Adversarial Robustness Through Local Lipschitzness

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Abstract
A standard method for improving the robustness of neural networks is adversarial training, where the network is trained on adversarial examples that are close to the training inputs. This produces classifiers that are robust, but it often decreases clean accuracy. Prior work even posits that the tradeoff between robustness and accuracy may be inevitable. We investigate this tradeoff in more depth through the lens of local Lipschitzness. In many image datasets, the classes are separated in the sense that images with different labels are not extremely close in \( \ell_\infty \) distance. Using this separation as a starting point, we argue that it is possible to achieve both accuracy and robustness by encouraging the classifier to be locally smooth around the data. More precisely, we consider classifiers that are obtained by rounding locally Lipschitz functions. Theoretically, we show that such classifiers exist for any dataset such that there is a positive distance between the support of different classes. Empirically, we compare the local Lipschitzness of classifiers trained by several methods. Our results show that having a small Lipschitz constant correlates with achieving high clean and robust accuracy, and therefore, the smoothness of the classifier is an important property to consider in the context of adversarial examples.

1. Introduction
A growing body of research shows that neural networks are vulnerable to adversarial examples, test inputs that have been modified slightly yet strategically to cause misclassification (Szegedy et al., 2013; Goodfellow et al., 2015). The standard defense against adversarial examples is adversarial training (Madry et al., 2018), where the neural network is trained directly on adversarial examples that are close to the training inputs. This produces classifiers that have high accuracy on adversarial inputs. Unfortunately, adversarial training and its variants often hurt clean accuracy on many datasets (Raghunathan et al., 2019; Madry et al., 2018). This observation has led prior works to claim that the tradeoff between robustness and accuracy may be inevitable for many classification tasks (Tsipras et al., 2019; Zhang et al., 2019).

If this is indeed the case, then robust machine learning technology is unlikely to be very useful in practice. The vast majority of instances encountered by practical systems will likely be natural examples, whereas adversaries are few and far between. A self-driving car will mostly encounter regular street signs and rarely come across adversarial ones. If increased robustness comes with a loss in performance on natural examples, then the system’s designer might be tempted to use a highly accurate classifier that is obtained through regular training and forego robustness altogether. For adversarially robust machine learning to be useful, accuracy needs to be achieved in conjunction with robustness.

We investigate when it is possible to achieve both high clean accuracy and high robustness to adversarial examples. Our starting point is the simple observation that many datasets have a natural separation property. For example, images from different classes often are somewhat far apart in \( \ell_2 \) or \( \ell_\infty \) from each other in feature space. We show that this implies that, in theory, there are robust classifiers with high clean accuracy that are also locally smooth. Of course, the classifiers have very non-linear decision boundaries, and the challenge lies in developing an efficient training procedure to find such classifiers when using large neural networks.

Motivated by this observation, we consider local smoothness as a guiding principle for achieving both accuracy and robustness. More precisely, classifiers such as neural networks are typically based on rounding a continuous function \( f \) by taking the sign for binary classification. The training process that achieves low training loss ensures that the underlying function \( f(x) \) is bounded away from zero on the training data. For the simplest case of linear classification with norm-bounded \( f \), this also guarantees that the training data has large spatial margin (it lies far away from the decision boundary). For neural networks, however, having a large

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1Code available at https://github.com/yangarbiter/robust-local-lipschitz.
We next aim to understand whether local Lipschitzness is achievable together – provided we use a training algorithm that produces a locally Lipschitz classifier. Previous work has not identified local Lipschitzness as an integral component of robustness, and we believe that our results will help guide the design of classifiers that achieve higher adversarial robustness and accuracy may indeed be correlated with improved local Lipschitzness. We observe this trend across multiple training methods, and in particular, we see that adversarial training (Madry et al., 2018) does improve local Lipschitzness, and so does imposing a locally linear regularization (Qin et al., 2019). The highest degree of local Lipschitzness is achieved by TRADES (Zhang et al., 2019), a regularization method that directly aims to impose a soft smoothness constraint on \( f \) in a small ball around training examples.

But which training methods lead to such functions? We next investigate this question empirically for several training methods that plausibly encourage smoothness. We see that natural training, as suspected, produces classifiers that have poor local Lipschitzness. A plausible way to achieve local Lipschitzness is gradient regularization, where the goal is to drive gradients to zero at training examples. Perhaps surprisingly, we see that this process does not produce very locally Lipschitz functions. We see that adversarial training (Madry et al., 2018) does improve local Lipschitzness, and so does imposing a locally linear regularization (Qin et al., 2019). The highest degree of local Lipschitzness is achieved by TRADES (Zhang et al., 2019), a regularization method that directly aims to impose a soft smoothness constraint on \( f \) in a small ball around training examples.

We next aim to understand whether local Lipschitzness is indeed correlated with both high robustness and accuracy in practice. Through an empirical study, we find that for several datasets, higher robustness and accuracy are indeed correlated with improved local Lipschitzness. We observe this trend across multiple training methods, and in particular, we see that adversarial training (Madry et al., 2018) and TRADES (Zhang et al., 2019) have favorable robustness-accuracy tradeoffs when they also have small Lipschitz constants. We also show that other methods, such as gradient regularization (Ross & Doshi-Velez, 2018) and local linearization (Qin et al., 2019), have much higher adversarial accuracy than natural training, but worse than adversarial training and TRADES. These other methods seem to interpolate between natural training and more robust methods. This suggests that robustness and accuracy may indeed be achievable together – provided we use a training algorithm that produces a locally Lipschitz classifier. Previous work has not identified local Lipschitzness as an integral component of robustness, and we believe that our results will help guide the design of classifiers that achieve higher adversarial robustness and clean accuracy on challenging datasets.

2. Robustness and Accuracy via Lipschitzness

We first prove that if the distribution of inputs is separated – the distance between points from different classes is larger than the amount of adversarial perturbation – then both robustness and perfect accuracy are achievable through a classifier based on rounding a locally Lipschitz function.

2.1. Definitions

We consider an underlying metric \( d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+ \) on the instance space \( \mathcal{X} \subseteq \mathbb{R}^d \); this is the metric in which robustness will be measured. Let \( x \in \mathcal{X} \) denote an input instance with associated label \( y \in \{-1, +1\} \). For simplicity, we present our theoretical results for binary classification; the extension to multiclass is straightforward, though notationally cumbersome. We refer to a continuous function underlying a classifier by \( g : \mathcal{X} \to \mathbb{R} \), which maps an instance in \( \mathcal{X} \) to a confidence value associated with being positive. The associated classifier is \( g = \text{sign}(f(\cdot)) \).

Robustness and Astuteness. We use the standard notation \( \mathcal{B}(x, \epsilon) \) to represent a neighborhood of \( x \): \( \{x' \in \mathcal{X} : d(x, x') \leq \epsilon\} \). A classifier \( g \) is robust at an example \( (x, y) \) with radius \( r \) if for all \( x' \in \mathcal{B}(x, r) \), \( g(x') = g(x) \). Such a \( g \) is also astute at \( (x, y) \) if in addition, \( g(x) = y \) has the correct label (Wang et al., 2018). The goal of robust classification is find a \( g \) with the highest astuteness.

Local Lipschitzness. We first define local Lipschitzness theoretically, while Section 3 provides an empirical way to estimate this quantity.

Definition 1. A function \( f : \mathcal{X} \to \mathcal{R} \) is \( L \)-Locally Lipschitz in a radius \( r \) around a point \( x \in \mathcal{X} \), if for all \( x' \) such that \( d(x, x') \leq r \), we have \( |f(x) - f(x')| \leq L \cdot d(x, x') \).

Figure 1 depicts a case where higher astuteness is obtained...
by only enforcing smoothness close to actual inputs (i.e., locally Lipschitz around the training data).

It would be reasonable to consider classifiers that satisfy a global Lipschitz property in function space. Unfortunately, it seems that globally Lipschitz classifiers are either not rich enough to fit the data accurately or the optimization process for these classifiers is suboptimal (Anil et al., 2019; Huster et al., 2018). In either case, this restriction appears to be too severe. Hence, we relax the smoothness criteria, and we aim instead for classifiers that are locally Lipschitz close to the input data.

2r-Separated Data. Unlike general distributions, we assume for the purpose of this work that inputs with positive and negative labels do not overlap. This assumption is valid for a number of real-world tasks, such as image classification. We use the notation $X^+ \subseteq X$ to denote the support of the positive examples, and $X^- \subseteq X$ to denote the support of the negative examples under the data distribution; observe that for data distributions with non-overlapping classes, $X^+ \cap X^- = \emptyset$. In the sequel, we talk about data distributions with separated classes, defined as follows.

**Definition 2.** We say that a data distribution is 2r-separated if $d(X^+, X^-) \geq 2r$, where

$$d(X^+, X^-) = \min_{a \in X^+, b \in X^-} d(a, b).$$

Many real classification tasks comprise of separated classes; for example, if $d$ is the $\ell_{\infty}$ norm, then images with different categories (e.g., dog, cat, panda, etc) will be 2r-separated for some value $r > 0$ depending on the feature space.

### 2.2. Existence Proof

We now show that it is theoretically possible to achieve both robustness and accuracy for 2r-separated data. More precisely, assuming that the classes are 2r-separated, we demonstrate the existence of a classifier that is based on a locally Lipschitz function, and has astuteness 1 at radius $r$. Our result makes no other assumptions about the input data. Here we present the results for binary classification, and we extend these results to multiple classes in Appendix A.

We begin by showing that accurate classifiers that are based on locally Lipschitz functions that are bounded away from zero on an input $x$ are also astute at $x$.

**Lemma 2.1.** Let $f : X \rightarrow R$ be a function and let $x \in X$. If (a) $f$ is $\frac{1}{r}$-Locally Lipschitz in a radius $r$ around $x$ (b) $|f(x)| \geq 1$ and (c) $g(x)$ has the same sign as $y$, then the function $g = \text{sign}(f)$ is astute at $x$ with radius $r$.

**Proof.** Suppose $x' \in X$ satisfying $d(x, x') \leq r$. Without loss of generality, suppose that $f(x) > 0$. By the assumptions that $f$ is $\frac{1}{r}$-Locally Lipschitz and $|f(x)| \geq 1$, we have

$$|f(x')| = |f(x)| \cdot \frac{d(x, x')}{r} \leq \frac{1}{r} \cdot d(x, x') \leq \frac{d(x, x')}{r}.$$

Lemma 2.1 suggests that for classifier $f = \text{sign}(g)$, astuteness can be encouraged by enforcing local Lipschitzness – or, a gradual ramp from low to high confidence.

A natural question is whether a perfectly astute classifier exists; in fact, our next theorem shows that there is always such a classifier when the data distribution is 2r-separated.

**Theorem 2.2.** Suppose the data distribution is 2r-separated. Then, there exists a function $f$ such that (a) $f$ is $\frac{1}{r}$-Locally Lipschitz in a ball of radius $r$ around all $x \in X^+ \cup X^-$ and (b) $g = \text{sign}(f)$ has astuteness 1.

**Proof.** We first show that if the distribution is 2r-separated, then there exists a function $f : X \rightarrow R$ satisfying:

1. If $x \in X^+ \cup X^-$, then $f(x)$ is $\frac{1}{r}$-Locally-Lipschitz in a ball of neighbor $r$ around $x$.
2. If $x \in X^+$, $f(x) \geq 1$; if $x \in X^-$, $f(x) \leq -1$.

Let $f(x) = \frac{d(x, X^-) - d(x, X^+)}{2r}$. Then, for any $x$, we have:

$$f(x) - f(x') = \frac{d(x, X^-) - d(x, X^+) - d(x', X^-) + d(x', X^+)}{2r}\leq \frac{2d(x, x')}{2r}.$$

where the first step follows from two applications of the triangle inequality. This establishes (1). To establish (2),
suppose without loss of generality that $x \in \mathcal{X}^+$. Then, $f(x) = \frac{d(x, x^-)}{2} \geq \frac{d(x^+, x^-)}{2} \geq 1$, since the two classes are well-separated. The case of $x \in \mathcal{X}^-$ is symmetric.

Now observe that by construction, $f$ satisfies all three conditions in Lemma 2.1 at all $x \in \mathcal{X}^+ \cup \mathcal{X}^-$; thus, applying Lemma 2.1, we get that $g = \text{sign}(f)$ has astuteness 1 under the data distribution.

A visualization of the function (and resulting classifier) from Theorem 2.2 appears in Figure 2. Dark colors indicate high confidence (far from decision boundary) and lighter colors indicate the gradual change from one label to the next.

Observe that the classifier $g = \text{sign}(f)$ guaranteed by this theorem will predict the label based on which decision region (positive or negative) is closer to the input example. While $\text{sign}(f)$ bears some similarity to the 1-nearest-neighbor classifier, it is actually different on any finite sample, and the classifiers only coincide in the limit when the supports of the two classes are exactly known. In practice, however, we only have a sparse sample of the underlying decision regions. Therefore, even though the locally Lipschitz function $f$ exists, it may be quite challenging to find. In the next section, we consider some plausible optimization methods that may be used to find such functions.

3. Empirical Results

So far we have theoretically shown that local Lipschitzness is connected to adversarial robustness and accuracy. We now empirically validate this connection. In particular, this section considers the following questions:

- Is local Lipschitzness correlated with robustness and accuracy in practice?
- Which training methods produce classifiers that are based on locally Lipschitz functions?

These questions are considered in the context of one synthetic and four real datasets, as well as several plausible training methods for improving adversarial robustness.

3.1. Experimental Methodology

We evaluate train/test clean accuracy, test adversarial accuracy, and test local lipschitzness of neural networks trained using different methods. We also measure generalization gaps: the difference between train and test clean accuracy (or between train and test adversarial accuracy). Experiments run with NVIDIA GeForce RTX 2080 Ti GPUs. The experiment code can be found in a public repository.\(^2\)

3.1.1. Baselines

We consider neural networks trained via Natural training (Natural), Gradient Regularization (GR) (Finlay & Oberman, 2019), Locally Linear Regularization (LLR) (Qin et al., 2019), Adversarial Training (AT) (Madry et al., 2018), and TRADES (Zhang et al., 2019).

**Gradient Regularization (GR).** The Gradient Regularization (GR) is in the form of soft regularization. We use the latest work by Finlay & Oberman (2019) for our experiments. In general, GR models can be formulated as adding a regularization term on the norm of gradient of the loss function:

$$
\min_f \mathbb{E}\left\{ L(f(X), Y) + \beta \left\| \nabla_X L(f(X), Y) \right\|^2 \right\}.
$$

Finlay & Oberman (2019) compute the gradient term through a finite difference approximation. Let $d = \frac{\nabla f(X)}{\|\nabla f(X)\|_2}$ and $h$ be the step size. Then,

$$
\left\| \nabla f(X) \right\|^2 \approx \left( \frac{L(f(X + hd), Y) - L(f(X), Y)}{h} \right) - L(f(X), Y)
$$

We use the publicly available implementation.\(^3\)

**Locally-Linear Regularization model (LLR).** Qin et al. (2019) propose to regularize the local linearity through the motivation that AT with PGD increases the model’s local linearity. The authors first formulate the function $g$ to evaluate the local linearity of a model.

$$
g(f, \delta, X) = |L(f(X + \delta), Y) - L(f(X), Y)| - \delta^T \nabla_X L(f(X), Y)|
$$

Define $\gamma(\epsilon, X) = \mathbb{E}\left\{ \max_{\delta \in B(X, \epsilon)} g(f, \delta, X) \right\}$ and also $\delta_{LLR} = \mathbb{E}\left\{ \text{argmax}_{\delta \in B(X, \epsilon)} g(f, \delta, X) \right\}$. The loss function for Locally-Linear Regularization (LLR) model is

$$
\mathbb{E}\left\{ L(f(X), Y) + \lambda \gamma(\epsilon, X) + \mu \left\| \delta_{LLR}^T \nabla_X L(f(X), Y) \right\| \right\}
$$

We use our own implementation of LLR.

**Adversarial training (AT).** Adversarial training is a successful defense by Madry et al. (2018) that trains based on adversarial examples:

$$
\min_f \mathbb{E}\left\{ \max_{X \in B(X, \epsilon)} L(f(X'), Y) \right\}.
$$

In this section, we particularly refer to adversarial training the model of (1) by Madry et al. (Madry et al., 2018).

**Locally-Lipschitz models (TRADES).** One of the best methods for robustness via smoothness is TRADES (Zhang

\(^2\)https://github.com/yangarbiter/robust-local-lipschitz

\(^3\)https://github.com/cfinlay/tulip
We estimate this through a PGD-like procedure, where we iteratively take a step towards the gradient direction (Krizhevsky et al., 2009) and Restricted ImageNet (Tsipras et al., 2019) uses the publicly available implementation.

The MT attack results appear in Appendix B.1.

3.1.2. ADVERSARIAL ATTACKS

PGD. We use projected gradient descent (PGD) (Kurakin et al., 2017) to evaluate the adversarial training and testing accuracy in this section. The step size is set to $\epsilon/5$ and we perform a total of 10 steps.

$$\arg\max_{X'} \mathcal{L}(f(X'), Y)$$

Multi-Targeted Attacks The empirically stronger Multi-Targeted Attack (MT) (Gowal et al., 2019) uses a surrogate loss (Equation (2)) along with projected gradient descent to approximate this loss. The inner maximization is solved using projected gradient descent with the step size set to $\epsilon/10$ and perform a total of 20 steps.

$$\arg\max_{X' \in B(X, \epsilon)} \max_{Y' \in C} f_{Y'}(X') - f_Y(X)$$

The MT attack results appear in Appendix B.1.

3.1.3. MEASURING LOCAL LIPSHCITNESS

For each classifier, we empirically measure the local Lipschitzness of the underlying function by the quantity

$$\frac{1}{n} \sum_{i=1}^{n} \max_{x_i' \in B(x_i, \epsilon)} \frac{\|f(x_i) - f(x_i')\|_1}{\|x_i - x_i'\|_\infty}.$$  \hspace{1cm} (3)

We estimate this through a PGD-like procedure, where we iteratively take a step towards the gradient direction $(\nabla_{x_i'} \|f(x_i) - f(x_i')\|_1)$ with radius $\epsilon$ being the perturbation radius of the given dataset, step size $\epsilon/5$ and a total of 10 steps. We study the correlation of this quantity (3) and accuracy/robustness.

3.1.4. DATASETS

We evaluate the various algorithms on one synthetic dataset: Staircase (Raghunathan et al., 2019) and four real datasets: MNIST (LeCun), SVHN (Netzer et al., 2011), CIFAR-10 (Krizhevsky et al., 2009) and Restricted ImageNet (Tsipras et al., 2019). We consider adversarial $\ell_\infty$ perturbations for all datasets. More details are in Appendix C.

Synthetic Staircase setup. As a toy example, we first consider a synthetic regression dataset, which is known to show that adversarial training can seriously overfit when the sample size is too small (Raghunathan et al., 2019). We use the code provided by the authors to reproduce the result for natural training and AT, and we add results for GR, LLR, and TRADES. The model for this dataset is linear regression in a kernel space using cubic B-splines as the basis. Let $F$ be the hypothesis set and the regularization term $\|f\|^2$ is the RKHS norm of the weight vector in the kernel space. The regularization term is set to $\lambda = 0.1$ and the result is evaluated using the mean squared error (MSE). For GR, we set $\beta = 10^{-4}$ and for LLR, we only use the local linearity $\gamma$ for regularization and the regularization strength is $10^{-2}$. The perturbation set $P(x, \epsilon) = \{x - \epsilon, x, x + \epsilon\}$ considers only the point-wise perturbation.

MNIST setup. We use two different convolutional neural networks (CNN) with different capacity. The first CNN (CNN1) has two convolutional layers followed by two fully connected layers and the second larger CNN (CNN2) has four convolutional layers followed by two fully connected layers. We set the perturbation radius to 0.1.

SVHN setup. We use the wide residual network WRN-40-10 (Zagoruyko & Komodakis, 2016) and set the perturbation radius to 0.031. The initial learning rate is set to 0.01 except LLR. We are not able to get decent performance for LLR with initial learning rate set to 0.01, thus, we set it to 0.001.

CIFAR10 setup. Following (Madry et al., 2018; Zhang et al., 2019), we use the wide residual network WRN-40-10 (Zagoruyko & Komodakis, 2016) and set the perturbation radius to 0.031. We test the model with and without data augmentation. When performing data augmentation, we randomly crop the image to $32 \times 32$ with 4 pixels of padding then perform random horizontal flips.

Restricted ImageNet setup. Following (Tsipras et al., 2019), we set the perturbation radius $\epsilon = 0.005$, use the residual network (ResNet50) (He et al., 2016) and use Adam (Kingma & Ba, 2014) to optimize the network. Data augmentation is performed. During training, each image is resized to $72 \times 72$ and randomly cropped to $64 \times 64$ with 8 pixels of padding. When evaluating, the image is resized to $72 \times 72$, we crop in the center resulting a $64 \times 64$ image. We set the initial learning rate to 0.01.

3CNN1 is retrieved from pytorch repository https://github.com/pytorch/examples/blob/master/mnist/main.py

6CNN2 is retrieved from TRADES (Zhang et al., 2019) github repository https://github.com/yaodongyu/TRADES/blob/master/models/small_cnn.py

4https://github.com/yaodongyu/TRADES

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3.2. Results

Synthetic Staircase Results. We begin with the small synthetic Staircase dataset ($n = 40$), where the goal is to minimize the clean/robust MSE. Prior work specifically constructs this dataset as an example where AT overfits the data (Raghunathan et al., 2018). We see that all methods have small training MSE (between .0059 and .0064), but they vary quite a bit in terms of test MSE, adversarial MSE, and Lipschitzness. Natural training has the smallest test MSE (.0192), as expected for this example, while it also has the highest local Lipschitzness (1.287) and it comes in at the middle in terms of adversarial test MSE (.1926). Adversarial training has the second highest Lipschitzness (.5396), the largest test MSE (.3360), and second largest adversarial test MSE (.3720). Hence, adversarial training performs badly on this dataset. For the three TRADES models, the Lipschitz constant decreases monotonically as we increase the TRADES parameter $\beta$ from one to six. The Lipschitz constant for TRADES($\beta=6$) is much smaller (.3870) than the other methods. LLR achieves a middle ground, with test MSE and Lipschitzness both in the middle. In this sense, LLR interpolates between the extremes of natural training and TRADES($\beta=6$), which is a trend we also see in real datasets. Overall, there is a consistent correlation between low clean/adversarial MSE and small Lipschitz constant. We will further investigate this connection on multiple real datasets. This toy experiment points out a salient difference between AT and TRADES, where AT overfits on this dataset with high test MSE and high adversarial MSE, whereas TRADES performs very well. We note that this difference between AT and TRADES is only apparent on the small Staircase dataset, and we show in the appendix that both methods perform well with more samples, which is consistent with the findings of Raghunathan et al. (2018).

MNIST Results. We evaluate all the training methods on MNIST in Tables 2 and 3, where we consider a small network (CNN1) and a larger network (CNN2). For both CNNs, natural training leads to high Lipschitz constants, while the other training methods lead to much smaller Lipschitz constants. Just as with the Staircase dataset, we observe that having smaller Lipschitz values correlates with higher adversarial training/testing accuracy. Also, LLR generally has the same clean accuracy as natural training but higher robustness. For this dataset, AT performs fairly well with both CNNs. In the case of CNN2, TRADES($\beta=6$) has the highest adversarial test accuracy (98.73), and also the smallest Lipschitz constant; it also achieves a very high clean test accuracy (99.59). We also see that increasing the TRADES parameter drives down the Lipschitz constant.

SVHN Results. Table 4 shows the results for the SVHN dataset. We again see fairly consistent correlation between accuracy, Lipschitzness, and adversarial accuracy. Natural training leads to the highest clean accuracy, and the highest Lipschitz constants, while having low adversarial test accuracy. GR performs somewhat poorly on this dataset. LLR has a clean test accuracy very close to natural training, but delivers a significantly better adversarial test accuracy. AT and TRADES both perform fairly well, where their adversarial accuracy increases and their Lipschitzness decreases compared to the other methods. This is consistent with our findings in MNIST. On the other hand, the gaps and adversarial gaps for this dataset are much higher compared to MNIST, for all of the methods.
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|                          | train accuracy | test accuracy | adv test accuracy | test lipschitz | gap  | adv gap |
|--------------------------|----------------|---------------|-------------------|----------------|------|--------|
| Natural                  | 100.00         | 88.62         | 0.00              | 356.46         | 11.38| 0.00   |
| GR                       | 99.71          | 71.68         | 13.73             | 33.27          | 28.03| 9.22   |
| LLR                      | 100.00         | 85.83         | 20.44             | 79.08          | 14.17| 11.04  |
| AT                       | 99.98          | 74.00         | 33.09             | 16.90          | 25.98| 68.60  |
| TRADES(\(\beta = 1\))   | 100.00         | 81.61         | 38.06             | 30.79          | 18.39| 61.94  |
| TRADES(\(\beta = 3\))   | 99.99          | 79.98         | 37.03             | 27.38          | 20.01| 62.95  |
| TRADES(\(\beta = 6\))   | 99.79          | 78.34         | 37.33             | 19.79          | 21.45| 62.42  |

Table 5. CIFAR-10 without data augmentation.

|                          | train accuracy | test accuracy | adv test accuracy | test lipschitz | gap  | adv gap |
|--------------------------|----------------|---------------|-------------------|----------------|------|--------|
| Natural                  | 100.00         | 93.81         | 0.00              | 425.71         | 6.19 | 0.00   |
| GR                       | 94.90          | 80.74         | 21.32             | 28.53          | 14.16| 3.94   |
| LLR                      | 100.00         | 91.44         | 22.05             | 94.68          | 8.56 | 4.50   |
| AT                       | 99.84          | 83.51         | 43.51             | 26.23          | 16.33| 49.94  |
| TRADES(\(\beta = 1\))   | 99.76          | 84.96         | 43.66             | 28.01          | 14.80| 44.60  |
| TRADES(\(\beta = 3\))   | 99.78          | 85.55         | 46.63             | 22.42          | 14.23| 47.67  |
| TRADES(\(\beta = 6\))   | 98.93          | 84.46         | 48.58             | 13.05          | 14.47| 42.65  |

Table 6. CIFAR-10 with data augmentation.

CIFAR Results. Moving on to CIFAR in Tables 5 and 6, we again see correlation between accuracy, Lipschitzness, and adversarial accuracy, which remains present with and without data augmentation. Overall, data augmentation improves the clean and adversarial accuracy for most methods. GR and LLR have much higher adversarial accuracy than natural training, but they perform worse than AT and TRADES. When using data augmentation, TRADES(\(\beta = 6\)) achieves the highest adversarial test accuracy (48.58), and also the lowest Lipschitz constant (13.05), which is consistent with the previous results. We note that AT also achieves high clean and adversarial accuracy on CIFAR, and it also has a lower Lipschitz value compared to natural training. Interestingly, the results show that TRADES may not always perform better than AT, but it seems like a very effective method to produce classifiers with small Lipschitz constants.

Restricted ImageNet Results. On the Restricted ImageNet dataset, the accuracy results are more mixed among the different methods, partially because ImageNet is a more challenging classification task than MNIST and CIFAR. Again natural training has the largest Lipschitz constant and the smallest adversarial accuracy, while LLR interpolates between AT and TRADES. The best performing methods for this dataset are AT and TRADES(\(\beta = 6\)), which have significantly smaller Lipschitz constants compared to GR and LLR, and they also have higher adversarial accuracy. For this large dataset, it seems that TRADES(6) underfits more than AT, which leads to slightly lower training accuracies, and also corroborates the trade-off between accuracy and robustness that is originally noted by (Zhang et al., 2019). On the other hand, the increased robustness of LLR indicates that it is possible to achieve high clean accuracy and moderate adversarial test accuracy at the same time.

3.3. Discussion

Our experimental results provide many insights into the role that Lipschitzness plays in classifier accuracy and robustness. For most datasets and training methods, locally smooth functions achieve a higher adversarial accuracy than functions with large Lipschitz constants. This highlights that Lipschitzness is just as important as training with adversarial examples when it comes to improving the adversarial robustness. Our second experimental goal involves understanding which training methods result in locally smooth functions. For all of the datasets, TRADES always leads to significantly smaller Lipschitz constants than most methods, and the smoothness increases with the TRADES parameter. Thus, the TRADES loss function is a very effective way to encourage the classifier to be smooth while preserving accuracy. On the other hand, the correlation between smoothness and robustness suffers from diminishing returns, and hence, it is not optimal to minimize the Lipschitzness as much as possible. For example, on ImageNet, TRADES models begin to drop in clean accuracy as \(\beta\) increases even though the Lipschitz constant continues to go down.

Is the accuracy-robustness trade-off really necessary? The main downside of AT and TRADES is that the clean accuracies sometimes suffer as a result of the increased robustness. We believe that this issue may not be inherent to robustness, but rather it may be possible to achieve the best of both worlds. We notice that LLR is consistently more robust than natural training, while simultaneously achieving state-of-the-art clean test accuracy. Furthermore, adversarial training and TRADES often have the highest adversarial accuracy, but suffer from lower clean accuracy. This leaves open the possibility of combining the benefits of both LLR and AT/TRADES into a classifier that does well across the board. We posit that this requires both an appropriate loss function and a finely-tuned optimization algorithm. In particular, one serious issue to address is underfitting. For example, in the MNIST experiments in Tables 2 and 3, TRADES(\(\beta = 6\)) underfits with a smaller network (CNN1) while it performs the best with a larger network (CNN2). In terms of smoothness, TRADES with
CNN2 has a more significant correlation between Lipschitz constant and accuracy compared to CNN1. In all of the experiments, when the TRADES parameter increases, it starts to underfit, so it seems that TRADES is a form of regularization, and it requires a larger network with good parameter tuning to perform well.

Robustness requires some local Lipschitzness. A central takeaway from our experiments is that very high Lipschitz constants imply that the classifier is vulnerable to adversarial examples. We see this most clearly with natural training, but it is also evidenced by GR and LLR on certain datasets. For MNIST and CIFAR, the experiments show that minimizing the Lipschitzness goes hand-in-hand with maximizing the adversarial accuracy. We observe that there is inherent similarity in computing Lipschitzness and computing adversarial accuracy. More precisely, both rely on the direction with the biggest change in function value, as this is often the best way to change the classifier label. This partially explains why the TRADES loss function is a good surrogate for adversarial robustness – optimizing for local Lipschitzness makes it more difficult to change the label with a small perturbation. We leave open the possibility that adversarial training leads to classifiers that benefit from less understood properties of the training process. This becomes more apparent on larger datasets, and there might be some form of implicit regularization involved. Overall, local Lipschitzness is an important property to consider in the context of adversarial robustness, and any robust classifier should be more locally smooth than natural training.

Generalization gaps. For the MNIST experiments with CNN2, we see that TRADES($\beta=6$) has a small drop in accuracy going from clean training accuracy to clean test accuracy (difference 0.02 = 99.61 – 99.59), and also going from adversarial training accuracy to adversarial test accuracy (diff. 0.80 = 99.53 – 98.73). In contrast, we see larger drops for AT in both clean (diff. 0.50 = 99.98 – 99.48) and adversarial (diff. 1.92 = 99.95 – 98.03) accuracies. However, this trend becomes much less clear on the other datasets. The gaps become larger, and there is less correlation with Lipschitzness. Thus, local Lipschitzness seems to be a good quantity to consider for accuracy and robustness, but it is not a very consistent indicator of generalization.

Why not globally Lipschitz? The smoothest classifier would be one with a small global Lipschitz constant, over the whole input space. However, previous work has shown that globally Lipschitz methods suffer from poor expressive power and low accuracy (Anil et al., 2019; Huster et al., 2018). In contrast, adversarial training and locally Lipschitz methods (e.g., TRADES) enjoy better expressibility, achieving high clean and robust accuracy on real datasets.

4. Related Work

Most previous work on adversarial robustness has focused on developing increasingly sophisticated attacks and defenses (Carlini & Wagner, 2017; Liu et al., 2017; Szegedy et al., 2013; Lowd & Meek, 2005; Madry et al., 2018; Papernot et al., 2017, 2015; Sinha et al., 2018). While some prior work has noted that robustness is sometimes accompanied by a loss in accuracy, the phenomenon remains ill-understood. Tsipras et al. (2019) reports that for neural networks trained with adversarial training, increased robustness is accompanied by a decrease in accuracy, and posits that this tradeoff might be inevitable. Theoretically, they present a simple example where a tradeoff is indeed necessary; however, the data distribution in their example is not 2\(r\)-separated. Raghunathan et al. (2019) provides a synthetic problem where adversarial training overfits, which we have already studied in Section 3. Bubeck et al. (2018) provides an example where finding a robust and accurate classifier is significantly more computationally challenging than finding one that is simply accurate.

Prior work shows a connection between adversarial robustness and local or global Lipschitzness of neural networks. Anil et al. (2019); Qian & Wegman (2018); Huster et al. (2018) provide methods for imposing global Lipschitzness constraints on neural networks; however, the state-of-the-art methods for training such networks do not lead to highly expressive functions. Weng et al. (2018) provides an efficient way of calculating a lower bound on the local Lipschitzness coefficient. Finlay et al. (2018) considers gradient regularization for Lipschitzness. However, our work is the first to make the connection that accuracy and robustness can both be achieved for separated data with Lipschitz functions.

Li et al. (2018); Cohen et al. (2019); Pinot et al. (2019); Salman et al. (2019) consider a randomized notion of local smoothness, and they show that enforcing it can lead to certifiably robust classifiers. While their techniques often achieve higher adversarial accuracy than adversarial training, they also come with a loss in clean training accuracy (Gao et al., 2020; Mohapatra et al., 2020). Moreover, the main focus of their work is on resilience to \(\ell_2\) perturbations, whereas we prove results for general metrics, and we evaluate empirically for the \(\ell_\infty\) distance. In fact, recent work shows that randomized smoothing may be inherently ineffective for \(\ell_\infty\) perturbations (Blum et al., 2020; Yang et al., 2020), whereas we show that local Lipschitzness is an effective indicator of \(\ell_\infty\) robustness in many cases.

Outside the context of adversarial robustness, Luxburg & Bousquet (2004) provide a framework for large margin classification in metric spaces using globally Lipschitz functions. Their results do not directly apply to our setting, and generalizing them is an interesting future direction.
5. Conclusion

Motivated by understanding when it is possible to achieve both accuracy and robustness, we studied several training methods through the lens of local Lipschitzness. We found that a small Lipschitz constant correlates well with having better adversarial accuracy and clean accuracy. Moreover, we saw that TRADES is an effective method for maximizing robustness and minimizing Lipschitzness without sacrificing too much clean accuracy. We provided further evidence for the importance of local Lipschitzness with our theoretical results, where we proved that there is always exists a classifier with low Lipschitzness and high accuracy and robustness on separated data. Our results suggest a hopeful possibility, where there may not be an inherent trade-off between robustness and accuracy, but rather there could be other methods that lead to the best of both worlds.

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References

Anil, C., Lucas, J., and Grosse, R. Sorting out lipschitz function approximation. In International Conference on Machine Learning, 2019.

Blum, A., Dick, T., Manoj, N., and Zhang, H. Random smoothing might be unable to certify $\ell_\infty$ robustness for high-dimensional images. arXiv preprint arXiv:2002.03517, 2020.

Bubeck, S., Price, E., and Razenshteyn, I. Adversarial examples from computational constraints. arXiv preprint arXiv:1805.10204, 2018.

Carlini, N. and Wagner, D. Towards evaluating the robustness of neural networks. IEEE Symposium on Security and Privacy, 2017.

Cohen, J. M., Rosenfeld, E., and Kolter, J. Z. Certified adversarial robustness via randomized smoothing. arXiv preprint arXiv:1902.02918, 2019.

Finlay, C. and Oberman, A. M. Scaleable input gradient regularization for adversarial robustness. arXiv preprint arXiv:1905.11468, 2019.

Finlay, C., Calder, J., Abbasi, B., and Oberman, A. Lipschitz regularized deep neural networks generalize and are adversarially robust. arXiv preprint arXiv:1808.09540, 2018.

Gao, Y., Rosenberg, H., Fawaz, K., Jha, S., and Hsu, J. Analyzing accuracy loss in randomized smoothing defenses. arXiv preprint arXiv:2003.01595, 2020.

Goodfellow, I. J., Shlens, J., and Szegedy, C. Explaining and harnessing adversarial examples. In International Conference on Learning Representations, 2015.

Gowal, S., Uesato, J., Qin, C., Huang, P.-S., Mann, T., and Kohli, P. An alternative surrogate loss for PGD-based adversarial testing. arXiv preprint arXiv:1910.09338, 2019.

He, K., Zhang, X., Ren, S., and Sun, J. Deep residual learning for image recognition. In IEEE conference on computer vision and pattern recognition, pp. 770–778, 2016.

Huster, T., Chiang, C.-Y. J., and Chadha, R. Limitations of the lipschitz constant as a defense against adversarial examples. In Joint European Conference on Machine Learning and Knowledge Discovery in Databases, pp. 16–29, 2018.

Kingma, D. P. and Ba, J. Adam: A method for stochastic optimization. arXiv preprint arXiv:1412.6980, 2014.

Krizhevsky, A. et al. Learning multiple layers of features from tiny images. 2009.

Kurakin, A., Goodfellow, I., and Bengio, S. Adversarial machine learning at scale. In International Conference on Learning Representations, 2017.

LeCun, Y. The mnist database of handwritten digits. http://yann. lecun. com/exdb/mnist/.

Li, B., Chen, C., Wang, W., and Carin, L. Certified adversarial robustness with additive gaussian noise. arXiv preprint arXiv:1809.03113, 2018.

Liu, Y., Chen, X., Liu, C., and Song, D. Delving into transferable adversarial examples and black-box attacks. ICLR, 2017.

Lowd, D. and Meek, C. Adversarial learning. In SIGKDD, pp. 641–647, 2005.

Luxburg, U. v. and Bousquet, O. Distance-based classification with Lipschitz functions. Journal of Machine Learning Research, 5:669–695, 2004.
Adversarial Robustness Through Local Lipschitzness

Madry, A., Makelov, A., Schmidt, L., Tsipras, D., and Vladu, A. Towards deep learning models resistant to adversarial attacks. In *International Conference on Learning Representations*, 2018.

Mohapatra, J., Ko, C.-Y., Tsui-Wei, Weng, Liu, S., Chen, P.-Y., and Daniel, L. Rethinking randomized smoothing for adversarial robustness. *arXiv preprint arXiv:2003.01249*, 2020.

Netzer, Y., Wang, T., Coates, A., Bissacco, A., Wu, B., and Ng, A. Y. Reading digits in natural images with unsupervised feature learning. 2011.

Papernot, N., McDaniel, P., Wu, X., Jha, S., and Swami, A. Distillation as a defense to adversarial perturbations against deep neural networks. *arXiv preprint arXiv:1511.04508*, 2015.

Papernot, N., McDaniel, P., Goodfellow, I., Jha, S., Celik, B., and Swami, A. Practical black-box attacks against deep learning systems using adversarial examples. In *ASiACCS*, 2017.

Pinot, R., Meunier, L., Araujo, A., Kashima, H., Yger, F., Gouy-Pailler, C., and Atif, J. Theoretical evidence for adversarial robustness through randomization: the case of the exponential family. *arXiv preprint arXiv:1902.01148*, 2019.

Qian, H. and Wegman, M. N. L2-nonexpansive neural networks. *ArXiv*, abs/1802.07896, 2018.

Qin, C., Martens, J., Gowal, S., Krishnan, D., Dvijotham, K., Fawzi, A., De, S., Stanforth, R., and Kohli, P. Adversarial robustness through local linearization. In *Advances in Neural Information Processing Systems*, pp. 13824–13833, 2019.

Raghunathan, A., Steinhardt, J., and Liang, P. Certified defenses against adversarial examples. In *International Conference on Learning Representations*, 2018.

Raghunathan, A., Xie, S. M., Yang, F., Duchi, J. C., and Liang, P. Adversarial training can hurt generalization. *arXiv preprint arXiv:1906.06032*, 2019.

Ross, A. S. and Doshi-Velez, F. Improving the adversarial robustness and interpretability of deep neural networks by regularizing their input gradients. In *AAAI conference on Artificial Intelligence*, 2018.

Salman, H., Li, J., Razenshteyn, I., Zhang, P., Zhang, H., Bubeck, S., and Yang, G. Provably robust deep learning via adversarially trained smoothed classifiers. In *Advances in Neural Information Processing Systems*, pp. 11289–11300, 2019.

Sinha, A., Namkoong, H., and Duchi, J. Certifiable Distributional Robustness with Principled Adversarial Training. In *ICLR*, 2018. URL https://openreview.net/forum?id=Hk6kPqZA-.

Szegedy, C., Zaremba, W., Sutskever, I., Bruna, J., Erhan, D., Goodfellow, I., and Fergus, R. Intriguing properties of neural networks. *arXiv preprint arXiv:1312.6199*, 2013.

Tsipras, D., Santurkar, S., Engstrom, L., Turner, A., and Madry, A. Robustness may be at odds with accuracy. In *International Conference on Learning Representations*, 2019.

Wang, Y., Jha, S., and Chaudhuri, K. Analyzing the robustness of nearest neighbors to adversarial examples. In *International Conference on Machine Learning*, pp. 5133–5142, 2018.

Weng, T.-W., Zhang, H., Chen, P.-Y., Yi, J., Su, D., Gao, Y., Hsieh, C.-J., and Daniel, L. Evaluating the robustness of neural networks: An extreme value theory approach. *arXiv preprint arXiv:1801.10578*, 2018.

Yang, G., Duan, T., Hu, E., Salman, H., Razenshteyn, I., and Li, J. Randomized smoothing of all shapes and sizes. *arXiv preprint arXiv:2002.08118*, 2020.

Zagoruyko, S. and Komodakis, N. Wide residual networks. In *British Machine Vision Conference*, 2016.

Zhang, H., Yu, Y., Jiao, J., Xing, E. P., Ghaoui, L. E., and Jordan, M. I. Theoretically principled trade-off between robustness and accuracy. In *International Conference on Machine Learning*, 2019.
A. Extension of Theoretical Results to Multiclass

We extend our existence proof to the multiclass case, again showing that it is theoretically possible to achieve both robustness and accuracy for 2r-separated data. We exhibit a classifier based on a locally Lipschitz function, which has astuteness 1 at radius r. Our proof is a straightforward extension of the binary case in Theorem 2.2.

Let $|C| = \{1, 2, \ldots, C\}$ denote the possible labels for $C \geq 2$. We consider classifiers of the following form: If there are $C$ classes, we will have a vector-valued function $f : \mathcal{X} \to \mathbb{R}^C$ so that $f(x)$ is a $C$-dimensional real vector. We use $f(x)_i$, to denote the value of the $i$th coordinate of $f(x)$ for $i \in [C]$. Then, we define a classifier $g(x) \in [C]$ as

$$g(x) = \arg\min_{i \in [C]} f(x)_i.$$ 

We extend the definition of local Lipschitzness to vector-valued functions $f$ by bounding the distance in each coordinate:

**Definition 3.** Let $(\mathcal{X}, d)$ be a metric space. We say that a function $f : \mathcal{X} \to \mathbb{R}$ is $L$-Locally-Lipschitz at radius $r$ if for each $i \in [C]$, we have

$$|f(x)_i - f(x')_i| \leq L \cdot d(x, x')$$ 

for all $x'$ such that $d(x, x') \leq r$.

We consider separated input spaces using the following notation. Let the instance $\mathcal{X}$ contain $C$ disjoint classes $X^{(1)}, \ldots, X^{(C)}$, where all points in $X^{(i)}$ have label $i$ for $i \in [C]$. We say that $\mathcal{X}$ is $2r$-separated if $d(X^{(i)}, X^{(j)}) \geq 2r$ for all $i \neq j$. Recall that $d(x, X^{(i)}) = \min_{z \in X^{(i)}} d(x, z)$.

We start with the extension of the lemma showing that accuracy and local Lipschitzness implies astuteness.

**Lemma A.1.** Let $f : \mathcal{X} \to \mathbb{R}^C$ be a function, and consider $x \in \mathcal{X}$ with true label $y \in [C]$. If

- $f$ is $\frac{1}{r}$-Locally Lipschitz in a radius $r$ around $x$
- $f(x)_j - f(x)_y \geq 2$ for all $j \neq y$, and

then $g(x) = \arg\min_i f(x)_i$ is astute at $x$ with radius $r$.

**Proof.** Suppose $x' \in \mathcal{X}$ satisfies $d(x, x') \leq r$. By the assumptions that $f$ is $\frac{1}{r}$-Locally-Lipschitz and $f(x)_j - f(x)_y \geq 2$, we have that

$$f(x')_j \geq f(x)_j - 1 \geq f(x)_y + 1 \geq f(x')_y,$$

where the first and third inequalities use Lipschitzness, and the middle inequality uses that $f(x)_j - f(x)_y \geq 2$. As this holds for all $j \neq y$, we have that $\arg\min_i f(x')_i = \arg\min_i f(x)_i = y$. Therefore, we see that $g(x) = \arg\min_i f(x)_i$ correctly classifies $x$ while being robust with radius $r$. \qed

We now prove that there always exists a classifier based on such a function when the data distribution is $2r$-separated.

**Theorem A.2.** Suppose the data distribution $\mathcal{X}$ is $2r$-separated, denoting $C$ classes $X^{(1)}, \ldots, X^{(C)}$. There exists a function $f : \mathcal{X} \to \mathbb{R}^C$ such that (a) $f$ is $\frac{1}{r}$-locally-Lipschitz in a ball of radius $r$ around all $x \in \bigcup_{i \in [C]} X^{(i)}$ and (b) $g(x) = \arg\min_i f(x)_i$ has astuteness 1.

**Proof.** We first show that if the distribution is $2r$-separated, then there exists a function $f : \mathcal{X} \to \mathbb{R}^C$ satisfying:

1. If $x \in \bigcup_{i \in [C]} X^{(i)}$, then $f(x)$ is $\frac{1}{r}$-locally-Lipschitz in a ball of radius $r$ around $x$.
2. If $x \in X^{(y)}$, then $f(x)_j - f(x)_y \geq 2$ for all $j \neq y$.

Define the function

$$f(x) = \frac{1}{r} \cdot \left(d(x, X^{(1)}), \ldots, d(x, X^{(C)})\right).$$
In other words, we set \( f(x)_i = \frac{1}{r} \cdot d(x, X(i)) \). Then, for any \( x \), we have:

\[
f(x)_i - f(x')_i = \frac{d(x, X(i)) - d(x', X(i))}{r} \leq \frac{d(x, x')}{r}
\]

where we used the triangle inequality. This establishes (1). To establish (2), suppose without loss of generality that \( x \in X(y) \), which in particular implies that \( f(x)_y = d(x, X(y)) = 0 \). Then,

\[
f(x)_j - f(x)_y = \frac{d(x, X(j))}{r} \geq \frac{d(X(y), X(j))}{r} \geq 2
\]

because every pair of classes is \( 2r \)-separated.

Now observe that by construction, \( f \) satisfies all three conditions in Lemma A.1 at all \( x \in \bigcup_{i \in [C]} X(i) \). Thus, applying Lemma A.1, we get that \( g(x) = \arg\min_i f(x)_i \) has astuteness 1 over any distribution over points in \( x \in \bigcup_{i \in [C]} X(i) \). □

**B. Further Experimental Results**

**Result for Large Sample Size Synthetic Staircase Dataset.** Table 8 shows the result for the synthetic staircase dataset with large sample size. We sampled 30000 training examples and 15000 testing examples. We can see from the table that all methods perform well in terms of clean test Mean Squared Error (MSE). For adversarial test MSE, LLR, AT and TRADES perform similarly well and they also have a lower test Lipschitzness. This demonstrated that with enough data, it is possible to have the classifier perform well on both clean and adversarial evaluation.

| | train MSE | test MSE | adv test MSE | test lipschitz | gap | adv gap |
|---|---|---|---|---|---|---|
| Natural | 0.010 | 0.010 | 0.027 | 0.341 | 0.000 | 0.000 |
| GR | 0.010 | 0.010 | 0.050 | 0.377 | 0.000 | 0.001 |
| LLR | 0.010 | 0.010 | 0.015 | 0.272 | 0.000 | 0.001 |
| AT | 0.010 | 0.010 | 0.014 | 0.251 | 0.000 | 0.001 |
| TRADES(β=6) | 0.011 | 0.011 | 0.016 | 0.284 | 0.000 | 0.001 |
| TRADES(β=1) | 0.010 | 0.010 | 0.015 | 0.279 | 0.000 | 0.001 |
| TRADES(β=1) | 0.010 | 0.010 | 0.015 | 0.277 | 0.000 | 0.001 |

Table 8. Synthetic staircase dataset with large sample size. Results measured in Mean Squared Error (MSE), where lower is better.

**B.1. Multi-targeted Attack Results**

Certain prior works have suggested that the multi-targeted (MT) attack (Gowal et al., 2019) is stronger than PGD. For example, the MT attack is highlighted as a selling point for LLR (Qin et al., 2019). For completeness, we complement our empirical results from earlier by running all of the experiments using the MT attack. Tables 9 to 14 provide the results.

We verify that our discussion about accuracy, robustness, and Lipschitzness remains valid using this attack. Comparing with the results using the PGD attack (Tables 2 to 7 above), the results with the MT attack gives a slightly lower adversarial test accuracy for all methods. The drop in accuracy is usually around 1–5%. This is within our expectation as this attack is regarded as a stronger attack than PGD.

The MT results still justify the previous discussion from Section 3 in general. Training methods leading to models with higher adversarial test accuracy are more locally smooth (smaller local Lipschitz constant during testing). Overall, we believe that seeing consistent results between PGD and MT only strengthens our argument that robustness requires some local Lipschitzness, and moreover, that the accuracy-robustness trade-off may not be necessary for separated data.

**C. Experimental Setup: More Details**

We provide more details about the specific dataset and network parameters that were used for our experiments.

Details on the network structure.
### Table 9. MNIST on CNN001, multi-targeted attack

| Accuracy | Lipschitz gap | Adversarial gap |
|----------|---------------|-----------------|
| **Natural** | 0.80 | 0.53 |
| **GR** | 0.70 | 3.30 |
| **LLR** | 0.57 | 4.06 |
| **AT** | 0.67 | 2.65 |
| **TRADES(β=1)** | 0.80 | 0.53 |
| **TRADES(β=3)** | 0.70 | 4.25 |

### Table 10. MNIST on CNN002, multi-targeted attack

| Accuracy | Lipschitz gap | Adversarial gap |
|----------|---------------|-----------------|
| **Natural** | 0.49 | 0.87 |
| **GR** | 0.44 | 2.39 |
| **LLR** | 0.43 | 1.70 |
| **AT** | 0.50 | 1.94 |
| **TRADES(β=1)** | 0.49 | 0.87 |
| **TRADES(β=3)** | 0.44 | 2.39 |

### Table 11. SVHN, multi-targeted attack

| Accuracy | Lipschitz gap | Adversarial gap |
|----------|---------------|-----------------|
| **Natural** | 11.38 | 0.00 |
| **GR** | 14.16 | 2.88 |
| **LLR** | 14.17 | 5.19 |
| **AT** | 16.33 | 48.99 |
| **TRADES(β=1)** | 14.80 | 43.69 |
| **TRADES(β=3)** | 20.01 | 63.95 |
| **TRADES(β=6)** | 21.45 | 63.70 |

### Table 12. CIFAR-10 without data augmentation, multi-targeted attack

| Accuracy | Lipschitz gap | Adversarial gap |
|----------|---------------|-----------------|
| **Natural** | 6.19 | 0.00 |
| **GR** | 14.16 | 2.88 |
| **LLR** | 14.17 | 5.19 |
| **AT** | 16.33 | 48.99 |
| **TRADES(β=1)** | 14.80 | 43.69 |
| **TRADES(β=3)** | 20.01 | 63.95 |
| **TRADES(β=6)** | 21.45 | 63.70 |

### Table 13. CIFAR-10 with data augmentation, multi-targeted attack

| Accuracy | Lipschitz gap | Adversarial gap |
|----------|---------------|-----------------|
| **Natural** | 6.19 | 0.00 |
| **GR** | 14.16 | 2.88 |
| **LLR** | 14.17 | 5.19 |
| **AT** | 16.33 | 48.99 |
| **TRADES(β=1)** | 14.80 | 43.69 |
| **TRADES(β=3)** | 20.01 | 63.95 |
| **TRADES(β=6)** | 21.45 | 63.70 |

- CNN1 is retrieved from pytorch repository.7

7https://github.com/pytorch/examples/blob/master/mnist/main.py
Table 14. Restricted ImageNet, multi-targeted attack

| network structure | dataset          | train accuracy | test accuracy | adv test accuracy | test lipschitz | gap | adv gap |
|-------------------|------------------|----------------|---------------|-------------------|----------------|-----|--------|
| Natural           | MNIST            | 97.72          | 93.47         | 4.21              | 32228.51       | 4.25| -0.29  |
| GR                | SVHN             | 91.12          | 88.51         | 60.55             | 886.75         | 2.61| -0.18  |
| LLR               | CIFAR10          | 98.76          | 93.44         | 50.16             | 4795.66        | 5.32| -0.36  |
| AT                |Restricted ImageNet| 96.22          | 90.33         | 81.91             | 287.97         | 5.00| 8.27   |
| TRADES(β=1)      |                  | 97.39          | 92.27         | 79.44             | 2144.66        | 5.13| 6.61   |
| TRADES(β=3)      |                  | 95.74          | 90.75         | 82.00             | 396.75         | 2.61| -0.18  |
| TRADES(β=6)      |                  | 93.34          | 88.92         | 81.86             | 200.90         | 5.00| 6.33   |

Table 15. Experimental setup and relevant parameters for the four real datasets that we test on in this paper.

- CNN2 is retrieved from TRADES (Zhang et al., 2019) github repository. 8
- WRN-40-10 represents the wide residual network (Zagoruyko & Komodakis, 2016) with depth equals to forty and widen factor equals to ten.
- ResNet50 represents the residual network with 50 layers (He et al., 2016).

Learning rate schedulers for each dataset

- MNIST: We run 160 epochs on the training dataset, where we decay the learning rate by a factor 0.1 in the 40th, 80th 120th and 140th epochs.
- SVHN: We run 60 epochs on the training dataset, where we decay the learning rate by a factor 0.1 in the 30th and 50th epochs.
- CIFAR10: We run 120 epochs on the training dataset, where we decay the learning rate by a factor 0.1 in the 40th, 80th and 100th epochs.
- Restricted ImageNet: We run 70 epochs on the training dataset, where we decay the learning rate by a factor 0.1 in the 40th and 60th epochs.

C.0.1. SPIRAL DATASET

Here we provide the details for generating the spiral dataset in Figure 2.

We take $x$ as a uniform sample $[0, 4.33\pi]$, the noise level is set to 0.75 (uniform $[0, 0.75]$).

We construct the negative examples using the transform:

$$(-x \cos x + \text{uniform}(noise), x \sin x + \text{uniform}(noise))$$

We construct the positive examples using the transform:

$$(-x \cos x + \text{uniform}(noise), -x \sin x + \text{uniform}(noise))$$

8https://github.com/yaodongyu/TRADES/blob/master/models/small_cnn.py