An account of transforms on $\mathcal{A}/\mathcal{G}$

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Abstract

In this article we summarize and describe the recently found transforms for theories of connections modulo gauge transformations associated with compact gauge groups. Specifically, we put into a coherent picture the so-called loop transform, the inverse loop transform, the coherent state transform and finally the Wick rotation transform which is the appropriate transform that incorporates the correct reality conditions of quantum gravity when formulated as a dynamical theory of connections while preserving the simple algebraic form of the Hamiltonian constraint.

1 Introduction

Recently, there has been significant progress in the development of calculus on the space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations for compact gauge groups [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Roughly, we can say that if we take the Abelian $C^*$ algebra generated by the Wilson loop functions (that is, the traces of holonomies around loops for smooth connections) and compute its Gel’fand spectrum then we obtain the space $\mathcal{A}/\mathcal{G}$ of distributional connections which are equipped with the topology defined by requiring that the Wilson loop functionals are continuous [1]. This space is not some abstract construction lacking any geometric interpretation, it can be very concretely characterized as the space of homomorphisms from the group of piecewise analytic loops (the group structure is obtained by identifying the inverse loop with the tracing of the original loop in the opposite direction) to the gauge group [2]. It should be said at this point that we are restricting ourselves to the analytic category for the sake of simplicity although all the results can be transferred to the smooth category [14] subject to some fairly technical modifications. Since the Gel’fand spectrum is a compact Hausdorff space, measures thereon are in one to one correspondence with positive linear functionals on the space of continuous functionals on $\mathcal{A}/\mathcal{G}$. Beautiful, diffeomorphism-invariant measures have been

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constructed \[ A/G \].

The space \( A/G \) can also be characterized as the projective limit of a certain self-consistent family of measure spaces \[ A/G \].

Differential geometry on \( A/G \) can be constructed as well \[ A/G \]. The framework can be applied to gauge theories \[ A/G \] and to quantum gravity \[ A/G \] when formulated as a theory of connections \[ A/G \].

In this article we focus on summarizing the theory underlying the recently found transforms associated with \( A/G \). All statements will be given without proof, the details can be found in the original literature.

In section 2 we review the loop transform which was heuristically defined for the first time in \[ A/G \] and later made precise in \[ A/G \]. This transform maps us from functions of connections to functions of loops, that is, (singular) knots.

Section 3 defines the inverse loop transform \[ A/G \]. We display here the loop network version of that transform, however, there is also an edge-network version of this theorem \[ A/G \] (compare also \[ A/G \] for a definition of spin-networks).

Section 4 reviews the coherent state transform \[ A/G \]. This transform maps us from functions of real connections to functions of complex connections.

Section 5 outlines the Wick rotation transform \[ A/G \] which does not only map us to functions of some complex connection but to a complex connection which corresponds to a particular complexification of the real phase space. It is only with this transform at our disposal that we can use all the techniques developed for compact gauge groups. In particular, only then do we incorporate the correct reality conditions while keeping the algebraic form of the Hamiltonian (constraint) simple.

An interesting modification of this transform is considered in \[ A/G \] which we outline briefly.

We display the explicit form of the transform for the case of canonical quantum gravity.

## 2 The loop transform

We provide only the absolutely necessary information in order to fix the notation. For further details see \[ A/G \] and references therein.

The space \( A/G \) of generalized connections modulo gauge transformations is the Gel’fand spectrum of the Abelian \( C^* \) algebra generated by the Wilson-loop functionals for smooth connections, that is, traces of the holonomy for piecewise analytic loops in the base manifold \( \Sigma \). As such it is a compact Hausdorff space and therefore measures on that space are in one to one correspondence with positive linear functionals on \( C(A/G) \).

A certain natural measure \( \mu_0 \) will play a very crucial role in this article so that we go now into more details:

In what follows, \( \gamma \subset \Sigma \) will always denote a finite, unoriented, closed (i.e. each vertex is at least 2-valent), piecewise analytic graph, meaning that it is the union of a finite number of analytic edges and vertices. Its fundamental group \( \pi_1(\gamma) \) is then finitely generated by some loops \( \beta_1(\gamma), \ldots, \beta_n(\gamma)(\gamma) \) which we fix once and for
all together with some orientation and which are based at some arbitrary but fixed basepoint \( p \in \Sigma \), \( n(\gamma) \) the number of generators of the fundamental group of \( \gamma \). A function \( f \) on \( A/\mathcal{G} \) is said to be cylindrical with respect to a graph \( \gamma \), \( f \in \text{Cyl}_{\gamma}(A/\mathcal{G}) \), if it is a function only of the finite set of arguments \( p_{\gamma}(A) := (h_{\beta_1}(A), \ldots, h_{\beta_n}(A)) \) where \( h_{\alpha}(A) \) is the holonomy of \( A \) along the loop \( \alpha \). A measure \( \mu \) is now specified by its finite joint distributions \( \mu_{\gamma} \) which are defined by

\[
\int_{A/\mathcal{G}} \mu_{\gamma}(A) f(A) = \int_{G^n} \mu_{\gamma}(g_1, \ldots, g_n) f_{\gamma}(g_1, \ldots, g_n)
\]  

(2.1)

where \( f = f_{\gamma} \circ p_{\gamma} \) and \( f_{\gamma} : G^n \to \mathbb{C} \). In order that this definition makes sense we have to make sure that if we write \( f = f_{\gamma} \circ p_{\gamma} = f_{\gamma'} \circ p_{\gamma'} \) in two different ways as a cylindrical function where \( \gamma \subset \gamma' \) is a subgraph of \( \gamma' \), then we should have that the so-called consistency conditions

\[
\int_{G^n} \mu_{\gamma} f_{\gamma} = \int_{G^{n'}} \mu_{\gamma'} f_{\gamma'}
\]

(2.2)

are satisfied.

The natural measure \( \mu_0 \) is the induced Haar measure, meaning that \( \mu_{0,\gamma}(g_1, \ldots, g_n) = d\mu_H(g_1).d\mu_H(g_n) \). One can check that the consistency conditions are satisfied and that the so defined cylindrical measure has a \( \sigma \)-additive extension \( \mu_0 \) on the projective limit measurable space \( \overline{A/\mathcal{G}} \) of the family of measurable spaces \( A/\mathcal{G}_{\gamma} \) \([5]\).

The space \( \overline{A/\mathcal{G}} \) is defined to be the set of all conjugacy classes of homomorphisms from the loop group restricted to \( \gamma \) into the gauge group while \( \overline{A/\mathcal{G}} \) is the set of all conjugacy classes of homomorphisms from the loop group into \( G \) (the conjugation corresponds to the gauge transformations at the basepoint). Note that the semigroup of loops with respect to compositions of loops can be given a group structure by identifying paths that are traversed in the opposite direction with the inverse of the original path.

Let us now recall some basic facts from harmonic analysis on compact gauge groups \([21]\).

Recall that every representation of a compact group is equivalent to a unitary one, so that we may restrict ourselves to unitary representations in the sequel. Also, every such representation is completely reducible.

**Definition 2.1** Let \( \{\pi\} \) denote the set of all finite dimensional, non-equivalent, unitary, irreducible representations of the compact gauge group \( G \), let \( d_\pi \) be the dimension of \( \pi \) and let \( \mu_H \) be the normalized Haar measure on \( G \).

For any \( f \in L_1(G, d\mu_H) \) define the Fourier transform of \( f \) by

\[
\hat{f}_{\pi}^{ij} := \int_G d\mu_H(g) \sqrt{d_\pi} \pi_{ij}(g) f(g), \ i, j = 1, \ldots, d_\pi.
\]

(2.3)

Note that this definition makes sense because the matrix elements of \( \pi(g) \) are bounded by 1.

**Definition 2.2** The Fourier transform of a function is said to be \( \ell_1 \) or \( \ell_2 \) respectively iff

\[
||\hat{f}||_1 := \sum_\pi \sum_{i,j=1}^{d_\pi} \sqrt{d_\pi} |f_{\pi}^{ij}| < \infty \quad \text{or} \quad ||\hat{f}||_2 := \sum_\pi \sum_{i,j=1}^{d_\pi} |f_{\pi}^{ij}|^2 < \infty.
\]

(2.4)
The Fourier series associated with a function \( f \) on \( G \) such that \( \hat{f} \in \ell_1 \) is given by
\[
\tilde{f}(g) := \sum_{\pi} \sum_{i,j=1}^{d_{\pi}} \hat{f}_{ij}(g) \sqrt{d_{\pi}}.
\] (2.5)

The analogue of the Plancherel theorem for \( \mathbb{R}^n \) is the Peter&Weyl theorem.

**Theorem 2.1 (Peter&Weyl)**

1) The functions \( g \rightarrow \sqrt{d_{\pi}} \pi_{ij}(g) \), \( i, j = 1, \ldots, d_{\pi} \)
form a complete and orthonormal system on \( L_2(G, d\mu_H) \).

2) For any \( f \in L_2(G, d\mu_H) \) it holds that \( f = \tilde{f} \) in the norm \( \| \cdot \|_2 \) and the Fourier transform is a unitary map \( \wedge : L_2(G, d\mu_H) \rightarrow \ell_2 \).

Next we introduce a new notion necessary to capture the gauge invariant cylindrical functions.

**Definition 2.3**

i) A loop network is a triple \((\gamma, \vec{\pi}, \pi)\) consisting of a graph \( \gamma \), a vector \( \vec{\pi} = (\pi_1, \ldots, \pi_{n(\gamma)}) \) of irreducible representations of \( G \) and an irreducible representation \( \pi \) of \( G \) which takes values in the set of irreducible representations of \( G \) contained in the decomposition into irreducibles of the tensor product \( \otimes_{k=1}^{n(\gamma)} \pi_k \).

ii) A loop-network state is a map from \( \mathbb{A}/G \) into \( \mathbb{C} \) defined by
\[
T_{\gamma,\vec{\pi},\pi}(A) := \text{tr}[\otimes_{k=1}^{n(\gamma)} \pi_k(h_{\beta_k(\gamma)}(A)) \cdot c(\vec{\pi}, \pi)]
\] (2.6)
where the matrix \( c \) is defined by
\[
c(\vec{\pi}, \pi) := \sqrt{\prod_{k=1}^{n(\gamma)} d_{\pi_k}} \pi(1).
\] (2.7)

Loop network states satisfy the following important properties.

**Lemma 2.1**

i) Given a graph \( \gamma \), the set of all loop network states provides for an orthonormal basis of \( L_2(\mathbb{A}/G, d\mu_0, \gamma) = L_2(\mathbb{A}/G, d\mu_0) \cap \text{Cyl}_\gamma(\mathbb{A}/G) \).

ii) Given a graph \( \gamma' \), consider all its subgraphs \( \gamma < \gamma' \). Remove all the loop network states on \( \gamma' \) which are pull-backs of loop-network states on \( \gamma \). The collection of all loop-network states so obtained provides for an orthonormal basis of \( L_2(\mathbb{A}/G, d\mu_0) \).

The proof of this lemma relies on the Peter&Weyl theorem. We are now in the position to define the Fourier transform of a measure on \( \mathbb{A}/G \).

**Definition 2.4**

The loop transform (Fourier transform, characteristic functional) of a measure \( \mu \) on \( \mathbb{A}/G \) is defined by
\[
\chi_\mu(\gamma, \vec{\pi}, \pi) := <\bar{T}_{\gamma,\vec{\pi},\pi}> := \int_{\mathbb{A}/G} d\mu(A) \bar{T}_{\gamma,\vec{\pi},\pi}(A)
\] (2.8)

This definition differs from the one given in [1, 8, 16], however, both definitions are equivalent in the sense that they allow for a reconstruction of \( \mu \) according to the Riesz-Markov theorem [22]. Namely, the former definition is based on the vacuum expectation value of products of Wilson loop functionals, and according to [1, 23], these functions are an overcomplete set of functions on \( \mathbb{A}/G \) (that is, they are subject to Mandelstam identities) so that we can reexpress them in terms of loop networks and vice versa which are linearly independent.

Note that the loop transform of a measure on \( \mathbb{A}/G \) is the analog of the Fourier transform of a measure on the quantum configuration space of quantum scalar field theory [10, 24] and the Riesz-Markov theorem substitutes the Bochner theorem [25].
3 The inverse loop transform

Now let be given a functional \( \chi \) on loop-networks. Provided it is positive (note that there are no Mandelstam relations between loop network states any more and that the product of loop network states is a linear combination of loop network states) we know by the Riesz-Markov theorem that there is a measure \( \mu \) whose Fourier transform is given by \( \chi \). This measure will be known if we know its finite joint distributions \( d\mu_\gamma \) (i.e. the loop transforms with respect to any finite graph \( \gamma \)) which are automatically form a self-consistent system of measures whose projective limit (known to exist) gives us back \( \mu \). We now wish to compute these joint distributions. This task is not entirely trivial because of the following : first it is not clear whether the finite joint distributions of a measure on \( \mathbb{A}/G \) are completely regular with respect to the product Haar measure. But even if we assume it was, that is, a formula like

\[
d\mu_\gamma = \rho_\gamma d\mu_{0,\gamma}
\]

is true with \( \rho_\gamma \) a positive \( L^1 \) function (because the constant functions are integrable) then we still do not know whether \( \rho_\gamma \in L^2(\mathbb{A}/G, d\mu_{0,\gamma}) \). If that were true then we could simply make use of the fact that loop networks provide for an orthonormal basis of \( L^2(\mathbb{A}/G, d\mu_0) \) to conclude theorem 3.3 directly from the gauge invariant version of the Peter&Weyl theorem. This is, however, not necessarily the case.

**Lemma 3.1** If the Fourier transform of a (complex) regular Borel measure \( \mu \) on a compact gauge group \( G \) is in \( \ell_1 \) then it is absolutely continuous with respect to the Haar measure on \( G \).

The theorem can obviously extended to any finite number of variables.

Next we need to prove the analogue of the inverse Fourier transform for compact groups [17]. This theorem answers the question whether a function which is only \( L^1 \) can be represented by its Fourier transform.

**Theorem 3.1** Let \( f \in L_1(g, d\mu_H) \) be such that also \( \hat{f} \in \ell_1 \). Then \( f(g) = \hat{f}(g) \) in the sense of \( || ||_1 \).

So the logic is as follows : given a positive linear functional compute its Fourier transform. If it is in \( \ell_1 \) then we can write the finite joint distributions of the measure corresponding to \( \chi \) as a positive \( L_1 \) function times the product Haar measure. By the preceding theorem we know that this function coincides in the \( L_1 \) sense with the associated Fourier series.

**Theorem 3.2** Let \( \chi \) be a positive linear functional on \( C(A/G) \). Then \( \chi \) is the loop transform of a positive regular Borel measure \( \mu \) on \( A/G \). If for a given graph \( \gamma \) with \( n \) generators the sequence \( \{\chi(\gamma, \bar{\pi}, \pi)\sqrt{d\pi \prod_{k=1}^n d\pi_k}\} \) is in \( \ell_1 \) then the finite joint distributions of \( \mu_\gamma \) are \( \mu_{0,\gamma} \) a.e. given by

\[
\frac{d\mu_\gamma(A)}{d\mu_{0,\gamma}(A)} = \sum_{\bar{\pi}} \sum_{\pi \in \mathbb{D}_{k=1}^n \pi_k} \chi(\gamma, \bar{\pi}, \pi) T_{\gamma, \bar{\pi}, \pi}(A) .
\]

The proof is a straightforward application of the inverse Fourier transform. In order to determine whether a given function \( \chi \) from (singular) knots into the complex numbers arises as the loop transform of a measure one has to check two
things:

1) All the identities that are satisfied by products of traces of holonomies of loops have to be satisfied Mandelstam identities [22]. Alternatively, it has to be true that \( \chi \) can be written purely in terms of loop-network states.

2) It is a positive linear functional on any cylindrical subspace of \( C(A/G) \).

An example of a (singular) knot function that satisfies these criteria is of course the Fourier transform of any \( \sigma \)-additive measure on \( A/G \). Let us look at the Fourier transform of the measure \( \mu_0 \) which is even diffeomorphism invariant so that \( \chi \) is a singular knot invariant:

\[
\chi_{\mu_0}(\gamma, \bar{\pi}, \pi) = \delta_{\bar{\pi}, \bar{0}} \delta_{\pi, 0} \tag{3.2}
\]

where 0 denotes the trivial representation. In other words, \( \chi \) is non-vanishing only on the trivial loop network 1. Therefore we find for the finite joint distribution precisely \( \rho_\gamma(g_1, ..., g_n) = 1 \).

4 The coherent state transform

For applications in canonical quantum gravity, which when formulated as a theory of connections [15] leads naturally to a theory of complexified \( SU(2) \) connections in order that the associated Hamiltonian constraint takes a simple algebraic form, all what we have said so far is inapplicable since it was fundamental to our approach that the gauge group was compact. Therefore we are interested in a transform which maps us from a theory of real connections to a theory of complex connections in such a way that the physics is unchanged.

The mathematical input that led to the definition of this transform came from the beautiful paper [26] which generalizes the classical Segal-Bargman transform (which is a unitary transform from the Schrödinger representation \( L_2(\mathbb{R}^n, dx) \) into the holomorphic representation \( L_2(\mathbb{C}^n, d\nu(z, \bar{z})) \cap \text{Hol}(\mathbb{C}^n) \) of square-integrable functions (with respect to a certain measure \( \nu \)) which are holomorphic) to the case where copies of \( \mathbb{R} \) are replaced by copies of a compact gauge group \( G \).

The details can be described by an analogy with the classical transform:

Suppose we have a Hilbert space \( \mathcal{H} = L_2(\mathbb{R}, dx) \) and decide to work in a representation \( \mathcal{H}_\mathbb{C} \) in which the holomorphic extension \( z = x_\mathbb{C} \) of \( x \) is a diagonal. We wish to do this in such a way that the two Hilbert spaces are isometric as not to change the physics. The way that this can be done is as follows:

There is the Laplacian \( \Delta := -\frac{1}{2} \frac{\partial^2}{\partial x^2} \) on \( \mathbb{R} \) and associated with it we construct the heat kernel \( \hat{R}_t := \exp(-t\Delta) \), \( t > 0 \). We now just define the coherent state transform to be kernel convolution followed by analytic extension, that is

\[
(\hat{C}_t\psi)(z) := < x | \hat{R}_t | \psi >_{|x \to z|} := \left[ \int_{\mathbb{R}} dy \rho_t(x, y) \psi(y) \right]_{x = z} \tag{4.1}
\]

where \( \rho_t(x, y) = \exp(-(x - y)^2/2t)/\sqrt{2\pi t} \) is the kernel of \( \hat{R}_t \) which is real analytic and therefore allows for a unique analytic extension.

The measure \( \nu_t \) which turns this transform into a unitary one is constructed by taking the Laplacian on \( \mathbb{C} \), that is, \( \Delta_\mathbb{C} = -\frac{1}{2}(\partial_1^2 + \partial_2^2) \), construct the kernel \( \mu_t(z_1, z_2) = \mu_t(z_1 - z_2) \) associated with it and average over the real direction in \( \mathbb{C} \) with the
Lebesgue measure $dx$ to produce

$$\nu_t(z, \bar{z}) := \int_\mathbb{R} \mu_t(z-x)dx, \quad d\nu_t(z, \bar{z}) := \nu_t(z, \bar{z})dz \wedge d\bar{z}. \quad (4.2)$$

The coherent state transform for any unimodular, compact gauge group is now constructed simply by translating step by step all the structures that we have defined so far into the terminology of group theory :

The Lebesgue measure $dx$ is the unique (up to a positive constant) translation invariant measure on the real line. Likewise, the same is true for the Haar measure $d\mu_H$ on $G$. Accordingly we replace $L_2(\mathbb{R}, dx)$ by $\mathcal{H} := L_2(G, d\mu_H)$. Next we notice that the real line is the additive group of real numbers and so the translation invariant vector field $\partial_x$ is replaced by the left invariant vector field $X_i := \text{tr}(g\tau_i \partial_g)$ where translation now means group multiplication and $\tau_i$ are the generators of the Lie algebra. Obviously the Laplacian on $G$ is now defined to be $\Delta := -\frac{1}{2}\delta^{ij}X_iX_j$ and the associated heat kernel by $\hat{R}_t := \exp(-t\hat{\Delta})$. The analytic extension of the variable $x$ is by going to $z = x + iy$. Notice that the additive group of real numbers is its own Lie algebra generated by 1 and that $\mathfrak{g}$ is generated by 1, $i$. Therefore in the case of $G$ we take the complexification of the Lie algebra generated by $\tau_j, i\tau_j$, i.e. we go from $x^j\tau_j$ to $(x^j + iy^j)\tau_j$ and exponentiate it to get the analytic extension $G^\mathfrak{g}$ of $G$. Likewise we define the Laplacian on $G^\mathfrak{g}$ to be $\Delta_{\mathfrak{g}} := -\frac{1}{2}\delta^{ij}[X_i^{\mathfrak{g}}X_j^{\mathfrak{g}} + X_i^{\mathfrak{g}}X_j^{\mathfrak{g}}]$ where $X_i^{\mathfrak{g}}$ is the analytic extension of the left invariant vector fields $X_i$ on $G$ and the bar means complex conjugation). We can now compute $\rho_t(g, h) = \rho_t(gh^{-1})$ and $\mu_t(g\mathfrak{g}, h\mathfrak{g}) = \mu_t(g\mathfrak{g}, h^{-1}\mathfrak{g})$ as before and arrive at

$$\nu_t(g\mathfrak{g}, \bar{g}\mathfrak{g}) := \int_\mathbb{R} \mu_t(g\mathfrak{g}g^{-1})d\mu_H(g), \quad d\nu_t := \nu_t(g\mathfrak{g}, \bar{g}\mathfrak{g})d\mu_H(g\mathfrak{g}, \bar{g}). \quad (4.3)$$

where, of course, $d\mu_H^\mathfrak{g}$ is the Haar measure on $\mathfrak{g}$. This completes the construction of the coherent state transform for one copy of $G$ [13].

We now wish to apply this machinery to our graph theoretic framework. Thus, we are led to take a graph $\gamma$ equipped with $n$ generators $\beta_i$ of its fundamental group and we assign to it a Laplacian on $G^n$ defined by

$$\Delta_\gamma(A) := \sum_{i=1}^n I(\beta_i)\Delta_I \quad (4.4)$$

where $\Delta_I$ is the Laplacian associated with $g_I := h_{\beta_i}(A)$, the holonomy around $\beta_I$ as defined above and $I(\beta_I)$ is a function from loops into the positive real numbers (so that the generator of the transform is still a positive operator on $G^n$).

We now plug $\Delta_\gamma$ into the machinery explained above (extended in an obvious way to $G^n$, that is, we do everything for each copy of the group separately). In particular, we obtain heat kernel measures $\rho_{t,\gamma}(g_1, ..., g_n)$ and $\nu_{t,\gamma}(g\mathfrak{g}, \bar{g}\mathfrak{g})$, however, in order that these measures define cylindrical measures the function $l$ cannot be entirely arbitrary. We need to check the consistency conditions that arise from the fact that we may write a cylindrical function on different graphs. As one can easily see, if a function $f$ is cylindrical with respect to a graph $\gamma$ then it is also cylindrical with respect to any bigger graph $\gamma' > \gamma$, i.e. $f = p^*_{\gamma'} f_{\gamma'} = p^*_{\gamma'} f_{\gamma'}$ and it is easy to see that $\gamma'$ can be obtained from $\gamma$ by a finite number of steps of the following type :
a) we just add one more generator $\beta_{n+1}$ independent of $\beta_1, \ldots, \beta_n$,
b) one of the generators of $\gamma$, $\beta_1$ say, is a composition of generators of $\gamma'$, so a formula like $\beta_1 = \beta'_i \circ \beta'_j$ holds and
c) one generator just appears inverted, $\beta_1 = (\beta'_1)^{-1}$.

Now the requirement $\int d\rho \gamma f_\gamma = \int d\rho \gamma' f_{\gamma'}$ leads to restrictions on $l$ as follows:

Requirement a) does not lead to a restriction since the heat kernel measures are non-interacting and normalized but b) leads to $l(\alpha \circ \beta) = l(\alpha) + l(\beta)$ and c) to $l(\alpha^{-1}) = l(\alpha)$.

This completes the construction of the coherent state transform.

However, our framework is unsatisfactory for two reasons:

a) Recall that the purpose of the coherent state transform is to map us from a theory of real connections to a theory of complex connections. Since two connections, real or complex, have the same transformation properties under diffeomorphisms, our transform should better be generated by a diffeomorphism covariant operator. While $\Delta_I$ is covariantly defined since it depends only on the holonomy of the connection around $\beta_I$, the function $l$ clearly breaks diffeomorphism covariance as it is independent of the connection, it is just kinematically defined, and not diffeomorphism invariant. So at best we can expect to work in a diffeomorphism-gauge fixed context.

b) Actually in [13] also a diffeomorphism-covariant coherent state transform is constructed based on measures introduced by Baez, however, those measures are not faithful and thus do not serve to provide inner products (they provide sesquilinear forms which are not positive definite) so that they cannot be used for our purposes.

b) The second disadvantage is that the only reason why to go from real to complex connections, at least in the context of quantum gravity, was to simplify the Hamiltonian constraint which is quite messy if we start, for instance, with the spin-connection representation [28]. However, the present transform $\hat{R}_t$ when viewed as a transform on $H := L_2(\mathcal{A}/G, d\mu_0)$ does not map us from the spin connection $\Gamma^i_a$ to the Ashtekar connection $\Gamma^i_a - iK^i_a$ [13] since it commutes with $\Gamma^i_a$ and so does not lead to a simplification of the Hamiltonian constraint at all. So the virtue of the complex representation is lost. Moreover, the spin-connection can be shown to be a bad coordinate for the triads [19].

5 The Wick rotation transform

The Wick rotation transform is geared at fixing both problems mentioned in the last paragraph in one stroke. In fact, it turns out that the coherent state transform can be viewed from a broader perspective as the unique solution of how to identify the particular complexification of a real variable with its analytic extension.

In the next subsection we will give an outline of the algorithm of incorporating the correct reality conditions into the quantum theory for a general theory. The procedure naturally leads to a generalized coherent state transform for whose underlying measure $\nu_t$, analogous to the ones discussed in the previous section, we will give a general formula. We comment on the available quantization strategies which automatically incorporate the correct reality conditions while keeping the constraints simple. Finally, we apply the algorithm to canonical quantum gravity.
5.1 The Complexifier and the Wick rotation

Suppose we are given some real phase space $\Gamma$ (finite or infinite dimensional) coordinatized by a canonical pair $(A, P)$ (we suppress all labels like indices or coordinates) where we would like to think of $A$ as the configuration variable and $P$ as its conjugate momentum. Let the Hamiltonian (constraint) on $\Gamma$ be given by a function $H(A, P)$ which has a quite complicated algebraic form and suppose that it turns out that it can be written in polynomial form if we write it in terms of a certain complex canonical pair $(A^\mathbb{C}, P^\mathbb{C}) := W^{-1}(A, P)$, that is, the function $H^\mathbb{C} := H \circ W$ is polynomial in $(A^\mathbb{C}, P^\mathbb{C})$ (the reason why we begin with the inverse of the invertible map $W$ will become clear in a moment). We will not be talking about kinematical constraints like Gauss and diffeomorphism constraint etc. which take simple algebraic forms in any kind of variables.

The requirement that the complex pair $A^\mathbb{C}, P^\mathbb{C}$ is still canonical is fundamental to our approach and should be stressed at this point. It should also be stressed from the outset that we are not complexifying the phase space, we just happen to find it convenient to coordinatize it by a complex valued set of functions. The reality conditions on these functions are encoded in the map $W$.

Of course, the theory will be much easier to solve (for instance computing the spectrum (kernel) of the Hamiltonian (constraint) operator) in a holomorphic representation $\mathcal{H}^\mathbb{C}$ in which the operator corresponding to $A^\mathbb{C}$ is diagonal rather than in the real representation $\mathcal{H}$ in which the operator corresponding to $A$ is diagonal. According to the canonical commutation relations and in order to keep the Hamiltonian (constraint) as simple as possible, we are naturally led to represent the operators on $\mathcal{H}$ corresponding to the canonical pair $(A, P)$ by $(\hat{A}\psi)(A) = A\psi(A), (\hat{P}\psi)(A) = -i\hbar\delta\psi(A)/\delta A$. Note that then in order to meet the adjointness condition that $(\hat{A}, \hat{P})$ be self-adjoint on $\mathcal{H}$, we are forced to choose $\mathcal{H} = L^2(\mathcal{C}, d\mu_0)$ where $\mathcal{C}$ is the quantum configuration space of the underlying theory and $d\mu_0$ is the unique (up to a positive constant) uniform measure on $\mathcal{C}$, that is, the Haar measure.

In order to avoid confusion we introduce the following notation throughout this section:

Denote by $\hat{K} : \mathcal{H} \to \mathcal{H}^\mathbb{C}$ and $\hat{K}^{-1}$ the operators of analytic continuation and restriction to real arguments respectively. The operators corresponding to $A^\mathbb{C}, P^\mathbb{C}$ can be represented on the two distinct Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^\mathbb{C}$. On $\mathcal{H}$ we just define them by some ordering of the function $W^{-1}$, namely $(\hat{A}^\mathbb{C}, \hat{P}^\mathbb{C}) := W^{-1}(\hat{A}, \hat{P})$. On $\mathcal{H}^\mathbb{C}$, the fact that $A^\mathbb{C}, P^\mathbb{C}$ enjoy canonical brackets allows us to define their operator versions simply by (and this is why it is important to have a canonical complex pair) $(\hat{A}', \hat{P}') = (\hat{K} \hat{A} \hat{K}^{-1}, \hat{K} \hat{P} \hat{K}^{-1})$, i.e. they are just the analytic extension of $\hat{A}, \hat{P}$, that is, $(\hat{A}'\psi)(A^\mathbb{C}) = A^\mathbb{C}\psi(A^\mathbb{C}), (\hat{P}'\psi)(A^\mathbb{C}) = -i\hbar\delta\psi(A^\mathbb{C})/\delta A^\mathbb{C}$.

But how do we know that the operators $\hat{A}', \hat{P}'$ on $\mathcal{H}$ and $\hat{A}', \hat{P}'$ on $\mathcal{H}^\mathbb{C}$ are the quantum analogues of the same classical functions $A^\mathbb{C}, P^\mathbb{C}$ on $\Gamma$? To show that there is essentially only one answer to this question is the first main result of this section.

Namely, when can two operators defined on different Hilbert spaces be identified as different representations of the same abstract operator? They can be identified iff their matrix elements coincide. Due to the canonical commutation relations we have to identify in particular also the matrix elements of the identity operator, that is,
scalar products between elements of the Hilbert spaces. The only way that this is possible is to achieve that the Hilbert spaces are related by a unitary transformation and that the two operators under question are just images of each other under this transformation.

In order to find such a unitary transformation we have to relate the two sets of operators $\hat{A}', \hat{P}'$ and $\hat{A}, \hat{P}$ via their common origin of definition, namely through the set $\hat{A}, \hat{P}$.

The first hint of how to do this comes from the observation that both pairs $(A, P)$ and $(\hat{A}', \hat{P})$ enjoy the same canonical commutation relations, i.e. they are related by a canonical complexification. Therefore the map $W$ must be a complex symplectomorphism, that is, an automorphism $(W \cdot O)(A, P) = (O \circ W)(A, P)$ of the Poisson algebra of (not necessarily real-valued) functions $O$ over $\Gamma$ (in particular $A, P$) which preserves the algebra structure (linear and symplectic structure) but, of course, not the reality structure. Let $iC$ be the infinitesimal generator of this automorphism (we do not require that $C$ be a bounded, positive or at least real functional), that is,

$$O_C = W^{-1} \cdot O = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}{\{O, C\}}^{(n)} \quad (5.1)$$

where, as usually, the multiple Poisson bracket is iteratively defined by ${\{O, C\}}^{(0)} = f$, ${\{O, C\}}^{(n+1)} = {\{O, C\}}^{(n)} \cdot O$. It follows immediately from (5.1) that the reality conditions for $O$ are given by (the bar denotes complex conjugation)

$$\bar{O}_C = \sum_{n=0}^{\infty} \frac{i^n}{n!}{\{O_C, C + \bar{C}\}}^{(n)} . \quad (5.2)$$

We are now going to assume that the functional $C$ has a well-defined quantum analogue, that is, the Complexifier $\hat{C}$ will be a (not necessarily bounded, not necessarily positive, not necessarily self-adjoint) densely defined operator on $\mathcal{H}$. Further, we would like to take equations (5.1), (5.2) over to quantum theory, that is, we replace Poisson brackets by commutators times $1/\hbar$ and we replace complex conjugation by the adjoint operation on $\mathcal{H} = L_2(\mathbb{C}, d\mu_0)$.

So let $\hat{O} = O(\hat{A}, \hat{P})$ be any operator on $\mathcal{H}$ where $O$ is an analytical function. Using the operator identity $e^{-\hat{g}e^{\hat{f}}} = \sum_{n=0}^{\infty} \frac{1}{n!}[\hat{g}, \hat{f}]^{(n)}$ and defining on $\mathcal{H}$

$$\hat{W}_t := \exp(-t\hat{C}) \quad (5.3)$$

we find that on $\mathcal{H}$ we may define the quantum version of (5.1) by

$$\hat{O}_C := O(\hat{A}', \hat{P}') = \hat{W}_t \hat{O} \hat{W}_t^{-1} \quad (5.4)$$

with $t = 1/\hbar$. That is, the generator $\hat{C}$ provides for a natural ordering of the function $W^{-1}(A, P)$.

$\hat{W}_t$ is called the Wick rotator (due to its role in quantum gravity).

We are now in the position to make the identification of the two sets $\hat{A}', \hat{P}'$ and $\hat{A}, \hat{P}$ precise, by writing the operators on $\mathcal{H}_\Phi$ in terms of the operators on $\mathcal{H}$. We have

$$(\hat{A}', \hat{P}') = (\hat{K} \hat{A} \hat{K}^{-1}, \hat{K} \hat{P} \hat{K}^{-1}) = (\hat{U}_t \hat{A} \hat{U}_t^{-1}, \hat{U}_t \hat{P} \hat{U}_t^{-1}) \quad (5.5)$$
where we have defined
\[ \hat{U}_t := \hat{K}\hat{W}_t. \quad (5.6) \]

So if we could achieve that \( \hat{U}_t \) is a unitary operator from \( \mathcal{H} \) to \( \mathcal{H}_C \) then our identification would be complete!

The way to do that is, of course, by constructing an appropriate measure \( \nu_t \) on the complexified quantum configuration space \( \mathcal{C}_q \) thereby fixing the Hilbert space \( \mathcal{H}_q \) to be the space of square-integrable functions on \( \mathcal{C}_q \) with respect to \( d\nu_t \) which are also holomorphic, that is, \( \mathcal{H}_q := L_2(\mathcal{C}_q, d\nu_t) \cap \text{Hol}(\mathcal{C}_q) \). We see that \( \hat{U}_t \) precisely coincides with the coherent state transform of the previous section in case that \( \hat{C} \) is the Laplacian on the group so that we call it the generalized coherent state transform and, due to its role in quantum gravity, we will also refer to it as the Wick (rotation) transform. In other words, the Wick transform can be viewed as the unique answer to our question. Any other unitary transformation \( \hat{u}_t \) between the Hilbert spaces necessarily corresponds to a different complexification \( \hat{w}_t = \hat{K}^{-1}\hat{u}_t \) of the classical phase space in which the Hamiltonian (constraint) takes a more complicated appearance. Note that any real canonical transformation corresponds to a unitary transformation in quantum theory, so the coherent state transform can also be characterized as the “unitarization” of the complex canonical transformation that we are dealing with.

Another characterization of the coherent state transform \( \hat{U} \) follows from the simple formula \( \hat{K} = \hat{U}_t\hat{W}_t^{-1} \) : it is the unique solution to the problem of how to identify analytic extension with the particular choice of complex coordinates \( A^\psi, P^\psi \) on the real phase space \( \Gamma \) as defined by \( W^{-1} \).

It should be stressed at this point that if \( \hat{C} \) is not a positive self-adjoint operator (so that \( \hat{W}_t \) is not a well-defined operator on \( \mathcal{H} \) for \( t > 0 \) which explains why we started with \( W^{-1} \) rather than \( W \)), we will assume that \( \hat{U}_t \) for positive \( t \) can still be densely defined (note that \( \hat{U}_{-t} \neq \hat{U}_t^{-1} \) due to the analytic continuation involved), that is, there is a dense subset \( \Phi \) of \( \mathcal{H} \) so that the analytic continuation of the elements of its image \( \hat{W}_t\Phi \) under \( \hat{W}_t \) are elements of a dense subset of \( \mathcal{H}_q \). We do not assume that \( \hat{U}_t \) itself can be densely defined on \( \Phi \) as an operator on \( \mathcal{H} \) ! Intuitively what happens here is that while \( \hat{W}_t\phi \) may not be normalizable with respect to \( \mu_0 \) for any \( \phi \in \Phi \), its analytic continuation will be normalizable with respect to \( \nu_t \) by construction since \( \hat{U}_t \) is unitary and thus bounded, just because the measure \( \nu_t \) falls off much stronger at infinity than \( \mu_0 \). So we see that going to the complex representation could be forced on us. This is a second characterization of the coherent state transform : not only is it a unique way to identify a particular complexification with analytic continuation, it also provides us with the necessary flexibility to choose a better behaved measure \( \nu_t \) which enables us to work in a representation in which \( \hat{W}_t \) or, rather, \( \hat{W}_t' \) is well-defined which is important because only then do we quantize the original theory defined by \( (\mathcal{H}, \hat{H}) \).

As a bonus, our adjointness relations are trivially incorporated ! Namely, because any operator \( \hat{O}_q \) on \( \mathcal{H} \) written in terms of \( A^\psi, P^\psi \) is defined by \( \hat{W}_t^{-1}\hat{O}\hat{W}_t \) where \( \hat{O} \) is written in terms of \( \hat{A}, \hat{P} \) and because \( \hat{O}_q \) is identified on \( \mathcal{H}_q \) with \( \hat{O}' = \hat{K}\hat{O}\hat{K}^{-1} = \hat{U}_t\hat{O}_q\hat{U}_t^{-1} \) we find due to unitarity that
\[ \hat{(\hat{O})}^\dagger = \hat{U}_t\hat{O}_q^\dagger\hat{U}_t^{-1} \quad (5.7) \]
where the adjoints involved on the left and right hand side of this equation are taken on $\mathcal{H}_q$ and $\mathcal{H}$ respectively. Therefore, $(\hat{O}')^\dagger$ is identified with $\hat{O}_q^\dagger$ as required.

Note that the adjoint of $\hat{O}_q$ on $\mathcal{H}$

$$\hat{O}_q^\dagger = [\hat{W}_q^\dagger \hat{W}_t] \hat{O}_q [\hat{W}_t^\dagger \hat{W}_q]^{-1}$$

(5.8)

follows unambiguously from the known adjoints of $\hat{A}, \hat{P}$ and coincides to zeroth order in $\hbar$ with the complex conjugate of its classical analogue (5.2) and therefore can be seen to be one particular operator-ordered version of the adjointness relations that follow from the requirement that the classical reality conditions (5.2) should be promoted to adjointness-relations in the quantum theory.

Interestingly, $\hat{W}_q^\dagger = \hat{W}_t$ on $\mathcal{H}$ corresponding to the fact that classically the complexifier is unchanged if we replace $A, P$ by $A_q, P_q$.

Finally we see that in extending the algebraic programme [12] from a real representation to the holomorphic representation of the Weyl relations we only have one additional input, everything else follows from the machinery explained below and can be summarized as follows:

Input A : define an automorphism $W$ (preferrably such that the constraints simplify).

Task A : determine the infinitesimal generator $C$ of $W$.

Input B : define a real $^*$ representation $\mathcal{H}$.

Task B : determine a holomorphic representation $\mathcal{H}_q$ so that $\hat{U} = \hat{K}\hat{W}$ is unitary.

Note that input B is also part of the programme if one were dealing only with the real representation so that input A is the only additional one. Task A is necessary if we want to express a given $W$ in terms of the phase space variables which is unavoidable in order to define $\hat{W}$.

In the next subsection we display a standard solution to Task B so that Task A is the only non-trivial problem left.

Remark :

Note that the existence of $W$ does not imply that classical solutions are mapped into solutions! That is, assume that we have found a physical solution $H(A_0, P_0) = E = \text{const.}$, then in general $H_q(A_0, P_0) = H(W(A_0, P_0)) \neq \text{const.}$ (this has nothing to do with the fact that $W$ does not preserve the reality conditions, rather it follows from the fact that $\{H, C\} \neq 0$ by construction since $W$ is supposed to turn the complicated algebraic form of $H$ into a simpler one). However, it will turn out that the quantum analogue of $W$ maps generalized eigenfunctions into generalized eigenfunctions!

### 5.2 Isometry

The construction of $\nu_t$ for a general theory differs considerably from the techniques applied in [13] and the previous section which turn out to be insufficient in dealing with an operator $\hat{C}$ which is neither positive nor self-adjoint. For instance, in applications to quantum gravity we need to apply the more general theory given below.

We will only display the result here, the details can be found in [19].

Isometry means that for any $\psi, \xi \in \mathcal{H}$ we have

$$\int_{\mathcal{M}/\mathcal{G}} d\mu_0(A) \overline{\psi}(A)\xi(A) = \int_{\mathcal{M}/\mathcal{G}} d\nu_t(A^q, A^\overline{q})[\hat{U}_t\psi](A^q)[\hat{U}_t\xi](A^q).$$

(5.9)
Denote by $\mu_0^\phi(A^\phi)$ the holomorphic extension of $\mu_0$ and by $\bar{\mu}_0^\phi(A^\bar{\phi})$ its anti-holomorphic extension which due to the positivity of $\mu_0$ are just complex conjugates of each other.

We now make the ansatz

$$d\nu_t(A^\phi, A^\bar{\phi}) = d\mu_0^\phi(A^\phi) \otimes d\bar{\mu}_0^\phi(A^\bar{\phi}) \nu_t(A^\phi, A^\bar{\phi}),$$

(5.10)

where $\nu_t$ is a distribution. Then we find that equation (5.10) can be solved by requiring

$$\nu_t(A^\phi, A^\bar{\phi}) := \left( W^\perp_t \right) \bar{\left( W^\perp_t \right)} \delta(A^\phi, A^\bar{\phi})$$

(5.11)

where the distribution involved in (5.11) is defined by

$$\int_{A^\phi \otimes G^\phi} d\mu_0^\phi(A^\phi) d\bar{\mu}_0^\phi(A^\bar{\phi}) f(A^\phi, A^\bar{\phi}) \delta(A^\phi, A^\bar{\phi}) = \int_{A \otimes G} d\mu_0(A) f(A, A).$$

(5.12)

Here the adjoints are taken with respect to $\mu_0$, the overline denotes complex conjugation of the whole expression of the operator (in particular $A^\phi \to A^\bar{\phi}$, $\delta/\delta A^\phi \to \delta/\delta A^\bar{\phi}$) and not anti-analytic extension and finally the prime means analytic extension $A \to A^\phi$, $\delta/\delta A \to \delta/\delta A^\phi$ as before. Whenever (5.11) exists and the steps to obtain this formula can be justified we have proved existence of an isometricity inducing positive measure on $A^\phi \otimes G^\phi$ by explicit construction. The rigorous proof for this [23] is by proving existence of (5.11) on cylindrical subspaces, so strictly speaking $d\nu_t$ is only a cylindrical measure. The measure is self-consistently defined because the operator $\hat{C}$ is.

Note that the proof is immediate in the case in which $\hat{C}$ is a positive and self-adjoint and therefore can be viewed by itself as an interesting extension of [13]. In particular it coincides with the method introduced by Hall [26] in those cases when $\hat{C}$ is the Laplacian on $G$ but our technique allows for a more straightforward computation of $\nu_t$.

### 5.3 Quantization

We are now equipped with two Hilbert spaces $\mathcal{H} := L_2(A^\phi \otimes G^\phi, d\mu_0)$ and $\mathcal{H}_\phi := L_2(A^\phi \otimes G^\phi, d\nu_t) \cap \text{Hol}(A^\phi \otimes G^\phi)$ which are isometric and faithfully implement the adjointness relations among the basic variables. $\mathcal{H}$ will be called the real representation and $\mathcal{H}_\phi$ the holomorphic or complex representation.

The last step in the algebraic quantization programme is to solve the theory, that is, to find the spectrum of the Hamiltonian (or the kernel of the Hamiltonian constraint) and observables, that is, operators that commute with the physical constraint operators. In more concrete terms it means the following [12]:

Let $\hat{H}_\phi := H_\phi(\hat{A}, \hat{P})$ be a convenient ordering of $H_\phi(A, P)$ such that its adjoint on $\mathcal{H}$ corresponds to the complex conjugate of its classical analogue (that is, write $H_\phi = a + ib$ where $a, b$ are real and order $a, b$ symmetrically) and let $\hat{H}' = H_\phi(\hat{A}', \hat{P}')$ be its analytic extension. Choose a topological vector space $\Phi(\Phi_\phi)$ and denote by $\Phi'(\Phi'_\phi)$ its topological dual. These two spaces are paired by means of the measure $\mu_0(\nu_t)$, for instance $f[\phi] := \int_{A^\phi \otimes G^\phi} d\mu_0(A) \hat{f}[A] \phi[A]$. $\Phi(\Phi_\phi)$ is by construction dense in its Hilbert space completion $\mathcal{H}(\mathcal{H}_\phi)$. We will be looking for generalized eigenvectors $f_\lambda(\phi_\phi)$ [27], that is, elements of $\Phi'(\Phi'_\phi)$ with the property that there exists a complex
number $\lambda$ such that $f_\lambda[\hat{H}_C\phi] = \lambda f_\lambda[\phi]$ for any $\phi \in \Phi(\Phi_C)$. Given this general setting we have at least two strategies at our disposal:

Strategy I:

We start working on $\mathcal{H}_\Phi$. This means that we would try to find a convenient ordering of the operator $\hat{H}' := H_\Phi(\hat{A}', \hat{P}')$. The Hamiltonian (constraint) on $\mathcal{H}$ now is defined to be the image under the inverse coherent state transform $\hat{H} := \hat{U}_t^{-1}\hat{H}'\hat{U}_t$ which to zeroth order in $\hbar$ coincides with one ordering of $H(A, P)$ but in general will involve an infinite power series in $\hbar$. That is, we have made use of the freedom that we always have in defining the quantum analog of a classical function, namely to add arbitrary terms which are of higher order in $\hbar$.

Of course, since the constraint is simple on $\mathcal{H}_\Phi$ we solve it in this representation as well as the problem of finding observables. After that we can go back to $\mathcal{H}$ which is technically easier to handle and compute spectra of the observables found and so on, thus making use of the powerful calculus on $\mathcal{H}_\Phi$ that has already been developed in $\mathcal{H}_\Phi$. This calculus can in particular be used to find a regularization in which $\hat{C}, \hat{H}_C = H_\Phi(\hat{A}_C, \hat{P}_C)$ are self-adjoint on $\mathcal{H}$ if they classically correspond to real functions because it then follows that also their images under $\hat{W}_t$, that is, $\hat{C}, \hat{H}$, are self-adjoint on $\mathcal{H}$ and then, by unitarity, the same holds for $\hat{C}', \hat{H}'$ on $\mathcal{H}_\Phi$. In this way, $\mathcal{H}_\Phi$ mainly arises as an intermediate step to solve the spectral problem.

Strategy II:

The following strategy is suggested by Abhay Ashtekar [20] in the restricted context of quantum gravity. Namely, just stick with the real representation all the time! The general idea of working only with real connections goes back to [13] and was revived by Barbero in [30]. This strategy now seems feasible because now we have a key new ingredient at our disposal – the Wick transform (compare next subsection) – which enables the real representation to simplify both, the reality conditions and the constraints.

Here we consider this strategy in the general case. That means, we look for a convenient ordering of $\hat{H}_\Phi := H_\Phi(\hat{A}, \hat{P})$, then the physical Hamiltonian (constraint) is defined by $\hat{W}_t\hat{H}_\Phi\hat{W}_t$ and agrees to zeroth order in $\hbar$ with some ordering of $H(A, P)$. The advantage of this approach is obvious: we never need to construct the measure $\nu_t$ which is only cylindrical so far while the measure $\mu_0$ is known to be $\sigma$–additive. For instance in the case of quantum gravity, although we continue to work with a connection dynamics formulations, the complex connection, drops out of the game altogether! All the results in $\mathcal{H}_\Phi$ are immediately available while in strategy II one could do so only after having solved the spectral problem (constraint).

Why then, would we ever try to quantize along the lines of strategy I? This is because there can be in general obstructions to find the physical spectrum or kernel directly from $\hat{H}_\Phi$. This is also the reason why we extended the programme as to construct the coherent state transform. Such obstructions can arise as follows: $H_\Phi(A, P)$ will be in general neither positive nor real valued even if $H(A, P)$ is, in which case it is questionable whether one is in principle able to find the correct spectrum from the former Hamiltonian (constraint). This is so because both $\hat{W}_t$ and $\hat{U}_t$ preserve the spectrum wherever they are defined, meaning that if we fail to have coinciding spectra for $\hat{H}, \hat{H}_\Phi$ then $\hat{W}_t$ is ill-defined as a map on $\mathcal{H}$ or on the
A fortunate case is when the topological vector space $\Phi$ is preserved by $\hat{W}_t$ : then generalized eigenvectors $f_\lambda$ of $\hat{H}_q$ are mapped (as elements of the topological dual space $\Phi'$) into generalized eigenvectors $\hat{W}_t^\dag f_\lambda$ of $\hat{H}$ with the same eigenvalue. The proof is easy : we have for any $\phi \in \Phi$ : 
$$\hat{W}_t^\dag f_\lambda [\hat{H} \phi] = f_\lambda [\hat{W}_t \hat{H} \phi] = f_\lambda [\hat{H}_q \hat{W}_t \phi] = \lambda \hat{W}_t^\dag f_\lambda [\phi]$$
as claimed. Note that it was crucial in this argument that $\hat{W}_t \phi \in \Phi$.

There are indications [23] that we are lucky in the case of quantum gravity.

Another way out might be the following : if $\hat{W}_t$ does not preserve $\Phi$ then we may just reduce the size of $\Phi$, thereby enlarging $\Phi'$, as to turn all the formal solutions to the constraints into rigorously defined generalized eigenvectors.

A, minor, disadvantage of strategy II) is as follows : while we can find physical observables $\hat{O}$ by looking for operators $\hat{O}_q$ that commute with $\hat{H}_q$ and then map them according to $\hat{O} := \hat{W}_t \hat{O}_q \hat{W}_t^\dag$, since $\hat{W}_t$ is not unitary on $\mathcal{H}$ we need to compute the expectation values $<\psi, \hat{W}_t^{-1} \hat{O}_q \hat{W}_t \xi>$ rather than just $<\psi, \hat{O}_q \xi>$. Via strategy I) we could compute everything either on $\mathcal{H}$ or $\mathcal{H}_q$, whatever is more convenient, because $\hat{U}_t$ is unitary and so does not change expectation values.

To summarize : 
If we proceed along strategy I) then we quantize the same physical Hamiltonian (constraint) $\hat{H}' := \hat{H}_q (\hat{A}', \hat{P}')$ and $\hat{U}_t^{-1} \hat{H} \hat{U}_t$ in two different but unitarily equivalent representations $\mathcal{H}_q$ and $\mathcal{H}$. The more convenient representation is $\mathcal{H}_q$ because the Hamiltonian (constraint) adopts a simple form. This procedure is guaranteed to lead to the correct physical spectrum of observables while it is technically more difficult to carry out since we are asked to construct the measure $\nu_t$.

If we proceed along strategy II) then we quantize the two distinct Hamiltonians $\hat{H}_q := \hat{H}_q (\hat{A}, \hat{P})$ and $\hat{H} := \hat{W}_t^{-1} \hat{H}_q \hat{W}_t$ (of which the latter is the physical one) in the same representation $\mathcal{H}$. While this procedure is technically easier to perform, as explained above, its validity depends on the strong condition that $\hat{W}_t$ can be densely defined on $\mathcal{H}$ which is often not the case!

5.4 A transform for quantum gravity

Denote by $q_{ab}, K_{ab}$ the induced metric and extrinsic curvature of a spacelike hypersurface $\Sigma$, introduce a triad $e_a^i$ which is an $SU(2)$ valued one-form by $q_{ab} = e_a^i e_b^j \delta_{ij}$ and denote by $e_a^i$ its inverse. Then we introduce the canonical pair of Palatini gravity by $(K^a_i := K_{ab} e^b_i, P^a_i := 1/\kappa \sqrt{\text{det}(q) e_a^i})$ where $\kappa$ is Newton’s constant.

We will now employ our algorithm to find the coherent state transform for quantum gravity.

The important observation due to Ashtekar [15] is that if we write the theory in terms of the complex canonical pair $A^i_a := \Gamma^i_a + iK^i_a, P^a_i := iP^a_i$ where $\Gamma$ is the spin-connection associated with $P$, then the Hamiltonian constraint adopts the very simple polynomial form $\hat{H}_q (A_q, P^q) = - \text{tr}(F_a^b [P^a_q, P^b_q])$ where $F^a_q$ is the curvature of $A^q$. The importance of this observation is that $(A_q, P^q)$ is a canonical pair which relies on the discovery that the spin connection is integrable with generating functional $F = \int_{\Sigma} d^3x \Gamma^i_a P^a_i$. Ashtekar and later Barbero [30] also considered the real
canonical pair \((A^j_a, P^a_j, P^a_i)\) in which, however, the Hamiltonian takes a much more complicated non-polynomial, algebraic appearance which becomes polynomial only after multiplying by a power of \(\det(P^a_i)\) which turns the constraint into an unmanageable form. After neglecting a term proportional to the Gauss constraint this Hamiltonian can be written as 

\[ H_C(A^a_C, P^a_C) \equiv H(A, P) = \text{tr}(\{F_{ab} - 2R_{ab}\}[P^a_i, P^b_j]) \]

in which \(F, R\) are respectively the curvatures of \(A, P\).

The real and complex canonical pairs are related by a chain of three canonical transformations 

\[(A^j_a, K^a_j, P^a_j) \rightarrow (K^a_j, P^a_j) \rightarrow (-iK^a_j, iP^a_j) \rightarrow (A^a_C, j^a_j, P^a_C, j^a_j)\]

of which the first and the last have as its infinitesimal generator the functional \(-F\) and \(iF\) respectively. The new result to is that we are able to derive the infinitesimal generator of the middle simplectomorphism \((K^a_j, P^a_j) \rightarrow (-iK^a_j, iP^a_j)\) which is a phase space Wick rotation.

This generator is uniquely given by (compare [19] for a systematic derivation)

\[ C := \frac{\pi}{2} \int_\Sigma d^3x K_i^a P_i^a \]

which is easily recognized as \((\pi/2\kappa)\) times the integral over the densitized trace of the extrinsic curvature of \(\Sigma\).

The key observation in proving this is that the Poisson bracket of \(C\) with the spin-connection \(\Gamma_i^a\) vanishes. The elegant way of seeing it is by noticing that \(\hat{C}\) generates constant scale transformations and remembering that \(\Gamma_i^a\) is a homogenous rational function in \(P_i^a\) and its spatial derivatives of degree zero. Summarizing, the Ashtekar variables \((A^a_C^i, j^a_j) = \Gamma_j^a j^a_j, P^a_C^i, j^a_j = iP^a_i, j^a_j)\) are the result of a Wick rotation generated by \(C\) in the sense of (5.1), namely

\[ A^a_C^i(x) = \sum_{n=0}^\infty \frac{(-i)^n}{n!} \{A^i_a(x), C\}_n = \Gamma^i_a(x) + \sum_{n=0}^\infty \frac{(-i\pi/2)^n}{2} K^i_a(x) \]

and similarly for \(P^a_i\).

The unphysical Hamiltonian constraint \(H_q(A, P) = -\text{tr}(F_{ab}[P^a_i, P^b_j])\) is up to the negative sign just the Hamiltonian constraint of the formal Hamiltonian formulation of Euclidean gravity (it is easy to see that our Wick rotated Lorentzian action equals that of Euclidean gravity if we replace the lapse by its negative and the shift and Lagrange multiplier of the Gauss constraint by \(-i\) times themselves), however it should be stressed that what we are doing here is not the quantization of Euclidean gravity : there is no analytic continuation of the time coordinate involved for which there is no natural choice anyway.

What is important is that \(\hat{C}, \hat{H}_q\) on \(H\) can be chosen to be self-adjoint operators and regularized with the techniques already available in the literature because it is a classically real expressions. It is a pecularity of the gravitational Hamiltonian that \(H\) and \(H_q\) are both real.

According to our general programme we have completed Input A, namely we have chosen the Ashtekar complexification, and task A, we have found the complexifier \(C\). The next Input B is to choose a representation. We choose \(H := L_2(A/\mathcal{G}, d\mu_0)\), that is \(C = A/\mathcal{G}\) is the quantum configuration space of real distributional connections introduced in section 2 and \(d\mu_0\) is precisely the induced Haar measure on \(A/\mathcal{G}\).
introduced in section 2 which is uniquely selected by the requirement that loop and strip operators are essentially self-adjoint [12].

The task B left to do is to construct the measure $\nu_t$ for the Hilbert space $\mathcal{H}_C := L_2(\mathcal{A}^C/\mathcal{G}^C, d\nu_t) \cap \text{Hol}(\mathcal{A}^C/\mathcal{G}^C)$, the set of square integrable functions of complexified connections which are holomorphic. Here $\mathcal{A}^C/\mathcal{G}^C$ is the quantum configuration space of complexified connections modulo gauge transformations [13]. This task requires to find an appropriate regularization of the operator version of $C$. This seems to be a hopeless thing to do because when written in terms of $A, P$ it involves the spin-connection which is a non-polynomial expression. However, using that for $\sqrt{|\det(E)|}$ already a well-defined regularization exists [31], in [29] a regularization is considered which defines a self-adjoint operator, well-defined on diffeomorphism-invariant states (in the same sense as in [32]) and which leaves every cylindrical subspace separately invariant. This operator is not positive so the latter property is quite important if one wants to exponentiate it (for instance the regularization used in [32] does not have this property).

We can then define the (Wick or) “generalized coherent state transform” associated with the “heat kernel” $\hat{W}_t$ for the operator $\hat{C}$ to be the following map

$$\hat{U}_t : \mathcal{H} \to \mathcal{H}_C, \hat{U}_t[f](A^C) := \langle A^C, \hat{W}_t f \rangle := \langle A, \hat{W}_t f \rangle|_{A \to A^C} \quad (5.15)$$

which on functions cylindrical with respect to a graph generated by $n$ loops $\beta_I$ reduces to (because $\hat{C}$ leaves that subspace invariant)

$$\hat{U}_{t,\gamma}[f_{\gamma}](g_1^q, \ldots, g_n^q) := \int_{G^n} d\mu_{0,\gamma}(g_1, \ldots, g_n) \rho_{t,\gamma}(g_1^q, \ldots, g_n^q ; g_1, \ldots, g_n) f_{\gamma}(g_1, \ldots, g_n) \quad (5.16)$$

where $g_I := h_{\beta_I}(A), g_I^q := h_{\beta_I}(A^q)$ are the holonomies along the loop $\beta_I$. Here $\rho_{t,\gamma}(g_t ; h_I) := \langle g_1, \ldots, g_n, \exp(-t\hat{C}_\gamma)h_1, \ldots, h_n \rangle$ is the kernel of $\hat{W}_t$ and $\hat{C}_\gamma$ is the projection of $\hat{C}$ to the given cylindrical subspace of $L_2(\mathcal{A}/\mathcal{G}, d\mu_0)$. The kernel, if it exists, is real analytic on $G^n$ and therefore has a unique analytic extension. Note that the transform is consistently defined on cylindrical subspaces of the Hilbert space because its generator $\hat{C}$ acts primarily on the connection and does not care how we write a given cylindrical function on graphs that are related to each other by inclusion.

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