Dirac Mass Matrices in Gauge Field Theory of Horizontal Symmetry

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We investigate Dirac mass matrices derived in the gauge field theory of a horizontal symmetry generated by a central extension of the Pauli algebra. Through numerical analyses of the observed data of the charged fermion masses and the flavor mixing matrix of quarks, values of free parameters in the mass matrices are determined and several empirical relations are found among the Yukawa coupling constants.

Subject Index: 140

§1. Introduction

The Standard Model (SM) of particle physics has no principle to restrict the pattern of Yukawa interactions. Nine complex coupling constants are treated as free parameters in every four sectors consisting of three generations of quarks and leptons. One way to describe order and variety of the generational structure is to postulate a gauge symmetry called the horizontal symmetry. In a previous paper, one of the present authors has proposed a gauge field theory of a new horizontal symmetry. For the new theory to be suitable for the low-energy flavor physics, the mass matrices derived in the theory must successfully describe the fermion mass spectra and the flavor mixing matrices (FMMs). As is well recognized, however, it is very difficult to reproduce the observed values of the quark FMM in the theory with a (local or gauge) flavor symmetry. The purpose of this study is to examine the validity of the gauge field theory of the horizontal symmetry by numerically investigating the Dirac mass matrices of the theory.

The horizontal (H) symmetry of the theory is postulated to be described by the Lie group generated by a central extension of the Pauli algebra, which was found when investigating the FMM of quarks and leptons. The algebra consisting of four generators has the central element identified with the democratic matrix, which can create hierarchical mass spectra of fundamental fermions. The number of Yukawa coupling constants of the theory is reduced to $4/9$ of that of the SM. Through spontaneous breakdowns of the H and electroweak (EW) symmetries, the theory leads to the Dirac mass matrices $M_f$ for the sector $f$ with definite electric charge.

The matrix $M_f$ with unique non-Hermitian structure possesses four unknown complex parameters. To deduce information on the $f$ sector, we must solve the eigen-
value problem of the Hermitian matrix $M_f M_f^\dagger$. It should be noted that unspecified quantities involved in these Hermitian matrices are reduced to ten real parameters. Consequently, numerical analysis of the experimental data\(^9\) on the six masses and four FMM parameters of the quark sector enables us to determine the values of ten real parameters, from which we find several empirical relations among the Yukawa coupling constants.

The Dirac mass matrices deduced from the Lagrangian density of the Yukawa interactions are shown explicitly in §2. We describe formalisms for the eigenvalue problem of $M_f M_f^\dagger$ in §§3 and 4. Different mass orderings in the up and down quark sectors are explained in §5. In §6, values of the parameters in the mass matrices are determined by numerical analysis, and specific empirical relations are found among the Yukawa coupling constants. Discussion is given in §7. We provide a brief survey on the central extension of the Pauli algebra and a short discussion of the setup of the gauge field theory of the H symmetry and of the derivation of the Dirac mass matrices in Appendix A, and examine the functional dependence among parameters and mass eigenvalues in Appendix B.

\section{2. Dirac mass matrices}

In the new gauge field theory,\(^4\) the Lagrangian density of the fermion-scalar interaction possesses four Yukawa coupling constants $Y_{fi}$ ($i = 1, \ldots, 4$), for each EW sector ($f = u, d, \nu, e$), which specify four invariant terms of the H and EW symmetries. The H symmetry is broken at a certain high-energy scale. Subsequently, the effects of the renormalization group must be considered to go down around the weak scale $v = 246$ GeV.\(^10\) The Yukawa coupling constants run down to the values at the scale $v$. Then, as briefly shown in Appendix A, the breakdown of the EW symmetry leads to the effective Lagrangian density for Dirac masses. Using the same symbols $Y_{fi}$ for the Yukawa constants including the renormalization effects, we obtain\(^4\)

$$
\mathcal{L}_M^Y = \sum_{f=u,d} \bar{\Psi}_f^L M_f \Psi_f^R + \sum_{f=\nu,e} \bar{\Psi}_f^L M_f \Psi_f^R + \text{h.c.}, \quad (2.1)
$$

in which $\Psi_f^L$ and $\Psi_f^R$ are chiral fermion fields and $M_f$ is the Dirac mass matrix. For the up sectors ($f = u, \nu$) of the EW symmetry, the mass matrix is given by

$$
M_f = a_f I + \frac{1}{\sqrt{3}} b_{f1} \begin{pmatrix}
1 & 1 & 1 \\
-1 & -1 & -1 \\
0 & 0 & 0
\end{pmatrix} + \frac{1}{3} b_{f2} \begin{pmatrix}
-1 & -1 & 2 \\
-1 & -1 & 2 \\
-1 & -1 & 2
\end{pmatrix} + c_f \tilde{D}, \quad (2.2)
$$

where

$$
\tilde{D} = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} \quad (2.3)
$$

is the democratic element.\(^8\) The four coefficients in the mass matrix are expressed in terms of the Yukawa coupling constants and the vacuum expectation value $v$ of
the scalar field as \( a_f = Y_{f1} v, b_{f1} = -Y_{f2} v, b_{f2} = Y_{f3} v, \) and \( c_f = 3Y_{f4} v. \) For the down sectors \((f = d, e)\), we find

\[
\mathcal{M}_f = a_f I + b_{f1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{\sqrt{3}} b_{f2} \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} + c_f \bar{D},
\]

where \( a_f = Y_{f1} v, b_{f1} = Y_{f2} v, b_{f2} = Y_{f3} v, \) and \( c_f = 3Y_{f4} v. \)

To diagonalize the non-Hermitian mass matrix \( \mathcal{M}_f \), it is necessary to have recourse to the bi-unitary transformation

\[
V_L^f \mathcal{M}_f V_R^f = \mathcal{M}_{f \text{diagonal}}.
\]

To derive the mass eigenvalues and diagonalizing matrix \( V_L^f \), it is necessary to solve the eigenvalue problem for the self-adjoint matrix \( \mathcal{M}_f \mathcal{M}_f^\dagger \) as

\[
\mathcal{M}_f \mathcal{M}_f^\dagger |v^{(f)i}\rangle = m_i^{(f)2} |v^{(f)i}\rangle
\]

for each charged fermion sector \((f = u, d, e)\). The diagonalizing matrix \( V_L^f \) is obtained in terms of the eigenvectors \( |v^{(f)i}\rangle \), and the FMM of the quark sector is constructed by

\[
V = V_L^{u\dagger} V_L^d = \left( \langle v^{(u)i}|v^{(d)j}\rangle \right),
\]

provided that the eigenvectors are arranged in increasing orders of the masses for up and down sectors.

§3. Eigenvalue problem 1: Quark FMM

To solve the eigenvalue problem for \( \mathcal{M}_f \mathcal{M}_f^\dagger \), it is convenient to use the basis vectors

\[
|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad |2\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad |3\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

which are eigenvectors of the democratic element \( \bar{D}. \) With these bases, the eigenvector in (2.6) is expanded as

\[
|v^{(f)}\rangle = x_f |1\rangle + y_f |2\rangle + z_f |3\rangle.
\]

For the up quark sector, (2.6) is rewritten for the coefficients of \( |v^{(u)}\rangle \) as

\[
\begin{pmatrix} |a_u|^2 + 2|b_{u1}|^2 & 0 & \sqrt{2}C_u^* b_{u1} \\ 0 & |a_u|^2 & -\sqrt{2}a_u b_{u2}^* \\ \sqrt{2}C_u b_{u1}^* & -\sqrt{2}a_u b_{u2} & |C_u|^2 + 2|b_{u2}|^2 \end{pmatrix} \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} = m^{(u)2} \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix},
\]

(3.3)
where \( C_u = c_u + a_u \). Similarly, for the down quark sector, we obtain

\[
\begin{pmatrix}
|a_d|^2 & 0 & \sqrt{2}a_d b_{d2}^* \\
0 & |a_d|^2 + 2|b_{d1}|^2 & -\sqrt{2}C_d^* b_{d1} \\
\sqrt{2}a_d^* b_{d2} & -\sqrt{2}C_d b_{d1}^* & |C_d|^2 + 2|b_{d2}|^2
\end{pmatrix}
\begin{pmatrix} x_d \\ y_d \\ z_d \end{pmatrix} = m^{(d)2} \begin{pmatrix} x_d \\ y_d \\ z_d \end{pmatrix}
\]

(3.4)

for the coefficients of \( |u^{(d)}| \), where \( C_d = c_d + a_d + b_{d1} \).

To clarify the counting of independent parameters in these equations and the FMM, we define phase factors by

\[
C_u b_{u1} = |C_u b_{u1}| e^{i\mu_u}, \quad a_u b_{u2} = |a_u b_{u2}| e^{i\nu_u}, \quad a_d b_{d2} = |a_d b_{d2}| e^{i\mu_d}, \quad C_d^* b_{d1} = |C_d b_{d1}| e^{i\nu_d}
\]

(3.5)

and introduce the diagonal phase matrix

\[
P^f = \text{diag}(e^{i\mu_f}, e^{i\nu_f}, 1)
\]

(3.6)

for the \( f \) sector to adjust phase factors. By separating the diagonal phase matrices, the eigenvectors of (3.3) and (3.4) are found, respectively, in the forms

\[
P^u u^u_j = P^u N^u_j \begin{pmatrix} \sqrt{2}|C_u b_{u1}|(m_j^{(u)2} - |a_u|^2) \\ -\sqrt{2}|a_u b_{u2}|(m_j^{(u)2} - |a_u|^2 - 2|b_{u1}|^2) \\ (m_j^{(u)2} - |a_u|^2)(m_j^{(u)2} - |a_u|^2 - 2|b_{u1}|^2) \end{pmatrix}
\]

(3.7)

and

\[
P^d u^d_j = P^d N^d_j \begin{pmatrix} \sqrt{2}|a_d b_{d2}|(m_j^{(d)2} - |a_d|^2 - 2|b_{d1}|^2) \\ -\sqrt{2}|C_d b_{d1}|(m_j^{(d)2} - |a_d|^2) \\ (m_j^{(d)2} - |a_d|^2)(m_j^{(d)2} - |a_d|^2 - 2|b_{d1}|^2) \end{pmatrix},
\]

(3.8)

where \( m_j^{(f)2} \) represents eigenvalues of squared masses and \( N_j^f \) indicates the normalization constants. Then, with the orthogonal matrices

\[
O^f_L = \begin{pmatrix} u^f_1, & u^f_2, & u^f_3 \end{pmatrix}
\]

(3.9)

consisting of the vectors \( u^f_j \), the FMM for the quark sector is calculated to be

\[
V = O^u_L P O^d_L
\]

(3.10)

with the diagonal phase matrix

\[
P = \text{diag}(e^{i\mu}, e^{i\nu}, 1) = P^u_p P^d, \quad \mu = \mu_d - \mu_u, \quad \nu = \nu_d - \nu_u.
\]

(3.11)

This FMM includes unknown parameters of eight real numbers and two phases. The secular equations for the eigenvalue problems in (3.3) and (3.4) work to fix six real parameters in terms of the mass eigenvalues. Consequently, two real numbers and two phases remain unspecified in the FMM for the quark sector.
§4. Eigenvalue problem 2: Mass spectra of charged fermions

Since the secular equations for both the eigenvalue problems in (3.3) and (3.4) take the same form, the suffix \( f \) is omitted for all the quantities in this section and the Appendix. In terms of the shifted variable \( s = m^2 - |a|^2 \), the secular equation is obtained as

\[
s^3 - (|C|^2 - |a|^2 + 2|b|^2) s^2 - 2(|ab|^2 - 2|b_1 b_2|^2) s + 4|ab_1 b_2|^2 = 0, \tag{4.1}
\]

where

\[
|b|^2 = |b_1|^2 + |b_2|^2. \tag{4.2}
\]

Let us solve this equation by the Cardano method. Introducing the dimensionless quantities

\[
P = \frac{1}{3} \frac{|C|^2 - |a|^2 + 2|b|^2}{|ab_1 b_2|^2}, \quad Q = \frac{2}{3} \frac{|a|^2 |b|^2 - 2|b_1 b_2|^2}{|ab_1 b_2|^2}, \tag{4.3}
\]

and changing the variable by \( s = |ab_1 b_2|^2 \frac{3}{2} (t + P) \), we obtain the reduced proper equation without the second-order term as

\[
t^3 - 3(P^2 + Q)t - 2P^3 - 3PQ + 4 = 0. \tag{4.4}
\]

One solution of this equation is determined as \( t = t_+ + t_- \) by the sum of two quantities \( t_+ \) and \( t_- \), which are subject to the relations

\[
t_+^3 = \frac{1}{2} \left( 2P^3 + 3PQ - 4 \pm i\sqrt{|D|} \right), \tag{4.5}
\]

where

\[
D = -16P^3 - 3P^2 Q^2 - 24PQ - 4Q^3 + 16. \tag{4.6}
\]

In the analysis below, it is appropriate to use the polar representation \( t_+ = \rho e^{i\theta} \) and \( t_- = \rho e^{-i\theta} \) in which \( \rho \) and \( \theta \) are expressed, in terms of \( P \) and \( Q \), as

\[
\rho = \sqrt{P^2 + Q}, \quad \tan 3\theta = \frac{\sqrt{|D|}}{2P^3 + 3PQ - 4}. \tag{4.7}
\]

Then, the three solutions of (4.4) are derived to be

\[
\begin{align*}
t_1 &= \omega t_+ + \omega^2 t_- = 2\rho \cos(\theta + \frac{2\pi}{3}), \\
t_2 &= \omega^2 t_+ + \omega t_- = 2\rho \cos(\theta + \frac{4\pi}{3}), \\
t_3 &= t_+ + t_- = 2\rho \cos \theta,
\end{align*} \tag{4.8}
\]

where \( \omega = \exp(i2\pi/3) \).

In this way, we have solved the eigenvalue problems in (3.3) and (3.4) obtaining the squared masses as follows:

\[
\begin{align*}
m_1^2 &= |a|^2 + \frac{1}{3} \left( |C|^2 - |a|^2 + 2|b|^2 \right) \left[ 1 + 2\sqrt{1 + \delta \cos(\theta + \frac{2\pi}{3})} \right], \\
m_2^2 &= |a|^2 + \frac{1}{3} \left( |C|^2 - |a|^2 + 2|b|^2 \right) \left[ 1 + 2\sqrt{1 + \delta \cos(\theta + \frac{4\pi}{3})} \right], \\
m_3^2 &= |a|^2 + \frac{1}{3} \left( |C|^2 - |a|^2 + 2|b|^2 \right) \left[ 1 + 2\sqrt{1 + \delta \cos \theta} \right], \tag{4.9}
\end{align*}
\]
where the parameter
\[
\delta = 6 \frac{|a|^2 |b|^2 - 2 |b_1 b_2|^2}{(|C|^2 - |a|^2 + 2 |b|^2)^2}
\]  
(4.10)
is introduced to simplify the expressions.

As confirmed below, the magnitude of the angle $|\theta|$ must be sufficiently small for the mass spectra to have hierarchical structure. Here, it is crucially important to note that the squared masses have orderings $m_1^2 < m_2^2 < m_3^2$ and $m_2^2 < m_1^2 < m_3^2$, respectively, for $\theta > 0$ and $\theta < 0$.

The squared masses in (4.9) depend on the four real quantities $|a|^2$, $|b_1|^2$, $|b_2|^2$, and $|C|^2$. In the present analysis, we interpret inversely that $|b_1|^2$, $|b_2|^2$, and $|C|^2$ are functions of the masses and the parameter $|a|^2$. Then, all the quantities for the quark FMM are determined in terms of the observed quark masses and the independent parameter $|a|$. (See Appendix B).

In the following sections, we numerically analyze the quark FMM using experimental values of quark masses as inputs and by adjusting the independent parameters $|a_u|$ and $|a_d|$.

\section*{§5. Mass orderings of the up and down quark sectors}

For numerical analyses below, we use the quark masses at the energy scale of the $Z$ boson, i.e., $m_Z = 91.2$ GeV. The values calculated using the renormalization group equations\textsuperscript{12}) are given as\textsuperscript{13})
\[
m_u = 1.27^{+0.50}_{-0.42} \text{MeV}, \quad m_c = 0.619 \pm 0.084 \text{GeV}, \quad m_t = 171.7 \pm 3.0 \text{GeV},
\]
\[
m_d = 2.90^{+1.24}_{-1.19} \text{MeV}, \quad m_s = 55^{+16}_{-15} \text{MeV}, \quad m_b = 2.89 \pm 0.09 \text{GeV}. \quad (5.1)
\]

The observed FMM of the quark sector has the prominent feature that the matrix elements decrease in number rapidly for each step away from the diagonal. To reproduce such characteristics, both the orthogonal matrices $O^u_L$ and $O^d_L$ in (3.9) must approximately be close to the unit matrix. Accordingly, as a step for the FMM analysis, it is reasonable to examine numerically which of the solutions characterized by $\theta > 0$ and $\theta < 0$ in (4.9) and the associated eigenvectors in (3.7) and (3.8) can bring the orthogonal matrix nearer to the unit matrix.

Let us apply the positive-$\theta$ solution with the experimental mass values in (5.1) to examine the orthogonal matrices. For the down quark sector, it is proved that all the diagonal elements of the matrix $O^d_L$ can approach the unit matrix for a small value of $|a_d|$. Contrastingly, for the up quark sector, some of the diagonal elements of the matrix $O^u_L$ is shown to be smaller than 1 for any value of $|a_u|$. The situation reverses completely for the negative-$\theta$ solution. Numerical calculations with the negative-$\theta$ solution show that, while $O^u_L$ approaches the unit matrix by adjusting $|a_u|$, $O^d_L$ cannot be made close to the unit matrix for any value of $|a_d|$.

Accordingly, it is necessary to choose the positive- and negative-$\theta$ solutions, respectively, for the down and up quark sectors to reproduce the experimental results of the quark FMM. The masses and state vectors of the observed down quark members ($d$, $s$, $b$) must be described using the positive-$\theta$ solution with normal ordering.
as follows:

\[ m_d = m_1^{(d)}, \quad m_s = m_2^{(d)}, \quad m_b = m_3^{(d)}; \]
\[ |v_d⟩ = |v^{(d)1}⟩, \quad |v_s⟩ = |v^{(d)2}⟩, \quad |v_b⟩ = |v^{(d)3}⟩. \]  

(5.2)

As for the observed up quark members \((u, c, t)\), their masses and state vectors have to be identified with the quantities of the negative-\(\theta\) solutions with partly reversed ordering as follows:

\[ m_u = m_2^{(u)}, \quad m_c = m_1^{(u)}, \quad m_t = m_3^{(u)}; \]
\[ |v_u⟩ = |v^{(u)2}⟩, \quad |v_c⟩ = |v^{(u)1}⟩, \quad |v_t⟩ = |v^{(u)3}⟩. \]  

(5.3)

In the present theory, it is crucial to accept these interpretations of the solutions of the eigenvalue problems in (2.6).

§6. Hierarchical structure of the Yukawa coupling constants

As shown in Appendix B, the quantities \(|b_{f1}|, |b_{f2}|, \text{and } |C_f|\) are expressed in terms of the masses and the adjustable parameter \(|a_f|^2\) in the hierarchical approximation \((m_3^{(f)} \gg m_1^{(f)}, m_2^{(f)})\). Using these results and accepting the interpretations in (5.2) and (5.3), we numerically analyze the quark FMM. The best fit to the FMM data of the Particle Data Group\(^9\) is obtained with the following values of the four parameters as

\[ |a_u| = 30.4 \text{ MeV}, \quad |a_d| = 13.2 \text{ MeV}, \quad \mu = 0.96, \quad \nu = 2.32 \]  

(6.1)

which lead to the magnitude of the elements of the quark FMM and the Jarlskog invariant measure for the \(CP\) violation\(^14\) as

\[ |V| = \begin{pmatrix} 0.974210 & 0.225705 & 0.003595 \\ 0.225473 & 0.973343 & 0.041511 \\ 0.008723 & 0.040746 & 0.999132 \end{pmatrix}, \quad J = 3.1 \times 10^{-5}. \]  

(6.2)

The values of all the parameters for the best fit found above are listed in Table I.

Results in Table I show clearly that the real parameters satisfy the hierarchical orderings \(|a_f|^2 \ll |b_{f1}|^2 \ll |b_{f2}|^2 \ll |C_f|^2\) for each quark sector. By making a more careful comparison among them, we find approximate relations

\[ \frac{|b_{u1}|}{|b_{d2}|} \sim 1, \quad \frac{|b_{d2}|}{|b_{d1}|} \sim 9^1, \quad \frac{|b_{u2}|}{|b_{u1}|} \sim 9^2. \]  

(6.3)

| Table I. Values of ten parameters. |
|-----------------------------------|
| up quark (MeV) | down quark (MeV) | phases |
|----------------|--------------------|--------|
| \(|a_u| = 30.4 | |a_d| = 13.2 | \mu = 0.96 |
| \(|b_{u1}| = 831 | |b_{d1}| = 92.1 | \nu = 2.32 |
| \(|b_{u2}| = 63800 | |b_{d2}| = 818 |
| \(|C_u| = 146000 | |C_d| = 2650 |
we can do nothing but determine in terms of seven quantities which verify all the relations in (6.3), and reinvestigate the quark masses and FMM among the four Yukawa coupling constants.

Thus far, the Yukawa coupling constants of quark sectors are calculated so as to recreate the observed data of the quark FMM. This approach is not applicable, as it stands, to the lepton sector. Here, let us find an approximate scheme with less numbers of adjustable parameters based on the benefit of hindsight on the empirical relations in (6.4) and apply it to analyze the masses of charged leptons.

Using \( v = 246 \text{ GeV} \) and the data in Table I, we can fix the magnitudes of the Yukawa coupling constants of the quark sectors as in Table II. In the somehow crude approximation for \( |c_f| \), the constant \( |Y_{f4}| \) has a comparatively large uncertainty.

Thus far, the Yukawa coupling constants of quark sectors are calculated so as to recreate the observed data of the quark FMM. This approach is not applicable, as it stands, to the lepton sector. Here, let us find an approximate scheme with less numbers of adjustable parameters based on the benefit of hindsight on the empirical relations in (6.4) and apply it to analyze the masses of charged leptons.

For its purpose, we introduce a new variable \( \beta \) using the equations

\[
|b_{d1}| = 9\beta, \quad |b_{d2}| = |b_{u1}| = 9^2\beta, \quad |b_{u2}| = 9^4\beta, \quad (6.5)
\]

which verify all the relations in (6.3), and reinvestigate the quark masses and FMM in terms of seven quantities \( |a_u|, |a_d|, |C_u|, |C_d|, \mu, \nu, \text{ and } \beta \). It turns out possible to explain all the observed data within the range of experimental errors. In fact, by using the values of seven quantities in Table III, we obtain

\[
m_u = 1.20 \text{ MeV}, \quad m_e = 0.628 \text{ GeV}, \quad m_t = 169.6 \text{ GeV},
\]
\[
m_d = 2.87 \text{ MeV}, \quad m_s = 53.8 \text{ MeV}, \quad m_b = 2.86 \text{ GeV}, \quad (6.6)
\]
Fig. 1. Yukawa coupling constants $|Y_{f1}|$, $|Y_{f2}|$ and $|Y_{f3}|$.

for the six quark masses and

$$|V| = \begin{pmatrix}
0.973984 & 0.226589 & 0.003509 \\
0.226452 & 0.973161 & 0.040962 \\
0.008616 & 0.040199 & 0.999155
\end{pmatrix}, \quad J = 2.9 \times 10^{-5} \quad (6.7)$$

for the magnitudes of the quark FMM elements and the Jarlskog parameter.

For both sectors of the down quark and the charged lepton, the Dirac mass matrices take the common forms in (2.4). To analyze the masses of the charged leptons in this approximate scheme, we assume that the empirical equations related to the down quarks in (6.3) hold also in the charged lepton sector. Then, following the equations in (6.5), we introduce a new variable $\beta_e$ using the relations

$$|b_{e1}| = 9\beta_e, \quad |b_{e2}| = 9^2\beta_e \quad (6.8)$$

and analyze the charged lepton masses in terms of the three parameters $|a_e|$, $|C_e|$, and $\beta_e$. Numerical estimation shows readily that the charged lepton masses

$$m_e = 0.4866 \text{ MeV}, \quad m_\mu = 102.7 \text{ MeV}, \quad m_\tau = 1746 \text{ MeV} \quad (6.9)$$

can be reproduced by adjusting the parameters as follows:

$$|a_e| = 8.537 \text{ MeV}, \quad |C_e| = 1197 \text{ MeV}, \quad \beta_e = 11.06 \text{ MeV} \quad (6.10)$$

from which the Yukawa coupling constants for the charged lepton sector can be fixed provided that $|c_e| \approx |C_e|$. The results are included in Table II. Figure 1 shows the behavior of the Yukawa coupling constants of charged fermion sectors.

§ 7. Discussion

We have investigated the non-Hermitian mass matrices $M_f$ in (2.2) and (2.4) deduced in the gauge field theory of the horizontal symmetry generated by the central
extension of the Pauli algebra. While the matrix $M_f$ possesses four complex unknown parameters, the Hermitian combination $M_f M_f^\dagger$ depends on four real numbers and two phases. Owing to this fact, we are able to numerically analyze the problem of the six quark masses and the quark FMM with four parameters.

To solve the problem effectively, six real parameters are interpreted as functions of quark masses and the remaining two parameters. Using the observed masses as inputs and adjusting the values of two independent parameters and two phases, we have reproduced the observed FMM and succeeded in estimating the values of the effective Yukawa coupling constants in the low-energy regime.

Here, it should be noted\(^5\) that it is not easy for the theory with a flavor symmetry to explain the observed values of the quark FMM. Therefore, the successful results of the present analysis have shown that the gauge field theory of the H symmetry\(^4\) is one of the tenable basic frameworks that can provide an effective phenomenology for flavor physics.

By careful examination of the numerical results, empirical relations in (6.4) are found among the coupling constants for the quark sector. The estimated values of the Yukawa coupling constants in Table II show orderings with a sizable amount of variation. It is beyond the scope of the present theory to elucidate the physical implications of those relations.

All the results of numerical analyses so far were obtained in the hierarchical approximation ($m_3^{(f)2} \gg m_1^{(f)2}, m_2^{(f)2}$) using the formulae in Appendix B, which express $|b_f_1|, |b_f_2|$, and $|C_f|$ as functions of $|a_f|^2$. It should be noted, however, that we can find almost the same results by directly adjusting the values of real quantities $|b_f_1|, |b_f_2|, |C_f|$, and $|a_f|^2$ and two phases $\mu$ and $\nu$.

The squared masses of the charged fermions are derived in the exact formula in (4.9). Depending on the sign of the angle $\theta$, the formula has the solutions with normal and partly reversed orderings in (5.2) and (5.3). As confirmed in §5, we have to describe the down and up quark states using the normal and partly reversed solutions, respectively, to reproduce the observed FMM. This interpretation is necessary in the present formalism where the same parametrization is used for the mass expressions of up and down quark sectors.

We made a speculative analysis of the charged lepton masses by postulating hypothetical relations (6.8) as analogues of the equations for the down quark sector in (6.5). To proceed with a full investigation of the charged and neutral lepton sectors, however, it is necessary to solve the combined eigenvalue problem, which involves not only the Dirac mass matrices but also the Majorana mass matrix deduced for the neutrino sector.\(^4\) We will study this problem in the near future.

**Appendix A**

*Derivation of the Dirac Mass Matrices*

The Lie group being identified with the H symmetry group in this paper is generated by the central extension of the Pauli algebra, whose elements are represented...
in terms of the Gell-Mann matrices $\lambda_j$ ($j = 1, 2, \cdots, 8$) as follows:\(^4\)

\[
\begin{align*}
\bar{D} &= \frac{1}{3}(I + \lambda_1 + \lambda_4 + \lambda_6), \quad \bar{\tau}_1 = \frac{1}{\sqrt{3}}(\lambda_3 - \lambda_4 + \lambda_6), \\
\bar{\tau}_2 &= \frac{1}{\sqrt{3}}(\lambda_2 - \lambda_5 + \lambda_7), \quad \bar{\tau}_3 = \frac{1}{3}(-2\lambda_1 + \lambda_4 + \lambda_6 + \sqrt{3}\lambda_8).
\end{align*}
\] (A.1)

The fundamental representation of the H symmetry group forms spinor triplets. For an arbitrary spinor triplet $T$, the sum of its elements expressed by

\[
\{T\} = \sum_{i=1}^{3} T_i = \sqrt{3} \langle 3 | T \rangle
\] (A.2)

is invariant under the transformation of $SU_H(2)$ subgroup generated by the subalgebra \{\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3\}.

In a high-energy regime where the H symmetry holds, three generations of fermions are postulated to form the spinor triplet of the H symmetry as

\[
\Psi^f_h(x) = \left( \begin{array}{c} \psi^f_{k_1}(x) \\ \psi^f_{k_2}(x) \\ \psi^f_{k_3}(x) \end{array} \right),
\] (A.3)

where $f (= q, u, d; l, \nu, e)$ distinguishes EW multiplets and $h (= L, R)$ refers to chiral components. Specifically, $\Psi^f_L(=\Psi^q_L, \Psi^d_L)$ is the H triplet of EW doublets and $\Psi^f_R(=\Psi^u_R, \Psi^d_R, \Psi^\nu_R, \Psi^e_R)$ is that of EW singlets.

Our theory\(^4\) possesses high- and low-energy scales, $\bar{v}$ and $v$. To break the H symmetry at the high-energy scale, we introduce a Higgs H triplet consisting of the EW singlet scalar fields. This Higgs triplet does not couple with the fermion triplet except for the Majorana-type interaction. The EW symmetry is broken at the low-energy scale $v$ by another Higgs H triplet, $\tilde{\Phi}$, composed of the EW doublet scalar fields. To form the Yukawa interaction for the up quark sector, we introduce an associate scalar triplet of $\Phi$ defined by $\tilde{\Phi} = (i\bar{\tau}_2)(i\bar{\tau}_2)\Phi$. The triplets $\Phi$ and $\tilde{\Phi}$ have the same transformation property under the group action of the H and EW symmetries.

The Lagrangian density $\mathcal{L}^f_Y$ of the Yukawa interaction is constructed by summing up all the EW×H invariants consisting of the bilinear forms of fermion triplets with the Higgs triplets $\Phi$ and $\tilde{\Phi}$. The density is given by

\[
\begin{align*}
\mathcal{L}^f_Y &= Y_{f_1} \bar{\Psi}^{f_f} i\bar{\tau}_2 \{\Phi^*\} \Psi^f_R + Y_{f_2} \bar{\Psi}^{f_f} \tilde{\Phi} \{\Psi^f_R\} \\
&\quad + Y_{f_3} \{\bar{\Psi}^{f_f}_L\} i\bar{\tau}_2 \Psi^f_R + Y_{f_4} \{\bar{\Psi}^{f_f}_L\} i\bar{\tau}_2 \{\Phi^*\} \{\Psi^f_R\} + \text{h.c.}
\end{align*}
\] (A.4)

for the EW up sectors ($f' = q, f = u$) and ($f' = l, f = \nu$), and

\[
\begin{align*}
\mathcal{L}^f_Y &= Y_{f_1} \bar{\Psi}^{f_f} \{\Phi\} \Psi^f_R + Y_{f_2} \bar{\Psi}^{f_f} \Phi \{\Psi^f_R\} \\
&\quad + Y_{f_3} \{\bar{\Psi}^{f_f}_L\} i\bar{\tau}_2 \Psi^f_R + Y_{f_4} \{\bar{\Psi}^{f_f}_L\} \{\Phi\} \{\Psi^f_R\} + \text{h.c.}
\end{align*}
\] (A.5)
for the down sectors \((f' = q, f = d)\) and \((f' = l, f = e)\). Here, \(Y_{fi}(i = 1, \ldots, 4)\) are the Yukawa coupling constants and the symbol \(\hat{t}\) indicates to take the transposition operation for the H degrees of freedom.

To deduce an effective theory for flavor physics in the low-energy regime, we assume that the Yukawa coupling constants are affected by the renormalization group evolution (RGEV) from the high-energy scale \(\hat{v}\) to the low-energy scale \(v\). In the present analysis, it is important to recognize that although the quark mass values are somehow sensitive, their ratios in each (up and down) sector are rather insensitive to the RGEV. Analysis by Kielanowski et al.\(^{15}\) had demonstrated, at one loop evolution, that the CKM matrix depends on the energy only through one function of energy and that the corrections are of the relative order \(\lambda^5\) where \(\lambda = \sin \theta_{\text{Cabbibo}} \approx 0.22\). By assuming that the neutral component of the third doublet for the H triplet \(\Phi\) has a value \(v\) in (A.4) and (A.5), we derive the effective Lagrangian (2.1) with the Dirac mass matrices (2.2) and (2.4). It is necessary to interpret that the Yukawa coupling constants in the mass matrices (2.2) and (2.4) include the effects of the RGEV. For the sake of simplicity, however, we use the same symbols for the Yukawa coupling constants in both expressions for the Lagrangian density in the high-energy regime and in the Dirac mass matrices in the low-energy regime.

Appendix B

--- Dependence of \(|b_1|, |b_2|, \text{ and } |C|\) on \(|a|\) ---

In this scheme, we interpret the real quantities \(|b_1|, |b_2|, \text{ and } |C|\) as functions of the quark masses and the parameter \(|a|\). For this purpose, it is appropriate to define a mass \(M\) for reference using the relation

\[
m_3^2 + m_2^2 + m_1^2 = |C|^2 + 2|b|^2 + 2|a|^2 = \frac{3M^2}{2}.
\]

(B.1)

The observed data in (5.1) clearly show hierarchical orderings of squared masses, \(M^2, m_3^2 \gg m_2^2, m_1^2\), for each sector. To deduce convenient relations for numerical analyses, we express all the quantities as power series of \(M^2\).

In the hierarchical limit, \(P\) and \(|C|^2\) take very large values. Equation (4.7) shows that \((\tan 3\theta)^2\) can be decomposed in the power series of \(M^{-2}\), leading to the approximate relation

\[
(3\theta)^2 \simeq 4 |ab_1b_2|^2 \frac{M^6}{M^6} + \frac{1}{3} \left( |a|^2 |b|^2 - 2|b_1b_2|^2 \right)^2 \frac{M^8}{M^8}.
\]

(B.2)

Similarly, the quantity \(\delta\) defined by (4.10) is approximated by

\[
\delta \simeq \frac{2}{3} \frac{|a|^2 |b|^2 - 2|b_1b_2|^2}{M^4}.
\]

(B.3)

Decomposition of the eigenvalues \(m_2^2\) and \(m_1^2\) in (4.9) with respect to \(M^{-2}\) results in the following expressions:

\[
m_2^2 + m_1^2 \simeq 2 |a|^2 - M^2 \delta, \quad m_2^2 - m_1^2 \simeq 2\sqrt{3}M^2 \theta.
\]

(B.4)
Eliminating $\theta$ and $\delta$ from these equations, we obtain the relations
\[ |b_1 b_2|^2 = \frac{3}{4|a|^2} (m_2^2 - |a|^2)(|a|^2 - m_1^2) M^2, \quad |b|^2 = \frac{3}{2} \frac{|a|^4 - m_1^2 m_2^2}{|a|^4} M^2. \] (B.5)

Then, substitution of the last expression for $|b|^2$ into the defining equation (B.1) allows to express $|C|^2$ in the form
\[ |C|^2 = \frac{3}{4} \frac{m_1^2 m_2^2}{|a|^4} M^2 - 2|a|^2. \] (B.6)

Finally, to distinguish between $|b_1|$ and $|b_2|$, we introduce an extra quantity $\kappa$ as
\[ |b_1|^2 = \frac{1}{2} |b|^2 - M\kappa, \quad |b_2|^2 = \frac{1}{2} |b|^2 + M\kappa. \] (B.7)

From (B.5), $\kappa$ is calculated to be
\[ \kappa = \sqrt{\frac{3}{4|a|^2} \left( \frac{3}{4} \frac{|a|^4 - m_2^2 m_1^2}{|a|^6} M^2 - (m_2^2 - |a|^2)(|a|^2 - m_1^2) \right)}. \] (B.8)

Consequently, the quantities $|C|$, $|b_1|$, and $|b_2|$ are determined as the functions of quark masses and the parameter $|a|$ for each sector. The quark FMM is now expressible in terms of the observed mass values and the four adjustable parameters $|a_u|$, $|a_d|$, $\mu$, and $\nu$.

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