A TWO-PIECE PROPERTY FOR FREE BOUNDARY MINIMAL SURFACES IN THE BALL

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Abstract. We prove that every plane passing through the origin divides an embedded compact free boundary minimal surface of the euclidean 3-ball in exactly two connected surfaces. We also show that if a region in the ball has mean convex boundary and contains a nulhomologous diameter, then this region is the intersection of two closed halfballs. Crucial to both results is the regularity at the corners of currents minimizing a partially free boundary problem, which we get by following ideas by Gr"uter and Simon. Our first result gives evidence to a conjecture by Fraser and Li.

1. Introduction

A beautiful theorem by A. Ros [30] states that every equator of the (round) 3-sphere divides an embedded closed minimal surface in exactly two open connected pieces. An interesting fact is that this result can be seen as a consequence of a (still open) conjecture due to Yau - which says that the first nonzero eigenvalue of the Laplacian of an embedded closed minimal surface in the 3-sphere is equal to 2 - together with the Courant nodal domain theorem. Hence Ros’s result can be seen as an evidence to the conjecture.

The analogy between the theory of closed minimal surfaces of the 3-sphere and the theory of compact free boundary minimal surfaces of the unit Euclidean 3-ball $B^3$ is well-known and has been well explored in many recent works, see for instance [2, 4, 11, 12, 14, 27, 32]. In this paper, inspired by this analogy, we prove the analog of Ros’s result in the context of free boundary minimal surfaces.

Theorem A (The two-piece property). Every plane in $\mathbb{R}^3$ passing through the origin divides an embedded compact free boundary minimal surface of the unit 3-ball $B^3$ in exactly two connected surfaces.

To prove this theorem we need the following result which is also the analog of another result by Ros in [30].

Theorem B. Let $W \subset B^3$ be a connected closed region with mean convex boundary such that $W$ meets $S^2$ orthogonally along its boundary. Suppose $W$ contains a straight line segment joining two antipodal points of $S^2$, which is nulhomologous in $W$ (see Definition 3). Then $W$ is the intersection of two closed halfballs. In particular, if $\partial W$ is smooth, then $W$ is a closed halfball.

We say that a surface $\Sigma \subset B^3$ links a curve $\Gamma$, if $\Sigma$ does not meet $\Gamma$ and it is homotopically non-trivial (relative to $\partial B^3$) in $B^3 \setminus \Gamma$ (see Figure 1). An interesting consequence of Theorem B is the following corollary which is the analog of a result in $S^3$ due to Solomon [35].

Corollary. Every embedded compact free boundary minimal surface of $B^3$ either meets or links each straight line passing through the origin.
In the proof of both Theorem A and Theorem B, we need the existence and regularity of a minimizer for a partially free boundary problem. Namely, let $\gamma$ be a compact curve contained in $\overline{B}^3$ such that $\gamma \cap S^2$ is either empty or consists of a finite number of points where $\gamma$ meets $S^2$ orthogonally (the corners), and consider the class of surfaces $M$ such that $\partial M \setminus \gamma \subset S^2$; we look for a surface $\Sigma$ which has least area among all such surfaces. The existence of such surface $\Sigma$ follows from general compactness results about currents, and the regularity of $\Sigma$ away from the corners was proved in [18, 21]. It was reported in [19] that the regularity at the corners would be settled in a work of Gr"uter and Simon (unpublished). Here we give the details of this proof by following the ideas contained in [19].

The study of free boundary minimal surfaces (in Euclidean domains) has attracted significant attention for several decades (see for instance classical works as [6, 20] or more recent results as [2–5, 7, 15, 25–29, 31, 32, 34, 36, 37] and references therein). Recently there was an increase in interest for free boundary minimal surfaces in the unit Euclidean 3-ball $B^3$ due to the work by Fraser and Schoen [12] (see also [13, 14]) where they made a connection between these objects and the Steklov eigenvalue problem. In analogy to Yau’s conjecture mentioned above, Fraser and Li [11] conjectured that the first nonzero Steklov eigenvalue of an embedded compact free boundary minimal surface in $B^3$ is equal to 1. This conjecture together with the Courant nodal domain theorem for the Steklov problem (stated for instance in [16], Section 6) implies the two-piece property for free boundary minimal surfaces in $B^3$ (see Remark 2). Hence our result in Theorem A can be seen as an evidence to the conjecture by Fraser and Li.

In the last few years there have been many important studies about free boundary minimal surfaces. Ambrozio, Carlotto and Sharp [3] established compactness theorems for free boundary minimal hypersurfaces. Maximo, Nunes and Smith [28] proved the existence of free boundary minimal annuli through a degree argument. Li and Zhou [26] developed a min-max theory for free boundary minimal hypersurfaces.

We should mention that the class of free boundary minimal surfaces in $B^3$ is very rich. In fact, many techniques have been developed to construct new examples of free boundary minimal surfaces in $B^3$. For instance, Fraser and Schoen [12, 14] constructed examples with genus 0 and any number of boundary components. Using gluing methods, Folha, Pacard and Zolotareva [10] constructed examples with genus 1 and any large number of boundary components, and also obtained examples of genus 0 and large number of boundary components displaying similar asymptotic behavior to Fraser-Schoen family. Examples with large genus and 3 boundary components were constructed by Ketover [24], where he also obtained examples with the symmetry group of the Platonic solids, both using min-max methods. Kapouleas and Li [22] also produced examples with large
genus and 3 boundary components, and examples with dihedral symmetry. Using gluing methods, Kapouleas and Wiygul [23] constructed examples with one boundary component and large genus, converging to an equatorial disk with multiplicity 3, as the genus goes to infinity.

This paper is organized as follows. In the second section we will prove both Theorem A and Theorem B, and in Section 3 we will show the proof of the regularity at the corners of a minimizer for the partially free boundary problem mentioned above.

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2. The two-piece property and other results

Throughout the paper we say that a curve in $\mathbb{B}^3$ is a diameter if it is a straight line segment joining two antipodal points of $S^2$; and we will define an equatorial disk as the intersection of $\mathbb{B}^3$ with a plane passing through the origin.

Given a surface $\Sigma$ in $\mathbb{B}^3$, we will write its boundary as $\partial \Sigma = \gamma_I \cup \gamma_S$ where $\text{int}(\gamma_I) \subset \text{int}(\mathbb{B}^3)$ and $\gamma_S \subset S^2$.

Definition 1. Let $\Sigma$ be a compact surface properly immersed in $\mathbb{B}^3$.

We say that $\Sigma$ is a minimal surface with free boundary if the mean curvature vector of $\Sigma$ vanishes and $\Sigma$ meets $S^2 = \partial \mathbb{B}^3$ orthogonally along $\partial \Sigma$ (in particular, $\gamma_I = \emptyset$).

We say that $\Sigma$ is a minimal surface with partially free boundary if the mean curvature vector of $\Sigma$ vanishes and its boundary $\partial \Sigma = \gamma_I \cup \gamma_S$ satisfies that $\Sigma$ meets $S^2$ orthogonally along $\gamma_S$ and $\gamma_I \neq \emptyset$.

From now on, given a (partially) free boundary minimal surface $\Sigma \subset \mathbb{B}^3$ with boundary $\partial \Sigma = \gamma_I \cup \gamma_S$, we will call $\gamma_I$ its fixed boundary and $\gamma_S$ its free boundary.

Definition 2. Let $\Sigma$ be a partially free boundary minimal surface in $\mathbb{B}^3$ with piecewise smooth boundary $\partial \Sigma = \gamma_I \cup \gamma_S$. We say that $\Sigma$ is stable if for any function $f \in C^\infty(\Sigma)$ such that $f|_{\gamma_I} \equiv 0$ we have

$$-\int_{\Sigma} (f \Delta f + |A| f^2) d\Sigma + \int_{\gamma_S} \left( f \frac{\partial f}{\partial \nu} - f^2 \right) ds \geq 0,$$

where $\nu$ is the outward normal vector field to $\gamma_S$.

Lemma 1. Let $\Sigma$ be a compact orientable immersed partially free boundary stable minimal surface in $\mathbb{B}^3$ with piecewise smooth boundary $\partial \Sigma = \gamma_I \cup \gamma_S$. Suppose $\gamma_I$ is contained in an equatorial disk. Then $\Sigma$ is totally geodesic. The same result holds in the case where $\Sigma$ has isolated singularities in $\gamma_I$.

Remark 1. Throughout this paper, the isolated singularities that $\gamma_I$ might have are of $n$-prong type, i.e., if $p \in \gamma_I$ is an isolated singularity then in a neighborhood of $p$, $\gamma_I$ consists of the union of $2k$ curves, $k \geq 1$, passing through $p$ and making equal angles. This is exactly the type of singularity that appears in the intersection of two distinct minimal surfaces that are tangent at a point.

Proof. Let $\Sigma$ be as in the hypotheses and denote by $D$ the equatorial disk that contains its fixed boundary $\gamma_I$.

Let us first assume that the fixed boundary of $\Sigma$ does not have singularities.

Let $v \in S^2$ be a vector orthogonal to the disk $D$ and consider the function $f(x) = \langle x, v \rangle$, $x \in \Sigma$. By hypothesis we know that $f|_{\gamma_I} \equiv 0$, so (2.1) holds. Moreover, since $\Sigma$ is minimal,
it is well-known that
\[ \Delta_{\Sigma} f(x) = 0. \]  

(2.2)

On the other hand,
\[ \frac{\partial f}{\partial \nu}(x) = \nu \langle x, v \rangle = \langle \nabla_{\nu} x, v \rangle = \langle \nu(x), v \rangle = \langle x, v \rangle = f(x), \]

(2.3)
since $\Sigma$ is free boundary on $\gamma_S$.

Using (2.2) and (2.3) in (2.1), we get
\[ |A_{\Sigma}|^2(x, v) = 0 \text{ for any } x \in \Sigma. \]  

(2.4)

If $|A_{\Sigma}| \equiv 0$ then $\Sigma$ is totally geodesic and we are done.

If $|A_{\Sigma}|(x) > 0$ for some $x \in \Sigma$, then we can find a neighborhood $U$ of $x$ in $\Sigma$ such that $|A_{\Sigma}|$ is strictly positive. By (2.4), this implies $\langle y, v \rangle = 0$ for any $y \in U$, that is, $U$ is contained in the disk $D$. Therefore, $\Sigma$ is entirely contained in the disk $D$; in particular, it is totally geodesic.

Now let us suppose that the fixed boundary of $\Sigma$ has isolated singularities. In this case, for $\varepsilon > 0$ and $f$ as above, we consider the surface $\Sigma_{\varepsilon} = |f|^{-1}([\varepsilon, +\infty))$, which is a smooth surface for almost every $\varepsilon > 0$, and define the smooth function $f_{\varepsilon} = |f| - \varepsilon$ on $\Sigma_{\varepsilon}$ and $f_{\varepsilon} = 0$ on $\Sigma \setminus \Sigma_{\varepsilon}$. We have $\Delta_{\Sigma} f_{\varepsilon} = 0$ and $\frac{\partial f_{\varepsilon}}{\partial \nu} = |f| = f_{\varepsilon} + \varepsilon$. Hence, applying $f_{\varepsilon}$ to (2.1) we get
\[ -\int_{\Sigma} |A_{\Sigma}|^2 f_{\varepsilon}^2 d\Sigma + \int_{\gamma_S} \varepsilon f_{\varepsilon} ds \geq 0. \]

Since the last integral converges to zero as $\varepsilon$ approaches zero, we can take limits when $\varepsilon$ goes to zero and apply the same arguments as above to get the same conclusion.

\[ \square \]

**Remark 2.** Let $M$ be an embedded free boundary minimal surface in $B^3$. Recall that a nodal domain of a function is a maximally connected subset of the domain where the function does not change sign, and the Courant nodal domain theorem for the Steklov problem says that an eigenfunction corresponding to the $n$-th nonzero Steklov eigenvalue has at most $n + 1$ nodal domains. Let $P$ be a plane passing through the origin and let $v \in S^2$ be a vector orthogonal to $P$. The Jacobi function $f : M \to \mathbb{R}, f(x) = \langle x, v \rangle$, defined in the proof of Lemma 1, is an eigenfunction with eigenvalue 1 for the Steklov problem. Hence, assuming Fraser-Li conjecture, it follows that $f$ has at most two nodal domains. Moreover, we can use the (interior and boundary) maximum principle with equatorial disks to conclude that $f$ has in fact two nodal domains, that is, the plane $P$ divides $M$ in exactly two connected surfaces. Hence our result in Theorem 2 can be seen as an evidence to the conjecture by Fraser and Li.

**Definition 3.** Let $W$ be a region in $B^3$ and let $\alpha \subset W$ be a diameter. We say that $\alpha$ is nullhomologous in $W$ if there exists a compact surface $M \subset W$ such that $\partial M = \alpha \cup \gamma$, where $\gamma \subset S^2$ (see Figure 2).

The boundary of the region $W$ can be written as $U \cup V$, where $\text{int}(U) \subset \text{int}(B^3)$ and $V \subset S^2$. In the next theorem we will denote by $\partial W$ the closure of the component $U$, that is, $\partial W = \overline{U}$.

**Theorem 1.** Let $W \subset B^3$ be a connected closed region with (non-strict) mean convex boundary such that $W$ meets $S^2$ orthogonally along its boundary. If $W$ contains a diameter $\alpha$, and $\alpha$ is nullhomologous in $W$, then $W$ is the intersection of two closed halfballs and $\alpha \subset \partial W$. In particular, if $\partial W$ is smooth, then $W$ is a closed halfball.
Proof. Up to a rotation of $\alpha$ around the origin, we can assume that $\alpha \cap \partial W$ is nonempty. Since $\alpha$ is nullhomologous in $W$, we can consider the class of curves $\alpha \cup \gamma$, where there exists a surface contained in $W$ with boundary $\alpha \cup \gamma$, $\gamma \subset S^2$. Hence we can minimize area for the partially free boundary problem (see Section 3) and we get a compact embedded (orientable) stable partially free boundary minimal surface $\Sigma \subset W$ which minimizes area among compact surfaces in $W$ with boundary on the class $\alpha \cup \gamma$; in particular, its fixed boundary is exactly $\alpha$.

Then, since $\alpha$ is contained in an equatorial disk, Lemma 1 implies that $\Sigma$ is necessarily a half disk. By the (interior and boundary version of) the maximum principle (for surfaces with nonnegative mean curvature meeting $S^2$ orthogonally), we know that either $\Sigma \subset \partial W$ or $\Sigma \cap \partial W \subset \alpha$.

Suppose $\Sigma \cap \partial W \subset \alpha$. Then rotating $\Sigma$ around $\alpha$ in the two possible directions, we will get a first contact point with $\partial W$, then applying (either the interior or boundary version of) the maximum principle, we conclude that these last rotated images of $\Sigma$ are both contained in $\partial W$. Since the union of the two half disks is a surface without boundary, it must coincide with the whole $\partial W$. We get this same conclusion with the other case where $\Sigma \subset \partial W$. Therefore, the theorem is proved.

An equatorial disk $D$ divides the ball $B^3$ into two (open) halfballs. We will denote these two halfballs by $B^+$ and $B^-$, and we have $B^3 \setminus D = B^+ \cup B^-$. In the next proposition we will summarize some simple facts about partially free boundary minimal surfaces in $B^3$ which we will use in the proof of Theorem 2.

Proposition 1. (i) Let $D$ be an equatorial disk and let $\Sigma$ be a partially free boundary minimal surface in $B^3$ contained in one of the closed halfballs determined by $D$ and such that $\gamma_1 \subset D$ (if $\gamma_1 \neq \emptyset$). If $\Sigma$ is not an equatorial disk, then $\Sigma$ has necessarily nonempty fixed boundary and nonempty free boundary.

(ii) The only (partially) free boundary minimal surface that contains an arc segment of a great circle in its free boundary is an equatorial disk.

Proof. (i) If the free boundary were empty, we could apply the (interior) maximum principle with the family of planes parallel to the disk $D$ and conclude that $\Sigma$ should be a disk. On the other hand, if the fixed boundary were empty, then we would have a minimal surface entirely contained in a halfball without fixed boundary, hence we could apply the (interior or boundary version of) maximum principle with the family of equatorial disks that are rotations of $D$ around a diameter and conclude that $\Sigma$ should be a disk as well.

(ii) Let $D$ be an equatorial disk and suppose that $\Sigma$ is a (partially) free boundary minimal surface such that $\Sigma \cap D$ contains an arc segment $\alpha$ in $S^2$; in particular, since they
are both free boundary, we know they are tangent along α. Hence, given a point \(x \in α\) there exists a neighborhood \(U\) of \(x\) in \(D\) where either \(Σ\) is on one side of \(D\) or \(Σ ∩ U\) is given by a collection of \(2k\) curves, \(k \geq 1\), passing through \(x\) and making equal angles. In this last case, for any point in \((α ∩ U) \setminus \{x\}\), we will have a neighborhood where \(Σ\) is on one side of \(D\); therefore, in either case, applying the boundary maximum principle we can conclude that \(Σ\) should be an equatorial disk. \(\Box\)

Now we can prove the two-piece property for free boundary minimal surfaces in \(B^3\).

**Theorem 2.** Let \(M\) be a compact embedded free boundary minimal surface in \(B^3\). Then for any equatorial disk \(D\), \(M ∩ B^+\) and \(M ∩ B^-\) are connected.

**Proof.** If \(M\) is an equatorial disk, then the result is trivial. So let us assume this is not the case.

Suppose that, for some equatorial disk \(D\), \(M ∩ B^+\) is a disjoint union of two nonempty open subsurfaces \(M_1\) and \(M_2\), \(M_1\) being connected. Notice that by Proposition 1(i) both \(M_1\) and (all components of) \(M_2\) have non empty fixed boundary and non empty free boundary.

Let us denote by \(Γ\) the boundary of \(M_1\), which is not necessarily connected. We can write \(Γ = γ_I \cup γ_S\), where \(γ_I\) is its fixed boundary (\(int(γ_I) ⊂ int(D)\)) and \(γ_S\) is its free boundary (\(γ_S ⊂ S^2\)). Since \(M\) and \(D\) are two minimal surfaces, either \(M\) and \(D\) are transverse or the intersection \(M ∩ D\) contains (at least) one \(n\)-prong singularity. Observe that, by applying (either the interior or boundary version) of the maximum principle, we know that \(M ∩ D\) does not contain any isolated point in \(D\) and by Proposition 1(ii), \(M ∩ D\) does not contain any arc segment in \(S^2\).

Denote by \(W\) and \(W'\) the closures of the two components of \(B^3 \setminus M\). They are compact domains with mean convex boundary, and observe that the curve \(Γ\) is the boundary of an orientable surface contained in them (in fact, \(M_1\) is orientable and \(M_1 ∂W, W'\)). Hence we can consider the compact surface \(Σ\) in \(W\) that is the solution to the partially free boundary problem with fixed boundary \(γ_I\) (see Section 3). Then \(Σ\) is an embedded stable (orientable) partially free boundary minimal surface, with possible isolated singularities in \(γ_I ⊂ D\). By Lemma 1, since \(γ_I\) is contained in \(D\), we know that each component of \(Σ\) is a piece of an equatorial disk. Since \(∂W\) is mean convex, we can apply (the interior or boundary version of) the maximum principle to conclude that any component of \(Σ\) is either contained in \(∂W\) or meets \(∂W\) only at points of \(Γ\). The first case can not happen because this would imply that \(M\) is a disk, and we are assuming it is not. Therefore, only the second case can happen, that is, any component of \(Σ\) meets \(∂W\) only at points of \(Γ\). Observe that each component of \(Σ\) that is not bounded by a diameter is necessarily contained in \(D\). If some component of \(Σ\) were bounded by a diameter, then we could apply Theorem 1 and would conclude that \(M\) is an equatorial disk, which is not the case. Then \(Σ\) is entirely contained in \(D\) and, since \(Σ ∩ ∂W ⊂ Γ\), \(M ∂W\) and \(M ∩ D\) does not contain any segment on \(S^2\), we have \(Σ ∩ M = γ_I\).

Doing the same procedure as in the last paragraph for \(W'\), we can construct another compact subsurface \(Σ'\) of \(D\) with fixed boundary \(∂Σ' = γ_I\) and such that \(Σ' ⊂ W'\) and \(Σ' ∩ M = γ_I\). Notice that \(Σ ∪ Σ'\) is a subsurface without fixed boundary of \(D\), therefore \(Σ ∪ Σ' = D\). In particular, \(M ∩ D = γ_I\), which implies that \(M_2 = M ∩ B^+ \setminus M_1\) has no fixed boundary, a contradiction (by Proposition 1(i)); therefore, the theorem is proved. \(\Box\)

As a consequence of the theorems above (and their proofs) we get the following results.
Proof. Denote by \( W \) and \( W' \) the closure of the two components of \( \overline{B^+} \setminus M \). They are mean convex regions.

(a) If \( M \) is not a half disk, then the arguments in the proof of Theorem 2 show that the solution \( \Sigma \) (resp. \( \Sigma' \)) to the partially free boundary problem in \( W \) (resp. \( W' \)) with fixed boundary \( \gamma_I \) is \( \Sigma = W \cap D \) (resp. \( \Sigma' = W' \cap D \)). In particular, \( \text{area}(M) > \text{area}(\Sigma) \), \( \text{area}(\Sigma') \). Since \( \Sigma \cup \Sigma' = D, \Sigma \cap \Sigma' = \gamma_I \) and \( \text{area}(D) = \pi \), we get \( \text{area}(M) > \pi/2 \).

(b) In this case either \( W \) or \( W' \) contains a closed half disk and, in particular, contains a nulhomologous diameter. Then, by Theorem 1, \( M \) is a half disk.

(c) Denote by \( W'' \) the closure of the component of \( \overline{B^+} \setminus M \cup M' \) that contains \( M \) and \( M' \) at its boundary. \( W'' \) is a mean convex region. Consider \( \Sigma \) the solution to the partially free boundary problem in \( W'' \) with fixed boundary \( \gamma_I \). As in the proof of Theorem 1 we can show that \( \Sigma \) is a half disk; in particular, we conclude that \( \gamma_I \) is a diameter. Hence, since \( \gamma_I \subset \Sigma \subset W'' \) is nulhomologous in \( W'' \), then by Theorem 1 the assertion follows.

\[ \square \]

Corollary 4. Every embedded compact free boundary minimal surface \( M \) of \( B^3 \) either meets or links each diameter.

Proof. Let \( \alpha \) be a diameter with end points \( p, q \), and suppose \( M \) does not meet \( \alpha \). Write \( B^3 \setminus M = W \cup W' \), where \( W \) contains \( \alpha \), and the decomposition is disjoint. Suppose, by contradiction, that \( M \) is homotopically trivial (relative to \( \partial B^3 \)) in \( B^3 \setminus \alpha \). We will prove that \( \alpha \) is nulhomologous in \( W \).

In fact, since \( M \) is homotopically trivial in \( B^3 \setminus \alpha \), we have that \( \partial M \cap S^2 \) is a finite collection of simple closed curves in \( S^2 \) which are homotopically trivial in \( S^3 \setminus \{p, q\} \). Hence, there is a curve \( \gamma \subset W \cap S^2 \) joining \( p \) and \( q \); and \( \alpha \cup \gamma \) bounds a (topological) disk \( V \) in \( B^3 \). If \( V \subset W \), we are done. If that is not the case, by deforming \( V \), if necessary, we can suppose that \( V \) and \( M \) are transverse. Let \( N \) be the unit normal vector field to \( M \) pointing into \( W \), and denote by \( M_\epsilon \) the intersection of \( B^3 \) with the boundary of a one-sided tubular neighbourhood of \( M \) (in the direction of \( N \)) of radius \( \epsilon \). We can choose \( \epsilon \) small enough such that \( \partial M_\epsilon \cap (\alpha \cup \gamma) = \emptyset \), and \( M_\epsilon \) is transverse to \( V \). The intersection \( M_\epsilon \cap V \) consists of a finite number of simple closed curves which bound open discs \( U_1, \ldots, U_n \) in \( V \) (since \( V \) is a disk). The set \( \Sigma = M_\epsilon \cup (V \cup \bigcup_{i=1}^n U_i) \) is a topological surface with \( \alpha \subset \Sigma \subset W \) and \( \partial \Sigma \setminus \alpha \subset S^2 \). Therefore \( \alpha \) is nulhomologous in \( W \).

Then, by Theorem 1, we conclude that \( W \) is a closed halfball (once \( \partial W \cap \text{int}(B^3) \) is smooth); in particular, \( M \) is an equatorial disk. However, this contradicts the fact that \( M \cap \alpha = \emptyset \). Therefore, \( M \) links \( \alpha \) necessarily.

\[ \square \]

3. Solution to a partially free boundary problem

3.1. Terminology. Let \( U \subset \mathbb{R}^{n+k} \) be an open set. We define

\[ D^n(U) = \{ C^\infty - n\text{-forms } \omega; \text{spt } \omega \subset U \} \]
with the usual topology of uniform convergence of all derivatives on compact subsets. Its dual space is denoted by \( D_n(U) \) and the elements of \( D_n(U) \) are called \( n \)-currents in \( U \). If \( T \in D_n(U) \), and \( W \subset U \) is open, the mass of \( T \) in \( W \) is defined by

\[
M_W(T) := \sup(T(\omega)); \quad \omega \in D^n(U), \ spt \omega \subset W, \ |\omega| \leq 1 \leq +\infty.
\]

The boundary of \( T \) is the \((n-1)\)-current \( \partial T \in D_{n-1}(U) \) given by

\[
\partial T(\omega) := T(d\omega),
\]

where \( d \) denotes the exterior derivative operator.

Given a sequence \( \{T_j\}_{j \in \mathbb{N}} \in D_n(U) \), we say that \( T_j \) converges to \( T \in D_n(U) \) as \( j \to \infty \), if

\[
T_j(\omega) \to T(\omega), \text{ as } j \to \infty, \forall \omega \in D^n(U).
\]

Let \( \mathcal{H}^n \) denote the \( n \)-dimensional Hausdorff measure. A set \( M \subset \mathbb{R}^{n+k} \) is called countably \( n \)-rectifiable if \( M \) is \( \mathcal{H}^n \)-measurable and if

\[
M \subset \bigcup_{j=0}^{\infty} M_j,
\]

where \( \mathcal{H}^n(M_0) = 0 \) and for \( j \geq 1 \), \( M_j \) is an \( n \)-dimensional \( C^1 \)-submanifold of \( \mathbb{R}^{n+k} \). Such \( M \) possesses \( \mathcal{H}^n \)-a.e. an approximate tangent space \( T_xM \).

A current \( T \in D_n(U) \) is called integer multiplicity rectifiable, if

\[
T(\omega) = \int_M \langle \omega, \xi \rangle \theta \, d\mathcal{H}^n, \quad \omega \in D^n(U),
\]

where \( M \subset U \) is countably \( n \)-rectifiable, \( \theta \geq 0 \) is a locally \( \mathcal{H}^n \)-integrable integer valued function and, for \( \mathcal{H}^n \)-a.e. \( x \in M \), \( \xi(x) = e_1 \wedge \cdots \wedge e_n \), where \( \{e_1, \cdots, e_n\} \) is an orthonormal basis of the approximate tangent space \( T_xM \). In this case, we write \( T = \tau(M, \theta, \xi) \). Also, we denote by \( \mu_T = \mathcal{H}^n \langle \cdot, \theta \rangle \) the Radon measure induced by the current \( T \).

A \( n \)-varifold in \( U \) is a Radon measure on \( G_{n,k}(U) := U \times G(n+k,n) \), where \( G(n+k,n) \) is the Grassmannian of \( n \)-hyperplanes in \( \mathbb{R}^{n+k} \). An integer multiplicity rectifiable \( n \)-varifold \( V = v(M, \theta) \) is defined by

\[
V(f) = \int_M f(x, T_xM) \, d\mathcal{H}^n, \quad f \in C_c(G_{n,k}(U), \mathbb{R}),
\]

where \( M \subset U \) is countably \( n \)-rectifiable and \( \theta \geq 0 \) is a locally \( \mathcal{H}^n \)-integrable integer valued function. In particular, given an integer multiplicity rectifiable current, forgetting the orientation we have an associated integer multiplicity rectifiable varifold, and vice-versa.

3.2. Minimizing Currents with Partially Free Boundary. Let \( S \subset \mathbb{R}^3 \) be a connected, closed, oriented \( C^2 \)-surface; in particular, \( S \) is the boundary of a bounded region \( U \). Let \( \gamma \) be a compact curve contained in \( U \) such that \( \gamma \cap S \) is either empty or consists of a finite number of points where \( \gamma \) meets \( S \) orthogonally. We shall call \( \gamma \) the fixed boundary and the points of \( \gamma \cap S \) by corners.

Define the class \( \mathcal{C} \) of admissible currents by

\[
\mathcal{C} = \{ T \in D_2(\mathbb{R}^3); \ T \text{ is integer multiplicity rectifiable,} \ spt T \text{ is compact and } spt([\gamma] - \partial T) \subset S \},
\]

where \([\gamma]\) is the current associated to \( \gamma \). We want to minimize area in \( \mathcal{C} \), that is, we are looking for \( T \in \mathcal{C} \) such that

\[
M(T) = \inf \{ M(\tilde{T}); \ \tilde{T} \in \mathcal{C} \}.
\]
The existence of the fixed boundary ensures that $\mathcal{C} \neq \emptyset$. If $T \in \mathcal{C}$ is a solution to this variational problem we have

$$M(T) \leq M(T + X),$$

for any integer multiplicity current $X \in D_2(\mathbb{R}^3)$ with compact support and such that $\text{spt } \partial X \subset S$.

It follows from [8], 5.1.6(1), that the variational problem introduced above has a solution (see also [17]). Concerning the regularity of the minimizer $T$ we know the following: in a neighborhood of each $x \in \text{spt } T \setminus \text{spt } \partial T$, $T$ is given by $(m$-times, $m \in \mathbb{N})$ integration over an embedded minimal surface - this follows from the classical interior regularity theory developed by DeGiorgi (since $n = 2 < 7$); the regularity near a point $x$ at the fixed part of the boundary away from the corners follows from the work of Hardt and Simon [21]; and the regularity at the free part of the boundary (away from the corners) was shown by Grüter in [18] (here again the condition $n = 2 < 7$ is necessary). Therefore, away from the corners we conclude that $T$ is supported in a connected oriented embedded minimal $C^2$-surface, which meets $S$ orthogonally along $\text{spt } (\llbracket \gamma \rrbracket - \partial T)$.

It remains the question about the regularity of $T$ at the corners. In [19], Grüter reports joint work with L. Simon (unpublished) where they would settle this question. We develop here the ideas present in [19] to prove the regularity.

Arguing as in Section 3 of [18], we can reduce the problem of local regularity at a corner to the following situation. Applying a translation and a dilation if necessary we can suppose one of the corners is located at the origin $0 \in \mathbb{R}^3$ and the ball $B_3(0)$ (centered at the origin with radius 3) is decomposed by $S$ into two open 3-cells, that is

$$B_3(0) = B^-_3 \cup (S \cap B_3(0)) \cup B^+_3,$$

where $B^-_3$ and $B^+_3$ are diffeomorphic to the 3-dimensional unit ball and the decomposition is disjoint.

Consider a rectifiable $T \in D_2(B_3(0))$ of integer multiplicity satisfying:

$$\text{spt } T \subset \overline{B^-_3}, \ 0 \in \text{spt } T,$$

$$\text{spt } (\llbracket \gamma \rrbracket - \partial T) \subset S,$$

$$M(T) < +\infty,$$

$$\theta_T = 1, \ \mu_T - \text{a.e},$$

$$M_W(T) \leq M_W(T + X),$$

for every open set $W \subset B_3(0)$ and for any integer multiplicity current $X \in D_2((B_3(0))$ such that $\text{spt } X \subset W$ and $\text{spt } \partial X \subset S$. It also holds $\mu_T(S \cap W) = 0$, for every open set $W \subset B_3(0)$.

Define the reflection $\Phi : B_3(0) \to \mathbb{R}^3$ across $S$ by

$$\Phi(y) = 2\Pi(y) - y,$$

where $\Pi(y)$ is defined as the unique point in $S$ such that $\text{dist}(y, S) = |y - \Pi(y)|$. Since our questions are local, we can ensure that $\Pi$ is well defined (after a dilation if necessary) and continuously differentiable. Geometrically we can see $\Phi$ as follows: the line through
Given $\Phi(y) = x$ with direction $\xi(x)$ (a unit normal vector to $S$ at $x$) is parametrized by $t \mapsto x + t\xi(x)$, so if $y = x + t\xi(x)$, we have $\Phi(y) = x - t\xi(x)$. It is easy to see that $\Phi^2 = \text{Id}$.

Define $T' = T - \Phi_\#(T)$. Thus $T' \in D_2(V)$ has multiplicity one and $\text{spt} \partial T' \subset \gamma \cup \Phi(\gamma)$, where

$$V = B^+_3 \cup (S \cap B_3(0)) \cup \Phi(B^+_3).$$

Moreover, $M(T') < +\infty$ and $B = B_1(0) \subset V$. Denote $T' \setminus B$ by $\tilde{T}$. Hence $\tilde{T} \in D_2(B)$ has integer multiplicity, finite mass and the support of its boundary is contained in $\gamma \cup \Phi(\gamma)$.

Now, we will prove that $\tilde{T}$ is regular at $0$ and, of course, this implies the regularity of $T$ at $0$. For this purpose, we will first adapt some ideas of [18] to show that $\tilde{T}$ has a tangent cone at $0$ which is area-minimizing, and then we will use the results in [21] to show the regularity.

**Lemma 2.** Consider $y = x + r\xi(x)$, where $x \in S$ and $\xi(x)$ is a unit normal vector to $S$ at $x$. Then, for $r$ small enough, the derivative of $\Phi$ satisfies

$$1 - c_1 r \leq |D\Phi(y)| \leq 1 + c_1 r,$$

where $c_1$ is a positive constant.

**Proof.** Given $x \in S$ and $v \in T_x S$, consider a curve $\alpha : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = x$ and $\alpha'(0) = v$. Then

$$D\Phi(x) \cdot v = \frac{d}{ds} \bigg|_{s=0} \Phi(\alpha(s)) = \frac{d}{ds} \bigg|_{s=0} \alpha(s) = v,$$

$$D\Phi(x) \cdot \xi = \frac{d}{ds} \bigg|_{s=0} \Phi(x + s\xi(x)) = \frac{d}{ds} \bigg|_{s=0} (x - s\xi(x)) = -\xi(x),$$

$$D^2\Phi(x) \cdot (\xi, \xi) = \frac{d^2}{ds^2} \bigg|_{s=0} \Phi(x + s\xi(x)) = \frac{d^2}{ds^2} \bigg|_{s=0} (x - s\xi(x)) = 0,$$

$$D^2\Phi(x) \cdot (\xi, v) = \frac{\partial^2}{\partial t \partial s} \bigg|_{t=0, s=0} \Phi(\alpha(s) + t\xi(\alpha(s))) = \nabla_v^3 \xi.$$  

So, $|D\Phi(x)| = 1$, and by Taylor’s theorem we have for $r$ small enough

$$|D\Phi(x)(x + r\xi(x))| = 1 + \langle D\Phi(x), D^2\Phi(x) \cdot (\xi, \cdot) \rangle r + O(r),$$

where $|O(r)| \leq \lambda r$, for some constant $\lambda > 0$. By the computation above,

$$|\langle D\Phi(x), D^2\Phi(x) \cdot (\xi, \cdot) \rangle| \leq \kappa,$$

where $\kappa$ is the supremum of the norm of the second fundamental form of $S$. Therefore, the result follows.  

□

**Lemma 3.** There exists $\Theta(\mu_T, 0) = \lim_{r \to 0} \frac{\mu_T(B_r(0))}{\pi r^2}$.  

Proof. Define $\tilde{B}_r(0) = \Phi^{-1}(B_r(0)) = \{y \in \mathbb{R}; \ |\Phi(y)| < r\}$. We have

$$
\mu_\tilde{T}(B_r(0)) = \mu_\tilde{T}(B_r(0) \cap B^+) + \mu_\tilde{T}(B_r(0) \cap \Phi(B^+))
$$

$$
= \mu_T(B_r(0) \cap B^+) + \mu_{\Phi \circ \tilde{T}}(B_r(0) \cap \Phi(B^+))
$$

$$
= \mu_T(B_r(0)) + \mu_{\Phi \circ \tilde{T}}(\Phi(B_r(0) \cap B^+)). \tag{3.7}
$$

Using Lemma 2, for $r$ small enough we obtain

$$(1 - c_1 r)^2 \mu_T(\tilde{B}_r(0)) \leq \mu_{\Phi \circ \tilde{T}}(\Phi(\tilde{B}_r(0) \cap B^+)) \leq (1 + c_1 r)^2 \mu_T(\tilde{B}_r(0)). \tag{3.8}$$

On the other hand, adapting the arguments of Section 2.2 of [20], we obtain a constant $\tau > 0$ such that for $r \leq 1/2$, $l_1(r) = 1 + \frac{2r \tau}{1 - \tau r}$ and $l_2(r) = 1 - 3\tau r$, we have

$$B_{r l_2(r)}(0) \subset \tilde{B}_r(0) \subset B_{r l_1(r)}(0). \tag{3.9}$$

Moreover, again arguing similarly as in [20] (Corollary 3.2) we conclude that there exists

$$\Theta(\mu_T, 0) = \lim_{r \to 0} \frac{\mu_T(B_r(0))}{\pi r^2}. \tag{3.10}$$

By (3.9),

$$\frac{\mu_T(B_{r l_2(r)}(0))}{\pi [r l_2(r)]^2} [l_2(r)]^2 \leq \frac{\mu_T(\tilde{B}_r(0))}{\pi r^2} \leq \frac{\mu_T(B_{r l_1(r)}(0))}{\pi [r l_1(r)]^2} [l_1(r)]^2. \tag{3.11}$$

Since $\lim_{r \to 0} l_i(r) = 1$, for $i = 1, 2$, it follows that

$$\lim_{r \to 0} \frac{\mu_T(\tilde{B}_r(0))}{\pi r^2} = \Theta(\mu_T, 0). \tag{3.11}$$

Now, (3.8), (3.10) and (3.11) together with equation (3.7) implies that

$$\lim_{r \to 0} \frac{\mu_T(B_r(0))}{\pi r^2} = 2\Theta(\mu_T, 0). \tag{3.12}$$

\[\square\]

**Definition 4.** Consider the map defined by $\eta_{x_0, \lambda} = \lambda^{-1}(x - x_0), x \in \mathbb{R}^{n+1}$. If $x_0 = 0$, we simply write $\eta_\lambda$. Suppose $T \in D_n(U)$ is integer multiplicity and $x_0 \in \text{spt} \ T$. If there exists a sequence $\{\lambda_j\}_{j \in \mathbb{N}}$ converging to 0 and an integer multiplicity current $C \in D_n(\mathbb{R}^{n+1})$ such that

$$(\eta_{x_0, \lambda_j})_\#T \to C, \text{ and } (\eta_{x_0, \lambda_j})_\#C = C, \ \forall \ \lambda > 0,$$

we call $C$ an oriented tangent cone to $T$ at $x_0$.

**Theorem 5.** There is an oriented tangent cone $C$ to $\tilde{T}$ at $0$ such that $\text{spt} \ \partial C$ is an oriented straight line. Moreover,

1. $C$ is minimizing in $\mathbb{R}^3$, that is, for any open set $W \subset \subset \mathbb{R}^3$ and any integer multiplicity current $X \in D_2(\mathbb{R}^3)$ satisfying $\text{spt} \ X \subset W$ and $\partial X = 0$, we have

$$M_W(C) \leq M_W(C + X).$$

2. If $T_j = (\eta_{\lambda_j})_\#\tilde{T} \to C$ we have

$$\mu_{T_j} \to \mu_C, \text{ and } \Theta(C, 0) = \Theta(\mu_{\tilde{T}}, 0).$$
Proof. Consider a sequence \( \{\lambda_j\}_{j \in \mathbb{N}} \) converging to 0. By Lemma 3, we have
\[
M_{B_r(0)}(\{(\eta_{\lambda_j}) \# \tilde{T}\}) = \lambda_j^{-2} M_{B_r(0)}(\tilde{T}) = \lambda_j^{-2} \mu_{\tilde{T}}(B_{r\lambda_j}(0)) \leq cr^2.
\]
Moreover, since \( \gamma \) meets \( S \) orthogonally at \( 0 \), we have that \( \gamma \cup \Phi(\gamma) \) is a \( C^{1,1} \)-curve. So, there exists
\[
\lim_{r \to 0} \frac{\mu_{\partial \tilde{T}}(B_r(0))}{r} < +\infty,
\]
therefore,
\[
M_{B_r(0)}(\partial(\{(\eta_{\lambda_j}) \# \tilde{T}\})) = M_{B_r(0)}((\eta_{\lambda_j}) \# \partial \tilde{T}) = \lambda_j^{-1} M_{B_r(0)}(\partial \tilde{T}) \leq \tilde{c} r.
\]
Thus, from the compactness theorem for integer multiplicity currents (see [8]; 4.2.17), a subsequence of \( \{(\eta_{\lambda_j}) \# \tilde{T}\} \) converges to a current \( C \in D_2(\mathbb{R}^3) \) and the boundaries \( \partial((\eta_{\lambda_j}) \# \tilde{T}) \) converge to \( \partial C \in D_1(\mathbb{R}^3) \) (in the subsequence). Furthermore, since \( \gamma \cup \Phi(\gamma) \) is a \( C^{1,1} \)-curve, it has an oriented tangent line \( \Gamma \) at \( 0 \). In particular, \( \partial C = \Gamma \). At the end of this proof we will show that \( C \) is a cone.

Now, let \( V \) be as in (3.5). Let \( \tilde{W} \subseteq V \) be an open set such that
\[
\tilde{W} \cup \Phi(\tilde{W}) \subseteq B_R(x_0) \subseteq B_3(0), \tag{3.13}
\]
for some \( x_0 \in S \cap B_3(0) \). Let \( X \in D_2(B) \) be of integer multiplicity, such that \( \text{spt} \ X \subseteq \tilde{W} \) and \( \partial X = 0 \).

We can write
\[
X = X_1 + X_2 = X \setminus \left( B^+ \cup (S \cap B(0)) \right) + X \setminus \Phi(B^+),
\]
where \( \text{spt} \partial X_i \subseteq S, \ i = 1, 2 \), since \( \partial X = 0 \). By (3.4), we have
\[
M_{\tilde{W}}(\tilde{T} + X) = M_{\tilde{W}}(T + X_1) + M_{\tilde{W}}(\Phi # T - X_2)
\]
\[
\geq M_{\tilde{W}}(T) + M_{\tilde{W}}(\Phi # (T - \Phi # X_2))
\]
\[
\geq M_{\tilde{W}}(T) + (1 - c_1 R)^2 M_{\Phi(\tilde{W})}(T - \Phi # X_2)
\]
\[
\geq M_{\tilde{W}}(T) + (1 - c_1 R)^2 M_{\Phi(\tilde{W})}(T)
\]
\[
\geq M_{\tilde{W}}(T) + (1 - c_1 R)^4 M_{\tilde{W}}(\Phi # T)
\]
\[
\geq M_{\tilde{W}}(T) + (1 - c_2 R) M_{\tilde{W}}(\Phi # T)
\]
\[
= M_{\tilde{W}}(\tilde{T}) - c_2 R M_{\tilde{W}}(\Phi # T)
\]
\[
\geq M_{\tilde{W}}(\tilde{T}) - c_2 R M_{\tilde{W}}(\tilde{T}).
\]
Thus,
\[
M_{\tilde{W}}(\tilde{T}) \leq M_{\tilde{W}}(\tilde{T} + X) + c_2 R M_{\tilde{W}}(\tilde{T}). \tag{3.14}
\]

Let \( W \subseteq \mathbb{R}^3 \) be an open set, and fix \( r > 0 \) such that \( W \cup \Phi(W) \subseteq B_r(0) \). Let \( X \in D_2(\mathbb{R}^3) \) be integer multiplicity such that \( \text{spt} \ X \subseteq W \) and \( \partial X = 0 \). Choose a sequence
\{ \lambda_j \} \text{ with } \lambda_j \to 0 \text{ and such that }

\[ T_j := (\eta_{\lambda_j})^\# \tilde{T} \to C. \]

Define \( X_j = (\eta_{\lambda_j})_{\#}^{-1} X \). We have \( (\eta_{\lambda_j})^{-1}(W) \subset B_{\lambda_j r}(0) \); hence, for \( j \) large enough, \( \lambda_j r \leq 3 \) and \( (\eta_{\lambda_j})^{-1}(W) \subset B \). Then, it follows from (3.14) that

\[
M_W(T_j) = M_W((\eta_{\lambda_j})^\# \tilde{T}) = \lambda_j^{-2}M((\eta_{\lambda_j})^{-1}(W)) \leq \lambda_j^{-2}[M((\eta_{\lambda_j})^{-1}(W)) \tilde{T} + X_j] + c_2 \lambda_j r M((\eta_{\lambda_j})^{-1}(W)) \leq M_W(T_j + X) + c_2 \lambda_j r M_W(T_j)
\]

\[
\leq M_W(T_j + X) + c_3 \lambda_j r^3,
\]  

(3.15)

where we used Lemma 3 in the last inequality.

Since \( T_j \to C \), \( \lambda_j \to 0 \) and because of the lower-semi-continuity of mass, a standard argument using (3.15) yields that

\[
M_W(C) \leq M_W(C + X).
\]

Next, we will prove \( \mu_{T_j} \to \mu_C \) in the sense of Radon measures.

Consider a compact set \( K \subset \mathbb{R}^3 \) and an open set \( W \subset \mathbb{R}^3 \) containing \( K \). For any \( \epsilon > 0 \), define \( \phi_\epsilon : W \to [0, 1] \) such that \( \phi_\epsilon \equiv 1 \) in some neighborhood of \( K \) and

\[
\text{spt } \phi_\epsilon \subset \{ x \in \mathbb{R}^3; \text{dist}(x, K) \leq \epsilon \}.
\]

Consider

\[
W_{\alpha, \epsilon} = \{ x \in \mathbb{R}^3; \phi_\epsilon > \alpha \};
\]

hence, for \( 0 \leq \alpha < 1 \), we have \( K \subset W_{\alpha, \epsilon} \subset \mathbb{R}^3 \).

By (3.15)

\[
\limsup M_{W_{\alpha, \epsilon}}(T_j) \leq M_{W_{\alpha, \epsilon}}(C),
\]

and since \( K \subset W_{\alpha, \epsilon} \subset \{ x; \text{dist}(x, K) < \epsilon \} \) by construction, we obtain

\[
\limsup \mu_{T_j}(K) \leq M_{\{ x; \text{dist}(x, K) < \epsilon \}}(C).
\]

Hence, letting \( \epsilon \to 0 \), it follows that

\[
\limsup \mu_{T_j}(K) \leq \mu_C(K).
\]

(3.16)

By the lower semi-continuity of mass with respect to weak convergence, we have

\[
\mu_C(K) \leq \liminf \mu_{T_j}(K).
\]

(3.17)

Since (3.16) and (3.17) hold for arbitrary compact \( K \) and open \( W \subset \mathbb{R}^3 \), it follows by a standard approximation argument that

\[
\mu_{T_j} \to \mu_C.
\]

(3.18)

Finally, choose \( r > 0 \) such that \( \mu_C(\partial B_r(0)) = 0 \) (which is true except possibly for countably many \( r \)). Then, (3.18) implies that

\[
\frac{\mu_C(B_r(0))}{\pi r^2} = \lim_{j \to \infty} \frac{\mu_{T_j}(B_r(0))}{\pi r^2} = \Theta(\mu_{\tilde{T}}, 0).
\]
Thus,

$$\Theta(\mu_C, 0) = \frac{\mu_C(B_r(0))}{\pi r^2} = \Theta(\mu_{\tilde{T}}, 0).$$  \hspace{1cm} (3.19)

Arguing as in [33], Theorem 19.3, we conclude that $C$ is a cone.

\[ \square \]

**Theorem 6.** There is a neighborhood $W$ of 0 in $V$, such that $W \cap \text{spt} \tilde{T}$ is an embedded oriented $C^{1,1}$-surface with boundary.

**Proof.** Arguing as in [21], 1.5(3), we conclude that $\Theta(\mu_T, 0) = m - 1/2$ and

$$C = L_{\#} \left[ mE^2 \sqcup \{(a, b); b > 0\} \cup (m - 1)E^2 \sqcup \{(a, b); b < 0\} \right], \hspace{1cm} (3.20)$$

where $m \in \mathbb{N}$, $L : \mathbb{R}^2 \to \mathbb{R}^3$ is an injective linear transformation and $E^2$ denotes the multiplicity one 2-current whose support is the Euclidean plane. In fact, by the interior regularity, the interior of $C$ is supported in a union of half-planes. Now, suppose (3.20) is not true, so there are at least two different half-planes contained in $\text{spt} C$ which are not co-planar. Consider two such half-planes and denote by $C_*$ the current obtained from $C$ restricting to the half-planes. Consider a plane $P$ orthogonal to $\partial C$. By [8] 5.4.8, the current obtained from $C_*$ by intersecting with $P$ is length-minimizing, however since the chosen half-planes are not co-planar this contradicts the triangular inequality. Therefore, (3.20) holds. By [8] 4.1.31(2) we have $\Theta(\mu_T, 0) = m - 1/2$.

Now, we can proceed as in the step I of Theorem 11.1 of [21] to finish the proof. Let $x_1, x_2, \cdots, x_n$ denote the canonical coordinates of $\mathbb{R}^n$. Applying a reflection and a rotation if necessary we can suppose that the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^3$ mentioned on the last paragraph is given by $L(x_1, x_2) = (x_1, x_2, 0)$, so that $\text{spt} C = \{(x_1, x_2, 0); x_1, x_2 \in \mathbb{R}\}$.

Choose a sequence of positive numbers $\{\lambda_j\}$ such that $\lambda_j \to 0$ and

$$\frac{1}{3} \text{dist}(0, \partial V) > \lambda_j, \forall j$$

$$(\eta_{\lambda_j})_{\#} \tilde{T} \to C.$$

Define $p : \mathbb{R}^2 \to \mathbb{R}^3$, $p(x_1, x_2, x_3) = (x_1, x_2)$, and

$$C_r = \{x \in \mathbb{R}^3; |p(x)| \leq r\}.$$

By [8] 5.4.2, we have

$$\lim_{j \to \infty} \sup_{B_{\lambda_j}(0) \cap \text{spt} \tilde{T}} \lambda_j^{-1}x_3 = 0$$

$$\lim_{j \to \infty} \lambda_j^{-2}M \left( p_{\#} [\tilde{T} \cup B_{3\lambda_j}(0) \cap C_{\lambda_j}] \right) = \left( m - \frac{1}{2} \right) \pi.$$

Thus, we conclude as in corollary 9.3 of [21] (adapting the arguments) that there is $\epsilon > 0$ such that for some fixed (sufficiently large) $j$

$$p^{-1}(V_\epsilon) \cap \text{spt}(\eta_{\lambda_j})_{\#} \tilde{T} = \bigcup_{i=1}^{m} \text{graph}(v_i)$$

$$p^{-1}(W_\epsilon) \cap \text{spt}(\eta_{\lambda_j})_{\#} \tilde{T} = \bigcup_{i=1}^{m-1} \text{graph}(w_i),$$
where

\[ V_{\epsilon} = \{ (x_1, x_2); x_1^2 + x_2^2 < \epsilon \} \cap \{ x = (x_1, x_2); x_2 > (x_1^2 + x_2^2)^{15/28} \} \]

\[ W_{\epsilon} = \{ (x_1, x_2); x_1^2 + x_2^2 < \epsilon \} \cap \{ x = (x_1, x_2); x_2 < (x_1^2 + x_2^2)^{15/28} \} , \]

and \( v_i \in C^{1, 1}(V_{\epsilon}) \), \( w_i \in C^{1, 1}(W_{\epsilon}) \), such that \( v_i|_{V_{\epsilon}} \), \( w_i|_{W_{\epsilon}} \) satisfy an elliptic equation, \( Dv_i(0, 0) = 0 = Dw_i(0, 0) \) and

\[ v_1 \leq v_2 \leq \cdots \leq v_m, \quad w_1 \leq w_2 \leq \cdots \leq w_{m-1}. \]

**Case 1 -** \( m = 1 \): We have \( \Theta(\mu_T, 0) = 1/2 \), thus the result follows from the Regularity Theorem for varifolds proved in section 4 of [1].

**Case 2 -** \( m > 1 \): Using the Hopf type maximum principle, lemma 10.1 of [21] we infer

\[ v_1 = v_2 = \cdots = v_m, \quad w_1 = w_2 = \cdots = w_{m-1}. \]

Let \( G \) be the component of the regular points of \( B_{\lambda_j}(0) \cap (\text{spt} \tilde{T} \setminus \text{spt} \partial T) \) which contains \( \eta_{\lambda_j}^{-1}(\text{graph} v_1) \). By [8] 4.1.31(2) it follows that the density function \( \Theta(\mu_{\tilde{T}}) \) is identically \( m \) on \( G \). Denote \( S = m^{-1}(\tilde{T}|_G) \). Since

\[ \mu_{\tilde{T}} = \mu_S + \mu_{\tilde{T} - S}, \]

we have that \( S \) is absolutely area minimizing.

Setting \( U = B_{\lambda_j}(0) \cap C_{\lambda_j} \), using [9] and [8] 4.1.21, we conclude that

\[ \text{spt}[(\partial S) \cup U] \subset \text{spt}[(\partial \tilde{T}) \cup U]. \]

The constancy theorem (see [8] 4.1.7) implies

\[ p_#[(\partial S) \cup U] = [\partial m^{-1} p_#(\tilde{T} \cup G \cap U)] \cup p(U) = [\partial p_#(\tilde{T} \cup U)] \cup p(U) = p_#[(\partial \tilde{T}) \cup U]. \]

Since \( p \) is a \( C^\infty \) diffeomorphism it follows that \( (\partial S) \cup U = (\partial T) \cup U \). Therefore we can apply the interior regularity theory [8] 5.3.18 to \( T - S \) and conclude that

\[ W \cap \text{spt} \tilde{T} = W \cap \text{spt}(\tilde{T} - S) \]

is a smooth embedded surface in some open neighbourhood \( W \) of \( 0 \).

\[ \square \]

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