ISCO, Lyapunov exponent and Kolmogorov-Sinai entropy for Kerr-Newman Black hole

Partha Pratim Pradhan

Department of Physics
Vivekananda Satabarshiki Mahavidyalaya
Manikpara, Paschim Medinipur
West Bengal 721513, India

Abstract

We compute the principal Lyapunov exponent and Kolmogorov-Sinai(KS) entropy for Kerr-Newman black hole space-times and investigate the stability and instability of the equatorial circular geodesics via these exponents. We also show that the principal Lyapunov exponent and KS entropy can be expressed in terms of the radial equation of ISCO(innermost stable circular orbit) for timelike circular geodesics. The other aspect we have studied that among the all possible circular geodesics, which encircle the central black-hole, the timelike circular geodesics has the longest orbital period i.e. $T_{\text{timelike}} > T_{\text{photon}}$, than the null circular geodesics (photon sphere) as measured by asymptotic observers. Thus, the timelike circular geodesics provide the slowest way to circle the Kerr-Newman black-hole. In fact, any stable timelike circular geodesics other than the ISCO traverses more slowly than the null circular geodesics.

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1 Introduction

Geodesic properties of blackholes have been examined for many years [9, 10, 12] to probe the stability or instability of circular geodesics using the effective potential. From the best of my knowledge, we know there were no attempt to made a link between non-linear Einstein’s general theory of relativity and nonlinear dynamics. Particularly, Lyapunov exponent and KS entropy are two of them. In this article, we shall focus on analytical calculations involving Lyapunov exponent and KS entropy in terms of the radial equation of circular geodesics around a black-hole space-time. This equatorial circular geodesics around a black-hole play an important role in general relativity for classification of the

1E-mail: pppradhan77@gmail.com, pppradhan@vsm.org.in
orbits. It also determines the important characteristics of the space-times and gives important information on the background geometry.

The author in [8] computed the Lyapunov exponent to probe the instability of circular null geodesics in terms of the quasi-normal modes (QNMs) for spherically symmetric space-times, but the focus there is on null circular geodesics. It has been shown in this reference that by computing the Lyapunov exponent, which is the inverse of the instability time scale associated with the geodesic motion and in the eikonal limit QNMs of blackholes is determined by the parameters of the circular null geodesics.

Note however that the principal Lyapunov exponents ($\lambda$) have been computed in [8, 4] using a coordinate time $t$, where $t$ is measured by the asymptotic observers. Thus, these exponents are explicitly coordinate dependent and therefore have a degree of unphysicality. Here we compute the principal Lyapunov exponent ($\lambda$) and Kolmogorov-Sinai entropy ($h_{KS}$) analytically by using the proper time which is coordinate invariant. Using Lyapunov exponent we investigate the stability and instability of equatorial circular geodesics for Kerr-Newman black-hole space-times. Another interesting point we have studied here is that Lyapunov exponent and KS entropy may be expressed in terms of the radial equation of ISCO.

The author of [11] observed that the null circular geodesics is characterized by the shortest possible orbital period as measured by the asymptotic observers among the all possible circular geodesics which encircle the central Kerr black-hole, thus null circular geodesics provide the fastest way to circle black-hole. Here we would like to add that among the all possible circular geodesics, which encircle the central Kerr-Newman black-hole, the timelike circular geodesics (ISCO) has the longest orbital period than the null circular geodesics as measured by the asymptotic observers. Therefore, the timelike circular geodesics provide the slowest way to circle the black-hole.

The plan of the Letter is as follows: in section 2 we give the basic definition of Lyapunov exponent and also show that it may be expressed in terms of the radial effective potential. In section 3 we provide the relation between Lyapunov exponent and KS entropy. In section 4 we describe reciprocal of Critical exponent can be expressed in terms of the effective radial potential. In section 5 we fully describe the equatorial circular geodesics, both time-like and null case for Kerr-Newman spacetimes. In section 6 the Lyapunov exponent can be expressed in terms of the ISCO equation and studied stability of timelike circular geodesics. In section 7 we show that reciprocal of Critical exponent can also be expressed in terms of the ISCO equation and we finally conclude with discussions in section 8.

## 2 Lyapunov exponent and Radial potential:

In any classical phase space the Lyapunov exponent gives a measure of the average rates of expansion and contraction of a trajectories surrounding it. They are the key indicators
of chaos in any dynamical systems and also they are the asymptotic quantities defined
locally in state space, and describe the exponential rate at which a perturbation to a
trajectory of a system grows or decays with time at a certain location in the state space.
A positive Lyapunov exponent indicates a divergence between two nearby geodesics, i.e.
the paths of such a system are extremely sensitive to changes of the initial conditions. A
negative Lyapunov exponent implies a convergence between two nearby geodesics. Lyap-
unov exponents can also distinguish among fixed points, periodic motions, quasi-periodic
motions, and chaotic motions.

In classical physics, an n-dimensional autonomous smooth dynamical system is gov-
erned by the differential equation[17] of the form

\[
\frac{dx}{dt} = F(x; M).
\]  

where \( t \) is defined usually as time parameter. Following [2, 6], chaos may be quantified
in terms of Lyapunov exponents when the following prescriptions are maintained: a) the
system is autonomous; b) the relevant part of the phase space is bounded; c) the invariant
measure is normalizable; d) the domain of the time parameter is infinite. This definition
signifies that the Lyapunov exponent is invariant under space diffeomorphisms of the form
\( u = \psi(x) \). As a result, chaos is a property of the physical system and does not depend on
the coordinates used to describe the system.

In general relativity(GR), there is no concept of absolute time, therefore the time
parameter forces us to consider equation (1) under spacetime diffeomorphism: \( u = \psi(x),
\)
\( d\tau = \eta(x)dt \). Thus the classical indicators of chaos like Lyapunov exponent and KS
entropy explicitly depend on the choice of the time parameter. This noninvariant charac-
terization implies that chaos is a property of the coordinate system rather than a property
of the physical system.

Motivated by the work of Motter[6], we find that chaos, as characterized by posi-
tive Lyapunov exponents and positive Kolmogorov-Sinai entropy. They are coordinate
invariants and transform according to

\[
\lambda_i^\tau = \frac{\lambda_i^t}{<\eta>^t}.
\]  

and

\[
h_{ks}^\tau = \frac{h_{ks}^t}{<\eta>^t}.
\]  

where \( 0 < (<\eta>) < \infty \) is the time average of \( \eta = \frac{d\tau}{dt} \) over typical trajectory
and \( i = 1, ..., n \), \( n \) is the phase-space dimension. Transformation like \( u = \psi(x), d\tau = \eta(x)dt \)
$\eta(x)dt$ is composed of a time re-parametrization followed by a space diffeomorphism. It is well known that the Lyapunov exponents and Kolmogorov-Sinai Entropy are invariant under space diffeomorphism[16]. In our previous work [9], we have analysed in detail the derivation of Lyapunov exponents using proper time. Following this the Lyapunov exponent may be expressed in terms of the radial potential

$$\lambda = \pm \sqrt{\left(\dot{r}^2\right)''}. \quad (4)$$

where we may defined $\dot{r}^2$ as radial potential or effective radial potential. In general the Lyapunov exponent come in $\pm$ pairs to conserve the volume of phase space. The circular orbit is unstable when the $\lambda$ is real, the circular orbit is stable when the $\lambda$ is imaginary and the circular orbit is marginally stable when $\lambda = 0$.

### 3 Kolmogorov-Sinai entropy and Lyapunove exponent:

An important quantity which is related to the Lyapunov exponents is so called Kolmogorov-Sinai [1] entropy ($h_{ks}$), gives a measure of the amount of information lost or gained by a chaotic orbit as it evolves. Alternatively it determines how a system is chaotic or disorder when $h_{ks} > 0$ and non-chaotic for $h_{ks} = 0$ [16].

Following Pesin [3] it is equal to the sum of the positive Lyapunov exponents i.e

$$h_{ks} = \sum_{\lambda_i > 0} \lambda_i. \quad (5)$$

In 2-dimensional phase-space, there are two Lyapunov exponent, since $h_{ks}$ is equal to the sum of positive Lyapunov exponent, therefore here the Kolmogorov-Sinai entropy in terms of effective radial potential is given by

$$h_{ks} = \sqrt{\left(\dot{r}^2\right)''}. \quad (6)$$

This entropy have played a crucial role in dynamical system to check whether a trajectory is in disorder or not when it evolves with time. It is some sense different from the physical or statistical entropy, for example the entropy of the 2nd law of thermodynamics or blackhole entropy. Formally it is defined somewhat like entropy in statistical mechanics i.e it involves a partition of phase space.
4 Critical exponent and Radial potential:

Following Pretorius and Khurana[7], we can define Critical exponent which is the ratio of Lyapunov time scale $T_\lambda$ and Orbital time scale $T_\Omega$ may be written as

$$\gamma = \frac{\Omega}{2\pi\lambda} = \frac{T_\lambda}{T_\Omega} = \frac{\text{Lyapunov Timescale}}{\text{Orbital Timescale}}. \quad (7)$$

where we have introduced $T_\lambda = \frac{1}{\lambda}$ and $T_\omega = \frac{2\pi}{\Omega}$, which is important for black-hole merger in the ring down radiation. In terms of the square of the proper radial velocity $(\dot{r}^2)$, Critical exponent can be written as

$$\gamma = \frac{T_\lambda}{T_\Omega} = \frac{1}{2\pi} \sqrt{\frac{2\Omega^2}{(\dot{r}^2)'}}. \quad (8)$$

Alternatively the reciprocal of critical exponent is proportional to the effective radial potential which is given by

$$\frac{1}{\gamma} = \frac{T_\Omega}{T_\lambda} = 2\pi \sqrt{\frac{(\dot{r}^2)''}{2\Omega^2}}. \quad (9)$$

5 Equatorial circular geodesics of the Kerr-Newman black-hole:

The Einstein-Maxwell field of a stationary, axisymmetric, charged spinning body with mass $M$, charge $Q$ and angular momentum parameter $a$ is described by the Kerr-Newman(KN) metric which in Boyer-Lindquist coordinates may be written as

$$ds^2 = -\frac{\Delta}{\rho^2} \left[ dt - a \sin^2 \theta \, d\phi \right]^2 + \frac{\sin^2 \theta}{\rho^2} \left[ (r^2 + a^2) \, d\phi - a dt \right]^2 + \rho^2 \left[ \frac{dr^2}{\Delta} + (d\theta)^2 \right]. \quad (10)$$

where

$$a \equiv \frac{J}{M}, \quad \rho^2 \equiv r^2 + a^2 \cos^2 \theta, \quad \Delta \equiv r^2 - 2Mr + a^2 + Q^2 \equiv (r - r_+)(r - r_-). \quad (11)$$

The metric is identical to the Schwarzschild black hole when $a = 0$, $Q = 0$, Reissner Nordström black-hole when $a = 0$ and Kerr black-hole when $Q = 0$. The horizon occurs at $g_{rr} = \infty$ or $\Delta = 0$ i.e.

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2} \quad (12)$$
The outer horizon $r_+$ is called event horizon and the inner horizon $r_-$ is called Cauchy horizon.

To compute the geodesics in the equatorial plane for the Kerr-Newman space-time we follow [14]. To determine the geodesic motions of a test particle in this plane we set $\theta = 0$ and \( \theta = constant = \frac{\pi}{2} \).

Therefore the necessary Lagrangian for this motion is given by

\[
\mathcal{L} = \frac{1}{2} \left[ - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \dot{t}^2 - \left(\frac{4aM}{r} - \frac{2aQ^2}{r^2}\right) \dot{t} \dot{\phi} + \frac{r^2}{\Delta} \dot{r}^2 + \left(r^2 + a^2 + \frac{2Ma^2}{r} - \frac{a^2Q^2}{r^2}\right) \dot{\phi}^2 \right].
\]

(13)

The generalized momenta can be derived from it are

\[
p_t = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \dot{t} + \left(\frac{2aM}{r} - \frac{aQ^2}{r^2}\right) \dot{\phi} = -E = Const .
\]

(14)

\[
p_\phi = - \left(\frac{2aM}{r} - \frac{aQ^2}{r^2}\right) \dot{t} + \left[r^2 + a^2 + \frac{2Ma^2}{r} - \frac{a^2Q^2}{r^2}\right] \dot{\phi} = L = Const .
\]

(15)

\[
p_r = \frac{r^2}{\Delta} \dot{r} .
\]

(16)

Here \((\dot{t}, \dot{r}, \dot{\phi})\) denotes differentiation with respect to proper time(\(\tau\)). Since the Lagrangian does not depends on ‘t’ and ‘\(\phi\)’, so \(p_t\) and \(p_\phi\) are conserved quantities. The independence of the Lagrangian on ‘t’ and ‘\(\phi\)’ manifested, the stationarity and the axisymmetric character of the Kerr-Newman space-time. The Hamiltonian is given by \(\mathcal{H} = p_t \dot{t} + p_\phi \dot{\phi} + p_r \dot{r} - \mathcal{L}\).

In terms of the metric the Hamiltonian is

\[
2\mathcal{H} = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \dot{t}^2 - \left(\frac{4aM}{r} - \frac{2aQ^2}{r^2}\right) \dot{t} \dot{\phi} + \frac{r^2}{\Delta} \dot{r}^2 + \left[r^2 + a^2 + \frac{2Ma^2}{r} - \frac{a^2Q^2}{r^2}\right] \dot{\phi}^2.
\]

(17)

Since the Hamiltonian is independent of ‘t’, therefore we can write it as

\[
2\mathcal{H} = - \left[\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \dot{t} + \left(\frac{2aM}{r} - \frac{aQ^2}{r^2}\right) \dot{\phi}\right] \dot{t} + \frac{r^2}{\Delta} \dot{r}^2 +
\]

\[
\left[- \left(\frac{2aM}{r} - \frac{aQ^2}{r^2}\right) \dot{t} + \left[r^2 + a^2 + \frac{2Ma^2}{r} - \frac{a^2Q^2}{r^2}\right] \dot{\phi}\right] \dot{\phi}.
\]

(18)

\[
= -E \dot{t} + L \dot{\phi} + \frac{r^2}{\Delta} \dot{r}^2 = \epsilon = const .
\]

(19)
Here $\epsilon = -1$ for time-like geodesics, $\epsilon = 0$ for light-like geodesics and $\epsilon = +1$ for spacelike geodesics. Solving equations (14) and (15) for $\dot{\phi}$ and $\dot{t}$, we find
\begin{align*}
\dot{\phi} &= \frac{1}{\Delta} \left[ \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) L + \left( \frac{2aM}{r} - \frac{aQ^2}{r^2} \right) E \right]. \\
\dot{t} &= \frac{1}{\Delta} \left[ \left( r^2 + a^2 + \frac{2Ma^2}{r} - \frac{a^2Q^2}{r^2} \right) E - \left( \frac{2aM}{r} - \frac{aQ^2}{r^2} \right) L \right].
\end{align*}

Inserting these solutions in equations (19), we obtain the radial equation for Kerr-Newman space-time which is given by
\begin{equation}
\dot{r}^2 = r^2 E^2 + \left( \frac{2M}{r} - \frac{Q^2}{r^2} \right) (aE - L)^2 + (a^2E^2 - L^2) + \epsilon \Delta. \tag{22}
\end{equation}

### 5.1 Circular null geodesics

For null geodesics $\epsilon = 0$, the radial equation (22) becomes
\begin{equation}
\dot{r}^2 = r^2 E^2 + \left( \frac{2M}{r} - \frac{Q^2}{r^2} \right) (aE - L)^2 + (a^2E^2 - L^2). \tag{23}
\end{equation}

The equations finding the radius of $r_c$ of the unstable circular ‘photon orbit’ at $E = E_c$ and $L = L_c$ are
\begin{align*}
E_c^2 r_c^2 &+ \left( \frac{2M}{r_c} - \frac{Q^2}{r_c^2} \right) (aE_c - L_c)^2 + (a^2E_c^2 - L_c^2) = 0. \tag{24} \\
2r_c E_c^2 &+ \left( -\frac{2M}{r_c^2} + \frac{2Q^2}{r_c^3} \right) (aE_c - L_c)^2 = 0. \tag{25}
\end{align*}

Now introducing the impact parameter $D_c = \frac{L_c}{E_c}$, the above equations may be written as
\begin{align*}
r_c^2 &+ \left( \frac{2M}{r_c} - \frac{Q^2}{r_c^2} \right) (a - D_c)^2 + (a^2 - D_c^2) = 0. \tag{26} \\
r_c - \left( \frac{M}{r_c^2} - \frac{Q^2}{r_c^3} \right) (a - D_c) = 0. \tag{27}
\end{align*}

From equation (27) we have
\begin{equation}
D_c = a \mp \frac{r_c^2}{\sqrt{Mr_c - Q^2}}. \tag{28}
\end{equation}
The equation (26) is valid if and only if $|D_c - a| > a$. For counter rotating orbit, we have $|D_c - a| = -(D_c - a)$, which corresponds to upper sign in the above equation and co-rotating $|D_c - a| = +(D_c - a)$, which corresponds to lower sign in the above equation. Inserting equation (28) in (26) we find an equation for the radius of null circular orbit

$$r_c^2 - 3Mr_c \pm 2a\sqrt{Mr_c - Q^2 + 2Q^2} = 0.$$  (29)

When $Q = 0$ we recover the well known result[14]. Another important relation can be derived using equations (26) and (28) for null circular orbits are

$$D_c^2 = a^2 + r_c^2 \left(\frac{3Mr_c - 2Q^2}{Mr_c - Q^2}\right).$$  (30)

Now we will derive an important physical quantity associated with the null circular geodesics is the angular frequency measured by asymptotic observers which is denoted by $\Omega_c$

$$\Omega_c = \frac{\left[\left(1 - \frac{2M}{r_c} + \frac{Q^2}{r_c^2}\right)D_c + \left(\frac{2M}{r_c} - \frac{Q^2}{r_c^2}\right)a\right]}{\left[r_c^2 + a^2 - \frac{2Ma^2}{r_c} - \frac{a^2Q^2}{r_c^2}\right]} = \frac{1}{D_c}. \quad \text{(31)}$$

Using equations (28) and (26) we show that the angular frequency $\Omega_c$ of the circular null geodesics is inverse of the impact parameter $D_c$, which generalizes the result of Kerr case[14] to the Kerr-Newman black-hole space-time. It proves that this is a general feature of any stationary space-time.

The equation (23) governing null geodesics in terms of impact parameter at $E = E_c$ and $L = L_c$ can be reduces to

$$\dot{u}^2 = E_c^2 u^4 (D_c - a)^2 (u - u_c)^2 \left[M(2u + u_c) - Q^2(u + u_c)^2\right]. \quad \text{(32)}$$

where

$$u = \frac{1}{r}, \quad u_c = \frac{1}{r_c}$$

$$r_c = \frac{3M}{2} \left[\frac{D_c - a}{D_c + a} \pm \sqrt{\left(\frac{D_c - a}{D_c + a}\right)^2 - \frac{8}{9} \left(\frac{Q}{M}\right)^2 \left(\frac{D_c - a}{D_c + a}\right)}\right]$$

$$r_c = \frac{3M}{2} \left(\frac{D_c - a}{D_c + a}\right) \Xi \quad \text{(33)}$$

$$\Xi = \left[1 \pm \sqrt{1 - \frac{8}{9} \left(\frac{D_c + a}{D_c - a}\right) \left(\frac{Q}{M}\right)^2}\right]. \quad \text{(34)}$$
Integrating equation (32) gives

\[ \tau [E_c(D_c - a)] = \pm \int \frac{du}{u^2(u - u_c)\sqrt{M(2u + u_c) - Q^2(u + u_c)^2}} \]. (35)

To manifested the orbit in the equatorial plane, we can combine it with the equation

\[ \dot{\phi} = \frac{E_c u^2}{3M\Xi(a^2u^2 + a^2Q^2 - 2Mu + 1)u_c} \left[ 3MD_cu_c\Xi + 2u(D_c + a)(Q^2u - 2M) \right] \]. (36)

(which follows directly from equation (20)) to obtain

\[ \frac{du}{d\phi} = \frac{2a^2(D_c + a)(u - u_c)(u - u_+)(u - u_-)\sqrt{M(2u + u_c) - Q^2(u + u_c)^2}}{[3MD_cu_c\Xi + 2u(D_c + a)(Q^2u - 2M)]}. \] (37)

In the limit \( Q = 0, \ \Xi = 2, \ \frac{du}{d\phi} \) is identical to (14). Integrating yields to obtain the trajectory for \( \phi \)

\[ \phi = \pm \frac{1}{2a^2(D_c + a)} \int \frac{[3MD_cu_c\Xi + 2u(D_c + a)(Q^2u - 2M)]}{(u - u_+)(u - u_-)(u - u_c)\sqrt{M(2u + u_c) - Q^2(u + u_c)^2}} du \] (38)

where \( u_\pm = \frac{1}{r_\pm} \). The integral on the right hand side of equation (38) is contains partial fractions.

### 5.2 Circular time-like geodesics

For circular time-like geodesics equation (22) can be written as by setting \( \epsilon = -1 \)

\[ r^2\dot{r}^2 = r^2E^2 + \left( \frac{2M}{r} - \frac{Q^2}{r^2} \right)(aE - L)^2 + (a^2E^2 - L^2)^2 - \Delta. \] (39)

where \( E \) is the energy per unit mass of the particle describes the trajectory.

Now we shall find the radial equation of ISCO that governing the time-like circular geodesics in terms of reciprocal radius \( u = 1/r \) as the independent variable, can be expressed as

\[ \mathcal{V} = u^{-4}\dot{u}^2 = E^2 + 2Mu^3(aE - L)^2 - u^4Q^2(aE - L)^2 + (a^2E^2 - L^2)u^2 - (a^2 + Q^2)u^2 + 2Mu - 1. \] (40)

The conditions for the occurrence of circular orbits are at \( r = r_0 \) or reciprocal radius \( u = u_0 \)

\[ \mathcal{V} = 0. \] (41)
and

\[ \frac{dV}{du} = 0 . \]  (42)

Now setting \( x = L_0 - aE_0 \), where \( L_0 \) and \( E_0 \) are the values of energy and angular momentum for circular orbits at the radius \( r_0 = \frac{1}{u_0} \). Therefore using (40, 42) we get the following equations

\[ -x^2 Q^2 u_0^4 + 2MX^2 u_0^3 - (x^2 + 2axE_0)u_0^2 - (a^2 + Q^2)u_0^2 + 2Mu_0 - 1 + E_0^2 = 0 . \]  (43)

and

\[ -2x^2 Q^2 u_0^3 + 3Mx^2 u_0^2 - (x^2 + 2axE_0)u_0 - (a^2 + Q^2)u_0 + M = 0 . \]  (44)

Using (43, 44) we find an equation for \( E_0^2 \) as

\[ E_0^2 = 1 - Mu_0 + Mx^2 u_0^3 - x^2 Q^2 u_0^4 . \]  (45)

with the aid of equation (45), equation (44) gives us

\[ 2axE_0u_0 = x^2[3Mu_0^2 - 2Q^2 u_0^3 - u_0] - [(a^2 + Q^2)u_0 - M] . \]  (46)

Eliminating \( E_0 \) between these equations, we have obtained the following quadratic equation for \( x^2 \) i.e

\[ Ax^4 + Bx^2 + C = 0 . \]  (47)

where

\[ A = u_0^2 \left[ (3Mu_0 - 1) - 2Q^2 u_0^3 \right] - 4a^2(Mu_0^3 - Q^2 u_0^4) \]
\[ B = -2u_0 \left[ (3Mu_0 - 1 - 2Q^2 u_0^3)((a^2 + Q^2)u_0 - M) + 2a^2 u_0(1 - Mu_0) \right] \]
\[ C = \left[ (a^2 + Q^2)u_0 - M \right]^2 \]

The solution of the equation (47) is

\[ x^2 = \frac{-B \pm D}{2A} . \]  (48)

where the discriminant of this equation is

\[ D = 4au_0 \Delta u_0 \sqrt{Mu_0 - Q^2 u_0^2} . \]  (49)
and

\[ \Delta u_0 = (a^2 + Q^2)u_0^2 - 2Mu_0 + 1. \]  \tag{50} 

The solution becomes simpler form by writing

\[ [(3Mu_0 - 1) - 2Q^2u_0^2]^2 - 4a^2(Mu_0^3 - Q^2u_0^4) = Z_+ Z_. \]  \tag{51} 

where

\[ Z_+ = (1 - 3Mu_0 + 2Q^2u_0^2) \pm 2a\sqrt{Mu_0^3 - Q^2u_0^4}. \]  \tag{52} 

Thus we get the solution as

\[ x^2u_0^2 = \frac{-B \pm D}{Z_+ Z_-}. \]  \tag{53} 

Thus we find

\[ x^2u_0^2 = \frac{\Delta u - Z_+}{Z_+}. \]  \tag{54} 

Again we can write

\[ \Delta u_0 - Z_+ = u_0 \left[ a\sqrt{u_0} \pm \sqrt{M - Q^2u_0}\right]^2. \]  \tag{55} 

Therefore the solution for \( x \) thus may be written as

\[ x = -\frac{a\sqrt{u_0} \pm \sqrt{M - Q^2u_0}}{\sqrt{u_0}Z_\pm}. \]  \tag{56} 

Here the upper sign in the foregoing equations applies to counter-rotating orbit, while the lower sign applies to co-rotating orbit. Replacing the solution (56) for \( x \) in equation (45), we obtain the energy

\[ E_0 = \frac{1}{\sqrt{u_0}Z_\pm} \left[ 1 - 2Mu_0 \mp au_0\sqrt{Mu_0 - Q^2u_0^2}\right]. \]  \tag{57} 

and the value of angular momentum associated with the circular orbit is given by

\[ L_0 = \mp \frac{1}{\sqrt{u_0}Z_\pm} \left[ \sqrt{M - Q^2u_0} \left( 1 + a^2u_0^2 \pm 2au_0\sqrt{Mu_0 - Q^2u_0^2}\right) \pm aQ^2\sqrt{u_0}\right]. \]  \tag{58}
As we previously defined $E_0$ and $L_0$ followed by equations (57) and (58) are the energy and the angular momentum per unit mass of a particle describing a circular orbit of radius $u_0$. Therefore the minimum radius for a stable circular orbit will be obtained at a point of inflection of the function $\mathcal{V}$ i.e we have to supply equations (41, 42) with the further equation

$$\frac{d^2 \mathcal{V}}{du^2}|_{u=u_0} = 0.$$ (59)

Now we have to calculate

$$\frac{d^2 \mathcal{V}}{du^2} = \frac{1}{u} \left[ 6 M x^2 u^2 - 8 Q^2 u^3 x^2 - 2 M \right].$$ (60)

Using (54) we find

$$\frac{d^2 \mathcal{V}}{du^2}|_{u=u_0} = \frac{2}{u_0 Z_\mp} \left[ (3M - 4Q^2 u_0) \Delta_{u_0} + (4Q^2 u_0 - 4M) Z_\mp \right].$$ (61)

Therefore the ISCO occurs at the reciprocal radius

$$\frac{2}{u_0 Z_\mp} \left[ (3M - 4Q^2 u_0) \Delta_{u_0} + (4Q^2 u_0 - 4M) Z_\mp \right] = 0.$$ (62)

or this can be written as

$$(3M u_0^2 - 4Q^2 u_0^3) (a^2 + Q^2) + 8Q^4 u_0^3 - 12MQ^2 u_0^2 \pm 8a (M - Q^2 u_0) \sqrt{M u_0^3 - Q^2 u_0^4} + 6M^2 u_0 - M = 0.$$ (63)

Reverting to the variable $r_0$, we obtain the equation of ISCO for non-extremal Kerr-Newman black-hole is given by

$$M r_0^3 - 6M^2 r_0^2 - 3M a^2 r_0 + 9MQ^2 r_0 = 8a (M r_0 - Q^2)^{3/2} + 4Q^2 (a^2 - Q^2) = 0.$$ (64)

Let $r_0 = r_{ISCO}$ be the smallest real root of the equation, which will be the innermost stable circular orbit of the black-hole. Here (−) sign indicates for direct orbit and (+) sign indicates for retrograde orbit.

**Special cases:**

- When $Q = 0$, we recover the equation of ISCO for Kerr black-hole[14] which is given by

$$r_0^2 - 6 M r_0 + 8 a \sqrt{M r_0 - 3 a^2} = 0.$$ (65)

The smallest real root of this equation gives the radius of ISCO.
• When $a = 0$, we find the equation of ISCO for Reissner Nordstrøm black hole\(^{14}\) which is given by
\[ M r_0^3 - 6M^2 r_0^2 + 9MQ^2 r_0 - 4Q^4 = 0 . \] (66)

The radius of the ISCO can be obtained by finding the smallest real root of the above equation.

• When $a = 0$, $Q = 0$, we get the radius of ISCO for Schwarzschild black hole is given by
\[ r_0 - 6M = 0 . \] (67)

For completeness, we include a treatment for trajectory of the time-like case. Let $L_0$ and $E_0$ are the corresponding values of angular momentum and energy for circular orbit at the reciprocal radius $u = u_0$. Therefore the equation (39), governing the time-like circular geodesics in terms of double root at $u = u_0$ can be reduces to
\[ u^{-4} \dot{u}^2 = 2M(L_0 - aE_0)^2(u - u_0)^2 \left[ u + 2u_0 - (u + u_0)^2 \frac{Q^2}{2M} - \frac{L_0^2 - a^2E_0^2 + a^2 + Q^2}{2M(L_0 - aE_0)^2} \right] . \] (68)

For $L_0$ and $E_0$ given by equations (58) and (57) (for $u = u_0$) we find
\[ \frac{L_0^2 - a^2E_0^2 + a^2 + Q^2}{2M(L_0 - aE_0)^2} = \frac{1 + 3a^2u_0^2 - Q^2u_0^2 \pm 4au_0\sqrt{Mu_0 - Q^2u_0^2}}{2(a\sqrt{u_0} \pm \sqrt{M - Q^2u_0^2})^2} . \] (69)

and
\[ u_0 - \frac{L_0^2 - a^2E_0^2 + a^2 + Q^2}{2M(L_0 - aE_0)^2} = - \frac{\Delta u_0}{2(a\sqrt{u_0} \pm \sqrt{M - Q^2u_0^2})^2} . \] (70)

Therefore the equation (68) can be rewritten as
\[ \dot{u}^2 = 2Mu^4(L_0 - aE_0)^2(u - u_0)^2(-Q^2u^2 + bu + c) . \] (71)

or this can be written as
\[ \dot{u}^2 = 2Mu^4x_0^2(u - u_0)^2(-Q^2u^2 + \alpha u + \beta) . \] (72)
\[ Q_*^2 = \frac{Q^2}{2M} \]
\[ x_0 = L_0 - aE_0 \]
\[ \alpha = 1 - 2u_0Q_*^2 \]
\[ \beta = -u_* - Q_*^2u_0^2 \]
\[ u_* = -u_0 + \frac{\Delta u_0}{2(a\sqrt{u_0} \pm \sqrt{M - Q^2u_0})^2}. \]  
\hspace{1cm} (73)

It can be easily seen that \( u_* \) defines the reciprocal radius of the orbit of the 2nd kind. Therefore the appropriate solution of equation (72) is
\[ \tau = \frac{1}{x_0\sqrt{2M}} \int \frac{du}{u^2(u - u_0)\sqrt{-Q_*^2u^2 + \alpha u + \beta}}. \]  
\hspace{1cm} (74)

Again using equation (72) with
\[ \dot{\phi} = \frac{u^2}{\Delta u} \left[ L_0 - (2Mu - Q^2u^2)x_0 \right]. \]  
\hspace{1cm} (75)

we find the trajectory
\[ \phi = \frac{1}{x_0(a^2 + Q^2)\sqrt{2M}} \int \frac{[L_0 - (2Mu - Q^2u^2)x_0]}{(u - u_+)(u - u_-)(u - u_0)\sqrt{-Q_*^2u^2 + \alpha u + \beta}} du. \]  
\hspace{1cm} (76)

### 5.2.1 Angular Velocity of Time-like Circular Orbit

Now we compute the orbital angular velocity for time-like circular geodesics at \( r = r_0 \) is given by
\[ \Omega_0 = \frac{\dot{\phi}}{\dot{t}} = \frac{\left[ \left( 1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2} \right) L_0 + \left( \frac{2M}{r_0} - \frac{Q^2}{r_0^2} \right) aE_0 \right]}{\left[ r_0^2 + a^2 + \frac{2Ma^2}{r_0^2} - \frac{a^2Q^2}{r_0^2} \right] E_0 - a \left( \frac{2M}{r_0} - \frac{Q^2}{r_0^2} \right) L_0}. \]  
\hspace{1cm} (77)

Again this can be rewritten as
\[ \Omega_0 = \frac{[L_0 - 2Mu_0x + Q^2u_0^3x] u_0^2}{(1 + a^2u_0^2)E_0 - 2aM u_0^3x + aQ^2u_0^4x}. \]  
\hspace{1cm} (78)

Now the previously mentioned expression can be simplified as
\[ L_0 - 2Mu_0x + Q^2u_0^3x = \pm \frac{\sqrt{M - Q^2u_0}}{\sqrt{u_0Z}} \Delta u_0 . \]  
\hspace{1cm} (79)

\[ (1 + a^2u_0^2)E_0 - 2aM u_0^3x + aQ^2u_0^4x = \frac{\Delta u_0}{Z} \left( 1 \pm a\sqrt{M u_0^3 - Q^2u_0} \right). \]  
\hspace{1cm} (80)
Substituting (79) and (80) into (78) we get the angular velocity for circular time-like geodesics is given by

$$\Omega_0 = \mp \frac{\sqrt{Mu_0^3 - Q^2u_0^4}}{1 \mp a\sqrt{Mu_0^3 - Q^2u_0^4}}. \quad (81)$$

Reverting to the variable $r_0$, we obtain the angular velocity for time like circular orbit is

$$\Omega_0 = \mp \frac{\sqrt{Mr_0 - Q^2}}{r_0^2 \mp a\sqrt{Mr_0 - Q^2}}. \quad (82)$$

Correspondingly the time period for time like circular orbit is

$$T_0 = 2\pi \frac{2}{\Omega} = \mp 2\pi r_0^2 \mp a\sqrt{Mr_0 - Q^2} \sqrt{Mr_0 - Q^2}. \quad (83)$$

In the limit $a = Q = 0$, this equation verifies the relativistic Kepler’s law $T_0^2 \propto r_0^3$ for Schwarzschild black-hole.

The rotational velocity with respect to the locally non-rotating observers is given by

$$v^\phi = \frac{\sqrt{Mu_0 - Q^2u_0^2} \left(1 + a^2u_0^2 + 2au_0\sqrt{Mu_0 - Q^2u_0^2}\right) \pm aQ^2\sqrt{u_0^3}}{1 \mp a\sqrt{Mu_0^3 - Q^2u_0^4} \sqrt{\Delta u_0}}. \quad (84)$$

Here we may note that we recover the photon spheres equation (29) for KN spacetimes by taking the limit $E_0 \to \infty$, when

$$Z_\pm = 1 - 3Mu_0 + 2Q^2u_0^2 \pm 2a\sqrt{Mu_0^3 - Q^2u_0^4} = 0. \quad (85)$$

or alternatively

$$r_0^2 - 3Mr_0 \pm 2a\sqrt{Mr_0 - Q^2} + 2Q^2 = 0. \quad (86)$$

The above equation describe the radius of circular photon sphere equation (29) at $r_0 = r_c$. Here (−) sign indicates for direct orbit and (+) sign indicates for retrograde orbit. The real positive root of the equation is the closest circular photon orbit of the black-hole.

5.2.2 Ratio of Angular velocity of time like circular orbit to null circular Orbit

Since we have already proved that for time-like circular geodesics the angular velocity is given by from equation (81)

$$\Omega_0 = \mp \frac{\sqrt{Mu_0^3 - Q^2u_0^4}}{1 \mp a\sqrt{Mu_0^3 - Q^2u_0^4}}. \quad (87)$$
Again we obtained for circular null geodesics $\Omega_c = \frac{1}{D_c}$, so we can deduce similar expression for it is given by

$$\Omega_c = \pm \sqrt{M u_c^3 - Q^2 u_c^4} \over 1 \mp a \sqrt{M u_c^3 - Q^2 u_c^4}. \quad (88)$$

Resultantly we obtain the ratio of angular frequency for time-like circular geodesics to the angular frequency for null circular geodesics is

$$\frac{\Omega_0}{\Omega_c} = \left( \frac{\sqrt{M r_0 - Q^2}}{\sqrt{M r_c - Q^2}} \right) \left( \frac{r_c^2 \mp a \sqrt{M r_c - Q^2}}{r_0^2 \mp a \sqrt{M r_0 - Q^2}} \right). \quad (89)$$

For $r_0 = r_c$, $\Omega_0 = \Omega_c$, i.e., when the radius of time-like circular geodesics is equal to the radius of null circular geodesics, the angular frequency corresponds to that geodesic are equal, which demands that the intriguing physical phenomena could occur in the curved space-time, for example, possibility of exciting quasi normal modes (QNM) by orbiting particles, possibly leading to instabilities of the curved space-time [8].

For $r_0 > r_c$, we have already proved that for Schwarzschild black-hole and Reissner Nordström black-hole [9] the null circular geodesics have the largest angular frequency as measured by asymptotic observers than the time-like circular geodesics. We therefore conclude that null circular geodesics provide the fastest way to circle black holes [11]. This generalizes the case of spherically symmetry Schwarzschild black-hole and Kerr Black-hole [11] to the more general case of stationary, axi-symmetry Kerr-Newman space-times.

Now the ratio of time period of time-like circular geodesics to the time period of null circular geodesics is given by

$$\frac{T_0}{T_c} = \left( \frac{\sqrt{M r_c - Q^2}}{\sqrt{M r_0 - Q^2}} \right) \left( \frac{r_c^2 \mp a \sqrt{M r_c - Q^2}}{r_0^2 \mp a \sqrt{M r_0 - Q^2}} \right). \quad (90)$$

This ratio is valid for $r_0 \neq r_c$. For $r_0 = r_c$, $T_0 = T_c$, i.e. time period of both geodesics are equal, which possibly leading to the excitations of QNM. For $r_0 > r_c$, $T_0 > T_c$, which implies that the orbital period of time-like circular geodesics is greater than the orbital period of null circular geodesics. For $r_0 = r_{ISCO}$ and $r_c = r_{photon}$, the ratio of time period for ISCO ($r_0 = r_{ISCO}$) to the time period for photon-sphere ($r_c = r_{photon}$) for Kerr-Newman black-hole is given by

$$\frac{T_{ISCO}}{T_{photon}} = \left( \frac{\sqrt{M r_c - Q^2}}{\sqrt{M r_0 - Q^2}} \right) \left( \frac{r_c^2 \mp a \sqrt{M r_c - Q^2}}{r_0^2 \mp a \sqrt{M r_0 - Q^2}} \right). \quad (91)$$

This implies that $T_{ISCO} > T_{photon}$, this implies that the orbital period of time-like circular geodesics is larger than the null circular geodesics. Thus we conclude that timelike circular geodesics (ISCO) provide the slowest way to circle the Kerr-Newman black-hole among all circular geodesics.
5.2.3 Marginally bound circular orbit

When a particle at rest at infinity falling towards the black-hole, we call the situation is marginally bound circular orbit. Using equations (57) and (56), the radius of the marginally bound circular orbit with $E_0^2 = 1$ is given by

$$1 = x^2 u_0^2 = \frac{[au_0 \pm \sqrt{Mu_0 - Q^2 u_0^2}]^2}{Z_\mp}.$$  \hspace{1cm} (92)

or

$$Z_\mp = \frac{Mu_0 - Q^2 u_0^2}{Mu_0} [au_0 \pm \sqrt{Mu_0 - Q^2 u_0^2}]^2.$$  \hspace{1cm} (93)

or

$$Mu \left[(1 - 3Mu_0 + 2Q^2 u_0^2) \pm 2a \sqrt{Mu_0^3 - Q^2 u_0^4}\right] = (Mu_0 - Q^2 u_0^2) \left[au_0 \pm \sqrt{Mu_0 - Q^2 u_0^2}\right]^2.$$  \hspace{1cm} (94)

After simplification we obtain the following form for marginally bound circular orbit is given by

$$(a^2 - Q^2)Q^2 u_0^3 + M(4Q^2 - a^2)u_0^2 - 4M^2 u_0 \mp (4aMu_0 - 2aQ^2 u_0^2) \sqrt{Mu_0 - Q^2 u_0^2} + M = 0.$$.  \hspace{1cm} (95)

In terms of $r_0$ we can written it as

$$Mr_0^3 - 4M^2 r_0^2 - Ma^2 r_0 + 4MQ^2 r_0 \mp (4aMr_0 - 2aQ^2) \sqrt{Mr_0 - Q^2} + Q^2(a^2 - Q^2) = 0.$$  \hspace{1cm} (96)

Let $r_0 = r_{mb}$ be the real smallest root of the above equation, which will be the closest bound circular orbit to the black-hole.

**Special Cases:**

In the above equation if we take the following limits:

- When $Q = 0$, we obtain the equation of marginally bound circular orbit for Kerr black-hole which is given by

$$r_0^2 - 4Mr_0 \mp 4a \sqrt{Mr_0} - a^2 = 0.$$  \hspace{1cm} (97)

The smallest real root of this equation gives the marginally bound circular orbit to the black-hole.

- When $a = 0$, we find the equation of marginally bound circular orbit for Reissner Nordstrom Black hole which is given by

$$Mr_0^3 - 4M^2 r_0^2 + 4MQ^2 r_0 - Q^4 = 0.$$  \hspace{1cm} (98)

The radius of the marginally bound circular orbit $r_0 = r_{mb}$ can be obtained by finding the smallest real root of the above equation.
• When $a = 0$, $Q = 0$, we get the radius of marginally bound circular orbit for Schwarzschild black hole which is given by

$$r_0 - 4M = 0. \quad (99)$$

6 Lyapunov exponent and Equation of ISCO:

Now we evaluate the Lyapunov exponent and KS entropy in terms of the radial equation of ISCO as follows, using equation (4) one obtains

$$\lambda = h_{ks} = \sqrt{-\frac{\left(Mr_0^3 - 6M^2r_0^2 - 3Ma^2r_0 + 9MQ^2r_0 \mp 8a \left(Mr_0 - Q^2\right)^{3/2} + 4Q^2(a^2 - Q^2)\right)}{r_0^3 \left(r_0^2 - 3Mr_0 \mp 2a\sqrt{Mr_0 - Q^2 + 2Q^2}\right)}} \quad (100)$$

Circular geodesic motion of the test particle to be exists when both energy (57) and angular momentum (58) are real and finite, therefore we must have $r_0^2 - 3Mr_0 \mp 2a\sqrt{Mr_0 - Q^2 + 2Q^2} > 0$ and $r_0 > \frac{Q^2}{M}$.

It can be easily seen from the above relation that L.H.S is the expression for Lyapunov exponent and KS entropy and R.H.S is the equation of ISCO for Kerr-Newman black-hole. So the time-like circular geodesics of Kerr-Newman black-hole are stable when

$$Mr_0^3 - 6M^2r_0^2 - 3Ma^2r_0 + 9MQ^2r_0 \mp 8a \left(Mr_0 - Q^2\right)^{3/2} + 4Q^2(a^2 - Q^2) > 0. \quad (101)$$

such that $\lambda$ or $h_{KS}$ is imaginary, the circular geodesics are unstable when

$$Mr_0^3 - 6M^2r_0^2 - 3Ma^2r_0 + 9MQ^2r_0 \mp 8a \left(Mr_0 - Q^2\right)^{3/2} + 4Q^2(a^2 - Q^2) < 0. \quad (102)$$

i.e $\lambda$ or $h_{KS}$ is real and the time like circular geodesics is marginally stable when

$$Mr_0^3 - 6M^2r_0^2 - 3Ma^2r_0 + 9MQ^2r_0 \mp 8a \left(Mr_0 - Q^2\right)^{3/2} + 4Q^2(a^2 - Q^2) = 0. \quad (103)$$

such that $\lambda$ or $h_{KS}$ is zero.

Special Cases:

• For Kerr black hole $Q = 0$, the Lyapunov exponent and KS entropy for timelike circular geodesics are

$$\lambda_{Kerr} = h_{ks} = \sqrt{-M \left(r_0^2 - 6M^2r_0 \mp 8a\sqrt{Mr_0^3 - 3a^2}\right)} \div r_0^3 \left(r_0^2 - 3Mr_0 \mp 2a\sqrt{Mr_0}\right). \quad (104)$$
For Reissner Nordstrøm black-hole $a = 0$, the Lyapunov exponent and KS entropy for timelike circular geodesics are

$$\lambda_{RN} = h_{ks} = \sqrt{-\frac{(Mr_0^3 - 6M^2r_0^2 + 9MQ^2r_0 - 4Q^4)}{r_0^3(r_0^2 - 3Mr_0 + 2Q^2)}}. \quad (105)$$

For Schwarzschild black hole $a = Q = 0$, the Lyapunov exponent and KS entropy in terms of ISCO equation are

$$\lambda_{Sch} = h_{ks} = \sqrt{-\frac{M(r_0 - 6M)}{r_0^3(r_0 - 3M)}}. \quad (106)$$

### 6.1 Lyapunov exponent and Null circular geodesics:

For null circular geodesics the Lyapunov exponent and KS entropy are given by

$$\lambda_{Null} = h_{ks} = \sqrt{\frac{(L_c - aE_c)^2(3Mr_c - 4Q^2)}{r_c^6}}. \quad (107)$$

Since $(L_c - aE_c)^2 \geq 0$ and $r_c > \frac{4Q^2}{3M}$, therefore $\lambda_{Null}$ is real so the null circular geodesics are unstable.

**Special Cases:**

- For Kerr black hole $Q = 0$, the Lyapunov exponent and KS entropy for null circular geodesics are

$$\lambda_{Null} = h_{ks} = \sqrt{\frac{3M(L_c - aE_c)^2}{r_c^5}}. \quad (108)$$

- For Reissner Nordstrøm black-hole $a = 0$, so the Lyapunov exponent and KS entropy for null circular geodesics are

$$\lambda_{Null} = h_{ks} = \sqrt{\frac{L_c^2(3Mr_c - 4Q^2)}{r_c^6}}. \quad (109)$$

So the geodesics are unstable since $\lambda_{Null}$ is real for $r_c > \frac{4Q^2}{3M}$.

- For Schwarzschild black hole $a = Q = 0$, the Lyapunov exponent and KS entropy are

$$\lambda_{Null} = h_{ks} = \sqrt{\frac{3ML_c^2}{r_c^5}}. \quad (110)$$

It can be easily check that for $r_c = 3M$, $\lambda_{Null}$ is real, so the Schwarzschild photon sphere are unstable.
7 Critical exponent and Equation of ISCO:

Now we compute the reciprocal of Critical exponent in terms of ISCO equation for Kerr-Newman blackhole to be, by using equation (9)

\[
\frac{1}{\gamma} = 2\pi \frac{\sqrt{- (Mr_0^3 - 6M^2r_0^2 - 3Ma^2r_0 + 9MQ^2r_0 \mp 8a (Mr_0 - Q^2)^{3/2} + 4Q^2(a^2 - Q^2)) (r_0^2 \mp a\sqrt{Mr_0 - Q^2})}}{\sqrt{r_0^4(Mr_0 - Q^2)(r_0^2 - 3Mr_0 \mp 2a\sqrt{Mr_0 - Q^2} + 2Q^2)}}.
\]

Special Cases:

- For Kerr black hole \( Q = 0 \), the reciprocal of Critical exponent in terms of ISCO equation are

\[
\frac{1}{\gamma} = 2\pi \frac{\sqrt{r_0^2 - 6Mr_0 \mp 8a\sqrt{Mr_0} - 3a^2}}{\sqrt{r_0^3(r_0^2 - 3Mr_0 \mp 2a\sqrt{Mr_0})}}.
\] (112)

- For Reissner Nordstrøm black-hole \( a = 0 \), the reciprocal of Critical exponent in terms of ISCO equation are

\[
\frac{1}{\gamma} = \sqrt{- \frac{Mr_0^3 - 6M^2r_0^2 + 9MQ^2r_0 - 4Q^4}{(Mr_0 - Q^2)(r_0^2 - 3Mr_0 + 2Q^2)}}.
\] (113)

- For Schwarzschild black hole \( a = Q = 0 \), the reciprocal of Critical exponents in terms of ISCO equation are

\[
\frac{1}{\gamma} = 2\pi \frac{\sqrt{r_0^2 - 6M}}{\sqrt{r_0 - 3M}}.
\] (114)

7.1 Critical exponent and Null circular geodesics:

The reciprocal of Critical Exponent in terms of null circular geodesics is given by

\[
\left( \frac{1}{\gamma} \right)_{\text{Null}} = 2\pi \frac{\sqrt{(L_c - aE_c)^2(3Mr_c - 4Q^2)(r_c^2 \mp a\sqrt{Mr_c - Q^2})}}{r_c^5(Mr_c - Q^2)}.
\] (115)

Special Cases:

- For Kerr black hole \( Q = 0 \), the reciprocal of Critical exponent for null circular geodesics are

\[
\left( \frac{1}{\gamma} \right)_{\text{Null}} = 2\pi \frac{\sqrt{3(L_c - aE_c)^2(r_c\sqrt{r_c} \mp a\sqrt{Mr_c - Q^2})}}{r_c^5}.
\] (116)
• For Reissner Nordstrøm black-hole \( a = 0 \), the reciprocal of Critical exponent for null circular geodesics are

\[
\left( \frac{1}{\gamma} \right)_{\text{Null}} = 2\pi \sqrt{\frac{L_c^2(3Mr_c - 4Q^2)}{r_c^2(Mr_c - Q^2)}}. \tag{117}
\]

• For Schwarzschild black hole \( a = Q = 0 \), and the reciprocal of Critical exponent are

\[
\left( \frac{1}{\gamma} \right)_{\text{Null}} = 2\pi \sqrt{\frac{3L_c^2}{r_c^2}}. \tag{118}
\]

8 Discussion

We have demonstrated that the Lyapunov exponent, KS entropy and Critical exponent can be used to give a full description of time-like circular geodesics and null circular geodesics in Kerr Newman space-time. We then explicitly derived it in terms of the radial equation of the ISCO. We proved that the Lyapunov exponent can be used to determine the stability and instability of equatorial circular geodesics, both massive and massless particles for Kerr Newman space-time. The other point we have studied that for circular geodesics around the central black-hole, timelike circular geodesics is characterized by the smallest angular frequency as measured by the asymptotic observers—no other circular geodesics can have a smallest angular frequency. Thus such types of space-times always have \( \Omega_{\text{timelike}} < \Omega_{\text{photon}} \) for all time-like circular geodesics. Alternatively it was shown that the orbital period of time-like circular geodesics is characterized by the longest orbital period than the null circular geodesics i.e. \( T_{\text{timelike}} > T_{\text{photon}} \). This implies that the timelike circular geodesics provide the slowest way to circle the black hole. In fact, any stable timelike circular geodesics other than the ISCO traverses more slowly than the null circular geodesics.

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