Conformal Covariance Subalgebras

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Abstract

We give a direct Lie algebraic characterisation of conformal inclusions of chiral current algebras associated with compact, reductive Lie algebras. We use quantum field theoretic arguments and prove a longstanding conjecture of Schellekens and Warner on grounds of unitarity and positivity of energy. We explore the structures found to characterise conformal covariance subalgebras and coset current algebras.

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1 Introduction

Conformal inclusions of chiral current algebras are of interest for a large variety of reasons. Their classification was undertaken some time ago, because they are particularly relevant to string theory for their making string compactification possible without altering conformal covariance. Using general arguments this task was transferred to checking maximal inclusions of reductive Lie algebras in simple Lie algebras, for which a classification was available already. The classification of conformal inclusions was thus achieved, looking at the central charge of the respective stress-energy tensors, by several authors [AGO87, BB87, SW86].

Many of the conformal inclusions were found to correspond to symmetric spaces (cf. [GNO85, Dab96] in particular), and isotropy irreducibility of the coset space proved a useful yet neither necessary nor sufficient criterion for an inclusion being conformal. We undertake a complete characterisation of conformal inclusions by means of straightforward arguments familiar in (axiomatic) quantum field theory. On the course we prove a longstanding\(^1\) conjecture of Schellekens and Warner [SW86].

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\(^1\)To the author’s surprise, there does not seem to be a proof available yet.
We use properties of any Wightman quantum field theory\textsuperscript{2}: positivity of energy, separating property of the vacuum for local quantum fields, and unitarity. Our analysis clarifies the situation in natural group theoretical terms and in direct correspondence to quantum field theoretical notions. Moreover, there is no need to specialise in maximal subalgebras and our approach is rather direct in that respect.

The point of view taken in this work arose from a more general question: how does the inner-implementing representation $U_A$, uniquely associated with every covariant subtheory $\mathcal{A}$ of a chiral conformal theory $\mathcal{B}$ by means of the Borchers-Sugawara construction [Kös02], act on the observables of the larger theory $\mathcal{B}$? An answer to the more general question is given in an independent work [Kös03].

We proceed as follows: In the next section we introduce notations and conventions, prove the conjecture of Schellekens and Warner and provide a direct argument for conformal inclusions being necessarily restricted to level 1. The third section is about studying conformal covariance subalgebras associated to Lie algebra inclusions, these being intermediate to the original inclusion, if not trivial. The section will be closed by a simple characterisation of coset currents, i.e. current subalgebras commuting with the given current subalgebra.

\section{Characterisation of conformal inclusions}

We study a current algebra associated with a simple\textsuperscript{3}, compact Lie algebra $\mathfrak{g}$ consisting of symmetric operator valued distributions, the currents. We will treat these as fields on the chiral light ray. Basis elements of $\mathfrak{g}$ will be denoted by $T^a$; they give the colour of the corresponding current $j^a$. The current algebra is given by the following commutation relations:

\begin{equation*}
[j^a(x), j^b(y)] = i f^{ac} j^c(x) \delta(x-y) + k g^{ab}_g \frac{i}{2\pi} \delta'(x-y) .
\end{equation*}

$g_g$ denotes the Killing metric of $\mathfrak{g}$, $f^{abc}$ its structure constants and $k$ the current algebra’s level; $k$ is a positive integer.

By embedding a reductive Lie subalgebra $\mathfrak{h}$ into $\mathfrak{g}$ via an injective homomorphism $\iota : \mathfrak{h} \hookrightarrow \mathfrak{g}$ we have an associated current subalgebra. $\mathfrak{h}$ consists of several simple ideals, denoted (for the time being) by $\mathfrak{h}_\alpha$, and an abelian ideal of dimension $n \geq 0$. The inclusions $\iota(\mathfrak{h}_\alpha) \subset \mathfrak{g}$ are partly characterised by their

\textsuperscript{2}Wightman’s axioms are manifestly fulfilled for all current algebras which are available as quark models [BH71] (corresponding to abelian Lie algebras $\mathbb{R}^n$ and to the classical Lie algebras of type $A_n$, $B_n$, $C_n$, $D_n$ and to the exceptional Lie algebra $G_2$). For the remaining four cases (corresponding to the exceptional Lie algebras of type $E_6$, $E_7$, $E_8$, $F_4$) Wightman’s axioms appear implicitly in the literature [Kac90], [GW84], [TL97].

\textsuperscript{3}General reasoning leads to an extension of the following discussion to inclusions of reductive subalgebras in reductive Lie algebras, cf. eg. AGO87.
Dynkin index $I_\alpha$. Denoting the Killing metric of $\mathfrak{h}_\alpha$ by $g_\alpha$ we have the following commutation relations for currents associated with colours in $\iota(\mathfrak{h}_\alpha)$:

$$\left[ j^{(a)}(x), j^{(b)}(y) \right] = if^{(a)(b)}_{\iota(c)} j^{(c)}(x) \delta(x-y) + I_\alpha k g^{ab}_\alpha \frac{i}{2\pi} \delta'(x-y) .$$

The infinitesimal conformal transformations are implemented by the adjoint action of the Sugawara stress-energy tensor $\Theta^\theta$. It is given by:

$$\Theta^\theta(x) = \frac{\pi}{k + v^\theta} g^{\theta}_{ab} : j^a j^b : (x) .$$

$v^\theta$ is the dual Coxeter number of $\mathfrak{g}$. The commutation relation of $\Theta^\theta$ with a current reads as:

$$\left[ \Theta^\theta(x), j^c(y) \right] = ij^c(x) \delta'(x-y) .$$

Restricting to colours in $\iota(\mathfrak{h}_\alpha)$ the same construction yields a stress-energy tensor $\Theta^\alpha$ having the same commutation relation with currents associated with colours in $\iota(\mathfrak{h}_\alpha)$:

$$\Theta^\alpha(x) = \frac{\pi}{I_\alpha k + v^\alpha} g^{\alpha}_{ab} : j^{(a)} j^{(b)} : (x) .$$

For the abelian ideal we adopt the following conventions: $I_{\mathbb{R}^n} := 1$, $v^{\mathbb{R}^n} := 0$, $g^{ij}_{\mathbb{R}^n} := g^{\iota(i)(j)}_{\theta}$. Using these as input all the formulas above apply to currents associated with colours in $\iota(\mathbb{R}^n)$. We drop the distinction between simple and abelian ideals of $\mathfrak{h}$ and use the symbol $\mathfrak{h}_\alpha$ for any simple or the abelian ideal from now on.

With this general notation the action of a stress-energy tensor $\Theta^\alpha$ on an arbitrary current $j^c$ reads as:

$$\left[ \Theta^\alpha(x), j^c(y) \right] = \frac{\pi}{I_\alpha k + v^\alpha} g^{\alpha}_{ab} i f^{(b)c}_{\iota(d)} j^{(a)} j^d + j^d j^{(a)} : (x) \delta(x-y)$$

$$+ i \frac{k}{I_\alpha k + v^\alpha} j^{(a)}(x) g^{\alpha}_{ab} g^{(b)c}_\theta \delta'(x-y)$$

$$+ i \frac{1}{2(I_\alpha k + v^\alpha)} j^d(x) (C^\alpha_2)_{d}^{c} \delta'(x-y) . \tag{1}$$

This equation is obtained by applying the current algebra and the normal ordering prescription for currents [FST89]. The matrix $C^\alpha_2$ stands for the second Casimir element of $\mathfrak{h}_\alpha$ in the representation $\operatorname{Ad}_\theta \circ \iota|_{\mathfrak{h}_\alpha}$, if $\mathfrak{h}_\alpha$ is a simple ideal. In any case we have:

$$(C^\alpha_2)_{d}^{c} = g^{\alpha}_{ab} i f^{(b)e}_{\iota(d)} i f^{(e)c} = g^{\alpha}_{ab} (\operatorname{Ad}_{T^{(e)}} \operatorname{Ad}_{T^{(a)}})_{d}^{c} .$$

Taking the trace of this matrix one may readily see that it does not vanish for the abelian ideal.

Now we are prepared to state and prove our main result. Schellekens and Warner conjectured it in their discussion closing [SW86].
Theorem 1  The following holds true for the weighted Casimir element $\tilde{C}_2^{\alpha}(h)$ of $\iota(h)$ ($P_{\alpha}$ stands for the projection onto $\iota(h_\alpha)$):

$$\tilde{C}_2^{\alpha}(h) := \sum_{\alpha} \frac{2I_{\alpha}kP_{\alpha} + C_{2\alpha}}{2(I_{\alpha}k + v^\alpha)} \leq 1.$$  \hspace{1em} (2)

This inequality is saturated if and only if $\iota(h) \subset g$ yields a conformal inclusion, i.e. $\sum_{\alpha} \Theta^\alpha =: \Theta^g = \Theta^h$.

**Proof:** By invariance of $g^\theta$ the orthocomplementation $g = \iota(h) + \iota(h)^\perp$ provides a reduction of the representation $Ad_g \circ \iota$. We have $C_{2\alpha}|_{\iota(h)} = 2v^\alpha P_{\alpha}$, i.e. $\tilde{C}_2^{\alpha}(h)|_{\iota(h)} = 1$, and the inequality only remains to be proven for colours orthogonal to $\iota(h)$, where $P_{\alpha}|_{\iota(h)^\perp} = 0$. Because all CASIMIR elements commute and all are positive operators, we assume as well that $T^c$ is a common eigenvector for all linear mappings $C_{2\alpha}$.

We prove the inequality by looking at specific expectation values of the coset HAMILTONian $L_0^g - L_0^h$. This is a positive operator, which is given by the coset stress-energy tensor $\Theta^g - \Theta^h$ smeared with the test function $\xi_{L_0^g}(x) = \frac{1}{2}(x^2 + 1)$. The infinitesimal action of a conformal HAMILTON operator on the test function of a smeared field covariant with respect to it shall be abbreviated by $l_0$, i.e. we have

$$[L_0^g, j^c(g)] = i \int dx g'(x) \xi_{L_0^g}(x) j^c(x) \equiv i \int dx l_0 g(x) j^c(x) = i j^c(l_0 g).$$

Using the general commutation relation \[1\], calculating two and three point functions of currents (cf. \[FST89\]), observing that some group-theoretical tensors involved are null for reasons of permutation symmetry/ antisymmetry and carefully taking into account the normal ordering of currents \[FST89\] one arrives at the following formula:

$$0 \leq \langle \Omega, j^c(g)^{\dagger}(L_0^g - L_0^h)j^c(g)\Omega \rangle = i \left(1 - \sum_{\alpha} C_{2\alpha}^{\alpha} \frac{C_{2\alpha}^c}{2(I_{\alpha}k + v^\alpha)}\right) \langle \Omega, j^c(g)^{\dagger}j^c(l_0 g)\Omega \rangle.$$  \hspace{1em} (3)

The desired inequality may be established through division by $i \langle \Omega, j^c(g)^{\dagger}j^c(l_0 g)\Omega \rangle$, which does not vanish for generic $g$ and is positive as an expectation value of $L_0^g - L_0^h \geq 0$.

If we have $\Theta^g = \Theta^h$, \[2\] is saturated on $\iota(h)$ trivially and because of \[3\] on $\iota(h)^\perp$ as well, hence on all of $g$. The conclusion in the opposite direction is, actually, a consequence of equation \[1\] in proposition \[3\]. This leads to trivial commutation relations for $\Theta^g - \Theta^h$, especially to $c_g = c_h$, which yields, by a variant of the REEH-SCHLIEDE styl e theorem\[4\], $\Theta^g - \Theta^h = 0$.

\[\square\]

\[4\] See for example \[Jos65\] (lemma 2, section V.3.B); for an argument directly referring to the Virasoro algebra see \[GW85\], \[Gom86\].
**Corollary 2** An embedding $\iota(h) \subset g$ can give rise to a conformal inclusion of the associated current algebras only, if the current algebra associated with $g$ has level $k = 1$.

**Proof:** Highest-weight representations of current algebras may be characterised uniquely by a vector of lowest energy which is a highest-weight vector with respect to the horizontal subalgebra of currents $j^a([1])$ smeared with the constant test-function $[1](x) = 1$. We look at the representation defined by the highest weight $\psi_\theta$ of the adjoint representation of $g$. Since $\psi_\theta$ has, by the usual convention, length 2, this representation is in accordance with the Weyl alcove condition \[FST89\] (4.51) for unitary representations of current algebras for $k \geq 2$. The following argument applies, therefore, to all but level 1.

Actually, we may restrict attention to the action of $L_0^g - L_0^h$ on $g\psi_\theta$, the highest-weight module of $g$ generated from the vector with lowest energy and highest weight $\psi_\theta$. Here we have:

$$0 \leq \left(L_0^g - L_0^h\right)_{\psi_\theta} = \frac{v^g}{k + v^g} \mathbb{1} - \sum_a \frac{C_a^2}{2(I_\alpha k + v^a)} .$$

This implies a strictly sharper bound than inequality (2) and by theorem \[\Box\] immediately yields the desired result.

\[\Box\]

**3 Covariant and invariant colours**

After we have given a characterisation of conformal inclusions $\iota(h) \subset g$, we now pursue further the structures in colour space which are associated with the action of $\Theta^b$ on currents with colours in $g$. We find that **covariant** and **invariant** colours form reductive Lie algebras, the first being intermediate to the original embedding $\iota(h) \subset g$, the second being orthogonal to and commuting with it. All these results are in terms of the **weighted Casimir element** $\tilde{\mathcal{C}}_{2(h)}^2$ of the Lie algebra $\iota(h)$, which already appeared in the previous section.

The following is the main ingredient of the results in this section:

**Proposition 3** For an arbitrary colour $T^c \in g$ we have:

$$\|\left[\left(\Theta^a - \Theta^b\right)(f), j^c(g)\right]\| \Omega \|^2 = 8k\pi^2(1 - \tilde{\mathcal{C}}_2^b)T^c, \tilde{\mathcal{C}}_2^bT^c)_{\theta} \tilde{\Delta}^4(f \cdot g, f \cdot g) + k(1 - \tilde{\mathcal{C}}_2^b)T^c, (1 - \tilde{\mathcal{C}}_2^b)T^c)_{\theta} \tilde{\Delta}^2(f \cdot g', f \cdot g') .$$

Here $\langle ., . \rangle_{\theta}$ stands for the scalar product on $g$ induced by the Killing form. We define

$$\Phi^c(x) := \sum_a \frac{1}{2(I_\alpha k + v^a)} g^\alpha_{ab} \mu^{(b)c}_d : j^a(x) j^d + j^d j^a(x) : (x) .$$

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The two-point function of $\Phi^c$ is given by:

$$\langle \Omega, \Phi^c(x)\Phi^c(y)\Omega \rangle = 2k\tilde{\Delta}^4(x-y)\langle(1 - \tilde{C}_2^b)T^c, \tilde{C}_2^bT^c\rangle_g.$$  \hfill (5)

The numerical distributions in these formulae are given by:

$$\tilde{\Delta}^4(\tilde{f} \cdot g, f \cdot g) = (2\pi)^{-4}\int dx\,dy\, (i[(x - y) - i\varepsilon])^{-4}\tilde{f} \cdot g(x)f \cdot g(y),$$

$$\tilde{\Delta}^2(\tilde{f} \cdot g', f \cdot g') = (2\pi)^{-2}\int dx\,dy\, (i[(x - y) - i\varepsilon])^{-2}\tilde{f} \cdot g'(x)f \cdot g'(y).$$

**Proof:** We will not give the derivation of these formulae in detail. We rather indicate their verification. First, one may restrict attention to colours $T^c \in \iota(h)$ since the weighted CASIMIR respects the orthogonal decomposition $g = \iota(h) + \iota(h)^\perp$ with respect to $Ad \circ \iota$ and $\Theta^b - \Theta^b$ commutes with all currents whose colours are in $\iota(h)$. “All” that one has to do is to apply the general commutation relation (1) restricted to colours from $\iota(h)^\perp$, follow carefully the normal ordering of currents, observe symmetries of group theoretical coefficients, keep in mind $T^c \in \iota(h)^\perp$, calculate some $n$-point functions of currents following the scheme in [EST80], use JACOBI’s identity a few times and recognise the second CASIMIR element in the adjoint representation, which amounts to twice the dual COXETER number. With all that, it is a straightforward algebraic exercise.

□

**Remark:** Taking $g$ as the test function of constant value 1, equation (1) implies $\tilde{C}_2^b(1 - \tilde{C}_2^b) \geq 0$, from which we immediately get inequality (2), and the second statement in theorem (4) follows from (1), too.

**Definition 4** A current $j^c$ is said to transform covariantly with respect to $\Theta^b$, if and only if $[\Theta^b(f), j^c(g)] = [\Theta^b(f), j^c(g)] \forall f, g$.

**Corollary 5** A current $j^c$ transforms covariantly with respect to $\Theta^b$, if and only if its colour fulfills: $\tilde{C}_2^{b}(b)T^c = T^c$. These covariant colours form a reductive LIE algebra, $\mathfrak{k}$, containing $\iota(h)$ as a subalgebra. If $\mathfrak{k} \neq \iota(h)$, then the level of the current algebra associated with $g$ has to be $k = 1$.

**Proof:** If we have $\tilde{C}_2^{b}(b)T^c = T^c$, we know from the variant of the REEH-SCHLIEDER theorem (see footnote [4] and proposition [3] above, that $j^c$ and the coset stress-energy tensor commute. This is another way of saying: $j^c$ transforms covariantly with respect to $\Theta^b$.

Conversely: If $j^c$ is covariant with respect to $\Theta^b$, the group theoretical scalar products in equation (1) have to be zero, since the numerical distributions involved are linearly independent. The second one of these is the norm of $(1 - \tilde{C}_2^{b}(b))T^c$, which makes the equation $\tilde{C}_2^{b}(b)T^c = T^c$ valid. Now, if $T^a$ and $T^b$ are covariant colours, then so is $-i[T^a, T^b]$. This is clear, if one observes $f^{ab}c^i(g) = -i[j^a([1]), j^b(g)]$, where $[1]$ is a constant test function: $[1](x) = 1$. 

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The reductivity of $\mathfrak{k}$ is not difficult to prove, either. $\mathfrak{k}$ is a subspace of $\mathfrak{g}$, endowed with an invariant scalar product, which is given by the restriction of the Killing form on $\mathfrak{g}$. By invariance of this scalar product on $\mathfrak{k}$ with respect to $Ad_{\mathfrak{k}}$ (this being a mere restriction of invariance under $Ad_{\mathfrak{g}}$) any invariant subspace of $\mathfrak{k}$ has an invariant orthogonal complement. Now this is complete reducibility of $Ad_{\mathfrak{k}}$ and thus $\mathfrak{k}$ is reductive [Cor89] (25.3.a).

Since one can reduce the problem of understanding all conformal inclusions to the studies of reductive inclusions in simple Lie algebras (cf. eg. [AGO87]) the last part follows immediately from corollary 2, as $\iota(\mathfrak{h}) \subset \mathfrak{k}$ is, by construction of $\mathfrak{k}$, a conformal inclusion and Dynkin indices of the simple ideals in $\mathfrak{k}$ are greater than or equal to 1.

□

Corollary 6 A current $j^c$, whose colour $T^c$ lies in $\iota(\mathfrak{h})^\perp$ and fulfills $\tilde{C}_2^{\iota(\mathfrak{h})}T^c = 0$, commutes with the entire current algebra associated with $\iota(\mathfrak{h})$. These colours form a reductive Lie algebra, the algebra of invariant colours; we call their current algebra coset current algebra.

Proof: We set $V_0 := \ker(\tilde{C}_2^{\iota(\mathfrak{h})}) \cap \iota(\mathfrak{h})^\perp$. $V_0$ is an invariant subspace with respect to the action of $\mathfrak{h}$ on $\mathfrak{g}$ via $Ad_{\mathfrak{g}} \circ \iota$. In fact, it is the representation space for the trivial subrepresentation on $\iota(\mathfrak{h})^\perp$: For any simple ideal $\mathfrak{h}_\alpha$ we have by complete reducibility $C^{\mathfrak{h}}_{2}\mid_{V_0} = \sum \langle \Lambda + 2\rho, \Lambda \rangle_{\mathfrak{h}_\alpha} = 0$. Since both the Weyl vector $\rho$ and the contributing highest-weight vectors $\Lambda$ are dominant, we have $\Lambda = 0$. For the abelian ideal the irreducible subrepresentations on $V_0$ are given by common eigenvectors, such that $C^{\mathfrak{h}}_{2}v = g^{\theta}_{\iota(\mathfrak{h})}\lambda^i\lambda^j v = 0$. This gives the same result. This means, that all of $V_0$ commutes with $\iota(\mathfrak{h})$, i.e. $V_0 \subset \iota(\mathfrak{h})'$. We gain directly: $V_0 = \iota(\mathfrak{h})' \cap \iota(\mathfrak{h})^\perp$.

By Jacobi’s identity and invariance of the Killing metric, $\iota(\mathfrak{h})' \cap \iota(\mathfrak{h})^\perp$ forms a Lie subalgebra of $\mathfrak{g}$. This is reductive by the same argument as in the proof of corollary 5.

□

Two concluding remarks: Generically, the coset theory is not generated by coset currents. Obviously $\iota(\mathfrak{h}) \oplus (\iota(\mathfrak{h})' \cap \iota(\mathfrak{h})^\perp) \subset \mathfrak{g}$ has to be a conformal inclusion for that to be the case, since the coset stress-energy tensor has to be the Sugawara stress-energy tensor of the current algebra associated with $\iota(\mathfrak{h})' \cap \iota(\mathfrak{h})^\perp$. Casimir elements of $\iota(\mathfrak{h})' \cap \iota(\mathfrak{h})^\perp$ give, when transferred to the corresponding horizontal subalgebra, charge operators of the coset theory. These will, in general, fail to separate the representations of the coset theory. The same goes for the Cartan subalgebra of $\iota(\mathfrak{h})' \cap \iota(\mathfrak{h})^\perp$, whose spectrum defines characters of the representations of the coset theory. The coset current algebra is trivial for all inclusions with minimal coset theory: Here the coset theory is generated by the coset stress-energy tensor and this theory contains nothing but this field [Car98].

Triviality of coset current algebra ought to be regarded as the generic situation.
Currents $j^c$ with vanishing covariance field $\Phi^c$ are linear combinations of covariant and coset currents. This is obvious, since a decomposition of $T^c$ into eigenvectors of $\tilde{C}_2^b$ with distinct eigenvalues $\lambda$ yields (cf. equation 5):

$$\langle(1 - \tilde{C}_2^b)T^c, \tilde{C}_2^bT^c\rangle_g = \sum_{\lambda} \lambda(1 - \lambda)\langle T^c_{\lambda}, T^c_{\lambda}\rangle_g .$$

As $0 \leq \lambda \leq 1$ this scalar product vanishes, if and only if just 0 and 1 contribute. This means, that there are no currents with a “simple” intermediate transformation behaviour with respect to the action of $\Theta^b$. Typically, a current $j^c$ has $\Phi^c \neq 0$, i.e. a “complicated” transformation behaviour. By the analysis in [Kös03] this behaviour is known to be physically satisfactory, still.

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