The initial drift of a 2D droplet at zero temperature

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Abstract

We consider the 2D stochastic Ising model evolving according to the Glauber dynamics at zero temperature. We compute the initial drift for droplets which are suitable approximations of smooth domains. A specific spatial average of the derivative at time 0 of the volume variation of a droplet close to a boundary point is equal to its curvature multiplied by a direction dependent coefficient. We compute the explicit value of this coefficient.

Key words. 2D Ising model, Glauber dynamics, zero temperature, Markov process, mean curvature, velocity.

Mathematics subject Classification 2000. 60K35, 82C22

Acknowledgments. The authors wish to thank Patrice Assouad and Sophie Lemaire for useful discussions. The first version of this work dealt only with the deterministic initial condition; we are very grateful to Herbert Spohn for explaining to us the relevance of the randomness in the initial condition.
Evolution of a square droplet
Introduction

The phenomenological theory asserts that the evolution of the shape of a droplet of one phase immersed in another phase is governed by the motion by mean curvature. We are still far from being able to verify this assertion starting from a genuine microscopic dynamics. Very interesting results have been obtained in a series of works in the context of the Ising model with Kac potentials [4, 5, 6, 7]. However, motion by mean curvature is recovered in some scaling limit where the range of the interactions diverges to infinity: the model becomes somehow close to a mean-field model and the ensuing motion is isotropic. For the true Ising model with only nearest-neighbour interactions, it is expected that an interface between the minus and the plus phase evolves according to an anisotropic motion by mean curvature, that is, each point \( x \) of the interface has velocity

\[
v(x) = -c(\nu(x)) \xi \nu(x)
\]

where \( \nu(x) \) is the vector normal to the interface at \( x \), \( \xi \) is the curvature of the interface at \( x \) and \( c(\nu) \) is a coefficient depending on the direction of \( \nu \). This anisotropy stems from the anisotropy of the cubic lattice.

In this paper, we consider the zero temperature Glauber dynamics for the 2D Ising model. Although we do not succeed in deriving the full motion by mean curvature, we manage to compute the initial drift for droplets which approximate suitably smooth domains and we believe this is a crucial step. Four works are directly relevant. In [10], Spohn claims to establish rigorously the mean curvature motion in the context of the 2D Ising model at zero temperature for interfaces which can be represented as the graph of a function. Although his results do not apply directly to the case of a full droplet, he succeeds in deriving an explicit formula for the coefficient \( c(\nu) \). We recover this result here with a different approach. The computation we present here can be considered to be a refinement of the observation of [2]. Chayes, Schonmann and Swindle proved a Lifshitz law for the volume of a two-dimensional droplet at zero temperature. Instead of looking at the total volume of the droplet, we shall concentrate here on the volume variation of the droplet in a small ball attached to its boundary. In [3], by interpreting the interface as a one dimensional exclusion process, Chayes and Swindle manage to prove that, starting from a square droplet, the evolution of the shape of one corner is described in the hydrodynamical limit by an appropriate Stefan problem. Finally, Sowers develops in [9] a framework of geometric measure theory to obtain the hydrodynamical limit. His convergence theorem is conditional on the verification of several assumptions, some of them concerning the structure of the interface. It might be that these estimates are the missing pieces to complete the picture.

Let us turn now to the description of our result. We work with the stochastic Ising model evolving according to the Glauber dynamics at zero temperature. We consider the diffusive limit where space is rescaled by a factor \( N \) and time is speeded up by a factor \( N^2 \). We start with a plus droplet immersed in the minus phase, whose boundary is a \( C^1 \) simple Jordan curve \( \gamma \):
the initial configuration at step $N$ is a suitable approximation of the smooth droplet, drawn on the square lattice $\mathbb{Z}^2/N$. We consider two cases:

**Deterministic initial condition.** The approximating set at step $N$ consists of the squares of the lattice $\mathbb{Z}^2/N$ which intersect the interior of $\gamma$.

**Spohn’s initial condition.** The approximating set at step $N$ is random. Its boundary converges in probability towards $\gamma$ as $N$ goes to $\infty$ and its law $\mu_N$ is given by the invariant measure of the associated zero range process.

The droplet is immersed in the minus phase, hence all the sites of the approximating set are initially set to plus, while the other sites of the lattice are set to minus. We then look at the process $(\sigma_{N^2t}, t \geq 0)$ and we denote by $A_\sigma^N(t)$ the plus droplet at time $N^2t$ starting from $\sigma$. Let $x$ be a point of $\gamma$. We study the variation of the magnetization inside the ball $B(x, r)$ centered at $x$ with radius $r$, for $r$ small. Equivalently, we look at the volume $\text{vol}(B(x, r) \cap A_\sigma^N(t))$ of the plus droplet in this ball and we aim at computing its derivative

$$
\lim_{t \to 0} \frac{1}{t} \left( \text{vol}(B(x, r) \cap A_\sigma^N(t)) - \text{vol}(B(x, r) \cap A_\sigma^N(0)) \right).
$$

Several problems arise. Since the dynamics proceeds by jumps, we have to take the expectation to get a differentiable quantity. Next we wish to link the infinitesimal volume variation with the curvature of the droplet’s boundary at $x$. To achieve this, we need to recover the slope of the continuous curve from its approximation. We perform a spatial averaging. Letting $x_0, x_1$ be the two points of $\gamma$ which belong to the sphere $\partial B(x, r)$, we consider the domain

$$
S(x, r, \alpha_1, \alpha_2) = B(x, r) \cup B(x_0, \alpha_1) \cup B(x_1, \alpha_2),
$$

and we denote by $S_N$ its discretization at step $N$. The quantity of primary interest to link the volume variation and the curvature is

$$
A_{N\gamma}^\delta(x, r, \delta) = \frac{1}{\delta^2} \int_0^\delta \int_0^\delta \lim_{t \to 0} \frac{1}{t} E \left( \left( \text{vol}(A_\sigma^N(t) \cap S_N) - \text{vol}(A_\sigma^N(0) \cap S_N) \right) d\alpha_1 d\alpha_2. \right.
$$

Let $\theta$ be the angle of the tangent to $\gamma$ at $x$ and let $\xi_\gamma(x)$ be the curvature of $\gamma$ at $x$. Our main result states that, for the deterministic initial condition,

$$
\lim_{r \to 0} \lim_{\delta \to 0} \liminf_{N \to \infty} \frac{1}{2r} A_{N\gamma}^\delta(x, r, \delta) = \lim_{r \to 0} \limsup_{N \to \infty} \frac{1}{2r} A_{N\gamma}^\delta(x, r, \delta) = -\frac{1}{2} |\cos(2\theta)| \xi_\gamma(x)
$$

while for Spohn’s initial condition,

$$
\lim_{r \to 0} \lim_{\delta \to 0} \liminf_{N \to \infty} \frac{1}{2r} \mu_N(A_{N\gamma}^\delta(x, r, \delta)) = \lim_{r \to 0} \limsup_{N \to \infty} \frac{1}{2r} \mu_N(A_{N\gamma}^\delta(x, r, \delta)) = -\frac{\xi_\gamma(x)}{2(|\cos \theta| + |\sin \theta|)^2}.
$$
In fact, we compute the above limits for a more general class of initial conditions, which includes the two cases above. The physically relevant case should be the one studied by Spohn, it corresponds to the equilibrium state of the zero range process. This indicates that the limit $(A(t), t \geq 0)$ of any decently converging subsequence of the stochastic motion $(A^N(t), t \geq 0)$ should satisfy the equation, for any $s > 0$ and for any $x \in \partial A(s)$,

$$
\lim_{r \to 0} \lim_{\delta \to 0} \frac{1}{2r \delta^2} \int_0^\delta \int_t^s \lim_{t \to s} \frac{1}{t-s} \mathbb{E} \left( \text{vol}(A(t) \cap S) - \text{vol}(A(s) \cap S) \right) \, d\alpha_1 \, d\alpha_2
$$

$$
= - \frac{\xi_{\partial A(s)}(x)}{2(|\cos \theta| + |\sin \theta|)^2}
$$
or at least a weaker variant of it. Here $(A(t), t \geq 0)$ is a random process describing the evolution of the shape of the droplet. A standard computation shows that the deterministic motion by mean curvature satisfies this equation. However we do not know whether it is the only solution to this equation; we have not investigated the corresponding theory so far. For instance, can one get rid of the expectation? Anyway, we are still far from establishing that the hydrodynamical limit of the droplet process satisfies the above equation. An important issue is to control dynamically the proportion of the corners in a microscopic random interface when its average slope is known. This would probably require some additional probabilistic input.

1 The model

We consider a zero-temperature 2D-stochastic Ising model. More precisely it is a continuous time Markov process $(\sigma_t)_{t \geq 0}$ taking values in $\{-1, +1\}^\mathbb{Z}^2$ with generator $L$ which acts on each local function $f : \{-1, +1\}^\mathbb{Z}^2 \to \mathbb{R}$ as

$$(Lf)(\sigma) = \sum_{x \in \mathbb{Z}^2} c(x, \sigma)(f(\sigma^x) - f(\sigma)).$$

Here, for $\sigma \in \{-1, +1\}^\mathbb{Z}^2$ and $x \in \mathbb{Z}^2$, we define

$$\forall y \in \mathbb{Z}^2 \quad \sigma^x(y) = \begin{cases} 
\sigma(y) & \text{if } y \neq x, \\
-\sigma(y) & \text{if } y = x,
\end{cases}$$

and $c(x, \sigma)$ is the rate with which the spin at site $x$ flips when the configuration is $\sigma$. The rates $c(x, \sigma)$ define the dynamics. For the zero-temperature 2D–Ising model, the rates $c(x, \sigma)$ are given by

$$
c(x, \sigma) = \begin{cases} 
1 & \text{if more than 2 neighbors of } x \text{ have a spin opposite to } x, \\
\alpha & \text{if exactly 2 neighbors of } x \text{ have a spin opposite to } x, \\
0 & \text{otherwise},
\end{cases}
$$

where $0 < \alpha \leq 1$ is a fixed parameter. For technical reasons, we will take $\alpha = \frac{1}{2}$ in the sequel.
2 Notation

Let $N$ be a fixed positive integer. We denote by $\mathbb{Z}_N^2$ the grid $\mathbb{Z}_N^2$. For $x = (x_1, x_2) \in \mathbb{Z}_N^2$, $\Lambda_{x/N}$ is the box defined as

$$
\Lambda_{x/N} = \left\{ (u_1, u_2) \in \mathbb{R}^2, \quad -\frac{1}{2N} \leq u_1 - \frac{x_1}{N} < \frac{1}{2N}, \quad -\frac{1}{2N} \leq u_2 - \frac{x_2}{N} < \frac{1}{2N} \right\}. \tag{1}
$$

The family of boxes $(\Lambda_x, x \in \mathbb{Z}_N^2)$, as defined by (1), forms a partition of $\mathbb{R}^2$:

$$
\mathbb{R}^2 = \bigcup_{x \in \mathbb{Z}_N^2} \Lambda_x, \quad \forall x, y \in \mathbb{Z}_N^2 \quad x \neq y \Rightarrow \Lambda_x \cap \Lambda_y = \emptyset.
$$

Hence, for each $u = (u_1, u_2) \in \mathbb{R}^2$ there exists a unique $u_N \in \mathbb{Z}_N^2$ such that $u \in \Lambda_{u_N}$. Moreover $\|u - u_N\|_\infty \leq \frac{1}{2N}$, where $\|u\|_\infty = \max(|u_1|, |u_2|)$.

To each bounded set $S$ of $\mathbb{R}^2$, we associate the set $S_N$ defined by

$$
S_N = \bigcup_{x \in \mathbb{Z}_N^2 : \Lambda_x \cap S \neq \emptyset} \Lambda_x.
$$

The set $S$ is included in the set $S_N$ with polygonal boundary.
For $\sigma \in \{-1,+1\}^{\mathbb{Z}^2}$ and for $x \in \mathbb{Z}^2$, we denote by $s(\sigma, x)$, the number of the neighbors of $x$ having a spin opposite to $x$ in the configuration $\sigma$:

$$s(\sigma, x) = \frac{1}{2} \sum_{y \in \mathbb{Z}^2, |x-y|=1} |\sigma(x) - \sigma(y)|,$$

where $|x| = \sqrt{x_1^2 + x_2^2}$ for $x = (x_1, x_2)$.

Let $N$ be a fixed positive integer, we define the set

$$\mathcal{A}_N^\sigma = \bigcup_{x \in \mathbb{Z}^2, \sigma(x)=+1} \Lambda_{x/N}.$$ 

For $x \in \mathbb{Z}^2$, $\sigma(x) = +1$ if and only if $x \in N \mathcal{A}_N^\sigma$.

Let $\gamma$ be a curve of $\mathbb{R}^2$. We define for $s \in \gamma$ and for $r, \alpha_1, \alpha_2$ positive real numbers, the set

$$\mathcal{S}(s, r, \alpha_1, \alpha_2) = B(s, r) \cup B(x_0, \alpha_1) \cup B(x_1, \alpha_2),$$

where $B(s, r)$ is the closed ball centered at $s$ with radius $r$ chosen sufficiently small, so that $\partial B(s, r) \cap \gamma$ contains exactly 2 points $x_0$ and $x_1$. We suppose that $x_0$, $s$ and $x_1$ are arranged counterclockwise.

Let

$$L_N^{\sigma, \gamma}(s, r, \alpha_1, \alpha_2) = \lim_{t \to 0} \frac{1}{t} \left( \mathbb{E}_\sigma(\text{vol}(\mathcal{A}_N^{\sigma N^2} \cap \mathcal{S}_N)) - \text{vol}(\mathcal{A}_N^\sigma \cap \mathcal{S}_N) \right),$$

where $\mathcal{S}_N = (\mathcal{S}(s, r, \alpha_1, \alpha_2))_N = (B(s, r) \cup B(x_0, \alpha_1) \cup B(x_1, \alpha_2))_N$ and vol denotes the planar Lebesgue measure.
The set \((B(s, r) \cup B(x_0, \alpha_1) \cup B(x_1, \alpha_2))_N\).

Finally, we define the average

\[
A_N^{\sigma, \gamma}(s, r, \delta) = \frac{1}{\delta^2} \int_0^\delta \int_0^\delta L_N^{\sigma, \gamma}(s, r, \alpha_1, \alpha_2) \, d\alpha_1 \, d\alpha_2.
\]

### 3 Results

We first control the quantity \(A_N^{\sigma, \gamma}(s, r, \delta)\) for deterministic sets \(A_N^\sigma\) defined as follows.

**Deterministic initial condition.** Let \(\gamma\) be a Jordan curve of \(\mathbb{R}^2\). Suppose that \(\gamma\) encloses a connected, compact and bounded set \(\Omega\) of \(\mathbb{R}^2\), so that \(\gamma = \partial\Omega\). Let \(N\) be a fixed positive integer. We define the spin configuration \(\sigma\) at time 0 as:

\[
\forall \, x \in \mathbb{Z}^2 \quad \sigma(x) = \begin{cases} 
+1 & \text{if } \Lambda_{x/N} \cap \Omega \neq \emptyset, \\
-1 & \text{otherwise},
\end{cases}
\]

where, for \(x \in \mathbb{Z}^2\) and \(N \in \mathbb{N}^*\), \(\Lambda_{x/N}\) is the box as defined by (1). We will say that \(\sigma\) is the spin configuration associated to the curve \(\gamma\) at step \(N\).
Having both the initial condition and the generator, the Markov process \((\sigma_t)_{t \geq 0}\) at step \(N\) is well defined.

The curve \(\gamma = \partial \Omega\) and the set \(\mathcal{A}^\sigma_N\).

**Proposition 1** Let \(\gamma\) be a Jordan curve of \(\mathbb{R}^2\) of class \(C_2\). Suppose that \(\gamma\) encloses a connected, compact and bounded set \(\Omega\) of \(\mathbb{R}^2\). Let \(s\) be a point of \(\gamma\). Let \(\sigma\) be the spin configuration associated to the curve \(\gamma\) at step \(N\). Then,

\[
\lim_{r \to 0} \lim_{\delta \to 0} \liminf_{N \to +\infty} \frac{1}{2r} A^\sigma_{N,\gamma}(s, r, \delta) = \lim_{r \to 0} \lim_{\delta \to 0} \limsup_{N \to +\infty} \frac{1}{2r} A^\sigma_{N,\gamma}(s, r, \delta) = -\frac{1}{2} \cos 2\theta \| \xi_{\gamma}(s) \|
\]

where \(\xi_{\gamma}(s)\) is the curvature of \(\gamma\) at \(s\) and \(\theta\) is the angle between the horizontal axis and the tangent to the curve \(\gamma\) at \(s\).

We suppose next that the sets \(\mathcal{A}^\sigma_N\) are random and that locally the height function associated to \(\partial \mathcal{A}^\sigma_N\) obeys to Spohn’s initial condition described as follows.

**Spohn’s initial condition.** Let \(\gamma\) be a Jordan curve of \(\mathbb{R}^2\). Suppose that \(\gamma\) encloses a connected, compact and bounded set \(\Omega\) of \(\mathbb{R}^2\), so that \(\gamma = \partial \Omega\). Let \(s\) be a point of \(\gamma\). Suppose that, on a neighborhood \(V_s\) of \(s\), the contour \(\gamma\) is the graph of a monotone differentiable function \(f\) defined on a segment \([a, b]\). For each positive integer \(N\), and for each random boundary \(\partial \mathcal{A}^\sigma_N \cap V_s\), let \(\Phi_N\) be the random height function associated to \(\partial \mathcal{A}^\sigma_N \cap V_s\) above \(\frac{\pi}{N} \cap [a, b]\), defined by

\[
\forall u \in \frac{\mathbb{Z}}{N} \cap [a, b] \quad \Phi_N(u) = \sup\{v : (u, v) \in \partial \mathcal{A}^\sigma_N\}.
\]

Let \(\mu_N\) be the initial distribution of \(\Phi_N\). We suppose that, under \(\mu_N\), the increments

\[
\Phi_N\left(\frac{k + 1}{N}\right) - \Phi_N\left(\frac{k}{N}\right), \quad \frac{k}{N} \in [a, b] \cap \frac{\mathbb{Z}}{N},
\]

are independent and their laws are such that

- If \(f\) is nondecreasing, then for \(l \in \mathbb{Z}\),

\[
\mu_N\left(\Phi_N\left(\frac{k + 1}{N}\right) - \Phi_N\left(\frac{k}{N}\right) = \frac{l}{N}\right) = \begin{cases} (f'(\frac{k}{N}))^l (1 + f'(\frac{k}{N}))^{-l-1} & \text{if } l \geq 0 \\ 0 & \text{if } l < 0 \end{cases}
\]
If $f$ is nonincreasing, then for $l \in \mathbb{Z}$

$$
\mu_N \left( \Phi_N \left( \frac{k + 1}{N} \right) - \Phi_N \left( \frac{k}{N} \right) = \frac{l}{N} \right) = \begin{cases} 
(|f'| \left( \frac{k}{N} \right))^{\|l\|} (1 + |f'| \left( \frac{k}{N} \right))^{-\|l\| - 1} & \text{if } l \leq 0 \\
0 & \text{if } l > 0
\end{cases}
$$

Proposition 2 Let $\gamma$ be a Jordan curve of $\mathbb{R}^2$ of class $C_2$. Let $s$ be a point of $\gamma$. Suppose that, for any positive real numbers $r$ and $\delta$ sufficiently small, the curve $\gamma \cap S(s, r, \delta, \delta)$ is the graph of a monotone function $f$ defined on a segment $[a, b]$ of $\mathbb{R}$. Let $\mu_N$ be the measure as defined above. Suppose that,

$$
\forall \varepsilon > 0 \quad \lim_{N \to +\infty} \mu_N (|\Phi_N(a_N) - f(a)| \geq \varepsilon) = 0,
$$

where $a_N$ is a point of $[a, b] \cap \mathbb{Z}^N$ such that $|a - a_N| \leq \frac{1}{N}$. Then

$$
\lim_{r \to 0} \lim_{\delta \to 0} \liminf_{N \to +\infty} \frac{1}{2r} \mu_N(\mathcal{A}_N^{a, \gamma}(s, r, \delta)) = \lim_{r \to 0} \lim_{\delta \to 0} \limsup_{N \to +\infty} \frac{1}{2r} \mu_N(\mathcal{A}_N^{a, \gamma}(s, r, \delta))
$$

$$
= \frac{1}{2(|\cos \theta| + |\sin \theta|)^2} \xi_\gamma(s),
$$

where $\xi_\gamma(s)$ is the curvature of $\gamma$ at $s$ and $\theta$ is the angle between the horizontal axis and the tangent to the curve $\gamma$ at $s$.

The limits obtained in propositions 1 and 2 are very different because the initial conditions differ. Spohn’s velocity is recovered in proposition 2 (cf. (4.26) of Spohn (1993)). The choice of the measure $\mu_N$ is the good one, since as noticed by Spohn (1993), the height differences are governed by the zero-range process with rate function $c(n) = \mathbb{I}_{n \geq 1}$. The product measure $\mu_N$ with geometric distribution is invariant for the zero range process (cf. Andjel (1982)). Motion by mean curvature for the sets $(\mathcal{A}_N^{a, \gamma})$ corresponds then to the hydrodynamic limit for the zero range process.

Propositions 1 and 2 are consequences of the following theorem 2, which handles the initial conditions described thereafter. The distance between a point $a \in \mathbb{R}^2$ and a subset $B$ of $\mathbb{R}^2$ is $d(a, B) = \inf_{b \in B} |a - b|$; the Hausdorff distance $d_H$ between two subsets $A$ and $B$ of $\mathbb{R}^2$ is

$$
d_H(A, B) = \max \left( \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right).
$$

Initial condition. Let $\gamma$ be a Jordan curve of $\mathbb{R}^2$ of class $C_1$. Suppose that $\gamma$ encloses a connected, compact and bounded set $\Omega$ of $\mathbb{R}^2$. Let $s$ be a point of $\gamma$. Let $r$ be a positive real number sufficiently small such that $\partial B(s, r) \cap \gamma$ contains exactly two points $x_0$ and $x_1$. Suppose that $x_0$, $s$ and $x_1$ are arranged counterclockwise. Let $\theta_1 \in [0, 2\pi]$ (respectively $\theta_0 \in [0, 2\pi]$) be
the oriented angle between the half horizontal axis $[0, +\infty[$ and $T_{x_1}\gamma$ (respectively $T_{x_0}\gamma$). We suppose that there exists a neighborhood $V_s$ of $s$ and a probability measure $\nu_N$ such that

$$\forall \varepsilon > 0 \lim_{N \to +\infty} \nu_N (d_H (A_N^\sigma \cap V_s, \Omega \cap V_s) \geq \varepsilon) = 0,$$

and that, with probability one, the boundaries $\gamma$ and $\partial A_N^\sigma$ are, in $V_s$, either both non-increasing or either both non-decreasing.

The polygonal curve $\partial A_N^\sigma$ behaves in $V_s$ as $\gamma$.

Let, for $x \in \gamma \cap V_s$ and $\delta > 0$,

$$C_N(x, \delta) = \sum I_{\sigma(y) = +1, s(\sigma, y) = 2},$$

where the sum is taken over all $y \in \mathbb{Z}^2$ for which $\frac{y}{N}$ is a point of $(B(x, \delta))_N \setminus B(s, |x - s|)$. The quantity $C_N(x, \delta)$ is equal to half of the number of the corners of the polygonal line $\partial A_N^\sigma$ belonging to $(B(x, \delta))_N \setminus B(s, |x - s|)$. 
We first suppose that $\gamma$ is a polygon and that $s$ is a corner point of $\gamma$. In this case, the following theorem proves that, for $r$ and $\delta$ sufficiently small, the limit as $N$ goes to infinity of $\nu_N(A_N^{\sigma,\gamma}(s, r, \delta))$ exists under a suitable behavior of the expected proportions of corners $\frac{1}{N}\nu_N(C_N(x_k, \delta))$, for $k \in \{0, 1\}$.

**Theorem 1** Let $\gamma, s, r, \delta$ and $\nu_N$ be as described in the previous initial condition. Suppose that $\gamma$ is a polygon and that for $k = 0, 1$ and for $r, \delta$ sufficiently small, the following limit holds:

$$
\lim_{N \to \infty} \frac{1}{N} \nu_N(C_N(x_k, \delta)) = \delta C(\theta_k).
$$

(4)

Then, for $r$ and $\delta$ sufficiently small, one has

$$
\lim_{N \to \infty} \nu_N(A_N^{\sigma,\gamma}(s, r, \delta)) =
= -\frac{1}{2} \sgn(\tan \theta_0) \left( \cos^2 \theta_0 + C(\theta_0) (|\sin \theta_0| - |\cos \theta_0|) \right)
+ \frac{1}{2} \sgn(\tan \theta_1) \left( \cos^2 \theta_1 + C(\theta_1) (|\sin \theta_1| - |\cos \theta_1|) \right)
+ \mathbf{I}_{\sin \theta_0 \sin \theta_1 > 0} (\sgn(\theta_1 - \theta_0) \mathbf{I}_{\cos \theta_0 \cos \theta_1 > 0} + \sgn(\tan \theta_0) \mathbf{I}_{\cos \theta_0 \cos \theta_1 < 0}).
$$

(5)

Suppose that $C(\theta) = f(|\sin \theta|, |\cos \theta|)$, where $f$ is a positive function defined on $[0, 1] \times [0, 1]$ and that $s$ is not a corner point of the polygon $\gamma$. Then, $\nu_N(A_N^{\sigma,\gamma}(s, r, \delta))$ vanishes (since in this case $\theta_1 = \theta_0 \pm \pi$). This constatation is not surprising since the inverse of the curvature of a straight line vanishes.

The following theorem extends theorem 1 to Jordan curves.

**Theorem 2** Let $\gamma, s, r$ and $\nu_N$ be as described in the previous initial condition. Suppose that $\gamma$ is a polygon and that for $r$ sufficiently small and for $k = 0, 1$, the following limits exist:

$$
\lim_{N \to +\infty} \frac{1}{\delta N} \nu_N(C_N(x_k, \delta)) = \lim_{\delta \to 0} \frac{1}{\delta N} \nu_N(C_N(x_k, \delta)) = C(\theta_k).
$$

(6)

Then

$$
\lim_{\delta \to 0} \lim_{N \to +\infty} \nu_N(A_N^{\sigma,\gamma}(s, r, \delta)) = \lim_{N \to +\infty} \nu_N(A_N^{\sigma,\gamma}(s, r, \delta)),
$$

the common value is as in (5) with the function $C(.)$ given by (6).
4 Proofs

We first prove theorems 1 and 2. Next, we prove the two propositions. For the proof of the theorems, we need the following preliminary lemma.

**Lemma 1** Let $S$ be a compact set of $\mathbb{R}^2$. Let $\sigma \in \{-1, +1\} \mathbb{Z}^2$ be fixed. Then

$$\lim_{t \to 0} \frac{1}{t} \left( \mathbb{E}_\sigma(\text{vol}(A_N^{\sigma} \cap S_N)) - \text{vol}(A_N^\sigma \cap S_N) \right) =$$

$$\sum_{x \in \mathbb{Z}^2 : \Lambda \sigma \subset S_N} \left( \mathbf{1}_{\sigma(x)=-1, s(\sigma,x) \geq 3} - \mathbf{1}_{\sigma(x)=+1, s(\sigma,x) \geq 3} \right) + \alpha \sum_{x \in \mathbb{Z}^2 : \Lambda \sigma \subset S_N} \left( \mathbf{1}_{\sigma(x)=-1, s(\sigma,x)=2} - \mathbf{1}_{\sigma(x)=+1, s(\sigma,x)=2} \right).$$

**Proof of lemma 1.** Let $f_N(\sigma) = \text{vol}(A_N^\sigma \cap S_N)$ and $S(t)f_N(\sigma) = \mathbb{E}_\sigma(\text{vol}(A_N^{\sigma t} \cap S_N))$. We deduce from

$$\lim_{t \to 0} \frac{1}{t} \left( S(t)f_N - f_N \right) = Lf_N,$$

that

$$\lim_{t \to 0} \frac{1}{t} \left( S(t)f_N(\sigma) - f_N(\sigma) \right) = \sum_{x \in \mathbb{Z}^2} c(x, \sigma)(f_N(\sigma^x) - f_N(\sigma)). \quad (7)$$

Now,

$$f_N(\sigma^x) - f_N(\sigma) = \frac{1}{N^2} \mathbf{1}_{\Lambda \sigma \subset S_N} \left( \mathbf{1}_{\sigma(x)=-1} - \mathbf{1}_{\sigma(x)=1} \right),$$

this fact together with (7) gives

$$\lim_{t \to 0} \frac{1}{t} \left( S(tN^2)f_N(\sigma) - f_N(\sigma) \right) = \sum_{x \in \mathbb{Z}^2 : \Lambda \sigma \subset S_N} c(x, \sigma) \left( \mathbf{1}_{\sigma(x)=-1} - \mathbf{1}_{\sigma(x)=1} \right),$$

which proves lemma 1 since $c(x, \sigma) = \mathbf{1}_{s(\sigma,x) \geq 3} + \alpha \mathbf{1}_{s(\sigma,x)=2}$. \qed

4.1 Evaluation of $\nu_N\left(L_N^{\sigma,\gamma}(s, r, \alpha_1, \alpha_2)\right)$

Throughout this step, we consider the set

$$S_N = (S(s, r, \alpha_1, \alpha_2))_N = (B(s, r) \cup B(x_0, \alpha_1) \cup B(x_1, \alpha_2))_N,$$

where $\alpha_1, \alpha_2$ are positive real numbers less than $\delta$, the positive real numbers $r$ and $\delta$ are small enough so that $\partial B(s, r) \cap \gamma$ contains exactly 2 points $x_0$ and $x_1$.

The boundary of $A_N^\sigma$ which is included in $S_N$ can be described as a sequence $v_1, \ldots, v_r$ of horizontal or vertical vectors of norm $\frac{1}{N}$, enumerated counterclockwise. We denote by $e_N^1(\alpha_1)$, $e_N^2(\alpha_2)$ the two unit vectors defined by

$$e_N^1(\alpha_1) = Nv_1, \quad e_N^2(\alpha_2) = Nv_r,$$

$$e_N(\alpha_1) = Nv_1, \quad e_N(\alpha_2) = Nv_r.$$
and by $\mathcal{L}_N^\sigma$ the maximal subgraph of $\partial A_N^\sigma$ included in $\mathcal{S}_N$:

$$\mathcal{L}_N^\sigma = (v_1, \ldots, v_r). \quad (10)$$

The polygonal line $\mathcal{L}_N^\sigma = (v_1, \ldots, v_r)$.

Here $\mathcal{S}_N = (B(s, r) \cup B(x_0, \alpha_1) \cup B(x_1, \alpha_2))_N$.

We now need the following definition and notation.

**Definition 1** We say that $\mathcal{L}_N$ is a path on $\mathbb{Z}_N^2$ if $\mathcal{L}_N$ is a finite sequence of consecutive vectors $(v_i)_{1 \leq i \leq r}$ (this means that the endpoint of $v_i$ is the starting point of $v_{i+1}$ for $1 \leq i < r$) of norm $1/N$, drawn on the grid $\mathbb{Z}_N^2$, and such that the endpoints of these vectors (resp. the starting points) are distinct.
The following family of vectors \((v_1, \ldots, v_r)\) is a path on the grid \(\mathbb{Z}_N^2\).

**Notation.** Let \(\mathcal{L}_N = (v_1, v_2, \ldots, v_r)\) be a path on \(\mathbb{Z}_N^2\). We define

\[
N_+ (\mathcal{L}_N) = \text{card} \left\{ i : \hat{\omega} (v_i, v_{i+1}) = -\frac{\pi}{2} \right\}, \quad N_- (\mathcal{L}_N) = \text{card} \left\{ i : \hat{\omega} (v_i, v_{i+1}) = +\frac{\pi}{2} \right\},
\]

where \((v_i, v_{i+1})\) denotes the oriented angle between \(v_i\) and \(v_{i+1}\).

The purpose of the following proposition is to establish the relation between \(N_-(\mathcal{L}_N^\alpha) - N_+ (\mathcal{L}_N^\alpha)\) and \(L^\sigma_{N^2}(s, r, \alpha_1, \alpha_2)\), for the path \(\mathcal{L}_N^\alpha\) as defined by (10).

**Proposition 3** Let \(N\) be a fixed positive integer. Let \(\mathcal{L}_N^\alpha\) be the random path as defined by (10). Then

\[
\nu_N (L^\sigma_{N^2}(s, r, \alpha_1, \alpha_2)) = \frac{1}{2} \nu_N (N_-(\mathcal{L}_N^\alpha) - N_+ (\mathcal{L}_N^\alpha)).
\]

**Proof of proposition 3.** Let \(N \in \mathbb{N}^*\) be fixed and \(S_N = (B(s, r) \cup B(x_0, \alpha_1) \cup B(x_1, \alpha_2))_N\). Let \(f\) be the function defined from \(\{0, 1, \ldots, 4\}\) to \(\{0, 1, 2\}\) by

\[
f(s(\sigma, x)) = \begin{cases} 
1 & \text{if } s(\sigma, x) = 2 \\
2 & \text{if } s(\sigma, x) = 3 \\
0 & \text{otherwise.}
\end{cases}
\]

On the one hand, by definition of \(N_-(\mathcal{L}_N^\alpha)\) and \(N_+ (\mathcal{L}_N^\alpha)\), we have

\[
\sum_{x \in \mathbb{Z}_N^2 \cap NS_N} \sigma(x) f(s(\sigma, x)) = N_+ (\mathcal{L}_N^\alpha) - N_-(\mathcal{L}_N^\alpha),
\]

on the other hand, we deduce from the definition of the function \(f\),

\[
\sum_{x \in \mathbb{Z}_N^2 \cap NS_N} \sigma(x) f(s(\sigma, x)) = - \sum_{x \in \mathbb{Z}_N^2, \Lambda_{x/N} \subset S_N} \left( \mathbb{I}_{\sigma(x) = -1, s(\sigma, x) = 2} - \mathbb{I}_{\sigma(x) = +1, s(\sigma, x) = 2} \right) - 2 \sum_{x \in \mathbb{Z}_N^2, \Lambda_{x/N} \subset S_N} \left( \mathbb{I}_{\sigma(x) = -1, s(\sigma, x) = 3} - \mathbb{I}_{\sigma(x) = +1, s(\sigma, x) = 3} \right).
\]
We combine the last formula, lemma 1 (with $\alpha = \frac{1}{2}$) together with the fact that $I_{s(\sigma,x)=4} = 0$, and we obtain
\[
\nu (L^\sigma \gamma (s, r, \alpha_1, \alpha_2)) = -\frac{1}{2} \nu \left( \sum_{x \in \mathbb{Z}^2 \cap S_N} \sigma(x) f(s(\sigma,x)) \right).
\]
(14)

The statement of proposition 3 follows from (13) and (14) by taking the expectation with respect to $\nu_N$. 

In view of proposition 3, in order to control $\nu (L^\sigma \gamma (s, r, \alpha_1, \alpha_2))$, it remains to evaluate $\nu (N_+ (L^\sigma_N) - N_- (L^\sigma_N))$. For this, we begin by controlling the quantity $N_+ (L^\sigma_N) - N_- (L^\sigma_N)$ for monotone deterministic paths $L^\sigma_N$ defined as follows.

**Definition 2** A path on $\mathbb{Z}^2_N$ is said to be monotone if all its horizontal as well as all its vertical vectors are oriented in the same sense.

![A monotone path on the grid $\mathbb{Z}^2_N$.](image)

The following lemma evaluates $N_+ (L^\sigma_N) - N_- (L^\sigma_N)$, whenever $L^\sigma_N$ is a monotone path on $\mathbb{Z}^2_N$.

**Lemma 2** Let $(v_i)_{1 \leq i \leq r}$ be a sequence of $r$ consecutive vectors drawn on the grid $\mathbb{Z}^2_N$. These vectors are enumerated beginning from $N^{-1}u_e := v_1$ until $N^{-1}u_s := v_r$. We suppose that they form a monotone path on $\mathbb{Z}^2_N$, say $L^\sigma_N$. Let $[u_e \wedge u_s] = (u_e \cdot i)(u_s \cdot j) - (u_e \cdot j)(u_s \cdot i)$. Then

$$N_+ (L^\sigma_N) - N_- (L^\sigma_N) = [u_e \wedge u_s].$$

**Remark.** Let us note that for any path $L^\sigma_N = (v_1, \ldots, v_r)$, we have

$$\left( \overline{u_e, u_s} \right) = \frac{\pi}{2} \left( N_- (L^\sigma_N) - N_+ (L^\sigma_N) \right),$$

where $u_e = Nv_1$ and $u_s = Nv_r$.

**Proof of lemma 2.** We denote by $L^\sigma_N(r) = (v_1, \ldots, v_r)$ a monotone path on $\mathbb{Z}^2_N$. The proof of lemma 2 is done by induction on $r$.

For $r = 1$, we have $N_- (L^\sigma_N(1)) - N_+ (L^\sigma_N(1)) = 0$ which corresponds to $[u_e \wedge u_s]$, since in this case $N^{-1}u_e = N^{-1}u_s = v_1$.

We suppose now that the property is true at step $r \geq 1$ and we prove it at step $r + 1$. We consider the path $L^\sigma_N(r + 1)$. Since $L^\sigma_N(r + 1)$ is monotone, we can suppose without loss of generality that

$$\forall l \in \{1, \ldots, r + 1\}, \quad (Nv_l) \cdot i \in \{0, -1\}, \quad (Nv_l) \cdot j \in \{0, -1\}.$$
For this monotone path $\mathcal{L}_N$, we have $u_e = (-1, 0)$ and $u_s = (0, -1)$, hence $[u_e \wedge u_s] = 1$. On the other hand $N_+ (\mathcal{L}_N) - N_- (\mathcal{L}_N) = 1 - 1 + 1 - 1 + 1 - 1 + 1 = 1$.

Once the hypothesis (H) is assumed, we have only three cases to discuss on the expression of $(v_r, v_r+1)$,

- If $v_r = v_{r+1}$, then $N_+ (\mathcal{L}_N(r+1)) - N_- (\mathcal{L}_N(r+1)) = N_+ (\mathcal{L}_N(r)) - N_- (\mathcal{L}_N(r))$, and the inductive assumption gives

$$N_+ (\mathcal{L}_N(r+1)) - N_- (\mathcal{L}_N(r+1)) = [Nv_1 \wedge N v_{r+1}]$$

- If $(Nv_r) \cdot j = -1 = (Nv_{r+1}) \cdot i$, then $(v_r, v_{r+1}) = \frac{\pi}{2}$ and $N_+ (\mathcal{L}_N(r+1)) - N_- (\mathcal{L}_N(r+1)) = N_+ (\mathcal{L}_N(r)) - N_- (\mathcal{L}_N(r)) - 1$. Together with the inductive assumption, this gives

$$N_+ (\mathcal{L}_N(r+1)) - N_- (\mathcal{L}_N(r+1)) = -(Nv_1) \cdot i - 1 = (Nv_1) \cdot j = [Nv_1 \wedge N v_{r+1}]$$

- If $(Nv_r) \cdot i = -1 = (Nv_{r+1}) \cdot j$, then $(v_r, v_{r+1}) = -\frac{\pi}{2}$, $N_+ (\mathcal{L}_N(r+1)) - N_- (\mathcal{L}_N(r+1)) = N_+ (\mathcal{L}_N(r)) - N_- (\mathcal{L}_N(r)) + 1$ and

$$N_+ (\mathcal{L}_N(r+1)) - N_- (\mathcal{L}_N(r+1)) = (Nv_1) \cdot j + 1 = -(Nv_1) \cdot i = [Nv_1 \wedge N v_{r+1}]$$

The equality $N_+ (\mathcal{L}_N(r+1)) - N_- (\mathcal{L}_N(r+1)) = [Nv_1 \wedge N v_{r+1}]$ is then always valid and lemma 2 is proved. \(\square\)

The following lemma generalizes lemma 2. Its purpose is to evaluate $N_+ (\mathcal{L}_N) - N_- (\mathcal{L}_N)$ for a path $\mathcal{L}_N$ constructed by concatenating two monotone paths.

**Lemma 3** Let $\mathcal{L}_N = (v_1, \ldots, v_r, w_1, \ldots, w_s)$ be a path on $\mathbb{Z}_N^2$. Suppose that $(v_1, \ldots, v_r)$ (respectively $(w_1, \ldots, w_s)$) forms a monotone path on $\mathbb{Z}_N^2$ and that $v_r \cdot w_1 = 0$. Let $a_1, a_2, b_1, b_2 \in \{-1,+1\}$. Suppose that for each $1 \leq i \leq r$ (resp. $1 \leq j \leq s$), the vector $N v_i$ (resp. $N w_j$) is either $(a_1, 0)$ (resp. $(b_1, 0)$) or $(0, a_2)$ (resp. $(0, b_2)$). Then,

$$N_-(\mathcal{L}_N) - N_+(\mathcal{L}_N) = -a_2(Nv_1) \cdot i + b_2(Nw_s) \cdot i + f(a_1, a_2, b_1, b_2), \quad (15)$$
where $i$ is the unit vector $(1, 0)$, $\cdot$ is the usual scalar product in $\mathbb{R}^2$ and

$$f(a_1, a_2, b_1, b_2) = \begin{cases} 2a_1a_2 & \text{if } a_2b_2 = -1, \quad (Nv_r) \cdot i = a_1 \text{ or } a_1b_1 = 1 \\ 2b_1a_2 & \text{if } a_2b_2 = -1, \quad (Nv_r) \cdot i = 0 \\ 0 & \text{if } a_2b_2 = 1. \end{cases}$$

**Proof of lemma 3.** We deduce, applying lemma 2 to the monotone paths $(v_1, \ldots, v_r)$, $(w_1, \ldots, w_s)$ and $(v_r, w_1)$ that, for $L_N = (v_1, \ldots, v_r, w_1, \ldots, w_s)$,

$$N_+ (L_N) - N_- (L_N) = [Nv_1 \land Nv_r] + [Nv_r \land Nw_1] + [Nw_1 \land Nw_s]. \quad (16)$$

In the following picture, we have $Nv_1 = (1, 0)$, $Nv_r = (0, -1)$, $Nw_1 = (-1, 0)$, $Nw_s = (-1, 0)$. Hence $[Nv_1 \land Nv_r] + [Nw_1 \land Nw_s] + [Nv_r \land Nw_1] = -2$. On the other hand, we have $N_+ (L_N) - N_- (L_N) = -2$.

We deduce from

$$(Nv_l) \cdot i \in \{0, a_1\}, \quad \text{and} \quad (Nv_l) \cdot j \in \{0, a_2\},$$

for $1 \leq l \leq r$, that

$$a_1(Nv_l) \cdot i + a_2(Nv_l) \cdot j = 1.$$  

This fact gives

$$[Nv_1 \land Nv_r] = a_2(Nv_1) \cdot i - a_2(Nv_r) \cdot i. \quad (17)$$

In the same way, we deduce that for any $1 \leq l \leq s$,

$$b_1(Nw_l) \cdot i + b_2(Nw_l) \cdot j = 1, \quad [Nw_1 \land Nw_s] = b_2(Nw_1) \cdot i - b_2(Nw_s) \cdot i. \quad (18)$$

We also have, since $v_r \cdot w_1 = 0$,

$$[Nv_r \land Nw_1] = b_2(Nv_r) \cdot i - a_2(Nw_1) \cdot i. \quad (19)$$
We obtain, collecting (17), (18), (19) and (16),

\[ N_+ (\mathcal{L}_N) - N_- (\mathcal{L}_N) = a_2(Nv_1) \cdot i - b_2(Nw_s) \cdot i + (b_2 - a_2)((Nv_r) \cdot i + (Nw_1) \cdot i). \]

From the last equality we deduce the following,

- If \( a_2 = b_2 \) i.e. \( a_2b_2 = 1 \), then \( N_- (\mathcal{L}_N) - N_+ (\mathcal{L}_N) = -a_2(Nv_1) \cdot i + b_2(Nw_s) \cdot i \).
- If \( a_2b_2 = -1 \) then since \( v_r \cdot w_1 = 0 \), \( (Nv_r) \cdot i + (Nw_1) \cdot i \in \{ a_1, b_1 \} \) and

\[ N_- (\mathcal{L}_N) - N_+ (\mathcal{L}_N) + a_2(Nv_1) \cdot i - b_2(Nw_s) \cdot i \]

is either \( 2a_1a_2 \) or \( 2b_1a_2 \). \( \square \)

The following corollary evaluates \( N_- (\mathcal{L}_N) - N_+ (\mathcal{L}_N) \) for a path \( \mathcal{L}_N \) behaving like a polygonal line. It will be very useful for the control of \( L^\sigma\gamma (s, r, \alpha_1, \alpha_2) \).

**Corollary 1** Let \( s_0, s_1 \) and \( s_2 \) be three points in \( \mathbb{R}^2 \). Let \( \theta_0 \) (resp. \( \theta_1 \)) be the oriented angle between the half horizontal axis \([0, +\infty[ \) and the segment \([s_1, s_0[ \) (respectively \([s_1, s_2[ \) ). Let \( \mathcal{L}_N = (v_1, \ldots, v_r, w_1, \ldots, w_s) \) be a path on \( \mathbb{Z}_N^2 \). Suppose that the family \( (v_1, \ldots, v_r) \) (respectively \( (w_1, \ldots, w_s) \)) forms a monotone path on \( \mathbb{Z}_N^2 \) and that \( v_r \cdot w_1 = 0 \). Suppose moreover that \( (v_1, \ldots, v_r) \) and \([s_0, s_1[ \) (respectively \( (w_1, \ldots, w_s) \) and \([s_1, s_2[ \) ) are either both non-increasing or either both non-decreasing. Then

\[ N_- (\mathcal{L}_N) - N_+ (\mathcal{L}_N) = \text{sgn}(\sin \theta_0)(Nv_1) \cdot i + \text{sgn}(\sin \theta_1)(Nw_s) \cdot i + f(\theta_1, \theta_0), \quad (20) \]

where

\[ f(\theta_1, \theta_0) = \begin{cases} 2\text{sgn}(\theta_1 - \theta_0) & \text{if } \sin \theta_0 \sin \theta_1 > 0, \cos \theta_0 \cos \theta_1 > 0, \\ 2\text{sgn}(\tan \theta_0) & \text{if } \sin \theta_0 \sin \theta_1 > 0, \cos \theta_0 \cos \theta_1 < 0, \\ 0 & \text{otherwise}. \end{cases} \]

We illustrate the conclusion of the previous corollary with the help of the following pictures.

\[ \mathcal{L}_N \] is the circuit \( (v_1, \ldots, v_r, w_1, \ldots, w_s) \).

Here, \( f(\theta_1, \theta_0) = 2\text{sgn}(\theta_1 - \theta_0) = 2 \).
\[ \mathcal{L}_N \text{ is the circuit } (v_1, \ldots, v_r, w_1, \ldots, w_s). \text{ Here } \]
\[ f(\theta_1, \theta_0) = 2\text{sgn}(\tan \theta_0) = -2. \]

In this picture, \( f(\theta_1, \theta_0) = 0. \)

**Proof of corollary 1.** We first check that for any \( 1 \leq l \leq r, \) \((Nv_l) \cdot i \in \{0, -\text{sgn}(\cos \theta_0)\}, \) and \((Nv_l) \cdot j \in \{0, -\text{sgn}(\sin \theta_0)\}. \) In the same way, we have \( 1 \leq l \leq s, \)

\((Nw_l) \cdot i \in \{0, \text{sgn}(\cos \theta_1)\}, \) and \((Nw_l) \cdot j \in \{0, \text{sgn}(\sin \theta_1)\}. \)

Lemma 3 gives then

\[ N_-(\mathcal{L}_N) - N_+(\mathcal{L}_N) = \text{sgn}(\sin \theta_0)(Nv_1) \cdot i + \text{sgn}(\sin \theta_1)(Nw_s) \cdot i + f(\theta_1, \theta_0), \]
where

- If \(a_2b_2 = -\text{sgn}(\sin \theta_0)\text{sgn}(\sin \theta_1) > 0\), then \(f(\theta_1, \theta_0) = 0\).
- If \(\text{sgn}(\sin \theta_0)\text{sgn}(\sin \theta_1) > 0\) and \(\text{sgn}(\cos \theta_0)\text{sgn}(\cos \theta_1) < 0\), then \(f(\theta_1, \theta_0) = 2\text{sgn}(\tan \theta_0)\).

In fact this case corresponds to \(a_2b_2 = -1\), \(a_1 = -\text{sgn}(\cos \theta_0) = \text{sgn}(\cos \theta_1) = b_1\).

Now we have to discuss the case \(\text{sgn}(\sin \theta_0)\text{sgn}(\sin \theta_1) > 0\) and \(\text{sgn}(\cos \theta_0)\text{sgn}(\cos \theta_1) > 0\)

i.e. when \(a_2b_2 = -1\) and \(a_1b_1 = -1\). We distinguish all the cases on the possible values of \((a_1, a_2)\) and we deduce the following: \((Nv_r) \cdot i = 0\) if and only if \((a_1a_2 < 0\) and \(\theta_0 < \theta_1)\) or \((a_1a_2 > 0\) and \(\theta_0 > \theta_1)\). So \((Nv_r) \cdot i = 0\) if and only if \(a_1a_2\text{sgn}(\theta_0 - \theta_1) > 0\). We apply again lemma 3 and we deduce that in this last case \(f(\theta_1, \theta_0) = 2\text{sgn}(\theta_1 - \theta_0)\).

We have now all the ingredients in order to evaluate \(\nu_N(N_-(L^\gamma_N) - N_+(L^\gamma_N))\) for the random path \(L^\gamma_N\) as defined by (10). The curve \(\gamma\) is of class \(C_1\), hence for \(r\) small enough, the part of \(\gamma\) situated between \(x_0\) and \(s\) (resp. between \(s\) and \(x_1\)) is either nondecreasing or nonincreasing.

We conclude from the assumptions of theorem 1 that, for \(N\) large enough and with probability one, the random path \(L^\gamma_N\) respects the behavior of the curve \(\gamma\), thus \(L^\gamma_N\) is either monotone or it is constructed by concatenating two monotone paths, say \(L^\gamma_N = (L^\gamma_{1,N}, L^\gamma_{2,N})\). These monotone paths are such that, noting by \(s'\) the point of \(T_{x_0}\gamma \cap T_{x_1}\gamma\), \(L^\gamma_{1,N}\) and \([x_0, s']\) (resp. \(L^\gamma_{2,N}\) and \([s', x_1]\)) are either both nondecreasing or both nonincreasing. Corollary 1 applies and gives, for \(N\) large enough,

\[
\nu_N(N_-(L^\gamma_N) - N_+(L^\gamma_N)) = \text{sgn}(\sin \theta_0)\nu_N(e^1_N(\alpha_1) \cdot i) + \text{sgn}(\sin \theta_1)\nu_N(e^2_N(\alpha_2) \cdot i) + f(\theta_1, \theta_0),
\]

the function \(f(\theta_1, \theta_0)\) is defined in corollary 1, the angles \(\theta_1, \theta_0\) are those defined by theorem 1, the random vectors \(e^1_N(\alpha_1), e^2_N(\alpha_2)\) are the two unit vectors as defined by (9). We then deduce from proposition 3 that there exists \(N_0\) depending only on \(\gamma\) such that, for any \(N \geq N_0\), we have,

\[
\nu_N(L^\gamma_{1,N}(s, r, \alpha_1, \alpha_2)) = \frac{1}{2}\text{sgn}(\sin \theta_0)\nu_N(e^1_N(\alpha_1) \cdot i) + \frac{1}{2}\text{sgn}(\sin \theta_1)\nu_N(e^2_N(\alpha_2) \cdot i) + \frac{1}{2}f(\theta_1, \theta_0).
\]

### 4.2 Evaluation of \(\int_0^\delta \int_0^\delta \nu_N(L^\gamma_N(s, r, \alpha_1, \alpha_2))\ d\alpha_1 d\alpha_2\)

By the previous formula, in order to evaluate the quantity

\[
\int_0^\delta \int_0^\delta \nu_N(L^\gamma_N(s, r, \alpha_1, \alpha_2))\ d\alpha_1 d\alpha_2,
\]

for \(\delta\) and \(r\) small enough, it suffices to evaluate the terms

\[
\nu_N\left(\int_0^\delta e^1_N(\alpha) \cdot i\ d\alpha\right), \quad \nu_N\left(\int_0^\delta e^2_N(\alpha) \cdot i\ d\alpha\right).
\]

We begin by the first quantity, for this we need some further notations.
**Notation.** For a vector $v$ drawn on the grid $\mathbb{Z}^2_N$, we denote by $R(v)$ the union of the two boxes of the family $(\Lambda_{x/N})_{x \in \mathbb{Z}^2}$ having $v$ as an edge vector.

The two blocks $R(v)$ and $R(w)$.

Let $\mathcal{L}^\sigma_N = (v_1, \ldots, v_r)$ be the oriented path as defined by (10). Let $\mathcal{L}^\sigma_{1,N} = (v_1, \ldots, v_s)$ be the subgraph of $\mathcal{L}^\sigma_N$ included in $\partial A^N_{\sigma} \cap (B(x_0, \delta))_N$ such that the vector $v_s$ is the entering vector in $(B(s, r))_N$.

To each vector $v_l$ ($1 \leq l \leq s$), we associate the block $R_{s-l+1} := R(v_l)$. These blocks $(R_l)_{1 \leq l \leq s}$ are enumerated according to their distances to $x_0$, $R_1$ being the block containing $v_s$. Let $(a_l)_{1 \leq l \leq s}$ be the sequence of vertices such that

$$d_l := d(x_0, R_l) = |a_l - x_0|,$$

then this sequence of vertices $(a_l)_{1 \leq l \leq s}$ is $L^1$ connected and the vector $a_l a_{l+1}$ is either vertical or horizontal. Finally, let $\mathcal{H}_N$ be the set of indices $l \in \{1, \ldots, s\}$ for which $v_l$ is horizontal.

For $N$ large enough, the vector $a_l a_{l+1}$ is either horizontal or vertical, and $|a_l - a_{l+1}| = \frac{1}{N}$.

With probability one, the path $\mathcal{L}^\sigma_{1,N}$ is monotone and behaves, on a neighborhood of $x_0$, as $T_{x_0 \gamma}$. This fact ensures that, with probability one, $e_N^\alpha \cdot i \in \{0, -\text{sgn}(\cos \theta_0)\}$. Now, by construction
$e_1^N(\alpha) \cdot i = -\text{sgn}(\cos \theta_0)$ if and only if there exists $l \in \mathcal{H}_N$ such that $\alpha \in [d_l, d_{l+1}]$ (such an index is necessarily unique). With probability one,

$$\left| \int_0^\delta e_1^N(\alpha) \cdot i \, d\alpha + \text{sgn}(\cos \theta_0) \sum_{l \in \mathcal{H}_N(x_0, \delta)} (d_{l+1} - d_l) \right| \leq \frac{2}{N}, \quad (21)$$

where $\mathcal{H}_N(x_0, \delta)$ is the set of all the horizontal edges of $\partial \mathcal{A}_N^\sigma$ included in $(B(x_0, \delta))_N \setminus B(s, r)$.

In order to evaluate $d_{l+1} - d_l$, we need the following lemma.

**Lemma 4** Let $u$ and $v$ be two vectors such that $\|u\| \leq \|v\|$. Then

$$\|u + v\| - \|v\| = \frac{(u + v) \cdot u}{\|u + v\|} - \frac{\|u\|^2}{\|v\|} \frac{\sin^2 \theta}{1 + \sqrt{1 - \frac{\|u\|^2}{\|v\|^2} \sin^2 \theta}},$$

where $\theta$ is the angle between $u$ and $u + v$.

**Proof of lemma 4.** Let $u$, $v$ and $\theta$ be as defined in lemma 4.

We have

$$\|u + v\|^2 = L^2 + H^2$$

$$= \cos^2 \theta \|u + v\|^2 + \|v\|^2 \left( (\cos \theta \|u + v\| - \|u\|)^2 \right)$$

$$= \|v\|^2 + 2 \cos \theta \|u\| \times \|u + v\| - \|u\|^2.$$ 

The quantity $\|u + v\|$ is then a positive solution of an algebraic equation of degree two. We deduce from $\|u\| \leq \|v\|$, that

$$\|u + v\| = \|u\| \cos \theta + \sqrt{\|v\|^2 - \sin^2 \theta \|u\|^2}.$$ 

Hence

$$\|u + v\| - \|v\| = \|u\| \cos \theta + \|v\| \left( \sqrt{1 - \sin^2 \theta \frac{\|u\|^2}{\|v\|^2}} - 1 \right)$$

$$= \|u\| \cos \theta - \frac{\|u\|^2}{\|v\|} \frac{\sin^2 \theta}{1 + \sqrt{1 - \frac{\|u\|^2}{\|v\|^2} \sin^2 \theta}}.$$ 

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The last equality together with the fact that \( \|u\| \cos \theta = \frac{(u + v) \cdot u}{\|u + v\|} \) proves lemma 4. □

We continue the proofs of theorems 1 and 2. We apply lemma 4 with \( u = a_i a_{i+1}, v = x_0 a_t \) and we get

\[
\frac{(u + v) \cdot u}{\|u + v\|} = \frac{(x_0 a_{i+1}) \cdot (a_i a_{i+1})}{|x_0 - a_{i+1}|}.
\]

Moreover, we deduce from lemma 4,

\[
\left| d_{i+1} - d_i - \frac{(x_0 a_{i+1}) \cdot (a_i a_{i+1})}{|x_0 - a_{i+1}|} \right| \leq \min \left( \frac{1}{N^2|x_0 - a_i|}, \frac{2}{N} \right). \tag{22}
\]

We first evaluate the sum over \( l \in \mathcal{H}_N(x_0, \delta) \) of the right hand side of the last inequality. Let \( \phi(l) \) be the cardinality of the set \( \mathcal{H}_N(x_0, \delta) \cap \{1, \ldots, l\} \). For \( l \in \mathcal{H}_N(x_0, \delta) \), we have \(|(x_0 a_i) \cdot i| \geq N^{-1}(\phi(l) - 1)\), whence

\[
\sum_{l \in \mathcal{H}_N(x_0, \delta)} \min \left( \frac{1}{N^2|x_0 - a_i|}, \frac{2}{N} \right) \leq \frac{1}{N} \sum_{l \in \mathcal{H}_N} \min \left( \frac{1}{\phi(l) - 1}, 2 \right) \leq \frac{2 + \ln |\mathcal{H}_N|}{N}. \tag{23}
\]

With probability one, we have

\[
\text{sgn}(\cos \theta_0)(a_i a_{i+1}) \cdot i + \text{sgn}(\sin \theta_0)(a_i a_{i+1}) \cdot j = \frac{1}{N},
\]

whence

\[
\sum_{l \in \mathcal{H}_N(x_0, \delta)} \frac{(x_0 a_{i+1}) \cdot (a_i a_{i+1})}{|x_0 - a_{i+1}|} = \frac{\text{sgn}(\cos \theta_0)}{N} \sum_{l \in \mathcal{H}_1, N(x_0, \delta)} \frac{(x_0 a_{i+1}) \cdot i}{|x_0 - a_{i+1}|} + \frac{\text{sgn}(\sin \theta_0)}{N} \sum_{l \in \mathcal{H}_2, N(x_0, \delta)} \frac{(x_0 a_{i+1}) \cdot j}{|x_0 - a_{i+1}|}, \tag{24}
\]

where

\[
\mathcal{H}_1, N(x_0, \delta) = \{ l \in \mathcal{H}_N(x_0, \delta) : (a_i a_{i+1}) \cdot i = 0 \},
\]

\[
\mathcal{H}_2, N(x_0, \delta) = \{ l \in \mathcal{H}_N(x_0, \delta) : (a_i a_{i+1}) \cdot i = 0 \}. \tag{25}
\]

We now distinguish the case of the polygons and the case of the Jordan curves.

### 4.3 End of the proof for polygons (theorem 1).

**Lemma 5** For \( \delta \) small enough, we have

\[
\lim_{N \to \infty} \nu_N \left( \frac{1}{N} \sum_{l \in \mathcal{H}_1, N(x_0, \delta)} \frac{(x_0 a_{i+1}) \cdot i}{|x_0 - a_{i+1}|} \right) = \nu_N \left( \frac{\mathcal{H}_1, N(x_0, \delta)}{N} \right) \cos \theta_0 = 0,
\]

\[
\lim_{N \to \infty} \nu_N \left( \frac{1}{N} \sum_{l \in \mathcal{H}_2, N(x_0, \delta)} \frac{(x_0 a_{i+1}) \cdot j}{|x_0 - a_{i+1}|} \right) = \nu_N \left( \frac{\mathcal{H}_2, N(x_0, \delta)}{N} \right) \sin \theta_0 = 0.
\]

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Proof of lemma 5. We only prove the first limit since the argument for the second limit is similar. Let $u$ be a unit vector tangent to $\gamma$ at $x_0$ and let $v$ be such that $(u, v)$ is a direct basis. For $\varepsilon > 0$, let $\mathcal{R}(\varepsilon)$ be the strip of width $2\varepsilon$ centered on the tangent line $T_{x_0} \gamma$, i.e.,

$$\mathcal{R}(\varepsilon) = \{ x \in \mathbb{R}^2 : |x_0 x \cdot v| \leq \varepsilon \}.$$

The condition (3) implies that for $\delta$ small enough,

$$\forall \varepsilon > 0 \quad \lim_{N \to +\infty} \nu_N (\partial A_N^x \cap B(x_0, \delta) \subset \mathcal{R}(\varepsilon)) = 1.$$

For $\delta$ small enough, $\lim_{N \to +\infty} \nu_N (\partial A_N^x \cap B(x_0, \delta) \subset \mathcal{R}(\varepsilon)) = 1$.

Let $\delta > 0$ be small enough so that the above limit holds. Let $\delta_0, \varepsilon$ such that $0 < \varepsilon < \delta_0 < \delta$ and let $x \in \mathcal{R}(\varepsilon) \setminus B(x_0, \delta_0)$. We have

$$x_0 x \cdot i = (x_0 x \cdot u) \cos \theta_0 - (x_0 x \cdot v) \sin \theta_0,$$

$$|x_0 - x|^2 = (x_0 x \cdot u)^2 + (x_0 x \cdot v)^2,$$

whence

$$\frac{x_0 x \cdot i}{|x_0 - x|} = \left(1 - \frac{(x_0 x \cdot v)^2}{|x_0 - x|^2}\right)^{1/2} \cos \theta_0 - \frac{(x_0 x \cdot v)}{|x_0 - x|} \sin \theta_0$$

and

$$\left| \frac{x_0 x \cdot i}{|x_0 - x|} - \cos \theta_0 \right| \leq 1 - \sqrt{1 - \frac{\varepsilon^2}{\delta_0^2}} + \frac{\varepsilon}{\delta_0} \leq \frac{2 \varepsilon}{\delta_0}.$$

If the event $\{ \partial A_N^x \cap B(x_0, \delta) \subset \mathcal{R}(\varepsilon) \}$ occurs, then for $l \in \mathcal{H}_N(x_0, \delta) \setminus \mathcal{H}_N(x_0, \delta_0)$, we have $a_{l+1} \in \mathcal{R}(\varepsilon) \setminus B(x_0, \delta_0)$, and thus

$$\limsup_{N \to \infty} \nu_N \left( \sup_{l \in \mathcal{H}_N(x_0, \delta) \setminus \mathcal{H}_N(x_0, \delta_0)} \left| \frac{(x_0 a_{l+1} \cdot i)}{|x_0 - a_{l+1}|} - \cos \theta_0 \right| \right) \leq \frac{2 \varepsilon}{\delta_0}.$$

Moreover, we have $|\mathcal{H}_N(x_0, \delta_0)| \leq 2N\delta_0$, whence, by splitting the sum over $\mathcal{H}_N(x_0, \delta_0)$ and $\mathcal{H}_N(x_0, \delta) \setminus \mathcal{H}_N(x_0, \delta_0)$, we obtain

$$\limsup_{N \to \infty} \nu_N \left( \frac{1}{N} \sum_{l \in \mathcal{H}_N(x_0, \delta)} \frac{(x_0 a_{l+1} \cdot i)}{|x_0 - a_{l+1}|} \right) - \nu_N \left( \frac{|\mathcal{H}_N(x_0, \delta)|}{N} \cos \theta_0 \right) \leq \frac{4 \delta \varepsilon}{\delta_0} + 4 \delta_0.$$

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We conclude by sending successively $\varepsilon$ to 0 and $\delta_0$ to 0. $\square$

We obtain, combining (24) and lemma 5, that for $\delta$ small enough,

$$
\lim_{N \to \infty} \nu_N \left( \sum_{l \in \mathcal{H}_N(x_0, \delta)} \frac{(x_0a_{l+1}) \cdot (a_la_{l+1})}{|x_0 - a_{l+1}|} \right)
- \left( \nu_N \left( \frac{\mathcal{H}_1,N(x_0, \delta)}{N} \right) |\cos \theta_0| + \nu_N \left( \frac{\mathcal{H}_2,N(x_0, \delta)}{N} \right) |\sin \theta_0| \right) = 0 .
$$

(26)

Our purpose now is to evaluate, for $N$ large enough, the expectations over $\nu_N$ of $\frac{\mathcal{H}_N(x_0, \delta)}{N}$, $\frac{\mathcal{H}_1,N(x_0, \delta)}{N}$ and $\frac{\mathcal{H}_2,N(x_0, \delta)}{N}$. For this, we prove the following lemma.

**Lemma 6** For $\delta$ small enough, one has

$$
\lim_{N \to +\infty} \nu_N \left( \frac{\mathcal{H}_N(x_0, \delta)}{N} \right) = \delta |\cos \theta_0|.
$$

(27)

**Proof of lemma 6.** We denote by $a$ and $x'$ the points of $\partial B(x_0, \delta) \setminus B(s, r)$ belonging respectively to $\partial \mathcal{A}_N^\sigma$ and to $T_{x_0, \gamma}$. Let $b$ be the point of $\partial \mathcal{A}_N^\sigma \cap \partial B(s, r) \setminus B(x_1, \delta)$. We suppose without loss of generality that $(ba) \cdot i \geq 0$.

In this case, $\gamma$ is a polygon. The proportion of the horizontal edges of $\partial \mathcal{A}_N^\sigma$, which are in $B(x_0, \delta) \setminus B(s, r)$ is controlled by $\delta |\cos \theta_0|$. 

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We have, by definition of $\mathcal{H}_N(x_0, \delta)$,
\[
\left| \frac{\mathcal{H}_N(x_0, \delta)}{N} - \frac{ba \cdot i}{N} \right| \leq \frac{2}{N}.
\]

We use the same notation as in the proof of lemma 5. We have
\[
\bigcap_{\varepsilon > 0} \mathcal{R}(\varepsilon) \cap \partial B(x_0, \delta) \setminus B(s, r) = T_{x_0 \gamma} \cap \partial B(x_0, \delta) \setminus B(s, r) = \{x'\},
\]
\[
\bigcap_{\varepsilon > 0} \mathcal{R}(\varepsilon) \cap \partial B(s, r) \setminus B(x_1, \delta) = T_{x_0 \gamma} \cap \partial B(s, r) \setminus B(x_1, \delta) = \{x_0\}.
\]

Let $\alpha > 0$. By the above identities, there exists $\varepsilon > 0$ such that, if $\{ \partial \mathcal{A}_N \cap B(x_0, \delta) \subset \mathcal{R}(\varepsilon) \}$, then $|b - x_0| < \alpha$ and $|a - x'| < \alpha$. Now, the condition (3) implies that for $\delta$ small enough,
\[
\lim_{N \to +\infty} \nu_N(\partial \mathcal{A}_N \cap B(x_0, \delta) \subset \mathcal{R}(\varepsilon)) = 1.
\]

Putting together the previous facts, we obtain that
\[
\limsup_{N \to +\infty} \nu_N\left( \frac{|\mathcal{H}_N(x_0, \delta)|}{N} - |x_0 x' \cdot i| \right) \leq 2\alpha.
\]

Remarking that $|x_0 x' \cdot i| = \delta |\cos \theta_0|$, we conclude the proof by sending $\alpha$ to 0. \(\Box\)

Now, let $\mathcal{L}_{1,N}^{\nu}$ be the monotone path $(v_1, \ldots, v_n)$ as defined in the subsection 4.2. We obtain using the definition of $\mathcal{H}_{2,N}(x_0, \delta)$, that $|\mathcal{H}_{2,N}(x_0, \delta)|$ is either $N_{\nu}(\mathcal{L}_{1,N}^{\nu})$ or $N_{\nu}(\mathcal{L}_{1,N}^{\nu})$. This fact together with the constatation that $|N_{\nu}(\mathcal{L}_{1,N}) - N_{\nu}(\mathcal{L}_{1,N})| \leq 1$, gives
\[
\left| \frac{|\mathcal{H}_{2,N}(x_0, \delta)|}{N} - \frac{N_{\nu}(\mathcal{L}_{1,N}^{\nu})}{N} \right| \leq \frac{1}{N}.
\]

Condition (4) together with the last inequality ensures, since $||\mathcal{H}_{2,N}(x_0, \delta)| - C_N(x_0, \delta)| \leq 1$,
\[
\lim_{N \to +\infty} \nu_N\left( \frac{|\mathcal{H}_{2,N}(x_0, \delta)|}{N} \right) = \delta C(\theta_0).
\] (28)

The two sets of indices $\mathcal{H}_{1,N}(x_0, \delta)$ and $\mathcal{H}_{2,N}(x_0, \delta)$ form a partition of $\mathcal{H}_N(x_0, \delta)$, hence
\[
\lim_{N \to +\infty} \nu_N\left( \frac{|\mathcal{H}_{1,N}(x_0, \delta)|}{N \delta} \right) = (|\cos \theta_0| - C(\theta_0)) = 0.
\] (29)

We obtain, collecting (21), (22), (23), (26), (28), (29) that
\[
\lim_{N \to \infty} \frac{1}{N} \nu_N\left( \int_{0}^{\delta} e_1 N(\alpha) \cdot i \, d\alpha \right) + \text{sgn}(\cos \theta_0) \left( \cos^2 \theta_0 + C(\theta_0) \left( |\sin \theta_0| - |\cos \theta_0| \right) \right) = 0.
\]

Using the same method, we prove that
\[
\lim_{N \to \infty} \frac{1}{N} \nu_N\left( \int_{0}^{\delta} e_2 N(\alpha) \cdot i \, d\alpha \right) - \text{sgn}(\cos \theta_1) \left( \cos^2 \theta_1 + C(\theta_1) \left( |\sin \theta_1| - |\cos \theta_1| \right) \right) = 0.
\]

We finish the proof of theorem 1 by combining proposition 3 together with the two last limits. \(\Box\)
4.4 End of the proof for Jordan curves (Theorem 2).

To extend the proofs to Jordan curves, we have to generalize lemmas 5 and 6 as follows.

Lemma 7 We have

\[ \lim_{\delta \to 0} \lim_{N \to \infty} \nu_N \left( \frac{1}{\delta N} \sum_{l \in H_1(N(x_0, \delta))} \frac{(x_0a_{l+1}) \cdot i}{|x_0 - a_{l+1}|} \right) - \nu_N \left( \frac{|H_1(N(x_0, \delta))|}{N} \cos \theta_0 \right) = 0, \]

\[ \lim_{\delta \to 0} \lim_{N \to \infty} \nu_N \left( \frac{1}{\delta N} \sum_{l \in H_2(N(x_0, \delta))} \frac{(x_0a_{l+1}) \cdot j}{|x_0 - a_{l+1}|} \right) - \nu_N \left( \frac{|H_2(N(x_0, \delta))|}{N} \sin \theta_0 \right) = 0. \]

Proof of lemma 7. We only prove the first limit since the argument for the second limit is similar. Let \( u \) be a unit vector tangent to \( \gamma \) at \( x_0 \) and let \( v \) be such that \( (u, v) \) is a direct basis. For \( \varepsilon > 0 \), let \( \mathcal{R}(\varepsilon) \) be the strip of width \( 2\varepsilon \) centered on the tangent line \( T_{x_0}\gamma \), i.e.,

\[ \mathcal{R}(\varepsilon) = \{ x \in \mathbb{R}^2 : |x_0 x \cdot v| \leq \varepsilon \}. \]

Since \( T_{x_0}\gamma \) is the tangent to \( \gamma \) at \( x_0 \), we have

\[ \lim_{\delta \to 0} \frac{1}{\delta} d_H \left( \gamma \cap B(x_0, \delta), T_{x_0}\gamma \cap B(x_0, \delta) \right) = 0. \]

(30)

Let \( 0 < \varepsilon < 1 \), there exists \( \delta_0 > 0 \) such that, for \( \delta < \delta_0 \),

\[ d_H \left( \gamma \cap B(x_0, \delta), T_{x_0}\gamma \cap B(x_0, \delta) \right) \leq \varepsilon \delta / 4. \]

This fact together with condition (3) implies that there exists \( \delta_1 > 0 \) such that

\[ \forall \delta < \delta_1 \lim_{N \to \infty} \nu_N (\partial A_N^\varepsilon \cap B(x_0, \delta) \subset \mathcal{R}(\varepsilon \delta / 2)) = 1. \]

Let \( \delta > 0 \) be such that \( \delta < \min(\delta_0, \delta_1) \). Let \( x \in \mathcal{R}(\varepsilon \delta) \setminus B(x_0, \sqrt{\varepsilon} \delta) \). We have

\[ x_0 \cdot i = (x_0 \cdot u) \cos \theta_0 - (x_0 \cdot v) \sin \theta_0, \]

\[ |x_0 - x|^2 = (x_0 \cdot u)^2 + (x_0 \cdot v)^2, \]

whence

\[ \frac{x_0 \cdot i}{|x_0 - x|} = \left( 1 - \frac{(x_0 \cdot v)^2}{|x_0 - x|^2} \right)^{1/2} \cos \theta_0 - \frac{(x_0 \cdot v)}{|x_0 - x|} \sin \theta_0 \]

and

\[ \frac{x_0 \cdot i}{|x_0 - x|} - \cos \theta_0 \leq 1 - \sqrt{1 - \varepsilon} + \sqrt{\varepsilon} \leq 2\sqrt{\varepsilon}. \]

If the event \( \{ \partial A_N^\varepsilon \cap B(x_0, \delta) \subset \mathcal{R}(\varepsilon \delta / 2) \} \) occurs, then \( a_{l+1} \in \mathcal{R}(\varepsilon \delta) \setminus B(x_0, \sqrt{\varepsilon} \delta) \) for \( l \in \mathcal{H}(x_0, \delta) \setminus \mathcal{H}(x_0, \sqrt{\varepsilon} \delta) \), and thus

\[ \limsup_{N \to \infty} \nu_N \left( \sup_{l \in \mathcal{H}(x_0, \delta) \setminus \mathcal{H}(x_0, \sqrt{\varepsilon} \delta)} \left| \frac{(x_0a_{l+1}) \cdot i}{|x_0 - a_{l+1}|} - \cos \theta_0 \right| \right) \leq 2\sqrt{\varepsilon}. \]
Moreover, we have $|\mathcal{H}_N(x_0, \sqrt{\varepsilon}\delta)| \leq 2N\delta\sqrt{\varepsilon}$, whence, by splitting the sum over $\mathcal{H}_N(x_0, \sqrt{\varepsilon}\delta)$ and $\mathcal{H}_N(x_0, \delta) \setminus \mathcal{H}_N(x_0, \sqrt{\varepsilon}\delta)$, we obtain

$$\limsup_{N \to \infty} \nu_N \left( \frac{1}{\delta N} \sum_{l \in \mathcal{H}_{1,N}(x_0,\delta)} (x_0 a_{l+1}) \cdot i \right) - \nu_N \left( \frac{|\mathcal{H}_{1,N}(x_0,\delta)|}{\delta N} \cos \theta_0 \right) \leq 8\sqrt{\varepsilon}.$$  

This inequality being valid for all $\delta$ small enough, the proof is completed. $\blacksquare$

**Lemma 8** We have

$$\lim_{\delta \to 0} \limsup_{N \to +\infty} \nu_N \left( \frac{|\mathcal{H}_N(x_0,\delta)|}{\delta N} \right) - |\cos \theta_0| = 0.$$  

(31)

**Proof of lemma 8.** We denote, as in the proof of lemma 6, by $a$ and $x'$ the points of $\partial B(s, r) \setminus B(x_1, \delta)$ belonging respectively to $\partial A^*_N$ and to $\gamma$. Let $b$ be the point of $\partial A^*_N \cap \partial B(s, r) \setminus B(x_1, \delta)$.

The random points $a$ and $b$ are approximated, for $N$ large enough, respectively by $x'$ and $x_0$.

We have, by definition of $\mathcal{H}_N(x_0, \delta)$,

$$\left| \frac{|\mathcal{H}_N(x_0, \delta)|}{N} - |ba \cdot i| \right| \leq \frac{2}{N}.$$  

We suppose that $r$ is small enough so that $T_{x_0}\gamma$ is not tangent to the circle $\partial B(s, r)$. Let $\alpha > 0$. There exists $\varepsilon > 0$ depending on $\alpha$ and the angle of the tangent $T_{x_0}\gamma$ with $\partial B(s, r)$ such that

\[ \forall \delta > 0 \quad \{ \partial A^*_N \cap B(x_0, \delta) \subset \mathcal{R}(\varepsilon\delta) \} \quad \Rightarrow \quad |b - x_0| < \alpha \delta, \quad |a - x'| < \alpha \delta. \]
Now as in the proof of lemma 7, the condition (3) together with (30) implies that for \( \delta \) small enough,
\[
\lim_{N \to +\infty} \nu_N \left( \partial \mathcal{A}_N \cap B(x_0, \delta) \right) \subset \mathcal{R}(\varepsilon \delta) = 1.
\]

Putting together the previous facts, we obtain that
\[
\limsup_{N \to +\infty} \left| \nu_N \left( \mathcal{H}_N(x_0, \delta) \right) - |x_0x' \cdot i| \right| \leq 2\alpha \delta.
\]

Remarking that \( |x_0 - x'| = \delta \), and that
\[
\lim_{\delta \to 0} \frac{|x_0x' \cdot i|}{|x_0 - x'|} = |\cos \theta_0|,
\]
we conclude the proof by sending \( \alpha \) to 0.

\[\blacksquare\]

**Corollary 2** We have
\[
\lim_{\delta \to 0} \limsup_{N \to +\infty} \left| \nu_N \left( \mathcal{H}_{1,N}(x_0, \delta) \right) \right| - \left( |\cos \theta_0| - C(\theta_0) \right) = 0. \tag{32}
\]
\[
\lim_{\delta \to 0} \limsup_{N \to +\infty} \left| \nu_N \left( \mathcal{H}_{2,N}(x_0, \delta) \right) \right| - C(\theta_0) = 0. \tag{33}
\]

**Proof of corollary 2.** The limit in (33) is deduced from the condition (6) since by definition
\[
||\mathcal{H}_{2,N}(x_0, \delta)| - C_N(x_0, \delta)| \leq \frac{1}{N}.
\]
The first limit is deduced by combining (33) and the result of lemma 8, since \( \mathcal{H}_{2,N}(x_0, \delta) \) and \( \mathcal{H}_{1,N}(x_0, \delta) \) form a partition of \( \mathcal{H}_N(x_0, \delta) \).

\[\blacksquare\]

We obtain, collecting (21), (22), (23), (24), lemma 7, (32) and (33) that
\[
\lim_{\delta \to 0} \limsup_{N \to +\infty} \left| \frac{1}{\delta} \nu_N \left( \int_0^{\delta} e_1(\alpha) \cdot i \, d\alpha \right) + \text{sgn}(\cos \theta_0) \left( \cos^2 \theta_0 + C(\theta_0) \left( |\sin \theta_0| - |\cos \theta_0| \right) \right) \right| = 0.
\]

Using the same method, we prove that
\[
\lim_{\delta \to 0} \limsup_{N \to +\infty} \left| \frac{1}{\delta} \nu_N \left( \int_0^{\delta} e_2(\alpha) \cdot i \, d\alpha \right) - \text{sgn}(\cos \theta_1) \left( \cos^2 \theta_1 + C(\theta_1) \left( |\sin \theta_1| - |\cos \theta_1| \right) \right) \right| = 0.
\]

We get the expression of \( \lim_{\delta \to 0} \limsup_{N \to +\infty} \nu_N(\mathcal{A}_N^{s,r}(s, r, \delta)) \) of theorem 2 by combining proposition 3 together with the two last limits. The \( \liminf \) can be handled similarly. \[\blacksquare\]
4.5 Proof of proposition 1

In this case the condition 3 of theorem 1 is satisfied and we have only to check the limit in (6) and to precise the value of the function $C$ defined there. For this, we need the following lemma.

**Lemma 9** Let $A$ and $B$ be two points of $\mathbb{R}^2$. For each fixed integer $N$, let $\mathcal{L}_N$ denote one of the two maximal subpaths of $\partial([AB])_N$ not crossing the line $(AB)$. Let $N_+(\mathcal{L}_N)$ be as defined in (11). Then

$$\lim_{N \to +\infty} \frac{N_+(\mathcal{L}_N)}{N} = |(AB) \cdot i| \wedge |(AB) \cdot j|,$$

where $a \wedge b = \min(a, b)$.

**Proof of lemma 9.** We suppose without loss of generality that $AB \cdot i$ and $AB \cdot j$ are positive. Let $\theta$ denote the angle between $AB$ and $i$. We consider only the case $0 \leq \tan \theta < 1$, since the proofs for the cases $\tan \theta > 1$ and $\tan \theta = 1$ are similar. We denote by $A_N$ and $B_N$ the extreme points of $\mathcal{L}_N$. Our task is to prove that

$$(A_N B_N) \cdot j = \frac{N_+(\mathcal{L}_N)}{N}. \tag{34}$$

The identity (34) will prove lemma 9 since $\lim_{N \to +\infty} (A_N B_N) \cdot j = (AB) \cdot j$ and $0 \leq \tan \theta < 1$.

We first prove the equality (34) for $N_+(\mathcal{L}_N) = 1$. When $N_+(\mathcal{L}_N) = 1$, the path $\mathcal{L}_N$ contains a unique monotone path $\mathcal{L}'_N = (v_1, w_1, \ldots, w_r)$ such that $v_1 \cdot j = 0$, $v_1 \cdot w_1 = 0$ and $w_1 \cdot i = \ldots = w_r \cdot i = 0$. These vectors are drawn on the lattice $\mathbb{Z}_N^2$ and arranged according to the direct sense.

Let $C_1$, $C_2$ be the two points of $(AB)$ such that $(C_1 C_2) \cdot i = 1/N$ and that the path $\mathcal{L}'_N$ cover the segment $[C_1, C_2]$. By construction

$$(C_1 C_2) \cdot j > \frac{r - 1}{N},$$
hence

\[ \tan \theta = \frac{(C_1C_2) \cdot j}{(C_1C_2) \cdot i} > r - 1. \]

Since \( 0 \leq \tan \theta < 1 \), we deduce that \( r = 1 \). Now let \( B_1, B_2 \) be the two points of \((AB)\) belonging to the boundary of the box of \([AB]_N\) that contains the point \( B \). Since \( 0 \leq \tan \theta < 1 \), we have

\[ |(B_1B_2) \cdot j| < \frac{1}{N}. \]

This fact together with \( r = 1 \) proves that the path \( \mathcal{L}_N \) is equal to \((e_1, \ldots, e_m, w_1, f_1, \ldots, f_n)\), where the vectors \((e_i)\) and \((f_i)\) are copies of the vector \( v_1 \), so that they are all horizontal. Hence \((A_NB_N) \cdot j = 1/N\). The general case when \( N_+(\mathcal{L}_N) > 1 \) is proved by induction on \( N_+(\mathcal{L}_N) \). \( \square \)

**Proofs for polygons.** Lemma 9 together with theorem 1 yield the control of \( A^\sigma_{N}(s,r,\delta) \) for a class of regular polygons \( \Gamma \) defined as follows.

**m-smooth polygons.** Let \( s_1, \ldots, s_m \) be \( m \) points of \( \mathbb{R}^2 \). We denote by \( \Gamma(s_1, \ldots, s_m) \) or by \( \Gamma \), if there is no ambiguities, the polygon in \( \mathbb{R}^2 \) linking the points \([s_1, s_2, \ldots, s_m, s_1]\); the points \( s_1, s_2, \ldots, s_m \) are then the corner points of \( \Gamma \). We suppose that the points \( s_1, s_2, \ldots, s_m \) are arranged counterclockwise. By convention, we set \( s_0 = s_m \). To each site \( s_i \), we associate two oriented angles \( \theta_i(s_i) \) and \( \theta_{i-1}(s_i) \) such that \( \theta_{i-1}(s_i) \) (respectively \( \theta_i(s_i) \)) is the oriented angle between the half horizontal axis \([0, +\infty]\) and the segment \([s_i, s_{i-1}]\) (respectively \([s_i, s_{i+1}]\) ).
Finally, we suppose that $\Gamma$ encloses a connected, compact, bounded set $U$ of $\mathbb{R}^2$ i.e. $\Gamma = \partial U$ and that $\Gamma \cap \mathbb{Z}^2/N = \emptyset$ for all $N \geq 1$.

**Initial condition.** We will consider $\sigma$ the spin configuration associated to the polygon $\Gamma$ at step $N$. 

A polygon $\Gamma$ and the configuration $\sigma$
Lemma 9 allows to apply theorem 1 with \( C(\theta_k) = |\sin \theta_k| \wedge |\cos \theta_k| \). Doing so, we get the following proposition.

**Proposition 4** Let \( \Gamma \) be an \( m \)-smooth polygon in \( \mathbb{R}^2 \) associated to the \( m \) points \( s_1, \ldots, s_m \) and let \( \sigma_0 := \sigma_{0,N} \) be the associated initial configuration at step \( N \). Let \( \theta_i \in [0, 2\pi] \) (respectively \( \theta_{i-1} \in [0, 2\pi] \)) be the oriented angle between the half horizontal axis \([0, +\infty[ \) and the segment \([s_i, s_{i+1}] \) (respectively \([s_i, s_{i-1}] \)) with the convention that \( s_0 = s_m \). Then, for each \( i = 1, \ldots, m \), and for any positive real numbers \( r, \delta \) small enough, one has

\[
\lim_{N \to +\infty} A_N^{\sigma, \Gamma}(s_i, r, \delta) = \frac{1}{4} \sin 2\theta_{i-1} \left( 2 \mathbf{1}_{|\sin \theta_{i-1}| < |\cos \theta_{i-1}|} - 1 \right) - \frac{1}{4} \sin 2\theta_i \left( 2 \mathbf{1}_{|\sin \theta_i| < |\cos \theta_i|} - 1 \right) \\
+ \frac{1}{2} \left( \text{sgn}(\tan \theta_i) \mathbf{1}_{|\sin \theta_i| < |\cos \theta_i|} - \text{sgn}(\tan \theta_{i-1}) \mathbf{1}_{|\sin \theta_{i-1}| < |\cos \theta_{i-1}|} \right) \\
+ \mathbf{1}_{\sin \theta_{i-1} \sin \theta_i > 0} \left( \text{sgn}(\theta_i - \theta_{i-1}) \mathbf{1}_{\cos \theta_{i-1} \cos \theta_i > 0} + \text{sgn}(\tan \theta_i) \mathbf{1}_{\cos \theta_{i-1} \cos \theta_i < 0} \right).
\]

Hence,

- if \( (\theta_{i-1}, \theta_i) \in [(2k + 1)\frac{\pi}{4}, (2k + 3)\frac{\pi}{4}] \times [(2k + 5)\frac{\pi}{4}, (2k + 7)\frac{\pi}{4}] \), with \( k \in \{0, 2\} \), then

\[
\lim_{N \to +\infty} A_N^{\sigma, \Gamma}(s_i, r, \delta) = \frac{1}{4} (\sin 2\theta_i - \sin 2\theta_{i-1}).
\]

- if \( (\theta_{i-1}, \theta_i) \in [(2k + 1)\frac{\pi}{4}, (2k + 3)\frac{\pi}{4}] \times [(2k + 5)\frac{\pi}{4}, (2k + 7)\frac{\pi}{4}] \), with \( k \in \{1, 3\} \), then

\[
\lim_{N \to +\infty} A_N^{\sigma, \Gamma}(s_i, r, \delta) = \frac{1}{4} (\sin 2\theta_{i-1} - \sin 2\theta_i).
\]

- if \( (\theta_{i-1}, \theta_i) \in [(2k + 1)\frac{\pi}{4}, (2k + 3)\frac{\pi}{4}]^2 \), with \( k \in \{0, 2\} \), then

\[
\lim_{N \to +\infty} A_N^{\sigma, \Gamma}(s_i, r, \delta) = \frac{1}{4} (4 \text{sgn}(\theta_i - \theta_{i-1}) + \sin 2\theta_i - \sin 2\theta_{i-1}).
\]

- if \( (\theta_{i-1}, \theta_i) \in [(2k + 1)\frac{\pi}{4}, (2k + 3)\frac{\pi}{4}]^2 \), with \( k \in \{1, 3\} \), then

\[
\lim_{N \to +\infty} A_N^{\sigma, \Gamma}(s_i, r, \delta) = \frac{1}{4} (4 \text{sgn}(\sin(\theta_i - \theta_{i-1})) + \sin 2\theta_{i-1} - \sin 2\theta_i).
\]

- if \( (\theta_{i-1}, \theta_i) \in [(2k + 1)\frac{\pi}{4}, (2k + 3)\frac{\pi}{4}] \times [(2k + 3)\frac{\pi}{4}, (2k + 5)\frac{\pi}{4}] \cup [(2k + 1)\frac{\pi}{4}, (2k + 3)\frac{\pi}{4}] \times [(2k + 3)\frac{\pi}{4}, (2k + 5)\frac{\pi}{4}] \), with \( k \in \{0, 2\} \), then

\[
\lim_{N \to +\infty} A_N^{\sigma, \Gamma}(s_i, r, \delta) = \begin{cases} 
\frac{1}{4} (2 - \sin 2\theta_i - \sin 2\theta_{i-1}) & \text{if } |\tan \theta_i| \leq 1, \ |\tan \theta_{i-1}| \geq 1 \\
\frac{1}{4} (-2 + \sin 2\theta_{i-1} + \sin 2\theta_i) & \text{if } |\tan \theta_i| \geq 1, \ |\tan \theta_{i-1}| \leq 1.
\end{cases}
\]
\( \cdot \) if \((\theta_{i-1}, \theta_i) \in [(2k+1)\frac{\pi}{4}, (2k+3)\frac{\pi}{4}] \times [(2k+3)\frac{\pi}{4}, (2k+5)\frac{\pi}{4}] \cup [(2k+3)\frac{\pi}{4}, (2k+5)\frac{\pi}{4}] \times [(2k+1)\frac{\pi}{4}, (2k+3)\frac{\pi}{4}],\)

\[ \text{with } k \in \{1, 3\}, \text{ then} \]

\[
\lim_{N \to +\infty} A^\sigma_{\Gamma N}(s_i, r, \delta) = \begin{cases} 
\frac{1}{4} (-2 - \sin 2\theta_i - \sin 2\theta_{i-1}) & \text{if } |\tan \theta_i| \leq 1, |\tan \theta_{i-1}| \geq 1 \\
\frac{1}{4} (2 + \sin 2\theta_{i-1} + \sin 2\theta_i) & \text{if } |\tan \theta_i| \geq 1, |\tan \theta_{i-1}| \leq 1.
\end{cases}
\]

**Remark.** We denote by \( L_{\Gamma}(s_i) = \lim_{N \to +\infty} A^\sigma_{\Gamma N}(s_i, r, \delta) \), where \( \Gamma \) is a polygon as described by proposition 4. Then we can check the following comparison criterion.

If \( U \cap B(s_i, r) \subset U' \cap B(s_i, r) \) for some \( r > 0 \) and \( s_i \in \Gamma \cap \Gamma' \),

\[ L_{\Gamma}(s_i) \leq L_{\Gamma'}(s_i). \]
We illustrate the results of proposition 4 with the help of the following pictures.

Here $\lim_{N \to +\infty} A^{\sigma F}_N (s_i, r, \delta) = \frac{1}{4} (\sin 2\theta_i - \sin 2\theta_{i-1})$.

In the first picture this limit is negative, while for the second one it is positive.
In the following picture, we have \( \lim_{N \to +\infty} A_N^{\sigma} \Gamma (s_i, r, \delta) = \frac{1}{4} (\sin 2\theta_{i} - \sin 2\theta_{i-1}) \).

This limit is negative.

Here \( \lim_{N \to +\infty} A_N^{\sigma} \Gamma (s_i, r, \delta) = \frac{1}{4} (4 + \sin 2\theta_{i} - \sin 2\theta_{i-1}) \).

This limit is positive.
In the following picture, we have \( \lim_{N \to +\infty} A_N^{\sigma \Gamma}(s_i, r, \delta) = \frac{1}{4} (2 \sin 2\theta_{i-1} - \sin 2\theta_i) \).

This limit is positive.

Here \( \lim_{N \to +\infty} A_N^{\sigma \Gamma}(s_i, r, \delta) = \frac{1}{4} (-2 \sin 2\theta_i - \sin 2\theta_{i-1}) \).

This limit is negative.
Proofs for Jordan curves. We consider now the case of Jordan curves. In order to apply theorem 2, we have to check the condition (6). For this, we generalize the lemma 9 as follows.

Lemma 10 Let \( f \) be a monotone function of class \( C_1 \) defined on \([a, b]\). Let \( \mathcal{L}_N \) denote one of the two maximal subpaths of \( \mathbb{Z}_N^2 \) covering \( f \). Let \( N_+(\mathcal{L}_N) \) be as defined in (11). Then

\[
\lim_{N \to +\infty} \frac{N_+(\mathcal{L}_N)}{N} = \int_a^b (|f'(x)| \wedge 1) \, dx.
\]

Proof of lemma 10. We suppose without loss of generality that the function \( f \) is nondecreasing on \([a, b]\). Let \( (I_i)_{i \in I} \) be the collection of the open intervals where \( f' - 1 \) is nonzero. Setting \( I_i = [x_{i-1}, x_i] \) for \( i \in I \), we have

\[
\left( \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right) \wedge 1 = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} (f'(x) \wedge 1) \, dx.
\]

We denote by \( f_i \) the restriction of \( f \) to \([x_{i-1}, x_i]\) and by \( \mathcal{L}^{(i)}_N \) the associated polygonal line. We deduce from the suitable construction of the intervals \((I_i)_{i \in I}\) and arguing as in the proof of lemma 9, that

\[
\lim_{N \to +\infty} \frac{N_+(\mathcal{L}^{(i)}_N)}{N} = (x_i - x_{i-1}) \wedge (f(x_i) - f(x_{i-1})).
\]

Hence

\[
\lim_{N \to +\infty} \frac{N_+(\mathcal{L}_N)}{N} = \sum_{i \in I} (x_i - x_{i-1}) \wedge (f(x_i) - f(x_{i-1})).
\]

Lemma 10 is proved by collecting the last bound together with (35).\( \square \)

We define a monotone function \( f \), such that the part of \( \gamma \) limited by \( x_0 \) and \( x_0(\delta) \) (where \( x_0(\delta) \) is the point of \( \gamma \cap \partial B(x_0, \delta) \setminus B(s, r) \)) is equal to the graph \( \{(x, y) : y = f(x)\} \) and we apply lemma 10 to the monotone path \( \mathcal{L}_N \) covering the part of \( \gamma \) limited by \( x_0 \) and \( x_0(\delta) \). We deduce, since \( |N_+(\mathcal{L}_N) - C_N(x_0, \delta)| \leq 1 \), that

\[
\lim_{N \to +\infty} \frac{C_N(x_0, \delta)}{N\delta} = \frac{1}{\delta} \int_{I_\delta} (|f'(x)| \wedge 1) \, dx
\]

\[
= |\cos \theta(\delta)| \frac{1}{I_\delta} \int_{I_\delta} (|f'(x)| \wedge 1) \, dx,
\]

where \( I_\delta \) is the segment \([x_0 \cdot i, x_0(\delta) \cdot i]\). We obtain, taking the limit over \( \delta \to 0 \) in the last equality,

\[
\lim_{\delta \to 0} \lim_{N \to +\infty} \frac{C_N(x_0, \delta)}{N\delta} = |\cos \theta_0| (|f'(x_0 \cdot i)| \wedge 1) = |\cos \theta_0| \wedge |\sin \theta_0|.
\]

We then obtain from the conclusion of theorem 2, that for \( r \) small enough,

\[
\lim_{\delta \to 0} \liminf_{N \to +\infty} A_N^{\sigma, \gamma}(s, r, \delta) = \lim_{\delta \to 0} \limsup_{N \to +\infty} A_N^{\sigma, \gamma}(s, r, \delta)
\]  

(36)
= \frac{1}{4} \sin 2\theta_0 \left(2|\sin \theta_0| - |\cos \theta_0| - 1\right) - \frac{1}{4} \sin 2\theta_1 \left(2|\sin \theta_1| - |\cos \theta_1| - 1\right) \\
+ \frac{1}{2} \left(\text{sgn}(\tan \theta_1) |\sin \theta_1| - \text{sgn}(\tan \theta_0) |\sin \theta_0|\right) \\
+ |\sin \theta_0 \sin \theta_1| > 0 \left(\text{sgn}(\theta_1 - \theta_0) |\cos \theta_0 \cos \theta_1| + \text{sgn}(\tan \theta_0) |\cos \theta_0 \cos \theta_1|\right).

**End of the proof of proposition 1.** In order to prove proposition 1, we suppose first that \( \theta \) takes a value different from \((2k + 1)\frac{\pi}{4}\), for \( k \in \mathbb{N} \). Since the curve \( \gamma \) admits a tangent at the point \( s \), then for \( r \) small enough, \((\theta_0, \theta_1)\) belongs to \(((2k + 1)\frac{\pi}{4}, (2k + 3)\frac{\pi}{4}] \times [(2k + 5)\frac{\pi}{4}, (2k + 7)\frac{\pi}{4}]\), for some \( k \in \mathbb{N} \). We then deduce from (36) that,

- if \((\theta_0, \theta_1)\) \in \[\frac{(2k + 1)\pi}{4}, (2k + 3)\frac{\pi}{4}\] \times \[(2k + 5)\frac{\pi}{4}, (2k + 7)\frac{\pi}{4}\], with \( k \in \{0, 2\} \), then

\[
\lim_{\delta \to 0} \lim_{N \to +\infty} \limsup_{\varepsilon \to 0} A^\gamma_{\alpha \delta N}(s, r, \delta) = \frac{1}{4} (\sin 2\theta_1 - \sin 2\theta_0).
\]

- if \((\theta_0, \theta_1)\) \in \[\frac{(2k + 1)\pi}{4}, (2k + 3)\frac{\pi}{4}\] \times \[(2k + 5)\frac{\pi}{4}, (2k + 7)\frac{\pi}{4}\], with \( k \in \{1, 3\} \), then

\[
\lim_{\delta \to 0} \limsup_{N \to +\infty} A^\gamma_{\alpha \delta N}(s, r, \delta) = \frac{1}{4} (\sin 2\theta_0 - \sin 2\theta_1).
\]

We now need the following lemma.

**Lemma 11** Let \( \gamma \) be a Jordan curve of \( \mathbb{R}^2 \) of class \( C_2 \). Let \( s \) be a fixed point of \( \gamma \). Let \( r \) be a positive real number sufficiently small such that \( \partial B(s, r) \cap \gamma \) contains exactly two points \( x_0 \) and \( x_1 \). Suppose that \( x_0, s \) and \( x_1 \) are arranged counterclockwise. Let \( s' \) be the common point to \( T_{x_0}\gamma \) and \( T_{x_1}\gamma \). Let \( \theta_1 \in [0, 2\pi] \) (respectively \( \theta_0 \in [0, 2\pi] \)) be the oriented angle between the half horizontal axis \([0, +\infty[\) and the segment \([s', x_1]\) (respectively \([s', x_0]\)). Then

\[
\lim_{r \to 0} \frac{\sin (\theta_0 - \theta_1)}{2r} = \xi_\gamma(s),
\]

and

\[
\lim_{r \to 0} \cos (\theta_0 + \theta_1) = -\cos 2\theta,
\]

where \( \theta \) is the angle between the half horizontal axis \([0, +\infty[\) and \( T_{s'}\gamma \).

Lemma 11, together with the two equalities just above Lemma 11 and the fact \( \sin 2a - \sin 2b = 2\sin(a - b) \cos(a + b) \), gives

\[
\lim_{r \to 0} \limsup_{N \to +\infty} \frac{1}{2r} A^\gamma_{\alpha \delta N}(s, r, \delta) = \begin{cases} 
\frac{1}{2} (\cos 2\theta) \xi_\gamma(s) & \text{if } \theta \in [(1 + 4k)\frac{\pi}{4}, (3 + 4k)\frac{\pi}{4}] \\
-\frac{1}{2} (\cos 2\theta) \xi_\gamma(s) & \text{if } \theta \in [(3 + 4k)\frac{\pi}{4}, (5 + 4k)\frac{\pi}{4}].
\end{cases}
\]

which proves theorem 1 when \( \theta \) is different from \((2k + 1)\frac{\pi}{4}\), for \( k \in \mathbb{N} \). Now, suppose that \( \theta = \frac{\pi}{4} \) and that for any \( r \) small enough \((\theta_0, \theta_1)\) \in \[(\frac{\pi}{4}, 3\frac{\pi}{4}] \times [3\frac{\pi}{4}, 5\frac{\pi}{4}]\) (the arguments for the
proof for the other values of \( \theta \) and the corresponding values of \( \theta_1, \theta_0 \) will be similar). We have in that case,

\[
\lim_{\delta \to 0} \limsup_{N \to +\infty} A_N^{\gamma}(s, r, \delta) = \frac{1}{4} (2 - \sin 2\theta_1 - \sin 2\theta_0)
\]

\[
= \frac{1}{2} \left( \sin \left( \frac{\pi}{4} - \theta_1 \right) \cos \left( \frac{\pi}{4} + \theta_1 \right) + \sin \left( \frac{\pi}{4} - \theta_0 \right) \cos \left( \frac{\pi}{4} + \theta_0 \right) \right).
\]

(38)

Now the method of the proof of lemma 11 gives

\[
\lim_{r \to 0} \frac{\sin (\theta - \theta_1)}{r} = \lim_{r \to 0} \frac{\sin (\theta - \theta_0)}{r} = - \xi_\gamma(s).
\]

This fact, together with (38), leads to

\[
\lim_{r \to 0} \limsup_{\delta \to 0} \frac{1}{r} A_N^{\gamma}(s, r, \delta) = 0,
\]

which is the conclusion of theorem 1 for \( \theta = \pm \frac{\pi}{4} \).

**Proof of lemma 11.** We begin by giving the definition of the curvature of \( \gamma \) at any \( s \in \gamma \).

**Definition.** Let \( \gamma \) be a smooth Jordan curve of \( \mathbb{R}^2 \). Suppose that \((\phi(t))_{t \in [-1, 1]}\) is a parametrization of the curve \( \gamma \). Let \( s = \phi(t) = (x(t), y(t)) \) be a fixed point of \( \gamma \). The **curvature** of \( \gamma \) at the point \( s \) is defined by

\[
\xi_\gamma(s) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'^2(t) + y'^2(t))^{3/2}}.
\]

Let \( s, x_0 \) and \( x_1 \) be as defined in lemma 11. Let \( t, t_0 \) and \( t_1 \) be three real numbers of \([-1, 1]\) such that \( s = \phi(t) = (x(t), y(t)) \), and for \( i \in \{0, 1\} \), \( x_i = \phi(t_i) = (x(t_i), y(t_i)) \). We have

\[
r^2 = (x(t_i) - x(t))^2 + (y(t_i) - y(t))^2,
\]

for \( i \in \{0, 1\} \). Hence

\[
\lim_{t \to t_0, t_0 < t} \frac{r}{t - t_0} = \sqrt{x'^2(t) + y'^2(t)}, \quad \lim_{t \to t_1, t < t_1} \frac{r}{t_1 - t} = \sqrt{x'^2(t) + y'^2(t)}.
\]

For any \( \tau \in [-1, 1] \), define \( f(\tau) = \frac{x'(\tau)}{\sqrt{x'^2(\tau) + y'^2(\tau)}} \). We have

\[
f'(\tau) = \frac{x''(\tau)}{\sqrt{x'^2(\tau) + y'^2(\tau)}} - x'(\tau) \frac{x'(\tau)x''(\tau) + y'(\tau)y''(\tau)}{(x'^2(\tau) + y'^2(\tau))^{3/2}}.
\]

Hence

\[
\cos \theta_0 = - \frac{x'(t_0)}{\sqrt{x'^2(t_0) + y'^2(t_0)}}
\]

\[
= - \frac{x'(t)}{\sqrt{x'^2(t) + y'^2(t)}} + (t - t_0)f'(t) + o(|t - t_0|).
\]

(39)
\[
\cos \theta_1 = \frac{x'(t)}{\sqrt{x'^2(t) + y'^2(t)}} + (t_1 - t) f'(t) + o(|t_1 - t|).
\] (40)

We obtain, combining the last two equalities
\[
\lim_{t_1 \to t, t_0 < t < t_1} \cos \theta_0 + \cos \theta_1 = \frac{2x''(t)}{x'^2(t) + y'^2(t)} - \frac{2x'(t)x''(t) + y'(t)y''(t)}{(x'^2(t) + y'^2(t))^2}.
\]

The last limit together with
\[
\lim_{t_1 \to t, t < t_1} \sin \theta_1 = \frac{y'(t)}{\sqrt{x'^2(t) + y'^2(t)}},
\]
ensures
\[
\lim_{t_1 \to t, t_0 < t < t_1} \frac{1}{r} \sin \theta_1 \left( \cos \theta_0 + \cos \theta_1 \right) = \frac{2x''(t)y'(t)}{(x'^2(t) + y'^2(t))^3/2} - \frac{2x'(t)x''(t) + y'(t)y''(t)}{(x'^2(t) + y'^2(t))^{5/2}}.
\]

In the same way, we prove that
\[
\lim_{t_1 \to t, t_0 < t < t_1} \frac{1}{r} \cos \theta_1 \left( \sin \theta_0 + \sin \theta_1 \right) = \frac{2x'(t)y''(t)}{(x'^2(t) + y'^2(t))^{3/2}} - \frac{2x'(t)x''(t) + y'(t)y''(t)}{(x'^2(t) + y'^2(t))^{5/2}}.
\]

The last two limits together with
\[
\sin (\theta_0 - \theta_1) = \cos \theta_1 (\sin \theta_1 + \sin \theta_0) - \sin \theta_1 (\cos \theta_0 + \cos \theta_1),
\]
prove that
\[
\lim_{t_1 \to t, t_0 < t < t_1} \frac{1}{2r} \sin (\theta_0 - \theta_1) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'^2(t) + y'^2(t))^{5/2}}.
\]

Now the equality
\[
\cos (\theta_0 + \theta_1) = \cos \theta_0 \cos \theta_1 - \sin \theta_0 \sin \theta_1,
\]

Together with the limits (39), (40), yields
\[
\lim_{t_1 \to t, t_0 < t < t_1} \cos (\theta_0 + \theta_1) = \frac{y'^2(t) - x'^2(t)}{x'^2(t) + y'^2(t)}.
\]

The last limit is equal to \(-\cos 2\theta\), where \(\theta\) is the angle between the horizontal axis and \(T_s \gamma\). □
4.6 Proof of proposition 2

Our purpose is to apply theorem 2. So we have to check, the requirements of theorem 2. We first prove the condition (3). We claim that, for all $\varepsilon > 0$,

$$
\lim_{N \to \infty} \mu_N(\sup \Phi_N(x_N) - f(x_N) | \geq \varepsilon) = 0,
$$

(41)

where the supremum is taken over $x_N \in [a, b] \cap \mathbb{Z}$.

Proof of (41). For $l \in \mathbb{Z}$, we denote by $\eta(l)$ the height difference $\eta(l) = \Phi_N(\frac{l+1}{N}) - \Phi_N(\frac{l}{N})$. Without loss of generality, we will take $a = 0$. We write, for $\frac{k}{N} \in [0, b] \cap \mathbb{Z}$,

$$
\Phi_N(\frac{k}{N}) - f(\frac{k}{N}) = \sum_{l=0}^{k-1} \left( \eta(\frac{l}{N}) - \mu_N \left( \eta(\frac{l}{N}) \right) \right) - \sum_{l=0}^{k-1} \left( f(\frac{l+1}{N}) - f(\frac{l}{N}) \right) - \frac{1}{N} |f'(\frac{l}{N})| + (\Phi_N(0) - f(0)).
$$

The last equality gives, since $\mu_N(\eta(l)) = \frac{1}{N} |f'|((\frac{l}{N}))$,

$$
\Phi_N(\frac{k}{N}) - f(\frac{k}{N}) = \sum_{l=0}^{k-1} \left( \eta(\frac{l}{N}) - \mu_N \left( \eta(\frac{l}{N}) \right) \right) - \sum_{l=0}^{k-1} \left( f(\frac{l+1}{N}) - f(\frac{l}{N}) \right) - \frac{1}{N} |f'(\frac{l}{N})| + (\Phi_N(0) - f(0)).
$$

We deduce from the last equality, assumption (2) of proposition 2 and the fact

$$
\sum_{l=0}^{k-1} \left| f(\frac{l+1}{N}) - f(\frac{l}{N}) \right| \leq \frac{b}{N} \|f''\|_{\infty},
$$

that (41) is proved as soon as,

$$
\lim_{N \to \infty} \mu_N \left( \sup_{0 \leq k \leq Nb} \left| \sum_{l=0}^{k-1} \left( \eta(\frac{l}{N}) - \mu_N \left( \eta(\frac{l}{N}) \right) \right) \right| \geq \varepsilon \right) = 0.
$$

(42)

For this, we use a Markov inequality, the independence of the random variables $(\eta(\frac{l}{N}))_{l \in \mathbb{Z}}$ and a Rosenthal inequality (cf. section 2.6.19 and Theorem 2.9 of Petrov (1995)). We get, for an universal constant $C$,

$$
\mu_N \left( \sup_{0 \leq k \leq Nb} \left| \sum_{l=0}^{k-1} \left( \eta(\frac{l}{N}) - \mu_N \left( \eta(\frac{l}{N}) \right) \right) \right| \geq \varepsilon \right)
\leq \frac{1}{\varepsilon^3} \mu_N \left( \sup_{0 \leq k \leq Nb} \left| \sum_{l=0}^{k-1} \left( \eta(\frac{l}{N}) - \mu_N \left( \eta(\frac{l}{N}) \right) \right) \right|^3 \right)
\leq C \left\{ \sum_{l=0}^{Nb} \text{Var}_{\mu_N} \left( \eta(\frac{l}{N}) \right)^{3/2} + \sum_{l=0}^{Nb} \left| \eta(\frac{l}{N}) - \mu_N \left( \eta(\frac{l}{N}) \right) \right|^3 \right\}.
$$

43
The last estimations and the fact that, for some constant $C$

\[
\text{Var}_{f_N} \eta \left( \frac{1}{N} \right) = \frac{1}{N^2} |f'(\frac{1}{N})|^2 \left( 1 + |f'(\frac{1}{N})|^2 \right), \quad \mu_N \left( \left| \eta \left( \frac{1}{N} \right) \right|^3 \right) \leq C \frac{1}{N^3}
\]

give

\[
\mu_N \left( \sup_{0 \leq k \leq Nb} \left\{ \sum_{i=0}^{k-1} \left( \eta \left( \frac{i}{N} \right) - \mu_N \left( \eta \left( \frac{i}{N} \right) \right) \right) \right\} \geq \varepsilon \right) = O \left( \left( \frac{1}{N} \right)^{3/2} \right),
\]

which proves (42) and then (41). Now (41) allows to deduce the condition (3). We deduce from the definition of $\mu_N$, that for any $N \in \mathbb{N}^*$

\[
\forall k \in [Na, Nb] \cap \mathbb{Z} \quad \mu_N \left( \text{sgn}(\eta \left( \frac{k}{N} \right) f' \left( \frac{k}{N} \right)) < 0 \right) = 0.
\]

Since the graph of the monotone function $f$ coincides with the restriction of $\gamma$ over $[a, b]$, we conclude from the above formula that $\partial A_\theta^N \cap S(s, r, \delta, \delta)$ and $\gamma \cap S(s, r, \delta, \delta)$ are both nondecreasing or both nonincreasing.

Our task now is to check the condition (6) and to precise the value of the corresponding function $C$. Recall that $f$ and $\Phi_N$ are both increasing or decreasing. Therefore

\[
C_N(x_0, \delta) = \sum_{x_0 - i \leq k \leq \delta} \mathbb{1}_{\eta \left( \frac{k}{N} \right) \neq 0},
\]

where the quantity $C_N(x_0, \delta)$ is defined just before theorem 1. We have

\[
\frac{1}{N\delta} \mu_N (C_N(x_0, \delta)) = \frac{1}{N\delta} \sum_{x_0 - i \leq k \leq \delta} \mu_N (\eta \left( \frac{k}{N} \right) \neq 0) = \frac{1}{N\delta} \sum_{x_0 - i \leq k \leq \delta} \frac{|f'(\frac{k}{N})|}{1 + |f'(\frac{k}{N})|}.
\]

The last equality gives

\[
\lim_{N \to \infty} \frac{1}{N\delta} \mu_N (C_N(x_0, \delta)) = \frac{1}{\delta} \int_{x_0 - i}^{\delta} \frac{|f'(x)|}{1 + |f'(x)|} \, dx.
\]

Hence

\[
\lim_{\delta \to 0} \lim_{N \to \infty} \frac{1}{N\delta} \mu_N (C_N(x_0, \delta)) = |\cos \theta_0| \frac{|f'(x_0 \cdot i)|}{1 + |f'(x_0 \cdot i)|} = |\cos \theta_0| \frac{|\tan \theta_0|}{1 + |\tan \theta_0|} = \frac{|\sin (2\theta_0)|}{2(|\sin \theta_0| + |\cos \theta_0|)} = C(\theta_0).
\]

We have assumed in proposition 2 that the curve $\gamma$ is monotone in $B(s, r) \cup B(x_0, \delta) \cup B(x_1, \delta)$. This fact allows to deduce that,

\[
\sin \theta_0 \sin \theta_1 \leq 0, \quad \cos \theta_0 \cos \theta_1 \leq 0.
\]
We use the last constatation together with the conclusion of theorem 2 to obtain,

\[ \lim_{\delta \to 0} \limsup_{N \to \infty} \mu_N(A_N^\gamma(s, r, \delta)) = \lim_{\delta \to 0} \liminf_{N \to \infty} \mu_N(A_N^\gamma(s, r, \delta)) \]

\[ = \frac{1}{2} \operatorname{sgn}(\tan \theta_0) \left( \cos^2 \theta_1 - \cos^2 \theta_0 \right) \]

\[ + \frac{1}{2} \operatorname{sgn}(\tan \theta_0) \left( \frac{|\sin(2\theta_1)|}{2(|\sin \theta_1| + |\cos \theta_1|)} (|\sin \theta_1| - |\cos \theta_1|) \right) \]

\[ - \frac{|\sin(2\theta_0)|}{2(|\sin \theta_0| + |\cos \theta_0|)} (|\sin \theta_0| - |\cos \theta_0|) \right) \].

We have

\[ \operatorname{sgn}(\tan \theta_0) \left( \frac{|\sin(2\theta_1)|}{2(|\sin \theta_1| + |\cos \theta_1|)} (|\sin \theta_1| - |\cos \theta_1|) - \frac{|\sin(2\theta_0)|}{2(|\sin \theta_0| + |\cos \theta_0|)} (|\sin \theta_0| - |\cos \theta_0|) \right) \]

\[ = \frac{\sin \theta_1 \cos \theta_1}{(|\sin \theta_1| + |\cos \theta_1|)} (|\sin \theta_1| - |\cos \theta_1|) - \frac{\sin \theta_0 \cos \theta_0}{(|\sin \theta_0| + |\cos \theta_0|)} (|\sin \theta_0| - |\cos \theta_0|) \]

\[ = -\sin(\theta_0 - \theta_1) \]

\[ \frac{(|\sin \theta_1| + |\cos \theta_1|)}{(|\sin \theta_1| + |\cos \theta_1|)} - \frac{(|\sin \theta_0| + |\cos \theta_0|)}{(|\sin \theta_0| + |\cos \theta_0|)} \]

\[ (\cos^2 \theta_1 - \cos^2 \theta_0)(\sin \theta_0 \cos \theta_1 + \sin \theta_1 \cos \theta_0 + \cos \theta_0 \cos \theta_1 \operatorname{sgn}(\tan \theta_0) + \sin \theta_0 \sin \theta_1 \operatorname{sgn}(\tan \theta_0)) \]

\[ = \frac{-\sin(\theta_0 - \theta_1)}{(|\sin \theta_1| + |\cos \theta_1|)} - \frac{(\cos^2 \theta_1 - \cos^2 \theta_0)}{(|\sin \theta_0| + |\cos \theta_0|)} \]

We conclude from (43) together with the last equalities,

\[ \lim_{\delta \to 0} \limsup_{N \to \infty} \mu_N(A_N^\gamma(s, r, \delta)) = \lim_{\delta \to 0} \liminf_{N \to \infty} \mu_N(A_N^\gamma(s, r, \delta)) \]

\[ = \frac{-\sin(\theta_0 - \theta_1)}{2(|\sin \theta_1| + |\cos \theta_1|)(|\sin \theta_0| + |\cos \theta_0|)}. \]

The last limit together with lemma 11 completes the proof of proposition 2. \( \square \)

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