Dynamical quantum determinants and Pfaffians

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Dedicated to Vyjayanthi Chari in honor of her 60th birthday

Abstract. We introduce the dynamical quantum Pfaffian on the dynamical quantum general linear group and prove its fundamental transformation identity. Hyper quantum dynamical Pfaffian is also introduced and formulas connecting them are given.

1. Introduction

Dynamical quantum groups are important generalization of quantum groups introduced by Etingof and Varchenko [3] in connection with the elliptic quantum groups [5, 6]. See the review [2] for the background and related literature as well as comparison with usual quantum groups (see [1]). The dynamical quantum group is in fact some quantum groupoid, thus also related to the deformation of the Poisson groupoid [16, 19, 21]. In this paper we essentially follow [3] to define the dynamical quantum group with some modification [14].

As discussed in [4] in general, given an R-matrix one can associate certain quantum semigroup \(A(R)\) via the RTT formulation. Let \(V\) be the complex \(n\)-dimensional vector space with basis \(v_i\) and dual basis \(\lambda_i\) for \(V^*\). We consider the dynamical R-matrix \(R(\lambda)\) defined on \(V \otimes V\) as follows.

\[
R(\lambda) = q \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{i<j}^{n} e_{ii} \otimes e_{jj} + \sum_{i>j}^{n} g(\lambda_i - \lambda_j) e_{ii} \otimes e_{jj} + \sum_{i \neq j}^{n} h(\lambda_i - \lambda_j) e_{ii} \otimes e_{jj}
\]

(1.1)

where \(\lambda = (\lambda_1, \ldots, \lambda_n)\), \(e_{ij}\) are the unit matrix elements inside \(\text{End}(V)\) such that \(e_{ij}v_k = \delta_{jk}v_i\), and \(g, h\) are certain \(q\)-analog functions on \(V\) (see (2.1)). The associated bialgebra \(F_R(M(n))\) is called the dynamical quantum group.
in the general linear type, which generalizes the usual quantum general linear semigroup $M_q(n)$. The non-dynamical quantum semigroup $M_q(n)$ becomes the quantum group $GL_q(n)$ with the help of a special central element called the quantum determinant $\det_q$.

On the quantum semigroup $M_q(n)$ one can develop a theory of quantum linear algebra [8], and introduce quantum determinants and minors. One can prove key equations such as the quantum Cramer identity, Cayley-Hamilton identity etc [22, 17]. For a unified treatment using Manin’s quadratic algebras, see [11, 13].

Correspondingly on the dynamical quantum general linear semigroupoid $\mathcal{F}_R(M(n))$, we can also introduce the dynamical quantum determinant, minors and prove that they also enjoy similar favorable properties [9, 14, 15]. It turns out that the quantum dynamical determinant is also a central group-like element, and the Laplace expansions for quantum dynamical minors also are satisfied in a manner similar to the non-dynamical quantum situation. In particular the quantum dynamical determinant also turns $\mathcal{F}_R(M(n))$ into a dynamical quantum groupoid [18, 20].

The goal of this paper is to introduce the quantum dynamical Pfaffian and show that it enjoys favorable properties similar to the quantum group situation [10]. Our main technique is to use quadratic algebras or quantum de Rham complexes [17] to study quantum determinants and quantum Pfaffians, and express them as the scaling constants of quantum differential forms (cf. [11]). In particular, we prove that the dynamical quantum Pfaffian satisfies the transformation property:

\[
\Pf(ABA^t) = \det(A)\Pf(B)
\]

even though the identity $\Pf(A) = \sqrt{\det(A)}$ no longer holds for the quantum anti-symmetric matrix.

The paper is organized as follows. In section two, we introduce the dynamical quantum general linear group via the generalized quantum Yang-Baxter R-matrix and review the basic information on quantum dynamical minors and determinants. In section three, we give a factorization formula for the dynamical quantum determinant in terms of quasi-determinant of Gelfand and Retakh. In section four, we study the dynamical quantum Pfaffians using $q$-forms. In the last section, quantum dynamical hyper-Pfaffians are given and their fundamental properties and identities are discussed.

2. Dynamical analogue of the quantum algebra $M(n)$.

In this section, we recall some basic facts about dynamical quantum groups [14].

Let $h^*$ be the dual space of the $n$-dimensional commutative Lie algebra $h$ and we fix a linear basis $\{e_i\}$ of $h^*$, so $h^*$ can be identified with $\mathbb{C}^n$. For $[1, n] = \{1, 2, \ldots, n\}$, define $\omega : [1, n] \to h^*$ by $\omega(i) = e_i$.

Fix a generic $q \in \mathbb{C}^\times$. For $\lambda \in h^*$, the functional $q^\lambda : h \to \mathbb{C}$ is defined as usual by $v \mapsto q^\lambda(v), v \in h$. We denote by $h(\lambda)$ and $g(\lambda)$ the following
special meromorphic functionals on $\mathfrak{h}$:
\[
h(\lambda) = q^{\frac{-2\lambda - q^{-2}}{q^{-2\lambda} - 1}},
\]
(2.1)
\[
g(\lambda) = h(\lambda)h(-\lambda) = \frac{(q^{-2\lambda} - q^{-2})(q^{-2\lambda} - q^2)}{(q^{-2\lambda} - 1)^2}.
\]

Let $M_{\mathfrak{h}^*}$ be the space of meromorphic functionals on $\mathfrak{h}^*$. In particular, the above $f(\lambda), g(\lambda)$ are elements inside $M_{\mathfrak{h}^*}$. Let $\mathfrak{h}$-algebra $\mathcal{F}_R(M(n))$ be the associative algebra generated by the elements $t_{ij}, 1 \leq i, j \leq n$ together with two copies of $M_{\mathfrak{h}^*}$. The elements of the two copies $M_{\mathfrak{h}^*}$ are $f(\lambda) = f(\lambda_1, \ldots, \lambda_n)$ and $f(\mu) = f(\mu_1, \ldots, \mu_n)$, embedded as subalgebras. Here $\lambda_i$ (resp. $\mu_i$) is a function on $\mathfrak{h}$. The defining relations of $\mathcal{F}_R(M(n))$ consist of two types. The first group of relations are given by
\[
f_1(\lambda)f_2(\mu) = f_2(\mu)f_1(\lambda),
\]
(2.2)
\[
f(\lambda)t_{ij} = t_{ij}f(\lambda + \omega(i)),
\]
\[
f(\mu)t_{ij} = t_{ij}f(\mu + \omega(j)),
\]
where $f, f_1, f_2 \in M_{\mathfrak{h}^*}$. The second set of relations are
\[
h(\mu_i - \mu_j)t_{ik}t_{il} = t_{ik}t_{il}, \quad k < l
\]
\[
h(\lambda_i - \lambda_j)t_{jk}t_{ik} = t_{ik}t_{jk}, \quad i < j
\]
(2.3)
\[
t_{ik}t_{jl} = t_{jl}t_{ik} + (h(\lambda_j - \lambda_i) - h(\mu_k - \mu_l))t_{jk}t_{i}, \quad i < j, k < l
\]
\[
g(\mu_i - \mu_j)t_{ik}t_{jl} = g(\lambda_i - \lambda_j)t_{il}t_{ik}
\]
\[+ (h(\mu_i - \mu_k) - h(\lambda_i - \lambda_j))t_{il}t_{jk}, \quad i < j, k < l.
\]

The algebra $\mathcal{F}_R(M(n))$ has a bigradation defined as follows. Let $deg(t_{ij}) = (\omega(i), \omega(j)) = (e_i, e_j) \in \mathbb{N}^n \times \mathbb{N}^n$ and extend this multiplicatively. Then
\[
\mathcal{F}_R(M(n)) = \bigoplus_{(m, p) \in \mathbb{N}^n \times \mathbb{N}^n \cup \{(0, 0)\}} \mathcal{F}_{m, p}
\]
where the summand $f(\lambda), f(\mu) \in \mathcal{F}_{00} = M_{\mathfrak{h}^*}^\otimes$, and $\mathcal{F}_{m, p} = \{t \mid deg(t) = (m, p) \in \mathbb{N}^n \times \mathbb{N}^n\}$, and the moment maps are given by $\mu_1(f) = f(\lambda), \mu_\tau(f) = f(\mu)$.

For $\alpha \in \mathfrak{h}^*$ we denote by $T_\alpha : M_{\mathfrak{h}^*} \rightarrow M_{\mathfrak{h}^*}$ the automorphism $(T_\alpha f)(\lambda) = f(\lambda + \alpha)$ for all $\lambda \in \mathfrak{h}^*$. The algebra $\mathcal{F}_R(M(n))$ has a comultiplication $\Delta : \mathcal{F}_R(M(n)) \rightarrow \mathcal{F}_R(M(n)) \otimes \mathcal{F}_R(M(n))$ given by
\[
\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj},
\]
(2.4)
\[
\Delta(f(\lambda)) = f(\lambda) \otimes 1,
\]
\[
\Delta(f(\mu)) = 1 \otimes f(\mu).
\]
and the counit $\varepsilon$ given by $\varepsilon(t_{ij}) = \delta_{ij}T(\omega(i)), \varepsilon(f(\lambda)) = \varepsilon(f(\mu)) = f$ and the map is extended as a homomorphism and we equip $\mathcal{F}_R(M(n))$ with the structure of an $\mathfrak{h}$-bialgebroid.
DEFINITION 2.1. An \( h \)-space \( V \) is a vector space over \( M_{h^*} \), equipped with a diagonalizable \( h \)-module, i.e. \( V = \sum_{\alpha \in h^*} V_\alpha \), with \( M_{h^*} V_\alpha \in V_\alpha \) for all \( \alpha \in h^* \). A morphism of \( h \)-spaces is an \( h^* \)-invariant \( h \)-linear map.

We next define the tensor product of an \( h \)-bialgebroid \( A \) and an \( h \)-space \( V \). Put \( V \otimes A = \bigoplus_{\alpha, \beta \in h^*} (V_\alpha \otimes h^*, A_{\alpha \beta}) \) where \( \otimes h^* \) denotes the usual tensor product modulo the relations \( v \otimes \mu_1(f) a = f v \otimes a \). The grading \( V_\alpha \otimes h^* A_{\alpha \beta} \subseteq (V \otimes A)_\beta \) for all \( \alpha \) and \( f(v \otimes a) = v \otimes \mu_\tau(f) a \) makes \( V \otimes A \) an \( h \)-space. Analogously \( A \otimes V = \bigoplus_{\alpha, \beta \in h^*} (A_{\alpha \beta} \otimes h^*, V_\beta) \) where \( \otimes h^* \) denotes the usual tensor product modulo the relations \( \mu_\tau(f) a \otimes v = a \otimes f v \). The grading \( A_{\alpha \beta} \otimes h^* V_\beta \subseteq (A \otimes V)_\alpha \) and \( f(a \otimes v) = \mu_\tau(f) a \otimes v \) makes \( A \otimes V \) an \( h \)-space.

We now construct two special \( \mathcal{F}_R(M(n)) \)-comodules. Let \( W = M_{h^*} \langle w_i \rangle \) be the unital associative algebra generated by the elements \( w_i, 1 \leq i \leq n \) and \( M_{h^*} \), its elements denoted by \( f(\Delta) \), subject to the relations

\[
\begin{align*}
\forall i \leq n, \\
w_i^2 &= 0, \\
w_i w_j &= -h(\lambda_j - \lambda_i) w_i w_j, 1 \leq j \leq n,
\end{align*}
\]

(2.5)

as well as the relation \( f(\Delta)w_i = w_i f(\Delta + \omega(i)) \) for all \( f \in M_{h^*} \).

Let \( V = M_{h^*} \langle v_i \rangle \) be the unital associative algebra generated by the elements \( v_i, 1 \leq i \leq n \) and a copy of \( M_{h^*} \), embedded as a subalgebra, its elements denoted by \( f(\Delta) \) subject to the relations

\[
\begin{align*}
\forall i \leq n, \\
v_i^2 &= 0, \\
v_i v_j &= -h(\lambda_i - \lambda_j) v_i v_j, 1 \leq j \leq n,
\end{align*}
\]

(2.6)

plus that \( f(\Delta)v_i = v_i f(\Delta + \omega(i)) \) for all \( f \in M_{h^*} \).

The following result is easy to see.

THEOREM 2.2. \( \text{[14]} \) Define \( \alpha_R(1) = 1 \otimes 1 \), \( \alpha_R(w_i) = \sum_{j=1}^{n} w_j \otimes t_{ji} \), \( \alpha_L(1) = 1 \otimes 1 \), \( \alpha_L(v_i) = \sum_{j=1}^{n} t_{ij} \otimes v_j \). Then \( \alpha_L \) extends uniquely to \( \alpha_L : V \to \mathcal{F}_R(M(n)) \otimes V \) such that \( V \) is a left \( h \)-comodule algebra for \( \mathcal{F}_R(M(n)) \) and \( \alpha_R \) extends uniquely to \( \alpha_R : W \to W \otimes \mathcal{F}_R(M(n)) \) such that \( W \) is a right \( h \)-comodule algebra for \( \mathcal{F}_R(M(n)) \).

Let \( I \) be a subset of \([1,n]\) with entries \( i_1 < i_2 < \cdots < i_r \) and \( S_r \) be the symmetric group in \( r \) letters. For an element \( \sigma \in S_r \), we use \( l(\sigma) \) denote the length of \( \sigma \). The generalized sign functions \( S(\sigma, I) \) and \( \tilde{S}(\sigma, I) \) are defined as follows:

\[
S(\sigma, I)(\Delta) = \prod_{1 \leq k < l \leq r; \sigma(k) > \sigma(l)} (-h(\lambda_{i_{\sigma(k)}} - \lambda_{i_{\sigma(l)}}))
\]

\[
= (-q)^{l(\sigma)} \prod_{1 \leq k < l \leq r; \sigma(k) > \sigma(l)} \frac{q^{2\lambda_{i_{\sigma(k)}}} - q^{-2}q^{-2\lambda_{i_{\sigma(l)}}}}{q^{-2\lambda_{i_{\sigma(k)}}} - q^{-2\lambda_{i_{\sigma(l)}}}}
\]

(2.7)
\[ S(\sigma, I)(\lambda) = \prod_{k<l: \sigma(k) > \sigma(l)} (-h(\lambda_{\sigma(l)} - \lambda_{\sigma(k)})) = \frac{1}{S(\sigma, I)(\lambda + 1)}. \]

where \( \mathbf{1} = (1, \ldots, 1) \).

For two subsets \( I, J \subset [1, n] \) with \( |I| = |J| = r \), the dynamical quantum column minor determinants \( \xi^I_J \) and row minor determinants \( \eta^I_J \) and are defined as follows:

\[ \xi^I_J = \mu_r(S(\rho, J)^{-1}) \sum_{\sigma \in S_r} \mu_l(S(\sigma, I)) t_{i_{\sigma(1)}j_{\rho(1)}} \cdots t_{i_{\sigma(r)}j_{\rho(r)}} \]

\[ \eta^I_J = \mu_l(S(\rho, I)^{-1}) \sum_{\sigma \in S_r} \mu_r(S(\sigma, J)) t_{i_{\rho(1)}j_{\sigma(1)}} \cdots t_{i_{\rho(r)}j_{\sigma(r)}} \]

where \( \rho \in S_r \). Using the comodule structures of \( \alpha_L \) and \( \alpha_R \) the following result can be easily obtained.

\[ \text{Theorem 2.3.} \quad \xi^I_J = \eta^I_J \text{ in } \mathcal{F}_R(M(n)). \]

The element \( \det = \xi^{\{1,2,\ldots,n\}}_{\{1,2,\ldots,n\}} \) will be called the (quantum dynamic) determinant of \( \mathcal{F}_R(M(n)) \).

For ordered subsets \( I, J \) one has that

\[ \alpha_L(v_I) = \sum_{|K|=|I|} \xi^I_K \otimes v_K, \]

\[ \alpha_R(w_J) = \sum_{|K|=|J|} w_K \otimes \xi^K_J, \]

\[ \Delta(\xi^I_J) = \sum_{|K|=|I|} \xi^K_I \otimes \xi^K_J. \]

For disjoint ordered subsets \( I_1, I_2 \) of \( \{1, 2, \ldots, n\} \), we introduce the quantum sign element \( \text{sign}(I_1, I_2) \) inside \( \mathcal{M}_h \) by

\[ \text{sign}(I_1, I_2) = \prod_{k>l: k \in I_1, l \in I_2} (-h(\lambda_k - \lambda_l)). \]

\[ \text{Proposition 2.4.} \quad \xi^I_J = \mu_r(\text{sign}(I_1; J_2)) \xi^I_J \xi^{I_1} \xi^{I_2}. \]

\[ \xi^I_J = \sum_{I_1 \cup I_2 = I} \xi^I_{I_1} \mu_r(\text{sign}(J_2; J_1)) \xi^I_{I_2}. \]
It is easy to see from Proposition 2.4 that for any $i, j \in [1, n]$

$$
\begin{align*}
\delta_{ij} \det &= \sum_{k=1}^{n} \frac{\text{sign}(\{k\}; \hat{k})}{\text{sign}(\{i\}; i)} (\lambda)^{k} t_{kj} \xi_{i}^{k}, \\
\delta_{ij} \det &= \sum_{k=1}^{n} t_{jk} \frac{\text{sign}(\hat{i}; \{i\})}{\text{sign}(\hat{k}; \{k\})} (\mu)^{i} \xi_{j}^{i}, \\
\delta_{ij} \det &= \sum_{k=1}^{n} \frac{\text{sign}(\hat{k}; \{k\})}{\text{sign}(\hat{i}; \{i\})} (\mu)^{k} \xi_{i}^{k} t_{kj}, \\
\delta_{ij} \det &= \sum_{k=1}^{n} \xi_{j}^{k} \frac{\text{sign}(\{i\}; \hat{j})}{\text{sign}(\{k\}; k)} (\mu)^{i} (\lambda)^{j} t_{kj}.
\end{align*}
$$

(2.17)

**Lemma 2.5.** In $\mathcal{F}_{R}(M(n))$, the determinant commutes with all quantum minor determinants. In particular, $\det$ commutes with all generators $t_{ij}$. Moreover, $\Delta(\det) = \det \otimes \det$ and $\varepsilon(\det) = T^{-1}$, with $\mathbf{1} = (1, \ldots, 1) \in h^*$. 

**Proposition 2.6.** The $h$-bialgebroid $\mathcal{F}_{R}(M(n))$ is an $h$-Hopf algebra with the antipode $S$ defined on the generators by $S(\det^{-1}) = \det$, $S(\mu_{r}(f)) = \mu_{l}(f), S(\mu_{l}(f)) = \mu_{r}(f)$ for all $f \in M_{h^n}$ and

$$
S(t_{ij}) = \det^{-1} \frac{\mu_{l}(\text{sign}(\hat{j}; \{j\}))}{\mu_{r}(\text{sign}(\hat{i}; \{i\}))} \xi_{j}^{i}
$$

and extended as an algebra antihomomorphism.

**3. Quasideterminants and Dieudonné determinants**

Throughout this section we work with rings of fractions of noncommutative rings.

**Definition 3.1.** Let $X = (x_{ij})$ be an $n \times n$ matrix over a ring with identity such that its inverse matrix $X^{-1}$ exists, and the $(j, i)$th entry of $X^{-1}$ is an invertible element of the ring. Then the $(ij)$th quasideterminant of $X$ is defined by the formula

$$
\begin{vmatrix}
  x_{11} & \cdots & x_{1j} & \cdots & x_{1n} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  x_{i1} & \cdots & x_{ij} & \cdots & x_{in} \\
  \cdots & \cdots & \cdots & \ddots & \vdots \\
  x_{n1} & \cdots & x_{nj} & \cdots & x_{nn}
\end{vmatrix} = (X^{-1})^{-1}_{ji},
$$

where the first or the second notation with $\boxed{x_{ij}}$ denotes the quasideterminant.

When $n \geq 2$, and let $X^{ij}$ be the $(n - 1) \times (n - 1)$-matrix obtained from $X$ by deleting the $i$th row and $j$th column. In general $X^{i_{1} \cdots i_{r}, j_{1} \cdots j_{r}}$
denotes the submatrix obtained from $X$ by deleting the $i_1, \ldots, i_r$-th rows, and $i_1, \ldots, i_r$-th columns. Then

$$|X|_{ij} = x_{ij} - \sum_{i',j'} x_{ij'}(|X|_{j'}^{i'})x_{j'j},$$

where the sum runs over $i' \notin I \setminus \{i\}, j' \notin J \setminus \{j\}$.

**Theorem 3.2.** Let $T$ be the matrix of generators $t_{ij}$ of $\mathcal{F}_R(M(n))$, $\sigma = i_1 \ldots i_n$ and $\tau = j_1 \ldots j_n$ be two permutations of $S_n$. In the ring of fractions of $\mathcal{F}_R(M(n))$, one has that

$$\text{det}(T) = \frac{\prod_{k=1}^{n} \mu_t(\text{sign}(I_k^c; i_k))}{\prod_{k=1}^{n} \mu_t(\text{sign}(J_k^c; j_k))} t_{i_1 j_1} \cdots |T|_{i_n j_n},$$

where $I_k = \{i_1, \ldots, i_k\}$, $J_k = \{j_1, \ldots, j_k\}$.

**Proof.** By definition the quasi-determinants of $T$ are inverses of the entries of $S(T)$,

$$|T|_{ij} = S(t_{ji})^{-1} = \xi_j^{-1} \mu_t(\text{sign}(\hat{j}; \{j\})) \mu_t(\text{sign}(\hat{i}; \{i\})) \text{det}(T),$$

then

$$\text{det}(T) = \frac{\mu_t(\text{sign}(\hat{i}; \{i\}))}{\mu_t(\text{sign}(\hat{j}; \{j\}))} \xi_j^i |T|_{ij}.$$  

Eqs (3.1) follows from induction on $n$. \hspace{1cm} \Box

**Remark 3.3.** If $i_k = j_k = n + 1 - k$ for any $k$, all the factors on the right hand side of (3.1) commute with each other. In general the factors do not commute.

### 4. Dynamical quantum Pfaffians

First we review the general theory of the Pfaffian [11, 12], and we assume the minimum condition here. Let $\mathcal{B}$ be the algebra generated by the elements $b_{ij}$ for $1 \leq i < j \leq 2n$, and a copy of $M_{2n}$, embedded as a subalgebra, its elements denoted by $f_2(\lambda)$. The dynamical quantum Pfaffian is defined by

$$\text{Pf}(B) = \sum_{\sigma \in \Pi} S(\sigma)b_{\sigma(1)\sigma(2)}b_{\sigma(3)\sigma(4)} \cdots b_{\sigma(2n-1)\sigma(2n)},$$

where $S(\sigma) = S(\sigma, [1, 2n])$, $\Pi$ is the set of permutations $\sigma$ of $2n$ such that $\sigma(2i-1) < \sigma(2i), i = 1, \ldots, n.$

For any two disjoint subsets $I_1, I_2$ of $[1, 2n]$, we define the dynamical quantum sign functions $\text{sign}(I_1, I_2)$ and $\tilde{\text{sign}}(I_1, I_2)$ by

$$\text{sign}(I_1, I_2) = \prod_{k > l, k \in I_1, l \in I_2} (-h(\lambda_k - \lambda_l)),$$

$$\tilde{\text{sign}}(I_1, I_2) = \prod_{k < l, k \in I_1, l \in I_2} (-h(\lambda_k - \lambda_l)).$$

$$\text{sign}(I_1, I_2) = \prod_{k > l, k \in I_1, l \in I_2} (-h(\lambda_k - \lambda_l)),$$

$$\tilde{\text{sign}}(I_1, I_2) = \prod_{k < l, k \in I_1, l \in I_2} (-h(\lambda_k - \lambda_l)).$$

$$\text{sign}(I_1, I_2) = \prod_{k > l, k \in I_1, l \in I_2} (-h(\lambda_k - \lambda_l)),$$

$$\tilde{\text{sign}}(I_1, I_2) = \prod_{k < l, k \in I_1, l \in I_2} (-h(\lambda_k - \lambda_l)).$$
Let $I = \{i_1, i_2, \ldots, i_k\}$ with $1 \leq i_1 < i_2 < \ldots, i_k \leq 2n$. Denote by $B_I$ the submatrix of $B$ with the rows and columns indexed by $I$. The following result gives an iterative algorithm to compute the dynamic Pfaffian.

**Proposition 4.1.** For each $0 \leq t \leq n$ we have that

$$\text{Pf}(B) = \sum_I \text{sign}(I; I^c) \text{Pf}(B_I) \text{Pf}(B_{I^c}),$$

where the sum runs over all subsets $I$ of $[1, 2n]$ such that $|I| = 2t$ and $I^c$ is the complement of $I$.

**Proof.** Define the tensor product of $W \otimes B$ to be the usual tensor product modulo the relations $fw \otimes b = w \otimes fb$. Let $\Omega = \sum_{i<j} w_i w_j \otimes b_{ij}$, then

$$\bigwedge^n \Omega = w_1 \wedge \cdots \wedge w_{2n} \otimes \text{Pf}(B).$$

On the other hand,

$$\bigwedge^n \Omega = \Omega^t \bigwedge^{n-t}$$

$$= \sum_{I,J} (w_I \otimes \text{Pf}(B_I))(w_J \otimes \text{Pf}(B_J))$$

$$= \sum_{I,J} w_I w_J \otimes \text{Pf}(B_I) \text{Pf}(B_J).$$

(4.4)

Note that $w_I w_J$ vanishes unless $J = I^c$, therefore we have that

$$\text{Pf}(B) = \sum_I \text{sign}(I; I^c) \text{Pf}(B_I) \text{Pf}(B_{I^c}).$$

(4.5)

The following transformation formula establishes the relation between the quantum dynamic Pfaffian and determinant.

**Theorem 4.2.** Denote by $\mathcal{F}_R(M(2n)) \otimes B$ the usual tensor product modulo the relations $\mu_t(f) t \otimes b = t \otimes fb$ and $f(t \otimes b) = \mu_t(f) t \otimes b$. Let $c_{kl}^{ij}$ be the $2 \times 2$-dynamical quantum minors in $\mathcal{F}_R(M(2n))$, $b_{ij}$ the generators of $B$, and $c_{ij} = \sum_{k<l} c_{kl}^{ij} \otimes b_{kl}$. Then in $\mathcal{F}_R(M(2n)) \otimes B$ we have

$$\text{Pf}(C) = \text{det}(T) \otimes \text{Pf}(B).$$

**Proof.** Let $w \otimes t \otimes b$ be an element of $W \otimes \mathcal{F}_R(M(n)) \otimes B$, then

$$f(w \otimes t \otimes b) = w \otimes \mu_t(f) t \otimes b = w \otimes f(t \otimes b),$$

$$f(w \otimes t) \otimes b = w \otimes \mu_t(f) t \otimes b = w \otimes t \otimes f(b).$$

(4.5)

Let $\delta_i = \sum_{j=1}^{2n} w_j \otimes t_{ji}$, and consider the element $\Omega = \sum_{i,j} w_i w_j \otimes c_{ij}$. It is clear that

$$\Omega^n = w_1 \cdots w_{2n} \otimes \text{Pf}(C),$$

(4.6)
where the product is the wedge product among $w_i$. On the other hand, 
$\Omega = \sum \delta_i \delta_j \otimes b_{ij}$. Then

\begin{equation}
\Omega^n = \delta_1 \cdots \delta_{2n} \otimes \text{Pf}(B) = w_1 \cdots w_{2n} \otimes \det(T) \otimes \text{Pf}(B).
\end{equation}

Comparing (4.13) and (4.14) we conclude that

$$\text{Pf}(C) = \det(T) \otimes \text{Pf}(B).$$

\[\square\]

Let $\tilde{B}$ be the algebra generated by the elements $\tilde{b}_{ji}$ for $1 \leq i < j \leq 2n$, and a copy of $M_{b_{ii}}$ embedded as a subalgebra, its elements denoted by $f(\lambda)$. The dynamical quantum Pfaffian is defined by

$$\tilde{\text{Pf}}(\tilde{B}) = \sum_{\sigma \in \Pi} S(\sigma) \tilde{b}_{\sigma(2n-1)} \cdots \tilde{b}_{\sigma(2n-2)} \cdots \tilde{b}_{\sigma(1)} \sigma(2),$$

where $\Pi$ is the set of shuffle permutations $\sigma$ of $2n$ such that $\sigma(2i-1) < \sigma(2i)$, $i = 1, \ldots, n$.

For any two disjoint subsets $I_1, I_2$ of $\{1, 2, \ldots, 2n\}$, we define the dynamical quantum sign by

\begin{equation}
\text{sign}(I_1, I_2) = \prod_{k<l, k \in I_1, l \in I_2} (-h(\lambda_k - \lambda_l)).
\end{equation}

Similarly, we have the following statements.

**Proposition 4.3.** For any $0 \leq t \leq n$, we have that

\begin{equation}
\tilde{\text{Pf}}(\tilde{B}) = \sum_I \text{sign}(I; I^c) \tilde{\text{Pf}}(\tilde{B}_I) \tilde{\text{Pf}}(\tilde{B}_{I^c}),
\end{equation}

where the sum runs through all subsets $I$ of $[1, 2n]$ such that $|I|=2t$.

**Proof.** Define the tensor product of $\tilde{B} \otimes V$ to be the usual tensor product modulo the relations $fb \otimes v = b \otimes fv$. Let $\bar{\Omega} = \sum_{i>j} b_{ij} \otimes v_i v_j$, then

\begin{equation}
\tilde{\text{Pf}}(\tilde{B}) = \bar{\text{Pf}}(\tilde{B}) \otimes v_n \wedge \cdots \wedge v_1.
\end{equation}

On the other hand,

\begin{equation}
\tilde{\text{Pf}}(\tilde{B}) = \bar{\text{Pf}}(\tilde{B}) \otimes v_1 \wedge \cdots \wedge v_n.
\end{equation}

It is easy to see that $v_I v_J$ vanishes unless $J = I^c$. Thus we conclude that

$$\tilde{\text{Pf}}(\tilde{B}) = \sum_I \text{sign}(I; I^c) \text{Pf}(\tilde{B}_I) \text{Pf}(\tilde{B}_{I^c}).$$
THEOREM 4.4. Denotes by $\tilde{B}\otimes\mathcal{F}_R(M(2n))$ the usual tensor product modulo the relations $b \otimes \mu(f)t = fb \otimes t$ and $f(b \otimes t) = b \otimes \mu_r(f)t$. Let $\xi_{ij}^k$ be the dynamical quantum minor in $\mathcal{F}_R(M(n))$, $c_{ji} = \sum_{k \leq i} \tilde{b}_{ik} \otimes \xi_{ij}^k$. Then in $\tilde{B}\otimes\mathcal{F}_R(M(2n))$ we have $\tilde{\text{Pf}}(C) = \tilde{\text{Pf}}(\tilde{B}) \otimes \det(T)$.

PROOF. Let $b \otimes t \otimes v$ be an element of $\tilde{B}\otimes\mathcal{F}_R(M(n))\tilde{\otimes}V$, then

\begin{equation}
\begin{aligned}
    b \otimes t \otimes fv &= b \otimes \mu_r(f)t \otimes v = f(b \otimes t) \otimes v, \\
    b \otimes f(t \otimes v) &= b \otimes \mu_l(f)t \otimes v = f(b) \otimes t \otimes v.
\end{aligned}
\end{equation}

Let $\delta_i = \sum_{j=1}^{2n} t_{ij} \otimes v_j$, and consider the element $\tilde{\Omega} = \sum c_{ji} \otimes v_j v_i$. It is clear that

\begin{equation}
\tilde{\Omega}^n = \tilde{\text{Pf}}(C) \otimes v_{2n} \cdots v_1.
\end{equation}

On the other hand, $\tilde{\Omega} = \sum b_{lk} \otimes \delta_l \delta_k$. Then

\begin{equation}
\tilde{\Omega}^n = \tilde{\text{Pf}}(C) \otimes \delta_{2n} \cdots \delta_1 = \tilde{\text{Pf}}(\tilde{B}) \otimes \det(T) \otimes \delta_{2n} \cdots \delta_1.
\end{equation}

Comparing (4.13) and (4.14) we conclude that

\[ \tilde{\text{Pf}}(C) = \tilde{\text{Pf}}(\tilde{B}) \otimes \det(T). \]

\[ \square \]

We now generalize the notion of the dynamical quantum Pfaffian to the dynamical quantum hyper-Pfaffian. Let $\mathcal{B}$ be the algebra generated by the elements $b_{i_1 \cdots i_m}, i_1 < i_2 < \cdots < i_m, 1 \leq i_k \leq mn, k = 1, \ldots, m$, and a copy of $M_{\bar{r}}$, embedded as a subalgebra, its elements denoted by $f(\underline{\lambda})$. The dynamical quantum hyper-Pfaffian is defined by

\[ \text{Pf}_m(B) = \sum_{\sigma \in \Pi} S(\sigma) b_{\sigma(1)} \cdots b_{\sigma(mn)}, \]

where $\Pi$ is the set of permutations $\sigma$ of $mn$ such that $\sigma((k-1)m+1) < \sigma((k-1)m+2) < \cdots < \sigma(km), k = 1, \ldots, n$.

Note that the dynamical hyper-Pfaffian uses only the entries $b_{i_1 \cdots i_m}$, where $i_1 < \cdots < i_m$. Clearly $\text{Pf}_2(B) = \text{Pf}(B)$, the quantum dynamical Pfaffian.

PROPOSITION 4.5. For any $0 \leq t \leq n$,

\begin{equation}
\text{Pf}_m(B) = \sum_I \text{sign}(I; I^c) \text{Pf}_m(B_I) \text{Pf}_m(B_{I^c}),
\end{equation}

where the sum is taken over all subset $I$ of $[1, mn]$ such that $|I| = mt$.

Let $I = \{i_1, i_2, \cdots, i_m\}$ such that $i_1 < i_2 < \cdots < i_m$. We denote by $b_I$ the element $b_{i_1 \cdots i_m}$.
Theorem 4.6. Denotes by $\mathcal{F}_R(M(mn))\otimes B$ the usual tensor product modulo the relations $\mu_i(f)t \otimes b = t \otimes fb$ and $f(t \otimes b) = \mu_i(f)t \otimes b$. Let $\xi^I_j$ be the dynamical quantum minor in $\mathcal{F}_R(M(n))$, $c_I = \sum_j \xi^I_j \otimes b_J$. Then in $\mathcal{F}_R(M(mn))\otimes B$ we have $\text{Pf}_m(C) = \det(T) \otimes \text{Pf}_m(B)$.

Let $\tilde{B}$ be the algebra generated by the elements $\tilde{b}_{i_1\cdots i_m}, i_1 > i_2 > \cdots > i_m, 1 \leq i_k \leq mn, k = 1, \ldots, m$. Define dynamical quantum hyper-Pfaffian by

$$\tilde{\text{Pf}}_m(\tilde{B}) = \sum_{\sigma \in \Pi} \tilde{S}(\sigma) \tilde{b}_{\sigma(mn)\cdots \sigma(mn-m+1)} \tilde{b}_{\sigma(mn-m)\cdots \sigma(mn-2m+1)} \cdots \tilde{b}_{\sigma(m)\cdots \sigma(1)},$$

where $\Pi$ is the set of permutations $\sigma$ of $mn$ such that $\sigma((k-1)m+1) < \sigma((k-1)m+2) < \cdots < \sigma(km), k = 1, \ldots, n$.

Clearly $\tilde{\text{Pf}}_2(\tilde{B}) = \tilde{\text{Pf}}(\tilde{B})$ discussed before.

Proposition 4.7. For any $0 \leq t \leq n$,

$$(4.16) \quad \tilde{\text{Pf}}_m(\tilde{B}) = \sum_I \tilde{\text{sign}}(I; I^c) \tilde{\text{Pf}}_m(\tilde{B}_I) \tilde{\text{Pf}}_m(\tilde{B}_{I^c}),$$

where the sum is taken over all subset $I$ of $[1, mn]$ such that $|I| = mt$.

Let $I = \{i_1, i_2, \cdots, i_m\}$ such that $i_1 < i_2 < \cdots < i_m$. We denote by $\tilde{b}_I$ the element $\tilde{b}_{i_1\cdots i_m}$. The following result is proved by the similar method as in the case of $m = 2$.

Theorem 4.8. Denotes by $\tilde{B} \otimes \mathcal{F}_R(M(mn))$ the usual tensor product modulo the relations $b \otimes \mu_i(f)t = fb \otimes t$ and $f(b \otimes t) = b \otimes \mu_r(f)t$. Let $\xi^I_j$ be the dynamical quantum minor in $\mathcal{F}_R(M(n))$, $c_I = \sum_j \tilde{b}_J \otimes \xi^I_j$. Then in $\tilde{B} \otimes \mathcal{F}_R(M(mn))$ we have $\tilde{\text{Pf}}_m(C) = \tilde{\text{Pf}}_m(\tilde{B}) \otimes \det(T)$.

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