\[ \Upsilon(nS) \rightarrow B_c\rho, B_cK^* \] decays with perturbative QCD approach

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Abstract

Inspired by the potential prospects of \( \Upsilon(nS) \) data samples \((n = 1, 2, 3)\) at LHC and SuperKEKB, \( \Upsilon(nS) \rightarrow B_c\rho, B_cK^* \) decays are studied phenomenologically with pQCD approach. Branching ratios for \( \Upsilon(nS) \rightarrow B_c\rho \) and \( B_cK^* \) decays are estimated to reach up to \( \mathcal{O}(10^{-11}) \) and \( \mathcal{O}(10^{-12}) \), respectively. Given the identification and detection efficiency of final states, searching for these weak decay modes should be fairly challenging experimentally in the future.
I. INTRODUCTION

The spin-triplet $S$-wave $b\bar{b}$ states $\Upsilon(1S), \Upsilon(2S),$ and $\Upsilon(3S)$ have some common features. They all lie below the open bottom threshold, and carry the same quantum numbers of $I^G J^{PC} = 0^- 1^{--}$ \cite{1}. For each of them, the mass is ten times as large as proton, but the full decay width is very narrow, only a few keV. Based on the above-mentioned facts, here we will use a notation $\Upsilon(nS)$ to represent special $\Upsilon(1S), \Upsilon(2S),$ and $\Upsilon(3S)$ mesons for simplicity if it is not specified explicitly. Thanks to the unremitting endeavor and splendid performance from experimental groups of CLEO, CDF, D0, BaBar, Belle, LHCb, ATLAS, and so on, great achievements have been made in understanding of bottomonium properties \cite{2}. The $\Upsilon(nS)$ decays through the strong interaction, electromagnetic interaction and radiative transition, have been extensively studied. The rapid accumulation of $\Upsilon(nS)$ data samples with high precision will enable a realistic possibility to search for $\Upsilon(1S)$ weak decay at the LHC and SuperKEKB. In this paper, we will study the $\Upsilon(nS) \to B_c V$ weak decays ($V = \rho, K^*$) with perturbative QCD (pQCD) approach \cite{3,4,5} to offer a ready reference for the future experimental research.

Both $b$ and $\bar{b}$ quarks in $\Upsilon(nS)$ meson can decay individually via the weak interaction. It is well known that a clear hierarchy of the quark-mixing Cabibbo-Kabayashi-Maskawa (CKM) matrix elements opts favorably for the $b \to c$ transition, so $\Upsilon(nS)$ weak decay into final states containing a $b\bar{c}$ or $b\bar{c}$ bound state should have a relatively large branching fraction. Recently, we have studied the nonleptonic $\Upsilon(nS) \to B_c^{(*)} P$ decays ($P = \pi, K, D$) with pQCD approach \cite{6,7,8,9,10}, and our estimation of branching ratio for $\Upsilon(1S) \to B_c P$ decays is basically consistent with previous results using other theoretical models \cite{11,12,13,14}. This positive fact encourages us to investigate other $\Upsilon(nS)$ weak decay modes. The amplitudes for $\Upsilon(nS) \to B_c V$ decays are relatively complicated because of the $s, p, d$ wave contributions rather than only $p$ wave contribution for $\Upsilon(nS) \to B_c P$ decays. In addition, the $\Upsilon(nS) \to B_c V$ decays offer another plaza to further explore the underlying dynamical mechanism of heavy quarkonium weak decay.

This paper is organized as follows. In section II we present the theoretical framework and the amplitudes for $\Upsilon(nS) \to B_c V$ decay. The numerical results and discussion are given in section III. The last section is a summary.
II. THEORETICAL FRAMEWORK

A. The effective Hamiltonian

Phenomenologically, assisted with the operator product expansion and renormalization group (RG) technique, the effective weak Hamiltonian accounting for $\Upsilon(nS) \to B_c V$ decay has the following structure [14],

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{cb} V_{uq}^* \left\{ C_1(\mu) Q_1(\mu) + C_2(\mu) Q_2(\mu) \right\} + \text{h.c.},$$

(1)

where $G_F \simeq 1.166 \times 10^{-5} \text{ GeV}^{-2}$ [1] is the Fermi constant. Using the Wolfenstein parameterization, the CKM factors are written approximately in term of $A$ and $\lambda$, i.e.,

$$V_{cb} V_{ud}^* = A \lambda^2 - \frac{1}{2} A \lambda^4 - \frac{1}{8} A \lambda^6 + \mathcal{O}(\lambda^8),$$

(2)

for $\Upsilon(nS) \to B_c \rho$ decay, and

$$V_{cb} V_{us}^* = A \lambda^3 + \mathcal{O}(\lambda^8),$$

(3)

for $\Upsilon(nS) \to B_c K^*$ decay. The local operators are expressed as

$$Q_1 = [\bar{c}_\alpha \gamma_\mu (1 - \gamma_5) b_\alpha][\bar{q}_\beta \gamma^\mu (1 - \gamma_5) u_\beta],$$

(4)

$$Q_2 = [\bar{c}_\alpha \gamma_\mu (1 - \gamma_5) b_\beta][\bar{q}_\beta \gamma^\mu (1 - \gamma_5) u_\alpha].$$

(5)

where $\alpha$ and $\beta$ are color indices, and $q$ denotes $d$ and $s$.

In Eq. (1), the auxiliary scale $\mu$ factorizes physical contributions into two parts. The physical contributions above $\mu$ are integrated into the Wilson coefficients $C_{1,2}$, which has been reliably calculated to the next-to-leading order with the RG-improved perturbation theory [14]. The physical contributions below $\mu$ are embodied in hadronic matrix elements (HME), where the local operators are sandwiched between initial and final hadron states. The incorporation of long distance contributions make HME very challenging and complicated to evaluate. HME is not yet fully understood so far. However, to obtain decay amplitudes, one has to treat HME with certain comprehensible approximation or assumptions, which result in a number of uncertainties.

B. Hadronic matrix elements

Based on factorization ansatz [15–17] and hard-scattering approach [18–22], HME has a simple structure, and is commonly expressed as a convolution of hard scattering kernel func-
tion $T$ with distribution amplitudes (DAs). Only DAs are nonperturbative inputs, which, on the other hand, are process independent, i.e., DAs determined by nonperturbative methods or extracted from experimental data can be employed to make predictions. With the collinear approximation, hard scattering kernels for annihilation contributions and spectator interactions can not provide sufficient endpoint suppression \[23 - 25\]. In order to admit a perturbative treatment for HME, the intrinsic transverse momentum of valence quarks is kept explicitly and a Sudakov factor for each DAs is introduced with pQCD approach \[3 - 5\]. Finally, a pQCD amplitude is written as a convolution integral of three parts: Wilson coefficients $C_i$, hard scattering kernel $T$ and wave functions $\Phi$,

$$
\int dk C_i(t) T(t, k) \Phi(k) e^{-S},
$$

where $t$ is a typical scale, $k$ is the momentum of valence quarks and $e^{-S}$ is a Sudakov factor.

C. Kinematic variables

In the center-of-mass frame of $\Upsilon(nS)$, kinematic variables are defined as follows.

$$
p_{\Upsilon} = p_1 = \frac{m_1}{\sqrt{2}}(1, 1, 0),
$$

$$
p_{B_c} = p_2 = (p_2^+, p_2^-, 0),
$$

$$
p_V = p_3 = (p_3^-, p_3^+, 0),
$$

$$
k_i = x_i p_i + (0, 0, \vec{k}_{i\perp}),
$$

$$
\epsilon_i^\parallel = \frac{p_i}{m_i} - \frac{m_i}{p_i n_+} n_+,
$$

$$
\epsilon_i^\perp = (0, 0, \vec{1}),
$$

$$
n_+ = (1, 0, 0),
$$

$$
p_i^\pm = (E_i \pm p)/\sqrt{2},
$$

$$
s = 2 p_2 \cdot p_3 = m_1^2 - m_2^2 - m_3^2,
$$

$$
t = 2 p_1 \cdot p_2 = m_1^2 + m_2^2 - m_3^2 = 2 m_1 E_2,
$$

$$
u = 2 p_1 \cdot p_3 = m_1^2 - m_2^2 + m_3^2 = 2 m_1 E_3,
$$

$$
st + su - tu - 4 m_1^2 p^2 = 0,
$$
where $x_i$ and $\vec{k}_{i\perp}$ are the longitudinal momentum fraction and transverse momentum of valence quark, respectively; $\epsilon_i^\parallel$ and $\epsilon_i^\perp$ are the longitudinal and transverse polarization vectors, respectively, satisfying relationship $\epsilon_i^2 = -1$ and $\epsilon_i \cdot p_i = 0$; $n_+$ is a positive null vector; the subscript $i = 1, 2, 3$ on variables $(p_i, E_i, m_i)$ and $(\epsilon_i)$ corresponds to $\Upsilon(nS), B_c$ and $V$ mesons, respectively; $s, t$ and $u$ are Lorentz-invariant variables. The notation of momentum is displayed in Fig[2](a).

D. Wave functions

With the notation in [26, 27], meson wave functions are defined as

$$\langle 0|b_i(z)\bar{b}_j(0)|\Upsilon(p_1, \epsilon_1^\parallel)\rangle = \frac{f_T}{4} \int d^4 k_1 e^{-i k_1 \cdot z} \left\{ \epsilon_1^\parallel \left[ m_1 \Phi^V_1(k_1) - \hat{p}_1 \Phi^T_1(k_1) \right] \right\}_{ji},$$

$$\langle 0|b_i(z)\bar{b}_j(0)|\Upsilon(p_1, \epsilon_1^\perp)\rangle = \frac{f_T}{4} \int d^4 k_1 e^{-i k_1 \cdot z} \left\{ \epsilon_1^\perp \left[ m_1 \Phi^V_1(k_1) - \hat{p}_1 \Phi^T_1(k_1) \right] \right\}_{ji},$$

$$\langle B_c(p_2)|\bar{c}_i(z)b_j(0)|0\rangle = \frac{i}{4} f_{B_c} \int d k_2 e^{i k_2 \cdot z} \left\{ \gamma_5 \left[ \hat{p}_2 \Phi^0_{B_c}(k_2) + m_2 \Phi^p_{B_c}(k_2) \right] \right\}_{ji},$$

$$\langle V(p_3, \epsilon_3^\parallel)|u_i(0)\bar{q}_j(z)|0\rangle = \frac{1}{4} \int_0^1 d k_3 e^{i k_3 \cdot z} \left\{ \epsilon_3^\parallel m_3 \Phi^V_3(k_3) + \epsilon_3^\parallel \hat{p}_3 \Phi^T_3(k_3) + m_3 \Phi^s_3(k_3) \right\}_{ji},$$

where $f_T$ and $f_{B_c}$ are decay constants; $\Phi^V_i$ and $\Phi^p_{B_c}$ are twist-2; $\Phi^T_i$ and $\Phi^s_{B_c}$ are twist-3.

The expressions of DAs for double heavy $\Upsilon(nS)$ and $B_c$ mesons are

$$\phi^V_{\Upsilon(1S)}(x) = \phi^T_{\Upsilon(1S)}(x) = A \bar{x} \exp \left\{ - \frac{m_b^2}{8 \beta_i^2 x \bar{x}} \right\},$$

$$\phi^T_{\Upsilon(1S)}(x) = B (\bar{x} - x)^2 \exp \left\{ - \frac{m_b^2}{8 \beta_i^2 x \bar{x}} \right\},$$

$$\phi^V_{\Upsilon(1S)}(x) = C \{ 1 + (\bar{x} - x)^2 \} \exp \left\{ - \frac{m_b^2}{8 \beta_i^2 x \bar{x}} \right\},$$

$$\phi^V_{\Upsilon(2S)}(x) = D \phi^V_{\Upsilon(1S)}(x) \left\{ 1 + \frac{m_b^2}{2 \beta_i^2 x \bar{x}} \right\},$$

$$\phi^V_{\Upsilon(3S)}(x) = E \phi^V_{\Upsilon(1S)}(x) \left\{ \left( 1 - \frac{m_b^2}{2 \beta_i^2 x \bar{x}} \right)^2 + 6 \right\},$$

$$\phi^V_{\Upsilon(1S)}(x) = A \bar{x} \exp \left\{ - \frac{m_b^2}{8 \beta_i^2 x \bar{x}} \right\},$$
\[ \phi_{B_c}^a(x) = F x \bar{x} \exp\left\{ - \frac{\bar{x} m_c^2 + x m_b^2}{8 \beta_2^2 x \bar{x}} \right\}, \quad (29) \]

\[ \phi_{B_c}^b(x) = G \exp\left\{ - \frac{\bar{x} m_c^2 + x m_b^2}{8 \beta_2^2 x \bar{x}} \right\}, \quad (30) \]

where \( \bar{x} = 1 - x; \beta_i \simeq m_i \alpha_s(m_c) \) according to nonrelativistic quantum chromodynamics (NRQCD) power counting rules \textsuperscript{28–30}; parameters \( A, B, C, D, E, F, G \) are normalization coefficients satisfying the conditions

\[ \int_0^1 dx \phi_{\Upsilon}^{\nu,V,T}(x) = 1, \quad \int_0^1 dx \phi_{B_c}^{a,p}(x) = 1. \quad (31) \]

The shape lines of DAs for \( \Upsilon(nS) \) and \( B_c \) mesons are showed in Fig. 1. It is clearly seen that (1) DAs for \( \Upsilon(nS) \) and \( B_c \) are basically consistent with a picture that valence quarks share momentum fractions according to their masses; (2) DAs fall quickly down to zero at endpoint \( x, \bar{x} \to 0 \) due to suppression from exponential functions, which are bound to offer a natural and effective cutoff for soft contributions.

For the light vector mesons, only three wave functions \( \Phi_v^v \) and \( \Phi_v^{V,A} \) are involved in actual calculation (see Appendix). Their asymptotic forms are \textsuperscript{26, 27}:

\[ \phi_v^v(x) = 6 x \bar{x}, \quad (32) \]

\[ \phi_v^V(x) = \frac{3}{4} \left\{ 1 + (\bar{x} - x)^2 \right\}, \quad (33) \]
\[ \phi_V^A(x) = \frac{3}{2} (\bar{x} - x). \] 

(34)

FIG. 2: Feynman diagrams for \( \Upsilon(nS) \rightarrow B_c \rho \) decay, where (a,b) are factorizable topologies, (c,d) are nonfactorizable topologies.

E. Decay amplitudes

The Feynman diagrams for \( \Upsilon(nS) \rightarrow B_c \rho \) decay are shown in Fig. 2 including factorizable emission topologies (a) and (b) where gluon connects initial \( \Upsilon(nS) \) with recoiled \( B_c \) mesons, and nonfactorizable emission topologies (c) and (d) where gluon attaches the spectator quark with emitted vector mesons.

After a straightforward calculation, amplitude for \( \Upsilon(nS) \rightarrow B_c V \) decay can be decomposed as below,

\[ \mathcal{A}(\Upsilon(nS) \rightarrow B_c V) = \mathcal{A}_L(\epsilon_1^\parallel, \epsilon_3^\parallel) + \mathcal{A}_N(\epsilon_1^\perp \cdot \epsilon_3^\perp) + i \mathcal{A}_T \varepsilon_{\mu \nu \alpha \beta} \epsilon_1^\mu \epsilon_3^\nu p_1^\alpha p_3^\beta, \] 

(35)

which is conventionally written as helicity amplitudes,

\[ \mathcal{A}_0 = -C \sum_i \mathcal{A}_{i,L}(\epsilon_1^\parallel, \epsilon_3^\parallel), \] 

(36)

\[ \mathcal{A}_\parallel = \sqrt{2} C \sum_i \mathcal{A}_{i,N}, \] 

(37)

\[ \mathcal{A}_\perp = \sqrt{2} C m_1 p \sum_i \mathcal{A}_{i,T}, \] 

(38)

\[ C = \frac{G_F \pi C_F}{\sqrt{2} N_c} f_{B_c} f_V V_{ub} V_{uq}^*, \] 

(39)

where \( C_F = 4/3 \) and the color number \( N_c = 3 \); the first superscript \( i \) on \( \mathcal{A}_{i,(N,T)} \) corresponds to the indices of Fig. 2. The detailed analytical expressions of building blocks \( \mathcal{A}_{i,(N,T)} \) are displayed in Appendix.
III. NUMERICAL RESULTS AND DISCUSSION

In the rest frame of \( \Upsilon(nS) \), decaying into \( B_c \) and light vector \( V \) mesons, branching ratio is defined as

\[
Br = \frac{1}{12\pi} \frac{p}{m_T^2 \Gamma_T} \left\{ |A_0|^2 + |A_\parallel|^2 + |A_\perp|^2 \right\},
\]

(40)

where \( p \) is the center-of-mass momentum of final states.

TABLE I: The numerical values of input parameters.

| CKM parameter [1] | \( A = 0.814^{+0.023}_{-0.024} \) | \( \lambda = 0.22537^{+0.00061}_{-0.00061} \) |
|-------------------|--------------------------|--------------------------|
| mass, width and decay constant |
| \( m_{\Upsilon(1S)} = 9460.30\pm0.26 \text{ MeV} \) [1], \( \Gamma_{\Upsilon(1S)} = 54.02\pm1.25 \text{ keV} \) [1], \( f_{\Upsilon(1S)} = 676.4\pm10.7 \text{ MeV} \) [7], |
| \( m_{\Upsilon(2S)} = 10023.26\pm0.31 \text{ MeV} \) [1], \( \Gamma_{\Upsilon(2S)} = 31.98\pm2.63 \text{ keV} \) [1], \( f_{\Upsilon(2S)} = 473.0\pm23.7 \text{ MeV} \) [7], |
| \( m_{\Upsilon(3S)} = 10355.2\pm0.5 \text{ MeV} \) [1], \( \Gamma_{\Upsilon(3S)} = 20.32\pm1.85 \text{ keV} \) [1], \( f_{\Upsilon(3S)} = 409.5\pm29.4 \text{ MeV} \) [7], |
| \( m_{B_c} = 6275.6\pm1.1 \text{ MeV} \) [1], \( m_b = 4.78\pm0.06 \text{ GeV} \) [1], \( m_c = 1.67\pm0.07 \text{ GeV} \) [1], |
| \( f_{B_c} = 434\pm15 \text{ MeV} \) [31], \( f_\rho = 216\pm3 \text{ MeV} \) [27], \( f_{K^*} = 220\pm5 \text{ MeV} \) [27] |

TABLE II: Branching ratio for \( \Upsilon(nS) \to B_c\rho, B_cK^* \).

| \( 10^{11} \times Br(\Upsilon(1S)\to B_c\rho) \) | \( 10^{11} \times Br(\Upsilon(2S)\to B_c\rho) \) | \( 10^{11} \times Br(\Upsilon(3S)\to B_c\rho) \) | \( 10^{12} \times Br(\Upsilon(1S)\to B_cK^*) \) | \( 10^{12} \times Br(\Upsilon(2S)\to B_cK^*) \) | \( 10^{12} \times Br(\Upsilon(3S)\to B_cK^*) \) |
|------------------|------------------|------------------|------------------|------------------|------------------|
| this work | Ref. [11] | Ref. [12] | Ref. [13] | this work | Ref. [11] | Ref. [12] | Ref. [13] | this work | Ref. [11] | Ref. [12] | Ref. [13] |
| \( 13.25^{+1.04+1.14+0.91}_{-0.63}-1.14-0.87 \) | 17.6 | 13.0 | 15.3 | \( 8.88^{+0.64+0.67+0.61}_{-0.40-0.74-0.58} \) | ... | ... | ... | \( 8.46^{+0.61+0.71+0.58}_{-0.37-0.68-0.56} \) | ... | ... | ... | \( 7.97^{+0.62+0.65+0.59}_{-0.38-0.67-0.57} \) | 10.0 | 7.0 | 8.75 | \( 5.28^{+0.38+0.48+0.39}_{-0.24-0.45-0.37} \) | ... | ... | ... | \( 4.98^{+0.36+0.44+0.37}_{-0.22-0.43-0.35} \) | ... | ... | ... |

The values of input parameters are listed in Table II. If it is not specified explicitly, their central values will be used as default inputs. Our numerical results are presented in Table III, where the uncertainties come from scale \((1\pm0.1)t_i, m_b \) and \( m_c \), and CKM parameters, respectively. The following are some comments.
(1) By and large, our results are consistent with previous estimation on branching ratio for $\Upsilon(1S) \rightarrow B_c\rho$, $B_cK^*$ decays. The hierarchical structure of CKM factors $|V_{cb}V_{ud}^*| > |V_{cb}V_{us}^*|$ leads to the general rank-size relationship among branching ratios $\mathcal{B}(\Upsilon(nS)\rightarrow B_c\rho) > \mathcal{B}(\Upsilon(nS)\rightarrow B_cK^*)$. Normally, there should be $\mathcal{B}(\Upsilon(3S)\rightarrow B_cV) > \mathcal{B}(\Upsilon(2S)\rightarrow B_cV) > \mathcal{B}(\Upsilon(1S)\rightarrow B_cV)$ for the same final $V$ meson, due to the fact that $m_{\Upsilon(3S)} > m_{\Upsilon(2S)} > m_{\Upsilon(1S)}$ and $\Gamma_{\Upsilon(3S)} < \Gamma_{\Upsilon(2S)} < \Gamma_{\Upsilon(1S)}$. However, the numbers in Table II are beyond expectation. Why is it that? In addition to form factors, one of the possible factors is

$$\mathcal{B}(\Upsilon(3S)\rightarrow B_cV) : \mathcal{B}(\Upsilon(2S)\rightarrow B_cV) : \mathcal{B}(\Upsilon(1S)\rightarrow B_cV) \propto \frac{f_{\Upsilon(3S)}^2}{m_{\Upsilon(3S)}^2 \Gamma_{\Upsilon(3S)}} : \frac{f_{\Upsilon(2S)}^2}{m_{\Upsilon(2S)}^2 \Gamma_{\Upsilon(2S)}} : \frac{f_{\Upsilon(1S)}^2}{m_{\Upsilon(1S)}^2 \Gamma_{\Upsilon(1S)}} \approx 0.8 : 0.7 : 1. \quad (41)$$

(2) Branching ratio for $\Upsilon(nS) \rightarrow B_c\rho$ decay can reach up to $\mathcal{O}(10^{-11})$. The $\Upsilon(nS)$ production cross section in p-Pb collision is about a few $\mu b$ at LHCb [32] and ALICE [33]. Over $10^{11}$ $\Upsilon(nS)$ data samples per $ab^{-1}$ data collected at LHCb and ALICE are in principle available, corresponding to dozens of $\Upsilon(nS) \rightarrow B_c\rho$ events. If the experimental identification of final states is considered, for example, the best experimental identification of $B_c$ meson is through $B_c \rightarrow J/\psi\mu^+\nu_\mu$ or $J/\psi\pi$ decays with branching ratios $\mathcal{O}(10^{-3}) \sim \mathcal{O}(10^{-4}) [34-36]$ and detection efficiency about $\mathcal{O}(10^{-2}) [36,37]$, then the feasibility of observation of $\Upsilon(nS) \rightarrow B_cV$ decays is very small.

![FIG. 3: Contributions to branching ratio from different region of $\alpha_s/\pi$ (abscissa axis), where the numbers over histogram denote percentage of the corresponding contributions.](image)

(3) From Fig[2] the spectator is a heavy bottom quark in the $\Upsilon(nS) \rightarrow B_c$ transition. It is assumed that the bottom quark is near on-shell and the gluon attaching to the spectator might be soft. It is natural to question the validity of perturbative calculation with pQCD approach. So, it is necessary to check how many shares come from the perturbative region. The contributions to branching ratio $\mathcal{B}(\Upsilon(nS)\rightarrow B_c\rho)$ from different region of $\alpha_s/\pi$ are...
displayed in Fig.3. It is clearly seen that more than 85% (some 95%) contributions to branching ratio come from \(\alpha_s/\pi \leq 0.2\) (0.3) regions, which implies that the calculation with pQCD approach is feasible. Compared with contributions from \(\alpha_s/\pi \in [0.1, 0.2]\) region, one of crucial reasons for a small percentage in the region \(\alpha_s/\pi \leq 0.1\) is that the absolute values of Wilson coefficients \(C_{1,2}\), parameter \(a_1\) and coupling \(\alpha_s\) decrease along with the increase of renormalization scale.

(4) Besides uncertainties listed in Table II, decay constants \(f_\Upsilon\) and \(f_{B_c}\) can bring some 8%, 12%, 16% uncertainties for \(\Upsilon(1S), \Upsilon(2S), \Upsilon(3S)\) decays, respectively. These are two ways to reduce theoretical uncertainty. One is to construct some relative ratios of branching ratios, for example, \(\text{Br}(\Upsilon(nS)\rightarrow B_cK^*)/\text{Br}(\Upsilon(nS)\rightarrow B_c\rho)\) and \(\text{Br}(\Upsilon(mS)\rightarrow B_c\rho)/\text{Br}(\Upsilon(nS)\rightarrow B_c\rho)\). The other is to consider higher order corrections to HME, relativistic effects on DAs, and so on. Here, our results just provide an order of magnitude estimation.

IV. SUMMARY

Besides the predominant strong and electromagnetic decay modes, \(\Upsilon(nS)\) can also decay through the weak interaction within the standard model. Study of \(\Upsilon(nS)\) weak decay is theoretically interesting and experimentally feasible. In this paper, we investigated the bottom- and charm-changing \(\Upsilon(nS) \rightarrow B_c\rho, B_cK^*\) decays with phenomenological pQCD approach. It is found that branching ratio for \(\Upsilon(nS) \rightarrow B_c\rho\) and \(B_cK^*\) decays can reach up to \(\mathcal{O}(10^{-11})\) and \(\mathcal{O}(10^{-12})\), respectively, and their actual detection at the future LHC and SuperKEKB experiments should be quite challenging.

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Appendix A: Building blocks of decay amplitudes

The amplitude for the $\Upsilon(nS) \to B_c V$ decays ($V = \rho, K^*$) are constituted of a linear combination of building block $A_{i,j}$, where the first subscript $i$ corresponds to the indices of Fig 2 and the second subscript $j = L, N, T$ denotes to three different helicity amplitudes. The expressions of $A_{i,j}$ are written as follows.

$$A_{a,L} = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \, H_f(\alpha_g, \beta_a, b_1, b_2) \, E_f(t_a) \, \alpha_s(t_a) \, a_1(t_a)$$

$$\times \phi_T^v(x_1) \left\{ \phi_{B_c}^a(x_2) \left[ m_1^2 s - (4 m_1^2 p^2 + m_2^2 u) \tilde{x}_2 \right] + \phi_{B_c}^p(x_2) m_2 m_b u \right\}, \quad (A1)$$

$$A_{a,N} = m_1 m_3 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \, H_f(\alpha_g, \beta_a, b_1, b_2) \, E_f(t_a) \, \alpha_s(t_a)$$

$$\times a_1(t_a) \phi_T^v(x_1) \left\{ \phi_{B_c}^a(x_2) (2 m_2 \tilde{x}_2 - t) - \phi_{B_c}^p(x_2) 2 m_2 m_b \right\}, \quad (A2)$$

$$A_{a,T} = 2 m_1 m_3 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \, H_f(\alpha_g, \beta_a, b_1, b_2) \, E_f(t_a)$$

$$\times \alpha_s(t_a) a_1(t_a) \phi_T^v(x_1) \phi_{B_c}^a(x_2), \quad (A3)$$

$$A_{b,L} = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \, H_f(\alpha_g, \beta_b, b_2, b_1) \, E_f(t_b) \, \alpha_s(t_b) \, a_1(t_b)$$

$$\times \left\{ \phi_T^v(x_1) \left[ \phi_{B_c}^a(x_2) \left[ m_1^2 (s - 4 p^2) \tilde{x}_1 - m_2^2 u \right] + \phi_{B_c}^p(x_2) 2 m_2 m_c u \right]$$

$$+ \phi_T^v(x_1) \left[ \phi_{B_c}^p(x_2) 2 m_1 m_2 (s - u \tilde{x}_1) - \phi_{B_c}^a(x_2) m_1 m_c s \right] \right\}, \quad (A4)$$

$$A_{b,N} = m_3 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \, H_f(\alpha_g, \beta_b, b_2, b_1) \, E_f(t_b) \, \alpha_s(t_b)$$

$$\times a_1(t_b) \left\{ \phi_T^v(x_1) m_1 \left[ \phi_{B_c}^a(x_2) (2 m_2^2 - t \tilde{x}_1) - \phi_{B_c}^p(x_2) 4 m_2 m_c \right]$$

$$+ \phi_T^v(x_1) \left[ \phi_{B_c}^p(x_2) m_c t + \phi_{B_c}^a(x_2) m_2 (4 m_1^2 \tilde{x}_1 - 2 t) \right] \right\}, \quad (A5)$$

$$A_{b,T} = -2 m_3 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \, H_f(\alpha_g, \beta_b, b_2, b_1) \, E_f(t_b) \, \alpha_s(t_b)$$

$$\times a_1(t_b) \left\{ \phi_T^v(x_1) \phi_{B_c}^a(x_2) m_1 \tilde{x}_1 + \phi_T^v(x_1) \left[ \phi_{B_c}^a(x_2) m_c - \phi_{B_c}^p(x_2) 2 m_2 \right] \right\}, \quad (A6)$$

$$A_{c,L} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \, H_n(\alpha_g, \beta_c, b_2, b_3)$$

$$\times \delta(b_1 - b_2) \, E_n(t_c) \, \alpha_s(t_c) \left\{ \phi_T^v(x_1) \phi_{B_c}^a(x_2) u (t x_1 - 2 m_2^2 x_2 - s \tilde{x}_3)$$

$$+ \phi_T^v(x_1) \phi_{B_c}^p(x_2) m_1 m_2 (s x_2 + 2 m_3^2 \tilde{x}_3 - u x_1) \right\} \phi_T^v(x_3) C_2(t_c), \quad (A7)$$

$$A_{c,N} = \frac{m_3}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \, H_n(\alpha_g, \beta_c, b_2, b_3)$$

$$\times \delta(b_1 - b_2) \left\{ \phi_T^v(x_1) \phi_{B_c}^a(x_2) \phi_T^v(x_3) m_1 \left( 2 s \tilde{x}_2 + 4 m_2^2 x_2 - 2 t x_1 \right)$$

$$+ \phi_T^v(x_1) \phi_{B_c}^p(x_2) m_2 \left[ \phi_T^v(x_3) (2 m_1^2 x_1 - t x_2 - u \tilde{x}_3) \right.$$$$+ \phi_T^v(x_3) 2 m_1 p (x_2 - \tilde{x}_3) \right\} \right\} E_n(t_c) \, \alpha_s(t_c) C_2(t_c), \quad (A8)$$
\[ A_{c,T} = \frac{m_3}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \, H_n(\alpha_g, \beta_c, b_2, b_3) \]
\[ \times \delta(b_1 - b_2) \left\{ \phi_T^V(x_1) \phi_B^a(x_2) \phi_V^A(x_3) (2 s \bar{x}_3 + 4 m_2^2 x_2 - 2 t x_1) \right. \]
\[ + \phi_T^T(x_1) \phi_B^p(x_2) \left[ \phi_V^A(x_3) m_2/m_1 (2 m_1^2 x_1 - t x_2 - u \bar{x}_3) \right. \]
\[ + \phi_V^T(x_3) 2 m_2 p (x_2 - \bar{x}_3) \} E_n(t_c) \alpha_s(t_c) C_2(t_c), \] (A9)

\[ A_{d,L} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \, H_n(\alpha_g, \beta_d, b_2, b_3) \]
\[ \times \delta(b_1 - b_2) \left\{ \phi_T^V(x_1) \phi_B^a(x_2) m_1 m_2 (s x_2 + 2 m_3^2 x_3 - u x_1) \right. \]
\[ + \phi_T^T(x_1) \phi_B^p(x_2) 4 m_2^2 p^2 (x_3 - x_2) \} E_n(t_d) \alpha_s(t_d) C_2(t_d), \] (A10)

\[ A_{d,N} = \frac{m_2 m_3}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \, \delta(b_1 - b_2) \]
\[ \times H_n(\alpha_g, \beta_d, b_2, b_3) \left\{ \phi_T^V(x_1) \phi_B^a(x_2) m_2 (2 m_1^2 x_1 - t x_2 - u x_3) \right. \]
\[ + \phi_T^T(x_1) \phi_B^p(x_2) 2 m_1 p (x_2 - x_3) \} \phi_T^T(x_1) \phi_B^p(x_2) C_2(t_d), \] (A11)

\[ A_{d,T} = \frac{m_2 m_3}{N_c m_1} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \, \delta(b_1 - b_2) \]
\[ \times H_n(\alpha_g, \beta_d, b_2, b_3) \left\{ \phi_T^V(x_1) \phi_B^a(x_2) m_2 (2 m_1^2 x_1 - t x_2 - u x_3) \right. \]
\[ + \phi_T^T(x_1) \phi_B^p(x_2) 2 m_1 p (x_2 - x_3) \} \phi_T^T(x_1) \phi_B^p(x_2) C_2(t_d), \] (A12)

where \( x_i \) and \( \bar{x}_i = 1 - x_i \) are longitudinal momentum fractions of valence quarks; \( b_i \) is the conjugate variable of the transverse momentum \( k_{i\perp} \); \( a_1 = C_1 + C_2/N_c \); \( N_c = 3 \) is the color number; \( C_{1,2} \) are the Wilson coefficients.

The Sudakov factor \( E_{f,n} \) and function \( H_{f,n} \) are defined as follows, where the subscript \( f \) (\( n \)) corresponds to (non)factorizable topologies.

\[ E_f(z) = \exp\{-S_T(z) - S_{B_c}(z)\}, \] (A13)

\[ E_n(z) = \exp\{-S_T(z) - S_{B_c}(z) - S_V(z)\}, \] (A14)

\[ S_T(z) = s(x_1, p_1^+, 1/b_1) + 2 \int_{1/b_1}^z \frac{d\mu}{\mu} \gamma_q, \] (A15)

\[ S_{B_c}(z) = s(x_2, p_2^+, 1/b_2) + 2 \int_{1/b_2}^z \frac{d\mu}{\mu} \gamma_q, \] (A16)

\[ S_V(z) = s(x_3, p_3^+, 1/b_3) + s(\bar{x}_3, p_3^+, 1/b_3) + 2 \int_{1/b_3}^z \frac{d\mu}{\mu} \gamma_q, \] (A17)

\[ H_f(\alpha, \beta, b_i, b_j) = K_0(b_i \sqrt{-\alpha}) \left\{ \delta(b_i - b_j) K_0(b_i \sqrt{-\beta}) I_0(b_j \sqrt{-\beta}) + (b_i \leftrightarrow b_j) \right\}, \] (A18)
\[ H_n(\alpha, \beta, b_2, b_3) = \left\{ \frac{\theta(-\beta)}{2} K_0(b_3\sqrt{-\beta}) + \frac{\pi \theta(\beta)}{2} \left[ iJ_0(b_3\sqrt{\beta}) - Y_0(b_3\sqrt{\beta}) \right] \right\} \]
\[ \times \left\{ \theta(b_2 - b_3) K_0(b_2\sqrt{-\alpha}) I_0(b_3\sqrt{-\alpha}) + (b_2 \leftrightarrow b_3) \right\}, \quad (A19) \]

where the form of \( s(x, Q, 1/b) \) can be found in Ref. [3]; \( \gamma_q = -\alpha_s/\pi \) is the quark anomalous dimension; \( I_0, J_0, K_0 \) and \( Y_0 \) are Bessel functions; the gluon virtuality \( \alpha_g \), the quark virtuality \( \beta_i \), and scale \( t_i \) are defined as follows.

\[
\alpha_g = \bar{x}_1 m_1^2 + \bar{x}_2 m_2^2 - \bar{x}_1 \bar{x}_2 t, \quad (A20) \\
\beta_a = m_1^2 - m_2^2 + \bar{x}_2 m_2^2 - \bar{x}_2 t, \quad (A21) \\
\beta_b = m_2^2 - m_c^2 + \bar{x}_1 m_1^2 - \bar{x}_1 t, \quad (A22) \\
\beta_c = x_1^2 m_1^2 + x_2^2 m_2^2 + \bar{x}_3 m_3^2 \]
\[
- x_1 x_2 t - x_1 \bar{x}_3 u + x_2 \bar{x}_3 s, \quad (A23) \\
\beta_d = x_1^2 m_1^2 + x_2^2 m_2^2 + x_3^2 m_3^2 \]
\[
- x_1 x_2 t - x_1 x_3 u + x_2 x_3 s, \quad (A24) \\
t_{a(b)} = \max(\sqrt{-\alpha_g}, \sqrt{-\beta_{a(b)}}, 1/b_1, 1/b_2), \quad (A25) \\
t_{c(d)} = \max(\sqrt{-\alpha_g}, \sqrt{\beta_{c(d)}}, 1/b_2, 1/b_3). \quad (A26) \\
\]

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