1. Introduction

We consider the Tensor Isomorphism Problem (TIP), which essentially asks whether two grids of numbers are equivalent up to change of basis. The problem arises in areas ranging from classifications of algebraic structures [4, 5, 10, 16] to computational complexity [14, 17] to quantum phases of matter [8, 19]. Previous attacks on the problem also exploit ideas from a variety of areas, including $*$-algebras, Lie and Jordan algebras, numerical methods, and invariant theory. The method we propose here uses a recently discovered ternary Galois connection [9] to generalize several existing approaches to the problem.

Fix a coefficient field $K$, a valence $\gamma$ (Hebrew letter “vav”), and the dimensions $d_1, \ldots, d_t$. A grid—hereafter a tensor—is an array $t(i_1, \ldots, i_t) \in K$, where $i_a \in [d_a] := \{1, \ldots, d_a\}$. Indexing the $a^{th}$ axis of the grid by a basis $\{e_a(i) \mid i \in [d_a]\}$ of $K^{d_a}$, we associate to $t$ a multi-linear form $(t|) \in (K^{d_1} \otimes \cdots \otimes K^{d_t})^*$ which evaluates at $\{v_1, \ldots, v_t\} \in K^{d_1} \otimes \cdots \otimes K^{d_t}$ as follows:

$$
\langle t|v_1, \ldots, v_t \rangle = \sum_{\sigma \in [t]} \sum_{i_a \in [d_a]} t(i_1, \ldots, i_t)v_{\sigma(1)}(i_1) \cdots v_{\sigma(t)}(i_t), \quad v_a = \sum_{i_a \in [d_a]} v_a(i_a)e_a(i_a).
$$

A tensor $t$ is nondegenerate if for each $0 \neq v_a \in K^{d_a}$ there exists $v_b \in K^{d_b}$ (for each $b \neq a$) such that $(t|v_1, \ldots, v_t) \neq 0$. Tensors $s$ and $t$ are isomorphic, written $s \cong t$, if there exists $\varphi \in \prod_{a \in [t]} \text{GL}_{d_a}(K)$ with

$$
\langle s|v_1, \ldots, v_t \rangle = \langle t|\varphi v_1, \ldots, v_t \rangle := \langle t|v_1, \ldots, \varphi v_t \rangle;
$$

that is, $s$ and $t$ lie in the same $\prod_{a \in [t]} \text{GL}_{d_a}(K)$-orbit of $(K^{d_1} \otimes \cdots \otimes K^{d_t})^*$.

TIP asks for an efficient algorithm that, given tensors $s$ and $t$, finds $\varphi$ with $s = t\varphi$. For $\gamma > 2$ the only known solution is for finite fields and checks all possible $\varphi$. One does substantially better than brute force for special classes of tensors: random tensors over finite fields are treated in [17], and tensors with large automorphism groups are considered in [2, 5, 14].

Our method uses an algebra associated to tensors to compress the ambient space in which $s$ and $t$ live, thereby speeding up the orbit search. Its antecedents in [5, 21]
used tensor products over associative algebras to build compressed search spaces. Here we use the following Lie algebra of derivations

$$\text{Der}(t) = \left\{ \delta \in \prod_{a=1}^{1} \mathfrak{gl}_{d_a}(K) \left| 0 = \langle t| \delta v_1, \ldots, v_1 \rangle + \cdots + \langle t| v_1, \ldots, \delta v_1 \rangle \right\} ,$$

and to replace tensor products we use the densor (derivation tensor) space

$$\langle \otimes \rangle = \left\{ s \in (K^{d_1} \otimes \cdots \otimes K^{d_1})^* \left| \text{Der}(t) \subseteq \text{Der}(s) \right\} .$$

Our computational model follows de Graaf [7] in that we assume bases for every vector space, including algebras. Our methods apply to fields $K$ that are either finite fields or finite extensions of $\mathbb{Q}$ (number fields). In the latter case we use $\mathbb{ff}$-algorithms [15]: these utilize oracles to factor in $\mathbb{Z}$ and $\mathbb{Q}[x]$. Recall, a Las Vegas algorithm returns a correct answer but may, with bounded probability, abort.

For reasons of brevity, when we write ‘algorithm’ in the paper we shall mean ‘Las Vegas algorithm’ if $K$ is finite, and ‘Las Vegas ff-algorithm’ if $K$ is a number field.

Our main result provides an efficient isomorphism test for tiny densors (tensors having 1-dimensional densor spaces). Note, a Lie algebra $L$ has Chevalley type if $[L, L]$ has a Chevalley basis.

**Main Theorem.** Let $K$ be a field with either $K = 6K$ finite or $K/\mathbb{Q}$ finite. There is an algorithm that, given $s, t \in (K^{d_1} \otimes \cdots \otimes K^{d_1})^*$ where $\text{Der}(s)$ has Chevalley type and $\dim \otimes = 1$, decides if $s \cong t$ using $(d_1 + \cdots + d_1)^{O(1)}$ steps.

When $K = 6K$ is finite, an implementation is available [3]. Further, in this case our method may be adapted to construct automorphism groups of tiny densors.

A critical component of our approach is an efficient algorithm to decide conjugacy of semisimple Chevalley Lie subalgebras of $\mathfrak{gl}(V)$ represented irreducibly (Theorem 5.4). These conditions are not easily relaxed, for J. Grochow has shown that general conjugacy of semisimple Lie algebras is at least as hard as graph isomorphism [11]. The tiny densor hypothesis ensures the conditions of Theorem 5.4 are satisfied in our application (Proposition 5.3).

Tiny densors are natural base cases for divide-and-conquer mechanisms in TIP, and are interesting special case in their own right. Indeed in Section 6 we present an infinite family of tiny densors arising from the representation theory of Lie algebras.

### 2. Algebraic Tensor Compression

Our new isomorphism test for tensors generalizes several existing methods that each proceeds by constructing a ring of linear operators acting on a space of tensors, and then compresses the search space to a tensor product over this ring.

To illustrate, the bilinear dot-product $\langle u|v \rangle = \sum u_i v_i$ can be treated as an element of the 1-dimensional space $(K^n \otimes \mathbb{M}_n(K) K^n)^*$ instead of the obvious $n^2$-dimensional space $(K^n \otimes K K^n)^*$. Thus, isomorphism against a dot-product now takes place in a vastly smaller space. Also, an isomorphism acting there makes an outer automorphism of $\mathbb{M}_n(K)$; by the Skolem–Noether theorem it is scalar.

In the literature, tensor isomorphism tests have exploited associative rings—like $\mathbb{M}_n(K)$ in the dot-product example—to compress the search space [5, 16, 21]. The rings used were each defined by corresponding universal properties. However, the recent work of [9] places the theory in a broader context, and demonstrates that universality in this setting involves Lie algebras (derivations) and their tensor space.
analogues (densors). This is the theory that underpins our new isomorphism test, which we call the derivation-densor method.

2.1. The adjoint-tensor method. The first algebraic tensor compression method was introduced in [16] and soon after extended and generalized [2, 5, 21]. Given a bilinear map \((t) : K^{d_2} \times K^{d_1} \rightarrow K^{d_0} \) (\(\rightarrow\) denotes multilinear), its adjoint algebra is

\[
\text{Adj}(t) = \{ \delta \in \mathbb{M}_{d_2}(K)^{op} \times \mathbb{M}_{d_1}(K) \mid \langle t|v_2\delta_2, v_1 \rangle = \langle t|v_2, \delta_1 v_1 \rangle \},
\]

and its associated tensor space is \(K^{d_2} \otimes_{\text{Adj}(t)} K^{d_1}\). Then \(\{t\}\) naturally factors through \(\otimes_{\text{Adj}(t)} : K^{d_2} \times K^{d_1} \rightarrow K^{d_2} \otimes_{\text{Adj}(t)} K^{d_1}\) [5, Theorem 2.11].

The adjoint-tensor method solves the isomorphism problem between \(s\) and \(t\) by first deciding if there exists \(\mu\) conjugating \(\text{Adj}(s)\) to \(\text{Adj}(t)\), and then carrying out a search within the compressed space \(\text{Hom}(K^{d_2} \otimes_{\text{Adj}(t)} K^{d_1}, K^{d_0})\)—in which both \(\mu\) and \(t\) now reside—under the action of the potentially much smaller group normalizing \(\text{Adj}(t)\), modulo \(\text{Adj}(t)^{\times}\). This is captured concisely as follows:

\[
(\exists \varphi)(s \varphi = t) \iff (\exists \mu)(\exists \nu) \begin{cases} 
\text{Adj}(s)^{\mu} := \mu^{-1} \text{Adj}(s)\mu = \text{Adj}(t), \\
\text{Adj}(t)^{\nu} = \text{Adj}(t), \text{ and} \\
(s\mu)^{\nu} = t \in \text{Hom}(K^{d_2} \otimes_{\text{Adj}(t)} K^{d_1}, K^{d_0}).
\end{cases}
\]

Note, this method distinguishes \(K^{d_0}\) due to its role as the codomain. However, one could just as easily consider \(s, t\) as tensors in \((K^{d_2} \otimes K^{d_1} \otimes (K^{d_0})^{\ast})^{\ast}\). With this interpretation the compressed tensor space is \((K^{d_2} \otimes_{\text{Adj}(t)} K^{d_1} \otimes (K^{d_0})^{\ast})\), which now seems like an arbitrary choice. In [21], attempting to reconcile the apparent asymmetry, the third author introduced a generalization involving operations between all pairs of \(K^{d_0}\) and \(K^{d_0}\). The philosophy of the new approach presented here is to move away from binary tensor products entirely.

2.2. A broader view. Our new approach appeals to a Galois correspondence described in [9, Theorem A]. Its constituents are subspaces of \(T := (K^{d_2} \otimes \cdots \otimes K^{d_1})^{\ast}\), ideals of \(K[X] := K[x_1, \ldots, x_N]\), and algebraic subsets of \(\Omega := \prod_a \mathbb{M}_{d_a}(K)\), called transverse operators. The key feature is that each operator \(\omega \in \Omega\) endows \(T\) with a \(K[X]\)-module structure. This leads to three natural constructions using pairs of elements (or subsets): \(S\) of \(T, \mathcal{Y}\) of \(\Omega\), and \(P\) of \(K[X]\).

First, to each pair \((S, \mathcal{Y})\) we associate the annihilator ideal \(\mathfrak{A}(S, \mathcal{Y})\), thus generalizing the concept of minimal polynomials.

Secondly, to each pair \((P, \mathcal{Y})\) we associate the subspace \(\mathfrak{R}(P, \mathcal{Y})\) of tensors annihilated by \(\mathcal{Y}\) with annihilator containing \(P\). The sets \(\mathfrak{R}(P, \mathcal{Y})\) generalize tensor products. For example, if \(p : A \rightarrow \mathbb{M}_{d_a}(K)^{op} \times \mathbb{M}_{d_1}(K)\) is an associative algebra representation, then \((K^{d_2} \otimes_A K^{d_1})^{\ast} = \mathfrak{R}(x_2 - x_1, A)\):

\[
(\forall t \in (K^{d_2} \otimes_A K^{d_1})^{\ast})(\forall (\varphi_2, \varphi_1) \in A\rho)(0 = \langle t|\varphi_2 \otimes 1 - 1 \otimes \varphi_1|v_2, v_1 \rangle).
\]

Finally, to each pair \((S, P)\) we associate the algebraic set \(\mathfrak{Z}(S, P)\) of operators acting on \(S\) with annihilator containing \(P\). These sets may come equipped with algebraic structure external to their definition. For example,

\[
\text{Adj}(S) = \mathfrak{Z}(S, x_2 - x_1) \quad \text{Der}(S) = \mathfrak{Z}(S, x_1 + \cdots + x_1)
\]

are, respectively, associative and Lie algebras.
The ternary Galois correspondence relating these three constructions affords
closures, of which the densor space is an example:
\[ \mathfrak{N}(d, \text{Der}(t)) = \mathfrak{N}(d, \mathfrak{Z}(t, d(X))), \quad d = x_1 + \cdots + x_1, \]

Adjoint-tensors \((K^{d_2} \otimes_{\text{Adj}(t)} K^{d_1} \otimes (K^{d_0})^*)^* = \mathfrak{N}(x_2 - x_1, \mathfrak{Z}(t, x_2 - x_1))\) are another. For every closure, we get a tensor compression method in [9, Proposition 7.1]:

\[
\begin{aligned}
(\exists \varphi)(s\varphi = t) \iff (\exists \mu)(\exists \nu) \left\{ \begin{array}{l}
3(s, P)^\mu = 3(t, P), \\
3(t, P)^\nu = 3(t, P), \quad \text{and} \\
(s\mu)\nu = t \in \mathfrak{N}(P, 3(t, P)).
\end{array} \right.
\end{aligned}
\]

2.3. The derivation-densor method. There are many possible ideals one can consider to seed the mechanism in (2.1). To narrow the candidate pool we insist first that \(3(t, P)\) has an algebraic structure like \(\text{Adj}(t)\) and \(\text{Der}(t)\), secondly that the choice of \(P\) is independent of the given tensor \(t\), and thirdly that elements of \(3(t, P)\) can be constructed efficiently.

First, there is a characterization of the ideals \(P\) for which \(3(t, P)\) is a subalgebra of \(\Omega\) under products of the form \((X_a + Y_a)_{a \in [t]} = (\alpha_a X_a + \beta_a Y_a)_{a \in [t]}, \) for some choice of \(\alpha_a, \beta_a \in K\) [9, Theorem D]. We call such \(P\) algebraic ideals. Secondy, all associative algebras arising this way, like \(\text{Adj}(t)\), embed into \(\text{Der}(t)\) [1, Theorem A], and the densor space embeds into all other \(P\)-closures for which \(P\) is an algebraic ideal [9, Theorem C]. Finally, since \(d = \sum a x_a\) is linear, constructing \(\text{Der}(t) = 3(t, d)\) requires only linear algebra. Thus, \(P = (d)\) is the natural choice.

Algorithm 1 gives a high level view of the resulting isomorphism test.

\begin{algorithm}
\textbf{Algorithm 1 Derivation-Densor Method}
\begin{itemize}
  \item \textbf{Input:} tensors \(s, t \in (K^{d_k} \otimes \cdots \otimes K^{d_1})^*\).
  \item \textbf{Output:} \(s, t \in \prod_{a=1}^n \text{GL}_{d_a}(K)\) with \(s\varphi = t\), if such exists.
  \item 1: Compute the derivation algebras \(\text{Der}(s)\) and \(\text{Der}(t)\) of \(s\) and \(t\), respectively.
  \item 2: if \((\exists \mu)(\exists \nu)\) then \(\text{Der}(t)^\mu = \text{Der}(s)\) then
  \item 3: Build the densor space \(\mathcal{U} \cong K^m\)
  \item 4: Build \(N(\text{Der}(t))|_{\mathcal{U}}\) where \(N(\text{Der}(t)) := \{\nu \mid \text{Der}(t)^\nu = \text{Der}(t)\}\).
  \item 5: if \((\exists \mu \in N(\text{Der}(t)))(s\mu)^\nu = t\) then return \(\mu\nu\).
  \item 6: else Report \(s \not\sim t\) as \(s\mu \not\sim t\) in \(\mathcal{U}\).
  \item 7: else Report \(s \not\sim t\) as derivation algebras not conjugate.
\end{itemize}
\end{algorithm}

We perform steps 1 and 3 by solving a system of Sylvester-type equations [3]. Line 4 needs only to exhibit elements that generate the action of \(N(\text{Der}(t))\) on \(\mathcal{U}\) and their pre-images. In Line 5, instead of the default \((K^{d_k} \otimes \cdots \otimes K^{d_1})^*\), we work in \(\mathcal{U}\) whose dimension can be significantly smaller. In fact, in our main theorem \(m = 1\), and this step is settled by linear algebra. This leaves just the Lie subalgebra conjugacy in Line 2, which is difficult in general. Fortunately, for tiny tensors considerable restrictions exist on the algebras we consider (Proposition 5.3).

3. Algebra Conjugacy as Module Similarity

We have seen that a critical step in algebraic tensor compression is to resolve a conjugacy problem for algebras. For adjoint-tensor, conjugacy of associative algebras is needed, while the derivation-densor method requires conjugacy of Lie
algebras. Although conjugacy problems in the two settings have obvious differences, there are important special cases in which essentially the same techniques may be used to attack them. These techniques, moreover, can be applied to conjugacy of other types of algebras (such as Jordan algebras and ∗-algebras) that arise in tensor problems with symmetry.

We formulate algebra conjugacy problems as similarity of modules. Then we define transverse actions on simple modules and demonstrate a polynomial-time tensor decomposition algorithm for these actions (Theorem 3.5). All irreducible representations of associative and Lie algebras are transverse. One consequence of this is a polynomial-time reduction from semisimple-imprimitive actions to simple-primitive ones (Corollary 3.6). Similarity of simple primitive modules is treated in Section 4 for Chevalley Lie algebras (the context of our main theorem).

3.1. Conversion to module similarity. For $A$ belonging to a class $\mathfrak{X}$ of $K$-algebras, a right $A$-module is a $K$-vector space with a $K$-linear map $\rho: A \to \text{End}(U)$ such that $A \rtimes_{\rho} U := A \times U$, equipped with product $(a, u) * (b, v) = (ab, u(b') + v)$, is again an algebra in $\mathfrak{X}$. For convenience, write $va$ for $v(a^\rho)$. For associative algebras this simply means that $\rho$ is a ring homomorphism; for Lie algebras, $\rho$ is a homomorphism into $\mathfrak{gl}(U) := \text{End}(U)$ with product $[\varphi, \tau] = \varphi\tau - \tau\varphi$.

For $A, B \in \mathfrak{X}$, a semilinear map from an $A$-module $U$ to $B$-module $V$ is a pair $(\varepsilon: A \to B, \varphi: U \to V)$, where $\varepsilon$ is an algebra homomorphism and

$$ (\forall a \in A)(\forall u \in U)((ua)\varphi = u\varphi a^\varepsilon). $$

An invertible semilinear map is a similarity, implying that for $A, B \subseteq \text{End}(U)$, condition (3.1) translates as $\varphi^{-1} A\varphi = A^\varepsilon = B$. This captures the conjugacy problems arising in adjoint-tensor and derivation-densor as module similarity problems.

3.2. The associative cyclic case. Let $\mathfrak{C}$ be the class of cyclic associative algebras, namely those algebras generated by a single element. This is a case for which a solution to the module similarity problem is known. Algorithms are given in [6] first to find a single generator of a given algebra in $\mathfrak{C}$, and secondly to solve the similarity problem for such algebras over finite fields. The latter uses algorithms that have since been generalized to number fields [15, pp. 211–212], so we state the result here in this greater generality.

**Theorem 3.2** ([6, Theorem 1.3]). Fix a field $K$ that is finite or finite over $\mathbb{Q}$. There is a polynomial-time algorithm to decide if an associative algebra $A \subseteq \text{End}_K(V)$ is cyclic and another to settle similarity for such algebras.

For an algebraic ideal $P \subseteq K[X]$ and a tensor $t$, $A_t := \{t, P\}$ is an algebra. The **associative enveloping algebra** $K\langle A_t \rangle$ is the (possibly non-unital) subalgebra of $\prod_{a \in [t]} \text{End}(V_a)$ generated by $A_t$. We will need the next consequence of Theorem 3.2.

**Corollary 3.3.** Let $P$ be an algebraic ideal. Given $s, t \in (K^{d_1} \otimes \cdots \otimes K^{d_k})^*$, and characteristic ideals $C_s = K\langle C_s \rangle$ of $A_s$, and $C_t = K\langle C_t \rangle$ of $A_t$, in polynomial time one can construct $\varphi \in \prod_{a \in [t]} \text{GL}(V_a)$ such that $C_s^\varphi = C_t$.

3.3. Tensor decomposition of simple modules. We return to the general setting of Section 3.1, where $A$ is an algebra in the class $\mathfrak{X}$. Let $U$ be a faithful simple $A$-module. In the associative algebra case, this implies that $A$ is a simple algebra. In general, however, $A$ may possess proper nontrivial ideals, so we consider the case where $A = M_1 \oplus \cdots \oplus M_r$ is a product of proper nontrivial minimal ideals of $A$. 

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Under mild constraints, we characterize these modules as iterated tensor products using a blend of Clifford theory and Morita condensation.

The existence of such decompositions is known for Lie algebras of Chevalley type over $\mathbb{C}$, but our treatment extends to the modular case. Furthermore, our approach is constructive: we present a polynomial-time algorithm to construct a tensor decomposition of an $M_1 \oplus \cdots \oplus M_r$-module.

For an $A$-module $U$, a decomposition $A = M \oplus N$ into ideals is transverse if

$$\forall m \in M)(\forall u \in N)(\forall v \in U)((um)n = (un)m).$$

We also say that $A$ acts transversely on $U$ when such a decomposition exists. Condition (3.4) always holds when $A$ is associative since $MN = 0 = NM$. For Lie algebras, every $A$-module has this property by the Jacobi identity.

For $S \subset U$, by definition $SA := SK \langle A \rangle$ is the smallest $A$-submodule of $U$ containing $S$. We impose one further constraint: we insist that for each minimal ideal $M$ of $A$, $K \langle M \rangle$ is central simple. Evidently $K \langle M \rangle \cong \mathcal{M}_{f \times f}(\Delta)$ for some integer $f$, so the additional constraint demands that $\Delta$ is a field. This allows us to exploit the powerful recognition algorithms of [15].

**Theorem 3.5.** There is a polynomial-time algorithm that, given a simple $A$-module $U$ and a transverse decomposition $A = M \oplus N$ with $M$ minimal and $K \langle M \rangle$ central simple, returns an $M$-submodule $S \subseteq U$, an $N$-submodule $T \subseteq U$, and an isomorphism $U \to T \otimes_E S$ where $E = \text{End}_M(S)$.

**Proof.** As $K$-vector spaces, $U = SA = SM + SN \subseteq S + SN$. By the transverse property, $(SM)N = (SN)M \leq SN$, so $SN$ is an $M$-module. As $S$ is simple, $S \cap SN$ is either $S$ or $0$. In the former case, $U = S$, and we use $T := E = \text{End}_M(S)$, which is a field by hypothesis. Otherwise, $S \cap SN = 0$, $U = S \oplus SN$ as $M$-modules, and there exists an $n \in N$ such that $nS \neq 0$. By the transverse property, this defines an $M$-module embedding $S \to SN$ mapping $s \mapsto sn$. By induction on dimension, there exists $r > 1$ such that $SN \cong S^{\otimes (r-1)}$; so, $U \cong S^r = E^r \otimes_E S$. Using the recognition algorithm of [15, Theorem 1], locate a primitive idempotent $e$ of $B := K \langle M \rangle$. Notice $eBe = E$, $S$ is a left $E^{op}$-module, and $Se \cong E$. Set $T := Ue = E^r \otimes_E Se \cong E^r \otimes_E E$. By transversality, $T$ is an $N$-module. Finally, $T \otimes_E S = E^r \otimes_E Se \otimes_E S \cong E^r \otimes_E S \cong U$. \hfill \Box

3.4. **Reduction to the primitive case.** We use Theorem 3.5 within a semilinear isomorphism algorithm for modules to reduce from the *imprimitive case*, where $r > 1$ and $A = M_1 \oplus \cdots \oplus M_r$ acts on a simple module, to the *primitive* case, where $A$ is a simple algebra. For convenience, we record the result we need.

**Corollary 3.6.** If $A = M_1 \oplus \cdots \oplus M_r$ is a decomposition into nontrivial minimal ideals each with $K \langle M_i \rangle$ central simple acting transversely on a simple $A$-module $U$, then for each $i \in [r]$, there is a simple $M_i$-submodule $S_i \subseteq U$ such that $U \cong S_1 \oplus \cdots \oplus S_r$. Each of these submodules can be constructed in polynomial time.

4. **Similarity of simple modules of simple Chevalley Lie algebras**

The goal of this section is to prove the following:

**Theorem 4.1.** Let $K$ be a field with $K = 6K$ finite or $K/\mathbb{Q}$ finite. There is a polynomial-time algorithm that, given two primitive simple Lie modules of Chevalley type over $K$, decides if they are similar.
Throughout this section $L$ is assumed to be a simple Chevalley Lie $K$-algebra and $\rho: L \to \mathfrak{gl}(V)$ an irreducible representation. Our notation follows Humphreys [13]. Fix a root system $\Phi$ for $L$, and let $\Delta \subseteq \Phi$ be a base. For $\alpha \in \Phi$, let $x_\alpha$ and $h_\alpha$ be elements of $L$ satisfying $[x_\alpha, x_{-\alpha}] = h_\alpha$. If $\alpha, \beta, \alpha + \beta \in \Phi$ and there exists $c_{\alpha\beta} \in \mathbb{Z}$ with $[x_\alpha, x_\beta] = c_{\alpha\beta} x_{\alpha+\beta}$, then the union of $\mathcal{X} := \{x_\alpha \mid \alpha \in \Phi\}$ and $\mathcal{H} := \{h_\delta \mid \delta \in \Delta\}$ is a Chevalley basis of $L$.

4.1. Crystal graphs. Our preferred data structure for deciding similarity of representations $\rho: L \to \mathfrak{gl}(V)$ is a graph whose vertices form a basis for $V$ compatible with the weight space decomposition. These graphs are inspired by work of M. Kashiwara, who established their existence and uniqueness; cf. [12, Section 4.2].

**Definition 4.2.** For $\alpha \in \Phi$ set $X_\alpha = x_\alpha^0$, and for $\delta \in \Delta$, set $H_\delta = h_\delta^0$. A crystal graph for the representation $\rho: L \to \mathfrak{gl}(V)$ is an edge-labeled digraph $\Gamma(\rho, \Delta)$ with edge labels $\Delta$, whose vertex set is a basis, $\mathcal{V}$, for $V$ with the following property: for all $u \in \mathcal{V}$ and for all $\delta \in \Delta$ we have $H_\delta(u) \in Ku$ and

$$u \overset{\delta}{\longrightarrow} v \iff 0 \neq X_\delta(u) \in Ku \text{ and } 0 \neq X_{-\delta}(v) \in Ku.$$  \hfill (4.3)

Crystal graphs exist [12, Theorem 5.1.2] and are unique (up to labeled graph isomorphisms) [12, Theorem 5.2.1]. The following result says that they can be built efficiently given a Chevalley basis for the algebra.

**Algorithm 2** (Crystal Graph)

**Input:** a Chevalley basis $\mathcal{X} = \{x_\alpha \mid \alpha \in \Phi\}$, $\mathcal{H} = \{h_\delta \mid \delta \in \Delta\}$ and $\rho: L \to \mathfrak{gl}(V)$.

**Output:** a crystal graph $\Gamma(\rho, \Delta)$ for $\rho$.

1. $Ku^+ \leftarrow \bigcap_{\delta \in \Delta} \ker X_{-\delta}$
2. Initialize $\mathcal{V} \leftarrow \{u^+\}$ and $\mathcal{E} \leftarrow \emptyset$
3. while $|\mathcal{V}| < \dim V$
4. Find $u \in \mathcal{V}$, $\delta \in \Delta$ such that $u' = X_\delta(u) \notin \langle \mathcal{V} \rangle$ and $0 \neq X_{-\delta}(u') \in Ku$
5. $\mathcal{V} \leftarrow \mathcal{V} \cup \{u'\}$
6. for $v \in \mathcal{V}$ with $X_\gamma(v) = u'$ and $0 \neq X_{-\gamma}(u') \in Ku$
7. $\mathcal{E} \leftarrow \mathcal{E} \cup \{v \overset{\gamma}{\longrightarrow} u'\}$
8. Return the edge-labelled graph $\Gamma = (\mathcal{V}, \mathcal{E})$.

**Lemma 4.4.** Given a Chevalley basis $\mathcal{X} = \{x_\alpha \mid \alpha \in \Phi\}$, $\mathcal{H} = \{h_\delta \mid \delta \in \Delta\}$ of $L$ and $\rho: L \to \mathfrak{gl}(V)$, in polynomial time one can construct a crystal graph $\Gamma(\rho, \Delta)$.

**Proof.** We use the procedure described in Algorithm 2. In line 1, we locate a cyclic generator $v^+$ that is annihilated by $\{x_\delta \mid \delta \in \Delta\}$, which by definition makes $v^+$ a maximal vector. Next, “$\mathcal{V}$ is linearly independent” is an invariant of the loop starting on line 3, so the algorithm exits with a basis for $V$. Each iteration extends $\mathcal{V}$ (line 5) by a linearly independent vector $u'$, and line 6 adds in the edges from the vertices in $\mathcal{V}$ reaching $u'$. Hence, the algorithm exits with a labeled digraph $\Gamma$ in $(\dim V)^2|\Delta|$ rounds of linear algebra in $V$. Line 4 guarantees condition (4.3); that a solution exists is the content of Kashiwara’s theorem [12, Theorem 5.1.2]. □

**Example 4.5** (Crystal bases for irreducible representations of type $A_2$). Let $L$ be the Lie algebra of type $A_2$, and let $\Delta = \{\alpha, \beta\}$ be a base for the root system of $L$. In the crystal graphs in Figure 4.1, arrows with a single barb are labelled by $\alpha$,
while those with double barbs are labelled by $\beta$. We invite the reader to construct these crystal graphs for themselves using Algorithm 2.

(A) The natural representation. Identify $L$ with its natural representation $\mathfrak{sl}_3(K)$ as $(3 \times 3)$-matrices of trace 0 with entries in a field $K$. Denote by $E_{ij}$ the matrix with a $1$ in $(i, j)$ entry and $0$ elsewhere. Put $x_\alpha := E_{12} = X_\alpha$ and $x_\beta := E_{23} = X_\beta$. Then $v^+ = (1, 0, 0)$ is a maximal vector of $V = K^3$ as it is the intersection of the nullspaces of $X_\alpha^\dagger = E_{21}$ and $X_\beta^\dagger = E_{32}$.

(B) The symmetric square representation. Let $V$ be the $K$-space of symmetric $3 \times 3$ matrices, and $\rho: \mathfrak{sl}_3(K) \to \mathfrak{gl}(V)$, where $M y^\rho = y M + y^\dagger$, so $v^+ = E_{33}$ is a maximal vector.

(C) The adjoint representation. Let $\text{ad}: L \to \mathfrak{gl}(L)$ be the adjoint representation, where $\text{ad} x(y) = [y, x]$. Setting $\gamma = \alpha + \beta$, a maximal vector is $v^+ = x_{-\gamma}$.

Remark 4.6. Consider the act of interchanging the single and double arrows in each of the graphs. Observe that only in the last graph do we get an isomorphic crystal graph. Also, when there are multiple paths from one node to another, the multiset of edge labels one accrues in traversing the different paths is independent of choice.

4.2. Lifting automorphisms. Fix representations $\rho_i: L_i \to \mathfrak{gl}(V_i)$. Without loss of generality, $V = V_1 = V_2$ as vector spaces. We further assume Chevalley bases for $L_1$ and $L_2$, and hence that we have fixed a Lie algebra isomorphism $\varepsilon: L_1 \to L_2$. We must construct a compatible $\varphi$ or prove that no such maps exist.

The material problem is that while $V_1$ may be similar to $V_2$ for some semilinear map $(\varepsilon': L_1 \to L_2, \varphi': V_1 \to V_2)$, we cannot in general assume that there is a semilinear isomorphism $(\varepsilon, \varphi: V_1 \to V_2)$, for the given $\varepsilon$. The work is to adjust $\varepsilon$ to an appropriate $\varepsilon' := \varepsilon \tau$, $\tau \in \text{Aut}(L_1)$. We say $\tau \in \text{Aut}(L_1)$ lifts to $V_1$ if there exists $\varphi$ such that $(\tau, \varphi: V_1 \to V_1)$ is a semilinear automorphism. We work through Steinberg’s classification of Lie algebra automorphisms (inner automorphisms, diagonal automorphisms, and graph automorphisms) to find those that lift. The cosets of $\tau$’s that do not lift are those we use to adjust our given $\varepsilon$. One could also consider Galois automorphisms, but we have limited our problem to $K$-linear similarity.
4.2.1. Inner automorphisms. Our first task is to prove the inner automorphisms of a Lie algebra $L$ lifts to each irreducible representation $\rho : L \to \mathfrak{gl}(V)$. By definition, the inner automorphism group $\text{Inn}(L)$ of $L$ is the subgroup of $\text{GL}_K(L)$ generated by the exponentials of the root subspaces $x_\alpha(t)$. So, in characteristic 0 one simply takes $I := \langle \exp(x_\alpha(t)^{\rho}) \mid t \in K, \alpha \in \Phi \rangle \triangleleft \text{GL}(V)$. In positive characteristic not dividing 6, the Chevalley basis can still be exponentiated to act on $L$ as the group $\text{Inn}(L)$, but some images $x_\alpha(t)^{\rho}$ may now have ad-nilpotence degree larger than the characteristic, thus making $\exp(x_\alpha(t)^{\rho})$ undefined. Therefore, to prove our main theorem for $K = 6K$, we appeal instead to the observation that $\text{Inn}(L)$ is generated by involutions, which can be lifted to act on $V$.

**Proposition 4.7.** There is a polynomial-time algorithm that, given a representation $\rho : L \to \mathfrak{gl}(V)$, where $V$ is a vector space over $K = 6K$, constructs a subgroup $\text{Inn}_\rho(L) \triangleleft \text{GL}(V)$ containing the image of the action by $\text{Inn}(L)$ of automorphisms.

**Proof.** Observe that $\text{Inn}(L)$ is the connected reduced Chevalley group of the given type. Hence, given our restrictions on $K$, $\text{Inn}(L)$ is a quasisimple group. In the rank 1 cases, the group is generated by 2 involutions by direct inspection, and in the larger rank cases, we combine these with the involutions of the Dynkin diagram.

Let $X$ be a fixed set of involutions generating $\text{Inn}(L)$. For each $\sigma \in X$, either our initial choice of split Cartan subalgebra $H$ already contains $\sigma$ (as is the case for the involutions involved in our Dynkin diagram), or else it lies in some other Cartan subalgebra $H'$. In the latter case, using a new Chevalley basis relative to $H'$, construct its corresponding crystal graph $\Gamma'$ using Lemma 4.4. Since the graphs $\Gamma$ and $\Gamma'$ are isomorphic, one obtains an isomorphism $\varphi : V \to V$. As $K = 2K$, $\sigma$ is diagonalizable; let $M_\sigma$ be the matrix it induces on the vertices of $\Gamma'$.

To see that $\text{Inn}_\rho(L) := \langle M_\sigma^{\varphi} \mid \sigma \in X \rangle$ contains the induced image of $\text{Inn}(L)$, notice that $\text{Inn}(L)$ is transitive on Cartan subalgebras, and every element of $\text{Inn}(L)$ is a product of the given involutions. The action of each Cartan algebra uniquely defines a crystal graph. Thus, we simply transfer the action of $\text{Inn}(L)$ on the Cartan subalgebras to the crystal graphs of $V$. As $\text{Inn}(L)$ is simple, $\text{Inn}(L) \hookrightarrow \text{Inn}_\rho(L)$. □

4.2.2. Diagonal automorphisms. Each diagonal automorphism scales the Chevalley basis elements $x_\alpha$ by some $\xi_\alpha \in K$. There are two types of diagonal automorphisms: we refer to them as inner and outer. For example,

$$
(4.8) \quad \begin{bmatrix}
\xi & . & . \\
. & 1/\xi & . \\
. & . & 1
\end{bmatrix}, \quad \begin{bmatrix}
\xi & . & . \\
. & 1 & . \\
. & . & 1
\end{bmatrix}
$$

both induce (by conjugation) diagonal automorphisms of $\mathfrak{sl}_2(K)$: the first being inner and the second being outer. Lifts of the inner diagonal automorphisms exist already within the group of inner automorphisms.

**Remark 4.9.** This case is only relevant to non-algebraically closed fields. For example $\text{PGL}_n(K) \supsetneq \text{PSL}_n(K) = \text{Inn}(\mathfrak{sl}_n(k))$ is a strict inequality in general, but for algebraically closed fields, $K$ it becomes an equality. This explains why works like [11], that focus on $\mathbb{C}$, omit the diagonal case.

We aim to show that outer diagonal automorphisms always lift from $\text{Aut}(L)$ to $\text{Aut}_L(V)$. To that end, it suffices to show that (an appropriate analogue of) the distinguished automorphism on the right in (4.8) lifts.
Lemma 4.10. For distinct \( \alpha, \beta \in \Delta \), there exists \( \tau \in \text{Inn}(L) \) acting as \( \xi \in K^\times \) on \( x_\alpha \), as \( 1/\xi \) on \( x_\beta \), and fixing the remaining \( x_\delta \) for \( \delta \in \Delta - \{ \alpha, \beta \} \).

Let \( u, v \) be vertices of a crystal graph \( \Gamma = \Gamma(\rho, \Delta) \). For each path \( P \) from \( u \) to \( v \), let \( w_P \) denote the edge word with (formal) letters from \( \Delta \) satisfying

1. \( u = v \) implies \( w_P \) is the empty word, and
2. if \( u \xrightarrow{\alpha} u' \) in \( P \), and \( P' \) is the remaining path from \( u' \) to \( v \), then \( w_P = \delta \cdot w_{P'} \).

In Remark 4.6, we alluded to a general phenomenon regarding the multiset of edge labels in a crystal graph. We now prove this fact, establishing the key condition needed to lift diagonal automorphisms.

Lemma 4.11. Let \( \rho: L \rightarrow \mathfrak{gl}(V) \) be a representation, \( \Delta \) a base for \( L \), and \( \Gamma = \Gamma(\rho, \Delta) \) the associated crystal graph. Then distinct paths \( P \) and \( P' \) in \( \Gamma \) from nodes \( u \) to \( v \) have the same multisets of symbols in their edge words.

Proof. If this holds for \( u = v^+ \), some maximal vector of \( L \), then the general case follows. Let \( X \) and \( X' \) be the elements of \( \text{End}(V) \) obtained by evaluating \( w = w_P \) and \( w' = w_{P'} \) at \( \{ X_\delta \mid \delta \in \Delta \} \). Then for some \( \xi, \xi' \in K^\times \), \( v = \xi(v^+X) = \xi'(v^+X') \).

Applying \( g \in \text{Inn}_\rho(L) \) to \( v^+X \) and \( v^+X' \), we see that

\[ \xi(v^+g)X^g = vg = \xi'(v^+g)X'^g. \]

If \( g \) is the lift of an inner diagonal automorphism, then for some \( \gamma, \gamma' \in K^\times \), \( X^g = \gamma X \) and \( X'^g = \gamma' X' \). Furthermore, \( g \) stabilizes the nullspaces of every \( X_{-\alpha} \) and hence fixes \( (v^+) \). Now for some \( \lambda \in K^\times \), \( v^+g = \lambda v^+ \). Hence,

\[ \gamma v = \frac{\lambda \gamma \xi}{\lambda} v^+ X = \frac{\xi}{\lambda} v^+ g X^g = \frac{\xi'}{\lambda} v^+ g X'^g = \frac{\xi' \gamma'}{\lambda} v^+ X' = \gamma' v \]

It follows that \( \gamma = \gamma' \). By Lemma 4.10, the multisets of symbols in \( w \) and \( w' \) must coincide in order for this to hold for all diagonal inner automorphisms. \qed

Proposition 4.12. Every diagonal automorphism of \( L \) lifts to \( V \).

Proof. A diagonal automorphism of \( L \) induces a scalar on each \( x_\delta \), so the putative lift of the automorphism induces the same scalar on \( X_\delta \). The scalar induced by that lift on a vector \( v \) in our basis \( V \) is computed as the product of the scalars induced on the constituent symbols of the edge word from \( v^+ \) to \( v \). Lemma 4.11 ensures that the answer we get is independent of the path to \( v \) in the crystal graph. Repeating for each \( v \in V \), we obtain the lift of the diagonal automorphism. \qed

4.2.3. Graph automorphisms. An automorphism \( \varphi_\sigma \) of \( L \) inducing a graph automorphism of the Dynkin diagram induces a permutation \( \sigma \) of \( \Delta \). Thus, an edge labeled by \( \alpha \) in the crystal graph is mapped to an edge labeled by \( \sigma(\alpha) \), preserving orientation. As every path in the crystal graph can be identified with a \( \mathbb{Z} \)-combination of simple roots, it follows that \( \sigma \) induces an automorphism of the crystal graph as a labeled digraph; this condition is also sufficient.

Proposition 4.13. Let \( \rho: L \rightarrow \mathfrak{gl}(V) \) be an irreducible representation of a simple Chevalley Lie algebra \( L \) with fixed base \( \Delta \), and let \( \Gamma = \Gamma(\rho, \Delta) \) be its associated crystal graph. Then a graph automorphism \( \varphi_\sigma \) of \( L \) lifts to \( \rho \) if, and only if, \( \Gamma = \Gamma'' \).
4.3. **Proof of Theorem 4.1.** Let \( \rho_i : L_i \rightarrow \mathfrak{gl}(V_i) \) be the given irreducible representations of the simple Lie algebras \( L_i \) of Chevalley type.

First, construct a Chevalley basis for each \( L_i \) using a polynomial-time algorithm \([7, 18, 20]\). This affords root data that induces an isomorphism \( \varepsilon : L_1 \rightarrow L_2 \). Otherwise, we report that \( L_1 \not\cong L_2 \).

Next, we construct the crystal graphs \( \Gamma_i \) in polynomial time using Lemma 4.4. We loop over each graph automorphism \( \sigma \) of \( L_1 \) and decide if \( \Gamma^\sigma_1 \cong \Gamma_2 \). If so then the graph isomorphism identifies the vertices which label the bases of \( V_1 \) and \( V_2 \) producing an invertible linear map \( \phi : V_1 \rightarrow V_2 \), so we return \( (\varepsilon, \phi) \).

From Propositions 4.7 and 4.12, the crystal graphs are not affected by applying either an inner or diagonal automorphism of \( L_1 \). Thus, the algorithm exhausts all of the—at most 6—graph automorphisms. Hence, the algorithm correctly decides similarity in polynomial time.

5. **Testing isomorphism of tiny densors**

Equipped with similarity tests in the foregoing sections, we turn now to the isomorphism problem for tiny densors. Define the *Lie tensor space* of a Lie representation \( \rho : L \rightarrow \prod_a \mathfrak{gl}(V_a) \) as:

\[
\otimes V_1, \ldots, V_1 \otimes_L = \mathfrak{gl}(x_1 + \cdots + x_1, L^a)
\]

(5.1)

Densor spaces are the special case \( \otimes \) := \( \otimes V_1, \ldots, V_1 \otimes_{\text{Der}(t)} \).

**5.1. Properties of tiny densors.** In order to use the results of Section 3 and 4, we must show that the necessary conditions are satisfied, namely that we end up with transverse modules for Chevalley Lie algebras. This is a consequence of Elton’s Lemma, which follows from the following fact.

**Lemma 5.2.** Let \( L \) be a Lie subalgebra of \( \prod_a \mathfrak{gl}(V_a) \) with \( \dim \otimes V_1, \ldots, V_1 \otimes_L > 0 \). If \( U_a \) is a proper nontrivial \( L \)-submodule of \( V_a \), then there is a nonzero \( t \in \otimes V_1, \ldots, V_1 \otimes_L \) such that \( (t|U_a, V_a) = 0 \).

**Proof.** Let \( e : V_a \rightarrow V_a \) be an idempotent with kernel \( U_a \) and \( (t|v) := (t|ev_a, v_a) \).

Then for each \( \delta \in L, (\delta\bar{e}\otimes \delta V_a) \in \text{Der}(t) \) so that \( (t|U_a, V_a) = 0 \).

**Proposition 5.3** (Elton’s Lemma). If \( t \in (V_1 \otimes \cdots \otimes V_1)^* \) with \( \dim \otimes t = 1 \), then Der(\( t \)) is reductive and every \( V_a \) is a simple Der(\( t \))-module.

**Proof.** Since \( \dim \otimes t \neq 0 \), it follows that \( t \neq 0 \). If \( n \) is a proper nilpotent ideal of Der(\( t \)), then \( nV_a \) is a proper nontrivial Der(\( t \))-submodule for some \( a \in [1] \). By Lemma 5.2, there exists \( 0 \neq s \in \otimes t \) with \( (s|nV_a, V_a) = 0 \). As \( \dim \otimes t = 1 \), \( t \) must be degenerate, which we assumed it was not. Hence, no such nilpotent ideal \( n \) exists; so \( L \) is reductive. The rest follows from Lemma 5.2.

The conclusion that Der(\( t \)) is reductive for a tiny densor cannot be strengthened. Consider the matrix multiplication tensor, \( (t) : \mathbb{M}_{a \times b}(K) \times \mathbb{M}_{b \times c}(K) \rightarrow \mathbb{M}_{a \times c}(K) \).

Here, Der(\( t \)) \( \cong (\mathfrak{gl}_a(K) \oplus \mathfrak{gl}_b(K) \oplus \mathfrak{gl}_c(K))/K \) and \( \dim \otimes t = 1 \) \([21, \text{Theorem} 7.1]\). In general, one expects that Der(\( t \)) will contain many proper ideals. For example, each kernel of the (exponential number of) homomorphisms described in the exact sequences of \([1]\) yield proper ideals of Der(\( t \)). However, modular Lie theory can become quite involved. Modular derivations are \( p \)-restricted, which helps, but we further limit our attention to tensors \( t \) where Der(\( t \)) is of Chevalley type. Note that if \( t \) is degenerate, then Der(\( t \)) has a proper nontrivial nil radical, implying
that Der(t) would not be reductive and, hence, not of Chevalley type. So implicit within our statement of the main theorem is that the tensors are nondegenerate.

5.2. Proof of Main Theorem. We apply the derivation-densor procedure described in Algorithm 1 to nondegenerate tensors \( t_1, t_2 \in (V_1 \otimes \cdots \otimes V_l)^* \) having 1-dimensional densor spaces. The proof references specific lines in Algorithm 1.

First, the derivation algebras \( L_i := \text{Der}(t_i) \) (Line 1) are constructed in polynomial time using [9, Theorem F]. By Elton’s Lemma, each \( L_i \) is reductive which allows us to decompose \( L_i = M_{i1} + \cdots + M_{ir} \), into nontrivial minimal ideals (see [15, Theorem 1] and the more general finite field case discussed in [15, pp. 211–212]). If \( r \neq r' \), then Der\((t_1)\) is not conjugate to Der\((t_2)\).

Elton’s lemma also implies that for each \( a \in [\ell] \), \( L_i \) restricted to \( V_a \) is irreducible. Use Corollary 3.6 to tensor-decompose \( V_a = S_{i1} \otimes \cdots \otimes S_{im_a} \). For each minimal ideal \( M_j \), locate the unique nontrivial module \( S_{1k} \). If none exist, then \( M_{1j} \) is in the kernel, so set \( \varphi_j = 1_{V_a} \). Otherwise, use Theorems 3.2 and 4.1 to construct a similarity \((\epsilon_j, \tilde{\varphi}_j)\) from \( (M_{1j}, S_{1k}) \) to some \((M_{2j}, S_{2k});\) reporting instead that \( t_1 \not\sim t_2 \) if there is no such matching. Let \( \varphi_j := \tilde{\varphi}_j \otimes 1 \in \text{GL}(V_a) \), where 1 is the identity on all other \( S_{1\ell} \), for \( \ell \neq k \). Set \( \varphi_a := \prod_j \varphi_j \in \text{GL}(V_a) \). It follows that \((L_1|_{V_a})^\varphi = L_2|_{V_a}\). The action of \( L_i \) on \((V_1 \otimes \cdots \otimes V_l)\) is transverse so setting \( \varphi := \varphi_1 \otimes \cdots \otimes \varphi_1 \) gives \( L_1^\varphi = L_2 \in \text{End}((V_1 \otimes \cdots \otimes V_l)^*) \). This gives Line 2.

Since \( \dim(\langle t_i \rangle) = 1 \), we do not need to induce images of normalizers. Therefore, we proceed to line 4, where the task is merely to decide if \( t_1 = \lambda t_2 \) for some scalar \( \lambda \). This is settled by solving a tiny linear equation. The result follows.

5.3. Further results. There are a number of similar results attainable by modest adaptation of our methods. The proof of our main theorem also proves:

**Theorem 5.4.** For fields \( K = 6K \) that are finite or finite over \( \mathbb{Q} \), similarity of simple modules of reductive Lie algebras of Chevalley type can be settled in polynomial time if the abelian factor is cyclic.

We are careful to avoid constructing \( N(\text{Der}(t)) \) in general because over infinite fields \( K^x \) may not have a finite generating set. For finite fields, however, we can give generators for \( K^x \) and, consequently, also for \( N(\text{Der}(t)) \).

**Theorem 5.5.** For a finite field \( K \) with \( K = 6K \), if \( t \in (K^d \otimes \cdots \otimes K^d)^* \) satisfies the hypotheses of our main theorem, then generators for the group \( \text{Aut}(t) \) can be constructed in polynomial time.

It can happen that the Lie algebras \( \text{Der}(s) \) are reductive and irreducible on each \( V_a \) without having a tiny densor. In that case we are left to search the orbit of the outer part of the normalizer of \( \text{Der}(s) \) acting on \( \langle s \rangle \). This search is often much better than the alternative.

When the derivation algebras are represented reducibly on the axes, one is confronted with familiar difficulties when matching simple factors. Grochow has shown that a general solution to the conjugacy problem for semisimple Lie algebras over any field requires solving Graph Isomorphism [11]. Nevertheless, this difficulty is not so pronounced with a bounded number of simple modules.

The situation when the derivation algebras have nontrivial nil radicals is worse. Indeed, this is a problem even for associative algebras [5]. Although the presence of a flag suggests that an inductive process may succeed, all actions must also normalize the radical. This removes the transitivity properties we have used.
6. Application to algebra isomorphism

We conclude with a natural construction of an infinite family of nilpotent $K$-algebras whose distributive product is a tiny denser. These algebras come from the classical representation theory of $\mathfrak{sl}_n$-modules, so it is likely that similar ideas can be used to build more such families.

Throughout this section, $K$ will denote a field that is either finite or finite over $\mathbb{Q}$. Let $n$ be a positive integer such that if char$(K) = p > 0$ then $p \nmid (n + 1)$. If $m$ is another positive integer, $d(m, n)$ is the number of divisors of $m$ no larger than $n$.

**Theorem 6.1.** For any $K$ there are infinitely many positive integers $n$ such that, for all positive integers $m$, there are at least $d(m, n)$ pairwise non-isomorphic $K$-algebras whose distributive product is a tiny denser.

**Remark 6.2.** Evidently, derivation-denser decides isomorphism for this family in polynomial time, but no existing methods are sub-exponential. For instance, a consequence of our construction is that Adj$(t) \cong K$, so the adjoint-tensor method is no better than brute force.

We fix some more notation. Let $L = \mathfrak{asl}_{n+1}(K)$, the simple Lie algebra of type $A_n$, and $M$ a finite-dimensional simple $L$-module. The Lie module operation is a $K$-bilinear map, $(t) : M \times L \to M$. We begin with the most fundamental step, the proof of which follows from the definition of a Lie module.

**Lemma 6.3.** Der$(t)$ contains a simple subalgebra isomorphic to $\mathfrak{sl}_{n+1}$.

The simple $L$-module $M$ contains a unique vector of highest weight $\lambda$. We write $M = V(\lambda)$ if $M$ is an $L$-module with highest weight $\lambda$, where $\lambda$ is a partition with $n$ parts, possibly equal to 0. Write $\lambda = (\ell_1, \ldots, \ell_n)$ if $\sum \ell_i = m$. We need to determine the number of irreducible submodules of $V(\lambda) \otimes V(\mu)$ isomorphic to $V(\nu)$, which are the Littlewood–Richardson numbers for type $A$, denoted by $c_{\lambda,\mu}^{\nu}$. These numbers can be computed by algorithms on Young tableaux, similar to the well-known $\mathfrak{sl}_n$ case. We follow closely the notation used in [12].

We denote by $Y$ a Young diagram of type $\lambda \vdash m$. Let $\mathcal{B}(Y)$ be the set of semi-standard Young tableaux obtained by filling in the boxes of the diagram $Y$ with integers $\{n + 1\}$ such that each row is weakly increasing and each column is strictly increasing. A tableau is standard if the integers 1 through $m$ appear once.

For a Young diagram $Y$ of type $\lambda = (\ell_1, \ldots, \ell_n)$, define a new Young diagram

\begin{equation}
Y[j] = \begin{cases} (\ell_1, \ldots, \ell_j + 1, \ldots, \ell_n) & j \leq n, \\
(\ell_1 - 1, \ldots, \ell_n - 1) & j = n + 1.
\end{cases}
\end{equation}

For $m \geq 2$, we define $Y[b_1, \ldots, b_m]$ recursively so that

$$Y[b_1, \ldots, b_m] = Y[b_1, \ldots, b_{m-1}][b_m],$$

provided $Y[b_1, \ldots, b_i]$ is a Young diagram for all $i \in [m - 1]$. The next theorem states how this operation can be used to determine $c_{\lambda,\mu}^{\nu}$.

For a Young diagram of type $\lambda$ with $n$ parts, we identify $\mathcal{B}(Y)$ with its $\mathfrak{sl}_{n+1}$-crystal graph. So we write $\mathcal{B}(Y) \oplus \mathcal{B}(Y')$ to be the (disjoint) union of the crystal graphs. The tensor product is given by the tensor product rule [12, Chapter 4].

**Theorem 6.5 ([12, Theorem 8.6.6]).** Let $\lambda \vdash \ell$ and $\mu \vdash m$ be partitions with $n$ parts, and let $Y$ and $Y'$ be the corresponding Young diagrams. Then there exists
an $\mathfrak{sl}_n$-crystal isomorphism
\[ B(Y) \otimes B(Y') \cong \bigoplus_{b_1 \otimes \cdots \otimes b_m \in B(Y')} B(Y[b_1, \ldots, b_m]). \]

Note, if $Y[b_1, \ldots, b_m]$ is not a Young diagram, then $B(Y[b_1, \ldots, b_m]) = 0$.

Proposition 6.6. With $n \geq 1$, set $\mu = (2, 1, \ldots, 1) \vdash n + 1$. If $\lambda = (\ell_1, \ldots, \ell_n)$ is a partition, then $c^\lambda_{\lambda, \mu} = |\{\ell_i \mid 1 \leq i \leq n, \ell_i > 0\}|$.

Proof. Write $\lambda = (n_1, \ldots, n_1, n_2, \ldots, n_2, \ldots, n_k, \ldots, n_k)$ for some $k \leq n$, where $n_i > n_{i+1}$ for $i \in [k-1]$. Let $Y$ and $Y'$ be the Young diagrams corresponding to $\lambda$ and $\mu$ respectively. We count the number of summands equal to $B(Y)$ in $B(Y) \otimes B(Y')$. From Theorem 6.5 these correspond to tableaux $T := b_1 \otimes \cdots \otimes b_{n+1} \in B(Y')$ such that $Y[b_1, \ldots, b_{n+1}] = Y$. The latter condition implies that $T$ is a standard Young tableau of type $\mu = (2, 1, \ldots, 1)$, so $b_1 = b_2 = 1$. If $n_k = 0$, since $b_1 \neq n+1$ there are $k-1$ choices for $b_1$ such that $Y[b_1]$ is a Young tableau. If $n_k > 0$, there are $k$ choices for $b_1$. Since $1 = b_2 < b_3 < \cdots < b_{n+1} \leq n+1$, the remaining $b_i$ in both cases are uniquely determined. Thus, $c^\lambda_{\lambda, \mu} \in \{k-1, k\}$ depending only on whether $n_k = 0$ or $n_k > 0$; the result follows.

Proof of Theorem 6.1. Set $L = \mathfrak{sl}_{n+1}$. By Lemma 6.3, $\text{Der}(t)$ contains a simple subalgebra $D \subset \text{Der}(t)$ isomorphic to $L$. Setting $d = x_2 + x_1 - x_0$, we consider only $\mathfrak{g}(d, D)$ in place of $\langle d \rangle$. We will show that $\dim \mathfrak{g}(d, D) = 1$, so that $\langle d \rangle = \mathfrak{g}(d, D)$. Since $L$ and $M$ are irreducible $L$-modules, $\langle d \rangle$ is nondegenerate. Since $D$ is a simple Chevalley Lie algebra, our main theorem applies. Hence, isomorphism between such tensors is decided in polynomial time. By a functorial construction of Brabahap [21, Section 9.1], each tensor induces a $K$-algebra on $M \oplus L \oplus M$.

Since $M$ and $L$ are irreducible $L$-modules, they are irreducible $D$-modules. Every tensor contained in $\langle d \rangle$ determines a $\text{Der}(t)$-module homomorphism $M \to M \otimes L$, which must also be a $D$-module homomorphism. Each irreducible $L$-module has a unique vector of highest weight, so there exist partitions $\lambda$ and $\mu$ such that $M \cong V(\lambda)$ and $L \cong V(\mu)$ as $D$-modules. Since $L$ is the adjoint module, $\mu = (2, 1, \ldots, 1) \vdash n + 1$. By irreducibility, the number of $K$-linearly independent $D$-module homomorphisms $M \to M \otimes L$ is equal to the generalized Littlewood–Richardson number for type $A$, namely $c^\lambda_{\lambda, \mu}$. For $m \geq 1$ and for all positive integers $\ell$ such that $\ell \mid m$ and $\ell \leq n$, let $\lambda \vdash m$ with parts of size $\ell$ and $0$. From Proposition 6.6, $c^\lambda_{\lambda, \mu} = 1$. There are at least $d(m, n)$ such partitions $\lambda$, which proves the theorem.

Acknowledgements. We thank W.A. de Graaf and J. Grochow for answers to questions about the conjugacy of Lie matrix algebras.

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