Partial randomized benchmarking and noise

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In practical randomized benchmarking of logical gates for quantum computation, partial averaging is sometimes used to simplify the implementation. For instance, for two-qubit gates, averaging over only single-qubit operations is much easier to realize than the full averaging. We analyze such simplified, partial averaging and demonstrate that, unlike for the standard randomized benchmarking, the measured decay law of fidelity is a linear combination of several exponentials with different decay factors (3 for two qubits and single-bit averaging). The evolution with the sequence length is given by an iteration matrix, whose eigenvalues give the decay factors. For generic two-qubit gates two contributions decay fast and are barely visible, while the third factor characterizes a combination of gate errors in three channels. While the decay factor of this slowest exponential is close to that in the standard randomized benchmarking, we demonstrate that they differ in general and find this correction to the leading order. Furthermore, via a connection to the local invariants of two-qubit gates we find all exceptional gates, with more than one slow exponential, and analyze possibilities to extract their decay factors from the measured curves.

I. INTRODUCTION

Efficient implementation of quantum algorithms requires high-fidelity quantum logic gates. At the same time inevitable entanglement with the environmental degrees of freedom and inaccuracies during control of a quantum-information device affect the fidelity of quantum logical operations. It is important to quantitatively describe the physical origin, structure, and magnitude of such errors in order to be able to further suppress them to the needed level.

Various approaches have been suggested in order to characterize these errors. In general, a non-ideal quantum gate is a superoperator, acting on density matrices, rather than a unitary operator. Due to its matrix structure, its deviation from the ideal gate is described by a collection of numbers, and the size of this collection grows very fast with the number of qubits.

Randomized benchmarking (RB) is a method of testing fidelity of quantum logical gates or gate sets, which provides a single overall metric for the error level. Random sequences of gates are tested in RB, with the fidelity of the sequence decaying exponentially with its length. This method allows one to separate gate errors from the errors in state preparation and measurement (the so called SPAM errors) as well as to measure small errors in current and prospective quantum-information devices, since the measured gate error is enhanced with the length of the sequence.

To characterize a set of unitary gates, random sequences of these gates are studied. If a specific gate needs to be tested, the interleaved RB is used, in which case, e.g., a sequence of random unitaries is interleaved with applications of this specific gate. Effectively, this implies averaging of the tested gate over unitary rotations. The resulting, isotropic averaged gate (a superoperator) is just a dilatation and is characterized by a single number. This number quantifies the fidelity of the tested gate.

Note, however, that the complexity of the RB sequence grows very fast with the number of qubits. It implies generation of a random unitary gate for each term in the sequence. However, implementation of such gates may be complicated, since in typical situations they need to be decomposed in elementary gates for a specific qubit realization. For instance, already for two qubits implementation of a random two-qubit gate may require a sequence of a few elementary two-qubit gates and single-qubit gates. This not only increases the complexity of qubit manipulations, but also extends the time of the RB sequence and suppresses its accuracy due to the larger number of gates and typically lower fidelity of two-qubit elementary gates.

It can be shown that full averaging is achieved already with gates from a discrete Clifford group instead of the full unitary group, and the standard RB procedures rely on implementation of random Clifford gates. This is based of the fact that the Clifford group is a 2-design. In a typical experiment, all gates from the group need to be implemented based on the set of available elementary operations in a given implementation. However, the size of the Clifford group grows very fast with the number of qubits. While for a single qubit it contains 24 elements, for two qubits its size (11520) exceeds $10^4$. For this reason, sometimes a simplified approach is used, with averaging over 24x24 single-qubit gates on each qubit instead of the full two-qubit unitary $U(4)$ or Clifford group. This includes, in particular, experiments with superconducting quantum bits in Refs. It is apparent from the experimental data and expectations that this approach based on the single-qubit subgroup produces substantial averaging. However, it
is not necessarily complete, and the question arises, which information it provides.

We analyze such partial averaging for the case of two qubits in this article. Related questions for multiqubit systems were studied recently. The approach, developed in the current article, allowed us not only to analyze the benchmarking of generic gates but also to find all exceptional gates and fully analyze the IRB in all these exceptional cases with the possibility to extract more information from a simpler experimental procedure.

We demonstrate that the dynamics of the noise-averaged evolution operator as a function of the length of the RB gate sequence can be described with a linear markovian operator. We find this operator explicitly, using local invariants of the gate. Its eigenvalues provide the decay factors for the measured fidelity in partial-RB experiments. For generic two-qubit gates, the fidelity decay is dominated by a single exponential, which is close but not identical to the result of the standard RB, and we find the deviation of these quantities. Furthermore, we complete the exhaustive analysis by finding all exceptional gates, when more than one exponent may be visible in the decay curves, and demonstrate how the RB fidelities can be extracted from the data in these cases. In the opposite limit, we find a family of two-qubit gates, which can be viewed as especially suited for the partial RB, since only a single decay factor is non-vanishing. The presented analysis can be generalized to situations with more qubits or larger subsystems.

II. FULL INTERLEAVED RANDOMIZED BENCHMARKING

In preparation to the analysis of the partial RB, in this section we summarize a description of the standard, full RB procedure. Consider an arbitrary gate $W$ to be tested. Normally, a gate is a unitary operation $U_0$ (we consider its action by conjugation on density matrices, $W_0[\rho] = w_0\rho w_0^\dagger$, and sometimes use the same notation for the unitary gate and the respective superoperator $W_0$), but one can also consider arbitrary superoperators $W[\rho]$, acting on density matrices $\rho$. We assume that ideally, $W$ should realize a (unitary) gate $W_0$, but in reality it contains an error, so that $W = W_0\Lambda$. Here $\Lambda$ is a superoperator, assumed to be close to the identity operator $1$, that is the error is small. The expression $W = W_0\Lambda$ should be understood as follows: in each particular realization of noise/errors the action of the gate is random, and its average over realizations is denoted $W_0\Lambda$. There is a slight ambiguity in notation: $W$ sometimes refers to a specific realization of the gate and sometimes to the noise-averaged value; the exact meaning should be clear from the context.

Interleaved randomized benchmarking (IRB) studies sequences of the type

$$\Lambda_n = FWV_n \ldots W_3WV_2WV_1,$$

where $V_i$ are random gates (here we assume gates $V_i$ to be drawn randomly and uniformly, in the Haar measure, from the unitary group $U(d)$; in variations of the problem other possibilities can be considered, cf. a comment at the end of this section). The final gate $F$ is chosen such that in the absence of errors the product reduces to $1$, that is $F = (W_0V_n \ldots W_3W_0V_2W_0V_1)^n$. In other words, without errors the resulting operation is $\Lambda_n = 1$, and any deviation from the identity indicates some kind of error. To quantify such deviations, one typically measures the action of $\Lambda_n$ by preparing an initial state $\rho_0$, acting by the sequence $\Lambda_n$ and measuring the resulting state. In the error-free case, the final state should coincide with $\rho_0$, while in a realistic situation the probability to find the system in the same final state $\rho_0$ is below 1 and decays with the sequence length $n$. The rate of this decay, in fact, quantifies the fidelity of the gate $W$, see below. Note that this procedure implies multi-fold repetition of the experiment.

There is a number of factors that influence the resulting operation $\Lambda_n$. Noise and inaccuracies during each instance of $W$ in the sequence force it to deviate from $W_0$. For the purposes of this paper, we assume that these noise contributions are uncorrelated for different instances of $W$ even within each sequence (short noise correlation time). Further, the total error of the sequence contains independent contributions from all terms in Eq. (\ref{eq:lr}). Since we assume that the errors in each $W$ in Eq. (\ref{eq:lr}) are uncorrelated, averaging of $\Lambda_n$ over repetitions results in replacement of each $W$ by its noise-averaged value. Exactly this noise-averaged value we denote as $W_0\Lambda$. Moreover, we assume that the random gates $V_i$ are ideal for the purposes of this paper. In principle, contribution of the errors in $V_i$’s is accounted for in the standard IRB procedure by, first, testing sequences of only $V_i$’s. We neglect this contribution since we focus here on a different phenomenon; moreover, this neglect is especially justified for the case of most interest to us, when $W$ is a two-qubit gate, while $V_i$’s are single-qubit gates, typically, with much lower errors.

If one substitutes $W$ by $W_0\Lambda$ in the sequence, one can rewrite it as a product of conjugate $\Lambda$’s:

$$\Lambda_n = FW_0AV_n \ldots W_0AV_3W_0AV_2W_0AV_1 = (U_n^\dagger AU_n)(U_{n-1}^\dagger AU_{n-1}) \ldots (U_2^\dagger AU_2)(U_1^\dagger AU_1),$$

where the unitaries $U_i$ are related to $V_i$ via:

$$V_1 = U_1, \quad V_iW_0 = U_iU_{i-1}^\dagger \quad \text{for } i = 2 \ldots n.$$
(We assume that we deal with \( n \) equivalent implementations of the same gate \( W \).) Note that the rhs of (2) clearly reduces to the identity in the absence of errors, \( \Lambda = \hat{1} \), as it should. One can easily see that if \( V_i \)'s are independent random gates, uniformly distributed over the unitary group, then so are \( U_i \)'s (here we use the fact that \( W_0 \) belongs to the same unitary group as \( V_i \)'s; cf. the next section, where this is not the case). This implies that averaging of the sequence \( \Lambda_n \) over the random choice of \( V_i \)'s reduces to the independent averaging of each term in the product on the rhs of Eq. (3).

Let us denote one term in the product on the rhs of Eq. (2) as \( \Lambda_U \). In other words, when applied to a state with the density matrix \( \rho \), this superoperator gives:

\[
\Lambda_U[\rho] \equiv U^\dagger \Lambda[U\rho U^\dagger]U.
\]

(4)

Its average over realizations of the random \( U \) (we use both notations, the over-bar and the angular brackets) is:

\[
\bar{\Lambda}[\rho] \equiv \langle \Lambda \rangle[\rho] = \langle \Lambda_U[\rho] \rangle_U.
\]

(5)

Here the subscript \( U \) denotes averaging over the unitary group with the Haar measure. One can see that the averaged gate \( \bar{\Lambda} \) is isotropic with respect to rotations from the unitary group (or the corresponding subgroup), it does not change under an arbitrary unitary rotation \( L \) of the basis:

\[
\bar{\Lambda}[L\rho L^\dagger] = L\bar{\Lambda}[\rho]L^\dagger.
\]

(6)

The isotropy strongly constrains the degrees of freedom in the gate \( \bar{\Lambda} \). It conserves the scalar trace-one density matrix \( \hat{1}/d \) as any physically relevant map, but otherwise, as one can easily verify, it is just a dilatation with the center at the isotropic density matrix \( \hat{1}/d \), where \( d \) is the (even) dimension of the Hilbert space:

\[
\bar{\Lambda}[\rho] = (1 - \mu)\frac{\hat{1}}{d} + \mu\rho.
\]

(7)

With varying \( 0 \leq \mu \leq 1 \), it interpolates between the identity map (at \( \mu = 1 \)) and a constant map to \( \hat{1}/d \) (at \( \mu = 0 \)).

As a consequence, \( \Lambda_n \) is isotropic for any \( n \), and the randomized benchmarking investigates how \( \mu_n \equiv \mu(\Lambda_n) \) depends on \( n \). Since according to Eq. (2) one has \( \langle \Lambda_n \rangle = \langle \bar{\Lambda} \rangle^n \), using Eq. (7) one finds that \( \mu_n = \mu^n \). In other words, \( \mu_n \) decays exponentially with the sequence length \( n \). By measuring this exponential decay one can extract \( \mu \), the fidelity of the tested gate \( W \).

Apart from realization of the sequence (1), any measurement involves preparation of the initial state \( \rho_0 \) and measurement of the final state, which both can be faulty and contribute to the error. However, their contribution does not depend on \( n \) and does not alter the exponential decay, and thus the procedure of RB allows one to separate the gate fidelity from these SPAM (state preparation and measurement) errors. This is another useful property of the RB method.

An important remark is in order. In the considerations above we assumed that the random gates \( V_i \) are drawn from the full unitary group. In particular, this ensured that the isotropic average gate is just a dilatation, characterized by a single parameter \( \mu \) (this can be referred to as complete averaging). However, it is important and convenient in practice that one can use a smaller, even a finite group instead. \( V_i \)'s can be picked from a Clifford group, which also ensures complete averaging, since the Clifford group is a 2-design. For smaller, or other, subgroups of \( U(d) \) the averaging may be incomplete or, equivalently, partial.

### III. AVERAGING A TWO-QUBIT OPERATION OVER SINGLE-QUBIT GATES

Let us consider how the complexity of the RB procedure grows with the size of the tested system. For instance, for a system of two qubits, to test a two-qubit gate one needs to generate random two-qubit gates \( V_i \). In practice, to implement each drawn \( V_i \) one typically needs to decompose it in terms of elementary operations for a given realization of qubits, for instance, in terms of CNOT and single-qubit gates. Thus, realization of \( V_i \) requires a few CNOT’s (or other elementary two-qubit gates) and single-qubit gates. This may strongly suppress the fidelity of the sequence, since two-qubit gates typically take longer and have a lower fidelity.

In some experimental work on randomized benchmarking with two qubits, for the sake of simplicity, repetition of a tested two-qubit gate is interleaved with random single-qubit operations only. This, of course, simplifies manipulations and keeps the fidelity of \( V_i \)'s at a higher level. But single-qubit gates do not generate the whole set of two-qubit gates, \( SU(4) \). So a question arises if in such experiments one indeed measures the average fidelity of the tested gate by fitting the decay curve with an exponential.
On one hand, the randomization with only single-qubit gates appears to be still sufficiently powerful. However, it is not obvious, if it is complete in the sense specified above. There is a number of questions, which we analyze below. Does the fidelity of the sequence decay exponentially? If not, what kind of decay should one expect and which information about the gate can one extract from this decay? If one fits the decay curve with an exponential, how is the extracted exponent related to the RB-fidelity in a complete RB experiment (i.e., in an experiment with random unitary two-qubit gates $V_i$ and complete averaging)?

We discuss these questions for two-qubit gates with single-qubit randomization as described above and show that the decay is characterized by three exponentials, and then show how to complete the analysis for the case of two qubits. These results can be generalized to a more general situation of partial averaging over a subgroup.

To begin the analysis, note that in the case of partial averaging the considerations of the previous section fail. More specifically, Eq. (3) does not map from the set of independent elements $V_i$ of the corresponding subgroup (the single-qubit gates in our primary example) to independent $U_i$ in this subgroup, and averaging over $U_i$’s in Eq. (2) cannot be done straightforwardly anymore. Instead, we can rewrite the sequence (1) as follows:

$$\Lambda_n = F_n(\hat{U}_n^\dagger W_0 \Lambda \hat{U}_n) \ldots (\hat{U}_2^\dagger W_0 \Lambda \hat{U}_2)(\hat{U}_1^\dagger W_0 \Lambda \hat{U}_1),$$

where

$$V_1 = \hat{U}_1, \quad V_i = \hat{U}_i \hat{U}_i^\dagger \text{ for } i = 2 \ldots n,$$

and

$$\tilde{F}_n = \left[(\hat{U}_1^\dagger W_0 \hat{U}_n) \ldots (\hat{U}_2^\dagger W_0 \hat{U}_2)(\hat{U}_1^\dagger W_0 \hat{U}_1)\right]^\dagger.$$

In this case $\hat{U}_i$’s belong to the same subgroup (the group of single-qubit gates of two qubits in our primary example) and are independent random gates. Note that they are also involved in the final gate $\tilde{F}_n$, see the rhs of Eq. (10). However, if $W_0$ is also a single-qubit gate, Eq. (2) can be applied, and this is used below in Section III A.

A. Testing a trivial two-qubit operation

Let us begin our analysis of this expression from the case of $W_0 = \hat{I}$, that is when the identity gate is tested. (Testing the identity gate may be not a pure formality: it verifies the ability of the system to store a quantum state over a finite time interval, that is, probes the influence of noise or decoherence.) Then

$$\Lambda_n = \hat{\Lambda}^n.$$

What is the most general form of a gate $\hat{\Lambda}$, which is isotropic w.r.t. the group of single-qubit rotations? In other words, it is invariant w.r.t. to single-qubit rotations, or locally invariant. What is a locally invariant two-qubit operation $\hat{\Lambda}$? It maps a two-qubit density matrix $\rho = \frac{1}{4} + \frac{1}{2} s \sigma^{(1)} + \frac{1}{2} p \sigma^{(2)} + \frac{1}{2} \beta \sigma^{(3)}$ to a matrix of the same form. One can easily see that in terms of $s, p, \beta$ the most general locally invariant mapping is:

$$s \mapsto a s, \quad p \mapsto b p, \quad \beta \mapsto c \beta$$

with three independent real factors $a, b, c$. Hence

$$\hat{\Lambda}_n : \quad s \mapsto a^n s, \quad p \mapsto b^n p, \quad \beta \mapsto c^n \beta.$$

For an operation with given $a, b, c$, if we average it over the whole SU(4), what value of $\mu$ would we obtain? Apparently, $\mu$ would be a linear combination $x a + y b + z c$. Since for $a = b = c$ they coincide with $\mu$, one finds that $x + y + z = 1$. Furthermore, averaging, e.g., with the gate CZ (or CNOT) replaces $a, b, c$ with $\frac{2}{3} a + \frac{1}{3} a, \frac{2}{3} c + \frac{1}{3} b, \frac{5}{6} c + \frac{2}{3} a + \frac{1}{3} b$, but should keep the same $\mu$. All this allows us to find that $x = y = \frac{1}{6}, z = \frac{1}{3} x$, and thus:

$$\mu = \frac{a + b + 3 c}{5}.$$

This is the value, which the standard randomized benchmarking (with complete averaging over all unitary two-qubit gates $V_i$) would measure.

For instance, the initial state $00$ has $s_z = p_z = 1/2$ and $\beta_{zz} = 1/4$, so that the probability to find the same state after $n$ rounds decays as $(1 + a^n + b^n + c^n)/4$. This latter value can be measured, and then one can extract the three
parameters $a, b, c$. To simplify extraction of these parameters, one can measure the result of the operation for various initial states and measure probabilities of various final states. For instance, if the system is prepared in the initial state $|↑↑⟩$, then by repeating the experiment one can measure the probabilities $P_{↑↑}, P_{↑↓}, P_{↓↑}, P_{↓↓}$ to find in each of the four computational-basis states after application of the IRB-sequence. From these one can find the three decaying exponentials directly:

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} P_{↑↑} \\ P_{↑↓} \\ P_{↓↑} \\ P_{↓↓} \end{pmatrix} = \begin{pmatrix} e^n \\ b^n \\ a^n \\ 1 \end{pmatrix}. \quad (15)$$

This allows one to extract three decay factors, $a, b, c$, separately (they are all close to 1 in the case of small errors).

Obviously, when two qubits are decoupled and uncorrelated, $a$ characterizes single-qubit errors on the first qubit, $b$ describes errors on the second qubit. If only single-qubit errors are present, $c = ab$. Hence, the difference $c - ab$ describes errors associated with interaction/cross-talk between the qubits, or any other kind of correlated noise experienced by them.

### B. Testing an arbitrary two-qubit gate

Averaging over single-qubit gates for a two-qubit system is a particular case of averaging over a subgroup. Another simple example is averaging only over rotation around one axis (say, the $z$-axis) for a single qubit. In such cases, in contrast to averaging over the whole unitary group, the averaged operation $\bar{\Lambda}$ is not necessarily characterized by a single dilatation parameter $\mu$, but in general by more parameters (see above).

Because of the fact that a number of experimental studies of randomized benchmarking are conducted with partial averaging only, we analyze which information does one learn from such studies.

Let us derive an expression for the averaged operation after many repetitions in the case of the interleaved RB. By rewriting the expression (8) for the interleaved sequence, one finds the following recurrence relation for the operation $\Lambda_n$, the result of the $n$-step interleaved RB:

$$\bar{\Lambda}_{n+1} = \langle W_0^\dagger \bar{\Lambda}_n W_0 \rangle \cdot \bar{\Lambda} \rangle, \quad (16)$$

where the angular brackets denote averaging over the subgroup $G$. Clearly, $\bar{\Lambda}_n$ is subgroup-invariant and characterized by the corresponding parameters (three numbers $a_n, b_n, c_n$ both for single-qubit averaging and two-qubit $W$ as well as for $z$-averaging and a single-qubit $W$). This recurrence relation is a central result in the analysis. Our further goal is to solve this recurrence relation.

Before doing that, we note that Eq. (16) simplifies in the case when the gate $W_0$ belongs to the group, which we are averaging over. This includes the cases of averaging over the whole unitary (or Clifford) group, i.e., the case of standard RB, as well as the case of the ‘waiting’ gate $W_0 = \hat{1}$. In these cases $W_0$ drops out of Eq. (16), and since $\bar{\Lambda}_n$ is locally invariant, it can be taken out of the averaging, which implies that $\bar{\Lambda}_n = \lambda^n$. However, in general, for arbitrary averaging group and gate $W_0$, the average product in (16) does not reduce to the product of averages. Below we analyze this expression in this generic situation.

Each averaged error operator $\bar{\Lambda}_n$ is characterized by a triple of numbers, which can be combined into a vector

$$f_n \equiv \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad (17)$$

and Eq. (16) is a linear relation between $f_{n+1}$ and $f_n$, which we can describe by an iteration matrix $M$:

$$f_{n+1} = M f_n. \quad (18)$$

Hence, the $n$-dependence of $f_n$, and thus results of any measurement in an RB experiment, are determined by the eigenvalues of the $3 \times 3$ matrix $M$. In the following we analyze the spectrum of this matrix.

The error-free (identity) operation $\Lambda = \hat{1}$ corresponds to $f = (1, 1, 1)$. Thus, neglecting preparation errors, we find that $f_n = M^n. (1, 1, 1)$.

While the exact spectrum of $M$ depends on the properties of the error-operator $\Lambda$, in the zero-order approximation, dropping the first factor $\Lambda$ on the rhs of Eq. (16), we obtain an error-free iteration matrix $M^0$. It corresponds to the following recurrence relation:

$$M^0 : \quad \bar{\Lambda}_{n+1} = \langle W_0^\dagger \bar{\Lambda}_n W_0 \rangle. \quad (19)$$
We begin our analysis from the spectrum of $M^0$, and the spectrum of $M$ can then be found perturbatively in small errors. In particular, the spectra of $M$ and $M^0$ are close (assuming the errors in the gates are small).

Introducing matrix elements of $W_0$ and $\Lambda$, we have found from Eq. (19) the matrix elements of $M$ by direct calculation in terms of the matrix elements of $\Lambda$. In particular, in the error-free case $\Lambda = 1$, we found certain relations between these elements. Let us demonstrate them and comment on these relations.

The first set of three relations,

$$
M^0_{i1} + M^0_{i2} + M^0_{i3} = 1, \quad i = 1, 2, 3,
$$

(20)

follows from the fact that error-free operator $\bar{\Lambda}_n$ remains error-free after application of (19). In other words, $(1, 1, 1)$ is the eigenvector of $M^0$ with eigenvalue 1.

Further, due to trace conservation

$$
M^0_{11} + M^0_{22} + 3M^0_{31} = 1, \\
M^0_{12} + M^0_{21} + 3M^0_{32} = 1, \\
M^0_{13} + M^0_{23} + 3M^0_{33} = 3.
$$

(21)

Trace conservation here implies that the traces of $\bar{\Lambda}_{n+1}$ and $\bar{\Lambda}_n$ coincide, which follows directly from (19).

Relations (20), (21) already strongly constrain the structure of the iteration matrix $M^0$. However, there is a further relation: the matrix $M^0$ remains intact under qubit transposition as we show in Section IV with the use of local invariants. This entails the fact that $M^0$ is symmetric in the sense that $M^0_{ij}$ remains the same if in the subscript $ij$ each 1 is replaced by 2 and 2 by 1, or explicitly:

$$
M^0_{12} = M^0_{21}, \\
M^0_{13} = M^0_{23}, \\
M^0_{31} = M^0_{32}, \\
M^0_{11} = M^0_{22}.
$$

(22)

Together with the previous relations, we find the general form of the iteration matrix $M^0$:

$$
M^0 = \begin{pmatrix}
    m_1 & m_2 & 1 - m_1 - m_2 \\
    m_2 & m_2 & 1 - m_1 - m_2 \\
    1 - m_1 - m_2 & 1 - m_1 - m_2 & 1 + 2m_1 + 2m_2
\end{pmatrix}.
$$

(23)

These findings allow us to find the spectrum of the iteration matrix $M^0$:

$$
\text{Spectrum of } M^0: 1, m_1 - m_2, \frac{5m_1 + 5m_2 - 2}{3}.
$$

(24)

The corresponding eigenvectors (isotropic superoperators) coincide for all gates $W$: $(1, 1, 1)$ for the identity superoperator, $(1, -1, 0)$ and $(3, 3, -2)$ for the antisymmetric/symmetric traceless superoperators. This implies that iteration matrices $M^0$ for all gates commute with each other.

There are no further relations between the matrix elements of the iteration matrix. It is illustrated in Fig. 1, in which the values of $(m_1, m_2)$ are plotted for various possible gates. They all fall within an area, limited by four curves, which correspond to the following relations:

$$
0 \leq m_1 \leq 1, \quad 0 \leq m_2 \leq 1, \\
m_1 + m_2 \geq \frac{1}{3}, \\
\sqrt{m_1} + \sqrt{m_2} \leq 1.
$$

(25, 26, 27)

These relations can be directly checked analytically, for instance, by using an explicit representation of two-qubit gates $W_0$ from all possible equivalence classes with the same local invariants [14,15]. In terms of these invariants $G_1$, $G_2$ (cf. Section IV) they read:

$$
2|G_1| + 1 \geq |G_2|, \\
|G_1| \geq 0, \\
(G_2)^2 + 3 \geq 12|G_1|.
$$

(28, 29, 30)
FIG. 1: Distribution of possible values of the entries $m_1$, $m_2$ of the iteration matrix $M^0$ for possible two-qubit gates, drawn uniformly from the unitary group $U(4)$. Each point $(m_1, m_2)$ corresponds to a family of two-qubit gates. Large solid dots correspond to (the families of) the gates SWAP, iSWAP, $\sqrt{\text{SWAP}}$, identity, and CNOT. Boundaries of this region are discussed in the text. The dashed line indicates the family in Eq. (40) with the solid square showing the gate $W_\lambda$ especially suited for partial RB — with only a single decay factor (here $\cos \lambda \pi = -1/5$), see discussion around Eq. (40). Inset: the partial-RB decay factors for each of these gates, that is the eigenvalues of the respective iteration matrix $M^0$.

### IV. ITERATION MATRIX AND LOCAL INVARIANTS

The special form of Eq. (19) entails an important property of the mapping and the corresponding matrix $M^0$. Indeed, the mapping from $\Lambda_n$ to $\Lambda_{n+1}$ is the same for two two-qubit gates $w_A^0$ and $w_B^0$ if they differ only by application of single-qubit gates before and/or after the gate: $w_A^0 = S_1 S_2 w_B^0 T_1 T_2$, where $S_1$, $S_2$, $T_1$, $T_2$ are single-qubit unitary gates (applied to the qubit 1 or 2 as indicated by the subscript). Such two gates are referred to as locally equivalent, or equivalent up to local transformations, and have the same correlation, or entanglement properties. It has been shown that two gates are locally equivalent if and only if they have the same value of the so called local invariants. A complete set of such invariants has been found in Ref. [16]. These invariants of a two-qubit gate, a complex number $G_1$ and a real number $G_2$, are given by explicit expressions in terms of the matrix of the gate and thus can be easily calculated for each particular gate $W_0$. They turn out to be useful in the analysis of optimal decompositions of complex unitary operations in terms of elementary quantum logic gates for specific physical realizations of qubits [17–19].

Explicitly the invariants can be found using the $4 \times 4$ matrix $Q$ of transformation to the Bell basis [16]: first, one finds the matrix of the gate in the Bell basis, $w_B = Q^\dagger w Q$, then the product $\omega = w_B^T w_B$, and finally, for a unitary gate the invariants are given by

$$G_1 = \frac{\text{tr}^2 \omega}{16}, \quad G_2 = \frac{\text{tr}^2 \omega - \text{tr} \omega^2}{4}. \quad (31)$$

The observation above about the local invariance of the iteration matrix $M^0$ implies that it is completely determined by the local invariants $G_1$, $G_2$. This observation has useful consequences for our analysis. First, one can show explicitly that the gates $W_0$ and SWAP-$W_0$-SWAP, which differ only by the transposition of the two qubits, have the same values of the local invariants. Hence they are locally equivalent and, in particular, have the same iteration matrix $M^0$. Here the SWAP gate is a standard operator, which exchanges the states of two qubits. On the other hand, it is obvious, and can be checked directly, that the $M^0$-matrices for $W_0$ and SWAP-$W_0$-SWAP differ by the exchange of the first two basis vectors, that is by the transposition of the first two columns and first two lines. This immediately proves the relation [22].
Further, one can find an explicit expression for the iteration matrix $M^0$ in terms of the local invariants:

$$m_1 = \frac{2|G_1| + G_2 + 1}{6},$$  \hspace{1cm} (32)$$
$$m_2 = \frac{2|G_1| - G_2 + 1}{6}. \hspace{1cm} (33)$$

One can verify these identities by various methods. For example, one can use the fact that the family of the gates $W_0 = \exp\left(\frac{i}{2}(c_x\sigma_x^1\sigma_z^2 + c_y\sigma_y^1\sigma_z^2 + c_z\sigma_z^3\sigma_z^2)\right)$, with real $c_x,y,z$ contains representatives with all possible values of the local invariants (hence, any two-qubit gate is locally equivalent to some gate in the family)\textsuperscript{[13]} One can directly calculate the lhs and rhs of these identities for this family in order to verify them.

Thus, we found that the iteration matrix $M^0$ is completely determined by the local invariants of the gate $W_0$. However, we notice that the matrix, in fact, depends only on $G_2$ and the absolute value $|G_1|$ of the complex $G_1$, but not on the phase of $G_1$. Thus, a question arises which two-qubit gates form this fixed-iteration-matrix family with the same $|G_1|$ and $G_2$, but various $\arg G_1$. To analyze it, it is convenient to introduce the parameters $k_x = \cos 2c_x$, $k_y = \cos 2c_y$, $k_z = \cos 2c_z$. In the $k$-space each point in the cube $-1 \leq k_x, k_y, k_z \leq 1$ represents two local-equivalence classes of two-qubit gates with the same $G_2$ and complex conjugate values of $G_1$. One can verify that

$$G_2 = k_x + k_y + k_z \quad \text{and} \quad |G_1| = \left(2 + G_2^2 - k^2\right)/8. \hspace{1cm} (34)$$

(In fact, Re $G_1 = (k_x + k_y + k_z + k_x k_y k_z)/4$ and Im $G_1 = \pm \sqrt{(1 - k_x^2)(1 - k_y^2)(1 - k_z^2)/4}$.) Thus, the family with fixed $|G_1|$ and $G_2$ is a circle in $k$-space, orthogonal to the main diagonal $(1,1,1)$ and with a center on this diagonal.

More precisely, the family only covers the part of circle within the cube. At the same time, $\arg G_1$ does not assume all values in $[0,2\pi]$ on this circle. In fact, using the expression\textsuperscript{[13]} for the uniform (Haar) measure in terms of the local invariants $G_1$, $G_2$, one finds that it can be expressed as $\propto dm_1 dm_2 d(\arg G_1)$ in terms of the entries $m_1$, $m_2$ of the iteration matrix and the phase of $G_1$, and thus the density of points in Fig. 1 shows directly, which fraction of the full interval $[0,\pi]$ is covered by admissible values of $\arg G_1$ for given $|G_1|$ and $G_2$. In particular, at each point on the main diagonal of the cube in $k$-space the phase $\arg G_1$ has only one fixed value, and thus the density of points at the upper boundary in Fig. 1 vanishes. We note further that the circles for the identity and the SWAP gates are just the points $(1,1,1)$ and $(-1,-1,-1)$ on the main diagonal.

V. DECAY FACTORS

The spectrum of $M$, which is close to the spectrum of $M^0$, determines the decay factors in the decay curves measured in an IRB experiment as discussed above. As one can see from the explicit expression\textsuperscript{[24]}, one eigenvalue of $M$ is always close to 1 (since one eigenvalue of $M^0$ is always exactly 1), while two other eigenvalues are in general away from 1. (It follows from Eq. (24), Eqs. (25)-(27) and Fig. 1 that the eigenvalues cannot exceed 1 by absolute value; they always close to 1 (since one eigenvalue of $M$ is completely determined by the local invariants of the gate $W_0$). This implies that the measured decay curve is a linear combination of three decaying exponentials, where one of these exponentials decays slowly, while two others decay fast and vanish already at small values of the sequence length $n$. A question remains, how this weakly decaying exponential is related to that observed in full IRB with complete averaging over the whole unitary group $SU(4)$.

Note, however, that exceptions to this picture are possible, and our approach allows us to find and analyze all exceptional cases. In such exceptional situations more than one exponent with the decay factor close to 1 by absolute value may appear. Analysis, based on the explicit expressions for the spectrum, demonstrates that this happens only near the points $m_1 = 1, m_2 = 0$ and $m_1 = 0, m_2 = 1$. Translation to the language of the local invariants allows us to find that the exceptional gates are those close to the identity and the SWAP gate, as well as to those locally equivalent to them. In other words, the exceptional cases are single-qubit gates and combinations of one SWAP gate with single-qubit gates.

In the exceptional cases more than one exponential is visible in the decay curve. Still by proper analysis one can extract the relevant values. Indeed, the second or third eigenvalue in Eq. (24) can reach 1 in absolute value only for $m_1 = 1, m_2 = 0$ and $m_1 = 0, m_2 = 1$. These two pairs correspond to the identity gate and the SWAP gate, respectively (plus the gates, locally equivalent to these). In other words, these are single-qubit gates (they are locally equivalent to identity) and the products $S$-SWAP-$S'$, where $S$ and $S'$ are single-qubit gates.

Thus, in the generic case only one decay factor defines the IRB decay curve. As for the exceptional situations, near the identity gate (or single-qubit gates) there are three decay factors $a, b, c$, close to 1, which determine the decay, and they can be extracted, for instance, using the procedure described near Eq. (15). In the remaining exceptional case, for the SWAP gate and the gates, locally equivalent to SWAP (cf. above), the matrix $M^0$ has eigenvalues $(1,1,-1)$,
and hence the perturbed matrix $M$ has eigenvalues close to these. They also can be extracted, using a procedure, similar to that for the identity gate above. In fact, in this case the transformation matrix is

$$M = \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & c \end{pmatrix},$$

where the parameters $a$, $b$, $c$ can be expressed via matrix elements of the $\Lambda$ superoperator in the basis of Pauli matrices $\sigma_i \otimes \sigma_j$ in the space of density matrices: for instance, if we define the basis vectors as $\sigma_i \otimes \sigma_0$, $\sigma_0 \otimes \sigma_j$, $\sigma_i \otimes \sigma_j$, $\sigma_0 \otimes \sigma_0$ (with $i, j = x, y, z$), then

$$a = \frac{1}{3} \sum_{\alpha=1,2,3} \Lambda_{\alpha\alpha}, \quad b = \frac{1}{3} \sum_{\alpha=4,5,6} \Lambda_{\alpha\alpha}, \quad c = \frac{1}{9} \sum_{n=7}^{15} \Lambda_{\alpha\alpha}.$$  \hspace{1cm} (36)

Then we find that after $n$ steps the error superoperator $\Lambda_n$ is described by $f_n = ((ab)^{n/2}, (ab)^{n/2}, c^n)$ for even $n$ and $f_n = (a(ab)^{(n-1)/2}, b(ab)^{(n-1)/2}, c^n)$ for odd $n$. This gives us a simple procedure for extracting all three decay factors, $a$, $b$, $c$ from combinations in Eq. (15):

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} P_{11} \\ P_{1+} \\ P_{+1} \\ P_{++} \end{pmatrix} = \begin{pmatrix} e^n \\ a^{k}b^{k+p} \\ a^{k+p}b^{k} \\ 1 \end{pmatrix} \quad \text{for } n = 2k + p, \ p = 0/1. \hspace{1cm} (37)$$

Here the powers on the rhs depend on the parity of the length sequence $n$, represented as $n = 2k + p$ with parity $p = 0$ or $1$. Thus, one can easily extract the decay factors of the gate by using the fact that the upper entry of the column $[37]$ is multiplied by $c$ with extension of the random sequence by one step, while the second and third entries are multiplied alternatively by $a$ and $b$ on odd and even steps. Having extracted $a$, $b$, and $c$, one can also obtain the decay factor $\mu = (a + b + 3c)/5$, which would be measured in the standard, more complex RB experiment with complete averaging.

Now let us account, perturbatively, for the deviations of the error operator $\Lambda$ from identity: $\Lambda_{\alpha\beta} = \delta_{\alpha\beta} + \epsilon_{\alpha\beta}$ with $\epsilon_{\alpha\beta} \ll 1$. Then we find for the highest eigenvalue of $M$ in the non-degenerate case:

$$\mu = 1 + \frac{1}{15} \text{Tr} \hat{\epsilon} = \frac{1}{15} \text{Tr} \hat{\Lambda}.$$  \hspace{1cm} (38)

We note that exactly this value (to the first order in $\epsilon$) would occur in a full RB procedure with complete averaging over all random two-qubit operations.

Thus, the highest eigenvalue of $M$ coincides with the decay factor for the full RB, to the first order in the errors $\epsilon$. This means that the decay factor extracted from the partial RB experiment would coincide with that in the full RB, and hence the simpler partial RB is efficient.

To study this further, we found the second-order correction to the highest eigenvalue $\mu$ in the case of the error operator, isotropic w.r.t. single-qubit rotations, $\Lambda = \text{diag}(1 - \alpha, 1 - \beta, 1 - \gamma)$:

$$\mu = 1 - \frac{\alpha + \beta + 3\gamma}{5} + \frac{(m_1 - m_2)(\alpha - \beta)^2}{10(1 - m_1 + m_2)} - \frac{3(-2 + 5m_1 + 5m_2)(\alpha + \beta - 2\gamma)^2}{250(-1 + m_1 + m_2)}. \hspace{1cm} (39)$$

Thus, due to these corrections a difference appears between the results of the full RB and simple partial RB (with single-qubit averaging only). Hence, strictly speaking the partial RB does not reproduce the result of full RB, although corrections are typically weak. Note that the second-order correction diverges at the exceptional gates (those, locally equivalent to the identity or SWAP), where $m_1 = 1$, $m_2 = 0$ or $m_1 = 0$, $m_2 = 1$, and hence for gates $W$ close to such exceptional gates deviations between full and partial RB are stronger.

We note a special role of the gate family with $m_1 = m_2 = 1/5$. In this case the second and third eigenvalues of $M^{\alpha}$ in (24) vanish, and the decay curve is a single exponential to a high accuracy. In other words, these gates are especially suited for partial randomized benchmarking. Furthermore, for these gates next-order corrections also vanish. For example, such a gate can be implemented by turning on the qubit-qubit coupling $H_{\text{cpl}} = g[1 - (1 + \lambda)\sigma_1^x\sigma_2^x - (1 - \lambda)\sigma_1^y\sigma_2^y - \sigma_1^z\sigma_2^z]$ for a finite period $t = h/(16g)$, provided that $\cos \lambda \pi = -1/5$:

$$W_{\lambda} = \exp \left( -i \frac{H_{\text{cpl}}}{\hbar} t \right) = \begin{pmatrix} \cos \frac{\lambda \pi}{4} & 0 & 0 & i \sin \frac{\lambda \pi}{4} \\ 0 & 1^{1/2} & i^{1/2} & 0 \\ 0 & i^{1/2} & 1^{1/2} & 0 \\ i \sin \frac{\lambda \pi}{4} & 0 & 0 & \cos \frac{\lambda \pi}{4} \end{pmatrix}. \hspace{1cm} (40)$$

For general $\lambda$ the gates in Eq. (40) describe a family with $m_1 = m_2 = \frac{1}{24}(5 + \cos \lambda \pi)$ between $1/6$ and $1/4$. 9
VI. CONCLUSIONS

We analyzed the process of partial randomized benchmarking with the focus on the case of testing a two-qubit quantum gate with averaging only over single-qubit rotations (in other words, with interleaving only with random single-qubit gates). We demonstrated that in this case, unlike for the standard randomized benchmarking, the decay of the fidelity as a function of the length of the gate sequence is not purely exponential, but is a combination of three exponential contributions with three different decay factors. These three exponents can be extracted from the experiment and provide information about the errors of the tested quantum gate.

To analyze these decay factors, we showed that the dynamics of the realization-averaged RB sequence as a function of its length may be described as linear and markovian with the use of a $3 \times 3$ iteration matrix $M$.

In the absence of errors, we found a complete description of the iteration matrix $M^0$. We expressed it in terms of the local invariants of the tested two-qubit gate. This allows one to efficiently find the matrix and its spectrum for a given gate, and thus to analyze partial-RB experiments.

It turns out that for generic gates only one of the three decay factors is close to 1 in absolute value, while the other two are smaller. As a consequence, already for not too long sequences only one exponential survives, and the experimental dependence of fidelity on the sequence length is just exponential to a high accuracy (we even found a family [40] of two-qubit gates, especially suited for partial RB, where the second and third decay factors vanish).

Furthermore, the decay factor of this exponential is very close to that, which one would obtain in a full RB experiment with complete averaging. Thus, a simplified partial RB provides the same information as the full-scale RB, which is harder to implement experimentally.

However, we found that there are corrections to this statement: while the slowest decay factor in partial RB coincides with the full-RB decay factor to the leading order in the size of the errors, there are second-order corrections, as we demonstrated.

Moreover, there exist exceptional quantum gates: for these gates, more than one of the three decay factors are close to one in absolute value, and hence the decay curve is not a simple exponential. Using the relation to the local invariants of the gates, we found and analyzed all the exceptional gates. These gates are the identity gate, the SWAP gate, and all the gates, locally equivalent to these two. Naturally, gates close to these also have similar properties.

Three decay factors for a generic gate, if measured, can be viewed as a fingerprint of the tested two-qubit gate. This fingerprint to a certain degree determines the gate, as we showed, up to local single-qubit operations. More precisely, this fingerprint determines the local invariants only up to the phase of the invariant $G_1$, and thus defines not a unique gate but a one-parameter family of gates (or rather, a family of local equivalence classes).

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