Relaxed Large Economies with Infinite-Dimensional Commodity Spaces: The Existence of Walrasian Equilibria∗

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Abstract

Whereas “convexification by aggregation” is a well-understood procedure in mathematical economics, “convexification by randomization” has largely been limited to theories of statistical decision-making, optimal control and non-cooperative games. In this paper, in the context of classical Walrasian general equilibrium theory, we offer a comprehensive treatment of relaxed economies and their relaxed Walrasian equilibria: our results pertain to a setting with a finite or a continuum of agents, and a continuum of commodities modeled either as an ordered separable Banach space or as an $L^\infty$-space. As a substantive consequence, we demonstrate that the convexity hypothesis can be removed from the original large economy under the saturation hypothesis, and that existing results in the antecedent literature can be effortlessly recovered.

Keywords: Relaxed large economy; Walrasian equilibrium; Saturated measure space; Lyapunov convexity theorem; Purification principle; Relaxed control.

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Contents

1 Introduction 2

2 Preliminaries 5
   2.1 Relaxed Controls ................................. 6
   2.2 The Purification Principle in Saturated Measure Spaces .... 7
   2.3 Gelfand Integrals in $L^\infty$ ........................ 9

3 Relaxed Large Economies 9
   3.1 Relaxation of Large Economies ........................ 9
   3.2 Relaxed Demand Sets ............................... 11
   3.3 Relaxed Walrasian Equilibria ........................ 12
   3.4 Existence of Walrasian Equilibria with Free Disposal .... 14
   3.5 Existence of Walrasian Equilibria on $L^\infty$ .............. 17

4 Concluding Summary 19
1 Introduction

Large economies were introduced by Aumann (1966) as a prototype of perfect competition, and he demonstrated the existence of a Walrasian equilibrium in the setting of a finite-dimensional commodity space and a continuum of agents modeled as a nonatomic finite measure space. In his theorem, Aumann dispensed with the assumption of convexity of preferences: an insight rather remarkable for its time. Since then, there have been several attempts to extend the theorem to a setting with an infinite-dimensional commodity space, and they have all stumbled on the well-known failure of the Lyapunov convexity theorem in infinite dimensions, and thereby failed to subdue the possible nonconvexity of the aggregate demand set. This is the reason why convexity assumptions on preferences are pervasive and inevitable even under the nonatomicity hypothesis on the set of agents, as in Khan and Yannelis (1991); Noguchi (1997) for ordered separable Banach spaces, and in Bewley (1991) for $l^\infty$. This has led to efforts to strengthen the hypothesis along the intuition that there be “many more agents than commodities”: in addition to Martins-da-Rocha (2003); Rustichini and Yannelis (1991), see Podczeck (1997) for its formalization of the intuition as a condition on the nonatomic disintegration of the population measure of agents.

However, recent work has established the validity of the Lyapunov convexity theorem, and as its corollary, the convexity property of the integral of a multifunction, under the reasonable assumption that the underlying measure space of agents is a saturated space. Both separable Banach spaces and the dual of separable Banach spaces have been considered, and indeed, this saturation property has been shown to be both necessary and sufficient for the results; see Khan and Sagara (2013, 2015, 2016); Podczeck (2008); Sun and Yannelis (2008). It is thus natural then that one looks for a generalization of Aumann’s theorem under the assumption of a saturated measure space of agents by exploiting the (exact) convexity of the aggregate demand.
set in the setting of an infinite-dimensional commodity space even when the individual demand sets are not convex. Such a result has now been satisfactorily executed by Lee (2013), and he also handles the difficulty arising from the failure of the joint continuity of the valuation functional.\(^1\) However, it bears emphasis that the result in Lee (2013) does not apply to economies with a finite set of agents, but only to those with a continuum.\(^2\)

In this paper, we take an approach alternative to “convexification by aggregation”, by drawing on “convexification by randomization”, a completely different operation from aggregation, and one that is also valid for economies with a finite set of agents. This procedure, and its terminology of “relaxed controls”, is well-established in optimal control theory, and explored in, by now classical, work of McShane (1967); Warga (1972); Young (1969). To be sure, this is not an altogether novel direction, and it has been pursued with varying emphases in Prescott and Townsend (1984), Nowak (1992), and Balder (2008).\(^3\) This work takes up the notion of “relaxation” of the relevant solution concept by focusing on a randomized choice by the optimizing agent. As specifically illustrated in Prescott and Townsend (1984), this randomization device is recognized to be more or less artificial, but that it copes with the nonconvex constraints which stem from idiosyncratic shocks to each agent.\(^4\)

In this paper, our sole concern is Walrasian general equilibrium theory, and more specifically, on Aumann’s theorem with non-convex preferences, and we incorporate the refinements of the relaxation technique as offered in Sagara (2016).

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\(^1\)As is by now well-known and well-understood, this was the third mathematical difficulty emphasized by Bewley (1991, pp. 224–225) when “there are both infinitely many commodities and infinitely many participants in an economy.” The other two difficulties that were identified concerned the facts that (i) “there does not exist an infinite dimensional version of Fatou’s lemma”, and that (ii) “budget sets in \(L^\infty\) are not typically norm bounded and hence weak-star compact, even when they are defined by price systems in \(L^1\)”. For a comprehensive discussion of these issues in the context of economies with a finite set of agents, as well as additional difficulties arising from the non-emptiness of the interior of the positive cone of a commodity space, see Mas-Colell and Zame (1991).

\(^2\)To be precise, the result applies to economies with atoms only under the convexity hypothesis on their preferences.

\(^3\)Prescott and Townsend (1984) have incentive compatibility in economies with a finite set of agents as their basic thrust; Nowak (1992) is concerned with correlated equilibria; and Balder (2008) is after a synthesis of the Nash mixed-strategy theorem in games with a continuum of agents and of general equilibrium theory with externalities and price-dependent preferences. He assumes, as he must, the hypotheses of convexity or single-valuedness of individual demands; but see Footnote 8 below.

\(^4\)Contrary to Cass and Shell (1983), randomness under consideration is different from extrinsic uncertainty that is unrelated to preferences and endowments of an economy, which results in state-dependent equilibria under the convexity hypothesis.
The procedure for the relaxation of economies that is pursued in this paper is as follows.

- The preferences of each agent possess a utility representation on a common consumption set $X$ formalized as a Polish subset of a Banach space.

- Utility functions on $X$ are then extended to the set $\Pi(X)$ of probability measures on $X$, and are thereby an affine extension on $\Pi(X)$. Extended preferences on $\Pi(X)$ are consistent with the expected utility hypothesis.

- Each probability measure in $\Pi(X)$ is regarded as a randomized commodity (a lottery) over $X$. Given a market price, each agent can purchase the barycenter of a probability measure, which is a convex combination of commodities in $X$ with respect to a probability measure in $\Pi(X)$ under the budget constraint.

- Barycentric commodities under the budget constraint, so to speak, are evaluated in terms of expected utilities and constitute the relaxed demand set of each agent. It is thereby defined as a closed convex subset of $\Pi(X)$.

- Relaxed allocations are well-defined in a way that the aggregate of the barycentric commodities of each agent does not exceed the total endowment. Relaxed Walrasian equilibria for relaxed economies are thereby formulated in a consistent manner.

- Dirac measures in $\Pi(X)$ reduce to the usual notion of a commodity in $X$, and consequently, Walrasian equilibria for the original economy are identified with relaxed (purified) Walrasian equilibria for the relaxed economy.

We then offer for classical Walrasian general equilibrium theory a supplementation to the usual convexification method: under the saturation hypothesis, we can always construct a Walrasian equilibrium for the original economy from a relaxed Walrasian equilibrium for the relaxed economy. The point is that the existence of the latter is easily established on account of the fact that the relaxed economy is already a convexification of the original economy. This is done through a “purification principle”, as in Khan and Sagara (2014); Sagara (2016), a powerful tool whose utility in non-cooperative game theory and statistical decision theory is well-understood. As to the difficulty

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5Since the preferences of each agent are not assumed to be convex, it should be noted that the individual demand set for the original economy lacks convexity.
of the joint continuity of the valuation functional for price-commodity pairs in infinite dimensions when one attempts to apply the fixed point theorem,\(^6\) we adapt the technique employed in Lee (2013); Podczeck (1997) to relaxed demand sets. In short summary, we can remove the convexity hypothesis from Khan and Yannelis (1991), and recover the existence result in Lee (2013); Podczeck (1997); Rustichini and Yannelis (1991) under the saturation hypothesis, all in the context of a separable Banach space. Furthermore, in \(L^\infty\)-spaces, we can also remove the convexity hypothesis from Bewley (1991) and derive the existence of Walrasian equilibria with free disposal under the saturation hypothesis.

As applications, we offer two examples. We illustrate how our existence result yields the existence of Pareto optimal, envy-free allocations in large economies with infinite-dimensional commodity spaces; this sharpens the classical result of Varian (1974). Curiously, envy-freeness is naturally interpreted as incentive compatibility in random economies where each agent incurs an idiosyncratic shock that characterizes his/her type along the lines of Prescott and Townsend (1984). We also demonstrate that our existence result is valid for economies with indivisible commodities along the lines of Khan and Yamazaki (1981); this presents an alternative approach to the existence result on economies with indivisible commodities investigated in Danilov et al. (2001) and Svensson (1984) via the alternative convexification technique.

\section{Preliminaries}

We develop four ideas in three subsections: relaxed controls in optimal control theory, the purification principle now also pervasive in application to the theory of non-atomic games, the saturation property and Gelfand integration of functions taking values in the space of essentially bounded measurable functions.

\(^6\)We note the use of the infinite-dimensional version of the Gale–Nikaido lemma and Michael’s selection theorem in Yannelis (1985). We also note that this difficulty of joint continuity of the valuation functional does not arise when one is working in a finite-dimensional space. Thus, in Balder (2008), even though the commodity space of probability measures on the finite-dimensional commodity space is infinite dimensional, the valuation functional is jointly continuous by virtue of the price space being a finite-dimensional Euclidean space.
2.1 Relaxed Controls

We denote by $\Pi(X)$ the set of probability measures on a Polish space $X$ furnished with the Borel $\sigma$-algebra $\text{Borel}(X)$. We endow $\Pi(X)$ with the topology of weak convergence of probability measures, which is the coarsest topology on $\Pi(X)$ for which the integral functional $P \mapsto \int v dP$ on $\Pi(X)$ is continuous for every bounded continuous function $v : X \to \mathbb{R}$. Then $\Pi(X)$ is also a Polish space; see Aliprantis and Border (2006, Theorem 15.15). Let $(T, \Sigma, \mu)$ be a finite measure space. (Throughout the paper, we always assume that it is complete.) By $M(T, X)$ we denote the space of measurable functions from $T$ to $X$ and by $R(T, X)$ the space of measurable functions from $T$ to $\Pi(X)$. Each element in $M(T, X)$ is called a control and that in $R(T, X)$ is called a relaxed control (a Young measure, a stochastic kernel, or a transition probability), which is a probability measure-valued control. For every function $\lambda : T \to \Pi(X)$, the real-valued function $t \mapsto \lambda(t)(C)$ is measurable for every $C \in \text{Borel}(X)$ if and only if $\lambda$ is measurable; see Podczeck (2009, Lemma 2). By $\Delta(X)$, we denote the set of Dirac measures on $X$, i.e., $\delta_x \in \Delta(X)$ whenever for every $C \in \text{Borel}(X)$: $\delta_x(C) = 1$ if $x \in C$ and $\delta_x(C) = 0$ otherwise. Each control $f \in M(T, X)$ is identified with the Dirac measure valued control $\delta_{f(t)} \in R(T, X)$ satisfying $\delta_{f(t)} \in \Delta(X)$ for every $t \in T$.

A real-valued function $u : T \times X \to \mathbb{R}$ is a Carathéodory function if $t \mapsto u(t, x)$ is measurable for every $x \in X$ and $x \mapsto u(t, x)$ is continuous for every $t \in T$. The Carathéodory function $u$ is jointly measurable; see Aliprantis and Border (2006, Lemma 4.51). A Carathéodory function $u$ is integrably bounded if there exists $\varphi \in L^1(\mu)$ such that $|u(t, x)| \leq \varphi(t)$ for every $(t, x) \in T \times X$. Denote by $C^1(T \times X, \mu)$ the space of integrably bounded Carathéodory functions on $T \times X$. For each $u \in C^1(T \times X, \mu)$, define the integral functional $I_u : R(T, X) \to \mathbb{R}$ by $I_u(\lambda) = \int \int u(t, x) \lambda(t, dx) d\mu$. The weak topology on $R(T, X)$ is defined as the coarsest topology for which every integral functionals $I_u$ is continuous for every $u \in C^1(T \times X, \mu)$. If $T$ is a singleton, then the set $R(T, X)$ coincides with the set $\Pi(X)$. In this case $C^1(T \times X, \mu)$ coincides with the space $C_b(X)$ of bounded continuous functions on $X$ and the weak topology of $R(T, X)$ is the topology of weak convergence of probability measures in $\Pi(X)$. Denote by $\overline{\mathcal{K}}^w$ the weak closure of $\mathcal{K} \subset R(T, X)$.
2.2 The Purification Principle in Saturated Measure Spaces

A finite measure space \((T, \Sigma, \mu)\) is said to be \textit{essentially countably generated} if its \(\sigma\)-algebra can be generated by a countable number of subsets together with the null sets; \((T, \Sigma, \mu)\) is said to be \textit{essentially uncountably generated} whenever it is not essentially countably generated. Let \(\Sigma_S = \{A \cap S \mid A \in \Sigma\}\) be the \(\sigma\)-algebra restricted to \(S \in \Sigma\). Denote by \(L^1_S(\mu)\) the space of \(\mu\)-integrable functions on the measurable space \((S, \Sigma_S)\) whose element is identified with a restriction of a function in \(L^1(\mu)\) to \(S\). An equivalence relation \(\sim\) on \(\Sigma_S\) is given by \(A \sim B \iff \mu(A \triangle B) = 0\), where \(A \triangle B\) is the symmetric difference of \(A\) and \(B\) in \(\Sigma\). The collection of equivalence classes is denoted by \(\Sigma(\mu) = \Sigma/\sim\) and its generic element \(\hat{A}\) is the equivalence class of \(A \in \Sigma\). We define the metric \(\rho\) on \(\Sigma(\mu)\) by \(\rho(\hat{A}, \hat{B}) = \mu(A \triangle B)\). Then \((\Sigma(\mu), \rho)\) is a complete metric space (see Aliprantis and Border (2006, Lemma 13.13) or Dunford and Schwartz (1958, Lemma III.7.1)) and \((\Sigma(\mu), \rho)\) is separable if and only if \(L^1(\mu)\) is separable (see Aliprantis and Border (2006, Lemma 13.14)). The density of \((\Sigma(\mu), \rho)\) is the smallest cardinal number of the form \(|U|\), where \(U\) is a dense subset of \(\Sigma(\mu)\).

**Definition 2.1.** A finite measure space \((T, \Sigma, \mu)\) is \textit{saturated} if \(L^1_S(\mu)\) is nonseparable for every \(S \in \Sigma\) with \(\mu(S) > 0\). We say that a finite measure space has the \textit{saturation property} if it is saturated.

Saturation implies nonatomicity and several equivalent definitions for saturation are known; see Fajardo and Keisler (2002); Fremlin (2012); Hoover and Keisler (1984); Keisler and Sun (2009). One of the simple characterizations of the saturation property is as follows. A finite measure space \((T, \Sigma, \mu)\) is saturated if and only if \((S, \Sigma_S, \mu)\) is essentially uncountably generated for every \(S \in \Sigma\) with \(\mu(S) > 0\). The saturation of finite measure spaces is also synonymous with the uncountability of the density of \(\Sigma_S(\mu)\) for every \(S \in \Sigma\) with \(\mu(S) > 0\); see Fremlin (2012, 331Y(e)). An germinal notion of saturation already appeared in Kakutani (1944); Maharam (1942). The significance of the saturation property lies in the fact that it is necessary and sufficient for the weak compactness and the convexity of the Bochner integral of a multifunction as well as the Lyapunov convexity theorem in Banach spaces; see Khan and Sagara (2013, 2015, 2016); Podczeck (2008); Sun and Yannelis (2008).

Let \(E\) be a Banach space and \(L^1(\mu, E)\) be the space of Bochner integrable functions from \(T\) to \(E\). We say that a function \(\Phi : T \times X \to E\) is \textit{integrably bounded} if there exists \(\varphi \in L^1(\mu)\) such that \(\|\Phi(t, x)\| \leq \varphi(t)\) for every \((t, x) \in T \times X\). Hence, \(\Phi(\cdot, x) \in L^1(\mu, E)\) for every \(x \in X\) whenever \(\Phi\) is integrably bounded.
bounded and measurable, and \( E \) is separable. Except for Subsections 2.3 and 3.5, and Section B, the integration of \( E \)-valued functions with respect to the finite measure \( \mu \) and probability measures in \( \Pi(X) \) is always supposed to be in the Bochner sense.

The following result is an immediate consequence of Khan and Sagara (2014, Theorem 5.1), whose proof hinges on the Lyapunov convexity theorem in separable Banach spaces obtained in Khan and Sagara (2013) under the saturation hypothesis.

**Proposition 2.1** (purification principle). Let \((T, \Sigma, \mu)\) be a saturated finite measure space, \( E \) be a separable Banach space, and \( X \) be a compact Polish space. If \( \Phi : T \times X \to E \) is an integrably bounded measurable function such that \( \Phi(t, \cdot) : X \to E \) is continuous in the weak topology of \( E \) for every \( t \in T \) and \( U : T \to X \) is a multifunction with \( \text{gph} U \in \Sigma \otimes \text{Borel}(X) \), then for every \( \lambda \in \mathcal{R}(T, X) \) with \( \lambda(t)(U(t)) = 1 \) a.e. \( t \in T \), there exists \( f \in \mathcal{M}(T, X) \) with \( f(t) \in U(t) \) a.e. \( t \in T \) such that

\[
\int_T \int_X \Phi(t, x) \lambda(t, dx) d\mu = \int_T \Phi(t, f(t)) d\mu.
\]

A control-theoretic interpretation of Proposition 2.1 means that any “relaxed” control system \( t \mapsto \hat{\Phi}(t, \lambda(t)) := \int \Phi(t, x) \lambda(t, dx) \) operated by \( \lambda \in \mathcal{R}(T, X) \) consistent with the control set \( U(t) \) is realized by adopting a “purified” control system \( t \mapsto \Phi(t, f(t)) \) operated by \( f \in \mathcal{M}(T, X) \) with the feasibility constraint \( f(t) \in U(t) \) in such a way that its Bochner integral over \( T \) is preserved with \( \int \hat{\Phi}(t, \lambda(t)) d\mu = \int \Phi(t, f(t)) d\mu \). An application of Proposition 2.1 to nonconvex variational problems with infinite-dimensional control systems is explored in Khan and Sagara (2014).

**Remark 2.1.** For the case with \( E = \mathbb{R}^n \), Proposition 2.1 holds under the nonatomicity hypothesis, which is a well-known result in control theory attributed to Warga (1972, Theorem IV.3.14); see also Arkin and Levin (1972, Theorem 2.5). In particular, when \( X \) is a finite or countably infinite set, Warga’s result corresponds to the classical result of Dvoretzky, Wald and Wolfowitz (1951); see also Khan and Rath (2009); Khan et al. (2006). The case for \( E = \mathbb{R}^N \) with \( X \) a compact Polish space is covered in Loeb and Sun (2006, Theorem 2.2), Loeb and Sun (2009, Theorem 2.2), and Podczeck (2009, Theorem 2) under the saturation hypothesis. As well as applications in optimal control theory along the lines of Arkin and Levin (1972); Balder (1984); Berliocchi and Lasry (1973); Khan and Sagara (2014); Sagara (2016); Sainte-Beuve (1978); Warga (1972), the purification principle of this type also justifies the elimination of randomness in statistical decision theory as
in Balder (1985); Dvoretsky et al. (1951); Feinberg and Pionovskiy (2006); González-Hernández and Hernández-Lerma (2005), and the purification of mixed strategies for games with incomplete information with finite players, as in Askoura et al. (2013); Khan and Rath (2009); Khan et al. (2006); Loeb and Sun (2006); Milgrom and Weber (1985); Noguchi (2014); Radner and Rosenthal (1982).

### 2.3 Gelfand Integrals in $L^\infty$

Let $(\Omega, \mathcal{F}, \nu)$ be a $\sigma$-finite measure space. A function $f : T \rightarrow L^\infty(\nu)$ is weakly* scalarly measurable if the scalar function $\langle \varphi, f(\cdot) \rangle$ on $T$ is measurable for every $\varphi \in L^1(\nu)$, where the duality between $L^1(\nu)$ and $L^\infty(\nu)$ is given by $\langle \varphi, \psi \rangle = \int \varphi \psi d\nu$ for $\varphi \in L^1(\nu)$ and $\psi \in L^\infty(\nu)$. We say that weakly* scalarly measurable functions $f$ and $g$ are weakly* scalarly equivalent if $\langle \varphi, f(t) - g(t) \rangle = 0$ for every $\varphi \in L^1(\nu)$ a.e. $t \in T$ (the exceptional $\mu$-null set depending on $\varphi$). We say that a weakly* scalarly measurable function $f : T \rightarrow L^\infty(\nu)$ is weakly* scalarly integrable if the scalar function $\langle \varphi, f(\cdot) \rangle$ is integrable for every $\varphi \in L^1(\nu)$. A weakly* scalarly measurable function $f$ is Gelfand integrable over $A \in \Sigma$ if there exists $\psi_A \in L^\infty(\nu)$ such that $\langle \varphi, \psi_A \rangle = \int_A \langle \varphi, f(t) \rangle d\mu$ for every $\varphi \in L^1(\nu)$. The element $\psi_A$ is called the Gelfand integral (or the weak* integral) of $f$ over $A$ and denoted by $\int_A f d\mu$. Every weakly* scalarly integrable function is weakly* integrable; see Aliprantis and Border (2006, Theorem 11.52). Denote by $G^1(\mu, L^\infty(\nu))$ the equivalence classes of Gelfand integrable functions with respect to weakly* scalarly equivalence.

Equipped with the notion of Gelfand integration, we turn to the development of a straightforward variant of Proposition 2.1. It is a special case of Sagara (2016, Theorem 3.3) to a setting where the integration of $L^\infty(\nu)$-valued functions with respect to the finite measure $\mu$ and probability measures in $\Pi(X)$ is always supposed to be in the sense of a Gelfand integral.

**Proposition 2.2** (purification principle in $L^\infty$). *Proposition 2.1 is valid in the sense of Gelfand integrals when the separable Banach space $E$ is replaced by $L^\infty(\nu)$ endowed with the weak*-topology, where $(\Omega, \mathcal{F}, \nu)$ is a countably generated $\sigma$-finite measure space.*

### 3 Relaxed Large Economies

#### 3.1 Relaxation of Large Economies

The set of agents is given by a complete finite measure space $(T, \Sigma, \mu)$. The commodity space is given by a separable Banach space $E$. The preference
relation $\succsim(t)$ of each agent $t \in T$ is a complete, transitive binary relation on a common consumption set $X \subset E$, which induces the preference map $t \mapsto \succsim(t) \subset X \times X$. We denote by $x \succsim(t) y$ the relation $(x, y) \in \succsim(t)$. The indifference and strict relations are defined respectively by $x \sim(t) y \iff x \succsim(t) y$ and $y \succsim(t) x$, and by $x \succ(t) y \iff x \succsim(t) y$ and $x \not\succsim(t) y$. Each agent possesses an initial endowment $\omega(t) \in X$, which is the value of a Bochner integrable function $\omega : T \to E$. The economy $E$ consists of the primitives $E = \left\{ (T, \Sigma, \mu), X, \succsim, \omega \right\}$.

The standing assumption on $E$ is described as follows.

**Assumption 3.1.** (i) $X$ is a weakly compact subset of $E$.

(ii) $\succsim(t)$ is a weakly closed subset of $X \times X$ for every $t \in T$.

(iii) For every $x, y \in X$ the set $\{ t \in T \mid x \succsim(t) y \}$ is in $\Sigma$.

The weak compactness assumption in condition (i) is made in Khan and Yannelis (1991); Lee (2013); Martins-da-Rocha (2003); Noguchi (1997); Podczeck (1997); Rustichini and Yannelis (1991) for the uncommon consumption set of each agent. Since $E$ is separable, the weakly compact set $X \subset E$ is metrizable for the weak topology (see Dunford and Schwartz (1958, Theorem V.6.3)), and hence, the common consumption set $X$ is a compact Polish space. The preference relation $\succsim(t)$ is said to be *continuous* if it satisfies condition (ii). The measurability of the preference mapping in condition (iii) is introduced in Aumann (1969).

It follows from Aumann (1969, Proposition 1) that there exists a Carathéodory function $u : T \times X \to \mathbb{R}$ such that

$$\forall x, y \in X \forall t \in T : x \succsim(t) y \iff u(t, x) \geq u(t, y).$$

Moreover, this representation in terms of Carathéodory functions is unique up to strictly increasing, continuous transformations in the following sense: If $F : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $t \mapsto F(t, r)$ is measurable and $r \mapsto F(t, r)$ is strictly increasing and continuous, then $x \succsim(t) y \iff F(t, u(t, x)) \geq F(t, u(t, y))$, where $t \mapsto F(t, u(t, x))$ is a Carathéodory function. In the sequel, we may assume without loss of generality that the preference map $t \mapsto \succsim(t)$ is represented by a Carathéodory function $u$ that is unique up to strictly increasing, continuous transformations.

Following Sagara (2016), we introduce the notion of “relaxation” of preferences for large economies. Given a continuous preference $\succsim(t)$ on $X$, its

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7While Aumann (1969) treated the case where $X$ is the nonnegative orthant of a finite-dimensional Euclidean space, the proof is obviously valid as it stands for the case where $X$ is a separable metric space.
continuous affine extension \( \succsim_R(t) \) to \( \Pi(X) \) is obtained by convexifying (randomizing) the individual utility function \( u(t, \cdot) \) in such a way

\[
\forall P, Q \in \Pi(X) \quad \forall t \in T : P \succsim_R(t) Q \iff \int_X u(t, x) dP \geq \int_X u(t, x) dQ. \quad (3.2)
\]

The continuous extension \( \succsim_R(t) \) of \( \succsim(t) \) from \( X \) to the relaxed consumption set \( \Pi(X) \) is called a relaxed preference relation on \( \Pi(X) \). Thus, the restriction of \( \succsim_R(t) \) to \( \Delta(X) \) coincides with \( \succsim(t) \) on \( X \). Indifference relation \( \sim_R(t) \) and strict relation \( \succ_R(t) \) are defined in a way analogous to the above.

The extension formula (3.2) conforms to the relaxation technique explored in McShane (1967); Warga (1972); Young (1969). It is noteworthy that relaxed preferences also conform to the “expected utility hypothesis” and the continuous function \( u(t, \cdot) \) corresponds to the “von Neumann–Morgenstern utility function” for \( \succsim_R(t) \). That is, \( \succsim_R(t) \) is a continuous preference relation on \( \Pi(X) \) satisfying the “independence axiom” introduced in von Neumann and Morgenstern (1953).

**Independence** For every \( P, Q, R \in \Pi(X) \) and \( \alpha \in [0, 1] \): \( P \sim_R(t) Q \) implies \( \alpha P + (1 - \alpha) R \sim_R(t) \alpha Q + (1 - \alpha) R \).

Conversely, for every \( t \in T \) any continuous binary relation on \( \Pi(X) \) satisfying the independence axiom is representable in terms of the continuous von Neumann–Morgenstern utility function \( u(t, \cdot) \) for which (3.2) is satisfied; see Grandmont (1972, Theorem 3). Furthermore, this representation is unique up to positive affine transformations.

Denote by \( \mathcal{E}_R = \{(T, \Sigma, \mu), \Pi(X), \succsim_R, \delta_{\omega(t)}\} \) the relaxed economy induced by the original economy \( \mathcal{E} = \{(T, \Sigma, \mu), X, \succsim, \omega\} \), where the initial endowment \( \omega(t) \in X \) of each agent is identified with a Dirac measure \( \delta_{\omega(t)} \in \Delta(X) \), and hence, \( \delta_{\omega(t)} \in \mathcal{R}(T, X) \).

## 3.2 Relaxed Demand Sets

Given a price \( p \in E^* \setminus \{0\} \), for each agent \( t \in T \), as usual we define the budget set by \( B(t, p) = \{ x \in X \mid \langle p, x \rangle \leq \langle p, \omega(t) \rangle \} \) and the demand set by \( D(t, p) = \{ x \in X \mid x \succsim(t) y \forall y \in B(t, p) \} \). Let \( \iota_X \) be the identity map on \( X \). Similarly, the relaxed budget set of each agent is defined by

\[
B_R(t, p) = \left\{ P \in \Pi(X) \mid \int_X \langle p, \iota_X(x) \rangle dP \leq \langle p, \omega(t) \rangle \right\}
\]

and the relaxed demand set is given by

\[
D_R(t, p) = \{ P \in B_R(t, p) \mid P \succsim_R(t) Q \forall Q \in B_R(t, p) \}.
\]
We denote by $\int \mathbb{1}_X dP$ the Bochner integral of $\mathbb{1}_X$ with respect to the probability measure $P \in \Pi(X)$. Since $\int (p, \mathbb{1}_X(x))dP = \langle p, \int \mathbb{1}_X dP \rangle$ in view of the Bochner integrability of $\mathbb{1}_X$, the “barycentric commodity” $\int \mathbb{1}_X dP$ of $P \in B_R(t, p)$ is in $X$ whenever $X$ is convex (which we do not assume), and affordable under the relaxed budget constraint, and the relaxed commodity $P$ is evaluated in terms of the expected utility represented in (3.2).

A remarkable, but natural connection between the market behavior of each agent in the original economy and that in the relaxed economy is that the maximization of expected utility subject to the relaxed budget constraint is “consistent” with the deterministic utility maximization subject to the budget constraint. Specifically, we have the following characterization on the relaxed demand set.

**Proposition 3.1.** Let $(T, \Sigma, \mu)$ be a finite measure space and $E$ be a separable Banach space. Suppose that the economy $\mathcal{E}$ satisfies Assumption 3.1. Then for every $p \in E^* \setminus \{0\}$ and $t \in T$: $P \in D_R(t, p)$ if and only if $P(D(t, p)) = 1$.

### 3.3 Relaxed Walrasian Equilibria

To deal with the equilibrium concept with or without free disposal simultaneously, following Mordukhovich (2006, Chapter 8), we introduce “market constraints” for the definition of (relaxed) allocations.

**Definition 3.1.** Let $C$ be a nonempty subset of $E$.

(i) An element $f \in L^1(\mu, E)$ is an *allocation* for $\mathcal{E}$ if it satisfies:

$$\int_T f(t)d\mu - \int_T \omega(t)d\mu \in C \quad \text{and} \quad f(t) \in X \text{ a.e. } t \in T.$$

(ii) An element $\lambda \in \mathcal{R}(T, X)$ is a *relaxed allocation* for $\mathcal{E}_R$ if it satisfies:

$$\int_T \int_X \mathbb{1}_X(x)\lambda(t, dx)d\mu - \int_T \omega(t)d\mu \in C.$$

In particular, when $C = \{0\}$, the definition reduces to the (relaxed) allocations “without” free disposal; when $-C$ is a convex cone and $E$ is endowed with the cone order $\leq$ defined by $x \leq y \iff y - x \in -C$, the definition reduces to the (relaxed) allocations “with” free disposal. Denote by $\mathcal{A}(\mathcal{E})$ the set of allocations for $\mathcal{E}$ and by $\mathcal{A}(\mathcal{E}_R)$ the set of relaxed allocations for $\mathcal{E}_R$. If $\lambda$ is a relaxed allocation for $\mathcal{E}_R$ such that $\lambda(t) = \delta_{f(t)} \in \Delta(X)$ for every $t \in T$ and $f \in L^1(\mu, E)$, then it reduces to the usual feasibility constraint $\int f d\mu - \int \omega d\mu \in C$ for $\mathcal{E}$. This means that $\mathcal{A}(\mathcal{E}) \subset \mathcal{A}(\mathcal{E}_R)$. 

12
Definition 3.2.  
(i) A price-allocation pair \((p, f) \in (\mathcal{E}^* \setminus \{0\}) \times \mathcal{A}(\mathcal{E})\) is a Walrasian equilibrium for \(\mathcal{E}\) if a.e. \(t \in T\): \(f(t) \in B(t, p)\) and \(f(t) \succeq (t) x\) for every \(x \in B(t, p)\).

(ii) A price-relaxed allocation pair \((p, \lambda) \in (\mathcal{E}^* \setminus \{0\}) \times \mathcal{A}(\mathcal{E}_R)\) is a relaxed Walrasian equilibrium for \(\mathcal{E}_R\) if a.e. \(t \in T\): \(\lambda(t) \in B_R(t, p)\) and \(\lambda(t) \succeq_R (t) P\) for every \(P \in B_R(t, p)\).

Denote by \(\mathcal{W}(\mathcal{E})\) the set of Walrasian allocations for \(\mathcal{E}\) and by \(\mathcal{W}(\mathcal{E}_R)\) the set of relaxed Walrasian allocations for \(\mathcal{E}_R\).

Any Walrasian equilibrium for the original economy is regarded as a “purified” relaxed Walrasian equilibrium for the relaxed economy. Under the saturation hypothesis, the converse result holds as well. That is, any relaxed Walrasian equilibrium for the relaxed economy can be purified as a Walrasian equilibrium for the original economy.

Proposition 3.2. Let \((T, \Sigma, \mu)\) be a finite measure space and \(E\) be a separable Banach space. Suppose that the economy \(\mathcal{E}\) satisfies Assumption 3.1. If \((p, f)\) is a Walrasian equilibrium for \(\mathcal{E}\), then \((p, \delta_{f(t)})\) is a relaxed Walrasian equilibrium for \(\mathcal{E}_R\). Conversely, if \((p, \lambda)\) is a relaxed Walrasian equilibrium for \(\mathcal{E}_R\), then there exists a Walrasian equilibrium \((p, f)\) for \(\mathcal{E}\) such that \(\lambda(t) \sim_R (t) \delta_{f(t)}\) a.e. \(t \in T\) whenever \((T, \Sigma, \mu)\) is saturated.

Another significant aspect on saturation is the density property of allocations and Walrasian allocations.

Proposition 3.3 (density property). Let \((T, \Sigma, \mu)\) be a saturated finite measure space and \(E\) be a separable Banach space. Suppose that the economy \(\mathcal{E}\) satisfies Assumption 3.1. Then \(\mathcal{A}(\mathcal{E}_R) = \overline{\mathcal{A}(\mathcal{E})}^w\) and \(\mathcal{W}(\mathcal{E}_R) = \overline{\mathcal{W}(\mathcal{E})}^w\).

Remark 3.1. It is Warga (1972, Theorem IV.2.6) who established the density theorem \(\mathcal{R}(T, X) = \overline{\mathcal{M}(T, X)}^w\) for compact polish spaces under the nonatomicity hypothesis. As noted in Khan and Sagara (2014, Remark 6.1), Proposition 3.3 holds under the nonatomicity hypothesis whenever \(E = \mathbb{R}^n\), in which case the classical Lyapunov convexity theorem is sufficient for the density property. For another variant of the density property with the finite-dimensional setting, see, e.g., Balder (1984, Corollary 3), Berliocchi and Lasry (1973, Proposition II.7), and Sainte-Beuve (1978, Theorem 7 and Corollary 4).
3.4 Existence of Walrasian Equilibria with Free Disposal

For a substantive validation of the equivalence in Proposition 3.2, it suffices to demonstrate the existence of relaxed Walrasian equilibria for the relaxed economy $E_R$ instead of Walrasian equilibria for the original economy $E$. Following Khan and Yannelis (1991); Lee (2013); Martins-da-Rocha (2003); Noguchi (1997); Podczeck (1997); Rustichini and Yannelis (1991), we consider (relaxed) Walrasian equilibria with free disposal in which the commodity space $E$ is an ordered separable Banach space such that the norm interior of the positive cone $E_+$ is nonempty. Denote by $E^*_+$ be the set of elements $x^* \in E^*$ with $\langle x^*, x \rangle \geq 0$ for every $x \in E_+$. An element in $E^*_+ \setminus \{0\}$ is said to be positive. A maximal element in $X$ for $\succsim(t)$ is called a satiation point for $\succsim(t)$. Under Assumption 3.1, satiation points for $\succsim(t)$ exist for every $t \in T$.

Assumption 3.2. (i) $X$ is a weakly compact subset of $E_+$.

(ii) For every $t \in T$ there exists $z(t) \in X$ such that $\omega(t) - z(t)$ belongs to the norm interior of $E_+$.

(iii) If $x \in X$ is a satiation point for $\succsim(t)$, then $x \geq \omega(t)$.

(iv) If $x \in X$ is not a satiation point for $\succsim(t)$, then $x$ belongs to the weak closure of the upper contour set $\{y \in X \mid y \succsim(t) x\}$.

Condition (ii) is due to Khan and Yannelis (1991), which guarantees that for every positive price the value of the initial endowment of each agent is strictly positive. Condition (iii) is introduced in Podczeck (1997) and imposed also in Lee (2013). Condition (iv) is a variant of “local nonsatiation” originated in Hildenbrand (1968) and is imposed also in Lee (2013); Podczeck (1997).

We now present the first substantive result of this paper.

Theorem 3.1. Let $(T, \Sigma, \mu)$ be a finite measure space and $E$ be an ordered separable Banach space such that the norm interior of $E_+$ is nonempty. Then for every economy $E$ satisfying Assumptions 3.1 and 3.2:

(i) There exists a relaxed Walrasian equilibrium with free disposal for $E_R$ with a positive price.

(ii) There exists a Walrasian equilibrium with free disposal for $E$ with a positive price whenever $(T, \Sigma, \mu)$ is saturated.
A sharp contrast to the literature on large economies, such as Aumann (1966); Hildenbrand (1974); Khan and Yannelis (1991); Lee (2013); Martins-da-Rocha (2003); Noguchi (1997); Podczeck (1997); Rustichini and Yannelis (1991), is that the saturation (or even the nonatomicity) hypothesis is unnecessary to guarantee the existence of relaxed Walrasian equilibria for the relaxed economies as well as the convexity hypothesis. Thus, whenever the set $T$ of agents is finite and $\mu$ is a counting measure, the first assertion of Theorem 3.1 reduces to the existence of relaxed Walrasian equilibria for a relaxed finite economies without convexity assumptions. This means that relaxed Walrasian equilibria always exist even though the original economy fails to possess Walrasian equilibria. Since the vernacular of “relaxed” economies and “relaxed” Walrasian equilibria is also used in Balder (2008), we invite the reader to compare Theorem 3.1 with the relevant result in Balder (2008).

Given Proposition 3.2, the second assertion of Theorem 3.1 simply drops the convexity hypothesis from Khan and Yannelis (1991) under the saturation hypothesis and recovers the existence result of Lee (2013); Podczeck (1997); Rustichini and Yannelis (1991) under the framework of economies with a common consumption set. Indeed, when $E$ is a finite-dimensional Euclidean space, the validity of the second assertion of Theorem 3.1 for nonatomic finite measure space of agents follows from Proposition 2.1; see Remark 2.1. Therefore, to repeat, if $(T, \Sigma, \mu)$ is a nonatomic finite measure space and $E = \mathbb{R}^n$, the existence of Walrasian equilibria with free disposal for $\mathcal{E}$ with a positive price is guaranteed under Assumptions 3.1 and 3.2. On the other hand, when $X$ is a finite subset of $E_+$, the conditions on (non)satiation points for $\succsim(t)$ are unnecessary for the existence result and Assumption 3.2 can be replaced by the following.

**Assumption 3.3.**

(i) $X$ is a finite subset of $E_+$.

(ii) For every $t \in T$ there exists $z(t) \in X$ such that $\omega(t) - z(t)$ belongs to the norm interior of $E_+$.

This allows us to present the second substantive result of the paper.

**Theorem 3.2.** Let $(T, \Sigma, \mu)$ be a finite measure space and $E$ be an ordered separable Banach space such that the norm interior of $E_+$ is nonempty. Then for every economy $\mathcal{E}$ satisfying Assumptions 3.1 and 3.3:

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8Indeed, Balder (2008) has the priority for the usage of this terminology in mathematical economics, but as mentioned in Footnote 3, rather than classical Walrasian general equilibrium theory, his concern is with a synthetic treatment that allows externalities and price-dependent preferences with a finite-dimensional commodity space.
There exists a relaxed Walrasian equilibrium with free disposal for $E_R$ with a positive price.

(ii) There exists a Walrasian equilibrium with free disposal for $E$ with a positive price whenever $(T, \Sigma, \mu)$ is saturated.

We now conclude this subsection with two applications.

Example 3.1 (envy-freeness/incentive compatibility). An allocation $f \in \mathcal{A}(E)$ is said to be envy-free if $f(t) \succeq (t) f(t')$ for a.e. $t, t' \in T$. Let $\bar{\omega}(t) = \int \omega d\mu/\mu(T)$ and assume that $\bar{\omega}(t) \in X$ for every $t \in T$. If Assumptions 3.1 and 3.2 are satisfied for the economy $\mathcal{E} = \{(T, \Sigma, \mu), \succeq, X, \bar{\omega}\}$ with the same initial endowment among agents, then Theorem 3.1 guarantees that $\mathcal{E}$ possesses a Walrasian equilibrium that is also Pareto optimal and envy-free; see Varian (1974). When $T$ is regarded as the set of random shocks drawn from the probability measure $\mu$, where each element $t \in T$ is an idiosyncratic shock that characterizes the type of agent, the envy-free condition is reduced to the “truth revelation principle”, i.e., the “incentive compatibility” condition studied in Prescott and Townsend (1984). This reduces to, and implies, the existence of Walrasian equilibria with incentive compatibility for economies with private information under the saturation hypothesis.

Example 3.2 (indivisible commodities). Suppose that there are $n$ indivisible commodities each of which can be consumed in integer units and that the common consumption set of such commodities is finite for all agents. The resulting economy $E$ with indivisible commodities is described in our framework as follows. Let $\mathbb{Z}_+$ be the set of nonnegative integers, $E$ the Euclidean space $\mathbb{R}^n$ with the Euclid norm, and $X$ a finite subset of $\mathbb{Z}_+^n$. Let $\bar{\omega}(t)$ be a preference on $X$ represented by a Carathéodory function $u : T \times X \rightarrow \mathbb{R}$. Assume further that the endowment function $\omega : T \rightarrow \mathbb{R}^n$ is integrable such that $\omega(t)$ belongs to $X$ and each of its coordinates is a positive integer for every $t \in T$. Then Assumptions 3.1 and 3.3 are automatically satisfied, and Theorem 3.2 then guarantees that there exists a relaxed Walrasian equilibrium with free disposal for the relaxed economy $E_R$ with a positive price. In particular, if $(T, \Sigma, \mu)$ is nonatomic, then there exists a Walrasian equilibrium with free disposal for $E$ with a positive price. The crucial difference of this consequence from the existence result in Khan and Yamazaki (1981) is that it dispenses with the introduction of divisible commodities and with the local nonsatisfaction of preferences, though at the cost of the finiteness of the consumption set.
3.5 Existence of Walrasian Equilibria on $L^\infty$

In this subsection, we turn to economies that are modeled with $L^\infty$ as a commodity. This extension is important for both substantive and technical reasons: substantively because it, and its dual, was identified by Bewley (1972) as the canonical space for Walrasian general equilibrium theory; technically because it leads to to shift the emphasis of analysis to the predual $L^1$ rather than the dual, and thereby from Bochner integration to Gelfand integration.

Towards this end, let $(\Omega, \mathcal{F}, \nu)$ be a countably generated, $\sigma$-finite measure space. The norm dual of $L^\infty(\nu)$ is $ba(\nu)$, the space of finitely additive signed measures on $\mathcal{F}$ of bounded variation that vanishes on $\nu$-null sets with the duality given by $\langle \pi, \psi \rangle = \int \psi d\pi$ for $\pi \in ba(\nu)$ and $\psi \in L^\infty(\nu)$; see Dunford and Schwartz (1958, Theorem IV.8.14). We consider a (Gelfand) economy $E^G = \{(T, \Sigma, \mu), X, \succ, \omega\}$ for which the commodity space is $L^\infty(\nu)$ and the price space is $ba(\nu)$ with $\omega \in G^1(\mu, L^\infty(\nu))$ and $\omega(t) \in X$ for every $t \in T$ satisfying the following conditions.

Assumption 3.4.  
(i) $X$ is a weakly* compact subset of $L^\infty(\nu)$.
(ii) $\succ(t)$ is a weakly* closed subset of $X \times X$ for every $t \in T$.
(iii) For every $x, y \in X$ the set $\{t \in T \mid x \succ(t) y\}$ is in $\Sigma$.

Since $L^1(\nu)$ is separable, the weak* compact set $X \subset L^\infty_+(\nu)$ is metrizable for the weak* topology of $L^\infty(\nu)$ (see Dunford and Schwartz (1958, Theorem V.5.1)), and hence, the common consumption set $X$ is a compact Polish space. Therefore, the preference representation in (3.1) is valid for $E^G$. Consequently, the preference representation (3.2) is also valid for its relaxed economy $E^G_R = \{(T, \Sigma, \mu), \Pi(X), \succ_R, \omega\}$.

Next, we develop the analogue for Definition 3.1.

Definition 3.3. Let $C$ be a nonempty subset of $L^\infty(\nu)$.

(i) An element $f \in G^1(\mu, L^\infty(\nu))$ is an allocation for $E^G$ if it satisfies:
$$\int_T f(t)d\mu - \int_T \omega(t)d\mu \in C \quad \text{and} \quad f(t) \in X \text{ a.e. } t \in T.$$

(ii) An element $\lambda \in R(T, X)$ is a relaxed allocation for $E^G_R$ if it satisfies:
$$\int_T \int_X t_X(x)\lambda(t, dx)d\mu - \int_T \omega(t)d\mu \in C.$$
Denote by \( A(\mathcal{E}^G) \) the set of Gelfand integrable allocations for \( \mathcal{E}^G \) and by \( A(\mathcal{E}^G_R) \) the set of relaxed allocations for \( \mathcal{E}^G_R \).

Given a price \( \pi \in \mathfrak{ba}(\nu) \setminus \{0\} \), we can define (relaxed) budget set and (relaxed) demand set for each agent as in the previous section. Thus, (relaxed) Walrasian equilibria with free disposal for \( \mathcal{E}^G \) (resp. \( \mathcal{E}^G_R \)) are introduced in an obvious way.

**Definition 3.4.**

(i) A price-allocation pair \( (\pi,f) \in (\mathfrak{ba}(\nu) \setminus \{0\}) \times A(\mathcal{E}^G) \) is a Walrasian equilibrium for \( \mathcal{E}^G \) if a.e. \( t \in T \): \( f(t) \in B(t,\pi) \) and \( f(t) \succeq (t,x) \) for every \( x \in B(t,\pi) \).

(ii) A price-relaxed allocation pair \( (\pi,\lambda) \in (\mathfrak{ba}(\nu) \setminus \{0\}) \times A(\mathcal{E}^G_R) \) is a relaxed Walrasian equilibrium for \( \mathcal{E}^G_R \) if a.e. \( t \in T \): \( \lambda(t) \in B_R(t,\pi) \) and \( \lambda(t) \succeq_R (t,P) \) for every \( P \in B_R(t,\pi) \).

Denote by \( \mathcal{W}(\mathcal{E}^G) \) the set of Walrasian allocations for \( \mathcal{E}^G \) and by \( \mathcal{W}(\mathcal{E}^G_R) \) the set of relaxed Walrasian allocations for \( \mathcal{E}^G_R \).

It is clear now that Proposition 3.1 is valid for \( E = L^\infty(\nu) \) and \( X \subset L^\infty(\nu) \) under Assumption 3.4. Corresponding to Proposition 3.2, we obtain the following characterization under the saturation hypothesis, whose proof is same with that of Proposition 3.2 if one simply replaces the Bochner integrals by Gelfand ones.

**Proposition 3.4.** Let \( (T,\Sigma,\mu) \) be a saturated finite measure space and \( (\Omega,\mathcal{F},\nu) \) be a countably generated \( \sigma \)-finite measure space. Suppose that the economy \( \mathcal{E}^G \) satisfies Assumption 3.4. If \( (\pi,f) \) is a Walrasian equilibrium for \( \mathcal{E}^G \), then \( (\pi,\delta_f) \) is a relaxed Walrasian equilibrium for \( \mathcal{E}^G_R \). Conversely, if \( (\pi,\lambda) \) is a relaxed Walrasian equilibrium for \( \mathcal{E}^G_R \), then there exists a Walrasian equilibrium \( (\pi,f) \) for \( \mathcal{E}^G \) such that \( \lambda(t) \sim_R (t) \delta_{f(t)} \) a.e. \( t \in T \) whenever \( (T,\Sigma,\mu) \) is saturated.

Furthermore, under the same hypothesis with Proposition 3.4, the density property in \( L^\infty(\nu) \) is also valid for the Gelfand integral setting, i.e., \( \mathcal{A}(\mathcal{E}^G_R) = \overline{\mathcal{A}(\mathcal{E}^G)}^w \) and \( \mathcal{W}(\mathcal{E}^G_R) = \overline{\mathcal{W}(\mathcal{E}^G)}^w \). The proof of this fact is same with that of Proposition 3.3 if one simply replaces the Bochner integrals by Gelfand ones invoking Proposition 2.2.

Next, we turn to the analogues of our substantive Theorems 3.1 on the existence of (relaxed) Walrasian equilibria in large economies with free disposal modeled on \( L^\infty(\nu) \) as a commodity space. Since the norm interior of the positive cone \( L^\infty_+(\nu) \) of \( L^\infty(\nu) \) is nonempty, under the additional assumption below, we can recover every result in Subsection 3.4 for the case with \( E = L^\infty(\nu) \) and \( E^* = \mathfrak{ba}(\nu) \) with the suitable replacement of the weak
topology by the weak* topology and the Bochner integrals by the Gelfand integrals.

Assumption 3.5. (i) \( X \) is a weakly* compact subset of \( L_+^\infty(\nu) \).

(ii) For every \( t \in T \) there exists \( z(t) \in X \) such that \( \omega(t) - z(t) \) belongs to the norm interior of \( L_+^\infty(\nu) \).

(iii) If \( x \in X \) is a satiation point for \( \succ(t) \), then \( x \geq \omega(t) \).

(iv) If \( x \in X \) is not a satiation point for \( \succ(t) \), then \( x \) belongs to the weak* closure of the upper contour set \( \{ y \in X \mid y \succ(t) x \} \).

While the norm dual \( ba(\nu) \) of \( L^\infty(\nu) \) is larger than \( L^1(\nu) \), as emphasized in Bewley (1972), the price systems in \( ba(\nu) \) lack a reasonable economic interpretation unless they belong to \( L^1(\nu) \) (i.e., they are countably additive); see also Mas-Colell and Zame (1991). To derive positive equilibrium prices with free disposal in \( L^1(\nu) \) for the relaxed economy from those in \( ba(\nu) \), the Yosida–Hewitt decomposition of finitely additive measures is crucial in our framework, similar to Bewley (1972, 1991).

Theorem 3.3. Let \( (T, \Sigma, \mu) \) be a finite measure space and \( (\Omega, \mathcal{F}, \nu) \) be a countably generated \( \sigma \)-finite measure space. Then for every economy \( \mathcal{E}^G \) satisfying Assumptions 3.4 and 3.5:

(i) There exists a relaxed Walrasian equilibrium with free disposal for \( \mathcal{E}^G_R \) with a positive price in \( L^1(\nu) \).

(ii) There exists a Walrasian equilibrium with free disposal for \( \mathcal{E}^G \) with a positive price in \( L^1(\nu) \) whenever \( (T, \Sigma, \mu) \) is saturated.

The second assertion of Theorem 3.3 removes the convexity and monotonicity of preferences from Bewley (1972, 1991) and introduces free disposability for Walrasian equilibria with the commodity space of \( L^\infty(\nu) \).

4 Concluding Summary

In this paper we have presented three results as our contribution to the existence question of classical Walrasian general equilibrium with a continuum of commodities and a finite or a continuum of agents. We have developed these results through four propositions emphasizing “convexification by randomization” as opposed to “convexification by aggregation”, and drawn on relaxation techniques pioneered, and now pervasive, in optimal control theory as well as in statistical decision-theory, and the theory of non-cooperative
games. To be sure, the substantive power of these techniques lies in their being supplemented by a purification principle that eliminates randomization in the testing of statistical hypothesis and in the replacement of mixed strategies by their “equivalent” pure strategies in both non-atomic and atomic game theory. All this being said, our thrust is squarely on classical Walrasian general equilibrium theory as formulated by Aumann in 1964–1966. It is this focus that leads us to ignore a possible third convexification procedure due to Hart–Hildenbrand–Kohlberg; see Hildenbrand (1974) and its references. This approach involves a substantive shift from an anonymous to a non-anonymous form for both games and economies, and a technical shift from measurable functions to their induced distributions. It substitutes a “symmetrization principle” for the purification principle. We leave a consideration of this developing and rich literature for future work.

A Appendix 1

A.1 Proof of Proposition 3.1

Choose any $P \in D_R(t,p)$. Given the preference representation (3.2), if $P(D(t,p)) < 1$, then

$$\int_X u(t,x)dP = \int_{D(t,p)} u(t,x)dP + \int_{X \setminus D(t,p)} u(t,x)dP < \max_{y \in B(t,p)} u(t,y) \tag{A.1}$$

because $u(t,x) = \max_{y \in B(t,p)} u(t,y)$ for every $x \in D(t,p)$ and $u(t,x) < \max_{y \in B(t,p)} u(t,y)$ for every $x \in X \setminus D(t,p)$. On the other hand, for every $y \in B(t,p)$ we have

$$\int_X u(t,x)dP = \max_{Q \in B_R(t,p)} \int_X u(t,x)dQ \geq \int_X u(t,x)d\delta_y = u(t,y)$$

in view of $\delta_y \in B_R(t,p)$. Hence, we obtain a contradiction because of $\int u(t,x)dP \geq \max_{y \in B(t,p)} u(t,y)$.

For the converse implication, suppose that $P(D(t,p)) = 1$. Since $\langle p, x \rangle \leq \langle p, \omega(t) \rangle$ for every $x \in D(t,p)$, we have

$$\int_X \langle p, t_X(x) \rangle dP = \int_{D(t,p)} \langle p, t_X(x) \rangle dP \leq \int_{D(t,p)} \langle p, \omega(t) \rangle dP = \langle p, \omega(t) \rangle.$$

Thus, if $P$ does not belong to $D_R(t,p)$, then there exists $Q \in B_R(t,p)$ such that $\int u(t,x)dQ > \int u(t,x)dP$. Note also that

$$\int_X u(t,x)dP = \int_{D(t,p)} u(t,x)dP = \int_{D(t,p)} \max_{y \in B(t,p)} u(t,y)dP = \max_{y \in B(t,p)} u(t,y).$$
Furthermore, $Q(D(t,p)) = 1$ a.e. $t \in T$; for otherwise, we have $\int u(t,x)dQ < \max_{y \in B(t,p)} u(t,y)$ as derived in (A.1), a contradiction. We thus obtain

$$\int_X u(t,x)dQ = \int_{D(t,p)} u(t,x)dQ = \int_{D(t,p)} \max_{y \in B(t,p)} u(t,y)dQ = \max_{y \in B(t,p)} u(t,y).$$

This is a contradiction to the initial hypothesis. Therefore, $P \in D_R(t,p)$.  

\section*{A.2 Proof of Proposition 3.2}

Pick any Walrasian equilibrium $(p,f)$ for $\mathcal{E}$. If the price-relaxed allocation pair $(p,\delta_{f(T)})$ is not a relaxed Walrasian equilibrium for $\mathcal{E}_R$, then there exists $A \in \Sigma$ of positive measure such that for every $t \in A$ there exists $P \in B_{\mathcal{R}}(t,p)$ with $P \succ_{\mathcal{R}}(t) \delta_{f(T)}$. Given the preference representation (3.2), this means the inequality $\int u(t,x)dP > u(t,f(t)) = \max_{y \in B(t,p)} u(t,y)$. We then have $P(D(t,p)) = 1$ for every $t \in A$; for otherwise, $\int u(t,x)dP < \max_{y \in B(t,p)} u(t,y)$ for some $t \in A$, a contradiction. On the other hand, the equalities

$$\int_X u(t,x)dP = \int_{D(t,p)} u(t,x)dP = \max_{y \in B(t,p)} u(t,y)$$

for every $t \in T$ yield a contradiction to the above inequality. Therefore, $(p,\delta_{f(T)})$ is a relaxed Walrasian equilibrium for $\mathcal{E}_R$.

Take any relaxed Walrasian equilibrium $(p,\lambda)$ for $\mathcal{E}_R$. Let $gph B(\cdot,p)$ (resp. $gph D(\cdot,p)$) be the graph of the multifunction $B(\cdot,p) : T \to X$ (resp. $D(\cdot,p) : T \to X$) and denote by Borel($E,w$) the Borel $\sigma$-algebra generated by the weak topology of $E$. Since

$$D(t,p) = \left\{ x \in X \mid u(t,x) = \max_{y \in B(t,p)} u(t,y) \right\}$$

with $gph B(\cdot,p) \in \Sigma \otimes \text{Borel}(E,w)$, the measurable maximum theorem (see Hildenbrand (1974, Proposition 3, p.60)) guarantees that $gph D(\cdot,p) \in \Sigma \otimes \text{Borel}(E,w)$. By Proposition 3.1, we have $\lambda(t)(D(t,p)) = 1$ a.e. $t \in T$. It follows from Proposition 2.1 that there exists $f \in \mathcal{M}(T,X) \subset L^1(\mu,E)$ with $f(t) \in D(t,p)$ a.e. $t \in T$ such that $\int f d\mu = \int_X \lambda(t,dx)d\mu$. Therefore, $(p,f)$ is a Walrasian equilibrium for $\mathcal{E}$. Since $\lambda(t)(D(t,p)) = 1$ a.e. $t \in T$ by Theorem 3.1, we have $\int u(t,x)\lambda(t,dx) = \max_{y \in B(t,p)} u(t,y) = u(t,f(t))$ a.e. $t \in T$. Therefore, $\lambda(t) \sim_{\mathcal{R}} (t) \delta_{f(T)}$ a.e. $t \in T$.  

\section*{A.3 Proof of Proposition 3.3}

Let $\lambda_0 \in \mathcal{A}(\mathcal{E}_R)$ be arbitrarily and $\mathcal{N}_0$ be its any neighborhood. By definition of the weak topology, there exists $u_1,\ldots,u_k$ in $\mathcal{C}^1(T \times X,\mu)$ such that
that \(|I_u(\nu) - I_u(\nu_0)| < 1\), \(i = 1, \ldots, k\) implies \(\nu \in \mathcal{N}_0\). Define \(\Phi : T \times X \to E \times \mathbb{R}^k\) in Proposition 2.1 by \(\Phi = (t_X, u_1, \ldots, u_k)\). Then there exists \(f \in \mathcal{M}(T, X)\) such that \(\int \int \Phi(t, x)\lambda(t, dx)d\mu = \int \Phi(t, f(t))d\mu\). This means that \(f \in L^1(\mu, E), \int \int t_X(x)\lambda(t, dx)d\mu = \int f(t)d\mu\), and \(I_{u_i}(\nu_0) = I_{u_i}(\delta_f)\) for \(i = 1, \ldots, k\). Therefore, \(f \in \mathcal{A}(\mathcal{E})\) and \(\delta_f \in \mathcal{N}_0\). Since the choice of \(\lambda_0\) and \(\mathcal{N}_0\) is arbitrary, \(\mathcal{A}(\mathcal{E})\) is dense in \(\mathcal{A}(\mathcal{E}_\mathcal{R})\). Next, let \(\lambda_0 \in \mathcal{W}(\mathcal{E}_\mathcal{R})\) be arbitrarily. Since \(\lambda_0 \in D_{\mathcal{R}}(t, p)\) a.e. \(t \in T\) for some \(p \in E^* \setminus \{0\}\), it follows from Proposition 3.1 that \(\lambda_0(t)(D(t, p)) = 1\) a.e. \(t \in T\). Let \(U(t) \equiv D(t, p)\) in Proposition 2.1 and \(\mathcal{N}_0\) be any neighborhood of \(\lambda_0\). Then as in the above there exists \(f \in \mathcal{A}(\mathcal{E})\) with \(f(t) \in D(t, p)\) a.e. \(t \in T\) such that \(I_{u_i}(\nu_0) = I_{u_i}(\delta_f)\) for \(i = 1, \ldots, k\). Therefore, \(f \in \mathcal{W}(\mathcal{E})\) and \(\delta_f \in \mathcal{N}_0\), and hence, \(\mathcal{W}(\mathcal{E})\) is dense in \(\mathcal{W}(\mathcal{E}_\mathcal{R})\).

\section*{A.4 Proof of Theorem 3.1}

The set of normalized price functionals is given by \(S^* = \{p \in E^*_+ \mid \langle p, v \rangle = 1\}\), where \(v \in E_+\) is taken from the norm interior of \(E_+\). Then the Banach–Alooglu theorem guarantees that \(S^*\) is weakly* compact; see Mas-Colell and Zame (1991, p. 1859).

\begin{lemma}
\(D_{\mathcal{R}} : T \times S^* \to \Pi(X)\) is a compact, convex-valued multifunction with \(\text{gph} \, D_{\mathcal{R}}(\cdot, p) \in \Sigma \otimes \text{Borel}(\Pi(X))\) for every \(p \in S^*\).
\end{lemma}

\textit{Proof.} The compactness and convexity of \(D_{\mathcal{R}}(t, p)\) follows from the continuity and affinity of the relaxed utility function \(P \mapsto \int u(t, x)dP\) in (3.2). Fix \(p \in S^*\) arbitrarily and define \(\theta_p : T \times \Pi(X) \to \mathbb{R}\) by \(\theta_p(t, P) = \int \langle p, t_X(x) \rangle dP - \langle p, \omega(t) \rangle\). Then \(t \mapsto \theta_p(t, P)\) is measurable for every \(P \in \Pi(X)\). Since \(x \mapsto \langle p, t_X(x) \rangle\) is a bounded continuous function on \(X\), the function \(P \mapsto \theta_p(t, P)\) is continuous for every \(t \in T\) in view of the definition of the topology of weak convergence of probability measures. Thus, \(\theta_p\) is a Carathéodory function, and hence, it is jointly measurable. Denote by \(\text{Borel}(\Pi(X))\) the Borel \(\sigma\)-algebra of \(\Pi(X)\). We then have

\[\text{gph} \, B_{\mathcal{R}}(\cdot, p) = \{(t, P) \in T \times \Pi(X) \mid \theta_p(t, P) \leq 0\} \in \Sigma \otimes \text{Borel}(\Pi(X)).\]

For the sake notational simplicity, define \(\tilde{u} : T \times \Pi(X) \to \mathbb{R}\) by \(\tilde{u}(t, P) = \int u(t, x)dP\). Then \(\tilde{u}\) is a Carathéodory function, and hence, it is jointly measurable. Given the representation of the relaxed preferences in (3.2), we have

\[D_{\mathcal{R}}(t, p) = \left\{P \in B_{\mathcal{R}}(t, p) \mid \tilde{u}(t, P) = \max_{Q \in B_{\mathcal{R}}(t, p)} \tilde{u}(t, Q)\right\}.
\]
Hence, by the measurable maximum theorem (see Hildenbrand (1974, Proposition 3, p. 60)), we have \( \text{gph} \, D_R(\cdot, p) \in \Sigma \otimes \text{Borel}(\Pi(X)) \). \( \square \)

A difficulty might arise in the derivation of the upper semicontinuity of \( p \mapsto D_R(t, p) \) because of the failure of the joint continuity of the valuation functional \( (p, P) \mapsto \int \langle p, \iota_X(x) \rangle dP \) on \( S^* \times \Pi(X) \) whenever \( S^* \) is endowed with the weak* topology of \( E^* \), which is analogous to the well-known failure of the joint continuity of the valuation functional \( (p, x) \mapsto \langle p, x \rangle \) on \( S^* \times \mathbb{R} \) whenever \( E \) is endowed with the weak topology; e.g., see Mas-Colell and Zame (1991). To overcome the difficulty of the upper semicontinuity of \( D_R(t, \cdot) : S^* \to \Pi(X) \), we “enlarge” the relaxed demand set by introducing the multifunction \( \Gamma : T \times S^* \to \Pi(X) \) defined by

\[
\Gamma(t, p) = \{ P \in \Pi(X) \mid P \gtrless_R(t) Q \forall Q \in B_R(t, p) \}
\]

adapting the device used in Lee (2013); Podczeck (1997) to the relaxation framework.\(^9\) By definition, \( D_R(t, p) \subseteq \Gamma(t, p) \) for every \( (t, p) \in T \times S^* \).

**Lemma A.2.** \( \Gamma : T \times S^* \to \Pi(X) \) is a compact, convex-valued multifunction with \( \text{gph} \, \Gamma(\cdot, p) \in \Sigma \otimes \text{Borel}(\Pi(X)) \) for every \( p \in S^* \) such that \( p \mapsto \Gamma(t, p) \) is upper semicontinuous for the weak* topology of \( S^* \) for every \( t \in T \).

**Proof.** The fact that \( \Gamma \) has compact convex values is obvious. To show the upper semicontinuity, fix \( t \in T \) arbitrarily and let \( \{p_\alpha\} \) be a net in \( S^* \) with \( p_\alpha \to p \) weakly* and choose any \( P_\alpha \in \Gamma(t, p_\alpha) \) for each \( \alpha \) with \( P_\alpha \to P \). We need to show that \( P \in \Gamma(t, p) \). Suppose, to the contrary, that \( P \not\in \Gamma(t, p) \). Then there exists \( Q \in \Pi(X) \) such that \( Q \gtrless_R(t) P \) and \( \int \langle p, \iota_X(x) \rangle dQ \leq \langle p, \omega(t) \rangle \). It follows from the continuity of \( \gtrless_R(t) \) and Assumption 3.2(ii) that \( Q \) is taken such that \( Q \gtrless_R(t) P \) and \( \int \langle p, \iota_X(x) \rangle dQ < \langle p, \omega(t) \rangle \). Thus, for all sufficiently large \( \alpha \), we have \( Q \gtrless_R(t) P \) and \( \langle p_\alpha, \int \iota_X dQ \rangle = \int \langle p_\alpha, \iota_X(x) \rangle dQ < \langle p, \omega(t) \rangle \), which contradicts the fact that \( P_\alpha \in \Gamma(t, p_\alpha) \). Therefore, \( P \in \Gamma(t, p) \). Since

\[
\Gamma(t, p) = \left\{ P \in \Pi(X) \mid \tilde{u}(t, P) \geq \max_{Q \in B_R(t, p)} \tilde{u}(t, Q) \right\}
\]

and the marginal function \( t \mapsto \max_{Q \in B_R(t, p)} \tilde{u}(t, Q) \) is measurable by the measurable maximum theorem, we have \( \text{gph} \, \Gamma(\cdot, p) \in \Sigma \otimes \text{Borel}(\Pi(X)) \) for every \( p \in S^* \). \( \square \)

\(^9\)See also Khan and Yannelis (1991) for another technique to evade the difficulty of joint continuity in the original economy.
**Lemma A.3.** Define the multifunction $I_\Gamma : T \times S^* \rightarrow E$ by

$$I_\Gamma(t,p) = \left\{ \int_X i_X(x) dP \mid P \in \Gamma(t,p) \right\}.$$ 

Then $I_\Gamma$ is a weakly compact, convex-valued multifunction such that its range $I_\Gamma(T \times S^*)$ is bounded and $p \mapsto I_\Gamma(t,p)$ is upper semicontinuous for the weak$^*$ topology of $S^*$ and the norm topology of $E$ for every $t \in T$.

**Proof.** It follows from the weak compactness of $X$ that $\sup_{x \in X} \|x\| \leq a$ for some $a \geq 0$. Thus, $\sup_{P \in \Gamma(t,p)} \| \int i_X dP \| \leq a$ for every $(t,p) \in T \times S^*$. Hence, $I_\Gamma(T \times S^*)$ is bounded. Since $\Gamma(t,p)$ is a convex subset of $\Pi(X)$ by Lemma A.2, the convexity of $I_\Gamma(t,p)$ is obvious. To show the weak compactness of $I_\Gamma(t,p)$, fix $(t,p) \in T \times S^*$ arbitrarily and choose a net $y_\alpha \in I_\Gamma(t,p)$ for each $\alpha$. Then there exists $P_\alpha \in \Gamma(t,p)$ such that $y_\alpha = \int i_X dP_\alpha$ for each $\alpha$. Since $\Gamma(t,p)$ is compact by Lemma A.2, we can extract a subnet from $\{P_\alpha\}$ (which we do not relabel) converging to $P \in \Gamma(t,p)$. Hence, the barycenter $\int i_X dP$ belongs to $I_\Gamma(t,p)$. It follows from the definition of the topology of the weak convergence of probability measures that for every $x^* \in E^*$, we have

$$\langle x^*, y_\alpha \rangle = \int_X \langle x^*, i_X(x) \rangle dP_\alpha \rightarrow \int_X \langle x^*, i_X(x) \rangle dP = \left\langle x^*, \int i_X(x) dP \right\rangle$$

because $x \mapsto \langle x^*, i_X(x) \rangle$ is a bounded continuous function on $X$. This means that $y_\alpha \rightarrow \int i_X dP$ weakly in $E$. Thus, $I_\Gamma(t,p)$ is weakly compact.

To show the upper semicontinuity, fix $t \in T$ arbitrarily and let $\{p_\alpha\}$ be a net in $S^*$ with $p_\alpha \rightarrow p$ weakly$^*$ and choose any $y_\alpha \in I_\Gamma(t,p_\alpha)$ for each $\alpha$ with $y_\alpha \rightarrow y$ strongly in $E$. We need to show that $y \in I_\Gamma(t,p)$. Suppose, to the contrary, that $y \notin I_\Gamma(t,p)$. Then for each $\alpha$ there exists $P_\alpha \in \Gamma(t,p_\alpha)$ such that $y_\alpha = \int i_X dP_\alpha$. Denote by $\overline{\omega} X$ be the closed convex hull of $X$. Then the barycenters $\int i_X dP_\alpha$ belong to $\overline{\omega} X$; see Diestel and Uhl (1977, Corollary II.2.8). Hence, we have $y \in \overline{\omega} X$. It follows from Choquet's theorem (see Phelps (2001, Proposition 1.2)) that there exists $P \in \Pi(X)$ such that $\langle x^*, y \rangle = \int \langle x^*, i_X(x) \rangle dP = \langle x^*, \int i_X dP \rangle$ for every $x^* \in E^*$. This means that $y = \int i_X dP$. In view of $y \notin I_\Gamma(t,p)$, we have $P \notin \Gamma(t,p)$. As demonstrated in the proof of Lemma A.2, there exists $Q \in \Pi(X)$ such that $Q \succ_R(t) P$ and $(p_\alpha, \int i_X dQ) = \int \langle p_\alpha, i_X(x) \rangle dQ < \langle p, \omega(t) \rangle$ for all sufficiently large $\alpha$, which contradicts the fact that $P_\alpha \in \Gamma(t,p_\alpha)$. Therefore, $y \in I_\Gamma(t,p)$.

What is significant in the next lemma is that the upper semicontinuity of $p \mapsto \int I_\Gamma(t,p) d\mu$ is preserved under integration without any assumption
on the finite measure space \((T, \Sigma, \mu)\) due to the fact that the upper semicontinuous multifunction \(p \mapsto I_\Gamma(t, p)\) has weakly compact convex values. This observation permits us to invoke fixed point theorems in the sequel.

**Lemma A.4.** The Bochner integral \(\int I_\Gamma(t, p)d\mu\) of the multifunction \(I_\Gamma(\cdot, p) : T \to E\) is nonempty, weakly compact, and convex for every \(p \in S^*\), and the multifunction \(p \mapsto \int I_\Gamma(t, p)d\mu\) is upper semicontinuous for the weak* topology of \(S^*\) and the norm topology of \(E\).

**Proof.** The nonemptiness and convexity of \(\int I_\Gamma(t, p)d\mu\) are obvious because for every \(p \in S^*\) the Bochner integrable selectors of \(I_\Gamma(\cdot, p) : T \to E\) are precisely of the form \(\int \mathbb{1}_E \lambda(t, dx) \in I_\Gamma(t, p)\) with \(\lambda(t) \in \Gamma(t, p)\) a.e. \(t \in T\) and \(\lambda \in \mathcal{R}(T, X)\). The weak compactness of \(\int I_\Gamma(t, p)d\mu\) follows from Yannelis (1991, Theorem 6.1).

To show the upper semicontinuity, introduce the support functional of \(C \subset E\) and define \(s(\cdot, C) : E^* \to \mathbb{R} \cup \{+\infty\}\) by \(s(x^*, C) = \sup_{x \in C} \langle x^*, x \rangle\). Then the weakly compact convex-valued multifunction \(p \mapsto \int I_\Gamma(t, p)d\mu\) is upper semicontinuous if and only if \(p \mapsto s(x^*, \int I_\Gamma(t, p)d\mu)\) is upper semicontinuous for every \(x^* \in E^*\); see Aliprantis and Border (2006, Theorem 17.41). Since \(s(x^*, \int I_\Gamma(t, p)d\mu) = \int s(x^*, I_\Gamma(t, p))d\mu\) for every \(p \in S^*\) (see Aubin and Frankowska (1990, Proposition 8.6.2)), it suffices to show the upper semicontinuity of \(p \mapsto \int s(x^*, I_\Gamma(t, p))d\mu\) for every \(x^* \in E^*\). Since \(S^*\) is metrizable with respect to the weak* topology, we can resort to sequential convergence in \(S^*\). Let \(\{p_n\}\) be a sequence in \(S^*\) with \(p_n \rightharpoonup p\) weakly*. Since the weak compact convex valued multifunction \(p \mapsto I_\Gamma(t, p)\) is upper semicontinuous for every \(t \in T\) by Lemma A.3, the function \(p \mapsto s(x^*, I_\Gamma(t, p))\) is upper semicontinuous for every \(x^* \in E^*\) and \(t \in T\). Fix \(x^* \in E^*\) arbitrarily. Since the sequence of functions \(t \mapsto s(x^*, I_\Gamma(t, p_n))\) is bounded, we obtain

\[
\limsup_{n \to \infty} \int_T s(x^*, I_\Gamma(t, p_n))d\mu \leq \int_T \limsup_{n \to \infty} s(x^*, I_\Gamma(t, p_n))d\mu \\
\leq \int_T s(x^*, I_\Gamma(t, p))d\mu
\]

where the first equality follows from Fatou’s lemma and the second inequality exploits the upper semicontinuity of \(p \mapsto s(x^*, I_\Gamma(t, p))\). Therefore, \(p \mapsto \int s(x^*, I_\Gamma(t, p))d\mu\) is upper semicontinuous for every \(x^* \in E^*\). \(\square\)

A maximal element in \(\Pi(X)\) for \(\succcurlyeq_R(t)\) is called a satiation point for \(\succcurlyeq_R(t)\). Corresponding to the conditions (iii) and (iv) of Assumption 3.2 on the original preferences, the condition on (non)satiation points for the relaxed preferences takes the following form.
Lemma A.5.  (i) If \( P \in \Pi(X) \) is a satiation point for \( \succsim_R(t) \), then \( \int_X dP \geq \omega(t) \).

(ii) If \( P \in \Pi(X) \) is not a satiation point for \( \succsim_R(t) \), then \( P \) belongs to the closure of the upper contour set \( \{Q \in \Pi(X) \mid Q \succsim_R(t) P\} \).

Proof. (i): Let \( U(t) \subset X \) be the set of satiation points for \( \succsim(t) \). Given the preference representation (3.2), we have \( \max_{x \in X} u(t, x) > u(t, y) \) for every \( y \in X \setminus U(t) \). We first claim that \( P \in \Pi(X) \) is a satiation point for \( \succsim_R(t) \) if and only if \( P(U(t)) = 1 \). Suppose that \( P \) is satiated for \( \succsim(t) \). We then have \( \int u(t, x)dP \geq \int u(t, x)dQ \) for every \( Q \in \Pi(X) \). Assume that \( P(U(t)) < 1 \). If we choose \( Q \in \Pi(X) \) satisfying \( Q(U(t)) = 1 \), then \( \int u(t, x)dP < \max_{x \in X} u(t, x) = \int u(t, x)dQ \), a contradiction. Conversely, if \( P(U(t)) = 1 \), then \( \int u(t, x)dP = \max_{x \in X} u(t, x) \geq \int u(t, x)dQ \) for every \( Q \in \Pi(X) \). Hence, \( P \) is a satiation point for \( \succsim_R(t) \). Since \( \int_X(x) - \omega(t) \leq 0 \) for every \( x \in U(t) \) in view of Assumption 3.2(iii), integrating this inequality over \( U(t) \) with respect to any satiated \( P \) yields \( \int_X dP - \omega(t) \leq 0 \).

(ii): Take any nonsatiation point \( P \in \Pi(X) \) for \( \succsim(t) \). We need to show that there exists a sequence \( \{P_n\} \) in \( \Pi(X) \) with \( P_n \succsim_R(t) P \) for each \( n \) and \( P_n \to P \). Since the convex hull of \( \Delta(X) \) is dense in \( \Pi(X) \) (see Aliprantis and Border (2006, Theorem 15.10)) and \( \Pi(X) \) is separable, \( P \) is approximated arbitrarily by a sequence of the convex combinations of Dirac measures of the form \( Q = \sum_{i \in I} \alpha_i \delta_{x^i} \in \Pi(X) \), where \( x^i \in X \), \( \alpha_i > 0 \), and \( \sum_{i \in I} \alpha_i = 1 \) with a finite index set \( I \). Since \( P(U(t)) < 1 \), we may assume without loss of generality that \( x^i \in X \setminus U(t) \) for some \( i \in I \) whenever \( Q \) is close enough to \( P \). For each \( x^i \in X \setminus U(t) \), choose a sequence \( y^i_n \in X \) with \( y^i_n \succsim_R(t) x^i \) for each \( n \) and \( y^i_n \to x^i \) weakly, which is possible by Assumption 3.2(iv). Define the probability measure by

\[
P_n = \sum_{\{i \in I \mid x^i \in U(t)\}} \alpha_i \delta_{x^i} + \sum_{\{i \in I \mid x^i \in X \setminus U(t)\}} \alpha_i \delta_{y^i_n}.
\]

By construction, we have

\[
\int_X u(t, x)dP_n = \sum_{\{i \in I \mid x^i \in U(t)\}} \alpha_i u(t, x^i) + \sum_{\{i \in I \mid x^i \in X \setminus U(t)\}} \alpha_i u(t, y^i_n)
\]

\[
> \sum_{i \in I} \alpha_i u(t, x^i) = \int_X u(t, x)dQ.
\]

Hence, \( P_n \succsim_R(t) Q \) for each \( n \). Since \( P_n \to Q \) and \( Q \) can be taken close arbitrarily to \( P \), we obtain the desired conclusion. \( \square \)
Lemma A.6. $\int \langle p, i_X(x) \rangle dP \geq \langle p, \omega(t) \rangle$ for every $(t, p) \in T \times S^*$ and $P \in \Gamma(t, p)$.

Proof. If $P$ is a satiation point for $\preceq_R(t)$, then by Lemma A.5(i), $\int i_X dP \geq \omega(t)$, and hence, $\int \langle p, i_X(x) \rangle dP = \langle p, \int i_X dP \rangle \geq \langle p, \omega(t) \rangle$ a.e. $t \in T$. If $P$ is not a satiation point $\preceq_R(t)$ and $\int \langle p, i_X(x) \rangle dP < \langle p, \omega(t) \rangle$, it follows from Lemma A.5(ii) that there exists $Q \succ_R(t) P$ such that $\int \langle p, i_X(x) \rangle dQ < \langle p, \omega(t) \rangle$, which contradicts the fact that $P \in \Gamma(t, p)$. \hfill $\square$

Proof of Theorem 3.1. (i): Define the multifunction $\xi : S^* \to E$ by

$$\xi(p) = \int_T I_{\Gamma}(t, p) d\mu - \int_T \omega(t) d\mu.$$  

Then by Lemma A.4, $\xi$ is upper semicontinuous for the weak* topology of $S^*$ and the norm topology of $E$ with weakly compact, convex values. We claim that for every $p \in S^*$ there exists $z \in \xi(p)$ such that $\langle p, z \rangle \leq 0$. To this end, fix $p \in S^*$ arbitrarily. By Lemma A.1, there exists a measurable function $\lambda_p : T \to \Pi(X)$ such that $\lambda_p(t) \in D_R(t, p) \subset \Gamma(t, p)$ a.e. $t \in T$. Since $\int i_X \lambda_p(t, dx) \in I_{\Gamma}(t, p)$, we have $\int \int i_X \lambda_p(t, dx) d\mu \in \int I_{\Gamma}(t, p) d\mu$. Integrating the relaxed budget constraint $\int \langle p, i_X(x) \rangle \lambda_p(t, dx) - \langle p, \omega(t) \rangle \leq 0$ over $T$ yields $\int \langle p, \int i_X \lambda_p(t, dx) d\mu - \int \omega(t) d\mu \rangle \leq 0$. Hence, the vector $z = \int \int i_X \lambda_p(t, dx) d\mu - \int \omega(t) d\mu \in \xi(p)$ satisfies $\langle p, z \rangle \leq 0$.

It follows from the infinite-dimensional version of the Gale–Nikaido Lemma (see Yannelis (1985, Theorem 3.1)) that there exists $p \in S^*$ such that $\xi(p) \cap (\mathbb{R}_+ E) \neq \emptyset$. Hence, there exists a measurable function $\lambda : T \to \Pi(X)$ with $\lambda(t) \in \Gamma(t, p)$ a.e. $t \in T$ satisfying $\int \int i_X \lambda(t, dx) d\mu \in \int I_{\Gamma}(t, p) d\mu$ and $\int \int i_X \lambda(t, dx) d\mu - \int \omega(t) d\mu \leq 0$. This means that $\lambda \in \mathcal{A}(E_R)$ and

$$\int_T \langle p, \int_X i_X(x) \lambda(t, dx) \rangle d\mu \leq \int_T \langle p, \omega(t) \rangle d\mu.$$  

On the other hand, by Lemma A.6,

$$\langle p, \int_X i_X(x) \lambda(t, dx) \rangle \geq \langle p, \omega(t) \rangle \quad \text{a.e. } t \in T.$$  

Combining these inequalities yields the equality $\int \langle p, i_X(x) \rangle \lambda(t, dx) = \langle p, \omega(t) \rangle$ a.e. $t \in T$ for the relaxed budget constraint. Note also that by construction we have

$$D_R(t, p) = \Gamma(t, p) \cap \left\{ P \in \Pi(X) \mid \int_X \langle p, i_X(x) \rangle dP = \langle p, \omega(t) \rangle \right\}.$$
This means that $\lambda(t)$ belongs to $D_R(t, p)$ a.e. $t \in T$. Therefore, the price-relaxed allocation pair $(p, \lambda) \in S^* \times R(T, X)$ is a relaxed Walrasian equilibrium for $\mathcal{E}_R$.

(ii): This follows from the assertion (i) and Proposition 3.2. \hfill \blacksquare

A.5 Proof of Theorem 3.2

(i): When $X$ is a finite set, the difficulty of the joint continuity of the valuation functional $(p, x) \mapsto \langle p, x \rangle$ on $S^* \times X$ never arises because the relative topology of $X$ inherited from the weak topology of $E$ is a discrete topology. Indeed, let $(p_\alpha, x_\alpha)$ be a net in $S^* \times X$ converging to $(p, x) \in S^* \times X$. Since $x_\alpha \rightharpoonup x$ weakly means that $x_\alpha = x$ for every $\alpha \geq \alpha_0$ with some $\alpha_0$, we have $\lim_\alpha \langle p_\alpha, x_\alpha \rangle = \lim_\alpha \langle p_\alpha, x \rangle = \langle p, x \rangle$. Hence, the valuation functional is continuous on $S^* \times X$ for the weak* topology of $E^*$ and the weak topology of $E$. This implies that the relaxed demand multifunction $D_R(t, \cdot)$ is upper semicontinuous on $S^*$ for every $t \in T$. Let $I_{D_R} : T \times S^* \rightarrow E$ be a multifunction defined by

$$I_{D_R}(t, p) = \left\{ \int_X i_X(x) dP \mid P \in D_R(t, p) \right\}.$$  

Replacing $\Gamma$ by $D_R$ in the proof of Lemma A.4 yields that the Bochner integral $\int I_{D_R}(t, p) d\mu$ is nonempty, weakly compact, and convex for every $p \in S^*$, and the multifunction $p \mapsto \int I_{D_R}(t, p) d\mu$ is upper semicontinuous for the weak* topology of $S^*$ and the norm topology of $E$.

Define the multifunction $\xi : S^* \rightarrow E$ by

$$\xi(p) = \int_T I_{D_R}(t, p) d\mu - \int_T \omega(t) d\mu.$$  

Then $\xi$ is upper semicontinuous for the weak* topology of $S^*$ and the norm topology of $E$ with weakly compact, convex values. As in the proof of Proposition 3.1, for every $p \in S^*$ there exists $z \in \xi(p)$ such that $\langle p, z \rangle \leq 0$. It follows from the infinite-dimensional version of the Gale–Nikaido Lemma (see Yannelis (1985, Theorem 3.1)) that there exists $p \in S^*$ such that $\xi(p) \cap (-E_+) \neq \emptyset$. Hence, there exists a measurable function $\lambda : T \rightarrow \Pi(X)$ with $\lambda(t) \in D_R(t, p)$ a.e. $t \in T$ satisfying $\int \int i_X \lambda(t, dx) d\mu \in \int I_{D_R}(t, p) d\mu$ and $\int \int i_X \lambda(t, dx) d\mu - \int \omega d\mu \leq 0$. This means that the price-relaxed allocation pair $(p, \lambda) \in S^* \times \mathcal{R}(T, X)$ is a relaxed Walrasian equilibrium for $\mathcal{E}_R$.

(ii): This follows from the assertion (i) and Proposition 3.2. \hfill \blacksquare
B Appendix 2

B.1 Proof of Theorem 3.3

Given the duality $L^\infty(\nu)^* = ba(\nu)$, denote by $\sigma(ba, L^\infty)$ the weak* topology of $ba(\nu)$. Define the normalized price space by $S^* = \{ \pi \in ba_+(\nu) \mid \langle \pi, \psi \rangle = 1 \}$, where $\psi$ is taken from the norm interior of $L_\infty^\infty(\nu)$. Then $S^*$ is $\sigma(ba, L^\infty)$-compact (i.e., weakly* compact) and convex.

**Lemma B.1.** There exists a sequence $\{ \varphi_n \}$ in $L^1(\nu)$ such that $ba(\nu) = \{ \varphi_n \}_{\sigma(ba,L^\infty)}$.

**Proof.** Consider the natural embedding $L^1(\nu) \subset L^1(\nu)^* = ba(\nu)$ and note that $L^1(\nu)$ is $\sigma(ba,L^\infty)$-dense subset of $ba(\nu)$; see Dunford and Schwartz (1958, Corollary V.4.6). Since $L^1(\nu)$ is separable in view of the countable generation of $\mathcal{F}$, there exists a countable dense set $\{ \varphi_n \}$ of $L^1(\nu)$ with respect to the norm topology. Since $L^1(\nu) = \{ \varphi_n \}_{\sigma(\nu)} \subset \{ \varphi_n \}_{\sigma(ba,L^\infty)}$, where $\{ \varphi_n \}_{\sigma(\nu)}$ is the norm closure of $\{ \varphi_n \}$ in $L^1(\nu)$ and $\{ \varphi_n \}_{\sigma(ba,L^\infty)}$ is the weak* closure of $\{ \varphi_n \}$ in $ba(\nu)$, we have $ba(\nu) = \overline{L^1(\nu)}_{\sigma(ba,L^\infty)} \subset \{ \varphi_n \}_{\sigma(ba,L^\infty)}$. Hence, $ba(\nu) = \{ \varphi_n \}_{\sigma(ba,L^\infty)}$. \hfill $\square$

Given the technique explored in Subsection A.4, the existence of equilibrium prices in $ba_+(\nu)$ is more or less routine because it is again a direct application of the Gale–Nikaido lemma in $L^\infty(\nu)$. A key result is Theorem B.1 below, for which we outline the proof.

**Observation B.1.** Lemma A.1 holds as it stands for $E = L^\infty(\nu)$ and $X \subset L^\infty_\infty(\nu)$.

**Observation B.2.** To see the validity of Lemma A.2, it suffices to show that $\langle \pi, \int t_X dP \rangle = \int_X \langle \pi, t_X(x) \rangle dP$ for every $P \in \Pi(X)$ and $\pi \in ba(\nu)$, where $\int t_X dP \in L^\infty(\nu)$ is the Gelfand integral of $t_X$ with respect to $P$. By Lemma B.1, for every $\pi \in ba(\nu)$ there exists a sequence $\varphi_n \in L^1(\nu)$ such that $\varphi_n \to \pi$ for $\sigma(ba,L^\infty)$-topology. We then have $\langle \varphi_n, \int t_X dP \rangle \to \langle \varphi_n, \int_X t_X(x) dP \rangle$ for each $n$. Taking the limit in the both side of this equality yields

$$
\langle \pi, \int_X t_X(x) dP \rangle = \lim_{n \to \infty} \langle \varphi_n, \int_X t_X(x) dP \rangle = \lim_{n \to \infty} \int_X \langle \varphi_n, t_X(x) \rangle dP = \int_X \lim_{n \to \infty} \langle \varphi_n, t_X(x) \rangle dP = \int_X \langle \pi, t_X(x) \rangle dP
$$

where the third equality in the above employs the Lebesgue dominated convergence theorem in view of the boundedness of $X \subset L^\infty_\infty(\nu)$.
Lemma B.2. Define the multifunction $I_{\Gamma}: T \times S^* \to L^\infty(\nu)$ by

$$I_{\Gamma}(t, \pi) = \left\{ \int_X i_X(x)dP \mid P \in \Gamma(t, \pi) \right\}.$$ 

Then $I_{\Gamma}$ is a weakly* compact, convex-valued multifunction such that its range $I_{\Gamma}(T \times S^*)$ is bounded and $\pi \mapsto I_{\Gamma}(t, \pi)$ is upper semicontinuous for the weak* topology of $S^*$ and the norm topology of $L^\infty(\nu)$ for every $t \in T$.

Proof. It follows from the weak* compactness of $X$ that $\sup_{x \in X} \|x\| \leq a$ for some $a \geq 0$. Thus, $\sup_{P \in \Gamma(t, \pi)} \|\int i_X dP\| \leq a$ for every $(t, \pi) \in T \times S^*$. Hence, $I_{\Gamma}(T \times S^*)$ is bounded, and hence, it is weakly* relatively compact. Since $\Gamma(t, \pi)$ is a convex subset of $\Pi(X)$ by Lemma A.2, the convexity of $I_{\Gamma}(t, \pi)$ is obvious. To show the weak* compactness of $\Gamma(t, \pi)$, it suffices to show that $\Gamma(t, \pi)$ is weakly* closed. To this end, fix $(t, \pi) \in T \times S^*$ arbitrarily and choose a net $\psi_\alpha \in I_{\Gamma}(t, \pi)$ for each $\alpha$. Then there exists $P_\alpha \in \Gamma(t, \pi)$ such that $\psi_\alpha = \int i_X dP_\alpha$ for each $\alpha$. Since $\Gamma(t, \pi)$ is compact by Lemma A.2, we can extract a subnet from $\{P_\alpha\}$ (which we do not relabel) converging to $P \in \Gamma(t, \pi)$. Hence, the barycenter $\int i_X dP$ belongs to $I_{\Gamma}(t, \pi)$. It follows from the definition of the topology of the weak convergence of probability measures that for every $\varphi \in L^1(\nu)$, we have

$$\langle \varphi, \psi_\alpha \rangle = \int_X \langle \varphi, i_X(x) \rangle dP_\alpha \to \int_X \langle \varphi, i_X(x) \rangle dP = \langle \varphi, \int i_X(x) dP \rangle$$

because $x \mapsto \langle \varphi, i_X(x) \rangle$ is a bounded continuous function on $X$. This means that $\psi_\alpha \to \int i_X dP$ weakly* in $L^\infty(\nu)$. Thus, $I_{\Gamma}(t, p)$ is weakly* closed.

To show the upper semicontinuity, fix $t \in T$ arbitrarily and let $\{\pi_\alpha\}$ be a net in $S^*$ with $\pi_\alpha \to \pi$ weakly* and choose any $\psi_\alpha \in I_{\Gamma}(t, \pi_\alpha)$ for each $\alpha$ with $\psi_\alpha \to \psi$ strongly in $L^\infty(\nu)$. We need to show that $\psi \in I_{\Gamma}(t, \pi)$. Suppose, to the contrary, that $\psi \notin I_{\Gamma}(t, \pi)$. Then for each $\alpha$ there exists $P_\alpha \in \Gamma(t, \pi_\alpha)$ such that $\psi_\alpha = \int i_X dP_\alpha$. Denote by $\overline{\overline{\varphi}}^{w^*} X$ be the weakly* closed convex hull of $X$. Then the barycenters $\int i_X dP_\alpha$ belong to $\overline{\overline{\varphi}}^{w^*} X$; see Sagara (2016, Lemma 3.1) and Khan and Sagara (2014, Lemma 2.1). Hence, we have $\psi \in \overline{\overline{\varphi}}^{w^*} X$. It follows from Choquet’s theorem (see Phelps (2001, Proposition 1.2)) that there exists $P \in \Pi(X)$ such that $\langle \varphi, \psi \rangle = \int \langle \varphi, i_X(x) \rangle dP = \langle \varphi, \int i_X dP \rangle$ for every $\varphi \in L^1(\nu)$. This means that $\psi = \int i_X dP$. In view of $\psi \notin I_{\Gamma}(t, \pi)$, we have $P \notin \Gamma(t, \pi)$. As demonstrated in the proof of Lemma A.2, there exists $Q \in \Pi(X)$ such that $Q \succ_{\mathcal{R}}(t) P$ and $\langle \pi_\alpha, \int i_X dQ \rangle = \int \langle \pi_\alpha, i_X(x) \rangle dQ < \langle \pi, \omega(t) \rangle$ for all sufficiently large $\alpha$, which contradicts the fact that $P_\alpha \in \Gamma(t, \pi_\alpha)$. Therefore, $\psi \in I_{\Gamma}(t, \pi)$.

□
Lemma B.3. The Gelfand integral \( \int I_{\Gamma}(t, \pi) d\mu \) of the multifunction \( I_{\Gamma} : T \times S^* \rightarrow L^\infty(\nu) \) is nonempty, weakly* compact, and convex for every \( \pi \in S^* \), and the multifunction \( \pi \mapsto \int I_{\Gamma}(t, \pi) d\mu \) is upper semicontinuous for the weak* topology of \( S^* \) and the norm topology of \( L^\infty(\nu) \).

Proof. The nonemptiness and convexity of \( \int I_{\Gamma}(t, \pi) d\mu \) are obvious because for every \( \pi \in S^* \) the Gelfand integrable selectors of \( I_{\Gamma}(\cdot, \pi) : T \rightarrow L^\infty(\nu) \) are precisely of the form \( \int t_X \lambda(t, dx) \in I_{\Gamma}(t, \pi) \) with \( \lambda(t) \in \Gamma(t, \pi) \) a.e. \( t \in T \) and \( \lambda \in \mathcal{R}(T, X) \).

To show the weak* compactness of \( \int I_{\Gamma}(t, \pi) d\mu \), introduce the support functional of \( C \subset L^\infty(\nu) \) and define \( s(\cdot, C) : L^1(\nu) \rightarrow \mathbb{R} \cup \{+\infty\} \) by \( s(\varphi, C) = \sup_{\psi \in C} \langle \varphi, \psi \rangle \). Since \( t \mapsto I_{\Gamma}(t, \pi) \) is an integrably bounded multifunction with weakly* compact, convex values by Lemma B.2, it suffices to show that \( t \mapsto I_{\Gamma}(t, \pi) \) is weakly* scalarly measurable in the sense that \( t \mapsto s(\varphi, I_{\Gamma}(t, \pi)) \) is measurable for every \( \varphi \in L^1(\nu) \); see Khan (1985, Claim 2 to the proof of Theorem 2) or Cascales et al. (2011, Proposition 2.3(i) and Theorem 4.5). To this end, it suffices to show that \( t \mapsto \sup_{P \in \Gamma(t, \pi)} \int \langle \varphi, t_X(x) \rangle dP \) is measurable for every \( \varphi \in L^1(\nu) \) because

\[
s(\varphi, I_{\Gamma}(t, \pi)) = \sup_{P \in \Gamma(t, \pi)} \left\langle \varphi, \int_X t_X(x) dP \right\rangle = \sup_{P \in \Gamma(t, \pi)} \int_X \langle \varphi, t_X(x) \rangle dP
\]

in view of the Gelfand integrability of \( t_X \). Since \( \text{gph} \Gamma(\cdot, \pi) \in \Sigma \otimes \text{Borel}(\Pi(X)) \) by Lemma A.2 and \( P \mapsto \int \langle \varphi, t_X(x) \rangle dP \) is continuous because \( x \mapsto \langle \varphi, t_X(x) \rangle \) is a bounded continuous function on \( X \), it follows from the measurable maximum theorem that the marginal function \( t \mapsto \sup_{P \in \Gamma(t, \pi)} \int \langle \varphi, t_X(x) \rangle dP \) is measurable.

The weakly* compact convex-valued multifunction \( \pi \mapsto \int I_{\Gamma}(t, \pi) d\mu \) is upper semicontinuous if and only if \( \pi \mapsto s(\varphi, \int I_{\Gamma}(t, \pi) d\mu) \) is upper semicontinuous for every \( \varphi \in L^1(\nu) \); see Aliprantis and Border (2006, Theorem 17.41). Since \( s(\varphi, \int I_{\Gamma}(t, \pi) d\mu) = \int s(\varphi, I_{\Gamma}(t, \pi)) d\mu \) for every \( \pi \in S^* \) (see Cascales et al. (2011, Proposition 2.3(i) and Theorem 4.5)), it suffices to show the upper semicontinuity of \( \pi \mapsto \int s(\varphi, I_{\Gamma}(t, \pi)) d\mu \) for every \( \varphi \in L^1(\nu) \). The rest of the proof is same with the proof of Lemma A.4.

Observation B.3. Lemma A.5 holds as it stands. Lemma A.6 holds by the same reason with Observation B.1.

An analogue of the first assertion of Theorem 3.1 with the commodity space of \( L^\infty(\nu) \) with the Gelfand integral setting is provided as follows.
Theorem B.1. Let \((T, \Sigma, \mu)\) be a finite measure space and \((\Omega, \mathcal{F}, \nu)\) be a countably generated \(\sigma\)-finite measure space. Then for every economy \(\mathcal{E}_G^G\) satisfying Assumptions 3.4 and 3.5, there exists a relaxed Walrasian equilibrium with free disposal for \(\mathcal{E}_R^G\) with a positive price.

Proof. Define the multifunction \(\xi : S^* \rightarrow L^\infty(\nu)\) by

\[
\xi(\pi) = \int_T I_\Gamma(t, \pi) d\mu - \int_T \omega(t) d\mu.
\]

Then by Lemma B.3, \(\xi\) is upper semicontinuous for the weak\(^*\) topology of \(S^*\) and the norm topology of \(L^\infty(\nu)\) with weakly\(^*\) compact, convex values. As in the proof of Theorem 3.1(i), we can show that for every \(\pi \in S^*\) there exists \(z \in \xi(\pi)\) such that \(\langle \pi, z \rangle \leq 0\). Hence, it follows from the infinite-dimensional version of the Gale–Nikaido Lemma (see Yannelis (1985, Theorem 3.1)) that there exists \(\pi \in S^*\) such that \(\xi(\pi) \cap (-L^\infty(\nu)) \neq \emptyset\). The rest of the proof is same with the proof of Theorem 3.1(i) with replacing the Bochner integrals by Gelfand ones and the use of Lemma A.6 with Observation B.3.

Proof of Theorem 3.3. (i): Let \((\pi, \lambda) \in ba_+(\nu) \times \mathcal{A}(\mathcal{E}_R^G)\) be a relaxed Walrasian equilibrium with free disposal for \(\mathcal{E}_R^G\) assured in Theorem B.1. By the Yosida–Hewitt decomposition of finitely additive measures (see Yosida and Hewitt (1952, Theorems 1.22 and 1.24)), \(\pi\) is decomposed uniquely into \(\pi = \pi_1 + \pi_2\), where \(\pi_1 \geq 0\) is countably additive and \(\pi_2 \geq 0\) is purely finitely additive. (Here, \(\pi_2\) is purely finitely additive if every countably additive measure \(\pi'\) on \(\mathcal{F}\) satisfying \(0 \leq \pi' \leq \pi_2\) is identically zero.) Furthermore, there exists a sequence \(\{\Omega_n\}\) in \(\mathcal{F}\) such that (a) \(\Omega_n \subset \Omega_{n+1}\) for each \(n = 1, 2, \ldots\); (b) \(\lim_n \pi_1(\Omega \setminus \Omega_n) = 0\); (c) \(\pi_2(\Omega_n) = 0\) for each \(n = 1, 2, \ldots\).

We claim that \((\pi_1, \lambda)\) a relaxed Walrasian equilibrium with free disposal for \(\mathcal{E}_R^G\). To this end, suppose that \(P \succ_R(t) \lambda(t)\). We need to demonstrate that \(\int \langle \pi_1, i_X(x) dP \rangle > \langle \pi_1, \omega(t) \rangle\). It follows from the definition of relaxed Walrasian equilibria that \(\int \langle \pi, i_X(x) dP \rangle > \langle \pi, \omega(t) \rangle\). Define \(X_n = \{\psi \in X \mid \psi(s) = 0 \ \forall s \in \Omega \setminus \Omega_n\}\). Then \(X_n \subset X_{n+1}\) for each \(n\) by virtue of condition (a) and \(P(\bigcup X_n) = P(X) = 1\). Without loss of generality we may assume that \(P(X_n) > 0\) for each \(n\). Let \(P_n \in \Pi(X)\) be the conditional probability measure of \(X_n\) defined by \(P_n(Z) = P(Z \cap X_n)/P(X_n)\) with \(Z \in \text{Borel}(X, w^*)\), where the relevant Borel \(\sigma\)-algebra of \(X \subset L^\infty(\nu)\) is with respect to the weak\(^*\) topology of \(L^\infty(\nu)\). By construction, \(P_n(X_n) = 1\) for each \(n\). Since each \(P_n\) is absolutely continuous with respect to \(P\), there is a Radon–Nikodym derivative \(w_n \in L^1(P)\) of \(P_n\). Since \(P_n(Z) \rightarrow P(Z)\) for every \(Z \in \text{Borel}(X, w^*)\) by condition (b), it is easy to see that \(w_n \rightarrow \chi_X\) strongly in \(L^1(P)\). Choose any continuous function \(v\) on \(X\). It follows from the Lebesgue
dominated convergence theorem that \( \int vdP_n = \int vw_n dP \to \int vdP \), and hence, \( P_n \to P \) in \( \Pi(X) \). By the continuity of \( \succ_R(t) \), we have \( P_n \succ_R(t) \lambda(t) \) and \( \langle \pi, \iota(x) \rangle dP_n > \langle \pi, \omega(t) \rangle \) for all sufficiently large \( n \). Since \( X_n \) is closed and convex, and \( P_n(X_n) = 1 \), we have \( \int \iota_X dP_n \in X_n \) by Sagara (2016, Lemma 3.1). Let \( \psi_n := \int \iota_X dP_n \). Since \( \langle \pi_2, \psi_n \rangle = \int \psi_n d\pi_2 = \int \Omega_n \psi_n d\pi_2 = 0 \) by condition (c), we have \( \langle \pi, \psi_n \rangle = \langle \pi_1, \psi_n \rangle + \langle \pi_2, \psi_n \rangle = \langle \pi_1, \psi_n \rangle \). In view of \( \int \iota_X dP \geq \int \iota_X dP_n \), we obtain
\[
\int_X \langle \pi_1, \iota_X(x) \rangle dP = \langle \pi_1, \int_X \iota_X(x) dP \rangle \geq \langle \pi_1, \psi_n \rangle = \langle \pi, \psi_n \rangle > \langle \pi, \omega(t) \rangle \\
\geq \langle \pi_1, \omega(t) \rangle
\]
as desired. This also implies that \( \pi_1 \neq 0 \). Since \( \pi \) is absolutely continuous with respect to \( \nu \), the Radon Nikodym derivative of \( \pi_1 \) is an equilibrium price in \( L^1(\nu) \).

(ii): This follows from the assertion (i) and Proposition 3.4.

\[ \square \]

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