On volume functions of special flow polytopes

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Abstract

In this paper, we consider the volume of a special kind of flow polytope. We show that its volume satisfies a certain system of differential equations, and conversely, the solution of the system of differential equations is unique up to a constant multiple. In addition, we give an inductive formula for the volume with respect to the rank of the root system of type A.

1 Introduction

The number of lattice points and the volume of a convex polytope are important and interesting objects and have been studied from various points of view (see, e.g., [4]). For example, the number of lattice points of a convex polytope associated to a root system is called the Kostant partition function, and it plays an important role in representation theory of Lie groups (see, e.g., [7]).

In this paper, we consider a convex polytope associated to the root system of type $A$, which is called a flow polytope. As explained in [2, 3], the cone spanned by the positive roots is divided into several polyhedral cones called chambers, and the combinatorial property of a flow polytope depends on a chamber. Moreover, there is a specific chamber called the nice chamber, which plays a significant role in [9]. Also in [2, 3], a number of theoretical results related to the Kostant partition function and the volume function of a flow polytope can be found. In particular, it is shown that these functions for the nice chamber are written as iterated residues ([3, Lemma 21]). We also refer to [1] for similar formulas for other chambers in more general settings.

The purpose of this paper is to characterize the volume function of a flow polytope for the nice chamber in terms of a system of differential equations, based on a result in [3]. In order to state the main results, we give some notation. Let $e_i$ be the standard basis of $\mathbb{R}^{r+1}$ and let

$$A_+^r = \{e_i - e_j \mid 1 \leq i < j \leq r+1\}$$

be the positive root system of type $A$ with rank $r$. We assign a positive integer $m_{i,j}$ to each $i$ and $j$ with $1 \leq i < j \leq r+1$. Let us set $m = (m_{i,j})$ and $M = \sum_{1 \leq i < j \leq r+1} m_{i,j}$. For $a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \in \mathbb{R}^{r+1}$, where $a_i \in \mathbb{R}_{\geq 0} \ (i = 1, \ldots, r)$, the following polytope $P_{A^r_+,m}(a)$ is called the flow polytope:

$$P_{A^r_+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^M \mid 1 \leq i < j \leq r+1, \ 1 \leq k \leq m_{i,j}, \ y_{i,j,k} \geq 0, \ \sum_{1 \leq i < j \leq r+1} \sum_{1 \leq k \leq m_{i,j}} y_{i,j,k} (e_i - e_j) = a \right\}.$$
Note that the flow polytopes in \([3]\) include the case that some of \(m_{i,j}\)'s are zero, whereas we exclude such cases in this paper. We denote the volume of \(P_{A^r,m}(a)\) by \(v_{A^r,m}(a)\).

The open set
\[
\mathfrak{c}_{\text{nice}} := \{ a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \in \mathbb{R}^{r+1} \mid a_i > 0, i = 1, \ldots, r \}
\]
in \(\mathbb{R}^{r+1}\) is called the nice chamber. We are interested in the volume \(v_{A^r,m}(a)\) when \(a\) is in the closure of the nice chamber, and then it is written by \(v_{A^r,m,\mathfrak{c}_{\text{nice}}} (a)\). It is a homogeneous polynomial of degree \(M - r\). The first result of this paper is the following.

**Theorem 1.1** Let \(a = \sum_{i=1}^{r} a_i (e_i - e_{r+1}) \in \mathfrak{c}_{\text{nice}}\), and let \(v_{A^r,m,\mathfrak{c}_{\text{nice}}} (a)\) be the volume of \(P_{A^r,m}(a)\). Then \(v = v_{A^r,m,\mathfrak{c}_{\text{nice}}} (a)\) satisfies the system of differential equations as follows:

\[
\begin{align*}
\partial_{r+1}^m v &= 0 \\
(\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^m v &= 0 \\
& \vdots \\
(\partial_1 - \partial_{r-1})^{m_{1,r-1}} (\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_{r-1}^{m_{r-1,r-1}} v &= 0,
\end{align*}
\]

where \(\partial_i = \frac{\partial}{\partial a_i}\) for \(i = 1, \ldots, r\). Conversely, the polynomial \(v = v(a)\) of degree \(M - r\) satisfying the above equations is equal to a constant multiple of \(v_{A^r,m,\mathfrak{c}_{\text{nice}}} (a)\).

We remark that it is known the volume function \(v_{A^r,m}(a)\) of \(P_{A^r,m}(a)\), as a distribution on \(\mathbb{R}^r\), satisfies the differential equation

\[
Lv_{A^r,m}(a) = \delta (a) \tag{1.1}
\]
in general, where \(L = \prod_{i<j} (\partial_i - \partial_j)^{m_{i,j}}\) and \(\delta (a)\) is the Dirac delta function on \(\mathbb{R}^r\) \([6, 9]\). Note that \(\partial_{r+1}\) in the definition of \(L\) is supposed to be zero. The above theorem characterizes the function \(v_{A^r,m,\mathfrak{c}_{\text{nice}}} (a)\) on \(\mathfrak{c}_{\text{nice}}\) more explicitly.

In addition, in Theorem 3.6, we show the volume \(v_{A^r,m,\mathfrak{c}_{\text{nice}}} (a)\) is written by a linear combination of \(v_{A_{r-1}^{r+1},m',\mathfrak{c}_{\text{nice}}'} (a)\) and its partial derivatives, where \(m' = (m_{i,j})_{2 \leq i < j \leq r+1}\) and \(\mathfrak{c}_{\text{nice}}'\) is the nice chamber of \(A_{r-1}^{r+1}\).

This paper is organized as follows. In Section 2, we recall the iterated residue, the Jeffrey-Kirwan residue, and the nice chamber based on \([2, 3, 5, 8]\). Also, we give the some examples of \(P_{A^r,m}(a)\) and the calculations of the volume \(v_{A^r,m,\mathfrak{c}_{\text{nice}}} (a)\). In Section 3, we prove the main theorems.

## 2 Preliminaries

In this section, we set up the tools to prove the main theorems based on \([2, 3, 5, 8]\).
2.1 Flow polytopes and its volumes

Let \( e_1, \ldots, e_{r+1} \) be the standard basis of \( \mathbb{R}^{r+1} \), and let

\[
V = \left\{ a = \sum_{i=1}^{r+1} a_i e_i \in \mathbb{R}^{r+1} \mid \sum_{i=1}^{r+1} a_i = 0 \right\}.
\]

We consider the positive root system of type \( A \) with rank \( r \) as follows:

\[
A_r^+ = \{ e_i - e_j \mid 1 \leq i < j \leq r + 1 \}.
\]

Let \( C(A_r^+) \) be the convex cone generated by \( A_r^+ \):

\[
C(A_r^+) = \{ a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \mid a_1, \ldots, a_r \in \mathbb{R}_{\geq 0} \}.
\]

We assign a positive integer \( m_{i,j} \) to each \( i \) and \( j \) with \( 1 \leq i < j \leq r + 1 \), and it is called a multiplicity. Let us set \( m = (m_{i,j}) \) and \( M = \sum_{1 \leq i < j \leq r+1} m_{i,j} \).

**Definition 2.1** Let \( a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \in C(A_r^+) \). We consider the following polytope:

\[
P_{A_r^+, m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^M \mid 1 \leq i < j \leq r + 1, 1 \leq k \leq m_{i,j},
\right.
\]

\[
y_{i,j,k} \geq 0, \quad \sum_{1 \leq i < j \leq r+1} \sum_{1 \leq k \leq m_{i,j}} y_{i,j,k} (e_i - e_j) = a \right\},
\]

which is called the flow polytope.

**Remark 2.2** The flow polytopes in [3] include the case that \( m_{i,j} = 0 \) for some \( i \) and \( j \).

The elements of \( A_r^+ \) generate a lattice \( V_Z \) in \( V \). The lattice \( V_Z \) determines a measure \( da \) on \( V \).

Let \( du \) be the Lebesgue measure on \( \mathbb{R}^M \). Let \( [\alpha_1, \ldots, \alpha_M] \) be a sequence of elements of \( A_r^+ \) with multiplicity \( m_{i,j} \), and let \( \varphi \) be the surjective linear map from \( \mathbb{R}^M \) to \( V \) defined by \( \varphi(e_k) = \alpha_k \). The vector space \( \ker(\varphi) = \varphi^{-1}(0) \) is of dimension \( d = M - r \) and it is equipped with the quotient Lebesgue measure \( du/da \). For \( a \in V \), the affine space \( \varphi^{-1}(a) \) is parallel to \( \ker(\varphi) \), and thus also equipped with the Lebesgue measure \( du/da \). Volumes of subsets of \( \varphi^{-1}(a) \) are computed for this measure. In particular, we can consider the volume \( v_{A_r^+, m}(a) \) of the polytope \( P_{A_r^+, m}(a) \).

2.2 Total residue and iterated residue

Let \( A_r = A_r^+ \cup (-A_r^+) \), and let \( U \) be the dual vector space of \( V \). We denote by \( R_{A_r} \) the ring of rational functions \( f(x_1, \ldots, x_r) \) on the complexification \( U_C \) of \( U \) with poles on the hyperplanes \( x_i - x_j = 0 \) (\( 1 \leq i < j \leq r + 1 \)) or \( x_i = 0 \) (\( 1 \leq i \leq r \)). A subset \( \sigma \) of \( A_r \) is called a basis of \( A_r \) if the elements \( \alpha \in \sigma \) form a basis of \( V \). In this case, we set

\[
f_{\sigma}(x) := \frac{1}{\prod_{\alpha \in \sigma} \alpha(x)}
\]
and call such an element a *simple fraction*. We denote by $S_{A_r}$ the linear subspace of $R_{A_r}$ spanned by simple fractions. The space $U$ acts on $R_{A_r}$ by differentiation: $(\partial(u)f)(x) = (\frac{d}{dx})f(x + \varepsilon u)|_{\varepsilon = 0}$. We denote by $\partial(U)R_{A_r}$ the space spanned by derivatives of functions in $R_{A_r}$. It is shown in [3, Proposition 7] that

$$R_{A_r} = \partial(U)R_{A_r} \oplus S_{A_r}.$$  

The projection map $\text{Tres}_{A_r} : R_{A_r} \to S_{A_r}$ with respect to this decomposition is called the total residue map.

We extend the definition of the total residue to the space $\hat{R}_{A_r}$ consisting of functions $P/Q$ where $Q$ is a finite product of powers of the linear forms $\alpha \in A_r$ and $P = \sum_{k=0}^{\infty} P_k$ is a formal power series with $P_k$ of degree $k$. As the total residue vanishes outside the homogeneous component of degree $-r$ of $A_r$, we can define $\text{Tres}_{A_r}(P/Q) = \text{Tres}_{A_r}(P_{q-r}/Q)$, where $q$ is degree of $Q$. For $a \in V$ and multiplicities $m = (m_{i,j}) \in (\mathbb{Z}_{\geq 0})^M$ of elements of $A_r^+$, the function

$$F := \frac{e^{a_1x_1 + \cdots + a_rx_r}}{\prod_{i=1}^{r} x_i^{m_{i,r}+1} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}}$$

is in $\hat{R}_{A_r}$. We define $J_{A_r^+,m}(a) \in S_{A_r}$ by

$$J_{A_r^+,m}(a) = \text{Tres}_{A_r}F.$$

Next, we describe the iterated residue.

**Definition 2.3** For $f \in R_{A_r}$, we define the iterated residue by

$$\text{Ires}_{x=0}f = \text{Res}_{x_1=0}\text{Res}_{x_2=0} \cdots \text{Res}_{x_r=0}f(x_1, \ldots, x_r).$$

Since the iterated residue $\text{Ires}_{x=0}f$ vanishes on the space $\partial(U)R_{A_r}$ as in [3], we have

$$\text{Ires}_{x=0}J_{A_r^+,m}(a) = \text{Ires}_{x=0}F. \quad (2.1)$$

### 2.3 Chambers and Jeffrey–Kirwan residue

**Definition 2.4** Let $C(\nu)$ be the closed cone generated by $\nu$ for any subset $\nu$ of $A_r^+$ and let $C(A_r^+)_{\text{sing}}$ be the union of the cones $C(\nu)$ where $\nu$ is any subset of $A_r^+$ of cardinal strictly less than $r = \dim V$. By definition, the set $C(A_r^+)_{\text{reg}}$ of $A_r^+$-regular elements is the complement of $C(A_r^+)_{\text{sing}}$. A connected component of $C(A_r^+)_{\text{reg}}$ is called a *chamber*.

The Jeffrey–Kirwan residue [8] associated to a chamber $c$ of $C(A_r^+)$ is a linear form $f \mapsto \langle\langle c, f \rangle\rangle$ on the vector space $S_{A_r}$ of simple fractions. Any function $f$ in $S_{A_r}$ can be written as a linear combination of functions $f_\sigma$, with a basis $\sigma$ of $A_r$ contained in $A_r^+$. To determine the linear map $f \mapsto \langle\langle c, f \rangle\rangle$, it is enough to determine it on this set of functions $f_\sigma$. So we assume that $\sigma$ is a basis of $A_r$ contained in $A_r^+$.
Definition 2.5 For a chamber $c$ and $f_\sigma \in S_{A_r}$, we define the Jeffrey–Kirwan residue $\langle \langle c, f_\sigma \rangle \rangle$ associated to a chamber $c$ as follows:

- If $c \subset C(\sigma)$, then $\langle \langle c, f_\sigma \rangle \rangle = 1$.
- If $c \cap C(\sigma) = \emptyset$, then $\langle \langle c, f_\sigma \rangle \rangle = 0$,

where $C(\sigma)$ is the convex cone generated by $\sigma$.

Remark 2.6 More generally, as in [3, Definition 11], the Jeffrey–Kirwan residue $\langle \langle c, f_\sigma \rangle \rangle$ is defined to be $\frac{1}{\text{vol}(\sigma)}$ if $c \subset C(\sigma)$, where vol$(\sigma)$ is the volume of the parallelepiped $\oplus_{\alpha \in \sigma}[0, 1] \alpha$, relative to our Lebesgue measure $da$. In our case, the volume vol$(\sigma)$ is equal to 1 since $A_r$ is unimodular.

The volume $v_{A_r^+, m}(a)$ of the flow polytope $P_{A_r^+, m}(a)$ is written by the function $J_{A_r^+, m}(a)$ and the Jeffrey–Kirwan residue in the following.

**Theorem 2.7 ([3])** Let $c$ be a chamber of $C(A_r^+)$. Then, for $a \in \overline{c}$, the volume $v_{A_r^+, m}(a)$ of $P_{A_r^+, m}(a)$ is given by

$$v_{A_r^+, m}(a) = \langle \langle c, J_{A_r^+, m}(a) \rangle \rangle.$$ 

We denote by $v_{A_r^+, m, c}(a)$ the polynomial function of $a$ coinciding with $v_{A_r^+, m}(a)$ when $a \in \overline{c}$. It is a homogeneous polynomial of degree $M - r$.

**2.4 Nice chamber**

Definition 2.8 The open subset $\mathfrak{c}_{\text{nice}}$ of $C(A_r^+)$ is defined by

$$\mathfrak{c}_{\text{nice}} = \{a \in C(A_r^+) \mid a_i > 0 \ (i = 1, \ldots, r)\}.$$ 

The set $\mathfrak{c}_{\text{nice}}$ is in fact a chamber for the root system $A_r^+$ ([3]). The chamber $\mathfrak{c}_{\text{nice}}$ is called the nice chamber.

**Lemma 2.9 ([3])** For the nice chamber $\mathfrak{c}_{\text{nice}}$ of $A_r^+$ and $f \in S_{A_r}$, we have

$$\langle \langle \mathfrak{c}_{\text{nice}}, f \rangle \rangle = \text{Ires}_{x=0} f.$$ 

From Theorem 2.7, Lemma 2.9 and (2.1), we have the following corollary.

**Corollary 2.10** Let $a \in \overline{\mathfrak{c}_{\text{nice}}}$. Then the volume function $v_{A_r^+, m, \mathfrak{c}_{\text{nice}}}(a)$ is given by

$$v_{A_r^+, m, \mathfrak{c}_{\text{nice}}}(a) = \text{Ires}_{x=0} F.$$
2.5 Examples

In this subsection, we give some examples of the flow polytopes for $A_1$, $A_2$, and $A_3$, and calculate their volumes.

Example 2.11 When $r = 1$, the nice chamber of $A_1^+$ is $c_{\text{nice}} = \{a = a_1(e_1 - e_2) \mid a_1 > 0\}$.
For $a = a_1(e_1 - e_2) \in c_{\text{nice}}$,
\[ P_{A_1^+,m}(a) = \{(y_{i,j,k}) \in \mathbb{R}^{m_{1,2}} \mid y_{i,j,k} \geq 0, \ y_{1,1,1} + y_{1,2,2} + \cdots + y_{1,2,m_{1,2}} = a_1\} \]
From Corollary 2.10, we have
\[ v_{A_1^+,m,c_{\text{nice}}}(a) = \text{Res}_{x_1=0} \left( \frac{e^{a_1 x_1}}{x_1^{m_{1,2}}} \right) \]
\[ = \frac{1}{(m_{1,2} - 1)!} a_1^{m_{1,2}-1} \]

Example 2.12 When $r = 2$, there are two chambers $c_1, c_2$ of $A_2^+$ as below, and the nice chamber $c_{\text{nice}}$ of $A_2^+$ is $c_1$.

![Diagram of the chamber of $A_2^+$](image)

For example, we set $m_{1,2} = n$ ($n \in \mathbb{Z}_{>0}$), $m_{1,3} = 1$, and $m_{2,3} = 1$. For $a = a_1 e_1 + a_2 e_2 - (a_1 + a_2)e_3 \in c_{\text{nice}}$,
\[ P_{A_2^+,m}(a) = \{(y_{i,j,k}) \in \mathbb{R}^{n+2} \mid y_{i,j,k} \geq 0, \ y_{1,1,1} + y_{1,2,2} + \cdots + y_{1,2,n} + y_{1,3,1} = a_1 \]
\[ -y_{1,2,1} - y_{1,2,2} - \cdots - y_{1,2,n} + y_{2,3,1} = a_2 \} \]
From Corollary 2.10, we have
\[ v_{A_2^+,m,c_{\text{nice}}}(a) = \text{Ires}_{x_1=0} \left( \frac{e^{a_1 x_1 + a_2 x_2}}{x_1 x_2 (x_1 - x_2)^n} \right) \]
\[ = \text{Res}_{x_1=0} \text{Res}_{x_2=0} \left( \frac{e^{a_1 x_1 + a_2 x_2}}{x_1 x_2 (x_1 - x_2)^n} \right) \]
\[ = \frac{1}{n!} a_1^n \]
Example 2.13 When \( r = 3 \), there are seven chambers of \( A_3^+ \) as below (II), and the nice chamber \( c_{\text{nice}} \) of \( A_3^+ \) is \( c_1 \).

For example, we set \( m_{1,2} = 1, m_{1,3} = 1, m_{1,4} = 2, m_{2,3} = 1, m_{2,4} = 2, \) and \( m_{3,4} = 2 \).

For \( a = \sum_{i=1}^{3} a_i(e_i - e_4) \in c_{\text{nice}} \),

\[
P_{A_3^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^9 \mid \begin{array}{l}
y_{i,j,k} \geq 0, \\
y_{1,2,1} + y_{1,3,1} + y_{1,4,1} + y_{1,4,2} = a_1, \\
y_{1,2,1} - y_{2,3,1} + y_{2,4,1} + y_{2,4,2} = a_2, \\
y_{3,1} - y_{2,3,1} + y_{3,4,1} + y_{3,4,2} = a_3, \\
y_{i,j,k} \geq 0, \\
y_{1,2,1} + y_{1,3,1} + y_{1,4,1} + y_{1,4,2} = a_1, \\
y_{1,2,1} - y_{2,3,1} + y_{2,4,1} + y_{2,4,2} = a_2, \\
y_{3,1} - y_{2,3,1} + y_{3,4,1} + y_{3,4,2} = a_3, \\
y_{i,j,k} \geq 0,
\end{array} \right\}
\]

From Corollary 2.10, we have

\[
v_{A_3^+,m,c_{\text{nice}}}(a) = \text{Ires}_{x=0} \left( \frac{e^{a_1 x_1 + a_2 x_2 + a_3 x_3}}{x_1^2 x_2^2 x_3^2 (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} \right)
= \frac{1}{360} a_1^3 (a_1^3 + 6a_1^2 a_2 + 3a_1^2 a_3 + 15a_1 a_2^2 + 15a_1 a_2 a_3 + 10a_2^3 + 30a_2^2 a_3).
\]

3 Main theorems

In this section, we prove the main theorems of this paper. Let \( c_{\text{nice}} \) be the nice chamber of \( A_r^+ \) and let \( a = \sum_{i=1}^{r} a_i(e_i - e_{r+1}) \in c_{\text{nice}} \).

Theorem 3.1 For \( a \in c_{\text{nice}} \), let \( P_{A_r^+,m}(a) \) be the flow polytope as in Definition 2.1 and let \( v_{A_r^+,m,c_{\text{nice}}}(a) \) be the volume of \( P_{A_r^+,m}(a) \). Then \( v = v_{A_r^+,m,c_{\text{nice}}}(a) \) satisfies the system of
differential equations as follows:

\[
\begin{align*}
\partial_r^{m_r,r+1} v &= 0 \\
(\partial_{r-1} - \partial_r)^{m_{r-1},r} \partial_{r-1}^{m_{r-1},r+1} v &= 0 \\
&\vdots \\
(\partial_1 - \partial_r)^{m_{1,2}}(\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} v &= 0,
\end{align*}
\]

(3.1)

where \( \partial_i = \frac{\partial}{\partial a_i} \) for \( i = 1, \ldots, r \).

Proof. Let \( F = \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{\prod_{i=1}^r x_i^{m_i,r+1} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \). It is easy to see that

\[
P(\partial_1, \ldots, \partial_r)(I_{x=0} F) = I_{x=0}(P(\partial_1, \ldots, \partial_r) F) = I_{x=0}(P(x_1, \ldots, x_r) F)
\]

where \( P \) is a polynomial. Therefore, from Corollary 2.10, we obtain

\[
\partial_r^{m_r,r+1} v = \partial_r^{m_r,r+1} I_{x=0} F = I_{x=0}(\partial_r^{m_r,r+1} F)
\]

\[
= I_{x=0} \left( \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{\prod_{i=1}^r x_i^{m_i,r+1} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \right)
\]

\[
= 0,
\]

and

\[
(\partial_{r-1} - \partial_r)^{m_{r-1},r} \partial_{r-1}^{m_{r-1},r+1} v
\]

\[
= I_{x=0}(\partial_{r-1} - \partial_r)^{m_{r-1},r} \partial_{r-1}^{m_{r-1},r+1} F
\]

\[
= I_{x=0}(\partial_{r-1} - \partial_r)^{m_{r-1},r} \left( \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{x_r^{m_r,r+1} \prod_{i=1}^{r-2} x_i^{m_i,r+1} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \right)
\]

\[
= I_{x=0} \left( \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{x_r^{m_r,r+1} \prod_{i=1}^{r-2} x_i^{m_i,r+1} \prod_{1 \leq i < j \leq r} (x_i - x_{i+1})^{m_{i,j}}} \right)
\]

\[
= \text{Res}_{x_1} \cdots \text{Res}_{x_{r-1}} \left( \frac{e^{a_r x_r}}{x_r^{m_r,r+1} \prod_{i=1}^{r-2} (x_i - x_r)^{m_{i,r}}} \right)
\]

\[
= 0,
\]

where we used

\[
\text{Res}_{x_k} \left( \frac{e^{a_1 x_1 + \cdots + a_k x_k}}{\prod_{i=1}^{k-1} x_i^{m_i,k+1} \prod_{1 \leq i < j \leq k} (x_i - x_j)^{m_{i,j}}} \right) = 0
\]

for \( k = 1, \ldots, r \). Similarly, we can check the left expressions. □
Remark 3.2 In general, it is known that the volume function $v_{A^r,m}(a)$ of $P_{A^r,m}(a)$, as a distribution on $V$, satisfies the differential equation

$$Lv_{A^r,m}(a) = \delta(a),$$

where $L = \prod_{i<j}(\partial_i - \partial_j)^{m_{i,j}}$ and $\delta(a)$ is the Dirac delta function on $V$ (see [10]). Note that $\partial_{r+1}$ in the definition of $L$ is supposed to be zero. The above theorem, together with Proposition 3.3 and Theorem 3.4 as below, characterizes the function $v_{A^r,m,c_{\text{nice}}}(a)$ on $\mathfrak{c}_{\text{nice}}$ more explicitly.

Let $M_l = \sum_{i=l+1}^{r+1} m_{i,l}$ ($l = 1, \ldots, r$). Then we have the following proposition.

**Proposition 3.3** The coefficient of $a_1^{M_1-1}a_2^{M_2-1}\cdots a_{r-1}^{M_{r-1}-1}a_r^{M_r-1}$ in the volume function $v_{A^r,m}(a)$ is given by

$$\frac{1}{(M_1-1)!(M_2-1)!\cdots(M_{r-1}-1)!(M_r-1)!}.$$

**Proof.** From Corollary 2.10, we have

$$v_{A^r,m,c_{\text{nice}}}(a) = \sum_{|i|=l-r} a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}\text{Res}_{x=0} \left( \frac{x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r}}{\prod_{i=1}^{l} x_i^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \right),$$

where $|i| = i_1 + \cdots + i_r$. When $i_l = M_l - 1$ for $l = 1, \ldots, r$,

$$\text{Res}_{x=0} \left( \frac{x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r}}{\prod_{i=1}^{l} x_i^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \right) = \text{Res}_{x_1=0} \cdots \text{Res}_{x_{r-1}=0} \text{Res}_{x_r=0} \left( \frac{x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r}}{\prod_{i=1}^{l} x_i^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \right)$$

$$= \text{Res}_{x_1=0} \cdots \text{Res}_{x_{r-1}=0} \left( \frac{x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r}}{\prod_{i=1}^{l} x_i^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \right)$$

$$= \text{Res}_{x_1=0} \frac{1}{x_1} = 1.$$ 

Thus we obtain the proposition. $\Box$

**Theorem 3.4** Let $\phi_r = \phi(a_1, \ldots, a_r)$ be a homogeneous polynomial of $a_1, \ldots, a_r$ with degree $d$ and let $M = \sum_{1 \leq i < j \leq r+1} m_{i,j}$. Suppose $\phi_r$ satisfies the system of differential equations as follows:

$$\begin{cases} 
\partial_r^{m_{r,r+1}}\phi_r = 0 \\
(\partial_{r-1} - \partial_r)^{m_{r-1,r}}\partial_r^{m_{r-1,r+1}}\phi_r = 0 \\
\vdots \\
(\partial_1 - \partial_2)^{m_{1,2}}(\partial_1 - \partial_3)^{m_{1,3}}\cdots(\partial_1 - \partial_r)^{m_{1,r}}\partial_1^{m_{1,r+1}}\phi_r = 0.
\end{cases} \tag{3.2}$$
(1) If $M - r < d$, then $\phi_r = 0$.

(2) If $0 \leq d \leq M - r$, then there is a non trivial homogeneous polynomial $\phi_r$ satisfying (3.2).

(3) If $d = M - r$ in particular, $\phi_r$ is equal to a constant multiple of $v = v_{A^+_1,m,\text{nice}}(a)$.

Proof. We argue by induction on $r$. In the case that $r = 1$, we write

$$\phi_1 = \phi(a_1) = pa_1^d,$$

where $p$ is a constant. If $m_{1,2} - 1 < d$ and $\phi_1$ satisfies the differential equation $\partial_1^{m_{1,2}} \phi_1 = 0$, then $p = 0$ and hence $\phi_1 = 0$. If $0 \leq d \leq m_{1,2} - 1$, then for any $p \neq 0$, $\partial_1^{m_{1,2}} \phi_1 = 0$. Also, if $d = m_{1,2} - 1$, in particular, then $\phi_1 = pa_1^{m_{1,2}-1}$, while $v = (\frac{1}{m_{1,2}})^{m_{1,2}-1}$ as in Example 2.11. Hence $\phi_1$ is equal to a constant multiple of $v$.

We assume that the statement of this theorem holds for $r - 1$. We write $\phi_r$ as

$$\phi_r = \phi(a_1, \ldots, a_r) = g_d(a_2, \ldots, a_r) + a_1g_{d-1}(a_2, \ldots, a_r) + \cdots + a_1^dg_0(a_2, \ldots, a_r),$$

where $g_k$ is a homogeneous polynomial of $a_2, \ldots, a_r$ with degree $k$ for $k = 0, 1, \ldots, d$. Then for $k = 0, 1, \ldots, d$, $g_k$ satisfies the differential equations as follows:

$$
\begin{cases}
\partial_r^{m_{r,r+1}} g_k = 0 \\
(\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} g_k = 0 \\
\vdots \\
(\partial_2 - \partial_3)^{m_{2,3}}(\partial_2 - \partial_4)^{m_{2,4}} \cdots (\partial_2 - \partial_r)^{m_{2,r}} \partial_{2}^{m_{2,r+1}} g_k = 0.
\end{cases}
$$

We set $h = (\sum_{2 \leq i < j \leq r+1} m_{i,j}) - (r - 1)$. From the inductive assumption, if $0 \leq k \leq h$, then $g_k$ is a homogeneous polynomial. On the other hand, if $h + 1 \leq k \leq d$, then $g_k = 0$, namely,

$$g_d(a_2, \ldots, a_r) = g_{d-1}(a_2, \ldots, a_r) = \cdots = g_{h+1}(a_2, \ldots, a_r) = 0. \quad (3.4)$$

(1) We consider the case of $M - r < d$. Let $M_1 = \sum_{i=2}^{r+1} m_{1,i}$. Now we compare the coefficients of $a_1^{d-h-M_1+n}$ in $(\partial_1 - \partial_2)^{m_{1,2}}(\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} \phi_r$ for $n = 0, \ldots, h$. For $q = 1, \ldots, M_1 - m_{1,r+1}$, we define

$$D_q = \sum_{2 \leq i_1 \leq r} \binom{m_{1,i_1}}{q} \partial_{i_1}^q + \cdots + \sum_{2 \leq i_1 < \cdots < i_k \leq r} \prod_{1 \leq i \leq k} \binom{m_{1,i}}{p_i} \partial_{i_1}^{p_1} \partial_{i_2}^{p_2} \cdots \partial_{i_k}^{p_k}$$

$$+ \cdots + \sum_{2 \leq i_1 < \cdots < i_q \leq r} \prod_{1 \leq i \leq q} \binom{m_{1,i}}{1} \partial_{i_1} \partial_{i_2} \cdots \partial_{i_q}. \quad (3.5)$$
Then we have the following equation:

\[
\frac{(d-h+n)!}{(d-h-M_1+n)!} g_{h-n}(a_2, \ldots, a_r) - \frac{(d-h+n-1)!}{(d-h-M_1+n)!} D_1 g_{h-n+1}(a_2, \ldots, a_r) \\
+ \cdots \pm \frac{(d-h+n-j)!}{(d-h-M_1+n)!} D_j g_{h-n+j}(a_2, \ldots, a_r) \\
\pm \cdots \pm \frac{(d-h+n-(M_1-m_{1,r+1})!}{(d-h-M_1+n)!} D_{M_1-m_{1,r+1}} g_{h-n+M_1,r}(a_2, \ldots, a_r) \\
= 0.
\] (3.6)

When \( n = 0 \), from (3.4) and (3.6), we have

\[ g_h(a_2, \ldots, a_r) = 0. \]

When \( n = 1 \), we have

\[ \frac{(d-h+1)!}{(d-h-M_1+1)!} g_{h-1}(a_2, \ldots, a_r) - \frac{(d-h)!}{(d-h-M_1+1)!} D_1 g_h(a_2, \ldots, a_r) = 0. \]

Thus we have

\[ g_{h-1}(a_2, \ldots, a_r) = 0. \]

Similarly, we have

\[ g_{h-2}(a_2, \ldots, a_r) = g_{h-3}(a_2, \ldots, a_r) = \cdots = g_0(a_2, \ldots, a_r) = 0 \]

and hence \( \phi_r = 0 \).

(2) We consider the case of \( 0 \leq d \leq M - r \). By the inductive assumption, there is a non trivial homogeneous polynomial \( g_{h-n_1+i} \) satisfying (3.3) for \( i = 1, \ldots, n_1 \), where \( n_1 = M - r - d + 1 \). We can take

\[ g_{h-n_1+i}(a_2, \ldots, a_r) \neq 0. \]

When \( n = n_1 \), from (3.4) and (3.6),

\[ g_{h-n_1}(a_2, \ldots, a_r) \\
= \frac{(d-h+n_1-1)!}{(d-h+n_1)!} D_1 g_{h-n_1+1}(a_2, \ldots, a_r) - \frac{(d-h+n_1-2)!}{(d-h+n_1)!} D_2 g_{h-n_1+2}(a_2, \ldots, a_r) \\
+ \cdots \pm \frac{(d-h)!}{(d-h+n_1)!} D_{n_1} g_h(a_2, \ldots, a_r). \]

When \( n = n_1 + 1 \),

\[ g_{h-(n_1+1)}(a_2, \ldots, a_r) \\
= \frac{(d-h+n_1)!}{(d-h+n_1+1)!} D_1 g_{h-n_1}(a_2, \ldots, a_r) - \frac{(d-h+n_1-1)!}{(d-h+n_1+1)!} D_2 g_{h-n_1+1}(a_2, \ldots, a_r) \\
+ \cdots \pm \frac{(d-h)!}{(d-h+n_1+1)!} D_{n_1+1} g_h(a_2, \ldots, a_r). \]
Similarly, for \( n = n_1 + 2, \ldots, h \), we can express \( g_{h-j}(a_2, \ldots, a_r) \) \((j = n_1, n_1 + 1, \ldots, h)\) in terms of \( g_{h-i}(a_2, \ldots, a_r) \) \((i = 1, \ldots, j)\) and their partial derivatives. Namely, we can express \( \phi_r \) in terms of \( g_{h-n_1+i}(a_2, \ldots, a_r) \) and their partial derivatives. It follows that \( \phi_r \neq 0 \) when \( 0 \leq d \leq M - r \).

(3) If \( d = M - r \) in particular, then \( n_1 = 1 \), and \( g_{h-j} \) \((j = 1, \ldots, h)\) becomes linear combination of \( g_h \) and their partial derivatives. Therefore \( \phi_r \) is uniquely determined by \( g_h \). Moreover, from the inductive assumption, \( g_h = C \cdot v_{A_{r-1}^{+}, m', c'_{\text{nice}}} \), where \( C \) is a constant, \( m' = (m_{i,j})_{2 \leq i < j \leq r+1} \), and \( c'_{\text{nice}} \) is a nice chamber of \( A_{r-1}^{+} \). Hence the solution of (3.2) is unique up to a constant multiple. On the other hand, by Theorem 3.1, \( v_{A_{r-1}^{+}, m, c_{\text{nice}}} \) satisfies the system of differential equations (3.2). Hence \( \phi_r \) is equal to a constant multiple of \( v_{A_{r-1}^{+}, m, c_{\text{nice}}} \). \( \square \)

**Remark 3.5** Let \( M_1 = \sum_{i=2}^{r+1} m_{1,i} \) and let \( D_q \) \((q = 1, \ldots, h)\) be as in (3.5). When \( d = M - r \), from the proof of Theorem 3.4(3), \( g_{h-j} \) \((j = 1, \ldots, h)\) is uniquely determined as follows:

\[
\begin{align*}
g_{h-1} &= \frac{(M_1-1)!}{M_1!} D_1 g_h \\
g_{h-2} &= \frac{(M_1-1)!}{(M_1+1)!} (D_1^2 - D_2) g_h \\
g_{h-3} &= \frac{(M_1-1)!}{(M_1+2)!} (D_1^3 - 2D_1D_2 + D_3) g_h \\
& \vdots \\
g_0 &= \frac{(M_1-1)!}{(M-r)!} (D_1^h - (h-1)D_1^{h-2}D_2 + \cdots \pm D_h) g_h.
\end{align*}
\]

Let \( m' = (m_{i,j})_{2 \leq i < j \leq r+1} \), \( c'_{\text{nice}} \) a nice chamber of \( A_{r-1}^{+} \) and \( \alpha' = \sum_{i=2}^{r} a_{i}(e_i - e_{r+1}) \in \overline{c}_{\text{nice}} \). From Proposition 3.3 and Remark 3.5, we obtain the following theorem.

**Theorem 3.6** Let \( h = (\sum_{2 \leq i < j \leq r+1} m_{i,j}) - (r - 1) \) and let \( D_q \) \((q = 1, \ldots, h)\) be as in (3.5). Then \( v_{A_{r-1}^{+}, m', c'_{\text{nice}}} \) \((\alpha')\) is written by linear combination of \( v_{A_{r-1}^{+}, m', c'_{\text{nice}}} \) \((\alpha')\) and its partial derivatives as follows:

\[
v_{A_{r-1}^{+}, m', c'_{\text{nice}}} \left( \frac{1}{(M_1-1)!} a_1^{M_1-1} + \frac{1}{M_1} a_1^M D_1 + \frac{1}{(M_1+1)!} a_1^{M_1+1}(D_1^2 - D_2) \\
+ \frac{1}{(M_1+2)!} a_1^{M_1+2}(D_1^3 - 2D_1D_2 + D_3) + \cdots \\
+ \frac{1}{(M-r)!} a_1^{M-r}(D_1^h - (h-1)D_1^{h-2}D_2 + \cdots \pm D_h) \right) v_{A_{r-1}^{+}, m', c'_{\text{nice}}} \left( \alpha' \right).
\]  

**Example 3.7** Let \( r = 3 \), let \( a = \sum_{i=1}^{3} a_i(e_i - e_4) \in \overline{c}_{\text{nice}} \) and let \( \alpha' = \sum_{i=2}^{3} a_{i}(e_i - e_4) \in \overline{c}_{\text{nice}} \). We set \( m_{1,2} = 1, m_{1,3} = 1, m_{1,4} = 2, m_{2,3} = 1, m_{2,4} = 2 \) and \( m_{3,4} = 2 \) as in Example 2.13. Then we have

\[
v_{A_{3}^{+}, m, c_{\text{nice}}} \left( \frac{1}{360} a_1^3(a_1^3 + 6a_1^2a_2 + 3a_1^2a_3 + 15a_1a_2^2 + 15a_1a_2a_3 + 10a_2^3 + 30a_2^2a_3) \right).
\]
We can check that $v = v_{A^+_3, m, \epsilon_{nice}}(a)$ satisfies the system of differential equations as follows:

$$\begin{cases}
\partial_3^2 v = 0 \\
(\partial_2 - \partial_3)\partial_2^2 v = 0 \\
(\partial_1 - \partial_2)(\partial_1 - \partial_3)\partial_1^2 v = 0.
\end{cases}$$

Also, from Proposition 3.2, the coefficient of the term $a_1^3a_2^2a_3$ is $\frac{1}{3!2!1!} = \frac{1}{12}$. When $r = 2$,

$$v_{A^+_2, m, \epsilon_{nice}}(a') = \frac{1}{6}a_2^3(a_2 + 3a_3).$$

Therefore, we have

$$\begin{align*}
\left\{ \frac{1}{6}a_1^3 + \frac{1}{24}a_1^4D_1 + \frac{1}{120}a_1^5(D_1^2 - D_2) + \frac{1}{720}a_1^6(D_1^3 - 2D_1D_2 + D_3) \right\} v_{A^+_2, m, \epsilon_{nice}}(a') \\
= \frac{1}{36}a_1^3a_2^3 + \frac{1}{12}a_1^3a_2^2a_3 + \frac{1}{24}a_1^4a_2^2 + \frac{1}{24}a_1^4a_2a_3 + \frac{1}{60}a_1^5a_2 + \frac{1}{120}a_1^5a_3 + \frac{1}{360}a_1^6 \\
= v_{A^+_3, m, \epsilon_{nice}}(a).
\end{align*}$$

Hence when $r = 3$, we can check the equation (3.7) in Theorem 3.6.

Acknowledgements

The third author was supported by JSPS KAKENHI Grant Number JP16K05137.

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