Supersonic quantum communication

J. Eisert\textsuperscript{1,2} and D. Gross\textsuperscript{\textsuperscript{2,3}}

\textsuperscript{1}Institute of Physics and Astronomy, University of Potsdam, 14476 Potsdam, Germany
\textsuperscript{2}Quantum Optics and Laser Science, Imperial College London, London SW7 2PE, UK
\textsuperscript{3}Institut für Mathematische Physik, Technische Universität Braunschweig, 38106 Braunschweig, Germany

When locally exciting a quantum lattice model, the excitation will propagate through the lattice. The effect is responsible for a wealth of non-equilibrium phenomena, and has been exploited to transmit quantum information through spin chains. It is a commonly expressed belief that for local Hamiltonians, any such propagation happens at a finite “speed of sound”. Indeed, the Lieb-Robinson theorem states that in spin models, all effects caused by a perturbation are limited to a causal cone defined by a constant speed, up to exponentially small corrections. In this work we show that for translationally invariant bosonic models with nearest-neighbor interactions, this belief is incorrect: We prove that one can encounter excitations which accelerate under the natural dynamics of the lattice and allow for reliable transmission of information faster than any finite speed of sound. The effect is only limited by the model’s range of validity (eventually by relativity). It also implies that in non-equilibrium dynamics of strongly correlated bosonic models far-away regions may become quickly entangled, suggesting that their simulation may be much harder than that of spin chains even in the low energy sector.

Quantum spin chains—or more generally, quantum spin models on a lattice—are ubiquitous in condensed matter physics and quantum optics. They share the fundamental feature that perturbations will propagate through the lattice at a characteristic “speed of sound” \cite{1,2}. This effect plays an important role for a wealth of non-equilibrium phenomena in many-body systems, e.g., for the dynamics of relaxation processes towards equilibrium \cite{3,4}. In the context of quantum information science, it has been noted that excitations propagating through a spin chain may be used to transmit quantum information—thus turning a spin chain into a quantum channel. Here, the appealing feature is that the transport is not facilitated by engineered quantum gates, but rather by the natural time evolution of the lattice system \cite{5,6,7}.

Because in lattice models only neighboring systems interact with each other directly, it is intuitive to assume that the maximal propagation speed of excitations (i.e., the speed of sound) is finite and given by a value characteristic for each model. Indeed, an analogous statement is clearly true for relativistic systems, where a perturbation can have no influence outside its causal cone. Mathematical physics provides a rigorous justification for this observation in the form of Lieb-Robinson bounds \cite{11}: in spin lattice systems, perturbations can spread only linearly in time, up to exponentially small corrections. Recently, an analogous result has been proven to hold for a class of bosonic systems \cite{8}.

Interestingly, familiar as the belief that propagation of excitations in local models happens with a finite velocity may be: it is not quite right. We demonstrate that certain well-defined local bosonic models allow excitations to accelerate to arbitrarily high velocities. The effect is only limited by the range of validity of the model (which must certainly break down with the onset of relativistic effects). It occurs even for single excitations with bounded energy, traveling along a one-dimensional chain of bosons with translationally invariant nearest-neighbor interactions. From the quantum information perspective, we show that the quantum channel associated with this chain has a strictly positive information capacity, even after a time sub-linear in the length of the chain. While the presented models are non-integrable, we derive the results rigorously, without resorting to numerical means. We do so by considering single excitation spaces and—in this context unusual—invoke ideas from convex optimization.

There are several conceptual consequences of these results. It is now clear that any analysis of non-equilibrium processes in bosonic models must incorporate the possibility of far-away regions exchanging information on short time scales. In particular, it seems likely that simulating short-term dynamics even of the low-energy sector of bosonic models is much harder than for spin chains (where Lieb-Robinson bounds are the basis for efficient algorithms \cite{2,10}). Further, the results highlight the non-triviality of Lieb-Robinson bounds for spin chains with finite-dimensional constituents.

More practically—while the models we present very strongly violate any bound on propagation speeds—they have reasonable physical properties. Related models with similar features could well be realized by tuning the parameters of suitable physical systems, for example in arrays of coupled cavities with polariton excitations \cite{11}. This opens up the possibility of observing accelerating excitations experimentally and, potentially, of using bosonic chains as fast channels for quantum communication.
Local Hamiltonians and causality in spin chains. – A local Hamiltonian on \( n \) sites is of the form

\[
H = \sum_{j=1}^{n} h_j, \quad (1)
\]

where \( h_j \) acts non-trivially only on a finite number of adjacent sites. In what follows, we will restrict attention to the most relevant case of nearest-neighbor interactions. Quantum information transmission through spin chains with Hamiltonians as above has been extensively studied in the literature.

Before turning to bosonic models, let us first recall the precise situation for spin chains (\( d \)-level systems). The fact that there always exists a speed of sound—a maximal speed of information propagation—is the content of the following Lieb-Robinson bound \([1]\): If \( A \) and \( B \) are operators which act non-trivially only on some (distinct) regions of the chain, then there exist constants \( \mu, C > 0 \) and a velocity \( v > 0 \) such that

\[
\| [A(t), B] \| \leq C \| A \| \| B \| e^{-\mu (\text{dist}(A, B) - vt)}, \quad (2)
\]

for all times \( t \). Here, \( A(t) = e^{iHt} A e^{-iHt} \) is the time-evolved observable, \( \| \cdot \| \) the operator norm, and \( \text{dist}(A, B) \) denotes the number of sites between the supports of \( A \) and \( B \) (see Fig. 1). The above form may seem somewhat awkward at first sight. To get a more physical statement, one may verify that Eq. (2) implies that any effect a perturbation \( A \) can have on a distant observable \( B \) is exponentially suppressed outside the causal cone defined by \( |t| \geq \text{dist}(A, B)/v \). In particular, any non-exponentially suppressed quantum communication using this spin chain can happen at most with velocity \( v \) \([2]\). Due to the intuitive nature of this statement the above bound is often taken for granted or even dismissed as being “trivial”.

Supersonic communication. – Roughly, we say that a model allows for “supersonic” communication, if its dynamics can carry information over distances \( m \) in time \( t(m) \) which scales sub-linearly in \( m \). We will make this concept precise below.

The setting is a chain of \( n \) bosonic systems with nearest-neighbor interactions and open boundary conditions. The interactions should be translationally invariant \((h_i = h_j \text{ in Eq. (1)})\), up to the obvious modifications at the boundary. We refer to the left sites \( 1, \ldots, a \) as section \( A \) of the chain, whereas sites \( a + m, \ldots, n \) form part \( B \). We assume that the system is initially in some factorizing, translationally invariant pure state \( |\psi\rangle = |\psi_0\rangle^\otimes n \). A party in control of region \( A \) may now try to communicate with a party at \( B \) by either creating some excitations in her end of the chain, or else leaving the system untouched. More precisely, in the first case party \( A \) could apply a unitary operator \( U_A \) to region \( A \). At the receiving end, party \( B \) waits for some time \( t \) before probing whether a signal corresponding to some POVM element \( O_B \) is detected. The statistics are influenced by \( A \)’s decision and given by

\[
P_1 = \text{tr}[O_B e^{-itH} U_A |\psi\rangle \langle \psi| U_A^\dagger e^{itH}] \]

in case \( A \) has excited the chain and

\[
P_0 = \text{tr}[O_B e^{-itH} |\psi\rangle \langle \psi| e^{itH}] \]

in case \( A \) has not done so. The classical information capacity of the channel thus defined is a function of the signal strength \( \delta = |P_0 - P_1| \). If \( \delta \) scales as \( 1/\text{poly}(m) \), standard protocols involving polynomially many channels used in parallel may be employed to, say, “transmit radio signals through the quantum chain” with arbitrarily high fidelity. The last relevant quantity is the energy scale of the states involved, measured, e.g., by the variances \( E_0^2 = \langle \psi| H^2 |\psi\rangle \), \( E_1^2 = \langle \psi| U_A^\dagger H^2 U_A |\psi\rangle \). Low values for \( E_0, E_1 \) imply that the states are largely contained in the low energy sector of \( H \) \([13]\).

Set \( \varepsilon = \max\{E_0, E_1\} \). Below, we define three increasingly strong ways in which bosonic models could potentially violate finite bounds on the maximum propagation speed for signals. We go on to establish the main result: even the strongest scenario can be realized by reasonable Hamiltonians.

(i) Models which allow for arbitrarily fast transmission of information, using polynomial resources. More precisely, for every signal velocity \( m/t \), there should be suitable encoding operations \( U_A(m) \) and observables \( O_B(m) \) such that the signal strength \( \delta(m) \) is of order \( 1/\text{poly}(m) \). To obtain a reasonable protocol, the energy scale \( \varepsilon(m) \) should grow only polynomially in \( m \). While models of this type are interesting objects of study, it may be argued that their existence would not be too surprising. Indeed, as energy and time take reciprocal roles in quantum mechanics, it is plausible that adding “more energy” to the system may lead to faster dynamics. This motivates the next, more stringent, situation.

(ii) Models for which the signal velocity scales faster than the inverse energy. In addition to the definitions above, we demand that \( m \delta(m)/(\varepsilon(m)) \to \infty \) as \( m \to \infty \). For such models, the phenomenon cannot just be explained by the fact that unbounded Hamiltonians allow for signals with higher energies and thus faster dynamics.

In scenario (i), (ii) above, information propagates at arbitrarily high velocities—yet the distance covered is still linear in time (so the causal regions are cones with arbitrarily wide opening angles). The final situation is more demanding, requiring that excitations “speed up” as they propagate.

(iii) Models allowing for accelerating signals. Here, we require that the signal strength \( \delta \), the energy scale \( \varepsilon \) and, in fact, the encoding operation \( U_A \) do not depend on the distance \( m \), while the time \( t \) should scale sub-linearly in \( m \). In the next section, we discuss situations exhibiting behavior of type (iii) (and hence also of types (i,ii)) in a quite radical fashion.

Models. – The type of models we subsequently allow for are governed by nearest-neighbor Hamiltonians of the form

\[
H = \sum_{j=1}^{n-1} f_{j,j+1} + \sum_{j=1}^{n} g_j
\]

with interaction term \( f_{j,j+1} \) and on-site term \( g_j \). For \( f_{j,j+1} = (a^\dagger_j + a_j)(a^\dagger_{j+1} + a_{j+1}) \), and \( g_j = \mu a^\dagger_j a_j \) for \( \mu > 0 \) this is an instance of a harmonic chain. For an on-site interaction \( g_j = \mu a^\dagger_j a_j + U a^\dagger_0 a_j (a^\dagger_0 a_j - 1) \) and a hopping \( f_{j,j+1} = J(a^\dagger_j a_{j+1} + h.c.) \) this gives rise to the Bose-Hubbard model.

Let us spend some time to develop the physical intuition behind the constructions below. Consider a Hamiltonian with
harmonic hopping terms of the form $f_{j,j+1} = J(a_j^\dagger a_{j+1} + h.c.)$ and initial state $|\psi\rangle = |0, \ldots, 0\rangle$ (using the Fock basis). Diagonalizing the operators $f_{j,j+1}$, we see that the coupling strength between the sites grows as higher Fock layers become populated. Now consider an on-site interaction $g_j$ which does not preserve the Fock basis. If $g_j$ is e.g. a (low-order) polynomial in $a_j, a_j^\dagger$, then any application of $g_j$ will couple the state $|\psi(t)\rangle$ to higher and higher Fock layers. For the sake of the argument, one may be tempted to model the spectrum of local reductions to undergo a random walk on the local Fock basis, starting at $|0\rangle$ and gradually spreading to higher levels. Therefore, for such Hamiltonians, the coupling strength between sites could be expected to grow as time proceeds. Terms coupling different Fock states are common in physical models (as long as the Fock basis does not represent massive particles). An obvious example is given by the harmonic chain $f_{j,j+1} = (a_j^\dagger + a_j)(a_{j+1}^\dagger + a_{j+1})$, whose Hamiltonian contains terms of the form $a_j a_{j+1}$. In particular, as Hamiltonian dynamics by definition conserves energy, these terms do not inject energy into the system.

It is unclear at this point whether this increase in coupling strength gives rise to an acceleration of the dynamics. Our objective below is to show that this can indeed happen, by constructing a model which exhibits an extreme violation of the usual causality bounds — allowing for exponentially accelerating signals of constant strength. Other models compatible with the above intuition would naturally be expected to show similar accelerations, albeit not necessarily exponential ones. The model is constructed to violate the bounds in the strongest possible fashion while still being soluble. It serves as a proof of principle and as a worst case estimate for applications of Lieb-Robinson bounds for the simulation of dynamics.

Specific "exchange interaction" model. — We will pay special attention to the following model, defined for bosons with spin $1$ (so associated with the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}) \otimes \mathbb{C}^3$). We define for site $j$ the operators $A_{j,k,l} = [k, \downarrow] \langle l, \downarrow |$ and $B_{j,k,l} = [k, \downarrow] \langle l, \downarrow |$. The Hamiltonian is specified by

$$
\begin{align*}
 f_{j,j+1} &= \sum_{k,l=1}^{\infty} (2k-1)(iA_{j,k,l}^\dagger B_{j+1,l,k} + h.c.), \\
g_j &= 2\sum_{k=1}^{\infty} \langle k|\{k+1, \uparrow\}(k, \downarrow | + h.c.) + |0, \downarrow\rangle \langle 0, \downarrow |.
\end{align*}
$$

Note that $f_{j,j+1}$ may be looked at as a variant of the familiar exchange interaction. Clearly, $H$ is a legitimate bosonic Hamiltonian with translationally invariant nearest-neighbor interactions. We will prove our claim three steps.

1. Mapping to an excitation Hamiltonian. — To start with, $|0, \downarrow\rangle, |0, \downarrow\rangle, \ldots, |0, \downarrow\rangle$ is an eigenstate of the Hamiltonian. If we now place a single excitation with spin $\uparrow$ at the first site — so start with the initial state vector $|1, \uparrow; 0, \downarrow; \ldots; 0, \downarrow\rangle$ — we see that time evolution will only couple this to state vectors of the form $|l, \downarrow\rangle$.

2. Moments. — In this single excitation sector, time evolution corresponds to $\rho(t) = e^{-itE} \rho(0) e^{itE}$, for states on $\mathcal{K}$. It proves expedient to introduce the operators $X = \sum_{l=0}^{\infty} (l+1)|l\rangle\langle l|$, $P = \sum_{l=1}^{\infty} (l+1)|l\rangle\langle l| + h.c.$.

Note that $X$ corresponds to a discrete position operator, measuring twice the distance of the original model. There is a lot of structure in this model: the commutation relations between these operators form a closed algebra. Indeed,

$$
i[E, X] = P, \quad i[E, P] = 4X - 2I,$$

the algebraic completion of which being that of the Lie-group $su(2)$. Using the familiar Baker-Hausdorff formula, the Heisenberg picture time evolution of $X$ under $E$ is given by

$$X(t) = e^{itE} X e^{-itE} = X + t[i[E, X]] + \frac{t^2}{2}[i[E, [i[E, X]]] + \ldots.$$

One obtains a closed-form expression for $X(t)$ by exploiting the relations in the algebra $\mathcal{A} = \{E, X, P, I\}$ to iteratively solve these nested commutators. Explicitly

$$X(t) = X + P \sum_{l=1}^{\infty} \frac{t^{2l-1}4^{l-1}}{(2l-1)!} + (4X - 2I) \sum_{l=1}^{\infty} \frac{t^{2l}4^{l-1}}{(2l)!} = X - \frac{P}{2} \sinh(2t) + (X - \frac{1}{2}I)(\cosh(2t) - 1),$$

For the time evolution of $X$ starting from the single excitation $|0\rangle$, we hence find

$$\langle 0|X(t)|0\rangle = \frac{1}{2}(1 + \cosh(2t)).$$

Thus, the expectation value of $X$ is increasing exponentially in $t$. This fact alone, however, is not enough to show that we have signaling: It could be that the excitation develops a long asymptotic tail that leads to large first moments, but carries a small weight. Thus, further information
is needed. It turns out that knowledge of the second moments $\langle 0|X(t)^2|0\rangle - \langle 0|X(t)|0\rangle^2$ is sufficient to prove signaling using the convex optimization ideas below. An analogous—if more tedious—calculation arrives at

$$\langle 0|X(t)^2|0\rangle = \cosh^2(t) \cosh(2t).$$

3. **Hitting time from a convex optimization problem.**—For some site $m > 1$ define the hitting operator

$$T = \sum_{l=2m-1}^{\infty} ||l|| \langle l|l||.$$

So $P_1 = \text{tr}[\rho(t)T]$ is the signal a distant observer $m$ sites away from the origin may receive, compared to $P_0 = 0$. Hence, the set $A = \{1\}$ is the single first site, whereas $B = \{m + 1, \ldots, \infty\}$ is the natural right part of the chain. Let us set $M = 2m - 2$. We will now bound this expectation value by analytically solving a convex optimization problem:

- minimize $\sum_{l=0}^{\infty} p_l$,
- subject to $\sum_{l=0}^{\infty} (l + 1)p_l = a(t), \sum_{l=0}^{\infty} (l + 1)^2 X_l = b(t),$

$$\sum_{l=0}^{\infty} p_l = 1,$$

and $p_l \geq 0$ for all $l$, where $a(t) = (1 + \cosh(2t))/2$ and $b(t) = \cosh^2(t) \cosh(2t)$. So we minimize the signal, given first and second moments, as a worst case analysis. This is an (infinite-dimensional) linear program, and hence an instance of a convex optimization problem. We can readily get a bound to the optimal solution by identifying a suitable solution to the Lagrange dual problem \([15]\), which is

- maximize $-d^Ty$,
- subject to $FT y \geq -c,$

where $F_{1,j} = j, F_{2,j} = j^2, F_{3,j} = 1$ for $j = 1, 2, \ldots,$ and $d(t) = (a(t), b(t), 1)^T$. Also, $c = (0, 0, 1, 1, 1, \ldots)^T$, as a vector starting with $M$ zeros. The latter is an optimization problem over $y$. Any solution we can identify of the dual will give a lower bound to the primal. It can be shown explicitly that the subsequent vector $y^* = (-2(1 + M)/M^3, 1/M^4, (1 + M^2/M^2 - 1)^T$ is always a feasible solution of the dual problem. Taking the time $t^* = \log(M)$ gives

$$c^Ty^* \geq -d^Ty^* = 2a(\log(M))(1 + M)/M^3 - b(\log(M))/M^4 - (1 + M^2/M^2 + 1 =: g(M).$$

For this function we have that $\lim_{M \to \infty} g(M) = 3/8$, and $g(M) > 1/5$ for $M \geq 9$. The solution of the dual will hence give rise to a lower bound of our primal problem. Hence, after a time $t^*$ logarithmic in $m$, a signal of constant strength $\text{tr}[T \rho(t^*)] > 1/5$ will have reached party $B$! This means, of course—within the validity of the model—that we can in principle signal at any speed over arbitrary distances: The signal will not even decay, and the Holevo-$\chi$ and the classical information capacity of the associated quantum channel are indeed constant. One can communicate with an exponentially accelerating signal of type (iii) in the above classification.

**Summary and Outlook.**—In this work, we have shown that in systems of locally interacting bosons, excitations may accelerate and carry information faster than any finite speed of sound. Many results on the simulatability of dynamics and ground-state properties of spin systems (in particular using DMRG techniques) rely on Lieb-Robinson bounds \([16]\). Exceeding the usual difficulties associated with reasoning about infinite-dimensional systems, our examples imply further difficulties any generalization of such results to bosonic systems has to deal with. This remains true even if one restricts attention to low-energy sectors. Interestingly, while analytical and numerical treatments of such models face formidable difficulties, they may be simulated using analogue physical systems, such as coupled cavity arrays \([11]\) where massless polaritonic excitations are not subject to particle number conservation.

**Acknowledgments.**—This work has been supported by the EU (QAP, COMPAS, CORNER), the EPSRC, QIP-IRC, Microsoft Research, and the EURYI.

---

[1] E.H. Lieb and D.W. Robinson, Commun. Math. Phys. 28, 251 (1972); M.B. Hastings and T. Kom, ibid. 265, 781 (2006); B. Nachtergaele and R. Sims, ibid. 265, 119 (2006); M.B. Hastings, Phys. Rev. B 69, 104431 (2004).

[2] J. Eisert and T.J. Osborne, Phys. Rev. Lett. 97, 150404 (2006); S. Bravyi, M.B. Hastings, and F. Verstraete, ibid. 97, 050401 (2006).

[3] M. Cramer, C.M. Dawson, J. Eisert, and T.J. Osborne, Phys. Rev. Lett. 100 (2008); M. Cramer, A. Flesch, I.A. McCulloch, U. Schollwöck, and J. Eisert, ibid. 101, 063006 (2008).

[4] P. Calabrese and J. Cardy, Phys. Rev. Lett. 96 (2006) 136801; A. Lauchli and C. Kollath, J. Stat. Mech. (2008) P05018; N. Schuch, M.M. Wolf, K.G.H. Vollbrecht, J.I. Cirac, New J. Phys. 10, 033032 (2008).

[5] S. Bose, Contemp. Phys. 48, 13 (2007).

[6] M. Christandl, N. Datta, A. Ekert, and A.J. Landahl, Phys. Rev. Lett. 92, 187902 (2004); T.S. Cubitt, F. Verstraete, and J.I. Cirac, Phys. Rev. A 71, 052308 (2005); D. Burgarth, V. Giovannetti, and S. Bose, ibid. A 75, 063237 (2007).

[7] J. Eisert, M.B. Plenio, S. Bose, J. Hartley, Phys. Rev. Lett. 93, 190402 (2004); M.B. Plenio, J. Hartley, and J. Eisert, New J. Phys. 6, 36 (2004); M.J. Hartmann, M.E. Reuter, and M.B. Plenio, ibid. 8, 94 (2006).

[8] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims, [arXiv:0712.3820]. M. Cramer, A. Serafini, and J. Eisert, in *Quantum information and many body quantum systems* (Edizioni della Normale, Pisa 2008); O. Buerschaper, M.M. Wolf, J.I. Cirac, in preparation; A. Hamma et al., Phys. Rev. Lett. 102, 017204 (2009).

[9] C.K. Burrell and T.J. Osborne, Phys. Rev. Lett. 99, 167201 (2007).

[10] T.J. Osborne, Phys. Rev. Lett. 97, 157202 (2006); M.B. Hast-
The initial state vector is taken to be $|\psi\rangle = |0, \downarrow\rangle \otimes^n$. This is obviously an eigenstate of the Hamiltonian. Now we excite site 1, forming set $A = \{1\}$, by setting it to $|1, \uparrow\rangle$, while keeping the rest of the chain unaltered. We then see that the initial state vector after the excitation $|1, \uparrow\rangle|0, \downarrow\rangle \otimes^{(n-1)}$ couples to $|0, \downarrow\rangle|1, \downarrow\rangle|0, \downarrow\rangle \otimes^{(n-2)}$, and so forth to the former and to $|0, \downarrow\rangle|2, \uparrow\rangle|0, \downarrow\rangle \otimes^{(n-2)}$.

Now the pattern is clear: All excitations are contained in $\text{span} \{ |0, \downarrow\rangle \otimes^k |k, \downarrow\rangle |0, \downarrow\rangle \otimes^{(n-k-1)},
|0, \downarrow\rangle \otimes^k |k+1, \uparrow\rangle |0, \downarrow\rangle \otimes^{(n-k-1)} \}$, and each basis vector—except at the ends of the chain—couples to two further basis vectors. It is now a straightforward exercise to verify that in this subspace of excitations, one arrives at the above given effective Hamiltonian.

### Appendix B: Convex optimization problem

The matrix $F$ in the convex optimization problem defining our hitting time problem is given by

$$F = \begin{bmatrix} 1 & 2 & 3 & 4 & \ldots \\ 1 & 4 & 9 & 16 & \ldots \\ 1 & 1 & 1 & 1 & \ldots \end{bmatrix}.$$

The first row captures the first moments, the second the second moments, whereas normalization of the probability distribution is enforced by the third one—together with the constraint that all entries of $x$ are positive. The vector $c = (0, \ldots, 0, 1, \ldots, 1)^T$, as a vector starting with $M$ zeros. The primal problem can hence be written as

$$\begin{align*}
\text{minimize} & \quad c^T x, \\
\text{subject to} & \quad Fx = d, \\
& \quad x \geq 0,
\end{align*}$$

where $d(t) = (a(t), b(t), 1)^T$. This is a linear program in standard form, with a matrix inequality constraint, and positive elements of the objective vector $x$. In this form it is easiest to identify the dual problem. It is given by

$$\begin{align*}
\text{maximize} & \quad -d^Ty, \\
\text{subject to} & \quad F^Ty \geq -c,
\end{align*}$$

where now $y$ is not constrained to be positive. The constraints on the vector $y$ can equally be written as

$$\begin{align*}
jy_1 + j^2y_2 + y_3 + \delta_{j>M} & \geq 0
\end{align*}$$

for all $j$, where $\delta_{j>M}$ takes the value 1 if $j > M$ and is zero otherwise. The question is: Can we find for each $M$ a choice...
of \((y_1, y_2, y_3)\) and a time \(t\) such that we can meaningfully find a positive solution to the dual problem?

We can take any solution to the dual problem. We choose
\[
\begin{align*}
y_1 &= -2(1 + M)/M^3, \\
y_2 &= 1/M^4, \\
y_3 &= (1 + M)^2/M^2 - 1,
\end{align*}
\]
and will verify that it is indeed a solution with the appropriate properties. The intuition behind this construction is as follows: Consider the quadratic function \(f : [0, \infty) \to \infty\) as
\[
f(x) = x y_1 + x^2 y_2 + y_3.
\]
This is identical with the left hand side of Eq. (3), up to being defined on \([0, \infty)\). We require that \(f\) takes the value 0 for the first time exactly at \(x = M\), and that \(f(x) > -1\) for all \(x\).

Taking \(y_2 = 1/M^4\) then gives rise to the above construction, satisfying Eq. (3) for all \(M\) and all \(j\). We have \(j y_1 + j^2 y_2 + y_3 = 0\) exactly at \(j = M\), which is a desirable feature to get a meaningful bound to the original problem at hand. This is hence always a feasible solution to the dual problem. We will therefore obtain a lower bound to the optimal solution of the primal problem \(c^T x^*\) as
\[
c^T x^* \geq -d^T y \\
= 2a(t)(1 + M)/M^3 - b(t)/M^4 \\
- (1 + M)^2/M^2 + 1 =: g(M).
\]
Choosing the time \(t^* = \log(M)\), logarithmic in the distance, we find
\[
c^T x^* \geq 2a(\log(M))(1 + M)/M^3 - b(\log(M))/M^4 \\
- (1 + M)^2/M^2 + 1 =: g(M).
\]
Now it is not difficult to see that indeed, in the limit of large \(M\) we have
\[
\lim_{M \to \infty} g(M) = 3/8,
\]
and that \(g(M) > 1/5\) for all \(M \geq 9\). This means that for each large \(m\)—and hence each large \(M\)—we can arrive at a constant signal for an appropriate constant signal.

**Appendix C: Bose-Hubbard-type models**

In a sketchier fashion, we will now briefly address the question whether even Bose-Hubbard-type models
\[
H = \sum_{j=1}^{n} \left( (a_j^\dagger a_{j+1} + h. c.) + h(a_j^\dagger a_j) \right),
\]
with some function \(h : \mathbb{R}^+ \to \mathbb{R}^+\) can display such a behavior. Indeed, it can be seen that if \(h\) increases in \(n\) sufficiently strongly, leading to highly interacting particles, then there exists for each chain length \(n\) an \(N\) (polynomial in \(n\)) such that the initial state \(|\psi\rangle = |N + 1\rangle|N\rangle^{\otimes (n-1)}\)—so an additional particle at site \(A = \{1\}\) compared to \(|N\rangle^{\otimes n}\)—will lead to a signal at site \(B = \{n\}\) that is at most polynomially suppressed in \(n\). The signal increases linearly in \(N\). Hence, hopping models can also display violations of any finite bound on propagation speeds, at least in the sense (i) of the above classification.

For each chain length \(n\), we will consider a relevant \(n\)-dimensional subspace of the original Hilbert space
\[
\mathcal{L}_N = \text{span}\{|j_1, \ldots, j_n\}|j_k = N + 1, j_l = N \forall l \neq k\},
\]
for a suitable integer \(N\), and denote with \(\mathcal{L}_N^\perp = \mathcal{H} \setminus \mathcal{L}_N\) the orthogonal complement of the infinite-dimensional Hilbert space. The initial state vector of the chain of length \(n\) is then taken to be
\[
|\psi\rangle = |N + 1\rangle|N\rangle^{\otimes (n-1)} \in \mathcal{L}_N.
\]
This means that we will fill up all number states with bosons at each site up to \(N\), and add an additional particle at site \(A = \{1\}\) as a local excitation. We will now see that we can then approximate the dynamics of the chain within a spin chain restricted to a single excitation \(\mathcal{L}_N\) arbitrarily well. Define for site \(j\) the operator \(A_j = |N\rangle\langle N + 1|\), then the above chain is approximated in its dynamics by the dynamics of
\[
V = (N + 1) \sum_{j=1}^{n} \left( A_j^\dagger A_{j+1} + h. c. \right).
\]
This dynamics is specifically simple, and we can see that in a time \(t^*\) that is linear in \(N\) we will receive a signal at site \(B = \{n\}\) that is at most polynomially suppressed in \(n\). Clearly means that this signal will be received at an arbitrarily velocity, if \(N\) is sufficiently large. To see that this mapping can be done to arbitrary approximation, we need the subsequent observation:

**Lemma 1** For any Hermitian matrix \(M\), partitioned as
\[
M = \begin{bmatrix} A & B \\ B^\dagger & C \end{bmatrix},
\]
we have that for all \(t \in [0, \infty)\),
\[
\lim_{t \to \infty} \sup_{K \in \mathcal{M}_x} \|Q e^{i t K} O - e^{i t A}\| = 0
\]
where
\[
\mathcal{M}_x = \left\{ M + \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} : P \geq x I \right\}, \ O = \begin{bmatrix} I & 0 \end{bmatrix}.
\]
Here, \(\|\cdot\|\) denotes the operator matrix or vector norm. To prove this, note that
\[
\sum_{j=1}^{d} \lambda_j \left( \begin{bmatrix} A & B \\ B^\dagger & C + P \end{bmatrix} \right) = \inf_Q \text{tr} \left[ Q \left( \begin{bmatrix} A & B \\ B^\dagger & C + P \end{bmatrix} \right) \right],
\]
where the infimum is taken over \(Q\) that are projectors of rank \(d\), when \(A\) is a \(d \times d\)-matrix. This infimum exists for any
$M \in \mathcal{M}_z$, call it $Q_M$. Since $\|A\| = c$ is constant, we have that
\[
\lim_{x \to \infty} \sup_{M \in \mathcal{M}_z} \left\| Q_M - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\| = 0.
\]
Hence, since $|e^{i\lambda_j(M)}| = 1$, the assertion follows.

Subsequently, the submatrix $A$ will be identified with $H$ restricted to $L_N$, whereas $C$ is the restriction to the orthogonal complement $L_N^\perp$. Using this observation, we see that for each $\varepsilon > 0$ we can find a function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that the following property is satisfied: For each chain length $n$ there exists an $N$ such that for the above hitting time $t^*$
\[
\| e^{-it^*H} \psi \rangle - e^{-it^*V} \psi \rangle \| < \varepsilon
\]
for all $t \in [0, t^*]$. But this means that we can signal with this chain, in the sense of (iii) in the above classification, as the class of restricted models has this property.