A REMARK ON 'SOME NUMERICAL RESULTS IN COMPLEX DIFFERENTIAL GEOMETRY'

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Abstract. In this note we verify certain statement about the operator $Q_K$ constructed by Donaldson in [3] by using the full asymptotic expansion of Bergman kernel obtained in [2] and [4].

In order to find explicit numerical approximation of Kähler-Einstein metric of projective manifolds, Donaldson introduced in [3] various operators with good properties to approximate classical operators. See the discussions in Section 4.2 of [3] for more details related to our discussion. In this note we verify certain statement of Donaldson about the operator $Q_K$ in Section 4.2 by using the full asymptotic expansion of Bergman kernel derived in [2, Theorem 4.18] and [4, §3.4]. Such statement is needed for the convergence of the approximation procedure.

Let $(X, \omega, J)$ be a compact Kähler manifold of dim $\mathbb{C} = n$, and let $(L, h^L)$ be a holomorphic Hermitian line bundle on $X$. Let $\nabla^L$ be the holomorphic Hermitian connection on $(L, h^L)$ with curvature $R^L$. We assume that

\( \frac{\sqrt{-1}}{2\pi} R^L = \omega. \)

Let $g^{TX}(\cdot, \cdot) := \omega(\cdot, J\cdot)$ be the Riemannian metric on $TX$ induced by $\omega, J$. Let $dv_X$ be the Riemannian volume form of $(TX, g^{TX})$, then $dv_X = \omega^n/n!$. Let $d\nu$ be any volume form on $X$. Let $\eta$ be the positive function on $X$ defined by

\( dv_X = \eta d\nu. \)

The $L^2$–scalar product $\langle \quad \rangle_\nu$ on $\mathcal{C}^\infty(X, L^p)$, the space of smooth sections of $L^p$, is given by

\( \langle \sigma_1, \sigma_2 \rangle_\nu := \int_X \langle \sigma_1(x), \sigma_2(x) \rangle_{L^p} d\nu(x). \)

Let $P_{\nu,p}(x, x') (x, x' \in X)$ be the smooth kernel of the orthogonal projection from $\langle \mathcal{C}^\infty(X, L^p), \langle \quad \rangle_\nu \rangle$ onto $H^0(X, L^p)$, the space of the holomorphic sections of $L^p$ on $X$, with respect to $d\nu(x')$. Note that $P_{\nu,p}(x, x') \in L^p_x \otimes L^{p*}_{x'}$. Following [3, §4], set

\( K_p(x, x') := |P_{\nu,p}(x, x')|^2_{h^p_x \otimes \overline{h}^p_{x'}}, \quad R_p := (\dim H^0(X, L^p))/\text{Vol}(X, \nu), \)

here $\text{Vol}(X, \nu) := \int_X d\nu$. Set $\text{Vol}(X, dv_X) := \int_X dv_X$.

Let $Q_{K_p}$ be the integral operator associated to $K_p$ which is defined for $f \in \mathcal{C}^\infty(X)$,

\( Q_{K_p}(f)(x) := \frac{1}{R_p} \int_X K_p(x, y)f(y)d\nu(y). \)
Let $\Delta$ be the (positive) Laplace operator on $(X, g^{TX})$ acting on the functions on $X$. We denote by $|f|_{L^2}$ the $L^2$-norm on the function on $X$ with respect to $dv_X$.

**Theorem 1.** There exists a constant $C > 0$ such that for any $f \in \mathcal{C}^\infty(X)$, $p \in \mathbb{N}$,

\[
\begin{aligned}
(6) & \quad \left| \left( Q_{K_p} - \frac{\Vol(X, \nu)}{\Vol(X, dv_X)} \eta \exp \left( - \frac{\Delta}{4\pi p} \right) \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}, \\
(7) & \quad \left| \left( \frac{\Delta}{p} Q_{K_p} - \frac{\Vol(X, \nu)}{\Vol(X, dv_X)} \Delta \eta \exp \left( - \frac{\Delta}{4\pi p} \right) \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}.
\end{aligned}
\]

Moreover, \(6\) is uniform in that there is an integer $s$ such that if all data $h^L$, $dv$ run over a set which are bounded in $\mathcal{C}^s$ and that $g^{TX}$, $dv_X$ are bounded from below, then the constant $C$ is independent of $h^L$, $dv$.

**Proof.** We explain at first the full asymptotic expansion of $P_{\nu,p}(x, x')$ from [2] Theorem 4.18] and [4 § 3.4]. For more details on our approach we also refer the readers to the recent book [5].

Let $E = \mathbb{C}$ be the trivial holomorphic line bundle on $X$. Let $h^E$ the metric on $E$ defined by $|1|_{h^E}^2 = 1$, here 1 is the canonical unity element of $E$. We identify canonically $L^p$ to $L^p \otimes E$ by Section 1.

As in [4 § 3.4], let $h_{\omega}^E$ be the metric on $E$ defined by $|1|_{h_{\omega}^E}^2 = \eta^{-1}$, here 1 is the canonical unity element of $E$. Let $\langle \cdot, \cdot \rangle_\omega$ be the Hermitian product on $\mathcal{C}^\infty(X, L^p \otimes E) = \mathcal{C}^\infty(X, L^p)$ induced by $h^L, h_{\omega}^E, dv_X$ as in [3]. Then by [2],

\[
(7) \quad (\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle_\omega) = (\mathcal{C}^\infty(X, L^p), \langle \cdot, \cdot \rangle_p).
\]

Observe that $H^0(X, L^p \otimes E)$ does not depend on $g^{TX}$, $h^L$ or $h^E$. If $P_{\omega,p}(x, x')$, $(x, x' \in X)$ denotes the smooth kernel of the orthogonal projection $P_{\omega,p}$ from $(\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle_\omega)$ onto $H^0(X, L^p \otimes E) = H^0(X, L^p)$ with respect to $dv_X(x)$, from [2], as in [4 (3.38)], we have

\[
(8) \quad P_{\nu,p}(x, x') = \eta(x') P_{\omega,p}(x, x').
\]

For $f \in \mathcal{C}^\infty(X)$, set

\[
K_{\omega,p}(x, x') = |P_{\omega,p}(x, x')|^2_{(h^L \otimes h_{\omega}^E) \otimes (h^{L^p} \otimes h_{\omega}^E)^*},
\]

(9) \quad $(K_{\omega,p}f)(x) = \int_X K_{\omega,p}(x, y)f(y)dv_X(y)$.

By the definition of the metric $h^E, h_{\omega}^E$, if we denote by $1^*$ the dual of the section 1 of $E$, we know

\[
(10) \quad 1 = |1 \otimes 1^*|_{h^E \otimes h_{\omega}^E}^2(x, x') = |1 \otimes 1^*|_{h^E \otimes h_{\omega}^E}^2(x, x')\eta(x)\eta^{-1}(x').
\]

Recall that we identified $(L^p, h^{L^p})$ to $(L^p \otimes E, h^{L^p} \otimes h^E)$ by Section 1. Thus from [4], [8] and [11], we get

\[
(11) \quad K_p(x, x') = |P_{\nu,p}(x, x')|^2_{(h^L \otimes h_{\omega}^E) \otimes (h^{L^p} \otimes h_{\omega}^E)^*} = \eta(x) \eta(x') K_{\omega,p}(x, x'),
\]

and from [2], [5] and [11],

\[
(12) \quad Q_{K_p}(f)(x) = \frac{1}{R_p} \int_X K_{\omega,p}(x, y)f(y)dv_X(y).
\]
Now for the kernel \( P_{\omega,p}(x,x') \), we can apply the full asymptotic expansion [2, Theorem 4.18]. In fact let \( \overline{\mathcal{D}}^{L_p \otimes E, \omega} \) be the formal adjoint of the Dolbeault operator \( \overline{\mathcal{D}}^{L_p \otimes E} \) on the Dolbeault complex \( \Omega^0 \cdot (X, L_p \otimes E) \) with the scalar product induced by \( g^{TX}, h^L, h^E, dv_X \) as in [3], and set

\[
D_p = \sqrt{2(\overline{\mathcal{D}}^{L_p \otimes E} + \overline{\mathcal{D}}^{L_p \otimes E, \omega})}.
\]

Then \( H^0(X, L_p \otimes E) = \text{Ker} \: D_p \) for \( p \) large enough, and \( D_p \) is a Dirac operator, as

\[
g^{TX}(\cdot, \cdot) = \omega(\cdot, J \cdot) \text{ is a Kähler metric on } TX.
\]

Let \( \nabla^E \) be the holomorphic Hermitian connection on \((E, h^E)\). Let \( \nabla^{TX} \) be the Levi-Civita connection on \((TX, g^{TX})\). Let \( R^E, R^{TX} \) be the corresponding curvatures.

Let \( a^X \) be the injectivity radius of \((X, g^{TX})\). We fix \( \varepsilon \in ]0, a^X/4[ \). We denote by \( B^X(x, \varepsilon) \) and \( B^{TX}(0, \varepsilon) \) the open balls in \( X \) and \( TX \) with center \( x \) and radius \( \varepsilon \). We identify \( B^X(x, \varepsilon) \) with \( B^X(x, \varepsilon) \) by using the exponential map of \((X, g^{TX})\).

We fix \( x_0 \in X \). For \( z \in B^{TX}(0, \varepsilon) \) we identify \((L_z, h^E_Z), (E_z, h^E_Z) \) and \((L_z, h^E_Z) \) to \((L_{x_0}, h^E_{x_0}), (E_{x_0}, h^E_{x_0}) \) and \((L, h^E) \) by parallel transport with respect to the connections \( \nabla^L, \nabla^E \) along the curve \( \gamma_z : [0, 1] \ni u \mapsto \exp_{x_0}(uZ) \). Then under our identification, \( P_{\omega,p}(Z, Z') \) is a function on \( Z, Z' \in T_{x_0}X, |Z|, |Z'| \leq \varepsilon \), we denote it by \( P_{\omega,p}(Z, Z') \). Let \( \pi : TX \times_X TX \to X \) be the natural projection from the fiberwise product of \( TX \) on \( X \). Then we can view \( P_{\omega,p}(Z, Z') \) as a smooth function on \( TX \times_X TX \) (which is defined for \(|Z|, |Z'| \leq \varepsilon \)) by identifying a section \( S \in \mathcal{C}^\infty(TX \times_X TX, \pi^* \text{End}(E)) \) with the family \((S_z)_{z \in X} \), \( S_z = S|_{\pi^{-1}(z)} \), \( \text{End}(E) = \mathbb{C} \).

We choose \( \{w_i\}_{i=1}^n \) an orthonormal basis of \( T_{x_0}X \), then \( e_{2j} = \frac{1}{\sqrt{2}} (w_j + \overline{w}_j) \) and \( e_{2j} = \frac{1}{\sqrt{2}} (w_j - \overline{w}_j), j = 1, \ldots, n \) forms an orthonormal basis of \( T_{x_0}X \). We use the coordinates on \( T_{x_0}X \cong \mathbb{R}^{2n} \) where the identification is given by

\[
(14) \quad (Z_1, \ldots, Z_{2n}) \in \mathbb{R}^{2n} \longrightarrow \sum_i Z_i e_i \in T_{x_0}X.
\]

In what follows we also introduce the complex coordinates \( \zeta = (z_1, \ldots, z_n) \) on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \). By [2, (4.114)] (cf. [4, (1.91)]), set

\[
(15) \quad P^N(Z, Z') = \exp \left( -\frac{\pi}{2} \sum_i (|z_i|^2 + |z_i'|^2 - 2z_i \overline{z}_i') \right).
\]

Then \( P^N \) is the classical Bergman kernel on \( \mathbb{C}^n \) (cf. [4, Remark 1.14]) and

\[
(16) \quad |P^N(Z, Z')|^2 = e^{-\pi|Z-Z'|^2}.
\]

By [2, Proposition 4.1], for any \( l, m \in \mathbb{N} \), \( \varepsilon > 0 \), there exists \( C_{l,m,\varepsilon} > 0 \) such that for \( p \geq 1, x, x' \in X \),

\[
(17) \quad |P_{\omega,p}(x, x')|_{\mathcal{E}^m(X \times X)} \leq C_{l,m,\varepsilon} p^{-l} \quad \text{if } d(x, x') \geq \varepsilon.
\]

Here the \( \mathcal{E}^m \)-norm is induced by \( \nabla^L, \nabla^E, \nabla^{TX} \) and \( h^L, h^E, g^{TX} \).

By [2, Theorem 4.18], there exist \( J_r(Z, Z') \) polynomials in \( Z, Z' \), such that for any \( k, m, m' \in \mathbb{N} \), there exist \( N \in \mathbb{N}, C_0 > 0, C_0 > 0 \) such that for \( \alpha, \alpha' \in \mathbb{N}^n, |\alpha| + |\alpha'| \leq m, \)
There exist function on an operator $L$, the polynomials coefficients are polynomials in $R$. Now, from [2, Theorem 5.1] (or [4, (1.87), (1.97)]), equation of (19).

Here $C^{m'}(X)$ is the $C^{m'}$ norm for the parameter $x_0 \in X$. The term $O(p^{-\infty})$ means that for any $l, l_1 \in \mathbb{N}$, there exists $C_{l,l_1} > 0$ such that its $C^{l_1}$-norm is dominated by $C_{l,l_0}p^{-l}$. (In fact, by [2, Theorems 4.6 and 4.17, (4.117)] (cf. [4, Theorem 1.18, (1.31)]), the polynomials $J_r(Z, Z')$ have the same parity as $r$ and $\deg J_r(Z, Z') \leq 3r$, whose coefficients are polynomials in $R^{\nabla X}$, $R^E$ and their derivatives of order $\leq r - 1$).

Now we claim that in [18],

$$J_0 = 1, \quad J_1(Z, Z') = 0.$$  

In fact, let $dv_{T_{x_0}X}$ be the Riemannian volume form on $(T_{x_0}X, g_{T_{x_0}X})$, and $\kappa$ be the function defined by

$$dv_X(Z) = \kappa(x_0, Z)dv_{T_{x_0}X}(Z).$$

Then (also cf. [4] (1.31))

$$\kappa(x_0, Z) = 1 + \frac{1}{6} \langle R_{T_{x_0}X}(Z, e_i)Z, e_i \rangle_{x_0} + O(|Z|^3).$$

As we only work on $C^{\infty}(X, L^p \otimes E)$, by [2] (4.115), we get the first equation in (19).

Recall that in the normal coordinate, after the rescaling $Z \to Z/t$ with $t = 1/\sqrt{p}$, we get an operator $\mathcal{L}$ from the restriction of $D^2_p$ on $C^{\infty}(X, L^p \otimes E)$ which has the following formal expansion (cf. [2] (1.104), [4] Theorem 1.4),

$$\mathcal{L} = \mathcal{L} + \sum_{r=1}^{\infty} Q_r t^r.$$

Now, from [2] Theorem 5.1] (or [4] (1.87), (1.97)),

$$\mathcal{L} = \sum_{j=1}^{\infty} (-2\frac{\partial}{\partial z_j} + \pi z_j)(2\frac{\partial}{\partial z_j} + \pi z_j), \quad Q_1 = 0.$$

(In fact, $P^N(Z, Z')$ is the smooth kernel of the orthogonal projection from $L^2(\mathbb{R})$ onto Ker($\mathcal{L}$)). Thus from [2] (4.107) (cf. [4] (1.111)), [21] and [23] we get the second equation of (19).

Note that $|P_{\omega, p, x_0}(Z, Z')|^2 = P_{\omega, p, x_0}(Z, Z')P_{\omega, p, x_0}(Z, Z')$, thus from [4], [18] and [19], there exist $J'_r(Z, Z')$ polynomials in $Z, Z'$ such that

$$\sup_{Z, Z'} \left| 1 \right| p^{n+1} \Delta Z \left( K_{\omega, p, x_0}(Z, Z') - \left( 1 + \sum_{r=2}^{k} p^{-r/2} J'_r(\sqrt{p}Z, \sqrt{p}Z') e^{-\pi p |Z-Z'|^2} \right) \right| \leq C p^{-(k+1)/2} (1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-C_0 \sqrt{p} |Z-Z'|) + O(p^{-\infty}).$$

For a function $f \in C^{\infty}(X)$, we denote it as $f(x_0, Z)$ a family (with parameter $x_0$) of function on $Z$ in the normal coordinate near $x_0$. Now, for any polynomial $Q_{x_0}(Z')$, we
define the operator

\[ (Q_p f)(x_0) = p^n \int_{|Z'| \leq \varepsilon} Q_{x_0}(\sqrt{p} Z') e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z'). \]

Then we observe that there exists \( C_1 > 0 \) such that for any \( p \in \mathbb{N}, f \in C^\infty(X) \), we have

\[ |Q_p f|_{L^2} \leq C_1 |f|_{L^2}. \]

In fact,

\[ |Q_p f|_{L^2}^2 \leq \int_X dv_X(x_0) \left\{ p^n \left( \int_{|Z'| \leq \varepsilon} |Q_{x_0}(\sqrt{p} Z')| e^{-\pi p|Z'|^2} dv_X(x_0, Z') \right)^2 \right\} \leq C' \int_X dv_X(x_0) p^n \int_{|Z'| \leq \varepsilon} |Q_{x_0}(\sqrt{p} Z')| e^{-\pi p|Z'|^2} f(x_0, Z')^2 dv_X(x_0, Z') \leq C_1 |f|_{L^2}^2. \]

Observe that in the normal coordinate, at \( Z = 0, \Delta_Z = -\sum_{j=1}^{2n} \frac{\partial^2}{\partial Z_j^2} \). Thus

\[ (\Delta_Z e^{-\pi p|Z-Z'|^2})|_{Z=0} = 4\pi p (n - \pi p|Z'|^2) e^{-\pi p|Z'|^2}. \]

Thus from \( 16, 18, 19, 24 \) and \( 26 \), we get

\[ \left| p^{-n} K_{\omega, p} f - p^n \int_{|Z'| \leq \varepsilon} e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}, \]

\[ \left| p^{-n-1} \Delta K_{\omega, p} f - 4\pi p^n \int_{|Z'| \leq \varepsilon} (n - \pi p|Z'|^2) e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}. \]

Set

\[ K_{\eta, \omega, p}(x, y) = \langle d(\eta(x), d_x K_{\omega, p}(x, y), g, T_X), \]

\[ (K_{\eta, \omega, p} f)(x) = \int_X K_{\eta, \omega, p}(x, y) f(y) dv_X(y). \]

Then from \( 18, 19 \) and \( 26 \), we get

\[ \left| p^{-n-1} K_{\eta, \omega, p} f - 2\pi p^n \int_{|Z'| \leq \varepsilon} \sum_{i=1}^{2n} \frac{\partial}{\partial Z_i}(\eta)(x_0, 0) Z_i e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}. \]

Let \( e^{-u\Delta}(x, x') \) be the smooth kernel of the heat operator \( e^{-u\Delta} \) with respect to \( dv_X(x') \). Let \( d(x, y) \) be the Riemannian distance from \( x \) to \( y \) on \( (X, g, T_X) \). By the heat kernel expansion in \( 11 \) Theorems 2.23, 2.26, there exist \( \Phi_i(x, y) \) smooth functions on \( X \times X \) such that when \( u \to 0 \), we have the following asymptotic expansion

\[ \left| \frac{\partial}{\partial u} \left( e^{-u\Delta}(x, y) - (4\pi u)^{-n} \sum_{i=0}^k u^i \Phi_i(x, y) e^{-\frac{1}{4u}(d(x, y)^2)} \right) \right|_{\psi^m(X \times X)} = \tilde{O}(u^{k-n-l-\frac{m}{2}+1}), \]
If we still use the normal coordinate, then by (32), there exist $\phi_{i,x_0}(Z') := \Phi_i(0, Z')$ such that uniformly for $x_0 \in X$, $Z' \in T_{x_0}X$, $|Z'| \leq \varepsilon$, we have the following asymptotic expansion when $u \to 0$,

$$\left| \frac{\partial^l}{\partial u^l} \left( e^{-u\Delta}(0, Z') - (4\pi u)^{-n} \left( 1 + \sum_{i=1}^{k} u^i \phi_{i,x_0}(Z') \right) e^{\frac{-1}{4\pi u}|Z'|^2} \right) \right| \leq \mathcal{O}(u^{k-n-l+1}), \quad (34)$$

and

$$\left| \left( d\eta(x_0), d_{x_0} e^{-u\Delta} \right)_{g^{tr}X}(0, Z') \right| - (4\pi u)^{-n} \sum_{i=1}^{2n} \left( \frac{\partial}{\partial Z_i} \eta \right)(x_0, 0) \frac{Z'_i}{2u} \left( 1 + \sum_{i=1}^{k} u^i \phi_{i,x_0}(Z') \right) e^{\frac{-1}{4\pi u}|Z'|^2} \right| \leq \mathcal{O}(u^{k-n+\frac{1}{2}}), \quad (35)$$

Observe that

$$\frac{1}{p} \Delta \exp \left( - \frac{\Delta}{4\pi p} \right) = \left. - \frac{1}{p} \left( \frac{\partial}{\partial u} e^{-u\Delta} \right) \right|_{u=\frac{1}{4\pi p}}. \quad (36)$$

Now from (26), (29)–(36), we get

$$\left| \left( p^{-n} K_{\omega,p} - \exp \left( - \frac{\Delta}{4\pi p} \right) \right) f \right|_{L^2} \leq \frac{C}{p} \| f \|_{L^2}, \quad (37)$$

and

$$\left| \left( \frac{1}{p} \left( p^{-n} \Delta K_{\omega,p} - \Delta \exp \left( - \frac{\Delta}{4\pi p} \right) \right) f \right|_{L^2} \leq \frac{C}{p} \| f \|_{L^2}. \quad (38)$$

Note that

$$\left( \Delta \eta K_{\omega,p} \right)(x, y) = (\Delta \eta)(x) K_{\omega,p}(x, y) + \eta(x) \Delta_x K_{\omega,p}(x, y) - 2\left( d\eta(x), d_x K_{\omega,p}(x, x') \right)_{g^{tr}X}, \quad (39)$$

and $R_p = \frac{\text{Vol}(X, d\nu_p)}{\text{Vol}(X, \nu)} p^n + \mathcal{O}(p^{n-1})$. From (12), (37)–(39), we get (6).

To get the last part of Theorem 1, as we noticed in [2, §4.5], the constants in (18) will be uniformly bounded under our condition, thus we can take $C$ in (6), (37) and (38) independent of $h^k$, $d\nu$.

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