LOGARITHMIC DECOMPOSITION OF CONNECTIONS ON A RELATIVE PUNCTURED DISK

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Abstract. Let $C$ be an algebraically closed field of characteristic zero and $R = C[[t]]$ be the ring of power series. We study the structure of modules with connection on the relative punctured disk over $\text{Spec}(R)$. Under the existence of the strong Turrittin-Levelt-Jordan decomposition with a separated condition of irregular values, we show that such a connection decomposes into sum of a regular singular connection and a linear diagonalizable endomorphism with eigenvalues being the irregular values.

1. Introduction

Let $C$ be an algebraically closed field of characteristic 0 and $x$ be a variable. The formal punctured disk is by definition $\text{Spec} \ C((x))$, which is equipped with the logarithmic derivation $\vartheta := x \frac{d}{dx}$. After the works of Turrittin and Levelt, the structure of modules with connection on the formal punctured disk is well understood. Recall that a module with connection on $C((x))$ is a finite $C((x))$–vector space $E$ equipped with a derivation $\nabla$ obeying the Leibniz’s rule (see Definition 2.1). The category of all connections (with morphisms defined in a natural way) will be denoted by $\mathbf{MC}(C((x))/C)$. If $(E, \nabla)$ has regular singularity at 0, it is well known (see [Ma65] and [Del70]) that there uniquely exists a $C[[x]]$–lattice $\mathcal{E}$ for $E$ with respect to $\tau$. Such $\mathcal{E}$ is called Deligne-Manin lattice of $(E, \nabla)$. In the irregular case, B.Malgange [Mal96] established a canonical lattice which has the property generalizing of Deligne-Manin latices. The existence of these lattices is quite significant in the study of irregular connections, for example, it helps us to apply GAGA to meromorphic bundles, as already remarked in [Mal96].

The main structure theorem says that for an object $(E, \nabla)$ in $\mathbf{MC}(C((x))/C)$, the connection $\nabla$ decomposes as

$$\nabla = S + N,$$

where $S$ is a connection of $E$ and $N$ is a $C((x))$–linear nilpotent endomorphism of $E$, further, these two maps commute each other and, after an appropriate base change to $C((x^\frac{1}{2}))$, the connection $(E, S)$ decomposes into direct sum of rank one objects. This decomposition can be seen as an analog of the Jordan decomposition for $(E, \nabla)$. Indeed, when restricted to the full subcategory $\mathbf{MC}_{\text{rs}}(C((x))/C)$ of regular singular connections, we can establish an equivalence of this category with the category $\text{rep}_C(\mathbb{Z})$ of $C$–linear representations of the group $\mathbb{Z}$.

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To understand the full category $\mathbf{MC}(C((x))/C)$ of all connections, one needs to go further to a ramified extenion $C((x^\frac{1}{s}))$ of $C((x))$. Over an appropriate extension, $(E, \nabla)$ decomposes into direct sum of isotopic components, which are indexed by the classes of eigenvalues of the representing matrix modulo $\frac{1}{s} \mathbb{Z} + x^\frac{1}{s} C[x^\frac{1}{s}]$ (see Definition 2.10). This yields the logarithmic decomposition

$$\nabla = \nabla^{rs} + P_{\nabla},$$

where $\nabla^{rs}$ is a regular singular connection on $E$, while $P_{\nabla}$ is an $C((x))$–linear endomorphism, which is diagonalizable with eigenvalues being the irregular values (see Theorem 2.14 for precise conditions). The decomposition (1) allows us to define the Deligne-Manin lattice for $(E, \nabla^{rs})$ with respect to $\tau$ which is Deligne-Manin-Malgrange lattice of $(E, \nabla)$ as in Section 2.7.

This work aim to understand the structure of $\mathbf{MC}(R((x))/R)$, where $R = C[\lbrack t \rbrack]$ – the power series in variable $t$. It is the continuation of [HdST22], where the structure of $\mathbf{MC}_{rs}(R((x))/R)$ was studied in which the Deligne-Manin lattices are quite significant. Our objective is to look for the logarithmic decomposition for connections over $R((x))$. We show that under the existence of the strong Turrittin-Levelt-Jordan decomposition and invertible conditions of characters (the descent condition of \cite[Theorem 17.5.3]{ABC20}), such a connection over $R((x))$ admits a logarithmic decomposition. This decomposition allows us to find the Deligne-Manin-Malgrange lattice for connections over $R((x))$ via the Deligne-Manin lattice of regular singular part, see Proposition 4.10. The general case requires further study and is the subject of our future work.

The paper is organized as follows. In section 2 we review the classical Turrittin-Levelt theory for connections. We recall the main results which are needed for our work. According to the Turrittin-Levelt-Jordan decomposition in the sense of \cite{ABC20}, we then obtain the logarithmic decomposition which defines in a canonical way the regular singular part of a connection. In section 3 we extend the classical results to the case, when the connection are equipped with an action of a local artinian algebra. The ideal obtained in Section 2 and 3 is used to derive the main results of our work in Section 4. In Theorem 4.9 we employ the Turrittin-Levelt-Jordan decomposition of connections with some extra condition to establish the logarithmic decomposition

$$\nabla = \nabla^{rs} + P_{\nabla}$$

for connection over $R((x))$ which is much the same as the decomposition in Theorem 2.14 for connection over $C((x))$.

**Notation and conventions.**

1. In this text, $C$ stands for an algebraically closed field of characteristic zero.
2. Given a $C$-algebra $R$, we let $R((x)) := R[\lbrack x \rbrack][x^{-1}]$ and the derivation

$$\vartheta \left( \sum_{n \geq k} a_n x^n \right) = \sum_{n \geq k} n a_n x^n.$$

3. For any $C$-algebra $R$ and $s \in \mathbb{N}^*$, we let $x_s$ stand for a primitive $s$-th root of $x$.

The natural inclusion $R((x)) \to R((x_s))$ by sending $x$ to $(x_s)^s$. We extend $\vartheta$ to a
derivation on $R((x_s))$ and denote it by the same symbol. Thus we have

$$\vartheta = s^{-1}x_s \frac{d}{dx_s}.$$  

(4) Let $A_i \in \text{Mat}_{r_i}(R), \quad i = 1, \ldots, r$. We write the diagonal matrix of the form

$$\text{diag}(A_1, \ldots, A_r) = \begin{pmatrix}
A_1 & 0 & 0 & \cdots & 0 \\
0 & A_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & A_r
\end{pmatrix}.$$  

(5) Given any ring $R$, for each $f \in R$ and each positive integer $r$, let

$$J_r(f) = \begin{pmatrix}
f & 0 & \cdots & \cdots & 0 \\
* & \ddots & \ddots & & \vdots \\
0 & \ddots & \ddots & & \vdots \\
& \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & * & f
\end{pmatrix}$$

be the Jordan block of size $r$ and eigenvalue $f$, where $* \in \{0, 1\}$. For a multi-index of positive integers $r = (r_1, \ldots, r_\ell)$, let

$$J_r(f_1, \ldots, f_\ell) = \text{diag}(J_{r_1}(f_1), \ldots, J_{r_\ell}(f_\ell))$$

be the associated Jordan matrix.

(6) We fix a subset $\tau \subset C$ of representatives of the co-sets $C/\mathbb{Z}$.

2. Review of Levelt-Turrittin theory

2.1. Basic material. For the convenience of the reader and to ease referencing, we recall some standard definitions and results.

**Definition 2.1.** Let $R$ be any commutative ring. The category of connections, $\text{MC}(R((x))/R)$, has for

- **objects** those couples $(M, \nabla)$ consisting of a finite $R((x))$-module and an $R$-linear endomorphism $\nabla : M \to M$, called the derivation, satisfying Leibniz’s rule $\nabla(fm) = \vartheta(f)m + f\nabla(m)$, and the
- **arrows** from $(M, \nabla)$ to $(M', \nabla')$ are $C((x))$-linear morphisms $\varphi : M \to M'$ such that $\nabla'\varphi = \varphi\nabla$.

The category of logarithmic connections, $\text{MC}_{\log}(R[[x]]/R)$, has for

- **objects** those couples $(M, \nabla)$ consisting of a finite $R[[x]]$-modules and a $C$-linear endomorphism, called the derivation, $\nabla : M \to M$ satisfying Leibniz’s rule $\nabla(fm) = \vartheta(f)m + f\nabla(m)$, and
- **arrows** from $(M, \nabla)$ to $(M', \nabla')$ are $R[[x]]$-linear morphisms $\varphi : M \to M'$ such that $\nabla'\varphi = \varphi\nabla$.

We possess an evident $R$-linear functor

$$\gamma : \text{MC}_{\log}(R[[x]]/R) \longrightarrow \text{MC}(R((x))/R).$$
**Definition 2.2.** An object \( M \in \text{MC}(R((x))/R) \) is said to be regular-singular if it is isomorphic to a certain \( \gamma(M) \). The full category of \( \text{MC}(R((x))/R) \) whose objects are regular-singular shall be denoted by \( \text{MC}_{rs}(R((x))/R) \).

Given \( M \in \text{MC}_{rs}(R((x))/R) \), any object \( \mathcal{M} \in \text{MC}_{\log}(R[[x]]/R) \) such that \( \gamma(M) \simeq M \) is called a logarithmic model of \( M \). In case the model \( \mathcal{M} \) is, in addition, a free \( R[[x]] \)-module, we shall speak of a logarithmic lattice.

In the rest of this section, we shall consider the case \( R = C \). Recall that \( \tau \) is by definition the image of a section to the map \( C \longrightarrow C/Z \).

**Definition 2.3.** Let \((M, \nabla) \in \text{MC}_{rs}(C((x))/C)\) be given. A logarithmic lattice \( \mathcal{M} \in \text{MC}_{\log}(C[[x]]/C)\) of \((M, \nabla)\) is call Deligne-Manin with respect to \( \tau \) if all the exponents of \( \mathcal{M} \) lie in \( \tau \).

For \((M, \nabla) \in \text{MC}(C((x))/C)\), it is customary to call \( \dim_{C((x))} M \) its rank. Let \((M, \nabla) \in \text{MC}(C((x))/C)\) be of rank \( n \) and let \( m = \{m_1, ..., m_n\} \) be a basis. The \( n \times n \) matrix \( A = (a_{ij}) \) determined by

\[
\nabla m_j = \sum_{i=1}^{n} a_{ij} m_i.
\]

shall be denoted by \( \text{Mat}(\nabla, m) \). With respect to this basis, \( \nabla \) has the form

\[
\nabla = x \frac{d}{dx} + A.
\]

Let \( m' = \{m'_1, ..., m'_n\} \) be another \( C((x)) \)-basis of \( M \) and suppose

\[
m'_j = \sum_{i=1}^{n} t_{ij} m_i,
\]

where the matrix \( T = (t_{ij}) \) is invertible. Let \( B = \text{Mat}(\nabla, m') \). Then we have the gauge transformation rule:

\[
B = T^{-1} \partial T + T^{-1} AT.
\]

**2.2. Connections on ramified extensions of \( C((x)) \).** Given a positive integer \( s \), we consider a new category

\[
\text{MC}_s(C((x_s))/C)
\]

whose objects are couples \((M, \nabla)\) consisting of a finite dimensional \( C((x_s)) \)-space \( M \) and a \( C \)-linear endomorphism \( \nabla : M \to M \) satisfying

\[
\nabla(gm) = \vartheta(g)m + g\nabla(m)
\]

for all \( g \in C((x_s)) \) and \( m \in M \). Arrows are defined in the evident way. See Notations for the convention on \( \vartheta \).

There is a natural functor

\[
(-)_{(s)} : \text{MC}(C((x))/C) \longrightarrow \text{MC}_s(C((x_s))/C),
\]

which, on the level of objects, is \((M, \nabla) \longmapsto (M_{(s)}, \nabla)\) with \( M_{(s)} = C((x_s)) \otimes M \) and the connection obeys the rule:

\[
\nabla(g \otimes m) = \vartheta(g) \otimes m + g \otimes \nabla(m).
\]

In parallel to Definitions 2.1 and 2.2 we have:
Definition 2.4. We define the category $\text{MC}_{log,s}(C[[x]]/C)$ having as objects couples $(M, \nabla)$ where $M$ is a finite $C[[x]]$-module and $\nabla : M \to M$ is a $C$-linear arrow satisfying $\nabla(fm) = \vartheta(f)m + f\nabla(m)$. (Morphisms between objects are defined in the evident way.) The category of regular-singular connections is the full subcategory of $\text{MC}_s(C((x))/C)$ whose objects are in the image of the obvious functor $\gamma : \text{MC}_{log,s}(C[[x]]/C) \to \text{MC}_s(C((x))/C)$.

Among regular-singular connections we have the euler connections, defined as follows.

Definition 2.5. Let $\text{End}_C$ be the category whose

- objects are couples $(V, A)$ consisting of a finite dimensional $C$-space $V$ and a $C$-linear endomorphism $A : V \to V$, and whose
- arrows from $(V, A)$ and $(V', A')$ are $C$-linear morphisms $\varphi : V \to V'$ such that $A'\varphi = \varphi A$.

Let $(V, A) \in \text{End}_C$, we define a logarithmic connection on $C[[x]] \otimes_C V$ by means of the formula

$$D_A(f \otimes v) = \vartheta f \otimes v + f \otimes Av.$$ 

In [HdST22, Section 3] (for the case $s = 1$) this connections is denoted by $\gamma_{eul}(V, A)$, which is an object of $\text{MC}(C((x))/C)$. Working over $C((x))$ we shall write $\gamma_{eul}(V, A)$. Note that there is a canonical isomorphism

$$\gamma_{eul,s}(V, A) \simeq \gamma_{eul}(V, A)(s).$$

2.3. Objects of rank one. For each $f \in C((x))$, define a connection on a rank one $C((x))$-module

$$\mathfrak{L}_f = (C((x))f, df)$$

by $df(f) = ff$. By equation (2), we see that $\mathfrak{L}_f \simeq \mathfrak{L}_g$ if and only if

$$f - g = u^{-1}\vartheta u,$$

for some $u \in C((x)) \setminus \{0\}$.

Notice that

$$d\text{Log} := \{u^{-1}\vartheta u : u \in C((x)) \setminus \{0\}\} = \left\{ \sum_{n \geq 0} a_n x^n \in C[[x]] : a_0 \in \mathbb{Z} \right\}.$$ 

Consider it as an additive subgroup of $C((x))$.

Now let $X$ stand for the set of isomorphism classes of rank one connections and endow it with the group structure obtained by the tensor product, it follows that

$$\frac{C((x))}{d\text{Log}} \longrightarrow X, \quad f + d\text{Log} \longmapsto \mathfrak{L}_f$$

is a group isomorphism. Consequently, the inclusion $C[x^{-1}] \to C((x))$ gives rise to an isomorphism

$$C[x^{-1}]/\mathbb{Z} \longrightarrow X, \quad f + \mathbb{Z} \longmapsto \mathfrak{L}_f.$$ 

For any $f \in C[x^{-1}]$, we shall denote its class in $C[x^{-1}]/\mathbb{Z}$ by $[f]$. We shall also use the notation $\mathfrak{L}_{[f]}$ to denote the isomorphic class of $\mathfrak{L}_f$. 


For an element \( f \in C((x)) \), its principal part, denoted by \( p(f) \) is an element of 
\( x^{-1}C[x^{-1}] \) such that \( f - p(f) \in C[x] \). This defines a projection 
\[ p : C((x)) \longrightarrow x^{-1}C[x^{-1}], \]
which induces a map, also denoted by \( p \):
\[ p : C[x^{-1}]/\mathbb{Z} \longrightarrow x^{-1}C[x^{-1}]; \quad [f] \longmapsto p(f). \]

For a class \([f]\) in \( C[x^{-1}]/\mathbb{Z} \), there exists a unique element \( c_f \in \tau \) such that 
\( c_f + p(f) \) represents \([f]\). The image of \( c_f \) in \( C/\mathbb{Z} \) is called the Turrittin exponent of 
the connection \( \mathcal{E}[f] \), see [ABC20].

All this can be taken up in the context of the category \( \text{MC}_s(C((x_s))/C) \) with simple modifications. Letting the rank of an object \((M, \nabla) \in \text{MC}_s(C((x_s))/C)\) be simply \( \dim_{C((x_s))} M \), we note that 
\[ X_s := \left\{ \begin{array}{l}
\text{group of isomorphism classes of objects} \\
\text{of rank one in } \text{MC}_s(C((x_s))/C)
\end{array} \right\} \]
can be determined as follows. Let 
\[ d\log_s := \{ u^{-1} \partial u : u \in C((x_s)) \setminus \{0\} \} \]
\[ = \left\{ \sum_{n \geq 0} a_n x_n^s \in C[[x_s]] : a_0 \in s^{-1}\mathbb{Z} \right\}. \]

Associating to each \( f \in C((x_s)) \) the object \((\mathcal{E}_f, d_f) \in \text{MC}_s(C((x_s))/C)\) as in (4), we conclude that 
\[ C[x_s^{-1}]/s^{-1}\mathbb{Z} \xrightarrow{\sim} C((x_s))/d\log_s \xrightarrow{\sim} X_s. \]
The choice of working on \( C((x_s)) \) with the extension \( \partial \) pays off here: The diagram 
\[ \begin{array}{ccc}
X & \xrightarrow{(-)_{(s)}} & X_s \\
\sim & & \sim \\
C[x^{-1}]/\mathbb{Z} & \overset{\text{obvious}}{\longrightarrow} & C[x_s^{-1}]/s^{-1}\mathbb{Z}
\end{array} \]
commutes.

2.4. Jordan-Levelt decompositions. Let \((E, \nabla) \in \text{MC}(C((x))/C)\). We say that 
\((E, \nabla)\) is diagonalizable in \( \text{MC}(C((x))/C) \) if it is the direct sum of rank one objects. The connection \((E, \nabla)\) is called semi-simple if there exists a number \( s \in \mathbb{N} \) such that \((E, \nabla)_{(s)}\) is a diagonalizable connection in \( \text{MC}(C((x_s))/C)\). Note that if \((E, \nabla)\) is diagonalizable then it admits a \( C((x))\)-basis \((e)\) such that the matrix \( \text{Mat}(\nabla, e) \) is diagonal.

The following is theorem of the main outputs of [Lev75]. It is reminiscent of the Jordan–Chevalley decomposition for a linear operator.

**Theorem 2.6 ([Lev75, Section 1, Theorem I]).** Let \((E, \nabla) \in \text{MC}(C((x))/C)\). Then, there exists a decomposition 
\[ \nabla = S_{\nabla} + N_{\nabla} \]
satisfying the following conditions:

(i) \((E, S_{\nabla})\) is a semi-simple connection in \( \text{MC}(C((x))/C)\).
(ii) $N_\nabla : E \rightarrow E$ is a $C((x))$–linear nilpotent endomorphism.

(iii) $S_\nabla$ and $N_\nabla$ commute.

Moreover, if $\nabla = S + N$ is another decomposition of $\nabla$ having properties (i)–(iii), then $S = S_\nabla$ and $N = N_\nabla$.

\[ \square \]

**Definition 2.7.** For given $(E, \nabla) \in \text{MC}(C((x))/C)$, let $(E, S_\nabla)$ be the semi-simple connection in Theorem 2.6.

(i) The **Turrittin index** of $(E, \nabla)$ is the smallest number $s \in \mathbb{N}$ such that the connection $(E, S_\nabla)_s$ is diagonalizable. If the Turrittin index of $(E, \nabla)$ equals 1, it is called **unramified**.

(ii) The decomposition (6) is called the **Jordan-Levelt decomposition** for $(E, \nabla)$.

**Example 2.8.** Let $(V, A) \in \text{End}_C$. We define on $C((x)) \otimes_C V$ a connection $\nabla_A$ given by $\nabla_A(1 \otimes v) = 1 \otimes Av$; in [HdST22, Section 3], this was denoted by $\gamma_{eul}(V, A)$. (The notation $eul(V, A)$ was reserved to the associated logarithmic connection.) Now, if $A_s$ is the semi-simple part of $A$ then it is diagonalizable because $C$ is an algebraically closed field. It then follows that the connection

$\nabla_{A_s} : C((x)) \otimes_C V \rightarrow C((x)) \otimes_C V$

is diagonalizable which is the direct sum of connections of rank one corresponding to the eigenvalues of $A_s$. Uniqueness of the Jordan-Levelt decomposition immediately shows that $S_{\nabla_A} = \nabla_{A_s}$.

2.5. **Turrittin-Levelt-Jordan decomposition.** Let $(E, \nabla) \in \text{MC}(C((x))/C)$ and $s \in \mathbb{N}$ its Turrittin index. Let us write $S_\nabla + N_\nabla$ its the Jordan-Levelt decomposition, see Theorem 2.6. By definition, the object $(E, S_\nabla)_s \in \text{MC}_s(C((x_s))/C)$ is the direct sum of objects of rank one. According to the description of the group $X_s$ offered in section 2.3, there exists a finite set $\Phi \subset C[x_s^{-1}]/s^{-1}\mathbb{Z}$ together with subobjects $\{E_{(s)\varphi}\}_{\varphi \in \Phi}$ of $E_{(s)}$ such that:

$$(E, S_\nabla)_s = \bigoplus_{\varphi \in \Phi}(E_{(s)\varphi}, S_\nabla),$$

where for each $\varphi \in C[x_s^{-1}]/s^{-1}\mathbb{Z}$, the object $(E_{(s)\varphi}, S_\nabla) \in \text{MC}_s(C((x_s))/C)$ is isomorphic to a direct sum of non-trivial powers of $\mathcal{L}_{f_\varphi}$, where $[f_\varphi] = \varphi$. Furthermore, we have $\nabla(E_{(s)\varphi}) \subset E_{(s)\varphi}$ for each $\varphi \in C[x_s^{-1}]/s^{-1}\mathbb{Z}$.

Since $S_\nabla$ and $N_\nabla$ commute, $E_{(s)\varphi}$ is closed under the action of $N_\nabla$. Hence we have a decomposition

$$(E_{(s)}, \nabla) = \bigoplus_{\varphi \in \Phi}(E_{(s)\varphi}, \nabla_\varphi)$$

of objects in $\text{MC}_s(C((x_s))/C)$.

In fact, $E_{(s)\varphi}$ can be determined from $\nabla$ as follows.

**Lemma 2.9** (Lev75, Section 4, Lemma). For each class $\varphi \in \Phi$ and each element $f_\varphi \in C((x_s))$ with $[f_\varphi] = \varphi$, there exists $q \in \mathbb{N}$ such that the kernel of $(\nabla - f_\varphi)^q$ spans the $C((x_s))$–space $E_{(s)\varphi}$. In addition, $E_{(s)\varphi}$ is stable under $\nabla$, $S_\nabla$ and $N_\nabla$.

\[ \square \]

**Definition 2.10** (Turrittin-Levelt-Jordan decomposition). The composition

$$(E_{(s)}, \nabla) = \bigoplus_{\varphi \in \Phi}(E_{(s)\varphi}, \nabla_\varphi)$$

(7)
is called the Turrittin-Levelt-Jordan decomposition of \((E, \nabla)\). Each component \((E_{(s)}\varphi, \nabla_\varphi)\) in \((7)\) is called the isotypic component of type \(\varphi\) of \((E_{(s)}, \nabla)\).

**Example 2.11.** Let \((E, \nabla) \in \text{MC}_{\text{st}}(C((x))/C)\). According to [HdST22, Corollary 4.3], there exists \((A, V) \in \text{End}_C\) with \(\text{Spec}(A) \subset \tau\) such that \((E, \nabla) \simeq \gamma_{\text{eul}}(A, V)\). Using [H72, Section 4.2, Proposition], the Jordan-Chevalley decomposition of \((A, V)\) allows us to write

\[ V = \bigoplus_{\varphi \in \text{Spec}(A)} G(A, \varphi), \]

where \(G(A, \varphi)\) denotes the generalized eigenspace of \(A\) associated to \(\varphi\). Then, the decomposition \((8)\) induces

\[ (E, \nabla) = \bigoplus_{\varphi \in \text{Spec}(A)} \gamma_{\text{eul}}(G(A, \varphi), A_\varphi) \]

which is the Turrittin-Levelt-Jordan decomposition of \((E, \nabla)\), where \(A_\varphi = A|_{G(A, \varphi)}\).

Let us consider \((E, \nabla) \in \text{MC}(c((x))/C)\) and its Turrittin-Levelt-Jordan decomposition

\[ (E_{(s)}, \nabla) = \bigoplus_{\varphi \in \Phi} (E_{(s)}\varphi, \nabla_\varphi). \]

For each \(\varphi \in \Phi\), according to [Lev75, Section 6.a], there is a \(C((x_s))\)-basis \((e_\varphi)\) of \(E_{(s)}\varphi\) such that the matrix

\[ \text{Mat} (\nabla_\varphi, e_\varphi) = J_{r_\varphi}(f_\varphi), \]

where \(f_\varphi \in C[x_s^{-1}]\) satisfies \([f_\varphi] = \varphi\). By setting \(S_\varphi = S_{\gamma}|_{E_{(s)}\varphi}\), it is easy to see that \(\text{Mat}(S_\varphi, e_\varphi) = f_\varphi I_{r_\varphi}\). Moreover, we get the following lemma:

**Lemma 2.12.** Let \(v_\varphi\) be a \(C((x_s))\)-basis of \(E_{(s)}\varphi\) such that

\[ \text{Mat} (S_\varphi, v_\varphi) = \text{diag} (g_1^\varphi, \cdots, g_{r_\varphi}^\varphi), \]

where \(g_1^\varphi, \cdots, g_{r_\varphi}^\varphi \in C((x_s))\). Then, \(p(g_i^\varphi) = p(f_\varphi)\) for all \(i = 1, \cdots, r_\varphi\).

**Proof.** Let \(Q = (q_{ij})_{r_\varphi}\) be the matrix of base change from \(e_\varphi\) to \(v_\varphi\) of \(E_{(s)}\varphi\). We then obtain that

\[ \text{Mat}(S_\varphi, v_\varphi) = f_\varphi I_{r_\varphi} + Q^{-1} \vartheta(Q). \]

It implies that \(Q^{-1} \vartheta(Q) = \text{diag}(g_1^\varphi - f_\varphi, \cdots, g_{r_\varphi}^\varphi - f_\varphi)\), and hence

\[ \vartheta(Q) = Q \text{diag}(\lambda_1, \cdots, \lambda_{r_\varphi}), \]

where \(\lambda_i = g_i^\varphi - f_\varphi\). Therefore,

\[ \vartheta(q_{ij}) = \lambda_j q_{ij}, \quad i, j = 1, \cdots, r_\varphi. \]

Because \(Q \in \text{GL}_{r_\varphi}(C((x_s)))\), there exists \(q_{ik} \neq 0\) for each \(k = 1, \cdots, r_\varphi\). It implies that \(\lambda_k \in C[x_s]\). Hence, we get \(p(g_i^\varphi) = p(f_\varphi)\) for all \(i = 1, \cdots, r_\varphi\). \(\square\)

By proceeding above with all \(\varphi \in \Phi\), there is a \(C((x_s))\)-basis \((e)\) of \(E_{(s)}\) such that

\[ J = \text{Mat} (\nabla, e) = \text{diag} (J_{r_\varphi}(f_\varphi))_{\varphi \in \Phi}. \]

In what follows we shall refer to the matrix in \((9)\) as Jordan-Levelt form of \((E, \nabla)\).
Remark 2.13. Assume that \( \dim_{C((x))} E = n \). The set \( \{ p(\varphi) : \varphi \in \Phi \} \) is defined uniquely, that is, it depends only on \((E, \nabla)\) (see And20, Théorème 2.3.1 for example). It is called the set of irregular values of \( \nabla \) and shall be denoted by \( \text{Irr}(E, \nabla) \) (see [Moc09a, p.228]). This set is defined in [Moc09a, Lemma 2.5, p.229] as follows: for any \( H \in \text{GL}_n(C[[x]]) \) and \( A = \text{Mat}_n(\nabla, e.H) \),

\[
\text{Irr}(\nabla) = \{ p(\alpha_i) : \alpha_i \in \text{Spec}(A) \}.
\]

2.6. Logarithmic decomposition. Let us now go further in Levelt’s theory which allows us to establish the following proposition.

Theorem 2.14 (Logarithmic decomposition). Let \((E, \nabla) \in \text{MC}(C((x))/C)\) be given and \( s \in \mathbb{N} \) be its Turrittin index. Then there exists a unique canonical decomposition

\[
\nabla = \nabla^{rs} + P_{\nabla}
\]

having the following properties:

(i) \((E, \nabla^{rs})\) is an object in the category \(\text{MC}_{rs}(C((x))/C)\).

(ii) \(P_{\nabla}\) preserves the Turrittin-Levelt-Jordan decomposition of \((E, \nabla^{rs})\).

(iii) For each isotypical component \(E_\theta\) of \((E, \nabla^{rs})\), there exists \(C((x)_s)\)–basis \((e_\theta)\) of \(E_{\theta(s)}\) such that

\[
\text{Mat}(\nabla^{rs}|_{E_{\theta(s)}}, e_\theta) = J_{r_\theta}(c_\theta), \quad \text{Mat}(P_{\nabla}|_{E_{\theta(s)}}, e_\theta) = f_\theta I_{r_\theta}
\]

where \(c_\theta \in C\), \(f_\theta \in x_s^{-1}C[x_s^{-1}]\) and \(r_\theta = \dim_{C((x))} E_\theta\).

In addition, if \(\nabla = \nabla^{rs} + \dot{P}\) is another decomposition having the properties (i)–(iii), then \(\dot{\nabla} = \nabla^{rs}\) and \(\dot{P} = P_{\nabla}\).

Proof. Existence. Consider

\[
(E(s), \nabla) = \bigoplus_{\varphi \in \Phi} (E(s)_\varphi, \nabla_\varphi)
\]

the Turrittin-Levelt-Jordan decomposition of \((E, \nabla)\). For each \(\varphi \in \Phi\), let us choose \(c_\varphi \in C\) such that \([c_\varphi + p(\varphi)] = \varphi\), cf. section 2.3. Now we define on the \(C((x)_s)\)–space \(E_{(s)_\varphi}\) a connection by

\[
\nabla_\varphi^{rs} := \nabla_\varphi - p(\varphi)\text{id};
\]

We now prove that \(\nabla_\varphi^{rs} \in \text{MC}_{rs}(C((x)_s)/C)\). Indeed, let \((e_\varphi)\) be a \(C((x)_s)\)–basis of \(E_{(s)_\varphi}\) over \(C((x)_s)\) such that

\[
\text{Mat}(S_{\nabla}, e_\varphi) = (c_\varphi + p(\varphi))\text{id},
\]

where \(\nabla = S_{\nabla} + N_{\nabla}\) is the Jordan-Levelt decomposition of \((E, \nabla)\). Since

\[
\text{End}_{\text{MC}_s}(H_{c_\varphi + p(\varphi)}) = \text{Cid},
\]

we conclude that \(N_{\nabla}\), which is an endomorphism of \((E_{(s)_\varphi}, S_{\nabla})\), preserves \(Ce_\varphi\) – the \(C\)-subspace of \(E_{(s)_\varphi}\) spanned by \((e_\varphi)\). Let \(N_{e_\varphi}\) be the matrix of \(N_{\nabla}\) in the basis \(e_\varphi\); it has coefficients in \(C\). We then obtain that

\[
\text{Mat}(\nabla_\varphi^{rs}, e_\varphi) = c_\varphi I + N_{e_\varphi} \in \text{Mat}(C).
\]

Let \(\zeta\) be a primitive \(s^{th}\) root of unity and \(\sigma : C((x)_s) \to C((x)_s)\) be the automorphism determined by \(x_s \to \zeta x_s\); the Galois group \(G\) of the extension \(C((x)_s)/C((x))\)
is generated by $\sigma$. Let us also abuse notation and write $\sigma$ for the map $\sigma \otimes \text{id}$ from $E'$ to itself. Clearly, $\nabla \circ \sigma = \sigma \circ \nabla, S_\sigma \circ \sigma = \sigma \circ S_\sigma$, and $\sigma E(s) \varphi = E'(s) \sigma(\varphi)$, where $\sigma(\varphi)$ is defined in the obvious manner. Now, $p(\sigma \varphi) = \sigma p(\varphi)$, and we conclude that

$$\sigma \circ \bigoplus_{\varphi \in \Phi} \nabla_{\nabla \varphi} = \bigoplus_{\varphi \in \Phi} \nabla_{\nabla \varphi} \circ \sigma.$$ 

Hence, by Galois descent, there exists a connection $\nabla_{\nabla \varphi}$ on $E$ such that $\bigoplus_{\varphi \in \Phi} \nabla_{\nabla \varphi}$ is its extension to $E(s)$. We define $P_\nabla := \nabla - \bigoplus_{\varphi \in \Phi} \nabla_{\nabla \varphi}$. It is a $C(\!(x)\!)$–linear endomorphism on $E$. More precisely, identifying $E$ with $\{e \in E(s) : \sigma e = e\}$, we see that $\bigoplus_{\varphi \in \Phi} \nabla_{\nabla \varphi}$ becomes just the restriction. It is not hard to see that $\bigoplus_{\varphi \in \Phi} \nabla_{\nabla \varphi}$ is regular singular over $C(\!(x)\!)$. 

Let us define on $\bigoplus_{\varphi \in \Phi} E(s) \varphi$ the $C(\!(x)\!)$–linear map $\bigoplus_{\varphi \in \Phi} p(\varphi) \text{id}$. Then, the connection $\nabla$ on $E(s)$ is defined by

$$\nabla = \bigoplus_{\varphi \in \Phi} \nabla_{\nabla \varphi} + \bigoplus_{\varphi \in \Phi} p(\varphi) \text{id}.$$ 

Since $\bigoplus_{\varphi \in \Phi} \nabla_{\nabla \varphi}$ and $\nabla$ commute with $\sigma : E(s) \to E(s)$, so the extension of the $C(\!(x)\!)$–linear map $P_\nabla$ to $E(s)$ is $\bigoplus_{\varphi \in \Phi} p(\varphi) \text{id}$. In other words, $P_\nabla : E(s) \to E(s)$ is a diagonalizable $C(\!(x)\!)$–linear map whose eigenvalues are all on $x_s^{-1}C[\![x_s^{-1}]\!]$.

Let us write $E = \bigoplus_{\theta \in \Theta} E_{\theta}$ the Turrittin-Levelt-Jordan decomposition of $(E, \nabla_{\nabla \varphi})$, where $\Theta \subset C/\mathbb{Z}$. We note that

$$E_{\theta} = \{e \in \bigoplus_{\varphi \in \Phi'} E(s) \varphi : \sigma e = e\},$$

where $\Phi' \subset \Phi$ is a union of some orbits of $G$. Because $P_\nabla$ preserves each $E(s) \varphi$, so $\bigoplus_{\varphi \in \Phi'} E(s) \varphi$ is invariant under $P_\nabla$. Note that $p(\sigma \varphi) = \sigma p(\varphi)$, and we conclude that the action of the Galois group preserves $\bigoplus_{\varphi \in \Phi' : [c_\varphi] = \theta} E(s) \varphi$. Hence, $P_\nabla(E_{\theta}) \subset E_{\theta}$ for each $\theta \in \Theta$.

**Uniqueness.** According to Example 2.11, the connection $(E, \nabla)$ admits a Turrittin-Levelt-Jordan decomposition

$$(E, \nabla) = \bigoplus_{\gamma \in \Gamma} E_\gamma$$

over $C(\!(x)\!)$, where $\Gamma \subset C/\mathbb{Z}$ is a finite set. For each $\gamma \in \Gamma$, using conditions (ii) and (iii), there exist a $C(\!(x)\!)$–basis $(e_\gamma) = \{e_{\gamma,1}, \ldots, e_{\gamma, \gamma}\}$ of $E_\gamma$ such that

$$(12) \quad \text{Mat}(\nabla, e_\gamma) = J_{r_\gamma}(c_\gamma) \quad \text{and} \quad \text{Mat}(\hat{P}, e_\gamma) = f_\gamma I_{r_\gamma},$$

where $c_\gamma \in C$ such that $[c_\gamma] = \gamma$ in $C/\mathbb{Z}$ and $f_\gamma \in x_s^{-1}C[\![x_s^{-1}]\!]$.

Let us define a connection $\hat{S}_\gamma : E_{\gamma(s)} \to E_{\gamma(s)}$ by declaring

$$(13) \quad \text{Mat}(\hat{S}_\gamma, e_\gamma) = c_\gamma I_{r_\gamma} + \text{Mat}(\hat{P}, e_\gamma) = (c_\gamma + f_\gamma) I_{r_\gamma}$$

which is diagonalizable over $C(\!(x)\!)$. Since (12) and (13), we obtain that $\nabla_\gamma = \nabla|_{E_{\gamma(s)}} - \hat{S}_\gamma$ is a $C(\!(x)\!)$–linear endomorphism of $E_{\gamma(s)}$ which, on the basis $e_\gamma$, has
nilpotent matrix in $\text{Mat}_{r_\gamma}(C)$:

$$\text{Mat}(\hat{N}_\gamma, e_\gamma) = \text{Mat}(\hat{\nabla}, e_\gamma) - c_\gamma I_{r_\gamma} = J_{r_\gamma}(0).$$

Let now us check that $\hat{S}_\gamma \hat{N}_\gamma = \hat{N}_\gamma \hat{S}_\gamma$. For each $e \in E_{\gamma}(s)$, by writing $e = a_1 e_{\gamma 1} + \cdots + a_{r_\gamma} e_{r_\gamma}$, where $a_1, \cdots, a_{r_\gamma} \in C((x_s))$. Then, we get

$$\hat{S}_\gamma \hat{N}_\gamma(e) = \sum_{i=1}^{r_\gamma-1} (\vartheta(a_i) + a_i \varphi_{\gamma}) e_i^{i+1} = \hat{N}_\gamma \hat{S}_\gamma(e).$$

Therefore, $\nabla|_{E_{\gamma}(s)} = \hat{S}_\gamma + \hat{N}_\gamma$ is the Jordan-Levelt decomposition of $\nabla|_{E_{\gamma}(s)}$. We then obtain that

$$\nabla = \bigoplus_{\gamma \in \Gamma} \big( \hat{S}_\gamma \big) + \bigoplus_{\gamma \in \Gamma} \big( \hat{N}_\gamma \big)$$

is the Jordan-Levelt decomposition of $\nabla$ over $C((x_s))$. According to Theorem 2.6, we conclude that $S_{\nabla} = \bigoplus \hat{S}_\gamma$. Hence, $S_{\nabla}$ preserves each $E_\gamma$ and so does $N_{\nabla}$. Moreover,

$$\nabla|_{E_{\gamma}(s)} = S_\gamma \quad \text{for all } \gamma \in \Gamma. \tag{14}$$

By putting (13) and (14) together, we obtain that $P_{\nabla}|_{E_{\gamma}(s)} = \hat{P}|_{E_{\gamma}(s)}$. Hence, using Galois descent, we obtain that $\hat{\nabla} = \nabla^{rs}$ and $\hat{P} = P_{\nabla}$. \hfill \square

We note that the condition Theorem 2.14.(iii) is necessary for the uniqueness. The following is an illustration.

**Example 2.15.** We set $E = C((x)) e_1 \oplus C((x)) e_2$, and the connection $\nabla$ is defined as follows:

$$\nabla(e_1) = \frac{1}{x} e_1 + e_2; \quad \nabla(e_2) = -\frac{1}{x} e_2.$$ 

According to [Lev75, Section 2, Lemma], there uniquely exist $G = \begin{pmatrix} 1 & g_{12} \\ g_{21} & 1 \end{pmatrix}$ in $\text{GL}_2(C((x)))$ such that

$$\begin{pmatrix} \frac{1}{x} + a_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{x} + a_2 \end{pmatrix} = G^{-1} \begin{pmatrix} \frac{1}{x} & 0 \\ 1 & -\frac{1}{x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} G + G^{-1} \vartheta(G),$$

where $a_1, a_2 \in xC[x]$. Let $H = \begin{pmatrix} \exp(-\int x^{-2}a_1)dx & 0 \\ 0 & \exp(-\int x^{-2}a_2)dx \end{pmatrix}$. Then, we obtain that

$$\begin{pmatrix} \frac{1}{x} & 0 \\ 0 & -\frac{1}{x} \end{pmatrix} = H^{-1} \begin{pmatrix} \frac{1}{x} + a_1 & 0 \\ 0 & -\frac{1}{x} + a_2 \end{pmatrix} H + H^{-1} \vartheta(H).$$

Let $\nabla^{rs}$ be the connection on $E$ having the matrix in the basis $e$. $GH$ is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The decomposition $\nabla = \nabla^{rs} + P_{\nabla}$ satisfies the conditions in Theorem 2.14.

On the other hand, let $\nabla^{rs}_1$ be the connection on $E$ having the matrix with respect to $e = \{e_1, e_2\}$ is $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We then obtain a decomposition $\nabla = \nabla^{rs}_1 + P_1$ which
satisfies the conditions (i) and (ii) in Theorem 2.14. It is easy to see that \( g_{21} \neq 0 \), hence \( P_1 \neq P_2 \). Note that, the decomposition \( \nabla = \nabla^{rs}_1 + P_1 \) doesn’t satisfy the condition (iii) of theorem.

2.7. Deligne-Manin-Malgrange lattices. Let \((E, \nabla)\) be an object in the category \( \text{MC}((x)/C) \). There exists a Deligne-Manin lattice \( \mathcal{E} \) in \( \text{MC}_{\log}(C[[x]]/C) \) for \((E, \nabla^{rs})\) having all exponents in \( \tau \), see [Del87, Section 15.32]. Let us denote \( s \in \mathbb{N} \) for the Turrittin index, \( G \) for the Galois group of the extension \( C((x_s))/C((x)) \) and \( \Phi \subset X_s \) for the set of irregular values of \( \nabla \). The Turrittin-Levelt-Jordan decomposition of \((E, \nabla)\):

\[
(E(s), \nabla) = \bigoplus_{\varphi \in \Phi} (E(s)_{\varphi}, \nabla_{\varphi}).
\]

The object \( \mathcal{E}_{(s)} = \mathcal{E} \otimes C[[x]] C[[x_s]] \) is automatically a Deligne-Manin lattice for \((E(s), \nabla^{rs})\). The follow gives a description on the structure of this lattice.

**Proposition 2.16.** Let us write \( \nabla = \nabla^{rs} + P_\nabla \) the logarithmic decomposition of \((E, \nabla)\). Then

(i) The lattice \( \mathcal{E}_{(s)} \) is closed under the action of \( G \).

(ii) There exists a decomposition \( \mathcal{E}_{(s)} = \bigoplus \mathcal{E}_{(s)\varphi} \) such that each \( \mathcal{E}_{(s)\varphi} \) is the Deligne-Manin lattice for \((E(s)_{\varphi}, \nabla^{rs}_{\varphi})\) with respect to \( \tau \), where \( \nabla^{rs}_{\varphi} = \nabla^{rs}|_{E(s)} \).

**Proof.** Let \( \zeta \) be a primitive \( s \)-th root of unity and \( \sigma : C((x_s)) \to C((x_s)) \) be the automorphism determined by \( x_s \to \zeta x_s \). Let use notation \( \sigma \) for the map from \((E(s), \nabla^{rs})\) to itself. By using [HdST22, Theorem 7.8], \( \sigma \) is extended to an action on \( \mathcal{E}_{(s)} \) satisfying the following diagram

\[
\begin{array}{ccc}
\gamma_s(\mathcal{E}_{(s)}) & \xrightarrow{\gamma_s(\sigma)} & \gamma_s(\mathcal{E}_{(s)}) \\
\downarrow & & \downarrow \\
(E(s), \nabla^{rs}) & \xrightarrow{\sigma} & (E(s), \nabla^{rs}).
\end{array}
\]

commutes. This shows that there exists an action of \( G \) on the \( C[[x_s]] \)-module \( \mathcal{E}_{(s)} \) which is extended by the action of \( G \) on \((E(s), \nabla)\).

Let \( \mathcal{F}_{\varphi} \in \text{MC}_{\log,s}(C[[x_s]]/C) \) be a Deligne-Manin lattice of \((E(s)_{\varphi}, \nabla^{rs}_{\varphi})\) with respect to \( \tau \). According to [HdST22, Theorem 6.8], there exists an isomorphism

\[ \mathcal{E}_{(s)} \simeq \bigoplus_{\varphi \in \Phi} \mathcal{F}_{\varphi} \]

in \( \text{MC}_{\log}(C[[x_s]]/C) \) which is the extension of \( \text{id}_{E(s)} \). In particular, we get

\[ \mathcal{E}_{(s)} = \bigoplus_{\varphi \in \Phi} \mathcal{E}_{(s)\varphi}, \]

where each \( \mathcal{E}_{(s)\varphi} \) is a Deligne-Manin lattice for \((E(s)_{\varphi}, \nabla^{rs}_{\varphi})\) associated to \( \tau \). \( \square \)

**Remark 2.17.** According to [Mal96], T. Mochizuki calls \( \mathcal{E} \) the Deligne-Malgrange lattice [Moc09a, Moc09b]. In what follows we shall refer to it as Deligne-Manin-Malgrange lattice.
3. Connections with an action of an Artinian ring

We fix a local $C$-algebra $\Lambda$ with maximal ideal $\mathfrak{I}$ and whose dimension, as a vector space, is finite. In this section, we prove the results in the previous section for connections over $C((x))$ equipped with an action of $\Lambda$. We first recall the following definitions (see [HdS20] and [HdST22]).

**Definition 3.1.** We define $\text{MC}(C((x))/C)_\Lambda$ as the category whose objects are couples $((E, \nabla), \alpha)$ with $(E, \nabla) \in \text{MC}(C((x))/C)$ and

$$\alpha : \Lambda \to \text{End}((E, \nabla))$$

is a morphism of $C$-algebras, and an arrow from $((E, \nabla), \alpha)$ to $((E_{(s)}, \nabla'), \alpha')$ is a morphism $\varphi : (E, \nabla) \to (E_{(s)}, \nabla')$ such that $\alpha'(\lambda) \circ \varphi = \varphi \circ \alpha(\lambda)$ for all $\lambda \in \Lambda$.

Given $(E, \nabla) \in \text{MC}(C((x))/C)_\Lambda$. Throughout this section, let $s \in \mathbb{N}$ be the Turrittin index of $(E, \nabla)$ and $G$ be the Galois group of the extension $C((x_s))/C((x))$. Consider the scalar extension $(E_{(s)}, \nabla)$ of $(E, \nabla)$ to $C((x_s))$. Then $\Lambda$ acts on $(E_{(s)}, \nabla)$ in an obvious way making it an object in $\text{MC}(C((x_s))/C)_\Lambda$.

The action of $\Lambda$ on $(E, \nabla)$ induces a connection on $\overline{E} = E/\mathfrak{I}E$, denoted by $\nabla$. Similarly, we have connection $(\overline{E_{(s)}}, \nabla)$. This two process commute, that is, there is a canonical isomorphism $\Phi : \overline{E_{(s)}} \to \overline{E_{(s)}}$ in $\text{MC}(C((x_s))/C)_\Lambda$.

Consider now the Turrittin-Levelt-Jordan decomposition

$$\tag{16} (E_{(s)}, \nabla) = \bigoplus_{\varphi \in \Phi} (E_{(s)}\varphi, \nabla_{\varphi})$$

of $(E, \nabla)$. By means of Lemma 2.9, we see that each $(E_{(s)}\varphi, \nabla_{\varphi})$ is closed under the action of $\Lambda$, and hence is an object in $\text{MC}(C((x_s))/C)_\Lambda$. Taking reduction modulo $\mathfrak{I}$ we get a decomposition

$$\tag{17} (\overline{E_{(s)}}, \nabla) = \bigoplus_{\varphi \in \Phi} (\overline{E_{(s)}\varphi}, \overline{\nabla_{\varphi}}).$$

**Proposition 3.2.** Let $(E, \nabla)$ be an object in $\text{MC}(C((x))/C)_\Lambda$ and

$$(E_{(s)}, \nabla) = \bigoplus_{\varphi \in \Phi} (E_{(s)}\varphi, \nabla_{\varphi})$$

be the Turrittin-Levelt-Jordan decomposition of $(E, \nabla)$ and $s \in \mathbb{N}$ its Turrittin index. Then, the following claims hold:

1. If $E_{(s)}\varphi \neq \{0\}$ then $\overline{E_{(s)}\varphi} \neq \{0\}$.
2. The decomposition (17) is the Turrittin-Levelt-Jordan decomposition of $(\overline{E}, \nabla)$.

**Proof.** According to Lemma 2.9, the $C((x_s))$-space $E_{(s)}\varphi$ is generated by $\ker(\nabla - f_{\varphi})^q$, where $f_{\varphi} \in C[x_s^{-1}]$ is such that $[f_{\varphi}] = \varphi$. The discussion above tells us that $E_{(s)}\varphi$ is closed under the action of $\Lambda$. Since $\mathfrak{I} \subset \Lambda$ is nilpotent, so $\mathfrak{I}E_{(s)}\varphi \subset E_{(s)}\varphi$. Hence, if $E_{(s)}\varphi \neq \{0\}$ then $\overline{E_{(s)}\varphi} \neq \{0\}$. The second claim then follows automatically by the very definition of the Turrittin-Levelt-Jordan decomposition (cf. Definition 2.10). \qed

Putting Theorem 2.14 and Proposition 3.2 together, we arrive at
Theorem 3.3. Let \((E, \nabla) \in \MC(C((x))/C)(\Lambda)\) be given. Then there exists a unique decomposition up to isomorphism
\[
\nabla = \nabla^{rs} + P_{\nabla}
\]
having the following properties:

(i) \((E, \nabla^{rs})\) is an object in the category \(\MC_{rs}(C((x))/C)(\Lambda)\).

(ii) \(P_{\nabla}\) preserves the Turrittin-Levelt-Jordan decomposition of \((E, \nabla^{rs})\).

(iii) For each isotypical component \(E_\theta\) of \((E, \nabla^{rs})\), there exists \(C((x_s))\)–basis \(\{e_\theta\}\)
of \(E_{\theta(s)}\) such that
\[
\text{Mat}(\nabla^{rs}|_{E_{\theta(s)}}, e_\theta) = J_{r_\theta}(c_\theta), \quad \text{Mat}(P_{\nabla}|_{E_{\theta(s)}}, e_\theta) = f_\theta I_{r_\theta}
\]
where \(c_\theta \in C\), \(f_\theta \in x_s^{-1}C[x_s^{-1}]\) and \(r_\theta = \text{dim}_{C((x))} E_\theta\).

In addition, if \(\hat{\nabla} = \hat{\nabla} + \hat{P}\) is another decomposition having the properties (i)–(iii), then \(\hat{\nabla} = \nabla^{rs}\) and \(\hat{P} = P_{\nabla}\).

Proof. Let \(r_\theta = \text{dim}_{C((x))} E_\theta\). Theorem 2.14 shows that there exists a decomposition \(\nabla = \nabla^{rs} + P_{\nabla}\) having the following properties:

(i) \((E, \nabla^{rs})\) \(\in \MC_{rs}(C((x))/C)\).

(ii) \(P_{\nabla}\) preserves the Turrittin-Levelt-Jordan decomposition of \((E, \nabla^{rs})\).

(iii) For each isotypical component \(E_\theta\) of \((E, \nabla^{rs})\), there exists \(C((x_s))\)–basis \(\{e_\theta\}\) of \(E_{\theta(s)}\) such that
\[
\text{Mat}(\nabla^{rs}|_{E_{\theta(s)}}, e_\theta) = J_{r_\theta}(c_\theta), \quad \text{Mat}(P_{\nabla}|_{E_{\theta(s)}}, e_\theta) = f_\theta I_{r_\theta}
\]
where \(c_\theta \in C\), \(f_\theta \in x_s^{-1}C[x_s^{-1}]\).

Firstly, each \((E_s(\varphi), \nabla_\varphi)\) is equipped with an action of \(\Lambda\) since Proposition 3.2. Hence, \((E_s(\varphi), \nabla^{rs})\) is an object of \(\MC_{rs}(C((x))/C)(\Lambda)\), and hence we obtain that \((E, \nabla^{rs}) \in \MC_{rs}(C((x))/C)(\Lambda)\) by taking Galois descent. In addition, \(P_{\nabla}\) is a semi-simple \(C((x))\)–linear endomorphism because \(P_{\nabla}|_{E_s(\varphi)} \nabla_\varphi = \nabla_\varphi^{rs} = p(\varphi) \text{id}\). Finally, the uniqueness of the decomposition is given by Theorem 2.14. \(\square\)

We end this section by using Theorems 3.3 to get the following result.

Theorem 3.4 (Deligne-Manin-Malgrange lattices). Let \((E, \nabla) \in \MC(C((x))/C)(\Lambda)\) be given and let \(E\) be the Deligne-Manin lattice of \((E, \nabla^{rs})\). Then \((E, \nabla^{rs})\) is closed under the action of \(\Lambda\). Further we have the decomposition
\[
(E_{s(\varphi)}, \nabla_\varphi^{rs}) = \bigoplus_{\varphi \in \Phi} (E_{s(\varphi)}, \nabla_\varphi^{rs}),
\]
where \((E_{s(\varphi)}, \nabla_\varphi^{rs})\) is the Deligne-Manin lattice of \((E_{s(\varphi)}, \nabla_\varphi^{rs})\) for each \(\varphi \in \Phi\).

Proof. Let us write \(\nabla = \nabla^{rs} + P_{\nabla}\) as in Theorem 3.3. Then, the object \((E, \nabla^{rs})\) is a connection in \(\MC_{rs}(C((x))/C)(\Lambda)\). Let \(E\) be the Deligne-Manin lattice for \((E, \nabla^{rs})\). Using [HdST22, Theorem 7.8], the natural morphism
\[
\text{End}_{\MC_{\log}}(E, \nabla^{rs}) \longrightarrow \text{End}_{\MC_{\log}}(E, \nabla^{rs})
\]
is bijective. Hence, by using [HdST22, Theorem 7.8-(2)], \(E\) is closed under the action of \(\Lambda\) on \((E, \nabla^{rs})\). Therefore, \((E, \nabla^{rs}) \in \MC_{\log}(C[[x]]/C)(\Lambda)\) is a Deligne-Manin lattice for \((M, \nabla^{rs}) \in \MC(C((x))/C)(\Lambda)\). In addition, since the uniqueness in [HdST22, Theorem 7.8], \((E_{s(\varphi)}, \nabla_\varphi^{rs})\) is the Deligne-Manin lattice for \((E_{s(\varphi)}, \nabla_\varphi^{rs})\). \(\square\)
4. Logarithmic decomposition for connections parameterized by a complete discrete valuation ring

In this section we let $R = C[[t]]$ be a complete discrete valuation ring and $K$ is its the field of fractions. We are mainly interested in the structure of connections in $\text{MC}(R((x))/R)$ whose underlying modules are free. Notice that, according to Theorem 8.16 of [HdST22], a connection is $R((x))$-flat if and only if it is $R$-flat. On the other hand, a finite and flat (that is, locally free) modules over $R((x))$ are always free, see [Po02, Theorem 1] and [BR83, Proposition 4.1]. We state this in a theorem for later references.

**Theorem 4.1.** Let $(E, \nabla) \in \text{MC}(R((x))/R)$ be given. Then if $E$ is $R$-flat then it is $R((x))$-free.

In the rest of this section, we shall assume that $E$ is a free $R((x))$-module and fix $p$ the projection

$$p : R[x^{-1}]/\mathbb{Z} \to x^{-1}R[x^{-1}].$$

The classification of objects of rank one in $\text{MC}(R((x))/R)$ works in close analogy with Section 2.3. Let us give a brief review below.

For each $s \in \mathbb{N}$, let $x_s = x^{1/s}$ and

$$d\text{Log}_s := \{ u^{-1}\vartheta u : u \in R((x_s))^\times \}.$$  

Note that, for each $m \in \mathbb{Z}$, the element $u = u_mx_s^m + u_{m+1}x_s^{m+1} + \cdots \in R((x_s))^\times$ if and only if $u_m \in R^\times$. Then,

$$d\text{Log}_s = \left\{ \sum_{n \geq 0} a_n x_s^n \in R[[x_s]] : a_0 \in s^{-1}\mathbb{Z} \right\}.$$  

Indeed, assume that $v = \frac{m}{s} + v_1x_s + v_2x_s^2 + \cdots + v_sx_s^s \in s^{-1}\mathbb{Z} + x_sR[[x_s]]$. We find $u = x_s^m + u_{m+1}x_s^{m+1} + \cdots$ in $R((x_s))$ satisfying $\vartheta(u) = uv$. In order to find such $u$ we solve the following equations

$$\frac{m+1}{s} u_{m+1} = \frac{m}{s} u_{m+1} + v_1;$$

$$\cdots$$

$$\frac{m+\ell}{s} u_{m+\ell} = \frac{m}{s} u_{m+\ell} + v_1 u_{m+\ell-1} + \cdots + v_\ell;$$

$$\cdots$$

By induction on the $u_{m+\ell}$, there exist $u \in R((x_s))^\times$ satisfying $u^{-1}\vartheta(u) = v$.

For each $f \in R((x_s))$, define a connection on a rank one $R((x_s))$-module

$$\mathcal{L}_f = (R((x_s))f, df)$$

by $df(f) = ff$. Then, $\mathcal{L}_f \simeq \mathcal{L}_g$ if and only if $f - g \in d\text{Log}_s$. By setting

$$X_s := \left\{ \text{group of isomorphism classes of objects of rank one in } \text{MC}_s(R((x_s))/R) \right\},$$

we have a classification of rank one objects in $\text{MC}(R((x_s))/R)$ as follows

$$R[x_s^{-1}]/s^{-1}\mathbb{Z} \to R((x_s))/d\text{Log}_s \to X_s.$$
For each multi-index of positive integers \( r = (r_1, \ldots, r_n) \) and \( f \in R((x_s)) \), let us recall the notation
\[
J_r(f) = \begin{pmatrix} f & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & f \end{pmatrix}, \quad * \in \{0, 1\}.
\]

Let \((E, \nabla) \in \text{MC}(R((x))/R)\) and \( s \in \mathbb{N} \). For each \( \varphi \in R[x_s^{-1}]/s^{-1}\mathbb{Z} \), let \( f_\varphi \in R((x_s)) \) be a representative. The **isotypical component** of \((E(s), \nabla)\) with respect to type \( \varphi \) is the \( R((x_s))\)-connection of \((E(s), \nabla)\) generated by \( \ker(\nabla - f_\varphi)^{q_\varphi} \) for some \( q_\varphi \in \mathbb{N} \).

**Definition 4.2.** Let \((E, \nabla) \in \text{MC}(R((x))/R)\) and \( s \in \mathbb{N} \). A **Turrittin-Levelt-Jordan decomposition** of \((E, \nabla)\) over \( R((x_s)) \) is a direct sum
\[
(E(s), \nabla) = \bigoplus_{\varphi \in \Phi} (E(s)_\varphi, \nabla_\varphi)
\]
indexed by a finite set \( \Phi \subset R[x_s^{-1}]/s^{-1}\mathbb{Z} \), the set of called characters of \((E, \nabla)\), where \( E(s)_\varphi = R((x_s)) \otimes_{R((x))} E \) and \((E(s)_\varphi, \nabla_\varphi)\) is the isotypical component of type \( \varphi \) of \((E(s), \nabla)\). We say that \((E, \nabla)\) is \textit{unramified over} \( R((x_s)) \) if it admits a Turrittin-Levelt-Jordan decomposition over \( R((x_s)) \). In special, if \( s = 1 \) then \((E, \nabla)\) is called \textit{unramified over} \( R((x)) \).

Let us consider a special class of connections which exists Jordan forms.

**Definition 4.3.** Let \((E, \nabla) \in \text{MC}(R((x))/R)\) and \( s \in \mathbb{N} \). The Turrittin-Levelt-Jordan decomposition over \( R((x_s)) \) of \((E, \nabla)\) is called **strong** if each \((E(s)_\varphi, \nabla_\varphi)\) in \(\text{[19]}\) has the following property: there exist an \( R((x_s))\)-basis \( \{e_\varphi\} \) of \((E(s)_\varphi)\) such that
\[
\text{Mat}(\nabla_\varphi, e_\varphi) = J_{r_\varphi}(f_\varphi),
\]
where \( f_\varphi \in R((x_s)) \) satisfies \([f_\varphi] = \varphi\).

We note that in the case \( R = C \) the Turrittin-Levelt-Jordan decomposition of a connection is exactly strong. In general, it is not true.

**Example 4.4.** Let \( E = R((x)) \cdot e_1 \oplus R((x)) \cdot e_2 \) and define \( \nabla(e_1) = 0; \nabla(e_2) = t e_1 \). It is an unramified connection over \( R((x)) \). We obverse that \((E, \nabla)\) has only an isotypical component of type 0. However, the matrix of \( \nabla \) in the basis \( \{e_1, e_2\} \) is
\[
\text{Mat}(\nabla, e) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}
\]
which shows that there does not exist the strong Turrittin-Levelt-Jordan decomposition of \((E, \nabla)\) over \( R((x)) \).

**Remark 4.5.** Let \((E, \nabla) \in \text{MC}(R((x))/R)\) be a connection over \( R((x)) \) which admits a strong Turrittin-Levelt-Jordan decomposition over \( R((x_s)) \) with \( s \in \mathbb{N} \). Let us fix its Jordan decomposition as in eq. \(\text{[19]}\).
Definition 4.6. The images of the constants term of \( f_\varphi \)'s in \( R/s^{-1}Z \) are called Turrittin exponents of \((E, \nabla)\). The set \( \{p(\varphi) : \varphi \in \Phi\} \) is called the set of irregular values of \( \nabla \) and shall be denoted by \( \text{Irr}(E, \nabla) \).

Example 4.7. Let \((E, \nabla) \in \text{MC}_{rs}(R((x))/R)\). According to [HdST22, Theorem 8.15] and [HdST22, Corollary 8.13], there exists a finite free \( R \)-module \( V \) and \( A \in \text{End}_R(V) \) such that \((E, \nabla) \simeq \gamma\text{eul}(V, A)\). Let \( v \) be a certain \( R \)-basis of \( V \) such that

\[
\text{Mat}(A, v) = \text{diag}(J_{r_1}(\lambda_1), \cdots, J_{r_n}(\lambda_n)),
\]

where \( \lambda_\ell \in R \). Hence, we obtain

\[
\text{Mat}(\nabla, v) = \text{diag}(J_{r_1}(\lambda_1), \cdots, J_{r_n}(\lambda_n)).
\]

Therefore, \((E, \nabla)\) admits a strong Turrittin-Levelt-Jordan decomposition over \( R((x)) \).

Furthermore, according to the proof of [H72, Section 4.2, Proposition], we obtain the following result.

Proposition 4.8. Let \((E, \nabla) \in \text{MC}_{rs}(R((x))/R)\) and a finite free module \( V \) over \( R \) equipped with an endomorphism \( A \in \text{End}_R(V) \) such that

\[
(E, \nabla) \simeq \gamma\text{eul}(V, A),
\]

(cf. [HdST22, Theorem 8.15]). Assume that all eigenvalues of \( A \) belong to \( R \) and their differences are invertible elements in \( R \) if they are not zero. Then there exists a Turrittin-Levelt-Jordan decomposition over \( R((x)) \) for \((E, \nabla)\).

Proof. Let us denote \( \chi_A(T) \) for the characteristic polynomial of \( A \). Then,

\[
\chi_A(T) = \prod_{i=1}^m (T - \lambda_i)^{q_i},
\]

where \( \lambda_1, \ldots, \lambda_m \in R \). According to Example 4.7, it is enough to show that there exists a Jordan-Chevalley decomposition of \((V, A)\) over \( R \). Since the pairwise differences \( \lambda_i - \lambda_j \) are invertible in \( R \) for all \( i \neq j \), the polynomials \( (T - \lambda_i)^{q_i} \) are comaximal in \( R[T] \). Using the Chinese Remainder Theorem, we can find a polynomial \( p(T) \) satisfying the congruences

\[
p(T) - \lambda_i \pmod{(T - \lambda_i)^{q_i}}
\]

for all \( i = 1, \cdots, m \). Because \( 1 = \sum_{i=1}^m \frac{\pi_{j \neq i}(T - \lambda_j)}{\pi_{j \neq i}(\lambda_i - \lambda_j)} \) in \( R[T] \), so

\[
V = \bigoplus_{i=1}^m V_i,
\]
where $V_i = \ker(A - \lambda_i)^n$. Hence, the restriction $A_s = p(A) - \lambda_i \text{id}|_{V_i}$ is zero for each $i = 1, \ldots, m$. Therefore, $p(A)$ acts diagonally on $V_i$ with single eigenvalue $\lambda_i$. On the other hand, by definition, the endomorphism $A_n = A - A_s$ is nilpotent and commutes with $A_s$. Consequently, the decomposition $A = A_s + A_n$ is the Jordan-Chevalley decomposition of $A$ over $R$. 

We now arrive at

**Theorem 4.9** (Logarithmic decomposition). Let $(E, \nabla) \in \text{MC}(R((x))/R)$ admit a strong Turrittin-Levelt-Jordan decomposition

$$\nabla = \nabla^\text{rs} + P_N$$

such that the following conditions are satisfied:

(i) $(E, \nabla^\text{rs}) \in \text{MC}_\text{rs}(R((x))/R)$ and admits a strong Turrittin-Levelt-Jordan decomposition over $R((x))$.

(ii) $P_N$ preserves the strong Turrittin-Levelt-Jordan decomposition of $(E, \nabla^\text{rs})$.

(iii) For each isotypical component $E_\theta$ of $(E, \nabla^\text{rs})$, there exists its $R((x))$–basis $(e_\theta)$ such that

$$\text{Mat}(\nabla^\text{rs}|_{E_\theta}, e_\theta) = J_{r_\theta}(c_\theta), \quad \text{Mat}(P_N|_{E_\theta}, e_\theta) = f_\theta I_{r_\theta}$$

where $c_\theta \in R$, $f_\theta \in x^{-1}R[x^{-1}]$ and $r_\theta = \text{rank}_{R((x))}E_\theta$.

**Proof. Existence.** For each $\varphi \in \Phi$, we define $\nabla^\text{rs}_\varphi := \nabla - p(\varphi)$ on $E_\varphi$. We shall prove that $(E_\varphi, \nabla^\text{rs}_\varphi) \in \text{MC}_\text{rs}(R((x))/R)$. Indeed, there exist an $R((x))$–basis $(e_\varphi)$ such that

$$\text{Mat}(\nabla^\text{rs}_\varphi, e_\varphi) = J_{r_\varphi}(f_\varphi).$$

Hence, we have

$$\text{Mat}(\nabla^\text{rs}_\varphi, e_\varphi) = J_{r_\varphi}(c_\varphi),$$

where $c_\varphi = f_\varphi - p(\varphi) \in R$. Therefore, $(E_\varphi, \nabla^\text{rs}_\varphi) \in \text{MC}_\text{rs}(R((x))/R)$ is a regular singular connection over $R((x))$. Let us define a connection $\nabla^\text{rs}$ on $E$ which is defined by $\nabla^\text{rs}_\varphi$ on each $E_\varphi$ and $P_\nabla = \nabla - \nabla^\text{rs}$. In addition, the equation (22) shows that $(E, \nabla^\text{rs})$ is a successive extension of rank one objects. Moreover, $(E, \nabla)$ admits a strong Turrittin-Levelt-Jordan decomposition $E = \bigoplus_{\theta \in \Theta} E_\theta$. Finally, it is not difficult to see that the $R((x))$–linear map $P_N$ satisfies the condition (iii).

**Uniqueness.** Because $(E, \nabla) = \bigoplus_{\varphi \in \Phi} (E_\varphi, \nabla_\varphi)$ is a strong Turrittin-Levelt-Jordan decomposition of $(E, \nabla)$, so we can write

$$\nabla = S_\nabla + N_\nabla,$$
where \( S_\nabla \in \text{MC}(R((x))/R) \) is diagonalizable and \( N_\nabla \in \text{End}_{R((x))}(E) \) is a nilpotent \( R((x)) \)-linearly endomorphism of \( E \). In addition, \([S_\nabla, N_\nabla] = 0 \) in \( \text{End}_{R}(E) \). We note that the Jordan-Levelt decomposition of \( \nabla \) on \( E_{K((x))} \) is defined by
\[
\nabla = S_\nabla + N_\nabla.
\]
Hence, using the descent condition \( \text{(ABC20, 17.5.4)} \), it implies that the decomposition \( (23) \) is uniquely defined.

Finally, by adapting the proof of the last part of the proof of Theorem 2.14 with the following replacement: the uniqueness of Jordan-Levelt decomposition (Theorem 2.6) is replaced by the uniqueness the decomposition \( (23) \) above, we obtain that the decomposition \( (21) \) is unique. \( \square \)

In \( \text{(HdST22)} \), Deligne-Manin models are quite significant in the study regular singular connections over \( R((x)) \). Such lattice can be defined by the following assertion.

**Proposition 4.10** (Deligne-Manin-Malgrange lattice). Let \((E, \nabla) \in \text{MC}(R((x))/R)\) admit a strong Turrittin-Levelt-Jordan decomposition \( (E, \nabla) = \bigoplus_{\varphi \in \Phi} (E_\varphi, \nabla_\varphi) \) over \( R((x)) \), where \( \varphi \in \Phi \subset R[x^{-1}]/\mathbb{Z} \) are characters of \((E, \nabla)\). Assume that \( f_\varphi \in R[x^{-1}] \) such that \([f_\varphi] = \varphi \) and their differences are invertible in \( R((x)) \) unless they are zero. There exists a \( R[x] \)-lattice \( \mathcal{E} \) for \( E \) having the following properties:

(i) \( \mathcal{E} \) is the Deligne-Manin lattice for \((E, \nabla^{rs})\) with respect to \( \tau \) (cf. \( \text{[HdST22]} \)), where \( \nabla^{rs} \) is the regular singular part of \( \nabla \) in the sense of Theorem 4.9.

(ii) There exists a decomposition \( \mathcal{E} = \bigoplus_{\varphi \in \Phi} \mathcal{E}_\varphi \) of \( \mathcal{E} \) such that \( \mathcal{E}_\varphi \) is the Deligne-Manin lattice for \((E_\varphi, \nabla^{rs}_\varphi)\) with respect to \( \tau \).

**Proof.** The \( R[x] \)-module \( \mathcal{E} \) is the Deligne-Manin lattice for the connection \((E, \nabla^{rs})\) which defined by Theorem 4.9. \( \square \)

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