Existence theorems for a generalized Chern-Simons equation on finite graphs

Jia Gao

Department of Applied Mathematics, College of Science, China Agricultural University, Beijing, 100083, P.R. China

Songbo Hou∗

Department of Applied Mathematics, College of Science, China Agricultural University, Beijing, 100083, P.R. China

Abstract

Denote by $G = (V,E)$ a finite graph. We study a generalized Chern-Simons equation

$$
\Delta u = Ae^u(e^u - 1) + 4\pi \sum_{j=1}^{N} \delta_{p_j}
$$

on $G$, where $\lambda$ and $b$ are positive constants; $N$ is a positive integer; $p_1, p_2, \cdots, p_N$ are distinct vertices of $V$ and $\delta_{p_j}$ is the Dirac delta mass at $p_j$. We prove that there exists a critical value $\lambda_c$ such that the equation has a solution if $\lambda \geq \lambda_c$ and the equation has no solution if $\lambda < \lambda_c$. We also prove that if $\lambda > \lambda_c$ the equation has at least two solutions which include a local minimizer for the corresponding functional and a mountain-pass type solution.

Key words: finite graph, Chern-Simons equation, upper and lower solution, mountain pass theorem

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1. Introduction

The study on the Abelian Chern-Simons model is an interesting topic. In this model, the Yang-Mills or the Maxwell field term are replaced by the specific Higgs potential. The Chern-Simons gauge field yields that vortices are magnetically and electrically charged. A natural question is how to get the existence of condensates or periodic multivortices. Under the assumption that the Higgs potential takes a sextic form, Caffarelli and Yang [2] reduced the self-dual Chern-Simons equations to the following form

$$
\Delta u = Ae^u(e^u - 1) + 4\pi \sum_{j=1}^{N} \delta_{p_j}
$$

(1.1)
on $\Omega$, where $\lambda > 0$, $\Omega \subset \mathbb{R}^2$ is a doubly periodic domain, $\delta_p$ is the Dirac distribution concentrated at $p \in \Omega$. Then they got the existences of solutions by the rigorous mathematical analysis. The solutions of Eq. (1.1) are of great significance in many fields of physics. The topological solutions, non-topological solutions, and solutions over a doubly periodic domain have been extensively studied [25, 21, 22, 4, 24, 5].

Chen and Han [3] studied a generalized Chern-Simons equation

$$\Delta u = \lambda e^u(e^{bu} - 1) + 4\pi \sum_{j=1}^{N} \delta_{p_j}, \quad x \in \Omega,$$

where $\lambda > 0$, $b > 0$, $\Omega \subset \mathbb{R}^2$ is a doubly periodic domain, $\delta_p$ is the Dirac distribution concentrated at $p \in \Omega$. They extended the results in [2]. As an application, they solved an open problem about the existence of solutions of the equation arising in the self-dual Maxwell-Chern-Simons model [6].

In this paper, we consider the discrete form of Eq. (1.2) on a finite graph. A finite graph $G = (V, E)$ is composed of vertices $V$ and edges $E$. We assume that $G$ is connected throughout the paper. We call $G$ is weighted if each edge $xy \in E$ is assigned a weight $\omega_{xy}$ which is positive and symmetric. Let $\mu : V \rightarrow \mathbb{R}^+$ be a finite measure. If vertex $y$ is adjacent to vertex $x$, we write $y \sim x$. For any function $u : V \rightarrow \mathbb{R}$, we define the $\mu$-Laplace operator by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x)).$$

The associated gradient form reads

$$\Gamma(u, \upsilon) = \frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))(\upsilon(y) - \upsilon(x)).$$

Write $\Gamma(u) = \Gamma(u, u)$. The length of its gradient is denoted by

$$|\nabla u(x)| = \sqrt{\Gamma(u)(x)} = \left( \frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))^2 \right)^{1/2}.$$

For any function $f : V \rightarrow \mathbb{R}$, the integral of $f$ over $V$ is defined by

$$\int_V f d\mu = \sum_{x \in V} \mu(x)f(x).$$

Define the Sobolev space

$$W^{1,2}(V) = \left\{ u \mid u : V \rightarrow \mathbb{R}, \int_V (|\nabla u|^2 + u^2) d\mu < +\infty \right\}.$$

The norm of $u \in W^{1,2}(V)$ is

$$\|u\|_{W^{1,2}(V)} = \left( \int_V (|\nabla u|^2 + u^2) d\mu \right)^{1/2}.$$
We consider the generalized Chern-Simons equation derived in [3], i.e.,
\[ \Delta u = \lambda e^u (e^{bu} - 1) + 4\pi \sum_{j=1}^{N} \delta_{p_j} \]  
(1.4)
on a graph \( G = (V, E) \), where \( \lambda \) and \( b \) are positive constants; \( N \) is a positive integer; \( p_1, p_2, \cdots, p_N \) are distinct vertices of \( V \) and \( \delta_{p_j} \) is the Dirac delta mass at \( p_j \).

We prove the following existence results for Eq. (1.4) on \( G \).

**Theorem 1.1.** There exists a critical value \( \lambda_c \) depending on the graph satisfying
\[ \lambda_c \geq \frac{(b + 1)^{b+1} 4\pi N}{|V|}, \]
such that
(i) Eq. (1.4) has a solution if \( \lambda \geq \lambda_c \), however, Eq. (1.4) has no solution if \( \lambda < \lambda_c \).
(ii) Eq. (1.4) has at least two solutions if \( \lambda > \lambda_c \).

**Remark 1.** If \( b = 1 \), then Eq. (1.4) becomes Eq. (1.1). There also exist multiple solutions for Eq. (1.1) on graphs. This completes the results of Huang, Lin and Yau [16].

There are a lot of research on theories of partial differential equations on graphs. For the discrete Chern-Simons equations, Huang, Lin and Yau [16] studied Eq. (1.1) on graphs and got the existence results except the critical case. We solved the critical case [15], and got the existence results for a generalized Chern-Simons-Higgs equation, which was also studied in [20]. Huang, Wang and Yang [17] considered the relativistic Abelian Chern-Simons equations and established the existence of multiple solutions. The results for Yamabe type equations include [12, 16, 27, 15, 28]. The results for Kazdan-Warner equations include [11, 8, 26, 7, 19]. One may refer to [13, 14] for the biharmonic equations.

The rest of the paper is arranged as follows. In Section Two, we first use the upper and lower solutions method to get the existence of the single solution for the non-critical case. Then we use the prior estimates to prove the existence for the critical case. In Section Three, we use the mountain pass theorem to get the existence of multiple solutions.

**2. Single Solution**

Denote \( u = u_0 + \nu \), where \( u_0 \) is the solution of the equation
\[ \Delta u = -\frac{4\pi N}{|V|} + 4\pi \sum_{j=1}^{N} \delta_{p_j}. \]  
(2.1)

Then we can reduce Eq. (1.4) into
\[ \Delta \nu = \lambda e^{u_0+\nu} (e^{b(u_0+\nu)} - 1) + \frac{4\pi N}{|V|}. \]  
(2.2)

We introduce the definition of upper and lower solutions. If a function \( \nu_s \) satisfies
\[ \Delta \nu_s \leq \lambda e^{u_0+\nu_s} (e^{b(u_0+\nu_s)} - 1) + \frac{4\pi N}{|V|}, \]

Then we can reduce Eq. (1.4) into
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We introduce the definition of upper and lower solutions. If a function \( \nu_s \) satisfies
\[ \Delta \nu_s \leq \lambda e^{u_0+\nu_s} (e^{b(u_0+\nu_s)} - 1) + \frac{4\pi N}{|V|}, \]
on $V$, we call it an upper solution of (2.2). Analogously, a function $\nu_-$ is said to be a lower solution of (2.2) if it satisfies

$$\Delta \nu_- \geq \lambda e^{b(u_0 + \nu_-)} \left(e^{(b(u_0 + \nu_- - 1)} - 1\right) + \frac{4\pi N}{|V|}$$

(2.3)
on $V$. We introduce Lemma 4.1 in [16], which is referred to as the maximum principle.

**Lemma 2.1.** Let $G = (V, E)$ be a finite graph. If for some constant $K > 0$, $\Delta u(x) - Ku(x) \geq 0$ for all $x \in V$, then $u \leq 0$ on $V$.

Set $u_0 = -u_0$. Let $\{u_n\}$ be the iterative sequence determined by the scheme

$$\begin{cases}
(\Delta - K) u_n = \lambda e^{b(u_0 + u_{n-1})} \left(\frac{Ku_{n-1}}{|V|} + \frac{4\pi N}{|V|}\right), \\
n = 1, 2, ..., 
\end{cases}$$

(2.4)

where $K$ denotes a positive constant.

**Lemma 2.2.** If $K > b\lambda$, then the sequence $\{u_n\}$ satisfies

$$u_0 > u_1 > u_2 > \cdots > u_n > \cdots > \nu_-,$$

where $\nu_-$ is any lower solution of (2.2). Hence, if (2.2) admits a lower solution, $\{u_n\}$ converges to a solution of (2.2), which is the maximal solution.

**Proof.** We use the inductive method. By (2.4), we obtain

$$(\Delta - K) u_1 = -Ku_0 + \frac{4\pi N}{|V|},$$

which together with (2.1) yields

$$(\Delta - K) (u_1 - u_0) = 4\pi \sum_{j=1}^{N} \delta_{\rho_j} \geq 0.$$ (2.5)

By Lemma 2.1, we get $u_1(x) \leq u_0(x)$ for all $x \in V$. Furthermore, we can prove $u_1 < u_0$ on $V$ by contradiction. Suppose $(u_1 - u_0)(x_0) = 0$ for some $x_0 \in V$. Obviously, $x_0$ is the maximum point of $u_1 - u_0$ on $V$. In view of (2.5), we have

$$\Delta (u_1 - u_0)(x_0) \geq K(u_1 - u_0)(x_0) \geq 0.$$ 

Noting the definition of the $\mu$-Laplace, we conclude that if $x \sim x_0$, $(u_1 - u_0)(x) = (u_1 - u_0)(x_0)$. Since $G$ is connected, we infer that $(u_1 - u_0)(x) = (u_1 - u_0)(x_0)$ for any $x \in V$, which contradicts (2.5). Hence there holds $u_1 < u_0$ on $V$.

Suppose that

$$u_0 > u_1 > \cdots > u_k.$$

By (2.4) and noting $K > b\lambda$, we calculate

$$(\Delta - K)(u_{k+1} - u_k) = \lambda \left[ e^{b(u_0 + u_k)} - e^{b(u_0 + u_{k-1})} \right] - K(u_k - u_{k-1})$$

$$= \lambda e^{b(u_0 + u_k)} \left(\frac{Ku_{k-1}}{|V|} + \frac{4\pi N}{|V|}\right) - K(u_k - u_{k-1})$$

$$\geq (b\lambda - K)(u_k - u_{k-1}) > 0.$$
where the second equality follows from the mean value theorem, \( u_0 + v_k \leq \xi \leq u_0 + v_{k-1} \), and we have used \( e^\xi \leq e^{u_{k+1}+\xi} \leq 1, e^{\delta \xi} \leq 1 \). It follows from Lemma 2.1 that \( v_{k+1} \leq v_k \). Applying the same method as in proving \( v_1 < v_0 \), we obtain \( v_{k+1} < v_k \). Hence, we establish

\[
0 > v_0 > v_1 > v_2 > \cdots > v_n > \cdots.
\]

We will also prove that \( v_k > v_- \) for all \( k \geq 0 \) by induction. Noting (2.4) and the definition of the lower solution, we have

\[
\Delta (v_- - v_0) \geq \lambda e^{v_- - v_0} (e^{b(v_- - v_0)} - 1) + 4\pi \sum_{j=1}^{N} \delta_{p_j} \tag{2.6}
\]

where \( c \) lies between \( v_- - v_0 \) and 0. Denote \( v_-(x_0) - v_0(x_0) = \max_{x \in V} [v_-(x) - v_0(x)] \) for some \( x_0 \in V \). Supposing \( (v_-(x_0) - v_0(x_0)) \geq 0 \), then by (2.6), we have

\[
\Delta (v_- - v_0)(x_0) \geq 0.
\]

It yields that if \( x \sim x_0 \), then \( v_-(x_0) - v_0(x) = (v_- - v_0)(x_0) \). Furthermore, \( (v_- - v_0)(x) \equiv (v_- - v_0)(x_0) \) for all \( x \in V \) since \( G \) is connected. Hence there is a contradiction with (2.6) at \( p_j \). It holds that \( v_0 > v_- \) on \( V \).

Suppose that for some \( k \geq 0, v_k > v_- \). In view of (2.4), (2.4) and the fact \( K > b\lambda \), we obtain

\[
(\Delta - K)(v_- - v_{k+1}) \geq \lambda \left[ e^{v_- - v_0} \left( e^{b(v_- - v_0)} - 1 \right) - e^{v_- - v_0} \left( e^{b(v_- - v_0)} - 1 \right) \right] - K(v_- - v_k)
\]

\[
= \lambda e^\xi \left[ b + 1 \right] e^{\delta \xi} - 1 (v_- - v_k) - K(v_- - v_k)
\]

\[
\geq (b\lambda - K)(v_- - v_k)
\]

\[
> 0,
\]

where we have used the mean value theorem, \( v_0 - v_0 \leq \xi \leq v_k - v_0 \). We have \( v_{k+1} > v_- \) by taking the same procedure as before. Combining (2.4) and the monotonicity of \( \{v_n\} \), we conclude that \( \{v_n\} \) converges to a solution of (2.2) pointwisely. Such a solution is bigger than any other solution. This finishes the proof of Lemma 2.2.

In order to get the existence of the solution of Eq. (2.2), we only need to find a lower solution. We have the following lemma.

**Lemma 2.3.** If \( \lambda \) is sufficiently large, there is a solution of Eq. (2.2).

**Proof.** Since \( G \) is finite, \( u_0 \) is a bounded function. There exists a positive constant \( c \) such that \( u_0 - c < 0 \). Set \( v_0 = -c \). We see that if \( \lambda \) is sufficiently large, then

\[
\Delta v_- = 0 > \lambda e^{u_0-c} \left( e^{b(u_0-c)} - 1 \right) + \frac{4\pi N}{|V|}
\]

This implies that \( v_- \) is a lower solution of (2.2). Consequently, Eq. (2.2) admits a solution. We complete the proof. \[\square\]
Define $f(t) = e^t(e^{2t} - 1)$, $t \in (-\infty, 0]$. A simple computation implies that $f$ has a unique minimum $\frac{b}{(b+1)e^2}$. If $\nu$ is a solution of Eq. (2.2), it is also a lower solution of Eq. (2.2). Then Lemma 2.2 implies that $u_0 + \nu < 0$. Thus, we obtain

$$\Delta \nu \geq \frac{b \lambda}{(b+1)^{b+1}} + \frac{4 \pi N}{|V|}.$$  

Integrating both sides of the above inequality over $V$, we get

$$\lambda \geq \frac{(b+1)^{b+1} 4 \pi N}{b |V|}.$$  

(2.7)

If Eq. (2.2) has a solution, then $\lambda$ must satisfy (2.7).

**Lemma 2.4.** There is a critical value $\lambda_c$ satisfying

$$\lambda_c \geq \frac{(b+1)^{b+1} 4 \pi N}{b |V|},$$

such that Eq. (2.2) admits a solution for $\lambda > \lambda_c$, while Eq. (2.2) admits no solution for $\lambda < \lambda_c$.

**Proof.** Denote

$$\Lambda = \{ \lambda > 0 \mid \lambda \text{ is such that (2.2) has a solution} \}.$$  

We claim that $\Lambda$ is an interval. In fact, if $\lambda' \in \Lambda$, we can show that $[\lambda', +\infty) \subset \Lambda$. Let $\nu'$ be the solution of Eq. (2.2) with $\lambda = \lambda'$. If $\lambda > \lambda'$, in view of $u_0 + \nu' < 0$, we have

$$\Delta \nu' = \lambda' e^{u_0 + \nu'} \left( e^{b(u_0 + \nu')} - 1 \right) + \frac{4 \pi N}{|V|} > \lambda^{u_0 + \nu'} \left( e^{b(u_0 + \nu')} - 1 \right) + \frac{4 \pi N}{|V|},$$

which implies that $\nu'$ is a lower solution of Eq. (2.2) for $\lambda > \lambda'$. Therefore, Eq. (2.2) has a solution if $\lambda > \lambda'$ and $[\lambda', +\infty) \subset \Lambda$.

Define $\lambda_c = \inf \{ \lambda \mid \lambda \in \Lambda \}$. Noting (2.7), we obtain

$$\lambda_c \geq \frac{(b+1)^{b+1} 4 \pi N}{b |V|}$$

by taking the limit $\lambda \rightarrow \lambda_c$. \hfill \Box

Now, we deal with the critical case $\lambda = \lambda_c$. In the following lemma, we get the monotonicity of the solution of (2.2) with respect to $\lambda$, which will be used later.

**Lemma 2.5.** Let $\{u_\lambda \mid \lambda > \lambda_c\}$ be the family of maximal solutions of (2.2). Then there holds $u_{\lambda_1} > u_{\lambda_2}$ if $\lambda_1 > \lambda_2 > \lambda_c$.

**Proof.** Assume that $\lambda_1 > \lambda_2$ and the associated solutions are $u_{\lambda_1}, u_{\lambda_2}$ respectively. Then by (2.2), we have

$$\Delta u_{\lambda_2} = \lambda_2 e^{u_{\lambda_2} + u_{\lambda_2}} \left( e^{b(u_0 + u_{\lambda_2})} - 1 \right) + \frac{4 \pi N}{|V|}$$

$$= \lambda_1 e^{u_{\lambda_2} + u_{\lambda_2}} \left( e^{b(u_0 + u_{\lambda_2})} - 1 \right) + \frac{4 \pi N}{|V|}$$

$$\quad + (\lambda_2 - \lambda_1) e^{u_{\lambda_2} + u_{\lambda_2}} \left( e^{b(u_0 + u_{\lambda_2})} - 1 \right)$$

$$\geq \lambda_1 e^{u_{\lambda_2} + u_{\lambda_2}} \left( e^{b(u_0 + u_{\lambda_2})} - 1 \right) + \frac{4 \pi N}{|V|}.$$
Thus, \( u_{i2} \) is a lower solution of (2.2) with \( \lambda = \lambda_1 \). Then by Lemma 2.2, we obtain \( u_{i1} \geq u_{i2} \). Furthermore,

\[
\Delta(u_{i2} - u_{i1}) = \lambda_2 e^{u_{i2}} \left( e^{(u_{i0} + u_{i1})} - 1 \right) - \lambda_1 e^{u_{i1}} \left( e^{(u_{i0} + u_{i1})} - 1 \right) \\
= \lambda_1 e^{u_{i1}} \left( e^{(u_{i0} + u_{i1})} - 1 \right) - \lambda_1 e^{u_{i1}} \left( e^{(u_{i0} + u_{i1})} - 1 \right) \\
+ (\lambda_2 - \lambda_1) e^{u_{i2}} \left( e^{(u_{i0} + u_{i1})} - 1 \right) \\
> \lambda_1 \left[ e^{u_{i1}} \left( e^{(u_{i0} + u_{i1})} - 1 \right) - e^{u_{i1}} \left( e^{(u_{i0} + u_{i1})} - 1 \right) \right] \\
= \lambda_1 e^{(b+1)\rho} - 1 \left| (u_{i2} - u_{i1}) \right| \\
\geq b \lambda_1 (u_{i2} - u_{i1}),
\]

where \( u_{i0} + u_{i2} \leq \xi \leq u_{i0} + u_{i1} \). Suppose that \( u_{i2} - u_{i1} \) achieves 0 at some point \( x_0 \in \Omega \). Then by (2.8), we get

\[ \Delta(u_{i2} - u_{i1})(x_0) > 0. \]

However, \( \Delta(u_{i2} - u_{i1})(x_0) \leq 0 \) according to the definition of the \( \mu \)-Laplace. There is a contradiction. Hence, \( u_{i2} - u_{i1} < 0 \) and we prove that \( u_{i1} > u_{i2} \) if \( \lambda_1 > \lambda_2 > \lambda_c \).

Now, we estimate the bound of solutions of (2.2) in \( W^{1,2}(\Omega) \). We first introduce the Poincaré inequality and the Trudinger-Moser inequality which were proved in [11].

**Lemma 2.6. (the Poincaré inequality)**

Let \( G = (V,E) \) be a finite graph. Then there exists some constant \( C \) depending only on \( G \) such that

\[
\int_V u^2 \, d\mu \leq C \int_V |\nabla u|^2 \, d\mu,
\]

for all \( u \in V^E \) with \( \int_V u \, d\mu = 0 \), where \( V^E = \{u \, | \, u \text{ is a real function : } V \to \R \} \).

**Lemma 2.7. (the Trudinger-Moser inequality)**

Let \( G = (V,E) \) be a finite graph. For any \( \alpha > 0 \) and all functions \( u \in V^E \) with \( \int_V |\nabla u|^2 \, d\mu \leq 1 \) and \( \int_V u \, d\mu = 0 \), there exists some constant \( C > 0 \) depending only on \( \alpha \) and \( G \) such that

\[
\int_V e^{\alpha u^2} \, d\mu \leq C.
\]

Applying the above lemmas and the integration method, we get the following lemma.

**Lemma 2.8.** For any maximal solution \( u_{i1} \) of (2.2), we split it into two parts, namely, \( u_{i1} = \bar{v}_{i1} + v'_i \), where \( \bar{v}_{i1} = \frac{1}{\mu(V)} \int_V u_{i1} \, d\mu \) and \( v'_i = u_{i1} - \bar{v}_{i1} \). Then \( v'_i \) satisfies

\[
\| \nabla v'_i \|_2 \leq C \lambda_1,
\]

where \( C \) is a positive constant depending only on \( |V| \). Furthermore, we obtain

\[
|\bar{v}| \leq C(1 + \lambda + \lambda^2)
\]

and

\[
\| u_{i1} \|_{W^{2,1}(\Omega)} \leq C(1 + \lambda + \lambda^2).
\]
Proof. Noting $\nu = \bar{\nu} + \nu'$, we multiply both sides of Eq. (2.2) by $\nu'$. Then integrating over $V$ and using the Poincaré inequality, we have

$$\|\nabla \nu'\|_2^2 = A \int_V e^{u_0 + \nu' + \nu} (1 - e^b(u_0 + \nu')) \nu' \, d\mu$$

$$\leq A \int_V |\nu'| \, d\mu \leq C A |V|^{1/2} \|\nabla \nu'\|_2,$$

which yields

$$\|\nabla \nu'\|_2 \leq C A. \quad (2.9)$$

In view of $u_0 + \nu = u_0 + \bar{\nu} + \nu' < 0$, integrating over $V$ again, we get the upper bound of $\bar{\nu}$.

$$\bar{\nu} < \frac{1}{|V|} \int_V u_0(x) \, d\mu. \quad (2.10)$$

Now, we need to prove that $\bar{\nu}$ has a lower bound. Integrating both sides of (2.2) over $V$, we get

$$\lambda \int_V e^{u_0 + \nu' + \nu} \, d\mu = \lambda \int_V e^{(b+1)(u_0 + \nu')} \, d\mu + 4\pi N > 4\pi N. \quad (2.11)$$

Using the Hölder inequality and the Trudinger-Moser inequality, we calculate

$$\int_V e^{u_0 + \nu' + \nu} \, d\mu = \int_V e^{u_0 + \bar{\nu} + \nu'} \, d\mu$$

$$\leq e^{\bar{\nu}} \max_{x \in V} e^{u_0} \int_V e^{\nu'} \, d\mu$$

$$\leq C e^{\bar{\nu}} \int_V e^{\|\nabla \nu'\|_2^2 + \nu^2} \, d\mu$$

$$\leq C e^{\bar{\nu}} e^{\|\nabla \nu'\|_2^2}. \quad (2.12)$$

In view of (2.11) and (2.12), we have

$$e^{\bar{\nu}} \geq C \lambda^{-1} e^{-\|\nabla \nu'\|_2^2}$$

which together with (2.9) and (2.10) yields

$$|\bar{\nu}| \leq C (1 + \lambda + \lambda^2).$$

Furthermore,

$$\|\nu'\|_{W^{1,2}(V)} \leq C (1 + \lambda + \lambda^2).$$

Lemma 2.9. The equation (2.2) at $\lambda = \lambda_c$ admits a solution.
Proof. Suppose \( \lambda_c < \lambda < \lambda_c + 1 \). Then we see that \( \{v_\lambda\} \) has a uniform bound in \( W^{1,2}(V) \) by Lemma 2.8. The space \( W^{1,2}(V) \) is precompact since it is finite dimensional. Noting the monotonicity of \( \{v_\lambda\} \) with respect to \( \lambda \), we conclude that there is a function \( v_* \) in \( W^{1,2}(V) \) such that

\[
v_\lambda \to v_*
\]
as \( \lambda \to \lambda_c \), and the convergence is pointwise. The face \( u_0 + v_\lambda < 0 \) implies \( u_0 + v_* < 0 \).

By the convergence of \( \{v_\lambda\} \), we have

\[
\Delta v_\lambda \to \Delta v_*
\]
and

\[
\lambda e^{\alpha v_\lambda + \beta} (e^{(b+1)u_0} - 1) \to \lambda_* e^{\alpha v_* + \beta} (e^{(b+1)u_0} - 1)
\]
as \( \lambda \to \lambda_c \). Hence \( v_* \) is a solution of (2.2). This completes the proof.

3. Multiple solutions

In this section, we will use the variation method to prove the existence of the second solution. Define the functional

\[
J(\nu) = \frac{1}{2} \int_V |\nabla \nu|^2 \mu + \frac{\lambda}{b + 1} \int_V e^{(b+1)u_0} \nu \mu - \lambda \int_V e^{\alpha \nu + \beta} \mu + \frac{4\pi N}{|V|} \int_V \nu \mu.
\]

(3.1)

Lemma 3.1. For every \( \lambda > \lambda_c \), there exists a solution of (2.2) in \( W^{1,2}(V) \). Such a solution is a local minimum point of \( J(\nu) \).

Proof. For every \( \lambda > \lambda_c \), define

\[
\Sigma = \{ \nu \in W^{1,2}(V) | \nu \geq v_* \text{ in } V \},
\]

(3.2)

where \( v_* \) is the solution of (2.2) at \( \lambda = \lambda_c \). By the Young inequality, we have

\[
e^{\alpha \nu + \beta} \leq \frac{e^{(b+1)u_0 + \nu}}{b + 1} + \frac{b}{b + 1}.
\]

Hence,

\[
J(\nu) \geq \frac{1}{2} \int_V |\nabla \nu|^2 \mu - \frac{b\lambda}{b + 1} |V| + \frac{4\pi N}{|V|} \int_V \nu \mu.
\]

(3.3)

We see that \( J \) has a lower bound on \( \Sigma \).

Denote

\[
\eta_0 = \inf [J(\nu) | \nu \in \Sigma].
\]

(3.4)

Next, we prove that there exists \( w_\lambda \in \Sigma \) such that \( J(w_\lambda) = \eta_0 \).

There is a sequence \( \{w_\lambda\} \) in \( \Sigma \) such that

\[
J(w_\lambda) \to \eta_0.
\]

(3.5)

As in Lemma 2.8, we write \( \nu_\lambda \) as \( \nu_\lambda = \nu'_\lambda + \tilde{\nu}_n \), where \( \tilde{\nu}_n = \frac{1}{|V|} \int_V \nu_\lambda \mu \) and \( \nu'_\lambda = \nu_\lambda - \tilde{\nu}_n \). By (3.5), we know that \( J(\nu_\lambda) \) is bounded. In view of (3.3), we get

\[
J(\nu_\lambda) + \frac{b\lambda}{b + 1} |V| \geq 4\pi N \tilde{\nu}_n,
\]

as \( \lambda \to \lambda_c \).
which together with the fact $\nu_n \geq \nu_*$ gives the bound of $\nu_n$. Furthermore, $\{\nu_n\}$ is bounded in $W^{1,2}(V)$.

Since $W^{1,2}(V)$ is precompact, we obtain

$$\nu_n \rightarrow w_\lambda$$

as $n \rightarrow \infty$ and $w_\lambda$ is a solution to problem \(3.4\). We use the same method as in \([23]\) to prove that $w_\lambda$ is a solution of \(\mathcal{P}_2\). Define

$$w_t = \max\{w_\lambda + t\phi, \nu_*\} \in \Sigma,$$

where $\phi \in W^{1,2}(V)$ and $t \in (0, 1)$. Letting $u_t = \max\{\nu_* - (w_\lambda + t\phi), 0\}$, we write

$$w_t = w_\lambda + t\phi + u_t.$$

Calculate

$$0 \leq \frac{1}{t} (J(w_t) - J(w_\lambda))$$

$$\begin{align*}
&= \frac{1}{2t} \int_V (|\nabla w_t|^2 - |\nabla w_\lambda|^2) \mu + \frac{\lambda}{t(b + 1)} \int_V \left[ \phi^{b+1}(\nu_0 + w_t) - \phi^{b+1}(\nu_0 + w_\lambda) \right] \mu \\
&= \frac{1}{t} \int_V \left( e^{\nu_0 + w_t} - e^{\nu_0 + w_\lambda} \right) \mu + \frac{4\pi N}{|V|} \int_V \left( w_t - w_\lambda \right) \mu \\
&= \frac{1}{2t} \int_V \left| \nabla (t\phi + u_t) \right|^2 \mu + \frac{1}{t} \int_V \nabla w_\lambda \cdot \nabla (t\phi + u_t) \mu + \frac{\lambda}{t} \int_V \left( e^{\nu_0 + w_\lambda} - 1 \right) t\phi \mu \\
&\quad + \frac{1}{t(b + 1)} \int_V \left( e^{b+1}(\nu_0 + w_\lambda) - e^{b+1}(\nu_0 + w_t) - (b + 1) \phi e^{b+1}(\nu_0 + w_t) t\phi \right) \mu \\
&\quad - \frac{\lambda}{t} \int_V \left( e^{\nu_0 + w_\lambda} - e^{\nu_0 + w_t} - e^{\nu_0 + w_\lambda} t\phi \right) \mu + \frac{4\pi N}{|V|} \int_V \phi \mu + \frac{1}{t} \int_V u_t \mu.
\end{align*}$$

As a result,

$$\begin{align*}
&\int_V \nabla w_\lambda \cdot \nabla \varphi \mu + \lambda \int_V e^{\nu_0 + w_\lambda} \left( \phi^{b+1}(\nu_0 + w_\lambda) - 1 \right) \mu + \frac{4\pi N}{|V|} \int_V \varphi \mu \\
&\geq -\frac{t}{2} ||\nabla \phi||^2 - \int_V \nabla \varphi \cdot \nabla u_t \mu - \frac{1}{2t} ||\nabla u_t||^2 - \frac{1}{t} \int_V \nabla w_\lambda \cdot \nabla u_t \mu \\
&\quad - \frac{\lambda}{t(b + 1)} \int_V \left( e^{b+1}(\nu_0 + w_\lambda) - e^{b+1}(\nu_0 + w_t) - (b + 1) \phi e^{b+1}(\nu_0 + w_t) t\phi \right) \mu \\
&\quad + \frac{1}{t} \int_V \left( e^{\nu_0 + w_\lambda} - e^{\nu_0 + w_t} - e^{\nu_0 + w_\lambda} t\phi \right) \mu + \frac{4\pi N}{|V|} \int_V u_t \mu.
\end{align*}$$

Furthermore,

$$\begin{align*}
O(t) + \frac{1}{t} \int_V \nabla (-t\phi - w_\lambda + u_t) \cdot \nabla u_t \mu - \frac{1}{2t} ||\nabla u_t||^2 - \frac{1}{t} \int_V \nabla \varphi \cdot \nabla u_t \mu \\
&= \frac{\lambda}{t} \int_V e^{\nu_0 + w_\lambda} \left( \phi^{b+1}(\nu_0 + w_\lambda) - 1 \right) u_t \mu + \frac{4\pi N}{|V|} \int_V u_t \mu \\
&= \frac{\lambda}{t(b + 1)} \int_V \left( e^{b+1}(\nu_0 + w_\lambda) - e^{b+1}(\nu_0 + w_t) - (b + 1) \phi e^{b+1}(\nu_0 + w_t) (t\phi + u_t) \right) \mu \\
&\quad + \frac{1}{t} \int_V \left( e^{\nu_0 + w_\lambda} - e^{\nu_0 + w_t} - e^{\nu_0 + w_\lambda} (t\phi + u_t) \right) \mu.
\end{align*}$$
\[
+ \frac{\lambda}{t} \int_V \left[ e^{(b+1)(u_0+w_0)} - e^{(b+1)(u_0+w_0)} \right] u d\mu - \frac{\lambda}{t} \int_V (e^{u_0+w_0} - e^{u_0+w_0}) u d\mu.
\]

Noting that \( u_\ast \) is the lower solution of (2.2), if \( \lambda > \lambda_c \), we obtain
\[
- \frac{1}{t} \int_V \nabla u_\ast \cdot \nabla u d\mu - \frac{\lambda}{t} \int_V e^{(u_0+w_0)}(e^{(u_0+w_0)} - 1)u d\mu = \frac{4\pi N}{|V|} \int_V u d\mu \geq 0. \tag{3.7}
\]

Since
\[
\int_V \nabla(-t\varphi - w_\lambda + u_\ast) \cdot \nabla u \varphi \geq ||\nabla u\varphi||^2_2,
\]
we have
\[
\frac{1}{t} \int_V \nabla(-t\varphi - w_\lambda + u_\ast) \cdot \nabla u \varphi - \frac{1}{2t} ||\nabla u\varphi||^2_2 \geq 0. \tag{3.8}
\]

If \( u_\lambda = 0 \), we get
\[
|t\varphi + u_\lambda| \leq t|\varphi|.
\]

If \( u_\lambda > 0 \), we have
\[
w_\lambda + t\varphi < u_\ast
\]
and
\[
t\varphi + u_\lambda = t\varphi + u_\ast - (w_\lambda + t\varphi) = u_\ast - w_\lambda \leq 0.
\]

Consequently,
\[
|t\varphi + u_\lambda| \leq w_\lambda - u_\ast < -t\varphi = t|\varphi|.
\]

Hence, we always have
\[
|t\varphi + u_\lambda| \leq t|\varphi|. \tag{3.9}
\]

It follows that
\[
\frac{\lambda}{t(b+1)} \int_V \left[ e^{(b+1)(u_\lambda+w_\lambda)} - e^{(b+1)(u_\lambda+w_\lambda)} - (b+1)e^{(b+1)(u_\lambda+w_\lambda)}(t\varphi + u_\lambda) \right] d\mu = O(t) \tag{3.10}
\]
and
\[
\frac{\lambda}{t} \int_V \left[ e^{u_\lambda+w_\lambda} - e^{u_\lambda+w_\lambda} - e^{u_\lambda+w_\lambda}(t\varphi + u_\lambda) \right] d\mu = O(t) \tag{3.11}
\]
by the Taylor expansion.

Define
\[
V_t = \{ x \in V : w_\lambda(x) + t\varphi(x) - u_\ast(x) < 0 \}
\]
and
\[
V_0 = \{ x \in V : w_\lambda(x) = u_\ast(x) \}.
\]

Noting (3.9) implies \(|u_\lambda| \leq 2|\varphi|\), we have
\[
\left| \frac{\lambda}{t} \int_{V_t} \left[ e^{(b+1)(u_\lambda+w_\lambda)} - e^{(b+1)(u_\lambda+w_\lambda)} \right] u d\mu - \frac{\lambda}{t} \int_{V_0} (e^{u_\lambda+w_\lambda} - e^{u_\lambda+w_\lambda}) u d\mu \right|
\leq \frac{\lambda}{t} \int_{V_t \setminus V_0} 2 |e^{(b+1)(u_\lambda+w_\lambda)} - e^{(b+1)(u_\lambda+w_\lambda)} - e^{u_\lambda+w_\lambda} + e^{u_\lambda+w_\lambda}| |t\varphi| \tag{3.12}
\leq C \int_{V_t \setminus V_0} |\varphi| \to 0.
\]
Lemma 3.2. Any sequence \( \{v_n\} \subset W^{1,2}(V) \) satisfying

1. \( J(v_n) \to c \) as \( n \to \infty \),
2. \( \|J'(v_n)\| \to 0 \) as \( n \to \infty \),

admits a convergent subsequence.
Proof. We have

\[
\frac{1}{2} \int_V |\nabla \mu|{}^2 d\mu + \frac{\lambda}{b + 1} \int_V \mu^{(b+1)(\mu_0 + \nu_0)} d\mu - \lambda \int_V \mu^{\nu_0 + \nu} d\mu + \frac{4\pi N}{|V|} \int_V \nu_0 d\mu = c + o_n(1),
\]

(3.17)

\[
\left| \int_V \nabla \mu \cdot \nabla \varphi + \lambda \int_V \mu^{(b+1)(\mu_0 + \nu_0)} (\mu_0 - 1) \varphi + \frac{4\pi N}{|V|} \int_V \varphi \right| \leq \epsilon_n \|\varphi\|_{W^{1,2}(V)}, \quad \epsilon_n \to 0,
\]

(3.18)

as \( n \to \infty \). By taking \( \varphi = -1 \) in (3.18), we conclude that there exists a positive integer \( N \) such that if \( n \geq N \),

\[
0 < c \leq \int_V \mu^{\nu_0 + \nu} (1 - e^{\delta(\mu_0 + \nu_0)}) \leq C,
\]

(3.19)

where \( c \) and \( C \) are constants depending only on \( \lambda \). Now we prove that \( \nu_n(x) \) is bounded for any fixed \( x \in V \) by contradiction. Suppose not, then we conclude that there exists \( x' \) and \( \{\nu_n\} \subset \{\nu_n\} \) satisfying \( \nu_n(x') \to +\infty (-\infty) \), as \( k \to +\infty \). Without loss of generality, we assume that \( \nu_n(x') \) tends to \(+\infty\) as \( k \to +\infty \).

From now on, we don’t distinguish the sequence and the subsequence. We prove that there is a point \( \bar{x} \in V \) such that \( \{\nu_n(\bar{x})\} \) is bounded. Supposing not, we obtain that for any \( y \in V \), \( \nu_n(y) \) tends to \(+\infty (-\infty) \), as \( k \to +\infty \), which contradicts with (3.19).

Noting \( \nu_n(x') \to +\infty \), assume that

\[
\Lambda_k = \max_{x \in V} |\nu_n(x)| = \nu_n(x_k).
\]

Then we get \( \nu_n(x_k) \to +\infty \) as \( k \to +\infty \). In view of (3.17), we have

\[
\frac{1}{2} \int_V |\nabla \mu|{}^2 d\mu + \frac{\lambda}{b + 1} \int_V \mu^{(b+1)(\mu_0 + \nu_0)} d\mu - \lambda \int_V \mu^{\nu_0 + \nu} d\mu + \frac{4\pi N}{|V|} \int_V \nu_0 d\mu = c + o_n(1).
\]

(3.20)

Noting (3.23) and denoting \( M = \frac{\lambda}{4\pi N} |V| \), we obtain

\[
c \geq \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} \omega_{xy} |\nu_n(y) - \nu_n(x)|^2 - M - 4\pi N \Lambda_k + o_n(1).
\]

(3.21)

Consequently,

\[
|\nu_n(x) - \nu_n(y)| \leq C \sqrt{c + M + 4\pi N \Lambda_k + o_n(1)},
\]

(3.22)

if \( x \sim y \).

Noting \( G \) is finite and connected, we can find a path connecting \( \bar{x} \) and \( x_k \), namely,

\[
\bar{x} = x_1 \sim x_2 \sim x_3 \sim \cdots \sim x_l = x_k.
\]

(3.23)

Combining (3.22) and (3.23), we obtain

\[
\nu_n(\bar{x}) \geq \nu_n(x_k) - (l - 1) \sqrt{c + M + 4\pi N \Lambda_k + o_n(1)}
\]

\[
= \Lambda_k - (l - 1) \sqrt{c + M + 4\pi N \Lambda_k + o_n(1)}
\]

\[
\to +\infty, \text{ as } k \to +\infty.
\]

(3.24)

Since we have proved that \( \{\nu_n(\bar{x})\} \) is bounded, there is a contradiction. Hence, \( \nu_n(x) \) is bounded for any fixed \( x \in V \). Furthermore, \( \{\nu_n(x)\} \) is bounded in \( W^{1,2}(V) \) and there exists a function \( \hat{\nu} \) such that

\[
\nu_n(x) \to \hat{\nu}(x),
\]

for all \( x \in V \), as \( k \to +\infty \). Then proof is completed. \( \square \)
Next we use the mountain pass theorem to get the second solution of (2.2). Since $\nu_1$ is a local minimum point for $J$ by the assumption, there exists a constant $r_0 > 0$ such that

$$\inf_{\|\nu - \nu_1\|_{W^{1,2}(V)} = r_0} J(\nu) > J(\nu_1).$$

(3.25)

For any $\tau > 0$, we have

$$J(\nu_1 - \tau) - J(\nu_1) = \frac{\lambda}{b+1} \int_V \left[ e^{(b+1)(u_0+\nu_1-\tau)} - e^{(b+1)(u_0+\nu_1)} \right] d\mu$$

$$- \lambda \int_V \left( e^{u_0+\nu_1-\tau} - e^{u_0+\nu_1} \right) d\mu - 4\pi N \tau$$

$$\to -\infty$$

(3.26)

as $\tau \to +\infty$. Hence, there exists a constant $\tau_0$ such that $\|\nu_1 - \tau_0\|_{W^{1,2}(V)} > r_0$ and

$$J(\nu_1 - \tau_0) < J(\nu_1) - 1 < J(\nu_1).$$

(3.27)

Define

$$\Gamma = \{ \gamma : [0, 1] \to W^{1,2}(V) \text{ continuous } \gamma(0) = \nu_1, \gamma(1) = \nu_1 - \tau_0 \}$$

and

$$c_0 = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} J(\gamma(t)).$$

(3.28)

By (3.25), we get $c_0 > J(\nu_1)$. Then the mountain pass theorem [1] implies that $c_0$ is a critical value of $J$. We obtain the second solution of (2.2), which is the critical point of $J$.

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