The time-dependent coupled oscillator model for the motion of a charged particle in the presence of a time-varying magnetic field

Salah Menouar¹, Mustapha Maamache¹ and Jeong Ryeol Choi ²

¹ Laboratoire de Physique Quantique et Systèmes Dynamiques, Département de Physique, Faculté des Sciences, Université Ferhat Abbas de Sétif, Sétif 19000, Algeria
² Department of Radiologic Technology, Daegu Health College, Taejeon 1-dong, Buk-gu, Daegu 702–722, Republic of Korea

E-mail: menouar_salah@yahoo.fr and choiardor@hanmail.net

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Abstract
The dynamics of the time-dependent coupled oscillator model for the motion of a charged particle subjected to a time-dependent external magnetic field is investigated. We use the canonical transformation approach for the classical treatment of the system, whereas the unitary transformation approach is used in managing the system in the framework of quantum mechanics. For both approaches, the original system is transformed into a much more simple system that is the sum of two independent harmonic oscillators with time-dependent frequencies. We therefore easily identify the wavefunctions in the transformed system with the help of an invariant operator of the system. The full wavefunctions in the original system are derived from the inverse unitary transformation of the wavefunctions associated with the transformed system.

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1. Introduction
Time-dependent harmonic oscillators have attracted considerable interest in the literature thanks to their usefulness for describing the dynamics of many physical systems. After Bateman’s [1] proposition concerning the use of the time-dependent harmonic oscillator model for describing dissipative systems, much attention has been paid to the quantum behavior of non-conservative and nonlinear systems.

In the meantime, coupled oscillators have become powerful modeling tools and, consequently, are frequently used for modeling a wide range of physical phenomena. With further progress in research, one may be interested in knowing what would happen if the two-dimensional harmonic oscillator is elaborated through the coupling of two additive potentials. As far as we know, the first to deal with this issue were Kim et al [2–5] 30 years ago. Abdalla demonstrated how to treat time-dependent coupled oscillators in the context of quantum mechanics [6]. The propagator for time-dependent coupled and driven harmonic oscillators with time-varying frequencies and masses was investigated by Benamira and Ghechi [7] using path integral methods.

Among the various systems that can be modeled by time-dependent coupled oscillators, the dynamics of charged particle motion in the presence of time-varying magnetic fields has played an important role in condensed matter physics and plasma physics. There are plenty of applications for this system, such as magnetoresistance [8], the Aharonov–Bohm effect [9], magnetic confinement devices for fusion plasmas [10], electromagnetic lenses with variable magnetic fields [11], cyclotron resonance [12] and entanglement of a two-qubit Heisenberg XY model [13]. Although all of these problems are interesting, we can find their exact analytic solutions only for a few special cases due to their complex mathematical structures.

The quantum properties of a free electron, which has a time-dependent effective mass under the influence of an external magnetic field, have been investigated in both the
Landau and the symmetric gauges [14, 15]. Laroze and Rivera [16] studied the dynamical behavior of electrons in the presence of a uniform time-dependent magnetic field and presented the time evolution of the corresponding wavefunctions for the case when the initial state is a superposition of Landau levels. The propagators of a charged particle subjected to a time-dependent magnetic field were studied using the linear and the quadratic invariants [17].

Kim et al [2–5] proposed a problem: what would actually happen if two harmonic oscillators are coupled so that the potential becomes \( V(X_1, X_2) = \frac{1}{2}(c_1 X_1^2 + c_2 X_2^2 + c_3 X_1 X_2) \), where \( c_3 \) is a coupling constant? They studied the corresponding density matrix in order to establish the Wigner function. In this work, we are interested in the problem of a Hamiltonian that involves the coupling term \( X_1 X_2 \) in the presence of a magnetic field. This system can be regarded as a generalization of the Hamiltonian model given in [14, 18]. Although the coupling among two or more oscillators is one of the most basic concepts when dealing with gyroscopic motions, interactions and complex structures, the related theory has scarcely been developed so far. This class of coupled harmonic oscillators can be used for describing numerous physical systems, among which are the Bogoliubov transformation model of superconductivity [19], two-mode squeezed light [20] and the Lee model in quantum field theory [21]. One of the main focuses of the research by Zhang et al [22, 23] in connection with time-dependent coupled oscillators including the term \( X_1 X_2 \) was specific problems of time-dependent coupled electronic circuits.

We will use the invariant methods [24–26] to derive the exact wavefunctions for time-dependent coupled oscillators in a variable magnetic field. The invariant operator method for describing the quantum features of time-dependent harmonic oscillators was first introduced by Lewis [24] and has now become a very useful tool for developing quantum theory for the case where the Hamiltonian of a system is explicitly dependent on time.

In section 2, we formulate our problem by introducing a general time-dependent Hamiltonian describing the complicated motion of a charged particle in the presence of an arbitrary time-dependent magnetic field. The classical treatment of the system based on the canonical transformation method is presented in section 3. In section 4, quantum analysis of the system using the unitary transformation approach is presented. The unitary transformation method enables us to transform the original Hamiltonian (which is somewhat complicated) into a more simple system such as the ordinary harmonic oscillator. We derive the quantum solutions of the system in section 5 by starting with the invariant operator associated with the transformed system described in section 4. Finally, we give our concluding remarks in the last section.

2. Formulation of the problem

For the dynamical system of interest in this work, the Hamiltonian has the form

\[
H(X_1, X_2, t) = \frac{\Pi_1^2}{2m_1(t)} + \frac{\Pi_2^2}{2m_2(t)} + \frac{1}{2}(c_1(t)X_1^2 + c_2(t)X_2^2 + c_3(t)X_1X_2),
\]

where \( \Pi_1 \) and \( \Pi_2 \) are the conjugate momenta. Note that \( \Pi_1 \) and \( \Pi_2 \) can be simplified by choosing an appropriate gauge. Actually, in the symmetric gauge with \( \tilde{A}(\frac{-\hbar}{2m_1(t)} X_2, \frac{\hbar}{2m_2(t)} X_1, 0) \), they are given by

\[
\Pi_1 = P_1 - \frac{eB(t)}{2} X_2, \quad \Pi_2 = P_2 + \frac{eB(t)}{2} X_1.
\]

The parameters \( m_1(t), m_2(t), c_1(t), c_2(t) \) and \( c_3(t) \) are arbitrary functions of time, \( (X_1, X_2) \) is the pair of position variables and \( (P_1, P_2) \) are the canonical conjugate momentum variables.

The main difference between the present study and [16] is that we have considered the coupling term \( X_1 X_2 \) in the Hamiltonian. Regarding the expressions for \( \Pi_1 \) and \( \Pi_2 \), the Hamiltonian in equation (1) can be recast as

\[
H(X_1, X_2, t) = \frac{P_1^2}{2m_1(t)} + \frac{P_2^2}{2m_2(t)} + \frac{1}{2}(c_1(t)X_1^2 + c_2(t)X_2^2 + c_3(t)X_1X_2) + \frac{1}{2}(\omega_{1e}(t)P_2X_1 - \omega_{1c}(t)P_1X_2),
\]

where the new time-dependent functions \( c_1(t), c_2(t) \) and \( c_3(t) \) read as

\[
c_1(t) = C_1(t) + m_2(t) \frac{\omega_{1e}(t)}{4}, \quad c_2(t) = C_2(t) + m_1(t) \frac{\omega_{1c}(t)}{4}, \quad c_3(t) = C_3(t),
\]

with the cyclotron frequencies

\[
\omega_{1e}(t) = \frac{eB(t)}{m_1(t)}, \quad \omega_{1c}(t) = \frac{eB(t)}{m_2(t)}.
\]

3. Classical treatment

The time-dependent canonical transformation approach is in fact very powerful for investigating the properties of dynamical systems described by a time-dependent Hamiltonian. In many cases, we can convert a given Hamiltonian into a simple and desired one by means of canonical transformation. Therefore, in order to recast the solutions of this problem in a more soluble form, it is convenient to use the canonical transformation method.

To simplify the Hamiltonian given in equation (3), let us transform the variables \( (X_1, X_2, P_1, P_2) \) into new variables \( (x_1, x_2, p_{x_1}, p_{x_2}) \) such that

\[
x_1 = \left( \frac{m_1(t)}{m_2(t)} \right)^{1/4} X_1, \quad x_2 = \left( \frac{m_2(t)}{m_1(t)} \right)^{1/4} X_2,
\]

\[
p_{x_1} = \left( \frac{m_2(t)}{m_1(t)} \right)^{1/4} P_1, \quad p_{x_2} = \left( \frac{m_1(t)}{m_2(t)} \right)^{1/4} P_2.
\]

Replacing all the canonical variables in equation (3) with the above ones, we have

\[
H(x_1, x_2, t) = \frac{1}{2m(t)} (p_{x_1}^2 + p_{x_2}^2) + \frac{1}{2} \left( d_1(t)x_1^2 + d_2(t)x_2^2 + d_3(t)x_1x_2 \right) + \frac{\omega(t)}{2}(x_1 p_{x_2} - x_2 p_{x_1}).
\]
where \( d_1 \), \( d_2 \) and \( d_3 \) are new time-dependent functions of the form
\[
d_1(t) = c_1(t) \left( \frac{m_2(t)}{m_1(t)} \right)^{1/2},
\]
\[
d_2(t) = c_2(t) \left( \frac{m_1(t)}{m_2(t)} \right)^{1/2},
\]
\[
d_3(t) = c_3(t) = C_3(t),
\]
with the unique mass \( m(t) = (m_1(t)m_2(t))^{1/2} \) and the cyclotron frequency \( \omega_c(t) = (\omega_{c1}(t)\omega_{c2}(t))^{1/2} = eB(t)/m(t) \).

To simplify the Hamiltonian of equation (8), we perform the following canonical transformation:
\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    p_{x1} \\
    p_{x2}
\end{pmatrix} = \begin{pmatrix}
    \cos \phi(t) & \sin \phi(t) \\
    -\sin \phi(t) & \cos \phi(t)
\end{pmatrix} \begin{pmatrix}
    q_1 \\
    q_2 \\
    p_1 \\
    p_2
\end{pmatrix},
\]
where
\[
\phi(t) = -\frac{1}{2} \int \omega_c(t) \, dt.
\]
If \( (q_1, q_2, p_1, p_2) \) are the canonical coordinates, there should exist a new Hamiltonian \( H(q_1, q_2, t) \) that is determined only in terms of the Hamiltonian given in equation (8) with the aid of the linear transformation shown in equations (12) and (13). The variables \( (x_1, x_2, p_{x1}, p_{x2}) \) and \( (q_1, q_2, p_1, p_2) \) in two representations must satisfy the following relation [27]:
\[
p_1q_1 + p_2q_2 - H(q_1, q_2, t) = p_{x1}x_1 + p_{x2}x_2 - H(x_1, x_2, t) + \frac{\partial F_1}{\partial t},
\]
where \( F_1 \) is a time-dependent generating function in phase space, which should be determined afterwards.

From the fundamental equations known in classical mechanics [27],
\[
p_{x1} = \frac{\partial}{\partial x_1} F_1(x_1, x_2, p_1, p_2, t), \\
p_{x2} = \frac{\partial}{\partial x_2} F_1(x_1, x_2, p_1, p_2, t),
\]
the generating function associated with the transformation is found to be
\[
F_1(x_1, x_2, p_1, p_2, t) = (p_1 \cos \phi + p_2 \sin \phi) x_1 \\
+ (-p_1 \sin \phi + p_2 \cos \phi) x_2,
\]

In terms of the new conjugate variables \( (q_1, q_2, p_1, p_2) \), the Hamiltonian of equation (8) becomes
\[
H(q_1, q_2, t) = \frac{1}{2m(t)} \left( p_1^2 + p_2^2 \right) \\
+ \frac{1}{2} \left( \lambda_1(t)q_1^2 + \lambda_2(t)q_2^2 + \lambda_3(t)q_1q_2 \right),
\]
where
\[
\lambda_1(t) = d_1(t) \cos^2 \phi + d_2(t) \sin^2 \phi - d_3(t) \sin \phi \cos \phi,
\]
\[
\lambda_2(t) = d_2(t) \cos^2 \phi + d_1(t) \sin^2 \phi + d_3(t) \sin \phi \cos \phi,
\]
\[
\lambda_3(t) = 2 \left( d_1(t) - d_2(t) \right) \sin \phi \cos \phi + d_3(t)(\cos^2 \phi - \sin^2 \phi).
\]
To eliminate the coupling term \( q_1q_2 \), we now perform the following canonical transformation [7, 22, 23]:
\[
\begin{pmatrix}
    q_1 \\
    q_2 \\
    p_1 \\
    p_2
\end{pmatrix} = \begin{pmatrix}
    \cos \frac{\theta(t)}{2} & \sin \frac{\theta(t)}{2} \\
    -\sin \frac{\theta(t)}{2} & \cos \frac{\theta(t)}{2}
\end{pmatrix} \begin{pmatrix}
    Q_1 \\
    Q_2 \\
    P_1 \\
    P_2
\end{pmatrix},
\]
where \( \theta(t) \) is an arbitrary function of time. Note that equations (24) and (25) do not always represent the canonical transformation [27] between variables \( (q_1, p_1) \) (i = 1, 2) and \( (Q_i, P_i) \). If \( (Q_i, P_i) \) are canonical coordinates, there should exist a new Hamiltonian that is determined only by the Hamiltonian of equation (20) and the linear transformation given in equations (24) and (25). The relationship between the variables \( (q_i, p_i) \) and \( (Q_i, P_i) \) in the two representations is [27]
\[
\sum_{i=1}^{2} P_iQ_i - H_Q = \sum_{i=1}^{2} p_iq_i - H_q + \frac{\partial F}{\partial t},
\]
where \( F \) is another time-dependent generating function in phase space.

Using the basic equations
\[
p_i = \frac{\partial}{\partial q_i} F(q_1, q_2, P_1, P_2, t),
\]
\[
Q_i = \frac{\partial}{\partial P_i} F(q_1, q_2, P_1, P_2, t),
\]
where \( i = 1, 2 \), we find that the generating function is given by
\[
F(q_1, q_2, P_1, P_2, t) = \sqrt{m(t)} \left( P_1 \cos \frac{\theta(t)}{2} + P_2 \sin \frac{\theta(t)}{2} \right) q_1 \\
+ \sqrt{m(t)} \left( -P_1 \sin \frac{\theta(t)}{2} + P_2 \cos \frac{\theta(t)}{2} \right) q_2 - \frac{1}{4} \dot{m}(t) \left( q_1^2 + q_2^2 \right),
\]
(28)
Then, in terms of the new conjugate variables \((Q_1, P_1)\), the Hamiltonian can be represented in the form

\[
H_Q(Q_1, Q_2, t) = \frac{1}{2} (P_1^2 + P_2^2) + \frac{1}{2} \Omega_1^2(t) Q_1^2 + \frac{1}{2} \Omega_2^2(t) Q_2^2 \\
+ \delta(t) [P_1 Q_2 - P_2 Q_1] + \delta(t) Q_1 Q_2.
\]  

(29)

Here, the time-dependent coefficients \(\Omega_1(t)\), \(\Omega_2(t)\) and \(\delta(t)\) are given by

\[
\Omega_1(t) = \left( \frac{\omega_1^2(t)}{2} \cos^2 \left( \frac{\theta(t)}{2} \right) + \frac{\omega_2^2(t)}{2} \sin^2 \left( \frac{\theta(t)}{2} \right) - \frac{\lambda_3(t) \sin \theta(t)}{m(t)} \right)^{1/2},
\]

(30)

\[
\Omega_2(t) = \left( \frac{\omega_1^2(t)}{2} \sin^2 \left( \frac{\theta(t)}{2} \right) + \frac{\omega_2^2(t)}{2} \cos^2 \left( \frac{\theta(t)}{2} \right) + \frac{\lambda_3(t) \sin \theta(t)}{m(t)} \right)^{1/2},
\]

(31)

\[
\delta(t) = \frac{1}{2} (\omega_1^2(t) - \omega_2^2(t)) \sin \theta(t) + \frac{\lambda_3(t) \cos \theta(t)}{m(t)},
\]

(32)

where

\[
\omega_1^2(t) = \frac{\lambda_1(t)}{m(t)} + \frac{1}{4} \left( \frac{m_1(t)}{m(t)} - 2 \frac{\dot{m}(t)}{m(t)} \right),
\]

(33)

\[
\omega_2^2(t) = \frac{n_2(t)}{m(t)} + \frac{1}{4} \left( \frac{m_2(t)}{m(t)} - 2 \frac{\dot{m}(t)}{m(t)} \right).
\]

(34)

If we take the choice \(\theta(t) = \text{const.}\), the terms \(P_1 Q_2\) and \(P_2 Q_1\) in equation (29) will be cancelled out so that the Hamiltonian becomes

\[
H_Q(Q_1, Q_2, t) = \frac{1}{2} (P_1^2 + P_2^2) + \frac{1}{2} \Omega_1^2(t) Q_1^2 + \frac{1}{2} \Omega_2^2(t) Q_2^2 + \delta(t) Q_1 Q_2.
\]

(35)

Note that, with the above canonical transformation, the coupling \(\delta(t)\) is a function of the parameters of the original system. It is therefore clear that the separation of variables in equation (35) requires that \(\delta(t) = 0\), i.e.

\[
\lambda_3(t) = \left( \omega_1^2(t) - \omega_2^2(t) \right) m(t) \tan \theta,
\]

(36)

and consequently

\[
\tan \theta = \frac{l_3(t)}{m(t)(\omega_1^2(t) - \omega_2^2(t))}.
\]

(37)

By taking into account equation (36), the Hamiltonian in equation (35) is rewritten as

\[
H_Q(Q_1, Q_2, t) = \frac{1}{2} (P_1^2 + P_2^2) + \frac{1}{2} \Omega_1^2(t) Q_1^2 + \frac{1}{2} \Omega_2^2(t) Q_2^2.
\]

(38)

Then, equation (38) represents the sum of two independent Hamiltonians of the simple harmonic oscillators with the time-dependent frequencies \(\Omega_1(t)\) and \(\Omega_2(t)\).

\subsection*{4. Quantum treatment}

The canonical transformations in classical mechanics, treated in the previous section, are an analogue of the unitary transformations in quantum mechanics. In this section, we demonstrate the relationship between the two transformations and show how to obtain the quantum-mechanical Hamiltonian from the classical one. To manage the system in the context of quantum physics, we replace the canonical variables \((X_1, X_2)\) in equation (3) by quantum operators \((\hat{X}_1, \hat{X}_2)\). Then the corresponding Hamiltonian has the form

\[
\hat{H}(\hat{X}_1, \hat{X}_2, t) = \frac{\hat{P}_1^2}{2m_1(t)} + \frac{\hat{P}_2^2}{2m_2(t)} + \frac{1}{2} (c_1(t) \hat{X}_1^2 + c_2(t) \hat{X}_2^2) + \frac{1}{2} (c_3(t) \hat{X}_1 \hat{X}_2).
\]

(39)

In this quantum case, the pair of momentum operators is given by \((\hat{P}_1 = -i\hbar \partial / \partial X_1, \hat{P}_2 = -i\hbar \partial / \partial X_2)\). The Schrödinger equation in the original system is

\[
i \hbar \frac{\partial}{\partial t} \Psi(X_1, X_2, t) = \hat{H}(\hat{X}_1, \hat{X}_2, t) \Psi(X_1, X_2, t).
\]

(40)

To simplify the Hamiltonian in equation (39), we perform the unitary transformation such that

\[
\Psi(X_1, X_2, t) = \hat{U}_1(t) \psi(X_1, X_2, t),
\]

(41)

where \(\hat{U}_1(t)\) is a time-dependent unitary operator of the form

\[
\hat{U}_1(t) = \exp \left[ \frac{i}{2\hbar} \left( \hat{P}_1 \hat{X}_1 + \hat{X}_1 \hat{P}_1 \right) \ln \left( \frac{m_1(t)}{m_2(t)} \right)^{1/4} \right],
\]

\[
\times \exp \left[ \frac{i}{2\hbar} \left( \hat{P}_2 \hat{X}_2 + \hat{X}_2 \hat{P}_2 \right) \ln \left( \frac{m_2(t)}{m_1(t)} \right)^{1/4} \right].
\]

(42)

In this case, the Hamiltonian, equation (39), can be rewritten as

\[
\hat{H}_1(\hat{X}_1, \hat{X}_2, t) = \frac{1}{2m(t)} (\hat{P}_1^2 + \hat{P}_2^2) + \frac{1}{2} (d_1(t) \hat{X}_1^2 + d_2(t) \hat{X}_2^2 + d_3(t) \hat{X}_1 \hat{X}_2) + \frac{\hbar \omega(t)}{2} (\hat{P}_2 \hat{X}_1 - \hat{P}_1 \hat{X}_2).
\]

(43)

It is easy to confirm that the commutation relations, \([L_Z, \hat{X}_1^2 + \hat{X}_2^2] = 0\) and \([L_Z, \hat{P}_1^2 + \hat{P}_2^2] = 0\), hold \((L_Z\) is the angular momentum operator). This implies that there are common eigenfunctions between \(L_Z\) and \(\hat{X}_1^2 + \hat{X}_2^2\) and between \(L_Z\) and \(\hat{P}_1^2 + \hat{P}_2^2\). However, \(L_Z\) does not commute with \(\hat{X}_1, \hat{X}_2\): \([L_Z, \hat{X}_1, \hat{X}_2] \neq 0\) and consequently \([L_Z, \hat{H}] \neq 0\). If we regard that \(L_Z, \hat{H}\) do not have the same eigenfunctions, it is not possible to simplify the Schrödinger equation

\[
i \hbar \frac{\partial}{\partial t} \psi(X_1, X_2, t) = \hat{H}_1(\hat{X}_1, \hat{X}_2, t) \psi(X_1, X_2, t)
\]

(44)
by decomposing it. However, we can overcome this difficulty via the transformation of the Hamiltonian of equation (39) into a simple form by introducing appropriate unitary transformation operators. In the first step, we perform the following unitary transformation:

$$\tilde{\psi}(X_1, X_2, t) = \hat{U}_2(t)\psi(X_1, X_2, t),$$  \hspace{1cm} (45)

where

$$\hat{U}_2(t) = \exp \left( -\frac{i}{2\hbar} \left( \hat{P}_1 \hat{X}_1 - \hat{P}_1 \hat{X}_2 \right) \int \sigma_c(t) \, dt \right) \exp \left( -\frac{i}{\hbar} \hat{L}_Z \int \sigma_c(t) \, dt \right).$$  \hspace{1cm} (46)

Under this transformation, the Schrödinger equation (41) is mapped to

$$i\hbar \frac{\partial}{\partial t} \tilde{\psi}(X_1, X_2, t) = \hat{H}_2(\hat{X}_1, \hat{X}_2, t)\tilde{\psi}(X_1, X_2, t),$$  \hspace{1cm} (47)

where the new Hamiltonian $\hat{H}_2(\hat{X}_1, \hat{X}_2, t)$ has the form

$$\hat{H}_2(\hat{X}_1, \hat{X}_2, t) = \frac{1}{2m(t)}(\hat{P}_1^2 + \hat{P}_2^2) + \frac{1}{2}(\lambda_1(t)\hat{X}_1^2 + \lambda_2(t)\hat{X}_2^2 + \lambda_3(t)\hat{X}_1\hat{X}_2).$$  \hspace{1cm} (48)

Now, the term involving $\hat{L}_Z$ has disappeared. This means that the magnetic field term is removed in the new frame rotating with the time-dependent phase $\phi(t) = -\frac{1}{2} \int \sigma_c(t) \, dt$.

To decouple the Hamiltonian of equation (48), we take another unitary transformation such that

$$\psi(X_1, X_2, t) = \hat{V}(t)\chi(X_1, X_2, t),$$  \hspace{1cm} (49)

where the unitary operator $\hat{V}(t)$ is given by

$$\hat{V}(t) = \hat{V}_1(t)\hat{V}_2(t)\hat{V}_3(t),$$  \hspace{1cm} (50)

with

$$\hat{V}_1(t) = \exp \left( \frac{i}{\hbar} \left( \hat{P}_1 \hat{X}_1 + \hat{X}_1 \hat{P}_1 \right) \right) \exp \left( \frac{i}{\hbar} \left( \hat{P}_2 \hat{X}_2 + \hat{X}_2 \hat{P}_2 \right) \right),$$  \hspace{1cm} (51)

$$\hat{V}_2(t) = \exp \left( -\frac{i}{\hbar} \frac{\theta}{2} \left( \hat{P}_2 \hat{X}_1 - \hat{P}_1 \hat{X}_2 \right) \right),$$  \hspace{1cm} (52)

$$\hat{V}_3(t) = \exp \left( -\frac{i}{4\hbar} \frac{\theta}{2} \left( \hat{X}_1^2 + \hat{X}_2^2 \right) \right).$$  \hspace{1cm} (53)

After some algebra, the substitution of equations (48) and (49) into equation (47) yields a transformed Hamiltonian that represents the sum of two uncoupled simple harmonic oscillators with frequencies $\Omega_1(t)$ and $\Omega_2(t)$ and the unit mass

$$\hat{H}_3(\hat{X}_1, \hat{X}_2, t) = \hat{V}^{-1}(t)\hat{H}_2(\hat{X}_1, \hat{X}_2, t)\hat{V}(t) - i\hbar \hat{V}^{-1}(t) \frac{\partial}{\partial t} \hat{V}(t)$$

$$= \frac{1}{2} (\hat{P}_1^2 + \hat{P}_2^2) + \frac{1}{2} \Omega_1^2(t)\hat{X}_1^2 + \frac{1}{2} \Omega_2^2(t)\hat{X}_2^2.$$  \hspace{1cm} (54)

At this stage, it is possible to confirm that the classically transformed Hamiltonian given in equations (38) is correct, since the above equation is consistent with it. Note that $\hat{U}_1(t)$ and $\hat{V}_1(t)$ given in equations (42) and (51) are the squeeze operators, whereas $\hat{U}_2(t)$ and $\hat{V}_2(t)$ given in equations (46) and (52) are the rotation operators characterized by the time-varying angles $\phi(t)$ and $\theta(t)/2$, respectively.

5. Quantum solutions

It can be seen that there exists an invariant for the harmonic oscillator with time-dependent mass and/or frequency [24]. In our case, the transformed system consists of two independent harmonic oscillators that have a time-dependent frequency. It is easy to verify, from the Liouville–von Neumann equation for the invariant $\hat{I}$ given by

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial \hat{\rho}} + \frac{1}{i\hbar}[\hat{I}, \hat{H}]=0,$$  \hspace{1cm} (55)

that the invariant associated with the Hamiltonian of a two-dimensional harmonic oscillator is

$$\hat{I}(\hat{X}_1, \hat{X}_2, t) = \hat{I}(\hat{X}_1, t) + \hat{I}(\hat{X}_2, t)$$

$$= \frac{1}{2} \left( \frac{\hat{X}_1}{\rho_1} \right)^2 + (\rho_1\hat{X}_1 - \rho_1\hat{X}_1^2)$$

$$+ \frac{1}{2} \left( \frac{\hat{X}_2}{\rho_2} \right)^2 + (\rho_2\hat{X}_2 - \rho_2\hat{X}_2^2),$$  \hspace{1cm} (56)

where $\rho_1(t)$ and $\rho_2(t)$ are $c$-number quantities obeying the auxiliary equations

$$\dot{\rho}_1 + \Omega_1^2(t)\rho_1 = 1/\rho_1^3,$$  \hspace{1cm} (57)

$$\dot{\rho}_2 + \Omega_2^2(t)\rho_2 = 1/\rho_2^3.$$  \hspace{1cm} (58)

To guarantee the Hermiticity of equation (56) ($\hat{I}^\dagger = \hat{I}$), we choose only the real solutions of the above two equations. It is clear that $I(\hat{X}_1, \hat{X}_2, t)$ satisfies the Liouville–von Neumann equation. We now derive a complete orthonormal set of eigenfunctions $\xi_{n_1n_2}(X_1, X_2, t)$ of $\hat{I}(\hat{X}_1, \hat{X}_2, t)$ to form the eigenvalue equation

$$\hat{I}(\hat{X}_1, \hat{X}_2, t)\xi_{n_1n_2}(X_1, X_2, t) = \lambda_{n_1n_2}\xi_{n_1n_2}(X_1, X_2, t),$$  \hspace{1cm} (59)

where $\lambda_{n_1n_2}$ are time-independent eigenvalues. Through a straightforward evaluation after inserting equation (56) into the above equation, we obtain the eigenvalues and eigenfunctions such that

$$\lambda_{n_1n_2} = \hbar \left( n_1 + \frac{1}{2} \right) + \hbar \left( n_2 + \frac{1}{2} \right),$$  \hspace{1cm} (60)

$$\xi_{n_1n_2}(X_1, X_2, t) = \left[ \frac{1}{\sqrt{\pi n_1!n_2!2^{n_1n_2}\rho_1\rho_2}} \right]^{1/2} H_{n_1} \left( \frac{X_1}{\hbar^{1/2}\rho_1} \right) \times H_{n_2} \left( \frac{X_2}{\hbar^{1/2}\rho_2} \right) \exp \left[ \frac{i}{2\hbar} \left( \frac{\rho_1}{\rho_1} + \frac{i}{2\hbar} \frac{\rho_1}{\rho_1} \right) X_1^2 + \frac{i}{2\hbar} \left( \frac{\rho_2}{\rho_2} + \frac{i}{2\hbar} \frac{\rho_2}{\rho_2} \right) X_2^2 \right],$$  \hspace{1cm} (61)
where $H_{n_1}$ and $H_{n_2}$ are the usual Hermite polynomials of order $n_1$ and $n_2$, respectively.

The solutions of the Schrödinger equation

$$i\hbar \frac{\partial \chi_{n_1 n_2}(X_1, X_2, t)}{\partial t} = \hat{H}_3(\hat{X}_1, \hat{X}_2, t)\chi_{n_1 n_2}(X_1, X_2, t),$$

(62)

can be written as

$$\chi_{n_1 n_2}(X_1, X_2, t) = e^{i\alpha_{n_1 n_2}(t)}\xi_{n_1 n_2}(X_1, X_2, t),$$

(63)

where the phase functions $\alpha_{n_1 n_2}(t)$ satisfy the equation

$$\frac{\partial}{\partial t} \alpha_{n_1 n_2}(t) = \frac{1}{\hbar} \left[ \xi_{n_1 n_2}(X_1, X_2, t) \right] \frac{\partial}{\partial t} - \hat{H}_3(\hat{X}_1, \hat{X}_2, t)\xi_{n_1 n_2}(X_1, X_2, t).$$

(64)

According to equations (61) and (63), the solutions $\chi_{n_1 n_2}(X_1, X_2, t)$ of the Schrödinger equation (62) in the transformed system become

$$\chi_{n_1 n_2}(X_1, X_2, t) = e^{i\alpha_{n_1 n_2}(t)} \left[ \frac{1}{\pi \hbar n_1 n_2^{2n_1+n_2} \rho_1 \rho_2} \right]^{1/2}$$

$$\times H_{n_1}\left( \frac{X_1}{\hbar^{1/2} \rho_1} \right) H_{n_2}\left( \frac{X_2}{\hbar^{1/2} \rho_2} \right)$$

$$\times \exp \left[ \frac{i}{2\hbar} \left( \frac{\rho_1}{\rho_1} + \frac{i}{\rho_1} \right) X_1^2 + \frac{i}{2\hbar} \left( \frac{\rho_2}{\rho_2} + \frac{i}{\rho_2} \right) X_2^2 \right],$$

(65)

where the time-dependent phase functions are given by

$$\alpha_{n_1 n_2}(t) = -\left( n_1 + \frac{1}{2} \right) \int_0^t \frac{dr'}{\rho_1^2(r')} - \left( n_2 + \frac{1}{2} \right) \int_0^t \frac{dr'}{\rho_2^2(r')}.$$  

(66)

The relationship between the wavefunctions, $\Psi_{n_1 n_2}(X_1, X_2, t)$, in the original system described by the Hamiltonian of equation (3) and the wavefunction $\chi_{n_1 n_2}(X_1, X_2, t)$ in the transformed system is

$$\Psi_{n_1 n_2}(X_1, X_2, t) = \hat{U}_1(t)\hat{U}_2(t)\hat{V}(t)\chi_{n_1 n_2}(X_1, X_2, t)$$

$$= \hat{U}_1(t)\hat{U}_2(t)\hat{V}_1(t)\hat{V}_2(t)\chi_{n_1 n_2}(X_1, X_2, t).$$

(67)

Using equations (42), (46), (50) and (65), we derive the full-wave functions in the form

$$\Psi_{n_1 n_2}(X_1, X_2, t) = \left[ \frac{\sqrt{m_1 m_2}}{\pi \hbar n_1 n_2^{2n_1+n_2} \rho_1 \rho_2} \right]^{1/2}$$

$$\times H_{n_1}\left( \frac{\sqrt{m_1} \cos(\phi + \theta/2)}{\hbar^{1/2} \rho_1} X_1 - \sqrt{m_2} \sin(\phi + \theta/2) X_2 \right)$$

$$\times H_{n_2}\left( \frac{\sqrt{m_1} \sin(\phi + \theta/2)}{\hbar^{1/2} \rho_2} X_1 + \sqrt{m_2} \cos(\phi + \theta/2) X_2 \right)$$

$$\times \exp \frac{im_1}{2\hbar} \left( \frac{\gamma}{2} + \frac{\beta}{2} + \left( \frac{\beta}{2} - \frac{\gamma}{2} \right) \sin(\theta + 2\phi) \right) X_1^2$$

$$\times \exp \frac{im_2}{2\hbar} \left( \frac{\gamma}{2} + \frac{\beta}{2} - \left( \frac{\beta}{2} - \frac{\gamma}{2} \right) \sin(\theta + 2\phi) \right) X_2^2,$$

(68)

where the time-dependent coefficients $\gamma(t)$ and $\beta(t)$ are given as

$$\gamma(t) = \left( \frac{\rho_1}{\rho_1} + \frac{i}{\rho_1} \right) - \frac{1}{2} \frac{d}{dt} \left( \frac{1}{\sqrt{m_1 m_2}} \right),$$

(69)

$$\beta(t) = \left( \frac{\rho_2}{\rho_2} + \frac{i}{\rho_2} \right) - \frac{1}{2} \frac{d}{dt} \left( \frac{1}{\sqrt{m_1 m_2}} \right).$$

(70)

The full solutions in the original system, given in equation (68), are exact because we did not use approximation or perturbation methods. Although these solutions are somewhat complicated, they are very useful in predicting the quantum behavior of the system. A merit of such analytical solutions is that they can be employed for deriving the evolution of the probability distribution, regardless of the change in the system’s parameters. However, the numerical solutions in this field, such as the one obtained from the finite difference time domain (FDTD) method [28], are somewhat inconvenient as inputs for further analyses, since one should recalculate the results whenever the parameters of the system change. Using equation (68), one can easily make a complete description of the charged particle motion even when the parameters of the system vary from time to time provided that the classical solutions of equations (57) and (58) are known.

6. Conclusion

We investigated the quantal problem of the time-dependent coupled oscillator model associated with the charged particle motion in the presence of a time-dependent magnetic field. Although the behavior of a charged particle in a magnetic field drew a great deal of attention from both quantum and classical viewpoints, research in this line is currently concentrated more on static problems that can be modeled by a time-independent harmonic oscillator.

The system treated in this work, however, is a more generalized one. It is summarized as follows:

(i) We supposed that the effective mass of a charged particle varies explicitly with time under the influence of a time-dependent magnetic field. If electrons or holes in the condensed matter interact with the environment or various excitations such as pressure, energy, temperature and stress, then their effective mass may naturally vary with time [14]. Moreover, random changes in the external field in the heterojunctions and solid solutions give rise to a variation of effective mass in accordance with a fluctuation in the composition of the system [29].

(ii) We let the external magnetic field $B(t)$ be an arbitrary function of time. Therefore, the application of our theory is not confined to a special system that has a specific class of time dependence for $B(t)$. In fact, we can apply it to a wide range of practical systems with a flexible choice of the type of $B(t)$. 

(iii) Our system is further generalized by adding a coupling term $X_1X_2$ to the Hamiltonian.

Through these generalizations, the system became a somewhat complicated one that is described in terms of the time-dependent Hamiltonian. Since the treatment of the original Hamiltonian system is not an easy task in this case, we transformed our system into a much more simplified one by using two different techniques. In the first one, we carried out canonical transformations in order to simplify the problem relevant to the original classical Hamiltonian given in equation (1). After the transformation, the Hamiltonian reduced to a simple form associated with two uncoupled harmonic oscillators that each have time-dependent frequencies $\Omega_1(t)$ and $\Omega_2(t)$. In the second technique, we used an alternative approach based on the unitary transformation method. With the choice of unitary operators $\hat{U}_1(t)$, $\hat{U}_2(t)$ and $\hat{V}(t)$, the quantum Hamiltonian (39) was transformed into an equally simple one as that of the canonical transformation performed previously, but within the realm of quantum mechanics.

Since the Hamiltonian in the transformed system is very simple, we easily constructed a dynamical invariant operator $\hat{I}(\hat{X}_1, \hat{X}_2, t)$ associated with the transformed system, as given in equation (55). The eigenstates $\xi_{n_1n_2}(X_1, X_2, t)$ of this invariant operator are represented in terms of the Hermite polynomial. The Schrödinger solutions $\chi_{n_1n_2}(X_1, X_2, t)$ in the transformed system are the same as $\xi_{n_1n_2}(X_1, X_2, t)$ except for the time-dependent phase factor $e^{i\omega_{n_1n_2}t}$. From the inverse transformation of $\chi_{n_1n_2}(X_1, X_2, t)$ with unitary operators, we derived the full wavefunctions (quantum solutions) in the original system (see equation (68)). The quantum solutions are expressed in terms of $\rho_1$ and $\rho_2$, which are the two independent solutions of the classical equation of motion given in equations (56) and (57), respectively. Although we represented the quantum solutions in terms of the classical solutions associated with the transformed system, it is also possible to represent them in terms of the classical solutions associated with the original system. The wavefunctions given in equation (68) can be used to investigate various quantum properties of the system such as the fluctuations of canonical variables, the evolution of quantum energy and probability densities even when the parameters of the system vary from time to time. This is the advantage of such analytical solutions over numerical solutions obtained, for example, using the FDTD method [28].

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