High Temperature Response Functions
and the Non-Abelian Kubo Formula*

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Abstract

We describe the relationship between time-ordered and retarded response functions in a plasma. We obtain an expression, including the proper $i\epsilon$-prescription, for the induced current due to hard thermal loops in a non-Abelian theory, thus giving the non-Abelian generalization of the Kubo formula. The result is closely related to the eikonal for a Chern-Simons theory and is relevant for a gauge-invariant description of Landau damping in the quark-gluon plasma at high temperature.

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1 Introduction

The behavior of electromagnetic fields in a plasma of charged particles is described by the polarization tensor $\Pi^{\mu\nu}(x, y)$, which is the two-point current correlation function; perturbatively this is a one-charged-particle-loop diagram with two external photon lines. The real part of this tensor describes phenomena such as Debye screening and propagation of plasma waves; the imaginary part describes the damping of fields in the plasma (Landau damping)\[1\]. If we integrate out the charged fields in a functional integral for the theory, the polarization tensor naturally emerges as the thermal average of the time-ordered product of two currents. However, there are situations where the response of the plasma to the electromagnetic field is described as the average of the retarded commutator of currents. For the real part of the response function, the retarded commutator and the time-ordered product coincide. We calculate the imaginary part for both and discuss their relationship in terms of spectral representations\[2\].

In a non-Abelian plasma, such as the one arising perhaps from quarks and gluons, the response of the plasma to fields is more complicated. At high temperature, a perturbative approach can be used. Nevertheless, it is necessary to go beyond the two-point function; one must include at least part of the contribution from all higher point functions for reasons of gauge invariance. There is also the need for a resummation of perturbation theory\[3\]. One must first sum up ‘hard thermal loops’ to define new effective propagators and vertices. Hard thermal loops are thermal one-loop diagrams in which the external momenta are relatively soft ($\sim gT$, where $g$ is the coupling constant and $T$ is the temperature, taken to be high) while the loop momentum is relatively hard ($\sim T$). One must reorganize perturbation theory in terms of effective vertices and propagators defined by these hard thermal loops, before integration on small values ($\lesssim gT$) of loop momenta can be carried out\[4\]. This is necessary so that all contributions to a given order in the coupling constant can be consistently included.

Recently it was shown\[5\] that the generating functional for hard thermal loops, or equivalently the effective action which describes Debye screening, plasma waves, etc., is given in terms of the eikonal for a Chern-Simons theory, which had been constructed earlier in a non-thermal context\[6\]. We now describe how this result is extended to include some of the decay and damping effects. Our discussion is still at the level of hard thermal loops; we do not address effects that arise by incorporating effective propagators and vertices in subsequent soft momentum integrations. Our result can be considered as a non-Abelian generalization of the Kubo formula\[7\].

In Section II we present the time-ordered and retarded polarizations for the Abelian plasma at high temperature. The physical significance of the imaginary part is explained in Section III. The induced current due to hard thermal loops in a non-Abelian theory is constructed in Section IV.
2 Response Functions for the Abelian Plasma

We consider QED with the fermion current \( J^\mu = \bar{\psi} \gamma^\mu \psi \) and interaction Lagrangian \( \mathcal{L}_{\text{int}} = -A_\mu J^\mu \). The equations of motion for the gauge field \( A_\mu \) have \( J^\mu \) as a source term. The calculation of the current \( J^\mu \) is thus a suitable way of studying response functions. The scattering operator \( S \) is given by \( T \exp[-i \int d^4 x A_\mu j^\mu] \) where \( j^\mu = \bar{\psi} \gamma^\mu \psi \) is the current in the interaction picture, denoted by the subscript \( I \). We can write \( J^\mu \) in terms of \( S \) as

\[
J^\mu(x) = iS^{-1} \frac{\delta S}{\delta A_\mu(x)} \tag{2.1}
\]

The equation of motion for the field \( A_\mu \) is

\[
\partial_\nu F^{\nu\mu}(x) = iS^{-1} \frac{\delta S}{\delta A_\mu(x)} \tag{2.2}
\]

[While no gauge choice is specified in (2.2), it can be fixed for example by adding \(-\frac{1}{2} (\partial \cdot A)^2\) to the Lagrange density, whereupon the left side would become \( \square A^\mu \) (Feynman gauge).] We can use (2.1) to obtain \( J^\mu \) as a power series in \( A_\mu \). In particular, a Taylor expansion of (2.1) to linear order in \( A_\mu \) gives

\[
J^\mu(x) = j^\mu(x) - i \int d^4 y \, \theta(x^0 - y^0) [j^\mu(x), j^\nu(y)] A_\nu(y) \tag{2.3}
\]

Using this in (2.2) and taking a thermal average we can write

\[
\partial_\nu F^{\nu\mu}(x) = \int d^4 y \, \Pi^{\mu\nu}_R(x, y) A_\nu(y) \tag{2.4}
\]

where

\[
\Pi^{\mu\nu}_R(x, y) = -i \theta(x^0 - y^0) \langle [j^\mu(x), j^\nu(y)] \rangle \tag{2.5}
\]

The angular brackets denote thermal averaging, with the unperturbed density matrix \( e^{-H_0/T} \), so that \( \langle j^\mu \rangle \) vanishes. (The Boltzmann constant is set to unity.) Thus the response function (2.5), \textit{viz.} the average of the retarded commutator [or equation (2.4)], is appropriate to the situation where we perturb the plasma by the field and ask how the field evolves. Equation (2.5) is the Kubo formula.

The effective action \( \Gamma(A) \), which is the sum of fermion loops, is given by

\[
e^{i \Gamma(A)} = \langle Te^{-i \int A_\mu j^\mu} \rangle \tag{2.6}
\]

The current that is here naturally defined as \( \mathcal{J}^\mu(x) = -\frac{\delta \Gamma(A)}{\delta A_\mu(x)} \) is given to linear order in \( A_\mu \) by

\[
\mathcal{J}^\mu(x) = \int d^4 y \, \Pi^{\mu\nu}_T(x, y) A_\nu(y) \tag{2.7}
\]

where

\[
\Pi^{\mu\nu}_T(x, y) = -i \langle T j^\mu(x) j^\nu(y) \rangle \tag{2.8}
\]

Clearly, the two response functions do not coincide. One difference is immediately evident: The time ordered product is symmetric in its labels \( (\mu, \nu)(x, y) \) [as it must be since \( \Pi^{\mu\nu}_T \) arises
from the generating functional $\Gamma(A) \sim 1 - \frac{1}{2} \int d^4x d^4y A_{\mu}(x) \Pi_{\mu\nu}(x,y) A_{\nu}(y) + \cdots$ while the retarded commutator does not enjoy any such symmetry.

To evaluate (2.3) and (2.8), we make use of the free-field commutator.

\[
\{ \psi(x), \tilde{\psi}(y) \} = \int \frac{d^3p}{(2\pi)^3} \frac{\gamma \cdot p}{p_0} \cos p(x-y) \tag{2.9}
\]

\[(p_0 = \sqrt{p^2}; \text{ since we are interested in the high temperature contribution, fermion masses are negligible by comparison.}) \text{ The averages needed in the computation can be evaluated using the following expression for the propagator.}

\[
S(x, y) = \langle T \psi(x) \tilde{\psi}(y) \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{\gamma \cdot p}{2p_0} \left\{ \theta(x^0 - y^0) \left( \alpha_p e^{-ip(x-y)} + \beta_p e^{ip(x-y)} \right) - \theta(y^0 - x^0) \left( \beta_p e^{-ip(x-y)} + \alpha_p e^{ip(x-y)} \right) \right\} \tag{2.10}
\]

where

\[
\alpha_p = 1 - n_p, \quad \beta_p = n_p
\]

\[
n_p = \frac{1}{e^{p_0/T} + 1} \tag{2.11}
\]

(The bar refers to antifermion distributions.) In evaluating the two-point functions, we shall use (2.9), (2.11) and carry out the time-integrations first, introducing as needed convergence factors $e^{\pm \epsilon y^0}$, $\epsilon$ small and positive. We then get energy-denominators with $\epsilon$-terms. Further simplification and extraction of the imaginary part can be easily done at this stage. We find for the polarization tensor,

\[
\Pi_{\mu\nu}(x, y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \Pi_{\mu\nu}(k) \tag{2.12a}
\]

\[
\Pi_{\mu\nu}(k) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q_0} \left[ T_{\mu\nu}(p, q) \left( \frac{\alpha_p \beta_q}{p_0 - q_0 - k_0 - i\epsilon} - \frac{\alpha_q \beta_p}{p_0 - q_0 - k_0 + i\eta} \right) + T_{\mu\nu}(p, q') \left( \frac{\alpha_p \alpha_q}{p_0 + q_0 - k_0 - i\epsilon} - \frac{\beta_p \beta_q}{p_0 + q_0 - k_0 + i\eta} \right) \right. \right.
\]

\[
+ T_{\mu\nu}(p', q) \left( \frac{\alpha_p' \alpha_q}{p_0 + q_0 + k_0 - i\eta} - \frac{\beta_p' \beta_q}{p_0 + q_0 + k_0 + i\eta} \right) \left. \right. \right. \left. \right. \]

\[
+ T_{\mu\nu}(p', q') \left( \frac{\alpha_p' \beta_q}{p_0 - q_0 + k_0 - i\epsilon} - \frac{\beta_p' \alpha_q}{p_0 - q_0 + k_0 + i\eta} \right) \tag{2.12b}
\]

where

\[
T_{\mu\nu}(p, q) \equiv \text{tr} \left( \gamma^\mu \gamma^\nu \gamma^\gamma \gamma^\gamma \gamma^\gamma \right) \cdot q \cdot p \tag{2.13}
\]

\[
p_\mu = (p^0, -\mathbf{p}), \quad q_\mu = (q^0, -\mathbf{q}), \quad p^0 = |\mathbf{p}|, \quad q^0 = |\mathbf{q}| \quad \text{and} \quad \mathbf{p} = \mathbf{q} + \mathbf{k}.
\]

Further, $\eta = \epsilon$ for $T$ products, i.e. for $\Pi_T^{\mu\nu}$, and $\eta = -\epsilon$ for the retarded function $\Pi_R^{\mu\nu}$. Notice that the retarded function can be obtained by continuing the real part by the rule
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$k_0 \to k_0 + i\epsilon$. Formula (2.12b) agrees with Ref. [3] where time-ordered products were used. (We have changed the sign of $k$ in our definition of the Fourier transform relative to Ref. [3].)

The real part of $\Pi^{\mu\nu}(k)$ is the same for both the time-ordered and the retarded functions and has long been familiar [1]. Here we concentrate on the imaginary part which is obtained from (2.12b) with the help of

$$\frac{1}{z - i\epsilon} = P\frac{1}{z} + i\pi\delta(z)$$

(2.14)

Consider first the contribution due to the $T^{\mu\nu}(p, q')$ and $T^{\mu\nu}(p', q)$ terms in (2.12b). The imaginary parts carry the $\delta$-functions, $\delta(p_0 + q_0 \pm k_0)$. It is easily seen that these are subdominant at high temperature, for $k$ small compared to $T$. [Their contributions are $O(T)$ or less.] The dominant contributions to the imaginary part of $\Pi^{\mu\nu}$ come from the $T^{\mu\nu}(p, q)$, $T^{\mu\nu}(p', q')$ terms. For $\Pi^{\mu\nu}_R$, we find from (2.12b)

$$\text{Im } \Pi^{\mu\nu}_R = \pi \int \frac{d^3q}{(2\pi)^3} \frac{1}{2p_0} \frac{1}{2q_0} \left[ T^{\mu\nu}(p, q) (n_q - n_p) \delta(p_0 - q_0 - k_0) 
- T^{\mu\nu}(p', q') (\bar{n}_q - \bar{n}_p) \delta(p_0 - q_0 + k_0) \right]$$

(2.15)

$$+ T^{\mu\nu}(p', q') (\bar{n}_q - \bar{n}_p) \delta(p_0 - q_0 + k_0)$$

(2.16)

For $\Pi^{\mu\nu}_T$, we find

$$\text{Im } \Pi^{\mu\nu}_T = \text{Im } \Pi^{\mu\nu}_R + 2\pi \int \frac{d^3q}{(2\pi)^3} \frac{1}{2p_0} \frac{1}{2q_0} \left[ T^{\mu\nu}(p, q)n_p(1 - n_q)\delta(p_0 - q_0 - k_0) 
+ T^{\mu\nu}(p', q')\bar{n}_q(1 - \bar{n}_p)\delta(p_0 - q_0 + k_0) \right]$$

(2.17)

In the high temperature limit, $|k|$ is small compared to $|p|, |q|$, and there are simplifications [3].

$$T^{\mu\nu}(p, q) \simeq 8q_0^2 Q^\mu Q^\nu$$

$$T^{\mu\nu}(p', q') \simeq 8q_0^2 Q'^\mu Q'^\nu$$

$$p_0 - q_0 - k_0 \simeq -k \cdot Q$$

$$p_0 - q_0 + k_0 \simeq k \cdot Q'$$

(2.18)

with $Q^\mu, Q'^\mu$ being the light-like four-vectors

$$Q^\mu = (1, \hat{q}) , \quad Q'^\mu = (1, -\hat{q})$$

(2.19)

Equation (2.16) simplifies as

$$\text{Im } \Pi^{\mu\nu}_R \simeq \frac{k_0 P^{\mu\nu}}{2\pi^2} \int_0^\infty dq \ q (n_q + \bar{n}_q)$$

$$= \frac{k_0 T^2}{12} P^{\mu\nu}$$

(2.20)

where

$$P^{\mu\nu} = \int d\Omega \ \delta(k \cdot Q) Q^\mu Q^\nu$$

(2.21)
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and we take \( n = \bar{n} \). The integration in (2.21) is over the orientations of the unit vector \( Q = \hat{q} \). Note that from its definition, \( P^{\mu\nu} \) is transverse and traceless, while the \( \delta \)-function enforces \( k \) to be space-like. Explicit evaluation gives

\[
P^{\mu\nu} = -k^2 \theta(-k^2) \frac{6\pi}{|k|^2} \left[ \frac{1}{3} P_{1}^{\mu\nu} + \frac{1}{2} P_{2}^{\mu\nu} \right] \tag{2.22}
\]

where

\[
P_{1}^{\mu\nu} = \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \tag{2.23a}
\]

and

\[
P_{2}^{\mu\nu} = 0, \quad P_{2}^{ij} = \delta^{ij} - \frac{k^i k^j}{|k|^2} \tag{2.23b}
\]

Simplifying (2.16) similarly, we find

\[
\text{Im} \, \Pi_{T}^{\mu\nu} \simeq \text{Im} \, \Pi_{R}^{\mu\nu} + \frac{P^{\mu\nu}}{\pi^2} \int_0^\infty dq \, q^2 n_q (1 - n_q) = \text{Im} \, \Pi_{R}^{\mu\nu} + \frac{\pi}{6} P^{\mu\nu} \tag{2.24}
\]

This relationship between \( \Pi_{T}^{\mu\nu} \) and \( \Pi_{R}^{\mu\nu} \) can be understood in the following way. \( \Pi_{R}^{\mu\nu} \), being retarded, obeys a spectral representation of the form [2]

\[
\Pi_{R}^{\mu\nu} = \Pi_{\text{sub}}^{\mu\nu} + \int dk' \frac{\rho^{\mu\nu}(k', k)}{k' - k - i\epsilon} \tag{2.25}
\]

for some spectral function \( \rho^{\mu\nu}(k) \). \( \Pi_{\text{sub}}^{\mu\nu} \) is a ‘subtraction term’ that can arise in the real part of \( \Pi_{R}^{\mu\nu} \). For \( \Pi_{T}^{\mu\nu} \), we then have [2]

\[
\Pi_{T}^{\mu\nu} = \Pi_{\text{sub}}^{\mu\nu} + \int dk' \frac{\rho^{\mu\nu}(k', k)}{k' - k - i\epsilon} + \frac{2\pi i f(k_0) \rho^{\mu\nu}(k_0, k)}{k_0^2 - k_0 - i\epsilon} \tag{2.26}
\]

where

\[
f(k_0) = \frac{1}{e^{k_0/T} - 1} \tag{2.27}
\]

The bosonic distribution function \( f(k_0) \) appears because \( \Pi_{T}^{\mu\nu} \) is ultimately part of the bosonic (i.e. photon) propagator, and also because it is given by the thermal average of the \( T \)-product of two bosonic operators: the two currents \( j^\mu \) and \( j^\nu \). From (2.25) and (2.26) we find

\[
\text{Im} \, \Pi_{R}^{\mu\nu} = \pi \rho^{\mu\nu}(k) \quad \text{Im} \, \Pi_{T}^{\mu\nu} = \text{Im} \, \Pi_{R}^{\mu\nu} + 2\pi f(k_0) \rho^{\mu\nu}(k) = \pi \coth \frac{k_0}{2T} \rho^{\mu\nu}(k) \tag{2.28}
\]
The essence of our results (2.20) and (2.24) is that the high-temperature spectral function is

\[
\rho^{\mu\nu}(k) \simeq \frac{k_0 T^2}{12\pi} P^{\mu\nu} \tag{2.29}
\]

and the difference in the high-temperature behavior between \( \text{Im} \, \Pi^{\mu\nu}_R \) and \( \text{Im} \, \Pi^{\mu\nu}_T \) \([\mathcal{O}(T^2) \text{ vs. } \mathcal{O}(T^3)]\) is attributed to the presence in the latter of \(2\pi f(k_0)\rho^{\mu\nu}\), which according to (2.27) and (2.29) tends to \(T^3/6 P^{\mu\nu} \).

The spectral function (2.29) also determines the high-temperature behavior for the (common) real part of \( \Pi^{\mu\nu}_R \) and \( \Pi^{\mu\nu}_T \), apart from possible subtraction terms. Inserting (2.29) into the dispersion formula (2.25) or (2.26) shows that the well-known high-\( T \) asymptote for \( \Pi^{\mu\nu} \) of (2.12b)

\[
- \Re \Pi^{\mu\nu} \simeq \frac{T^2}{6} P^{\mu\nu}_2 + \frac{T^2 k^2}{|k|^2} \left[ 1 + \frac{k_0}{2|k|} \ln \left| \frac{k_0 - |k|}{k_0 + |k|} \right| \right] \left[ \frac{1}{3} P^{\mu\nu}_1 + \frac{1}{2} P^{\mu\nu}_2 \right] \tag{2.30}
\]

is reproduced with an appropriate subtraction term.

Our result (2.20), (2.30) for the retarded function \( \Pi^{\mu\nu}_R \) agrees with various previous calculations [1], [8]. It is noteworthy that these early calculations in the Soviet literature, based on the Boltzmann and Vlasov equations of kinetic theory, are here regained in quantum field theory at one-loop order.

Another correlation function that is frequently considered is the imaginary time one. It too is given by a dispersive integral [3].

\[
\Pi^{\mu\nu}_\text{imaginary} = \Pi^{\mu\nu}_\text{sub} + \int dk^{\prime} \frac{\rho^{\mu\nu}(k^{\prime}, k)}{k^{\prime}_0 - \omega_n} \tag{2.31}
\]

Because the “external energy” \( \omega_n \) is temperature dependent in imaginary time, it makes sense to speak of high temperature behavior only for the \( n = 0 \) mode, (effectively reducing dimensionality to three) where the spectral function enforces an \( \mathcal{O}(T^2) \) large-\( T \) behavior.

The foregoing discussion emphasizes that finite-temperature field theory can be described by different correlation functions, with differing large-\( T \) behavior. In particular the retarded commutator is what is relevant for the operator equations of motion, Landau damping, etc. Because of the asymmetry in its time-arguments, the retarded commutator cannot be obtained by varying an effective action, which necessarily involves the symmetric, time-ordered products. Correspondingly, only the retarded commutators behave as \( T^2 \) for large \( T \), while the time ordered products and the real-time effective action have a more complicated large-\( T \) behavior. Finally, the imaginary time correlations again behave as \( T^2 \) for large \( T \), provided the “external energies” are restricted to the \( n = 0 \) mode, or are continued away from the temperature-dependent values \( 2\pi inT \). Such analytic continuation from the imaginary time expression will produce the retarded functions in real time, and not the time-ordered ones, which possess non-analytic \([\sim f(k_0)\rho^{\mu\nu}]\) contributions.

In spite of this variety, a universally true statement can be made about the high-temperature asymptote of the spectral function \( \rho^{\mu\nu} \), which determines quantities of physical interest through the various dispersive representations. In the present context, such a result is given in Eq. (2.29).
3 Physical Significance of $\text{Im } \Pi_R^{\mu\nu}$

The imaginary part of $\Pi_R^{\mu\nu}$ describes Landau damping, which can occur for fields with space-like momenta [8]. Explicitly we consider

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}A_\mu(k)$$

(3.1a)

with

$$A_\mu(k) = \delta(k_0 - \omega(k))A_\mu(k) + \delta(k_0 + \omega(k))A_\mu^\dagger(k)$$

(3.1b)

and $\omega^2 < k^2$. The amplitudes $A_\mu(k)$, $A_\mu^\dagger(k)$, respectively of the positive and negative frequency terms, correspond to absorption and emission processes. The decay of the field in the plasma arises from absorption by fermions and antifermions. The amplitude for absorption by fermions is given, to the lowest order in coupling constant, by

$$A = i\bar{u}_p(\gamma \cdot A)u_q(2\pi)^4\delta^{(4)}(p - q - k)\sqrt{n_q(1 - n_p)}$$

(3.2)

where $u_p, u_q$ are wave functions for outgoing and incoming fermions, respectively. The factor $\sqrt{n_q(1 - n_p)}$ arises since the initial fermion is chosen from a state of occupation number $n_q$ and the final fermion is scattered into a state of occupation number $n_p$. Single quantum absorption as in (3.2) is kinematically allowed for spacelike momenta; it is in fact the inverse of Čerenkov radiation. One can also have creation of the mode $(\omega, k)$ by (Čerenkov) radiation from fermions, given by a formula like (3.2) with $p \leftrightarrow q$ and $A \leftrightarrow A^\dagger$. There are similar contributions from antifermions. For the net absorption probability per unit spacetime volume, denoted by $\gamma$, we then find, with summation over all fermion states,

$$\gamma = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2p_0} \frac{1}{2q_0} A_\mu^\dagger(k) \left[ T^{\mu\nu}(p, q) [n_q(1 - n_p) - n_p(1 - n_q)] 2\pi\delta(p_0 - q_0 - k_0) \right.$$

$$+ T^{\mu\nu}(p', q') [\bar{n}_p(1 - \bar{n}_q) - \bar{n}_q(1 - \bar{n}_p)] 2\pi\delta(p_0 - q_0 + k_0) \left.] A_\nu(k \right)$$

(3.3)

where $p_0 = |p|$, $q_0 = |q|$. Comparing this with (2.12) or (2.16) we see that

$$\gamma = 2A_\mu^\dagger(k) \left[ \text{Im } \Pi_R^{\mu\nu}(k_0) \right] A_\nu(k)$$

(3.4)

We can parametrize the field $A_\mu$ as

$$A = \frac{k k_0}{k^2} A_0 + A_T$$

(3.5)

where $\mathbf{k} \cdot A_T = 0$. $A_T$ gives rise to the transverse electric field and the magnetic field; $\phi = \left(1 - \frac{k_0^2}{k^2}\right) A_0$ gives the longitudinal component of the electric field via $E_L = -\nabla \phi$. In terms of the parametrization (3.5) we can write

$$\gamma = T^2 \frac{\pi \omega}{|\mathbf{k}|} \left[ \frac{1}{3} \phi^\dagger \phi + \frac{1}{6} \left(1 - \frac{\omega^2}{k^2}\right) A_T^\dagger \cdot A_T \right]$$

(3.6)

$\gamma$ is positive, as expected for net absorption or damping.
4 Induced Current in a Non-Abelian Plasma

We now consider the non-Abelian plasma and the induced current due to the hard thermal loops.

As far as the two-point function is concerned, there is no significant difference between the Abelian case of electrodynamics and QCD. For the latter, the contribution of $N_F$ flavors of quarks at high temperature is

$$\text{Im } \Pi_{\mu\nu,ab}^R \simeq \delta^{ab} \frac{N_F T^2}{2} P^{\mu\nu}$$  \hspace{1cm} (4.1)

The real part of the high-T two-point function is given by

$$\text{Re } \Pi_{\mu\nu,ab}^R \simeq \delta^{ab} \frac{N_F T^2}{24\pi} \left[ 4\pi g^{\mu_0} g^{\nu_0} - \int d\Omega \frac{k_0 Q^\mu Q^\nu}{k \cdot Q} \right]$$  \hspace{1cm} (4.2)

The singularity when $k \cdot Q$ vanishes is defined as a principal value, and then the integral reproduces (2.30), apart from the group factors.

Notice that (4.1) and, with appropriate modifications, (2.20) are obtained by continuing (4.2) by $k_0 \rightarrow k_0 + i\epsilon$. We expect this to be true in general, including the gluon contributions.

Thus we expect

$$\text{Im } \Pi_{\mu\nu,ab}^R \simeq \delta^{ab} \left( 2N + \frac{N_F}{2} \right) \frac{k_0 T^2}{12} P^{\mu\nu}$$  \hspace{1cm} (4.3)

for $SU(N)$ gauge theory with $N_F$ flavors of fermions in the fundamental representation.

We now consider the higher point functions in QCD. The evolution of fields in the plasma is still described by an equation similar to (2.2), viz. in the Feynman gauge

$$\Box A^\mu_a(x) = iS^{-1} \frac{\delta S}{\delta A^\mu_a(x)}$$  \hspace{1cm} (4.4)

This is still the operator equation of motion, reexpressed using the scattering operator $S$. The operator $S$ is now more complicated; correspondingly the current defined by (4.4) is a more involved expression, including ghost and gluon terms. Nevertheless, when we expand (4.4) in powers of $A^\mu_a$, we get only retarded terms, since $S^{-1} \frac{\delta S}{\delta A^\mu_a(x)}$ does not involve fields in the future of $A^\mu_a(x)$. In fact, writing $U(x^0, y^0) = T \exp(i \int_{y^0}^{x^0} d^4x L_{\text{int}})$, $S = U(\infty, -\infty)$, we find

$$J^\mu_a(x) = iS^{-1} \frac{\delta S}{\delta A^\mu_a(x)} = U(-\infty, x^0) j^\mu_a(x) U(x^0, -\infty)$$  \hspace{1cm} (4.5)

where $\left. \frac{\delta S}{\delta A^\mu_a(x)} \right|_f = j^\mu_a(x)$. When we differentiate (4.4) with respect to the potentials, we get products of currents, and ‘contact’ terms like $\delta j^\mu_a(x) / \delta A^\mu_b(x)$. Such terms eventually give rise to expressions like $f^{abc} A^\mu_b F^\nu_{\mu} c$ needed to write (4.4) with a gauge covariant derivative $D_{\nu} F^\nu_{\mu}$, along with ghost and gauge-fixing terms. | Strictly speaking, we need suitable $T^*$-products in the definition of $S$ to make the contact terms together with the commutator terms come
out covariant [7]. Our conclusion will not be affected by this needed qualification, since we use only the retardation property.] Ignoring contact terms for the moment, we find
\[
\frac{\delta^n J^\mu_a(x)}{\delta A_{\nu_1}^b(y_1) \cdots \delta A_{\nu_n}^b(y_n)} = (-i)^n \sum_p \theta(x^0 - y_{1}^0) \theta(y_{1}^0 - y_{2}^0) \cdots \theta(y_{n-1}^0 - y_{n}^0)
\times \cdots \left[ \cdots \left[ j^\mu_a(x), j^\nu_{b_1}(y_1) \right], j^\nu_{b_2}(y_2) \right] \cdots
\]
where the summation is over all permutations of \((\nu_1 \, b_1 \, y_1, \, \nu_2 \, b_2 \, y_2, \ldots)\). The right hand side of (4.6) is the definition of the multiple retarded commutator. Using this, we arrive at
\[
J^\mu_a(x) = j^\mu_a(x) + \sum_{n=1}^{\infty} (-i)^n \int d^4y_1 \cdots d^4y_n \theta(x^0 - y_1^0) \theta(y_1^0 - y_2^0) \cdots \theta(y_{n-1}^0 - y_{n}^0)
\times \cdots \left[ \cdots \left[ j^\mu_a(x), j^\nu_{b_1}(y_1) \right], j^\nu_{b_2}(y_2) \right] \cdots A_{\nu_1}^b(y_1) \ldots A_{\nu_n}^b(y_n)
\]
We must take thermal averages on the right hand side of (4.4). If we write
\[
A_{\nu_1}^b(y_1) = \int \frac{d^4k_1}{(2\pi)^4} e^{-ik_1y_1} A_{\nu_1}^b(y_1), \text{ etc.},
\]
the retardation property of (4.7) indicates that the integration over the time-components of \(y_i\)'s requires \(k_i^0 \to k_i^0 + i\epsilon\) for convergence. This will generally be true, since it follows from the retardation property and does not depend specifically on the expansion (4.7). We can thus make a rule for the \(n\)-point functions: Calculate the thermal average of the current, first ignoring the \(i\epsilon\)'s in the propagators or energy-denominators. Then for each momentum \(k_i\) at each of the \(y_i\)'s (but not at the unintegrated point \(x\)), make the replacement \(k_i^0 \to k_i^0 + i\epsilon\). The averaged equation of motion (4.4)
\[
\Box A^a_\mu(x) = \langle J^\mu_a(x) \rangle
\]
gives the evolution of fields in the plasma, \(\langle J^\mu_a(x) \rangle\) being calculated as above.

We can now apply this argument to the induced current due to hard thermal loops in QCD. The hard thermal loop contribution, found in Ref. [3], is determined in the following fashion. We begin with the functional \(I(A)\) defined on Euclidean 2-space [complex coordinates \(z\) and \(\bar{z}\)] and depending on the Lie-algebra valued vector potential \(A\), with a single (spatial) component [10].
\[
I(A) = i \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \int \frac{d^2z_1}{\pi} \cdots \frac{d^2z_n}{\pi} \text{ tr } A(z_1, \bar{z}_1) \cdots A(z_n, \bar{z}_n)
\]
\[
\bar{z}_{ij} \equiv \bar{z}_i - \bar{z}_j
\]
This is the Chern-Simons eikonal, \(i.e.\) it is the exponent in a WKB wave functional. In a Chern-Simons gauge theory on \((2+1)\) dimensional space-time, the Schrödinger-picture state, which is defined on 2-space (at fixed time) and satisfies the Chern-Simons Gauss law, has the form \(e^{iI(A)}\), with \(A\) being one of the two spatial gauge potentials of the theory and the other being conjugate to \(A\). Owing to the simple symplectic structure of this theory, the eikonal/WKB approximation is exact, and \(I(A)\) coincides with the eikonal. \(I(A)\) is
also recognized as the Polyakov-Wiegman determinant for two-dimensional, single chirality fermions \[4\]. The structure \[4.10\] is relevant to high-T field theory owing to the fact that its gauge transformation properties are closely related to those of the generating functional for hard thermal loops \[5\], and the latter is completely determined by its response to gauge transformations.

To present the thermal generating functional \[4.10\] must be elaborated upon and continued from the Euclidean space \((z, \bar{z})\) to Minkowski space-time. This is accomplished by replacing \(A\) in \[4.10\] with \(A_\perp \equiv \frac{1}{2} A \cdot Q\), while \(z\) is replaced by \(x \cdot Q', \bar{z} \equiv x \cdot Q\). Also all the \(A\)'s \(\rightarrow \frac{1}{2} A \cdot Q\) depend on four variables: \(z \rightarrow x \cdot Q', \bar{z} \rightarrow x \cdot Q\) and \(x_\perp\), which is orthogonal to \(Q\) and \(Q'\). All the \(A\)'s carry the same value of \(x_\perp\), which therefore is a fixed parameter in the functional \[4.10\], and is subjected to a single integration: \(I(A) \rightarrow \int d^2x_\perp I(A) \equiv I'(A)\)

The hard thermal loop generating functional is now given by

\[
\Gamma = -\left(N + \frac{N_F}{2}\right) \frac{T^2}{6\pi} \int d\Omega \left[ \int d^4x \, \text{tr} \, A_+ A_- + i\pi I(A_+) + i\pi I'(A_-) \right]
\]

where \(I'(A_-)\) is obtained from \(I(A_+)\) by interchanging \(Q\) and \(Q'\), and \(A_- \equiv \frac{1}{2} A \cdot Q'\).

The integrals in Minkowski space-time are now singular: when transformed to momentum space they contain denominators involving \(k \cdot Q\) and \(k \cdot Q'\), which can vanish. If these zeroes are ignored and the integrals are evaluated formally, one regains the hard thermal loop contributions, but they do not have the appropriate analyticity properties, and cannot be interpreted as time-ordered products, retarded products, etc.

Here we give the prescription which results in an evaluation of the retarded current, relevant to the non-Abelian gauge theory.

The contribution of \(I(A_+)\) in \[4.10\] to the current is given in Euclidean space by

\[
-\frac{\delta I(\frac{1}{2} A \cdot Q)}{\delta A_\mu^a(x)} = -\left(N + \frac{N_F}{2}\right) \frac{T^2}{6\pi} \int d\Omega \sum_{n=1}^{\infty} (-1)^{n+1} \int \frac{d^2z_1}{\pi} \cdots \frac{d^2z_n}{\pi} \, \frac{\text{tr} \left[ \left( \frac{T_a Q^\mu}{2} \right) A_+(x_1) \cdots A_+(x_n) \right]}{(z_1 - \bar{z}) (z_2 - \bar{z}_1) \cdots (z_n - \bar{z}_{n-1}) (\bar{z}_n - \bar{z})}
\]

where \(T_a\) is the anti-Hermitian group generator. Using \[4.8\] and

\[
\frac{1}{\bar{z}} = \frac{2\pi}{i} \int \frac{d^2p}{(2\pi)^2} \frac{e^{ipx}}{\bar{p}}
\]

we can write \[4.12\] in [Euclidean] momentum space. The term with \(n\) potentials is given by

\[
-\frac{\delta I_n(k)}{\delta A_\mu^a} = \left(N + \frac{N_F}{2}\right) \frac{T^2}{6\pi} \frac{(2i)^{n+1}}{4} \, \text{tr} \left[ \left( \frac{T_a Q^\mu}{2} \right) A_+(k_1) \cdots A_+(k_n) \right] F(k_1, \ldots, k_n)
\]

where

\[
F = -4\pi \int \frac{d^2p}{(2\pi)^2} \frac{1}{(p + \bar{q}_0)(\bar{p} + \bar{q}_1) \cdots (\bar{p} + \bar{q}_n)}
\]

\[
\bar{q}_0 = 0, \quad \bar{q}_i = \sum_{j=1}^{i} \bar{k}_j
\]

Using the identities

\[
\int \frac{d^2p}{(2\pi)^2} \frac{\partial}{\partial \bar{p}} \left[ p \Pi_i \frac{1}{(\bar{p} + \bar{q}_i)} \right] = 0
\]
\[
\frac{\partial}{\partial p} \left( \frac{1}{\bar{p} + q} \right) = \pi \delta^{(2)}(p + q) \tag{4.16b}
\]

we can evaluate \( F \) as

\[
F(k_1, k_2, \ldots, k_n) = \sum_{i=0}^{n} \frac{-q_i}{(\bar{q}_0 - \bar{q}_i)(\bar{q}_1 - \bar{q}_i) \cdots (\bar{q}_{i-1} - \bar{q}_i)(\bar{q}_{i+1} - \bar{q}_i) \cdots (\bar{q}_n - \bar{q}_i)} \tag{4.17}
\]

We can now continue to Minkowski space by \( \bar{k} \rightarrow k \cdot Q, k \rightarrow k \cdot Q' \). Since the \( A \)'s have \( e^{-ikx} \) factors, the retardation condition requires \( k_0^0 \rightarrow k_0^0 + i\epsilon_j \). We thus define \( F \) in Minkowski space as the expression (4.17) with

\[
\bar{q}_i = \sum_{j=1}^{i} (k_j \cdot Q + i\epsilon_j), \quad q_i = \sum_{j=1}^{i} k_j \cdot Q'.
\tag{4.18}
\]

The expression for the current, which follows, is therefore written as [11]

\[
J^\mu_a(x) = \sum_{n=1}^{\infty} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} e^{-i(\sum k) \cdot x} J^\mu_{a,n}(k)
\tag{4.19a}
\]

\[
J^\mu_{a,n}(k) = \left( N + \frac{N_F}{2} \right) \frac{T^2}{6\pi} \int d\Omega \left[ \frac{(T_a Q^\mu}{2}) A_-(k_1) + A_+(k_1) \left( \frac{T_a Q^\mu}{2} \right) \right] \delta_{n,1} \tag{4.19b}
\]

\[
\quad + \frac{(2i)^{n+1}}{4} \left\{ \frac{T_a Q^\mu}{2} A_+(k_1) \cdots A_+(k_n) \right\} F(k_1, \ldots, k_n) + (Q \leftrightarrow Q')
\]

The equations of motion for the potential with the hard thermal loop corrections is

\[
(D_\mu F^\mu_\nu)_\alpha = J^\mu_a
\tag{4.20}
\]

with \( J^\mu_a \) from (4.19). (Evidently we have not displayed gauge-fixing and ghost terms.) Equation (4.20) describes Landau damping of fields in the quark-gluon plasma as well as Debye screening and propagation of plasma waves in a gauge covariant way. [Of course, as mentioned in the Introduction, these equations still do not include the soft momentum loop-integrations which have to be done in terms of effective vertices and propagators defined by equations (4.19), (4.20).]

The two-point function (i.e. \( n = 1 \)) as defined by (4.19) agrees with our previous considerations. The three-point function (i.e. \( n = 2 \)) as given again by (4.19) is

\[
J^\mu_{a,2}(k) = 2i \left( N + \frac{N_F}{2} \right) \frac{T^2}{6\pi} \int d\Omega \left\{ \frac{(T_a Q^\mu}{2}) A_+(k_2)A_+(k_3) \right\} \tag{4.21}
\]

\[
\times \frac{(k_2 \cdot Q k_3 \cdot Q' - k_2 \cdot Q' k_3 \cdot Q)}{(k_2 \cdot Q + i\epsilon)(k_3 \cdot Q + i\epsilon)[(k_2 + k_3) \cdot Q + i\epsilon]} + (Q \leftrightarrow Q')
\]
For quark loops we do not have to consider $T^*$-products or contact terms like $\frac{\delta j^\mu}{\delta \epsilon^a}$ since $j^{\mu,a} = i\bar{q}\gamma^\mu T^a q_I$. Equation (4.8) holds as it is written and we can calculate the 3-point function (i.e. two $A$'s) by evaluating the retarded commutators. One can verify that this does indeed give equation (4.21) with the $i\epsilon$'s as indicated.

Our emphasis has been on the operator equations of motion and on how retarded functions arise in such a description. There are of course many other real time formalisms for thermal field theory. For example, in the Keldysh approach [8] one considers time to run along a contour, which passes from $-\infty$ to $\infty$, and then folds back and returns from $\infty$ to $-\infty$. We have checked that the evolution equations for field configurations in this formalism lead to the retarded functions as in our discussion.

Prescriptions for obtaining the imaginary parts of Green’s functions in terms of ‘cutting rules’ have been discussed in Ref. [12].

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[10] The first term in the series involves a product of distributions $-\frac{1}{(\bar{z}_{12})^2}$; this is defined as $\frac{\partial}{\partial \bar{z}_{1}} \frac{1}{(\bar{z}_{12})^2}$.

[11] A summed formula for the current in Euclidean space can be given, see Ref. [6].

\[
J_+ \propto A_+ - g^{-1} \partial_+ g
\]
\[
J_- \propto A_- - h^{-1} \partial_- h
\]

Here $A_+ = h^{-1} \partial_+ h$, $A_- = g^{-1} \partial_- g$ and $\pm$ refers to the $(1) \pm i(2)$ components. However, no compact expression in Minkowski space encodes the proper $i\epsilon$ prescription.

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