SIMILARITY VERSUS SYMMETRIES.
An excuse for revising a theory of time-varying “constants”.

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In this paper we compare the dimensional method with the Lie groups tactic in order to show the limitations and advantages of each technique. For this purpose we study in detail a perfect fluid cosmological model with time-varying “constants” by using dimensional analysis and the symmetry method. We revise our previous conclusion about the variation of the fine structure constant finding for example that in the radiation predominance era if α varies is only due to the variation of $e^{0.10}$ since $ch = const$ in this era.

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I. INTRODUCTION

The main goal of this paper is to show how fruitful is to work with dimensional techniques (DA) as compared to other more sophisticated methods as the Lie groups (LG). For this purpose we study some cosmological models that consider all the physical “constants” i.e. $G,c,\Lambda$ and $h$ as functions depending on time $t$. In particular we will study models which matter content is modeled by a perfect fluid. Therefore this paper is devoted to compare the dimensional method with the Lie group tactic, by showing the limitations and advantages of each technique.

We understand that the dimensional method is in actually a method of Lie groups, this method has such algebraic structure, but the Pi-Theorem only finds scaling symmetries while the Lie method finds all the possible symmetries. One of our purposes in this paper will be to show this fact.

Cosmological models with time-varying “constants” have been studied for quite some time ever since Dirac proposed a theory with a time-varying gravitational constant $G$. Several works have investigated cosmological models with variable cosmological constant within a framework of dissipative thermodynamics as well as in the case of perfect fluids.

The purpose of this work is to perform a detailed study of all the possible symmetries of a perfect fluid model with time varying constants showing that in this case it is possible to find more solutions in addition to the scaling one (obtained through DA). In order to carry out this study, we begin in section 2 by outlining the equations that govern the model as well as the notation employed. We present three models. The first of them will be formulated without the condition $\text{div}(T^i_j) = 0$. In order to make that “constant” $h$ appears into the field equations we impose an adequate equation of state for the energy density, in this way through this condition we will be able to formulate a very general equation that contains to all the “constants”. The second and third of the models verify the condition $\text{div}(T^i_j) = 0$. As in the first of our models we will consider two cases in each model, to take a general form for the energy density and the case which verifies the equation of state $\rho = a0^4$ (black body radiation). In order to get rid of the entropy problem in our third model we will consider adiabatic matter creation, showing that for an adequate value of a β-parameter this model could be reduced to our second model. Therefore this will be our more general model.

In section 3, we review the scaling solution obtained in previous works by highlighting the “assumed” hypotheses that we need to make in order to obtain a solution using dimensional analysis, these are: $\text{div}(T^i_j) = 0$, conservation principle, and that the relation $G/c^2$ remain constant for all values of $t$ (cosmic time). In this section we will find a solution for the field equations through two different dimensional ways. In the first of them we apply the Pi-theorem in order to solve our most general model, the third one, and as we will able to see this is, in our opinion, the best method to solve the equations since it is the simplest one and it lets us to obtain a complete solution. We discuss some interesting relationships that arise with the similarity method. We will revise our conclusions obtained in previous works trying to improve them, in this way we arrive to the conclusion that the fine structure constant varies but we cannot know which constant or constants are the responsible of such variation since we can only calculate the behaviour of the constants $G,c,\Lambda$ and $h$ but no of any electromagnetic constant, $e,\varepsilon_0,...$. With regard to the second dimensional tactic we will try to solve the first of our model, which does not verify the condition $\text{div}(T^i_j) = 0$. In order to do it we shall need to impose some hypotheses to reduce the number of unknown quantities in the field equations. We arrive to the conclusion that the solution obtained verifies the general conservation principle (this is a limitation of the dimensional tactic) and that it is less general than the solution obtained with our first dimensional method. This solu-
tion is only valid for the case of radiation predominance i.e. $\omega = 1/3$ while the above solution is valid for all value of $\omega$ that is to say for all kind of matter and not only for radiation.

In Section 4, we work towards finding other possible solutions to the field equations using the Lie group method. We start this section by rewriting the field equations in such a way that we can use the standard Lie procedure that allow us to find more symmetries. With this tactic we only will be able the study the second and the third model but without the possibility of considering the case where $\rho = a\theta^4$ that is to say we are not able to study the general case where $\hbar$ appears. After outlining the equation and the constraint, we proceed to study some cases. The first one is the obtained previously by using dimensional analysis since dimensional analysis is just a special class of symmetry (scaling symmetry). We would like to emphasize that the Lie method show us that one of the assumptions made with the dimensional method, $G/c^2 = \text{const.}$, is at least correct from the mathematical point of view. This result allow us to validate completely the solution obtained through similarity. Nevertheless there are more symmetries, some of them correspond to our second and third solutions.

We conclude the paper by summarizing the results in section 5 and emphasizing why the dimensional method works so well. In the appendix we try to show how tedious is to work with the Lie method if one tries to be rigorous.

II. THE MODEL

In this section we will outline the field equations of three models. In the first of them, its energy-momentum tensor does not verify the conservation principle. The second model and the third do it but in the third one we will consider adiabatic matter creation. In all these cases we will consider two possibilities, a general form for the energy density $\rho$ and a specific equation of state for this $\rho = a\theta^4$, in order to introduce into the field equations the radiation “constant” $a$.

We will use in all the models the field equations in the form:

$$R_{ij} - \frac{1}{2}g_{ij}R = \frac{8\pi G(t)}{c^4(t)}T_{ij} + \Lambda(t)g_{ij},$$  \hspace{1cm} (1)

where the energy momentum tensor is:

$$T_{ij} = (\rho + p)u_iu_j - pg_{ij},$$ \hspace{1cm} (2)

and $p = \omega\rho$ in such a way that $\omega \in (-1, 1)$, that is to say, our universe is modeled by a perfect fluid. The line element is defined by:

$$ds^2 = -c^2dt^2 + d\Omega^2,$$ \hspace{1cm} (3)

with

$$d\Omega^2 = f^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],$$ \hspace{1cm} (4)

we consider “only” a flat model i.e. $k = 0$, as the most recent observations suggest us.$^{1-4}$.

The cosmological equations are now:

$$2H' + 3H^2 = -\frac{8\pi G(t)}{c^4(t)}p + c(t)^2\Lambda(t),$$ \hspace{1cm} (5)

$$3H^2 = \frac{8\pi G(t)}{c^4(t)}\rho + c(t)^2\Lambda(t),$$ \hspace{1cm} (6)

where $H = (f'/f)$ is the Hubble function.

Applying the covariance divergence to the second member of equation (1) we get:

$$\text{div}\left(\frac{8\pi G}{c^4}T_i^j + \delta_i^j\Lambda\right) = 0,$$ \hspace{1cm} (7)

that simplified is:

$$T^j_{i;j} = \left(\frac{4c^4}{G}\right)T_i^j - \frac{c^4(t)G_i^j}{8\pi G} = 0,$$ \hspace{1cm} (8)

which yields:

$$\rho' + 3(\omega + 1)\rho H = -\frac{N'i^4}{8\pi G} - \rho\frac{G'}{G} + 4\rho\frac{c'}{c},$$ \hspace{1cm} (9)

where it is noted that this is the main difference with respect to other approaches.

Therefore our first model is governed by the following equations:

$$2H' + 3H^2 = -\frac{8\pi G(t)}{c^4(t)}p + c(t)^2\Lambda(t),$$ \hspace{1cm} (10)

$$3H^2 = \frac{8\pi G(t)}{c^4(t)}\rho + c(t)^2\Lambda(t),$$ \hspace{1cm} (11)

$$\rho' + 3(\omega + 1)\rho H = -\frac{N'i^4}{8\pi G} - \rho\frac{G'}{G} + 4\rho\frac{c'}{c},$$ \hspace{1cm} (12)

In order to incorporate the “Planck constant $\hbar$” we chose or impose an adequate equation of state for the energy density, we use the black body equation of state $\rho = a\theta^4$ where $a = \frac{\sqrt{k_4m^2}}{15c^5\hbar^3}$ so that equation (9) is now:

$$4\frac{\theta'}{\theta} - 3\left[\frac{c'}{c} + \frac{\hbar'}{\hbar}\right] + 3(\omega + 1)H = \frac{15\Lambda'c^7h_4^3}{8\pi^3Gk_4^2\theta} + \frac{G'}{G} - 4\frac{c'}{c},$$ \hspace{1cm} (13)

in this way our first modified model is governed by the following equations:
The second class of models that we study verifies the principle of conservation for its energy-momentum tensor i.e. we assume that $\text{div}(T^j_i) = 0$, then equation (9) is reduced to:

$$\rho' + 3(\omega + 1)\rho H = 0,$$

or if $\rho = a\theta^4$, equation (13) reads now:

$$\frac{4}{\theta} - 3 \left[ \frac{c'}{c} + \frac{k'}{h} \right] + 3(\omega + 1)H = 0,$$

$$\frac{15 \Lambda c^7 h^3}{8 \pi^3 G k_B^2 \theta^4} + \frac{G'}{G} - 4 \frac{c'}{c} = 0.$$  

Hence the field equations for our second model are (for a general form of $\rho$):

$$2H' + 3H^2 = -\frac{8\pi G(t)}{c^2(t)} p + \Lambda c^2,$$

$$3H^2 = \frac{8\pi G}{c^2} \rho + \Lambda c^2,$$

$$\rho' + 3(\omega + 1)\rho H = 0,$$

$$-\frac{\Lambda c^4}{8 \pi \rho G} \frac{G'}{G} + 4 \frac{c'}{c} = 0.$$  

and for the black body equation of state $\rho = a\theta^4$ :

$$2H' + 3H^2 + \frac{8\pi G}{c^2} p - \Lambda c^2 = 0,$$

$$3H^2 - \frac{8\pi G}{c^2} \rho - \Lambda c^2 = 0,$$

$$\frac{4}{\theta} - 3 \left[ \frac{c'}{c} + \frac{k'}{h} \right] + 3(\omega + 1)H = 0,$$

$$\frac{15 \Lambda c^7 h^3}{8 \pi^3 G k_B^2 \theta^4} + \frac{G'}{G} - 4 \frac{c'}{c} = 0.$$  

To end we study briefly the important case in which adiabatic matter creation $5-8$ can be taken into account, in order to get rid of the entropy problem. This will be our third model. The matter creation theory is based on an interpretation of the matter energy-stress tensor in open thermodynamic systems, which leads to the modification of the adiabatic energy conservation law and as a result including the irreversible matter creation. The matter creation theory is based on $\frac{\rho}{\rho}$ or if $\rho = a\theta^4$, equation (13) reads now:

$$\frac{4}{\theta} - 3 \left[ \frac{c'}{c} + \frac{k'}{h} \right] + 3(\omega + 1)H = 0,$$

$$\frac{15 \Lambda c^7 h^3}{8 \pi^3 G k_B^2 \theta^4} + \frac{G'}{G} - 4 \frac{c'}{c} = 0.$$  

The field equations that now govern our model are as follows:

$$2H' + 3H^2 = -\frac{8\pi G(t)}{c^2(t)} (p + p_c) + c^2(t)\Lambda(t),$$

$$3H^2 = \frac{8\pi G(t)}{c^2(t)} \rho + c^2(t)\Lambda(t),$$

$$\rho' + 3(\omega + 1)\rho H = (\omega + 1)\rho \frac{\psi}{n},$$

and taking again into account our general assumption on the conservation principle i.e. equation (8) with $T^j_i = 0$, we obtain the two equations

$$\rho' + 3 (\rho + p + p_c) H = 0,$$

$$\frac{\Lambda c^4}{8 \pi G \rho} + \frac{G'}{G} - 4 \frac{c'}{c} = 0,$$

and where $n$ is the particle number density, $\psi$ is the function that measures the matter creation, $H = f'/f$ represents the Hubble parameter ($f$ is the scale factor that appears in the metric), $p$ is the thermostatic pressure, $\rho$ is energy density and $p_c$ is the pressure that generates the matter creation.

The creation pressure $p_c$ depends on the function $\psi$. For adiabatic matter creation this pressure takes the following form:

$$p_c = -\left[ \frac{\rho + p}{3nH} \psi \right].$$

The state equation that we next use is the well-known expression $p = \omega \rho$, where $\omega = \text{const.}$ and $\omega \in (-1, 1]$.

Therefore, the new set of field equations is now:

$$2H' + 3H^2 = -\frac{8\pi G(t)}{c^2(t)} (p + p_c) + c^2(t)\Lambda(t),$$

$$3H^2 = \frac{8\pi G(t)}{c^2(t)} \rho + c^2(t)\Lambda(t),$$

$$\rho' + 3(\omega + 1)\rho H = (\omega + 1)\rho \frac{\psi}{n},$$

$$\frac{\Lambda c^4}{8 \pi G \rho} + \frac{G'}{G} - 4 \frac{c'}{c} = 0.$$  

Similarity versus Symmetries.
If we study from the dimensional point of view the equations (35-38) it is found the following relationship between the quantities:

\[
\begin{align*}
\pi_1 &= \frac{Gpt^2}{c^2}, \\
\pi_2 &= \frac{Gp_c t^2}{c^2}, \\
\pi_3 &= \frac{G\rho t^2}{c^2}, \\
\pi_4 &= c^2 \Lambda t^2, \\
\pi_5 &= \frac{n}{\psi t^4}, \\
\pi_6 &= \frac{G' \rho}{N' c^4}, \\
\pi_7 &= \frac{G\rho c'}{N' c^5},
\end{align*}
\]

(39)

where \( \pi_1 \) and \( \pi_2 \) have been obtained from equation (35) while \( \pi_3 \) and \( \pi_4 \) have been obtained from equations (36), and \( \pi_5 \) from (37). The \( \pi \) \( - \) \( \text{monomias} \) \( \pi_6 \) and \( \pi_7 \) have been obtained from equation (38).

In this way we have the following conclusions:

1. From the monomias \( \pi_1 - \pi_3 \), we see that the quantities \( p, p_c \) and \( \rho \) behave in a similar way i.e.

\[
p \approx p_c \approx \rho.
\]

(40)

as we already know, since we have imposed the equation of state \( p \approx \rho \), but the DA of the field equations suggests us that \( p_c \approx \rho \), i.e. that the pressure \( p_c \) behaves as the energy density.

2. From \( \pi_4 = c^2 \Lambda t^2 \) we obtain a clear behaviour for the cosmological constant:

\[
\Lambda \approx \frac{1}{c^2 t^2}
\]

(41)

3. From \( \pi_5 \) it is obtained that

\[
\psi \approx nH,
\]

(42)

i.e., the DA suggest us that \( \psi \) must be proportional to the Hubble parameter. Of course other possibilities are allowed, see for example the model presented by Prigogine and collaborators, where \( \psi \approx H^2 \).

4. And from the monomias \( \pi_6 \) and \( \pi_7 \) we see that

\[
\frac{G'}{G} \approx \frac{c'}{c}.
\]

(43)

It is noted that the DA does not understand of numerical factors, only relations between quantities.

Since the dimensional method has suggested \( (\psi \approx nH) \), we assume that the matter creation function follows the law (the same than Lima et al.):

\[
\psi = 3\beta nH,
\]

(44)

where \( \beta \in [0, 1] \) is a dimensionless constant (if \( \beta = 0 \) then there is no matter creation since \( \psi = 0 \)). The generalized principle of conservation \( T_{ij}^\mu = 0 \), for the stress-energy tensor (32) leads us to:

\[
\rho' + 3(\omega + 1) (1 - \beta) \rho H = 0.
\]

(45)

In this way, the set of field equations for the third model are now:

\[
\begin{align*}
2H' + 3H^2 + \frac{8\pi G(t)}{c^2(t)} (p + p_c) - c^2(t)\Lambda(t) &= 0, \\
3H^2 - \frac{8\pi G(t)}{c^2(t)} \rho - c^2(t)\Lambda(t) &= 0, \\
\rho' + 3(\omega + 1) (1 - \beta) \rho H &= 0,
\end{align*}
\]

or equivalently for \( \rho = a\theta^4 \):

\[
\begin{align*}
2H' + 3H^2 + \frac{8\pi G(t)}{c^2(t)} (p + p_c) - c^2(t)\Lambda(t) &= 0, \\
3H^2 - \frac{8\pi G(t)}{c^2(t)} \rho - c^2(t)\Lambda(t) &= 0, \\
\frac{d\theta'}{\theta} - 3 \left[ \frac{c'}{c} + \frac{H'}{H} \right] + 3(\omega + 1) (1 - \beta) H &= 0,
\end{align*}
\]

(52)

\[
\frac{15\Lambda c^4 H^3}{8\pi^3 G k_B^3 T^4} + \frac{G'}{G} - \frac{c'}{c} = 0.
\]

(53)

We must emphasize that these models with \( \beta = 0 \) (no matter creation) reduces to the second class of models.

Therefore these are the more general field equations. In the next sections we will find a solution for all these field equations through different ways beginning with the dimensional one and ending with the Lie method.

### III. DIMENSIONAL METHODS.

In this section we will find a solution to the field equations through two different dimensional ways\(^9-14\). We will begin in the next subsection, the simplest method, finding a solution to eqs. (50-53) applying the Pi-theorem, while in the following subsection, the not so simple method, we will find a solutions to the field eqs. (14-16) applying a dimensional tactic which consists in reducing the number of variables into the equations making a simple hypotheses obtained through DA. In this way it is obtained a simple differential equations that admits a trivial integration.

#### A. The simplest method.

Our purpose in this subsection is to integrate eqs. (50-53) that is to say to find a solution to these equations. In this case we apply the Pi-theorem. In order to apply the Pi-theorem in the first place we need to fix the set of governing parameters \( \varphi = \varphi (C, \mathfrak{R}) \) which it is composed of the set of fundamental constants \( C \) and the set of fundamental quantities \( \mathfrak{R} \). The set of fundamental constants \( C \) contains the physical constants and the characteristic constants of the model. This model evidently
has no physical “constants” since these vary. Our first characteristic constant will be obtained by integrating equation (45), that brings us to obtain the following relation between the energy density and the scale factor and which is more important, the constant of integration that we shall need for our subsequent calculations:

\[ \rho = A_{\omega,\beta} f^{-3(\omega+1)(1-\beta)}, \]  

(54)

where \( A_{\omega,\beta} \) is the integration constant that depends on the equation of state that we need to consider i.e. constant \( \omega \) and constant \( \beta \) that controls the matter creation, \([A_{\omega,\beta}] = L^{3(\omega+1)(1-\beta)-1}MT^{-2}\), where we are using a dimensional base \( \mathfrak{B} = \{L, M, T\} \), see\(^\text{15}\) for details. To apply the Pi-theorem we need another constant. This new constants is obtained by making a simple hypothesis about the behaviour of the “constants” \( G \) and \( c \). We suppose that the relation \( G/e^2 = B, [B] = LM^{-1}T^0 \), remain constant in spite of both “constants” vary, but in such a way that this relationship remain constant for all \( t \), the universal time. Furthermore, in this way we are guaranteeing that it is verified the covariance principle. With the Lie group tactic we will show that at least this relationship has mathematical meaning since it is deduced as solution of the field equations and not imposed as hypothesis. Therefore the set of fundamental constants is: \( \mathcal{C} = \{A_{\omega,\beta}, B\} \). Now, the set of fundamental quantities has only one quantity, the universal time, \( t \), as it can be trivially deduced through the Killing equations. Therefore \( \mathfrak{M} = \mathfrak{M}(t) \).

Our purpose is to show that no more hypothesis are necessary to solve the differential equations that govern the model. Therefore the set of governing parameters is now: \( \varphi = \varphi(A_{\omega,\beta}, B, t) \), that brings us to obtain the next relations:

\[
\begin{array}{c|cccc}
  & G & A_{\omega,\beta} & B & t \\
  L & 3 & \gamma & 1 & 0 \\
  M & -1 & 1 & -1 & 0 \\
  T & -2 & -1 & 0 & 1 \\
\end{array}
\]

\[ G \propto A_{\omega,\beta}^{2\omega+1} B^{2+\omega(1-\gamma)} t^{3(1-\gamma)}, \]

(55)

In this way, it can be easily obtained the rest of quantities, obtaining:

\[
\begin{align*}
  c & \propto A_{\omega,\beta}^{\omega+\frac{1}{2}} B^{\frac{1}{2}t^{-\gamma}} t^{\frac{(1+\gamma)}{2}}, \\
  h & \propto A_{\omega,\beta}^{\omega+\frac{1}{2}} B^{-\frac{1}{2}t^{\gamma}} t^{-\frac{3}{2}}, \\
  m_1 & \propto A_{\omega,\beta}^{\omega+\frac{1}{2}} B^{-\frac{1}{2}t^{\gamma}} t^{-\frac{3}{2}}, \\
  \rho & \propto B^{-1} t^{-2}, \\
  f & \propto A_{\omega,\beta}^{\omega+\frac{1}{2}} B^{-\frac{1}{2}t^{\gamma}} t^{-\frac{3}{2}}, \\
  k_B & \propto A_{\omega,\beta}^{\omega+\frac{1}{2}} B^{\frac{1}{2}t^{\gamma}} t^{-\frac{3}{2}}. \\
\end{align*}
\]

(56)

\[ \alpha^{-1/4} s \propto t^{2(3-\omega-\gamma)/7}, \]

(57)

we can check that the next results are verified: we see that \( \frac{h}{c} = B \propto \beta \) (trivially), \( \rho = \alpha^2 \propto \beta \), \( f = ct \propto \beta \), \( \Lambda \propto 1 \propto f^{-2} \) while the relation \( h \propto \text{const} \) since it depends on \( \beta \) (if \( \beta = 0 \), then \( h = \text{const} \) but only when \( \omega = 1/3 \) i.e. in the radiation predominance era). We emphasize this relationship between these two “constants” because as it is known from the definition of the fine structure constants \( \alpha \),

\[ \alpha = \frac{e^2}{4\pi \varepsilon_0 \hbar c}, \]

(58)

if \( \alpha \) varies (in this epoch and only in this one) this variation only can be caused by \( e^2/\varepsilon_0 \). While in other epoch \( h \neq \text{const} \) independently of \( \beta \), being very difficult to explain the origin of such variation or more exactly which one or which ones of the constant are the cause of such variation. With this model we can only calculate the variation of the “constants” \( c \) and \( h \) (and if we are strictly rigorous we would have to say that only for the time of radiation predominance) and we cannot calculate the behaviour of any electromagnetic quantity. We believe that this must be the behaviour between this “constants” because as we have shown in a previous paper\(^\text{16}\) other relation as \( h \approx c \) brings us to a static universe. Furthermore, the behavior obtained here works well in the framework of the quantum cosmology as we have pointed out in reference\(^\text{17}\).

In previous works we calculate the behaviour of \( e^2\varepsilon_0^{-1} \) as a function of \( A_{\omega,\beta}, B, t \) in such a way that \( \alpha \) (dimensionless quantity) remain constant in spite of the fact of that all “constants” vary but in a conspire way since they vary but keep alpha constant. Now we believe that the relationship, for \( e^2\varepsilon_0^{-1} \) was wrong in spite of the fact that there is a relation between \( c \) and \( \varepsilon_0 \), \( c^2 = (\mu_0\varepsilon_0)^{-1} \). We cannot obtain an expression for electromagnetic quantities only in function of \( A_{\omega,\beta}, B, t \) (obtained in the framework of standard cosmology). This fact has been pointed out in a previous work (but in a different form since in that work \( c = \text{const}) \) where we showed that it is impossible to reconcile the cosmological quantities with the electromagnetic quantities (see\(^\text{18}\) for details). Here we have the same situation. Therefore alpha can vary but we do not know which of the constants are the responsible of such variation as already it has been pointed out by M. E. Tobar\(^\text{19}\).

We also can check that our model has no the so called Planck’s problem since the Planck system behaves now as:

\[
\begin{align*}
  l_p &= \left(\frac{\hbar c}{e^2}\right)^{1/2} \approx f(t), \\
  m_p &= \left(\frac{\hbar}{e}\right)^{1/2} \approx f(t), \\
  t_p &= \left(\frac{\hbar}{\mu_0 e}\right)^{1/2} \approx t,
\end{align*}
\]

(59)

since the radius of the Universe \( f(t) \) at Planck’s epoch coincides with the Planck’s length \( f(t_p) \approx l_p \), while
the energy density at Planck’s epoch coincides with the Planck’s energy density \( \rho(t_p) \approx \rho_p \approx t^{-2} \), where \( \rho_p = m_p c^2 / l_\pi^3 \). See \(^{15}\) for more details and the followed method etc...

It is observed from (56) that if we make \( \beta = 0 \) the following set of solutions are obtained:

\[
\begin{array}{|c|c|c|c|c|c|}
\hline 
\omega & 1 & 2/3 & 1/3 & 0 & -1/3 & -2/3 \\
\hline 
\rho & 1/3 & 2/5 & 1/2 & 2/3 & 1 & 2 \\
\hline 
\theta & -2 & -2 & -2 & -2 & -2 & -2 \\
\hline 
\phi & -1 & -4/5 & -1/2 & 0 & 1 & 4 \\
\hline 
\mu & -1/2 & -3/10 & 0 & 1/2 & 3/2 & 9/2 \\
\hline 
\omega & -2/3 & -3/5 & -1/2 & -1/3 & 0 & 1 \\
\hline 
\omega & 0 & 1/5 & 1/2 & 1 & 2 & 5 \\
\hline 
\omega & 1/3 & 2/5 & 1/2 & 2/3 & 1 & 2 \\
\hline 
\omega & -2/3 & -4/5 & -1 & -4/3 & -2 & -4 \\
\hline 
\omega & 5/2 & 3/2 & 1 & 1/2 & 0 & -1/2 \\
\hline 
\end{array}
\]

with: \( \omega = -1 \) corresponds to de Sitter (false vacuum) represented by the cosmological constant (special case), \( \omega = -\frac{2}{3} \) for domain walls, \( \omega = -\frac{1}{3} \) for strings, \( \omega = 0 \) for dust (matter predominance), \( \omega = \frac{2}{3} \) for radiation or ultrarelativistic gases (radiation predominance), \( \omega = \frac{3}{2} \) for perfect gases, \( \omega = 1 \) for ultra-stiff matter. For example, if we take the case \( \omega = 0 \) the table (60) tells us that

\[
f \propto t^{2/3}, \quad \rho \propto t^{-2}, \quad \theta \propto t^0 \text{ const.}, \ldots \quad G \propto t^{-2/3}, \quad c \propto t^{-1/3}, \quad h \propto t, \ldots \text{ etc.}
\]

This table tells us that if we want that our universe accelerates then we have to impose that \(-1 \leq \omega < -1/3\) but we must be careful since with this parameter we see that the temperature increases.

We can try to generalize this scenario taking into account various kinds of matter. The idea is as follow. We can define a general energy density \( \bar{\rho} \) as:

\[
\bar{\rho} = \sum_{i=0}^{6} \rho_i
\]

where \( \rho_i \) stands for each kind of energy density, and the parameter \( i = 0, 1, ..., 6 \) in such a way that: \( i = 0 \) correspond to \( \omega = -1 \) (the false vacuum), \( i = 1 \) correspond to domain walls i.e. to \( \omega = -2/3, i = 2 \) to \( \omega = -1/3, i = 3 \) to \( \omega = 0, i = 4 \) to \( \omega = 1/3, i = 5 \) to \( \omega = 2/3 \) and finally \( i = 6 \) to \( \omega = 1 \); but in such a way that each type of matter verifies the relation

\[
\rho_i = \omega_i \rho_i
\]

in this way we define the total pressure as:

\[
\bar{\rho} = \sum_{i=0}^{6} \rho_i
\]

but in this case we do not impose that it is verified for each kind of matter the relation:

\[
\rho_i = A_{\omega_i} f^{-3(\omega+1)}
\]

Our purpose is as follows: we impose that the relation \( \bar{\rho}' + 3(\bar{\rho} + \bar{\rho})H = 0 \) taken into account that \( \rho_i = \omega_i \rho_i \) then

\[
\bar{\rho}' + 3 H \sum_{i=0}^{6} (\omega_i + 1) \rho_i = 0
\]

with \( m = 3 \sum_{i=0}^{6} (\omega_i + 1) \). It is proven in a trivial way that if we consider only one type of matter we then recuperate the above results i.e., \( m = 3(\omega + 1) \).

Therefore with the next set of governing quantities \( \bar{\rho} = \rho (A_m, B, t) \) we arrive to obtain the following table of results:

\[
\begin{align*}
G & \propto A_m^{-1} B^{\frac{1}{3} - 1} t^{\frac{1}{3} - 2} \\
c & \propto A_m^{-1} B^{\frac{1}{3} - 1} t^{\frac{1}{3} - 1} \\
\lambda & \propto A_m^{-1} B^{\frac{1}{3} - 1} t^{\frac{1}{3} - 1} \\
\h & \propto A_m^{-1} B^{\frac{1}{3} - 1} t^{\frac{1}{3} - 1} \\
f & \propto A_m^{-1} B^{\frac{1}{3} - 1} t^{\frac{1}{3} - 1} \\
\rho & \propto B^{-1} t^{-2} \\
k_B \theta & \propto A_m^{-1} B^{\frac{1}{3} - 1} t^{\frac{1}{3} - 1} \\
\alpha^{-1/4} s & \propto A_m^{-1} B^{\frac{1}{3} - 1} t^{\frac{1}{3} - 1} \\
\omega & = \frac{x+1}{x-1} - 1
\end{align*}
\]

where \( x + 1 = m = 3 \sum_{i=0}^{6} (\omega_i + 1) \). If for example we consider an universe with dust \( (\omega = 0) \) and radiation \( (\omega = 1/3) \) then \( m = 9 \), we obtain a very surprising results as we can see

\[
\begin{align*}
G & \propto t^{-8/5}, \quad c \propto t^{-4/5}, \quad h \propto t^{-2/5}, \quad \Lambda \propto t^{-2/5}, \\
\rho & \propto t^{-2}, \quad f \propto t^{1/5} \text{ etc...}
\end{align*}
\]

As we have seen, it is obtained a complete solution of the field equations making only one hypothesis i.e. \( G/c^2 = B \). The method, a direct application of the Pi-theorem, allows us to obtain, in a trivial way, the behavior of all the quantities under study. Furthermore, the followed tactic allows us to generalize the model considering a mixed of fluids (but without interactions between them). This method is therefore, simple but powerful as we will see in the next sections since with the rest of the tactics we will not be able to solve this so complex model.

\\

B. Not so simple method.

In this subsection we will try to solve the first of our models eqs. (14-16), i.e. a model where all its “constants” are time-varying and its energy-momentum tensor does not verify the conservation principle \( \text{div}(T_{\mu}^\nu) \neq 0 \).
0. In particular we are interested in solving the eq. (16) which contains all the information about the model. As this equation has 6 unknown quantities, we will need to make some hypotheses in order to simplify the original equation and to try to integrate it. In this case we will impose some hypotheses about the behavior between some of the quantities through DA in such a way that these hypotheses allows us to reduce the number of variables in the equation under study.

To solve equation (16) we imposed two simplifying hypotheses, the first one, that the relation $G/c^2 = B$ (where $B$ is a const.) remains constant, and the second one, that the cosmological “constant” verifies the relation $\Lambda \propto \frac{d}{c^2}$ with $d \in \mathbb{R}$ (this is a very strong condition), while $\text{div}(T^i_i) \neq 0$, in this way the equation was solved perfectly. Since the constant $B$ has dimensions $[B] = LM^{-1}$ we can get the dimensionless monomia $\pi_1 = \frac{4c^4}{\pi b^2}$ where $b \in \mathbb{R}$. With these hypotheses and $\pi_1$ equation (16) simplifies to:

\[
\frac{4\theta'}{\theta} - 3 \left[ \frac{c'}{c} + \frac{h'}{h} \right] + 3(\omega + 1)H - \frac{15d}{4\pi^3} \frac{c'[t + c]}{Gk_B^4[\theta^4t^3 + \frac{G'}{G} - 4 \frac{c'}{c}] = 0, \tag{71}
\]

that has no immediate integration. We have to take into account the field equations (15)

\[
3H^2 = 8\pi bt^{-2} + dt^{-2}, \tag{72}
\]

i.e. one equation with 3 unknowns.

If we want to integrate equation (73) we have to take a decision on the behavior of the constant $h$.

Taking $ch = \text{const.} \approx h = \frac{4c}{\pi}$ then $\frac{h}{B} = -\frac{c}{B}$ yielding:

\[
\frac{4\theta'}{\theta} - 3 \left[ \frac{c'}{c} + \frac{h'}{h} \right] + 3(\omega + 1)\kappa \frac{t}{t} - \frac{15d}{4\pi^3} \frac{c'[t + c]}{3(\omega + 1)\kappa} - 2 \frac{c'}{c} = 0. \tag{74}
\]

Also if $\rho = a\theta^4$ and $\rho = \frac{b}{\pi^2} \Rightarrow \frac{b}{\pi^2} = a\theta^4$ we have:

\[
k_B\theta = \left( \frac{15c^3(t)h^3(t)b^{1/4}}{\pi^2Bt^2} \right)^{1/4} = \left( \frac{15A^3b^1}{\pi^2B} \right)^{1/4} t^{-1/2}. \tag{75}
\]

And substituting into the previous equation we get:

\[
- \frac{2}{t} \left[ \frac{3(\omega + 1)\kappa}{t} - \frac{15d}{4\pi^3} \frac{c'[t + c]}{3(\omega + 1)\kappa} \right] - 2 \frac{c'}{c} = 0, \tag{76}
\]

and simplifying

\[
- \frac{2}{t} \left[ \frac{3(\omega + 1)\kappa}{t} - \frac{d}{4\pi b} \left[ \frac{c'}{c} + \frac{1}{t} \right] \right] - 2 \frac{c'}{c} = 0, \tag{77}
\]

therefore we get a very simple differential equation:

\[
\frac{c'}{c} = \left[ \frac{12\pi b(\omega + 1)\kappa - 8\pi b - d}{8\pi b + d} \right] \frac{1}{t}, \tag{78}
\]

from this we get $f = K_\kappa t^\kappa$ where $\kappa = \left( \frac{8\pi b + d}{3} \right)^{1/2}$ and substituting in (71) together with $G = Be^2$ we get

integrating it we obtain easily:

\[
c = K_\xi t^\xi \tag{79}
\]

where $\xi = \left[ \frac{12\pi b(\omega + 1)\kappa - 8\pi b - d}{8\pi b + d} \right]$. We can consider another possibility. Take the group of governing quantities $\varphi = \{K_\kappa, A, t\}$ where $K_\kappa$ is the proportionality constant obtained from $f = K_\kappa t^\kappa$ and $A$ is the constant establishing the relation between $h$ and $c$. The results obtained by means of the gauge relations are:

\[
G \propto K_\kappa^6 A^{-1} t^{6\kappa - 4}, \quad c \propto K_\kappa t^{\kappa - 1}, \quad h \propto K_\kappa^{-1} A t^{1-\kappa}, \quad k_B\theta \propto K_\kappa^{-1} A t^{-\kappa}, \quad \rho \propto K_\kappa^{-4} A t^{-4-\kappa}, \quad m_i \propto K_\kappa^3 A^{-3+\kappa}, \quad \Lambda \propto K_\kappa^{-2} t^{-2-\kappa} \tag{80}
\]

where $m_i$ comes from the energy density definition $\rho_E = \frac{nm_i c^2}{f}$ ($n$ stands for the particles number). We can check that we recover the general covariance property $\frac{\text{d}g}{\text{d}f} = \text{const.}$ if $\kappa = \frac{1}{2}$. Similarly we can see that the following relations are satisfied: $\rho = a\theta^4$, $\rho = Af^{-4}$ (equivalent to $\text{div}(T_i^i) = 0$), $\Lambda \propto f^{-2}$ and $f = ct$ (no horizon problem).
With the value \( \kappa = \frac{1}{3} \) we get
\[
\begin{align*}
  c & \propto t^{-1/2}, \quad h \propto t^{1/2}, \quad G \propto t^{-1}, \quad k_B \theta \propto t^{-1/2} \\
  f & \propto t^{1/2}, \quad \rho \propto t^{-2}, \quad m_i \propto t^{1/2}
\end{align*}
\] (81)

To obtain this solution we have needed three hypotheses:
1. \( G/c^2 = B \),
2. \( \Lambda \approx t^{-2} \),
3. \( c h = \text{const.} \)

Some of them are very restrictive since they induce a scaling solution (power law solution) and as we can see we have obtained the same solution than in the previous subsection for a model with \( \omega = 1/3 \) and \( \beta = 0 \), i.e. with radiation predominance and without matter creation. But, How it is possible?. We have tried to solve eq. (16) i.e. a model that, in principle, it does not verify the condition \( \text{div}(T^i_j) = 0 \) however our solution verifies such condition. As we will see in the last section, the DA always is related to conservation principles, for this reason in spite of working with eq. (16) our hypotheses obtained through DA have taken us to solve our second model which verifies the condition \( \text{div}(T^i_j) = 0 \).

Furthermore, we must emphasize that with this tactic we have obtained less information since our solution is only valid for the case \( \omega = 1/3 \), while with the naive method, Pi-theorem, the solution is valid for all kind of matter.

Therefore our first dimensional method, the simplest one, needs fewer simplifying hypotheses and its solution is much more general than the obtained one with this other dimensional method.

\section*{IV. LIÈRE METHOD}

As we have seen earlier, the \( \pi - \text{monomia} \) is the main object in dimensional analysis. It may be defined as a product of quantities which are invariant under changes of fundamental units. \( \pi - \text{monomia} \) are dimensionless quantities, their dimensions are equal to unity. Dimensional analysis has the structure of a Lie group\(^{20}\). The \( \pi - \text{monomia} \) are invariant under the action of the similarity group. On the other hand, we must mention that the similarity group is only a special class of the mother group of all symmetries that can be obtained using the Lie method. For this reason, when one uses dimensional analysis, only one of the possible solutions to the problem is obtained.

As we have been able to find a solution through dimensional analysis, it is possible that there are other symmetries of the model, since dimensional analysis is a reminiscent of scaling symmetries, which obviously are not the most general form of symmetries. Hence, we shall study the model through the method of Lie group symmetries, showing that under the assumed hypotheses there are other solutions of the field equations. In this section we shall show how the lie method allows us to obtain different solutions for the field equations. In particular we seek the forms of \( G \) and \( c \) for which our field equations admit symmetries i.e. are integrable (see\(^{21-30}\)).

An alternative use of the Lie groups have been performed by M. Szydlowki et. al.\(^{31-32}\) where they study the Friedman equations in order to find the correct equation of state following pioneer works of Collins\(^{33}\).

In order to use the Lie method, we rewrite the field equations as follows. From (21) – (22), we obtain
\[
2\frac{f''}{f} - 2\left(\frac{f'}{f}\right)^2 = -\frac{8\pi G}{c^2} (p + \rho),
\] (82)
and therefore
\[
2(H)' = -\frac{8\pi G}{c^2} (p + \rho).
\] (83)

From equation (23), we can obtain
\[
H = -\frac{\rho'}{3((\omega + 1)\rho)},
\] (84)
therefore
\[
\left(\frac{\rho'}{\rho}\right)' = 12\pi (\omega + 1)^2 \frac{G}{c^2} \rho.
\] (85)

Taking \( 12\pi (\omega + 1)^2 = A \) and then expanding, we obtain
\[
\rho'' = \frac{\rho'^2}{\rho} + A \frac{G}{c^2} \rho'.
\] (86)

If we consider the case in which there is matter creation, the resulting equation to study is now:
\[
\rho'' = \frac{\rho'^2}{\rho} + \tilde{A} \frac{G}{c^2} \rho'.
\] (87)

with \( 12\pi (\omega + 1)^2 (1 - \beta)^2 = \tilde{A} \). Therefore, we obtain the same equation that we have obtained, namely equation (86) except the constant \( \tilde{A} \), that incorporates all the parameters that controls the matter creation.

Now, we apply the standard Lie procedure to this equation. A vector field \( X \)
\[
X = \xi(t, \rho) \partial_t + \eta(t, \rho) \partial_\rho,
\] (88)

is a symmetry of (86) iff
Similarity versus Symmetries.

\[-\xi f_t - \eta f_\rho + \eta t + (2\eta_\rho - \xi_t) \rho' + (\eta_{pp} - 2\xi_{t\rho}) \rho'^2 - \xi_{pp} \rho'^3 +

+ (\eta_\rho - 2\xi_t - 3\rho' \xi_\rho) f - [\eta_t + (\eta_\rho - \xi_t) \rho' - \rho'^2 \xi_\rho] f_\rho' = 0. \quad (89)\]

By expanding and separating (89) with respect to powers of \(\rho'\), we obtain the overdetermined system:

\[
\begin{align*}
\xi_{pp} + \rho^{-1} \xi_\rho &= 0, \\
\eta_{pp} - 2\xi_{t\rho} + \rho^{-2} \eta - \rho^{-1} \eta_\rho &= 0, \\
2\eta_\rho - \xi_t - 3A \frac{G}{c^2} \rho^2 \xi_\rho - 2\rho^{-1} \eta_\rho &= 0, \\
\eta_t - A \left( \frac{G'}{c^2} - 2G \frac{c'}{c^3} \right) \rho^2 \xi - 2\eta A \frac{G}{c^2} \rho + (\eta_\rho - 2\xi_t) A \frac{G}{c^2} \rho^2 &= 0. \quad (93)
\end{align*}
\]

Solving (90-93), we find that

\[
\xi(t, \rho) = -2ct + a, \quad \eta(t, \rho) = (bt + d) \rho, \quad (94)
\]

subject to the constrain

\[
\frac{G'}{G} = 2 \frac{c'}{c} + \frac{bt + d - 4e}{2ct - a}, \quad (95)
\]

with \(a, b, e,\) and \(d\) as constants. In order to solve (95), we consider the following cases.

A. Case I: \(b = 0\) and \(d - 4e = 0\)

In this case, the solution (95) reduces to

\[
\frac{G'}{G} = 2 \frac{c'}{c} \Rightarrow \frac{G}{c^2} = B = \text{const}. \quad (96)
\]

which means that “constants” \(G\) and \(c\) vary but in such a way that the relation \(\frac{G}{c^2}\) remains constant.

The solution obtained through Dimensional Analysis needs to make this relations as hypothesis in order to obtain a complete solution for the field equations. This case shows us that such hypothesis is correct (at least has mathematical sense).

The knowledge of one symmetry \(X\) might suggest the form of a particular solution as an invariant of the operator \(X\) i.e. the solution of

\[
\frac{dt}{\xi(t, \rho)} = \frac{d\rho}{\eta(t, \rho)}, \quad (97)
\]

this particular solution is known as an invariant solution (generalization of similarity solution), therefore the energy density is obtained as

\[
\frac{dt}{-2ct + a} = \frac{d\rho}{4\rho} \Rightarrow \rho = \frac{1}{(2ct - a)^2}, \quad (98)
\]

for simplicity we adopt

\[
\rho = \rho_0 t^{-2}, \quad (99)
\]

Once we have obtained \(\rho\), we can obtain \(f\) (the scale factor) from

\[
\rho = A_\omega f^{-3(\omega + 1)} \Rightarrow f = (A_\omega t)^{\frac{2}{3(\omega + 1)}}, \quad (100)
\]

in this way we find \(H\) and from eq. (22), we obtain the behaviour of \(\Lambda\) as:

\[
c^2 \Lambda = 3H^2 - \frac{8\pi G}{c^2} \rho, \quad (101)
\]

and therefore,

\[
\Lambda = (3\beta^2 - 8\pi B \rho_0) \frac{1}{c^2 t^2} = \frac{l}{c^2 t^2}. \quad (102)
\]

If we replace all these results into eq. (24), then we obtain the exact behaviour for \(c\), i.e.,

\[
- \left( \frac{1}{t} + \frac{c'}{c} \right) \lambda = \frac{c'}{c}, \quad (103)
\]

where \(\lambda = \frac{l}{c^2 \rho_0}\), with \(\lambda \in \mathbb{R}^+\), i.e. is a positive real number and thus,

\[
c = c_0 t^{-\alpha}, \quad (104)
\]

with \(\alpha = \left( \frac{\lambda}{1 + \lambda} \right)\).
Hence, in this case we have found that (see fig.1):

\[ G = G_0 t^{-\alpha}, c = c_0 t^{-\alpha}, \Lambda = \Lambda_0 t^{-2(\alpha-1)}, \]
\[ f = (A_\omega t)^{\frac{\omega}{\omega-1}}, \rho = \rho_0 t^{-2}. \]  
\[ (105) \]

This is the solution that we have obtained with dimensional analysis in the previous section.

B. Case II, \( b = a = 0 \)

In this case, we find that

\[ \frac{G}{c^2} = \tilde{B}t^{\alpha}, \]  
\[ (106) \]

where \( \alpha = \delta - 2 \) and \( \delta = \frac{d}{2\omega} \). On following the same procedure as above, we find that

\[ \frac{dt}{\xi} = \frac{d\rho}{\eta} \implies \rho = \rho_0 t^{-\delta}, \]  
\[ (107) \]

we must impose the condition \( \text{sign}(d) = \text{sign}(e) \), i.e., \( \delta \in \mathbb{R}^+ \), in order that the solution has some physical meaning that the energy density is a decreasing function of time \( t \). It is observed that if \( d = 4e \) then we obtain same solution that the obtained one in the case I. The scale factor is found to be

\[ f = K_f t^{\frac{\delta}{\omega-1}}, \]  
\[ (108) \]

where \( K_f \) is an integration constant, and therefore, the Hubble parameter is:

\[ H = \frac{\delta}{3(\omega + 1) t}, \]  
\[ (109) \]

which is similar to the scale factor obtained in case I. To obtain the behaviour of the “constants” \( G, c \) and \( \Lambda \), we follow the same steps as in case I, i.e., from

\[ c^2 \Lambda = 3H^2 - \frac{8\pi G}{c^2} \rho, \]  
\[ (110) \]

we obtain the behaviour of \( \Lambda \) being:

\[ \Lambda = \frac{l}{c^2 t^2}, \]  
\[ (111) \]

where, \( l = (K_1 - K_2), K_1 = \frac{c^2}{3(\omega + 1)} \) and \( K_2 = 8\pi \rho_0 \tilde{B} \) i.e., \( l \in \mathbb{R}^+ \). Therefore,

\[ \Lambda' = \frac{2l}{c^2 t^2} \left( \frac{c'}{c} + \frac{1}{t} \right) \]  
\[ (112) \]

If we substitute all this results into the next equation

\[ \frac{\Lambda'c^4}{8\pi G\rho} + \frac{G'}{G} - 4\frac{c'}{c} = 0, \]  
\[ (113) \]

we obtain an ODE for \( c \), i.e.,

\[ \frac{c'}{c} (\lambda - 2) = - (\lambda - 2 + \delta) \frac{1}{t} \]  
\[ (114) \]

where, \( \lambda = \left( -\frac{t}{4\pi \rho_0 \tilde{B}} \right), \lambda \in \mathbb{R}^-, \) which leads to

\[ c = c_0 t^{-\alpha} \]  
\[ (115) \]

with \( \alpha = \left( 1 + \frac{\delta}{\lambda - 2} \right) \) such that \( \alpha \in [0, 1) \). In this way we can find the rest of quantities:

\[ G = G_0 t^{-2(\alpha+1)+\delta}, \quad \Lambda = \Lambda_0 t^{-2(\alpha-1)}, \]  
\[ (116) \]

note that \( \alpha < 1 \). The case \( \alpha = 1 \Leftrightarrow \delta = 0 \) is forbidden and \( \alpha = 0 \) brings us to the limiting case of the \( G, \Lambda \) variable cosmologies.

We notice that this solution is very similar to the case I but in this case all the parameters are perturbed by \( \delta \) and more important is the result, \( \frac{G}{c^2} = \tilde{B}t^{\alpha} \) (see figs.2, 3 and 4).

C. Case III, \( b = c = 0 \)

Following the same procedure as above, we find in this case that such restrictions imply \( \xi(t, \rho) = a, \eta(t, \rho) = d\rho \) and therefore:

\[ \frac{G'}{G} = 2\frac{c'}{c} - \frac{d}{a}, \]  
\[ (117) \]

which brings us to:

\[ \frac{G}{c^2} = K \exp(-\alpha t), \]  
\[ (118) \]

where \( \frac{d}{a} = \alpha \) and note that \( [K] = [B] \) i.e has the same dimensional equation,

\[ \frac{dt}{\xi(t, \rho)} = \frac{d\rho}{\eta(t, \rho)} \implies \frac{dt}{\rho} = \frac{d\rho}{d\rho} \implies \rho = \rho_0 \exp(\alpha t), \]  
\[ (119) \]

this expression only has sense if \( \alpha \in \mathbb{R}^- \), note that \( [\alpha] = T^{-1} \).

The scale factor \( f \) satisfies the relationship:

\[ \rho = A_\omega f^{-3(\omega+1)} \implies f = K_f \exp(\alpha t) \frac{1}{\rho_0}, \]  
\[ (120) \]

that is to say, it is a growing function without singularity.

In this way, we find that

\[ H = -\frac{\alpha}{3(\omega + 1)} = \text{conts.} \quad H > 0. \]  
\[ (121) \]

The cosmological “constant” is obtained as

\[ c^2 \Lambda = \frac{\alpha^2}{3(\omega + 1)^2} - 8\pi K \rho_0 \implies c^2 \Lambda = l, \]  
\[ (122) \]
FIG. 1: We see the behavior of $G, c$ and $\Lambda$ for the first class of solutions for different values of $\alpha$: $\alpha = 0.5$ (solid curve), $\alpha = 0.9$ (dotted curve), $\alpha = 0.3$ (matter era) (dashed curve). In all cases the constants are decreasing functions.

FIG. 2: Time variation of $c(t)$ for the second class of solutions for different values of $\alpha$: $\alpha = 0.5$ (solid curve), $\alpha = 0.9$ (dotted curve), $\alpha = 0.1$ (dashed curve) and $\alpha = 0.000001$ (long dashed curve), the last solution describes the case $c(t) = \text{const.}$

FIG. 3: The variation of the gravitational “constant” $G(t)$, for different values of $\alpha$ and $\delta$ : $\alpha = 0.5$ and $\delta = 1$ (solid curve), $\alpha = 0.9$ and $\delta = 1$ (dotted curve), $\alpha = 0.1$ and $\delta = 1$ (dashed curve) and $\alpha = 0.000001$ and $\delta = 5$ (long dashed curve), the last curve describes a growing solution.

FIG. 4: Time variation of $\Lambda(t)$ for the second class of solutions for different values of $\alpha$: $\alpha = 0.5$ (solid curve), $\alpha = 0.9$ (dotted curve), $\alpha = 0.1$ (dashed curve) and $\alpha = 0.000001$ (long dashed curve). In all cases, $\Lambda(t)$ is a decreasing function.

and hence,

$$c = K \exp(c_0 t),$$

where $c_0 = \frac{\alpha}{(\pi K \rho_0 + 2)}$ with $c_0 \in \mathbb{R}^-$ since $\alpha \in \mathbb{R}^-$, that is, $c$ is a decreasing function on time $t$.

In this case, we have found

$$c = K \exp(c_0 t), \quad G = G_0 \exp((-\alpha + 2c_0) t),$$

$$\Lambda = l \exp(c_0 t)^{-2},$$

therefore the solutions for this case are (see fig. 5):

$$G = G_0 \exp((-\alpha + 2c_0) t), \quad c = K \exp(c_0 t),$$

$$\Lambda = l \exp(c_0 t)^{-2}, \quad \rho = \rho_0 \exp(\alpha t),$$

$$f = K_f \exp(\alpha t)^{\frac{1}{p-1}}.$$ (128)

We have only obtained three solutions but playing with the constants $a, b, d$ and $e$ more solutions can be obtained (without forgetting that we are only interested in solutions with physical sense). As we indicated in the introduction and throughout all the paper, we have obtained in the first of our solutions (Case 1), the same solution that the obtained one with the dimensional methods. In
this occasion we have not needed to make a previous hypothesis in order to arrive a complete solution, this is therefore one of the advantages of this method as opposed to the dimensional one. We have also studied two other cases which can be considered as physically relevant solutions since \( f \) is a growing function on time and \( \rho \) is a decreasing function on time. They could describe very early cosmological solutions (inflationary ones).

We see that with this sophisticated method we can only solve the models 2 and 3 in which the energy density is expressed in a generic form \( i.e. \) we cannot consider the complicated case in which appears the Planck’s constant \( h \) (the disadvantage). Although we have not been excessively scrupulous (formal in the procedure (see appendix)) the method becomes more and more complicated in function of the complexity of the equation under study.

V. WHY THE DIMENSIONAL METHOD IS SO WONDERFUL AND CONCLUSIONS.

As have been pointed out by Carr and Coley, the existence of self-similar solutions (Barenblatt and Zeldovich) is related to conservation laws and to the invariance of the problem with respect to the group of similarity transformations of quantities with independent dimensions. This can be characterized within general relativity by the existence of a homotetic vector field and for this reason one must distinguish between geometrical and physical self-similarity. Geometrical similarity is a property of the spacetime metric, whereas physical similarity is a property of the matter fields (our case). In the case of perfect fluid solutions admitting a homotetic vector, geometrical self-similarity implies physical self-similarity.

As we show in this section as well as in previous works, the assumption of self-similarity reduces the mathematical complexity of the governing differential equations. This makes such solutions easier to study mathematically. Indeed self-similarity in the broadest Lie sense refers to an invariance which allows such a reduction.

Perfect fluid space-times admitting a homotetic vector within general relativity have been studied by Eardley. In such space-times, all physical transformations occur according to their respective dimensions, in such a way that geometric and physical self-similarity coincide. It is said that these space-times admit a transitive similarity group and space-times admitting a non-trivial similarity group are called self-similar. Our model \( i.e. \) a flat FRW model with a perfect fluid stress-energy tensor has this property and as already have been pointed out by Wainwright, this model has a power law solution.

Under the action of a similarity group, each physical quantity \( \phi \) transforms according to its dimension \( q \) under the scale transformation. For space-times with a transitive similarity group, dimensionless quantities are therefore spacetime constants. This implies that the ratio of the pressure of the energy density is constant so that the only possible equation of state is the usual one in cosmology \( i.e. \) \( p = \omega \rho \), where \( \omega \) is a constant. In the same way, the existence of homotetic vector implies the existence of conserved quantities.

In this paper we have studied the behaviours of time-varying “constants” \( G, c \) and \( \Lambda \) in a perfect fluid model. We began reviewing the scaling solution obtained through dimensional analysis.

To obtain this solution, we imposed the assumption, \( \text{div}(T^i_j) = 0 \), from which we obtained the dimensional constant \( A_0 \) that relates \( \rho \propto f^{-3(\omega+1)} \) and the relationship \( G/c^2 = \text{const.} = B \) remaining constants for all value of \( t \), \( i.e. \) \( G \) and \( c \) vary but in such a way that \( G/c^2 \) remain constant. With these two hypothesis, we have obtained a scaling solution for all the quantities. In this context, the solution obtained through dimensional analysis show us that the “constants” \( G, c \) and \( \Lambda \) are decreasing functions of time, but in this case decrease slowly than in the radiation predominance era, while \( \rho \) and \( f \) behave as in the FRW model solving the horizon problem.
Therefore the DA is very simple and allows us to study very complex models. Nevertheless it has very serious limitations, we are restricted to models that verify the conservation principle $div(T_i^j) = 0$ and work only with flat models i.e. $k = 0$, for models with $k \neq 0$ there are not scaling solutions.

Since we have been able to found a solution through similarity, i.e. through dimensional analysis, it is possible that there are other symmetries of the model, since dimensional analysis is a reminiscent of scaling symmetries, which obviously are not the most general form of symmetries. Therefore, we studied the model through the method of Lie group symmetries, showing that under the assumed hypotheses, there are other solutions of the field equations (an advantage of this method).

The first solution obtained is the already obtained one through similarity, but in this case we have showed the condition $G/c^2$ arises as a result and not as an ad-hoc condition. We also have studied two other cases which can be considered as physically relevant solutions since $f$ is a growing function on time and $\rho$ is a decreasing function on time. They could describe very early cosmological solutions (inflationary ones).

The lie method maybe is the most powerful but has drawbacks, it is very complicate. We cannot, we do not know, solve the most general of our models, but it allows us to find more solutions than with de DA method and without the necessity of making previous hypotheses.

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**APPENDIX A: COMPLETE LIE METHOD FOR THE CASE I (IV A)**

In this section we try to show how the Lie method works in order to obtain a complete solution to equation

$$ \rho'' = \frac{\rho'^2}{\rho} + K\rho^2, \quad (A1) $$

taking $K = AB$ with $12\pi(\omega + 1)^2 = A$ and $B = G/c^2 = const$. We will follow the ways of the canonical coordinates.

The first step is to determine the admissible algebra $L_2$. As in section (IVA) the standard procedure brings us to obtain the overdetermined system:

\[
\begin{align*}
\xi_{\rho \rho} + \rho^{-1} \xi_\rho &= 0, \quad (A2) \\
\eta_{\rho \rho} - 2\xi_\eta + \rho^{-2} \eta - \rho^{-1} \eta_\rho &= 0, \quad (A3) \\
2\eta_\rho - \xi_\eta - 3K\rho^2 \xi_\rho - 2\rho^{-1} \eta_\rho &= 0, \quad (A4) \\
\eta_{\eta \eta} - 2\eta K\rho + (\eta_\rho - 2\xi_\eta) K\rho^2 &= 0. \quad (A5)
\end{align*}
\]

Solving (A2-A5), we find that

$$ \xi(t, \rho) = b + at, \quad \eta(t, \rho) = -2a\rho, \quad (A6) $$

where $a$ is a constant. Thus equation (A1) admits two linearly independent operators

$$ X_1 = \partial_t, \quad X_2 = t\partial_t - 2\rho \partial_\rho, \quad (A7) $$

The second step consists of determining the type of algebra that form these two operators. We see that

$$ [X_1, X_2] = X_1 \quad \text{with} \quad \xi_{\eta \eta} - 2\eta_\xi = -2\rho \neq 0, \quad (A8) $$

therefore $X_1$ and $X_2$ span a solvable non abelian algebra type III (see Ibragimov (1999) for details, theorem 12.6. page. 287).

The next step consists in determining the integrating change of variable. Upon introducing the canonical variables for $X_1 (x_1 s = 0, x_1 u = 1)$ given by

$$ s = \rho \quad \text{and} \quad u(s) = t, \quad (A9) $$

and therefore

$$ t = u(s) \quad \rho = s. \quad (A10) $$

We transform the operator $X_1$ and $X_2$ to the form:

$$ X_1 = \partial_u, \quad X_2 = -2s\partial_s + u\partial_u. \quad (A11) $$

their difference from the corresponding operators of type III by the factor $-2$ in $X_2$ does no hinder the integration.

We rewrite equation (A1) in the new variables (A9) obtaining:

$$ u'' = -\frac{(1 + Ks^3u^2)}{s} u', \quad (A12) $$

the canonical variables allow us to reduce the order of our ode. If we follow this tactic, then it is obtained the next first order ode (type Bernoulli),

$$ z' = \frac{s}{s - Kz^3 s^2}, \quad (A13) $$

where $s = \rho$ and $z = du/ds = (1/\rho')$ and which solution is:

$$ z(s) = \pm \frac{1}{s\sqrt{2Ks + C_1}}. \quad (A14) $$

The last step consists in obtaining the solution in the original variables, being this:

$$ \rho' = \rho\sqrt{2K\rho + C_1} \quad (A15) $$

this last ode has the following solution

$$ t + 2\arctan\left(\frac{2\rho + C_2}{C_1}\right) + C_2 = 0 \quad (A16) $$
and therefore
\[ \rho = \frac{1 + \tan \left( \frac{t \xi}{2K} \right)^2}{2K \xi^2}. \]  
(A17)

Therefore this is the most general solution to equation (A1) but as it can be observed this seems to be non-physical, for this reason we will seek more change of variables in order to find a better (physical) solutions.

We would like to emphasize how the Lie procedure brings us to solve equation (A13). Of course this ode can be trivially integrated but, at this point we prefer to integrate it following two tedious ways. In the first of them we study equation (A13) through the Lie method i.e. following all the procedure shown above and with the second method we try to find and integration factor (Lie integration factor) which brings us to a direct integration of our equation.

Therefore studying equation (A13) through the Lie method it is obtained the overdetermined system:

\[ \eta + (\xi - \xi_x) \left( \frac{-z}{s} - Kz^3s^2 \right) - \xi_x \left( \frac{-z}{s} - Kz^3s^2 \right)^2 - \]

\[ -\xi \left( \frac{z}{s^2} - 2Kz^3s \right) - \eta \left( \frac{-1}{s} - 3Kz^2s^2 \right) = 0 \]  
(A18)

Solving (A18), we find that equation (A13) admits the following linearly independent operators

\[ X_1 = z^3s^2 \partial_z, \quad X_2 = (z - 2Kz^3s^3) \partial_z, \]

\[ X_3 = \partial_x - \frac{z}{s} \partial_z, \quad X_4 = s \partial_s - \frac{3}{2}z^2 \partial_z. \]  
(A19)

With one of these fields, we obtain a new set of variables which brings us to obtain a new ode which will be trivially integrated. For example, from \( X_3 \) it is obtained

\[ i = z(s)s, \quad \text{and} \quad h(i) = s, \]  
(A20)

and the inverse transformation is:

\[ s = h(i) \quad \text{and} \quad z(s) = \frac{i}{h(i)}, \]  
(A21)

in these new variables equation (A13) yields

\[ h' = \frac{-1}{Kz^3}, \]  
(A22)

which is integrated by quadratures

\[ h = \frac{1}{2Kz^2} + C_1 \]  
(A23)

Now we obtain the solution of the equation in the original variables, finding that

\[ z = \pm \frac{1}{s\sqrt{2ks + C_1}} \]  
(A24)

Our second way consists in obtaining a Lie integration factor \( \mu \) (for our ode (A13)) in such a way that \( \mu \cdot \text{ode} \) is an exact equation. We find the following integration factor:

\[ \mu = \frac{1}{z^3s^2} \]  
(A25)

which bring us to the following solution

\[ z = \pm \frac{1}{s\sqrt{2ks + C_1}} \]  
(A26)

Now we see other changes of variables induced by other operators. If for example we calculate the canonical coordinates it is reduced the order in the following way i.e.

\[ z' = z(1 + zs + (Ks - 2) s^2z^2), \]  
(A30)

which is first-order ode type Abel, where \( z = \frac{du}{ds} \),

\[ s = \rho t^2 \quad \text{and} \quad z = \frac{1}{t^2(\rho t + 2\rho)}. \]  
(A31)

The solution to eq. (A30) is:

\[ C_1 - \frac{\sqrt{-4z^2s^2 + 2Ks^3z^2 + 4zs - 1}}{sz} + 2 \arctan \left( \frac{1 - 2zs}{\sqrt{-4z^2s^2 + 2Ks^3z^2 + 4zs - 1}} \right) = 0. \]  
(A32)

To end we will see another transformation. We are interesting into knowing the transformation that produces the operator \( X = t \partial_t + \rho \partial_\rho \). Knowing that the canonical variables are obtained by solving the equations \( (Xs = 0, Xu = 1) \) we find that

\[ s = \frac{\rho}{t} \quad \text{and} \quad u(s) = \ln t, \]  
(A33)

and therefore

\[ t = e^{u(s)} \quad \text{and} \quad \rho = se^{u(s)} \]  
(A34)
with these new variables, equation (A1) simplifying yields
\[ u'' = -u'^2 - \frac{u'}{s} - \left(1 + Ke^{3s(s)}\right) su^3 \] (A35)
which has a particular solution
\[ u = \frac{1}{3} \ln \left(\frac{2}{Ks}\right) \] (A36)
finding in this way that in the original variables it yields
\[ \rho = \frac{2}{Kt^2} \] (A37)
which is the solution obtained in section (IV A).

Once again if one insist in solving equation (A1) through DA (applying the Pi-theorem, see\cite{18} for details) it is found that with respect to the dimensional base \( \mathcal{B} = (\rho, T) \) each quantity has the following new dimensional equation \( [\rho] = \rho, [t] = t \) and \( [K] = \rho^{-1}t^{-2} \). Therefore, we find in a trivial way that:
\[
\begin{pmatrix}
\rho \\
K \\
t
\end{pmatrix}
\begin{pmatrix}
1 & -1 & 0 \\
0 & -2 & 1
\end{pmatrix}
\Rightarrow
\rho \approx \frac{1}{Kt^2}.
\] (A38)

It is observed that if for example we work with a new dimensional base like \( \mathcal{B} = (L, M, T) \), i.e. the usual one, the above quantities have the following new dimensional equations: \( [\rho] = L^{-3}M^{-1}T^{-2}, [t] = T \) and \( [K] = LM^{-1} \). Therefore, we have a new overdetermined system of equations which determine \( \rho \),
\[
\begin{pmatrix}
\rho \\
K \\
t
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
M & -1 & 0 \\
T & 0 & 1
\end{pmatrix}
\Rightarrow
\rho \approx \frac{1}{Kt^2}.
\] (A39)

As it is observed we have obtained the same solution than the obtained one in the previous sections, but now making only a simple dimensional considerations.

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