Asymptotically optimal strategies in a diffusion approximation of a repeated betting game

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Abstract

We construct a diffusion approximation of a repeated game in which agents make bets on outcomes of i.i.d. random vectors and their strategies are close to an asymptotically optimal strategy. This model can be interpreted as trading in an asset market with short-lived assets. We obtain sufficient conditions for a strategy to maintain a strictly positive share of total wealth over the infinite time horizon. For the game with two players, we find necessary and sufficient conditions for the wealth share process to be transient or recurrent in this model, and also in its generalization with Markovian regime switching.

Keywords: repeated betting, diffusion approximation, asymptotic optimality, survival strategies, capital growth, regime switching.

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1. Introduction

In the present paper, we consider a dynamic game-theoretic model in which agents make bets on outcomes of random events or random variables. Investigation of this model is motivated by applications in analysis of asymptotic performance of investment strategies in a multi-agent financial market. The main aim of the paper is to construct and study a continuous-time approximation of the model which arises when all agents make “almost optimal” bets.

To facilitate the exposition, let us begin with an example. Let (Ω, F, P) be a probability space and {A^1_i, . . . , A^N_i}, i = 1, 2, . . ., a sequence of independent partitions of Ω into N random events. Assume that the probabilities P(A^p_i) are equal for all i. Suppose that at each time i ≥ 0, an agent bets a proportion λ_i ∈ [0, 1] of her capital on the occurrence of A^p_{i+1}, where λ_1 + . . . + λ_N = 1 (for further analysis it is important that the whole capital is bet). At time i + 1 the pool is divided between the winning bets proportionally to their sizes, changing the distribution of capital between the agents. We are interested in determining who of the agents will have more wealth in the long run. In this paper, we consider only fixed-mix (constant) strategies which are given exogenously; in particular, they need not to form a Nash equilibrium.

In the above example, it is known that as i → ∞ the entire market wealth will be held by the agent whose strategy has the smallest Kullback–Leibler divergence D(p∥λ) = \sum_n p_n \ln(p_n/λ_n) from the distribution p = (p_1, . . . , p_N), p_n = P(A^p_n) (Blume and Easley,∗Steklov Mathematical Institute of the Russian Academy of Sciences. 8 Gubkina St., Moscow, Russia. Email: mikhailzh@mi-ras.ru.
This model has a natural interpretation of an asset market consisting of $N$ Arrow securities (instruments with unit payoffs in only one random state) with endogenous prices determined from one-period equilibrium of variable asset demand, which depends on agents’ strategies, and fixed asset supply, see Remark 1. Many generalizations and extensions of this model have been obtained in the literature. Among works in this direction which are closely related to the material of the present paper, let us mention the paper of Evstigneev et al. (2002), in which Arrow securities are replaced with assets that make random i.i.d. payoffs simultaneously. Further extensions were obtained by Amir et al. (2005) who proved similar results for a model with Markov payoffs, and Amir et al. (2013) who considered general payoff sequences. The main results of the mentioned papers consist in proving the existence of an “unbeatable” strategy which allows an agent to survive in the market in the sense of maintaining a share of the total market wealth strictly bounded away from zero over the infinite time horizon. Under some additional conditions, such an agent turns out to be a single survivor and accumulates in the limit the entire market wealth.

If no agent uses this optimal strategy, there might be several survivors, even if some agent uses a strategy which is strictly closer to the optimal strategy than the strategies of the other agents. This possibility depends in an essential way on whether a market is complete or incomplete. In a complete market there is always a single survivor, except some uninteresting cases (the above example with independent partitions is a complete market). A simple example of coexistence of surviving agents in an incomplete market is provided by Evstigneev et al. (2009, Ch. 9.3.3). Let us also mention the works of Bottazzi and Dindo (2014); Bottazzi and Giachini (2017, 2019) who studied survival and coexistence in a related setting but with agents’ strategies depending on one-step equilibrium asset prices. There is also a large number of results on selection of agents by market forces in the framework of general equilibrium, see, for example, Blume and Easley (2006); Sandroni (2000) and references therein.

In the present paper we are interested in conditions for survival of agents with fixed-mix strategies in a general (incomplete) market model, and focus on the situation when strategies of agents are close to an optimal strategy. The closeness is understood in the sense that we consider series of models in which agents’ strategies converge to the optimal strategy. This allows to approximate the dynamics of the model by a system of stochastic differential equations and investigate the solution of this system. From the point of view of economic modeling, such an approximation is reasonable, since in the long run we can leave out agents who make “less correct” predictions as their share in the market wealth and influence on the dynamics of the model will diminish with time. Although we do not obtain formal mathematical conditions when an agent can be left out from the model, let us mention that this idea is known in economics since long ago, see, e.g., Alchian (1950) (however later studies show that it is not always applicable, see, e.g., Blume and Easley (2006); De Long et al. (1990)).

Analytically, our approximation has an advantage over the pre-limit discrete-time models, since it is easier to work with an SDE rather than a recursive sequence defining the dynamics in discrete time. In particular, this approximation becomes especially convenient in the case of two agents and allows to thoroughly analyze the asymptotic behavior of the wealth process.

The main results of the paper are as follows. First we prove the convergence of the discrete-time model to the continuous-time model driven by a system of SDEs. Then we obtain sufficient conditions for an agent to dominate or survive in the continuous-time model. By survival we mean that the limit superior of her share of total market wealth
is strictly positive with probability 1 as time goes to infinity. By dominance we mean that the limit of the share of wealth is 1, i.e. this agent is a single survivor. These conditions are obtained for the model with arbitrary number of agents. When there are only two agents, we can go further and provide necessary and sufficient conditions for survival and dominance, and, in the case when both of the agents survive, show that the process of the share of wealth is recurrent, determine when it is null or positive recurrent and find the ergodic distribution. The latter result has a tight link with the stochastic replicator equation of Fudenberg and Harris (1992).

The paper is organized as follows. In Section 2, we describe the discrete-time model and recall the main results known for it in the literature. In Section 3, we consider series of discrete-time models and pass to the limit obtaining a continuous-time model driven by a system of stochastic differential equations. Section 4 contains the main results about asymptotic performance of agents’ strategies. We first consider the case of many agents, and then refine the obtained results in the case of two agents. Illustrations and numerical examples are provided in Section 5. In Section 6, we study an extension of the two-agent case in which the market is modeled by the same SDE but with switching between two regimes. The Appendix contains a theorem on convergence in distribution of a discrete-time sequence to a diffusion process in a form convenient for our purposes.

2. A discrete-time model

Let \((\Omega, \mathcal{F}, P)\) be a probability space on which all random variables will be defined. Equalities and inequalities for random variables will be understood to hold with probability 1, unless else is stated.

There are \(M \geq 2\) agents and \(N \geq 2\) assets in the model. The time is discrete, \(i = 0, 1, 2, \ldots\) At each moment of time \(i \geq 1\), the assets yield random payoffs \(X^n_i\), \(n = 1, \ldots, N\), which are divided between the agents proportionally to the amount of wealth each agent invests in an asset. The random vectors \(X_i = (X^1_i, \ldots, X^N_i)\) are i.i.d.

The wealth of the agents is represented by random sequences \(Y^m_i, m = 1, \ldots, M, i \geq 0\), which are defined inductively as follows. The initial wealth \(Y^m_0\) is non-random and strictly positive. At each moment of time, agent \(m\) splits the available wealth for investing in the assets in proportions \(\lambda^m = (\lambda^m_1, \ldots, \lambda^m_N)\). The vector \(\lambda^m\) represents the strategy of this agent. The whole wealth is reinvested, so \(\lambda^m\) belongs to the standard \(N\)-simplex \(\Delta_N = \{\lambda \in \mathbb{R}^N_+: \sum_n \lambda^n = 1\}\). We consider only constant strategies which depend neither on time nor on a random outcome. Then the wealth sequences \(Y^m_i\) are defined by the equation (see Remark 1 below for an interpretation)

\[
Y^m_{i+1} = \sum_{n=1}^N \lambda^m_{nm} Y^m_i X^n_i. \tag{1}
\]

We will assume that there is at least one agent who allocates a strictly positive proportion of wealth in every asset, i.e.

\[
\lambda^m_{nm} > 0 \text{ for some } m \text{ and all } n.
\]

Under this conditions, \(Y^m_i > 0\) for all \(i\), the total market wealth \(\sum_m Y^m_i\) does not depend on the agents’ strategies and is equal to \(\sum_n X^n_i\).

We will be interested in the asymptotic behavior of the relative wealth of agents

\[
R^m_i = \frac{Y^m_i}{\sum_k Y^k_i}.
\]
It is easy to see that
\[
\frac{R_{i+1}^m}{R_i^m} = \sum_{n=1}^{N} \left( \frac{\lambda_{mn}}{\sum_k \lambda_{kn}R_i^k} \cdot \frac{X_i^n}{\sum_l X_i^l} \right).
\]

In particular, the relative wealth does not change under scaling of the vector \( X_i \) and therefore in what follows we will assume that
\[
\sum_{n=1}^{N} X_i^n = 1 \text{ for all } i \geq 1, \quad \sum_{m=1}^{M} Y_i^m = 1.
\]

Under this assumption we have \( R_i^m = Y_i^m \) and \( \sum_m Y_i^m = 1 \).

**Remark 1.** One can interpret the model defined by equation (1) as an asset market in which at every moment of time agents can buy \( N \) assets which yield random payoffs at the next moment of time. If their prices \( P_i^n \) are determined from the equality of supply and demand, then agent \( m \) buys \( x_i^{mn} = \lambda_{mn}Y_i^m/P_i^n \) units of asset \( n \). On the other hand, the equality of supply and demand implies that \( P_i^m = \sum_m \lambda_{mn}Y_i^m/S_n \), where \( S_n \) is the supply of asset \( n \). Hence agent \( m \) will receive the payoff \( x_i^{mn}X_i^{n+1}/S_n \), where \( X_i^{n+1}/S_n \) is the payoff per one unit of asset \( n \). This gives equation (1).

It should be noted that this model assumes the assets are short-lived in the sense that they are bought by the agents, yield payoffs at the next moment of time, then expire and get replaced by new assets (so they live for just one period). Such assets can used to model standardized contracts, for example, derivative securities, agreements to produce or deliver goods or services, etc. The model differs from a usual stock market model in mathematical finance, see, e.g., Evstigneev et al. (2016) for details and interpretations.

Observe that the model described in the introduction, which corresponds to the case of a complete market, is obtained if
\[
P(X_i \in \{e_1, \ldots, e_N\}) = 1,
\]
where \( e_i = (0, \ldots, 1, \ldots, 0) \) are the standard basis vectors.

To motivate further discussion, let us state a result on asymptotically optimal strategies in the model under consideration. We will say that there are no redundant assets in the market if there is no non-trivial linear combination \( c_1X_i^1 + \cdots + c_NX_i^N \) equal to a constant vector with probability 1.

**Proposition 1.** Suppose agent \( m \) uses the strategy \( \lambda_m = \hat{\lambda} := (EX_1^1, \ldots, EX_i^N) \). Then for any strategies of the other agents it holds that (with probability 1)
\[
\inf_{i \geq 0} Y_i^m > 0.
\]

If agent \( m \) uses a strategy different from \( \hat{\lambda} \), then it is possible to find strategies \( \lambda_k \) of agents \( k \neq m \) (one can take \( \lambda_k = \hat{\lambda} \)) such that
\[
\lim_{i \to \infty} Y_i^m = 0.
\]

If there are no redundant assets and at least one agent uses the strategy \( \hat{\lambda} \), then for any agent \( k \) who uses a different strategy it holds that
\[
\lim_{i \to \infty} Y_i^k = 0.
\]
These statements were proved by Evstigneev et al. (2002); see also Amir et al. (2005) for similar results in a model with Markov payoff sequences, and Amir et al. (2013) for a model with general payoffs. Observe that in model (2), the strategy \( \hat{\lambda} \) has the components \( \hat{\lambda}^n = P(X^*_t = 1) \), and therefore is sometimes called the Kelly strategy after Kelly (1956) (the Kelly strategy consists in “betting one’s beliefs”, i.e. in proportion to probabilities of outcomes).

Note that the formula for the optimal strategy \( \hat{\lambda} \) appearing in Proposition 1 has a relatively simple form largely due to the assumption that the whole capital is reinvested in the assets. If agents are allowed to keep part of their wealth not invested (i.e. \( \sum \lambda_{mn} \leq 1 \)), the optimal strategy will not be constant; see Drokin and Zhitlukhin (2020); Zhitlukhin (2021a) for details; an extension to continuous time can be found in Zhitlukhin (2020, 2021b).

Proposition 1 shows that the strategy \( \hat{\lambda} \) drives other strategies out of the market in the long run, which can be regarded as a form of asymptotic optimality. However this strategy requires an agent to have precise estimates of the expected payoffs \( E \), which may be difficult to achieve. In view of that and as discussed in the introduction, it becomes interesting to consider a model where agents have “almost” precise estimates and study it in the limit when the estimation error vanishes. The next section describes such a model using an appropriate approximation with diffusion processes.

3. Approximation by a diffusion process

Consider series of the above discrete-time models indexed by a parameter \( \delta > 0 \) with asset payoffs and investors strategies satisfying the relations

\[
E X^\delta,n_t = \mu_n + a_n \sqrt{\delta} + c_n(\delta), \tag{3}
\]

\[
\lambda_{mn} = \mu_m + b_{mn} \sqrt{\delta} + d_{mn}(\delta), \tag{4}
\]

\[
\text{cov}(X^\delta,n_t, X^\delta,l_t) = \sigma_{nl} + e_{nl}(\delta), \tag{5}
\]

where \( \mu_n > 0, \sum_n \mu_n = 1, \sum_n a_n = \sum_n b_{mn} = \sum_n c_n(\delta) = \sum_n d_{mn}(\delta) = 0 \), the matrix \( \sigma = (\sigma_{nl}) \in \mathbb{R}^{N \times N} \) is symmetric and non-negative definite, the functions \( c_n(\delta), d_{mn}(\delta), e_{nl}(\delta) \) have the limits \( \lim_{\delta \downarrow 0} c_n(\delta)/\sqrt{\delta} = \lim_{\delta \downarrow 0} d_{mn}(\delta)/\sqrt{\delta} = \lim_{\delta \downarrow 0} e_{nk}(\delta) = 0 \).

Denote by \( \hat{Y}^\delta_t = (\hat{Y}^\delta,1_t, \ldots, \hat{Y}^\delta,M_t) \) the wealth sequences of the agents (tildes will be used in the notation to distinguish discrete-time objects). The vector of initial wealth \( \hat{Y}_0 \) is assumed to be the same for all \( \delta \), with \( \hat{Y}^m_0 > 0 \) for all \( m \). Denote by \( Y^\delta_t \) the piecewise-constant embedding of \( \hat{Y}^\delta_t \) into continuous time with step \( \delta \), i.e.

\[
Y^\delta_t = \hat{Y}^\delta_{\lfloor t/\delta \rfloor}. \tag{6}
\]

The next theorem contains the main result about the convergence of the discrete-time models to a continuous-time model. Everywhere the convergence will be understood as the weak convergence of distributions on the Skorokhod space. Let \( \bar{b}(y) : \Delta_M \rightarrow \Delta_N \) denote the weighted coefficient \( b \) of the strategies of the agents with a vector of weights \( y \), i.e.

\[
\bar{b}_n(y) = \sum_{m=1}^M y^m b_{mn}. \tag{7}
\]

Consider the system of \( M \) stochastic differential equations (\( m = 1, \ldots, M \))

\[
dY^m_t = Y^m_t \sum_{n=1}^N \frac{1}{\mu_n} \left( (b_{mn} - \bar{b}_n(Y_t))(a_n - \bar{b}_n(Y_t))dt + (b_{mn} - \bar{b}_n(Y_t))dW^n_t \right), \tag{8}
\]
where $W^n_t$ are correlated Brownian motions with zero mean and covariance
\[
E(W^n_tW^n_t) = \sigma_{nt} t.
\] (7)

**Theorem 1.** Equation (6) has a unique strong solution $Y$ for any initial condition $Y_0 \in \Delta_N$ and Law($Y^\delta_t$, $t \geq 0$) $\to$ Law($Y_t$, $t \geq 0$) as $\delta \to 0$.

Before giving a proof, which will be based on verification of some technical conditions, let us provide an intuitively clear (but not formally rigorous) argument explaining why one can expect equation (6) in the limit. Expanding (1) in the Taylor series up to order $\delta$, we get
\[
\tilde{Y}^\delta_{i+1} \approx \tilde{Y}^\delta_i + \sqrt{\delta} \sum_{n=1}^N \sum_{i=1}^\delta \frac{1}{\mu_n} (b_{mn} - \bar{b}_n(Y^\delta_i))(a_n - \bar{b}_n(Y^\delta_i))
\]

As for the term $\delta F^\delta_t$, since $X^\delta_t$ is a sequence of i.i.d. vectors, we can approximate it with a sequence of increments of a multidimensional Brownian motion with appropriate correlation matrix. Namely,
\[
\sqrt{\delta} G^\delta_t \approx \sum_{n=1}^N \int_0^t \frac{Y^n_s}{\mu_n} (b_{mn} - \bar{b}_n(Y_s))(a_n - \bar{b}_n(Y_s)) ds.
\]

### Proof of Theorem 1

It is easy to see that for any initial condition $Y_0 \in \Delta_M$, a solution of (6) must always stay in $\Delta_M$. Indeed, $Y_t$ is clearly non-negative. The equality $\sum_{m} Y^m_t = 1$ follows from that $\sum_{m} y^m(b_{mn} - \bar{b}_n(y)) = \bar{b}_n(y)(1 - \bar{y})$, where $\bar{y} = \sum_{m} y^m$, and hence
\[
d\bar{Y}_t = \sum_{n=1}^N \frac{1}{\mu_n} \bar{b}_n(Y_t)(1 - \bar{Y}_t)((a_n - \bar{b}_n(Y_t))) dt + dB^n_t.
\]
So, if $\bar{Y}_0 = 1$, then $\bar{Y}_t = 1$ for all $t \geq 0$.

Consequently, without loss of generality, the coefficients of (6) can be modified outside of $\Delta_M$ and replaced with functions $f (y)$ and $g(y)$ (the drift and diffusion coefficients, respectively) which are smooth on $\mathbb{R}^M$, have a bounded support, and for $y \in \Delta_M$

$$f^n (y) = y^n \sum_{n=1}^{N} \frac{1}{\mu_n} (b_{mn} - \bar{b}_n (y)) (a_n - \bar{b}_n (y)), \quad g^n (y) = y^n \sum_{n=1}^{N} \frac{1}{\mu_n} (b_{mn} - \bar{b}_n (y)).$$

The existence and uniqueness of a strong solution of (6) follows from classical Ito’s theorem. To prove the convergence of distributions, we will apply Proposition 2 from the Appendix. Conditions (a), (b) of this proposition are clearly met. Condition (c) holds with the function $F(t) = t \max_y (\sum_m |f^n (y)| + |\text{tr}(g(y) \sigma g(y)^T)|)$.

Let us check (d). Since $\tilde{Y}_t^\delta = Y_t^\delta$ is a homogeneous Markov sequence, we can find functions $f^\delta (y) : \mathbb{R}^M \rightarrow \mathbb{R}^M$ such that

$$E(\Delta Y_t^\delta | F_t) = f^\delta (Y_t^\delta),$$

where $F_t = \sigma (Y_s^\delta; s \leq t)$. For $\alpha \in D(\mathbb{R}^+_+; \Delta_M)$, define

$$B_t^\delta (\alpha) = \sum_{1 \leq i \leq [t/\delta]} f^\delta (\alpha_{i,1}^\delta), \quad B_t (\alpha) = \int_0^t f (\alpha_s) ds,$$

so that $B_t^\delta (Y_t^\delta)$ and $B_t (Y_t)$ are the first predictable characteristics of the processes $Y_t^\delta$ and $Y_t$ (see (27), (28)). It will be enough to show that for any sequence of functions $\alpha^\delta \in D(\mathbb{R}^+_+; \Delta_M)$ which are piecewise-constant on intervals $[i\delta, (i+1)\delta)$ and any $t \geq 0$ we have

$$\lim_{\delta \rightarrow 0} \sup_{s \leq t} \| B_t^\delta (\alpha^\delta) - B_t (\alpha^\delta) \| = 0. \quad (9)$$

To compute $f^\delta$, observe that (1) and (4) imply (with $\bar{d}_n (y) = \sum_m d_{mn} y^n$)

$$\Delta Y_{i,1}^\delta = Y_{i,1}^\delta \sum_{n=1}^{N} \frac{\mu_n + b_{mn} \sqrt{\delta} + d_{mn} (\delta)}{\mu_n + \bar{b}_n (Y_{i,1}^\delta) \sqrt{\delta} + \bar{d}_n (Y_{i,1}^\delta)} X_{i,1}^{\delta,n} - 1$$

$$= Y_{i,1}^\delta \sum_{n=1}^{N} \left( \frac{\sqrt{\delta}}{\mu_n} (b_{mn} - \bar{b}_n (Y_{i,1}^\delta)) + \frac{\delta}{\mu_n^2} (K_n (Y_{i,1}^\delta) - b_{mn} \bar{b}_n (Y_{i,1}^\delta)) \right) X_{i,1}^{\delta,n} + \delta \rho_n (\delta; i), \quad (10)$$

where $\rho (\delta; i) = \rho (\omega; \delta; i)$ is a family of vectors in $\mathbb{R}^M$ with $\text{esssup}_\omega \sup_{i \geq 1} \| \rho (\delta; i) \| \rightarrow 0$ as $\delta \rightarrow 0$. The above equation was obtained by expanding the denominator in Taylor series and using the relation $\sum_n X_{i,1}^{\delta,n} = 1$. Consequently, using (3) we find

$$f^\delta (y) = \delta y^n \sum_{n=1}^{N} \frac{1}{\mu_n} (b_{mn} - \bar{b}_n (y)) (a_n - \bar{b}_n (y)) + \delta \rho (\delta; y),$$

where $\rho (\delta; y)$ is a function with values in $\mathbb{R}^M$ such that $\rho (\delta; y) \rightarrow 0$ uniformly in $y \in \Delta_M$ as $\delta \rightarrow 0$. Here we used the relation $\sum_n b_{mn} = \sum_n \bar{b}_n (y) = \sum_n d_{mn} = \sum_n \bar{d}_n (y) = 0$. Thus, for any $t = j \delta$, we have

$$\| B_t^\delta (\alpha^\delta) - B_t (\alpha^\delta) \| \leq \delta \sum_{1 \leq i \leq j} \| \rho (\delta; \alpha^\delta_{i,1}^\delta) \|,$$
which implies (9).

In a similar way, we can verify condition (e). Let $g^\delta: \mathbb{R}^M \to \mathbb{R}^{M \times M}$ be defined by

$$g^{\delta,mk}(y) = \mathbb{E}(\Delta Y^{\delta,m}_{i\delta} \Delta Y^{\delta,k}_{i\delta} \mid Y^{\delta}_{(i-1)\delta} = y) - \mathbb{E}(\Delta Y^{\delta,m}_{i\delta} \mid Y^{\delta}_{(i-1)\delta} = y) \mathbb{E}(\Delta Y^{\delta,k}_{i\delta} \mid Y^{\delta}_{(i-1)\delta} = y)$$

$$= \delta y^m y^k \sum_{n,l=1}^N \frac{\sigma_{nl}}{\mu_{n\mu l}} (b_{mn} - \bar{b}_n(y))(b_{kl} - \bar{b}_l(y)) + \delta \rho^{mk}(\delta; y),$$

where $\rho(\delta; y)$ is (another) function converging to 0 uniformly in $y \in \Delta M$ as $\delta \to 0$. The second modified predictable characteristics of $Y^\delta_t$ and $Y_t$ are, respectively, $C^\delta_t(Y^\delta)$ and $C_t(Y)$, where

$$C^\delta_t(\alpha) = \sum_{1 \leq i \leq [t/\delta]} g^\delta(\alpha^{\delta}_{[i-1]\delta}); \quad C_t(\alpha) = \int_0^t g(\alpha_s)\sigma g(\alpha_s)^T ds.$$

Hence for $t = j\delta$ and $\alpha^\delta \in D(\mathbb{R}_+; \Delta M)$ which are piecewise-constant on $[i\delta, (i + 1)\delta)$ we have

$$\|C^\delta_t(\alpha^\delta) - C_t(\alpha^\delta)\| \leq \delta \sum_{1 \leq i \leq j} \|\rho(\delta, \alpha^\delta_{[i-1]\delta})\| \to 0,$$

which gives (e).

Finally, condition (f) holds because if $h(y)$ is a function vanishing in a neighborhood of zero, then for sufficiently small $\delta$ all the jumps $\Delta Y^\delta_i$, $i \geq 0$, lie in such a neighborhood with probability 1, see (10).

4. Asymptotic relative performance of strategies

4.1. General results for an arbitrary number of agents

In the rest of the paper we will work within the continuous-time model obtained in the previous section and identify agents’ strategies with vectors $b_m = (b_{m1}, \ldots, b_{MN})^T$ from (4). Denote also $a = (a_1, \ldots, a_N)^T$, where $a_n$ are the coefficient from (3). We will be primarily interested in the relative performance of strategies as $t \to \infty$.

**Definition 1.** We shall say that in a strategy profile $b = (b_1, \ldots, b_M)$ with initial wealth $Y_0 = (Y^{1}_0, \ldots, Y^{M}_0)$ agent 1

- vanishes if $\lim_{t \to \infty} Y^1_t = 0$ a.s.;
- survives\(^1\) if $\limsup_{t \to \infty} Y^1_t > 0$ a.s.;
- dominates if $\lim_{t \to \infty} Y^1_t = 1$ a.s.

The next theorem provides sufficient conditions for an agent to survive, dominate, or vanish. For brevity of notation, introduce the matrices

$$\mathcal{M} = \text{diag} \left( \frac{1}{\mu_1}, \ldots, \frac{1}{\mu_N} \right), \quad \mathcal{S} = \left( \frac{\sigma_{nl}}{\mu_{n\mu l}} \right)_{n,l=1}^N.$$

\(^1\)In some works survival means $\liminf_{t \geq 0} Y^1_t > 0$. 

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Theorem 2. Fix a strategy profile \((b_1, \ldots, b_M)\) and a vector of initial wealth \((Y_{10}^1, \ldots, Y_{0M}^M)\) with \(Y_{10}^1 > 0\). Let \(B = \text{conv}(b_2, \ldots, b_M)\) denote the convex hull of the strategies of agents \(m \geq 2\). Define the coefficients
\[
\theta_0 = \inf_{\tilde{b} \in B} \left( (a - b_1)^T \mathcal{M}(b_1 - \tilde{b}) + \frac{1}{2} (b_1 - \tilde{b})^T (2\mathcal{M} - \mathcal{S})(b_1 - \tilde{b}) \right),
\]
\[
\theta_1 = \inf_{\tilde{b} \in B} \left( (a - b_1)^T \mathcal{M}(b_1 - \tilde{b}) + \frac{1}{2} (b_1 - \tilde{b})^T \mathcal{S}(b_1 - \tilde{b}) \right).
\]

Then \(\theta_0 \geq \theta_1\) and the following statements are true:

(a) if \(\theta_0 > 0\) or \(\theta_1 \geq 0\), then agent 1 survives;

(b) if \(\theta_1 > 0\), then agent 1 dominates;

(c) if \((a - b_1)^T \mathcal{M}(b_1 - b_m) \geq 0\) for \(m = 2, \ldots, M\), then there exists \(\lim_{t \to \infty} Y_{1t}^1 > 0\) a.s., and, in particular, agent 1 survives.

Remark 2. As will be seen from the proof of the theorem, the coefficients \(\theta_0\) and \(\theta_1\) give lower bounds for the drift coefficient of the process \(\ln(Y_{1t}^1/(1 - Y_{1t}^1))\) when \(Y_{1t}^1\) is close to 0 or 1, respectively. See (20) below.

Observe that we have \(\theta_0 = \theta_1\) if the matrix \(\mathcal{S}\) has the form
\[
S_{nl} = -1 \text{ for } n \neq l, \quad S_{nn} = \frac{1}{\mu_n} - 1,
\]
(14)
because in this case \(\mathcal{M} - \mathcal{S}\) is the matrix of all units, so \((b_1 - \tilde{b})^T (\mathcal{M} - \mathcal{S})(b_1 - \tilde{b}) = (\sum_n (b_{1n} - \tilde{b}_n))^2 = 0\), which implies \(\theta_0 = \theta_1\). In particular, (14) takes place if the pre-limit discrete-time models represent a complete market in the sense of (2).

In order to prove Theorem 2, we will first establish the following auxiliary inequality of a general nature.

Lemma 1. Let \(X\) be a random vector in \(\Delta_N\), \(\mu_n = E X_n > 0\), \(\sigma_{nl} = \text{cov}(X_n, X_l)\), and define \(\mathcal{M}, \mathcal{S}\) as in (11). Then for any \(c \in \mathbb{R}^N\) such that \(\sum_n c_n = 0\) we have
\[
c^T (\mathcal{M} - \mathcal{S}) c \geq 0.
\]
(15)

Proof. Since any distribution can be approximated by a discrete one, it is sufficient to prove the lemma in the case when \(X\) has a discrete distribution.

Fix a set \(\{x_1, \ldots, x_K\} \subset \Delta_N\). We are going to show that inequality (15) holds true for any distribution \(p = (p_1, \ldots, p_K)\), \(p_k = P(X = x_k)\), such that \(E X_n = \mu_n\), \(\text{cov}(X_n, X_l) = \sigma_{nl}\). Observe that
\[
c^T (\mathcal{M} - \mathcal{S}) c = c^T \mathcal{M} c - E(c^T \mathcal{M} X)^2 = c^T \mathcal{M} c - \sum_{k=1}^K (c^T \mathcal{M} x_k)^2 p_k.
\]
Consider the following linear programming problem with variables $p_1, \ldots, p_K$:

\[
\begin{align*}
\text{minimize} & \quad v(p) := c^T M c - \sum_{k=1}^{K} (c^T M x_k)^2 p_k \\
\text{subject to} & \quad \sum_{k=1}^{K} x_k n p_k = \mu_n, \quad n = 1, \ldots, N, \quad (16) \\
& \quad \sum_{k=1}^{K} p_k = 1, \quad (17) \\
& \quad p_k \geq 0, \quad k = 1, \ldots, K. \quad (18)
\end{align*}
\]

Since the constraint set is non-empty and compact, the minimizer $p^*$ exists. We need to show that $v(p^*) \geq 0$. To that end, consider the dual problem with variables $q = (q_1, \ldots, q_{N+1})$ which correspond to equality constraints (16)–(17) (see, e.g., Boyd and Vandenberghe (2004, Ch. 5.2.1)):

\[
\begin{align*}
\text{maximize} & \quad d(q) := c^T M c - \sum_{n=1}^{N} \mu_n q_n - q_{N+1} \\
\text{subject to} & \quad \sum_{n=1}^{N} x_k n q_n + q_{N+1} - (c^T M x_k)^2 \geq 0, \quad k = 1, \ldots, K. \quad (19)
\end{align*}
\]

Let $q = (q_1, \ldots, q_{N+1})$ be defined by

\[q_n = \frac{c_n^2}{\mu_n^2} \text{ for } n = 1, \ldots, N, \quad q_{N+1} = 0.\]

It is easy to check that $d(q) = 0$ and $q$ satisfies constraints (19) (this follows from applying Jensen’s inequality and treating each vector $x_k \in \Delta_N$ as coefficients of a convex combination). Then $v(p^*) \geq d(q) = 0$ in view of the duality. \hfill \Box

**Proof of Theorem 2.** The inequality $\theta_0 \geq \theta_1$ follows from Lemma 1 with $c = b_1 - \tilde{b}$.

To prove claims (a) and (b), let $Z_t = \ln(Y_t^1/(1-Y_t^1))$ and denote by $\tilde{b}_t$ the weighted strategy of agents $m \geq 2$:

\[\tilde{b}_t = \sum_{m=2}^{M} \frac{Y_t^m}{1-Y_t^1} b_m.\]

By Ito’s formula, we have

\[dZ_t = \gamma_t dt + (b_1 - \tilde{b}_t)^T M dW_t, \quad (20)\]

where

\[\gamma_t = (a - \tilde{b}_t)^T M (b_1 - \tilde{b}_t) - \frac{1}{2} (b_1 - \tilde{b}_t)^T M (b_1 - \tilde{b}_t) + Y_t^1 (b_1 - \tilde{b}_t)^T (S - M) (b_1 - \tilde{b}_t).\]

Notice that $\theta_0$ and $\theta_1$ give the minimum possible values for $\gamma_t$ when the value of $Y_t^1$ approaches 0 and 1, respectively.

Suppose $\theta_0 > 0$. Then on the set $\Omega' = \{\lim_{t \to \infty} Y_t^1 = 0\} = \{\lim_{t \to \infty} Z_t = -\infty\}$ we have

\[\lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} \gamma_s ds > \frac{\theta_0}{2} > 0,\]
while by the strong law of large numbers for martingales we have with probability 1
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t (b - \tilde{b}_t)^T \mathcal{M} dW_t = 0.
\]
Hence on \( \Omega' \) we have \( \lim_{t \to \infty} t^{-1} Z_t > 0 \) a.s., which by the definition of \( \Omega' \) implies \( P(\Omega') = 0 \), and proves claim (a) in the case \( \theta_0 > 0 \). Claim (b) is proved in a similar way, but using that \( \lim_{t \to \infty} t^{-1} \int_0^t \gamma_t ds \geq \theta_1 > 0 \) a.s.

To prove (a) when \( \theta_0 = \theta_1 = 0 \), observe that in this case \( \gamma_t \geq 0 \), so the process \( Z_t \) is a submartingale, and a continuous submartingale cannot have the limit \(-\infty\) with positive probability.\footnote{The process \( Z_t = Z_{t \wedge \tau} \), where \( \tau_t = \inf \{ t \geq 0 : Z_t \geq a \} \) is a submartingale bounded from above, hence it has a finite limit. Therefore, \( \lim_{t \to \infty} \tau_t \) exists and is finite on the set \( \{ \sup_{t \geq 0} Z_t < \infty \} \), while on the complementary set we have \( \limsup_{t \to \infty} Z_t = +\infty \).}

To prove claim (c), consider the process \( \ln Y_t^1 \). By Ito’s formula,
\[
d\ln Y_t^1 = (1 - Y_t^1) \left\{ (a - b_1)^T \mathcal{M} (b_1 - \tilde{b}_t) + \frac{1}{2} (1 - Y_t^1) (b_1 - \tilde{b}_t)^T (2 \mathcal{M} - \mathcal{S}) (b_1 - \tilde{b}_t) \right\} dt + (b_1 - \tilde{b}_t)^T \mathcal{M} dW_t.
\]
If the condition of the claim holds true, then Lemma 1 implies that the drift coefficient is non-negative, so \( \ln Y_t^1 \) is a submartingale. Since it is bounded from above, there limit \( \lim_{t \to \infty} \ln Y_t^1 \) is finite, hence \( \lim_{t \to \infty} Y_t^1 > 0 \).

4.2. The case of two agents

When \( M = 2 \), it is possible to give a more thorough characterization of the asymptotics of the agents’ wealth by using standard results on ergodicity of Markov processes (see, e.g., Gikhman and Skorokhod (1972, §16.18) for details on results that are needed below). In this case the wealth dynamics is determined by the one-dimensional equation for the wealth of one agent and we have
\[
dY_t^1 = Y_t^1 (1 - Y_t^1) \left\{ (a - b_2)^T \mathcal{M} (b_1 - b_2) - Y_t^1 (b_1 - b_2)^T \mathcal{M} (b_1 - b_2) \right\} dt + \nu dW_t,
\]
where \( \nu^2 = (b_1 - b_2)^T \mathcal{S} (b_1 - b_2) \) and \( W_t \) is a new one-dimensional standard Brownian motion (it is obtained as \( \nu^{-1} (b_1 - b_2)^T \mathcal{M} W_t \) for the old \( N \)-dimensional Brownian motion \( W_t^1 \) from (6)–(7)). The coefficients \( \theta_0 \) and \( \theta_1 \) from Theorem 2 simplify to
\[
\theta_0 = (a - b_1)^T \mathcal{M} (b_1 - b_2) + \frac{1}{2} (b_1 - b_2)^T (2 \mathcal{M} - \mathcal{S}) (b_1 - b_2),
\]
\[
\theta_1 = (a - b_1)^T \mathcal{M} (b_1 - b_2) + \frac{1}{2} (b_1 - b_2)^T \mathcal{S} (b_1 - b_2).
\]

Theorem 3. Suppose \( M = 2 \).

I. If \( \nu^2 > 0 \), then the following statements hold true.

(I.a) If \( \theta_1 > 0 \), then agent 1 dominates.

(I.b) If \( \theta_0 \geq 0 \) and \( \theta_1 \leq 0 \), then both of the agents survive, \( \liminf_{t \to \infty} Y_t^1 = 0 \), \( \limsup_{t \to \infty} Y_t^1 = 1 \) a.s., the process \( Y_t^1 \) is recurrent and has the invariant measure \( F(dy) = \rho(y)dy \) with density
\[
\rho(y) = y^{\frac{2 \theta_0}{\nu^2} - 1} (1 - y)^{-\frac{2 \theta_1}{\nu^2} - 1}, \quad y \in (0, 1).
\]
Moreover, if $\theta_0 \theta_1 < 0$, then $Y_t^1$ is positive recurrent, $F([0, 1]) = B(\frac{2\theta_0}{v^2}, -\frac{2\theta_1}{v^2}) < \infty$, and $Y_t^1 \to F/F([0, 1])$ in distribution as $t \to \infty$ ($B$ is the beta function). If $\theta_0 \theta_1 = 0$, then $Y_t^1$ is null recurrent and $F([0, 1]) = \infty$.

(I.c) If $\theta_0 < 0$, then agent 1 vanishes.

II. If $v^2 = 0$, then $Y_t^1$ is a non-random process and the following statements hold true.

(II.a) If $\theta_0 > 0$ and $\theta_1 \geq 0$, then agent 1 dominates.

(II.b) If $\theta_0 > 0$ and $\theta_1 < 0$, then both of the agents survive and $\lim_{t \to \infty} Y_t^1 = \frac{\theta_0}{\theta_0 - \theta_1}$.

(II.c) If $\theta_0 \leq 0$ and $\theta_1 < 0$, then agent 1 vanishes.

(II.d) If $\theta_0 = \theta_1 = 0$ then $Y_t^1$ is constant for all $t \geq 0$.

Proof. Claims (I.a) and (I.c) immediately follow from Theorem 2 (for (I.c) note that the second agent has the corresponding coefficient $\tilde{\theta}_1 = -\theta_0$). To prove (I.b), let $Z_t = \ln(Y_t^1/(1 - Y_t^1))$. Define

$$f(z) = \theta_0 + \frac{\theta_1 - \theta_0}{1 + e^{-z}}, \quad (22)$$

so that (cf. (20))

$$dZ_t = f(Z_t)dt + v dW_t.$$

Let $s(z)$ and $m(dz)$ be the scale function and the speed measure of $Z_t$,

$$s(z) = \int_0^z \exp\left(-\int_0^y \frac{2f(u)}{v^2}du\right)dy = C\int_1^{e^z} (1 + u) \frac{2(\theta_0 - \theta_1)}{v^2} u^{-1 - \frac{2\theta_0}{v^2}} du,$$

$$m(dz) = \frac{2}{v^2 s'(z)}dz = \frac{2C}{v^2} (1 + e^z)^{-\frac{2(\theta_0 - \theta_1)}{v^2}} e^{-\frac{2\theta_0}{v^2} z},$$

where $C = 4 \frac{\theta_1 - \theta_0}{v^2}$. In view of the conditions $\theta_0 \geq 0$, $\theta_1 \leq 0$ we have $s(\pm \infty) := \lim_{z \to \pm \infty} s(z) = \pm \infty$, which implies that the process $Z_t$ is recurrent and its speed measure is the unique (up to multiplication by a constant) invariant measure. If $\theta_0 > 0$ an $\theta_1 < 0$, we have $m(\mathbb{R}) < \infty$, and then the process is positive recurrent and ergodic, so $\lim_{t \to \infty} Z_t^1 = m/m(\mathbb{R})$ in distribution, implying the claimed result for $Y_t^1$. If $d_0 d_1 = 0$, then $m(\mathbb{R}) = \infty$ and $Z_t$ is null recurrent, so $Y_t^1$ is also null recurrent.

If $v = 0$, then $Z_t$ has no Brownian part and claims (II.a)-(II.d) easily follow from analysis of the solution of the corresponding ODE.

Corollary 1. In the general model $(M \geq 2)$, the strategy $\hat{b} = a$ is the unique strategy which guarantees survival of an agent using it in any strategy profile with any (positive) initial wealth.

Proof. A strategy of agent 1 surviving in any strategy profile must also survive when all the other agents use the strategies $b_m = a$. In this case those agents can be considered as a single agent, and then $\theta_0 = -\frac{1}{2}(b_1 - a)^T S(b_1 - a) \leq 0$. By (I.c) and (II.c) of Theorem 3, survival is possible only when $\theta_0 = 0$, which implies $v^2 = 0$ and $\theta_1 = -(a - b_1)^T M (a - b_1)$. By (II.c), (II.d), for survival it must hold that $\theta_1 = 0$, hence $b_1 = a$. \qed
Remark 3. It is worth mentioning that the process $Y^i_t$ satisfies the stochastic replicator equation of Fudenberg and Harris (1992) (see also Taylor and Jonker (1978) for the seminal work on the deterministic replicator equation, and Foster and Young (1990) for another form of the stochastic equation).

Recall that the corresponding model can be formulated as follows. Consider a symmetric two-player game with two pure strategies and a payoff matrix $A = (A_{ij}) \in \mathbb{R}^{2 \times 2}$. There are two continuum populations of players who are programmed to use, respectively, strategies 1 or 2 (e.g. strategies are phenotypes of biological species). Suppose the players are randomly matched against each other. Let $S_i = (S^i_1, S^i_2)$ denote the size of the populations $i = 1, 2$. Then the average payoff of a player from population $i$ in a game against a random adversary is $(AS)_i$. The model states that the population growth rates satisfy the equation

$$\frac{dS^i_t}{S^i_t} = (A S^i)_i dt + \sigma_i dW^i_t,$$

where $W^i_t$ are independent standard Brownian motions. Let $Y^i_t = S^i_t/(S^1_t + S^2_t)$ denote the proportion of players of type $i$. By Ito’s formula, we have

$$dY^i_t = Y^i_t Y^2_t \left \{ ((-a_{22} + a_{12} + \sigma^2_2) + (a_{11} - a_{21} - \sigma^2_1 + a_{22} - a_{12} - \sigma^2_2)Y^1_t) dt + \sigma Y^1_t dW^1_t \right \},$$

where $v = \sqrt{\sigma^2_1 + \sigma^2_2}$, and $W^i_t = v^{-1}(\sigma_1 W^1_t + \sigma_2 W^2_t)$ is a new Brownian motion. If $\sigma_1 = \sigma_2 = 0$, one gets the non-random replicator equation of Taylor and Jonker (1978).

It is straightforward to check that equation (21) is a particular case of (23) with

$$A = \begin{pmatrix} \theta_1 & 0 \\ 0 & -\theta_0 \end{pmatrix}, \quad \sigma^2_1 = \sigma^2_2 = \frac{1}{2}(b_1 - b_2)^T S (b_1 - b_2).$$

Note that (23) admits one more type of asymptotic behavior, which does not appear in our model because $\theta_0 > \theta_1$, namely when $P(Y^1_t \to 1) > 0$ and $P(Y^1_t \to 0) > 0$, see Proposition 1 of Fudenberg and Harris (1992).

5. Examples

As an illustration of the survival and dominance conditions in terms of the coefficients $\theta_0, \theta_1$, consider the model with two agents and two assets. Assume $\mu_1 = \mu_2 = 1/2$. Then the strategies of agents 1 and 2 are given respectively by the vectors $(b_1, -b_1)$ and $(b_2, -b_2)$. Denote the coefficient $a_1$ of asset 1 simply by $a$, so that the coefficient $a_2 = -a$. Let $\sigma^2 = \text{Var} X_1$. Then

$$S = \begin{pmatrix} 4\sigma^2 & -4\sigma^2 \\ -4\sigma^2 & 4\sigma^2 \end{pmatrix}.$$  

Note that we have $0 \leq \sigma^2 \leq 1/4$; the maximum variance corresponds to the case of a complete market (2), the zero variance to the case of non-random payoffs. The coefficients $\theta_0, \theta_1, v$ become

$$\theta_0 = 4(a - b_1)(b_1 - b_2) + 4(1 - 2\sigma^2)(b_1 - b_2)^2,$$

$$\theta_1 = 4(a - b_1)(b_1 - b_2) + 8\sigma^2(b_1 - b_2)^2,$$

$$v = 4\sigma|b_1 - b_2|.$$
The conditions $\theta_0 \geq 0$ and $\theta_1 \leq 0$ turn into

$$\begin{align*}
\theta_0 \geq 0 & \iff (1 - 2\sigma^2)|b_1 - b_2| \geq (b_1 - a) \operatorname{sgn}(b_1 - b_2), \\
\theta_1 \leq 0 & \iff 2\sigma^2|b_1 - b_2| \leq (b_1 - a) \operatorname{sgn}(b_1 - b_2).
\end{align*}$$

These inequalities define regions with linear boundaries with slope $1$, $2\sigma^2/(2\sigma^2 - 1)$, and $(2\sigma^2 - 1)/(2\sigma^2)$. In Figure 1, we depict them for $\sigma^2 = 1/4$, $1/8$, $0$ and $a = 0$. The green region is where agent 1 dominates, the red region is where agent 2 dominates, in the blue region both of the agents survive. The color of the boundaries bears the same meaning.

Figure 2 shows simulated paths of the process $Y^1_t$ in the case $\sigma^2 = 1/8$ for the three pairs of the agents’ strategies $(b_1, b_2)$: $(-1/4, 1)$, $(-1/3, 1)$, $(-1/2, 1)$. In the first pair, agent 1 dominates ($\theta_1 > 0$) and the process $Y^1_t$ is transient. In the second pair both of the agents survive, the process $Y^1_t$ is null recurrent ($\theta_0 > 0$, $\theta_1 = 0$). In the third pair also both of the agents survive, but $Y^1_t$ is positive recurrent ($\theta_0 > 0$, $\theta_1 < 0$). The same realization of the Brownian motion $W_t$ was used in the simulations.
6. Extension: the two-agent model with regime switching

In this section we consider an extension of the two-agent model from Section 4.2 in which the distribution of the payoffs changes between two regimes at random moments of time.

Let $Q_t$ be a continuous-time ergodic Markov chain with two states and the generator matrix

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

where $g_{12} = -g_{11} > 0$ and $g_{21} = -g_{22} > 0$. Let $\pi = (\pi_1, \pi_2)$ denote its invariant distribution given by

$$\pi_1 = \frac{g_{21}}{g_{12} + g_{21}}, \quad \pi_2 = \frac{g_{12}}{g_{12} + g_{21}}.$$

Suppose that when $Q_t$ is in state $i \in \{1, 2\}$, the model is governed by equations (6)–(7) with parameters $\mu(i), a(i) \in \mathbb{R}^N$, $\sigma(i) \in \mathbb{R}^{N \times N}$. The agents’ strategies $b_1, b_2$ are constant. Then the wealth of agent 1 satisfies the SDE (cf. (21))

$$dY_t^1 = Y_t^1(1 - Y_t^1)\left\{\left( (a(Q_t) - b_2)^T \mathcal{M}(Q_t)(b_1 - b_2) \\
- Y_t^1(b_1 - b_2)^T \mathcal{M}(Q_t)(b_1 - b_2) \right) dt + v(Q_t)dW_t \right\},$$

where $\mathcal{M}(Q_t), \mathcal{S}(Q_t)$ are defined as in (11), and $v(Q_t) = \sqrt{(b_1 - b_2)^T \mathcal{S}(Q_t)(b_1 - b_2)}$.

Let $\theta_0(i), \theta_1(i)$ denote the coefficients from (12)–(13) corresponding to state $i$. Define

$$\tilde{\theta}_j = \mathbb{E}^\pi(\theta_j(Q_t)) := \sum_{i=1}^2 \theta_j(i)\pi_i, \quad j = 1, 2,$$

where $\mathbb{E}^\pi$ is the expectation with respect to the stationary distribution of $Q_t$. Note that $\tilde{\theta}_0 \geq \tilde{\theta}_1$ by Theorem 2.

**Theorem 4.** Assume that $v(i) > 0$ for $i = 1, 2$.

(a) If $\tilde{\theta}_1 > 0$, then agent 1 dominates.

(b) If $\tilde{\theta}_0 > 0$ and $\tilde{\theta}_1 < 0$, then both of the agents survive, the process $Y_t^1$ is positive recurrent, and $\limsup_{t \to \infty} Y_t^1 = 1, \liminf_{t \to \infty} Y_t^1 = 0$.

(c) If $\tilde{\theta}_0 < 0$, then agent 1 vanishes.

**Proof.** Let $Z_t = \ln(Y_t^1/(1 - Y_t^1))$. Similarly to (22), define

$$f(z, i) = \theta_0(i) + \frac{\theta_1(i) - \theta_0(i)}{1 + e^{-z}}.$$

Then

$$dZ_t = f(Z_t, Q_t)dt + v(Q_t)dW_t.$$

Note that always $\theta_1(i) \leq f(z, i) \leq \theta_0(i)$. In case (a), using the ergodicity of $Q_t$ and the strong law of large numbers applied to the martingale $\int_0^t v(Q_s)dW_s$, we find

$$\lim_{t \to \infty} \frac{Z_t}{t} \geq \mathbb{E}^\pi(\theta_1(Q_t)) = \tilde{\theta}_1 > 0,$$

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hence \( Z_t \to \infty \) and \( Y_t^1 \to 1 \) as \( t \to \infty \). Similarly, in case (c) we have \( Z_t \to -\infty \).

Consider case (b). In order to prove the positive recurrence it is enough to prove it for a set of the form \( I \times \{1, 2\} \), where \( I \) is an interval \((z_1, z_2) \subset \mathbb{R} \) (see, e.g., Yin and Zhu (2009, Sec. 3.2)). Since \( \theta_0 > 0 \) and \( \theta_1 < 0 \), we can find \( z_1 \leq z_2 \) such that

\[
E^T f(z_1, Q_t) > 0, \quad E^T f(z_2, Q_t) < 0.
\]

Consider an initial condition \( z_0 \leq z_1 \) and let \( \tau = \inf \{ t \geq 0 : Z_t = z_1 \} \). We are going to find a function \( V(z, i) \) such that \( V(z, i) \geq 0 \) and \( LV(z, i) \leq -u \) for some \( u > 0 \) and all \( z \leq z_1 \), \( i = 1, 2 \), where \( L \) is the infinitesimal generator of \((Z, Q)\):

\[
LV(z, i) = \frac{v^2(i)}{2} V''(z, i) + f(z, i)V'(z, i) + V(z, 1)g_{i1} + V(z, 2)g_{i2}.
\]

Observe that it is possible to find \( \varepsilon > 0 \) such that for \( c = \frac{f(z_2, 2)}{2g_{21}} - \varepsilon \) we have

\[
\begin{align*}
f(z_1, 1) + 2cg_{12} & > 0, \quad (24) \\
f(z_1, 2) - 2cg_{21} & > 0. \quad (25)
\end{align*}
\]

Indeed, for \( \varepsilon = 0 \) we have the equality to zero in (25) and the strict inequality in (24), which follows from that \( E^T f(z_1, Q) = (g_{12} + g_{21})^{-1}(f(z_1, 1)g_{21} + f(z_1, 2)g_{12}) > 0 \). Hence such \( \varepsilon \) can be found by continuity. Since the function \( f(z, i) \) is non-increasing in \( z \), inequalities (24)–(25) hold for all \( z \leq z_1 \).

For \( \gamma \in (0, |c|^{-1}) \), consider the function \( V(z, i) \) defined by

\[
V(z, 1) = (1 + c\gamma)e^{-\gamma z}, \quad V(z, 2) = (1 - c\gamma)e^{-\gamma z}.
\]

We have

\[
\begin{align*}
LV(z, 1) &= \frac{v^2(i)}{2} \gamma^2 (1 + c\gamma) - \gamma(f(z, 1)(1 + c\gamma) + 2cg_{12})e^{-\gamma z}, \\
LV(z, 2) &= \frac{v^2(i)}{2} \gamma^2 (1 - c\gamma) - \gamma(f(z, 1)(1 - c\gamma) - 2cg_{21})e^{-\gamma z}.
\end{align*}
\]

In view of (24)–(25), taking \( \gamma \) sufficiently small, it is possible to make \( LV(z, 1) \leq -u \) and \( LV(z, 2) \leq -u \) for some \( u > 0 \) and all \( z \leq z_1 \). By Ito’s formula, under the initial condition \( Z_0 = z_0 \) and \( Q_0 = i_0 \), we find

\[
EV(Z_{\tau \wedge \Lambda}, Q_{\tau \wedge \Lambda}) = V(z_0, i_0) + E \int_0^{\tau \wedge \Lambda} LV(Z_s, Q_s)ds \leq V(z_0, i_0) - uE(\tau \wedge t).
\]

By the monotone convergence theorem applied with \( t \to \infty \), we have \( E\tau \leq u^{-1}V(z_0, i_0) \). Hence the set \( I \times \{1, 2\} \) can be reached from a point \( z_0 \leq z_1 \) in time with finite expectation. In a similar way, we consider points \( z_0 \geq z_2 \), and establish the positive recurrence of the set \( I \times \{1, 2\} \).

\begin{remark}
Asymptotic behavior of a solution of a replicator equation with regime switching was studied by Vlasic (2015), although his assumptions are more complicated than ours (see Assumption 4.1 of that paper).
\end{remark}
Appendix. Conditions for convergence to a diffusion process

This appendix states a result about convergence in distribution of discrete-time processes with uniformly bounded jumps to a diffusion process in a form convenient for our needs.

Consider a stochastic differential equation
\[ dY_t = f(Y_t)dt + g(Y_t)dW_t, \quad Y_0 = y_0 \in \mathbb{R}^M, \tag{26} \]
where \( f : \mathbb{R}^M \to \mathbb{R}^M, g : \mathbb{R}^M \to \mathbb{R}^{M \times M} \) are measurable functions and \( W_t \) is a Brownian motion in \( \mathbb{R}^M \) with covariance matrix \( \sigma = (\sigma_{mk}) \in \mathbb{R}^{M \times M} \), i.e., \( \mathbb{E}W^m_tW^k_t = \sigma_{mk}t \). Assume that equation (26) has a unique weak solution and denote by \( \text{Law}(Y) \) the distribution which it generates on the Skorokhod space \( D(\mathbb{R}_+; \mathbb{R}^M) \) of càdlàg function \( \alpha : \mathbb{R}_+ \to \mathbb{R}^M \) (the support of this distribution lies in the subspace of continuous functions).

Let \( Y^\delta_t \) be piecewise-constant càdlàg processes in \( \mathbb{R}^M \) with the same initial values \( Y^\delta_0 = y_0 \), which are indexed by a parameter \( \delta > 0 \). Assume that \( Y^\delta_t \) is constant on intervals \( [i\delta, (i + 1)\delta), i \geq 0 \), with jumps \( \|\Delta Y^\delta_t\| \leq c \) for some constant \( c \) which is the same for all \( \delta \). Let \( \mathcal{F}^\delta_t = \sigma(Y^\delta_s; s \leq t) \) be the natural filtration of \( Y^\delta_t \). We are interested in conditions for the weak convergence
\[ \text{Law}(Y^\delta) \to \text{Law}(Y), \quad \delta \to 0. \]

To formulate these conditions, introduce the predictable processes \( B^\delta_t, C^\delta_t \) with values in \( \mathbb{R}^M \) and \( \mathbb{R}^{M \times M} \), respectively, defined by
\[ B_t^\delta = \sum_{1 \leq i \leq \lfloor t/\delta \rfloor} \mathbb{E}(\Delta Y^\delta_{i\delta} | \mathcal{F}_{(i-1)\delta}), \tag{27} \]
\[ C_{t,k}^\delta = \sum_{1 \leq i \leq \lfloor t/\delta \rfloor} \left( \mathbb{E}(\Delta Y^\delta_{i\delta} \Delta Y^\delta_{i\delta}^T | \mathcal{F}_{(i-1)\delta}) - \mathbb{E}(\Delta Y^\delta_{i\delta} | \mathcal{F}_{(i-1)\delta}) \mathbb{E}(\Delta Y^\delta_{i\delta}^T | \mathcal{F}_{(i-1)\delta}) \right). \]

These processes are the first and modified second predictable characteristics without truncation of \( Y^\delta_t \), see Jacod and Shiryaev (2002, IX.3.25). Also, on the space \( D(\mathbb{R}_+; \mathbb{R}^M) \) define the functionals
\[ B_t(\alpha) = \int_0^t f(\alpha_s)ds, \quad C_t(\alpha) = \int_0^t g(\alpha_s)\sigma g(\alpha_s)^T ds, \quad \alpha \in D(\mathbb{R}_+; \mathbb{R}^M), \tag{28} \]
so that \( B_t(\alpha) \) and \( C_t(\alpha) \) are the first and second predictable characteristics of \( Y_t \).

The next proposition follows from Theorem IX.3.27 of Jacod and Shiryaev (2002).

**Proposition 2.** Suppose the following conditions hold true:

(a) equation (26) has a unique weak solution;
(b) the functions \( f(y) \) and \( g(y) \) are continuous;
(c) there exists a non-random continuous increasing function \( F(t) \) such that for any \( \alpha \in D(\mathbb{R}_+; \mathbb{R}^M) \) the function \( F(t) - \int_0^t \sum_m |f^m(\alpha_s)|ds - \text{tr}(C_t(\alpha)) \) is increasing;
(d) \( \sup_{s \leq t} ||B_s^\delta - B_s(Y^\delta)|| \to 0 \) in probability as \( \delta \to 0 \) for all \( t > 0 \);
(e) \( C_t^\delta - C_t(Y^\delta) \to 0 \) in probability as \( \delta \to 0 \) for all \( t > 0 \);
(f) \( \sum_{1 \leq i \leq \lfloor t/\delta \rfloor} \mathbb{E}(h(\Delta Y^\delta_{i\delta}) | \mathcal{F}_{(i-1)\delta}) \to 0 \) in probability for all \( t > 0 \) and any continuous bounded function \( h(y) : \mathbb{R}^M \to \mathbb{R} \) which vanishes in a neighborhood of zero.

Then \( \text{Law}(Y^\delta) \) weakly converges to \( \text{Law}(Y) \) as \( \delta \to 0 \).
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