The generalized inverses of tensors via the C-Product

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Abstract

This paper studies the issues about the generalized inverses of tensors under the C-Product. The aim of this paper is threefold. Firstly, this paper present the definition of the Moore-Penrose inverse, Drazin inverse of tensors under the C-Product. Moreover, the inverse along a tensor is also introduced. Secondly, this paper gives some other expressions of the generalized inverses of tensors by using several decomposition forms of tensors. Finally, the algorithms for the Moore-Penrose inverse, Drazin inverse of tensors and the inverse along a tensor are established.

Keywords: C-Product, Tensors, Moore-Penrose, Drazin inverse, Inverse along a tensor

AMS classification: 15A18, 15A69.

1 Introduction

In recent years, the studies of tensors or the multidimensional array have become more popular. A complex tensor can be regarded as a multidimensional array of data, which takes the form \( A = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_p}. \) The order of a tensor is the number of dimensions which is also called ways or modes. Therefore, the well-known vectors and matrices are called first-order tensors and second-order tensors. This paper studies the third-order tensors.

Higher-order tensors have been used in various fields, such as psychometrics[1], chemometrics[2], face recognition[3] and image and signal processing[4, 5, 6, 7, 8, 9], etc. Sun et al.[10] introduced the notion of the inverse of an even-order tensor under the Einstein product and called it as the Moore-Penrose inverse of tensors. Sun et al.[11] defined the \( \{i\}\)-inverse and group inverse of tensors based on a general product of tensors, and investigated properties of the generalized inverses of tensors. Miao et al.[12] investigate the tensor similar relationship and propose the T-Jordan canonical form based on the tensor T-product. Meanwhile, the T-polar, T-LU, T-QR and T-Schur decompositions of tensors are obtained. Besides, the T-group inverse and T-Drazin inverse are studied. Jin et al.[13] established the generalized inverse of tensors by using tensor equations. Moreover, the authors investigated the least squares solutions of tensor equations. Behera et al.[14] had a further study on the generalized inverses of tensors. Several characterizations of generalized inverses of tensors are provided. Besides, a new method for computing the Moore-Penrose inverse of a tensor was obtained. Ji et al.[15] extended the notion of the Drazin inverse of a square matrix to an even-order square tensor. Also, the authors obtained the expression of the Drazin inverse by using the core-nilpotent decomposition. Behera et al.[16] further elaborated the theory of the Drazin inverse and W-weighted Drazin inverse of tensors. Moreover, different types of methods were built to compute the Drazin inverse of tensors. Benítez et al.[17] studied one-sided (b, c)-inverses of arbitrary matrices as well as one-sided inverses along a (not necessarily square) matrix. In addition, the (b, c)-inverse and the inverse along an element were also researched in the context of rectangular matrices. Kolda et al.[18] provided an overview of higher-order tensor decompositions and their applications. Two particular tensor decompositions: the CP decomposition and the Tucker decomposition were introduced.

Kernfeld et al.[19] defined a new tensor-tensor product—Cosine Transform Product, referred to as C-product for short. And it has been shown that the C-product can be implemented efficiently using DCTs. In addition, the authors indicate that one can use C-product to conveniently specify a discrete image blurring model and the image restoration model. Based on this background, we will study the theory of the generalized inverses of tensors via the C-product in this paper.

This paper is organized as follows. In Section 2, we give the terms and symbols needed to be used in this paper. Then, we introduce the C-product of two tensors and some properties of it. In Section 3, we first define the Moore-Penrose inverse of tensors via the C-product. Then, we provide some decompositions of the tensor, including C-SVD, C-QR decomposition, C-Schur decomposition, C-full rank decomposition, C-QDR decomposition and C-HS decomposition. Furthermore, we use these decompositions to give the expressions for the Moore-Penrose inverse of tensors. In Section 4, we study the Drazin inverse of the tensor under the C-product. This part gives the definition and a few properties for...
the Drazin inverse of tensors, and provide several expressions for the Drazin inverse of tensors. In Section 5, we define the inverse along a tensor under the C-product. Some expressions of the class of the inverse are obtained. Moreover, an algorithm for computing the inverse along a tensor is built. In the last section, we establish an application on higher-order Markov Chains concerning the group inverse of the tensor.

2 Preliminaries

In this paper, we denote vectors, matrices, three or higher order tensors like \( a, A, \mathcal{A} \), respectively. Also, \( a_i, A_{ij} \) and \( A_{i_1i_2 \ldots i_p} \) are the elements of the vector \( a \), matrix \( A \) and tensor \( \mathcal{A} \), respectively. The frontal slice of tensor \( \mathcal{A} \) is \( \mathcal{A}_{(:, :, i)} \). We denote the frontal slice as \( \mathcal{A}^{(i)} \) for simplicity. When fixing two indices of the third order tensor, we can get the fiber. The mode-3 fiber is also called tube, denoted as \( \mathcal{A}(i, j, :) \). We denote \( \mathbf{\mathcal{T}} \) the tube of the tensor \( \mathcal{A} \). We can vectorize a tube by \( \mathbf{a} = \text{vec}(\mathbf{\mathcal{T}}) \).

2.1 C-product

**Definition 2.1** [19] Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) and \( B \in \mathbb{C}^{n_2 \times l \times n_3} \). The **face-wise product** \( A \triangle B \) is defined as

\[
(A \triangle B)^{(i)} = A^{(i)} B^{(i)}.
\]

**Definition 2.2** [19] Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \). \( A^{(1)}, A^{(2)}, \ldots, A^{(n_3)} \) are its frontal slices. Then we use \( \text{mat}(A) \) to denote the block Toeplitz-plus-Hankel matrix

\[
\text{mat}(A) = \begin{bmatrix}
A^{(1)} & A^{(2)} & \cdots & A^{(n_3-1)} & A^{(n_3)} \\
A^{(2)} & A^{(1)} & \cdots & A^{(n_3-2)} & A^{(n_3-1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A^{(n_3-1)} & A^{(n_3-2)} & \cdots & A^{(1)} & A^{(2)} \\
A^{(n_3)} & A^{(n_3-1)} & \cdots & A^{(2)} & A^{(1)}
\end{bmatrix} + \begin{bmatrix}
A^{(2)} & A^{(3)} & \cdots & A^{(n_3)} & O \\
A^{(3)} & A^{(4)} & \cdots & O & A^{(n_3)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A^{(n_3)} & O & \cdots & A^{(4)} & A^{(3)} \\
O & A^{(n_3)} & \cdots & A^{(3)} & A^{(2)}
\end{bmatrix},
\]

(1)

where \( O \) is \( n_1 \times n_2 \) zero matrix.

**Definition 2.3** [19] Let \( \text{ten}(\cdot) \) be the inverse operation of the \( \text{mat}(\cdot) \), i.e.,

\[
\text{ten}(\text{mat}(A)) = A.
\]

**Definition 2.4** [19] Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) and \( B \in \mathbb{C}^{n_2 \times l \times n_3} \). The **cosine transform product**, which is called C-product for short, is defined as

\[
A \circ_c B = \text{ten}(\text{mat}(A) \text{mat}(B)).
\]

Let \( \mathbf{y} \) be a \( 1 \times 1 \times n_3 \) tensor, then \( \text{mat}(\mathbf{y}) \) is a \( 1 \cdot n_3 \times 1 \cdot n_3 \) Toeplitz-plus-Hankel matrix as defined in [11], which each blocks are \( 1 \times 1 \). Let \( \mathbf{C}_{n_3} \) denote the \( n_3 \times n_3 \) orthogonal DCT matrix as defined in [2], which can be computed in Matlab by using \( \mathbf{C}_{n_3} = \text{dct}(\text{eye}(n_3)) \). Moreover, one has

\[
\mathbf{C}_{n_3} \text{mat}(\mathbf{y}) \mathbf{C}_{n_3}^T = \mathbf{D} = \text{diag}(\mathbf{d}),
\]

where \( \mathbf{d} = \mathbf{W}^{-1}(\mathbf{C}_{n_3} \text{mat}(\mathbf{y}) \mathbf{e}_1) \), \( \mathbf{W} = \text{diag}(\mathbf{C}_{n_3}(:, 1)) \), \( \mathbf{e}_1 = [1, 0, \ldots, 0]^T \).

Notice that, \( \text{mat}(\mathbf{y}) \mathbf{e}_1 = (\mathbf{I} + \mathbf{Z}) \text{vec}(\mathbf{y}) \), where \( \text{vec}(\mathbf{y}) \) means the vectorization of \( \mathbf{y} \), \( \mathbf{Z} \) is the \( n_3 \times n_3 \) singular circulant upshift matrix, which can be computed in Matlab by using \( \mathbf{Z} = \text{diag}((\text{ones}(n_3-1, 1), 1)) \). Hence, we have

\[
\mathbf{d} = \mathbf{W}^{-1} \mathbf{C}_{n_3} (\mathbf{I} + \mathbf{Z}) \text{vec}(\mathbf{y}) = \mathbf{M} \text{vec}(\mathbf{y}).
\]

**Definition 2.5** [19] Let \( L : \mathbb{C}^{l \times n_2 \times n_3} \rightarrow \mathbb{C}^{l \times n_2 \times n_3} \) is an invertible linear transform. Define

\[
\text{vec}(L(\mathbf{y})) = \mathbf{M} \text{vec}(\mathbf{y}),
\]

where \( \mathbf{y} = \text{vec}(\mathbf{y}) \), \( \mathbf{M} = \mathbf{W}^{-1} \mathbf{C}_{n_3} (\mathbf{I} + \mathbf{Z}) \).

Notice that an \( n_1 \times n_2 \times n_3 \) tensor can be seen as an \( n_1 \times n_2 \) matrix whose \((i, j)\)th element \( \mathbf{\pi}_{ij} = (\mathbf{A})_{ij} \) are the tube fibers in \( \mathbb{C}^{l \times 1 \times n_3} \).

**Definition 2.6** [19] Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \). Then, \( L(A) = \hat{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) with tube fibers

\[
\hat{a}_{ij} = (\hat{A})_{ij} = L(\mathbf{\pi}_{ij}), \quad i = 1, 2, \ldots, n_1, \quad j = 1, 2, \ldots, n_2,
\]

where \( \mathbf{\pi}_{ij} \) are the tube fibers of \( A \).
Definition 2.7 \[13\] The mode-3 product of a tensor \(A \in \mathbb{C}^{n_1 \times n_2 \times n_3}\) with a matrix \(U \in \mathbb{C}^{l \times n_3}\) is denoted by \(A \times_3 U\). More precise, we have

\[
(A \times_3 U)_{i_1,i_2,i_3} = \sum_{i_3=1}^{n_3} A_{i_1 i_2 i_3} U_{j i_3}.
\]

Let the frontal slice of \(A \in \mathbb{C}^{n_1 \times n_2 \times n_3}\) are

\[
A^{(1)} = \begin{bmatrix}
A_{111} & A_{121} & \cdots & A_{1n_2} \\
A_{211} & A_{221} & \cdots & A_{2n_2} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n_11} & A_{n_12} & \cdots & A_{n_1n_2}
\end{bmatrix}, \ldots, A^{(n_3)} = \begin{bmatrix}
A_{1n_3} & A_{12n_3} & \cdots & A_{1n_2n_3} \\
A_{2n_3} & A_{22n_3} & \cdots & A_{2n_2n_3} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n_1n_3} & A_{n_12n_3} & \cdots & A_{n_1n_2n_3}
\end{bmatrix}.
\]

Then, the mode-3 unfolding of \(A\), denoted \(A_{(3)}\), is

\[
A_{(3)} = \begin{bmatrix}
A_{111} & A_{121} & \cdots & A_{1n_2} & A_{121} & A_{221} & \cdots & A_{2n_2} & \cdots & A_{1n_2n_3} \\
A_{112} & A_{122} & \cdots & A_{1n_2} & A_{212} & A_{222} & \cdots & A_{2n_2} & \cdots & A_{21n_2n_3} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
A_{1n_1} & A_{12n_2} & \cdots & A_{1n_1n_3} & A_{21n_3} & A_{22n_3} & \cdots & A_{2n_2n_3} & \cdots & A_{n_1n_2n_3}
\end{bmatrix}. \tag{3}
\]

Notice that \(A \times_3 U\) can be computed using the following matrix-matrix product. See \[13\] for details.

\[
y = A \times_3 U \iff y_{(3)} = UA_{(3)}. \tag{4}
\]

Observe that

\[
L(A) = A \times_3 M \tag{5}
\]

and

\[
L^{-1}(A) = A \times_3 M^{-1}. \tag{6}
\]

Lemma 2.1 \[19\] Let \(A \in \mathbb{C}^{n_1 \times n_2 \times n_3}\). Then,

\[
(C_{n_3} \otimes I_{n_1}) \mat(A)(C_{n_3}^{-1} \otimes I_{n_2}) = \begin{bmatrix}
L(A)^{(1)} \\
L(A)^{(2)} \\
\vdots \\
L(A)^{(n_3)}
\end{bmatrix},
\]

where \(C_{n_3}\) is the \(n_3 \times n_3\) orthogonal DCT matrix.

Lemma 2.2 \[19\] Let \(A \in \mathbb{C}^{n_1 \times n_2 \times n_3}\) and \(B \in \mathbb{C}^{n_2 \times l \times n_3}\). Then,

1. \(\mat(A *_c B) = \mat(A) \mat(B)\).
2. \(A *_c B = L^{-1}(L(A) \triangle L(B))\).

The C-product of \(A \in \mathbb{C}^{n_1 \times n_2 \times n_3}\) and \(B \in \mathbb{C}^{n_2 \times l \times n_3}\) can be computed using the following Algorithm from \[19\].

**Algorithm 2.1: Compute the C-product of two tensors**

**Input:** \(n_1 \times n_2 \times n_3\) tensor \(A\) and \(n_2 \times l \times n_3\) tensor \(B\)

**Output:** \(n_1 \times l \times n_3\) tensor \(C\)

1. \(\hat{A} = L(A), \hat{B} = L(B)\)
2. for \(i = 1, \ldots, n_3\)
   \(\hat{c}^{(i)} = \hat{A}^{(i)} \hat{B}^{(i)}\)
end
3. \(C = L^{-1}(\hat{c})\)

Lemma 2.3 \[19\] If \(A, B, C\) are order-3 tensors of proper size, then the following statements are true:

1. \(A *_c (B + C) = A *_c B + A *_c C\);
2. \((A + B) *_c C = A *_c C + B *_c C\);
Definition 2.8 \[19\] Let \( L(\vec{j}) = \vec{j} \in \mathbb{C}^{n \times n \times n} \) be such that \( \vec{j}^{(i)} = I_n \), \( i = 1, 2, ..., n_3 \). Then \( L = L^{-1}(\vec{j}) \) is the identity tensor.

Lemma 2.4 \[19\] Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) and \( J \in \mathbb{C}^{n_1 \times n_1 \times n_3} \) is the identity tensor. Then,
\[ J \ast c \ast A = A \ast c \ast J = A. \]

Proof: It is clear that
\[ L(J \ast c \ast A) = L(J) \triangle L(A) = L(A) \triangle L(J) = L(A \ast c \ast J). \]
Thus, \( J \ast c \ast A = A \ast c \ast J = A. \)

Definition 2.9 Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) and \( B \in \mathbb{C}^{n_2 \times n_1 \times n_3} \). If
\[ A \ast c \ast B = \mathbb{I} \quad \text{and} \quad B \ast c \ast A = \mathbb{I}, \]
then \( A \) is said to be invertible and \( B \) is the inverse of \( A \), which is denoted by \( A^{-1} \).

It is easy to see the inverse of a tensor, if exists, is unique. The conjugate transpose of tensors can be defined as follows.

Definition 2.10 \[19\] If \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \), then the conjugate transpose of \( A \), which is denoted by \( A^H \), is such that
\[ L(A^H)^{(i)} = (L(A^{(i)}))^H, \quad i = 1, 2, ..., n_3. \]

Lemma 2.5 \[19\] Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) and \( B \in \mathbb{C}^{n_2 \times n_1 \times n_3} \). It holds that
\[ (A \ast c \ast B)^H = B^H \ast c \ast A^H. \]

Definition 2.11 Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \). \( A \) is said symmetric if \( A^H = A \).

Definition 2.12 \[19\] Let \( Q \in \mathbb{C}^{n_1 \times n_2 \times n_3} \). \( Q \) is said unitary if \( Q^H \ast Q = Q \ast Q^H = \mathbb{I} \).

Definition 2.13 Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \). Then, \( A \) is called an F-diagonal/F-upper/F-lower tensor if all frontal slices \( A^{(i)} \), \( i = 1, 2, ..., n_3 \) of \( A \) are diagonal/upper triangular/lower triangular matrices.

Lemma 2.6 Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \). Then, \( L(A) \) is an F-diagonal/F-upper/F-lower tensor if and only if \( A \) is an F-diagonal/F-upper/F-lower tensor.

Proof: We only prove the case of the F-upper tensor for the sake of the F-diagonal tensor is one special case of the F-upper tensor and the F-lower tensor can be proved similarly.

Let \( B = L(A) \). Then, by using (1) and (3), one has \( A = L^{-1}(B) = B \times_3 M^{-1} \) and \( A_{(3)} = M^{-1}B_{(3)} \), where \( M \) is defined in (2). Since \( B \) is an F-upper tensor, by (3), one has
\[ B_{(3)} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n_2} & B_{111} & B_{112} & \cdots & B_{1(n_2-1)1} & 0 & 0 & \cdots & B_{2(n_2-2)1} & 0 & 0 & \cdots \\ B_{12} & B_{22} & \cdots & B_{1n_2} & B_{21} & B_{22} & \cdots & B_{2(n_2-1)2} & 0 & 0 & \cdots & B_{3(n_2-2)2} & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{n_2} & B_{n_2} & \cdots & B_{n_2} & B_{n_2} & B_{n_2} & \cdots & B_{n_2} & 0 & B_{31} & B_{32} & \cdots & B_{3(n_2-2)n_3} & 0 & 0 & \cdots \end{bmatrix} \]

By using the matrices product, it is easy to see
\[ A_{(3)} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n_2} & A_{111} & A_{112} & \cdots & A_{1(n_2-1)1} & 0 & 0 & \cdots & A_{2(n_2-2)1} & 0 & 0 & \cdots \\ A_{12} & A_{22} & \cdots & A_{1n_2} & A_{21} & A_{22} & \cdots & A_{2(n_2-1)2} & 0 & 0 & \cdots & A_{3(n_2-2)2} & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{n_2} & A_{n_2} & \cdots & A_{n_2} & A_{n_2} & A_{n_2} & \cdots & A_{n_2} & 0 & A_{31} & A_{32} & \cdots & A_{3(n_2-2)n_3} & 0 & 0 & \cdots \end{bmatrix} \]

Then, we have all the frontal slices of \( A \) are upper triangular matrices, which means \( A \) is an F-upper tensor.

Conversely, if \( A \) is an F-upper tensor, then \( A_{(3)} \) has the above form. Also, we have
\[ L(A) = A \times_3 M \Leftrightarrow L(A)_{(3)} = MA_{(3)} \]
by (1) and (5). Then,
\[ L(A)_{(3)} = \begin{bmatrix} L_{111} & L_{112} & \cdots & L_{1n_21} & L_{211} & L_{212} & \cdots & L_{2(n_2-1)1} \\ L_{112} & L_{122} & \cdots & L_{1n_22} & L_{212} & L_{222} & \cdots & L_{2(n_2-1)2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n_1} & L_{n_2} & \cdots & L_{n_2} & L_{n_3} & L_{n_3} & \cdots & L_{n_3} \end{bmatrix} \]
Theorem 3.1

By Lemma 2.1, one has

Proof:

then

Definition 3.1

\[ L \]

\( S \)

(7). In this case,

For each

an algorithm to compute the Moore-Penrose inverse based on the C-SVD of a tensor.

In this part, we will give some expressions of the Moore-Penrose inverse by using the C-SVD, C-QR decomposition, C-Schur decomposition, C-full rank decomposition, C-QDR decomposition and C-HS decomposition. Then, we establish

3 The Moore-Penrose inverse of tensors under the C-product

Theorem 3.2

The Moore-Penrose inverse of an arbitrary tensor

For

and

denote

If there exists a tensor

such that

is an

A

C-upper tensor. We call this decomposition the C-SVD of

Theorem 3.1 [19] Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \). Then there exist unitary tensors \( U \in \mathbb{C}^{n_1 \times n_1 \times n_3} \) and \( V \in \mathbb{C}^{n_2 \times n_2 \times n_3} \) such that

\[
A = U *_c S *_c V^H,
\]

where \( S \) is an \( n_1 \times n_2 \times n_3 \) F-diagonal tensor. We call this decomposition the C-SVD of \( A \).

Theorem 3.2 The Moore-Penrose inverse of an arbitrary tensor \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) exists and is unique.

Proof: By Lemma 2.1 one has

\[
(C_{n_3} \otimes I_{n_1})(C_{n_3}^{-1} \otimes I_{n_2}) = \begin{bmatrix}
L(A)^{(1)} \\
L(A)^{(2)} \\
& & \ddots \\
& & & & L(A)^{(n_3)}
\end{bmatrix}.
\]

Let \( L(A)^{(i)} = U_i \Sigma_i V_i^H \) be the singular value decomposition of \( L(A)^{(i)} \), \( i = 1, \ldots, n_3 \). Thus, we have

\[
(C_{n_3} \otimes I_{n_1})(C_{n_3}^{-1} \otimes I_{n_2}) = \begin{bmatrix}
L(A)^{(1)} \\
L(A)^{(2)} \\
& & \ddots \\
& & & & L(A)^{(n_3)}
\end{bmatrix}
= \begin{bmatrix}
U_1 \Sigma_1 V_1^H \\
U_2 \Sigma_2 V_2^H \\
& & \ddots \\
& & & & U_{n_3} \Sigma_{n_3} V_{n_3}^H
\end{bmatrix}.
\]

For each

\[
\Sigma_i = \begin{bmatrix}
\sigma_i & & \\
& \ddots & \ddots \\
& & \sigma_i
\end{bmatrix},
\]

which implies \( L(A) \) is an F-upper tensor. \( \square \)
\(\sigma^j_i, j = 1, 2, \ldots, r_i, r_i = \text{rank}(L(A)^{(i)})\) are singular values of \(L(A)^{(i)}\). We define the matrices \(R_i, i = 1, \ldots, n_3\), as

\[
R_i = \begin{bmatrix}
\frac{1}{\sigma_1} & & \\
& \ddots & \\
& & \frac{1}{\sigma_{r_i}} \\
& & 0 \\
& & \ddots \\
& & & 0
\end{bmatrix}.
\]

Observe that \(R_i = \Sigma_i^i\) for \(i = 1, \ldots, n_3\). Let \(X_i = V_i R_i U_i^H\) for \(i = 1, \ldots, n_3\). Now, we have

\[
\begin{bmatrix}
X_1 \\
\vdots \\
X_{n_3}
\end{bmatrix} = \begin{bmatrix}
V_1 \\
\vdots \\
V_{n_3}
\end{bmatrix} \begin{bmatrix}
R_1 \\
\vdots \\
R_{n_3}
\end{bmatrix} \begin{bmatrix}
U_1^H \\
\vdots \\
U_{n_3}^H
\end{bmatrix}.
\]

Thus,

\[
\text{ten}((C_{n_3}^{-1} \otimes I_{n_2}) \begin{bmatrix}
X_1 \\
\vdots \\
X_{n_3}
\end{bmatrix}) = \text{ten}((C_{n_3}^{-1} \otimes I_{n_2}) \begin{bmatrix}
V_1 \\
\vdots \\
V_{n_3}
\end{bmatrix}) \times \text{ten}((C_{n_3}^{-1} \otimes I_{n_2}) \begin{bmatrix}
R_1 \\
\vdots \\
R_{n_3}
\end{bmatrix}) \times \text{ten}((C_{n_3}^{-1} \otimes I_{n_2}) \begin{bmatrix}
U_1^H \\
\vdots \\
U_{n_3}^H
\end{bmatrix})
\]

that is \(X = V \ast_c R \ast_c U^H\). It is easy to check that \(X\) satisfies (7), which means the Moore-Penrose inverse of a tensor \(A\) exists.

On the other hand, suppose \(X_1\) and \(X_2\) both are the solutions of (7). Then, we have

\[
\begin{aligned}
X_1 &= X_1 \ast_c A \ast_c X_1 = X_1 \ast_c (A \ast_c X_2 \ast_c A) \ast_c X_1 = X_1 \ast_c (A \ast_c X_2)^H \ast_c (A \ast_c X_1)^H \\
&= X_1 \ast_c (A \ast_c X_1 \ast_c A \ast_c X_2)^H = X_1 \ast_c (A \ast_c X_2)^H \\
&= X_1 \ast_c A \ast_c X_2 \\
&= X_1 \ast_c (A \ast_c X_2 \ast_c A) \ast_c X_2 = (X_1 \ast_c A)^H \ast_c (X_2 \ast_c A)^H \ast_c X_2 \\
&= (X_2 \ast_c A \ast_c X_1 \ast_c A)^H \ast_c X_2 = (X_2 \ast_c A)^H \ast_c X_2 \\
&= X_2 \ast_c A \ast_c X_2 = X_2.
\end{aligned}
\]

Therefore, the Moore-Penrose inverse of \(A\) is unique. \(\square\)

**Theorem 3.3** Let \(A \in \mathbb{C}^{n_1 \times n_2 \times n_3}\) and \(A = U \ast_c S \ast_c V^H\) be C-SVD of \(A\). Then,

\[A^\dagger = V \ast_c S^\dagger \ast_c U^H.\]

**Proof:** It is easy to check that \(V \ast_c S^\dagger \ast_c U^H\) holds for the four equations of (7). \(\square\)

**Theorem 3.4** Let \(A \in \mathbb{C}^{n_1 \times n_2 \times n_3}\). Then there exist a unitary tensor \(Q \in \mathbb{C}^{n_1 \times n_2 \times n_3}\) and an F-upper tensor \(R \in \mathbb{C}^{n_1 \times n_2 \times n_3}\) such that

\[A = Q \ast_c R,
\]

which is called the C-QR decomposition of \(A\).

**Proof:** Let \(\widehat{A} = L(A), \widehat{Q} = L(Q)\) and \(\widehat{R} = L(R)\). Suppose \(\widehat{A}^{(i)} = Q_i R_i = \widehat{Q}^{(i)} \widehat{R}^{(i)}, i = 1, 2, \ldots, n_3\), are the QR decomposition of \(\widehat{A}^{(i)}\). Hence, \(A = Q \ast_c R\). Furthermore, one has \(L(Q \ast_c Q^H) = L(Q) \Delta L(Q^H)\). Thus,

\[L(Q)^{(i)} L(Q^H)^{(i)} = L(Q)(Q^H) = I_{n_1} = L(I)^{(i)}, i = 1, 2, \ldots, n_3.
\]

This implies \(Q \ast_c Q^H = I\), that is \(Q\) is a unitary tensor. On the other hand, \(R_i\) are upper triangular matrices and so are \(\widehat{R}^{(i)}\). This implies \(R\) is an F-upper tensor. \(\square\)
Theorem 3.5 Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ and $\mathcal{A} = \mathcal{Q} \ast_c \mathcal{R}$ be the C-QR decomposition of $\mathcal{A}$. Then,
$$\mathcal{A}^\dagger = \mathcal{R}^\dagger \ast_c \mathcal{Q}^H.$$ 

PROOF: It is easy to check that $\mathcal{R}^\dagger \ast_c \mathcal{Q}^H$ holds for the four equations of [7]. \qed

Theorem 3.6 Let $\mathcal{A} \in \mathbb{C}^{n \times n \times n \times n_3}$. Then there exist a unitary tensor $\mathcal{Q} \in \mathbb{C}^{n \times n \times n}$ and an F-upper tensor $\mathcal{T} \in \mathbb{C}^{n \times n \times n_3}$ such that
$$\mathcal{A} = \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q},$$
which is called the C-Schur decomposition of $\mathcal{A}$.

PROOF: Let $\mathcal{A} = \mathcal{L}(\mathcal{A})$, $\mathcal{Q} = \mathcal{L}(\mathcal{Q})$ and $\mathcal{T} = \mathcal{L}(\mathcal{T})$. Suppose $\mathcal{\hat{A}}^{(i)} = \mathcal{Q}_i^H \mathcal{T}_i \mathcal{Q}_i = (\mathcal{\hat{Q}}^{(i)})^H (\mathcal{\hat{T}}^{(i)}) \mathcal{Q}_i$, $i = 1, 2, ..., n_3$, are the Schur decomposition of $\mathcal{\hat{A}}^{(i)}$. Thus, $\mathcal{A} = \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q}$. By the proof of Theorem 3.3 $\mathcal{Q}$ is a unitary tensor. On the other hand, $\mathcal{T}_i$ are upper triangular matrices and so are $\mathcal{\hat{T}}^{(i)}$. This implies $\mathcal{T}$ is an F-upper tensor. \qed

Theorem 3.7 Let $\mathcal{A} \in \mathbb{C}^{n \times n \times n \times n_3}$ and $\mathcal{A} = \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q}$ be the C-Schur decomposition of $\mathcal{A}$. Then,
$$\mathcal{A}^\dagger = \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q},$$

PROOF: Now, we will check that $\mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q}$ holds for the four equations of [7]: Let $\mathcal{X} = \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q}$, we will have
$$\mathcal{A} \ast_c \mathcal{X} \ast_c \mathcal{A} = \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q} \ast_c \mathcal{Q} \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q} = \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q} \ast_c \mathcal{Q} = \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q} = \mathcal{A},$$
$$\mathcal{X} \ast_c \mathcal{A} \ast_c \mathcal{X} = \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q} \ast_c \mathcal{Q} \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q} \ast_c \mathcal{Q} = \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q} \ast_c \mathcal{Q} \ast_c \mathcal{Q} = \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q} = \mathcal{X},$$
and
$$\mathcal{(A \ast_c X)^H} = (\mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q} \ast_c \mathcal{Q} \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q})^H = (\mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q})^H \ast_c \mathcal{Q}^H \ast_c \mathcal{T} \ast_c \mathcal{Q} = \mathcal{A} \ast_c \mathcal{X}.$$ 

From now on, we denote
$$\text{DCT(mat}(\mathcal{A})) = (\mathcal{C}_{n_3} \otimes \mathcal{I}_{n_1}) \text{mat}(\mathcal{A})(\mathcal{C}_{n_3}^{-1} \otimes \mathcal{I}_{n_2})^\dagger = \begin{bmatrix} L(\mathcal{A})^{(1)} & & \\ & L(\mathcal{A})^{(2)} & \\ & & \ddots \end{bmatrix},$$
and
$$\text{ten(IDCT)}(\begin{bmatrix} L(\mathcal{A})^{(1)} & & \\ & L(\mathcal{A})^{(2)} & \\ & & \ddots \end{bmatrix}) = \mathcal{A}.$$ 

In the following, we give the full rank decomposition of the tensor. Notice that not all the tensors have the full rank decomposition we defined.

Definition 3.2 Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$. If $\mathcal{A}$ can be decomposed into
$$\mathcal{A} = \mathcal{M} \ast_c \mathcal{N},$$
where
$$\mathcal{M} = \text{ten(IDCT}(\begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_{n_3} \end{bmatrix})) \in \mathbb{C}^{n_1 \times r \times n_3}, \quad \mathcal{M}_i \in \mathbb{C}^{n_1 \times r}, \quad i = 1, 2, ..., n_3$$

and
$$\mathcal{N} = \text{ten(IDCT}(\begin{bmatrix} N_1 & & \\ & \ddots & \\ & & N_{n_3} \end{bmatrix})) \in \mathbb{C}^{r \times n_2 \times n_3}, \quad \mathcal{N}_i \in \mathbb{C}^{r \times n_2}, \quad i = 1, 2, ..., n_3,$$
then we call this decomposition the C-full rank decomposition of $\mathcal{A}$. 

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Note: Let $\hat{A} = L(A)$, $\hat{M} = L(M)$ and $\hat{N} = L(N)$. Suppose $\hat{A}^{(i)} = M_i N_i = \hat{M}^{(i)} \hat{N}^{(i)}$, $M_i \in \mathbb{C}^{p_i \times r}$, $N_i \in \mathbb{C}^{r \times n_i}$, $i = 1, 2, ..., n_3$, are the full rank decomposition of $\hat{A}^{(i)}$. We deduce when $\text{rank}(\hat{A}^{(i)}) = r$, $i = 1, 2, ..., n_3$, one has the decomposition of the definition established.

**Theorem 3.8** Let $A \in \mathbb{C}^{n_1 \times n_2 \times n_3}$. Suppose $A$ has the C-full rank decomposition $A = M \ast_c N$. Then,

$$A^t = N^H \ast_c (M^H \ast_c A \ast_c N^H)^{-1} \ast_c M^H.$$ 

**Proof:** We will check that $N^H \ast_c (M^H \ast_c A \ast_c N^H)^{-1} \ast_c M^H$ holds for the four equations of (7). Let $X = N^H \ast_c (M^H \ast_c A \ast_c N^H)^{-1} \ast_c M^H$. Then, we have

$$A \ast_c X \ast_c A = M \ast_c N \ast_c N^H \ast_c (M^H \ast_c A \ast_c N^H)^{-1} \ast_c M^H \ast_c M \ast_c N$$

$$= M \ast_c N \ast_c N^H \ast_c (M^H \ast_c M \ast_c N \ast_c N^H)^{-1} \ast_c M^H \ast_c M \ast_c N$$

$$= M \ast_c N \ast_c N^H \ast_c (N \ast_c N^H)^{-1} \ast_c (M^H \ast_c M)^{-1} \ast_c M^H \ast_c M \ast_c N$$

$$= M \ast_c N = A,$$

$$X \ast_c A \ast_c X = N^H \ast_c (M^H \ast_c A \ast_c N^H)^{-1} \ast_c M^H \ast_c A \ast_c N^H \ast_c (M^H \ast_c A \ast_c N^H)^{-1} \ast_c M^H$$

$$= N^H \ast_c (M^H \ast_c A \ast_c N^H)^{-1} M^H = X,$$

$$(A \ast_c X)^H = [M \ast_c N \ast_c N^H \ast_c (M^H \ast_c A \ast_c N^H)^{-1} \ast_c M^H]^H$$

$$= [M \ast_c N \ast_c N^H \ast_c (N \ast_c N^H)^{-1} \ast_c (M^H \ast_c M)^{-1} \ast_c M^H]^H$$

$$= M \ast_c (M^H \ast_c M)^{-1} \ast_c M$$

$$= M \ast_c N \ast_c N^H \ast_c (N \ast_c N^H)^{-1} \ast_c (M^H \ast_c M)^{-1} \ast_c M$$

$$= M \ast_c (M^H \ast_c A \ast_c N^H)^{-1} \ast_c M = A \ast_c X$$

and

$$(X \ast_c A)^H = [N^H \ast_c (M^H \ast_c A \ast_c N^H)^{-1} \ast_c M^H \ast_c M \ast_c N]^H$$

$$= [N^H \ast_c (N \ast_c N^H)^{-1} \ast_c (M^H \ast_c M)^{-1} \ast_c M^H \ast_c M \ast_c N]^H$$

$$= N^H \ast_c (N \ast_c N^H)^{-1} \ast_c (M^H \ast_c M)^{-1} \ast_c M \ast_c N$$

$$= N^H \ast_c (M^H \ast_c A \ast_c N^H)^{-1} \ast_c M = X \ast_c A.$$

\[\square\]

**Definition 3.3** Let $A \in \mathbb{C}^{n_1 \times n_2 \times n_3}$. If $A$ can be decomposed as

$$A = Q \ast_c D \ast_c R,$$

where

$$Q = \text{ten}(\text{IDCT}([Q_1 \ldots Q_{n_3}])), \quad Q_i \in \mathbb{C}^{n_1 \times r}, \quad i = 1, 2, ..., n_3,$$

$D \in \mathbb{C}^{r \times r \times n_3}$ is an invertible $F$-diagonal tensor

and

$$R = \text{ten}(\text{IDCT}([R_1 \ldots R_{n_3}])), \quad R_i \in \mathbb{C}^{r \times n_2}, \quad i = 1, 2, ..., n_3$$

is an $F$-upper tensor, then we call this decomposition the C-QDR decomposition of $A$.

**Note:** Let $\hat{A} = L(A)$, $\hat{Q} = L(Q)$, $\hat{D} = L(D)$ and $\hat{R} = L(R)$. Suppose

$$\hat{A}^{(i)} = Q_i \hat{D}_i R_i = \hat{Q}^{(i)} \hat{D}^{(i)} \hat{R}^{(i)}, \quad Q_i \in \mathbb{C}^{p_i \times r}, \quad D_i \in \mathbb{C}^{r \times r}, \quad R_i \in \mathbb{C}^{r \times n_2}, \quad i = 1, 2, ..., n_3,$$

are the QDR decomposition of $\hat{A}^{(i)}$ [20]. We deduce when $\text{rank}(\hat{A}^{(i)}) = r$, $i = 1, 2, ..., n_3$, one has the decomposition of the definition established. Since $D_i$ are invertible diagonal matrices, we have $\hat{D}^{(i)}$ also are invertible diagonal matrices. Meanwhile, $R_i$ are upper triangular matrices and so are $\hat{R}^{(i)}$. This implies $D$ is an invertible $F$-diagonal tensor and $R$ is an $F$-upper tensor.
Theorem 3.9 Let $A \in \mathbb{C}^{n_1 \times n_2 \times n_3}$. Suppose $A^H$ has the C-QDR decomposition $A^H = \Omega * c \ D * c \ R$. Then,

$$A^\dagger = \Omega * c \ (R * c \ A * c \ \Omega)^{-1} * c \ R. \quad \square$$

**Proof:** Let $X = \Omega * c \ (R * c \ A * c \ \Omega)^{-1} * c \ R$. Thus, one has

\[
\begin{align*}
A * c \ X * c \ A &= R^H * c \ D^H * c \ Q^H * c \ \Omega * c \ (R * c \ A * c \ \Omega)^{-1} * c \ R * c \ R * c \ H * c \ D * c \ Q^H \\
&= R^H * c \ D^H * c \ Q^H * c \ \Omega * c \ (R * c \ H * c \ D * c \ Q^H)^{-1} * c \ R * c \ H * c \ D * c \ Q^H \\
&= R^H * c \ D^H * c \ Q^H = A,
\end{align*}
\]

\[
\begin{align*}
X * c \ A * c \ X &= \Omega * c \ (R * c \ A * c \ \Omega)^{-1} * c \ R * c \ A * c \ Q * c \ (R * c \ A * c \ \Omega)^{-1} * c \ R \\
&= \Omega * c \ (R * c \ A * c \ \Omega)^{-1} * c \ R = X,
\end{align*}
\]

\[
(A * c \ X)^H = [R^H * c \ D^H * c \ Q^H * c \ \Omega * c \ (R * c \ A * c \ \Omega)^{-1} * c \ R]^H \\
= [R^H * c \ D^H * c \ Q^H * c \ \Omega * c \ (Q^H * c \ \Omega)^{-1} * c \ (D^H)^{-1} * c \ (R * c \ H)^{-1} * c \ R]^H \\
= R^H * c \ (R * c \ H)^{-1} * c \ R \\
= R^H * c \ D^H * c \ Q^H * c \ \Omega * c \ (R * c \ A * c \ \Omega)^{-1} * c \ R = A * c \ X
\]

and

\[
(X * c \ A)^H = [\Omega * c \ (R * c \ A * c \ \Omega)^{-1} * c \ R * c \ R^H * c \ D^H * c \ Q^H]^H \\
= [\Omega * c \ (Q^H * c \ \Omega)^{-1} * c \ (D^H)^{-1} * c \ (R * c \ H)^{-1} * c \ R * c \ R^H * c \ D^H * c \ Q^H]^H \\
= \Omega * c \ (Q^H * c \ \Omega)^{-1} * c \ Q^H \\
= \Omega * c \ (R * c \ A * c \ \Omega)^{-1} * c \ R * c \ R^H * c \ D^H * c \ Q^H = X * c \ A.
\]

Therefore, $X = A^\dagger$.

For a tensor $A \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, which the block form is

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where $A_1 \in \mathbb{C}^{s \times t \times n_3}, A_2 \in \mathbb{C}^{s \times (n_2-t) \times n_3}, A_3 \in \mathbb{C}^{(n_1-s) \times t \times n_3}, A_4 \in \mathbb{C}^{(n_1-s) \times (n_2-t) \times n_3}$. Let

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \in \mathbb{C}^{n_2 \times n_4 \times n_3},$$

where $B_1 \in \mathbb{C}^{t \times k \times n_3}, B_2 \in \mathbb{C}^{s \times (n_4-k) \times n_3}, B_3 \in \mathbb{C}^{(n_1-s) \times k \times n_3}, B_4 \in \mathbb{C}^{(n_1-s) \times (n_2-t) \times n_3}$. It is easy to check that

$$A * c \ B = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} * c \ \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} A_1 * c \ B_1 + A_2 * c \ B_3 & A_1 * c \ B_2 + A_2 * c \ B_4 \\ A_3 * c \ B_1 + A_4 * c \ B_3 & A_3 * c \ B_2 + A_4 * c \ B_4 \end{bmatrix}.$$ 

Suppose $A \in \mathbb{C}^{n \times n \times n}$ and $A = U * c \ S * c \ V^H$ is the C-SVD of $A$. When $\text{rank}(S^{(1)}) = \text{rank}(S^{(2)}) = \cdots = \text{rank}(S^{(n)}) = r$, the decomposition of $A$ can be written as

$$A = U * c \ \begin{bmatrix} S_r & 0 \\ 0 & 0 \end{bmatrix} * c \ V^H,$$

where $S_r \in \mathbb{C}^{r \times r \times n_3}, U \in \mathbb{C}^{n \times n \times n_3}, V \in \mathbb{C}^{n \times n \times n_3}$. Let

$$V^H * c \ U = \begin{bmatrix} K & L \\ M & N \end{bmatrix}, \text{ where } K \in \mathbb{C}^{r \times r \times n_3}.$$

Thus, we have

$$A = U * c \ \begin{bmatrix} S_r & 0 \\ 0 & 0 \end{bmatrix} * c \ V^H = U * c \ \begin{bmatrix} S_r & 0 \\ 0 & 0 \end{bmatrix} * c \ \begin{bmatrix} K & L \\ M & N \end{bmatrix} * c \ U^H = U * c \ \begin{bmatrix} S_r * c \ K & S_r * c \ L \\ 0 & 0 \end{bmatrix} * c \ U^H.$$

Since $V^H * c \ U$ is unitary, one can arrive $K * c \ K^H + L * c \ L^H = I_r$, where $I_r \in \mathbb{C}^{r \times r \times n_3}$. We call this decomposition the C-HS decomposition of $A$. 

\[9\]
Theorem 3.10 Let $A \in \mathbb{C}^{n \times n \times n}$. Suppose $A$ has the C-HS decomposition. Then,

$$A^\dagger = \mathcal{U} *_{c} \left[ \begin{array}{cc} \mathcal{X}^H & *_{c} \mathcal{S}^{-1}_{r} \\ \mathcal{L}^H & *_{c} \mathcal{S}^{-1}_{r} \end{array} \right] *_{c} \mathcal{U}^H. \quad (8)$$

Proof: Let $A = \mathcal{U} *_{c} \left[ \begin{array}{cc} S_r *_{c} \mathcal{X} & S_r *_{c} \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{array} \right] *_{c} \mathcal{U}^H$ and $X = \mathcal{U} *_{c} \left[ \begin{array}{cc} \mathcal{X}^H & *_{c} \mathcal{S}^{-1}_{r} \\ \mathcal{L}^H & *_{c} \mathcal{S}^{-1}_{r} \end{array} \right] *_{c} \mathcal{U}^H$. Then,

$$A *_{c} X *_{c} A = \mathcal{U} *_{c} \left[ \begin{array}{cc} S_r *_{c} \mathcal{X} & S_r *_{c} \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{array} \right] *_{c} \mathcal{U}^H *_{c} U *_{c} \left[ \begin{array}{cc} \mathcal{X}^H & *_{c} \mathcal{S}^{-1}_{r} \\ \mathcal{L}^H & *_{c} \mathcal{S}^{-1}_{r} \end{array} \right] *_{c} \mathcal{U}^H = \mathcal{U} *_{c} \left[ \begin{array}{cc} \mathcal{X}^H & *_{c} \mathcal{S}^{-1}_{r} \\ \mathcal{L}^H & *_{c} \mathcal{S}^{-1}_{r} \end{array} \right] *_{c} \mathcal{U}^H = X.$$

$$A *_{c} X = \mathcal{U} *_{c} \left[ \begin{array}{cc} S_r *_{c} \mathcal{X} & S_r *_{c} \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{array} \right] *_{c} \mathcal{U}^H *_{c} U *_{c} \left[ \begin{array}{cc} \mathcal{X}^H & *_{c} \mathcal{S}^{-1}_{r} \\ \mathcal{L}^H & *_{c} \mathcal{S}^{-1}_{r} \end{array} \right] *_{c} \mathcal{U}^H = \mathcal{U} *_{c} \left[ \begin{array}{cc} \mathcal{X}^H & *_{c} \mathcal{S}^{-1}_{r} \\ \mathcal{L}^H & *_{c} \mathcal{S}^{-1}_{r} \end{array} \right] *_{c} \mathcal{U}^H = (A *_{c} X)^H.$$

$$X *_{c} A = \mathcal{U} *_{c} \left[ \begin{array}{cc} \mathcal{X}^H & *_{c} \mathcal{S}^{-1}_{r} \\ \mathcal{L}^H & *_{c} \mathcal{S}^{-1}_{r} \end{array} \right] *_{c} \mathcal{U}^H *_{c} U *_{c} \left[ \begin{array}{cc} S_r *_{c} \mathcal{X} & S_r *_{c} \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{array} \right] *_{c} \mathcal{U}^H = \mathcal{U} *_{c} \left[ \begin{array}{cc} \mathcal{X}^H & *_{c} \mathcal{S}^{-1}_{r} \\ \mathcal{L}^H & *_{c} \mathcal{S}^{-1}_{r} \end{array} \right] *_{c} \mathcal{U}^H = (X *_{c} A)^H.$$

Therefore, $A^\dagger = \mathcal{U} *_{c} \left[ \begin{array}{cc} \mathcal{X}^H & *_{c} \mathcal{S}^{-1}_{r} \\ \mathcal{L}^H & *_{c} \mathcal{S}^{-1}_{r} \end{array} \right] *_{c} \mathcal{U}^H$. \quad \square$

3.2 The algorithm for computing the Moore-Penrose inverse of a tensor

In the following, we have Algorithm 3.1 provided the procedure for the Moore-Penrose inverse operation.

Algorithm 3.1: Compute the Moore-Penrose inverse of a tensor $A$

Input: $n_1 \times n_2 \times n_3$ tensor $A$

Output: $n_2 \times n_1 \times n_3$ tensor $X$

1. $\widehat{A} = L(A) = A \times_3 M$, where $M$ is defined in (2)
2. for $i = 1, \ldots, n_3$
   $$\widehat{X}^{(i)} = \text{pinv}(\widehat{A}^{(i)}); \text{ where pinv}(\widehat{A}^{(i)}) \text{ is the Moore-Penrose inverse of } \widehat{A}^{(i)}$$
end
3. $X = L^{-1}(\widehat{X}) = \widehat{X} \times_3 M^{-1}$

Example 3.1 Let $A \in \mathbb{C}^{3 \times 3 \times 4}$ with frontal slices

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 2 & 3 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 5 \end{bmatrix}, \quad A^{(3)} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 3 \\ 4 & 0 & 0 \end{bmatrix}, \quad A^{(4)} = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \end{bmatrix}.$$

Then, by using Algorithm 3.1, we have

$$A^\dagger^{(1)} = \begin{bmatrix} 1.6666 & 1.3333 & 9.7778 \\ 1.3333 & 1 & 7.5556 \\ 0 & 0 & -0.3333 \end{bmatrix}, \quad A^\dagger^{(2)} = \begin{bmatrix} -1.2722 & -1.0482 & -8.2780 \\ -1.2295 & -0.7384 & -6.2015 \\ 0.1057 & -0.0651 & 0.2724 \end{bmatrix},$$

$$A^\dagger^{(3)} = \begin{bmatrix} 0.7451 & 0.7255 & 5.0065 \\ 1.1372 & 0.3529 & 3.4837 \\ -0.2353 & 0.1568 & -0.0196 \end{bmatrix}, \quad A^\dagger^{(4)} = \begin{bmatrix} -0.2723 & -0.3815 & -1.6113 \\ -0.5629 & -0.0718 & -1.0905 \\ 0.1057 & -0.0651 & -0.0610 \end{bmatrix}.$$
4 The Drazin inverse of tensors under the C-product

In this section, we will give some expressions of the Drazin inverse of tensors. Then, an algorithm is established for the Drazin inverse of a tensor.

4.1 The expressions of the Drazin inverse of tensors

Recall that the index of a matrix $A$ is defined as the smallest nonnegative integer $k$ such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$, which is denoted by $\text{Ind}(A)$. Now, let us define the index of a tensor $\mathcal{A}$.

**Definition 4.1** Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$. The index of the tensor $\mathcal{A}$ is defined as

$$\text{Ind}(\mathcal{A}) = \text{Ind} (\text{mat}(\mathcal{A})).$$

**Lemma 4.1** Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$. Suppose that $\mathcal{A}$ can be expressed as

$$\text{DCT}(\text{mat}(\mathcal{A})) = \begin{bmatrix} L(A)^{(1)} & L(A)^{(2)} & \cdots & L(A)^{(n_3)} \\ \end{bmatrix}.$$ 

Then, $\text{Ind}(\mathcal{A}) = \max_{1 \leq i \leq n_3} \{\text{Ind}(L(A)^{(i)})\}$.

**Proof:** Notice that

$$\text{mat}(\mathcal{A}) = (C_{n_3}^{-1} \otimes I_{n_1}) \begin{bmatrix} L(A)^{(1)} & L(A)^{(2)} & \cdots & L(A)^{(n_3)} \end{bmatrix} (C_{n_3} \otimes I_{n_1}).$$

Thus,

$$(\text{mat}(\mathcal{A}))^k = (C_{n_3}^{-1} \otimes I_{n_1}) \begin{bmatrix} L(A)^{(1)} & L(A)^{(2)} & \cdots & L(A)^{(n_3)} \end{bmatrix}^k (C_{n_3} \otimes I_{n_1})$$

and

$$\begin{bmatrix} L(A)^{(1)} & L(A)^{(2)} & \cdots & L(A)^{(n_3)} \end{bmatrix} \\text{is } \max_{1 \leq i \leq n_3} \{\text{Ind}(L(A)^{(i)})\}.$$ Therefore, \(\text{Ind} (\text{mat}(\mathcal{A})) = \text{Ind}(\mathcal{A}) = \max_{1 \leq i \leq n_3} \{\text{Ind}(L(A)^{(i)})\}. \)

Next, we will give the definition of the Drazin inverse of a tensor. Before that we note that $\mathcal{A}^k = \underbrace{\mathcal{A} \ast_e \cdots \ast_e \mathcal{A}}_k$.

**Definition 4.2** Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ and $\text{Ind}(\mathcal{A}) = k$. Then, the tensor $\mathcal{X} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ satisfying

$$\mathcal{A}^{k+1} \ast_e \mathcal{X} = \mathcal{A}^k, \quad \mathcal{X} \ast_e \mathcal{A} \ast_e \mathcal{X} = \mathcal{X}, \quad \mathcal{A} \ast_e \mathcal{X} = \mathcal{X} \ast_e \mathcal{A},$$

is called the **Drazin inverse** of the tensor $\mathcal{A}$ and is denoted by $\mathcal{A}^D$. Especially, when $k=1$, $\mathcal{X}$ is called the **Group inverse** of the tensor $\mathcal{A}$ and is denoted by $\mathcal{A}^\#$.

**Lemma 4.2** Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ and

$$\text{DCT}(\text{mat}(\mathcal{A})) = \begin{bmatrix} L(A)^{(1)} & L(A)^{(2)} & \cdots & L(A)^{(n_3)} \end{bmatrix}.$$ 

If $\text{Ind}(\mathcal{A}) = k$, then the Drazin inverse of $\mathcal{A}$ exists and is unique.
PROOF: Since \( \text{Ind}(A) = k \), one has that the matrices \( L(A)^{(1)}, \ldots, L(A)^{(n_3)} \) are Drazin invertible. Let \( X_i = (L(A)^{(i)})^D, \ i = 1, 2, \ldots, n_3. \) Then,

\[
X = \text{ten}(\text{IDCT}([X_1 \ \cdots \ X_{n_3}]))
\]
satisfies the three equations of (3). It is trivial to see that \( X \) is the Drazin inverse of \( A \).

Suppose both tensors \( X \) and \( \gamma \) are the solutions of (3). Let

\[
\text{DCT(mat}(X)) = \begin{bmatrix}
L(X)^{(1)} & L(X)^{(2)} & \cdots & L(X)^{(n_3)}
\end{bmatrix}
\]

and

\[
\text{DCT(mat}(A)) = \begin{bmatrix}
L(\gamma)^{(1)} & L(\gamma)^{(2)} & \cdots & L(\gamma)^{(n_3)}
\end{bmatrix}
\]

It follows \( L(X)^{(i)} = (L(A)^{(i)})^D \) and \( L(\gamma)^{(i)} = (L(A)^{(i)})^D, \ i = 1, 2, \ldots, n_3. \) Therefore, \( X \) and \( \gamma \) coincide since \( L(X)^{(i)} \) and \( L(\gamma)^{(i)} \) are the same.

Theorem 4.1 Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) and \( \text{Ind}(A) = k. \) Then,

\[
A^D = A^k \ast_c (A^{2k+1})^1 \ast_c A^k.
\]

In particular,

\[
A^D = A^k \ast_c (A^{2k+1})^\dagger \ast_c A^k.
\]

PROOF: By the definition of the Drazin inverse, one has

\[
A^k = A^{k+1} \ast_c A^D = A^{k+2} \ast_c (A^D)^2 = \ldots = A^{2k} \ast_c (A^D)^k = A^{2k+1} \ast_c (A^D)^{k+1}.
\]

Let \( \mathcal{X} = A^k \ast_c (A^{2k+1})^1 \ast_c A^k. \) Therefore, we have

\[
A^{k+1} \ast_c \mathcal{X} = A^{k+1} \ast_c A^k \ast_c (A^{2k+1})^1 \ast_c A^k = A^{2k+1} \ast_c (A^{2k+1})^1 \ast_c A^{2k+1} \ast_c (A^D)^{k+1}
\]

\[
= A^{2k+1} \ast_c (A^D)^{k+1} = A^k,
\]

\[
\mathcal{X} \ast_c A \ast_c \mathcal{X} = A^{k} \ast_c (A^{2k+1})^1 \ast_c A^k \ast_c A \ast_c A^k \ast_c (A^{2k+1})^1 \ast_c A^k
\]

\[
= A^{k} \ast_c (A^{2k+1})^1 \ast_c A^k = \mathcal{X}.
\]

Moreover,

\[
A \ast_c \mathcal{X} = A \ast_c A^{k} \ast_c (A^{2k+1})^1 \ast_c A^k = A \ast_c A^{2k} \ast_c (A^D)^k \ast_c (A^{2k+1})^1 \ast_c A^{2k+1} \ast_c (A^D)^{k+1}
\]

\[
= (A^D)^k \ast_c A^{2k+1} \ast_c (A^{2k+1})^1 \ast_c A^{2k+1} \ast_c (A^D)^{k+1} = (A^D)^k \ast_c A^{2k+1} \ast_c (A^D)^{k+1},
\]

and

\[
\mathcal{X} \ast_c A = A^k \ast_c (A^{2k+1})^1 \ast_c A^{k+1} = A^{2k+1} \ast_c (A^D)^{k+1} \ast_c (A^{2k+1})^1 \ast_c A \ast_c A^{2k} \ast_c (A^D)^k
\]

\[
= (A^D)^{k+1} \ast_c A^{2k+1} \ast_c (A^{2k+1})^1 \ast_c A^{2k+1} \ast_c (A^D)^k = (A^D)^{k+1} \ast_c A^{2k+1} \ast_c (A^D)^k.
\]

which implies \( A \ast_c \mathcal{X} = \mathcal{X} \ast_c A. \) Thus, we obtain \( A^D = A^k \ast_c (A^{2k+1})^1 \ast_c A^k. \) By taking \((A^{2k+1})^\dagger\) for \((A^{2k+1})^1\), we have \( A^D = A^k \ast_c (A^{2k+1})^\dagger \ast_c A^k. \)
Theorem 4.2 Let $A \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ and $\text{Ind}(A) = k$. Suppose $A^k$ has the C-QDR decomposition $A^k = Q \ast_c D \ast_c R$. Then,

$$A^D = Q \ast_c (R \ast_c A \ast_c Q)^{-1} \ast_c R.$$ 

PROOF: Let

$$\text{DCT(mat}(A)) = \begin{bmatrix}
L(A)^{(1)} & L(A)^{(2)} & \cdots & L(A)^{(n_3)}
\end{bmatrix}.$$ 

Since $A^k = Q \ast_c D \ast_c R$ is the C-QDR decomposition, we conclude that $(L(A)^{(i)})^k = Q_i D_i R_i$, $Q_i \in \mathbb{C}^{n_1 \times r}$, $D_i \in \mathbb{C}_r^{r \times r}$, $R_i \in \mathbb{C}_r^{r \times n_2}$, $i = 1, 2, \ldots, n_3$ are the QDR decomposition of $(L(A)^{(i)})^k$. Notice that $(L(A)^{(i)})^k = (Q_i D_i R_i) = Q_i (D_i R_i)$ are full rank decomposition of $(L(A)^{(i)})^k$. By [23] Theorem 2.1, we have $R_i L(A)^{(i)} Q_i D_i$ and $D_i R_i L(A)^{(i)} Q_i$, $i = 1, 2, \ldots, n_3$ are invertible. So are $R \ast_c A \ast_c Q \ast_c D$ and $D \ast_c R \ast_c A \ast_c Q$. Hence, we have $R \ast_c A \ast_c Q$ is invertible.

On the other hand, by [24], we conclude that

$$(Q_i D_i R_i L(A)^{(i)} Q_i D_i R_i)^{(i)} = (D_i R_i)^{(i)} (R_i L(A)^{(i)} Q_i)^{-1} (Q_i D_i)^{(i)}$$

due to $R_i L(A)^{(i)} Q_i$ are invertible, $D_i R_i$ are full row rank and $Q_i D_i$ are full column rank. Therefore,

$$\text{Ind}(A) = k \ast_c (Q \ast_c D \ast_c R) = (D \ast_c R) \ast_c (R \ast_c A \ast_c Q)^{-1} \ast_c (Q \ast_c D)^{(i)}.$$

By Theorem 4.1 we have

$$A^D = A^k \ast_c (A^{2k+1}) \ast_c A^k = A^k \ast_c (A^k \ast_c A \ast_c A^k)^{(i)} \ast_c A^k$$

$$= Q \ast_c D \ast_c R \ast_c (Q \ast_c D \ast_c R \ast_c A \ast_c Q \ast_c D \ast_c R)^{(i)} \ast_c Q \ast_c D \ast_c R$$

$$= Q \ast_c D \ast_c R \ast_c (D \ast_c R \ast_c (Q \ast_c A \ast_c Q)^{-1} \ast_c (Q \ast_c D)^{(i)} \ast_c Q \ast_c D \ast_c R$$

$$= Q \ast_c (R \ast_c A \ast_c Q)^{-1} \ast_c R.$$ 

In the following, we will establish another expression of the Drazin inverse by using the core-nilpotent decomposition of the tensors.

Definition 4.3 Let $A \in \mathbb{C}^{n_1 \times n_1 \times n_3}$. Then,

$$C_A = A^2 \ast_c A^D$$

is called the core part of the tensor $A$.

Lemma 4.3 Let $A \in \mathbb{C}^{n_1 \times n_1 \times n_3}$, $\text{Ind}(A) = k$ and $C_A \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ is the core part of the tensor $A$. Define $N_A = A - C_A$. Then,

$$N_A^k = 0 \text{ and } \text{Ind}(N_A) = k.$$ 

PROOF: When $\text{Ind}(A) = 0$, we have $A$ is invertible. Then, $N_A = 0$ and $\text{Ind}(N_A) = 0$.

When $\text{Ind}(A) \geq 1$,

$$N_A^k = (A - A^2 \ast_c A^D)^k = A^k \ast_c (J - A \ast_c A^D)^k = A^k \ast_c (J - A \ast_c A^D) = A^k - A^k = 0.$$ 

On the other hand, $N_A^{l+1} \ast_c A^D \neq 0$ for $l < k$. Hence, we have $\text{Ind}(N_A) = k$. The $N_A$ we defined is call the nilpotent part of the tensor $A$.

Definition 4.4 Let $A \in \mathbb{C}^{n_1 \times n_1 \times n_3}$, $C_A$ be the core of $A$ and $N_A = A - C_A$. Then,

$$A = C_A + N_A$$ 

is called the core-nilpotent decomposition of the tensor $A$.

Theorem 4.3 [22] Let $A \in \mathbb{C}^{n_1 \times n_1}$, $\text{Ind}(A) = k$, and $A = C_A + N_A$ is the core-nilpotent decomposition of $A$. Then, there exists an invertible matrices $P \in \mathbb{C}^{n_1 \times n_1}$ such that

$$A = P \begin{bmatrix} C & O \\ O & N \end{bmatrix} P^{-1},$$

where $C_A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}$, $N_A = P \begin{bmatrix} O & O \\ O & N \end{bmatrix} P^{-1}$, $C \in \mathbb{C}^{r \times r}$, $N \in \mathbb{C}^{(n_1 - r) \times (n_1 - r)}$. Besides,

$$A^D = P \begin{bmatrix} C^{-1} & O \\ O & O \end{bmatrix} P^{-1}.$$ 

.
Theorem 4.4 Let $A \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ and $\text{Ind}(A) = k$. Then
\[ A = \mathcal{P} \ast_c \Phi \ast_c \mathcal{P}^{-1}, \]
(10)
where $\mathcal{P} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ is an invertible tensor,
\[ \Phi = \text{ten}(\text{IDCT}(\left[ \begin{array}{ccc} C_1 & O & \vdots \\ O & N_1 & \vdots \\ \vdots & \vdots & \vdots \\ C_{n_3} & O & N_{n_3} \end{array} \right])), \]
\[ \left[ \begin{array}{ccc} C_i & O \\ O & N_i \end{array} \right] = \mathcal{P}^{-1} (C_{A_i} + N_{A_i}) \mathcal{P}, \]
where $C_{A_i}$ and $N_{A_i}$ are the core and nilpotent part of $L(A)^{(i)}$, $i = 1, 2, ..., n_3$, respectively. Furthermore, if $\text{rank}(C_i) = r$, $i = 1, 2, ..., n_3$, then
\[ A = \mathcal{P} \ast_c \left[ \begin{array}{ccc} C & O \\ O & N \end{array} \right] \ast_c \mathcal{P}^{-1}. \]

Besides,
\[ A^D = \mathcal{P} \ast_c \left[ \begin{array}{ccc} C^{-1} & O \\ O & O \end{array} \right] \ast_c \mathcal{P}^{-1}. \]

Proof: Suppose
\[ \text{DCT(mat}(A)) = \left[ \begin{array}{ccc} L(A)^{(1)} & L(A)^{(2)} \\ & \ddots & \\ & & L(A)^{(n_3)} \end{array} \right]. \]
Then, by using Theorem 4.3, we have
\[ \left[ \begin{array}{ccc} L(A)^{(1)} \\ & \ddots & \\ & & L(A)^{(n_3)} \end{array} \right] = \mathcal{P}_1 \left[ \begin{array}{ccc} C_1 & O \\ O & N_1 \end{array} \right] \mathcal{P}_1^{-1} \]
\[ = \mathcal{P}_1 \mathcal{P}_n \left[ \begin{array}{ccc} C_{n_3} & O \\ O & N_{n_3} \end{array} \right] \mathcal{P}_n^{-1} \]
\[ = \mathcal{P}_1 \mathcal{P}_n \left[ \begin{array}{ccc} L(A)^{(1)} \\ & \ddots & \\ & & L(A)^{(n_3)} \end{array} \right] \mathcal{P}_n^{-1} \].

Executing $\text{ten(IDCT)}(\cdot)$ on the tensors of the both sides of the equation, we have
\[ A = \mathcal{P} \ast_c \Phi \ast_c \mathcal{P}^{-1}, \]
where
\[ \Phi = \text{ten(IDCT}(\left[ \begin{array}{ccc} C_1 & O & \vdots \\ O & N_1 & \vdots \\ \vdots & \vdots & \vdots \\ C_{n_3} & O & N_{n_3} \end{array} \right])). \]
Again by using Theorem 4.3, we have
\[ \left[ \begin{array}{ccc} (L(A)^{(1)})^D \\ & \ddots & \\ & & (L(A)^{(n_3)})^D \end{array} \right] \]
\[
\begin{align*}
(P_1 \begin{bmatrix} C_1 & O \\ O & N_1 \end{bmatrix} P_1^{-1})^D &= \begin{bmatrix} (P_n \begin{bmatrix} C_n & O \\ O & N_n \end{bmatrix} P_n^{-1})^D \\
\end{bmatrix} \\
(P_1 \begin{bmatrix} C_1 & O \\ O & N_1 \end{bmatrix} P_1^{-1}) &= \begin{bmatrix} C_1 & O \\ O & N_1 \end{bmatrix} \begin{bmatrix} (P_n \begin{bmatrix} C_n & O \\ O & N_n \end{bmatrix} P_n^{-1})^D \\
\end{bmatrix}
\end{align*}
\]

Executing \text{ten(IDCT)}(\cdot) on the tensors of both sides of the equation, we have

\[
A^D = \mathcal{P} *_c \Phi^D *_c \mathcal{P}^{-1},
\]

where

\[
\Phi^D = \text{ten(IDCT)}( \begin{bmatrix} C^{-1}_1 & O \\ O & O \\ & \ddots \\ & & C^{-1}_n & O \\ O & O \end{bmatrix} )
\]

When \(\text{rank}(C_1) = \text{rank}(C_2) = \cdots = \text{rank}(C_n) = r\), one has

\[
\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} = \text{ten(IDCT)}( \begin{bmatrix} C_1 & O \\ O & N_1 \\
\end{bmatrix} \cdots \begin{bmatrix} C_n & O \\ O & N_n \end{bmatrix} ), \text{ where } C \in \mathbb{C}^{r \times r \times n}, N \in \mathbb{C}^{(n_1-r) \times (n_1-r) \times n_3}.
\]

Hence,

\[
A = \mathcal{P} *_c \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} *_c \mathcal{P}^{-1}.
\]

Since

\[
A^D = \mathcal{P} * \text{ten(IDCT)}( \begin{bmatrix} C^{-1}_1 & O \\ O & O \\ & \ddots \\ & & C^{-1}_n & O \\ O & O \end{bmatrix} ) * \mathcal{P}^{-1},
\]

it is trivial to see

\[
A^D = \mathcal{P} *_c \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{P}^{-1} \end{bmatrix} *_c \mathcal{P}^{-1}.
\]

\[\square\]

**Theorem 4.5** Let \(\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}\). Suppose \(\mathcal{A}\) has the C-HS decomposition. Then,

\[
A^D = \mathcal{U} *_c \begin{bmatrix} (S_r *_c \mathcal{K})^D \\ 0 \end{bmatrix} + ((S_r *_c \mathcal{K})^D *_c S_r *_c \mathcal{L}) *_c \mathcal{U}^H.
\]

**Proof:** Let

\[
\mathcal{A} = \mathcal{U} *_c \begin{bmatrix} S_r *_c \mathcal{K} & S_r *_c \mathcal{L} \\ 0 & 0 \end{bmatrix} *_c \mathcal{U}^H,
\]

\[
= \mathcal{U} *_c \begin{bmatrix} (S_r *_c \mathcal{K})^D \\ 0 \end{bmatrix} + ((S_r *_c \mathcal{K})^D *_c S_r *_c \mathcal{L}) *_c \mathcal{U}^H.
\]
4.2 The algorithm for computing the Drazin inverse of a tensor

In the following, we construct an algorithm to compute the Drazin inverse of a tensor based on Theorem 4.1.

**Algorithm 4.1: Compute the Drazin inverse of a tensor \( A \)**

**Input:** \( n_1 \times n_2 \times n_3 \) tensor \( A \)

**Output:** \( n_2 \times n_1 \times n_3 \) tensor \( \tilde{X} \)

1. \( \hat{A} = L(A) = A \times_3 M \), where \( M \) is defined in \( \mathbb{2} \)
2. \( k = \max \{ \text{Ind}(\hat{A}^{(i)}) \} \)
3. \( \hat{B} = L(A^k) = A^k \times_3 M \), \( \hat{C} = L(A^{2k+1}) = A^{2k+1} \times_3 M \)
4. for \( i = 1, \ldots, n_3 \)
   \( \hat{\mathcal{C}}^{(i)} = \text{pinv}(\hat{\mathcal{C}}^{(i)}) \); where \( \text{pinv}(\hat{\mathcal{C}}^{(i)}) \) is the Moore-Penrose inverse of \( \hat{\mathcal{C}}^{(i)} \)
end
5. for \( i = 1, \ldots, n_3 \)
   \( \hat{\mathcal{C}}^{(i)} = \hat{B}^{(i)} \hat{\mathcal{C}}^{(i)} \hat{B}^{(i)} \)
end
6. \( \tilde{X} = L^{-1}(\hat{X}) = \hat{X} \times_3 M^{-1} \)
Example 4.1 Let $A \in \mathbb{C}^{3 \times 3}$ with frontal slices

$$
A^{(1)} = \begin{bmatrix}
2 & 0 & 0 \\
1 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad A^{(2)} = \begin{bmatrix}
1 & 3 & 3 \\
0 & 4 & 5 \\
3 & 0 & 0
\end{bmatrix}, \quad A^{(3)} = \begin{bmatrix}
3 & 2 & 0 \\
0 & 1 & 3 \\
2 & 0 & 1
\end{bmatrix}.
$$

Then, by using Algorithm 4.1, we have

$$(A^D)^{(1)} = \begin{bmatrix}
0.0007 & 0.0123 & -0.1008 \\
-0.1030 & 0.0358 & 0.0223 \\
-0.0036 & -0.0617 & 0.0042
\end{bmatrix}, \quad (A^D)^{(2)} = \begin{bmatrix}
0.2056 & -0.0473 & 0.6283 \\
0.0145 & 0.0637 & -0.1531 \\
0.1721 & 0.0365 & 0.0585
\end{bmatrix},$$

and

$$(A^D)^{(3)} = \begin{bmatrix}
-0.1937 & 0.0317 & -0.5392 \\
0.1115 & -0.1005 & 0.0693 \\
-0.2316 & 0.0415 & -0.0040
\end{bmatrix}.$$  

5 The inverse along a tensor under the C-product

In this section, we firstly define the inverse along a tensor under the C-product and then give some representations of this inverse. Moreover, an algorithm is built to compute the inverse along a tensor.

5.1 The expressions of the inverse along a tensor

Definition 5.1 Let $A \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ and $G \in \mathbb{C}^{n_2 \times n_1 \times n_3}$. If there exist tensors $X \in \mathbb{C}^{n_2 \times n_1 \times n_3}$, $U \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ and $V \in \mathbb{C}^{n_2 \times n_2 \times n_3}$ such that

$$X \ast_c A \ast_c G = G, \quad G \ast_c A \ast_c X = G, \quad X = G \ast_c U, \quad X^H = G^H \ast_c V,$$  

then $X$ is called the inverse along $G$ and is denoted by $A^{|G|}$.

Theorem 5.1 Let $A \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ and $G \in \mathbb{C}^{n_2 \times n_1 \times n_3}$. If $A$ is invertible along $G$, then the inverse of $A$ along $G$ is unique.

**Proof:** Let $X_1, X_2 \in \mathbb{C}^{n_2 \times n_1 \times n_3}$ be two inverses of $A$ along $G$. There exist tensors $U_1, U_2, V_1, V_2$ of adequate size such that

$$X_i \ast_c A \ast_c G = G, \quad G \ast_c A \ast_c X_i = G, \quad X_i = G \ast_c U_i, \quad X_i^H = G^H \ast_c V_i,$$

for $i = 1, 2$. Now we have

$$X_1 = G \ast_c U_1 = X_2 \ast_c A \ast_c G \ast_c U_1 = X_2 \ast_c A \ast_c X_1 = V_2^H \ast_c G \ast_c A \ast_c X_1 = V_2^H \ast_c G = X_2,$$

The proof is finished. \[\square\]

Theorem 5.2 Let $A \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, $G \in \mathbb{C}^{n_2 \times n_1 \times n_3}$. If $A$ is invertible along $G$, then

$$A^{|G|} = G \ast_c (G \ast_c A \ast_c G)^\dagger \ast_c G.$$

**Proof:** Suppose

$$\text{DCT(mat}(A)) = \begin{bmatrix}
L(A)^{(1)} \\
\cdot \cdot \\
L(A)^{(n_3)}
\end{bmatrix}$$

and

$$\text{DCT(mat}(G)) = \begin{bmatrix}
L(G)^{(1)} \\
\cdot \cdot \\
L(G)^{(n_3)}
\end{bmatrix}.$$  

Let $A_i = L(A)^{(i)}$ and $G_i = L(G)^{(i)}$. By [17], we have

$$A_i^{|G_i|} = G_i(\mathfrak{C}_i A_i \mathfrak{C}_i)^\dagger \mathfrak{C}_i, \quad i = 1, 2, \ldots, n_3.$$
Then,

\[
\text{DCT}(\text{mat}(A^{|G|})) = \begin{bmatrix}
\overline{A}_1 | \overline{G}_1 \\
\vdots \\
\overline{A}_{n_3} | \overline{G}_{n_3}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\overline{G}_1(A_1 \overline{A}_1 \overline{G}_1)^\dagger \overline{G}_1 \\
\vdots \\
\overline{G}_{n_3}(A_{n_3} \overline{A}_{n_3} \overline{G}_{n_3})^\dagger \overline{G}_{n_3}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\overline{G}_1 \\
\vdots \\
\overline{G}_{n_3}
\end{bmatrix}
\begin{bmatrix}
(A_1 \overline{A}_1)^\dagger \\
\vdots \\
(A_{n_3} \overline{A}_{n_3})^\dagger
\end{bmatrix}
\begin{bmatrix}
\overline{G}_1 \\
\vdots \\
\overline{G}_{n_3}
\end{bmatrix}.
\]

Therefore, implementing \text{ten}(\text{IDCT})(\cdot) on both sides of the equation above, we get \(A^{\|G\|} = \mathcal{G} \ast_c (\mathcal{G} \ast_c A \ast_c \mathcal{G})^\dagger \ast_c \mathcal{G}. \quad \square
\]

**Theorem 5.3** Let \(A \in \mathbb{C}^{n_1 \times n_2 \times n_3}, \mathcal{G} \in \mathbb{C}^{n_2 \times n_1 \times n_3}. \) Suppose \(\mathcal{G} = M \ast_c N\) is the \(C\)-full rank decomposition of \(\mathcal{G}. \) If \(A\) is invertible along \(\mathcal{G},\) then

\[
A^{\|G\|} = M \ast_c (N \ast_c A \ast_c M)^{-1} \ast_c N.
\]

**Proof:** Let

\[
\text{DCT}(\text{mat}(A)) = \begin{bmatrix}
L(A)^{(1)} \\
\vdots \\
L(A)^{(n_3)}
\end{bmatrix}
\]

and

\[
\text{DCT}(\text{mat}(\mathcal{G})) = \begin{bmatrix}
L(\mathcal{G})^{(1)} \\
\vdots \\
L(\mathcal{G})^{(n_3)}
\end{bmatrix}.
\]

On the other hand,

\[
\text{DCT}(\text{mat}(M \ast_c N)) = \begin{bmatrix}
L(M)^{(1)}L(N)^{(1)} \\
\vdots \\
L(M)^{(n_3)}L(N)^{(n_3)}
\end{bmatrix}
\]

Let \(\overline{A}_i = L(A)^{(i)}, \overline{G}_i = L(\mathcal{G})^{(i)}, \overline{M}_i = L(M)^{(i)}, \overline{N}_i = L(N)^{(i)}, i = 1, 2, \ldots, n_3. \) Thus, we have \(\overline{G}_i = \overline{M}_i \overline{N}_i, i = 1, 2, \ldots, n_3,\) which are the full rank decomposition of \(\overline{G}_i. \) By [127], we have

\[
\overline{A}_i \overline{G}_i = \overline{M}_i (\overline{N}_i \overline{A}_i \overline{M}_i)^{-1} \overline{N}_i, \quad i = 1, 2, \ldots, n_3.
\]

Therefore,

\[
\text{DCT}(\text{mat}(A^{\|G\|})) = \begin{bmatrix}
\overline{A}_1 \overline{G}_1 \\
\vdots \\
\overline{A}_{n_3} \overline{G}_{n_3}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\overline{M}_1 (\overline{N}_1 \overline{A}_1 \overline{M}_1)^{-1} \overline{N}_1 \\
\vdots \\
\overline{M}_{n_3} (\overline{N}_{n_3} \overline{A}_{n_3} \overline{M}_{n_3})^{-1} \overline{N}_{n_3}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\overline{M}_1 \\
\vdots \\
\overline{M}_{n_3}
\end{bmatrix}
\begin{bmatrix}
(\overline{N}_1 \overline{A}_1 \overline{M}_1)^{-1} \\
\vdots \\
(\overline{N}_{n_3} \overline{A}_{n_3} \overline{M}_{n_3})^{-1}
\end{bmatrix}
\begin{bmatrix}
\overline{N}_1 \\
\vdots \\
\overline{N}_{n_3}
\end{bmatrix}.
\]

Performing \text{ten}(\text{IDCT})(\cdot) on both sides of the equation above, one has \(A^{\|G\|} = M \ast_c (N \ast_c A \ast_c M)^{-1} \ast_c N. \quad \square
\]
Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) and \( G \in \mathbb{C}^{n_2 \times n_1 \times n_3} \) be the C-SVD of \( G \). Suppose that \( \text{rank}(\mathcal{S}(i)) = r_i, \ i = 1, 2, \ldots, n_3 \). If \( A \) is represented as

\[
A = \mathcal{V} \ast_c \text{ten}(\text{DCT}(\begin{bmatrix}
X_1 & \ast & \ast \\
& \ddots & \ast \\
& & \ast
\end{bmatrix}))) \ast_c \mathcal{U}^H,
\]

then \( A \| \mathcal{G} \) exists if and only if \( X_i, \ i = 1, 2, \ldots, n_3 \), are nonsingular. In particular, if \( \text{rank}(\mathcal{S}(i)) = r, \ i = 1, 2, \ldots, n_3 \), then

\[
A \| \mathcal{G} = \mathcal{U} \ast_c \begin{bmatrix}
X^{-1} & \ast & \ast \\
& \ddots & \ast \\
& & \ast
\end{bmatrix} \ast_c \mathcal{V}^H.
\]

**Proof:** Let

\[
\text{DCT(mat}(A) = \begin{bmatrix}
L(A)^{(1)} & \ast & \ast \\
& \ddots & \ast \\
& & L(A)^{(n_3)}
\end{bmatrix}
\]

and

\[
\text{DCT(mat}(G) = \begin{bmatrix}
L(G)^{(1)} & \ast & \ast \\
& \ddots & \ast \\
& & L(G)^{(n_3)}
\end{bmatrix}.
\]

Denote \( \overline{A}_i = L(A)^{(i)} \) and \( \overline{G}_i = L(G)^{(i)} \). Thus,

\[
\text{DCT(mat}(A \| \mathcal{G}) = \begin{bmatrix}
\overline{A}_1 \| \overline{G}_1 \\
& \ddots & \ast \\
& & \overline{A}_{n_3} \| \overline{G}_{n_3}
\end{bmatrix}.
\]

So, \( A \| \mathcal{G} \) exists if and only if \( \overline{A}_i \| \overline{G}_i \) exists, \( i = 1, 2, \ldots, n_3 \). Since \( G = \mathcal{U} \ast_c \mathcal{S} \ast_c \mathcal{V}^H \) is the C-SVD of \( G \), we have

\[
\text{DCT(mat}(G) = \begin{bmatrix}
L(G)^{(1)} & \ast & \ast \\
& \ddots & \ast \\
& & L(G)^{(n_3)}
\end{bmatrix} \]

\[
= \begin{bmatrix}
\overline{G}_1 \\
& \ddots \\
& & \overline{G}_{n_3}
\end{bmatrix} = \text{DCT(mat}(\mathcal{U}) \text{DCT(mat}(\mathcal{G}) \text{DCT(mat}(\mathcal{V}^H))
\]

\[
= \begin{bmatrix}
L(\mathcal{U})^{(1)} & \ast & \ast \\
& \ddots & \ast \\
& & L(\mathcal{U})^{(n_3)}
\end{bmatrix} \begin{bmatrix}
L(\mathcal{G})^{(1)} & \ast & \ast \\
& \ddots & \ast \\
& & L(\mathcal{G})^{(n_3)}
\end{bmatrix} \begin{bmatrix}
L(\mathcal{V})^{(1)} & \ast & \ast \\
& \ddots & \ast \\
& & L(\mathcal{V})^{(n_3)}
\end{bmatrix}^H.
\]

Let \( \mathcal{U}_i = L(\mathcal{U})^{(i)}, \ \mathcal{S}_i = L(\mathcal{G})^{(i)} \) and \( \mathcal{V}_i = L(\mathcal{V})^{(i)} \). Hence,

\[
\overline{G}_i = \mathcal{U}_i \begin{bmatrix}
\mathcal{S}_i & \mathcal{O} \\
\mathcal{O} & \mathcal{O}
\end{bmatrix} \mathcal{V}_i^H \] are the SVD of \( \overline{G}_i \), where \( \mathcal{S}_i \in \mathbb{C}^{r_i \times r_i}, \ i = 1, 2, \ldots, n_3 \).

Suppose

\[
\overline{A}_i = \mathcal{V}_i \begin{bmatrix}
X_i & \ast & \ast \\
& \ddots & \ast \\
& & \ast
\end{bmatrix} \mathcal{U}_i^H, \] where \( X_i \in \mathbb{C}^{r_i \times r_i}, i = 1, 2, \ldots, n_3 \).
By \cite{17}, $\overline{A}_i|^{|G|}$ exist if and only if $X_i$, $i = 1, 2, ..., n_3$, are nonsingular. In this case,

$$\overline{A}_i|^{|G|} = U_i \begin{bmatrix} X_i^{-1} & 0 \\ 0 & 0 \end{bmatrix} \overline{V}_i^H, \quad i = 1, 2, ..., n_3.$$ 

Thus, $A|^{|G|}$ exists if and only if $X_i$, $i = 1, 2, ..., n_3$, are nonsingular. Also, we have

$$\text{DCT(mat}(A|^{|G|}) = \begin{bmatrix} A_1|^{|G|} \\ \vdots \\ A_{n_3}|^{|G|} \end{bmatrix} = \begin{bmatrix} U_1 \begin{bmatrix} X_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \overline{V}_1^H \\ \vdots \\ U_{n_3} \begin{bmatrix} X_{n_3}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \overline{V}_{n_3}^H \end{bmatrix} = \begin{bmatrix} U_1 \begin{bmatrix} X_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \overline{V}_1^H \\ \vdots \\ U_{n_3} \begin{bmatrix} X_{n_3}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \overline{V}_{n_3}^H \end{bmatrix} \begin{bmatrix} \overline{V}_1^H \\ \vdots \\ \overline{V}_{n_3}^H \end{bmatrix}.$$ 

If $\text{rank}(\tilde{G}^{(i)}) = r$, $i = 1, 2, ..., n_3$, then one has $\text{rank}(X_1) = \text{rank}(X_2) = \cdots = \text{rank}(X_{n_3}) = r$. Implementing $\text{ten(IDCT)}(\cdot)$ on both sides of the equation above, we have

$$A|^{|G|} = U \ast_c \begin{bmatrix} \overline{X}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \ast_c \overline{V}^H.$$ 

$\square$

### 5.2 The algorithm for computing the inverse along a tensor

In the following, we establish an algorithm to compute the inverse along a tensor by using Theorem \cite{54}.

**Algorithm 5.1: Compute the inverse of $A$ along a tensor $G$**

**Input:** $n_1 \times n_2 \times n_3$ tensor $A$ and $n_2 \times n_1 \times n_3$ tensor $G$

**Output:** $n_2 \times n_1 \times n_3$ tensor $\widehat{X}$

1. $\widehat{A} = L(A) = A \times_3 M$, $\widehat{G} = L(G) = G \times_3 M$, where $M$ is defined in \cite{2}

2. for $i = 1, \ldots, n_3$

   svd($\tilde{G}^{(i)}$) = $U_i \Sigma_i V_i^H$;

   rank($\Sigma_i$) = $r_i$;

   $V_i^H \tilde{A}^{(i)} U_i = \begin{bmatrix} X_i & \ast & \ast \\ \ast & \ast & \ast \end{bmatrix}$, where $X_i \in \mathbb{C}^{r_i \times r_i}$;

   If $X_i$ is nonsingular, $\tilde{W}^{(i)} = \begin{bmatrix} X_i^{-1} & 0 \\ 0 & 0 \end{bmatrix}$;

   $\widehat{G}^{(i)} = U_i \tilde{W}^{(i)} V_i^H$;

   $i = i + 1$;

   else Output: $A$ is not invertible along $G$.

end

3. $\widehat{X} = L^{-1}(\widehat{G}) = \widehat{G} \times_3 M^{-1}$

end

**Example 5.1** Let $A, G \in \mathbb{C}^{3 \times 3 \times 3}$ with frontal slices

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & 0 & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & 0 & 3 \\ 5 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^{(3)} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 3 \end{bmatrix},$$
which implies that the group inverse of $L$ (but not necessarily in one step). See [27] for details.

Let $\Omega$ be a set whose elements are states in which every state of $\Omega$ is accessible from every other state of $\Omega$, but some state outside $\Omega$ is accessible from each state of $\Omega$.

A Markov chain is ergodic if the transition tensor of the chain is irreducible, or equivalently, the states of this chain form a single ergodic set. An ergodic chain is regular if its transition tensor $A$ has the following property: exists a natural number $k$ such that $A^k > 0$.

A state is absorbing if the chain enters in this state, it can never be left. A chain is an absorbing chain if it has at least one absorbing state and, in addition, from every state of this chain it is possible to enter in an absorbing state (but not necessarily in one step). See [27] for details.

Theorem 6.1. If $\mathcal{P} \in \mathbb{R}^{n \times n \times n}$ is any transition tensor and $A = \mathcal{J} - \mathcal{P}$, then $A^#$ exists.

Proof: Let

$$\text{DCT(mat}(\mathcal{P})) = \begin{bmatrix} L(\mathcal{P})(1) & \cdots & L(\mathcal{P})(n) \end{bmatrix}.$$ 

Since $A = \mathcal{J} - \mathcal{P}$, then

$$\text{DCT(mat}(A)) = \begin{bmatrix} L(A)(1) & \cdots & L(A)(n) \end{bmatrix} = \begin{bmatrix} L(\mathcal{J})(1) - L(\mathcal{P})(1) & \cdots & L(\mathcal{J})(n) - L(\mathcal{P})(n) \end{bmatrix}.$$ 

Hence, we have $L(A)(i) = L(\mathcal{J})(i) - L(\mathcal{P})(i)$ for $i = 1, 2, \ldots, n$. By using [27] Theorem 8.2.1, one has $\text{Ind}(L(A)(i)) = 1$, which implies that the group inverse of $L(A)(i)$ exists. Then the group inverse of $A$ exists.

6 Applications to higher-order Markov Chains

Let $\mathcal{P} \in \mathbb{R}^{n \times n \times n}$ be a tensor and

$$\text{DCT(mat}(\mathcal{P})) = \begin{bmatrix} L(\mathcal{P})(1) & \cdots & L(\mathcal{P})(n) \end{bmatrix}.$$ 

A higher order Markov chain is an extension of a finite Markov chain, in which the stochastic process $X_0, X_1, \ldots$ with values in $\{1, 2, \ldots, n\}$, has the transition probabilities

$$0 \leq \mathcal{P}_{i_1i_2i_3} = \text{Prob}(X_t = i_1 | X_{t-1} = i_2, X_{t-2} = i_3) \leq 1,$$

where $\sum_{i=1}^{n} L(\mathcal{P})_{i_kj_k} = 1$, $i = 1, \ldots, n$, $1 \leq k \leq n$. We call the tensor $\mathcal{P}$ a transition tensor.

Let $F$ be a subset of $\mathbb{R}$ and let $\{X_t : t \in F\}$ be a set of random variables. If $F$ is countable and if the range of each $X_t$ is the same finite set, then the chain is said to be a finite Markov chain. Let us denote $\{G_1, \ldots, G_m\}$ the range of any $X_t$.

It is useful to have in mind that $X_k$ is the outcome of the chain on the $k$th step. The probability of $X_k$ being in state $G_j$ provided that $X_{k-1}$ was in state $G_i$ is $L(\mathcal{P})_{i_kj_k}(s) = \text{Prob}(X_t = G_k | X_{t-1} = G_j)$, $i = 1, \ldots, n$. These probabilities are said to be the one-step transition probabilities. If each of the one-step transition probabilities does not depend on $s$ (does not depend on time), i.e., $L(\mathcal{P})(j_k) = \mathcal{P}_{jki}$, for any $s = 1, 2, \ldots, i = 1, \ldots, n$, then we say that the chain is homogeneous.

In the sequel, we will focus our attention to finite homogeneous Markov chains and will simply write ‘Markov chain’ or ‘chain’ to denote a finite homogeneous Markov chain.

An ergodic set $\Omega$ is a set of states in which every state of $\Omega$ is accessible from any other state of $\Omega$ and, in addition, no state outside $\Omega$ is accessible from any state of $\Omega$.

A transient set $\Omega$ is a set whose elements are states in which every state of $\Omega$ is accessible from every other state of $\Omega$, but some state outside $\Omega$ is accessible from each state of $\Omega$.

A Markov chain is ergodic if the transition tensor of the chain is irreducible, or equivalently, the states of this chain form a single ergodic set. An ergodic chain is regular if its transition tensor $A$ has the following property: exists a natural number $k$ such that $A^k > 0$.

A state is absorbing if the chain enters in this state, it can never be left. A chain is an absorbing chain if it has at least one absorbing state and, in addition, from every state of this chain it is possible to enter in an absorbing state (but not necessarily in one step). See [27] for details.
\textbf{Theorem 6.2} Let $\mathcal{P} \in \mathbb{R}^{n \times n \times n}$ be the transition tensor of a chain and let $A = I - \mathcal{P}$. Then

\[
\begin{align*}
J - A \ast_c A^\# &= \begin{cases}
\lim_{n \to \infty} \frac{J + (\alpha J + (1 - \alpha)\mathcal{P})^n}{n}, & \text{for every transition tensor } \mathcal{P} \\
\lim_{n \to \infty} \frac{J + (\alpha J + (1 - \alpha)\mathcal{P})^n}{n}, & \text{for every transition tensor } \mathcal{P} \text{ and } 0 < \alpha < 1 \\
\lim_{n \to \infty} \mathcal{P}^n, & \text{for every regular chain} \\
\lim_{n \to \infty} \mathcal{P}^n, & \text{for every absorbing chain.}
\end{cases}
\end{align*}
\]

\textbf{Proof:} Let

\[
\text{DCT(mat}(A)) = \begin{bmatrix}
L(A)^{(1)} \\
\vdots \\
L(A)^{(n)}
\end{bmatrix}.
\]

Since $A = I - \mathcal{P}$, by Theorem 6.1, the group inverse of $A$ exists. Hence,

\[
\text{DCT(mat}(A^\#)) = \begin{bmatrix}
(L(A)^{(1)})^\# \\
\vdots \\
(L(A)^{(n)})^\#
\end{bmatrix}.
\]

Now, it is easy to see

\[
\text{DCT(mat}(J - A \ast_c A^\#)) = \begin{bmatrix}
L(J)^{(1)} - (L(A)^{(1)})(L(A)^{(1)})^\# \\
\vdots \\
L(J)^{(n)} - (L(A)^{(n)})(L(A)^{(n)})^\#
\end{bmatrix}.
\]

Denote $L(J) = I_i$, $L(A) = A_i$ and $L(P) = P_i$. By using [27, Theorem 8.2.2], we have

\[
I_i - A_i A_i^\# = \begin{cases}
\lim_{n \to \infty} \frac{I_i + (\alpha I_i + (1 - \alpha)P_i)^n}{n}, & \text{for every transition matrix } P_i \\
\lim_{n \to \infty} \frac{I_i + (\alpha I_i + (1 - \alpha)P_i)^n}{n}, & \text{for every transition matrix } P_i \text{ and } 0 < \alpha < 1 \\
\lim_{n \to \infty} P_i^n, & \text{for every regular chain} \\
\lim_{n \to \infty} P_i^n, & \text{for every absorbing chain, } i = 1, \ldots, n,
\end{cases}
\]

which implies that

\[
J - A \ast_c A^\# = \begin{cases}
\lim_{n \to \infty} \frac{J + (\alpha J + (1 - \alpha)\mathcal{P})^n}{n}, & \text{for every transition tensor } \mathcal{P} \\
\lim_{n \to \infty} \frac{J + (\alpha J + (1 - \alpha)\mathcal{P})^n}{n}, & \text{for every transition tensor } \mathcal{P} \text{ and } 0 < \alpha < 1 \\
\lim_{n \to \infty} \mathcal{P}^n, & \text{for every regular chain} \\
\lim_{n \to \infty} \mathcal{P}^n, & \text{for every absorbing chain.}
\end{cases}
\]

\[\square\]

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