Reconstruction of complete interval tournaments. II.

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Abstract. Let \( a, b \) (\( b \geq a \)) and \( n \) (\( n \geq 2 \)) be nonnegative integers and let \( \mathcal{T}(a, b, n) \) be the set of such generalised tournaments, in which every pair of distinct players is connected at most with \( b \), and at least with \( a \) arcs. In [40] we gave a necessary and sufficient condition to decide whether a given sequence of nonnegative integers \( D = (d_1, d_2, \ldots, d_n) \) can be realized as the out-degree sequence of a \( T \in \mathcal{T}(a, b, n) \). Extending the results of [40] we show that for any sequence of nonnegative integers \( D \) there exist \( f \) and \( g \) such that some element \( T \in \mathcal{T}(g, f, n) \) has \( D \) as its out-degree sequence, and for any \( (a, b, n) \)-tournament \( T' \) with the same out-degree sequence \( D \) hold \( a \leq g \) and \( b \geq f \). We propose a \( \Theta(n) \) algorithm to determine \( f \) and \( g \) and an \( O(d_n n^2) \) algorithm to construct a corresponding tournament \( T \).

1 Introduction

Let \( a, b \) (\( b \geq a \)) and \( n \) (\( n \geq 2 \)) be nonnegative integers and let \( \mathcal{T}(a, b, n) \) be the set of such generalised tournaments, in which every pair of distinct players is connected at most with \( b \), and at least with \( a \) arcs. The elements of \( \mathcal{T}(a, b, n) \) are called \( (a, b, n) \)-tournaments. The vector \( D = (d_1, d_2, \ldots, d_n) \) of the out-degrees of \( T \in \mathcal{T}(a, b, n) \) is called the score vector of \( T \). If the elements of \( D \) are in nondecreasing order, then \( D \) is called the score sequence of \( T \).

An arbitrary vector \( D = (d_1, d_2, \ldots, d_n) \) of nonnegative integers is called graphical vector, iff there exists a loopless multigraph whose degree vector is

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D, and D is called *digraphical vector* (or *score vector*) iff there exists a loopless directed multigraph whose out-degree vector is D.

A nondecreasingly ordered graphical vector is called *graphical sequence*, and a nondecreasingly ordered digraphical vector is called *digraphical sequence* (or *score sequence*).

The number of arcs of T going from player P_i to player P_j is denoted by m_{ij} (1 \leq i, j \leq n), and the matrix M = [1..n, 1..n] is called *point matrix* or *tournament matrix* of T.

In the last sixty years many efforts were devoted to the study of both types of vectors, resp. sequences. E.g. in the papers [8, 16, 18, 19, 20, 21, 26, 30, 32, 34, 36, 45, 68, 84, 85, 88, 90, 98] the graphical sequences, while in the papers [1, 2, 3, 7, 8, 11, 17, 27, 28, 29, 31, 33, 37, 49, 48, 50, 55, 58, 57, 60, 61, 62, 64, 65, 66, 69, 78, 79, 82, 94, 86, 87, 97, 100, 101] the score sequences were discussed.

Even in the last two years many authors investigated the conditions, when D is graphical (e.g. [4, 9, 12, 13, 22, 23, 24, 25, 38, 39, 43, 47, 51, 52, 59, 75, 81, 92, 93, 95, 96, 104]) or digraphical (e.g. [5, 35, 40, 46, 54, 56, 63, 67, 70, 71, 72, 73, 74, 83, 87, 89, 102]).

In this paper we deal only with directed graphs and usually follow the terminology used by K. B. Reid [79, 80]. If in the given context a, b and n are fixed or non important, then we speak simply on *tournaments* instead of generalised or (a, b, n)-tournaments.

We consider the loopless directed multigraphs as generalised tournaments, in which the number of arcs from vertex/player P_i to vertex/player P_j is denoted by m_{ij}, where m_{ij} means the number of points won by player P_i in the match with player P_j.

The first question: how one can characterise the set of the score sequences of the (a, b, n)-tournaments. Or, with another words, for which sequences D of nonnegative integers does exist an (a, b, n)-tournament whose out-degree sequence is D. The answer is given in Section 2.

If T is an (a, b, n)-tournament with point matrix M = [1..n, 1..n], then let E(T), F(T) and G(T) be defined as follows: E(T) = \max_{1 \leq i,j \leq n} m_{ij}, F(T) = \max_{1 \leq i < j \leq n} (m_{ij} + m_{ji}), and g(T) = \min_{1 \leq i < j \leq n} (m_{ij} + m_{ji}). Let \Delta(D) denote the set of all tournaments having D as out-degree sequence, and let e(D), f(D) and g(D) be defined as follows: e(D) = \{min E(T) | T \in \Delta(D)\}, f(D) = \{min F(T) | T \in \Delta(D)\}, and g(D) = \{max G(T) | T \in \Delta(D)\}. In the sequel we use the short notations E, F, G, e, f, g, and \Delta.

Hulett et al. [39, 99], Kapoor et al. [44], and Tripathi et al. [91, 92] investigated the construction problem of a minimal size graph having a prescribed
degree set \[57, 103\]. In a similar way we follow a mini-max approach formu-
lating the following questions: given a sequence \(D\) of nonnegative integers,

- How to compute \(e\) and how to construct a tournament \(T \in \Delta\) charac-
terised by \(e\)? In Section 3 a formula to compute \(e\), and an algorithm to
construct a corresponding tournament are presented.

- How to compute \(f\) and \(g\)? In Section 4 an algorithm to compute \(f\) and
\(g\) is described.

- How to construct a tournament \(T \in \Delta\) characterised by \(f\) and \(g\)? In
Section 5 an algorithm to construct a corresponding tournament is pre-
sented and analysed.

We describe the proposed algorithms in words, by examples and by the
pseudocode used in [14].

Researchers of these problems often mention different applications, e.g. in
biology [55], chemistry Hakimi [32], and Kim et al. in networks [47].

2 Existence of a tournament with arbitrary degree
sequence

Since the numbers of points \(m_{ij}\) are not limited, it is easy to construct a
\((0, d, n)\)-tournament for any \(D\).

**Lemma 1** If \(n \geq 2\), then for any vector of nonnegative integers \(D = (d_1,
d_2, \ldots, d_n)\) there exists a loopless directed multigraph \(T\) with out-degree vector
\(D\) so, that \(E \leq d_n\).

**Proof.** Let \(m_{n1} = d_n\) and \(m_{i,i+1} = d_i\) for \(i = 1, 2, \ldots, n - 1\), and let the
remaining \(m_{ij}\) values be equal to zero. ■

Using weighted graphs it would be easy to extend the definition of the
\((a, b, n)\)-tournaments to allow arbitrary real values of \(a, b,\) and \(D\). The fol-
lowing algorithm **Naive-Construct** works without changes also for input
consisting of real numbers.

We remark that Ore in 1956 [66] gave the necessary and sufficient conditions
of the existence of a tournament with prescribed in-degree and out-degree
vectors. Further Ford and Fulkerson [17, Theorem 11.1] published in 1962
necessary and sufficient conditions of the existence of a tournament having
prescribed lower and upper bounds for the in-degree and out-degree of the
vertices. They results also can serve as basis of the existence of a tournament having arbitrary out-degree sequence.

2.1 Definition of a naive reconstructing algorithm

Sorting of the elements of $D$ is not necessary.

**Input.** $n$: the number of players ($n \geq 2$);

$D = (d_1, d_2, \ldots, d_n)$: arbitrary sequence of nonnegative integer numbers.

**Output.** $M = [1..n, 1..n]$: the point matrix of the reconstructed tournament.

**Working variables.** $i, j$: cycle variables.

**Naive-Construct**($n, D$)

01 for $i \leftarrow 1$ to $n$
02 \hspace{1em} for $j \leftarrow 1$ to $n$
03 \hspace{2em} do $m_{ij} \leftarrow 0$
04 $m_{n1} \leftarrow d_n$
05 for $i \leftarrow 1$ to $n - 1$
06 \hspace{1em} do $m_{i,i+1} \leftarrow d_i$
07 return $M$

The running time of this algorithm is $\Theta(n^2)$ in worst case (in best case too). Since the point matrix $M$ has $n^2$ elements, this algorithm is asymptotically optimal.

3 Computation of $e$

This is also an easy question. From here we suppose that $D$ is a nondecreasing sequence of nonnegative integers, that is $0 \leq d_1 \leq d_2 \leq \ldots \leq d_n$. Let $h = [d_n/(n-1)]$.

Since $\Delta(D)$ is a finite set for any finite score vector $D$, $e(D) = \min\{E(T)|T \in \Delta(D)\}$ exists.

**Lemma 2** If $n \geq 2$, then for any sequence $D = (d_1, d_2, \ldots, d_n)$ there exists a $(0,b,n)$-tournament $T$ such that

$$E \leq h \quad \text{and} \quad b \leq 2h,$$

(1)

and $h$ is the smallest upper bound for $e$, and $2h$ is the smallest possible upper bound for $b$. 
Proof. If all players gather their points in a uniform as possible manner, that is
\[
\max_{1 \leq i \leq n} m_{ij} - \min_{1 \leq j \leq n, i \neq j} m_{ij} \leq 1 \quad \text{for } i = 1, 2, \ldots, n,
\] (2)
then we get \( E \leq h \), that is the bound is valid. Since player \( P_n \) has to gather \( d_n \) points, the pigeonhole principle [6, 15, 42] implies \( E \geq h \), that is the bound is not improvable. \( E \leq h \) implies \( \max_{1 \leq i < j \leq n} m_{ij} + m_{ji} \leq 2h \). The score sequence \( D = (d_1, d_2, \ldots, d_n) = (2n(n-1), 2n(n-1), \ldots, 2n(n-1)) \) shows, that the upper bound \( b \leq 2h \) is not improvable. ■

Corollary 1 If \( n \geq 2 \), then for any sequence \( D = (d_1, d_2, \ldots, d_n) \) holds \( e(D) = \lceil d_n/(n-1) \rceil \).

Proof. According to Lemma 2 \( h = \lceil d_n/(n-1) \rceil \) is the smallest upper bound for \( e \). ■

3.1 Definition of a construction algorithm

The following algorithm constructs a \((0, 2h, n)\)-tournament \( T \) having \( E \leq h \) for any \( D \).

Input. \( n \): the number of players (\( n \geq 2 \));
\( D = (d_1, d_2, \ldots, d_n) \): arbitrary sequence of nonnegative integer numbers.

Output. \( M = [1. \ldots n, 1. \ldots n] \): the point matrix of the tournament.

Working variables. \( i, j, l, k \): cycle variables;
\( d_n \): the number of the "larger parts" in the uniform distribution of the points.

\text{Pigeonhole-Construct}(n, D)

01 \textbf{for} \ i \leftarrow 1 \ \textbf{to} \ n
02 \quad \textbf{do} \ m_{ii} \leftarrow 0
03 \quad \quad \ k \leftarrow d_i - (n-1)\lfloor d_i/(n-1) \rfloor
04 \quad \textbf{for} \ j \leftarrow 1 \ \textbf{to} \ k
05 \quad \quad \textbf{do} \ l \leftarrow i + j \text{ (mod } n)
06 \quad \quad \quad m_{il} \leftarrow \lceil d_n/(n-1) \rceil
07 \quad \textbf{for} \ j \leftarrow k+1 \ \textbf{to} \ n-1
08 \quad \quad \textbf{do} \ l \leftarrow i + j \text{ (mod } n)
09 \quad \quad \quad m_{il} \leftarrow \lfloor d_n/(n-1) \rfloor
10 \textbf{return} \ M

The running time of \text{Pigeonhole-Construct} is \( \Theta(n^2) \) in worst case (in best case too). Since the point matrix \( M \) has \( n^2 \) elements, this algorithm is asymptotically optimal.
4 Computation of $f$ and $g$

Let $S_i$ ($i = 1, 2, \ldots, n$) be the sum of the first $i$ elements of $D$, $B_i$ ($i = 1, 2, \ldots, n$) be the binomial coefficient $n(n-1)/2$. Then the players together can have $S_n$ points only if $fB_n \geq S_n$. Since the score of player $P_n$ is $d_n$, the pigeonhole principle implies $f \geq \lceil d_n/(n-1) \rceil$.

These observations result the following lower bound for $f$:

$$f \geq \max \left( \left\lfloor \frac{S_n}{B_n} \right\rfloor, \left\lfloor \frac{d_n}{n-1} \right\rfloor \right). \quad (3)$$

If every player gathers his points in a uniform as possible manner then

$$f \leq 2 \left\lfloor \frac{d_n}{n-1} \right\rfloor. \quad (4)$$

These observations imply a useful characterisation of $f$.

**Lemma 3** If $n \geq 2$, then for arbitrary sequence $D = (d_1, d_2, \ldots, d_n)$ there exists a $(g, f, n)$-tournament having $D$ as its out-degree sequence and the following bounds for $f$ and $g$:

$$\max \left( \left\lfloor \frac{S_n}{B_n} \right\rfloor, \left\lfloor \frac{d_n}{n-1} \right\rfloor \right) \leq f \leq 2 \left\lfloor \frac{d_n}{n-1} \right\rfloor, \quad (5)$$

$$0 \leq g \leq f. \quad (6)$$

**Proof.** (5) follows from (3) and (4), (6) follows from the definition of $f$. \hfill \Box

It is worth to remark, that if $d_n/(n-1)$ is integer and the scores are identical, then the lower and upper bounds in (5) coincide and so Lemma 3 gives the exact value of $F$.

In connection with this lemma we consider three examples. If $d_i = d_n = 2c(n-1)$ ($c > 0$, $i = 1, 2, \ldots, n-1$), then $d_n/(n-1) = 2c$ and $S_n/B_n = c$, that is $S_n/B_n$ is twice larger than $d_n/(n-1)$. In the other extremal case, when $d_i = 0$ ($i = 1, 2, \ldots, n-1$) and $d_n = cn(n-1) > 0$, then $d_n/(n-1) = cn$, $S_n/B_n = 2c$, so $d_n/(n-1)$ is $n/2$ times larger, than $S_n/B_n$.

If $D = (0, 0, 0, 40, 40, 40)$, then Lemma 3 gives the bounds $8 \leq f \leq 16$. Elementary calculations show that Figure 1 contains the solution with minimal $f$, where $f = 10$.

In [40] we proved the following assertion.
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| Player/Player | P₁ | P₂ | P₃ | P₄ | P₅ | P₆ | Score |
|---------------|----|----|----|----|----|----|-------|
| P₁            | —  | 0  | 0  | 0  | 0  | 0  | —     |
| P₂            | 0  | —  | 0  | 0  | 0  | 0  | —     |
| P₃            | 0  | 0  | —  | 0  | 0  | 0  | —     |
| P₄            | 10 | 10 | 10 | —  | 5  | 5  | 40    |
| P₅            | 10 | 10 | 10 | 5  | —  | 5  | 40    |
| P₆            | 10 | 10 | 10 | 5  | 5  | —  | 40    |

Figure 1: Point matrix of a $(0,10,6)$-tournament with $f = 10$ for $D = (0,0,0,40,40,40)$.

**Theorem 1** For $n \geq 2$ a nondecreasing sequence $D = (d₁,d₂,\ldots,dₙ)$ of nonnegative integers is the score sequence of some $(a,b,n)$-tournament if and only if

$$aB_k \leq \sum_{i=1}^{k} d_i \leq bB_n - L_k - (n - k)d_k \quad (1 \leq k \leq n),$$

where

$$L₀ = 0, \text{ and } L_k = \max \left( L_{k-1}, bB_k - \sum_{i=1}^{k} d_i \right) \quad (1 \leq k \leq n).$$

The theorem proved by Moon [61], and later by Kemnitz and Dolff [46] for $(a,a,n)$-tournaments is the special case $a = b$ of Theorem 1. Theorem 3.1.4 of [22] is the special case $a = b = 2$. The theorem of Landau [55] is the special case $a = b = 1$ of Theorem 1.

4.1 Definition of a testing algorithm

The following algorithm INTERVAL-Test decides whether a given $D$ is a score sequence of an $(a,b,n)$-tournament or not. This algorithm is based on Theorem 1 and returns $W = \text{TRUE}$ if $D$ is a score sequence, and returns $W = \text{FALSE}$ otherwise.

**Input.** $a$: minimal number of points divided after each match;
$b$: maximal number of points divided after each match.

**Output.** $W$: logical variable ($W = \text{TRUE}$ shows that $D$ is an $(a,b,n)$-tournament.

**Local working variables.** $i$: cycle variable;
$L = (L₀, L₁, \ldots, Lₙ)$: the sequence of the values of the loss function.
Global working variables. \( n \): the number of players \((n \geq 2)\);
\( D = (d_1, d_2, \ldots, d_n) \): a nondecreasing sequence of nonnegative integers;
\( B = (B_0, B_1, \ldots, B_n) \): the sequence of the binomial coefficients;
\( S = (S_0, S_1, \ldots, S_n) \): the sequence of the sums of the i smallest scores.

\text{Interval-Test}(a, b)

\begin{verbatim}
01 for i ← 1 to n
02 \hspace{1em} do \( L_i ← \max(L_{i-1}, bB_n - S_i - (n - i)d_i) \)
03 \hspace{2em} if \( S_i < aB_i \)
04 \hspace{3em} then \( W ← \text{FALSE} \)
05 \hspace{2em} return \( W \)
06 \hspace{2em} if \( S_i > bB_n - L_i - (n - i)d_i \)
07 \hspace{3em} then \( W ← \text{FALSE} \)
08 \hspace{2em} return \( W \)
09 return \( W \)
\end{verbatim}

In worst case \text{Interval-Test} runs in \( \Theta(n) \) time even in the general case \( 0 < a < b \) (in the best case the running time of \text{Interval-Test} is \( \Theta(n) \)). It is worth to mention, that the often referenced Havel–Hakimi algorithm \cite{32,36} even in the special case \( a = b = 1 \) decides in \( \Theta(n^2) \) time whether a sequence \( D \) is digraphical or not.

4.2 Definition of an algorithm computing \( f \) and \( g \)

The following algorithm is based on the bounds of \( f \) and \( g \) given by Lemma \cite{3} and the logarithmic search algorithm described by D. E. Knuth \cite{53} page 410.

Input. No special input (global working variables serve as input).
Output. \( b \): \( f \) (the minimal \( F \));
\( a \): \( g \) (the maximal \( G \)).

Local working variables. \( i \): cycle variable;
\( l \): lower bound of the interval of the possible values of \( F \);
\( u \): upper bound of the interval of the possible values of \( F \).

Global working variables. \( n \): the number of players \((n \geq 2)\);
\( D = (d_1, d_2, \ldots, d_n) \): a nondecreasing sequence of nonnegative integers;
\( B = (B_0, B_1, \ldots, B_n) \): the sequence of the binomial coefficients;
\( S = (S_0, S_1, \ldots, S_n) \): the sequence of the sums of the i smallest scores;
\( W \): logical variable (its value is \text{TRUE}, when the investigated \( D \) is a score sequence).

\text{MinF-MaxG}
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01 $B_0 \leftarrow S_0 \leftarrow L_0 \leftarrow 0$ \hfill \textit{Initialisation}
02 \textbf{for} $i \leftarrow 1$ \textbf{to} $n$
03 \hspace{1em} \textbf{do} $B_i \leftarrow B_{i-1} + i - 1$
04 \hspace{1em} $S_i \leftarrow S_{i-1} + d_i$
05 \hspace{1em} $l \leftarrow \max([S_n/B_n], [d_n/(n-1)])$
06 \hspace{1em} $u \leftarrow 2 \left[ d_n/(n-1) \right]$
07 $W \leftarrow \text{TRUE}$ \hfill \textit{Computation of f}
08 \textit{Interval-Test}(0, l)
09 \hspace{1em} \textbf{if} $W = \text{TRUE}$
10 \hspace{2em} $b \leftarrow l$
11 \hspace{2em} \textbf{go to} 21
12 \hspace{1em} $b \leftarrow \lceil (l + u)/2 \rceil$
13 \hspace{1em} \textit{Interval-Test}(0, f)
14 \hspace{1em} \textbf{if} $W = \text{TRUE}$
15 \hspace{2em} \textbf{then} \textbf{go to} 17
16 \hspace{1em} $l \leftarrow b$
17 \hspace{1em} \textbf{if} $u = l + 1$
18 \hspace{2em} $b \leftarrow u$
19 \hspace{2em} \textbf{go to} 21
20 \hspace{1em} \textbf{go to} 14
21 $l \leftarrow 0$ \hfill \textit{Computation of g}
22 $u \leftarrow f$
23 \textit{Interval-Test}(b, b)
24 \hspace{1em} \textbf{if} $W = \text{TRUE}$
25 \hspace{2em} $a \leftarrow f$
26 \hspace{2em} \textbf{go to} 37
27 \hspace{1em} $a \leftarrow \lceil (l + u)/2 \rceil$
28 \hspace{1em} \textit{Interval-Test}(0, a)
29 \hspace{1em} \textbf{if} $W = \text{TRUE}$
30 \hspace{2em} $l \leftarrow a$
31 \hspace{2em} \textbf{go to} 33
32 $u \leftarrow a$
33 \hspace{1em} \textbf{if} $u = l + 1$
34 \hspace{2em} $a \leftarrow l$
35 \hspace{2em} \textbf{go to} 37
36 \hspace{1em} \textbf{go to} 27
37 \textbf{return} $a, b$

\textit{MinF-MaxG} determines $f$ and $g$. 
Lemma 4 Algorithm MinG-MaxG computes the values $f$ and $g$ for arbitrary sequence $D = (d_1, d_2, \ldots, d_n)$ in $O(n \log(d_n/\ell(n)))$ time.

Proof. According to Lemma 3 $F$ is an element of the interval $[\lceil d_n/(n-1) \rceil, \lceil 2d_n/(n-1) \rceil]$ and $g$ is an element of the interval $[0, f]$. Using Theorem B of [53, page 412] we get that $O(\log(d_n/\ell(n)))$ calls of INTERVAL-TEST is sufficient, so the $O(n)$ run time of INTERVAL-TEST implies the required running time of MinF-MaxG.  

4.3 Computing of $f$ and $g$ in linear time

Analysing Theorem 1 and the work of algorithm MinF-MaxG one can observe that the maximal value of $G$ and the minimal value of $F$ can be computed independently by Linear-MinF-MaxG.

Input. No special input (global working variables serve as input).

Output. $b$: $f$ (the minimal $F$).

Global working variables. $n$: the number of players ($n \geq 2$); $D = (d_1, d_2, \ldots, d_n)$: a nondecreasing sequence of nonnegative integers; $B = (B_0, B_1, \ldots, B_n)$: the sequence of the binomial coefficients; $S = (S_0, S_1, \ldots, S_n)$: the sequence of the sums of the $i$ smallest scores.

Local working variables. $i$: cycle variable.

Linear-MinF-MaxG
01 $B_0 \leftarrow S_0 \leftarrow L_0 \leftarrow 0$ $\triangleright$ Initialisation
02 for $i \leftarrow 1$ to $n$
03 do $B_i \leftarrow B_{i-1} + i - 1$
04 $S_i \leftarrow S_{i-1} + d_i$
05 $a \leftarrow 0$
06 $b \leftarrow \min 2 \lceil d_n/(n-1) \rceil$
07 for $i \leftarrow 1$ to $n$ $\triangleright$ Computation of $g$
08 do $a_i \leftarrow \lceil 2S_i/(n^2 - n) \rceil$
09 if $a_i > a$
10 then $a \leftarrow a_i$
11 for $i \leftarrow 1$ to $n$ $\triangleright$ Computation of $f$
12 do $L_i \leftarrow \max(L_{i-1}, bB_n - S_i - (n - i)d_i)$
13 $b_i \leftarrow (S_i + (n - i)d_i + L_i)/B_i$
14 if $b_i < b$
15 then $b \leftarrow b_i$
16 return $a, b$
Lemma 5 Algorithm Linear-MinG-MaxG computes the values $f$ and $g$ for arbitrary sequence $D = (d_1, d_2, \ldots, d_n)$ in $\Theta(n)$ time.

Proof. Lines 01–03, 07, and 18 require only constant time, lines 04–06, 09–12, and 13–17 require $\Theta(n)$ time, so the total running time is $\Theta(n)$.

5 Tournament with $f$ and $g$

The following reconstruction algorithm Score-Slicing2 is based on balancing between additional points (they are similar to ,,excess", introduced by Brauer et al. [10]) and missing points introduced in [40]. The greediness of the algorithm Havel–Hakimi [32, 36] also characterises this algorithm.

This algorithm is an extended version of the algorithm Score-Slicing proposed in [40].

5.1 Definition of the minimax reconstruction algorithm

The work of the slicing program is managed by the following program Mini-Max.

Input. No special input (global working variables serve as input).

Output. $M = [1 \ldots n, 1 \ldots n]$: the point matrix of the reconstructed tournament.

Local working variables. $i, j$: cycle variables.

Global working variables. $n$: the number of players ($n \geq 2$);
$D = (d_1, d_2, \ldots, d_n)$: a nondecreasing sequence of nonnegative integers;
$p = (p_0, p_1, \ldots, p_n)$: provisional score sequence;
$P = (P_0, P_1, \ldots, P_n)$: the partial sums of the provisional scores;
$M[1 \ldots n, 1 \ldots n]$: matrix of the provisional points.

Mini-Max

\begin{verbatim}
01 MinF-MaxG  // Initialisation
02 $p_0 \leftarrow 0$
03 for $i \leftarrow 1$ to $n$
04     do for $j \leftarrow 1$ to $i - 1$
05         do $M[i, j] \leftarrow b$
06     for $j \leftarrow i$ to $n$
07         do $M[i, j] \leftarrow 0$
08     $p_i \leftarrow d_i$
09 if $n \geq 3$  // Score slicing for $n \geq 3$ players
\end{verbatim}
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10 then for k ← n downto 3
11 do Score-Slicing2(k, pk, M)
12 if n = 2
13 then m_{1,2} ← p_1
14 m_{2,1} ← p_2
15 return M

5.2 Definition of the score slicing algorithm

The key part of the reconstruction is the following algorithm Score-Slicing2 [40].

During the reconstruction process we have to take into account the following bounds:

\[ a \leq m_{i,j} + m_{j,i} \leq b \quad (1 \leq i < j \leq n); \] (9)
modified scores have to satisfy \( m_{i,j} \leq p_i \quad (1 \leq i, j \leq n, i \neq j); \) (11)
the monotonicity \( p_1 \leq p_2 \leq \ldots \leq p_k \) has to be saved \( (1 \leq k \leq n) \) (12)
\[ m_{ii} = 0 \quad (1 \leq i \leq n). \] (13)

Input. \( k \): the number of the actually investigated players \((k > 2)\);
\( p_k = (p_0, p_1, p_2, \ldots, p_k) \) \((k = 3, 4, \ldots, n)\): prefix of the provisional score sequence \( p \);
\( M[1 \ldots n, 1 \ldots n] \): matrix of provisional points.

Output. \( M[1 \ldots n, 1 \ldots n] \): matrix of provisional points;
\( p_k = (p_0, p_1, p_2, \ldots, p_k) \) \((k = 2, 3, 4, \ldots, n - 1)\): prefix of the provisional score sequence \( p \).

Local working variables. \( A = (A_1, A_2, \ldots, A_n) \): the number of the additional points;
\( M \): missing points (the difference of the number of actual points and the number of maximal possible points of \( P_k \));
\( d \): difference of the maximal decreasable score and the following largest score;
\( y \): minimal number of sliced points per player;
\( f \): frequency of the number of maximal values among the scores \( p_1, p_2, \ldots, p_{k-1} \);
\( i, j \): cycle variables;
\( m \): maximal amount of sliceable points;
\( P = (P_0, P_1, \ldots, P_n) \): the sums of the provisional scores;
\( x \): the maximal index \( i \) with \( i < k \) and \( m_{i,k} < b \).
Global working variables. $n$: the number of players ($n \geq 2$);  
$B = (B_0, B_1, B_2, \ldots, B_n)$: the sequence of the binomial coefficients;  
$a$: minimal number of points divided after each match;  
b: maximal number of points divided after each match.

SCORE-SLICING2($k, p_k, M$)

01 $P_0 \leftarrow 0$  \> Initialisation  
02 for $i \leftarrow 1$ to $k - 1$  
03 do $P_i \leftarrow P_{i-1} + p_i$  
04 $A_i \leftarrow P_i - aB_i$  
05 $M \leftarrow (k - 1)b - p_k$  
06 while $M > 0$ and $A_{k-1} > 0$  \> There are missing and additional points  
07 do $x \leftarrow k - 1$  
08 while $r_{x,k} = b$  
09 do $x \leftarrow x - 1$  
10 $f \leftarrow 1$  
11 while $p_{x-f+1} = p_{x-f}$  
12 do $f = f + 1$  
13 $d \leftarrow p_{x-f+1} - p_{x-f}$  
14 $m \leftarrow \min(b, d, \lceil A_x/f \rceil, \lfloor M/f \rfloor)$  
15 for $i \leftarrow f$ downto 1  
16 do $y \leftarrow \min(b - m_{x+1-i,k}, m, M, A_{x+1-i}, p_{x+1-i})$  
17 $m_{x+1-i,k} \leftarrow m_{x+1-i,k} + y$  
18 $p_{x+1-i} \leftarrow p_{x+1-i} - y$  
19 $m_{k,x+1-i} \leftarrow m_{k,x+1-i} - m_{x+1-i,k}$  
20 $M \leftarrow M - y$  
21 for $j \leftarrow i$ downto 1  
22 $A_{x+1-i} \leftarrow A_{x+1-i} - y$  
23 while $M > 0$ and $A_{k-1} = 0$  \> No additional points  
24 do for $i \leftarrow k - 1$ downto 1  
25 $y \min(m_{k,i}, M, m_{k,i+m_i,k-a})$  
26 $m_{ki} \leftarrow m_{k,i} - y$  
27 $M \leftarrow M - y$  
28 return $p_k, M$

Let’s consider an example. Figure 2 shows the point table of a $(2, 10, 6)$-tournament $T$.

The score sequence of $T$ is $D = (9, 9, 19, 20, 32, 34)$. In [40] the algorithm SCORE-SLICING2 resulted the point table represented in Figure 3.

The algorithm MINI-MAX starts with the computation of $f$. MINF-MAXG
called in line 01 begins with initialisation, including provisional setting of the elements of $M$ so, that $m_{ij} = b$, if $i > j$, and $m_{ij} = 0$ otherwise. Then $\text{MINF-MAXG}$ sets the lower bound $l = \max(9, 7) = 9$ of $f$ in line 05 and tests it in line 08 by $\text{INTERVAL-TEST}$. The test shows that $l = 9$ is large enough so $\text{Mini-Max}$ sets $b = 9$ in line 12 and jumps to line 21 and begins to compute $g$. $\text{Interval-Test}$ called in line 23 shows that $a = 9$ is too large, therefore $\text{MINF-MAXG}$ continues with the test of $a = 5$ in line 27. The result is positive, therefore comes the test of $a = 7$, then the test of $a = 8$. Now $u = l + 1$ in line 33, so $a = 8$ is fixed, and the control returns to line 02 of $\text{Mini-Max}$.

Lines 02–08 contain initialisation, and $\text{Mini-Max}$ begins the reconstruction of a $(8, 9, 6)$-tournament in line 9. The basic idea is that $\text{Mini-Max}$ successively determines the won and lost points of $P_6$, $P_5$, $P_4$ and $P_3$ by repeated calls of $\text{Score-Slicing2}$ in line 11, and finally it computes directly the result of the match between $P_2$ and $P_1$ in lines 12–14.

At first $\text{Mini-Max}$ computes the results of $P_6$ calling $\text{Score-Slicing2}$ with parameter $k = 6$. The number of additional points of the first five players is $A_5 = 89 - 8 \cdot 10 = 9$ according to line 04, the number of missing points of $P_6$ is
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$M = 5 \cdot 9 - 34 = 11$ according to line 05. Then SCORE-SLICING2 determines the number of maximal numbers among the provisional scores $p_1$, $p_2$, $\ldots$, $p_5$ ($f = 1$ according to lines 10–12) and computes the difference between $p_5$ and $p_4$ ($d = 12$ according to line 13). In line 14 we get, that $m = 9$ points are sliceable, and $P_5$ gets these points in the match with $P_6$ in line 17, so the number of missing points of $P_6$ decreases to $M = 11 - 9 = 2$ (line 20) and the number of additional point decreases to $A = 9 - 9 = 0$. Therefore the computation continues in lines 23–28 and $m_{64}$ and $m_{63}$ will be decreased by 1 resulting $m_{64} = 8$ and $m_{63} = 8$ as the seventh line and seventh column of Figure 4 show. The returned score sequence is $p_5 = (9, 9, 19, 20, 23)$.

| Player/Player | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ | Score |
|---------------|-------|-------|-------|-------|-------|-------|-------|
| $P_1$         | —     | 4     | 4     | 1     | 0     | 0     | 9     |
| $P_2$         | 4     | —     | 4     | 1     | 0     | 0     | 9     |
| $P_3$         | 4     | 4     | —     | 7     | 4     | 0     | 19    |
| $P_4$         | 7     | 7     | 1     | —     | 5     | 0     | 20    |
| $P_5$         | 8     | 8     | 4     | 3     | —     | 9     | 32    |
| $P_6$         | 9     | 9     | 8     | 8     | 0     | —     | 34    |

Figure 4: The point table of $T$ reconstructed by Mini-Max.

Second time Mini-Max calls SCORE-SLICING2 with parameter $k = 5$, and get $A_4 = 9$ and $M = 13$. At first $P_4$ gets 1 point, then $P_3$ and $P_4$ get both 4 points, reducing $M$ to 4 and $A_4$ to 0. The computation continues in line 23 and results the further decrease of $m_{54}$, $m_{53}$, $m_{52}$, and $m_{51}$ by 1, resulting $m_{54} = 3$, $m_{53} = 4$, $m_{52} = 8$, and $m_{51} = 8$ as the sixth row of Figure 4 shows. The returned score sequence is $p_4 = (9, 9, 15, 15)$.

Third time Mini-Max calls SCORE-SLICING2 with parameter $k = 4$, and get $A_3 = 11$ and $M = 11$. At first $P_3$ gets 6 points, then $P_3$ further 1 point, and $P_2$ and $P_1$ also both get 1 point, resulting $m_{34} = 7$, $m_{43} = 2$, $m_{42} = 8$, $m_{24} = 1$, $m_{14} = 1$ and $m_{14} = 8$, further $A_3 = 0$ and $M = 2$. The computation continues in lines 23–28 and results a decrease of $m_{43}$ by 1 point resulting $m_{43} = 1$, $m_{42} = 7$, and $m_{41} = 7$, as the fifth row and fifth column of Figure 4 show. The returned score sequence is $p_3 = (8, 8, 8)$.

Fourth time Mini-Max calls SCORE-SLICING2 with parameter $k = 3$, and gets $A_2 = 8$ and $M = 10$. At first $P_1$ and $P_2$ get 4 points, resulting $m_{13} = 4$, and $m_{23} = 4$, and $M = 2$, and $A_2 = 0$. Then Mini-Max sets in lines 23–26 $m_{31} = 4$ and $m_{32} = 4$. The returned score sequence is $p_2 = (4, 4)$. 
Finally Mini-Max sets \( m_{12} = 4 \) and \( m_{21} = 4 \) in lines 14–15 and returns the point matrix represented in Figure 4.

The comparison of Figures 3 and 4 shows a large difference between the simple reconstruction of SCORE-SLICING2 and the minimax reconstruction of Mini-Max: while in the first case the maximal value of \( m_{ij} + m_{ji} \) is 10 and the minimal value is 2, in the second case the maximum equals to 9 and the minimum equals to 8, that is the result is more balanced (the given \( D \) does not allow to build a perfectly balanced \((k,k,n)\)-tournament).

### 5.3 Analysis of the minimax reconstruction algorithm

The main result of this paper is the following assertion.

**Theorem 2** If \( n \geq 2 \) is a positive integer and \( D = (d_1, d_2, \ldots, d_n) \) is a non-decreasing sequence of nonnegative integers, then there exist positive integers \( f \) and \( g \), and a \((g,f,n)\)-tournament \( T \) with point matrix \( M \) such, that

\[
f = \min(m_{ij} + m_{ji}) \leq b,
\]

\[
g = \max m_{ij} + m_{ji} \geq a
\]

for any \((a,b,n)\)-tournament, and algorithm Linear-MinF-MaxG computes \( f \) and \( g \) in \( \Theta(n) \) time, and algorithm Mini-Max generates a suitable \( T \) in \( O(d_n n^2) \) time.

**Proof.** The correctness of the algorithms SCORE-SLICING2, MinF-MaxG implies the correctness of Mini-Max.

Lines 1–46 of Mini-Max require \( O(\log(d_n/n)) \) uses of MinG-MaxF, and one search needs \( O(n) \) steps for the testing, so the computation of \( f \) and \( g \) can be executed in \( O(n \log(d_n/n)) \) times.

The reconstruction part (lines 47–55) uses algorithm SCORE-SLICING2, which runs in \( O(bn^3) \) time. Mini-Max calls SCORE-SLICING2 \( n - 2 \) times with \( f \leq 2\lceil d_n/n \rceil \), so \( n^3 d_n/n = d_n n^2 \) finishes the proof.

The property of the tournament reconstruction problem that the extremal values of \( f \) and \( g \) can be determined independently and so there exists a tournament \( T \) having both extremal features is called linking property. This concept was introduced by Ford and Fulkerson in 1962 [17] and later extended by A. Frank in [22].
6 Summary

A nondecreasing sequence of nonnegative integers $D = (d_1, d_2, \ldots, d_n)$ is a score sequence of a $(1,1,1)$-tournament, iff the sum of the elements of $D$ equals to $B_n$ and the sum of the first $i$ ($i = 1, 2, \ldots, n - 1$) elements of $D$ is at least $B_i$ [55].

$D$ is a score sequence of a $(k,k,n)$-tournament, iff the sum of the elements of $D$ equals to $kB_n$, and the sum of the first $i$ elements of $D$ is at least $kB_i$ [46] [60].

$D$ is a score sequence of an $(a,b,n)$-tournament, iff (7) holds [40].

In all 3 cases the decision whether $D$ is digraphical requires only linear time.

In this paper the results of [40] are extended proving that for any $D$ there exists an optimal minimax realization $T$, that is a tournament having $D$ as its out-degree sequence, and maximal $G$, and minimal $F$ in the set of all realizations of $D$.

In a continuation [41] of this paper we construct balanced as possible tournaments in a similar way if not only the out-degree sequence but the in-degree sequence is also given.

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