Higher-derivative terms in one-loop effective action
for general trajectories of D-particles
in Matrix theory

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Abstract

The one-loop effective action for general trajectories of D-particles in Matrix theory is calculated in the expansion with respect to the number of derivatives up to six, which gives the equation of motion consistently. The result shows that the terms with six derivatives vanish for straight-line trajectories, however, they do not vanish in general. This provides a concrete example that non-renormalization of twelve-fermion terms does not necessarily imply that of six-derivative terms.
1. Introduction

To construct a consistent quantum theory including gravity is one of the most important problems in theoretical physics today. The most natural approach, namely the second quantization of the metric field, has turned out to suffer from non-renormalizability. We have to search for another way to describe gravity which reconciles controllable behavior in the short-distance region with general covariance in the low-energy region.

Matrix theory [1], which was proposed as a matrix model for M theory [2], provided a novel possibility of quantum description of gravity based on the description of Dirichlet branes [3] in terms of the super Yang-Mills theory [4, 5]. The effective action of D-particles in Matrix theory calculated in the loop expansion is in precise agreement with that obtained from the eleven-dimensional supergravity in the low-energy region up to two loops [6, 7, 8, 9]. However, we only understand the reason why gravity could emerge in Matrix theory indirectly through the superstring theory which underlies Matrix theory. It is desirable to understand the reason within the framework of the super Yang-Mills theory. The supersymmetries of the model certainly play an important role there. However, it is still uncertain to what extent the effective action of Matrix theory is constrained by them. It would be necessary to investigate the nature of interactions described by Matrix theory from various viewpoints for deeper understanding.

The one-loop effective action of Matrix theory has been discussed from many aspects. In particular, it was shown that it produces interactions of the linearized supergravity between an arbitrary pair of M-theory excitations [10] including the effects of arbitrary background configurations of the fermionic field in Matrix theory [11]. Besides the contribution which corresponds to the supergravity, the one-loop effective action of Matrix theory contains higher-derivative corrections. This is not unexpected since Matrix theory is a model for M theory, not for the supergravity: It must contain corrections to the supergravity in the short-distance region. The higher-derivative corrections at one loop can be considered as a part of such corrections. The primary purpose of the present paper is to investigate these corrections at one loop.

In investigating them, we want to extract information on interaction Lagrangian from the scattering phase shift, which is the quantity we can first obtain from Matrix-theory
calculations, with the criterion that the Lagrangian should produce the phase shift correctly. We first point out an ambiguity in this procedure which is related to the difference between the time in Matrix theory and that in the Lagrangian, and then fix it with an assumption that the two coincide. This assumption ensures that the equation of motion of D-particles \[9\] is consistently derived from the Lagrangian. The result indicates that the interaction Lagrangian contains terms with six derivatives at one loop.

This observation raises an interesting question related to the constraints imposed by the supersymmetries. It was shown by Paban, Sethi and Stern \[12, 13\] that eight- and twelve-fermion terms can appear only at one loop and at two loops, respectively. Since eight- and twelve-fermion terms belong to the same multiplets of the supersymmetries as four- and six-derivative terms, respectively, their results indicate that the latter terms are strongly constrained by the supersymmetries as well. However, if there really exist six-derivative terms at one loop, the mechanism which constrains the six-derivative terms would be more complicated than that for the twelve-fermion terms.

These motivate us to develop systematic methods to investigate higher-derivative terms in the effective action of Matrix theory. We explicitly calculate the one-loop effective action for general trajectories of D-particles in the expansion with respect to the number of derivatives up to six hoping that the result provides a basis for future studies to understand the role of the supersymmetries and the reason why gravity could emerge in Matrix theory.

The organization of the paper is as follows. In Section 2, after pointing out the ambiguity in determining the interaction Lagrangian from the phase shift, we argue that there exist six-derivative terms at one loop under the assumption that the time in Matrix theory coincides with that in the Lagrangian. We then determine the complete form of the interaction Lagrangian up to six derivatives by calculating the effective action for general trajectories of D-particles in Section 3. Section 4 is devoted to the conclusions and discussions.
2. Extraction of interaction Lagrangian from scattering phase shift

Let us reconsider the procedure to extract information on the interaction Lagrangian from the scattering phase shift of D-particles in Matrix theory. Matrix theory is defined by the action of the super Yang-Mills theory dimensionally reduced from 9+1 dimensions to 0+1 dimension [1]

\[
S = \int_{-\infty}^{\infty} dt \left[ \frac{1}{2} \text{tr} D_t X^n D_t X^n + \frac{1}{4} g^2 \text{tr}[X^n, X^m][X^n, X^m] + \frac{1}{2} \text{tr}(i\theta^T D_t \theta + g\theta^T \gamma^n[X^n, \theta]) \right],
\]  

(2.1)

with

\[
D_t X^n = \partial_t X^n - ig[A, X^n], \quad D_t \theta = \partial_t \theta - ig[A, \theta],
\]

(2.2)

where \( g \) is the Yang-Mills coupling constant and \( n, m = 1, 2, \cdots, 9 \) stand for transverse dimensions. \( X^n(t), A_{ij}(t) \) and \( \theta_{ij}(t) \) are \( N \times N \) Hermitian-matrix fields. Eigenvalues of the bosonic field \( X^n(t) \) are interpreted as transverse coordinates of D-particles and the fermionic field \( \theta(t) \) is an \( SO(9) \) Majorana spinor which represents spins of D-particles. We take a representation of \( SO(9) \) gamma matrices \( \gamma^n \) such that \( \gamma^n \) are real and symmetric satisfying \( \{\gamma^n, \gamma^m\} = 2\delta^{nm} \).

We will perform our computations in Euclidean formulation, defining the Euclidean time \( \tau \) and gauge field in Euclidean time \( \tilde{A} \) as

\[
\tau = it, \quad \tilde{A} = -iA.
\]

(2.3)

The Euclidean action is then

\[
\bar{S} = \int_{-\infty}^{\infty} d\tau \left[ \frac{1}{2} \text{tr} D_\tau X^n D_\tau X^n - \frac{1}{4} g^2 \text{tr}[X^n, X^m][X^n, X^m] + \frac{1}{2} \text{tr}(\theta^T D_\tau \theta + g\theta^T \gamma^n[X^n, \theta]) \right],
\]

(2.4)

with

\[
D_\tau X^n = \partial_\tau X^n - ig[\tilde{A}, X^n], \quad D_\tau \theta = \partial_\tau \theta - ig[\tilde{A}, \theta].
\]

(2.5)

We consider diagonal background configurations \( B^n(\tau) \) of the bosonic field \( X^n(\tau) \)

\[
X^n = \frac{1}{g} B^n + Y^n, \quad B^n_i(\tau) = \delta_{ij} r^n_j(\tau),
\]

(2.6)
and use the standard background field gauge condition

\[- \partial_\tau \tilde{A} + i [B^n, Y^n] = 0. \tag{2.7}\]

The one-loop effective action \(\tilde{\Gamma}^{(1)}\) is obtained from the functional determinant of the quadratic part of the action \(\tilde{S}_{(2)}\) expanded around the background (2.6) after the gauge fixing as follows:

\[
\exp[-\tilde{\Gamma}^{(1)}] = \int D\bar{Y}n D\tilde{A} D\bar{c} Dc D\theta \exp[-\tilde{S}_{(2)}], \tag{2.8}\]

\[
\tilde{S}_{(2)} = \int_{-\infty}^{\infty} d\tau \left[ \frac{1}{2} Y^n_{ij} (-\partial_\tau^2 + r_{ij}(\tau)^2) Y^n_{ji} + \frac{1}{2} \tilde{A}_{ij} (-\partial_\tau^2 + r_{ij}(\tau)^2) \tilde{A}_{ji} - 2i \partial_\tau r^n_{ij}(\tau) \tilde{A}_{ij} Y^n_{ji} + \bar{c}_{ij} (-\partial_\tau^2 + r_{ij}(\tau)^2) c_{ji} + \frac{1}{2} (\theta^T g_{ij}, \theta) \right], \tag{2.9}\]

where we defined

\[
r^n_{ij}(\tau) = r^n_i(\tau) - r^n_j(\tau), \quad r_{ij}(\tau) = \sqrt{r^n_{ij}(\tau) r^n_{ji}(\tau)}, \quad \theta_{ij}(\tau) = \gamma^n r^n_{ij}(\tau). \tag{2.10}\]

After the Gauss integrations, we have

\[
\tilde{\Gamma}^{(1)} = \sum_{i<j} \left[ \text{tr} \ln \left( 1 - 4 \partial_\tau r^n_{ij}(\tau) - \partial_\tau^2 + r_{ij}(\tau)^2 \right) \frac{1}{\partial_\tau^2 + r_{ij}(\tau)^2} \right] - \frac{1}{2} \text{Tr} \ln \left( 1 + \partial_\tau \theta_{ij}(\tau) \frac{1}{-\partial_\tau^2 + r_{ij}(\tau)^2} \right), \tag{2.11}\]

where tr denotes the trace over the functional space and Tr is that over the functional space and the spinor indices. Note that we do not assume that the background (2.6) satisfies the equation of motion in deriving (2.11) so that we can calculate the effective action for general trajectories of D-particles based on (2.11). We will concentrate on a pair of D-particles and omit the subscripts such as \(ij\) in what follows.

The one-loop effective action for the configuration which represents the straight-line trajectories of D-particles

\[
r^n(\tau) = v^n \tau + x^n, \tag{2.12}\]

\[\text{†}\] Similar calculations have recently been done in the reference [14] in the context of the generalized conformal symmetry [13, 14, 15].
is exactly evaluated \( \tilde{\Gamma} \) using the proper-time integration as
\[
\tilde{\Gamma}^{(1)} = -\int_{-\infty}^{\infty} d\tau \int_{0}^{\infty} \frac{d\sigma}{\sigma} 16 \sinh^4 \frac{\sigma v}{2} \Delta(\sigma, \tau, \tau),
\]
(2.13)
where the proper-time propagator \( \Delta(\sigma, \tau_1, \tau_2) \) is defined by
\[
\Delta(\sigma, \tau_1, \tau_2) \equiv \exp[-\sigma(-\partial^2_{\tau_1} + r(\tau_1)^2)] \delta(\tau_1 - \tau_2),
\]
(2.14)
and its explicit form when \( \tau_1 = \tau_2 = \tau \) is
\[
\Delta(\sigma, \tau, \tau) = \sqrt{\frac{v}{2\pi \sinh(2\sigma v)}} \exp \left[-v \left(\tau + \frac{x \cdot v}{v^2}\right)^2 \tanh(\sigma v) - \sigma \left(x^2 - \frac{(x \cdot v)^2}{v^2}\right)\right].
\]
(2.15)
The integration over \( \tau \) is easily carried out and that over \( \sigma \) can also be performed after expanding the integrand with respect to \( v \)
\[
\tilde{\Gamma}^{(1)} = -\frac{v^3}{b^6} + 0 \frac{v^5}{b^{10}} - \frac{3}{2} \frac{v^7}{b^{14}} + O(v^9),
\]
(2.16)
where \( b \) is the impact parameter
\[
b = \sqrt{x^2 - \frac{(x \cdot v)^2}{v^2}}.
\]
(2.17)
This phase shift (2.16) precisely coincides with the one coming from the following interaction Lagrangian in the eikonal approximation:
\[
\mathcal{L} = C_4 \frac{v^4}{r^7} + C_6 \frac{v^6}{r^{11}} + C_8 \frac{v^8}{r^{15}} + O(v^{10}).
\]
(2.18)
In fact, the integration over \( \tau \) after substituting the straight-line trajectory (2.12) into (2.18) gives
\[
\int_{-\infty}^{\infty} d\tau \mathcal{L} = \frac{16C_4 v^3}{15} \frac{1}{b^6} + \frac{256C_6 v^5}{315} \frac{1}{b^{10}} + \frac{2048C_8 v^7}{3003} \frac{1}{b^{14}} + O(v^9).
\]
(2.19)
The coefficients of the terms in the Lagrangian (2.18) can be determined by comparing two expressions (2.16) and (2.19). Thus, we obtain the Lagrangian
\[
\mathcal{L} = -\frac{15}{16} \frac{v^4}{r^7} + 0 \frac{v^6}{r^{11}} - \frac{9009}{4096} \frac{v^8}{r^{15}} + O(v^{10}),
\]
(2.20)
which yields the phase shift (2.16).
However, this is not the unique Lagrangian which gives the phase shift \((2.16)\). For example, there are four possible terms which contain six derivatives if we allow interactions of the form \(v \cdot r\):

\[
\int_{-\infty}^{\infty} d\tau \left[ A \frac{v^6}{r^{11}} + B \frac{v^4(v \cdot r)^2}{r^{13}} + C \frac{v^2(v \cdot r)^4}{r^{15}} + D \frac{(v \cdot r)^6}{r^{17}} \right]. \tag{2.21}
\]

Since the integration over \(\tau\) after substituting the straight-line trajectory \((2.12)\) into \((2.21)\) gives

\[
\frac{256}{45045} (143A + 13B + 3C + D) \frac{v^5}{b^{10}}, \tag{2.22}
\]

the vanishing contribution proportional to \(v^5/b^{10}\) in \((2.16)\) only requires a relation among the coefficients

\[
143A + 13B + 3C + D = 0. \tag{2.23}
\]

The term which we took in \((2.20)\) is only a special one which satisfies the relation \((2.23)\). Thus, the criterion that the interaction Lagrangian should produce the phase shift \((2.16)\) correctly alone cannot determine the form of the Lagrangian uniquely. We need some assumptions or further physical inputs to determine them.

It should be noted that this ambiguity is related to the difference between the time \(\tau\) in the Matrix-theory effective action \((2.13)\) and that in the Lagrangian such as \((2.20)\) or \((2.21)\). For example, the value of the Lagrangian \((2.20)\) for the configuration \((2.12)\) depends on \(\tau\) only through a combination \(r(\tau) = \sqrt{v^2 \tau^2 + 2v \cdot x \tau + x^2}\). The \(\tau\)-dependence of \(\tilde{\Gamma}^{(1)}\) \((2.13)\) is different from the above form and therefore the two \(\tau\)'s in \(\tilde{\Gamma}^{(1)}\) \((2.13)\) and \((2.20)\) are different when we interpret that the phase shift \((2.16)\) is coming from the Lagrangian \((2.21)\).

Now, realizing the difference between the two \(\tau\)'s, a natural assumption in determining the form of the interaction Lagrangian is that the time \(\tau\) in the interaction Lagrangian should be equal to \(\tau\) appeared in \(\tilde{\Gamma}^{(1)}\). This assumption ensures that the equation of motion of D-particles calculated in the one-loop approximation in \([1]\) is consistently derived from the resultant Lagrangian. In general, it is not obvious how the equation of motion derived from another Lagrangian such as \((2.20)\) is related to the variational equation \(\delta \tilde{\Gamma}/\delta r^m(\tau) = 0\) in Matrix theory because of the difference of the time. This assumption seems plausible naively because the action of Matrix theory originates in the low-energy effective action.
of D-particles in the type IIA superstring theory in the gauge that the parametrization
of the trajectories of D-particles is equal to the time in ten-dimensional space-time.

Under the above assumption, the form of the Lagrangian is uniquely determined. The
integrand of $\tilde{\Gamma}(1)$ (2.13) should be rewritten in the form which depends on $\tau$ through the
following combinations:

$$r(\tau)^2 = v^2 \tau^2 + 2v \cdot x \tau + x^2,$$
$$v \cdot r(\tau) = v^2 \tau + v \cdot x.$$ (2.24)

It follows that $x^2$ should be replaced with $r^2 - v^2 \tau^2 - 2v \cdot x \tau$ and the remaining $v \cdot x$ with $v \cdot r - v^2 \tau$. After these replacements, $\tau$ must appear only within $r(\tau)^2$ or $v \cdot r(\tau)$ in order for the assumption to be consistent. It is indeed the case for $\Delta(\sigma, \tau, \tau)$

$$\Delta(\sigma, \tau, \tau) = \sqrt{\frac{v}{2\pi \sinh(2\sigma v)} \exp \left[ -\frac{(v \cdot r(\tau))^2}{v^3} \right]},$$ (2.25)

and $\tilde{\Gamma}(1)$ (2.13) is rewritten as follows:

$$\tilde{\Gamma}(1) = -\int_{-\infty}^{\infty} d\tau \int_{0}^{\infty} d\sigma \frac{4v^4}{\sigma^3} \sinh^2(2\sigma v) \exp \left[ -\frac{(v \cdot r(\tau))^2}{v^3} \right].$$ (2.26)

We can obtain the explicit form of the Lagrangian in the expansion with respect to $v$, which is

$$\tilde{\Gamma}(1) = \int_{-\infty}^{\infty} d\tau \left[ -\frac{15}{16} v^4 + \frac{315}{128} \frac{v^6}{r^7} - \frac{3465}{128} \frac{v^4(v \cdot r)^2}{r^{13}} + O(v^8) \right].$$ (2.27)

The terms with six derivatives in (2.27) satisfy the relation (2.23) so that the phase shift (2.16) is reproduced correctly up to $v^6$ as it should be.

The first term in (2.27) proportional to $v^4/r^7$ is the familiar one which corresponds
to the supergravity contribution [1, 6, 7]. The appearance of the terms which contain six
derivatives is interesting from the aspect of the supersymmetric constraint as mentioned in
Section 1. It was shown that twelve-fermion terms without derivatives only appear at two-
loop effective action [13] and this statement is often referred as the non-renormalization
of twelve-fermion terms. This in particular implies that there are no twelve-fermion
terms at one loop. Although the six-derivative terms belong to the same multiplet of
the supersymmetries as the twelve-fermion terms because the number of derivatives plus half of the number of fermions is the same, the appearance of the six-derivative terms at one loop does not contradict the non-renormalization of twelve-fermion terms if the supersymmetric completion of the six-derivative terms is achieved without twelve-fermion terms. This can happen and in fact the tree-level effective action provides a simpler example of such cases

\[ \tilde{\Gamma}^{(0)} = \int_{-\infty}^{\infty} d\tau \left[ \frac{1}{2g^2} v^2 + \frac{1}{2g^2} \psi \dot{\psi} \right], \]  

(2.28)

where \( \psi \) is the fermionic background. There is a two-derivative term but no four-fermion terms. The same thing can happen for four- or six-derivative terms at higher loops and such possibility is not excluded by the arguments put forward so far. Therefore, the non-renormalization of eight- or twelve-fermion terms itself does not ensure the non-renormalization of four- or six-derivative terms, respectively. If there really exist the six-derivative terms in the one-loop effective action (2.27), this provides a concrete example. This argument shows the importance of the question whether there exist six-derivative terms in the one-loop effective action.

The expression (2.27) indicates that there are six-derivative terms, however, it is not conclusive yet: The terms might be arranged to a total derivative if there are appropriate terms which contain acceleration. We can present evidence in favor of the fact that there do exist six-derivative terms at one loop from the result of the one-point proper function calculated in the one-loop approximation [9]. The one-point proper function is nothing but the recoil acceleration \( \delta \alpha^n(\tau) \) [9]:

\[ \delta \alpha^n(\tau) = \left. \frac{\delta \tilde{\Gamma}^{(1)}}{\delta r^n(\tau)} \right|_{r^n(\tau)=v^n r+x^n} = \int_{0}^{\infty} d\sigma \left[ 32 r^n(\tau) \sinh^4 \frac{\sigma v}{2} \Delta(\sigma, \tau, \tau) + 32 \frac{v^n}{v} \cosh \frac{\sigma v}{2} \sinh^3 \frac{\sigma v}{2} \partial_\tau \Delta(\sigma, \tau, \tau) \right]. \]  

(2.29)

Using the expression of \( \Delta(\sigma, \tau, \tau) \) in terms of \( r(\tau)^2 \) and \( v \cdot r(\tau) \) (2.25), the explicit form of \( \delta \alpha^n(\tau) \) in the expansion with respect to \( v \) is given by

\[ \delta \alpha^n(\tau) = \frac{105}{16} \frac{v^4 r^n}{r^9} - \frac{105}{4} \frac{v^2 (r \cdot v) v^n}{r^9}. \]
The first two terms in (2.30) precisely coincide with the contributions coming from the Euler-Lagrange equation derived from the first term in (2.27). The fact that there are non-vanishing terms with six derivatives in (2.30) shows that the six-derivative terms in (2.27) cannot be arranged to a total derivative.

We want to comment here that the equation of motion derived from another Lagrangian such as (2.20) does not coincide with (2.30) because of the difference of the time. It may be possible to make them coincide if we properly redefine the time in (2.30). However, the time in the other Lagrangians such as (2.20) is not simply related to that in $\tilde{\Gamma}^{(1)}$ (2.13) in general. In fact, an effective way to obtain (2.20) is the following transformation of $\tau$ after exchanging the order of integrations between $\tau$ and $\sigma$ in $\tilde{\Gamma}^{(1)}$:

$$
\tilde{\tau} = \sqrt{\text{tanh}(\sigma v)} \left( \tau + \frac{x \cdot v}{v^2} \right) - \frac{x \cdot v}{v^2}. \quad (2.31)
$$

Then, $\tilde{\Gamma}^{(1)}$ is expressed using $\tilde{\tau}$ as

$$
\tilde{\Gamma}^{(1)} = - \int_{0}^{\infty} \frac{d\sigma}{\sigma} \int_{-\infty}^{\infty} d\tilde{\tau} \sqrt{\frac{\sigma v}{\text{tanh}(\sigma v)}} \left[ 16 \sinh^4 \frac{\sigma v}{2} \sqrt{\frac{v}{2\pi \sinh(2\sigma v)}} \exp[\sigma r(\tilde{\tau})^2] \right] - \int_{-\infty}^{\infty} d\tilde{\tau} \left[ - \int_{0}^{\infty} d\sigma \frac{16v \sinh^4 \frac{\sigma v}{2} e^{-\sigma r(\tilde{\tau})^2}}{\sinh(\sigma v) \sqrt{4\pi \sigma}} \right], \quad (2.32)
$$

with $r(\tilde{\tau}) = \sqrt{v^2 \tilde{\tau}^2 + 2v \cdot x \tilde{\tau} + x^2}$. The coefficients in (2.20) is reproduced by expanding the integrand of (2.32) with respect to $v$ and performing the integration over $\sigma$. The peculiar relation (2.31) which depends on $v$ and $\sigma$ enforces a significant change of the usual interpretation that eigenvalues of $\langle X^n(\tau) \rangle$ represent the transverse positions of D-particles at the time $\tau$ since we have to use $\tilde{\tau}$ as time when we adopt the Lagrangian (2.20).

It is uncertain whether such change of the interpretation makes sense, but we believe that our assumption that the time in the Lagrangian coincides with that in the effective action of Matrix theory is the natural one which automatically ensures the consistency between the Lagrangian and the equation of motion.

The argument presented here shows that there do exist six-derivative terms in the one-loop effective action $\tilde{\Gamma}^{(1)}$. However, the calculations performed so far cannot determine
the complete form of the six-derivative terms since the effective action for the straight-line trajectories cannot detect terms with second or higher derivative of coordinate. In the next section, we will compute the effective action for general trajectories and determine them, which is possible under our assumption that the time in $\tilde{\Gamma}^{(1)}$ is equal to that in the Lagrangian.

### 3. One-loop effective action for general trajectories in derivative expansion

We cannot evaluate the one-loop effective action (2.11) exactly for general background configurations so that we calculate it in the expansion with respect to the number of derivatives up to six. Let us introduce the following abbreviations to simplify expressions:

$$
\dot{r}^n \equiv \partial_\tau r^n(\tau), \quad \ddot{r}^n \equiv \partial_\tau^2 r^n(\tau), \quad \cdots, \\
\Delta \equiv \frac{1}{-\partial_\tau^2 + r(\tau)^2}.
$$

(3.1)

Then, the one-loop effective action (2.11) is concisely expressed as

$$
\tilde{\Gamma}^{(1)} = \text{tr} \ln(1 - 4\dot{r}^n \Delta \ddot{r}^n) - \frac{1}{2} \text{Tr} \ln(1 + \dot{r} \Delta). 
$$

(3.2)

To determine the form of six-derivative terms, we expand the effective action with respect to the number of $\Delta$'s and evaluate the following contributions:

$$
\tilde{\Gamma}^{(1)} = \tilde{\Gamma}_{2}^{(1)} + \tilde{\Gamma}_{4}^{(1)} + \tilde{\Gamma}_{6}^{(1)} + O(\partial_\tau^8),
$$

(3.3)

where

$$
\tilde{\Gamma}_{2}^{(1)} \equiv -4\text{tr}(\dot{r}^n \Delta \ddot{r}^n) + \frac{1}{4} \text{Tr}(\dot{r} \Delta)^2, \\
\tilde{\Gamma}_{4}^{(1)} \equiv -8\text{tr}(\dot{r}^n \Delta \ddot{r}^n)^2 + \frac{1}{8} \text{Tr}(\dot{r} \Delta)^4, \\
\tilde{\Gamma}_{6}^{(1)} \equiv -\frac{64}{3} \text{tr}(\dot{r}^n \Delta \ddot{r}^n)^3 + \frac{1}{12} \text{Tr}(\dot{r} \Delta)^6.
$$

(3.4, 3.5, 3.6)

In evaluating them, it is convenient to “normal order” the expressions first. By “normal ordering”, we mean ordering expressions which contain functions of $\tau$, derivatives with
respect to $\tau$ and $\Delta$'s to the form $f(\tau)\partial^\tau \Delta^m$ using the commutation relation $[\partial_{\tau}, \tau] = 1$. The following formulas are useful in the normal ordering:

\[
[\partial_{\tau}, f(\tau)] = \dot{f}(\tau),
\]

\[
[\Delta, f(\tau)] = \Delta(\dot{f}(\tau) + 2\dot{f}(\tau)\partial_{\tau})\Delta,
\]

\[
[\Delta, \partial_{\tau}] = 2\Delta(r \cdot \dot{r})\Delta.
\]

Note that these commutators increase the number of derivatives by at least one. Then, the expressions $\Delta f(\tau)$ and $\Delta \partial_{\tau}$ are normal ordered as follows:

\[
\Delta f(\tau) = f(\tau)\Delta + 2\dot{f}(\tau)\partial_{\tau}\Delta^2 + \ddot{f}(\tau)(1 + 4\partial^2_{\tau})\Delta^2 + O(\partial^3_{\tau}),
\]

\[
\Delta \partial_{\tau} = \partial_{\tau}\Delta + 2(r \cdot \dot{r})\Delta^2 + O(\partial^2_{\tau}).
\]

We should make a technical but important remark here. When we count the number of derivatives, we should not count derivatives acting directly on $\Delta$'s since they do not necessarily generate derivatives of coordinate in the final form as we will see. For example, we should count the second term on the right-hand side of (3.10) as $O(\partial_{\tau})$, not as $O(\partial^2_{\tau})$.

Let us begin with $\tilde{\Gamma}^{(1)}_2$ and $\tilde{\Gamma}^{(1)}_6$. $\tilde{\Gamma}^{(1)}_2$ is shown to vanish

\[
\tilde{\Gamma}^{(1)}_2 = 0,
\]

which follows from

\[
\gamma^a_{ab} \gamma^m_{ba} = 16\delta^m_{nm}.
\]

$\tilde{\Gamma}^{(1)}_6$ is easily normal ordered as follows:

\[
\tilde{\Gamma}^{(1)}_6 = -\frac{64}{3}\text{tr}[\dot{r}^6 \Delta^6] + \frac{1}{12}\text{Tr}[\dot{r}^6 \Delta^6] + O(\partial^7_{\tau}).
\]

Then using $\phi \phi = a^2 \mathbf{1}$, we have

\[
\tilde{\Gamma}^{(1)}_6 = -20\text{tr}[\dot{r}^6 \Delta^6] + O(\partial^7_{\tau}).
\]

We go on to the last one, $\tilde{\Gamma}^{(1)}_4$. It is transformed using

\[
\gamma^k_{ab} \gamma^\ell_{bc} \gamma^m_{cd} \gamma^n_{da} = 16(\delta^{k\ell} \delta^{mn} - \delta^{km} \delta^{\ell n} + \delta^{kn} \delta^{\ell m}),
\]

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to the following form:

\[ \tilde{\Gamma}^{(1)}_4 = -4\text{tr}(\dot{r}^n \Delta \dot{r}^n \Delta \dot{r}^m \Delta \dot{r}^m \Delta) - 2\text{tr}(\dot{r}^n \Delta \dot{r}^m \Delta \dot{r}^m \Delta \dot{r}^m \Delta). \]  

(3.17)

Let us normal order the expression \( \text{tr}(\dot{r}^k \Delta \dot{r}^l \Delta \dot{r}^m \Delta \dot{r}^n \Delta) \). There are three \( \Delta \)'s to be moved to the right. One way of doing that is to move the three one by one in the order that the right one, the middle one and the left one. Then, it is normal ordered as follows:

\[
\begin{align*}
\text{tr}(\dot{r}^k \Delta \dot{r}^l \Delta \dot{r}^m \Delta \dot{r}^n \Delta) &= \text{tr}[\dot{r}^k \dot{r}^l \dot{r}^m \dot{r}^n \Delta^4] \\
&= \text{tr}[\dot{r}^k \dot{r}^l \dot{r}^m \dot{r}^n \Delta^4] \\
&+ 2\{\dot{r}^k \dot{r}^l \dot{r}^m [\partial_r, \dot{r}^n] + \dot{r}^k \dot{r}^l [\partial_r, \dot{r}^m \dot{r}^n] + \dot{r}^k [\partial_r, \dot{r}^l \dot{r}^m \dot{r}^n]\} \partial_r \Delta^5 \\
&+ \{\dot{r}^k \dot{r}^l \dot{r}^m [\partial_r, \dot{r}^n] + \dot{r}^k \dot{r}^l [\partial_r, \dot{r}^m \dot{r}^n] + \dot{r}^k [\partial_r, \dot{r}^l \dot{r}^m \dot{r}^n]\}(1 + 4\partial_r^2 \Delta) \Delta^5 \\
&+ \{12\dot{r}^k \dot{r}^l \dot{r}^m [\partial_r, \dot{r}^n] + 8\dot{r}^k \dot{r}^l \dot{r}^m [\partial_r, \dot{r}^n] + 4\dot{r}^k \dot{r}^l \dot{r}^m [\partial_r, \dot{r}^m \dot{r}^n]\}(r \cdot \dot{r}) \Delta^6 \\
&+ 4\{\dot{r}^k \dot{r}^l \dot{r}^m [\partial_r, \dot{r}^n] + \dot{r}^k [\partial_r, \dot{r}^l \dot{r}^m \dot{r}^n] + \dot{r}^k [\partial_r, \dot{r}^l \dot{r}^m \dot{r}^n]\}\partial_r^2 \Delta^6 \\
&+ O(\partial_r^7). \tag{3.18}
\end{align*}
\]

This is one of possible normal-ordered expressions. If we perform the normal ordering in another way, the result will apparently differ from this one. However, the difference is an integral of a total derivative and hence it does not affect the final result. The normal-ordered expression of \( \tilde{\Gamma}^{(1)}_4 \) is easily derived using the formula (3.18), which is

\[
\begin{align*}
\tilde{\Gamma}^{(1)}_4 &= \text{tr}[\dot{r}^k \Delta \dot{r}^l \Delta \dot{r}^m \Delta \dot{r}^n \Delta - 6\dot{r}^4 \Delta^4 - 72\dot{r}^2 (\dot{r} \cdot \dot{r}) \partial_r \Delta^5 - 240\dot{r}^2 (\dot{r} \cdot \dot{r})(r \cdot \dot{r}) \Delta^6 \\
&\quad + \{36\dot{r}^2 (\dot{r} \cdot \dot{r}) - 20\dot{r}^2 \dot{r}^2 - 28(\dot{r} \cdot \dot{r})^2\} \partial_r \Delta^5 \\
&\quad + \{-240\dot{r}^2 (\dot{r} \cdot \dot{r}) - 160\dot{r}^2 \dot{r}^2 - 200(\dot{r} \cdot \dot{r})^2\}\partial_r^2 \Delta^6 + O(\partial_r^7). \tag{3.19}
\end{align*}
\]

Next thing which we have to do is to evaluate the quantity \( \langle \tau_1 | \Delta^n | \tau_2 \rangle \). It is expressed in the proper-time representation as

\[
\langle \tau_1 | \Delta^n | \tau_2 \rangle = \frac{1}{\left[-\partial^2_{\tau_1} + r(\tau_2)\right]^n} \delta(\tau_1 - \tau_2) = \frac{1}{\Gamma(n)} \int_0^\infty d\sigma \sigma^{n-1} e^{-\sigma \left[-\partial^2_{\tau_1} + r(\tau_2)\right]} \delta(\tau_1 - \tau_2). \tag{3.20}
\]

We will arrange the operator \( e^{-\sigma \left[-\partial^2_{\tau} + r(\tau)^2\right]} \) in the following form:

\[
e^{-\sigma \left[-\partial^2_{\tau} + r(\tau)^2\right]} = X e^{-\sigma r^2} e^{\sigma \partial^2_{\tau}}, \tag{3.21}
\]

13
where an operator $X$ is defined by

$$X \equiv e^{-\sigma[-\partial^2_r + r(\tau)^2]} e^{-\sigma\partial^2_{\tau}} e^{\sigma(\tau)^2}$$

$$= e^{B+A} e^{-B} e^{-A},$$

(3.22)

with $A \equiv -\sigma r(\tau)^2$ and $B \equiv \sigma\partial^2_{\tau}$. Using the Baker-Campbell-Hausdorff’s formula twice

$$e^X e^Y = \exp \left\{ X + Y + \frac{1}{2} [X, Y] + \frac{1}{12} ([X, [X, Y]] + [Y, [Y, X]]) + \cdots \right\},$$

(3.23)

$X$ can be arranged to the following form:

$$X = \exp \left[ -\frac{1}{2} [A, B] - \frac{1}{3} [A, [A, B]] - \frac{1}{6} [B, [A, B]] + O(\partial^3_{\tau}) \right]$$

$$= \exp \left[ -\sigma^2 (\dot{r}^2 + r \cdot \dot{r}) - 2\sigma^2 (r \cdot \dot{r}) \partial_{\tau} - \frac{8}{3} \sigma^3 (\dot{r}^2 - 4) \sigma^3 (\dot{r}^2 + r \cdot \dot{r}) \partial^2_{\tau} + O(\partial^3_{\tau}) \right].$$

(3.24)

The terms of the form $[A, [A, \cdots [A, B] \cdots]]$ could potentially contribute to $O(\partial^2_{\tau})$ but they did not since $[A, [A, [A, B]] = 0$. Note also that the remark on the counting of the number of derivatives which we made below (3.11) should be applied to the operators $\partial_{\tau}$ here as well because of the same reason. The operator $\exp[-\sigma(-\partial^2_{\tau} + r(\tau)^2)]$ is finally expressed as

$$e^{-\sigma[-\partial^2_r + r(\tau)^2]} = \left[ 1 - \sigma^2 (\dot{r}^2 + r \cdot \dot{r}) - 2\sigma^2 (r \cdot \dot{r}) \partial_{\tau} - \frac{8}{3} \sigma^3 (\dot{r}^2 + r \cdot \dot{r}) \partial_{\tau} - \frac{4}{3} \sigma^3 (\dot{r}^2 + r \cdot \dot{r}) \partial^2_{\tau} + 2\sigma^4 (r \cdot \dot{r})^2 \partial^2_{\tau} + O(\partial^3_{\tau}) \right] e^{-\sigma(\tau)^2} e^{\sigma\partial^2_{\tau}}.$$

(3.25)

Now the quantity $\langle \tau_1 | \Delta^n \tau_2 \rangle$ is easily calculated based on the expression

$$\langle \tau_1 | e^{\sigma\partial^2_{\tau}} | \tau_2 \rangle = \frac{1}{\sqrt{4\pi\sigma}} \exp \left[ -\frac{1}{4\sigma}(\tau_1 - \tau_2)^2 \right].$$

(3.26)

We need the explicit forms of $\langle \tau | f(\tau) \Delta^4 | \tau \rangle$, $\langle \tau | f(\tau) \Delta^5 | \tau \rangle$, $\langle \tau | f(\tau) \Delta^6 | \tau \rangle$, $\langle \tau | f(\tau) \partial_{\tau} \Delta^5 | \tau \rangle$ and $\langle \tau | f(\tau) \partial_{\tau}^2 \Delta^6 | \tau \rangle$ where $f(\tau)$ is an arbitrary function:

$$\langle \tau | f(\tau) \Delta^4 | \tau \rangle$$

$$= \frac{1}{6} \int_0^\infty d\sigma \sigma^3 f(\tau) \left[ 1 - \frac{1}{3} \sigma^2 (\dot{r}^2 + r \cdot \dot{r}) + \frac{1}{3} \sigma^3 (r \cdot \dot{r})^2 + O(\partial^2_{\tau}) \right] \frac{e^{-\sigma(\tau)^2}}{\sqrt{4\pi\sigma}},$$

(3.27)

$$\langle \tau | f(\tau) \Delta^5 | \tau \rangle = \frac{1}{24} \int_0^\infty d\sigma \sigma^4 f(\tau) \left[ 1 + O(\partial_{\tau}) \right] \frac{e^{-\sigma(\tau)^2}}{\sqrt{4\pi\sigma}},$$

(3.28)
\[
\langle \tau | f(\tau) \Delta^6 | \tau \rangle = \frac{1}{120} \int_0^\infty d\sigma \sigma^5 f(\tau) \left[ 1 + O(\partial_\tau^2) \right] \frac{e^{-\sigma r(\tau)^2}}{\sqrt{4\pi \sigma}}, \tag{3.29}
\]
\[
\langle \tau | f(\tau) \partial_\tau \Delta^5 | \tau \rangle = \frac{1}{24} \int_0^\infty d\sigma \sigma^4 f(\tau) \left[ -\sigma (r \cdot \dot{r}) + O(\partial_\tau^2) \right] \frac{e^{-\sigma r(\tau)^2}}{\sqrt{4\pi \sigma}}, \tag{3.30}
\]
\[
\langle \tau | f(\tau) \partial_\tau^2 \Delta^6 | \tau \rangle = \frac{1}{120} \int_0^\infty d\sigma \sigma^5 f(\tau) \left[ \frac{1}{2} \sigma^4 + O(\partial_\tau^2) \right] \frac{e^{-\sigma r(\tau)^2}}{\sqrt{4\pi \sigma}}. \tag{3.31}
\]

The last of the formulas (3.31) is an example of the fact that derivatives acting directly on \( \Delta \)'s do not necessarily generate derivatives of coordinate as we mentioned before.

The final form of the one-loop effective action \( \tilde{\Gamma}^{(1)} \) up to six derivatives is obtained by applying these formulas to the normal-ordered expressions (3.12), (3.15) and (3.19). The result simplified by a partial integration is

\[
\tilde{\Gamma}^{(1)}(1) = \int_{-\infty}^\infty d\tau \int_0^\infty d\sigma \left[ -\sigma^3 \dot{r}^4 + \frac{1}{6} \sigma^5 \dot{r}^6 + \frac{1}{3} \sigma^5 \dot{r}^4 (r \cdot \ddot{r}) - \frac{1}{3} \sigma^6 \dot{r}^4 (r \cdot \dot{r})^2 \\
+ \frac{1}{3} \sigma^4 \dot{r}^2 \ddot{r}^2 + \frac{2}{3} \sigma^4 (r \cdot \dot{r})^2 + O(\partial_\tau^7) \right] \frac{e^{-\sigma r(\tau)^2}}{\sqrt{4\pi \sigma}}, \tag{3.32}
\]

or after integrating over \( \sigma \)

\[
\tilde{\Gamma}^{(1)}(1) = \int_{-\infty}^\infty d\tau \left[ -\frac{15}{16} \frac{\dot{r}^4}{r^7} + \frac{315}{128} \frac{\dot{r}^6}{r^{11}} + \frac{315}{64} \frac{\dot{r}^4 (r \cdot \ddot{r})}{r^{11}} - \frac{3465}{128} \frac{\dot{r}^4 (r \cdot \dot{r})^2}{r^{13}} \\
+ \frac{35}{32} \frac{\dot{r}^2 \ddot{r}^2}{r^9} + \frac{35}{16} \frac{(r \cdot \dot{r})^2}{r^9} + O(\partial_\tau^7) \right]. \tag{3.33}
\]

The expressions (3.32) and (3.33) are our final results for the interaction Lagrangian. There are no four-derivative terms other than the one which is already found in the effective action for straight-line trajectories (2.27). Thus, this term is valid not only for straight-line trajectories but also general ones. This result is consistent with that in [10, 11] where the effective action for arbitrary background configurations (not restricted to diagonal ones) is calculated, or that in [14] where four-derivative terms are determined by calculating the effective action for particular trajectories. The complete form of six-derivative terms is our new result. It is not difficult to show that this expression cannot be expressed as a total derivative. As a check of our calculations, we confirmed that the Euler-Lagrange equation derived from this Lagrangian precisely produces the recoil acceleration (2.30) where the third term on the right-hand side of (3.33) which contains acceleration does contribute to (2.30).
4. Conclusions and discussions

We calculated the one-loop effective action for general trajectories of D-particles in Matrix theory in the expansion with respect to the number of derivatives up to six. We determined the form of the interaction Lagrangian under the assumption that the time in Matrix theory coincides with that in the Lagrangian which ensures that the equation of motion is consistently derived. We found that there are non-vanishing terms which contain six derivatives although the value of them at straight-line trajectories vanishes. Our result provides a concrete example that non-renormalization of twelve-fermion terms does not necessarily imply that of six-derivative terms.

It is possible to calculate higher-derivative terms in the one-loop effective action at any order following the method developed in Section 3. Such calculations provide predictions from Matrix theory that M theory should have these corrections to the supergravity approximation. Although these interactions are understood as the remaining $\alpha'$ corrections in the limit discussed in \cite{5, 18, 19} in the viewpoint of the type IIA superstring theory, it is interesting to identify the origin of them in M theory.

Although we did not use the supersymmetries manifestly in the calculations, the result must be supersymmetric if correctly calculated. It is desirable to perform the supersymmetric completion of the six-derivative terms and see if it does not require twelve-fermion terms.

Our result indicates that the mechanism which ensures the non-renormalization of terms with four or six derivatives, if it exists, is more complicated than that of eight- or twelve-fermion terms which was shown in \cite{12, 13}. Something more will be required to prove the non-renormalization of four- or six-derivative terms, in particular, in multi-body systems as discussed in \cite{20}, or in \cite{21} where the possibility of the existence of six-derivative terms in four-body scattering at three loops is argued. However, these arguments of course do not claim that terms with four or six derivatives cannot be constrained only by the supersymmetries, at least for two-body interactions. In fact, the method taken in \cite{12, 13} makes use of only small part of the potential power of the supersymmetries. It would be necessary to exploit more powerful methods to constrain the effective action using the supersymmetries.
As a possible extension of our work, it is interesting to discuss the influence of the higher-derivative terms in the one-loop effective action on the generalized conformal symmetry proposed in [15] and developed in [16, 17, 14]. The question whether the conformal transformation is modified by them seems important since the conformal transformation is closely related to the property of the background metric which is not manifest in Matrix theory.

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