Limiting Distributions of Scaled Eigensections in a GIT-Setting

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Abstract

Let $L \to X$ be a base point free $T = T^C$-linearized hermitian line bundle over a compact variety $X$ where $T = (S^1)^m$ is a real torus. The main focus of this paper is to describe the asymptotic behavior of a certain class of sequences $(s_n)$ of $T$-eigensections $s_n \in H^0(X, L^n)$ as $n \to \infty$, introduced by Shiffman, Tate and Zelditch, and its connection to the geometry of the Hilbert quotient $\pi : X^{ss}_\xi \to X^{ss}_\xi // T$ where $\xi \in t^*$. Using these sequences $(s_n)$ we will first define a naturally associated sequence of probability measures $(\nu_n(y))_n$ on each fiber $\pi^{-1}(y)$ of the Hilbert quotient where $y$ varies in a Zariski dense subset $Y_0$ of the base $X^{ss}_\xi // T$. In the main part of this paper we will then show that $(\nu_n(y))_n$ converges uniformly over $Y_0$ to a Dirac fiber measure whose support is completely determined by the hermitian bundle metric $h$ and the asymptotic geometry of the rescaled weight vectors $(n^{-1}\xi_n)_n$ given by the initial sequence $(s_n)_n$.

The essential step of the proof is based on the work of D. Barlet whose results provide us with the construction of an equivariant, dimensional-theoretical flattening $\Pi : \tilde{X} \to \tilde{Y}$ of the Hilbert quotient $\pi : X^{ss}_\xi \to X^{ss}_\xi // T$ which turns out to be crucial in order to guarantee uniform estimates over all of $Y_0$.

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I Introduction and Statement of Results

The aim of this thesis is to examine the asymptotic geometry of a certain class of sequences of eigensections of a line bundle by describing the convergence properties of a naturally associated measure sequence. In this discussion, we let $X$ be a projective, normal, purely m-dimensional variety and $T = T^\C$ a complex torus with the unique maximal compact subgroup $T = (S^1)^m$. We equip $X$ with an algebraic action $\phi: T \times X \to X$ of the complex torus $T$ and assume $\phi$ to be compatible with the holomorphic structure of $X$. Furthermore, we will fix a based point free line bundle $p: L \to X$ over $X$ and an algebraic $T$-action $\hat{\phi}: T \times L \to L$ which projects down to $\phi$ so that the corresponding morphisms of the fibers $L|_x$ are linear transformations. In the sequel, we will refer to $L$ as a $T$-linearized line bundle. Moreover, let $h$ be a $T$-invariant smooth, hermitian, positive bundle metric on $L$ and let $t$ be the Lie algebra of the compact form $T \subset T^\C$. In this context, there exists (cf. [GU-ST]) a naturally associated moment map $\mu: X \to t^*$ with respect to the Kähler form $p^* \omega = -\frac{i}{2} \partial \bar{\partial} | \cdot |_h^2$ given by the formula

$$p^* \mu^\xi = -\frac{1}{2} d^c \log | \cdot |_h^2 \hat{X}_\xi$$

where $\hat{X}_\xi$ denotes the fundamental vector field of $\hat{\phi}$ on the total space $L$ (in the sequel we will use the notation $| \cdot |_h^2 = | \cdot |^2$).

If $\xi \in \text{Im}(\mu)$, it is possible to define an equivalence relation on the Zariski open, $T$-invariant set of semistable points

$$X_\xi^{ss} = \{ x \in X : \text{cl}(T.x) \cap \mu^{-1}(\xi) \neq \emptyset \}$$
Given by

\[ z_0 \sim z_1 : \iff \text{cl}(\mathbb{T}, z_0) \cap \text{cl}(\mathbb{T}, z_1) \neq \emptyset. \]

Having defined \( \sim \), it is possible (cf. [He-Hu2], pp. 310-349) to equip the induced quotient \( X_\xi^{ss} / \sim \) with a unique, holomorphic structure of a complex space denoted by \( X_\xi^{ss} / \mathbb{T} \) so that the quotient map \( \pi : X_\xi^{ss} \to X_\xi^{ss} / \mathbb{T} \) is holomorphic and has the following two characteristic properties:

1. If \( \mathcal{O}_X \) denotes the sheaf of holomorphic functions on \( X \) and \( \mathcal{O}_{X^{ss} / \mathbb{T}} \) and is the sheaf of holomorphic functions on \( X_\xi^{ss} / \mathbb{T} \), then \( (\pi_*, \mathcal{O}_{X_\xi^{ss}})^\mathbb{T} = \mathcal{O}_{X_\xi^{ss} / \mathbb{T}} \).

2. We have \( \text{cl}(\mathbb{T}, x) \cap \text{cl}(\mathbb{T}, y) \neq \emptyset \) if and only if \( \pi(x) = \pi(y) \).

As a generalization of the work of Shiffman, Tate and Zelditch (cf. [S-T-Z]) and the results of Huckleberry, Seb"{e}rt (cf. [Hu-SE]), we link the geometry of the sequence of \( \mathbb{T} \)-representation spaces, given by \( H^0(X,L^n) \), as \( n \to \infty \) to the geometry of the quotient \( \pi : X_\xi^{ss} \to X_\xi^{ss} / \mathbb{T} =: Y \). To be more precise, we show that for each choice of \( \xi \in \text{Im}(\mu) \), it is possible to construct a convergent measure sequence which localizes along \( \mu^{-1}(\xi) \) by using sequences of \( \mathbb{T} \)-eigensections \( s_n \in H^0(X,L^n) \), i.e.

\[ \exp(\eta) s_n = e^{2\pi \sqrt{-1} \zeta_n(\eta)} s_n \text{ where } \eta \in t, \xi_n \in t_*^* \]

whose rescaled weights \( \frac{\xi_n}{n} \) asymptotically approximate the chosen \( \xi \in t_*^* \) appropriately as \( n \to \infty \).

As a starting point, we show (cf. Theorem 1) that, given \( \xi \in \text{Im}(\mu) \), there exists a finite cover \( \{X_i\}_{i \in I} \) of \( X_\xi^{ss} \) consisting of open, \( \pi \)-saturated subsets and a finite collection

\[ \{\{s_{ni}\}_{i \in I}\}_{i \in I}, \text{ where } s_{ni} \in H^0(X,L^n) \]

of sequences consisting of \( \mathbb{T} \)-eigensections such that the following properties are fulfilled:

1. \( \frac{\xi_n}{n} \to \xi \) where \( |\xi_n - n \xi| \in O(1) \) for each \( i \).
2. \( X_i \subset X(s_{ni}) := \{ x \in X : s_{ni}(x) \neq 0 \} \) for all \( n \) big enough.
3. \( X_i \cap \mathbb{T}, \mu^{-1}(\xi) = X_i \cap \mathbb{T}, \mu^{-1}(n^{-1} \xi_n) \) for all \( n \geq N_0 \).

The construction of such a tame collection will be the first step of the present work and involves the combinatorial analysis of the sets \( \mu(\text{cl}(\mathbb{T}, x)) \), \( x \in X_\xi^{ss} \), which are known to be convex polytopes in \( t_* \) (cf. [AT]). The crucial step of the existence proof is to control the dependence of the geometry of \( \mu(\text{cl}(\mathbb{T}, x)) \) as \( x \) varies in \( X_\xi^{ss} \). Even in the special case \( X_\xi^{ss} = X_\xi^{ss} \), i.e. where every \( \pi \)-fiber is given by a \( \mathbb{T} \)-orbit, the shape and position of \( \mu(\text{cl}(\mathbb{T}, x)) \) can in general vary considerably.

After having proven the existence of \( \{\{s_{ni}\}_{i \in I}\}_{i \in I} \) with the aforementioned properties, it is possible to define for each finite, open cover \( \Lambda = \{U_i\}_{i \in I} \) subordinate to \( \{\pi(X_i)\}_{i \in I} \), i.e. \( U_i \subset \pi(X_i) \), a finite collection \( \{\nu_n^i\}_{n \in \mathbb{N}} = \{\nu_n^i\}_{i \in I} \) of sequences of \( \pi \)-fiber probability measures on Zariski open subsets of \( X_\xi^{ss} \) by using the corresponding norm functions \( |s_{ni}|^2 \) with respect to the hermitian bundle metric \( h \). The precise construction of the collection \( \{\nu_n\}_{n \in \mathbb{N}} \) is based
on the observation that the ambiguity of the norm sequence \((|s_n^i|^2)_n\), which is only well-determined up to scalar multiplication, can be abolished if one considers the normalized sequence given by \(\|s_n^i\|^{-2}|s_n^i|^2\). Here \(\|s_n^i\|^2\) denotes the fiber integral of the function \(|s_n^i|^2\) over \(\pi_y := \pi^{-1}(y)\). The collection \(\{\nu_n^i\}\) associated to the tame collection \(\{s_n^i\}\) is then given by

\[
\nu_n^i(y) (A) := \int_A \|s_n^i\|^{-2}|s_n^i|^2 d[\pi_y] \text{ for } A \text{ measurable and } y \in U_i \cap Y_0
\]

where \(\int_A d[\pi_y]\) is defined to be the integral over \(A \subset \pi^{-1}(y)\) of the restriction \(\omega^{\dim_{\pi_y}}|\pi_y\) with a certain multiplicity\(^1\). Since the dimension \(k_y = \dim_{\pi^{-1}(y)}\) of a fiber \(\pi^{-1}(y)\) for \(y \in Y\) can change as \(y\) moves in \(Y\) and since the construction of each \(\nu_n^i(y)\) involves \(k_y\), one can not expect that \(\nu_n^i\) defines a uniform object over the full base \(Y\). However, it is possible to find a Zariski dense subset \(Y_0 \subset Y\), so that \(\pi^{-1}(y)\) for \(y \in Y_0\) are purely \(k\)-dimensional complex varieties (set \(X_0 := \pi^{-1}(Y_0)\)). Over this set, it is reasonable to examine the convergence properties of the measure sequence \(\nu_n^i(\cdot)\) for each \(i \in I\). More precisely, if \(f \in C^0(X)\) is a continuous function and if \(f_{\text{red}}\) denotes the reduced function on the base \(Y\) given by the restriction \(\int_{\mu^{-1}(\xi)}\) of the averaged function \(\bar{f}(x) = \int_T f(t.x)\) with respect to the Haar measure \(\nu_T\), we prove the following theorem.

**Theorem 3. [Uniform Convergence of the Tame Measure Sequence]**

For every tame collection \(\{s_n^i\}\) there exists a finite cover \(\Upsilon\) of \(Y\) with \(U_i \subset \pi(X)\) so that the collection of fiber probability measures \(\{\nu_n^i\}\) associated to \(\{s_n^i\}\) converges uniformly on \(Y_0\) to the fiber Dirac measure of \(\mu^{-1}(\xi)\cap X_0\), i.e. for every \(i \in I\) and every \(f \in C^0(X)\), we have

\[
(y \mapsto \int_{\pi^{-1}(y)} f d\nu_n^i(y)) \xrightarrow{\text{f_{\text{red}} uniformly on } U_i \cap Y_0} f_{\text{red}}
\]

Furthermore, if \(\{D_n^i\} = \{D_n^i\}\) denotes the corresponding collection of cumulative fiber probability densities given by

\[
D_n^i(y, \cdot) : t \mapsto D_n^i(y, t) := \int_{\left\{\frac{|s_n^i|^2}{\|s_n^i\|^2} \geq t\right\} \cap \pi^{-1}(y)} d[\pi_y] \text{ for } y \in U_i \cap Y_0
\]

the following convergence result can be proved.

**Theorem 4. [Uniform Convergence of the Tame Distribution Sequence]**

For every \(t \in \mathbb{R}\) and every tame collection \(\{s_n^i\}\) there exists a finite cover \(\Upsilon\) of \(Y\) with \(U_i \subset \pi(X)\) so that the collection of cumulative fiber probability densities \(\{D_n^i(\cdot, t)\}\) associated to \(\{s_n^i\}\) converges uniformly on \(Y_0\) to the zero function on \(Y_0\), i.e. for every \(i \in I\) we have

\[
(y \mapsto D_n^i(y, t)) \xrightarrow{0 \text{ uniformly on } U_i \cap Y_0} \pi(X)\]
I INTRODUCTION AND STATEMENT OF RESULTS

In this sense, we have shown that each tame sequence of eigensections \((s_n^i)\) attached to a prescribed weight \(\xi\), gives rise to a sequence of fiber measures over \(Y_0\), which independently of the choice of \((s_n^i)\), localizes uniformly along the critical \(\mu^{-1}(\xi)\). The localization property of the measure sequence \(\nu_n^i(\cdot)\) attached to the tame collection \(\{(s_n^i)\}\) is a consequence of the fact that the corresponding sequences of strictly plurisubharmonic functions \(\varphi_n^i: X^i \to \mathbb{R}\) given by

\[
\varphi_n^i := -\frac{1}{n} \log |s_n^i|^2
\]

converge (along with all derivatives) uniformly on compact sets to a strictly plurisubharmonic function \(\varphi^i\). It is crucial to note that the restriction \(\varphi_n^i|_{\pi^{-1}(y)}\) of \(\varphi^i\) to each fiber of the projection \(\pi: X^s \to Y\) takes on its T-invariant minimum along the uniquely defined T-orbit \(T.x_y \subset \pi^{-1}(y)\) given by

\[
T.x_y = \mu^{-1}(\xi) \cap \pi^{-1}(y).
\]

Using this observation, it is then possible to deduce estimates of the magnitude of \(\varphi_n^i\) and hence of \(e^{-n \varphi_n^i} = |s_n^i|^2\) outside a T-invariant, relatively compact tube of \(\mu^{-1}(\xi)\) as \(n\) tends to infinity (cf. Theorem 2).

Apart from the determination of the asymptotic behavior of \(\varphi_n^i\), which plays an essential role in the proof of Theorem 3, 4, it is also necessary to deal with the following issue: The fact that \(\nu_n^i(\cdot)\) is defined over a non-compact base makes a direct application of the standard convergence theorems of measure theory considerably more difficult. Therefore, a good portion of the proof of the above convergence theorem will be devoted to resolving this issue by constructing a new quotient \(\Pi: \tilde{X} \to \tilde{Y}\) which extends the restricted quotient \(\pi: X_0 \to Y_0\), so that the following diagram commutes

\[
\begin{array}{ccc}
X_0 & \xrightarrow{\pi} & \tilde{X} \\
\downarrow & & \downarrow \Pi \\
Y_0 & = & \tilde{Y}
\end{array}
\]

and so that all fibers of \(\Pi\) are compact and of pure dimension \(k\).

The construction of this equivariant, dimensional-theoretical flattening \(\Pi: \tilde{X} \to \tilde{Y}\), which is based on results of D. Barlet, allows us to realize the fiber measure sequence \(\nu_n^i(\cdot)\) as a restriction of a measure sequence defined on \(\tilde{X}\). Since \(\tilde{X}\) and all its fibers are compact, it is then possible to show the above mentioned convergence properties of \(\{\nu_n^i\}\) by applying results concerning the continuity of fiber integration.

In the last part of this work, we return to the initial sequence of \(T\)-eigensections \((s_n)\) and examine the convergence properties of the fiber measure sequence induced by \(|s_n|^2\|s_n\|^{-2}\). Unlike in the tame case, we first have to face the problem that the function \(\|s_n\|^{-2}\) is only well defined for all \(y \in Y\) with the property \(s_n|_{\pi^{-1}(y)} \neq 0\). The task of defining a maximal, n-stable set in \(Y\), on which we can consistently write down a measure sequence \((\nu_n)\) for all \(n \in \mathbb{N}\) big enough attached to \((s_n)\), naturally leads to the notion of a removable singularity.

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Having available this concept, which is based on the idea that certain singularities can be "divided out" by multiplying $s_n$ with locally defined, invariant holomorphic functions, we are able to uniquely extend the measure sequence beyond its original set of definition for all $n \geq N_0$ on an open subset of $Y_0$ which is given by $R_{N_0} \cap Y_0$. Here, $R_{N_0}$ denotes the open subset of all singularities $y \in Y$ which are removable for all $n \geq N_0$. Once the maximal set of definition given by $R_{N_0} \cap Y_0$ is found, we continue our discussion by analyzing the convergence properties of the measure sequence $(\nu_n)_n$ over $Y_0 \cap R_{N_0}$ and obtain the following two results.

**Theorem 5.b. [Uniform Convergence of the Initial Distribution Sequence]**

For fixed $t \in \mathbb{R}$ the sequence $(D_n(\cdot, t))_n$ converges uniformly on $Y_0 \cap R_{N_0}$ to the zero function.

**Theorem 6.b. [Uniform Convergence of the Initial Measure Sequence]**

For $f \in C^0(X)$ the sequence

$$\left( y \mapsto \int_{\pi^{-1}(y)} f \, d\nu_n(y) \right)_n$$

converges uniformly over $Y_0 \cap R_{N_0}$ to the reduced function $f_{\text{red}}$.

Since the deviation of the initial measure sequence $(\nu_n)$ induced by $(s_n)_n$ from a tame, locally defined measure sequence $(\nu^i_n)_n$ is completely described by the locally defined sequence of functions $(\Delta^i_n)_n$ given by $s_n = \Delta^i_n \cdot s_0$, it is not surprising that the proof of both theorems is based on technics we already used before when proving **Theorem 3, 4**. These are combined with certain analytic facts about the growth properties of $\Delta^i_n$ as $\rho \to \infty$ on $\pi^{-1}(y)$ and as $n \to \infty$.

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II Existence of Tame Sequences

The aim of this section is to prove the following theorem.

**Theorem 1.** [Existence of Tame Sequences]

If \( L \to X \) is as above and \( \xi \in \text{Im}(\mu) \), then we can find a finite cover
\[
\{X^i\}_{i \in I}
\]
consisting of open, \( \pi \)-saturated subsets and a finite collection
\[
\{(s^i_n)\}_{i \in I}, \text{ where } s^i_n \in H^0(X, L^n)
\]
of sequences consisting of \( T \)-eigensections such that the following properties are fulfilled:

1. \( \xi^i_n \to \xi \) where \( |\xi^i_n - n\xi| \in \mathcal{O}(1) \) for each \( i \).
2. \( X^i \subset X(s^i_n) \) for all \( n \) big enough.
3. \( X^i \cap T.\mu^{-1}(\xi) = X^i \cap T.\mu^{-1}\left(\frac{\xi^i_n}{n}\right) \) for all \( n \geq N_0 \).

In the sequel, we will refer to a collection \( \{(s^i_n)\}_{i} \) of sequences of \( T \)-eigensections \( s^i_n \in H^0(X, L^n) \) with the aforementioned properties as tame. Before we proceed with the proof of Theorem 1, let us consider an example of a tame collection.

**Example II.1.** Let \( X = \mathbb{CP}^1 \times \mathbb{CP}^1 \) with Kähler form \( \omega = p_1^*\omega_{\mathbb{CP}^1} + p_2^*\omega_{\mathbb{CP}^1} \), where \( p_i : X \to \mathbb{CP}^1 \) denotes the \( i \)-th projection, \( i \in \{0, 1\} \). Equip \( X \) with the \( T \cong \mathbb{C}^* \)-action given by
\[
t([z_0 : z_1], [\zeta_0 : \zeta_1]) = ([t z_0: z_1], [t \zeta_0 : \zeta_1])
\]
and consider the \( T \)-linearization on \( L = p_0^*O_{\mathbb{CP}^1}(1) \otimes p_1^*O_{\mathbb{CP}^1}(1) \to X \) induced by the \( T \)-action on \( \mathbb{CP}^4 \) given by
\[
t[u_0 : u_1 : u_2 : u_3 : v] = [tu_0 : u_1 : u_2 : t^{-1}u_3 : v].
\]
Here we view \( L \) embedded in \( O_{\mathbb{CP}^3}(1) \cong \mathbb{CP}^4 \setminus \{0:0:0:1\} \) realized as the cone over
\[
X \cong \{[u] \in \mathbb{CP}^4 : u_0u_3 - u_1u_2 = 0, v = 0\} \subset \{[u] \in \mathbb{CP}^4 : v = 0\} \cong \mathbb{CP}^3.
\]
Furthermore, let \( H \) denote the standard hermitian metric on \( O_{\mathbb{CP}^3}(1) \) which is given by \( H([u:v]) = ||u||^2||v||^2 \) for \( u = (u_0, u_1, u_2, u_3) \neq 0 \) and which is \( T \)-invariant with respect to the above action. Define \( h := H|X \).

If \( \xi = 0 \) and \( \mu \) denotes the moment map induced by the hermitian bundle metric \( h \), it follows that
\[
\mu^{-1}(0) = \{(z_0: z_1), [\zeta_0: \zeta_1]) : z_0\zeta_0 - z_1\zeta_1 = 0\} \subset \mathbb{CP}^1 \times \mathbb{CP}^1.
\]
A calculation shows that the only eigensections \( s_n \in H^0(X, L^n) \) which can be part of a tame sequence in the sense of Theorem 1 are given by the collection
\[
\left\{ s_{n,k} = z_0^{n-k}z_s^kz_0^{n-k}, \quad 0 \leq k \leq n \right\} \subset H^0(X, L^n).
\]
A tame collection is for example given by
\[
\left\{ (z_0^n\zeta^n)_n, (z_1^n\zeta_0^n)_n \right\}.
\]
□□□

For the proof of the existence of tame sequences, we will first restate an important fact about the geometry of an arbitrary \( T \)-orbit closure \( \text{cl}(T.x) \) where \( x \in X \):

Every image \( \mu(\text{cl}(T.x)) \subset t^* \) of an arbitrary orbit closure \( \text{cl}(T.x) \subset X \) is the convex hull of the image of finitely many fixed points \( x_i \in \text{Fix}^T : \{ x \in X : t.x = x \text{ for all } t \in T \} \) (cf. [AT]), i.e.
\[
\mu(\text{cl}(T.x)) = \text{Conv} \left\{ \mu(\mathcal{S}_x) \right\}
\]
where
\[
\mathcal{S}_x = \left\{ \sigma_{x,jx} \in \text{cl}(T.x) \cap \text{Fix}^T, \ 1 \leq jx \leq m_x \right\} \subset \text{Fix}^T
\]
is a finite set and \( \text{Conv} \left\{ \mu(\mathcal{S}_x) \right\} \) denotes the convex hull of the corresponding image \( \mu(\mathcal{S}_x) \).

Consider the decomposition of the \( T \)-representation space \( H^0(X, L) \) in its eigenspaces \( C_{s_k} \) where \( 1 \leq k \leq m := \dim \mathbb{C} H^0(X, L) \). For any non-empty \( J \subset \{1, \ldots, m\} \) let \( \mathcal{S}_J := \{ s_j : j \in J \} \subset C_{s_k} \) and let \( \mathcal{E}_J \subset t^* \) be the set of all characters corresponding to the set \( \mathcal{S}_J \).

We introduce the following notation: For a non-empty \( J \subset \{1, \ldots, m\} \), let
\[
\mathcal{M}_J := \left\{ x \in \mu^{-1}(\xi) : s_j(x) \neq 0 \text{ for all } j \in J, s_j(x) = 0 \text{ for all } j \in \mathbb{C} \setminus J \right\}.
\]
Note that \( \mathcal{M}_J \subset \mu^{-1}(\xi) \) is a \( T \)-invariant, open subset which might be empty for certain non-empty subindices \( J \subset \{1, \ldots, m\} \). Furthermore, the collection \( \{ \mathcal{M}_J \}_{J \neq \emptyset} \) is finite and cover \( \mu^{-1}(\xi) \). The first claim follows by the fact that \( \dim \mathbb{C} H^0(X, L) < \infty \) and the second claim is a direct consequence of the assumption that \( L \) is base point free.

The next lemma establishes a connection between the geometry of the image of \( T.x \) under the moment map \( \mu \) where \( x \in \mathcal{M}_J \) and the convex set \( \text{Conv} \ \mathcal{E}_J \).

**Lemma II.1.** Let \( x \in \mathcal{M}_J \), then it follows that
\[
\mu(\text{cl}(T.x)) = \text{Conv} \ \mathcal{E}_J
\]
and, if \( x \) is not a \( T \)-fixed point,
\[
\xi \in \mu(T.x) = \text{Relint} \ \text{Conv} \ \mathcal{E}_J
\]
where \( \text{Relint} \ \text{Conv} \ \mathcal{E}_J \) denotes the relative interior of \( \mathcal{E}_J \subset t^* \).

**Proof.** First of all note that since the image of \( \text{cl}(T.x) \) is known to be the convex hull of
the image $\mu(S_x)$ where $S_x$ are the $T$-fixed points in $\text{cl}(T.x)$, the first claim follows as soon as we have shown that $\mu(\sigma_{x,j_x}) \in \mathcal{S}_J$ for all $\sigma_{x,j_x} \in S_x$.

Let $\sigma_{x,j_x} \in S_x$. Since $L$ is assumed to be base point free there exists at least one $T$-eigensection $s$ which does not vanish at $\sigma_{x,j_x}$. As $x \in M_1$, such an $s$ is necessarily given by $s = s_j$ for $j \in J$. If $\xi_j$ denotes its corresponding character, we deduce $\mu(\sigma_{x,j_x}) = \xi_j$ which is a direct consequence of the following reasoning: Let $D: \mathcal{A}^0(L) \to \mathcal{A}^1(L)$ be the uniquely defined, hermitian connection associated to $h$ and recall that we have the formula (cf. [Gu-St])

$$D_Xs + 2\sqrt{-1}\mu^\eta s = \frac{d}{dt}|_{t=0}\exp(\sqrt{-1}t\eta)s.$$  \hspace{1cm} (II.2)

If we apply formula (II.2) to $s = s_j$ at the fixed point $\sigma_{x,j_x}$, we deduce $\mu^\eta(\sigma_{x,j_x}) = \xi_j(\eta)$ for all $\eta \in t$. So it follows $\mu(\sigma_{x,j_x}) = \xi_j$ and hence

$$\mu(\text{cl}(T.x)) = \text{Conv} \mathcal{S}_J$$

as claimed.

The second claim follows from the fact that $\xi = \mu(x)$ for $x \in M_1 \subset \mu^{-1}(\xi)$ and the general fact that $\text{cl}(f(A)) = f(\text{cl}(A))$ for a continuous map $f: X \to Y$ where $X$ is compact and $A \subset X$.

Q.E.D.

For the proof of the next proposition, which will be crucial for the existence of tame sequences, we need the following technical lemma.

**Lemma II.2.** Let $\mathcal{P} = \text{Conv} \{(q_1, \ldots, q_m)\}$, $q_1, \ldots, q_m \in \mathbb{R}^n$ and let

$$\mathcal{P}_{n-1}\mathbb{N}^0 := \text{Conv}_{n-1}\mathbb{N}^0 \{q_1, \ldots, q_m\} := \left\{ p = \sum_j \nu_j q_j : \sum_j \nu_j = 1, \nu_j \in n^{-1}\mathbb{N}^0 \right\}.$$  \hspace{1cm} (II.3)

If $\xi \in \mathcal{P}$, then there exists a sequence $\xi_n = \sum_j \nu_{j,n} q_j$ so that the following conditions are fulfilled:

1. $\xi_n \to \xi$
2. $|\xi_n - n\xi| \in O(1)$
3. $\xi_n \in \mathcal{P}_{n-1}\mathbb{N}^0$ for all $n$ big enough.
4. There exists $N_0 \in \mathbb{N}$ and a partition $J = J_0 \cup J_1$, $J_1 \neq \emptyset$ of $J = \{1, \ldots, m\}$ so that $\nu_{j,n} = 0$ for all $j \in J_0$, $n \geq N_0$ and $\nu_{j,n} > 0$ for all $j \in J_1$, $n \geq N_0$.
5. The sequences $(\nu_{j,n})_n$ are convergent so that the limits are strictly positive for all $j \in J_1$.

**Proof.** Let $\mathcal{P}_0 \subset \mathbb{R}^m$ denote the convex hull of the standard basis $\{e_1, \ldots, e_m\}$ in $\mathbb{R}^m$, then

$$\mathcal{P} := \text{Conv} \{q_1, \ldots, q_m\} = M(\mathcal{P}_0)$$  \hspace{1cm} (II.3)
for the matrix $M$ whose columns are given by $(q_1, \ldots, q_m)$. By II.3 we find $\nu \in \mathfrak{P}_0$ with $\xi = M(\nu)$. Without restriction of generality, we can assume that

$$\nu \in \text{Relint Conv} \{e_\ell, \ldots, e_m\}, \quad (\text{II.4})$$

where $1 \leq \ell \leq m - 1$ because otherwise, $\xi = q_{j_0}$ for one $1 \leq j_0 \leq m$ and the claim follows immediately by choosing $\nu_{j_0,n} = 1$ and $\nu_{j,n} = 0$ for all $j \neq j_0$.

Now, it is direct to see that we can choose a sequence $(\nu_n)_n$ in $\text{Conv} \{e_\ell, \ldots, e_m\}$ so that $\nu_n \to \nu$ where all limits are strictly positive. Furthermore, we can always guarantee that $|\nu_n - n \nu| \in O(1)$. Set

$$(\xi_n)_n = (M(\nu_n))_n \subset \mathfrak{P}_{n-1}\mathfrak{P}_0.$$

The first claim follows by $\xi_n = M(\nu_n) \to M(\nu) = \xi$, the second claim is a direct consequence of $|\xi_n - n\xi| \leq \|M\|_\infty |\nu_n - n\nu| \in O(1)$ and the third, resp. fifth claim follows by construction. The fourth claim results from equation II.4: We have $\nu_{j,n} = 0$ for all $j$ with $1 \leq j \leq \ell - 1$ and $\nu_{j,n} > 0$ for all $j$ with $\ell \leq j \leq m$ and $n$ big enough which follows by $\nu_n \to \nu \in \text{Relint Conv} \{e_\ell, \ldots, e_m\}$. Hence, we have $J_0 = \{1, \ldots, \ell - 1\}$ and $J_1 = \{\ell, \ldots, m\}$.

Q. E. D.

The following proposition will be the essential step in order to prove Theorem 1.

**Proposition II.1.** If $L \to X$ is as above and $\xi \in \text{Im}(\mu)$, then for each $x \in M_J$ there exists a sequence $(s^J_n)_n, s^J_n \in H^0(X, L^n)$ of $\xi^J_n$-eigensections with the following characteristic properties:

1. $\xi^J_n \to \xi$ where $|\xi^J_n - n \xi| \in O(1)$
2. The set $X(s^J_n) := \{x \in X: s^J_n(x) \neq 0\}$ is independent of $n$ for $n$ big enough.
3. $X^J := \pi^{-1}(\pi(M_J)) \subset X(s^J_n)$.
4. $X^J \cap T.\mu^{-1}(\xi) = X^J \cap T.\mu^{-1}\left(\frac{\xi^J_n}{n}\right)$ for all $n$ big enough.

**Proof.** By Lemma II.1 we know that $\mu(\text{cl} (T.x)) = \text{Conv } \mathfrak{G}_J$. By applying Lemma II.2 we find a sequence

$$(\xi^J_n)_n \subset \text{Conv } \mathfrak{G}_J$$

so that

$$\xi^J_n = \sum_{j=1}^{\text{Card } J} \nu^J_{j,n} \xi_j$$

where $n \cdot \nu^J_{j,n} \in \mathbb{N} \cup \{0\}$ for all $n \in \mathbb{N}$ and $(\xi^J_n)_n$ converges to $\xi$ so that $|\xi^J_n - n \xi| \in O(1)$. 

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Note that we have \( n \cdot \sum_{j=1}^{\text{Card} J} \nu_{j,n}^J = n \) and that

\[
\xi_n^J = \sum_{j=1}^{\text{Card} J} \nu_{j,n}^J \xi_j = \sum_{j \in J_1} \nu_{j,n}^J \xi_j
\]

(recall that, according to Lemma II.2 we have a partition \( J = J_0 \cup J_1, J_0 \neq \emptyset \), where \( \nu_{j,n}^J = 0 \) for \( j \in J_0 \)). Consider

\[
s_n^J := \prod_{j=1}^{\text{Card} J} s_j^{n \nu_{j,n}^J} \in H^0(X, L^n \cdot \sum_{j=1}^{\text{Card} J} \nu_{j,n}^J) = H^0(X, L^n),
\]

which defines a sequence of eigensections whose associated weight vectors \( \xi_n^J \) approximate \( \xi \):

\[
\frac{\xi_n^J}{n} = \sum_{j=1}^{\text{Card} J} \nu_{j,n}^J \xi_j \to \xi.
\]

Furthermore, by the fourth claim of Lemma II.2 we know that \( \nu_{j,n} = 0 \) for all \( j \in J_0 \) and \( \nu_{j,n} > 0 \) for all \( j \in J_1 \) for all \( n \) big enough. Therefore we deduce

\[
X (s_n^J) = \emptyset \cup \{ s_j = 0 \}
\]

for all \( n \) big enough which proves the second claim.

The third property can be proved as follows: If \( z \in \pi^{-1} (\pi (M_J)) \), then we have \( \text{cl}(\mathbb{T}z) \cap \mathbb{T}M_J \neq \emptyset \). Since \( \text{cl}(\mathbb{T}z) \) is \( \mathbb{T} \)-invariant, it follows that \( \mathbb{T}M_J \subset \text{cl}(\mathbb{T}z) \). As \( s_n^J |_{\mathbb{T}M_J} \neq 0 \) for all \( n \) big enough, it follows that \( s_n^J |_{\text{cl}(\mathbb{T}z)} \neq 0 \) and hence \( s_n^J (z) \neq 0 \) for all \( n \) big enough.

It remains to verify the fourth claim. For this, let \( x \in X^J \cap \mathbb{T} \mu^{-1} (\xi) \). Note that we can assume that \( x \) is not a \( \mathbb{T} \) fixed point: If \( x \) is a \( \mathbb{T} \) fixed point, then it follows \( x \in M_J \subset \mu^{-1} (\xi) \) and hence \( \xi_j = \xi \) for all \( j \in J \). In particular, we find \( \xi = n^{-1} \xi_n^J \) for all \( n \) big enough, so the fourth claim is immediate. In the sequel, let \( x \in X^J \cap \mathbb{T} \mu^{-1} (\xi) \) be not a \( \mathbb{T} \) fixed point. Observe that \( x \in \mathbb{T}M_J \) and hence \( \xi \in \mu (\mathbb{T}x) \) or

\[
\xi \in \text{Relint} \mu (\text{cl}(\mathbb{T}x)) = \text{Relint} \text{Conv} \mathcal{E}_J
\]

by Lemma II.1. Since \( n^{-1} \xi_n^J \to \xi \) and \( \xi \in \text{Relint} \text{Conv} \mathcal{E}_J \), we have

\[
n^{-1} \xi_n^J \in \text{Relint} \mu (\text{cl}(\mathbb{T}x)) = \text{Relint} \text{Conv} \mathcal{E}_J
\]

for all \( n \) big enough as well. Therefore we deduce

\[
x \in \mathbb{T} \mu^{-1} (n^{-1} \xi_n^J)
\]

for all \( n \) big enough, which proves the inclusion "\( \subset \)".
II EXISTENCE OF TAME SEQUENCES

Now, let \( x \in X \cap T.\mu^{-1}(n^{-1}\xi_n) \). Note that, as in the previous case, we can assume that \( x \) is not a \( T \)-fixed point. Moreover, let \( T.z_x \) be the unique closed orbit in \( \pi^{-1}(\pi(x)) \) i.e.

\[
\pi^{-1}(\pi(x)) \cap T.\mu^{-1}(\xi) = T.z_x.
\]

Note that \( z_x \in T.M_J \). If the claim were false, i.e. if \( T.x \neq T.z_x \), then we would deduce

\[
\xi \in \text{Conv} S_J = \mu(\text{cl}(T.z_x)) \subset \text{bd} \mu(\text{cl}(T.x))
\]

where \( n^{-1}\xi_n \in \mu(\text{cl}(T.x)) \) for all \( n \). In particular, by the above inclusion, it then follows that \( n^{-1}\xi_n' \notin \mu(T.x) \) for all \( n \) in contradiction to \( x \in X \cap T.\mu^{-1}(n^{-1}\xi_n) \) for all \( n \) big enough. Therefore, the assumption is false and it follows that \( x \in T.M_J \) and hence \( \xi \in \mu(T.x) \) or \( x \in T.\mu^{-1}(\xi) \) which proves \( "\supset" \).

Q. E. D.

After having shown Proposition II.1, we can prove that the set of semistable points \( X_{ss} \xi \) can be covered by the \( n \)-stable complements of finitely many sequences \( (s_n^i) \) of eigensections whose associated weight sequences \( (\xi_n^i) \) approximate the ray \( R_{\geq 0}\xi \).

**Theorem 1. [Existence of Tame Sequences]**

If \( L \to X \) is as above and \( \xi \in \text{Im}(\mu) \), then we can find a finite cover

\[
\{X^i\}_{i \in I} \text{ of } X_{ss} \xi
\]

consisting of open, \( \pi \)-saturated subsets and a finite collection

\[
\{(s_n^i)\}_{i \in I}, \text{ where } s_n^i \in H^0(X, L^n)
\]

of sequences consisting of \( T \)-eigensections such that the following properties are fulfilled:

1. \( \xi_n^i \to \xi \) where \( |\xi_n^i - n\xi| \in O(1) \) for each \( i \).
2. \( X^i \subset X(s_n^i) \) for all \( n \) big enough.
3. \( X^i \cap T.\mu^{-1}(\xi) = X^i \cap T.\mu^{-1}\left(\frac{\xi_n^i}{n}\right) \) for all \( n \geq N_0 \).

**Proof.** First of all choose an indexing \( I \) of all non-empty \( M_J \) and recall that the collection \( \{M_i\}_{i \in I} \) defines a finite cover \( \mu^{-1}(\xi) \). Hence, the corresponding collection \( \{X^i\}_{i \in I} \) of open, \( \pi \)-saturated subset \( X^i = \pi^{-1}(\pi(M_i)) \) defines a finite cover of \( X_{ss} \xi \) and the claim is a direct consequence of Proposition II.1.

Q. E. D.

Before we proceed with the proof of Proposition II.2, we consider the following example of Theorem 1.

**Example II.2.** Let \( X = \Sigma_m, m \in \mathbb{N} \), be the \( m \)-th Hirzebruch-Surface (cf. [Hir]) which is defined as the projectivization \( \mathbb{P}(O_{\mathbb{C}P^1}(1) \oplus O_{\mathbb{C}P^2}(m)) \) and which is isomorphic to the hypersurface \( \{z_0^m\z_1 - z_1^m\z_2 = 0\} \subset \mathbb{C}P^1 \times \mathbb{C}P^2 \).
Consider the $T = \mathbb{C}^*$-action on $\mathbb{CP}^5$ given by $t \cdot [u] = [u_0 : t u_1 : t u_2 : u_3 : t u_4 : t u_5]$ which pulls back to the $\mathbb{C}^*$-action on $\mathbb{CP}^1 \times \mathbb{CP}^2$ given by

$$t \cdot ([z_0 : z_1], [\zeta_0 : \zeta_1 : \zeta_2]) = ([z_0 : z_1], [\zeta_0 : t \zeta_1 : t \zeta_2])$$

via the Segre-embedding $\sigma_{1,2} : \mathbb{CP}^1 \times \mathbb{CP}^2 \hookrightarrow \mathbb{CP}^5$. Moreover, fix the $\mathbb{C}^*$-linearization on $\mathcal{O}_{\mathbb{CP}^5}(1) \to \mathbb{CP}^5$ given by $t \cdot [u, \zeta] = [t \cdot u : \zeta]$ where we have used the identification

$$\mathcal{O}_{\mathbb{CP}^5}(1) \cong \mathbb{CP}^6 \setminus \{[0 : \ldots : 0 : 1]\}.$$}

It is direct to check that this $\mathbb{C}^*$-linearization can be pulled back to a $\mathbb{C}^*$-linearization on $L := (\sigma_{1,2}^* \mathcal{O}_{\mathbb{CP}^5}(1))|_{\Sigma_m}$. Furthermore, the moment map of the $\mathbb{C}^*$-linearization on $\mathcal{O}_{\mathbb{CP}^5}(1) \to \mathbb{CP}^5$ associated to the standard hermitian metric $h$ on $\mathcal{O}_{\mathbb{CP}^5}(1)$ defined by

$$h((u_0, \zeta_0), (u_0, \zeta_0)) = \|u\|^2 \zeta_0 \overrightarrow{\zeta_1}$$

yields a moment map $\mu : X \to t^*$ on $X$ which is given by

$$\mu([z_0 : z_1], [\zeta_0 : \zeta_1 : \zeta_2]) = \frac{1}{2} \frac{|z_0 \zeta_1|^2 + |z_0 \zeta_2|^2 + |z_1 \zeta_1|^2 + |z_1 \zeta_2|^2}{|z_0 \zeta_0|^2 + |z_0 \zeta_2|^2 + |z_0 \zeta_2|^2 + |z_1 \zeta_1|^2 + |z_1 \zeta_2|^2}.$$}

If $\xi = \frac{1}{2 \sqrt{2}} \in \text{Im} \mu = [0, \frac{1}{2}]$, then

$$\mu^{-1}(\xi) = \left\{ \left(\sqrt{2} - 1\right) \left(|z_0 \zeta_1|^2 + |z_0 \zeta_2|^2 + |z_1 \zeta_1|^2 + |z_1 \zeta_2|^2\right) - |z_0 \zeta_0|^2 - |z_1 \zeta_2|^2 = 0 \right\} \cap \Sigma_m$$

and

$$X_{\xi}^{ss} = X \setminus \left\{ \{\zeta_0 = 0 \vee \zeta_1 = \zeta_2 = 0\}\right\} \subset \mathbb{CP}^1 \times \mathbb{CP}^2$$

where $\pi : X_{\xi}^{ss} \to Y \cong \mathbb{CP}^1$ can be identified with the restriction $p_{\mathbb{CP}^1}|_{\Sigma_m} = \pi$ of the projection map $p_{\mathbb{CP}^1} : \mathbb{CP}^1 \times \mathbb{CP}^2 \to \mathbb{CP}^1$. A further analysis of the example shows that $6 = \dim_{\mathbb{C}} H^0(\Sigma_m, L)$ where

$$s_1 = z_0 \zeta_0, \quad s_2 = z_0 \zeta_1, \quad s_3 = z_0 \zeta_2, \quad s_4 = z_1 \zeta_0, \quad s_5 = z_1 \zeta_1, \quad s_6 = z_1 \zeta_2$$

$$\xi_1 = 0, \quad \xi_2 = 1, \quad \xi_3 = 1, \quad \xi_4 = 0, \quad \xi_5 = 1, \quad \xi_6 = 1.$$}

Moreover, it is direct to check that $M_J \neq \emptyset$ if and only if $J \in \{\{1, 3\}, \{4, 5\}, J\}$. A tame collection is for example given by

$$\{X^{i}\}_{i=1,2} = \{\pi^{-1}(\mathbb{CP}^1 \setminus \{[1 : 0]\}), \pi^{-1}(\mathbb{CP}^1 \setminus \{[0 : 1]\})\}$$

where

$$\{(s^{i}_{n})_{n}\}_{i=1,2} = \left\{ \left(s_{1}^{-1} \left[ \frac{\bar{s}_{1}}{s_{3}} \right] \frac{\bar{s}_{3}}{s_{2}} \right)_{n}, \left(s_{4}^{-1} \left[ \frac{\bar{s}_{4}}{s_{5}} \right] \frac{\bar{s}_{5}}{s_{2}} \right)_{n} \right\}. \square$$

We complete this section by proving the following proposition.
Proposition II.2. If \((s^i_n)_n\) is a tame collection, then the associated collection of sequences of strictly plurisubharmonic potentials \((\varphi^i_n)_n\) given by

\[
\varphi^i_n = -\frac{1}{n} \log |s^i_n|^2
\]

converges uniformly on every compact set of \(X^i\) to a smooth strictly plurisubharmonic function \(\varphi^i: X^i \to \mathbb{R}\). Moreover, the same is true for all its derivatives.

Proof. First of all recall (cf. proof of Proposition II.1) that each sequence \((s^i_n)_n\) is given by

\[
s^i_n = \prod_{j=1}^{\text{Card} J} s^i_{n,j}^{\nu_{j,n}},
\]

where \(J \subset \{1, \ldots, m\}\) is a finite, suitable index set and \((\nu_{j,n})_n\) a sequence of integers such that

\[
\sum_{j=1}^{\text{Card} J} \nu_{j,n}^i s^j \to \xi
\]

where \(J = J_0 \cup J_1, J_1 \neq \emptyset, \nu_j = 0\) for all \(j \in J_0\) and all \(n\) big enough, resp. \(\nu_j > 0\) for all \(j \in J_1\) and all \(n\) big enough. Hence, it follows that

\[
\varphi^i_n = -\sum_{j \in J_1} \nu_{j,n}^i \log |s^j|^2, \nu_{j,n}^i > 0 \quad \text{for} \quad j \in J_1
\]

for all \(n\) big enough. Recall that \(X^i (s^i_n) = \bigcup_{j \in J_1} \{s^j = 0\}\) for all \(n\) big enough. As the sequences \((\nu_{j,n}^i)_n\) are convergent with strictly positive limits for all \(j \in J_1\), it follows that \(\varphi^i_n\) converges uniformly on compact subsets of \(X^i\) in all derivatives to the smooth s.p.s.h. (:= strictly plurisubharmonic) function \(\varphi^i: X^i \to \mathbb{R}\) given by

\[
\varphi^i = -\sum_{j \in J_1} \lim_{n \to \infty} \nu_{j,n}^i \log |s^j|^2.
\]

Q. E. D.

III Uniform Localization Proposition

In this section fix a tame sequence \((s^i_n)_n\) and let \((\varphi^i_n)_n\) be the associated sequence of strictly plurisubharmonic functions. Moreover, recall that by Theorem 1, the subset \(X^i\) is \(\pi\)-saturated and contained in the complement of the zero set \(X^i (s^i_n)\) of \(s^i_n\) for all \(n\) big enough.

We start this section by recalling the following basic fact (cf. [HE-HU2], pp. 310-349).

Lemma III.1. Let \(\varphi^i: X^i \to \mathbb{R}^\geq\) and let \(Y^i = \pi (X^i)\) be as above, then for each \(y \in Y^i\)
there exists a \( \pi \)-saturated, open subset \( \mathcal{V} \) of \( X^s \) where \( \mathcal{W} \) is open so that \( \varrho^i \times \pi \) is proper.

Note that it is always possible to assume that \( \mathcal{W} \) is a compact neighborhood which we will do form now on. Moreover, since \( Y \) is compact finitely many of those compact neighborhoods \( \mathcal{W} \) will cover \( Y \).

In the sequel, we will work with the normalized s.p.s.h function \( \varrho^i \), resp. \( \varrho^i_n \) on \( X^i \) defined by

\[
\varrho^i := \varrho^i - \pi^* \varrho^i_{\text{red}} \quad \text{resp.} \quad \varrho^i_n := \varrho^i_n - \pi^* \varrho^i_{n,\text{red}}.
\]

Since \( \pi^* \varrho^i \) is continuous and since \( \mathcal{W} \) was chosen to be a compact neighborhood, the above lemma remains valid for \( \varrho^i \), resp. for \( \varrho^i_n \). In the sequel, we will set \( \varrho^i = \varrho^i \) and \( \varrho^i_n = \varrho^i_n \).

We define

\[
T((\varepsilon, \mathcal{W})) := (\varrho^i \times \pi)^{-1}((0, \varepsilon) \times \mathcal{W}).
\]

By the above lemma, \( T((\varepsilon, \mathcal{W})) \) is a relatively compact subset of \( X^i \subset X^s \) which is \( T \)-invariant by its definition. Furthermore, define

\[
T^c((\varepsilon, \mathcal{W})) := (\varrho^i \times \pi)^{-1}((\varepsilon, \infty) \times \mathcal{W}).
\]

**Remark III.1.** Note that there exists \( N_0(\varepsilon) \in \mathbb{N} \) so that

\[
\mu^{-1}\left(n^{-1}\xi_n^i \cap \pi^{-1}(\mathcal{W}) \right) \subset T((\varepsilon, \mathcal{W})) \quad \text{for all } n \geq N_0(\varepsilon).
\]

Otherwise there would exist a sequence \( (x_n)_n \) in \( \mu^{-1}\left(n^{-1}\xi_n^i \cap \pi^{-1}(\mathcal{W}) \right) \) so that \( x_n \notin T((\varepsilon, \mathcal{W})) \) for all \( n \in \mathbb{N} \) big enough. Since \( X^i \) is compact, we can find a convergent subsequence \( (x_{n_j})_j \) so that \( x_{n_j} \rightarrow x_0 \) where \( \pi(x_0) \in \mathcal{W} \) and \( x_0 \in \mu^{-1}(\xi) \) which follows by \( x_{n_j} \in \mu^{-1}\left(n^{-1}\xi_n^i \right) \) and \( n_j^{-1}\xi_n^i \rightarrow \xi \). Hence, we have \( x_0 \in \mu^{-1}(\xi) \cap \pi^{-1}(\mathcal{W}) \). On the other hand, we have assumed that \( x_{n_j} \notin T((\varepsilon, \mathcal{W})) \) for all \( n \) big enough. However, since \( T((\varepsilon, \mathcal{W})) \) is an open neighborhood of \( \mu^{-1}(\xi) \cap \pi^{-1}(\mathcal{W}) \) in \( \pi^{-1}(\mathcal{W}) \) this yields a contradiction to \( x_0 \in \mu^{-1}(\xi) \cap \pi^{-1}(\mathcal{W}) \).

Before we prove **Theorem 2**, we need the following preparation.

**Lemma III.2.** Let \( (s^i_n)_n \) be a tame sequence as above, \( \pi^{-1}(y) \subset X^i \) and let \( T.z^i \) be the unique closed orbit in \( \pi^{-1}(y) \) then it follows that \( \varrho^i_n|\pi^{-1}(y) \cap T.z^i \) takes on a unique minimum along the set

\[
T.\mu^{-1}\left(n^{-1}\xi_n^i \cap \pi^{-1}(y) \right)
\]

which is contained in \( T.\mu^{-1}(\xi) \cap \pi^{-1}(y) \).

**Proof.** Note that since \( (s^i_n)_n \) is a tame sequence we have

\[
T.\mu^{-1}(\xi) \cap X^i = T.\mu^{-1}\left(n^{-1}\xi_n^i \cap X^i \right).
\]
by the third claim of Theorem 1. This shows that the set
\[ T_\mu^{-1}(n^{-1}\xi_n) \cap \pi^{-1}(y) \]
is contained in \( T_\mu^{-1}(\xi) \cap \pi^{-1}(y) \) so the second claim is proved.

The first claim is a direct consequence of the fact that the unique minimum of \( \bar{\varphi}_n \) on
\[ T_\mu^{-1}(n^{-1}\xi_n) \cap \pi^{-1}(y) \]
is known to be equal to \( T_\mu^{-1}(n^{-1}\xi_n) \cap \pi^{-1}(y) \).

Q. E. D.

We can now prove the uniform Localization Proposition.

**Theorem 2.** [Uniform Localization of the Potential Functions]

Let \((\varphi_n^i)\) be a tame sequence and \((\bar{\varphi}_n^i)\) the associated sequence of strictly plurisubharmonic functions. Let \( W^i \subset \pi(X^i) \) and \( \epsilon > 0 \) be as above and let \( \delta > 0 \) be given. Then there exists \( N_0 \in \mathbb{N} \) so that
\[ \bar{\varphi}_n^i(x) \geq \epsilon - \delta \]
for all \( x \in T^c(\epsilon, W^i) \) and all \( n \geq N_0 \).

**Proof.** First of all, by Remark III.1 we can assume that
\[ \mu^{-1}(n^{-1}\xi_n) \cap \pi^{-1}(W^i) \subset T(\epsilon, W^i) \quad \text{for all } n \geq N_0(\epsilon). \] (III.1)

Moreover, note that \( \bar{\varphi}_n^i \) converges uniformly on relatively compact sets and hence on \( T(\epsilon, W^i) \).
Therefore, we can find an \( N_0 \in \mathbb{N} \) so that
\[ \bar{\varphi}_n^i(x) \geq \epsilon - \delta \] (III.2)
for all \( x \) in the compact subset \((\varphi^i \times \pi)^{-1}(\{\epsilon\} \times W^i)\) and all \( n \geq N_0 \). We continue the proof by considering the following two cases:

1) First of all let, \( x \in T^c(\epsilon, W^i) \cap T.z_x \) where \( T.z_x \) is the unique closed orbit in the fiber \( \pi^{-1}(\pi(x)) \). We have to show that
\[ \bar{\varphi}_n^i|T^c(\epsilon, W^i) \cap T.z_x > \epsilon - \delta. \]

Applying Lemma III.2, we know that the restriction \( \bar{\varphi}_n^i|T.z_x \) of the strictly plurisubharmonic function \( \bar{\varphi}_n^i \) on the unique closed orbit \( T.z_x \) in the fiber \( \pi^{-1}(\pi(x)) \subset X^i \) takes on its minimum along
\[ \mu^{-1}(n^{-1}\xi_n) \cap \pi^{-1}(\pi(x)) \subset T.z_x. \]

If \( m_x := \text{Ann} \ t_x \subset t^* \) denotes the annihilator \( \{\eta \in t^* : \langle \eta, \xi \rangle = 0 \text{ for all } \xi \in t_x\} \) of the isotropy \( t_x \), then \( T.z_x \) is isomorphic to the homogenous vector bundle with typical fiber \( m_x \):
\[ T.z_x \cong T \times^{T^*} m_z. \]
III UNIFORM LOCALIZATION PROPOSITION

Note that the zero section in $T \times T^*_x m_z$ is mapped under the above identification to the $T$-orbit

$$T.\mu^{-1}(\xi) \cap T.x.$$ 

Moreover, using this identification, it follows that the restriction of $\phi^1_n$ on $T.z_x$ yields a smooth function on $T \times T^*_x m_z$ whose restriction on each fiber $\{(t) \times \{m_n\}\}$, $t \in T$ is a strictly convex function with a unique minimum given by $\{(t) \times \{m_n\}\}$ for $m_n \in m_x$. Note that the tube $T(\epsilon, W^i) \cap T.z_x$ is isomorphic to a tube of the zero section in $T \times T^*_x m_x$ and also note that this tube contains $[T \times \{m_n\}]$ for all $n$ big enough by the remark at the beginning of this proof which is based on Remark III.1.

We continue the proof by connecting $x = [(t) \times \{\eta_n\}]$, for $\eta_n \in m_z$ suitable, and the unique minimum $m_n = [(t) \times \{m_n\}]$ of $g^1_n$ with a line $\Lambda: \mathbb{R} \to [(t) \times \{m_n\}]$ in the vector space $\{(t) \times m_z\}$ so that $\Lambda(0) = m_n$ and $\Lambda(1) = x$. Note that this line intersects $g^1_n^{-1}(\epsilon)$ in, say $\Lambda(\tau_n) = y_n$, where $0 < \tau_n < x$, because the minimum of $g^1_n$ is contained in the tube $T(\epsilon, W^i)$ (for all $n$ big enough) whereas $x$ is not by our assumption. To sum up, we have a convex function $\Lambda^* g^1_n$ on $\mathbb{R}$ with a unique minimum at 0 so that $\Lambda^* g^1_n(\tau_n) \geq \epsilon - \delta$ for all $n$ big enough by equation III.2. Hence, it follows that $g^1_n(x) \geq \epsilon - \delta$ as well for all $n$ big enough.

2) The next step is to show that the inequality $g^1_n(x) \geq \epsilon - \delta$ also holds for all $x \in T^c(\epsilon, W^i)$ so that $x$ is not contained in the unique closed orbit $T.x_x$ of the fiber $\pi^{-1}(\pi(x))$.

Let us assume that this is not true. By the Hilbert Lemma (cf. [KRA]), we can find a one parameter group $\gamma: C \to T$ so that

$$\lim_{t \to 0} \gamma(t).x \in T.z_x.$$ 

Note that neither the pull back $\gamma^* g^1$ nor the pull back $\gamma^* g^1_n$ attains its $S^1$-invariant minimum on $C^*$ since otherwise it would follow that $g^1_n|\text{Im}(\gamma|C^*)$ and $g^1_n|\text{Im}(\gamma|C^*)$ attain their $S^1$-invariant minimum on $T.x$. However, this would yield a contradiction to the assumption $T.x \neq T.z_x$ and the claim of Lemma III.2. Hence, it follows that $t \mapsto \gamma^* g^1_n(t)$ and $t \mapsto \gamma^* g^1(t)$ where $t \in C^*$ are strictly monotone decreasing when $|t| \to 0$ (they can not be strictly increasing since $s^1_n$ does not vanish on $T.z_x$). The present case can be subdivided into the following cases:

2.a) Assume that

$$x_0 = \lim_{t \to 0} \gamma(t).x \in T.z_x \cap T^c(\epsilon, W^i),$$

then we have

$$g^1_n(x_0) \geq \epsilon - \delta$$

for all $n \geq N_0$ by case 1). Since $\gamma^* g^1_n$ is monotone decreasing for $|t| \to 0$ we deduce that

$$\delta - \epsilon \geq \gamma^* g^1_n(1) = g^1_n(x)$$

for all $n \geq N_0$ as claimed.

2.b) Assume that

$$x_0 = \lim_{t \to 0} \gamma(t).x \in T.z_x \cap T(\epsilon, W^i),$$

then we have

$$g^1_n(x_0) \geq \epsilon - \delta$$

for all $n \geq N_0$ by case 1). Since $\gamma^* g^1_n$ is monotone decreasing for $|t| \to 0$ we deduce that

$$\delta - \epsilon \geq \gamma^* g^1_n(1) = g^1_n(x)$$

for all $n \geq N_0$ as claimed.
then there exists $t \in \mathbb{C}^*$ so that $\gamma(t) \cdot x \in \varrho^{i-1}(\epsilon)$ and hence $\varrho^{i}_{n}(\gamma(t) \cdot x) = \epsilon - \delta$ for all $n \geq N_0$ by III.2. As in case 2.a), it then follows that $\varrho^{i}_{n}(x) \geq \delta - \epsilon$ for all $n \geq N_0$ as claimed. Q. E. D.

We close this section with the following corollary which slightly generalizes the claim of Theorem III.

In the context of Theorem III, fix $m_0 \in \mathbb{N}$ and consider the sequence $(s_{i_n}^{i-m_0})_{n}$ of meromorphic $\eta_{n}^{i-m_0} := \xi_{n}^{i} - \xi_{m_0}^{i}$-eigensections. Since we have $s_{i_n}^{i}(x) \neq 0$ for all $x \in X^i$ and all $n \in \mathbb{N}$ and hence in particular for $n = m_0$, we can define

$$\varrho_{n}^{i,m_0} := -\frac{1}{n}\log |s_{i_n}^{i} \cdot s_{m_0}^{i-1}|^2.$$  

It is direct to check that the above definition yields a sequence of smooth, strictly plurisubharmonic, $T$-invariant functions for all $n \geq m_0$ on the $\pi$-saturated open set $X^i$. Furthermore, it is known that $\varrho_{n}^{i,m_0}|\pi^{-1}(y)$ for $y \in \pi(\{X^i\})$ takes on its unique minimum along the set $\mu^{-1}(n^{-1}\eta_{n}^{i,m_0}) \cap \pi^{-1}(y)$.

Therefore, after having applied the argumentation of Remark III.1, we can assume, as at the beginning of the proof of Theorem III, that

$$\mu^{-1}(n^{-1}\eta_{n}^{i,m_0}) \cap \pi^{-1}(W^i) \subset T(\epsilon, W^i) \text{ for all } n \text{ big enough.}$$

The proof of Theorem III now translates verbatim to the sequence $(\varrho_{n}^{i,m_0})_{n}$ and yields the following corollary.

**Corollary III.1.** If $\delta > 0$, $m_0 \in \mathbb{N}$ fixed, then it follows $\varrho_{n}^{i,m_0}(x) \geq \epsilon - \delta$ for all $x \in T^c(\epsilon, W^i)$ and all $n$ big enough.

## IV Fiber Probability Measure Sequence

### IV.1 Definition of $\hat{\pi}: \hat{X} \to Y$ and $Y_0$

As before, let $X$ be a purely $m$-dimensional, normal $T$-variety with Kähler structure $\omega$ and let $Y = X^{ss}/T$ be the associated Hilbert quotient. Note that we can always assume $Y$ to be purely dimensional, i.e. $n = \dim T Y$: Since $X$ is normal by our assumption, it follows (cf. [He-Hu1], p. 124) that $Y$ is normal as well. In particular, it follows that $Y$ is locally of pure dimension (cf. [Gr-Re], p. 125) and by considering the connected components of $Y$, which are finite in number, we can confine ourselves to the case where $Y$ is of pure
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dimension\( n = \dim_C Y \).

Now, recall that \( X_{\xi}^{ss} \) is Zariski open and Zariski dense in \( X \). Consider the compact variety \( \hat{X} \) defined by the normalization of \( \text{cl} \Gamma_\pi \), where \( \Gamma_\pi \) denotes the graph of \( \pi \), i.e.

\[
\Gamma_\pi := \{ (x, y) \in X_{\xi}^{ss} \times Y : \pi(x) = y \}.
\]

Furthermore, we define the algebraic map \( \hat{\pi} : \hat{X} \to Y \) by \( \hat{\pi} := p_Y |_{\text{cl} \Gamma_\pi} \circ \zeta \), where \( \zeta : \hat{X} = (\text{cl} \Gamma_\pi)^{\text{nor}} \to \text{cl} \Gamma_\pi \) denotes the normalization map and \( p_Y : X \times Y \to Y \). Moreover, endow \( \hat{X} \) with the \( T \)-action induced by the lift (for its existence see [Gr-Re], pp. 164 f.) of the \( T \)-action on \( \text{cl} \Gamma_\pi \), defined by \( t(x, y) = (t, x, y) \) and equip \( \hat{X} \) with a smooth \( (2, 2) \)-form \( \omega' \) given by \( \omega' := (p_X |_{\text{cl} \Gamma_\pi} \circ \zeta)^* \omega \). For the sake of completeness, we note the following remark.

**Remark IV.1.** The graph \( \Gamma_\pi \subset X \times Y \) is \( T \)-invariant with respect to the action on \( X \times Y \) given by \( t(x, y) = (t, x, y) \). Since \( \Gamma_\pi \) is a Zariski open and Zariski dense subset of \( X \), it follows that \( \text{cl} \Gamma_\pi \) is \( T \)-invariant. Moreover, as \( \zeta \) is \( T \)-equivariant, all fibers \( \hat{\pi}^{-1}(y) \), \( y \in Y \) are \( T \)-invariant as well.

Note that since \( X \) is assumed to be of pure dimension \( m \), it follows that \( \text{cl} \Gamma_\pi \) is likewise a purely \( m \)-dimensional subvariety of \( X \times Y \). As \( \zeta \) is finite, we deduce that \( \hat{X} \) is of pure dimension \( m \) too.

The next step is to find a Zariski open subset \( Y_0 \subset Y \) so that the fibers of the restricted projection \( \hat{\pi} : \hat{\pi}^{-1}(Y_0) : \hat{\pi}^{-1}(Y_0) \to Y_0 \) are all purely \( k \)-dimensional varieties. The existence of \( Y_0 \) is a direct consequence of known facts in complex analysis and algebraic geometry. By a theorem of CARTAN and REMMERT (cf. [Loj], p. 271 f.), it follows that the subset \( E \subset \hat{X} \) defined by

\[
E := \left\{ x \in \hat{X} : k < \dim_{C,x} \hat{\pi}^{-1}(\hat{\pi}(x)) \right\} \subset \hat{X}
\]

is a proper analytic subset of \( \hat{X} \) where \( k = m - n = \dim_C \hat{X} - \dim_C Y \). Hence, by CHOW’s Theorem it follows that \( E \) is a proper algebraic subset. Applying the Direct Image Theorem (cf. [Gr-Re], p. 207), one deduces that the image \( \hat{\pi}(E) \) is a proper analytic subset of \( Y \). In particular it is a proper algebraic subvariety of \( Y \). Now, set \( Y_0 := \hat{\pi}(E) \) and note that all fibers of \( \hat{\pi} \) over \( Y_0 \) are purely \( k \)-dimensional by construction.

For later use, we introduce the following notation: Let \( X, Y \) be purely dimensional complex spaces \( (m = \dim_C X, n = \dim_C Y) \) where \( Y \) is assumed to be normal and let \( F : X \to Y \) be a holomorphic map so that all non-empty fibers \( F^{-1}(y) \) are of pure dimension \( k = m - n \). Then \( F \) is called a \( k \)-fibering. Note that \( \pi|\pi^{-1}(Y_0) : \pi^{-1}(Y_0) \to Y_0 \) is a \( k \)-fibering.

An example for the construction of \( \hat{X} \) as described above is given by the next example.

**Example IV.1.** Let \( X = \mathbb{CP}^1 \times \mathbb{CP}^1 \) with the Kähler form \( \omega = p_1^* \omega_{FS} + p_2^* \omega_{FS} \) where \( p_i : X \)
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→ \mathbb{CP}^1 denotes the i-th projection, i ∈ \{0, 1\} equipped with the T ∼ \mathbb{C}^\ast-action given by

\[ t \cdot ([z_0 : z_1], [\zeta_0 : \zeta_1]) = ([t \cdot z_0 : z_1], [t \cdot \zeta_0 : \zeta_1]) \]

and consider the moment map given by (cf. Example II.1)

\[ \mu ([z_0 : z_1], [\zeta_0 : \zeta_1]) = \frac{|z_1|^2}{\|z\|^2} - \frac{|\zeta_0|^2}{\|\zeta\|^2}. \]

In particular, we have

\[ \mathcal{X}^{ss}_{\xi} = \mathcal{X} \setminus \{\{z_1 = 0\} \cup \{\zeta_1 = 0\} \cup \{([1:0], [0:1])\}\} \]

where \( \pi : \mathcal{X}^{ss}_0 \to \mathcal{X}^{ss}/T = \mathcal{Y} \cong \mathbb{CP}^1 \) is given by the map \( \pi : ([z_0 : z_1], [\zeta_0 : \zeta_1]) \to z_0 \cdot \zeta_1 \cdot z_1 \cdot \zeta_0 \).

Furthermore, a calculation shows that

\[ \bar{\mathcal{X}} = \text{cl} \Gamma_\pi = \{z_0 \cdot \zeta_1 - z_1 \cdot \zeta_0 \cdot \zeta_0 = 0\} \subset \mathcal{X} \times \mathbb{CP}^1 \]

if \([\zeta_0, \zeta_1]\) are the homogeneous coordinates of \( \mathbb{CP}^1 \). A further analysis of the geometry of \( \bar{\mathcal{X}} \) and \( \mathcal{Y} \) reveals that \( \mathcal{Y}_0 = \mathcal{Y} \). Note that we have

\[ p_\mathcal{X} (\pi^{-1} ([\zeta_0, \zeta_1])) = \text{cl} (\pi^{-1} ([\zeta_0, \zeta_1])) \text{ for all } [\zeta_0, \zeta_1] \text{ such that } \zeta_{0,1} \neq 0. \]

For \([\zeta_0] = [1:0]\) and \(\zeta_1 = [0:1]\), we have a proper inclusion

\[ p_\mathcal{X} (\pi^{-1} ([\zeta_i])) \supset \text{cl} (\pi^{-1} ([\zeta_i])) \]

for all \(i \in \{0, 1\}. \square \)

IV.2 The Fiber Probability Measure Sequence (Tame Case)

Let \( \mathcal{U} = \{\mathcal{U}_i\}_i \) be a finite cover of \( \mathcal{Y} \) then, after having chosen a finite refinement \( \mathcal{U}' \) of \( \mathcal{U} \) (for convenience set \( \mathcal{U} = \mathcal{U}' \)), it follows by Section II that there exists a tame collection \(\{ (s^i_n) \}_i \) so that \( \mathcal{U}_j \subset \pi (\mathcal{X}_i) = \mathcal{Y}_i \). By changing the index set \( J \) of the finite cover \( \mathcal{U} \) we can always assume that \( \mathcal{U}_j = \mathcal{U}_i \).

Let \( \mathcal{Y}_0 \) be as in Section IV.1 and set

\[ \|s^i_n\|^2 : \mathcal{X}_i \cap \mathcal{Y}_0 \to \mathbb{R}^\geq 0, \|s^i_n\|^2 (x) := \int_{\pi^{-1}(\pi(x))} |s^i_n|^2 d[\pi_y] \]

where \( \int_{\pi^{-1}(\pi(x))} d[\pi_y] \) denotes the fiber integral of \( \omega^k \) with respect to the k-fibering \( \pi : \mathcal{X}_0 \to \mathcal{Y}_0 \) as defined in the work of J. KING (cf. [KIN]).
Remark IV.2. Since we have
\[
\int_{\pi^{-1}(y)} |s^i_n|^2 (\omega|\pi^{-1}(y))^k \leq \int_{\hat{\pi}^{-1}(y)} (\zeta \circ p_X)^* |\hat{s}^i_n|^2 (\omega'|\pi^{-1}(y))^k
\]
it follows by the compactness of \( \hat{\pi}^{-1}(y) \) where \( y \in Y \), that
\[
\|s^i_n\|^2 < \infty \text{ for all } y \in Y_0 \subset Y.
\]

The next step is to define a sequence of collections of fiber probability densities attached to \( \mathcal{U} = \{U_i\}_i \) as follows.

**Definition IV.1.** Let \( \mathcal{U} = \{U_i\}_i \) and \( \{s^i_n\}_i \) be as above, i.e. \( U_i \subset \pi(X^i) \), then define a sequence of collections of fiber distribution densities on \( \pi^{-1}(U_i) \cap X_0 \) by
\[
\phi^i_n : x \mapsto \phi^i_n(x) = \|s^i_n\|^2(x) |\hat{s}^i_n|^2(x).
\]

In the sequel, the terminology \( \{\phi^i_n\} = \{\phi^i_n\}_i \) will be used. Furthermore, \( \{\phi^i_n\} \) will be referred to as the collection of fiber distribution densities associated to \( \{s^i_n\}_i \).

Having defined \( \{\phi^i_n\} \), it is self-evident to introduce

**Definition IV.2.** Let \( \{\phi^i_n\} \) be a sequence of collections of fiber distribution densities associated to a tame collection \( \{s^i_n\}_i \), then define a sequence of collections of fiber probability measures over \( Y_0 \) by
\[
\nu^i_n(y) : A \mapsto \nu^i_n(y)(A) := \int_A d\nu^i_n := \int_A \phi^i_n d[\pi_y]
\]
where \( y \in U_i \cap Y_0 \) and \( A \subset \pi^{-1}(y) \) measurable.

As in **Definition IV.1** set \( \nu^i_n(U_i) = \{\nu^i_n\}_i \) and refer to \( \nu^i_n(U_i) \) as the collection of fiber probability measures associated to \( \{s^i_n\}_i \).

We complete our definitions with

**Definition IV.3.** Let \( \{\phi^i_n\} \) be a sequence of collections of fiber distribution densities associated to a tame collection \( \{s^i_n\}_i \), then define a sequence of collections of cumulative fiber probability densities over \( Y_0 \) by
\[
D^i_n(y, \cdot) : t \mapsto D^i_n(y, t) := \int_{\{\phi^i_n \geq t\} \cap \pi^{-1}(y)} d[\pi_y]
\]
where \( y \in U_i \cap Y_0 \).

As in the aforementioned definitions, we set \( D^i_n(U_i) = \{D^i_n\}_i \) and refer to \( D^i_n(U_i) \) as the collections of cumulative fiber probability densities associated to \( \{s^i_n\}_i \).

In **Section VII** we will give a prove of the following two convergence results.
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**Theorem 3. [Uniform Convergence of the Tame Measure Sequence]**

For every tame collection \( \{s_n^i \}_i \) there exists a finite cover \( \mathcal{U} \) of \( Y \) with \( U_i \subset \pi(X^i) \) so that the collection of fiber probability measures \( \{\nu_n^i\} \) associated to \( \{s_n^i\}_i \) converges uniformly on \( Y_0 \) to the fiber Dirac measure of \( \mu^{-1}(\xi) \cap X_0 \), i.e. for every \( i \in I \) and every \( f \in C^0(X) \), we have

\[
(y \mapsto \int_{\mu^{-1}(y)} f \, d\nu_n^i(y)) \quad \xrightarrow{\text{f_{red uniform}} on U_i \cap Y_0.}
\]

**Theorem 4. [Uniform Convergence of the Tame Distribution Sequence]**

For every \( t \in \mathbb{R} \) and every tame collection \( \{s_n^i\}_i \) there exists a finite cover \( \mathcal{U} \) of \( Y \) with \( U_i \subset \pi(X^i) \) so that the collection of cumulative fiber probability densities \( \{D_{n}^i(\cdot, t)\} \) associated to \( \{s_n^i\}_i \) converges uniformly on \( Y_0 \) to the zero function on \( Y_0 \), i.e. for every \( i \in I \) we have

\[
(y \mapsto D_{n}^i(y, t)) \quad \xrightarrow{\text{0 uniform}} on U_i \cap Y_0 \subset \pi(X^i).
\]

**IV.3 The Fiber Probability Measure Sequence (Non-Tame Case)**

Let \( (s_n)_n \) be a sequence of \( T \)-eigensections so that \( |\xi_n - n\xi| \in \mathcal{O}(1) \). Moreover, let

\[
\pi:X^i_{\xi} \to Y = X^i_{\xi}/T
\]

be the projection map attached to the Hilbert quotient associated to the level subset \( \mu^{-1}(\xi) \) and \( \pi_n:X^i_{\xi_n} \to Y_n = X^i_{\xi_n}/T \) the corresponding Hilbert quotient associated to the level subset \( \mu^{-1}_{\xi_n}(n^{-1}\xi_n) \).

The aim of this subsection is to define a sequence \( (\nu_n)_n \) of fiber probability measures over the base \( Y_0 \) by

\[
\nu_n(y):(A \ni \nu_n(y)(A) := \int_{A} \frac{|s_n|^2}{\|s_n\|^2} \, d[\pi_y].
\]

However, since \( X(s_n) = \{x \in X : s_n(x) \neq 0\} \) moves as \( n \to \infty \), the above measure is not well defined on all of \( Y \): For example if \( y \in Y \) is so that \( \pi^{-1}(y) \subset \{s_n = 0\} \), then the above measure is not defined for \( n \) at \( y \). In some cases, it is however possible to circumvent this problem. For this we will first introduce the following definition.

**Definition IV.4. [Removable Singularity]**

A point \( y \in Y \) is defined to be a removable singularity of order \( N_0 \) for the fiber measure \( \nu_n \) if there exists an open neighborhood \( U_y \subset Y \) of \( y \), a sequence \( (f_{y,n})_n \) of local holomorphic functions \( f_n \in \mathcal{O}(U_y) \) and \( N_0 \in \mathbb{N} \) so that

\[
\hat{s}_{f_{y,n}} := s_{n} \cdot \pi^* f_{y,n}^{-1}
\]

defines a local holomorphic section on \( \pi^{-1}(U_y) \) for all \( n \geq N_0 \) which does not vanish identically on \( \pi^{-1}(y') \) for all \( y' \in U_y \).

In the sequel, we will denote the set of all removable singularities of order \( N_0 \) by \( \mathbb{R}_{N_0} \).

**Remark IV.3.** The local extension \( \hat{s}_{f_{y,n}} \) is again a \( \xi_n \)-eigensection.

As a corollary of the definition, we deduce
Corollary IV.1. If \( y \in Y_0 \) is a removable singularity for \( \nu_0 \) of order \( N_0 \in \mathbb{N} \), then after having shrunken \( U_y \) appropriately, the quotient \( \frac{\|s_{f_{y,n}}\|^2}{\|s_{\hat{f}_{y,n}}\|^2} \) is independent of the local scaling functions \( (f_{y,n})_n \), i.e. we have

\[
\frac{\|s_{f_{y,n}}\|^2}{\|s_{\hat{f}_{y,n}}\|^2} = \frac{\|s_{\hat{f}_{y,n}}\|^2}{\|s_{f_{y,n}}\|^2}
\]

over \( U_y \subset Y_0 \cap R_{N_0} \).

Proof. First of all, since \( y \) is contained in the open set \( Y_0 \cap R_{N_0} \), we can assume that \( U_y \subset Y_0 \).

Let \( (f_{y,n}^i)_n, f^i_{y,n} \in \mathcal{O}(U_Y) \) where \( i \in \{0, 1\} \). As in Definition [IV.4], we have \( \hat{s}_{f_{y,n}} = s_n \cdot \pi^* f_{y,n}^{-1} \). Throughout this proof, we use the abbreviation \( f_{y,n} = f_n \) for \( i \in \{0, 1\} \). Using this notation, it follows that

\[
\hat{s}_{h_{0,n}} = \pi^*(f_{1,n} f_{0,n}^{-1}) \cdot \hat{s}_{f_{1,n}} \tag{IV.1}
\]

where \( h_{0,1,n} := f_{1,n}^{-1} f_{0,n} \) yields a sequence of meromorphic function defined on \( U_y \). We claim that after having shrunken \( U_y \), each \( h_{0,1,n} \) is bounded from above and below away from zero on \( U_y \).

For this note the following: Since \( \hat{s}_{f_{1,n}} \mid \pi^{-1}(y) \neq 0 \) there exists one \( x \in \pi^{-1}(y) \) so that \( \hat{s}_{f_{1,n}}(x) \neq 0 \) and hence \( |\hat{s}_{f_{1,n}}|^2(x) > 0 \). In particular, we find an open neighborhood \( V \subset X_0 \) of \( x \) so that

\[
|\hat{s}_{f_{1,n}}|^2(x') \geq c > 0 \tag{IV.2}
\]

for all \( x' \in V \). By continuity we also have

\[
|\hat{s}_{f_{0,n}}|^2(x') \leq C < \infty \tag{IV.3}
\]

for all \( x' \in V \).

Since \( V \subset X_0 \) and since \( \pi|X_0 : X_0 \to Y_0 \) is a k-fibered and hence an open map (cf. [LO], p. 297 f.), we can assume (after having shrunken \( V \) and \( U_y \) appropriately) that \( \pi(V) = U_y \). Combining [IV.2] [IV.3] and [IV.1] we deduce \( |h_{0,1,n}|^2(y') \leq C' < \infty \) for all \( y' \in U_y \). Reversing the roles of \( f_0 \) and \( f_1 \) shows (after having shrunken \( U_y \) again) that \( |h_{0,1,n}|^2(y') \geq c' > 0 \). Hence, the meromorphic function \( h_{0,1,n} = f_{1,n}^{-1} f_{0,n} \) is bounded from above and below away from zero on \( U_y \). As the base \( Y \) is assumed to be normal, we can apply Riemann’s Extension Theorem (cf. [Gr-Re], p. 144) in order to deduce that \( h_{0,1,n} \) yields a non-vanishing holomorphic function on \( U_y \). All in all we deduce

\[
\frac{\|s_{f_{y,n}}\|^2}{\|s_{\hat{f}_{y,n}}\|^2} \cdot \frac{\pi^* |h_{0,1,n}|^2}{\pi^* |h_{0,1,n}|^2} = \frac{\|s_{\hat{f}_{y,n}}\|^2}{\|s_{f_{y,n}}\|^2}
\]

over \( U_y \) as claimed.

Q. E. D.

In this sense, the quotient \( |s_n|^2 |s_n|^{-2} \) can be uniquely extended onto \( R_{N_0} \cap Y_0 \). Assume
now that \( Y_0 \cap R_{N_0} \neq \emptyset \). Using the independence of \( \| \tilde{s}_{r_n} \|^2 \| \tilde{s}_{r_n} \|^{-2} \) of the chosen sequence \((f_{y,n})_n, f_{y,n} \in O(U_y)\) shown in Corollary [IV.1] we can define

**Definition IV.5.** Let \((s_n)_n\) be sequence of \( \xi_n \)-eigensections whose rescaled weights approximate \( \xi \), then define a sequence of fiber distribution densities over \( Y_0 \cap R_{N_0} \) by

\[
\phi_n : x \mapsto \phi_n(x) := \| \tilde{s}_{r_n} \|^2(x) \]

where \((f_{y,n})_n\) is a sequence of holomorphic functions \( f_{y,n} \in O(U_y) \) as in Definition [IV.4].

As in the same case we introduce the following two definitions.

**Definition IV.6.** Let \((\phi_n)_n\) be the sequence of fiber distribution densities associated to \((s_n)_n\) as defined in Definition [IV.5], then we define a sequence of fiber probability measures parametrized over \( Y_0 \cap R_{N_0} \) by

\[
\nu_n(y) : A \mapsto \nu_n(y)(A) := \int_A d\nu_n := \int_A \phi_n d[\pi_y]
\]

for \( A \subset \pi^{-1}(y) \) measurable where \( y \in Y_0 \cap R_{N_0} \).

**Definition IV.7.** Let \((s_n)_n\) be sequence of \( \xi_n \)-eigensections whose rescaled weights approximate \( \xi \), then we define a sequence of cumulative fiber probability densities over \( Y_0 \cap R_{N_0} \) by

\[
D_n(y, \cdot) : t \mapsto D_n(y, t) := \int_{\{\phi_n \geq t\} \cap \pi^{-1}(\pi(x))} d[\pi_y]
\]

where \( y \in Y_0 \cap R_{N_0} \).

The aim of Section [VII] is to give a proof of the following two convergence results.

**Theorem 5.b.** [Uniform Convergence of the Initial Distribution Sequence]
For fixed \( t \in \mathbb{R} \) the sequence \((D_n(\cdot, t))_n\) converges uniformly on \( Y_0 \cap R_{N_0} \) to the zero function.

**Theorem 6.b.** [Uniform Convergence of the Initial Measure Sequence]
Let \( f \in C^0(X) \) then the sequence

\[
\left( y \mapsto \int_{\pi^{-1}(y)} f d\nu_n(y) \right)_n
\]

converges uniformly over \( Y_0 \cap R_{N_0} \) to the reduced function \( f_{\text{red}} \).

We close this section by showing that in general there exists no \( N_0 \in \mathbb{N} \) so that \( Y = R_{N_0} \). Furthermore, the example shows that even the extreme case \( R_{N_0} = \emptyset \) for all \( N_0 \in \mathbb{N} \) is possible.
Example IV.2. Let $X \subset \mathbb{CP}^k \times \mathbb{CP}^k$ equipped with the $T = \mathbb{C}^*$ action given by
\[
t([\zeta_0: \ldots : \zeta_k], [z_0: \ldots : z_k]) = ([\zeta_0: \ldots : \zeta_k], [t^{-1}z_0: \ldots : t^{-1}z_{k-1}: z_k]) .
\]
In the sequel, we will consider the $T$-linearization $L = p_0^*O_{\mathbb{CP}^k}(1) \otimes p_1^*O_{\mathbb{CP}^k}(1)$ of the $T$-action on $X$ induced by the trivial $T$-action on the first factor $O(1)_{\mathbb{CP}^k} \to \mathbb{CP}^k$ given by
\[
t([\zeta_0: \ldots : \zeta_k : \zeta]) = ([\zeta_0: \ldots : \zeta_k : \zeta])
\]
where we have used the identification $\pi$ of the second factor.

Note that we have $\pi$ by and it is also direct to verify that one can identify $Y$.

In the sequel, we will consider the $T$-linearization $L = p_0^*O_{\mathbb{CP}^k}(1) \otimes p_1^*O_{\mathbb{CP}^k}(1)$ of the $T$-action on $X$ induced by the trivial $T$-action on the first factor $O(1)_{\mathbb{CP}^k} \to \mathbb{CP}^k$ given by
\[
t([z_0: \ldots : z_k : \zeta]) = [t^{-1}z_0: \ldots : t^{-1}z_{k-1}: z_k : \zeta].
\]

A calculation shows that the $\xi = 0$-level of the associated moment map is given by the set
\[
\mu^{-1}(0) = \mathbb{CP}^k \times \{0: \ldots : 0:1\}
\]
and it is also direct to verify that one can identify $Y = X_{\xi=0}/T \simeq \mathbb{CP}^k$ where $X_{ss} \simeq \mathbb{CP}^k \times \mathbb{C}^k$. Furthermore, each fiber of the quotient map $\pi$ is isomorphic to $\mathbb{C}^k$ equipped with the inverse diagonal action.

We will now consider the sequence $(s_n)_n$ whose rescaled weights converge to $\xi = 0$ defined by
\[
s_n = \sum_{i=0}^{k-1} c^n_i z_i z_k^{n-1} \in H^0(X, L^n) .
\]

Note that we have
\[
s_n|\pi^{-1}(0: \ldots : 0:1)) \equiv 0 \text{ for all } n \in \mathbb{N}.
\]

Using the homogenous standard coordinates $\zeta'_i = \frac{\zeta_i}{\zeta}, z'_i = \frac{z_i}{z_k}, 0 \leq i \leq k-1$ on the open subset
\[
U_{k,k} = \{(z, [\zeta]) : \zeta_k \neq 0, z_k \neq 0\} \subset X_{ss}^0 \subset \mathbb{CP}^k \times \mathbb{CP}^k ,
\]
the restriction $\pi|U_{k,k} \to V_k \simeq \mathbb{C}^k$ of the quotient map $\pi : X_{ss}^0 \to \mathbb{CP}^k$ is given by the projection map $\pi|U_{k,k} : U_{k,k} \simeq \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{C}^k$. With respect to this trivialization the sequence $s_n|U_{k,k}$ is given by
\[
s_n|U_{k,k} = \sum_{i=0}^{k-1} c^{t,n}_i z'_i
\]
where $\pi^{-1}(0: \ldots : 0:1)) = \{0\} \times \mathbb{C}^k$. To shorten notation, we will just write $s_n|U_{k,k} = s_n$ throughout the rest of this example and set $\zeta'_i = \zeta_i, z'_i = z_i$ for all $0 \leq i \leq k-1$.

Assume now that $\zeta_0 = 0: \ldots : 0:1$ is a removable singularity for the fiber measure $\nu_n$ (we will fix $n$ henceforth), i.e. there exists a non-vanishing holomorphic function $f \in \mathcal{O}(U)$
defined on an open neighborhood \( U \subset \mathbb{C}^k \) of \( \zeta_0 \) so that \( \tilde{s}_n := s_n \cdot \pi^* f^{-1} \) is holomorphic and does not vanish identically on \( \pi^{-1}(\zeta') \) for all \( y' \in U \). Note that, after having shrunk \( U \), we can assume that \( \zeta_0 \) is an isolated zero of the function \( f \). In particular, the restriction \( \tilde{s}_n|_{\pi^{-1}(\zeta')} \) defines a non-vanishing linear form on \( \pi^{-1}(\zeta') \cong \mathbb{C}^k \) for all \( \zeta' \in U \). Using this, it follows that for each sequence \( (\zeta_m)_m \) in \( U \) converging to \( \zeta_0 \), the sequence of one-codimensional subspaces in \( \mathbb{C}^k \cong \pi^{-1}(\zeta_m) \) given by \( \mathcal{H}(\zeta_m) = \{ x \in \mathbb{C}^k : \tilde{s}_n(\zeta_m) = 0 \} \) must converge to a uniquely defined one-codimensional subspace which is independent of the choice of \( (\zeta_m)_m \). However, this is a contradiction to the equation \( \zeta_n = s_n \cdot \pi^* f^{-1} \) and the fact that \( f \) is non-zero on \( U \setminus \{ \zeta_0 \} \). For example, consider the collection of sequences given by

\[
\{ (\zeta_m)_i \} = \{(0, \ldots, m^{-1}, \ldots, 0) \}_i,
\]

then we have

\[
\mathcal{H}(\zeta_m)_i = \{ x \in \mathbb{C}^k : \tilde{s}_n(\zeta_m)_i = 0 \} \rightarrow \{ x \in \mathbb{C}^k : z_i = 0 \}
\]

so the limit is not independent of the chosen sequence. Hence, we deduce a contradiction and it follows that \( \zeta_0 \) is a not a removable singularity for any \( N_0 \in \mathbb{N} \) in the sense of \textbf{Definition IV.4}, i.e., we have \( \zeta_0 \notin \mathcal{R}_{N_0} \) for all \( N_0 \).

Furthermore, by slightly changing the above sequence \( (s_n)_n \), we can show that \( \mathcal{R}_{N_0} = \emptyset \) for each \( N_0 \): Choose a dense sequence \( (\zeta_n)_n \) in the quotient \( Y \cong \mathbb{C}^p \) and let \( (\Phi_n)_n \), \( \Phi_n \in \text{Aut}(Y) \) be a sequence of projective transformations so that \( \Phi_n(\zeta_0) = \zeta_n \). Define a new sequence of eigensections by

\[
s'_n([\zeta], [z]) := s_1([\Phi_n(\zeta)], [z])
\]

and consider the sequence given by

\[
s_n := \prod_{i=1}^n s'_i.
\]

It is direct to see that \( \mathcal{R}_{N_0} = \emptyset \) because for each open neighborhood \( U \) of any point \( y \in Y \), the subset \( U \cap \{ \zeta_n \} \) is dense by construction and \( \zeta_n \) is non-removable. \( \square \)

As a consequence of this example we conclude

\textbf{Remark IV.4.} There are examples of approximating sequences \( (s_n)_n \) so that \( \mathcal{R}_{N_0} = \emptyset \) for all \( N_0 \in \mathbb{N} \).

As a further-reaching question, one could ask whether the set of all \( \xi \)-approximating sequences of \( \zeta_n \)-eigensections \( (s_n)_n \), with the property that \( \mathcal{R}_{N_0} = \emptyset \) for all \( N_0 \), is ”thin” as a subset of \( \bigcup_{n=0}^\infty \mathcal{H}^0(X, L^n) \).

It turns out that there is no definitive answer to this question: In the context of \textbf{Example II.1} one can show that each singularity of \( s \in \mathcal{H}^0(X, L^n) \) is removable and hence \( Y = \mathcal{R}_{N_0} \) for all \( N_0 \in \mathbb{N} \) and any choice of \( (s_n)_n \). On the other hand, if \( k = 2 \) in \textbf{Example IV.2}, it turns out that a \( \xi_n \)-eigensection \( s_n \in \mathcal{H}^0(X, L^n) \), which has been randomly chosen with respect to a choice of a Lebesgue measure on \( \mathcal{H}^0(X, L^n) \) (induced by a choice of a basis),
has almost surely at least one non-removable singularity $y_n \in Y$. Moreover, if one randomly chooses a $\xi$-approximating sequence of $\xi_n$-eigensections in this setting, it turns out that the set $\{y_n\}_{n \in \mathbb{N}}$ is almost surely dense in $Y$.

V The k-Fibering $\Pi: \tilde{X} \to \tilde{Y}$

V.1 Construction of the k-Fibering $\Pi: \tilde{X} \to \tilde{Y}$

Let $\hat{\pi}: \hat{X} \to Y$ be the holomorphic map between the purely dimensional varieties $\hat{X}$ and $Y$ as defined in Section IV.1 and recall that there exits a Zariski-dense subset $Y_0 \subset Y$ so that all fibers $\hat{\pi}^{-1}(y)$ for $y \in Y_0$ are purely k-dimensional, compact subvarieties not necessarily irreducible. Recall that we have assumed $X$ to be normal and hence, it follows (cf. [HE-HU1], p. 124) that $Y$ is normal and therefore, in particular, the open subset $Y_0$ as well.

By [BAR1] the following is known in the above context: There exists a holomorphic map $\varphi^{\hat{\pi}}: Y_0 \to C^k(\hat{X})$ into the cycle space $C^k(\hat{X})$ of all compact k-dimensional cycles

$$\mathcal{C} = \sum_{i \in I} n_i C_i, \ n_i \in \mathbb{N}, C_i \subset \hat{X}$$

globally irreducible subspaces of $\hat{X}$ of dimension k so that the support $|\varphi^{\hat{\pi}}(y)|$ of the cycle $\mathcal{C}_y := \varphi^{\hat{\pi}}(y)$ for $y \in Y_0$ is equal to the set theoretic fiber $\hat{\pi}^{-1}(y)$, i.e. we have

$$|\mathcal{C}_y| = \hat{\pi}^{-1}(y) \text{ for all } y \in Y_0. \quad (V.1)$$

Furthermore, since $\hat{X}$ and $Y$ are compact, there exists (cf. [BAR1]) a proper modification $\sigma: \tilde{Y} \to Y$ with center $Y \setminus Y_0$, a proper modification $\Sigma: \tilde{X} \to \hat{X}$ with center $\hat{\pi}^{-1}(Y \setminus Y_0)$ and a surjective holomorphic map $\Pi: \tilde{X} \to \tilde{Y}$ so that the following diagram commutes:

$$\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{\pi}} & \tilde{X} \\
\Sigma \downarrow & & \Pi \downarrow \\
\tilde{Y} & \xleftarrow{\sigma} & Y
\end{array}$$

The compact, complex space $\tilde{Y}$ is given by

$$\tilde{Y} := \text{cl} \left\{ (y, \mathcal{C}) \in Y_0 \times C^k(\hat{X}): \varphi^{\hat{\pi}}(y) = \mathcal{C} \right\} \subset Y \times C^k(\hat{X}) \quad (V.2)$$

and the holomorphic map $\sigma: \tilde{Y} \to Y$ is defined by $\sigma := p_Y \tilde{Y}$. Moreover, if $\mathcal{X} \subset C^k(\hat{X}) \times \hat{X}$ denotes the universal space defined by $\mathcal{X} := \{(\mathcal{C}, x) \in C^k(\hat{X}) \times \hat{X} : x \in |\mathcal{C}| \}$, then $\hat{X}$

---

2 The support $|\mathcal{C}|$ of a cycle $\mathcal{C} = \sum_{i \in I} n_i C_i$ is defined by $|\mathcal{C}| = \bigcup_{i \in I} C_i$. 

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is compact and given by
\[ \tilde{X} = (Y \times x) \cap (\tilde{Y} \times \tilde{X}) \]
where \( \Sigma \) is the restriction of the projection \( p : Y \times C^k(\tilde{X}) \times \tilde{X} \to \tilde{X} \). Note that the above construction, whose details can be found in [BAR1], implies that the fiber \( \Pi^{-1}(y, C) \in \tilde{X} \) identifies with \( |C| \subseteq \tilde{X} \). Using this identification, we will simply write \( \Pi^{-1}(y, C) = |C| \) henceforth.

**Remark V.1.** Note that \( \Pi : \tilde{X} \to \tilde{Y} \) is a holomorphic map whose fibers are purely \( k \)-dimensional by construction. Furthermore, \( \tilde{X} \) and \( \tilde{Y} \) are both purely dimensional where \( \dim \tilde{X} = \dim X \) and \( \dim \tilde{Y} = \dim Y \) (which is a well known fact of the theory of proper modifications, cf. [GR-RE], p. 214).

Moreover, we have the following lemma.

**Lemma V.1.** The support \( \Pi^{-1}(y, C) = |C| \) is \( T \)-invariant subset of \( \tilde{\pi}^{-1}(y) \).

**Proof.** By the commutativity of the previous diagram, we deduce that \( |C| \subseteq \tilde{\pi}^{-1}(y) \), hence it remains to verify that \( |C| \) is \( T \)-invariant. In order to prove this, we can proceed as follows: Since the set
\[ \{(y, C) \in Y_0 \times C^k(\tilde{X}) : \varphi^{\tilde{\pi}}(y) = C\} \]
is Euclidean dense in \( \tilde{Y} \), we can choose a sequence \( (y_n) \) in \( Y_0 \) so that \( (y_n, C_{y_n}) \to (y, C) \) where \( C_{y_n} \) are the cycles whose underlying sets are equal to \( \tilde{\pi}^{-1}(y_n) \) by property [V.1]. The \( T \)-invariance of the limit cycle \( C \) follows by the following reasoning: Let \( x = t.x_0 \in T.C \) where \( x_0 \in C \). Then, since \( C_n \to C \) means convergence in the Hausdorff topology of the underlying support, there exists a sequence \( x_n \in C_n \) so that \( x_n \to x_0 \). Since \( |C_n| = \tilde{\pi}^{-1}(y) \) for all \( y \in Y_0 \), it follows that \( |C_{y_n}| = \tilde{\pi}^{-1}(y_n) \) for all \( n \in \mathbb{N} \). Using the \( T \)-invariance of \( \tilde{\pi}^{-1}(y) \) (cf. Remark [V.1]), we deduce that \( t.x_n \in |C_{y_n}| \). By the continuity of the action it follows that \( t.x_n \to t.x_0 \). So \( (t.x_n) \) is a convergent sequence with limit \( t.x_0 \) where \( t.x_n \in C_n \) and \( C_n \to C \). By the definition of the Hausdorff topology, it then follows that \( t.x_0 \in C \) and hence \( T.C \subseteq C \) which proves the claim \( T.C = C \). \quad \text{Q. E. D.} \)

We close this section with the following remark and example.

**Remark V.2.** In general (cf. Example [V.1]) \( |C| \) is a proper subset of \( \tilde{\pi}^{-1}(y) \).

**Example V.1.** Let \( X = \mathbb{C}P^3 \) equipped with the \( T = \mathbb{C}^* \) action given by
\[ t.[z_0 : z_1 : z_2 : z_3] = [t^{-1}z_0 : tz_1 : tz_2 : tz_3] \]
and consider the Hilbert quotient
\[ \pi : X_0^{ss} = \mathbb{C}P^3 \setminus \{[1 : 0 : 0 : 0] \cup \{z_0 = z_3 = 0\}\} \to Y \cong \mathbb{C}P^2 \]
associated to the \( 0 = \xi \)-level set of the moment map
\[ \mu : [z] \mapsto \|z\|^{-2}(-|z_0|^2 + |z_1|^2 + |z_2|^2) \].
The corresponding projection map $\pi$ is given by $\pi: [z] \mapsto [\zeta] = [z_0 z_1 : z_0 z_2 : z_2^2]$ where $[\zeta] \in Y \cong \mathbb{CP}^2$. Note that all fibers $\hat{\pi}^{-1}([\zeta])$ over $Y_0 = \mathbb{CP}^2 \setminus \{[0:0:1]\}$ are of pure dimension one and of degree two. Moreover, it is direct to verify that these fibers can be parameterized by

$$\gamma_{[\zeta]}: t \mapsto [\zeta_2 t^2 : \zeta_0 t_0^2 : \zeta_1 t_0^2 : \zeta_2 t_0 t_1]$$

for $\zeta \in Y_0$ and that they are given as the zero set of the following system of equations

$$\zeta_2 z_0 z_1 - \zeta_0 z_2^2 = 0, \quad \zeta_2 z_0 z_2 - \zeta_1 z_2^2 = 0 \text{ where } \zeta \in Y_0.$$

Let $U_2 = \{[\zeta] \in \mathbb{CP}^2 : \zeta_2 \neq 0\}$ and set $c_0 := \zeta_2^{-1} \zeta_0, c_1 := \zeta_2^{-1} \zeta_1$ and consider $U_2^* := Y_0 \cap U_2$ which we can identify with $\mathbb{C}^2 \setminus \{0\}$. As mentioned before, there exists a holomorphic map $\varphi^\#: U_2^* \to C^1(\tilde{X})$. It turns out that all fibers of $\tilde{\pi}$ are compact subvarieties of degree 2 in $\mathbb{CP}^3$. Hence, it follows that the image of $\varphi^\#$ is contained in the cycle space component which can be identified with the compact connected Chow Variety $\mathcal{C}_{1,2}(\mathbb{CP}^3)$ of all 1-dimensional cycles in $\mathbb{CP}^3$ of degree 2 which itself is realized as closed variety in the projective space $\mathbb{CP}^{2,1,2}$ (for a rigorous definition cf. [SHA]).

Recall that the Chow coordinates of a cycle $\mathcal{C}$ in $X \subset \mathbb{CP}^m$ of degree $d$ and dimension $k$ are given by the coefficients of the Chow form $\mathfrak{C}_{\mathcal{C}, \mathbb{CP}^m}$, i.e. by the coefficients of a polynomial homogenous in $k + 1$ groups $\zeta^{(i)}$, $i \in \{0, \ldots, k\}$ of $m + 1$ indeterminates of degree $d$ modulo multiplication with a non-vanishing complex number $\lambda \in \mathbb{C}^*$ (cf. [SHA]). In the sequel, let $\mathbf{c}_e$ for $e \in \mathbb{C}^2 \setminus \{0\}$ and let $\mathfrak{C}_{\mathcal{C}, \mathbb{CP}^3}$ be the corresponding Chow form. A calculation shows that

$$\mathfrak{C}_{\mathcal{C}, \mathbb{CP}^3} \left( \xi_0^{(0)}, \xi_1^{(0)}, \xi_2^{(0)}, \xi_3^{(0)}, \xi_0^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)} \right)$$

$$= c_0^2 \xi_0^{(0)} \xi_1^{(1)} + c_1^2 \xi_1^{(0)} \xi_2^{(1)} + c_2^2 \xi_2^{(0)} \xi_3^{(1)} + c_3^2 \xi_3^{(0)} \xi_0^{(1)} + c_0 \xi_0^{(0)} \xi_2^{(1)} + c_1 \xi_1^{(0)} \xi_3^{(1)} + c_2 \xi_2^{(0)} \xi_0^{(1)} + c_3 \xi_3^{(0)} \xi_1^{(1)}$$

$$+ 2 c_0 c_1 \xi_0^{(0)} \xi_2^{(1)} \xi_3^{(1)} - c_0 \xi_0^{(0)} \xi_3^{(1)} \xi_1^{(1)} - c_1 \xi_1^{(0)} \xi_3^{(1)} \xi_2^{(1)} - c_2 \xi_2^{(0)} \xi_0^{(1)} \xi_3^{(1)} - c_3 \xi_3^{(0)} \xi_0^{(1)} \xi_2^{(1)}$$

so the map $\varphi^\#: \mathbb{C}^2 \setminus \{0\} \to \mathcal{C}_{1,2}(\mathbb{CP}^3)$ is given by

$$\varphi^\#: \mathbb{C}^2 \setminus \{0\} \ni (c_1, c_2) \mapsto [c_0^2 : c_1^2 : -2 c_0^2 : c_2^2 : c_3^2 : -2 c_0 c_1 : 2 c_0 c_1 : -2 c_0 c_1 : -c_0 : -c_0 : -c_0 : c_0 : c_1 : c_1 : c_1 : 0 : \ldots : 0].$$
The closure of the graph
\[ \left\{ (c, \mathbf{C}) : c \in \mathbb{C}^2 \setminus \{0\}, \varphi^\pi (c) = \mathbf{C} \right\} \subset \mathbb{C}^2 \times \mathcal{C}_{1,2} (\mathbb{C}P^3) \]
of the map \( \varphi^\pi \) turns out to be isomorphic to the blow-up \( \mathcal{B} \mathcal{I} (0, \mathbb{C}^2) \) of the origin \( 0 \in \mathbb{C}^2 \).
This can be seen in the following way: Let

\[ \varphi : \mathbb{C}^2 \setminus \{0\} \ni (c_0, c_1) \mapsto ((c_0, c_1), [c_0 : c_1]) \in \mathcal{B} \mathcal{I} (0, \mathbb{C}^2) \]

where \( \mathcal{B} \mathcal{I} (0, \mathbb{C}^2) = \{((c_0, c_1), [c'_0 : c'_1]) : c_0c'_1 - c_1c'_0 = 0\} \subset \mathbb{C}^2 \times \mathbb{C}P^1 \). We have

\[ \text{cl} \left( \varphi \left( \mathbb{C}^2 \setminus \{0\} \right) \right) = \mathcal{B} \mathcal{I} (0, \mathbb{C}^2). \]

We can embed \( \mathcal{B} \mathcal{I} (0, \mathbb{C}^2) \subset \mathbb{C}^2 \times \mathbb{C}P^1 \subset \mathbb{C}P^2 \times \mathbb{C}P^1 \) into \( \mathbb{C}P^5 \) using the SEGRE map. The composition of \( \varphi \) with the SEGRE embedding yields an embedding \( \tilde{\varphi} : \mathbb{C}^2 \setminus \{0\} \hookrightarrow \mathbb{C}P^5 \) given by

\[ \tilde{\varphi} : \mathbb{C}^2 \setminus \{0\} \mapsto [c_0 : c_1 : c'_0 : c'_0c_1 : c_0c_1 : c_1^2]. \]

It is direct to see that there exists a projective transformation \( \Phi \) of \( \mathbb{C}P^{5,1,2} \) so that the following diagram commutes:

\[ \begin{array}{ccc}
\mathbb{C}^2 \setminus \{0\} & \xrightarrow{\tilde{\varphi}} & \mathcal{C}_{1,2} (\mathbb{C}P^3) \subset \mathbb{C}P^{5,1,2} \\
\downarrow & & \downarrow \Phi \\
\mathcal{B} \mathcal{I} (0, \mathbb{C}^2) & \subset & \mathbb{C}P^2 \times \mathbb{C}P^1, \text{SEGRE map} \quad \mathbb{C}P^5 \subset \mathbb{C}P^{5,1,2}
\end{array} \]

Let \( ((c_n, \mathbf{c}_{c_n}))_n \) be the sequence in \( \tilde{\mathbf{Y}} \cap \sigma^{-1} (\mathbb{C}^2 \setminus 0) \) given by \( c_n = n^{-1} (c'_0, c'_1) \). The above formulas show that \( (\mathbf{c}_{c_n})_n \) converges to

\[ \mathbf{c}_c = [0 : 0 : 0 : 0 : 0 : 0 : -c'_0 : -c'_0 : c'_0 : -c'_1 : c'_1 : c'_1 : 0 : \ldots : 0] \]

which corresponds to the point \( ((0, 0), [c'_1 : c'_2]) \in \mathcal{B} \mathcal{I} (0, \mathbb{C}^2) \) under the above identification. In particular we have \( \sigma (0, \mathbf{c}_c) = [0 : 0 : 1] \in \mathbb{C}^2 \). In order to determine the limit cycle

\[ \mathbf{c}_c \in \tilde{\mathbf{Y}} \cap \sigma^{-1} (U_2) \]

we consider the Chow form \( \tilde{\mathbf{c}}_{c}, \mathbb{C}P^3 \)

\[ \tilde{\mathbf{c}}_{c}, \mathbb{C}P^3 \left( \xi_0^{(0)}, \xi_1^{(0)}, \xi_2^{(0)}, \xi_3^{(0)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)} \right) \]

\[ = c'_0 \xi_0^{(0)} \xi_1^{(0)} \xi_3^{(1)} + c'_1 \xi_0^{(0)} \xi_2^{(0)} \xi_3^{(1)} + c'_0 \xi_0^{(0)} \xi_1^{(1)} \xi_2^{(1)} + c'_1 \xi_0^{(0)} \xi_1^{(1)} \xi_2^{(1)} - c'_0 \xi_0^{(0)} \xi_3^{(0)} \xi_1^{(1)} \xi_2^{(1)} \]

\[ - c'_1 \xi_0^{(0)} \xi_3^{(0)} \xi_1^{(1)} \xi_2^{(1)} - c'_0 \xi_1^{(0)} \xi_3^{(0)} \xi_0^{(1)} \xi_3^{(1)} - c'_1 \xi_2^{(0)} \xi_3^{(0)} \xi_0^{(1)} \xi_3^{(1)} \]
which turns out to be reducible:

\[
\vec{\mathcal{e}}_{c', \mathbb{C}P^3} \left( c_0 \xi_1(0) \xi_3(1), c_1 \xi_2(0) \xi_3(1) - c_0 \xi_3(0) \xi_1(1) - c_1 \xi_3(0) \xi_2(1) \right) = \left( c_0' \xi_1(0) \xi_3(1), c_1' \xi_2(0) \xi_3(1) - c_0' \xi_3(0) \xi_1(1) - c_1' \xi_3(0) \xi_2(1) \right) \cdot \left( \xi_0(0) \xi_3(1) - \xi_3(0) \xi_0(1) \right)
\]

A direct computation shows that \( \vec{\mathcal{e}}_{c', \mathbb{C}P^3} \) is the Chow form associated to the line

\[
\tilde{\mathcal{e}}_{c'} = \{ z_0 = 0, z_4 = 0, c_2' z_1 - c_1' z_2 = 0 \}
\]

and \( \vec{\mathcal{e}}_{c_0, \mathbb{C}P^3} \) is the corresponding Chow form of the line

\[
\mathcal{C}_0 = \{ z_1 = 0, z_2 = 0, z_3 = 0 \}
\]

where \( \mathcal{C}_c = \tilde{\mathcal{C}}_{c'} + \mathcal{C}_0 \). In particular, note that \( \mathcal{C}_c \neq \tilde{\pi}^{-1}([0:0:1]) \) where

\[
\tilde{\pi}^{-1}([0:0:1]) = \{ z_0 = 0 \} \cup \{ z_1 = z_2 = z_3 = 0 \}.
\]

V.2 Fiber Integral Properties of the k-Fibering II: \( \tilde{\mathbf{X}} \to \tilde{\mathbf{Y}} \)

As in Section III let \( \mathbf{X}^i \) be the \( \mathbb{T} \)-invariant Zariski open subset of \( \mathbf{X} \left( s^i_n \right) \subset \mathbf{X}^{ss}_\xi \) of Theorem 1, where \( \mathbf{X} \left( s^i_n \right) \) is the n-stable complement of the zero set of the tame sequence \( (s^i_n)_n \). As before, let \( \varphi^i: \mathbf{X}^i \to \mathbb{R} \) be the normalized s.p.s.h. limit function and recall the definition of the compact tube \( T(\epsilon, \mathbf{W}) \subset \mathbf{X}^i \) for \( \mathbf{W}^i \subset \pi(\mathbf{X}^i) \) a compact neighborhood and \( \epsilon > 0 \) given in Section III

\[
T(\epsilon, \mathbf{W}) = (\varphi^i \times \pi)^{-1}([0, \epsilon] \times \mathbf{W}^i).
\]

Define the corresponding tube \( \tilde{T}(\epsilon, \mathbf{W}) \) in \( \tilde{\mathbf{X}} \) by

\[
\tilde{T}(\epsilon, \mathbf{W}) := \Sigma^{-1}(T(\epsilon, \mathbf{W}))
\]

where we have used the fact that \( \mathbf{X}^{ss}_\xi \supset T(\epsilon, \mathbf{W}) \) is naturally embedded in \( \tilde{\mathbf{X}} \) via \( \zeta^{-1}|\mathbf{X}^{ss}_\xi \) (recall that \( \zeta|\zeta^{-1}(\mathbf{X}^{ss}_\xi) \) is biholomorphic because \( \mathbf{X}^{ss}_\xi \) is assumed to be normal). Note that \( \tilde{T}(\epsilon, \mathbf{W}) \) projects down via \( \Pi \) onto the compact neighborhood \( \tilde{\mathbf{W}}^i := \sigma^{-1}(\mathbf{W}^i) \).

The first aim of this section is to show that the fiber integral \( \text{vol}(\pi^{-1}(y)) = \int_{\pi^{-1}(y)} d[\pi_y] \) is bounded as \( y \) varies in \( \mathbf{Y}_0 \).

Lemma V.2. For all \( y \in \mathbf{W}^i \cap \mathbf{Y}_0 \) we have

\[
\text{vol}(T(\epsilon, \mathbf{W}) \cap \pi^{-1}(y)) = \text{vol}(\tilde{T}(\epsilon, \mathbf{W}) \cap \Pi^{-1}(y, \mathcal{E}_y)).
\]
where the right hand side is the fiber integral of the projection taken with respect to \( \Omega^k \) where 
\( \Omega := \Sigma^* \omega' \).

Furthermore, we have 
\[
\text{vol} \left( \pi^{-1}(y) \right) \leq \text{vol} \left( \Pi^{-1}(y, \mathcal{C}_y) \right)
\]
for all \( y \in Y_0 \).

**Proof.** This is a direct consequence of the following reasoning: Recall that \( \Sigma \) is a modification with center \( \hat{\pi}^{-1}(Y \setminus Y_0) \) and \( \zeta|^{-1}(X^s) \) is an isomorphism (as before we consider \( X^s \) embedded in \( \Gamma \)). Hence, it follows that the open subset 
\[
\Pi^{-1} \left( \zeta^{-1}(X_0) \right), \text{ where } X_0 = \pi^{-1}(Y_0)
\]
in \( \hat{X} \) is mapped isomorphically on \( \pi^{-1}(Y_0) \) by \( \zeta \circ \Sigma \). Therefore, if \( y \in Y_0 \), we deduce that \( \pi^{-1}(y) \) is biholomorphic to 
\[
\Pi^{-1}(y, \mathcal{C}_y) \cap \Sigma^{-1} \left( \zeta^{-1}(X^s) \right)
\]
via \( (\zeta \circ \Sigma)^{-1} \). Using the fact that \( \tilde{T}(\epsilon, W^i) \) is defined as the pull back of \( T(\epsilon, W^i) \) and that \( \Omega = (\zeta \circ \Sigma)^* \omega \), it follows that the volume of \( T(\epsilon, W^i) \cap \pi^{-1}(y) \) with respect to \( \omega \) is equal to the volume of \( \tilde{T}(\epsilon, W^i) \cap \Pi^{-1}(y, \mathcal{C}_y) \) with respect to \( \Omega \), which proves the first claim.

The second claim is an immediate consequence of the above argumentation: Via \( (\zeta \circ \Sigma)^{-1} \), the fiber \( \pi^{-1}(y) \), where \( y \in Y_0 \), is biholomorphic to 
\[
\Pi^{-1}(y, \mathcal{C}_y) \cap \Sigma^{-1} \left( \zeta^{-1}(X^s) \right)
\]
in particular, it can be seen as a subset of \( \Pi^{-1}(y, \mathcal{C}_y) \) realized by \( (\zeta \circ \Sigma)^{-1} \). \( \text{ Q. E. D. } \)

**Remark V.3.** In general, the inequality in Lemma V.2 is a strict inequality. This is exhibited in Example IV.1 where 
\[
\text{cl} \left( \pi^{-1}([1:0]) \right) = \{z_1 = 0\} \text{ and } \text{cl} \left( \pi^{-1}([0:1]) \right) = \{\zeta_1 = 0\}
\]
on the one hand and 
\[
\text{pr}_X \left( \hat{\pi}^{-1}([1:0]) \right) = \{z_1 = 0\} \cup \{\zeta_0 = 0\} \text{ resp.}
\]
\[
\text{pr}_X \left( \hat{\pi}^{-1}([0:1]) \right) = \{\zeta_1 = 0\} \cup \{z_0 = 0\}
\]
on the other hand. Hence, \( \text{cl} \left( \pi^{-1}([\zeta_i]) \right) \) \( i \in \{0,1\} \) is properly contained as an irreducible component in \( \text{pr}_X \left( \hat{\pi}^{-1}([\zeta_i]) \right) \).

As a direct consequence of Lemma V.2 we deduce the following two corollaries.
Corollary V.1. Let $Y_0$ be as in Section [IV.1] then there exists a constant $C > 0$ so that

$$\text{vol} (\pi^{-1}(y)) = \int_{\pi^{-1}(y)} d[\pi_y] \leq C$$

for all $y \in Y_0$.

Proof. By the second claim of Lemma [V.2] we have

$$\text{vol} (\pi^{-1}(y)) \leq \text{vol} (\Pi^{-1}(y, \mathcal{C}_y))$$

for all $y \in Y_0$.

Since the projection of $\tilde{Y}$ on $\mathcal{C}^k(\tilde{X})$ is a compact subset, the claim then follows by the fact that the volumes of all cycles which are contained in a compact subset of $\mathcal{C}^k(\tilde{X})$ are uniformly bounded from above (cf. \text{[Bar2]}). Q. E. D.

Lemma V.3. Let $(y, \mathcal{C}) \in \tilde{Y}$, then $\Pi^{-1}(y, \mathcal{C}) \cap \tilde{T}(0, W_i)$ is of measure zero concerning the measure induced by $\Omega$.

Moreover, the restriction of the form $\Omega$ on $\Pi^{-1}(y, \mathcal{C}) \cap \tilde{T}(\epsilon, W_i)$ where $\epsilon > 0$ is non-zero.

Proof. First of all note, that we can view $\Pi^{-1}(y, \mathcal{C}) \cap \tilde{T}(\epsilon, W_i)$ as a $T$-invariant, closed, $k$-dimensional complex subspace in $\pi^{-1}(y) \cap T(\epsilon, W_i)$. In fact, each fiber $\Pi^{-1}(y, \mathcal{C})$ is given by the $T$-invariant, $k$-dimensional subvariety $|\mathcal{C}| \subset \tilde{X}$. The identification is then induced by $\text{p}_{X}|\text{cl}(\Gamma_{\pi}) \circ \zeta$ which is biholomorphic over $X^{ss}_x \supset T(\epsilon, W_i)$. Now, the second claim of the lemma is an immediate consequence of this fact combined with $\omega' = (\text{p}_{X}|\text{cl}(T_{\pi}) \circ \zeta)^* \omega$.

The first claim follows from the fact that the minimal closed, $T$-invariant complex space of $\pi^{-1}(y) \cap T(\epsilon, W_i)$ containing

$$\pi^{-1}(y) \cap T(0, W_i) = \pi^{-1}(y) \cap \mu^{-1}(\xi)$$

is given by the unique closed orbit $T.z_y$ which contains $\pi^{-1}(y) \cap T(0, W_i)$ as a total real submanifold. Q. E. D.

The rest of this section is devoted to the proof of the existence of uniform estimates concerning the fiber volume.

Proposition V.1. Let $\Delta > 0$ and $W_i \subset \pi(X_i)$ be as above, then there exists $\epsilon_\Delta > 0$ so that

$$\text{vol} (\pi^{-1}(y) \cap T(\epsilon_\Delta, W_i)) \leq \Delta$$

for all $y \in W_i \cap Y_0$.

Moreover, if $\epsilon > 0$, then there exists $\delta > 0$ so that

$$\delta \leq \text{vol} (\pi^{-1}(y) \cap T(\epsilon, W_i))$$

for all $y \in W_i \cap Y_0$. 

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\textbf{V} \ THE K-FIBERING $\widetilde{\Pi}: \widetilde{X} \to \widetilde{Y}$

\textit{Proof.} Choose a sequence $\epsilon_n \to 0$ and a sequence $(\psi_n)_n$ of smooth cut-off functions on $\widetilde{X}$ so that
$$\psi_n|\widetilde{T}(\epsilon_n, W^i) \equiv 1 \text{ and supp } \psi_n \cap \widetilde{T}^c(\epsilon_{n+1}, W^i) = \emptyset$$
which is possible since $\widetilde{T}(\epsilon_n, W^i)$ is relatively compact. In particular, we have
$$\Pi^{-1}(\omega_i) \cap \text{supp } \psi_n \subset \widetilde{T}(\epsilon_{n+1}, W^i)$$
and therefore
$$\Pi^{-1}(\omega_i) \cap \text{supp } \psi_n \downarrow \widetilde{T}(0, W^i) \text{ as } n \to \infty.$$

Hence, by the first claim of \textbf{Lemma V.3} the intersection of the set supp $\psi_n$ and a fiber of $\Pi$ over $\widetilde{W}^i$ converges monotonically decreasing to a set of measure zero with respect to the measure induced by $\Omega$. Set $\Omega_n := \psi_n \Omega^k$.

To sum up, $(\Omega_n)_n$ is a sequence of smooth (k, k)-forms on $\widetilde{X}$ with compact support where $\Pi: \widetilde{X} \to \widetilde{Y}$ is a holomorphic map between purely dimensional compact complex spaces so that each fiber $\Pi^{-1}(y)$ is purely k-dimensional. Furthermore, for each $(y, C) \in \widetilde{W}^i \subset \widetilde{Y}$ we know that
$$\int_{\Pi^{-1}(y, C)} \Omega_n \downarrow 0.$$

In order to finish the proof of \textbf{Proposition V.1} we need the following lemma, which we will proved at the end of this section.

\textbf{Lemma V.4.} Let $\Pi: \widetilde{X} \to \widetilde{Y}$ be a k-fibering between compact spaces and let $\widetilde{W} \subset \widetilde{Y}$ be a closed subset.

1. Let $(\Omega_n)_n$ be a sequence of smooth, positive (k, k)-forms on $\widetilde{X}$ and assume that
$$\int_{\Pi^{-1}(y)} \Omega_n \downarrow 0$$
for each $y \in \widetilde{W} \subset \widetilde{X}$ and let $\Delta > 0$. Then there exists $n_\Delta \in \mathbb{N}$ so that
$$\int_{\Pi^{-1}(y)} \Omega_n \leq \Delta \text{ for all } y \in \widetilde{W} \text{ and all } n \geq n_\Delta.$$

2. If $\Omega$ is a smooth, positive (k, k)-form on $\widetilde{X}$ so that
$$\int_{\Pi^{-1}(y)} \Omega > 0$$
for all $y \in \widetilde{W}$. Then there exists $\delta > 0$ so that
$$\delta \leq \int_{\Pi^{-1}(y)} \Omega \text{ for all } y \in \widetilde{W}.$$
Applying the first statement of the above lemma to the fiber integral yields the existence of \( n_\Delta \in \mathbb{N} \) so that

\[
\left( y \mapsto \int_{\Pi^{-1}(y, \mathcal{C})} \Omega_n \right) \leq \Delta \text{ for all } n \geq n_\Delta \text{ and all } (y, \mathcal{C}) \in \widetilde{W}^i.
\]

By the choice \( \psi_n|_{\mathcal{T}} \equiv 1 \), we deduce

\[
(y, \mathcal{C}) \mapsto \int_{\Pi^{-1}(y, \mathcal{C})} \Omega_n = \int_{\Pi^{-1}(y, \mathcal{C})} \psi_n \Omega^k
\]

\[
\geq \text{vol} \left( \mathcal{T} (\epsilon_n, \mathcal{W}^i) \cap \Pi^{-1} (y, \mathcal{C}_y) \right)
\]

for all \((y, \mathcal{C}) \in \widetilde{W}^i\). Now, set \( n = n_\Delta \), resp. \( \epsilon = \epsilon_{n_\Delta} \) and note that for all \( y \in \mathcal{W}^i \cap Y_0 \) we have

\[
\text{vol} \left( T (\epsilon, \mathcal{W}^i) \cap \pi^{-1} (y) \right) = \text{vol} \left( \mathcal{T} (\epsilon, \mathcal{W}^i) \cap \Pi^{-1} (y, \mathcal{C}_y) \right)
\]

by Lemma \[\text{V.2}\]. Hence, it follows that

\[
\text{vol} \left( T (\epsilon, \mathcal{W}^i) \cap \pi^{-1} (y) \right) \leq \Delta \text{ for all } y \in \mathcal{W}^i \cap Y_0
\]

as claimed.

The second claim is a direct consequence of the second claim of the above lemma applied to the smooth form \( \Omega_{n_0} \), where \( n_0 \in \mathbb{N} \) is chosen so that \( \epsilon_{n_0+1} < \epsilon \). In fact, we have

\[
(y, \mathcal{C}) \mapsto \int_{\Pi^{-1}(y, \mathcal{C})} \Omega_{n_0} = \int_{\Pi^{-1}(y, \mathcal{C})} \psi_{n_0} \Omega^k
\]

\[
\leq \text{vol} \left( \mathcal{T} (\epsilon, \mathcal{W}^i) \cap \Pi^{-1} (y, \mathcal{C}_y) \right)
\]

for all \((y, \mathcal{C}) \in \widetilde{W}^i\) which follows by \( \Pi^{-1}(\widetilde{W}^i) \cap \text{supp} \psi_n \subset \mathcal{T} (\epsilon_{n+1}, \mathcal{W}^i) \). Note that \( \Omega_{n_0} \) has compact support and that the left hand side of \[\text{V.3}\] is non-zero for all \((y, \mathcal{C}) \in \widetilde{W}^i\) which follows by the second claim of Lemma \[\text{V.3}\] applied to \( \epsilon = \epsilon_{n_0} \):

\[
0 < \text{vol} \left( \mathcal{T} (\epsilon_{n_0}, \mathcal{W}^i) \cap \Pi^{-1} (y, \mathcal{C}_y) \right) \leq \int_{\Pi^{-1}(y, \mathcal{C})} \Omega_{n_0} \text{ for all } (y, \mathcal{C}) \in \widetilde{W}^i.
\]

Applying the equality

\[
\text{vol} \left( T (\epsilon, \mathcal{W}^i) \cap \pi^{-1} (y) \right) = \text{vol} \left( \mathcal{T} (\epsilon, \mathcal{W}^i) \cap \Pi^{-1} (y, \mathcal{C}_y) \right)
\]

for all \( y \in \mathcal{W}^i \cap Y_0 \) and using the second claim of Lemma \[\text{V.4}\] completes the proof.

Q. E. D.
It remains to prove Lemma V.4.

Proof. (of Lemma V.4) Let \( \xi: \widetilde{X}^{\text{nor}} \to \widetilde{Y} \) be the normalization of \( \widetilde{Y} \) and consider the pull back space \( \xi^* \widetilde{X} \) of \( \widetilde{X} \) which is defined to be the complex space given by

\[
\xi^* \widetilde{X} := \{(x, \hat{y}) : \Pi(x) = \xi(\hat{y})\} \subset \widetilde{X} \times \widetilde{Y}^{\text{nor}}.
\]

Note that we have a projection map \( \xi^*\Pi: \xi^* \widetilde{X} \to \widetilde{Y}^{\text{nor}} \) which is given by the map \( \Pi \times \text{Id} \widetilde{Y}^{\text{nor}} \) and whose fibers are purely \( k \)-dimensional since they are exactly given by the fibers of the projection associated to the universal space \( \widetilde{X} \). Moreover, note that each smooth \((k, k)\)-form \( \Omega \) defined on \( \widetilde{X} \) induces a smooth \((k, k)\)-form on \( \xi^* \widetilde{X} \) via its pull back with respect to the projection map \( p_{\widetilde{X}}|_{\xi^* \widetilde{X}}: \xi^* \widetilde{X} \to \widetilde{X} \). In the sequel, we will label this form by \( \Omega^{\text{nor}} \). Since \( \xi^* \widetilde{X} \) is compact as a closed subset of the compact space \( \widetilde{X} \times \widetilde{Y}^{\text{nor}} \), it follows that \( \Omega^{\text{nor}} \) has compact support. Furthermore, the inverse image \( \tilde{W}^{\text{nor}} := \xi^{-1}(\tilde{W}) \) of the compact subset \( W \) is likewise compact for the same reason.

We will now prove the first claim of the lemma. For this, note that

\[
\int_{\Pi^{-1}(\tilde{y})} \Omega_n = \int_{\Pi^{\text{nor}, -1}(\tilde{y})} \Omega_n^{\text{nor}} \text{ for all } \tilde{y} \text{ with } \xi(\tilde{y}) = y. \tag{V.4}
\]

Since \( \Omega_n^{\text{nor}} \) defines a sequence of smooth \((k, k)\)-forms with compact support and since the map fulfills all requirements of theorem in [Kin], pp. 185-220, we deduce that the fiber integral

\[
\tilde{y} \mapsto \int_{\Pi^{\text{nor}, -1}(\tilde{y})} \Omega_n^{\text{nor}}
\]

defines a continuous function over the base \( \widetilde{Y}^{\text{nor}} \). So the sequence given by

\[
\left( \tilde{y} \mapsto \int_{\Pi^{\text{nor}, -1}(\tilde{y})} \Omega_n^{\text{nor}} \right)_n
\]

defines a sequence of continuous functions over \( \widetilde{Y}^{\text{nor}} \) which will converge for each point by the assumption of the lemma to the zero function. By Dini’s convergence theorem of strictly decreasing sequence of continuous functions, it follows that this sequence of functions converges uniformly over the compact subset \( \tilde{W}^{\text{nor}} \) to the zero function. Hence, for each \( \Delta > 0 \) we can find \( n_\Delta \in \mathbb{N} \) so that

\[
\int_{\Pi^{\text{nor}, -1}(\tilde{y})} \Omega_n^{\text{nor}} \leq \Delta \text{ for all } \tilde{y} \in \tilde{W}^{\text{nor}} \text{ and all } n \geq n_\Delta.
\]

By equation V.4 and the fact that \( \tilde{W}^{\text{nor}} = \xi^{-1}(\tilde{W}) \) the first claim is shown.

The second claim is a direct consequence of the fact that \( \Omega^{\text{nor}} \) has compact support and
by the assumption of the lemma which is given by

\[ 0 < \int_{\Pi^{-1}(y)} \Omega \text{ for all } y \in \tilde{W}. \]

Hence, we deduce that

\[ 0 < \int_{\Pi_{\text{nor}}^{-1}(\tilde{y})} \Omega_{\text{nor}} \text{ for all } \tilde{y} \text{ so that } \xi(\tilde{y}) = y \in \tilde{W} \]

by V.4. Note that \( \tilde{y} \mapsto \int_{\Pi_{\text{nor}}^{-1}(\tilde{y})} \Omega_{\text{nor}} \) is a continuous function which does not vanish over \( \tilde{W}_{\text{nor}} \) by the above inequality. Hence, there exists \( \delta > 0 \) so that \( \delta \leq \int_{\Pi_{\text{nor}}^{-1}(\tilde{y})} \Omega_{\text{nor}} \) for all \( \tilde{y} \) in the compact subset \( \tilde{W}_{\text{nor}} \). This proves the claim by applying V.4 again. \textbf{Q. E. D.}

For the sake of completeness, we will close this section by stating the theorem of J. King, which we have used in the proof of the preceding proposition.

\textbf{Theorem V.1.} (cf. [Kin], pp. 185-220)

Let \( F : X \to Y \) be a \( k \)-fibering between complex purely dimensional spaces \( X, Y \) where \( m = \dim C X, n = \dim C Y \) and assume that \( Y \) is normal. If \( \Omega \) is a continuous, complex valued \((k,k)\)-form on \( X \) with compact support, then the fiber integral

\[ y \mapsto \int_{F^{-1}(y)} \Omega \, d[Fy] := \int_{F^{-1}(y)} \nu_F |F^{-1}(y)| \]

defines a continuous function on \( Y \) where \( \nu_F \) denotes the order\(^3\) of the \( k \)-fibering \( F : X \to Y \).

VI Uniform Convergence in the Tame Case

VI.1 Uniform Convergence of the Fiber Probability Measures

The aim of this section is to prove \textbf{Theorem 3}. For this recall, that for each \( x \in \mu^{-1}(\xi) \) there exists a \( T \)-invariant Zariski open subset \( X^i \subset X^s_{\xi} \) which is contained in the n-stable complement of the zero set of the tame sequence \( (s^i_n)_n \) given by \( X (s^i_n) \subset X^s_{\xi} \) (cf. \textbf{Theorem 1}). Moreover, let \( g^i : X^i \to \mathbb{R} \) be the normalized s.p.s.h. limit function as defined at the beginning of \textbf{Section III} and recall that there exists a compact neighborhood \( W^i \subset \pi(X^i) \) so that \( T(\epsilon, W^i) \subset X^i \) given by

\[ T(\epsilon, W^i) = (g^i \times \pi)^{-1}([0, \epsilon] \times W^i) \] (cf. \textbf{Section III})

defining a compact \( T \)-invariant tube for all \( \epsilon \geq 0.\)

\(^3\)For a rigorous definition of the order of a \( k \)-fibering in the point \( x \in X \) cf. [Kin].
We begin the proof with the following lemma.

**Lemma VI.1.** Let $f \in C^0(X)$ be a $T$-invariant, continuous function. Given $\sigma > 0$, there exists $\epsilon_\sigma > 0$ so that

$$|f|\left(\pi^{-1}(y) \cap T(\epsilon_\sigma, W^i)\right) - f_{\text{red}}(y) \leq \sigma$$

for all $y \in W^i \subset Y^i$.

**Proof.** Let us assume that $\epsilon_\sigma > 0$ with the above property does not exist. Then we can find a sequence $(\epsilon_\sigma, n)_n$ of positive numbers with $\epsilon_\sigma, n \to 0$, a sequence $(y_n)_n$ in $W^i$ and a lifted sequence $(x_n)_n$ in $\pi^{-1}(W^i)$ (i.e. $\pi(x_n) = y_n$) where

$$x_n \in T(\epsilon_\sigma, n, W^i)$$

and so that

$$|f(x_n) - f_{\text{red}}(y_n)| \geq \epsilon > 0$$

for all $n$.

By the compactness of $W^i$, we can assume that $y_n \to y \in W^i$. Furthermore, by the compactness of $T(\epsilon_\sigma, n, W^i)$, we can assume that the lifted sequence $(x_n)_n$, which is contained in the sequence of the compact nested subsets given by $T(\epsilon_\sigma, n, W^i)$, is convergent as well, i.e. $x_n \to x$. Since

$$T(\epsilon_\sigma, n, W^i) \downarrow \mu^{-1}(\xi) \cap \pi^{-1}(W^i)$$

we have $x_n \to x \in \mu^{-1}(\xi) \cap \pi^{-1}(W^i)$.

So, all in all, we have found a convergent sequence $(x_n)_n$ in $\pi^{-1}(W^i)$ so that

$$x_n \to x \in \mu^{-1}(\xi) \cap \pi^{-1}(W^i)$$

and

$$|f(x_n) - f_{\text{red}}(\pi(x_n))| = |f(x_n) - f_{\text{red}}(y_n)| \geq \epsilon > 0$$

for all $n$. From the continuity of $\pi$, $f$ and the fact that $f_{\text{red}}(\pi(x)) = f_{\text{red}}(y) = f(x)$ we deduce a contradiction. 

**Q. E. D.**

After this preparation, we can prove the uniform convergence of the measure sequence with respect to the weak topology. For this recall that $\overline{f}$ for $f \in C^0(X)$ is defined to be the averaged function given by $\overline{f}(x) = \int_T f(t, x) \, d\nu_T$ where denotes the HAAR measure. Moreover, recall the definition of the sequence $(\nu_n^i)_n$ of fiber probability measures attached to the tame sequence $(s_n)_n$ (cf. Definition IV.2).

**Proposition VI.1.** Let $f \in C^0(X)$ and $W^i \subset Y^i = \pi(X^i)$ as before. Then the sequence of functions on $W^i \cap Y_0$ given by

$$y \mapsto \int_{\pi^{-1}(y)} f \, d\nu_n^i(y)$$
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converges uniformly on $W^i \cap Y_0$ to the reduced function $f_{\text{red}}|W^i \cap Y_0 : W^i \cap Y_0 \to \mathbb{R}$.

Proof. First, let us show that it is enough to prove the claim for continuous $T$-invariant functions: Let us assume that the claim is valid for all continuous functions which are $T$-invariant and assume that $f \in C^0(X)$ is arbitrary. Using the $T$-invariance of $\nu_n$ and $\nu_T$ and the fact $\int_T d\nu_T = 1$ it follows that:

$$\int_{\pi^{-1}(y)} t^* f d\nu_n(y) \, d\nu_T = \int_{\pi^{-1}(y)} f d\nu_n(y).$$

Interchanging the order of integration yields

$$\int_{\pi^{-1}(y)} f d\nu_n(y) = \int_{\pi^{-1}(y)} \left( \int_{t \in T} t^* f d\nu_T \right) d\nu_n(y) = \int_{\pi^{-1}(y)} \overline{f} d\nu_n(y). \quad (VI.1)$$

Since $\overline{f}$ is $T$-invariant and continuous and since we have assumed that the claim is true for those functions, we deduce by $[VI.1]$ that

$$\left( \int_{\pi^{-1}(y)} f d\nu_n(y) \right) = \left( y \mapsto \int_{\pi^{-1}(y)} \overline{f} d\nu_n(y) \right)$$

converges uniformly on $W^i \cap Y_0$ to the reduced function $f_{\text{red}} = |W^i \cap Y_0 : W^i \cap Y_0 \to \mathbb{R}$. Hence, it remains to verify the claim for continuous, $T$-invariant functions.

If $\sigma > 0$ and $f \in C^0(X)$, then by Lemma $[VI.1]$ there exists $\epsilon_\sigma > 0$ so that

$$|f| (\pi^{-1}(y) \cap \mathbb{T}(\epsilon_\sigma, W^i)) - f_{\text{red}}(y) \leq \frac{\sigma}{2}$$

for all $y \in W^i \subset \mathcal{U}$. By Theorem 2 we can find $N_0 \in \mathbb{N}$ so that for all $n \geq N_0$ we have

$$\tilde{\nu}_n^i(x) \geq \frac{\epsilon_\sigma}{2}$$

for all $x \in T^{c}(\epsilon_\sigma, W^i)$. Therefore, we deduce that

$$\left| \int_{\pi^{-1}(y)} f d\nu_n^i(y) - f_{\text{red}}(y) \right| \leq \frac{\sigma}{2} \int_{\pi^{-1}(y) \cap \mathbb{T}(\epsilon_\sigma, W^i)} |s_n^i|^2 \, d[\pi_y]$$

$$\quad \quad + e^{-n} C(f) \int_{\pi^{-1}(y) \cap \mathbb{T}(\epsilon_\sigma, W^i)} d[\pi_y]$$

for all $y \in W^i \cap Y_0$ where $C(f) := \max_{x \in X} |f - \pi^* f_{\text{red}}| < \infty$.

By Proposition $[II.2]$ the sequence $\tilde{\nu}_n^i$ converges uniformly on compact subsets of
VI UNIFORM CONVERGENCE IN THE TAME CASE

\( \pi^{-1}(Y^i) = X^i \) to \( \varrho^i \) and hence, for each \( \delta > 0 \), there exists \( N'_0 \in \mathbb{N} \) so that

\[ \varrho^i_n \leq \epsilon' + \delta \]

for all \( x \in T(\epsilon', W^i) \). Here we will choose \( \delta > 0 \) and \( \epsilon' > 0 \) so that \( \epsilon' + \delta < \frac{\varrho}{2} \). It then follows for all \( y \in W^i \cap Y_0 \)

\[ \|s^i_n\|^2(y) \geq \int_{\pi^{-1}(y) \cap T(\epsilon', W^i)} |s^i_n|^2 \ d[\pi_y] \]

\[ \geq e^{-n(\epsilon'+\delta)} \int_{\pi^{-1}(y) \cap T(\epsilon', W^i)} d[\pi_y]. \]

So we deduce

\[ \left| \int_{\pi^{-1}(y)} f \, d\nu^i_n(y) - f_{\text{red}}(y) \right| \leq \frac{\sigma}{2} \frac{\int_{\pi^{-1}(y) \cap T(\epsilon', W^i)} |s^i_n|^2 \ d[\pi_y]}{\|s^i_n\|^2(y)} \]

\[ + e^{-n(\frac{\varrho}{2}-\epsilon' - \delta)} C(f) \int_{\pi^{-1}(y) \cap T(\epsilon', W^i)} d[\pi_y] \]

for all \( y \in W^i \cap Y_0 \). Using Corollary \[V.1\] we know that the dominator of the last term is bounded for all \( y \in Y_0 \) and by the second claim of Proposition \[V.1\] we know that the denominator is bounded away from zero as \( y \) varies in \( W^i \cap Y_0 \). Therefore, we find a constant \( \Gamma < \infty \) so that

\[ \left| \int_{\pi^{-1}(y)} f \, d\nu^i_n(y) - f_{\text{red}}(y) \right| \leq \frac{\sigma}{2} + e^{-n(\frac{\varrho}{2}-\epsilon' - \delta)} C(f) \Gamma. \]

Since

\[ \frac{\int_{\pi^{-1}(y) \cap T(\epsilon, W^i)} |s^i_n|^2 \ d[\pi_y]}{\|s^i_n\|^2(y)} \leq 1 \]

for all \( n \in \mathbb{N} \) and all \( y \in W^i \cap Y_0 \), we find

\[ \left| \int_{\pi^{-1}(y)} f \, d\nu^i_n(y) - f_{\text{red}}(y) \right| \leq \frac{\sigma}{2} + e^{-n(\frac{\varrho}{2}\epsilon' - \delta)} C(f) \Gamma. \]

Since \( \frac{\varrho}{2} - \epsilon' - \delta > 0 \) there exists \( N''_0 \geq N'_0 \) so that

\[ \left| \int_{\pi^{-1}(y)} f \, d\nu^i_n(y) - f_{\text{red}}(y) \right| \leq \sigma \]
for all \(n \geq N_0\) as claimed. 

Q. E. D.

We sum up the results in

**Theorem 3. [Uniform Convergence of the Tame Measure Sequence]**

For every tame collection \(\{s^i_n\}_i\) there exists a finite cover \(\mathcal{U}\) of \(Y\) with \(U_i \subset \pi(X^i)\) so that the collection of fiber probability measures \(\{\nu^i_n\}\) associated to \(\{s^i_n\}_i\) converges uniformly on \(Y_0\) to the fiber Dirac measure of \(\mu^{-1}(\xi) \cap X_0\), i.e. for every \(i \in I\) and every \(f \in C^0(X)\), we have

\[
(\ y \mapsto \int_{\pi^{-1}(y)} f \ d\nu^i_n(y)) \rightarrow f_{\text{red}} \text{ uniformly on } U_i \cap Y_0.
\]

Proof. Let \(\{s^i_n\}_i\) be a tame collection and \(\mathcal{U}\) be a finite open cover of \(Y\) so that

\[
U^i \subset W^i \subset Y^i = \pi(X^i)
\]

where \(W^i\) is a compact neighborhood as defined at the beginning of Section III.

By Proposition VI.1 there exists \(N_0 \in \mathbb{N}\) so that

\[
\left| \int_{\pi^{-1}(y)} f \ d\nu^i_n(y) - f_{\text{red}}(y) \right| \leq \epsilon
\]

for all \(y \in Y_0 \cap W^i\) and all \(n \geq N_0\) which proves the claim. 

Q. E. D.

**VI.2 Uniform Convergence of the Fiber Distribution Densities**

In this section we will give a proof of Theorem 4. For this, let \((D^i_n(\cdot,t))_n\) be the sequence of cumulative fiber distribution functions on \(W^i \cap Y_0\) as defined in Definition IV.3 associated to a tame sequence \(\{s^i_n\}_n\) and let \(W^i \subset \pi(X^i)\) be a compact neighborhood so that the T-invariant tube \(T(\epsilon, W^i) \subset X^i\) as defined in Section III is compact.

**Proposition VI.2.** The sequence \((D^i_n(\cdot,t))_n\) of distribution functions on \(W^i \cap Y_0\) converges uniformly on \(W^i \cap Y_0\) to the zero function for all \(t \geq 0\).

Proof. First of all fix \(\sigma > 0\). Then by the first part of Proposition V.1 there exists \(\epsilon_\sigma > 0\) so that

\[
\text{vol} (\pi^{-1}(y) \cap T(\epsilon_\sigma, W^i)) \leq \sigma
\]

for all \(y \in W^i \cap Y_0\). Hence, we are finished as soon as we have proved that there exists \(N_0 \in \mathbb{N}\) so that

\[
\left\{ x \in \pi^{-1}(W^i \cap Y_0) : \frac{|s_n|^2}{\|s_n\|}(x) \geq t \right\} \subset T(\epsilon_\sigma, W^i)
\]

for all \(n \geq N_0\).
By Theorem 2 we find $N_0 \in \mathbb{N}$, so that $g_n^i (x) \geq \frac{\epsilon}{2}$ for all $x \in T^c (\epsilon_\sigma, W^i)$ and all $n \geq N_0$. Therefore, we deduce

$$\frac{|s_n|^2}{\|s_n\|^2} \leq \frac{e^{-n \frac{\epsilon}{2}}}{\|s_n\|^2}$$  \quad (VI.2)

for all $n \geq N_0$ and all $x \in T^c (\epsilon_\sigma, W^i) \cap \pi^{-1} (Y_0)$. Moreover, since $\varrho_n^i$ converges uniformly to $\varrho^i$ on the compact subsets like $T (\epsilon_\sigma, W^i)$ (cf. Proposition II.2), there exists $N_0 \geq N_0$ so that

$$\varrho_n^i (z) \leq \varrho^i (z) + \frac{\epsilon_\sigma}{5}$$

for all $z \in T (\epsilon_\sigma, W^i)$ and all $n \geq N_0$, i.e.

$$\varrho_n^i (z) \leq \frac{2}{5} \epsilon_\sigma$$

on $T (\epsilon_\sigma, W^i)$. So we deduce

$$\|s_n\|^2 (y) \geq e^{-n \frac{\epsilon_\sigma}{5}} \int_{\pi^{-1} (y) \cap T (\epsilon_\sigma, W^i)} d \pi_y$$

for all $y \in W^i \cap Y_0$. If we substitute this into equation (VI.2) it follows that

$$\frac{|s_n|^2}{\|s_n\|^2} (x) \leq \frac{e^{-n \frac{\epsilon_\sigma}{5}}}{\int_{\pi^{-1} (\pi (x)) \cap T (\epsilon_\sigma, W^i)} d \pi_y}$$

for all $x \in T^c (\epsilon_\sigma, W^i) \cap \pi^{-1} (Y_0)$ and all $n \geq N_0$. By the second part of Proposition V.1 we know that the denominator is bounded away from zero as $x$ varies in $\pi^{-1} (W^i \cap Y_0)$ and therefore we find $N_0 \in \mathbb{N}$, $N_0 \geq N_0$ so that

$$\frac{|s_n|^2}{\|s_n\|^2} (x) \leq t$$

for all $x \in T^c (\epsilon_\sigma, W^i) \cap \pi^{-1} (Y_0)$ and all $n \geq N_0$. So it follows that

$$\left\{ x \in \pi^{-1} (W^i \cap Y_0) : \frac{|s_n|^2}{\|s_n\|^2} (x) \geq t \right\} \subset T (\epsilon_\sigma, W^i)$$

for all $n \geq N_0$ and therefore

$$D_n (y, t) \leq \text{vol} \left\{ \pi^{-1} (y) \cap T (\epsilon_\sigma, W^i) \right\} \leq \sigma.$$  \quad (Q. E. D.)

To sum up, we have proved

**Theorem 4.** [Uniform Convergence of the Tame Distribution Sequence]

For every $t \in \mathbb{R}$ and every tame collection $\{ s^i_n \}$ there exists a finite cover $\mathcal{U}$ of $Y$ with $U_i \subset$
\[ \pi(X^i) \mathrm{ so that the collection of cumulative fiber probability densities } \{ D_n^i(\cdot, t) \} \mathrm{ associated to } \{ s_n^i \}_{i} \mathrm{ converges uniformly on } Y_0 \mathrm{ to the zero function on } Y_0, \mathrm{ i.e. for every } i \in I \mathrm{ we have} \]

\[ (y \mapsto D_n^i(y, t)) \to 0 \mathrm{ uniformly on } U_i \cap Y_0 \subset \pi(X^i). \]

**Proof.** As in the proof of Theorem 3, using the compactness of \( Y \), we can find by means of Theorem 1 a tame collection \( \{ s_n^i \}_{i} \) and an finite open cover \( \mathcal{U} \) of \( Y \) so that

\[ U^i \subset W^i \subset Y^i = \pi(X^i) \]

where \( W^i \) is a compact neighborhood as defined at the beginning of Section III. The claim then follows by applying Proposition VI.1 to the sequence \( (D_n^i(\cdot, t))_{i} \).

Q. E. D.

## VII Uniform Convergence in the Non-Tame Case

Throughout this section we make the following assumption.

**General Agreement.** There exists \( N_0 \in \mathbb{N} \) so that \( R_{N_0} \subset Y \) is non-empty.

Recall that \( R_{N_0} \cap Y_0 \) is open. Furthermore, if \( y \in R_{N_0} \cap Y_0 \), there exists an open neighborhood \( U_y \subset R_{N_0} \cap Y_0 \) and a sequence of local, holomorphic \( \xi_n \)-eigensections \( \hat{s}_f^n \) defined on the open \( \pi \)-saturated subset \( \pi^{-1}(U_y) \) so that \( \hat{s}_f^n|\pi^{-1}(y') \neq 0 \) for all \( n \geq N_0 \) and all \( y' \in U_y \). Here \( \hat{s}_f^n \) is given by (cf. Section IV.3)

\[ \hat{s}_f^n = s_n \cdot \pi^* f^n \] for all \( y' \in U_y \subset Y \)

where \( (f^n)_n \) is a sequence of holomorphic functions \( f^n \in \mathcal{O}(U_y) \).

Using Theorem 2, there exists a tame sequence \( \{ s^i_n \} \) so that \( y \in Y^i = \pi(X^i) \). Note that after having shrunken \( U_y \), we can always assume that \( U_y \subset Y^i \). We now define a sequence \( \{ \Delta^i_{y,n} \} \) of holomorphic \( \xi_n - \xi_n^i \)-eigenfunctions on \( \pi^{-1}(U_y) \) by

\[ s_n^i \cdot \Delta^i_{y,n} = \hat{s}_f^n. \]

Note that \( \Delta^i_{y,n}|\pi^{-1}(y') \neq 0 \) for all \( n \geq N_0 \) and all \( y' \in U_y \).

Moreover, recall that the sequence

\[ \phi_n : x \mapsto \phi_n(x) := ||\hat{s}_f^n||^{-2}(x)|\hat{s}_f^n|^2(x) \]

is independent of the choice of \( (f^n)_n, f^n \in \mathcal{O}(U_y) \), over \( U_y \subset R_{N_0} \cap Y_0 \) (cf. Corollary IV.1). Therefore, we will not longer specify \( (f^n)_n \) and just write \( \Delta^i_{y,n} = \Delta^i_{y,n} \) and \( s_n = \hat{s}_f^n \).
VII.1 Analysis of $\Delta^i_n|\pi^{-1}(y)$

The decomposition $s_n = s^i_n \cdot \Delta^i_{y,n}$ introduced above will play a crucial role in the proof of Theorem 5.a, b and Theorem 6.a, b. Since the sequence of local functions given by $\Delta^i_{y,n}$ can be seen as measure of the difference between the initial sequence of eigensections $s_n$ and the tame sequence $(s^i_n)_n$, it is desirable to control the growth of $\Delta^i_{y,n}$. However, in general the restrictions of the sequence of functions given by $\Delta^i_{y,n}$ to the fibers of the quotient map $\pi$ turns out to be unbounded. The aim of this section is to prove (cf. Proposition VII.1) that there always exists $m_0 \in \mathbb{N}$ so that $s^i_{m_0} \cdot \Delta^i_{y,n}$ is uniformly bounded over the quotient and takes on its maximum on the fibers in a given neighborhood of $\mu^{-1}(\xi)$ for all $n$ big enough.

**Proposition VII.1.** Let $W \subset Y$ be a compact neighborhood (we can assume that $W \subset Y^1$), then there exists $m_0 \in \mathbb{N}$ so that the restriction of $|s^i_{m_0} \cdot \Delta^i_{y,n}|$ on $\pi^{-1}(y)$ takes on its maximum in $\pi^{-1}(y) \cap T(\epsilon, W)$ for all $n$ big enough and all $y \in W \cap R_{N_0}$.

Before we prove the above claim, we first have to show the following lemma.

**Lemma VII.1.** If $(\eta_m)_m$ is an arbitrary sequence in $t^*$ so that $\frac{1}{m} \eta_m \to \xi$, then there exists $N_0 \in \mathbb{N}$ so that

$$X^ss_{m^{-1} \eta_m} \subset X^ss_\xi,$$

for all $m \geq N_0$.

Furthermore, for all $m \geq N_0$, each fiber of $\pi_m$ is entirely contained in a fiber of $\pi$.

**Proof.** Since $m^{-1} \eta_m \to \xi$ it follows that the sequence of compact sets given by $\mu^{-1}(m^{-1} \eta_m)$ converges to $\mu^{-1}(\xi)$. From the fact $X^ss_\xi$ is an open neighborhood of $\mu^{-1}(\xi)$ we deduce the existence of $N_0 \in \mathbb{N}$ so that $\mu^{-1}(m^{-1} \eta_m) \subset X^ss_\xi$ for all $m \geq N_0$. As a consequence, we conclude

$$X^ss_{m^{-1} \eta_m} \subset X^ss_\xi$$

for all $m \geq N_0$: Let $x \in X^ss_{m^{-1} \eta_m}$ (in the sequel fix $m \geq N_0$). Then by the definition of set of semistable points, we can find $y$ so that

$$y \in cl(T.x) \cap \mu^{-1}(m^{-1} \eta_m).$$

Since $y \in \mu^{-1}(m^{-1} \eta_m) \subset X^ss_\xi$, it follows that

$$cl(T.y) \cap \mu^{-1}(\xi) \neq \emptyset.$$

However, since $cl(T.y) \subset cl(T.x)$, it follows that $cl(T.x) \cap \mu^{-1}(\xi) \neq \emptyset$ which proves that $x \in X^ss_\xi$ and hence $X^ss_{m^{-1} \eta_m} \subset X^ss_\xi$ as claimed.

Using the fact that $X^ss_{m^{-1} \eta_m} \subset X^ss_\xi$ and the universality of the Hilbert quotient map $\pi_m$, there exists an algebraic map $\varphi_m : Y_m := X^ss_{m^{-1} \eta_m}/T \to Y$ for each $m \geq N_0$ so that the
following diagram commutes:

\[ \begin{array}{ccc}
X_{ss} & \xrightarrow{\pi_m} & Y_m \\
\downarrow \pi & & \downarrow \varphi_m \\
X_\xi & \xrightarrow{\eta_m} & Y \\
\end{array} \]

This proves the second claim. Q. E. D.

Before we proceed with the proof of Proposition VII.1, we note the following remark.

**Remark VII.1.** Let \( U \subset Y_m \) be an open subset, \( V := \pi_m^{-1}(U) \) and let

\[ \sigma \in H^0(V, L^m|V) \]

be a local \( \eta_m \)-eigensection over \( V \). In this situation, it follows for \( y \in U \) that the restriction of \( |\sigma|^2 \) to \( \pi_m^{-1}(y) \) takes on its maximum on \( \mu^{-1}(m^{-1}\eta_m) \cap \pi_m^{-1}(y) \). To see this, recall that by the construction of the algebraic Hilbert quotient, we can always find a global \( N \cdot \eta_m \)-eigensection \( \sigma' \in H^0(X, L^{N \cdot m}) \) for \( N \) big enough so that \( \sigma'(x) \neq 0 \) for all \( x \in \pi_m^{-1}(y) \). Therefore, \( \sigma'^{-1} \cdot \sigma \) defines a holomorphic \( T \)-invariant function on \( \pi_m^{-1}(y) \).

The next step is to consider two possibilities: If \( \sigma'^{-1} \cdot \sigma \equiv 0 \) then it follows that \( \sigma \equiv 0 \) and hence the claim is true. Otherwise, it follows that \( \sigma(x) \neq 0 \) for all \( x \in \pi_m^{-1}(y) \). In this case, the claim follows by the theory of the Hilbert quotient because \( -\log |\sigma|^2 \) defines a smooth plurisubharmonic potential on \( \pi_m^{-1}(y) \) of the shifted moment map data and in this case the claim is known.

**Proof.** (of Proposition VII.1) Let \( y \in W \cap R_{N_0} \). First of all, note that \( s_m^i \cdot \Delta_{y,n}^i \) defines a local holomorphic \( \eta_{m,n} := \xi_m^i + (\xi_n^i - \xi_{m}^i) \) eigensection over \( \pi_m^{-1}(U_y) \cap \pi_m^{-1}(y) \) for all \( n \). As \( |\xi_n - \xi_{m}^i| \in O(1) \), it follows that the set \( \{ \xi_n - \xi_{m}^i \}_{n \in \mathbb{N}} \subset t_{\mathbb{C}}^* \) is finite. Hence, it is enough to prove the claim under the assumption that \( \xi_n - \xi_{m}^i \) is a constant weight \( \xi_0 \in t^* \). In the sequel, set \( \eta_{m} := \xi_m^i + \xi_0 \). Since \( m^{-1}\eta_m \rightarrow \xi \) we can apply Lemma VII.1 in order to find \( m_0 \in \mathbb{N} \) so that \( X_{ss}^{m_0, \eta_{m_0}} \subset X_{\xi}^{ss} \). Set

\[ \Delta_{y,m_0,n}^i := s_{m_0}^i \cdot \Delta_{y,n}^i \]

and note that \( \Delta_{y,m_0,n}^i \) induces a local holomorphic \( \xi_{m_0} + \xi_0 \)-eigensection over the open, \( \pi_{m_0} \)-saturated subset

\[ V_{m_0,y} := (\varphi_{m_0} \circ \pi_{m_0})^{-1}(U_y) \subset X_{ss}^{m_0, \eta_{m_0}} \]

for all \( n \) big enough. In fact: By the above assumption, \( \Delta_{y,n}^i \) is of fixed weight \( \xi_0 \) for all \( n \in \mathbb{N} \).
Note that by Remark VII.1 it follows that the strictly plurisubharmonic function given by 
\[- \log |\Delta_{y,m_0,n}^i|^2\] takes on its uniquely defined minimum on \(\pi_{m_0}^{-1}(\tilde{y}) \cap \mu_{m_0}^{-1}(\eta_{m_0})\) for all \(n\) big enough and all \(\tilde{y} \in \varphi_{m_0}^{-1}(U_y)\). Equivalently, the restriction of \(|\Delta_{y,m_0,n}^i|^2\) to on \(\pi_{m_0}^{-1}(\tilde{y})\) takes on its uniquely defined maximum on \(\pi_{m_0}^{-1}(\tilde{y}) \cap \mu_{m_0}^{-1}(\eta_{m_0})\) for all \(n\) big enough and all \(\tilde{y} \in \varphi_{m_0}^{-1}(U_y)\).

By the commutative diagram of Lemma VII.1 it then follows that the restriction of \(|\Delta_{y,m_0,n}^i|^2\) to \(\pi_{-1}^{-1}(y') \cap X^{ss}_{m_0-1,\eta_{m_0}}\) takes on its maximum in \(\mu_{m_0}^{-1}(\eta_{m_0}) \cap \pi_{-1}^{-1}(y') \cap X^{ss}_{m_0-1,\eta_{m_0}}\) for all \(y' \in U_y\) and all \(n\) big enough. Since \(X^{ss}_{m_0-1,\eta_{m_0}}\) is Zariski dense in \(X^\xi\) it follows by continuity that the restriction of \(|\Delta_{y,m_0,n}^i|^2\) to \(\pi_{-1}^{-1}(y')\) takes on its maximum on

\[\mu_{m_0}^{-1}(\eta_{m_0}) \cap \pi_{-1}^{-1}(y')\]

for all \(y' \in U_y\) and all \(n\) big enough - in particular this holds for \(y \in U_y\) itself. The claim then follows by the fact that we can always assume that

\[\mu_{m_0}^{-1}(\eta_{m_0}) \cap \pi_{-1}^{-1}(W) \subset T(\epsilon, W)\]

for \(m_0\) big enough.

We close this section with the proof of

**Lemma VII.2.** Let \(W \subset Y\) be a compact neighborhood of \(y_0 \in Y\) (we can assume that \(W \subset Y^1\)), and \(\epsilon > 0\), then there exists \(N_0 \in \mathbb{N}\) so that

\[\max_{x \in \pi_{-1}(y) : T(\epsilon, W)} |\Delta_{y,n}^i|^2(x) > 0\]

for all \(y \in W \cap R_{N_0}\) and all \(n\) big enough.

In particular, it follows that

\[\max_{x \in \pi_{-1}(y) : T(\epsilon, W)} |s_{m_0}^i \cdot \Delta_{y,n}^i|^2(x) > 0\]

for all \(y \in W \cap R_{N_0}\) and all \(n\) big enough.

**Proof.** First of all note that the second claim is a direct consequence of the first because \(|s_{m_0}^i|^2(x) > 0\) for all \(x \in X^i\) and all \(m_0 \in \mathbb{N}\).

We already know that \(\Delta_{y,n}^i|^{-1}(y) \neq 0\) for all \(n \in \mathbb{N}\) and all \(y \in W \cap R_{N_0}\). Throughout the proof we will fix \(y\) and let

\[\pi_{-1}^{-1}(y) = \bigcup_j C_{j,y}\]

be the decomposition of \(\pi_{-1}^{-1}(y)\) in its global irreducible components \(C_{j,y}\). It is direct to see that these components are \(T\)-invariant and hence each \(C_{j,y}\) intersects \(\mu_{m_0}^{-1}(\xi)\) non-trivially: In fact, let \(z \in C_{j,y}\) and choose a one parameter subgroup \(\gamma : \mathbb{C}^* \rightarrow \mathbb{T}_{xy}\) so that

\[z_0 = \lim_{t \rightarrow 0} \gamma(t) . z \in \mathbb{T}_{xy}\]
where $T_x(y)$ is the unique closed orbit in the fiber $\pi^{-1}(y)$. Since $C_{j,y}$ is closed and $T$ invariant it follows that $z_0 \in T_x(y) \cap C_{j,y}$. Again by the invariance of $C_{j,y}$ and the fact that $\mu^{-1}(\xi) \cap T_x(y) = \mu^{-1}(\xi) \cap \pi^{-1}(y)$ it follows that $\mu^{-1}(\xi) \cap C_{j,y} \neq \emptyset$. In particular, the open set 

$$T(\epsilon, W) \cap C_{j,y}$$

is always non-empty in $C_{j,y}$. So if

$$\max_{x \in \pi^{-1}(y) \cap T(\epsilon, W)} |\Delta_{y,n}^i(x)|^2 = 0$$

for $y \in W \cap B_N$, it would follow that $\Delta_{y,n}^i$ would vanish identically on each non-empty open subset $T(\epsilon, W) \cap C_{j,y}$ of the irreducible component $C_{j,y}$. Hence, it would follow by the Identity Principle that $\Delta_{y,n}^i |C_{j,y} \equiv 0$ for all $j$ and therefore $\Delta_{y,n}^i |\pi^{-1}(y) \equiv 0$ in contradiction to $\Delta_{y,n}^i |\pi^{-1}(y) \neq 0$.

**Q. E. D.**

### VII.2 A Local Proposition Concerning Fiber Integration

In this section we will prove a technical proposition (cf. Proposition VII.2) which will be of crucial importance when proving Theorem 5.a, b and Theorem 6.a, b.

A first step towards Proposition VII.2 is the following technical lemma.

**Lemma VII.3.** Let

$$z^d + \alpha_{d-1}z^{d-1} + \cdots + \alpha_0, \text{ where } \alpha_i \in \mathbb{C} \text{ for } 0 \leq i \leq d-1$$

be a monic polynomial of degree $d$ and let $\zeta_i, 1 \leq i \leq d$ the corresponding roots, then there exists a constant $c_d > 0$ which only depends of the degree $d$, so that

$$\sum_{i=1}^{d} |\zeta_i|^2 \geq c_d \left( \sum_{i=0}^{d-1} |\alpha_i|^2 \right)^{\frac{1}{d}}.$$

**Proof.** Consider the holomorphic map $F: \mathbb{C}^d \to \mathbb{C}^d$ defined by

$$F: (\zeta_1, \ldots, \zeta_d) \mapsto (\Psi_0(\zeta_1, \ldots, \zeta_d), \ldots, \Psi_{d-1}(\zeta_1, \ldots, \zeta_d))$$

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where \( \{ \mathfrak{P}_\ell \}_{0 \leq \ell \leq d-1} \) is the set of all elementary symmetric polynomials, i.e.

\[
\mathfrak{P}_{d-1} (\zeta_1, \ldots, \zeta_d) = (-1)^{d} \sum_{1 \leq j \leq d} \zeta_j \\
\mathfrak{P}_{d-2} (\zeta_1, \ldots, \zeta_d) = (+1)^{d} \sum_{1 \leq j < j_2 \leq d} \zeta_j \zeta_{j_2} \\
\vdots \\
\mathfrak{P}_{d-\ell} (\zeta_1, \ldots, \zeta_d) = (-1)^{\ell} \sum_{1 \leq j_1 < \cdots < j_\ell \leq d} \zeta_{j_1} \cdots \zeta_{j_\ell} \\
\vdots \\
\mathfrak{P}_0 (\zeta_1, \ldots, \zeta_d) = (-1)^{d} \zeta_1 \cdots \zeta_m
\]

If \( \alpha = (\alpha_0, \ldots, \alpha_{d-1}) \in \mathbb{C}^d \), then by Vieta’s Formula we deduce that

\[
F^{-1} (\alpha) = \left\{ \zeta \in \mathbb{C} : \zeta^d + \alpha_{d-1} \zeta^{d-1} + \alpha_{d-2} \zeta^{d-2} + \cdots + \alpha_0 = 0 \right\}
\]

so the inverse image of \( \alpha \in \mathbb{C}^d \) contains exactly all roots of the polynomial with coefficients \( \alpha_\ell \) for \( 0 \leq \ell \leq d-1 \). In particular, it follows that \( F^{-1} (0) = \{0\} \) and by continuity there exists \( c_d > 0 \) so that \( \Delta_{c_d} \subset \text{Int} F^{-1} (\Delta_1) \) where \( \Delta_\lambda \subset \mathbb{C}^d \) is the closed ball of radius \( \lambda > 0 \) in \( \mathbb{C}^d \). It is direct to check that

\[
F (\lambda \zeta_1, \ldots, \lambda \zeta_d) = \left( \lambda^d \mathfrak{P}_0 (\zeta_1, \ldots, \zeta_d), \ldots, \lambda \mathfrak{P}_{d-1} (\zeta_1, \ldots, \zeta_d) \right)
\]

for all \( \lambda \in \mathbb{C} \) and hence, we deduce that

\[
\Delta_{\lambda \pm c_d} \subset \text{Int} F^{-1} (\Delta_\lambda)
\]

for all \( \lambda \geq 0 \). This can be reformulated as follows: If \( z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_0 \) is a monic polynomial of degree \( d \) so that \( \sum_{i=0}^{d-1} |\alpha_i|^2 = \lambda \), i.e. \( \alpha \in \text{bd} \Delta_\lambda \), then \( F^{-1} (\alpha) \notin \Delta_{\lambda \pm c_d} \), i.e.

\[
\sum_{i=1}^{d} |\zeta_i|^2 \geq \lambda \pm c_d
\]

where \( F^{-1} (\alpha) = \{ \zeta_i \}_{1 \leq i \leq d} \) is the set of the corresponding roots and therefore

\[
\sum_{i=1}^{d} |\zeta_i|^2 \geq c_d \left( \sum_{i=0}^{d-1} |\alpha_i|^2 \right)^{\frac{1}{d}}
\]

as claimed.

\[ \text{Q. E. D.} \]

Let \( F : X \to Y \) be \( k \)-fibering, i.e. a holomorphic map between purely dimensional complex spaces where \( m = \dim_{\mathbb{C}} X \) and \( n = \dim_{\mathbb{C}} Y \) so that \( F^{-1} (y) \) is a purely dimensional complex space of dimension \( k = m - n \) for all \( y \in Y \). The relevant examples of \( k \)-fiberings, which we have in mind, are given by \( \hat{\pi} : \hat{X} \to Y \), \( \pi |_{X_0} : X_0 \to Y_0 \) where \( X_0 = \pi^{-1} (Y_0) \) and
II: \( \widetilde{X} \to \widetilde{Y} \).

If \( x_0 \in X \) and \( F: X \to Y \) a \( k \)-fibered, then there exists (cf. [KIN], p. 205) an open neighborhood \( U \subset X \) of \( x_0 \) which can be realized via an isomorphism \( \Phi \) as a closed analytic subset \( Z \subset Q = Q_0 \times Q_1 \subset \mathbb{C}^k \times \mathbb{C}^k \) of a relatively compact, open product set \( Q_0 \times Q_1 \) and an open neighborhood \( B \subset Y \) of \( y_0 = F(x_0) \in Y \) so that the following holds: If

\[
z^{(0)} = (z_1^{(0)}, \ldots, z_k^{(0)}), \ \text{resp.} \ z^{(1)} = (z_1^{(1)}, \ldots, z_k^{(1)}).
\]

are the standard coordinates on \( \mathbb{C}^k \), resp. on \( \mathbb{C}^k \), then the restriction of the projection map \( p: \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{C}^k \) to the closed \( k \)-dimensional space \( Z_y := \Phi \left( \mathbb{F}^{-1} \left( y \right) \cap U \right) \) of \( Q = Q_0 \times Q_1 \) induces a \( d \)-sheeted covering map \( p|_{Z_y}: Z_y \to Q_1 \) onto \( Q_1 \) for all \( y \in B \subset Y \). Moreover, by the theory of finite \( d \)-sheeted coverings the following is known (cf. [GR-RE], pp. 133-146): For each \( d \)-sheeted covering map \( p|_{Z_y}: Z_y \to Q_1 \) where \( y \in B \), the inclusion

\[
\mathcal{O}(Q_1) \subset (p|_{Z_y})_* \mathcal{O}(Z_y)
\]

is a finite, integral ring extension so that for all for each \( f \in \mathcal{O}(Z_y) \), there exists \( \alpha_{f,j} \in \mathcal{O}(Q_1) \), \( 0 \leq j \leq d - 1 \) with

\[
f^d + (p|_{Z_y})^* \alpha_{f,d-1} f^{d-1} + \cdots + (p|_{Z_y})^* \alpha_{f,0} = 0 \tag{VII.1}
\]
on \( Z_y \).

**Example VII.1.** Let \( \widetilde{X} = \{z_0 \xi_1 - z_1 \zeta_0 \xi_0 = 0\} \subset X \times \mathbb{CP}^1 \) be as in example Example 11.1 where \( \widetilde{Y} \cong Y \cong \mathbb{CP}^1 \) and \( \Pi = \pi = \pi_{\mathbb{CP}^1} \) and consider the open neighborhood \( U \subset \widetilde{X} \) of \( \{1:0\}, \{0:1\}, \{1:0\} \) which is isomorphic to the affine variety \( \{\xi - z \zeta = 0\} \subset \mathbb{C}^3 \) where \( z = z_0^{-1} z_1, \zeta = \zeta_1^{-1} \zeta_0, \xi = \xi_0^{-1} \xi_1 \). After a linear change of coordinates it follows that \( \widetilde{X} \cap U \cong \{\xi - z^{j_2} - \zeta^{j_2} = 0\} \) and we can choose \( \mathbb{C}^2 \times \mathbb{C} = B_0 \times B_1 \) where \( \zeta', \xi \) are the coordinates of the first factor and \( z' \) of the second one in order to deduce a two sheeted, global projection \( p|_{Z_\xi}: Z_\xi \to Q_1 \) onto \( Q_1 = \mathbb{C} \) for all \( \xi \in Y \setminus \{0:1\} \).

If \( f \in \mathcal{O}(Z_\xi) \) is given by the restriction of the polynomial \( p(z') = \sum_{j=0}^m c_j z'^j \) to \( Z_\xi \) one can verify that

\[
\alpha_{f,1} (z') = -2 \sum_{j=0}^m c_j \left( z'^2 - \xi \right)^{\frac{j}{2}}
\]

and

\[
\alpha_{f,0} (z') = \sum_{j=0}^m (-1)^j c_j^2 \left( z'^2 - \xi \right)^{\frac{j}{2}} + 2 \sum_{0 \leq i < k \atop j, k \text{ even}} c_i c_k \left( z'^2 - \xi \right)^{\frac{j}{2}(j+k)} \cdots
\]

Now let \( V \subset \mathbb{C}^k \times \mathbb{C}^k \) be an open neighborhood of \( x = 0 \in \mathbb{C}^k \times \mathbb{C}^k \) containing \( Q \) and equip \( V \) with a smooth Kähler form \( \omega \).

With the help of Lemma VII.3 and the existence of VII.1 we deduce the following lemma.
Lemma VII.4. Let \( y \in B \) and \( f \in \mathcal{O}(Z_y) \) where \( \alpha_{f,j} \in \mathcal{O}(Q_1), 1 \leq j \leq d-1 \) are as above. Then it follows
\[
\int_{Z_y} |f|^2 d[Z_y] \geq d \cdot c_d \int_{Q_1} \left( \sum_{j=0}^{d-1} |\alpha_{f,j}|^2 \right)^{\frac{1}{2}} \omega_0^q
\]
where \( \omega_0 \) denotes the standard Kähler form, \( c_d \) the constant of Lemma VII.3 and \( d \) the degree of the covering map.

Proof. It is known (cf. [KIN], pp. 185-220) that the right hand side of the above inequality can be be bounded from below by
\[
d \int_{Q_1} \left( z^{(1)} \mapsto \sum_{j=1}^{d} \left| \zeta_{f,j} \left( z^{(1)} \right) \right|^2 \right) \omega_0^q
\]
where \( \zeta_{f,j} \left( z^{(1)} \right) \) are the \( d \) roots of the polynomial equation
\[
z^d + \alpha_{f,d-1}(z^{(1)}) z^{d-1} + \cdots + \alpha_{f,0}(z^{(1)}) = 0.
\]
The claim then follows by using the inequality proved in Lemma VII.3. Q. E. D.

After this preparation, we can now prove the announced proposition.

Proposition VII.2. Let \((y_n)_n \) be a sequence in \( B \) converging to \( y_0 \in B \) and let \((f_n)_n \), \( f_n \in \mathcal{O}(Z_{y_n}) \) be a sequence of uniformly bounded holomorphic functions so that
\[
\int_{Z_n} |f_n|^2 d[Z_n] \to 0.
\]

Then it follows that for each compact subset \( K \subset Q_1 \) and each \( \epsilon > 0 \) there exists \( N_\epsilon(K) \in \mathbb{N} \) so that
\[
|f_n|^2 \leq \epsilon \quad \text{on} \quad p^{-1}(K) \cap Z_n \quad \text{for all} \quad n \geq N_\epsilon(K).
\]

Proof. First of all set \( \alpha_{n,j} := \alpha_{f,n,j} \in \mathcal{O}(Z_{y_n}) \) for \( 0 \leq j \leq d-1 \). By the assumption combined with Lemma VII.4 we deduce that
\[
d \cdot c_d \int_{Q_1} \left( \sum_{j=0}^{d-1} |\alpha_{n,j}|^2 \right)^{\frac{1}{2}} \omega_0^q \to 0 \quad \text{(VII.2)}
\]
where
\[
f_n^d + (p|Z_n)^* \alpha_{n,d-1} f_n^{d-1} + \cdots + (p|Z_n)^* \alpha_{n,0} = 0. \quad \text{(VII.3)}
\]
Since the sequence \((f_n)_n \) is uniformly bounded, i.e. there exists \( C > 0 \) so that \(|f_n|^2 \leq C \) on
Z_n for all n, it follows by equation [VII.3] in combination with Lemma VII.3 that
\[ d \cdot C \geq c_d \left( \sum_{j=0}^{d-1} |\alpha_{n,j}|^2 \right)^{\frac{1}{d}} \]
on Q_1 for all n.

Hence, the sequences (\alpha_{n,j})_n, where \alpha_{n,j} \in O(Q_1), are uniformly bounded as well for all 1 \leq j \leq d - 1. By the theorem of Montel (after having chosen a subsequence) it follows that
\[ \alpha_{n,j} \to \alpha_j \in O(K) \]
uniformly on the compact subset K \subset Q_1.

The next step is to show that \alpha_j \equiv 0 for all j. Let us assume that this is false, i.e. there exist at least one \alpha_\ell \not\equiv 0 for 0 \leq \ell \leq d - 1. Since \alpha_{n,j} \to \alpha_j converges uniformly on K, it follows by VII.2 that
\[ 0 = \lim_{n \to \infty} c_d \int_K \left( \sum_{j=0}^{d-1} |\alpha_{n,j}|^2 \right)^{\frac{1}{2}} \omega_0^q = c_d \int_K \left( \sum_{j=0}^{d-1} |\alpha_j|^2 \right)^{\frac{1}{2}} \omega_0^q \]
which yields a contradiction to the assumption that \alpha_\ell \not\equiv 0 on K for at least one \ell. Hence, we deduce that \alpha_j \equiv 0 on K for all 0 \leq j \leq d - 1. As the convergence is uniform, we find for an arbitrary \Gamma > 0 an integer N(\Gamma) \in \mathbb{N} so that |\alpha_{d-j}|^2 \leq \Gamma for all \Gamma \geq N(\Gamma) on K. By equation [VII.3] combined with the general fact that if \zeta \in \mathbb{C} is a root of \zeta^d + \alpha_{d-1} \zeta^{d-1} + \cdots + \alpha_0 = 0 then |\zeta| \leq 2 \max_{1 \leq j \leq d} |\alpha_{d-j}|^{\frac{1}{d}} (cf. [DE]), we deduce
\[ |f_n| \leq 2 \max_{1 \leq j \leq d} \Gamma_\epsilon^{\frac{1}{d}} \]
on p^{-1}(K) \cap Z_n for all n \geq N(\Gamma)
which proves the claim: Choose \Gamma_\epsilon > 0 so that 2 \max_{1 \leq j \leq d} \Gamma_\epsilon^{\frac{1}{d}} = \epsilon and set N_\epsilon(K) := N(\Gamma_\epsilon).
Q.E.D.

VII.3 Local Uniform Convergence on R_{N_0} \cap Y_0

Recall that the set of all removable singularities R_{N_0} of the measure sequence (\nu_n)_n induced by the \xi-approximating sequence (s_n)_n of \xi_n-eigensections s_n \in H^0(X, L^n) is assumed to be non-empty throughout this section (cf. General Agreement at the beginning of page 42). The first step towards the proof of Theorem 5.a and Theorem 6.a is the following proposition.

Proposition VII.3. Let W \subset Y_0 \cap R_{N_0} be a compact neighborhood of y_0 \in Y_0 \cap R_{N_0} (we can assume that W \subset Y^1) and \epsilon > 0, then after having shrunken W, there exists a constant
C > 0 and \( m_0 \in \mathbb{N} \) so that
\[
\max_{x \in \pi^{-1}(y)} \frac{|s_{m_0}^i \cdot \Delta^i_{y,n}|^2 \int_{\pi^{-1}(y) \cap T(\epsilon, W)} |s_{m_0}^i \cdot \Delta^i_{y,n}|^2 \, d[\pi_y]}{\max_{x \in \pi^{-1}(y) \cap T(\epsilon, W)} |s_{m_0}^i \cdot \Delta^i_{y,n}|^2} < C
\]
for all \( n \geq m_0 \) and all \( y \in W \).

**Proof.** First of all by Proposition [VII.1] we know that there exists \( m_0 \) so that
\[
\max_{x \in \pi^{-1}(y)} |s_{m_0}^i \cdot \Delta^i_{y,n}|^2 = \max_{x \in \pi^{-1}(y) \cap T(\epsilon, W)} |s_{m_0}^i \cdot \Delta^i_{y,n}|^2 \quad (\text{VII.4})
\]
for all \( n \) big enough and all \( y \in W \cap R_{N_0} = W \) (by the above assumption, \( W \) is contained in \( R_{N_0} \cap Y_0 \)). Furthermore, by lemma Lemma [VII.2] we deduce that
\[
\max_{x \in \pi^{-1}(y) \cap T(\epsilon, W)} |s_{m_0}^i \cdot \Delta^i_{y,n}|^2 > 0
\]
for all \( n \) big enough and all \( y \in W \cap R_{N_0} = W \). Hence,
\[
\int_{\pi^{-1}(y) \cap T(\epsilon, W)} \frac{|s_{m_0}^i \cdot \Delta^i_{y,n}|^2 \, d[\pi_y]}{\max_{x \in \pi^{-1}(y) \cap T(\epsilon, W)} |s_{m_0}^i \cdot \Delta^i_{y,n}|^2} > c
\]
is well defined for all \( n \) big enough and all \( y \in W \). Note that the claim is shown, as soon as we have shown that, after having shrunk \( W \), there exists \( c > 0 \) (set \( C := c^{\epsilon^{-1}} \)) so that
\[
\int_{\pi^{-1}(y) \cap T(\epsilon, W)} \frac{|s_{m_0}^i \cdot \Delta^i_{y,n}|^2 \, d[\pi_y]}{\max_{x \in \pi^{-1}(y) \cap T(\epsilon, W)} |s_{m_0}^i \cdot \Delta^i_{y,n}|^2} < c
\]
for all \( n \) big enough and all \( y \in W \).

Let us assume that this is not the case, then there exists a sequence \( y_n \in W \) converging to \( y_0 \) so that
\[
\int_{\pi^{-1}(y_n) \cap T(\epsilon, W)} \frac{|s_{m_0}^i \cdot \Delta^i_{y,n}|^2 \, d[\pi_{y_n}]}{\max_{x \in \pi^{-1}(y_n) \cap T(\epsilon, W)} |s_{m_0}^i \cdot \Delta^i_{y,n}|^2} \to 0. \quad (\text{VII.5})
\]
To shorten notation we will set
\[
f_n := \frac{s_{m_0}^i \cdot \Delta^i_{y,n}}{\max_{x \in \pi^{-1}(y_n) \cap T(\epsilon, W)} |s_{m_0}^i \cdot \Delta^i_{y,n}|} \pi^{-1}(y_n)
\]
throughout the rest of this proof where \( f_n \in H^0(\pi^{-1}(y_n), L_{m_0} |\pi^{-1}(y_n)|) \). By equation [VII.4]
it follows that
\[
\max_{x \in \pi^{-1}(y_n)} |f_n|^2 = \max_{x \in \pi^{-1}(y_n) \cap T(\frac{\epsilon}{2}, W)} |f_n|^2 = 1
\]
for all \(n\) big enough where \(y_n \to y_0 \in \text{Int } W\). Therefore we can find a lift of \((y_n)_n\), i.e. a sequence \((x_n)_n\) in \(T(\frac{\epsilon}{2}, W)\) so that \(\pi(x_n) = y_n\) with the property
\[
|f_n|^2(x_n) = 1 = \max_{x \in \pi^{-1}(y_n)} |f_n|^2.
\]
(VII.6)

As \(x_n \in T(\frac{\epsilon}{2}, W)\) where \(y_n = \pi(x_n) \to y_0 \in \text{Int } W\), we can assume using the compactness of \(T(\frac{\epsilon}{2}, W)\) (after having chosen a subsequence) that \(x_n \to x_0 \in \text{Int } T(\epsilon, W)\). Hence, there exists an open neighborhood \(U\) of \(x_0\) which is contained in the interior of \(T(\epsilon, W)\) so that \(x_n \in U\) for all \(n\) big enough.

Since we have \(x_0 \in \pi^{-1}(y_0) \subset \pi^{-1}(Y_0)\), we can assume that the open neighborhood \(U\) (after having shrunked it) is isomorphic via an holomorphic embedding \(\Phi: U \to \mathbb{C}^k \times \mathbb{C}^k\) (let \(\Phi(x_0) = 0 = (0, 0) \in Q_0 \times Q_1\)) to a closed analytic subset in a relatively compact product neighborhood \(Q = Q_0 \times Q_1 \subset \mathbb{C}^k \times \mathbb{C}^k\) so that each \(\Phi(\pi^{-1}(y) \cap U) = Z_y\), where \(y\) varies in an open neighborhood \(B \subset Y_0\) of \(y(x_0)\), is a closed analytic subset which yields a surjective, \(d\)-sheeted covering of \(Q_1\) given by \(p|Z_y \to Q_1\). The existence of such a neighborhood \(U\) has been treated in Section VII.2.

After having shrunked \(U\) again, we can also assume that \(L^m|U \cong U \times \mathbb{C}\). In particular, the norm \(|s_{m_0}^i|^2\) of the holomorphic sections \(s_{m_0}^i\) over \(U\) is then given by a smooth, strictly positive function \(\alpha \in \mathcal{C}^\infty(U)\), independent of \(n \in \mathbb{N}\), so that \(|f_n|^2 = |\Delta_{y_n,n}|^2\). It is important to note that \(|f_n|^2\) denotes the norm of the restricted local section \(f_n\) with respect to the hermitian bundle metric \(h\), whereas \(|\Delta_{y_n,n}|^2\) is the norm induced by the absolute value of the complex numbers. In the sequel, we will use the abbreviation \(g_n := \Delta_{y_n,n} \in \mathcal{O}(Z_{y_n})\), i.e. we have \(|f_n|^2 = \alpha \cdot |g_n|^2\) on \(Z_{y_n}\). Note that we have
\[
\max_{x \in Z_{y_n}} |f_n|^2|Z_{y_n}|^2 = |f_n(x_n)|^2 = 1
\]
because \(x_n \in U\) for all \(n\) big enough where \(\Phi(\pi^{-1}(y_n) \cap U) = Z_{y_n}\). In other words, if \(A > 0\), resp. \(a > 0\) denotes the maximum, resp. the minimum of \(\alpha\) on \(U\) we deduce that
\[
a^{-1} \geq \max_{x \in Z_{y_n}} |g_n|^2|Z_{y_n}|^2 \quad \text{and} \quad |g_n|^2(x_n) \geq A^{-1} > 0
\]
(VII.7)
for all \(n\) big enough. Furthermore, by assumption VII.5 it follows that
\[
\int_{Z_{y_n}} |f_n|^2 d[Z_n] \geq a \int_{Z_{y_n}} |g_n|^2 d[Z_{y_n}] \to 0 \quad \text{and hence} \quad \int_{Z_{y_n}} |g_n|^2 d[Z_{y_n}] \to 0.
\]

We are now in the situation of Proposition VII.2 We have a sequence \(Z_n = Z_{y_n}\) and
a sequence \((g_n)_n\) of uniformly bounded holomorphic functions on \(Z_n\) given by \(g_n\) so that
\[
\int_{Z_n} |g_n|^2 \, d[Z_n] \to 0.
\]
In particular, if \(K \subset Q_1\) is a compact neighborhood of \(0 \in K\) and if \(\epsilon = \frac{1}{2}A^{-1}\) we deduce, by Proposition VII.2 that there exists \(N_{\frac{1}{2}A^{-1}}(K)\) so that
\[
|g_n|^2 \leq \frac{1}{2}A^{-1} \text{ on } p^{-1}(K) \cap Z_n \text{ for all } n \geq N_{\frac{1}{2}A^{-1}}(K). \tag{VII.8}
\]
However, since \(x_n \to x_0 = 0 \in p^{-1}(K)\), it follows that \(x_n \in p^{-1}(K) \cap Z_n\) for all \(n\) big enough. According to the second inequality of VII.7, we have \(|g_n|^2 (x_n) \geq A^{-1}\) and hence a contradiction to VII.8. Therefore, the assumption VII.5 is false and the claim is proven. Q. E. D.

**Theorem 5.a. [Locally Uniform Convergence of the Initial Distribution Sequence]**

Let \(y \in Y_0 \cap R_{N_0}\), \(t \in \mathbb{R}\) and let \(W \subset Y_0 \cap R_{N_0}\) be a compact neighborhood\(^4\) of \(y_0\). Then after having shrunk \(W\), the sequence \((D_n(\cdot, t))_n\) converges uniformly to the zero function over \(W\).

**Proof.** Let \(\epsilon > 0\), then by the first part of Proposition VII.1 there exists \(\sigma_\epsilon > 0\) so that
\[
\text{vol}(\pi^{-1}(y) \cap T(\sigma_\epsilon, W)) \leq \epsilon \tag{VII.9}
\]
for all \(y \in W \cap Y_0 = W\). Hence, it is enough to show that
\[
\phi_n = \frac{|s_n|^2}{||s_n||^2} \leq t \tag{VII.10}
\]
on \(T^c(\sigma_\epsilon, W)\) for all \(n\) big enough. Recall that \(|s_n|^2||s_n||^{-2}\) is an abbreviation for the local description given by \(|s_{y,n}|^2||s_{y,n}||^{-2}\) on \(\pi^{-1}(U_y)\). The first step is to write
\[
\phi_n = \frac{|s_n|^2}{||s_n||^2} \leq t
\]
on \(T^c(\sigma_\epsilon, W)\) for all \(n\) big enough. Recall that \(|s_n|^2||s_n||^{-2}\) is an abbreviation for the local description given by \(|s_{y,n}|^2||s_{y,n}||^{-2}\) on \(\pi^{-1}(U_y)\). The first step is to write
\[
\phi_n = \frac{|s_n|^2}{||s_n||^2} \leq t
\]
for arbitrary \(m \in \mathbb{N}\) which is possible because \(s_m^i (x) \neq 0\) for all \(x \in X^i\) and all \(m \in \mathbb{N}\). Using Lemma VII.2 we deduce that
\[
\max_{x \in \pi^{-1}(y) \cap T(\sigma_\epsilon, W)} |s_m^i \cdot \Delta_{y,n}^i|^2 > 0 \tag{VII.11}
\]

\(^4\)From now on, we will always assume that \(W \subset Y^i\) which is possible without restriction of generality.
for all $n$ big enough and all $y \in W$ and hence we can estimate

$$
\phi_n \leq \frac{|s^i_n \cdot s^{i-1}_m|^2 \cdot |s^i_m \cdot \Delta^i_{y,n}|^2}{\int_{\pi^{-1}(y) \cap T(\mathcal{X}, W)} |s^i_n \cdot s^{i-1}_m|^2 \cdot |s^i_m \cdot \Delta^i_{y,n}|^2 \, d[\pi_y]}
$$  \hspace{1cm} (VII.12)

for all $n, m \in \mathbb{N}$ and all $x \in \pi^{-1}(W)$.

Note that $\phi^m_n := -\frac{1}{n} \log |s^i_n \cdot s^{i-1}_m|^2 = \phi - \frac{1}{n} \log |s^i_m - 1|^2$ defines for a fixed $m$ and for all $n \geq m$ a strictly plurisubharmonic function on $\pi^{-1}(W)$ which converges uniformly on compact subsets to $\phi$. Therefore, for each $m$, there exists $N_m \in \mathbb{N}$ so that $\phi^m_n(x) \leq \frac{5}{8} \sigma_\epsilon$ for all $x \in T(\mathcal{X}, W)$ and for all $n \geq N_m$ or equivalently

$$
|s^i_n \cdot s^{i-1}_m|^2(x) \geq e^{-\frac{5}{8} \pi \sigma_\epsilon}.
$$  \hspace{1cm} (VII.13)

for all $x \in T(\mathcal{X}, W)$ and for all $n \geq N_m$. Using inequality (VII.13) and (VII.12) we deduce

$$
\phi \leq e^{-\frac{5}{8} \pi \sigma_\epsilon} \cdot \frac{|s^i_n \cdot s^{i-1}_m|^2 \cdot |s^i_m \cdot \Delta^i_{y,n}|^2}{\int_{\pi^{-1}(y) \cap T(\mathcal{X}, W)} |s^i_m \cdot \Delta^i_{y,n}|^2 \, d[\pi_y]}
$$  \hspace{1cm} (VII.14)

for all $n \geq N_m$ an all $x \in \pi^{-1}(W)$.

Note that by Corollary (III.1) we know that $\phi^m_n(x) \geq \frac{7}{8} \sigma_\epsilon$ or equivalently

$$
|s^i_n \cdot s^{i-1}_m|^2(x) \leq e^{-\frac{7}{8} \pi \sigma_\epsilon}.
$$  \hspace{1cm} (VII.15)

for all $x \in T^c(\sigma_\epsilon, W)$ and all $n \geq N_m$. So if we combine (VII.15) and (VII.14) we deduce

$$
\phi \leq e^{-\frac{7}{8} \pi \sigma_\epsilon} \cdot \frac{|s^i_m \cdot \Delta^i_{y,n}|^2}{\int_{\pi^{-1}(y) \cap T(\mathcal{X}, W)} |s^i_m \cdot \Delta^i_{y,n}|^2 \, d[\pi_y]}
$$  \hspace{1cm} (VII.16)

for all $x \in T^c(\sigma_\epsilon, W)$ and all $n \geq \max \{N_m, N'_m\}$. After having shrunken $W$ there exists (cf. Proposition VII.3) $C > 0$ and $m_0 \in \mathbb{N}$ so that

$$
\max_{x \in \pi^{-1}(y)} \frac{|s^i_{m_0} \cdot \Delta^i_{y,n}|^2}{\int_{\pi^{-1}(y) \cap T(\mathcal{X}, W)} |s^i_{m_0} \cdot \Delta^i_{y,n}|^2 \, d[\pi_y]} < C
$$

for all $n$ big enough and all $y \in W$. In particular combining this with inequality (VII.16) we deduce

$$
\phi_n \leq e^{-\frac{7}{8} \pi \sigma_\epsilon} \cdot C
$$  \hspace{1cm} (VII.17)

for all $n$ big enough (i.e. at least $n \geq \max \{N_{m_0}, N'_{m_0}\}$) and all

$$
x \in T^c(\sigma_\epsilon, W) \cap \pi^{-1}(W) = T^c(\sigma_\epsilon, W).
$$
In other words we have
\[ \phi_n \leq t \]  \hspace{1cm} (VII.18)
on \mathcal{T}^c(\sigma_\varepsilon, \mathbf{W}) \) for all \( n \) big enough which proves the claim because of equation VII.10.

Q. E. D.

As a direct consequence of the above theorem we deduce **Theorem 6.a.**

**Theorem 6.a. [Locally Uniform Convergence of the Initial Measure Sequence]**

Let \( y_0 \in \mathbf{Y}_0 \cap \mathbf{R}_{N_0} \), \( t \in \mathbb{R} \) and let \( \mathbf{W} \subset \mathbf{Y}_0 \cap \mathbf{R}_{N_0} \) be a compact neighborhood of \( y_0 \). Moreover, let \( f \in \mathcal{C}^0(\mathbf{X}) \). Then after having shrunken \( \mathbf{W} \), the sequence

\[
\left( y \mapsto \int \pi^{-1}(y) f \, d\nu_n(y) \right)_n
\]

converges uniformly on \( \mathbf{W} \) to the reduced function \( f_{\text{red}} \).

**Proof.** By the same argumentation as in the proof of **Proposition VI.1**, it is enough to prove the claim for all \( f \in \mathcal{C}^0(\mathbf{X}) \) which are \( T \)-invariant. Hence, in the sequel let \( f \) be continuous, \( T \)-invariant function on \( \mathbf{X} \) and let \( \varepsilon > 0 \). Then by **Lemma VI.1** there exists \( \sigma_\varepsilon > 0 \) so that

\[
|f\left(\pi^{-1}\left(y\right) \cap T(\sigma_\varepsilon, \mathbf{W})\right) - f_{\text{red}}(y)| \leq \frac{\varepsilon}{2} \]  \hspace{1cm} (VII.19)
for all \( y \in \mathbf{W} \). Furthermore, note that we have

\[
\int_{\pi^{-1}(y) \cap T(\sigma_\varepsilon, \mathbf{W})} (f - \pi^*f_{\text{red}}) \, d\nu_n(y)
\]

\[
= \int_{\pi^{-1}(y) \cap T(\sigma_\varepsilon, \mathbf{W})} (f - \pi^*f_{\text{red}}) \phi_n \, d[\pi y]
\]

\[
+ \int_{\pi^{-1}(y) \cap T^c(\sigma_\varepsilon, \mathbf{W})} (f - \pi^*f_{\text{red}}) \phi_n \, d[\pi y].
\]

Since \( f \) is bounded as continuous function on a compact space, we find \( \Gamma := \max \{ f - \pi^*f_{\text{red}} \} < \infty \). Moreover, we can assume that \( f - \pi^*f_{\text{red}} \neq 0 \) and therefore \( \Gamma^{-1} < \infty \) is defined. Furthermore, using **Corollary V.1** there exists a constant \( C > 0 \) so that

\[
\int_{\pi^{-1}(y) \cap T^c(\sigma_\varepsilon, \mathbf{W})} d[\pi y] \leq C
\]
for all \( y \in \mathbf{W} \subset \mathbf{Y}_0 \cap \mathbf{R}_{N_0} \). According to the proof of **Theorem 5.a**, we know by VII.17 that, after having replaced \( \mathbf{W} \) by a smaller compact neighborhood,

\[
\phi_n \leq C^{-1} \cdot \Gamma^{-1} \cdot \frac{\varepsilon}{4} \text{ on } \mathcal{T}^c(\sigma_\varepsilon, \mathbf{W}) \text{ for all } n \text{ big enough.}
\]
VII. UNIFORM CONVERGENCE IN THE NON-TAME CASE

Hence, it follows that

\[ \left| \int_{\pi^{-1}(y)} (f - \pi^* f_{\text{red}}) \, d\nu_n (y) \right| \leq \int_{\pi^{-1}(y)} |f - \pi^* f_{\text{red}}| \, d\nu_n (y) \]

\[ \leq \int_{\pi^{-1}(y) \cap T(\sigma, W)} |f - \pi^* f_{\text{red}}| \, \phi_n \, d[\pi y] + \frac{\epsilon}{4} \]

for all \( n \) big enough and all \( y \in W \). Using inequality VII.19, we deduce

\[ \left| \int_{\pi^{-1}(y)} f \, d\nu_n (y) - f_{\text{red}} (y) \right| \leq \frac{\epsilon}{2} \int_{\pi^{-1}(y) \cap T(\sigma, W)} \phi \, d[\pi y] + \frac{\epsilon}{4} \]

for all \( y \in W \) and all \( n \) big enough. Since \( \int_{\pi^{-1}(y) \cap T(\sigma, W)} \phi_n \leq 1 \) for all \( n \) we find

\[ \left| \int_{\pi^{-1}(y)} f \, d\nu_n (y) - f_{\text{red}} (y) \right| \leq \frac{3}{4} \epsilon \]

for all \( y \in W \) and all \( n \) big enough and therefore the theorem is proven. Q. E. D.

VII.4 Global Uniform Convergence on \( \mathbb{R}_{N_0} \cap Y_0 \)

The strategy of the proof of the globally uniform convergence theorems on \( \mathbb{R}_{N_0} \cap Y_0 \) runs along the following lines: If \( y_0 \in \text{cl} (Y_0 \cap \mathbb{R}_{N_0}) \) then we find an open neighborhood \( W \subset Y \) of \( y_0 \) so that we have uniform convergence over \( W \cap Y_0 \cap \mathbb{R}_{N_0} \). Since \( \text{cl} (Y_0 \cap \mathbb{R}_{N_0}) \) is compact, we can cover \( Y_0 \cap \mathbb{R}_{N_0} \) by finitely many of such compact neighborhoods and the claim follows.

We begin this section by proving an extended version Proposition VII.3

**Proposition VII.4.** Let \( W \subset Y \) be a compact neighborhood of \( y_0 \in \text{cl} (Y_0 \cap \mathbb{R}_{N_0}) \) (recall: we have assumed that \( W \subset Y^1 \)) and \( \epsilon > 0 \), then after having shrunken \( W \), there exists a constant \( C > 0 \) and \( m_0 \in \mathbb{N} \) so that

\[ \max_{x \in \pi^{-1}(y)} \frac{|s_{m_0} \cdot \Delta_{y,n}^1|^2}{\int_{\pi^{-1}(y) \cap T(\epsilon, W)} |s_{m_0} \cdot \Delta_{y,n}^1|^2 \, d[\pi y]} < C \]

for all \( n \) big enough and all \( y \in W \cap Y_0 \cap \mathbb{R}_{N_0} \).

**Proof.** Let us assume that this is not the case, then as in the proof of Proposition VII.3

\[ \text{Recall that we have assumed that } \mathbb{R}_{m_0} \neq \emptyset; \text{ in particular } Y_0 \cap \mathbb{R}_{N_0} \text{ is non-empty and euclidean open.} \]

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we can find a sequence \((y_n)_n\) in \(y_n \in Y_0 \cap R_{N_0}\) converging to \(y_0 \in cl (Y_0 \cap R_{N_0})\) so that

\[
\int_{\pi^{-1}(y_n) \cap T(\epsilon, W)} \frac{|s_{i_{mo}} \cdot \Delta^i_{y_n,n}|^2}{\max_{x \in \pi^{-1}(y_n) \cap T(\epsilon, W)} |s_{i_{mo}} \cdot \Delta^i_{y_n,n}|^2} \, d [\pi y_n] \to 0. \tag{VII.20}
\]

Using the same notation as in the proof of Proposition VII.3 we can write

\[
\int_{\pi^{-1}(y_n) \cap T(\epsilon, W)} |f_n|^2 \, d [\pi y_n] \to 0. \tag{VII.21}
\]

Furthermore, as as in the proof of Proposition VII.3 we can assume that

\[
\max_{x \in \pi^{-1}(y_n)} |f_n|^2 = \max_{x \in \pi^{-1}(y_n) \cap T(\epsilon, W)} |f_n|^2 = 1
\]

for all \(n\) big enough where \(y_n \to y_0 \in Int W\). Moreover, as before we find a lift \((x_n)_n\) of the sequence \((y_n)_n\), i.e. a sequence \((x_n)_n\) in \(T(\hat{\epsilon}, W) \cap \pi^{-1}(Y_0 \cap R_{N_0})\) so that \(\pi (x_n) = y_n\) where

\[
|f_n|^2 (x_n) = 1 = \max_{x \in \pi^{-1}(y_n)} |f_n|^2 = \max_{x \in \pi^{-1}(y_n) \cap T(\epsilon, W)} |f_n|^2. \tag{VII.22}
\]

Recall that we have the following commutative diagram:

\[
\begin{array}{ccc}
X^ss \subset X & \xrightarrow{p X | cl \Gamma_\pi \circ \zeta} & \hat{X} \\
\downarrow \pi & & \downarrow \Sigma \\
Y & \xleftarrow{\sigma} & \hat{Y}
\end{array}
\]

Set \(P := p X | cl \Gamma_\pi \circ \zeta \circ \Sigma\) and define \(\tilde{L} := P^* L\). Since \(\Sigma\) is surjective and the above diagram commutes, we can find a lift \((\tilde{x}_n)_n\) in \(\tilde{X}\) of \((x_n)_n\) under \(P\), i.e. we have \(P (\tilde{x}_n) = x_n\). In the sequel, let \((\tilde{y}_n)_n\) be the sequence in \(\tilde{Y}\) given by \(\tilde{y}_n = \Pi (\tilde{x}_n)\) and note that \(\sigma (\tilde{y}_n) = y_n\). Moreover, we have \(\tilde{x}_n \in T (\hat{x}, W) \subset T (\epsilon, W)\) (recall: \(\tilde{W} := \sigma^{-1} (W)\)) for all \(n\). Arguing as before, since \(\tilde{x}_n \in T (\hat{x}, W)\) for all \(n\), we can assume that (after having chosen a subsequence)

\[
\tilde{x}_n \to \tilde{x}_0 \in Int T (\epsilon, W).
\]

Hence, there exists an open neighborhood \(U \subset T (\epsilon, W)\) of \(\tilde{x}_0\) so that \(\tilde{x}_n \in U\) for all \(n\) big enough. In the sequel, let \(\tilde{y}_0 := \Pi (\tilde{x}_0) \in \tilde{Y}\).

We will now define \(\tilde{f}_n\) on \(T (\epsilon, W) \cap \Pi^{-1} (\tilde{y}_n)\) by the pull back of the restriction of \(f_n\) to \(T (\epsilon, W) \cap \pi^{-1} (y_n)\). Note that we have \(|\tilde{f}_n|^2 (\tilde{x}_n) = 1\) for all \(n\) and more precisely

\[
\max_{x \in T (\epsilon, W) \cap \Pi^{-1} (\tilde{y}_n)} |\tilde{f}_n|^2 = |\tilde{f}_n|^2 (\tilde{x}_n) = 1 \tag{VII.23}
\]

which is a direct consequence of [VII.22]
Since \( \text{II: } \widetilde{X} \to \widetilde{Y} \) is a k-fiber, we can proceed as in the proof of Proposition VII.3. After having shrunken \( U \) we can assume that the open neighborhood \( U \) is isomorphic to a closed analytic subset in a relatively compact product neighborhood \( Q = Q_0 \times Q_1 \subset C^k \times C^k \), i.e., there exists an isomorphism \( \Phi: U \to C^k \times C^k \) (let \( \Phi (\tilde{x}_0) = 0 \)) so that for each \( \tilde{y} \) in an open neighborhood \( B \subset \widetilde{Y} \) of \( \Pi (\tilde{x}_0) = \tilde{y}_0 \) each \( \Phi (\Pi^{-1}(\tilde{y}) \cap U) = Z_{\tilde{y}} \) is a closed analytic subset which yields a d-sheeted covering onto \( Q_1 \) given by the restriction \( p|Z_{\tilde{y}_n} \to Q_1 \). We can now finish the proof exactly like the proof of Proposition VII.3. First of all after having shrunken \( W \) we can assume that \( \tilde{f}_n \) is represented by \( \tilde{f}_n = \tilde{\alpha} \cdot \tilde{g}_n \) where \( \tilde{\alpha} \in C^\infty (U) \) and \( \tilde{g}_n \in O(Z_{\tilde{y}_n}) \). Furthermore, we also have

\[
a^{-1} \geq \max_{x \in Z_{\tilde{y}_n}} |\tilde{g}_n|_{Z_{\tilde{y}_n}}^2 \text{ and } |\tilde{g}_n|_{Z_{\tilde{y}_n}}^2 (\tilde{x}_n) \geq A^{-1} > 0 \tag{VII.24}\]

for all \( n \) big enough, where \( A > 0 \), resp. \( a > 0 \) denotes the maximum, resp. the minimum of \( \tilde{\alpha} \) on \( U \). We can now apply Proposition VII.2 to the sequence \( \tilde{g}_n \): Each \( \tilde{g}_n \) is a holomorphic function on \( \tilde{Z}_n := Z_{\tilde{y}_n} \). Furthermore, this sequence is uniformly bounded by \( a^{-1} \) and by assumption VII.21 we have

\[
\int_{\tilde{Z}_n} |\tilde{g}_n|^2 \, d[\tilde{Z}_n] \to 0.
\]

Using Proposition VII.2, it follows that for a compact subset \( K \subset Q_1 \) and for \( \epsilon = \frac{1}{2}A^{-1} \) we have

\[
|\tilde{g}_n|^2 \leq \frac{1}{2}A^{-1} \text{ on } p^{-1}(K) \cap \tilde{Z}_n \text{ for all } n \geq N_{\frac{1}{2}A^{-1}}(K). \tag{VII.25}
\]

However, as \( \tilde{x}_n \to x_0 = 0 \in p^{-1}(K) \) we know that \( \tilde{x}_n \in p^{-1}(K) \cap \tilde{Z}_n \) for all \( n \) big enough. By the second inequality of VII.24 we know that \( |\tilde{g}_n|^2 (\tilde{x}_n) \geq A^{-1} \) which yields a contradiction to VII.25. Consequently, the assumption VII.21 is false and the claim of the proposition holds.

Q. E. D.

As a consequence of Proposition VII.4 we deduce a generalization of Theorem 5.a.

**Theorem 5.b.** [Uniform Convergence of the Initial Distribution Sequence]

*For fixed \( t \in \mathbb{R} \) the sequence \((D_n (\cdot, t))_n\) converges uniformly on \( Y_0 \cap R_{N_0} \) to the zero function.*

**Proof.** Note that by the compactness of \( cl (Y_0 \cap R_{N_0}) \) the claim follows as soon as we have shown that for each \( y_0 \in cl (Y_0 \cap R_{N_0}) \), there exists a compact neighborhood \( W \subset Y \) of \( y_0 \) (after having shrunken \( W \) we can assume that \( W \subset Y') \) so that \( D_n (\cdot, t) \) converges uniformly on \( W \cap Y_0 \cap R_{N_0} \) to the zero function. The proof of the latter claim is similar to the proof of Theorem 5.a: First of all fix \( y_0 \in cl (Y_0 \cap R_{N_0}) \) and \( \epsilon > 0 \). As in the proof of Theorem 5.a, we apply Proposition V.1 in order to find \( \sigma_\epsilon > 0 \) so that

\[
\text{vol } (\pi^{-1}(y) \cap T (\sigma_\epsilon, W)) \leq \epsilon \tag{VII.26}
\]
for all \( y \in W \cap Y_0 \). Hence, it is enough to show that after having shrunken \( W \) we have

\[
\phi_n = \frac{|s_n|^2}{||s_n||^2} \leq t
\]  

(VII.27)

on \( T^c(\sigma, W) \cap \pi^{-1}(Y_0 \cap R_{N_0}) \) for all \( n \) big enough. Note that we can now perform the same steps as in the proof Theorem 5.a over the set \( W \cap Y_0 \cap R_{N_0} \).

First of all, we write

\[
\phi_n = \frac{|s_n^1 \cdot s_m^{-1}|^2 \cdot |s_m^1 \cdot \Delta_{y,n}^1|^2}{\int_{\pi^{-1}(y) \cap T(\frac{\sigma}{2}, W)} |s_n^1 \cdot s_m^{-1}|^2 \cdot |s_m^1 \cdot \Delta_{y,n}^1|^2 \, d[\pi_y]}
\]

where \( m \in \mathbb{N} \) is arbitrary and by Lemma [VII.2] we know that

\[
\max_{x \in \pi^{-1}(y) \cap T(\frac{\sigma}{2}, W)} |s_m^1 \cdot \Delta_{y,n}^1|^2 > 0
\]  

(VII.28)

for all \( n \) big enough and all \( y \in W \cap Y_0 \). Therefore we deduce the estimate

\[
\phi_n \leq \frac{|s_n^1 \cdot s_m^{-1}|^2 \cdot |s_m^1 \cdot \Delta_{y,n}^1|^2}{\int_{\pi^{-1}(y) \cap T(\frac{\sigma}{2}, W)} |s_n^1 \cdot s_m^{-1}|^2 \cdot |s_m^1 \cdot \Delta_{y,n}^1|^2 \, d[\pi_y]}
\]  

(VII.29)

for all \( n, m \in \mathbb{N} \) and all \( x \in \pi^{-1}(W \cap R_{N_0} \cap Y_0) \). Note that the right hand side is well defined because the denominator of the term on the right hand side of the above inequality is not zero according to equation VII.28.

As before we know that \( g_n^m := -\frac{1}{n} \log |s_n^1 \cdot s_m^{-1}|^2 = g_n^i - \frac{1}{n} \log |s_m^{-1}|^2 \) is a strictly plurisubharmonic function on \( \pi^{-1}(W) \) for \( m \) fixed and for all \( n \geq m \) which converges uniformly on compact subsets to \( \phi^i \). In particular we deduce that \( g_n^m(x) \leq \frac{5}{8} \sigma \) for all \( x \in T(\frac{\sigma}{2}, W) \) and all \( n \geq N_m \in \mathbb{N} \) big enough where \( m \) is fixed. Combined this with inequality VII.29 it follows that

\[
\phi_n \leq e^{\frac{5}{8} \sigma} \frac{|s_n^1 \cdot s_m^{-1}|^2 \cdot |s_m^1 \cdot \Delta_{y,n}^1|^2}{\int_{\pi^{-1}(y) \cap T(\frac{\sigma}{2}, W)} |s_n^1 \cdot s_m^{-1}|^2 \cdot |s_m^1 \cdot \Delta_{y,n}^1|^2 \, d[\pi_y]}
\]  

(VII.30)

on \( \pi^{-1}(R_{N_0} \cap Y_0 \cap W) \) for all \( n \geq N_m \).

On the other hand, using Corollary [III.1] we have \( g_n^m(x) \geq \frac{7}{8} \sigma \) or equivalently

\[
|s_n^1 \cdot s_m^{-1}|^2(x) \geq e^{-\frac{7}{8} n \sigma}
\]  

(VII.31)

for all \( x \in T^c(\sigma, W) \) and all \( n \geq N'_m \). Combining VII.31 and VII.30 we deduce

\[
\phi_n \leq e^{-n \frac{7}{8} \sigma} \frac{|s_m^1 \cdot \Delta_{y,n}^1|^2}{\int_{\pi^{-1}(y) \cap T(\frac{\sigma}{2}, W)} |s_m^1 \cdot \Delta_{y,n}^1|^2 \, d[\pi_y]}
\]  

(VII.32)

for all \( x \in T^c(\sigma, W) \cap \pi^{-1}(Y_0 \cap R_{N_0}) \) and all \( n \geq \max \{N_m, N'_m\} \). Since \( y_0 \in \text{cl}(Y_0 \cap \text{cl}(W) \cap \pi^{-1}(Y_0 \cap R_{N_0}) \cap \pi^{-1}(Y_0 \cap R_{N_0}) \)
Theorem 6.b. [Uniform Convergence of the Initial Measure Sequence]

Let \( f \in C^0(X) \) then sequence
\[
\left( y \mapsto \int_{\pi^{-1}(y)} f \, d\nu_n(y) \right)_n
\]
converges uniformly over \( Y_0 \cap R_{N_0} \) to the reduced function \( f_{\text{red}} \).

Proof. As before (cf. proof of Theorem 6.a) it is enough to consider the case where \( f \in C^0(X) \) is \( T \)-invariant.

Let \( \epsilon > 0 \). Again, by the compactness of \( \text{cl} (Y_0 \cap R_{N_0}) \), the proof of the claim reduces to the following statement: If \( y_0 \in \text{cl} (Y_0 \cap R_{N_0}) \), then there exists a compact neighborhood \( W \) so that the claim is true over the set \( W \cap Y_0 \cap R_{N_0} \). In order to show this, we proceed similar as in the proof of Theorem 6.a: First of all, by Lemma VI.1 there exists \( \sigma \epsilon > 0 \) so that
\[
|f| \left( \pi^{-1}(y) \cap T(\sigma \epsilon, W) \right) - f_{\text{red}}(y) \leq \frac{\epsilon}{2}
\]
for all \( y \in W \). Furthermore, as before we have the decomposition

\[
\int_{\pi^{-1}(y)} (f - \pi^* f_{\text{red}}) \, d\nu_n (y)
= \int_{\pi^{-1}(y) \cap T(\sigma_\epsilon, W)} (f - \pi^* f_{\text{red}}) \, \phi_n \, d[\pi y]
+ \int_{\pi^{-1}(y) \cap T^c(\sigma_\epsilon, W)} (f - \pi^* f_{\text{red}}) \, \phi_n \, d[\pi y].
\]

Again we have \( \Gamma := \max \{ f - \pi^* f_{\text{red}} \} < \infty \) because \( f \) is continuous and without restriction of generality we can assume that \( f - \pi^* f_{\text{red}} \not\equiv 0 \) so that \( \Gamma^{-1} < \infty \) exists. Moreover, as in the proof of Theorem 5.b, using Corollary V.1, there exists \( C > 0 \) so that

\[
\int_{\pi^{-1}(y) \cap T^c(\sigma_\epsilon, W)} d[\pi y] \leq C
\]
for all \( y \in W \cap Y_0 \). By inequality VII.33 in the proof of Theorem 5.b we know, after having shrunken \( W \), that \( \phi_n \leq C^{-1} \cdot \Gamma^{-1} \cdot \frac{1}{4} \) on \( T^c(\sigma_\epsilon, W) \cap \pi^{-1}(Y_0 \cap R_{N_0}) \) for all \( n \in \mathbb{N} \) big enough and hence

\[
\left| \int_{\pi^{-1}(y)} (f - \pi^* f_{\text{red}}) \, d\nu_n (y) \right| \leq \int_{\pi^{-1}(y)} |f - \pi^* f_{\text{red}}| \, d\nu_n (y)
\leq \int_{\pi^{-1}(y) \cap T(\sigma_\epsilon, W)} |f - \pi^* f_{\text{red}}| \, \phi_n \, d[\pi y] + \frac{\epsilon}{4}
\]
for all \( n \) big enough and all \( y \in W \cap Y_0 \cap R_{N_0} \). By VII.35 we have

\[
\left| \int_{\pi^{-1}(y)} f \, d\nu_n (y) - f_{\text{red}} (y) \right| \leq \frac{\epsilon}{2} \int_{\pi^{-1}(y) \cap T(\sigma_\epsilon, W)} \phi \, d[\pi y] + \frac{\epsilon}{4}
\]
for all \( y \in W \cap Y_0 \cap R_{N_0} \) and all \( n \) big enough. As \( \int_{\pi^{-1}(y) \cap T(\sigma_\epsilon, W)} \phi_n \leq 1 \) for all \( n \) we deduce

\[
\left| \int_{\pi^{-1}(y)} f \, d\nu_n (y) - f_{\text{red}} (y) \right| \leq \frac{3}{4} \epsilon
\]
for all \( y \in W \cap Y_0 \cap R_{N_0} \) and all \( n \) big enough as claimed. So with \( W \subset Y \) we have found a compact neighborhood of \( y_0 \in \text{cl} (Y_0 \cap R_{N_0}) \) so that the claim is true over \( W \cap Y_0 \cap R_{N_0} \) and hence the claim follows by the introducing statement of this proof. Q. E. D.

We close this section by giving an alternative formulation of our results in the language of operators. For this let \( f \in C^0(X) \) and consider the continuous, \( T \)-invariant function \( \overline{T} \) on
Example IV.2

Even if it follows that \( \Phi_m \to \infty \) as \( m \to \infty \) and consequently \( \Phi \) is not possible to extend \( \Phi : C^0(X) \to C^0(Y) \) to an operator \( \Phi : C^0(X) \to C^0(Y) \): Using Example IV.2 one can find a continuous function \( f \in C^0(X) \) so that \( \Phi_n(f) \) has no

Let \( f \in C^0(X) \), \( K := \max \{|f|\} = \max \{|\Phi|\} \) and \( y \in R_{N_0} \cap Y_0 \). Choose a compact neighborhood \( W \subset R_{N_0} \cap Y_0 \) of \( y \) and let \( \epsilon_m \in \mathbb{R}^{\geq 0} \) be a strictly increasing sequence converging to \( \infty \). Furthermore, choose a sequence \( \{\psi_m\}_m \) of smooth, \( T \)-invariant cut-off functions defined on \( X \) so that

\[
\psi_m|T(\epsilon_m, W) \equiv 1, \quad \text{supp} \psi_m \cap T^c(\epsilon_{m+1}, W) = \emptyset \quad \text{and supp} \psi_m \subset \pi^{-1}(Y_0 \cap R_{N_0}).
\]

Since \( \pi|X_0 : X_0 \to Y_0 \) is a k-fibering, it follows by Theorem 5.a that \( \Phi_n(\psi_m f) \in C^0(W) \) for all \( m \in \mathbb{N} \). We calculate

\[
|\Phi_n(\psi_m f) - \Phi_n(f)(y)| = \left| \int_{\pi^{-1}(y)} (\psi_m \Phi - \Phi) \, d\nu_n(y) \right|
\leq 2K \int_{\pi^{-1}(y) \cap T^c(\epsilon_m, W)} |\psi_m - 1| \, \phi_n d[\pi y]
\leq 2K \int_{\pi^{-1}(y) \cap T^c(\epsilon_m, W)} \phi_n d[\pi y].
\]

Using equation VII.17 (for \( \sigma = \sigma_m \)) in the proof of Theorem 5.a, we deduce that

\[
|\Phi_n(\psi_m f) - \Phi_n(f)(y)| \leq 2 \cdot e^{-n^T \epsilon_m} \cdot C \cdot K
\]

for all \( n \) big enough and all \( y \in W \). Hence, for a fixed \( n \in \mathbb{N} \) big enough, the sequence of continuous functions on \( W \) given by \( \{\Phi_n(\psi_m f) W\}_m \) converges uniformly to \( \Phi_n(f) W \) as \( m \to \infty \) and consequently \( \Phi_n(f) W \in C^0(W) \). Since \( y \in R_{N_0} \cap Y_0 \) was chosen arbitrarily, it follows that \( \Phi_n(f) \in C^0(R_{N_0} \cap Y_0) \) for all \( f \in C^0(X) \) and all \( n \in \mathbb{N} \) big enough as claimed.

Q. E. D.

Remark VII.2. Even if \( \tilde{C}(R_{N_0} \cap Y_0) \) is a proper analytic subset of \( Y \), it is in general not possible to extend \( \Phi_n : C^0(X) \to C^0(R_{N_0} \cap Y) \) to an operator \( \tilde{\Phi}_n : C^0(X) \to C^0(Y) \): Using Example IV.2 one can find a continuous function \( f \in C^0(X) \) so that \( \Phi_n(f) \) has no
continuous extension in $[0: \ldots :0:1] \in Y = \mathbb{CP}^k$.

If $\mathcal{R}: C^0(X) \to C^0_{bd}(R_{N_u} \cap Y_0)$ denotes the operator given by $\mathcal{R}(f) := T_{\text{red}}f|_{R_{N_u} \cap Y_0}$, then we deduce the following corollary.

**Corollary VII.1.** The operator sequence $(\Phi_n)_n$ converges to $\mathcal{R}$ with respect to the topology induced by the supremum norm on $C^0(X)$ and $C^0_{bd}(R_{N_u} \cap Y)$.

**Proof.** Apply Theorem 6.b. Q. E. D.
Index of Notation

\( C^k(\widehat{X}) \): The complex space of all k-dimensional cycles \( C \) in \( \widehat{X} \), S. 26

\( C \): k-dimensional cycle \( \sum_n n_i C_i \), \( n_i \in \mathbb{N} \), \( \dim C_i = k \) in \( \widehat{X} \), S. 26

\( C_\gamma \): k-dimensional cycle associated to the fiber \( \widehat{\pi}^{-1}(\gamma) \) for \( \gamma \in \widehat{Y}_0 \), S. 26

|\( C \)|: Support |\( C \)\( = \bigcup_{i \in \mathbb{N}} C_i \) of the cycle |\( C = \sum_n n_i C_i \), S. 26

|\( \text{cl}(\mathbb{T}x) \)|: Zariski closure of the \( T \)-orbit \( \mathbb{T}x \) where \( x \in X \), S. 7

\( \text{Conv} A \): Convex hull of a subset \( A \subset t^* \), S. 7

\( \text{D}_n(\cdot, t) \): Sequence of cumulative distribution densities associated to the tame sequence \( (s_n)_{n=1}^\infty \), S. 20

\( \text{D}_n^\mu(\cdot, t) \): Sequence of distribution functions associated to the open cover \( \Omega \), S. 20

\( \Delta^{i}_{\tau, n} (= \Delta^{i}_{\tau, n}) \): Holomorphic \( \xi_n - \xi_n \)-eigenfunction on \( \pi^{-1}(\mathbb{U}) \) where \( y \in \mathbb{R}^n \) defined by \( s_n \cdot \Delta^{i}_{\tau, n} = \delta_{\tau, n}, \) S. 42

\( \text{Fix}^T \): Set of all \( T \)-fixed points in \( X \), S. 7

\( \tilde{f} \): Averaged function defined by \( \tilde{f}(x) = \int_T f(t.x) \, d\nu_T \), S. 3

\( f_{\text{red}} \): Function on the quotient \( Y = X_{\xi}^{ss} \square \mathbb{T} \) induced by the restriction \( \tilde{f}\vert_{\pi^{-1}(\mathbb{U})} \) of the averaged function \( \tilde{f} \), S. 40

\( \widehat{s}_C,T \circ \text{cp} \): Chow form associated to a cycle \( C \) in \( \mathbb{C}^{\text{cp}} \), S. 28

\( \Gamma_{\pi} \): Graph of the quotient map \( \pi : X_{\xi}^{ss} \to X_{\xi}^{ss} \square \mathbb{T} = Y \) in \( X \times Y \), S. 18

\( h \): Hermitian, positive, \( T \)-invariant bundle metric on \( L \), S. 1

\( f_{\text{F}^{-1}(\gamma)} f d[F_y] \): Fiber integral of \( f \) with respect to a \( k \)-fibering \( F : X \to Y \), S. 36

\( M_j \): Subset of all points \( x \in \mu^{-1}(\gamma) \) so that \( s_j(x) \neq 0 \) for all \( j \in J \) and \( s_j(x) \neq 0 \) for all \( j \in J^c \), S. 7

\( \mu \): Moment map associated to the hermitian, positive, \( T \)-invariant bundle metric \( h \), S. 1

\( \nu_{\pi} \): Sequence of fiber measures associated to the tame sequence \( (s_n)_{n=1}^\infty \), S. 20

\( \nu_{\pi}^\mu \): Sequence of fiber measures associated to the open cover \( \Omega \), S. 20

\( \nu_{\pi}(x) \): Order of a \( k \)-fibering \( F : X \to Y \) at a point \( x \in X \), S. 36

\( \rho \): Naturally associated Kähler form given by \( \rho = -\frac{1}{2} \partial \bar{\partial} \log | \cdot |^2 \), S. 11

\( \rho' \): Smooth (2,2)-form on \( \widehat{X} \) given by \( \rho' = \langle p_X \circ \text{cl}(\Gamma_{\pi}) \circ \xi \rangle^* \rho \), S. 18

\( \Omega \): Smooth (2,2)-form on the compact variety \( \widehat{X} \) given by \( \Omega := \Sigma^* \omega' \), S. 31

\( \pi \): Algebraic projection map \( \pi : X_{\xi}^{ss} \to X_{\xi}^{ss} \square \mathbb{T} \) of the Hilbert Quotient, S. 4

\( \pi \): Algebraic projection map from the compact variety \( \widehat{X} \) to \( Y \) given by \( \pi = p_Y \circ \text{cl}(\Gamma_{\pi}) \circ \xi \), S. 18

\( \Pi \): Surjective, holomorphic k-fibering from \( \widehat{X} \) to \( \widehat{Y} \), S. 26

\( \varphi^\pi \): Holomorphic map from \( \widehat{Y}_0 \) into the cycle space \( C^k(\widehat{X}) \) induced by the k-fibering \( \pi|_{\pi^{-1}(Y_0)} : Y_0 \to Y_0 \), S. 26
| **Symbol** | **Mathematical Description** |
|-----------|-----------------------------|
| $\Phi_n$  | Continuous operator from $C^0(X)$ to $C^0_{bd}(R_{N_0} \cap Y_0)$ induced by $\nu_n$, S. 62 |
| Relint $A$ | Relative interior of a convex subset $A \subset t^*$, S. 7 |
| $R_{N_0}$ | The set of all removable singularities of order $N_0 \in \mathbb{N}$ of the fiber measure sequence $(\nu_n)_n$, S. 21 |
| $\sigma^i$ | S.p.s.h limit function associated to the tame sequence $(s_n^i)_n$, S. 13 |
| $\sigma^i_n$ | Sequence of s.p.s.h functions associated to the tame sequence $(s_n^i)_n$, S. 13 |
| $s^i_n$ | $i$-th tame sequence associated to the ray $\mathbb{R}^\infty$ |
| $s_{\tilde{t}^n}$ | Extension of $s_n$ given by $s_{\tilde{t}^n} = s_n \cdot \pi^* f_{\tilde{t}^n}$ for all $n \geq N_0 \in \mathbb{N}$ where $f_{y,n} \in \mathcal{O}(U_y)$ and $y \in R_{N_0}$, S. 21 |
| $\sigma$ | Proper, surjective holomorphic map from $\tilde{Y}$ to $Y$ given by $p_Y|\tilde{Y}$, S. 26 |
| $\Sigma$ | Proper, surjective holomorphic map from $\tilde{X}$ to $\tilde{X}$, S. 26 |
| $|s|^2$ | Norm function of a section $s \in H^0(X,L)$ with respect to the bundle metric $h$, S. 11 |
| $||s_n^i||^2$ | Fiber integral of $|s_n^i|^2$ over the subset $Y_0 \subset Y$ with respect to $\pi|\pi^{-1}(Y_0)$, S. 19 |
| $S_x$ | Set of all $T$-fixed points contained in the Zariski closure $T.x$ where $x \in X$, S. 7 |
| $S_J$ | Set of all $T$-eigensections indexed by $J \subset \{1, \ldots, m\}$ where $m = \dim_{\mathbb{C}} H^0(X,L)$, S. 7 |
| $\mathcal{G}_J$ | Set of all characters $\xi_j \in t_x^*$ attached to the collection of all $T$-eigensections given by $S_J$, S. 7 |
| $T(\epsilon, W^i)$ | Compact $\epsilon$-neighborhood tube around $\mu^{-1}(\xi)$ over $W^i \subset Y$, S. 14 |
| $T^c(\epsilon, W^i)$ | Complement of $T(\epsilon, W^i)$ in $\pi^{-1}(W^i)$, S. 14 |
| $\tilde{T}(\epsilon, W^i)$ | Compact $\epsilon$-neighborhood tube in $\tilde{X}$ induced by $T(\epsilon, W^i)$, S. 30 |
| $V^i$ | Inverse image of the compact neighborhood $W^i \subset Y$ under the projection $\pi$, S. 14 |
| $\text{vol}(\pi^{-1}(y))$ | Volume of the fiber $\pi^{-1}(y)$ for $y \in Y_0$ with respect to the form $\omega^k|\pi^{-1}(y)$ counted with multiplicities, S. 30 |
| $W^i$ | Compact neighborhood contained in $Y^i$, S. 14 |
| $\tilde{W}^i$ | Compact neighborhood in $\tilde{Y}$ induced by $W^i$ via $W^i = \sigma^{-1}(W^i)$, S. 30 |
| $\tilde{X}$ | Compact variety defined as the normalization of the compact variety subvariety $c_{\Gamma_x} \subset X \times Y$, S. 18 |
| $\tilde{X}$ | Compact variety given by $\tilde{X} = (Y \times X) \cap (\tilde{Y} \times \tilde{X})$, S. 26 |
| $X_0$ | Inverse image of $Y_0$ with respect to the projection map $\pi : X^{ss} \rightarrow Y$, S. 4 |
| $X^i$ | Open, $\pi$-saturated subset of $X^{ss} \cap X(s_n^i)$ given by $X^i := \pi^{-1}(\pi(M_i))$, S. 6 |
| $X(s_n^i)$ | $n$-stable, Zariski open subset of $X$ given by $\{x \in X : s_n^i(x) \neq 0\}$, S. 22 |
| $X^{ss}_\xi$ | Set of semistable points with respect to the level subset $\mu^{-1}(\xi)$, S. 11 |
VII UNIFORM CONVERGENCE IN THE NON-TAME CASE

\(X^{ss}_{\xi} / T\) Hilbert Quotient with respect to the level subset \(\mu^{-1}(\xi)\), S. 22

\(\hat{X}_\xi\) Fundamental vector field on \(L\) generated by the flow induced by \(\xi \in \mathfrak{k}\), S. 4

\(\mathfrak{X}\) Universal space defined by \(\mathfrak{X} := \{(\mathfrak{c}, x) \in \mathcal{C}^k(\hat{X}) \times \hat{X} : x \in |\mathfrak{c}|\}\), S. 26

\(\xi\) Normalization map \(\xi : \tilde{Y}^{\text{nor}} \to \tilde{Y}\) of \(\tilde{Y}\), S. 35

\(\xi^1\) Sequence of weights associated to a tame sequence \((s^1_n)_n\), S. 6

\((y, \mathfrak{c}_y)\) Point in \(\sigma^{-1}(Y_0) \subset \tilde{Y} \subset Y \times \mathcal{C}^k(\hat{X})\) where \(\varphi^\pi(y) = \mathfrak{c}_y\) for \(y \in Y_0\), S. 27

\(Y\) Abbreviation for the Hilbert Quotient \(X^{ss}_{\xi} / T\), S. 22

\(Y^i\) Image of \(X^i\) under the projection map \(\pi : X^{ss}_{\xi} \to Y\), S. 13

\(Y_0\) Subset of all points in \(Y\) so that \(\hat{\pi}^{-1}(y)\) is a \(k\)-dimensional subvariety of \(\hat{X}\), S. 18

\(\tilde{Y}\) Complex subspace of \(Y \times \mathcal{C}^k(\hat{X})\) given by the closure of the graph of \(\varphi^\pi : Y_0 \to \mathcal{C}^k(\hat{X})\), S. 26

\(\zeta\) Normalization map \(\zeta : \hat{X} = (\text{cl} \Gamma_\pi)^{\text{nor}} \to \text{cl} \Gamma_\pi\) of the compact variety \(\text{cl} \Gamma_\pi \subset X \times Y\), S. 18
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