Approximate Lie Group Analysis of a Model Advection Equation on an Unstructured Grid

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Abstract

Purpose: To investigate the Lie groups of transformations (symmetries) approximately admitted by a finite difference discretization of a nonlinear advection equation on an unstructured mesh.

Method: A recently developed technique of “approximate group analysis” is applied to a differential approximation (otherwise referred to as an equivalent differential equation) corresponding to the above finite difference approximation.

Results and Conclusions: We determine which groups from the infinite variety of groups admitted by a nonlinear advection equation “survive” the discretization. The situations arising for different choices of an arbitrary function (local speed of propagation) are also studied.

1 Introduction

Invariance of differential equations with respect to one-parameter groups of transformations in the space of independent and dependent variables carries important information about the fundamental properties of the physical system that these equations describe, such as conservation laws. Knowledge of the groups admitted by the system of differential equations may also allow one to reduce the order of the system, to find important particular solutions, or to produce new solutions from a single solution which is known [1].

Preserving the group properties of differential equations in their discretizations seems to be a natural requirement which would ensure that the fundamental properties of the approximated differential equations are preserved in the discrete models [2, 3, 4, 5]. This idea is often followed in the design of approximations used in CFD by making sure that the scheme preserves steady-state or lower-dimensional solutions exactly. Further illustrating this statement, the author of [2] refers to the result from [5] where it was demonstrated that the undesirable process of self-oscillation observed in the solutions obtained using Harlow’s scheme [6] is caused by the non-invariance of the scheme with respect to the Galileo transformation. Unfortunately, in general, replacing the system of differential equations with their finite difference, finite volume or finite element discretizations introduces the non-invariance into the discretization. This non-invariance stems from the fact that performing the discretization requires utilizing a discrete set of nodes in the space of independent variables, i.e. a computational grid. Formally, this implies that the group of transformations acting on the discrete variables needs to leave invariant not only the approximating discrete system, but also a set of algebraic equations defining the grid [5]. Although some of the important groups of transformations arising in applications do satisfy this property, generally speaking, the discrete system loses many of the groups of the original differential equations.

The natural way to avoid this restrictiveness referred to above is to weaken the requirement of invariancy of the discrete system. One of the possibilities is to consider the differential approximation [2] or equivalent hyperbolic differential equation (EHDE) [3] corresponding to the discrete system. By formally applying a
Taylor series expansion to the approximating discrete system, one can obtain a modified differential equation which can be treated as a perturbation of the original differential equation due to discretization.

The purpose of this work is to extend the above approach using a recently developed concept of approximate invariance [10], combining the methods of group analysis and perturbation theory.

2 Method

2.1 Nonlinear advection equation and its symmetries

We consider a nonlinear advection equation

\[ L(x, t, u, u_t, u_x) = u_t + \varphi(u) u_x = 0, \]  

where \( \varphi(u) \) is a given arbitrary smooth function, characterizing the local speed of propagation of small disturbances.

Let \( z = (t, x, u) \) be a point in the space of independent variables and unknown functions. Consider the following one–parameter family of transformations acting on \( z \):

\[ z' = f(z, a), \]

where \( f = (f_1, f_2, f_3) \) and \( a \) is a scalar parameter of this family of transformations. We say that a transformation (2) is a one–parameter group (o.p.g.) (or a symmetry) with respect to the parameter \( a \) if (i) \( f(z, a) = z \) for all \( z \), if and only if \( a = 0 \), and (ii) \( f(f(z, a), b) = f(z, a + b) \) for all \( a, b \) and \( z \). It is convenient to associate each o.p.g. with its infinitesimal operator

\[ X = \xi^T \partial_z = \xi^{(t)}(x, t, u) \partial_t + \xi^{(x)}(x, t, u) \partial_x + \eta(x, t, u) \partial_u, \]

where \( \xi = (\xi^{(t)}, \xi^{(x)}, \eta)^T \). Lie theorem [1] establishes a one–to–one correspondence between an o.p.g. and its infinitesimal operator.

Once the o.p.g. of transformations (2) in the space of independent variables and unknown functions is specified, one can easily derive the corresponding transformations of the partial derivatives of unknown functions with respect to the independent variables. For equation (1), this procedure applied to partial derivatives of first order, results in an o.p.g. of transformations acting in the space of variables \((x, t, u, u_t, u_x)\), which is called the first prolongation of the o.p.g (2). One of the conveniences of using the infinitesimal operator to describe the o.p.g. is that there exist a compact prolongation formulæ [1] Theorem 2.36], which gives the components of the first prolongation \( X^{(1)} \) of the infinitesimal operator in terms of the components of the operator \( X \). For the case under consideration, these formulæe give:

\[ X^{(1)} = X + \zeta^{(t)} \partial_{u_t} + \zeta^{(x)} \partial_{u_x}, \]

where

\[ \zeta^{(t)} = D_t(\eta - \xi^{(t)} u_t - \xi^{(x)} u_x) + \xi^{(t)} u_{tt} + \xi^{(x)} u_{tx}, \quad \zeta^{(x)} = D_x(\eta - \xi^{(t)} u_t - \xi^{(x)} u_x) + \xi^{(t)} u_{tx} + \xi^{(x)} u_{xx}, \]

and \( D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{tx} \partial_{u_x} + \ldots, \quad D_x = \partial_x + u_x \partial_u + u_{tx} \partial_{u_t} + u_{xx} \partial_{u_x} + \ldots, \) are the first total derivatives.

The main problem of group analysis of differential equations is to find all o.p.g. with respect to which the given differential equation is invariant. The solution to this problem can generally be found using infinitesimal criterion of invariance [1] Theorem 2.31] which, for the present case, reduces to the following statement. Equation (1) is invariant with respect to an o.p.g. (2) if and only if

\[ X^{(1)}(u_t + \varphi(u) u_x)|_{u_x} = -\varphi(u) u_x = 0. \]

Direct calculation of the left–hand side of the above equation and substitution \( u_t = -\varphi(u) u_x \) result in an expression containing the first partial derivatives of \( \xi^{(t)}, \xi^{(x)} \) and \( \eta \) and also the factors \( u_x \) and \( u_{xx}^2 \). Since the resulting expressions must equal zero for all values of \( u_x \), the terms with different powers of \( u_x \) must
be equated to zero separately. Having grouped the terms by powers of \( u_x \), one notices that the terms with \( u_x^2 \) cancel each other. Equating the terms without \( u_x \) and the terms with \( u_x \) to zero leads to the following system of linear PDEs for the unknown components of \( X \):

\[
\begin{align*}
\eta_t + \varphi(u)\eta_x & = 0, \quad (3) \\
\varphi(u)\xi^{(t)}_t - \xi^{(x)}_t - \varphi(u)\xi^{(x)}_x + \varphi^2(u)\xi^{(t)}_x + \varphi'(u)\eta & = 0. \quad (4)
\end{align*}
\]

Having solved the above system, one obtains the following infinitesimal operator describing all o.p.g. admitted by the nonlinear advection equation:

\[
X = \xi^{(t)} \partial_t + (A + \varphi(u)\xi^{(t)} + t\varphi'(u)B) \partial_x + B \partial_u,
\]

where \( \xi^{(t)}(t, x, u) \), \( A = A(u, x - \varphi(u)t) \) and \( B = B(u, x - \varphi(u)t) \) are arbitrary smooth functions of the indicated arguments.

2.2 Finite difference discretization and differential approximation

A discretization of (1) is performed on a stencil composed of three (non–collinear) nodes of an unstructured grid – \((t_0, x_0)\), \((t_1, x_1)\) and \((t_2, x_2)\). Denoting the values of the grid function at the nodes by \( u_i \) and also \( \varepsilon \tau_i = t_i - t_0 \), \( \varepsilon h_i = x_i - x_0 \), \((i = 1, 2)\), where \( \varepsilon \) is a small parameter, the finite difference discretization can be written as:

\[
L_h(t_0, x_0, u_0, \varepsilon \tau_1, \varepsilon h_1, u_1, \varepsilon \tau_2, \varepsilon h_2, u_2) \equiv \frac{h_2(u_1 - u_0) - h_1(u_2 - u_0)}{\tau_1 h_2 - \tau_2 h_1} + \varphi(u)\frac{\tau_2(u_1 - u_0) - \tau_1(u_2 - u_0)}{\tau_2 h_1 - \tau_1 h_2} = 0. \quad (6)
\]

One can eliminate \( u_1 \) and \( u_2 \) from (6) using Taylor series expansions centered at the node \((t_0, x_0)\). Omitting the index 0 in the resulting expression, one can write the result as

\[
L_0 + \varepsilon L_1 + o(\varepsilon) = 0, \quad (7)
\]

where

\[
L_0 = L(x, x, u, u_x) \equiv u_t + \varphi(u)u_x,
\]

\[
L_1 = \frac{h_2 - \varphi(u)\tau_2}{2(\tau_1 h_2 - \tau_2 h_1)}(\tau_2^2 u_{tt} + 2\tau_1 h_1 u_{tx} + h_1^2 u_{xx}) - \frac{h_1 - \varphi(u)\tau_1}{2(\tau_1 h_2 - \tau_2 h_1)}(\tau_2^2 u_{tt} + 2\tau_2 h_2 u_{tx} + h_2^2 u_{xx}).
\]

Equation (7) is called a first differential approximation or an equivalent differential equation, and it can be interpreted as a partial differential equation with a small parameter which models the perturbations of the original advection process due to the effects of finite difference discretization. Along with the small parameter \( \varepsilon \), equation (7) has the parameters \( \tau_i \) and \( h_i \) \((i = 1, 2)\), which are not necessarily small.

2.3 Approximate Group Analysis of the First Differential Approximation

Following [14], consider a small perturbation of the infinitesimal operator \( X = X_0 + \varepsilon X_1 + o(\varepsilon) \). This perturbed operator corresponds to an approximate o.p.g. (a.o.p.g.) of transformations \( z' = f_0(z, a) + \varepsilon f_1(z, a) + o(\varepsilon) \) by virtue of an “approximate” Lie theorem [14]. The main problem of approximate group analysis is to find all a.o.p.g. of transformations such that the equation (7) remains invariant up to \( o(\varepsilon) \) terms under these transformations. As in the case of “exact” group analysis, this problem can be solved through the approximate infinitesimal criterion of invariance [14], Theorem 3, p.98] which, for this case, reduces to the solution of the problem

\[
\left(X_0^{(2)} + \varepsilon X_1^{(2)}\right)(L_0 + \varepsilon L_1)|_{L_0 + \varepsilon L_1 = o(\varepsilon)} = o(\varepsilon), \quad (8)
\]

where

\[
X_1^{(2)} = \xi^{(t)} \partial_t + \xi^{(x)} \partial_x + \eta_t \partial_u + \xi^{(t)} \partial_u + \xi^{(x)} \partial_u,
\]

\[
+ \xi^{(t)} \partial_{uu} + \xi^{(x)} \partial_{uu} + \xi^{(t)} \partial_{xx} + \xi^{(x)} \partial_{xx} + \xi^{(t)} \partial_{tx} + \xi^{(x)} \partial_{tx} + \xi^{(t)} \partial_{xt} + \xi^{(x)} \partial_{xt},
\]

\[
+ \xi^{(h)} \partial_{h1} + \xi^{(h)} \partial_{h2},
\]
operators to the second derivatives participating in (7), we must also prolong it to the parameters $\tau_i$, $h_1$ and $h_2$ (the grid step sizes), because the latter are transformed according to the transformations of the independent variables. Consider, for instance, $\tau_1$. Since $\tau_1 = t_1 - t_0$, we get:

$$\varepsilon \xi^{(\tau_1)} = \varepsilon (\xi_0^{(\tau_1)} + \varepsilon \xi_1^{(\tau_1)} + o(\varepsilon)) = \xi^{(t)}(t_1, x_1, u_1) - \xi^{(t)}(t_0, x_0, u_0) = \varepsilon (\tau_1 \delta \xi_0^{(t)} + h_1 D_x \xi_0^{(t)}) + o(\varepsilon). \quad (9)$$

Equating the terms of the first order of $\varepsilon$ and performing the same calculations for $\tau_2, h_1$ and $h_2$, one obtains:

$$\xi_0^{(\tau_2)} = \tau_2 D_t \xi_0^{(t)} + h_2 D_x \xi_0^{(t)}, \quad \xi_0^{(h_1)} = \tau_1 D_t \xi_0^{(x)} + h_1 D_x \xi_0^{(x)}, \quad \xi_0^{(h_2)} = \tau_2 D_t \xi_0^{(x)} + h_2 D_x \xi_0^{(x)}.$$

Using expansions up to the second order in $\varepsilon$ in (8) one can obtain the expressions for $\xi_1^{(\tau_1)}, \xi_1^{(\tau_2)}, \xi_1^{(h_1)}$ and $\xi_1^{(h_2)}$ in a similar way.

Substituting the above relations into (8) and using (7) to eliminate $u_t = -\varphi(u)u_x - \varepsilon L_1 + o(\varepsilon)$, the left–hand side of (8) can be rewritten as

$$(X_0 L_0 + \varepsilon (X_0 L_1 + X_1 L_0)) = \eta_0 + \varphi(u)\eta_ux + u_x (\varphi(u)\xi_0^{(t)} - \xi_0^{(x)^t} - \varphi(u)\xi_0^{(x)}) + \varphi^2(u)\xi_0^{(x)^{tt}} + \varphi'(u)\eta_0) + \varepsilon L_1 (u_x \xi_0^{(x)} - \varphi(u)u_x \xi_0^{(t)} - \eta_0 u + \xi_0^{(t)} + \varphi(u)\xi_0^{(t)}) + \varepsilon X_0 L_1 + \varepsilon X_1 L_0 + o(\varepsilon). \quad (10)$$

Equating the leading terms in the above expression to zero, as required by (8), and noticing that the resulting equation must hold for all $u_x$, one obtains that the functions $\xi_0^{(t)}, \xi_0^{(x)}$ and $\eta_0$ must satisfy the system of equations (8)–(8), resulting from application of infinitesimal criterion of invariance to the original equation (8).

Continuing decomposition by powers of $\varepsilon$ in (10), consider the equation obtained by equating the $O(\varepsilon)$ terms to zero. First notice that one can eliminate $u_t$ from the resulting equation using the substitution $u_t = -\varphi(u)u_x$, because taking the term $\varepsilon L_1$ from (7) into account contributes only to the terms $O(\varepsilon^2)$ which are not considered here. After $u_t$ is eliminated, one is left with a partial differential equation for the unknown functions $\xi_0^{(t)}$ and $\eta_1$ which depend on independent variables $t, x$ and $u$. The equation also contains $\tau_i, h_i, (i = 1, 2)$ and $u_x, u_{xt}, u_{tx}, u_{xx}$. Since (8) has already been used to eliminate $u_t$, the resulting equation must hold for all values of the parameters $\tau_i, h_i, (i = 1, 2)$, $u_x, u_{xt}, u_{tx}, u_{xx}$. This allows one to decompose the equation further. To this end, notice that among all $O(\varepsilon)$ terms in (10), only the term $\varepsilon X_1 L_0$ is independent of $\tau_i, h_i, (i = 1, 2)$. Therefore, the equation can be decomposed to obtain the following system:

$$X_1 L_0 = 0, \quad (11)$$

$$L_1 (u_x \xi_0^{(x)} - \varphi(u)u_x \xi_0^{(t)} - \eta_0 u + \xi_0^{(t)} + \varphi(u)\xi_0^{(t)}) + X_0 L_1 = 0. \quad (12)$$

where $u_t$ is replaced by $-\varphi(u)u_x$. It follows from equation (11) that in order for the a.o.p.g. $X_0 + \varepsilon X_1 + o(\varepsilon)$ to leave equation (7) invariant up to $O(\varepsilon)$ terms, it is necessary that the components of $X_1$ satisfy the same system of equations (12) as the system that an infinitesimal operator of an o.p.g. admitted by advection equation (8) satisfies. It was also established above that $X_0$ must be admitted by the original equation (8). Notice that (12) imposes additional conditions on $X_0$. Knowledge of these conditions allows one to select those of the o.p.g. (8) of the advection equation which can be made a.o.p.g. of the first differential approximation at the price of adding a small correction $\varepsilon X_1$ to the group in order to account for the effects of discretization. Following the terminology of (10), these groups are said to be inherited by a perturbed equation (8). Notice that an infinitesimal operator $X^{(1)}$ with all zero components satisfies (10). Therefore, an operator which is inherited by (8) leaves this equation invariant up to $O(\varepsilon)$ terms even without a small correction. The purpose of the following is to derive this set of conditions on $X_0$ and to describe all groups inherited by (8).

Substituting the expression for $L_1$ into the left–hand side of (12) and multiplying both sides of the resulting equation by $2(\tau_1 h_2 - \tau_2 h_1)$, one obtains an equation with zero on the right–hand side and a number of terms on the left–hand side. Each of these terms has a factor of the form $\tau_1^k \tau_2^l h_1^m h_2^n$, where $k, l, m$ and $n$
can be 0, 1 or 2, and the terms with the same factors can be grouped together. Since (12) must hold for all values of \( \tau_1, \tau_2, h_1 \) and \( h_2 \), each group of terms must be equated to zero independently. This procedure gives the following system of equations:

\[
\begin{align*}
\zeta^{(tt)}_0 + 2u_{tt}D_\xi^{(t)}_0 + 2u_{tx}D_\xi^{(x)}_0 - \varphi(u)u_{ttx} + u_{utx}u_{x0t} - u_{ttt} & = 0, \\
-\varphi(u)(\zeta^{(tt)}_0 + 2u_{tt}D_\xi^{(t)}_0 + 2u_{tx}D_\xi^{(x)}_0) + \varphi(u)u_{tx} + u_{tx}u_{x0t} - \varphi(u)u_{ttx} + u_{utx}u_{x0t} & = 0, \\
\zeta^{(tx)}_0 + u_{tx}D_\xi^{(t)}_0 + u_{tx}D_\xi^{(x)}_0 + u_{xx}D_\xi^{(x)}_0 - \varphi(u)u_{tx}u_{x0t} + u_{tx}u_{x0t} & = 0, \\
-\varphi(u)(\zeta^{(tx)}_0 + u_{tx}D_\xi^{(t)}_0 + u_{tx}D_\xi^{(x)}_0 + u_{xx}D_\xi^{(x)}_0 - \varphi(u)u_{tx}u_{x0t} + u_{tx}u_{x0t}) & = 0, \\
2\varphi(u)u_{tx}u_{x0t} - \varphi(u)u_{tx}u_{x0t} & = 0, \\
\phi^{(tx)}_0 + 2u_{tx}D_\xi^{(t)}_0 + 2u_{xx}D_\xi^{(x)}_0 + \varphi(u)u_{xx}u_{x0t} + u_{xx}u_{x0t} & = 0, \\
-\varphi(u)(\phi^{(tx)}_0 + 2u_{tx}D_\xi^{(t)}_0 + 2u_{xx}D_\xi^{(x)}_0 + \varphi(u)u_{xx}u_{x0t} + u_{xx}u_{x0t}) & = 0, \\
\phi^{(x)}_0 + 2u_{tx}D_\xi^{(t)}_0 + 2u_{xx}D_\xi^{(x)}_0 + \varphi(u)u_{xx}u_{x0t} + u_{xx}u_{x0t} & = 0, \\
-\varphi(u)(\phi^{(x)}_0 + 2u_{tx}D_\xi^{(t)}_0 + 2u_{xx}D_\xi^{(x)}_0 + \varphi(u)u_{xx}u_{x0t} + u_{xx}u_{x0t}) & = 0, \\
\end{align*}
\]

After the expressions for \( \zeta^{(tt)}_0, \zeta^{(tx)}_0 \) and \( \zeta^{(x)}_0 \) and the total derivatives occurring in the above equations are substituted, and \( u_t \) is replaced by \( -\varphi(u)u_x \), one finds that in each equation the terms containing the second partial derivatives of \( u \) cancel each other. Proceeding as before and decomposing each of the above equations results in the following set of necessary and sufficient conditions, under which \( X_0 \) is inherited by \( \xi \):

\[
\begin{align*}
\eta_{tt} & = 0, \\
\eta_{tx} & = 0, \\
\eta_{xx} & = 0, \\
2\varphi(u)\eta_{tu} - \varphi(u)\xi^{(t)}_{0tt} + \xi^{(x)}_{0tt} & = 0, \\
\eta_{tu} - \xi^{(x)}_{0xx} - \varphi(u)\eta_{0xx} + \varphi(u)\xi^{(t)}_{0xx} & = 0, \\
2\eta_{ux} - \xi^{(x)}_{0xx} + \varphi(u)\xi^{(t)}_{0xx} & = 0, \\
2\xi_{0ux} - 2\varphi(u)\xi^{(t)}_{0ux} + \varphi(u)\eta_{0uu} & = 0, \\
-\varphi(u)\xi^{(t)}_{0ux} - \xi^{(x)}_{0ux} - \varphi(u)\eta_{0uu} + \varphi(u)\xi^{(t)}_{0uu} & = 0, \\
2\varphi(u)\xi^{(t)}_{0ux} - 2\xi^{(x)}_{0xx} + \eta_{0uu} & = 0, \\
\xi^{(x)}_{0uu} - \varphi(u)\xi^{(t)}_{0uu} & = 0.
\end{align*}
\]

Since \( \zeta^{(tt)}_0, \zeta^{(tx)}_0 \) and \( \zeta^{(x)}_0 \) satisfy (3) – (4), one can write \( \zeta^{(t)}_0 = \zeta^{(t)}_0(t, x, u), \zeta^{(x)}_0 = A(u, x - \varphi(u)t) + \varphi(u)\xi^{(t)}_0(t, x, u) + t\varphi'(u)B(u, x - \varphi(u)t) \) and \( \eta_0 = B(u, x - \varphi(u)t) \), where \( \xi^{(t)}, A \) and \( B \) are arbitrary smooth functions of their arguments. Introducing new independent variables \( x_1 = u, x_2 = x - \varphi(u)t, x_3 = t \) and denoting the partial derivatives of the above arbitrary functions by subindices (eg. \( A_{12} = A_{x_1x_2} \) etc), system (13) – (22) can be rewritten as:

\[
\begin{align*}
B_{22} & = 0, \\
2\varphi B_{12} + 4\varphi' B_2 & = \varphi A_{22} = 0, \\
-2\varphi B_{12} + 2\varphi' B_2 & = \varphi A_{22} = 0, \\
A_{22} - 2B_{12} & = 0, \\
-2\varphi' A_2 - 2\varphi A_{12} + 2t\varphi' A_{22} + 2\varphi' \xi^{(t)}_{03} - 2t\varphi' \xi^{(t)}_{02} + 2\varphi'' B & = 0, \\
+2\varphi' B_1 - (4t\varphi')^2 + 3t\varphi' \xi^{(t)}_{02} & = 0, \\
2\varphi A_{12} + \varphi' A_2 + 2t\varphi' A_{22} - \varphi' \xi^{(t)}_{03} + 2\varphi' \xi^{(t)}_{02} - \varphi' B & = 0, \\
-\varphi' B_1 + (2t\varphi')^2 + 3t\varphi' \xi^{(t)}_{02} & = 0, \\
2A_{12} + 2t\varphi' A_{22} + 2\varphi' \xi^{(t)}_{02} + 3t\varphi' B_2 + 4t\varphi' B_{12} & = 0.
\end{align*}
\]
\[ A_{11} - 2t \varphi' A_{12} - t \varphi'' A_{22} + \ell^2 (\varphi')^2 A_{22} + \varphi'' \xi_0^{(t)} + 2 \varphi' \xi_0^{(t)} - 2t (\varphi')^2 \xi_0^{(t)} + t \varphi'' B + 2t \varphi' B_1 - 3t^2 \varphi' \varphi'' B_2 + t \varphi' B_{11} - 2t^2 (\varphi')^2 B_{12} = 0, \]

It follows from (24) and (25) that \( \varphi' B_2 = 0 \). According to this, consider two possibilities: \( \varphi' = 0 \) or \( B_2 = 0 \).

### 2.3.1 Case \( \varphi' = 0 \) (linear advection)

Denote \( \varphi(u) = c \equiv \text{const} \). Using (23), write \( B = B^{(0)}(x_1) + B^{(1)}(x_1)x_2 \), where \( B^{(0)}(x_1) \) and \( B^{(1)}(x_1) \) are unknown functions. Then (24), (26), (25) give \( A = (B^{(1)}(x_1))^2 x_2^2 + g'(x_1)x_2 + l(x_1) \) with \( g(x_1) \) and \( l(x_1) \) being unknown functions. Substitution of the above expressions into the left-hand side of (29) results in an expression which is linear in \( x_2 \) with its coefficients being functions of \( x_1 \). This expression can be zero only if its coefficients are zero. This results in two equations \((B^{(0)}(x_1) - 2g(x_1))'' = 0 \) and \((B^{(1)}(x_1))'' = 0 \), and therefore \( B^{(0)}(x_1) = 2g(x_1) + C x_1 + D, B^{(1)}(x_1) = Ex_1 + F \) and \( A = Ex_2^2 + (g(x_1))'x_2 + l(x_1) \) where \( C, D, E, F \) and \( x_2 \) are arbitrary constants. Finally, (30) implies that \( A_{11} = 0 \), and after substituting the above expression for \( A \), one obtains \((g(x_1))'' = 0 \) and \((l(x_1))'' = 0 \). Therefore \( g(x_1) = M x_2^2 + N x_1 + P \) and \( l(x_1) = K x_1 + L \), with arbitrary constants \( K, L, M, N \) and \( P \).

Collecting the above yields the following expressions for \( A \) and \( B \):

\[
A = \quad Ex_2^2 + (2M + N)x_2 + K x_1 + L, \\
B = \quad 2Mx_2^2 + (2N + C)x_1 + D + (Ex_1 + F)x_2.
\]

Notice that, as follows from (23) – (26), \( \xi_0^{(t)} \) can still be an arbitrary function.

Since the expressions for \( A \) and \( B \) are linear with respect to eight arbitrary constants, the set of infinitesimal operators inherited by (30) is spanned by the following eight infinitesimal operators

\[
X_{(1)} = u \partial_u, \quad X_{(5)} = u \partial_x, \\
X_{(2)} = \partial_u, \quad X_{(6)} = \partial_x, \\
X_{(3)} = (x - ct)^2 \partial_x + u(x - ct) \partial_u, \quad X_{(7)} = u(x - ct) \partial_x + (x - ct)^2 \partial_u, \\
X_{(4)} = (x - ct) \partial_u, \quad X_{(8)} = (x - ct) \partial_x + 2u \partial_u,
\]

and an infinite-dimensional set of infinitesimal operators

\[
X_{(\infty)} = \xi_0^{(t)} \partial_t + a \xi_0^{(t)} \partial_x,
\]

where \( \xi_0^{(t)} = \xi_0^{(t)}(t, x, u) \) is an arbitrary function.

### 2.3.2 Case \( \varphi' \neq 0, B_2 = 0 \)

In this case \( B = B^{(0)}(x_1) \), where \( B^{(0)}(x_1) \) is an unknown function, and (24) immediately gives \( A_{22} = 0 \), and therefore \( A = A^{(0)}(x_1) + x_2 A^{(1)}(x_1) \). Substituting the expressions for \( A \) and \( B \) into (29) gives \( \xi_0^{(t)} = x_2 ((B^{(0)})'' = 2(A^{(1)})' + (2\varphi') + f(x_1, x_2), where \( f(x_1, x_2) \) is an unknown function. After this expression for \( \xi_0^{(t)} \) is substituted into (27), one obtains \(- \varphi' A^{(1)} + \varphi' f_3 + \varphi'' B^{(0)} + \varphi'' B^{(0)})' = 0 \). All terms in this equation, except the second one, depend solely on \( x_1 \). As a consequence, we must have \( f_3 = g(x_1) \) and \( f = g(x_1)x_2 + l(x_1) \) where \( g(x_1) \) and \( l(x_1) \) are unknown functions. Using this result, one can write \( A^{(1)} = g + aB^{(0)} + (B^{(0)})' \), where \( a = \varphi'' / \varphi' \).

Collecting together the above results, one obtains

\[
A = \quad A^{(0)} + (g + aB^{(0)} + (B^{(0)})')x_2, \\
B = \quad B^{(0)}, \\
\xi_0^{(t)} = \quad G(x_1)x_2 + g(x_1)x_3 + l(x_1),
\]

where \( G(x_1) = -2g - 2(aB^{(0)})' - (B^{(0)})'' \). Proceeding with the above, substitute the resulting expressions for \( A, B \) and \( \xi_0^{(t)} \) into (23) and (26). The first equation will be satisfied identically, while the second will result in an equation which has zero on its right-hand side and a linear expression with respect to the independent
variables \( x_2 \) and \( x_3 \) with its coefficients being functions of \( x_1 \) only, on its right–hand side. Equating these coefficients to zero results in the following conditions on the unknown functions \( A^{(0)}, A^{(1)}, B^{(0)}, g \) and \( l \):

\[
- \varphi'' A^{(1)} + \varphi'' g + 2\varphi' g' + \varphi''' B^{(0)} + 2\varphi'' (B^{(0)})' = 0, \quad (33)
\]

\[
(A^{(1)})'' + \varphi'' G + 2\varphi' G' = 0, \quad (34)
\]

\[
(A^{(0)})'' + \varphi'' l + 2\varphi' l' = 0. \quad (35)
\]

Substitution of the expressions for \( A^{(1)} \) and \( G \) into \( (33) \) and \( (34) \) results in a system of two differential equations with two unknown functions, \( B^{(0)} \) and \( g \). Integrating this pair of equations yields

\[
a B^{(0)} + 2g = C_1, \quad (36)
\]

\[
b^2 B^{(0)} = C_2, \quad (37)
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants and \( b = a^2/2 - a' \). Equation \( (33) \) can be integrated to find \( l \) in terms of \( A^{(0)} \):

\[
l = C_3 (\varphi')^{-\frac{3}{2}} - \frac{1}{2} (\varphi')^{-\frac{1}{2}} \int (\varphi')^{-\frac{3}{2}} (A^{(0)})'' dx_1, \quad (38)
\]

where \( C_3 \) is an arbitrary constant. In order to formulate the final result, consider two cases: \( b = 0 \) and \( b \neq 0 \).

**Case \( b = 0 \).**

In this case \( \varphi = K(L - u)^{-1} + M \), where \( L \) and \( M \) are arbitrary constants and \( K \) is a non–zero constant, since in the present case \( \varphi' \neq 0 \). As follows from \( (37) \), \( B^{(0)} = B^{(0)}(x_1) \) can be an arbitrary function and \( C_2 = 0 \). Using \( (33) \) and \( (34) \), one finds that the linear space of the infinitesimal operators inherited by \( (3) \) in this case is spanned by the following infinitesimal operators:

\[
X_{(1)} = t \partial_t + x \partial_x, \quad (39)
\]

\[
X_{(2)} = (\varphi')^\frac{1}{2} \partial_t + \varphi(\varphi')^\frac{1}{2} \partial_x, \quad (40)
\]

\[
X_{(3)} = \tilde{l} \partial_t + ((x - \varphi t)A^{(0)} + \varphi \tilde{l}) \partial_x, \quad (41)
\]

\[
X_{(4)} = \left( -\frac{a b^{(0)} (a' t - \varphi) + (B^{(0)})''}{2} (x - \varphi t) - \frac{1}{2} a (B^{(0)})' \right) \partial_t + \frac{1}{2} a B^{(0)} t \varphi + t \varphi' B^{(0)} \partial_x
\]

\[
+ B^{(0)} \partial_u. \quad (42)
\]

where \( \tilde{l} \) is related to \( A^{(0)} \) by \( (35) \) with \( C_3 = 0 \). Unlike \( X_{(1)} \) and \( X_{(2)} \), the operators \( X_{(3)} \) and \( X_{(4)} \) each represent an infinite family of operators, parametrized by two arbitrary functions \( B^{(0)} = B^{(0)}(u) \) and \( A^{(0)} = A^{(0)}(u) \).

**Case \( b \neq 0 \).**

In this case, \( (33) \) and \( (34) \) immediately give \( B^{(0)} = C_2 b^{-\frac{1}{2}} \) and \( g = \frac{1}{2} C_1 - \frac{1}{2} C_2 a b^{-\frac{1}{2}} \). The functions \( l \) and \( A^{(0)} \), as in the previous case, are related through \( (35) \). Using these expressions, one can see that, as in the cases above, all infinitesimal operators inherited by \( (3) \) form a linear space. For the given case, the basis in this space can be chosen as follows:

\[
X_{(1)} = t \partial_t + x \partial_x, \quad (43)
\]

\[
X_{(2)} = \left( -\frac{b - \frac{1}{2}}{8 \varphi'} (4 a b^2 - 2 a b + 3 a'^2 - 2 b b'') (x - \varphi t) - \frac{1}{2} a b^{-\frac{1}{2}} t \right) \partial_t
\]

\[
+ (t \varphi b^{-\frac{1}{2}} - \frac{1}{2} t a b^{-\frac{1}{2}} + (1) b^{-\frac{1}{2}} (a'' - 2 a a' + \frac{a^3}{2}) - \frac{\varphi b^{-\frac{1}{2}}}{8 \varphi'} (4 a b^2 - 2 a b + 3 a'^2 - 2 b b'') (x - \varphi t)) \partial_x
\]

\[
+ b^{-\frac{1}{2}} \partial_u, \quad (44)
\]

\[
X_{(3)} = (\varphi')^\frac{1}{2} \partial_t + \varphi(\varphi')^\frac{1}{2} \partial_x, \quad (45)
\]

\[
X_{(4)} = \tilde{l} \partial_t + ((x - \varphi t) A^{(0)} + \varphi \tilde{l}) \partial_x, \quad (46)
\]

where \( \tilde{l} \) and \( A^{(0)} \) are related through \( (35) \) with \( C_3 = 0 \). Unlike the case \( b = 0 \), only \( X_{(4)} \) is an infinite family parametrized by an arbitrary function \( A^{(0)} \).
3 Results and Conclusion

One parameter groups of transformations approximately admitted by a first differential approximation corresponding to a finite difference approximation of a nonlinear advection equation have been studied.

It was demonstrated that the necessary condition for such a group to be approximately admitted by a first differential approximation is that this group is admitted (exactly) by the original advection equation. Sufficient conditions (13)–(22), imposing additional restrictions upon the groups of the original equation, were also derived.

The description of a family of transformation groups admitted by a nonlinear advection equation (5) contains three arbitrary functions. The requirement that the group must be inherited by a first differential approximation (7) narrows this arbitrariness in the following way.

For the most general case \( \varphi' \neq 0, b \neq 0 \), the family of transformation groups inherited by (7) is linear, and its basis may be chosen to consist of three groups (43)–(45) and an infinite family of groups (46) parametrized by an arbitrary function of \( u \) only.

For the case \( b = 0 \), the family of inherited groups expands and can be described as a linear set spanned by two groups (41)–(43) and two infinite families of groups (41)–(42) each parametrized by a different arbitrary function of \( u \) only.

Finally, in the linear advection case when \( \varphi' = 0 \), the set of inherited transformation groups has as its basis eight groups (41) and one infinite family of groups (52) parametrized by an arbitrary function of \( t, x \) and \( u \).

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