Development of the distinct lattice spring model with polyhedral particles

Zhe Li and Gao-Feng Zhao*
State Key Laboratory of Hydraulic Engineering Simulation and Safety, School of Civil Engineering, Tianjin University, Tianjin, 300350, China

Corresponding author: gaofeng.zhao@tju.edu.cn

Abstract. Distinct Lattice Spring Model (DLSM) is a microscopic model to overcome the mismatch problem of degrees of freedom when coupling between discrete element method (DEM) and finite element method (FEM). In the original DLSM, the lattice model is composed of a number of same spherical particles, which has the problem of inaccurate geometric description. A polyhedral 3D DLSM was developed to overcome some limitations of the original DLSM. A new pre- and post-processing method is introduced to extend the ability of DLSM on precise geometry description. Following this, a pre-processing method is proposed to generate the lattice model from a 3D Finite Element (FE) Mesh or a 3D Numerical Manifold Method (NMM) Mesh. It is required that the generated mesh can present the precise 3D geometry and consider the spatial morphology of joint surfaces. The 3D FE mesh is generated by using the general mesh software. The FE mesh is generated according to the free surfaces and the joint surfaces defined in the model. To get the 3D NMM mesh, a 3D surface mesh Boolean algorithm is developed. The surface mesh Boolean algorithm is used for solving the Boolean operations between mathematic covers and physical covers in NMM. By using the algorithm, the 3D closed geometry cut by 3D surface and the union of multiple 3D surfaces can be solved. The 3D NMM mesh is ultimately generated through the Boolean intersection of mathematical covers and physical covers, the physical covers include the free surfaces and joint faces. The polyhedral particles generated by the FE mesh or NMM mesh have different shapes and sizes, the springs between polyhedral particles will also have different lengths. The effectiveness, necessity, and correctness of these enrichments are demonstrated from a number of specially designed numerical examples.

1. Introduction

Due to advantages on repeatability, economic, ease of data accessing, and predictability, numerical methods have been widely used in scientific research and engineering application. Nowadays, many numerical methods have been developed for various scientific problems from simulation of black hole to atomic behavior of materials. Jing [1] conducted a comprehensive review on numerical methods for rock mechanics and rock engineering, in which these methods were categorised as continuum-based models (CbMs), discontinuum-based models (DbMs) and hybrid model according to their abilities of handling discontinuous media. However, the barrier between the DbMs and CbMs becomes vague for many newly developed models. It is hard to distinguish the Finite Element Method (FEM)/Smoothed Particle Hydrodynamics (SPH) and Discrete Element Model (DEM) from its computational model and ability of solving discontinuous media. Contact detection and contact treatment were believed as the
distinct features of DbMs, such as DEM developed by Cundall [2] and the Discontinuous Deformation Analysis (DDA) developed by Shi [3]. The Lattice Spring Model (LSM) developed by Hrennikoff [4] was initially to solve partial differential equations (PDEs) of elasticity at a fixed Poisson's ratio (1/3). It is a CbM when it is used to discrete/approximate the PDEs. In meantime, it can also be viewed as a DbM to reconstruct the mechanical response from springs. There are two steps for a CbM to describe the physical world: a) build up the PDEs; and b) discrete/approximate PDEs, whereas, DbM has only one step that is to discrete/approximate the physical world directly. This article will focus on a fully DbM.

DbMs have been applied to solve many problems of rock mechanics and rock engineering. However, there is lack of solid and systematic theoretical development of DbMs compared with CbMs. DbMs are still regarded as undeveloped and un-proven methods by many researchers. For example, DEM has been successfully applied to model fracturing problems of rock and can obtain realistic failure patterns under most conditions. The ability of DEM on modelling pre-failure stage of the rock is not well studied. The Poisson's ratio limitation of LSM (Hrennikoff 1941) is still partially inherited in DEM. The main purpose of this article is to provide a positive message of DbMs. It covers the latest development of the Distinct Lattice Spring Model (DLSM). DLSM directly discrete the physical world as a group of particles linked through spring bonds which is a fully DbM. If all the particles have a same shape, it is hard to represent the complicated physical world precisely. The advantages of FEM and Numerical Manifold Method (NMM) were introduced to DLSM in this work, in which the numerical model is built precisely and then discretized. In FEM, the tetrahedral mesh generation method is utilized to easily generate a tetrahedral mesh for arbitrary geometry and are widely used in geotechnical modeling [5]. Shewchuk [6] firstly reported a tetrahedral mesh generation method based on Delaunay refinement. For the NMM, it was originally proposed by Shi [7, 8, 9], which combines the advantages of both the continuum-based FEM and the discontinuum-based DDA through two cover systems. The NMM heavily considers the complex joint surfaces and the failures of materials that are common in geotechnical engineering. Due to the performance on crack propagation, NMM is widely used in rock dynamic crack analysis [10]. Kang et al [11] simulated the meso-scale deformation and damage behaviors of polymer bonded explosive using 2D-NMM. Yang et al [12] and Yang et al [13] analyzed 3D crack propagation by the 3D-NMM. The NMM is based on DDA, in which the basic elements are polyhedral blocks. Zheng et al [14] investigated all the contact conditions of polyhedral blocks in 3D-DDA. Wu et al [15] generated the 3D-DDA blocks by using three groups of joint faces. According to the literature review above, the polyhedral blocks are used as the basic elements in DDA. It can also represent the geometry as well. In this work, the methods of FE mesh and NMM mesh are choose to generate the DLSM model, the basic element in the model is actually a polyhedral particle. A brief overview of DLSM will be introduced first to show the principle and fundamental difference between CbMs. Then, FE mesh and NMM based pre-processing will be presented. Next, advantages of DLSM over CbMs will be highlighted through a number of numerical examples. The fundamental advantages of DLSM on modelling continuous deformation, crack initiation and dynamic cracking are illustrated from numerical examples.

2. Distinct Lattice Spring Model (DLSM)
DLSM was developed in early 2009 as a microscopic model to overcome the DOFs mismatch problem when coupling between DEM and FEM. The original description of DLSM in Zhao et al. (2011) [16] intended to present the model in an easy understanding way. In the following, a more fundamental description would be introduced.

2.1. Discretization
As shown in figure 1, the modelling target is represented as $\Omega$. The first step is to divide the domain into a number of sub domains $\Omega_i$. Mathematical relationship between sub domains and the main domain can be represented as
Figure 1. Discretization strategy adopted in DLSM.

\[ \Omega = \bigcup_{i=1}^{n} \Omega_i \]

\[ \Omega_i \cap \Omega_j = 0 \quad (2) \]

For a given sub domain \( \Omega_i \), its neighbors satisfy the following equation

\[ \forall \Omega_{j \neq i}, \| \Omega_i, \Omega_j \| < \delta \quad (3) \]

\( \| \Omega_i, \Omega_j \| \) is to calculate the distance (e.g., Euler distance) between two sub domains. A bond is formed between \( \Omega_i \) and its neighbors

\[ L = \{ \Omega_i, \Omega_j \} \quad (4) \]

\( L \) takes account the interaction between the two sub domains. For each sub-domain, its neighbors and itself made up a cloud as

\[ \Theta_i = \{ \Omega_i, \text{Neighbours}(\Omega_i) \} \quad (5) \]

The sub domain, bond, and cloud made up the geometric skeleton of DLSM.

2.2. Multi-body interaction

In DLSM, mechanical response of the physical model is represented as interaction of a group of bonds. Assume the deformation matrix of the bond is

\[ F = \begin{pmatrix}
1 + \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & 1 + \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & 1 + \frac{\partial w}{\partial z}
\end{pmatrix} \quad (6) \]

Which can be further decomposed as

\[ F = QU \quad (7) \]

in which \( Q \) is the rigid rotation matrix and \( U \) is the deformation matrix without rigid rotation. Under small deformation assumption, the rigid body rotation can be given as

\[ u(x) = \omega \times x \quad (8) \]

where \( \omega \) is the angular displacement vector with components \( [\omega_x, \omega_y, \omega_z]^T \). The true gradient of this displacement field is
The calculated strain is
\[
\varepsilon = \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}{2}
\]  
(10)
Finally, the shear deformation of the bond can be obtained as
\[
\hat{\mathbf{u}}_s = \left[ \varepsilon \right] \cdot \mathbf{n} - \left( \left[ \varepsilon \right] \cdot \mathbf{n} \right) \cdot \mathbf{n}
\]  
(11)
The normal deformation is
\[
\hat{\mathbf{u}}_n = \left( \left[ \varepsilon \right] \cdot \mathbf{n} \right) \cdot \mathbf{n}
\]  
(12)
Which can also be calculated through
\[
\mathbf{u}_n = \left( \left( \mathbf{u}_j - \mathbf{u}_i \right) \cdot \mathbf{n} \right) \mathbf{n}
\]  
(13)
The local strain of the bond can only be approximated, therefore, in DLSM, equation (13) was used to calculate the normal deformation. However, the shear deformation can't be determined using two particle information, equation (11) was used instead. It was called as multi-body shear spring due it was determined from deformation states of the particles in the clouds of two linked particles. The bond strain is given as
\[
\left[ \varepsilon \right]_{bond} = \frac{\varepsilon_i + \varepsilon_j}{2}
\]  
(14)
The normal and shearing forces within a bond are calculated as
\[
\mathbf{F}_y^s = k_s \hat{\mathbf{u}}_y^s
\]  
(15)
\[
\mathbf{F}_y^n = k_n \mathbf{u}_y^n
\]  
(16)
Through strain energy analysis, micro-parameters of the DLSM can be derived from macro-parameters. The relationship between them can be calculated through
\[
k_n = \frac{3E}{\alpha_{3D}^M (1 - 2\nu)}
\]  
(17)
\[
k_s = \frac{3(1 - 4\nu)E}{\alpha_{3D}^M (1 + \nu)(1 - 2\nu)}
\]  
(18)
\[
\alpha_{3D} = \sum \frac{l^2}{V}
\]  
(19)
Therefore, using the DLSM to solve elastic problems, only macro-parameters, Elastic modulus, Poisson’s ratio, and density, are needed to input in the model.

3. Model generation
In DLSM, the numerical model is composed of a large number of particles bonded by springs. The particles actually participate in the motion cycle as mass points. Each mass point has volume and contact detection radius. The volume is the volume of the sub-domain represented by the mass point. The radius is used for contact detection between mass points. In the original DLSM, the physical model is divided into a large number of same cubes, and displayed as spherical particles to show the contact detection radius. This is effective for regular simple geometries which are composed of several planes, such as rock sample with cube shape. If the geometry is composed of some curved surfaces, such as a ball or a disc. When using cubes to represent the physical model, it cannot describe the model precisely and will cause large errors. The error can reduce by decreasing the elements size, but this may cause an
uneconomic computing performance. To enhance the ability of DLSM on precise geometry presentation, in this work, we tried two methods to generate the model with complex geometry.

3.1. Using FE mesh
Since the development of FEM, the method on mesh generation has also made great success. As shown in figure 2(a), a tetrahedral mesh of a ball is generated. To build the DLSM model, the mesh vertices are used as the mass points, the edges are used as the spring bonds, and form the initial contact list according to the edges.

\[ \mathbf{c}_i = \frac{\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4}{4} \]  

where \( \mathbf{p} \) is vertex position, \( \mathbf{c}_i \) is the mass center of tetrahedron. Then, mass centers of all faces can be calculated

\[
\begin{align*}
\mathbf{c}_f &= \frac{\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4}{4} \\
\mathbf{c}_{fl} &= \frac{\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3}{3} \\
\mathbf{c}_{fr} &= \frac{\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_4}{3} \\
\mathbf{c}_{fr} &= \frac{\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{p}_4}{3}
\end{align*}
\]  

where \( \mathbf{c}_f \), \( \mathbf{c}_{fl} \), \( \mathbf{c}_{fr} \), \( \mathbf{c}_{fr} \), are the mass center of \( \Omega_1 \), \( \Omega_2 \), \( \Omega_3 \), \( \Omega_4 \), respectively. The mass centers of all edges can be calculated

\[ \mathbf{c}_{ij} = 0.5(\mathbf{p}_i + \mathbf{p}_j), \quad (i = 1, 2, 3, 4 \text{ & } j = 1, 2, 3, 4 \text{ & } i \neq j) \]

in which \( \mathbf{c}_{ij} \) is the mass center of edge \( l_{\mathbf{p}_i \mathbf{p}_j} \). Finally, build the faces in the tetrahedron according to these mass centers as illustrated in figure 2(b). Then, the controlled sub-domain of each vertex is determined, the volume of a sub-domain can be calculated by simplex method. The sub-domains have the relationship

\[
\begin{align*}
\Omega &= \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \\
\Omega &= \Omega_i \cap \Omega_j \quad i, j = 1, 2, 3, 4 \text{ & } i \neq j
\end{align*}
\]  

where \( \Omega \) is the tetrahedron. Every tetrahedron in the FE mess is divided into 4 parts like above. All the sub-domains of the vertex constitute the control domain of the mass point. Generally, there will be two kinds of FE based elements, elements inside the model and elements on the surface, as illustrated in...
For each of the elements, the volume of the controlled domain is used, and the minimum distance from the mass point to the newly generated triangular patches is used as the contact detection radius. Finally, the polyhedral DLSM model can be generated from the FE mesh, as shown in figure 4.

![Polyhedron element in model](image1)

![Polyhedron element on surface](image2)

**Figure 3.** Generate polyhedron element from FE mesh.

**Figure 4.** Polyhedral DLSM model from FE mesh.

### 3.2. Using NMM mesh

The NMM combines the advantages of both the FEM and the DDA through two cover systems. The NMM considers joints and the failures of materials, which are common in geotechnical engineering. The NMM mesh do not need to coincide with the crack face and material boundary, which makes NMM mesh method very suitable for constructing DLSM model with complex geometrical boundaries. The whole process is mainly divided into three steps: a) Generate the basic DLSM model as the mathematical covers and generate the contact list through the existing method in DLSM; b) Generate the NMM mesh based on mathematical covers and physical covers (material geometry); c) Compute the elements’ volume and contact detection radius and generate the new polyhedral DLSM model and contact list.

To build a dense and stable DLSM model, the cubic-II lattice model is used as the basic model and the mathematical covers of NMM mesh, and the contact list is generated from the basic model, the discretization and lattice model are demonstrated in figure 5.

![Boolean intersection](image3)

**Figure 5.** Boolean intersection between the basic DLSM model and physical covers are calculated. In order to perform the Boolean operation of spatial geometry, a surface mesh consists of triangular faces with normal direction are utilized to represent the 3D geometries. And the method on intersection line tracing is introduced in this work [17, 18, 19]. Two common 3D geometries are illustrated in figure 6.
If $\Omega_1$ and $\Omega_2$ intersect, the intersection line between them can be found by traversing triangular faces of them. If two triangular faces $\Gamma_1$ and $\Gamma_2$ intersect, the intersection line $AB$ and the attribution are recorded. The relationship between them as shown in figure 7.

**Figure 5.** Basic DLSM model: Cube-II.

**Figure 6.** Geometries presented by triangular faces.

**Figure 7.** Relationship between intersecting line, triangular face, and polyhedron.

**Figure 8.** 2D Delaunay triangulation of a triangular face based on all intersecting lines.
After all the intersection lines are found, the triangular face which has the intersection lines will be re-meshed through 2D Delaunay triangulation method, as shown in figure 8. All the new faces will build the new surface meshes (see figure 9(a)), and they can be divided into 4 groups, \( \Gamma_{11} \) is faces of \( \Omega_1 \) outside \( \Omega_2 \), \( \Gamma_{12} \) is faces of \( \Omega_1 \) in \( \Omega_2 \), \( \Gamma_{22} \) is faces of \( \Omega_2 \) outside \( \Omega_1 \), \( \Gamma_{21} \) is faces of \( \Omega_2 \) in \( \Omega_1 \).

According to the face’s groups, the Boolean difference, Boolean intersection, and Boolean union are illustrated in figure 9(b, c, d). The minus superscript of faces means the faces with opposite direction.

![Figure 9. Boolean operations of 3D Polyhedron.](image)

For the NMM mesh, the Boolean intersection is used. As shown in figure 10, the elements on the surface of physical covers are irregular, and the internal elements are same as the basic model. If the result of Boolean intersection is null, the element and the springs bonds will be removed. For the irregular elements, the mass center of the polyhedron is used as the mass point position, and the minimum distance between the mass center to the boundary is used as the element contact detection radius.

![Figure 10. Polyhedral DLSM model from NMM mesh.](image)
The boundary elements may be extremely small than the basic cube. To reduce the influence of small elements on the numerical results and computational stability, the small elements are regard as part of large elements. For the element of which the volume is small enough (default threshold is 5% of basic cube), it will combine with the largest particle bonded with it. The volume of the small one will be added to the volume of the large one, they will have a same motion state and base information, the new lattice network model will be reformed using the modified model. Then, the polyhedral DLSM model can be generated from the NMM mesh.

4. Numerical Examples
According to the pre-processing methods above, the models of eccentric holed discs with different hole radius are built. The model discretization from FE and NMM mesh are demonstrated in figure 11. The top of the disc is applied a constant velocity loading and the displacement is fixed at the bottom. The radius of the disc is 25mm, the thickness is 4mm. For the small eccentric hole, the radius is 4mm, the distance from the hole center to the disc center is 13mm. For the large eccentric hole, the radius is 7mm, the distance between two center is 9mm.

**Figure 11.** Polyhedral DLSM model from FE mesh and NMM mesh.

**Figure 12.** Crack predictions by FE mesh and NMM mesh based DLSM.
The volume of the holed disc with the hole radius 7mm is 7238.23mm$^3$. For the polyhedral DLSM model based on FE mesh, the number of elements is 12564, the total volume of all the elements is 7238.12mm$^3$, the volume error is $-1.5 \times 10^{-3}$%. The average diameter of the FE based model is 0.7654mm, for the basic DLSM model generated with the average diameter, the number of elements is 15424, the total volume is 6916.12 mm$^3$ (error -4.45%). For the polyhedral DLSM model based on NMM mesh, the basic mathematical covers’ size is 0.9mm, the number of elements is 11840, the total volume is 7238.23mm$^3$, the volume error is $+2.1 \times 10^{-5}$%. The average diameter of the NMM based model is 0.7579mm, for the basic DLSM model generated with the average diameter, the number of elements is 15850, the total volume is 6900.25 mm$^3$ (error -4.67%). The volume of the holed disc with the hole radius 4mm is 7652.92mm$^3$. For the polyhedral DLSM model based on FE mesh, the number of elements is 13496, the total volume of all the elements is 7652.83mm$^3$, the volume error is $-1.1 \times 10^{-2}$%. The average diameter of the FE based model is 0.7996mm, for the basic DLSM model generated with the average diameter, the number of elements is 14965, the total volume is 7650.59 mm$^3$ (error -0.03%). For the polyhedral DLSM model based on NMM mesh, the same basic mathematical covers are used, the number of elements is 12420, the total volume is 7652.30mm$^3$, the volume error is $+8.1 \times 10^{-3}$%. The average diameter of the NMM based model is 0.7632mm, for the basic DLSM model generated with the average diameter, the number of elements is 16412, the total volume is 7295.86 mm$^3$ (error -4.66%). The volume errors and figure 11 show that both of the FE mesh and NMM mesh based DLSM can precisely describe the physical model well, and the correctness and accuracy of the Boolean algorithm of surface mesh are verified.

The numerical results show that all the numerical results show good agreements to the experiments (see figure 12). Compared the model generated from FE mesh with the model generated from NMM mesh, the crack predictions of them have some differences. Due to the unstructured tetrahedron mesh method are utilized to generate the FE mesh, the model is heterogeneous. The NMM mesh is based on cubic-II DLSM model, the elements in the model except on the surface are homogeneous. For the crack in the small eccentric holed disc, the NMM mesh based DLSM is more similar to the experience results, however, for the large eccentric holed disc, the FE mesh based DLSM is better. The differences between the results are due to the influence of mesh. When increasing the resolution of the model, the predictions of the two models will be more similar, and the predictions will also be more similar to the experiments.

5. Conclusions
In this work, the pre- and post-process methods of DLSM are developed based on FE and NMM mesh method. The mesh based DLSM can precisely describe arbitrary geometry in the 3D space. The numerical examples show the correctness of the mesh based DLSM. There is no need to improve the accuracy at the cost of computational performance. the DLSM solver is robust to the mesh methods. For the FE mesh based DLSM, unstructured tetrahedron mesh technology is utilized to generate FE mesh, the vertices of the mesh are regarded as the center of the DLSM elements, the edges are regarded as the lattice springs. The mass center of the tetrahedrons, faces and edges which connect to an element center are utilized to compute the control domain of the element. The elements generated from FE mesh are commonly complex polyhedron and are heterogeneous. For the NMM mesh based DLSM, the cubic-II model is regarded as the mathematical covers, the material is the physical covers. The intersection line tracing method is developed realize the Boolean operations of 3D geometry. The NMM mesh are the Boolean intersection result of mathematical covers and physical covers. Then, the blocks are utilized as the DLSM elements, the lattice springs are generated from the cube-II model. The accuracy and performance of mesh based DLSM are verified by modelling the Brazilian splitting test on eccentric holed discs.

Acknowledgments
This research is financially supported by the National Natural Science Foundation of China (Grant No. 11772221) and the Tianjin Municipal Science and Technology Bureau (Grant No. 19JCZDJC39400).
References
[1] Jing L, 2003. A review of techniques, advances and outstanding issues in numerical modelling for rock mechanics and rock engineering. *International Journal of Rock Mechanics and Mining Sciences*. 40(3), 283-353.
[2] Hart R, Cundall P, 1988. Formulation of a three-dimensional distinct element model—Part II. Mechanical calculations for motion and interaction of a system composed of many polyhedral blocks. *International Journal of Rock Mechanics and Mining Sciences & Geomechanics Abstracts*. 25(3), 117-125.
[3] Shi, G.H., Goodman, R.E., 1989. Generalization of two-dimensional discontinuous deformation analysis for forward modeling. *International Journal for Numerical and Analytical Methods in Geomechanics*. 13, 359-380.
[4] Hrennikoff A, 1941. Solution of problems of elasticity by the frame work method. *Journal of Applied Mechanics*. 8, A619-A715.
[5] Wang B-W, Mei G, Xu N-X, 2020. Method for generating high-quality tetrahedral meshes of geological models by utilizing CGAL. *MethodsX*. 7, 101061.
[6] Shewchuk J.R., 1998. Tetrahedral Mesh Generation by Delaunay Refinement. *ACM Press the fourteenth annual symposium*. Minneapolis, Minnesota, United States, 86-95.
[7] Shi G-H, 1991. Manifold method of material analysis. *Transactions of the 9th army conference on applied mathematics and computing*. Minneapolis, Minnesota, USA, 51-76.
[8] Shi G-H, 1992. Modeling rock joints and blocks by manifold method. *Rock mechanics: proceedings of the 33th US symposium*. American Rock Mechanics Association (ARMA), USA, 639-648.
[9] Shi G-H, 1995. Numerical manifold method. *Proceedings of the 1st international conference on analysis of discontinuous deformation (ICADD-1)*. Chungli, Taiwan, China, 187-222.
[10] Ma G-W, An X-M, He L, 2010. The numerical manifold method: a review. *International Journal of Computational Methods*. 7(1), 1-32.
[11] Kang G, Chen P-W, Guo X, Ma G-W, Ning Y-J, 2018. Simulations of meso-scale deformation and damage of polymer bonded explosives by the numerical manifold method. *Engineering Analysis with Boundary Elements*. 96, 123-137.
[12] Yang S-K, Cao M-S, Ren X-H, Ma G-W, Zhang J-X, Wang H-J, 2018. 3D crack propagation by the numerical manifold method. *Computers and Structures*. 194, 116-129.
[13] Yang Y-T, Tang X-H, Zheng H, Liu Q-S, He L, 2016. Three-dimensional fracture propagation with numerical manifold method. *Engineering Analysis with Boundary Elements*. 72, 65-77.
[14] Zheng F, Jiao Y-Y, Leung Y-F, Zhu J-B, 2018. Algorithmic robustness for contact analysis of polyhedral blocks in discontinuous deformation analysis framework. *Computers and Geotechnics*. 104, 288-301.
[15] Wu W, Zhu H-H, Lin J-S, Zhuang X-Y, Ma G-W, 2018. Tunnel stability assessment by 3D DDA-key block analysis. *Tunnelling and Underground Space Technology*. 71, 210-214.
[16] Zhao G-F, Fang J, Zhao J, 2011. A 3D distinct lattice spring model for elasticity and dynamic failure. *International Journal for Numerical and Analytical Methods in Geomechanics*. 35, 859-885.
[17] Guo K-B, Zhang L-C, Wang C-J, Huang S-H, 2006. Implementation of boolean operations on STL models. *Journal of Huazhong University of Science and Technology (Nature Science Edition)*. 34(7), 96-99.
[18] Wang H-J, Kan S-T, Zhang X-L, Lu X-M, Zhou L-Q, 2018. Robust Boolean operations algorithm on regularized triangular mesh and implementation. *Multimedia Tools and Applications*. 79, 5301-5320.
[19] Zhang Q-H, Su H-D, Lin S-Z, Shi G-H, 2020. Algorithm for three-dimensional curved block cutting analysis in solid modeling. *Computer Methods in Applied Mechanics and Engineering*. 360, 112721.
[20] Ma X, Zhou B, Xue S-F, 2021. A meshless Hermite weighted least-square method for
piezoelectric structures. *Applied Mathematics and Computation*. **400**, 126073.

[21] Nakamura K, Matsumura S, Mizutani T, 2021. Particle-to-surface frictional contact algorithm for material point method using weighted least squares. *Computers and Geotechnics*. **134**, 104069.

[22] Tang L-W, Lu Y-Y, 2020. Study of the grey Verhulst model based on the weighted least square method. *Physica A*. **545**, 123615.