STABILIZATION FOR VIBRATING PLATE WITH SINGULAR STRUCTURAL DAMPING

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Abstract. We consider the dynamic elasticity equation, modeled by the Euler-Bernoulli plate equation, with a locally distributed singular structural (or viscoelastic) damping in a boundary domain. Using a frequency domain method combined, based on the Burq’s result [8], combined with an estimate of Carleman type we provide precise decay estimate showing that the energy of the system decays logarithmically as the type goes to the infinity.

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with a sufficiently smooth boundary $\partial \Omega = \Gamma = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let $\omega$ be an nonempty and open subset of $\Omega$ with smooth boundary $\partial \omega = I \cup \Gamma_1$ such that $\Gamma_1 \cap I = \emptyset$ and $\Gamma_0 \cap I = \emptyset$ and (see Figure 1).

Consider the damping plate system

\begin{align*}
\partial_t^2 u + \Delta^2 u - \text{div}(a(x) \nabla \partial_t u) &= 0, \quad \Omega \times (0, +\infty), \\
u = \Delta u &= 0, \quad \partial \Omega \times (0, +\infty), \\
u(x, 0) &= u^0, \quad \partial_t u(x, 0) = u^1(x), \quad \Omega,
\end{align*}

where $a(x) = d \mathbf{1}_\omega(x)$ and $d > 0$ is a constant. This condition ensures that the damping term is singular and effective on the set $\omega$. System (1.1) - (1.3), involving a constructive viscoelastic damping $\text{div}(a(x) \nabla u_t)$, models the vibrations of an elastic body which has one part made of viscoelastic material. The study of the stabilization of problem involving constructive viscoelastic damping has attached a lot of attention in recent years e.g. [1, 2, 3, 10, 13, 14, 19, 20, 21, 25, 26] for the case of the Kelvin-Voigt damping and [11, 22, 27] for the case of the locally distributed structural damping. Noting that the main difference between these two kinds of damping from a mathematical point of view is that the Kelvin-Voigt damping is an operator of the same order of the leading elastic term while the structural order is of the half of the order of the principal operator.

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The undamped plate equation with \( a = 0 \) occurs as a linear model for vibrating stiff objects where the potential energy involves curvature-like terms which lead to the bi-Laplacian \((-\Delta)^2\) as the main “elastic” operator. (In the one-dimensional case one obtains the Euler–Bernoulli beam equation). In this model, energy dissipation is neglected and the equation has no smoothing effect as the governing semigroup is unitary on the canonical \( L^2 \)-based phase space. One adds damping terms to incorporate the loss of energy. Structural damping describes a situation where higher frequencies are more strongly damped than low frequencies. Here the damping term has “half of the order” of the leading elastic term.

From a theoretical point of view, the resulting system can be seen as a transmission problem of mixed type: while the structurally damped plate equation is of parabolic nature, the undamped part is of dissipative nature. Below we will see that the damping is strong enough (independent of the size of the damped part) to obtain logarithm stability for the semigroup of the coupled system. The analogue result for a coupled system of plates was obtained in the study by Denk and Kammerlander [11] for clamped (Dirichlet) boundary conditions. It is shown in this work that the damping supported near the whole boundary is strong enough to produce uniform exponential decay of the energy of the coupled system. Noting as well the paper of Denk et al. [22] in which they consider a transmission problem where a structurally damped plate equation is coupled with a damped or undamped wave equation by transmission conditions. They show that exponential stability holds in the damped-damped situation and polynomial stability (but no exponential stability) holds in the damped-undamped case. However, in this work we deal with damping supported near an arbitrary small part of the boundary. So in particular here we aim to prove the logarithm stabilization of problem (1.1)-(1.3). Our approach consists first to transform the resolvent problem respect to the semigroup operator to a transmission system, then applying a special Carleman estimate adopted to a such coupled system in order to obtain a resolvent estimate with at most exponential growth finally the Burq’s result [8] we find out the decay rate of the energy.

We define the natural energy of \( u \) solution of (1.1)-(1.3) at instant \( t \) by

\[
E(u, t) = \frac{1}{2} \left( \int_{\Omega} |\partial_t u(t, x)|^2 \, dx + \int_{\Omega} |\Delta u(t, x)|^2 \, dx \right), \quad \forall t \geq 0.
\]

Simple formal calculations gives

\[
E(u, 0) - E(u, t) = -d \int_{0}^{t} \int_{\omega} |\nabla \partial_t u(x, s)|^2 \, dx \, ds, \quad \forall t \geq 0,
\]

and therefore, the energy is non-increasing function of the time variable \( t \).
Theorem 1.1. For any $k \in \mathbb{N}^*$ there exists $C > 0$ such that for any initial data $(u^0, u^1) \in D(A^k)$ the solution $u(x, t)$ of (1.1) starting from $(u^0, u^1)$ satisfying
\[
E(u, t) \leq \frac{C}{(\ln(2 + t))^{2k}}\| (u^0, u^1) \|_{D(A^k)}^2, \quad \forall t > 0,
\]
where $(\mathcal{A}, D(\mathcal{A}))$ is defined in Section 3.

This paper is organized as follows. In Section 2, we give the proper functional setting for systems (1.1)-(1.3), then we prove that this system is well-posed and strong stability of the semigroup. In Section 3 we study the stabilization for (1.1)-(1.3) by resolvent method and give the explicit decay rate of the energy of the solutions of (1.1)-(1.3).

2. Well-posedness and strong stability

We define the energy space by $\mathcal{H} = H^2(\Omega) \cap H^1_0(\Omega) \times L^2(\Omega)$ which is endowed with the usual inner product
\[
\langle (u_1, v_1); (u_2, v_2) \rangle = \int_{\Omega} \Delta u_1(x) \Delta v_2(x) \, dx + \int_{\Omega} v_1(x) v_2(x) \, dx.
\]

We next define the linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ by
\[
D(\mathcal{A}) = \{(u, v) \in \mathcal{H} : v \in H^2(\Omega) \cap H^1_0(\Omega), \; \Delta^2 u - \text{div}(a \nabla v) \in L^2(\Omega), \; \Delta u|_{\partial \Omega} = 0 \}
\]

and
\[
\mathcal{A}(u, v)^t = (v, -\Delta^2 u + \text{div}(a \nabla v))^t
\]

Then, putting $v = \partial_\nu u$, we can write (1.1)-(1.3) into the following Cauchy problem
\[
\frac{d}{dt} (u(t), v(t))^t = \mathcal{A}(u(t), v(t))^t, \; (u(0), v(0)) = (u^0(x), u^1(x)).
\]

Theorem 2.1. The operator $\mathcal{A}$ generates a $C_0$-semigroup of contractions on the energy space $\mathcal{H}$.

Proof. Firstly, it is easy to see that for all $(u, v) \in D(\mathcal{A})$, we have
\[
\text{Re} \langle \mathcal{A}(u, v); (u, v) \rangle = -\int_{\Omega} a |\nabla v|^2 \, dx,
\]
which show that the operator $\mathcal{A}$ is dissipative.

Next, for any given $(f, g) \in \mathcal{H}$, we solve the equation $\mathcal{A}(u, v) = (f, g)$, which is recast on the following way
\[
\begin{aligned}
\left\{ \begin{array}{l}
v = f, \\
-\Delta^2 u + \text{div}(a \nabla f) = g.
\end{array} \right.
\end{aligned}
\]

It is well known that by Lax-Milgram’s theorem the system (2.1) admits a unique solution $u \in H^2(\Omega) \cap H^1_0(\Omega)$. Moreover by multiplying the second line of (2.1) by $\pi$ and integrating over $\Omega$ and using Cauchy-Schwarz inequality we find that there exists a constant $C > 0$ such that
\[
\int_{\Omega} |\Delta u(x)|^2 \, dx \leq C \left( \int_{\Omega} |\Delta f(x)|^2 \, dx + \int_{\Omega} |g(x)|^2 \, dx \right).
\]

It follows that for all $(u, v) \in D(\mathcal{A})$ we have
\[
\| (u, v) \|_{\mathcal{H}} \leq C \|(f, g)\|_{\mathcal{H}}.
\]

This imply that $0 \in \rho(\mathcal{A})$ and by contraction principle, we easily get $R(\lambda I - \mathcal{A}) = \mathcal{H}$ for sufficient small $\lambda > 0$. The density of the domain of $\mathcal{A}$ follows from \cite[Theorem 1.4.6]{23}. Then thanks to Lumer-Phillips Theorem (see \cite[Theorem 1.4.3]{23}), the operator $\mathcal{A}$ generates a $C_0$-semigroup of contractions on the Hilbert $\mathcal{H}$. \hfill \Box
Theorem 2.2. The semigroup $e^{tA}$ is strongly stable in the energy space $\mathcal{H}$, i.e,
$$ \lim_{t \to +\infty} \| e^{tA}(u_0, v_0) \|_{\mathcal{H}} = 0, \forall (u_0, v_0) \in \mathcal{H}. $$

Proof. To show that the semigroup $(e^{tA})_{t \geq 0}$ is strongly stable we only have to prove that the intersection of $\sigma(A)$ with $i\mathbb{R}$ is an empty set. Since the resolvent of the operator $A$ is not compact (see [19, 21]) but $0 \in \rho(A)$ we only need to prove that $(i\mu I - A)$ is a one-to-one correspondence in the energy space $\mathcal{H}$ for all $\mu \in \mathbb{R}^*$. 

i) Let $(u, v) \in \mathcal{D}(A)$ such that
$$ (A(u, v))^t = i\mu(u, v)^t. $$
Then taking the real part of the scalar product of (2.2) with $(u, v)$ we get
$$ \Re(i\mu \| (u, v) \|_{\mathcal{H}}^2) = \Re \langle A(u, v), (u, v) \rangle = -\int_{\Omega} a(x)|\nabla v|^2 dx = 0. $$
which implies that
$$ \nabla v = 0 \quad \text{in} \ \omega. $$
Inserting (2.3) into (2.2), we obtain
$$ \begin{cases}
-\mu^2 u + \Delta^2 u = 0 \quad \text{in} \ \Omega, \\
\nabla u = 0 \quad \text{in} \ \omega \\
u = \Delta u = 0 \quad \text{on} \ \Gamma.
\end{cases} $$

We set $w = \Delta u - |\mu|u$ then from (2.4) one follows
$$ \Delta w + |\mu|w = 0 \quad \text{in} \ \Omega. $$
We denote by $w_j = \partial_{x_j} w$ and we derive (2.5) and the second equation of (2.4), one gets
$$ \begin{cases}
\Delta w_j + |\mu|w_j = 0 \quad \text{in} \ \Omega, \\
w_j = 0 \quad \text{in} \ \omega.
\end{cases} $$

Hence, from the unique continuation theorem we deduce that $w_j = 0$ in $\Omega$ and therefore $u_j = \partial_{x_j} u$ satisfies to the following equation
$$ \Delta u_j - |\mu|u_j = 0 \quad \text{in} \ \Omega. $$
Since $u_j \equiv 0$ in $\omega$ once again the unique continuation theorem implies that $u_j \equiv 0$ in $\Omega$. Hence, $u$ is constant in $\Omega$ then from the boundary condition $u|_\Gamma = 0$ we follow that $u \equiv 0$ in $\Omega$. We have thus proved that Ker$(i\mu I - A) = 0$.

ii) Now given $(f, g) \in \mathcal{H}$, we solve the equation
$$ (A - i\mu I)(u, v) = (f, g) $$
Or equivalently,
$$ \{ 
\begin{align*}
\Delta u_j - |\mu|u_j &= 0 \quad \text{in} \ \Omega, \\
\nabla u &= 0 \quad \text{in} \ \omega \\
u &= \Delta u = 0 \quad \text{on} \ \Gamma.
\end{align*}
\right. $$

Hence, from the unique continuation theorem we deduce that $w_j = 0$ in $\Omega$ and therefore $u_j = \partial_{x_j} u$ satisfies to the following equation
$$ \Delta u_j - |\mu|u_j = 0 \quad \text{in} \ \Omega. $$
Since $u_j \equiv 0$ in $\omega$ once again the unique continuation theorem implies that $u_j \equiv 0$ in $\Omega$. Hence, $u$ is constant in $\Omega$ then from the boundary condition $u|_\Gamma = 0$ we follow that $u \equiv 0$ in $\Omega$. We have thus proved that Ker$(i\mu I - A) = 0$.

Let’s define the operator
$$ A : \mathcal{D}(A) \to L^2(\Omega) $$
where $\mathcal{D}(A) = \{ u \in H^2(\Omega) : u|_\Gamma = \Delta u|_\Gamma = 0 \}$. It is well known that $A$ a defined positive and self adjoint operator. The square root of the operator $A$ is given by
$$ A^{\frac{1}{2}} : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega) $$
We define the bounded operator \( Su = -A^{-1}(\operatorname{div}(a \nabla u)) \) in \( H^2(\Omega) \cap H^1_0(\Omega) \) and since \( S \) is a self-adjoint operator then we have \( 0 \in \rho(I + i \mu S) \).

On the one hand, the second line of (2.6) can be written as follow
\[(I + i \mu S)u - \mu^2 A^{-1}u = -A^{-1} [g + i \mu f - \operatorname{div}(a \nabla f)].\]

Let \( u \in \text{Ker}(I - \mu^2(I + i \mu S)^{-1}A^{-1}) \), then \( \mu^2 u - A(I + i \mu S)u = 0 \) and it is clear that \( u \in \mathcal{D}(A) \).

It follows that
\[(2.8) \quad \mu^2 u - \Delta^2 u + i \mu \operatorname{div}(a \nabla u) = 0.\]

Multiplying (2.8) by \( \overline{\mu} \) and integrating over \( \Omega \), then by Green’s formula we obtain
\[
m^2 \int_{\Omega} |u(x)|^2 \, dx - \int_{\Omega} |\Delta u(x)|^2 \, dx - i \mu \int_{\omega} |\nabla u(x)|^2 \, dx = 0.
\]

By taking its imaginary part it follows
\[
d \int_{\omega} |\nabla u(x)|^2 \, dx = 0,
\]
and this implies that \( \nabla u = 0 \) in \( \omega \). Inserting this last equation into (2.8) we get
\[
\mu^2 u - \Delta^2 u = 0, \quad \text{in } \Omega.
\]

Following the steps of the first part of this proof we can prove that \( u = 0 \) and this imply that \( \text{Ker}((I - \mu^2(I + i \mu S)^{-1}A^{-1}) = \{0\} \). On the other hand, the compactness of the injection \( H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^2(\Omega) \) implies the compactness of the operator \( A^{-\frac{3}{2}} \) and consequently the compactness of the operator \( A^{-1} \) as well. Therefore thanks to Fredholm’s alternative, the operator \( (I - \mu^2(I + i \mu S)^{-1}A^{-1}) \) is bijective in \( L^2(\Omega) \). Then by setting
\[
Au = \mu^2 A^{-1}u - (I + i \mu S)u = (I + i \mu S)(\mu^2(I + i \mu S)^{-1}A^{-1} - I)u.
\]
we deduce that \( A \) is a bijection in \( H^2(\Omega) \cap H^1_0(\Omega) \). It is not difficult to see that equation of (2.7) is equivalent to the following equation
\[
Au = A^{-1}(g + i \mu f - \operatorname{div}(a \nabla f)).
\]
So that, equation (2.7) have a unique solution in \( H^2(\Omega) \cap H^1_0(\Omega) \) and it is clear that \( u \in \mathcal{D}(A) \).

This prove that the operator \( (i \mu I - A) \) is surjective in the energy space \( \mathcal{H} \).

The proof is thus complete. \( \square \)

3. Stabilization result

In this section, we will prove the logarithmic stability of the system (1.1). To this end, we establish a particular resolvent estimate precisely we will show that for some constant \( C > 0 \) we have
\[(3.1) \quad ||(A - i \mu I)^{-1}||_{\mathcal{L}(\mathcal{H})} \leq C e^{C|\mu|}, \quad \forall |\mu| \gg 1,
\]
and then by Burq’s result [8] and the remark of Duyckaerts [12 section 7] (see also [2]) we obtain the expected decay rate of the energy. Let \( \mu \) be a real number such that \( |\mu| \) is large, and assume that
\[(3.2) \quad (A - i \mu I)(u, v)^t = (f, g)^t, \quad (u, v) \in \mathcal{D}(A), \quad (f, g) \in \mathcal{H}.
\]
which can be written as follow
\[
\begin{align*}
\begin{cases}
\begin{align*}
v - i \mu u &= f &\text{in } \Omega \\
-\Delta^2 u + \operatorname{div}(a(x) \nabla v) - i \mu v &= g &\text{in } \Omega,
\end{align*}
\end{cases}
\end{align*}
\]
o or equivalently,
\[(3.3) \quad \begin{cases}
\begin{align*}
v &= f + i \mu u \\
-\Delta^2 u + i \mu \operatorname{div}(a(x) \nabla u) + \mu^2 u &= g + i \mu f - \operatorname{div}(a(x) \nabla f) &\text{in } \Omega.
\end{align*}
\end{cases}
\]
Multiplying the second line of (3.3) by \( \pi \) and integrating over \( \Omega \) then by Green’s formula we obtain

\[
\int_\Omega (g + i \mu f) \pi \, dx + \int_\omega a \nabla f \cdot \nabla \pi \, dx = \mu^2 \int_\Omega |u|^2 \, dx - \int_\Omega |\Delta u|^2 \, dx - i \mu \int_\omega a |\nabla u|^2 \, dx.
\]

Taking the imaginary part of (3.4) we obtain

\[
|u| \int_\omega a |\nabla u|^2 \, dx \leq \left( \int_\Omega (g + i \mu f)^2 \, dx \right)^{\frac{1}{2}} \left( \int_\Omega |u|^2 \, dx \right)^{\frac{1}{2}} + \left( \int_\Omega |\nabla f|^2 \, dx \right)^{\frac{1}{2}} \left( \int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}}
\]

(3.5)

Then by setting \( u = u_1 1_\omega + u_2 1_{\Omega \setminus \omega}, v = v_1 1_\omega + v_2 1_{\Omega \setminus \omega} \), \( f = f_1 1_\omega + f_2 1_{\Omega \setminus \omega} \), and \( g = g_1 1_\omega + g_2 1_{\Omega \setminus \omega} \) system (3.3) is transformed to the following transmission equation

\[
\begin{cases}
  v_1 = i \mu u_1 + f_1 & \text{in } \omega, \\
  v_2 = i \mu u_2 + f_2 & \text{in } \Omega \setminus \omega, \\
  -\Delta^2 u_1 + i \mu \Delta u_1 + \mu^2 u_1 = g_1 + i \mu f_1 - d \Delta f_1 & \text{in } \omega, \\
  -\Delta^2 u_2 + \mu^2 u_2 = g_2 + i \mu f_2 & \text{in } \Omega \setminus \omega,
\end{cases}
\]

(3.6)

where the following the transmission conditions

\[
\begin{cases}
  u_1 = u_2 & \text{in } I_1, \\
  \partial_n u_1 = \partial_n u_2 & \text{in } I, \\
  \Delta u_1 = \Delta u_2 & \text{in } I_1, \\
  \partial_n (\Delta u_1 - i \mu u_1 - d f_1) = \partial_n \Delta u_2 & \text{on } I,
\end{cases}
\]

(3.7)

follow from the regularity of the state, and with the boundary conditions

\[
\begin{cases}
  u_1 = \Delta u_1 = 0 & \text{on } \Gamma_1, \\
  u_2 = \Delta u_2 = 0 & \text{on } \Gamma_0,
\end{cases}
\]

(3.8)

where \( \nu(x) \) denote the outer unit normal to \( \Omega \setminus \pi \) on \( \Gamma_0 \) and on \( I \) (see Figure 1).

Now we can prove the resolvent estimate (3.1). We set \( w_1 = \Delta u_1 + (|\mu| - i \mu) u_1 \) and \( w_2 = \Delta u_2 + |\mu| u_2 \), then the system (3.6) - (3.8) can be recast as follow

\[
\begin{cases}
  -\Delta w_1 + |\mu| w_1 = \Phi_1 & \text{in } \omega, \\
  -\Delta w_2 + |\mu| w_2 = \Phi_2 & \text{in } \Omega \setminus \omega,
\end{cases}
\]

(3.9)

the transmission conditions

\[
\begin{cases}
  w_1 = w_2 + \phi_1 & \text{on } I, \\
  \partial_n w_1 = \partial_n w_2 + \phi_2 & \text{on } I,
\end{cases}
\]

(3.10)

and the boundary conditions

\[
\begin{cases}
  w_1 = 0 & \text{on } \Gamma_1, \\
  w_2 = 0 & \text{on } \Gamma_0,
\end{cases}
\]

(3.11)

where we have denoted by \( \Phi_1 = g_1 + i \mu f_1 - d \Delta f_1 - i d |\mu| \mu u_1, \Phi_2 = g_2 + i \mu f_2, \phi_1 = -i d \mu u_1 \) and \( \phi_2 = d \partial_n f_1 \).

We denoted by \( B_r \) a ball of radius \( r > 0 \) in \( \omega \) and \( B_r^c \) its complementary such that \( B_{4r} \subset \omega \).

Let’s introduce the cut-off function \( \chi \in C^\infty(\omega) \) by

\[
\chi(x) = \begin{cases} 
  1 & \text{in } B_{3r}^c, \\
  0 & \text{in } B_{2r}. 
\end{cases}
\]

Next, we denote by \( \tilde{w}_1 = \chi w_1 \) then from the first line of (3.5), one sees that

\[
-\Delta \tilde{w}_1 + |\mu| \tilde{w}_1 = \tilde{\Phi}_1 & \text{in } \omega,
\]

(3.12)

where \( \tilde{\Phi}_1 = \chi \Phi_1 - [\Delta, \chi] w_1. \) We denote by \( \Omega_1 = \omega \setminus B_r \) and \( \Omega_2 = \Omega \setminus B_r. \)
Our proof of (3.1) is based on a Carleman estimate established in [1] by Ammari, Hassine and Robbiano and recalled here in the following theorem.

**Theorem 3.1.** [1] Theorem 3.2] Consider a bounded smooth open set $U$ of $\mathbb{R}^n$ with boundary $\partial U = \gamma$. We set $U_1$ and $U_2$ two smooth open subsets of $U$ with boundaries $\partial U_1 = \gamma_0$ and $\partial U_2 = \gamma_0 \cup \gamma$ such that $\gamma_0 \cup \gamma = \emptyset$. We denote by $\nu(x)$ the unit outer normal to $U_2$ if $x \in \gamma_0 \cup \gamma$.

For $\tau$ a large parameter and $\varphi_1$ and $\varphi_2$ two weight functions of class $C^\infty$ in $\overline{U}_1$ and $\overline{U}_2$ respectively such that $\varphi_{1\mid\gamma_0} = \varphi_{2\mid\gamma_0}$ we denote by $\varphi(x) = \text{diag}(\varphi_1(x), \varphi_2(x))$ and let $\alpha$ be a non null complex number. We set the differential operator

$$P = \text{diag}(P_1, P_2) = \text{diag}(-\Delta + \tau, -\Delta + \tau),$$

and its conjugate operator

$$P(x, D, \tau) = e^{r\varphi}P e^{-r\varphi} = \text{diag}(P_1(x, D, \tau), P_2(x, D, \tau)),$$

with principal symbol $p(x, \xi, \tau)$ given by

$$p(x, \xi, \tau) = \text{diag}(p_1(x, \xi, \tau), p_2(x, \xi, \tau)) = \text{diag}(\langle |\xi|^2 + 2i\tau \xi \nabla \varphi_1 - \tau^2 |\nabla \varphi_1|^2, |\xi|^2 + 2i\tau \xi \nabla \varphi_2 - \tau^2 |\nabla \varphi_2|^2 \rangle).$$

We define the tangential operators $\text{op}(B_1)$ and $\text{op}(B_2)$ by

$$\text{op}(B_1) u = u_{1\mid\gamma_0} - u_{2\mid\gamma_0}$$

and

$$\text{op}(B_2) u = \partial_\nu u_{1\mid\gamma_0} - \partial_\nu u_{2\mid\gamma_0}.$$

Assume that the weight function $\varphi$ defined on $U$ satisfies

(3.14) $|\nabla \varphi_k(x)| > 0$, $\forall x \in \overline{U}_k$, $k = 1, 2,$

(3.15) $\partial_\nu \varphi_{1\mid\gamma_0}(x) < 0$,

(3.16) $\partial_\nu \varphi_{2\mid\gamma_0}(x) > 0$, $k = 1, 2,$

(3.17) $(\partial_\nu \varphi_{1\mid\gamma_0}(x))^2 - (\partial_\nu \varphi_{2\mid\gamma_0}(x))^2 > 1$,

and the sub-ellipticity condition

(3.18) $\exists c > 0$, $\forall (x, \xi) \in \overline{U}_k \times \mathbb{R}^n$, $p_k(x, \xi) = 0 \implies \{\text{Re}(p_k), \text{Im}(p_k)\}(x, \xi, \tau) \geq c(\xi, \tau)^3$.

Then there exist $C > 0$ and $\tau_0 > 0$ such that we have the following estimate

(3.19) $\tau^3 \|e^{r\varphi}u\|^2_{L^2(\overline{U}_1)} + \tau \|e^{r\varphi}\nabla u\|^2_{L^2(\overline{U}_1)}$

$$\leq C \left( \|e^{r\varphi}Pu\|^2_{L^2(\overline{U}_1)} + \tau^2 \|e^{r\varphi}\text{op}(B_1)u\|^2_{H^1(\gamma_0)} + \tau \|e^{r\varphi}\text{op}(B_2)u\|^2_{L^2(\gamma_0)} \right)$$

for all $\tau \geq \tau_0$ and $u = (u_1, u_2) \in H^2(\overline{U}_1) \times H^2(\overline{U}_2)$ such that $u_{1\mid\gamma} = 0$.

Following to [3] or [13] or [15] we can find four weight functions $\varphi_{1,1}$, $\varphi_{1,2}$, $\varphi_{2,1}$ and $\varphi_{2,2}$, a finite number of points $x^i_{j,k}$, where $B(x^i_{j,k}, 2\varepsilon) \subset \Omega_j$ for all $j, k = 1, 2$ and $i = 1, \ldots, N_{j,k}$ such that

$$\bigcup_{i=1}^{N_{j,k}} B(x^i_{j,k}, 2\varepsilon) \bigcap_{i=1}^{N_{j,k}} B(x_{j,k}^i, 2\varepsilon) = \emptyset$$

and by denoting $\check{U}_{j,k} = \Omega_j \bigcap_{i=1}^{N_{j,k}} B(x^i_{j,k}, \varepsilon)$ the weight function $\varphi_k = \text{diag}(\varphi_{1,k}, \varphi_{2,k})$ verifying the assumption (3.14)-(3.18) in $U_{1,k} \cup U_{2,k}$ with $\gamma_0 = \mathcal{L}$. Moreover, $\varphi_{j,k} < \varphi_{j,k+1}$ in $\bigcup_{i=1}^{N_{j,k}} B(x^i_{j,k}, 2\varepsilon)$ for all $j, k = 1, 2$ where we have denoted by $\varphi_{j,3} = \varphi_{j,1}$.

Let $\chi_{j,k}$ (for $j, k = 1, 2$) four cut-off functions equal to 1 in $\bigcup_{i=1}^{N_{j,k}} B(x^i_{j,k}, 2\varepsilon)$ and supported in $\bigcup_{i=1}^{N_{j,k}} B(x^i_{j,k}, \varepsilon)$ (in order to eliminate the critical points of the weight functions $\varphi_{j,k}$). We
set \( w_{1,1} = \chi_{1,1} \bar{w}_1, w_{1,2} = \chi_{1,2} \bar{w}_1, w_{2,1} = \chi_{2,1} w_2 \) and \( w_{2,2} = \chi_{2,2} w_2 \). Then from system (3.10) and equations (3.8) and (5.12), then for \( k = 1, 2 \) we obtain

\[
\begin{aligned}
&-\Delta w_{1,k} + |\mu| w_{1,k} = \Psi_{1,k} \quad \text{in } \omega \\
&-\Delta w_{2,k} + |\mu| w_{2,k} = \Psi_{2,k} \quad \text{in } \Omega \setminus \mathcal{M} \\
&\quad w_{1,k} = w_{2,k} + \phi_1 \quad \text{on } \mathcal{I} \\
&\quad \partial_\nu w_{1,k} = \partial_\nu w_{2,k} + \phi_2 \quad \text{on } \mathcal{I} \\
&\quad w_{1,k} = 0 \quad \text{on } \Gamma_1 \\
&\quad w_{2,k} = 0 \quad \text{on } \Gamma_0,
\end{aligned}
\]

(where)

\[
\begin{aligned}
\Psi_{1,k} &= \chi_{1,k} \Phi_1 - [\Delta, \chi_{1,k}] \bar{w}_1 \\
\Psi_{2,k} &= \chi_{2,k} \Phi_2 - [\Delta, \chi_{2,k}] w_2.
\end{aligned}
\]

Applying now Carleman estimate (3.19) to the system (3.20) with \( \tau = |\mu| \) then for \( k = 1, 2 \) we have

\[
\begin{aligned}
&\tau^3 \sum_{j=1,2} \|e^{\tau \varphi_{j,k}} w_{j,k}\|_{L^2(U_{j,k})}^2 + \tau \sum_{j=1,2} \|e^{\tau \varphi_{j,k}} \nabla w_{j,k}\|_{L^2(U_{j,k})}^2 \\
&\leq C \left( \|e^{\tau \varphi_{1,k}} \Psi_{1,k}\|_{L^2(U_{1,k})}^2 + \|e^{\tau \varphi_{2,k}} \Psi_{2,k}\|_{L^2(U_{2,k})}^2 + \tau^2 \|\phi_1\|_{H_\mathcal{I}^2(I)}^2 + \tau \|\phi_2\|_{L^2(I)}^2 \right).
\end{aligned}
\]

From the expression of \( \Psi_{1,k} \) and \( \Psi_{2,k} \) in (3.21), then we can write

\[
\begin{aligned}
&\tau^3 \sum_{j=1,2} \|e^{\tau \varphi_{j,k}} w_{j,k}\|_{L^2(U_{j,k})}^2 + \tau \sum_{j=1,2} \|e^{\tau \varphi_{j,k}} \nabla w_{j,k}\|_{L^2(U_{j,k})}^2 \leq C \left( \|\phi_1\|_{L^2(U_{1,k})}^2 \\
&+ \|\Phi_2\|_{L^2(U_{2,k})}^2 + \|\Delta, \chi_{1,k}\|_{L^2(U_{1,k})}^2 + \|\Delta, \chi\|_{L^2(U_{1,k})}^2 \right). \end{aligned}
\]

Adding the two last estimates and using the property of the weight functions \( \varphi_{j,1} < \varphi_{j,2} \) in \( \bigcup_{i=1}^{N_j} B(x_{j,i},2\varepsilon) \) and \( \varphi_{j,2} < \varphi_{j,1} \) in \( \bigcup_{i=1}^{N_{j,2}} B(x_{j,i},2\varepsilon) \) for all \( j = 1, 2 \), then we can absorb first order the terms \( |\Delta, \chi_{1,k}\|_{\bar{w}_1} \) and \( |\Delta, \chi_{2,k}\| w_2 \) at the right hand side into the left hand side for \( \tau > 0 \) sufficiently large, mainly we obtain

\[
\begin{aligned}
&\tau^3 \int_{\Omega_1} (e^{2\tau \varphi_{1,1}} + e^{2\tau \varphi_{1,2}}) |\bar{w}_1|^2 \, dx + \tau^3 \int_{\Omega_2} (e^{2\tau \varphi_{2,1}} + e^{2\tau \varphi_{2,2}}) |w_2|^2 \, dx \\
&+ \tau \int_{\Omega_1} (e^{2\tau \varphi_{1,1}} + e^{2\tau \varphi_{1,2}}) \nabla |\bar{w}_1|^2 \, dx + \tau \int_{\Omega_2} (e^{2\tau \varphi_{2,1}} + e^{2\tau \varphi_{2,2}}) |\nabla w_2|^2 \, dx \\
&\leq C \left( \int_{\Omega_1} (e^{2\tau \varphi_{1,1}} + e^{2\tau \varphi_{1,2}}) |\Phi_1|^2 \, dx + \int_{\Omega_2} (e^{2\tau \varphi_{2,1}} + e^{2\tau \varphi_{2,2}}) |\Phi_2|^2 \, dx \right) \\
&+ \tau^2 \left( \|\phi_1\|_{H_\mathcal{I}^2(I)}^2 + \|\phi_2\|_{H_\mathcal{I}^2(I)}^2 \right) + \tau \left( \|\varphi_{1,1}\|_{L^2(I)}^2 + \|\varphi_{1,2}\|_{L^2(I)}^2 \right) \\
&+ \int_{\Omega_1} (e^{2\tau \varphi_{1,1}} + e^{2\tau \varphi_{1,2}}) |\Delta, \chi| w_1^2 \, dx \right).
\end{aligned}
\]
Since \( \chi \equiv 1 \) outside \( B_{3r} \) then using the expressions of \( \phi_1 \) and \( \phi_2 \) we obtain
\[
(3.22) \quad \tau^3 \int_{\omega \setminus B_{3r}} \left( e^{2\tau \varphi_{1,1}} + e^{2\tau \varphi_{1,2}} \right) |w_1|^2 \, dx + \tau^3 \int_{\Omega \setminus \omega} \left( e^{2\tau \varphi_{2,1}} + e^{2\tau \varphi_{2,2}} \right) |w_2|^2 \, dx
\]
\[
\tau \int_{\omega \setminus B_{3r}} \left( e^{2\tau \varphi_{1,1}} + e^{2\tau \varphi_{1,2}} \right) |\nabla w_1|^2 \, dx + \tau \int_{\Omega \setminus \omega} \left( e^{2\tau \varphi_{2,1}} + e^{2\tau \varphi_{2,2}} \right) |\nabla w_2|^2 \, dx
\]
\[
\leq C \left( \int_{\omega \cup B_{3r}} \left( e^{2\tau \varphi_{1,1}} + e^{2\tau \varphi_{2,1}} \right) |\Phi_1|^2 \, dx + \int_{\Omega \cup B_{3r}} \left( e^{2\tau \varphi_{2,1}} + e^{2\tau \varphi_{2,2}} \right) |\Phi_2|^2 \, dx \right.
\]
\[
+ \tau \left( \left\| e^{2\tau \varphi_{1,1}} u_1 \right\|^2_{H^s(\Omega)} + \left\| e^{2\tau \varphi_{1,2}} u_1 \right\|^2_{H^s(\Omega)} \right) + \tau \left( \left\| e^{2\tau \varphi_{2,1}} \partial_x f_1 \right\|^2_{L^2(\Omega)} + \left\| e^{2\tau \varphi_{2,2}} \partial_x f_1 \right\|^2_{L^2(\Omega)} \right)
\]
\[
+ \tau \left( \left\| e^{2\tau \varphi_{1,1}} \partial_y f_1 \right\|^2_{L^2(\Omega)} + \left\| e^{2\tau \varphi_{2,2}} \partial_y f_1 \right\|^2_{L^2(\Omega)} \right) + \tau \int_{\Omega \setminus \omega} \left( e^{2\tau \varphi_{2,1}} + e^{2\tau \varphi_{2,2}} \right) |\Delta \chi| |w_1|^2 \, dx \right).
\]
Taking the maximum of \( \varphi_{1,1}, \varphi_{1,2}, \varphi_{2,1} \) and \( \varphi_{2,2} \) in the right hand side of (3.22) and their minimum in the left hand side, next since the operator \( |\Delta \chi| \) is of the first order then by Poincaré’s inequality, the trace formula and the expressions of \( \Phi_1 \) and \( \Phi_2 \), we follow
\[
\left\| w_1 \right\|^2_{L^2(\omega \setminus B_{3r})} + \left\| w_2 \right\|^2_{L^2(\Omega \setminus \overline{B}_{3r})} + \left\| \nabla w_1 \right\|^2_{L^2(\omega \setminus B_{3r})} + \left\| \nabla w_2 \right\|^2_{L^2(\Omega \setminus \overline{B}_{3r})}
\]
\[
\leq C e^{C \tau} \left( \left\| \nabla w_1 \right\|^2_{L^2(\omega)} + \left\| f_1 \right\|^2_{L^2(\omega)} + \left\| \Delta f_1 \right\|^2_{L^2(\omega)} + \left\| f_2 \right\|^2_{L^2(\omega)} + \left\| g_1 \right\|^2_{L^2(\Omega)} + \left\| g_2 \right\|^2_{L^2(\Omega)} + \left\| u_1 \right\|^2_{H^s(\omega)} \right)
\]
\[
\leq C e^{C \tau} \left( \left\| \nabla w_1 \right\|^2_{L^2(\omega)} + \left\| f_1 \right\|^2_{L^2(\omega)} + \left\| \Delta f_1 \right\|^2_{L^2(\omega)} + \left\| f_2 \right\|^2_{L^2(\omega)} + \left\| g_1 \right\|^2_{L^2(\Omega)} + \left\| g_2 \right\|^2_{L^2(\Omega)} + \left\| u_1 \right\|^2_{H^s(\omega)} \right).
\]
Now let \( \overline{B}_r \) a ball of reduce \( r \) such that \( \overline{B}_{3r} \subset \omega \) and \( \overline{B}_{4r} \cap \overline{B}_{3r} = \emptyset \). We resume the same work with \( \overline{B}_r \) instead of \( B_r \) we obtain a similar estimate as (3.22) namely, one gets
\[
\left\| w_1 \right\|^2_{L^2(\omega \setminus B_{3r})} + \left\| w_2 \right\|^2_{L^2(\omega \setminus B_{3r})} + \left\| \nabla w_1 \right\|^2_{L^2(\omega \setminus B_{3r})} + \left\| \nabla w_2 \right\|^2_{L^2(\omega \setminus B_{3r})}
\]
\[
\leq C e^{C \tau} \left( \left\| \nabla w_1 \right\|^2_{L^2(\omega)} + \left\| f_1 \right\|^2_{L^2(\omega)} + \left\| \Delta f_1 \right\|^2_{L^2(\omega)} + \left\| f_2 \right\|^2_{L^2(\omega)} + \left\| g_1 \right\|^2_{L^2(\Omega)} + \left\| g_2 \right\|^2_{L^2(\Omega)} + \left\| u_1 \right\|^2_{H^s(\omega)} \right)
\]
\[
\leq C e^{C \tau} \left( \left\| \nabla w_1 \right\|^2_{L^2(\omega)} + \left\| f_1 \right\|^2_{L^2(\omega)} + \left\| \Delta f_1 \right\|^2_{L^2(\omega)} + \left\| f_2 \right\|^2_{L^2(\omega)} + \left\| g_1 \right\|^2_{L^2(\Omega)} + \left\| g_2 \right\|^2_{L^2(\Omega)} + \left\| u_1 \right\|^2_{H^s(\omega)} \right).
\]
Summing up the two estimates (3.27) and (3.28) and using the fact that \( B_{3r} \cap \overline{B}_{3r} = \emptyset \), we follow that
\[
\left\| w_1 \right\|^2_{L^2(\omega)} + \left\| w_2 \right\|^2_{L^2(\Omega \setminus \overline{B}_{3r})} + \left\| \nabla w_1 \right\|^2_{L^2(\omega)} + \left\| \nabla w_2 \right\|^2_{L^2(\omega \setminus B_{3r})}
\]
\[
\leq C e^{C \tau} \left( \left\| \nabla w_1 \right\|^2_{L^2(\omega)} + \left\| f_1 \right\|^2_{L^2(\omega)} + \left\| \Delta f_1 \right\|^2_{L^2(\omega)} + \left\| f_2 \right\|^2_{L^2(\omega)} + \left\| g_1 \right\|^2_{L^2(\Omega \setminus \overline{B}_{3r})} + \left\| g_2 \right\|^2_{L^2(\Omega \setminus \overline{B}_{3r})} + \left\| u_1 \right\|^2_{H^s(\omega)} \right).
\]
Noting that \( u_1 \) and \( u_2 \) are solution of the following problem
\[
\begin{cases}
\Delta u_1 + |\mu| u_1 = w_1 + id|\mu| \cdot \mu u_1 & \text{in } \omega \\
\Delta u_2 + |\mu| u_2 = w_2 & \text{in } \Omega \setminus \overline{\omega},
\end{cases}
\]
the transmission conditions
\[
\begin{cases}
u_1 = u_2 & \text{on } \mathcal{I} \\
\partial_n u_1 = \partial_n u_2 & \text{on } \mathcal{I},
\end{cases}
\]
and the boundary conditions
\[
\begin{cases}
u_1 = 0 & \text{on } \Gamma_1 \\
u_2 = 0 & \text{on } \Gamma_0,
\end{cases}
\]
then as done with \( w_1 \) and \( w_2 \) we can apply Carleman estimate to \( u_1 \) and \( u_2 \) and we get an estimate of the same kind as (3.24), namely we have
\[
\left\| w_1 \right\|^2_{L^2(\omega)} + \left\| w_2 \right\|^2_{L^2(\Omega \setminus \overline{B}_{3r})} + \left\| \nabla w_1 \right\|^2_{L^2(\omega)} + \left\| \nabla w_2 \right\|^2_{L^2(\omega \setminus B_{3r})}
\]
\[
\leq C e^{C |\mu|} \left( \left\| \nabla w_1 \right\|^2_{L^2(\omega)} + \left\| w_1 \right\|^2_{L^2(\omega)} + \left\| u_2 \right\|^2_{L^2(\omega \setminus B_{3r})} \right),
\]
which imply in particular that

\[
\| \nabla u_2 \|_{L^2(\Omega \setminus \overline{\omega})}^2 \leq C e^{C|\mu|} \left( \| \nabla u_1 \|_{L^2(\omega)}^2 + \| w_2 \|_{L^2(\Omega \setminus \overline{\omega})}^2 \right).
\]

From (3.29), performing now the following calculation

\[
\| \nabla w_1 \|_{L^2(\omega)}^2 + \| \nabla w_2 \|_{L^2(\omega)}^2 = - \langle \Delta w_1, w_1 \rangle_{L^2(\omega)} - \langle \Delta w_2, w_2 \rangle_{L^2(\Omega \setminus \omega)} - \langle \partial_\nu w_2, w_2 \rangle_{L^2(\Gamma)} + \langle \partial_\nu w_1, w_1 \rangle_{L^2(\Gamma)}
\]

\[
= \langle \Phi_1, w_1 \rangle_{L^2(\omega)} + \langle \Phi_2, w_2 \rangle_{L^2(\Omega \setminus \omega)} - |\mu| \left( \| w_2 \|_{L^2(\omega)}^2 + \| w_2 \|_{L^2(\Omega \setminus \omega)}^2 \right) = - \langle \partial_\nu w_2, w_2 \rangle_{L^2(\Gamma)} + \langle \partial_\nu w_1, w_1 \rangle_{L^2(\Gamma)}.
\]

Using the transmission conditions (3.10) we obtain

\[
\langle \partial_\nu w_1, w_1 \rangle_{L^2(\Gamma)} - \langle \partial_\nu w_2, w_2 \rangle_{L^2(\Gamma)} = id \mu \langle \partial_\nu w_1, w_1 \rangle_{L^2(\Gamma)} + d \langle \partial_\nu f_1, w_1 + id \mu u_1 \rangle_{L^2(\Gamma)}
\]

\[
= -id \mu \left( \langle \Delta w_1, u_1 \rangle_{L^2(\omega)} + \langle \nabla w_1, \nabla u_1 \rangle_{L^2(\omega)} \right) + d \langle \partial_\nu f_1, w_1 + id \mu u_1 \rangle_{L^2(\Gamma)}
\]

\[
= -id \mu \left( |\mu| \langle w_1, u_1 \rangle_{L^2(\omega)} + \langle \nabla w_1, \nabla u_1 \rangle_{L^2(\omega)} - \langle \Phi_1, u_1 \rangle_{L^2(\omega)} \right) + d \langle \partial_\nu f_1, w_1 + id \mu u_1 \rangle_{L^2(\Gamma)}.
\]

Putting together the two last equalities we find

\[
\| \nabla w_1 \|_{L^2(\omega)}^2 + \| \nabla w_2 \|_{L^2(\omega)}^2 = \langle \Phi_1, w_1 \rangle_{L^2(\omega)} + \langle \Phi_2, w_2 \rangle_{L^2(\Omega \setminus \omega)} - |\mu| \left( \| w_2 \|_{L^2(\omega)}^2 + \| w_2 \|_{L^2(\Omega \setminus \omega)}^2 \right)
\]

\[
= - id \mu \left( |\mu| \langle w_1, u_1 \rangle_{L^2(\omega)} + \langle \nabla w_1, \nabla u_1 \rangle_{L^2(\omega)} - \langle \Phi_1, u_1 \rangle_{L^2(\omega)} \right) + d \langle \partial_\nu f_1, w_1 + id \mu u_1 \rangle_{L^2(\Gamma)}.
\]

The Poincaré inequality, the trace formula and the Young’s inequality imply

\[
|\mu| \left( \| w_1 \|_{L^2(\omega)}^2 + \| w_2 \|_{L^2(\Omega \setminus \omega)}^2 \right) + \| \nabla w_1 \|_{L^2(\omega)}^2 + \| \nabla w_2 \|_{L^2(\Omega \setminus \omega)}^2 \leq C \left( \| \Phi_1 \|_{L^2(\omega)}^2 + \| \Phi_2 \|_{L^2(\Omega \setminus \omega)}^2 + |\mu|^4 \| \nabla u_1 \|_{L^2(\omega)}^2 + \| f_1 \|_{L^2(\omega)}^2 \right)
\]

\[
\leq C \mu^2 \left( \| f_1 \|_{L^2(\omega)}^2 + \| f_2 \|_{L^2(\Omega \setminus \omega)}^2 \right) + \| g_1 \|_{L^2(\omega)}^2 + \| g_2 \|_{L^2(\Omega \setminus \omega)}^2 + |\mu|^4 \| \nabla u_1 \|_{L^2(\omega)}^2 \right)
\]

Combining (3.24) and (3.27) we follow

\[
\| w_1 \|_{L^2(\omega)}^2 + \| w_2 \|_{L^2(\Omega \setminus \omega)}^2 + C e^{C|\mu|} \left( \| w_2 \|_{L^2(\omega)}^2 + \| w_2 \|_{L^2(\Omega \setminus \omega)}^2 \right)
\]

\[
\leq C_1 e^{C_1|\mu|} \left( \| f_1 \|_{L^2(\omega)}^2 + \| g_1 \|_{L^2(\omega)}^2 + \| g_2 \|_{L^2(\Omega \setminus \omega)}^2 + \| \nabla u_1 \|_{L^2(\omega)}^2 \right)
\]

From this last estimates and (3.25) we find

\[
\| w_1 \|_{L^2(\omega)}^2 + \| w_2 \|_{L^2(\Omega \setminus \omega)}^2 + 2|\mu| \| \nabla u_2 \|_{L^2(\omega)}^2 \leq C_1 e^{C_1|\mu|} \left( \| f_1 \|_{L^2(\omega)}^2 + \| g_1 \|_{L^2(\omega)}^2 + \| g_2 \|_{L^2(\Omega \setminus \omega)}^2 + \| \nabla u_1 \|_{L^2(\omega)}^2 \right).
\]

Evoking $u_1$ and $u_2$ through the expressions of $w_1$ and $w_2$ and using the transmission conditions (3.7) and the boundary conditions (3.8) to perform the following integration by parts

\[
\| w_1 \|_{L^2(\omega)}^2 + \| w_2 \|_{L^2(\Omega \setminus \omega)}^2 = -2d\mu \text{Im} \langle \partial_\nu u_1, u_1 \rangle_{L^2(\Gamma)} + \| \Delta u_1 \|_{L^2(\omega)}^2 + \| \Delta u_2 \|_{L^2(\Omega \setminus \omega)}^2 + |\mu|^2 \left( (1 + d^2) \| u_1 \|_{L^2(\omega)}^2 + \| u_2 \|_{L^2(\Omega \setminus \omega)}^2 \right) - 2|\mu| \left( \| \nabla u_1 \|_{L^2(\omega)}^2 + \| \nabla u_2 \|_{L^2(\Omega \setminus \omega)}^2 \right).
\]
From the Young’s inequality we obtain
\begin{equation}
\|w_1\|_{L^2(\omega)}^2 + \|w_2\|_{L^2(\Omega \setminus \omega)}^2 \geq \|\Delta u_1\|_{L^2(\omega)}^2 + \|\Delta u_2\|_{L^2(\Omega \setminus \omega)}^2 \\
+ \|\mu\|^2 \left((1 + d^2)\|u_1\|_{L^2(\omega)}^2 + \|u_2\|_{L^2(\Omega \setminus \omega)}^2\right) \\
- 2\|\mu\| \left(\|\nabla u_1\|_{L^2(\omega)}^2 + \|\nabla u_2\|_{L^2(\Omega \setminus \omega)}^2\right) \\
- \frac{d^2\|\mu\|^2}{\varepsilon} \|u_1\|_{H^1(\omega)}^2 - \varepsilon \|u_1\|_{H^2(\omega)}^2.
\end{equation}
Combining (3.28) and (3.29), taking $\varepsilon$ small enough and using the Poincaré inequality, one gets
\begin{equation}
\|\Delta u_1\|_{L^2(\omega)}^2 + \|\Delta u_2\|_{L^2(\Omega \setminus \omega)}^2 \leq C e^{C \mu} \left(\|\Delta f_1\|_{L^2(\omega)}^2 + \|\Delta f_2\|_{L^2(\Omega \setminus \omega)}^2 \\
+ \|g_1\|_{L^2(\omega)}^2 + \|g_2\|_{L^2(\Omega \setminus \omega)}^2 + \|\nabla u_1\|_{L^2(\omega)}^2\right),
\end{equation}
which implies
\begin{equation}
\|\Delta u\|_{L^2(\omega)}^2 \leq C e^{C \mu} \left(\|\Delta f\|_{L^2(\omega)}^2 + \|g\|_{L^2(\omega)}^2 + \|\nabla u\|_{L^2(\omega)}^2\right).
\end{equation}
Using (3.30) and (3.31) we follow
\begin{equation}
\|\Delta u\|_{L^2(\omega)}^2 \leq C e^{C \mu} \left(\|\Delta f\|_{L^2(\omega)}^2 + \|g\|_{L^2(\omega)}^2 + \|\nabla u\|_{L^2(\omega)}^2\right).
\end{equation}
By Poincaré inegalité, one has
\begin{equation}
\|\Delta u\|_{L^2(\omega)}^2 \leq C e^{C \mu} \left(\|\Delta f\|_{L^2(\omega)}^2 + \|g\|_{L^2(\omega)}^2\right).
\end{equation}
We refer to the expression of $v$ in the first line of (3.3) and using the fact that
\begin{equation}
\|u\|_{L^2(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)}
\end{equation}
then estimate (3.31) gives
\begin{equation}
\|v\|_{L^2(\Omega)}^2 \leq C e^{C \mu} \left(\|\Delta f\|_{L^2(\omega)}^2 + \|g\|_{L^2(\omega)}^2\right).
\end{equation}
So that, the estimate (3.1) is obtained by the combination of the two estimates (3.31) and (3.32). And this completes the proof.

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