A Meshless method for the numerical solution of Coupled Drinfeld’s-Sokolov-Wilson System

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Abstract

This paper applies meshless method of lines, which uses radial basis functions (RBFs) as a spatial collocation scheme to solve the Coupled Drinfeld’s-Sokolov-Wilson System. Runge-Kutta method is used for time integration of the system of ODEs obtained as a result of spatial discretization in contrast to usual RBFs or finite difference methods. Accuracy ($L_2$ and $L_\infty$) is compared with the existing results from other methods available in the literature.

Keywords: Meshless RBFs, Coupled Drinfeld’s-Sokolov-Wilson System, Multiquadric (MQ), Gaussian (GA), Method of Lines.

1. Introduction

Nonlinear partial differential equations model many important physical, chemical and biological phenomena, and their use recently has spread into economics, finance, image processing, medicine and other fields. In order to investigate the prediction of these models, it is often necessary to approximate their solution numerically. Numerous numerical methods are available in the literature; the mesh based methods including, finite difference (FDM), finite element (FEM) and finite volume method (FVM), are most widely used, however, the construction of an appropriate mesh in arbitrary geometry is hard and expensive.

Consequently, to avoid the mesh generation, in recent years, meshless techniques have attracted the attention of researchers as alternatives to traditional finite element, finite volume and finite difference methods. In a meshless (meshfree) method a set of scattered nodes, with no connectivity information required among the set of points, is used instead of meshing the domain of the problem. Examples of some meshless schemes are the elementfree Galerkin method, the reproducing kernel particle, the local point interpolation, etc (e.g. see [? ] and references therein).

Over the last two decades, the radial basis functions method is known as a powerful tool for scattered data interpolation problems. The use of radial basis functions as a meshless procedure for numerical solution of PDEs is based on the collocation scheme. Due to the collocation technique, this method does not need to evaluate any integral. The main advantage of numerical procedures that use radial basis functions over traditional techniques is the meshless property of these methods. Radial basis functions are actively used for solving PDEs ([? ], and reference therein). In the above cited work, RBFs are used to replace the function and its spatial derivatives, while finite difference scheme is used to march in time. This method was first introduced by Kansa [? ] in 1990 for the numerical solutions of the PDEs. Kansa used the Multiquadric (MQ) RBF to solve the elliptic and parabolic PDEs. Recently, Flyer and Wright [? ] indicated RBFs allowed for a much lower spatial resolution, while being able to take unusually large time-steps to achieve the same accuracy compared to other methods.

In this work we use method of lines coupled with RBFs to study the numerical solution of Coupled Drinfeld’s-Sokolov-Wilson System:

\[ \begin{align*}
u_t + 3v v_x &= 0, \\
v_t + 2v_{xxx} + 2u v_x + u_x v &= 0,
\end{align*} \tag{1.1} \]

subject to following initial and boundary conditions

\[ u(x,0) = h(x), \quad v(x,0) = g(x), \quad a \leq x \leq b, \quad \tag{1.2} \]

\[ u(a,t) = h_1(t), \quad u(b,t) = h_2(t), \quad v(a,t) = g_1(t), \quad v(b,t) = g_2(t), \quad \tag{1.3} \]

inspired by a recent approach of meshless MOL [15]. We generalize this method to a system of coupled nonlinear PDEs.

The method of lines (MOL) [? ] is generally recognized as a comprehensive and powerful approach to the numerical solution of time-dependent PDEs. This method comprised of two steps: first, approximating the spatial derivatives; Second, then resulting system of semi-discrete ordinary differential equations (ODEs) is integrated in time. Hence the method of lines approximates the solution of PDEs using ODEs integrators. In this paper, we will use the radial basis functions combined with the MOL to solve the system (1.1) inspired by [15]. As it evident from our results that this method possesses high accuracy and ease of implementation. The computed results are compared with the analytic solutions and good agreement is indicated.

We find the numerical solution using radial basis functions (RBFs), particularly Hardy’s multiquadric (MQ) and Guassian
(GA), as a spatial approximation for $u$, $v$ and their derivatives in the coupled Drinfeld’s-Sokolov-Wilson Equation. This transforms the system of partial differential equations to a system of first order differential equations in time. The solution can be obtained using a high order time integration scheme like the fourth order Runge Kutta method. Numerical results indicate that this method offers a highly accurate approximate solution, and it is easy to implement. This method does not require the grid generation as in the finite difference method, and the computation domain is composed of scattered collocation points. Higher order derivatives of $u$ and $v$ can be computed using the derivatives of infinitely continuously differentiable radial basis functions.

This paper is organized as follows. In Section II we introduce the MOL combined with RBF method (MOL–RBF) for the coupled Drinfeld’s-Sokolov-Wilson Equation. Section III presents the implementation scheme and numerical experiments to demonstrate the effectiveness of the method proposed to solve the coupled Drinfeld’s-Sokolov-Wilson equation. Numerical results and comparison are given in Section IV and we conclude our work in Section V.

2. Formulation of MOL–RBF method

Radial basis function is a kind of function with the independent variable $r_i = r(x, x_i) = ||x - x_i||$. Some of the commonly used RBFs in the literature are:

- $\phi(x) = (c^2 + r_i^2)^{1/2}$, Multiquadrics (MQ),
- $\phi(x) = (c^2 + r_i^2)^{-1/2}$ Inverse Multiquadrics (IMQ),
- $\phi(x) = e^{-c r_i^2}$ Gaussian (GA),

where free parameter $c$ is called the shape parameter of RBF. In the above definition $x = (x, y)$ are the cartesian coordinates in $\Omega \subset \mathbb{R}^2$, and the radius is given by

$$r_j = ||x - x_j|| = \sqrt{((x - x_j)^2 + (y - y_j)^2)}^2,$$

where $(x_j, y_j)$ is called the $j^{th}$ source point of the RBF and is denoted by $x_j$. We choose $N$ nodes $(x_i, i = 1, 2, \ldots, N)$ in $\mathcal{P}_s \subseteq \Omega \cup \partial \Omega$. Any given smooth function can be represented as a linear combination of RBFs:

$$u^N(x) = \sum_{i=1}^{N} \lambda_i \phi_i = \Phi^T(x) \lambda,$$  \hspace{1cm} (2.1)

where

$$\Phi(x) = \begin{bmatrix} \phi_1(x), \phi_2(x), \ldots, \phi_N(x) \end{bmatrix}^T,$$

$$\lambda = \begin{bmatrix} \lambda_1, \lambda_2, \ldots, \lambda_N \end{bmatrix}^T.$$

Which can be written as

$$A\lambda = u,$$

where

$$u = [u_1, u_2, \ldots, u_N]^T,$$

and the matrix

$$A = \begin{bmatrix} \Phi^T(x_1) & \Phi^T(x_2) & \cdots & \Phi^T(x_N) \\ \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_N(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_N(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_1(x_N) & \phi_2(x_N) & \cdots & \phi_N(x_N) \end{bmatrix}$$

is called the interpolation matrix, consisting of functions forming the basis of the approximation space. It follows from Eq. (2.1) and $A\lambda = u$ that

$$u^N(x) = \Phi^T(x) A^{-1} u = V(x) u,$$

where

$$V(x) = \Phi^T(x) A^{-1}.$$

The convergence of RBF interpolation is given by the theorems in [13, 20]:

Assuming $\{x_i\}_{i=1}^N$ are $N$ source points in $\mathcal{P}_s$ which is convex, the radial distance is defined as

$$\delta := \delta(\Omega, \mathcal{P}_s) = \max_{x_i \in \Omega} \min_{j \leq N} ||x - x_i||_2,$$  \hspace{1cm} (2.2)

we have

$$\|u^N(x) - u(x)\| = O(\eta^{\delta}),$$  \hspace{1cm} (2.3)

where $0 < \eta < 1$ is a real number and $\eta = \exp(-\theta)$ with $\theta > 0$. From (2.3), it is clear that the parameter $c$ and radial distance $\delta$ affect the rate of convergence.

The exponential convergence proofs in applying RBFs in Sobolov space was given by Yoon [16], spectral convergence of the method in the limit of flat RBFs was given by Fornberg et al. [2]. The exponential convergence rate was verified numerically by Fedoseev et al. [17]. The exponential convergence cited above is limited to certain classes of functions that are smooth enough and well-behaved in the domain of approximation.

In 1971, Hardy [5] developed multi-quadric MQ to approximate two-dimensional geographical surfaces. In Franke’s [3] review paper, the MQ was rated one of the best methods among 29 scattered data interpolation schemes based on their accuracy, stability, efficiency, ease of implementation, and memory requirement. Further, the interpolation matrix for MQ is invertible. In 1990, since Kansa [7] modified the MQ for the solution of elliptic, parabolic and hyperbolic type PDEs, radial basis functions has been used to solve partial differential equations numerically [1, 8, 12, 13, 15]. The accuracy of MQ depends on the choice of a user defined parameter $c$ called the shape parameter that affects the shape of the RBFs. Golberg, Chen, and Karur [4] and Hickernell and Hon [14] applied the technique of cross validation to obtain an optimal value of the shape parameter $c$.

The non-singularity of the collocation matrix $A$ depends on the properties of RBFs used. According to [18], the matrix $A$ is conditionally positive definite for MQ radial basis functions. This fact guarantees the non-singularity of the matrix $A$ for distinct supporting points. This section includes details of the semi-discrete method, a method of line combined with the RBFs as in recent work [13, 5].
First, we approximate the space derivatives using radial basis functions. The domain \( \Omega \) is divided into collocation points \( x_i, i = 1, 2, \ldots, N \). Out of these points \( x_i, i = 1, N \) belongs to the boundary of \( \Omega \).

Let
\[
\begin{align*}
\phi^N(x_i) &= \sum_{j=1}^{N} \phi_j \phi(r_{ij}) = \phi(x)^T A_1, \\
\nu^N(x_i) &= \sum_{j=1}^{N} \phi_j \phi(r_{ij}) = \phi(x)^T A_2,
\end{align*}
\]

(2.4)

(2.5)

where \( r_{ij} = \|x_i - x_j\| \). In the matrix form
\[
u = A \Lambda_1, \quad \nu = A \Lambda_2,
\]

(2.6)

where \( u = [u_1, u_2, \ldots, u_N] \) and \( v = [v_1, v_2, \ldots, v_N] \) and
\[
A = \begin{bmatrix}
\phi^T(x_1) \\
\phi^T(x_2) \\
\vdots \\
\phi^T(x_N)
\end{bmatrix} = \begin{bmatrix}
\phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_N(x_1) \\
\phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_N(x_2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1(x_N) & \phi_2(x_N) & \cdots & \phi_N(x_N)
\end{bmatrix}
\]

Therefore, from (2.4) and (2.6), if the collocation matrix \( A \) is non-singular, it follows that
\[
\begin{align*}
u^N(x) &= \phi^T(r)A^{-1} - u = W(x)u, \\
u^N(x) &= \phi^T(r)A^{-1} - u = W(x)v,
\end{align*}
\]

(2.7)

where \( W(x) = \phi^T(r)A^{-1} \). The non-singularity of the collocation matrix \( A \) depends on the properties of RBFs used. According to [18], the matrix \( A \) is conditionally positive definite for MQ RBFs. This fact guarantees the non-singularity of the matrix \( A \) for distinct supporting points.

Applying (2.7) on (1.1), and collocation on each node \( x_i \), we obtain
\[
\begin{align*}
du_i/dt + 3v_i(W_i(x)v) &= 0, \\
dv_i/dt + 2(W_i(x)v) + 2u(W_i(x)v) + (W_i(x)u)v &= 0,
\end{align*}
\]

(2.8)

(2.9)

where \( u_i(t) = u_i, \quad v_i(t) = v_i \) and
\[
\begin{align*}
W_i(x_i) &= [W_{1i}(x_i) \ W_{1i}(x_i) \ \ldots \ W_{1i}(x_i)], \\
W_{ij}(x_i) &= \frac{\partial}{\partial x_j} W_{ij}(x_i), \\
W_{iij}(x_i) &= [W_{iij}(x_i) \ \ldots \ W_{iij}(x_i)], \\
W_{iij}(x_i) &= \frac{\partial^2}{\partial x^2} W_{ij}(x_i),
\end{align*}
\]

for \( j = 1, 2, \ldots, N \). Then we rewrite the system (2.3) as
\[
\begin{align*}
dU/dt + 3V*(W_i V) &= 0, \\
dV/dt + 2(W_{xx}V) + 2U*(W_i V) + (W_i U)*V &= 0,
\end{align*}
\]

(2.10)

(2.11)

where \( * \) shows the component by component multiplication. Equivalently, this system can be written as
\[
\begin{align*}
dU/dt = F_1(U), \\
dV/dt = F_2(V),
\end{align*}
\]

(2.12)

where
\[
F_1(U) = -3V*(W_i V)
\]
and
\[
F_2(V) = -2(W_{xx}V) + 2U*(W_i V) + (W_i U)*V.
\]

The initial and boundary conditions translate to:
\[
\begin{align*}
U(t_0) &= [u_i^0(x_1), u_i^0(x_2), \ldots, u_i^0(x_N)]^T, \\
V(t_0) &= [v_i^0(x_1), v_i^0(x_2), \ldots, v_i^0(x_N)]^T.
\end{align*}
\]

(2.13)

(2.14)

(2.15)

To solve Eq. (2.12–2.15), the fourth order Runge-Kutta scheme is applied as:
\[
\begin{align*}
K_{11} &= F_1(U^n, t_n), \\
K_{12} &= F_1(U^n + \Delta t K_{11}), \\
K_{13} &= F_1(U^n + \Delta t K_{12}), \\
K_{14} &= F_1(U^n + \Delta t K_{13}),
\end{align*}
\]

(2.16)

\[
\begin{align*}
K_{21} &= F_2(V^n, t_n), \\
K_{22} &= F_2(V^n + \Delta t K_{21}), \\
K_{23} &= F_2(V^n + \Delta t K_{22}), \\
K_{24} &= F_2(V^n + \Delta t K_{23}).
\end{align*}
\]

(2.17)

\[
\begin{align*}
U^{n+1} &= U^n + \frac{\Delta t}{6}(K_{11} + 2K_{12} + 2K_{13} + K_{14}), \\
V^{n+1} &= V^n + \frac{\Delta t}{6}(K_{21} + 2K_{22} + 2K_{23} + K_{24}).
\end{align*}
\]

(2.18)

Then, with the proper initial and boundary conditions, the computation can be carried out step by step. In the next section we implement our scheme to compute the solution of two examples of nonlinear coupled partial differential equations.

3. Numerical Experiments

In this section we apply the MOL-RBF method to the Coupled Drinfeld’s-Sokolov-Wilson equations and compare our results with other work in literature.

Example . Consider the Coupled Drinfeld’s-Sokolov-Wilson Equation with the following initial
\[
\begin{align*}
u(x, 0) &= (c - 4k)/2 + 3k^2 sech^2(kx), \\
u(x, 0) &= 2k \sqrt{c}/2 sech(kx),
\end{align*}
\]

and boundary conditions
\[
\begin{align*}
u(a, t) &= (c - 4k)/2 + 3k^2 sech^2(k(a - ct)), \\
u(b, t) &= (c - 4k)/2 + 3k^2 sech^2(k(b - ct)), \\
u(a, t) &= 2k \sqrt{c}/2 sech(k(a - ct)), \\
u(b, t) &= 2k \sqrt{c}/2 sech(k(b - ct)).
\end{align*}
\]

(2.19)

The analytical solution [13] is
\[
\begin{align*}
u(x, t) &= (c - 4k)/2 + 3k^2 sech^2(k(x - ct)), \\
u(x, t) &= 2k \sqrt{c}/2 sech(k(x - ct))
\end{align*}
\]

(2.20)
Numerical simulations are carried out on the interval \([-4, 4]\) from \(t = 0\) to \(t = 0.5\) with step size \(h = 0.1\) and time step \(0.0001\), using Gaussian, Multiquadric, and Inverse Multi-quadric RBFs. Initial and boundary conditions are obtained from the exact solution in [14]. Table 1–4 shows the error for different values of \(c\) and \(k\) for GA and MQ.

4. Results and discussion

We have applied meshless MoL to solve the coupled Drinfeld’s-Sokolov-Wilson system using MoL-RBF and results are compared in Tables 1–4 for different values of \(c\) and \(k\). Accuracy was compared in terms of \(L_{\infty}\) and RMS errors. As seen in Table 1 GA has accuracy higher than MQ. Since the accuracy depends upon the number of nodes and the value of the shape parameter \(c\) and we have obtained these values numerically by observing condition number of the matrix \(A\) and the error by keeping the number of nodes fixed.

5. Conclusion

In this work we constructed a MoL scheme combined with RBFs to solve system of two nonlinear partial differential equations. This system has been solved in the literature using the Variational Iteration Method (VIM), Homotopy Analysis Method (HAM) and Homotopy Perturbation Methods (HPM). We have compared our scheme with the exact solutions taken from the cited literature.
GA 0.3 1

property of RBFs, as the mesh generation is expensive for many

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Table 1: Error comparison $a(x, t)$ with the exact solution for $c = 1, k = 0.001$

Table 2: Error comparison for Drenfeld system with the exact solution for $c = 4, k = 0.01$

Although the optimal value of shape parameter is open prob-

Further investigations can be carried out for larger number of nodes, optimal value of shape parameter, using adaptive step

Runge-Kutta method or other multistep time integrators. Recent-

lysome work has been done on adaptive RBFs methods.

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Table 3: $v(x,t)$ Error comparison for Drenfield system with the exact solution for $c = 0.1, k = 0.001$

| $/c$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|------|-----|-----|-----|-----|-----|
| MQ   | 0.2 | 0.3637 $\times 10^{-13}$ | 1.3637 $\times 10^{-13}$ | 4.3637 $\times 10^{-13}$ | 1.4363 $\times 10^{-13}$ | 4.4363 $\times 10^{-13}$ |
| C = 0.0001 | 0.5 | 2.0827 $\times 10^{-13}$ | 6.0827 $\times 10^{-13}$ | 1.6083 $\times 10^{-13}$ | 4.3083 $\times 10^{-13}$ | 1.4308 $\times 10^{-13}$ |
| C = 10^{-7} | 0.5 | 3.1415 $\times 10^{-12}$ | 9.1415 $\times 10^{-12}$ | 2.7143 $\times 10^{-12}$ | 6.7143 $\times 10^{-12}$ | 1.6714 $\times 10^{-12}$ |

Table 4: $v(x,t)$ Error comparison for Drenfield system with the exact solution for $c = 0.0001, k = 0.01$

| $/c$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|------|-----|-----|-----|-----|-----|
| MQ   | 0.2 | 0.3615 $\times 10^{-13}$ | 1.3615 $\times 10^{-13}$ | 4.3615 $\times 10^{-13}$ | 1.4361 $\times 10^{-13}$ | 4.4361 $\times 10^{-13}$ |
| C = 0.0001 | 0.5 | 2.0827 $\times 10^{-13}$ | 6.0827 $\times 10^{-13}$ | 1.6083 $\times 10^{-13}$ | 4.3083 $\times 10^{-13}$ | 1.4308 $\times 10^{-13}$ |
| C = 10^{-7} | 0.5 | 3.1415 $\times 10^{-12}$ | 9.1415 $\times 10^{-12}$ | 2.7143 $\times 10^{-12}$ | 6.7143 $\times 10^{-12}$ | 1.6714 $\times 10^{-12}$ |

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Figure 3: $c = 0.1$

Figure 4: $c = 0.0001$

Figure 5: $c = 1$

Figure 6: $c = 4$
Figure 7: $u(x,t)$ with GA

Figure 8: $v(x,t)$ with GA

Figure 9: $u(x,t)$ with MQ

Figure 10: $v(x,t)$ with MQ
Figure 11: \( u(x, t) \) with IMQ

Figure 12: \( v(x, t) \) with IMQ

Figure 13: Error vs. Shape Parameter

Figure 14: Error in \( u \) with GA
Figure 15: Optimal Shape for GA, $k = 0.01$, $c = 0.0001$

Figure 16: Optimal Shape for MQ, $k = 0.01$, $c = 0.0001$
Max error at t=0.5

Shape Parameter
Max error at $t=3$
Shape Parameter

Max error at t=0.5
Max error at \( t=3 \)
MeshFree and exact solution
MoL–RBF Solution for $u(x,t)$
MoL–RBF Solution for $v(x,t)$
MoL–RBF Solution for $u(x,t)$
MeshFree and exact solution

\[ u(x,t) \]

- MeshFree Solution
- Exact Solution
MeshFree and exact solution

\[ v(x, t) \]

\( x \times 10^{-3} \)

MeshFree Solution
Exact Solution