CANONICAL FORESTS IN DIRECTED FAMILIES

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Abstract. Two uniqueness results on representations of sets constructible in a directed family of sets are given. In the unpackable case, swiss cheese decompositions are unique. In the packable case, they are not unique but admit a quasi-ordering under which the minimal decomposition is unique. Both cases lead to a one-dimensional elimination of imaginaries in VC-minimal and quasi-VC-minimal theories.

1. Introduction

In this paper, we study canonical forms for sets constructible in a directed family of sets, in the sense of Adler [1]. Every such set is realizable as a disjoint union of swiss cheeses, that is, balls with (finitely many) holes removed. However, the uniqueness of such a presentation can fail. The dividing line is given by the notion of packability. In an unpackable directed family, we show in Section 2 that the swiss cheese decomposition is unique. If the family is packable, on the other hand, it is still possible to canonically choose an ‘optimal’ decomposition. This is described in Section 3.

Directed families arise in logic as the building blocks of the VC-minimal theories. Introduced by Adler [1], these theories have garnered interest for being well-situated in the realm of model-theoretic tameness. Classical examples such as strongly minimal and o-minimal theories are VC-minimal. VC-minimal theories can also be seen as a natural ‘simplest case’ among the dependent theories. Some fundamental model-theoretic machinery has already been developed, for example Cotter and Starchenko’s recent analysis of forking in VC-minimal theories [2].

A prototypical example is given by algebraically closed valued fields, from which much of the language of directed families is derived. Our primary goal is to give suitable generalizations of Holly’s study of definable sets in algebraically closed valued fields [6,7], including the elimination of imaginaries in one dimension which is detailed in Section 4. We also point out how these results can be adapted to the somewhat weaker quasi-VC-minimal setting.

Throughout the paper, we work with a directed family of sets as defined below. Note that this notion is not in accordance with some other uses of the term ‘directed’, such as in category theory.
1.1. Directed families. For any set $\mathcal{U}$, a family $\mathcal{B} \subseteq \mathcal{P}(\mathcal{U}) \setminus \{\emptyset\}$ of nonempty subsets of $\mathcal{U}$ is directed if, for all $B_0, B_1 \in \mathcal{B}$, one of the following holds:

(i) $B_0 \subseteq B_1$,
(ii) $B_1 \subseteq B_0$,
(iii) $B_0 \cap B_1 = \emptyset$.

$\mathcal{U}$ is the universe of $\mathcal{B}$. The members of $\mathcal{B}$ are called balls, and a constructible set is a (finite) boolean combination of balls. A directed family is called unpackable if no ball is a finite union of proper sub-balls.

If $\mathcal{B}$ is directed, then $(\mathcal{B}, \subseteq)$ is easily seen to be a forest, that is, a union of trees whose roots are the maximal sets in $\mathcal{B}$. Moreover, if $\mathcal{B}$ is directed, then so is $\mathcal{B} \cup \{\mathcal{U}\}$.

Thus we may assume that $\mathcal{U} \in \mathcal{B}$, as will often be necessary. A simple application of directedness gives the following:

**Lemma 1.1.** Fix $B \in \mathcal{B}$ and finite $C \subseteq \mathcal{B}$. Suppose that $B \subseteq \bigcup C$ and that for no $C \in C$ do we have $B \subset C$. Then $B = \bigcup C_0$ for some $C_0 \subseteq C$.

2. Swiss cheese and unpackability

In this section, we study the representation of constructible sets as boolean combinations of balls and the relation of these representations to unpackability.

Fix a directed family $\mathcal{B} \subseteq \mathcal{P}(\mathcal{U}) \setminus \{\emptyset\}$ with $\mathcal{U} \in \mathcal{B}$. A swiss cheese is a subset of $\mathcal{U}$ of the form $S = A \setminus (B_1 \cup \ldots \cup B_n)$, where $A$ is a ball and $B_1, \ldots, B_n \subseteq A$ are proper sub-balls of $A$. In this expression, $n$ may be 0, but it is required that the expression is nonredundant in the sense that for no $i \neq j$ is $B_i \subseteq B_j$. $A$ is called a wheel of $S$ and each $B_i$ is a hole. Note that this notion of wheels and holes is not immediately intrinsic to the set $S$, but depends on its presentation as a swiss cheese. In fact, we will show in Theorem 2.3 that as long as the holes are pairwise disjoint, the unique determination of wheels and holes is equivalent to unpackability.

A constructible set can be canonically derived from any finite set of balls. Consider a finite set $\mathcal{S} \subseteq \mathcal{B}$. Then $\mathcal{S}$ is partially ordered by $\subseteq$, and as noted in the introduction this ordering is a (finite) forest. Define the levels $\text{Lev}_n(\mathcal{S})$ inductively for $n \geq 0$ by

$$\text{Lev}_n(\mathcal{S}) = \left\{ B \in \mathcal{S} \mid B \text{ is } \subseteq\text{-maximal in } \mathcal{S} \setminus \bigcup_{i < n} \text{Lev}_i(\mathcal{S}) \right\}. $$

For $B \in \text{Lev}_n(\mathcal{S})$, also define

$$\text{Sub}(B, \mathcal{S}) = \{ C \in \text{Lev}_{n+1}(\mathcal{S}) \mid C \subseteq B \} , $$

$$\mathcal{G}(B, \mathcal{S}) = B \setminus \bigcup \text{Sub}(B, \mathcal{S}).$$

Finally, from $\mathcal{S}$ we construct the set

$$(2.1) \qquad \text{Ch}(\mathcal{S}) = \bigcup_{B \in \text{Lev}_{2n}(\mathcal{S}), n \geq 0} \mathcal{G}(B, \mathcal{S}).$$

It is clear that $\text{Ch}(\mathcal{S})$ is a disjoint union of swiss cheeses. The balls on the even levels of $\mathcal{S}$ are the wheels, and the holes are the wheels’ immediate predecessors in $(\mathcal{S}, \subseteq)$. An example of this construction is given in Figure 1.
Remark 2.1. For any finite set of balls $S$, we have

$$\text{Ch}(S) = \left( \bigcup \text{Lev}_0(S) \right) \setminus \text{Ch}(S \setminus \text{Lev}_0(S))$$

since $\text{Lev}_{n+1}(S) = \text{Lev}_n(S \setminus \text{Lev}_0(S))$ for all $n$.

If $S$ is a finite set of balls and $X = \text{Ch}(S)$, we call $S$ a swiss cheese decomposition of $X$. This definition differs slightly from that found in Holly [6] and elsewhere. The motivation for this change will be made clear in Theorem 2.3. In algebraically closed valued fields and other unpackable families, the wheel and holes of a swiss cheese are uniquely determined. But more generally, it becomes necessary for a swiss cheese decomposition to specifically carry the additional data of the wheels and holes involved.

Lemma 2.2. Every constructible subset $X \subseteq \mathcal{U}$ has a swiss cheese decomposition.

Proof. Let $T \subseteq B$ be minimal (in the sense of $\subseteq$) such that $\mathcal{U} \in T$ and $X$ is a boolean combination of elements of $T$. To begin, write $X$ as a boolean combination of balls in $T$ of the form

$$X = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} B_{i,j}^{e(i,j)}$$

(where $e(i,j) \in \{0, 1\}$, $B^1 = B$, and $B^0 = \mathcal{U} \setminus B$). We may assume that $T = \{B_{i,1}, \ldots, B_{i,n_i}\}$ for each $i$ (and in particular, that $n_i = n_j$ for all $i, j$). If not, say $A \notin \{B_{i,1}, \ldots, B_{i,n_i}\}$, we may replace $\bigcap_{j=1}^{n_i} B_{i,j}^{e(i,j)}$ with

$$\left( \bigcap_{j=1}^{n_i} B_{i,j}^{e(i,j)} \cap A \right) \cup \left( \bigcap_{j=1}^{n_i} B_{i,j}^{e(i,j)} \cap (\mathcal{U} \setminus A) \right).$$

Considering one of the disjuncts $\bigcap_{j=1}^{n_i} B_{i,j}^{e(i,j)}$, let $C_1, \ldots, C_r$ be those balls that appear positively (i.e. $B_{i,j}$ for $e(i,j) = 1$) and $D_1, \ldots, D_s$ those that appear negatively. Due to the assumption that $\mathcal{U} \in T$, we can always assume that $r \geq 1$. Furthermore, by the intersection property of balls, either $C_1 \cap \ldots \cap C_r = \emptyset$ or, for some $i$ and all $j$, $C_i \subseteq C_j$. In the former case, the disjunct is empty and can be removed altogether. In the latter case, note that again the disjunct is empty unless for each $j \leq s$ either $D_j \cap C_i = \emptyset$ or $D_j \subseteq C_i$.

Because each $D \in \text{Sub}(C_i, T)$ is among $D_1, \ldots, D_s$, we conclude that each nonempty disjunct

$$\bigcap_{j=1}^{n_i} B_{i,j}^{e(i,j)} = \mathcal{G}(C_i, T)$$
for some $C \in \mathcal{T}$. Thus, we have

\begin{equation}
X = \bigcup_{C \in S_0} \mathcal{G}(C, \mathcal{T})
\end{equation}

for some $S_0 \subseteq \mathcal{T}$. Note that the only balls required to write (2.2) are those in $S_0$ and those in Sub($C, \mathcal{T}$) for $C \in S_0$.

We show that for all $n$, Lev$_n(\mathcal{T}) \subseteq S_0$ or Lev$_n(\mathcal{T}) \cap S_0 = \emptyset$, and this alternates with $n$. This is trivial for $n = 0$, as Lev$_0(\mathcal{T}) = \{\mathcal{U}\}$. Now, suppose that $C \in S_0$ and $C' \in$ Sub($C, \mathcal{T}$). It is easy to show that

\[ \mathcal{G}(C, \mathcal{T} \setminus \{C'\}) = \mathcal{G}(C, \mathcal{T}) \cup \mathcal{G}(C', \mathcal{T}). \]

Therefore, if $C' \in S_0$, then (2.2) gives

\[ X = \bigcup_{C \in S_0 \setminus \{C'\}} \mathcal{G}(C, \mathcal{T} \setminus \{C'\}), \]

contradicting the minimality of $\mathcal{T}$. Thus Sub($C, \mathcal{T}$) $\cap S_0 = \emptyset$.

Conversely, suppose that $C \notin S_0$ and $C' \in$ Sub($C, \mathcal{T}$). If $C' \notin S_0$, then as observed above $C'$ is not needed in (2.2), again contradicting the minimality of $\mathcal{T}$. Therefore Sub($C, \mathcal{T}$) $\subseteq S_0$ and the claim follows.

Finally, if $\mathcal{U} \in S_0$, then set $S = \mathcal{T}$ and set $S = \mathcal{T} \setminus \{\mathcal{U}\}$ otherwise. This ensures that Lev$_0(S) \subseteq S_0$. Since we have just showed that $S_0$ consists of precisely the even levels of $S$, it follows that $X = \text{Ch}(S)$ as required.

While Lemma 2.2 says that every constructible set has a swiss cheese decomposition, it may occur that Ch($S$) $\neq$ Ch($\mathcal{T}$) even though $S \neq \mathcal{T}$. The notion of unpackability is central in obtaining uniqueness of the decomposition. In fact, it is equivalent. In the next theorem, (4) was first proved by Holly [6] in the case of algebraically closed valued fields. The observation that Holly’s theorem holds in the more general unpackable setting is due to Dolich (unpublished).

**Theorem 2.3.** For a directed family $\mathcal{B}$ containing its universe $\mathcal{U}$, the following are equivalent:

1. $\mathcal{B}$ is unpackable.
2. If $A, B_1, \ldots, B_n$ are balls such that $A \subseteq \bigcup_{i=1}^{n} B_i$, then $A \subseteq B_i$ for some $i$.
3. If

\[ S = A_1 \setminus (B_{1,1} \cup \ldots \cup B_{1,m}) = A_2 \setminus (B_{2,1} \cup \ldots \cup B_{2,n}) \]

is a swiss cheese and $B_{i,j} \cap B_{i,k} = \emptyset$ for $i \in \{1, 2\}$, $j \neq k$, then $A_1 = A_2$ and

\[ \{B_{1,1}, \ldots, B_{1,m}\} = \{B_{2,1}, \ldots, B_{2,n}\}. \]

4. Every constructible set admits a unique swiss cheese decomposition.

**Proof.** We first show the equivalence of (1)-(3) and then show the equivalence of these three conditions with (4).

1$\Rightarrow$2: Suppose $A \subseteq \bigcup_{i=1}^{n} B_i$. We may assume that $A \cap B_i \neq \emptyset$ for each $i$. But then either $B_i \subseteq A$ or $A \subseteq B_i$. If $B_i \subseteq A$ for every $i$, then by Lemma 1.1 $A$ would be a finite union of proper sub-balls, contradicting unpackability.

2$\Rightarrow$3: Since $A_1 \subseteq A_2 \cup B_{1,1} \cup \ldots \cup B_{1,m}$, (2) gives $A_1 \subseteq A_2$ or $A_1 \subseteq B_{1,i}$, for some $i$. But the latter cannot occur, as the holes in a swiss cheese are presumed to
be proper sub-balls of the wheel. Therefore \( A_1 \subseteq A_2 \), and by symmetry, \( A_1 = A_2 \). It follows that
\[
B_{1,1} \cup \ldots \cup B_{1,m} = B_{2,1} \cup \ldots \cup B_{2,n}
\]
and hence \( B_{1,i} \subseteq B_{2,j} \), for some \( j \leq n \). Similarly, \( B_{2,j} \subseteq B_{1,k} \). But here we must have \( k = i \), since \( B_{1,i} \cap B_{1,k} = \emptyset \) for \( i \neq k \). Thus \( B_{1,i} = B_{2,j} \) and
\[
\{B_{1,1}, \ldots, B_{1,m}\} \subseteq \{B_{2,1}, \ldots, B_{2,n}\}.
\]
Again, (3) follows by symmetry.

\( 3 \Rightarrow 1 \): Suppose a ball \( A \) were the disjoint union of proper sub-balls \( B_1, \ldots, B_n \). Necessarily \( n > 1 \). Then the swiss cheese \( B_1 = A \setminus \bigcup (B_2 \cup \ldots \cup B_n) \) contradicts (3).

\( 4 \Rightarrow 1 \): This is similarly clear. If, for example, the ball \( A \) were the disjoint union of proper sub-balls \( B_1, \ldots, B_n \), then \( X = A \) could be decomposed either as simply \( \text{Ch}\{\{A\}\} \) or as \( \text{Ch}\{\{B_1, \ldots, B_n\}\} \).

\( 1 \Rightarrow 4 \): In light of Lemma 2.2, there remains only to prove uniqueness. Suppose we have two decompositions, \( X = \text{Ch}(S) = \text{Ch}(T) \). Write \( \text{Ch}(S) = S_1 \cup \ldots \cup S_r \) and \( \text{Ch}(T) = T_1 \cup \ldots \cup T_s \) as the disjoint union of swiss cheeses
\[
S_i = A_i \setminus (A_{i,1} \cup \ldots \cup A_{i,m}),
\]
\[
T_i = B_i \setminus (B_{i,1} \cup \ldots \cup B_{i,n})
\]
as in 2.1. We work by induction on \( r \). If \( r = 0 \), then \( X = \emptyset \). Since no \( B_i \) can be the union of its proper sub-balls \( B_{i,1}, \ldots, B_{i,n} \), no \( T_i \) can be empty. Thus, \( s = 0 \) as well.

For \( r > 0 \), note that \( S_1 \subseteq X \) implies
\[
A_1 \subseteq \left( \bigcup_{i=1}^{s} B_i \right) \cup \left( \bigcup_{j=1}^{m_1} A_{1,j} \right).
\]
By (2), it follows that \( A_1 \) is a subset of one of these balls. But since \( S_1 \neq \emptyset \), we must have \( A_1 \subseteq B_i \) for some \( i \leq s \). By the same reasoning, \( B_i \subseteq A_j \) for some \( j \leq r \). This can be repeated until one of the balls appears twice, giving equality. So, renaming for convenience, let us say that \( A_1 = B_1 \).

Now we claim that \( S_1 = T_1 \). To this end, note first that if \( S_1 \) has no holes, then \( S_1 = A_1 = B_1 \), and \( T_1 \) can have no holes either. To see this, suppose we have a hole \( B_{1,1} \) of \( T_1 \). Since \( B_{1,1} \subseteq A_1 \subseteq X \), \( B_{1,1} \) must be covered by \( \text{Sub}(B_{1,1}, T) \), contradicting unpackability.

Otherwise, suppose \( S_1 \) has at least one hole \( A_{1,1} \). Every element \( x \in A_{1,1} \) is either
- not in \( X \), in which case \( x \in A_1 = B_1 \) implies \( x \in B_{1,i} \) for some \( i \), or
- in \( S_j \) for some \( j \neq 1 \). In this case, since \( S_1 \cap S_j = \emptyset \) and \( A_j \neq A_{1,1} \), as before we must have \( A_j \subseteq A_{1,1} \).

Altogether,
\[
A_{1,1} \subseteq \left( \bigcup_{i=1}^{n_1} B_{1,i} \right) \cup \left( \bigcup_{A_j \subseteq A_{1,1}} A_j \right),
\]
from which (2) implies \( A_{1,1} \subseteq B_{1,i} \) for some \( i \). The same argument applies to the other holes of \( S_1 \) and \( T_1 \), with the result that
\[
A_{1,1} \cup \ldots \cup A_{1,m_1} = B_{1,1} \cup \ldots \cup B_{1,n_1}
\]
and \( S_1 = T_1 \). Now (3) gives \( \{A_{1,i} \mid i \leq m_1\} = \{B_{1,i} \mid i \leq n_1\} \).
Finally, the induction hypothesis applied to
\[ S_2 \cup \ldots \cup S_r = T_2 \cup \ldots \cup T_s \]
finishes the proof. \qed

3. Forests and packable families

For this section, fix a set \( \mathcal{U} \) and \( B \subseteq \mathcal{P}(\mathcal{U}) \setminus \{\emptyset\} \) directed, not necessarily unpack-
able. We again require \( \mathcal{U} \in B \). Consider \( \text{Ch} \) as defined before. Since \( B \) is potentially packable, we may have finite \( S, T \subseteq B \) distinct but \( \text{Ch}(S) = \text{Ch}(T) \). Nevertheless, in this section we describe a way to choose a canonical \( S \) representing \( X \).

Define, on the set of all finite forests, a quasi-ordering \( \preceq \) so that \( S \preceq T \) iff:

(i) \( |S| = |T| \) and, for all \( n \), \( |\text{Lev}_n(S)| = |\text{Lev}_n(T)| \) or
(ii) \( |S| < |T| \) or
(iii) \( |S| = |T| \) and for some \( n \) and all \( i < n \), \( |\text{Lev}_i(S)| = |\text{Lev}_i(T)| \), but \( |\text{Lev}_n(S)| > |\text{Lev}_n(T)| \).

So, roughly speaking, forests are ordered first by cardinality, then by top-heaviness, as depicted in Figure 2.

\[ \ldots \preceq \ldots \preceq \ldots \preceq \ldots \approx \ldots \preceq \ldots \preceq \ldots \]

\textbf{Figure 2.} Some forests of size 4, quasi-ordered by \( \preceq \)

We use this order to get a uniqueness of decomposition result:

**Theorem 3.1.** Let \( S \) and \( T \) be finite sets of balls such that \( \text{Ch}(S) = \text{Ch}(T) \). If \( (S, \subseteq) \) and \( (T, \subseteq) \) are both \( \preceq \)-minimal among all swiss cheese decompositions of \( \text{Ch}(S) \) in \( B \), then \( S = T \).

**Proof.** We proceed by induction on \( N = |S| \). If \( N = |S| = 0 \), then \( S = T = \emptyset \) and we are done.

Assume that \( N > 0 \). We aim to prove
\[ \text{Lev}_0(S) = \text{Lev}_0(T). \]

By induction and Remark 2.1, this suffices. Recall that, for \( B \in S \),
\[ G(B, S) = B \setminus \bigcup \text{Sub}(B, S). \]

Note that since \( G(B, S) = \emptyset \) or \( G(C, T) = \emptyset \) clearly violates minimality of \( N \), we can rule this possibility out.

So, consider a ball \( B \in \text{Lev}_0(S) \). Since \( G(B, S) \subseteq \text{Ch}(S) = \text{Ch}(T) \), there must exist \( C \in \text{Lev}_{2n}(T) \) for some \( n \) such that \( G(B, S) \cap G(C, T) \neq \emptyset \). It follows that \( B \cap C \neq \emptyset \). If \( C \notin \text{Lev}_0(T) \), then replace \( C \) with the ball containing it in \( \text{Lev}_0(T) \).

In other words, we have shown that for any \( B \in \text{Lev}_0(S) \), there is \( C \in \text{Lev}_0(T) \) such that \( B \cap C \neq \emptyset \). Since \( B \) is directed, \( B \subseteq C \) or \( C \subseteq B \). Likewise, for any \( C \in \text{Lev}_0(T) \), there is \( B \in \text{Lev}_0(S) \) such that \( B \subseteq C \) or \( C \subseteq B \).

Now, if we did not have \( \text{Lev}_0(S) = \text{Lev}_0(T) \), then by the above observation there would be \( B \in \text{Lev}_0(S) \) and \( C \in \text{Lev}_0(T) \) such that \( B \subseteq C \) or \( C \subseteq B \). Thus say, for instance, that we have found \( B \nsubseteq C \). Note that as \( B \in \text{Lev}_0(S) \), there can be no \( B' \in S \) for which \( C \subseteq B' \).
Define $X = \text{Ch}(S) = \text{Ch}(T)$ and

$$S' = \{B' \in \text{Lev}_0(S) \mid B' \subseteq C, \text{ but } B' \not\subseteq C' \text{ for any } C' \in \text{Sub}(C, T)\},$$

$$T' = \{C' \in \text{Sub}(C, T) \mid C' \cap B' = \emptyset \text{ for all } B' \in S'\}.$$

**Claim 1.** $C \setminus \bigcup T' = \bigcup S'$.

First, if $x \in \bigcup S'$, then $x \in C$. If $C' \in T'$, then $C' \cap (\bigcup S') = \emptyset$ gives $x \notin C'$. Therefore, $x \in C \setminus \bigcup T'$.

Conversely, suppose $x \in C \setminus \bigcup T'$. If $x \notin X$, then since $x \in C$ we must have $x \in C'$ for some $C' \in \text{Sub}(C, T)$. But $C' \not\subseteq T'$, so $C' \cap B' \neq \emptyset$ for some $B' \in S'$. Now $C' \subsetneq B'$, so that $x \in \bigcup S'$. If on the other hand $x \in X$, then $x \in B'$ for some $B' \in \text{Lev}_0(S)$. Suppose $B' \not\subseteq S'$. Since $B' \subseteq C$ by choice of $C$, it follows that $B' \subseteq C'$ for some $C' \in \text{Sub}(C, T)$. Since $x \in C'$ but not $\bigcup T'$, $C' \cap B'' \neq \emptyset$ for some $B'' \in S'$. Now $B' \subseteq C' \subseteq B''$, but $B' \neq B''$ since $B'' \in S'$. This contradicts $B' \in \text{Lev}_0(S)$, and the claim is proven.

Next, let

$$S^* = (S \setminus S') \cup \{C\} \cup T',$$

$$T^* = (T \setminus (\{C\} \cup T')) \cup S',$$

noting that $S' \subseteq \text{Lev}_0(T^*)$, $C \in \text{Lev}_0(S^*)$, and $T' \subseteq \text{Lev}_1(S^*)$.

**Claim 2.** $\text{Ch}(S^*) = X$.

Suppose first that $x \in X$. Then $x$ resides in a chain

$$x \in B_{2n} \subseteq B_{2n-1} \subseteq \ldots \subseteq B_0$$

with $B_i \in \text{Lev}_i(S)$, $x \notin \bigcup \text{Sub}(B_{2n}, S)$. There are several cases to consider:

- If $B_0 \notin S'$, then either
  - $B_0 \cap C = \emptyset$, in which case $B_i \in \text{Lev}_i(S^*)$ as well and $x \in \text{Ch}(S^*)$, or
  - $B_0 \subseteq C$ and, since $B_0 \notin S'$, $B_0 \subseteq C'$ for some $C' \in \text{Sub}(C, T)$. This $C'$ must be in $T'$ since $\bigcup S' = C \setminus \bigcup T'$. So in this case we have $B_i \in \text{Lev}_{i+2}(S^*)$ in (3.1) and $x \in \text{Ch}(S^*)$.

- If $B_0 \in S'$, then $B_0 \subseteq C$, but $B_0 \cap C' = \emptyset$ for all $C' \in T'$. It follows that

$$x \in B_{2n} \subseteq \ldots \subseteq B_1 \subseteq C$$

with $B_i \in \text{Lev}_i(S^*)$, and $x \in \text{Ch}(S^*)$.

This shows that $X \subseteq \text{Ch}(S^*)$.

For the converse, again suppose (3.1) but this time with $B_i \in \text{Lev}_i(S^*)$, $x \notin \bigcup \text{Sub}(B_{2n}, S^*)$. If $B_0 \in S \setminus S'$, then by choice of $C$, $B_0 \cap C = \emptyset$ and $x \notin \text{Ch}(S) = X$. The other possibility is that $B_0 = C$. Again there are several cases:

- If $B_1 \in T'$, then $B_2 \in \text{Lev}_0(S)$ and $x \in \text{Ch}(S)$.
- If $B_1 \notin T'$, then $x \in C \setminus \bigcup T' = \bigcup S'$, but $B_1 \notin S'$ implies that $B_1 \notin \text{Lev}_0(S)$. Since $B_1 \in \text{Lev}_1(S^*)$, it follows that $B_1 \in \text{Lev}_1(S)$ as well. So there is $B \in \text{Lev}_0(S)$, $B_1 \subseteq B$ and

$$x \in B_{2n} \subseteq \ldots \subseteq B_1 \subseteq B$$

with $B_i \in \text{Lev}_i(S)$, giving $x \in \text{Ch}(S) = X$.

We thus have shown $\text{Ch}(S^*) = X$.

**Claim 3.** $\text{Ch}(T^*) = X$. 
Suppose, similarly, \( x \in X \) with a chain

\[(3.2) \quad x \in C_{2n} \subseteq \ldots \subseteq C_0,\]

\(C_i \in \text{Lev}_i(\mathcal{T}), x \notin \bigcup \text{Sub}(C_{2n}, \mathcal{T}). \) So:

- If \(C_0 = C\), then either
  - \(C_1 \in \mathcal{T}'\), in which case \(C_1 \cap B' = \emptyset\) for every \(B' \in \mathcal{S}'\), and \(C_i \in \text{Lev}_{i-2}(\mathcal{T}')\) for \(i \geq 2\). This shows \(x \in \text{Ch}(\mathcal{T}')\).
  - \(C_1 \notin \mathcal{T}'\), so that \(x \in C \setminus \bigcup \mathcal{T}' = \bigcup \mathcal{S}'\). So \(x \in B'\) for some \(B' \in \mathcal{S}'\), and the definition of \(\mathcal{S}'\) gives \(C_1 \subseteq B'\). Thus the chain
    \(x \in C_{2n} \subseteq \ldots \subseteq C_1 \subseteq B'\)
    with \(C_i \in \text{Lev}_i(\mathcal{T}')\) gives \(x \in \text{Ch}(\mathcal{T}')\).

- If \(C_0 \neq C\), then in (3.2), \(C_i \in \text{Lev}_i(\mathcal{T}')\) as well, so again \(x \in \text{Ch}(\mathcal{T}')\).

Conversely, suppose (3.2) but now with \(C_i \in \text{Lev}_i(\mathcal{T}')\), \(x \notin \bigcup \text{Sub}(C_{2n}, \mathcal{T}')\). If \(C_0 \in \mathcal{S}'\), then \(x \in C \setminus \bigcup \mathcal{T}'\). Since \(C_1 \subseteq C_0\), \(C_1 \cap C' = \emptyset\) for all \(C' \in \mathcal{T}'\). Thus we have in \(\mathcal{T}\) the (maximal) chain

\(x \in C_{2n} \subseteq \ldots \subseteq C_1 \subseteq C\)

with \(C_i \in \text{Lev}_i(\mathcal{T})\).

On the other hand, suppose \(C_0 \in \mathcal{T} \setminus \mathcal{S}'\). Then since \(C_0 \in \text{Lev}_0(\mathcal{T}')\), \(C_0 \cap B' = \emptyset\) for all \(B' \in \mathcal{S}'\). It follows that either \(C_0 \cap C = \emptyset\) or \(C_0 \subseteq C'\) for some \(C' \in \mathcal{T}'\). In the first case, the chain \(C_i\) is the same in \(\mathcal{T}\) and \(x \in \text{Ch}(\mathcal{T})\). In the second case, since \(C_0 \notin \mathcal{T}'\), the chain becomes

\(x \in C_{2n} \subseteq \ldots \subseteq C_0 \subseteq C' \subseteq C\)

with \(C_i \in \text{Lev}_{i+2}(\mathcal{T})\). This again shows that \(x \in \text{Ch}(\mathcal{T})\), and the claim is proven.

Finally, depending on the relative sizes of \(|\mathcal{S}'|\) and \(|\mathcal{T}'|\), we derive a contradiction to \(\leq\)-minimality of \(\mathcal{S}\) and \(\mathcal{T}\) as follows:

- If \(|\mathcal{S}'| < |\mathcal{T}'| + 1\), then \(\mathcal{T}'\) is a swiss cheese decomposition of \(X\) having strictly fewer balls than \(\mathcal{T}\).
- If \(|\mathcal{S}'| > |\mathcal{T}'| + 1\), then \(\mathcal{T}'\) is a swiss cheese decomposition of \(X\) having strictly fewer balls than \(\mathcal{S}\).
- If \(|\mathcal{S}'| = |\mathcal{T}'| + 1\), say \(\mathcal{S}' = \{B\}\), then \(C \setminus \bigcup \mathcal{T}' = C = B\) contradicts our choice of \(C\).
- If \(|\mathcal{S}'| = |\mathcal{T}'| + 1 \geq 2\), then \(\mathcal{T}'\) is a swiss cheese decomposition of \(X\) with \(N = |\mathcal{S}|\) balls, but with \(|\text{Lev}_0(\mathcal{S}')| > |\text{Lev}_0(\mathcal{S})|\).

The contradiction gives \(\text{Lev}_0(\mathcal{S}) = \text{Lev}_0(\mathcal{T})\), and the result follows. \(\square\)

4. VC-MINIMALITY

While the previous sections relied purely on the combinatorial properties of directed families, the questions originated in logic with the notion of VC-minimality.

Given a theory \(\mathcal{T}\) and a set of formulas \(\Psi = \{\psi_i(x; \bar{y}_i)\}\) in the language of \(\mathcal{T}\), \(\Psi\) is directed if the family of sets

\[\Psi(\mathcal{M}) = \{\psi(\mathcal{M}; \bar{a}) \mid \psi(x; \bar{y}) \in \Psi, \bar{a} \in \mathcal{M}^{\bar{y}}\}\]

is directed for every \(\mathcal{M} \models \mathcal{T}\). Note that the length of the tuples \(\bar{y}\) may vary with \(\psi(x; \bar{y}) \in \Psi\), but \(x\) is exclusively a single variable. We also call a single formula \(\delta(x; \bar{y})\) directed if the set \(\{\delta(x; \bar{y})\}\) is directed.
Now \( T \) is VC-minimal if there is a directed family of formulas \( \Psi \) such that for all \( \mathfrak{M} \models T \), every definable subset of \( \mathfrak{M} \) is a constructible set of \( \Psi(\mathfrak{M}) \). In this case, \( \Psi \) is called a generating family for \( T \).

The terminology around directed families carries over naturally to VC-minimal theories. As with directed families, for our purposes it will be most convenient to stick to the convention that \( x = x \in \Psi \) but \( x \neq x \notin \Psi \); i.e. the whole universe is always a ball, and the empty set is never a ball.

Likewise, Theorems 2.3 and 3.1 can immediately be applied to the family of balls \( \Psi(\mathfrak{M}) \) generated by any model \( \mathfrak{M} \) of a VC-minimal theory.

4.1. VC-minimality and imaginaries. In this subsection, we outline an application which extends the analogy to Holly’s work with algebraically closed valued fields [7]. The main observation is that the canonical representation of definable sets from Theorem 3.1 leads to a one-dimensional elimination of imaginaries.

A theory is said to eliminate imaginaries if for every model \( \mathfrak{M} \), \( n \in \mathbb{N} \), and definable set \( X \subseteq \mathfrak{M}^n \), there is a formula \( \varphi(\bar{x}; \bar{y}) \) and tuple \( \bar{a} \in \mathfrak{M}^{|\bar{y}|} \) such that for all \( \bar{b} \in \mathfrak{M}^{|\bar{y}|} \), \( \varphi(\bar{x}; \bar{b}) \) defines \( X \) iff \( \bar{b} = \bar{a} \). In this case, \( \bar{a} \) is called a code (or canonical parameter) of \( X \). The existence of codes allows one, in a sense, to treat definable sets as elements of the model. See [8] for further discussion.

It should be noted that it is always possible to expand a model \( \mathfrak{M} \) to a (usually multi-sorted) structure \( \mathfrak{M}^{na} \) which eliminates imaginaries by explicitly adding to the language a code for every definable set. This suffices for many applications, but in other situations one may gain a better understanding of the definable sets in a structure by finding a way to expand the language to achieve the elimination of imaginaries in a more efficient, or natural, way.

The notion of codes also specializes naturally to definable sets of a certain dimension. To this end, say a theory has \( n \)-prototypes if there is a family \( \Phi = \{ \varphi(\bar{x}; \bar{y}) \} \) with \( |\bar{x}| = n \) such that for every model \( \mathfrak{M} \) and every definable set \( X \subseteq \mathfrak{M}^n \), there is exactly one \( \varphi \in \Phi \) and \( \bar{a} \in \mathfrak{M}^{|\bar{y}|} \) such that \( \varphi(\bar{x}; \bar{a}) \) defines \( X \). Holly proves in [7] that a theory eliminates imaginaries iff it has \( n \)-prototypes for every \( n \geq 1 \). It is also clear from the proof that a theory has 1-prototypes iff every definable subset of a model (in one variable) has a code.

Returning to VC-minimal theories, the definable sets in more than one variable are not yet well understood. The favorite example of algebraically closed valued fields indicates that the situation can be quite complex (see for instance [5]). However, regarding the question of 1-prototypes, the work of the preceding sections does the trick. We present this in two forms.

Suppose \( T \) is VC-minimal, with generating family \( \Psi \). Expand the language of \( T \) to add to any model \( \mathfrak{M} \models T \) a new sort consisting of the finite sets of balls. Add also, for each \( \langle \psi_1(x; \bar{y}_1), \ldots, \psi_n(x; \bar{y}_n) \rangle \in \Psi^n \), a new function symbol from the main sort to the new sort,

\[
f_{\langle \psi_1, \ldots, \psi_n \rangle}(\bar{y}_1, \ldots, \bar{y}_n): \langle \bar{a}_1, \ldots, \bar{a}_n \rangle \mapsto \{ B_1, \ldots, B_n \},
\]

taking the tuple of parameters \( \langle \bar{a}_1, \ldots, \bar{a}_n \rangle \) to the set of balls \( B_i = \psi_i(\mathfrak{M}; \bar{a}_i) \). Let \( T^* \) be the theory of a model of \( T \) expanded in this way.

**Theorem 4.1.** If \( T \) is VC-minimal, then \( T^* \) has 1-prototypes.
We show only that every definable set $X \subseteq \mathcal{M}$ has a code. Assume $X \neq \mathcal{M}$ or $\emptyset$; that $\mathcal{M}$ and $\emptyset$ have codes is obvious. By Theorem 3.1 there is a finite set $S = \{B_1, \ldots, B_n\}$ of balls so that $\text{Ch}(S) = X$, and if $\text{Ch}(T) = X$ and $(T, \subseteq) \cong (S, \subseteq)$ (as forests), then $S = T$.

Now, for $i \leq n$ let $\psi_i \in \Psi$ and $\bar{a}_i$ be such that $B_i = \psi_i(\mathcal{M}; \bar{a}_i)$. Let $\varphi(x; \gamma)$ be the formula stating that

$$\forall \langle \bar{y}_1, \ldots, \bar{y}_n \rangle \in f_{[\psi_1, \ldots, \psi_n]}^{-1}(\gamma) \left( \left\{ \psi_i(\mathcal{M}; \bar{y}_i) \right\}_{i \leq n}, \subseteq \right) \cong (S, \subseteq) \quad \rightarrow \quad x \in \text{Ch} \left( \left\{ \psi_i(\mathcal{M}; \bar{y}_i) \right\}_{i \leq n} \right).$$

Since $X \neq \mathcal{M}$, if $\varphi(x; T)$ defines $X$, then $(T, \subseteq) \cong (S, \subseteq)$ and $\text{Ch}(T) = \text{Ch}(S)$. Thus, by the choice of $S$, we must have $T = S$. So $S$ is in fact a code for $X$. 

This language is complicated somewhat by the need to allow for all finite sets of balls. This is necessary, as it is not generally possible, for example, to distinguish $\langle B_1, B_2 \rangle$ from $\langle B_2, B_1 \rangle$ in terms of the definable set represented by this pair of balls. This phenomenon is commonplace enough to earn its own terminology.

A theory weakly eliminates imaginaries if for every model $\mathcal{M}$, $n \in \mathbb{N}$, and definable set $X \subseteq \mathcal{M}^n$, there is a formula $\varphi(x; \bar{y})$ and nonempty finite set $A \subseteq \mathcal{M}[\bar{y}]$ such that for all $\bar{b} \in \mathcal{M}[\bar{y}]$, $\varphi(x; \bar{b})$ defines $X$ if $\bar{b} \in A$. Analogously, a theory has weak 1-prototypes if there is a family $\Phi$ such that for every model $\mathcal{M}$ and every definable set $X \subseteq \mathcal{M}$, there is exactly one $\varphi(x; \bar{y}) \in \Phi$ and finitely many $\bar{a} \in \mathcal{M}[\bar{y}]$ such that $\varphi(x; \bar{a})$ defines $X$.

Now from $T$ construct an expanded theory $T^\circ$ by adding a new sort consisting of the balls, as well as for each $\psi \in \Psi$ a new function symbol $f_\psi$ defined by

$$f_\psi^{\mathcal{M}} : \bar{a} \mapsto B = \psi(\mathcal{M}, \bar{a}).$$

As the codes from Theorem 4.1 depended only on the set of balls $\{B_1, \ldots, B_n\}$, we may replace them with an ordered tuple of balls $\langle B_1, \ldots, B_n \rangle$ at the expense of allowing as many as $n!$ potential codes rather than only one. We thus obtain

**Corollary 4.2.** If $T$ is VC-minimal, then $T^\circ$ has weak 1-prototypes.

### 4.2. Quasi-VC-minimality.

A theory $T$ is called quasi-VC-minimal if there exists a directed family $\Psi$ of formulas such that every parameter-definable formula of a single free variable is $T$-equivalent to a boolean combination of instances of formulas from $\Psi$ and $\emptyset$-definable formulas. An example of a quasi-VC-minimal but not VC-minimal theory is Presburger arithmetic, $\text{Th}(\mathbb{Z}; +, \leq)$. See 3 for details.

We outline how the above results can be adapted to apply to quasi-VC-minimal theories. As the main differences in the proofs are notational annoyances, these are omitted.

Given $\mathcal{M} \models T$ and $\emptyset$-definable $Q \subseteq \mathcal{M}$, the restriction of the balls to $Q$, $\{B \cap Q \mid B \text{ a ball}\}$ is again a directed family. Thus a boolean combination of balls intersected with $Q$ admits a canonical, $\leq$-minimal Swiss cheese decomposition as in Theorem 8.1. Here, the balls themselves are not uniquely defined, but only their intersection with $Q$. For a finite set $S = \{B_i\}$ of balls, write $S \cap Q = \{B_i \cap Q\}$.

Now, given a formula $\varphi(x; \bar{y})$, by compactness there are formulas $\theta_1(x), \ldots, \theta_k(x)$ over $\emptyset$ such that every instance of $\varphi$ is a boolean combination of balls and $\theta_1, \ldots, \theta_k$. 
For $e : \{1, \ldots, k\} \to \{0, 1\}$, write

$$\theta^e(x) = \bigwedge_{i=1}^{k} \theta_i(x)^{e(i)}.$$ 

Then as in Lemma 2.2 it is proved that every instance of $\varphi$ can be written as

$$\bigvee_e (\theta^e(x) \land \sigma_e(x, \bar{a}_e)),$$

where $\sigma_e$ defines a swiss cheese decomposition. Again, the balls in this swiss cheese decomposition are not uniquely determined, but by Theorem 3.1 they can be chosen so that their intersections with $\theta^e$ are:

**Theorem 4.3.** If $X \subseteq \mathcal{M}$ is definable, then there are pairwise disjoint $\emptyset$-definable $Q_1, \ldots, Q_k \subseteq \mathcal{M}$ partitioning $\mathcal{M}$ and, for each $i \leq k$, a finite set of balls $S_i$ such that

1. $X \cap Q_i = \text{Ch}(S_i \cap Q_i)$, and
2. for any $T$, if also $X \cap Q_i = \text{Ch}(T \cap Q_i)$ and $(T \cap Q_i, \subseteq) \cong (S_i \cap Q_i, \subseteq)$, then $T \cap Q_i = S_i \cap Q_i$.

Finally, for quasi-VC-minimal $T$, let $T^\sharp$ be the theory obtained by adding

- a new sort consisting of the intersections of balls with $\emptyset$-definable sets, and
- for each $\psi(x; \bar{y}) \in \Psi$ and each formula $\theta$ over $\emptyset$, a new function symbol $f_{\theta, \psi}$ for which

$$f_{\theta, \psi}^\mathcal{M} : \bar{a} \mapsto \theta(\mathcal{M}) \land \varphi(\mathcal{M}; \bar{a}).$$

We then obtain as in Corollary 4.4.

**Corollary 4.4.** $T^\sharp$ has weak 1-prototypes.

**4.3. Uniform definability of levels.** We conclude with the observation that the levels of a canonical decomposition as in Theorem 3.1 are uniformly definable. This fact will be useful in type counting arguments in VC-minimal theories (see [4]).

Fix a formula $\varphi(x; \bar{y})$ in a VC-minimal theory $T$. By compactness, there exist a single directed $\delta(x; \bar{z})$ and $N < \omega$ so that all instances of $\varphi$ are a boolean combination of at most $N$ instances of $\delta(x; \bar{z})$. (More precisely, compactness gives finitely many $\psi \in \Psi$; then standard coding tricks can be used to combine them into a single directed $\delta$.) As we will be working only with instances of $\varphi$, we disregard $\Psi$ and work instead in the directed family of instances of $\delta$.

There are only finitely many forests of size at most $N$; call this set $F_N$. For each $F \in F_N$, let $\psi_F(\bar{y})$ denote the formula which says that there exists $\bar{z}_f$ for each $f \in F$ so that

1. $f \leq f'$ in $F$ iff $\forall x (\delta(x; \bar{z}_f) \rightarrow \delta(x; \bar{z}_{f'}))$;
2. $\varphi(x; \bar{y})$ is $T$-equivalent to

$$\text{Ch}(\{\delta(x; \bar{z}_f) \mid f \in F\}) = \bigvee_{f \in \text{Lev}_{2n}(F), \ n < N} \left( \delta(x; \bar{z}_f) \land \bigwedge_{f' \in \text{Lev}_{2n+1}(F)} \neg \delta(x, \bar{z}_{f'}) \right).$$

Finally, for any $n < N$, let $\gamma_n(x; \bar{y})$ denote the formula that says for the $\leq$-least $F \in F_N$ such that $\psi_F(\bar{y})$ holds, there exist witnesses $\bar{z}_f$ for $f \in F$ as above such that $\delta(x; \bar{z}_f)$ holds for some $f \in \text{Lev}_n(F)$. Thus, for any $\bar{b}$, $\gamma_n(x; \bar{b})$ holds if and
only if \( x \) appears in the \( n \)th level of a \( \preceq \)-minimal decomposition \( \{ \delta(x; \bar{c}_f) \mid f \in F \} \) of \( \varphi(x; \bar{b}) \). However, by Theorem 3.1 the set \( \{ \delta(x; \bar{c}_f) \mid f \in F \} \) is unique up to \( T \)-equivalence. Therefore, \( \gamma_n(x; \bar{b}) \) holds if and only if \( x \) is in the \( n \)th level of the \( \preceq \)-minimal decomposition.

We summarize in the following theorem:

**Theorem 4.5.** If \( \varphi(x; \bar{y}) \) is any formula in a VC-minimal theory \( T \), there exists a directed \( \delta(x; \bar{z}) \), \( N_{\varphi} < \omega \) and \( \gamma_{\varphi, n}(x; \bar{y}) \) for all \( n < N_{\varphi} \) such that:

(i) For all \( \bar{b} \), \( \gamma_{\varphi, n}(x; \bar{b}) \) is \( T \)-equivalent to a disjoint union of at most \( N_{\varphi} \) instances of \( \delta \).

(ii) \( \varphi(x; \bar{y}) \) is \( T \)-equivalent to

\[
\bigvee_{n < N_{\varphi}} (\gamma_{\varphi, 2n}(x; \bar{y}) \land \neg \gamma_{\varphi, 2n+1}(x; \bar{y})) .
\]

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