The noncommutative BTZ black hole in polar coordinates

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Abstract

Based on the equivalence between the three-dimensional gravity and the Chern–Simons theory, we obtain a noncommutative BTZ black hole solution as a solution of $U(1, 1) \times U(1, 1)$ noncommutative Chern–Simons theory using the Seiberg–Witten map. The Seiberg–Witten map is carried out in noncommutative polar coordinates whose commutation relation is equivalent to the usual canonical commutation relation in the rectangular coordinates up to first order in the noncommutativity parameter $\theta$. The solution exhibits a characteristic of noncommutative polar coordinates in such a way that the apparent horizon and the Killing horizon coincide only in the nonrotating limit showing the effect of noncommutativity between the radial and angular coordinates.

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1. Introduction

Quantum theory of gravity has been the main issue of theoretical physics since the quantum theory succeeded to describe most of physical phenomena except gravity. Currently, string theory is widely regarded as the most promising candidate for quantum theory of gravity. Still there remain many obstacles to overcome for string theory to become the theory of gravity in the real world. Among other attempts for quantum gravity, the notion of quantized spacetime has been around for a long time since the work of Snyder [1] more than half a century ago. From conventional Einstein’s viewpoint, gravity is regarded as the dynamics of spacetime, and thus upon quantization of gravity it is natural to consider the notion of quantized spacetime in which the coordinates become noncommutative. However, the notion of noncommutative spacetime was not quite popular until the notion appeared in the string theory context about ten years ago [2, 3]. Since then there has appeared lots of works on noncommutative (deformed) spacetime in the context of field theory and gravity itself.
The most common commutation relation for noncommutative spacetime, which we will call canonical, is modeled on quantum mechanics,

\[ [\hat{x}^\alpha, \hat{x}^\beta] = i\theta_{\alpha\beta}, \]

where \( \theta_{\alpha\beta} = -\theta_{\beta\alpha} \) are constants. It has been known that a theory on the deformed spacetime with the above given commutation relation is equivalent to another theory on commutative spacetime in which a product of any two functions on the original noncommutative spacetime is replaced with a deformed \((\star)\) product of the functions on commutative spacetime, the so-called Moyal product [4]:

\[ (f \star g)(x) \equiv \exp \left[ \frac{i}{2} \theta_{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y^\beta} \right] f(x)g(y) \bigg|_{x=y}. \]  

(2)

Using the Moyal product many works on noncommutative spacetime have been carried out and especially in [3] a map between a gauge theory on noncommutative spacetime and one on commutative spacetime, the so-called Seiberg–Witten map, was established. The Seiberg–Witten map became a very useful tool for understanding various properties of noncommutative gauge theories. Though there appeared many works on noncommutative gravity side, the progress has been rather slow compared with that of field theory side. There might be several factors for this but if we just count two of them: one is that noncommutative gravity itself is not quite established yet, and the other is that gravity is not exactly a gauge theory, thus one cannot use the Seiberg–Witten map to noncommutative gravity directly. One way of evading this is to regard the Einstein’s gravity as the Poincaré gauge theory and apply the Seiberg–Witten map for its noncommutative extension [5] or take the twisted Poincaré algebra approach [6–8] based on [9]. Only in the three-dimensional case, can one directly deal with the gravity using the Seiberg–Witten map in the conventional Einstein’s framework thanks to the equivalence between the three-dimensional gravity theory and the Chern–Simons theory [10, 11]. The noncommutative extension of this equivalence was investigated in [12, 13].

Noncommutative black holes have been investigated by many [14]. In most of these works, solutions were not obtained from field equations directly. Rather they were obtained under certain guidelines emerging from noncommutativity in the name of noncommutative-inspired. On the other hand, in [15] noncommutative AdS\(^3\) vacuum and conical solutions were obtained directly from field equations using the three-dimensional gravity Chern–Simons equivalence and the Seiberg–Witten map. There the Seiberg–Witten map was carried out in the rectangular coordinates with the canonical commutation relation, then after the mapping the solutions were expressed in the polar coordinates.

In this paper, based on the three-dimensional equivalence between gravity and the Chern–Simons theory and using the Seiberg–Witten map with the commutative BTZ solution [16], we study the BTZ black hole solution on noncommutative AdS\(^3\) in the polar coordinates \((\hat{r}, \hat{\phi}, t)\) with the following commutation relation\(^3\):

\[ [\hat{r}, \hat{\phi}] = i\theta \hat{r}^{-1}, \quad \text{others} = 0. \]

(3)

In fact, in [17], using the same approach that we adopt in this paper, a noncommutative BTZ solution had been worked out also in the polar coordinates but with the different commutation relation:

\[ [\hat{r}, \hat{\phi}] = i\theta, \quad \text{others} = 0. \]

(4)

However, the above commutation relation adopted there is not equivalent to the canonical commutation relation (5) even by the dimensional count as we shall see below. In the four-dimensional case, noncommutative solutions were obtained using the Poincaré gauge theory

\(^3\) When \( r \to 0 \), the commutation relation (3) is not well defined. However, here we are only concerned with the region \( 0 < r \), as usual for black holes.
approach [5] in [18] for the Schwarzschild black hole case, and in [19, 20] for the charged black hole case.

This paper is organized as follows. In section 2, we explain the relationship between the polar coordinates and the rectangular coordinates in noncommutative space. In section 3, we review the equivalence between gravity and the $U(1, 1) \times U(1, 1)$ Chern–Simons theory in three-dimensional noncommutative spacetime and revisit the classical(commutative) BTZ solution. In section 4, we work out the Seiberg–Witten map for the $U(1, 1) \times U(1, 1)$ Chern–Simons theory and obtain a noncommutative BTZ solution. In section 5, we conclude with a discussion.

2. Polar coordinates in noncommutative space

In this section, we show that our commutation relations (3) are equivalent to the canonical commutation relation of the rectangular coordinates

\[ [\hat{x}, \hat{y}] = i\theta, \quad \text{others} = 0, \]  

up to first order in the noncommutativity parameter $\theta$.

To see this, we first assume that the usual relation between the rectangular and polar coordinates holds in noncommutative space\(^4\),

\[ \hat{x} = \hat{r} \cos \hat{\phi}, \quad \hat{y} = \hat{r} \sin \hat{\phi}, \]  

then check how the two commutation relations (4) and (5) are related.

We begin with the evaluation of $\hat{x}^2 + \hat{y}^2$ in polar coordinates with the relation (6) and the commutation relation (3), and see how it differs from $\hat{r}^2$ and check its consistency with the commutation relation (5) of the rectangular coordinates.

\[ \hat{x}^2 + \hat{y}^2 := \hat{r} \cos \hat{\phi} \cos \hat{r} \cos \hat{\phi} + \hat{r} \sin \hat{\phi} \hat{r} \sin \hat{\phi}. \]  

Using the Campbell–Hausdorff formula $e^{AB}e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \cdots$, (7) can be expressed as

\[ \hat{x}^2 + \hat{y}^2 := \hat{r} \left( 1 - \frac{1}{2!} [\hat{\phi}, [\hat{\phi}, \hat{r}]] + \cdots \right) = \hat{r}^2 - \frac{1}{2!} \theta^2 \hat{r}^{-2} + \cdots, \]  

where we used the commutation relation $[\hat{\phi}, \hat{r}^{-1}] = i\theta \hat{r}^{-3}$ coming from (3). In the above calculation, the first-order terms in $\theta$ are cancelled out. Thus, we can say that the commutation relations (3) and (5) are equivalent up to first order in $\theta$. For a consistency check, with the commutation relation (5), we have

\[ [\hat{x}^2 + \hat{y}^2, \hat{x}] = [\hat{y}^2, \hat{x}] = -2i\theta \hat{y} = -2i\theta \hat{r} \sin \hat{\phi}, \]  

and using (8) the above can be reexpressed as

\[ [\hat{r}^2 + O(\theta^2), \hat{x}] \equiv [\hat{r}^2, \hat{r} \cos \hat{\phi}] = \hat{r} [\hat{r}^2, \cos \hat{\phi}] = -2i\theta \hat{r} \sin \hat{\phi}. \]  

In the last step, we used the following commutation relation which is equivalent to the commutation relation (3):

\[ [\hat{r}^2, \hat{\phi}] = 2i\theta. \]  

The two commutation relations (5) and (11) correspond to the Poisson tensors, $(\partial \wedge \partial)_\mu \equiv \theta^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta}, 2\theta \left( \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial \phi} \right)$, respectively. Note that the two Poisson tensors are also equivalent up to first order in $\theta$.

\(^4\) Since $r$-coordinate commutes with all the other coordinates, here we deal with noncommutative space rather than noncommutative spacetime.
In solving the Seiberg–Witten map, we will use the commutation relation (11) instead of (3) for calculational convenience. Note also that the commutation relation (4) in [17] is not equivalent to the canonical relation (5) if we assume the usual relationship (6) between the rectangular and polar coordinates:

\[ [\hat{x}, \hat{y}] = [\hat{r} \cos \phi, \hat{r} \sin \phi] = i0\hat{r}. \]  

(12)

3. Noncommutative Chern–Simons gravity

It has been well known that the three-dimensional Einstein’s gravity is equivalent to the three-dimensional Chern–Simons theory [10, 11]. The noncommutative version of this equivalence was investigated in [12, 13]. In [13], the (2 + 1)-dimensional \( U(1, 1) \times U(1, 1) \) noncommutative Chern–Simons theory was worked out. There it was shown that in the commutative limit this theory becomes equivalent to the three-dimensional Einstein’s gravity plus two decoupled \( U(1) \) theories. Even if one begins with the commutative \( SU(1, 1) \times SU(1, 1) \) Chern–Simons theory, its noncommutative extension has to contain \( U(1) \) elements. Thus, the noncommutative extension of the \( SU(1, 1) \times SU(1, 1) \) Chern–Simons theory has to be the \( U(1, 1) \times U(1, 1) \) noncommutative Chern–Simons theory. Therefore, one can regard the \( U(1, 1) \times U(1, 1) \) noncommutative Chern–Simons theory as a noncommutative extension of the three-dimensional Einstein’s gravity. Since Chern–Simons theory is a gauge theory, one can use the Seiberg–Witten map [3] to get a solution of noncommutative Chern–Simons theory from its commutative counterpart. In this section, we review the \( U(1, 1) \times U(1, 1) \) noncommutative Chern–Simons theory as a \( (2 + 1) \)-dimensional noncommutative gravity.

The action of the \( (2 + 1) \)-dimensional noncommutative Chern–Simons theory with the negative cosmological constant \( \Lambda = -1/\ell^2 \) is given by up to boundary terms,

\[ \hat{S}(\hat{A}^+, \hat{A}^-) = \hat{S}_+ + \hat{S}_-, \]  

(13)

\[ \hat{S}_\pm(\hat{A}^\pm) = \beta \int \text{Tr} \left( \hat{A}^\pm \wedge d\hat{A}^\pm + \frac{2}{3} \hat{A}^\pm \wedge \hat{A}^\pm \wedge \hat{A}^\pm \right), \]  

(14)

where \( \beta = l/16\pi G_N \) and \( G_N \) is the three-dimensional Newton constant. The deformed wedge product \( \hat{\wedge} \) denotes that

\[ A \hat{\wedge} B = A_{\mu} \wedge B_{\nu} \, dx^\mu \wedge dx^\nu, \]  

(15)

where the star (\( * \)) means the Moyal product defined in (2). The noncommutative \( U(1, 1) \times U(1, 1) \) gauge fields \( \hat{A} \) consist of noncommutative \( SU(1, 1) \times SU(1, 1) \) gauge fields \( \hat{A} \) and two \( U(1) \) fluxes \( \hat{B} \):

\[ \hat{A}^\pm = \hat{A}^{\pm}_{A_\alpha} \tau_A = \hat{A}^{a\pm}_{\alpha} \tau_A + \hat{B}^{\pm}_{\alpha}, \]  

(16)

where \( A = 0, 1, 2, 3, a = 0, 1, 2 \), and \( \hat{A}^{a\pm} = \hat{A}^{a\pm} \), \( \hat{A}^{3\pm} = \hat{B}^{\pm} \). The noncommutative \( SU(1, 1) \times SU(1, 1) \) gauge fields \( \hat{A} \) are expressed in terms of the triad \( \hat{e} \) and the spin connection \( \hat{\omega} \) as

\[ \hat{A}^{a\pm} := \hat{\omega}^a \pm \hat{e}^a / l. \]  

(18)

\[ \tau_A \)'s satisfy the following relations:

\[ \text{Tr}(\tau_A \tau_B) = \frac{1}{2} \eta_{AB}, \quad \text{Tr}(\tau_A, \tau_B) = \delta_{AB}, \quad \text{Tr}(\tau_A) = 0, \quad \tau_A = \xi_{abc} e^a \tau^c + \frac{1}{2} \eta_{ab} \]  

(17)

where \( \eta_{AB} = \text{diag}(1, -1, 1, -1) \) and \( e^{012} = -e^{012} = 1 \). The bases we take are \( \tau_0 = \frac{1}{2} \sigma_3, \tau_1 = \frac{1}{2} \sigma_1, \tau_2 = \frac{1}{2} \sigma_2, \tau_3 = \frac{1}{2} \mathbb{I} \), where \( \sigma_a \) are the Pauli matrices and \( \mathbb{I} \) denotes an identity.
Substituting (16) into (13), the action becomes [13]
\[
S = \frac{1}{8\pi G_N} \int \left( \hat{\varepsilon}^a \hat{\hat{\omega}}_a + \frac{1}{6l^2} \epsilon_{abc} \hat{\varepsilon}^a \hat{\hat{\omega}}_b \hat{\hat{\omega}}_c \right)
- \frac{\beta}{2} \int \left( \hat{B}^+ \hat{\hat{\omega}} + d\hat{B}^- + \frac{i}{3} \hat{B}^+ \hat{\hat{B}} \right)
+ \frac{\beta}{2} \int \left( \hat{B}^- \hat{\hat{\omega}} + d\hat{B}^+ + \frac{i}{3} \hat{B}^- \hat{\hat{B}} \right)
+ \frac{i\beta}{2} \int \left( \hat{B}^+ - \hat{B}^- \right) \hat{\omega} \left( \hat{\omega}^a \hat{\hat{\omega}}_a + \frac{1}{l^2} \hat{\varepsilon}^a \hat{\hat{\omega}}_a \right)
+ \frac{i\beta}{2l} \int \left( \hat{B}^+ + \hat{B}^- \right) \hat{\omega} \left( \hat{\omega}^a \hat{\hat{\omega}}_a + \hat{\varepsilon}^a \hat{\hat{\omega}}_a \right),
\]
up to surface terms, where the curvature \(\hat{R}^a = d\hat{\omega}^a + \frac{1}{2} \epsilon_{abc} \hat{\omega}_b \hat{\omega}_c \) is the noncommutative version of the spin curvature 2-form. Note that the noncommutative \(SU(1,1) \times SU(1,1)\) gauge fields \(\hat{A}\) are coupled with the two noncommutative \(U(1)\) fluxes \(\hat{B}\) nontrivially.

The equation of motion can be reexpressed in terms of the noncommutative \(U(1,1) \times U(1,1)\) curvature as follows:
\[
\hat{F}^\pm \equiv d\hat{A}^\pm + \hat{A}^\pm \hat{\hat{A}}^\pm = 0.
\]
In the commutative limit we have, the \(SU(1,1) \times SU(1,1)\) curvature vanishes,
\[
F^\pm \equiv dA^\pm + A^\pm A^\pm = 0,
\]
and the \(SU(1,1) \times SU(1,1)\) gauge fields decouple from the two \(U(1)\) fluxes:
\[
R^a + \frac{1}{2l^2} \epsilon_{abc} e_b \wedge e_c = 0,
\]
\[
T^a \equiv de^a + \epsilon_{abc} \omega_b \wedge e_c = 0,
\]
\[
d\lambda^a = 0.
\]

The noncommutative equations (20) are not easy to solve directly. It was shown in [13, 22] that free noncommutative Chern–Simons theory has one-to-one correspondence with its commutative one. The commutative black hole solution which corresponds to the \(SU(1,1) \times SU(1,1)\) Chern–Simons theory expressed with the triad \(e_a\) and the spin connection \(\omega_a\) is given by [23]
\[
e^0 = m \left( \frac{r^+_n}{L} \ dt - r_+ \ d\phi \right), \quad e^1 = \frac{1}{n} \ dm, \quad e^2 = n \left( r_+ d\phi - \frac{r_-}{L} \ dt \right),
\]
\[
\omega^0 = -\frac{m}{L} \left( r_+ d\phi - \frac{r_-}{L} \right), \quad \omega^1 = 0, \quad \omega^2 = -\frac{n}{L} \left( \frac{r_+}{L} \ dt - r_- \ d\phi \right),
\]
where \(m^2 = (r^2 - r_+^2)/(r^2 - r_-^2)\) and \(n^2 = (r^2 - r_+^2)/(r^2 - r_-^2)\).

4. Noncommutative solution via the Seiberg–Witten map

On the basis of the equivalence between the \(SU(1,1) \times SU(1,1)\) Chern–Simons theory and the three-dimensional Einstein gravity shown in the previous section, we will get a solution of the noncommutative \(U(1,1) \times U(1,1)\) Chern–Simons theory from the solution of the
commutative $SU(1, 1) \times SU(1, 1)$ Chern–Simons theory together with two $U(1)$ fluxes via the Seiberg–Witten map.

The original Seiberg–Witten map solution was given in [3] with the canonical commutation relation (1) as

$$\hat{A}_\nu (A) = A_\nu + \hat{A}_\nu = A_\nu - \frac{i}{4} \theta^{\alpha \beta \gamma \delta} [A_\alpha, \partial_\beta A_\gamma + \mathcal{F}_{\beta\gamma}] + \mathcal{O}(\theta^2),$$

(27)

$$\hat{\lambda} (\lambda, A) = \lambda + \lambda' = \lambda + \frac{i}{4} \theta^{\alpha \beta} [a_\alpha, A_\beta] + \mathcal{O}(\theta^2),$$

(28)

where $(A, \lambda)$ and $(\hat{A}, \hat{\lambda})$ are commutative and noncommutative gauge fields and parameters, respectively.

As we discussed in the beginning sections the commutation relations are different in different coordinate systems in noncommutative space even though they are equivalent in commutative space. Therefore, one must be careful when the Seiberg–Witten map is used in a different coordinate systems in noncommutative space even though they are equivalent in respect.

The twist element $\theta_{\alpha \beta}$ for the noncommutative space is given by

$$\theta_{\alpha \beta} = \theta_{\alpha \beta} + \theta_{\beta \alpha} = \frac{1}{4} \theta^{\alpha \beta \gamma \delta} [A_\gamma, \partial_\delta A_\gamma] + \mathcal{O}(\theta^2),$$

(29)

where $(A, \lambda)$ and $(\hat{A}, \hat{\lambda})$ are commutative and noncommutative gauge fields and parameters, respectively.

From the commutative $SU(1, 1) \times SU(1, 1)$ gauge fields $A_{\mu}^{a b}$, the solution of equation (21), and two $U(1)$ flux fields $B_{\mu}^{\pm}$ satisfying $d B_{\mu}^{\pm} = 0$ the noncommutative $SU(1, 1) \times U(1)$ solution via the Seiberg–Witten map is given by

$$A_{\mu}^{\pm} = -
\frac{i}{4} \left[ \frac{1}{2} (A_{\mu}^{a b} \partial_\phi A_{\mu}^{b c} - A_{\phi}^{a b} \partial_\phi A_{\mu}^{b c}) + \mathcal{O}(\theta^2) \right] + \mathcal{O}(\theta^2),$$

(31)

where $A_{\mu}^{a b} = A_{\mu}^{a b}$, $a = 0, 1, 2$ and $A_{\mu}^{a b} = B_{\mu}^{a b}$. From (25) and (26), the commutative $SU(1, 1) \times SU(1, 1)$ gauge fields $A_{\mu}^{a b}$ are given by

$$A_{\mu}^{0 \pm} = \pm \frac{m(r \pm r_{-})}{l^2} (dr \mp i d\phi),$$

$$A_{\mu}^{1 \pm} = \pm \frac{m}{n},$$

$$A_{\mu}^{2 \pm} = \frac{n(r \pm r_{-})}{l^2} (dr \pm i d\phi).$$

(32)

Since we deal with the Seiberg–Witten map with $(t, R, \phi)$, we need to express the metric in the $(t, R, \phi)$ coordinates. The metric in $(t, R, \phi)$ coordinates is given by

$$ds^2 = -N^2 dt^2 + \frac{N^{-2}}{4R} dR^2 + R (d\phi + N^\phi dt)^2,$$

(33)

where $N^2 = \frac{(R-r_{-}^2)(R-r_{+}^2)}{R}$ and $N^\phi = -\frac{r_{-} r_{+}}{R}$.  

6 One can also deduce the same deformed product by using the twist element [9, 21]

$$\mathcal{F}_\nu = \exp \left[ -i \theta \left( \frac{\partial}{\partial r^2} \Phi \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \phi} \Phi \frac{\partial}{\partial r^2} \right) \right].$$

(29)

The twist element $\mathcal{F}_\nu$ yields the commutation relation $[r^2, \phi]_\nu = r^2 \phi - \phi r^2 = 2i \theta$.  

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For simplicity, we consider the $U(1) \text{ flux } B^\pm_\mu = B \phi \text{ with constant } B$. Then, the noncommutative solution $\mathcal{A}^\pm_\mu$ is given by

$$\mathcal{A}^\pm_\mu = \hat{A}^{\mu \pm}_\mu \tau_\sigma + \hat{B}^{\mu \pm}_\nu \tau_3 = \left( A^{\mu \pm}_\mu - \frac{\theta}{2} B^\pm_\phi \partial_\phi A^{\mu \pm}_\mu \right) \tau_\sigma + B^{\mu \pm}_\mu \tau_3 + \mathcal{O}(\theta^2). \quad (34)$$

Here, the noncommutative solutions $\hat{A}^{\mu \pm}_\mu$ are explicitly given by

$$\hat{A}^0_\pm = -\frac{1}{l} (r_+ \pm r_-) \left( m - \frac{\theta B}{2} m' \right) \left( d\phi \mp \frac{dt}{l} \right),$$

$$\hat{A}^1_\pm = \pm \left( \frac{m'}{n} - \frac{\theta B}{2} \left( \frac{m'}{n} \right) \right) dR,$$

$$\hat{A}^2_\pm = \pm \frac{1}{l} (r_+ \pm r_-) \left( n - \frac{\theta B}{2} n' \right) \left( d\phi \mp \frac{dt}{l} \right). \quad (35)$$

One can check that the noncommutative solution $\hat{A}^\pm_\mu$ in (34) satisfies equation (20) as follows, the components of the noncommutative $U(1) \times U(1)$ curvature in (20) can be written as

$$\hat{F}^\pm_\mu = \partial_\mu \hat{A}^\pm_\nu - \partial_\nu \hat{A}^\pm_\mu + [\hat{A}^\pm_\mu, \hat{A}^\pm_\nu], \quad (36)$$

where $[A, B]_\mu = A \star B - B \star A$. We note that $[\hat{A}^\pm_\mu, \hat{A}^\pm_\nu]_\sigma = [\hat{A}^\pm_\mu, \hat{A}^\pm_\nu]_\sigma = [\hat{A}^\pm_\mu, \hat{A}^\pm_\nu]_\sigma$ because $\hat{A}^\pm_\mu$ are functions of $r$ only and $[\tau_\sigma, \tau_3] = 0$, $[\tau_3, \tau_3] = 0$. Thus using (34) the independent components of $\hat{F}^\pm_\mu$ in (36) become

$$\hat{F}^\pm_\phi = \left( \partial_\phi \hat{A}^0_\phi + \hat{A}^1_\phi \hat{A}^2_\phi \right) \tau_0 + \left( \partial_\phi \hat{A}^2_\phi + \hat{A}^1_\phi \hat{A}^0_\phi \right) \tau_2 = 0,$$

$$\hat{F}^\pm_R = -\left( \partial_\phi \hat{A}^0_R + \hat{A}^1_R \hat{A}^2_R \right) \tau_0 + \left( \partial_\phi \hat{A}^2_R + \hat{A}^1_R \hat{A}^0_R \right) \tau_2 = 0,$$

$$\hat{F}^\pm_t = \left( \hat{A}^0_t - \hat{A}^2_t \right) \tau_1 = 0, \quad (37)$$

where we used the relation $n' = nm'/n$ and $'$ denotes the differentiation with respect to $R$.

From the noncommutative solution (35) and using the relation (18), the noncommutative triad and spin connection are given by

$$\hat{e}^0 = \left( m - \frac{\theta B}{2} m' \right) \left( \frac{r_+}{l} dt - r_- d\phi \right) + \mathcal{O}(\theta^2),$$

$$\hat{e}^1 = l \left[ \frac{m'}{n} - \frac{\theta B}{2} \left( \frac{m'}{n} \right) \right] dR + \mathcal{O}(\theta^2),$$

$$\hat{e}^2 = \left( n - \frac{\theta B}{2} n' \right) \left( r_+ d\phi - \frac{r_-}{l} dt \right) + \mathcal{O}(\theta^2), (38)$$

$$\hat{\omega}^0 = -\frac{1}{l} \left( m - \frac{\theta B}{2} m' \right) \left( r_+ d\phi - \frac{r_-}{l} \right) + \mathcal{O}(\theta^2),$$

$$\hat{\omega}^1 = \mathcal{O}(\theta^2),$$

$$\hat{\omega}^2 = -\frac{1}{l} \left( n - \frac{\theta B}{2} n' \right) \left( \frac{r_+}{l} dt - r_- d\phi \right) + \mathcal{O}(\theta^2), \quad (39)$$

where $'$ also denotes the differentiation with respect to $R = r^2$.

The metric in the noncommutative case can be defined by

$$d\tilde{s}^2 \equiv \eta_{ab} \hat{e}^a_\mu \hat{e}^b_\nu dx^\mu dx^\nu. \quad (40)$$

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Since the commutative BTZ black hole solution has only $R$-dependence, the $\star$-product of the triads becomes $\hat{e}_\mu \star \hat{e}_\nu = \hat{e}_\mu \hat{e}_\nu$. Then the metric expressed in terms of $r$ is given by

$$d\hat{s}^2 = -f^2 dt^2 + \hat{N}^{-2} dr^2 + 2r^2 N^\phi dt d\phi + \left( r^2 + \frac{\theta B}{2} \right) d\phi^2 + O(\theta^2),$$

where

$$N^\phi = -r_+ r_- / l^2,$$

$$f^2 = -\left( r^2 - r_+^2 - r_-^2 \right) / l^2 + \frac{\theta B}{2l^2},$$

$$\hat{N}^2 = \frac{1}{r^2 r^\pm} \left[ (r^2 - r_+^2)(r^2 - r_-^2) - \frac{\theta B}{2} (2r^2 - r_+^2 - r_-^2) \right].$$

The above solution shows an interesting feature which does not exist in the commutative case: the apparent and Killing horizons do not coincide except for the nonrotating case. The apparent horizon is defined as a hypersurface on which the norm of the vector normal to the surface $r =$ constant is null: $g^{\mu\nu} \partial_\mu r \partial_\nu r|_{r=r_H} = 0$. The Killing horizon is a hypersurface on which the norm of the Killing vector $\chi = \partial_t + \Omega H \partial_\phi$ vanishes, where the horizon angular velocity $\Omega H$ is defined by $\Omega H = -g_{\phi t} / g_{\phi \phi}$ at $r = r_H$. Hence, the apparent horizon is determined by the following relation:

$$\hat{g}^{rr} \hat{g}^{-1} = \hat{N}^2 = 0.$$  

Solving the above equation up to first order in $\theta$, we obtain two apparent horizons at

$$\hat{r}_\pm^2 = r_\pm^2 + \frac{\theta B}{2} + O(\theta^2).$$  

The Killing horizon is determined by

$$\hat{\chi}^2 = \hat{g}_{tt} - \hat{g}_{\phi t} / \hat{g}_{\phi \phi} = 0,$$

and we obtain the Killing horizons at

$$\hat{r}_\pm^2 = r_\pm^2 \pm \frac{\theta B}{2} \left( \frac{r_+^2 + r_-^2}{r_+^2 - r_-^2} \right) + O(\theta^2).$$

Note that the two Killing horizons are not equally shifted unlike the apparent horizons. Namely, the apparent and Killing horizons do not coincide, they coincide only in the nonrotating limit in which the inner horizon collapses, $r_- = 0$. We understand this as the effect of the noncommutativity among the radial ($\hat{r}$) and angular ($\hat{\phi}$) coordinates. In the commutative case, the apparent horizons and the Killing horizons coincide [24]. Here, in the nonrotating case, the apparent horizon is determined by the null vector given by the translation generator along the $\hat{r}$-direction, while the Killing horizon is determined by the null vector given by the translation generator along the time direction. Therefore, the noncommutative effect will not change the relation between the two horizons from the commutative case. However, in the rotating case, the Killing horizon is determined by the null vector given by the translation generators along the time- and $\hat{\phi}$-directions, while the apparent horizon is determined by the null vector given by the translation generator along the $\hat{r}$-direction. Since the effects of the translation generators along the $\hat{r}$- and $\hat{\phi}$-directions interfere with each other in the noncommutative case, the relation between the two horizons will differ from the commutative case. Therefore, we expect that the apparent and Killing horizons do not coincide in the rotating case when the $\hat{r}$ and $\hat{\phi}$ coordinates do not commute.
5. Conclusion

In this paper, we obtained a noncommutative BTZ black hole solution as a solution of the $U(1, 1) \times U(1, 1)$ noncommutative Chern–Simons theory using the Seiberg–Witten map. This is based on the following two previously known relations: (1) the equivalence between the BTZ black hole solution and the solution of the $SU(1, 1) \times SU(1, 1)$ Chern–Simons theory in the commutative case. (2) In the commutative limit, the $U(1, 1) \times U(1, 1)$ noncommutative Chern–Simons theory becomes the three-dimensional Einstein gravity and two decoupled $U(1)$ theories.

In order to use the commutative BTZ solution which is given in the polar coordinates, we have to solve the Seiberg–Witten map in the polar coordinates. This is what we do in this paper. In [17], the same task has been done. However, the commutation relation used there $[\hat{r}, \hat{\phi}] = i\theta$ is only equivalent to the canonical one $[\hat{x}, \hat{y}] = i\theta$ at a fixed radius, and thus the two commutation relations are dimensionally different as we explained in section 1. Instead, we use the commutation relation $[\hat{r}, \hat{\phi}] = i\theta\hat{r}^{-1}$, which is equivalent to the canonical one up to linear order in the noncommutativity parameter $\theta$. In our solution, the apparent horizon and the Killing horizon do not coincide except for the nonrotating limit. This feature was also appeared in [17] dubbed as smeared black hole. We understand this result due to the noncommutativity between the two coordinates $(\hat{r}, \hat{\phi})$. In the rotating case, the Killing vector which determines the Killing horizon is dependent on the translation generator along the $\hat{\phi}$-direction, while the apparent horizon is determined by the null vector given by the translation generator along the radial $\hat{r}$-direction. Hence, in the rotating case the relation between the two horizons is affected by the noncommutativity between the two coordinates $(\hat{r}, \hat{\phi})$, and will differ from the commutative case. The two horizons will not coincide. In the nonrotating case, the Killing vector does not depend on the translation generator along the $\hat{\phi}$-direction, thus the relation between the two horizons will not differ from the commutative case.

Finally, a critical comment is in order: the solution of noncommutative gauge theory obtained using the Seiberg–Witten map would be different if one adopts different coordinate systems in evaluating the Seiberg–Witten map, even though the coordinate systems used are classically equivalent in the commutative limit. Namely, our solution is different from what we would get from the Seiberg–Witten map using the rectangular coordinates commutation relation as in the work of Pinzul and Stern [15]. In [15], the solution was obtained via the Seiberg–Witten map with the rectangular coordinates commutation relation, then converted into the polar coordinates using the classical equivalence relation such as $x = r \cos \phi$, etc. The difference is due to the fact that the deformed commutation relations, for instance the rectangular and the polar cases, are not exactly equivalent to each other. We further investigate this aspect in [25].

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