MODULI CORRECTIONS TO GAUGE AND GRAVITATIONAL COUPLINGS
IN FOUR DIMENSIONAL SUPERSTRINGS

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ABSTRACT

We study one-loop, moduli-dependent corrections to gauge and gravitational couplings in supersymmetric vacua of the heterotic string. By exploiting their relation to the integrability condition for the associated CP-odd couplings, we derive general expressions for them, both for (2, 2) and (2, 0) models, in terms of tree level four-point functions in the internal $N = 2$ superconformal theory. The (2, 2) case, in particular symmetric orbifolds, is discussed in detail.

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1 Introduction

Understanding the structure of the low-energy effective Lagrangian, which governs the dynamics of the massless modes of the heterotic superstring, is clearly an important issue. In particular the dependence of the gauge couplings on the moduli of compactified space \[1\] plays an essential role in this context, having also implications on the problem of Supersymmetry breaking.

This dependence has been studied extensively in the case of the symmetric orbifold compactifications of the heterotic string for untwisted moduli \[2\] and was shown to satisfy a non renormalization theorem \[3\]; namely, it is given entirely at the one loop level and is determined by the violation of the integrability condition of the corresponding \(\Theta\)-angles with respect to the moduli. In the context of the effective field theory, this violation of the integrability condition is related to one loop anomalous graphs involving gauginos and matter fermions \[2, 4\]. An expression for these anomalies has also been obtained using effective field theory for general heterotic vacua \[5, 6, 7\].

In this work, starting from the full string theory amplitude, we obtain a general formula for the violation of the integrability condition in terms of quantities computable from the internal, superconformal theory. We also study the example of blowing up modes for symmetric orbifolds. Finally, we use the above method to generalize the results of ref.\[8\] for the gravitational couplings.

A general four dimensional heterotic string vacuum is obtained by tensoring four free bosons \(X^\mu\) representing the space-time coordinates together with their left-moving fermionic superpartners \(\psi^\mu\) and an internal conformal field theory with central charge \((c, \bar{c}) = (9, 22)\). \(N = 1\) space-time supersymmetry implies that the internal conformal field theory must have \(N = 2\) superconformal symmetry in the left-moving sector. The \(N = 2\) algebra involves in addition to the energy momentum tensor \(T_B\), a \(U(1)\) current \(J\) and a complex supercurrent \(T_F^\pm\) carrying \(U(1)\) charge \((\pm 1)\). Massless chiral or anti-chiral space-time scalars correspond, respectively, to chiral or anti-chiral \(N = 2\) supermultiplets. Their lower components are primary fields \(\Psi_\pm\) having dimensions \((\frac{1}{2}, 1)\) and \(U(1)\) charges \((\pm 1)\) while their upper components \(\Phi_\pm\) are neutral and have dimensions \((1, 1)\). Their
relevant operator product expansions (OPE) are

\[ T_F^\pm(z) \Psi_\pm(w) = \text{regular} \]
\[ T_F^\pm(z) \Psi_\mp(w) = \frac{1}{z-w} \Phi_\pm + \text{regular} \]
\[ T_F^\pm(z) \Phi_\pm(w) = \text{regular} \]
\[ T_F^\pm(z) \Phi_\mp(w) = \partial_w \left( \frac{\Psi_\pm(w)}{z-w} \right) + \text{regular} \]

(1.1)

2 Θ-term and threshold corrections

Consider a modulus field \( T \) (\( \bar{T} \)) corresponding to a chiral (anti-chiral) marginal operator. \( \Theta_T \) is defined from the CP-odd part of the on-shell three-point function of two gauge fields and the modulus \( T \). The one-loop contribution is given by:

\[
\epsilon^{\mu\nu\lambda\rho} p_1 p_2 \delta^{ab} \Theta_T = \int \frac{d^2 \tau}{\tau_2} \prod_{i=1}^3 d^2 z_i \left< V_A^{a\mu}(p_1, z_1) V_A^{b\nu}(p_2, z_2) V_T^{(-1)}(p_3, z_3) T_T(z) \right>_{\text{odd}}
\]

(2.1)

where \( \tau = \tau_1 + i\tau_2 \) is the Teichmuller parameter of the world-sheet torus and \( \Gamma \) its fundamental domain. \( V_A \)'s are the gauge vertices in zero ghost picture

\[
V_A^{a\mu}(p, z) = : \bar{J}^a(\bar{z}) (\partial \mathcal{X}^\mu + ip \cdot \psi \psi^\mu) e^{ip \cdot \mathcal{X}} : \]

(2.2)

with \( J^a \) the Kac-Moody currents. \( V_T^{(-1)} \) is the \( T \)-vertex operator in the \((-1)\) ghost picture

\[
V_T^{(-1)}(p, z) = : \bar{\Psi}_+(z, \bar{z}) e^{ip \cdot \mathcal{X}} : \]

(2.3)

where the \( N = 1 \) supercurrent insertion \((T_F = T_F^+ + T_F^- + \text{space-time part})\) and the \((-1)\) picture for one of the vertices is due to the CP-odd part of the amplitude, which receives contributions only from the odd spin structure.

The four space-time zero-modes required in the odd spin structure come from the two gauge vertices giving rise to the kinematic factor in (2.1). Furthermore, the two-point correlator of the Kac-Moody currents is

\[
\langle \bar{J}^a(\bar{z}_1) \bar{J}^b(\bar{z}_2) \rangle = -k \delta^{ab} \partial_{\bar{z}_1}^2 \ln \theta(\bar{z}_1 - \bar{z}_2) + Q^a Q^b
\]

(2.4)

where \( k \) is the level of the Kac-Moody algebra, \( Q^a \)'s are the charges of the propagating states, and \( \theta(z) \) is the Jacobi’s theta-function \( \theta_1(\tau, z) \). Finally, from the OPE relations
it follows that the $z$-dependence in (2.1) can have at most a first order pole at $z_3$ and therefore it must be constant, due to the periodicity of $T_F$. The $z_i$ integrations then give

$$\Theta_T = \int \frac{d^2 \tau}{\tau_2} \int d^2 z \tilde{\eta}^{-2} \langle (Q_2^2 - \pi \frac{k_a}{\tau_2}) T_F^- (w) \Psi_+ (z) \rangle_{\text{odd}}$$

(2.5)

where $\eta$ is the Dedekind eta function.

Now, using (1.1) and the OPE between the $N = 2 U(1)$ current $J$ and the supercurrent $T_F^{mp}$, one can see that

$$T_F^\pm (w) \Psi_\pm (z) = \mp J (w) \Phi_\pm (z) \pm \oint_C T_F^\pm (x) dx J (w) \psi_\pm (z)$$

(2.6)

where the contour $C$ encloses both $w$ and $z$. Substituting (2.6) in (2.5), one sees that the term with the contour integral vanishes by contour deformation, since the supercurrent $T_F^\pm$ is single valued on the torus in the odd spin structure. Thus, only the first term of the r.h.s. of (2.6) contributes in (2.5) and the result is

$$\Theta_T = i \partial_T \Delta$$

with

$$\Delta = -i \int \frac{d^2 \tau}{\tau_2} \tilde{\eta}^{-2} T_{R'} (-1)^F F (Q_2^2 - \pi \frac{k_a}{\tau_2}) q^{L_0 - \frac{\tau}{\tau_2}} q^{\bar{L}_0 - \frac{\tau}{\tau_2}}$$

(2.7)

where $q = e^{2\pi i r}$ and $T_{R'}$ denotes taking trace over massive states in the internal conformal theory. Here we used the fact that the contribution of massless states is independent of $T$ and that the insertion of $\Phi_\pm$ being the marginal operator corresponding to the modulus $T$ just gives the the $T$-variation of $\Delta$. Repeating the same analysis for the anti-chiral modulus $T$, one obtains $\Theta_T = -i \partial_T \Delta$ where the difference of sign arises from (2.3). This shows that $\Delta$ can be identified with the one loop gauge coupling constant $\beta$. It is also clear from this discussion that non harmonicity of gauge couplings is given by the violation of the integrability condition for the corresponding angles $\Theta$. In the following we evaluate this violation.

As a parenthetical remark concerning (2.7), it is interesting to notice that the quantity $\text{Tr} (-)^F F e^{-\beta H}$, for the total $F = F_L - F_R$, has been recently studied in the off-critical case in $[1]$, where it is also pointed out that in the case of our interest here, conformal and
chiral, $F = F_L$, it gives a stringy generalization of the (complex manifolds) Ray-Singer torsion.

Going back to the original expression (2.1) for $\Theta_T$, its variation with respect to $T$ is obtained formally by inserting in the three point function the vertex $V_T^{(0)}$ in zero ghost picture at zero momentum and subtracting all the one-particle reducible graphs. However, in order to regularize the short distance singularities in the $z$-integrals, one must start with non zero momenta and take the zero momentum limit only after performing the $z$-integrations. Indeed, one can explicitly check in the orbifold case for untwisted moduli that the above prescription gives the correct answer. As for the one particle reducible graphs, the only ones that appear in the above on-shell four point amplitude involve the antisymmetric tensor as intermediate state, which couples to the two gauge fields at the tree level and to the two moduli at the one loop level. These graphs are independent of the gauge group and therefore they drop out when one considers differences of $\frac{1}{k} \Theta_T$’s for different gauge groups. To simplify our discussion, in the following we will consider such differences and comment on individual gauge groups only at the end.

3 The integrability condition

The variation of $\Theta_T$ with respect to $T$ is given by the same four point amplitude except that now $T$-vertex appears in the $(-1)$ ghost picture while the $T$-vertex in zero ghost picture. $V_T^{(0)}$ is given by

$$V_T^{(0)}(p,z) = \oint dz' T_F(z')V^{(-1)}_T(p,z)$$

where the contour is around $z$. The violation of the integrability condition for $\Theta$, namely

$$\partial_T \Theta_T - \partial_T \Theta_T \equiv I,$$

is then given by the difference

$$I = \langle V_A V_A V_T^{(-1)} V_T^{(0)} T_F \rangle - \langle V_A V_A V_T^{(0)} V_T^{(-1)} T_F \rangle$$

Now using (3.1) in the first term of r.h.s. of (3.2) we perform the contour integral by deforming it. Due to the periodicity of $T_F$ in the odd spin structure, the only contribution comes when the contour encircles all other insertions. When it encircles $V_T^{(-1)}$,
one gets a contribution which cancels the second term of the r.h.s. of (3.2), while the contour integrals around \( V \)'s give total derivatives with respect to their positions and, consequently, vanish after \( z_i \)-integrations. Thus, the only term left over is the contour integral around the \( T_F \)-insertion, which gives rise to the stress energy tensor \( T_B \):

\[
I = \int \langle V_A(z_1)V_A(z_2)V^{(-1)}_T(z_3)V^{(-1)}_T(z_4)T_B(z) \rangle \quad (3.3)
\]

where the integral is over \( z_i \)'s and \( \tau \).

To evaluate (3.3) we extract the \( z \)-dependence of the above correlator [10]:

\[
\langle V_A(z_1)V_A(z_2)V^{(-1)}_T(z_3)V^{(-1)}_T(z_4)T_B(z) \rangle = -\sum_i h_i \partial^2_z \ln \theta(z - z_i)G + \sum_i \partial_z \ln \theta(z - z_i) \partial z_i G + 2\pi i \partial \tau G \quad (3.4)
\]

where \( G \) is the four-point correlator \( \langle V_A(z_1)V_A(z_2)V^{(-1)}_T(z_3)V^{(-1)}_T(z_4) \rangle \), and \( h_i \)'s are the left-moving conformal weights. Using translation invariance, one can fix one of the positions, say \( z_1 \), and perform first the \( z \)-integral. The first term in the r.h.s. of (3.4) is total derivative in \( z \) and, therefore, upon integration one picks up the boundary contribution \( \pi \sum_i h_i G \). The second term is also a total derivative in \( z \), but this time one has also contributions from the branch cuts. Using the translational invariance of \( G \), and putting together the previous contribution, a straightforward calculation yields

\[
\pi \sum_{i=2}^{4} \partial z_i \{[(z_i - z_1) - (\bar{z}_i - \bar{z}_1)]G \}.
\]

These terms are total derivatives with respect to \( z_i \)'s and the corresponding integrations pick up only boundary contributions, which can be identified with the \( \tau \)-variation of the domains of the \( z_i \)-integrals of \( G \). Combining with the last term of (3.4), one finally obtains

\[
I = 2\pi i \int d^2 \tau \partial \tau \int \prod_{i=1}^{3} d^2 z_i G \quad (3.5)
\]

The fact that the above equation involves a total derivative in \( \tau \) is not surprising. Indeed, from (3.2) the integrand of \( I \) is the difference between two different choices of ghost pictures of a physical amplitude, which is expected to be a total derivative. For non zero external momenta \( I \) would have vanished after integration over \( \tau \). However, as we will see below, setting the momenta to zero before the \( \tau \)-integration gives a non trivial result.
The gauge vertices provide the four space-time fermion zero modes required in the odd
spin structure giving rise to the kinematic factor in (2.1). Moreover the Kac-Moody cur-
rents are replaced by their zero modes as we are considering differences of gauge groups.
Since $T$ and $\overline{T}$ are marginal operators their vertices have no first order pole in the anti-
holomorphic sector (i.e. bosonic sector). Therefore we can set the external momenta to
zero and the short distance singularities can be handled by angular integration. Integration
over $z_1$ and $z_2$ then gives two powers of $\tau_2$ which cancel the powers of $\tau_2$ coming from
the integration over the loop momentum. One is then left with $\tau$ derivative of the two
point function of $T$ and $\overline{T}$ in the internal superconformal theory:

$$I \sim \int d^2\tau \bar{\eta}^{-2} \partial_{\tau} \int d^2z \langle V_T^{(-1)}(z) V^{(-1)}(0) \overline{T} \rangle_{\text{odd}}$$  (3.6)

$\tau$ integration then gives only the boundary term, i.e. $\tau_2 \to \infty$ limit, giving rise to a sum
over four point functions on the sphere:

$$I \sim \lim_{\tau_2 \to \infty} \int d\tau_1 \bar{\eta}^{-2} \text{Tr}(-1)^F Q^2 q^{L_0 - \frac{3}{2}} \bar{q}^{\bar{L}_0 - \frac{3}{2}} \int d^2x x^{-\frac{3}{2}} \langle V(0) V_T^{(-1)}(x) V_T^{(-1)}(1) \overline{V}(\infty) \rangle$$  (3.7)

where the trace is over all states of the internal conformal theory in the Ramond sector
with vertices $V$ and $\overline{V}$ (conjugate vertices) in the $(-\frac{1}{2})$ ghost picture. The integration
of $x$ is in an annulus within radii $|q|^\frac{3}{2}$ and $|q|^{-\frac{3}{2}}$ and the factor $x^{-\frac{3}{2}}$ comes from the
transformation of the torus coordinates to the annulus coordinates (recall that $V_T^{(-1)}$ has
dimension $(\frac{1}{2}, 1)$). If the $x$ integration is regular then it is clear from (3.7) that only
the trace over massless states contribute. The only possibility for a massive state to
contribute in this limit is if the behavior of $x$ integral near the boundaries of annulus
diverge. Consider for instance a massive state, i.e. with $L_0 > \frac{3}{8}$ and $\bar{L}_0 > 1$. The
behaviour of the correlation function as $x$ goes to zero is $x^{-L_0 - \frac{1}{2} + h - \bar{L}_0 - 1 + \bar{h}}$ where $(h, \bar{h})$
are the conformal weights of the intermediate state. The angular integration of $x$ in (3.7)
is non zero only if $-L_0 - 1 + h = -\bar{L}_0 - 1 + \bar{h}$. If $h = L_0$, the $x$ integration has only
logarithmic divergence in $|q|$. The integration of $\tau_1$ enforces the level macching condition
and the result vanishes due to $|q|^m$ suppression, with $m$ being the mass. If $h \neq L_0$, the
$x$ integral after including the remaining terms in (3.7) behaves as

$$\frac{1}{L_0 - h} q^{\frac{1}{2}(L_0 - h - \frac{3}{2})} \bar{q}^{\frac{1}{2}(L_0 + \bar{h} - 2)}$$  (3.8)
Note that as the trace is over Ramond sector both $L_0$ and $h$ are greater than or equal to $\frac{3}{8}$. Performing the $\tau_1$ integration and taking the limit $\tau_2$ to infinity, once again the result vanishes due to the mass suppression, assuming that there is no accumulation point of masses at zero which should be the case for compact internal spaces. Therefore, at generic points in moduli space only massless states contribute in (3.7).

The only subtlety arises if there are special subspaces of moduli space where there are extra massless states. Away from these subspaces, these extra states do not contribute to (3.7). However as one approaches these regions of moduli spaces, the masses of these extra particles become arbitrarily small. In particular, if one sits at a point on this subspace, the extra massless states do contribute to (3.7). For example, if $T$, $\overline{T}$ correspond to vector fields which are tangent to this subspace, then the contribution of the extra massless states is finite just on it, and vanishes away from it due to the limit $|q|\to 0$ as $|q|\to 0$. This gives rise to a discontinuity in $\Delta_T\overline{T}$. On the other hand, if one considers moduli $T$, $\overline{T}$ that are normal to the subspace, then the contribution of these states gives rise to a log $|q|$ divergence, as it appears from (3.8). This is a problem which arises also in field theory when doing mass perturbation expansion around zero mass. A possible way to handle this problem is to introduce an infrared cutoff $\Lambda$ in the $\tau_2$ integration; the result will then be continuous and duality invariant. The above discussion applies also to the case of $x$ near infinity.

In the following we will ignore this subtleties and consider the case where there is no divergence. Thus (3.7) becomes

$$I \sim \text{Tr}(-1)^F Q^2 \int d^2 x \ x^{-\frac{3}{8}} \langle V_R(0) V_T^{(-1)}(x) V_T^{(-1)}(1) \overline{V}_R(\infty) \rangle$$  \hspace{1cm} (3.9)

where the trace is only over massless states in the Ramond sector.

Now we will relate the above four point amplitude to that of a physical 4-boson amplitude. Consider first a physical amplitude involving four scalars

$$A = \langle V^{(-1)}_+(p_1, z_1) V^{(0)}_+(p_2, z_2) V^{(0)}_-(p_3, z_3) V^{(-1)}_-(p_4, z_4) \rangle$$  \hspace{1cm} (3.10)

where the subscripts + and - refer to the $N = 2$ chirality and

$$V^{(-1)}_=(p) = \Psi_+ e^{ip \cdot X}$$
Using the above in (3.10), we obtain

\[ A = \langle V^{(-1)}_-(p_1, z_1) \Phi_+ e^{ip_2 \cdot X} (z_2) \rangle \oint dz \frac{z - z_1}{z_3 - z_1} T^+(z) V^{(-1)}_-(p_3, z_3) V^{(-1)}_+(p_4, z_4) \]

\[ - \frac{p_2 \cdot p_3}{z_2 - z_3} \langle V^{(-1)}_- (p_1, z_1) V^{(-1)}_+ (p_2, z_2) V^{(-1)}_- (p_3, z_3) V^{(-1)}_+ (p_4, z_4) \rangle \]  

(3.11)

Now pulling the contour around and noting that there is no singularity at infinity we see that the only contribution comes from the contour integral around \( z_2 \). The result is

\[ A = - \frac{p_2 \cdot p_4}{z_3 - z_1} \langle V^{(-1)}_- (p_1, z_1) V^{(-1)}_- (p_2, z_2) V^{(-1)}_- (p_3, z_3) V^{(-1)}_+ (p_4, z_4) \rangle \]  

(3.12)

To relate the amplitude \( A \) to that of two fermions and two bosons, we use the space-time supersymmetry relation

\[ \Psi_\pm (w) = \oint dz (z - w)^{-\frac{1}{4}} e^{\pm i \frac{\sqrt{2}}{2} H(z)} \Psi^R_\pm (w) \]  

(3.13)

where \( \Psi^R_\pm \) is the Ramond vertex of the internal \( N = 2 \) superconformal theory with \( U(1) \) charges \( \mp \frac{1}{2} \), and \( H \) bosonises the corresponding \( U(1) \) current \( \sqrt{3} \partial H \). Note that \( e^{\pm i \frac{\sqrt{2}}{2} H} \) is just the internal part of the space-time supersymmetry current. Using (3.13) we can replace the bosonic vertices at \( z_1 \) and \( z_4 \) with two contour integrals \( \oint dz \) and \( \oint dw \), respectively, around Ramond vertices. The \( z \) and \( w \) dependence can be extracted explicitly from the \( U(1) \) charges of all vertices with the result

\[ \langle V^{(-1)}_- (p_1, z_1) V^{(-1)}_+ (p_2, z_2) V^{(-1)}_- (p_3, z_3) V^{(-1)}_+ (p_4, z_4) \rangle = \oint dz \oint dw \frac{\frac{1}{4} (z - z_4)^{\frac{1}{4}} (w - z_1)^{\frac{1}{4}} (z - w)^{\frac{1}{4}} (z - z_3)^{\frac{1}{4}} (w - z_2)^{\frac{1}{4}} (w - z_3)^{\frac{1}{4}}}{(z - z_1)(w - z_4)} \]

\[ \langle V^R (p_1, z_1) V^{(-1)}_- (p_2, z_2) V^{(-1)}_- (p_3, z_3) V^R (p_4, z_4) \rangle \]  

(3.14)

where \( V^R_\pm (p) = \Psi^R_\pm e^{ip \cdot X} \). Substituting the above equation in (3.12), performing the contour integrals and fixing \( SL(2, C) \) invariance by setting \( z_1 = 0, z_2 = x, z_3 = 1 \) and \( z_4 = \infty \) we obtain

\[ x^{-\frac{1}{2}} \langle V_-^R (p_1, 0) V_+^{(-1)} (p_2, x) V_-^{(-1)} (p_3, 1) V_+^R (p_4, \infty) \rangle = \]

\[ -\frac{1}{u} \langle V_-^{(-1)} (p_1, 0) V_+^{(0)} (p_2, x) V^{(0)}_+ (p_3, 1) V_-^{(-1)} (p_4, \infty) \rangle \]  

(3.15)
where $u = p_2 \cdot p_4$. Similarly, by exchanging the role of 0 and $\infty$ one gets

$$\frac{1}{s} \langle (p_1, 0) \rangle V^{(-1)}_+ \langle p_2, x \rangle V^{(0)}_+ \langle p_3, 1 \rangle V^{(-1)}_-(p_4, \infty) \rangle = \frac{1}{s} \langle (p_1, 0) \rangle V^{(0)}_+ \langle p_2, x \rangle V^{(0)}_+ \langle p_3, 1 \rangle V^{(-1)}_-(p_4, \infty) \rangle$$

(3.16)

where $s = p_2 \cdot p_1$ and the relative sign is due to the exchange of space-time fermions. Note that the terms appearing on the l.h.s. of the above two equations give precisely the contribution of matter fields in (3.9).

The terms appearing in the r.h.s. of (3.15) and (3.16) are four boson tree-level amplitudes and they can be calculated at the level of the effective field theory. Since we are interested in the zero-momentum limit, only the terms involving two derivatives are relevant and the result is [11]:

$$A(C(p_1)T(p_2)\bar{T}(p_3)C(p_4)) = s(R_{C\bar{C}TT} + \frac{u}{t} G_{C\bar{C}g_{TT}}) + V_{C\bar{C}TT}$$

(3.17)

where $C$ ($\overline{C}$) are chiral (anti-chiral) matter fields (not necessarily complex conjugate), $G_{C\overline{C}} = \partial_C \partial_{\overline{C}} K$ and $g_{TT} = \partial_{T} \partial_{\overline{T}} K$ are the matter and moduli metrics, respectively, with $K$ being the Kahler potential and $R_{C\bar{C}TT}$ is the corresponding Riemann tensor. $V_{C\bar{C}TT}$ denotes the potential contribution which generate a mass to the fields $C, \overline{C}$ from the $T$ expectation value. If this term is non zero, it induces a divergence in (3.9) via (3.13) and (3.16), as also discussed before. As also previously stated we will not consider such a situation. The second term in the r.h.s. of (3.17) accounts for graviton exchange in the $t$-channel. When there is enhanced gauge symmetry, there is an additional contribution in the $t$-channel due to the exchange of gauge bosons which couple simultaneously to $T, \overline{T}$ and $C, \overline{C}$ fields. However this term is proportional to $\text{Tr}_C Q'_C$, where $Q'$ is the charge with respect to the enhanced gauge symmetry, and it vanishes unless there is an anomalous $U(1)$ factor. In the following we restrict ourselves to moduli which are neutral under such anomalous $U(1)$ factors.

Similarly, (3.13) involves $A(\overline{C}(p_1)T(p_2)\overline{T}(p_3)C(p_4))$ which is obtained from (3.17) by interchanging $s$ and $u$. Substituting these expressions in (3.16) and (3.13) one has to consider their difference in order to take into account $(-1)^F$ in (3.4). Moreover, the amplitude (3.9) obtained from the degeneration of the torus is a forward amplitude and
therefore the correct zero-momentum limit is to first take \( t \to 0 \) and then set the momenta equal to zero. The final expression is

\[
I_{\text{matter}} = \sum_{C, \overline{C}} \mathcal{T}(C) G^{C\overline{C}} (2R_{C\overline{C}1T} - G_{C\overline{C}1T} g_{1T})
\]

(3.18)

where \( I_{\text{matter}} \) is the contribution of matter fields to \( I \) and \( \mathcal{T}(C) \) is the index of the representation of the \( C \) field. The remaining contribution to the trace in (3.9) is due to the gauginos, which can be explicitly calculated using the fact that the internal part of their vertices is \( e^{\pm i \frac{\sqrt{3}}{2} H \hat{J}} \), \( \hat{J} \) being the Kac-Moody current. The answer is

\[
I_{\text{gauge}} = \mathcal{T}(\text{Ad}) g_{1T}
\]

(3.19)

where \( \text{Ad} \) denotes adjoint representation.

4 \quad (2, 2) \text{ models and orbifolds}

In the case of symmetric \((2, 2)\) compactifications, the gauge group is \( E_6 \times E_8 \) and the matter fields transform as \( 27 \) or \( \overline{27} \) under \( E_6 \) and they are in one-to-one correspondence with the moduli: \( 27 \)'s are related to \((1, 1)\) moduli and \( \overline{27} \)'s to \((1, 2)\) moduli. Furthermore, the moduli metric is block-diagonal with respect to these two types of moduli. In the following \((1, 1)\) and \((1, 2)\) moduli will be collectively denoted by \( T_1 \) and \( T_2 \) and their metrics by \( g^1 \) and \( g^2 \). Equation (3.18) is then further simplified due to relations among various four-point string amplitudes, emerging from the right moving \( N = 2 \) internal supersymmetry [11]. In fact, relating amplitudes involving two matter and two moduli fields with those involving four moduli, one obtains for \((1, 1)\) moduli

\[
G^{C\overline{C}} R_{C\overline{C}1T_1} T_1 = R^1_{T_1 T_1} - \frac{1}{3} (h_{(1,1)} - h_{(1,2)}) g^1_{T_1 T_1}
\]

(4.1)

where \( h_{(1,1)} \) and \( h_{(1,2)} \) denote the number of \((1, 1)\) and \((1, 2)\) moduli, respectively, and \( R_1 \) is the Ricci-tensor constructed with the metric \( g_1 \), \( R^1_{TT} = \partial_T \partial_T \ln \det g_1 \). A similar expression is obtained for \((1, 2)\) moduli by interchanging \((1 \leftrightarrow 2)\) and \( h_{(1,1)} \leftrightarrow h_{(1,2)} \).
Substituting (4.1) in (3.18) and adding (3.19) one gets

\[
I_1(E_6) = [\mathcal{T}(E_6) - \mathcal{T}(27)(5h_{(1,1)} + 3h_{(1,2)})]g^1 + 2\mathcal{T}(27)R^1
\]

\[
I_1(E_8) = \mathcal{T}(E_8)g^1
\]

(4.2)

Here \( \mathcal{T}(27) = 3, \mathcal{T}(E_6) = 12 \) and \( \mathcal{T}(E_8) = 30 \). A similar expression is obtained for \( I_2 \) by interchanging \((1 \leftrightarrow 2)\) and \( h_{(1,1)} \leftrightarrow h_{(1,2)} \). Equation (4.2) is identical with that obtained in [7] using effective field theory anomalous graphs. Note that the result (4.2) continues to hold when there is an enhanced gauge symmetry. Although the relationship between the four-moduli Riemann tensor and the one with two matter and two moduli indices gets modified by additional terms due to the exchange of additional gauge bosons, these terms being proportional to \( Q'_C \) (additional gauge charges of \( C \)'s) drop out after taking the trace in order to obtain (4.1). The above expressions can be further related to Yukawa couplings by using the relation [11, 12]

\[
R^1_{i\bar{\jmath}} = (h_{(1,1)} + 1)g^1_{i\bar{j}} - e^{2K_1} |W^1|^2_{i\bar{\jmath}}
\]

(4.3)

where \( W^1 \) and \( W^2 \) are holomorphic functions of the moduli fields and give the Yukawa couplings of 27 and \( \bar{27} \) families, respectively. A similar relation is valid for \( R^2 \) by interchanging \((1 \leftrightarrow 2)\) and \( h_{(1,1)} \leftrightarrow h_{(1,2)} \).

We now apply the above general results to the case of \( (2, 2) \) orbifolds. Consider first the untwisted moduli. For simplicity, we discuss the case when the three internal planes are orthogonal to each other and obtain the dependence of the gauge couplings on the moduli associated to these three planes. The dependence on the remaining untwisted moduli can then be obtained using the full duality group. Let \( T \) denote the \((1, 1)\) modulus associated with the first plane; its vertex at zero-momentum being \( \partial X_1 \bar{\partial} \bar{X}_1 \), where \( X_1 \) is the complex coordinate in the plane normalized so that the double pole in the two point function \( \langle \partial X_1 \partial \bar{X}_1 \rangle \) comes with the coefficient \( \frac{1}{T+\bar{T}} \). The moduli metric is then given by \( g_{i\bar{\jmath}} = (T + \bar{T})^{-2} \). To compute the Ricci tensor entering in (4.2) we also need the \( T \)-dependence of the remaining components of the moduli metric. The components of the metric along the moduli associated to the other two planes are independent of \( T \). However, the untwisted moduli related to the angles between the first and the other
two planes with vertices being of the form $\partial X_1 \bar{\partial} X_{2,3}$, lead to a metric proportional to $(T + \overline{T})^{-1}$. Let $n_1$ be the number of such untwisted moduli. As for the twisted moduli (blowing-up modes), only the $(1, 1)$ type which carry non zero charge with respect to the $U(1)$ current $\bar{J}_1 = \bar{\psi}_1 \psi_1^*$ have a $T$-dependent metric proportional to $(T + \overline{T})^{-1}$ \[1\]. $\bar{\psi}_1$ is the right-moving $N = 2$ superpartner of the coordinate $X_1$. Let $b_1$ be the number of such twisted $(1, 1)$ moduli. Then, the Ricci-tensor is

$$R_{1T} = (2 + n_1 + b_1) g_{1T}$$

Substituting this expression in (4.2) and taking the difference between $E_6$ and $E_8$ contributions, one gets

$$I_1(E_6) - I_1(E_8) = \left[ \frac{1}{3} (h_{(1,1)} - h_{(1,2)}) - 4 - 2N_1 \right] T(27) \frac{1}{(T + \overline{T})^2} \quad (4.4)$$

where $N_1$ is the total number of $(1, 1)$ moduli which have no singular OPE with $T$ and $\overline{T}$. One can explicitly check in examples that if $T$ corresponds to a plane which is twisted under all non trivial elements of the orbifold group the r.h.s. of (4.4) vanishes, in agreement with known results \[2\]. In the case when some non trivial element of the orbifold group leaves this plane untwisted, one gets a non vanishing coefficient related to the $\beta$-function of the corresponding $N = 2$ subsector. More precisely, in this case one has

$$I_1(E_8) - I_1(E_6) = \frac{2}{\text{ind}} (\hat{b}_8 - \hat{b}_6) \frac{1}{(T + \overline{T})^2} \quad (4.5)$$

in which ind is the index of the little subgroup of the plane in the full orbifold group, and $\hat{b}_8, \hat{b}_6$ are the $\beta$-function coefficients of the corresponding $N = 2$ subsectors.

In the case of blowing-up modes ($B, \overline{B}$), one can calculate the dependence of the gauge couplings on these moduli in a perturbative expansion. For instance, the first non trivial term proportional to $B \overline{B}$ is just given by the violation of the integrability condition (4.2) for $B, \overline{B}$. From the initial expression (3.3) this involves a correlation function on the torus involving $B$ and $\overline{B}$, which can be non zero only if $B$ and $\overline{B}$ come from opposite twisted

*Recall that in the case of $(2, 2)$ orbifolds there is an enhanced gauge symmetry: besides the $SO(10)$ there are three Cartan generators $\bar{J}_i = \bar{\psi}_i \psi_i^*$, $i = 1, 2, 3$, corresponding to the three planes. Of these, the diagonal generator is the $N = 2 U(1)$ current which together with the $SO(10)$ gives rise to $E_6$. \[12\]
sectors and fixed points. $\overline{B}$ is then the complex conjugate of $B$. The untwisted moduli dependence of the metric $g_{\overline{B}B}$ has already been discussed in the previous paragraph, and, on the other hand, the corresponding Ricci tensor can be calculated using \( (4.3) \), so that one has:

\[
\Delta E_6 = \frac{2}{\text{ind}} \hat{b}_6 \log(|\eta(iT)|^4(T+\overline{T}))+B\overline{B}\{g_{\overline{B}B}\{T(E_6)+T(2\tilde{T})(\frac{\chi}{3}+2)\}-2T(2\tilde{T})e^{2K_1}|W^1|\}
\]

where $\chi = h_{(1,1)} - h_{(1,2)}$ is the Euler number, and the Yukawa couplings are given for instance in \([13]\).

## 5 Gravitational couplings

We now discuss the moduli dependence of gravitational couplings, generalizing the results of ref\([8]\) which were obtained for orbifolds. In this case, the role of the gauge couplings and the $\Theta$-angles is played by the coefficients of the Gauss-Bonnet combination and the $R\tilde{R}$ term, respectively. The study of the moduli dependence of this coupling involves the calculation of the violation of the integrability condition for $\Theta^{\text{grav}}$. As before, we start from the CP-odd part of the on-shell three-point function of two gravitons and one modulus. Equation \((2.1)\) is now replaced by

\[
\epsilon^{\mu\nu\lambda\rho}p_1\lambda p_2\rho\gamma T_{\text{grav}} = \int_{\Gamma} \frac{d^2\tau}{\tau_2} \int \prod_{i=1}^{3} d^2z_i \left( V_h^{\alpha\mu}(p_1, z_1) V_h^{\beta\nu}(p_2, z_2) V^{(-1)}_T(p_3, z_3) T_F(z) \right)_{\text{odd}}
\]

where $V_h$’s are the graviton vertices obtained from \((2.2)\) by replacing $\bar{J}^a$ with $\bar{\partial}X^\alpha$. The four space-time zero-modes are provided by the graviton vertices, giving rise to the kinematic factor $\epsilon^{\mu\nu\lambda\rho}p_1\lambda p_2\rho$. The remaining space-time part of the correlator \((5.1)\) is

\[
\int d^2z_1 \int d^2z_2 \langle \bar{\partial}X^\alpha \epsilon^{ip_1X}(z_1) \bar{\partial}X^\beta \epsilon^{ip_2X}(z_2) \epsilon^{ip_3X}(z_3) \rangle
\]

as only the internal part of $T_F$ contributes in \((5.1)\). Of all possible ways of contractions, the only non vanishing one is when $\bar{\partial}X^\alpha$ is contracted with $\epsilon^{ip_2X}$ and $\bar{\partial}X^\beta$ with $\epsilon^{ip_1X}$: the remaining ones are total derivatives (on-shell) in $\bar{z}_1$ or $\bar{z}_2$ and, consequently, vanish.
upon integration. Then, (5.2) provides the kinematic factor \( p_2^2 p_1^3 \) multiplying \( \tilde{\tau} \)
\[
\tau_2 \int d^2 z_1 \langle \overline{\partial} X(z_1) X(0) \rangle^2 = -2\pi i \tau_2 \partial_\tau (\frac{1}{\tau_2 \tilde{\eta}^2})
\] (5.3)

The internal part of (5.1) being the same as in the gauge case, one obtains for \( \Theta^{\text{grav}} \) an expression identical to (2.5) after replacing \( Q_a^2 \) with
\[
Q_{\text{grav}}^2 \equiv -\partial_\tau \ln(\tilde{\eta}^2)
\] (5.4)

The rest of the analysis can now be repeated exactly as for the gauge couplings leading to (2.7) for \( \Delta^{\text{grav}} \) after replacing \( Q_a^2 \) with \( Q_{\text{grav}}^2 \). Furthermore, considering the differences \( \Theta^{\text{grav}} - \frac{1}{k} \Theta^{\text{gauge}} \) one finds equation (3.7) for the violation of the integrability condition, and once again only the massless states contribute in the trace. There are two possible ways to get these massless states:

(a) From an excitation of the space-time part, i.e. gravitinos and dilatino corresponding to the first term of the expansion of \( \tilde{\eta}^2 \) in (5.3), giving rise to \( Q_{\text{grav}}^2 = \frac{11}{6} \).

(b) From the lowest order term in (5.3) corresponding to gauginos and matter fermions with \( Q_{\text{grav}}^2 = -\frac{1}{12} \).

Equation (3.9) then follows with the above values of \( Q_{\text{grav}}^2 \). The contribution of gravitinos, dilatino and gauginos can be determined by a direct calculation as in the case of the adjoint representation for gauge couplings with the result
\[
I^{\text{grav}}_V = \left( \frac{11}{6} - \frac{1}{12} n_V \right) g_{TT}
\] (5.5)
where \( n_V \) is the number of massless vectors. The contribution of matter fields can also be calculated by the same steps as in the gauge case, resulting
\[
I^{\text{grav}}_S = -\frac{1}{12} \left( 2 \sum_{a,\alpha} G^{a\alpha} R_{a\alpha TTT} - n_S g_{TT} \right)
\] (5.6)
where the sum is over all massless complex scalar fields (excluding dilaton), \( G_{a\alpha} \) being the corresponding metric and \( n_s \) their number.

In the case of (2, 2) models, (5.3) and (5.6) together become (for example for (1, 1) moduli)
\[
I^{\text{grav}}_1 = \left[ \frac{11}{6} + \frac{1}{12} (n_S - n_V) + \frac{3}{2} (h_{1,1} - h_{1,2}) \right] g_{TT} - \frac{14}{3} R_{TT} - \frac{1}{6} \sum_{a,\alpha} G^{a\alpha} R_{a\alpha TTT}
\] (5.7)
where the sum is restricted to the $E_6 \times E_8$ singlets which are not $(1, 1)$ or $(1, 2)$ moduli. Unfortunately, this expression cannot be further simplified due to the present lack of understanding of these extra massless singlet states. Of course for specific models such as orbifolds and Gepner models these quantities can be computed.

### 6 Examples and conclusions

For instance, consider $(2, 2)$ orbifolds, where the three internal planes are orthogonal to each other. Associated to them there are $U(1)$ charges $F^i$ in the fermionic sector (the $N = 2$, $U(1)$ charge being the sum of the $F^i$'s). Let then $T$ denote the $(1, 1)$ modulus corresponding to the first plane: the right moving part of the vertex operator for it is then $\bar{\partial}X_1 (\partial X_1)$ where $X_1, \bar{X}_1$ are complex coordinates on the first plane which diagonalize the orbifold group action: $g : X_1 \rightarrow e^{2\pi i \alpha_1} X_1$, $\alpha_1$ being a rational number $0 \leq \alpha < 1$.

The non-moduli singlet states are generally constructed by applying certain numbers of right moving oscillator modes on the twist fields. Let $n_i (-\pi_i)$ be the numbers of $\bar{\partial}X_i (\partial X_i)$ oscillator modes in a given state.

The contribution to the curvature term in (5.7) from the non-moduli singlets, from the sector where the first plane is twisted can be calculated directly from the 4-point function and is:

$$-\frac{1}{6} \{ \sum [(1 - \alpha_1) + n_1] - b_1 \},$$

(6.1)

where $b_1$ are the twisted moduli that have singular OPE with $T$, and the sum is over chiral states ($\sum F_i = 1$) in these sectors. Similarly the contribution from the sectors that leave $X_1$ untwisted (i.e. $\alpha_1 = 0$) is given by:

$$-\frac{1}{6} \sum F_1,$$

(6.2)

where the sum is over the chiral states in such sectors.

For specific orbifold models one can get all the above numbers from the spectrum. One can check that for models with no $N = 2$ subsector (such as $Z_3$ and $Z_7$ orbifolds), $I_1^{grav} + 2I_{E_8} = 0$ (note that this is the correct modular invariant combination). On the
other hand for models with $N = 2$ subsector one obtains:

$$I_1^{\text{grav}} + 2I_8 = \frac{2}{\text{ind}} \left( \frac{1}{2} \hat{b}_{\text{grav}} + 2\hat{b}_8 \right) \frac{1}{(T + \bar{T})^2},$$

(6.3)

where as before ind is the index of the little group of the plane, $\hat{b}_{\text{grav}}$ and $\hat{b}_8$ are the trace anomaly and $\beta$-function coefficients for the corresponding $N = 2$ space-time supersymmetric theory obtained when the orbifold is taken to be the above little group.

For example, take the $Z_4$ orbifold; the orbifold group is generated by an element $g$ with $(\alpha_1, \alpha_2, \alpha_3) = (1/4, 1/4, 1/2)$. The gauge group is $E_8 \times E_6 \times SU(2) \times U(1)$ and the left-moving chiral $E_6$ singlets are shown below, with the relevant quantum numbers $(F_1, F_2, F_3)$ and $(n_1, n_2, n_3)$, for untwisted and twisted sectors. From the untwisted sector: $(F_1, F_2, F_3) = (1, 0, 0)$, or $(0, 1, 0)$, $SU(2)$ doublets.

From the $g$-twisted sector: $(F_1, F_2, F_3) = (1/4, 1/4, 1/2)$ and for $(n_1, n_2, n_3)$ one has the following possibilities: $16(2, 0, 0)$, $16(1, 1, 0), 16(0, 2, 0), 16(0, 0, 1), 16(0, 0, -1), all singlets, and $16(1, 0, 0), 16(0, 1, 0)$, which are doublets.

From the $g^2$-twisted sector: $(F_1, F_2, F_3) = (1/2, 1/2, 0)$ one has for $(n_1, n_2, n_3)$ the following cases: $10(1, 0, 0), 10(0, 1, 0), 6(1, 0, 0), 6(0, 1, 0), 16(0, 0, 0)$, which are doublets, and $16(0, 0, 0)$ singlets.

Plugging these values in (6.1) and (6.2) for the case where $T$ is the $(1, 1)$ modulus corresponding to the first plane, one finds $I_1^{\text{grav}} + 2I_8 = 0$, as expected because this plane is twisted by all the nontrivial elements of the orbifold group. On the other hand, if $T$ corresponds to the $(1, 1)$ modulus of the third plane, one finds: $I_1^{\text{grav}} + 2I_8 = \frac{1}{2} \hat{b}_{\text{grav}} + 2\hat{b}_8$, consistent with (6.3) as the index of the little group of the third plane is 2. Here $\hat{b}_{\text{grav}}$ and $\hat{b}_8$ are the trace anomaly and $E_8$ $\beta$-function for the $N = 2$ spacetime SUSY theory corresponding to the $Z_2$ orbifold generated by $g^2$.

Although we have discussed here the examples of $(2, 2)$ orbifolds, one can easily generalize this to $(2, 0)$ orbifolds. The only difference is that now that the relation (1.1) between the mixed moduli-matter and moduli Riemann tensors is no longer valid. However one can use relations similar to (6.1) and (6.2) to evaluate the contribution of matter fields.

To conclude, the main result here is that the 1-loop violation of the integrability
condition for Θ-angles (gauge and gravitation) and consequently non-harmonicity of the

gauge and gravitational couplings is expressed in terms of tree level four-point amplitudes.

For (2, 2) models this quantity for the gauge coupling case, was further related to the

metric on the moduli space due to relations among various four point amplitudes.

One of the issues which needs further study is that of special subspaces of the Moduli

Space where one has extra massless states. As discussed in in section 2, if one evaluates

threshold corrections at a generic point (i.e. away from these subspaces) these extra

massless states do not contribute, however on these special subspaces they do contribute.

This could introduce a discontinuity in the threshold correction as function of moduli.

For example, many of the massless singlets in orbifold backgrounds become massive when

the blowing-up modes are turned on; consequently if one approaches the orbifold point

from smooth manifolds the result would be different from the one obtained at the orbifold

point. As suggested also in section 2, a possible starting point to analyze this problem

could be to introduce an infrared cut-off (e.g. a cut-off in the τ_2 integration); the resulting

threshold corrections would then be continuous and duality invariant in moduli. A full

understanding of this issue requires however further study.

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