Study into Lorenz dynamic system

G.V. Belskii, I.A. Prihodko, V.B. Vtorov, E.A. Vasiliev
Saint Petersburg Electrotechnical University LETI, Saint Petersburg

gvbelskiy@etu.ru

Abstract. The examination of chaotic phenomena origination and their dynamics is a promising area of studies into technical malfunction forecasting. This paper provides analysis of the Lorenz system on a basis of the expansion with the ‘Caterpillar’-SSA method. The trend, periodic and non-cyclic components are determined. The necessary number of principal components for time series reconstruction on a basis of Lorenz system is obtained.

1. Introduction
There are known complex dynamical systems where chaotic phenomena are possible. Chaotic processes are found in radio engineering, electronics and electric power facilities, for example [1]. Chaotic signals are used in radio electronics for covert transmission of information (chaotic masking, chaotic shift keying) as well as in cryptographic applications, pattern recognition, navigation, hydrodynamics, mechanics, chemistry, aviation monitoring, nonlinear admixing in telecommunication systems, computer networks etc. Chaotic processes can cause a breakdown or malfunctioning of components of automatic systems. The examination of chaotic phenomena origination and their dynamics is a promising area of studies into technical malfunction forecasting.

This paper covers the results of numerical analysis of the behavior of chaotic submotions of the Lorenz model [2], assessment of the possibility of reconstructing a non-periodic time series based on the method of singular spectrum analysis ‘Caterpillar’-SSA [3], [4].

2. Study into Lorenz Dynamical System with Strange Attractor
Lorenz system equations [2]

\[
\begin{align*}
\dot{x} &= -\sigma (x - y), \\
\dot{y} &= -xz - y + \rho x, \\
\dot{z} &= -bz + xy.
\end{align*}
\] (1)

Typically, the following parameters are assumed: \(\sigma = 10, \ b = 8/3\). The type of attractor depends on the control parameter \(\rho\), the Rayleigh coefficient. The calculation is provided for the values \(\rho = 22\) and \(\rho = 28\).

Before evaluating the possibility of using the spectral analysis method for the analysis of non-periodic systems, we briefly give preliminary information on the structure of the Lorenz dynamical system with a strange attractor [2].

For \(0 < \rho < 1\), the origin (point \(0, 0, 0\)) is an equilibrium state of global attraction. For \(\rho > 1\), the point \((0, 0, 0)\) loses stability, a pair of new stable equilibrium positions appears. Starting from the value
\( \rho \approx 13,927 \), the attractor changes in the phase space. For values \( 1 < \rho < 24.74 \), two nonzero fixed points are stable and attractive (Fig. 1, Fig. 2).

For \( \rho > 24.74 \) all three fixed points are unstable [2], the only attractive set is the strange Lorenz attractor (Fig. 3, Fig. 4).

While passing a loop in the vicinity of one fixed point, the trajectory unexpectedly and unpredictably makes a transition to the other.
While spinning in a spiral in the vicinity of one of the fixed points, the trajectory of the system due to the dissipativity of the Lorenz system remains within the boundaries of a certain limit set.

Fig. 4. Transient processes, \( \rho = 28 \)

The problem of reconstruction and forecasting of time series in various applications has recently been successfully solved on the basis of the ‘Caterpillar’ -SSA method [5], [6]. This is a time series analysis method based on transforming a one-dimensional series into a multidimensional one using a single parameter shift procedure, examining the resulting multidimensional trajectory through the principal component analysis (singular value decomposition) and reconstructing the series from selected principal components.

Applying the method results in the decomposition of a time series into simple components on the basis generated by the function: trends, periodic components and noise. The resulting decomposition can be used as a foundation for the analysis of series components and forecasting. The method does not require the series to be stationary, it allows working with noisy data [3].

Assume a time series \( F_N = (f_0, ..., f_{N-1}) \) corresponding to a set of vectors composed of moving average line segments of the series with a selected length \( L \) (the length of the ‘caterpillar’), \( 1 < L < N \) [3]. The embedding procedure converts the initial time series into a sequence of multidimensional embedding vectors \( X_i = (f_{i-L}, ..., f_{i+L-2})^T, 1 \leq i \leq K \) where \( K \) is the number of embedding vectors. The trajectory matrix \( X \) of the time series \( F \) consists of embedding vectors as columns.

\[
X = \left( x_{ij} \right)_{i=1}^{L,K} = \begin{bmatrix}
    f_0 & f_1 & f_2 & ... & f_{K-1} \\
    f_1 & f_2 & f_3 & ... & f_K \\
    f_2 & f_3 & f_4 & ... & f_{K+1} \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    f_{L-1} & f_L & f_{L+1} & ... & f_{N-1}
\end{bmatrix}
\]

According to the ‘Caterpillar’-SSA method, it is necessary to perform a singular decomposition of the trajectory matrix of the series. Let us denote \( S = XX^T, \lambda_1, ..., \lambda_d \) – eigenvalues of matrix \( S \) taken in non-decreasing order \( (\lambda_1 \geq ... \geq \lambda_d \geq 0) \), \( V_i = XU_i / \sqrt{\lambda_i}, i = 1, d \), \( U_1, ..., U_d \) – eigenvectors of matrix
$S, U_i \in \mathbb{R}^L, V_i \in \mathbb{R}^K$. Singular value decomposition of the matrix $X$ can be written as $X = X_1 + \ldots + X_d$, where $X_i = \sqrt{\lambda_i} U_i V_i^T$. Each of the matrices $X_i$ has rank 1.

The next stage is grouping up decomposition components. As a result of dividing ${1, \ldots, d}$ into $m$ disjoint subsets $I_j$, the following matrix is formed: $X = X_{I_1} + X_{I_2} + \ldots + X_{I_m}$.

The final stage is reconstructing ranks $F_N^{(j)}$ per grouped matrices $X_{I_j}$. As a result, we have the initial time series $F_N = F_N^{(1)} + \ldots + F_N^{(m)}$ decomposed.

The diagram of the transient process for the Lorenz system (1) with coefficient $\rho = 22$ is shown in Fig. 5.

![Fig. 5. Time series, $\rho = 22$](image)

Depending on the initial conditions, the chaotic oscillations of the transient process (for $\rho = 22$), which are characteristic of the Lorenz system, break off at some point in time and enter the damped oscillation mode.

The studies were carried out with time series forecasting software using ‘Caterpillar’-SSA method, Caterpillar SSA 3.40, Standard Edition, developed at St. Petersburg State University [7].

For the initial time series with a duration of $N = 2,950$, the length of the ‘caterpillar’, which is the main parameter of the method, is assumed $L = 900$. The number of embedding vectors is $K = 2,051$. From the analysis of the graph of the logarithm of the eigenvalues of the trajectory matrix $X$, the number of principal components was chosen equal to 150.

According to the ‘Caterpillar’-SSA method, it is necessary to determine components related to the trend and those related to the periodic component, and non-periodic components. The identification of the components of the series on the basis of the singular value decomposition of its trajectory matrix was carried out visually based on the graphical presentation of the results (Fig. 6, 7) and the available theoretical information [3], [5].

Slowly changing eigenvectors are significant, the first one is 73.285%, the second is 4.766%. It should be taken into account that other eigenvectors also remain significant, the third being 1.71%, the tenth being 0.769%, the fiftieth being 0.052%, etc.

We also indicate component values of the periodic time series considered in [6]: The first eigenvector is 98.687%, the second is 0.436%, the ninth is 0.009%. The comparison implies the need to use a considerably larger number of principal components for the analysis of the time series based on the Lorenz system than for the periodic series.

The separated 150 principal components comprise more than 99% of the time series. To detect a trend, it is required to group eigen triplets with slowly changing singular vectors. The first and second vector (Fig. 6) change slowly, although they have an additive oscillating component.
This is due to the inaccurate divisibility of the series by a trend, a periodic component and noise when the component of the series that sets up the slowly varying part of the singular vectors is mixed with the harmonic [5].

![Principal component diagram](image)

**Fig. 6.** Principal component diagram

To determine the periodic component, two-dimensional eigenvector diagrams are used, on which the elements of the first vector from a pair are plotted along one axis, and the elements of the second vector are plotted along the other. As an example, two-dimensional diagrams of some eigenvectors are given (Fig. 7).

According to the figure, a pair of eigenvectors (Fig. 7, a) 3–4, 10–11, 34–35, and also 13–14; 20-21; 22-23; 36-37; 38-39; 42-43; 49-50; 51–52; 58-59; 61-62; 63-64; 66-67; 68-69; 70-71; 72-73; 74-75; 83-84; 86-87; 107-108; 112-113; 115-116; 122-123; 125-126; 127-128; 130-131; 134-135; 137-138; 141-142; 143-144; 148–149 form fairly distinct spirals. Hence it means [3] that the series has harmonics corresponding to these vectors, which can be attributed to the periodic component.

![Two-dimensional eigenvector diagrams](image)

**Fig. 7.** Two-dimensional eigenvector diagrams

All other unaccounted above eigenvectors belong to non-periodic components (Fig. 7b).

The accuracy of the reconstruction is estimated with the analysis of the reconstruction residuals.

It is important to note that the analysis of the principal components allows us to predict the nature of the future movement of the time series (for a short period). For the time series under consideration, the damping of chaotic oscillations manifests itself earlier in 5 (1,613 %), 8 (0,885 %), 15 (0,497%), 18 (0,459 %), 28 (0,245 %), 44 (0,081 %) principal components.

The results of the time series reconstruction on the basis of Lorenz attractor (1) and reconstruction residuals are shown in Fig. 8. The graphs refer to changed conditions as opposed to Fig. 5 referring to the initial ones.
The paper estimates the accuracy of reconstruction for various values of the length of the ‘caterpillar’ $L$ and the number of principal components. Decreasing $L$ down to 450 even along with increasing the number of principal components up to 250 resulted in a growth of the reconstruction error.

The expected result of an unsuccessful forecasting attempt for the Lorenz system with the Rayleigh coefficient $\rho = 22$ at the damping section of the transient process is shown in Fig. 9. Changing the number of principal components, the length of the ‘caterpillar’ and grouping parameters does not yield a positive result.

![Fig. 8. Time series reconstruction, $\rho = 22$](image1)

![Fig. 9. Time series forecast, $\rho = 22$](image2)
The time series reconstruction graph based on the Lorenz system with the parameter \( \rho = 28 \), selecting the ‘caterpillar’ length \( L = 613 \) and the number of principal components being 250 with practically zero error is shown in Fig. 10.

Forecast quality: standard deviation: 1.403; maximum deviation: 4.008 [7].

The plot of the time series reconstruction based on the Lorenz system with the parameter \( \rho = 28 \) and a series of residuals are presented in Fig. 10, where the original time series is shown in red and the restored one in blue. With a smaller caterpillar length \( L = 613 \), a result comparable to the previous one (Fig. 8) was obtained.

An increase in the number of principal components to 250, despite the insignificance of the components not taken into account above, makes it possible to obtain practically zero reconstruction error.

**Conclusion**

The comparison of the results of the periodic time series analysis [6] and the chaotic Lorenz system based on the spectrum analysis method shows the need to use a much higher number of principal components in the second case; it is necessary to take into account eigenvectors of insignificant percentage and take into account non-periodic additive components. In the example [6], the use of eighteen principal components ensures a high forecast accuracy; for a successful solution of the problem of reconstructing a time series based on the Lorenz equations, the minimum required number of principal components is approximately 150 (Fig. 8).

For higher accuracy of reconstruction, the length of the ‘caterpillar’ must be slightly less than \( N/2 \).

For \( 22 < \rho < 24.74 \) when chaotic oscillations become damped, chaotic behavior can occur, what rules out forecasting even for the region of damping oscillations.

The paper estimates the reconstruction accuracy for various values of the caterpillar length and the number of principal components. The best result is achieved if both values are 980 for \( \rho = 22 \). For the time series based on the Lorenz system with the parameter \( \rho = 28 \), nearly zero error was obtained when choosing the length of the ‘caterpillar’ \( L = 613 \) and the number of principal components being 250.

It is important to note that the analysis of the principal components allows us to predict the nature of the future movement of the time series (for a short period).
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