An Adaptive Characteristic-wise Reconstruction WENO scheme for Gas Dynamic Euler Equations

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Abstract

Due to its excellent shock-capturing capability and high resolution, the WENO scheme family has been widely used in varieties of compressive flow simulation. However, for problems containing strong shocks and contact discontinuities, such as the Lax shock tube problem, the WENO scheme still produces numerical oscillations. To avoid such numerical oscillations, the characteristic-wise construction method should be applied. Compared to component-wise reconstruction, characteristic-wise reconstruction leads to much more computational cost and thus is not suite for large scale simulation such as direct numeric simulation of turbulence. In this paper, an adaptive characteristic-wise reconstruction WENO scheme, i.e. the AdaWENO scheme, is proposed to improve the computational efficiency of the characteristic-wise reconstruction method. The new scheme performs characteristic-wise reconstruction near discontinuities while switching to component-wise reconstruction for smooth regions. Meanwhile, a new calculation strategy for the WENO smoothness indicators is implemented to reduce over-all computational cost. Several one dimensional and two dimensional numerical tests are performed to validate and evaluate the AdaWENO scheme. Numerical results show that AdaWENO maintains essentially non-oscillatory flow field near discontinuities as the characteristic-wise reconstruction method. Besieds, compared to component-wise reconstruction, AdaWENO is about...
40% faster which indicates its excellent efficiency.

*Keywords:* WENO scheme, characteristic-wise reconstruction, adaptive method, Euler equations

1. Introduction

Numerical simulation of compressible flow lever engineering and scientific researches by providing detailed and high-quality flow field information. For high-resolution and accurate simulation of compressible flow, numerical methods being applied shall be able to capture all important features, e.g. turbulence and shockwave, in the flow field.

The family of weighted essentially non-oscillatory (WENO) finite difference schemes [1,2] has been widely used in compressible flow simulations due to its high resolution of small structures and good shock-capturing capability. Within the general framework of smoothness indicators and non-linear weights proposed by Jiang and Shu [2], many efforts have been made to improve the accuracy and efficiency of the WENO scheme. Henrick et al. [3] improved the accuracy of the WENO scheme at critical points by suggesting a mapping function. Borges et al. [4,5] proposed the WENO-Z scheme which calculates the non-linear weights with a high order smoothness indicator. The WENO-Z scheme achieves lower dissipation and higher resolution than the classical WENO scheme of Jiang and Shu and has lower computational cost than the mapping function method of Henrick et al.. The accuracy of the WENO-Z scheme was further improved by Yamaleev and Carpenter [6,7] and Fan et al. [8] by introducing higher order smoothness indicators. Fu et al. [9] proposed a family of high-order targeted ENO schemes which combines the idea of both the ENO scheme and the WENO scheme. Except for these improvements for the fifth order WENO scheme, higher order WENO schemes (higher than fifth order) were also developed [10,11,12].

To develop and improve a WENO scheme, two issues should be addressed at the same time: maintaining high order accuracy at smooth region and capturing discontinuity non-oscillatorily. Aforementioned methods mainly focus on the improvement of the accuracy of WENO schemes at smooth regions. Shen and Zha [13] showed that at transitional points, which connect smooth region and discontinuity, the accuracy of fifth order WENO schemes is second order and a multi-step weighting method [14,15] was developed to improve the accuracy.
In spite of their excellent performances for scalar problems, WENO schemes still produce numerical oscillations for problems like the Lax shock tube problem of the gas dynamic Euler equations. To get rid of such oscillations, the characteristic-wise reconstruction method [16, 2, 17] should be applied. Compared to the component-wise reconstruction method, the characteristic-wise reconstruction method results in much more computational cost. Therefore, for efficiency consideration, in practical large scale simulations, the component-wise reconstruction method is always preferred, e.g. [18], that some numerical oscillations are tolerable. However, such compromise may reduce the reliability of the simulation result that numerical oscillations disturb the flow field and may change the whole flow structure. To prevent numerical oscillations and avoid the use of characteristic-wise reconstruction, He et al. [19] analyzed the WENO weights and proposed a new method to calculate the final smoothness indicators. This method reduces but is not free of numerical oscillations. Puppo [20] proposed an adaptive method to combine the component-wise reconstruction method and the characteristic-wise reconstruction method and showed good performance and efficiency. Nevertheless, the flagging criterion in Puppo’s method may misclassify smooth cells and thus reduces the computational efficiency of the scheme. Hu et al. [21] proposed a discontinuity detector to combine characteristic-wise WENO with low dissipation linear scheme that several free parameters were introduced.

In this paper, we focus on developing an efficient and accurate characteristic-wise reconstruction WENO scheme which is of practical value for large scale simulation of compressible flow. A new adaptive characteristic-wise reconstruction WENO scheme utilizing a highly accurate switch function is proposed, and a new strategy for smoothness indicators calculation is also introduced to further reduce the computational cost of the WENO scheme. This paper is organized as following. In Section 2 the WENO scheme and its component-wise and characteristic-wise reconstruction implementations are introduced. In section 3 a new adaptive method is developed based on the analysis of the two implementations. In section 4 several numerical tests are presented to illustrate the performance and efficiency of the new method. Concluding remarks are given in Section 5.
2. Solving the gas dynamic Euler equations with WENO

2.1. The gas dynamic Euler equations

The one dimensional Euler equations of inviscid ideal gas is given by:

\[ \frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}}{\partial x} = 0 \]  

in which \( \vec{U} \) and \( \vec{F} \) are the conserved variable and the convective flux vector:

\[ \vec{U} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad \vec{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{bmatrix}, \]  

where \( \rho \) is the density, \( u \) is the velocity, and \( E = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2 \) is the total energy with \( \gamma = 1.4 \).

For the convective flux vector \( \vec{F} \), its Jacobian matrix \( A \) is defined as:

\[ \frac{\partial \vec{F}(\vec{U})}{\partial x} = A \frac{\partial \vec{U}}{\partial x}, \]  

where \( A \) is written as:

\[ A = R \lambda L = \begin{bmatrix} 0 & 1 & 0 \\ \frac{3-\gamma}{2} u^2 & (3-\gamma)u & \gamma - 1 \\ \frac{2}{\gamma - 1} u^3 - \frac{u c^2}{\gamma - 1} & \frac{c^2}{\gamma - 1} + \frac{3-\gamma}{2} u^2 & \gamma u \end{bmatrix}, \]  

where \( c = \sqrt{\gamma p/\rho} \) is the sound speed. Here, \( \lambda \) is the eigen matrix of \( A \):

\[ \lambda = \begin{bmatrix} u - c \\ u \\ u + c \end{bmatrix}, \]  

and \( L \) and \( R \) are the left and right eigen vectors:

\[ L = \begin{bmatrix} l_0 \\ l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} \frac{\gamma - 1}{4} u^2 + \frac{1}{2} \frac{c^2}{\gamma - 1} & -\frac{\gamma - 1}{2} u & -\frac{1}{2} \frac{c^2}{\gamma - 1} \\ 1 - \frac{\gamma - 1}{2} \frac{u^2}{c^2} & \frac{\gamma - 1}{2} u & -\frac{\gamma - 1}{2} \frac{c^2}{\gamma - 1} \\ \frac{\gamma - 1}{4} u^2 - \frac{1}{2} \frac{c^2}{\gamma - 1} & -\frac{\gamma - 1}{2} u & +\frac{1}{2} \frac{c^2}{\gamma - 1} \end{bmatrix}, \]  

\[ R = [r_0, r_1, r_2] = \begin{bmatrix} 1 & 1 & 1 \\ u - c & u & u + c \\ \frac{u^2}{2} + \frac{c^2}{\gamma - 1} - uc & \frac{u^2}{2} & \frac{u^2}{2} + \frac{c^2}{\gamma - 1} + uc \end{bmatrix}. \]
To introduce correct upwinding, the flux $\vec{F}$ is generally split into two parts:

$$\vec{F} = \vec{F}^+ + \vec{F}^-$$  \hspace{1cm} (8)

where

$$\frac{d\vec{F}^+}{d\vec{U}} \geq 0, \quad \frac{d\vec{F}^-}{d\vec{U}} < 0.$$  

In this paper, the Lax-Friedrich splitting method [2] is used:

$$\vec{F}_i^\pm = \frac{1}{2}(\vec{F}_i \pm \alpha_i \vec{U}_i). \hspace{1cm} (9)$$

For the Euler equations, $\alpha_i$ is taken as:

$$\alpha_i = \alpha = \max_i (|u_i| + c_i) \hspace{1cm} (10)$$

for simplicity and robustness as discussed in [2, 19].

2.2. The WENO scheme

To introduce the WENO scheme, let us consider the semi-discrete form of eq.(1) on equally spaced grid, i.e. $\Delta x = x_{i+1} - x_i$:

$$\frac{\partial \vec{U}}{\partial t} = -\frac{\partial \vec{F}}{\partial x} \approx \frac{-1}{\Delta x} (\vec{F}_{i+1/2} - \vec{F}_{i-1/2}) \hspace{1cm} (11)$$

where $\vec{F}_{i+1/2} = \vec{F}_{i+1/2}^+ + \vec{F}_{i+1/2}^-$ is the numerical flux at cell interface. Each component of the numerical fluxes $\vec{F}_{i+1/2}^+$ and $\vec{F}_{i+1/2}^-$, i.e. $k \hat{f}_{i+1/2}^\pm$, is then reconstructed by the WENO scheme. For simplicity, $k$ and $\pm$ in the superscript are dropped in the following parts of this paper.

The numerical flux component $\hat{f}_{i+1/2}$ can be obtained by high order upwind schemes. The fifth order upwind scheme is written as:

$$\hat{f}_{i+1/2} = \frac{2}{60} f_{i-2} - \frac{13}{60} f_{i-1} + \frac{47}{60} f_i + \frac{27}{60} f_{i+1} - \frac{3}{60} f_{i+2}. \hspace{1cm} (12)$$

It is a convex combination of three third order upwind schemes over three sub-stencils $S_0 = (x_{i-2}, x_{i-1}, x_i)$, $S_1 = (x_{i-1}, x_i, x_{i+1})$, and $S_2 = (x_i, x_{i+1}, x_{i+2})$:

$$\hat{f}_{0,i+1/2} = \frac{1}{3} f_{i-2} - \frac{7}{6} f_{i-1} + \frac{11}{6} f_i, \hspace{1cm} (13)$$

$$\hat{f}_{1,i+1/2} = -\frac{1}{6} f_{i-1} + \frac{5}{6} f_i + \frac{1}{3} f_{i+1}, \hspace{1cm} (14)$$

$$\hat{f}_{2,i+1/2} = \frac{1}{3} f_i + \frac{5}{6} f_{i+1} - \frac{1}{6} f_{i+2}. \hspace{1cm} (15)$$
with linear weights $c_0 = 0.1$, $c_1 = 0.6$, and $c_2 = 0.3$ respectively. By substituting the linear weights with the non-linear WENO weights, we have the fifth order WENO scheme:

$$\text{WENO5 : } \hat{f}_{i+1/2} = \omega_0 \hat{f}_{0,i+1/2} + \omega_1 \hat{f}_{1,i+1/2} + \omega_2 \hat{f}_{2,i+1/2}. \quad (16)$$

The non-linear WENO weights $\omega_k$ are given by:

$$\omega_k = \frac{\alpha_k}{\sum_i \alpha_i}, \quad (17)$$

$$\alpha_k = \frac{c_k}{(\varepsilon + \beta_k)^p}, k = \{0, 1, 2\}, p = 1, 2, ... \quad (18)$$

where $\varepsilon$ is a small number to avoid dividing by zero. In this paper, $\varepsilon$ is taken to be $1.0e-6$. The smoothness indicators $\beta_i$ are

$$\beta_0 = \frac{13}{12} (f_{i-2} - 2f_{i-1} + f_i)^2 + \frac{1}{4} (f_{i-2} - 4f_{i-1} + 3f_i)^2, \quad (19)$$

$$\beta_1 = \frac{13}{12} (f_{i-1} - 2f_i + f_{i+1})^2 + \frac{1}{4} (f_{i-1} - f_{i+1})^2, \quad (20)$$

$$\beta_2 = \frac{13}{12} (f_i - 2f_{i+1} + f_{i+2})^2 + \frac{1}{4} (3f_i - 4f_{i+1} + f_{i+2})^2. \quad (21)$$

As shown by Henrick et al. [3], for the fifth order WENO scheme of Jiang and Shu [2], the non-linear weights do not satisfy the necessary and sufficient conditions for fifth order convergence. A mapping function was introduced to improve the accuracy of the final weights (the WENO-M scheme). In [4], Borges et al. introduced a parameter $\tau_5 = |\beta_0 - \beta_2|$ to calculate the weights as:

$$\alpha_k = c_k \left(1 + \left(\frac{\tau_5}{\beta_k + \varepsilon}\right)^q\right), k = \{0, 1, 2\}, q = 1, 2, ... \quad (22)$$

This new scheme (the WENO-Z scheme) is less computational expensive than the WENO-M scheme. In this paper, the WENO-Z scheme is used as the base WENO scheme.

There are two ways to calculate the numerical fluxes of the Euler equations with WENO schemes, namely, the component-wise reconstruction method and the characteristic-wise reconstruction method. The former method reconstructs the numerical flux vector component-by-component while the latter method performs the reconstruction in the characteristic space. In the following two subsections [2.3 and 2.4], details of these two methods will be presented.
2.3. Component-wise reconstruction

By implementing the WENO reconstruction for the numerical flux vector \( \vec{F} \) component-by-component, the resulted numerical flux at cell interface can be written as:

\[
\vec{F}_{i+1/2}^{CP} = \begin{bmatrix}
0 \\
\hat{f}_{i+1/2}^{CP} \\
1 \\
\hat{f}_{i+1/2}^{CP} \\
2 \\
\hat{f}_{i+1/2}^{CP}
\end{bmatrix} = \sum_k \omega_k^{CP} \hat{f}_{k,i+1/2}^{CP}
\]

(23)
in which

\[
\omega_k^{CP} = \begin{bmatrix}
0 \\
\omega_k^{CP} \\
1 \\
\omega_k^{CP} \\
2 \\
\omega_k^{CP}
\end{bmatrix}
\]

(24)
and

\[
\hat{f}_{k,i+1/2}^{CP} = \begin{bmatrix}
0 \\
\hat{f}_{k,i+1/2}^{CP} \\
1 \\
\hat{f}_{k,i+1/2}^{CP} \\
2 \\
\hat{f}_{k,i+1/2}^{CP}
\end{bmatrix}, \quad k = 0, 1, 2.
\]

(25)
The WENO weights in Eq. (24) are calculated according to the corresponding flux component at each stencil:

\[
s\omega_k^{CP} = s\omega_k^{CP} (s f_{i+k-2}, \cdots, s f_{i+k}), \quad k = 0, 1, 2, \quad s = 0, 1, 2,
\]

(26)
where \( s \) is the component index and \( k \) is the stencil index. The right superscript ‘CP’ stands for component-wise reconstruction.

It can be observed that the component-wise reconstruction method is easy to implement that only one single WENO reconstruction subroutine is needed in one’s code. However, numerical oscillations may present in solutions obtained by the component-wise reconstruction method. In the following parts, the component-wise method will be referred to as the CP method.

2.4. Characteristic-wise reconstruction

Compared to the CP method, the characteristic-wise method does not produce numerical oscillations. To perform reconstruction in the characteristic space, the flux vector \( \vec{F} \) should firstly be projected onto the left eigenvector of its Jacobian on cell interface \( x_{i+1/2} \). The left eigenvectors on cell interface are obtained from Roe-averaged primitive variables:

\[
\bar{u} = \frac{\sqrt{\rho_i} u_i + \sqrt{\rho_{i+1}} u_{i+1}}{\sqrt{\rho_i} + \sqrt{\rho_{i+1}}}
\]

(27)
\[ h = \frac{\sqrt{\rho_i h_i} + \sqrt{\rho_{i+1} h_{i+1}}}{\sqrt{\rho_i} + \sqrt{\rho_{i+1}}} \quad (28) \]

\[ \bar{c} = \sqrt{(\gamma - 1)(\bar{h} - \frac{1}{2} \bar{u}^2)} \quad (29) \]

\[ h = \frac{p}{(\gamma - 1)\rho} + \frac{1}{2} \bar{u}^2 + \frac{p}{\rho} \quad (30) \]

The left eigenvector matrix is therefore written as:

\[ \bar{L}_{i+1/2} = \begin{bmatrix} \bar{l}_0 & \bar{l}_1 & \bar{l}_2 \end{bmatrix} = \begin{bmatrix} \gamma - \frac{1}{4} \frac{\bar{c}^2}{\bar{u}^2} + \frac{1}{2} \frac{\bar{c}}{\bar{u}} - \frac{1}{2} \frac{\bar{c}}{\bar{u}} & -\frac{1}{2} \frac{\bar{c}}{\bar{u}} & -\frac{1}{2} \frac{\bar{c}}{\bar{u}} \\ 1 - \frac{1}{2} \frac{\bar{c}^2}{\bar{u}^2} & \frac{\bar{c}^2}{\bar{u}^2} - \frac{1}{2} \frac{\bar{c}}{\bar{u}} & -\frac{1}{2} \frac{\bar{c}}{\bar{u}} \\ -\frac{1}{4} \frac{\bar{c}^2}{\bar{u}^2} - \frac{1}{2} \frac{\bar{c}}{\bar{u}} & -\frac{1}{2} \frac{\bar{c}}{\bar{u}} + \frac{1}{2} \frac{\bar{c}}{\bar{u}} & \frac{\bar{c}^2}{\bar{u}^2} \end{bmatrix} \quad (31) \]

The WENO reconstruction is performed component-by-component to the projected variable \( \hat{w} \):

\[ \hat{w}_{k,i+1/2}^{CH} = \begin{bmatrix} 0 \hat{w}_{k,i+1/2} \\ 1 \hat{w}_{k,i+1/2} \\ 2 \hat{w}_{k,i+1/2} \end{bmatrix} = \bar{L}_{i+1/2} \hat{f}_{CP}^{k,i+1/2}, k = 0, 1, 2 \quad (32) \]

\[ \tilde{W}_{i+1/2}^{CH} = \sum_{k=0}^{2} \omega_k^{CH} \hat{w}_{k,i+1/2}^{CH} \quad (33) \]

Different from the CP method, the WENO weights are computed according to the projected variables on each stencil:

\[ \omega_k^{CH} = \begin{bmatrix} 0 \omega_k^{CH} \\ 1 \omega_k^{CH} \\ 2 \omega_k^{CH} \end{bmatrix} \quad (34) \]

\[ s \omega_k^{CH} = s \omega_k^{CH} (\bar{I}_s \bar{F}_{i+k-2}, \cdots, \bar{I}_s \bar{F}_{i+k}), \quad k = 0, 1, 2, \quad s = 0, 1, 2 \quad (35) \]

After the WENO reconstruction of the projected variables, the obtained values need to be projected back to the physics space by projecting onto the right eigenvectors on cell interface:

\[ \tilde{F}_{i+1/2}^{CH} = \begin{bmatrix} 0 \tilde{f}_{i+1/2}^{CH} \\ 1 \tilde{f}_{i+1/2}^{CH} \\ 2 \tilde{f}_{i+1/2}^{CH} \end{bmatrix} = \bar{R}_{i+1/2} \tilde{W}_{i+1/2}^{CH} \quad (36) \]
The right eigenvectors are given by:

\[
\tilde{\mathbf{R}}_{i+1/2} = [\bar{\mathbf{r}}_0, \bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2], = \begin{bmatrix}
1 & 1 & 1 \\
\bar{u} - \bar{c} & \bar{u} & \bar{u} + \bar{c} \\
\bar{u}^2 + \bar{c}^2 - \bar{u}\bar{c} & \bar{u}^2 & \bar{u}^2 + \bar{c}^2 - \bar{u}\bar{c} \\
\end{bmatrix}
\]  

(37)

The whole process of the characteristic-wise WENO reconstruction can be summarized into one formula:

\[
\tilde{\mathbf{F}}_{i+1/2}^{CH} = \begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix}_{j+i+1/2} \tilde{\mathbf{R}}_{i+1/2} \mathbf{\tilde{L}}_{i+1/2} = 2 \sum_{k=0}^{2} \tilde{\omega}^C_{k,i+1/2} \tilde{\mathbf{F}}_k^{CP},
\]

(38)

where the superscript 'CH' stands for characteristic-wise reconstruction. In the following parts, the characteristic-wise method will be referred to as the CH method.

Compared to the CP method, the CH method requires several matrix constructions and projections resulting in much more computational cost. Although the computational cost of the CH method is high, it does not produce numerical oscillations.

3. The new scheme

On one hand, in spite of its high computational cost, the merit of the CH method is its non-oscillatory property for discontinuities. On the other hand, the advantage of the CP method is its computational efficiency regardless of the numerical oscillations it brings. Therefore, a straightforward strategy is to combine these two methods to get rid of their drawbacks and utilize their advantages. In this section, the two methods will be analyzed to show where their differences and similarities lie. A new adaptive method that combines these two method will be introduced according to our analysis.

3.1. Comparison of the two reconstruction methods

Eq. (38) can be written in a more compact form by considering:

\[
\tilde{\omega}^C_k = \tilde{\mathbf{R}}_{i+1/2} \mathbf{\tilde{L}}_{i+1/2}.
\]

(39)

Substituting Eq. (39) into Eq. (38), we have:

\[
\tilde{\mathbf{F}}_{i+1/2}^{CH} = \begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix}_{j+i+1/2} \tilde{\omega}^C_k \tilde{\mathbf{F}}_k^{CP},
\]

(40)
Comparing Eq.(23) and Eq.(40), one immediately finds that the difference between the CP method and the CH method lies in their WENO weight matrices: $\tilde{\omega}_k^{CH}$ and $\omega_k^{CP}$.

For smooth region, the WENO weights approximate the linear weights. Therefore, the WENO weight matrix $\omega_k^{CP}$ of the CP method becomes:

$$\omega_k^{CP} = c_k I,$$

where $I$ is an identity matrix. For the CH method, its weight matrix $\tilde{\omega}_k^{CH}$ reads:

$$\tilde{\omega}_k^{CH} = \tilde{R}_{i+1/2} c_k \tilde{LL}_{i+1/2} = c_k \tilde{R}_{i+1/2} \tilde{LL}_{i+1/2},$$

Considering that $\tilde{R}_{i+1/2} = \tilde{L}_{i+1/2}^{-1}$,

we have:

$$\omega_k^{CP} = \tilde{\omega}_k^{CH}.$$

Eq.(44) reveals that for smooth flow the numerical fluxes obtained by the two methods are the same:

$$\begin{align*}
\omega_k^{CP} &= \tilde{R}_{i+1/2} \omega_k^{CH} \tilde{L}_{i+1/2} \\
\tilde{F}_{CP}^{i+1/2} &= \tilde{F}_{CH}^{i+1/2}
\end{align*}$$

For discontinues region, it is a different story. Let us rewrite (39) as:

$$\begin{align*}
\tilde{\omega}_k^{CH} &= \tilde{R}_{i+1/2} \omega_k^{CH} \tilde{L}_{i+1/2} \\
&= \begin{bmatrix}
\tilde{r}_{00} & \tilde{r}_{01} & \tilde{r}_{02} \\
\tilde{r}_{10} & \tilde{r}_{11} & \tilde{r}_{12} \\
\tilde{r}_{20} & \tilde{r}_{21} & \tilde{r}_{22}
\end{bmatrix} \begin{bmatrix}
0 & \omega_k^{CH} & 1 \\
1 & \omega_k^{CH} & 2 \omega_k^{CH}
\end{bmatrix} \begin{bmatrix}
\tilde{l}_{00} & \tilde{l}_{01} & \tilde{l}_{02} \\
\tilde{l}_{10} & \tilde{l}_{11} & \tilde{l}_{12} \\
\tilde{l}_{20} & \tilde{l}_{21} & \tilde{l}_{22}
\end{bmatrix}
\end{align*}$$

Assume that numerical fluxes obtained by the two methods in discontinuous situation are still the same, this means that Eq.(45) holds:

$$\omega_k^{CP} = \tilde{\omega}_k^{CH} = \tilde{R}_{i+1/2} \omega_k^{CH} \tilde{L}_{i+1/2}.$$

Inserting Eq.(46) into Eq.(47), we have:

$$\begin{align*}
\begin{bmatrix}
0 & \omega_k^{CP} & 1 \\
1 & \omega_k^{CP} & 2 \omega_k^{CP}
\end{bmatrix} &= \begin{bmatrix}
\tilde{r}_{00} & \tilde{r}_{01} & \tilde{r}_{02} \\
\tilde{r}_{10} & \tilde{r}_{11} & \tilde{r}_{12} \\
\tilde{r}_{20} & \tilde{r}_{21} & \tilde{r}_{22}
\end{bmatrix} \begin{bmatrix}
0 & \omega_k^{CH} & 1 \\
1 & \omega_k^{CH} & 2 \omega_k^{CH}
\end{bmatrix} \begin{bmatrix}
\tilde{l}_{00} & \tilde{l}_{01} & \tilde{l}_{02} \\
\tilde{l}_{10} & \tilde{l}_{11} & \tilde{l}_{12} \\
\tilde{l}_{20} & \tilde{l}_{21} & \tilde{l}_{22}
\end{bmatrix}
\end{align*}$$
Given that $\vec{R}_{i+1/2} \vec{L}_{i+1/2} = \vec{I}$, Eq. (48) becomes:

$$
\begin{bmatrix}
\bar{l}_{00} & \bar{l}_{01} & \bar{l}_{02} \\
\bar{l}_{10} & \bar{l}_{11} & \bar{l}_{12} \\
\bar{l}_{20} & \bar{l}_{21} & \bar{l}_{22}
\end{bmatrix}
\begin{bmatrix}
0 \omega_{k}^{CP} \\
1 \omega_{k}^{CP} \\
2 \omega_{k}^{CP}
\end{bmatrix}
= 
\begin{bmatrix}
0 \omega_{k}^{CH} \\
1 \omega_{k}^{CH} \\
2 \omega_{k}^{CH}
\end{bmatrix}
\begin{bmatrix}
\bar{l}_{00} & \bar{l}_{01} & \bar{l}_{02} \\
\bar{l}_{10} & \bar{l}_{11} & \bar{l}_{12} \\
\bar{l}_{20} & \bar{l}_{21} & \bar{l}_{22}
\end{bmatrix}
$$

(49)

After some manipulations, we obtain the following relation:

$$
\begin{bmatrix}
\bar{l}_{00} 0 \omega_{k}^{CP} - \bar{l}_{00} 0 \omega_{k}^{CH} \\
\bar{l}_{10} 0 \omega_{k}^{CP} - \bar{l}_{10} 1 \omega_{k}^{CP} - \bar{l}_{11} 1 \omega_{k}^{CH} \\
\bar{l}_{20} 0 \omega_{k}^{CP} - \bar{l}_{20} 2 \omega_{k}^{CP} - \bar{l}_{21} 2 \omega_{k}^{CH}
\end{bmatrix}
= 0,
$$

(50)

which gives:

$$
0 \omega_{k}^{CP} = 1 \omega_{k}^{CP} = 2 \omega_{k}^{CP} = 0 \omega_{k}^{CH} = 1 \omega_{k}^{CH} = 2 \omega_{k}^{CH}.
$$

(51)

It can be easily verified that (51) does not always hold as the degrees of discontinuity of each flow component are not always the same like the contact discontinuity. Therefore, for discontinuous regions, the numerical fluxes calculated by the two methods are different. As the CH method does not produce numerical oscillations near discontinuities, it is the desirable method for such circumstances.

3.2. Adaptive characteristic-wise reconstruction

According to the analysis above, the CP method and the CH method are the same in smooth region and different in discontinuous region. As the CH method does not result in numerical oscillations, it is preferred in discontinuous region. Although the CP method produces oscillations in some discontinuous regions, it is more computationally efficient than the CH method.

A straightforward strategy to combine the advantages of these two methods is to use the CP method in smooth region and to use the CH method in discontinuous region. Therefore, a shock indicator or switch function is needed to discern these regions. To discern discontinuity accurately, we employ the switch function proposed in [23]:

$$
\theta(x, z) = \frac{1}{1 + \frac{1}{z^2}}, \quad z \geq 1.
$$

(52)

Fig. illustrates $\theta$ with different values of parameters. It varies from 1 to 0 rapidly and smoothly with increasing $x$. Different from tradition switches
proposed for hybrid schemes, $\theta$ is free of empirical parameters and has a high accuracy. Details about this function can be found in [23].

Here, we take:

$$x = \left( \sum_k \alpha_k - 1 \right)$$

and get

$$\theta = \frac{1}{1 + (\sum_k \alpha_k - 1)^z}$$

(53)

where $\alpha_k$ is the WENO non-linear weights Eq.(22). Note that for smooth region:

$$\sum_k \alpha_k \approx 1,$$

(54)

and for discontinuous region:

$$\sum_k \alpha_k \to \infty.$$  

(55)
This property leads to

\[ \theta = \begin{cases} 
1, & \text{smooth} \\
0, & \text{discontineous}. 
\end{cases} \]  
(56)

Taking advantage of Eq. (56), we propose the following adaptive approach:

\[ \tilde{F}_{i+1/2} = \begin{cases} 
\tilde{F}_{C \, i+1/2}, & \theta > \theta_0 \\
\tilde{F}_{C \, i+1/2}, & \theta \leq \theta_0 
\end{cases} \]  
(57)

where \( \theta_0 \) is a threshold which can be simply taken to be:

\[ \theta_0 = \theta(1, z) = 0.5. \]  
(58)

Bearing in mind that the \( \theta \) function can be interpreted as how much we can believe the function being measure is smooth, it is not hard to understand the choice of \( \theta_0 \) made above.

Given the adaptive method Eq. (57), an issue to be addressed is that based on which variable to calculate the non-linear weights \( \alpha_k \) and \( \theta \). In [20], a set of WENO weights for all fluxes to be reconstructed are computed based on a set of new global smoothness indicators which are combinations of smoothness indicators of each field. Although this method reduced the number of WENO weights needed to be computed, it still calculates all the smoothness indicators. A similar method can be found in [19]. In this paper, we propose the following variable as the input of the WENO weights:

\[ G^\pm = \rho + (\rho u^2 + p \pm \alpha \rho u). \]  
(59)

Note that the \( \pm \) sign denotes \( G \) for the positive and the negative fluxes Eq. (9) respectively. Based on \( G^\pm \), the smoothness indicators \( \beta_k^\pm \) for the fluxes are then calculated. Finally, Two set of uniform WENO weights \( \omega_k^{\pm, CP} \) are calculated based on \( \beta_k^\pm \) for all the components of the fluxes \( \bar{F}_i^\pm \):

\[ 0, \omega_k^{\pm, CP} = \omega_k^{\pm, CP} = \omega_k^{\pm, CP} = \omega_k^{\pm, CP}, \quad k = 0, 1, 2. \]  
(60)

In this way, we only have to calculate two sets of smoothness indicators and two sets of WENO weights. Noted that there are also two \( \theta \)s should be calculated: \( \theta^\pm \).

The final adaptive characteristic-wise WENO scheme (referred to as AdaWENO) algorithm is given in Algorithm [1]
Algorithm 1 The AdaWENO algorithm

1: Calculate $G^\pm$ for the positive and the negative splited fluxes $\vec{F}^\pm$ respectively according to Eq.(59);
2: for Each stencil $S_i = (i-2, i-1, i, i+1, i+2), i = 1$ to $n$ do
3: Calculate $\beta^\pm_{k,i}$ for the positive and the negative splited fluxes $\vec{F}^\pm$ based on the calculated $G^\pm$ according to Eq.(19)-(22);
4: Calculate $\omega^\pm_{k,i}, CP$ and $\theta^\pm_{i}$ according to Eq.(21) and Eq.(53).
5: for Each stencil $S_i = (i-2, i-1, i, i+1, i+2), i = 1$ to $n$ do
6: if $\theta^\pm_{i} \geq 0.5$ then
7: $s_{i+1/2}^\pm = \sum_k \omega^\pm_{k,i} s_{k,i+1/2}^\pm, s = 0, 1, 2$
8: else
9: $s_{i+1/2}^\pm = s_{i+1/2}^\pm, s = 0, 1, 2$

4. Numerical tests

In this section, several numerical tests including one dimensional and two dimensional problems are considered to validate and evaluate the performance of the new method. Numerical results are compared between the CP method, the CH method, and AdaWENO.

The third order TVD Runge-Kutta method [24] is used for time advancing:

$$u^{(1)} = u^n + \Delta t L(u^n),$$
$$u^{(2)} = \frac{3}{4} u^n + \frac{1}{4} u^{(1)} + \frac{1}{4} \Delta t L(u^{(1)}),$$
$$u^{n+1} = \frac{1}{3} u^n + \frac{2}{3} u^{(2)} + \frac{2}{3} \Delta t L(u^{(2)}).$$

4.1. One dimensional cases

4.1.1. The Sod problem

The initial condition of the Sod problem is given by:

$$(\rho, u, p) = \begin{cases} 
  (1, 0, 1) & x \leq 0 \\
  (0.125, 0, 0.1) & x > 0 
\end{cases}$$

The final solution time is $t = 0.14$. The number of uniform grids is $N = 200$. The reference result is calculated with the CH method with $N = 2000$. 

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Figure 2: Density distributions of the Sod problem at t=0.14, N=200
Fig. 2 shows density distributions of different methods at $t = 0.14$. It can be seen that the CP method produces numerical oscillations near the contact discontinuity while the other two methods do not. Zoomed distributions are shown in Fig. 3. AdaWENO captures the discontinuity without obvious oscillations as with the CH method.

4.1.2. The Lax problem

The initial condition of the Lax problem is given by:

$$\begin{align*}
(\rho, u, p) &= \begin{cases} 
(0.445, 0.698, 3.528) & x \leq 0 \\
(0.5, 0, 0.571) & x > 0 
\end{cases} 
\end{align*}$$

(65)

Solutions are integrated to $t = 0.13$. The number of grid points is $N = 200$. The reference result is computed by the CH method with $N = 2000$. 

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Figure 4: Density distributions of the Lax problem at $t=0.13$, $N=200$
Figure 5: Zoomed density distributions near the contact discontinuity and the shock of the Lax problem

Density distributions calculated by different methods at $t = 0.13$ are shown in Fig.4. As with being shown in the Sod problem, the CP method also produces numerical oscillations in vicinity of the contact discontinuity while the other two do not. Zoomed results near the contact discontinuity and the shock wave are shown in Fig.5. The new method captures the contact discontinuity and the shock wave very well. AdaWENO and the CH method show almost identical distribution.

4.1.3. Shu-Osher problem

The initial condition of the Shu-Osher problem is given by:

$$
(\rho, u, p) = \begin{cases} 
\left( \frac{27}{7}, \frac{4\sqrt{15}}{9}, \frac{31}{3} \right) & x < -4 \\
\left( 1 + \frac{1}{5} \sin 5x, 0, 1 \right) & x \geq -4
\end{cases}
$$

(66)
Solutions are integrated to $t = 1.8$. The number of grid points is $N = 200$. The reference result is computed by the CH method with $N = 2000$.

Figure 6: Density distributions of the Shu-Osher problem at $t=1.8$, $N=200$
4.2. Two dimensional cases

4.2.1. Shock-vertex interaction

This problem describes the interaction of a stationary shock and a vertex. The computational domain is set to be $[0, 2] \times [0, 1]$. A Mach 1.1 shock is positioned at $x = 0.5$ and normal to the $x$-axis. Its left state is $(\rho, u, v, p) = (1, 1.1\sqrt{\gamma}, 0, 1)$ and the right states is $(\rho, u, v, p) = (1.1691, 1.1133, 0, 1.245)$. A small vortex is superposed to the flow left to the shock and centers at
\((x, y_c) = (0.25, 0.5)\). The initial condition to the left of the shock is given by:

\[
\rho = \left(1 - (\gamma - 1) \frac{\varepsilon^2}{4\alpha\gamma} e^{2\alpha(1-\tau^2)}\right)^{1/(\gamma-1)},
\]

\(u = 1.1\sqrt{\gamma + \varepsilon \tau} e^{\alpha(1-\tau^2)} \frac{(y - y_c)}{r},\)

\(v = -\varepsilon \tau e^{\alpha(1-\tau^2)} \frac{(x - x_c)}{r},\)

\(p = \rho^\gamma,\)

where \(\tau = r/r_c\) and \(r = \sqrt{(x - x_c)^2 + (y - y_c)^2}\). \(\varepsilon = 0.3\) indicates the strength of the vortex, \(\alpha = 0.204\) controls the decay rate of the vortex, and \(r_c = 0.05\) is the critical radius for which the vortex has the maximum strength. A grid of \((N_x, N_y) = (250, 100)\) is used. The reference result is computed on a refined grid of \((N_x, N_y) = (2000, 400)\) with the CH method.
(a) Component-wise reconstruction

(b) Characteristic-wise reconstruction
Figure 8: Pressure contours of the shock-vortex interaction problem at $t = 0.6$ with 90 equally separated levels ranging form 1.19 to 1.37.
Figure 9: Pressure distributions at $y=0.5$, one of every three grid points is displayed.
Figure 10: Zoomed pressure distributions at y=0.5

(a) Near the shock

(b) Vortex center
Fig. 8 shows the pressure contours of different methods at $t = 0.6$. The present method obtains an almost identical result to the CH method. The CP method also gives a similar result, however, with more numerical noises. Pressure distributions of different results at $y = 0.5$ are illustrated in Fig. 9. It can be seen that all methods capture the shock and the vortex well. Zoomed pressure distributions near the shock and the center of the vortex are shown in Fig. 10. We find that the CH method and AdaWENO is slightly better than the CP method in these regions.

4.2.2. Double Mach reflection

The double mach reflection test is a mimic of the planar shock reflection in the air from wedges. It is a widely used benchmark to test the ability of shock capturing and the small scale structure resolution of a certain scheme. In the present simulation, the computation domain is taken as $[0, 4] \times [0, 1]$. The lower boundary is set to be a reflecting wall starting from $x = \frac{1}{6}$. At $t = 0$, a right-moving 60° inclined Mach 10 shock is positioned at $(\frac{1}{6}, 0)$. The upper boundary is set to describe the exact motion of the Mach 10 shock. The left boundary at $x = 0$ is assigned with post-shock values. Zero gradient outflow condition is set at $x = 4$. Readers may refer to [25, 26] for detailed descriptions of the double Mach reflection problem. An uniform grid is used with $\Delta x = \Delta y = \frac{1}{240}$. The reference result is given by the CH method with $\Delta x = \Delta y = \frac{1}{480}$.
Figure 11: Density contours of the double Mach reflection problem at $t=0.2$, ranging from $\rho = 2.1379$ to 24 with 90 equally separated levels.
Fig. 11 shows density contours of different methods at $t = 0.2$. All methods capture discontinuities. The result of the CP method shows more numerical noises behind the reflected shock than the other two. Density contours of the up-rolling region of different methods are shown in Fig. 12. The CP method result shows obvious oscillations near the triple-wave interaction point while the CH method and the present method gives a similar and clear
**Table 1: Averaged CPU times (in seconds) per time step for the double Mach reflection problem**

| Method   | CPU time(s)/time step |
|----------|-----------------------|
| CP       | 0.59                  |
| CH       | 1.09                  |
| AdaWENO  | 0.37                  |

structure. Note that the two characteristic methods show few K-H instability vortices near the contact line than the CP method. It seems that the numerical oscillations produced by the CP method disturb the contact line and therefore result in more instability structures. However, the characteristic methods produce less oscillations which are not strong enough to give rise to the K-H instability structures. CPU times of different methods are given in Tab.1 AdaWENO is about 3 times faster than the CH method and is about 40% faster than the CP method.

4.2.3. Shock/shear layer interaction

The shock wave impingement problem is designed to measure the resolution of schemes when shock waves interact with vortices \[27\]. A Mach 0.6 shear layer evolves and impacts on an oblique shock. The vortices produced by the shear layer instability pass, firstly, through the oblique shock and then a second shock reflected from the slip wall at the lower boundary. The computation domain is \([0, 200] \times [-20, 20]\). At \(x = 0\), the inlet condition is specified as:

\[
u = 2.5 + 0.5\tanh(2y)\tag{71}\]

For the upper stream \((y > 0)\), \(\rho = 1.6374\), \(p = 0.3327\) and for lower stream \((y < 0)\), \(\rho = 0.3626\), \(p = 0.3327\). Post shock condition \((\rho, u, v, p) = (2.1101, 2.9709, -0.1367, 0.4754)\) is set at the upper boundary, and slip wall condition is applied at the lower boundary. Besides, fluctuations are added to the vertical velocity component at the inlet:

\[
v' = \sum_{k=1}^{2} a_k \cos(2\pi kt/T + \phi_k) \exp(-y^2/b)\tag{72}\]

\[b = 10, a_1 = a_2 = 0.05, \phi_1 = 0, \phi_2 = \pi/2\tag{73}\]
in which $T = \lambda / u_c$ is the period, $\lambda = 30$ is the wavelength, $u_c = 2.68$ is the convective velocity. The two dimensional Navier-Stokes equations are solved with the Reynolds number $Re = 500$ and the Prandtl Number $Pr = 0.72$. A equally spaced grid with grid number $(N_x, N_y) = (500, 100)$ is used. The reference result is calculated on a refined grid with $(N_x, N_y) = (2000, 400)$ by the CH method.
Density contours are shown in Fig. 13 at $t = 120$. It can be observed that the CH method and AdaWENO obtain similar results which is very comparable to the reference result, while the self-similar structure of the
downstream vortices are twisted in the CP method result. CPU times of different methods are given in Table 2, as having been shown in the double Mach reflection case, AdaWENO is about 3 times faster than the CH method and is about 40% faster than the CP method.

5. Conclusion

In this paper, an adaptive characteristic-wise reconstruction WENO scheme (AdaWENO) is proposed. The new method uses a high-accuracy switch function to determine the reconstruction method to be used. A new way of calculating the WENO weights is also proposed so as to reduce the overall computational cost of the WENO scheme. Numerical tests show that AdaWENO achieves high efficiency without producing any obvious numerical oscillation. AdaWENO is about 3 times faster than the CH method and about 40% faster than the CP method. With good performance and high efficiency, AdaWENO is suitable for large scale compressible flow simulations.

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