A HEAT KERNEL VERSION OF MIYACHI’S THEOREM FOR THE LAGUERRE HYPERGROUP

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Abstract. Let \( K = [0, +\infty] \times \mathbb{R} \) the Laguerre Hypergroup. In this paper, we are going to formulate and prove an analogue of Miyachi’s uncertainty principle for the Laguerre-Hypergroup Fourier transform. Our version will be in terms of the heat kernel associated to the radial part of the sub-Laplacian on the Heisenberg group.

Keywords: Miyachi’s theorem, Laguerre-Hypergroup, Fourier Laguerre transform

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1 Introduction

A wonderful aspect of quantum physics is that, we cannot measure the position and momentum of a particle simultaneously with high precision. The mathematical formulation of this rule is that, we cannot at the same time localize the value of a function and its Fourier transform. There are many formulations of this idea, previously developed by Heisenberg in 1927 [3]. Later, in 1933 Hardy have obtained a new formulation of this principle [2]. After, in 1997 Miyachi [8] proved the following theorem for the real line:

**Theorem 1.1.** Let \( f \) be an integrable function on \( \mathbb{R} \) such that

\[
e^{ax^2} f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}),
\]

Further assume that

\[
\int_{\mathbb{R}} \log^+ \left( \frac{|e^{b\lambda^2} \hat{f}(\lambda)|}{\delta} \right) d\lambda < \infty,
\]

for some numbers \( a, b, \delta > 0 \).

(i) If \( ab > \frac{1}{4} \), then \( f = 0 \) a.e.

(ii) If \( ab = \frac{1}{4} \), then \( f \) is a constant multiple of the gaussian \( e^{-ax^2} \).

For the Laguerre-Hypergroup Fourier transform in [4] H.Jizheng and L.Heping proved the Hardy’s theorem, and in [6] they demonstrated Beurling’s theorem. In this paper we are going to give a version of Miyachi’s theorem for the Laguerre-Hypergroup.

2 Harmonic Analysis for Laguerre Hypergroup

We consider the following partial differential operator

\[
\begin{cases}
D = \frac{\partial}{\partial t}, \\
\mathcal{L} = \frac{\partial^2}{\partial x^2} + 2\alpha + 1 \frac{\partial}{\partial x} + x^2 \frac{\partial^2}{\partial x^2},
\end{cases}
\]

\((x,t) \in [0, +\infty] \times \mathbb{R} \) and \( \alpha \in [0, +\infty] \).

For \( \alpha = n - 1, n \in \mathbb{N}^* \), the operator \( \mathcal{L} \) is the radial part of the sub-Laplacian on the Heisenberg group \( \mathbb{H}_n \).

For \((\lambda, m) \in \mathbb{R} \times \mathbb{N}\), the initial value problem
\[
\begin{cases}
Du = i\lambda u, \\
Lu = -4|\lambda|(m + \alpha + 1)u, \\
u(0,0) = 1, \ \frac{\partial u}{\partial x}(0,t) = 0 \quad \text{for all} \quad t \in \mathbb{R},
\end{cases}
\]

has a unique solution \(\phi_{\lambda,m}\) given by

\[
\phi_{\lambda,m}(x,t) = e^{i\lambda t}L_m^{(\alpha)}(\sqrt{|\lambda| x}), \quad (x,t) \in \mathbb{K},
\]

where \(L_m^{(\alpha)}\) is the Laguerre function on \(\mathbb{R}_+\) defined by

\[
L_m^{(\alpha)}(x) = e^{-x^2}L_m^{(\alpha)}(x)/L_m^{(\alpha)}(0)
\]

and \(L_m^{(\alpha)}\) is the Laguerre polynomial of degree \(m\) and order \(\alpha\) defined in terms of the generating function by (see [1]):

\[
\sum_{m=0}^{\infty} s^m L_m^{(\alpha)}(x) = \frac{1}{(1-s)^{\alpha+1}} \exp(-\frac{xs}{1-s})
\]

Set

\[
\varphi_m^{(\alpha)}(x) = e^{-\frac{x^2}{2}}L_m^{(\alpha)}(x^2)
\]

Lemma 2.1. For any \(\lambda \neq 0\), the system

\[
\{(\frac{2|\lambda|^{\alpha+1}m!}{\Gamma(m+\alpha+1)})^\frac{1}{2} \varphi_m^{(\alpha)}(\sqrt{|\lambda| x}); m \in \mathbb{N}\}
\]

forms an orthonormal basis of space \(L^2([0, +\infty[.x^{2\alpha+1}dx)\).

For \((\lambda, m) \in \mathbb{R} \times \mathbb{N}\), we put

\[
\psi_{\lambda}(x,t) = \frac{m!\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)}e^{i\lambda t}\varphi_m^{(\alpha)}(\sqrt{|\lambda| x}).
\]

Let \(\alpha \geq 0\) be a fixed number, \(\mathbb{K} = [0, +\infty[ \times \mathbb{R}\) and \(m_\alpha\) the weighted Lebesgue measure on \(\mathbb{K}\), given by

\[
dm_\alpha(x,t) = \frac{x^{2\alpha+1}dxd\theta}{\pi \Gamma(\alpha+1)}, \quad \alpha \geq 0.
\]

For \((x,t) \in \mathbb{K}\), the generalized translation operator \(T_{(x,t)}^{(\alpha)}\) is defined by

\[
T_{(x,t)}^{(\alpha)}f(y,s) = \begin{cases}
\frac{1}{2\pi} \int_0^{2\pi} f(\sqrt{x^2+y^2+2xycos\theta}, s+t+xysin\theta)d\theta, & \text{if} \quad \alpha = 0 \\
\frac{\alpha}{\pi} \int_0^{2\pi} \frac{1}{r} f(\sqrt{x^2+y^2+2xycos\theta}, s+t+xysin\theta)r(1-r^2)^{\alpha-1}drd\theta, & \text{if} \quad \alpha > 0
\end{cases}
\]

Let \(M_b(\mathbb{K})\) denote the space of bounded Radon measures on \(\mathbb{K}\).
The convolution on $M_b(\mathbb{K})$ is defined by

$$(\mu \ast \nu)(f) = \int_{\mathbb{K} \times \mathbb{K}} T^\alpha(x,t)f(y,s)d\mu(x,t)d\nu(y,s).$$

It is seen that $\mu \ast \nu = \nu \ast \mu$. If $f, g \in L^1(\mathbb{K})$ and $\mu = f m_\alpha$, $\nu = g m_\alpha$, then $\mu \ast \nu = (f \ast g)m_\alpha$, where $f \ast g$ is the convolution of functions $f$ and $g$, defined by

$$(f \ast g)(x,t) = \int_{\mathbb{K} \times \mathbb{K}} T^\alpha_{x,t}f(y,s)g(y,-s)dm_\alpha(y,s).$$

For every $1 \leq p < \infty$, we denote by $L^p(\mathbb{K}) = L^p(\mathbb{K}, dm_\alpha)$ the space of complex-valued functions $f$, measurable on $\mathbb{K}$ such that

$$\|f\|_{L^p(\mathbb{K})} = \left(\int_{\mathbb{K}} |f(x,t)|^pd\mu(x,t)\right)^{\frac{1}{p}}, \text{ if } p \in [1, +\infty[.$$

and

$$\|f\|_{L^\infty(\mathbb{K})} = \text{ess sup}_{(x,t) \in \mathbb{K}} |f(x,t)|.$$

The following proposition summarizes some basic properties of functions $\psi_{\lambda, m}$ (see [7]).

**Lemma 2.2.** The functions $\psi_{\lambda, m}$ satisfy that

- $\|\psi_{\lambda, m}\|_{L^\infty} = \psi_{\lambda, m}(0,0)$,
- $T^{(\alpha)}_{x,t}\psi_{\lambda, m}(y,s) = \psi_{\lambda, m}(x,t)\psi_{\lambda, m}(y,s)$,
- $L\psi_{\lambda, m} = 2|\lambda|(2m + \alpha + 1)\psi_{\lambda, m}$.

**Fourier Laguerre transform**

Let $f \in L^1(\mathbb{K})$, the generalized Fourier transform of $f$ is defined by

$$\hat{f}(\lambda, m) = \int_{\mathbb{K}} f(x,t)\psi_{(-\lambda, m)}dm_\alpha(x,t).$$

We note that

$$\hat{f}(\lambda, m) = \frac{m!}{\pi \Gamma(m + \alpha + 1)} \int_{0}^{+\infty} f^\lambda(x)\varphi_m^\alpha(\sqrt{\lambda|x|}x^{2\alpha+1})dx$$

where

$$f^\lambda(x) = \int_{-\infty}^{+\infty} f(x,t)e^{-i\lambda t}dt, \quad (2.3)$$

is the classical Fourier transform of $f(x,t)$ in the variable $t$.

Let $d_{\gamma_\alpha}$ be the positive measure defined on $\mathbb{R} \times \mathbb{N}$ by

$$\int_{\mathbb{R} \times \mathbb{N}} g(\lambda, m)d_{\gamma_\alpha}(\lambda, m) = \sum_{m=0}^{\infty} \frac{\Gamma(m + \alpha + 1)}{m!\Gamma(\alpha + 1)} \int_{\mathbb{R}} g(\lambda, m)|\lambda|^{\alpha+1}d\lambda.$$

Write $L^p(\overline{\mathbb{K}})$ instead of $L^p(\mathbb{R} \times \mathbb{N}, \gamma_\alpha)$.

We have the following Plancherel formula:

$$\|f\|_{L^2(\mathbb{K})} = \|\hat{f}\|_{L^2(\overline{\mathbb{K}})}; \text{ for } f \in L^1(\mathbb{K}) \cap L^2(\mathbb{K}).$$
We also have the inverse formula of the generalized Fourier transform:

\[ f(x, t) = \int_{\mathbb{R} \times \mathbb{N}} \hat{f}(\lambda, m) \psi(\lambda, m)(x, t) d\gamma_\alpha(\lambda, m), \]

provided \( \hat{f} \in L^1(\mathbb{R}) \).

**Heat kernel**

Let \( \{H^s : s > 0\} = \{e^{-s\mathcal{L}} : s > 0\} \) be the heat semigroup generated by \( \mathcal{L} \).

There is a unique smooth function \( h((x, t), s) = h_s(x, t) \) on \( \mathbb{K} \times ]0, +\infty[ \), such that

\[ H^s f(x, t) = f * h_s(x, t). \]

\( h_s \) is called the heat kernel associated to \( \mathcal{L} \).

By the definition of the generalized Fourier transform and lemma 2.2, it is known that

**Lemma 2.3.**

\[ \hat{\mathcal{L}}f = 2|\lambda|(2m + \alpha + 1)\hat{f}(\lambda, m), \]

\[ (\hat{f} * \hat{g})(\lambda, m) = \hat{f}(\lambda, m)\hat{g}(\lambda, m). \]

Therefore

\[ \hat{h}_s(\lambda, m) = e^{-|\lambda|(2m+\alpha+1)s}, \]

\[ h_{s_1} * h_{s_2} = h_{s_1+s_2}, \]

\[ h_s(x, t) = s^{-(\alpha+2)}h_1(\frac{x}{\sqrt{s}}, \frac{t}{s}). \]

Although the heat kernel \( h_s(x, t) \) is not explicitly known, we have an explicit expression of \( h_s \) in terms of Euclidean Fourier transform with respect to the variable \( t \).

**Lemma 2.4.** The Fourier transform of the Laguerre heat kernel is given by,

\[ h_s(x, t) = \int_{\mathbb{R}} \frac{\lambda}{2\sinh(2\lambda s)}^{\alpha+1} e^{-\frac{1}{2}M\coth(2\lambda s)x^2} e^{iM} d\lambda. \]

The pointwise estimate of the heat kernel \( h_s(x, t) \) can be derived from its fourier transform expression, we have the next lemma,

**Lemma 2.5.** There exists \( A > 0 \) such that

\[ 0 < h_s(x, t) \leq Cs^{-\alpha+2}e^{-\frac{4}{9}(|x|^2+|t|)}. \]

**Proof.** For the demonstration we can see [4]. \( \square \)

Now we turn to the Hankel transform. For \( z \in \mathbb{C} \), the Bessel function of first kind and order \( \alpha \) is defined by

\[ J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{-a - 2k} z^{\alpha + 2k}}{\Gamma(k + 1) \Gamma(k + \alpha + 1)} = \frac{2^{-a} z^\alpha}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_{-1}^{1} e^{iz(1 - s^2)^{\alpha - \frac{1}{2}}} ds. \]
Suppose that $\alpha \geq 0$.

The Hankel transform of order $\alpha$ of $f \in L^1([0, +\infty[, x^{2\alpha + 1}dx)$ is defined by

$$\langle H_\alpha f \rangle(y) = \int_0^{+\infty} f(x) \frac{J_\alpha(xy)}{(xy)^\alpha} x^{2\alpha + 1} dx.$$  

The functions $\varphi_m(x)$ defined in 2.1 are the eigenfunctions of the Hankel transform, that is,

$$\langle H_\alpha \varphi_m \rangle(x) = (-1)^m \varphi_m(x).$$

3 Miyachi’s theorem

In the proof of Miyachi’s uncertainty principle, in the most cases of Fourier transform we use the following approach, first, the transform is an entire function, in the second we prove that, this transform verified the conditions of Miyachi’s lemma, after we conclude the result. But for the Laguerre-Hypergroup Fourier transform is not a holomorphic function, so for that we are going to use a new trick that we write the Laguerre-hypergroup Fourier transform of a function satisfying the condition 3.6 as an infinite sum of elements of the basis in the lemma 2.1 from this we will try to find the conditions of the Miyachi’s lemma 3.2.

To prove our main result, we need the following lemmas,

We begin by the following lemma proved by A. Miyachi in 3.

**Lemma 3.1.** Let $f$ be an entire function on $\mathbb{C}$ and suppose there exist constants, $A, B > 0$ such that :

$$|f(z)| \leq Ae^{B(Re(z))^2} \text{ for all } z \in \mathbb{C}$$

Also suppose

$$\int_{-\infty}^{+\infty} \log^+ |f(t)| dt < +\infty,$$

Where $\log^+(x) = \log(x)$ if $x > 1$ and $\log^+(x) = 0$ if $x < 1$.

Then $f$ is a constant function.

**Lemma 3.2.** Let $h$ be an entire function on $\mathbb{C}$ and $\alpha \geq \frac{1}{2}$, such that

$$\forall z \in \mathbb{C}, |h(z)| \leq Ae^{B(Re(z))^2};$$

for some constants $A, B > 0$, and

$$\int_{-\infty}^{+\infty} \log^+ |h(t)||t|^{2\alpha + 1} dt < +\infty,$$

Then $h$ is a constant function.

**Proof.** We have

$$\int_{-\infty}^{+\infty} \log^+ |h(t)| dt = \int_{-1}^{+1} \log^+ |h(t)| dt + \int_{|t|>1} \log^+ |h(t)| dt, \quad (3.1)$$

and

$$\int_{|t|>1} \log^+ |h(t)| dt \leq \int_{|t|>1} \log^+ |h(t)||t|^{2\alpha + 1} dt < +\infty, \quad (3.2)$$

$h$ is an entire, particularly is continuous, then

$$\int_{-1}^{+1} \log^+ |h(t)| dt < +\infty, \quad (3.3)$$
we deduce from 3.2 and 3.3 that
\[
\int_{-\infty}^{+\infty} \log^+ |h(t)| dt < +\infty,
\]
the lemma 3.1 finishes the proof. □

We put
\[
S_a = \{ \lambda \in \mathbb{C} / |\text{Im}\lambda| < 4aA \},
\]
where \( A \) is the constant in heat kernel estimate 2.5.

The purpose of the following lemma is to prove that, if we have a function \( f \) satisfied the condition
\[
h^{-1}_{\frac{1}{4a}} f \in L^1(\mathbb{K}) + L^\infty(\mathbb{K}),
\]
so \( f^\lambda \) is in the space \( L^2([0, +\infty), x^{2\alpha+1} dx) \).

**Lemma 3.3.** Let \( f \) be a measurable function and \( a \) is a positive constant, such that,
\[
h^{-1}_{\frac{1}{4a}} f \in L^1(\mathbb{K}) + L^\infty(\mathbb{K})
\]
for \( \lambda \in S_a \), we have
\[
|f^\lambda(x)| \leq Ce^{-4aAx^2}.
\]
where \( C \) is a positive constant.

**Proof.** Let \( h^{-1}_{\frac{1}{4a}} f \in L^1(\mathbb{K}) + L^\infty(\mathbb{K}) \),

Then, there are two functions \( u \in L^1(\mathbb{K}) \) and \( v \in L^\infty(\mathbb{K}) \), such that:
\[
h^{-1}_{\frac{1}{4a}} (x, t) f(x, t) = u(x, t) + v(x, t),
\]
So
\[
f(x, t) = h_{\frac{1}{4a}} (x, t) u(x, t) + h_{\frac{1}{4a}} (x, t) v(x, t),
\]
by 2.5 we have
\[
f^\lambda(x) = \int_{-\infty}^{+\infty} h_{\frac{1}{4a}} (x, t) u(x, t) e^{-i\lambda t} dt + \int_{-\infty}^{+\infty} h_{\frac{1}{4a}} (x, t) v(x, t) e^{-i\lambda t} dt
\]
for \( \lambda \in \mathbb{C} \) with \( (\lambda = \xi + i\eta) \), we have
\[
|\int_{-\infty}^{+\infty} h_{\frac{1}{4a}} (x, t) u(x, t) e^{-i\lambda t} dt|
\]
\[
\leq C \int_{-\infty}^{+\infty} e^{-4aA(\xi^2 + |t|)} |u(x, t)| e^{\eta t} dt
\]
\[
\leq Ce^{-4aAx^2} \left( \int_{-\infty}^{0} e^{4aA|\eta| t} |u(x, t)| dt + \int_{0}^{+\infty} e^{-4aA|\eta| t} |u(x, t)| dt \right)
\]
\[
= Ce^{-4aAx^2} \left( \int_{-\infty}^{0} e^{(4aA+\eta) |t|} |u(x, t)| dt + \int_{0}^{+\infty} e^{(4aA-\eta) |t|} |u(x, t)| dt \right)
\]
Then \( |\int_{-\infty}^{+\infty} h_{\frac{1}{4a}} (x, t) e^{-i\lambda t} dt| < +\infty \) if and only if
\[
-4aA < \eta < 4aA
\]
So, for \( \lambda \in S_a \) we have
\[
|\int_{-\infty}^{+\infty} h_{\frac{1}{4a}} (x, t) u(x, t) e^{-i\lambda t} dt| \leq Ce^{-4aAx^2},
\] (3.4)
and we have
\[ \left| \int_{-\infty}^{+\infty} h_{\frac{1}{4\alpha}}(x,t)v(x,t)e^{-i\lambda t}dt \right| \]
\[ \leq C \int_{-\infty}^{+\infty} e^{-4aA|x^2+|t||}|v(x,t)|e^{\eta t}dt \]
\[ \leq C e^{-4aAx^2} \int_{-\infty}^{+\infty} e^{-4aA|t|}v(x,t)|e^{\eta t}dt \]
\[ \leq C\|v\|_{\infty}e^{-4aAx^2} \left( \int_{-\infty}^{0} e^{4aAt+\eta t}dt + \int_{0}^{+\infty} e^{-4aAt+\eta t}dt \right) \]

Then \[ \left| \int_{-\infty}^{+\infty} h_{\frac{1}{4\alpha}}(x,t)v(x,t)e^{-i\lambda t}dt \right| < +\infty \text{ if and only if } -4aA < \eta < 4aA, \]
and therefore if \( \lambda \in S_a \), we have
\[ \left| \int_{-\infty}^{+\infty} h_{\frac{1}{4\alpha}}(x,t)v(x,t)e^{-i\lambda t}dt \right| \leq C e^{-4aAx^2} \]
by the inequalities (3.4) and (3.5) we deduce that
\[ |f^\lambda(x)| \leq C e^{-4aAx^2}. \]

As in the paper [4], we have, when the function \( f \) satisfies the condition (3.6), we have the following estimation for the Hankel transform of the function \( f \),

**Lemma 3.4.** Let \( f \) be a measurable function and \( \alpha \) is a positive constant, such that
\[ h_{\frac{1}{4\alpha}}^{-1}f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}), \]
for \( \lambda \in S_a \), we have,
\[ |\mathcal{H}_\alpha(f^\lambda)(z)| \leq Ce^{\frac{(Im(z))^2}{4a\alpha}} \text{ for all } z \in \mathbb{C}. \]
where \( C \) is a positive constant.

Our main results is the following theorem:

**Theorem 3.5.** Let \( f \) be a measurable function on \( \mathbb{R} \) such that
\[ h_{\frac{1}{4\alpha}}^{-1}f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}), \]
and
\[ \int_{\mathbb{R}} \log^+ \left( \frac{|\mathcal{H}_\alpha(f^\lambda)(y)e^{by^2}|}{\delta} \right) |y|^{2a+1}dy < +\infty. \]
where \( \delta \) is positive constant. where \( a, b > 0 \).
If \( ab > \frac{1}{4} \) then, \( f(x,t) = 0 \) a.e.

**Proof.** First, let \( g \) a function defined by \( g(x) = e^{\frac{x^2}{2\delta}}\mathcal{H}_\alpha(f^\lambda)(x) \)
* If \( ab > \frac{1}{4} \) we have
\[
\int_{-\infty}^{+\infty} \log^+ |g(x)||x|^{2\alpha + 1} \, dx \leq \\
\leq \int_{-\infty}^{+\infty} \log^+ (|\mathcal{H}_\alpha(f^\lambda)(x)e^{bx^2}|)|x|^{2\alpha + 1} \, dx + \int_{-\infty}^{+\infty} e^{(\frac{1}{4a} - b)x^2}|x|^{2\alpha + 1} \, dx
\]
< +\infty

As \( z \mapsto \mathcal{H}_\alpha(f^\lambda)(z) \) is an entire function, so \( z \mapsto g(z) \) is also entire. For \( \lambda \in S_a \) applying the lemma 3.2 we get \( g \) is a constant. Thus, for \( ab > \frac{1}{4} \), there is a constant \( C \) such that

\[
g(x) = C \iff e^{\frac{1}{4a}x^2}\mathcal{H}_\alpha(f^\lambda)(x) = C
\]

then

\[
\mathcal{H}_\alpha(f^\lambda)(x) = Ce^{-\frac{1}{4a}x^2}
\]

but in this case, the relation

\[
\int_{-\infty}^{+\infty} \log^+ \left( \frac{\mathcal{H}_\alpha(f^\lambda)(x)e^{bx^2}}{\delta} \right)|x|^{2\alpha + 1} \, dx < +\infty
\]

holds only whenever \( C = 0 \).

So

\[
\mathcal{H}_\alpha(f^\lambda)(x) = 0
\]

implies that, for all \( \lambda \in S_a \): \( f^\lambda(x) = 0 \) (because \( \mathcal{H}_\alpha \) is injective).

The function \( f^\lambda \) is entire, so we get that then \( f^\lambda = 0 \) for all \( \lambda \in \mathbb{R} \).

then we have

\[
\int_{-\infty}^{+\infty} f(x,t)e^{-i\lambda t} \, dt = 0
\]

thus \( f(x,t) = 0 \) a.e.

This complete the proof of our main result.

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