ON THE EXISTENCE OF A GENERALIZED CLASS OF RECURRENT MANIFOLDS

ABSOS ALI SHAIKH, INDRANIL ROY AND HARADHAN KUNDU

Abstract. The present paper deals with the proper existence of a generalized class of recurrent manifolds, namely, hyper-generalized recurrent. It also deals with the existence of properness of various generalized curvature restricted geometric structures. For example, the existence of manifolds which are non-recurrent but Ricci recurrent, conharmonically recurrent and semisymmetric; not weakly symmetric but weakly Ricci symmetric, conformally weakly symmetric and conharmonically weakly symmetric; non-Einstein but Ricci simple; not satisfy $P \cdot P = 0$ but fulfill the condition $P \cdot P = \frac{1}{3}Q(S, P)$ etc., $P$ being the projective curvature tensor. For this purpose we have presented a metric and computed its curvature properties and finally we have checked various geometric structures admitting by the metric.

1. Introduction

Let $M$, dim $M = n \geq 3$, be a connected semi-Riemannian manifold endowed with a semi-Riemannian metric $g$ of signature $(s, n - s)$, $0 \leq s \leq n$. If $s = 0$ or $n$ then $M$ is a Riemannian manifold and if $s = 1$ or $n - 1$ then $M$ is a Lorentzian manifold. A semi-Riemannian manifold has mainly three notions of curvature tensors, namely Riemann-Christoffel curvature tensor $R$ (simply called curvature tensor), the Ricci tensor $S$ and the scalar curvature $r$. Let $\nabla$ be the Levi-Civita connection on $M$, which is the unique torsion free metric connection on $M$. Symmetry plays an important role in the geometry. It is well known that $M$ is called locally symmetric ([1], [3]) if $\nabla R = 0$, which can be stated as the local geodesic symmetries at each point of $M$ are isometry. The study of generalization of locally symmetric manifolds was started in 1946 and continued till date in different directions such as semisymmetric manifolds by Cartan [6] (which was classified by Szabó ([73]-[75])), $\kappa$-space or recurrent space by Ruse ([46], [47], [48]), 2-recurrent manifolds by Lichnerowicz [35], weakly symmetric manifolds by Selberg [49], generalized recurrent manifolds by Dubey [11], quasi-generalized recurrent manifolds by Shaikh and Roy [69], hyper-generalized recurrent manifolds by Shaikh and Patra [68], weakly generalized recurrent manifolds by Shaikh and Roy [70], pseudosymmetric manifolds by Chaki.
pseudosymmetric manifolds by Deszcz ([17], [23]), weakly symmetric manifolds by Tamássy and Binh [76], conformally recurrent manifolds by Adati and Miyazawa [1], projectively recurrent manifolds by Adati and Miyazawa [2]. It also be mentioned that the notion of weakly symmetric manifold by Selberg is different from that by Tamássy and Binh, and pseudosymmetric manifold by Chaki is also different from pseudosymmetric manifold by Deszcz. However recently Shaikh et. al. [53] studied the equivalency of Deszcz pseudosymmetry with Chaki pseudosymmetry as well as weak symmetry by Tamássy and Binh.

Again the manifold \( M \) is said to be Ricci symmetric if \( \nabla S = 0 \). The notion of Ricci symmetry was also weakened in various ways such as Ricci recurrent by Patterson [45], Ricci pseudosymmetric by Deszcz [15], Ricci semisymmetric by Szabó ([73], [74], [75]), pseudo Ricci symmetric by Chaki [8], weakly Ricci symmetric by Tamássy and Binh [77]. Weakly symmetric and weakly Ricci symmetric manifolds by Tamássy and Binh were investigated by Shaikh and his coauthors (see, [9], [10], [32], [54], [55], [56], [57], [58], [60], [61], [62], [63], [64], [71] and also references therein). Recently in [66] Shaikh and Kundu studied the characterization of warped product weakly symmetric and weakly Ricci symmetric manifolds.

Again, a semi-Riemannian manifold is Einstein if its Ricci tensor is constant multiple to the metric tensor. As a generalization of Einstein manifold, the notion of quasi-Einstein manifold arose during the study of exact solutions of the Einstein’s field equation as well as during the investigation of quasi-umbilical hypersurfaces.

By the decomposition of the covariant derivative \( \nabla S \), Gray [31] obtained two classes \( \mathcal{A} \) and \( \mathcal{B} \) of Riemannian manifolds which are properly lie between the class of Ricci symmetric manifolds and the manifolds of constant scalar curvature. The class \( \mathcal{A} \) (resp. \( \mathcal{B} \)) is the class of Riemannian manifolds whose Ricci tensor is cyclic parallel (resp. Codazzi type). Every Ricci symmetric manifold is of class \( \mathcal{A} \) and also of class \( \mathcal{B} \) but not conversely. We note that every Einstein manifold and hence every manifold of constant curvature is of class \( \mathcal{A} \) as well as \( \mathcal{B} \). The existence of both the classes are given in [52] (see also [25]).

Hence a natural question arises:

**Q. 1** Does there exist a hyper-generalized recurrent manifold which is

(i) not locally symmetric but semisymmetric,

(ii) not weakly symmetric but weakly Ricci symmetric as well as conformally weakly symmetric,

(iii) not recurrent but Ricci recurrent as well as conformally recurrent,

(iv) neither class \( \mathcal{A} \) nor \( \mathcal{B} \) but of constant scalar curvature,

(v) not Einstein but Ricci simple i.e., special quasi-Einstein,

(vi) does not satisfy \( P \cdot P = 0 \) but fulfills the condition \( P \cdot P = LQ(S, P) \) for some scalar \( L \).
This paper provides the answer of this question as affirmative by an explicit example which induces a new class of semi-Riemannian manifolds (see Theorem 3.1 and Remark 3.1).

The paper is organized as follows. The definitions of various notions and their interrelations are given in Section 2 as preliminaries. In the last section we compute the curvature properties of the metric given by

$$ds^2 = g_{ij}dx^idx^j = e^{x^1+x^3}(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + e^{x^1}(dx^4)^2, \quad i, j = 1, 2, 3, 4,$$

which produces the answer of Q. 1 as affirmative, and hence we obtain a generalized class of recurrent manifold realizing the conditions of Q.1.

2. Preliminaries

Let us consider a connected semi-Riemannian manifold $M$ of dimension $n(\geq 3)$ (these conditions are assumed through the paper). The manifold $M$ is said to be Einstein (briefly, $E_n$) if its Ricci tensor $S$ of type $(0, 2)$ satisfies $S = \frac{r}{n}g$. Consider the subset $U_S = \{ x \in M : (S - \frac{r}{n}g)_x \neq 0 \}$. Then the manifold is said to be quasi-Einstein (briefly, $QE_n$) ([14], [18], [19], [26], [27], [28], [30], [59], [65], [72]) if on $U_S \subset M$, we have

$$S - \alpha g = \beta A \otimes A,$$

where $A$ is an 1-form on $U_S$ and $\alpha, \beta$ are two scalars on $U_S$, $\otimes$ is the tensor product. It is clear that the 1-form $A$ as well as the function $\beta$ are non-zero at every point on $U_S$. It is noted that if $\alpha$ vanishes identically then a quasi-Einstein manifold turns into a Ricci simple manifold. Hence a semi-Riemannian manifold $(M, g)$, $n \geq 3$ is said to be Ricci simple if the following condition (2.1)

$$S = \beta A \otimes A$$

holds, where $\beta$ is a scalar and $A \in \chi^*(M)$, $\chi^*(M)$ being the Lie algebra of all smooth 1-forms on $M$.

The manifold $M$ is said to be a Codazzi type Ricci tensor ([29], [50]) (resp. cyclic Ricci parallel [31]), if it satisfies

$$(\nabla_{X_1}S)(X_2, X_3) = (\nabla_{X_2}S)(X_1, X_3)$$

(resp. $$(\nabla_{X_1}S)(X_2, X_3) + (\nabla_{X_2}S)(X_3, X_1) + (\nabla_{X_3}S)(X_1, X_2) = 0$$)

for all vector fields $X_1, X_2, X_3 \in \chi(M)$, where $\chi(M)$ being the Lie algebra of all smooth vector fields on $M$. We note that throughout this paper we consider $X, Y, X_i \in \chi(M)$, $i = 1, 2, 3, \cdots$.

It is well-known that the conformal transformation is an angle preserving mapping, the projective transformation is a geodesic preserving mapping whereas concircular transformation is
the geodesic circle preserving mapping and conharmonic transformation is a harmonic function preserving mapping and each transformation induces a curvature tensor as invariant such as conformal curvature tensor $C$, projective curvature tensor $P$, concircular curvature tensor $W$ and conharmonic curvature tensor $K$ ([14], [20], [30], [33], [79]), and are respectively given as:

$$C(X_1, X_2, X_3, X_4) = \left[ R - \frac{1}{n-2}(g \wedge S) + \frac{r}{2(n-2)(n-1)}(g \wedge g) \right] (X_1, X_2, X_3, X_4),$$

$$P(X_1, X_2, X_3, X_4) = R(X_1, X_2, X_3, X_4) - \frac{1}{n-1}[g(X_1, X_4)S(X_2, X_3) - g(X_2, X_4)S(X_1, X_3)],$$

$$W(X_1, X_2, X_3, X_4) = \left[ R - \frac{r}{2n(n-1)}(g \wedge g) \right] (X_1, X_2, X_3, X_4),$$

$$K(X_1, X_2, X_3, X_4) = \left[ R - \frac{1}{n-2}(g \wedge S) \right] (X_1, X_2, X_3, X_4),$$

where ‘$\wedge$’ denotes the Kulkarni-Nomizu product, for two $(0,2)$-tensors $E$ and $F$, which is defined as (see e.g. [20], [30], [34]):

$$(E \wedge E)(X_1, X_2, X_3, X_4) = E(X_1, X_4)E(X_2, X_3) + E(X_2, X_3)E(X_1, X_4) - E(X_1, X_3)E(X_2, X_4) - E(X_2, X_4)E(X_1, X_3).$$

We recall that the manifold $M$ is locally symmetry (briefly, $S_n$) and resp., Ricci symmetric (briefly, $S$-$S_n$) if $\nabla R = 0$ and $\nabla S = 0$ respectively. If we replace $R$ by other curvature tensors we get the corresponding symmetric structures, e.g., conformally symmetric (briefly, $C$-$S_n$), projectively symmetric (briefly, $P$-$S_n$), concircularly symmetric (briefly, $W$-$S_n$), conharmonically symmetric (briefly $K$-$S_n$) etc.

Let $U_L = \{x \in M : (R)_x \neq 0 \text{ and } (\nabla R)_x \neq 0 \}$ and $U_N = \{x \in M : (S)_x \neq 0 \text{ and } (\nabla S)_x \neq 0 \}$. The manifold $M$ is said to be recurrent (briefly, $K_n$) ([78]) (resp., Ricci recurrent (briefly, $S$-$K_n$) ([45]) if on $U_L \subset M$ (resp., $U_N \subset M$) we have the following:

$$\nabla R = A \otimes R \quad (\text{resp. } \nabla S = A \otimes S)$$

for some $A \in \chi^*(M)$, called the associated 1-form of recurrency. Again replacing $R$ by some other $(0,4)$-tensors we get the corresponding recurrent structures, e.g., conformally recurrent (briefly, $C$-$K_n$), projectively recurrent (briefly, $P$-$K_n$), concircularly recurrent (briefly, $W$-$K_n$), conharmonically recurrent (briefly, $K$-$K_n$) etc. Now we present some generalization of recurrency.

Let $U_Q = \{x \in M : (R)_x \neq 0 \text{ and } (\nabla R - \xi \otimes R)_x \neq 0 \text{ for all 1-forms } \xi \}$. Then $M$ is said to generalized recurrent ([11]) (briefly, $GK_n$), quasi-generalized recurrent ([69]) (briefly, $QGK_n$), hyper generalized recurrent ([68]) (briefly, $H GK_n$) and weakly generalized recurrent ([70]) (briefly,
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$W G K_n$) if on $U_Q \subset M$ the conditions

$$\nabla R = A \otimes R + B \otimes G$$

$$\nabla R = A \otimes R + B \otimes [G + g \wedge (\eta \otimes \eta)]$$

$$\nabla R = A \otimes R + B \otimes (S \wedge g)$$

$$\nabla R = A \otimes R + B \otimes \frac{1}{2}(S \wedge S)$$

hold respectively for some $A, B$ and $\eta \in \chi^*(M)$. Recently Olszak and Olszak [44] showed that every generalized recurrent manifold turns into a recurrent manifold and hence such a generalized notion does not exist. It is interesting to note that although $G K_n$ does not exist but the proper existence of $W G K_n$ is already presented in [51] and in this paper we established the proper existence of $H G K_n$. We note that on a quasi-Einstein manifold with associated scalars $\alpha$ and $\beta$, if $\alpha = \beta$ then $W G K_n$ and $Q G K_n$ are equivalent, and if $2\alpha = \beta$ then $H G K_n$ and $Q G K_n$ are equivalent.

Also there is a generalized notion of Ricci recurrent, namely, generalized Ricci recurrent. The manifold $M$ is said to be generalized Ricci recurrent (briefly, $S-G K_n$) if on $U_T = \{ x \in M : (S)_x \neq 0 \text{ and } (\nabla S - \xi \otimes S)_x \neq 0 \text{ for all 1-forms } \xi \}$

$$\nabla S = A \otimes S + B \otimes g$$

holds for some $A, B \in \chi^*(M)$.

Again generalizing the notion of local symmetry, Chaki [7] introduced the notion of pseudosymmetry. The manifold $M$ is said to be pseudosymmetric in sense of Chaki or simply Chaki pseudosymmetric (briefly, $C P S_n$) [7] if on $U_L \subset M$ its curvature tensor $R$ satisfies the relation

$$(\nabla_X R)(X_1, X_2, X_3, X_4) = 2A(X)R(X_1, X_2, X_3, X_4) + A(X_1)R(X_1, X, X_3, X_4)$$

$$+ A(X_2)R(X_1, X, X_3, X_4) + A(X_3)R(X_1, X_2, X, X_4) + A(X_4)R(X_1, X_2, X_3, X),$$

$\forall X, X_i \in \chi(M)$ and a non-zero 1-form $A$. Now replacing $R$ by other $(0, 4)$-tensors we obtain various Chaki pseudosymmetry structures, e.g., Chaki conformally pseudosymmetric (briefly, $C-C P S_n$), Chaki projectively pseudosymmetric (briefly, $P-C P S_n$), Chaki concircularly pseudosymmetric (briefly, $W-C P S_n$), Chaki conharmonically pseudosymmetric (briefly, $K-C P S_n$) etc.

Again in 1988, Chaki [8] introduced the notion of pseudo Ricci symmetric (briefly, $S-C P S_n$) manifolds defined as follows:

The manifold $M$ is said to be pseudo Ricci symmetric if on $U_N \subset M$ its Ricci tensor $S$ is not
identically zero and satisfies the relation

\[
(\nabla_X S)(X_1, X_2) = 2A(X)S(X_1, X_2) + A(X_1)S(X, X_2) + A(X_2)S(X_1, X),
\]

\forall X, X_i \in \chi(M) and a non-zero 1-form \(A\).

The manifold \(M\) is said to be weakly symmetric by Tamássy and Binh (briefly, \(WS_n\)) \([76]\) if on \(U_Q \subset M\) we have

\[
\nabla_X R(X_1, X_2, X_3, X_4) = A(X)R(X_1, X_2, X_3, X_4) + B(X_1)R(X_1, X_2, X_3, X_4)
\]

\[
+ B(X_2)R(X_1, X_3, X_4) + D(X_3)R(X_1, X_2, X_4) + D(X_4)R(X_1, X_2, X_3, X)
\]

\forall X, X_i \in \chi(M) (i = 1, 2, 3, 4) and some \(A, B, D \in \chi^*(M)\). Now replacing \(R\) by some other \((0,4)\)-tensors we obtain the corresponding weak symmetry structures, e.g., conformally weak symmetry (briefly, \(C-WS_n\)), projectively weak symmetry (briefly, \(P-WS_n\)), concircularly weak symmetry (briefly, \(W-WS_n\)), conharmonically weak symmetry (briefly, \(K-WS_n\)) etc.

Again in 1993 Tamássy and Binh \([77]\) introduced the notion of weakly Ricci symmetric manifold defined as follows:

The manifold \(M\) is said to be weakly Ricci symmetric (briefly, \(S-WS_n\)) if on \(U_T \subset M\) its Ricci tensor \(S\) is not identically zero and satisfies the condition

\[
(\nabla_X S)(X_1, X_2) = A(X)S(X_1, X_2) + B(X_1)S(X, X_2) + D(X_2)S(X_1, X),
\]

\forall X, X_i \in \chi(M) and some \(A, B, D \in \chi^*(M)\). Again to get similar structures we can replace \(S\) by some other \((0,2)\)-tensors. We note that recurrency and pseudosymmetry are special cases of weak symmetry for any particular \((0,4)\) or \((0,2)\)-tensor. Recently in \([53]\) Shaikh et. al. studied on weak symmetry structure and presented the reduced form of the defining condition of weak symmetry for various \((0,4)\) and \((0,2)\)-tensors.

For a \((0,4)\)-tensor \(H\) and a symmetric \((0,2)\)-tensor \(E\) we can define two endomorphisms \(X \wedge_E Y\) and \(\mathcal{H}(X, Y)\) by \([14], [20], [30]\)

\[
(X \wedge_E Y)X_1 = E(Y, X_1)X - E(X, X_1)Y, \quad \mathcal{H}(X, Y)X_1 = H(X, Y)X_1
\]

respectively, where \(H\) is the corresponding \((1,3)\)-tensor of \(H\). We note that the corresponding endomorphism of \(R\) is given by \(\mathcal{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\), where the square bracket is the Lie bracket over \(\chi(M)\).

For a \((0,k)\)-tensor \(T, k \geq 1\), a \((0,4)\)-tensor \(H\) and a symmetric \((0,2)\)-tensor \(E\) we define two
(0, k + 2)-tensors \( H \cdot T \), and \( Q(E, T) \) by \([14, 20, 22, 30]\)
\[
(H \cdot T)(X_1, \ldots, X_k, X, Y) = (\mathcal{H}(X, Y) \cdot T)(X_1, \ldots, X_k)
\]
\[
= -T(\mathcal{H}(X, Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, \mathcal{H}(X, Y)X_k),
\]
\[
Q(E, T)(X_1, \ldots, X_k; X, Y) = ((X \wedge E Y) \cdot T)(X_1, \ldots, X_k)
\]
\[
= -T((X \wedge E Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge E Y)X_k).
\]

Putting in the above formulas \( T = R, S, C, P, W, K \) and \( E = g \) or \( S \), we obtain the tensors: \( R \cdot R, R \cdot S, R \cdot C, R \cdot P, R \cdot W, R \cdot K, C \cdot R, C \cdot S, C \cdot C, C \cdot P, C \cdot W, C \cdot K, P \cdot R, P \cdot C, P \cdot P, P \cdot W, P \cdot K, Q(g, R), Q(g, S), Q(g, C), Q(g, P), Q(g, W), Q(g, K), Q(S, R), Q(S, C), Q(S, P), Q(S, W), Q(S, K) \) etc. The tensor \( Q(E, T) \) is called the Tachibana tensor of the tensors \( E \) and \( T \), or the Tachibana tensor for short \([22]\).

If the manifold \( M \) satisfying the condition \( R \cdot R = 0 \) (resp., \( R \cdot S = 0 \), \( R \cdot C = 0 \), \( R \cdot W = 0 \), \( R \cdot P = 0 \), \( R \cdot K = 0 \)) then it is called semisymmetric \([6, 73]\) (briefly, \( SS_n \)) (resp., Ricci semisymmetric (briefly, \( S-SS_n \)), conformally semisymmetric (briefly, \( C-SS_n \)), projectively semisymmetric (briefly, \( P-SS_n \)), concircularly semisymmetric (briefly, \( W-SS_n \)), conharmonically semisymmetric (briefly, \( K-SS_n \)).

The manifold \( M \) is said to be pseudosymmetric in sense of Deszcz or simply Deszcz pseudosymmetric (briefly, \( DPS_n \)) (resp., Ricci pseudosymmetric (briefly, \( S-DPS_n \)) \([23]\) if the condition
\[
R \cdot R = L_R Q(g, R) \quad \text{(resp.,} R \cdot S = L_S Q(g, S))
\]
holds on \( U_R = \{ x \in M : \left( R - \frac{s}{n(n-1)} G \right)_x \neq 0 \} \) (resp., \( U_S = \{ x \in M : (S - \frac{s}{n} g)_x \neq 0 \} \)) for a function \( L_R \) (resp., \( L_S \)) on \( U_R \) (resp., \( U_S \)).

The manifold \( M \) such that its dimension \( n \geq 4 \), is said to be a manifold with pseudosymmetric Weyl conformal curvature tensor \([16, 24]\) (resp., pseudosymmetric Weyl projective curvature tensor) if
\[
C \cdot C = L_C Q(g, C) \quad \text{(resp.,} P \cdot P = L_P Q(g, P))
\]
on \( U_C = \{ x \in M : C \neq 0 \text{ at } x \} \) (resp., \( U_P = \{ x \in M : Q(g, P) \neq 0 \text{ at } x \} \)), where \( L_C \) (resp., \( L_P \)) is a function on \( U_C \) (resp., \( U_P \)). We note that \( U_C \subset U_R \).

Again \( M \) is said to be Ricci-generalized pseudosymmetric \([12, 13]\) if at every point of \( M \), the tensor \( R \cdot R \) and the Tachibana tensor \( Q(S, R) \) are linearly dependent. Hence \( M \) is Ricci-generalized pseudosymmetric if and only if
\[
(2.4) \quad R \cdot R = L Q(S, R)
\]
holds on \( U = \{ x \in M : Q(S, R) \neq 0 \text{ at } x \} \), where \( L \) is some function on this set. An important subclass of Ricci-generalized pseudosymmetric manifolds is formed by the manifolds realizing the condition \([12], [24]\)

\[
(2.5) \quad R \cdot R = Q(S, R).
\]

The manifold \( M \) such that its dimension \( n \geq 4 \), is said to be a manifold with Ricci generalized pseudosymmetric Weyl conformal curvature tensor \([16], [24]\) (resp., Ricci generalized pseudosymmetric Weyl projective curvature tensor) if

\[
(2.6) \quad C \cdot C = L_1 Q(S, C) \quad \text{(resp., } P \cdot P = L_2 Q(S, P))
\]
on \( U_1 = \{ x \in M : Q(S, C) \neq 0 \text{ at } x \} \) (resp., \( U_2 = \{ x \in M : Q(S, P) \neq 0 \text{ at } x \} \)), where \( L_1 \) (resp., \( L_2 \)) is a function on \( U_1 \) (resp., \( U_2 \)).

A symmetric \((0,2)\)-tensor \( E \) on \( M \) is called Riemann compatible or simply \( R \)-compatible \([37], [38]\) if on \( M \) we have

\[ R(\mathcal{E}X_1, X, X_2, X_3) + R(\mathcal{E}X_2, X, X_3, X_1) + R(\mathcal{E}X_3, X, X_1, X_2) = 0, \]

\( \forall X, X_1, X_2, X_3 \in \chi(M) \), where \( \mathcal{E} \) is the endomorphism on \( \chi(M) \) defined as \( g(\mathcal{E}X_1, X_2) = E(X_1, X_2) \).

Again a vector field \( Y \) is called \( R \)-compatible (also known as Riemann compatible \([39]\)) if

\[ A(X_1)R(Y, X, X_2, X_3) + A(X_2)R(Y, X, X_3, X_1) + A(X_3)R(Y, X, X_1, X_2) = 0, \]

\( \forall X, X_1, X_2, X_3 \in \chi(M) \), where \( A \) is the corresponding 1-form of \( Y \), i.e., \( A(X) = g(X, Y) \).

We note that if \( Y \) is Riemann compatible, then \( A \otimes A \) is a Riemann compatible tensor. In above definitions of Riemann compatibility we can replace \( R \) by \( C, W \) and \( K \) respectively, and get conformal compatibility (also known as Weyl compatibility see, \([21]\) and \([39]\)), concircular compatibility, conharmonic compatibility respectively for both a vector and a \((0,2)\)-tensor.

Recently Mantica and Suh \([40], [41], [42] \) and \([43]\) presented a curvature restriction which is necessary and sufficient for the recurrency of the curvature 2-forms \( \Omega^m_{(R)l} = R^m_{jkl} dx^j \wedge dx^k \) \([3], [36]\) and Ricci 1-forms \( \Lambda_{(S)l} = S_{lm} dx^m \), where \( \wedge \) indicates the exterior product. They showed that \( \Omega^m_{(R)l} \) are recurrent (i.e., \( \mathcal{D} \Omega^m_{(R)l} = A \wedge \Omega^m_{(R)l} \), \( \mathcal{D} \) is the exterior derivative and \( A \) is the associated 1-form) if and only if

\[
\begin{align*}
\nabla_{X_1} R(X_2, X_3, X, Y) + \nabla_{X_2} R(X_3, X_1, X, Y) + \nabla_{X_3} R(X_1, X_2, X, Y) &= \\
A(X_1)R(X_2, X_3, X, Y) + A(X_2)R(X_3, X_1, X, Y) + A(X_3)R(X_1, X_2, X, Y)
\end{align*}
\]
and \( \Lambda_{(S)} \) are recurrent (i.e., \( \mathcal{D}\Lambda_{(S)} = A \wedge \Lambda_{(S)} \)) if and only if
\[
\nabla_{X_1} S(X_2, X) - \nabla_{X_2} S(X_1, X) = A(X_1) S(X_2, X) - A(X_2) S(X_1, X)
\]
for a 1-form \( A \). We can replace \( R \) by some other curvature tensors such as conformal, concircular, conharmonic curvature tensor etc. and \( S \) by any other symmetric (0,2) tensors and get the corresponding results. If on \( M \) the curvature 2-forms are recurrent for \( R \) and \( C \), then we denote these structures as \( K_n(2) \) and \( C-K_n(2) \) respectively. Again if the Ricci 1-forms are recurrent then we denote it by \( S-K_n(1) \).

### 3. Existence of \( HGK_n \)

Let \( M \) be an open connected subset of \( \mathbb{R}^4 \) such that \( x_1, x_2, x_3, x_4 > 0 \) endowed with the Riemannian metric
\[
ds^2 = g_{ij} dx^i dx^j = e^{x_1+x^3} (dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + e^{x_1} (dx^4)^2, \quad i, j = 1, 2, 3, 4.
\]
Then the non-zero components of the Christoffel symbols of second kind (upto symmetry) are given by:
\[
\Gamma^2_{11} = -\Gamma^3_{11} = \Gamma^2_{13} = \frac{e^{x_1+x^3}}{2}, \quad \Gamma^4_{14} = \frac{1}{2}, \quad \Gamma^2_{44} = -\frac{e^{x_1}}{2}.
\]
The non-zero components of curvature tensor and Ricci tensor (upto symmetry) are given by:
\[
R_{1313} = -\frac{1}{2} e^{x_1+x^3}, \quad R_{1414} = -\frac{e^{x_1}}{4}, \quad S_{11} = \frac{1}{4} (1 + 2e^{x_1+x^3}).
\]
The scalar curvature of this metric is given by \( r = 0 \) and thus constant. Again the non-zero components \( R_{hijk,l} \) and \( S_{ij,l} \) of the covariant derivatives of curvature tensor and Ricci tensor (upto symmetry) are given by:
\[
R_{1313,1} = -\frac{e^{x_1+x^3}}{2} = R_{1313,3}, \quad S_{11,1} = S_{13,3} = -\frac{e^{x_1+x^3}}{2}.
\]
The non-zero components of the conformal curvature tensor and its covariant derivative (upto symmetry) are given by:
\[
\begin{align*}
C_{1313} &= \frac{1}{8} (1 - 2e^{x_1+x^3}), \quad C_{1414} = -\frac{1}{8} e^{x_1} (1 - 2e^{x_1+x^3}), \\
C_{1313,1} &= C_{1313,3} = -\frac{e^{x_1+x^3}}{4}, \quad C_{1414,1} = C_{1414,3} = \frac{e^{2x_1+x^3}}{4}.
\end{align*}
\]
The non-zero components of the conharmonic curvature tensor and its covariant derivative (upto symmetry) are given by:

\begin{equation}
\begin{aligned}
K_{1313} &= \frac{1}{8}(1 - 2e^{x^1 + x^3}), & K_{1414} &= -\frac{1}{8}e^{x^1}(1 - 2e^{x^1 + x^3}), \\
K_{1313,1} &= K_{1313,3} = -\frac{e^{x^1} + x^3}{4}, & K_{1414,1} &= K_{1414,3} = \frac{e^{2x^1 + x^3}}{4}.
\end{aligned}
\end{equation}

The non-zero components of the projective curvature tensor and its covariant derivative (upto symmetry) are given by:

\begin{equation}
\begin{aligned}
P_{1211} &= \frac{1}{12} \left(2e^{x^1 + x^3} + 1\right), & P_{1313} &= \frac{1}{12} \left(1 - 4e^{x^1 + x^3}\right), & P_{1331} &= \frac{1}{2}e^{x^1 + x^3}, \\
P_{1414} &= \frac{1}{6}e^{x^1}(e^{x^1 + x^3} - 1), & P_{1441} &= \frac{e^{x^1}}{4},
\end{aligned}
\end{equation}

\begin{equation}
P_{1211,1} = P_{1211,3} = -P_{1313,1} = -P_{1313,3} = P_{1331,1} = P_{1331,3} = \frac{3}{e^1}P_{1414,1} = \frac{3}{e^1}P_{1441,3} = \frac{1}{2}e^{x^1 + x^3}.
\end{equation}

The non-zero components of the concircular curvature tensor and its covariant derivative (upto symmetry) are given by:

\begin{equation}
W_{1313} = -\frac{1}{2}e^{x^1 + x^3}, \quad W_{1414} = -\frac{e^{x^1}}{4} \quad \text{and} \quad W_{1313,1} = W_{1313,3} = -\frac{1}{2}e^{x^1 + x^3}.
\end{equation}

From (3.2), to (3.7) it can be easily shown that the tensors $R \cdot R$, $C \cdot R$, $P \cdot R$, $P \cdot S$, $P \cdot C$, $P \cdot Z$, $P \cdot K$, $W \cdot R$ and $K \cdot R$ are identically zero. Thus from [67] we can conclude that $R \cdot H$, $C \cdot H$, $W \cdot H$ and $K \cdot H$ are all identically zero for $H = C, W$ and $K$.

The non-zero components of $P \cdot P$ and $Q(S, P)$ are (upto symmetry) given below:

\begin{equation}
(P \cdot P)_{131131} = \frac{1}{144} \left(1 + 2e^{x^1 + x^3}\right)^2, \quad (P \cdot P)_{141141} = \frac{1}{144}e^{x^1} \left(1 + 2e^{x^1 + x^3}\right)^2.
\end{equation}

\begin{equation}
Q(S, P)_{141114} = e^{x^1}Q(S, P)_{131113} = \frac{e^{x^1}}{48}(1 + 2e^{x^1 + x^3})^2.
\end{equation}

In terms of local coordinates the defining condition of a $HGK_n$ can be written as

\begin{equation}
R_{hijk,l} = A_lR_{hijk} + B_l[S_{hkg_{ij}} + S_{ij}g_{hk} - S_{hj}g_{ik} - g_{hj}S_{ik}], \quad 1 \leq h, i, j, k, l \leq n.
\end{equation}
Then by virtue of (3.1), (3.2), (3.3) and (3.10) it can be check that the manifold is a \(HGK_4\) with associated 1-form

\[
\begin{align*}
A_i(x) &= \begin{cases} 
-\frac{2e^{x^1} + x^3}{1-2e^{x^1} + x^3} & \text{for } i = 1, 3 \\
0 & \text{otherwise,}
\end{cases} \\
B_i(x) &= \begin{cases} 
\frac{2e^{x^1} + x^3}{1-4e^{x^1} + 2x^3} & \text{for } i = 1, 3 \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]  

(3.11)

If we consider the 1-form \(\eta\) as

\[
\eta_i(x) = \begin{cases} 
-\frac{\sqrt{1+2e^{x^1} + x^3}}{2\sqrt{\beta}} & \text{for } i = 1 \\
0 & \text{otherwise,}
\end{cases}
\]

then from (2.1) and (3.2) it follows that the manifold satisfies \(S = \beta \eta \otimes \eta\) for any positive value of \(\beta\) i.e., the manifold is Ricci simple.

From (3.2) and (3.3) it can be check that the manifold is Ricci recurrent with the associated 1-form

\[
A_i(x) = \begin{cases} 
\frac{2e^{x^1} + x^3}{1+2e^{x^1} + x^3} & \text{for } i = 1, 3 \\
0 & \text{otherwise.}
\end{cases}
\]

Again from (3.3) it follows that the manifold is conharmonically recurrent with the associated 1-form

\[
A_i(x) = \begin{cases} 
-\frac{2e^{x^1} + x^3}{1-2e^{x^1} + x^3} & \text{for } i = 1, 3 \\
0 & \text{otherwise.}
\end{cases}
\]

Now from the values of \(R\) it can be easily check that \(M\) satisfies the condition

\[
A(X_1)R(X_2, X_3, X, Y) + A(X_2)R(X_3, X_1, X, Y) + A(X_3)R(X_1, X_2, X, Y) = 0
\]

for the 1-form

\[
A_i(x) = \begin{cases} 
\frac{2e^{x^1} + x^3}{1+2e^{x^1} + x^3} & \text{for } i = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Thus from (2.7) we can conclude that the the curvature 2-forms \(\Omega^m_{\rho(R)i}\), are recurrent as \(R\) satisfies the second Bianchi identity.

The general format of Riemann compatible tensor \(E\) with local components \(E_{ij}\), of \(M\) can be easily evaluated as

\[
\begin{pmatrix}
E_{11} & E_{12} & E_{13} & E_{14} \\
E_{21} & 0 & 0 & 0 \\
E_{31} & 0 & E_{33} & 2e^{x^1} + x^3 E_{43} \\
E_{41} & 0 & E_{43} & E_{44},
\end{pmatrix}
\]
where $E_{ij}$’s being arbitrary. Thus from above we can also evaluate the general format of Riemann compatible vector $Y$ with associated 1-form $A$ is given by

$$\{A_1, 0, 0, A_4\} \text{ or } \{0, A_2, A_3, 0\}.$$ 

Similarly we can get the conformal compatible, concircular compatible and conharmonic compatible tensors respectively of the following form:

$$\begin{pmatrix}
E_{11} & E_{12} & E_{13} & E_{14} \\
E_{21} & 0 & 0 & 0 \\
E_{31} & 0 & E_{33} & E_{34} \\
E_{41} & 0 & -E_{44} & E_{44}
\end{pmatrix}, \quad \begin{pmatrix}
E_{11} & E_{12} & E_{13} & E_{14} \\
E_{21} & 0 & 0 & 0 \\
E_{31} & 0 & E_{33} & 2e^{x_1+x_3}E_{43} \\
E_{41} & 0 & E_{43} & E_{44}
\end{pmatrix} \text{ and } \begin{pmatrix}
E_{11} & E_{12} & E_{13} & E_{14} \\
E_{21} & 0 & 0 & 0 \\
E_{31} & 0 & E_{33} & E_{34} \\
E_{41} & 0 & -E_{44} & E_{44}
\end{pmatrix}.$$ 

Again from above we can evaluate the following format of compatible vectors for conformal, concircular and conharmonic curvature tensors as follows:

$$\{A_1, 0, 0, A_4\} \text{ or } \{0, A_2, A_3, 0\}$$ 

for all cases.

Hence from the above results we can state the following:

**Proposition 3.1.** Let $(M^4, g)$ be a semi-Riemannian manifold equipped with the metric $(3.1)$. Then $(M^4, g)$ is:

(i) Ricci simple and hence a special quasi Einstein manifold,

(ii) manifold of constant scalar curvature,

(iii) Ricci recurrent and hence weakly Ricci symmetric and the Ricci 1-forms are recurrent (see [53] and [43]),

(iv) conharmonically recurrent and hence conformally recurrent (see Corollary 6.2 of [67]) as well as conharmonically and conformally weakly symmetric, and the curvature 2-forms for $C$ and $K$ are recurrent,

(v) hyper-generalized recurrent and hence generalized Ricci recurrent,

(vi) semisymmetric and hence Ricci semisymmetric and also realize semisymmetry conditions for $C$, $P$, $W$ and $K$,

(vii) weakly Ricci symmetric and weakly conharmonic symmetric such that all the associated 1-forms are different (This result is support to the non uniqueness of associated 1-forms of weak symmetry structures, see Section 3 of [53]),

(viii) recurrent curvature 2-form, and

(ix) fulfills the condition $P \cdot P = -\frac{1}{3}Q(S, P)$, i.e, Ricci generalized pseudosymmetric Weyl projective curvature tensor.
From the above values of various tensors, we can get the following:

**Proposition 3.2.** The 4-dimensional semi-Riemannian manifold $M$ equipped with the metric given in (3.1) does not satisfy the following structures:

(i) Einstein, (ii) class $A$ or $B$, (iii) pseudosymmetric in sense of Chaki for $R$, $C$, $P$, $W$ and $K$, (iv) pseudo Ricci symmetric, (v) quasi generalized recurrent, (vi) weakly generalized recurrent, (vii) $P \cdot P = 0$, (viii) Codazzi type Ricci tensor, (ix) cyclic Ricci parallel, (x) weakly symmetric for $R$ and $Z$.

We know that various geometric structures are generalization of some other structures i.e., one implies some others but not conversely. For example, (i) recurrent structure implies Ricci recurrent, conformally recurrent and also conharmonic recurrent; (ii) Chaki pseudosymmetry for $R$, $S$, $C$, $P$, $W$ and $K$ implies weak symmetry for $R$, $S$, $C$, $P$, $W$ and $K$ respectively but the converse cases are not true. Now from above two corollaries and the dependency and equivalency of various structures we have the following interesting results on the properness of various generalized notions.

**Theorem 3.1.** Let $(M^4, g)$ be a semi-Riemannian manifold equipped with the metric (3.1). Then $(M^4, g)$ is

(i) not locally symmetric but semisymmetric,
(ii) not weakly symmetric but weakly Ricci symmetric as well as conformally weakly symmetric,
(iii) not recurrent but Ricci recurrent as well as conformally recurrent,
(iv) not recurrent but hyper-generalized recurrent,
(v) neither class $A$ nor $B$ but of constant scalar curvature,
(vi) not Einstein but Ricci simple i.e., special quasi-Einstein,
(vii) not recurrent but the curvature 2-forms are recurrent.
(viii) does not satisfy $P \cdot P = 0$ but fulfills the condition $P \cdot P = LQ(S, P)$.

**Remark 3.1.** Then from above theorem we get an affirmative answer to $Q. 1$. Also from this theorem we can say that

(i) semisymmetry is a proper generalization of local symmetry and the result is also true for the case $S$, $C$, $P$, $W$ and $K$,
(ii) Ricci recurrence and conharmonic recurrence are both proper generalizations of recurrence,
(iii) weak Ricci symmetry is a proper generalization of pseudo Ricci symmetry and the result is also true for conharmonic and concircular curvature tensors,
(iv) the classes $A$ or $B$ are properly lie between Ricci symmetry and manifold of constant scalar curvature,
(v) hyper-generalized recurrency is proper generalization of recurrency,
(vi) recurrency of curvature 2-forms is a proper generalization of recurrency.

The above remark can be stated in a diagram as follows:

Figure 3.1. Paths of generalizations of local symmetry

where ‘implication’ signs denote the generalization of the notion of the corresponding structures, and the above example ensures the properness of some of them, which are indicated by the gray filled ones.

Again, if we consider the signature of the metric (3.1) as semi-Riemannian or Lorentzian given by

\[ ds^2 = g_{ij} dx^i dx^j = e^{x_1^1+x^3_3} (dx^1)^2 + 2dx^1 dx^2 + (dx^3)^2 - e^{x_1^1} (dx^4)^2, \]

\[ ds^2 = g_{ij} dx^i dx^j = e^{x_1^1+x^3_3} (dx^1)^2 + 2dx^1 dx^2 - (dx^3)^2 - e^{x_1^1} (dx^4)^2, \]

\[ ds^2 = g_{ij} dx^i dx^j = -e^{x_1^1+x^3_3} (dx^1)^2 + 2dx^1 dx^2 - (dx^3)^2 - e^{x_1^1} (dx^4)^2, \]

\[ ds^2 = g_{ij} dx^i dx^j = e^{x_1^1+x^3_3} (dx^1)^2 + 2dx^1 dx^2 - (dx^3)^2 + e^{x_1^1} (dx^4)^2, \]

\[ ds^2 = g_{ij} dx^i dx^j = -e^{x_1^1+x^3_3} (dx^1)^2 + 2dx^1 dx^2 + (dx^3)^2 + e^{x_1^1} (dx^4)^2, \]

\[ ds^2 = g_{ij} dx^i dx^j = -e^{x_1^1+x^3_3} (dx^1)^2 + 2dx^1 dx^2 + (dx^3)^2 - e^{x_1^1} (dx^4)^2, \]

\[ ds^2 = g_{ij} dx^i dx^j = -e^{x_1^1+x^3_3} (dx^1)^2 + 2dx^1 dx^2 + (dx^3)^2 + e^{x_1^1} (dx^4)^2, \]

where \( i, j = 1, 2, 3, 4 \), then it can be easily shown that the obtained results remain unchanged.

Again if we consider the metric as:

\[ (3.12) \quad ds^2 = g_{ij} dx^i dx^j = x^2 x^3 (dx^1)^2 + 2dx^1 dx^4 + x^1 [(dx^3)^2 + (dx^4)^2], \]
where the metrics (3.12) - (3.22) are also satisfied. Then if the signature of the metrics (3.12) - (3.16) are considered as semi-Riemannian or Lorentzian, then also the results will be same.

By extending the dimension of the metrics (3.1), (3.6) - (3.7), respectively given as
\[
(3.17) \quad ds^2 = g_{ij}dx^i dx^j = e^{x^1 + x^3}(dx^1)^2 + 2dx^1 dx^2 + (dx^3)^2 + e^{x^4}(dx^4)^2 + x^1 \delta_{ab} dx^a dx^b, \\
(3.18) \quad ds^2 = g_{ij}dx^i dx^j = x^2 x^3 (dx^1)^2 + 2dx^1 dx^4 + (dx^3)^2 + (dx^4)^2 + x^1 \delta_{ab} dx^a dx^b, \\
(3.19) \quad ds^2 = g_{ij}dx^i dx^j = x^1 x^2 (dx^1)^2 + 2dx^1 dx^3 + (dx^2)^2 + e^{x^4}(dx^4)^2 + x^1 \delta_{ab} dx^a dx^b, \\
(3.20) \quad ds^2 = g_{ij}dx^i dx^j = x^1 x^3 (dx^1)^2 + 2dx^1 dx^3 + (dx^3)^2 + e^{x^4}(dx^4)^2 + x^1 \delta_{ab} dx^a dx^b, \\
(3.21) \quad ds^2 = g_{ij}dx^i dx^j = x^1 x^3 (dx^1)^2 + 2dx^1 dx^3 + (dx^3)^2 + e^{x^4}(dx^4)^2 + x^1 \delta_{ab} dx^a dx^b, \\
(3.22) \quad ds^2 = g_{ij}dx^i dx^j = x^2 x^4 (dx^1)^2 + 2dx^1 dx^3 + (dx^3)^2 + (dx^4)^2 + x^1 \delta_{ab} dx^a dx^b, \\
\]

where \( \delta_{ab} \) denotes the Kronecker delta, \( 5 \leq a, b \leq n \) and \( i, j = 1, 2, ..., n \), it is easy to check that the metrics (3.17) - (3.22) are also satisfy Q. 1. This leads to the following:

**Theorem 3.2.** Let \( (M^n, g) \), \( n \geq 4 \), be a semi-Riemannian manifold equipped with any one metric given in (3.13) - (3.22). Then \( (M^n, g) \) is (i) Ricci simple, (ii) manifold of constant scalar curvature, (iii) Ricci recurrent, (iv) conharmonic recurrent, (v) hyper-generalized recurrent, (vi) semisymmetric, (vii) weakly Ricci symmetric and weakly conharmonic symmetric such that all the associated 1-forms are different, (viii) recurrent curvature 2-form and (ix) fulfills the condition \( P \cdot P = -\frac{1}{2}Q(S, P) \).

**Conclusion.** From the above results and discussion we conclude that we obtain a new class of semi-Riemannian manifolds which is \( H \Gamma K_n \), Ricci recurrent, conharmonically recurrent and
thus conformally recurrent, Ricci simple (thus special quasi Einstein), weakly Ricci symmetric, conformally weakly symmetric, conharmonically weakly symmetric, semisymmetric (thus Ricci semisymmetry and conformally semisymmetric) and fulfills $P \cdot P = -\frac{1}{3}Q(S, P)$ but does not satisfy any one of the conditions (i)-(x) of Proposition 3.2.

Acknowledgement. We have made all the calculations by Wolfram mathematica. The last named author gratefully acknowledges to CSIR, New Delhi, India (File No. 09/025 (0194)/2010-EMR-I) for the financial support.

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Department of Mathematics,

University of Burdwan, Golapbag,

Burdwan-713104,

West Bengal, India

*E-mail address*: aask2003@yahoo.co.in, aashaikh@math.buruniv.ac.in

*E-mail address*: royindranil1@gmail.com

*E-mail address*: kundu.haradhan@gmail.com