A MAXIMAL BOOLEAN SUBLATTICE THAT IS NOT THE RANGE OF A BANASCHEWSKI FUNCTION

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Dedicated to Jara Cimrman on the occasion of his 50th birthday.

Abstract. We construct a countable bounded sublattice of the lattice of all subspaces of a vector space with two non-isomorphic maximal Boolean sublat-
tice. We represent one of them as the range of a Banschewski function and we prove that this is not the case of the other. Hereby we solve a problem of F. Wehrung.

1. Introduction

In [14] Friedrich Wehrung defined a Banaschewski function on a bounded com-
plemented lattice \( L \) as an antitone (i.e., order-reversing) map sending each element of \( L \) to its complement, being motivated by the earlier result of Bernhard Ba-

naschewski that such a function exists on the lattice of all subspaces of a vector space [1]. Wehrung extended Banaschewski’s result by proving that every countable complemented modular lattice has a Banaschewski function with a Boolean range and that all the possible ranges of Banaschewski functions on \( L \) are isomorphic [14, Corollary 4.8].

Still in [14] Wehrung defined a ring-theoretical analogue of Banaschewski func-
tion that, for a von Neumann regular ring \( R \), is closely connected to the lattice-

theoretical Banaschewski function on the lattice \( L(R) \) of all finitely generated right ideals of \( R \). He made use of these ideas to construct a unit-regular ring \( S \) (in fact of bounded index 3) of size \( \aleph_1 \) with no Banaschewski function [15].

Furthermore in [14] Wehrung defined notions of a Banaschewski measure and a Banaschewski trace on sectionally complemented modular lattices and he proved that a sectionally complemented lattice which is either modular with a large 4-
frame or Arguesian with a large 3-frame is coordinatizable (i.e. isomorphic to \( L(R) \) for a possibly non-unital von Neumann regular ring \( R \)) if and only if it has a Banaschewski trace. Applying this results, he constructed a non-coordinatizable sectionally complemented modular lattice, of size \( \aleph_1 \), with a large 4-frame [14, Theorem 7.5].

The aim of our paper is to solve the second problem from [14].
**Problem** (Problem 2 from [14]). *Is every maximal Boolean sublattice of an at most countable complemented modular lattice $L$ the range of some Banaschewski function on $L$? Are any two such Boolean sublattices isomorphic?*

We construct a countable complemented modular lattice $S$ with two non-isomorphic maximal Boolean sublattices $B$ and $E$. We represent $E$ as the range of a Banaschewski function on $S$ and we prove that $B$ is not the range of any Banaschewski function. Finally we represent the lattice $S$ as a bounded sublattice of the subspace-lattice of a vector space.

2. **Basic concepts**

We start with recalling same basic notions as well as the precise definition of the Banaschewski function adopted from [14]. Next we outline the Schmidt’s $M_3[L]$ construction, which we then apply to define the bounded modular lattice $S$ containing a pair of non-isomorphic maximal Boolean sublattices.

2.1. **Some standard notions, notation, and the Banaschewski function.** A lattice $L$ is *bounded* if it has both the least element and the greatest element, denoted by $0_L$ and $1_L$, respectively. A bounded sublattice of a bounded lattice is its sublattice containing the bounds. Given elements $a, b, c$ of a lattice $L$ with zero, we will use the notation $c = a \oplus b$ when $a \land b = 0_L$ and $a \lor b = c$. A complement of an element $a$ of a bounded lattice $L$ is an element $a'$ of $L$ such that $a \oplus a' = 1_L$. A lattice $L$ is said to be *complemented* provided that it is bounded and each element of $L$ has a (not necessarily unique) complement. A lattice $L$ is *relatively complemented* if each of its interval is complemented. Note that a relatively complemented lattice is not necessarily bounded.

We say that a lattice $L$ is *uniquely complemented* if it is bounded and each element of $L$ has a unique complement. By a Boolean lattice we mean a lattice reduct of a Boolean algebra, that is, a distributive uniquely complemented lattice. For the clarity, let us recall the formal definition of the Banaschewski function [14, Definition 3.1]:

**Definition 2.1.** Let $L$ be a bounded lattice. A Banaschewski function on $L$ is a map $f: L \to L$ such that both

1. $x \leq y$ implies $f(x) \geq f(y)$, for all $x, y \in L$, and
2. $f(x) \oplus x = 1_L$ for all $x \in L$,

hold true.

2.2. **The $M_3[L]$-construction.** Let $L$ be a lattice. We will call a triple $\langle a, b, c \rangle \in L^3$ balanced, if it satisfies

$$a \land b = a \land c = b \land c$$

and we denote by $M_3[L]$ the set of all balanced triples. It is readily seen that $M_3[L]$ is a meet-subsemilattice of the cartesian product $L^3$. However, it is not necessarily a join-subsemilattice, for one easily observes that the join of balanced triples may not be balanced. The $M_3[L]$-construction was introduced by E. T. Schmidt [12, 13] for a bounded distributive lattices $L$. He proved [13, Lemma 1] that in this case $M_3[L]$ is a bounded modular lattice and that it is a congruence-preserving extension of the distributive lattice $L$. This result was later extended by Grätzer and Schmidt in various directions [2, 3]. In particular, in [2] they proved
that every lattice with a non-trivial distributive interval has a proper congruence-preserving extension. This was further improved by Grätzer and Wehrung in [7], where they introduced a modification of the $M_3(L)$-construction, called $M_3(L)$-construction. Using this new idea they proved that every non-trivial lattice admits a proper congruence-preserving extension.

The lattice constructions $M_3(L)$ and $M_3(L)$ appeared in the series of papers by Grätzer and Wehrung [4, 5, 6, 7, 8, 9, 10] dealing with semilattice tensor product and its related structures, namely the box product and the lattice tensor product [6, Definition 2.1 and definition 3.3]. Indeed, $M_3 \boxtimes L \simeq M_3(\ast)$ for every lattice $L$ and $M_3 \otimes L \simeq M_3(L)$ whenever $L$ has zero and $M_3 \otimes L$ is a lattice (see [10, Theorem 6.5] and [5, Corollary 6.3]). In particular, the latter is satisfied when the lattice $L$ is modular with zero. Note also, that if $L$ is a bounded distributive lattice both the constructions $M_3(L)$ and $M_3(L)$ coincide. In our paper we get by with this simple case.

Let $L$ be a distributive lattice. Given a triple $(a, b, c) \in L^3$, we define

$$\mu(a, b, c) = (a \land b) \lor (a \land c) \lor (b \land c)$$

and we set

$$(a, b, c) = (a \lor \mu(a, b, c), b \lor \mu(a, b, c), c \lor \mu(a, b, c)) .$$

Using the distributivity of $L$ one easily sees that $(a, b, c)$ is the least balanced triple $\geq (a, b, c)$ in $L^3$ and that the map $(\overline{\text{-}}) : L^3 \to L^3$ determines a closure operator on the lattice $L^3$ (see [13, Lemma 2.3] for a refinement of this observation). It is also clear that

$$a \lor \mu(a, b, c) = a \lor (b \land c),$$

$$b \lor \mu(a, b, c) = b \lor (a \land c),$$

$$c \lor \mu(a, b, c) = c \lor (a \land b).$$

A triple $(a, b, c) \in L^3$ is closed with respect to the closure operator if and only if it is balanced. Therefore the set of all balanced triples, denoted by $M_3(L)$, forms a lattice [13, Lemma 2.1], where

$$(a, b, c) \lor (a', b', c') = (a \lor a', b \lor b', c \lor c')$$

and

$$(a, b, c) \land (a', b', c') = (a \land a', b \land b', c \land c') .$$

By [5, Lemma 2.9] the lattice $M_3(L)$ is modular if and only if the lattice $L$ is distributive. The “if” part of the equivalence is included in the above mentioned [13, Lemma 1].

3. The lattice

Fix an infinite cardinal $\kappa$. As it is customary, we identify $\kappa$ with the set of all ordinals of cardinality less than $\kappa$. Let us denote by $\mathcal{P}(\kappa)$ the Boolean lattice of all subsets of $\kappa$ and set

$$\mathcal{F}(\kappa) := \{ X \subseteq \kappa \mid X \text{ is finite or } \kappa \setminus X \text{ is finite} \}.$$ 

It is well-known that $\mathcal{F}(\kappa)$ is a bounded Boolean sublattice of $\mathcal{P}(\kappa)$. Next, let us define

$$T = \{ (A, B, C) \in \mathcal{F}(\kappa)^3 \mid C \setminus \mu(A, B, C) \text{ is finite} \} .$$

**Lemma 3.1.** The set $T$ forms a bounded join-subsemilattice of $\mathcal{F}(\kappa)^3$. 

Proof. Being a lattice polynomial, the map \( \mu : \mathcal{P}(\kappa)^3 \to \mathcal{P}(\kappa) \) is monotone. It follows that for all \( \langle A, B, C \rangle, \langle A', B', C' \rangle \in \mathcal{P}(\kappa) \), the inclusion
\[
\mu(\langle A \cup A', B \cup B', C \cup C' \rangle) \supseteq \mu(\langle A, B, C \rangle) \cup \mu(\langle A', B', C' \rangle)
\]
holds, whence also
\[
(\langle C \cup C' \rangle \setminus \mu(\langle A \cup A', B \cup B', C \cup C' \rangle)) \subset (\langle C \cup C' \rangle \setminus (\mu(\langle A, B, C \rangle) \cup \mu(\langle A', B', C' \rangle))) \subset (C \setminus \mu(\langle A, B, C \rangle) \cup (C' \setminus \mu(\langle A', B', C' \rangle))).
\]
Thus if both \( \langle C \setminus \mu(\langle A, B, C \rangle) \rangle \) and \( \langle C' \setminus \mu(\langle A', B', C' \rangle) \rangle \) are finite, then \( \langle C \cup C' \rangle \setminus \mu(\langle A \cup A', B \cup B', C \cup C' \rangle) \) is finite as well. It follows that \( T \) is join-subsemilattice of \( \mathcal{F}(\kappa)^3 \). Finally, it is clear that both \( 0_{\mathcal{F}(\kappa)^3} = \langle \emptyset, \emptyset, \emptyset \rangle \) and \( 1_{\mathcal{F}(\kappa)^3} = \langle \kappa, \kappa, \kappa \rangle \) belong to \( T \). ∎

Let \( S := T \cap \mathcal{M}_3[\mathcal{F}(\kappa)] \) denote the set of all balanced triples from \( T \).

**Lemma 3.2.** The join-semilattice \( T \) is closed under the \( \langle - \rangle \) operation.

Proof. Let \( \langle A, B, C \rangle \in T \). Since \( \mathcal{F}(\kappa) \) is a lattice, we have that all \( A \cup \mu(A, B, C) \), \( B \cup \mu(A, B, C) \) and \( C \cup \mu(A, B, C) \) belong to \( \mathcal{F}(\kappa) \). Since the map \( \mu : \mathcal{P}(\kappa)^3 \to \mathcal{P}(\kappa) \) is monotone, the inclusion \( \mu(A, B, C) \subseteq \mu(A \cup A', B \cup B', C \cup C') \) holds. It follows that
\[
(\langle C \cup \mu(A, B, C) \rangle \setminus \mu(A, B, C)) \subseteq C \setminus \mu(A, B, C),
\]
which is finite due to \( \langle A, B, C \rangle \) being an element of \( T \). ∎

**Lemma 3.3.** The set \( S \) forms a bounded sublattice of the lattice \( \mathcal{M}_3[\mathcal{F}(\kappa)] \).

Proof. Applying Lemmas 3.1 and 3.2, we deduce that \( S \) is a bounded join-subsemilattice of \( \mathcal{F}(\kappa)^3 \). Therefore, it suffices to verify that \( S \) is a meet-subsemilattice of \( \mathcal{F}(\kappa)^3 \). It is easy to observe that if at least one of \( \langle A, B, C \rangle, \langle A', B', C' \rangle \in \mathcal{P}(\kappa)^3 \) is balanced, then
\[
\mu(\langle A \cap A', B \cap B', C \cap C' \rangle) = \mu(\langle A, B, C \rangle \cap \mu(\langle A', B', C' \rangle).
\]
From this we get that if \( \langle A, B, C \rangle, \langle A', B', C' \rangle \in S \), then
\[
\langle C \cap C' \rangle \setminus \mu(\langle A \cap A', B \cap B', C \cap C' \rangle) \subseteq \langle C \setminus \mu(\langle A, B, C \rangle) \cup (C' \setminus \mu(\langle A', B', C' \rangle),
\]
so the set \( \langle C \cap C' \rangle \setminus \mu(\langle A \cap A', B \cap B', C \cap C' \rangle) \) is finite. This concludes the proof. ∎

As discussed in Section 2, since the lattice \( \mathcal{F}(\kappa) \) is distributive, the lattice \( \mathcal{M}_3[\mathcal{F}(\kappa)] \) is modular. Observe that the mapping \( A \mapsto \langle A, A, A \rangle \) embeds \( \mathcal{F}(\kappa) \) into \( S \), from which we deduce that
\[
|\mathcal{F}(\kappa)| \leq |S| \leq |\mathcal{F}(\kappa)^3|.
\]
Since the size of both \( \mathcal{F}(\kappa) \) and \( \mathcal{F}(\kappa)^3 \) is \( \kappa \), we get that \( |S| = \kappa \). Let us sum up these observations in the following corollary to Lemma 3.3.

**Corollary 3.4.** For \( \kappa \) countable infinite, \( S \) forms a countable bounded modular lattice.

**Remark 3.5.** Note that unlike \( S \), the lattice \( T \) is not a meet-subsemilattice of \( \mathcal{F}(\kappa)^3 \). Indeed, both \( \langle \kappa, \emptyset, \emptyset \rangle, \langle \emptyset, \kappa, \kappa \rangle \in T \) while \( \langle \kappa, \emptyset, \emptyset \rangle \land \langle \emptyset, \kappa, \kappa \rangle = \langle \emptyset, \emptyset, \emptyset \rangle \notin T \).
4. A Banaschewski function on $S$

In this section we define a Banaschewski function $f: S \to S$ and describe, element-wise, its range.

**Lemma 4.1.** The map $f: S \to S$ defined by

\[(4.1)\]

$$f(A, B, C) := \langle \kappa \setminus A, \kappa \setminus (B \cup C), \kappa \setminus (A \cup B \cup C) \rangle,$$

for all $\langle A, B, C \rangle \in S$, is a Banaschewski function on $S$.

**Proof.** First we prove that $S$ contains the range of the map $f$. Observe that if we put $A' := \kappa \setminus A$ and $B' := \kappa \setminus (B \cup C)$, then $f(A, B, C) = \langle A', B', A' \cap B' \rangle$. Since $\mathcal{F}(\kappa)$ is a Boolean lattice, the sets $A', B'$ and $A' \cap B'$ all belong to $\mathcal{F}(\kappa)$. Furthermore, we have that

$$A' \cap B' = \mu(A', B', A' \cap B') = \mu f(A, B, C).$$

In particular, $A' \cap B' \setminus \mu f(A, B, C) = \emptyset$, whence $f(A, B, C) \in S$.

It is clear from (4.1) that the map $f$ is antitone. Finally, we check that

$$1_S = \langle \kappa, \kappa, \kappa \rangle = \langle A, B, C \rangle \oplus f(A, B, C), \quad \text{for all } \langle A, B, C \rangle \in S.$$

It follows immediately from the definition of $f$ that

$$\langle A, B, C \rangle \land f(A, B, C) = \langle \emptyset, \emptyset, \emptyset \rangle = 0_S.$$

To prove that $\langle A, B, C \rangle \lor f(A, B, C) = 1_S$, let us verify that

\[(4.2)\]

$$\kappa = \mu(A \cup (\kappa \setminus A), B \cup (\kappa \setminus (B \cup C)), C \cup (\kappa \setminus (A \cup B \cup C))).$$

Note that each element of $\kappa$ that is not contained in $C$ belongs to $B \cup (\kappa \setminus (B \cup C))$. Together with $A \cup (\kappa \setminus A) = \kappa$, we get that (4.2) holds, which concludes the proof. □

**Lemma 4.2.** Let $E$ denote the range of the Banaschewski function $f: S \to S$. Then

$$E = \{ \langle A, B, A \cap B \rangle \mid A, B \in \mathcal{F}(\kappa) \}$$

and the mapping

\[(4.3)\]

$$\langle A, B, A \cap B \rangle \mapsto \langle A, B \rangle$$

determines an isomorphism from $E$ onto the Boolean lattice $\mathcal{F}(\kappa) \times \mathcal{F}(\kappa)$.

**Proof.** While proving Lemma 4.1 we have observed that

\[(4.4)\]

$$E \subseteq \{ \langle A, B, C \rangle \in S \mid C = A \cap B \} = \{ \langle A', B', A' \cap B' \rangle \mid A', B' \in \mathcal{F}(\kappa) \}.$$

A straightforward computation gives that $f(\langle A', B', A' \cap B' \rangle) = \langle A', B', A' \cap B' \rangle$, so the lattice $E$ is equal to the right-hand side of (4.3). Finally, it is readily seen that the correspondence (4.3) determines an isomorphism $E \to \mathcal{F}(\kappa) \times \mathcal{F}(\kappa)$. □

It was noted in [14] that if the range of a Banaschewski function on a lattice $L$ is Boolean, then it is a *maximal* Boolean sublattice of $L$. Thus we derive from Theorem 4.2 that $E$ is a maximal Boolean sublattice of $B$. 
5. The counter-example

In the present section, we construct another maximal Boolean sublattice \( B \) of the lattice \( S \). We show that the lattices \( B \) and \( S \) are not isomorphic and we prove directly that the lattice \( B \) is not the range of any Banaschewski function on \( S \).

Lemma 5.1. The assignment \( (A, C) \mapsto g\langle A, C\rangle := \langle A, A \cap C, C\rangle \) defines a bounded lattice embedding \( g: \mathcal{F}(\kappa) \times \mathcal{F}(\kappa) \to M_3[\mathcal{F}(\kappa)] \). In particular, the range of \( g \) is a bounded Boolean sublattice of \( M_3[\mathcal{F}(\kappa)] \) isomorphic to \( \mathcal{F}(\kappa) \times \mathcal{F}(\kappa) \).

Proof. It is clear from the definition of the map \( g \) that it is injective and that its range is included in \( M_3[\mathcal{F}(\kappa)] \). Further, for any \( A, A', C, C' \subseteq \kappa \), the equality

\[
g\langle A, C\rangle \land g\langle A', C'\rangle = g\langle A \cap A', C \cap C'\rangle
\]

holds by (2.3), while

\[
g\langle A, C\rangle \lor g\langle A', C'\rangle = g\langle A \cup A', C \cup C'\rangle
\]

can be easily deduced from (2.1) and (2.2). Finally, observe that \( g\langle \kappa, \kappa\rangle = \langle \kappa, \kappa, \kappa\rangle \) and \( g\langle \emptyset, \emptyset\rangle = \langle \emptyset, \emptyset, \emptyset\rangle \), which concludes the proof.

For any \( A, C \in \mathcal{F}(\kappa) \), we say that \( \langle A, C\rangle \) is finite if both \( A \) and \( C \) are finite, and we say that \( \langle A, C\rangle \) is co-finite if both \( \kappa \setminus A \) and \( \kappa \setminus C \) are finite. Let us write \( A \sim C \) if \( \langle A, C\rangle \) is either finite or co-finite. Note that there are pairs \( A, C \in \mathcal{F}(\kappa) \) such that \( \langle A, C\rangle \) is neither finite nor co-finite; namely, \( A \sim C \) if and only if the symmetric difference \( (A \setminus C) \cup (C \setminus A) \) is finite.

Lemma 5.2. The set

\[
A = \{\langle A, C\rangle \in \mathcal{F}(\kappa) \mid A \sim C\}
\]

form a bounded Boolean sublattice of \( \mathcal{F}(\kappa) \times \mathcal{F}(\kappa) \).

Proof. Let \( \langle A, C\rangle, \langle A', C'\rangle \) be a pair of elements from \( A \). If at least one of them is finite, then \( \langle A \cap A', C \cap C'\rangle \) is clearly finite as well. If both \( \langle A, C\rangle \) and \( \langle A', C'\rangle \) are co-finite, then so is \( \langle A \cap A', C \cap C'\rangle \). In either case, \( \langle A \cap A', C \cap C'\rangle \in A \).

If at least one of the pairs \( \langle A, C\rangle, \langle A', C'\rangle \) is co-finite, then \( \langle A \cup A', C \cup C'\rangle \) is co-finite, while if both \( \langle A, C\rangle \) and \( \langle A', C'\rangle \) are finite, then so is \( \langle A \cup A', C \cup C'\rangle \). In particular, \( \langle A \cup A', C \cup C'\rangle \in A \) whenever \( \langle A, C\rangle, \langle A', C'\rangle \in A \).

We have shown that \( A \) is a sublattice of \( \mathcal{F}(\kappa) \times \mathcal{F}(\kappa) \). To complete the proof, observe that \( \langle \emptyset, \emptyset\rangle \) is finite and \( \langle \kappa, \kappa\rangle \) is co-finite and that the unique complement in \( \mathcal{F}(\kappa) \times \mathcal{F}(\kappa) \) of each \( \langle A, C\rangle \in A \), namely \( \langle \kappa \setminus A, \kappa \setminus C\rangle \) belongs to \( A \).

Lemma 5.3. The g-image \( B = g(A) \) of \( A \) is a bounded Boolean sublattice of \( S \).

Proof. Due to Lemma 5.1 and Lemma 5.2 the set \( B \) is a bounded Boolean sublattice of \( M_3[\mathcal{F}(\kappa)] \). Thus in view of Lemma 5.1 it suffices to verify that \( B \subseteq S \), that is, that \( C \setminus (A \cap C) \) is finite for every \( \langle A, C\rangle \in A \). This is clear when \( \langle A, C\rangle \) is finite. If \( \langle A, C\rangle \) is co-finite, then \( C \setminus (A \cap C) = C \setminus A \subseteq \kappa \setminus A \) is finite and we are done.

Observe that if \( \langle A, B, C\rangle \) is a balanced triple then \( B \subseteq A \) if and only if \( B = A \cap B = A \cap C \). It follows that

\[
B = \{\langle A, B, C\rangle \in S \mid A \sim C \text{ and } B \subseteq A\}.
\]
Lemma 5.4. Let \( \langle A, B, C \rangle \in S \setminus B \) and let \( \langle A', B', C' \rangle \) be a complement of \( \langle A, B, C \rangle \) in \( S \). If \( B \subseteq A \), then \( B' \not\subseteq A' \).

Proof. Since \( \langle A, B, C \rangle \notin B \) and \( B \subseteq A \), it follows from (5.2) that \( A \sim C \). Hence exactly one of the two sets \( A, C \) is finite. From \( B \subseteq A \) and \( C \setminus B \) being finite we conclude that \( C \) and \( \kappa \setminus A \) are finite. It follows that the set \( B = B \cap C \) is finite as well.

Suppose now that \( B' \subseteq B' \). Since \( \langle A, B, C \rangle \cap \langle A', B', C' \rangle = 0_S \), we have that \( A \cap A' = \emptyset \), whence the set \( A' \subseteq \kappa \setminus A \) is finite. A fortiori, the set \( B' \) is also finite due to the assumption that \( B' \subseteq A' \). As \( C' \setminus B' = C' \setminus \langle B' \cap A' \rangle = C' \setminus \mu \langle A', B', C' \rangle \) is also finite, we conclude that so is \( C' \). But then

\[
\mu \langle A \cup A', B \cup B', C \cup C' \rangle \subseteq B \cup B' \cup C \cup C'
\]

is a finite set, which contradicts the assumption that \( \langle A, B, C \rangle \lor \langle A', B', C' \rangle = \langle \kappa, \kappa, \kappa \rangle = 1_S \).

Corollary 5.5. Every complemented bounded sublattice \( C \) of \( S \) such that \( B \subseteq C \) contains an element \( \langle A, B, C \rangle \) with \( B \nsubseteq A \).

Proof. Let \( \langle A, B, C \rangle \in C \setminus B \) and let \( \langle A', B', C' \rangle \) be its complement in \( C \). Applying Lemma 5.4, we get that either \( B \nsubseteq A \) or \( B' \nsubseteq A' \).

Proposition 5.6. The lattice \( B \) is a maximal Boolean sublattice of \( S \).

Proof. Let \( C \) be a complemented bounded sublattice of \( S \) satisfying \( B \subseteq C \). There is \( \langle A, B, C \rangle \in C \) with \( B \nsubseteq A \) by Corollary 5.5. We can pick a finite nonempty \( F \subseteq (B \setminus A) \). Since the triple \( \langle A, B, C \rangle \) is balanced,

\[
\emptyset = F \cap A = F \cap B \cap A = F \cap B \cap C = F \cap C.
\]

Now observe that both \( g \langle F, \emptyset \rangle \) and \( g \langle \emptyset, F \rangle \) are in \( B \). Applying (5.1) and (5.3), we get that

\[
\langle A, B, C \rangle \land (g \langle F, \emptyset \rangle \lor g \langle \emptyset, F \rangle) = \langle A, B, C \rangle \land g \langle F, F \rangle = \langle \emptyset, F, \emptyset \rangle,
\]

while

\[
(\langle A, B, C \rangle \land g \langle F, \emptyset \rangle) \lor (\langle A, B, C \rangle \land g \langle \emptyset, F \rangle) = \langle \emptyset, \emptyset, \emptyset \rangle.
\]

It follows from (5.1) and (5.5) that the lattice \( C \) is not distributive, a fortiori it is not Boolean.

Proposition 5.7. The sublattice \( B \) of \( S \) is not the range of any Banaschewski function on \( S \).

Proof. The range of a Banaschewski function on \( S \) must contain a complement of each element of \( S \). We show that no complement of \( \langle \kappa, \emptyset, \emptyset \rangle \) in \( S \) belongs to \( B \).

Suppose the contrary, that is, that there is \( \langle A, B, C \rangle = g \langle A, C \rangle \in B \) satisfying \( \langle \kappa, \emptyset, \emptyset \rangle \lor \langle A, B, C \rangle = 1_S \). Then \( A = A \cap \kappa = \emptyset \), and by (5.2) also \( B = \emptyset \). Then from \( B = \emptyset \) and \( \langle \kappa, \emptyset, \emptyset \rangle \lor \langle A, B, C \rangle = 1_S \), one infers that \( C = \kappa \). It follows that \( \langle A, B, C \rangle \notin S \); indeed, \( C \setminus \mu \langle A, B, C \rangle = C \setminus \emptyset = \kappa \) is not finite. Thus \( \langle A, B, C \rangle \notin B \), which is a contradiction.

Remark 5.8. Note that for the particular case of \( \kappa = \aleph_0 \), the assertion of Proposition 5.7 follows from Proposition 5.3 together with [14] Corollary 4.8, which states that the ranges of two Boolean Banaschewski functions on a countable complemented modular lattice are isomorphic.
Proposition 5.9. The lattices \( B \) and \( E \) are not isomorphic.

Proof. In \( B \), every finite element \( g \langle A, C \rangle \) is a join of a finite set of atoms, namely

\[
g \langle A, C \rangle = \left( \bigvee_{\alpha \in A} g(\{\alpha\}, \emptyset) \right) \vee \left( \bigvee_{\gamma \in C} g(\emptyset, \{\gamma\}) \right),
\]

and, dually, every co-finite element is a meet of a finite set of co-atoms. On the other hand, there are elements in \( \mathcal{F}(\kappa) \times \mathcal{F}(\kappa) \) that are neither finite joins of atoms nor finite meets of co-atoms. Recall that in Lemma [12], we have observed that the lattice \( E \) is isomorphic to \( \mathcal{F}(\kappa) \times \mathcal{F}(\kappa) \). Therefore the lattices \( B \) and \( E \) are not isomorphic. \( \square \)

6. Representing \( S \) in a subspace-lattice

Although the construction in the three previous sections was performed for an infinite cardinal \( \kappa \), the results of the present section on embedding the lattice \( M_3[\mathcal{P}(\kappa)] \) into \( \text{Sub}(V) \) (namely Theorem [13]) work just as well for \( \kappa \) finite. In particular, Proposition [6.3] (an enhancement of [5, Lemma 2.9]) holds for lattices of any cardinality.

Let \( F \) be an arbitrary field and let \( V \) denote the vector space over the field \( F \) presented by generators \( x_\alpha, y_\alpha, z_\alpha, \alpha \in \kappa \), and relations \( x_\alpha + y_\alpha + z_\alpha = 0 \). For a subset \( X \) of the vector space \( V \) we denote by \( \text{Span}(X) \) the subspace of \( V \) generated by \( X \). Given subspaces of \( V \), say \( X \) and \( Y \), we will use the notation \( X + Y = \text{Span}(X \cup Y) \). Let \( \text{Sub}(V) \) denote the lattice of all subspaces of the vector space \( V \).

For all \( A, B, C \subseteq \kappa \) we put \( X_A = \text{Span}\{x_\alpha \mid \alpha \in A\} \), \( Y_B = \text{Span}\{y_\beta \mid \beta \in B\} \), and \( Z_C = \text{Span}\{z_\gamma \mid \gamma \in C\} \).

We define a map \( F : \mathcal{P}(\kappa)^3 \to \text{Sub}(V) \) by the correspondence

(6.1) \[
\langle A, B, C \rangle \mapsto X_A + Y_B + Z_C.
\]

Each of the sets \( \{x_\alpha \mid \alpha \in \kappa\} \), \( \{y_\beta \mid \beta \in \kappa\} \), and \( \{z_\gamma \mid \gamma \in \kappa\} \) is clearly linearly independent. It follows that \( X_{A \cup A'} = X_A + X_{A'} \) for all \( A, A' \subseteq \kappa \) and, similarly, \( Y_{B \cup B'} = Y_B + Y_{B'} \) and \( Z_{C \cup C'} = Z_C + Z_{C'} \) for all \( B, B', C, C' \subseteq \kappa \). A straightforward computation gives the following lemma:

Lemma 6.1. The map \( F : \mathcal{P}(\kappa)^3 \to \text{Sub}(V) \) is a bounded join-homomorphism.

Proof. Clearly \( F(\emptyset, \emptyset, \emptyset) = 0 \) and \( F(\kappa, \kappa, \kappa) = V \). Following the definitions, we compute

\[
F(\langle A, B, C \rangle) + F(\langle A', B', C' \rangle) = X_A + Y_B + Z_C + X_{A'} + Y_{B'} + Z_{C'} = X_{A \cup A'} + Y_{B \cup B'} + Z_{C \cup C'} = F((A \cup A', B \cup B', C \cup C')).
\]

\( \square \)

Let \( G : \text{Sub}(V) \to \mathcal{P}(\kappa)^3 \) be a map defined by

\[
W \mapsto \langle \{\alpha \mid x_\alpha \in W\}, \{\beta \mid y_\beta \in W\}, \{\gamma \mid z_\gamma \in W\} \rangle,
\]

for all \( W \in \text{Sub}(V) \).

It is straightforward that \( G \) is a bounded meet-homomorphism and that it is the right adjoint of \( F \) (i.e., replacing the lattice \( \text{Sub}(V) \) with its dual, the maps \( F \) and \( G \) form a Galois correspondence [13]). Indeed, one readily sees that

\[ F(\langle A, B, C \rangle) \subseteq W \text{ iff } \langle A, B, C \rangle \leq G(W). \]

The maps \( F \) and \( G \) induce a closure operator \( GF \) on \( \mathcal{P}(\kappa)^3 \).
Lemma 6.2. The composition \( GF : \mathcal{P}(\kappa)^3 \to \mathcal{P}(\kappa)^3 \) is precisely the closure operator \( \langle - \rangle \) on \( \mathcal{P}(\kappa)^3 \) defined by (2.1).

Proof. We shall prove that \( GF \langle A, B, C \rangle = \overline{A, B, C} \) for every \( \langle A, B, C \rangle \in \mathcal{P}(\kappa)^3 \).

By symmetry, it suffices to prove that

\[
\{ \alpha \in \kappa \mid x_\alpha \in F \langle A, B, C \rangle \} = A \cup (B \cap C).
\]

Let \( \alpha \in A \cup (B \cap C) \). If \( \alpha \in A \), then \( x_\alpha \in F \langle A, B, C \rangle \) by the definition (6.1), while if \( \alpha \in B \cap C \), then \( x_\alpha = -y_\alpha - z_\alpha \in F \langle A, B, C \rangle \) by (6.1) and the defining relations of \( V \). It follows that \( A \cup (B \cap C) \subseteq \{ \alpha \in \kappa \mid x_\alpha \in F \langle A, B, C \rangle \} \).

In order to prove the opposite inclusion, take any \( \xi \in \kappa \setminus A \) satisfying \( x_\xi \in F \langle A, B, C \rangle \); if there is one, there is nothing to prove. We need to show that then \( \xi \in B \cap C \). Certainly

\[
x_\xi = \sum_{\alpha \in A} a_\alpha x_\alpha + \sum_{\beta \in B} b_\beta y_\beta + \sum_{\gamma \in C} c_\gamma z_\gamma
\]

for suitable \( a_\alpha, b_\beta, \) and \( c_\gamma \in \mathbb{F} \) such that all but finitely many of them are zero. We set \( a_\alpha = 0 \) for \( \alpha \notin A \), \( b_\beta = 0 \) for \( \beta \notin B \), and \( c_\gamma = 0 \) for \( \gamma \notin C \). Since \( z_\gamma + x_\alpha + y_\beta = 0 \) for every \( \gamma \in \kappa \), it follows from (6.2) that

\[
x_\xi = \left( \sum_{\alpha \in A} a_\alpha x_\alpha - \sum_{\gamma \in C} c_\gamma x_\gamma \right) + \left( \sum_{\beta \in B} b_\beta y_\beta - \sum_{\gamma \in C} c_\gamma y_\gamma \right).
\]

It easily follows from the defining relations of \( V \) that \( \{ x_\alpha, y_\alpha \mid \alpha \in \kappa \} \) forms a basis of \( V \). Thus, applying (6.3) we get that

\[
a_\xi c_\xi = 1 \text{ and } b_\xi - c_\xi = 0.
\]

Since by our assumption \( \xi \notin A \), we get from (6.2) that \( a_\xi = 0 \). Substituting to (6.4) we get that \( b_\xi = c_\xi = -1 \), hence \( \xi \in B \cap C \). This concludes the proof that \( A \cup (B \cap C) \blacktrianglerighteq \{ \alpha \in \kappa \mid x_\alpha \in F \langle A, B, C \rangle \} \). \( \square \)

The next lemma shows that \( F \upharpoonright M_3[\mathcal{P}(\kappa)] \) preserves meets. Note that with Lemma 6.1 this means that \( F \upharpoonright M_3[\mathcal{P}(\kappa)] \) is a lattice embedding of \( M_3[\mathcal{P}(\kappa)] \) into the lattice \( \text{Sub}(V) \).

Lemma 6.3. Let \( \langle A, B, C \rangle, \langle A', B', C' \rangle \in M_3[\mathcal{P}(\kappa)] \) be balanced triples. Then

\[
F \langle A, B, C \rangle \cap F \langle A', B', C' \rangle = F \langle A \cap A', B \cap B', C \cap C' \rangle.
\]

Proof. Since, by Lemma 6.1, \( F \) is a join-homomorphism, it is monotone, whence \( F \langle A \cap A', B \cap B', C \cap C' \rangle \subseteq F \langle A, B, C \rangle \cap F \langle A', B', C' \rangle \). Thus it remains to prove the opposite inclusion.

Let \( v \in F \langle A, B, C \rangle \cap F \langle A', B', C' \rangle \) be a non-zero vector. Then \( v \) can be expressed as

\[
v = \sum_{\alpha \in A} a_\alpha x_\alpha + \sum_{\beta \in B} b_\beta y_\beta + \sum_{\gamma \in C} c_\gamma z_\gamma = \sum_{\alpha \in A'} a'_\alpha x_\alpha + \sum_{\beta \in B'} b'_\beta y_\beta + \sum_{\gamma \in C'} c'_\gamma z_\gamma.
\]

Consider such an expression of \( v \) with

\[
|\{ \alpha \mid a_\alpha \neq 0 \}| + |\{ \beta \mid b_\beta 
eq 0 \}| + |\{ \gamma \mid c_\gamma 
eq 0 \}|
\]

minimal possible. Put \( a_\alpha = 0 \) for \( \alpha \notin A \), \( b_\beta = 0 \) for \( \beta \notin B \), and \( c_\gamma = 0 \) for \( \gamma \notin C \). By symmetry, we can assume that \( a_\alpha = 0 \) for some \( \alpha \in A \). Suppose for a contradiction that \( \alpha \notin A' \). Since the triple \( \langle A', B', C' \rangle \) is balanced, \( B' \cap C' \subseteq A' \),

\[
|\{ \alpha \mid a_\alpha 
eq 0 \}| + |\{ \beta \mid b_\beta 
eq 0 \}| + |\{ \gamma \mid c_\gamma 
eq 0 \}|
\]
whence $\alpha \notin B' \cap C'$. Without loss of generality we can assume that $\alpha \notin B'$. If all $a_\alpha, b_\alpha$, and $c_\alpha$ were non-zero, we could replace $c_\alpha z_\alpha$ with $-c_\alpha x_\alpha - c_\alpha y_\alpha$ and reduce the value of the expression in (6.6) which is assumed minimal possible. Thus either $b_\alpha = 0$ or $c_\alpha = 0$ (recall that we assume that $a_\alpha \neq 0$). We will deal with these two cases separately. If $b_\alpha = 0$, then the equality
\begin{equation}
(6.7)
\quad a_\alpha x_\alpha + c_\alpha z_\alpha = \alpha_\alpha z_\alpha
\end{equation}
must hold true. Since $x_\alpha$ and $z_\alpha$ are linearly independent, it follows from (6.7) that $a_\alpha = 0$ which contradicts our choice of $\alpha$. The remaining case is when $c_\alpha = 0$. Under this assumption we have that
\begin{equation}
(6.8)
\quad a_\alpha x_\alpha + b_\alpha y_\alpha = \alpha_\alpha z_\alpha.
\end{equation}
It follows that
\begin{equation}
(6.9)
\quad a_\alpha x_\alpha = \alpha_\alpha z_\alpha - b_\alpha y_\alpha = -\alpha_\alpha x_\alpha - (\alpha_\alpha + b_\alpha) y_\alpha.
\end{equation}
Since $x_\alpha$ and $y_\alpha$ are linearly independent, we infer from (6.9) that $a_\alpha = -\alpha_\alpha = b_\alpha$. Then we could reduce the value of (6.6) by replacing $a_\alpha x_\alpha + b_\alpha y_\alpha$ with $\alpha_\alpha z_\alpha$ in (6.5). This contradicts the minimality of (6.6). \hfill \Box

Combining Lemma 6.1, Lemma 6.2, and Lemma 6.3 we conclude:

**Theorem 6.4.** The restrictions $F \upharpoonright M_3[\mathcal{P}(\kappa)] : M_3[\mathcal{P}(\kappa)] \to \text{Sub}(V)$ and, a fortiori, $F \upharpoonright S : S \to \text{Sub}(V)$ are bounded lattice embeddings. In particular, the lattice $S$ is isomorphic to a bounded sublattice of the subspace-lattice of a vector space.

It is well-known that a distributive lattice $L$ embeds (via a bounds-preserving lattice embedding) into the lattice $\mathcal{P}(\kappa)$, where $\kappa$ is the cardinality of the set of all maximal ideals of $L$. Such embedding induces an embedding $M_3[L] \hookrightarrow M_3[\mathcal{P}(\kappa)]$ (cf. Lemma 6.3). By Theorem 6.4 the lattice $M_3[\mathcal{P}(\kappa)]$ embeds into the lattice $\text{Sub}(V)$ for a suitable vector space $V$ (note again that we now also admit finite $\kappa$). Since the lattice $\text{Sub}(V)$ is Arguesian, so are $M_3[\mathcal{P}(\kappa)]$ and $M_3[L]$.

On the other hand, [5, Lemma 2.9] states that a lattice $L$ is distributive if and only if $M_3[L]$ is modular. Hence, if $M_3[L]$ is modular, it follows that $L$ is distributive, and, by the above argument, $M_3[L]$ is even Arguesian. We have thus proven the following strengthening of [5, Lemma 2.9]:

**Proposition 6.5.** Let $L$ be a lattice. Then $L$ is distributive iff the lattice $M_3[L]$ is modular iff $M_3[L]$ is Arguesian. If this is the case, then $M_3[L]$ can be embedded into the lattice of all subspaces of a suitable vector space over any given field.

**References**

1. B. Banaschewski, *Totalgeordnete moduln*, Arch. Math. **7** (1957), 430–440.
2. G. Grätzer and E. T. Schmidt, *A lattice construction and congruence-preserving extensions*, Acta Math. Hungar. **66** (1995), 275–288.
3. , *On the independence theorem of related structures for modular (arguesian) lattices*, Studia Sci. Math. Hungar. **40** (2003), 1–12.
4. G. Grätzer and F. Wehrung, *Flat semilattices*, Colloq. Math. **79** (1999), 185–191.
5. , *The $M_3[D]$ construction and n-modularity*, Algebra Universalis **41** (1999), 87–114.
6. , *A new lattice construction: the box product*, J. Algebra **221** (1999), 5893–5919.
7. , *Proper congruence-preserving extension of lattices*, Acta Math. Hungar. **85** (1999), 169–179.
8. , *Tensor product and transferability of semilattices*, Canad. J. Math. **51** (1999), 792–815.
9. ______, Tensor product and semilattices with zero, revisited, J. Pure Appl. Algebra 147 (2000), 273–301.
10. ______, A survey of tensor product and related structures in two lectures, Algebra Universalis 45 (2001), 117–143.
11. Ø. Ore, Galois Connexions, Trans. Amer. Math. Soc. 55 (1944), 493–513.
12. E. T. Schmidt, Zur charakterisierung der kongruenzverbände der verbände, Mat. Časopis Sloven. Akad. Vied. 18 (1968), 3–20.
13. ______, Every finite distributive lattice is the congruence lattice of a modular lattice, Algebra Universalis 4 (1974), 49–57.
14. F. Wehrung, Coordination of lattices by regular rings without unit and Banaschewski functions, Algebra Universalis 64 (2010), 49–67.
15. ______, A non-coordinatizable sectionally complemented modular lattice with a large Jónsson four-frame, Adv. in Appl. Math. 47 (2011), 173–193.

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