Study of unstable particle through the spectral function in $O(4)\phi^4$ theory*

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We test application of the maximum entropy method to decompose the states contributing to the unstable $\sigma$ correlation function through the spectral function in the four dimensional $O(4)\phi^4$ theory. Reliable results are obtained for the $\sigma$ mass and two-particle $\pi\pi$ state energy using only the $\sigma$ correlation function. We also find that the property of the $\sigma$ particle is different between the unstable ($m_\sigma/m_\pi > 2$) and stable ($m_\sigma/m_\pi < 2$) cases.

1. Introduction

The decay of particles, e.g., $\rho$ meson decay is not understood well on the lattice. One of the difficulties is that an unstable particle and its intermediate states, such as the $\rho$ meson and the $\pi\pi$ state in the isospin-1 channel, have the same quantum numbers, leading to a multi-exponential form of the unstable particle correlation function. To obtain the energies of the states we have to decompose the states in the correlation function. In a finite volume it is possible to decompose the states with the spectral function, which can be numerically reconstructed from the correlation function with the maximum entropy method (MEM). MEM has been recently applied to meson correlation functions in quenched lattice QCD [1]. Here we explore application of MEM to calculate the energies of states in unstable particle systems.

2. Model and parameters

Since the calculation of the unstable particle is difficult in QCD, we use the four dimensional $O(4)\phi^4$ theory in this work. In this theory the $\sigma$ particle and two-pion state in the isospin-0 channel have the same quantum numbers. These states overlap in the $\sigma$ correlation function.

We calculate the correlation functions for $\sigma$ and $\pi\pi$ with zero momentum for the two cases, one where $\sigma$ is unstable ($m_\sigma/m_\pi \approx 3.7$) and the other stable ($m_\sigma/m_\pi \approx 1.8$). Several spatial lattice sizes in the range $10^3 - 28^3$ are employed to study the volume dependences of the spectral functions, while the temporal lattice size is fixed to 64. We perform 0.6 to 1.2 million iterations per simulation point and lattice size. The method of simulation and measurements of the correlation functions follow ref. [2]. We calculate $\langle (O(\tau) - O(\tau + 1))O(0) \rangle$ to subtract the vacuum contribution where $O$ denotes $\sigma$ or $\pi\pi$ operator.

3. Results

3.1. Spectral function data

In Fig.1 we illustrate the $\sigma$ and $\pi\pi$ correlation functions at two volumes in the unstable case. For smaller volume the slopes of the two correlation functions agree, while for larger volume the slopes are different and the $\sigma$ correlation function is of multi-exponential form. This means that the overlap of the $\pi\pi$ state dominates the $\sigma$ correlation function for small volume, while it decreases as volume increases.

The spectral function reconstructed from the correlation function at various volumes in the unstable case is presented in Fig. 2. The parameters for MEM are compiled in ref. [3]. We find the spectral functions to be a sum of $\delta$-function peaks as expected.

The first peak corresponds to the $\pi\pi$ state with zero momentum and the second peak to the $\sigma$
state. As seen from the peak height, the overlap of the $\pi\pi$ state in the $\sigma$ spectral function decreases with increasing volume. This explains the difference of behavior of the $\sigma$ correlation function in Fig. 1. The $\sigma$ peak in the $\pi\pi$ spectral function also decreases as the volume increases.

In the $\sigma$ spectral function the downward arrow marks the position where the $\pi\pi$ state with relative momentum $p = 2\pi/L$ should appear. The state is not observed, showing that the overlap is too small in these small volumes.

### 3.2. Sigma mass and two-pion energy

In the unstable case, the $\sigma$ mass and $\pi\pi$ energy are obtained from the positions of the peaks in both the $\sigma$ and $\pi\pi$ spectral functions. We compare results from this analysis with those obtained with the diagonalization of the correlation function matrix [4]. The results are shown in Fig. 3: MEM($\sigma$) and MEM($\pi\pi$) denote data obtained from the $\sigma$ and $\pi\pi$ spectral functions, respectively. For larger volumes the $\sigma$ mass from MEM($\pi\pi$) and the $\pi\pi$ energy from MEM($\sigma$) suffer from large errors or do not appear, because the overlaps of the states are small for large volumes. We find the $\sigma$ mass from MEM to be reasonably consistent with that from the diagonalization. For the $\pi\pi$ energy all results agree well.

In the stable case we also obtain the $\sigma$ mass and $\pi\pi$ energy with MEM and diagonalization. The results are presented in Fig. 4. The two set of data are reasonably consistent.

Comparing Fig. 3 with Fig. 4, we find that there is an essential difference in the volume dependence of the $\sigma$ mass and $\pi\pi$ energy between the unstable and stable cases. In the unstable case the $\sigma$ mass decreases as the volume increases while it increases in the stable case. The $\pi\pi$ energy also shows a different volume dependence for the two cases.

### 3.3. Volume dependence of $m_\sigma$

In order to understand the volume dependence of $\sigma$ mass, we fit our data from diagonalization assuming the perturbative form in a finite volume given by $m_\sigma(L) = m_\sigma + g_R(\Delta m_\sigma(L) - \Delta m_\sigma(\infty))$ where $\Delta m_\sigma(L) = \frac{1}{\sin^2\frac{\pi}{L}} \sum_{\vec p} \left[ \frac{D_{\sigma\sigma}(p)}{W_{\sigma\sigma}(p)} + \frac{D_{\pi\pi}(p)}{W_{\pi\pi}(p)} + 2C_\pi(p) + 6C_\sigma(p) \right]$ with $D = 1 - \frac{3(m_\sigma^2 - m_\pi^2)}{m_\sigma^2}, \quad C_\alpha(p) = \frac{m_\sigma^2 - m_\pi^2}{W_{\alpha\alpha}(p)(m_\sigma^2 - W_{\alpha\alpha}(p))},$ and $W_{\alpha\alpha}(p) = 2\sqrt{m_\alpha^2 + \vec p^2}, \quad \hat p_i = 2\sin(p_i/2), \quad p_i = 2\pi n_i/L$. Here $m_\sigma$ is $\sigma$ mass in the infinite volume and $g_R$ is the renormalized coupling, which are the fit parameters. We employ simulation results for $m_\sigma$ obtained from the pion correlation function at $L = 28$. The fit results are shown by dashed lines in Fig. 3 and Fig. 4; they agree quite well with the simulation results. We note that the difference in volume dependence is caused by
the sign of $C_{\pi}(p)$ between the unstable and stable cases.

3.4. Volume dependence of $E_{\pi\pi}$

To understand the volume dependence of $\pi\pi$ energy, we use the scattering length $a_0$ that relates to the difference of $E_{\pi\pi}$ from $2m_\pi$ through Lüscher’s formula [5] given by $E_{\pi\pi} - 2m_\pi = -\frac{4\pi a_0}{m_\pi L^2} \left(1 + c_1 \frac{a_0}{L} + c_2 \frac{a_0^2}{L^2}\right)$, with $c_1 = -2.837297$, $c_2 = 6.375183$. We employ the $\pi\pi$ energy obtained with diagonalization. The results are presented in Table 1. The scattering length exhibits an opposite sign between the two cases.

We calculate the perturbative scattering length from the perturbative phase shift [2] and find $a_0 = \frac{g_R m_\pi}{96\pi m_N} \frac{2R^2 + 8}{R - 4}$, where $R = m_\sigma/m_\pi$. The renormalized coupling $g_R$ is already determined in the $\sigma$ mass fit. Alternatively one may use the perturbative relation $g_R = 3Z_\pi (m_\pi^2 - m_\sigma^2)/v^2$, where $m_\pi$, $Z_\pi$ and $v$ are taken from the simulation results for $L = 28$. We employ the $\sigma$ mass determined from the $\sigma$ mass fit for $m_\sigma$. We compile $a_0$ and $g_R$ from simulations and from perturbation theory in Table 1.

We find results from the simulation to be reasonably consistent with the perturbative results. The sign of the perturbative scattering length depends only on whether the $\sigma$ particle is unstable or stable. From these signs the $\pi\pi$ energy in a finite volume is expected to increase or decrease in the unstable or stable cases, respectively. The behavior of the $\pi\pi$ energy also agrees with our simulation results.

|     | unstable | stable |
|-----|----------|--------|
| $a_0$ | 0.289(9) | -2.49(19) |
| $g_R$ (fit) | 0.383(3) | 14(1) -1.98(23) 9(1) |
| $g_R$ (def.) | 0.496(1) | 18.9(2) -3.62(11) 17.8(1) |

Table 1
Scattering lengths obtained from simulation (S) and perturbative formula (P).

4. Conclusion

We have demonstrated that the maximum entropy method is a simple technique for decomposing the states in the unstable particle correlation function and determining their energies. In future it will be interesting and challenging to apply this method to studies of the $\rho$ meson decay and the $\sigma$ particle in QCD.

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