Testing Asymptotic Scaling and Nonabelian Symmetry Enhancement.

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Abstract

We determine some points on the finite size scaling curve for the correlation length in the two dimensional $O(3)$ and icosahedron spin models. The Monte Carlo data are consistent with the two models possessing the same continuum limit. The data also suggest that the continuum scaling curve lies above the estimate of Kim [1] and Caracciolo et al [2] and thus leads to larger thermodynamic values of the correlation length than previously reported.

In 1993 Kim [1] proposed using finite size scaling to obtain the thermodynamic value of the correlation length $\xi$ in the two dimensional ($2D$) nonlinear $\sigma$ model at given inverse temperature $\beta$. He claimed that he could predict correctly (less than 2% error) values as large as 15000 from measurements taken on lattices not larger than $283 \times 283$. Kim’s method was adopted by Caracciolo et al [2], who, after refining it and collecting data at a 180 pairs of $L$ and $\beta$ values claimed to have such control over the scaling curve that they
could predict values of the correlation length as large as $10^5$ even though their largest lattice was only $512 \times 512$.

Of course at fixed $x = \xi(L)/L$ the value of $\xi(2L)/\xi(L)$ depends upon $L$ and the continuum value is the limit of this ratio as $L \to \infty$. Both Kim and Caracciolo et al claimed that within their statistical accuracy, they could not detect any systematic drift of $\xi(2L)/\xi(L)$ with $L$ within the range of values of $L$ they studied; so they concluded that they had reached the continuum limit. (The latter authors also investigated in more detail the approach to the continuum limit in a different model, the two-dimensional $SU(3)$ principal chiral model [8] and convinced themselves that their continuum extrapolation was supported by their data (private communication from Alan Sokal)).

Shortly after the appearance of these papers, we criticized both Kim's paper [4] and the Caracciolo et al paper [5] by pointing out that we know rigorously [6] that in the true continuum limit large spin fluctuations have to occur for any value of $x > 0$. Indeed it can be shown that the spins must become decorrelated over distances that are fixed, nonvanishing multiples of the infinite volume correlation length, while in the regime in which these authors took their data, at least at $x > 0.75$, the lattices were so well ordered that second order lattice perturbation theory (PT) could reproduce their Monte Carlo results within the errors.

In this paper we significantly improve Caracciolo et al's statistics, and go to larger lattices (their largest $L$-$2L$ pair has $L = 256$, ours $640$). Thereby we find (for $\xi(L)/L = 0.5$) some statistically significant corrections to scaling even at the largest $L$ values, which in our view signifies that the data of Caracciolo et al were rather far from the continuum limit even at $x = 0.5$ (of course, it is to be expected, and the present study corroborates that, that the continuum limit for larger values of $x$ is reached at larger values of $L$).

Our interest in this problem was revived by the recent discussions of symmetry enhancement of discrete nonabelian groups. Namely both we [7] [8] and Hasenfratz and Niedermayer [9] pointed out that numerics suggest that the dodecahedron and icosahedron spin models have the same continuum limit as the $O(3)$ model. Hasenfratz and Niedermayer interpreted this as meaning that these discrete spin models were asymptotically free (AF). In fact in our paper we [8] had measured the Lüscher-Weisz-Wolff running coupling constant [10] and showed that it did not vanish at the critical point, as required by AF. A theoretical argument was also published shortly thereafter [11] claiming to show the impossibility of the Hasenfratz-Niedermayer scenario of AF in a discrete spin model.
This latter paper contains an interesting suggestion: the observed agreement between the discrete spin models and $O(3)$ is a transient phenomenon, which should disappear at larger correlation length. The observation stems from the fact that if one accepts the standard scenario of $O(3)$ being AF, then the discrete spin models could be regarded as a perturbed $O(3)$ model, and in the accepted PT scheme around the Gaussian fixed point, this would be a relevant perturbation, hence the two models could not possibly be equivalent. However the authors find that this perturbation becomes relevant only for $\beta$ sufficiently large. They translate their estimate of this value of $\beta$ into a correlation length of approximately 200 and suggest that perhaps the numerics will show that as the correlation length is increased, for continuum observables, the agreement between $O(3)$ and the icosahedron or dodecahedron spin models improves for a while, then, as $\xi$ exceeds approximately 200, it starts deteriorating.

We decided to investigate numerically this possibility by comparing the finite size scaling curve of $\xi(L)$ in the two models. We studied the icosahedron spin model and the $O(3)$ model at $x = 0.25$, $x = 0.5$ and $x = 0.75$. For the icosahedron we also took data at our estimated $\beta_{\text{crit}} = 1.803$ [8]. The results are recorded in Tabs.1-7 and displayed in the figures.

Before discussing the figures, let us specify in detail what we did and summarize what we find. Our systems consist of two dimensional square arrays of spins of size $L \times L$ with periodic boundary conditions. The spin at each site is of unit length and takes values either on the sphere $S^2$ or on the vertices of an inscribed regular icosahedron. Each spin interacts ferromagnetically only with its 4 nearest neighbours at inverse temperature $\beta$.

We used the same definition of the correlation length $\xi$ as [2]: let $P = (p, 0)$, $p = \frac{2n\pi}{L}$, $n = 0, 1, 2, ..., L - 1$. Then

$$\xi = \frac{1}{2\sin(\pi/L)} \sqrt{(G(0)/G(1) - 1)} \quad (1)$$

where

$$G(p) = \frac{1}{L^2} \langle |\hat{s}(P)|^2 \rangle; \quad \hat{s}(P) = \sum_x e^{iPx} s(x) \quad (2)$$

At a given $L$, we adjusted $\beta$ so that the ratio $\xi(L)/L$ was approximately 0.25, 0.5 respectively 0.75 (we allowed differences only smaller than $2 \times 10^{-3}$). Leaving $\beta$ unchanged, we then doubled $L$ and measured $\xi(2L)$ and therefore $\xi(2L)/\xi(L)$ at our $x$. (We used the slope of the scaling curve of [2] to correct
for the fact that our $\xi(L)/L$ is not exactly equal to the desired values of 0.25, 0.5 and 0.75; since this correction is tiny, it does not matter whether their scaling curve represents exactly the truth). In our study $L$ was varied from 20 to 640 or to 1280 and the data were produced using the usual one cluster algorithm. For each $\beta L$ pair we performed several runs (up to 200), one run consisting of 100,000 clusters used for thermalization and 1,000,000 for taking measurements. Each run started from a new randomly chosen configuration. The errors were computed from the results produced by the different runs, using the jack-knife method.

In Fig.1 we show our results for $O(3)$ (full squares) and for the icosahedron (open circles) at $x = 0.5$. For comparison, we also plotted the results of Caracciolo et al for $O(3)$ (full triangles). The choice of the abscissa $1/(\ln L + c)$, was motivated by the fact that in studying lattice artefacts for the LWW step scaling function we found that such an ansatz seemed to describe the data pretty well (see ref. [12] Fig.7). The value of the parameter $c = 0.7$ was obtained from a joint fit to our $O(3)$ and icosahedron data with a common limiting value for $L \to \infty$; it is of acceptable quality ($\chi^2/dof = 11.6/8$). The solid curve is a Symanzik type fit ($a + b/L^2 + c\ln L/L^2$) to the $O(3)$ data with, however, an unacceptable $\chi^2/dof = 10.7/3$. Several facts are suggested by this figure:

- Our MC data agree very well with those of Caracciolo et al. However while the largest $L$ value investigated by these authors was $L = 256$, ours is $L = 640$. Also our error bars are much smaller.

- The $O(3)$ data suggest a systematic increase of $\xi(2L)/\xi(L)$ with $L$. This is true about both our data and those of Caracciolo et al, the latter however, having much larger error bars, could also be interpreted as showing no $L$ dependance for $L > 64$.

- The data suggest that the continuum value is definitely larger than the value predicted by Caracciolo et al 1.5420(7).

- There is no obvious reason to suspect that the continuum limit in the two models is different.

- The dashed lines represent linear extrapolations in $1/(\ln L + 0.7)$ and they intersect at $\xi(2L)/\xi(L) = 1.584$. The extrapolations used and the value they produce should be regarded only as an illustration, since we
Figure 1: Scaling function $\xi(2L)/\xi(L)$ versus $1/(\ln L + 0.7)$ for $O(3)$ (full triangles: ref.\cite{2} and full squares: our data) and icosahedron (open circles) at $\xi(L)/L = 0.5$. The straight lines are fits.

have no way of knowing whether our $L$ values are truly asymptotic and the extrapolation ansatz is correct.

For a better test of the approach to the continuum, we investigated a smaller value of $x$, namely $x = 0.25$, where one would expect the continuum limit to be reached at smaller $L$ values. The data are shown in Fig. 2.

They suggest the following facts:

- Indeed the continuum limit seems to be reached at lower $L$ values, perhaps as low as $L = 320$ corresponding to a correlation length $\xi = 80$.
- The data are consistent with the two models sharing a common continuum limit value for the step scaling function. This value seems
Figure 2: Scaling function $\xi(2L)/\xi(L)$ versus $1/(\ln L + 0.7)$ for $O(3)$ (full squares) and icosahedron (open circles) at $\xi(L)/L = 0.25$

to be around 1.031, higher than the value quoted by Caracciolo et al (1.0255(20)).

In Fig.3 we present the same results as in Fig.1 but at $x = 0.75$. To go to this increased value of $x$, we had to increase $\beta$. Consequently it is to be expected that at this $x$ value, asymptopia will set in at a much larger value of $L$. This must be so especially for the $O(3)$ model, which will be described by PT up to much larger values of $L$ (as we emphasized above, in the true continuum limit, the system cannot be in a PT regime, since in the continuum limit the spins decorrelate over distances proportional to $L$). Two things can nevertheless be learned from this figure:

• There is no reason to rule out that the continuum value is the same in the two models.
The continuum limit may very well be again higher than the prediction of Caracciolo et al (1.8810(3)), which most likely comes from a transient, PT dominated regime; with present day’s computers one cannot study reliably the larger $L$ values needed to settle this issue.

Our conclusion is that there is no evidence for the crossover phenomenon in the discrete nonabelian symmetry enhancement suggested by Caracciolo et al. While it follows from the accepted PT calculations, their scenario seems bizarre from the point of view of the discrete spin model. Indeed the latter may simulate a continuous symmetry only if the spins are correlated over a sufficiently large portion of the lattice. Fluctuations alone are not sufficient to enhance the symmetry, as can be seen by the high temperature expansion, which clearly is different for the icosahedron and the $O(3)$ models. Symmetry

Figure 3: Scaling function $\xi(2L)/\xi(L)$ versus $1/(\ln L + 0.7)$ for $O(3)$ (full squares: our data, full triangles: ref.\[2\]) and icosahedron (open circles) at $\xi(L)/L = 0.75$
enhancement requires the collective effect of many discrete spins over the
typical distance of a correlation length; therefore one should expect that
the discrete spins are capable of approximating the continuous ones better
and better with increasing correlation length. Thus it would be bizarre if
as the correlation length increases, beyond some value, correlation functions
at distances measured in units of the correlation length would become less
symmetric. However, being based only on some numerics, the present work
cannot rule out the Caracciolo et al scenario.

What the present work does show though is that the scaling curve pre-
dicted by Caracciolo et al in 1995 is not correct and that the lattice
artefacts are larger than they claimed. If one accepts the universality of the
$O(3)$ and icosahedron spin model, then it appears that one could get a much
better approximation of the continuum scaling curve by studying the icosahedron spin model. For instance the scaling curve produced by Caracciolo et
al never reached 2, presumably because the best it could do is reach the
value predicted by perturbation theory. Now in the icosahedron model one
can work at $\beta_{crt}$. The results, shown in Tab.7 show that $\xi(2L)/\xi(L)$ does
actually reach 2 (definition of $\beta_{crt}$) and that this happens at $x \sim 1$ (non-
trivial information). If the scaling curve were known, one could repeat the
Kim-Caracciolo et al procedure and produce some thermodynamic values for
$\xi(\beta)$. It should be emphasized though that with that procedure, the error
compounds and one can lose control very fast. For instance if the scaling
curve predicted by Caracciolo et al is changed by adding to their fit for-
mula for $\xi(2L)/\xi(L)$ the term $0.02x$ and the starting values were $L = 100$
$\xi = 75$ then the predicted thermodynamic value changes from 2000 to 2670.
Please notice that the modification we introduced vanishes for $x \to 0$ and it
amounts to an increase of less than 1% for any $x$, yet the change in the pre-
dicted value is about 33%. Since in their 1995 paper Caracciolo et al stated
themselves that they did observe scaling violations of about 1.5% (and the
data shown in Fig.1 suggest that that was a gross underestimate), we believe
that their claim of having verified AF at correlation lengths $10^5$ at the 4%
level is unjustified.

Note Added in Proof:

After the completion of this paper, S. Caracciolo drew our attention to
some recent work done by himself with A. Montanari and A. Pelissetto con-
cerning the issue of nonabelian discrete symmetry enhancement. In
this work Caracciolo et al claim to have shown that in fact the continuum
limit of the model perturbed by a term enjoying only icosadral symmetry is
different from that of the $O(3)$ model; but in contrast to their earlier estimate
that the difference should show up at correlation length of about 200 they
now say that they have to go to correlation length of about 10$^5$.

Unfortunately the authors base their conclusion only on data taken on
relatively small lattices (while their estimate of the thermodynamic corre-
lation length is $\approx 10^5$, their largest lattice for which they determine the step
scaling function only is $150 \times 150$. As we already stated both in previous
papers [4], [5], [6] and repeated throughout the present paper, such data can-
ot be considered as representing the true continuum limit since they place
the system in a perturbative regime, known rigorously not to occur in the
true continuum limit. It is also not clear if the perturbed system with the
icosahedral symmetry is not in its magnetized phase at the values of the
parameters chosen (the authors do not provide any information on this).

The data reported in the present paper, at $x = 0.25$ and $x = 0.5$, do
not suffer from this limitation and are consistent with discrete symme-
try enhancement. In support of their claim, Caracciolo et al state that their data
indicate that the difference between the model enjoying a discrete symmetry
and $O(3)$ is increasing with the size of the lattice $L$. We have two comments
to this observation:

- Even though the continuum limit of the two models may be the same,
  there is no guarantee that this limit must be reached in a monotonic
  fashion, hence the observed increased discrepancy with increased $L$
could be a transient phenomenon.

- In some cases (Fig.2 of [13], Figs. 4 and 6 of [14]) the data at their
  largest $L$ value $L = 150$ seem to lie below the data at $L = 90$, contrary
to their claim.

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conducted, for its hospitality.
Tab.1:
ξ(L) and ξ(2L) for the O(3) model at ξ(L)/L ≈ .25

| L   | 20  | 40  | 80  | 160 | 320  |
|-----|-----|-----|-----|-----|------|
| β   | 1.332| 1.488| 1.6135| 1.728| 1.838 |
| ξ(L)| 5.011(3) | 10.070(7) | 19.919(11) | 39.881(25) | 79.988(48) |
| ξ(2L)| 5.164(3) | 10.395(5) | 20.529(10) | 41.097(26) | 82.458(47) |

Tab.2:
ξ(L) and ξ(2L) for the O(3) model at ξ(L)/L ≈ .5

| L   | 20  | 40  | 80  | 160 | 320  | 640  |
|-----|-----|-----|-----|-----|------|------|
| β   | 1.595| 1.7143| 1.825| 1.935| 2.047| 2.16 |
| ξ(L)| 10.007(9) | 19.999(17) | 40.002(28) | 79.884(52) | 159.73(13) | 319.29(37) |
| ξ(2L)| 15.205(13) | 30.629(27) | 61.451(41) | 123.079(91) | 246.99(26) | 493.94(68) |

Tab.3:
ξ(L) and ξ(2L) for the O(3) model at ξ(L)/L ≈ .75

| L   | 20  | 40  | 640  |
|-----|-----|-----|------|
| β   | 2.092| 2.18| 2.63 |
| ξ(L)| 15.010(16) | 29.607(30) | 470.07(86) |
| ξ(2L)| 28.260(32) | 55.608(66) | 882.8(2.5) |

Tab.4:
ξ(L) and ξ(2L) for the icosahedron model at ξ(L)/L ≈ .25

| L   | 20  | 40  | 80  | 160 | 320  |
|-----|-----|-----|-----|-----|------|
| β   | 1.319| 1.457| 1.556| 1.6295| 1.6836 |
| ξ(L)| 5.016(3) | 10.043(6) | 19.960(14) | 39.955(15) | 80.278(57) |
| ξ(2L)| 5.164(3) | 10.348(4) | 20.583(11) | 41.237(13) | 82.829(41) |

Tab.5:
ξ(L) and ξ(2L) for the icosahedron model at ξ(L)/L ≈ .5

| L   | 20  | 40  | 80  | 160 | 320  | 640  |
|-----|-----|-----|-----|-----|------|------|
| β   | 1.5382| 1.618| 1.6767| 1.718| 1.7469| 1.7664 |
| ξ(L)| 9.991(9) | 19.996(17) | 40.057(25) | 80.131(74) | 160.04(20) | 318.62(32) |
| ξ(2L)| 14.955(14) | 30.141(20) | 61.076(42) | 122.58(13) | 246.14(35) | 487.17(70) |

Tab.6:
ξ(L) and ξ(2L) for the icosahedron model at ξ(L)/L ≈ .75
| \(L\) | 20 | 40 | 80 | 160 | 320 | 640 |
|---|---|---|---|---|---|---|
| \(\beta\) | 1.733 | 1.758 | 1.774 | 1.788 | 1.794 | 1.797 |
| \(\xi(L)\) | 15.114(31) | 29.936(68) | 58.99(11) | 121.20(21) | 242.68(39) | 479.40(5) |
| \(\xi(2L)\) | 27.279(43) | 54.227(083) | 107.69(17) | 222.97(33) | 448.06(73) | 882.2(1.6) |

**Tab. 7a:**

\(\xi(L)\) for the icosahedron model at \(\beta = 1.802\)

| \(L\) | 20 | 40 | 80 | 160 | 320 |
|---|---|---|---|---|---|
| \(\xi(L)\) | 19.946(49) | 39.50(11) | 78.11(21) | 155.77(38) | 305.20(51) |

**Tab. 7b:**

\(\xi(L)\) for the icosahedron model at \(\beta = 1.803\)

| \(L\) | 320 | 640 |
|---|---|---|
| \(\xi(L)\) | 318.23(76) | 642.2(1.4) |

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