New solutions to an old equation

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Abstract
We present a brief historical overview of the classical theory of a radiating point charge, described by the Lorentz-Dirac equation. A recent development is the discovery of tunnelling of a charge through a potential barrier, in a completely classical context. Also, a concrete example is discussed of the existence of several physically acceptable solutions for a range of initial data. We end by pointing out some open problems in connection with D-brane and monopole physics.

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1. Introduction

More than one hundred years ago, Lorentz determined that the force exerted on a particle (charge $e$, no spin, but we will call it “electron” nevertheless) by an electromagnetic field is given by

$$F_\mu = eF_{\mu\nu}(x^\alpha) \dot{x}^\nu.$$  \hspace{1cm} (1.1)

The electromagnetic field in eq. (1.1) is to be taken at the position of the charge. However, a charge generates an electromagnetic field that is singular. Hence the expression for the force in eq. (1.1) cannot be taken literally: somehow it should not include the field generated by the particle itself, which would render the expression meaningless. On the other hand, just leaving it out is not satisfactory either, since it would miss an important physical effect: a charge may radiate, and this radiation acts back on the motion. The resolution of this question dates from the beginning of this century, and resulted in an equation, nowadays called the Lorentz-Dirac equation (LDE), that takes into account this back-reaction. In the next section we review some highlights in the history of this equation. It can be formulated as a third order differential equation, which entails some peculiar features of the associated particle trajectories. Among these peculiarities we mention (in section 3) the well known runaways, and call attention to the fact that there is an indeterminacy in the theory: for given initial conditions (combined with physical asymptotic conditions), multiple trajectories may be possible.

There has been a steady (if limited) interest in this subject that never completely faded. It may come therefore as a surprise that recently we discovered a whole new class of solutions, which, if not very relevant experimentally, nevertheless poses some intriguing theoretical questions. This class describes tunnelling, in a completely classical context: it is possible for a charge to traverse a classically forbidden region, provided this is accomplished in a

\footnote{Even though Lorentz didn’t, of course, we use relativistic notation: $x^\nu$ is the space-time position of the particle, and the dot denotes a derivative with respect to its proper time.}
sufficiently short time. We illustrate this in section 4. We end with an assortment of additional remarks connecting this old chapter to more modern developments, like monopoles and D-branes.

2. The Lorentz-Dirac equation: a short historic overview

Here is the Lorentz-Dirac equation

\[ \ddot{x}^\mu = \frac{e}{mc} F^{\mu\nu} \dot{x}_\nu + \frac{2e^2}{3mc^3} \left( \ddot{x}^2 \dot{x}^\mu + \ddot{x}^\mu \right) \]  

The first term is the Lorentz force. The second term results from the non-relativistic theory of the radiation reaction as developed by Lorentz and Abraham in the (eighteen-)nineties (the electron was discovered experimentally at the end of that period). It takes into account the energy loss due to radiation, as should be familiar. It was also found to be in agreement with considerations of extended models for a charged particle (Poincaré), in the limit that its size tends to zero. This involves a classical renormalisation of the mass of the charged particle. That the third term in eq. (2.1) is a necessary is clear when one checks that \( \ddot{x}^2 = 1 \) is preserved. It follows from a proper relativistic treatment as given first by Schott (1915), and is (perhaps surprisingly) called the “Schott term”.

Facing some peculiarities of the solutions of eq. (2.1) — to which we come back soon —, over the years, several alternatives have been (re-)proposed. For point electrons, these typically run into difficulties, with energy-momentum conservation. Indeed, Dirac showed\(^1\) in 1938 that the equation as it stands follows from very general considerations on the conservation of the energy-stress tensor everywhere up to the immediate vicinity of the electron world line.\(^3\) Therefore, the only alternative is an extended model, but this is not without problems. A rigid model is incompatible with special relativity.

\(^2\)See \[3\] for general references.

\(^3\)Dirac’s derivation eventually attached his name to the equation.
Dirac’s own deformable model of 1962 (which he hoped would “explain” the muon) was shown in 1978 \[2\] to be instable\[6\] The situation was summed up in the sixties in the classical work of Rohrlich\[4\]

In the meantime, with the improved understanding of the physical (i.e. quantum mechanical) electron, the quantum field techniques were applied to this problem, without shocking results: the equation (2.1) also follows if one\[6\] applies the new renormalisation techniques. A singularly missing piece of the puzzle however is a direct connection of the equation to quantum electrodynamics.

3. Solutions

Now we look at some solutions of eq. (2.1). Already for the motion of the electron in a region without external electromagnetic fields there is a surprise: apart from the familiar linear solutions there is a solution with exponential rising rapidity, like \( \dot{x}^{\mu} = c \left( \cosh Y, \sinh Y, 0, 0 \right) \) with \( Y = \exp \left( \frac{\tau}{\tau_0} \right) \). Here, \( \tau_0 = \frac{2e^{3}}{3mc^3} \) is the natural time unit for this “runaway” phenomenon, its value for an electron is \( \simeq 0.62 \times 10^{-23} \) s. This phenomenon is not observed in nature.

In quantum theory, for an electron, a typical timescale would be given by the Compton radius divided by the light velocity, \( \frac{\hbar}{mc^2} = \tau_0 \frac{3}{2\alpha} \). Therefore, a way out is to state that one should not take seriously the classical theory down to this timescale. Be that as it may, one should eliminate these solutions from the classical theory. This is achieved by an asymptotic (large time) condition: one decrees that, if the particle ends up in a force free region, it does not accelerate asymptotically. While eliminating the runaways, this procedure in turn gives rise to two new surprises. The first appears if we solve eq. (2.1) for a particle moving into the vicinity of a potential step: it starts to accelerate before it reaches the edge of the step. The pre-acceleration time, i.e. the characteristic time of this onset of the acceleration, is again \( \tau_0 \).

\[4\]This is illustrative of the slow but steady progress in this outpost of classical physics.

\[5\]At present, it is still an excellent starting point, even though it is incomplete (see further). The paper by T. Erber\[5\] contains complementary information.
Although some may consider this a violation of causality, the phenomenon is less shocking if one takes into consideration that the Coulomb field, with its singularity at the position of the particle, is noticed in the region of the force field long before the particle arrives there. The second is the existence of multiple solutions of eq. (2.1) for given initial position and velocity. Although in [4] the question of uniqueness was called “one of the most important unsolved problems of the theory” [7], the older literature already contains examples of non-uniqueness, as well as conditions for uniqueness. During the seventies (see for example [8, 9]) this aspect was stressed at the occasion of (often numerical) studies of the LDE in various circumstances. A recent addition [11] is the proof of uniqueness in a constant magnetic field, a study issued from the ill-fated SSC-laboratory. More on this non-uniqueness topic in the next section.

During the seventies, an interesting interpretation of eq. (2.1) was worked out (see [13]) in terms of a bound momentum

$$p^\mu = m \dot{x}^\mu - \frac{2e^2}{3c^3} \ddot{x}^\mu$$  \hspace{1cm} (3.1)

which contains, apart from the usual velocity term, also a contribution from the acceleration when the particle is charged. This quantity is a property of the particle’s motion at a fixed time, and corresponds to a consistent split of the energy-momentum tensor in a part “bound” to the electron, and a part radiating away. In terms of this momentum, transferring the third term on the right hand side of eq. (2.1) to the left, the equation of motion assumes a completely conventional interpretation: the rate of change of the momentum is equal to the applied force, where the latter consists of two pieces, the externally applied force (term (1)) and the radiation reaction force (term (2)). This interpretation is very useful, as we shall see, to guide the intuition.

\textsuperscript{6} which time does not permit us to go into, but see [10].
4. Tunnelling

In spite of the venerable history of the subject, a whole new class of solutions lay undiscovered until recently\cite{12}. The surprising fact is that, according to eq. (2.1), a classical particle may cross a potential barrier provided it can do so in a time of the order of $\tau_0$. The demonstration is completely elementary for a particle crossing a rectangular potential barrier, in one dimension. The equation, in natural units, reduces to

$$\dot{p}=\dot{v}-\ddot{v}=F.$$  \hspace{1cm} (4.1)

This may be viewed as the non-relativistic approximation (where $v$ is the velocity), but the equation is in fact exact also relativistically if $v$ is taken to be the rapidity. \footnote{The presentation we follow in the sequel is non-relativistic, but only because of the more transparent relation between distance and velocity in that case. None of our conclusions depend on it.} The electron only experiences a force when crossing the boundaries of the regions of constant potential. The solutions in the separate force-free regions, which are easy to write down explicitly, are connected using the following matching condition\footnote{This can be checked using the formal equation $\dot{p}=\Delta V \cdot \delta(x)$. The validity of the rectangular barrier idealisation is discussed in \cite{12}.} on the momentum, or the acceleration, the position and the velocity being continuous:

$$\Delta p = -\Delta \dot{v} = -\Delta V/v.$$  \hspace{1cm} (4.2)

This results in the following set of equations relating the initial and final velocities to the time $T$ spent in the barrier region of width $w$ and height $V$:

$$w = v_f T - \frac{V}{v_f} (e^{-T} - 1 + T),$$

$$v_i = v_f - \frac{V}{v_f} + \frac{V}{v_f - \frac{V}{v_f}(1-e^{-T})}.$$  \hspace{1cm} (4.3)

Although the analysis of these equations in general is not very difficult, it becomes particularly simple when the final electron energy is equal to half
the barrier height, \( V = v_f^2 \). An explicit example is, for \( V = 144, w = 3 \):

\[
x = \begin{cases} 
-7(e^t - 1) + 16t & t < 0, \\
9(e^t - 1) & 0 < t < T, \\
3 + 12(t - T) & T < t,
\end{cases}
\]

(4.4)

with \( T = \log 4/3 \). The matching condition eq.(4.2) implies jumps of 16 and 
\(-12\) units in the acceleration at \( t = 0 \) and \( t = T \). Tunneling occurs, the 
initial energy is equal to 128, a fraction \( 1/9 \) below the barrier height.

The motion can be described intuitively by following the bound momen-
tum, eq. (4.1), while the electron crosses the barrier. Its value is piecewise
constant, but the velocity component of the momentum and the acceleration
component may change continuously, \( v = p + \text{constant. exp}(\tau) \). This is akin
to the pre-acceleration phenomenon. In the regions outside the barrier the
momentum is equal to its asymptotic value. Under the barrier itself the value
is \( v_f - V/v_f \). Thus, the momentum may be in the same direction or opposite
to the velocity.\(^9\) Within a time of the order of the pre-acceleration time \( \tau_0 \)
this situation may be “rectified”, bringing momentum and velocity back in
line. However, in the meantime the electron may have reached the other side
of the barrier. In that case, it tunnels through.

Whereas the details of this example are of course special, the tunnelling
phenomenon is actually quite generic. A decisive parameter is the width of
the potential. If the electron can cross the barrier within a time of order 1,
i.e. the pre-acceleration time \( \tau_0 \), tunnelling occurs. The smaller the width,
the wider the range of initial velocities for which the electron will tunn

Since a sudden step in the potential actually corresponds to an infinite
force, one may well be worried that the tunnelling solutions are an artifact
of that unphysical feature. This is not the case. We have made a detailed
investigation of what happens at a (steep) ramp, a potential step being an
idealisation of this. The results are shown in figure [1], taken from [12].

\(^9\)The explicit example given above is special in that the momentum vanishes in the
intermediate region.
Figure 1: Plot of the initial velocity vs. the final velocity for the solution of the Lorentz-Dirac equation in a linearly rising step potential (note the difference in scale). The dotted lines leave out the radiation reaction. The inset shows the potential and kinetic energy as a function of position for four representative examples. The four types of motion are discussed in the text. For the plots, a step height $V = 9$ was used, and a slope width $\epsilon = 0.5$. 
In keeping with the solution strategy that corresponds to integrating the equations backwards (so as to use the asymptotic condition with maximal efficiency), the plot is of incoming velocity (rapidity) vs. outgoing velocity. The inset shows the four types of motion around the ramp, the main figure shows the relation between the asymptotic velocities. The branch labeled IV is investigated in very great detail in [14]. The branch II, in the limit of an infinitely steep potential, is investigated numerically in [15], where it is concluded that for a certain range of initial velocities no solution exists. Whereas this is true for the infinitely steep case, we see that the missing range of small initial velocities is covered by our branch I, and it is therefore an unphysical feature of the infinite -force approximation. We see clearly that it corresponds to the particle turning back under the sloping region, which is reduced to zero width in that approximation. For more details on figure I, we refer to the original paper. The complete figure is a good illustration of the fact that for specific initial velocities several different solution may exist: up to five in this case, on branches I,II and IV. A complementary remark is that actually for all initial values a solution is found. We do not know of a proof of this fact in any generality, nor have we attempted to construct one. It would clearly be worrying if some initial velocities would turn out to be impossible, but happily this is not the case here.

5. Remarks

For the electron, as mentioned in section 3, the pre-acceleration time (times c) is a factor $\alpha$ smaller than the Compton wavelength, and therefore one expects quantum effects to mask any classical effect on that timescale. Lacking any solid investigation, it is an open question whether a full quantum theory, QED for instance, does give rise to the LDE in some suitable limit. If so, the classical tunnelling solutions we found (and the undeterminacy of final velocities) should have a quantum counterpart and correspond to physical features of the electron theory.
For magnetic monopoles, this argument may be reversed\textsuperscript{10} since in that case the size of the coupling is the inverse of the electric coupling, quantum theory does not bail out the classical theory, so the “strange” features of the LDE and its solutions should find a “resolution” within the classical theory. Happily, purely classical field theories are available where a monopole exists\textsuperscript{16} as a soliton. The monopole is extended, and the fields have a finite energy so that no infinite renormalisation is needed. Since another length scale comes in due to the finite extension, the final form of the effective equation of motion does not have to reproduce eq. (2.1) exactly. It would be interesting to investigate\textsuperscript{11} to what extent it does, and especially what happens to the unexpected features like tunnelling.

As soon as a substructure of the electron is considered, other possibilities lie open. Apart from mechanical models, probably the oldest variation on this theme is the modification of electrodynamics known as Born-Infeld electrodynamics. In currently fashionable developments in connection with string and M-theory, actions of the Born-Infeld type govern the dynamics of D-branes. Point charge solutions in the classical theory got a new lease of life\textsuperscript{12} following their use\textsuperscript{18, 19} for the description of strings ending on these branes. In such an interpretation, an electrostatic self-energy divergence is re-interpreted as the (infinite) length of a string stretching from a 3-brane to infinity. A finite energy solution is obtained by having the string end on another brane, at a finite distance. Therefore, from a vantage point on the brane, this configuration provides a finite relativistic model for a point charge as well. A serious drawback, for our purposes, of the Born-Infeld dynamics is its non-linearity. It is not clear whether a consistent split can be made between the field of the point charge and the external fields, or between the energy-momentum “bound” to this charge and the energy-momentum of

\textsuperscript{10}This remark discusses a point raised by G. ’t Hooft at the meeting.
\textsuperscript{11}In \textsuperscript{17}, in a the nonrelativistic approximation, a correction to eq. (2.1) is found proportional to fourth order derivatives.
\textsuperscript{12}Old solutions to new equations.
radiation: these were instrumental in the development of the Lorentz-Dirac equation. It would certainly be amusing if string theory would not only open new vistas on traditional field theory, but would improve the status of classical electron theory as well.

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