Inflationary solutions and inhomogeneous Kaluza-Klein cosmology in $4+n$ dimensions

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Abstract

We analyze the existence of inflationary solutions in an inhomogeneous Kaluza-Klein cosmological model in $4+n$ dimensions. It is shown that the 5-dimensional case is the exception rather than the rule, in the sense that the system is integrable (under the assumption of the equation of state $p=k\rho$) for any value of $k$. It is also shown that the cases $k=0$ and $k=1/3$ are integrable if and only if $n=1$.

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Much of the recent work related to unification of the fundamental interactions involve theories that are formulated in a spacetime with dimension $d>4$. Superstrings \cite{1} and M-theory \cite{2} are examples of such theories. Although with different motivations, the idea of a higher dimensional spacetime has been advocated by different authors \cite{3,4} since Kaluza \cite{5} put forward a theory that unified relativity and electromagnetism in a 5-dimensional spacetime. If one admits that the extra dimensions have a physical reality (and are not a

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mere mathematical device to carry out the dimensional reduction of the theory, as in [3]),
then one must account for the fact that our universe is, in the present era, manifestly 4-
dimensional. Usually, one adopts the view that the vacuum of the theory has undergone
a spontaneous compactification [8], in which the extra dimensions are compactified to the
Planck size. However, it must be verified that this assumption is consistent with cosmology.
That is, there must exist cosmological solutions of the theory that, starting from a spacetime
in which all dimensions had comparable size, exhibit the desirable feature of an unobservable
internal space at later times. Solutions of this kind have been found for maximally symmetric
spaces [10], homogeneous models [11–13], and also for anisotropic models [14,15], but little
attention has been paid to higher dimensional inhomogeneous models. These are important
in the light of the findings of the COBE [16], which reveal the existence of inhomogeneities
in the early Universe. In turn, these might be accounted for through the existence of
inhomogeneous extra dimensions in an early phase of the Universe [17].

Several papers have been devoted to inhomogeneous Kaluza-Klein cosmology in the case
of one extra dimension [17–20]. Here we deal with the more general case of an arbitrary
number of extra dimensions in the presence of matter with a given equation of state.

We assume a flat and homogeneous 3-space, and we introduce the inhomogeneity in the
$n$ extra dimensions. The metric tensor is such that the $4 + n$-dimensional interval takes the
form

$$ds^2 = dt^2 - R^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) - A^2(r,t)dy^2$$

(1)

where $dy^2 \equiv \sum_{i=4}^{4+n} dy_n^2$. The nonzero components of the stress-energy tensor are

$$T^0_0 = \rho(r,t) + \Lambda$$
$$T^1_1 = T^2_2 = T^3_3 = -p(r,t) + \Lambda$$
$$T^4_4 = ... = T^n_n = -p_n(r,t) + \Lambda$$

(2)

($\Lambda$ is the $4 + n$ cosmological constant, and $p_n$ is the internal pressure).

The Einstein field equations for the metric (1) take the form

$$\frac{\dot{A}}{A} - \frac{\dot{RA'}}{RA} = 0$$

(3a)
\[
\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + 2n\frac{\dot{R}A}{RA} + n\frac{\ddot{A}}{A} + \frac{n(n-1)\dot{A}^2}{2A^2} - \frac{2n}{r} \frac{A'}{AR^2} + \frac{n(1-n)}{2} \frac{A^2}{A^2R^2} = \Lambda - 8\pi p \tag{3b}
\]

\[
\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + 2n\frac{\dot{R}A}{RA} + n\frac{\ddot{A}}{A} + \frac{n(n-1)\dot{A}^2}{2A^2} - \frac{n}{r} \frac{A'}{AR^2} + \frac{n(1-n)}{2} \frac{A^2}{A^2R^2} = \Lambda - 8\pi p \tag{3c}
\]

\[
\frac{3\ddot{R}^2}{R^2} + \frac{n(n-1)\dot{A}^2}{2A^2} + 3n\frac{\dot{R}A}{RA} - \frac{n}{2} \frac{A''}{AR^2} + \frac{n(1-n)}{2} \frac{A'^2}{A^2R^2} - \frac{2n}{r} \frac{A'}{AR^2} = \Lambda + 8\pi p \tag{3d}
\]

\[
(n-1)\frac{\ddot{A}}{A} + (n-1)(n-2)\frac{\dot{A}^2}{A^2} + 3(n-1)\frac{\dot{R}\dot{A}}{RA} + 3\frac{\ddot{R}}{R} + 3\frac{\dot{R}^2}{R^2} - (n-1)\frac{A''}{AR^2}
- (n-1)(n-2)\frac{A'^2}{A^2R^2} - \frac{2(n-1)}{r} \frac{A'}{AR^2} = \Lambda - 8\pi p_n \tag{3e}
\]

where a dot (a prime) denotes differentiation with respect to time (radial) coordinate. We also assume an equation of state of the form \( p = k\rho \). Equation (3a) can be integrated and its solution is

\[
A(r, t) = F(r)R(t) + G(t) \tag{4}
\]

where \( F(r) \) and \( G(t) \) are arbitrary functions.

From (3b) and (3c) we can obtain

\[
A(r, t) = f(t)br^2 + g(t) \tag{5}
\]

where \( b \) is an arbitrary constant. Comparing (4) and (5) it follows that

\[
A(r, t) = br^2R(t) + g(t) \tag{6}
\]

Now using (3) and the equation of state in equations (3b) and (3d) we obtain the following system of equations:

\[
\left[ \frac{k}{2} f + \frac{n}{2} (n + 3) + 1 \right] \ddot{R}^2 - (1 + k)\Lambda R^2 + (n + 2)\dot{R}\dot{R} = 0 \tag{7a}
\]

1 The system (3a)-(3e) reduces to the equations given in [17] in the case \( n = 1 \).
\[nR\dot{g} + n[k + (k + 1)(n + 1)]\ddot{R}g + \left\{ \frac{\dot{R}^2}{R} [2(n + 1) + 3k(n + 2)] - 2\Lambda(1 + k)R + (n + 4)\dddot{R} \right\} g - 2bn[n + 1 + 1] = 0 \quad (7b)\]

\[\left[ \dddot{R}^2(1 + 3k) + 2R\dddot{R} - \Lambda(1 + k)\dddot{R}^2 \right] g^2 + \frac{n}{2}(1 + k)(n - 1)R^2 \dddot{g}^2 + n(3k + 2)R \dddot{R}g \dddot{g} + nR^2 \dddot{g} - 2nb(2 + 3k)Rg = 0 \quad (7c)\]

where \(f = (n + 2)(n + 3)\). In the case \(n = 1\), equation (7c) reduces to an identity.

The metric given in (1) will be a solution of Einstein equations if and only if equations (7a), (7b) and (7c) are compatible. If this is the case, equations (3c), (3d) and (3e) act as definitions of \(\rho\), \(p\) and \(p_n\) respectively.

The general solution to equation (7a) is

\[R(t) = a \left( \frac{\exp \left[ \frac{(1+k)\sqrt{2f \Lambda}}{n+2} t \right] - c}{\exp \left[ \frac{(1+k)\sqrt{\Lambda f/2}}{n+2} t \right]} \right)^{2(n+2)/f(k+1)} \quad (8)\]

where \(\Lambda > 0\), and \(a\) and \(c\) are arbitrary constants.

Without loss of generality, we take \(c = 1\), which corresponds to zero volume of the 3-space at \(t = 0\). Replacing the expression (8) in (7a) we get

\[\ddot{g} + \sqrt{\frac{2\Lambda}{f}} \coth(2\beta t) \dddot{g} - \frac{\Lambda}{f} [n + 3 + (n + 1) \coth^2(2\beta t)]g = \frac{2(n + 1)b}{a} \left[ 2 \sinh(2\beta t) \right]^{\frac{2}{n+3}} \quad (9)\]

where \(\beta = \sqrt{2\Lambda f/4(n + 2)}\). It has not been possible up to now to solve this equation neither in the case of arbitrary values for \(k\) and \(n\) nor with a given \(k\) and arbitrary \(n\).\[\]

From now on we consider the case \(c = 0\) which corresponds to an inflationary solution for the 3-d space with a singularity at infinite past. The scale factor \(R(t)\) takes the form

\[R(t) = a \exp \left( \sqrt{\frac{2\Lambda}{f}} t \right) \quad (10)\]

After replacing this equation in (7a), we get a differential equation for \(g(t)\) that can be integrated. Its solution is

\[\]

\[\]

\[\]

\[2\] Note however that a solution has been found in the case \(k = 0, n = 1\) by Chatterjee et al [17].
\[ g(t) = C_1 \exp \left( \sqrt{\frac{2\Lambda}{f}} t \right) + C_2 \exp \left[ -\sqrt{\frac{2\Lambda}{f}} (n+2)(k+1)t \right] + \frac{fb}{2n\Lambda} \exp \left( -\sqrt{\frac{2\Lambda}{f}} t \right) \] (11)

where \( C_1 \) and \( C_2 \) are integration constants.

Now we have to look for the conditions under which equation (7c) is satisfied. Upon replacement of (8) and (11) in (7c) we get the following equations in \( n \):

\[
(k^2 + 2k + 1)(-n^4 - 6n^3 - 9n^2 + 4n + 12) = 0 \quad (12)
\]
\[
(k + 1)(n^3 + 4n^2 + n - 6) = 0 \quad (13)
\]

\[
n^3[6k(1 + k) + 2(1 + k^3)] + 2n^2(3k^3 + 11k^2 + 13k + 5) + 2n((2k^2 + 5k + 3) - 2(9 + 21k + 16k^2 + 4k^3) = 0 \quad (14)
\]

The value \( n = 1 \) is a root of the three polynomials, irrespective of the value of \( k \). Besides, if we restrict the values of \( k \) to 0 or 1/3, we see that there is only one root of the three polynomials that is a natural number, and that is again \( n = 1 \) \( ^3 \).

We conclude then that the model described by the metric (1), the stress-energy tensor (4), and the equation of state \( p = k\rho \) is integrable in the case of a 5-dimensional space-time for any value of \( k \) (with \( c = 0 \)). Furthermore, the cases of dust \( (k = 0) \) and radiation \( (k = 1/3) \) are integrable (with \( c = 0 \)) if and only if \( n = 1 \).

Finally, let us mention that the model studied here can be generalized by introducing inhomogeneity in \( R(t) \), and also by adding some degree of anisotropy in the extra dimensions. Work along this lines is currently in progress.

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\( ^3 \) Solutions for \( k = 0 \) and \( k = 1/3 \) with \( n = 1 \) are given in [20].
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