Zombie fires in peatlands disappear from the surface, smoulder underground during the winter, and ‘come back to life’ in the spring. They can release hundreds of megatonnes of carbon into the atmosphere per year and are believed to be caused by surface wildfires. Here, we propose rate-induced tipping (R-tipping) to a subsurface hot metastable state in bioactive peat soils as a main cause of Zombie fires. Our hypothesis is based on a conceptual soil-carbon model subjected to realistic changes in weather and climate patterns, including global warming scenarios and summer heatwaves. Mathematically speaking, R-tipping to the hot metastable state is a genuine nonautonomous instability, due to crossing an elusive quasi-threshold, in a multiple-timescale dynamical system. To explain this instability, we provide a framework combining a special compactification technique with concepts from geometric singular perturbation theory. This framework allows us to reduce an R-tipping problem due to crossing a quasi-threshold to a heteroclinic orbit problem in a singular limit. We identify generic cases of tracking–tipping transitions via: (i) unfolding of a codimension-two heteroclinic folded-saddle-node type-I singularity for global warming and (ii) analysis of a codimension-one saddle-to-saddle heteroclinic orbit for summer heatwaves, in turn revealing new types of excitability quasi-thresholds.

1. Introduction

Tipping points, or critical transitions, are instabilities known to occur in natural and human systems subjected to changing external conditions, or external inputs. They may be explained in layman’s terms as an abrupt and
large change in the state of a system in response to a small or slow change in the external input. The change in the state of the system may be permanent or transient.

Climate change is an important factor for tipping points in natural systems. An immense amount of research is being conducted to predict and prevent its worst effects. Hand in hand with this research effort, interest in tipping has accelerated due to theorized [1,2] and observed [3,4] tipping points in the earth system caused by climate change. Many of the tipping elements identified in [1], for example the loss of Arctic sea ice or the shutdown of the Atlantic Meridional Overturning Circulation, can be captured by elaborate, high-resolution mathematical models referred to as general circulation models (GCMs). In this paper, we use a conceptual soil-carbon model with time-varying climate as an external input to describe a tipping element that is not captured by GCMs: a release of gigatonnes of carbon from temperature-sensitive peat soils into the atmosphere via so-called ‘Zombie fires’ [5] that disappear from the surface, smoulder underground during the winter, and ‘come back to life’ in the spring.

Owing to the explicit time dependence of the external input, the ensuing dynamical system is nonautonomous. This means that analysis of tipping points requires, in general, techniques beyond classical autonomous stability theory [6–8]. Nonetheless, it is useful to consider the corresponding autonomous frozen system with fixed-in-time inputs. In the frozen system, we identify a desired stable state, and refer to this state as the base state. When the external input changes over time, the shape and position of the base state may change too, and the nonautonomous system will try to track the moving base state. However, sometimes tracking is not possible and tipping occurs. For example, the base state may lose stability or disappear in a classical bifurcation at some critical level of the input. If this bifurcation is dangerous [9], the system tips to a different stable state, referred to as an alternative stable state. We then say the system undergoes bifurcation-induced tipping, or in short B-tipping [6,9,10]. Another, arguably more interesting example is when the external input changes faster than some critical rate, the nonautonomous system deviates too far from the changing base state, crosses some threshold or quasi-threshold [11] and tips to an alternative state. Such tipping is caused solely by the rate of change of the external input and we say the system undergoes rate-induced tipping, or in short R-tipping [6,8]. Crucially, unlike B-tipping, R-tipping can occur to an alternative transient state that lasts for a finite time, after which the system returns to the base state [12–15]. Such R-tipping is referred to as ‘reversible’ in [8]. Systems that exhibit reversible R-tipping are also known as excitable systems [16]. The peat soil instability studied in this paper is an example of reversible R-tipping to an alternative metastable state that lasts for a long but finite time, that occurs due to crossing an elusive quasi-threshold.

The main motivation for our study is a combination of two environmental features of the Arctic. First is the organic carbon content. Estimates for the organic carbon contained in Northern and Arctic permafrost peat soil alone range from approximately 500 Gt [17] to approximately 1700 Gt [18]. For comparison, the atmospheric carbon pool is estimated to be approximately 850 Gt [19]. Second is the rate of atmospheric warming and the increasing trends in summer heatwaves. Due to so-called ‘Arctic Amplification’ [20], both the so-far observed and future predicted warming for Arctic regions are approximately double the global mean [21]. This means that the Arctic is home to massive deposits of ancient peat carbon and is the fastest warming region on the planet. To visualize this combination of environmental features we combine in figure 1 (colourscale) recent rates of global warming and (greyscale) the global distribution of peat soils. In addition to the increase in the mean global temperature, there is an increase in the intensity, frequency and duration of summer heatwaves in the Northern Hemisphere [25], with the Arctic temperature record a scorching 38°C in 2020 [26]. Since peat soils are bioactive and thus temperature sensitive [27], such conditions can lead to thermal runaway in the soil. This is the reason why northern latitude peat soils were identified as a potential tipping element in [1]. To the best of our knowledge, the first example of R-tipping in peat soils: thermal runaway triggered by the rate of atmospheric warming, was reported, but not emphasized, by Khvorostyanov et al. [28],

1 This is in contrast to an alternative stable state that lasts forever.

2 Note that these recent short-term rates of global warming exceed the long-term rates in the CMIP5 outputs.
Figure 1. (Colourscale) Historical global rates of warming from the period 1951–1980 to the period 2017–2021 obtained using observations compiled by Berkeley Earth [22,23], together with (black and grey) the global distribution of peat soils obtained from PEATMAP [24]. Note the largest warming rates at higher latitudes, in areas with a significant concentration of peat soils, such as (blue arrow) Cherskii in Northern Siberia.

Fig. 4(a)). Later, Luke & Cox [29] proposed a conceptual soil-carbon model that exhibits a short-lived explosive release of soil carbon into the atmosphere above some critical rate of global warming, which they termed the ‘compost-bomb instability’; see also [30]. The dynamical mechanism responsible for this R-tipping instability was explained by Wieczorek et al. [12]. Additionally, it has been known that spontaneous combustion is the main cause of fires at composting facilities [31], which can then spread, e.g. the recent Wennington fire in London [32].

In the first, mathematical modelling part of the paper in §§2 and 3, we:

— Modify the conceptual model introduced by Luke & Cox [29] with a more realistic microbial soil respiration function.
— Show that the modified conceptual model has a new alternative hot metastable state and reproduces the key features of the medium-complexity model from Khvorostyanov et al. ([28], Fig. 4(a)).
— Demonstrate R-tipping to the hot metastable state in the modified conceptual model for realistic climate change scenarios including global warming and summer heatwaves.
— Based on the above, propose an explanation for ‘Zombie fires’ in peatlands [5,33–35].

In the second, mathematical analysis part of the paper in §§4–7, we identify non-obvious dynamical mechanisms that are responsible for the R-tipping instability to the hot metastable state. The main obstacles to analysis of this instability are twofold: the conceptual soil-carbon model is a nonautonomous dynamical system, so it does not have any equilibria or compact invariant sets, and the R-tipping is a quantitative change ([8], Sec.3.3) due to crossing an elusive quasi-threshold in the phase space [13,36]. To overcome these obstacles, we combine three different strategies. Firstly, we consider external inputs that decay to a constant at infinity. Secondly, we compactify the problem to include the equilibrium base states for the autonomous limit systems from infinity [37]. These two strategies alone work for R-tipping due to crossing regular thresholds that are anchored at infinity by unstable compact invariant sets called regular R-tipping edge states ([8], Sec.4). However, quasi-thresholds do not contain such edge states ([8], Sec.8). Therefore, thirdly, we exploit large timescale separation in the model and apply concepts from geometric singular perturbation theory (GSPT). Specifically, we:

— Define R-tipping due to crossing a quasi-threshold in the nonautonomous system in terms of slow manifolds and canard trajectories [38] for the autonomous compactified system.
— Identify singular R-tipping edge states: special points called folded singularities [39] that arise in the reduced (slow) system, and new saddle equilibria that arise in the layer (fast) system.
— Reduce an R-tipping problem due to crossing a quasi-threshold to a heteroclinic orbit connecting the base state for the past limit system to a singular R-tipping edge state. We then use this reduction to identify four different cases of tracking–tipping transition:
— For global warming, three (slow) cases are identified via unfolding of a codimension-two non-central heteroclinic folded-saddle-node type-I singularity that arises in the reduced system.
— For a summer heatwave, a fourth (fast) case is identified via analysis of a codimension-one heteroclinic orbit connecting the base state for the past limit system to a new saddle equilibrium that arises in the layer system.
— Show that a quasi-threshold gives rise to critical ranges of rate of change of the external input rather than isolated critical rates. Furthermore, we reveal new types of quasi-thresholds through analysis of canard trajectories associated with singular R-tipping edge states.

2. The nonautonomous soil-carbon model

The starting point of our analysis is a discussion of the conceptual soil-carbon model introduced by Luke & Cox [29]. This model describes the time evolution of the soil temperature $T$ and soil carbon concentration $C$ in peat soils

$$
\mu \frac{dT}{dt} = -\lambda(T - T_a(rt)) + A C R_s(T)
$$

(2.1)

and

$$
\frac{dC}{dt} = \Pi - CR_s(T).
$$

(2.2)

The model incorporates three soil processes and one time-varying external input. The first process describes balancing of the soil, $T$, and atmospheric, $T_a$, temperatures towards a thermal equilibrium according to Newton’s Law of Cooling, at a rate that depends on the soil-to-atmosphere heat transfer coefficient $\lambda$ and the specific heat capacity of the soil $\mu$. The second process describes a linear increase in the soil carbon concentration $C$ over time at a rate $\Pi$, due to carbon generated from decaying plant litter and other processes referred to as ‘Gross Primary Production’ [40]. The third and only nonlinear process in the model describes temperature-sensitive microbial activity in the soil in terms of the soil respiration function $R_s(T)$. This process couples the dynamics of $T$ and $C$, and is discussed in detail in §2(a). Atmospheric temperature $T_a(rt)$ is a time-varying external input, which represents weather anomalies or climate variation. The rate parameter $r > 0$ quantifies the timescale of climatic variability and is the key input parameter in the model. An important aspect of our study is that we use realistic values of the soil parameters based on Luke & Cox [29], which are given in table S1 in the electronic supplementary material, a realistic soil respiration function $R_s(T)$ introduced in §2(a), and realistic climate-change scenarios $T_a(rt)$ based on real weather data from Cherskii in Siberia [41]; see the arrow in figure 1.

To facilitate the analysis, we introduce a small parameter $\epsilon = \mu/A \ll 1$ and rewrite the soil-carbon model (2.1)–(2.2) as a fast-slow nonautonomous dynamical system [12]:

$$
\epsilon \frac{dT}{dt} = f_1(T, C, T_a(rt)) := -\lambda(T - T_a(rt)) + A C R_s(T)
$$

(2.3)

$$
\frac{dC}{dt} = f_2(T, C) := \Pi - CR_s(T),
$$

(2.4)

where we define $f_1$ and $f_2$ for convenience. Note that system (2.1)–(2.2) or (2.3)–(2.4) does not have any equilibria (stationary solutions) owing to the time-varying external input $T_a(rt)$.

3Note that $dC/dt$ is the ‘Net Primary Production’ in the model.
Figure 2. (Black) The modified non-monotone microbial soil respiration function $R^*_s(T)$ in (2.6) with $c = 10$ and the die-off temperature $T^* = 50^\circ$C and $70^\circ$C. (Grey) In the limit $T^* \to \infty$, we recover the unmodified monotone microbial soil respiration function $R^+_s(T)$ in (2.5). See table S1 in the electronic supplementary material for other parameter values.

(a) Modified microbial respiration

Our contribution to the model introduced by Luke and Cox is a modification of the microbial soil respiration function from Luke & Cox [29]. At low to moderate soil temperatures $T$, microbial soil respiration can be described by a $Q_{10}$ exponential function [42]:

$$R^+_s(T) = R_s(0) e^{\alpha T}$$

with $\alpha = \ln (Q_{10})/10$, (2.5)

where the dimensionless, soil-specific parameter $Q_{10}$ may be estimated from experimental data [42]. While the monotone $R^+_s(T)$ in (2.5) captures thermal runaway—the process responsible for the R-tipping instability, it becomes unrealistic at high $T$. Specifically, $T$ quickly increases to unrealistically high levels due to microbial soil respiration alone [12,29]. To address this issue, we account for an important limitation, that is, soil microbes die above some die-off temperature $T^*$. Specifically, we construct a modified non-monotone microbial soil respiration function

$$R^*_s(T) = R_s(0) \frac{e^{ab} + e^{-cab}}{e^{-a(T-b)} + e^{a(T-b)}} \quad \text{with} \quad b = T^* + \frac{\ln c}{a + ca},$$

(2.6)

that agrees with the $Q_{10}$ exponential growth (2.5) for $T < T^*$, has a maximum at $T = b \approx T^*$, and decays exponentially for $T > T^*$ (figure 2). Such $R^*_s(T)$ captures thermal runaway and, additionally, stops it at high but realistic levels of $T$. This construction introduces two additional parameters, namely the die-off temperature $T^*$, and the ratio $c$ of the exponential decay rate for $T > T^*$ and exponential growth rate $\alpha$ for $T < T^*$. In the remainder of the paper, we use

$$R_s(T) = R^*_s(T).$$

(b) The autonomous frozen system and the moving equilibrium

To gain insight into the tipping mechanisms in the nonautonomous system (2.3)–(2.4), or equivalently system (2.1)–(2.2), we set $r = 0$ and consider properties of the resulting autonomous frozen system with fixed-in-time $T_a$. The frozen system has just one equilibrium

$$e(T_a) = (T^*(T_a), C^*(T_a)) = \left( T_a + \frac{A \Pi}{\lambda}, \frac{\Pi}{R_s(T_a) + (A \Pi/\lambda)} \right),$$

(2.7)

which is the ‘base state’ described in the introduction. The position of $e(T_a)$ in the phase plane $(T, C)$ changes with $T_a$, but the equilibrium remains linearly stable and globally attractive (attracts all initial conditions) within the realistic range of $T_a$ used in our study. Thus, we can exclude the possibility of B-tipping from $e(T_a)$ [6]. Next, we consider the stable equilibrium of the frozen
system parameterized by time \( t \) for a given input \( T_a = T_a(rt) \), which we denote

\[
e(T_a(rt)),
\]

and refer to as the moving stable equilibrium [7,8]. The moving stable equilibrium is not a solution to the nonautonomous system (2.3)–(2.4). However, it can serve as a useful point of reference. We follow the approach of O’Keeffe & Wieczorek [7], Wieczorek et al. [8] and Ashwin et al. [43] and relate solutions of the nonautonomous system (2.3)–(2.4) to \( e(T_a(rt)) \) for different rates \( r > 0 \). For small \( r \), solutions to (2.3)–(2.4) started near \( e(T_a(rt)) \) are guaranteed to stay near or track \( e(T_a(rt)) \) ([8], Th.7.1). However, a nonautonomous R-tipping instability in the form of a large transient departure from \( e(T_a(rt)) \) may appear for larger \( r \).

3. R-tipping to a hot metastable state and zombie fires

The more realistic \( R_s(T) = R^*_s(T) \) introduced in §2 gives rise to a hot metastable state at high soil temperatures \( T \approx T^* \). This state is reminiscent of ‘Zombie fires’ observed in tropical and arctic peatlands [5,33–35]: such fires appear to be extinguished, but smoulder underground throughout the winter and re-emerge the following year [33]. Zombie fires are generally believed to happen as a result of surface wildfires [5]. However, as far as we know, there have been no attempts to explain this phenomenon. Here, we propose a hypothesis that R-tipping to a subsurface hot metastable state, due to rates of atmospheric warming, is a main cause of Zombie fires. Our hypothesis is based on two remarkable results of the soil-carbon model (2.3)–(2.4) for realistic soil parameters and different climate-change scenarios \( T_a(rt) \) based on real weather data [41].

R-tipping to a hot metastable state in the conceptual model (2.3)–(2.4) can be triggered by realistic climate patterns ranging from summer heatwaves to global warming scenarios. Figure 3 shows the soil temperature change in response to seasonal variations of the atmospheric temperature. Rather amazingly, a summer heatwave in year 10 breaks the seasonal response pattern and triggers a sudden transition to a hot metastable state that lasts for over a decade and releases most of the carbon from the soil into the atmosphere. Following this, the system settles to a lower than initial soil temperature pattern for a refractory period of over a century, during which time both soil carbon and soil temperature slowly increase back to their initial seasonal patterns. Figure 4 shows the mean soil temperature change in response to a slow increase in the mean atmospheric temperature of 4°C over 200 years.4 In this scenario, a sudden transition to a hot metastable state that lasts for over a decade is triggered, and astonishingly occurs after only a modest mean temperature increase of \( \approx 1.1 \)°C. Following this, the system settles to a lower than initial soil temperature for a refractory period of over a century, during which time both soil carbon and soil temperature slowly increase to their moving equilibrium levels.

R-tipping to a hot metastable state in the conceptual model (2.3)–(2.4) shows qualitative and quantitative agreement with the results of intermediate-complexity PDE models. The conceptual ordinary differential equation (ODE) model (2.3)–(2.4) with the more realistic \( R_s(T) = R^*_s(T) \) captures the key nonlinearities of the soil-carbon system. This is evidenced by the ODE model reproducing both qualitatively and quantitatively the peat soil instability predicted by a medium-complexity partial differential equation (PDE) model of Siberian permafrost carbon dynamics under climate change [28,48]. Specifically, figure 5 shows that the ODE model manages to capture the four key features of the PDE dynamics: permafrost thawing, R-tipping to the hot metastable state that lasts for half a century, followed by a sudden soil cooling to slightly above the air temperature.

4We use 4°C in this example because the global climate change mitigation target is an increase in the global mean temperature of 2°C by 2100 compared to pre-industrial levels (1850–1900) [46,47]. The 2°C target is within the range of uncertainty of CMIP5 outputs under two greenhouse gas concentration pathways: the ‘very stringent’ RCP2.6 that gives a 1.5°C increase, and the ‘intermediate’ RCP4.5 that gives a 2.4°C increase [21]. Due to the well documented phenomenon of ‘Arctic amplification’ [20,21], this target corresponds to an increase of roughly 4°C in the Arctic mean temperature over the same period.
Figure 3. (a) Seasonal variations of atmospheric temperature $T_a(rt)$ with a summer heatwave in year 10, based on observations at Cherskii in Northern Siberia [41]. (b) Time evolution of (black) moving equilibrium soil temperature $T_e(T_a(rt))$ from (2.7) and (red) the actual soil temperature $T(t)$ obtained by solving (2.3)–(2.4) with the input $T_a(rt)$ from (a). The model is initialized at $t = -10$ yr, and initial soil temperature is $T(−10) = −10\degree C$. Initial soil carbon $C(−10) = 120$ kg m$^{-2}$ corresponds to $\approx 3.64$ m soil depth assuming a volumetric carbon density of $33$ kg m$^{-3}$ [28,44]. $R_s(T)$ is the same as in figure 3. See the electronic supplementary material for other parameter values in table S1 and details of $T_a(rt)$ in section S1.3.

Figure 4. (a) A realization within the range of uncertainty of the ‘very stringent’ low-emissions RCP2.6 global warming scenario for Arctic regions [21,45], based on observations at Cherskii in Northern Siberia [41]. (b) Time evolution of (black) the moving equilibrium soil temperature $T_e(T_a(rt))$ from (2.7) and (red) the actual soil temperature $T(t)$ obtained by solving (2.3)–(2.4) with the input $T_a(rt)$ from (a). Initial soil temperature is $T(0) = T_a^− + (AΠ/λ) \approx −1.95\degree C$. Initial soil carbon $C(0) = 120$ kg m$^{-2}$ and $R_s(T)$ is the same as in figure 3. See the electronic supplementary material for other parameter values in table S1 and details of $T_a(rt)$ in section S1.4.

4. The multiscale autonomous compactified system

To analyse and understand non-obvious dynamical mechanisms that are responsible for the R-tipping instabilities in figures 3 and 4, we:

— Consider external inputs $T_a(rt)$ that tend exponentially to a constant at infinity.
— Reformulate the ensuing two-dimensional nonautonomous system (2.3)–(2.4) as a three-dimensional autonomous compactified system [37].
— Identify three different timescales in the soil-carbon system.

We choose to work with external inputs $T_a(rt)$ that decay exponentially to a constant $T_a^\pm$ as time $t$ tends to $\pm\infty$. To be precise, we follow Wieczorek et al. ([8], Def.6.1) and

**Definition 4.1.** We say $T_a(rt)$ is **exponentially bi-asymptotically constant** if, for all $r > 0$,

$$
\lim_{t \to \pm\infty} T_a(rt) = T_a^\pm \in \mathbb{R} \quad \text{and} \quad \lim_{t \to \pm\infty} \frac{dT_a(rt)/dt}{e^{\pm rt}} \in \mathbb{R},
$$

for some decay coefficient $\rho > 0$. 
Figure 5. (a) The output of a PDE model [48] parameterized by data from near Cherskii in Siberia to simulate a soil column with a depth of 12 m and a volumetric carbon density of 33 kg m\(^{-3}\), reproduced from Khvorostyanov et al. ([28], Fig. 4(a)). Shown are time evolution of (black) air temperature at 2 m above the ground, and (red) soil temperature at a depth of 5 m. (b) Time evolution of (red) the soil temperature \(T(t)\) obtained by solving (2.3)–(2.4) with the (black) input \(T_a(rt)\). For comparison with (a), we use \(\Pi = 0.09\text{ kg m}^{-1}\text{ yr}^{-1}\), \(Rs(T) = 0\) for \(T \leq 0\) as in [28], and \(Rs(T) = R^*_s(T)\) given in (2.6) with \(T^* = 30^\circ\text{C}\) for \(T > 0\). Initial soil temperature is \(T(0) = -6^\circ\text{C}\). Initial soil carbon is \(C(0) = 396\text{ kg m}^{-3}\), which corresponds to a soil column with a depth of 12 m and a volumetric carbon density of 33 kg m\(^{-3}\) matching [28]. See the electronic supplementary material for other parameter values in table S1 and details of \(T_a(rt)\) in section S1.5.

Thus, we can define the autonomous past limit system with \(T_a(rt) = T_a^-\),

\[
\epsilon \frac{dT}{dt} = f_1(T, C, T_a^-) \quad \text{and} \quad \frac{dC}{dt} = f_2(T, C),
\]

and the autonomous future limit system with \(T_a(rt) = T_a^+\),

\[
\epsilon \frac{dT}{dt} = f_1(T, C, T_a^+) \quad \text{and} \quad \frac{dC}{dt} = f_2(T, C),
\]

which are examples of the frozen system. We note that, unlike the nonautonomous system (2.3)–(2.4), the autonomous past (4.1) and future (4.2) limit systems contain the equilibria

\[
e^\pm := e(T_a^\pm),
\]

respectively, and \(e(T_a(rt)) \to e^\pm\) as \(t \to \pm \infty\) for any \(r > 0\).

In the next step, we include these limit systems and their equilibria in the model (see [8,37] and section S2 of the electronic supplementary material for full details). We introduce a bounded dependent variable

\[
s = g_s(rt) = \tanh\left(\frac{\nu}{2} rt\right) \in (-1, 1),
\]

and reformulate the two-dimensional nonautonomous system (2.3)–(2.4) on \(\mathbb{R}^2\) as a three-dimensional autonomous compactified system on the extended phase space \(\mathbb{R}^2 \times [-1, 1]\),

\[
\epsilon \frac{dT}{dt} = f_1(T, C, T_a^+(s)),
\]

\[
\frac{dC}{dt} = f_2(T, C),
\]

and

\[
\frac{1}{r} \frac{ds}{dt} = \frac{\nu}{2} (1 - s^2),
\]
with the continuously extended external input

\[
T_a^v(s) = \begin{cases} 
T_a(g_v^{-1}(s)) & \text{for } s \in (-1, 1), \\
T_a^+ & \text{for } s = 1, \\
T_a^- & \text{for } s = -1,
\end{cases}
\] (4.8)

and compactification parameter \( v \). A particular advantage of compactification is that the flow-invariant planes

\[
\mathbb{R}^2 \times \{-1\} \quad \text{and} \quad \mathbb{R}^2 \times \{1\},
\] (4.9)

of the extended phase space contain equilibria \( e^- \) and \( e^+ \) of the autonomous past (4.1) and future (4.2) limit systems, respectively. When embedded in the extended phase space, \( e^- \) gains one unstable eigendirection with positive eigenvalue \( v r > 0 \) and becomes a hyperbolic saddle,

\[
\tilde{e}^- = (e^-, -1),
\] (4.10)

whereas \( e^+ \) gains one additional stable eigendirection with negative eigenvalue \( -v r < 0 \) and becomes a hyperbolic sink, which is the only attractor for the compactified system (4.5)–(4.7),

\[
\tilde{e}^+ = (e^+, +1).
\] (4.11)

Furthermore, we note that the moving equilibrium \( e(T_a(rt)) \) with \( t \in \mathbb{R} \) corresponds to

\[
\tilde{e}(s) := \left( e\left(T_a\left(g_v^{-1}(s)\right)\right), s\right) \quad \text{with } s \in (-1, 1),
\]
in (4.5)–(4.7), and \( \tilde{e}(s) \to \tilde{e}^\pm \) as \( s \to \pm 1 \).

The left-hand side of the compactified system (4.5)–(4.7) shows that the soil-carbon system may evolve on up to three different timescales, depending on the rate parameter \( r \). The slow time \( t \) is the timescale of the slow variable \( C \). The fast time \( \tau = t/\epsilon \) is the timescale of the fast variable \( T \). The third time \( u = rt \) is the timescale of the external input \( T_a(rt) \), which is represented by the additional variable \( s \). The magnitude of the third time \( u \) relative to \( t \) and \( \tau \) depends on the magnitude of the rate parameter \( r \). Here, we consider different two-timescale limits of this three-timescale problem. A more unifying view of the system dynamics can be obtained through analysis of the three-timescale problem [49] (section S6 in the electronic supplementary material), which is beyond the scope of this paper.

5. Defining R-tipping due to crossing a quasi-threshold

The multiple-timescale soil-carbon system can be viewed as a singular perturbation problem [50]. In this section, we combine compactification with concepts and techniques of GSPT to:

- Define R-tipping due to crossing a quasi-threshold in the soil-carbon system (2.3)–(2.4).
- Give intuition for when to expect such R-tipping.

It is convenient to start the discussion with the autonomous frozen system obtained by setting \( r = 0 \) in system (2.3)–(2.4), so that \( T_a \) becomes a fixed-in-time input parameter. The frozen system is a 1-fast 1-slow singular perturbation problem with a small parameter \( \epsilon \). Taking the limit \( \epsilon \to 0 \) in the slow time \( t \) gives a singular reduced problem

\[
\frac{dC}{dt} = f_2(T, C),
\] (5.1)

with slow timescale solutions restricted to the one-dimensional critical manifold

\[
S(T_a) = \{(T, C) : f_1(T, C, T_a) = 0\} \subset \mathbb{R}^2.
\]

Taking the limit \( \epsilon \to 0 \) in the fast time \( \tau = t/\epsilon \) gives a regular layer problem

\[
\frac{dT}{d\tau} = f_1(T, C, T_a),
\] (5.2)
with fast timescale solutions for a fixed-in-time $C$. Note that, for a given $T_a$, the critical manifold $S(T_a)$ consists of all branches of equilibria of the layer problem (5.2) parameterized by $C$. Hence, stability analysis of these equilibria gives stability of different parts of $S(T_a)$. Specifically, $S(T_a)$ has two normally hyperbolic attracting branches, denoted $S_1(T_a)$ and $S_2(T_a)$, which are separated from a normally hyperbolic repelling branch $S_3(T_a)$ by two non-hyperbolic fold points, denoted $F_1(T_a)$ and $F_2(T_a)$. The attracting branch $S_1(T_a)$ contains the base state $e(T_a)$. The other attracting branch, $S_3(T_a)$, is the hot metastable state that arises from the non-monotone microbial respiration function (2.6). The aim of GSPT is to combine the slow and fast timescale solutions for $e = 0$, shown in figure 6a, into slow–fast composite solutions for $0 < \epsilon \ll 1$, shown in figure 6b. We point out one particular combination: the (blue) special candidate trajectory $\theta$.

For $0 < \epsilon \ll 1$, the normally hyperbolic parts of $S(T_a)$ are guaranteed by the ‘first’ Fenichel theorem to perturb smoothly to nearby normally hyperbolic and locally invariant slow manifolds [38,50–53]. To be specific, $S_1(T_a)$ and $S_3(T_a)$ perturb to nearby attracting slow manifolds $S_{1,\epsilon}(T_a)$ and $S_{3,\epsilon}(T_a)$. Similarly, $S_2(T_a)$ perturbs to a nearby repelling slow manifold $S_{2,\epsilon}(T_a)$. Each slow manifold is usually non-unique in the sense that there is a family of such slow manifolds that lie exponentially close in $e$ to each other (see [38], Th.3.1 and [50], Th.3.1.4). The strategy is to fix a representative for each of these manifolds and work with these representatives. Near the non-hyperbolic fold points $F_1(T_a)$ and $F_2(T_a)$, the hyperbolic branches $S_{1,\epsilon}(T_a)$, $S_{3,\epsilon}(T_a)$ and $S_{2,\epsilon}(T_a)$ typically split, meaning that they typically become slow manifolds with either one or two boundaries. In particular, $S_{1,\epsilon}(T_a)$ has one inflow boundary near $F_1(T_a)$, $S_{3,\epsilon}(T_a)$ has one outflow boundary near $F_2(T_a)$, while $S_{2,\epsilon}(T_a)$ has one inflow boundary near $F_1(T_a)$ and one outflow boundary near $F_2(T_a)$. Local invariance means that trajectories can enter or leave a slow manifold only through its boundary. Furthermore, the splitting of $S_{1,\epsilon}(T_a)$ and $S_{2,\epsilon}(T_a)$ near $F_1(T_a)$ gives rise to a narrow continuum of special solutions, referred to as canards, that move along $S_{2,\epsilon}(T_a)$ for some time $t$ [38,54–56](figure 6b). In the remainder of the paper, we follow Wechselberger [57], Benoît et al. [58] and Wechselberger et al. ([38], Sec.3.2.4) and identify three different types of canards.

**Definition 5.1.** In the autonomous frozen system and in the autonomous compactified system (4.5)–(4.7):

---

Note that this stable branch does not exist for the monotone respiration function in [29].
(i) Canards ‘without head’ are solutions, or the corresponding trajectories, that move slowly along $S_{2,e}$ for $t = O(1)$ before moving quickly and directly to $S_{1,e}$.

(ii) Canards ‘with head’ are solutions, or the corresponding trajectories, that move slowly along $S_{2,e}$ for $t = O(1)$, then move quickly and directly to $S_{3,e}$, then move slowly along $S_{3,e}$, before converging to $S_{1,e}$.

(iii) A maximal canard is a special solution, or the corresponding trajectory, that enters $S_{2,e}$ through its inflow boundary.

Examples of a maximal canard, denoted $\theta_\epsilon$, a canard ‘without head’ and a canard ‘with head’ in the frozen system are shown in blue, cyan and magenta, respectively, in figure 6b. The (blue) maximal canard $\theta_\epsilon$ is a special example of a canard without head, and a perturbation of the special candidate trajectory $\theta$. In practice, a maximal canard is computed by choosing a suitable arclength and finding the trajectory along $S_2(T_\theta)$ that takes the longest time to trace out this arclength.

The above discussion identifies two obstacles to defining R-tipping to the hot metastable state. First, the hot metastable state is a transient and thus quantitative phenomenon. In the long term, the system converges to the same stable state $e^+$ for any rate $r > 0$. Second, there is no unique threshold for R-tipping to the hot metastable state [8]. Rather, one speaks of a ‘quasi-threshold’ comprising a family of canards. GSPT together with compactification allow us to overcome both of these obstacles.

We focus on R-tipping from $e^-$ and relate nonautonomous and compactified system dynamics. Using the notation $x = (T,C) \in \mathbb{R}^2$ for the state variable of the soil-carbon system, we write

$$x^{[r]}(t, e^-) \in \mathbb{R}^2,$$

(5.3)

to denote the unique solution to the nonautonomous system (2.3)–(2.4) at time $t$, that limits to $e^-$ as $t \to -\infty$ for a given rate parameter $r$. We also write $W_0^{u,[r]}(e^-)$ to denote the unique rate-dependent one-dimensional unstable invariant manifold of the hyperbolic saddle $\tilde{e}^-$ in the autonomous compactified system (4.5)–(4.7). It follows from Wieczorek et al. ([8], Prop.6.4(a)) that $W_0^{u,[r]}(\tilde{e}^-)$ contains $x^{[r]}(t, e^-)$ in the sense that

$$W_0^{u,[r]}(\tilde{e}^-) \supset \{ (x,s) : x = x^{[r]}(t, e^-), s = g_\rho(t) \}_{t \in \mathbb{R}}.$$

(5.4)

This relation allows us to define tracking and R-tipping for the nonautonomous system (2.3)–(2.4) in terms of $W_0^{u,[r]}(\tilde{e}^-)$ and slow manifolds in the autonomous compactified system (4.5)–(4.7):

**Definition 5.2.** Consider the nonautonomous system (2.3)–(2.4) with exponentially bi-asymptotically constant $T_\rho(rt)$ and decay coefficient $\rho > 0$, and the corresponding autonomous compactified system (4.5)–(4.7) with the compactification parameter $\nu \in (0, \rho]$. For a fixed $r > 0$:

(i) We say $x^{[r]}(t, e^-)$ **tracks** the moving stable equilibrium $e(T_\rho(rt))$ in system (2.3)–(2.4) if $W_0^{u,[r]}(\tilde{e}^-)$ connects to $\tilde{e}^+$ directly, i.e. without visiting $S_{2,e}$ or $S_{3,e}$, in system (4.5)–(4.7).

(ii) We say $x^{[r]}(t, e^-)$ **R-tips** in system (2.3)–(2.4), or say system (2.3)–(2.4) **R-tips** from $e^-$, if $W_0^{u,[r]}(\tilde{e}^-)$ visits $S_{3,e}$ before connecting to $\tilde{e}^+$ in system (4.5)–(4.7).

Since R-tipping is not the complement of tracking, the nonautonomous system (2.3)–(2.4) with $0 < \epsilon \ll 1$ does not have unique R-tipping thresholds separating trajectories that track the moving stable equilibrium from those that R-tip, and does not have isolated critical rates $r$. More precisely:

**Definition 5.3.** Consider the nonautonomous system (2.3)–(2.4) with exponentially bi-asymptotically constant $T_\rho(rt)$ and decay coefficient $\rho > 0$, and the corresponding autonomous compactified system (4.5)–(4.7) with the compactification parameter $\nu \in (0, \rho]$.

---

6 This solution can be understood as a local pullback attractor for (2.3)–(2.4) ([43], Th.2.2).

7 For convenience, we leave out the dependence on the compactification parameter $\nu$ in the notation for the unstable manifold.

8 In the sense that canards ‘without head’ visit $S_{2,e}$ but not $S_{3,e}$ and thus correspond to neither tracking nor R-tipping.
Figure 7. Phase portraits of the frozen system with $\epsilon = 0$ for three different values of $T_a$ show when to expect R-tipping in the nonautonomous system (2.3)–(2.4) with $0 < \epsilon \ll 1$ and a time-varying $T_a$. Note the change in the position of the (blue) special candidate trajectory $\theta(T_a)$ for different values of $T_a$ relative to the fixed initial base state $e(0)$ at $T_a = 0$.

(i) We define a critical range of $r$ as an interval of $r$ for which $x[r](t,e^{-})$ neither tracks the moving stable equilibrium $e(T_a(rt))$ nor R-tips.

(ii) For a fixed $r > 0$, we define an R-tipping quasi-threshold in system (4.5)–(4.7) as a family of canards ‘without head’ including a maximal canard, and in system (2.3)–(2.4) as a family of solutions $x[r](t)$ or trajectories corresponding to this family of canards.

Remark 5.4. There are important differences between R-tipping quasi-thresholds defined above and regular R-tipping thresholds introduced in Wieczorek et al. ([8], Def.5.3). We refer to section S3 of the electronic supplementary material for more details.

To give intuition for when to expect R-tipping due to crossing a quasi-threshold in the nonautonomous system (2.3)–(2.4), we extend the concept of threshold instability, introduced in Wieczorek et al. ([8], Def.4.5) for regular thresholds, to quasi-thresholds. Consider phase portraits of the frozen system in the limit $\epsilon = 0$ for three different values of $T_a$ in figure 7. Suppose that $T_a = 0^\circ C$ and the system is settled at the base state $e(0)$ (figure 7a). Then, there is a special value of $T_a$, derived in section S4 of the electronic supplementary material,

$$T_a = T_a^{\text{inst}} \approx -\frac{1}{\alpha} \left( 1 + \log \left( \frac{A\Pi}{\lambda} \right) - \alpha \frac{A\Pi}{\lambda} \right),$$

such that the base state for $T_a = 0$ crosses the special candidate trajectory for $T_a = T_a^{\text{inst}} > 0$, that is $e(0) \in \theta(T_a^{\text{inst}})$ (figure 7b). If $T_a$ switches discontinuously from 0 to $T_a^{\text{max}}$, then $e(0)$ becomes the initial condition for the frozen system with $T_a = T_a^{\text{max}}$. Thus, following a switch to $T_a^{\text{max}} > T_a^{\text{inst}}$, $e(0)$ will find itself on the other side of the special candidate trajectory $\theta(T_a^{\text{max}})$ and the system will undergo R-tipping (figure 7c). In §§6 and 7, we relate condition (5.5) to R-tipping conditions for $0 < \epsilon \ll 1$ and continuously varying $T_a(rt)$.

6. R-tipping mechanisms for global warming

The R-tipping instability in figure 4 arises because time-variation of the mean atmospheric temperature $T_a$ interacts with the slow timescale of soil carbon $C$. Thus, to uncover the underlying dynamical mechanisms, we consider the 1-fast 2-slow system (4.5)–(4.7) with the rate parameter $r \lesssim 1$, where $u = rt \lesssim t$ becomes another slow time and $s$ becomes another slow variable.

The singular reduced problem, obtained by setting $\epsilon = 0$ for the slow time $t$ in (4.5)–(4.7),

$$\frac{dC}{dt} = f_2(T,C)$$

and

$$\frac{ds}{dt} = vr \frac{(1 - s^2)}{2},$$

where $r$ and $v$ are parameters.
Figure 8. (a) R-tipping diagram for nonautonomous system (2.3)–(2.4) with global warming scenario (6.5), in the plane of the shift amplitude \(T_a\) and the rate parameter \(r\), for \(\epsilon \approx 0.064\). Shown are regions of (green) tracking, (cyan, blue) critical range, and (magenta, red) R-tipping from \(\hat{e}^-\). (b) (Colour) The unstable invariant manifold \(W_u(r)(\hat{e}^-)\) of the saddle \(\hat{e}^-\) for the compactified system (4.5)–(4.7) with \(\epsilon \approx 0.064\) and global warming scenario (6.6) with \(T_a = 5^\circ\text{C}\), for three different values of the rate parameter: (green) \(r_1 = 0.02\), (blue) \(r_2 \approx 0.0454218\) and (red) \(r_3 = 0.1\); see the black dots in (a). Included for reference is (grey) the critical manifold \(S\) defined in (6.3). See table S1 in the electronic supplementary material for other parameter values.

\[
S = \{(T, C, s) : f_1(T_\nu^a(s)) = 0\} \subset \mathbb{R}^2 \times \{−1, 1\}. \tag{6.3}
\]

S consists of two normally hyperbolic attracting sheets, \(S_1\) containing \(\hat{e}(s)\), and \(S_3\) being the hot metastable state, which are separated from a normally hyperbolic repelling sheet \(S_2\) by two non-hyperbolic fold curves \(F_1\) and \(F_2\) (figure 8b). The regular layer problem, obtained by setting \(\epsilon = 0\) for the fast time \(\tau = t/\epsilon\),

\[
\frac{dT}{d\tau} = f_1(T, C, T_\nu^a(s)), \tag{6.4}
\]

gives fast timescale solutions along straight lines for fixed-in-time \(C\) and \(s\).

As a model of the global warming scenario, we consider a slow nonlinear shift from \(T_a = 0^\circ\text{C}\) to a given \(T_a > 0\), that decays exponentially with the decay coefficient \(\rho = 2\) as per definition 4.1,

\[
T_a(\tau t) = \frac{T_a^+}{2} \left(\tanh(\rho \tau t) + 1\right). \tag{6.5}
\]

We then fix the compactification parameter \(\nu = 1\) and apply the inverse of the compactification transformation (4.4)–(6.5) to obtain \(T_\nu^a(s) = T_\nu^a(s) = \frac{T_a^+}{2} \left(1 + s\right)^2 \left(1 + s^2\right)\).

\[
T_\nu^a(s) = T_a^\nu(s) = \frac{T_a^+}{2} \left(1 + s\right)^2 \left(1 + s^2\right). \tag{6.6}
\]

To give an overview of the dynamics near transitions from tracking to R-tipping, we plot in figure 8a an R-tipping diagram in the plane \((T_a^+, r)\) of the input parameters. The diagram was obtained by computing \(W_u(r)(\hat{e}^-)\) in system (4.5)–(4.7) with \(0 < \epsilon < 1\) for different values of \(T_a^+\) and \(r\), and using definitions 5.1–5.3 to identify different dynamical regions for system (2.3)–(2.4). There are two R-tipping regions, and one tracking–tipping transition found for a large enough shift magnitude \(T_a^+\). The smaller R-tipping region is a curious R-tipping tongue. This tongue gives rise to two (cyan) critical ranges of \(r\) for a fixed \(T_a^+\) (one of them being very narrow), and different tracking–tipping transitions for low and high magnitudes \(T_a^+\) of the shift.

To gain geometric insight into the R-tipping instability caused by global warming, we plot in figure 8b the unstable manifold \(W_u(r)(\hat{e}^-)\) for a fixed \(T_a^+ = 5^\circ\text{C}\) and three different slow rates \(0 < r_1 < r_2 < r_3\) (see the black dots in figure 8a), together with the critical manifold \(S\) for reference.

\[\text{The inverse of (4.4) is given by equation (7) in section S2 of the electronic supplementary material.}\]
Figure 8b shows that, as \( r \) is increased, tracking of \( \tilde{e}(s) \) by (green) \( W^u[r_1](\tilde{e}^-) \) is lost via canard trajectories, including the maximal canard contained in (blue) \( W^u[r_1](\tilde{e}^-) \) that crosses \( F_1 \) and moves along \( S_2 \) for the longest time. This is followed by R-tipping at higher \( r \) as shown by (red) \( W^u[r_1](\tilde{e}^-) \) that crosses \( F_1 \) and approaches the hot metastable state \( S_{3,e} \) along the fast \( T \)-direction before connecting to \( \tilde{e}^+ \). Since R-tipping due to global warming (6.5) occurs when the slow timescale segment of \( W^u[r_1](\tilde{e}^-) \) on \( S \) crosses \( F_1 \), it should be possible to explain the R-tipping diagram in figure 8a, including the curious R-tipping tongue, in terms of slow timescale solutions of the two-dimensional reduced problem alone [12,14,36].

(a) The reduced problem and desingularization

The reduced problem (6.1)–(6.2) for the 1-fast 2-slow system (4.5)–(4.7) with global warming (6.6) describes the evolution of the slow variables \( s \) and \( t \) on the two-dimensional critical manifold \( S \) defined in (6.3). However, the onset of R-tipping involves a sudden jump in the fast variable \( T \), without any noticeable change in \( C \). Thus, it is convenient to consider the evolution of the fast variable \( T \) in the slow time \( t \) on \( S \), which is obtained by differentiating the critical-manifold condition in (6.3) with respect to \( t \). Furthermore, we use the critical-manifold condition to project the slow flow within \( S \) onto the plane \((T,s)\), i.e. eliminate the dependence on \( C \), and reformulate the reduced problem as

\[
\frac{dT}{dt} = -\frac{\partial f_1}{\partial C} \cdot f_2 + \frac{\partial f_1}{\partial T^i} \cdot \frac{dT^i}{ds} \cdot \frac{ds}{dt} \bigg|_S \tag{6.7}
\]

and

\[
\frac{ds}{dt} = \frac{r}{2}(1-s^2), \tag{6.8}
\]

where \( |_S \) denotes restriction to \( S \). In the reduced problem (6.7)–(6.8), the question of loss of tracking boils down to whether \( W^u[r_1](\tilde{e}^-) \) leaves \( S_1 \) through \( F_1 \). To address this question, we note that the denominator in (6.7) changes sign at a fold

\[
\frac{\partial f_1}{\partial T} \bigg|_S = \begin{cases} 
< 0 & \text{for } (T,C,s) \in S_1 \cup S_3, \\
0 & \text{for } (T,C,s) \in F_1 \cup F_2, \\
> 0 & \text{for } (T,C,s) \in S_2.
\end{cases} \tag{6.9}
\]

It then follows that, depending on the numerator in (6.7), there are two types of fold points:

- A jump point [39] is a point \( p \) on a fold such that

\[
\left( \frac{\partial f_1}{\partial C} \cdot f_2 + \frac{\partial f_1}{\partial T^i} \cdot \frac{dT^i}{ds} \cdot \frac{ds}{dt} \right)_{p \in F_1 \cup F_2} \neq 0. \tag{6.10}
\]

If a solution of (6.7)–(6.8) approaches a jump point, the denominator in (6.7) approaches zero while the numerator remains finite, so that \( dT/dt \) tends to infinity, and \( T(t) \) blows up in \( t \). This means that the solution reaches a jump point in finite time and ceases to exist within \( S \). Jump points are typically found on open subsets of a fold [39].

- A folded singularity [39] is a point \( q \) on a fold such that

\[
\left( \frac{\partial f_1}{\partial C} \cdot f_2 + \frac{\partial f_1}{\partial T^i} \cdot \frac{dT^i}{ds} \cdot \frac{ds}{dt} \right)_{q \in F_1 \cup F_2} = 0. \tag{6.11}
\]

If a solution of (6.7)–(6.8) approaches a folded singularity in a special direction where the numerator in (6.7) approaches zero faster than or at the same rate as the denominator in (6.7), then \( dT/dt \) remains finite, and this solution either grazes or crosses the fold at \( q \) with a finite speed. If the crossing is from an attracting to an unstable part of \( S \), the solution, or the corresponding trajectory, is termed a singular canard [54,58]. Folded singularities, which are typically found as isolated fold points, are examples of ‘singular R-tipping edge states’ described in the introduction.
Thus, in the singular limit \( \epsilon = 0 \) for the slow time \( t \), transitions from tracking to R-tipping caused by global warming can be understood in terms of singular canards through folded singularities [12,13,36]. The obstacle to analysis of folded singularities and their singular canards is that the right-hand side of (6.7) is undefined on \( F_1 \) and \( F_2 \). This obstacle is overcome by a desingularization [39,59] in the form of a state-dependent time rescaling:

\[
dt = -dt \left. \frac{\partial f_I}{\partial T} \right|_S ,
\]

which gives the desingularized system

\[
\frac{dT}{dt} = R_s^*(T) \left( T - \frac{\lambda}{A} (T - T^1(s)) \right) + \frac{r\lambda T^2_s + (1 - s^2)^2}{2A} \left( 1 + \frac{1}{s^2} \right)
\]

and

\[
\frac{ds}{dt} = \frac{r\lambda}{2A} (1 - s^2) \left( 1 + \alpha(T - T^1_s(s)) c^{e^{\alpha(T-b)} - e^{-\alpha(T-b)}} \right),
\]

defined everywhere on \( S \). The main advantages of desingularization are:

(a1) Regular equilibria for the reduced problem (6.7)–(6.8) remain regular equilibria for the desingularized system (6.13)–(6.14). Folded singularities for (6.7)–(6.8) become (new) regular equilibria for (6.13)–(6.14). Hence the classification of folded singularities into ‘folded nodes’, ‘folded foci’, ‘folded saddles’ and ‘folded saddle-nodes’, based on their classification into different types of equilibria in (6.13)–(6.14) [39].

(a2) According to (6.9) and (6.12), the new time \( t \) in (6.13)–(6.14) flows in the same direction as \( t \) on \( S_1 \) and \( S_2 \), passes infinitely faster on \( F_1 \) and \( F_2 \), and reverses direction on \( S_2 \). Thus, a phase portrait for (6.7)–(6.8) can be obtained by producing the corresponding phase portrait for (6.13)–(6.14), reversing the direction of time (the arrows on trajectories) on \( S_2 \), and relabelling the (new) equilibria on \( F_1 \) and \( F_2 \) as folded singularities.

(a3) It follows from points (a1) and (a2) above that a singular canard through a folded singularity in (6.7)–(6.8) can be obtained by smoothly concatenating two trajectories tangent to a stable eigendirection of the corresponding equilibrium in (6.13)–(6.14). We refer to section S5 of the electronic supplementary material for the discussion of different singular canards associated with different folded singularities.

(b) Three slow cases in the reduced problem

To be precise and consistent with definitions 5.2 and 5.3 for \( 0 < \epsilon \ll 1 \), we now define tracking, R-tipping and critical rates for slow external inputs in the limit \( \epsilon = 0 \) as follows:

**Definition 6.1.** In the reduced problem (6.7)–(6.8), we say that:

(i) Tracking occurs when \( W_{\epsilon \rightarrow 0}^u([\hat{e}^-]) \) connects to \( \hat{e}^+ \) directly, that is without visiting \( F_1 \).

(ii) R-tipping occurs when \( W_{\epsilon \rightarrow 0}^u([\hat{e}^-]) \) reaches a jump point on \( F_1 \) and stops existing in \( S \).

(iii) A critical rate is an isolated value of \( r \) that gives neither tracking nor R-tipping.

Then, we use relations (a1)–(a3) between the desingularized and reduced system dynamics to identify different tracking–tiping transitions in the singular reduced problem (6.7)–(6.8) through analysis of the unstable manifold \( W_{\epsilon \rightarrow 0}^u([\hat{e}^-]) \) in the regular desingularized system (6.13)–(6.14). We note that, when \( r \) is sufficiently small, tracking occurs because the whole of \( F_1 \) is repelling, so \( W_{\epsilon \rightarrow 0}^u([\hat{e}^-]) \) must remain on \( S_1 \) and connect directly to \( \hat{e}^+ \). As the rate parameter \( r \) is increased, there is a saddle-node bifurcation of equilibria on \( F_1 \) at some \( r = r_{SN} \) (in (6.13)–(6.14)). This bifurcation corresponds to the appearance of a folded-saddle-node type-I (FSN-I) singularity [60,61] in (6.7)–(6.8). As \( r \) is increased past \( r_{SN} \), FSN-I bifurcates into a FS and a folded node (FN). Thus, \( W_{\epsilon \rightarrow 0}^u([\hat{e}^-]) \)

---

10For \( 0 < \epsilon \ll 1 \), the corresponding slow–fast composite solution leaves the attracting slow manifold \( S_{1,\epsilon} \) via its outflow boundary and approaches \( S_{3,\epsilon} \) along the fast \( T \)-direction, which is consistent with definition 5.2 of R-tipping.
may interact with singular canards of FS and FN to cross $F_1$ and cause loss of tracking. Analysis of heteroclinic orbits in (6.13)–(6.14), where $W_\text{t}^u[\epsilon^{-}]$ connects $\tilde{e}^-$ to an equilibrium on $F_1$, reveals three different cases of loss of tracking in (6.7)–(6.8):

(i) The simple slow case: $W_\text{t}^u[\epsilon^{-}]$ coalesces with the folded-saddle (FS) singular canard $\tilde{\gamma}^S$, and thus crosses from $S_1$ to $S_2$ via FS, at some critical rate $r = r_c > r_{SN}$.

(ii) The complicated slow case: $W_\text{u}^u[\epsilon^{-}]$ grazes $F_1$ at FSN-I when $r = r_{SN}$, coalesces with a FN weak singular canard when $r \in (r_{SN}, r_{NS})$, coalesces with the FN strong singular canard $\tilde{\gamma}^N$ at some $r = r_{NS} > r_{SN}$, and thus crosses from $S_1$ to $S_2$ via FN when $r \in (r_{SN}, r_{NS})$.

(iii) The degenerate slow case: $W_\text{u}^u[\epsilon^{-}]$ coalesces with the folded-saddle-node singular canard $\tilde{\gamma}^SN$, and thus crosses from $S_1$ to $S_2$ via FSN-I, at a critical rate $r_c = r_{SN}$.

These three cases of ‘loss of tracking’ give rise to three different cases of ‘tracking–tipping transition’, which we refer to as the three slow cases. Cases (i) and (ii) are typical and closely related to the two cases of non-obvious tipping thresholds described in separate systems in [36]. The new case (iii) is special and separates the typical cases (i) and (ii). Below, we identify all related to the two cases of non-obvious tipping thresholds described in separate systems in [36].

The problem at hand, consolidates the results of Perryman & Wieczorek [36], and is of interest in its three cases in an unfolding of a.

The new case (iii) is special and separates the typical cases (i) and (ii). Below, we identify all related to the two cases of non-obvious tipping thresholds described in separate systems in [36].

To highlight interactions of $W_\text{u}^u[\epsilon^{-}]$ with different singular canards, we colour $S_1$ in the phase portraits as follows. Solutions initialized in the green region remain in this region and converge directly to $\tilde{e}^+$ (tracking). Solutions initialized in the dark red region reach a jump point of $F_1$ and cease to exist within $S$ (R-tipping). Remaining solutions (neither tracking nor R-tipping) include: weak FN singular canards initialized in the yellow region (singular funnel), the FN strong singular canard $\tilde{\gamma}^N$ separating the red and yellow regions, and the FS singular canard $\tilde{\gamma}^S$ separating the green and red regions. Finally, in the figures, we show projections of $S$ onto the plane $(T, s)$.

(i) The simple slow case

As $r$ is increased past $r_{SN}$ for $T_{\text{d}}^+ = 1.5$ in (6.7)–(6.8), the appearance of FS and FN via FSN-I gives rise to an FS singular canard $\tilde{\gamma}^S$, FN singular canards including the FN strong singular canard $\tilde{\gamma}^N$, and new dynamics (figure 9a,b). Of particular interest is an attracting interval of jump points on $F_1$ between FS and FN, and the corresponding (dark red) region of solutions between $\tilde{\gamma}^S$ and $\tilde{\gamma}^N$ that reach one of these jump points from $S_1$ and cease to exist within $S$. In this case, $W_\text{u}^u[\epsilon^{-}]$ does not interact with FSN-I, meaning that the FSN-I is local. $W_\text{u}^u[\epsilon^{-}]$ does not interact with the FN singular canards either because it is separated from them by $\tilde{\gamma}^S$. Rather, when $r = r_c$, $W_\text{u}^u[\epsilon^{-}]$ and $\tilde{\gamma}^S$ coalesce (figure 9c). At this point, tracking is lost since $W_\text{u}^u[\epsilon^{-}]$ crosses $F_1$ from $S_1$ to $S_2$ via FS. For $r > r_c$, $W_\text{u}^u[\epsilon^{-}]$ reaches a jump point on $F_1$ and ceases to exist within $S$ so R-tipping occurs (figure 9d).

Thus, in the simple slow case for $\epsilon = 0$, there is a critical rate $r = r_c$. This critical rate is the value of $r$ that gives a codimension-one heteroclinic orbit connecting $\tilde{e}^-$ to the corresponding saddle on $F_1$ in the desingularized system (6.13)–(6.14).

(ii) The complicated slow case

As $r$ is increased past $r_{SN}$ for $T_{\text{d}}^+ = 3.5$, the local bifurcation scenario is the same as in the simple case: appearance of FS and FN via FSN-I gives rise to FS and FN singular canards, an attracting interval of jump points on $F_1$ between FS and FN, and the corresponding (dark red) region of solutions between $\tilde{\gamma}^S$ and $\tilde{\gamma}^N$ that reach one of these jump points from $S_1$ and cease to exist within $S$ (figure 10). However, the global dynamics are different in that $W_\text{u}^u[\epsilon^{-}]$ interacts with various FN singular canards. When $r = r_{SN}$, $W_\text{u}^u[\epsilon^{-}]$ grazes $F_1$ at FSN-I, giving rise to a codimension-one central heteroclinic FSN-I (figure 10b). In other words, the FSN-I is global. At this point tracking is lost, but R-tipping cannot occur for $r$ just above $r_{SN}$ because $W_\text{u}^u[\epsilon^{-}]$ passes through the yellow singular funnel and crosses $F_1$ from $S_1$ to $S_2$ via FN (figure 10c). Thus, $W_\text{u}^u[\epsilon^{-}]$ contains a FN
Figure 9. The simple slow case of tracking–tipping transition in phase portraits of the reduced problem (6.7)–(6.8) with nonlinear shift (6.6), obtained using the desingularized system (6.13). Note tracking in spite of a codimension-one local FSN-I in (a), and loss of tracking via a codimension-one heteroclinic orbit connecting $\tilde{e}^-$ to FS in (c) that leads to R-tipping in (d). $T^+_a = 1.5$ and the rate parameter takes values (a) $r = r_{SN} \approx 0.107194$, (b) $r \approx 0.108119$, (c) $r = r_c \approx 0.111459$ and (d) $r = 0.12$; see the black dots in the left inset of figure 12c. See table S1 in the electronic supplementary material for other parameter values.

Thus, in the complicated slow case for $\epsilon = 0$, there is a critical range of $r \in [r_{SN}, r_{NS}]$. The minimum of the critical range is the value of $r$ that gives a codimension-one heteroclinic orbit connecting $\tilde{e}^-$ to the corresponding saddle-node on $F_1$ along the centre eigendirection in the desingularized system (6.13)–(6.14). The maximum of the critical range is the value of $r$ that gives a codimension-one heteroclinic orbit connecting $\tilde{e}^-$ to the corresponding stable node on $F_1$ along the strong eigendirection in (6.13)–(6.14). For $r$ in the interior of the critical range, there is a codimension-zero heteroclinic orbit connecting $\tilde{e}^-$ to the corresponding stable node on $F_1$ along the weak eigendirection in (6.13)–(6.14).

(iii) The degenerate slow case

A natural question to ask is: what separates the simple and complicated slow cases? It turns out that there is a special value $T^+_a = T_{a,c}$. As $r$ is increased past $r_{SN}$ for this special value of $T^+_a$, the local bifurcation scenario is the same as in the simple and complicated slow cases. However, the global dynamics are different in that $W^{u,[r]}(\tilde{e}^-)$ interacts with the folded-saddle-node singular canard $\tilde{\gamma}^{SN}$. By contrast to the simple and complicated slow cases, if $r = r_{SN}$ and $T^+_a = T_{a,c}$, $W^{u,[r]}(\tilde{e}^-)$ and $\tilde{\gamma}^{SN}$ coalesce, giving rise to a codimension-two non-central heteroclinic
Figure 10. The complicated slow case of tracking–tipping transition in phase portraits of the reduced problem (6.7)–(6.8) with nonlinear shift (6.6), obtained using the desingularized system (6.13)–(6.14). Note tracking in (a), loss of tracking via a codimension-one central heteroclinic FSN-I in (b), and a codimension-one heteroclinic orbit connecting $\tilde{e}^-$ to FN in (d) that leads to R-tipping in (e). $T_a + a = 3.5$ and the rate parameter takes values (a) $r = 0.045$, (b) $r = r_{SN} \approx 0.050086$, (c) $r \approx 0.050669$, (d) $r = r_c \approx 0.052266$ and (e) $r = 0.06$; see the black dots in the right inset of figure 12c. See table S1 in the electronic supplementary material for other parameter values.

FSN-I (figure 11b). In other words, the FSN-I is global, but different from the complicated slow case. At this point, tracking is lost since $W^{u,r}(\tilde{e}^-)$ crosses $F_1$ from $S_1$ to $S_2$ via FSN-I. For $r > r_{SN}$, $W^{u,r}(\tilde{e}^-)$ reaches a jump point on $F_1$ and R-tipping occurs (figure 11c).

Thus, in the degenerate case for $\epsilon = 0$, there is neither a critical range nor a critical rate. Instead, there is a critical pair $(T_a + a, r_c)$. This critical pair is the combination of $r$ and $T_a$ that give a codimension-two heteroclinic orbit connecting $\tilde{e}^-$ to the corresponding saddle-node on $F_1$ along the stable eigendirection in the desingularized system (6.13)–(6.14).11

(c) Bringing it together: unfolding of non-central heteroclinic FSN-I

The aim of this section is twofold. First, we summarize our results from §6(b) in the singular R-tipping diagram for $\epsilon = 0$ in figure 12c. Second, we use the singular R-tipping diagram to explain the regular R-tipping diagram for $\epsilon \approx 0.064$ in figure 8a.

The singular R-tipping diagram is obtained by the unfolding of a codimension-two non-central heteroclinic FSN-I in the plane $(T_a + a, r)$ of the input parameters in the reduced problem (6.7)–(6.8). This unfolding, in turn, is obtained by the unfolding of a codimension-two non-central saddle-node heteroclinic bifurcation in the desingularized system (6.13)–(6.14).12 The diagram is partitioned into regions of (green) tracking, (red) R-tipping, and (yellow) neither tracking nor R-tipping by two curves that are tangent at the special point of codimension-two non-central heteroclinic

11Note the difference from typical codimension-one heteroclinic orbits connecting $\tilde{e}^-$ to a saddle-node along the centre eigendirection as in the complicated slow case.

12This unfolding is reminiscent of the unfolding of a codimension-two non-central saddle-node homoclinic bifurcation in [62].
Figure 11. The **degenerate slow case** of tracking–tipping transition in phase portraits of the reduced problem (6.7)–(6.8) with nonlinear shift (6.6), obtained using the desingularized system (6.13)–(6.14). Note tracking in (a), and loss of tracking via a codimension-two non-central heteroclinic FSN-I in (b) that leads to R-tipping in (c). $T_a^+ = 2.15938$ and the rate parameter takes values (a) $r \approx 0.0766488$, (b) $r = r_{SN} = r_c \approx 0.0766488$ and (c) $r \approx 0.0776488$; see the black dots in the middle inset of figure 12. System parameter values are given in table S1 in the electronic supplementary material.

Figure 12. Singular R-tipping diagram for the reduced problem (6.7)–(6.8) with global warming scenario (6.6), obtained using the desingularized system (6.13)–(6.14). (Green) region of tracking, (red) region of R-tipping and (yellow) critical range of neither tracking nor R-tipping are separated by: (blue) $\tilde{\epsilon}^{-}\text{-to-FS}$, $\tilde{\epsilon}^{-}\text{-to-FSN-I}_s$ and $\tilde{\epsilon}^{-}\text{-to-FN}_s$, along which $W^{u,r}(\tilde{\epsilon}^-)$ contains the singular canards $\tilde{\gamma}^S$, $\tilde{\gamma}^S_{FS}$ and $\tilde{\gamma}^N$, respectively, and (black) $\tilde{\epsilon}^{-}\text{-to-FSN-I}_c$, along which $W^{u,r}(\tilde{\epsilon}^-)$ contains the weak folded saddle-node singular canard. See table S1 in the electronic supplementary material for other parameter values.

FSN-I, denoted $\tilde{\epsilon}$-to-FSN-I$_c$ in the middle inset. Each curve has two branches emanating from the special point. Both branches of the black curve of FSN-I were obtained by computing the saddle-node bifurcation of equilibria on $F_1$ in (6.13)–(6.14); we will return to this curve later. The left branch of the blue curve, denoted $\tilde{\epsilon}$-to-FS, was obtained by computing a codimension-one heteroclinic orbit connecting $\tilde{\epsilon}^{-}$ to the saddle on $F_1$ in (6.13)–(6.14). This connection gives a critical rate in the simple slow case for the reduced problem (6.7)–(6.8), where $W^{u,r}(\tilde{\epsilon}^-) \supset \tilde{\gamma}^S$ (figure 9c). Note that $\tilde{\epsilon}$-to-FS has a vertical asymptote $T_a^+ = T_{a}^{\text{inst}} \approx 0.423364$, which is given by condition (5.5) illustrated in figure 7. The right branch of the blue curve, denoted $\tilde{\epsilon}$-to-FN$_s$, was obtained by computing a codimension-one heteroclinic orbit connecting $\tilde{\epsilon}^{-}$ to the stable node on $F_1$ along the strong eigendirection in (6.13)–(6.14). This connection gives the upper boundary of the critical range in the complicated slow case for (6.7)–(6.8), where $W^{u,r}(\tilde{\epsilon}^-) \supset \tilde{\gamma}_s^N$ (figure 10d). The black curve, denoted $\tilde{\epsilon}$-to-FSN-I$_c$, was obtained by computing a codimension-one heteroclinic orbit connecting $\tilde{\epsilon}^{-}$ to the saddle-node on $F_1$ along the centre eigendirection in (6.13)–(6.14). This
connection gives the lower boundary of the critical range in the complicated slow case for (6.7)–(6.8), where $W^{u,1}(\tilde{e}^-)$ grazes $F_1$ at FSN-I (figure 10b). In the (yellow) region between $\tilde{e}$-to-FSN-I, and $\tilde{e}$-to-FN, which is the interior of the critical range, there is a codimension-zero heteroclinic orbit connecting $\tilde{e}^-$ to the stable node on $F_1$ along the weak eigendirection in (6.13)–(6.14). Here, $W^{u,1}(\tilde{e}^-)$ contains a FN weak singular canard, meaning that it crosses from $S_1$ to $S_2$ via FN, and also contains the faux FS singular canard, meaning that it crosses back to $S_1$ via FS; see the red trajectory in figure 10b. The special point $\tilde{e}$-to-FSN-I, where the black and blue curves touch, corresponds to a codimension-two non-central saddle-node heteroclinic bifurcation that involves a heteroclinic orbit connecting $\tilde{e}^-$ to the saddle-node on $F_1$ along the stable eigendirection in (6.13)–(6.14). This connection gives the degenerate case for (6.7)–(6.8), where $W^{u,1}(\tilde{e}^-) \supset \tilde{y}^{SN}$. Next, we use the singular R-tipping diagram in figure 12c together with rigorous results on persistence of singular canards as maximal canards for $0 < \epsilon \ll 1$ [39,56,60,61,63] to explain the regular R-tipping diagram in figure 8a. In the simple slow case, the FS singular canard $\tilde{y}^S$ perturbs to a family of canards when $0 < \epsilon \ll 1$. The family contains one maximal canard, namely the folded-saddle maximal canard $\tilde{y}^S_e$, and associated canards ‘without head’ and ‘with head’ [39]. Thus, these canards explain the simple part of the regular R-tipping diagram at lower values of $T_+^a$, including the (cyan) critical range and the vertical asymptote of the blue curve of maximal canards. In the context of excitable systems, we use definition 5.3(ii) to relate a family of canards ‘without head’ associated with a codimension-one normally hyperbolic repelling slow manifold near FS to a simple excitability quasi-threshold; see also [12,13].

In the complicated slow case, the perturbed dynamics for $0 < \epsilon \ll 1$ are far less straightforward due to the presence of many FN maximal canards in addition to the FS maximal canard. The number and type of FN maximal canards depend on the distance between FN and FS, and on the ratio of the weak and strong eigenvalues of FN, denoted $\mu \in (0,1)$. The FN strong singular canard $\tilde{y}^{SN}_e$ perturbs to a family of canards for all $\mu \in (0,1)$. This family contains the FN strong maximal canard $\tilde{y}^{SN}_e$ and associated canards ‘without head’ and ‘with head’. Depending on $\mu$, the family of FN weak singular canards from the singular funnel may perturb to a single FN weak maximal canard $\tilde{y}^{SN}_{w,e}$ [38,63]. Additionally, there can be a number of secondary FN maximal canards that lie between $\tilde{y}^{SN}_e$ and $\tilde{y}^{SN}_{w,e}$ [36,39,63,64]. Furthermore, the FS singular faux canard perturbs to FS faux canards, which may interact with secondary FN maximal canards near FSN-I [65]. The interplay between the FN and the FS can give rise to real-faux canards described in [36,61], that are a perturbation of the red trajectory from figure 10b, and to composite canards identified in [36], that follow the FS maximal canard and then ‘switch’ to and follow one of the FN maximal canards. For example, we have checked that as $r$ is increased for a fixed $T_+^a \geq 9.3^\circ C$ in figure 8a, $W^{u,1}(\tilde{e}^-)$ coalesces with a secondary FN canard at the bottom boundary of the R-tipping tongue, then with a composite canard at the upper boundary of the tongue, and finally with the FN strong maximal canard at the bottom boundary of the main R-tipping region [66]. Thus, it is FN maximal canards and composite canards, together with the associated canards ‘without head’ and ‘with head’, that give rise to the complicated tracking–tipping transitions including R-tipping tongue(s), at higher values of $T_+^a$. In the context of excitable systems, we use definition 5.3(ii) to relate a family of canards ‘without head’ associated with a codimension-one normally hyperbolic repelling slow manifold near FN to a new type of excitability quasi-threshold. This quasi-threshold is expected to have an intricate shape as indicated by the computations of slow manifolds in [67,68].

7. The R-tipping mechanism for summer heatwaves

The R-tipping instability in figure 3 arises because time-variation of the atmospheric temperature $T_a$ interacts with the fast timescale of soil temperature $T$. Thus, to uncover the underlying dynamical mechanisms, we consider the 2-fast 1-slow system (4.5)–(4.7) with the rate parameter $r \lesssim 1/\epsilon$, where $u = rt \lesssim \tau$ becomes another fast time and $s$ becomes another fast variable.

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13The left branch of the black curve, denoted FSN-I, does not contribute to tracking–tipping transitions.
14For higher $T_+^{max}$, we found additional R-tipping tongues, not shown in the figures, whose boundaries involve different secondary and composite canards ([66], Fig. 5.10).
Figure 13. (a) R-tipping diagram for nonautonomous system (2.3)–(2.4) with summer heatwave (7.6), in the plane of the impulse amplitude $T_{\text{max}}$ and the rate parameter $r$, for $\epsilon \approx 0.064$; note the logarithmic scale for $r$. Shown are regions of (green) tracking, (cyan, blue) critical range and (magenta, red) R-tipping from $\tilde{e}^-$. (b) (Colour) The unstable invariant manifold $W_u^{\nu}[\tilde{e}^-]$ of the saddle $\tilde{e}^-$ for the compactified system (4.5)–(4.7) with $\epsilon \approx 0.064$, summer heatwave (7.7) with $T_{\text{max}}^a = 15^\circ C$ and $\nu = 1/2$, and three different values of the rate parameter: (green) $r = 20$, (blue) $r \approx 12.9123$ and (red) $r = 10$; see the black dots in (a). Included for reference are (dark grey) the two disconnected components $S^-\text{ and }S^+$ of the critical manifold, (light grey) the layer $L$ defined by $C = C^e(0)$, and (light blue) four new equilibria $\tilde{m}_2^\pm$ and $\tilde{m}_3^\pm$ for the layer problem (7.3)–(7.4). See table S1 in the electronic supplementary material for other parameter values.

The singular reduced problem, obtained by setting $\epsilon = 1/r = 0$ for the slow time $t$ in (4.5)–(4.7),

$$\frac{dC}{dt} = f_2(T, C),$$

(7.1)

gives slow timescale solutions evolving on a critical manifold that consists of two disconnected one-dimensional components

$$S = S^- \cup S^+, \quad S^\mp = \{(T, C, s) : f_1(T, C, 0) = 0, \quad s = \mp 1\}.$$  

(7.2)

Specifically, $S^- = S_1^- \cup F_1^- \cup S_2^- \cup F_2^- \cup S_3^-$ is the critical manifold of the past limit system (4.1) with $T_a^\nu = 0^\circ C$, embedded in the compactified phase space where it gains one unstable direction. Similarly, $S^+ = S_1^+ \cup F_1^+ \cup S_2^+ \cup F_2^+ \cup S_3^+$ is the critical manifold of the future limit system (4.2) with $T_a^\nu = 0^\circ C$, embedded in the compactified phase space where it gains one stable direction. The attracting branch $S_3^+$ is the hot metastable state (figure 13b). The layer problem, obtained by defining $r = \tilde{r}/\epsilon$ for some constant $\tilde{r} = O(1)$ and taking the limit $\epsilon \to 0$ for the fast time $\tau = t/\epsilon$,

$$\frac{dT}{d\tau} = f_1(T, C, T_a^\nu(s))$$

(7.3)

and

$$\frac{ds}{d\tau} = \frac{\tilde{r} \nu}{2} (1 - s^2),$$

(7.4)

gives fast timescale solutions on a two-dimensional layer with a fixed-in-time $C$,

$$L = \{(T, C, s) : C = \text{const.}\}. \quad (7.5)$$

As a model of a summer heatwave, we sacrifice seasonal variations to simplify the analysis\(^{15}\) and consider a fast impulse rising from $T_a^\nu = 0^\circ C$ to a given $T_a^\text{max} > 0^\circ C$ and then dropping back to $0^\circ C$, that decays exponentially with the decay coefficient $\rho = 1$ as per definition 4.1,

$$T_a(rt) = T_a^\text{max} \sech(rt). \quad (7.6)$$

\(^{15}\)An extension of the compactification to asymptotically periodic inputs, such as in figure 3a, is left for future research.
We then fix the compactification parameter $\nu = 1/2$ and apply the inverse of the compactification transformation (4.4) to (7.6) to obtain

$$T_\nu(s) = T_\nu^{1/2}(s) = \frac{2T_{\text{max}}^2(1-s)^2}{(1+s)^4 + (1-s)^4}. \quad (7.7)$$

To give a full overview of transitions from tracking to R-tipping for impulse inputs, and to make connections to §6, we plot an R-tipping diagram in the plane $(T_{\text{max}}^2, r)$ of the input parameters in figure 13a for a wide range of the rate parameter $r$. The diagram was obtained by computing $W^u[s](\tilde{e}^-)$ in system (4.5)–(4.7) with $0 < \epsilon \ll 1$ for different values of $T_{\text{max}}^2$ and $r$, and using definitions 5.1–5.3 to identify different dynamical regions for system (2.3)–(2.4). The families of canards were considered relative to the evolving slow manifold, from two-dimensional $S$ in (6.3) to one dimensional $S^+$ in (7.2), as $r$ was increased. The shape of the R-tipping region in figure 13a is rather different to that obtained in figure 8a. There are two R-tipping tongues, one large akin to that in O’Keeffe & Wieczorek ([7], Sec.6) and one small akin to that in figure 8a, each enclosing a separate (magenta, red) region of R-tipping. The lower part of the diagram corresponds to slow impulses that last for decades (1-fast 2-slow system). The tracking–tipping transitions in this part occur during the rise of the impulse $T_a(r)$, and are of the same type as those described in §6. The upper part of the diagram corresponds to fast impulses (2-fast 1-slow system) and is the focus of this section. The rate parameter $r$ in the range of 10–15 yr$^{-1}$ gives a pulse duration in the range of three to two months, in line with realistic summer heatwaves [25].

Thus, the single tipping-tracking transition found in the upper part of the diagram quantifies the intensity and duration of summer heatwaves that trigger R-tipping to the hot metastable state in the soil-carbon system (2.3)–(2.4).

To gain geometric insight into the R-tipping instability caused by a summer heatwave, we plot in figure 13b the unstable manifold $W^u[s](\tilde{e}^-)$ for a fixed $T_{\text{max}}^2 = 15°C$ and three different fast rates $r_1 > r_2 > r_3 > 0$ (see the black dots in figure 13a), together with the (light grey) layer $L$ of constant $C = C^r(0)$ and the (dark grey) critical manifold $S = S^- \cup S^+$. Figure 13b shows that, as $r$ is decreased (the duration of the heatwave is increased), tracking by (green) $W^u[s](\tilde{e}^-)$ is lost via canard trajectories, including the maximal canard contained in (blue) $W^u[a](\tilde{e}^-)$ that approaches $S^-$ and moves along $S^+_2$ for the longest time. This is followed by R-tipping at lower $r$ as shown by (red) $W^u[a](\tilde{e}^-)$ that visits the hot metastable state $S^+_3$ before connecting to $\tilde{e}^+ \in S^+_1$. Since tracking–tipping transitions due to a summer heatwave (7.6) occur when the fast timescale segment of $W^u[a](\tilde{e}^-)$ on $L$ approaches the hyperbolic saddle branch $S_2^+$ of $S^+$, it should be possible to explain the upper part of the R-tipping diagram in figure 13 in terms of fast timescale solutions of a suitably chosen two-dimensional layer problem alone.

(a) One fast case in the layer problem

The suitable layer parameter for the 2-fast 1-slow system (4.5)–(4.7) with a summer heatwave (7.7) is obtained by fixing $C$ in (7.3)–(7.4) and (7.5) at the equilibrium soil-carbon concentration $C^r(0)$ for the past and future limit systems. Crucially, such a layer problem (7.3)–(7.4) has six equilibria. Two of these equilibria, namely the saddle $\tilde{e}^- \in S^-_1$ and the base-state sink $\tilde{e}^+ \in S^+_1$, are the equilibria of the compactified system (4.5)–(4.7). The four new equilibria, that do not exist in (4.5)–(4.7), are found at the intersections of $L$ with $S^+_2$. These are: the source $\tilde{m}_2^- \in S^-_2$, the saddle $\tilde{m}_3^- \in S^-_3$, the saddle $\tilde{m}_2^+ \in S^+_2$, and the hot-state sink $\tilde{m}_3^+ \in S^+_3$; see the (blue) dots in figure 13b.

To be precise and consistent with definitions 5.2 and 5.3 for $0 < \epsilon \ll 1$, we now define tracking, R-tipping and critical rates for fast external inputs in the limit $\epsilon = 0$ as follows:

Definition 7.1. In the layer problem (7.3)–(7.4), we say that,

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16 The inverse of (4.4) is given by equation (7) in section S2 of the electronic supplementary material appendix.
17 The pulse duration is obtained using the formula $2 \ln(2 + \sqrt{3})/r \approx 2.6/r$ for the full width at half maximum of the hyperbolic secant sech($rt$).
In other words, in the layer problem (7.3)–(7.4), the question of loss of tracking boils down to whether or not $W^{u,\epsilon}_{\epsilon}(\tilde{e}^-)$ connects to the saddle $\tilde{m}^+_2$. Thus, the saddle $\tilde{m}^+_2$ is another example of the ‘singular R-tipping edge state’ described in the introduction.

The dynamics of the layer problem (7.3)–(7.4) on $L$ are shown in a series of phase portraits in figure 14, where we fix $T^\text{max}_a = 15 > T^\text{inst}_a$ and decrease $\tilde{r}$ across the critical rate. The basin of attraction of $\tilde{e}^+$ is plotted in green, and the basin of attraction of $\tilde{m}^+_3$ is plotted in dark red. 

The rate-dependent (blue) stable manifold of $\tilde{m}^+_3$, denoted $W^s(\tilde{m}^+_3)$, is contained in the basin boundary separating these two basins of attraction. For $\tilde{r}$ sufficiently large, (red) $W^{u,\epsilon}_{\epsilon}(\tilde{e}^-)$ lies in the basin of attraction of $\tilde{e}^+$ and connects to $\tilde{e}^+$ (figure 14a). When $\tilde{r} = \tilde{r}_c$, $W^{u,\epsilon}_{\epsilon}(\tilde{e}^-)$ coalesces with $W^s(\tilde{m}^+_3)$ along the basin boundary. At this point, tracking is lost since $W^{u,\epsilon}_{\epsilon}(\tilde{e}^-)$ connects to the saddle $\tilde{m}^+_2$ (figure 14b). For $\tilde{r} < \tilde{r}_c$, $W^{u,\epsilon}_{\epsilon}(\tilde{e}^-)$ lies in the basin of attraction of $\tilde{m}^+_3$ and connects to $\tilde{m}^+_2$. In other words, R-tipping occurs for $r < \tilde{r}_c$ (figure 14c).

Thus, in the layer problem (7.3)–(7.4) with $C = C^e(0)$, there is just one case of a tracking–tipping transition, where tracking is lost at a critical rate $\tilde{r} = \tilde{r}_c$. We refer to this case as the fast case to distinguish it from the three slow cases identified in §6(b). The critical rate is the value of $\tilde{r}$ that gives a codimension-one heteroclinic orbit connecting $\tilde{e}^-$ to the new saddle $\tilde{m}^+_2$ for the layer problem (7.3)–(7.4). For additional insight into R-tipping caused by a summer heatwave, we refer to section S6 of the electronic supplementary material.

(b) Bringing it together: additional saddle-to-saddle heteroclinic orbit

In this section, we first produce the singular R-tipping diagram for $\epsilon = 0$ in figure 15, and then use it to explain the regular R-tipping diagram for $\epsilon \approx 0.064$ in figure 13a.

In the singular R-tipping diagram in figure 15, the lower-$\epsilon$ tracking–tipping transitions were obtained in the same manner as outlined in §6, that is by computing different heteroclinic orbits in the desingularized system for the slow impulse $T^1/2_a(s)$ in (7.7). To avoid repetition, we skip the derivations and refer to §6(b),(c) for more details.

The higher-$\epsilon$ tipping-tracking transition, which is the focus of this section, is given by the (blue) curve denoted $\tilde{e}^-\to\tilde{m}^+_2$. This curve was obtained by computing a codimension-one heteroclinic orbit connecting the saddle $\tilde{e}^-$ to the saddle $\tilde{m}^+_2$ in the layer problem (7.3)–(7.4) (figure 14b). This
Figure 15. Singular R-tipping diagram obtained using a combination of two problems. (The upper curve $\tilde{e}^\leftarrow$-to-$\tilde{m}_2^+$) the two-dimensional layer problem (7.3)–(7.4) with summer heatwave (7.7), rate parameter $\tilde{r} = \epsilon r$ shown on the left vertical axis, and $\nu = 1/2$. (The lower curves) the two-dimensional reduced problem (6.7)–(6.8) with slow impulse (7.7) and rate parameter $r$ shown on the right vertical axis. Note the logarithmic scales for $r$ and $\tilde{r}$. At higher $\tilde{r}$, the (red) region of R-tipping is separated from the (green) region of tracking by the (blue) curve $\tilde{e}^\leftarrow$-to-$\tilde{m}_2^+$, along which $W_{u,s}\left[\tilde{r}\right](\tilde{e}^-)$ intersects $W_{s,s}\left[\tilde{r}\right](\tilde{m}_2^+)$ in (7.3)–(7.4). For lower $r$, see the caption of figure 12 (c). See table S1 in the electronic supplementary material for other parameter values.

8. Conclusion and outlook

In this paper, we obtain two kinds of results. First, we demonstrate that sufficiently fast atmospheric warming can cause R-tipping to a subsurface hot metastable state in bioactive
peat soils, using a conceptual process-based ODE model with realistic soil parameter values and contemporary climate patterns. This gives rise to the hypothesis that such R-tipping is a main cause of ‘Zombie fires’ observed in peatlands, that disappear from the surface, smoulder underground during the winter, and ‘come back to life’ in the spring. Second, we recognize that such R-tipping is a nonautonomous instability that occurs due to crossing an elusive quasi-threshold in the phase space of a multiple-timescale dynamical system, and thus cannot be explained by traditional autonomous stability theory. Therefore, to understand the underlying dynamical mechanisms, we provide a mathematical framework that is underpinned by a compactification technique for asymptotically autonomous dynamical systems and concepts from GSPT, such as slow manifolds, canard trajectories and folded singularities. This framework explains R-tipping to the hot metastable state in the soil-carbon system and, more generally, identifies generic cases of tracking–tipping transitions due to crossing a quasi-threshold. Furthermore, we show that a quasi-threshold gives rise to critical ranges of the rate of change of the external input rather than isolated critical rates, and reveal new types of quasi-thresholds. These results pose two types of challenges for future research.

First, to strengthen our hypothesis, it would be interesting to build on the excellent agreement with the medium-complexity PDE model in figure 5 and extend the conceptual ODE model in different directions. For example, include terms describing peat soil ignition processes at higher temperatures [69]. This would allow investigation of existence of additional (quasi-)thresholds for the onset of flames. Include other physical processes, in addition to soil temperature, that contribute to microbial soil respiration. For example, the inclusion of soil hydrology would extend the applicability of the model; see for example DigiBog_Hydro [70,71]. Include random fluctuations to account for the fact that climate and weather patterns are inherently noisy. This would give a more accurate description of the tipping being a combination of R-tipping and noise-induced tipping (N-tipping). Understanding how the two tipping mechanisms interact is non-trivial [72,73] and an area of ongoing research for quasi-thresholds. Account for spatio-temporal dynamics by extending the conceptual ODE model to a PDE model. Vertical diffusion has already been studied in the extended Luke and Cox model with interesting results confirming validity of the conceptual ODE model [30]. Finally, it might be interesting to couple the conceptual model to spatially extended land-surface models such as JULES [74] or DigiBog [75]. Currently, such models often neglect heat produced by microbial decomposition which is a key factor for the R-tipping instability to the hot metastable state described in this paper.

Second, there are challenges related to extending the mathematical framework and obtaining rigorous results. One is the extension of the compactification technique to asymptotically periodic inputs and usage of geometrical singular perturbation theory to analyse external inputs with seasonal variations. Another is a general theory of R-tipping due to crossing quasi-thresholds in multiple-timescale systems, for which the definitions of R-tipping and a quasi-threshold introduced here for the soil-carbon system are a good starting point. Of particular interest to scientists and mathematical modellers are rigorous yet easily testable criteria for the occurrence of such R-tipping, akin to those derived in [8] for R-tipping due to crossing regular thresholds.

Data accessibility. The data are provided in electronic supplementary material [76].

Authors’ contributions. E.O’S.: conceptualization, formal analysis, investigation, visualization, writing—original draft, writing—review and editing; K.M.: conceptualization, formal analysis, investigation, supervision, validation, writing—original draft, writing—review and editing; S.W.: conceptualization, formal analysis, investigation, supervision, writing—original draft, writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Conflict of interest declaration. We declare we have no competing interests.

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