We generalize in this appendix Theorem 1.5 to nontrivial coefficients on varieties $V$ which are neither smooth nor projective. We thank Alexander Beilinson, Luc Illusie and Takeshi Saito for very helpful discussions.

The notations are as in the article. Thus $K$ is a local field with finite residue field $k$, $R \subset K$ is the ring of integers, $\Phi$ is a lifting of the geometric Frobenius in the Galois group of $K$. We consider $\ell$-adic sheaves on schemes of finite type defined over $K$ in the sense of [3], (1.1). One generalizes the definition [2], Définition 5.1 of $T$-integral $\ell$-adic sheaves on schemes of finite type defined over finite fields to $\ell$-adic sheaves on schemes of finite type defined over local fields with finite residue field. Recall to this aim that if $\mathcal{C}$ is a $\ell$-adic sheaf on a $K$-scheme $V$ and $v$ is a closed point of $V$, then the stalk $\mathcal{C}_v$ of $\mathcal{C}$ at $\bar{v}$ is a $\text{Gal} (\bar{K}/K_v)$-module, where $K_v \supset K$ is the residue field of $v$, with residue field $\kappa(v) \supset k$. On $\bar{\mathcal{C}}_v$ the inertia $I_v = \text{Ker}(\text{Gal}(\bar{K}/K_v) \to \text{Gal}(\kappa(v)/k))$ acts quasi-unipotently ([4]). Consequently the eigenvalues of a lifting $\Phi_v \in \text{Gal}(\bar{K}/K_v)$ of the geometric Frobenius $F_v \in \text{Gal}(\kappa(v)/\kappa(v))$ are, up to multiplication by roots of unity, well defined ([3], Lemma (1.7.4)).

Let $T \subset \mathbb{Z}$ be a set of prime numbers.

**Definition 0.1.** The $\ell$-adic sheaf $\mathcal{C}$ is $T$-integral if the eigenvalues of $\Phi_v$ acting on $\bar{\mathcal{C}}_v$ are integral over $\mathbb{Z}[\frac{1}{t}, t \in T]$ for all closed points $v \in V$.

**Theorem 0.2.** Let $V$ be a scheme of finite type defined over $K$, and let $\mathcal{C}$ be a $T$-integral $\ell$-adic sheaf on $V$. Then if $f : V \to W$ is a morphism to another $K$-scheme of finite type $W$ defined over $K$, the $\ell$-adic sheaves $R^i f_* \mathcal{C}$ are $T$-integral as well. More precisely, if $w \in W$ is a closed point, then both $F_w$ and $|\kappa(w)|^{n-1} F_w$ acting on $(R^i f_* \mathcal{C})_{\bar{w}}$ are integral over $\mathbb{Z}[\frac{1}{t}, t \in T]$, with $n = \dim(f^{-1}(w))$.

**Proof.** Let $w$ be a closed point of $W$. By base change for $Rf_*$ and by $\{w\} \hookrightarrow W$, one is reduced to the case where $W$ is the spectrum of a finite extension $K'$ of $K$. If $V := V \otimes_K K'$, for $K'$ an algebraic closure...
of $K'$, one has to check the integrality statements for the eigenvalues of a lifting of Frobenius on $H^i_c(\overline{V}, \mathcal{C})$.

Let us perform the same reductions as in [2] p. 24. Note that in loc. cit. $U$ can be shrunk so as to be affine, with the sheaf smooth on it (= a local system). This reduces us to the cases where $V$ is of dimension zero or is an affine irreducible curve smooth over $W = \text{Spec}(K')$ and $\mathcal{C}$ is a smooth sheaf.

The integrality statement to be proven is insensitive to a finite extension of scalars $K^{''}/K'$. The 0-dimensional case reduces in this way to the trivial case where $V$ is a sum of copies of $\text{Spec}(K')$. In the affine curve case, $H^0_c$ vanishes, while, as in [2] Lemma 5.2.1, there is a 0-dimensional $Z \subset V$ such that the natural ($\Phi$-equivariant) map from $H^0_c(Z, \mathcal{C})(-1)$ to $H^2_c(\overline{V}, \mathcal{C})$ is surjective, leaving us only $H^1_c$ to consider.

Let $\mathcal{C}_{Z^\ell}$ be a smooth $\mathbb{Z}/\ell$-sheaf from which $\mathcal{C}$ is deduced by $\otimes \mathbb{Q}_\ell$, and let $\mathcal{C}_\ell$ be the reduction modulo $\ell$ of $\mathcal{C}_{Z^\ell}$. For some $r$, it is locally (for the étale topology) isomorphic to $(\mathbb{Z}/\ell)^r$. Let $\pi: V' \to V$ be the étale covering of $V$ representing the isomorphisms of $\mathcal{C}_\ell$ with $(\mathbb{Z}/\ell)^r$. It is a $\text{GL}(r, \mathbb{Z}/\ell)$-torsor over $V$. As $H^*(V, \mathcal{C})$ injects into $H^*(V', \pi^*\mathcal{C})$, renaming irreducible components of $V''$ as $V$ and $K'$ as $K$, we may and shall assume that $W = \text{Spec}(K)$ and that $\mathcal{C}_{\ell}$ is a constant sheaf.

Let $V_1$ be the projective and smooth completion of $V$, and $Z := V_1 \setminus V$. Extending scalars, we may and shall assume that $Z$ consists of rational points and that $V_1$, marked with those points, has semi-stable reduction. It hence is the general fiber of $X$ regular and proper over $\text{Spec}(R)$, smooth over $\text{Spec}(R)$ except for quadratic non-degenerate singular points, with $Z$ defined by disjoint sections $z_\alpha$ through the smooth locus.

Let $Y$ be the special fiber of $X$, and $\overline{Y}$ be $Y \times_k \bar{k}$, for $\bar{k}$ the residue field of the algebraic closure $\bar{K}$ of $K$.

The cohomology with compact support $H^1_c(\overline{V}, \mathcal{C})$ is $H^1(V_1, j_!\mathcal{C})$, and vanishing cycles theory relates this $H^1$ to the cohomology groups on $\overline{Y}$ of the nearby cycle sheaves $\psi^j(j_!\mathcal{C})$, which are $\ell$-adic sheaves on $\overline{Y}$, with an action of $\text{Gal}(\bar{K}/K)$ compatible with the action of $\text{Gal}(\bar{K}/K)$ (through $\text{Gal}(\bar{k}/k)$) on $\overline{Y}$. The choice of a lifting of Frobenius, i.e. of a lifting of $\text{Gal}(\bar{k}/k)$ in $\text{Gal}(\bar{K}/K)$, makes them come from $\ell$-adic
sheaves on $Y$, to which the integrality results of [2] apply. Using the exact sequence
\[ 0 \to H^1(\bar{Y}, \psi^0(j_!\mathcal{C})) \to H^1(V_1, j_!\mathcal{C}) \to H^0(\bar{Y}, \psi^1(j_!\mathcal{C})) \]
and [2] Théorème 5.2.2, we are reduced to check integrality of the sheaves $\psi^i(j_!\mathcal{C})$ ($i = 0, 1$). It even suffices to check it at any $k$-point $y$ of $Y$, provided we do so after any unramified finite extension of $K$.

Let $X_y$ be the henselization of $X$ at $y$, and $Y_y, V_1(y)$ and $V(y)$ be the inverse image of $Y, V_1$ or $V$ in $X_y$. There are three cases:

(1) $y$ singular on $Y$  
(2) $y$ on a $z_\alpha$  
(3) general case.

The restriction of $\psi^i$ to $y$ depends only on the restriction of $\mathcal{C}$ to $V(y)$, and short exact sequences of sheaves give rise to long exact sequences of $\psi$.

Because $\mathcal{C}_\ell$ is a constant sheaf, $\mathcal{C}$ is tamely ramified along $Y$ and the $z_\alpha$. More precisely, it is given by a representation of the pro-$\ell$ fundamental group of $V_y$. It is easier to describe the group deduced from the profinite fundamental group by pro-$\ell$ completing only the kernel of its map to $\hat{\mathbb{Z}} = \text{Gal}(\bar{k}/k)$. By Abhyankhar’s lemma, this group is an extension of $\hat{\mathbb{Z}}$, generated by Frobenius, by $\mathbb{Z}_\ell(1)^2$ in case (1) or (2) or $\mathbb{Z}_\ell(1)$ in case (3). The representation is given by $r \times r$ matrices congruent to 1 mod $\ell$. For $\ell \neq 2$, such a matrix, if quasi-unipotent, is unipotent. Indeed, it is the exponential of its logarithm and the eigenvalues of its logarithm are all zero. For $\ell = 2$, the same holds if the congruence is mod 4, hence if $\mathcal{C}_{\mathbb{Z}_2}$ mod 4 is constant, a case to which one reduces by the same argument we used mod 2.

By Grothendieck’s argument [6] p.515, the action of $\mathbb{Z}_\ell(1)$ or $\mathbb{Z}_\ell(1)^2$ is quasi-unipotent, hence unipotent, and we can filter $\mathcal{C}$ on $V_y$ by smooth sheaves such that the successive quotients $Q$ extend to smooth sheaves on $X_y$. If $Q$ extends to a smooth sheaf $\mathcal{L}$ on $X_y$, the corresponding $\psi$ are known by Picard-Lefschetz theory: $\psi^0$ is $\mathcal{L}$ restricted to $Y_y$ in cases (1) and (3), and $\mathcal{L}$ outside of $y$ extended by zero in case (2); $\psi^1$ is non-zero only in case (1), where it $\mathcal{L}(-1)$ on $\{y\}$ extended by zero.

By dévissage, this gives the required integrality. □

**Corollary 0.3.** Let $V$ be smooth scheme of finite type defined over $K$. Then the eigenvalues of $\Phi$ on $H^i(V, \mathbb{Q}_\ell)$ are integral over $\mathbb{Z}$.
Proof. If $K$ has characteristic zero, there is a good compactification $j : V \hookrightarrow W$, with $W$ smooth proper over $K$ and $D = W \setminus V = \cup D_i$ a strict normal crossing divisor. Then the long exact sequence

$$\ldots \to H^i_D(\bar{W}, \mathbb{Q}_\ell) \to H^i(W, \mathbb{Q}_\ell) \to H^i(\bar{V}, \mathbb{Q}_\ell) \to \ldots$$

and Theorem 0.2 applied to the cohomology of $W$ reduces to showing integrality for $H^i_D(\bar{W}, \mathbb{Q}_\ell)$. As in (3.3) of the article, the Mayer-Vietoris spectral sequence

$$E_1^{-a+1,b} = \bigoplus_{|I|=a} H^b_{D_I}(W, \mathbb{Q}_\ell) \Rightarrow H^{1-a+b}(W, \mathbb{Q}_\ell),$$

with $D_I = \cap_{i \in I} D_i$, reduces to the case where $D$ is smooth projective of codimension $\geq 1$. Then purity together with Theorem 0.2 allow to conclude. If $K$ has equal positive characteristic, we apply de Jong’s theorem [1], Theorem 6.5 to find $\pi : V' \to V$ generically finite and $j : V' \hookrightarrow W$ a good compactification. As $\pi^* : H^i(V, \mathbb{Q}_\ell) \hookrightarrow H^i(\bar{V}, \mathbb{Q}_\ell)$ is injective, we conclude as above. \qed

Corollary 0.3 gives some flexibility as we do not assume that $V$ is projective. In particular, one can apply the same argument as in the proof of Theorem 2.1 of the article in order to show an improved version of Theorem 1.5, (ii) there:

**Corollary 0.4.** Let $V$ be a smooth scheme of finite type over $K$, and $A \subset V$ be a codimension $\kappa$ subscheme. Then the eigenvalues of $\Phi$ on $H^i_A(\bar{V}, \mathbb{Q}_\ell)$ are divisible by $|k|^\kappa$ as algebraic integers.

**Proof.** One has a stratification $\ldots \subset A_i \subset A_{i-1} \subset \ldots A_0 = A$ by closed subschemes defined over $K$ with $A_{i-1} \setminus A_i$ smooth. The $\Phi$-equivariant long exact sequence

$$\ldots \to H^m_{A_i}(\bar{V}, \mathbb{Q}_\ell) \to H^m_{A_{i-1}}(\bar{V}, \mathbb{Q}_\ell) \to H^m_{(A_{i-1}\setminus A_i)}(\bar{V} \setminus A_i, \mathbb{Q}_\ell) \to \ldots$$

together with purity and Corollary 0.3 allow to conclude by induction on the codimension. \qed

**Remark 0.5.** One has to pay attention that even if Theorem 0.2 generalizes Theorem 1.5 i) of the article to $V$ not necessarily smooth, there is no such generalization of Theorem 1.5 ii) to the non-smooth case, even on a finite field. Indeed, let $V$ be a rational curve with one node. Then $H^1(\bar{V}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(0)$ as we see from the normalization sequence, yet $H^1_{\text{node}}(\bar{V}, \mathbb{Q}_\ell) = H^1(\bar{V}, \mathbb{Q}_\ell)$ as the localization map $H^1(\bar{V}, \mathbb{Q}_\ell) \to H^1(\bar{V} \setminus \text{node}, \mathbb{Q}_\ell)$ factorizes through $H^1(\text{normalization}, \mathbb{Q}_\ell) = 0$. So we can’t improve the integrality statement to a divisibility statement in general. In order to force divisibility, one needs the divisor supporting the cohomology to be in good position with respect to the singularities.
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