SINGULAR LINK CONCORDANCE IMPLIES LINK HOMOTOPY
IN CODIMENSION $\geq 3$

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We prove the analogue of the Concordance Implies Isotopy in Codimension $\geq 3$ Theorem for link maps, together with some other its singular analogues.

1. INTRODUCTION AND STATEMENT OF RESULTS

An interesting and deep question of geometric topology is to determine under which conditions two given maps from a special class (for example link maps, immersions, embeddings, close embeddings) are connected by a continuous deformation in this class (respectively by link homotopy, regular homotopy, isotopy, small isotopy). Sometimes this problem of map equivalence can be reduced to a problem of existence of (extensions of) maps of the same class. A reduction of isotopy to relative embeddability was achieved in 60s by Smale and Haefliger [10, 1.2], Zeeman [38], Lickorish [23, Th. 6] and Hudson [12], [13], who showed that concordant embeddings are ambient isotopic in codimension $\geq 3$, in smooth and PL categories (see [30], [27], [2, §7] for alternative proofs, and [31, last historic remark], [11, proof of Cor. 1] for typical applications of this reduction). The similar question for link maps (does singular link concordance imply link homotopy in codimension $\geq 3$?) has remained open [21, p. 303].

Throughout this paper, let $X = X_1 \sqcup \cdots \sqcup X_k$ be a compact polyhedron ($X_i$’s are fixed, but not necessarily connected) and $Q$ a PL manifold. A continuous map $f: X \to Q$ is called a (generalized) link map if $fX_i \cap fX_j = \emptyset$ whenever $i \neq j$. A (singular) link concordance – not to be confused with (embedded) concordance – between link maps $f_0, f_1: X \to Q$ is a link map $F: X \times I \to Q \times I$ such that $F(x, i) = (f_i(x, i))$ for $i = 0, 1$ and each $x \in X$. A level-preserving (i.e. such that $F(X \times t) \subset Q \times t$ for each $t \in I$) link concordance is called a link homotopy.

Under link concordance and link homotopy we also mean the equivalence relations they generate on the set of link maps $X \to Q$. In this section all maps are supposed to be continuous unless the contrary is stated. Isotopy means ambient isotopy.

The relation of link homotopy was introduced by Milnor for classical links of circles in $\mathbb{R}^3$ [26]. Scott considered spherical link maps (i.e. of spheres into a sphere) in higher dimensions up to link homotopy [33], and Nezhinskij – up to link concordance (see [28, §1]). Since mid-80s the problem of classification of spherical and generalized link maps up to link homotopy and link concordance has been studied widely (see for instance [4], [9], [16]–[22], [28], [29], [32], [35]).

One may have an intuitive feeling for link maps up to link homotopy as for ‘links modulo knots’, that is, ignoring knotting phenomena and concentrating on
interaction of different components, but it is well also to bear in mind the following. Some embedded links with unknotted components, which are link homotopic to the trivial link, are not isotopic to it (e.g. the Whitehead link or the link on Fig. 1a; see [25, 7.7], [7, §2] for higher dimensional examples). Also, leaving alone the case of link maps $X^n \to Q^m$ with negative codimension $m - n$, observe that there are link maps not link homotopic to embedded links (see Fig. 1b and codimension-two examples in [4], [16], [17, 2.22], [22, §3.4], [25, 7.9]).

Figure 1

It appears that for classical links $S^1 \sqcup \cdots \sqcup S^1 \looparrowright S^3$ in PL category embedded concordance implies link homotopy [5], [6]. Notice that it does not imply isotopy, see Fig. 1a: to obtain an unlinking concordance, make a bridge as indicated by dotted line, isotop one of two appeared circles far from others and glue it up by a disk, cf. [37]. Moreover, singular link concordance implies link homotopy for link maps $S^1 \sqcup \cdots \sqcup S^1 \to S^3$ [24] (see also [8], [36]).

Does link concordance imply link homotopy in general? Koschorke proved that it does for link maps $S^{n_1} \sqcup \cdots \sqcup S^{n_k} \to S^m$, where $n_1 \leq m - 3$ and $n_i \leq \frac{m}{3}m - 1$ for $i > 1$ [20], and for smooth embedded links $S^{n_1} \sqcup \cdots \sqcup S^{n_k} \looparrowright S^m$, where $n_i \leq m - 3$ for all $i$, $n_i \leq l(m - 2) - (n_1 + \cdots + n_k)$ for $i > 1$, and all strict sublinks are link homotopically trivial [21, 8.4c]. For two-component link maps $S^p \sqcup S^q \to S^m$ link concordance was shown to coincide with link homotopy in the range $2p + 2q \leq 3m - 5$ [9]. Teichner announced that link concordance implies link homotopy for spherical link maps in codimension $\geq 2$ [36, Rem. to Lem. 3.1], [21, footnote on p. 303]. Meanwhile Sayakhova obtained an example of link concordant but not link homotopic link maps $S^1 \sqcup S^1 \sqcup S^2 \to S^3$ [32]. In this paper we prove the following

**Theorem 1.1.** Let $X^n = X_1 \sqcup \cdots \sqcup X_k$ be a compact polyhedron, $Q^m$ a PL manifold, $m - n \geq 3$, and $f_0, f_1 : X \to Q$ link maps. If $f_0$ and $f_1$ are link concordant, then they are link homotopic.

Thus in codimension $\geq 3$ any classification of link maps up to link concordance (in particular, ones in [29], [35]) is that up to link homotopy, and any invariant of link homotopy is that of link concordance. In contrast, the sufficiency of link concordance for link homotopy in [9], [20], [21] was obtained as a corollary of completeness and link concordance invariance of certain link homotopy invariants.
To prove Theorem 1.1 we use a different approach, independent on the recent progress in the theory of link maps. The proof goes in PL category, and the tool we use for codimension $\geq 3$ is a version of Zeeman’s ‘sunny collapsing’. Actually, our method develops some ideas from a part [13, 5.1] (see also [12, 9.5]) of Hudson’s proof of the Concordance Implies Isotopy Theorem. However, the proof of 1.1 seems to be the singular analogue of a new proof (to appear in a subsequent paper) of the Concordance Implies Isotopy Theorem rather than of Hudson’s or of either alternative proof mentioned above. Apparently, the ideas of this paper do not suffice to prove 1.1 in codimension 2, where Concordance Implies Isotopy fails. To make a brief introduction into the proof of 1.1, we sketch its idea later in this section.

Given link maps $f_0, f_1: S^{n_1} \sqcup \cdots \sqcup S^{n_k} \to S^m$, where $n_i \leq m - 3$, one defines a (set-valued) connected sum $f_0 \# f_1: S^{n_1} \sqcup \cdots \sqcup S^{n_k} \to S^m$ as follows. Choose a basepoint $a_i$ in each $S^{n_i}$. Push the image of $f_0$ into the southern hemisphere of $S^m$, so that each component $f_0(S^{n_i})$ meets the equator precisely in one point $f_0(a_i)$, and push similarly the image of $f_1$ into the northern hemisphere so that, in addition, $f_1$ agrees with $f_0$ on $a_i$’s (this is easy to achieve using general position). Now define $f_0 \# f_1$ by shrinking the equator of each $S^{n_i}$ onto $f_0(a_i) = f_1(a_i)$ and mapping the southern and northern hemispheres using (shifted) $f_0$ and $f_1$ respectively.

Connected sum induces an operation (also called connected sum) on the set $LM^m_{n_1, \ldots, n_k}$ of link homotopy classes of link maps $S^{n_1} \sqcup \cdots \sqcup S^{n_k} \to S^m$ (by picking up arbitrary representatives of link homotopy classes). This operation is known to be well-defined and single-valued when $n_i \leq m - 3$ [19, 25, 33]. Generally, this is not the case in codimension two [22, Fig. 4.1], except for some special situations, such as two-component [17, 2.3] and base-point preserving [19, 1.4] link maps (although in the latter case connected sum fails to be commutative [19, 3.12]).

A reflection ($-f$) of a link map $f: S^{n_1} \sqcup \cdots \sqcup S^{n_k} \to S^m$ is the composition $R \circ f \circ r$, where $r$ and $R$ are reflections of $S^{n_i}$’s and of $S^m$ in their equators. Clearly, reflections of link homotopic link maps are link homotopic (by the ‘reflected’ link homotopy). Thus, reflection induces a well-defined operation (also called reflection) on $LM^m_{n_1, \ldots, n_k}$.

**Corollary 1.2.** Suppose that $n_i \leq m - 3$ for each $i = 1, \ldots, k$. Then connected sum (regarded as addition) and reflection (regarded as inverse) generate abelian group structure on $LM^m_{n_1, \ldots, n_k}$.

In fact, a statement similar to 1.2 can be found in [33]. But, as it was pointed out in [17, 2.4], the proof of this statement had contained a gap: it had been actually shown that reflection is inverse up to link concordance, not up to link homotopy. Theorem 1.1 covers this gap; the rest of the proof of 1.2 goes as in [33], [19, 1.4] (see also [10], [17, §2], [18, 1.19], [23, Th. 7], [25, 5.2], [28]).

Corollary 1.2 generalizes [17, 2.3], [18, 1.19], [19, 1.4+1.7] and answers [9, Question 1] in the case of codimension $\geq 3$. Compare 1.2 to group structures on link maps $S^p \sqcup S^q \to S^m$ up to link concordance [28], on embedded links up to link homotopy [25], on smooth embeddings of spheres up to diffeotopy [10].

Analogues of link maps, called ‘doodles’, were introduced by Fenn and Taylor in ‘search for a method of cancelling triple points’ [3] (see also [15]). We call a continuous map $f: X_1 \sqcup \cdots \sqcup X_k \to Q$ a (generalized) doodle if $fX_i \cap fX_j \cap fX_k = \emptyset$ for any distinct $i, j, k$. In the obvious way one can define doodle concordance and doodle homotopy (or, in terms of [3], cobordism and isotopy of doodles). See [3] for examples of doodles $S^1 \sqcup \cdots \sqcup S^1 \to \mathbb{R}^2$, in particular, of a doodle which is doodle
concordant but not doodle homotopic to the trivial doodle. In codimension $\geq 3$, our method works to prove the following

**Theorem 1.3.** Let $X^n = X_1 \sqcup \cdots \sqcup X_k$ be a compact polyhedron, $Q^m$ a PL manifold, and $f_0, f_1: X \to Q$ doodles. If $f_0$ and $f_1$ are doodle concordant, then they are doodle homotopic, provided $m - n \geq 3$.

One can define $l$-doodle to be a map $f: X_1 \sqcup \cdots \sqcup X_k \to Q$ such that images of any $l$ distinct $X_i$'s have no point in common. Then '$l$-doodle concordance implies $l$-doodle homotopy' theorem is stated and proved analogously. We prove the following more general statement on making a map preserve levels almost without introducing new singularities:

**Theorem 1.4.** Let $X^n$ be a compact polyhedron, $Q^m$ a PL manifold, $m - n \geq 3$, and $F: X \times I \to Q \times I$ a map such that $F(X \times i) \subset Q \times i$, $i = 0, 1$. For each $\varepsilon > 0$ there is a level-preserving map $\Phi: X \times I \to Q \times I$ such that $\Phi(x, i) = F(x, i)$ for $i = 0, 1$ and each $x \in X$, and such that for any $x_1, \ldots, x_l \in X$ the following holds: $\Phi(x_1 \times I) \cap \cdots \cap \Phi(x_l \times I) \neq \emptyset$ only if $F(N_{\varepsilon}(x_1) \times I) \cap \cdots \cap F(N_{\varepsilon}(x_l) \times I) \neq \emptyset$.

Here $N_{\varepsilon}(x)$ denotes the $\varepsilon$-neighborhood of $x$ in some fixed metric on $X$. Theorem 1.4 is proved in essentially the same ideas as its partial cases 1.2 and 1.3. Notice that the condition on $\Phi$ in 1.4 is equivalent to the requirement that, for each positive integer $l$, the ‘singular set’ of $\Phi$ lies in an arbitrarily small neighborhood of the ‘singular set’ of $F$, both ‘singular sets’ being considered as subsets of the configuration space $X \times \cdots \times X$ of $l$-tuples of generators $x \times I$. The ‘singular set’ of $F$ always contains the diagonal of $X \times \cdots \times X$, hence even if $F$ is an embedding, $\Phi$ may be not an embedding. Therefore the (PL/smooth) Concordance Implies Isotopy Theorem is not a partial case of (the PL/smooth version of) Theorem 1.4. Instead, we have Corollary 1.5 below, which can be thought of as the singular version of Concordance Implies Isotopy.

It is said that $X$ is quasi-embeddable in $Q$, if for each $\varepsilon > 0$ there exists an $\varepsilon$-map $f_\varepsilon: X \to Q$, i.e. such map that $\text{diam } f_\varepsilon^{-1}(q) < \varepsilon$ for each $q \in f(X)$. Quasi-embeddability and embeddability of $X^n$ into $Q^m$ are equivalent in case $m \geq \frac{3(n+1)}{2}$ by [11, Th. 1], and distinct for all pairs $(m, n)$ such that $3 < m < \frac{3(n+1)}{2}$, see [34].

**Corollary 1.5.** Let $X^n$ be a compact polyhedron, $Q^m$ a PL manifold, $m - n \geq 3$, and $f_0, f_1: X \to Q$ embeddings. If $f_0$ and $f_1$ are quasi-concordant, then they are quasi-isotopic.

Any map $f: X \to Q$ is homotopic, by an arbitrarily small homotopy, to a PL map and, moreover, the homotopy can be chosen to fix any subpolyhedron $Y$ such that $f|_Y$ is PL [39]. Therefore link concordant link maps $f_0, f_1: X \to Q$ are link homotopic to PL link concordant PL link maps $f'_0, f'_1: X \to Q$. Thus Theorem 1.1 is reduced to its PL version (analogously for Theorems 1.3, 1.4), and what we prove below is these PL versions. Actually, 1.1 – 1.5 are equivalent to their PL versions (for level-preserving approximations can be made) and to their smooth versions (for similar approximations in smooth category can be made).
Let us outline the idea of proof of Theorem 1.1. Let \( F: X \times I \to Q \times I \) be the given link concordance between \( f_0 \) and \( f_1 \). Suppose that sun emits its rays along the \( I \)-fibers of \( Q \times I \), upside down. The idea is to (singularly) reparametrize the product \( X \times I \) by means of a map \( H: X \times I \to X \times I \) (fixing \( X \times \{0,1\} \), but not necessarily fiber-preserving or homeomorphism), so that the \( F \)-image of each ‘fiber’ \( H(X \times t) \) is not self-overshadowing (although possibly self-intersecting). Then simple vertical shifting of each \( F \circ H(X \times t) \) into \( Q \times t \) introduces no new singularities, while the levels get preserved under shifted \( F \circ H \) (compare this to a proof of link concordance invariance of the \( \alpha \)-invariant in [35]).

To obtain such reparametrization, notice first that without loss of generality \( F \) is PL and by general position (§2) we can also assume that the set \( S \) of points in \( X \times I \), which \( F \)-images lie in the same sunray with some other ones, is of codimension \( \geq 2 \) in \( X \times I \). Hence \( Y \times I \setminus S \) is connected for each connected component \( Y \) of \( X \), and it follows (§3) that we can collapse \( X \times I \) onto \( X \times 0 \) so that the simplexes of \( S \) are collapsed in any order of decreasing dimension, in particular, in the order their \( F \)-images overshadow each other. In other words, there is a collapse \( X \times I \searrow X \times 0 \) with the following ‘sunny’ property: ‘\( F(a) \) overshadows \( F(b) \)’ implies ‘\( a \) is collapsed before \( b \)’ for any points \( a, b \in X \times I \). By a special care (§4; see remark in the next paragraph) this collapse can be improved to satisfy the following ‘stable sunny’ condition: ‘\( F(a) \) overshadows \( F(b) \)’ implies ‘\( a \) is collapsed before a neighborhood of \( b \)’ for any points \( a, b \in X \times I \). This slight improvement is, however, the key step. Indeed, by it we achieve that the \( F \)-image of the frontier (in \( X \times I \)) of points, already collapsed at the moment, is not self-overshadowing (for each moment \( t \in I \) during the collapse). Thus the mapping of \( X \times I \) onto itself, defined by \( (x, t) \mapsto \) ‘the \( t \)-moment image of \( x \times 1 \) under the collapse’ (which lies in the \( t \)-moment frontier), is a proper reparametrization.

Actually, only (allowable) new self-intersections of each component \( F \circ H(X_i \times t) \) can appear, if we weaken the conditions of reparametrization and of stable sunny collapse so that shadows, casted by each \( F(X_i \times I) \) onto itself, are not taken into account. To simplify the proof, we obtain only these weakened properties. The same method works to prove 1.3, while in the proof of 1.5 we do not take into account the shadows between sufficiently close points only.

Hereafter we assume all spaces to be polyhedra and all maps to be piecewise-linear, unless the contrary is stated. We follow [31] for the PL notation. We denote simplicial complexes and their bodies by the same letters.

2. General position

**Definition.** We call \( S(f) = \{ a \in A \mid f^{-1}f(a) \neq a \} \) the **singular set** of a PL map \( f: A \to B \) (not to be confused with the ‘singular set’ in the configuration space, mentioned in §1; hereafter we use only the last definition). A subpolyhedron \( L \) of a polyhedron \( K \) is **locally of codimension** \( \geq k \) in \( K \), if every \( n \)-simplex of \( L \) faces some \((n+k)\)-simplex of \( K \) for any triangulation of the pair \((K,L)\) (cf. [23]). Let us say that a map \( F: X \times I \to Q \times I \) is a **general position map** with respect to the projections \( P: Q \times I \to Q \), \( p: X \times I \to X \) if

1) \( S(P \circ F) \) is locally of codimension \( \geq m - n - 1 \) in \( X \times I \), and

2) \( p|_{S(P \circ F)} \) is non-degenerate.

**Lemma 2.1.** Let \( X^n \) be a compact polyhedron, \( Q^m \) a PL manifold, \( m - n \geq 2 \) and \( F: X \times I \to Q \times I \) a PL map. For each \( \varepsilon > 0 \), \( F \) is \( \varepsilon \)-homotopic to a PL general
position map $\tilde{F}: X \times I \to Q \times I$ such that $\tilde{F}^{-1}(Q \times t) = F^{-1}(Q \times t)$ for each $t \in I$.

Proof. For each point $p \in X \times I$, we fix the $I$-coordinate of $F(p)$ and change its $Q$-coordinate, following Bing [1, proof of 2.1] and assuming the following remarks. The property (2), although not included into the statement of [1, 2.1], was actually achieved in its proof (see [1, 2.5]). The additional restriction $m - n \geq 3$ in [1, 2.1] was used only to prove some properties additional to (1) and (2), hence it can be omitted here. (However, in the sequel we use only the codimension $\geq 3$ case of Lemma 2.1.) □

The analogous to 2.1 result for embedding $F$, with condition (2) dropped, was obtained in [14, Lem. 1] (compare to [13, 5.2]).

3. Sunny collapsing

Definition. Let us think of the second factor of $Q \times I$ as of heigh (that is, a point $(q_1,t_1)$ lies below a point $(q_2,t_2)$ if $q_1 = q_2$ and $t_1 < t_2$). If $V \subset Q \times I$, let $sh V$ denote shadow of $V$, the set of points of $Q \times I$ lying (strictly) below some point of $V$. We say that a collapse $V \searrow W$ in $Q \times I$ is a simple sunny collapse, if no point of $V \setminus W$ lies in $sh V$ (equivalently, $V \cap sh V \subset W$). A sequence of simple sunny collapses is called a sunny collapse (this concept is due to Zeeman [38]). It is convenient to generalize this for the case of any map $F: K \to Q \times I$, $K$ being a polyhedron. For any $V \subset K$ define $F$-shadow of $V$ by $sh_F V = F^{-1}(sh F(V))$. A collapse $V \searrow W$ in $K$ is called a simple $F$-sunny collapse if $V \cap sh_F \subset W$, and an $F$-sunny collapse if it is a sequence of simple $F$-sunny ones.

Lemma 3.1. Let $X^n$ be a compact polyhedron, $Q^m$ a PL manifold and $m - n \geq 3$. If $F: X \times I \to Q \times I$ is a PL general position map such that $F(X \times 0) \subset Q \times 0$, then there is an $F$-sunny collapse $X \times I \searrow X \times 0$.

For the case of embedding $F$ this was proved in [14, Lem. 2] (the condition $F(X \times 0) \subset Q \times 0$ was erroneously omitted in the statement). The proof goes in general case with only minor changes; nevertheless for completeness we sketch it below (see also [38, proof of Lem. 9] for detailed proof of a similar statement).

Proof. Assume that Lemma 3.1 is true when $\dim X < n$ and prove it for $\dim X = n$ (if $n = 0$ then $S(P \circ F) = \emptyset$ and any collapse $\{x_1, ..., x_m\} \times I \searrow \{x_1, ..., x_m\} \times 0$ is $F$-sunny, even simple). Triangulate $X \times I$ and $Q \times I$ so that $F$ and the projections $P: Q \times I \to Q$, $p: X \times I \to X$ are simplicial. Let $Y$ be the $(n - 1)$-skeleton of $X$ and $S = S(P \circ F)$. By general position $\dim S \leq n - 1$, hence $S \subset Y \times I$. Notice that although the cylindric collapse $X \times I \searrow X \times 0 \cup Y \times I$ is $F$-sunny (even simple) and decreases the dimension, it, however, increases by 1 the codimension of $S$ and the inductive step can not be applied. To overcome this, we should arrange some device for collapsing away the top-dimensional simplexes of $S$.

Let $A_i$ be a simplex of $S$. Since $p|_S$ is simplicial and non-degenerate, there is a unique simplex $B$ of $Y$ such that $\text{Int} A_i \subset (\text{Int} B) \times I$. Since $A_i$ is locally of codimension $\geq 2$ in $X \times I$, $B$ faces some $n$-simplex, say $C$, of $X$. Let $a$ be the barycenter of $A_i$. Suppose that $A_i \not\subset X \times 0$ and define some points near $a$. If $A_i \subset X \times \text{Int} I$, choose $a_\uparrow$ directly above $a$ (with respect to $p$), $a_\downarrow$ directly below $a$ and $a_{\rightarrow}$ in $\text{Int}(C \times I)$. If $A_i \subset X \times 1$, let $a_\uparrow = a$, choose $a_\downarrow$ directly below $a$ and $a_{\rightarrow}$ in $\text{Int} C$. Define a blister $J_i = a_\uparrow * a_\downarrow * a_{\rightarrow} * \partial A_i$ over $A_i$ (here ‘*’ means join), its bad face $K_i = a_\uparrow * a_\downarrow * \partial A_i$ and its good face $L_i = (a_\uparrow \cup a_\downarrow) * a_{\rightarrow} * \partial A_i$. For
each simplex $A_i$ of $S$ construct $J_i$ in so small neighborhood of $A_i$, that each blister
meets $Y \times I$ only in its bad face and $J_i \cap J_j = \partial A_i \cap \partial A_j$ for each $i \neq j$. Let $J, K$
and $L$ be the unions of all $J_i, K_i$ and $L_i$, respectively. Notice that enough room to
construct blisters is due to local codimension $\geq 2$ of $S$.

Since the blisters are small, we can collapse all the top-dimensional prisms, but
leaving the blisters. This gives an $F$-sunny collapse $X \times I \searrow X \times 0 \cup Y \times I \cup J$.
Now each blister $J_i$ has the bad face $K_i$ as a free face, which we may collapse
it from. Therefore the blisters $J_i$ can be collapsed onto their good faces in any
order, particularly, in the order the corresponding $A_i$’s $F$-overshadow each other.

Although we omitted simplexes, lying in $X \times 0$, the condition $F(X \times 0) \subset Q \times 0$
implies that they $F$-overshadow nothing. Hence the obtained collapse $J \searrow L$ or,
equivalently, $Y \times I \cup J \searrow (Y \times I \setminus K) \cup L = Z$, is $F$-sunny. Thus there is an $F$-sunny
collapse $X \times I \searrow Z \cup X \times 0$. □

4. Lagging collapse

Definition. Let $F: K \to Q \times I$ be a map. A collapse $V \searrow W$ in $K$ is a simple
stable $F$-sunny collapse if $V \setminus \text{Int} W$ is not $F$-overshadowed by $V$ (or, equivalently,
$V \cap \text{sh}_F V \subset \text{Int} W$; here ‘Int’ denotes topological interior in $K$). Define stable
$F$-sunny collapse to be a sequence of simple stable collapses. If $K = K_1 \sqcup \cdots \sqcup K_l$,
let $F$-link-shadow of $V \subset K_i$, denoted $\text{ls}_{F} V$, be the union of $(\text{sh}_F (V \cap K_i)) \setminus K_i$
for all $i = 1, \ldots, l$ (speaking informally, link-shadow is the analogue of shadow for
the case when each $F(K_j)$ is ‘visible’ from $F(K_j)$, $j \neq i$, but ‘transparent’ from

Figure 2

By mapping $a \mapsto a \_\_\,$ for each $A_i$ and connecting linearly with $\text{id}|_{Z \cap Y \times I}$ we
obtain a homeomorphism $\varphi: Y \times I \to Z$. Notice that $Z$ meets its own $F$-shadow in
$S' = S \setminus X \times 0$ which is, by the construction of blisters, locally of codimension
$\geq 2$ in $Z$. Also, $p|_{S'} = p|_{\varphi^{-1} S'}$ is non-degenerate. Hence $F \circ \varphi: Y \times I \to Q \times I$
is in general position, and by the inductive hypothesis $Y \times I$ collapses $(F \circ \varphi)$-sunny
onto $Y \times 0$. Since $\varphi$ is a homeomorphism, $Z$ collapses $F$-sunny onto $Y \times 0$, hence
we obtain an $F$-sunny collapse $X \times I \searrow Z \cup X \times 0 \searrow X \times 0$. □
F(K1)). Define (simple) (stable) F-link-sunny collapse using F-link-shadow instead of F-shadow. Evidently, any stable F-(link-)sunny collapse is F-(link-)sunny, and any (stable) F-sunny collapse is (stable) F-link-sunny, but not vice versa.

**Lemma 4.1.** Let X be a compact polyhedron, Q a PL manifold and F: X × I → Q × I a PL link map such that F(X × 0) ⊂ Q × 0. If there is an F-link-sunny collapse X × I ̸→ X × 0, then there is a stable F-link-sunny collapse X × I ̸→ X × 0.

**Sublemma 4.2.** (compare to [12, Claim on p. 188], [13, Lem. 5.3]) Let K = K1 ∪ · · · ∪ Kn be a finite simplicial complex, Q a combinatorial manifold, Q × I be triangulated so that the projection P: Q × I → Q is simplicial, and F: K → Q × I be a simplicial link map. Then there is a second derived subdivision K′′ of K such that lshF V ⊂ W implies lshF NKn(V) ⊂ Int NKn(W) for any subcomplexes V and W of K.

Here NKn(L) denotes (for any subcomplex L of K) the second derived neighborhood of L in K, which is the simplicial neighborhood of L′′ in K′′.

**Proof of Sublemma 4.2.** Let K′ be the barycentrically derived subdivision of K and let us construct a derived subdivision K′′ of K′. Let A1, . . . , Ap be the simplexes of K′, arranged in an order of increasing dimension. Assuming that A′′ i is already defined for each i < j, define A′′ j as follows. Let aj be the barycentre of Ai and Π: Q × I → I the horizontal projection. Define a map fAj : Aj → Π by ∂Aj ց −1, aj ց Π ∩ F(aj) + 1/2 and extending linearly (1/2 can be replaced by any fixed positive number). For any (already defined) derivation point b of a simplex B of (∂Aj)′, define a derivation point of aj * B to be (aj * b) ∩ f−1 Aj(0).

Any point x ∈ K is contained in the interior of a unique simplex Cx of K. If c x is the barycentre of Cx and x ̸= c x, there is a unique point px ∈ ∂Cx such that x ∈ px * c x. Notice that for any subcomplex L of K the following criterion holds: x ∈ NKn(L) \ L (respectively x ∈ Int NKn(L) \ L) if and only if x ̸∈ L, px ∈ NKn(L) (resp. px ∈ Int NKn(L)) and fCx(x) ≤ 0 (resp. fCx(x) < 0).

Suppose that a point y ∈ K is F-overshadows a point z ∈ Kj, i ̸= j, and prove the statement: ‘y ∈ NKn(V) implies z ∈ Int NKn(W)’ by induction on dim Cy. If dim Cy = 0, y is a vertex of K. Hence y ∈ NKn(V) implies y ∈ V, consequently z ∈ W ⊂ Int NKn(W). Let us prove the inductive step. Since P, F are simplicial, F(cy) overshadows F(cy) and F(px) either overshadows or equals F(px); since i ̸= j and F is a link map, F(px) ̸= F(px). We can assume that y ̸∈ V, then y ∈ NKn(V) implies fCx(y) ≤ 0. Since fCx(cy) < fCx(cy), we have that fCx(z) < fCx(y), consequently fCx(z) < 0. Also y ∈ NKn(V) implies px ∈ NKn(V), hence by the inductive hypothesis px ̸∈ Int NKn(W). Thus, if z ̸∈ W, we obtain z ∈ Int NKn(W) \ W. □

**Sublemma 4.3.** If V simplicially collapses onto W in a finite simplicial complex K, then NKn(V) collapses onto NKn(W).

**Proof.** We will need the following simple observation. Suppose that a simplex A (strictly) faces a simplex B. Since ball collapses to its face,

\[ N_{B′′}(A) \searrow N_{B′′}(A) \cup N_{B′′}(\partial A). \]  

(*)

Without loss of generality V \ W is a single simplex, say C. Let D be its free face, i.e. ∂C \ W. Applying (*) to A = C and B running over the faced by C simplexes of K, in order of decreasing dimension, we obtain NKn(V) \ V ⊂ NKn(W \ D).
Applying (**) to $A = D$ and $B$ running over the faced by $D$ simplexes of $K$ except for $C$, in order of decreasing dimension, we obtain $V \cup N_{K''}(W \cup D) \searrow V \cup N_{K''}(W)$. Finally, since ball collapses to its face, $V \cup N_{K''}(W) \searrow N_{K''}(W)$. □
Addendum 6.2 to Lemma 3.1. For each collapse \( X \times I \) there is a stable \( \mathcal{V} \)-homotopy using Lemma 6.1. Triangulate \( X \times I \) so that \( K_1, \ldots, K_m \) are its subcomplexes, and both \( P \) and \( F \) are simplicial in some triangulation of \( Q \times I \). Let \( U_i = N_{(X \times I)'}(K_i), i = 0, \ldots, m, \) where the subdivision \( (X \times I)' \) is yielded by 4.2. Then \( \text{lsf} U_i \subset \text{Int} U_{i+1} \) (hereafter ‘Int’ means topological interior in \( X \times I \)). Furthermore, \( F(X \times 0) \subset Q \times 0 \) implies that \( N_{(X \times I)'}(X_j \times 0) \) can \( F \)-overshadow only the points of \( X_j \times I \). Hence the \( F \)-link-shadow of \( U_m \) is empty. By 4.3, \( U_i \subset U_{i+1} \), and since \( U_m \) is a regular neighborhood of \( X \times 0 \) (or by an argument analogous to the proof of 4.3), \( U_m \setminus X \times 0 \). Thus we obtain a stable \( F \)-link-sunny collapse \( X \times I = U_0 \setminus \ldots \setminus U_m \setminus X \times 0. \)

5. PROOFS OF 1.1 AND 1.3

Proof of Theorem 1.1. Let \( F: X \times I \to Q \times I \) be the given link concordance between \( f_0 \) and \( f_1 \). Without loss of generality \( F \) is PL and, by 2.1, in general position. Then 3.1 and 4.1 yield a stable \( F \)-link-sunny collapse \( X \times I \setminus X \times 0 \). This collapse gives a homotopy \( h_t: X \times I \to X \times I \) (obtained by retracting, for each elementary collapse \( V \setminus W \), of the ball \( V \setminus W \) onto its face \( V \setminus W \cap W \)). Conversely, for each \( t \in I \) the map \( h_t \) corresponds to some simple stable \( F \)-link-sunny collapse \( V \setminus W \). Then for any \( Y \subset X \times I \) we have \( h_t(Y) \subset V \) and \( h_t(Y) \cap \text{Int} W = Y \cap \text{Int} W \). Consequently, \( h_t(X \times 1) \subset V \cup ((\text{Int} W) \setminus X \times 1) \). On the other hand, since \( V \setminus W \) is a simple stable \( F \)-link-sunny collapse and \( F(X \times 1) \subset Q \times 1 \), we have that \( \text{lsf} F(V) \subset (\text{Int} W) \setminus X \times 1 \). Thus \( h_t(X \times 1) \) does not \( F \)-link-overshadow itself.

Define \( H: X \times I \to X \times I \) by \( H(x,t) = h_{2t}(x \times 1) \) for \( t \in [0,1] \) and \( H(x,t) = p \circ h_{2t}(x \times 1) \) for \( t \in [\frac{1}{2},1] \) (where \( p: X \times I \to X \) is the projection). Then \( H \) fixes \( X \times \{0,1\} \), while \( H(X \times t) \) does not \( F \)-link-overshadow itself. In other words, \( F \circ H(X_i \times t) \) and \( F \circ H(X_j \times t) \) do not overshadow each other whenever \( i \neq j \). Since \( F \) is a link map, they do not intersect as well. Hence a map \( \Phi: X \times I \to Q \times I \), defined by \( \Phi(x,t) = (P \circ F \circ H(x,t),t) \), is a link homotopy between \( f_0 \) and \( f_1 \).

Proof of Theorem 1.3. Using the above notation, let us show that if \( F \) is a doodle concordance, then \( \Phi \) is actually a doodle homotopy. By the above, \( F \circ H(X_i \times t) \) and \( F \circ H(X_j \times t) \) do not overshadow each other for any distinct \( i,j \). Since \( F \) is a doodle, they do not meet as well. Therefore \( \Phi(X_i \times t) \) does not meet \( \Phi(X_j \times t) \cap \Phi(X_{j_2} \times t) \). It follows that \( \Phi \) is a doodle homotopy.

6. PROOF OF 1.4

Definition. Let \( K \) be a polyhedron and \( F: K \to Q \times I \) a map. Define \( F_{\varepsilon} \)-shadow of \( V \subset K \) to be \( \text{sh}_F(V) \setminus N_{\varepsilon}(F^{-1}F(V)) \) and define (simple) (stable) \( F_{\varepsilon} \)-sunny collapse using \( F_{\varepsilon} \)-shadow instead of \( F \)-shadow. The proof of the following lemma is analogous to that of Lemma 4.1.

Lemma 6.1. For any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that the following holds. Let \( X \) be a compact polyhedron, \( Q \) a PL manifold and \( F: X \times I \to Q \times I \) a PL map such that \( F(X \times 0) \subset Q \times 0 \). If there is an \( F_{\delta} \)-sunny collapse \( X \times I \setminus X \times 0 \), then there is a stable \( F_{\varepsilon} \)-sunny collapse \( X \times I \setminus X \times 0 \). □

The below statements follow immediately from the proofs of 3.1 and 4.1.

Addendum 6.2 to Lemma 3.1. For each \( \varepsilon > 0 \) we can choose the \( F \)-sunny collapse \( X \times I \setminus X \times 0 \) so that the trace of \( x \times I \) lies in \( N_{\varepsilon}(x) \times I \) for any \( x \in X \). □
Addendum 6.3 to Lemma 6.1. The stable $F$-$\varepsilon$-sunny collapse $X \times I \searrow X \times 0$ can be chosen so that the trace of any point $p \in X \times I$ under this collapse lies in the $\varepsilon$-neighborhood of that under the given collapse. □

Proof of Theorem 1.4. Without loss of generality the given map $F: X \times I \to Q \times I$ is a PL general position map. Apply (3.1+6.2) and (6.1+6.3) to obtain a stable $F^{\frac{\varepsilon}{2}}$-sunny collapse $X \times I \searrow X \times 0$ such that the trace of $x \times I$ lies in $N_{\varepsilon/2}(x) \times I$ for any $x \in X$. Analogously to the proof of 1.1 we obtain a map $H: X \times I \to X \times I$ fixing $X \times \{0,1\}$ and such that

1) $H(x \times I) \subset N_{\varepsilon/2}(x) \times I$ for each $x \in X$, and

2) $X \times t$ does not $(F \circ H)^{-\frac{\varepsilon}{2}}$-overshadow itself for each $t \in I$.

Let us prove that $\Phi: X \times I \to Q \times I$, defined by $\Phi(x,t) = (P \circ F \circ H(x,t), t)$, is the required map. Indeed, suppose that $\Phi(x_1 \times I) \cap \cdots \cap \Phi(x_l \times I) \neq \emptyset$ for some $x_1, \ldots, x_l \in X$. Since $\Phi$ is level-preserving, $\Phi(x_1 \times t) = \cdots = \Phi(x_l \times t)$ for some $t \in I$. Then $F \circ H(x_1 \times t), \ldots, F \circ H(x_l \times t)$ lie in the same vertical line in $Q \times I$. By (2) $F \circ H(N_{\varepsilon/2}(x_1) \times t) \cap \cdots \cap F \circ H(N_{\varepsilon/2}(x_l) \times t) \neq \emptyset$. By (1) $F \circ H(x_i \times I) \subset F(N_{\varepsilon/2}(x_i) \times I)$ for $i = 1, \ldots, l$. Hence $F(N_{\varepsilon}(x_1) \times I) \cap \cdots \cap F(N_{\varepsilon}(x_l) \times I) \neq \emptyset$. □

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This preprint dates from 1998, when it was privately circulated and publicly presented (Pontryagin’s 90th Anniversary Conference, Moscow, September 1998; Rokhlin Topology Seminar of V. M. Nezhinsky, St.-Petersburg, March 1998; Geometric Topology Seminar of E. V. Shchepin, Moscow, Fall 1997). Being a second year undergraduate student at the time, I followed the advice by A. Skopenkov to submit the preprint to the journal Topology — where it was received in April 1998 and rejected in September 2000.

The results of this preprint were announced in Uspekhi Mat. Nauk 55:3 (2000), 183–184 (English transl.: Russ. Math. Surv. 55 (2000), 589–590). A more elaborate version of the main construction (the “lagging collapse”), which reproves the classical Concordance Implies Isotopy Theorem and proves its controlled version, appeared in Topol. Appl. 120 (2002), 105–156; arXiv:math.GT/0101047 (§5). Given that, and the fact that Theorem 1.1 is far from being optimal (as compared to the results of X.-S. Lin and P. Teichner) I was no longer keen to publicize the present preprint in its extant form. However, the referee of another paper, which uses Theorem 1.3, leaves me no choice.

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