Inequalities having Seven Means and Proportionality Relations

Inder J. Taneja
Departamento de Matemática
Universidade Federal de Santa Catarina
88.040-900 Florianópolis, SC, Brazil.
e-mail: ijtaneja@gmail.com
http://www.mtm.ufsc.br/~taneja

Abstract

In 2003, Eve [2], studied seven means from geometrical point of view. These means are Harmonic, Geometric, Arithmetic, Heronian, Contra-harmonic, Root-mean square and Centroidal mean. Some of these means are particular cases of Gini’s [3] mean of order $r$ and $s$. In this paper we have established some proportionality relations having these means. Some inequalities among some of differences arising due to seven means inequalities are also established.

Key words: Arithmetic mean, Geometric mean, Heronian mean, triangular discrimination, Hellingar’s distance

AMS Classification: 94A17; 26A48; 26D07.

1 Seven Geometrical Means

Let $a, b > 0$ be two positive numbers. In 2003, Eves [2] studied the geometrical interpretation of the following seven means:

1. Arithmetic mean: $A(a, b) = (a + b)/2$;
2. Geometric mean: $G(a, b) = \sqrt{ab}$;
3. Harmonic mean: $H(a, b) = 2ab/(a + b)$;
4. Heronian mean: $N(a, b) = \left(a + \sqrt{ab} + b\right)/3$;
5. Contra-harmonic mean: $C(a, b) = (a^2 + b^2)/(a + b)$
6. Root-mean-square: $S(a, b) = \sqrt{(a^2 + b^2)/2}$
7. Centroidal mean: $R(a, b) = 2(a^2 + ab + b^2)/(a + b)$

Except 4 and 7 the above means are particular cases of well-known Gini [3] mean of order $r$ and $s$ is given by

$$E_{r,s}(a, b) = \begin{cases} 
\left(\frac{a^r + b^r}{a^s + b^s}\right)^{\frac{1}{r-s}} & r \neq s \\
\sqrt{ab} & r = s = 0 \\
\exp\left(\frac{a^r \ln a + b^r \ln b}{a^r + b^r}\right) & r = s \neq 0
\end{cases}$$

(1)
In particular, we have $E_{-1,0} = H$, $E_{-1/2,1/2} = G$, $E_{0,1} = A$, $E_{0,2} = S$ and $E_{1,2} = R$. Since $E_{r,s} = E_{s,r}$, the Gini-mean $E_{r,s}(a,b)$ is an increasing function in $r$ or $s$ [1]. In view of this we have $H \leq G \leq A \leq S \leq C$. Moreover we can easily verify the following inequality having the above seven means:

$$H \leq G \leq N \leq A \leq R \leq S \leq C. \quad (2)$$

We can write, $M(a,b) = b f_M(a/b)$, where $M$ stands for any of the above seven means, then we have

$$f_H(x) \leq f_G(x) \leq f_N(x) \leq f_A(x) \leq f_R(x) \leq f_S(x) \leq f_C(x), \quad (3)$$

where $f_H(x) = \frac{2x}{x+1}$, $f_G(x) = \sqrt{x}$, $f_N(x) = \left(\frac{x+\sqrt{x}+1}{3}\right)$, $f_A(x) = \frac{x+1}{2}$, $f_R(x) = \frac{2(x^2+x+1)}{3(x+1)}$, $f_S(x) = \frac{\sqrt{(x^2+1)/2}}{2}$ and $f_C(x) = \frac{(x^2+1)/(x+1)}{2}$, $\forall x > 0$, $x \neq 1$. We have equality sign in (3) iff $x = 1$. For simplicity, let us write

$$D_{AB} = b f_{AB}(a,b), \quad (4)$$

where $f_{UV}(x) = f_U(x) - f_V(x)$, with $U \geq V$. Inequalities appearing in (2) admits 21 nonnegative differences. Some of these are equal with multiplicative constants as given below:

$$\Delta := 3D_{CR} = 2D_{AH} = 2D_{CA} = D_{CH} = 6D_{RA} = \frac{3}{2}D_{RH}, \quad (5)$$

$$h := 3D_{AN} = D_{AG} = \frac{3}{2}D_{NG} \quad (6)$$

and

$$D_{CG} = 3D_{RN}. \quad (7)$$

The measures $\Delta$ and $h$ appearing in (5) and (6) are respectively the triangular discrimination [5] and Hellingar’s distance [4] respectively and are given by

$$\Delta(a,b) = \frac{(a-b)^2}{a+b}$$

and

$$h(a,b) = \frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{2}.$$ 

More studied on these two measures can be seen in [6, 7, 8].

We shall improve considerably the inequalities given in (2). For this we need first the convexity of the difference of means. In total, we have 21 differences. Some of them are equal to each other with some multiplicative constants. Some of them are not convex and some of them are convex.

## 2 Convexity of Difference of Means

Let us prove now the convexity of some of the difference of means arising due to inequalities (2). In order to prove it we shall make use of the following lemma [6, 7].
Lemma 2.1. Let $f : I \subseteq R_+ \to R$ be a convex and differentiable function satisfying $f(1) = 0$. Consider a function

$$\varphi_f(a,b) = af\left(\frac{b}{a}\right), \quad a,b > 0,$$

then the function $\varphi_f(a,b)$ is convex in $R^2_+$. Additionally, if $f'(1) = 0$, then the following inequality hold:

$$0 \leq \varphi_f(a,b) \leq \left(\frac{b - a}{a}\right) \varphi_f(a,b).$$

In all the cases, it is easy to check that $f_{AB}(1) = f_A(1) - f_B(1) = 1 - 1 = 0$. According to Lemma 2.1, it is sufficient to show the convexity of the functions $f_{AB}(x)$. It requires only to show that the second order derivative of $f_{AB}(x)$ to be nonnegative for all $x > 0$. Here below are the second order derivatives of the convex functions:

$$f''_{CS}(x) = f''_C(x) - f''_S(x) = \frac{2 \left(2(2x^2 + 2)^{3/2} - (x + 1)^3\right)}{(x + 1)^3 (2x^2 + 2)^{3/2}},$$

$$f''_{CN}(x) = f''_C(x) - f''_N(x) = \frac{48x^{3/2} + (x + 1)^3}{12x^{3/2}(x + 1)^3} > 0,$$

$$f''_{CG}(x) = f''_C(x) - f''_G(x) = \frac{16x^{3/2} + (x + 1)^3}{4x^{3/2}(x + 1)^3} > 0,$$

$$f''_{SA}(x) = f''_S(x) - f''_A(x) = \frac{1}{(x^2 + 1) \sqrt{2x^2 + 2}} > 0,$$

$$f''_{SN}(x) = f''_S(x) - f''_N(x) = \frac{12x^{3/2} + (x^2 + 1) \sqrt{2x^2 + 2}}{12x^{3/2} (x^2 + 1) \sqrt{2x^2 + 2}} > 0,$$

$$f''_{SG}(x) = f''_S(x) - f''_G(x) = \frac{4x^{3/2} + (x^2 + 1) \sqrt{2x^2 + 2}}{4x^{3/2} (x^2 + 1) \sqrt{2x^2 + 2}} > 0,$$

$$f''_{SH}(x) = f''_S(x) - f''_H(x) = \frac{(x + 1)^3 + 4 (x^2 + 1) \sqrt{2x^2 + 2}}{(x + 1)^3 (x^2 + 1) \sqrt{2x^2 + 2}} > 0,$$

$$f''_{AG}(x) = f''_A(x) - f''_G(x) = \frac{1}{4x^{3/2}} > 0,$$

$$f''_{AH}(x) = f''_A(x) - f''_H(x) = \frac{4}{(x + 1)^3} > 0$$

and

$$f''_{RG}(x) = f''_R(x) - f''_G(x) = \frac{16x^{3/2} + 3 (x + 1)^3}{4x^{3/2} (x + 1)^3} > 0.$$

Since, $S \geq A$, this implies that $S^3 \geq A^3$, i.e., $\left(\sqrt{\frac{x^2 + 1}{2}}\right)^3 - (\frac{x + 1}{2})^3 \geq 0$. This gives $2(2x^2 + 2)^{3/2} - (x + 1)^3 \geq 0$. Thus we have $f''_{CS}(x) \geq 0$ for all $x > 0$. The difference means $D_{SR}$, $D_{NH}$ and $D_{GH}$ are not convex.
3 Inequalities among of Differences of Means

In this section we shall bring sequence of inequalities based on the differences arising due to (??). This we shall present in two parts. The results given in this section are based on the applications of the following lemma [6, 7]:

Lemma 3.1. Let $f_1, f_2 : I \subset \mathbb{R}_+ \to \mathbb{R}$ be two convex functions satisfying the assumptions:

(i) $f_1(1) = f_1'(1) = 0, f_2(1) = f_2'(1) = 0$;
(ii) $f_1$ and $f_2$ are twice differentiable in $\mathbb{R}_+$;
(iii) there exists the real constants $\alpha, \beta$ such that $0 \leq \alpha < \beta$ and

\[ \alpha \leq \frac{f_1''(x)}{f_2''(x)} \leq \beta, \quad f_2''(x) > 0, \]

for all $x > 0$ then we have the inequalities:

\[ \alpha \varphi_{f_2}(a, b) \leq \varphi_{f_1}(a, b) \leq \beta \varphi_{f_2}(a, b), \]

for all $a, b \in (0, \infty)$, where the function $\phi_{(\cdot)}(a, b)$ is as defined in Lemma 2.1.

The inequalities appearing in (2) admits 21 nonnegative differences. The differences satisfies some simple inequalities. These are given by the following pyramid:

\[
\begin{align*}
D_{GH} ; \\
D_{NG} & \leq D_{NH} ; \\
D_{AN} & \leq D_{AG} \leq D_{AH} ; \\
D_{RA} & \leq D_{RN} \leq D_{RG} \leq D_{RH} ; \\
D_{SR} & \leq D_{SA} \leq D_{SN} \leq D_{SG} \leq D_{SH} ; \\
D_{CS} & \leq D_{CR} \leq D_{CA} \leq D_{CN} \leq D_{CG} \leq D_{CH} ,
\end{align*}
\]

where the $D_{GH} = G - H$, $D_{NG} = N - G$, $D_{CS} = C - S$, etc. As we have seen above some of these differences are equals with multiplicative constants.

The difference means $D_{SR}$, $D_{NH}$ and $D_{GH}$ are not convex. The other convex measures satisfies some interesting inequalities with each other given by the theorem below.

Theorem 3.1. The following inequalities hold:

\[
D_{SA} \leq \left\{ \frac{3}{4} D_{SN} \right\} \leq \left\{ \frac{3}{4} D_{SN} \right\} \leq \left\{ \frac{3}{4} D_{CR} \right\} \leq \left\{ \frac{3}{4} D_{CR} \right\} \leq \frac{3}{2} D_{AN} .
\]

Proof. We shall prove above result by parts. Here we shall use frequently the second order derivatives given in section 2.
1. For $D_{SA} \leq \frac{3}{4} D_{SN}$: Let us consider the function $g_{SA,SN}(x) = \frac{f'_{SA}(x)}{f_{SN}(x)}$. This gives

$$g'_{SA,SN}(x) = -\frac{72 (x^2 - 1) \sqrt{x} \sqrt{2x^2 + 2}}{[24x^{3/2} + (2x^2 + 2)^{3/2}]^2} \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases}.$$

Also we have

$$\beta_{SA,SN} = \sup_{x \in (0,\infty)} g_{SA,SN}(x) = g_{SA,SN}(1) = \frac{3}{4}. \tag{9}$$

By the application Lemma 3.1 with (9) we get the required result.

2. For $D_{SA} \leq \frac{1}{3} D_{SH}$: Let us consider the function $g_{SA,SH}(x) = \frac{f'_{SA}(x)}{f_{SH}(x)}$. This gives

$$g'_{SA,SH}(x) = -\frac{12 (x - 1) (x + 1)^2 \sqrt{2x^2 + 2}}{[(x + 1)^3 + 4 (x^2 + 1) \sqrt{2x^2 + 2}]^2} \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases}.$$

Also we have

$$\beta_{SA,SH} = \sup_{x \in (0,\infty)} g_{SA,SH}(x) = g_{SA,SH}(1) = \frac{1}{3}. \tag{10}$$

By the application Lemma 3.1 with (10) we get the required result.

3. For $D_{SH} \leq \frac{9}{4} D_{CR}$: Let us consider the function $g_{SH,CR}(x) = \frac{f'_{SH}(x)}{f_{CR}(x)}$. This gives

$$g'_{SH,CR}(x) = -\frac{9 (x - 1) (x + 1)^2}{8 (x^2 + 1)^2 \sqrt{2x^2 + 2}} \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases}.$$

Also we have

$$\beta_{SA,SH} = \sup_{x \in (0,\infty)} g_{SA,SH}(x) = g_{SA,SH}(1) = \frac{1}{3}. \tag{11}$$

By the application Lemma 3.1 with (11) we get the required result.

4. For $D_{CR} \leq \frac{4}{7} D_{CN}$: Let us consider the function $g_{CR,CN}(x) = \frac{f'_{CR}(x)}{f_{CN}(x)}$. This gives

$$g'_{CR,CN}(x) = -\frac{48 \sqrt{x} (x - 1) (x + 1)^2}{[48x^{3/2} + (x + 1)^3]^2} \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases}.$$

Also we have

$$\beta_{CR,CR} = \sup_{x \in (0,\infty)} g_{CR,CR}(x) = g_{CR,CR}(1) = \frac{4}{7}. \tag{12}$$

By the application Lemma 3.1 with (11) we get the required result.
5. For $D_{CR} \leq \frac{2}{3}D_{SG}$: Let us consider the function $g_{CR\_SG}(x) = f''_{CR}(x)/f''_{SG}(x)$. This gives
\[ g'_{CR\_SG}(x) = -\frac{16(x - 1)\sqrt{x^2 + 2} \times v_1(x)}{(x + 1)^3 \left[4x^{3/2} + (x^2 + 1)\sqrt{2x^2 + 2}\right]^2} \left\{ \begin{array}{ll} > 0 & x < 1 \\ < 0 & x > 1 \end{array} \right. , \]
where
\[ v_1(x) = (x^2 + 1)^2 \sqrt{2x^2 + 2} - 8x^{5/2} = 8 \left[\left(\frac{x^2 + 1}{2}\right)^5 - (\sqrt{x})^5\right] > 0. \]
Above expression holds since $S > G, \forall x > 0, x \neq 1$. Also we have
\[ \beta_{CR\_SG} = \sup_{x \in (0,\infty)} g_{CR\_SG}(x) = g_{CR\_SG}(1) = \frac{2}{3}. \tag{13} \]
By the application Lemma 3.1 with (13) we get the required result.

6. For $D_{SN} \leq \frac{4}{3}D_{CN}$: We have $\beta_{SN\_CN} = f''_{SN}(1)/f''_{CN}(1) = \frac{4}{3}$. Now, we have to show that $\frac{4}{3}D_{CN} - D_{SN} \geq 0$, i.e., $\frac{4}{3}(4C + 3N - 7S) \geq 0$. We can write $\frac{4}{3}D_{CN} - D_{SN} = b f_{CN\_SN}(a/b)$, where
\[ f_{CN\_SN}(x) = \frac{4}{3}f_{SN}(x) - f_{CN}(x) = \frac{1}{14(x+1)} \times v_2(x), \]
where
\[ v_2(x) = 10x^2 + 10 + 4x + 2x^{3/2} + 2\sqrt{x} - 7(x + 1)\sqrt{2x^2 + 2}. \]
In order to prove the non-negativity of $v_2(x)$, let us consider the function
\[ h_2(x) = \left(10x^2 + 10 + 4x + 2x^{3/2} + 2\sqrt{x}\right)^2 - \left(7(x + 1)\sqrt{2x^2 + 2}\right)^2 \]
\[ = \left(2x^2 + 48x^{3/2} + 68x + 48\sqrt{x} + 2\right)(\sqrt{x} - 1)^4. \]
Since $h_2(x) \geq 0$, giving $v_2(x) \geq 0, \forall x > 0$. This implies that $f_{CN\_SN}(x) \geq 0, \forall x > 0$, hence proving the required result.

**Argument**: Let $a$ and $b$ two positive numbers, i.e., $a > 0$ and $b > 0$. If $a^2 - b^2 \geq 0$, then we can conclude that $a \geq b$ because $a - b = (a^2 - b^2)/(a + b)$. We have used this argument to prove $v_2(x) \geq 0, \forall x > 0$.

7. For $D_{SN} \leq \frac{2}{3}D_{SG}$: Let us consider the function $g_{SN\_SG}(x) = f''_{SN}(x)/f''_{SG}(x)$. This gives
\[ g'_{SN\_SG}(x) = -\frac{4\sqrt{x} (x^2 - 1)\sqrt{2x^2 + 2}}{\left[4x^{3/2} + (x^2 + 1)\sqrt{2x^2 + 2}\right]^2} \left\{ \begin{array}{ll} > 0 & x < 1 \\ < 0 & x > 1 \end{array} \right. . \]
Also we have
\[ \beta_{SBN\_SG} = \sup_{x \in (0,\infty)} g_{SN\_SG}(x) = g_{SN\_SG}(1) = \frac{2}{3}. \tag{14} \]
By the application Lemma 3.1 with (14) we get the required result.
8. For $D_{CN} \leq \frac{7}{3}D_{CS}$: We have $\beta_{CN,CS} = f''_{CN}(1)/f''_{CS}(1) = \frac{7}{3}$. Now, we have to show that $\frac{7}{3}D_{CS} - D_{CN} \geq 0$, i.e., $\frac{1}{3}(4C + 3N - 7S) \geq 0$. This is true in view of part 6.

9. For $D_{CS} \leq 3D_{AN}$: Let us consider the function $g_{CS,AN}(x) = f''_{CS}(x)/f''_{AN}(x)$. This gives

$$g'_{CS,AN}(x) = -\frac{18\sqrt{x}(x-1)x_3(x)}{(x^2+1)^2(x+1)^2\sqrt{2x^2+2}} \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases},$$

where

$$x_3(x) = 4(x^2+1)^2\sqrt{2x^2+2} - (x+1)^5$$

$$= 32\left[\left(\frac{x^2+1}{2}\right)^2 - \left(\frac{x+1}{2}\right)^5\right] > 0.$$

Above expression holds since $S > A, \forall x > 0, x \neq 1$. Also we have

$$\beta_{CS,AN} = \sup_{x \in (0,\infty)} g_{CS,AN}(x) = g_{CS,AN}(1) = 3.$$  \hspace{1cm} (15)

By the application Lemma 3.1 with (15) we get the required result.

10. For $D_{CN} \leq \frac{7}{5}D_{CG}$: Let us consider the function $g_{CN,CG}(x) = f''_{CN}(x)/f''_{CG}(x)$. This gives

$$g'_{CN,CG}(x) = -\frac{16\sqrt{x}(x-1)(x+1)^2}{16x^{3/2} + (x+1)^2} \begin{cases} > 0 & x < 1 \\ < 0 & x > 1 \end{cases}.$$

Also we have

$$\beta_{CN,CG} = \sup_{x \in (0,\infty)} g_{CN,CG}(x) = g_{CN,CG}(1) = \frac{7}{9}.$$  \hspace{1cm} (16)

By the application Lemma 3.1 with (16) we get the required result.

11. For $D_{SG} \leq \frac{6}{5}D_{RG}$: We have $\beta_{SG,RG} = f''_{SG}(1)/f''_{RG}(1) = \frac{6}{5}$. Now, we have to show that $\frac{6}{5}D_{RG} - D_{SG} \geq 0$, i.e., $\frac{1}{5}(6R - G - 5S) \geq 0$. We can write $\frac{6}{5}D_{RG} - D_{SG} = b f_{SG,RG}(a/b)$, where

$$f_{SG,RG}(x) = \frac{6}{5}f_{RG}(x) - f_{SG}(x) = \frac{1}{10(x+1)}x_3(x),$$

where

$$x_3(x) = 8(x^2 + x + 1) - 2\sqrt{x}(x+1) - 5(x+1)\sqrt{2x^2+2}.$$

In order to prove the non-negativity of $x_3(x)$, let us consider the function

$$h_3(x) = \left[8(x^2 + x + 1) - 2\sqrt{x}(x+1)\right]^2 - \left(5(x+1)\sqrt{2x^2+2}\right)^2$$

$$= \left(14x^2 + 24x^{3/2} + 44x + 24\sqrt{x} + 14\right)(\sqrt{x} - 1)^4.$$
Since $h_3(x) \geq 0$, giving $v_3(x) \geq 0, \forall x > 0$. The non-negativity of the expression $8 \left( x^2 + x + 1 \right) - 2\sqrt{x} (x + 1)$ can be shown easily following the same lines, i.e.

$$\left[ 4 \left( x^2 + x + 1 \right) \right]^2 - \left[ \sqrt{x} (x + 1) \right]^2 = 16x^4 + 31x^3 + 46x^2 + 31x + 16 > 0.$$ 

This implies that $f_{SG, RG}(x) \geq 0, \forall x > 0$, hence proving the required result.

12. **For $D_{CG} \leq \frac{9}{5} D_{RG}$:** Let us consider the function $g_{CG, RG}(x) = \frac{f''_{CG}(x)}{f''_{RG}(x)}$. This gives

$$g'_{CG, RG}(x) = -\frac{144 \sqrt{x} (x - 1) (x + 1) \left[ 16x^{3/2} + 3 (x + 1) \right]^2}{x < 1} \quad x > 1.$$

Also we have

$$\beta_{CG, RG} = \sup_{x \in (0, \infty)} g_{CG, RG}(x) = g_{CG, RG}(1) = \frac{9}{5}. \quad (17)$$

By the application Lemma 3.1 with (17) we get the required result.

13. **For $D_{RG} \leq 5D_{AN}$:** Let us consider the function $g_{RG, AN}(x) = \frac{f''_{RG}(x)}{f''_{AN}(x)}$. This gives

$$g'_{RG, AN}(x) = -\frac{24 \sqrt{x} (x - 1) (x + 1)}{(x + 1)^4} \quad x < 1.$$ 

Also we have

$$\beta_{RG, AN} = \sup_{x \in (0, \infty)} g_{RG, AN}(x) = g_{RG, AN}(1) = 5. \quad (18)$$

By the application Lemma 3.1 with (18) we get the required result.

**Remark 3.1.** The above 13 parts allows writing inequalities in their equivalent forms:

1. $\frac{2G + S}{3} \leq N$;
2. $\frac{2C + 7G}{9} \leq N$;
3. $\frac{S + 3N}{4} \leq A$;
4. $\frac{2S + H}{3} \leq A$;
5. $\frac{2C + 3N}{4} \leq R$;
6. $\frac{G + 5S}{6} \leq R$;
7. $\frac{4G + 5C}{9} \leq R$;
8. $S \leq \frac{4C + 3N}{7}$;
9. $9R + 4S \leq 9C + 4H$;
10. $2G + 3C \leq 2S + 3R$;
11. $3N + C \leq 3A + S$;
12. $5N + R \leq 5A + G$.

Based on these equivalent versions, here below is an improvement over inequalities appearing in (2):
Proposition 3.1. The following inequalities hold:

\[
H \leq G \leq \left\{ \frac{2G+S}{2G+7G} \right\} \leq N \leq \left\{ \frac{3A+S-C}{3} \right\} \leq A \leq \left\{ \frac{G+5S}{4G+5C} \right\} \leq R \leq \left\{ \frac{S\cdot 5\cdot N}{5A+G-5N} \right\} \leq \frac{9C+4H-9R}{4} \leq C \leq \frac{2S+3R-2G}{3}.
\]  

\tag{19}

Inequalities appearing (19) can be proved by using similar arguments of Theorem 3.1.

3.1 Proportionality Relations among Means

As a part of (8), let us consider the following inequalities:

\[
\begin{align*}
\frac{1}{4} \Delta & \leq \frac{3}{4} D_{CN} \leq \frac{1}{4} D_{CG} \leq \frac{3}{4} D_{RG} \leq h. \\
W_1 \leq W_2 \leq W_3 \leq W_4 \leq W_5.
\end{align*}
\]  

\tag{20}

The expression (20) has six means instead of seven. For simplicity, let us rewrite the expression (20):

\[
W_1 \leq W_2 \leq W_3 \leq W_4 \leq W_5,
\]  

\tag{21}

where for example \( W_1 = \frac{1}{4} \Delta, W_2 = \frac{3}{4} D_{CN}, W_9 = \frac{1}{4} D_{CG}, \) etc. The inequalities (21) again admits 10 nonnegative differences. These differences satisfies some natural inequalities given in a pyramid below:

\[
\begin{align*}
D_{W_2W_1}^1, \\
D_{W_3W_2}^2 \leq D_{W_3W_1}^3, \\
D_{W_4W_3}^4 \leq D_{W_4W_2}^5 \leq D_{W_4W_1}^6, \\
D_{W_5W_4}^7 \leq D_{W_5W_3}^8 \leq D_{W_5W_2}^9 \leq D_{W_5W_1}^{10},
\end{align*}
\]

where \( D_{W_iW_j}^k \) := \( W_j-W_i \), \( D_{W_7W_6}^{16} := W_7-W_6 \), etc. Interestingly, the above 10 nonnegative differences are equals to each other by some multiplicative constants:

\[
\begin{align*}
\frac{7}{2} D_{W_2W_1}^1 = & \frac{21}{8} D_{W_3W_2}^2 = \frac{3}{2} D_{W_3W_1}^3 = \frac{15}{8} D_{W_4W_3}^4 = \frac{35}{32} D_{W_4W_2}^5 = \frac{35}{32} D_{W_4W_1}^6 = \\
= & \frac{5}{6} D_{W_5W_4}^7 = \frac{5}{4} D_{W_5W_3}^8 = \frac{3}{4} D_{W_5W_2}^9 = \frac{7}{12} D_{W_5W_1}^{10} = \frac{1}{2} D_{W_5W_1}^1 = \frac{1}{2} \frac{(\sqrt{a}-\sqrt{b})^4}{a+b}.
\end{align*}
\]  

\tag{22}

Based on the expressions (4), (5), (6) and (22) we have the following proportionality relations among the six means:

1. \( 4A = 2(C + H) = 3R + H \);  
2. \( 3R = C + 2A = 2C + H \);  
3. \( 3N = 2A + G \);  
4. \( 3C + 2H = 3R + 2A \);  
5. \( C + 6A = H + 6R \);  
6. \( C + 3N = G + 3R \);  
7. \( 3N + 2A = 2C + 2H + G \);  
8. \( 27R + 2G = 14A + 9C + 6N \);
\[9. \quad 3(N+3R) = 8A + 3C + G; \quad 11. \quad 4G + 14H + 17C = 9R + 14A + 12N;\]
\[10. \quad 3G + 8H + 9C = 3R + 8A + 9N; \quad 12. \quad 5G + 24H + 31C = 21R + 24A + 15N.\]

**References**

[1] P. CZINDER and Z. PALES, Local monotonicity properties of two-variable Gini means and the comparison theorem revisited, *J. Math. Anal. Appl.*, 301(2005), 427-438.

[2] H. EVES, Means Appearing in Geometrical Figures, *Mathematics Magazine*, 76(4)(2003), 292-294.

[3] C. GINI, Di una formula compressiva delle medie, *Metron*, 13(1938) 3-22.

[4] E. HELLINGER, Neue Begründung der Theorie der quadratischen Formen von unendlichen vielen Veränderlichen, *J. Reine Ang. Math.*, 136(1909), 210-271.

[5] L. LeCAM, Asymptotic Methods in Statistical Decision Theory. New York: Springer, 1986.

[6] I.J. TANEJA, On symmetric and non-symmetric divergence measures and their generalizations, Chapter in: *Advances in Imaging and Electron Physics*, Ed. P.W. Hawkes, 138(2005), 177-250.

[7] I.J. TANEJA, Refinement of Inequalities among Means, *Journal of Combinatorics, Information and Systems Sciences*, 31(2006), 357-378.

[8] I.J. TANEJA, Bounds On Triangular Discrimination, Harmonic Mean and Symmetric Chi-square Divergences, *Journal of Concrete and Applicable Mathematics*, 4(1)(2006), 91-111.