A result on the phase diagram of a Ginzburg-Landau problem

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Abstract
Working with a particular modelization of Ginzburg-Landau phenomenological theory (see [Du01], [Du99] and Section II), we first recall the form of the phase diagram of this modelization as it usually drawn in the physical literature ([T], [Ki], [SST] and [Ge]).
We then study in detail the special case, when the critical Ginzburg Landau parameter $k$ is equal to $1/\sqrt{2}$. This allows us to prove that the critical magnetic field $H_{c1}(k)$ is strictly decreasing at $k = 1/\sqrt{2}$.
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I INTRODUCTION
In 1950 V. Ginzburg and L. Landau ([GL50]) have proposed a modelization for describing the various states of a superconducting material. They introduce a functional depending on a wave function $\phi$ and a magnetic potential vector $A$, whose local minima will describe the properties of the material; in this modelization $|\phi|^2$ represents the local density of superconducting electrons.

Abrikosov ([Ab]) has introduced a particular Ginzburg-Landau modelization, which predicts the periodic structure for the zeros of $\phi$, which was subsequently observed in experiments. His model depends on two positive parameters $k$ and $H_{ext}$, called Ginzburg-Landau parameter and external magnetic field. It also assumed that:

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1. The superconductor is infinite, homogeneous and isotropic.

2. The magnetic field \( H_{\text{ext}} = (0, 0, H_{\text{ext}}) \) is constant.

3. The energy functional \( F(\phi, A) \) has a Ginzburg-Landau form and depends on the Ginzburg-Landau parameter \( k \).

4. The pairs \( (\phi, A) \) considered are gauge invariant along the z-axis and also along a lattice of \( \mathbb{R}^2 \).

5. The lattice has a fixed shape and there is one quantum flux per unit cell of it.

After some change of variable, recalled in Section II, we obtain the following formulation of the problem:

Denote \( \mathcal{L} \) a lattice of \( \mathbb{R}^2 \), with fundamental domain \( \Omega \) of area 1. Define the vector bundle \( E_1 \) over \( \mathbb{R}^2/\mathcal{L} \) as the vector bundle, whose \( C^\infty \) sections are described by

\[
C^\infty(E_1) = \left\{ u : \mathbb{R}^2 \to \mathbb{C} \text{ s.t. } \forall (x, y) \in \mathbb{R}^2, \forall v = (v_x, v_y) \in \mathcal{L}, \ u((x, y) + v) = e^{i\pi(v_x y - v_y x)}u(x, y) \right\}
\]

The vector bundle \( E_1 \) is non-trivial; this implies that any section \( u \in C^\infty(E_1) \) has at least one zero in \( \mathbb{R}^2/\mathcal{L} \).

The potential vector \( a \) belongs to the space

\[ \{ a \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2) \text{ such that } \text{div } a = 0, \text{ a is } \mathcal{L}-\text{periodic and } \int_\Omega a = 0 \} \]

We denote by \( \mathcal{A} \) the space of all pairs \( (u, a) \) with \( u \) being a \( H^1_{\text{loc}} \) section of \( E_1 \) and \( a \) belonging to the above space.

Denote \( H_{\text{int}} \) the internal magnetic field and \( E_{k, H_{\text{int}}} \) the functional defined over \( \mathcal{A} \) by

\[
E_{k, H_{\text{int}}}(u, a) = \int_\Omega \left[ \frac{\mu}{2} \| i \nabla u + (A_0 + a)u \|^2 + \frac{1}{4}(1 - |u|^2)^2 + \frac{\mu^2}{2} |\text{curl } a|^2 \right]
\]

with \( \mu = \frac{H_{\text{int}}}{2\pi k} \) and \( A_0 = \pi \left( \begin{array}{c} -y \\ x \end{array} \right) \). We then define the energy of the superconductor as

\[
E_{k, H_{\text{ext}}}(H_{\text{int}}, u, a) = E_{k, H_{\text{int}}}(u, a) + \frac{1}{2}(H_{\text{int}} - H_{\text{ext}})^2.
\]

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The term \( \frac{1}{2}(H_{\text{int}} - H_{\text{ext}})^2 \) is a simple magnetic energy, while the term \( E_{k,H_{\text{int}}} \) is the internal energy of the superconductor. The energy \( E_{k,H_{\text{ext}}} \) is then defined as the minimum of \( E_{k,H_{\text{ext}}} \) over all magnetic field \( H_{\text{int}} \) and pairs \( (u,a) \in A \). Also we denote \( m_{E_{k,H_{\text{int}}}} \) the infimum of \( E_{k,H_{\text{int}}} \) over all pairs \( (u,a) \in A \).

For \( u = 0, a = 0 \) and \( H_{\text{int}} = H_{\text{ext}} \) one obtains the energy \( E_N = \frac{1}{4} \), which is the energy of the so called normal state. In the limit case \( H_{\text{int}} = 0 \), one obtains (see [Du99] or [Du01]) the energy \( E_P = \frac{H_{\text{ext}}^2}{2} \), which is the energy of the pure state. This leads us to introduce three sets in \( \mathbb{R}^*_+ \times \mathbb{R}^*_+ \):

\[
N = \{(k,H_{\text{ext}}) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \text{ s.t. } E_{k,H_{\text{ext}}} = E_N\},
\]
\[
P = \{(k,H_{\text{ext}}) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \text{ s.t. } E_{k,H_{\text{ext}}} = E_P\},
\]
\[
M = \{(k,H_{\text{ext}}) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \text{ s.t. } E_{k,H_{\text{ext}}} < \inf(E_P,E_N)\}.
\]

The set \( M \) is the complementary of \( P \cup N \) in \( \mathbb{R}^*_+ \times \mathbb{R}^*_+ \); if \( (k,H_{\text{ext}}) \in M \), then the superconductor is said to be in a mixed state.

Using this simple modelization we were able (see [Du99] and [Du01]) to prove the following monotonicity theorem.

**Theorem 1** (i) If \( (k,H_{\text{ext}}) \in P, k' \leq k \) and \( H'_{\text{ext}} \leq H_{\text{ext}} \) then \( (k',H'_{\text{ext}}) \in P \).

(ii) If \( (k,H_{\text{ext}}) \in N, k' \geq k \) and \( H'_{\text{ext}} \geq H_{\text{ext}} \) then \( (k',k'H'_{\text{ext}}) \in N \).

The existence of such a Theorem is possible only because the system is invariant by homotheties (see, for example, [DH] for the case of a superconductor restricted to a domain \( \mathcal{D} \) of \( \mathbb{R}^2 \)).

From this theorem we derived the existence of two functions \( k \mapsto H_{c1}(k) \) and \( k \mapsto H_{c2}(k) \) such that

\[
N = \{(k,H_{\text{ext}}), \text{ s.t. } H_{\text{ext}} \geq H_{c2}(k)\},
\]
\[
P = \{(k,H_{\text{ext}}), \text{ s.t. } H_{\text{ext}} \leq H_{c1}(k)\},
\]
\[
M = \{(k,H_{\text{ext}}), \text{ s.t. } H_{c1}(k) < H_{\text{ext}} < H_{c2}(k)\}.
\]

Using this modelization we obtained in [Du01] the qualitative form of the phase diagram depicted in Figure 1, which is recalled in Section IV.

This phase diagram is made of three curves:

(i) (boundary normal-pure) \( H_{\text{ext}} = H_{c1}(k) = H_{c2}(k) = \frac{1}{\sqrt{2}} \) with \( k \leq \frac{1}{\sqrt{2}} \),

(ii) (boundary normal-mixed) \( H_{\text{ext}} = H_{c2}(k) = k \) with \( k \geq \frac{1}{\sqrt{2}} \),

(iii) (boundary pure-mixed) \( H_{\text{ext}} = H_{c1}(k) \) with \( k \geq \frac{1}{\sqrt{2}} \).
The exact expression of curve (iii) is unknown. Those three curves meet at the triple point \( k = H_{\text{ext}} = \frac{1}{\sqrt{2}} \). A key point of the proof is that the case \( k = \frac{1}{\sqrt{2}} \) is exactly solvable thanks to the Bochner-Kodaira-Nakano formula explained in Section III. Using a more advanced analysis of the case \( k = \frac{1}{\sqrt{2}} \) in Section VI we prove in Section VII the following Theorem:

**Theorem 2** (i) There exist \( \delta > 0 \) and \( S > 0 \) such that for all \( h \) in \([0, \delta]\), we have

\[
-h \leq H_{c1}(\frac{1}{\sqrt{2}} + h) - \frac{1}{\sqrt{2}} \leq -Sh .
\]

(ii) The critical magnetic field \( H_{c1}(k) \) is strictly decreasing at \( k = \frac{1}{\sqrt{2}} \).

**II THE CHANGE OF VARIABLE**

In this Section, we recall the original formulation of the problem by V.Ginzburg and L.Landau in [GL50] and how it is related to our formulation. They proposed the following expression for the density of energy in superconductors

\[
\frac{1}{2}||ik^{-1}\nabla \phi + A \phi||^2 + \frac{1}{4}(1 - |\phi|^2)^2 + \frac{1}{2}(\text{curl } A - H_{\text{ext}})^2
\]

This expression belongs to \( L_{\text{loc}}^1(\mathbb{R}^3) \) if \((\phi, A)\) is in the Sobolev space \( H_{\text{loc}}^1(\mathbb{R}^3; \mathbb{C}) \times H_{\text{loc}}^1(\mathbb{R}^3; \mathbb{R}^3) \). It is invariant under the gauge transform

\[4]
We assumed that the problem is invariant under translation along the z-axis. This means that we consider pairs \((\phi, A)\), which satisfies: for every \(h \in \mathbb{R}\), the pair \((\phi, A)(x_1, x_2, x_3 + h)\) is gauge equivalent to the pair \((\phi, A)\).

In fact, as proved in [Du99] p. 17, we can assume that the pairs \((\phi, A)\) considered are independent of \(x_3\) and satisfy \(A_{x_3} = 0\). So, we can reduce ourself to a 2-dimensional problem.

We take \(\mathcal{L}\) a 2-dimensional lattice of \(\mathbb{R}^2\) with fundamental domain \(\Omega\) of area 1. We consider the dilated lattice : \(\mathcal{L}_{\lambda} = \sqrt{\lambda} \mathcal{L}\) with fundamental domain \(\Omega_{\lambda} = \sqrt{\lambda} \Omega\). Following Abrikosov, we choose \(\lambda\) in \(\mathbb{R}_+\) and restrict the analysis to pairs \((\phi, A)\), which are gauge periodic with respect to \(\mathcal{L}_{\lambda}\) \([AB]\).

This means that, for all \(v \in \mathcal{L}_{\lambda}\), there exists \(g_v \in H^2_{\text{loc}}(\mathbb{R}^2)\) such that
\[
\phi(z + v) = e^{ikg_v(z)}\phi(z) \quad \text{and} \quad A(z + v) = A(z) + \nabla g_v(z).
\]

Consequently, all the considered physical quantities are \(\mathcal{L}_{\lambda}\)-periodic. We denote by \(|\Omega_{\lambda}|\) the area of \(\Omega_{\lambda}\), which is actually equal to \(\lambda\).

A classic consequence (see [Du99], [BGT]) of gauge periodicity is that there exist \(d \in \mathbb{Z}\) satisfying to
\[
2\pi d = k \int_{\Omega_{\lambda}} \text{curl} \ A.
\]

We will then, according to Abrikosov, fix the quantization \(d\) per unit cell equal to 1.

The Ginzburg-Landau functional is obtained by integration of the local density over the fundamental domain \(\Omega_{\lambda}\) and division by \(|\Omega_{\lambda}|\). This gives:
\[
F(\phi, A) = \frac{1}{|\Omega_{\lambda}|} \int_{\Omega_{\lambda}} \frac{1}{2} \|ik^{-1} \nabla \phi + A \phi\|^2 + \frac{1}{4} (1 - |\phi|^2)^2 + \frac{1}{2} (\text{curl} \ A - H_{\text{ext}})^2,
\]
which should be understood as a mean energy.

We denote by \(H_{\text{int}} = \frac{1}{|\Omega_{\lambda}|} \int_{\Omega_{\lambda}} \text{curl} \ A\) the mean internal magnetic field induced by \(A\). The quantization relation is then rewritten as \(2\pi = k\lambda H_{\text{int}}\).

It is also a classical result (see [BGT], [YS] or [Du99], p. 21-29) that we can associate to the pair \((\phi, A)\), another pair \((\phi', A')\), with the same Ginzburg-Landau energy but satisfying to
(i) \( A' = \frac{H_{\text{int}}}{2\pi} A_0 + P \) with \( P \ \mathcal{L}_\lambda \)-periodic, \( \text{div } P = 0, \int_{\Omega_\lambda} P = 0 \),

(ii) \( \varphi'(z + v) = e^{ikg^v(z)} \varphi'(z) \) with \( g^v(x,y) = \frac{H_{\text{int}}}{2}(v_x y - v_y x) \) for all \( v \in \mathcal{L}_\lambda \).

This reduction is rather involved and is performed by a suitable gauge transform and a translation in \( x, y \). The relation relating \( \varphi'(z + v) \) to \( \varphi'(z) \) actually defines the sections of a complex line bundle over the torus \( \mathbb{R}^2 / \mathcal{L} \); above result is so, a classification result.

With this expression one gets

\[
\frac{1}{|\Omega_\lambda|} \int_{\Omega_\lambda} \frac{1}{2} (\text{curl } A - H_{\text{ext}})^2 = \frac{1}{|\Omega_\lambda|} \int_{\Omega_\lambda} \frac{1}{2} (\text{curl } P)^2 + \frac{1}{2} (H_{\text{int}} - H_{\text{ext}})^2.
\]

This leads to the simple expression

\[
F(\phi, A) = F^{\text{int}}(\phi, P) + \frac{1}{2} (H_{\text{int}} - H_{\text{ext}})^2
\]

with

\[
F^{\text{int}}(\phi, P) = \frac{1}{\lambda} \int_{\Omega_\lambda} \frac{1}{2} ||k^{-1}\nabla \phi + A \phi||^2 + \frac{1}{2} (1 - |\phi|^2)^2 + \frac{1}{2} (\text{curl } P)^2.
\]

The functional \( F^{\text{int}} \) is called internal energy and depends only on \( H_{\text{int}}, k, \phi \) and \( P \).

The quantities \( H_{\text{int}}, k \) and \( \lambda \) are related by the quantization relation \( 2\pi = k\lambda H_{\text{int}} \), which makes the analysis of \( F^{\text{int}} \) cumbersome. So, we reduce the complexity of the computation by the following change of variable and of functions:

\[
\begin{cases}
  u(x) = \phi(x \sqrt{\frac{2\pi}{k H_{\text{int}}}}), \\
  a(x) = \sqrt{\frac{2\pi k}{H_{\text{int}}}} [A - \frac{H_{\text{int}}}{2\pi} A_0](x \sqrt{\frac{2\pi}{k H_{\text{int}}}}) = \sqrt{\frac{2\pi k}{H_{\text{int}}}} A(x \sqrt{\frac{2\pi}{k H_{\text{int}}}}) - A_0(x).
\end{cases}
\]

We then obtain the formulation given in the introduction since the pair \((u, a)\) so defined belongs to \( \mathcal{A} \) and verifies \( E_{k, H_{\text{int}}}(u, a) = F^{\text{int}}(\phi, P) \).

### III THE FUNCTIONAL \( E_{k, H_{\text{int}}} \)

Let us now analyze the functional \( E_{k, H_{\text{int}}} \) by assuming here that \( k \) and \( H_{\text{int}} \) are fixed.

\( E_{k, H_{\text{int}}} \) is defined over \( \mathcal{A} \) since \((u, a)\) of class \( H^1 \) guarantees local integrability of the density, while the compactness of the torus \( \mathbb{R}^2 / \mathcal{L} \) guarantees its integrability.
In fact, the variational theory of the functional $E_{k,H_{\text{int}}}$ is easy (see [Du99]) since the torus $\mathbb{R}^2/L$ is compact and the non-linear partial differential equations obtained for the critical points are elliptic; the vector bundle adds only technical difficulties (see [LM]). More precisely one can prove successively that:

1. **Coerciveness**: for every $C \in \mathbb{R}$ there is a $C' > 0$ such that $E_{k,H_{\text{int}}}(u,a) < C$ implies $\|u\|_{H^1} + \|a\|_{H^1} \leq C'$.

2. **Lower semicontinuity**: If $(u_n,a_n) \in A$ converges weakly to $(u,a) \in A$, then $E_{k,H_{\text{int}}}(u,a) \leq \liminf_n E_{k,H_{\text{int}}}(u_n,a_n)$.

3. **Minimum**: The functional $E_{k,H_{\text{int}}}$ attains its minimum on at least one pair $(u,a) \in A$.

4. **Ginzburg-Landau equations**: The minimizing pairs satisfy to the following equation

$$\begin{cases}
\mu i \nabla + A_0 + a|^2u &= (1 - |u|^2)u \\
\Delta a &= \frac{1}{k^2} \text{Re}[\overline{u}(i \nabla u + (A_0 + a)u)]
\end{cases}$$

5. **Regularity**: The pairs $(u,a) \in A$ verifying the Ginzburg-Landau equations are in fact of class $C^\infty$.

6. **Maximum principle**: The pairs $(u,a) \in A$ verifying the Ginzburg-Landau equations satisfy $|u| \leq 1$.

We now explain the Bochner-Kodaira-Nakano formula for the functional $E_{k,H_{\text{int}}}$ (see [Dj], [JaTa] and [WY] for related formulas and results). This classical formula is also called Bogmol’nyi formula, Weitzenbock formula, Lichnerowicz formula (see [JT]) according to different scientific schools.

We set $C = A_0 + a$; we get $\text{curl } C = 2\pi + \text{curl } a$ and define

$$A_{+,H_{\text{int}}}(u,a) = \int_{\Omega} \frac{\mu}{2} |D_+ u|^2 + \frac{1}{4} \mu \text{ curl } C - (1 - |u|^2)^2,$$

where $\mu = \frac{H_{\text{int}}}{2\pi k}$ and $D_+ = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} + C_y - i C_x$.

**Theorem 3** (Bochner-Kodaira-Nakano)

For all $(u,a) \in A$, we have:

$$E_{1/\sqrt{2},H_{\text{int}}}(u,a) = \mu \pi - (\mu \pi)^2 + A_{+,H_{\text{int}}}(u,a).$$
Proof. We perform computations with smooth functions and then extend by density. After expansion, simplification and regrouping one obtains

\[ \{ A_+, H_{int} - \frac{E}{\sqrt{2}}, H_{int} \}(u, a) = \frac{1}{2} \int_{\Omega} \text{div} W - \mu \text{curl} C + \frac{\mu^2}{4} \int_{\Omega} |\text{curl} C|^2 - |\text{curl} a|^2 \]

with \( W = \left( \frac{i}{\sqrt{2}} \frac{\partial u}{\partial y} + C_y u \right) \). The vector field \( W \) being \( L \)-periodic, the integral of its divergence over \( \Omega \) is 0. The formula is then obtained by replacing \( \text{curl} C \) by \( 2\pi \text{curl} a \) and using \( \int_{\Omega} \text{curl} a = 0 \). \( \square \)

The magnetic Schrodinger operator is defined as \( H = [i\nabla + A_0]^2 \); its spectrum, called Landau levels, is recalled in next theorem.

**Theorem 4** (i) The operator \( H \) admits a self-adjoint extension over \( L^2(E_1) \), also denoted by \( H \), whose domain is \( H^2(E_1) \).

(ii) It can be expressed as \( H = L^*_+ L_+ + 2\pi \) with \( [L_+, L^*_+] = 4\pi \) and \( L_+ = 2\partial_z + \pi \).

(iii) Its spectrum is discrete, \( \text{sp}(H) = 2\pi + 4\pi \mathbb{N} \), and every eigenvalue is simple.

(iv) The eigenvector \( u_0 \) associated to \( \lambda = 2\pi \) satisfies \( L_+(u_0) = 0 \) and has a unique simple zero in \( \Omega \) denoted by \( z_0 \).

Proof. (i) and the discreteness of the spectrum follow from the fact that \( H \) is an elliptic pseudo-differential operator of order 2 defined over the vector bundle of a compact manifold (see [LM]).

Formula \( H = L^*_+ L_+ + 2\pi \) and \( [L_+, L^*_+] = 4\pi \) are proved by first computing with smooth functions and then extending by density.

If we proved that the equation \( L_+(u) = 0 \) has a unique solution \( u_0 \) up to scalar, then by the harmonic oscillator formalism we would get (iii).

In fact, if one writes, \( u_0(z) = e^{-|z|^2/2} s(z) \), then \( s(z) \) is analytic. Furthermore, without loss of generality, we can assume that \( L \) is generated by the vectors \( v_1 = (u,0) \) and \( v_2 = (w,r) \) with \( ru = 1 \). Then, after using gauge periodicity conditions, one finds the following expression for \( u_0 \):

\[ u_0(x, y) = e^{i\pi xy} \sum_{n \in \mathbb{Z}} e^{-\pi(y+nu)^2} e^{\pi n^2 iwu + 2\pi nuix} . \]

This expression is a theta function; it is known that such function have a unique simple zero in \( \Omega \) (see [Cha]). Another method of proof is the use of Rouché Theorem as done in [Du99]. \( \square \)
**Theorem 5** If \( k \geq \frac{1}{\sqrt{2}} \) and \( H_{\text{int}} \geq k \), then \( m_E(k, H_{\text{int}}) = \frac{1}{4} \). Furthermore, the minimum is met only by the pair \((0,0)\).

**Proof.** We use following expansion of the functional \( E_{k,H_{\text{int}}} \):

\[
E_{k,H_{\text{int}}}(u, a) \geq E_{\frac{1}{\sqrt{2}}, H_{\text{int}}}(u, a) \\
\geq (\mu \pi) - (\mu \pi)^2 + \int_{\Omega} \frac{1}{2} |D+.u|^2 + \frac{1}{4} |2\mu \pi - 1 + \mu \text{curl } a + |u|^2|^2 \\
\geq (\mu \pi) - (\mu \pi)^2 + \frac{(2\mu \pi - 1)^2}{4} + \frac{1}{4} \int_{\Omega} 2(2\mu \pi - 1)(\mu \text{curl } a + |u|^2) \\
+ \frac{1}{4} \int_{\Omega} |\mu \text{curl } a + |u|^2|^2 \\
\geq \frac{1}{4} + \frac{2\mu \pi - 1}{2} \int_{\Omega} |u|^2.
\]

Then using the hypothesis \( 2\mu \pi - 1 = \frac{H_{\text{int}}}{k} - 1 \geq 0 \), we get \( m_E(k, H_{\text{int}}) \geq \frac{1}{4} \) by positivity of terms of above equation.

Now assume that \( E_{k,H_{\text{int}}}(u, a) = \frac{1}{4} \); in fact, last computation give us the following equalities:

\[
\begin{align*}
0 &= \int_{\Omega} |u|^2, \\
0 &= \int_{\Omega} |\mu \text{curl } a + |u|^2|^2, \\
0 &= (k^2 - \frac{1}{2}) \int_{\Omega} |\text{curl } a|^2, \\
0 &= \int_{\Omega} |D+.u|^2.
\end{align*}
\]

The second equality give us \( \text{curl } a + |u|^2 = 0 \), which integrated over \( \Omega \) yields

\[
\int_{\Omega} |u|^2 = - \int_{\Omega} \text{curl } a = 0
\]

and then \( u = 0 \).

Now, using the equation \( \text{div } a = 0 \), one obtains the equality \( \text{curl}^* \text{curl } a = \Delta a = 0 \). The potential vector \( a \) is \( L \) periodic; so, it has to be constant. Now, the property \( \int_{\Omega} a = 0 \) yields \( a = 0 \).

\( \square \)

**IV THE PHASE DIAGRAM**

Let us first consider the special case when \( k = H_{\text{ext}} = \frac{1}{\sqrt{2}} \). We have the following Lemma:

**Lemma 6** One has

(i) \( E_{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}} (H_{\text{int}}, u, a) = \frac{1}{4} + A_{+, H_{\text{int}}}(u, a) \),

(ii) \( E_{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}} = \frac{1}{4} \),

(iii) \( (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \in \mathcal{P} \cap \mathcal{N} \).
Proof. (i) is in fact a rewriting of the Bochner-Kodaira-Nakano formula; it yields (ii) by positivity of \( A_{+} + H_{\text{int}} \), while (iii) is obtained by remarking that 
\[ E_{N} = \frac{1}{4} = \frac{1}{2} \left( \frac{1}{\sqrt{2}} \right)^2 = E_{P}. \]

\[ \square \]

Theorem 7 (Type I superconductors) If \( k \leq \frac{1}{\sqrt{2}} \), then:

(i) If \( H_{\text{ext}} \leq \frac{1}{\sqrt{2}} \), then \( \mathcal{E}_{k,H_{\text{ext}}} = E_{P} \) and \( (k,H_{\text{ext}}) \in \mathcal{P} \);

(ii) If \( H_{\text{ext}} \geq \frac{1}{\sqrt{2}} \), then \( \mathcal{E}_{k,H_{\text{ext}}} = E_{N} \) and \( (k,H_{\text{ext}}) \in \mathcal{N} \).

Proof. Lemma \( \text{6} \) combined with Theorem \( \text{1.(i)} \) give the result in the case \( H_{\text{ext}} \leq \frac{1}{\sqrt{2}} \).

In particular, if \( k \leq \frac{1}{\sqrt{2}} \) we have \( (k,\frac{1}{\sqrt{2}}) \in \mathcal{P} \) and so, \( \mathcal{E}_{k,\frac{1}{\sqrt{2}}} = E_{P} = \frac{1}{4} (\frac{1}{\sqrt{2}})^2 = \frac{1}{4} = E_{N} \); therefore Theorem \( \text{1.(ii)} \) gives the conclusion in case \( H_{\text{ext}} \geq \frac{1}{\sqrt{2}} \). \[ \square \]

Theorem 8 (Type II superconductors) If \( H_{\text{ext}} \geq k \geq \frac{1}{\sqrt{2}} \), then:

(i) If \( H_{\text{ext}} \geq k \), then \( \mathcal{E}_{k,H_{\text{ext}}} = E_{N} \) and \( (k,H_{\text{ext}}) \in \mathcal{N} \);

(ii) If \( H_{\text{ext}} < k \), then \( (k,H_{\text{ext}}) \notin \mathcal{N} \).

Proof. Lemma \( \text{6} \) combined with Theorem \( \text{1.(ii)} \) gives (i).

By setting \( H_{\text{int}} = H_{\text{ext}}, u = \alpha u_{0}, a = 0 \) and doing a development of order 2 around the pair \((0,0)\), one obtains

\[ E_{k,H_{\text{ext}}}(H_{\text{ext}},\alpha u_{0},0) = E_{k,H_{\text{ext}}}(\alpha u_{0},0) = \frac{1}{4} + \frac{1}{2} \left( \frac{H_{\text{ext}}}{k} - 1 \right) \alpha^2 + o(\alpha^2). \]

Since \( k > H_{\text{ext}} = H_{\text{int}} \), one obtains for \( \alpha \) small \( E_{k,H_{\text{ext}}}(H_{\text{ext}},\alpha u_{0},0) < \frac{1}{4} \); so, the energy will be lower than \( \frac{1}{4} \), i.e. \((k,H_{\text{ext}}) \notin \mathcal{N} \). \[ \square \]

V ANALYSIS OF THE CASE \( k = \frac{1}{\sqrt{2}} \)

In this section we will find all pairs \((u,a)\) verifying \( A_{+,H_{\text{int}}}(u,a) \), thus get the value of \( m_{E}(\frac{1}{\sqrt{2}},H_{\text{int}}) \). A similar study is done in [Al2] for a rectangular problem. In book [JaTa], the case considered is of \( u \) defined over \( \mathbb{R}^2 \), while in paper ([Ga]) the problem is considered over a Riemann surface. Also, in
it is proved that all critical points of the Ginzburg-Landau functional are solution of the Bogmol’nyi equations, but their proof does not apply to our case.

The papers ([KW]), ([CY]), ([WY]) are devoted to existence theorem concerning the Kazdan-Warner equation. They get as a byproduct existence Theorems for the self-dual equations.

**Theorem 9** (Kazdan-Warner, see [KW]) If \( h \) is a positive function, \( h \neq 0 \), and \( C^\infty(\mathbb{R}^2/L) \). If \( A > 0 \) then the equation
\[
-\Delta f + e^f h = A
\]
has a unique solution \( f \) in \( C^\infty(\mathbb{R}^2/L) \).

We define
\[
\begin{align*}
  u_{H_{\text{int}}} &= u_0 e^{f_{H_{\text{int}}}} \\
  a_{H_{\text{int}}} &= \left( \frac{\partial f_{H_{\text{int}}}}{\partial y}, -\frac{\partial f_{H_{\text{int}}}}{\partial x} \right),
\end{align*}
\]
with \( f_{H_{\text{int}}} \) being the unique solution of \( 1 - 2\mu \pi = |u_0|^2 e^{2f} - \mu \Delta f \) and \( \mu = \frac{H_{\text{int}}}{\pi \sqrt{2}} \).

Let us introduce first the following family of sections of \( E_1 \):
\[
u_h(x, y) = e^{i\pi(h_y x - h_x y)} u_0(z - h).
\]
Recall that \( z_0 \) is the zero of \( u_0 \) in \( \mathbb{R}^2/L \); the section \( u_h \) verifies the following easy properties
\[
\begin{align*}
  u_h &\in C^\infty(E_1), \\
  L_+(u_h) &= 2\pi h u_h, \\
  u_h(z) &= 0 \text{ if and only if } z \in z_0 + h + L.
\end{align*}
\]
Furthermore, for any \( h \in \mathbb{R}^2, v \in L \), there exists \( \alpha \in \mathbb{R} \) such that
\[
u_{h+v}(z) = e^{i\alpha} e^{2i\pi(v_y x - v_x y)} u_h(z).
\]

**Theorem 10** We assume \( H_{\text{int}} \leq \frac{1}{\sqrt{2}} \).

(i) If \( (u, a) \in \mathcal{A} \) satisfies \( A_{+,H_{\text{int}}}(u, a) = 0 \), then there exist \( c \in \mathbb{R} \) such that \( (u, a) = (e^{ic} u_{H_{\text{int}}}, a_{H_{\text{int}}}) \).

(ii) The pair \( (u_{H_{\text{int}}}, a_{H_{\text{int}}}) \) satisfies to
\[
\begin{align*}
  \int_\Omega (1 - |u_{H_{\text{int}}}|^2)^2 &= \mu^2 [(2\pi)^2 + \int_\Omega |\text{curl } a_{H_{\text{int}}}|^2] \\
  \int_\Omega \frac{\mu^2}{2} \| i \nabla u_{H_{\text{int}}} + (A_0 + a) u_{H_{\text{int}}})\|^2 + \frac{\mu^2}{2} |\text{curl } a_{H_{\text{int}}}|^2 &= (\mu \pi) - 2(\mu \pi)^2.
\end{align*}
\]
Proof. Let \((u, a) \in \mathcal{A}\) be a pair satisfying \(A_+ H_{\text{int}}(u, a) = 0\), it then verifies the following Bogmol’nyi equations

\[
D_+ u = L_+ u + (a_y - ia_x)u = 0 \quad \text{and} \quad 2\mu \pi + \mu \text{curl } a = 1 - |u|^2
\]

and, by Theorem 3, minimizes the functional \(E_{\frac{1}{\sqrt{2}} H_{\text{int}}}\). Therefore, by Section 3II it satisfies the Ginzburg-Landau equations and so, it is \(C^\infty\).

Since the vector bundle \(E_1\) is non trivial the section \(u\) possess at least one zero in \(\mathbb{R}^2/\mathcal{L}\), which we write as \(z_h = z_0 + h\).

The zero-set of the function \(u\) defined on \(\mathbb{R}^2\) contains \(z_h + \mathcal{L}\), while the zero-set of \(u_h\) is exactly \(z_h + \mathcal{L}\); so, one defines on \(\mathbb{R}^2 - (z_h + \mathcal{L})\) the function

\[
f = \frac{u}{u_h}.
\]

Since both \(u\) and \(u_h\) are section of the vector bundle \(E_1\), the function \(f\) is \(\mathcal{L}\)-periodic. The equation \(D_+ u = 0\) is rewritten on \(\mathbb{R}^2 - (z_h + \mathcal{L})\) as:

\[
0 = 2(\partial_z f)u_h + f D_+ u_h = 2(\partial_z f)u_h + [2\pi h f + (a_y - ia_x)f]u_h,
\]

Since \(u_h\) is not zero on \(\mathbb{R}^2 - (z_h + \mathcal{L})\) we obtain:

\[
\partial_z f = fw \quad \text{with} \quad w = \frac{1}{2}((-a_y - 2\pi h_x) + i(a_x - 2\pi h_y)).
\]

Note that the function \(w\) is defined on \(\mathbb{R}^2\), also it is \(C^\infty\) and \(\mathcal{L}\)-periodic.

We now want to extend \(f\) to \(\mathbb{R}^2\): it is a classic result of complex analysis that the equation \(\partial_z k = w\) has a \(C^\infty\) solution \(k\) on \(\mathbb{R}^2\).

The function \(g = fe^{-k}\) is defined on \(\mathbb{R}^2 - (z_h + \mathcal{L})\), satisfies \(\partial_z g = 0\) and is so, analytic. If \(m \in z_h + \mathcal{L}\) then \(u = O(z - m)\), since \(u\) is \(C^\infty\). The complex \(m\) is a simple zero of \(u_h\), consequently \(u_h^{-1} = O(|z - m|^{-1})\) and \(f = O(1)\) at \(m\).

The function \(g\) stay bounded around \(m\) and is analytic outside \(m\). By a classic result of complex analysis, we get that \(g\) can be extended to \(m\) in a complex analytic function. The function \(g\) is extended to \(\mathbb{C}\) and so, \(f\) too.

The function \(g\) is analytic and so, its zero set is discrete. There exist a translate \(\Omega'\) of \(\Omega\) such that the boundary \(\partial \Omega'\) of \(\Omega'\) does not meet any zero of \(g\).

By Rouch theorem the number \(n\) of zero of \(g\) in \(\Omega'\) is equal to:

\[
n = \frac{1}{2\pi i} \int_{\partial \Omega'} \frac{\partial_z g}{g} dz = \frac{1}{2\pi i} \int_{\partial \Omega'} \frac{\partial_z f}{f} dz = \frac{1}{2\pi i} \int_{\partial \Omega'} \frac{\partial_z k}{k} dz = \frac{1}{2\pi i} \int_{\partial \Omega'} \partial_z k dz.
\]
The integral of $\frac{\partial f}{\partial z}$ over $\partial \Omega'$ is zero, since $f$ is $\mathcal{L}$-periodic.

Now using Stokes theorem, we get:

$$n = \frac{-1}{2\pi i} \int_{\partial \Omega'} \partial_z k dz = \frac{-1}{2\pi i} \int_{\partial \Omega'} d(\partial_z k dz) = \frac{-1}{2\pi i} \int_{\partial \Omega'} \partial_z \partial_z k dz \wedge dz = \frac{-1}{2\pi i} \int_{\partial \Omega'} \partial_z k dz \wedge dz.$$

The function $w$ is $\mathcal{L}$-periodic; consequently the function $\partial_z w$ is a $\mathcal{L}$-periodic function, which has integral zero over $\Omega'$; so, $n = 0$.

Since $f = ge^k$, the function $f$ has no zero over $\mathbb{R}^2$. Since $\mathbb{R}^2$ is simply connected there exist a complex valued $C^\infty$ function $\psi$ such that $f = e^\psi$.

The function $\psi$ is not $\mathcal{L}$-periodic, but since the function $f$ is $\mathcal{L}$-periodic and $C^\infty$ there exist two integer $n_1, n_2$ such that

$$\psi(z + v_1) = \psi(z) + 2\pi i n_1 \quad \text{and} \quad \psi(z + v_2) = \psi(z) + 2\pi i n_2.$$

We pose $v' = n_1 v_2 - n_2 v_1$, the function $\psi_2(z) = \psi(z) - 2\pi i \det(z, v')$ is $\mathcal{L}$-periodic, and we have:

$$u(z) = f(z)u_h(z) = e^{\psi_2(z) + 2\pi i [xv'_y - yv'_x]}u_h(z) = e^{\psi_2(z) - i \alpha}u_{h + v'}(z)$$

with $\alpha \in \mathbb{R}$. We set $h_3 = h + v'$, $\psi_3 = \psi_2 - i \alpha$, and we rewrite $u$ as:

$$u(z) = e^{\psi_3(z)}u_{h_3}(z)$$

with $\psi_3$ a $\mathcal{L}$-periodic $C^\infty$ function. The Bogmol’nyi equations are rewritten as

$$\begin{cases} \frac{\partial \psi_3}{\partial x} = \frac{1}{2}((-a_y - \pi h_{3,y}) + i(a_x - \pi h_{3,x})), \\ 0 = 2\mu \pi - 1 + |u_{h_3}|^2 e^{2 \text{Re} \psi_3} + \mu \text{curl} \ a. \end{cases}$$

The real and imaginary part of first equation give us the expression of the potential vector:

$$\begin{cases} a_x = \pi h_{3,y} + \frac{\partial \text{Re} \psi_3}{\partial y} + \frac{\partial \text{Im} \psi_3}{\partial x}, \\ a_y = -\pi h_{3,x} - \frac{\partial \text{Re} \psi_3}{\partial x} + \frac{\partial \text{Im} \psi_3}{\partial y}. \end{cases}$$

The equation $\text{div} \ a = 0$ is then rewritten as $\Delta \text{Im} \ \psi_3 = 0$. Thus $\text{Im} \ \psi_3$ is constant, since it is $\mathcal{L}$-periodic. We now write $\psi_3 = f + ic$ with $f$ a real $C^\infty$, $\mathcal{L}$-periodic function; so, one has

$$a_x = \pi h_{3,y} + \frac{\partial f}{\partial y} \quad \text{and} \quad a_y = -\pi h_{3,x} - \frac{\partial f}{\partial x}.$$
The functions $a$, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ have zero integral over $\Omega$. So, we have $h_3 = 0$ and the zero of $u$ in $\Omega$ is $z_0$.

One then obtain $\text{curl } a = -\Delta f$ and the following equation for $f$:

$$0 = 2\mu \pi - 1 + |u_0|^2 e^{2f} - \mu \Delta f.$$ 

So, one gets $f = f_{\text{int}}$; now above equation rewrites as

$$-\mu \text{curl } a_{\text{int}} = 2\mu \pi - 1 + |u_{\text{int}}|^2 e^{2f} - \mu \Delta f.$$ 

It yields $\int_{\Omega} |u_{\text{int}}|^2 = 1 - 2\mu \pi$ and $\int_{\Omega} (1 - |u_{\text{int}}|^2)^2 = \mu^2 [(2\pi)^2 + \int_{\Omega} |\text{curl } a_{\text{int}}|^2]$, the second equation of (ii) is then obtained by Theorem 3. □

**Corollary 11.** For every positive $H_{\text{int}}$ one has:

$$m_E(\frac{1}{\sqrt{2}}, H_{\text{int}}) = \begin{cases} H_{\text{int}} - (\frac{H_{\text{int}}}{\sqrt{2}})^2 & \text{if } H_{\text{int}} \leq \frac{1}{\sqrt{2}} \\ \frac{1}{4} & \text{if } H_{\text{int}} \geq \frac{1}{\sqrt{2}} \end{cases}.$$ 

**Proof.** By Theorem 3 one has the inequality $m_E(\frac{1}{\sqrt{2}}, H_{\text{int}}) \geq H_{\text{int}} - (\frac{H_{\text{int}}}{\sqrt{2}})^2$, since $A_{+,H_{\text{int}}} \geq 0$. This lower bound is attained by the pair $(u_{H_{\text{int}}}, a_{H_{\text{int}}})$.

Theorem 3 give the result if $H_{\text{int}} \geq \frac{1}{\sqrt{2}}$. □

**Remark 12** It can be shown that the pair $(u_{H_{\text{int}}}, a_{H_{\text{int}}})$ depends continuously on $H_{\text{int}}$ and vanish for $H_{\text{int}} = \frac{1}{\sqrt{2}}$, i.e. it is a bifurcated state (see [Du99]).

**VI LOCAL STUDY**

We define

$$H_k(u, a) = \frac{1}{4\pi k} \int_{\Omega} [i \nabla u + (A_0 + a) u]^2 + \sqrt{\frac{1}{2} + \frac{1}{2(2\pi)^2} \int_{\Omega} |\text{curl } a|^2} \left[ \int_{\Omega} (1 - |u|^2)^2 \right].$$

**Theorem 13** If $k \geq \frac{1}{\sqrt{2}}$ then $H_{c1}(k) = \inf_{(u,a) \in \mathcal{A}} H_k(u, a)$. If this infimum is attained on a pair, say, $(u', a') \in \mathcal{A}$, then one has

$$E_{k,H_{c1}(k)}(H_{\text{int}}, u', a') = \frac{H_{c1}^2(k)}{2} \text{ with } H_{\text{int}} = \frac{1}{2} \sqrt{\frac{\int_{\Omega} (1 - |u'|^2)^2}{\frac{1}{2} + \frac{1}{2(2\pi)^2} \int_{\Omega} |\text{curl } a'|^2}}.$$
Proof. By Section II, we have \((k, H_{\text{ext}}) \in \mathcal{P}\) equivalent to:

\[
E_{k,H_{\text{int}}}(u,a) + \frac{1}{2}(H_{\text{int}} - H_{\text{ext}})^2 \geq \frac{H_{\text{ext}}^2}{2},
\]

which after simplification is equivalent to

\[
\left\{ \begin{array}{l}
H_{\text{int}} \left[ \frac{1}{2} + \frac{1}{2(2\pi)^2} \int_{\Omega} |\text{curl}\ a|^2 \right] + \frac{4H_{\text{int}}}{4\pi k} \int_{\Omega} (1 - |u|^2)^2 \\
+ \frac{1}{4\pi k} \int_{\Omega} \|\nabla u + (A_0 + a)u\|^2 \geq H_{\text{ext}}.
\end{array} \right.
\]

The minimum over \(H_{\text{int}} > 0\) of the above expression is attained for \(H_{\text{int}} = \frac{1}{2} \sqrt{\frac{\int_{\Omega} (1 - |u|^2)^2}{\frac{1}{2} + \frac{1}{2(2\pi)^2} \int_{\Omega} |\text{curl}\ a|^2}}\) which yields the Theorem. \(\square\)

The above expression of \(H_{c_1}(k)\) allow us to obtain \(H_{c_1}(k) = O\left(\frac{\ln k}{k}\right)\) (see [Du99]). From Theorem 7, one has \(H_{c_1}(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}\).

**Theorem 14** The set of pairs \((u,a) \in \mathcal{A}\) verifying \(H_{\frac{1}{\sqrt{2}}}(u,a) = \frac{1}{\sqrt{2}}\) is

\[(e^{ic}u_{H_{\text{int}}}, a_{H_{\text{int}}})\]

with \(c \in \mathbb{R}\) and \(0 < H_{\text{int}} \leq \frac{1}{\sqrt{2}}\).

**Proof.** If \((u,a) \in \mathcal{A}\) satisfies \(H_{\frac{1}{\sqrt{2}}}(u,a) = \frac{1}{\sqrt{2}}\), then one has

\[
E_{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}}(H_{\text{int}}, u,a) = \frac{1}{4} \quad \text{and} \quad H_{\text{int}} = \frac{1}{2} \sqrt{\frac{\int_{\Omega} (1 - |u|^2)^2}{\frac{1}{2} + \frac{1}{2(2\pi)^2} \int_{\Omega} |\text{curl}\ a|^2}}.
\]

By Lemma 19(i), first equation simplifies to \(A_{+,H_{\text{int}}}(u,a) = 0\), and then using Theorem 10 to \((u,a) = (e^{ic}u_{H_{\text{int}}}, a_{H_{\text{int}}})\).

When the expression of \((u,a)\) is substituted into the second equation, one obtains

\[
4H_{\text{int}}^2 = \frac{\int_{\Omega} (1 - |u_{H_{\text{int}}}|^2)^2}{\frac{1}{2} + \frac{1}{2(2\pi)^2} \int_{\Omega} |\text{curl}\ a_{H_{\text{int}}}|^2}.
\]

By Theorem 10(ii), this relation is always satisfied. \(\square\)

**Theorem 15** (i) There exist \(\delta > 0\) and \(S > 0\) such that for all \(h\) in \([0, \delta]\), we have

\[-h \leq H_{c_1}\left(\frac{1}{\sqrt{2}} + h\right) - \frac{1}{\sqrt{2}} \leq -Sh.\]

(ii) The critical magnetic field \(H_{c_1}(k)\) is strictly decreasing at \(k = \frac{1}{\sqrt{2}}\).
Proof. The expression of $H_{c_1}(k)$ obtained in Theorem 13 give us that the function $k \mapsto kH_{c_1}(k)$ is increasing; this yields the lower bound.

Now we will prove the upper bound by using the $(u_{H_{\text{int}}}, a_{H_{\text{int}}})$ as quasi-modes. If $k = \frac{1}{\sqrt{2}} + h$ then we will have

$$H_k(u_{H_{\text{int}}}, a_{H_{\text{int}}}) = \frac{1}{\sqrt{2}} - \frac{h}{2\pi} \int_{\Omega} \| i \nabla u_{H_{\text{int}}} + (A_0 + a_{H_{\text{int}}}) u_{H_{\text{int}}} \|^2 + o(h).$$

We get the following values of $S$ using Bochner-Kodaira-Nakano

$$S = \sup_{0 < H_{\text{int}} < \frac{1}{\sqrt{2}}} \frac{1}{2\pi} \int_{\Omega} \| i \nabla u_{H_{\text{int}}} + (A_0 + a_{H_{\text{int}}}) u_{H_{\text{int}}} \|^2$$

follows from Theorem 10(ii).

One may want now to know the exact value of $S$ at $\frac{1}{\sqrt{2}}$. Using numerical simulations we obtain that the function

$$\chi(H_{\text{int}}) = 1 - \sqrt{2}H_{\text{int}} - \frac{H_{\text{int}}}{2\pi^2 \sqrt{2}} \int_{\Omega} |\text{curl} a_{H_{\text{int}}}|^2$$

is decreasing and has a limit of approximately 0.78 at $H_{\text{int}} = 0$ for a square lattice.

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