Affine spherical homogeneous spaces
with good quotient by a maximal unipotent subgroup

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Abstract. For an affine spherical homogeneous space \( G/H \) of a connected semisimple algebraic group \( G \), we consider the factorization morphism by the action on \( G/H \) of a maximal unipotent subgroup of \( G \). We prove that this morphism is equidimensional if and only if the weight semigroup of \( G/H \) satisfies a simple condition.

Bibliography: 16 titles.

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§ 1. Introduction

Let \( G \) be a connected semisimple complex algebraic group and let \( U \) be a maximal unipotent subgroup of \( G \). Consider a homogeneous space \( G/H \) with a finitely generated algebra \( \mathbb{C}[G/H] = \mathbb{C}[G]^H \) of regular functions on it. In this situation, the algebra \( U \mathbb{C}[G/H] \), which consists of regular functions on \( G/H \) that are invariant under the action of \( U \) on the left, is also finitely generated (see [1], Theorem 3.1). Therefore one may consider the corresponding factorization morphism

\[
\pi_U : X = \text{Spec } \mathbb{C}[G/H] \to Y = \text{Spec } U \mathbb{C}[G/H].
\]

The algebra \( \mathbb{C}[G/H] \) is a rational \( G \)-module with respect to the action of \( G \) on the left and decomposes into a direct sum of finite-dimensional irreducible \( G \)-modules. The highest weights of irreducible \( G \)-modules that occur in this decomposition form a semigroup called the weight semigroup of \( G/H \). We denote this semigroup by \( \Lambda_+(G/H) \).

A subgroup \( H \subset G \) (or a homogeneous space \( G/H \)) is said to be spherical if a Borel subgroup \( B \subset G \) has an open orbit in \( G/H \). The algebra \( \mathbb{C}[G/H] \) for a spherical homogeneous space \( G/H \) is finitely generated (see [2]), therefore the morphism \( \pi_U \) is well defined. Further, it is known (see [3], Theorem 1) that the \( G \)-module \( \mathbb{C}[G/H] \) for a spherical homogeneous space \( G/H \) is multiplicity free (the converse is also true in the case of quasi-affine \( G/H \)), that is, every irreducible submodule occurs in this \( G \)-module with multiplicity at most 1. In this situation, if we fix a highest weight vector (with respect to \( U \)) in each irreducible submodule

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of $\mathbb{C}[G/H]$, then these vectors form a basis of the algebra $U_+\mathbb{C}[G/H]$ (regarded as a vector space over $\mathbb{C}$). Moreover, if we normalize these vectors in an appropriate way, we can establish a natural isomorphism between $U_+\mathbb{C}[G/H]$ and the semigroup algebra of the semigroup $\Lambda_+(G/H)$ (see [4], Theorem 2).

Let $\omega_1, \ldots, \omega_l$ be all the fundamental weights of $G$. For every dominant weight $\lambda = k_1 \omega_1 + \cdots + k_l \omega_l$, $k_i \in \{0, 1, 2, \ldots\}$, we introduce its support

$$\text{Supp } \lambda = \{\omega_i \mid k_i > 0\}.$$ 

A spherical homogeneous space $G/H$ is said to be excellent if it is quasi-affine and the semigroup $\Lambda_+(G/H)$ is generated by dominant weights $\lambda_1, \ldots, \lambda_m$ of $G$ satisfying $\text{Supp } \lambda_i \cap \text{Supp } \lambda_j = \emptyset$ for $i \neq j$. For example, consider the (affine) spherical homogeneous space $SL_{2n} / S(L_n \times L_n)$, where $S(L_n \times L_n)$ is the intersection of the group $SL_{2n}$ with the subgroup $GL_n \times GL_n \subset GL_{2n}$. Its weight semigroup is (freely) generated by the weights $\omega_1 + \omega_{2n-1}, \omega_2 + \omega_{2n-2}, \ldots, \omega_{n-1} + \omega_{n+1}, 2\omega_n$, where $\omega_i$ is the $i$th fundamental weight of $SL_{2n}$, hence this space is excellent. It follows from the definition that for an excellent spherical homogeneous space $G/H$ the semigroup $\Lambda_+(G/H)$ is free. Therefore the algebra $U_+\mathbb{C}[G/H]$ is also free and the variety $Y$, which is the spectrum of this algebra, is just the affine space $\mathbb{C}^r$, where $r$ is the rank of $\Lambda_+(G/H)$.

A spherical homogeneous space $G/H$ is said to be almost excellent if it is quasi-affine and the convex cone $\mathbb{Q}_+\Lambda_+(G/H)$, which consists of all linear combinations of elements in $\Lambda_+(G/H)$ with nonnegative rational coefficients, is generated (as a convex cone) by elements $\lambda_1, \ldots, \lambda_m \in \Lambda_+(G/H)$ satisfying $\text{Supp } \lambda_i \cap \text{Supp } \lambda_j = \emptyset$ for $i \neq j$. It is easy to see ([5], Corollary 1) that for a spherical homogeneous space $G/H$ the property of being almost excellent is local, that is, it depends only on the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$.

There is a close connection between excellent and almost excellent spherical homogeneous spaces. First, it is obvious that every excellent spherical homogeneous space is almost excellent. Second, the simply connected covering homogeneous space of every almost excellent spherical homogeneous space is excellent (see [5], Theorem 3).

The following theorem was in fact proved by Panyushev in 1999.

**Theorem 1.** Let $G/H$ be a quasi-affine spherical homogeneous space.

(a) (see [6], Theorem 5.5) If $G/H$ is excellent, then the morphism $\pi_U$ is equidimensional.

(b) (see [6], Theorem 5.1) If $Y \cong \mathbb{C}^r$ for some $r$ and $H$ contains a maximal unipotent subgroup of $G$, then the converse to (a) is also true.

**Remark 1.** When we say that a morphism is equidimensional, we have in mind that it is surjective.

**Remark 2.** In fact, Theorems 5.1 and 5.5 in [6] assert more general facts, formulated in other terms. The term ‘excellent spherical homogeneous space’ appeared in 2007 when Theorem 1 was proved again, independently, by E.B. Vinberg and S.G. Gindikin (unpublished).

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1In [5], the quasi-affinity condition was omitted by mistake in the definition of an almost excellent spherical homogeneous space.
In view of the above connection between excellent and almost excellent spherical homogeneous spaces, Theorem 1 implies the following similar result for almost excellent spherical homogeneous spaces (see also §3).

**Corollary 1.** Let $G/H$ be a quasi-affine spherical homogeneous space.

(a) If $G/H$ is almost excellent, then the morphism $\pi_U$ is equidimensional.

(b) If $H$ contains a maximal unipotent subgroup of $G$, then the converse to (a) is also true.

As can be seen, Theorem 1(b) (Corollary 1(b)) is the converse of Theorem 1(a) (Corollary 1(a)) for spherical homogeneous spaces $G/H$ such that $H$ contains a maximal unipotent subgroup of $G$ (such subgroups $H$ are said to be *horospherical*).

The goal of this paper is to establish the converse of Theorem 1(a) and Corollary 1(a) in the case of affine spherical homogeneous spaces $G/H$, that is, in the case where $H$ is reductive. Namely, in this paper we prove the following theorem.

**Theorem 2.** Let $G/H$ be an affine spherical homogeneous space.

(a) If the morphism $\pi_U$ is equidimensional, then $G/H$ is almost excellent.

(b) If the morphism $\pi_U$ is equidimensional and $Y \cong \mathbb{C}^r$ for some $r$, then $G/H$ is excellent.

It would undoubtedly be interesting to understand to what extent this result can be generalized to the case of arbitrary quasi-affine spherical homogeneous spaces.

Theorem 1(a), Corollary 1(a), and Theorem 2 imply the following geometric characterization of excellent and almost excellent affine spherical homogeneous spaces.

**Corollary 2.** Let $G/H$ be an affine spherical homogeneous space.

(a) $G/H$ is almost excellent if and only if the morphism $\pi_U$ is equidimensional.

(b) $G/H$ is excellent if and only if the morphism $\pi_U$ is equidimensional and $Y \cong \mathbb{C}^r$ for some $r$.

The paper is organized as follows. In §3 we reformulate Theorem 2 in a form which is more convenient to prove (see Theorem 3). In §§4–7 we collect all the concepts and results we need to prove Theorem 3. Namely, in §4 we recall the classification of affine spherical homogeneous spaces; in §5 we prove that under some restrictions on a homogeneous space $G/H$ the null fibre of the morphism $\pi_U$ is nonempty; in §6 we consider symmetric linear actions of tori; in §7 we recall the notion of the extended weight semigroup of a homogeneous space. We prove Theorem 3 in §§8, 9. More precisely, in §8 we reduce the proof of this theorem to the case of strictly irreducible spaces, which in turn is considered in §9.

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**§2. Some notation and conventions**

In this paper the base field is the field $\mathbb{C}$ of complex numbers, all topological terms relate to the Zarisky topology, all groups are assumed to be algebraic and their subgroups closed. The tangent algebras of groups denoted by upper case Latin letters are denoted by the corresponding lower case German letters. For every group $L$ $\mathcal{X}(L)$ denotes the character lattice of $L$. 
Throughout the paper, \( G \) stands for a connected semisimple algebraic group. We assume a Borel subgroup \( B \subset G \) and a maximal torus \( T \subset B \) to be fixed. We denote by \( U \) the maximal unipotent subgroup of \( G \) contained in \( B \). We identify the lattices \( X(B) \) and \( X(T) \) by restricting characters from \( B \) to \( T \).

The actions of \( G \) on itself by left translation \( ((g, x) \mapsto gx) \) and right translation \( ((g, x) \mapsto xg^{-1}) \) induce representations of \( G \) on the space \( \mathbb{C}[G] \) of regular functions on \( G \) by the formulae

\[
(gf)(x) = f(g^{-1}x), \quad (gf)(x) = f(xg),
\]

respectively. For brevity, we refer to these actions as the action on the left and on the right, respectively. For every subgroup \( L \subset G \), we let \( L \mathbb{C}[G] (\mathbb{C}[G] L) \) denote the algebra of functions in \( \mathbb{C}[G] \) that are invariant under the action of \( L \) on the left (on the right, respectively).

Two homogeneous spaces \( G_1/H_1 \) and \( G_2/H_2 \) are said to be locally isomorphic if their simply connected homogeneous covering spaces are isomorphic. This is equivalent to the existence of an isomorphism \( g_1 \rightarrow g_2 \) taking \( h_1 \) to \( h_2 \).

Without loss of generality, for every simply connected homogeneous space \( G/H \) we assume that \( G \) is simply connected and \( H \) is connected.

For a group \( L \), the notation \( L = L_1 \cdot L_2 \) signifies that \( L \) is an almost direct product of subgroups \( L_1, L_2 \), that is, \( L = L_1L_2 \), the subgroups \( L_1, L_2 \) commute elementwise and the intersection \( L_1 \cap L_2 \) is finite.

Notation:
- \( e \) is the identity element of an arbitrary group;
- \( \mathbb{C}^\times \) is the multiplicative group of the field \( \mathbb{C} \);
- \( V^* \) is the space of linear functions on a vector space \( V \);
- \( \Lambda_+(G) \subset X(B) \) is the semigroup of dominant weights of \( G \) with respect to \( B \);
- \( V(\lambda) \) is the irreducible \( G \)-module with highest weight \( \lambda \in \Lambda_+(G) \);
- \( v_\lambda \in V(\lambda) \) is a highest weight vector in \( V(\lambda) \) with respect to \( B \);
- \( \lambda^* \) is the highest weight of the irreducible \( G \)-module \( V(\lambda)^* \);
- \( L^0 \) is the connected component of the identity of a group \( L \);
- \( L' \) is the derived subgroup of a group \( L \);
- \( Z(L) \) is the centre of a group \( L \);
- \( N_L(K) \) is the normalizer of a subgroup \( K \) in a group \( L \);
- \( \text{diag}(a_1, \ldots, a_n) \) is the diagonal matrix of order \( n \) with elements \( a_1, \ldots, a_n \) on the diagonal.

§ 3. The reformulation of the main theorem

In this section we reduce proving Theorem 2 to proving the following theorem.

**Theorem 3.** Suppose that \( G/H \) is a simply connected affine spherical homogeneous space such that the morphism \( \pi_U \) is equidimensional. Then \( G/H \) is excellent.

Let \( G/H \) be a simply connected spherical homogeneous space. Every homogeneous space that is locally isomorphic to \( G/H \) has the form \( G/\tilde{H} \), where \( \tilde{H} \) is a finite extension of \( H \), that is, \( \tilde{H}^0 = H \). Put \( \tilde{Y} = \text{Spec} U \mathbb{C}[G/\tilde{H}] \) and consider the
following commutative diagram:

\[
\begin{array}{ccc}
G/H & \overset{\pi_U}{\longrightarrow} & Y \\
\downarrow & & \downarrow \\
G/\tilde{H} & \overset{\tilde{\pi}_U}{\longrightarrow} & \tilde{Y}
\end{array}
\]

In this diagram, the vertical arrows correspond to the factorization morphisms by the finite group $\tilde{H}/H$, which implies that the condition that $\pi_U$ be equidimensional is equivalent to the condition that $\tilde{\pi}_U$ be equidimensional. In view of this, Theorem 3 implies Theorem 2(a). Now we will deduce part (b) of Theorem 2 from part (a). To do this we note that an almost excellent spherical homogeneous space $G/H$ is excellent if and only if the semigroup $\Lambda_+(G/H)$ is free. But this is equivalent to the condition that the algebra $U_C[G/H]$ be free, that is, $Y \simeq C^r$ for some $r$.

§ 4. Classification of affine spherical homogeneous spaces

In this section we recall the known classification, up to a local isomorphism, of all affine spherical homogeneous spaces or, equivalently, the classification, up to an isomorphism, of all simply connected affine spherical homogeneous spaces. Before we proceed, we recall some concepts.

A direct product of spherical homogeneous spaces

\[(G_1/H_1) \times (G_2/H_2) = (G_1 \times G_2)/(H_1 \times H_2)\]

is again a spherical homogeneous space. Spaces of this type, and also spaces locally isomorphic to them, are said to be reducible and all others are said to be irreducible. A spherical homogeneous space $G/H$ is said to be strictly irreducible if the spherical homogeneous space $G/N_G(H)^0$ is irreducible (see [7], § 1.3.6).

Up to a local isomorphism, the list of all strictly irreducible affine spherical homogeneous spaces $G/H$ is known: in the case of simple $G$ it was obtained in [8] and in the case of nonsimple semisimple $G$ it was obtained in [9] and independently, by another method, in [10]. This list is collected in its entirety in Tables 1 and 2 in [5]. (Although different parts of this list can be found in many other papers including the original papers mentioned above.) In its final shape, the general procedure for obtaining arbitrary affine spherical homogeneous spaces starting from strictly irreducible ones is given\(^2\) in [11] (see the description below).

We note that for all simply connected strictly irreducible affine spherical homogeneous spaces $G/H$ the corresponding weight semigroups are known. For simple $G$ these subgroups were computed in [8] and in the case of nonsimple semisimple $G$ they were computed in [12]. (More precisely, in [12] the corresponding extended weight semigroups were computed; these are defined in § 7.)

\(^2\)Originally a procedure of this kind was suggested in [9], however it proved to be wrong. In [11] the error is pointed out and a correct version of the procedure given.
We subdivide the strictly irreducible affine spherical homogeneous spaces into the following three types.

Type I: \( \dim Z(H) = 1 \), the space \( G/H' \) is not spherical.

Type II: \( \dim Z(H) = 1 \), the space \( G/H' \) is spherical.

Type III: \( \dim Z(H) = 0 \).

We shall now describe the general procedure for constructing arbitrary simply connected affine spherical homogeneous spaces starting with the strictly irreducible ones. Let \( G_1/H_1, \ldots, G_n/H_n \) be simply connected affine spherical homogeneous spaces. Recall that we have taken the groups \( G_1, \ldots, G_n \) to be simply connected and the subgroups \( H_1, \ldots, H_n \) to be connected. Renumbering, if necessary, we may assume that for some \( p, q \), where \( 0 \leq p \leq q \leq n \), the spaces \( G_1/H_1, \ldots, G_p/H_p \) are of type I, the spaces \( G_{p+1}/H_{p+1}, \ldots, G_q/H_q \) are of type II, and the spaces \( G_{q+1}/H_{q+1}, \ldots, G_n/H_n \) are of type III. We put

\[
G = G_1 \times \cdots \times G_n, \quad \tilde{H} = H_1 \times \cdots \times H_n.
\]

Clearly,

\[
Z(\tilde{H}) = Z(H_1) \times \cdots \times Z(H_n), \quad \tilde{H}' = H_1' \times \cdots \times H_n'.
\]

Further, for all \( i = 1, \ldots, n \) we have \( \operatorname{rk} \mathcal{X}(H_i) = \dim Z(H_i) \), whence \( \mathcal{X}(\tilde{H}) \simeq \mathcal{X}(H_1) \oplus \cdots \oplus \mathcal{X}(H_q) \) is a lattice of rank \( q \). Let \( \chi_1, \ldots, \chi_p \) denote the images in \( \mathcal{X}(\tilde{H}) \) of basis elements of the lattices \( \mathcal{X}(H_1), \ldots, \mathcal{X}(H_p) \), respectively.

Let \( Z \) be a connected subgroup of \( Z(\tilde{H})^0 = Z(H_1) \times \cdots \times Z(H_q) \). Put \( H = Z \cdot (H_1' \times \cdots \times H_n') \subset \tilde{H} \). Then we can consider the character restriction map \( \tau : \mathcal{X}(\tilde{H}) \to \mathcal{X}(H) \).

**Theorem 4** ([11], Theorem 3 and Lemma 5). (a) The space \( G/H \) obtained using the above procedure is spherical if and only if the characters \( \tau(\chi_1), \ldots, \tau(\chi_p) \) are linearly independent in \( \mathcal{X}(H) \).

(b) Every simply connected affine spherical homogeneous space is isomorphic to one of the spaces \( G/H \) obtained using the above procedure.

§ 5. The null fibre of the morphism \( \pi_U \)

The main result in this section is Proposition 1.

The following isomorphism of \((G \times G)\)-modules is well known:

\[
\mathbb{C}[G] \simeq \bigoplus_{\lambda \in \Lambda_+(G)} V(\lambda) \otimes V(\lambda^*),
\]

where on the left-hand side the group \((G \times G)\) acts on the left and the right and in each summand on the right-hand side the left (right) factor of \((G \times G)\) acts on the left (right) tensor factor. Under this isomorphism, an element \( v \otimes w \in V(\lambda) \otimes V(\lambda^*) \) corresponds to the function in \( \mathbb{C}[G] \) whose value at an element \( g \in G \) is \( \langle g^{-1}v, w \rangle = \langle v, gw \rangle \), where \( \langle \cdot, \cdot \rangle \) is the natural pairing between \( V(\lambda) \) and \( V(\lambda^*) \simeq V(\lambda^*) \).

As can easily be seen, for a fixed subgroup \( H \subset G \) the subspace \( U \mathbb{C}[G/H] \lambda \subset U \mathbb{C}[G/H] \), consisting of all \( T \)-semi-invariant functions of weight \( \lambda \), corresponds to the subspace \( \langle v_\lambda \rangle \otimes V(\lambda^*)^H \subset V(\lambda) \otimes V(\lambda^*) \) under the isomorphism (1). In particular, this implies that for an element \( \lambda \in \Lambda_+(G) \) the condition \( \lambda \in \Lambda_+(G/H) \)
is equivalent to the condition that the subspace $V(\lambda^*)^H \subset V(\lambda^*)$, consisting of all $H$-invariant vectors, is nontrivial.

For an arbitrary homogeneous space $G/H$ we set $\mathcal{N}(G/H)$ to be the subset of $G/H$ such that all the functions in $U \mathbb{C}[G/H]$ that are $T$-semi-invariant and of nonzero weight vanish.

**Proposition 1.** Suppose that $G = G_1 \times \cdots \times G_s$, where each of the groups $G_i$ is simple. Let $H \subset G$ be a subgroup such that for every simple component $G_i$ of $G$ the projection of $H^0$ to $G_i$ is not unipotent. Then the set $\mathcal{N}(G/H) \subset G/H$ is nonempty.

**Proof.** For $i = 1, \ldots, s$ we put $T_i = T \cap G_i$ so that $T_i$ is a maximal torus in $G_i$ and $T = T_1 \times \cdots \times T_s$. We identify the lattice $\mathfrak{x}(T)$ with a sublattice in $t^*$ by taking each character $\chi \in \mathfrak{x}(T)$ to its differential $d\chi \in t^*$. We look at the rational subspace $t^*_Q = \mathfrak{x}(T) \otimes \mathbb{Q} \subset t^*$ and fix an inner product $(\cdot, \cdot)$ on it, invariant under the Weyl group $W = \tilde{W}_G(T)/T$. By means of this inner product we identify $t^*_Q$ with the rational subspace $t_Q = \{x \in t \mid \xi(x) \in \mathbb{Q} \text{ for all } \xi \in t^*_Q\} \subset t$.

Replacing $H$ by a conjugate subgroup, we may assume that the subgroup $T_H = (T \cap H)^0 \subset H$ is a maximal torus of $H$. We note that

$$\dim_{\mathbb{Q}}(t_Q \cap t_H) = \dim_{\mathbb{C}} t_H.$$ 

This, together with the hypothesis implies that there exists an element $z \in t_Q \cap t_H$ whose projection to each of the subspaces $t_1, \ldots, t_s$ is nonzero. The element $z$, regarded as an element of $\mathfrak{x}(T) \otimes \mathbb{Q}$, is contained in a Weyl chamber $C \subset \mathfrak{x}(T) \otimes \mathbb{Q}$. Again replacing $H$ by a conjugate subgroup (conjugate by a suitable element of $N_G(T)$), without loss of generality we may assume that $C$ is the dominant Weyl chamber.

Suppose that $\lambda \in \Lambda_+(G) \setminus \{0\}$. Consider the irreducible $G$-module $V(\lambda^*)$, with a lowest weight vector $w_{\lambda^*}$, and a $T$-invariant subspace $V'((\lambda^*))$ complementary to $w_{\lambda^*}$. Let $\nu$ be a linear function on $V(\lambda^*)$ taking $w_{\lambda^*}$ to a nonzero value and vanishing on $V'(\lambda^*)$. Then $\nu$ is a highest weight vector of the irreducible $G$-module $V((\lambda^*))^H \simeq V(\lambda)$. Let $f \in U \mathbb{C}[G/H]$ be a $T$-semi-invariant function of weight $\lambda$. Under the isomorphism (1), it corresponds to a vector $v_f \in V((\lambda^*))^H$ such that $f(g) = \nu(gv_f)$ for all $g \in G$. We will show that $v_f \in V'(\lambda^*)$. To do this, it suffices to check that the vector $w_{\lambda^*}$ is not $T_H$-invariant. This will hold if we show that $zw_{\lambda^*} \neq 0$. We have $zw_{\lambda^*} = (z, -\lambda^*)^*w_{\lambda^*} = -(z, \lambda)w_{\lambda^*}$. Recall the following well-known fact: any two fundamental weights $\omega, \omega'$ of $G$ satisfy the inequality $(\omega, \omega') \geq 0$, and equality is attained if and only if $\omega, \omega'$ are fundamental weights of different simple factors of $G$. Since $\lambda \neq 0$, $z \in C$, and the projection of $z$ on each of the subspaces $t_1, \ldots, t_s$ is nonzero, in view of the above fact, we obtain $(z, \lambda) > 0$, whence $zw_{\lambda^*} \neq 0$ and $v_f \in V'(\lambda^*)$. Therefore $f(eH) = 0$. Since $\lambda$ and $f$ are arbitrary, we obtain $eH \in \mathcal{N}(G/H)$.

Let $G/H$ be a homogeneous space such that the algebra $\mathbb{C}[G/H]$ is finitely generated. The *null fibre* of the morphism $\pi_U$ is the subset of $X$ such that all the functions in $U \mathbb{C}[G/H]$ that are $T$-semi-invariant and of nonzero weight vanish.
Corollary 3. In the assumptions of Proposition 1 suppose that the algebra \( \mathbb{C}[G/H] \) is finitely generated. Then the null fibre of \( \pi_U \) is nonempty and intersects \( G/H \).

§ 6. Symmetric linear actions of tori

Suppose we are given a linear action of a quasi-torus \( S \) on a vector space \( V \). We recall that the notation \( V//S \) stands for the categorical quotient for the action \( S : V \), that is, \( V//S = \text{Spec} \mathbb{C}[V]^S \). We also recall that the null fibre (that is, the fibre containing zero) of the factorization morphism \( V \to V//S \) is said to be the null cone.

For every character \( \chi \in \mathfrak{X}(S) \) we denote the weight subspace in \( V \) of weight \( \chi \) with respect to \( S \) by \( V_\chi \). We put

\[
\Phi = \{ \chi \in \mathfrak{X}(S) \mid V_\chi \neq 0 \} \setminus \{0\} \subset \mathfrak{X}(S).
\]

Then, evidently, we have \( V = V_0 \oplus \bigoplus_{\chi \in \Phi} V_\chi \).

Definition 1. The linear action \( S : V \) is said to be symmetric if \( \dim V_\chi = \dim V_{-\chi} \) for every \( \chi \in \Phi \).

For the rest of this section, we assume that \( S \) is a torus and the action \( S : V \) is linear and symmetric.

Set \( c = \dim V_0 \) and \( d = (\dim V - c)/2 \). For the action \( S : V \) there are (not necessarily different) elements \( \chi_1, \ldots, \chi_d \in \mathfrak{X}(S) \) such that there is an \( S \)-module isomorphism \( V \cong V_0 \oplus \bigoplus_{i=1}^d (\mathbb{C}_{\chi_i} \oplus \mathbb{C}_{-\chi_i}) \), where for \( \chi \in \mathfrak{X}(S) \) we take \( \mathbb{C}_\chi \) to be the one-dimensional \( S \)-module on which \( S \) acts by the character \( \chi \).

Definition 2. The action \( S : V \) is said to be excellent if the elements \( \chi_1, \ldots, \chi_d \) are linearly independent in \( \mathfrak{X}(S) \).

The description of linear actions of tori such that the factorization morphism is equidimensional (see [13], §8.1) implies that the action \( S : V \) is excellent if and only if the factorization morphism \( V \to V//S \) is equidimensional. Further, using the method of supports (see [13], §5.4), it is easy to show that the dimension of the null cone of the action \( S : V \) is equal to \( d \). As the dimension of an arbitrary fibre of the morphism \( V \to V//S \) does not exceed that of the null cone (see [13], Corollary 1 to Proposition 5.1), putting what we have said above together with the theorem on the dimensions of fibres of a dominant morphism imply the following proposition.

Proposition 2. The inequality \( \dim V//S \geq c + d \) holds, and equality is attained if and only if the action \( S : V \) is excellent.

§ 7. The extended weight semigroup of a homogeneous space

Let \( H \subset G \) be an arbitrary subgroup. For every character \( \chi \in \mathfrak{X}(H) \) we consider the subspace

\[
V_\chi = \{ f \in \mathbb{C}[G] \mid f(gh) = \chi(h)f(g) \text{ for all } g \in G, h \in H \}
\]

of the algebra \( \mathbb{C}[G] \). It is easy to see that the action of \( G \) on the space \( \mathbb{C}[G] \) on the left preserves \( V_\chi \) for every \( \chi \in \mathfrak{X}(H) \). All pairs of the form \( (\lambda, \chi) \), where \( \lambda \in \Lambda_+(G) \),
\( \chi \in \mathcal{X}(H) \), such that \( V_\chi \) contains the irreducible \( G \)-submodule with highest weight \( \lambda \) form a semigroup. This semigroup is said to be the extended weight semigroup of the homogeneous space \( G/H \). (For a more detailed description see [12], §1.2 or [14], §1.2.) We denote this semigroup by \( \hat{\Lambda}_+(G/H) \). Since \( V_0 = \mathbb{C}[G]^H = \mathbb{C}[G/H] \), we obtain

\[
\hat{\Lambda}_+(G/H) \simeq \{ (\lambda, \chi) \in \hat{\Lambda}_+(G/H) \mid \chi = 0 \}. \tag{2}
\]

We define the subgroup \( H_0 \subset H \) to be the common kernel of all characters of \( H \). This subgroup is normal in \( H \) and contains \( H' \), therefore the group \( H/H_0 \) is commutative (and is thereby a quasi-torus). If \( H \) is connected, then \( H/H_0 \) is also connected and is therefore a torus. In what follows, we identify the groups \( \mathcal{X}(H) \) and \( \mathcal{X}(H/H_0) \) via the natural isomorphism \( \mathcal{X}(H/H_0) \to \mathcal{X}(H) \). We have

\[
\bigoplus_{\chi \in \mathcal{X}(H)} V_\chi = \mathbb{C}[G]^{H_0} = \mathbb{C}[G/H_0].
\]

We note that the action of the group \( T \times H/H_0 \) determines a grading on \( U\mathbb{C}[G/H_0] \) by the semigroup \( \hat{\Lambda}_+(G/H) \) (\( T \) acts on the left and \( H/H_0 \) acts on the right).

We now turn to the situation where \( H \) is a spherical subgroup of \( G \). According to [3], Theorem 1, \( H \) being spherical is equivalent to the condition that the representation of \( G \) on \( V_\chi \) is multiplicity free for every \( \chi \in \mathcal{X}(H) \). This implies that the action of the group \( T \times H/H_0 \) on the space \( U\mathbb{C}[G/H_0] \) is multiplicity free. Further, for a spherical subgroup \( H \) the semigroup \( \hat{\Lambda}_+(G/H) \) is free (see [14], Theorem 2, and for the case of connected \( H \) see [12], Theorem 1), hence the algebra \( U\mathbb{C}[G/H_0] \) is free and isomorphic to the semigroup algebra of the semigroup \( \hat{\Lambda}_+(G/H) \). Consider the affine space \( Y_0 = \text{Spec } U\mathbb{C}[G/H_0] \simeq \mathbb{C}^n \), where \( n = \text{rk } \hat{\Lambda}_+(G/H) \). We equip \( Y_0 \) with the structure of a vector space in such a way that \( (T \times H/H_0) \)-semi-invariant functions that freely generate the algebra \( U\mathbb{C}[G/H_0] \) correspond to the coordinate functions on \( Y_0 \). The action of \( H/H_0 \) on \( U\mathbb{C}[G/H_0] \) on the right corresponds naturally to an action of this group on \( Y_0 \).

**Lemma 1.** For a spherical subgroup \( H \subset G \) the action \( H/H_0 : Y_0 \) is linear. Moreover, if \( (\lambda_1, \chi_1), \ldots, (\lambda_n, \chi_n) \) are all the indecomposable elements of the (free) semigroup \( \hat{\Lambda}_+(G/H) \), then there is an \( H/H_0 \)-module isomorphism \( Y_0 \simeq \mathbb{C}_{-\chi_1} \oplus \cdots \oplus \mathbb{C}_{-\chi_n} \).

**Proof.** Let \( f_1, \ldots, f_n \in U\mathbb{C}[G/H_0] \) be nonzero \( (T \times H/H_0) \)-semi-invariant functions corresponding to the elements \( (\lambda_1, \chi_1), \ldots, (\lambda_n, \chi_n) \) of \( \hat{\Lambda}_+(G/H) \), respectively. These functions freely generate the algebra \( U\mathbb{C}[G/H_0] \). Interpreting \( f_1, \ldots, f_n \) as coordinate functions on \( Y_0 \), we obtain the required result.

**Lemma 2.** For a reductive spherical subgroup \( H \subset G \) the linear action \( H/H_0 : Y_0 \) is symmetric.

**Proof.** Using the isomorphism (1), it is not hard to show that for \( \lambda \in \Lambda_+(G) \) and \( \chi \in \mathcal{X}(H) \) the element \( (\lambda, \chi) \) is contained in \( \hat{\Lambda}_+(G/H) \) if and only if the subspace \( V(\lambda^\ast)^{(H)}(\chi) \subset V(\lambda^\ast) \), consisting of \( H \)-semi-invariant vectors of weight \( \chi \), is one-dimensional. Suppose that \( (\lambda, \chi) \in \hat{\Lambda}_+(G/H) \). As \( H \) is reductive, we have \( V(\lambda^\ast) = V(\lambda^\ast)^{(H)}(\chi) \oplus W \) for some \( H \)-invariant subspace \( W \subset V(\lambda^\ast) \). Let \( \xi \in V(\lambda^\ast)^{(H)} \simeq V(\lambda) \).
be the linear function on \(V(\lambda^*)\) taking a basis vector of \(V(\lambda^*)^{(H)}\) to 1 and vanishing on \(W\). Then \(\xi\) is an \(H\)-semi-invariant element in \(V(\lambda)\) of weight \(-\chi\). Therefore \((\lambda^*, -\chi) \in \hat{\Lambda}_+(G/H)\). Thus the map \((\lambda, \chi) \mapsto (\lambda^*, -\chi)\) is an automorphism of the semigroup \(\hat{\Lambda}_+(G/H)\). In particular, under this automorphism indecomposable elements are taken into indecomposable elements. Hence in view of Lemma 1, we obtain the required result.

§ 8. Reduction of the proof of Theorem 3 to the case of strictly irreducible spaces

Suppose that \(G\) is simply connected and \(H \subset G\) is a connected reductive spherical subgroup. We recall that in § 7 we took \(H_0\) to be the common kernel of all characters of \(H\) and introduced the notation \(Y_0\) for the affine space \(\text{Spec} \mathbb{C}[G/H_0] \cong \mathbb{C}^n\), where \(n = \text{rk} \hat{\Lambda}_+(G/H)\). We put \(X_0 = \text{Spec} \mathbb{C}[G/H_0]\). In our situation, we have \(X = G/H, X_0 = G/H_0\).

The commutative diagram

\[
\begin{array}{ccc}
\mathbb{C}[G/H_0] & \xleftarrow{U} & \mathbb{C}[G/H_0] \\
\uparrow & & \downarrow \\
\mathbb{C}[G/H] & \xleftarrow{U} & \mathbb{C}[G/H]
\end{array}
\]  

(3)

of injective homomorphisms of algebras corresponds to the commutative diagram

\[
\begin{array}{ccc}
G/H_0 = X_0 & \xrightarrow{\varphi_U} & Y_0 \\
\psi_X \downarrow & & \psi \downarrow & \psi_Y \\
G/H = X & \xrightarrow{\pi_U} & Y
\end{array}
\]  

(4)

of dominant morphisms of the respective varieties. We recall that the affine space \(Y_0\) is equipped with the structure of a vector space (see § 7) and that the action \(H/H_0 : Y_0\) is linear (Lemma 1) and symmetric (Lemma 2). In this situation, \(\psi_Y\) is nothing other than the factorization morphism for the action \(H/H_0 : Y_0\).

**Proposition 3.** If the morphism \(\pi_U\) is equidimensional, then the action \(H/H_0 : Y_0\) is excellent.

**Proof.** Suppose that \(\pi_U\) is equidimensional. Since the morphism \(\psi_X\) is also equidimensional, it follows that the morphism \(\psi : X_0 \to Y\) is equidimensional. Assume that the action \(H/H_0 : Y_0\) is not excellent. Let \(c\) and \(d\) be the corresponding characteristics of this action (see § 6). In view of Proposition 2, we have \(\dim Y > c + d\), whence by the theorem on the dimensions of fibres of a dominant morphism the codimension of a generic fibre of \(\psi\) is greater than \(c + d\).

It follows from Theorem 4 that there are simply connected strictly irreducible affine spherical homogeneous spaces \(G_1/H_1, \ldots, G_n/H_n\) such that

\[
G = G_1 \times \cdots \times G_n, \quad H' \times \cdots \times H' \subset H \subset H_1 \times \cdots \times H_n.
\]
Then
\[ G/H_0 \simeq G_1/H'_1 \times \cdots \times G_n/H'_n. \]

For every \( i = 1, \ldots, n \) we put \( U_i = U \cap G_i, T_i = T \cap G_i \). In each algebra \( U_i \mathbb{C}[G_i/H'_i] \subset U \mathbb{C}[G/H_0] \), we fix a subset \( F_i \), consisting of \((T_i \times H_i/H'_i)\)-semi-invariant functions that freely generate this algebra. Then \( F = F_1 \cup \cdots \cup F_n \) is a set of \((T \times H/H_0)\)-semi-invariant functions that freely generate the algebra \( U \mathbb{C}[G/H_0] \).

Let \( \Phi \subset X(H/H_0) \) be the set of nonzero \( H/H_0 \)-weights of all functions in \( F \). Fix a hyperplane \( h \) in the space \( X(H/H_0) \otimes \mathbb{Z} \mathbb{Q} \) that contains no elements of \( \Phi \). Choose one of the half-spaces bounded by \( h \) and denote the set of weights in \( \Phi \) that are contained in this half-space by \( \Phi^+ \). Put \( \Phi^- = -\Phi^+ \) so that \( \Phi = \Phi^+ \cup \Phi^- \). Let \( F^+ \) (\( F^- \)) be the set of functions in \( F \) whose \( H/H_0 \)-weights belong to \( \Phi^+ \) (\( \{0\}, \Phi^- \), respectively). For every \( i = 1, \ldots, n \), also set
\[ F^+_i = F^+ \cap F_i, \quad F^0_i = F^0 \cap F_i, \quad F^-_i = F^- \cap F_i. \]

Clearly, \( c = |F^+| = |F^-| \) and \( d = |F^0| \). It is not hard to deduce that the subset of \( Y_0 \) where all the \( c + d \) functions in \( F^+ \cup F^0 \) vanish simultaneously is contained in the null fibre of \( \psi_Y \) (see [13], §5.4). Then the subset \( \mathcal{N} \) of \( X_0 \) where the same functions vanish is contained in the fibre \( \psi^{-1}(\psi_Y(0)) \) of the morphism \( \psi \) (below we refer to this fibre as the null fibre as well). Let us show that \( \mathcal{N} \neq \emptyset \). As can easily be seen, it is enough to show that the subset of \( G_i/H'_i \) where all the functions in \( F^+_i \cup F^0_i \) vanish is nonempty for every \( i = 1, \ldots, n \). To do that, we consider two possibilities.

1) The group \( H'_i \) is trivial. Then \( H_i \) is a torus. Inspecting the list of all simply connected strictly irreducible affine spherical homogeneous spaces we find that this is only possible for \( G_i = \text{SL}_2 \), \( H_i \simeq \mathbb{C}^\times \). In this case, \( F_i \) contains two functions having nonzero opposite weights with respect to the torus \( H_i/H'_i \). As \( \text{SL}_2 / \mathbb{C}^\times \) is a strictly irreducible affine spherical homogeneous space of type \( \Gamma \), by Theorem 4(a) the images of these weights in \( X(H/H_0) \) are also nonzero. Hence \( |F^+_i| = |F^-_i| = 1 \) and \( F_i = F^+_i \cup F^-_i \). A direct check shows that the subset of \( \text{SL}_2 \) where the unique function in \( F^+_i \) vanishes is nonempty.

2) The group \( H'_i \) is nontrivial. Inspecting the list of all simply connected strictly irreducible affine spherical homogeneous spaces we find that in this case the group \( H'_i \) satisfies the hypothesis of Proposition 1, hence even the subset of \( G_i/H'_i \) where all the functions in \( F_i \) vanish is nonempty.

Thus \( \mathcal{N} \neq \emptyset \). Since \( \mathcal{N} \) is contained in the null fibre of \( \psi \) and is defined in \( X_0 \) by the \( c + d \) functions vanishing, the codimension in \( X_0 \) of the null fibre of \( \psi \) is at most \( c + d \) and therefore is strictly less than the codimension of a generic fibre. Therefore the morphism \( \psi \) is not equidimensional, a contradiction.

**Proposition 4.** Suppose that \( G/H \) is a simply connected affine spherical homogeneous space such that the morphism \( \pi_U \) is equidimensional. Then \( G/H \) is a direct product of several (simply connected) strictly irreducible affine spherical homogeneous spaces.

**Proof.** By Theorem 4 there are simply connected strictly irreducible affine spherical homogeneous spaces \( G_1/H_1, \ldots, G_n/H_n \) such that:

1) \( G = G_1 \times \cdots \times G_n \);

2) \( H'_1 \times \cdots \times H'_n \subset H \subset H_1 \times \cdots \times H_n \).

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Without loss of generality we may assume that the space $G_i/H_i$ is of type I or II for $i \leq q$ and is of type III for $i > q$. For every $i=1,\ldots,n$, put $U_i = G_i \cap U$.

Let $\tau: \mathfrak{X}(H_1 \times \cdots \times H_n) \to \mathfrak{X}(H)$ be the character restriction map. For $i=1,\ldots,q$ denote a basis character in $\mathfrak{X}(H_i) = \mathfrak{X}(H_i/H'_i)$ by $\chi_i$. Renumbering if necessary, we may assume that $\tau(\chi_i) \neq 0$ for $i \leq p$ and $\tau(\chi_i) = 0$ for $p < i \leq q$. Since for every $i = 1,\ldots,p$ the (linear) action of the torus $H_i/H'_i$ on the affine space Spec $U_i \mathbb{C}[G_i/H'_i]$ is nontrivial, Proposition 3 implies that the weights $\tau(\chi_1), \ldots, \tau(\chi_p)$ are pairwise distinct and linearly independent in $\mathfrak{X}(H)$. It follows that

$$H = H_1 \times \cdots \times H_p \times H'_{p+1} \times \cdots \times H'_n,$$

$$G/H \simeq G_1/H_1 \times \cdots \times G_p/H_p \times G_{p+1}/H'_{p+1} \times \cdots \times G_n/H'_n.$$

§ 9. The case of strictly irreducible spaces

In view of Proposition 4, to complete the proof of Theorem 3 it remains to prove the following proposition.

**Proposition 5.** Suppose that $G/H$ is a simply connected strictly irreducible affine spherical homogeneous space that is not excellent. Then the morphism $\pi_U$ is not equidimensional.

**Proof.** Since we have at our disposal the classification (or more precisely, a complete list) of simply connected strictly irreducible affine spherical homogeneous spaces (see § 4), it suffices to consider all nonexcellent spaces among them case by case and prove the assertion by a direct check in each case. We recall that, up to an isomorphism, the list of all simply connected strictly irreducible affine spherical homogeneous spaces is collected in Tables 1 and 2 in [5]. In these tables it is also indicated whether each of the spaces is excellent or not. Hence we obtain a list of the spaces we need to consider in order to prove the proposition. Before we proceed to this list, we shall introduce some additional conventions and notation.

If $G$ is a product of several factors, then we denote by $\pi_i$, $\varphi_i$, and $\psi_i$ the $i$th fundamental weight of the first, second, and third factor, respectively, and we use the same numeration of the fundamental weights of simple groups as in the book [15].

We denote the identity matrix of order $m$ by $E_m$ and the matrix of order $m$ with ones on the antidiagonal and zeros elsewhere by $F_m$.

For every matrix denoted by a capital letter, the corresponding lower case letter with a double index $ij$ stands for the element in the $i$th row and the $j$th column of this matrix. For example, $P$ is a matrix and $p_{ij}$ is the element in its $i$th row and $j$th column.

A basis $e_1, \ldots, e_n$ of the space of the tautological linear representation of the group $SO_n$ is assumed to be chosen in such a way that the matrix of the invariant nondegenerate symmetric bilinear form is $F_n$. A basis $e_1, \ldots, e_{2m}$ of the space of the tautological linear representation of the group $Sp_{2m}$ is assumed to be chosen in such a way that the invariant nondegenerate skew-symmetric bilinear form has the matrix

$$\begin{pmatrix} 0 & F_m \\ -F_m & 0 \end{pmatrix}.$$
With these choices of bases we can (and will) assume that for every simple factor \( G \) of the groups \( B \cap G, U \cap G, \) and \( T \cap G \) consist of the upper-triangular, upper unitriangular, and diagonal matrices, respectively, contained in \( \overline{G} \).

We now proceed to look at all the simply connected strictly irreducible affine spherical homogeneous spaces \( G/H \) that are not excellent. We look at two separate cases depending on the value \( \dim Z(H) \).

**Case 1.** \( \dim Z(H) = 1 \). For every space \( G/H \) under consideration we indicate the indecomposable elements of the semigroup \( \Lambda_+(G/H) \). By Lemma 1 these elements determine the \( H/H_0 \)-module structure in \( Y_0 \) completely. For every space \( G/H \) it is easy to check that the action \( H/H_0 : Y_0 \) is not excellent, whence by Proposition 3 the morphism \( \pi_U \) is not equidimensional.

1) \( G = SL_{2n+1}, H = C^\times \times Sp_{2n}, n \geq 2 \). The factor \( Sp_{2n} \) of \( H \) is embedded in \( G \) as the upper left \( 2n \times 2n \) block, and the torus \( C^\times \) is embedded in \( G \) via the map \( s \mapsto \text{diag}(s, \ldots, s, s^{-2n}) \).

The information contained in rows 6 and 7 of Table 1 in [5] enables us to conclude that the semigroup \( \Lambda_+(G/H) \) is freely generated by the elements

\[
(\pi_1, n\chi), \ (\pi_2, -\chi), \ (\pi_3, (n-1)\chi), \ (\pi_4, -2\chi), \ldots, \ (\pi_{2n-1}, \chi), \ (\pi_{2n}, -n\chi)
\]

for some nonzero character \( \chi \in \mathcal{X}(H) \). As \( n \geq 2 \), it follows that the action \( H/H_0 : Y_0 \) is not excellent and the morphism \( \pi_U \) is not equidimensional.

2) \( G = Spin_{10}, H = C^\times \times Spin_7 \). This homogeneous space is uniquely determined by the homogeneous space \( G/H \), locally isomorphic to \( G/H \), where \( \tilde{G} = SO_{10}, \tilde{H} = C^\times \times Spin_7 \). The factor \( Spin_7 \) of \( \tilde{H} \) is regarded as a subgroup of \( SO_8 \) (the embedding \( Spin_7 \hookrightarrow SO_8 \) is given by the spinor representation), and the group \( SO_8 \) is embedded in \( \tilde{G} \) as the central \( 8 \times 8 \) block. The torus \( C^\times \) is embedded in \( \tilde{G} \) via the map \( s \mapsto \text{diag}(s, 1, 1, \ldots, 1, s^{-1}) \).

The information contained in row 16 of Table 1 in [5] and in row 13 of Table 1 in [16] enables us to conclude that the semigroup \( \Lambda_+(G/H) \) is freely generated by the elements

\[
(\pi_1, 2\chi), \ (\pi_1, -2\chi), \ (\pi_2, 0), \ (\pi_4, \chi), \ (\pi_5, -\chi)
\]

for some nonzero character \( \chi \in \mathcal{X}(H) \). It follows that the action \( H/H_0 : Y_0 \) is not excellent and the morphism \( \pi_U \) is not equidimensional.

3) \( G = SL_n \times SL_{n+1}, H = SL_n \times C^\times, n \geq 2 \). The factor \( SL_n \) of \( H \) is diagonally embedded in \( G \) as the upper left \( n \times n \) block in the factor \( SL_{n+1} \). The torus \( C^\times \) is embedded in the factor \( SL_{n+1} \) of \( G \) via the map \( s \mapsto \text{diag}(s, \ldots, s, s^{-n}) \).

In row 1 of Table 1 in [12] it is indicated that the semigroup \( \Lambda_+(G/H) \) is freely generated by the elements

\[
(\varphi_1, n\chi), \ (\pi_{n-1} + \varphi_2, (n-1)\chi), \ldots, \ (\pi_1 + \varphi_n, \chi),
(\pi_{n-1} + \varphi_1, -\chi), \ldots, \ (\pi_1 + \varphi_{n-1}, -(n-1)\chi), \ (\varphi_n, -n\chi)
\]

for some nonzero character \( \chi \in \mathcal{X}(H) \). It follows that the action \( H/H_0 : Y_0 \) is not excellent and the morphism \( \pi_U \) is not equidimensional.
4) \( G = SL_n \times Sp_{2m}, \quad H = \mathbb{C}^\times \cdot SL_{n-2} \times SL_2 \times Sp_{2m-2}, \quad n \geq 3, \quad m \geq 1 \). The factor \( SL_{n-2} \) of \( H \) is embedded in the factor \( SL_n \) of \( G \) as the upper left \((n-2) \times (n-2)\) block. The factor \( SL_2 \) of \( H \) is diagonally embedded in \( G \) as the lower right \(2 \times 2\) block in the factor \( SL_n \) and as the \(2 \times 2\) block corresponding to the first and the last rows and columns in the factor \( Sp_{2m} \). The factor \( Sp_{2m-2} \) of \( H \) is embedded in the factor \( Sp_{2m} \) of \( G \) as the central \((2m-2) \times (2m-2)\) block. Finally, the torus \( \mathbb{C}^\times \) is embedded in the factor \( SL_n \) of \( G \) via the map \( s \mapsto \text{diag}(s^{-2}, \ldots, s^{-2}, s^{n-2}, s^{n-2}) \) for odd \( n \) and via the map \( s \mapsto \text{diag}(s^{-1}, \ldots, s^{-1}, s^{(n-2)/2}, s^{(n-2)/2}) \) for even \( n \).

It is indicated in row 3 of Table 1 in [12] that the semigroup \( \Lambda_+(G/H) \) is freely generated by the elements \((\pi_{n-2}, 2\chi), (\varphi_2, 0)\) (this element is contained in the set of indecomposable elements for \( m \geq 2 \)), \((\pi_{n-1} + \varphi_1, \chi), (\pi_1 + \pi_{n-1}, 0)\) (this element is contained in the set of indecomposable elements for \( n \geq 4 \)), \((\pi_1 + \varphi_1, -\chi), (\pi_2, -2\chi)\) for some nonzero character \( \chi \in \mathfrak{X}(H) \). It follows that the action \( H/H_0 : Y_0 \) is not excellent and the morphism \( \pi_U \) is not equidimensional.

**Case 2.** \( \dim Z(H) = 0 \). For each of the spaces \( G/H \) considered below we fix a set of \( T \)-semi-invariant functions which freely generate the algebra \( U \mathbb{C}[G/H] \) and denote it by \( F \). The weights of these functions are the indecomposable elements of the (free) semigroup \( \Lambda_+(G/H) \simeq \widetilde{\Lambda}_+(G/H) \). We put \( |F| = r \). Note that \( r \) is the codimension of a generic fibre of the morphism \( \pi_U \). In all the cases the null fibre of \( \pi_U \) is nonempty by Proposition 1. To consider the spaces of series 2) and 4) we shall need the auxiliary lemma below, which follows from the theorem on the dimensions of fibres of a dominant morphism.

**Lemma 3.** Let \( \gamma : M \to N \) be a morphism of affine algebraic varieties. Suppose that a closed subvariety \( N_0 \subseteq N \) is such that the set \( \gamma(M) \cap N_0 \) is dense in \( N_0 \). Then \( \text{codim}_M \gamma^{-1}(N_0) \leq \text{codim}_N N_0 \).

Below we treat all the required spaces \( G/H \).

1) \( G = \text{Spin}_n \times \text{Spin}_{n+1}, \quad H = \text{Spin}_n, \quad n \geq 3 \). The homogeneous space \( G/H \) is uniquely determined by the homogeneous space \( \widetilde{G}/\widetilde{H} \), locally isomorphic to \( G/H \), where \( \widetilde{G} = SO_n \times SO_{n+1}, \quad \widetilde{H} = SO_n \), and the subgroup \( \widetilde{H} \) is diagonally embedded in \( \widetilde{G} \). Let \( \theta_m : SO_m \hookrightarrow SO_{m+1} \) be the embedding induced by the embedding \( \mathbb{C}^m \hookrightarrow \mathbb{C}^{m+1} \) sending the basis \( e_1, \ldots, e_m \) to the tuple

\[
e_1, \ldots, e_{m/2}, e_{m/2+2}, \ldots, e_{m+1}
\]

for even \( m \) and to the tuple

\[
e_1, \ldots, e_{(m-1)/2}, \frac{1}{\sqrt{2}}(e_{(m+1)/2} + e_{(m+3)/2}), e_{(m+5)/2}, \ldots, e_{m+1}
\]

for odd \( m \). The image of \( \theta_m \) is the stabilizer of the vector \( e_{m/2+1} \) for even \( m \) and the vector \( e_{(m+1)/2} - e_{(m+3)/2} \) for odd \( m \). We fix the embedding of \( \widetilde{H} \) in \( \widetilde{G} \) such that the image in \( \widetilde{G} \) of the matrix \( P \in \widetilde{H} \) is \((P, \theta_n(P)) \). The covering \( G/H \to \widetilde{G}/\widetilde{H} \) determines the natural embedding \( \mathbb{C}[\widetilde{G}/\widetilde{H}] \hookrightarrow \mathbb{C}[G/H] \). In view of this embedding, every regular function on \( \widetilde{G}/\widetilde{H} \) will also be considered as a regular function on \( G/H \).

For \( n = 3 \) there is an isomorphism between the homogeneous space \( G/H \) and the space \((SL_2 \times SL_2 \times SL_2)/SL_2\), where the subgroup \( SL_2 \) is diagonally embedded
in $\text{SL}_2 \times \text{SL}_2 \times \text{SL}_2$. This space is isomorphic to the space of series 4) (see below) with $n = m = l = 1$. For $n = 4$ the homogeneous space $G/H$ is isomorphic to the space of series 5) (see below) with $n = m = 1$. Therefore, below we assume that $n \geq 5$.

Suppose that $g = (P, Q) \in \tilde{G}$. We put $R = Q\theta_n(P)^{-1}$. For $n = 2k$ the set $F$ contains functions that are proportional to the following functions of $g$:

$$f_1 = r_{n+1,1}, \quad f_2 = r_{n+1,k+1}, \quad f_3 = f_1 r_{n,k+1} - f_2 r_{n,1};$$

for $n = 2k + 1$ the set $F$ contains functions that are proportional to the following functions of $g$:

$$f_1 = r_{n+1,1}, \quad f_2 = r_{n+1,k+1} - r_{n+1,k+2}, \quad f_3 = f_1 (r_{n,k+1} - r_{n,k+2}) - f_2 r_{n,1}$$

(see [12], §3.1, Case 2). In what follows, without loss of generality we shall assume that $f_1, f_2, f_3 \in F$. The $T$-weights of the functions $f_1, f_2$ are $\pi_1 + \varphi_1, \varphi_1$, respectively, and the $T$-weight of $f_3$ is $\pi_1 + \varphi_2$ for $n \geq 6$ and $\pi_1 + \varphi_2 + \varphi_3$ for $n = 5$. Since the condition $f_1 = f_2 = 0$ implies $f_3 = 0$, the subset of $G/H$ where all functions in $F$ vanish coincides with the subset of $G/H$ where all functions in $F \setminus \{f_3\}$ vanish. Thus the codimension of the null fibre of the morphism $\pi_U$ has codimension at most $r - 1$, hence $\pi_U$ is not equidimensional.

2) $G = \text{SL}_n \times \text{Sp}_{2m}, H = \text{SL}_{n-2} \times \text{SL}_2 \times \text{Sp}_{2m-2}, n \geq 5, m \geq 1$. The factor $\text{SL}_{n-2}$ of $H$ is embedded in the factor $\text{SL}_n$ of $G$ as the upper left $(n-2) \times (n-2)$ block. The factor $\text{SL}_2$ of $H$ is diagonally embedded in $G$, as the lower right $2 \times 2$ block in the factor $\text{SL}_n$ and as the $2 \times 2$ block in the first and last rows and columns in the factor $\text{Sp}_{2m}$. The factor $\text{Sp}_{2m-2}$ of $H$ is embedded in the factor $\text{Sp}_{2m}$ of $G$ as the central $(2m-2) \times (2m-2)$ block.

Suppose that $g = (P, Q) \in G$. We denote by $P_{ij}$ the $(i, j)$-cofactor of the matrix $P$. The set $F$ contains functions that are proportional to the following functions of $g$:

$$f_1 = p_{n,n-1}q_{2m,2m} - p_{n,n}q_{2m,1}, \quad f_2 = p_{n,n-1}P_{1,n-1} + p_{n,n}P_{1,n},$$

$$f_3 = q_{2m,1}P_{1,n-1} + q_{2m,2m}P_{1,n}$$

(see [12], §3.2, Case 4). The $T$-weights of the functions $f_1, f_2, f_3$ are $\pi_{n-1} + \varphi_1, \pi_1 + \pi_{n-1}, \pi_1 + \varphi_1$, respectively. Let $\gamma: G/H \to \mathbb{C}^6$ be the morphism defined by the functions $P_{1,n-1}, P_{1,n}, p_{n,n-1}, p_{n,n}, q_{2m,1}, q_{2m,2m}$. Let $N_0$ denote the subset in $\mathbb{C}^6$ defined by $f_1 = f_2 = f_3 = 0$. As can be easily seen, $\dim N_0 = 4$. Now, given nonzero numbers $a, b, c, d$ we consider the pair of matrices

$$P_0 = \begin{pmatrix} 0 & \ldots & 0 & b^{-1} \\ \vdots & \ddots & \vdots & \vdots \\ d^{-1} & \ldots & 0 & 0 \\ 0 & \ldots & bd & -ad \end{pmatrix}, \quad Q_0 = \begin{pmatrix} a^{-1}c^{-1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ -bc & \ldots & ac \end{pmatrix}$$

(the dots stand for zero entries). We have $(P_0, Q_0) \in G$ for any nonzero values of $a, b, c, d$, and the values of the functions $P_{1,n-1}, P_{1,n}, p_{n,n-1}, p_{n,n}, q_{2m,1}, q_{2m,2m}$ on this pair are $a, b, bd, -ad, -bc, ac$, respectively. It is also easy to check that
Each of the first three factors of $\gamma((P_0, Q_0)H) \in N_0$. It follows that $\dim(\gamma(G/H) \cap N_0) = 4$, that is, the set $\gamma(G/H) \cap N_0$ is dense in $N_0$. Then by Lemma 3 the codimension in $G/H$ of the subset where the functions $f_1, f_2, f_3$ vanish is at most 2. Hence the codimension in $G/H$ of the subset where all functions in $F$ vanish is at most $r - 1$. Therefore the morphism $\pi_U$ is not equidimensional.

3) $G = \Sp_{2n} \times \Sp_4$, $H = \Sp_{2n-4} \times \Sp_4$, $n \geq 3$. The first factor of $H$ is embedded in the first factor of $G$ as the central $(2n - 4) \times (2n - 4)$ block, and the second factor of $H$ is diagonally embedded in $G$, as the $4 \times 4$ block in rows and columns $1, 2, 2n - 1, 2n$ in the first factor.

Suppose that $g = (P, Q) \in G$. We put $R = PQ^{-1} \in \Sp_{2n}$ (the matrix $Q$ is embedded in $\Sp_{2n}$ as the $4 \times 4$ block in rows and columns $1, 2, 2n - 1, 2n$). Let $W$ denote the minor of order 3 of $R$ corresponding to the last three rows and columns $1, 2, 2n$. The set $F$ contains functions that are proportional to the functions $f_1, f_2, f_3$ of $g$, where $f_1 = r_{2n, 1}$, $f_2$ is the minor of order 3 of $R$ corresponding to the last three rows and columns $1, 2, 2n - 1$, and $f_3 = f_1 W + f_2 r_{2n, 2}$ (see [12], §3.2, Case 6). Further without loss of generality we assume that $f_1, f_2, f_3 \in F$. The $T$-weights of the functions $f_1, f_2, f_3$ are $\pi_1 + \varphi_1, \pi_3 + \varphi_1, \pi_1 + \pi_3 + \varphi_2$, respectively. As the condition $f_1 = f_2 = f_3 = 0$ implies $f_3 = 0$, the subset of $G/H$ where all the functions in $F$ vanish coincides with the subset of $G/H$ where all the functions in $F \setminus \{f_3\}$ vanish. Thus the codimension of the null fibre of the morphism $\pi_U$ is at most $r - 1$, hence $\pi_U$ is not equidimensional.

4) $G = \Sp_{2n} \times \Sp_{2m} \times \Sp_{2l}$, $H = \Sp_{2n-2} \times \Sp_{2m-2} \times \Sp_{2l-2} \times \Sp_2$, $n, m, l \geq 1$. Each of the first three factors of $H$ is embedded in the respective factor of $G$ as the central block of the corresponding size. The factor $\Sp_2$ of $H$ is diagonally embedded in $G$ as the $2 \times 2$ block in the first and the last rows and columns in each factor.

Suppose that $g = (P, Q, R) \in G$. The set $F$ contains functions that are proportional to the following functions of $g$:

\[
f_1 = p_{2n, 1} q_{2m, 2m} - p_{2n, 2n} q_{2m, 1}, \quad f_2 = q_{2m, 1} r_{2l, 2l} - q_{2m, 2m} r_{2l, 1}, \quad f_3 = p_{2n, 1} r_{2l, 2l} - p_{2n, 2n} r_{2l, 1}
\]

(see [12], §3.2, Case 7). The $T$-weights of the functions $f_1, f_2, f_3$ are $\pi_1 + \varphi_1$, $\varphi_1 + \psi_1, \pi_1 + \psi_1$, respectively. Let $\gamma : G/H \to \mathbb{C}^6$ be the morphism defined by the functions $p_{2n, 1}, p_{2n, 2n}, q_{2m, 1}, q_{2m, 2m}, r_{2l, 1}, r_{2l, 2l}$. Let $N_0$ denote the subset in $\mathbb{C}^6$ defined by $f_1 = f_2 = f_3 = 0$. As can be easily seen, $\dim N_0 = 4$. Each of the functions $p_{2n, 1}, p_{2n, 2n}, q_{2m, 1}, q_{2m, 2m}, r_{2l, 1}, r_{2l, 2l}$ is semi-invariant with respect to the action of the group $T \times T$, where the left factor acts on the left and the right factor acts on the right. The weights of all these functions are linearly independent in $\mathfrak{X}(T \times T)$. It follows that for every quadruple of nonzero numbers $a, b, c, d$ there is a triple of matrices $(P_0, Q_0, R_0) \in G$ such that the values of the functions $p_{2n, 1}, p_{2n, 2n}, q_{2m, 1}, q_{2m, 2m}, r_{2l, 1}, r_{2l, 2l}$ are $a, ad, b, bd, c, cd$, respectively. In addition, it is easy to check that $\gamma((P_0, Q_0, R_0)H) \in N_0$ for any nonzero values of $a, b, c, d$. This implies that $\dim(\gamma(G/H) \cap N_0) = 4$, that is, the set $\gamma(G/H) \cap N_0$ is dense in $N_0$. Then by Lemma 3 the codimension in $G/H$ of the subset where the functions $f_1, f_2, f_3$ vanish is at most 2. Therefore the codimension in $G/H$ of the subset where all the functions in $F$ vanish is at most $r - 1$. Hence the morphism $\pi_U$ is not equidimensional.
5) $G = \text{Sp}_{2n} \times \text{Sp}_4 \times \text{Sp}_{2m}$, $H = \text{Sp}_{2n-2} \times \text{Sp}_2 \times \text{Sp}_2 \times \text{Sp}_{2m-2}$, $n, m \geq 1$. The first factor of $H$ is embedded in the first factor of $G$ as the central $(2n-2) \times (2n-2)$ block. The fourth factor of $H$ is similarly embedded in the third factor of $G$. The second factor of $H$ is diagonally embedded in the first and second factors of $G$ as the $2 \times 2$ block in the first and the last rows and columns. The third factor of $H$ is diagonally embedded in the second and third factor of $G$, as the central $2 \times 2$ block in the second factor and as the $2 \times 2$ block in the first and the last rows and columns in the third factor.

Suppose that $g = (P, Q, R) \in G$. The set $F$ contains functions that are proportional to the following functions of $g$:

\[
\begin{align*}
&f_1 = p_{2n,1}q_{4,4} - p_{2n,2n}q_{4,1}, \quad f_2 = r_{2m,1}q_{4,3} - r_{2m,2m}q_{4,2}, \\
&f_3 = f_2(p_{2n,1}q_{3,4} - p_{2n,2n}q_{3,1}) - f_1(r_{2m,1}q_{3,3} - r_{2m,2m}q_{3,2})
\end{align*}
\]

(see [12], § 3.2, Case 8). We can assume without loss of generality that $f_1, f_2, f_3 \in F$. The $T$-weights of the functions $f_1, f_2, f_3$ are $\pi_1 + \varphi_1$, $\varphi_1 + \psi_1$, $\pi_1 + \varphi_2 + \psi_1$, respectively. As the condition $f_1 = f_2 = 0$ implies $f_3 = 0$, the subset of $G/H$ where all the functions in $F$ vanish coincides with the subset in $G/H$ where all the functions in $F \setminus \{f_3\}$ vanish. Thus the codimension of the null fibre of the morphism $\pi_U$ is at most $r - 1$, hence $\pi_U$ is not equidimensional.

The proof of Proposition 5 is completed.

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