APPROXIMATION BY A KANTOROVICH TYPE $q$-BERNSTEIN-STANCU OPERATORS

M. MURSALEEN*, KHURSHEED J. ANSARI AND ASIF KHAN

Abstract. In this paper, we introduce a Kantorovich type generalization of $q$-Bernstein-Stancu operators. We study the convergence of the introduced operators and also obtain the rate of convergence by these operators in terms of the modulus of continuity. Further, we study local approximation property and Voronovskaja type theorem for the said operators. We show comparisons and some illustrative graphics for the convergence of operators to a certain function.

1. Introduction

The applications of $q$-calculus in the area of approximation theory were initiated by Lupas [13], who first introduced $q$-Bernstein polynomials. Later, Phillips [24] proposed other $q$-analog of Bernstein polynomials which became very popular and several researchers obtained the interesting approximation properties for $q$-Bernstein polynomials. In the recent years, many researchers have studied the approximation properties for linear positive operators [1, 3, 7, 8, 15, 16, 17, 20, 18]. Mursaleen et al in [22, 23, 19, 21] have also obtained statistical approximation properties for new positive linear operators and some approximation theorems for generalized $q$-Bernstein-Schurer operators.

Initially, we start off with the basic definitions and notations of quantum calculus [10]. Let $q > 0$ be a fixed real number. For any $n \in \mathbb{N} \cup \{0\}$, the $q$-integer $[n] = [n]_q$ is defined by

$$[n] := \begin{cases} \frac{(1-q^n)}{(1-q)}, & q \neq 1 \\ n, & q = 1 \end{cases}$$

and $q$-factorial $[n]! = [n]_q!$ by

$$[n]! := \begin{cases} [n][n-1] \cdots [1], & n \geq 1 \\ 1, & n = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the $q$-binomial coefficient is defined by

$$\binom{n}{k} := \frac{[n]!}{[k]![n-k]!}.$$

The $q$-analogue of integration in the interval $[0, A]$ is defined by

$$\int_0^A f(t)d_qt := A(1-q) \sum_{n=0}^{\infty} f(Aq^n)q^n, \ 0 < q < 1.$$

In [4], Bernstein introduced the following well-known positive linear operators

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

and he showed that if $f \in C[0,1]$, then $B_n(f; x) \Rightarrow f(x)$ where ” $\Rightarrow$ ” represents the uniform convergence. One can find more details about the Bernstein polynomials in [12]. The $q$-generalization of the Bernstein polynomials was introduced by G.M. Phillips [24].

The classical Kantorovich operator $B_n^*, n = 1, 2, \cdots$ is defined by (cf. [12])

$$B_n^*(f; x) = (n+1) \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_{\frac{k+1}{n+1}}^{\frac{k+1}{n}} f(t)dt$$

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*Corresponding author.
(1.2) \[ = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_0^1 f \left( \frac{k+t}{n+1} \right) dt \]

Inspired by (1.2), Mahmudov \[14\] introduced a \(q\)-type generalization of Bernstein-Kantorovich operators as follows:

(1.3) \[ B^*_{n,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 f \left( \frac{[k] + q^k t}{[n] + 1} \right) dt \]

where \[ p_{n,k}(q;x) := \binom{n}{k} x^k (1-x)^{n-k}, \quad (1-x)_q^n = \prod_{s=0}^{n-1} (1-q^s x). \]

It can be seen that for \( q \to 1^- \) the \(q\)-Bernstein-Kantorovich operator becomes the classical Bernstein-Kantorovich operator (1.2).

In 2010, Gadjiiev and Gorbunov\[6\] introduced the following construction of Bernstein-Stancu type polynomials with shifted knots:

(1.4) \[ S_{n,\alpha,\beta}(f;x) = \left( \frac{n + \beta_2}{n} \right) n \sum_{r=0}^{n} f \left( \frac{r + \alpha_1}{n + \beta_1} \right) \left( \frac{n}{r} \right) \left( x - \frac{\alpha_2}{n + \beta_2} \right)^r \left( \frac{n + \alpha_2}{n + \beta_2} - x \right)^{n-r} \]

where \( \frac{\alpha_2}{n + \beta_2} \leq x \leq \frac{n + \alpha_2}{n + \beta_2} \) and \( \alpha_k, \beta_k (k = 1, 2) \) are positive real numbers provided \( 0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \). It is clear that for \( \alpha_2 = \beta_2 = 0 \), then polynomials (1.4) turn into the Bernstein-Stancu polynomials (1.2) and if \( \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 \) then these polynomials turn into the classical Bernstein polynomials.

Motivated by (1.4), Içöz\[9\] introduced a Kantorovich type generalization of Bernstein-Stancu polynomials as follows:

(1.5) \[ S^*_{n,\alpha,\beta}(f;x) = (n+\beta_1+1) \left( \frac{n + \beta_2}{n} \right) n \sum_{r=0}^{n} f \left( \frac{r + \alpha_1}{n + \beta_1} \right) \left( \frac{n}{r} \right) \left( x - \frac{\alpha_2}{n + \beta_2} \right)^r \left( \frac{n + \alpha_2}{n + \beta_2} - x \right)^{n-r} \int_{x}^{x+\alpha_1+\beta_1} f(s)ds. \]

where \( \frac{\alpha_2}{n + \beta_2} \leq x \leq \frac{n + \alpha_2}{n + \beta_2} \) and \( \alpha_k, \beta_k (k = 1, 2) \) are positive real numbers provided \( 0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \). It is clear that for \( \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 \) then these polynomials turn into the Bernstein-Kantorovich operators.

2. CONSTRUCTION OF NEW OPERATORS AND SOME AUXILIARY RESULTS

We construct a Kantorovich type \(q\)-Bernstein-Stancu type polynomials as follows:

(2.1) \[ K^*_{n,q}(\alpha,\beta) = \left( \frac{[n] + \beta_2}{[n]} \right) n \sum_{k=0}^{n} \binom{n}{k} \left( x - \frac{\alpha_2}{[n] + \beta_2} \right)^k \left( \frac{[n] + \alpha_2}{[n] + \beta_2} - x \right)^{n-k} \int_0^1 f \left( \frac{[k] + q^k t + \alpha_1}{[n] + 1} + \beta_1 \right) dt \]

where \( \frac{\alpha_2}{[n] + \beta_2} \leq x \leq \frac{n + \alpha_2}{[n] + \beta_2} \) and \( \alpha_k, \beta_k (k = 1, 2) \) are positive real numbers provided \( 0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \). If we put for \( \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 \) in (2.1) then these polynomials turn into the Bernstein-Kantorovich operators (1.3) introduced by Mahmudov. Throughout this paper, \( \| \cdot \| \) denotes the sup-norm on \([0,1]\). Further, \( C \) denotes the absolutely positive constant not necessarily the same at each occurrence.

The aim of this paper is to study some approximation properties of Kantorovich type \(q\)-Bernstein-Stancu operators defined by (2.1). First, we prove the basic convergence of the introduced operators and also obtain the rate of convergence by these operators in terms of the modulus of continuity. Further, we study local approximation property and Voronovskaja type theorem for the said operators. With the help of the Matlab we show comparisons and some illustrative graphics for the convergence of operators to a function.

**Lemma 2.1.** Let \( K^*_{n,q}(\alpha,\beta) \) be given by (2.1). Then the following properties hold:

(i) \[ K^*_{n,q}(1;x) = 1; \]
Proof. (i) Using binomial coefficient

\[ K_{n,q}^{(n,\beta)}(1; x) = \left( \frac{n + \beta_2}{n} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( x - \frac{\alpha_2}{n + \beta_2} \right)_q^k \left( \frac{n + \alpha_2}{n + \beta_2} \right)_q - x \right]^{n-k}_q \int_0^1 d_q t \]

\[ = \left( \frac{n + \beta_2}{n} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( x - \frac{\alpha_2}{n + \beta_2} \right)_q^k \left( \frac{n + \alpha_2}{n + \beta_2} \right)_q - x \right]^{n-k}_q \int_0^1 \frac{[k] + \alpha_1 + q^k t}{[n + 1] + \beta_1} d_q t \]

\[ K_{n,q}^{(n,\beta)} = \left( \frac{n + \beta_2}{n} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( x - \frac{\alpha_2}{n + \beta_2} \right)_q^k \left( \frac{n + \alpha_2}{n + \beta_2} \right)_q - x \right]^{n-k}_q \left\{ \frac{[k] + \alpha_1}{[n + 1] + \beta_1} + \frac{q^k}{2([n + 1] + \beta_1)} \right\} \]

(ii) For \( f(t) = t \), we have

\[ K_{n,q}^{(n,\beta)} = \frac{1}{[n + 1] + \beta_1} \left( \frac{n + \beta_2}{n} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( x - \frac{\alpha_2}{n + \beta_2} \right)_q^k \left( \frac{n + \alpha_2}{n + \beta_2} \right)_q - x \right]^{n-k}_q \left\{ \frac{2q^k}{[2][k]} \right\} \]

(\text{using} \ q^k = 1 + (q - 1)[k])

\[ K_{n,q}^{(n,\beta)} = \frac{1}{[n + 1] + \beta_1} \left( \frac{n + \beta_2}{n} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( x - \frac{\alpha_2}{n + \beta_2} \right)_q^k \left( \frac{n + \alpha_2}{n + \beta_2} \right)_q - x \right]^{n-k}_q \left\{ \frac{2q^k}{[2][k]} \right\} \]

\[ + \left( \frac{\alpha_1}{[2]} + \frac{1}{[2]} \right) \left( \frac{n + \beta_2}{n} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( x - \frac{\alpha_2}{n + \beta_2} \right)_q^k \left( \frac{n + \alpha_2}{n + \beta_2} \right)_q - x \right]^{n-k}_q \left\{ \frac{2q^k}{[2][k]} \right\} \]

\[ + \frac{1}{[n + 1] + \beta_1} \left( \frac{n + \beta_2}{n} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( x - \frac{\alpha_2}{n + \beta_2} \right)_q^k \left( \frac{n + \alpha_2}{n + \beta_2} \right)_q - x \right]^{n-k}_q \left\{ \frac{2q^k}{[2][k]} \right\} \]

\[ = \frac{[n + \beta_2}{[n + 1] + \beta_1} \frac{2q}{[2]} \left( \frac{n + \beta_2}{n} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( x - \frac{\alpha_2}{n + \beta_2} \right)_q^k \left( \frac{n + \alpha_2}{n + \beta_2} \right)_q - x \right]^{n-k}_q \left\{ \frac{2q^k}{[2][k]} \right\} \]

\[ + \frac{1}{[n + 1] + \beta_1} \left( \frac{n + \beta_2}{n} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( x - \frac{\alpha_2}{n + \beta_2} \right)_q^k \left( \frac{n + \alpha_2}{n + \beta_2} \right)_q - x \right]^{n-k}_q \left\{ \frac{2q^k}{[2][k]} \right\} \]

\[ = \frac{n + \beta_2}{[n + 1] + \beta_1} \frac{2q}{[2]} \left( x - \frac{\alpha_2}{n + \beta_2} \right) + \frac{1}{[n + 1] + \beta_1} \left( \frac{\alpha_1}{[2]} + \frac{1}{[2]} \right) \]
(iii) For $f(t) = t^2$, in view of (2.1), we have

\[
K_{\alpha, \beta}^{(n, q)} = \left(\frac{[n] + \beta_2}{[n]}\right)^n \sum_{k=0}^{n} \left\lceil \frac{n}{k} \right\rceil \left( x - \frac{\alpha_2}{[n] + \beta_2} \right)^k \left( \frac{[n] + \alpha_2}{[n] + \beta_2} - x \right)^{n-k} \int_0^1 \left( \left\lceil \frac{k}{n + 1} + \beta_1 \right\rceil \cdot [x] \right) \, dq, t
\]

\[
= \frac{1}{(n + 1) + \beta_1^2} \left( \frac{[n] + \beta_2}{[n]} \right)^n \sum_{k=0}^{n} \left\lceil \frac{n}{k} \right\rceil \left( x - \frac{\alpha_2}{[n] + \beta_2} \right)^k \left( \frac{[n] + \alpha_2}{[n] + \beta_2} - x \right)^{n-k} \int_0^1 \left( [k] + (1 + (q - 1)(k) + \alpha_1) \right) \, dq, t
\]

\[
= \frac{1}{(n + 1) + \beta_1^2} \left( \frac{[n] + \beta_2}{[n]} \right)^n \sum_{k=0}^{n} \left\lceil \frac{n}{k} \right\rceil \left( x - \frac{\alpha_2}{[n] + \beta_2} \right)^k \left( \frac{[n] + \alpha_2}{[n] + \beta_2} - x \right)^{n-k} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} [k] \]

\[
+ \frac{2(q - 1)}{[3]} + 2\alpha_1 + \frac{2\alpha_1(q - 1)}{[2]} + \frac{2}{[2]} [k] + \alpha_1^2 + \frac{2\alpha_1}{[2]} \right\} \left( \frac{n + \beta_2}{n} \right)^n \sum_{k=0}^{n} \left\lceil \frac{n}{k} \right\rceil \left( x - \frac{\alpha_2}{[n] + \beta_2} \right)^k \left( \frac{[n] + \alpha_2}{[n] + \beta_2} - x \right)^{n-k} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \}
\]

\[
= \frac{1}{(n + 1) + \beta_1^2} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \left( \frac{[n] + \beta_2}{[n]} \right)^n \sum_{k=0}^{n} \left\lceil \frac{n}{k} \right\rceil \left( x - \frac{\alpha_2}{[n] + \beta_2} \right)^k \left( \frac{[n] + \alpha_2}{[n] + \beta_2} - x \right)^{n-k} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \}
\]

\[
= \frac{1}{(n + 1) + \beta_1^2} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \left( \frac{[n] + \beta_2}{[n]} \right)^n \sum_{k=0}^{n} \left\lceil \frac{n}{k} \right\rceil \left( x - \frac{\alpha_2}{[n] + \beta_2} \right)^k \left( \frac{[n] + \alpha_2}{[n] + \beta_2} - x \right)^{n-k} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \}
\]

\[
= \frac{1}{(n + 1) + \beta_1^2} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \left( \frac{[n] + \beta_2}{[n]} \right)^n \sum_{k=0}^{n} \left\lceil \frac{n}{k} \right\rceil \left( x - \frac{\alpha_2}{[n] + \beta_2} \right)^k \left( \frac{[n] + \alpha_2}{[n] + \beta_2} - x \right)^{n-k} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \}
\]

\[
= \frac{1}{(n + 1) + \beta_1^2} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \left( \frac{[n] + \beta_2}{[n]} \right)^n \sum_{k=0}^{n} \left\lceil \frac{n}{k} \right\rceil \left( x - \frac{\alpha_2}{[n] + \beta_2} \right)^k \left( \frac{[n] + \alpha_2}{[n] + \beta_2} - x \right)^{n-k} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \}
\]

\[
= \frac{1}{(n + 1) + \beta_1^2} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \left( \frac{[n] + \beta_2}{[n]} \right)^n \sum_{k=0}^{n} \left\lceil \frac{n}{k} \right\rceil \left( x - \frac{\alpha_2}{[n] + \beta_2} \right)^k \left( \frac{[n] + \alpha_2}{[n] + \beta_2} - x \right)^{n-k} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \}
\]

\[
= \frac{1}{(n + 1) + \beta_1^2} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \left( \frac{[n] + \beta_2}{[n]} \right)^n \sum_{k=0}^{n} \left\lceil \frac{n}{k} \right\rceil \left( x - \frac{\alpha_2}{[n] + \beta_2} \right)^k \left( \frac{[n] + \alpha_2}{[n] + \beta_2} - x \right)^{n-k} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \}
\]

\[
= \frac{1}{(n + 1) + \beta_1^2} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \left( \frac{[n] + \beta_2}{[n]} \right)^n \sum_{k=0}^{n} \left\lceil \frac{n}{k} \right\rceil \left( x - \frac{\alpha_2}{[n] + \beta_2} \right)^k \left( \frac{[n] + \alpha_2}{[n] + \beta_2} - x \right)^{n-k} \left\{ \left( 1 + (q - 1)^2 \right) + \frac{2(q - 1)}{[2]} \right\} \}
\]
Lemma 2.2. For all $x \in \left[\frac{\alpha_1}{n_1 + \beta_1}, \frac{\alpha_2}{n_1 + \beta_2}\right]$, we have

$$K^\alpha{\beta}(t - x)^2; x \leq \frac{2q^2 (2q + 1)}{2[3]} \frac{|n| (|n| + \alpha_2)}{|n| + \alpha_2} + \frac{q}{1 + q} \left(\frac{3 + 5q + 4q^2 + 4\alpha_1}{1 + q + q^2 + 4\alpha_1}\right) \frac{|n|}{(|n| + 1 + \beta_1)^2} + \frac{2}{1 + q} \frac{(2q|n| + 2\alpha_1 + 1)(|n| + \alpha_2)}{(|n| + 1 + \beta_1)(|n| + \beta_2)} + \left(\frac{|n| + \alpha_2}{|n| + \beta_2}\right)^2 + \left(\frac{1 + \alpha_1}{|n| + 1 + \beta_1}\right)^2$$

Proof. From Lemma 2.1, we have

$$K^\alpha{\beta}(t - x)^2; x = \left(\frac{q|n - 1|}{|n|} \left(1 + \frac{(q - 1)^2}{3} + \frac{2(q - 1)}{2}\right) \frac{|n| + \beta_2}{|n| + 1 + \beta_1} \right)^2 - \frac{4q}{1 + q} \frac{|n| + \beta_2}{|n| + 1 + \beta_1} + 1 \right) x^2$$

By using the monotonicity of positive linear operators $K^\alpha{\beta}$ over $\left[\frac{\alpha_1}{n_1 + \beta_1}, \frac{\alpha_2}{n_1 + \beta_2}\right]$, we have

$$K^\alpha{\beta}(t - x)^2; x \leq \left(\frac{1 - \frac{1}{|n|}}{1 + \frac{(q - 1)^2}{3} + \frac{2(q - 1)}{2}} \right) - \frac{2q|n - 1|}{|n|} \left(1 + \frac{(q - 1)^2}{3} + \frac{2(q - 1)}{2}\right) \frac{|n| + \beta_2}{|n| + 1 + \beta_1} + 1 \right) x^2$$

$$= \left(\frac{1 - \frac{1}{|n|}}{2[3]} \frac{|n| (|n| + \alpha_2)}{|n| + \alpha_2} + \frac{q}{1 + q} \left(\frac{3 + 5q + 4q^2 + 4\alpha_1}{1 + q + q^2 + 4\alpha_1}\right) \frac{|n|}{(|n| + 1 + \beta_1)^2} + \frac{2}{1 + q} \frac{(2q|n| + 2\alpha_1 + 1)(|n| + \alpha_2)}{(|n| + 1 + \beta_1)(|n| + \beta_2)} + \left(\frac{|n| + \alpha_2}{|n| + \beta_2}\right)^2 + \left(\frac{1 + \alpha_1}{|n| + 1 + \beta_1}\right)^2$$

$$\leq \frac{2q^2 (2q + 1)}{2[3]} \frac{|n| (|n| + \alpha_2)}{|n| + \alpha_2} + \frac{q}{1 + q} \left(\frac{3 + 5q + 4q^2 + 4\alpha_1}{1 + q + q^2 + 4\alpha_1}\right) \frac{|n|}{(|n| + 1 + \beta_1)^2} + \frac{2}{1 + q} \frac{(2q|n| + 2\alpha_1 + 1)(|n| + \alpha_2)}{(|n| + 1 + \beta_1)(|n| + \beta_2)} + \left(\frac{|n| + \alpha_2}{|n| + \beta_2}\right)^2 + \left(\frac{1 + \alpha_1}{|n| + 1 + \beta_1}\right)^2$$
which is the required result. 

\[ \text{Lemma 2.3.} \quad \text{Assume that } 0 < q_n < 1, \; q_n \to 1 \text{ and } q_n^a \to a \; (0 \leq a < 1) \text{ as } n \to \infty. \; \text{Then we have} \\
\lim_{n \to \infty} [n]_{q_n} K_{n,q_n}^{(\alpha,\beta)}(t-x) = -\frac{1 + a + 2(\beta_1 - \beta_2)}{2} x + \frac{1 + 2(\alpha_1 - \alpha_2)}{2}; \\
\lim_{n \to \infty} [n]_{q_n} K_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x) = (a + 2\beta_1 - 2\beta_2)x^2 + x. \]

\[ \text{Proof.} \quad \text{To prove the lemma we use formulae for } K_{n,q_n}^{(\alpha,\beta)}(t;x) \text{ and } K_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x) \text{ given in Lemma 2.1.} \]

\[ \lim_{n \to \infty} [n]_{q_n} K_{n,q_n}^{(\alpha,\beta)}(t-x;x) \]
\[ = \lim_{n \to \infty} \left( \frac{[n]_{q_n}}{(n+1)_{q_n} + \beta_1} \right) \frac{2q_n([n]_{q_n} + \beta_2)}{[2]_{q_n}} (1 + q_n) \left( \frac{[n]_{q_n} + \beta_1}{[2]_{q_n}} \right) \left( \frac{q_n - 1}{q_n} \right) \left( \frac{2q_n}{[2]_{q_n}} \right) \left( \frac{\alpha_1 + 1}{[2]_{q_n}} \right) \left( \frac{\beta_1}{[2]_{q_n}} \right) \left( \frac{\alpha_2}{[2]_{q_n}} \right) \left( \frac{\beta_2}{[2]_{q_n}} \right) \left( \frac{a}{[2]_{q_n}} \right) \left( \frac{x}{[2]_{q_n}} \right) \left( \frac{1 + 2(\beta_1 - \beta_2)}{2} \right) x + \frac{1 + 2(\alpha_1 - \alpha_2)}{2}. \]

\[ \lim_{n \to \infty} [n]_{q_n} K_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x) \]
\[ = \lim_{n \to \infty} \left( \frac{[n]_{q_n}}{(n+1)_{q_n} + \beta_1} \right) \frac{2q_n([n]_{q_n} + \beta_2)}{[2]_{q_n}} (1 + q_n) \left( \frac{[n]_{q_n} + \beta_1}{[2]_{q_n}} \right) \left( \frac{q_n - 1}{q_n} \right) \left( \frac{2q_n}{[2]_{q_n}} \right) \left( \frac{\alpha_1 + 1}{[2]_{q_n}} \right) \left( \frac{\beta_1}{[2]_{q_n}} \right) \left( \frac{\alpha_2}{[2]_{q_n}} \right) \left( \frac{\beta_2}{[2]_{q_n}} \right) \left( \frac{a}{[2]_{q_n}} \right) \left( \frac{x}{[2]_{q_n}} \right) \left( \frac{1 + 2(\beta_1 - \beta_2)}{2} \right) x + \frac{1 + 2(\alpha_1 - \alpha_2)}{2}. \]

\[ \text{3. Convergence results} \]

First we give the following theorem on convergence of \( K_{n,q_n}^{(\alpha,\beta)}(f;x) \) to \( f(x) \).

\[ \text{Theorem 3.1.} \quad \text{Let } q = q_n \in (0,1) \text{ be a sequence such that } q_n \to 1 \text{ as } n \to \infty \text{ and } f \text{ a continuous function on } [0,1]. \text{ Then} \\
\lim_{n \to \infty} \max_{0 \leq x \leq 1} \left| \frac{[n]_{q_n}}{(n+1)_{q_n} + \beta_1} \right| K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x) \right| = 0 \\
\text{Proof.} \quad \text{Taking into consideration the equalities in Lemma 1, for } v = 0, 1, 2 \text{ we can write} \\
(3.1) \quad \lim_{n \to \infty} \max_{0 \leq x \leq 1} \left| \frac{[n]_{q_n}}{(n+1)_{q_n} + \beta_1} \right| K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x) \right| = 0. \]

Now consider the sequence of operators
Theorem 3.2. \( K^\ast_{n,q_n}(f;\delta) = \begin{cases} K^\ast_{n,q_n}(x) & \text{if } \frac{\alpha_2}{|n| q_n + \beta_2} \leq x \leq \frac{|n| q_n + \alpha_2}{|n| q_n + \beta_2}; \\ f(x) & \text{if } x \in \left[0, \frac{\alpha_2}{|n| q_n + \beta_2}\right] \cup \left[\frac{|n| q_n + \alpha_2}{|n| q_n + \beta_2}, 1\right]. \end{cases} \)

Then obviously,

(3.2) \[ \|K^\ast_{n,q_n} - f\| = \max_{\alpha_2 \leq x \leq |n| q_n + \beta_2} |K^\ast_{n,q_n}(f;\delta) - f(x)| \]

and using (3.1) we obtain

\[ \lim_{n \to \infty} \|K^\ast_{n,q_n}(t^v;\delta) - x^v\|_{C[0,1]} = 0, \quad v = 0, 1, 2 \]

Applying the Korovkin theorem [11] (see also [2]) to the sequence of positive linear operators \( K^\ast_{n,q_n} \), we obtain

\[ \lim_{n \to \infty} \|K^\ast_{n,q_n}(f;\delta) - f(x)\|_{C[0,1]} = 0 \]

for every continuous function \( f \). Therefore (3.2) gives

\[ \lim_{n \to \infty} \max_{\alpha_2 \leq x \leq |n| q_n + \beta_2} |K^\ast_{n,q_n}(f;\delta) - f(x)| = 0 \]

and thus the result is obtained. \( \square \)

We use modulus of continuity to give quantitative error estimates for the approximation by positive linear operators.

**Theorem 3.2.** If \( f \in C[0,1] \) and \( 0 < q < 1 \), then

\[ \|K^\ast_{n,q}(f;\delta) - f(x)\| \leq 2\omega_f(\delta_n), \]

where

\[ \delta_n^2 = \frac{2q^2(2q + 1)}{2[3]} \frac{|n|(|n| + \alpha_2)}{|n + 1 + \beta_1|^2} + \frac{q}{1 + q} \frac{3 + 5q + 4q^2 + 4\alpha_1}{1 + q + q^2} \frac{|n|}{|n + 1 + \beta_1|^2} \]

\[ - \frac{2}{1 + q} \frac{(2q|n| + 2\alpha_1 + 1)(|n| + \alpha_2)}{|n + 1 + \beta_1)(|n| + \beta_2)} + \frac{|n| + \alpha_2}{|n + \beta_2|^2} + \left(1 + \alpha_1\right)^2 \frac{|n|}{|n + 1 + \beta_1|^2} \]

**Proof.** For any \( x, y \in [a, b] \), it is known that

\[ |f(y) - f(x)| \leq \omega_f(\delta) \left(\frac{y-x}{\delta^2}\right)^2 + 1 \]

Therefore, we get

\[ |K^\ast_{n,q}(f;\delta) - f(x)| \leq K^\ast_{n,q}(|f(t) - f(x)|;\delta) \leq \omega_f(\delta) \left(1 + \frac{1}{\delta^2} K^\ast_{n,q}((t-x)^2;\delta)\right) \]

By using Lemma 2.2, we can write

\[ |K^\ast_{n,q}(f;\delta) - f(x)| \leq \omega_f(\delta) \left\{ \frac{2q^2(2q + 1)}{2[3]} \frac{|n|(|n| + \alpha_2)}{|n + 1 + \beta_1|^2} + \frac{q}{1 + q} \frac{3 + 5q + 4q^2 + 4\alpha_1}{1 + q + q^2} \frac{|n|}{|n + 1 + \beta_1|^2} \right\} \]

Choosing

\[ \delta = \delta_n = \left\{ \frac{2q^2(2q + 1)}{2[3]} \frac{|n|(|n| + \alpha_2)}{|n + 1 + \beta_1|^2} + \frac{q}{1 + q} \frac{3 + 5q + 4q^2 + 4\alpha_1}{1 + q + q^2} \frac{|n|}{|n + 1 + \beta_1|^2} \right\} \]

\[ - \frac{2}{1 + q} \frac{(2q|n| + 2\alpha_1 + 1)(|n| + \alpha_2)}{|n + 1 + \beta_1)(|n| + \beta_2)} + \frac{|n| + \alpha_2}{|n + \beta_2|^2} + \left(1 + \alpha_1\right)^2 \frac{|n|}{|n + 1 + \beta_1|^2} \}

we have

\[ \|K^\ast_{n,q}(f;\delta) - f(x)\| \leq 2\omega_f(\delta_n). \]

Thus, we obtain the desired result. \( \square \)
4. LOCAL APPROXIMATION

We begin considering the following $K$-functional:

$$K_2(f, \delta^2) := \inf \{ \| f - g \| + \delta^2 \| g'' \|, \ g \in C^2[0,1] \}, \ \delta \geq 0,$$

where

$$C^2[0,1] := \{ g : \ g, g', g'' \in C[0,1] \}.$$ 

Then, in view of a known result [5], there exists an absolute constant $C_0 > 0$ such that

$$K_2(f, \delta^2) \leq C_0 \omega_2(f, \delta)$$

where

$$\omega_2(f, \delta) := \sup_{0 < h \leq \delta} \sup_{x \in [0,1]} | f(x - h) - 2f(x) + f(x + h) |$$

is the second modulus of smoothness of $f \in C[0,1]$.

**Theorem 4.1.** Let $f \in C[0,1]$ with $0 < q < 1$. Then for every $x \in \left[ \frac{a_n}{n+1} + \beta_1, \frac{a_n+\alpha_2}{n+1} + \beta_1 \right]$ we have

$$|K_{n,q}^{(\alpha,\beta)}(f; x) - f(x)| \leq C_0 \omega_2(f, \sqrt{\delta_n(x)}) + \omega(f, [(a_n - 1)x + b_n])$$

where $a_n = \frac{2q}{1+q} \frac{[n+\alpha_2][n+\beta_2]}{[n+1][n+\beta_1]}$, $b_n = \frac{1}{[n+1] + \beta_1} \left( \alpha_1 + \frac{1}{1+q} \right) - \frac{2q}{1+q} \frac{\alpha_2}{[n+1] + \beta_1}$ and

$$\delta_n(x) = \left\{ \begin{array}{l}
1 + 2q + 4q^2 + 5q^3 + \frac{[n+\beta_1]^2}{[n+1]+\beta_1} \frac{2}{1+q} \frac{[n+\beta_2]}{[n+1]+\beta_1} + 2 \\
+ \left\{ \frac{q^2(2q+1)}{1+q+q^2} \frac{\alpha_2}{[n+1]+\beta_1} - \frac{q}{1+q} \left( \frac{3+5q+4q^2}{1+q+q^2} + 4\alpha_1 \right) \frac{\alpha_2}{[n+1]+\beta_1} + 2 \right\} x^2 \\
\end{array} \right.$$

**Proof.** Let

$$K_{n,q}^{(\alpha,\beta)}(f; x) = K_{n,q}^{(\alpha,\beta)}(f; x) + f(x) - f(a_n x + b_n)$$

where $f \in C[0,1]$, $a_n = \frac{2q}{1+q} \frac{[n+\alpha_2]}{[n+1] + \beta_1}$ and $b_n = \frac{1}{[n+1] + \beta_1} \left( \alpha_1 + \frac{1}{1+q} \right) - \frac{2q}{1+q} \frac{\alpha_2}{[n+1] + \beta_1}$. Using the Taylor formula

$$g(t) = g(x) + g_x''(t-s)g''[0,1],$$

we have

$$K_{n,q}^{(\alpha,\beta)}(g; x) = g(x) + K_{n,q}^{(\alpha,\beta)} \left( \int_x^d (t-s)g''(s)ds; x \right) - \int_x^{a_n x + b_n} (a_n x + b_n - s)g''[0,1].$$
Hence

\[
\left| K_{n,q}^{(\alpha,\beta)}(g(x) - g(x)) \right| \leq K_{n,q}^{(\alpha,\beta)} \left( \left| \int_x^t (t-s)g''(s)\,ds \right| + \int_x^a \left| a_n x + b_n - s \right| \left| g''(s) \right| ds \right) \\
\leq K_{n,q}^{(\alpha,\beta)} \left( (t-x)^2 \| g'' \| + (a_n x + b_n - x)^2 \| g'' \| \right) \\
= \left\{ \frac{q[n-1]}{n} \left( \alpha + 1 \right) \left( \frac{[n+1]+\beta_1}{n+1} \right)^2 \right. \\
- \frac{q[n-1]}{n} \left( \frac{[n+1]+\beta_1}{n+1} \right)^2 - \frac{2q}{n} \left( \frac{[n+1]+\beta_1}{n+1} \right) \left( \frac{[n+1]+\beta_1}{n+1} \right) - \frac{1}{n} \left( q \frac{[n+1]+\beta_1}{n+1} \right) \\
\left. + \frac{2q}{n} \left( [n+1]+\beta_1 \right) \right\} \| g'' \| \\
= \left\{ \frac{2q}{n} \left( \frac{[n+1]+\beta_1}{n+1} \right) \right\} \| g'' \| \\
\leq \left\{ \frac{2q}{n} \left( \frac{[n+1]+\beta_1}{n+1} \right) \right\} \| g'' \| \leq \delta_n(x) \| g'' \| (3.3)
\]

Using (3.3) and the uniform boundedness of \( K_{n,q}^{(\alpha,\beta)} \), we get

\[
\left| K_{n,q}^{(\alpha,\beta)}(f; x) - f(x) \right| \leq \left| K_{n,q}^{(\alpha,\beta)}(g; x) \right| + \left| f(x) - g(x) \right| + \left| f(a_n x + b_n) - f(x) \right| \\
\leq 4 \| f \| + \delta_n(x) \| g'' \| + \omega(f, |a_n - 1| x + b_n)
\]

Taking the infimum on the right hand side over all \( g \in C^2[0,1] \), we obtain

\[
\left| K_{n,q}^{(\alpha,\beta)}(f; x) - f(x) \right| \leq C \omega_2(f, \sqrt{\delta_n(x)}) + \omega(f, |a_n - 1| x + b_n)
\]

This completes the proof.

**Corollary 4.2.** Assume that \( q_n \in (0,1) \), \( q_n \to 1 \) as \( n \to \infty \). For any \( f \in C^2[0,1] \) we have

\[
\lim_{n \to \infty} \left| K_{n,q}^{(\alpha,\beta)}(f) - f(x) \right| = 0.
\]

\[
\]
Furthermore, we estimate the rate of convergence for smooth functions. For this reason, we now state the following general estimate theorem obtained by Shisha and Mond [25] in terms of the modulus of continuity.

**Theorem 4.3.** Let \([c, d] \subseteq [a, b]\) and \((L_n)_{n \in \mathbb{N}}\) be a sequence of positive linear operators such that \[L_n : C[a, b] \to C[c, d]\]

If \(f \in C[a, b]\) and \(x \in [c, d]\), then we have

\[
|L_n(f; x) - f(x)| \leq |f(x)||L_n(1; x) - 1| + |f'(x)||L_n(t-x; x)| + \sqrt{L_n((t-x)^2; x)} \times \left\{ \sqrt{L_n(1; x)} + \frac{1}{\delta} \sqrt{L_n((t-x)^2; x)} \right\} \omega(f')
\]

where \(\omega\) is the modulus of continuity of the function \(f\) defined by

\[
\omega(f; \delta) = \sup \{|f(x) - f(y)| : x, y \in [0, 1], |x - y| \leq \delta\}
\]

for any positive number \(\delta\).

**Theorem 4.4.** For any \(f \in C^1[0, 1]\) and each \(x \in \left[\frac{n\alpha}{n+\beta_2} \left[\frac{n}{n+\beta_2} + \frac{2}{\beta_2} \right] \right]\), we have

\[
|K_{n,q}^{(\alpha, \beta)}(f; x)| \leq \left( \frac{2q}{1 + q} \frac{|n| + \beta_2}{|n| + 1 + \beta_1} - 1 \right) x + 1 + \frac{1 + \alpha + q\alpha - 2\alpha q_2}{(1 + q)(|n| + 1 + \beta_1)} |f'(x)| + 2 \sqrt{\delta_n(x)} \omega(f', \sqrt{\delta_n(x)})
\]

**Proof.** In view of Lemma 2.1, Lemma 2.2 & Theorem 4.3, and if we choose \(\delta = \sqrt{\delta_n(x)} = \sqrt{K_{n,q}^{(\alpha, \beta)}}((t-x)^2; x)\), we have

\[
|K_{n,q}^{(\alpha, \beta)}(f; x)| \leq \left( \frac{2q}{1 + q} \frac{|n| + \beta_2}{|n| + 1 + \beta_1} - 1 \right) x + 1 + \frac{1 + \alpha + q\alpha - 2\alpha q_2}{(1 + q)(|n| + 1 + \beta_1)} |f'(x)| + 2 \sqrt{\delta_n(x)} \omega(f', \sqrt{\delta_n(x)})
\]

Next we prove Voronovskaja type result for Kantorovich type \(q\)-Bernstein-Stancu operators.

**Theorem 5.1.** Assume that \(q = q_n \in (0, 1), q_n \to 1\) and \(q_n^a \to a (0 \leq a < 1)\) as \(n \to \infty\). For any \(f \in C^2[0, 1]\) the following equality holds

\[
\lim_{n \to \infty} [n]_{q_n} \left( K_{n,q_n}^{(\alpha, \beta)}(f; x) - f(x) \right) = f'(x) \left( -\frac{1 + a + 2(\beta_1 - \beta_2)}{2} x + \frac{1 + 2(\alpha_1 - \alpha_2)}{2} + \frac{1}{2} f''(x) \right) (a + 2\beta_1 - 2\beta_2) x^2 + x
\]

uniformly on \(x \in \left[\frac{\alpha}{|n| + \beta_2} \left[\frac{n\alpha}{|n| + \beta_2} + \frac{2}{\beta_2} \right] \right]\).

**Proof.** Let \(f \in C^2[0, 1]\) and \(x \in [0, 1]\) be fixed. By the Taylor formula we may write

\[
f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(t)(t-x)^2,
\]

where \(r(t; x)\) is the Peano form of remainder, \(r(.) \in C[0, 1]\) and \(\lim_{t \to x} = 0\). Applying \(K_{n,q_n}^{(\alpha, \beta)}(f; x)\) on both sides of (3.4), we obtain

\[
[n]_{q_n} \left( K_{n,q_n}^{(\alpha, \beta)}(f; x) - f(x) \right) = f'(x)[n]_{q_n} K_{n,q_n}^{(\alpha, \beta)}((t-x); x) + \frac{1}{2} f''(x)[n]_{q_n} K_{n,q_n}^{(\alpha, \beta)}((t-x)^2; x) + [n]_{q_n} K_{n,q_n}^{(\alpha, \beta)}(r(t; x)(t-x)^2; x).
\]

By the Cauchy-Schwartz inequality, we have

\[
K_{n,q_n}^{(\alpha, \beta)}((t-x)^2; x) \leq \sqrt{K_{n,q_n}^{(\alpha, \beta)}(r^2(t; x); x) K_{n,q_n}^{(\alpha, \beta)}((t-x)^4; x)}
\]
Observe that \( r^2(x, x) = 0 \) and \( r^2(\cdot; x) \in C[0, 1] \). Then it follows from Corollary 3.4 that

\[
(5.3) \quad \lim_{n \to \infty} K_{n,q_n}^{(\alpha, \beta)}(r^2(t; x); x) = r^2(x, x) = 0
\]

uniformly with respect to \( x \in \left[ \frac{\alpha_2}{[n]_{q_n} + \beta_2}, \frac{[n]_{q_n} + \alpha_2}{[n]_{q_n} + \beta_2} \right] \). Now from (3.5), (3.6) and Lemma 3.4 we get immediately

\[
\lim_{n \to \infty} [n]_{q_n} K_{n,q_n}^{(\alpha, \beta)}(r(t; x)(t-x)^2; x) = 0.
\]

The proof is completed. \( \square \)

Now we give the rate of convergence of the operators \( K_{n,q_n}^{(\alpha, \beta)} \) in terms of the elements of the usual Lipschitz class \( \text{Lip}_M(\alpha) \).

Let \( f \in C[0, 1], M > 0 \) and \( 0 < \alpha \leq 1 \). We recall that \( f \) belongs to the class \( \text{Lip}_M(\alpha) \) if the inequality

\[
|f(t) - f(x)| \leq M |t - x|^\alpha \quad (t, x \in [0, 1])
\]

is satisfied.

**Theorem 5.2.** Let \( q = q_n \in (0,1) \) such that \( \lim_{n \to \infty} q_n = 1 \). Then for each \( f \in \text{Lip}_M(\alpha) \) we have

\[
\|K_{n,q_n}^{(\alpha, \beta)} - f(x)\| \leq M\delta_n^\alpha
\]

where \( \| \cdot \| \) is the supremum norm over \( \left[ \frac{\alpha_2}{[n]_{q_n} + \beta_2}, \frac{[n]_{q_n} + \alpha_2}{[n]_{q_n} + \beta_2} \right] \) and

\[
\|K_{n,q_n}^{(\alpha, \beta)}(f) - f\| \leq M \left[ \frac{2q_n^2(2q_n + 1)}{[2]_{q_n}[3]_{q_n}} \left( \frac{[n]_{q_n} + \alpha_2}{[n]_{q_n} + \beta_2} \right)^2 + q_n \left( \frac{3 + 5q_n + 4q_n^2 + 4\alpha_1}{[n]_{q_n} + \beta_1} \right)^2 \right] \left( \frac{[n]_{q_n} + \alpha_2}{[n]_{q_n} + \beta_2} \right)^2 + \left( \frac{3 + 5q_n + 4q_n^2 + 4\alpha_1}{[n]_{q_n} + \beta_1} \right)^2 \left( \frac{[n]_{q_n} + \alpha_2}{[n]_{q_n} + \beta_2} \right)^2.
\]

**Proof.** Let us denote \( P_{n,k}^{(\alpha, \beta)}(x) = \left( \frac{\alpha_2}{[n]_{q_n} + \beta_2} \right)^k \sum_{k=0}^{n} \binom{n}{k} q_n \left( x - \frac{\alpha_2}{[n]_{q_n} + \beta_2} \right)^k \left( \frac{[n]_{q_n} + \alpha_2}{[n]_{q_n} + \beta_2} - x \right)^{n-k} \).

Then by the monotonicity of the operators \( K_{n,q_n}^{(\alpha, \beta)} \), we can write

\[
|K_{n,q_n}^{(\alpha, \beta)} - f(x)| \leq K_{n,q_n}^{(\alpha, \beta)}(|f(t) - f(x)|; x)
\]

\[
\leq \sum_{k=0}^{n} P_{n,k}^{(\alpha, \beta)}(x) \int_0^1 \left| f \left( \frac{[k]_{q_n} + q_n^kt + \alpha_1}{[n+1]_{q_n} + \beta_1} \right) - f(x) \right| d_qn t
\]

\[
\leq M \sum_{k=0}^{n} P_{n,k}^{(\alpha, \beta)}(x) \int_0^1 \left( \frac{[k]_{q_n} + q_n^kt + \alpha_1}{[n+1]_{q_n} + \beta_1} - x \right) d_qn t.
\]

On the other hand, by using the Hölder’s inequality for integrals with \( p = \frac{2}{\alpha} \) and \( q = \frac{2}{2-\alpha} \), we have

\[
|K_{n,q_n}^{(\alpha, \beta)} - f(x)| \leq M \sum_{k=0}^{n} P_{n,k}^{(\alpha, \beta)}(x) \left( \int_0^1 \left( \frac{[k]_{q_n} + q_n^kt + \alpha_1}{[n+1]_{q_n} + \beta_1} - x \right)^2 d_qn t \right)^{\frac{\alpha}{2}} \left( \int_0^1 1 d_qn t \right)^{\frac{2}{2-\alpha}}
\]

\[
= M \sum_{k=0}^{n} P_{n,k}^{(\alpha, \beta)}(x) \left( \int_0^1 \left( \frac{[k]_{q_n} + q_n^kt + \alpha_1}{[n+1]_{q_n} + \beta_1} - x \right)^2 d_qn t \right)^{\frac{\alpha}{2}} \left( \int_0^1 1 d_qn t \right)^{\frac{2}{2-\alpha}}.
\]


Now again applying the Hölder’s inequality for the sum with \( p = \frac{\alpha}{\alpha} \) and \( q = \frac{2}{\alpha - \alpha} \) and taking into consideration Lemma 2.1(i) and Lemma 2.2, we have

\[
\left| K_{n,q}^{(\alpha,\beta)}(f; x) - f(x) \right| \leq M \left( K_{n,q}^{(\alpha,\beta)}((t-x)^2; x) \right)^{\frac{\alpha}{\alpha}} \left( K_{n,q}^{(\alpha,\beta)}(1; x) \right)^{\frac{2}{\alpha - \alpha}}
\]

\[
\leq M \left\{ \left( \frac{2n^2}{[n]_{q_n}^2} + \frac{2q_n}{[2]_{q_n}} \right) \left( \frac{[n]_{q_n} + \beta_2}{[n + 1]_{q_n} + \beta_1} \right)^2 - \frac{4q_n}{1 + q_n} \left[ \frac{[n]_{q_n} + \beta_2}{[n + 1]_{q_n} + \beta_1} + 1 \right] \right\}
\]

Replacing \( x \) by \( \frac{[n]_{q_n} + \beta_2}{[n]_{q_n} + \beta_1} \) implies that

\[
\| K_{n,q_n}^{(\alpha,\beta)}(f) - f \| \leq M \left\{ \left( \frac{2q_n^2}{[2]_{q_n} [3]_{q_n}} + \frac{2q_n}{[n + 1]_{q_n} + \beta_1} \right) + \frac{q_n}{1 + q_n} \left( \frac{3 + 5q_n + 4q_n^2}{1 + q_n + q_n^2} + 4\alpha_1 \right) \left( \frac{[n]_{q_n}}{[n + 1]_{q_n} + \beta_1} \right)^2 \right. \]

\[
- \frac{2}{1 + q_n} \left( \frac{2q_n [n]_{q_n} + 2\alpha_1 + 1 ([n]_{q_n} + \alpha_2)}{[n + 1]_{q_n} + \beta_1 ([n]_{q_n} + \beta_2)} \right) + \left( \frac{[n]_{q_n} + \alpha_2}{[n]_{q_n} + \beta_2} \right)^2 + \left( \frac{1 + \alpha_1}{[n + 1]_{q_n} + \beta_1} \right)^2 \left[ \frac{\alpha_2}{\alpha_1} + \frac{2\alpha_1}{[2]_{q_n}} + \frac{1}{[3]_{q_n}} \right]
\]

Hence if we choose \( \delta := \delta_n \), then we arrive at the desired result.

\[ \square \]

6. Graphical Analysis

With the help of Matlab, we show comparisons and some illustrative graphics [21] for the convergence of operators (2.1) to the function \( f(x) = 1 - \cos(4e^x) \) under different parameters.

From figure 6.1, 4.2, 4.3, we can observe that as the value the \( n \) increases, Kantorovich type \( q \)-Bernstein-Stancu operators given by (2.1) converges towards the function.

![Convergence of operators to the function](image)

\[ \text{Figure 6.1.} \]

12
Similarly as the value the $q$ increases, convergence of operators to the function is shown in figure 4.4 with different values of parameters $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, and $n$. 

**Figure 6.2.**

**Figure 6.3.**

**Figure 6.4.**

13
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Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
E-mail address: mursaleenm@gmail.com; ansarijkhursheed@gmail.com; asifjnu07@gmail.com

14