Transfer Learning with Label Noise

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Abstract

Transfer learning aims to improve learning in the target domain with limited training data by borrowing knowledge from a related but different source domain with sufficient labeled data. To reduce the distribution shift between source and target domains, recent methods have focused on exploring invariant representations that have similar distributions across domains. However, existing methods assume that the labels in the source domain are uncontaminated, while in reality, we often only have access to a source domain with noisy labels. In this paper, we first analyze the effects of label noise in various transfer learning scenarios in which the data distribution is assumed to change in different ways. We find that although label noise has no effect on the invariant representation learning in the covariate shift scenario, it has adverse effects on the learning process in the more general target/conditional shift scenarios. To solve this problem, we propose a new transfer learning method to learn invariant representations in the presence of label noise, which also simultaneously estimates the label distributions in the target domain. Experimental results on both synthetic and real-world data verify the effectiveness of the proposed method.

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1 Introduction

In the classical transfer learning setting, given data points \( \{x_T^1, \ldots, x_T^n\} \) from the target domain, we aim to learn a function to predict the labels \( \{y_T^1, \ldots, y_T^n\} \) using labeled data \( \{(x_S^1, y_S^1), \ldots, (x_S^m, y_S^m)\} \) from a different but related source domain. In contrast to standard supervised learning, the joint distributions \( P_{XY}^S \) and \( P_{XY}^T \) are different across domains. For example, in the indoor WiFi localization problem [26], the signal distributions received by different phone models vary. Transferring knowledge from one phone model to another is desirable since it is laborious to obtain labeled data for all phone models. Further, in computer vision, collecting a labeled visual recognition dataset is expensive but it is easy to access a large number of loosely labeled web images and videos [5]. In this kind of problem, transfer learning can improve the generalization ability of models learned from the source domain by correcting domain mismatches.

Due to various assumptions about how the distribution shifts across domains, several transfer learning scenarios have been proposed. (1) Covariate shift is a traditional scenario where the marginal distribution \( P_X \) changes but the conditional distribution \( P_{Y|X} \) stays the same. In this situation, several methods have been proposed to correct the shift in \( P_X \); for instance, importance reweighting [7] and invariant representation [12]. (2) Model shift [25] assumes that the marginal distribution \( P_X \) and the conditional distribution \( P_{Y|X} \) change independently. In this case, successful transfer requires \( Y \) to be continuous, the change in \( P_{Y|X} \) to be smooth, and that some labeled data are available in the target domain. (3) Target shift [27] assumes that the marginal distribution \( P_Y \) shifts while \( P_{X|Y} \) stays the same. In this scenario, \( P_X \) and \( P_{Y|X} \) will change dependently because their changes are caused by the change in \( P_Y \). (4) Generalized target shift [27] assumes that \( P_{X|Y} \) and \( P_Y \) change independently across domains, causing \( P_X \) and \( P_{Y|X} \) to change dependently. An interpretation of the difference between these scenarios from a causal standpoint was also provided [20, 28].

Previous transfer learning methods usually assume that the source domain labels are “clean”. In practice, however, training set labeling tends to be expensive and time-consuming, especially for the large-scale datasets. Thus, cheap but imperfect surrogates, such as examples labeled by non-expert labelers or queried from search engines using imprecise keywords, have their advantages, but label noise is often inevitable. In this paper, we consider the case in which the observed labels in the source domain are noisy. The noise is assumed to be random and the flip rates are class-conditional (abbreviated as CCN [15]), which is a widely employed label noise model in the machine learning community. Since we have no access to the true source distribution when the labels are noisy, it might be problematic if we directly apply existing transfer learning methods to correct the mismatch between the noisy source domain and the target domain.

It is therefore essential to sort out how label noise can influence existing transfer learning methods in different scenarios. Obviously, label noise has no effect on transfer learning methods in the covariate shift scenario, as correcting the shift in \( P_X \) does not require label information. Unfortunately, label noise can adversely affect knowledge transfer in other
scenarios. Taking target shift as an example, in order to correct the shift in \(P_Y\), the labeled data points in the source domain are required to estimate the density ratio between \(P^T_Y\) and \(P^S_Y\). However, in the presence of label noise, it is unclear whether the density ratio \(P^T_Y / P^S_Y\) can be estimated from noisy data. Label noise also affects the learning in the model shift scenario, but we will not consider this case because we are concerned with discrete labels and the case in which there is no label in the target domain.

In this paper, we mainly focus on the generalized target shift scenario which is prevalent in many classification problems. In addition to the possible wrong estimate of \(P^T_Y / P^S_Y\), the estimates of conditional invariant representations would be inaccurate due to the presence of label noise. To resolve this challenge, we propose a new transfer learning method that can extract the conditional invariant representations \(X' = \tau(X)\) that have similar \(P_{X'|Y}\) across domains and identify the possible changes in clean distribution \(P_Y\). Specifically, we construct a new domain with distribution \(P^{new}_{X'|Y}\) from the noisy source distribution \(P^{\rho}_{X|Y}\), where we denote \(P_\rho\) as the distributions associated with label noise. We also develop a novel reweighting strategy to assign weights to noisy labels, which ensures matching between \(P^{new}_{X'|Y}\) and \(P^T_{X'|Y}\). By minimizing the domain discrepancy between \(P^{new}_{X'|Y}\) and \(P^T_{X'|Y}\), the conditional invariant components and the changes in \(P_Y\) are identifiable from the noisy observations. To verify the effectiveness of the proposed method, we conduct experiments on both synthetic and real-world data. The experimental results show the capability of the proposed method to transfer invariant knowledge across different domains when label noise is present.

2 Related Work

Classification in the presence of label noise. Learning with noisy labels in the traditional classification problem has been widely studied [13, 24]. These methods can be categorized into four categories dealing with unbiased losses [15], label noise-robust losses [9], label noise cleansing [11], and label noise fitting [23, 17], respectively. Our method is related to the importance reweighting [11] and the unbiased estimator [15] in classification. However, the problem considered in this paper is more challenging because the clean source domain distribution is not assumed to be identical to the target domain distribution. In contrast to [11] and [15], our method aims to learn the invariant components \(X'\) across different domains, where both \(P_Y\) and \(P_{X|Y}\) may change. Reports on the general results obtained in this setting are scarce.

Generalized target shift. Existing methods to address generalized shift usually assume that there exists a transformation \(\tau\), e.g., location-scale transformation [27, 6], such that the conditional distribution \(P_{\tau(X)|Y}\) is invariant across domains. In this paper, we also assume that the conditional invariant components (CIC) exist. We aim to learn them from the noisy distribution \(P^{\rho}_{X|Y}\), which makes the problem more challenging. We show that the performance of our method significantly surpasses the method in [6] if noisy labels are present. Directly computing \(\beta\) from noisy distributions by employing the method in [6] will give a largely deviated weight to the kernel mean value for each class, compromising
the ability to learn invariant components, which is empirical verified in our experiments.

3 Transfer Learning in the Presence of Label noise

In this section, we examine the effect of label noise in four different transfer learning scenarios, namely 1) covariate shift, 2) model shift, 3) target shift, and 4) generalized target shift. From a causal perspective, 1) and 2) assume that $X$ causes $Y$, indicating that $P_X$ and $P_{Y|X}$ contain no information about each other. In transfer learning, the causal relation implies that changes in $P_X$ are independent of changes in $P_{Y|X}$. If the change in $P_{Y|X}$ is large, then it is difficult to correct the shift in $P_{Y|X}$ because we often have no or scarce labels in the target domain. On the contrary, 3) and 4) assume that $Y$ is the cause for $X$, implying that changes in $P_Y$ and $P_{X|Y}$ are independent, while changes in $P_X$ and $P_{X|Y}$ depend on each other. Figure 1 represents the causal relations between variables in transfer learning using selection diagram defined in [18].

![Diagram](image1.png)

Figure 1: Possible situations of transfer learning in the presence of label noise. $V_s^1$ and $V_s^2$ are independent domain-specific selection variables, leading to changing $P_{XY}$ across domains. (a) Model shift: $V_s^1$ and $V_s^2$ change $P_X$ and $P_{Y|X}$, respectively. (b) Generalized Target shift: $V_s^1$ and $V_s^2$ change $P_Y$ and $P_{X|Y}$, respectively. In the first scenario, $X$ is a cause for $Y$, whilst in the second scenario, $Y$ is a cause of $X$. If $V_s^2$ is not present, (a) reduces to covariate shift and (b) reduces to target shift. Note that in our setting, the true labels $Y$ in the source domain is unobservable, we can only observe noisy labels $\hat{Y}$ in the source domain.

![Graphs](image2.png)

Figure 2: A simple illustration of the difficulties brought by noisy labels in the target shift situation. (a)-(c) show the class conditionals in the clean source domain, noisy source domain, and target domain, respectively.
• Covariate shift. In this situation, label noise has no effects on the correction of shift in $P_X$, because the changes in $P_X$ has nothing to do with $Y$. However, after correcting the shift in $P_X$, one needs to take the effects of label noise into account when training a classifier on the source domain [15, 11].

• Model shift. In the model shift scenario, since $P_X$ and $P_{Y|X}$ change independently, we can correct them separately. Similar to covariate shift, correcting $P_X$ is not affected by label noise. However, correcting shift in $P_X$ requires matching $P_{Y|X}$ and $P_{T|X}$, which can be seriously harmed by label noise. In this scenario, since a small number of clean labels are assumed to be available in the target domain, $P_{Y|X}$ is often assumed to change smoothly across domains to reduce the estimation error.

• Target shift. In this scenario, it is required that $P_{S|X|Y} = P_{T|X|Y}$. The changes in $P_Y$ are often corrected by matching the marginal distribution of the importance-reweighted source domain $P_{X}^{new} = \sum_{i=1}^{c} P_{Y=i} P_{X=i} \beta(Y = i)$ and the target domain $P_{T|X}$, where $\beta(Y = i) = P_{T|Y=i} / P_{S|Y=i}$ and $c$ is the class number. In the presence of label noise, unfortunately, we only have access to $P_{S|X|Y}$ and $P_{S|Y}$ in the source domain. As shown in Figure 2, $P_{S|X|Y=i}$ becomes a mixture of $P_{S|X|Y=1}$ and $P_{S|X|Y=2}$ and is no longer identical to $P_{T|X|Y=i}$. In this case, it is unknown whether $P_{T|X}$ is still identifiable from the noisy source domain. Directly applying the methods in [27, 8] on the noisy data will lead to wrong results. For example, suppose the clean source domain and target domain share the same distribution, i.e., $P_{T|Y} = P_{S|Y}$. As the sample size $N \to \infty$, a vector of ones, will be a trivial solution, resulting in $P_{T|Y} = P_{S|Y}$. However, $P_{S|Y}$ is usually different from $P_{S|Y}$, leading to a wrong estimate of $P_{T|Y}$. It seems that one can consider this problem as generalized target shift and adopt the method in [4] to find conditional invariant components $X' = \tau(X)$ that satisfy $P_{S|X'|Y} = P_{T|X'|Y}$. Unfortunately, since the changes in $P_{X|Y}$ do not cause changes in $P_X$, as implied in Figure 1(a)-(b), one can not correct the changes in $P_{X|Y}$ by minimizing the changes in $P_X$.

• Generalized target shift. In this situation, $P_{X|Y}$ also changes across domains, but it changes independently of $P_Y$. A widely employed approach is learning conditional invariant components that satisfy $P_{X'|Y} = P_{X'|Y}$. The learning of $X'$ needs $P_{S|X}$ and $P_{T|X}$. Similar to target shift, the estimate of invariant components and the label distribution $P_{T|Y}$ will be inaccurate if we directly use the noisy source distribution $P_{S|XY}$ to correct distribution shift.
4 Denoising Conditional Invariant Components

In this section, we elaborate the general target shift learning with label noise. We aim to learn the invariant components $\tau(X)$ and the changes in $P_Y$ based on the “noisy” observations $\{(x_1^n, y_1^n), \ldots, (x_m^n, y_m^n)\}$ in the source domain and data points $\{x^T, \ldots, x^n\}$ in the target domain. To solve this problem, even though both $P_{X|Y}$ and $P_Y$ change, we assume some unchanged knowledge exists in two domains enabling knowledge transfer.

**Assumption 1 (Conditional Invariant Components [6])**. For every $d$-dimensional raw data $X$, there exists a universe transformation $\tau : \mathbb{R}^d \rightarrow \mathbb{R}^d$, which satisfies,

$$P_{\tau(X)|Y} = P_{\tau(X)|Y}^S,$$

where $\tau(X) \in \mathbb{R}^d$ are the conditional invariant components across different domains.

Note that for simplicity, in some part of the paper, we will denote $\tau(X)$ as $X'$ if without ambiguity. According to the complexity of the input data space, the transformation $\tau$ varies from linear ones to non-linear ones in order to extract conditional invariant components. The choice of $\tau$ is not our main concern in this paper. Here we only consider the linear transformation, that is, $\tau(X) = W^T X$, where $W^T W = I_d$, and $W \in \mathbb{R}^{d \times d'}$.

We show that under mild conditions, matching the conditional distribution in Eq. (1) can be achieved by matching the marginal distribution $P_{X|Y}^T$, with a new distribution $P_{X|Y}^{new}$, which is marginalized from the reweighted distribution $P_{\rho_{X|Y}}^S$ as follows,

$$P_{X|Y}^{new} = \sum_{y'} \beta(y') P_{\rho_Y}^S(X', \hat{Y} = y') = \sum_{y'} \sum_{y} \beta(y') P_{\rho_Y}^S(X', Y = y, \hat{Y} = y'),$$

where $\beta$ are the weights for noisy labels; and $Y$ and $\hat{Y}$ are the variables for “clean” and “noise” labels, respectively. Note that when no ambiguity occurs, we only use $Y$ as the variable for both “clean” and “noisy” labels; otherwise, we will use $Y$ and $\hat{Y}$.

**Theorem 1.** Suppose the linear transformation $W$ satisfies that $P(W^T X | Y = i), i \in \{1, \cdots, c\}$ are linearly independent, and that the elements in the set $\{u_i P_S(W^T X | Y = i) + \lambda_i P^T (W^T X | Y = i); i \in \{1, \cdots, c\}; \forall u_i, \lambda_i (u_i^2 + \lambda_i^2 \neq 0)\}$ are linearly independent. Then, if $P_{X|Y}^{new} = P_{X|Y}^T$, we have $P_{X|Y}^T = P_{X|Y}^S; and \beta(Y = y) = \sum_{y'} P_S(\hat{Y} = y') / P_S(Y = y)$.

See the proof of Theorem 1 in the Appendix A.

Note that, let $u = [\beta(Y = 1), \cdots, \beta(Y = c)]^T$ and $u_\rho = [\beta_\rho(Y = 1), \cdots, \beta_\rho(Y = c)]^T$, we have $u = Q u_\rho$, where $Q$ is the transition matrix induced by label noise, and $Q_{ij} = P(\hat{Y} = j | Y = i), \forall i, j \in \{1, \cdots, c\}$. If the transition matrix is invertible, the relationship between $\beta_\rho$ and $\beta$ is uniquely determined. We also assume that the probability of one label flipping to another is smaller than that of keeping its original value. Thus, the transition matrix $Q$ is usually diagonally dominant, which ensures that $Q$ is invertible. In
this case, by matching the marginal distribution, we can learn $\beta_\rho$, which indicates that the changes in the clean distribution $P^T_Y$ is also identifiable. In practice, the transition matrix $Q$ is not available. In this paper, we will estimate it by employing the estimation method proposed in [11]. More discussions about the estimation can be found in Section 4.2.

4.1 Kernel Mean Matching with Label Noise

According to Theorem 1 even though learning with noisy labels, by constructing a new distribution $P^\text{new}_{X'} = P^T_{X'}$, we can still extract the conditional invariant components and identify the changes in $P_Y$. To enforce the matching between $P^\text{new}_{X'}$ and $P^T_{X'}$, we employ the kernel mean matching of these two distributions and minimize the squared maximum mean discrepancy (MMD):

$$
\|\mu_{P^\text{new}_{X'}}[\psi(X')] - \mu_{P^T_{X'}}[\psi(X')]\|^2 = \|\mathbb{E}_{X' \sim P^\text{new}_{X'}}[\psi(X')] - \mathbb{E}_{X' \sim P^T_{X'}}[\psi(X')]\|^2,
$$

where $\psi$ is a kernel mapping. According to Eq. (2), we have

$$
\mathbb{E}_{X' \sim P^\text{new}_{X'}}[\psi(X')] = \mathbb{E}_{(X', Y) \sim P^S_{X'Y}}[\beta_\rho(Y) \psi(X')].
$$

Therefore minimizing Eq. (3) is equivalent to minimizing

$$
\|\mathbb{E}_{(X', Y) \sim P^S_{X'Y}}[\beta_\rho(Y) \psi(X')] - \mathbb{E}_{X' \sim P^T_{X'}}[\psi(X')]\|^2.
$$

In the setting of the paper, we can only observe the corruptly labeled source data $\{(x_1, \hat{y}^S_1), \ldots, (x_m, \hat{y}^S_m)\}$ and the unlabeled data $\{x_1^T, \ldots, x_n^T\}$ in the target domain. Therefore, we approximate the expected kernel mean values by their empirical expectations. Then we have the following objective function:

$$
\|\frac{1}{m}\psi(W^T x^S)\beta_\rho(\hat{y}^S) - \frac{1}{n}\psi(W^T x^T)1\|^2,
$$

where $\beta_\rho(\hat{y}^S) = [\beta_\rho(\hat{y}_1), \ldots, \beta_\rho(\hat{y}_m)]^T$; $x$ represents the data design matrix; $1 \in \mathbb{R}^n$ is a vector of all ones.

However, we cannot directly optimize the objective function (4) w.r.t. $\beta_\rho(\hat{y}^S)$. This is because the resulting $\beta_\rho(\hat{y}^S)$ will violate the fact that for the same $\hat{y}$, $\beta_\rho(\hat{y})$ should be the same. Thus, we need to reparametrize $\beta_\rho(\hat{y}^S)$. To this end, we need to figure out the relationship between $\beta_\rho$ and $P^T_Y$. Note that $\beta_\rho(\hat{Y} = i) = \sum_{j=1}^c Q^{-1}_{ij} P^T_{(Y=i)}$ and that $[P^S(Y = 1), \ldots, P^S(Y = c)]Q = [P^S(\hat{Y} = 1), \ldots, P^S(\hat{Y} = c)]$. Given estimated $\hat{Q}$ and $[\hat{P}^S_{\rho}(Y = 1), \ldots, \hat{P}^S_{\rho}(Y = c)]^T$, we can construct the vectors $g_i = [\hat{Q}^{-1}_{i1}, \ldots, \hat{Q}^{-1}_{ic}], i \in \{1, \ldots, c\}$. If $\hat{y}_k = i, \forall k \in \{1, \ldots, m\}$, define the matrix $G \in \mathbb{R}^{m \times c}$, where the $k$-th row of $G$ is $g_i$. Let $\beta_\rho(\hat{y}^S) = G\alpha$. Then, we have that $\alpha$ is an estimation of $[P^T(\hat{Y} = 1), \ldots, P^T(\hat{Y} = c)]^T$. 


The objective function now can be reparametrized as

\[
\| \frac{1}{m} \psi(W^T x^S) G \alpha - \frac{1}{n} \psi(W^T x^T) 1 \|^2 = \frac{\alpha^T G^T K_W^S G \alpha - 21^T K_W^{TS} G \alpha + 1^T K_W^T 1}{mn} + \frac{1}{n^2},
\]

where \( K_W^S \) and \( K_W^T \) are the kernel matrix of \( W^T x^S \) and \( W^T x^T \), respectively; \( K_W^{TS} \) is the cross kernel matrix. In this paper, the Gaussian kernel, i.e., \( k(x_i, x_j) = \exp\left(-\frac{\|x_i-x_j\|^2}{2\sigma^2}\right) \) is applied, where \( \sigma \) is the kernel width. Let \( D(W, \alpha) = \| \frac{1}{m} \psi(W^T x^S) G \alpha - \frac{1}{n} \psi(W^T x^T) 1 \|^2 \), we come to the model

\[
\min_{W, \alpha} D(W, \alpha), \quad \text{s.t.} \quad W^T W = I; \quad \sum_{i=1}^{c} \alpha_i = 1; \quad \text{and} \quad \alpha_i \geq 0, \forall i \in \{1, \cdots, c\}.
\]

The alternating optimization method is applied to update \( W \) and \( \alpha \). Specifically, we apply the conjugate gradient algorithm on the Grassmann manifold to optimize \( W \), and use the quadratic programming to optimize \( \alpha \). In this way, we can learn the invariant components, and identify the \( \beta_p \) as well as \( \beta \).

### 4.2 Convergence Analysis

Without any assumption on \( P_{X|Y} \), the true transition matrix (induced by label noise flip rates) \( Q^* \) and the true class priors \( \alpha^* \) are not identifiable. In this subsection, we will study the convergence rates of the estimators to the true label noise rate and true class prior under the linearly independent property of \( P_{\psi(X)|Y} \) in the universal kernel Hilbert space. The convergence rate for estimating the label noise rate has been well studied under the “anchor set” condition that for any \( y \) there exist \( x \) in the domain of \( X \) such that \( P(Y = y | X) = 0 \), which is likely to be held in practice. Two estimators with convergence guarantees has been proposed in [11] and [21], respectively. Recently, [19] further exploited the “anchor set” condition in Hilbert space and designed estimators with careful convergence analysis. However, the results are still based on the “anchor set” condition and that the upper bounds contain some constants which would be hard to estimate. Inspired by the analyses on estimating class ratio in [27], we could show that the estimate of label noise rate will become easier under a weaker condition that \( P(\psi(X)|Y = i), i \in \{1, \cdots, c\} \) are linearly independent in the universal kernel Hilbert space, which is also the basic condition for the true label noise rate [21] and class ratio [3] to be identifiable.

We take the binary classification problem with symmetric random label noise as an example. We denote \( \pi \) here as the inverse flip rate, i.e. \( P(Y = 1| \hat{Y} = -1) = \pi_{-1} \) and \( P(Y = -1| \hat{Y} = 1) = \pi_{+1} \). Then we have \( P(\psi(X)|\hat{Y} = y) = (1 - \pi_y) P(\psi(X)|Y = y) + \pi_y P(\psi(X)|Y = -y) \). If the mapping \( \psi \) induces a universal reproducing kernel Hilbert space (RKHS) and \( P(\psi(X)|Y = y) \) and \( P(\psi(X)|Y = -y) \) are given, we could estimate the inverse flip rate \( \pi \) with a fast convergence rate guarantee [22] by simply matching the mean of the distributions \( P(\psi(X)|Y = y) \) and \( P(\psi(X)|Y = -y) \) with that of \( P(\psi(X)) \),
because there is an injection between the distributions and the mean feature vectors in the universal kernel Hilbert space. Then the noise rate can also be computed according to the Bayes' Theorem. However, to estimate $P(\psi(X)|Y = y)$, $P(\psi(X)|Y = -y)$, and $P(Y)$, we need some true labels, which is not available in this setting and would be considered in our future research. In this paper, we will employ the estimator designed in [11] to estimate the label noise rates, the effectiveness of which has been empirically verified by [17].

It is interesting to analyze the convergence rate of the estimated class prior parameter $\hat{\alpha}$ to the true $\alpha^*$ in the presence of label noise. We abuse the training samples $\{(x_1^S, \hat{y}_1^S), \cdots, (x_m^S, \hat{y}_m^S)\}$ and $\{x_1^T, \cdots, x_n^T\}$ as being i.i.d. variables, respectively. Let

$$\mathcal{D}(W, \alpha) = \|E_1^m \psi(W^\top x^S) G \alpha - E_1^n \psi(W^\top x^T) 1\|^2.$$ 

We analyze the convergence rate by deriving an upper bound for $\mathcal{D}(\hat{W}, \hat{\alpha}) - \mathcal{D}(W, \alpha^*)$ with fixed $Q$ and $W$.

**Theorem 2.** Given learned $\hat{Q}$ and $\hat{W}$, let the induced RKHS be universal and upper bounded that $\|\psi(\tau(x))\| \leq \wedge_W$, for all $x$ in the source and target domains, and let the entries of $G$ be bounded that $|G_{ij}| \leq \wedge_Q$ for all $i \in \{1, \cdots, m\}, j \in \{1, \cdots, c\}$. For any $\delta > 0$, with probability at least $1 - \delta$, we have

$$\mathcal{D}(\hat{W}, \hat{\alpha}) - \mathcal{D}(\hat{W}, \alpha^*) \leq 4(\wedge_Q + 1)^2 \wedge_W^2 \sqrt{8\sqrt{c} \over \sqrt{m}} + 8\sqrt{c} \over \sqrt{n} + \sqrt{8(1 \over m) + 1 \over n} \log 1 \over \delta.$$ 

See the proof of Theorem 2 in the Appendix B.

Although the bound in Theorem 2 involves two fixed parameters, the result is informative if $Q^*$ and $W^*$ are given or $\hat{Q}$ and $\hat{W}$ quickly converges to $Q^*$ and $W^*$, respectively. From previous analyses, we know that fast convergence rates for estimating label noise rate are guaranteed. However, the convergence of $\hat{W}$ to $W^*$ is not guaranteed because the objective function is non-convex w.r.t. $W$. How to identify the transferable components $W^* \top X$ should be further studied.

### 5 Experiments

To show the robustness of our method to label noise, we conduct comprehensive evaluations on both simulated and real data. We first compare our method, denoising conditional invariant components (abbr. as DCIC hereafter), with CIC [6] on identifying the changes in $P_Y$ given noisy observations. The effectiveness of extracting invariant representations are evaluated by the classification performances on both the synthetic and real data. The method for training the classifiers is elaborated in the Appendix C. We further compare DCIC with the domain invariant projection (DIP) method [2], transfer component analysis (TCA) [16], and CIC. In all experiments, we set the standard deviation parameter $\sigma$ of the Gaussian kernel to be the median value of the pairwise distances between all source examples.
5.1 Synthetic Data

We study the performance of DCIC in the following two situations, (a) the estimate of
density ratio $\beta$ in the target shift (TarS) situation given the true label flip rates; and (b) the
evaluation of the extracted invariant components in the generalized target shift (GeTarS)
situation, with various density ratios and different label flip rates. In all classification
experiments, the flip rates are estimated using the method proposed in [11]. We repeat
the experiments for 20 times and report the average performance.

We generate binary classification training and test data from a 2-dimensional mixture
of Gaussians [6], i.e.

$$x \sim \sum_{i=1}^{2} \pi_i N(\theta_i, \Sigma_i), \theta_{ij} \sim U(-0.25, 0.25), \text{ and } \Sigma_i \sim 0.01 * \mathcal{W}(2 \times I_2, 7),$$

where $\mathcal{U}(a, b)$ and $\mathcal{W}(\Sigma, df)$ are the uniform and Wishart distributions, respectively. The
class labels are the cluster indices. Under TarS, $P_{X|Y}$ remains the same. We only change
the class priors across domains. Under GeTarS, we apply location and scale transforma-
tions on the features to generate target domain data. To get the noisy observations, we
randomly flip the clean labels in the source domain with the same probability $\rho$.

First, we verify that with corrupted labels, the proposed DCIC can almost recover the
correct density ratio in the target shift scenario. We set the source class prior $P_S(Y = 1)$
to be 0.5. The target domain class prior $P_T(Y = 1)$ varies from 0.1 to 0.9 with step 0.1.
The corresponding class ratio $\beta(Y = 1) = P_T(Y = 1)/P_S(Y = 1)$ varies from 0.2 to 1.8
with step 0.2. Then, we compare the proposed method with CIC [6] on finding the true
density ratio $\beta^*$ with noisy labels in source domain. We evaluate the performance using
the density ratio estimation error $\|\beta_{est} - \beta^*\|/\|\beta^*\|$, where $\beta_{est}$ is the estimated density
ratio vector. Figure 3(a) shows that DCIC can find the solution close to the true $\beta^*$ for
various density ratios. In this experiment, given large label noise ($\rho = 0.4$), $\beta$ estimated
by CIC is close to the true one only when $\beta^*(Y = 1)$ is close to 0, 1, and 2. The estimation
of CIC is accurate at $\beta^*(Y = 1) = 1$ because we set the class prior $P_{Y=1}^S$ to be 0.5 in the
clean source domain, which happens to make $P_{Y=1}^S = P_Y^S$. If $P_{Y=1}^S \neq 0.5$, then $P_{Y=1}^S \neq P_Y^S$,
the estimated $\beta$ will be wrong (see Section 3). CIC gives accurate results when $\beta^*(Y = 1)$
is close to 0, 2 because target domain collapses to a single class, rendering the estimated
results trivially right. Figure 3(b) shows the superiority of the proposed method over CIC
at different levels of label noise. When $\rho > 0.1$, CIC seems to find the trivial solutions.
However, our method can find a good solution even when $\rho$ is close to 0.5. Figure 3(c)
shows that the estimate of $\beta$ improves as the sample size gets larger.

Second, under GeTarS, we evaluate whether our method can discover the invariant
representations given noisy labels in source domain using classification accuracies as the
performance measure. In these experiments, we fix the sample size to 500. The class prior
$P_S(Y = 1)$ is set to 0.5. The results in Figure 4 shows that our method is more robust to
the label noise than DIP, TCA, and CIC.
Figure 3: The estimation of density ratio $\beta$. (a), (b), and (c) present the estimate errors of $\beta$ with the increasing class ratio $\beta(Y = 1)$, the increasing flip rate $\rho$, and the increasing sample size $n$, respectively.

Figure 4: The effectiveness of invariant components extraction. (a), (b), and (c) present the classification error with increasing flip rate $\rho$ when $\beta_1 = 1.4, 1.6, and 1.8$, respectively.

5.2 Real Data

We further compare DCIC with DIP, TCA, and CIC on the cross-domain indoor WiFi localization dataset [26]. The problem is to learn the function between signals $X$ and locations $Y$. In this paper, we view it as a classification problem, where each location space is assigned with a discrete label. In the prediction stage, the label is then converted to the location information. We resample the training set to simulate the changes in $P_Y$. To resample the source domain training examples such that the density ratio is not a vector of all ones, we randomly select $c/2$ class and let their class ratio be 2.5. For the other $c/2$ classes, we set their $P(Y)$ to be equal. In this experiment, the flip rate from one class to another is set to $\frac{\rho}{c-1}$.

Table 1: Classification accuracies and their standard deviations for different transfer learning methods in the presence of label noise.

|                         | Softmax  | TCA     | DIP     | CIC     | DCIC     |
|-------------------------|----------|---------|---------|---------|----------|
| t1 $\rightarrow$ t2    | 60.73 ± 0.66 | 70.80 ± 1.66 | 71.40 ± 0.83 | 75.50 ± 1.02 | **79.28 ± 0.56** |
| t1 $\rightarrow$ t3    | 55.20 ± 1.22 | 67.43 ± 0.55 | 64.65 ± 0.32 | 69.05 ± 0.28 | **70.75 ± 0.91** |
| t2 $\rightarrow$ t3    | 54.38 ± 2.01 | 63.58 ± 1.33 | 66.71 ± 2.63 | 70.92 ± 3.86 | **77.28 ± 2.87** |
| hallway1                | 40.81 ± 12.05 | 42.78 ± 7.69 | 44.31 ± 8.34 | 51.83 ± 8.73 | **59.31 ± 12.30** |
| hallway2                | 27.98 ± 10.28 | 43.68 ± 11.07 | 44.61 ± 5.94 | 43.96 ± 6.20 | **60.50 ± 8.68** |
| hallway3                | 24.94 ± 9.89 | 31.44 ± 5.47 | 33.50 ± 2.58 | 32.00 ± 3.88 | **33.89 ± 5.94** |
We first learn the linear transformation \( W \in \mathbb{R}^{d \times d'} \) \((d' = 10)\) and extract the invariant components. Then a neural network with one hidden layer is trained to classify the signals in target domain. The output layer is a softmax with the cross-entropy loss. The activation functions in the hidden layer are the Rectified Linear Unit (ReLU). The size of hidden layer is set to 800. During training, learning rate is fixed to 0.1. After training, as in [6], we report the percentage of examples on which the difference between the predicted and true locations is within 3 meters. Here we also train a neural network with the raw features as the baseline. All the experiments are repeated 10 times and the average performances are reported. In Table 1, the three upper rows present the transfer across different time periods \( t_1, t_2, \) and \( t_3, \) where \( \rho = 0.4. \) The lower part shows the transfer across different devices, where \( \rho = 0.2. \) Since the input features in two domains are too complex in these cases, the invariant components cannot be well identified by a simple linear transformation, which also in turn obstructs the optimization of \( \alpha. \) Thus, the performance is relatively degraded. However, DCIC still outperforms other methods because it reduces the influence of label noise. All the results show DCIC can better transfer the invariant knowledge than other methods.

6 Conclusion

In this paper, we have studied the problem of transfer learning in the presence of label noise. We have found that the presence of labels is detrimental to the performance of existing transfer learning methods. In particular, when the label is the cause for the features, the estimate of target domain class distribution and conditional invariant representations can be unreliable. To alleviate the effects of label noise on transfer learning, we have proposed a novel reweighting method to assign weights to noisy labels, leading to significant improvement in estimation of both target domain label distribution and conditional invariant components. We have provided both theoretical and empirical studies to demonstrate the effectiveness of the proposed method. Future work will focus on investigating learning complex invariant representations using deep neural networks in the proposed framework.
A Proof of Theorem 1

Theorem 1. Suppose the linear transformation $W$ satisfies that $P(W^TX|Y = i), i \in \{1, \cdots, c\}$ are linearly independent, and that the elements in the set $\{v_iP^T(W^TX|Y = i) + \lambda_iP^T(W^TX|Y = i); i \in \{1, \cdots, c\}; \forall v_i, \lambda_i(v_i^2 + \lambda_i^2 \neq 0)\}$ are linearly independent. Then, if $P_{X'}^{new} = P_{X}^{T}$, we have $P_{X'|Y}^{T} = P_{X'|Y}^{S}$ and $\beta(Y = y) = \sum_{y'} P_{\hat{Y} = y'|Y = y}\beta_{\rho}(\hat{Y} = y'), \forall y, y' \in \{1, \cdots, c\}$, where $\beta(Y = y) = P_{Y = y}/P_{S}(Y = y)$. 

Proof. In this proof, $Y = y$ (resp. $\hat{Y} = y'$) is replaced by $y$ (resp. $y'$) for simplicity. For example, we let $P_{S}(\hat{Y} = y'|Y = y) = P_{S}(y'|y)$. According to Eq. (2), we have 

$$P_{X'}^{new} = \sum_{y} \sum_{y'} \beta_{\rho}(y')P_{S}(y'|X')P_{S}(X') = \sum_{y} P_{S}(X'|y)P_{S}(y)\sum_{y'} P_{S}(y'|y)\beta_{\rho}(y').$$

Because $P_{X'}^{T} = \sum_{y} P_{T}(X'|y)P_{T}(y)$, then combining with the above equation, we have 

$$\sum_{y} P_{T}(X'|y)P_{T}(y) = \sum_{y} P_{S}(X'|y)P_{S}(y)\sum_{y'} P_{S}(y'|y)\beta_{\rho}(y').$$

Because the linear transformation $W$ satisfies that $P(W^TX|Y = i), i \in \{1, \cdots, c\}$ are linearly independent, there exist no such non-zero $\gamma_1, \cdots, \gamma_c$ and $v_1, \cdots, v_c$ that $\sum_{i=1}^{c} \gamma_iP_{S}(X'|Y = i) = 0$ and $\sum_{i=1}^{c} v_iP_{T}(X'|Y = i) = 0$. According to the assumption in Theorem 1, the elements in the set $\{v_iP_{S}(W^TX|Y = i) + \lambda_iP_{T}(W^TX|Y = i); i \in \{1, \cdots, c\}; \forall v_i, \lambda_i(v_i^2 + \lambda_i^2 \neq 0)\}$ are also linearly independent. Then we have, $\forall y \in \{1, \cdots, c\}$, 

$$P_{T}(X'|y)P_{T}(y) - P_{S}(X'|y)P_{S}(y)\sum_{y'} P_{S}(y'|y)\beta_{\rho}(y') = 0.$$ 

Taking the integral of above equation w.r.t. $X'$, we have 

$$P_{T}(y) = P_{S}(y)\sum_{y'} P_{S}(y'|y)\beta_{\rho}(y'),$$

(A.1)

which further implies $P_{T}(X'|y) = P_{S}(X'|y), \forall y \in \{1, \cdots, c\}$. According to Eq. (A.1), we have 

$$\sum_{y'} P_{S}(y'|y)\beta_{\rho}(y') = P_{T}(y)/P_{S}(y) = \beta(y), \forall y \in \{1, \cdots, c\}.$$ 

The proof of Theorem 1 ends. 

B Proof of Theorem 2

Recall the objective function in Eq. (5) 

$$\hat{D}(W, \alpha) = \| \frac{1}{m}\psi(W^TX)G\alpha - \frac{1}{n}\psi(W^Ta)x1\|^2.$$
Let
\[ \mathcal{D}(W, \alpha) = \left\| \mathbb{E} \frac{1}{m} \psi(W^T x^S)G\alpha - \mathbb{E} \frac{1}{n} \psi(W^T x^T)1 \right\|^2, \]
where we abuse the training samples \{(x_1^S, \hat{y}_1^S), \ldots, (x_m^S, \hat{y}_m^S)\} and \{x_1^T, \ldots, x_n^T\} as being i.i.d. variables, respectively.

We analyze the convergence property of the learned \( \hat{\alpha} \) to the true one \( \alpha^* \) by analyzing the convergence from the expected objective function \( \mathcal{D}(\hat{W}, \hat{\alpha}) \) to \( \mathcal{D}(W, \alpha^*) \).

**Theorem 2.** Given learned \( \hat{Q} \) and \( \hat{W} \), let the induced RKHS be universal and upper bounded that \( \|\psi(\tau(x))\| \leq \wedge W \) for all \( x \) in the source and target domains, and let the entries of \( G \) be bounded that \( |G_{ij}| \leq \wedge Q \) for all \( i \in \{1, \ldots, m\}, j \in \{1, \ldots, c\} \). For any \( \delta > 0 \), with probability at least \( 1 - \delta \), we have
\[ \mathcal{D}(\hat{W}, \hat{\alpha}) - \mathcal{D}(\hat{W}, \alpha^*) \leq 4(\wedge Q + 1)^2 \wedge W^2 \sqrt{\frac{8\sqrt{c}}{\sqrt{m}} + \frac{8\sqrt{c}}{\sqrt{n}} + \frac{8(\frac{1}{m} + \frac{1}{n})\log \frac{1}{\delta}}{}}. \]

To prove Theorem 2, we need the following. Theorem 3.1, Lemma B.1 and Lemma B.2. Theorem 3.1 is about concentration inequality (McDiarmid’s inequality [4], also known as the bounded difference inequality). Lemma B.1 shows that the distance \( \mathcal{D}(\hat{W}, \hat{\alpha}) - \mathcal{D}(\hat{W}, \alpha^*) \) can be upper bounded even though we do not know the true \( \alpha^* \). Lemma B.2 upper bounds the Rademacher-like [3] term \( \mathbb{E} \sup_{\alpha \in \Delta} \| f(x^S, x^T, \alpha) \|^2 \).

**Theorem B.1.** Let \( X = [X_1, \ldots, X_n] \) be an independent and identically distributed sample and \( X^i \) a new sample with the \( i \)-th example in \( X \) being replaced by an independent example \( X'_i \). If there exists \( b_1, \ldots, b_n > 0 \) such that \( f : X^n \rightarrow \mathbb{R} \) satisfies the following conditions
\[ |f(X) - f(X^i)| \leq b_i, \forall i \in \{1, \ldots, n\}. \]
Then for any \( X \in X^n \) and \( \epsilon > 0 \), the following inequality holds
\[ P(\mathbb{E}f(X) - f(X) \geq \epsilon) \leq \exp \left( \frac{-2\epsilon^2}{\sum_{i=1}^{n} b_i^2} \right). \]

**Lemma B.1.** We denote \( \Delta \triangleq \{ \alpha | \alpha \geq 0, \| \alpha \|_1 = 1 \} \) and
\[ f(x^S, x^T, \alpha) \triangleq \mathbb{E} \left( \frac{1}{m} \psi(W^T x^S)G\alpha - \frac{1}{n} \psi(W^T x^T)1 \right) - \frac{1}{m} \psi(W^T x^S)G\alpha + \frac{1}{n} \psi(W^T x^T)1. \]

We have
\[ \mathcal{D}(\hat{W}, \hat{\alpha}) - \mathcal{D}(\hat{W}, \alpha^*) \leq 2 \sup_{\alpha \in \Delta} |\mathcal{D}(\hat{W}, \alpha) - \mathcal{D}(\hat{W}, \alpha)| \leq 4(\wedge Q + 1) \wedge W \sup_{\alpha \in \Delta} \| f(x^S, x^T, \alpha) \|. \]
Proof. We have
\[
\mathcal{D}(\hat{W}, \hat{\alpha}) - \mathcal{D}(\hat{W}, \alpha^*) \\
= \mathcal{D}(\hat{W}, \hat{\alpha}) - \mathcal{D}(\hat{W}, \hat{\alpha}) + \mathcal{D}(\hat{W}, \hat{\alpha}) - \mathcal{D}(\hat{W}, \alpha^*) + \mathcal{D}(\hat{W}, \alpha^*) - \mathcal{D}(\hat{W}, \alpha^*) \\
\leq \mathcal{D}(\hat{W}, \hat{\alpha}) - \mathcal{D}(\hat{W}, \hat{\alpha}) + \mathcal{D}(\hat{W}, \alpha^*) - \mathcal{D}(\hat{W}, \alpha^*) \\
\leq 2 \sup_{\alpha \in \Delta} |\mathcal{D}(\hat{W}, \alpha) - \mathcal{D}(\hat{W}, \alpha)|,
\]
where the first inequality holds because \(\hat{\alpha}\) is the empirical minimizer of \(\mathcal{D}(\hat{W}, \alpha)\) and thus \(\mathcal{D}(\hat{W}, \hat{\alpha}) \leq \mathcal{D}(\hat{W}, \alpha^*)\).

Further, we have
\[
|\mathcal{D}(\hat{W}, \alpha) - \mathcal{D}(\hat{W}, \alpha)| \\
= \left(\mathbb{E} \left( \frac{1}{m} \psi(W^T x^S)G\alpha - \frac{1}{n} \psi(W^T x^T)1 \right) + \frac{1}{m} \psi(W^T x^S)G\alpha - \frac{1}{n} \psi(W^T x^T)1 \right)^T \\
\leq 2(\wedge \hat{Q} + 1)\psi_{\text{max}} \\
\leq 2(\wedge \hat{Q} + 1)\wedge \hat{W} \\
\left| \left| \mathbb{E} \left( \frac{1}{m} \psi(W^T x^S)G\alpha - \frac{1}{n} \psi(W^T x^T)1 \right) - \frac{1}{m} \psi(W^T x^S)G\alpha + \frac{1}{n} \psi(W^T x^T)1 \right| \right|,
\]
where the last inequality holds because of Cauchy-Schwarz inequality.

Since
\[
f(x^S, x^T, \alpha) \triangleq \mathbb{E} \left( \frac{1}{m} \psi(W^T x^S)G\alpha - \frac{1}{n} \psi(W^T x^T)1 \right) - \frac{1}{m} \psi(W^T x^S)G\alpha + \frac{1}{n} \psi(W^T x^T)1,
\]
we have
\[
2 \sup_{\alpha \in \Delta} |\mathcal{D}(\hat{W}, \alpha) - \mathcal{D}(\hat{W}, \alpha)| \leq 4(\wedge \hat{Q} + 1)\wedge \hat{W} \sup_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\|.
\]
The proof ends.

Lemma B.2. Given learned \(\hat{Q}\) and \(\hat{W}\), let the induced RKHS be universal and upper bounded that \(\|\psi(\tau(x))\| \leq \wedge \hat{W}\) for all \(x\) in the source and target domains. Let the entries of \(G\) be bounded that \(|G_{ij}| \leq \wedge \hat{Q}\) for all \(i \in \{1, \cdots, m\}, j \in \{1, \cdots, c\}\). We have
\[
\mathbb{E} \sup_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\|^2 \leq 8(\wedge \hat{Q} + 1)^2 \wedge \hat{W} \sqrt{c} \left( \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}} \right).
\]

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Proof. Recall that when \( \hat{y}_k = i, \forall k \in \{1, \cdots, m\} \), the \( k \)-th row of \( G \in \mathbb{R}^{m \times c} \) is \( \frac{Q_{ij}^{-1}}{p^s_{(Y=1)}} \cdots \frac{Q_{ij}^{-1}}{p^s_{(Y=c)}} \). Given \( \hat{Q}, \hat{W} \) and the estimated \( \hat{P}^s(Y) \), we assumed that the entries of \( G \) is bounded, i.e., \( |G_{ij}| \leq \wedge \hat{Q} \), and that RKHS is upper bounded, i.e., \( -\psi_{\max} \leq \psi(\tau(x)) \leq \psi_{\max} \) and \( \|\psi_{\max}\| \leq \wedge \hat{W} \). Because \( \alpha \geq 0 \) and \( \|\alpha\|_1 = 1 \), we can conclude that for any training sample in the source domain, we have

\[
\|\frac{1}{m}\psi(W^T x^S)G\alpha\| \leq \wedge \hat{W} \wedge \hat{Q}.
\]

We there have \( \|f(x^S, x^T, \alpha)\| \leq 2(\wedge \hat{Q} + 1)\wedge \hat{W} \) and that

\[
\|f(x^S, x^T, \alpha)\|^2 \leq 2(\wedge \hat{Q} + 1)\wedge \hat{W} \|f(x^S, x^T, \alpha)\|.
\]

We then have

\[
\mathbb{E}_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\|^2 \leq 4(\wedge \hat{Q} + 1)\wedge \hat{W} \mathbb{E}_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\|. \tag{B.1}
\]

Furthermore, let \( x^{is} \) and \( x^{iT} \) be i.i.d. copies of \( x^S \) and \( x^T \), respectively. In the literature, \( x^{is} \) and \( x^{iT} \) are referred as ghost samples \[14]. We have

\[
\mathbb{E}_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\|
\]

\[
= \mathbb{E}_{\alpha \in \Delta} \left\| \mathbb{E} \left( \frac{1}{m}\psi(W^T x^S)G\alpha - \frac{1}{n}\psi(W^T x^T)1 \right) - \frac{1}{m}\psi(W^T x^S)G\alpha + \frac{1}{n}\psi(W^T x^T)1 \right\|
\]

\[
= \mathbb{E}_{x^S, x^T} \sup_{\alpha \in \Delta} \left\| \mathbb{E}_{x^{is}, x^{iT}} \left( \frac{1}{m}\psi(W^T x^{is})G\alpha - \frac{1}{n}\psi(W^T x^{iT})1 \right) - \frac{1}{m}\psi(W^T x^{is})G\alpha + \frac{1}{n}\psi(W^T x^{iT})1 \right\|
\]

\[
\leq \mathbb{E}_{x^S, x^T} \mathbb{E}_{x^{is}, x^{iT}} \sup_{\alpha \in \Delta} \left\| \left( \frac{1}{m}\psi(W^T x^{is})G\alpha - \frac{1}{n}\psi(W^T x^{iT})1 \right) - \frac{1}{m}\psi(W^T x^{is})G\alpha + \frac{1}{n}\psi(W^T x^{iT})1 \right\|, \tag{B.2}
\]

where the last inequality holds because of Jensen’s inequality and that every norm is a convex function.

Since \( x^{is} \) and \( x^{iT} \) be i.i.d. copies of \( x^S \) and \( x^T \), respectively, the random variable \( \frac{1}{m}\psi(W^T x^{is})G\alpha - \frac{1}{n}\psi(W^T x^{iT})1 - \frac{1}{m}\psi(W^T x^{is})G\alpha + \frac{1}{n}\psi(W^T x^{iT})1 \) is a symmetric random variable, which means its density function is even. Let \( \sigma_i \) be independent Rademacher variables, which are uniformly distributed from \( \{-1, 1\} \). Let

\[
\psi(W^T x^S, \sigma) \triangleq [\sigma_1 \psi(W^T x^S_1), \cdots, \sigma_m \psi(W^T x^S_m)]^T.
\]

Let

\[
\psi(W^T x^T, \sigma) \triangleq [\sigma_1 \psi(W^T x^T_1), \cdots, \sigma_m \psi(W^T x^T_m)]^T.
\]

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We have that the random variable \( \frac{1}{m} \psi(W^T x^S) G\alpha - \frac{1}{n} \psi(W^T x^T) 1 \) and the random variable \( \frac{1}{m} \psi(W^T x^S, \sigma) G\alpha - \frac{1}{n} \psi(W^T x^T, \sigma) 1 \) have the same distribution.

Then, we have

\[
\mathbb{E}_{x^S, x^T, x^S, x^T} \sup_{\alpha \in \Delta} \left\| \left( \frac{1}{m} \psi(W^T x^S) G\alpha - \frac{1}{n} \psi(W^T x^T) 1 \right) \right\| \leq \frac{1}{m} \psi(W^T x^S) G\alpha + \frac{1}{n} \psi(W^T x^T) 1 \]

\[
= \mathbb{E}_{x^S, x^T, x^S, x^T, \sigma} \sup_{\alpha \in \Delta} \left\| \left( \frac{1}{m} \psi(W^T x^S, \sigma) G\alpha - \frac{1}{n} \psi(W^T x^T, \sigma) 1 \right) \right\| \leq 2 \mathbb{E}_{x^S, x^T, \sigma} \sup_{\alpha \in \Delta} \left\| \frac{1}{m} \psi(W^T x^S, \sigma) G\alpha \right\| + 2 \mathbb{E}_{x^T, \sigma} \sup_{\alpha \in \Delta} \left\| \frac{1}{n} \psi(W^T x^T, \sigma) 1 \right\|,
\]

where the inequalities hold because of the triangle inequality.

We then upper bound \( \mathbb{E}_{x^S, \sigma} \sup_{\alpha \in \Delta} \left\| \frac{1}{m} \psi(W^T x^S, \sigma) G\alpha \right\| \) and \( \mathbb{E}_{x^T, \sigma} \left\| \frac{1}{n} \psi(W^T x^T, \sigma) 1 \right\| \), respectively. For example, we have

\[
\mathbb{E}_{x^S, \sigma} \sup_{\alpha \in \Delta} \left\| \frac{1}{m} \psi(W^T x^S, \sigma) G\alpha \right\| \leq \mathbb{E}_{x^S, \sigma} \sup_{\alpha \in \Delta} \frac{1}{m} \left\| G^T \left[ \sigma_1 \psi(W^T x^S_1), \ldots, \sigma_m \psi(W^T x^S_m) \right]^T, \alpha \right\|
\]

\[
\leq \mathbb{E}_{x^S, \sigma} \sup_{\alpha \in \Delta} \frac{1}{m} \left\| G^T \left[ \sigma_1 \psi(W^T x^S_1), \ldots, \sigma_m \psi(W^T x^S_m) \right]^T \right\|_1
\]

\[
\leq \mathbb{E}_{x^S, \sigma} \frac{1}{m} \left\| G^T \left[ \sigma_1 \psi(W^T x^S_1), \ldots, \sigma_m \psi(W^T x^S_m) \right]^T \right\|
\]

\[
\leq \frac{\hat{q} \hat{w}}{m} \mathbb{E}_\sigma \sqrt{\sum_{i=1}^m \sigma_i^2}
\]

\[
\leq \frac{\hat{q} \hat{w}}{m} \sqrt{\mathbb{E}_\sigma \left( \sum_{i=1}^m \sigma_i^2 \right)}
\]

\[
= \frac{\hat{q} \hat{w} \sqrt{c}}{\sqrt{m}},
\]

(B.4)
where $G \in \mathbb{R}^{m \times c}$, $c$ is the number of classes. The first inequality holds because of Cauchy-Schwarz inequality. The second inequality holds because $\|\alpha\| \leq \|\alpha\|_1$. The fourth inequality holds because of the Talagrand Contraction Lemma [10]. And the last inequality holds because of the Jensen’s inequality and that the function sqrt is a concave function. Similarly, we can prove that

$$\mathbb{E}_{x^T, \sigma} \left\| \frac{1}{n} \psi(W^T x^T, \sigma) 1 \right\| \leq \frac{\Lambda W}{\sqrt{n}}. \quad \text{(B.5)}$$

Combining Eq. (B.1), Eq. (B.2), Eq. (B.3), Eq. (B.4), and Eq. (B.5), we have

$$\mathbb{E} \sup_{\alpha \in \Delta} \| f(x^S, x^T, \alpha) \|^2 \leq 8(\wedge Q + 1) \wedge W \left( \frac{\Lambda Q \wedge W \sqrt{c}}{\sqrt{m}} + \frac{\wedge W}{\sqrt{n}} \right) \leq 8(\wedge Q + 1) ^2 \wedge W \sqrt{c} \left( \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}} \right).$$

The proof of Lemma B.2 ends.

Now, we are ready to prove Theorem 2.

**Proof of Theorem 2.** According to Lemma B.1, we have

$$\mathcal{D}(\hat{W}, \hat{\alpha}) - \mathcal{D}(\hat{W}, \alpha^*) \leq 2 \sup_{\alpha \in \Delta} | \mathcal{D}(\hat{W}, \alpha) - \mathcal{D}(\hat{W}, \alpha) | \leq 4(\wedge Q + 1) \Lambda W \sup_{\alpha \in \Delta} \| f(x^S, x^T, \alpha) \|.$$

Since $\| f(x^S, x^T, \alpha) \| \geq 0$, it holds that $\sup_{\alpha \in \Delta} \| f(x^S, x^T, \alpha) \| = \sqrt{\sup_{\alpha \in \Delta} \| f(x^S, x^T, \alpha) \|^2}$.

Then, we will employ McDiarmid’s inequality to upper bound the defect $\sup_{\alpha \in \Delta} \| f(x^S, x^T, \alpha) \|^2$.

We now check its bounded difference property.

Let $x^{Si}$ be a new sample in the source domain with the $i$-th example in $x^S$ being replaced by an independent example $x^{Ti}$, where $i \in \{1, \cdots, m\}$, and $x^{Ti}$ be a new sample in the source domain with the $i$-th example in $x^T$ being replaced by an independent example $x^{Ti}$, where $i \in \{1, \cdots, n\}$.

For any $i \in \{1, \cdots, m\}$, we have

$$\left| \sup_{\alpha \in \Delta} \| f(x^{Si}, x^T, \alpha) \|^2 - \sup_{\alpha \in \Delta} \| f(x^S, x^T, \alpha) \|^2 \right| \leq \sup_{\alpha \in \Delta} \left| \left( f(x^{Si}, x^T, \alpha) + f(x^S, x^T, \alpha) \right)^\top \left( f(x^{Si}, x^T, \alpha) - f(x^S, x^T, \alpha) \right) \right|$$

$$= \sup_{\alpha \in \Delta} \left| 2(\wedge Q + 1)\psi_{\max}^\top \left( f(x^{Si}, x^T, \alpha) - f(x^S, x^T, \alpha) \right) \right|$$

$$= \sup_{\alpha \in \Delta} \left| 2(\wedge Q + 1)\psi_{\max}^\top \left( \frac{1}{m} \psi(W^T x^{Si}) G\alpha - \frac{1}{m} \psi(W^T x^S) G\alpha \right) \right|$$

$$\leq \frac{4 \wedge Q (\wedge Q + 1)}{m} \left| \psi_{\max} \right|^\top \left| \psi_{\max} \right|$$

$$\leq \frac{4(\wedge Q + 1)^2 \wedge W}{m}.$$
Similarly, for any $i \in \{1, \cdots, n\}$, we have
\[
\sup_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\|^2 - \sup_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\|^2 \leq \sup_{\alpha \in \Delta} |(f(x^S, x^T, \alpha) + f(x^S, x^T, \alpha))^\top (f(x^S, x^T, \alpha) - f(x^S, x^T, \alpha))|
\]
\[
\leq \sup_{\alpha \in \Delta} \left|2(\hat{\chi} + 1)\psi_{\text{max}}^T (f(x^S, x^T, \alpha) - f(x^S, x^T, \alpha))\right|
\]
\[
= \sup_{\alpha \in \Delta} \left|2(\hat{\chi} + 1)\psi_{\text{max}}^T \left(\frac{1}{n} \psi(W^T x^T)1 - \frac{1}{n} \psi(W^T x^T)1\right)\right|
\]
\[
\leq \frac{4(\hat{\chi} + 1)}{n} |\psi_{\text{max}}|^2 |\psi_{\text{max}}|
\]
\[
\leq \frac{4(\hat{\chi} + 1)}{n} \Delta_w^2.
\]

Employing McDiarmid's inequality, we have that
\[
P(\sup_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\|^2 - \mathbb{E}_{x^S, x^T} \sup_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\|^2 \geq \epsilon) \leq \exp \left(\frac{-\epsilon^2}{8(\hat{\chi} + 1)^4 \Delta_w^4 \left(\frac{1}{m} + \frac{1}{n}\right)}\right).
\]

Let
\[
\delta = \exp \left(\frac{-\epsilon^2}{8(\hat{\chi} + 1)^4 \Delta_w^4 \left(\frac{1}{m} + \frac{1}{n}\right)}\right).
\]

For any $\delta > 0$, with probability at least $1 - \delta$, we have
\[
\sup_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\| \leq \sqrt{\mathbb{E} \sup_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\|^2 + 4(\hat{\chi} + 1)^2 \Delta_w^2 \sqrt{\frac{1}{2} \left(\frac{1}{m} + \frac{1}{n}\right) \log \frac{1}{\delta}}}.
\]

Combining the above inequality with those in Lemma B.1 and Lemma B.2, we have
\[
\mathcal{D}(\hat{\alpha}) - \mathcal{D}(\hat{\alpha}^*) \\
\leq 2 \sup_{\alpha \in \Delta} |\mathcal{D}(\hat{\alpha}) - \mathcal{D}(\hat{\alpha}^*)|
\]
\[
\leq 4(\hat{\chi} + 1) \Delta_w \sup_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\|
\]
\[
\leq 4(\hat{\chi} + 1) \Delta_w \sqrt{\mathbb{E} \sup_{\alpha \in \Delta} \|f(x^S, x^T, \alpha)\|^2 + 4(\hat{\chi} + 1)^2 \Delta_w^2 \sqrt{\frac{1}{2} \left(\frac{1}{m} + \frac{1}{n}\right) \log \frac{1}{\delta}}}.
\]
\[
\leq 4(\hat{\chi} + 1)^2 \Delta_w^2 \sqrt{8(\hat{\chi} + 1)^2 \Delta_w^2 \sqrt{\frac{1}{m} + \frac{1}{n}} + 4(\hat{\chi} + 1)^2 \Delta_w^2 \sqrt{\frac{1}{2} \left(\frac{1}{m} + \frac{1}{n}\right) \log \frac{1}{\delta}}}.
\]
which concludes the proof of Theorem 2.
C Learning a Classifier for the Target Domain

Suppose that the target domain has the same label noise model as the source domain and that we can learn a classifier $f^* = \arg\min_f \int \ell(X', Y, f) P^T(X', Y)dX'dY$ to predict $P^T(Y|X')$. Then, we could obtain $P^T(Y|X')$ because $[P^T(Y = 1|X'), \ldots, P^T(Y = c|X')]Q = [P^T(Y = 1|X'), \ldots, P^T(Y = c|X')]$ and that $Q$ is invertible. The problem then remains to learn the classifier $f^*$.

We have

$$f^* = \arg\min_f \int \ell(X', Y, f) P^T(X', Y)dX'dY$$

$$= \arg\min_f \int \frac{P^T(X', Y)}{P^S(X', Y)} \ell(X', Y, f) P^S(X', Y)dX'dY$$

$$= \arg\min_f \int \frac{P^T(X'|Y) P^T(Y)}{P^S(X'|Y) P^S(Y)} \ell(X', Y, f) P^S(X', Y)dX'dY.$$  

For the conditional invariant representations $X'$, we have $P^T(X'|Y) = P^S(X'|Y)$. Since we assumed that the target domain has the same label noise model as the source domain, we have $P^T(X'|Y) = P^S(X'|Y)$. Thus, we have

$$f^* = \arg\min_f \int \frac{P^T(Y)}{P^S(Y)} \ell(X', Y, f) P^S(X', Y)dX'dY.$$  

It is held that $[P^T(Y = 1), \ldots, P^T(Y = c)]Q = [P^T(Y = 1), \ldots, P^T(Y = c)]$. In Section 4.1, we also have that $\alpha$ is an estimation of $[P^T(Y = 1), \ldots, P^T(Y = c)]^\top$. Then, $[P^T(Y = 1), \ldots, P^T(Y = c)] = \alpha^\top Q$. Since we have $\alpha^\top Q$ and the source domain sample $\{(x_1^S, \hat{y}_1^S), \ldots, (x_m^S, \hat{y}_m^S)\}$ at hand, we can estimate $f^*$ by employing empirical risk minimization algorithms [14].
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