On $p$-adic properties of the Witten-Reshetikhin-Turaev invariant

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Abstract

We prove the Lawrence conjecture about $p$-adic convergence of the series of Ohtsuki invariants of a rational homology sphere to its $SO(3)$ Witten-Reshetikhin-Turaev invariant. Our proof is based on the surgery formula for Ohtsuki series and on the properties of the expansion of the colored Jones polynomial of a knot in powers of $q - 1$ and $q^\alpha - 1$, $\alpha$ being the color of the knot.

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1 Introduction

A quantum invariant of 3-manifolds (WRT invariant) discovered by E. Witten [29] and by N. Reshetikhin, V. Turaev [16], is an extension of the Jones polynomial from links in $S^3$ to links in any compact oriented 3-manifold $M$. For a link $\mathcal{L} \subset M$ with $L$ components, the WRT invariant $Z_\alpha(M, \mathcal{L}; K)$ is a complex number depending on positive integer parameters $K$ and $\alpha = (\alpha_1, \ldots, \alpha_L)$ such that $1 \leq \alpha_j \leq K - 1$. The WRT invariant is quite effective in distinguishing between the topologically inequivalent links and 3-manifolds. However its relation to the ‘classical’ topology remains mysterious, especially if we use the mathematically rigorous definition of [16], which relies on the representation theory of quantum group $su_q(2)$ rather than the non-rigorous path integral definition of [29].

One could try to determine the topological content of the WRT invariant $Z(M; K)$ independently for every value of $K$. R. Kirby and P. Melvin have tried this approach in [4], but they succeeded in relating $Z(M; K)$ to classical invariants only for few small values of $K$. An alternative approach is to study how $Z(M; K)$ depends on $K$ in a hope of identifying the topological invariants which parametrize this dependence. Two different methods of achieving this goal were developed – a number-theoretic and an analytic one. We showed in [23] that if $M$ is a rational homology sphere (QHS) then, surprisingly, both methods produce the same sequence of invariants.

The number-theoretic method was advanced by H. Murakami and T. Ohtsuki. They worked with the modified WRT invariant $Z'(M; K)$ introduced by Kirby and Melvin [4]. Murakami proved [11], [12] (see also [10]) that for a QHS $M$ and for an odd prime $K$, $Z'(M; K)$ belongs to the cyclotomic ring $\mathbb{Z}[q]$, $q = \exp(2\pi i / K)$. Therefore one can write

$$Z'(M; K) = \sum_{n=0}^{K-2} a_n(M; K) (q - 1)^n,$$

where $a_n(M; K) \in \mathbb{Z}$. (1.1)

The coefficients $a_n(M; K)$ still depend on $K$, however Ohtsuki observed [13], [14] that their remainders modulo $K$ are almost $K$-independent. He showed that there exist $K$-independent invariants $\lambda_n(M) \in \mathbb{Q}$, $n \geq 0$ such that

$$a_n(M; K) \equiv \left( \frac{|H_1(M, \mathbb{Z})|}{K} \right)_L \lambda_n(M) \pmod{K} \text{ for } n \leq \frac{K-3}{2}.$$  

(1.2)
Here $|H_1(M,\mathbb{Z})|$ is the order of the first homology of $M$ and $\left(\frac{|H_1(M,\mathbb{Z})|}{K}\right)_L$ is the Legendre symbol. If $\lambda_n(M)$ is not integer, then it should be viewed in eq.(1.2) as a $K$-adic integer.

Murakami showed that the first two invariants $\lambda_0$ and $\lambda_1$ are classical

$$
\lambda_0(M) = 1/|H_1(M,\mathbb{Z})|, \quad \lambda_1(M) = 3\lambda_{CW}(M)/|H_1(M,\mathbb{Z})|,
$$

where $\lambda_{CW}(M)$ is the Casson-Walker invariant of $M$.

R. Lawrence [5] conjectured that $\lambda_n(M)$ have much more control over $Z'(M;K)$, than that displayed by eq.(1.2). She suggested that

$$
\lambda_n(M) \in \mathbb{Z}[1/2, 1/|H_1(M,\mathbb{Z})|] \quad \text{(1.4)}
$$

and that the series $\left(\frac{|H_1(M,\mathbb{Z})|}{K}\right)_L \sum_{n=0}^{\infty} \lambda_n(M) (q - 1)^n$ converges $K$-adically to $Z'(M;K)$ in the $K$-adic completion $\mathbb{Z}_K[q]$ of the cyclotomic ring $\mathbb{Z}[q]$, that is,

$$
Z'(M;K) = \left(\frac{|H_1(M,\mathbb{Z})|}{K}\right)_L \sum_{n=0}^{\infty} \lambda_n(M) (q - 1)^n. \quad \text{(1.5)}
$$

Lawrence verified her conjecture for Seifert manifolds constructed by Dehn’s surgery on $(2, m)$ torus knots.

The goal of this paper is to prove the Lawrence conjecture for a general $\mathbb{Q}$HS. Our proof is based on a surgery formula for the invariants $\lambda_n(M)$. This formula has first appeared [19] in the study of the asymptotic properties of $Z(M;K)$ at large values of $K$. The same formula is relevant for the number-theoretic properties of $Z'(M;K)$ because the sum in the r.h.s. of eq.(1.5) considered as a formal power series in $(q - 1)$, represents the trivial connection contribution to $Z(M;K)$.

At the ‘physical’ level of rigor, the asymptotic properties of $Z(M;K)$ follow from Witten’s formula [29] which presents this quantum invariant as a path integral over $SU(2)$ connections in the (trivial) $SU(2)$ bundle over $M$. At large values of $K$ the integral can be evaluated in the stationary phase approximation, the stationary phase points being the flat $SU(2)$ connections. Therefore Witten conjectured that $Z(M;K)$ splits into a sum of contributions coming from connected components $\mathcal{M}^{(c)}$ of the moduli space $\mathcal{M}$ of flat $SU(2)$ connections.
on $M$

$$Z(M; K) \simeq \sum_c Z^{(c)}(M; K), \quad (1.6)$$

where $\simeq$ denotes asymptotic convergence. Each contribution $Z^{(c)}(M; K)$ is proportional to a 'classical exponential'

$$Z^{(c)}(M; K) = e^{(iK/2\pi)S_{\text{CS}}^{(c)}X^{(c)}(M; K)}. \quad (1.7)$$

Here $S_{\text{CS}}^{(c)}$ is the Chern-Simons invariant of connections in the component $\mathcal{M}^{(c)} \subset \mathcal{M}$ while $X^{(c)}(M; K)$ is an asymptotic series in powers of $K^{-1}$ whose coefficients are invariants of $M$. These coefficients are independent of $K$ and therefore they can be expected to be related to classical topology. If $M$ is a QHS, then the trivial $SU(2)$ connection is an isolated point in the moduli space $\mathcal{M}$. Therefore according to path integral predictions, it should yield a distinct contribution to $Z(M; K)$ of the form

$$Z^{(\text{tr})}(M; K) = |H_1(M, \mathbb{Z})|^{-1/2} \sum_{n=0}^{\infty} D_n(M) K^{-n}, \quad (1.8)$$

where $D_n(M)$ are complex-valued invariants of $M$.

Since the asymptotic properties (1.6), (1.7) of the WRT invariant have not been proven yet, we had to choose a different way to work with $Z^{(\text{tr})}(M; K)$. Following the approach of Reshetikhin and Turaev [16] to the WRT invariant, we defined the invariant $Z^{(\text{tr})}(M; K)$ of a QHS $M$ constructed by a surgery on a link $\mathcal{L} \subset S^3$ as a formal power series (1.8) whose coefficients $D_n(M)$ are expressed in terms of the coefficients of the colored Jones polynomial of $\mathcal{L}$ through special surgery formulas [19]-[22]. We call $Z^{(\text{tr})}(M; K)$ defined by these formulas the TCC invariant, TCC being the abbreviation of the trivial connection contribution. We proved that $Z^{(\text{tr})}(M; K)$ is well-defined by our surgery formulas, that is, it does not depend on the choice of a surgery link used to construct $M$. Then we gave a path integral explanation of why our invariant should indeed represent the trivial connection contribution to the path integral of [29]. We also proved [17], [18] that the WRT invariants of Seifert homology spheres have the asymptotic properties (1.6), (1.7) and that the surgery
formula for the TCC invariant of these manifolds yields the trivial connection contribution to their WRT invariants in the sense of eqs. (1.6), (1.7).

Our surgery formulas for \( Z^{(tr)}(M; K) \) showed that, as we hoped, the first coefficients \( D_n(M) \) in the sum of eq.(1.8) are classical topological invariants of \( M \)

\[
D_0(M) = 1/|H_1(M, \mathbb{Z})|, \quad D_1(M) = 6\pi i \lambda_{CW}(M)/|H_1(M, \mathbb{Z})|.
\] (1.9)

There is an apparent similarity between eqs.(1.9) and (1.3). In [23] we showed that for \( n \geq 2 \) the invariants \( D_n(M) \) and \( \lambda_n(M) \) are also essentially equivalent. More precisely, we proved the equation

\[
Z^{(tr)}(M; K) = \sum_{n=0}^{\infty} \lambda_n(M) (e^{2\pi i / K} - 1)^n,
\] (1.10)

which should be understood as a relation between two formal power series in \( K^{-1} \). In view of relation (1.10), the Lawrence conjecture (1.5) has a deeper meaning: it means that although the TCC invariant \( Z^{(tr)}(M; K) \) is only a part of the total WRT invariant \( Z(M; K) \), still it determines \( Z(M; K) \) for all prime \( K \).

Our proof of eq.(1.10) as well as an alternative proof of the Murakami and Ohtsuki results (1.1) and (1.3) which we provided in [23], was based on the properties of the expansion of the colored Jones polynomial in powers of \( (q - 1) \) and colors (this expansion was first considered in [9]). We used a previously established bound [21] on the powers of colors versus the powers of \( (q - 1) \) which exists for algebraically split links, i.e. links whose linking numbers are zero (recently T. Le [7] has used the same idea in order to prove the analogs of eqs.(1.1) and (1.3) for \( SU(N) \) WRT invariants). However, the bound itself did not allow us to relate \( \lambda_n(M) \) to \( Z'(M; K) \) beyond eq.(1.3).

In [25] we proved that the expansion of the colored Jones polynomial of a knot in \( S^3 \) in powers of \( (q - 1) \) and \( \alpha \) (\( \alpha \) being the color of the knot) can be rewritten as an expansion in powers of \( (q - 1) \) and \( (q^\alpha - 1) \) (\( \alpha \) being the color of the knot) with integer coefficients. The integrality of those coefficients allowed us [24] to give a simple proof of Lawrence’s conjecture for a \( \mathbb{Q} \)HS, constructed by a surgery on a knot in \( S^3 \). In [27] we generalized the
results of [25] to a knot in a QHS In this paper we will use the method of [24] in order to prove the Lawrence conjecture in the general case.

2 Notations, background and statement of results

First of all, let us introduce our multi-index, number-theoretic and topological notations.

We use multi-index notations for the colors of links. For a link \( \mathcal{L} \) with \( L \) components we denote \( \underline{x} = (x_1, \ldots, x_L) \) and

\[
\underline{x} + \underline{y} = (x_1 + y_1, \ldots, x_L + y_L), \quad \underline{y} \underline{x} = (yx_1, \ldots, yx_L), \quad \underline{x} \underline{y} = (x_1y_1, \ldots, x_Ly_L),
\]

\[
x^y = (x^y_1, \ldots, x^y_L), \quad x^y = (x_1^y, \ldots, x_L^y), \quad x^y = \prod_{j=1}^L x_j^y, \tag{2.1}
\]

\[
\{f(x)\} = \prod_{j=1}^L f(x_j), \quad |\underline{x}| = \sum_{j=1}^L x_j.
\]

When we consider two links \( \mathcal{L} \) and \( \mathcal{L}' \) simultaneously, we also use a multi-index notation

\[
\underline{y} = (y_1, \ldots, y_{L'}) \tag{2.2}
\]

for the colors of the \( L' \)-component link \( \mathcal{L}' \). We denote concatenation of multi-indices as

\[
\underline{x}, \underline{x} = (x_1, \ldots, x_L, x), \quad \underline{x}, \underline{y} = (x_1, \ldots, x_L, y_1, \ldots, y_{L'}). \tag{2.3}
\]

Also \( \underline{x} = \underline{x} \) means \( x_j = x \) for all \( 1 \leq j \leq L \).

We will use three variables \( K, q \) and \( h \) which are related by

\[
q = e^{2\pi i/K}, \quad h = q - 1. \tag{2.4}
\]

A rational power of \( q \) is defined as

\[
q^r = e^{2\pi ir/K}, \quad r \in \mathbb{Q}. \tag{2.5}
\]

In many cases we will have to specialize \( K \) to a positive integer. In order to indicate this explicitly we use ‘checked’ symbols

\[
\check{K} \in \mathbb{Z}_+, \quad \check{q} = e^{2\pi i/\check{K}}, \quad \check{h} = \check{q} - 1. \tag{2.6}
\]
Also while \(\alpha, \beta\) are variables, \(\check{\alpha}\) and \(\check{\beta}\) stand for positive integers. These elaborate notations will help us to avoid confusion between the formal parameters and their \(K\)-adic and cyclotomic versions.

For \(\check{K}\) being prime, \(\mathbb{Z}_{\check{K}}\) denotes the ring of \(\check{K}\)-adic integers, \(\mathbb{Z}[\check{q}]\) is the cyclotomic ring and \(\mathbb{Z}_{\check{K}}[\check{q}]\) is its \(K\)-adic completion. Note that since we set \(\check{q} = e^{2\pi i / \check{K}}\), then \(\mathbb{Z}[\check{q}]\) for us is a particular subring of \(\mathbb{C}\).

Denote by \(\mathbb{Z}_{(\check{K})} \subset \mathbb{Q}\) the ring of rational numbers whose denominators are not divisible by \(\check{K}\). There is a natural embedding

\[
\check{\nu} : \mathbb{Z}_{(\check{K})} \to \mathbb{Z}_{\check{K}}, \quad \lambda \mapsto \lambda^{\check{\nu}},
\]

which maps fractions into power series in \(\check{K}\). Whenever we use a notation \(\lambda^{\check{\nu}}\), we assume that \(\lambda \in \mathbb{Z}_{(\check{K})}\). The homomorphism (2.7) can be extended from \(\mathbb{Z}_{(\check{K})}\) to formal power series with coefficients in \(\mathbb{Z}_{(\check{K})}\)

\[
\check{\nu} : \mathbb{Z}_{(\check{K})}[[h]] \to \mathbb{Z}_{\check{K}}[\check{q}]
\]

by setting \(h \mapsto \check{h}\). Indeed, since

\[
(\check{q}^{\check{K}} - 1) / (\check{q} - 1) = 0,
\]

then

\[
\check{h}^{\check{K} - 1} = \check{K}x, \quad \text{for some } x \in \mathbb{Z}[\check{q}],
\]

As a result, any series of the form \(\sum \limits_{n=0}^{\infty} a_n \check{h}^n, a_n \in \mathbb{Z}_{\check{K}}\) has a \(\check{K}\)-adic limit in \(\mathbb{Z}_{\check{K}}[\check{q}]\)

\[
\sum \limits_{n=0}^{\infty} a_n \check{h}^n = A \in \mathbb{Z}_{\check{K}}[\check{q}],
\]

which means by definition that for any \(N_0 > 0\) there exists \(N'_0\) such that for any \(N > N'_0\)

\[
A = \sum \limits_{n=0}^{N} a_n \check{h}^n + \check{K}^{N_0 + 1}x, \quad x \in \mathbb{Z}_{\check{K}}[\check{q}],
\]
Therefore the image of a formal power series \( S(h) = \sum_{n=0}^{\infty} a_n h^n \), \( a_n \in \mathbb{Z}_K(\hat{\mathbb{K}}) \), under the homomorphism (2.8) can be defined as the \( K \)-adic limit of the corresponding series \( \sum_{n=0}^{\infty} a_n^\vee \hat{h}^n \):

\[ \forall : S(h) \mapsto S^\vee = \sum_{n=0}^{\infty} a_n^\vee \hat{h}^n \in \mathbb{Z}_K[\hat{\mathbb{q}}]. \tag{2.13} \]

We will need a few more number-theoretic notations. \( (\frac{x}{K})_L \) denotes a Legendre symbol of \( x \). If \( x \) is an integer not divisible by \( K \), then \( (\frac{x}{K})_L = 1 \) if there exists \( y \in \mathbb{Z} \) such that \( x \equiv y^2 \pmod{K} \), and \( (\frac{x}{K})_L = -1 \) otherwise. For a prime \( K \) and an integer \( n \) not divisible by a prime \( K \), let \( n^* \) denote any integer such that

\[ nn^* \equiv 1 \pmod{K}. \tag{2.14} \]

Also denote for an odd \( K \)

\[ \kappa = \begin{cases} 
1 & \text{if } K \equiv 1 \pmod{4} \\
-1 & \text{if } K \equiv -1 \pmod{4}.
\end{cases} \tag{2.15} \]

Finally, for \( x \in \mathbb{Z}_K \), we denote by \( [x]_N \in \mathbb{Z} \) the remainder of \( x \) modulo \( K^{N+1} \)

\[ [x]_N \equiv x \pmod{K^{N+1}}. \tag{2.16} \]

Now let us fix some standard topological notations. Let \( \mathcal{K} \) be a knot in a 3-manifold \( M \). A meridian of \( \mathcal{K} \) is a simple cycle on the boundary of its tubular neighborhood, which is contractible through that neighborhood. A parallel is a cycle on the same boundary which has a unit intersection number with the meridian. A knot is called framed if we made a choice of its parallel. A link is framed if all of its components are framed. We denote framed knots and links by putting a hat on top of their symbols: \( \hat{\mathcal{K}}, \hat{\mathcal{L}} \).

Suppose that a framed knot \( \hat{\mathcal{K}} \) is of finite order as an element of \( H_1(M, \mathbb{Z}) \) (we will denote the order of \( \hat{\mathcal{K}} \) as \( o \)). Then its self-linking number \( p \) is the linking number between the knot and its parallel, \( p \in \mathbb{Z}/o \). Dehn’s surgery on a framed knot \( \hat{\mathcal{K}} \subset M \) is a transformation of cutting out a tubular neighborhood of \( \hat{\mathcal{K}} \) and gluing it back in such a way that the meridian of the tubular neighborhood matches the parallel left on the boundary of the knot complement. A result of this procedure is a new 3-manifold \( M' \).
In order to shorten our formulas, we will denote the order of the first homology of a QHS $M$ as $h_1(M)$:

$$h_1(M) = |H_1(M, \mathbb{Z})|. \quad (2.17)$$

Let us recall the quantum invariants appearing in 3-dimensional topology. The colored Jones polynomial

$$J_{\hat{\boldsymbol{\alpha}}}(\hat{\mathcal{L}}; q) \in \mathbb{Z}[q^{1/2}, q^{-1/2}], \quad (2.18)$$

$$J_{\hat{\boldsymbol{\alpha}}}(\hat{\mathcal{L}}; q) \in \mathbb{Z}[q, q^{-1}] \quad \text{if } \hat{\boldsymbol{\alpha}} \text{ are odd} \quad (2.19)$$

is an invariant of a framed $L$-component link $\hat{\mathcal{L}} \subset S^3$. The positive integers $\hat{\boldsymbol{\alpha}}$ are called ‘colors’, they are assigned to components of $\hat{\mathcal{L}}$. The WRT invariant is an invariant of a framed link $\hat{\mathcal{L}}$ in a 3-manifold $M$. It depends on a positive integer $\hat{K}$, and we denote it as $Z_{\hat{\boldsymbol{\alpha}}}(M, \hat{\mathcal{L}}; \hat{K})$. N. Reshetikhin and V. Turaev defined the WRT invariant by a surgery formula and verified that $Z_{\hat{\boldsymbol{\alpha}}}(M, \hat{\mathcal{L}}; \hat{K})$ does not depend on the choice of a surgery link.

**Definition 2.1 (N. Reshetikhin, V. Turaev [16])** If $M = S^3$, then

$$Z_{\hat{\boldsymbol{\alpha}}}(S^3, \hat{\mathcal{L}}; \hat{K}) = J_{\hat{\boldsymbol{\alpha}}}(\hat{\mathcal{L}}; \hat{q}), \quad (2.20)$$

If $M$ is constructed by Dehn's surgery on a framed $L'$-component link $\hat{\mathcal{L}}' \subset S^3$, then

$$Z_{\hat{\boldsymbol{\alpha}}}(M, \hat{\mathcal{L}}; \hat{K}) = i^{-L'}(2\hat{K})^{-L'/2}e^{-(3\pi i/4)\text{sign}(\hat{\mathcal{L}}')}\hat{q}^{(3/4)\text{sign}(\hat{\mathcal{L}}')} \times \sum_{0 < \hat{\beta} < \hat{K}} \{\hat{q}^{\hat{\beta}/2} - \hat{q}^{-\hat{\beta}/2}\} J_{\hat{\alpha}\hat{\beta}}(\hat{\mathcal{L}} \cup \hat{\mathcal{L}}'; \hat{q}), \quad (2.21)$$

where $\text{sign}(\hat{\mathcal{L}}')$ is the signature of the linking matrix of $\hat{\mathcal{L}}'$.

For an empty link $\mathcal{L}$ we denote $Z_{\hat{\boldsymbol{\alpha}}}(M, \hat{\mathcal{L}}; \hat{K})$ simply as $Z(M; \hat{K})$. Note that we use the normalization in which

$$Z(S^3; \hat{K}) = 1. \quad (2.22)$$

The WRT invariant (and all other quantum invariants of links that we will consider in this paper) has a simple dependence on the choice of framing of $\hat{\mathcal{L}}$. As a result, if a component
̂K of a link ̂L ∪ ̂K has a finite order as an element of $H_1(M, \mathbb{Z})$ so that it has a well-defined self-linking number $p$, then we can introduce an invariant which does not depend on the framing of ̂K:

$$Z_{\Delta; \hat{\beta}}(M, \hat{L} \cup \hat{K}; \hat{K}) = q^{-(1/4)p(\beta^2 - 1)} Z_{\Delta; \hat{\beta}}(M, \hat{L} \cup \hat{K}; \hat{K}).$$  (2.23)

If ̂K is homologically trivial, then K can be interpreted as a knot with zero self-linking number, otherwise K is just a symbolic notation for the framing-independent invariant $Z_{\Delta; \hat{\beta}}(M, \hat{L} \cup K; \hat{K})$.

For an odd number ̂K, R. Kirby and P. Melvin introduced the $SO(3)$ WRT invariant $Z'(M; ̂K)$ which has a simple relation to $Z(M; ̂K)$

$$Z(M; K) = Z(M; 3) Z'(M; K) \quad \text{if } K \equiv -1 \pmod{4},$$

$$Z(M; K) = \overline{Z(M; 3)} Z'(M; K) \quad \text{if } K \equiv 1 \pmod{4}. \quad (2.24)$$

H. Murakami [11],[12] and T. Ohtsuki [13],[14] proved that $Z'(M; ̂K)$ satisfies the following properties.

**Theorem 2.2** If ̂K is an odd prime and M is a $\mathbb{Q}$HS such that the order of the first homology $h_1(M)$ is not divisible by ̂K, then

1. [Murakami]

$$Z'(M; ̂K) \in \mathbb{Z}[q], \quad (2.25)$$

so that

$$Z'(M; ̂K) = \sum_{n=0}^{K-2} a_n(M; ̂K) \hat{h}^n, \quad \text{where } a_n(M; ̂K) \in \mathbb{Z}; \quad (2.26)$$

2. [Ohtsuki] There exists a sequence of invariants $\lambda_n(M) \in \mathbb{Q}$, $n \geq 0$ such that if $n \leq \frac{K-3}{2}$ then $\lambda_n(M) \in \mathbb{Z}(̂K)$ and

$$a_n(M; ̂K) \equiv \left( \frac{h_1(M)}{̂K} \right)_L \lambda_n^\vee(M) \pmod{̂K}, \quad (2.27)$$

where $\left( \frac{·}{̂K} \right)_L$ is the Legendre symbol.
(3) [Murakami]
\[\lambda_0(M) = 1/h_1(M), \quad \lambda_1(M) = 3\lambda_{CW}(M)/h_1(M),\]  
\hspace*{3.0in} (2.28)

where \(\lambda_{CW}(M)\) is the Casson-Walker invariant of \(M\).

The goal of this paper is to prove the following conjecture made by R. Lawrence.

**Conjecture 2.3 (R. Lawrence [5])** For a QHS \(M\),

\[\lambda_n(M) \in \begin{cases} 
\mathbb{Z} & \text{if } h_1(M) = 1 \\
\mathbb{Z}[1/2, 1/h_1(M)] & \text{if } h_1(M) > 1
\end{cases}\]  
\hspace*{3.0in} (2.29)

and if \(\tilde{K}\) is an odd prime which does not divide \(h_1(M)\), then

\[Z'(M; \tilde{K}) = \left(\frac{h_1(M)}{\tilde{K}}\right) \sum_{n=0}^{\infty} \lambda_n^*(M) \tilde{h}^n \quad \text{in } \mathbb{Z}[\tilde{q}].\]  
\hspace*{3.0in} (2.30)

The proof of the Lawrence conjecture will be based on the properties of the invariant of QHS which we call the trivial connection contribution to the WRT invariant, or simply the TCC invariant. We introduced it and studied its properties in [19]-[22]. The TCC invariant of a QHS \(M\) is a formal power series in powers of \(K^{-1}\)

\[Z^{(tr)}_\alpha(M, \hat{L}; K) = h_1^{-1/2}(M) \sum_{m,n \geq 0} D_{m,n}(M, \hat{L}) \alpha^{2m+1} K^{-n},\]  
\hspace*{3.0in} (2.31)

whose coefficients \(D_{m,n}(M, \hat{L})\) are invariants of \(M\) and \(\hat{L}\). The invariant (2.31) can be defined by various equivalent surgery formulas. Here we will use the definition which is provided by the following

**Theorem 2.4 ([22])** There exists a unique invariant \(Z^{(tr)}_\alpha(M, \hat{L}; K)\) of a framed link \(\hat{L}\) in a QHS \(M\) (called the TCC invariant) which has the form (2.31) and satisfies the following three properties:

(1) If \(M = S^3\), then eq.(2.31) coincides with the Melvin-Morton expansion [9] of the colored Jones polynomial in powers of \(K^{-1}\) at fixed values of colors \(\alpha\), that is

\[Z^{(tr)}_\alpha(S^3, \hat{L}; \tilde{K}) = J_\alpha(\hat{L}; \tilde{q}).\]  
\hspace*{3.0in} (2.32)
(2) Let $\hat{\mathcal{L}}$ be a framed $L$-component link and $\hat{K}$ be a framed knot with self-linking number $p$ in a QHS $M$. Then the expression

$$Z_{\alpha, \beta}^{(\text{tr})}(M, \hat{\mathcal{L}} \cup \hat{K}; K) = q^{-p(\beta^2 - 1)/4} Z_{\alpha, \beta}^{(\text{tr})}(M, \hat{\mathcal{L}} \cup \hat{K}; K),$$

(2.33)
does not depend on the choice of the framing of $\hat{K}$ (that is, $Z_{\alpha, \beta}^{(\text{tr})}(M, \hat{\mathcal{L}} \cup \hat{K}; K)$ is well-defined for unframed knots). Moreover, if $m > n/2$, then $D_{m, m, n}(M, \hat{\mathcal{L}} \cup \hat{K}) = 0$ or, in other words,

$$Z_{\alpha, \beta}^{(\text{tr})}(M, \hat{\mathcal{L}} \cup \hat{K}; K) = h_1^{-1/2}(M) \sum_{m, m, n \geq 0 \atop m \leq n/2} \frac{D_{m, m, n}(M, \hat{\mathcal{L}} \cup \hat{K})}{q^{2m + 1} \beta^{2m + 1} K^{-n}}. (2.34)$$

(3) If a QHS $M'$ is constructed by Dehn's surgery on a knot $\hat{K}$ with self-linking number $p$ in a QHS $M$, then

$$Z_{\alpha}^{(\text{tr})}(M', \hat{\mathcal{L}}; K) = -i (8K)^{-1/2} e^{-(3\pi i/4) \text{sign}(p)} q^{(3/4) \text{sign}(p)}$$

$$\times \int_{[\beta = 0]} Z_{\alpha, \beta}^{(\text{tr})}(M, \hat{\mathcal{L}} \cup \hat{K}; K) (q^{\beta/2} - q^{-\beta/2}) d\beta,$$

(2.35)

(here $\int_{[\beta = 0]}$ denotes a stationary phase contribution of the point $\beta = 0$), or more precisely,

$$Z_{\alpha}^{(\text{tr})}(M', \hat{\mathcal{L}}; K) = -i (8K)^{-1/2} e^{-(3\pi i/4) \text{sign}(p)} q^{(3/4) \text{sign}(p)} h_1^{-1/2}(M)$$

$$\times \sum_{m, m, n \geq 0 \atop m \leq n/2} \left( D_{m, m, n}(M, \hat{\mathcal{L}} \cup \hat{K}) a^{2m + 1} K^{-n} \times \int_{-\infty}^{+\infty} e^{(i\pi/2K) p(\beta^2 - 1)} \beta^{2m + 1} (e^{i\pi/K} \beta - e^{-i\pi/K} \beta) d\beta \right).$$

(2.36)

Equation (2.36) is a well-defined relation between formal power series. Indeed, since

$$K^{-1/2} \int_{-\infty}^{+\infty} e^{(i\pi/2K) p(\beta^2 - 1)} \beta^{2m + 1} (e^{i\pi/K} \beta - e^{-i\pi/K} \beta) d\beta = O(K^m) \quad \text{as} \quad K \to \infty, (2.37)$$

then only the coefficients $D_{m, m, n}(M, \hat{\mathcal{L}} \cup \hat{K})$ with $n - m \leq n'$ participate in the expression for $D_{m, n'}(M', \hat{\mathcal{L}})$. In view of the bound $m \leq n/2$ in the sum of eq.(2.34), this means that the number of such coefficients $D_{m, m, n}(M, \hat{\mathcal{L}} \cup \hat{K})$ is finite for any fixed $n'$. 
We proved Theorem 2.4 in [22] by showing that eq.(2.34) (which indicates that the removal of the self-linking factor $q^{p(β^2 - 1)/4}$ reduces the power of $β$ versus the power of $K^{-1}$ in the expansion) is consistent with the surgery formula (2.35) and that the invariant $Z^{(tr)}_α(M, \hat{L}; K)$ does not depend on the choice of the sequence of knot surgeries which leads from $S^3$ to $M$.

We also proved in [27] (Theorem 1.15 and Remark 1.17) that if we switch in eq.(2.31) from powers of $K^{-1}$ to powers of $h = e^{2πi/K} - 1$ by writing

$$Z^{(tr)}_α(M, \hat{L}; K) = q^{1/4} \sum_{j=1}^L l_{jj}(α_j^2 - 1) Z^{(tr)}_α(M, \hat{L}; K),$$

(2.41)

then for fixed odd $α$

$$P_{α,n}(M, \hat{L}) = \sum_{m≥0, |m|≤n} d_{α,n}(M, \hat{L}) α_{2m+1},$$

(2.39)

(2.38)

Remark 2.5 In fact, we proved relation (2.40) in [27] for a 0-framed link $L$. However, since $l_{jj}$ being self-linking numbers of the link components $\hat{L}_j$, and since for odd $α$

$$q^{1/4} \sum_{j=1}^L l_{jj}(α_j^2 - 1) ∈ Z[1/h_1(M)] [[h]],$$

(2.42)

then relations (2.40) are also true for framed links $\hat{L} \subset M$.

Remark 2.6 Similarly to the TCC invariant $Z^{(tr)}(M; K)$ which we mentioned in Introduction, the definition of $Z^{(tr)}_α(M, \hat{L}; K)$ was motivated by an asymptotic approach to the study of the WRT invariant. The calculation of Witten’s path integral for $Z^{(tr)}_α(M, \hat{L}; \bar{K})$ in the stationary phase approximation at large values of $\bar{K}$ suggests that $Z^{(tr)}_α(M, \hat{L}; \bar{K})$ splits into a sum of contributions of connected components $M_c$ of the moduli space $M$ of flat $SU(2)$ connections on $M$

$$Z^{(tr)}_α(M, \hat{L}; \bar{K}) = \sum_c Z^{(c)}(M, \hat{L}; \bar{K}), \qquad Z^{(c)}_α(M, \hat{L}; \bar{K}) = e^{(i\bar{K}/2π)S^{(c)}_{α}(M, \hat{L}; \bar{K})}.$$

(2.43)

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Here $S^{(c)}_{\text{CS}}$ is the Chern-Simons invariant of connections of $\mathcal{M}_c$ and $X^{(c)}_\alpha(M, \hat{\mathcal{L}}; \hat{K})$ are asymptotic power series in $\hat{K}^{-1}$ (possibly with a fractional power of $K$ as a prefactor) whose coefficients depend on $M, \hat{\mathcal{L}}, c$ and $\hat{\alpha}$. If $M$ is a QHS, then the trivial connection is a separate point in $\mathcal{M}$ and we conjectured in [19] that $Z^{(\text{tr})}_\alpha(M, \hat{\mathcal{L}}; \hat{K})$ represents its contribution to the whole WRT invariant $Z\hat{\alpha}(M, \hat{\mathcal{L}}; \hat{K})$.

We proved in [19] and [21] that for an empty link $\hat{\mathcal{L}}$, 

$$P_0(M) = 1/h_1(M), \quad P_1(M) = 3\lambda_{\text{CW}}(M)/h_1(M).$$ (2.44)

Then we showed in [23] that an apparent similarity between eqs. (2.28) and (2.44) can be extended.

**Theorem 2.7** For an empty link $\hat{\mathcal{L}}$, the coefficients $P_n(M)$ of eq.(2.38) coincide with Ohtsuki’s invariants $\lambda_n(M)$, or in other words,

$$\sum_{n=0}^{\infty} \lambda_n(M) h^n = h_1^{1/2}(M) Z^{(\text{tr})}(M; K).$$ (2.45)

In this paper we will prove the Lawrence Conjecture 2.3 extended to the $SO(3)$ WRT invariant $Z'$ of links in QHS The definition of this invariant is given by the following

**Theorem 2.8** (cf. [4]) Let $\hat{\mathcal{L}}$ be a framed $L$-component link in a 3-manifold $M$. Suppose that $M$ is constructed by Dehn’s surgery on a framed $L'$-component link $\hat{\mathcal{L}}' \subset S^3$. Then for an odd integer $\hat{K}$ and for a set of odd colors $\hat{\alpha}$, the following expression

$$Z'_{\hat{\alpha}}(M, \hat{\mathcal{L}}; \hat{K}) = i^{-L'} \hat{K}^{-L'/2} e^{(i\pi/4)(\kappa+1)\text{sign}(\hat{\mathcal{L}}')} q^{3\gamma^* \text{sign}(\hat{\mathcal{L}}')}$$

$$\times \sum_{0 \leq \hat{\beta} \leq \hat{K} \atop \hat{\beta} \equiv \text{odd}} \{q^{\hat{\beta}/2} - q^{-\hat{\beta}/2}\} J_{\hat{\alpha},\hat{\beta}}(\hat{\mathcal{L}} \cup \hat{\mathcal{L}}'; \hat{q})$$

(2.46)

does not depend on the choice of the surgery link $\hat{\mathcal{L}}'$ which is used to construct $M$. We call $Z'_{\hat{\alpha}}(M, \hat{\mathcal{L}}; \hat{K})$ the $SO(3)$ invariant of $\hat{\mathcal{L}}$ and $M$. It satisfies the analog of eq.(2.24)

$$Z'_{\hat{\alpha}}(M, \hat{\mathcal{L}}; \hat{K}) = Z(M; 3) Z'_{\hat{\alpha}}(M, \hat{\mathcal{L}}; \hat{K}) \quad \text{if } \hat{K} \equiv -1 \pmod{4}$$

$$Z'_{\hat{\alpha}}(M, \hat{\mathcal{L}}; \hat{K}) = \overline{Z(M; 3) Z'_{\hat{\alpha}}(M, \hat{\mathcal{L}}; \hat{K})} \quad \text{if } \hat{K} \equiv 1 \pmod{4}$$

(2.47)
Proof. The independence of $Z'_\alpha(M, \hat{L}; K)$ on the choice of a surgery link $\hat{L}'$ can be proved in exactly the same way as the invariance of $Z'(M; K)$ in [4]. We will prove eq. (2.47) in Section 4. 

Main Theorem For a QHS $M$ let $\hat{K}$ be an odd prime which does not divide $h_1(M)$. If $\hat{L}$ is a framed link in $M$, then

$$Z'_\alpha(M, \hat{L}; \hat{K}) \in \mathbb{Z}[\hat{q}]$$

(2.48)

and

$$Z'_\alpha(M, \hat{L}; \hat{K}) = \left( \frac{h_1(M)}{\hat{K}} \right)_L \left( h_1^{1/2}(M) Z^{(tr)}_{\hat{\alpha}}(M, \hat{L}; K) \right)^\vee,$$  

(2.49)

or equivalently,

$$Z'_\alpha(M, \hat{L}; \hat{K}) = \left( \frac{h_1(M)}{\hat{K}} \right)_L \sum_{n=0}^{\infty} P_{\hat{n}}^{\vee}(M, \hat{L}) \hat{h}^n,$$  

(2.50)

where $P_{\hat{n}}(M, \hat{L})$ are the coefficients in the expansion (2.38) of $Z^{(tr)}_{\hat{\alpha}}(M, \hat{L}; K)$.

Remark 2.9 Since eqs. (2.30) and (2.50) imply that $\lambda_n(M) = P_n(M)$, then relation (2.29) follows from (2.40). In fact, (2.40) is a generalization of (2.29) for the case of a link $\hat{L} \subset M$ (see also eqs. (1.87), (1.90) and (1.93) in Theorem 1.8 of [27] for a slightly stronger statement about the denominators of $P_{\hat{n}}(M, \hat{L})$).

Sketch of the proof of Main Theorem. Suppose that a QHS $M$ is constructed by a surgery on a framed algebraically split link $\hat{L}' \subset S^3$ (i.e. all linking numbers between the components of $\hat{L}'$ are zero). We prove the theorem by induction in the number of components of $\hat{L}'$. An easy Proposition 4.4 establishes Main Theorem for $M = S^3$. It remains to prove that if Main Theorem is true for a QHS $M$, then it is also true for a QHS $M'$ constructed by a surgery on a homologically trivial framed knot $\hat{K} \subset M$. The key to the proof is Corollary 3.2 which describes the structure of $Z^{(tr)}_{\hat{\alpha}, \beta}(M, \hat{L} \cup \hat{K}; K)$ and Corollary 3.6 which establishes the similarity between the gaussian sum of the surgery formula (2.46) and the gaussian integral of the surgery formula (2.35) as they appear after the substitution (3.6).
Our proof of Main Theorem relies exclusively on the properties of the TCC invariant. We use neither the Murakami-Ohtsuki Theorem 2.2, nor Theorem 2.7. In fact, Main Theorem is stronger than these theorems (except eq. (2.28) which we will prove in Appendix).

3 Preliminary results

All our proofs are based on a particular case of Theorem 1.8 of [27].

Theorem 3.1 Let \( \hat{L} \) be a framed link and let \( \hat{K} \) be a framed knot with self-linking number \( p \) in a QHS \( M \). Let \( o \) be the order of \( K \) as an element of \( H_1(M, \mathbb{Z}) \). For odd colors \( \bar{\alpha} \) of \( \hat{L} \), there exist the polynomials

\[
P_{\bar{\alpha}}(M, \hat{L}, \hat{K}; t) \in \mathbb{Z}[t^{1/o}, t^{-1/o}, 1/2, 1/h_1(M)] \tag{3.1}
\]

such that

\[
P_{\bar{\alpha}}(M, \hat{L}, \hat{K}; t^{-1}) = P_{\bar{\alpha}}(M, \hat{L}, \hat{K}; t) \tag{3.2}
\]

and

\[
Z^{(tr)}_{\bar{\alpha}, \beta}(M, \hat{L} \cup \hat{K}; K) = h_1^{-1/2}(M) q^{(1/4)p(\beta^2 - 1)} \frac{q^{\beta/2o} - q^{-\beta/2o}}{q^{1/2} - q^{-1/2}} \sum_{n=0}^{\infty} \frac{P_{\bar{\alpha}}(M, \hat{L}, \hat{K}; q^\beta)}{\Delta_{A}^{2n+1}(M, \hat{K}; q^\beta)} h^n. \tag{3.3}
\]

Here \( \Delta_A(M, \mathcal{K}; t) \) is the Alexander polynomial of \( \mathcal{K} \) normalized in such a way that

\[
\Delta_A(M, \mathcal{K}; t^{-1}) = \Delta_A(M, \mathcal{K}; t), \tag{3.4}
\]

\[
\Delta_A(M, \mathcal{K}; t)|_{t=1} = h_1(M)/o. \tag{3.5}
\]

Proof. For the case of 0-framed link \( L \) and knot \( K \), this theorem is a special case of Theorem 1.8 and Remark 1.10 of [27]. Remark 2.5 allows us to extend it to framed links and knots.
Corollary 3.2 If $\hat{K}$ is a framed homologically trivial knot with self-linking number $p$ in a $\mathbb{Q}_{HS} M$, then for odd colors $\hat{\nu}$

$$Z_{\hat{\alpha},\beta}^{(tr)}(M, \hat{\mathcal{L}} \cup \hat{K}; \hat{K}) = h_1^{1/2}(M) q^{1/2} h^{-1} q^{(1/4)p(\beta^2-1)}$$

$$\times \sum_{m,n \geq 0} d_{\hat{\alpha},m,n}(M, \hat{\mathcal{L}}, \hat{K}) (q^{\beta/2} - q^{-\beta/2})^{2m+1} h^n$$

$$d_{\hat{\alpha},m,n}(M, \hat{\mathcal{L}}, \hat{K}) \in \mathbb{Z}[1/2, 1/h_1(M)] \quad (3.6)$$

Proof. Since $K$ is homologically trivial, then $o = 1$ and also

$$\Delta_A(M, K; t) \in \mathbb{Z}[t, t^{-1}].$$

Therefore relations (3.2), (3.4) and (3.5) imply that $\Delta_A(M, \hat{K}; t)$ and $P_{\hat{\alpha},n}(M, \hat{\mathcal{L}}, \hat{K}; t)$ are polynomials of $(t^{1/2} - t^{-1/2})^2$

$$\Delta_A(M, \hat{K}; t) = h_1(M) + \sum_{m \geq 1} a_m (t^{1/2} - t^{-1/2})^{2m}, \quad a_m \in \mathbb{Z},$$

$$P_{\hat{\alpha},n}(M, \hat{\mathcal{L}}, \hat{K}; t) = \sum_{m \geq 0} b_m (t^{1/2} - t^{-1/2})^{2m}, \quad b_m \in \mathbb{Z}[1/2, 1/h_1(M)]. \quad (3.9)$$

The expansion of denominators in the sum of eq.(3.3) in powers of $(q^{\beta/2} - q^{-\beta/2})^2$ leads to eq.(3.6). $\square$

To prove Main Theorem, we will also need some simple facts.

Lemma 3.3 (H. Murakami [11]) If $f(q) \in \mathbb{Z}[q, q^{-1}]$ and its expansion in powers of $h = q - 1$ is

$$f(q) = \sum_{n=0}^{\infty} a_n h^n,$$  \quad (3.11)

then for any $N > 0$

$$f(q) = \sum_{n=0}^{N} a_n h^n + h^{N+1} x, \quad \text{where } x \in \mathbb{Z}[q, q^{-1}]. \quad (3.12)$$

Lemma 3.4 If $\hat{K}$ is prime and $m, n \in \mathbb{Z}$, $n$ not divisible by $\hat{K}$, then there is an equality in $\mathbb{Z}_{K}[q]$

$$q^{mn^*} = (q^{m/n})^\vee, \quad (3.13)$$
where \( q^{m/n} \) is understood as a power series in \( h \)

\[
q^{m/n} = (1 + h)^{m/n} = \sum_{k=0}^{\infty} \left( \frac{m/n}{k} \right) h^k 
\]  
(3.14)

\[
\left( \frac{m/n}{k} \right) = \frac{\frac{m}{n} \cdot \left( \frac{m}{n} - 1 \right) \cdots \left( \frac{m}{n} - k + 1 \right)}{k!} = \frac{m(m-n) \cdots (m-n(k-1))}{n^k k!}. 
\]  
(3.15)

**Proof.** It is easy to see that if \( n \) is not divisible by \( \tilde{K} \), then

\[
\left( \frac{m/n}{k} \right) \in \mathbb{Z}_{\tilde{K}} 
\]  
(3.16)

and as a result

\[
\left( \frac{m/n}{k} \right)^\vee = \left( \frac{m(1/n)^\vee}{k} \right) \in \mathbb{Z}_{\tilde{K}}. 
\]  
(3.17)

Therefore according to eq. (3.14), the r.h.s. of eq. (3.13) can be presented in \( \mathbb{Z}_{\tilde{K}}[\tilde{q}] \) as a convergent power series in \( \tilde{h} \)

\[
(q^{m/n})^\vee = \sum_{k=0}^{\infty} \left( \frac{m/n}{k} \right)^\vee \tilde{h}^k = \sum_{k=0}^{\infty} \left( \frac{m(1/n)^\vee}{k} \right) \tilde{h}^k. 
\]  
(3.18)

Since \( \tilde{q}^{mn^*} \) does not depend on the choice of \( n^* \) which satisfies eq. (2.14), then according to definition (2.16), for any positive integer \( N \) we can set \( n^* = [(1/n)^\vee]_N \) and present the l.h.s. of eq. (3.13) in \( \mathbb{Z}_{\tilde{K}}[\tilde{q}] \) also as a convergent power series in \( h \)

\[
q^{mn^*} = \tilde{q}^{m[(1/n)^\vee]_N} = (1 + \tilde{h})^m[(1/n)^\vee]_N = \sum_{k=0}^{\infty} \left( \frac{m[(1/n)^\vee]_N}{k} \right) \tilde{h}^k. 
\]  
(3.19)

Now eq. (3.13) follows from the fact that each term of the series (3.19) converges \( \tilde{K} \)-adicly to the corresponding term of the series (3.18) as \( N \rightarrow \infty \). \( \square \)

Our proof of the Lawrence conjecture is based on a simple relation between gaussian sums and gaussian integrals. We introduce the following notation: for a function \( f(\tilde{\beta}) \),

\[
\sum'_{0 < \tilde{\beta} < \tilde{K}} f(\tilde{\beta}) = \sum_{\tilde{\beta} \text{ odd}} f(\tilde{\beta}) + \frac{1}{2} f(\tilde{K}). 
\]  
(3.20)
**Lemma 3.5** For $K$ being an odd prime and for $p, m \in \mathbb{Z}$, $p$ not divisible by $K$, consider the expression

$$X_{\text{cycl.}} = K^{-1/2} e^{(i\pi/4)(\kappa-1) \text{sign}(p)} \sum_{0 < \beta < K \atop \beta \text{ - odd}} \tilde{q}^{4^* p \beta^2} (\tilde{q}^{m \beta} + \tilde{q}^{-m \beta}) \in \mathbb{C} \quad (3.21)$$

and a function of $q$

$$X_{\text{asympt.}} = (8K)^{-1/2} |p|^{1/2} e^{-(i\pi/4) \text{sign}(p)} \int_{-\infty}^{+\infty} q^{(1/4)p \beta^2} (q^{m \beta} + q^{-m \beta}) d\beta. \quad (3.22)$$

which is well-defined for $0 < |q| < 1$. Then

$$X_{\text{cycl.}} = \left( \frac{|p|}{K} \right)_L \tilde{q}^{-m^2 p^*} \in \mathbb{Z}[\tilde{q}] \subset \mathbb{C}, \quad (3.23)$$

while (3.22) can be extended analyticly to the vicinity of $q = 1$ and expanded in power series of $h = q - 1$

$$X_{\text{asympt.}} = q^{-m^2/p} \in \mathbb{Z}[1/p] [[h]]. \quad (3.24)$$

Finally,

$$X_{\text{cycl.}} = \left( \frac{|p|}{K} \right)_L (X_{\text{asympt.}})^\vee. \quad (3.25)$$

**Proof.** We calculate the sum

$$\sum_{0 < \beta < K \atop \beta \text{ - odd}} \tilde{q}^{4^* p \beta^2} (\tilde{q}^{m \beta} + \tilde{q}^{-m \beta}) = \sum_{-K < \beta \leq K \atop \beta \text{ - odd}} \tilde{q}^{4^* p \beta^2 + m \beta} = \sum_{-K < \beta \leq K \atop \beta \text{ - odd}} \tilde{q}^{4^* p (\beta + 2mp^*)^2 - m^2 p^*}$$

$$= \tilde{q}^{-m^2 p^*} \sum_{-K < \beta \leq K \atop \beta \text{ - odd}} \tilde{q}^{4^* p \beta^2} = \tilde{q}^{-m^2 p^*} \sum_{-(K+1)/2 < \gamma \leq (K-1)/2} \tilde{q}^{4^* p (2\gamma + 1)^2}$$

$$= \tilde{q}^{-m^2 p^*} \sum_{-(K+1)/2 < \gamma \leq (K-1)/2} \tilde{q}^{p (\gamma + 2)^2} = \tilde{q}^{-m^2 p^*} \sum_{\gamma = 1}^{K} \tilde{q}^{p \gamma^2}$$

$$= \left( \frac{|p|}{K} \right)_L e^{(i\pi/4)(1-\kappa) \text{sign}(p)} K^{1/2} \tilde{q}^{-m^2 p^*} \quad (3.26)$$

In the last line we used the formula for the gaussian sum which can be found, for example, in [2], Chapter 6:

$$\sum_{\gamma = 1}^{K} \tilde{q}^{p \gamma^2} = \left( \frac{|p|}{K} \right)_L e^{(i\pi/4)(1-\kappa) \text{sign}(p)} K^{1/2}. \quad (3.27)$$
Equation (3.23) follows from eq.(3.26).

Next we calculate the gaussian integral

\[
\int_{-\infty}^{+\infty} q^{(1/4)p\beta^2} (q^{m\beta} + q^{-m\beta})\, d\beta = 2q^{-m^2/p} \int_{-\infty}^{+\infty} q^{(1/4)p(\beta+2m/p)^2}\, d\beta = 2q^{-m^2/p} \int_{-\infty}^{+\infty} e^{i\pi/2Kp\beta^2}\, d\beta = (8K)^{1/2} e^{i\pi/4} \text{sign}(p) |p|^{-1/2} q^{-m^2/p}. \tag{3.28}
\]

Relation (3.24) follows from this equation, because \(q^{-m^2/p} = (1 + h)^{-m^2/p} \in \mathbb{Z}[1/p][[h]]\).

Equation (3.25) follows from eqs. (3.23), (3.24) and from Lemma 3.4, because the r.h.s. of eq.(3.25) is equal to \(\left(\frac{|p|}{K}\right)_L (q^{-m^2/p})^\vee\).

\[\Box\]

**Corollary 3.6** For \(\tilde{K}\) being an odd prime, \(m, p \in \mathbb{Z}\), \(p\) not divisible by \(\tilde{K}\), consider the expression

\[
Y_{\text{cycl.}}(p, m, \tilde{q}) = \tilde{h}^{-1} \tilde{K}^{-1/2} e^{(i\pi/4)(\kappa-1)\text{sign}(p)} \sum_{0<\beta<\tilde{K} \atop \beta-\text{odd}} q^{4\beta^2} (q^{\beta/2} - q^{-\beta/2})^{2m+2} \in \mathbb{C} \tag{3.29}
\]

and a function

\[
Y_{\text{asympt.}}(p, m, q) = h^{-1} (8K)^{-1/2} |p|^{1/2} e^{-(i\pi/4)\text{sign}(p)}
\times \int_{-\infty}^{+\infty} q^{(1/4)p\beta^2} (q^{\beta/2} - q^{-\beta/2})^{2m+2}\, d\beta. \tag{3.30}
\]

which is well-defined for \(0 < |q| < 1\). Then \(Y_{\text{cycl.}}(p, m, \tilde{q}) \in \mathbb{Z}[\tilde{q}] \subset \mathbb{C}\) and it is \(\tilde{K}\)-adicly small

\[
Y_{\text{cycl.}}(p, m, \tilde{q}) = \tilde{h}^m x, \quad x \in \mathbb{Z}[\tilde{q}]. \tag{3.31}
\]

\(Y_{\text{asympt.}}(p, m, q)\) can be extended analytically to the vicinity of \(q = 1\) and expanded there in powers of \(h = q - 1\), so that \(Y_{\text{asympt.}}(p, m, q) \in \mathbb{Z}[1/p][[h]]\) and it is asymptotically small

\[
Y_{\text{asympt.}}(p, m, q) = h^m x', \quad x' \in \mathbb{Z}[1/p][[h]]. \tag{3.32}
\]

Finally,

\[
Y_{\text{cycl.}}(p, m, \tilde{q}) = \left(\frac{|p|}{K}\right)_L (Y_{\text{asympt.}}(p, m, q))^\vee \tag{3.33}
\]

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Proof. Obviously,

$$\hat{q}^{K/2} - \hat{q}^{-K/2} = 0,$$

so we can replace the sum \( \sum_{0 < \beta < K} \beta \) in the expression (3.29) for \( Y_{\text{cycl}} \) by the extended sum \( \sum_{0 < \beta < K} \beta \) for odd \( \beta \).

Since for \( m \geq 0 \)

$$ (q^{\beta/2} - q^{-\beta/2})^{2m+2} = \sum_{n=0}^{m+1} a_{m,n} (q^n + q^{-n}), \quad a_{m,n} \in \mathbb{Z}, $$

then we conclude from eq. (3.23) and (3.25) that

$$ \hat{h} Y_{\text{cycl}} \in \mathbb{Z}[\hat{q}], \quad h Y_{\text{asympt}} \in \mathbb{Z}[1/p] [[h]], \quad \tilde{h} Y_{\text{cycl}} = \left( \frac{|p|}{K} \right)_L (h Y_{\text{asympt}})^\vee $$

(3.36)

Since

$$ K^{-1/2} \int_{-\infty}^{+\infty} q^{(1/4)p\beta} (q^{\beta/2} - q^{-\beta/2})^{2m+2} d\beta = O(K^{-m-1}) \quad \text{as } K \to \infty, $$

(3.37)

then the expansion of \( h Y_{\text{asympt}} \) in powers of \( h \) starts at \( h^{m+1} \). Therefore \( Y_{\text{asympt}} \) is divisible by \( h^m \) in \( \mathbb{Z}[1/p] [[h]] \), \( Y_{\text{cycl}} \) is divisible by \( \hat{h}^m \) in \( \mathbb{Z}[\hat{q}] \) and the relations (3.31)–(3.33) follow from (3.36).

\( \square \)

4 Proof of Main Theorem

We start with Symmetry Principle formulated by R. Kirby and P. Melvin in [4].

**Theorem 4.1 (R. Kirby, P. Melvin [4])** Let \( \hat{L} \) be a framed \( L \)-component link in \( S^3 \). For a fixed integer \( j, 1 \leq j \leq L \), consider two sets of colors \( \hat{\alpha} \) and \( \hat{\alpha}' \) such that

$$ \hat{\alpha}'_k = \hat{\alpha}_k \quad \text{if } k \neq j, \quad \hat{\alpha}'_j = \hat{K} - \hat{\alpha}_j. $$

(4.1)

Then

$$ J_{\hat{L}; \hat{q}} = i^{l_{1j}} (-1)^{l_{1j} \hat{\alpha}_j + \sum_{1 \leq k < L \atop k \neq j} l_{jk}(\hat{\alpha}_k - 1)} J_{\hat{L}; \hat{q}}, $$

(4.2)

here \( l_{jk} \) are the linking numbers of \( \hat{L} \) while \( \kappa \) is defined by eq. (2.15).
Remark 4.2 This theorem follows easily from the Main Theorem of [26] which is a particular case of Theorem 1.8 of [27] (see Appendix C of [26] for the details).

Proof of eq. (2.47) of Theorem 2.8. Our proof is a slightly shortened version of the proof of eq. (2.24) in [4]. The Symmetry Principle applied to the colors \( \tilde{\beta} \) in the surgery formula (2.21) suggests that the sum over \( \tilde{\beta} \) can be split into a sum over the elements of the symmetry group generated by the transformations (4.1) acting on \( \tilde{\beta} \), and over the orbits of that action. Since according to our assumptions, \( \tilde{K} \) is odd, then each orbit contains a set of colors \( \tilde{\beta} \) which are all odd. Therefore for odd colors \( \tilde{\alpha} \) we can rewrite eq. (2.21) as

\[
Z_{\tilde{\alpha}}(M, \hat{L}, \tilde{K}) = i^{-L'} (2\tilde{K})^{-L'/2} e^{-\left(3\pi i/4\right) \text{sign}(\hat{L}') \tilde{q}^{(3/4) \text{sign}(\hat{L}')}} \\
\times X(\hat{L}'; \kappa) \sum_{0 \leq \hat{\beta} < \tilde{K} \atop \hat{\beta} \text{ - odd}} \{\tilde{q}^{\hat{\beta}/2} - \tilde{q}^{-\hat{\beta}/2}\} J_{\tilde{\alpha},\hat{\beta}}(\hat{L} \cup \hat{L}'; \tilde{q}),
\]

where

\[
X(\hat{L}'; \kappa) = \sum_{J \subseteq \{1, \ldots, L'\}} i^\kappa \left( \sum_{j \in J} l'_{jj} (\kappa + 3) \sum_{j,k \in J, j \leq k} l'_{j,k} \right)
\]

and \( l'_{ij} \) are the linking numbers of \( \hat{L}' \). Note that

\[
X(\hat{L}'; 1) = X(\hat{L}'; -1)
\]

and the sum \( X(\hat{L}'; \kappa) \) depends on \( K \) only through \( \kappa \). Consider the invariant \( Z(M; K) \) at \( K = 3 \). Since \( J_{\tilde{\alpha}}(\hat{L}; \tilde{q}) \bigg|_{\tilde{\alpha} = 1} = 1 \), then according to eq. (4.3),

\[
Z(M; 3) = 2^{-L'/2} e^{-\left(i\pi/4\right) \text{sign}(\hat{L}')} X(\hat{L}'; -1).
\]

A combination of eqs. (4.3), (4.5) and (4.6) shows that eq. (2.47) is satisfied if the invariant \( Z'_{\tilde{\alpha}}(M, \hat{L}, \tilde{K}) \) was given by the expression

\[
Z'_{\tilde{\alpha}}(M, \hat{L}, \tilde{K}) = i^{-L'} \tilde{K}^{-L'/2} e^{-\left(i\pi/4\right) (\kappa + 3) \text{sign}(\hat{L}') \tilde{q}^{(3/4) \text{sign}(\hat{L}')}} \\
\times \sum_{0 \leq \hat{\beta} < \tilde{K} \atop \hat{\beta} \text{ - odd}} \{\tilde{q}^{\hat{\beta}/2} - \tilde{q}^{-\hat{\beta}/2}\} J_{\tilde{\alpha},\hat{\beta}}(\hat{L} \cup \hat{L}'; \tilde{q}).
\]
In fact, since

$$\frac{1 - \kappa K}{4} \equiv 4^* \pmod{\tilde{K}}, \quad (4.8)$$

then it is easy to see that

$$\exp\left(-i\pi/4(\kappa + 3) \text{sign}(\tilde{L}')\right) \tilde{q}^{(3/4) \text{sign}(\tilde{L}')} = \exp\left(i\pi/4(\kappa + 1) \text{sign}(\tilde{L}')\right) \tilde{q}^{34^* \text{sign}(\tilde{L}')} \quad (4.9)$$

and eq.(4.7) is equivalent to eq.(2.46).

Proposition 4.3 Suppose that $\tilde{K}$ is an odd integer. Let $\hat{\mathcal{L}} \subset M$ be a framed knot with self-linking number $p \neq 0$. If $M'$ is constructed by Dehn's surgery on $\hat{\mathcal{L}}$, then for odd colors $\tilde{\alpha}$

$$Z'_{\tilde{\alpha}}(M', \hat{\mathcal{L}}; \tilde{K}) = \tilde{K}^{-1/2} \text{sign}(p) \exp\left(i\pi/4(\kappa - 1) \text{sign}(p)\right) \tilde{q}^{34^* \text{sign}(p)}$$

$$\times \sum_{0 < \beta < \tilde{K}} (q^{\beta/2} - q^{-\beta/2}) Z'_{\tilde{\alpha}, \beta}(M, \hat{\mathcal{L}} \cup \hat{\mathcal{K}}; \tilde{K}). \quad (4.10)$$

Proof. Suppose that $M$ is constructed by a surgery on a framed link $\hat{\mathcal{L}}' \subset S^3$. Then $M'$ can be constructed by a surgery on $\hat{\mathcal{L}}' \cup \hat{\mathcal{K}}$ with $\hat{\mathcal{K}} \subset S^3$ having a self-linking number

$$p_0 = p + \sum_{j,k=1}^{L'} l'_{0j} l'_{0k} (l')^{-1}_{jk}, \quad (4.11)$$

here $l'_{0j}$ are the linking numbers between $\mathcal{K}$ and components of $\hat{\mathcal{L}}'$ while $(l')^{-1}_{jk}$ is the inverse linking matrix of $\hat{\mathcal{L}}'$. Since

$$\text{sign}(\hat{\mathcal{L}}' \cup \hat{\mathcal{K}}) = \text{sign}(p) + \text{sign}(\tilde{L}') \quad (4.12)$$

and for $p \neq 0$

$$i^{-1} \exp\left(i\pi/4(\kappa + 1) \text{sign}(p)\right) = \text{sign}(p) \exp\left(i\pi/4(\kappa - 1) \text{sign}(p)\right), \quad (4.13)$$

then comparing eq.(2.46) for the surgeries on $\hat{\mathcal{L}}'$ and on $\hat{\mathcal{L}}' \cup \hat{\mathcal{K}}$ we arrive at eq.(4.10). \square

The following two propositions will allow us to prove Main Theorem by induction.
Proposition 4.4  The Main Theorem is true for $M = S^3$.

Proof. According to eq. (2.46) applied in the trivial case of an empty surgery link,

$$Z'_\Delta(S^3, \hat{\mathcal{L}}; \hat{K}) = J_{\Delta}(\hat{\mathcal{L}}; \bar{q}).$$  \hfill (4.14)

Therefore relation (2.48) follows from (2.19). Equation (2.32) indicates that $Z'_\Delta(S^3, \hat{\mathcal{L}}; K)$ is the expansion of $J_{\Delta}(\hat{\mathcal{L}}; q)$ in powers of $h$, so eq. (2.49) follows from Lemma 3.3 together with eq. (2.10) and the definition of $\hat{K}$-adic convergence (2.12).

Proposition 4.5  Suppose that a QHS $M'$ is constructed by a surgery on a homologically trivial framed knot $\hat{K}$ in another QHS $M$. If the Main Theorem is true for $M$, then it is true for $M'$.

Proof. Suppose that $h_1(M')$ is not divisible by an odd prime $\hat{K}$. Since

$$h_1(M') = |p|h_1(M),$$  \hfill (4.15)

where $p$ is the self-linking number of $\hat{K}$, then neither $p$ nor $h_1(M)$ is divisible by $\hat{K}$. We assume that eq. (2.50) is true for the link $\hat{\mathcal{L}} \cup \hat{K} \subset M$

$$Z'_{\Delta, \hat{\beta}}(M, \hat{\mathcal{L}} \cup \hat{K}; \hat{K}) = \left(\frac{h_1(M)}{\hat{K}}\right) \sum_{n=0}^{\infty} P_{\Delta, \hat{\beta}; n}(M, \hat{\mathcal{L}} \cup \hat{K}) \hat{h}^n.$$

On the other hand, if $\hat{\alpha}$ and $\hat{\beta}$ are odd, then in view of eqs. (2.38), (3.6) and Lemma 3.3, for any $N > 0$

$$\sum_{n=0}^{N} P_{\Delta, \hat{\beta}; n}(M, \hat{\mathcal{L}} \cup \hat{K}) \hat{h}^n$$

$$= q^{1/2} \bar{h}^{-1} \bar{q}^{(1/4)p(\beta^2-1)} d_{\Delta, m, n}(M, \hat{\mathcal{L}}, \mathcal{K}) \sum_{m, n \geq 0, m + n \leq N} (\bar{q}^{\beta/2} - \bar{q}^{-\beta/2})^{2m+1} \bar{h}^n + \hat{h}^{N+1} x_1,$$

where

$$d_{\Delta, m, n}(M, \hat{\mathcal{L}}, \mathcal{K}) \in \mathbb{Z}[1/2, 1/h_1(M)], \quad x_1 \in \mathbb{Z}[q, q^{-1}, 1/2, 1/h_1(M)].$$  \hfill (4.18)

Since we assumed that $\hat{K}$ does not divide $h_1(M)$, then the first relation of (4.18) implies that $d_{\Delta, m, n}(M, \hat{\mathcal{L}}, \mathcal{K}) \in \mathbb{Z}(\hat{K})$. Therefore $d_{\Delta, m, n}(M, \hat{\mathcal{L}}, \mathcal{K})$ is well-defined and we can also use
\[ d_{\Delta; m, n}(M, \hat{L}, \mathcal{K}) \] as its $\hat{K}$-adic approximation. Thus equations (4.16) and (4.17) together with the relations (2.10) and
\[
\tilde{q}^{(1/4)p(\beta^2-1)} = \tilde{q}^{4^*p(\beta^2-1)} \quad \text{if } \beta \text{ is odd,} \quad \tilde{q}^{1/2} = -\tilde{q}^{2^*}
\] (4.19)

imply that for any $N_0$ there exists $N'_0$ such that for any $N > N'_0$
\[
Z'_{\hat{K}, \beta}(M, \hat{L} \cup \hat{K}; \hat{K}) = -\left( \frac{h_1(M)}{K} \right)_L \tilde{q}^{2^*} \hat{h}^{-1} \tilde{q}^{4^*p(\beta^2-1)}
\times \sum_{\substack{m, n \geq 0 \atop m + n \leq N}} d_{\Delta; m, n}(M, \hat{L}, \mathcal{K}) \right)_N (\tilde{q}^{\beta/2} - \tilde{q}^{-\beta/2})^{2m+1} \hat{h} n + \hat{K}^{N_0+2} x_2, \quad x_2 \in \mathbb{Z}[\tilde{q}].
\] (4.20)

We substitute eq. (4.20) into eq. (4.10). Since
\[
\hat{K}^{-1/2} e^{(i\pi/4)(\kappa-1)} = \sum_{\gamma=1}^{K} \tilde{q}^{\gamma^2} \in \mathbb{Z}[\tilde{q}]
\] (4.21)

and for odd $\beta$
\[
\tilde{q}^{\beta/2} - \tilde{q}^{-\beta/2} = \tilde{q}^{1/2}(\tilde{q}^{(\beta-1)/2} - \tilde{q}^{-(\beta+1)/2}) = -\tilde{q}^{2^*}(\tilde{q}^{(\beta-1)/2} - \tilde{q}^{-(\beta+1)/2}) \in \mathbb{Z}[\tilde{q}],
\] (4.22)

then it is easy to see that the contribution of the term $\hat{K}^{N_0+2} x_2$ to $Z'_{\Delta}(M', \hat{L}; \hat{K})$ is equal to
\[
\hat{K}^{-1/2} \text{sign}(p) e^{(i\pi/4)(\kappa-1)} \text{sign}(p) \tilde{q}^{3^*} \sum_{0 < \beta < K} \text{sign}(p) (\tilde{q}^{\beta/2} - \tilde{q}^{-\beta/2}) \hat{K}^{N_0+2} x_2
= \hat{K}^{N_0+1} x_3, \quad x_3 \in \mathbb{Z}[\tilde{q}].
\] (4.23)

Thus we come to the following statement: for any $N_0 > 0$ there exists $N$ such that
\[
Z'_{\Delta}(M', \hat{L}; \hat{K}) = -\left( \frac{h_1(M')}{K} \right)_L \text{sign}(p) \tilde{q}^{4^*(2-p+3\text{sign}(p))} \sum_{\substack{m, n \geq 0 \atop m + n \leq N}} d_{\Delta; m, n}(M, \hat{L}, \mathcal{K}) \right)_N Y_{\text{cycl.}}(p, m, \tilde{q}) \hat{h} n
+ \hat{K}^{N_0+1} x_3, \quad x_3 \in \mathbb{Z}[\tilde{q}],
\] (4.24)

where $Y_{\text{cycl.}}(p, m, \tilde{q})$ is defined by eq.(3.29). Since eq.(3.31) implies that $Y_{\text{cycl.}}(p, m, \tilde{q}) \in \mathbb{Z}[\tilde{q}]$, then the claim (2.48) of Main Theorem for the manifold $M'$ follows easily from eq.(4.24).
It remains to prove eq.(2.49) for \( M' \). If we substitute the expression for \( Z^{(tr)}(M, \hat{\mathcal{L}} \cup \hat{\mathcal{K}}; K) \) from eq.(3.6) into the surgery formula (2.35) and use the relations (4.15) and

\[
i e^{-(i\pi/2)\text{sign}(p)} = \text{sign}(p) \quad \text{if} \quad p \neq 0, \tag{4.25}\]

then we find that

\[
h^{1/2}_1(M') Z^{(tr)}(M', \hat{\mathcal{L}}; K) = -\text{sign}(p) q^{(2-p+3\text{sign}(p))/4} \sum_{m, n \geq 0 \atop m + n \geq N} d_{\hat{\mathcal{L}}, m,n}(M, \hat{\mathcal{L}}, K) Y_{\text{asympt.}}(p, m, q) h^n + h^{N+1} x_4, \quad x_4 \in \mathbb{Z}[1/2, 1/h_1(M')][[h]], \tag{4.26}\]

where \( Y_{\text{asympt.}}(p, m, q) \) is defined by eq.(3.30). The remainder \( h^{N+1}x_4 \) in this formula comes from the terms \( d_{\hat{\mathcal{L}}, m,n}(M, \hat{\mathcal{L}}, K) Y_{\text{asympt.}}(p, m, q) \) with \( m + n > N \) which were not included in the sum of eq.(4.26). Their contribution is estimated with the help of relation (3.32).

Now if we compare eqs. (4.24) and (4.26), then eq. (2.49) for \( M' \) follows easily from eq.(3.33) if we recall eq.(2.16) which relates \( d_{\hat{\mathcal{L}}, m,n}(M, \hat{\mathcal{L}}, K) \) and \( \left[d_{\hat{\mathcal{L}}, m,n}(M, \hat{\mathcal{L}}, K)\right]_N \) and if we use the relation

\[
\left(q^{(2-p+3\text{sign}(p))/4}\right)^\vee = \bar{q}^{4^*(2-p+3\text{sign}(p))} \tag{4.27}\]

which follows from Lemma 3.4.

\[\square\]

**Corollary 4.6** Main Theorem is true for a QHS \( M \) which is constructed by Dehn’s surgery on an algebraically split framed link \( \hat{\mathcal{L}}' \subset S^3 \).

**Proof.** The proof is by induction in the number \( L' \) of components of \( \hat{\mathcal{L}}' \). Proposition 4.4 demonstrates that the claim is true for \( L' = 0 \), that is, when \( \hat{\mathcal{L}}' \) is an empty link and as a result \( M = S^3 \). Suppose that the claim is true for \( L' \)-component links. Consider an \( (L' + 1) \)-component algebraically split link \( \hat{\mathcal{L}}' \). The surgery on the first \( L' \) components of \( \hat{\mathcal{L}}' \) produces a QHS \( M' \), and the \( (L' + 1) \)-st component of \( \hat{\mathcal{L}}' \) is homologically trivial inside \( M' \). Therefore Main Theorem is true for the surgery on the whole link \( \hat{\mathcal{L}}' \) because of Proposition 4.5. \( \square \)

**Lemma 4.7** Suppose that a QHS \( M' = M \# L_{p,1} \) is a connected sum of a lens space \( L_{p,1} \) and of a QHS \( M \) which contains a framed link \( \hat{\mathcal{L}} \). Suppose that Main Theorem holds for \( M' \). If
an odd prime $K$ divides neither $p$, nor $h_1(M)$, then the claim of Main Theorem is true for $M$.

Proof. The lens space $L_{p,1}$ can be constructed by Dehn’s surgery on a framed unknot in $S^3$ with self-linking number $-p$. Since

$$J_\beta(\text{unknot}; q) = \tilde{q}^{-p(p^2-1)/4} \frac{q^{\tilde{\beta}/2} - q^{-\tilde{\beta}/2}}{q^{1/2} - q^{-1/2}},$$

then according to eq. (2.35)

$$Z^{(tr)}(L_{p,1}; K) = \left( \frac{|p|}{K} \right)^{1/2} Z^{(tr)}(L_{p,1}; K)^{\vee}$$

$$= \left( \frac{|p|}{K} \right)^{1/2} \text{sign}(p) q^{1/2 - 1/2} \frac{q^{1/2} - q^{-1/2}}{q^{1/2} - q^{-1/2}}.$$

Note that since $\tilde{q}^{2r} = (\tilde{q}^{2r})^p$, then

$$\frac{\tilde{q}^{2r} - \tilde{q}^{-2r}}{\tilde{q}^{2r} - \tilde{q}^{-2r}} \in \mathbb{Z}[\tilde{q}]$$

and as a result

$$\frac{1}{Z'(L_{p,1}; K)} \in \mathbb{Z}[\tilde{q}].$$

Suppose that an odd prime $K$ divides neither $p$, nor $h_1(M)$. Since $h_1(L_{p,1}) = |p|$, then in view of multiplicativity of $h_1(M)$ under the operation of connected sum,

$$h_1(M') = h_1(L_{p,1}) h_1(M) = |p| h_1(M)$$
and, as a result, $K$ does not divide $h_1(M')$. Therefore, according to our assumptions, the claims of Main Theorem are true for $M'$

$$Z'_\Delta(M', \hat{\mathcal{L}}; \hat{K}) = \left( \frac{h_1(M')}{K} \right)_L (h_1(M')^{1/2} Z'^{(\text{tr})}_{\Delta}(M', \hat{\mathcal{L}}; \hat{K}))^{\vee}. \tag{4.34}$$

Both quantum invariants $Z'_\Delta(M, \hat{\mathcal{L}}; K)$ and $Z'^{(\text{tr})}_{\Delta}(M, \hat{\mathcal{L}}; K)$ are also multiplicative under the operation of connected sum, so

$$Z'_\Delta(M, \hat{\mathcal{L}}; K) = Z'_\Delta(M', \hat{\mathcal{L}}; \hat{K})/Z'(L_{p,1}; \hat{K}), \tag{4.36}$$

$$Z'^{(\text{tr})}_{\Delta}(M, \hat{\mathcal{L}}; K) = Z'^{(\text{tr})}_{\Delta}(M', \hat{\mathcal{L}}; K)/Z'^{(\text{tr})}(L_{p,1}; K), \tag{4.37}$$

Therefore relation (2.48) follows from eq. (4.36) and from relations (4.34) and (4.32), while eq. (2.49) follows from eqs. (4.37), (4.35) and (4.30). $\square$

The last statement that we need in order to prove Main Theorem is the lemma that H. Murakami introduced in [12] at the suggestion of T. Ohtsuki.

**Lemma 4.8 (H. Murakami [12])** For a QHS $M$ and an odd prime $\hat{K}$ there exists an algebraically split framed link $\hat{\mathcal{L}}' \subset S^3$ such that Dehn’s surgery on $\hat{\mathcal{L}}'$ produces a QHS $M'$ which is a connected sum of $M$ and of a finite number of lens spaces $L_{p,j}$, $1 \leq j \leq N$,

$$M' = M \# L_{p,1} \# \cdots \# L_{p,N,1}, \tag{4.38}$$

such that neither of $p_j$ is divisible by $\hat{K}$.

**Proof of Main Theorem.** For a QHS $M$ let $\hat{K}$ be an odd prime number which does not divide $h_1(M)$. Consider the QHS $M'$ of Lemma 4.8 which can be constructed by Dehn’s surgery on a framed algebraically split link $\hat{\mathcal{L}}' \subset S^3$. Corollary 4.6 says that Main Theorem is true for $M'$ and then Lemma 4.7 implies that it is also true for $M$. $\square$

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Appendix

A Casson’s invariant as the second coefficient of the TCC invariant

In [19] we used path integral arguments in the framework of the Chern-Simons quantum field theory of [29] in order to derive a much weaker version of Theorem 3.1. Considering that result as a conjecture, we used it to derive relations (2.44). Now that we have Theorem 3.1 at our disposal, we repeat the arguments of [19] in order to present a rigorous proof of (2.44) which in conjunction with Theorem 2.7 constitutes an alternative proof of Murakami’s result (2.28).

The first of eqs. (2.44) was already established in [27], so it remains to show that for a QHS $M$

$$\frac{1}{3} h_1(M) P_1(M) = \lambda_{CW}(M). \quad (A.1)$$

We have to establish some preliminary facts. We proved in [27] that for a knot $K \subset M$

$$P_0(M, K; t) = 1, \quad (A.2)$$

here $P_n(M, K; t) = 1$ are the polynomials of eq.(3.3) in the case of an empty link $\hat{L}$. A simple corollary of eqs. (A.2) and (3.5) is that if $K$ is homologically trivial in $M$, then

$$d_{0,0}(M, K) = 1/h_1(M), \quad (A.3)$$

$$d_{1,0}(M, K) = \frac{1}{2} h_1^{-2}(M) \Delta''(M, K). \quad (A.4)$$

Here

$$\Delta''(M, K) = \frac{d^2}{dt^2} \Delta_A(M, K; t)|_{t=1} \quad (A.5)$$

and the coefficients $d_{m,n}(M, K)$ come from eq.(3.6) written for an empty link $\hat{L}$.

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Suppose that a QHS $M'$ is constructed by a surgery on a framed homologically trivial knot $\hat{K} \subset M$ with self-linking number $p$. Then according to eq. (2.35), (2.38) and (3.6)

$$P_0(M') + h P_1(M') = -i (8K)^{-1/2} e^{-(3\pi i/4) \text{sign}(p)} q^{3/4} \text{sign}(p) |p|^{1/2} (q^{1/2} - q^{-1/2})^{-1}$$

$$\times \int_{-\infty}^{+\infty} q^{(1/4)p(\beta^2-1)} (d_{0,0}(M,K) (q^{\beta/2} - q^{-\beta/2})^2 + d_{1,0}(M,K) (q^{\beta/2} - q^{-\beta/2})^4)$$

$$+ h d_{0,1}(M,K) (q^{\beta/2} - q^{-\beta/2})^2) d\beta + O(h^2)$$

(A.6)

and as a result,

$$P_1(M') = \frac{1}{|p|} \left[ \left( \frac{3}{4} \text{sign}(p) - \frac{p}{4} - \frac{1}{2p} \right) d_{0,0}(M,K) + d_{0,1}(M,K) - \frac{6}{p} d_{1,0}(M,K) \right],$$

(A.7)

or in view of eqs. (A.3), (A.4)

$$P_1(M') = \frac{1}{h_1(M')} \left( \frac{3}{4} \text{sign}(p) - \frac{p}{4} - \frac{1}{2p} - \frac{3}{4} \frac{\Delta_A''(M,K)}{h_1(M)} + h_1(M) d_{0,1}(M,K) \right).$$

(A.8)

**Proposition A.1** Equation (A.1) is true for a QHS $M$ constructed by a surgery on a framed algebraically split link $\hat{L}' \subset S^3$.

**Proof.** (cf. the proof of Corollary 4.6) We prove the proposition by induction in the number of components $L'$ of $\hat{L}'$.

If $L' = 0$, then $M = S^3$. Since

$$\lambda_{CW}(S^3) = 0,$$

(A.9)

then eq. (A.1) is true in view of eqs. (2.22) and (2.32).

Suppose that eq. (A.1) is true for all QHS constructed by surgeries on $L'$-component links in $S^3$. Let us prove that it is also true for manifolds constructed by surgeries on $(L' + 1)$ component links. It is enough to prove the following: if eq. (A.1) is true for a QHS $M$ then it is also true for a QHS $M'$ constructed by Dehn’s surgery on a framed homologically trivial knot $\hat{K} \subset M$.

Equations (2.38), (2.39) imply that

$$Z^{(\text{tr})}_{\beta}(M, \hat{K}; K) = h_1^{-1/2}(M) \sum_{m,n \geq 0, |m| \leq n/2} d_{m,n}(M,K) \beta^{2m+1} h^n.$$  

(A.10)
If we substitute $\beta = 1$ into this equation and take into account the relation

$$Z_{\beta}(M, \hat{K}; K) \big|_{\beta=1} = Z^{(\text{tr})}(M; K),$$

then we conclude that

$$d_{0;1}(M, \mathcal{K}) = P_1(M).$$

(A.12)

Assume that eq. (A.1) is true for $M$. Then in view of eq. (A.12)

$$d_{0;1}(M, \mathcal{K}) = 3\lambda_{CW}(M)/h_1(M),$$

and therefore according to eq. (A.8),

$$(1/3) h_1(M') P_1(M') = \frac{1}{4} \text{sign}(p) - \frac{p}{12} - \frac{1}{6p} - \frac{1}{p} \frac{\Delta^\mathcal{K}_{tr}(M, \mathcal{K})}{h_1(M)} + \lambda_{CW}(M).$$

(A.14)

According to [28], the r.h.s. of this equation is equal to $\lambda_{CW}(M')$, hence we proved eq. (A.1) for $M'$. \hfill \Box

Since a lens space $L_{p,1}$ is constructed by Dehn’s surgery on an unknot in $S^3$, then Proposition A.1 implies that eq. (A.1) is true for $L_{p,1}$. Suppose that eq. (A.1) holds for a manifold $M' = M \# L_{p,1}$, with $M$ being a QHS Since both $\lambda_{CW}(M)$ and $(1/3) h_1(M) P_1(M')$ are additive under the operation of connected sum (the latter additivity follows from the multiplicativity of $Z^{(\text{tr})}(M; K)$), then eq. (A.1) should also be true for $M$. In view of Lemma 4.8 and Proposition A.1, this proves eq. (A.1) for any QHS (cf. Proof of Main Theorem).

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