THE $SL(2,\mathbb{C})$ CHARACTER VARIETY OF A CLASS OF TORUS KNOTS

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Abstract. In this paper we present some families of polynomials and use them to find, using the techniques in [2], a defining polynomial for the $SL(2,\mathbb{C})$ character variety (as defined in [1]) of the torus knots of type $(m,2)$ with $m > 1$ being an odd integer.

1. THE CHARACTER VARIETY OF A FINITELY PRESENTED GROUP

Let us consider a finitely presented group

$$G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_s \rangle$$

and let be $\rho : G \rightarrow SL(2,\mathbb{C})$ be a representation, i.e., a group homomorphism. It is clear that $\rho$ is completely determined by the $n$-tuple $(\rho(x_1), \ldots, \rho(x_n))$ and thus we can define

$$R(G) = \{ (\rho(x_1), \ldots, \rho(x_n)) \mid \rho \text{ is a representation of } G \} \subseteq \mathbb{C}^{4n}$$

which is (see [1]) an (up to canonical isomorphism) well-defined affine algebraic set.

Recall that we define the character $\chi_\rho : G \rightarrow F$ (see [4]) of a representation $\rho : G \rightarrow GL(n,F)$ by $\chi_\rho(g) = \text{tr}(\rho(g))$, two representations $\rho$ and $\rho'$ having the same character if and only if they are equivalent; i.e., if there exists $P \in GL(n,F)$ such that $\rho'(g) = P^{-1} \rho(g) P$ for all $g \in G$. Now choose any $g \in G$ and define $\tau_g : R(G) \rightarrow \mathbb{C}$ by $\tau_g(\rho) = \chi_\rho(g)$. It is easily seen that $T = \{ \tau_g \mid g \in G \}$ is a finitely generated ring ([1] Proposition 1.4.1.) and moreover it can be shown using some identities holding in $SL(2,\mathbb{C})$ (see [2] Corollary 4.1.2.) that $T$ is generated by the set:

$$\{ \tau_{x_i}, \tau_{x_j}, \tau_{x_k}, \tau_{x_i x_m x_p} \mid 1 \leq i \leq n, 1 \leq j < k \leq n, 1 \leq l < m < p \leq n \}$$

Now choose $\gamma_1, \ldots, \gamma_\nu \in G$ such that $T = \{ \tau_{\gamma_i} \mid 1 \leq i \leq \nu \}$ and define the map $t : R(G) \rightarrow \mathbb{C}^\nu$ by $t(\rho) = (\tau_{\gamma_1}(\rho), \ldots, \tau_{\gamma_\nu}(\rho))$. Put $X(G) = t(R(G))$, then $X(G)$ is an algebraic variety ([1] Corollary 1.4.5.) which is well-defined up to canonical isomorphism and is called the $SL(2,\mathbb{C})$ character variety of the group $G$. Observe that $\nu = \frac{n(n^2 + 5)}{6}$.

For every $0 \leq j \leq n$ and for every $1 \leq i \leq s$ we have that $\tau_{x_i x_j} - \tau_{x_j}$ is a polynomial with rational coefficients in the variables $\{ \tau_{x_1}, \ldots, x_m \mid m \leq 3 \}$. With this definition we have that (see [2] Theorem 3.2.)

$$X(G) = \{ \boldsymbol{\tau} \in \mathbb{C}^\nu \mid p_{ij}(\boldsymbol{\tau}) = 0, \forall i, j \}$$
2. Torus knots

Recall that $\mathbb{R}^2$ is the universal covering of the torus $T^2$. We define the action $\phi : (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $\phi((m, n), (x, y)) = (x + m, y + n)$ and we have that $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z}) \cong T^2$. If we now take the family $\{r_p : y = px \mid p \in \mathbb{R}\}$ of straight lines passing through the origin, it is easily seen that if $p$ is irrational then $\mu(r_p)$ is dense in $T^2$ and if $p = \frac{m}{n}$ then $\mu(r_p) \subseteq T^2 \subseteq \mathbb{R}^3$ is a knot. We denote this knot by $K_{\frac{m}{n}}$ and call it the torus knot of type $(m,n)$ (see [5] Chapter 3 for further considerations).

If we denote, as usual, by $G(K)$ the fundamental group of the exterior of any knot $K$ we can see that $G(K_{\frac{m}{n}}) \cong \langle A, B \mid A^m = B^n \rangle$.

Now let us define the following group:

$$H_m = \langle x, y \mid xyxy \ldots yx = \underbrace{xyxy \ldots yx}_{\text{length } m} \rangle$$

Then we have

**Lemma 2.1.** Let $m \geq 1$ be an odd integer. Then $G(K_{\frac{m}{n}}) \cong H_m$.

**Proof.** We define $\varphi : H_m \rightarrow G(K_{\frac{m}{n}})$ by $\varphi(x) = B^{-1}A^{\frac{m+1}{2}}$ and $\varphi(y) = A^{-\frac{m-1}{2}}B$. On the other hand, define $\psi : G(K_{\frac{m}{n}}) \rightarrow H_m$ by $\psi(A) = yx$ and $\psi(B) = \underbrace{xyxy \ldots y}_{\text{length } m}$. Seeing that these homomorphisms are well defined and are each other’s inverse is straightforward. \qed

3. Some families of polynomials

We will start this section by defining recursively the following family of polynomials:

$$q_1(T) = T - 2$$
$$q_2(T) = T + 2$$
$$\prod_{d \mid n, 1 \neq d \mid n} q_d \left( X + \frac{1}{X} \right) = \frac{X^{n-1} + X^{n-2} + \cdots + X + 1}{X^{n-\frac{1}{2}}} \quad \text{if } n \text{ is odd}$$
$$\prod_{d \mid n, 1,2 \neq d \mid n} q_d \left( X + \frac{1}{X} \right) = \frac{X^{n-2} + X^{n-4} + \cdots + X^2 + 1}{X^{n-\frac{2}{2}}} \quad \text{if } n \text{ is even}$$

**Remark 1.** If we recall the recursive definition of the cyclotomic polynomials (see [3] Chapter 5) by

$$\prod_{d \mid n} g_d(T) = T^n - 1$$

then it is easily seen that $g_\varphi(X) = X^{\frac{\varphi(x)}{2}} q_{\varphi} \left( X + \frac{1}{X} \right)$ where $\varphi$ is the Euler function.
Now we introduce another family of polynomials:

\[ p_1(X) = X \]
\[ p_2(X) = X^2 - 2 \]
\[ p_n(X) = Xp_{n-1}(X) - p_{n-2}(X), \ \forall n \geq 3 \]

**Remark 2.** Let \( G \) be a group and \( \rho : G \to SL(2, \mathbb{C}) \) a representation. Then \( p_n(tr\rho(x)) = tr\rho(x^n) \) for every \( n \geq 1 \). For the sake of completeness we will set, where necessary, \( p_0(X) = 1 \).

We have the following relationship between the families we have just defined:

**Proposition 3.1.**

\[ p_n(X) - 2 = q_1(X) \prod_{1 \neq d \mid n} q_d^2(X) \quad \text{if } n \text{ is odd} \]
\[ p_n(X) - 2 = q_1(X)q_2(X) \prod_{1, 2 \neq d \mid n} q_d^2(X) \quad \text{if } n \text{ is even} \]

**Proof.** We will just show the result for an odd \( n \), the even case being completely analogous.

Consider the cyclic group \( G = \langle x \rangle \) and a representation \( \rho : G \to SL(2, \mathbb{C}) \). We can suppose, conjugating if necessary, that \( \rho(x) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \). In such a case it must be \( \rho(x^n) = \rho(x)^n = \begin{pmatrix} a^n & c \\ 0 & a^{-n} \end{pmatrix} \).

Set \( X = tr(\rho(x)) = a + a^{-1} \), then

\[ p_n(X) - 2 = tr(\rho(x^n)) - 2 = a^n + a^{-n} - 2 = \frac{(a^n - 1)^2}{a^n} = \frac{1}{a^n} \left( \prod_{d \mid n} \phi_d(a) \right)^2 \]
\[ = \frac{(a - 1)^2}{a^n} \left( \prod_{1 \neq d \mid n} g_d(a) \right)^2 \]
\[ = \frac{(a + a^{-1} - 2)a}{a^n} \left( \prod_{1 \neq d \mid n} a^{\varphi(d)} g_d(a + a^{-1}) \right)^2 \]
\[ = q_1(X) \prod_{1 \neq d \mid n} q_d^2(X) \]

where the identity \( \sum_{d \mid n} \varphi(d) = n \) was used. \( \square \)

**Remark 3.** The roots of \( p_n(X) - 2 \) are precisely the possible values of \( tr(\rho(x)) \) if \( \rho : G \to SL(2, \mathbb{C}) \) is a representation and \( x^n = 1 \).

Let \( R \) be any ring and take \( g(T) = \sum_{i=0}^{n} a_i T^i \in R[T] \). We define \( * : R[T] \to R[T] \) by \( g^*(T) = \sum_{i=0}^{n} (-1)^{n-i} a_i T^i \). In the next lemma we show some interesting properties of this application.
Lemma 3.1. Given $g, h \in R[T]$ we have:

a) $g^{**} = g$.

b) $(gh)^* = g^* h^*$.

c) If $g(T) = \sum_{i=0}^{n} a_i T^i$, then $g^* = g$ if and only if $a_i = 0$ for every $i$ such that $(n - i) \equiv 1 \pmod{2}$.

Proof. c) is trivial. a) and b) follow from the identity $g^*(T) = (-1)^{\deg g} g(-T)$. □

We can use the involution just defined to show another relation between our two families of polynomials.

Proposition 3.2. If $s \geq 1$ is an integer, then

$$
\sum_{i=0}^{s} (-1)^{i} p_{s-i}(Z) = \prod_{1 \neq d \mid 2s+1} q_d(Z)
$$

Proof. We observe that the degree of every term in $p_s(Z)$ has the same parity as $s = \deg p_s(Z)$. This fact together with the definition of $*$ shows that

$$
\left(\sum_{i=0}^{s} (-1)^{i} p_{s-i}(Z)\right)^* = \sum_{i=0}^{s} p_i(Z)
$$

Now, we claim that

$$
\sum_{i=0}^{s} p_i(Z) = \prod_{1 \neq d \mid 2s+1} q_d(Z)
$$

We will prove this by induction on $s$, the case $s = 1$ being trivial as $p_0(Z) + p_1(Z) = 1 + Z = q_1(Z)$. Now let $s > 1$ be an odd integer (the even case is similar), by hypothesis we have

$$
\sum_{i=0}^{s} p_i(Z) = \sum_{i=0}^{s-1} p_i(Z) + p_s(Z) = \prod_{1 \neq d \mid 2s-1} q_d(Z) + p_s(Z)
$$

and thus, setting $Z = X + \frac{1}{X}$ one obtains:

$$
\sum_{i=0}^{s} p_i \left( X + \frac{1}{X} \right) = \prod_{1 \neq d \mid 2s-1} q_d \left( X + \frac{1}{X} \right) + p_s \left( X + \frac{1}{X} \right)
$$

$$
\sum_{i=0}^{2s-2} X^i
$$

$$
= \frac{X^{s-1}}{X^{s-1}} + q_1 \left( X + \frac{1}{X} \right) \prod_{1 \neq d \mid s} q_d^2 \left( X + \frac{1}{X} \right) + 2
$$

$$
\sum_{i=0}^{2s-2} \frac{X^i}{X^{s-1}} = \frac{(X-1)^2}{X} \frac{\left( \sum_{i=0}^{s-1} X^i \right)^2}{X^{s-1}} + 2
$$

$$
\sum_{i=0}^{2s-2} \frac{X^i}{X^{s-1}} + \frac{X^{2s} + 1}{X^s} = \sum_{i=0}^{2s} X^i \frac{\left( \sum_{i=0}^{s} X^i \right)}{X^s} = \prod_{1 \neq d \mid 2s+1} q_d \left( X + \frac{1}{X} \right)
$$
The proof is now completed by applying \(3.1\) a), b).

4. The \(\text{SL}(2, \mathbb{C})\) Character Variety of the Knots \(K_m\)

The objective of this section is to give a generating family of polynomials for \(X(G)\) with \(G = G(K_m)\) with \(m > 1\) an odd integer. In \(2\) we show the isomorphism \(G(K_m) \cong H_m\) so it is enough to find such a family for \(X(H_m)\).

Before going into our main result we have to introduce another polynomial. We set \(h(X, Z) = X^2 - Z\) and \(k(X) = X^2 - 2\). Now we define

\[
\alpha_l(X, Z) = \begin{cases} h(X, Z) & \text{if } l \text{ is even} \\ k(X) & \text{if } l \text{ is odd} \end{cases}
\]

and finally we write for \(s \geq 1\)

\[
f_s(X, Z) = p_s(Z)(h(X, Z) - 1) + \sum_{i=1}^{s} (-1)^i p_{s-i}(Z) \alpha_i(X, Z)
\]

With these definitions we can prove the following

**Proposition 4.1.** If \(m > 1\) is and odd integer, then

\[
X(H_m) = \{(X, Z) \in \mathbb{C}^2 \mid f_s(X, Z) = 0\}
\]

**Proof.** We set \(w = xyxy \ldots yx y^{-1} x y^{-1} x^{-1} x^{-1} y^{-1} \ldots y^{-1}\). Then, using Theorem 3.2 in \(2\), we have

\[
X(H_m) = \{(X, Y, Z) \in \mathbb{C}^3 \mid p_0(X, Y, Z) = p_1(X, Y, Z) = p_2(X, Y, Z) = 0\}
\]

where

\[
X = \tau_x, \quad p_0(X, Y, Z) = \tau_w - \tau_1 \\
Y = \tau_y, \quad p_1(X, Y, Z) = \tau_{wx} - \tau_x \\
Z = \tau_{xy}, \quad p_2(X, Y, Z) = \tau_{wy} - \tau_y
\]

Now, \(w = x y x \ldots y x (x y x \ldots y)^{-1}\) so we have \(\tau_{wy} = \tau_x\) obtaining that

\[
p_2(X, Y, Z) = X - Y.
\]

On the other hand \(\tau_{wx} = \tau_w \tau_x - \tau_{wx^{-1}}\) and \(\tau_{wx^{-1}} = \tau_y\) so we get

\[
\tau_{wx^{-1}} = \tau_{w^{-1}} = \tau_y
\]

and thus \(p_1(X, Y, Z) = \tau_{wx} - \tau_x = \tau_w \tau_x - \tau_y - \tau_x = \tau_x(\tau_w - 1) - \tau_y = X p_0(X, Y, Z) + X - Y.
\]

Set now \(w_1 = (y x)^{m-1}_m y^{-1}_x\) and \(w_2 = (x y)^{m-1}_x y^{-1}_x\). Then it is easy to see that

\[
p_0(X, Y, Z) = \tau_w\]

vanishes if and only if \(f(X, Y, Z) = \tau_{w^{-2}} - \tau_{w^{-1}}\) does. Let us compute now this polynomial.

Firstly it is obvious by definition that \(\tau_{w^{-1}} = p_{m-1}(Z)\). In addition we have \(\tau_{w^{-2}} = \tau_{(xy)^{m-2}_x y^{-2}} = p_{m-3}(Z)(X Y - Z) - \tau_{(xy)^{m-3}_x y^{-2}}\). Moreover we see that

\[
\tau_{w^{-2}}(Z) = \tau_{(xy)^{m-2}_x y^{-2}} - \tau_{(xy)^{m-3}_x y^{-2}} = p_{m-3}(Z)(X^2 - 2) - \tau_{(xy)^{m-3}_x y^{-2}}
\]

so it is enough to iterate the process.

By now we have obtained

\[
X(H_m) = \{(X, Y, Z) \in \mathbb{C}^3 \mid f(X, Y, Z) = 0 = X - Y\}
\]

\[
\cong \{(X, Z) \in \mathbb{C}^2 \mid f(X, Z) = 0\}
\]
and this completes the proof as the equality $f(X, X, Z) = f_{m-1}(X, Z)$ is just a straightforward computation. □

Let us rewrite now the polynomial $f_s(X, Z)$ in a different way. In fact we can see that

$$f_s(X, Z) = (X^2 - Z - 2) \left( \sum_{i=0}^{s} (-1)^{i} p_{s-i}(Z) \right) + p_s(Z) + \sum_{i=1}^{s} (-1)^{i} \beta_i(Z) p_{s-i}(Z)$$

where $eta_k(Z) = \begin{cases} Z & \text{if } k \text{ is odd} \\ 2 & \text{if } k \text{ is even} \end{cases}$

**Lemma 4.1.** $p_s(Z) + \sum_{i=1}^{s} (-1)^{i} \beta_i(Z) p_{s-i}(Z) = 0$

**Proof.** It is enough to use the fact that $p_s(Z) - Zp_{s-1}(Z) = -p_{s-2}(Z)$. □

**Corollary 4.1.** If $m > 1$ is an odd integer, then

$$X(H_m) \cong \{(X, Z) \in \mathbb{C}^2 \mid (X^2 - Z - 2) \prod_{1 \neq d \mid m} q_d^*(Z) = 0\}$$

**Proof.** Just apply Proposition 3.2 and Lemma 4.1 to Proposition 4.1 □

**Lemma 4.2.** Let $\{a_1, a_2, \ldots, a_{\varphi(r)}, \bar{a}_{\varphi(r)}\}$ be set of the $\varphi(r)$ primitive $r$th roots of unity. Then

$$q_r(Z) = \prod_{i=1}^{\varphi(r)} (Z - 2\text{Re}(a_i))$$

**Proof.** Recall that, for $r > 2$ we have $g_r(X) = X^{\frac{\varphi(r)}{2}} q_r \left( X + \frac{1}{X} \right)$ with $g_r(X)$ being the $r$th cyclotomic polynomial. As for all $1 \leq j \leq \frac{\varphi(r)}{2}$ it holds that $\frac{1}{a_j} = \bar{a}_j$ we obtain that $q_r(Z)$ has exactly $\frac{\varphi(r)}{2}$ different roots, namely $\{2\text{Re}(a_1), \ldots, 2\text{Re}(a_{\varphi(r)})\}$. This together with the fact that the degree of $q_r(Z)$ is $\frac{\varphi(r)}{2}$ completes the proof. □

This lemma allows us to go one step further in our description of the curve $X(H_m)$.

**Corollary 4.2.** Let $m > 1$ be an odd integer. In the complex plane $(X, Z)$ the curve $X(H_m)$ consists of the parabola $Z = X^2 - 2$ and the union of $\frac{m-1}{2}$ horizontal lines of the form $Z = -2\text{Re}(w)$, being $1 \neq w$ an $m$th root of unity.

**Proof.** It is enough to apply the previous lemma together with the fact that given a polynomial $g$, then a number $a$ is a root of $g$ if and only if $-a$ is a root of $g^*$. □
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