Stability of solitons in $\mathcal{PT}$-symmetric couplers

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Families of analytical solutions are found for symmetric and antisymmetric solitons in the dual-core system with the Kerr nonlinearity and $\mathcal{PT}$-balanced gain and loss. The crucial issue is stability of the solitons. A stability region is obtained in an analytical form, and verified by simulations, for the $\mathcal{PT}$-symmetric solitons. For the antisymmetric ones, the stability border is found in a numerical form. Moving solitons of both types collide elastically. The two soliton species merge into one in the “supersymmetric” case, with equal coefficients of the gain, loss and inter-core coupling. These solitons feature a subexponential instability, which may be suppressed by periodic switching (“management”).

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Dissipative media featuring the parity-time ($\mathcal{PT}$) symmetry have recently drawn a great deal of attention. The introduction of this symmetry in optics followed works extending the canonical quantum theory to non-Hermitian Hamiltonians that may exhibit a real spectrum [1]. The Hamiltonian is $\mathcal{PT}$-symmetric if it includes a complex potential $V(x)$ which satisfies constraint $V(x) = V^*(-x)$. Such potentials were proposed [2]- [7] and realized [8,9] in optics, by juxtaposing spatially symmetric patterns of the refractive index and appropriately placed gain and loss elements, see Ref. [7] for a review. Nonlinear $\mathcal{PT}$ systems [10]- [15] and respective solitons [16] were introduced too.

A medium which is akin to $\mathcal{PT}$ systems is a dual-core waveguide with gain and loss acting separately in two cores, which are linearly coupled by the tunneling of light [17]. This system predicts stable 1D solitons in optical [17]- [19] and plasmonic [20] waveguides with the Kerr nonlinearity, as well as 2D dissipative solitons and vortices [21, 22]. The system is made $\mathcal{PT}$ symmetric by adopting equal strengths of the gain and loss in the cores.

A challenging problem is the stability of solitons, as stable pulses in the dual-core system were previously found far from the point of the $\mathcal{PT}$ symmetry [18-20]. We produce two families of exact soliton solutions for the $\mathcal{PT}$-symmetric system, which correspond to symmetric and antisymmetric solitons in the ordinary dual-core coupler [23]- [26]. For the former family, an exact stability border is found analytically, and verified by simulations.

For the $\mathcal{PT}$-antisymmetric solitons, the stability region is identified in a numerical form. In the “supersymmetric” limit, when the gain and loss coefficients coincide with the constant of the inter-core coupling, both families merge into solitons which are subject to a subexponential instability. We demonstrate that this instability can be suppressed by means of a “management” technique, i.e., periodic switching of the gain, loss, and coupling.

The transmission of light or plasmons in the dual-core waveguide is described by the linearly coupled equations for amplitudes $u(z,t)$ and $v(z,t)$ in the active and passive cores [17]- [20]:

\begin{align*}
    iu_2 + (1/2)u_{tt} + |u|^2 u - i\gamma u + \kappa v &= 0, \\
    iv_2 + (1/2)v_{tt} + |v|^2 v + i\Gamma v + \kappa u &= 0,
\end{align*}

where $z$ is the propagation distance and $t$ the reduced time or transverse coordinate in the temporal- or spatial-domain system. Coefficients accounting for the dispersion or diffraction, Kerr nonlinearity, and inter-core coupling, $\kappa$, are normalized to be 1, while $\gamma$ and $\Gamma$ are coefficients of the linear gain and loss in the two cores. The $\mathcal{PT}$ symmetry in Eqs. (1) corresponds to $\Gamma = \gamma$. Strictly speaking, the $\mathcal{PT}$-balanced gain and loss correspond to the border between stable and unstable systems: The zero solution, $u = v = 0$, is unstable at $\gamma > \Gamma$, while the stability region was found at $\gamma < \Gamma$ [17,18].

Obviously, any solution to the nonlinear Schrödinger (NLS) equation (with a frequency shift), $iu_2 + (1/2)u_{tt} + |U|^2 U + \sqrt{1-\gamma^2}U = 0$, gives rise to two exact solutions of the $\mathcal{PT}$-symmetric system, provided that $\gamma \leq 1$:

\begin{equation}
    v = \left(i\gamma \pm \sqrt{1-\gamma^2}\right)u = U(z,t)
\end{equation}

(recall we fix $\kappa \equiv 1$). For $\gamma = 0$, solutions (2) with $+$ and $-$ amount, respectively, to the symmetric and antisymmetric modes in the dual-core coupler [23]- [26], therefore we call the respective solutions (2) $\mathcal{PT}$-symmetric and $\mathcal{PT}$-antisymmetric ones. In the limit of $\gamma = 1$, exact solutions (2) reduce to a single one, $v = iu = U(z,t)$. In particular, $\mathcal{PT}$-symmetric and antisymmetric solitons, with arbitrary amplitude $\eta$, are generated by the NLS solitons,

\begin{equation}
    U(z,t) = \eta \exp\left(i \left(\eta^2/2 \pm \sqrt{1-\gamma^2}\right)z\right) \text{sech} (\eta t).
\end{equation}

As concerns stability of the solitons, it is again relevant to compare the $\mathcal{PT}$-symmetric system to its counterpart with $\gamma = 0$. The exact result for the latter system is that
the symmetric solitons are unstable against symmetry-breaking perturbations at $\eta^2 > \eta_{\text{max}}^2 (\gamma = 0) \equiv 4/3$ [23]. The antisymmetric solitons are unstable against oscillatory perturbations at $\eta^2 > \hat{\eta}_{\text{max}}^2 \approx 0.8$ [25]. The latter instability threshold was found in a numerical form.

The analysis of the instability against antisymmetric perturbations, $\delta u = -\delta v$, which leads to the exact result $\eta^2 (\gamma = 0) \equiv 4/3$, can be extended for the $\mathcal{PT}$-symmetric solitons. The respective perturbation $\delta u$ at the critical point, $\eta^2 = \eta_{\text{max}}^2$, obeys the linearized equation,

\[
\left\{ 4\sqrt{1 - \gamma^2} - d^2/dt^2 + \eta_{\text{max}}^2 \{ 1 - 6\text{sech}^2 (\eta_{\text{max}} t) \} \right\} \delta u = 0.
\]

This solvable equation yields

\[
\eta_{\text{max}}^2 (\gamma) = (4/3) \sqrt{1 - \gamma^2}.
\]

We have performed simulations of the evolution of perturbed solitons within the framework of Eqs. (1), aiming to verify the stability border (4) of the $\mathcal{PT}$-symmetric solitons, and identify a border for their antisymmetric counterparts. Perturbations were introduced by independently changing the initial amplitudes of both components by $\pm 3\%$. The results are summarized in Fig. 1 for both soliton families. The numerical stability border for the symmetric solitons goes somewhat below the analytical one (4) because the finite perturbations used in the simulations are actually rather strong. Accordingly, taking smaller perturbations, we obtain the numerical stability border approaching the analytical limit. For instance, at $\gamma = 0.5$, amplitude perturbations $\pm 5\%$, $\pm 3\%$, and $\pm 1\%$ give rise to the stability border at $\eta_{\text{max}}^2 = 1.02, 1.055,$ and 1.08, respectively, while Eq. (4) yields $\eta_{\text{max}}^2 \approx 1.15$ in the same case.

Examples of the stable propagation of perturbed solitons of both types are displayed in Fig. 2. These examples are shown for the solitons located close to the stability border, cf. Fig. 1 [in particular, Eq. (4) yields $\eta_{\text{crit}} (\gamma = 0.5) \approx 1.07$, cf. $\eta = 0.9$ in Fig. 2(a)]. Unlike the dual-core coupler [17]- [26], the $\mathcal{PT}$ system cannot support asymmetric solitons, as the balance between the gain and loss is impossible for them. Accordingly, the instability of solitons with $\eta > \eta_{\text{max}}$ leads to a blowup of field $u$ and decay of $v$ in the direct simulations (not shown here).

The Galilean invariance of Eqs. (1) suggests to consider collisions between stable solitons, setting them in motion by means of boosting, $\{u, v\} \rightarrow \{u, v\} \exp (\pm i\chi t)$, with frequency shift $\chi$. The results is that the collisions always seem elastic. A typical collision between $\mathcal{PT}$-symmetric and antisymmetric solitons is displayed in Fig. 3.
the linearization operator of the NLS equation. If the NLS solution is stable, the first equation (5) produces no instability, while the second gives rise to the resonance, as \((\delta u + i\delta v)\) is a zero mode of entire operator \(\hat{L}\), which includes term \(i\partial_z\). According to the linear-resonance theory, the respective perturbation \((\delta u - i\delta v)\) will be unstable, growing \(\sim z\) (rather than exponentially). Direct simulations of Eqs. (1) with \(\gamma = \kappa\) confirm that the solitons are unstable, the character of the instability being consistent with its subexponential character.

The “supersymmetric” solitons may be stabilized by means of the management technique [27,28], periodically reversing the common sign of \(\gamma = \Gamma = \kappa\), which does not break the supersymmetry of system (1). Flipping \(\gamma\) and \(\Gamma\) means switch of the gain between the two cores, which is possible if both are doped [29]. The coupling coefficient, \(\kappa\), cannot flip by itself, but the signal in one core may pass \(\pi\)-shifting plates, which is tantamount to the periodic sign reversal of \(\kappa\).

Ansatz (2) still yields exact solutions to Eqs. (1) with coefficients \(\Gamma = \gamma = \kappa\) subjected to the periodic management. On the other hand, the replacement of \(\kappa\) by the periodically flipping coefficient destroys the resonance in Eq. (5). In simulations, this management mode indeed provides for self-trapping of robust solitons from inputs significantly different from the exact soliton solution. An example is shown in Fig. 4 for the input with a relatively large perturbation, in the form of a 10% amplitude deviation from the exact solution. A detailed analysis of the management will be reported elsewhere.

![Fig. 4. (Color online) Self-trapping of the soliton in the “supersymmetric” system \((\Gamma = \gamma = \kappa = 1)\) subject to the management with period \(\Delta z = 2\). The input is \(u_0 = -i\xi_0 = 0.9\text{sech}(t)\).](image)

In conclusion, we have found the families of exact symmetric and antisymmetric solitons in the \(\mathcal{PT}\)-symmetric dual-core system with the Kerr nonlinearity. For the symmetric family, the stability border (4) was found in the analytical form, and corroborated by simulations. The stability region for the antisymmetric solitons was obtained in the numerical form. Collisions between solitons of both types are elastic. In the “supersymmetric” limit, with the equal gain, loss and inter-core-coupling coefficients, the two soliton species merge into one, which is subject to the subexponential instability, and can be stabilized by means of the periodic management. It may be interesting to extend the spatial-domain model by including a periodic linear or nonlinear potential, which may give rise to other soliton modes [30], and to include the intermode dispersion in the temporal domain [31].

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