A Condition for the Nullity of Quantum Discord

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The positivity of quantum discord is shown to be equivalent to the strong subadditivity of von-Neumann entropy. This leads us to a necessary and sufficient condition characterizing the set of states with zero quantum discord. This also gives us a mathematical definition of pointer states, as they are the states with zero discord. Finally, we suggest that strong subadditivity of entropy might delineate the boundaries of the set of quantum correlations.

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Identification, characterization and manipulation of non-classical correlations is at the heart of quantum information science. Canonically, this has meant the study of quantum entanglement [1, 2]. The role of entanglement is, however, far from clear in its most celebrated application, quantum computation. In particular, instances of mixed state quantum computation exist which outdo their best classical counterparts with limited amounts of entanglement [3, 4]. This motivated the study of other measures of capturing nonclassical correlations, the foremost of which has been quantum discord [5, 6]. It has been studied in a variety of systems [7–10] and circumstances [11–14]. This has subsequently led to the introduction of several related measures [15, 16], inspired by different physical [17] and operational principles [18].

Quantum discord aims at capturing all quantum correlations in a quantum state, including entanglement [5, 6]. Quantum mutual information is generally taken to be the measure of total correlations, classical and quantum, in a quantum state. For two systems, A and B, it is defined as \( I(A : B) = H(A) + H(B) - H(A, B) \), where \( H(\cdot) \) stands for the von Neumann entropy, \( H(\rho) = -\text{Tr}(\rho \log \rho) \). In our paper, all logarithms are taken to base 2. For a classical probability distribution, Bayes’ rule leads to an equivalent definition of the mutual information as \( I(A : B) = H(A) - H(A|B) \), where the conditional entropy \( H(A|B) \) is an average of the Shannon entropies of A, conditioned on the alternatives of B. It captures the ignorance in A once the state of B has been determined. For a quantum system, this depends on the measurements that are made on B. For a POVM given by the set \( \{\Pi_i\} \), the state of A after the measurement corresponding to the outcome \( i \) is given by

\[
\rho_{A|i} = \text{Tr}_B(\Pi_i \rho_{AB})/p_i, \quad p_i = \text{Tr}_A(\Pi_i \rho_{AB}).
\]

A quantum analogue of the conditional entropy can then be defined as \( \tilde{H}_{\{\Pi_i\}}(A|B) \equiv \sum_i p_i H(\rho_{A|i}) \), and an alternative version of the quantum mutual information can now be defined as \( J_{\{\Pi_i\}}(A : B) = H(A) - \tilde{H}_{\{\Pi_i\}}(A|B) \).

The above quantity depends on the chosen set of measurements \( \{\Pi_i\} \). To capture all the classical correlations present in \( \rho_{AB} \), we maximize \( J_{\{\Pi_i\}}(A : B) \) over all \( \{\Pi_i\} \), arriving at a measurement independent quantity \( J(A : B) = \max_{\{\Pi_i\}}(H(A) - \tilde{H}_{\{\Pi_i\}}(A|B)) = H(A) - H(A|B) \), where \( H(A|B) = \min_{\{\Pi_i\}} \tilde{H}_{\{\Pi_i\}}(A|B) \). Since the conditional entropy is concave over the set of POVMs, which is convex, the minimum is attained on the extreme points of the set of POVMs, which are rank 1 [19]. Then, quantum discord is finally defined as

\[
\mathcal{D}(A : B) = I(A : B) - J(A : B) = H(A) - H(A : B) + \min_{\{\Pi_i\}} \tilde{H}_{\{\Pi_i\}}(A|B),
\]

where \( \{\Pi_i\} \) are now, and henceforth in the paper, rank 1 POVMs. It is well known that the quantum discord is non-negative for all quantum states [6, 19]. In this paper, we present a new proof of this fact, based on the strong subadditivity of the von-Neumann entropy. More importantly, it allows us to derive the first necessary and sufficient condition for a state to have zero quantum discord. They are the ones that saturate strong subadditivity, and are in fact, quite easy to work with, as we see later. This leads us to suggest an experimental realization of pointer states in superconducting cavity QED systems. Finally, we propose a simple certificate for distinguishing classical, quantum and more general no-signalling theories based on strong subadditivity.

**Theorem 1.** Quantum discord is always positive, i.e., \( \mathcal{D}(A, B) \geq 0 \).

**Proof:** Consider the joint state \( \rho_{AB} \) subject to one dimensional orthogonal measurements \( \Pi_i = |e_j)(e_j| \) on B (extended to arbitrary, at most \( \text{dim}(B)^2 \) dimensions, by the Neumark extension). The post-measurement state is given by Eq. (1). Suppose now that a system C interacts with B so as to make the desired measurement \((U|e_j)(0) = |e_j)(f_j)\), leaving the state

\[
\rho'_{ABC} = \sum_{j,k} |e_j)(e_k| \otimes |e_j)(e_k| \otimes |f_j)(f_k|.
\]

If the eigendecomposition of \( \rho_{ABC} = \sum_l \lambda_l |r_l)(r_l| \), then

\[
\rho'_{ABC} = \sum_{j,k,l} \lambda_l |r_l, e_j)(r_l, e_j)(r_l, e_k| \otimes |e_j)(e_k| \otimes |f_j)(f_k| \]

\[
= \sum_l \lambda_l |e_l, r_l, f_l|e_l, r_l, f_l|.
whereby \( H(\rho'_{AB}) = H(\rho_{AB}) \). Also, from Eq. (3),

\[
\rho'_{AB} = \sum_j p_j \rho_{ABj} \otimes |e_j\rangle\langle e_j|,
\]

(4a)

\[
\rho'_{BC} = \sum_{j,k} |e_j\rangle\langle e_j| \rho_{BCj} (e_k) \otimes |f_k\rangle\langle f_k|,
\]

(4b)

\[
\rho'_B = \sum_j p_j |e_j\rangle\langle e_j|,
\]

(4c)

whereby \( H(\rho'_{AB}) = H(\rho_B) + \sum_j p_j H(A|j) \), \( H(\rho'_{BC}) = H(\rho_B) \) and \( H(\rho'_B) = H(\rho_B) \) respectively. These reduce the strong subadditivity of the von-Neumann entropy [20]

\[
H(\rho'_{ABC}) + H(\rho'_B) \leq H(\rho'_{AB}) + H(\rho'_{BC}),
\]

(5)

to \( H(\rho_{AB}) + H(\rho_B) = H(\rho(\rho_{AB}) + H(\rho_B) , \) whereby

\[
\tilde{H}(\Pi_j)(A|B) = \sum_j p_j H(A|j) \geq H(\rho_{AB}) - H(\rho_B) = H(A|B). \tag{6}
\]

This, being true for all measurements, also holds for the minimum, proving the theorem. 

This theorem also applies to infinite dimensional systems, since the von-Neumann entropy is strongly subadditive for such systems as well [20], albeit the summations in Eqs. (3) and the following could be infinite. Having proven that the quantum discord is always nonnegative, it is evident that the condition for zero discord can be reduced to that of the equality in strong subadditivity in Eq. (5). To that end, we will employ a result stated by Ruskai [21], and proven by Hayden et al. [22], which we quote below for completeness.

**Lemma 1** ([22]). A state \( \rho'_{ABC} \) on \( \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \) satisfies strong subadditivity (Eq. (5)) with equality if and only if there is a decomposition of system \( B \) as

\[
\mathcal{H}_B = \bigoplus_j \mathcal{H}_{B_j}^L \otimes \mathcal{H}_{B_j}^R
\]

into a direct sum of tensor products such that

\[
\rho'_{ABC} = \bigoplus_j \sum_i q_i \rho_{ABj}^{iL} \otimes \rho_{Bj}^{iR} \otimes \mathcal{H}_C
\]

with states \( \rho_{ABj}^{iL} \) on \( \mathcal{H}_A \otimes \mathcal{H}_{B_j}^L \) and \( \rho_{ABj}^{iR} \) on \( \mathcal{H}_{B_j}^R \otimes \mathcal{H}_C \), and a probability distribution \( \{ q_j \} \).

**Theorem 2.** \( D(A, B) = 0 \) if and only if the state \( \rho_{AB} \) is joint density operator is block-diagonal in the marginal eigenbasis of \( B \), that is

\[
\rho_{AB} = \sum_j P_j \rho_{AB}^{P_j} P_j
\]

where \( \rho_{AB} \) is the marginal density operator of \( \{ \tau_j \} \) a probability distribution [9].

**Proof:** For any measurement on \( B \) executed via \( C \), the state has the form

\[
\rho'_{ABC} = \sum_\alpha q_\alpha \rho_{AB|\alpha} \otimes \rho_B^\alpha,
\]

(7)

where \( \rho_B^\alpha = \Pi_\alpha \rho_B^\alpha \Pi_\alpha \), with \( \Pi_\alpha \) being projectors of the form \( \Pi_\alpha = \sum_j |E_{\alpha j}\rangle\langle E_{\alpha j}| \otimes |F_{\alpha j}\rangle\langle F_{\alpha j}| \). In our case, the state \( \rho'_{ABC} \) is invariant under the exchange of \( B \) and \( C \) relative to the measurement basis, here denoted by \( |E_{\alpha j}\rangle \) and \( |F_{\alpha j}\rangle \). Additionally, following Lemma. (1), for any measurement that saturates Eq. (5), there exists a decomposition of the Hilbert space of \( B \) that can be written as \( \mathbb{I}_B = \sum_\alpha \Pi_\alpha = \sum_\alpha \Pi_\alpha L \otimes \Pi_\alpha R \), and \( \Pi_\alpha \Pi_\beta = \delta_{\alpha \beta} \Pi_\alpha \). Thus,

\[
\rho'_{BC} = \sum_{j,k} \rho_{BCj}^{\alpha} (E_{\alpha j}|\otimes |F_{\alpha j}\rangle\langle F_{\alpha j}|) \quad \text{and} \quad \rho'_{AB} = \sum_\alpha q_\alpha \rho_{AB|\alpha} \otimes \rho_B^\alpha = \sum_\alpha \rho_{AB|\alpha} \otimes \rho_B^\alpha = \sum_\alpha \rho_{AB|\alpha} \otimes \rho_B^\alpha.
\]

Undoing the measurement, \( U|e_j\rangle \otimes |0\rangle = |e_j\rangle \otimes |f_j\rangle \), gives

\[
\rho_{AB} = \langle 0|U\rho'_{ABC}U|0\rangle = \sum_\alpha q_\alpha \rho_{AB|\alpha} \otimes \rho_B^\alpha.
\]

Diagonalizing \( \rho_B^\alpha = \sum_k \lambda_{jk}^\alpha |\lambda_{jk}^\alpha\rangle\langle \lambda_{jk}^\alpha| \), we get \( \rho_{AB} = \sum_{\lambda k} \lambda_{jk}^\alpha q_\alpha \rho_{AB|\alpha} \otimes |\lambda_{jk}^\alpha\rangle\langle \lambda_{jk}^\alpha|. \) Relabelling, we have that the discord is zero if and only if

\[
\rho_{AB} = \sum_j p_j \rho_{ABj} \otimes |\lambda_j\rangle\langle \lambda_j| \tag{8}
\]

in the basis that diagonalizes \( \rho_B \). The \( \alpha \) subspaces take into account that if the states \( \rho_{ABj} \) are the same for different \( j \), then we can attain zero discord by using any measurement in the subspace spanned by those values of \( j \). Diagonalising \( \rho_{ABj} = \sum_k \mu_{jk} \mu_{jk}\rangle\langle \mu_{jk}| \), we get that a state has zero discord if and only if

\[
\rho_{AB} = \sum_{jk} p_j \mu_{jk} \otimes |\mu_{jk}\rangle\langle \mu_{jk}| \langle \lambda_j|\langle \lambda_j|
\]

(9)

Thus the eigenbasis of \( \rho_{AB} \) has a tree product structure \( |\mu_{jk}\rangle\otimes |\lambda_j| \).

The condition for equality in strong subadditivity is that the state \( \rho'_{ABC} \) be what is known as a short Markov chain, that is, \( \mathcal{D}(A, B) = 0 \) if and only if the state \( \rho_{AB} \) is joint density operator is block-diagonal in the marginal eigenbasis of \( B \), that is

\[
\rho_{AB} = \sum_j p_j \rho_{ABj} P_j
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Discussions—Since the states with vanishing discord were identified with pointer states, our results also allow for their mathematical and constructive definition. Curiously, they form a set of measure zero within the space of density matrices in finite dimensions. Finally, recent efforts of identifying information-theoretic criterion to differentiate quantum theory from more general no-signalling theories have led to the notion of information causality [27], which is respected by quantum and classical theories, but not by more general theories. These violations have now been traced to a violation of the strong subadditivity of entropy [28, 29]. This leads us to propose that strong subadditivity of entropy demarcates the edges of quantum correlations, with the quantity

\[ H(AB) + H(BC) - H(ABC) - H(B) \]

being positive for quantum states, zero for classical states, and negative for states in more general probabilistic theories. We hope that the connections unraveled in this paper, between pointer states, strong subadditivity, quantumness and discord lead to better understandings of the quantum nature of the universe, at a conceptual as well as a practical level.

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