Exact results and scaling properties of small-world networks

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We study the distribution function for minimal paths in small-world networks. Using properties of this distribution function, we derive analytic results which greatly simplify the numerical calculation of the average minimal distance, \( \bar{\ell} \), and its variance, \( \sigma^2 \). We also discuss the scaling properties of the distribution function. Finally, we study the limit of large system sizes and obtain some analytic results.

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Recently Watts and Strogatz \cite{1} have studied a class of networks which can be ‘tuned’ from an ordered to a random state by varying a single parameter. For a range of parameter values, they found that the networks resembled ordered networks locally but random networks globally. They named this class of networks “small-world” networks because of their relevance to a well-known problem in sociology \cite{2}. Since their introduction, small-world networks and their properties have received considerable attention, \cite{3–18}, in part because of their possible applications to a broad range of systems, ranging from social networks \cite{2} to coupled oscillators \cite{3}.

Exact results and scaling properties of small-world networks has focused on the average minimal distance, \( \bar{\ell} \), and as a consequence obtain an

basic probability distributions for small-world networks, \cite{5,8,10,12,14,15}. In this work, we will focus on some

two scaling regimes has been the subject of debate \cite{8}. We start with a 1-

d period lat-

tice with \( L = 2N \) sites and nearest neighbor connections.

We then add short-cuts uniformly with probability \( p \) such that the average number of short-cuts added is \( x = pL \). We denote the distance between two sites, counted along the lattice using only nearest neighbor links, as the Euclidean distance. By contrast, the shortest distance between two sites, counted along any bond including short-cut bonds, is called the minimal distance.

Using these definitions, we now introduce the following probability functions: (i) \( P(n|m) \), the probability that two sites are separated by Euclidean distance \( n \) given that their minimal distance is \( m \); (ii) \( P(m|n) \), the probability that two sites have minimal separation \( m \) given that their Euclidean distance is \( n \); and (iii) \( Q(m) \), the probability that two randomly chosen sites have a minimal separa-

tion \( m \). Recently, Dorogovtsev et al \cite{13} have introduced two exactly solvable models similar to small-world networks. For these models they derive the explicit form of \( P(m|n) \), from which they obtain \( \bar{\ell} \) and other properties of their networks. In this Letter, we derive the general form of \( P(m|n) \) for small-world networks, and we confirm it numerically. Using this form, we derive an exact expression for \( \bar{\ell} \) and the variance of \( \bar{\ell} \), \( \sigma^2 \equiv \bar{\ell}^2 - \bar{\ell}^2 \). We also study the scaling properties of \( P(m|n) \) and obtain some approximate results for it in the limit of large \( L \).

Note that in describing the networks, we have considered the case of coordination number of \( 2k = 2 \) for each site. However, our arguments for the general form of \( P(m|n) \) are valid for arbitrary \( k \). For simplicity we will consider the case \( k = 1 \) in the rest of this Letter and generalizations to arbitrary \( k \) will be indicated as appropriate.

We begin by deriving the general form of \( P(n|m) \). First, since the minimal distance cannot exceed the Euclidean distance, \( P(n|m) = 0 \) for \( n < m \). For \( n > m \), the minimal path must use at least one short-cut. But taking a short-cut is equivalent to randomizing the position along the network, since the short-cuts are uniformly distributed. Hence, \( P(n|m) \) must be independent of \( n \) for all \( n > m \). Finally, for \( n = m \), it is not necessary to use any short-cuts in the minimal path; so the arguments invoked for \( n > m \) do not apply. Instead, \( P(n|m) \) is determined by the constraint that the probability distribution is normalized.

We now derive the general form of \( P(m|n) \). From elementary probability theory, we have

\[
\begin{align*}
P(n|m) * Q(m) &= \begin{cases} 
\frac{2}{L-1} P(m|n) & ; \quad n < N \\
\frac{1}{L-1} P(m|n) & ; \quad n = N
\end{cases}
\end{align*}
\]

From Eqn. (1) and the properties discussed in the previous paragraph, \( P(m < n|m) = f(m) \) is independent of \( n \), and \( P(m > n|m) = 0 \). Thus the general form of \( P(m|n) \) is

\[
P(m|n) = \Theta(n - m) f(m) + \left[ 1 - \sum_{m' < 1}^{n-1} f(m') \right] \delta_{m,n},
\]

1
where \( \Theta(x) \) is defined by \( \Theta(x) = 0 \) for \( x \leq 0 \) and \( \Theta(x) = 1 \) for \( x > 0 \). We have numerically confirmed the validity of this form, as shown in Fig. 1.

The fact that \( P(m|n) \) is completely determined by \( f(m) \) has some surprising consequences, regardless of the exact form of \( f(m) \). To explore these consequences, we examine some other properties of small-world networks. For example, besides \( \bar{\ell} \), the quantity \( \langle \ell(n) \rangle \), which is the average minimal distance separating two sites with Euclidean distance \( n \), has been discussed in the literature [4,5]. We can express both these quantities in terms of \( P(m|n) \) as follows:

\[
\langle \ell(n) \rangle = \sum_{m=1}^{n} m P(m|n) \quad (3)
\]

\[
\bar{\ell} = \frac{1}{L-1} \left[ \sum_{n=1}^{N-1} \sum_{m=1}^{n} m P(m|n) + \sum_{m=1}^{N} m P(m|N) \right] \quad (4)
\]

Similar expressions hold for \( \langle \ell^2(n) \rangle \), \( \langle \ell^3(n) \rangle \), and \( \bar{\ell}^2 \).

Substituting the form of \( P(m|n) \) [Eqn. (5)] into the expression for \( \bar{\ell} \) [Eqn. (4)], we obtain

\[
\bar{\ell} = \frac{1}{L-1} \left[ 2 \sum_{n=1}^{N-1} \sum_{m=1}^{n} m f(m) + 2 \sum_{n=1}^{N} n \left[ 1 - \sum_{m=1}^{n} f(m) \right] + \sum_{m=1}^{N-1} m f(m) + N \left[ 1 - \sum_{m=1}^{N-1} f(m) \right] \right] \quad (5)
\]

which can be simplified to give the following exact expression:

\[
\bar{\ell} = \frac{1}{L-1} \left[ \sum_{n=1}^{N-1} \left( \frac{n}{L} \right) + \sum_{n=1}^{N} \left( \frac{n}{L} \right) \right] + \frac{N}{L} \sum_{m=1}^{N-1} f(m)
\]

\[
= \frac{1}{L-1} \left[ \sum_{n=1}^{N-1} \left( \frac{n}{L} \right) + \sum_{n=1}^{N} \left( \frac{n}{L} \right) + \frac{N}{L} \left( \frac{N-1}{2} - \sum_{m=1}^{N-1} f(m) \right) \right]
\]

\[
= \frac{1}{L-1} \left[ \sum_{n=1}^{N-1} \left( \frac{n}{L} \right) + \sum_{n=1}^{N} \left( \frac{n}{L} \right) + \frac{N}{L} \left( \frac{N-1}{2} \right) \right]
\]

\[
= \frac{1}{L-1} \left[ \sum_{n=1}^{N-1} \left( \frac{n}{L} \right) + \sum_{n=1}^{N} \left( \frac{n}{L} \right) + \frac{N}{L} \left( \frac{N-1}{2} \right) \right]
\]

\[
= \frac{1}{L-1} \left[ \sum_{n=1}^{N-1} \left( \frac{n}{L} \right) + \sum_{n=1}^{N} \left( \frac{n}{L} \right) + \frac{N}{L} \left( \frac{N-1}{2} \right) \right]
\]

The surprising aspect of the above equations is that \( \bar{\ell} \) and \( \sigma^2 \), which are average properties of the entire network, are completely determined by the mean separation of ‘diametrically opposite sites’ (d.o.s.) \( \langle \ell(n) \rangle \), and its higher moments \( \langle \ell^2(N) \rangle \) and \( \langle \ell^3(N) \rangle \). This fact, in particular, makes for significant numerical simplification in the computation of \( \bar{\ell} \). Note that Eqns. (4) and (5) can readily be generalized to any \( k \) by performing the substitution \( L \rightarrow \lceil L/k \rceil \).

When the network has exactly one short-cut, we can calculate \( \bar{\ell} \) analytically using Eqn. (3). In this case, in the limit of large \( N \), we get \( \langle \ell(N) \rangle = \frac{2}{3} N \) and \( \langle \ell^2(N) \rangle = \frac{1}{2} N^2 \) which gives \( \bar{\ell} = \frac{5}{12} N \). As expected, this is in perfect agreement with the results obtained by Strang and Eriksen [12,20]. We have further confirmed Eqn. (3) by numerically computing \( \bar{\ell} \) using the following two procedures: (i) Averaging the minimal distance over all pairs of sites, and (ii) considering only pairs of d.o.s. and
Using Eqn. (3), the results, which are presented in Fig. 2, indicate that the two procedures are equivalent.

The results obtained so far have been independent of the functional form of $f(m)$. To gain further insight, we consider the scaling properties of $f(m)$, following the real-space renormalization group (RG) analysis of Newman and Watts [8]. This procedure consists of blocking pairs of adjacent sites while preserving the total number of short-cuts in the network. This gives for the transformed lattice: $N' = N/2$ and $p' = 2p$. We note the following features of this transformation [8]: (i) the geometry of the minimal paths is unchanged in almost all cases, and the number of site-pairs for which the geometry does change is negligible for large $L$ and small $p$, and (ii) the distance along the minimal path is halved, i.e. $m' = m/2$ for large $L$ and small $p$.

Now, by definition $f(m) = P(m|N)$ ($m < N$); thus to evaluate $f(m)$ we need to calculate the average number of minimal paths of length $m$ between two d.o.s. in the network. Furthermore we note that the RG-transformation maps two pairs of d.o.s. into a single pair. This fact, in conjunction with points (i) and (ii) above gives us

$$f'(\frac{m}{2}, \frac{N}{2}, 2p) = 2f(m, N, p) \quad (8)$$

For large $N$, taking the continuum limit, we can generalize the above expression to

$$f'(\frac{m\lambda}{\lambda}, \frac{N\lambda}{\lambda}, \lambda p) = \lambda f(m, N, p) \quad (9)$$

These observations can now be summarized in the following scaling form:

$$f(m, N, p) = \frac{1}{N} h(y, x) \quad (10)$$

where $y = \frac{m}{N}$, $x = 2pN$.

By fixing $x$, we have observed the scaling collapse of $f(m)$ for different values of $N$ and $p$. This is demonstrated numerically for $x = 10$ in Fig. 2. Our simulations indicate that for any given $x$, this scaling collapse holds for large enough $N$.

It is interesting to note that the scaling properties of $f(m)$ can be derived from the scaling form of $f(m)$. Using the definition of $\bar{f}$ (Eqn. (3)) and the scaling form for $f(m)$, we get

$$\bar{f} = \frac{L}{4} [1 - \int_0^1 dy (1 - y)^2 h(y, x)] \quad (11)$$

$$= L g(x) \quad (12)$$

which is consistent with the scaling form proposed in previous works. Similar scaling forms hold for $\bar{f}'$, $<\ell(N)>$, and $<\ell'(N)>$.

We now consider the limit of large system sizes such that $x \gg 1$. In this limit, we have observed numerically that we can approximate $f(m)$ by a gaussian distribution function:

$$f(m) = \frac{1}{\sqrt{2\pi} \sigma_g} e^{-\frac{(m - \mu_g)^2}{2\sigma_g^2}} \quad (13)$$

where $\mu_g$ and $\sigma_g^2$ are respectively the mean and variance of the distribution. The corresponding fit for $x = 250$ and $x = 500$ is shown in Fig. 4. Our simulations indicate that as $x$ increases, $(\mu_g/\sigma_g)$ also increases, as can be seen from the figure.

Using the gaussian approximation for $f(m)$, we are now able to calculate the function $<\ell(n)>$, which has

![Figure 3](image_url) This figure confirms the proposed scaling form of $f(m, N, p)$ (Eqn. (10)) for $x = pL = 10$ and system sizes: (a) $L = 500$ (○), (b) $L = 750$ (●), and (c) $L = 1000$ (▲). We have confirmed this scaling collapse for a wide range of $x$-values.

![Figure 4](image_url) $f(m)$ vs. $m$ for $x = pL = 250$ and $L = 2000$. The solid line is the gaussian fit to the calculated data. The inset shows the gaussian fit for $x = 500$ and $L = 2000$. Note that with increasing $x$, the gaussian becomes more sharply peaked.
FIG. 5. The mean distance \( <\ell(n)> \) between two sites having Euclidean separation \( n \) for \( x = pL = 250 \). Results are shown for: (a) numerical simulation (◦), and (b) analytic expression (Eqn. (14)) (solid line). The analytic expression is an excellent fit for \( x \gg 1 \).

We have derived some exact relations which will significantly aid computational efforts. We have obtained an approximate scaling form for this probability distribution in the limit of large system sizes. It is our hope that further efforts along these lines will provide a better understanding of the structure of small-world networks.

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