SYSTEMS OF INFINITE HORIZON AND ERGODIC BSDE ARISING IN REGIME SWITCHING FORWARD PERFORMANCE PROCESSES

YING HU†, GECHUN LIANG‡, AND SHANJIAN TANG§

Abstract. We introduce and solve a new type of quadratic backward stochastic differential equation systems defined in an infinite time horizon. Such systems arise naturally as candidate solutions to characterize forward performance processes and their associated optimal trading strategies in a regime switching market. We also study the asymptotic limit of the infinite horizon BSDE system, which gives rise to a novel ergodic BSDE system. In addition, we develop a connection between the solution of the ergodic BSDE system and the long-term growth rate of classical utility maximization problems.

Key words. Infinite horizon BSDE system, ergodic BSDE system, multidimensional comparison theorem, regime switching, forward performance.

AMS subject classifications. 60H30, 91G10, 93E20

1. Introduction. This paper introduces a new class of quadratic backward stochastic differential equation (BSDE for short) systems in an infinite time horizon. The systems are motivated by our study of forward performance processes for portfolio optimization problems in a regime switching market. We show that both infinite horizon BSDE systems and their asymptotic limit, known as ergodic BSDE systems, are natural candidates for the characterization of forward performance processes and associated optimal strategies in a financial market with multiple regimes.

Let us first recall that an infinite horizon BSDE typically takes the form

\[ dY_t = -F(t, Y_t, Z_t)dt + (Z_t)^{tr}dW_t, \]

for \( t \geq 0 \), where \( F \) is called the driver of the equation, and \( W \) is a \( d \)-dimensional Brownian motion as the driving noise of the equation. Different from the classical case where the equation is defined in a finite time interval, say \([0, T]\), the infinite horizon BSDE (1.1) is defined over all time horizons and may be ill posed, even if the driver \( F \) is Lipschitz continuous in both \( Y \) and \( Z \). This problem has been solved in [7], where a strictly monotone condition on the driver is imposed. A typical driver satisfying the strictly monotone condition is

\[ F(t, Y_t, Z_t) = f(t, Z_t) - \rho Y_t, \]

for some constant \( \rho > 0 \). Then, it has been shown in [7] that (1.1) admits a unique bounded solution \((Y, Z)\) if \( f \) is Lipschitz continuous in \( Z \). The case that \( f \) has quadratic growth in \( Z \) has been further treated in [5].

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In a Markovian framework, where \( f(t, Z_t) = f(V_t, Z_t) \) with \( V \) as an underlying forward process, it has been shown in [18] and subsequently [13] that, when \( \rho \to 0 \), (1.1) converges to the ergodic BSDE

\[
    dY_t = -f(V_t, Z_t)dt + (Z_t)^{tr}dW_t,
\]

for \( t \geq 0 \). Herein, the constant \( \lambda \), as a part of the solution to (1.2), has a stochastic control interpretation as the value of an ergodic control problem. The ergodic BSDE (1.2) has also been widely used to study the long-time behavior of their finite horizon counterparts (see, for example, [22] and [12]).

Both ergodic equation (1.2) and infinite horizon equation (1.1) have been found as natural candidates for the characterization of forward performance processes and their associated optimal portfolio strategies in portfolio optimization problems. Forward performance processes were introduced and developed in [32, 33, 34, 35]. They complement the classical expected utility paradigm in which the utility is a deterministic function chosen at a single terminal time. The value function process is, in turn, constructed backwards in time, as the dynamic programming principle yields. As a result, there is limited flexibility to incorporate updating of risk preferences, rolling horizons, learning, and other realistic “forward in nature” features if one requires that time-consistency is being preserved at all times. Forward performance processes alleviate some of these shortcomings and offer the construction of a genuinely dynamic mechanism for evaluating the performance of investment strategies as the market evolves across (arbitrary) trading horizons. See also [19] [26] [36] [37] [41] and [42] for their developments and various applications.

The construction of a forward performance process is, however, difficult, due to the ill-posed nature and degeneracy of the corresponding (stochastic) partial differential equations (see [17]). This difficulty has been recently overcome in [30], which shows that forward performance processes in homothetic form can be effectively constructed via the solutions of the equations like (1.1) and (1.2). It bypasses a number of aforementioned difficulties inherited in the associated SPDE. See also [11] for a further development of this method to study forward entropic risk measures.

Our aim herein is to generalize both (1.1) and (1.2) from scalar-valued to vector-valued equations, i.e. systems of equations. The corresponding BSDE systems are motivated by the construction of forward performance processes in a regime switching market. Due to the interactions of different market regimes through a given Markovian chain, the corresponding infinite horizon BSDE system for a forward performance process is expected to take the form

\[
    dY^i_t = -f^i(V_t, Z^i_t)dt - \sum_{k \in I} q^{ik}(e^{Y^k_t - Y^i_t} - 1)dt + (Z^i_t)^{tr}dW_t,
\]

for \( t \geq 0 \) and \( i \in I := \{1, 2, \ldots, m^0\} \), where \( q^{ik} \) is the transition rate from market regime \( i \) to \( k \). The second term on the right hand side of (1.3) couples all the equations together and represents the interaction of different market regimes. A similar feature has also appeared in [24] and [8], where the authors studied classical utility maximization in a regime switching framework and derived a finite horizon BSDE system.

However, different from the finite horizon case, the infinite horizon BSDE system (1.3) is ill posed. Indeed, in a single regime case, (1.3) then reduces to a scalar-valued BSDE, and the strictly monotone condition fails. To overcome this difficulty, we modify (1.3) by adding a discount term \( \rho Y^i_t \) in the driver (see (2.1) in section 2),...
which serves the role of strict monotonicity. Although this additional discount term makes the modified BSDE system well posed, it however distorts the original problem. As a result, the solution of the modified BSDE system will no longer correspond to a forward performance process.

As a first contribution, we construct regime switching forward performance processes in homothetic form via the modified infinite horizon BSDE system (2.1), together with a nonlinear ODE system (see Theorem 4.2). The ODE system is coupled through the solution of the infinite horizon BSDE system, and is used to correct the distortion effect produced by the discount terms. To the best of our knowledge, both BSDE and ODE systems (2.1) and (4.1) are new. They are introduced for the first time for the characterization of regime switching forward performance processes. In particular, we show that when there is a single regime, our representation of forward performance processes will recover the infinite horizon BSDE representation appearing in [30] (See Remark 2).

Our second contribution is about solvability of the infinite horizon BSDE system (2.1). Since the driver \( f \) has quadratic growth in \( Z \), the standard Lipschitz estimates do not apply to our system. Instead, we first apply a truncation technique and derive a priori estimates for the solutions, and subsequently show that the truncation constants coincide with the constants appearing in the a priori estimates. For this, we make an extensive use of the multidimensional comparison theorem for BSDE systems, which was firstly developed in [23]. An essential idea herein is to use the bounded solution of an auxiliary ODE (not system!) as a universal bound to control all the solution components of the BSDE systems.

We then derive a new ergodic BSDE system (see (3.7) in section 3) as the limit system of the infinite horizon BSDE system. This ergodic BSDE system, on one hand, characterizes the regime switching forward performance processes (without any additional parameter \( \rho \)) and, on the other hand, is also a natural extension of the ergodic equation introduced in [13]. Herein, a new feature is that all the equation components have a common ergodic constant \( \lambda \) as a part of the solution. Similar to [13], we apply the perturbation technique to construct a sequence of approximate solutions to the ergodic BSDE system. However, the commonly used Girsanov’s transformation method does not imply uniqueness of \( \lambda \) due to different probability measures induced by each equation component. Instead, we use the multidimensional comparison theorem to obtain the uniqueness of the ergodic constant \( \lambda \) in (3.7).

Our final contribution is about a stochastic control representation for the ergodic constant \( \lambda \). We show that it corresponds to the long-term growth rate of a risk-sensitive optimization problem in a regime switching framework. This, in turn, connects with the long-term growth rate of a regime switching utility maximization problem. Thus, our result also unveils an intrinsic connection between forward performance processes and classical expected utilities in a market with multiple regimes.

Turning to literature about the quadratic BSDE (systems), most of the existing results are only for a finite time horizon. The scalar equation with bounded terminal data was first solved in [27] and was applied to solve utility maximization problems in [20]. See also [9], [31], and [38] for extensions. The case with unbounded terminal data is more challenging and was solved in [13], [29], and [14], with [15] and [16] further showing the uniqueness of the solution. Their applications can be found in [11] and [21]. Recently, there have been a renewed interest in the corresponding quadratic BSDE systems due to their various applications in equilibrium problems, price impact models and non-zero sum games (see, for example, [10], [21], [24], and [29] with more references.
and adapted processes (2.1). Brownian motion on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Denote by \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\) the augmented filtration generated by \(W\). Throughout this paper, we will be using the superscript \(A^\text{tr}\) to denote the transpose of matrix \(A\). Consider the infinite horizon BSDE system

\[
dY^i_t = -f^i(V_t, Z_t^i)dt - \sum_{k \in I} q^{ik}(e^{Y_t^k} - 1)dt + \rho Y_t^i dt + (Z_t^i)^{tr}dW_t,
\]

for \(t \geq 0\) and \(i \in I := \{1, 2, \ldots, m^0\}\). By a solution to (2.1), we mean a pair of adapted processes \((Y^i, Z^i)_{i \in I}\) satisfying (2.1) in an arbitrary time horizon.

To solve (2.1), we impose the following assumptions on \(f^i\).

**Assumption 1.** There exist constants \(C_v, C_z, K_1, K_2\) such that, for \(i, k \in I\) and \(v, \bar{v}, z, \bar{z} \in \mathbb{R}^d\),

- (i) \(|f^i(v, z) - f^i(\bar{v}, z)| \leq C_v (1 + |z|)|v - \bar{v}|\);
- (ii) \(|f^i(v, z) - f^i(v, \bar{z})| \leq C_z (1 + |z| + |\bar{z}|)|z - \bar{z}|\);
- (iii) \(|f^i(v, 0)| \leq K_1\);
- (iv) \(|f^i(v, z) - f^k(v, z)| \leq K_2 (1 + |z|)\).

It follows from Assumption 1(ii) that \(f^i(v, z)\) satisfies a quadratic growth condition in \(z\). Thus, we are facing a system of quadratic BSDE defined in an infinite time horizon. The system is coupled through the coefficients \(q^{ik}\), for \(i, k \in I\), which satisfy

**Assumption 2.** The square matrix \(Q = \{q^{ik}\}_{i,k \in I}\) is a transition rate matrix satisfying (i) \(\sum_{k \in I} q^{ik} = 0\); (ii) \(q^{ik} \geq 0\) for \(i \neq k\). Let \(q^{\text{max}}\) be the maximal transition rate, i.e. \(q^{\text{max}} = \max_{i,k} q^{ik}\).

The infinite horizon BSDE system (2.1) is driven by a forward process \(V\) satisfying

**Assumption 3.** The underlying \(d\)-dimensional forward process \(V\) is given by the solution of the mean-reverting SDE

\[
dV_t = \eta(V_t)dt + \kappa dW_t,
\]

where the drift coefficients \(\eta(\cdot)\) satisfy a dissipative condition, namely, there exists a constant \(C_\eta > C_v\) such that, for \(v, \bar{v} \in \mathbb{R}^d\),

\[\eta(v) - \eta(\bar{v})|v - \bar{v}^\text{tr}| \leq -C_\eta |v - \bar{v}|^2.\]

Moreover, the volatility matrix \(\kappa \in \mathbb{R}^{d \times d}\) is positive definite and normalized to \(|\kappa| = 1\).

The main result of this section is the existence and uniqueness of the solution to (2.1).

**Theorem 2.1.** Suppose that Assumptions 1-3 are satisfied. Then, there exists a unique bounded solution \((Y^i, Z^i)_{i \in I}\) to the infinite horizon BSDE system (2.1) satisfying

\[
|Y_t^i| \leq K_y := \frac{K_1}{\rho} \text{ and } |Z_t^i| \leq K_z := \frac{C_v}{C_\eta - C_v}.
\]
The rest of this section is devoted to the proof of Theorem 2.1.

2.1. Idea of the proof. To construct a solution of (2.1), we follow a truncation procedure and a stability analysis. To this end, we first define two truncation functions \( p : \mathbb{R} \to \mathbb{R} \) and \( q : \mathbb{R}^d \to \mathbb{R}^d \) by

\[
(2.4) \quad p(y) := \max\{-K_y, \min\{y, K_y\}\} \quad \text{and} \quad q(z) := \frac{\min\{|z|, K_z\}}{|z|} z 1_{\{z \neq 0\}}.
\]

We consider the truncated system of (2.1), namely,

\[
(2.5) \quad dY^i_t = -f^i(V_t, q(Z^i_t))dt - \sum_{k \leq t} q^{ik}(e^{p(V^k_t)} - p(Y^i_t))dt + \rho Y^i_t dt + (Z^i_t)^{tr}dW_t,
\]

for \( t \geq 0 \) and \( i \in I \).

It then follows from Assumption (1(i)(ii)) that the function \( f^i(\cdot, q(\cdot)) \) is Lipschitz continuous, i.e.

\[
(2.6) \quad |f^i(v, q(z)) - f^i(\bar{v}, q(\bar{z}))| \leq \frac{C_v C_y}{C_y - C_v} |v - \bar{v}|,
\]

and

\[
(2.7) \quad |f^i(v, q(z)) - f^i(v, q(\bar{z}))| \leq C_z \frac{C_v + C_y}{C_y - C_v} |z - \bar{z}|.
\]

It is also immediate to verify that \( \sum_{k \leq t} q^{ik}(e^{p(v^k_t)} - p(v^k_t)) - \rho y^i_t \) is continuous and has bounded derivatives except at finite many points. Thus, the driver of the truncated system (2.5) is Lipschitz continuous.

If, moreover, we can show that (2.6) admits a solution, say \((Y^i, Z^i)_{i \in I}\), with \(|Y^i_t| \leq K_y\) and \(|Z^i_t| \leq K_z\), then \( p(Y^i_t) = Y^i_t \) and \( q(Z^i_t) = Z^i_t \), for \( t \geq 0 \) and \( i \in I \).

Next, we construct a solution to (2.5) by an approximation procedure. For \( m \geq 1 \) and \( t \in [0, m] \), we consider the finite horizon BSDE system

\[
Y^i_t(m) = \int_t^m \left[ f^i(V_s, q(Z^i_s(m))) + \sum_{k \leq s} q^{ik}(e^{p(Y^k_s(m))} - p(Y^i_s(m)) - 1) - \rho Y^i_s(m) \right] ds
- \int_t^m (Z^i_s)^{tr}dW_s.
\]

(2.8)

For \( t \geq m \), we define \( Y^i_t(m) = Z^i_t(m) \equiv 0 \). Note that (2.8) is a standard BSDE system with Lipschitz continuous driver, so its solution is unique, which is \((Y^i(m), Z^i(m))_{i \in I}\).

We will first establish uniform bounds (independent of \( m \)) for \( Y^i(m) \) and \( Z^i(m) \) in section 2.3. Subsequently, we will show in section 2.4 that the pair of processes \((Y^i(m), Z^i(m))_{i \geq 1}\) form a Cauchy sequence in an appropriate space, whose limit then provides a solution to the infinite horizon BSDE system (2.1). On the other hand, the uniqueness of the solution relies on the multidimensional comparison theorem introduced next.
2.2. Multidimensional comparison theorem. The main tool that we use is the multidimensional comparison theorem for systems of BSDE, which was firstly established in [23]. Herein, we present a slightly different version tailor made for our later use, and provide its proof in the appendix for the reader’s convenience.

**Lemma 2.2.** Let $T > 0$ be fixed. Consider a system of BSDE($\xi^i, F^i, G^i$) with the terminal data $\xi^i$ and the driver $(F^i, G^i)$, namely,

$$Y_t^i = \xi^i + \int_0^T [F^i_s(Z_s) + G^i_s(Y_s^i, Y_{s-}^i)] \, ds - \int_0^T (Z_s^i)^{tr} \, dW_s,$$

where $Y_t^{-i} := (Y_t^1, \ldots, Y_t^{i-1}, Y_t^{i+1}, \ldots, Y_t^m)$. Let $(\bar{Y}^i, \bar{Z}^i)$ be the solution of another system of BSDE($\xi^i, F^i, G^i$) with the corresponding terminal data $\bar{\xi}^i$ and the driver $(\bar{F}^i, \bar{G}^i)$. Suppose that

- (i) both $\xi^i$ and $\bar{\xi}^i$ are square integrable and satisfying $\xi^i \leq \bar{\xi}^i$ for $i \in I$;
- (ii) there exist constants $C_f$ and $C_g$ such that, for $i \in I$ and $z, \bar{z} \in \mathbb{R}^d$, $y = (y^i, y^{-i}), \bar{y} = (\bar{y}^i, \bar{y}^{-i}) \in \mathbb{R}^{m_d}$,

$$|F^i_s(z) - F^i_s(\bar{z})| \leq C_f |z - \bar{z}|,$$

$$|G^i_s(y^i, y^{-i}) - G^i_s(\bar{y}^i, \bar{y}^{-i})| \leq C_g |y - \bar{y}|;$$

- (iii) the driver $G^i_s(y^i, y^{-i})$ is nondecreasing in all of its components other than $y^i$, i.e. it is nondecreasing in $y^k$, for $k \neq i$;
- (iv) the following inequalities hold,

$$F^i_s(\bar{Z}_s^i) \leq \bar{F}^i_s(\bar{Z}_s^i),$$

$$G^i_s(\bar{Y}_s^i, \bar{Y}_{s-}^i) \leq \bar{G}^i_s(\bar{Y}_s^i, \bar{Y}_{s-}^i).$$

Then, $Y_t^i \leq \bar{Y}_t^i$, for $t \in [0, T]$ and $i \in I$.

2.3. A priori estimates. We show that the pair of processes $(Y^i(m), Z^i(m))_{i \in I}$, as the solution to the finite horizon BSDE system (2.8), have the estimates

$$|Y_t^i(m)| \leq K_y \quad \text{and} \quad |Z_t^i(m)| \leq K_z,$$

where the constants $K_y$ and $K_z$, independent of $m$, are given in Theorem 2.1

The boundedness of $Y^i(m)$. For $z \in \mathbb{R}^d$ and $y = (y^i, y^{-i}) \in \mathbb{R}^{m_d}$, let

$$F^i_s(z) := f_s(V_s, q(z)) \quad \text{and} \quad G^i_s(y^i, y^{-i}) := \sum_{k \in I} q^{ik}(e^{p(y^k) - p(y^i)} - 1) - \rho y^i.$$

Note that both $F^i_s(z)$ and $G^i_s(y^i, y^{-i})$ are Lipschitz continuous, and $G^i_s(y^i, y^{-i})$ is nondecreasing in $y^k$ for $k \neq i$. Moreover, by Assumption 2(iii), $F^i_s(0) \leq K_1$ and $G^i_s(0, \bar{y}^{-i}) = -\rho \bar{y}^i$, where $\bar{Y}^{-i} := (\bar{Y}_s^1, \ldots, \bar{Y}_s^{i-1})$ and $\bar{Y}$ solves the ODE

$$\bar{Y}_t = \int_t^m (K_1 - \rho \bar{Y}_s) \, ds.$$

Consequently, it follows from Lemma 2.2 that $Y_t^i(m) \leq \bar{Y}_t^i \leq \frac{K_y}{\rho}$, for $t \in [0, m]$ and $i \in I$. Likewise, we also obtain that $Y_t^i(m) \geq -\frac{K_y}{\rho}$, so $|Y_t^i(m)| \leq \frac{K_y}{\rho} = K_y$. Hence, we have $p(Y_t^i(m)) \equiv Y_t^i(m)$, i.e. the truncation function $p(\cdot)$ does not play a role in BSDE system (2.8).
The boundedness of $Z^i(m)$. Given $V_0 = v \in \mathbb{R}^d$, we use the notations $V^v_t, Y^i_t(v)$ and $Z^i_t(v)$ to emphasize their dependencies on the initial data $v$. For $t \in [0, m]$ and $v, \bar{v} \in \mathbb{R}^d$, let

$$\delta Y^i_t(m) := Y^i_t(v) - Y^i_t(\bar{v}) \text{ and } \delta Z^i_t(m) := Z^i_t(v) - Z^i_t(\bar{v}).$$

It then follows from (2.8) that

$$\delta Y^i_t(m) = \int_t^m \left[ f^i(V^v_s, q(Z^i_s(v))) - f^i(V^\bar{v}_s, q(Z^i_s(\bar{v}))) \right] ds$$

$$+ \int_t^m \sum_{k \in I} \left( q^{ik}(e^{Y^k_s(m) - Y^i_s(v)} - 1) - q^{ik}(e^{Y^k_s(m) - Y^i_s(\bar{v})} - 1) \right) ds$$

$$- \int_t^m \rho \delta Y^i_s(m) ds - \int_t^m (\delta Z^i_s(m))^t dW_s$$

(2.14)

where

$$F^i_s(z) = f^i(V^v_s, q(Z^i_s(v))) - f^i(V^\bar{v}_s, q(Z^i_s(\bar{v})))$$

$$+ f^i(V^v_s, q(z + Z^i_s(v))) - f^i(V^\bar{v}_s, q(Z^i_s(\bar{v}))),$$

and

$$G^i_s(y^i, y^{-i}) = \sum_{k \in I} q^{ik}(e^{y^k_s - y^i + Y^k_s(m) - Y^i_s(m)} - e^{Y^k_s(m) - Y^i_s(m)}) - \rho y^i,$$

for $z \in \mathbb{R}^d$ and $y = (y^i, y^{-i}) \in \mathbb{R}^{m_i}$, with $|y^i| \leq 2K_v$ for $i \in I$.

Note that $F^i_s(z)$ and $G^i_s(y^i, y^{-i})$ are Lipschitz continuous. Moreover, $F^i_s(0, 0^{-i}) = 0$ and, by Assumption (3.1) and the Lipschitz estimate (2.6),

$$|F^i_s(0)| = |f^i(V^v_s, q(Z^i_s(v))) - f^i(V^\bar{v}_s, q(Z^i_s(\bar{v})))|$$

$$\leq \frac{C_v C_q}{C_{\eta} - C_v} |V^v_s - V^\bar{v}_s| \leq \frac{C_v C_q}{C_{\eta} - C_v} e^{-C_{\eta} |v - \bar{v}|},$$

where the last inequality follows from the dissipative condition in Assumption (3.1). Thus, $(\delta Y^i(m), \delta Z^i(m))_{i \in I}$ is the unique solution to (2.8). Furthermore, note that $G^i_s(y^i, y^{-i})$ is nondecreasing in $y^k$ for $k \neq i$ and $G^i_s(\bar{Y}_s, \bar{Y}^{-i}) = -\rho \bar{Y}_s$, where $\bar{Y}$ solves the ODE

$$\bar{Y}_t = \int_t^m \left( \frac{C_v C_q}{C_{\eta} - C_v} e^{-C_{\eta} |v - \bar{v}|} - \rho \bar{Y}_s \right) ds.$$

Consequently, it follows from Lemma (2.2) that

$$\delta Y^i_t(m) \leq \bar{Y}_t = \frac{C_v C_q}{C_{\eta} - C_v} e^{\rho t} \left( e^{-|v - \bar{v}|} - e^{-|v - \bar{v}|} \right)$$

$$\leq \frac{C_v}{C_{\eta} - C_v} |v - \bar{v}|,$$

(2.15)
for \( t \in [0, m] \) and \( i \in I \). Likewise, we also have

\[
\delta Y^i_t(m) \geq \frac{-C_v}{C_\eta - C_v} |v - \bar{v}|.
\]

By the Markov property of \( Y^{i,v}(m) \), there exists a measurable function \( y^i(\cdot, m) \) such that \( Y^i_t(m) = y^i(t, V^v_t; m) \). In turn, (2.16) and (2.17) yield that

\[
|\nabla y^i(t, v; m)| \leq \frac{C_v}{C_\eta - C_v} = K_z,
\]

for \( v \in \mathbb{R}^d \) and \( i \in I \). Furthermore, using the relationship \( \kappa^{tr} \nabla y^i(t, V^v_t; m) = Z^{i,v}_t(m) \) and Assumption 3 on \( \kappa \), we conclude that \( |Z^{i,v}_t(m)| \leq \frac{C_v}{C_\eta - C_v} = K_z \) and, therefore, the \( a \) priori estimates (2.13) on \( Y^{i,v}(m) \) and \( Z^{i,v}(m) \) have been proved.

2.4. Proof of Theorem 2.1. Existence. We first prove that \((Y^i(m))_{m \geq 1}\) is a Cauchy sequence. For \( m \geq n \geq 1 \) and \( t \in [0, m] \), let

\[
\delta Y^i_t(m,n) := Y^i_t(m) - Y^i_t(n) \quad \text{and} \quad \delta Z^i_t(m,n) := Z^i_t(m) - Z^i_t(n).
\]

Since we have already shown in the last section that \( |Y^i_t(m)| \leq K_y \) and \( |Z^i_t(m)| \leq K_z \), the truncation functions \( p(\cdot) \) and \( q(\cdot) \) actually do not play any role in (2.18), and we have \((p(Y^i_t(m)), q(Z^i_t(m))) = (Y^i_t(m), Z^i_t(m))\).

In turn,

\[
\begin{align*}
\delta Y^i_t(m,n) &= \int_t^m [f^i(V_s, Z^i_s(n)) - f^i(V_s, Z^i_s(n))] \, ds + \int_t^m f^i(V_s, 0) \chi(s \geq n) \, ds \\
&\quad + \int_t^m \sum_{k \in I} \left( q^{ik} (e^{Y^i_s(m)} - Y^i_s(m) - 1) - q^{ik} (e^{Y^i_s(n)} - Y^i_s(n) - 1) \right) \, ds \\
&\quad - \int_t^m \rho Y^i_s(m,n) \, ds - \int_t^m (\delta Z^i_t(m,n))^{tr} dW_s \\
&= \int_t^m \left[ F^i_s(\delta Z^i_t(m,n)) + G^i_s(\delta Y^i_s(m,n), \delta Y^i_s(m,n)) \right] \, ds \\
&\quad - \int_t^m (\delta Z^i_t(m,n))^{tr} dW_s,
\end{align*}
\]

where

\[
F^i_s(z) = f^i(V_s, z + Z^i_s(n)) - f^i(V_s, Z^i_s(n)) + f^i(V_s, 0) \chi(s \geq n),
\]

and

\[
G^i_s(y^i, y^{-i}) = \sum_{k \in I} q^{ik} \left( e^{y^i + Y^i_s(n)} - e^{Y^i_s(n)} - e^{y^i + Y^i_s(n)} - e^{Y^i_s(n)} \right) - \rho y^i,
\]

for \( z \in \mathbb{R}^d \) and \( y = (y^i, y^{-i}) \in \mathbb{R}^{m_0} \), with \( |z| \leq 2K_z \) and \( |y^i| \leq 2K_y \) for \( i \in I \).

Following along similar arguments as in section 2.3, we deduce that (2.18) is with Lipschitz continuous driver and, therefore, \((\delta Y^i_t(m,n), \delta Z^i_t(m,n))_{t \in I}\) is the unique solution to (2.18). Moreover, by Assumption 1(iii), we have \( F^i_s(0) = f^i(V_s, 0) \chi(s \geq n) \leq K_1 \chi(s \geq n) \) and \( G^i_s(\bar{Y}_s, \bar{Y}^{-i}) = -\rho \bar{Y}_s \), with \( \bar{Y} \) solving the ODE

\[
\bar{Y}_t = \int_t^m (K_1 \chi(s \geq n) - \rho \bar{Y}_s) \, ds.
\]
Hence, using Lemma 2.2, we obtain

\[(2.19)\] \[\delta Y_t^i(m,n) \leq \bar{Y}_t \leq e^{-\rho(m-t)}(m-n)K_1,\]

for \(t \in [0,m]\) and \(i \in I\). Likewise, we also have

\[(2.20)\] \[\delta Y_t^i(m,n) \geq -e^{-\rho(m-t)}(m-n)K_1.\]

Sending \(m,n \to \infty\), we obtain that, for any \(T > 0\), \(\sup_{t \in [0,T]} |\delta Y_t^i(m,n)| \to 0\) and, therefore, there exits a limit process \(Y^i\) such that \(Y_t^i(m) \to Y_t^i\) for almost every \((t, \omega) \in [0, \infty) \times \Omega\), with \(|Y_t^i| \leq K_y\).

To prove that \(Z^i(m)\) is also a Cauchy sequence, we introduce the Banach space

\[\mathcal{L}^{2,\rho} := \left\{ (Z)_{t \geq 0} : Z \text{ is progressively measurable and } \mathbb{E}\left[ \int_0^\infty e^{-2\rho s} |Z_s|^2 ds \right] < \infty \right\}.\]

Applying Itô’s formula to \(e^{-\rho t}|\delta Y_t^i(m,n)|^2\) and using (2.18), we get

\[\begin{align*}
|\delta Y_t^i(m,n)|^2 + & \int_0^m e^{-2\rho s}|\delta Z_s^i(m,n)|^2 ds \\
= & \int_0^m 2e^{-2\rho s}\delta Y_s^i(m,n) \left[ f^i(V_s, Z_s^i(m)) - f^i(V_s, Z_s^i(n)) \right] ds \\
& + \int_0^m 2e^{-2\rho s}\delta Y_s^i(m,n)f^i(V_s,0)\chi_{\{s \geq n\}} ds \\
& + \int_0^m 2e^{-2\rho s}\delta Z_s^i(m,n) \sum_{k \in I} q^{ik} \left( e^{y^k_t(m)-Y^k_t(m)} - e^{y^k_s(n)-Y^k_s(n)} \right) ds \\
& - \int_0^m 2e^{-2\rho s}\delta Y_s^i(m,n)(\delta Z_s^i(m,n))^{tr} dW_s.
\end{align*}\]

Furthermore, we apply the elementary inequality \(2ab \leq \frac{1}{2}|a|^2 + \epsilon|b|^2\) to term \((I)\) and obtain

\[\begin{align*}
(I) \leq & \frac{1}{2}e^{-2\rho s}\left| f^i(V_s, Z_s^i(m)) - f^i(V_s, Z_s^i(n)) \right|^2 \\
& + \frac{C_2^2(1 + 2K_z)^2}{C_z^2(1 + 2K_z)^2}e^{-2\rho s}|\delta Y_s^i(m,n)|^2 \\
\leq & \frac{1}{2}e^{-2\rho s}|\delta Z_s^i(m,n)|^2 + 2C_2^2(1 + 2K_z)^2e^{-2\rho s}|\delta Y_s^i(m,n)|^2,
\end{align*}\]

where we also used Assumption [1]ii) and the \(a \ priori\) estimate (2.13) on \(Z^i(m)\) in the second equality.

In turn, taking expectation on both sides of (2.21) and using the \(a \ priori\) estimate (2.13) on \(Y^i(m)\) yield

\[\begin{align*}
\frac{1}{2} \mathbb{E}\left[ \int_0^m e^{-2\rho s}|\delta Z_s^i(m,n)|^2 ds \right] \\
\leq & 2C_2^2(1 + 2K_z)^2\mathbb{E}\left[ \int_0^m e^{-2\rho s}|\delta Y_s^i(m,n)|^2 ds \right] + 2K_1\mathbb{E}\left[ \int_0^m e^{-2\rho s}|\delta Y_s^i(m,n)| ds \right] \\
& + 4m^9q^{max}e^{2K_y}\mathbb{E}\left[ \int_0^m e^{-2\rho s}|\delta Y_s^i(m,n)| ds \right].
\end{align*}\]
The dominated convergence theorem then implies $\delta Z^i(m, n) \to 0$ in $L^2_\rho$ and, therefore, there exists a limit process $Z^i$ such that $Z^i(m) \to Z^i$ in $L^2_\rho$, with $|Z^i| \leq K_\rho$.

It is standard to check that the pair of limit processes $(Y^i, Z^i)_{i \in I}$ indeed satisfy the infinite horizon BSDE system (2.1). See, for example, section 5 of [7].

**Uniqueness.** Since both $Y^i$ and $Z^i$ are bounded, the uniqueness of the bounded solution $(Y^i, Z^i)_{i \in I}$ to (2.1) follows from the multidimensional comparison theorem in Lemma 2.2. Indeed, suppose $(Y^i, Z^i)_{i \in I}$ and $(\tilde{Y}^i, \tilde{Z}^i)_{i \in I}$ are two bounded solutions to (2.1). For $t \geq 0$, let

$$\delta Y^i_t := e^{-\rho t}(Y^i_t - \tilde{Y}^i_t) \text{ and } \delta Z^i_t := e^{-\rho t}(Z^i_t - \tilde{Z}^i_t).$$

For $T \geq t$, let $\varepsilon_T := 2K_\rho e^{-\rho T}$. Then, for $0 \leq t \leq T$,

$$\begin{align*}
\delta Y^i_t &= \int_t^T e^{-\rho s} \left[ f^i(V_s, Z^i_s) - f^i(V_s, \tilde{Z}^i_s) \right] ds \\
&\quad + \int_t^T e^{-\rho s} \sum_{k \in I} \left( q^{ik}(e^{Y^k_s-Y^i_s} - 1) - q^{ik}(e^{\tilde{Y}^k_s-\tilde{Y}^i_s} - 1) \right) ds \\
&\quad - \int_t^T (\delta Z^i)_{1T} dW_s \\
(2.22) &\quad = \delta Y^i_T + \int_t^T \left[ F^i_s(\delta Z^i_s) + G^i_s(\delta Y^i_s, \delta Y^{-i}_s) \right] ds - \int_t^T (\delta Z^i)_{1T} dW_s,
\end{align*}$$

where

$$F^i_s(z) = e^{-\rho s} \left[ f^i(V_s, e^{\rho s}z + \tilde{Z}^i_s) - f^i(V_s, \tilde{Z}^i_s) \right],$$

and

$$G^i_s(y^i, y^{-i}) = e^{-\rho s} \sum_{k \in I} q^{ik} \left( e^{\rho s(y^k-y^i)} + \tilde{Y}^k_s - \tilde{Y}^i_s - e^{\tilde{Y}^k_s-\tilde{Y}^i_s} \right),$$

for $z \in \mathbb{R}^d$ and $y = (y^i, y^{-i}) \in \mathbb{R}^{m_0}$, with $|z| \leq 2K_\rho$ and $|y^i| \leq 2K_\rho$ for $i \in I$.

We apply similar arguments as in section 2.3 to deduce that (2.22) is with Lipschitz continuous driver and, therefore, $(\delta Y^i, \delta Z^i)_{i \in I}$ is the unique solution to (2.1). Moreover, note that

$$|\delta Y^i_T| \leq 2K_\rho e^{-\rho T} = \varepsilon_T, \quad F^i_s(0) = 0 \quad \text{and} \quad G^i_s(\varepsilon_T, \varepsilon_T) = 0. \quad \text{(2.22)}$$

By Lemma 2.2, we deduce that $|\delta Y^i_T| \leq \varepsilon_T$ and, therefore, $\delta Y^i_T = 0$ by sending $T \to \infty$. Consequently, $\delta Z^i_T = 0$, which proves the uniqueness of the solution to the infinite horizon BSDE system (2.1).

**3. System of ergodic quadratic BSDE.** We study the asymptotics of the infinite horizon BSDE system (2.1) when $\rho \to 0$, and show that it gives arise to a new type of ergodic BSDE systems. To this end, we require that the transition rate matrix $Q$ in Assumption 2 satisfies some sort of irreducible property.

**Assumption 4.** The transition rate matrix $Q$ satisfies $q^{ik} > 0$, for $i \neq k$. Let $q^{\min} > 0$ be the minimal transition rate, i.e. $q^{\min} = \min_{i,k} q^{ik}$.

We show that the difference of any two components, say $Y^i$ and $Y^j$, of the solution to (2.1) is actually bounded uniformly in $\rho$.

**Lemma 3.1.** Suppose that Assumptions 1-4 are satisfied. For $i, j \in I$ and $t \geq 0$, let $\Delta Y^{ij}_t = Y^i_t - Y^j_t$. Then,

$$|\Delta Y^{ij}_t| \leq \frac{K_\rho C_\eta}{2q^{\min}(C_\eta - C_\nu)}.$$  

(3.1)
with the constants $K_2$ and $C_v$ as in Assumption \textit{(7)} and $C_R$ as in Assumption \textit{(5)}.

Proof. It suffices to prove that, for $m \geq 1$,

\begin{equation}
|\Delta Y_{ij}^m(m)| := |Y_i^m(m) - Y_j^m(m)| \leq \frac{K_2C_v}{2q^{\min}(C_R - C_v)}.
\end{equation}

Then, (3.1) follows by sending $m \to \infty$.

To this end, let $\Delta Z_{ij}^m(m) = Z_i^m(m) - Z_j^m(m)$. It is immediate to check that the pair of processes $(\Delta Y_{ij}^m(m), \Delta Z_{ij}^m(m))_{i,j \in I}$ satisfy

\begin{align*}
\Delta Y_{ij}^m(m) &= \int_t^m \left[ f^i(V_s, Z_s^i(m)) - f^j(V_s, Z_s^j(m)) \right] ds \\
&\quad + \int_t^m \sum_{k \in I} \left( q^{jk}(e^{Y_k^m(s)} - Y_j^m(s)) - 1 \right) ds \\
&\quad - \int_t^m \rho \Delta Y_{ij}^m(m) ds - \int_t^m (\Delta Z_{ij}^m(s))^{tr} dW_s \\
&\quad = \int_t^m \left[ F_s^i(\Delta Z_{ij}^m(s)) + G_s^i(\Delta Y_{ij}^m(s), \Delta Y_{ij}^{-i}(s)) \right] ds \\
&\quad - \int_t^m (\Delta Z_{ij}^m(s))^{tr} dW_s,
\end{align*}

where

\begin{align*}
F_s^i(z) &= f^i(V_s, z + Z_s^i(m)) - f^i(V_s, Z_s^j(m)) + f^i(V_s, Z_s^j(m)) - f^j(V_s, Z_s^i(m)), \\
G_s^i(y^{ij}, y^{-ij}) &= q^{ij}e^{-y^{ij}} - q^{ji}e^{y^{ij}} - \sum_{k \neq j} q^{ik}e^{y^{ki}} - \sum_{k \neq i} q^{jk}e^{-y^{jk}},
\end{align*}

for $z \in \mathbb{R}^d$ and $y = (y^{ij}, y^{-ij}) \in \mathbb{R}^m$, with $|z| \leq 2K_z$ and $|y^{ij}| \leq 2K_y$ for $i, j \in I$.

Note that $F_s^i(z)$ and $G_s^i(y^{ij}, y^{-ij})$ are Lipschitz continuous. Following along similar arguments as in section \textit{2.3}, we deduce that $(\Delta Y_{ij}^m(m), \Delta Z_{ij}^m(m))_{i,j \in I}$ is the unique solution to BSDE system (3.3). Moreover, by Assumption \textit{(1)(iv)}, we have

\begin{align*}
F_s^i(0) &= f^i(V_s, Z_s^i(m)) - f^i(V_s, Z_s^j(m)) \leq K_2(1 + K_z),
\end{align*}

and by the facts that $\sum_{k \neq j} q^{ik} = -q^{ij}$ and $\sum_{k \neq i} q^{ik} = -q^{ji}$, we also have

\begin{align*}
G_s^i(\bar{Y}_s, \bar{Y}^{-i}_s) &= -(q^{ij} + q^{ji})(e^{\bar{Y}_s} - e^{-\bar{Y}_s}) - \rho \bar{Y}_s,
\end{align*}

where $\bar{Y}$ solves the ODE

\begin{align*}
\bar{Y}_t &= \int_t^m (K_2(1 + K_z) - 2q^{\min}\bar{Y}_s) ds.
\end{align*}

Since $0 \leq \bar{Y}_t \leq \frac{K_2(1 + K_z)}{2q^{\min}}$, we further have

\begin{align*}
G_s^i(\bar{Y}_s, \bar{Y}^{-i}_s) &\leq -(q^{ij} + q^{ji})(e^{\bar{Y}_s} - e^{-\bar{Y}_s}) \\
&\leq -2q^{\min}(\bar{Y}_s + 1 - e^{-\bar{Y}_s}) \leq -2q^{\min}\bar{Y}_s,
\end{align*}
and, consequently, using Lemma 2.2, we deduce that
\[ \Delta Y_t^{ij}(m) \leq \bar{Y}_i \leq \frac{K_2(1 + K_s)}{2\rho\min}. \]

By the symmetric property, we also have \( \Delta Y_t^{ji}(m) \leq \frac{K_2(1 + K_s)}{2\rho\min} \), from which we obtain estimate (3.2). \( \square \)

Next, we send \( \rho \to 0 \) in the infinite horizon BSDE system (2.1). To emphasize the dependencies on \( \rho \) and \( V_0 = v \), we use the notations \( V_t^v, Y_t^{i,\rho, v} \) and \( Z_t^{i,\rho, v} \) in the rest of this section. Sending \( m \to \infty \) in the gradient estimate (2.17) yields that, for the first component \( Y_t^{i,\rho, v} = y^{i,\rho}(V_t^v) \) of the solution to (2.1),

\[ |\nabla y^{i,\rho}(\cdot)| \leq \frac{C_v}{C_\eta - C_v}. \]

Given a fixed reference point, say \( v_0 \in \mathbb{R}^d \), we define the processes \( \bar{Y}_t^{i,\rho, v} := Y_t^{i,\rho, v} - Y_0^{m_0,\rho, v} \) for \( t \geq 0 \) and \( i \in I \), and consider the perturbed version of the infinite horizon BSDE system (2.1), i.e.

\[
\bar{Y}_t^{i,\rho, v} = Y_T^{i,\rho, v} + \int_t^T \left[ \sum_{k \in I} q_{ik}(e^{ar{Y}_s^{i,\rho, v} - Y_s^{i,\rho, v}} - 1) - \rho Y_s^{i,\rho, v} + \rho Y_0^{m_0,\rho, v} \right] ds \\
+ \int_t^T f_i(V_s^v, Z_s^{i,\rho, v}) ds - \int_t^T (Z_s^{i,\rho, v})^T dW_s,
\]

for \( 0 \leq t \leq T < \infty \) and \( i \in I \). By the Markov property of \( Y_t^{i,\rho, v} \), we have \( \bar{Y}_t^{i,\rho, v} = \bar{y}^{i,\rho}(V_t^v) \) with \( \bar{y}^{i,\rho}(\cdot) := y^{i,\rho}(\cdot) - y^{m_0,\rho}(v_0) \).

Note that, by estimate (3.3), \( \bar{y}^{i,\rho}(\cdot) \) is Lipschitz continuous uniformly in \( \rho \), and by estimate (3.1), \( \bar{y}^{i,\rho}(v_0) = y^{i,\rho}(v_0) - y^{m_0,\rho}(v_0) \) is bounded uniformly in \( \rho \). In turn, we deduce that, for \( v \in \mathbb{R}^d \),

\[
|\bar{y}^{i,\rho}(v)| = |y^{i,\rho}(v) - y^{i,\rho}(v_0) + y^{i,\rho}(v_0) - y^{m_0,\rho}(v_0)| \\
\leq \frac{C_v}{C_\eta - C_v} |v - v_0| + \frac{K_2 C_\eta}{2\rho \min(C_\eta - C_v)}.
\]

Moreover, (2.3) implies that \( |\rho y^{m_0,\rho}(v_0)| \leq \rho K_y = K_1 \). Hence, by a standard diagonal procedure, there exists a sequence, denoted by \( \{\rho_n\}_{n \geq 1} \), such that, for \( v \in \mathbb{R}^d \),

\[
\lim_{\rho_n \to 0} \rho_n y^{m_0,\rho_n}(v_0) = \lambda, \quad \lim_{\rho_n \to 0} \bar{y}^{i,\rho_n}(v) = y^i(v),
\]

for some \( \lambda \in \mathbb{R} \) and the limit function \( y^i(v) \).

Since \( \bar{y}^{i,\rho}(\cdot) \) is Lipschitz continuous uniformly in \( \rho \), the limit function \( y^i(\cdot) \) can be further extended to a Lipschitz continuous function defined for all \( v \in \mathbb{R}^d \), i.e.

\[
\lim_{\rho_n \to 0} \bar{y}^{i,\rho_n}(v) = y^i(v).
\]

Thus, for the infinite horizon BSDE system (2.2), it holds that \( \lim_{\rho_n \to 0} \bar{Y}_t^{i,\rho_n, v} = y^i(V_t^v) \) and \( \lim_{\rho_n \to 0} \rho_n Y_t^{i,\rho_n, v} = 0. \)
As a result, by defining the processes \( Y_{t}^{i} := y'(V_{t}^{i}) \), for \( t \geq 0 \) and \( i \in I \), it is standard to show that (see [13] and [15]) there exist a limit process \( Z_{t}^{i,v} \) for \( t \geq 0 \) and \( i \in I \), such that \( Z_{t}^{i,v} \to Z_{t}^{i,v} \) in \( L^{2} \) as \( \rho_{n} \to 0 \), and \( (Y_{t}^{i,v}, Z_{t}^{i,v})_{i \in I, \lambda} \) solve the ergodic BSDE system

\[
(3.7) \quad dY_{t}^{i,v} = -f^{i}(V_{t}^{v}, Z_{t}^{i,v}) dt - \sum_{k \in I} q^{ik}(e^{Y_{t}^{k,v} - Y_{t}^{i,v}} - 1) dt + \lambda dt + (Z_{t}^{i,v})^{tr} dW_{t},
\]

for \( t \geq 0 \) and \( i \in I \).

The main result in this section is the existence and uniqueness of the solution to the ergodic BSDE system (3.7).

**Theorem 3.2.** Suppose that Assumptions [14] are satisfied. Then, there exists a solution \((\bar{Y}_{t}^{i,v}, \bar{Z}_{t}^{i,v})_{i \in I, \lambda}\) to the ergodic BSDE system (3.7) satisfying

\[
(3.8) \quad |Y_{t}^{i,v}| = |y'(V_{t}^{i})| \leq C_{y}(1 + |V_{t}^{i}|),
\]

\[
(3.9) \quad |Z_{t}^{i,v}| \leq K_{z} = \frac{C_{v}}{C_{y} - C_{v}}.
\]

Moreover, if \((\bar{Y}_{t}^{i,v}, \bar{Z}_{t}^{i,v})_{i \in I, \lambda}\) is another solution to (3.7) satisfying (3.8) and (3.9), then \( \lambda = \bar{\lambda} \).

**Proof.** We have already shown the existence of a solution to (3.7). The two estimates (3.8) and (3.9) follow, respectively, from (3.6) and (2.3) by sending \( \rho \to 0 \). Hence, we are left to show the uniqueness of the constant \( \lambda \). Indeed, for \( t \geq 0 \) and \( v \in \mathbb{R}^{d} \), let

\[
\delta Y_{t}^{i,v} := Y_{t}^{i,v} - \bar{Y}_{t}^{i,v}, \quad \delta Z_{t}^{i,v} := Z_{t}^{i,v} - \bar{Z}_{t}^{i,v} \quad \text{and} \quad \delta \lambda := \lambda - \bar{\lambda}.
\]

Then, it follows from (3.7) that, for any \( 0 \leq t \leq T < \infty \),

\[
\delta Y_{t}^{i,v} = \delta Y_{T}^{i,v} + \int_{t}^{T} \left[ f^{i}(V_{s}^{v}, Z_{s}^{i,v}) - f^{i}(V_{s}^{v}, \bar{Z}_{s}^{i,v}) \right] ds
\]

\[
+ \int_{t}^{T} \sum_{k \in I} \left( q^{ik}(e^{Y_{s}^{k,v} - Y_{s}^{i,v}} - 1) - q^{ik}(e^{\bar{Y}_{s}^{k,v} - \bar{Y}_{s}^{i,v}} - 1) \right) ds
\]

\[
- \int_{t}^{T} \delta \lambda ds - \int_{t}^{T} (\delta Z_{s}^{i,v})^{tr} dW_{s}
\]

\[
(3.10) \quad = \delta Y_{T}^{i,v} + \int_{t}^{T} \left[ F_{s}^{i}(\delta Y_{s}^{i,v}) + G_{s}^{i}(\delta Y_{s}^{i,v}) \right] ds - \int_{t}^{T} (\delta Z_{s}^{i,v})^{tr} dW_{s},
\]

where

\[
F_{s}^{i}(z) = f^{i}(V_{s}^{v}, z + \bar{Z}_{s}^{i,v}) - f^{i}(V_{s}^{v}, \bar{Z}_{s}^{i,v}),
\]

and

\[
G_{s}^{i}(y^{i}, y^{-i}) = \sum_{k \in I} q^{ik} \left( e^{y^{k,v} + \bar{Y}_{s}^{k,v} - \bar{Y}_{s}^{i,v}} - e^{Y_{s}^{k,v} - \bar{Y}_{s}^{i,v}} \right) - \delta \lambda,
\]

for \( z \in \mathbb{R}^{d} \) with \( |z| \leq 2K_{z} \), and \( y = (y^{i}, y^{-i}) \in \mathbb{R}^{n_{v}} \).

Note that \( F_{s}^{i}(z) \) is Lipschitz continuous in \( z \), but \( G_{s}^{i}(y^{i}, y^{-i}) \) is not Lipschitz continuous due to the unboundedness of \( y \). To overcome this difficulty, for \( n \geq 1 \), we define the stopping time

\[
\tau_{n} := \inf\{ t \geq 0 : \min\{|Y_{t}^{i,v}|, |\bar{Y}_{t}^{i,v}|\} \geq n \text{ for } i \in I \}.
\]
Then, \( G_t'(y', y^{-i}) \), for \( s \in [0, \tau_n] \), becomes Lipschitz continuous in \( y = (y', y^{-i}) \) with \( |y'| \leq 2n \) for \( i \in I \). Consequently, following along similar arguments as in section 2.3, we deduce that (3.10) has a unique solution on the time interval \([0, \tau_n]\), which is \((\delta Y^{i,v}, \delta Z^i)_t \in \mathbb{t} \).

To find an upper bound of \( \delta Y^{i,v} \), we consider another BSDE also defined on the interval \([0, \tau_n]\), namely,

\[
\delta Y^i_t = 2C'(1 + |V_{\tau_n}^i|) + \int_t^{\tau_n} \left[ C_{x} C_C + C_{v} \right] \left| \delta \mathbb{Z}_s^i \right| - \delta \lambda \right] ds
- \int_t^{\tau_n} \left( \delta \mathbb{Z}_s^i \right)^{tr} dW_s.
\]

Since at terminal time \( \tau_n \), \( \delta Y^{i,v}_t \leq 2C'(1 + |V_{\tau_n}^i|) \) and, for \( s \in [0, \tau_n] \), \( F^i_s(\delta \mathbb{Z}_s^i) \leq C_2 C_{x} C_{C} + C_{v} \left| \delta \mathbb{Z}_s^i \right| \) and \( G^i_s(\delta \mathbb{Y}_s^i, (\delta \mathbb{Y}_s^i)^{-1}) = \delta \lambda \), it follows from Lemma 2.2 that

\[
\delta Y^{i,v}_0 \leq 2C(1 + |V_{\tau_n}^i|) - \delta \lambda E[\tau_n],
\]

where the probability measure \( Q \) is defined as

\[
\frac{dQ}{dP} := \mathcal{E}_{\tau_n} \left( C_2 C_{x} C_{C} + C_{v} \int_0^{\tau_n} \chi_{\left\{ \left| \delta \mathbb{Z}_s^i \right| > 0 \right\}} \frac{1}{\left| \delta \mathbb{Z}_s^i \right|} \left( \delta \mathbb{Z}_s^i \right)^{tr} dW_s \right),
\]

with \( \mathcal{E}() \) denoting Doléans-Dade stochastic exponential. Moreover, Assumption 3 on \( V^i \) and Proposition 1 in [11] imply that \( E[1 + |V_{\tau_n}^i|] \leq K(1 + |v|) \), for some constant \( K > 0 \), and estimate (3.8) implies that

\[
\delta Y^{i,v}_0 - \delta Y^{i,0}_0 = Y^i(v) - Y^i(v) \geq -2C'(1 + |v|).
\]

Therefore, \( \delta \lambda E[\tau_n] \leq 2C'(1 + K)(1 + |v|) \). Since \( E[\tau_n] \to \infty \) as \( n \to \infty \), we deduce that \( \delta \lambda \leq 0 \). By the symmetric property, we also have \( -\delta \lambda = \lambda - \lambda \leq 0 \), from which we obtain the uniqueness of \( \lambda \).

**Remark 1.** Note that if we do not impose (3.8), the constant \( \lambda \) may not be unique. Indeed, if we take \( Z^i_t \equiv 0 \), (3.7) reduces to

\[
dY^{i,v}_t = -f^i(V^i_t, 0)dt - \sum_{k \in I} q^{ik}(e^{Y^{k,v}_t - Y^{i,v}_t} - 1)dt + \lambda dt,
\]

for \( t \geq 0 \) and \( i \in I \). It is obvious that there are infinitely many pairs \((Y^{i,v}, \lambda)\) satisfying the above ODE system. However, in such a case, \( Y^{i,v} \) is neither Markovian nor satisfies the linear growth property in (3.8).

**4. Application to regime switching forward performance processes.** Let \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) be the filtered probability space introduced in section 2. Assume the probability space also supports a Markov chain \( \alpha \) with its augmented filtration \( \mathbb{F} = \{ \mathcal{H}_t \}_{t \geq 0} \) independent of the Brownian filtration \( \mathbb{F} \). The Markov chain \( \alpha \) has the transition rate matrix \( Q \) as specified in Assumption 4 and admits the representation

\[
\alpha_t = \sum_{j \geq 1} \alpha_{t-1} \chi_{(T_{j-1}, T_j]}(t),
\]

where \( T_0 = 0 \) and \( T_1, T_2, \ldots \) are the jump times of the Markov chain \( \alpha \), with \((\alpha^j)_{j \geq 1}\) being a sequence of \( \mathcal{H}_{T_j} \)-measurable random variables representing the position of \( \alpha \).
in the time interval \([T_{j-1}, T_j]\). Without loss of generality, assume that \(\alpha^0_i = i \in I\).

Denote the smallest filtration generated by \(\mathbb{F}\) and \(\mathbb{H}\) as \(\mathbb{G} = \{\mathcal{G}_t\}_{t \geq 0}\), i.e. \(\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t\).

We consider a market consisting of a risk-free bond offering zero interest rate and \(n\) risky assets, with \(n \leq d\). The prices of the \(n\) risky assets are driven by the Markov chain \(\alpha\) and a \(d\)-dimensional stochastic factor process \(V\), which satisfies Assumption 4.

Each state \(i \in I\) of the Markov chain \(\alpha\) represents a market regime, and in regime \(i\), the corresponding market price of risk at time \(t\) is \(\theta^i(V_t)\). The \(n\)-dimensional price process \(S = (S^1, \ldots, S^n)^{tr}\) of the risky assets in regime \(i\) follows

\[
dS_t = \text{diag}(S_t)\sigma(V_t)(\theta^i(V_t)dt + dW_t),
\]

where \(\sigma(V_t) \in \mathbb{R}^{n \times d}\) is the volatility matrix of the risky assets at time \(t\), and \(\text{diag}(S_t) = \{\text{diag}(S_t)_{ij}\}_{1 \leq i, j \leq n}\), with \(\text{diag}(S_t)_{ii} = S^i_t\) and \(\text{diag}(S_t)_{ij} = 0\) for \(i \neq j\), represents the prices of the risky assets at time \(t\).

**Assumption 5.** The market coefficients of the \(n\) risky assets satisfy that

1. \(\sigma(v)\) is uniformly bounded in \(v \in \mathbb{R}^d\) and has full rank \(n\);
2. \(\theta^i(v)\) is uniformly bounded and Lipschitz continuous in \(v \in \mathbb{R}^d\).

### 4.1 Trading strategies.

In this market environment, an investor trades dynamically among the risk-free bond and the risky assets. Let \(\tilde{\pi} = (\tilde{\pi}^1, \ldots, \tilde{\pi}^n)^{tr}\) denote the (discounted by the bond) proportions of her wealth in the risky assets. They are taken to be self-financing and, thus, the (discounted by the bond) wealth process in regime \(i\) satisfies

\[
dx_i(\tilde{\pi}) = X_i(\tilde{\pi})\tilde{\pi}^{tr} \sigma(V_t) (\theta^i(V_t)dt + dW_t),
\]

with \(X_0 = x \in \mathbb{R}_+\).

As in [30], we work with the trading strategies rescaled by the volatility matrix, namely, \(\pi_t^{tr} := \tilde{\pi}^{tr} \sigma(V_t)\). Then, the wealth process in regime \(i\) satisfies

\[
dx_i(\pi) = X_i(\pi)\pi^{tr} (\theta^i(V_t)dt + dW_t).
\]

For any \(t \geq 0\), we denote by \(\mathcal{A}^G_{[0,t]}\) the set of admissible trading strategies in \([0, t]\), defined as

\[
\mathcal{A}^G_{[0,t]} := \{\pi \in \mathcal{L}^G_{\text{BMO}}[0,t] : \pi_s \in \Pi \text{ for } s \in [0,t]\},
\]

where \(\Pi\) is a closed subset in \(\mathbb{R}^d\), and

\[
\mathcal{L}^G_{\text{BMO}}[0,t] := \left\{ \pi_s = \pi_0 + \sum_{j \geq 1} \pi_{s}^{a_{j-1}} \chi_{[T_{j-1}, T_j]}(s), s \in [0,t] : \pi^j\right\}
\]

\(\pi^j\) is \(\mathbb{F}\)-progressively measurable and \(\int_0^t (\pi_s^{j})^{tr} dW_s\) is an \(\mathbb{F}\)-BMO martingale.

Recall \(\int_0^t (\pi_s^{j})^{tr} dW_s\) is an \(\mathbb{F}\)-BMO martingale (up to time \(t\)), if

\[
\text{ess sup}_\tau \mathbb{E} \left[ \int_\tau^t \|\pi_s^{j}\|^2 ds \right]_{\mathcal{F}_\tau} < \infty,
\]

for any \(\mathbb{F}\)-stopping time \(\tau \in [0,t]\).

The set \(\mathcal{A}^G_{[t,s]}\) for \(0 \leq t \leq s\), is defined in a similar way, and the set of admissible trading strategies for all \(t \geq 0\) is, in turn, defined as \(\mathcal{A}^G = \cup_{t \geq 0} \mathcal{A}^G_{[0,t]}\).
4.2. Regime switching forward performance processes. The investor uses a forward criterion to measure the performance for her admissible trading strategies. We introduce the definition of regime switching forward performance processes associated with this market.

**Definition 4.1.** A family of stochastic processes \((U^i(x,t))_{i \in I}\), for \((x,t) \in \mathbb{R}^2_+\), is a regime switching forward performance process if the following conditions are satisfied:

(i) For each \(i \in I\) and \(x \in \mathbb{R}_+\), \(U^i(x,t)\) is \(\mathbb{F}\)-progressively measurable;

(ii) For each \(i \in I\) and \(t \geq 0\), the mapping \(x \mapsto U^i(x,t)\) is strictly increasing and strictly concave;

(iii) Define the \(\mathcal{G}\)-progressively measurable process as

\[
U(x,t) = \sum_{j \geq 1} U^{\alpha^{j-1}}(x,t) \chi_{[T_{j-1}, T_j)}(t). \tag{4.3}
\]

Then, for all \(\pi \in \mathcal{A}^\mathcal{G}\) and \(0 \leq t \leq s\),

\[
U(X^i_t(\pi), t) \geq \mathbb{E}[U(X^i_s(\pi), s)|\mathcal{G}_t], \tag{4.4}
\]

and there exists an optimal \(\pi^* \in \mathcal{A}^\mathcal{G}\) such that

\[
U(X^i_t(\pi^*), t) = \mathbb{E}[U(X^i_s(\pi^*), s)|\mathcal{G}_t], \tag{4.5}
\]

with \(X(\pi), X(\pi^*)\) solving (4.2).

The above martingale condition can be restated as follows: For \(j \geq 1\), on the event \(\{T_{j-1} \leq t < T_j\}\),

\[
U(x,t) = U^{\alpha^{j-1}}(x,t) = \text{ess sup}_{\pi \in \mathcal{A}^\mathcal{G}_{[t,s]}} \mathbb{E} \left[ U^{\alpha^{j-1}}(X^i_s(\pi), s) \chi_{s < T_j} \right]
+ U^{\alpha^j}(X^i_{T_j}(\pi), T_j) \chi_{s \geq T_j} |\mathcal{F}_t, X_t = x
\]

and on \(\{t = T_j\}\), \(U(x,t)\) has a jump with size

\[
U(x, T_j) - U(x, T_j^-) = U^{\alpha^j}(x, T_j) - U^{\alpha^{j-1}}(x, T_j^-).
\]

Hence, we have the following decomposition formula for \(U(x,t)\) (recall that \(\alpha^0 = i\)):

\[
U(x,t) = U(x,0) + \sum_{j \geq 1} \left[ U(x, t \wedge T_{j-}) - U(x, t \wedge T_{j-1}) \right]
+ \sum_{j \geq 1} \left[ U(x, t \wedge T_j) - U(x, t \wedge T_{j-}) \right]
= U^i(x,0) + \sum_{j \geq 1} \left[ U^{\alpha^{j-1}}(x, t \wedge T_{j-}) - U^{\alpha^{j-1}}(x, t \wedge T_{j-1}) \right]
+ \sum_{j \geq 1} \left[ U^{\alpha^j}(x, t \wedge T_j) - U^{\alpha^{j-1}}(x, t \wedge T_{j-}) \right] \chi_{\{T_j \leq t\}}. \tag{4.6}
\]

The first sum on the right hand side of (4.6) is the continuous component of \(U(x,t)\), while the second term is the jump component of \(U(x,t)\).
4.3. Representation via system of infinite horizon BSDE. The main result in this section is about a characterization of regime switching forward performance processes via the infinite horizon BSDE system (2.1) introduced in section 2. For \( i \in I \) and \((v, z) \in \mathbb{R}^d \times \mathbb{R}^d\), we consider the driver

\[
f^i(v, z) = \frac{1}{2} \delta(\delta - 1) \text{dist}^2 \left( \Pi, \frac{z + \theta^i(v)}{1 - \delta} \right) + \frac{\delta}{2(1 - \delta)} |z + \theta^i(v)|^2 + \frac{|z|^2}{2}.
\]

It is easy to check that \( f^i(v, z) \) in (4.7) satisfies Assumption 1, so from Theorem 2.1 the infinite horizon BSDE system (2.1) admits a unique bounded solution \((Y^i, Z^i)_{i \in I}\).

**Theorem 4.2.** Suppose that Assumptions 1-3 and 5 are satisfied. Let \((Y^i, Z^i)_{i \in I}\) be the unique bounded solution of the infinite horizon BSDE system (2.1) with driver \( f^i \) as in (4.7). Then,

\[
U^i(x, t) = \frac{x^i}{\delta} e^{Y^i - g^i},
\]

for \( i \in I \), form a regime switching forward performance process, and in each regime \( i \),

\[
\pi_{i^*} = \text{Proj}_{\Pi} \left( \frac{Z^i + \theta^i(V_t)}{1 - \delta} \right)
\]

is the associated optimal trading strategy in this regime. Herein, \((g^i)_{i \in I}\) is the unique solution of the ODE system (4.10)

\[
g^i_t = \int_0^t \rho Y^i_s ds + \int_0^t \sum_{k \in I} q^i_k e^{Y^k_s - Y^i_s} (e^{g^k_s - g^i_s} - 1) ds
\]

for \( i \in I \) and \( t \geq 0 \).

**Remark 2.** If there is only a single regime, i.e. \( m^0 = 1 \), then the infinite horizon BSDE system (2.1) and the ODE system (4.10) reduce, respectively, to

\[
dY_t^1 = -f^1(V_t, Z_t^1) dt + \rho Y_t^1 dt + (Z_t^1)^{tr} dW_t,
\]

and \( g^1_t = \int_0^t \rho Y^1_s ds \). In this case, the forward performance process has the representation

\[
U^1(x, t) = \frac{x^1}{\delta} e^{Y^1 - \int_0^t \rho Y^1_s ds},
\]

which is precisely the representation formula in Theorem 3.6 of [30].

4.4. Proof of Theorem 4.2. We divide the proof into four steps. The first step shows the well-posedness of both systems (2.1) and (4.10). The next two steps derive, locally and globally, the stochastic dynamics of the regime switching forward performance process. The last step verifies its time-consistency property.

**Step 1.** According to Theorem 2.1 the infinite horizon BSDE system (2.1) admits a unique bounded solution. In addition, by the transformation \( \tilde{g}^i_t := \exp(-g^i_t) \), the ODE system (4.10) reduces to a linear ODE system with bounded coefficients, namely,

\[
\frac{dg^i_t}{dt} = \left( \sum_{k \in I} q^i_k e^{Y^k_t - Y^i_t} - \rho Y^i_t \right) \tilde{g}^i_t - \sum_{k \in I} q^i_k e^{Y^k_t - Y^i_t} \tilde{g}^k_t.
\]
for $i \in I$ and $t \geq 0$, which obviously admits a unique bounded solution in any finite time interval.

**Step 2.** For $j \geq 1$, in the time interval $[T_{j-1}, T_j)$, we have

$$U(x, t) = U^{\alpha^{j-1}}(x, t) = \frac{x^\delta}{\delta} e^{(Y-g)t^{\alpha^{j-1}}}.$$

On the other hand, for $t \in (T_{j-1}, T_j]$, note that any admissible trading strategy $\pi \in \mathcal{A}^G$ takes the form $\pi_t = \pi^{\alpha^{j-1}}_t$, with $\pi^{\alpha^{j-1}}$ being $\mathbb{F}$-progressively measurable. In turn, applying Itô’s formula and using the equations (2.1), (4.2) and (4.10), we obtain

$$\begin{align*}
\frac{\langle X_t \rangle \delta}{\delta} e^{(Y-g)t^{\alpha^{j-1}}} & = \int_{T_{j-1}}^{T_j} \frac{\langle X_s \rangle \delta}{\delta} e^{(Y-g)s^{\alpha^{j-1}}} \left[ f^{\alpha^{j-1}}(V_s, Z_s^{\alpha^{j-1}}; \pi_s^{\alpha^{j-1}}) - f^{\alpha^{j-1}}(V_s, Z_s^{\alpha^{j-1}}) \right] ds \\
& + \int_{T_{j-1}}^{T_j} \frac{\langle X_s \rangle \delta}{\delta} e^{(Y-g)s^{\alpha^{j-1}}} \sum_{k \in I} q^{\alpha^{j-1}k} \left[ 1 - e^{(Y-g)_k^{\alpha^{j-1}}-(Y-g)_s^{\alpha^{j-1}}} \right] ds \\
& + \int_{T_{j-1}}^{T_j} \frac{\langle X_s \rangle \delta}{\delta} e^{(Y-g)s^{\alpha^{j-1}}} \left( \delta \pi_s^{\alpha^{j-1}} + Z_s^{\alpha^{j-1}} \right)^{tr} dW_s,
\end{align*}$$

where

$$f^i(v, z; \pi) := \frac{1}{2} \delta(\delta - 1)|\pi|^2 + \delta \pi^T \theta^i(v) + \delta \pi^T z + \frac{1}{2} |z|^2,$$

for $i \in I$ and $(v, z, \pi) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$.

**Step 3.** In general, for $t \geq 0$ and $\pi \in \mathcal{A}^G$, i.e.

$$\pi_t = \pi^{\alpha^0}_{T_{T_j}} + \sum_{j \geq 1} \pi^{\alpha^{j-1}}_{T_{j-1}} \chi(T_{j-1}, T_j)(t),$$

using the decomposition formula (4.3), we further have

$$\begin{align*}
\frac{\langle X_t \rangle \delta}{\delta} e^{(Y-g)t^{\alpha^{j-1}}} & = \frac{x^\delta}{\delta} e^{(Y-g)_t^{\alpha^{j-1}}} \\
& = \sum_{j \geq 1} \left[ \frac{\langle X_{t \wedge T_{j-1}} \rangle \delta}{\delta} e^{(Y-g)_{t \wedge T_{j-1}}^{\alpha^{j-1}}} - \frac{\langle X_{T_{j-1}} \rangle \delta}{\delta} e^{(Y-g)_{T_{j-1}}^{\alpha^{j-1}}} \right] \\
& + \sum_{j \geq 1} \left[ \frac{\langle X_{T_{j-1}} \rangle \delta}{\delta} e^{(Y-g)_{T_{j-1}}} - \frac{\langle X_{T_{j-1}} \rangle \delta}{\delta} e^{(Y-g)_{T_{j-1}}^{\alpha^{j-1}}} \right] \chi(T_{j-1} \leq t) \\
& = (I) + (II).
\end{align*}$$

For the continuous component $(I)$, using (4.11) and the facts that $\alpha^{j-1} = \alpha^{j-1}$, $\pi_s = \pi^{\alpha^{j-1}}_s$, for $s \in (t \wedge T_{j-1}, t \wedge T_j)$, we obtain that

$$\begin{align*}
(I) & = \sum_{j \geq 1} \int_{t \wedge T_{j-1}}^{t \wedge T_j} \frac{\langle X_s \rangle \delta}{\delta} e^{(Y-g)s^{\alpha^{j-1}}} \left[ f^{\alpha^{j-1}}(V_s, Z_s^{\alpha^{j-1}}; \pi_s^{\alpha^{j-1}}) - f^{\alpha^{j-1}}(V_s, Z_s^{\alpha^{j-1}}) \right] ds \\
& + \sum_{j \geq 1} \int_{t \wedge T_{j-1}}^{t \wedge T_j} \frac{\langle X_s \rangle \delta}{\delta} e^{(Y-g)s^{\alpha^{j-1}}} \sum_{k \in I} q^{\alpha^{j-1}k} \left[ 1 - e^{(Y-g)_k^{\alpha^{j-1}}-(Y-g)_s^{\alpha^{j-1}}} \right] ds \\
& + \sum_{j \geq 1} \int_{t \wedge T_{j-1}}^{t \wedge T_j} \frac{\langle X_s \rangle \delta}{\delta} e^{(Y-g)s^{\alpha^{j-1}}} \left( \delta \pi_s^{\alpha^{j-1}} + Z_s^{\alpha^{j-1}} \right)^{tr} dW_s.
\end{align*}$$
In turn,

\[ (I) = \int_0^t \left( \frac{X_s(\pi)}{\delta} e^{(Y-g)_{t-}^a} \right) ds \left[ f^{\alpha_x}(V_s, Z_s^{\alpha_x}; \pi_s) - f^{\alpha_x}(V_s, Z_s^{\alpha_x}) \right] ds \\
+ \int_0^t \left( \frac{X_s(\pi)}{\delta} e^{(Y-g)_{t-}^a} \right) \sum_{k \in I} q^{\alpha_k} \left[ 1 - e^{(Y-g)_{t-}^a} \right] ds \\
\]

(4.13)

\[ + \int_0^t \left( \frac{X_s(\pi)}{\delta} e^{(Y-g)_{t-}^a} \right) \left( \delta \pi_s + Z_s^{\alpha_x} \right)^{tr} dW_s \\
\]

For the jump component \( (II) \), following [41] and [10], we may rewrite it as a stochastic integral against a sequence of independent Poisson processes, namely,

\[ (II) = \int_0^t \sum_{k, k' \in I} \left[ \frac{(X_s(\pi))}{\delta} e^{(Y-g)_s^a} - \frac{(X_s(\pi))}{\delta} e^{(Y-g)^{k'}_{s-} - (Y-g)^{k'}_{s-}} \right] \chi(\alpha_s = k') d\tilde{N}_s^{k_k}, \]

where \( (N^{k_k}_{k_k})_{k, k' \in I} \) are independent Poisson processes each with intensity \( q^{k_k} \).

Introducing the compensated Poisson martingales \( \tilde{N}_s^{k_k} = N^{k_k}_{k_k} - q^{k_k} t \) for \( t \geq 0 \), we can further decompose the stochastic integral in \( (4.13) \) as a martingale part and a finite variation part:

\[ (II) = \int_0^t \sum_{k, k' \in I} \left[ \frac{(X_s(\pi))}{\delta} e^{(Y-g)_s^a} - \frac{(X_s(\pi))}{\delta} e^{(Y-g)^{k'}_{s-} - (Y-g)^{k'}_{s-}} \right] \chi(\alpha_s = k') d\tilde{N}_s^{k_k} \\
+ \int_0^t \left( \frac{(X_s(\pi))}{\delta} e^{(Y-g)_s^a} \right) \sum_{k \in I} q^{\alpha_k} \left[ e^{(Y-g)_s^a} - (Y-g)_s^{a_x} - 1 \right] ds. \]

(4.15)

It then follows from \( (4.13) \) and \( (4.15) \) that

\[ \frac{(X_s(\pi))}{\delta} e^{(Y-g)_s^a} = \frac{(X_s(\pi))}{\delta} e^{(Y-g)_s^a} - \frac{(X_s(\pi))}{\delta} e^{(Y-g)_s^a} - \frac{(X_s(\pi))}{\delta} e^{(Y-g)_s^a} \]

\[ = \int_0^t \left( \frac{(X_s(\pi))}{\delta} e^{(Y-g)_s^a} \right) \left[ f^{\alpha_x}(V_s, Z_s^{\alpha_x}; \pi_s) - f^{\alpha_x}(V_s, Z_s^{\alpha_x}) \right] ds \\
+ \int_0^t \left( \frac{(X_s(\pi))}{\delta} e^{(Y-g)_s^a} \right) \left( \delta \pi_s + Z_s^{\alpha_x} \right)^{tr} dW_s \\
+ \int_0^t \left( \frac{(X_s(\pi))}{\delta} e^{(Y-g)_s^a} \right) \sum_{k, k' \in I} \left[ e^{(Y-g)_s^a} - (Y-g)_s^{a_x} - 1 \right] \chi(\alpha_s = k') d\tilde{N}_s^{k_k}, \]

and therefore,

\[ \frac{(X_s(\pi))}{\delta} e^{(Y-g)_s^a} = \frac{(X_s(\pi))}{\delta} e^{(Y-g)_s^a} \times \int_0^t f^{\alpha_x}(V_s, Z_s^{\alpha_x}; \pi_s) - f^{\alpha_x}(V_s, Z_s^{\alpha_x}) ds \\
\times \mathcal{E}_t \left( \int_0^t \left( \delta \pi_s + Z_s^{\alpha_x} \right)^{tr} dW_s \right) \\
\times \mathcal{E}_t \left( \int_0^t \sum_{k, k' \in I} \left[ e^{(Y-g)_s^a} - (Y-g)_s^{a_x} - 1 \right] \chi(\alpha_s = k') d\tilde{N}_s^{k_k}, \right), \]

for any \( \pi \in \mathcal{A}^G \), where \( \mathcal{E}(\cdot) \) denotes Doléans-Dade stochastic exponential.
Step 4. Finally, we verify the condition (iii) in Definition 4.1 (note that the conditions (i) and (ii) are straightforward). It follows from (4.7) and (4.12) that
\[
f^{\alpha_s}(V_s, Z_s^{\alpha_s}; \pi_s) - f^{\alpha_s}(V_s, Z_s^{\alpha_s}) \leq 0,
\]
for any \( \pi_s \in \Pi \), and with \( \pi_s^* = \text{Proj}_H \left( \frac{Z_s^{\alpha_s} + \theta^s(V_s)}{1 - \delta} \right) \), the equality then holds:
\[
f^{\alpha_s}(V_s, Z_s^{\alpha_s}; \pi_s^*) - f^{\alpha_s}(V_s, Z_s^{\alpha_s}) = 0.
\]

By the admissibility of \( \pi^i \) and boundedness of \( Z^i \), \( \int_0^T (\delta \pi_s^i + Z_s^{\alpha_s})^{\delta} dW_s \) is an \( \mathbb{F} \)-BMO martingale. Thus, \( \mathcal{E} \left( \int_0^T (\delta \pi_s^i + Z_s^{\alpha_s})^{\delta} dW_s \right) \) is a uniformly integrable \( \mathbb{F} \)-martingale for any \( i \in I \), and \( \mathcal{E} \left( \int_0^T (\delta \pi_s + Z_s^{\alpha_s})^{\delta} dW_s \right) \) is a uniformly integrable \( \mathbb{G} \)-martingale. On the other hand, let
\[
\eta_s^{k,k'} := \left[ e^{(Y_g)^k_s - (Y_g)^{k'}_s} - 1 \right] \chi_{\{\alpha_s = k'\}},
\]
which is bounded in any finite time interval. In turn,
\[
\mathcal{E}_t \left( \int_0^t \sum_{k,k' \in I} \left[ e^{(Y_g)^k_s - (Y_g)^{k'}_s} - 1 \right] \chi_{\{\alpha_s = k'\}} d\tilde{N}_s^{k,k'} \right) = \prod_{k,k' \in I} \mathcal{E}_t \left( \int_0^t \eta_s^{k,k'} (d\tilde{N}_s^{k,k'} - q^{k,k'} ds) \right) = \prod_{k,k' \in I} e^{-\int_0^t \eta_s^{k,k'} q^{k,k'} ds} \prod_{0 \leq s \leq t} \left( 1 + \eta_s^{k,k'} \Delta \tilde{N}_s^{k,k'} \right)
\]
which is also bounded, so \( \mathcal{E} \left( \int_0^T \sum_{k,k' \in I} \eta_s^{k,k'} d\tilde{N}_s^{k,k'} \right) \) is a bounded \( \mathbb{G} \)-martingale. Thus, we verify the martingale property in Definition 4.1(iii) of \( U(x,t) \), and conclude the proof.

4.5. Representation via system of ergodic BSDE. We further characterize the regime switching forward performance processes via the ergodic BSDE system (3.7) introduced in section 3. One advantage of such a characterization is that the corresponding representation does not involve any additional parameter \( \rho \).

Theorem 4.3. Suppose that Assumptions 4.3 are satisfied. Let \((Y^i, Z^i)_{t \in I}, \lambda)\) be a solution of the ergodic BSDE system (3.7) with driver \( f^i \) as in (4.7), and satisfy (3.8) and (3.9). Then,
\[
(4.16) U^i(x,t) = \frac{x^\delta}{\delta} e^{Y^i_t - g^i_t},
\]
for \( i \in I \), form a regime switching forward performance process, and in each regime \( i \),
\[
(4.17) \pi^{i,*}_{t+s} = \text{Proj}_H \left( \frac{Z_t^i + \theta^i(V_t)}{1 - \delta} \right)
\]
is the associated optimal trading strategy in this regime. Herein, \((\mathcal{G}^i)_{i \in I}\) solves the ODE system
\[
(4.18) \mathcal{G}^i_t = \lambda t + \int_0^t \sum_{k \in I} q^k e^{Y^k_s} - Y^i_s (e^{g^i_s - g^k_s} - 1) ds
\]
for \( i \in I \) and \( t \geq 0 \).

The proof follows along similar arguments as in Theorem 4.2 and it is thus omitted. We only note that, due to Lemma 3.1, the difference of any two components \( Y_k \) and \( Y_i \) is bounded:

\[
|Y_k^s - Y_i^s| \leq \frac{K_2 C_{\eta}}{2q_{\min}(C_{\eta} - C_v)}
\]

so the ODE system (4.18) is with bounded coefficients and admits a unique bounded solution in any finite time interval.

**Remark 3.** If there is only a single regime, i.e. \( m^0 = 1 \), then the ergodic BSDE system (3.7) and the ODE system (4.18) reduce, respectively, to

\[
dY^1_t = - f^1(V_t, Z^1_t) dt + \lambda dt + (Z^1_t)^{tr}dW_t,
\]

and \( G^1_t = \lambda t \). In this case, the forward performance process has the representation

\[
U^1(x, t) = \frac{x^\delta}{\delta} e^{Y^1_t - \lambda t},
\]

which is precisely the representation formula in Theorem 3.2 of [30].

**4.5.1. Connection with risk-sensitive optimization.** We provide an interpretation of the constant \( \lambda \), appearing in the representation of the forward performance process (4.16), as the solution of the risk-sensitive control problem (4.19) below. It turns out that the constant \( \lambda \) is also the optimal growth rate of the long-term utility maximization problem (see (4.20) below).

**Proposition 4.4.** Let \( T > 0 \) and \( \pi \in A_{G^i} \), and define the probability measure \( \mathbb{P}^{\pi} \) as

\[
\frac{d\mathbb{P}^{\pi}}{d\mathbb{P}} := \mathcal{E}_T \left( \int_0^T \delta \pi_u^{tr} dW_u \right).
\]

and the cost functional

\[
L^i(v; \pi) := \frac{1}{2} \delta(\delta - 1) |\pi|^2 + \delta \pi^{tr} \theta^i(v),
\]

for \( i \in I \) and \( (v, z) \in \mathbb{R}^d \times \mathbb{R}^d \).

Let \( (Y^i, Z^i)_{i \in I}, \lambda \) be a solution of the ergodic BSDE system (3.7) with driver \( f^i \) as in (4.7), and satisfy (3.8) and (3.9). Then, \( \lambda \) is the long-term growth rate of the risk-sensitive control problem

\[
\lambda = \sup_{\pi \in A_{G^i}} \lim_{T \to \infty} \frac{1}{T} \ln \mathbb{E}^{\mathbb{P}^\pi} \left[ e^{\int_0^T L^{\alpha_s} (V_s, \pi_s) ds} \right],
\]

or, alternatively,

\[
\lambda = \sup_{\pi \in A_{G^i}} \lim_{T \to \infty} \frac{1}{T} \ln \mathbb{E} \left[ \left( X_T(\pi) \right)^{\delta} \right].
\]

For both problems (4.19) and (4.20), the associated optimal control in each regime \( i \) is \( \pi^*_i \) as in (4.17).
Proof. We first observe that the driver \( f^i \) in (4.7) can be written as

\[
f^i(v, z) = \sup_{\pi \in \Pi} \left( L^i(v, \pi) + z^T \delta \pi \right) + \frac{1}{2} |z|^2.
\]

Therefore, for arbitrary admissible \( \tilde{\pi} \), we apply Itô’s formula to the ergodic BSDE system (3.7) on \([T_{j-1}, T_j]\), and obtain

\[
e^Y_{T_j -} - e^Y_{T_{j-1}} = \int_{T_{j-1}}^{T_j} - \sup_{\pi^{j-1} \in \Pi} \left( L^{\alpha^{j-1}}(V_s, \pi^{\alpha^{j-1}}_s) + (Z^{\alpha^{j-1}}_s)^T \delta \pi^{\alpha^{j-1}}_s \right) \, ds
\]

\[
+ \int_{T_{j-1}}^{T_j} \lambda \sum_{k \in I} q^{\alpha^{j-1}k} (e^{Y^k_s - Y_s} - 1) \, ds
\]

\[
+ \int_{T_{j-1}}^{T_j} (Z^{\alpha^{j-1}}_s)^T (dW_s - \delta \pi^{\alpha^{j-1}}_s \, ds).
\]

In general, we decompose \( e^Y_{T_j -} - e^Y_{T_{j-1}} \) into continuous and jump components as

\[
e^Y_{T_j -} - e^Y_{T_{j-1}} = \sum_{j \geq 1} \left[ e^{Y^{\alpha^{j-1}}_{T_j \wedge T} - Y^{\alpha^{j-1}}_{T_{j-1} \wedge T} \wedge T} + \sum_{j \geq 1} \left[ e^{Y^{\alpha^j}_{T_j \wedge T} - Y^{\alpha^{j-1}}_{T_{j-1} \wedge T} \wedge T} \right] \chi_{\{T_j \leq T\}}
\]

\[
= (I) + (II).
\]

It follows from the facts that \( \alpha_s = \alpha^{j-1}_s, \pi_s = \pi^{\alpha^{j-1}}_s \) and \( \tilde{\pi}_s = \tilde{\pi}^{\alpha^{j-1}}_s \) for \( s \in (T \wedge T_{j-1}, T \wedge T_j) \) that (I) has the expression

\[
(I) = \int_0^T e^{Y^{\alpha^{s-}}_s} \left[ - \sup_{\pi_s \in \Pi} \left( L^{\alpha^{s-}}(V_s, \pi_s) + (Z^{\alpha^{s-}}_s)^T \delta \pi_s + \lambda \right) \right] \, ds
\]

\[
- \int_0^T e^{Y^{\alpha^{s-}}_s} \sum_{k \in I} q^{\alpha^{s-}k} (e^{Y^k_s - Y_s} - 1) \, ds
\]

\[
+ \int_0^T e^{Y^{\alpha^{s-}}_s} (Z^{\alpha^{s-}}_s)^T (dW_s - \delta \tilde{\pi}_s \, ds).
\]

(4.21)

Analogous to the proof of Theorem 4.2, we introduce independent Poisson processes \( (N^{k,k'})_{k',k \in I} \), and deduce that (II) has the expression

\[
(II) = \int_0^T e^{Y^{\alpha^{s-}}_s} \sum_{k,k' \in I} \left( e^{Y^{\alpha^{s-}}_s - Y^{k',k}_s} - 1 \right) \chi_{\{\alpha_s = k'\}} d\tilde{N}^{k,k}_s
\]

\[
+ \int_0^T e^{Y^{\alpha^{s-}}_s} \sum_{k \in I} q^{\alpha^{s-}k} (e^{Y^k_s - Y^{k'}_s} - 1) \, ds.
\]

(4.22)

Consequently, combining (4.21) and (4.22), we obtain

\[
e^Y_{T_j -} - e^Y_{T_{j-1}} = \int_0^T e^{Y^{\alpha^{s-}}_s} \left[ - \sup_{\pi_s \in \Pi} \left( L^{\alpha^{s-}}(V_s, \pi_s) + (Z^{\alpha^{s-}}_s)^T \delta \pi_s + \lambda \right) \right] \, ds
\]

\[
+ \int_0^T e^{Y^{\alpha^{s-}}_s} (Z^{\alpha^{s-}}_s)^T (dW_s + \delta \tilde{\pi}_s \, ds)
\]

\[
+ \int_0^T e^{Y^{\alpha^{s-}}_s} \sum_{k,k' \in I} \left( e^{Y^k_s - Y^{k'}_s} - 1 \right) \chi_{\{\alpha_s = k'\}} \, d\tilde{N}^{k,k}_s,
\]
where the process $W^\pi_t := W_t - \int_0^t \delta \pi u \, du$, $t \geq 0$, is a Brownian motion under $\mathbb{P}^{\tilde{\pi}}$. In turn,

\[ e^{\mathcal{Y}^\pi_{T \mathbb{T}}} = e^{\mathcal{Y}^0_{0 \mathbb{T}}} \mathcal{E}_T \left( \int_0^T (\mathcal{Z}^{\alpha, -})_t^T dW^\pi_t \right) \mathcal{E}_T \left( \sum_{k,k' \in I} \left( e^{\mathcal{Y}^k_t - Y_t^{k'}} - 1 \right) \chi_{\{ \alpha_s = k' \}} d\tilde{N}^{k,k}_s \right) \]

\[ \times e^{-\int_0^T L^{\alpha, -}( V_s, \tilde{\pi}_s ) \, ds} \]

\[ \times e^{\int_0^T \left( ( L^{\alpha, -}( V_s, \tilde{\pi}_s ) + (Z^{\alpha, -})_s^T \delta \tilde{\pi}_s ) - \sup_{\pi_s \in \Pi} \{ L^{\alpha, -( V_s, \pi_s )} + (Z^{\alpha, -})_s^T \delta \pi_s \} \right) ds} . \]

Next, we observe that for any $\tilde{\pi} \in \mathcal{A}_G$, the last exponential term on the right hand side is bounded above by 1. Taking expectation under $\mathbb{P}^{\tilde{\pi}}$ then yields

\[ \mathbb{E}^{\tilde{\pi}} \left[ e^{\int_0^T L^{\alpha, -}( V_s, \tilde{\pi}_s ) \, ds} \right] e^{-\mathcal{Y}^0_{0 \mathbb{T}}} \leq \mathbb{E}^{\tilde{\pi}} \left[ e^{\mathcal{Y}_{T \mathbb{T}}} \mathcal{E}_T \left( \int_0^T (\mathcal{Z}^{\alpha, -})_t^T dW^\pi_t \right) \mathcal{E}_T \left( \sum_{k,k' \in I} \left( e^{\mathcal{Y}^k_t - Y_t^{k'}} - 1 \right) \chi_{\{ \alpha_s = k' \}} d\tilde{N}^{k,k}_s \right) \right] . \]

Define the probability measure $\mathbb{Q}^{\tilde{\pi}}$ as

\[ \frac{d\mathbb{Q}^{\tilde{\pi}}}{d\mathbb{P}^{\tilde{\pi}}} := \mathcal{E}_T \left( \int_0^T (\mathcal{Z}^{\alpha, -})_t^T dW^\pi_t \right) \mathcal{E}_T \left( \sum_{k,k' \in I} \left( e^{\mathcal{Y}^k_t - Y_t^{k'}} - 1 \right) \chi_{\{ \alpha_s = k' \}} d\tilde{N}^{k,k}_s \right) . \]

Then, it follows from the linear growth property of $\mathcal{Y}^\pi_T = \mathcal{Y}^\pi(T)$ and Assumption 3 on $V$ that

\[ \frac{1}{C} \leq \mathbb{E}^{\mathbb{Q}^{\tilde{\pi}}} \left( e^{-\mathcal{Y}^\pi_{T \mathbb{T}}} \right) \leq C, \]

for some constant $C$ independent of $T$ (see, for example, Proposition 1 in [11]). Consequently,

\[ \frac{1}{T} \ln \mathbb{E}^{\tilde{\pi}} \left[ e^{\int_0^T L^{\alpha, -}( V_s, \tilde{\pi}_s ) \, ds} \right] \leq \lambda + \frac{Y^\pi_0}{T} + \frac{1}{T} \ln \mathbb{E}^{\mathbb{Q}^{\tilde{\pi}}} \left( e^{-\mathcal{Y}^\pi_{T \mathbb{T}}} \right) . \]

Sending $T \to \infty$, we obtain, for any $\pi_s \in \mathcal{A}_G$,

\[ \lambda \geq \limsup_{T \to \infty} \frac{1}{T} \ln \mathbb{E}^{\pi} \left[ e^{\int_0^T L^{\alpha, -}( V_s, \pi_s ) \, ds} \right] , \]

with equality choosing $\pi_{s,T} = \pi^*_s$, with $\pi^*_s$ as in [11].

To show that $\lambda$ also solves (1.20), we observe that for $\pi \in \mathcal{A}_G$, we have

\[ \mathbb{E} \left[ \frac{(X^{j \pi}_{T \mathbb{T}})_\delta}{\delta} \right] = \frac{X^{j \pi}_0}{\delta} \mathbb{E} \left[ e^{\int_0^T L^{\alpha, -}( V_s, \pi_s ) \, ds} \mathcal{E}_T \left( \int_0^T \delta \pi_s^T \, dW_s \right) \right] \]

\[ = \frac{X^{j \pi}}{\delta} \mathbb{E}^{\mathbb{P}^{\pi}} \left[ e^{\int_0^T L^{\alpha, -}( V_s, \pi_s ) \, ds} \right] , \]

and the rest of the arguments follow. \[ \square \]
5. Conclusions. In this paper, we introduced and solved a new type of quadratic
BSDE systems in an infinite time horizon and, subsequently, derived their asymptotic
processes and their classical expected utility counterparts via the constant λ in the
corresponding ergodic BSDE system.

Finally, we would like to mention that the solution uniqueness of the ergodic
BSDE system [3,7] only refers to the uniqueness of the constant λ, which is in the sense
of uniqueness introduced in [18]. Ideally, we would also like to have the uniqueness of
\( (Y^i, Z^i)_{i \in I} \) (in the sense of uniqueness in [13]). This is left for the further research.

Appendix A. Proof of Lemma 2.2
The idea of the proof is adapted from the arguments used in [23]. For
\( t \in [0, T] \), let
\[
\delta Y^i_t := Y^i_t - \bar{Y}^i_t, \quad \delta Z^i_t := Z^i_t - \bar{Z}^i_t \quad \text{and} \quad \delta \xi^i := \xi^i - \bar{\xi}^i.
\]
Applying Itô’s formula to \( (\delta Y^i_t)^2 \) yields
\[
(\delta Y^i_t)^2 = (\delta \xi^i)^2 + \int_t^T 2 \delta Y^i_s [F^i_s(Z^i_s) - F^i_s(\bar{Z}^i_s)] ds
\]
\[
+ \int_t^T 2 \delta Y^i_s [G^i_s(Y^i_s, Y^{-i}_s) - \bar{G}^i_s(\bar{Y}^i_s, \bar{Y}^{-i}_s)] ds
\]
\[
- \int_t^T \chi_{\{\delta Y^i_s \neq 0\}} |\delta Z^i_s|^2 ds - \int_t^T 2 \delta Y^i_s (\delta Z^i_s)^{tr} dW_s.
\]
Using (2.9) and (2.11), we obtain
\[
F^i_s(Z^i_s) - F^i_s(\bar{Z}^i_s) = F^i_s(Z^i_s) - F^i_s(\bar{Z}^i_s) + F^i_s(\bar{Z}^i_s) - F^i_s(\bar{Z}^i_s) \leq C_f |\delta Z^i_s|.
\]
Using (2.10) and (2.12), together with the monotone condition of \( G^i_s \), we further obtain
\[
G^i_s(Y^i_s, Y^{-i}_s) - \bar{G}^i_s(\bar{Y}^i_s, \bar{Y}^{-i}_s)
\]
\[
= G^i_s(Y^i_s, Y^{-i}_s) - G^i_s(\bar{Y}^i_s, \bar{Y}^{-i}_s) + G^i_s(\bar{Y}^i_s, \bar{Y}^{-i}_s) - \bar{G}^i_s(\bar{Y}^i_s, \bar{Y}^{-i}_s)
\]
\[
\leq C_g \left( |\delta Y^i_s| + \sum_{k \neq i} \delta Y^{k+}_s \right).
\]
In turn, since \( \delta \xi^i = 0 \), we have
\[
\mathbb{E}[|\delta Y^i_t|^2] \leq \mathbb{E} \left[ \int_t^T \left( 2C_f \delta Y^i_s |\delta Z^i_s| + 2C_g \delta Y^i_s |\delta Y^{k+}_s| + \sum_{k \neq i} \delta Y^{k+}_s \right) ds \right]
\]
\[
\leq \mathbb{E} \left[ \int_t^T \chi_{\{\delta Y^i_s \neq 0\}} (|\delta Z^i_s|^2 + 2C_f \delta Y^i_s |\delta Z^i_s| - C_f^2 (\delta Y^i_s)^2) ds \right]
\]
\[
+ \mathbb{E} \left[ \int_t^T \left( 2C_g + C_f^2 \right) |\delta Y^i_s|^2 + C_f^2 (\delta Y^i_s)^2 + \sum_{k \neq i} |\delta Y^{k+}_s|^2 \right] ds \right).
\]
Thus, there exists a constant $C$ such that
\[
\sum_{i \in I} \mathbb{E}[\delta Y_{i,t}^2] \leq C \int_t^T \sum_{i \in I} \mathbb{E}[\delta Y_{i,s}^2] \, ds.
\]

It then follows from Gronwall’s inequality that $\mathbb{E}[\delta Y_{i,t}^2] = 0$, for $t \in [0, T]$ and $i \in I$, so $Y_{i,t} \leq \bar{Y}_{i,t}$ and we conclude.

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