δ-derivations of semisimple finite-dimensional structurable algebras.

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Abstract. In this paper we show the absence of nontrivial δ-derivations of semisimple finite-dimensional structurable algebras over an algebraically closed field of characteristic not equal to 2,3,5.

Keywords: δ-derivation, structurable algebra.

1. Introduction

The concept of δ-derivation first appeared in papers by V. Filippov \cite{Filippov1981, Filippov1982}, as a generalization of ordinary derivations. Recall that for a fixed δ of the main field \( F \), under δ-derivation of algebra \( A \) we understand a linear mapping \( \phi \), which satisfies the condition

\[
\phi(xy) = \delta(\phi(x)y + x\phi(y))
\]

for arbitrary elements \( x, y \in A \). In article \cite{Filippov1981} V. Filippov proved that every prime Lie algebra has no nonzero δ-derivations, if \( \delta \neq -1,0,\frac{1}{2},1 \). Subsequently in papers \cite{Shestakov1988, Shestakov1989, Shestakov1990} was given the description of δ-derivations of semisimple finite-dimensional Jordan algebras and δ-(super-)derivations of semisimple finite-dimensional Jordan superalgebras. Later, A. Shestakov generalized some results of \cite{Shestakov1988} and described ternary derivations of simple finite-dimensional Jordan algebras \cite{Shestakov1989}. δ-derivations of Lie superalgebras were considered by I. Kaygorodov \cite{Kaygorodov2010, Kaygorodov2011} and P. Zusmanovich \cite{Zusmanovich2011}. The analogue of δ-derivation for \( n \)-ary algebras was investigated in \cite{Kaygorodov2011, Kaygorodov2012}. We note that some generalizations of δ-derivations were proposed in articles \cite{Shestakov1990, Shestakov1991, Shestakov1992} and δ-derivations were used for studying prime Lie algebras by V. Filippov \cite{Filippov1981}, for construction of post-Lie structures by D. Burde and K. Dekimpe \cite{Burde2013}, for studying speciality of Jordan superalgebras by I. Kaygorodov and V. Zhelyabin \cite{Kaygorodov2011}, in analysis of the derivations of current Lie algebras by P. Zusmanovich \cite{Zusmanovich2011}, and for studying homotopes of Novikov algebras by V. Filippov and V. Sereda \cite{Filippov2012}.

2. Examples and definitions

The class of structurable algebras was introduced in 1978 by B. Allison \cite{Allison1978}. It contains such objects as tensor multiplication of composition algebras, 56-dimensional Freudenthal module over \( E_7 \) with natural binary operation, a 35-dimensional algebra \( T(C) \). Also, for these algebras a natural generalization of Kantor-Koecher-Tits construction holds.

Structurable algebras are algebras with the unit, which we are going to designate as \( e \), and involution \( \bar{\cdot} \), which satisfy the identity:

\[
[T_z,V_{x,y}] = V_{T_zx,y} - V_{x,T_zy}, \text{ where } T_z,V_{x,y} \in \text{End}(A),
\]

\[
V_{x,y}(z) = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y, T_z = V_{z,1} \text{ for } x, y, z \in A.
\]

Let \( (A,\bar{\cdot}) \) be an algebra with involution. We can consider vector space \( A \) as a direct sum \( A = H \oplus S \), where \( H = \{ a \in A|\bar{a} = a \} \) is the set of symmetric elements and \( S = \{ a \in A|\bar{a} = -a \} \) is the set of antisymmetric elements.

Let us give some examples of structurable algebras (see \cite{Kaygorodov2011, Kaygorodov2012}):

S1) Let \( (A,\bar{\cdot}) \) be an unital associative algebra with the involution. Then \( (A,\bar{\cdot}) \) is a structurable algebra.

S2) Let \( J \) be a Jordan algebra and \( \bar{\cdot} = \text{id} \). For \( (J,\bar{\cdot}) \) the identity \cite{Filippov1981} turns into the following identity

\[
[L_a,V_{a,b}] = V_{ca,b} - V_{a,cb}
\]

which holds in an arbitrary Jordan algebra. Therefore \( (J,\bar{\cdot}) \) is a structurable algebra.

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S3) Let \((E, \cdot)\) be an associative algebra with involution and \(W\) a left \(E\)-module relative to the operation \(\circ : E \times W \rightarrow W\).
Let \(W\) possesses a Hermitian form \(h\), that is a bilinear mapping \(h : W \times W \rightarrow E\) such that
\[
\begin{align*}
    h(w_1, w_2) &= h(w_2, w_1), \\
    h(e_1 \circ w_1, w_2) &= e_1 h(w_1, w_2)
\end{align*}
\]
for any \(w_1, w_2 \in W\) and \(e_1 \in E\). The operations of multiplication and involution on the space \(A = E \oplus W\) are defined as follows:
\[
\begin{align*}
    (e_1, w_1) \cdot (e_2, w_2) &= (e_1 e_2 + h(w_2, w_1)), e_2 \circ w_1 + \overline{e_1} \circ w_2), \\
    (e, w) &= (\overline{e}, w).
\end{align*}
\]

The algebra \((A, \cdot)\) is a structurable algebra. It is a generalization of Jordan algebra of bilinear form and is called the algebra of Hermitian form. The algebra of Jordan form is simple if the Hermitian form \(h\) on the left \(E\)-module is nondegenerate and \((A, \cdot)\) is a central simple algebra with involution.

S4) Let \((G_1, \cdot)\) and \((G_2, \cdot)\) be composition algebras with standard involution (for more about the composition algebras see, for example, [23]). Let us define on algebra \(A = G_1 \otimes G_2\) the involution \(x_1 \otimes \overline{x_2} = \overline{x_1} \otimes x_2\). Therefore \((A, \cdot)\) is a structurable algebra.

S5) Let \(J, J'\) be vector spaces over \(F\) with trilinear forms \(N\) and \(N'\) which are related by a nondegenerate bilinear form \(T : J \times J' \rightarrow F\). For \(j, k \in J, j', k' \in J',\) we define \(j \times k \in J'\) and \(j' \times k' \in J\) by the condition
\[
T(l, j \times k) = N(j, k, l), T(j' \times k', l') = N'(j', k', l')
\]
for any \(l \in J, l' \in J'\). On \(J\) and \(J'\) we define the operation \(#\) as:
\[
j^# = \frac{1}{2} j \times j, j'^# = \frac{1}{2} j' \times j', \text{ for } j \in J, j' \in J'.
\]
The triple \((T, N, N')\) is called an admissible triple defined on the pair of spaces \((J, J')\) if \(N\) and \(N'\) are nontrivial and satisfy the conditions:
\[
\begin{align*}
    (j^#)^# &= \frac{1}{6} N(j, j, j), \\
    (j'^#)^# &= \frac{1}{6} N'(j', j', j') j'
\end{align*}
\]
for \(j \in J, j' \in J'\).

Let \((T, N, N')\) be an admissible triple on a pair of spaces \((J, J')\). On a space of matrices
\[
A = \left\{ \begin{bmatrix} \alpha & j \\ j' & \beta \end{bmatrix} : \alpha, \beta \in F, j \in J, j' \in J' \right\}
\]
the multiplication
\[
\begin{bmatrix} \alpha & j \\ j' & \beta \end{bmatrix} \begin{bmatrix} \gamma & k \\ k' & \delta \end{bmatrix} = \begin{bmatrix} \alpha \gamma + T(j, k') & \alpha k + \delta j + j' \times k' \\ \gamma j' + \beta k' + j \times k & T(k, j') + \beta \delta \end{bmatrix}
\]
and involution
\[
\begin{bmatrix} \alpha & j \\ j' & \beta \end{bmatrix}^# = \begin{bmatrix} \beta & j \\ j' & \alpha \end{bmatrix}
\]
are defined. The obtained structurable algebra \((A, \cdot)\) is called an algebra constructed from an admissible triple \((T, N, N')\) on a pair of spaces \((J, J')\).

S6) Let \(C\) be a Cayley-Dickson algebra with an involution (about a Cayley-Dixon algebra see [23]). The set \(S\) of antisymmetric elements is a 7-dimensional simple non-Lie Malcev algebra relative to operation commutation \([\cdot, \cdot]\) in \(C\) (see [24]); it is \(S(-)\). On \(S(-)\) the symmetrical nondegenerate invariant bilinear form \((\cdot, \cdot)\) is such that for \(x, y \in S\) the condition
\[
[[x, y], y] = (y, y)x - (x, y)y
\]
is satisfied. Let \(M\) be a subspace of tensor product \(S \otimes S\) which is generated by the set \(\{s \otimes r - r \otimes s : s, r \in S\}\). We let \(H = S \otimes S/M\) and on the direct sum of spaces \(H \oplus S\) the commutative \(\circ\) and anticommutative \([\cdot, \cdot]\) operations are defined by the following conditions:
\[
[s_1, s_2] = [s_1, s_2],
\]
\[ [s_1 \otimes s_2, s_3 \otimes s_4] = \]
\[ = (s_1, s_3)[s_2, s_4] + (s_1, s_4)[s_2, s_3] + (s_2, s_3)[s_1, s_4] + (s_2, s_4)[s_1, s_3], \]
\[ s_1 \otimes s_2 = s_1 \otimes s_2, \]
\[ s_1 \otimes (s_1 \otimes s_2) = \frac{1}{2}(s_1, s_2)s + \frac{1}{4}(s_1)s_2 + \frac{1}{4}(s_2)s_1, \]
\[ (s_1 \otimes s_2) \odot (s_3 \otimes s_4) = \]
\[ = \frac{1}{4}[s_1, s_3] \otimes [s_2, s_4] + \frac{1}{4}[s_1, s_4] \otimes [s_2, s_3] + \frac{1}{2}(s_1, s_2)s_3 \otimes s_4 + \frac{1}{2}(s_3, s_4)s_1 \otimes s_2 \]
for \( s, s_1, s_2, s_3, s_4 \in S \), where on the right side of equality under \([\cdot, \cdot]\) we understand the commutator in algebra \( C \).

The operations of multiplication and involution in \( H \oplus S \) are defined in the following way:
\[ xy = x \odot y + \frac{1}{2}[x, y], \]
\[ h + s = h - s, \quad \text{where} \quad x, y \in H \oplus S, h \in H, s \in S. \]

So obtained algebra with involution which is built by means of Cayley-Dikson algebra \( C \) is designated as \( T(C) \).

Let us note that according to the results of [21, 22], the algebras of types \( S1 \)–\( S6 \) exhaust all simple finite-dimensional structurable algebras over algebraically closed fields of characteristic not equal to \( 2, 3, 5 \).

As was noted above, for a fixed element \( \delta \) from the main field under \( \delta \)-derivation of algebra \( A \) we understand a linear mapping \( \phi : A \rightarrow A \) which for arbitrary \( x, y \in A \) satisfies the condition
\[ \phi(xy) = \delta(\phi(x)y + x\phi(y)). \]

Centroid \( \Gamma(A) \) of algebra \( A \) is a set of linear mappings \( \chi : A \rightarrow A \) with condition \( \chi(ab) = \chi(a)b = a\chi(b) \). It is clear that any element of the centroid of an algebra is a \( \frac{1}{2} \)-derivation. Any endomorphism \( \phi \) of algebra \( A \) such that \( \phi(A^2) = 0 \) is a 0-derivation. Any derivation of algebra \( A \) is a 1-derivation. Nonzero \( \delta \)-derivation \( \phi \) is a nontrivial \( \delta \)-derivation if \( \delta \neq 0, 1 \) and \( \phi \notin \Gamma(A) \).

3. Main Lemmas

Here and below, all algebras are considered over an algebraically closed field \( F \) with the characteristic not equal to \( 2, 3, 5 \), although some of the results of the following lemmas will be correct with some relaxing of the required conditions on the field.

Make a trivial direct calculation and using [4, Theorem 2.1], we can obtain that if \( \phi \) is a nontrivial \( \delta \)-derivation of structurable algebra \( A \), then \( \delta = \frac{1}{2} \) and for any element \( x \in A \) it is true that \( \phi(x) = ax = xa \), when \( a \in A \) is a certain fixed element.

**Lemma 1.** Let \( A \) be an algebra of nondegenerate Hermitian form on a left \( E \)-module, where \( (E, \cdot) \) is a simple central associative algebra with the involution. Then every \( \frac{1}{2} \)-derivation of \( A \) is trivial.

**Proof.** We will denote the unit of the algebra \( A \) and of the algebra \( E \) as \( e \). Then we suppose
\[ \phi(e) = (ae + e^*, w^*), \quad \text{where} \quad a \in F, e^* \in E, w^* \in W \]
and the projection of \( e^* \) on the vector space generated by \( e \) is 0. It is essential to prove \( e^* = w^* = 0 \). Let us specify the mapping \( \psi = \phi - \phi_0 \), where \( \phi_0(x) = ax \). The mapping \( \psi \) is a \( \frac{1}{2} \)-derivation as the sum of two \( \frac{1}{2} \)-derivations. It is easy to see,
\[ \psi(a, 0) = \psi(e) \odot (a, 0) = (e^*a, a \circ w^*), \]
but from the other side
\[ \psi(a, 0) = (a, 0) \odot \psi(e) = (ae^*, \overline{a} \circ w^*). \]
We see that \( e^* \) is in the commutative center of the central algebra \( (E, \cdot) \) and hence, \( e^* = \beta e \), where \( \beta \in F \) and \( \beta = 0 \). Then we can consider \( \psi(e) = (0, w^*) \). Let us notice the following:
\[ \psi((a, 0) \odot (0, w^*)) = ((a, 0) \odot (0, w^*)) \odot \psi(e) = (h(w^*, \overline{a} \circ w^*), 0) = (h(w^*, w^*)a, 0), \]
(3)
and also
\[ \psi((a, 0) \bullet (0, w^*)) = \psi(e) \bullet (((a, 0) \bullet (0, w^*)) = (h(x \circ w^*, w^*), 0) = (\overline{\psi h}(w^*, w^*), 0). \]
Comparing the components from the algebra \( E \) we get:
\[ h(w^*, w^*)a = \overline{\psi h}(w^*, w^*). \tag{4} \]
Let us show that \( h(w^*, w^*) \) is in the commutative center of the algebra \( E \). It is easy to see that
\[ \psi((a, 0) \bullet (0, w^*)) = \frac{1}{2} \psi(a, 0) \bullet (0, w^*) + \frac{1}{2} (a, 0) \bullet \psi(0, w^*) = (\frac{1}{2} h(w^*, \circ w^*)a + \frac{1}{2} ah(w^*, w^*), 0). \]
Now, using (3) we can obtain that \( h(w^*, w^*) \) is in the commutative center \( Z(E) \).
Let us show that \( \overline{\psi h}(w^*, w^*) \) is an ideal in \( E \). It is easy to see that \( a \overline{\psi h}(w^*, w^*) \subseteq \overline{Eh}(w^*, w^*) \). Considering \( \overline{Eh}(w^*, w^*)a \) and using the associative property of the algebra \( E \) we have
\[ (\overline{Eh}(w^*, w^*)a = E(h(w^*, w^*)a) = E(ah(w^*, w^*)) \subseteq \overline{Eh}(w^*, w^*). \]
As \( E \) is a simple associative unital algebra, and for the ideal \( \overline{Eh}(w^*, w^*) \) either \( \overline{Eh}(w^*, w^*) = 0 \) or \( \overline{Eh}(w^*, w^*) = E \). The last case is impossible, because then the element \( h(w^*, w^*) \) not equal to zero, and consequently, is invertible. Then from (1) we can conclude that from \((a - \overline{\psi})h(w^*, w^*) = 0\) it follows that \( a - \overline{\psi} = 0 \). If the involution is identity then the algebra reduces to the case of a Jordan algebra, which is described in [3]. If we have non-identity involution, then \( h(w^*, w^*) = 0 \), which means that \( \psi(w^*) = 0 \). Let \( u \in W \) be an arbitrary element. Then
\[ \psi((0, u) \bullet (0, w^*)) = \frac{1}{2} \psi((0, u)) \bullet (0, w^*) \]
and
\[ ((0, u) \bullet (0, w^*)) \bullet (0, w^*) = 0. \]
Let us notice that \( h(W, Ew^*) \) is an ideal in \( E \). It is easy to see that:
\[ ah(W, E \circ w^*) = h(a \circ W, E \circ w^*) \subseteq h(W, E \circ w^*) \]
and
\[ h(W, E \circ w^*)a = \overline{\psi h}(E \circ w^*, W) \subseteq h(W, E \circ w^*). \]
As the Hermitian form \( h : W \times W \to E \) is nondegenerate, the ideal is nonzero and as the algebra \( E \) is simple, \( h(W, E \circ w^*) = E \). Then \( \sum_{i}(0, u_i) \bullet (0, a_i \circ w^*) = e \), and we may conclude that
\[ e = \sum_{i}(0, u_i) \bullet (0, a_i \circ w^*) = \sum_{i} h(a_i \circ w^*, u_i) = \sum_{i} a_i h(w^*, u_i). \]
We can say that \((0, w^*) \) is in the commutative center of algebra \( A \). It is easy to see that
\[ (0, w^*) = (0, (\sum_{i} a_i h(w^*, u_i) \circ w^*) = \sum_{i} (0, h(w^*, u_i) \circ w^*) \bullet (a_i, 0) = \sum_{i} (((0, u_i) \bullet (0, w^*)) \bullet (0, w^*)) \bullet (a_i, 0) = 0. \]
And so we get \( \phi(e) = ae \). The lemma is proved.

**Lemma 2.** The algebra of the tensor multiplication of composition algebras has no nontrivial \( \frac{1}{4} \)-derivations.

**Proof.** Let \( A, B \) be the composition algebras (about composition algebras see, for example, [23]) with the units \( 1_A, 1_B \) and \( e \) be the unit of the algebra \( A \otimes B \). If \( \phi \) is a \( \frac{1}{4} \)-derivation of the algebra \( A \otimes B \), then \( \phi(e) = \sum_{i} a_i \otimes b_i, a_i \in A, b_i \in B \).

It is easy to see that \( a_i \) and \( b_i \) are in the commutative centers of the algebras \( A \) and \( B \), respectively.

It is easy to see that
\[ \phi(ac \otimes 1_B) = \frac{1}{2} \phi(a \otimes 1_B)(c \otimes 1_B) + \frac{1}{2}(a \otimes 1_B)\phi(c \otimes 1_B). \]
From the previous relation we get
\[ (ac \otimes 1_B)\phi(1_A \otimes 1_B) = \phi(ac \otimes 1_B) = \phi((a \otimes 1_B)(c \otimes 1_B)) = \frac{1}{2}((a \otimes 1_B)\phi(1_A \otimes 1_B))(c \otimes 1_B) + \frac{1}{2}(a \otimes 1_B)((c \otimes 1_B)\phi(1_A \otimes 1_B)). \]
that is,
\[ \sum_i (ac) a_i \otimes b_i = \frac{1}{2} \sum_i (aa_i) c \otimes b_i + a(ca_i) \otimes b_i. \]

The last relation implies
\[ \sum_i (2ac) a_i - (aa_i)c - a(ca_i) \otimes b_i = 0 \quad \text{and} \quad 2(ac)a_i - (aa_i)c - a(ca_i) = 0. \]

Therefore, we can conclude that is, \((a, a, c) = 0\). As composition algebras are alternative, \((a, c, a_i) = 0 = (a, a, c)\). Thus, \(a_i\) is in commutative-associative center \(Z(A)\).

As the algebra \(A \otimes B\) is isomorphic to algebra \(B \otimes A\), analogously we get that \(b_i\) is in commutative-associative center \(Z(B)\).

Now we are going to show that the mapping \(\phi : A \otimes B \to A \otimes B\) which is defined by the rule

\[ \phi(x \otimes y) = (x \otimes y)(\sum_i a_i \otimes b_i), \] where \(a_i \in Z(A), b_i \in Z(B),\]

is the element of the centroid of the algebra \(A \otimes B\), that is a trivial \(\frac{1}{2}\)-derivation.

It is easy to see that
\[ \phi((a \otimes b)(c \otimes d)) = \phi(ac \otimes bd) = (ac \otimes bd)(\sum_i a_i \otimes b_i) = \sum_i aca_i \otimes bbd_i = \sum_i (aa_i \otimes bb_i)(c \otimes d) = \phi(a \otimes b)(c \otimes d). \]

Analogously, we show that \(\phi((a \otimes b)(c \otimes d)) = (a \otimes b)\phi(c \otimes d)\). The lemma is proved.

**Lemma 3.** The algebra of an admissible triple \((T, N, N')\) has nontrivial \(\frac{1}{2}\)-derivations.

**Proof.** Let \(\phi(e) = \begin{bmatrix} \alpha & j' \\ \beta & \delta \end{bmatrix} \) and \(x = \begin{bmatrix} \gamma & k' \\ k & \delta \end{bmatrix} \), where \(\alpha, \beta, \gamma, \delta \in F, k, j \in J; k', j' \in J'; k, j, k', j' \neq 0\). We have \(x\phi(e) = \phi(e)x\), so

\[ \phi(e)x = \begin{bmatrix} \alpha & j' \\ \beta & \delta \end{bmatrix} \cdot \begin{bmatrix} \gamma & k' \\ k & \delta \end{bmatrix} = \begin{bmatrix} \alpha \gamma + T(j, k') & \alpha k + \delta j + j' \times k' \\ \gamma j' + \beta k' + j \times k & T(k, j') + \beta \delta \end{bmatrix} \]

and

\[ x\phi(e) = \begin{bmatrix} \gamma & k' \\ k & \delta \end{bmatrix} \cdot \begin{bmatrix} \alpha & j' \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} \gamma \alpha + T(k, j') & \gamma j + \beta k + k' \times j' \\ \alpha k' + \delta j' + k \times j & T(j, k') + \delta \beta \end{bmatrix}. \]

Thus, it is necessary that

\[ \gamma j' + \beta k' + j \times k = \alpha k + \delta j + j' \times k. \]

Let us assume \(k = j, \gamma \neq \delta\). Then we get \(\alpha = \beta, j' = 0\). Similarly, from the condition

\[ \alpha k + \delta j + j' \times k' = \gamma j + \beta k + k' \times j', \]

assuming \(k' = j'\) and \(\delta \neq \gamma\), we get \(j = 0\). Hence, \(\phi(e) = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\). The lemma is proved.

**Lemma 4.** The algebra \(T(C)\) has no nontrivial \(\frac{1}{2}\)-derivations.

**Proof.** Let \(e_1, \ldots, e_8\) be the standard basis of algebra \(C\) (see, for example, \([23]\)). Let us recall that \(S^{(-)}\) is a simple 7-dimensional non-Lie Malcev algebra relative to commutation operation \([\cdot, \cdot]\) in \(C\) (about Malcev algebras see, for example, \([21]\)), \(S\) is the set of elements of algebra \(C\) that are antisymmetric relative to involution. It is easy to see that \(e_i \in S\) if \(i = 2, 3, \ldots, 8\).

Let \(\phi(e) = h^* + s^*,\) where \(h^* \in H, s^* \in S,\) and \(e\) is the unit of the algebra \(T(C)\). Then:

\[ [\phi(e), s] = 0 \quad \text{and} \quad [h^*, s] + [s^*, s] = 0. \]

Let us notice that \([s^*, s] = 0\). As the algebra \(S^{(-)}\) is simple, its annihilator is trivial. It is easy to see that \(s^*\) is in the annihilator of the algebra \(S^{(-)}\), that is \(s^* = 0\). Consequently, \([h^*, s] = 0\).

It is easy to see (see, for example, \([23]\)) that in Cayley-Dikson algebra \(C\), a basis can be chosen in such way that the multiplication of the basic elements of the algebra will be represented by the following table:
Let the element \( h^* \) be expressed by the basic elements in the following way:

\[
h^* = \sum_{i=2}^{8} \alpha_{ij} e_i \otimes e_j.
\]

We can say that \([h^*, x] = 0 = [x, h^*]\). Let us analyze the product \([e_j, h^*]\). From the condition

\[
[e_2, \sum_{i=2}^{8} \alpha_{ij} e_i \otimes e_j] = 0
\]

we will get the conditions on \(\alpha_{ij}\), and then the element of interest can be represented as follows:

\[
h^* = \alpha_{22} e_2 \otimes e_2 + \alpha_{33} e_3 \otimes e_3 + \alpha_{33} e_4 \otimes e_4 + \alpha_{45} e_4 \otimes e_5 + \alpha_{45} e_5 \otimes e_4 + \alpha_{55} e_5 \otimes e_5 - \alpha_{45} e_5 \otimes e_6 + \alpha_{45} e_4 \otimes e_6 + \alpha_{55} e_5 \otimes e_6 + \alpha_{37} e_3 \otimes e_7 + \alpha_{37} e_4 \otimes e_7 + \alpha_{57} e_5 \otimes e_7 + \alpha_{67} e_6 \otimes e_7 + \alpha_{77} e_7 \otimes e_7 + \alpha_{28} e_8 \otimes e_8 - \alpha_{47} e_3 \otimes e_8 - \alpha_{37} e_4 \otimes e_8 - \alpha_{67} e_6 \otimes e_8 + \alpha_{77} e_8 \otimes e_8.
\]

Performing the multiplication by \(e_j (j = 3, 4, 5, 6, 7)\), we will get that all \(\alpha_{ij} = 0\) when \(i \neq j\). Combining these results with those obtained from the condition \([e_8, h^*] = 0\), it is easy to notice that \(\alpha_{ii} = \alpha_{jj}\) for any \(i, j\).

Therefore, \(h^* = \sum_{i=2}^{8} \alpha e_i \otimes e_i\).

Let us show that \((e_i, e_i) = -4\) and \((e_i, e_j) = 0\) when \(i \neq j\):

\[
[e_j, e_i]e_i = (e_j, e_i)e_i = (e_j,e_i) = 2[e_j, e_i]e_i = 2(e_j e_i - e_i e_j) = 4(e_j e_i) = i(-4e_j).
\]

Using this fact and the multiplication in the algebra \(T(C)\), making direct substitution and verification we can easily see that the element \(e = -\sum_{i=2}^{8} e_i \otimes e_i\) is the unit of the algebra. Let \(\beta = -\frac{1}{8}\). Then \(h^* = \beta e\). The lemma is proved.

4. **Main results**

**Theorem 5.** A simple finite-dimensional structurable algebra over an algebraically closed field with the characteristic \(p \neq 2, 3, 5\) has no nontrivial \(\delta\)-derivations.

**Proof.** Following [22], we can say that a simple finite-dimensional structurable algebra over an algebraically closed field with the characteristic \(p \neq 2, 3, 5\) is isomorphic to one of the algebras of the types \(S1\)-\(S6\) listed in section 2. Let us notice that nontrivial \(\delta\)-derivations are possible only when \(\delta = \frac{1}{2}\). We have that 

1) algebras of types \(S1, S2\) do not admit nontrivial \(\frac{1}{2}\)-derivations (see [3, 4]);

2) algebras of types \(S3\)-\(S6\) do not admit nontrivial \(\frac{1}{2}\)-derivations (see lemmas 1-4).

Thus, the theorem is proved.

**Theorem 6.** A semisimple finite-dimensional structurable algebra over an algebraically closed field of characteristic \(p \neq 2, 3, 5\) has no nontrivial \(\delta\)-derivations.

**Proof.** According to [22], if \(A\) is a semisimple finite-dimensional structurable algebra over an algebraically closed field with the characteristic not equal to 2, 3, 5, then \(A = \bigoplus A_i\), where \(A_i\) is a simple structurable algebra. Let \(e_k\) be the unit of the algebra \(A_k\). If \(x_i \in A_i\), then \(\phi(x_i) = x_i^+ + x_i^-\), where \(x_i^+ \in A_i, x_i^- \notin A_i\). Let \(e^i = \sum e_k - e^i\) and \(\phi(e^i) = e^{i+} + e^{i-}\), where \(e^{i+} \in A_i, e^{i-} \notin A_i\). Then

\[
0 = \phi(x_i e^i) = \phi(x_i)e^i + x_i \phi(e^i)) = \phi((x_i^+ + x_i^-)e^i + x_i(e^{i+} + e^{i-})) = \delta(x_i^- + x_i e^{i+}),
\]

whence \(x_i^- = 0\). Thus \(\phi\) is invariant on \(A_i\). According to theorem 6, \(A_i\) has no nontrivial \(\delta\)-derivations. Then the semisimple algebra \(A\) has no nontrivial \(\delta\)-derivations. The theorem is proved.
Let us note that in \cite{20} the definition of the generalized \(\delta\)-derivation is introduced. The linear mapping \(\chi\) is called a generalized \(\delta\)-derivation if it is related with \(\delta\)-derivation \(\phi\) by the following correlations

\[
\chi(xy) = \delta(\chi(x)y + x\phi(y)) = \delta(\phi(x)y + x\chi(y)).
\]

According to the results of the work \cite{20}, unital algebras have no generalized \(\delta\)-derivations, that are not either generalized derivations or \(\delta\)-derivations. It was also was shown that if the generalized \(\delta\)-derivation is the generalized derivation then it is the sum of the derivation and the element of the centroid. Therefore, from theorem 6 follows:

**Theorem 7.** Let \(\chi\) be the generalized \(\delta\)-derivation of the semisimple finite-dimensional structurable algebra \(A\) over an algebraically closed field with the characteristic \(p \neq 2, 3, 5\), then \(\chi \in Der(A) + \Gamma(A)\), where \(Der(A)\) is the space of derivations and \(\Gamma(A)\) is the centroid.

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