TOWARDS A GL_n VARIANT OF THE HOHEISEL PHENOMENON

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Abstract. Let π be a unitary cuspidal automorphic representation of GL_n over a number field, and let \( \tilde{\pi} \) be contragredient to \( \pi \). We prove effective upper and lower bounds of the correct order in the short interval prime number theorem for the Rankin–Selberg L-function \( L(s, \pi \times \tilde{\pi}) \), extending the work of Hoheisel and Linnik. Along the way, we prove for the first time that \( L(s, \pi \times \tilde{\pi}) \) has an unconditional standard zero-free region apart from a possible Landau–Siegel zero.

1. Introduction

It is well-known that if there exists a constant \( 0 < \delta < \frac{1}{2} \) such that the Riemann zeta function \( \zeta(s) \) is nonzero in the region \( \text{Re}(s) \geq 1 - \delta \), then the primes are regularly distributed in intervals of length \( x^{1-\delta} \); that is,

\[
\sum_{x < p \leq x+h} \log p \sim h, \quad x^{1-\delta} \leq h \leq x.
\]

It was quite stunning when Hoheisel \[11\] proved that (1.1) holds unconditionally for any \( \delta \leq 1/33000 \); this has been improved to \( \delta \leq \frac{5}{12} \) \[8\]. Hoheisel proved (1.1) using the bound

\[
N(\sigma, T) := \#\{\rho = \beta + i\gamma: \beta \geq \sigma, |\gamma| \leq T, \zeta(\rho) = 0\} \ll T^4\sigma(1-\sigma)(\log T)^{13}
\]

(a zero density estimate for \( \zeta(s) \)) and an explicit version of Littlewood’s zero-free region

\[
\zeta(s) \neq 0, \quad \text{Re}(s) \geq 1 - \frac{c \log \log(|\text{Im}(s)| + e)}{\log(|\text{Im}(s)| + e)},
\]

where \( c > 0 \) is an absolute and effectively computable constant. This is an improvement over the “standard” zero-free region

\[
\zeta(s) \neq 0, \quad \text{Re}(s) \geq 1 - \frac{c \log \log(|\text{Im}(s)| + e)}{\log(|\text{Im}(s)| + e)}
\]

proved by de la Vallée Poussin.

Here, we study a broad generalization of Hoheisel’s work. Suppose that an object \( \pi \) (e.g., a number field, abelian variety, automorphic form) gives rise to a Dirichlet series

\[
L(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi}(n)}{n^s}
\]

satisfying the **Hoheisel property**; that is, the following conditions hold:

1. \( \lambda_{\pi}(n) \geq 0 \) for all \( n \).

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(2) We have an “explicit formula”
\[
\sum_{p \leq x} \lambda_\pi(p) \log p = x - \sum_{\beta \geq 0, \gamma \leq T \beta + i\gamma} \frac{x^{\beta+i\gamma} + O_\pi\left(x(\log T x)^2 \right)}{\beta + i\gamma} \gamma \leq T
\]
where \(1 \leq T \leq x\) and \(\beta + i\gamma\) denotes a zero of \(L(s, \pi)\).

(3) We have \(L(1+it, \pi) \neq 0\) for all \(t \in \mathbb{R}\). Also, there exists a constant \(a_\pi > 0\), depending at most on \(\pi\), such that \(L(s, \pi)\) is nonzero in the region
\[
\text{Re}(s) \geq 1 - \frac{a_\pi}{\log(|\text{Im}(s)| + 1)}
\]
apart from \(O_\pi(1)\) exceptional zeroes \(\beta + i\gamma\) that satisfy \(|\gamma| \ll 1\).

(4) If \(N_\pi(\sigma, T)\) is the number of zeroes \(\rho = \beta + i\gamma\) of \(L(s, \pi)\) such that \(\beta \geq \sigma\) and \(|\gamma| \leq T\), then there exist constants \(c_\pi > 0\), depending at most on \(\pi\), such that
\[
N_\pi(\sigma, T) \ll \pi T^{c_\pi(1-\sigma)}.
\]

(5) We have \(N(0, T) \ll T \log T\).

Moreno \cite{23} proved that if \(L(s, \pi)\) satisfies the Hoheisel property, then there exists a constant \(0 < \delta_\pi < 1\) (depending at most on \(\pi\)) such that
\[
\sum_{x < p \leq x + h} \lambda_\pi(p) \log p \gg_\pi h, \quad h \geq x^{1-\delta_\pi}.
\]
Moreno referred to this as the Hoheisel phenomenon.

A key point here is that while the “standard” zero-free region (1.4) is inferior to Littlewood’s region (1.3) in the dependence on \(|\text{Im}(s)|\), one can still prove a lower bound on (1.6) of the expected order when the zero density estimate (1.5) is \(\log\)-free, in the sense that there are no factors of \(\log T\) (in contrast with (1.2)). The absence of the logarithmic factors in (1.5) serves as a proxy for a zero-free region as strong as (1.3). However, even with a \(\log\)-free zero density estimate at one’s disposal, Moreno’s proof suggests that if \(h\) is as small as \(x^{1-\delta}\), then one must be able to take \(a_\pi\) arbitrarily large in (1.4) in order to replace the lower bound (1.6) with an asymptotic. Such a zero-free region appears to be well beyond the reach of current methods.

Akbary and Trudgian \cite{1} proved that certain \(L\)-functions arising from automorphic representations achieve the Hoheisel phenomenon. To describe their results, let \(\mathcal{A}_F\) be the ring of adèles over a number field \(F\), and for an integer \(n \geq 1\), let \(\mathfrak{F}_n\) be the set of cuspidal automorphic representations \(\pi\) of \(\text{GL}_n(\mathcal{A}_F)\) with arithmetic conductor \(q_\pi\) and unitary central character. We implicitly normalize the central character to be trivial on the product of positive reals when embedded diagonally into the archimedean places of the idèles \(\mathcal{A}_F^\times\), so that \(\mathfrak{F}_n\) is discrete. Let \(\tilde{\pi}\) be the representation contragredient to \(\pi\). To each \(\pi \in \mathfrak{F}_n\), there is an associated standard \(L\)-function
\[
L(s, \pi) = \sum_n \frac{\lambda_\pi(n)}{N_n^s} = \prod_p \prod_{j=1}^n (1 - \alpha_{j,\pi}(p) Np^{-j})^{-1}, \quad \text{Re}(s) > 1.
\]
Here, \(n\) (resp. \(p\)) runs through the nonzero integral (resp. prime) ideals of \(O_F\), the ring of integers of \(F\). The \(L\)-function \(L(s, \pi)\) has an analytic continuation and functional equation similar to that of \(\zeta(s)\)
Consider the Rankin–Selberg $L$-function

\[ L(s, \pi \times \widetilde{\pi}) = \sum_n \frac{\lambda_{\pi \times \widetilde{\pi}}(n)}{Nn^s} = \prod_p \prod_{j=1}^{\alpha} \prod_{j' = 1}^{\alpha'} (1 - \alpha_{j,j',\pi \times \widetilde{\pi}}(p)Np^{-s})^{-1}, \quad \text{Re}(s) > 1, \]

which also has an analytic continuation and functional equation. If $p \nmid q_\pi$, then we have

\[ \{\alpha_{j,j',\pi \times \widetilde{\pi}}(p) : 1 \leq j, j' \leq n\} = \{\alpha_{j,\pi}(p)\alpha_{j',\widetilde{\pi}}(p) : 1 \leq j, j' \leq n\}. \]

We define the numbers $\Lambda_{\pi \times \widetilde{\pi}}(n)$ by the Dirichlet series identity

\[ (1.7) \quad -\frac{L'}{L}(s, \pi \times \widetilde{\pi}) = \sum_n \frac{\Lambda_{\pi \times \widetilde{\pi}}(n)}{Nn^s} = \sum_p \sum_{k=1}^{\infty} \sum_{1 \leq j, j' \leq n} \alpha_{j,j',\pi \times \widetilde{\pi}}(p)^k \log Np, \]

where convergence is absolute for $\text{Re}(s) > 1$. The numbers $\Lambda_{\pi \times \widetilde{\pi}}(n)$ are nonnegative [10, Lemma a], and they equal zero when $n$ is not a power of a prime ideal. Note that $\Lambda_{\pi \times \widetilde{\pi}}(p) = \lambda_{\pi \times \widetilde{\pi}}(p) \log Np$. Also, if $p \nmid q_\pi$, then $\Lambda_{\pi \times \widetilde{\pi}}(p) = |\lambda_{\pi}(p)|^2$. The generalized Ramanujan conjecture (which we abbreviate to GRC) predicts that $|\alpha_{j,\pi}(p)| = 1$ whenever $p \nmid q_\pi$ and $|\alpha_{j,\pi}(p)| \leq 1$ otherwise.

Until now, a “standard” zero-free region of the shape (1.4) was known for $L(s, \pi \times \widetilde{\pi})$ only when $\pi$ is self-dual, so that $\pi = \widetilde{\pi}$ (see Brumley [12, Theorem A.1]), and a hypothesis slightly weaker than GRC suffices to prove such a zero-free region when $\pi \neq \widetilde{\pi}$ [12]. A zero-free region of the shape (1.3) seems to be currently out of reach when $n > 1$ unless $\pi$ is induced, via automorphic induction, by a one-dimensional representation over a cyclic Galois extension of $F$.

When $F = \mathbb{Q}$, Akbary and Trudgian [1] proved that if $L(s, \pi \times \widetilde{\pi})$ has a zero-free region of the shape (1.4) and there exists a constant $0 < \alpha_\pi < \frac{1}{2}$ such that the upper bound

\[ (1.8) \quad \sum_{x/Nn \leq x+h} \Lambda_{\pi \times \widetilde{\pi}}(n) \ll_{\pi} h, \quad x^{1-\alpha_\pi} \leq h \leq x \]

holds, then one can prove a log-free zero density estimate for $L(s, \pi \times \widetilde{\pi})$ like (1.5). This leads to a result of the following shape: there exists a constant $0 < \delta_\pi < 1$ such that

\[ (1.9) \quad \sum_{x/Nn \leq x+h} \Lambda_{\pi \times \widetilde{\pi}}(n) \asymp_{\pi} h, \quad x^{1-\delta_\pi} \leq h \leq x. \]

One may think of the hypothesis (1.8) as an average form of GRC. If we assume GRC in full, then the contribution from the terms for which $n = p^k$ with $k \geq 2$ is negligible, and

\[ (1.10) \quad \sum_{x/Np \leq x+h} |\lambda_\pi(p)|^2 \log Np \ll_{\pi} h, \quad x^{1-\delta_\pi} \leq h \leq x. \]

Around the same time as Akbary and Trudgian’s work, Motohashi [25] unconditionally proved a refined version of (1.10) when $F = \mathbb{Q}$, $n = 2$, and $\pi$ corresponds to a level 1 Hecke–Maaß cusp form. Shortly afterward, Lemke Oliver and Thorner [17] proved that (1.9) holds for $n \geq 1$ and all $F$ without appealing to (1.8), regardless of whether $\pi \in \mathfrak{F}_n$ is self-dual, provided that there exists a (noncuspidal) automorphic representation $\pi \otimes \widetilde{\pi}$ of $\text{GL}_{m^2}(\mathbb{A}_F)$ such that $L(s, \pi \otimes \widetilde{\pi}) = L(s, \pi \times \widetilde{\pi})$. This is predicted by Langlands functoriality but is only known in special cases. For instance, this is not even known for an arbitrary $\pi \in \mathfrak{F}_3$. 
2. Main results

In this paper, we prove an unconditional proof of \[ (1.9) \] in a more precise form. Our result also exhibits effective dependence on the analytic conductor \( C(\pi) \) of \( \pi \) (see \[ (3.3) \]) in the spirit of Linnik’s bound on the least prime in an arithmetic progression \[ [19] \].

**Theorem 2.1.** Let \( \pi \in \mathfrak{F}_n \). There exist positive, absolute, and effectively computable constants \( c_1, c_2, c_3, c_4, c_5, \) and \( c_6 \) such that the following are true.

1. The Rankin–Selberg \( L \)-function \( L(s, \pi \times \tilde{\pi}) \) has at most one zero in the region

\[
\Re(s) \geq 1 - \frac{c_1}{\log(C(\pi)n(|\Im(s)| + e^{\omega_2[F:Q]})).
\]

If such an exceptional zero \( \beta_1 \) exists, then it must be real and simple and satisfy \( \beta_1 \leq 1 - C(\pi)^{-c_2n} \).

2. Let \( A \geq c_3, \log \log C(\pi) \geq 4c_4n[F : Q]^2, \) and \( x \geq C(\pi)^{c_5A^2n^3[F : Q]|\log(en[F : Q])} \). If

\[
\delta = \frac{1}{16Ae^{2n^2[F : Q]\log(en[F : Q])}}
\]

and \( x^{1-\delta} \leq h \leq x \), then

\[
(2.1) \sum_{x < N \leq x + h} \Lambda_{\pi \times \tilde{\pi}}(n) = \begin{cases} 
\delta h(1 - \xi^{\beta_1 - 1})(1 + O(e^{-c_6A})) & \text{if } \beta_1 \text{ exists}, \\
h(1 + O(e^{-c_6A})) & \text{otherwise}.
\end{cases}
\]

The implied constant is absolute, and \( \xi \in [x, x+h] \) satisfies \( (x+h)^{\beta_1} - x^{\beta_1} = \beta_1 h \xi^{\beta_1 - 1} \).

**Remark.** Our assumed lower bound on \( C(\pi) \) simplifies several aspects of the proof. It can be removed with additional effort, but the dependence on \( n \) and \( [F : Q] \) will change.

Note that since \( \Lambda_{\pi \times \tilde{\pi}}(n) \geq 0 \) for all \( n \) and \( \Lambda_{\pi \times \tilde{\pi}}(p) = |\lambda_{\pi}(p)|^2 \log Np \) for \( p \nmid q_\pi \), Theorem 2.1 (along with the Luo–Rudnick–Sarnak bound \[ (3.3) \] to handle the \( p \nmid q_\pi \)) implies the bound

\[
\sum_{x < N \leq x + h} |\lambda_{\pi}(p)|^2 \log Np \leq \begin{cases} 
\delta h(1 - \xi^{\beta_1 - 1})(1 + O(e^{-c_6A})) & \text{if } \beta_1 \text{ exists}, \\
h(1 + O(e^{-c_6A})) & \text{otherwise}
\end{cases}
\]

under the hypotheses of Theorem 2.1. We do not have the corresponding lower bound because we cannot rule out the possibility that the contribution from prime powers is \( \gg h \) due to insufficient progress toward GRC. In contexts where a prime power contribution is expected to be small but GRC is not yet known, it often suffices to establish the “Hypothesis H” of Rudnick and Sarnak \[ [27] \], which asserts that for any fixed \( k \geq 2 \), we have

\[
\sum_{p} |\Lambda_{\pi}(p^k)|^2 Np^{-k} < \infty.
\]

(The original hypothesis is stated over \( \mathbb{Q} \), but the extension to number fields incurs no complications.) Hypothesis H is known when \( \pi \in \mathfrak{F}_n \) and \( 1 \leq n \leq 4 \), along with a few other special cases \[ [15, 27, 30] \]. While Hypothesis H on its own is not enough to ensure that the contributions from higher prime powers in \[ (2.1) \] are negligible, the progress toward Langlands functoriality that leads to proofs of Hypothesis H when \( n \leq 4 \) also leads to the following strong form of the Hoheisel phenomenon for \( \pi \in \mathfrak{F}_n \) with \( n \in \{1, 2, 3, 4\} \).
**Theorem 2.2.** Let $n \in \{1, 2, 3, 4\}$ and $\pi \in \mathcal{F}_n$. With the notation and hypotheses of Theorem 2.1, we have

$$
\sum_{x < N_0 \leq x + h} |\lambda_\pi(p)|^2 \log Np = \begin{cases} 
h(1 - \xi^{\beta_1 - 1})(1 + O(e^{-c_\pi A})) & \text{if } \beta_1 \text{ exists}, \\
h(1 + O(e^{-c_\pi A})) & \text{otherwise}.
\end{cases}
$$

**Remark.** Taking $n = 2$ and $F = \mathbb{Q}$, we recover Motohashi’s result in [25]. Also, if $n \geq 1$ and $\pi \in \mathcal{F}_n$ satisfies the averaged form of GRC in (7.2) below, then Theorem 2.2 will hold for $\pi$. However, the implied constant will depend on $n$ in accordance with the implied constant in (7.2).

As in [11, 17, 23, 25], one must have a standard zero-free region and a log-free zero density estimate for $L(s, \pi \times \overline{\pi})$. Our log-free zero density estimate is proved using the ideas in Soundararajan and Thorner [28]. However, an unconditional standard zero-free region for $L(s, \pi \times \overline{\pi})$ has not yet appeared in the literature. This is due to the fact that most previous proofs of a standard zero-free region for an $L$-function $L(s, \Pi)$, including [13, Proof of Theorem 5.10] (which is based on [9, Appendix]), have required that the Rankin–Selberg $L$-functions $L(s, \Pi \times \Pi)$, $L(s, \Pi \times \overline{\Pi})$, and $L(s, \Pi \times \overline{\Pi})$ exist, and this is not yet known to be the case when $\Pi$ is the Rankin–Selberg convolution $\pi \times \overline{\pi}$ except when $n \in \{1, 2\}$. The only exception is the aforementioned work of Brumley when $\pi$ is self-dual; this avoids these requirements but is contingent on the assumption that $\pi$ is self-dual. Humphries [12] recently proved the existence of a constant $c_\pi > 0$ (depending at most on $\pi$) such that if $|\alpha_{j, \pi}(p)| \leq 1$ for almost all $p$, then $L(s, \pi \times \overline{\pi})$ has a zero-free region of the shape

$$
\text{Re}(s) \geq 1 - \frac{c_\pi}{\log(|\text{Im}(s)| + 1)}, \quad \text{Im}(s) \neq 0.
$$

Our work addresses the remaining cases using the fact that $\pi \times \overline{\pi}$ is self-dual even if $\pi$ is not.

As part of our proofs, we supply an unconditional standard zero-free region for $L(s, \pi \times \overline{\pi})$ (apart from a possible Landau–Siegel zero) with good uniformity in the analytic conductor. In order to ensure that our results are completely effective (even if a Landau–Siegel zero exists), we also prove a uniform version of Deuring and Heilbronn’s observation that Landau–Siegel zeroes tend to repel other zeroes away from the line $\text{Re}(s) = 1$.

### 3. Properties of $L$-functions

We recall some standard facts about $L$-functions arising from automorphic representations and their Rankin–Selberg convolutions; see [2, 11, 14, 22, 23].

#### 3.1. Standard $L$-functions

Let $\pi = \bigotimes_p \pi_p \in \mathcal{F}_n$ be a cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$. Let $q_\pi$ be the conductor of $\pi$. The local standard $L$-function $L(s, \pi_p)$ at a prime ideal $p$ is defined in terms of the Satake parameters \{$\alpha_{1, \pi}(p), \ldots, \alpha_{n, \pi}(p)$\} by

$$
L(s, \pi_p) = \prod_{j=1}^n \left(1 - \frac{\alpha_{j, \pi}(p)}{Np^s}\right)^{-1} = \sum_{k=0}^{\infty} \frac{\lambda_{\pi}(p^k)}{Np^{ks}}.
$$

We have $\alpha_{j, \pi}(p) \neq 0$ for all $j$ whenever $p \nmid q_\pi$, whereas it may be the case that $\alpha_{j, \pi}(p) = 0$ for at least one $j$ when $p \mid q_\pi$. The standard $L$-function $L(s, \pi)$ associated to $\pi$ is of the form

$$
L(s, \pi) = \prod_p L(s, \pi_p) = \sum_n \frac{\lambda_{\pi}(n)}{Nn^s}.
$$
The Euler product and Dirichlet series converge absolutely when \( \text{Re}(s) > 1 \).

At each archimedean place \( v \) of \( F \), there are \( n \) Langlands parameters \( \mu_{j,\pi}(v) \in \mathbb{C} \), from which we define

\[
L(s, \pi_{\infty}) = \prod_v \prod_{j=1}^n \Gamma_v(s + \mu_{j,\pi}(v)), \quad \Gamma_v(s) := \begin{cases} 
\pi^{-s/2}\Gamma(s/2) & \text{if } F_v = \mathbb{R}, \\
2(2\pi)^{-s}\Gamma(s) & \text{if } F_v = \mathbb{C}.
\end{cases}
\]

Luo, Rudnick, and Sarnak [21] and Mueller and Speh [26] proved the uniform bounds

\[
|\alpha_{j,\pi}(p)| \leq Np^{\theta_n} \quad \text{and} \quad \text{Re}(\mu_{j,\pi}(v)) \geq -\theta_n, \quad \theta_n = \frac{1}{2} - \frac{1}{n^2 + 1}
\]

The generalized Selberg eigenvalue conjecture and GRC assert that we have \( \theta_n = 0 \) in (3.2).

Let \( \tilde{\pi} \in \mathcal{F}_n \) be the cuspidal automorphic representation contragredient to \( \pi \). We have \( q_\pi = q_{\tilde{\pi}} \), and for each \( p \mid q_\pi \), we have the equalities of sets \( \{\alpha_{j,\tilde{\pi}}(p)\} = \{\alpha_{j,\pi}(p)\} \).

The completed standard \( L \)-function

\[
\Lambda(s, \pi) = (D_F^n q_{\pi})^{s/2} L(s, \pi) L(s, \pi_{\infty})
\]

is entire of order 1, and there exists a complex number \( \varepsilon(\pi) \) of modulus 1 such that for all \( s \in \mathbb{C} \), we have the functional equation \( \Lambda(s, \pi) = \varepsilon(\pi)\Lambda(1 - s, \tilde{\pi}) \).

Let \( d(v) = 1 \) if \( F_v = \mathbb{R} \) and \( d(v) = 2 \) if \( F_v = \mathbb{C} \). We define the analytic conductor of \( \pi \) to be

\[
C(\pi, t) := D_F^n q_{\pi} \prod_v \prod_{j=1}^n (e + |it + \mu_{j,\pi}(v)|^{d(v)}), \quad C(\pi) := C(\pi, 0).
\]

### 3.2. Rankin–Selberg \( L \)-functions.

Let \( \pi \in \mathcal{F}_n \). The local Rankin–Selberg \( L \)-function \( L(s, \pi_p \times \tilde{\pi}_p) \) is defined at a prime ideal \( p \) by

\[
L(s, \pi_p \times \tilde{\pi}_p) = \prod_{j=1}^n \prod_{j'=1}^n (1 - \alpha_{j,j',\pi_p \times \tilde{\pi}_p}(p)) Np^{-s} = \sum_{k=0}^{\infty} \frac{\lambda_{\pi \times \tilde{\pi}}(p^k)}{Np^{k+s}}.
\]

for suitable Satake parameters \( \alpha_{j,j',\pi_p \times \tilde{\pi}_p}(p) \). If \( p \nmid q_\pi \), then we have the equality of sets

\[
\{\alpha_{j,j',\pi_p \times \tilde{\pi}_p}(p)\} = \{\alpha_{j,\pi}(p)\alpha_{j',\tilde{\pi}(p)}\} = \{\alpha_{j,\pi}(p)\alpha_{j',\pi}(p)\}.
\]

See [28] Appendix for a complete description of the numbers \( \alpha_{j,j',\pi_p \times \tilde{\pi}_p}(p) \) even when \( p \mid q_\pi \).

The Rankin-Selberg \( L \)-function \( L(s, \pi \times \tilde{\pi}) \) associated to \( \pi \) and \( \tilde{\pi} \) is of the form

\[
L(s, \pi \times \tilde{\pi}) = \prod_p L(s, \pi_p \times \tilde{\pi}_p) = \sum_n \frac{\lambda_{\pi \times \tilde{\pi}}(n)}{Nn^s}.
\]

Bushnell and Henniart [3] proved that the conductor \( q_{\pi \times \tilde{\pi}} \) divides \( q_\pi^{2n-1} \). At an archimedean place \( v \) of \( F \), there are \( \frac{n^2}{2} \) complex Langlands parameters \( \mu_{j,j',\pi \times \tilde{\pi}}(v) \), from which we define

\[
L(s, \pi_{\infty} \times \tilde{\pi}_{\infty}) = \prod_v \prod_{j=1}^n \prod_{j'=1}^n \Gamma_v(s + \mu_{j,j',\pi \times \tilde{\pi}}(v)).
\]

If \( \pi \) is unramified at \( v \), then we have the equality of sets

\[
\{\mu_{j,j',\pi \times \tilde{\pi}}(v)\} = \{\mu_{j,\pi}(v) + \mu_{j',\tilde{\pi}(v)}\}.
\]

Using the explicit descriptions of \( \alpha_{j,j',\pi_p \times \tilde{\pi}_p}(p) \) and \( \mu_{j,j',\pi \times \tilde{\pi}}(v) \) in [12, 28], one sees that

\[
|\alpha_{j,j',\pi_p \times \tilde{\pi}_p}(p)| \leq Np^{2\theta_n}, \quad \text{Re}(\mu_{j,j',\pi \times \tilde{\pi}}(v)) \geq -2\theta_n.
\]
The completed Rankin–Selberg $L$-function
\[
\Lambda(s, \pi \times \overline{\pi}) = (D_F^\nu Nq_{\pi \times \overline{\pi}})^{s/2} L(s, \pi \times \overline{\pi}) L(s, \pi_\infty \times \overline{\pi}_\infty)
\]
is entire of order 1, and there exists a number $\varepsilon(\pi \times \overline{\pi}) \in \{\pm 1\}$ such that $\Lambda(s, \pi \times \overline{\pi})$ satisfies the functional equation
\[
\Lambda(s, \pi \times \overline{\pi}) = \varepsilon(\pi \times \overline{\pi}) \Lambda(1 - s, \pi \times \overline{\pi}).
\]

As with $L(s, \pi)$, we define the analytic conductor
\[
C(\pi \times \overline{\pi}, t) := D_F^\nu Nq_{\pi \times \overline{\pi}} \prod_{v} \prod_{j,j' = 1}^{n} (e + |it + \mu_{j,j',\pi \times \overline{\pi}}(v)|^{d(v)}),
\]
\[
C(\pi \times \overline{\pi}) := C(\pi \times \overline{\pi}, 0).
\]
The work of Bushnell and Henniart [4] and the proofs in Brumley [12, Appendix] yields
\[
\text{(3.7) } C(\pi \times \overline{\pi}, t) \leq C(\pi \times \overline{\pi})(e + |t|)^{|F:Q| n^2}, \quad C(\pi \times \overline{\pi}) \leq e^{O(n)} C(\pi)^{2n}.
\]

4. A Brun–Titchmarsh bound

We require a Brun–Titchmarsh type bound for the coefficients $\Lambda_{\pi \times \overline{\pi}}(n)$. In [28, Theorem 2.4], it is shown that if $F = \mathbb{Q}$, $\pi \in \mathfrak{F}_m$, $x \gg_m C(\pi \times \overline{\pi})^{36m^2}$, and $1 \leq T \leq x^{\frac{1}{4m}}$, then
\[
\sum_{x < n \leq x e^{1/T}} \Lambda_{\pi \times \overline{\pi}}(n) \ll_m \frac{x}{T}.
\]
In this section, we prove a field-uniform version of this bound for $\pi \in \mathfrak{F}_n$ where the dependence of the implied constant on $n$ and $[F : \mathbb{Q}]$ is made clear. In [28], the implied constant was not pertinent, but here, the dependence of the implied constant on $n$ and $[F : \mathbb{Q}]$ impacts exactly how large we may take $\delta$ in Theorem 2.1.

**Proposition 4.1.** Let $\pi \in \mathfrak{F}_n$. Suppose that $\log \log C(\pi \times \overline{\pi}) \gg n^4 [F : \mathbb{Q}]^2$ with a sufficiently large implied constant. If
\[
x \geq e^{O(n^4 [F : \mathbb{Q}]^2)} C(\pi \times \overline{\pi})^{32n^2 [F : \mathbb{Q}]}, \quad \text{and} \quad 1 \leq T \leq x^{\frac{1}{16n^4 [F : \mathbb{Q}]^2}},
\]
then
\[
\sum_{x < n \leq x e^{1/T}} \Lambda_{\pi \times \overline{\pi}}(n) \ll n^2 [F : \mathbb{Q}] \frac{x}{T}.
\]

Much like the work in [28, Section 6], we use the Selberg sieve. The primary difference is the attention we pay to the dependence of implied constants on $n$ and $[F : \mathbb{Q}]$. We begin with an effective bound for $L(s, \pi \times \overline{\pi})$ with the dependence on $n$ and $[F : \mathbb{Q}]$ made clear.

**Lemma 4.2.** Let $\pi \in \mathfrak{F}_n$. If $\varepsilon > 0$ and
\[
\log \log C(\pi \times \overline{\pi}) \gg \frac{n^2 [F : \mathbb{Q}]}{\varepsilon}
\]
with a sufficiently large implied constant, then for $\frac{1}{2} \leq \sigma \leq 1$ and $t \in \mathbb{R}$, we have the bound
\[
\lim_{\sigma' \to \sigma^+} |(1 - \sigma') L(\sigma' + it, \pi \times \overline{\pi})| \ll e^{O(n^2 [F : \mathbb{Q}] \frac{1}{1 - \sigma})} C(\pi \times \overline{\pi}) (1 + |t|)^{n^2 [F : \mathbb{Q}] \frac{1}{1 - \sigma} + \varepsilon}.
\]
Proof. It follows from the work of Li [18, Theorem 2] (with straightforward changes in order to apply to arbitrary number fields $F$) that there exists an absolute constant $c_7 > 0$ such that
\[ \lim_{\sigma' \to \sigma^+} (1 - \sigma') L(\sigma', \pi \times \overline{\pi}) \ll \exp \left( c_7 n^2 [F : \mathbb{Q}] \frac{\log C(\pi \times \overline{\pi})}{\log \log C(\pi \times \overline{\pi})} \right), \quad 1 \leq \sigma \leq 3. \]
If $\log \log C(\pi \times \overline{\pi}) \geq 2c_7 n^2 [F : \mathbb{Q}]/\varepsilon$, then
\[ \exp \left( c_7 n^2 [F : \mathbb{Q}] \frac{\log C(\pi \times \overline{\pi})}{\log \log C(\pi \times \overline{\pi})} \right) \leq C(\pi \times \overline{\pi})^{\frac{3}{4}}. \]
It follows from work of Soundararajan and Thorner [28, Theorem 1.1 with $\delta = 0$] (with straightforward changes in order to apply to arbitrary number fields $F$) and the above analysis that
\[ |L(\frac{1}{2}, \pi \times \overline{\pi})| \ll e^{O(n^2 [F : \mathbb{Q}] |C(\pi \times \overline{\pi})|^{1/4} + \varepsilon)} \]
By the Phragmén–Lindelöf principle, we have the bound
\[ \lim_{\sigma' \to \sigma^+} \left| (1 - \sigma') L(\sigma', \pi \times \overline{\pi}) \right| \leq e^{O(n^2 [F : \mathbb{Q}] (1 - \sigma) \log \log \log C(\pi \times \overline{\pi}))} \]
Since our results are uniform in the analytic conductor $C(\pi \times \overline{\pi})$, and hence in the spectral parameters $\mu_{j,j',\pi \times \overline{\pi}}(v)$, we can shift all of the spectral parameters by $it$, thus proving that
\[ \lim_{\sigma' \to \sigma^+} \left| \frac{1 - \sigma' - it}{1 + \sigma' + it} L(\sigma' + it, \pi \times \overline{\pi}) \right| \leq e^{O(n^2 [F : \mathbb{Q}] (1 - \sigma) \log \log \log C(\pi \times \overline{\pi}, t))} \]
We conclude the desired result by invoking (3.7).

For a squarefree integral ideal $\mathfrak{d}$ of $\mathcal{O}_F$, define
\[ g_\mathfrak{d}(s, \pi \times \overline{\pi}) := \prod_{p \mid \mathfrak{d}} \left( 1 - L(s, \pi_p \times \overline{\pi}_p)^{-1} \right), \quad g(\mathfrak{d}) := g_\mathfrak{d}(1, \pi \times \overline{\pi}). \]
We require some estimates for $g_\mathfrak{d}(s, \pi \times \overline{\pi})$ and $g(\mathfrak{d})$.

**Lemma 4.3.** Suppose that $\log \log C(\pi \times \overline{\pi}) \gg n^4 [F : \mathbb{Q}]^2$ with a sufficiently large implied constant. Let $\mathfrak{d} \neq \mathcal{O}_F$ be a squarefree integral ideal. We have $0 < g(\mathfrak{d}) < 1$, $g(\mathcal{O}_F) = 1$, and
\[ |g_\mathfrak{d}(s, \pi \times \overline{\pi})| \leq C(\pi \times \overline{\pi})^{-\frac{1}{8n^2 [F : \mathbb{Q}]} \mathcal{N} \mathfrak{d}^{\frac{1}{4}}}, \quad \text{Re}(s) = 1 - \frac{1}{2n^2 [F : \mathbb{Q}]} \]
**Proof.** The bound on $g(\mathfrak{d})$ follows immediately from (3.6). Also, by (3.6) and the fact that $\mathcal{N} \mathfrak{p} \geq 2$ for all prime ideals $\mathfrak{p}$, we have the bound
\[ |g_\mathfrak{d}(1 - \frac{1}{2n^2 [F : \mathbb{Q}]} + it, \pi \times \overline{\pi})| \leq \prod_{\mathfrak{p} \mid \mathfrak{d}} \left( 1 + \left( 1 + \frac{\mathcal{N} \mathfrak{p}^{\frac{1}{2} - \frac{n^2 + 1}{n^2 - \frac{1}{2n^2 [F : \mathbb{Q}]}})}{\mathcal{N} \mathfrak{p}^{\frac{1}{2n^2 [F : \mathbb{Q}]}}} \right)^n \right) \leq \prod_{\mathfrak{p} \mid \mathfrak{d}} 2n^{2 + 2}. \]
The proof of [28, Lemma 1.13b] shows that for all $\varepsilon > 0$, the number of distinct prime ideal divisors of $\mathfrak{d}$ is bounded by $6e^{2/\varepsilon} [F : \mathbb{Q}] + \varepsilon \log \mathcal{N} \mathfrak{d}$. We apply this with $\varepsilon = \frac{1}{4(n^2 + 2)}$ to bound the above display by
\[ \mathcal{N} \mathfrak{d}^{\frac{1}{4}} e^{6e^{2/(n^2 + 2)}} (n^2 + 2)|F : \mathbb{Q}| \]
This is bounded as claimed when the implied constant for the lower bound on $C(\pi \times \overline{\pi})$ is made sufficiently large. \qed
Let $\Phi$ be a smooth nonnegative function supported in $(-2, 2)$, and let
\[ (4.1) \quad \tilde{\Phi}(s) = \int_{-\infty}^{\infty} \Phi(y) e^{sy} dy. \]
Then $\tilde{\Phi}(s)$ is entire, and for any integer $k \geq 1$, integration by parts yields
\[ (4.2) \quad |\tilde{\Phi}(s)| \leq \Phi_k e^{2|\text{Re}(s)|} |s|^{-k}. \]
Let $T \geq 1$. By Mellin inversion, we have
\[ \Phi(T \log x) = \frac{1}{2\pi i T} \int_{c-i\infty}^{c+i\infty} \tilde{\Phi}(s/T) x^{-s} ds \]
for any $x > 0$ and $c \in \mathbb{R}$.

**Lemma 4.4.** Let $\pi \in \mathfrak{F}_n$, and let $\mathfrak{d}$ be a squarefree integral ideal of $\mathcal{O}_F$. Let $x, T \geq 1$, and let $\log \log C(\pi \times \tilde{\pi}) \gg n^4[F:Q]^2$ with a sufficiently large implied constant. We have
\[ \sum_{\mathfrak{d} | n} \lambda_{\pi \times \tilde{\pi}}(n) \Phi\left(T \log \frac{Nn}{x}\right) - \kappa g(\mathfrak{d}) \frac{x}{T} \Phi(1/T) \ll x^{1 - \frac{1}{2n^2[F:Q]} T^{\frac{3}{2}} C(\pi \times \tilde{\pi})^{\frac{1}{2n^2[F:Q]} N \mathfrak{d}}} \]
where $\kappa > 0$ is the residue at $s = 1$ of $L(s, \pi \times \tilde{\pi})$.

**Proof.** The quantity to be estimated equals
\[ \frac{1}{2\pi i T} \int_{1 - \frac{1}{2n^2[F:Q]} - i\infty}^{1 - \frac{1}{2n^2[F:Q]} + i\infty} L(s, \pi \times \tilde{\pi}) \Phi(s/T) x^s g_\mathfrak{d}(s, \pi \times \tilde{\pi}) ds. \]
By Lemma 4.2 with $\varepsilon = \frac{1}{8n^2[F:Q]}$, Lemma 4.3 and (4.2) with $k = 0$ and $2$, this is
\[ \ll x^{1 - \frac{1}{2n^2[F:Q]} T^{\frac{3}{2}} C(\pi \times \tilde{\pi})^{\frac{1}{2n^2[F:Q]} N \mathfrak{d}}} \int_{-\infty}^{\infty} (1 + |t|)^{\frac{3}{8}} \min \left\{ 1, \frac{T^2}{(1 + |t|)^2} \right\} dt, \]
which is bounded as claimed. \hfill $\Box$

**Lemma 4.5.** Suppose that $\log \log C(\pi \times \tilde{\pi}) \gg n^4[F:Q]^2$ and $z \geq e^{O(n^2[F:Q])} C(\pi \times \tilde{\pi})^2$, each with a sufficiently large implied constant. If $x > 0$ and $T \geq 1$, then
\[ \sum_{p | n \Rightarrow Np > z} \lambda_{\pi \times \tilde{\pi}}(n) \Phi\left(T \log \frac{Nn}{x}\right) \leq \frac{3x}{T \log z} \Phi(1/T) + O\left(x^{1 - \frac{1}{2n^2[F:Q]} T^{\frac{3}{2}} C(\pi \times \tilde{\pi})^{\frac{1}{2n^2[F:Q]} z^5}} \right). \]

**Proof.** By proceeding as in the formulation of the Selberg sieve in [5] Theorem 7.1 (see also [29] Lemma 3.6) for a treatment with field uniformity), we find using Lemma 4.4 that
\[ \sum_{p | n \Rightarrow Np > z} \lambda_{\pi \times \tilde{\pi}}(n) \Phi\left(T \log \frac{Nn}{x}\right) \leq \exp \left( \frac{x}{T} \Phi(1/T) \left( \sum_{p | \prod_{Np \leq z} \frac{g(p)}{1 - g(p)}} \prod_{Np \leq z} \frac{g(p)}{1 - g(p)} \right) \right)
+ O\left(x^{1 - \frac{1}{2n^2[F:Q]} T^{\frac{3}{2}} C(\pi \times \tilde{\pi})^{\frac{1}{2n^2[F:Q]} \sum_{Np_1, Np_2 \leq z} N(\text{lcm}(\mathfrak{d}_1, \mathfrak{d}_2))^{\frac{1}{2}}} \right). \]
We use the bound
\[ \sum_{Nn \leq z} 1 \leq (2/\varepsilon)^{[F:Q] z^{1+\varepsilon}}, \quad z > 0, \quad 0 < \varepsilon < 1 \]
in [29, Lemma 1.12a] to bound the sum over \( \mathfrak{d}_1 \) and \( \mathfrak{d}_2 \) by \( O(e^O([F:Q])z^5) \). By the definitions of \( g(\mathfrak{d}) \) and \( L(s, \pi_p \times \pi_{p'}) \), we have the lower bound

\[
\sum_{\mathfrak{d} \mid \prod_{Np \leq z} p \mid \mathfrak{d}} \prod_{\mathfrak{d} \mid p \mid \mathfrak{d}} \frac{g(p)}{1 - g(p)} \geq \sum_{Nn \leq z} \prod_{j=1}^{\infty} \lambda_{\pi \times \pi}(p_j^j) \geq \sum_{Nn \leq z} \frac{\lambda_{\pi \times \pi}(n)}{Nn} \geq 1 + \sum_{\sqrt{z} < Nn \leq z} \frac{\lambda_{\pi \times \pi}(n)}{Nn}.
\]

Let \( 0 < \varepsilon_0 < \frac{1}{10} \), and let \( \Phi_1 \) be a fixed nonnegative smooth function supported on \([0, 1]\), with \( \Phi_1(t) = 1 \) for \( \varepsilon_0 < t < 1 - \varepsilon_0 \) and \( \Phi_1(t) \leq 1 \) for \( 0 \leq t \leq 1 \). By taking \( \mathfrak{d} = \mathcal{O}_F \) and \( T = 1 \) in Lemma \([4.4]\), we find that if \( y \geq 1 \), then

\[
\sum_{y \leq Nn \leq ey} \frac{\lambda_{\pi \times \pi}(n)}{Nn} \geq \frac{1}{ey} \sum_n \lambda_{\pi \times \pi}(n) \Phi_1 \left( \frac{\log Nn}{y} \right)
\]

\[
= \frac{e - 1 + O(\varepsilon_0)}{e} \kappa + O(y^{-\frac{1}{2n^2[F:Q]}C(\pi \times \pi)^{\frac{1}{2n^2[F:Q]}}} \lambda n \leq z + O(z^{-\frac{1}{2n^2[F:Q]}C(\pi \times \pi)^{\frac{1}{2n^2[F:Q]}}})
\]

once \( \varepsilon_0 \) is suitably small. If \( z \geq e^{O(n^2[F:Q])}C(\pi \times \pi)^2 \), then since \( \kappa > 0 \), it follows that

\[
\kappa \frac{x}{T} \Phi(1/T) \left( \sum_{\mathfrak{d} \mid \prod_{Np \leq z} p \mid \mathfrak{d}} \prod_{\mathfrak{d} \mid p \mid \mathfrak{d}} \frac{g(p)}{1 - g(p)} \right)^{-1} \leq \frac{x}{T} \Phi(1/T) \frac{3\kappa}{1 + \kappa \log z} \leq \frac{3x}{T \log z} \Phi(1/T).
\]

The result follows once we account for our lower bound for \( C(\pi \times \pi) \).

**Proof of Proposition 4.4**: We first fix \( \Phi \), requiring that \( 0 \leq \Phi(y) \leq 1 \) for all \( y \) and that \( \Phi(y) = 1 \) for \( y \in [0, 1] \). By \([4.1]\), we see that \( |\Phi(1/T)| \ll \Phi 1 \). We choose

\[
x \geq e^{O(n^2[F:Q])}C(\pi \times \pi)^{32n^2[F:Q]}, \quad 1 \leq T \leq x^{-\frac{1}{16n^2[F:Q]}} = z.
\]

The sum in Lemma \([4.5]\) includes all prime powers \( p^k \) with \( Np^k \in [x, xe^{1/T}] \) with \( Np > x^{1/(16n^2[F:Q])} \), hence

\[
\sum_{x \leq Np^k \leq xe^{1/T} \atop k \leq 16n^2[F:Q]} \lambda_{\pi \times \pi}(p^k) \ll n^2[F:Q] x \frac{x}{T \log x}.
\]

For a given \( p \), we compare coefficients in the formal identity

\[
\exp \left( \sum_{k=1}^{\infty} \frac{\Lambda_{\pi \times \pi}(p^k)}{k \log Np} X^k \right) = 1 + \sum_{k=1}^{\infty} \lambda_{\pi \times \pi}(p^k) X^k
\]

and use the nonnegativity of \( \Lambda_{\pi \times \pi}(p^k) \) and \( \lambda_{\pi \times \pi}(p^k) \) to deduce the bound

\[
\lambda_{\pi \times \pi}(p^k) \geq \frac{\Lambda_{\pi \times \pi}(p^k)}{k \log Np},
\]

hence

\[
\sum_{x \leq Np^k \leq xe^{1/T} \atop k \leq 16n^2[F:Q]} \Lambda_{\pi \times \pi}(p^k) \ll n^2[F:Q] x \frac{x}{T}.
\]
To handle the contribution when \( k > 16n^2[F : \mathbb{Q}] \), we appeal to the bound \( |\Lambda_{\pi \times \overline{\pi}}(p^k)| \leq n^2 Np^{\left(1 - \frac{2}{n^2 + 1}\right)k} \log Np \), which follows from (3.2) and (3.6). This, along with the trivial estimate
\[
\sum_{Np \leq x} 1 \ll [F : \mathbb{Q}] \frac{x}{\log x}
\]
for \( x \geq 3 \), implies that
\[
\sum_{\substack{x < Np^k \leq xe^{1/T} \\ k > 16n^2[F : \mathbb{Q}]}} \Lambda_{\pi \times \overline{\pi}}(p^k) \leq n^2 x^{1 - \frac{2}{n^2 + 1}} \sum_{\substack{Np^k \leq ex \\ k > 16n^2[F : \mathbb{Q}]}} \log Np
\]
\[
\ll n^2 [F : \mathbb{Q}] x^{1 - \frac{2}{n^2 + 1} + \frac{1}{16n^2[F : \mathbb{Q}]}} \log x \ll n^2 [F : \mathbb{Q}] \frac{x}{T}.
\]
Since \( \Lambda_{\pi \times \overline{\pi}}(n) = 0 \) whenever \( n \) is not a power of a prime ideal, this concludes our proof. □

5. Zeroes of \( L(s, \pi \times \overline{\pi}) \)

We require three results on the distribution of zeroes of \( L(s, \pi \times \overline{\pi}) \), which are analogous to the key ingredients in Linnik’s bound on the least prime in an arithmetic progression: a standard zero-free region with an effective bound on a possible Landau–Siegel zero, a log-free zero density estimate, and a quantification of the Deuring–Heilbronn zero repulsion phenomenon for Landau–Siegel zeroes.

5.1. **A standard zero-free region for** \( L(s, \pi \times \overline{\pi}) \). Let \( \pi \in \mathfrak{F}_n \). In [12], Humphries proved that if \( |\alpha_{j,\pi}(p)| \leq 1 \) (uniformly in \( j \)) for all except a density zero subset of prime ideals \( p \), then there exists a constant \( c_\pi > 0 \) dependent on \( \pi \) (and hence also on \( n \)) such that \( L(s, \pi \times \overline{\pi}) \neq 0 \) in the region
\[
\text{Re}(s) \geq 1 - \frac{c_\pi}{\log(|\text{Im}(s)| + e)}, \quad \text{Im}(s) \neq 0.
\]
This extended upon work of Goldfeld and Li [7], who proved a weaker zero-free region via a different method under the additional conditions that \( F = \mathbb{Q} \) and that \( \pi \) is everywhere unramified. Here, we prove an unconditional refinement with improved uniformity in \( \pi \).

**Theorem 5.1.** Let \( \pi \in \mathfrak{F}_n \). There exists an absolute constant \( c_8 > 0 \) such that the Rankin–Selberg \( L \)-function \( L(s, \pi \times \overline{\pi}) \) is nonvanishing in the region
\[
\text{Re}(s) \geq 1 - \frac{c_8}{\log(C(\pi \times \overline{\pi})(|\text{Im}(s)| + e)^{n^2[F : \mathbb{Q}]})}
\]
apart from at most one exceptional zero \( \beta_1 \). If \( \beta_1 \) exists, then it is both real and simple.

**Proof.** Let \( \rho = \beta + i\gamma \) be a zero of \( L(s, \pi \times \overline{\pi}) \) with \( \beta \geq 1/2 \) and \( \gamma \neq 0 \). We define
\[
\Pi := \pi \boxtimes \pi \otimes |\text{det}|^{i\gamma} \boxplus \pi \otimes |\text{det}|^{-i\gamma}.
\]
This is an isobaric (noncuspidal) representation of \( \text{GL}_{3n}(\mathbb{A}_F) \). The Rankin–Selberg \( L \)-function \( L(s, \Pi \times \overline{\Pi}) \) factorises as
\[
L(s, \pi \times \overline{\pi})^3 L(s + i\gamma, \pi \times \overline{\pi})^2 L(s - i\gamma, \pi \times \overline{\pi})^2 L(s + 2i\gamma, \pi \times \overline{\pi}) L(s - 2i\gamma, \pi \times \overline{\pi}).
\]
Since \( L(s, \pi \times \overline{\pi}) \) is meromorphic on \( \mathbb{C} \) with only a simple pole at \( s = 1 \), \( L(s, \Pi \times \overline{\Pi}) \) is a meromorphic function on \( \mathbb{C} \) with a triple pole at \( s = 1 \), double poles at \( s = 1 \pm i\gamma \), and simple poles at \( s = 1 \pm 2i\gamma \). Moreover, the functional equation for \( L(s, \pi \times \overline{\pi}) \) together with
the fact that $\pi \times \tilde{\pi}$ is self-dual (even if $\pi$ itself is not self-dual) implies that if $\rho$ is a zero of $L(s, \pi \times \tilde{\pi})$, then so is $\overline{\rho}$. Consequently, $s = \beta$ is a zero of $L(s, \Pi \times \tilde{\Pi})$ of order at least 4.

Define

$$\Lambda(s, \Pi \times \tilde{\Pi}) := \Lambda(s, \pi \times \tilde{\pi})^3 \Lambda(s + i\gamma, \pi \times \tilde{\pi})^2 \Lambda(s - i\gamma, \pi \times \tilde{\pi})^2 \Lambda(s + 2i\gamma, \pi \times \tilde{\pi}) \Lambda(s - 2i\gamma, \pi \times \tilde{\pi}).$$

Since $\Lambda(s, \Pi \times \tilde{\Pi})$ is entire of order 1 (regardless of whether $\gamma = 0$), it admits a Hadamard factorization

$$e^{a_{\Pi \times \tilde{\Pi}} + b_{\Pi \times \tilde{\Pi}} s} \prod_{L(\rho, \Pi \times \tilde{\Pi}) = 0} \left(1 - \frac{s}{\rho}\right)^{e^{s/\rho}}.$$ 

where $\rho$ runs through the nontrivial zeroes of $L(s, \Pi \times \tilde{\Pi})$. A standard calculation shows that

$$\sum_{\rho \neq 0, 1} \frac{1}{\sigma - \rho} = \text{Re}\left(\frac{L'(s, \Pi \times \tilde{\Pi})}{L(s, \Pi \times \tilde{\Pi})}\right) + \frac{3}{\sigma - 1} + \frac{3}{\sigma + i\gamma - 1} + \frac{1}{\sigma + 2i\gamma - 1} + \frac{1}{\sigma + 2i\gamma} + \frac{1}{\sigma - 2i\gamma - 1} + \frac{3}{\sigma - 2i\gamma} + \frac{2}{\sigma + i\gamma} + \frac{2}{\sigma - i\gamma} + \frac{2}{\sigma + 2i\gamma} + \frac{2}{\sigma - 2i\gamma - 1} + \frac{1}{\sigma + 2i\gamma - 1} + \frac{1}{\sigma - 2i\gamma} + \frac{1}{\sigma - 2i\gamma - 1} + 1\cdot \log(D_{\mathcal{F}}^{\text{reg}} N_{\Pi \times \tilde{\Pi}}) + \text{Re}\left(\frac{L'(s, \Pi \times \tilde{\Pi})}{L(s, \Pi \times \tilde{\Pi})}\right).$$

Restricting $\rho$ to the real zeroes $\beta > \frac{1}{2}$ of $L(s, \Pi \times \tilde{\Pi})$ and applying Stirling’s formula, we find that there exists an absolute and effectively computable constant $c_9 \geq 1$ such that

$$\sum_{\beta > \frac{1}{2}} \frac{1}{\sigma - \beta} \leq \text{Re}\left(\frac{L'(s, \Pi \times \tilde{\Pi})}{L(s, \Pi \times \tilde{\Pi})}\right) + \frac{3}{\sigma - 1} + \frac{2}{\sigma + i\gamma - 1} + \frac{2}{\sigma - i\gamma - 1} + \frac{1}{\sigma + 2i\gamma - 1} + \frac{1}{\sigma - 2i\gamma} + \frac{1}{\sigma - 2i\gamma - 1} + c_9 \log C(\Pi \times \tilde{\Pi}).$$

We now observe that if $\sigma > 1$, then

$$\frac{L'(s, \Pi \times \tilde{\Pi})}{L(s, \Pi \times \tilde{\Pi})} = -\sum_{n} \frac{\Lambda_{\pi \times \tilde{\pi}}(n)}{Nn^{\sigma}} (1 + 2\cos(\gamma \log Nn))^2 \leq 0.$$ 

This crucially relies on the nonnegativity of $\Lambda_{\pi \times \tilde{\pi}}(n)$, even if $\gcd(n, q_{\pi}) \neq \mathcal{O}_{\mathcal{F}}$ [10, Lemma 1]. Therefore, it follows that

$$\sum_{\beta > \frac{1}{2}} \frac{1}{\sigma - \beta} \leq \frac{3}{\sigma - 1} + \frac{4(\sigma - 1)}{(\sigma - 1)^2 + \gamma^2} + \frac{2(\sigma - 1)}{(\sigma - 1)^2 + 4\gamma^2} + c_9 \log C(\Pi \times \tilde{\Pi}).$$
By (3.7), we have the bound
\[
C(\Pi \times \overline{\Pi}) = C(\pi \times \overline{\pi})^3 C(\pi \times \overline{\pi}, \gamma)^2 C(\pi \times \overline{\pi}, -\gamma)^2 C(\pi \times \overline{\pi}, 2\gamma) C(\pi \times \overline{\pi}, -2\gamma)
\leq C(\pi \times \overline{\pi})^0 (2|\gamma| + e)^{6n^2[F:Q]},
\]
in which case
\[
\sum_{L(\beta, \Pi \times \overline{\Pi}) = 0} \frac{1}{\sigma - \beta} \leq \frac{3}{\sigma - 1} + \frac{4(\sigma - 1)}{(\sigma - 1)^2 + \gamma^2} + \frac{2(\sigma - 1)}{(\sigma - 1)^2 + 4\gamma^2} + 9c_9 \log(C(\pi \times \overline{\pi})(2|\gamma| + e)^{n^2[F:Q]}).
\]
(5.1)

Recall that $L(\beta + i\gamma, \pi \times \overline{\pi}) = 0$. By the discussion at the beginning of the proof, it follows that $\beta$ is a zero of $L(s, \Pi \times \overline{\Pi})$ with multiplicity at least 4. Therefore, by (5.1), the inequality
\[
\frac{4}{\sigma - \beta} \leq \frac{3}{\sigma - 1} + \frac{4(\sigma - 1)}{(\sigma - 1)^2 + \gamma^2} + \frac{2(\sigma - 1)}{(\sigma - 1)^2 + 4\gamma^2} + 9c_9 \log(C(\pi \times \overline{\pi})(2|\gamma| + e)^{n^2[F:Q]}).
\]
holds for all $1 < \sigma < \frac{3}{2}$. However, if $|\gamma|$ is less than a sufficiently small positive multiple of $1/\log C(\pi \times \overline{\pi})$, then for all $1 < \sigma < \frac{3}{2}$, (5.2) does not imply a nontrivial upper bound for $\beta$. By choosing
\[
\sigma = 1 + \frac{1}{28c_9 \log(C(\pi \times \overline{\pi})(2|\gamma| + e)^{n^2[F:Q]});}
\]
we ensure via (5.2) that
\[
\beta \leq 1 - \frac{1}{3108c_9 \log(C(\pi \times \overline{\pi})(2|\gamma| + e)^{n^2[F:Q]})}
\]
whenever
\[
|\gamma| \geq \frac{1}{7c_9 \log C(\pi \times \overline{\pi})}.
\]

To handle the case where $0 < |\gamma| < 1/(7c_9 \log C(\pi \times \overline{\pi}))$, we proceed as above, but with $L(s, \pi \times \overline{\pi})$ in place of $L(s, \Pi \times \overline{\Pi})$. As described above, if $L(\beta + i\gamma, \pi \times \overline{\pi}) = 0$, then $L(\beta + i\gamma, \pi \times \overline{\pi}) = 0$. By analysis that is essentially identical to before, we conclude that if $1 < \sigma < \frac{3}{2}$, $L(\beta + i\gamma, \pi \times \overline{\pi}) = 0$, $\beta > \frac{1}{2}$, and $0 < |\gamma| < 1/(c_9 \log C(\pi \times \overline{\pi}))$, then
\[
\sum_{L(\beta + i\gamma, \pi \times \overline{\pi}) = 0} \frac{\sigma - \beta}{(\sigma - \beta)^2 + \gamma^2} \leq \frac{1}{\sigma - 1} + c_9 \log C(\pi \times \overline{\pi}).
\]
(5.3)
If $\gamma \neq 0$ and $\beta + i\gamma$ is in the sum over zeroes in (5.3), then so is $\beta - i\gamma$. Therefore, we have
\[
2\frac{\sigma - \beta}{(\sigma - \beta)^2 + \gamma^2} \leq \frac{1}{\sigma - 1} + c_9 \log C(\pi \times \overline{\pi}).
\]
By choosing
\[
\sigma = 1 + \frac{1}{2c_9 \log C(\pi \times \overline{\pi})};
\]
we ensure that
\[
\beta \leq 1 - \frac{2\sqrt{1 - 9c_9^2\gamma^2(\log C(\pi \times \overline{\pi}))^2} - 1}{6c_9 \log C(\pi \times \overline{\pi})}.
\]
Our hypothesis that $0 < |\gamma| < 1/(7c_9 \log C(\pi \times \overline{\pi}))$ implies that

\begin{equation}
\beta \leq 1 - \frac{4\sqrt{10} - 7}{42c_9 \log C(\pi \times \overline{\pi})} \leq 1 - \frac{1}{3108c_9 \log(C(\pi \times \overline{\pi})(|\gamma| + e)n^2 [F : Q])}.
\end{equation}

Finally, we address the real zeroes of $L(s, \pi \times \overline{\pi})$. By (5.3), if $1 < \sigma < \frac{3}{2}$, then

\begin{equation}
\sum_{L(\beta, \pi \times \overline{\pi}) = 0} \frac{1}{\sigma - \beta} \leq \frac{1}{\sigma - 1} + c_9 \log C(\pi \times \overline{\pi}),
\end{equation}

where the zeroes $\beta$ are counted with multiplicity. Therefore, if $N$ is the total number of zeroes (with multiplicity) in (5.6), then

\[
\frac{N}{\sigma - 1} \leq \frac{1}{\sigma - 1} + c_9 \log C(\pi \times \overline{\pi}).
\]

With $\sigma$ as in (5.4), it follows that

\[
N \leq \frac{3}{2}.
\]

Since $N$ is an integer, we must have $N \in \{0, 1\}$. We have counted zeroes with multiplicity, so if $N = 1$, then the enumerated zero must be simple. \hfill Q.E.D.

5.2. **Log-free zero density estimate.** Our log-free zero density estimate is a version of the work of Soundararajan and Thörner [28, Theorem 1.2] in which we explicate the dependence of the implied constant on $n$ and $[F : Q]$. Because the details follow those in the proof of [28, Theorem 1.2] so closely, we only provide a sketch that outlines our minor adjustments. These adjustments use Proposition 4.1 and Theorem 5.1.

**Proposition 5.2.** Let $\pi \in \mathfrak{F}_n$, $T \geq 1$, and $0 \leq \sigma \leq 1$. Define

\[
N(\sigma, T) := \#\{\rho = \beta + i\gamma \neq \beta_1: \beta \geq \sigma, |\gamma| \leq T, L(\rho, \pi \times \overline{\pi}) = 0\}.
\]

If $\log \log C(\pi \times \overline{\pi}) \gg n^4 [F : Q]^2$ with a sufficiently large implied constant, then

\[
N(\sigma, T) \ll n^2 [F : Q] \log(C(\pi \times \overline{\pi})T^{[F : Q]})^{107n^2(1-\sigma)}.
\]

**Sketch of proof.** The following discussion assumes familiarity with the proofs in [28]. The analogue of [28, Lemma 2.3] over number fields is

\begin{equation}
\sum_{n} \frac{A_{\pi \times \overline{\pi}}(n)}{n^{1+\eta}} \leq \frac{1}{\eta} + \frac{1}{2} \log C(\pi \times \overline{\pi}) + O(n^2 [F : Q]),
\end{equation}

which is easily seen using the shape of $L(s, \pi_\infty \times \overline{\pi}_\infty)$. Since $L(s, \pi \times \overline{\pi})$ has a pole of order 1 at $s = 1$, a more careful look at the proof of [28, Lemma 3.1] yields the estimates

\[
\sum_{\rho} \frac{1 + \eta - \beta}{|1 + \eta + it - \rho|^2} \leq 2\eta \log(C(\pi \times \overline{\pi})(2 + |t|)^{n^2 [F : Q]}) + O(n^2 [F : Q] + \eta^{-1})
\]

and

\begin{equation}
\#\{\rho: |1 + it - \rho| \leq \eta\} \leq 10\eta \log(C(\pi \times \overline{\pi})(2 + |t|)^{n^2 [F : Q]}) + O(n^2 [F : Q] + \eta + 1)
\end{equation}

for $\eta > 0$. This leads to the choice

\[
\frac{c_8}{\log(C(\pi \times \overline{\pi})T^{n^2 [F : Q]})} < \eta \leq \frac{1}{200n^2 [F : Q]}
\]
when replicating the arguments in [28, Section 4] and the choice

\[ K > 2000n \log(C(\pi \times \bar{\pi})(2 + |t|)^{n^2[F:Q]}) + O(n^2[F:Q] \eta + 1) \]

in [28, Lemma 4.2]. In order to replace the use of [28, Theorem 2.4] by Proposition 4.1 in the estimation of [28, Equation 4.6], one chooses

\[ K = 4800n^2 \eta \log(C(\pi \times \bar{\pi})(2 + |t|)^{n^2[F:Q]}) + O(n^2[F:Q] \eta + 1). \]

Proposition 5.3 follows for \( \sigma \leq 1 - \frac{c_{N_0}}{2 \log(C(\pi \times \bar{\pi})(3^2[F:Q]))} \) from inserting these changes into the proof of [28, Theorem 1.2]. For \( \sigma \) in the complementary range, Theorem 5.1 ensures that \( N(\sigma, T) \leq 1 \).

5.3. Zero repulsion. If \( L(s, \pi \times \bar{\pi}) \) has a Landau–Siegel zero especially close to \( s = 1 \), then the standard zero-free region in Theorem 5.1 improves noticeably.

**Proposition 5.3.** Let \( \pi \in \mathfrak{P}_n \). If the exceptional zero \( \beta_1 \) in Theorem 5.1 exists, then here exist absolute and effectively computable constants \( c_{10}, c_{11} > 0 \) such that apart from the point \( s = \beta_1 \), \( L(s, \pi \times \bar{\pi}) \) is nonzero in the region

\[
\text{Re}(s) \geq 1 - c_{11} \frac{\log \left( \frac{(1 - \beta_1) \log(C(\pi \times \bar{\pi})(|\text{Im}(s)| + e)^{n^2[F:Q]}))}{\log(C(\pi \times \bar{\pi})(|\text{Im}(s)| + e)^{n^2[F:Q]})} \right)}{c_{10}}.
\]

**Remark.** A similar result appears in [24, Theorem 4.2], but with a weaker notion of the analytic conductor. We provide a self-contained proof for the sake of completeness.

**Proof.** Assume that the exceptional zero \( \beta_1 > 0 \) in Theorem 5.1 exists. We follow the ideas in [16, Theorem 5.1] as applied to the Dedekind zeta function \( \zeta_F(s) \). The nonnegativity of the Dirichlet coefficients \( \Lambda_{\pi \times \bar{\pi}}(n) \) will play a key role.

Since \((s - 1)L(s, \pi \times \bar{\pi})\) is entire of order 1, it has the Hadamard product representation

\[
(s - 1)L(s, \pi \times \bar{\pi}) = s^r e^{\alpha_1 + \alpha_2 s} \prod_{\omega \neq 0} \left( 1 - \frac{s}{\omega} \right) e^{s/\omega},
\]

where \( r \geq 1 \) is the order of the zero of \( L(s, \pi \times \bar{\pi}) \) at \( s = 0 \) and the product runs through all non-zero roots \( \omega \) of \( L(s, \pi \times \bar{\pi}) \). In the region \( \text{Re}(s) > 1 \), we take the \((2j - 1)\)-th derivative of \(-L'(s, \pi \times \bar{\pi})\) (trivial and nontrivial). In the region \( \text{Re}(s) > 1 \), we arrive at

\[
\frac{1}{(2j - 1)!} \sum_n \frac{\Lambda_{\pi \times \bar{\pi}}(n)}{N^n} (\log N)^{2j - 1} = \frac{1}{(s - 1)^{2j}} - \sum_{\omega} \frac{1}{(s - \omega)^{2j}},
\]

where the sum is over all zeroes including \( \omega = 0 \).

Let \( \beta + i \gamma' \neq \beta_1 \) be a zero of \( L(s, \pi \times \bar{\pi}) \) (trivial or nontrivial). By considering (5.9) at \( s = 2 \) and \( s = 2 + i \gamma' \), we conclude that

\[
\frac{1}{(2j - 1)!} \sum_n \Lambda_{\pi \times \bar{\pi}}(n)(\log N)^{2j - 1} N^{-2m} (1 + (Np^m)^{-i \gamma'})
\]

\[
= 1 + \frac{1}{(1 + i \gamma')^{2j}} - \frac{1}{(2 - \beta_1)^{2j}} - \frac{1}{(2 - \beta_1 + i \gamma')^{2j}} - \sum_{n=1}^{\infty} z_n^j.
\]
We sum (5.10) at $s = 2 + i\gamma'$ with (5.10) at $s = 2$, take real parts, and deduce via the nonnegativity of $\Lambda_{\pi \times \overline{\pi}}(n)$ that

$$0 \leq \frac{1}{(2j - 1)!} \sum_n \Lambda_{\pi \times \overline{\pi}}(n)(\log Nn)^{2j-1} (1 + \cos(\gamma' \log Nn))$$

(5.11)

$$= 1 - \frac{1}{(2 - \beta_1)^{2j}} + \text{Re} \left( \frac{1}{(1 + i\gamma')^{2j}} - \frac{1}{(2 - \beta_1 + i\gamma')^{2j}} \right) - \sum_{n=1}^{\infty} \text{Re}(z_n^j),$$

where the set $\{z_n : n \geq 1\}$ is equal to $\{(2 - \omega)^{-2}, (2 + i\gamma' - \omega)^{-2} : \omega \neq \beta_1\}$ with the labeling chosen such that $|z_1| \geq |z_n|$ for all $n \geq 1$; in particular, since there exists some $n$ for which $z_n = 2 + i\gamma' - \rho' = 2 - \beta'$, we must have that $|z_1| \geq (2 - \beta')^{-2}$. This lower bound on $|z_1|$ applied to (5.11) and a Taylor expansion imply that there exists a constant $c_{12} \geq 1$ such that

$$\sum_{n=1}^{\infty} \text{Re}(z_n^j) \leq 1 - \frac{1}{(2 - \beta_1)^{2j}} + \text{Re} \left( \frac{1}{(1 + i\gamma')^{2j}} - \frac{1}{(2 - \beta_1 + i\gamma')^{2j}} \right) \leq c_{12}j(1 - \beta_1).$$

We need a lower bound for the left hand side of (5.12). Define

$$L := |z_1|^{-1} \sum_{n=1}^{\infty} |z_n|.$$

For $t \in \mathbb{R}$, the number of nontrivial zeroes $\rho = \beta + i\gamma$ of $L(s, \pi \times \overline{\pi})$ satisfying $|\gamma - t| \leq 1$ is $O(\log C(\pi \times \overline{\pi}, t))$. Thus by [35,7], there exist constants $c_{13} \geq 1$ and $c_{14} \geq 1$ such that

$$L \leq c_{13}(2 - \beta')^2 \sum_{\omega} \left( \frac{1}{|2 - \omega|^2} + \frac{1}{|2 + i\gamma' - \omega|^2} \right) \leq c_{14} \log(C(\pi \times \overline{\pi})(e + |\gamma'|)^n)^{n^2[F:Q]}).$$

The lower bound for power sums in [16] Theorem 4.2] implies that there exists an integer $1 \leq j_1 \leq 24L$ such that

$$8 \sum_{n=1}^{\infty} \text{Re}(z_n^{j_1}) \geq (2 - \beta')^{-2j_1} \geq \exp(-2j_1(1 - \beta')).$$

We combine (5.12) and (5.14) to obtain the bound $\exp(-2j_1(1 - \beta')) \leq 8c_{12}j_1(1 - \beta_1)$. Since $j_1 \leq 24L \leq 24c_{14} \log(C(\pi \times \overline{\pi})(e + |\gamma'|)^n)^{n^2[F:Q]}$, we arrive at the bound

$$\exp(-48c_{14} \log(C(\pi \times \overline{\pi})(e + |\gamma'|)^n)^{n^2[F:Q]}(1 - \beta')) \leq 192c_{12}c_{14} \log(C(\pi \times \overline{\pi})(e + |\gamma'|)^n)^{n^2[F:Q]}(1 - \beta_1).$$

We obtain the desired result by solving the above inequality for $\beta'$. \hfill \square

We use Proposition 5.3 to determine an upper bound for $\beta_1$ (if it exists).

**Corollary 5.4.** There exists an absolute and effectively computable constant $c_{15} > 0$ such that if the exceptional zero $\beta_1$ in Theorem 5.1 exists, then $\beta_1 \leq 1 - C(\pi \times \overline{\pi})^{-c_{15}}$.

**Proof.** If $\beta' \neq \beta_1$ is a real zero of $L(s, \pi \times \overline{\pi})$ (trivial or nontrivial), then by Proposition 5.3

$$\beta' \leq 1 - \frac{c_{11} \log \frac{1}{1 - \beta_1} + c_{11} \log c_{10} - c_{11} \log \log(C(\pi \times \overline{\pi})e^{n^2[F:Q]})}{\log(C(\pi \times \overline{\pi})e^{n^2[F:Q]})}.$$

First, consider the case where $C(\pi \times \overline{\pi})e^{n^2[F:Q]}$ is large enough so that

$$\frac{- c_{11} \log c_{10} - c_{11} \log \log(C(\pi \times \overline{\pi})e^{n^2[F:Q]})}{\log(C(\pi \times \overline{\pi})e^{n^2[F:Q]})} \leq \frac{1}{2}.$$
This is satisfied when $C(\pi \times \overline{\pi}) e^{n^2[F:Q]}$ is larger than a certain absolute and effectively computable constant. By (5.14), we have
\[
\beta' \leq \frac{1}{2} - c_{11} \frac{\log \frac{1}{1-\beta_1}}{\log(C(\pi \times \overline{\pi}) e^{n^2[F:Q]}).}
\]
Define $c$ so that
\[
\beta_1 = 1 - (C(\pi \times \overline{\pi}) e^{n^2[F:Q]})^{-c}.
\]
It follows that $\beta' \leq \frac{1}{2} - c_{11}c$. If $c \geq 7/(2c_{11})$, then we have determined that $L(\sigma, \pi \times \overline{\pi}) \neq 0$ for all $\sigma > -3$. This contradicts the fact that the trivial zeroes of $\zeta_F(s)$ are included among the trivial zeroes of $L(s, \pi \times \overline{\pi})$, and $\zeta_F(s)$ has a trivial zero in the set $\{-2, -1, 0\}$. We conclude that $\beta_1 \leq 1 - (C(\pi \times \overline{\pi}) e^{n^2[F:Q]})^{-7/(2c_{11})}$. The Minkowski bound $[F : Q] \ll \log D_F$ yields $\beta_1 \leq 1 - C(\pi \times \overline{\pi})^{-c_{15}}$ once $c_{15}$ is sufficiently large.

If $C(\pi \times \overline{\pi}) e^{n^2[F:Q]}$ is not large enough to satisfy (5.16), then there exists an absolute and effectively computable constant $0 < B < 1$ such that any real zero $\beta$ of $L(s, \pi \times \overline{\pi})$ satisfies
\[
\beta \leq 1 - B = 1 - C(\pi \times \overline{\pi})^{-\frac{\log B^{-1}}{\log C(\pi \times \overline{\pi})}} \leq 1 - C(\pi \times \overline{\pi})^{-c_{15}}
\]
once $c_{15}$ is made sufficiently large. 

6. Proof of Theorem 2.1

We prove the result when the exceptional zero $\beta_1$ in Theorem 5.1 exists; the complementary case is easier. In order to apply Proposition 4.1, we make the initial restrictions
\[
(6.1) \quad \log \log C(\pi \times \overline{\pi}) \gg n^4[F : Q]^2, \quad x \geq C(\pi \times \overline{\pi})^{32n^2}, \quad 1 \leq T \leq x^{\frac{1}{16n^2[F:Q]}}.
\]
We begin with the version of Perron’s integral formula proved in [20, Corollary 2.2] applied to $L(s, \pi \times \overline{\pi})$:
\[
\sum_{Nn \leq x} \Lambda_\pi \Lambda_{\pi \times \overline{\pi}}(n) = \frac{1}{2\pi i} \int_{1+\frac{1}{\log x}+iT}^{1-\frac{1}{\log x}-iT} - \frac{L'}{L} (s, \pi \times \overline{\pi}) \frac{x^s}{s} ds
\]
\[
+ \frac{x}{\sqrt{T}} \sum_n \frac{\Lambda_{\pi \times \overline{\pi}}(n)}{Nn^{1+\frac{1}{\log x}}} + O\left( \sum_{|Nn-x| \leq \frac{x}{\sqrt{T}}} \Lambda_{\pi \times \overline{\pi}}(n) + \sum_{|Nn-x| \leq \frac{x}{\sqrt{T}}} \Lambda_{\pi \times \overline{\pi}}(n) \right).
\]
One constructs a contour given by the perimeter of the rectangle with vertices $1 + \frac{1}{\log x} \pm iT$ and $-U \pm iT$, where $U \geq 1$ is fixed, arbitrarily large, and chosen so that no trivial zero of $L(s, \pi \times \overline{\pi})$ lies within a distance of $\frac{1}{4\log x}$ from the line $\operatorname{Re}(s) = -U$. The residues in the interior of this rectangle arise from the trivial zeroes (which are handled by (3.18)), the nontrivial zeroes $\rho = \beta + i\gamma$ with $\gamma \in \mathbb{R}$ and $0 < \beta < 1$, and the residue at $s = 0$.

The bound (3.8) and Stirling’s formula, when applied to the logarithmic derivative of the Hadamard product for $L(s, \pi \times \overline{\pi})$, imply that if $-\frac{1}{2} \leq \operatorname{Re}(s) \leq 2$, then
\[
- \frac{L'}{L} (s, \pi \times \overline{\pi}) = \frac{1}{s} + \frac{1}{s-1} - \sum_{|s-\rho| < 1} \frac{1}{s-\rho}
\]
\[
- \sum_{|s+\mu_j,\pi \times \overline{\pi}(v)| < 1} \frac{1}{s+\mu_j,\pi \times \overline{\pi}(v)} + O(\log C(\pi \times \overline{\pi}, |\operatorname{Im}(s)|)).
\]
(See also [13, Proposition 5.7].) This can be extended to a wider strip using the functional equation for \( L(s, \pi \times \tilde{\pi}) \), leading to a strong bound on both the residue at \( s = 0 \) and the legs of the contour to the left of \( \text{Re}(s) = 1 + \frac{1}{\log x} \). Ultimately, with the help of (6.1), we arrive at

\[
\sum_{N \leq x} \Lambda_{\pi \times \tilde{\pi}}(n) = x - \sum_{0 < \beta < 1} \frac{x^\rho}{\rho} + O \left( \sum_{x - \frac{x}{\sqrt{T}} \leq n \leq x + \frac{x}{\sqrt{T}}} \Lambda_{\pi \times \tilde{\pi}}(n) + \frac{\sqrt{T}}{N n^{1 + \frac{1}{\log x}}} + \frac{x (\log x)^2}{T} \right). 
\]

(See also [20, Sections 4 and 5].) The bounds (5.7) and (6.1) imply that

\[
\sum_{N \leq x} \Lambda_{\pi \times \tilde{\pi}}(n) = x - \sum_{0 < \beta < 1} \frac{x^\rho}{\rho} + O \left( \sum_{x - \frac{x}{\sqrt{T}} \leq n \leq x + \frac{x}{\sqrt{T}}} \Lambda_{\pi \times \tilde{\pi}}(n) + \frac{x (\log x)^2}{T} \right). 
\]

By Proposition 4.1 and (6.1), we have

\[
(6.2) \quad \sum_{N \leq x} \Lambda_{\pi \times \tilde{\pi}}(n) = x - \sum_{0 < \beta < 1} \frac{x^\rho}{\rho} + O \left( \frac{x (\log x)^2}{\sqrt{T}} \right). 
\]

It follows from (6.2) that for \( 2 \leq h \leq x \), we have that

\[
\sum_{x < N \leq x + h} \Lambda_{\pi \times \tilde{\pi}}(n) = h - \sum_{0 < \beta < 1} \frac{(x + h)^\rho - x^\rho}{\rho} + O \left( \frac{x (\log x)^2}{\sqrt{T}} \right). 
\]

We observe that

\[
\left| \frac{(x + h)^\rho - x^\rho}{\rho} \right| \leq \min \left\{ h x^\beta - 1, \frac{3 x^\beta}{|\gamma|} \right\} 
\]

using the bounds

\[
\left| \frac{(x + h)^\rho - x^\rho}{\rho} \right| = \left| \int_x^{x + h} \tau^{\beta - 1} d\tau \right| \leq \int_x^{x + h} \tau^{\beta - 1} d\tau \leq h x^{\beta - 1} 
\]

and

\[
\left| \frac{(x + h)^\rho - x^\rho}{\rho} \right| \leq \frac{(2x)^\beta + x^\beta}{|\gamma|} \leq \frac{3 x^\beta}{|\gamma|}. 
\]

Finally, note that by the mean value theorem, there exists \( \xi \in [x, x + h] \) such that

\[
\frac{(x + h)^\beta - x^\beta}{\beta} = h \xi^{\beta - 1}. 
\]

Thus, there exists \( \xi \in [x, x + h] \) such that

\[
(6.3) \quad \left| \frac{1}{h} \sum_{x < N \leq x + h} \Lambda_{\pi \times \tilde{\pi}}(n) - 1 + \xi^{\beta - 1} \right| \ll \sum_{0 < \beta < 1} \frac{x^{\beta - 1}}{h} \sum_{0 < \beta < 1} \frac{x^{\beta - 1}}{|\gamma|} + \frac{x (\log x)^2}{h \sqrt{T}}, 
\]

where \( \sum' \) denotes a sum over nontrivial zeroes \( \rho = \beta + i \gamma \neq \beta_1 \). Next, we subdivide the zeroes into \( O(\log x) \) dyadic intervals \( c^j < |\gamma| \leq c^{j+1} \) and deduce that the right hand side of
(6.3) is

\[ (6.4) \quad \ll \sum' x^{\beta - 1} + \frac{x \log x}{h} \sup_{|\gamma| \leq \frac{\log x}{h}, 0 < \beta < 1} \frac{1}{M} \sum' x^{\beta - 1} + \frac{x (\log x)^2}{h \sqrt{T}}. \]

At this stage, we require some constraints on the relationships amongst \( x, T, h \), and \( C(\pi \times \tilde{\pi}) \). To help simplify future calculations, we choose \( A \geq 10^7 \) (the constant in the exponent in Proposition 5.2) and

\[ (6.5) \quad C(\pi \times \tilde{\pi}) \leq x^{\theta}, \quad x^{1 - \delta} \leq h \leq x, \quad T = x^{\frac{1}{4A^2[F : Q]}}, \quad \theta, \delta[F : Q] \in \left[ 0, \frac{1}{16A^n} \right]. \]

We will take \( A \) to be sufficiently large later on. Subject to (6.5), if \( \beta + i\gamma \) is a zero other than \( \beta_1 \), then by Theorem 5.1 we have that

\[ (6.6) \quad \beta \leq 1 - 2c_8A/\log x, \quad |\gamma| \leq T = x^{\frac{1}{4A^2[F : Q]}}. \]

By Proposition 5.3 we also have that

\[ (6.7) \quad \beta \leq 1 - 2c_1A \log \left( \frac{2Ac_{10}}{(1 - \beta_1) \log x} \right)/\log x, \quad |\gamma| \leq T = x^{\frac{1}{4A^2[F : Q]}}. \]

Finally, by Proposition 5.2 we have that

\[ (6.8) \quad N(\sigma, M) \ll n^2[F : Q]x^{\frac{1}{4A^2}}, \quad M \leq T = x^{\frac{1}{4A^2[F : Q]}}. \]

Now, if \( M \leq T \) and

\[ \eta = \max \left\{ c_8, c_{11} \log \left( \frac{2Ac_{10}}{(1 - \beta_1) \log x} \right) \right\}, \]

then by (6.6), (6.7), and (6.8), we have that

\[ \sum' x^{\beta - 1} \ll \log x \int_0^{1 - \frac{2A\eta}{\log x}} N(\sigma, M)x^{\sigma - 1} d\sigma \ll n^2[F : Q] \log x \int_0^{1 - \frac{2A\eta}{\log x}} x^{\frac{c_{11}A}{2}} d\sigma \]

\[ \ll n^2[F : Q]e^{-A\eta} \]

\[ = n^2[F : Q] \min \left\{ e^{-c_8A}, \left( \frac{(1 - \beta_1) \log x}{2Ac_{10}} \right)^{c_{11}A} \right\} \]

\[ \ll n^2[F : Q]e^{-c_8A} \min\{1, ((1 - \beta_1) \log x)^{c_{11}A}\}. \]

Since \( \min\{1, a^b\} \leq \min\{1, a\} \) for \( a > 0 \) and \( b \geq 1 \), it follows that if \( A > 1/c_{11} \), then (6.9) is

\[ \ll n^2[F : Q]e^{-c_8A} \min\{1, (1 - \beta_1) \log x\}. \]

We apply this bound to the two sums in (6.4) and invoke (6.3) and (6.5) to deduce the bound

\[ (6.10) \quad \left| \frac{1}{h} \sum_{x < N \leq x + h} \Lambda_{\pi \times \tilde{\pi}}(n) - 1 + \xi^{-\beta_1 - 1} \right| \]

\[ \ll n^2[F : Q]e^{-c_8A} \min\{1, (1 - \beta_1) \log x\} + x^{-\frac{1}{16A^2[F : Q]}} (\log x)^2. \]

If \( x \geq (An^2[F : Q])^{900A^2[F : Q]} \), then

\[ \log x \leq x^{\frac{1}{544A^2(F : Q)}}, \]
and (6.10) is

\[ (6.11) \quad \ll n^2[F : \mathbb{Q}] e^{-c s A} \min\{1, (1 - \beta_1) \log x\} + x^{-\frac{1}{17A n^2[F : \mathbb{Q}]}}. \]

If we also require that \( x \geq (e^{c s A} C(\pi \times \pi))^{17A n^2[F : \mathbb{Q}]}, \) then by Corollary 5.4, (6.11) is

\[ \ll n^2[F : \mathbb{Q}] e^{-c s A} \min\{1, (1 - \beta_1) \log x\} \ll n^2[F : \mathbb{Q}] e^{-c s A} \min\{1, (1 - \beta_1) \log \xi\}. \]

The bound \( \min\{1, (1 - \beta_1) \log \xi\} \ll (1 - \xi^{\beta_1-1}) \) is easily demonstrated. Theorem 2.1 follows once we invoke (3.7) to bound \( C(\pi \times \pi) \).

### 7. Proof of Theorem 2.2

In light of Theorem 2.1 and the nonnegativity of \( \Lambda_{\pi \times \pi}(n) \), it suffices to prove that if \( n \in \{1, 2, 3, 4\} \), \( \pi \in \mathfrak{F}_n \), and \( x > C(\pi) \), then

\[ (7.1) \quad \sum_{n \text{ composite}} \Lambda_{\pi \times \pi}(n) \ll n^2 x^{1 - \frac{1}{2(\alpha + 1)} (\log x)^3}. \]

Our main tools for this estimate are existing bounds towards GRC, namely (3.2) and (3.6), and the following result of Brumley.

**Lemma 7.1.** Let \( n \in \{1, 2, 3, 4\} \) and \( \pi \in \mathfrak{F}_n \). If \( \varepsilon > 0 \), then

\[ (7.2) \quad \prod_p \sum_{r=0}^{\infty} \frac{\max_{1 \leq j \leq n} |\alpha_{j,\pi}(p)|^{2r}}{N \pi^{(1+\varepsilon)}} \ll_{n,\varepsilon} C(\pi)^{\varepsilon}. \]

**Proof.** See [3] Theorem 1; the Dirichlet series to be bounded is denoted by \( L(s, \pi, |\max|^2) \) therein. For \( n \in \{1, 2, 3\} \), the proof relies on basic properties of \( L(s, \pi \times \pi) \). When \( n = 4 \), the proof relies on the automorphy of the exterior square lift from GL\(_4\) to GL\(_6\), as proved by Kim [15].

We begin with the observation that (7.1) is

\[ (7.3) \quad \leq x \sum_{\substack{n \text{ composite} \\text{and} \ n \leq 2x \\text{fixed}}} \frac{\Lambda_{\pi \times \pi}(n)}{Nn}. \]

Note that there are \( O(\log x) \) ramified prime ideals, each of which has norm at most \( x \). At each such prime ideal, we have the bound

\[ \Lambda_{\pi \times \pi}(n) \leq n^2 N n^{1 - \frac{2}{n+1}} \Lambda(n). \]

Therefore, the contribution from the ramified primes to (7.3) is

\[ \ll n^2(\log x) \sum_{\frac{2}{2} \leq r \leq \frac{2}{\log x}} \sum_{N \pi^{r \frac{2}{n+1}}} \ll n^2 (\log x)^2 x^{1/r} \ll n^2 (\log x)^2 \ll n^2 x^{1 - \frac{2}{n+1} (\log x)^3}. \]

For integers \( r \geq 1 \), it follows from the definition of \( \Lambda_{\pi \times \pi}(p^r) \) that if \( \gcd(p, q_x) = O_F \), then

\[ \Lambda_{\pi \times \pi}(p^r) \leq n^2 \max_{1 \leq j \leq n} |\alpha_{j,\pi}(p)|^{2r} \log Np. \]
Define
\[ \beta_p := Np^{-1} \max_{1 \leq j \leq n} |\alpha_{j,\pi}(p)|^2. \]

By (3.2), we have that \( \beta_p \leq Np^{-(n^2+1)/2} \). The contribution from composite \( n \) with \( \gcd(n, q_\pi) = O_F \) to (7.3) is
\[ \ll n^2 x (\log x) \sum_{r=2}^{\infty} \sum_{x^{1/r} \leq Np^{(2x)^1/r}} \beta_p^r \ll n^2 x (\log x) \sum_{2 \leq R \leq \log(2x)/\log 2} \sum_{x^{1/R} \leq Np^{(2x)^1/R}} \sum_{r=0}^{\infty} \beta_p^r. \]

(7.4)
\[ = n^2 x (\log x) \sum_{2 \leq R \leq \log(2x)/\log 2} \sum_{x^{1/R} \leq Np^{(2x)^1/R}} \beta_p^R \frac{1}{1 - \beta_p}. \]

In (7.4), the contribution from prime ideals with norm at most \( 2n^2 \) is \( O(1) \) (since \( n \leq 4 \)). If \( Np > 2n^2 \), then \( 1 - \beta_p \geq \frac{1}{2} \). Thus, (7.3) is
\[ \leq n^2 x \left( \frac{1}{n^2+1} \right) (\log x) \sum_{2 \leq R \leq \log(2x)/\log 2} \left( 2 \sum_{x^{1/R} \leq Np^{(2x)^1/R}} \beta_p + O(1) \right). \]

(7.5)
\[ \leq 2n^2 x \left( \frac{1}{2(n^2+1)} \right) (\log x) \sum_{2 \leq R \leq \log(2x)/\log 2} \left( 2 \sum_{x^{1/R} \leq Np^{(2x)^1/R}} \beta_p Np^{-\frac{1}{n^2+1}} + O(1) \right). \]

Note that \( x \leq 2 \log(x+1) \) for all \( 0 \leq x \leq 5/2 \). Since \( 0 \leq \beta_p Np^{-\frac{1}{n^2+1}} \leq 1 \) for all \( p \), it follows that (7.5) is
\[ \leq 2n^2 x \left( \frac{1}{2(n^2+1)} \right) (\log x) \sum_{2 \leq R \leq \log(2x)/\log 2} \left( 2 \sum_{x^{1/R} \leq Np^{(2x)^1/R}} \log(1 + \beta_p Np^{-\frac{1}{n^2+1}}) + O(1) \right). \]

(7.6)
\[ \ll n^2 x \left( \frac{1}{2(n^2+1)} \right) (\log x) \sum_{p} \log \left( \sum_{r=0}^{\infty} \beta_p Np^{-\frac{r}{n^2+1}} \right) \]
\[ = n^2 x \left( \frac{1}{2(n^2+1)} \right) (\log x) \log \left( \prod_{p} \sum_{r=0}^{\infty} \max_{1 \leq j \leq n} |\alpha_{j,\pi}(p)|^{2r} \right) \frac{1}{Np^{(1+\frac{1}{n^2+1})}}. \]

By Lemma 7.1 and our hypothesis that \( x \geq C(\pi) \), (7.6) is \( \ll x^{1-\frac{1}{2(n^2+1)}} (\log x)^2 \).

REFERENCES

[1] A. Akbary and T. S. Trudgian. A Log-Free Zero-Density Estimate and Small Gaps in Coefficients of L-Functions. Int. Math. Res. Not. IMRN, (12):4242–4268, 2015.
[2] F. Brumley. Effective multiplicity one on \( \text{GL}_N \) and narrow zero-free regions for Rankin-Selberg L-functions. Amer. J. Math., 128(6):1455–1474, 2006.
[3] F. Brumley. Second order average estimates on local data of cusp forms. Arch. Math. (Basel), 87(1):19–32, 2006.
[4] C. J. Bushnell and G. Henniart. An upper bound on conductors for pairs. J. Number Theory, 65(2):183–196, 1997.
[5] J. Friedlander and H. Iwaniec. *Opera de cribro*, volume 57 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2010.

[6] R. Godement and H. Jacquet. *Zeta functions of simple algebras*. Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin-New York, 1972.

[7] D. Goldfeld and X. Li. A standard zero free region for Rankin-Selberg L-functions. *Int. Math. Res. Not. IMRN*, (22):7067–7136, 2018.

[8] D. R. Heath-Brown. The number of primes in a short interval. *J. Reine Angew. Math.*, 389:22–63, 1988.

[9] J. Hoffstein and P. Lockhart. Coefficients of Maass forms and the Siegel zero. *Ann. of Math.* (2), 140(1):161–181, 1994. With an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman.

[10] J. Hoffstein and D. Ramakrishnan. Siegel zeros and cusp forms. *Internat. Math. Res. Notices*, (6):279–308, 1995.

[11] G. Hoheisel. Primzahl probleme in der Analysis. *S.-B. Preuss. Akad. Wiss.*, 8:580–588, 1930.

[12] P. Humphries. Standard zero-free regions for Rankin-Selberg L-functions via sieve theory. *Math. Z.*, 292(3-4):1105–1122, 2019. With an appendix by Farrell Brumley.

[13] H. Iwaniec and E. Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[14] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika. Rankin-Selberg convolutions. *Amer. J. Math.*, 105(2):367–464, 1983.

[15] H. H. Kim. Functoriality for the exterior square of GL(4) and the symmetric fourth of GL(2). *J. Amer. Math. Soc.*, 16(1):139–183, 2003. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.

[16] J. C. Lagarias, H. L. Montgomery, and A. M. Odlyzko. A bound for the least prime ideal in the Chebotarev density theorem. *Invent. Math.*, 54(3):271–296, 1979.

[17] R. J. Lemke Oliver and J. Thorner. Effective log-free zero density estimates for automorphic L-functions and the Sato-Tate conjecture. *Int. Math. Res. Not. IMRN*, (22):6988–7036, 2019.

[18] X. Li. Upper bounds on L-functions at the edge of the critical strip. *Int. Math. Res. Not. IMRN*, (4):727–755, 2010.

[19] U. V. Linnik. On the least prime in an arithmetic progression. *Rec. Math. [Mat. Sbornik] N.S.*, 15(57):139–178,347–368, 1944.

[20] J. Liu and Y. Ye. Perron’s formula and the prime number theorem for automorphic L-functions. *Pure Appl. Math. Q.*, 3(2, Special Issue: In honor of Leon Simon. Part 1):481–497, 2007.

[21] W. Luo, Z. Rudnick, and P. Sarnak. On the generalized Ramanujan conjecture for GL(n). In *Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996)*, volume 66 of *Proc. Sympos. Pure Math.*, pages 301–310. Amer. Math. Soc., Providence, RI, 1999.

[22] C. Moeglin and J.-L. Waldspurger. Le spectre r´esiduel de GL(n). *Ann. Sci. École Norm. Sup. (4)*, 22(4):605–674, 1989.

[23] C. J. Moreno. The Hoheisel phenomenon for generalized Dirichlet series. *Proc. Amer. Math. Soc.*, 40:47–51, 1973.

[24] C. J. Moreno. Analytic proof of the strong multiplicity one theorem. *Amer. J. Math.*, 107(1):163–206, 1985.

[25] Y. Motohashi. On sums of Hecke-Maass eigenvalues squared over primes in short intervals. *J. Lond. Math. Soc. (2)*, 91(2):367–382, 2015.

[26] W. Müller and B. Speh. Absolute convergence of the spectral side of the Arthur trace formula for GL_n. *Geom. Funct. Anal.*, 14(1):58–93, 2004. With an appendix by E. M. Lapid.

[27] Z. Rudnick and P. Sarnak. Zeros of principal L-functions and random matrix theory. *Duke Math. J.*, 81(2):269–322, 1996. A celebration of John F. Nash, Jr.

[28] K. Soundararajan and J. Thorner. Weak subconvexity without a Ramanujan hypothesis. *Duke Math. J.*, 168:1231–1268, 2019. With an appendix by Farrell Brumley.

[29] A. Weiss. The least prime ideal. *J. Reine Angew. Math.*, 338:56–94, 1983.

[30] J. Wu and Y. Ye. Hypothesis H and the prime number theorem for automorphic representations. *Funct. Approx. Comment. Math.*, 37(part 2):461–471, 2007.
