Strong Isospin-Breaking Effects in $K \to 2\pi$ at Next-to-Leading Order in the Chiral Expansion

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Strong isospin-breaking (IB) contributions to both the octet and 27-plet weak $K \to \pi\pi$ transitions are evaluated at next-to-leading order (NLO) in the chiral expansion. NLO contributions are shown to significantly reduce the leading order result for the potentially large contribution to the $\Delta I = 3/2$ amplitude resulting from strong isospin-breaking modifications to the weak $\Delta I = 1/2$ amplitude. The ratio of strong IB 27-plet to strong IB octet contributions is found to be small for all decay amplitudes. Combined with recent results on the corresponding electromagnetic contributions, we find that the ratio of the intrinsic strengths of octet and 27-plet effective weak operators can be taken to be that obtained from experimental data, analyzed ignoring isospin breaking, to an accuracy better than of order $\sim 10\%$.

13.20.Eb,11.30.Rd,11.30.Hv,14.40.Aq

I. INTRODUCTION

It appears likely that the large ratio ($\sim 20$) between octet $\Delta I = 1/2$ and 27-plet $\Delta I = 3/2$ amplitudes in hyperon and non-leptonic $K$ decay (the so-called $\Delta I = 1/2$ Rule) results from a compounding of long-distance and short-distance effects, and that the sources of both effects are now reasonably well understood. QCD dressing, in the regime of scales $> 1 \text{ GeV}^2$, for which perturbative QCD can be sensibly employed, contribute a factor of $3 - 4$ to the enhancement [1,2], while, in the case of non-leptonic $K$ decay, long-distance effects, including those of final state interactions (FSI) [3–5] also contribute significantly. Attempts to provide a sensible matching of short and long distance effects in a single theoretical framework now appear likely to account for the full observed enhancement [3,6–8]. The neglect of isospin breaking (IB), however, represents a potential problem for this putative understanding [9]. Indeed, since the ratio of magnitudes of the $\Delta I = 1/2$ and $\Delta I = 3/2$ amplitudes is $\sim 20$, IB at the typical few percent scale could lead to a “leakage” of the large weak $\Delta I = 1/2$ transition strength into the $\Delta I = 3/2$ channel with a strength $\sim 20 \times$ a few %. Were the experimental $\Delta I = 3/2$ amplitude to include such a large contribution, an isospin-conserving (IC) calculation which reproduced the experimental ratio of amplitudes could, in fact, be in error by as much as a factor of $\sim 2$.

At leading order in the chiral expansion, and for conventional field choices, strong IB in $K \to \pi\pi$ has only two sources: $\pi^0 - \eta$ mixing on the external $\pi^0$ legs, and IB in the squared $K$ masses (which produces “kinematic” contributions as a result of the momentum dependence of the weak transition amplitudes). At this order, the resulting $\Delta I = 1/2$ leakage contribution represents $\sim 15\%$ of the observed $\Delta I = 3/2$ amplitude [9,10]. Next-to-leading order (NLO) IC corrections are known to be important for the $I = 0$ final state (the $\Delta I = 1/2$ transition) [5] and hence are unlikely to be negligible for the IB corrections. Some phenomenological estimates [11], in fact, suggest that they are rather large: Ref. [10], for example, estimates that including the effect of mixing with the $\eta'$ (a pure NLO effect) raises the $\Delta I = 1/2$ leakage contribution to the $\Delta I = 3/2$ amplitude to $35\%$ of the total. (In contrast, recent evaluations of electromagnetic (EM) contributions to the $K \to \pi\pi$ amplitudes [13] find them to represent few to several percent effects in all three channels, i.e., strongly suppressed relative to the naive estimate given above.) There are, however, other strong-IB-induced NLO contributions not included in the estimate based only on the effect of $\eta'$ mixing. Since the

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method of effective chiral Lagrangians [12] (Chiral Perturbation Theory, or ChPT) provides a straightforward method of evaluating the sum of all such NLO contributions, we will, in this work, determine the strong IB contributions to the $K \to \pi\pi$ amplitudes, including the leakage contribution of the weak $\Delta I = 1/2$ transition, to NLO in ChPT.

II. THE STRONG ISOSPIN-BREAKING CONTRIBUTIONS TO $K \to \pi\pi$

IB has two sources in the Standard Model, electromagnetic (EM) and strong (due to $m_d \neq m_u$). EM IB has $I = 0, 1, 2$ components, and hence, in combination with the dominant $\Delta I = 1/2$ octet weak transition operator, produces contributions to $K \to \pi\pi$ with $\Delta I = 1/2, 3/2$ and $5/2$. These contributions have been recently studied in Refs. [13]. Strong IB, to $\mathcal{O}(m_d - m_u)$, is, in contrast, pure $I = 1$. The strong modifications of the basic $\Delta I = 1/2, 3/2$ transitions thus again produce $\Delta I = 1/2, 3/2$ and $5/2$ contributions. Due to the factor of $\sim 20$ difference in the octet $\Delta I = 1/2$ and 27-plet $\Delta I = 3/2$ weak operator strengths, one would expect strong IB octet contributions to dominate those associated with the weak 27-plet.

In the presence of IB (including the $\Delta I = 2$ component of EM, which couples the $I = 0$ and $I = 2 \pi\pi$ channels) the analogue of the standard isospin decomposition of the $K^+ \to \pi^+ \pi^0$, $K_S \to \pi^+ \pi^-$, $\pi^0 \pi^0$ decay amplitudes, $A_{+0}$, $A_{+-}$ and $A_{00}$, is [13]

$$
A_{00} = \sqrt{2} A_0 - \frac{2}{\sqrt{3}} A_2 = \sqrt{2} |A_0| e^{i(\Phi_0 + \gamma_0)} - \frac{2}{\sqrt{3}} |A_2| e^{i(\Phi_2 + \gamma_2)},
A_{+-} = \frac{2}{\sqrt{3}} A_0 + \frac{1}{\sqrt{3}} A_2 = \frac{2}{\sqrt{3}} |A_0| e^{i(\Phi_0 + \gamma_0)} + \frac{1}{\sqrt{3}} |A_2| e^{i(\Phi_2 + \gamma_2)},
A_{+0} = \frac{\sqrt{2}}{2} A_2 = \frac{\sqrt{2}}{2} |A_2| e^{i(\Phi_2 + \gamma_2)},
$$

where the $\Phi_I$ are the strong $\pi\pi$ (rescattering) phases. In the absence of $\Delta I = 2$ FSI, $\gamma_2 = \gamma_2$ and $\Phi_I + \gamma_I \equiv \Phi_I$ should be the physical isospin $I \pi\pi$ scattering phase, $\delta_I$. In general, $|A_2| \neq |A_2|$ as a consequence of EM- and strong-IB-induced $\Delta I = 5/2$ contributions. If one ignores IB, $A_2 = A_2$ and $\phi_I = \delta_I$.

The conventional IC analysis of $K \to \pi\pi$ involves first determining $|A_2|$ (assumed equal to $|A_2|$) from the $K^+ \to \pi^+ \pi^0$ decay rate, and then extracting $|A_0|$ using the IC relation

$$
2|A_{+-}|^2 + |A_{00}|^2 = 2|A_0|^2 + 2|A_2|^2.
$$

If IB is indeed negligible, then the relative phase, $\phi = \phi_0 - \phi_2$, of the $I = 0/I = 2$ interference terms in the two $K_S$ decay rates should equal $\delta_0 - \delta_2$. Fitting to the experimental decay rates [14,15] assuming IC, one finds

$$
\begin{align*}
|A_0| &= (4.70 \pm 0.01) \times 10^{-4} \text{ MeV} \\
|A_2| &= (2.11 \pm 0.04) \times 10^{-5} \text{ MeV} \\
\phi &= 0.98 \pm 0.06 \text{ rad}.
\end{align*}
$$

The large value of the ratio $|A_0/A_2| = 22.3$ reflects the well-known $\Delta I = 1/2$ Rule, while the deviation of the nominal value of $\phi$, $\phi_{\exp} \approx 56^\circ$, from $\delta_0 - \delta_2 = (42 \pm 4)^\circ$ presumably reflects the presence of neglected IB contributions.

In general, the two $K_S$ decay rates depend on three parameters, $|A_0|$, $|A_2|$, and $\phi$. Since, in the presence of $\Delta I = 5/2$ IB contributions, $|A_2|$ can no longer be determined in $K^+ \to \pi^+ \pi^0$, $\phi$ is not, in fact, experimentally measurable. The (assumed) IC analysis produces a nominal value, $\phi_{\exp}$, related to the actual value, $\phi$, by

$$
\cos (\phi_{\exp}) = \frac{|A_2|}{|A_2|} \cos (\phi) + \frac{(|A_0|^2 - |A_2|^2)}{2\sqrt{2}|A_0||A_2|}.
$$

In the presence of $\Delta I = 5/2$ transitions, the coefficient of the first term on the RHS is $\neq 1$, and the second (small) term is non-zero. $\phi_{\exp}$ can thus differ from $\delta_0 - \delta_2$, even if $\Delta I = 2$ EM FSI effects are negligible.

We now outline the ingredients needed to compute the strong IB contributions to the CP-even $K \to \pi\pi$ amplitudes in ChPT.

The low-energy representation of the strong interactions, sufficient to determine effects at NLO, is given by the 1-loop effective Lagrangian of Ref. [12]. Writing $\mathcal{L}_S = \mathcal{L}_S^{(2)} + \mathcal{L}_S^{(4)}$, where the superscripts denote chiral order, and setting the external vector and axial vector fields (not required for our purposes) to zero, one has
employ the weak deformation/factorization model (FM) estimates of Ref. [19] for the weak LEC’s, we will work through $E$ of Ref. [18]. We work with a form in which the former constraint has been used to eliminate $E$ mind that, in employing the GNC model [22,23] below to make estimates for the weak LEC’s, one must also impose $I$ including those required to separate $\Delta$ decays, whereas ours, following the notation of Ref. [5], refer to $G_L, D_1, r$ from those of Ref. [13]. This can be understood from a

\begin{equation}
\mathcal{L}_S^{(2)} = \frac{F^2}{4} \text{Tr}[\partial_\mu U \partial^\mu U] + \frac{F^2}{4} \text{Tr}[\chi U^\dagger + U \chi^\dagger] \tag{5}
\end{equation}

\begin{equation}
\mathcal{L}_S^{(4)} = L_1(\text{Tr}[\partial_\mu U \partial^\mu U^\dagger])^2 + L_2(\text{Tr}[\partial_\mu U \partial^\mu U]) \text{Tr}[\chi U^\dagger + U \chi^\dagger] + L_3 \text{Tr}[\partial_\mu U \partial^\mu U \partial_\nu U \partial^\nu U^\dagger] \tag{6}
\end{equation}

where $\chi = 2B_0 M_q$ (with $M_q$ the quark mass matrix), $U = \exp(i\lambda \cdot \pi/F)$ and $\{L_i\}, F$ and $B_0$ are the usual strong low-energy constants (LEC’s), in the notation of Ref. [12], for which we employ the values found in Ref. [16].

The low-energy representation of the CP-even part of the non-leptonic weak interactions is given by the Lagrangian, $\mathcal{L}_W$, of Ref. [17,18] (or the equivalent reduced forms of Refs. [19,20], which take into account constraints associated with the Cayley-Hamilton theorem and the leading order equation of motion). We work with the version in which the weak mass term present in the most general form of the leading (second) order part of $\mathcal{L}_W, \mathcal{L}_W^{(2)}$, has been removed by a field redefinition, and the NLO part of $\mathcal{L}_W, \mathcal{L}_W^{(4)}$, correspondingly modified (see Ref. [18] for details). With $\mathcal{L}_W^{(2)} = \mathcal{L}_W^{(2)}(8) + \mathcal{L}_W^{(2)}(27)$, where the subscripts label the flavor octet and 27-plet components, respectively, one has

\begin{equation}
\mathcal{L}_W^{(2)} = c_2 \text{Tr} \left[\lambda_0 D_\mu U^\dagger D^\mu U\right] \tag{7}
\end{equation}

\begin{equation}
\mathcal{L}_W^{(2)} = c_3 t \left(\text{Tr}[\bar{Q} L_\mu] \text{Tr}[\bar{Q} L^\mu]\right) \tag{8}
\end{equation}

where $L_\mu = iU^\dagger D_\mu U$, $c_2$ and $c_3$ are leading order weak LEC’s of order $G_F$, $D_\mu$ is the covariant derivative (which, for our purposes, reduces to the ordinary partial derivative), the matrix $\bar{Q}$ projects out the flavor octet components of any trace in which it occurs, and the tensor, $t$, combines two octets into a 27-plet. The explicit forms of $\bar{Q}$ and $t$, including those required to separate the $\Delta I = 1/2$ and $3/2$ components of the 27-plet, may be found in Ref. [21].

For the NLO weak contributions one has [18],

\begin{equation}
\mathcal{L}_W^{(4)} = \sum_{i=1}^{48} E_i O_i^{(8)} + \sum_{i=1}^{34} D_i O_i^{(27)} \tag{9}
\end{equation}

where the $E_i$ and $D_i$ are the weak NLO octet and 27-plet LEC’s (which have an implicit proportionality to $c_2, c_3$, respectively). The corresponding renormalized LEC’s are denoted $E_i^\gamma$ and $D_i^\gamma$. Their relations to the $E_i$ and $D_i$ are given in Ref. [18]. Explicit expressions for the operators $O_i^{(n)}$ may be found in Ref. [18]. Use of the Cayley-Hamilton theorem and the equation of motion allows one to remove certain of the terms in Eq. (9), as explained in Section 3 of Ref. [18]. We work with a form in which the former constraint has been used to eliminate $E_{14}^\gamma$ and the LEC’s $E_{10}^\gamma$ through $E_{13}^\gamma$ modified accordingly. (The constraint also allows elimination of $E_{44}^\gamma, E_{45}^\gamma$, and $D_{32}^\gamma$, but this is irrelevant for our purposes since the corresponding operators do not contribute to $K \rightarrow \pi \pi$ at NLO.) The reader should bear in mind that, in employing the GNC model [22,23] below to make estimates for the weak LEC’s, one must also impose this constraint, which has not been implemented in Ref. [23].

An alternate choice of operator basis for the NLO weak octet Lagrangian is that given in Ref. [19]. When we employ the weak deformation/factorization model (FM) estimates of Ref. [19] for the weak LEC’s, we will work

\footnote{Our definition of the $O(p^2)$ weak octet operator, and hence our normalization convention for $c_2$, agrees with that of Refs. [5,17,18]. The choice $c_2 > 0$ conforms to that of Ref. [5], but differs by a sign from that used in Ref. [21]. Our convention for the tensor $t$ is such that data then requires $c_3 > 0$, which differs by a sign from the convention of Ref. [18], and by both a sign and a factor of 2 from that of Refs. [5,17]. With $c_2 > 0, c_3 > 0$, our tree-level amplitudes $A_0$ and $A_2$ are negative. Our invariant amplitudes $A_0, A_2$ and $A_4$ differ by a factor of $-\sqrt{3}$ from those of Ref. [13]. This can be understood from a comparison of the expressions for the amplitudes, provided one takes into account the fact that the neutral $K$ decay amplitudes of Refs. [13] refer to $K^0$ decays, whereas ours, following the notation of Ref. [5], refer to $K^0$ decays.

\footnote{We concur with Ref. [20] in requiring an overall difference in the sign of the divergent parts of all 27-plet LEC’s, as compared to the results of Ref. [18].}

\footnote{Ref. [23] also employs a form of the strong NLO Lagrangian in which, in contrast to the conventional form given above, operators which could be omitted as a consequence of the leading order equation of motion have not been removed. In order to employ the results of Ref. [23] one must, therefore, first remove those operators, and then make the corresponding changes to the weak LEC’s. These modifications affect the values of $E_{15}, E_{22}, E_{33}, D_5, D_{10}$ of Ref. [23].}
with the reduced set of octet operators, \( W_i^{(8)} \), and corresponding LEC’s, \( N_i \), defined in that reference. For the corresponding weak 27-plet operators we follow Ref. [20], denoting the operators by \( \tilde{O}_i^{(27)} \) and the LEC’s by \( \tilde{D}_i \). The renormalized LEC’s are written, in obvious notation, \( \tilde{N}_i \) and \( \tilde{D}_i \), respectively.

Certain combinations of the weak LEC’s were determined in Ref. [5] by neglecting IB and fitting the calculated \( \mathcal{O}(p^4) \) amplitudes for \( K \to \pi \pi \) and \( K \to \pi \pi \pi \) to experimental data. (Sufficient data exists to allow such an IC fit, provided one neglects contributions suppressed by a relative factor of \( m_d^2/m_K^2 \) [5].) Since all IC octet (respectively, 27-plet) contributions are proportional to \( c_2 \) (respectively, \( c_3 \)), the presence of IB contributions can be accommodated in the fit by rescalings of \( c_2 \) and \( c_3 \). One, of course, expects a small rescaling for \( c_2 \), but potentially significant rescaling for \( c_3 \). As can be seen from the results below, the LEC combinations entering the IB contributions to \( A_0 \), \( A_2 \) and \( A'_2 \) are such that the total number of linearly independent IC and IB LEC combinations exceeds the existing number of \( K \to \pi \pi \) and \( K \to \pi \pi \pi \) observables, making an experimental determination of the new IB LEC values impossible. It is, therefore, necessary to estimate their values using models. We employ two models for this purpose, each representing the extension of a model successful in reproducing the empirical values of the strong LEC’s.

In the first of these models, the FM [19], a rescaled version of the factorization of the four-quark currents into products of two-quark currents is employed, the LEC contributions to the latter being given by resonance saturation

\[ A \text{ self-energy}, \Sigma \]

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Using the expressions above for the weak and strong effective Lagrangians, it is straightforward to compute the desired strong IB contributions to \( A_0 \), \( A_2 \) and \( A'_2 \). The leading \( \mathcal{O}(p^2) \) contributions are given in Table I. \(^4\) The NLO contributions are obtained by evaluating the graphs of Figs. 1(b)-(h). The notation for the Figures is as follows: internal lines represent any of the members of the pseudoscalar octet, solid circles the \( \pi \) and \( \eta \) fields at NLO, which are required to handle the effects of \( \pi-\eta \) mixing at this order, are taken from Ref. [26], while expressions for the \( \mathcal{O}(m_d - m_u) \) contributions to the wavefunction renormalization factors may be found in Ref. [21].

Since the expressions for the IB parts of the one-loop graphs (Figures 1(b)-(g)) are rather lengthy and unilluminating, we record here only their numerical values. \(^5\) The results (including the strong LEC contributions to the NLO mixing and wavefunction renormalizations (Fig. 1(c)), of which the \( L_7 \) term, which reflects the \( \eta \) mixing contribution at this order, is a part) are given in Table II. These results correspond, for definiteness, to the scale \( \mu^2 = m_N^2 \), and are given in the form \( \delta(f)A_i/c(f) \), where \( f = (8) \) or \( (27) \) labels the flavor of the weak transition operator, \( c^{(8)} \equiv c_2 \), \( c^{(27)} \equiv c_3 \), and \( A_i = A_0 \), \( A_2 \) or \( A'_2 \) (these combinations are independent of the specific values of the weak LEC’s \( c_2 \) and \( c_3 \)). The scale dependence of each such sum must, of course, cancel that of the corresponding weak LEC combination. Collectively, the finiteness and scale independence of each of the three \( K \to \pi \pi \) decay amplitudes provides a powerful cross-check on the calculations.

The weak LEC (counterterm) contributions, corresponding to Fig. 1(h), are given by

\[
\left[ \delta^{(8)} A_0 \right]_{LEC} = - \frac{\sqrt{6} B_0 (m_d - m_u)}{9 F^3} \left( m_K^2 J_1 - m_\pi^2 J_2 \right)
\]

\[
\left[ \delta^{(8)} A_2 \right]_{LEC} = \left[ \delta^{(8)} A'_2 \right]_{LEC} = \frac{2 B_0 (m_d - m_u)}{\sqrt{3} F^3} \left( m_K^2 J_3 - m_\pi^2 J_4 \right)
\]

\[
\left[ \delta^{(27)} A_0 \right]_{LEC} = - \frac{\sqrt{2} B_0 (m_d - m_u)}{4 \sqrt{3} F^3} \left( m_K^2 K_1 + m_\pi^2 K_2 \right)
\]

\(^4\)The IB \( \pi-\eta \) mixing and kinematic contributions turn out to exactly cancel for \( K \to \pi^0 \pi^0 \), in both the octet and 27-plet cases. The mixing contribution is, of course, absent for the \( K \to \pi^+ \pi^- \) amplitude.

\(^5\)Expressions for the octet one-loop IB contributions may be found in Appendix B of Ref. [21]; those for the 27-plet will be reported elsewhere [25].
\[ \left[ \delta^{(27)} A_2 \right]_{LEC} = \left( \frac{B_0(m_d - m_u)}{4\sqrt{3}F^3} \right) \left( m_K^2 K_3 + m_{\pi}^2 K_4 \right) \]
\[ \left[ \delta^{(27)} A_2^* \right]_{LEC} = \left( \frac{B_0 (m_d - m_u)}{2\sqrt{3}F^3} \right) \left( m_K^2 K_5 + m_{\pi}^2 K_6 \right) \] (10)

where, in the basis of Ref. [18],

\[ J_1 = -12E_1^r + 24E_3^r + 36E_5^r - 12E_9^r + 21E_{10}^r + 9E_{11}^r + 36E_{12}^r + 15E_{13}^r - 72E_{32}^r - 48E_{33}^r - 24E_{34}^r + 30E_{37}^r + 30E_{38}^r \]
\[ J_2 = -60E_1^r - 36E_3^r + 12E_5^r + 48E_9^r + 33E_{10}^r - 12E_{11}^r + 36E_{12}^r + 18E_{13}^r + 9E_{15}^r - 72E_{32}^r + 96E_{34}^r + 24E_{35}^r + 24E_{36}^r + 18E_{37}^r + 18E_{38}^r - 48E_{39}^r - 48E_{40}^r \]
\[ J_3 = -4E_1^r + 8E_3^r + 12E_5^r - 4E_9^r + E_{10}^r + 3E_{11}^r + 12E_{12}^r - E_{15}^r - 24E_{32}^r - 16E_{33}^r - 8E_{34}^r - 2E_{37}^r - 2E_{38}^r \]
\[ J_4 = -2E_1^r + 10E_5^r - 12E_9^r - 8E_{15}^r + 2E_{10}^r + 5E_{11}^r + 12E_{12}^r - 24E_{32}^r - 24E_{33}^r - 16E_{34}^r - 4E_{35}^r - 4E_{36}^r + 8E_{39}^r + 8E_{40}^r \]
\[ K_1 = 208D_1^r + 10D_4^r - 10D_5^r - 66D_6^r + 32D_7^r + 20D_{22}^r + 20D_{23}^r \]
\[ K_2 = -144D_1^r + 32D_5^r + 30D_4^r + 2D_5^r + 50D_6^r + 16D_7^r - 64D_{19}^r - 32D_{20}^r - 32D_{21}^r - 4D_{22}^r - 4D_{23}^r + 128D_{24}^r + 128D_{25}^r \]
\[ K_3 = -64D_1^r - 28D_4^r + 28D_5^r + 12D_6^r - 32D_7^r - 56D_{22}^r - 56D_{23}^r \]
\[ K_4 = -48D_1^r - 32D_5^r - 12D_9^r - 20D_6^r + 16D_7^r - 64D_{19}^r + 32D_{20}^r + 32D_{21}^r + 40D_{22}^r + 40D_{23}^r - 128D_{24}^r - 128D_{25}^r \]
\[ K_5 = -32D_1^r + 11D_4^r - 11D_5^r + 16D_6^r + 4D_7^r + 22D_{22}^r + 22D_{23}^r \]
\[ K_6 = 16D_1^r + 4D_4^r - 6D_5^r + 10D_6^r - 12D_7^r - 2D_7^r - 8D_{19}^r - 4D_{20}^r - 4D_{21}^r - 20D_{22}^r - 20D_{23}^r + 16D_{24}^r + 16D_{25}^r \] (11)

while in that of Ref. [19,20],

\[ J_1 = c_2 \left[ 7N_5^r + 6N_6^r + 4N_9^r + 5N_9^r - 4N_{10}^r - 8N_{12}^r - 12N_{13}^r \right] / F^2 \]
\[ J_2 = c_2 \left[ 11N_5^r + 6N_6^r - 6N_9^r - 2N_9^r + 3N_9^r - 20N_{10}^r - 12N_{11}^r - 4N_{12}^r - 12N_{13}^r \right] / F^2 \]
\[ J_3 = c_2 \left[ N_5^r + 6N_6^r - 2N_9^r - N_9^r + 4N_{10}^r - 8N_{12}^r - 12N_{13}^r \right] / F^2 \]
\[ J_4 = c_2 \left[ 2N_5^r + 6N_6^r - N_9^r - 2N_{10}^r - 10N_{12}^r - 12N_{13}^r \right] / F^2 \]
\[ K_1 = -2c_3 \left[ -104D_1^r - 5D_4^r + 5D_6^r + 33D_6^r - 16D_7^r \right] / F^2 \]
\[ K_2 = -2c_3 \left[ 72D_1^r + 16D_4^r - 15D_5^r - 25D_6^r - 8D_7^r \right] / F^2 \]
\[ K_3 = 4c_3 \left[ -16D_1^r - 7D_4^r + 7D_5^r + 3D_6^r - 8D_7^r \right] / F^2 \]
\[ K_4 = 4c_3 \left[ -12D_1^r + 8D_2^r - 3D_3^r - 5D_5^r + 4D_6^r - 4D_7^r \right] / F^2 \]
\[ K_5 = 3c_3 \left[ -32D_1^r + 11D_4^r - 11D_5^r + 16D_6^r + 4D_7^r \right] / F^2 \]
\[ K_6 = 3c_3 \left[ 16D_1^r - 4D_4^r - 6D_5^r + 10D_6^r - 12D_7^r + 2D_7^r \right] / F^2. \] (12)

It is worth commenting that, although the \( J_4 \) contribution to \( \left[ \delta^{(8)} A_2 \right]_{LEC} \) is suppressed by a factor of \( m_{\pi}^2/m_K^2 \) relative to that involving \( J_3 \), the ratio of the two contributions in fact ranges between 0.3 and 0.6 for the models discussed. One should, therefore, reserve some caution for the procedure of neglecting LEC contributions to the \( K \to \pi \pi \) and \( K \to \pi \pi \pi \) amplitudes which are suppressed by \( m_{\pi}^2/m_K^2 \).

**III. NUMERICAL RESULTS AND CONCLUSIONS**

Our numerical results are based on the following input: \( \pi \) and \( K \) masses and decay constants from Ref. [14]; strong NLO LEC’s from Ref. [16]; weak LEC values from the models noted above; and

\[ B_0(m_d - m_u) = \left( \frac{m_d - m_u}{m_d + m_u} \right) m_{\pi}^2 = 5248 \pm 674 \text{ MeV}^2, \] (13)
which follows from Leutwyler’s determination of the light quark mass ratios [27].

In determining the rescaling of the weak LEC’s, $c_2$ and $c_3$, from their IC values, we include not only our strong octet and 27-plet IB contributions, but also the EM IB contributions, as determined in the most constraining (dispersive) version of the analysis of Refs. [13]. The difference in the magnitudes of these rescalings determines the error in the extracted value of the ratio, $c_3/c_2$, of weak 27-plet to weak octet operator strengths made by neglecting IB effects in the analysis of experimental data. The fitted values of $c_2$ and $c_3$, together with the ratio

$$R_{IB} = \frac{c_3/c_2}{c_{3IC}/c_{2IC}},$$

which quantifies this error, are given in Table III, where the IC fit values of $c_2$ and $c_3$ have also been included for comparison. Note that a value of $R_{IB} < 1$ implies that the ratio of $\Delta I = 1/2$ to $\Delta I = 3/2$ operator strengths is larger than would be obtained in an IC analysis. After including the quoted errors on the EM contributions from Ref. [13], we find

$$R_{IB} = 0.963 \pm 0.029 \pm 0.010 \pm 0.034,$$

where the first error reflects the model dependence associated with the $O(p^4)$ weak LEC values, the second the uncertainty in $B_0(m_d - m_u)$, and the third the uncertainty in the EM contributions. The ratio $c_2/c_3$ can thus be taken to be that obtained in an IC analysis to an accuracy of better than $\sim 10\%$.

To understand the reason for this rather small IB shift, it is useful to examine separately the octet, 27-plet and EM IB contributions to the $K \to \pi \pi$ amplitudes. We denote by $\delta^{(s)} A_k$ the IB contribution to $A_k$, where $A_k$ is any of $A_0$, $A_2$, and $A_2'$, and (s) = (8), (27) or (EM) labels the source of IB. The results for the $\delta^{(s)} A_k$, are given in Table IV. The EM results and errors are those of Refs. [13], adapted to our conventions. The errors on the real parts of the strong IB contributions correspond to the range of values of the weak LEC contributions obtained from the different models above combined in quadrature with the error associated with the uncertainty in $B_0(m_d - m_u)$; the former turns out to be the dominant source of error.

A number of features of the results are worth further comment. First, in all cases the IB 27-plet contributions are a factor of $\sim 20$ smaller than than the IB octet, compatible with naive estimates based on the relative size of the $\Delta I = 1/2$ and $\Delta I = 3/2$ weak operator strengths. Second, the EM contributions are of order $\sim 50\%$ of the octet IB contributions for $A_0$ and $A_2'$, and of order $\sim 80\%$ for $A_2$, the two contributions adding constructively for $A_0$ and $A_2$, but destructively for $A_2'$. Third, while in all cases the strong IB contributions add constructively to the IC contributions, the EM contributions add constructively for $A_0$ and $A_2$, but destructively for $A_2'$. These features ensure that $|A_2|/|A_2'| > 1$, an effect which tends to make the nominal phase, $\phi_{exp}$, smaller than the actual phase difference, $\phi$. Because the IB 27-plet contributions are, as expected, small, this effect (associated with the presence of a $\Delta I = 5/2$ contribution in the $K \to \pi \pi$ amplitudes) is almost totally dominated by the EM component. In fact, as one can see from the near equality of $\delta^{(27)} A_2$ and $\delta^{(27)} A_2'$, the 27-plet-induced $\Delta I = 5/2$ component is strongly suppressed, in contrast to the situation for the EM contributions. As a result, though the 27-plet IB contribution to each of $A_2$ and $A_2'$ is at the $\sim 10\%$ level of the corresponding EM contribution, it has been reduced to the 1/2% level when one considers $|A_2| - |A_2'|$.

Let us return to the question of the IB modification of $c_3/c_2$, the ratio which parametrizes the $\Delta I = 1/2$ rule enhancement in the low energy effective theory. We have seen above that the IB effect is, in fact, quite modest. It is now possible to see why it is that this is the case. The results of Table III show that, as expected, $c_2$ is only slightly modified (at the $\sim 1\%$ level) by IB effects. The ratio $c_3/c_{3IC}$ is, however, much closer to 1 than the 15% deviation produced by including only the leading order strong IB octet contributions. This decrease in the IB effect on $c_3$ has two sources. First, as can be seen from Table IV, there is a significant cancellation between the octet and EM IB contributions to $A_2'$, which quantity dominates the determination of $c_3$. Second, this cancellation is facilitated by the fact that the $O(p^2)$ and $O(p^4)$ octet leakage contributions add destructively. This latter feature might seem unnatural given the observation that $\eta'$ mixing is expected to increase the leading order octet IB effect, but there is, in fact, a natural reason why this is not the case. In the strong interaction part of the low energy effective theory, the effects of the $\eta'$ are encoded entirely in the LEC $L_7^\eta$. A contribution proportional to $L_7^\eta$, associated with the effects of mixing on the external $\pi^0$ legs, is, of course, present in the results above, and indeed, on its own, would serve to significantly increase the leading order result. However, as can be seen from Eqs. (15)-(17) of Ref. [26], the LEC contributions to the relevant mixing angles occur in the combination $3L_7^\eta + L_6^\eta$, for which, empirically, there is an almost complete cancellation between the $L_7^\eta$ and $L_6^\eta$ terms. The cumulative effect of all NLO corrections, including the strong LEC

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6This observation has also been made in the context of an estimate of NLO mixing contributions to the IB correction, $\Omega_{IB}$,
corrections just discussed, is, in fact, to lower the magnitude of the leading order results; the estimate based only on the inclusion of \( \eta' \) mixing effects thus turns out to be misleading. One of the great advantages of the ChPT approach is that it allows one, in a straightforward manner, to include all contributions of a given chiral order which occur in the Standard Model.

We conclude with a brief comment about the relation of the nominal phase, \( \phi_{\text{exp}} \), and the actual relative phase, \( \phi \), between the \( I = 0 \) and \( I = 2 \) components of the two \( K_S \) amplitudes. In order to fully explicate the phase question, one would require both a determination of the IB contributions to the (in the presence of EM, coupled channel) \( \pi \pi \) scattering phases, and a determination, and subtraction, of non-\( \pi \pi \)-scattering IB effects in the processes in which the \( \pi \pi \) phases are nominally measured. Such expressions are not currently available, and a determination of them is beyond the scope of this paper. Without such expressions, however, the relation between \( \phi \) and the nominally determined experimental \( I = 0 \) and \( I = 2 \) \( \pi \pi \) phases is subject to IB corrections whose size is not, at present, known. In addition, one should bear in mind that the experimental data has yet to have applied to it the detector-dependent IR correction factor present in the expression for the \( K_S \to \pi^+ \pi^- \) cross-section (see Ref. [13] for a discussion of this point). Since the difference \( |A_{+-}|^2 - |A_{00}|^2 \), from which the interference term which determines \( \phi_{\text{exp}} \) is obtained, is \( \sim 10\% \) of the individual terms, even a 1\% IR correction can have a sizeable numerical impact. While the problems just discussed mean that uncertainties exist, both in our knowledge of the relation of \( \phi \) to the measured \( \pi \pi \) scattering phase difference, and in the experimental determination of \( \phi_{\text{exp}} \), our results, combined with those of Ref. [13], allow us to quantify the deviation of \( \phi_{\text{exp}} \) from \( \phi \) resulting from the presence of \( \Delta I = 5/2 \) strong and EM IB effects. We find, for the coefficient of \( \cos(\phi) \) in Eq. (4),

\[
\frac{|A_2|}{|A_2'|} = 1.094 \pm 0.039 .
\]

The second term in Eq. (4) is then \(-0.0015 \pm 0.0006\); its effect is thus tiny, and in any case, swamped by the error on \( |A_2|/|A_2'| \). As an example of the magnitude of the resulting effect, note that, were \( \phi \) to be 42\(^\circ \), one would then obtain \( \phi_{\text{exp}} = 35.8 \pm 2.9^\circ \). Recall that this effect is almost entirely EM in origin. The sign of the EM contributions is thus such as to significantly exacerbate the existing phase discrepancy.

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in \( \epsilon'/\epsilon \) [28]. A useful discussion of the resonance interpretation of the \( L_8^L \) contribution can be found in that reference.
the corresponding contributions to the \( K \) are described in the text. Entries are in units of \( 10^{-6} \) MeV and \( -6 \) MeV.

\[ c(27) = -\frac{1}{4}c_2 \]

Total \( (\delta A_0, \delta A_2, \delta A_0') \) correspond to the three models for the weak LEC's described in the text. The results quoted here correspond to central values of both \( B_0(m_d - m_u) \), as given in Eq. (13), and the EM contributions, as given in Ref. [12].

| (f) | \( \delta(\delta)A_0 \) | \( \delta(\delta)A_2 \) | \( \delta(\delta)A_0' \) |
|-----|-----------------|-----------------|-----------------|
| (8) | \( -\sqrt{\frac{2}{3}}c_2 \) | \( -\sqrt{\frac{2}{3}}c_2 \) | \( -\sqrt{\frac{2}{3}}c_2 \) |
| (27) | \( -2\sqrt{\frac{2}{3}}c_3 \) | \( -2\sqrt{\frac{2}{3}}c_3 \) | \( \frac{1}{2}\sqrt{\frac{2}{3}}c_3 \) |

TABLE II. Octet and 27-plet contributions to \( A_0, A_2, A_0' \) corresponding to the graphs of Figures 1(b)-(g). The quantities \( c(8) \equiv c_2 \) (for the octet case) and \( c(27) \equiv c_3 \) (for the 27-plet case) have been factored out, for the reasons described in the text. The entries correspond to the renormalization scale \( \mu^2 = m_d^2 \), and are in units of MeV^{-1}. As in Table 1, \( (f) \) represents the flavor of the weak transition operator. The fitted \( c_2 \) and \( c_3 \) values, needed in order to determine the actual numerical values of the corresponding contributions to the \( K \to \pi\pi \) amplitudes, are given in Table III.

| (f) | \( \delta(\delta)A_0/c(\delta) \) | \( \delta(\delta)A_2/c(\delta) \) | \( \delta(\delta)A_0'/c(\delta) \) |
|-----|-----------------|-----------------|-----------------|
| (8) | \( 0.00185 - 0.00538i \) | \( 0.00091 + 0.00078i \) | \( 0.00091 + 0.00078i \) |
| (27) | \( -0.00803 - 0.0119i \) | \( 0.0181 + 0.0110i \) | \( -0.0239 - 0.00775i \) |

| Fit type | \( c_2 \) | \( c_3 \) | \( R_{IB} \) |
|----------|---------|---------|---------|
| IC | \( 5.43 \times 10^{-4} \) | \( 7.23 \times 10^{-6} \) | 1.000 |
| IB, GNC A = 1 | \( 5.38 \times 10^{-4} \) | \( 6.91 \times 10^{-6} \) | 0.965 |
| IB, GNC A = 3 | \( 5.37 \times 10^{-4} \) | \( 7.09 \times 10^{-6} \) | 0.992 |
| IB, FM | \( 5.40 \times 10^{-4} \) | \( 6.71 \times 10^{-6} \) | 0.934 |

| Source | \( \delta^{(8)}A_0 \) | \( \delta^{(8)}A_2 \) | \( \delta^{(8)}A_0' \) |
|--------|-----------------|-----------------|-----------------|
| (8) | \( -4.11 \pm 1.22 \) - \( (2.89 \pm 0.37)i \) | \( -1.56 \pm 0.63 \) + \( (0.42 \pm 0.05)i \) | \( -1.56 \pm 0.63 \) + \( (0.42 \pm 0.05)i \) |
| (27) | \( -0.28 \pm 0.07 \) - \( (0.08 \pm 0.01)i \) | \( -0.08 \pm 0.05 \) + \( (0.07 \pm 0.01)i \) | \( -0.07 \pm 0.02 \) - \( (0.05 \pm 0.01)i \) |
| (EM) | \( -2.17 \pm 0.50 \) + \( (0.61 \pm 0.02)i \) | \( -1.27 \pm 0.40 \) - \( (1.28 \pm 0.02)i \) | \( 0.70 \pm 0.73 \) - \( (0.07 \pm 0.04)i \) |
| Total | \( -6.56 \pm 1.32 \) - \( (2.36 \pm 0.37)i \) | \( -2.91 \pm 0.75 \) - \( (0.79 \pm 0.05)i \) | \( -0.93 \pm 0.96 \) + \( (0.30 \pm 0.06)i \) |
FIG. 1. Feynman diagrams for $K \rightarrow \pi \pi$ up to $\mathcal{O}(p^4)$ in the chiral expansion. Closed circles represent $\mathcal{O}(p^2)$ strong vertices, open circles $\mathcal{O}(p^4)$ strong vertices, closed boxes $\mathcal{O}(p^2)$ weak vertices, and open boxes $\mathcal{O}(p^4)$ weak vertices. No one-line weak tadpoles occur because, in the weak effective Lagrangian employed, they have already been rotated away. Figures (b) and (c) should be understood to represent collectively the strong dressing on all the external lines.
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