} \right) > 0, \tag{20}
\]
where \( \beta_2 \) is given by Lemma 2.1.

One can show that the assumptions (A2)-(A4) related to the nonlinear part of the system (6) are acceptable for the following particular choices:
\[
f_{i1}(u_j \mid j \in I_{iq_1}) = \frac{c_3 u_i}{2 \left( 1 + \sum_{j \in I_{iq_1}} u_j^2 \right)}, \quad i \in \mathbb{Z}^n,
\]
\[
f_{2i}(u_j \mid j \in I_{iq_2}) = \sum_{h=0}^{h_2} \lambda_{hi} u_i^{2h+1} + \frac{\delta_2 u_i}{1 + \sum_{j \in I_{iq_2}} u_j^2}, \quad i \in \mathbb{Z}^n,
\]
where \( h_2 \) is a positive integer and there exists \( \lambda^* > 0 \) such that
\[
0 \leq \lambda_{hi} \leq \lambda^*, \quad h = 0, \ldots, h_2.
\]
Remark 1. For \( m = 1, 2 \), \((u_i)_{i \in \mathbb{Z}^n} \in l^2\) and \( i = (i_1, i_2, \cdots, i_n) \in \mathbb{Z}^n\), let us choose \( (A_m u)_i = \alpha_m ((A u)_i) \), where \( \alpha_m > 0 \), then to (12) and (16), we get

\[
(A u)_i = 2nu_{(i_1, i_2, \cdots, i_n)} - u_{(i_1-1, i_2, \cdots, i_n)} - u_{(i_1, i_2-1, \cdots, i_n)} - \cdots - u_{(i_1, i_2, \cdots, i_n-1)} - u_{(i_1+1, i_2, \cdots, i_n)} - u_{(i_1, i_2+1, \cdots, i_n)} - \cdots - u_{(i_1, i_2, \cdots, i_n+1)},
\]

then LDS (6) can be regarded as a spatial discretization of the following damped non-autonomous nonlinear hyperbolic family of equations with continuous spatial variable \( x \),

\[
u_{tt} - \alpha_1 \Delta u - \alpha_2 \Delta u + f_1(x, u_t) + f_2(x, u) = g(t, x), \quad x \in \mathbb{R}^n, \tau \in \mathbb{R}, t > \tau, g \in \mathcal{H}(\gamma_0).
\]

In some compact domain, Chepyzhov and Vishik [10, 11, 12] studied the existence of a uniform global attractor for similar non-autonomous systems. Some compactness results are not available whenever we study the existence of global attractors on unbounded domains and usually the global attractors are infinite dimensional, cf. [19]. Therefore, it is important to investigate the existence of the uniform global attractor for the LDS (6) since it can be regarded as a discrete analogue of the continuous nonlinear family of equations (21) on the unbounded domain \( \mathbb{R}^n \).

Here we present some well known results which will be frequently used within this work, for \( u = (u_i)_{i \in \mathbb{Z}^n} \in l^2 \) we have

\[
\left( \sum_{j \in I_{qm}} |u_j| \right)^2 \leq (2q_m + 1)^n \sum_{j \in I_{qm}} u_j^2, \tag{22}
\]

\[
\sum_{i \in \mathbb{Z}^n} \sum_{j \in I_{qm}} u_j^2 = (2q_m + 1)^n \|u\|^2, \tag{23}
\]

and for \( u = (u_i)_{i \in \mathbb{Z}^n}, v = (v_i)_{i \in \mathbb{Z}^n} \in l^2 \), the mean value theorem implies

\[
f_{2i}(u_j | j \in I_{q_k}) - f_{2i}(v_j | j \in I_{q_k}) = \sum_{k \in I_{q_k}} f'_{2i,k}(\theta_i u_j + (1 - \theta_i) v_j | j \in I_{q_k}) (u_k - v_k), \quad \theta_i \in (0, 1). \tag{24}
\]

Here we show that \( f_1 \) and \( f_2 \) map \( l^2 \) into \( l^2 \). Indeed, for \( (u_i)_{i \in \mathbb{Z}^n} \in l^2 \), considering (12) and (16), we get

\[
|f_{1i}(u_j | j \in I_{q_k})| \leq c_3 \sum_{j \in I_{q_k}} |u_j|, \tag{26}
\]

and from (22)-(23), we find

\[
\|f_1(u)\|^2 = \sum_{i \in \mathbb{Z}^n} f_{1i}^2(u_j | j \in I_{q_k}) \leq c_3^2 (2q_1 + 1)^n \sum_{i \in \mathbb{Z}^n} \sum_{j \in I_{q_k}} u_j^2 = c_3^2 (2q_1 + 1)^{2n} \|u\|^2. \tag{27}
\]

Using (12), (17), and (25), and the fact that \( M_2 \) is a positive continuous increasing function, it follows that

\[
|f_{2i}(u_j | j \in I_{q_k})| \leq M_2 \left( \sum_{j \in I_{q_k}} u_j^2 \right)^\frac{1}{2} \sum_{j \in I_{q_k}} |u_j|, \tag{28}
\]
and from (22)-(23), and again the fact that $M_2$ is a positive continuous increasing function, we obtain
\[
\|f_2(u)\|^2 = \sum_{i \in \mathbb{Z}^n} f_{2i}^2(u_j \mid j \in I_{i q_2}) \leq M_2^2 (\|u\|) (2q_2 + 1) 2^n \|u\|^2 .
\]  
\hspace*{1cm} (29)

Let
\[
\Theta_2(u) = \sum_{i \in \mathbb{Z}^n} F_{2i}(u_j \mid j \in I_{i q_2}) , \ u = (u_i)_{i \in \mathbb{Z}^n} \in l^2 .
\]  
\hspace*{1cm} (30)

Recalling (19) and using the mean value theorem for integrals, there exists $c_i$ between 0 and $u_i$ such that
\[
F_{2i}(u_j \mid j \in I_{i q_2}) = u_i \tilde{f}_{2i}(c_i, u_j \mid j \in I_{i q_2} \setminus \{i\}) ,
\]  
and following (28), we find
\[
|F_{2i}(u_j \mid j \in I_{i q_2})| \leq \frac{1}{2} (2q_2 + 1)^n u_i^2 + \frac{1}{2} M_2^2 (\|u\|) \sum_{j \in I_{i q_2}} u_j^2 ,
\]  
\hspace*{1cm} (31)

and
\[
|\Theta_2(u)| \leq \frac{1}{2} (2q_2 + 1)^n (1 + M_2^2 (\|u\|)) \|u\|^2 .
\]  
\hspace*{1cm} (32)

Using (13) and (15), we obtain
\[
\langle f_1(u), u \rangle \geq c_1 \sum_{\|i\|_0 > I_1} u_i^2 .
\]  
\hspace*{1cm} (33)

From (28), it is clear that
\[
|f_2(u_j \mid j \in I_{i q_2})| \leq (2q_2 + 1)^n r_2 M_2 (r_2) ,
\]  
\hspace*{1cm} (34)

and
\[
\sqrt{\sum_{j \in I_{i q_2}} u_j^2} \leq r_2 .
\]  

In such a case, using (14)-(15), there exists a constant $\beta_1 = \beta_1 (n, I_2, r_2, q_2, M_2) \geq 0$ such that
\[
\langle f_2(u), u \rangle \geq c_2 \sum_{\|i\|_0 > I_2} u_i^2 - \beta_1 ,
\]  
\hspace*{1cm} (35)

\[
2\Theta_2(u) \geq c_2 \sum_{\|i\|_0 > I_2} u_i^2 - \beta_1 .
\]  

The assumptions on the nonlinear operators $f_1$ and $f_2$ of the LDS (6) present the main difficulty of this work. Lemmas 2.1 and 2.2 will be helpful to overcome this difficulty.

**Lemma 2.1.** For $u \in l^2$ there exists a positive constant $\beta_2$ such that
\[
\langle f_1(u), u \rangle + \sum_{k=1}^n \|D_{1k}u\|^2 \geq \beta_2 \|u\|^2 ,
\]  
\hspace*{1cm} (36)

\[
\langle f_2(u), u \rangle + \sum_{k=1}^n \|D_{2k}u\|^2 \geq \beta_2 \|u\|^2 - \beta_1 ,
\]  
\hspace*{1cm} (37)

\[
2\Theta_2(u) + \sum_{k=1}^n \|D_{2k}u\|^2 \geq \beta_2 \|u\|^2 - \beta_1 .
\]  
\hspace*{1cm} (38)
where
\[ 0 < \beta_2 \leq \min \{ c_1, c_2 \}. \] (39)

**Proof.** Taking into account (33)-(35), the proof is similar to that of Lemma 2 in [1]. \qed

The following is a Gronwall-type lemma, cf. [6], which will be helpful to introduce global solutions, a uniform absorbing set, and uniform tail estimates of the solutions.

**Lemma 2.2.** Let \( X \) be a Banach space, \( \Pi : X \to \mathbb{R} \) be a given functional, and \( \varphi \in C ([\tau, \infty), X), \tau \in \mathbb{R} \). If the function \( t \to \Pi (\varphi (t)) \) is continuously differentiable such that
\[
\frac{d}{dt} \Pi (\varphi (t)) + a_1 \| \varphi (t) \|^2_X \leq a_2, \quad t - \tau > 0,
\]
\[
\Pi (\varphi (t)) \geq -b_1, \quad t - \tau \geq 0, \quad \text{and} \quad \Pi (\varphi (\tau)) \leq b_2,
\]
for some \( a_1, a_2 > 0, b_1, b_2 \geq 0 \). Then
\[
\Pi (\varphi (t)) \leq \sup_{y \in X} \{ \Pi (y) : a_1 \| y \|^2_X \leq 2a_2 \}, \quad t - \tau \geq \frac{b_1 + b_2}{a_2}.
\]

3. **Global solutions and uniform absorbing sets.** Let \( v = \dot{u}, \varphi = \left( \begin{array}{c} u \\ v \end{array} \right) \), and \( \varphi_\tau = \left( \begin{array}{c} u_{0\tau} \\ u_{1\tau} \end{array} \right) \). Then the LDS (6)-(7) can be written in the abstract form:
\[
\dot{\varphi} + \mathcal{C} \varphi + \mathcal{F} (t, \varphi) = 0, \quad t > \tau, \tau \in \mathbb{R}, g \in \mathcal{H} (g_0),
\]
\[
\varphi (\tau) = \varphi_\tau,
\]
where \( \mathcal{C} : H \to H \) is the bounded linear operator:
\[
\mathcal{C} = \left( \begin{array}{cc} 0 & -I \\ A_2 & A_1 \end{array} \right),
\]
\( I : l^2 \to l^2 \) is the identity operator, and \( \mathcal{F} : [\tau, +\infty) \times H \to H \) is the nonlinear operator:
\[
\mathcal{F} (t, \varphi) = \left( \begin{array}{c} 0 \\ f_1 (v) + f_2 (u) - g (t) \end{array} \right).
\]

**Lemma 3.1.** For \( g \in \mathcal{H} (g_0), \tau \in \mathbb{R}, \) and \( \varphi_\tau = \left( \begin{array}{c} u_{0\tau} \\ u_{1\tau} \end{array} \right) \in H \), there exists a unique local maximal classical solution \( \varphi = \left( \begin{array}{c} u \\ v \end{array} \right) \in H \) satisfying (40)-(41) in \( H \) such that \( \varphi \in C ([\tau, T), H) \cap C^1 ([\tau, T), H), \) for some \( T > \tau \). Moreover, if \( T < +\infty \), then
\[
\lim_{t \to T^-} \| \varphi (t) \|_H = +\infty.
\] (42)

**Proof.** Since \( g \in \mathcal{H} (g_0) \subset C_b (\mathbb{R}, l^2), A_1, A_2 : l^2 \to l^2 \) are bounded linear operators and from (16)-(17) we find that \( f_1, f_2 : l^2 \to l^2 \) are locally Lipschitz continuous. In such a case, following the standard theory of ordinary differential equations the proof is completed. \qed
Now we are ready to prove that if \( g \in \mathcal{H} (g_0), \tau \in \mathbb{R} \), and the initial data \( \varphi (\tau) \) belongs to a bounded set in \( H \), then the solution \( \varphi (t) \) of (40)-(41) in \( H \) is uniformly bounded with respect to \( g \in \mathcal{H} (g_0) \), for \( t-\tau \geq 0 \). In such a case, taking into account (42), the local solution \( \varphi (t) \) of (40)-(41) given by Lemma 3.1 exists globally, that is \( \varphi \in C([\tau, \infty) , H) \cap C^1([\tau, \infty) , H) \) and we can introduce a family of processes \( \{ \Phi^g (t, \tau) : t \geq \tau, \tau \in \mathbb{R} \} \) on \( H \) such that for \( g \in \mathcal{H} (g_0), t \geq \tau, \tau \in \mathbb{R} \), and \( \varphi \in H \), we have \( \Phi^g (t, \tau) \varphi = \varphi (t) \), where \( \varphi \) is the solution of (40)-(41). By the unique solvability of (40)-(41), the family of processes \( \{ \Phi^g (t, \tau) \} \) satisfies the multiplicative properties:

\[
\Phi^g (t, s) \Phi^g (s, \tau) = \Phi^g (t, \tau), \quad t \geq s \geq \tau, \tau \in \mathbb{R},
\]

\[
\Phi^g (\tau, \tau) = I, \quad \tau \in \mathbb{R},
\]

where \( I : H \to H \) is the identity operator. Moreover, the following translation identity holds for the processes and the translation semigroup \( \{ T (h) \}_{h \geq 0} \):

\[
\Phi^g (t + h, s + h) = \Phi^g (t, s), \quad h \geq 0, s \in \mathbb{R}, t \geq s,
\]

where

\[
T (h) g = g (\cdot + h), \quad g \in \mathcal{H} (g_0).
\]

**Lemma 3.2.** There exists a bounded ball of \( H \), \( O = O (0, R_0) \) with center 0 and radius \( R_0 \) such that \( O \) is a uniform absorbing set for the family of processes \( \{ \Phi^g (t, \tau) \}_{g \in \mathcal{H} (g_0)} \) corresponding to the LDS (40)-(41) with respect to \( g \in \mathcal{H} (g_0) \). That is, for \( R_1 > 0, g \in \mathcal{H} (g_0), \tau \in \mathbb{R} \), and \( \varphi = \left( \begin{array}{c} u_0 \\ v_1 \end{array} \right) \in H \) with \( \| \varphi \|_H \leq R_1 \), the solution \( \varphi (t) = \Phi^g (t, \tau) \varphi \) of (40)-(41) satisfies

\[
\| \varphi (t) \|_H \leq R_0, \quad t - \tau \geq T_0,
\]

for some constant \( T_0 = T_0 (R_1) \). Moreover, there exists a constant \( R_2 = R_2 (R_1) \) such that

\[
\| \varphi (t) \|_H \leq R_2, \quad t - \tau \in [0, T_0).
\]

**Proof.** Given \( R_1 > 0, g \in \mathcal{H} (g_0), \tau \in \mathbb{R} \), and \( \varphi = \left( \begin{array}{c} u_0 \\ v_1 \end{array} \right) \in H \) with \( \| \varphi \|_H \leq R_1 \), let \( \varphi (t) = \left( \begin{array}{c} u (t) \\ v (t) \end{array} \right) \) be the solution of (40)-(41) given by Lemma 3.1. Considering the inner product of (6) with \( \mu + \mu u \) in \( l^2 \) where \( 0 < \mu < 1 \) satisfies (20) and \( u = (u_i)_{i \in \mathbb{Z}_n} \), we get

\[
\frac{d}{dt} \Pi (\varphi (t)) + \Gamma (\varphi (t)) = 0, \quad t - \tau > 0,
\]

where

\[
\Pi (\varphi (t)) = \frac{1}{2} \| \dot{u} \|^2 + \frac{\mu}{2} \sum_{k=1}^{n} \| D_{1k} u \|^2 + \frac{1}{2} \sum_{k=1}^{n} \| D_{2k} u \|^2 + \Theta_2 (u) + \mu \langle \dot{u}, u \rangle,
\]

\[
\Gamma (\varphi (t)) = \frac{\mu}{2} \sum_{k=1}^{n} \| D_{1k} u \|^2 + \frac{1}{2} \sum_{k=1}^{n} \| D_{2k} u \|^2 + \Theta_2 (u) + \mu \langle \dot{u}, u \rangle.
\]
and

\[
\Gamma (\varphi (t)) = \langle f_1 (\dot{u}) , \dot{u} \rangle + \sum_{k=1}^{n} \| D_{1k} \dot{u} \|^2 + \mu \langle f_2 (u) , u \rangle
\]

\[
+ \mu \sum_{k=1}^{n} \| D_{2k} u \|^2 - \mu \| \dot{u} \|^2 + \mu \langle f_1 (\dot{u}) , u \rangle - \langle g (t) , \dot{u} + \mu u \rangle
\]

\[
- \sum_{i \in \mathbb{Z}^n} \sum_{k \in I_{i \mathbb{Z}^2 \setminus \{i\}}} F'_{2i,k} (u_j \mid j \in I_{i \mathbb{Z}^2}) \dot{u}_k
\]

(47)

Recalling (18) and (22), we find

\[
- \sum_{k \in I_{i \mathbb{Z}^2 \setminus \{i\}}} F'_{2i,k} (u_j \mid j \in I_{i \mathbb{Z}^2}) \dot{u}_k
\]

\[
\geq - \sum_{k \in I_{i \mathbb{Z}^2 \setminus \{i\}}} | F'_{2i,k} (u_j \mid j \in I_{i \mathbb{Z}^2}) \dot{u}_k |
\]

\[
\geq - \delta_2 \| u_i \| \sum_{k \in I_{i \mathbb{Z}^2}} | \dot{u}_k |
\]

\[
\geq - \frac{\delta_2^2}{2 \beta_2} (2q_2 + 1)^{2n} u_i^2 - \frac{\beta_2}{2} (2q_2 + 1)^{-2n} \left( \sum_{k \in I_{i \mathbb{Z}^2}} | \dot{u}_k | \right)^2
\]

\[
\geq - \frac{\delta_2^2}{2 \beta_2} (2q_2 + 1)^{2n} u_i^2 - \frac{\beta_2}{2} (2q_2 + 1)^{-n} \sum_{k \in I_{i \mathbb{Z}^2}} \dot{u}_k^2,
\] (48)

and from (23), we get

\[
- \sum_{i \in \mathbb{Z}^n} \sum_{k \in I_{i \mathbb{Z}^2 \setminus \{i\}}} F'_{2i,k} (u_j \mid j \in I_{i \mathbb{Z}^2}) \dot{u}_k \geq - \frac{\delta_2^2}{2 \beta_2} (2q_2 + 1)^{2n} \| u \|^2 - \frac{\beta_2}{2} \| \dot{u} \|^2.
\] (49)

Using (27), we find

\[
\mu \langle f_1 (\dot{u}) , u \rangle \geq - \mu \| f_1 (\dot{u}) \| \| u \| \geq - \mu c_3 (2q_1 + 1)^n \| \dot{u} \| \| u \|
\]

\[
\geq - \frac{\beta_2}{4} \| \dot{u} \|^2 - \frac{\mu c_3}{\beta_2} (2q_1 + 1)^{2n} \| u \|^2.
\] (50)

It is clear that

\[
- g_i (t) (\dot{u}_i + \mu u_i) \geq - \frac{\beta_2}{8} \dot{u}_i^2 - \mu^2 u_i^2 - \left( \frac{2}{\beta_2} + \frac{1}{4} \right) g_i^2 (t),
\] (51)

but we know that \( g \in \mathcal{H} (g_0) \subset C_b (\mathbb{R}, l^2) \), therefore there exists a positive constant \( M_0 \) independent of \( g \) such that

\[
\| g (t) \| \leq \sup_{s \in \mathbb{R}} \| g (s) \| = \sup_{s \in \mathbb{R}} \| g_0 (s) \| = M_0, \quad t \in \mathbb{R},
\] (52)

and from (51), we obtain

\[
- \langle g (t) , \dot{u} + \mu u \rangle \geq - \frac{\beta_2}{8} \| \dot{u} \|^2 - \mu^2 \| u \|^2 - \left( \frac{2}{\beta_2} + \frac{1}{4} \right) M_0^2.
\] (53)
Recalling (36)-(37), and putting (49), (50), (53) into (47), it follows that
\[ \Gamma (\varphi (t)) \geq \left( \frac{\beta_2}{8} - \mu \right) \| \dot{u} \|^2 + \left( \mu \beta_2 - \frac{\delta_2}{2 \beta_2} (2q_2 + 1)^2n - \mu^2 \left( 1 + \frac{c_2}{\beta_2} (2q_1 + 1)^2n \right) \right) \| u \|^2 \]
\[ - \left( \frac{2}{\beta_2} + \frac{1}{4} \right) M_0^2 - \mu \beta_1. \]  
(54)
Taking into account (20) and choosing
\[ a_1 = \min \left\{ \beta_2 - \mu, \mu \beta - \frac{\delta_2}{2 \beta_2} (2q_2 + 1)^2n - \mu^2 \left( 1 + \frac{c_2}{\beta_2} (2q_1 + 1)^2n \right) \right\} > 0, \]  
(55)
\[ a_2 = \left( \frac{2}{\beta_2} + \frac{1}{4} \right) M_0^2 + \mu \beta_1, \]  
(56)
we get
\[ \Gamma (\varphi (t)) \geq a_1 \| \varphi (t) \|^2_H - a_2, \]
and from (45), it follows that
\[ \frac{d}{dt} \Pi (\varphi (t)) + a_1 \| \varphi (t) \|^2_H \leq a_2, \quad t - \tau > 0. \]  
(57)
It is clear that
\[ |\mu \dot{u} u_\tau| \leq \frac{1}{4} \dot{u}_\tau^2 + \mu^2 u_\tau^2. \]  
(58)
From (38), (46), and (58), we obtain
\[ \Pi (\varphi (t)) \geq \frac{1}{4} \| u \|^2 + \left( \frac{\beta_2}{2} - \mu^2 \right) \| u \|^2 - \frac{\beta_1}{2}, \quad t - \tau \geq 0, \]  
(59)
and since $0 < \mu < 1$, the first inequality of (20) implies that
\[ \frac{\beta_2}{2} - \mu^2 > 0. \]
Putting
\[ b_0 = \min \left\{ \frac{1}{4} \frac{\beta_2}{2} - \mu^2 \right\} > 0, \quad b_1 = \beta_1/2, \]  
(60)
into (59), we find
\[ \Pi (\varphi (t)) \geq b_0 \| \varphi (t) \|^2_H - b_1, \quad t - \tau \geq 0, \]  
(61)
and
\[ \Pi (\varphi (t)) \geq -b_1, \quad t - \tau \geq 0. \]  
(62)
Using (9), (32), (46), and (58), it is clear that
\[ \Pi (\varphi (t)) \leq \frac{3}{4} \| \dot{u} (t) \|^2 + \left[ \mu^2 + \frac{\mu C_1^2}{2} + \frac{n C_2^2}{2} + \frac{1}{2} (2q_2 + 1)^n (1 + M_0 (\| u (t) \|)) \right] \| u (t) \|^2, \]  
(63)
and since $\| \varphi \|_H \leq R_1$, we have
\[ \Pi (\varphi (\tau)) \leq b_2, \]  
(64)
where
\[ b_2 = \frac{3}{4} R_1^2 + \left[ \mu^2 + \frac{\mu C_1^2}{2} + \frac{n C_2^2}{2} + \frac{1}{2} (2q_2 + 1)^n (1 + M_0 (R_1)) \right] R_1^2. \]  
(65)
Applying Lemma 2.2, taking into account that (57), (62), and (64) are satisfied, we obtain
\[ \Pi (\varphi (t)) \leq \sup_{y \in H} \left\{ \Pi (y) : a_1 \| y \|^2_H \leq 2a_2 \right\}, \quad t - \tau \geq T_0, \]  
(66)
Lemma 3.3. The family of processes \( \{ \Phi^g(t, \tau) \}_{g \in \mathcal{H}(g_0)} \) is \((E, H)\)-continuous, where \( E = H \times \mathcal{H}(g_0) \).

Proof. Given \( g, g_k \in \mathcal{H}(g_0) \subset C_b(\mathbb{R}, W) \) and \( \phi, \phi_k \in H \), where \( k = 1, 2, \ldots \), with \( g_k \rightarrow g \) and \( \phi_k \rightarrow \phi \) as \( k \rightarrow +\infty \), we want to show that for \( \tau \in \mathbb{R} \) and \( t \geq \tau \),
\[
\| \Phi^{g_k}(t, \tau) \phi_k - \Phi^g(t, \tau) \phi \|_H \rightarrow 0, \quad \text{as} \quad k \rightarrow +\infty.
\] (69)

Indeed, let \( \varphi_k(t) = \left( \begin{array}{c} u_k(t) \\ v_k(t) \end{array} \right) = \Phi^{g_k}(t, \tau) \phi_k, \varphi(t) = \left( \begin{array}{c} u(t) \\ v(t) \end{array} \right) = \Phi^g(t, \tau) \phi, \) and \( \psi_k(t) = \varphi_k(t) - \varphi(t) \). From (40), we have
\[
\psi_k(t) + C \psi_k(t) + F_k(t, \varphi_k(t)) - F(t, \varphi(t)) = 0, \quad t > \tau,
\] (70)
where
\[
F(t, \varphi(t)) = \begin{pmatrix} 0 \\ f_1(\dot{u}(t)) + f_2(u(t)) - g(t) \end{pmatrix},
\]
\[
F_k(t, \varphi_k(t)) = \begin{pmatrix} 0 \\ f_1(\dot{u_k}(t)) + f_2(u_k(t)) - g_k(t) \end{pmatrix}.
\]
Considering the inner product of (70) with \( \psi_k(t) \) in \( H \), we get
\[
1 \frac{d}{dt} \|\psi_k(t)\|_H^2 + \langle C \psi_k(t), \psi_k(t) \rangle_H + \langle F_k(t, \varphi_k(t)) - F(t, \varphi_k(t)) - F(t, \varphi(t)), \psi_k(t) \rangle_H = 0, \quad t > \tau.
\] (71)

But
\[
|\langle C \psi_k(t), \psi_k(t) \rangle_H| \leq \|C\|_O \|\psi_k(t)\|_H \leq \|C\|_O \|\psi_k(t)\|_H^2, \quad \text{at} \quad \tau \geq \tau,
\] (72)
where \( \|\cdot\|_O \) is the norm of a bounded linear operator from \( H \) into \( H \).

Following Lemma 3.2, since \( \{ \phi_k \}_{k=1}^{+\infty} \) is bounded in \( H \), there exists a constant \( R_3 > 0 \) such that
\[
\|\varphi(t)\|_H \leq R_3, \|\varphi_k(t)\|_H \leq R_3, \|\psi_k(t)\|_H \leq R_3, \quad k \geq 1, \quad t \geq \tau.
\]
In such a case, we have
\[
\|\mathcal{F}_k(t, \varphi_k(t)) - \mathcal{F}(t, \varphi(t)), \psi_k(t)\|_H
\]
\[
\leq (\|f_1(\dot{u}_k(t)) - f_1(\dot{u}(t))\|_H + \|f_2(u_k(t)) - f_2(u(t))\|).
\]
Taking into account (22)-(25), from (16) and (17) we find
\[
\|f_1(\dot{u}_k(t)) - f_1(\dot{u}(t))\| + \|f_2(u_k(t)) - f_2(u(t))\|.
\]
Putting (74) into (73), we obtain
\[
\|\mathcal{F}_k(t, \varphi_k(t)) - \mathcal{F}(t, \varphi(t)), \psi_k(t)\|_H
\]
\[
\leq (c_3(2q_1 + 1)^n + M_2(R_3)(2q_2 + 1)^n)\|\psi_k(t)\|_H^2 + R_3\|g_k(t) - g(t)\|.
\]
By (71), (72), and (75), it follows that for \(k \geq 1, t > \tau\)
\[
\frac{d}{dt}\|\psi_k(t)\|_H^2 - M_1\|\psi_k(t)\|_H^2 \leq 2R_3\|g_k(t) - g(t)\|,
\]
where
\[
M_1 = 2\|\mathcal{C}\|_O + 2(c_3(2q_1 + 1)^n + M_2(R_3)(2q_2 + 1)^n).
\]
That is, for \(k \geq 1, t > \tau\)
\[
\frac{d}{dt}\left(e^{-M_1 t}\|\psi_k(t)\|_H^2\right) \leq 2R_3 e^{-M_1 t}\|g_k(t) - g(t)\|.
\]
Integrating both sides of the last inequality from \(\tau\) to \(t\), we get for \(k \geq 1, t > \tau\)
\[
\|\psi_k(t)\|_H^2 \leq e^{M_1(t-\tau)}\|\psi_k(\tau)\|_H^2 + \frac{2R_3}{M_1} e^{M_1(t-\tau)}\|g_k(t) - g(t)\|_C,b,
\]
where \(\psi_k(\tau) = \phi_k - \phi\). From the last inequality, it is clear that if \(\phi_k \to \phi\) and \(g_k \to g\) as \(k \to +\infty\), then (69) is satisfied. Hence, the family of processes \(\{\Phi^g(t, \tau)\}_{t \in \mathcal{T}(g_0)}\) is \((E, H)\)-continuous.

4. Uniform global attractors. Here we present uniform estimates on the tails of solutions with respect to initial data from the uniform absorbing set \(O\) and \(g \in \mathcal{H}(g_0)\). Such estimates are needed to obtain the asymptotic compactness of the solution semigroup, then by the semigroup theory we present the uniform global attractor.

**Lemma 4.1.** Given \(\eta > 0\), there exist constants \(I = I(\eta)\) and \(T = T(\eta)\) such that for \(g \in \mathcal{H}(g_0), \tau \in \mathbb{R}, \) and \(\varphi_\tau \in O\), the solution \(\varphi(t) = (\varphi_i(t))_{i \in \mathbb{Z}^n} = \left(\begin{array}{c} u_i(t) \\ \dot{u}_i(t) \end{array}\right)_{i \in \mathbb{Z}^n}\)
of (40)-(41) in \(H\) satisfies
\[
\sum_{i: \|\varphi_i(t)\|_H^2 \leq \eta} |\varphi_i(t)|^2 \leq \eta, \quad t - \tau \geq T,
\]
where \(O\) is the uniform absorbing ball given by Lemma 3.2 and \(|\varphi_i(t)|^2 = u_i^2(t) + \dot{u}_i^2(t)\).
Proof. Consider a smooth increasing function \( \theta \in C^1(\mathbb{R}^+, \mathbb{R}) \) such that
\[
\begin{cases}
\theta(s) = 0, & 0 \leq s < 1, \\
0 \leq \theta(s) \leq 1, & 1 \leq s < 2, \\
\theta(s) = 1, & 2 \leq s,
\end{cases}
\]
and there exists a constant \( B \) such that
\[
|\theta'(s)| \leq B, \quad s \geq 0. \tag{77}
\]
From Lemma 3.2, there exists \( T_1 = T_1(R_0) > 0 \) such that for \( g \in H(g_0), \tau \in \mathbb{R}, \) and \( \varphi_\tau \in O, \) the solution \( \varphi(t) = (\varphi_i(t))_{i \in \mathbb{Z}^n} \) of (40)-(41) in \( H \) satisfies
\[
\|\varphi(t)\|_H = \left(\|u(t)\|^2 + \|\dot{u}(t)\|^2\right)^{1/2} \leq R_0, \quad t - \tau \geq T_1. \tag{78}
\]
Considering this solution, let \( w = (w_i)_{i \in \mathbb{Z}^n} \) where \( w_i = \theta\left(\frac{\|\|u\|}{L}\right)u_i \) and \( L \) be a positive integer such that
\[
L > \max \{I_1, I_2\}, \tag{79}
\]
where \( I_1 \) and \( I_2 \) are given by assumption (A2). In such a case considering the inner product of (6) with \( w + \mu w \) in \( l^2, \) we get
\[
\sum_{i \in \mathbb{Z}^n} \theta\left(\frac{\|\|u\|}{L}\right) \left( \frac{d}{dt} \Pi_i(\varphi_i(t)) + \Gamma_i(\varphi_i(t)) \right) + \sum_{i \in \mathbb{Z}^n} \sum_{k=1}^n \left( D_{1k}\dot{u}_i \right)_i \left( \left( D_{1k}\dot{w}\right)_i - \theta\left(\frac{\|\|u\|}{L}\right) \left( D_{1k}\dot{u}\right)_i \right) + \mu \left( \left( D_{1k}w\right)_i - \theta\left(\frac{\|\|u\|}{L}\right) \left( D_{1k}u\right)_i \right) \right)
\]
\[
+ \sum_{i \in \mathbb{Z}^n} \sum_{k=1}^n \left( D_{2k}u_i \right)_i \left( \left( D_{2k}\dot{w}\right)_i - \theta\left(\frac{\|\|u\|}{L}\right) \left( D_{2k}\dot{u}\right)_i \right) + \mu \left( \left( D_{2k}w\right)_i - \theta\left(\frac{\|\|u\|}{L}\right) \left( D_{2k}u\right)_i \right) \right) = 0, \tag{80}
\]
where
\[
\Pi_i(\varphi_i(t)) = \frac{1}{2} \dot{u}_i + \frac{\mu}{2} \sum_{k=1}^n \left( D_{1k}u_i \right)_i \left( D_{1k}u_i \right)_i + \frac{1}{2} \sum_{k=1}^n \left( D_{2k}u_i \right)_i \left( D_{2k}u_i \right)_i + F_{2i}(u_j \mid j \in I_{i_2}) + \mu u_i, \tag{81}
\]
and
\[
\Gamma_i(\varphi_i(t)) = f_{1i}(\dot{u}_j \mid j \in I_{i_1}) \dot{u}_i + \sum_{k=1}^n \left( D_{1k}u_i \right)_i \left( D_{1k}u_i \right)_i + \mu f_{2i}(u_j \mid j \in I_{i_2}) u_i + \mu \sum_{k=1}^n \left( D_{2k}u_i \right)_i \left( D_{2k}u_i \right)_i
\]
\[
- \mu D_{1i} + \mu f_{1i}(u_j \mid j \in I_{i_1}) u_i - g_i(t) \left( \dot{u}_i + \mu u_i \right) - \sum_{k \in I_{i_2} \setminus \{i\} \} F_{2i,k}(u_j \mid j \in I_{i_2}) \dot{u}_k. \tag{82}
\]
Recalling (15), (39), and (79), we find
\[
f_{1i}(\dot{u}_j \mid j \in I_{i_1}) \dot{u}_i + \mu f_{2i}(u_j \mid j \in I_{i_2}) u_i \geq c_1 u_i^2 + \mu c_2 u_i^2 \geq \beta_2 \left( u_i^2 + \mu u_i^2 \right), \quad \|i\|_0 \geq L. \tag{83}
\]
\[
\geq c_1 u_i^2 + \mu c_2 u_i^2 \geq \beta_2 \left( u_i^2 + \mu u_i^2 \right), \quad \|i\|_0 \geq L. \tag{84}
\]
For \( m = 1,2 \), from (23), (77)-(78), and the mean value theorem, we obtain
\[
\sum_{i \in \mathbb{Z}^n} \sum_{j \in I_{q_m}} \left[ \theta \left( \frac{\|i\|_0}{L} \right) - \theta \left( \frac{\|j\|_0}{L} \right) \right] \hat{u}_j^2 \leq (2q_m + 1)^n \frac{q_mBR_0^2}{L}, \quad t - \tau \geq T_1,
\]
that is,
\[
- \sum_{i \in \mathbb{Z}^n} \sum_{j \in I_{q_m}} \theta \left( \frac{\|i\|_0}{L} \right) \hat{u}_j^2 \geq - \sum_{i \in \mathbb{Z}^n} \sum_{j \in I_{q_m}} \left[ \theta \left( \frac{\|i\|_0}{L} \right) \hat{u}_j^2 + \left[ \theta \left( \frac{\|i\|_0}{L} \right) - \theta \left( \frac{\|j\|_0}{L} \right) \right] \hat{u}_j^2 \right.
\]
\[
\geq - (2q_m + 1)^n \left( \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_0}{L} \right) \hat{u}_i^2 + \frac{q_mBR_0^2}{L} \right), \quad t - \tau \geq T_1. \quad (85)
\]
In such a case, using (48) and (85), we get
\[
- \sum_{i \in \mathbb{Z}^n} \sum_{k \in I_{q_2} \setminus \{i\}} \theta \left( \frac{\|i\|_0}{L} \right) F_{2i,k} (u_j | j \in I_{q_2}) \dot{u}_k
\]
\[
\geq - \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_0}{L} \right) \frac{\delta_2^2}{2\beta_2} (2q_2 + 1)^n u_i^2 + \beta_2 \frac{u_i^2}{2} - \frac{q_2BR_0^2\beta_2}{2L}, \quad t - \tau \geq T_1. \quad (86)
\]
Along the lines of (69), in [25], it follows that there exists a positive constant \( R_3 = R_3(n,C_1,C_2,l_1,l_2,R_0,B) \) such that for \( t - \tau \geq T_1 \)
\[
\sum_{i \in \mathbb{Z}^n} \sum_{k=1}^{n} (D_{1k} \dot{u}_i) \left[ \left( D_{1k} \dot{u} \right) \left( D_{1k} \dot{u} \right) \right] + \mu \left[ \left( D_{1k} w \right) \left( D_{1k} w \right) \right] \geq \frac{R_3}{L}. \quad (87)
\]
From (26), we find
\[
\mu f_{1i} (\dot{u}_j | j \in I_{q_1}) u_i \geq - \mu c_3 |u_i| \sum_{j \in I_{q_1}} |\dot{u}_j|
\]
\[
\geq - \frac{\beta_2}{4} (2q_1 + 1)^{-2n} \left( \sum_{j \in I_{q_1}} |\dot{u}_j| \right)^2 - \mu^2 c_3^2 \beta_2 (2q_1 + 1)^{2n} u_i^2
\]
\[
\geq - \frac{\beta_2}{4} (2q_1 + 1)^{2n} \sum_{j \in I_{q_1}} u_j^2 - \mu^2 c_3^2 \beta_2 (2q_1 + 1)^{2n} u_i^2, \quad (88)
\]
and from (85) and (88), we get
\[ \mu \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_0}{L} \right) f_1, (u_j \mid j \in I_{i q_1}) u_i \]
\[ \geq - \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_0}{L} \right) \left[ \frac{\beta_2}{4} u_i^2 + \frac{\mu^2 c_3^2}{\beta_2} (2q_1 + 1)^{2n} u_i^2 \right] - \frac{q_1 R_0^2 \beta_2}{4L}, \quad t - \tau \geq T_1. \] (89)

Taking into account (82) and putting (51), (83), (86), (87), and (89) into (80), it follows that
\[ \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_0}{L} \right) \left( \frac{d}{dt} \Pi_i (\varphi_i (t)) + \left( \frac{\beta_2}{4} - \mu \right) u_i^2 \right) \]
\[ \leq \frac{1}{L} \left( R_3 + \frac{1}{2} B R_0^2 \beta_2 \left[ \frac{1}{2} (2q_1 + q_2) \right] + \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_0}{L} \right) \left( \frac{2}{\beta_2} + \frac{1}{4} \right) \varphi_i^2 (t), \quad t - \tau \geq T_1, \right. \] (90)
and
\[ \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_0}{L} \right) \left( \frac{d}{dt} \Pi_i (\varphi_i (t)) + a_1 |\varphi_i (t)|_H^2 \right) \leq a_3, \quad t - \tau \geq T_1, \] (91)
where \( a_1 \) is given by (55) and
\[ a_3 = \frac{1}{L} \left( R_3 + \frac{1}{2} B R_0^2 \beta_2 \left[ \frac{1}{2} (2q_1 + q_2) \right] + \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_0}{L} \right) \left( \frac{2}{\beta_2} + \frac{1}{4} \right) \varphi_i^2 (t). \right. \]

Using (15), (39), and (79), we find
\[ F_{2i} (u_j \mid j \in I_{i q_2}) = \int_0^{u_1} \tilde{F}_{2i} (r, u_j \mid j \in I_{i q_2} \setminus \{i\}) \, dr \geq c_2 \int_0^{u_1} r \, dr \geq \frac{\beta_2}{2} u^2_i, \|i\|_0 \geq L. \] (92)

Putting (58) and (92) into (81), we get
\[ \Pi_i (\varphi_i (t)) \geq \frac{1}{4} u_i^2 + \left( \frac{\beta_2}{2} - \mu^2 \right) u_i^2 \geq b_0 |\varphi_i (t)|_H^2, \quad \|i\|_0 \geq L, \]
where \( b_0 > 0 \) is given by (60). Therefore
\[ \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_0}{L} \right) \Pi_i (\varphi_i (t)) \geq b_0 \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_0}{L} \right) |\varphi_i (t)|_H^2, \] (93)
and
\[ \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_0}{L} \right) \Pi_i (\varphi_i (t)) \geq 0, \quad t - \tau \geq 0. \] (94)

Taking into account (8) and (11), for \( m = 1, 2 \), we have
\[ \sum_{i \in \mathbb{Z}^n} \sum_{k=1}^{n} \theta \left( \frac{\|i\|_0}{L} \right) (D_{mk} u)_i^2 \]
\[ \leq \left( 2l_m + 1 \right) C^2_m \sum_{i \in \mathbb{Z}^n} \sum_{k=1}^{l_m} \sum_{l=-l_m}^{l_m} \theta \left( \frac{\|i\|_0}{L} \right) u_{ik,l}^2 \]
\[ = \left( 2l_m + 1 \right) C^2_m \sum_{i \in \mathbb{Z}^n} \sum_{k=1}^{n} \sum_{l=-l_m}^{l_m} \left[ \theta \left( \frac{\|i\|_0}{L} \right) - \theta \left( \frac{\|i\|_0}{L} \right) \right] u_{ik,l}^2 + \theta \left( \frac{\|i\|_0}{L} \right) u_{ik,l}^2, \]
and now using (77)-(78) and the mean value theorem, we obtain
\[
\sum_{i \in \mathbb{Z}^n} \sum_{k=1}^{n} \theta \left( \frac{\| \mathbf{i} \|_0}{L} \right) (D_{mk}u_i)^2 \\
\leq n(2l_m + 1)^2 C_m^2 \frac{t_m}{L} BR_0^2 + (2l_m + 1) C_m^2 \sum_{i \in \mathbb{Z}^n} \sum_{k=1}^{n} \sum_{l=-l_m}^{l_m} \theta \left( \frac{\| k,l \|_0}{L} \right) u_i^2 \\
\leq n(2l_m + 1)^2 C_m^2 \left( \frac{t_m}{L} BR_0^2 + \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \mathbf{i} \|_0}{L} \right) u_i^2 \right), \quad t - \tau \geq T_1.
\]
(95)

From (31) and (78), we have
\[
\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \mathbf{i} \|_0}{L} \right) F_{2i}(u_j \mid j \in I_{i\eta_2}) \\
\leq \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \mathbf{i} \|_0}{L} \right) \left[ \frac{1}{2} (2q_2 + 1)^n u_i^2 + \frac{1}{2} M_2^2 (R_0) \sum_{j \in I_{i\eta_2}} u_j^2 \right], \quad t - \tau \geq T_1,
\]
and by (85), where we can replace $u_j$ by $u_j$, we obtain
\[
\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \mathbf{i} \|_0}{L} \right) F_{2i}(u_j \mid j \in I_{i\eta_2}) \leq \frac{1}{2} (2q_2 + 1)^n (1 + M_2^2 (R_0)) \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \mathbf{i} \|_0}{L} \right) u_i^2 \\
+ \frac{1}{2} M_2^2 (R_0) (2q_2 + 1)^n \left( q_0 BR_0^2 \right), \quad t - \tau \geq T_1.
\]
(96)

Considering (81) and using (58), (95), and (96), it follows that
\[
\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \mathbf{i} \|_0}{L} \right) \Pi_i(\varphi_i(t)) \\
\leq BR_0^2 \left[ \mu n (2l_1 + 1)^2 C_1^2 l_1 + n (2l_2 + 1)^2 C_2^2 l_2 + M_2^2 (R_0) q_2 (2q_2 + 1)^n \right] \\
+ \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \mathbf{i} \|_0}{L} \right) \left[ \frac{3}{4} u_i^2 + \left( \frac{\mu n}{4} (2l_1 + 1)^2 C_1^2 + \frac{n}{2} (2l_2 + 1)^2 C_2^2 \right) \right] u_i^2, \quad t - \tau \geq T_1,
\]
that is,
\[
\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \mathbf{i} \|_0}{L} \right) \Pi_i(\varphi_i(t)) \leq b_3 \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \mathbf{i} \|_0}{L} \right) |\varphi_i(t)|_H^2, \quad t - \tau \geq T_1,
\]
(97)

where
\[
b_3 = BR_0^2 \left[ \mu n (2l_1 + 1)^2 C_1^2 l_1 + n (2l_2 + 1)^2 C_2^2 l_2 + M_2^2 (R_0) q_2 (2q_2 + 1)^n \right], \quad (98)
\]
and
\[
b_4 = \max \left\{ \frac{3}{4}, \frac{\mu n}{2} (2l_1 + 1)^2 C_1^2 + \frac{n}{2} (2l_2 + 1)^2 C_2^2 + \frac{1}{2} (2q_2 + 1)^n (1 + M_2^2 (R_0)) + \mu^2 \right\},
\]
(99)

and from (78), we find
\[
\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \mathbf{i} \|_0}{L} \right) \Pi_i(\varphi_i(t)) \leq b_2, \quad t - \tau \geq T_1.
\]
(100)
where

$$b_5 = b_3 + b_4 R_b^2.$$  \hfill (101)

Recalling (90), (94), and (100), then along the lines of Lemma 2.2, we find that for
$$t - \tau \geq T, T = \max \left\{ T_1, \frac{b_3}{a_3} \right\},$$

$$\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \varphi \|_0}{L} \right) \Pi_i (\varphi_i (t))$$  \hfill (102)

$$\leq \sup_{y = (y_i)_{i \in \mathbb{Z}^n} \in H} \left\{ \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \varphi \|_0}{L} \right) \Pi_{i} (y_i) : a_1 \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \varphi \|_0}{L} \right) |y_i|^2 \leq 2a_3 \right\}.$$  \hfill (103)

In such a case, following (93), (97), and (102), we find

$$\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \varphi \|_0}{L} \right) |\varphi_i (t)|^2_H \leq \frac{1}{b_0} \left[ b_3 + \frac{2a_3 b_1}{a_1} \right], \quad t - \tau \geq T,$$  \hfill (104)

where $a_1, b_0, a_3, b_3,$ and $b_4,$ are given by (55), (60), (91), (98), and (99), respectively. Substituting the values of $a_3$ and $b_3$ into (104), we get

$$\sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \varphi \|_0}{L} \right) |\varphi_i (t)|^2_H \leq \frac{BR_3^2}{2Lb_0} \left[ \mu n (2l_1 + 1)^2 C^2_1 l_1 + n (2l_2 + 1)^2 C^2_2 l_2 + M^2_2 (R_0) q_2 (2q_2 + 1)^n \right]$$  \hfill (105)

$$+ \frac{2b_4}{b_0 a_1} \left( \frac{1}{L} \left( R_3 + \frac{1}{2} B R_0^2 \beta_2 \left[ \frac{1}{2} q_1 + q_2 \right] \right) + \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \varphi \|_0}{L} \right) \left( \frac{2}{\beta_2} + 1 + \frac{1}{4} \right) g^2_i (t) \right), \quad t - \tau \geq T.$$}

Since $g : \mathbb{R} \to \mathbb{R}^2$ is almost periodic, the set $\{(g_i (t))_{i \in \mathbb{Z}^n} : t \in \mathbb{R}\}$ is precompact in $\mathbb{R}^2,$ which implies that for given $\epsilon > 0,$ there exists a constant $N_1 = N_1 (g, \epsilon)$ depending on $g$ and $\epsilon$ such that

$$\sum_{\| \varphi \|_0 \geq N_1} (g_i (t))^2 \leq \epsilon, \quad t \in \mathbb{R}. \hfill (106)$$

But $g \in \mathcal{H} (g_0) = \{g_0 \cdot (\cdot + h) : h \in \mathbb{R}\}$ and the set $\mathcal{H} (g_0)$ is compact in $C_b (\mathbb{R}, \mathbb{R}^2),$ taking into account (106), it follows that for given $\epsilon > 0$ there exists $N_2 = N_2 (\epsilon)$ depending only on $\epsilon$ and independent of $g$ such that

$$\sum_{\| \varphi \|_0 \geq N_2} (g_i (t))^2 \leq \epsilon, \quad t \in \mathbb{R}, g \in \mathcal{H} (g_0). \hfill (107)$$

Given $\eta > 0,$ taking into account (107), we shall fix $L = L (\eta)$ to be sufficiently large such that

$$\frac{BR_3^2}{2Lb_0} \left[ \mu n (2l_1 + 1)^2 C^2_1 l_1 + n (2l_2 + 1)^2 C^2_2 l_2 + M^2_2 (R_0) q_2 (2q_2 + 1)^n \right]$$

$$+ \frac{2b_4}{b_0 a_1} \left( \frac{1}{L} \left( R_3 + \frac{1}{2} B R_0^2 \beta_2 \left[ \frac{1}{2} q_1 + q_2 \right] \right) + \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\| \varphi \|_0}{L} \right) \left( \frac{2}{\beta_2} + 1 + \frac{1}{4} \right) g^2_i (t) \right) \leq \eta, \quad t - \tau \geq T. \hfill (108)$$

It is clear that $\frac{b_3}{a_3},$ where $a_3$ and $b_3$ are given by (91) and (101), respectively, depends on $L (\eta).$ In such a case, for $T = T (\eta) = \max \left\{ T_1, \frac{b_3}{a_3} \right\}$ and $I = I (\eta) = 2L (\eta),$
taking into account (105) and (108), we find
\[
\sum_{|i| \geq I} |\varphi_i (t)|^2 \leq \sum_{|i| \geq I} \theta \left( \frac{\|i\|_0}{L} \right) |\varphi_i (t)|^2 \leq \sum_{i \in \mathbb{Z}^n} \theta \left( \frac{\|i\|_0}{L} \right) |\varphi_i (t)|^2 \leq \eta, \quad t - \tau \geq T.
\]
The proof is completed. \(\square\)

Here we study the existence of a uniform global attractor for the family of processes \(\{\Phi^\varrho (t, \tau) : t \geq \tau, \tau \in \mathbb{R}\}_{g \in \mathcal{H}(g_0)}\) associated with the LDS (40)-(41) in \(H\) by following the semigroup theory. Along the lines of [12], we define the nonlinear solution for the process \(\Phi\), that is, \((\varphi, g) \in E, t \geq 0\).

The kernel of the process \(\Phi\), solutions, that is, \((\varphi, g) \in E, t \geq 0\).

The kernel section of the process \(\Phi\), satisfies the semigroup identities:
\[
S (t) (\varphi, g) = (\Phi^\varrho (t, 0) \varphi, T (t) g), \quad (\varphi, g) \in E, \quad t \geq 0.
\]
where \(\{S (t)\}_{t \geq 0}\) satisfies the semigroup identities:
\[
S (t) S (s) = S (t + s), S (0) = I_E, \quad t \geq s \geq 0,
\]
and \(I_E : E \to E\) is the identity operator.

**Lemma 4.2.** The solution semigroup \(\{S (t)\}_{t \geq 0}\) associated with the LDS (40)-(41) is asymptotically compact, that is, \(\{((\varphi_n, g_n))\}_{n=1}^\infty\) is bounded in \(E\), and \(t_n \to \infty\), then \(\{S (t_n) (\varphi_n, g_n)\}_{n=1}^\infty\) is precompact in \(E\).

**Proof.** Using Lemmas 3.2 and 4.1, the proof is similar to that of Lemma 5.4 [20]. \(\square\)

**Definition 4.3.** In \(H\), a closed set \(A\) is called the uniform attractor for the family of processes \(\{\Phi^\varrho (t, \tau)\}_{g \in \mathcal{H}(g_0)}\) with respect to \(g \in \mathcal{H}(g_0)\) if:

(a) For any bounded set \(G \subset H\),
\[
\lim_{t \to \infty} \sup_{g \in \mathcal{H}(g_0)} \text{dist} (\Phi^\varrho (t, \tau) G, A) = 0, \forall \tau \in \mathbb{R}.
\]
(b) (Minimal property) If \(\bar{A}\) is any closed subset of \(H\) satisfying property (a), then \(A \subseteq \bar{A}\).

**Definition 4.4.** Given \(g \in \mathcal{H}(g_0)\), a curve \(t \to \varphi (t) \in H\) is said to be a complete solution for the process \(\Phi^\varrho (t, \tau)\), if it satisfies
\[
\Phi^\varrho (t, \tau) \varphi (\tau) = \varphi (t), \quad \forall \tau \in \mathbb{R}, t \geq \tau.
\]
The kernel of the process \(\Phi^\varrho (t, \tau)\) is the collection \(K_g\) of all its bounded complete solutions, that is,
\[
K_g = \{\varphi (\cdot) \in C_b (\mathbb{R}, H) : \varphi (\cdot) \text{ satisfies } (110)\}.
\]
The kernel section of the process \(\Phi^\varrho (t, \tau)\) at time \(s \in \mathbb{R}\) is the set
\[
K_g (s) = \{\varphi (s) : \varphi (\cdot) \in K_g\}.
\]
Following the uniform attractor theory [12], we obtain the following proposition, where \(\mathcal{F}_1\) and \(\mathcal{F}_2\), given below, are the projectors from \(E = H \times \mathcal{H}(g_0)\) onto \(H\) and \(\mathcal{H}(g_0)\), respectively.

**Proposition 1.** In \(E\), if the semigroup \(\{S (t)\}_{t \geq 0}\) is continuous, point dissipative, and asymptotically compact, then it has a compact global attractor \(A_S\). Furthermore, in \(H, A = \mathcal{F}_1 A_S\) is the compact uniform attractor for the family of processes \(\{\Phi^\varrho (t, \tau)\}_{g \in \mathcal{H}(g_0)}\). In addition,
(a) \(A_S = \bigcup_{g \in \mathcal{H}(g_0)} K_g (0) \times \{g\}\).
(b) $A = \bigcup_{g \in \mathcal{H}(g_0)} K_g (0)$,
(c) $\mathcal{F}_t A_S = \mathcal{H}(g_0)$.

**Theorem 4.5.** In $H$, the family of processes $\{\Phi^g(t,\tau)\}_{g \in \mathcal{H}(g_0)}$ associated with the LDS (40)-(41) has a compact uniform attractor $\mathcal{A}$ with respect to $g \in \mathcal{H}(g_0)$.

**Proof.** From Lemma 3.3, it is clear that the family of processes is continuous from $E$ into $H$. In such a case, using the continuity of translation semigroup $\{T(t)\}_{t \geq 0}$, we find that the solution semigroup $\{S(t)\}_{t \geq 0}$ associated with the LDS (40)-(41) is continuous in $E$. Choosing $O_S = O \times \mathcal{H}(g_0)$, where $O$ is the uniform absorbing set of the processes $\{\Phi^g(t,\tau)\}_{g \in \mathcal{H}(g_0)}$ given by Lemma 3.2, it follows that $O_S$ is a bounded absorbing set for the solution semigroup $\{S(t)\}_{t \geq 0}$ in $E$. Recalling Lemma 4.2 and Proposition 1, there is a compact global attractor $\mathcal{A}_S$ for the solution semigroup $\{S(t)\}_{t \geq 0}$ in $E$ and $\mathcal{A} = \mathcal{F}_1 \mathcal{A}_S$ is the compact uniform global attractor for the family of processes $\{\Phi^g(t,\tau)\}_{g \in \mathcal{H}(g_0)}$ in $H$ with respect to $g \in \mathcal{H}(g_0)$.

**Remark 2.** The upper semicontinuity of global attractors for infinite-dimensional autonomous LDSs [2, 5, 25] and non-autonomous LDSs [14, 20] have been studied. Following the same procedure one can prove the upper semicontinuity of the uniform global attractor $\mathcal{A}$ associated with the family of processes $\{\Phi^g(t,\tau)\}_{g \in \mathcal{H}(g_0)}$ of (40)-(41). That is, the uniform global attractor $\mathcal{A}$ generated by the infinite-dimensional non-autonomous LDS (40)-(41) can be approached by the uniform global attractors of finite-dimensional truncated ordinary differential systems.

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